1. Introduction

Let \( K \) be a number field of degree \( n \) over \( \mathbb{Q} \) and \( \alpha \in K \) a generator, i.e. \( K = \mathbb{Q}(\alpha) \), with minimal polynomial \( f = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \) and \( a_i \in \mathbb{Z} \), \( \gcd(a_0, \ldots, a_n) = 1 \). We call \( H(\alpha) = H(f) = \max(|a_0|, \ldots, |a_n|) \) the height of \( \alpha \). Our question is now: How large are small generators \( \alpha \) of \( K \) (where our measure is the height \( H(\alpha) \))?

A bound from below is given in the following Proposition 1.

For every \( n \in \mathbb{N} \) there is a real number \( c_n > 0 \) such that if \( \alpha \) generates a number field \( K \) of degree \( n \) and discriminant \( D_K \) then
\[
H(\alpha) \geq c_n |D_K|^{\frac{1}{2n-2}}.
\]

One can take \( c_n = \frac{1}{n \sqrt{n}} \).

Proof: Let \( f = a_n x^n + \cdots + a_0 \) be the minimal polynomial of \( \alpha \). According to [Cohen, p.216, exercise 15 (H. W. Lenstra)] the \( \mathbb{Z} \)-module
\[
R = \mathbb{Z} + (a_n \alpha) \mathbb{Z} + (a_n \alpha^2 + a_{n-1} \alpha) \mathbb{Z} + (a_n \alpha^3 + a_{n-1} \alpha^2 + a_{n-2} \alpha) \mathbb{Z} + \cdots + (a_n \alpha^{n-2} + \cdots + a_3 \alpha) \mathbb{Z} + (a_n \alpha^{n-1} + \cdots + a_2 \alpha) \mathbb{Z}
\]
is an order in \( K = \mathbb{Q}(\alpha) \) with discriminant \( D(f) \), the discriminant of the polynomial \( f \). So there is an \( m \in \mathbb{N} \) with \( D(f) = m^2 D_K \). On the other hand \( D(f) \) is homogeneous in \( a_0, \ldots, a_n \) of degree \( 2n-2 \) and there is a constant \( e_n > 0 \) such that
\[
|D(f)| \leq e_n H(f)^{2n-2}.
\]
This gives \( |D_K| \leq e_n H(\alpha)^{2n-2} \) and therefore
\[
H(\alpha) \geq e_n^{-\frac{1}{2n-2}} |D_K|^{\frac{1}{2n-2}}.
\]
Using the determinant representation of the discriminant one sees that one can take \( e_n = (n \sqrt{n})^{2n-2} \), which gives the desired explicit inequality.

A natural question is now:

Question 1. Is there a number \( d_n \) such that every number field \( K \) of degree \( n \) over \( \mathbb{Q} \) has a generator \( \alpha \) with
\[
H(\alpha) \leq d_n |D_K|^{\frac{1}{2n-2}}?
\]

The answer will be ‘yes’ for quadratic fields. We do not know what’s going on in general. We conclude with two examples.

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2. Imaginary Quadratic Fields

If \( K = \mathbb{Q}(\alpha) \) is imaginary quadratic with discriminant \( D \) and \( f = ax^2 + bx + c \) the minimal polynomial of \( \alpha \) then there is an \( m \in \mathbb{N} \) with \( b^2 - 4ac = D(f) = m^2 \cdot D \) and therefore \( 4H(\alpha)^2 \geq 4ac = b^2 - m^2D = b^2 + m^2|D| \geq |D| \), so

\[
H(\alpha) \geq \frac{1}{2} \sqrt{|D|}.
\]

Is is easy to write down examples which show that the inequality is best possible:

**Example:** Assume that \( m \in \mathbb{N} \) and \( 4m^2 - 1 = (2m - 1)(2m + 1) \) is square-free. Then \( D = 1 - 4m^2 \) is the discriminant of \( K = \mathbb{Q}(\sqrt{D}) \) and is generated by an element \( \alpha \) with minimal polynomial \( f = mx^2 + x + m \). So we get

\[
H(\alpha) = \frac{m}{\sqrt{4m^2 - 1}}
\]

which tends to \( \frac{1}{2} \) as \( m \) goes to \( \infty \).

To get an estimate in the other direction we look first at small integral elements. The result is given in the following lemma:

**Lemma 1.** Let \( d \) be a square-free integer \( < 0 \). The smallest integral generators of \( \mathbb{Q}(\sqrt{d}) \) have minimal polynomial

\[
x^2 - d \quad \text{for} \quad d \equiv 2, 3 \mod 4
\]

\[
x^2 \pm x + \frac{1 - d}{4} \quad \text{for} \quad d \equiv 1 \mod 4.
\]

The lemma is easy to prove. It shows that the height of integral generators is always \( \geq \frac{1}{4}|D| \) and is far away from what we are looking for.

For a quadratic number field with discriminant \( D \) let \( H_{\text{min}}(D) \) be the height of a generator of minimal height. The following table gives the discriminant \( D \) where \( \frac{H_{\text{min}}(D)}{\sqrt{|D|}} \) is maximal in the specified range. \( ax^2 + bx + c \) is the minimal polynomial of a minimal generator of \( \mathbb{Q}(\sqrt{D}) \).

| \( D \) | \( (a, b, c) \) | \( \frac{H_{\text{min}}(D)}{\sqrt{|D|}} \) |
|-------|-------------|------------------|
| \( 0 \geq D \geq -10000 \) | -163 | (1,1,41) | 3.2114 |
| \(-10000 \geq D \geq -20000 \) | -17467 | (47,39,101) | 0.7642 |
| \(-20000 \geq D \geq -30000 \) | -21379 | (55,29,101) | 0.6908 |
| \(-30000 \geq D \geq -40000 \) | -36523 | (73,59,137) | 0.7169 |
| \(-40000 \geq D \geq -50000 \) | -47947 | (83,39,149) | 0.6805 |
| \(-50000 \geq D \geq -60000 \) | -50395 | (89,35,145) | 0.6459 |
| \(-60000 \geq D \geq -70000 \) | -68707 | (127,127,167) | 0.6371 |
| \(-70000 \geq D \geq -80000 \) | -73747 | (109,41,173) | 0.6372 |
| \(-80000 \geq D \geq -90000 \) | -81859 | (121,93,187) | 0.6536 |
| \(-90000 \geq D \geq -100000 \) | -91795 | (127,91,197) | 0.6502 |

The table and some further hints suggest the following

**Conjecture 1.** If \( K \) is an imaginary quadratic field with discriminant \( D \) then there is \( \alpha \) with \( K = \mathbb{Q}(\alpha) \) and

\[
H(\alpha) \leq 3.22 \sqrt{|D|}.
\]
We have the following asymptotic result:

**Theorem 1.**

\[
\lim_{D \to -\infty} \frac{H_{\min}(D)}{\sqrt{|D|}} = \frac{1}{2}
\]

where \(D\) runs through the discriminants of imaginary quadratic fields.

The proof depends heavily on a result of Duke [Duke] which we shortly recall\(^1\):

Let 
\[F = \{z \in \mathbb{C} : \text{Im} z > 0, -\frac{1}{2} < \text{Re} z \leq \frac{1}{2}, |z| \geq 1 \text{ and } |z| > 1 \text{ if } \text{Re} z < 0\}\]

be the standard fundamental domain of \(\text{PSL}_2(\mathbb{Z})\) in the upper half plane \(\{z : \text{Im} z > 0\}\). Using the measure \(d\mu = \frac{3}{\pi} \frac{dx dy}{y^2}\) gives \(\mu(F) = 1\). Define for a discriminant \(D\)
\[\Lambda_D = \{z = \frac{b + \sqrt{D}}{2a} : a^2 - 4ac = D, z \in F, a, b, c \in \mathbb{Z}\}\].

If \(\Omega \subseteq F\) is convex (in the non-Euclidean sense) with a piece-wise smooth boundary then by [Duke, Theorem 1]
\[\lim_{D \to -\infty} \frac{\#\Lambda_D \cap \Omega}{\#\Lambda_D} = \mu(\Omega)\]
where \(D\) runs through the discriminants of imaginary quadratic fields. Now we are ready to prove the theorem:

**Proof of Theorem 1:** Let \(0 < \epsilon \leq 1\) be given and choose a convex set \(\Omega\) with piece-wise smooth boundary, \(\mu(\Omega) > 0\) and
\[\Omega \subseteq \{z \in F : 0 \leq \text{Re} z \leq \frac{1}{2}, 1 \leq \text{Im} z \leq 1 + \frac{1}{2}\epsilon\}\].

By the just mentioned result we find \(D_\epsilon\) such that \(\#\Lambda_D \cap \Omega \geq 1\) whenever \(D < D_\epsilon\).

Take \(D < D_\epsilon\) then we get an \(\alpha \in \Lambda_D \cap \Omega\), i.e. \(\alpha = \frac{b + \sqrt{D}}{2a}\) with \(a, b, c \in \mathbb{Z}\) and \(D = b^2 - 4ac\) such that
\[a > 0, \quad 0 \leq \frac{b}{2a} \leq \frac{1}{2} \epsilon, \quad 1 \leq \frac{\sqrt{|D|}}{2a} \leq 1 + \frac{1}{2} \epsilon\].

We see at once
\[|a| = a \leq \frac{1}{2} \sqrt{|D|} \quad \text{and} \quad |b| = b \leq a \epsilon \leq a \leq \frac{1}{2} \sqrt{|D|}\].

Finally
\[|c| = c = \frac{b^2 + |D|}{4a} \leq \frac{a^2 \epsilon^2 + |D|}{4a} = \frac{a^2 \epsilon^2 + \frac{1}{2} \sqrt{|D|}|D|}{2a} \leq \frac{1}{2} \sqrt{|D|}\epsilon^2 + \frac{1}{2} |D| \left(1 + \frac{1}{2} \epsilon^2\right) = \frac{1}{2} \sqrt{|D|}\left(1 + \frac{\epsilon^2}{4}\right) \leq \frac{1}{2} \sqrt{|D|}(1 + \epsilon)\].

As \(\alpha\) has minimal polynomial \(ax^2 - bx + c\) and generates \(\mathbb{Q}(\sqrt{D})\) we get (with the trivial estimate)
\[\frac{1}{2} \leq \frac{H_{\min}(D)}{\sqrt{|D|}} \leq \frac{H(\alpha)}{\sqrt{|D|}} \leq \frac{1}{2}(1 + \epsilon)\]

\(^1\) I would like to thank H.W.Lenstra who gave me the hint to Duke's paper.
which proves our claim. ■

It would be nice to have an effective version of Duke’s theorem in order to prove a statement like Conjecture 1.

3. REAL QUADRATIC FIELDS

Let \( K \) be a real quadratic field with discriminant \( D \) and \( \alpha \) a generator of \( K \) with minimal polynomial \( f = ax^2 + bx + c \). Then there is an \( m \in \mathbb{N} \) with \( b^2 - 4ac = m^2 D \) so that we obtain \( 0 < D \leq m^2 D = b^2 - 4ac \leq b^2 + 4|a||c| \leq 5H(\alpha)^2 \) and therefore

\[
H(\alpha) \geq \frac{1}{\sqrt{5}} \sqrt{D} = 0.4472\ldots \sqrt{D}.
\]

The following example shows that the estimate is best possible:

**Example:** Let \( m \in \mathbb{N} \) such that \( 5m^2 - 2m + 1 \) is square-free. Then an element \( \alpha \) with minimal polynomial \( f = mx^2 + (m - 1)x - m \) generates a real quadratic number field \( K \) with discriminant \( D = 5m^2 - 2m + 1 \) and

\[
H(\alpha) \leq \frac{m}{\sqrt{5m^2 - 2m + 1}}
\]
tends to \( \frac{1}{\sqrt{5}} \) as \( m \) goes to \( \infty \).

To get an estimate in the other direction the situation is much easier than in the imaginary quadratic case as we find small integral generators:

**Proposition 2.** If \( K \) is a real quadratic field with discriminant \( D \) there is an (integral) \( \alpha \in K \) with \( K = \mathbb{Q}(\alpha) \) and

\[
H(\alpha) < \sqrt{D}.
\]

**Proof:** Let \( m = \lceil \sqrt{D} \rceil \) and choose \( a = m \) or \( a = m - 1 \) such that \( a^2 \equiv D \mod 4 \). Take \( b \) with \( a^2 - 4b = D \). It is clear that \( b < 0 \). The assumption \( |b| \geq m \) would imply

\[
D = a^2 + 4|b| \geq (m - 1)^2 + 4m = m^2 + 2m + 1 = (m + 1)^2,
\]

so \( \sqrt{D} \geq m + 1 \), which contradicts the definition of \( m \). Therefore an element \( \alpha \) with minimal polynomial \( f = x^2 + ax + b \) has height \( < \sqrt{D} \) and proves the proposition. ■

Proposition 2 answers Question 1 for real quadratic fields in an effective way. For the rest of this section we want to study the asymptotic behavior of \( \frac{H_{\text{min}}(D)}{\sqrt{D}} \).

We know already

\[
\frac{1}{\sqrt{5}} \leq \frac{H_{\text{min}}(D)}{\sqrt{D}} < 1 \quad \text{and} \quad \liminf_{D \to \infty} \frac{H_{\text{min}}(D)}{\sqrt{D}} = \frac{1}{\sqrt{5}}.
\]

The following table lists discriminants \( D \) where \( \frac{H_{\text{min}}(D)}{\sqrt{D}} \) is maximal in the specified range. \( ax^2 + bx + c \) is the minimal polynomial of a minimal generator.
We will study \( \limsup_{D \to \infty} \frac{H_{\min}(D)}{\sqrt{D}} \) in two ways each of which depends on certain conjectures.

3.1. **Assuming the existence of primes with certain properties.** Let \( M_\varepsilon \) be the set of all discriminants \( D \) of real quadratic fields \( \mathbb{Q}(\sqrt{D}) \) such that there is an odd prime \( p \) with \( \left( \frac{D}{p} \right) = 1 \) and \( \frac{1}{2} \sqrt{D} \leq p \leq \left( \frac{1}{2} + \varepsilon \right) \sqrt{D} \).

Then we get the

**Lemma 2.** If \( D \in M_\varepsilon \) and \( 0 < \varepsilon \leq \frac{1}{2} \) the real quadratic field \( \mathbb{Q}(\sqrt{D}) \) has a generator \( \alpha \) with

\[
H(\alpha) \leq \left( \frac{1}{2} + \varepsilon \right) \sqrt{D}.
\]

**Proof:** By definition there is an odd prime \( p \) with

\[
\left( \frac{D}{p} \right) = 1 \quad \text{and} \quad \frac{1}{2} \sqrt{D} \leq p \leq \left( \frac{1}{2} + \varepsilon \right) \sqrt{D}.
\]

Choose \( b \in \mathbb{Z} \) with \( 0 \leq b \leq p \) and \( b^2 \equiv D \mod p \). The number \( p - b \) satisfies the same conditions so that we can assume \( b^2 \equiv D \mod 2p \) which implies \( b^2 \equiv D \mod 4p \) as \( D \equiv 0, 1 \mod 4 \). Define \( c \in \mathbb{Z} \) as \( c = \frac{b^2 - D}{4p} \). As \( b^2 < D \) we have \( c < 0 \). It follows

\[
|c| = -c = \frac{D - b^2}{4p} \leq \frac{D}{4p} \leq \frac{1}{2} \sqrt{D}.
\]

The element \( \alpha \) with minimal polynomial \( f = px^2 + bx + c \) generates \( \mathbb{Q}(\sqrt{D}) \) and satisfies

\[
H(\alpha) \leq \left( \frac{1}{2} + \varepsilon \right) \sqrt{D}
\]

which proves the lemma.

If \( \varepsilon > 0 \) is fixed and \( D \) is large there are many primes \( p \) with \( \frac{1}{2} \sqrt{D} \leq p \leq \left( \frac{1}{2} + \varepsilon \right) \sqrt{D} \). As \( \left( \frac{D}{p} \right) \) is a quadratic character the following conjecture seems plausible:

**Conjecture 2.** For every \( \varepsilon > 0 \) there is a \( D_\varepsilon \) such that \( D > D_\varepsilon \) implies \( D \in M_\varepsilon \) if \( D \) is a real quadratic discriminant.

After some computations we conjecture e.g. \( D > 981913 \Rightarrow D \in M_{0.1} \).

I do not know how to attack conjecture 2. A natural question seems to be:
Question 2. Can conjecture 2 be deduced from ERH (extended Riemann hypothesis)?

An immediate consequence of Lemma 2 is

**Corollary 1.** If conjecture 2 holds then

$$\limsup_{D \to \infty} \frac{H_{\min}(D)}{\sqrt{D}} \leq \frac{1}{2}$$

where $D$ runs through the discriminants of real quadratic fields.

3.2. **Assuming the existence of reduced elements with certain properties.**

Let $D$ be the discriminant of a real quadratic field and define the set of reduced elements by

$$\Lambda_D = \{ \alpha = \frac{b + \sqrt{D}}{2a} : b^2 - 4ac = D, \alpha > 1, -1 < \alpha' < 0, a, b, c \in \mathbb{Z} \}$$

(where $\alpha'$ is the conjugate of $\alpha$ and $\sqrt{D} > 0$). The map $\alpha \mapsto \frac{1}{\alpha - |\alpha|}$ induces a decomposition of $\Lambda_D$ in $h(D)$ cycles where $h(D)$ is the class number of $\mathbb{Q}(\sqrt{D})$ [Cohen, p.260]. It is also known that $\lim_{D \to \infty} \log |\Lambda_D| \log \sqrt{D} = 1$ [Lachaud] so the elements of $\Lambda_D$ play a similar role as the elements of $\Lambda_D$ in the imaginary quadratic case.

The smallest generators of a field need not be reduced as the following example shows:

**Example:** The real quadratic field $K = \mathbb{Q}(\sqrt{635})$ with discriminant $D = 2540$ has as a smallest generator $\alpha = \frac{15 + \sqrt{635}}{10}$ with minimal polynomial $10x^2 - 30x - 41$ and height $H(\alpha) = 41$. Among the reduced elements $\beta = \frac{25 + \sqrt{635}}{10}$ (with minimal polynomial $x^2 - 50x - 10$) has the smallest height $H(\beta) = 50$.

Nevertheless we have the following lemma:

**Lemma 3.** If $\alpha$ generates $K$ with discriminant $D$ and $H(\alpha) \leq 0.48\sqrt{D}$ then one of the elements $\alpha, \alpha', -\alpha, -\alpha'$ is reduced.

**Proof:** Let $ax^2 - bx + c$ be the minimal polynomial of $\alpha$ with $a > 0$. Changing from $\alpha$ to $-\alpha$ we can assume $b \geq 0$. Then we have is $f \in \mathbb{N}$ with $b^2 - 4ac = Df^2$. As $a, b, |c| \leq 0.48\sqrt{D}$ we get $Df^2 \leq 5 \cdot 0.48^2 D$ and therefore $f = 1$. So we have without restriction

$$\alpha = \frac{b + \sqrt{D}}{2a} \quad \text{and} \quad \alpha' = \frac{b - \sqrt{D}}{2a}.$$ 

As $D = b^2 - 4ac \leq b^2 + 4 \cdot 0.48^2 D$ we get $b \geq 0.28\sqrt{D}$ and $D \leq 0.48^2 D + 4 \cdot 0.48\sqrt{D} \cdot a$ gives $a \geq 0.40\sqrt{D}$. Therefore

$$\alpha = \frac{b + \sqrt{D}}{2a} \geq \frac{0.28\sqrt{D} + \sqrt{D}}{2 \cdot 0.48\sqrt{D}} \geq 1.33 > 1$$

and

$$0 > \alpha' = \frac{b - \sqrt{D}}{2a} \geq \frac{0.28\sqrt{D} - \sqrt{D}}{2a} = -\frac{0.72\sqrt{D}}{2a} \geq -\frac{0.72\sqrt{D}}{2 \cdot 0.40\sqrt{D}} > -1$$

which shows that $\alpha$ is reduced. $\blacksquare$
Lemma 4. Let $K$ be a real quadratic field with discriminant $D$. If $\alpha \in \Lambda_D$ then
\[
\frac{H(\alpha)}{\sqrt{D}} = \max\left(\frac{1}{\alpha - \alpha'}, \frac{\alpha + \alpha'}{\alpha - \alpha'}, \frac{\alpha(-\alpha')}{\alpha - \alpha'}\right).
\]

Proof: If the minimal polynomial of $\alpha$ is $ax^2 - bx + c$ then
\[
|a| = a = \sqrt{D} \cdot \frac{1}{\alpha - \alpha'}, \quad |b| = b = \sqrt{D} \cdot \frac{\alpha + \alpha'}{\alpha - \alpha'}, \quad |c| = -c = \sqrt{D} \cdot \frac{\alpha(-\alpha')}{\alpha - \alpha'},
\]
which gives the result. \[\Box\]

Let $G = \{(x, y) \in \mathbb{R}^2 : x > 1, -1 < y < 0\}$ and $\tilde{\Lambda}_D = \{(\alpha, \alpha') : \alpha \in \Lambda_D\}$, then $\tilde{\Lambda}_D \subseteq G$. Define for $0 < h < 1$ the set
\[
G_h = \{(x, y) \in G : \frac{1}{x - y} \leq h, \ \frac{x + y}{x - y} \leq h, \ \frac{x(-y)}{x - y} \leq h\},
\]
which can be written as
\[
G_h = \{(x, y) \in G : y \leq x - \frac{1}{h}, \ y \leq -\frac{1 - h}{1 + h} x, \ y \geq -\frac{hx}{x - h}\}.
\]
If $(\alpha, \alpha') \in \tilde{\Lambda}_D \cap G_h$ then the lemma gives $H(\alpha) \leq h\sqrt{D}$. It is not difficult to see that $G_h = \emptyset$ for $h < \frac{1}{\sqrt{5}}$. $G\frac{1}{\sqrt{5}} = \{\left(\frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}\right)\}$ and $G_h$ is the closure of an open non empty set for $h > \frac{1}{\sqrt{5}}$.

Now we formulate a conjecture for reduced real quadratic numbers:

Conjecture 3. If $U$ is an open non empty subset of $G$ then there is a $c_U$ such that $U \cap \tilde{\Lambda}_D \neq \emptyset$ for all $D > c_U$.

It would be interesting to know if Conjecture 3 can be deduced from Duke’s results for real quadratic fields [Duke].

Corollary 2. Assuming Conjecture 3 we get
\[
\lim_{D \to \infty} \frac{H_{\text{min}}(D)}{\sqrt{D}} = \frac{1}{\sqrt{5}},
\]
where $D$ runs through the discriminants of real quadratic fields.

Proof: Let $\epsilon > 0$ be sufficiently small. Then there is a $d_\epsilon$ such that $G\frac{1}{\sqrt{5}} + \epsilon \cap \tilde{\Lambda}_D \neq \emptyset$ for all discriminant $D > d_\epsilon$. For $D > d_\epsilon$ take $\alpha \in \Lambda_D$ with $(\alpha, \alpha') \in \tilde{\Lambda}_D \cap G\frac{1}{\sqrt{5}} + \epsilon$. Then
\[
\frac{1}{\sqrt{5}} \leq \frac{H(\alpha)}{\sqrt{D}} \leq \frac{1}{\sqrt{5}} + \epsilon,
\]
which implies at once our statement. \[\Box\]

We conclude this section with numerical examples. Let $H_{\text{min, red}}(D)$ be the minimal height of all elements in $\Lambda_D$. The following table gives the discriminant $D$ where $\frac{H_{\text{min, red}}(D)}{\sqrt{D}}$ is maximal in the specified range. $ax^2 - bx + c$ is the minimal
polynomial of a corresponding element of $\Lambda_D$. Finally *average* gives the average value of all $\frac{H_{min,red}(D)}{\sqrt{D}}$ in the given range.

| $D$               | $(a, b, c)$           | $\frac{H_{min,red}(D)}{\sqrt{D}}$ | average |
|-------------------|----------------------|-----------------------------------|---------|
| $1 \leq D \leq 10000$ | 908                  | (1.30,-2)                         | 0.9956  |
| $10000 \leq D \leq 20000$ | 14693               | (19.109,-37)                     | 0.8992  |
| $20000 \leq D \leq 30000$ | 24173               | (23.115,-119)                    | 0.7654  |
| $30000 \leq D \leq 40000$ | 37532               | (38.122,-149)                    | 0.7691  |
| $40000 \leq D \leq 50000$ | 49013               | (37.153,-173)                    | 0.7814  |
| $50000 \leq D \leq 60000$ | 54053               | (47.153,-163)                    | 0.7011  |
| $60000 \leq D \leq 70000$ | 69893               | (97.173,-103)                    | 0.6544  |
| $70000 \leq D \leq 80000$ | 79805               | (95.105,-181)                    | 0.6407  |
| $80000 \leq D \leq 90000$ | 87533               | (79.159,-197)                    | 0.6659  |
| $90000 \leq D \leq 100000$ | 95672               | (106.128,-187)                   | 0.6046  |
| $10^6 \leq D \leq 10^6 + 10000$ | 104093             | (83.195,-199)                    | 0.6168  |
| $10^6 \leq D \leq 10^6 + 10000$ | 1006232             | (463.194,-523)                   | 0.5214  |
| $10^7 \leq D \leq 10^7 + 10000$ | 10000973            | (1423.1029,-1571)                | 0.4968  |

4. Concluding Remarks

The following example shows that for every $n$ there are infinitely many number fields of degree $n$ such that an estimate as in question 1 holds.

**Example:** Let $n$ be an integer $\geq 2$ and $p, q$ primes with $p < q < 2p$. Let $\alpha$ be a zero of $f = px^n + q$ and $K = \mathbb{Q}(\alpha)$. Then $K$ has degree $n$ over $\mathbb{Q}$ and the primes $p$ and $q$ are totally ramified in $K$, so $p^{n-1}$ and $q^{n-1}$ divide $D_K$. Therefore we get the estimate

$$H(\alpha) = q < \sqrt{2pq} = \sqrt{2(p^{n-1}q^{n-1})^{\frac{1}{2n-2}}} \leq \sqrt{2} |D_K|^{\frac{1}{2n-2}}.$$

In the next example ‘small’ (integral) generators are constructed with Minkowski’s theorem.

**Example:** Let $K$ be a totally real number field of prime degree $n$, $\alpha_1, \ldots, \alpha_n$ an integral basis of $K$ and $\sigma_1, \ldots, \sigma_n$ the different embeddings $K \rightarrow \mathbb{R}$. As $|\det(\sigma_j, \alpha_j)| = \sqrt{|D_K|}$ there is $(x_1, \ldots, x_n) \in \mathbb{Z}^n \setminus \{0\}$ by Minkowski’s linear forms theorem [Hua, p.540] such that $\alpha = x_1\alpha_1 + \cdots + x_n\alpha_n$ satisfies

$$|\sigma_1\alpha| < 1 \quad \text{and} \quad |\sigma_2\alpha|, \ldots, |\sigma_n\alpha| \leq |D_K|^{\frac{1}{2n-2}}.$$  

The condition $|\sigma_1\alpha| < 1$ implies $\alpha \notin \mathbb{Z}$, so $K = \mathbb{Q}(\alpha)$. Let $f = x^n + a_1x^{n-1} + \cdots + a_n$ be the minimal polynomial of $\alpha$. Then for $1 \leq j \leq n - 1$ we get $|a_j| \leq \binom{n}{j} |D_K|^{\frac{1}{2n-2}} \leq 2^n |D_K|^{\frac{1}{2n-2}}$ and $|a_n| \leq \sqrt{|D_K|}$, therefore

$$H(\alpha) \leq 2^n |D_K|.$$  

In case $n \geq 3$ this is far away from what we would like to have.
REFERENCES

[Cohen] H. Cohen, A Course in Computational Algebraic Number Theory, GTM 138, Springer-Verlag 1993.

[Duke] W. Duke, Hyperbolic distribution problems and half-integral weight Maass forms, Invent. math. 92 (1988), 73-90.

[Hua] Hua Loo Keng, Introduction to Number Theory, Springer-Verlag 1982.

[Lachaud] G. Lachaud, On Real Quadratic Fields, Bull. of the AMS (N.S.) 17 (1987), 307-311.

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