Abstract—In this work, we used deep neural networks (DNNs) to solve a fundamental problem in differential geometry. One can find many closed-form expressions for calculating curvature, length, and other geometric properties in the literature. As we know these concepts, we are highly motivated to reconstruct them by using deep neural networks. In this framework, our goal is to learn geometric properties from examples. The simplest geometric object is a curve. Therefore, this work focuses on learning the length of planar sampled curves created by a sine waves dataset. For this reason, the fundamental length axioms were reconstructed using a supervised learning approach. Following these axioms a simplified DNN model, we call ArcLengthNet, was established. The robustness to additive noise and discretization errors were tested.

Index Terms—Differential geometry, length learning, planar Euclidean curves, supervised learning.

I. INTRODUCTION

The calculation of curve length is the major component in many classical and modern problems involving numerical differential geometry [1], [2]. For example, a handwritten signature involves the computation of the length along the curve [3]. Several numerical constraints affect the quality of the length calculation; additive noise, discretization error, and partial information. A robust approach to handle it is required.

Lastly, Machine Learning (ML) has become highly popular. It has achieved great success in solving many classification, regression and anomaly detection tasks [4]. A sub-field of ML is the Deep Neural Networks (DNN), which outperforms many classic methods by design deep architectures [4]. An efficient DNN architecture finds intrinsic properties by using a convolutional operator (and some more sophisticated operators) and generalize them. Their success is related to the enormous amount of data and their capability to optimize it by high computational available resources. In this work, we address a fundamental question in the field of differential geometry [5] and we aim to reconstruct a basic property using DNN. The simplest geometric object is a curve. To characterize a curve, one can define metrics such as length and curvature, and distinguish one curve from another. There are many close form expressions for calculation of the length, curvature, and other geometric properties in the classical literature [6]. However, since we know the powerful functions of DNN, we are highly motivated to reconstruct the fundamental length property for curves, the arclength, by designing a DNN. We focused on the two-dimensional Euclidean domain. The formulation of this task was done in a supervised learning method where a data-dependent learning-based approach was applied by feeding each example at a time through our DNN model and by minimizing a unique loss function that satisfies the length axioms. For simplicity, we focused on sine wave curves and created a dataset by tuning the wave amplitude, phase, translation and rotation to cover a wide-range of geometric representation. The resulted trained DNN was called ArcLengthNet. It obtains a 2D vector as an input, represents the planar Euclidean sampled curve, and outputs their respective length.

Related papers in the literature mainly address a higher level of geometric information by deep learning approach [7], [8]. Saying that, a fundamental property was reconstructed by DNN in [9], where a curvature-based invariant signature was learned by using a Siamese network configuration [10]. They presented the advantages of using DNN to reconstructing the curvature signature, among which it mainly results in robustness to noise and sampling errors.

The main contributions of this work is to reconstruct the length property. For that, two architectures were designed. One is based on Convolutional Neural Networks (CNNs), and the other is based on Recurrent Neural Networks (RNNs). We showed that the CNN-based architecture overcomes the RNN-based architecture, and by that establishing the ArcLengthNet as a CNN based architecture. The advantage of the ArcLengthNet is presented.

The remainder of the paper is organized as follows: Section 2 summarizes the geometric background of the length properties. Section 3 provides a detailed description of the learning approach were the two architectures are presented. Section 4 presents the results followed by the discussion. Section 5 gives the conclusions.

II. GEOMETRIC BACKGROUND OF ARCLength

In this section, the length properties are presented and the discretization error is reviewed.

A. Length properties

Consider a planar parametric differential curve in the Euclidean space, \( C(p) = \{ x(p), y(p) \} \in \mathbb{R}^2 \), where \( x \) and \( y \) are the curve coordinates parameterized by parameter \( p \in [0, N] \), where \( N \) is a partition parameter. The Euclidean length of the curve, is given by

\[
\ell_p = \int_0^N |C_p(\tilde{p})| \, d\tilde{p} = \int_0^N \sqrt{x_p^2 + y_p^2} \, d\tilde{p},
\]

(1)

where \( x_p = \frac{dx}{dp}, y_p = \frac{dy}{dp} \). Summing all the increments results in the total length of \( C \), given by

\[
L = \int_0^N |C_p(\tilde{p})| \, d\tilde{p}.
\]

(2)
Following the length definition, the main length axioms are provided.

**Additivity**: The length additives with respect to concatenation, where for any \( C_1 \) and \( C_2 \) the following holds
\[
\mathcal{L}(C_1) + \mathcal{L}(C_2) = \mathcal{L}(C_1 \cup C_2).
\]
\begin{equation}
(3)
\end{equation}

**Invariant**: length is invariant with respect to rotation (R) and translation (T),
\[
\mathcal{L}\left((\mathbf{T} + \mathbf{R})C\right) = \mathcal{L}(C).
\]
\begin{equation}
(4)
\end{equation}

**Monotonic**: length is monotone, where for any \( C_1 \) and \( C_2 \) the following holds
\[
\mathcal{L}(C_1) \leq \mathcal{L}(C_2), C_1 \subset C_2.
\]
\begin{equation}
(5)
\end{equation}

**Non-negativity**: The length of any curve is non-negative,
\[
\mathcal{L}(C) \geq 0.
\]
\begin{equation}
(6)
\end{equation}

In order to reconstruct the length property by DNN, a discretization of the curve should be applied. As a consequence, it prone to errors.

**B. Discretization error**

The curve \( C \) lies on a close interval \([\alpha, \beta] \). In order to find the length by a discretized process, a partition of the interval is done, where
\[
\mathcal{P} = \{\alpha = p_0 < p_1 < p_2 < \cdots < p_N = \beta\}.
\]
\begin{equation}
(7)
\end{equation}

For every partition \( \mathcal{P} \), the curve length can be represented by the sum
\[
s(\mathcal{P}) = \sum_{n=1}^{N} |C(p_n) - C(p_{n-1})|.
\]
\begin{equation}
(8)
\end{equation}

The discretization error is given by,
\[
ed = \mathcal{L} - s(\mathcal{P})
= \int_{0}^{N} |C_p(\mathcal{P})| \, dp - \sum_{n=1}^{N} |C(p_n) - C(p_{n-1})|,
\]
\begin{equation}
(9)
\end{equation}

where obviously, \( ed \to 0 \) when \( N \to \infty \) (for further reading, the reader refers to [11]). Fig. 1 illustrates a general curve with their discretized representation.
### LENGTH LEARNING FOR PLANAR EUCLIDEAN CURVES

#### Fig. 2. Curve examples

#### Fig. 3. ArcLengthNet architecture

example index, $\lambda$ is a regularization parameter, $\Theta_{ij}$ are the various DNN weight, and $\|\cdot\|_2$ is the two dimension norm. By minimizing $J_k$ by passing the examples through a model, the weights were tuned. The optimized model is characterized by the optimal weights that provided by

$$
\Theta^*_{ij} = \arg\min \Theta_{ij} \sum_k J_k. \quad (13)
$$

#### D. ArcLengthNet Architecture

In this subsection the ArcLengthNet is presented (the CNN-based architecture). The model architecture is very simplified, including a convolutional layer and two fully-connected layers with only one activation function. Each curve is represented by $N = 200$ points. This representation is inserted into a convolutional layer with a small kernel of size 3. It is processed into a fully connected layer that outputs only 10 weights through a Rectified Linear Unit (ReLU) activation function to another fully connected layer which finally outputs the length. The architecture is shown in Fig.3.

#### E. ArcLengthNet Training

The DNN was trained by passing many examples in small batches with a back-propagation method. The training process was carried out in batches of 200 examples for 100 epochs. The optimizer we used is Stochastic Gradient Descent (SGD) with momentum and weight decay [16]. Various parameters are provided in Table 1. Fig.4 shows a graph of the train and test losses as a function of the number of epochs.

#### F. LSTM-based Architecture

The task of length learning can be interpreted as a time-series based learning. Each point of the curve can be interpreted as a time step with $x$ and $y$ coordinates. The DNN aims to generalize the local properties into a global length. The typical architecture to deal with time-series data is the Recurrent Neural Network (RNN). One modification of RNN is the Long Short-Term Memory (LSTM) architecture, where has feedback connections (in opposite to the classical RNN). The LSTM is used in handwritten recognition tasks with a great success [17]. Motivated by this success, we designed an LSTM architecture, presented in Fig.5. Similar to ArcLengthNet, the LSTM-based architecture is simplified, including 200 blocks with 1 vector outputs. They concatenated and inserted to two fully connected layers (10 each) that outputs connected with a ReLU activation function. Fig.6 shows a graph of train and test losses as a function of the number of epochs.

#### IV. RESULTS AND DISCUSSION

Both architectures were trained in the same approach with the same database. As shown in Fig.3 and Fig.6 these architectures were well trained after 100 epochs. A holdout set was defined to test the performance of the architectures on unseen data. This set contains 5,000 examples that have not been used in train set or test test. The ArcLengthNet obtained a minimum MSE of 0.17, where the LSTM based

#### TABLE I

| Description          | Symbol | Value   |
|----------------------|--------|---------|
| Number of examples   | $K$    | 20,000  |
| Train/test ratio     |        | 80/20   |
| Regularization parameter | $\lambda$ | 0.01   |
| Partition parameter  | $N$    | 200     |
| Batch size           |        | 200     |
| Learning rate        | $\eta$ | 0.001   |
| Momentum             |        | 0.9     |
| Weight decay         |        | 0.0005  |
| Epochs               |        | 100     |
Fig. 5. LSTM-based architecture

Fig. 6. LSTM-based architecture training

Fig. 7. Monotonic property on holdout set (ArcLengthNet)

criterion: RMSE-over-Mean-Length-Ratio (RMLR), given by

$$RMLR = \frac{\text{RMSE}}{\mathbb{E}[L]},$$  \hspace{1cm} (14)

where $\mathbb{E}$ is the expected value operator. This measure provides a normalized error with respect to the curve length. Various curves of different lengths, we must appropriately weigh the errors.

V. CONCLUSION AND FURTHER WORK

A learning-based approach to reconstruct the length of curves was presented. The powerful of deep neural networks to reconstruct the fundamental axioms was demonstrated. The results can be further used to improve handwritten signatures, and reconstruct some more differential geometry properties and theorems.

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