A Simple Proof of an Inequality Connecting the Alternating Number of Independent Sets and the Decycling Number

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Abstract

If \( s_k \) denotes the number of independent sets of cardinality \( k \) and \( \alpha(G) \) is the size of a maximum independent set in graph \( G \), then \( I(G; x) = s_0 + s_1 x + \ldots + s_{\alpha(G)} x^{\alpha(G)} \) is the independence polynomial of \( G \) [8].

In this paper we provide an elementary proof of the inequality \(|I(G; -1)| \leq 2^{\varphi(G)}\), where \( \varphi(G) \) is the decycling number of \( G \).

Keywords: independent set, independence polynomial, decycling number, forest, cyclomatic number.

1 Introduction

Throughout this paper \( G = (V, E) \) is a finite, undirected, loopless and without multiple edges graph, with vertex set \( V = V(G) \) and edge set \( E = E(G) \). By \( G - W \) we mean the subgraph induced by \( V - W \). The set \( N(v) = \{w : w \in V \text{ and } vw \in E\} \) neighborhood of the vertex \( v \in V \), and \( N[v] = N(v) \cup \{v\} \). A leaf is a vertex having a unique neighbor.

A set of pairwise non-adjacent vertices is called independent. The independence number of \( G \), denoted by \( \alpha(G) \), is the cardinality of a maximum independent set.

If \( G \) has \( s_k \) independent sets of size \( k \), then

\[
I(G; x) = s_0 + s_1 x + s_2 x^2 + \ldots + s_{\alpha(G)} x^{\alpha(G)}
\]

is known as the independence polynomial of \( G \) [8]. Some properties of the independence polynomial are presented in [11] [14] [12] [13] [14].

The value of a graph polynomial at a specific point can give sometimes a very surprising information about the structure of the graph [2]. In the case of independence polynomials, let us notice that:

- \( I(G; 1) = s_0 + s_1 + s_2 + \ldots + s_{\alpha} \) equals the number of independent sets of \( G \). It is known as the Fibonacci number of \( G \) [11] [16] [17].
• $I(G; -1) = s_0 - s_1 + s_2 - \ldots + (-1)^n s_n$ is equal to difference of the numbers of independent sets of even and odd sizes. It is known as the alternating number of independent sets [3]. The value of $|I(G; -1)|$ can be any non-negative integer. For instance, $|I(K_{a,a,\ldots,a}; -1)| = n - 1$, where $K_{a,a,\ldots,a}$ is the complete n-partite graph.

• $I(G; -1) = -\chi(\text{Ind}(G))$, where $\chi(\Sigma)$ is the reduced Euler characteristic of the abstract simplicial complex $\Sigma$. Recall that an abstract simplicial complex on a finite vertex set $A$ is a subset $\Sigma$ of $2^A$ satisfying: $\{v\} \in \Sigma$ for every $v \in A$, and $A \subseteq B \in \Sigma$ implies $A \in \Sigma$. The elements of $\Sigma$ are faces and the dimension of a face $A$ is $|A| - 1$. For a simplicial complex with $s_i$ faces of dimension $i - 1$, the reduced Euler characteristic equals $-s_0 + s_1 - s_2 + s_3 - \ldots$. The family $\text{Ind}(G)$ of all independent sets of a graph $G = (V, E)$ forms a simplicial complex on $V$, called the independence complex of $G$ [9].

The cyclomatic number $\nu(G)$ of the graph $G$ is the dimension of the cycle space of $G$, i.e., the dimension of the linear space spanned by the edge sets of all the cycles of $G$. The decycling number [3] (or the feedback vertex number [15]) $\varphi(G)$ of a graph $G$ is the minimum number of vertices that need to be removed in order to eliminate all its cycles. While $\varphi(G)$ can be easily computed, since $\nu(G)$ is the decycling number of $\varphi(G)$, it is known that to compute $\varphi(G)$ is an NP-complete problem [10].

The inequality $|I(G; -1)| \leq 2^{\varphi(G)}$ has been established in [15], while a stronger result, namely, $|I(G; -1)| \leq 2^{\nu(G)}$ has been proved in [7].

In this paper we provide a simple proof of the inequality $|I(G; -1)| \leq 2^{\nu(G)}$ using only elementary arguments.

2 Results

Proposition 2.1 [5] If $v \in V(G)$, then $I(G; x) = I(G - v; x) + x \cdot I(G - N[v]; x)$.

Theorem 2.2 For any graph $G$ the alternating number of independent sets is bounded as follows

$$|I(G; -1)| \leq 2^{\varphi(G)},$$

where $\varphi(G)$ is the decycling number of $G$.

Proof. We establish the inequality by induction on $\varphi(G)$.

• If $\varphi(G) = 0$, then $G$ is a forest, and we have to show that $|I(G; -1)| \leq 1$.

We proceed by mathematical induction on $n = |V(G)|$.

For $n = 0$, $I(G; x) = 1$ and $I(G; -1) = 1$, while for $n = 1$, $I(G; x) = 1 + x$ and $I(G; -1) = 0$. Suppose that $G$ is a forest with $|V(G)| = n \geq 2$.

If $G$ has no leaves, then $I(G; x) = (1 + x)^n$ and $I(G; -1) = 0$. Otherwise, let $v$ be a leaf of $G$ and $N(v) = \{u\}$. According to Proposition 2.1 we obtain that

$$I(G; x) = I(G - u; x) + x \cdot I(G - N[u]; x) = (1 + x) \cdot I(G - \{u, v\}; x) + x \cdot I(G - N[u]; x).$$

Hence, by the induction hypothesis, we finally get

$$|I(T; -1)| = |(-1) \cdot I(T - N[u]; -1)| \leq 1.$$
Assume that the statement is true for graphs with the decycling number $\varphi(G) \leq k$.

Let $G$ be a graph with $\varphi(G) = k + 1$. Clearly, there exists some $v \in V(G)$, such that $\varphi(G - v) < \varphi(G)$. According to Proposition 2.1 we get:

$$I(G; -1) = I(G - v; -1) - I(G - N[v]; -1).$$

By the induction hypothesis, it assures that

$$|I(G; -1)| \leq |I(G - v; -1)| + |I(G - N[v]; -1)| \leq 2 \cdot 2^k = 2^{\varphi(G)},$$

and this completes the proof.

Notice that if $G = qK_3$, then $I(G; x) = (1 + 3x)^q$ and hence, $I(G; -1) = (-2)^{\varphi(G)}$.

**Conjecture 2.3** For every positive integer $k$ and each integer $q$ such that $|q| \leq 2^k$, there is a graph $G$ with $\varphi(G) = k$ and $I(G; -1) = q$.

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