Some infinite dimensional representations of reductive groups with Frobenius maps

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Abstract In this paper, we construct certain irreducible infinite dimensional representations of algebraic groups with Frobenius maps. In particular, a few classical results of Steinberg and Deligne & Lusztig on complex representations of finite groups of Lie type are extended to reductive algebraic groups with Frobenius maps.

Keywords infinite dimensional representation, reductive group, induced module

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0 Introduction

The construction of induced representations of Frobenius for finite groups has various generalizations for infinite groups. It seems that for infinite groups of Lie type the original form of construction of Frobenius was not used much. In this paper, we try to study abstract representations of algebraic groups by using the original construction of Frobenius directly. We are mainly interested in reductive groups with Frobenius maps. A few classical results of Steinberg and Deligne & Lusztig on complex representations of finite groups of Lie type are extended to reductive algebraic groups with Frobenius maps, see Propositions 2.3 and 2.4, Theorems 3.2 and 3.4, etc.

The paper is organized as follows. In Section 1 we give some trivial extensions for several results in representation theory of finite groups and introduce the concept of quasi-finite groups (see Subsection 1.8). A few general results on irreducibility of a representation for a quasi-finite group are established, if the representation is a “limit” of the irreducible representations (Lemmas 1.5 and 1.6). A partial generalization of Mackeys criterion on irreducibility of induced modules for quasi-finite groups is given (see Subsection 1.9).

In Section 2 we consider algebraic groups with split $BN$-pairs. The main objects of this section are induced representations of certain one-dimensional representations of a Borel subgroup of an algebraic group with split $BN$-pairs. In particular, the Steinberg module of the algebraic group is constructed (Proposition 2.3(b)). In Section 3 we consider reductive groups with Frobenius maps. The main results are Theorems 3.2 and 3.4. The first one says that the Steinberg module of a reductive group over an algebraically closed field of positive characteristic is irreducible when the base field of the Steinberg
module is the field of complex numbers or the ground field of the reductive group, the second one says that the induced representations of certain one-dimensional complex representations of a Borel subgroup are irreducible.

Gelfand-Graev modules of reductive groups are defined in Section 4, which are similar to those for finite groups of Lie type. In Section 5 a few questions are raised. In Section 6 we discuss type $A_1$. Section 7 is devoted to discussing representations of some infinite Coxeter groups and infinite dimensional groups of Lie type.

This work was partially motivated by trying to find an algebraic counterpart for Lusztig’s theory of character sheaves, the author is grateful to Professor G. Lusztig for his series of lectures on character sheaves delivered at the Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, in 2012.

1 General setting

1.1. In this section, we give some trivial extensions for several results in representation theory of finite groups. There are many good references, say [10] and [4].

Let $H$ be a subgroup of a group $G$ and $k$ a field. In this section all modules are assumed to be over $k$. For an $H$-module $M$, we can consider the naive induced module of $M$: $\text{Ind}_H^G M = kG \otimes_{kH} M$, where $kG$ and $kH$ are the group algebras of $G$ and $H$ over the field $k$ respectively.

As the case of finite groups, we can define the induced module in another way. Let $\mathcal{M}$ be the set of all functions $f : G \to M$ satisfying $f(gh) = h^{-1}f(g)$ for any $g \in G$ and $h \in H$. For $f \in \mathcal{M}$ and $x \in G$, set $xf(g) = f(x^{-1}g)$. This defines a $kG$-module structure on $\mathcal{M}$.

Let $G/H$ be the set of left cosets of $H$ in $G$. A function $f : G \to M$ is called to have finite support on $G/H$ if all $f(gH) = 0$ except for finitely many left cosets of $H$ in $G$. Let $\mathcal{M}_0$ be the subset of $\mathcal{M}$ consisting of all functions $f$ in $\mathcal{M}$ with finite support on $G/H$. It is clear that $\mathcal{M}_0$ is a $kG$-submodule of $\mathcal{M}$. The following result is known for finite groups.

Lemma 1.2. The $kG$-module $\mathcal{M}_0$ is isomorphic to the induced module $kG \otimes_{kH} M$.

Proof. Let $\{g_i\}_{i \in I}$ be a set of representatives of left cosets of $H$ in $G$. Then the map $f \mapsto \sum_i g_i \otimes f(g_i)$ defines an isomorphism of $kG$-module from $\mathcal{M}_0$ to $kG \otimes_{kH} M$.

The induced modules above are extremely important in representation theory of finite groups and Lie algebras, but seem not studied much for infinite groups of Lie type. We have some trivial properties for these induced modules, such as Frobenius’s reciprocity, etc..

Lemma 1.3. (a) Let $M$ be an $H$-module and $N$ be a $G$-module. Then we have

$$\text{Hom}_G(\text{Ind}_H^G M, N) \simeq \text{Hom}_H(M, \text{Res}_H N) \quad \text{and} \quad \text{Ind}_H^G(M \otimes \text{Res}_H N) \simeq \text{Ind}_H^G M \otimes N,$$

where $\text{Res}_H$ denotes the restriction functor from $G$-modules to $H$-modules.

(b) Let $H \subset K$ be subgroups of $G$ and $M$ an $H$-module. Then $\text{Ind}_K^G(\text{Ind}_H^G M)$ is isomorphic to $\text{Ind}_H^G M$.

The following result should be known.

Lemma 1.4. Assume that $G$ is commutative and each element of $G$ has finite order. If $k$ is algebraically closed, then any irreducible representation of $G$ over $k$ is one-dimensional.

Proof. Let $M$ be an irreducible $kG$-module. By Schur lemma, $\text{End}_G M$ is a division algebra over $k$. Since $G$ is commutative, for any $x \in G$, the map $\varphi_x : M \to M$, $a \mapsto xa$ is a $G$-homomorphism. Since $x$ has finite order, $\varphi_x$ is algebraic over $k$. Now $k$ is algebraically closed, so $\varphi_x$ must be in $k$, i.e., $x$ acts on $M$ by multiplication of a scalar in $k$. Thus $M$ must be one-dimensional since $M$ is irreducible. The lemma is proved.

Remark. The author is grateful to Binyong Sun for pointing out that the lemma above cannot be extended to arbitrary commutative groups, say, the field $\mathbb{C}$ is an irreducible representation of $\mathbb{C}^*$, but it is of infinite dimension over $\mathbb{Q}$. 
Lemma 1.5. Let $(I, \preceq)$ be a directed set, $\{A_i, f_{ij}\}$ be a direct system of algebras over $k$ (resp. groups) and $\{M_i, \varphi_{ij}\}$ be a direct system of vector spaces over $k$. Assume that $M_i$ is $A_i$-module for each $i$ and for any $i, j \in I$ with $i \preceq j$, the homomorphism $\varphi_{ij} : M_i \rightarrow M_j$ is compatible with the homomorphism $f_{ij} : A_i \rightarrow A_j$, i.e., $\varphi_{ij}(ux) = f_{ij}(u)\varphi_{ij}(x)$ for any $u \in A_i$ and $x \in M_i$. Then $M = \varinjlim M_i$ is naturally a module of $A = \varinjlim A_i$. Moreover, if $M_i$ is irreducible $A_i$-module for each $i \in I$, then $M$ is irreducible $A$-module.

Proof. The proof is easy. For convenience, we give the details. Let $\bigsqcup M_i$ be the disjoint union of all $M_i$. By definition, $M = \bigsqcup M_i / \sim$, here for $x' \in M_i$ and $y' \in M_j$, $x' \sim y'$ if and only if there exists some $r \in I$ such that $\varphi_{ir}(x') = \varphi_{jr}(y')$.

Let $x \in M$ (resp. $u \in A$). Choose $x' \in M_i$ (resp. $u' \in A_i$) such that $x' \sim y'$ (resp. $u \sim u'$). Choose $r \in I$ such that $i \preceq r$ and $j \preceq r$. Then set $ux$ to be the class containing $f_{ir}(u')\varphi_{ir}(x')$. One can verify that this defines an $A$-module structure on $M$.

Assume that $M_i$ is irreducible $A_i$-module for each $i$. To prove that $M$ is irreducible $A$-module it suffices to prove that $M = Ax$ for any nonzero element $x$ in $M$. Assume that $x$ and $y$ are two nonzero elements in $M$. Let $x' \in M_i$ (resp. $y' \in M_j$) be an element in the equivalence class $x$ (resp. $y$). Choose $r \in I$ such that $i \preceq r$ and $j \preceq r$. Then $x'' = \varphi_{ir}(x')$ (resp. $y'' = \varphi_{jr}(y')$) is in the class $x$ (resp. $y$).

Since $M_i$ is irreducible $A_i$-module, there exists $u' \in A_i$ such that $x'' = u'y''$. Let $u$ be the equivalence class containing $u'$. Then $u$ is element of $A$ and $ux = y$. The lemma is proved.

The following two simple lemmas will be used frequently.

Lemma 1.6. (a) Let $A$ be an algebra over $k$ and $M$ be an $A$-module. Assume that $A$ has a sequence of subalgebras $A_1, A_2, \ldots, A_n, \ldots$ and that $M$ has a sequence of $k$-subspaces $M_1, M_2, \ldots, M_n, \ldots$ such that $M$ is the union of all $M_i$, and for any positive integers $i, j$ there exists a positive integer $r$ such that $M_i$ and $M_j$ are contained in $M_r$. If $M_i$ is an irreducible $A_i$-submodule of $M$ for any $i$, then $M$ is an irreducible $A$-module.

(b) Let $G$ be a group and $M$ be a $G$-module. Assume that $G$ has a sequence of subgroups $G_1, G_2, \ldots, G_n, \ldots$ and that $M$ has a sequence of $k$-subspaces $M_1, M_2, \ldots, M_n, \ldots$ such that $M$ is the union of all $M_i$, and for any positive integers $i, j$ there exists an integer $r$ such that $M_i$ and $M_j$ are contained in $M_r$. If $M_i$ is an irreducible $G_i$-submodule of $M$ for any $i$, then $M$ is an irreducible $G$-module.

Proof. (a) We only need to prove that $M = Ax$ for any nonzero element $x$ in $M$. Assume that $x$ and $y$ are two nonzero elements in $M$. Then we can find some positive integer $i$ such that both $x$ and $y$ are contained in $M_i$. Since $M_i$ is irreducible $A_i$-submodule of $M$, there exists $u$ in $A_i$ such that $ux = y$. Therefore $M = Ax$.

(b) Applying (a) to $A = kG$ and $A_i = k[G_i]$, we see that (b) is a special case of (a).

The lemma is proved.

Lemma 1.7. Let $H$ be a subgroup of $G$ and $M$ a $kH$-module. Assume that $G$ has a sequence $G_1, G_2, \ldots, G_n, \ldots$ of subgroups such that $G$ is the union of all $G_i$, and for any positive integers $i, j$ there exists an integer $r$ such that $G_i$ and $G_j$ are contained in $G_r$. Then the following results hold:

(a) As $G_i$-modules, $kG_i \otimes_{k(G_i \cap H)} M$ is isomorphic to the $G_i$-submodule $Y_i$ of $kG \otimes_{kH} M$ generated by all $x \otimes m$, where $x \in kG_i$ and $m \in M$.

(b) $kG \otimes_{kH} M$ is the union of all $Y_i$.

(c) $kG \otimes_{kH} M$ is irreducible if each $Y_i$ is irreducible $G_i$-module.

(d) Let $M_i$ be an $H_i = H \cap G_i$-submodule of $M$. Then we have a natural homomorphism of $G_i$-module $\varphi_i : kG_i \otimes_{kH_i} M_i \rightarrow kG \otimes_{kH} M$. If $M$ is the union of all $M_i$ and $M_i$ is a subspace of $M_j$ whenever $G_i$ is a subgroup of $G_j$, then $kG \otimes_{kH} M$ is the union of all the images $\text{Im} \varphi_i$.

Proof. (a), (b) and (d) are clear, (c) follows from (b) and Lemma 1.6(b).

1.8. Let $A$ be an algebra over a field $k$. Assume that $A$ has a sequence of subalgebras $A_1, A_2, \ldots, A_n, \ldots$ such that $A$ is the union of all $A_i$, and for any positive integers $i, j$ there exists an integer $r$ such that $A_i$ and $A_j$ are contained in $A_r$. We can consider a category $\mathcal{F}$ of $A$-modules whose objects are those $A$-modules $M$ with a finite dimensional $A_i$-submodule $M_i$ for each $i$ such that $M$ is the union of all $M_i$ and
for any positive integers $i, j, M_i$ and $M_j$ are contained in $M_r$ whenever both $A_i$ and $A_j$ are contained in $A_r$. Let $(M, M_i)$ and $(N, N_i)$ be two objects in $\mathcal{F}$. The morphisms from $(M, M_i)$ to $(N, N_i)$ are just those homomorphisms of $A$-module from $M$ to $N$ such that $f(M_i) \subset N_i$ for all $i$. Clearly $\mathcal{F}$ is an abelian category.

Let $A$ be as above. We say that $A$ is quasi-finite if all $A_i$ are finite dimensional over $k$. Similarly we say that a group $G$ is quasi-finite if $G$ has a sequence $G_1, G_2, \ldots, G_n, \ldots$ of finite subgroups such that $G$ is the union of all $G_i$ and for any positive integers $i, j$ there exists an integer $r$ such that $G_i$ and $G_j$ are contained in $G_r$. The sequence $G_1, G_2, G_3, \ldots$ is called a quasi-finite sequence of $G$. A subgroup of a quasi-finite group is clearly quasi-finite. Clearly if a group $G$ is quasi-finite then the group algebra $kG$ is a quasi-finite algebra over $k$.

**Example.** (1) Let $W_n$ be a Weyl group of one type $A_n$ (resp. $B_n$ $(n \geq 2)$, $D_n$ $(n \geq 4)$). Then we have a canonical imbedding $W_n \rightarrow W_{n+1}$. Let $W_\infty = \bigcup_n W_n$. Then $W_\infty$ is a quasi-finite group and is also a Coxeter group. (2) Let $\mathbb{F}_q$ be a finite field of $q$ elements and $\overline{\mathbb{F}}_q$ be its algebraic closure. The additive group of $\overline{\mathbb{F}}_q$ is quasi-finite and is the union of all $\mathbb{F}_{q^a}$, $a = 1, 2, \ldots$ Also the multiplication group $\overline{\mathbb{F}}_q^\times$ is quasi-finite and is the union of all $\mathbb{F}_{q^a}^\times$, $a = 1, 2, \ldots$

(3) Let $G$ be an algebraic group defined over $\mathbb{F}_q$. By (2) we see that the $\mathbb{F}_q$-points $G(\overline{\mathbb{F}}_q)$ of $G$ is quasi-finite and is the union of all $G(q^a)$, $a \geq 1$, where $G(q^a)$ is the $\mathbb{F}_{q^a}$-points of $G$.

(4) Let $G_n$ be $GL_n(k)$ (resp. $SL_n(k)$, $SO_{2n}(k)$, $SO_{2n+1}(k)$, $Sp_{2n}(k)$). Then $G_n$ is naturally embedded into $G_{n+1}$. Let $G_{\infty}$ be the union of all $G_n$. If $k$ is finite then $G_\infty$ is quasi-finite.

More generally, direct union of quasi-finite groups is also quasi-finite, in particular, $G_\infty$ is quasi-finite if $k = \overline{\mathbb{F}}_q$. (The author is grateful to a referee for pointing out this fact.)

In the rest of this section we assume that all groups are quasi-finite unless other specifications are given. For a quasi-finite group $G$, we fix a quasi-finite sequence $G_1, G_2, G_3, \ldots$ For a subgroup $H$ of $G$, the quasi-finite sequence of $H$ is chosen to be $H \cap G_1, H \cap G_2, H \cap G_3, \ldots$, called the quasi-finite sequence of $H$ induced from the given quasi-finite sequence of $G$.

Assume that $N$ is a finitely generated $G$-module, say, generated by $x_1, \ldots, x_n$. For each positive integer $i$, let $N_i$ be the $G_i$-submodule of $M$ generated by $x_1, \ldots, x_n$. Then $N_i$ is a subspace of $N_j$ if $G_i$ is a subgroup of $G_j$, and $N$ is the union of all $N_i$.

We shall say that an irreducible module (or representation) $N$ of $G$ is quasi-finite (with respect to the quasi-finite sequence $G_1, G_2, G_3, \ldots$) if it has a sequence of subspaces $N_1, N_2, N_3, \ldots$ of $N$ such that (1) each $N_i$ is an irreducible $G_i$-submodule of $N$, (2) if $G_i$ is a subgroup of $G_j$, then $N_i$ is a subspace of $N_j$, and (3) $N$ is the union of all $N_i$. The sequence $N_1, N_2, N_3, \ldots$ will be called a quasi-finite sequence of $N$. If the intersection $\bigcap_i N_i$ of all $N_i$ is nonzero, then a nonzero element in the intersection $\bigcap_i N_i$ will be called primitive since such an element generates an irreducible $G_i$-submodule of $N$ for any $i$. It is often that $G_1$ is a subgroup of all $G_i$, in which case $N_1$ is the intersection of all $N_i$ and any nonzero element in $N_1$ is primitive.

**Question 1.** Is every irreducible $G$-module quasi-finite (with respect to a certain quasi-finite sequence of $G$)?

When the irreducible module $N$ is finite dimensional, the answer is affirmative, since the map $kG = \bigcup_i kG_i \rightarrow \text{End}_kN$ is surjective and $\text{End}_kN$ is finite dimensional. A weak version of the above question is the following.

**Question 2.** Assume that $N$ is an irreducible $G$-module. Does there exist an irreducible $G_i$-submodule $N_i$ of $N$ for each $i$ such that $N$ is the union of all $N_i$.

In the rest of this section $k$ has characteristic $0$.

**1.9.** For quasi-finite groups, a partial generalization of Mackey’s criterion on irreducibility is stated as follows.

Let $G$ be a quasi-finite group and $H$ be a subgroup of $G$. Let $M$ be a $kH$-module. Then $\text{Ind}_H^G M$ is irreducible $G$-module if the following two conditions are satisfied:
(1) $M$ is quasi-finitely irreducible (with respect to the quasi-finite sequence of $H$ induced from the given quasi-finite sequence of $G$).

(2) Let $M_1, M_2, M_3, \ldots$ be a quasi-finite sequence of $M$. For any positive integer $i$ and $s \in G_i - H \cap G_i$, the two representations $M_i, s$ and $M_i$ of $H_{s,i} = sH^{-1} \cap H \cap G_i$ have no common composition factors, where $M_i$ is regarded as $H_{s,i}$-module by restriction and $M_i, s$ is the $H_{s,i}$-module with $M_i$ as base space and the action of $g \in H_{s,i}$ on $M_s$ is the same action on $M$ of $s^{-1}gs$.

Proof. Assume the conditions (1) and (2) are satisfied. By Mackey’s criterion, we know that $kG_i \otimes_{kH_i} M_i$ is irreducible $G_i$-module. By Lemma 1.7(d) and Lemma 1.6(b) we see that $\text{Ind}_{H}^{G} M$ is irreducible. 

1.10. Let $A$ be a normal subgroups of a group $G$. Then for any representation $\rho : A \to GL(V)$ and $s \in G$, we can define a new representation $^s\rho : A \to GL(V)$ by setting $^s\rho(g) = \rho(s^{-1}gs)$ for any $g \in A$. In this way we get an action of $G$ on the set of representations of $A$.

Now assume that (1) $A$ is commutative and each element of $A$ has finite order, and (2) $G = H \ltimes A$ for some subgroup $H$ of $G$. By Lemma 1.4, any irreducible representation of $A$ is one-dimensional. Note that the set $X = \text{Hom}(A, k^*)$ is a group. We have seen that $H$ acts on $X$. Denote by $X/H$ the set of $H$-orbits in $X$. Let $(\chi_{\alpha})_{\alpha \in X/H}$ be a complete set of representatives of the $H$-orbits. For each $\alpha \in X/H$, let $A_{\alpha}$ be the subgroup of $H$ consisting of $h \in H$ with $h\chi_{\alpha} = \chi_{\alpha}$ and let $G_\alpha = AH_{\alpha}$. Define $\chi_{\alpha}(gh) = \chi_{\alpha}(g)$ for any $g \in A$ and $h \in H_{\alpha}$. In this way the representation $\chi_{\alpha}$ is extended to a representation of $G_{\alpha}$, denoted again by $\chi_{\alpha}$.

Let $\rho$ be an irreducible representation of $H_{\alpha}$. Through the homomorphism $G_{\alpha} \to H_{\alpha}$ we get an irreducible representation $\hat{\rho}$ of $G_{\alpha}$. The tensor product $\hat{\rho} \otimes \chi_{\alpha}$ then is an irreducible representation of $G_{\alpha}$. Let $\theta_{\alpha,\hat{\rho}} = \text{Ind}_{G_{\alpha}}^{G}(\hat{\rho} \otimes \chi_{\alpha})$.

Proposition 1.11. Assume that $G$ is quasi-finite. Keep the notation above. If $\rho$ is quasi-finite (with respect to the quasi-finite sequence of $H_{\alpha}$ induced from the given quasi-finite sequence of $G$), then

(a) $\theta_{\alpha,\hat{\rho}}$ is irreducible.

(b) If $\theta_{\alpha,\hat{\rho}}$ is isomorphic to $\theta_{\alpha',\hat{\rho}'}$, then $\alpha = \alpha'$, $\rho$ and $\rho'$ are isomorphic.

Proof. The argument is similar to that for [10, Proposition 25(a), (b)]. Let $G_1, G_2, G_3, \ldots$ be the quasi-finite sequence of $G$. Then every $G_i$ is finite, $G$ is the union of all $G_i$ and for any pair $i, j$ there exists an integer $r$ such that both $G_i$ and $G_j$ are contained in $G_r$. Set $H_{\alpha, i} = H_{\alpha} \cap G_i$. Let $M$ be the $kH_{\alpha, i}$-module affording the representation $\rho$ of $H_{\alpha}$ and $M_1, M_2, M_3, \ldots$ be a quasi-finite sequence of $M$ (with respect to the sequence $H_{\alpha, 1}, H_{\alpha, 2}, H_{\alpha, 3}, \ldots$). Let $V$ be the one-dimensional $kG_{\alpha}$-module affording representation $\chi_{\alpha}$. Let $A$ act on each $M_i$ trivially. Then $M_i$ becomes an irreducible $G_{\alpha, i}$-module. Regarding $V$ as a $G_{\alpha, i}$-module by restriction, then $M_i \otimes V$ is an irreducible $G_{\alpha, i}$-module.

We claim that $\text{Ind}_{H_{\alpha, i}}^{G_{\alpha}}(M_i \otimes V)$ is irreducible $G_{\alpha, i}$-module. For any $t \in G_i - H_{\alpha, i}$, there exists an $s$ in $G_i \cap H - G_i \cap H_{\alpha}$ such that $^t\chi_{\alpha} = \chi_{\alpha}$. For $s \in G_i \cap H - G_i \cap H_{\alpha}$, we have $^s\chi_{\alpha} \neq \chi_{\alpha}$. This implies that there exists some $a_s \in A$ such that $\chi_{\alpha}(a_s) \neq \chi_{\alpha}(s^{-1}a_s)$. Since $G$ is the union of all $G_i$ and for any pair $i, j$ there exists an integer $r$ such that both $G_j, G_j'$ are contained in $G_r$, we can find an $r$ such that $a_s$ is in $G_r$ for any $s$ in $G_i \cap H - G_i \cap H_{\alpha}$, thanks to all $G_j$ being finite. Thus $a_s$ is in $A_r = A \cap G_r$ for all $s$ in $G_i \cap H - G_i \cap H_{\alpha}$. For $s \in G_i \cap H - G_i \cap H_{\alpha}$, set $K_s = H_{\alpha, i}A_r \cap sH_{\alpha, i}A_r s^{-1}$. Note that the restriction to $H_{\alpha, i}A_r$ of the $H_{\alpha, i}A_r$-module $M_i \otimes V$ is irreducible. Through the two injections $K_s \to H_{\alpha, i}A_r$, $x \to x$ and $x \to s^{-1}xs$ we get two $K_s$-module structures on the vector space $M_i \otimes V$. The restriction of the first $K_s$-module structure on $M_i \otimes V$ to $A_r$ is the direct sum of some copies of $\text{Res}_{A_r} \chi_{\alpha}$, and the restriction of the second $K_s$-module structure on $M_i \otimes V$ to $A_r$ is the direct sum of some copies of $\text{Res}_{A_r} ^s\chi_{\alpha}$. Since $a_s$ is in $A_r$ and $\chi_{\alpha}(a) \neq \chi_{\alpha}(s^{-1}a_s)$, the restrictions of the two $K_s$-modules to $A_r$, are not isomorphic, hence the two $K_r$-modules are not isomorphic. By Mackey’s criterion on irreducibility, we see that $\text{Ind}_{H_{\alpha, i}}^{G_{\alpha}}(M_i \otimes V)$ is irreducible $G_{\alpha, i}$-module.

The natural map $\text{Ind}_{H_{\alpha, i}}^{G_{\alpha}}(M_i \otimes V) \to \text{Ind}_{G_{\alpha, i}}^{G}(M_i \otimes V)$ is homomorphism of $G_{\alpha, i}$-module, hence $\text{Ind}_{G_{\alpha, i}}^{G}(M_i \otimes V)$ is irreducible $G_{\alpha}$-module. Using Lemma 1.7(d) and Lemma 1.6(b) we see that $\theta_{\alpha, \hat{\rho}}$ is irreducible.

(2) The restriction of $\theta_{\alpha, \hat{\rho}}$ to $A$ is completely reducible and involves only characters in the orbit $H\chi_{\alpha}$ of $\chi_{\alpha}$, which shows that $\theta_{\alpha, \hat{\rho}}$ determines $\alpha$. Let $N$ be the subspace of $\text{Ind}_{G_{\alpha}}^{G}(M \otimes V)$ consisting of all $x \in \text{Ind}_{G_{\alpha}}^{G}(M \otimes V)$ such that $\theta_{\alpha, \hat{\rho}}(a)x = \chi_{\alpha}(a)x$ for all $a \in A$. The subspace $N$ is stable under $H_{\alpha}$ and
one checks easily that the representation of \( H_n \) in \( N \) is isomorphic to \( \rho \), hence \( \theta_{\alpha, \rho} \) determines \( \rho \).

The proposition is proved. \( \square \)

**Remark.** (1) The above proposition and argument are valid even if \( A \) is not commutative. The author is grateful to a referee for this observation.

(2) It is not clear whether any irreducible representation of \( G \) is isomorphic to a certain \( \theta_{\alpha, \rho} \).

1.12. Let \( G \) be a quasi-finite group. Assume that there exists a sequence \( \{1\} = G_0 \subset G_1 \subset \cdots \subset G_n = G \) such that all \( G_i \)'s are normal subgroups of \( G \) and \( G_i / G_{i-1} \) are abelian. Examples of such groups include Borel subgroups of a reductive group over \( \mathbb{F}_q \).

**Question.** Is each irreducible representation of \( G \) isomorphic to the induced representation of a one-dimensional representation of a subgroup of \( G \)?

2 Algebraic groups with split \( BN \)-pairs

2.1. In this section we assume that \( G \) is an algebraic group with a split \( BN \)-pair. By definition (see, for example, [3, p. 50]), \( G \) has closed subgroups \( B \) and \( N \) with the following properties:

(i) The set \( B \cup N \) generates \( G \), while \( T = B \cap N \) is a normal subgroup of \( N \) and all elements of \( T \) are semisimple.

(ii) The group \( W = N/T \) is generated by a set \( S \) of elements \( s_i, \ i \in I \), of order 2.

(iii) If \( n_i \in N \) maps to \( s_i \in S \) under the natural homomorphism \( N \rightarrow W \), then \( n_iBn_i \neq B \).

(iv) For each \( n \in N \) and each \( n_i \) we have \( n_iBn \subseteq B_n n_iB \cup BnB \).

(v) \( B \) has a closed normal unipotent subgroup \( U \) such that \( B = T \ltimes U \).

(vi) \( \bigcap_{n \in N} nBn^{-1} = T \).

It is known that \( W \) is a Weyl group. Let \( R \) be the root system of \( W \) and \( \alpha_i, \ i \in I \) be simple roots. For any \( w \in W \), \( U \) has two subgroups \( U_w \) and \( U'_w \) such that \( U = U'_w U_w \) and \( wU'_w w^{-1} \subseteq U \). If \( w = s_i \) for some \( i \), we simply write \( U_i \) and \( U'_i \) for \( U_w \) and \( U'_w \), respectively. For each \( w \in W \) we choose an element \( n_w \in N \) such that its natural image in \( W \) is \( w \) and let \( n_i \) stand for \( n_{s_i} \). The Bruhat decomposition says that \( G \) is a disjoint union of the double cosets \( Bn_w B, \ w \in W \). Note that \( G_i = B \cup Bn_iB \) is a subgroup of \( G \).

Any representation of \( T \) can be regarded naturally as a representation of \( B \) through the homomorphism \( B \rightarrow T \). Let \( k \) be a field. In this section all representations are assumed over \( k \). Let \( \theta \) be a one-dimensional representation of \( T \), we use the same letter when it is regarded as a representation of \( B \). Let \( k_\theta \) denote the corresponding \( B \)-module. We are interested in the induced module \( M(\theta) = kG \otimes_{k_B} k_\theta \).

Let \( P \supset B \) be a parabolic subgroup of \( G \) and \( L \) be a Levi subgroup of \( P \) containing \( T \). Let \( U_P \) be the unipotent radical of \( P \). Then \( P = L \ltimes U_P \). Moreover, \( B_L = B \cap L \) is a Borel subgroup and \( (B_L, N \cap L) \) forms a \( BN \)-pair of \( L \). By abusing notation, we also use \( k_\theta \) for its restriction to \( B_L \). Set \( M_L(\theta) = kL \otimes_{k_B} k_\theta \). Let \( U_P \) act on \( M_L(\theta) \) trivially. Then \( M_L(\theta) \) becomes a \( P \)-module. The following result is easy to check.

**Lemma 2.2.** \( M(\theta) \) is isomorphic to \( kG \otimes_{k_P} M_L(\theta) \).

If \( \theta \) is trivial we shall use \( M(tr) \) for \( M(\theta) \) and \( k_{tr} \) for \( k_\theta \), respectively. Let \( 1_{tr} \) be a nonzero element in \( k_{tr} \). For \( x \in kG \) we simply denote the element \( x \otimes 1_{tr} \) in \( M(tr) \) by \( x 1_{tr} \). For any element \( t \in T \) and \( n \in N \) we have \( nt1_{tr} = n1_{tr} \), so for \( w = nT \in W \), the notation \( w1_{tr} = n1_{tr} \) is well defined.

For any subset \( J \) of \( S \), we shall denote by \( W_J \) the subgroup of \( W \) generated by \( J \) and let \( w_J \) be the longest element of \( W_J \). Set \( \eta_J = \sum_{w \in W_J} (-1)^{|w|} w 1_{tr} \), where \( l(w) \) is the length of \( w \). The following result is a natural extension of [13, Theorem 1, p. 348] and part (a) seems new even for finite groups.

**Proposition 2.3.** Keep the notation above. Let \( J \) be a subset of \( S \). Then

(a) The space \( kUW\eta_J \) is a submodule of \( M(tr) \) and is denoted by \( M(tr)_J \).

(b) In particular, \( kU\eta_S, kU \sum_{w \in W} (-1)^{|w|} w 1_{tr} \) is a submodule of \( M(tr) \). This submodule will be called a Steinberg module of \( G \) and is denoted by \( St \).

**Proof.** The argument for [13, Theorem 1] works well here. Clearly \( kUW\eta_J \) is stable under the action of \( B \). Since \( G \) is generated by \( B \) and \( N \), it remains to check that \( kUW\eta_J \) is stable under the action of \( N \).
But \( N \) is generated by all \( n_i \) and \( T \), so we only need to check that \( n_i kUW \eta_j \subseteq kUW \eta_j \). We need to show that \( n_i u \eta_j \in kUW \eta_j \) for any \( u \in U \) and \( h \in W \). Let \( u = u' u_i \), where \( u_i \in U_i \) and \( u' \in U'_i \). Then \( n_i u \eta_j = n_i u'_i n_i^{-1} u_i \eta_j \). Since \( n_i u'_i n_i^{-1} \in U \), it suffices to check that \( n_i u_i \eta_j \in kUW \eta_j \). When \( u_i = 1 \), this is clear. Now assume that \( u_i \neq 1 \). Since \( s\eta_j = -\eta_j \) for any \( s \in J \), it is no harm to assume that \( l(w_j) = l(h) + l(w) \).

If \( hw_j \leq s_i h w_j \), then \( hw \leq s_i h w \) for all \( w \in W_j \). In this case, we have \( n_i u_i \eta_j = n_i h \eta_j \in kUW \eta_j \).

If \( s_i h \leq h \), then \( n_i u_i \eta_j = n_i u_i n_i(s_i h) \eta_j \). Note that \( n_i^2 \equiv T \). Since \( G = B \cup B_n B \) is a subgroup \( G \), if \( u_i \neq 1 \), we have \( n_i u_i n_i = n_i u_i n_i^{-1} n_i^2 = x n_i t y \) for some \( x, y \in U_i \) and \( t \in T \). Thus \( n_i u_i \eta_j = x n_i y(s_i h) \eta_j = x n_i y(s_i h) \eta_j = x h \eta_j \) since \( (s_i h) w_j \leq h w_j \).

Now assume that \( h \leq s_i h \) but \( s_i h w_j \leq h w_j \). Then we must have \( s_i h = h s_j \) for some \( s_j \in J \). If \( w \in W_j \) and \( w^{-1}(\alpha_j) \) is a positive root, then we have \( hw \leq s_i hw \), hence

\begin{align*}
(i) \quad n_i u_i hw_1 \tau_r &= n_i h w_1 \tau_r = s_i hw_1 \tau_r = s_i h w_1 \tau_r, \\
(ii) \quad hw_1 \tau_r &= x h w_1 \tau_r, \\
(iii) \quad n_i u_i h s_j w_1 \tau_r &= n_i u_i n_i^{-1} h w_1 \tau_r = x s_i h w_1 \tau_r = x h s_j w_1 \tau_r.
\end{align*}

Multiplying (i), (ii) and (iii) by \((-1)^{l(w)} \), \((-1)^{l(w)} \) and \((-1)^{l(s_j w)} \), respectively, adding them, then summing on all \( w \in W_j \) satisfying \( l(s_j w) = l(w) + 1 \), we get \((1-x+n_i u_i) h \eta_j = 0 \). Thus \( n_i u_i \eta_j = n_i u_i n_i^{-1} (x-1) \eta_j \in kUW \eta_j \). The proposition is proved.

An analogue of [5, Proposition 7.3] is the following result.

**Proposition 2.4.** Let \( \theta \) be a one-dimensional representation of \( T \). Then \( M(\theta) \otimes S \) is isomorphic to \( \text{Ind}^G_T k_\theta \).

**Proof.** Let \( 1_\theta \) be a nonzero element in \( k_\theta \) and \( \eta = \sum_{w \in W} (-1)^{l(w)} w_1 \tau_r \). Then it is easy to check that the map \( g_1 \tau_r \rightarrow g(1_\theta \otimes \eta) \) defines an isomorphism of \( G \)-module between \( \text{Ind}^G_T k_\theta \) and \( M(\theta) \otimes S \). The proposition is proved.

**Lemma 2.5.** For each \( n \in N = NG(T) \), \( kn_1 \theta \) is \( T \)-stable. If each \( U_i \) is infinite, then any \( T \)-stable one-dimensional subspace of \( M(\theta) \) is contained in \( \sum_{n \in N} kn_1 \theta \), which is of dimension \( |W| \).

**Proof.** It is clear.

**2.6.** Let \( J \) be a subset of \( S \) and let \( M(tr)_J \) be the sum of all \( M(tr)_K \) (see Proposition 2.3(a) for definition) with \( J \subseteq K \). Then \( M(tr)_J \) is a proper submodule of \( M(tr)_J \). Let \( E_J = M(tr)_J/M(tr)_J \).

**Proposition 2.7.** Assume that each \( U_i \) is infinite. If \( J \) and \( K \) are different subsets of \( S \), then \( E_J \) and \( E_K \) are not isomorphic.

**Proof.** For any \( w \in W \), let \( c_w = \sum_{y \leq w} (-1)^{l(y)} P_{y,w}(1) y_1 \tau_r \), where \( P_{y,w} \) are Kazhdan-Lusztig polynomials. Note that \( c_w = \eta_j \) if \( w = w_J \) for some subset \( J \) of \( S \).

We claim that \( M(tr)_J \) is the sum of all \( kUc_w \), \( w \in W \) with \( l(w) = l(w_J) \). Since \( M(tr)_J = kUW \eta_j = kUWc_w \), we only need to show that \( kWc_w \) is spanned by all \( c_w \), \( w \in W \) with \( l(w) = l(w_J) \). But this follows from and \([6, (2.3.a)], (2.3.c) \) and Proposition 2.4).

Let \( \tilde{c}_w \) be the image of \( c_w \) in \( E_J \). Let \( A_J \) be the subset of \( W \) consisting of all \( w \in W \) such that \( w \leq w_s \) for all \( s \in S - J \) and \( w \leq w_s \) for all \( s \in J \). Then \( \tilde{c}_w \) is nonzero if and only if \( w \in A_J \) and \( E_J \) is the sum of all \( kUc_w \). Since \( U_i \) is infinite for each \( i \), any \( T \)-stable line in \( E_J \) is contained in \( \sum_{w \in A_J} wc_w = E_J^T \). If there exists a \( G \)-isomorphism \( \phi : E_J \rightarrow E_K \), then we must have \( \phi(E_J^T) = E_K^T \). Thus \( \phi(\tilde{c}_w) = \sum_{w \in A_K} a_w \tilde{c}_w \), \( a_w \in k \). But \( \tilde{c}_w \neq 0 \) is uniquely determined by the following two conditions: (1) \( n_i \tilde{c}_w = -\tilde{c}_w \) if and only if \( s_i \in J \), and (2) \( U_i \tilde{c}_w = \tilde{c}_w \) if and only if \( s_i \notin J \). Therefore, \( J \neq K \) implies that any nonzero element in \( E_K^T \) does not satisfy the conditions for \( \tilde{c}_w \), hence \( \phi \) does not exist. The proposition is proved.

**2.8.** Let \( P \) be a parabolic subgroup of \( G \) with unipotent radical \( U_P \). Assume that \( L \) is a Levi subgroup of \( P \). Any \( kL \)-module \( E \) is naturally a \( kP \)-module through the homomorphism \( P \rightarrow L \). Then we can define the induced module \( \text{Ind}^G_T E = kG \otimes kP \). If \( P \) contains \( B \) and \( E \) is one-dimensional \( P \)-module, then \( \text{Ind}^G_T E = kG \otimes kP \) is a quotient module of some \( M(\theta) \).

Let \( P_J (J \subseteq S) \) be a standard parabolic subgroup of \( G \). Let \( P_J \) act on \( k \) trivially. Define \( 1^J_{P_J} = kG \otimes_{P_J} k \).
Clearly $1_{F_p}^G$ is a quotient module of $M(tr)$. Assume that each $U_i$ is infinite. By the discussion above we see that $\text{Hom}_G(1_{F_p}^G, 1_{F_p}^G)$ is nonzero if and only if $J$ is a subset of $K$.

3. Reductive groups with Frobenius maps

3.1. In this section we assume that $G$ is a connected reductive group defined over a finite field $\mathbb{F}_q$ of $q$ elements, where $q$ is a power of a prime $p$. Lang’s theorem implies that $G$ has a Borel subgroup $B$ defined over $\mathbb{F}_q$ and $B$ contains a maximal torus $T$ defined over $\mathbb{F}_q$. For any power $q^a$ of $q$, we denote by $G_{q^a}$ the $\mathbb{F}_{q^a}$-points of $G$ and shall identify $G$ with its $\mathbb{F}_{q^a}$-points, where $\mathbb{F}_q$ is an algebraic closure of $\mathbb{F}_q$. Then we have $G = \bigcup_{a \geq 1} G_{q^a}$. Similarly we define $B_{q^a}$ and $T_{q^a}$.

Let $N$ be the normalizer of $T$ in $G$. Then $B$ and $N$ form a $BN$-pair of $G$. Let $k$ be a field. For any one-dimensional representation $\theta$ of $T$ over $k$. As in Section 2 we can define the $kG$-module $M(\theta) = kG \otimes_{kB} k\theta$. When $\theta$ is trivial representation of $T$ over $k$, as in Section 2 we write $M(tr)$ for $M(\theta)$ and let $1_{tr}$ be a nonzero element in $k\theta$. We shall also write $x1_{tr}$ instead of $x \otimes 1_{tr}$ for $x \in kG$. Let $U$ be the unipotent radical of $B$.

Recall that for $w \in W = N/T$, the element $w1_{tr}$ is defined to be $n_w 1_{tr}$, where $n_w$ is a representative in $N$ of $w$ (cf. the paragraph below Lemma 2.2).

**Theorem 3.2.** (a) Assume that $k = \mathbb{C}$ is the field of complex numbers. Then $kU \sum_{w \in W} (-1)^{\theta(w)} w1_{tr}$ is an irreducible $G$-module.

(b) Assume that $k = \mathbb{F}_q$. Then $kU \sum_{w \in W} (-1)^{\theta(w)} w1_{tr}$ is an irreducible $G$-module.

**Proof.** (a) Let $U_{q^a}$ be the $\mathbb{F}_{q^a}$-points of $U$. Then $U = \bigcup_{a \geq 1} U_{q^a}$. Let $\eta = \sum_{w \in W} (-1)^{\theta(w)} w1_{tr}$. Then $\mathbb{C}[U_{q^a}]\eta$ is isomorphic to the Steinberg module of $G_{q^a}$, so it is an irreducible $G_{q^a}$-module. We have $\mathbb{C}U\eta = \bigcup_{a \geq 1} \mathbb{C}[U_{q^a}]\eta$ and $\mathbb{C}[U_{q^a}]\eta \subseteq \mathbb{C}[U_{q^b}]\eta$ for any integer $b \geq 1$. By Lemma 1.6(b), $\mathbb{C}U\eta$ is an irreducible $G$-module.

The argument for (b) is similar. The theorem is proved. □

3.3. According to [13, Theorems 2 and 3], the $G_{q^a}$-module $k[U_{q^a}] \sum_{w \in W} (-1)^{\theta(w)} w1_{tr}$ is irreducible if and only if $\theta$ does not divide $\sum_{w \in W} q^{al(w)}$. Therefore $kU \sum_{w \in W} (-1)^{\theta(w)} w1_{tr}$ is irreducible $G$-module if $\theta$ does not divide $\sum_{w \in W} q^{al(w)}$ for all positive integers $a$. Unfortunately, it is by no means easy to determine the prime factors of $\sum_{w \in W} q^{al(w)}$ even for type $A_1$ (in this case $W$ has only two elements). So it seems we need to find other ways to see whether $kU \sum_{w \in W} (-1)^{\theta(w)} w1_{tr}$ is irreducible $G$-module if $\theta$ is different from 0 and from char$\mathbb{F}_q = p$.

Let $\theta$ be a group homomorphism from $T$ to $k^*$. For any $w \in W$, define $w\theta : T \to k$, $t \to \theta(t)w^{-1}tw$.

**Theorem 3.4.** Assume that $k = \mathbb{C}$. Then $M(\theta)$ has at most $|W_\theta|$ composition factors, where $W_\theta = \{w \in W : w\theta = \theta\}$. In particular, if $w\theta \neq \theta$ for any $1 \neq w \in W$ (i.e., there exists $t \in T$ such that $\theta(w^{-1}tw) \neq \theta(t)$), then $M(\theta)$ is an irreducible $G$-module.

**Proof.** We can find an integer $a$ such that for any $b \geq a$ we have $W_\theta = \{w \in W : w\theta_{T_{q^b}} = \theta_{T_{q^b}}\}$, where $\theta_{T_{q^b}}$ denotes the restriction of $\theta$ to $T_{q^b}$. Assume that $0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_h = M(\theta)$ is a filtration of submodules of $M(\theta)$. Then there exist $x_i \in M_i - M_{i-1}$ for $i = 1, 2, \ldots, h$. Clearly there exists $c \geq a$ such that all $x_i$ are in $\mathbb{C}G_{q^c}(1 \otimes 1_\theta)$, where $1_\theta$ is nonzero element in $k\theta$. But it is known that $\mathbb{C}G_{q^c}(1 \otimes 1_\theta)$ has at most $W_\theta$ composition factors. The theorem is proved. □

**Proposition 3.5.** Let $\theta : T \to \mathbb{C}^*$ be a group homomorphism. Assume that $W_\theta$ is a parabolic subgroup $W_J$ of $W$. Then the element $\sum_{w \in W_J} (-1)^{\theta(w)} w1_{\theta}$ generates an irreducible submodule of $M(\theta)$ and the elements $(s - e)1_\theta, s \in W_J$ being simple reflections, generate a maximal submodule of $M(\theta)$, where $e$ is the neutral element of $W$.

**Proof.** It is known that the $kG_{q^a}$-submodule of $M(\theta)$ generated by $\sum_{w \in W_J} (-1)^{\theta(w)} w1_{\theta}$ is irreducible for all positive integers $a$ and the $kG_{q^a}$-submodule of $kG1_\theta$ generated by all $(s - e)1_\theta, s \in W_J$ being simple reflections, is a maximal submodule of $kG_{q^a}1_\theta$. The proposition then follows Lemma 1.6(b). □

3.6. Assume that $k = \mathbb{C}$. It is an interesting question to determine the composition factors of $M(\theta)$. Assume that $P$ is a parabolic subgroup containing $B$. Let $P$ act trivially on $\mathbb{C}$. Then $\text{Ind}_P^C kG \otimes_{kP} \mathbb{C}$
is a quotient module of $M(tr)$, so it has finitely many composition factors. If $P$ is a maximal parabolic subgroup, then $\text{Ind}_{\bar{P}}^G$ has much less composition factors than $M(tr)$.

Let $G$ be a connected reductive group over $\mathbb{F}_q$ such that its derived group is of type $A_n$. Let $P$ be a maximal parabolic subgroup of $G$ containing the $F$-stable Borel subgroup $B$ and assume that the derived subgroup of a Levi subgroup of $P$ has type $A_{n-1}$. Using Lemma 1.6(b) and representation theory for $G_{\mathbb{F}_q}$, it is easy to see that $\text{Ind}_{\bar{P}}^G$ has a unique irreducible quotient module which is trivial and a unique irreducible submodule.

It is known that there is a bijection between the composition factors of $G_{\mathbb{F}_q}$-submodule $CG_{\mathbb{F}_q}(tr)$ of $M(tr)$ and the composition factors of the regular module $CW$ of $W$, which preserves multiplicities. But this result cannot be extended to $M(tr)$ since by the proof for Proposition 2.7 it is easy to see that $M(tr)$ has at least $2^{|S|}$ composition factors which are pairwise non-isomorphic.

### 3.7. Assume that $k = \bar{\mathbb{F}}_q$. Then for each dominant weight $\lambda : T \rightarrow k^*$, we have Weyl module $V(\lambda)$ and its irreducible quotient $L(\lambda)$. Clearly $V(\lambda)$ is a quotient module of $M(\lambda)$. Also it is clear that the tensor product $M(\theta) \otimes V(\lambda)$ has a filtration of submodules such that the quotient modules of the filtration are some $M(\theta + \mu)$, where $\mu$ are weights of $V(\lambda)$. It is not clear whether some $M(\theta)$ have infinite composition factors. It might be interesting to study $\text{St} \otimes L(\lambda)$.

It is easy to see that the trivial module of $\bar{\mathbb{F}}_qU$ is the unique irreducible $\bar{\mathbb{F}}_qU$-module. A question comes naturally: is every irreducible $\bar{\mathbb{F}}_qB$-module one-dimensional?

If $\text{char} k$ is different from 0 and from $\text{char} \bar{\mathbb{F}}_q = p$, the structure of the modules $M(\theta)$ are more complicated.

### 4 Gelfand-Graev modules

4.1. Keep the notation in Subsection 3.1. Thus $G$ is a connected reductive group defined over $\mathbb{F}_q$, $B$ a maximal Borel subgroup of $G$ defined over $\mathbb{F}_q$ and $T$ a maximal torus in $B$ defined over $\mathbb{F}_q$. Let $U$ be the unipotent radical of $B$. The group $G$ and its subgroups are identified with their $\mathbb{F}_q$-points, so $G = G(\mathbb{F}_q)$, $B = B(\mathbb{F}_q)$, $T = T(\mathbb{F}_q)$, etc.

Let $R$ be the root system of $G$ and $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ be the set of simple roots corresponding to $B$. Denote by $R^+$ the set of positive roots. For each positive root $\alpha \in R$, let $U_\alpha$ be the corresponding root subgroup in $G$. We choose an isomorphism $\epsilon_\alpha : \mathbb{F}_q \rightarrow U_\alpha$ so that $t \epsilon_\alpha(a)t^{-1} = \epsilon_\alpha(\alpha(t)a)$ for any $a \in \mathbb{F}_q$ and $t \in T$. It is known that the subgroup $U'$ of $B$ generated by all $U_\alpha$, $\alpha \in R^+ - \Delta$, is a normal subgroup of $B$ and the quotient group $U/U'$ is isomorphic to the direct product $U_{\alpha_1} \times U_{\alpha_2} \times \cdots \times U_{\alpha_l}$ and $B/U'$ is isomorphic to the semidirect product $T \rtimes U/U'$.

Each irreducible representation of $B/U'$ gives rise naturally an irreducible representation of $B$. In general it is hard to get a classification of irreducible representations for groups $B$ and $U$.

4.2. Clearly, there is a bijection between one-dimensional representations of $U$ with $U'$ in the kernel and the sets $(\sigma_j)$, where $\sigma_j$ is a one-dimensional representation of $U_{\alpha_j}$. A one-dimensional representation of $U$ with $U'$ in its kernel is called non-degenerate if all $\sigma_j$ are non trivial. The group $T$ acts naturally on the set of irreducible representations of $U/U'$.

It is known that all non-degenerate one-dimensional complex representations of $U_q$ form a $T_q$-orbit if the center of $G$ is connected. But the set $\Phi$ of non-degenerate one-dimensional complex representations of $U$ is uncountable. This implies that the $T$-orbits in $\Phi$ is uncountable. For a non-degenerate one-dimensional complex representation $\sigma$ of $U$, we may consider the induced representation $\text{Ind}_{\bar{U}}^G\sigma$ and call it a Gelfand-Graev representation of $G$. It seems not easy to decompose these Gelfand-Graev representations (cf. [5, Section 10]).

### 5 Some questions

There are some natural questions.

1. Develop a theory of $kG$-modules for infinite quasi-finite groups. A particular question is to classify irreducible $kG$-modules for some interesting quasi-finite groups, say, reductive groups with Frobenius
6 Type A$_1$

In this section $G$ will denote $GL_2(\mathbb{F}_q)$ or $SL_2(\mathbb{F}_q)$. Let $T$ be the torus of $G$ consisting of diagonal matrices and $B$ be the Borel subgroup consisting of upper triangle matrices. Let $U$ be the unipotent radical of $B$. In this section we consider complex representations of these groups.

6.1. We first consider representations of $B$ over $\mathbb{C}$. We have $B = T \times U$. Let $X = \text{Hom}(U, \mathbb{C}^*)$. For $t \in T$, $\chi \in X$, define $t\chi : U \to \mathbb{C}^*$, $u \to \chi(t^{-1}ut)$. Then we get an action of $T$ on $X$. Note that $U$ is isomorphic to the additive group $\mathbb{F}_q$, so as abelian group, $U$ is a direct sum of countable copies of $\mathbb{F}_p$, where $p$ is the characteristic of $\mathbb{F}_q$. Therefore, the set $X$ of homomorphisms $U \to \mathbb{C}^*$ is uncountable. This implies the set of $T$-orbits in $X$ is uncountable.

Denote by $X/T$ the set of $T$-orbits in $X$ and let $(\chi_\alpha)_{\alpha \in X/T}$ be a complete set of representatives of the $T$-orbits. For each $\alpha \in X/T$, let $T_\alpha$ be the subgroup of $T$ consisting of $t \in T$ with $t\chi_\alpha = \chi_\alpha$ and let $B_\alpha = T_\alpha U$. Define $\chi_\alpha(tu) = \chi_\alpha(t)$ for any $t \in T$ and $u \in U$. In this way the representation $\chi_\alpha$ is extended to a representation of $B_\alpha$, denoted again by $\chi_\alpha$. Note that $T_\alpha$ is the center of $B$ if $\chi_\alpha$ is non-trivial, is the whole $T$ if $\chi_\alpha$ is trivial.

Let $\rho$ be an irreducible complex representation of $T_\alpha$, which is one-dimensional since $T_\alpha$ is commutative. Through the homomorphism $B_\alpha \to T_\alpha$ we get an irreducible representation $\tilde{\rho}$ of $B_\alpha$. The tensor product $\tilde{\rho} \otimes \chi_\alpha$ then is an irreducible representation of $G_\alpha$. Let $\theta_{\alpha,\rho}$ be the corresponding induced representation of $B$. According to Proposition 1.11 we have the following result.

**Lemma 6.2.** The complex representation $\theta_{\alpha,\rho}$ of $B$ is irreducible. Moreover, $\theta_{\alpha,\rho}$ is isomorphic to $\theta_{\alpha',\rho'}$ if and only if $\alpha = \alpha'$ and $\rho = \rho'$.

We can further induce $\theta_{\alpha,\rho}$ to $G$. By the lemma above, if $\chi_\alpha$ is trivial, then $B_\alpha = B$ and $\theta_{\alpha,\rho}$ is just $\tilde{\rho}$. Since the commutator group $[B, B]$ of $B$ is $U$, any homomorphism $\theta : B \to \mathbb{C}^*$ has the form $\tilde{\rho}$. According to Theorems 3.2 and 3.4, we have the following result.

**Proposition 6.3.** Let $\theta : B \to \mathbb{C}^*$ be a group homomorphism. Then

(a) $M(\theta) = CG \otimes_{\mathbb{C}B} \mathbb{C}_q$ is irreducible $G$-module if $\theta$ is not trivial.

(b) If $\theta$ is trivial, then $M(\theta)$ has a unique nonzero proper submodule and unique quotient module. The nonzero proper submodule is the Steinberg module. The quotient module is the trivial module of $G$.

6.4. Let $B_q$, $T_q$ and $U_q$ be the $F_q$-points of $B$, $T$ and $U$, respectively. Keep the notation in Subsection 6.1. Assume that $\chi_\alpha$ is nontrivial. If the restriction of $\chi_\alpha$ to $U_q$ is not trivial, we can consider the induced representation $\theta_{\alpha,\rho,q}$ of $B_q$ from the restriction of $\tilde{\rho} \otimes \chi_\alpha$ to $G_q \cap B_q$, which is irreducible. It is known that the action of $B_q$ on $\theta_{\alpha,\rho,q}$ can be extended to actions of $G_q$ and in this way one can get all cuspidal representations of $G_q$. But the author has not been able to see how to extend the $B$ action on $\theta_{\alpha,\rho}$ to an action of $G$. 
7 Miscellany

In this section we give some discussion to representations of the groups listed in 1.8 Example (1) and Example (4).

7.1. Let $W = W_\infty$ be the group defined in 1.8 Example (1). Since $W$ is a Coxeter group, we can use Kazhdan-Lusztig cells to construct representations of $W$ and its Hecke algebras. Let $s_1, \ldots, s_n$ be the simple reflections of $W_n$, and let $S$ be the set of all simple reflections.

(1) Assume that $W$ is of type $A$. Let $C_w$, $w \in W$, be the Kazhdan-Lusztig basis of $\mathbb{C}W$ (cf. [6, Theorem 1.1]). For each left cell $\sigma$ of $W$, let $I_\sigma$ be the subspace of $\mathbb{C}W$ spanned by all $C_w$, $w \in \sigma$. Denote by $I_{<\sigma}$ the subspace of $\mathbb{C}W$ spanned by all $C_w$, $w \leq u$ for some $u \in \sigma$ but $w \notin \sigma$. Then $\mathbb{C}W$ is the direct sum of all $I_\sigma$, and both $I_\sigma + I_{<\sigma}$, $I_{<\sigma}$ are left ideals of $\mathbb{C}W$. According to [6, Theorem 1.4] and Lemma 1.6(b), $L_\sigma = (I_\sigma + I_{<\sigma})/I_{<\sigma}$ is an irreducible $\mathbb{C}W$-module. When two left cells $\sigma$ and $\tau$ are in the same two-sided cell, the right star actions leads to an isomorphism between $L_\sigma$ and $L_\tau$. Moreover, $L_\sigma$ and $L_\tau$ are isomorphic $\mathbb{C}W$-modules if and only if $\sigma$ and $\tau$ are in the same two-sided cell of $W$. Similar results hold for Hecke algebra $H$ over $\mathbb{C}(q^\frac{1}{2})$ with parameter $q$ (here $q$ is an indeterminate).

According to the proof of [6, Theorem 1.4], any two-sided cell of $W$ contains some $w_P$, where $P$ is a finite subset of $S$ and $w_P$ is the longest element of the subgroup of $W$ generated by $P$. Let $\sigma_P$ be the left cell of $W$ containing $w_P$. For subsets of $P$ and $Q$ of $S$, $w_P$ and $w_Q$ are in the same two-sided cell of $W$ if and only if $w_P$ and $w_Q$ are in the same two-sided cell of $W_n$ whenever both $w_P$ and $w_Q$ are contained in $W_n$.

Unlike the group $W_n$, some irreducible $\mathbb{C}W$-modules are not isomorphic to any $L_\sigma$, for example, the sign representation of $W$ is not isomorphic to any $L_\sigma$. It is also less easy to discuss irreducible $\mathbb{C}W$-modules.

(2) Assume that $W$ is of type $B$. Let $H$ be the Hecke algebra of $W$ defined over $\mathbb{A} = \mathbb{C}[q^\frac{1}{2}, q^{-\frac{1}{2}}]$ ($q$ an indeterminate) with $\mathbb{A}$-basis $T_w$, $w \in W$, and multiplication relations $(T_w - q_i)(T_w + 1) = 0$ and $T_uT_w = T_{wu}$ if $l(wu) = l(w) + l(u)$, where $q_1 = q$, $q_i = q^2$ for all $i \geq 2$. Let $C_w$, $w \in W$, be the Kazhdan-Lusztig basis of $H$ defined in [7, Proposition 2]. The corresponding cells are called generalized cells ($\varphi$-cells in [7]). Regarding $C$ as an $\mathbb{A}$-module by specifying $q$ to 1, then we have $\mathbb{C}W = H \otimes_{\mathbb{A}} C$. By abuse notation we use also notation $C_w$ for its image in $\mathbb{C}W$. For each generalized left cell $\sigma$ of $W$, let $I_\sigma$ be the subspace of $\mathbb{C}W$ spanned by all $C_w$, $w \in \sigma$. Denote by $I_{<\sigma}$ the subspace of $\mathbb{C}W$ spanned by all $C_w$, $w \leq u$ for some $u \in \sigma$ but $w \notin \sigma$. Then $\mathbb{C}W$ is the direct sum of all $I_\sigma$, and both $I_\sigma + I_{<\sigma}$, $I_{<\sigma}$ are left ideals of $\mathbb{C}W$. According to [7, Theorem 11] and Lemma 1.6(b), $L_\sigma = (I_\sigma + I_{<\sigma})/I_{<\sigma}$ is an irreducible $\mathbb{C}W$-module. However, it seems not clear whether $L_\sigma$ and $L_\tau$ are isomorphic $\mathbb{C}W$-modules when the generalized left cells $\sigma$ and $\tau$ are in the same generalized two-sided cell of $W$. Similar results hold for the Hecke algebra $H = H \otimes_{\mathbb{A}} \mathbb{C}(q^\frac{1}{2})$. According to [7, Section 10], if one chooses $q_1 = q^3$, $q_i = q^2$ for all $i \geq 2$, the above results remain valid.

(3) Assume that $W$ is of type $D$. Let $C_w$, $w \in W$, be the Kazhdan-Lusztig basis of $\mathbb{C}W$ (cf. [6, Theorem 1.1]). For each left cell $\sigma$ of $W$, as in (1) we can define the subspaces $I_\sigma$ and $I_{<\sigma}$ of $\mathbb{C}W$ and the $\mathbb{C}W$-module $L_\sigma$. Unlike the case of types $A$ or $B$, $L_\sigma$ may not be irreducible. In [8, Chapter 12], Lusztig has proved that the $\mathbb{C}W_n$-module afforded by a left cell of $W_n$ is multiplicity free and the number of irreducible components in the $\mathbb{C}W_n$-module is a power of 2. So it is likely that the $\mathbb{C}W$-module $L_\sigma$ is semisimple (i.e., a direct sum of some irreducible submodules).

7.2. In the rest of this section $G_n$ and $G_\infty$ are as in 1.8 Example (4). Let $T$ be the subgroup of $G$ consisting of diagonal matrices in $G$ and $N$ be the normalizer of $T$ in $G$. We can choose naturally a subgroup $B$ of $G$ so that $B$ and $N$ form a $BN$-pair for $G$. For example, $B$ can be chosen to be the subgroup of $G$ consisting of upper triangular matrices in $G$ if $G = GL_\infty$ or $SL_\infty$. Let $U$ be the kernel of the natural homomorphism $B \to T$. It is easy to see that $W = N/T$ is just a group in 1.8 Example (1). Let $S = \{s_1, s_2, s_3, \ldots\}$ be the set of simple reflections of $W$. For each $s_i$, we choose a representative $n_i \in N$ of $s_i$. For each $i$, there exist subgroups $U_i$ and $U'_i$ of $U$ such that $U = U'_i U_i$ and $n_i U'_i n_i^{-1} \subset U$. Note that if $u_i \in U_i$, we have $n_i u_i n_i^{-1} = x u_i y$ for some $x, y \in U_i$ and $t \in T$.

Set $T_n = T \cap G_n$ and $B_n = B \cap G$. Let $N_n = N \cap G_n$, and $W_n = N_n/T_n$. Note that $W_n$ can be regarded as a subgroup of $W$ in a natural way.
Assume that \( k = \bar{k} \), by Lemma 1.6(b) the natural representation \( V \) of \( G \) is irreducible. We may consider to decompose the tensor product of \( m \) copies of \( V \). Many classical results for \( G_n \) can be extended to \( G_\infty \).

Let \( \lambda : T \to k^* \) be a character of \( T \). Assume that the restriction \( \lambda_n \) of \( \lambda \) to \( T_n \) is a dominant weight of \( G_n \) for each \( n \). Then we have an irreducible rational \( G_n \)-module \( V_n \) with highest weight \( \lambda_n \). Clearly, we have a natural embedding \( V_n \hookrightarrow V_{n+1} \) for each \( n \). Moreover, the embedding is a \( G_n \)-homomorphism. Let \( V(\lambda) \) be the union of all \( V_n \). Then \( V(\lambda) \) is naturally a \( G \)-module. By Lemma 1.6(b) it is an irreducible \( G \)-module.

It is known that a \( G_n \)-module \( M_n \) is called rational if for any \( x \in M \), the \( G \)-submodule of \( G \) generated by \( x \) is a finite dimensional rational \( G \)-module. We call a \( G \)-module \( M \) rational if the restriction of \( M \) to \( G_n \) is rational. Clearly, \( V(\lambda) \) a rational \( G \)-module in this sense.

7.3. Keep the notation in Subsection 7.2. For any homomorphism \( \theta : T \to K \), where \( K \) is a field, let \( K\theta \) be the corresponding one-dimensional \( B \)-module. As in Section 2 we may consider the induced module \( M(\theta) = KG \otimes_{KB} K\theta \). We can define Steinberg module for \( KG \) but which is not a submodule of \( M(tr) \).

Let \( St = KU\xi \) be a free \( KU \)-module generated by the element \( \xi \). Note that \( KG \) (resp. \( kU \)) is the union of all \( KG_n \) (resp. \( K(U \cap G_n) \)). By the proof for Proposition 2.3 we get the following result.

**Proposition 7.4.** The \( KU \)-module structure on \( St \) can be uniquely extended to a \( KG \)-module structure as follows:

1. \( tu\xi = tut^{-1}\xi \) for any \( t \in T \) and \( u \in U \),
2. \( n_iu\xi = -n_in_i^{-1}\xi \) if \( u \in U_i' \),
3. \( n_iu_i'\xi = n_iu_i'\xi^{-1}(x-1)\xi \) for \( u_i' \in U_i' \) and \( 1 \neq u_i \in U_i \), where \( x \in U_i \) is defined uniquely by the formula \( n_iu_i\xi^{-1} = xu_i'\xi \), \( t \in T \), \( y \in U_i \).

Naturally, we call the \( G \)-module \( St \) a Steinberg module of \( G \). Proposition 2.4 also has its counterpart here, i.e., \( M(\theta) \otimes St \) is isomorphic to \( \text{Ind}_G^G K\theta \).

Using [13, Theorem 2], Theorems 3.2 and 3.4 and Lemma 1.6(b), we get the following result.

**Theorem 7.5.** (a) Assume that (1) \( k = \mathbb{F}_q \) or \( \bar{\mathbb{F}}_q \), (2) \( K = \mathbb{C} \) or \( \bar{\mathbb{F}}_q \), then \( St \) is irreducible \( KG \)-module.

(b) Assume that \( K = \mathbb{C} \) and \( \theta : T \to \mathbb{C}^* \) is a character of \( T \). If \( W_\theta = \{ w \in W \mid w\theta = \theta \} \) has only one element, then \( M(\theta) \) is irreducible \( KG \)-module.

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