INTEGRAL PRESENTATIONS OF THE SHIFTED CONVOLUTION PROBLEM AND SUBCONVEXITY ESTIMATES FOR GL\textsubscript{n}-AUTOMORPHIC L-FUNCTIONS

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Abstract. Fix \( n \geq 2 \) an integer, and let \( F \) be a totally real number field. We reduce the shifted convolution problem for \( L \)-function coefficients of \( \text{GL}_n(\mathbb{A}_F) \)-automorphic forms to the better-understood setting of \( \text{GL}_2(\mathbb{A}_F) \) via new integral presentations derived from Fourier-Whittaker expansions of projected cusp forms. As one application of this reduction, we derive the following uniform asymptotic subconvexity bound for \( \text{GL}_n(\mathbb{A}_F) \)-automorphic \( L \)-functions twisted by Hecke characters. Let \( \Pi \) be an irreducible cuspidal automorphic representation of \( \text{GL}_n(\mathbb{A}_F) \), and let \( \chi \) be a Hecke character of \( F \) of conductor \( q \). Let \( \theta_0 \) be the best known approximation to the generalized Ramanujan-Petersson conjecture for \( \text{GL}_2(\mathbb{A}_F) \)-automorphic forms; hence \( \theta_0 = 0 \) is conjectured, and \( \theta_0 = 7/64 \) is admissible by the theorem of Blomer and Brumley. Writing \( L(s, \Pi \otimes \chi) \) to denote the finite part of the standard \( L \)-function of \( \Pi \otimes \chi \), normalized to have central value at \( s = 1/2 \), we show that for any \( \varepsilon > 0 \) we have the upper bound

\[
L(1/2, \Pi \otimes \chi) \ll_{\Pi, \chi_{\infty}, \varepsilon} N_q^{\theta_0/2 + \varepsilon} N_q^{\varepsilon},
\]

and even \( L(1/2, \Pi \otimes \chi) \ll_{\Pi, \chi_{\infty}, \varepsilon} N_q^{\theta_0/2 + \varepsilon} N_q^{\varepsilon} \) if \( n \geq 4 \). Here, the implied constants depend only on the representation \( \Pi \), the archimedean component \( \chi_{\infty} \) of \( \chi \), and the totally real field \( F \). This estimate appears to the the first of its kind in higher dimensions.

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1. Introduction

Let $F$ be a totally real number field of degree $d = [F : \mathbb{Q}]$ and adele ring $\mathbb{A}_F$. Let $\mathbb{A}_F^\times$ denote the ideles, with $F_\infty = F \otimes \mathbb{R} \approx \mathbb{R}^d$ the archimedean component, and $| \cdot |$ the idele norm. Fix an integer $n \geq 2$. Let $\Pi = \otimes_v \Pi_v$ be an irreducible cuspidal automorphic representation of $\text{GL}_n(\mathbb{A}_F)$ of unitary central character $\omega$. Let $\Lambda(s, \Pi) = L(s, \Pi_{\mathbb{A}}) L(s, \Pi)$ denote the standard $L$-function of $\Pi$, where for $\Re(s) > 1$ we write the Dirichlet series expansion of the Euler product over finite places $v$ of $F$ (identified with primes $v \subset \mathcal{O}_F$) as

$$L(s, \Pi) = \prod_{v \neq \infty} L(s, \Pi_v) = \sum_{m \subset \mathcal{O}_F} c_{\Pi}(m)Nm^{-s}.$$  

We show in this work essentially how to reduce the shifted convolution problem for sums of the $L$-function coefficients $c_{\Pi}$ to the better-understood setting of dimension $n = 2$, so that various arguments such as those of Blomer-Harcos [6], Templier-Tsimerman [33] and others for $\text{GL}_2(\mathbb{A}_F)$ apply to derive completely new estimates in dimensions $n \geq 3$. The key insight is that such shifted convolution sums can be realized as Fourier-Whittaker expansions of certain $L^2$-automorphic forms of trivial central character on $\text{GL}_2(\mathbb{A}_F)$ and its two-fold metaplectic cover $\widetilde{\text{GL}}(\mathbb{A}_F)$. More precisely, we derive integral presentations for sums of the coefficients $c_{\Pi}$ along progressions defined by linear shifts of arbitrary positive definite quadratic forms (Propositions 3.4 and 4.2 (A)), as well as for linear shifts with the $L$-function coefficients of an auxiliary cuspidal $\text{GL}_m(\mathbb{A}_F)$-automorphic representation for any dimension $m \geq 2$ (Propositions 3.5 and 4.2 (B)). To derive such presentations, we use the classical projection operator $\mathbb{P}^m_1$ sending cuspidal automorphic forms on $\text{GL}_n(\mathbb{A}_F)$ to cuspidal automorphic forms on the mirabolic subgroup $P_2(\mathbb{A}_F) \subset \text{GL}_2(\mathbb{A}_F)$, together withCogdell’s theory of Eulerian integral presentations for $L$-functions on $\text{GL}_n(\mathbb{A}_F) \times \text{GL}_1(\mathbb{A}_F)$, and the ability to choose a pure tensor $\varphi = \otimes_v \varphi_v \in V_\Pi$ whose nonarchimedean local components $\varphi_v$ are each essential Whittaker vectors (as we can at ramified places thanks to Matringe [28]). That is, we use purely representation theoretic methods to describe the shifted convolution problem for $\text{GL}_n(\mathbb{A}_F)$ in this unconventional way. The Fourier-Whittaker expansions of such projected pure tensors can be related via a shifted Mellin transform to the Dirichlet series $L(s, \Pi)$, as we explain for Theorem 2.2 and Corollary 2.3 below. Hence, the sums we wish to consider can be related to the Fourier-Whittaker coefficients of certain $L^2$-automorphic forms on the mirabolic subgroup $P_2(\mathbb{A}_F)$ of $\text{GL}_2(\mathbb{A}_F)$. Using strong approximation and Iwasawa decomposition to lift to a well-defined $L^2$-automorphic form of trivial weight and central character on $\text{GL}_2(\mathbb{A}_F)$ (in §2.3 below), relating Fourier-Whittaker coefficients (see Propositions 2.8, 2.9, 2.10, and 4.2 especially), we can then reduce the shifted convolution problem for $\text{GL}_n(\mathbb{A}_F)$ to the less mysterious setting of dimension $n = 2$. That is, decomposing a certain lifted $L^2$-automorphic form on $\text{GL}_2(\mathbb{A}_F)$ or $\widetilde{\text{GL}}(\mathbb{A}_F)$ which represents the shifted convolution problem spectrally, we can justify the convergence of the corresponding spectral coefficients by either a standard argument with Sobolev norms for the case of dimension $n = 2$ (Lemma 4.8), or else via a direct estimate in terms of the inner product with some Poincaré series (Proposition 4.9) in the style of [31, §2] for the case of dimensions $n \geq 3$. This in effect allows us to derive bounds for the shifted convolution problem for $\text{GL}_n(\mathbb{A}_F)$ in terms of existing estimates for Whittaker functions and generic bounds for Fourier-Whittaker coefficients of automorphic forms on $\text{GL}_2(\mathbb{A}_F)$ and $\widetilde{\text{GL}}(\mathbb{A}_F)$ as predicted by the generalized Ramanujan conjecture and the generalized Lindelöf hypothesis respectively. In this way, we derive the following uniform estimates for the shifted convolution problem for coefficients of automorphic $L$-function coefficients in all dimensions $n \geq 2$, which are completely new in dimensions $n \geq 3$. Let us write $\mathbb{F}_\infty^\times, + \cong \mathbb{R}^d_{>0}$ to denote the totally positive archimedean adeles.

**Theorem 1.1.** Fix $n \geq 2$ an integer. Let $\Pi = \otimes_v \Pi_v$ be a cuspidal $\text{GL}_n(\mathbb{A}_F)$-automorphic representation of unitary central character $\omega$ and $L$-function coefficients $c_{\Pi}$. Let $\omega$ be any nonzero $F$-integer, which we also identify with its image under the diagonal embedding $\alpha \mapsto (\alpha, \alpha, \cdots) \in \mathbb{A}_F^\times$. Let $Y_\infty \subset \mathbb{F}_\infty^\times$ be any archimedean idele of idele norm $|Y_\infty| > |\alpha|$. Writing $0 \leq \theta_0 \leq \frac{1}{2}$ to denote the best uniform approximation towards the generalized Ramanujan conjecture for $\text{GL}_2(\mathbb{A}_F)$-automorphic forms (with $\theta_0 = 0$ conjectured), and $0 \leq \delta_0 \leq \frac{1}{4}$ that towards the generalized Lindelöf hypothesis for $\text{GL}_2(\mathbb{A}_F)$-automorphic forms in the level aspect (with $\delta_0 = 0$ conjectured), we derive the following uniform estimates

(A) Let $f(a_1, \ldots, a_k)$ be an arbitrary positive definite $F$-rational quadratic form in $k \geq 1$ many variables, and $p(a_1, \cdots, a_k)$ a spherical polynomial for $f$ (possibly trivial). Let $W$ be any smooth function
of \( y_\infty \in F^X_{\infty, +} \cong \mathbb{R}^d_0 \) satisfying satisfying the moderate decay condition \( W(|y_\infty|) = O(|y_\infty|^\kappa) \) for \( |y_\infty| \to 0 \) for some \( 0 < \kappa < 1 \), which decays rapidly for \(|y_\infty| \to \infty\), and which is square integrable or else is compactly supported. Let us also assume that \( W^{(i)} \ll 1 \) for all \( i \geq 1 \). Then for any choice of \( \varepsilon > 0 \), we have the estimate

\[
\sum_{a = (a_1, \ldots, a_k) \in \mathcal{O}_F} p(a) \frac{c_H(f(a) + \alpha)}{|f(a) + \alpha|^s} W \left( \frac{f(a) + \alpha}{Y_\infty} \right) = O_{\Pi, f, p, W}(1) + O_{\Pi, \varepsilon} \left( |Y_\infty|^{-\frac{d_k}{2} + \frac{d_0}{2} + \varepsilon} |\alpha|^{\delta_0 + \frac{d_0}{2} - \varepsilon} \right).
\]

Here, the leading term \( O_{\Pi, f, p, W}(1) = M_{\Pi, f} I(W) \) is given by a constant \( M_{\Pi, f} \geq 0 \) times a linear functional \( I(W) \) in \( W \); this constant \( M_{\Pi, f} \) vanishes unless the \( F \)-integer \( \alpha \) is totally positive and the Dirichlet series corresponding to the shifted convolution sum (as described below) has a pole. In the setting where \( f(x) = x^2 \), this constant term vanishes unless the symmetric square \( L \)-function \( L(s, \text{Sym}^2 \Pi) \) has a pole at \( s = 1 \), equivalently unless \( \Pi \) is orthogonal (and hence self-dual). Finally, the exponent \( \delta_0 \) may be replaced with \( \theta_0 \) when the number of variables \( k \geq 2 \) is even.

(B) Let \( m \geq 2 \) be an integer, and \( \pi = \otimes \pi_v \) any cuspidal \( \text{GL}_m(A_F) \)-automorphic representation with \( L \)-function coefficients \( c_v \). Let \( W_j \) be smooth functions of \( y_\infty \in F^X_{\infty, +} \cong \mathbb{R}^d_0 \) for \( j = 1, 2 \), each satisfying the moderate growth condition \( W_j(|y_\infty|) = O(|y_\infty|^\kappa_j) \) as \(|y_\infty| \to 0 \) for some \( 0 < \kappa_j < 1 \) for each of \( j = 1, 2 \), which decay rapidly for \(|y_\infty| \to \infty\), and which are square integrable or else is compactly supported. Let us also assume that \( W_{i, j}^{(i)} \ll 1 \) for all \( i \geq 1 \). Then for any choice of \( \varepsilon > 0 \), we have the estimate

\[
\sum_{\gamma_1, \gamma_2 \in F^X_{\infty}} \frac{c_H(\gamma_1 \gamma_2)}{|\gamma_1 \gamma_2|^{s/2}} W_1 \left( \frac{\gamma_1}{Y_\infty} \right) W_2 \left( \frac{\gamma_2}{Y_\infty} \right) = O_{\Pi, \pi, W_1, W_2, \alpha}(1) + O_{\Pi, \pi, \varepsilon} \left( |Y_\infty|^{-\frac{d_0}{2} + \frac{d_0}{2} + \varepsilon} |\alpha|^{\delta_0 + \varepsilon} \right).
\]

Here, the constant term vanishes unless the Rankin-Selberg \( L \)-function \( L(s, \Pi \times \pi) \) has a pole.

Notice in particular that the exponents \( 0 \leq \theta_0 \leq \frac{1}{2} \) and \( 0 \leq \delta_0 \leq \frac{1}{2} \) describe the best existing approximations towards the generalized Ramanujan conjecture and the generalized Lindelöf hypothesis respectively for \( \text{GL}_2(A_F) \)-automorphic forms, rather than the corresponding exponents for \( \text{GL}_m(A_F) \)-automorphic forms \((!) \). Hence, we can take \( \theta_0 = 7/64 \) thanks to the theorem of Blomer-Brumley \([5]\), and \( \delta_0 = 103/512 \) thanks to Blomer-Harcos \([6, \text{Corollary 1}]\). Let us also remark that while we follow the works of Blomer-Harcos \([6]\) and Templier-Tsimerman \([33]\) (cf. also \([34]\)) closely in places to derive these bounds, the work is closer in spirit to those of Bernstein-Reznikov \([2]\), \([3]\), \([4]\) and Krötz-Stanton \([24]\) using analytic continuation of automorphic representations and other representation theoretic methods. Although we do not explore links to these seminal works here, we do give the following immediate applications to Dirichlet series and subconvexity estimates for automorphic \( L \)-functions on \( \text{GL}_m(A_F) \times \text{GL}_1(A_F) \) via more standard analytic arguments.

1.1. Analytic continuation of Dirichlet series. In the distinct setting where the \( F \)-integer \( \alpha \) is taken to be zero for the sums appearing in Theorem 1.1 \((1)\), so that there is no shift, we also explain in Corollary 5.3 below how to derive the analytic continuation of the Dirichlet series

\[
D(s, \Pi, f, p) := \sum_{a_1, \ldots, a_k \in \mathcal{O}_F \atop f(a_1, \ldots, a_k) \neq 0} p(a_1, \ldots, a_k) \frac{c_H(f(a_1, \ldots, a_k))}{|f(a_1, \ldots, a_k)|^s},
\]

which is defined a priori only for \( s \in \mathbb{C} \) with \( \Re(s) \) sufficiently large. Here, we also use integral presentations derived from the projection operator \( \mathcal{P}_1 \), together with a semi-classical unfolding argument, but omit a fully detailed account for simplicity. Let us remark however, as in the setting of \([6, \text{Theorem 3, Remark 14}]\), that Selberg \([32]\) posed the question of showing the analytic continuation of such Dirichlet series associated to shifted convolution sums. It would be interesting to give a more precise account of this variation of the
Rankin-Selberg theory, making clear the analytic properties (such as existence or location of poles) from the GL$_2(A_F)$ and metaplectic Eisenstein series for the cases of $k$ even and $k$ odd respectively.

1.2. Application to the subconvexity problem in higher dimensions. Using a variation of Theorem 1.1 (B) to estimate an amplified second moment in the style of Blomer-Harcos [6, §3.3] (cf. [36]) and Cogdell [15], as well as unpublished work of Cogdell-Piatetski-Shapiro-Sarnak [11], we can also derive the following estimate for central values of the $L$-functions of cuspidal GL$_n(A_F)$-automorphic representations twisted by idele class character $\omega$. In the main case we consider, where the form is continuous and compactly supported – but can be used to bound these spectral coefficients in terms of the spectral parameters of the corresponding or sufficiently smooth that its Sobolev norm converges, then a relatively standard argument (Lemma 4.8) to show that the coefficients in the corresponding spectral expansion are bounded. If the form is smooth

\[ L(1/2, \Pi \otimes \chi) \ll \Pi_{\chi, \varepsilon} N_{q}^{\frac{3}{2} + \frac{\varepsilon}{2}} + N_{q}^{\frac{3}{2} + (\frac{1}{2} + \theta_{0}) + \frac{(5 - 8\theta_{0}) + \frac{(5 - 8\theta_{0}) - 12\theta_{0}}{4\pi + (16 - 8\theta_{0})}}{2}}, \]

Note that in all of these estimates, $0 \leq \theta_{0} \leq 1/2$ denotes the best uniform approximation to the generalized Ramanujan conjecture for all GL$_2(A_F)$-automorphic forms at the finite places, and in particular that one cannot simply assume $\theta_{0} = 0$ if the generalized Ramanujan conjecture is known for the given cuspidal automorphic representation $\Pi = \otimes \chi \Pi_{\omega}$ of GL$_n(A)$ (e.g. for important spectral cases where $\Pi$ is cohomological). We give a high-level sketch of the proof of this latter application via amplified second moments following [6] in §6.1 below. As we explain in the next paragraph, the key step in our approach here is to reduce the problem to one of estimating the Fourier-Whittaker coefficients of some $L^2$-automorphic form on GL$_2(A_F)$ via the derivation of suitable integral presentations. One this key reduction step is achieved, the claimed estimates can be derived by a relatively straightforward technical generalization of the main theorems of Blomer-Harcos [6, Theorems 1 via Theorem 3]. The bound we derive in this way appears to be completely new in higher dimensions, with some scope to improve the exponent. This approach also appears to lead to some subtler questions about cuspidal automorphic forms on the mirabolic subgroup $P_{2}(A_F)$ and their inclusion in $L^2(GL_2(F) \setminus GL_2(A_F), 1)$. That is, certain extensions of $L^2$-automorphic forms on the mirabolic subgroup $P_{2}(A_F)$ to $L^2$-automorphic forms on GL$_2(A_F)$ play a crucial role in our reduction, as does the subsequent study of relations between Fourier-Whittaker coefficients and compact restrictions with respect to a chosen fundamental domain for the action of GL$_2(O_F)$ on GL$_2(F_\infty)$. Our findings here are by no means exhaustive, and there appears to be scope to develop a better theory of the arithmetic of such mirabolic automorphic forms. As well, it seems that our discussion of the analytic continuation could be developed to derive some triple-product integral presentations of the sums we consider. This in turn suggests that the techniques of Venkatesh [35] and Michel-Venkatesh [29] could perhaps also be extended to this higher-rank setting using variations of the ideas developed here.

1.3. Idea of proof. The proofs of Theorems 1.1 and 6.6 reduce the corresponding problems to deriving bounds for Fourier-Whittaker coefficients of certain continuous and compactly supported $L^2$-automorphic forms on GL$_2(A_F)$ or its two-fold metaplectic cover $\tilde{G}(A_F)$. Once such a reduction is established, it remains to show that the coefficients in the corresponding spectral expansion are bounded. If the form is smooth or sufficiently smooth that its Sobolev norm converges, then a relatively standard argument (Lemma 4.8) can be used to bound these spectral coefficients in terms of the spectral parameters of the corresponding basis forms. In the main case we consider, where the form is continuous and compactly supported – but not in general smooth – we introduce an auxiliary Poincaré series $P$ to derive bounds in terms of the
inner product with $P_\varphi$ (Proposition 4.9). Although the bounds we derive in this more general setting carry some dependence on the chosen real parameter $Y = |Y_\infty| > |\alpha|$, we can in fact proceed after this point to decompose the form spectrally to derive the stated bounds in terms of classical Whittaker functions and Fourier-Whittaker coefficients of automorphic forms on $GL_2(\mathbf{A}_F)$ and $\overline{G}(\mathbf{A}_F)$. Let us now describe the sequence of steps required to make such a reduction to $GL_2(\mathbf{A}_F)$, written in such a way that an expert could reconstruct the proof of the main results without much trouble from the existing theory for $GL_2(\mathbf{A}_F)$. Here, we shall assume we are in the generic case of dimension $n \geq 2$, where we make modifications to both the projected form $P_1^n\varphi$ and the second form (e.g. the metaplectic theta series) multiply with to derive the integral presentation for the corresponding shifted convolution problem.

(ii). The first observation is that the shifted convolution sums of Theorem 1.1 (A) and (B) can be encoded in the Fourier-Whittaker coefficients at $\alpha \in \mathcal{O}_F$ of certain $L^2$-automorphic forms on the mirabolic subgroup

$$P_2(\mathbf{A}_F) = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} : y \in \mathbf{A}_F^\times, x \in \mathbf{A}_F \right\} \subset GL_2(\mathbf{A}_F)$$

or its metaplectic cover via the classical projection operator $P_1^n\varphi$ taking $L^2$-automorphic forms on $GL_n(\mathbf{A}_F)$ to the $L^2$-automorphic forms on $P_2(\mathbf{A}_F)$. This operator $P_1^n$ could be thought of as a kind of partial Whittaker transform. To be more precise, writing $Y_{n,1}$ to denote the unipotent radical of the standard parabolic subgroup corresponding to the partition $(2,1,\ldots,1)$ of $n$, so that the standard unipotent subgroup $N_n \subset GL_n$ of upper triangular matrices decomposes into a semi-direct product $N_n \cong N_2 \rtimes Y_{n,1}$, this $P_1^n$ takes a cuspidal automorphic form $\varphi$ on $GL_n(\mathbf{A}_F)$ to the function on $p \in P_2(\mathbf{A}_F)$ defined by

$$P_1^n\varphi(p) = |\det(p)|^{-\frac{n-3}{2}} \int_{Y_{n,1}(F) \backslash Y_{n,1}(\mathbf{A}_F)} \varphi \left( \begin{pmatrix} y & \frac{p}{1} \\ 0 & 1 \end{pmatrix} \right) \psi^{-1}(y)dy.$$ 

Here, $\psi$ denotes the standard additive character on $\mathbf{A}_F$, extended in the usual way to $N_n$. Now, we can and do replace the cuspidal automorphic form $\varphi$ with a pure tensor $\varphi = \otimes_v \varphi_v \in V_\Pi$ in the corresponding representation space $V_\Pi$ of $\Pi = (\Pi, V_\Pi)$. That the shifted convolution sums of Theorem 1.1 (A) and (B) can be expressed in terms of the Fourier-Whittaker coefficients of certain $L^2$-automorphic forms on $P_2(\mathbf{A}_F)$ or its two-fold metaplectic cover by this operator is not obvious. However, it is relatively simple to deduce from the following powerful and arguably deep results applied the choice of pure tensor $\varphi = \otimes_v \varphi_v \in V_\Pi$:

(i). Using the surjectivity of the archimedean local Kirillov map, we can and do choose $\varphi_\infty = \otimes_v \varphi_\infty^v$ in such a way that the corresponding Whittaker function $W_{\varphi}(y_\infty)$ on $y_\infty \in F_\infty$ is given by the chosen weight function $W(y_\infty)$, or rather $W_{\varphi_j}(y_\infty)$ ($j = 1, 2$) for the case Theorem 1.1 (B).

(ii). Using the existence and theory of essential Whittaker vectors (completed recently by Matringe [28]), we can and do choose each of the nonarchimedean local vectors $\varphi_v \in V_\Pi_v$ to be an essential Whittaker vector. Essentially, this allows us to relate the local $L$-factor $L(s, \Pi_v)$ to a shifted Mellin transform of $\varphi_v$.

(iii). Using Cogdell’s theory of Eulerian integral presentations for the $L$-function $\Lambda(s, \Pi) = L(s, \Pi_\infty)L(s, \Pi)$, or more generally for $\Lambda(s, \Pi \otimes \xi) = L(s, \Pi \otimes \xi_\infty)L(s, \Pi \otimes \xi)$ with $\xi = \otimes_v \xi_v$ a Hecke character of $F$, we can then deduce from the corresponding integral presentation for the finite part

$$L(s, \Pi \otimes \xi) = \sum_{m \in \mathcal{O}_F} c_\Pi(m)\xi(m)N^m$$

as

$$L(s, \Pi \otimes \xi) = \prod_{\nu < \infty} L(s, \Pi_\nu) = \int_{\mathbf{A}_F^\times} W_{\varphi} \left( \begin{pmatrix} h_f & 0 \\ 0 & 1 \end{pmatrix} \right) \xi(h_f) |h_f|^s \psi^{-1}(y_\infty) dh_f$$

that the finite coefficients in the Fourier-Whittaker expansion of the projected pure tensor $P_1^n\varphi$ are given by shifts of the $L$-function coefficients $c_\Pi$. In particular, the projected form $P_1^n\varphi$ for $\varphi$ chosen according to (ii) and (iii) has the Fourier-Whittaker expansion for any idele $y = y_f y_\infty \in \mathbf{A}_F^\times \cong \mathbf{A}_{F,f}^\times \times F_\infty$ and adele $x \in \mathbf{A}_F$.
(Corollary 2.3): 
\[
P^n_1 \varphi \left( \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix} \right) = |y|^{-\left(\frac{d-2}{2}\right)} \sum_{\gamma \in F_*} c_\gamma (\gamma y_f) |\gamma y_f|^{-\frac{d-2}{2}} W_\varphi (\gamma y_\infty) \psi (\gamma x).
\]

It is easy to deduce integral presentations from this latter expansion (see Propositions 3.4 and 3.5). That is, writing \( \theta_{f,p} \) to denote the (metaplectic) theta series associated to the quadratic form \( f(a) = f(a_1, \cdots, a_k) \), with \( \tilde{\theta}_{f,p} = T_{-1} \theta_{f,p} \) its image under the Hecke operator at infinity sending \( x \to -x \in A_F \), the shifted convolution sum of Theorem 1.1 (A) can be realized as a Fourier-Whittaker coefficient

\[
\int_{A_F/F} \mathbb{P}_1^n \varphi \cdot \tilde{\theta}_{f,p} \left( \begin{pmatrix} 1 \\ y_\infty & x \\ 1 & 1 \end{pmatrix} \right) \psi (-\alpha x_\infty) dx
\]

Similarly, the shifted convolution sum of Theorem 1.1 (B) can be realized as a Fourier-Whittaker coefficient

\[
\int_{A_F/F} \mathbb{P}_1^n \varphi \cdot \mathbb{P}_1^n \theta_{f,p} \left( \begin{pmatrix} 1 \\ y_\infty & x \\ 1 & 1 \end{pmatrix} \right) \psi (-\alpha x_\infty) dx
\]

for some similarly chosen pure tensor \( \varphi' = \otimes \psi \varphi' \in V_\pi \) in the representation space corresponding to the cuspidal automorphic representation \( \pi = (\pi, V_\pi) \) on \( GL_m(A_F) \).

(2). The second observation (which applies in all dimensions \( n \geq 2 \)) is that we may use a combination of the strong approximation theorem and the Iwasawa decomposition of \( GL_2(F_\infty) \) to extend the \( L^2 \)-automorphic form \( \mathbb{P}_1^n \varphi \) on \( P_1(A_F) \) to an \( L^2 \)-automorphic form on \( GL_2(A_F) \) with trivial central character and trivial right action by the maximal compact subgroup \( K \). We note that this procedure works for an arbitrary automorphic form on \( GL_2(A_F) \) or its two-fold metaplectic cover \( \widetilde{GL_2(A_F)} \), and is not unlike taking the classical descent of such a form. To be more precise, we show in Theorem 2.5 how a combination of the strong approximation theorem with the Iwasawa decomposition allows us to derive a certain unique factorization of elements \( g \in GL_2(A_F) \), which we then use to define an extension

\[
\mathbb{P}_1^n \varphi \in L^2(GL_2(F) \backslash GL_2(A_F), 1)^K
\]

to \( GL_2(A_F) \) in Definition 2.6 and Proposition 2.7. This extension is analogous to the classical passage from Hilbert Maass forms on the \( d \)-fold upper-half plane \( \mathfrak{H}^d \cong P_2(F_\infty) \) to automorphic forms on \( GL_2(A_F) \), and requires that we fix a fundamental domain for the action of \( GL_2(O_F) \) on \( GL_2(F_\infty) \). This choice of fundamental domain corresponds to choosing a natural way to choose one collection of fundamental domains for the action of \( SL_2(\mathfrak{H}) \) on \( \mathfrak{H} \) indexed by the real places of \( F \). Moreover, it imposes some constraints on the archimedean idele \( y_\infty = (y_\infty, j_{(d-1)/2}) \in F_\infty^\times \) and adele \( x_\infty = (x_\infty, j_{(d-1)/2}) \in F_\infty \) coordinates of the mirabolic matrices \( P_2(F_\infty) \cong \mathfrak{H}^d \), and moreover the extended function \( \mathbb{P}_1^n \varphi \) is not smooth along the boundary of the chosen fundamental domain. However, we explain in \( \S 2 \) and \( \S 4 \) how to surmount these issues.

(3). The third step of the reduction is to use the various automorphy properties of both the mirabolic cusp form \( \mathbb{P}_1^n \varphi \) and its extension \( \mathbb{P}_1^n \varphi \to GL_2(A_F) \) to show (in Lemma 2.8, Proposition 2.9, and Proposition 4.2) that for any \( Y_\infty \in F_\infty^\times \) in our chosen fundamental domain, we can relate the unipotent integrals

\[
\int_{I \simeq [0, 1]^d \subset F_\infty} \mathbb{P}_1^n \varphi \cdot \tilde{\theta}_{f,p} \left( \begin{pmatrix} Y_\infty \\ x_\infty \\ 1 \end{pmatrix} \right) \psi (-\alpha x_\infty) dx
\]

\[
\int_{I \simeq [0, 1]^d \subset F_\infty} \mathbb{P}_1^n \varphi \cdot \mathbb{P}_1^n \theta_{f,p} \left( \begin{pmatrix} Y_\infty \\ x_\infty \\ 1 \end{pmatrix} \right) \psi (-\alpha x_\infty) dx
\]
and
\[
\int_{\mathbb{R}^d \times (-1,1)^d \subset F_\infty} \mathbb{P}_1^m \phi \cdot \mathbb{P}_1^n \phi \left( \left( \begin{array}{c} Y \n \n x \n 1 \end{array} \right) \right) \psi(-\alpha x) dx = \int_{\mathbb{R}^d \times (-1,1)^d \subset F_\infty} \mathbb{P}_1^m \phi \cdot \mathbb{P}_1^n \phi \left( \left( \begin{array}{c} Y \n \n x \n 1 \end{array} \right) \right) \psi(-\alpha x) dx.
\]

The arguments for this step are elementary, but delicate. For instance, we require the automorphy of the mirabolic form \( \mathbb{P}_1^m \phi \), and remark that this \( L^2 \)-automorphic form on \( P_2(\mathbb{A}_F) \) could not simply be replaced by a random function (e.g. a formal power series having values of the M"obius function as coefficients). Here, we need to assume in addition that \( W_\phi(y_\infty) = W_\phi(-y_\infty) \). We also need to fix a smooth partition of unity and dyadic decomposition to derive a more delicate integral presentation for the shifted convolution problem, as explained in Proposition 4.2 below.

(4). The penultimate step of the reduction, given in Theorem 2.10, Proposition 4.2, and Proposition 4.9, is to argue that the spectral coefficients of the function defined by

\[
\Phi = \begin{cases} 
\mathbb{P}_1^m \phi \cdot \tilde{\theta}_{f,p} & \text{for Theorem 1.1 (A)} \\
\mathbb{P}_1^n \phi \cdot \mathbb{P}_1^m \phi & \text{for Theorem 1.1 (B)}
\end{cases}
\]

are bounded. The theta series \( \tilde{\theta}_{f,p} \) in this latter definition is really some sort of descent defined in a similar way as the lift \( \mathbb{P}_1^m \phi \) via Iwasawa decomposition of the metaplectic theta series \( \theta_{f,p} \) introduced above. We refer to Propositions 4.1 and 4.2 below for more details. After this point, there are at least two ways to proceed. One is to take the inner product with some suitably chosen Poincaré series \( P_\sigma \) whose underlying smooth function \( \phi \in C^\infty(N_2(\mathbb{A}_F)Z_2(\mathbb{A}_F) \backslash GL_2(\mathbb{A}_F);\psi) \) is compactly supported, and moreover supported only on some domain \( J = J(\infty) \times I \subset P_2(F_\infty) = \mathfrak{S}^d \) where the function \( \Phi \) is smooth. As we show in Proposition 4.9 below, some relatively simple calculations with unfolding and orthogonality then allow us to reduce to the standard argument for \( n = 2 \) described in Lemma 4.8 to derive bounds for the coefficients. As explained in a subsequent remark, we could also consider the convolution \( \Phi * \mathfrak{K} \) with a smoothing kernel \( \mathfrak{K} \) to derive the same bounds for the spectral coefficients of \( \Phi \).

(5). Taking for granted the convergence of spectral coefficients, we can then decompose \( \Phi \) on \( \mathfrak{G}(\mathbb{A}_F) \) or \( GL_2(\mathbb{A}_F) \) spectrally to derive the claimed bounds. Again, this accounts for how all of the exponents appearing in the bounds we derive come from the best exponents in approximations towards the generalized Ramanujan conjecture for \( GL_2(\mathbb{A}_F) \)-automorphic forms and the generalized Lindelöf hypothesis for \( GL_2(\mathbb{A}_F) \)-automorphic forms in the level aspect.

(6). The application to subconvexity estimates is derived via a direct generalization of [6, Theorem 3] given in Corollary 6.5 and Theorem 6.6 below via the shifted convolution sums estimate of Theorem 1.1 (B). We give a sketch of the method in § 6.1 via amplified second moments (following [6, §3]), but note that it is a well-known and standard method for the corresponding shifted convolution problem. In a nutshell, a minor technical variation of Propositions 3.5 and 4.2 (B) above allows us to derive a suitable integral presentation for the corresponding off-diagonal term as a non-constant Fourier-Whittaker coefficient of some continuous and compactly supported \( L^2 \)-automorphic form on \( GL_2(\mathbb{A}_F) \). Decomposing this form spectrally according to the discussion above then allows us to extend the main estimate of [6, Theorem 3] to this setting to derive the stated subconvexity bounds.

Outline of the paper. We first review Fourier-Whittaker expansions in §2, leading to a description of the expansion of \( \mathbb{P}_1^m \phi \) for \( \phi \in V_1 \) a carefully chosen pure tensor, and give the key arguments with extensions to \( GL_2(\mathbb{A}_F) \) and relations of the Fourier-Whittaker coefficients. We then explain how to derive integral presentations via mirabolic coefficients in the §3, starting with the special case of the metaplectic theta series associated to the quadratic form \( q(x) = x^2 \) (Proposition 3.2), followed by the classical theta series associated to a positive definite binary quadratic form (Proposition 3.3), the general case of a theta series associated to a positive definite quadratic form (Proposition 3.4), and then the setting of linear shifts of two forms (Proposition 3.5). We then prove Theorem 1.1 in §4, starting with some discussion of dyadic

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1I am grateful to Peter Sarnak and to Akshay Venkatesh for suggesting these respective approaches to me.
decompositions and modified integral presentations (leading to Proposition 4.2) as required to deal with our fixed choice of fundamental domain for the lifting of the mirabolic form. This part of the paper also contains key arguments concerning spectral decompositions and uniform bounds for coefficients (namely Lemma 4.8 and Proposition 4.9). The remaining bounds for Whittaker functions and Fourier-Whittaker expansions are then relatively standard, but nevertheless given in detail. After showing these main bounds for the shifted convolution problem, we explain in §5 how to derive analytic continuations for the corresponding Dirichlet series. Finally in §6, we derive subconvexity bounds via amplified second moments in the style of Blomer-Harcos [6]. As we explain, this latter application is derived from a variation of Theorem 1.1 (B).

Notations. We write the set of real embeddings of $F$ as $(\tau_j)_{j=1}^d$, and embed $F$ as a $\mathbb{Q}$-algebra into $F_\infty = \mathbb{R}^d$ via these embeddings. We also write $F_{\infty,+}^\times = \mathbb{R}_{>0}^d$ to denote the set of totally positive elements of $F_\infty$, and $F_{\infty,+}^\mathrm{diag} = \{(x, \ldots, x) : x \in \mathbb{R}_{>0}\}$. We decompose the idele group $A_F^\times$ into its corresponding nonarchimedean and archimedean components as $A_F = A_F^\times F_{\infty}^\times$, so that $A_F^\times$ denotes the finite ideles. We write $| \cdot |$ to denote the idele norm, which on idele representatives of integral ideals $m \subset \mathcal{O}_F$ coincides with the absolute norm $Nm = [\mathcal{O}_F : m\mathcal{O}_F]$. We also write $\psi = \otimes \psi_v$ to denote the standard additive character of $A_F/F$. Hence, $\psi : A_F \to F$ is the unique continuous additive character on $A_F$ which is trivial on $F$, agrees with $x \mapsto \exp(2\pi i (x_1 + \cdots + x_d))$ on $F_{\infty}$, and at each nonarchimedean completion $F_v$ is trivial on the local inverse different $\mathcal{d}^{-1}_{F,v}$ but nontrivial on $p^{-1}\mathcal{d}^{-1}_{F,p}$. In general, we use many of the same notations and conventions as in Blomer-Harcos [6], but use $F$ instead of $\mathbb{K}$ to denote the totally real field. However, for each nonarchimedean place $v$ of $F$ where $\mathbb{F}_v$ is ramified, we choose a measure on $F_v^\times$ in such a way that the local zeta integrals attached to our chosen essential Whittaker vectors are characterized in terms of the local Euler factor $L(s,\Pi_v)$ for the special case of $m = 1$ corresponding to a twist by the trivial idele class character as in Matringe [28, Corollary 3.3] (see Theorem 2.2).

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2. Fourier-Whittaker expansions

Let $\Pi = \otimes_v \Pi_v$ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_n(A_F)$ of unitary central character $\omega = \otimes_v \omega_v$. Let $\varphi = \otimes \varphi_v \in V_\Pi$ be a pure tensor in the space of smooth vectors $V_\Pi$, chosen so that each of the nonarchimedean local vectors $\varphi_v$ is an essential Whittaker vector, as we can thanks to Matringe [28, Theorem 1.3]. Let us also fix a nontrivial additive character $\psi = \otimes_v \psi_v$ on $A_F/F$, which we extend in the usual way to the corresponding quotient $N_n(F)/N_n(A)$ of the maximal unipotent subgroup $N_n \subset \mathrm{GL}_n$ (see e.g. [13], [14]). Again, we shall take this $\psi$ to be the standard additive character throughout all of the discussion that follows. Recall that for $g \in \mathrm{GL}_n(A_F)$, we have the usual Fourier-Whittaker expansion

$$
\varphi(g) = \sum_{\gamma \in N_{n-1}(F)/\mathrm{GL}_{n-1}(F)} W_\varphi \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right),
$$

where

$$
W_\varphi(g) = W_{\varphi,\psi}(g) = \int_{N_n(F)\backslash N_n(A_F)} \varphi(ng)\psi^{-1}(n)dn.
$$

2.1. Projection operators and their expansions. We now review the construction and properties of the projection operator $\mathcal{P}_1^\Pi$, in particular as it relates to Cogdell’s theory of Eulerian integrals for automorphic $L$-functions on $\mathrm{GL}_n \times \mathrm{GL}_1$. We refer to [13, Lecture 5] and [14, §2.2.1] for details.

Let $Y_{n,1} \subset \mathrm{GL}_n$ denote the unipotent radical of the standard parabolic subgroup attached to the partition $(2,1,\ldots,1)$ of $n$. Note that we have the semi-direct product decomposition $N_n = N_2 \rtimes Y_{n,1}$. Note as well that our fixed additive character $\psi$ extends to the quotient $Y_{n,1}(F)/Y_{n,1}(A_F)$ via the inclusion $Y_{n,1} \subset N_n$,
and also that $Y_{n,1}$ is normalized by $GL_2 \subset GL_n$. Let $P_2 \subset GL_2$ denote the mirabolic subgroup determined by the stabilizer in $GL_2$ of $\psi$,

$$P_2 = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\} \subset GL_2 \rightarrow GL_2 \times GL_1 \times \cdots \times GL_1 \subset GL_n.$$

Fix a pure tensor $\varphi = \otimes_v \varphi_v \in V_1$ as above. We shall also view $\varphi$ as its corresponding cuspidal automorphic form on $GL_n(\mathbb{A}_F)$. The projection operator $P^n_1$ from the space of cuspidal automorphic forms on $GL_n(\mathbb{A}_F)$ to the space of cuspidal automorphic forms on $P_2(\mathbb{A}_F) \subset GL_2(\mathbb{A}_F)$ is defined for any $p \in P_2(\mathbb{A}_F)$ by the partial Whittaker integral

$$P^n_1 \varphi(p) = |\det(p)|^{-\frac{n-2}{2}} \int_{Y_{n,1}(F) \backslash Y_{n,1}(\mathbb{A}_F)} \varphi\left( y \begin{pmatrix} p & 1_{n-2} \\ 1_{n-1} & 1 \end{pmatrix} \right) \psi^{-1}(y) dy,$$

where $1_m$ for a positive integer $m$ denotes the $m \times m$ identity matrix. Note that the integral in (2) is taken over a compact domain, and hence converges absolutely. This projection has the following basic properties.

**Proposition 2.1.** Given a cuspidal automorphic form $\varphi$ on $GL_n(\mathbb{A}_F)$, the projection $P^n_1 \varphi$ defined by the integral (2) is a cuspidal automorphic form on $P_2(\mathbb{A}_F)$ having the Fourier-Whittaker expansion

$$P^n_1 \varphi(p) = |\det(p)|^{-\frac{n-2}{2}} \sum_{\gamma \in F^\times} W_\varphi\left( \begin{pmatrix} \gamma & 1_{n-1} \\ 1_{n-1} & 1 \end{pmatrix} \begin{pmatrix} p & 1_{n-2} \\ 1_{n-1} & 1 \end{pmatrix} \right) \psi(\gamma x).$$

In particular, for $x \in \mathbb{A}_F$ a generic adele and $y \in \mathbb{A}_F$ a generic idele, we have that

$$P^n_1 \varphi\left( \begin{pmatrix} y & x \\ 1 \end{pmatrix} \right) = |y|^{-\frac{n-2}{2}} \sum_{\gamma \in F^\times} W_\varphi\left( \begin{pmatrix} \gamma & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y & 1 \\ 1 \end{pmatrix} \right) \psi(\gamma x).$$

**Proof.** See [13, Lemma 5.2] or [14, §2.2.1] for the first two statements. The third is an easy consequence of specialization. To be clear, writing $\varphi'$ to denote the normalized function defined on $p \in P_2(\mathbb{A}_F)$ by

$$\varphi'(p) = |\det(p)|^{-\frac{n-2}{2}} P^n_1 \varphi(p),$$

we specialize the expansion of the second statement to $p = \begin{pmatrix} y & x \\ 1 \end{pmatrix}$. It is then easy to check that

$$P^n_1 \varphi\left( \begin{pmatrix} y & x \\ 1 \end{pmatrix} \right) = \sum_{\gamma \in F^\times} W_{\varphi'}\left( \begin{pmatrix} \gamma & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y & 1 \\ 1 \end{pmatrix} \right) \psi(\gamma x).$$

\[\square\]

2.2. Relation to $L$-function coefficients. Let us retain all of the setup described above. Given an idele class character $\xi = \otimes_v \zeta_v$ of $F$, we now consider the shifted Mellin transform

$$I(s, \varphi, \xi) = \int_{\mathbb{A}^\times/F^\times} P^n_1 \varphi\left( \begin{pmatrix} h & 1 \\ 1 \end{pmatrix} \right) \xi(h)|h|^{s-\frac{1}{2} - \frac{n-2}{2}} dh,$$

defined first for $s \in \mathbb{C}$ with $\Re(s) > 1$. As explained in Cogdell [13, Lecture 5] or [14, §2.2] (with $m = 1$), we can open up the Fourier-Whittaker expansion of $P^n_1 \varphi$ in this integral to derive the integral presentation

$$I(s, \varphi, \xi) = \int_{\mathbb{A}^\times/F^\times} \sum_{\gamma \in F^\times} W_\varphi\left( \begin{pmatrix} \gamma h & 1_{n-1} \\ 1_{n-1} & 1 \end{pmatrix} \right) \xi(h)|h|^{s-\frac{1}{2} - \frac{n-2}{2}} dh$$

$$= \int_{\mathbb{A}^\times/F^\times} W_{\varphi'}\left( \begin{pmatrix} h & 1_{n-1} \\ 1_{n-1} & 1 \end{pmatrix} \right) \xi(h)|h|^{s-\frac{1}{2} - \frac{n-1}{2}} dh.$$
Since we choose the pure tensor \( \varphi = \otimes_v \varphi_v \in V_{\Pi} \) in such a way that each nonarchimedean local component \( \varphi_v \) is an essential Whittaker vector (thanks to Matringe [28, Corollary 3.3]), we deduce from Cogdell’s theory of Eulerian integrals for \( \text{GL}_n \times \text{GL}_1 \) that we have the following exact integral presentation for the finite part \( L(s, \Pi \otimes \xi) \) of the standard \( \text{L} \)-function \( \Lambda(s, \Pi \otimes \xi) = L(s, \Pi_\infty \otimes \xi_\infty) L(s, \Pi \otimes \xi) \) of \( \Pi \otimes \xi \).

**Theorem 2.2.** Let \( \varphi = \otimes_v \varphi_v \in V_{\Pi} \) be a pure tensor whose local nonarchimedean components \( \varphi_v \) are each essential Whittaker vectors. Let \( \xi = \otimes_v \xi_v \) be an idele class character of \( F \), and consider the standard \( \text{L} \)-function \( \Lambda(s, \Pi \otimes \xi) = L(s, \Pi_\infty \otimes \chi_\infty) L(s, \Pi \otimes \xi) \) of \( \Pi \otimes \xi \). Then, the finite part \( L(s, \Pi \otimes \xi) \) of this \( \text{L} \)-function then has the following integral presentation:

\[
L(s, \Pi \otimes \xi) = \prod_{v < \infty} \int_{F_v^\times} \mathcal{W}_{\varphi_v} \left( \left( \begin{array}{c} h_v \\ 1_{n-1} \end{array} \right) \right) \xi_v(h_v)|h_v|^{s-(\frac{n-1}{2})} \, dh_v = \int_{\mathbb{A}_F^\times} \mathcal{W}_{\varphi} \left( \left( \begin{array}{c} h_f \\ 1_{n-1} \end{array} \right) \right) \xi(h_f)|h_f|^{s-(\frac{n-1}{2})} \, dh_f.
\]

Here, for each nonarchimedean place \( v \) of \( F \) not dividing the conductor of \( \Pi \) or \( \xi \), we choose a measure on \( F_v^\times \) according to Matringe [28, Corollary 3.3]. As well, for an idele \( h = (h_v)_v \in \mathbb{A}_F^\times \) write the corresponding decomposition into nonarchimedean and archimedean components as \( h = h_f h_\infty \) for \( h_f \in \mathbb{A}_F^\times \) and \( h_\infty \in F_\infty^\times \).

In particular, taking \( \xi \) to be the trivial character, we have the following relation of the specialized coefficients

\[
(3) \quad \rho_{\varphi}(h_f) := \mathcal{W}_{\varphi} \left( \left( \begin{array}{c} h_f \\ 1_{n-1} \end{array} \right) \right) \quad \text{and} \quad \rho_{\varphi}(h_\infty) := \mathcal{W}_{\varphi} \left( \left( \begin{array}{c} h_\infty \\ 1_{n-1} \end{array} \right) \right),
\]

to the coefficients in the Dirichlet series of \( L(s, \Pi) \) (first for \( \Re(s) > 1 \): Fixing a finite idele representative of each nonzero integral ideal \( \mathfrak{m} \subset O_F \), and using the same symbol to denote this so that \( \mathfrak{m} \in \mathbb{A}_F^\times \),

\[
(4) \quad L(s, \Pi) := \sum_{\mathfrak{m} \subset O_F} \frac{c_{\Pi}(\mathfrak{m})}{N(\mathfrak{m})} \sum \frac{\rho_{\varphi}(\mathfrak{m})}{N(\mathfrak{m})^{s-(\frac{n-1}{2})}}.
\]

**Proof.** The first claim is deduced from Cogdell’s theory of Eulerian integrals (see e.g. [13, Lecture 9]) with the theorem of Matringe [28, Corollary 3.3] to describe the local zeta integrals at each nonarchimedean place \( v \) for which \( \Pi_v \) or \( \xi_v \) is ramified. This latter result gives the identification

\[
L(s, \Pi_v \otimes \xi_v) = \int_{F_v^\times} \mathcal{W}_{\varphi_v} \left( \left( \begin{array}{c} h_v \\ 1_{n-1} \end{array} \right) \right) \xi_v(h_v)|h_v|^{s-(\frac{n-1}{2})} \, dh_v
\]

for each nonarchimedean place \( v \) of \( F \) where \( \Pi_v \) or \( \xi_v \) is ramified, the unramified case being well-understood (see e.g. [13, Lecture 7]). The second claim follows after specialization to the trivial character, comparing Mellin coefficients. \( \square \)

Using (4), we can now relate the Fourier-Whittaker coefficients of \( \mathbb{P}_1^\varphi \) to the \( \text{L} \)-function coefficients \( c_{\Pi} \):

**Corollary 2.3.** Let \( \varphi = \otimes_v \varphi_v \in V_{\Pi} \) be a pure tensor whose nonarchimedean local components are each essential Whittaker vectors. Given a generic adele \( x \in \mathbb{A}_F \) and a generic idele \( y \in \mathbb{A}_F^\times \), the projected cusp form \( \mathbb{P}_1^\varphi \) has the Fourier-Whittaker expansion

\[
\mathbb{P}_1^\varphi \left( \left( \begin{array}{c} y \\ x \\ 1 \end{array} \right) \right) = |y|^{-(\frac{n-2}{2})} \sum_{\gamma \in F^\times} \frac{c_{\Pi}(\gamma yf)}{|\gamma yf|^{\frac{n-1}{2}}} \mathcal{W}_{\varphi} (\gamma y_\infty) \psi(\gamma x).
\]

**Proof.** The expansion follows from Proposition 2.1, and decomposing Whittaker coefficients as in (3), and then using the relation to \( \text{L} \)-function coefficients implied by (4). \( \square \)

2.3. **Extensions to \( \text{GL}_2(\mathbb{A}_F) \).** Recall that a function \( \phi : \text{GL}_2(\mathbb{A}_F) \rightarrow \mathbb{C} \) is said to be an \( L^2 \)-automorphic form on \( \text{GL}_2(\mathbb{A}_F) \) of a given unitary central character \( \xi = \otimes_v \xi_v : \mathbb{A}_F^\times \rightarrow S^1 \) if

- It is measurable and \( \int_{\text{GL}_2(\mathbb{A}_F) \backslash \text{GL}_2(\mathbb{F})} |\phi(g)|^2 \, dg < \infty. \)

- It is left \( \text{GL}_2(\mathbb{F}) \)-invariant, so \( \phi(\gamma g) = \phi(g) \) for all \( \gamma \in \text{GL}_2(\mathbb{F}) \) and \( g \in \text{GL}_2(\mathbb{A}_F) \).
• The centre \(Z_2(\mathbf{A}_F) \cong \mathbf{A}_F^\times\) of \(\text{GL}_2(\mathbf{A}_F)\) acts via \(\xi\), so \(\phi(\xi g) = \xi(z)\phi(g)\) for all \(z \in Z_2(\mathbf{A}_F)\) and \(g \in \text{GL}_2(\mathbf{A}_F)\).

We write \(L^2(\text{GL}_2(F) \backslash \text{GL}_2(\mathbf{A}_F), \xi)\) to denote the corresponding Hilbert space of such functions. Note that \(\text{GL}_2(\mathbf{A}_F)\) acts naturally on this space by right translation, giving it the structure of a unitary representation. Let

\[
K = \prod_{v \leq \infty} K_v = O_2(F_\infty) \prod_{v < \infty} \text{GL}_2(O_{F_v})
\]

denote the maximal compact subgroup of \(\text{GL}_2(\mathbf{A}_F)\). Let us also remark that the discussion that follows applies to any dimension \(n \geq 2\), although we shall focus on the generic higher dimensional setting \(n \geq 3\) where the projection operator \(P_1^n\) is not trivial (and hence why such a discussion is needed). Keeping with the setup described above, we now show how to extend an \(L^2\)-automorphic form \(P_1^n \phi\) on the mirabolic subgroup \(P_2(\mathbf{A}_F) \subset \text{GL}_2(\mathbf{A}_F)\) to an \(L^2\)-automorphic form on \(\text{GL}_2(\mathbf{A}_F)\) of trivial central character which is also right \(K\)-invariant. To be more precise, we argue as follows that we can construct an extension of \(\mathbb{P}_1^n \phi\) to \(L^2(\text{GL}_2(F) \backslash \text{GL}_2(\mathbf{A}_F), 1)\) via the strong approximation theorem for \(\mathbf{A}_F^\times\) together with the Iwasawa decomposition for \(\text{GL}_2(F_\infty)\), analogous to the classical setting of Hilbert-Maass cusp forms on the \(d\)-fold upper-half plane. Let us write \(\iota : \text{GL}_2(F) \rightarrow \text{GL}_2(\mathbf{A}_F)\) to denote the diagonal embedding, and \(\nu : \text{GL}_2(F_\infty) \rightarrow \text{GL}_2(\mathbf{A}_F)\) the embedding sending a matrix \(x\) to the idele with archimedean component \(x\) and all other components trivial. We shall sometimes drop these notations when the context is clear. Recall that the strong approximation theorem for \(\mathbf{A}_F^\times\) gives us the identification

\[
\mathbf{A}_F^\times / F_\infty^\times F^\times \mathcal{O}_F^\times = \mathbf{A}_F^\times / F_\infty^\times F^\times \prod_{v < \infty} \mathcal{O}_{F_v}^\times \cong C(\mathcal{O}_F),
\]

where \(C(\mathcal{O}_F)\) denotes the ideal class group of \(F\). Hence, fixing a set of idele representatives \(\Delta\) of \(C(\mathcal{O}_F)\), the strong approximation theorem for \(\mathbf{A}_F^\times\) implies that

\[
D := F_\infty^\times \prod_{v < \infty} \mathcal{O}_{F_v}^\times \prod_{\zeta \in \Delta} \zeta
\]

is a fundamental domain for the idele class group \(\mathbf{A}_F^\times / F^\times\), and so we derive the disjoint union decomposition

(5)

\[
\mathbf{A}_F^\times = \coprod_{\alpha \in F^\times} \alpha D.
\]

Given a representative \(\zeta \in \Delta\), let us write \(h_\zeta = \left( \begin{array}{cc} \zeta & \varepsilon \\ 1 & \zeta \end{array} \right)\) to denote the corresponding diagonal matrix,

\[
\mathbf{A}_F^\times \cong \text{Z}_2(\mathbf{A}_F), \quad \zeta \mapsto h_\zeta := \left( \begin{array}{cc} \zeta & 0 \\ 0 & \zeta \end{array} \right).
\]

We have the following well-known application of this theorem to \(\text{GL}_2(\mathbf{A}_F)\) via the determinant map.

**Proposition 2.4 (Strong approximation).** We have that \(\text{GL}_2(\mathbf{A}_F) = \prod_{\zeta \in \Delta} \text{GL}_2(F) \text{GL}_2(F_\infty) K h_\zeta\).

**Proof.** Cf. [16, Appendix 3] and [19, Proposition 4.4.2] or [10, §3.3.1]. To derive the version we state here, observe that we can view \(\text{GL}_2(F) \text{GL}_2(F_\infty) K \subset \text{GL}_2(\mathbf{A}_F)\) as a subgroup, and that the homomorphism

\[
\text{GL}_2(\mathbf{A}_F) \xrightarrow{\text{det}} \mathbf{A}_F^\times \rightarrow C(\mathcal{O}_F)
\]

factors to given an identification

\[
\text{GL}_2(\mathbf{A}_F) / \text{GL}_2(F) \text{GL}_2(F_\infty) K \cong C(\mathcal{O}_F).
\]

The stated identification is then easy to deduce. \(\square\)

We can now use some version of the Iwasawa decomposition for \(\text{GL}_2(F_\infty)\) to deduce the following useful “unique factorization” result for elements of \(\text{GL}_2(\mathbf{A}_F)\) (cf. [19, Theorem 4.4.4]).
Theorem 2.5 (Unique decomposition). Fixing a fundamental domain for $GL_2(\mathcal{O}_F) \backslash GL_2(F_\infty)$, we obtain from the strong approximation decomposition $GL_2(\mathcal{A}_F) \cong \coprod_{\xi \in \Delta} GL_2(F)P_2(F_\infty)h_{\xi}Z_2(F_\infty)K$ described above that each element $g \in GL_2(\mathcal{A}_F)$ can be expressed uniquely as a product of matrices of the form

\[
g = \prod_{\xi \in \Delta} \gamma \cdot \begin{pmatrix} y_\infty & x_\infty \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} r_\infty & r_\infty \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \zeta & \zeta \\ 1 & 1 \end{pmatrix} \cdot k,
\]

or more precisely

\[
g = \prod_{\xi \in \Delta} \nu(\gamma) \cdot \nu \left( \begin{pmatrix} y_\infty & x_\infty \\ 1 & 1 \end{pmatrix} \right) \cdot \nu \left( \begin{pmatrix} r_\infty & r_\infty \\ 1 & 1 \end{pmatrix} \right) \cdot \nu \left( \begin{pmatrix} \zeta & \zeta \\ 1 & 1 \end{pmatrix} \right) \cdot k,
\]

where $\gamma \in GL_2(F)$ is a rational element,

\[
\begin{pmatrix} y_\infty & x_\infty \\ 1 & 1 \end{pmatrix} \in P_2(F_\infty)
\]

is a mirabolic element with archimedean coordinates,

\[
\begin{pmatrix} r_\infty & r_\infty \\ 1 & 1 \end{pmatrix} \in Z_2(F_\infty)
\]

is a central element with archimedean idele coordinates, and $k \in K$ is an element of the maximal compact subgroup. Here, after fixing a fundamental domain, the archimedean coordinates in this presentation (6) are constrained as follows so that the (6) presentation is unique: For each index $1 \leq j \leq d$, the adele $x_\infty = (x_{\infty,j})_{j=1}^d$ and the idele $r_\infty = (r_{\infty,j})_{j=1}^d$ satisfy the simultaneous constraints

\[
r_{\infty,j} > 0, \quad y_{\infty,j} > 0, \quad 0 \leq x_{\infty,j} \leq 1/2, \quad x_{\infty,j}^2 + y_{\infty,j}^2 \geq 1.
\]

Proof. Cf. [19, Theorem 4.4.4], where it is explained how to derive a similar (classical) result via the Iwasawa decomposition for $GL_2(F_\infty)$ in the strong approximation decomposition described above. To be more precise, let us first recall the Iwasawa decomposition for $GL_2(F_\infty)$, as described in [19, Proposition 4.1.1] (for instance). Hence, let us fix a fundamental domain for the action of $GL_2(\mathcal{O}_F)$ on $GL_2(F_\infty)$, as described in more detail below. Fixing such a choice of fundamental domain, a minor variation of the standard argument given for [19, Proposition 4.1.1] allows us to deduce that each matrix $g_\infty \in GL_2(F_\infty)$ can be expressed uniquely as

\[
g_\infty = \begin{pmatrix} 1 & x_\infty \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} y_\infty & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} r_\infty & r_\infty \\ 1 & 1 \end{pmatrix} \cdot k,
\]

with $k \in O_2(F_\infty)$, $x_\infty \in F_\infty$, and $y_\infty, r_\infty \in F_\infty^\times$ constrained as in the statement. To be more precise here, we consider for each real place of $F$ the identification

\[
\mathfrak{H} := \{ z = x + iy \in \mathbb{C} : x \in \mathbb{R}, y \in \mathbb{R}_{>0} \} \cong GL_2(\mathbb{R})/O_2(\mathbb{R}) \cdot \mathbb{R}^\times,
\]

so that $GL_2(\mathbb{Z}) \subset GL_2(\mathbb{R})$ acts on $\mathfrak{H}$ via left matrix multiplication. We then use the natural identification of matrices of the from appearing in (7) above with $x_\infty + iy_\infty \in \mathfrak{H}^d$ to form a fundamental domain following the classical example of $SL_2(\mathbb{Z}) \backslash \mathfrak{H}$ for each component of the adele coordinate $x_\infty = (x_{\infty,j})_{j=1}^d \in F_\infty \cong \mathbb{R}^d$ and the idele coordinate $y_\infty = (y_{\infty,j})_{j=1}^d \in F_\infty^\times \cong (\mathbb{R}^d)^\times$. That is, for each component $j$ corresponding to a real place of $F$, we consider the standard fundamental domain $\mathcal{D}$ for $SL_2(\mathbb{Z})$ acting on the complex upper-half plane $z = x + iy \in \mathfrak{H}$ given by

\[
\mathcal{D} = \left\{ z \in \mathfrak{H} : -\frac{1}{2} \leq \Re(z) \leq \frac{1}{2}, \Im(z) \geq 1 \right\}.
\]

Following a similar argument to what is given in [19, Theorem 4.4.4], we use the elementary matrix identity

\[
\begin{pmatrix} -1 & 1 \\ y & x \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} y & -x \\ 1 & 1 \end{pmatrix},
\]

for each $x \in \mathbb{R}$ and $y \in \mathbb{R}^\times$ to deduce that a fundamental domain for $GL_2(\mathbb{Z}) \backslash GL_2(\mathbb{R})$ is half of a fundamental domain for $SL_2(\mathbb{R}) \backslash \mathfrak{H}$ in each adele coordinate $x = x_{\infty,j}$. The constraints stated above, which correspond to choosing $\mathcal{D}$ with $0 \leq x_{\infty,j} \leq 1/2$ for each $1 \leq j \leq d$, are then easy to deduce. In particular, this justifies that the $x_\infty \in F_\infty$ and $y_\infty, r_\infty \in F_\infty^\times$ are constrained as stated.
Proposition 2.7. The function \( g \in GL_2(F_\infty) \) according (7) in the strong approximation theorem \( GL_2(A_F) = \prod_{\xi \in \Delta} GL_2(F) P_2(F_\infty) Kh \cdot Z_2(F_\infty) \) to deduce that each element \( g \in GL_2(A_F) \) can be expressed uniquely as
\[
g = \prod_{\xi \in \Delta} \gamma \cdot \left( \begin{smallmatrix} y_\infty & x_\infty \\ 1 & 1 \end{smallmatrix} \right) \cdot \left( \begin{smallmatrix} r_\infty \\ -r_\infty \end{smallmatrix} \right) \cdot \left( \begin{smallmatrix} \zeta \\ \zeta \end{smallmatrix} \right) \cdot k
\]
or more precisely
\[
g = \prod_{\xi \in \Delta} \iota(\gamma) \cdot \nu \left( \begin{smallmatrix} y_\infty & x_\infty \\ 1 & 1 \end{smallmatrix} \right) \cdot \nu \left( \begin{smallmatrix} r_\infty \\ -r_\infty \end{smallmatrix} \right) \cdot \left( \begin{smallmatrix} \zeta \\ \zeta \end{smallmatrix} \right) \cdot k
\]
with \( \gamma \in GL_2(F) \) and \( k \in K = O_2(F_\infty) \prod_{v \leq \infty} K_v \). This proves the stated claim.

We can now define an extension \( \widetilde{P}_1^\infty \) of the mirabolic form \( P_1^\infty \) to \( GL_2(A_F) \) via Theorem 2.5 as follows:

**Definition 2.6.** Let \( \varphi \in V_\Pi \) be a pure tensor for the irreducible cuspidal \( GL_n(A_F) \)-automorphic representation \( \Pi \), and let \( P_1^\infty \varphi \) denote its projection to the mirabolic subgroup \( P_2(A_F) \subset GL_2(A_F) \). We take \( \widetilde{P}_1^\infty \varphi \) to be the function defined on \( g \in GL_2(A_F) \) decomposed uniquely as in (6) above by the rule
\[
\widetilde{P}_1^\infty \varphi(g) := P_1^\infty \varphi \left( \begin{smallmatrix} y_\infty & x_\infty \\ 1 & 1 \end{smallmatrix} \right)
\]

**Remark** Note that this definition does not depend on the choice of fundamental domain for \( GL_2(O_F) \backslash GL_2(F_\infty) \).

We now check that this extension \( \widetilde{P}_1^\infty \varphi \) in fact determines and \( L^2 \)-automorphic form on \( GL_2(A_F) \) of trivial central character which is also right \( K \)-invariant.

**Proposition 2.7.** The function \( \widetilde{P}_1^\infty \varphi(g) = P_1^\infty \varphi \left( \begin{smallmatrix} y_\infty & x_\infty \\ 1 & 1 \end{smallmatrix} \right) \) from Definition 2.6 above determines an \( L^2 \)-automorphic form of trivial central character on \( GL_2(A_F) \) which is also \( K \)-finite (but not \( Z \)-finite),
\[
\widetilde{P}_1^\infty \varphi \in L^2 \left( GL_2(F) \backslash GL_2(A_F), 1 \right)^K \rightarrow L^2 \left( GL_2(F) \backslash GL_2(A_F), 1 \right).
\]

**Proof.** Cf. [19, Proposition 4.8.4]. Let us write \( \phi = \widetilde{P}_1^\infty \varphi \) to lighten notation. We first argue that \( \phi \) is measurable and convergent in the \( L^2 \)-norm as a consequence of the corresponding properties satisfied by the pure tensor \( \varphi \in V_\Pi \) on \( GL_2(A_F) \). To be more precise, writing \( || \cdot || \) to denote the standard matrix norm on \( g \in GL_n(A_F) \), we know that there exist constants \( C, B > 0 \) such that
\[
\varphi(g) = \sum_{\gamma \in N_{n-1}(F) \backslash GL_{n-1}(F)} W_\varphi \left( \begin{smallmatrix} \gamma \\ 1 \end{smallmatrix} \right) g < C \cdot ||g||^B.
\]
Comparing Fourier-Whittaker expansions, it is easy to deduce that at a similar bound holds for the projection:
\[
P_1^\infty \varphi(p) = |\det(p)|^{-\frac{q \cdot 2}{2}} \sum_{\gamma \in F^q} W_\varphi \left( \begin{smallmatrix} \gamma \\ 1_{n-1} \end{smallmatrix} \right) \left( \begin{smallmatrix} p \\ 1_{n-2} \end{smallmatrix} \right) < C \cdot ||\text{diag}(p, 1_{n-2})||^B.
\]

We now check the required invariance properties. It is easy to see from Definition 2.6 that \( \phi(\gamma \cdot g) = \phi(\gamma) \) for all \( \gamma \in GL_2(A_F) \) and \( g \in GL_2(A_F) \). To check invariance under the action of \( Z_2(A_F) \), we argue following the discussion of strong approximation (5) above that given \( z \in Z_2(A_F) \), we can find \( \alpha \in F_\infty^q \), \( w_\infty \in F_\infty^q \) and \( k \in K \) such that \( z \) can be expressed as
\[
z = t \left( \begin{smallmatrix} \alpha \\ \alpha \end{smallmatrix} \right) \cdot \nu \left( \begin{smallmatrix} w_\infty \\ w_\infty \end{smallmatrix} \right) \cdot k \cdot \prod_{\xi \in \Delta} h_\xi = \prod_{\xi \in \Delta} h_\xi \cdot t \left( \begin{smallmatrix} \alpha \\ \alpha \end{smallmatrix} \right) \cdot \nu \left( \begin{smallmatrix} w_\infty \\ w_\infty \end{smallmatrix} \right) \cdot k.
\]
It is then easy to check from Definition 2.6 that \( z \in Z_2(A_F) \) acts trivially (with \( g \cdot z \) in the right form):
\[
\phi(z \cdot g) = \phi(g \cdot z) = \phi \left( g \cdot \prod_{\xi \in \Delta} h_\xi \cdot t \left( \begin{smallmatrix} \alpha \\ \alpha \end{smallmatrix} \right) \cdot \nu \left( \begin{smallmatrix} w_\infty \\ w_\infty \end{smallmatrix} \right) \cdot k \right) = \phi(g).
\]
Finally, observe that the extension $\phi(g)$ is trivially $K$-finite, since
\[ \phi(g \cdot k) = \mathbb{P}^n_1 \varphi \left( \begin{array}{cc} y_\infty & x_\infty \\ 1 & 1 \end{array} \right) = \phi(g) \]
for any $g \in \text{GL}_2(A_F)$ and $k \in K$ as a consequence of Definition 2.6. Hence, it is trivial to deduce that the set \{ $\phi(gk) : k \in K$ \} of all right translates of $\phi$ generates a finite dimensional vector space.

2.4. Relations of Fourier-Whittaker coefficients. We now consider how to relate the Fourier-Whittaker coefficients of a mirabolic cusp form $\mathbb{P}^n_1 \varphi$ to those of its extension $\mathbb{P}^n_1 \varphi$ to $\text{GL}_2(A_F)$ according to Theorem 2.5, Definition 2.6, and Proposition 2.7 above. We also consider localizations of the latter functions to certain compact subdomains of the chosen fundamental domain of Theorem 2.5 for our subsequent arguments. Let us start with the following basic general observation.

Lemma 2.8. Let $\phi \in L^2(\text{GL}_2(F) \backslash \text{GL}_2(A_F), 1)$ be any $L^2$-automorphic form on $\text{GL}_2(A_F)$ of trivial central character $1$ which is also right $K$-invariant. Then for any idele $y \in A_F^\times$, we have the identification
\[ (8) \quad \phi \left( \begin{array}{cc} y & 1 \\ 1 & 1 \end{array} \right) = \phi \left( \begin{array}{cc} y^{-1} & 1 \\ 1 & 1 \end{array} \right). \]

Proof. Let us suppose more generally for illustration that $\phi \in L^2(\text{GL}_2(F) \backslash \text{GL}_2(A_F), \omega)$ is any $L^2$-automorphic form on $\text{GL}_2(A_F)$ of central character $\omega$ which is right $K$-invariant. A direct computation shows that
\[ (9) \quad \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) \left( \begin{array}{cc} y & 1 \\ 1 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) = \left( \begin{array}{cc} 1 & y \\ 1 & 1 \end{array} \right). \]
Since we can view the long Weyl element
\[ \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) \]
as an element of both the rational subgroup $\text{GL}_2(F)$ and the maximal compact subgroup $K$, it follows from the left $\text{GL}_2(F)$-invariance and right $K$-invariance of $\phi$ that we have the identifications
\[ \phi \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) \left( \begin{array}{cc} y & 1 \\ 1 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) = \phi \left( \begin{array}{cc} y & 1 \\ 1 & 1 \end{array} \right) = \phi \left( \begin{array}{cc} 1 & y \\ 1 & 1 \end{array} \right). \]
That is, the first identity follows from the invariance of $\phi$, while the second follows from the calculation (9) applied the the expression on the left. Factoring out by a central element on the right, we then deduce that
\[ \phi \left( \begin{array}{cc} y & 1 \\ 1 & 1 \end{array} \right) = \omega(y)^{-1} \phi \left( \begin{array}{cc} y^{-1} & 1 \\ 1 & 1 \end{array} \right). \]
This implies the stated identity (8) for the special case of trivial central character $\omega = 1$. □

Remark Notice that for any archimedean idele $y_\infty \in F_\infty^\times$, Lemma 2.8 implies we have the identification
\[ \mathbb{P}^n_1 \varphi \left( \begin{array}{cc} y_\infty^{-1} & 1 \\ 1 & 1 \end{array} \right) = \mathbb{P}^n_1 \varphi \left( \begin{array}{cc} y_\infty & 1 \\ 1 & 1 \end{array} \right). \]
Thus if $|y_\infty| > 1$ is contained in the fundamental domain of Theorem 2.5, we can consider the decomposition
\[ (10) \quad \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) \left( \begin{array}{cc} y_\infty^{-1} & 1 \\ 1 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) = \left( \begin{array}{cc} y_\infty^{-1} & y_\infty^{-1} \\ y_\infty & 1 \end{array} \right) \left( \begin{array}{cc} y_\infty & 1 \\ 1 & 1 \end{array} \right) \]
to deduce that the diagonal matrix on the left hand side of (10) is put into the correct form for the unique decomposition of Theorem 2.5 by the expression on the right hand side. Hence by Definition 2.6, we have
\[ (11) \quad \mathbb{P}^n_1 \varphi \left( \begin{array}{cc} y_\infty^{-1} & 1 \\ 1 & 1 \end{array} \right) = \mathbb{P}^n_1 \varphi \left( \begin{array}{cc} y_\infty & 1 \\ 1 & 1 \end{array} \right) = \mathbb{P}^n_1 \varphi \left( \begin{array}{cc} y_\infty & 1 \\ 1 & 1 \end{array} \right). \]
Let us now consider relations between the Fourier-Whittaker coefficients of a given mirabolic form $P_1^0\varphi$ and its extension $\tilde{P}_1^0\varphi$ as defined above. In particular, keeping the idele coordinate $y_\infty \in F_\infty^\times$ inside of the chosen fundamental domain of Theorem 2.5, we can show that there is a matching of each of the Fourier-Whittaker coefficients, provided that the pure tensor $\varphi = \otimes_v \varphi_v \in V_{\Pi}$ on $\text{GL}_n(\mathbb{A}_F)\text{ has the property that its local archimedean Whittaker function is even, i.e. so that}$

$$\varphi(12)$$

as functions of $y_\infty \in F_\infty^\times$. To justify this claim, we shall first describe the Fourier-Whittaker coefficients equivalently in semi-classical terms, as integrals over the compact domains $I \cong [0,1]^d \subset F_\infty \cong \mathbb{R}^d$ against the corresponding archimedean additive character $\psi_\infty(x_\infty) = \exp(2\pi i \text{Tr}(x_\infty)) = \exp(2\pi i (x_{\infty,1} + \cdots + x_{\infty,d}))$.

To be more precise, recall that given an $L^2$-automorphic form $\phi \in L^2(\text{GL}_2(F) \backslash \text{GL}_2(\mathbb{A}_F), \omega)$ evaluated on $g \in \text{GL}_2(\mathbb{A}_F)$ and $\alpha \in \mathcal{O}_F$ an $F$-integer, the coefficient in the Fourier-Whittaker expansion of $\phi(g)$ along the unipotent subgroup of upper triangular matrices $N_2 \subset \text{GL}_2$ at $\alpha$ is given by the unipotent integral

$$\int_{\text{A}_F/F} \phi \left( \left( \begin{array}{cc} 1 & x \\ 1 & 1 \end{array} \right) g \right) \psi(-\alpha x) \, dx.$$ 

This integral can be described equivalently as one over the domain $I \cong [0,1]^d \subset F_\infty \cong \mathbb{R}^d$, using that this forms a full orthogonal set for the additive characters $\psi(\alpha x_\infty) = \psi_\infty(\alpha x_\infty) = \exp(-2\pi i \text{Tr}(\alpha x_\infty))$.

Note that the identity (13) reflects the well-known equivalence of the Fourier-Whittaker expansion of a GL$_2(\mathbb{A}_F)$-automorphic form with that of its underlying GL$_2(F_\infty)$-automorphic form, or that of its underlying Maass form on the $d$-fold upper-half plane, and that we shall often take such identifications for granted.

**Proposition 2.9.** Let $\varphi = \otimes_v \varphi_v \in V_{\Pi}$ be any pure tensor in the representation space of the cuspidal automorphic representation $\Pi = \otimes_v \Pi_v$ of $\text{GL}_n(\mathbb{A}_F)$. Assume that the archimedean local Whittaker coefficient $W_\varphi$ satisfies condition (12), so that $W_\varphi(y_\infty) = W_\varphi(-y_\infty)$ as functions of $y_\infty \in F_\infty^\times$. Let $P_1^0\varphi$ denote the image of $\varphi$ under the projection operator $P_1^0$. Let $\tilde{P}_1^0\varphi \in L^2(\text{GL}_2(F) \backslash \text{GL}_2(\mathbb{A}_F), 1^K)$ denote the extension of $P_1^0\varphi$ to an $L^2$-automorphic form on $\text{GL}_2(\mathbb{A}_F)$ via Theorem 2.5, Definition 2.6, and Proposition 2.7. Let $y_\infty \in F_\infty^\times$ be any archimedean idele with $|y_\infty| > 1$ contained in our fixed fundamental domain of Theorem 2.5. Then for any $F$-integer $\alpha \in \mathcal{O}_F$, we have the corresponding identification of Fourier-Whittaker coefficients

$$\int_{I \cong [0,1]^d} \tilde{P}_1^0\varphi \left( \left( \begin{array}{cc} y_\infty & x_\infty \\ 1 & 1 \end{array} \right) \right) \psi(-\alpha x_\infty) \, dx_\infty = \int_{I \cong [0,1]^d} P_1^0\varphi \left( \left( \begin{array}{cc} y_\infty & x_\infty \\ 1 & 1 \end{array} \right) \right) \psi(-\alpha x_\infty) \, dx_\infty.$$

In other words, the Fourier-Whittaker coefficients of such an $L^2$-automorphic form $P_1^0\varphi$ evaluated on diagonal elements $\text{diag}(y_\infty,1) \in P_2(F_\infty)$ match those of its extension $\tilde{P}_1^0\varphi$ to GL$_2(\mathbb{A}_F)$ evaluated on diagonal elements $\text{diag}(y_\infty,1) \in \text{GL}_2(F_\infty) \cong P_2(F_\infty)Z_2(F_\infty)O_2(F_\infty)$.

**Proof.** Consider

$$\int_{I \cong [0,1]^d} \tilde{P}_1^0\varphi \left( \left( \begin{array}{cc} y_\infty & x_\infty \\ 1 & 1 \end{array} \right) \right) \psi(-\alpha x_\infty) \, dx_\infty.$$

Let us split the integral into intervals $I_1 \cong [0,1/2]^d \subset F_\infty$ and $I_2 \cong (1/2,1]^d \subset F_\infty$ as

$$\int_{I_1 \cong [0,1/2]^d} \tilde{P}_1^0\varphi \left( \left( \begin{array}{cc} y_\infty & x_\infty \\ 1 & 1 \end{array} \right) \right) \psi(-\alpha x_\infty) \, dx_\infty + \int_{I_2 \cong (1/2,1]^d} \tilde{P}_1^0\varphi \left( \left( \begin{array}{cc} y_\infty & x_\infty \\ 1 & 1 \end{array} \right) \right) \psi(-\alpha x_\infty) \, dx_\infty.$$ 

Since the coordinates in the first integral of the latter expression are well-defined with respect to the chosen fundamental domain of Theorem 2.5, it is easy to see from Definition 2.6 that

$$\int_{I_1 \cong [0,1/2]^d} \tilde{P}_1^0\varphi \left( \left( \begin{array}{cc} y_\infty & x_\infty \\ 1 & 1 \end{array} \right) \right) \psi(-\alpha x_\infty) \, dx_\infty = \int_{I_1 \cong [0,1/2]^d} P_1^0\varphi \left( \left( \begin{array}{cc} y_\infty & x_\infty \\ 1 & 1 \end{array} \right) \right) \psi(-\alpha x_\infty) \, dx_\infty.$$
To identify the second integral over $I_2$ in this way, let us first fix any adele $x_\infty = (x_{\infty,j})_{j=1}^d \in I_2 \cong (1/2, 1)^d$. Observe that since $\tilde{P}_1^n \varphi$ is left invariant by \( \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \text{GL}_2(F) \), we have the identification

\[ \tilde{P}_1^n \varphi \left( \begin{pmatrix} y_\infty & x_\infty \\ x_\infty & 1 \end{pmatrix} \right) = \tilde{P}_1^n \varphi \left( \begin{pmatrix} y_\infty & x_\infty - 1 \\ x_\infty - 1 & 1 \end{pmatrix} \right). \]

On the other hand, since the $\text{GL}_2(A_F)$-automorphic form $\tilde{P}_1^n \varphi$ is both left invariant by \( \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \in \text{GL}_2(F) \) and right invariant by \( \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \in K \), it is easy to deduce that

\[ \tilde{P}_1^n \varphi \left( \begin{pmatrix} y_\infty & x_\infty - 1 \\ x_\infty - 1 & 1 \end{pmatrix} \right) = \tilde{P}_1^n \varphi \left( \begin{pmatrix} -y_\infty & -x_\infty \\ x_\infty & 1 \end{pmatrix} \right), \]

where the latter function is well-defined with respect to the chosen fundamental domain of Theorem 2.5. That is, we then have by Definition 2.6 the identification

\[ \tilde{P}_1^n \varphi \left( \begin{pmatrix} y_\infty & 1 - x_\infty \\ y_\infty & 1 - x_\infty \end{pmatrix} \right) = \tilde{P}_1^n \varphi \left( \begin{pmatrix} y_\infty & 1 - x_\infty \\ y_\infty & 1 - x_\infty \end{pmatrix} \right). \]

Now, since the mirabolic form $P_1^n \varphi$ is left invariant by \( \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in P_2(F) \), we find that

\[ P_1^n \varphi \left( \begin{pmatrix} y_\infty & 1 - x_\infty \\ y_\infty & 1 - x_\infty \end{pmatrix} \right) = P_1^n \varphi \left( \begin{pmatrix} y_\infty & -x_\infty \\ y_\infty & 1 - x_\infty \end{pmatrix} \right). \]

Since the mirabolic form is also left invariant by \( \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \in P_2(F) \), we then find that

\[ P_1^n \varphi \left( \begin{pmatrix} y_\infty & -x_\infty \\ y_\infty & x_\infty \end{pmatrix} \right) = P_1^n \varphi \left( \begin{pmatrix} y_\infty & x_\infty \\ y_\infty & x_\infty \end{pmatrix} \right). \]

Now, it is easy to deduce from the condition (12) that we have the identification

\[ P_1^n \varphi \left( \begin{pmatrix} y_\infty & x_\infty \\ y_\infty & x_\infty \end{pmatrix} \right) = P_1^n \varphi \left( \begin{pmatrix} y_\infty & x_\infty \\ y_\infty & x_\infty \end{pmatrix} \right). \]

Indeed, opening up the Fourier-Whittaker expansion of the function on the left hand side, we check that

\[ \begin{align*}
\tilde{P}_1^n \varphi \left( \begin{pmatrix} y_\infty & x_\infty \\ y_\infty & x_\infty \end{pmatrix} \right) &= \sum_{\gamma \in F^\times} W_\varphi \left( \begin{pmatrix} y_\infty & x_\infty \\ y_\infty & x_\infty \end{pmatrix} \right) \psi(\gamma x_\infty) \\
&= |y_\infty|^{\frac{d+1}{2}} \sum_{\gamma \in F^\times} W_\varphi \left( \begin{pmatrix} y_\infty & x_\infty \\ y_\infty & x_\infty \end{pmatrix} \right) \psi(\gamma x_\infty) \\
&= P_1^n \varphi \left( \begin{pmatrix} y_\infty & x_\infty \\ y_\infty & x_\infty \end{pmatrix} \right),
\end{align*} \]
Hence, we have shown that

\[
\int_{I_2 \cong (1/2, 1]^d} \mathbb{P}_n^I \varphi \left( \begin{pmatrix} y_\infty & x_\infty \\ 1 \end{pmatrix} \right) \psi_\infty(-\alpha x_\infty) dx_\infty
= \int_{I_2 \cong (1/2, 1]^d} \mathbb{P}_n^I \varphi \left( \begin{pmatrix} y_\infty & x_\infty - 1 \\ 1 \end{pmatrix} \right) \psi_\infty(-\alpha x_\infty) dx_\infty
= \int_{I_2 \cong (1/2, 1]^d} \mathbb{P}_n^I \varphi \left( \begin{pmatrix} y_\infty & 1 - x_\infty \\ 1 \end{pmatrix} \right) \psi_\infty(-\alpha x_\infty) dx_\infty
= \int_{I_2 \cong (1/2, 1]^d} \mathbb{P}_n^I \varphi \left( \begin{pmatrix} y_\infty & -x_\infty \\ 1 \end{pmatrix} \right) \psi_\infty(-\alpha x_\infty) dx_\infty
= \int_{I_2 \cong (1/2, 1]^d} \mathbb{P}_n^I \varphi \left( \begin{pmatrix} y_\infty & x_\infty \\ 1 \end{pmatrix} \right) \psi_\infty(-\alpha x_\infty) dx_\infty.
\]

The claim follows after inserting this identification for the second integral in (14).

\[\square\]

**Remark** Observe that the argument given above requires the automorphy of the mirabolic cusp form \(\mathbb{P}_n^I \varphi\), and so would not simply work for a random function defined on \(P_2(F_\infty) \cong S^d\) extended to an \(L^2\)-automorphic form in the space \(L^2(\text{GL}_2(F) \backslash \text{GL}_2(A_F), 1)^K\) by Theorem 2.5, Definition 2.6, and Proposition 2.7.

Note that we also obtain from this identification of coefficients the following useful result.

**Corollary 2.10.** Fix \(y_\infty \in F_\infty^\times\) an idele in the chosen fundamental domain of Theorem 2.5. Then, the corresponding function of the adele coordinate \(x_\infty \in I \cong [0, 1]^d \subset F_\infty\)

\[\mathbb{P}_n^I \varphi \left( \begin{pmatrix} y_\infty & x_\infty \\ 1 \end{pmatrix} \right)\]

is continuous.

**Proof.** The claim is a direct consequence of the algorithmic calculation of Proposition 2.9, which shows that for any fixed adele \(x_\infty \in I_2 \cong (1/2, 1]^d \subset F_\infty\), we have the series of identifications

\[
\begin{align*}
\mathbb{P}_n^I \varphi\left( \begin{pmatrix} y_\infty & x_\infty \\ 1 \end{pmatrix} \right) &= \mathbb{P}_n^I \varphi\left( \begin{pmatrix} y_\infty & x_\infty - 1 \\ 1 \end{pmatrix} \right) \\
&= \mathbb{P}_n^I \varphi\left( \begin{pmatrix} y_\infty & 1 - x_\infty \\ 1 \end{pmatrix} \right) \\
&= \mathbb{P}_n^I \varphi\left( \begin{pmatrix} y_\infty & -x_\infty \\ 1 \end{pmatrix} \right) \\
&= \mathbb{P}_n^I \varphi\left( \begin{pmatrix} y_\infty & x_\infty \\ 1 \end{pmatrix} \right).
\end{align*}
\]

(15)

Since it is clear from the definition that the same identification (15) holds for any \(x_\infty \in I_1 \cong [0, 1/2]^d \subset F_\infty\), we deduce the claim. That is, we verify that the corresponding function on \(y_\infty \in F\) with the adele coordinate \(x_\infty \in F_\infty\) varying freely in the interval \(I \cong [0, 1]^d \subset F_\infty\) is continuous. This is because we have verified that it matches the underlying mirabolic function \(\mathbb{P}_n^I \varphi\) in this region, the mirabolic function being continuous in \(x_\infty \in I \cong [0, 1]^d \subset F_\infty\). \[\square\]
3. Integral presentations

We now show how to use the expansion of Corollary 2.3 to derive the following integral expansions for the shifted convolution problem for $\text{GL}_n(A_F)$. In this section, we shall work entirely with the mirabolic cusps form $\mathbb{P}_n^0 \varphi$ coming from a pure tensor $\varphi = \otimes_v \varphi_v \in V_\Pi$ on $\text{GL}_n(A_F)$ with proscribed local vectors $\varphi_v$ as described above, but defer considering relations to the extended form $\mathbb{P}_n^0 \varphi$ following Proposition 2.9 until later. The key property we shall use here is the surjectivity of the archimedean local Kirillov map, which can be viewed as a vector space isomorphism $V_\Pi \cong L^2(F_\infty^\times)$, $\phi \rightarrow W_\phi$. To be more precise, we shall use the following key result due to Jacquet and Shalika.

**Proposition 3.1.** Let $W$ be any smooth and compactly supported function on $F_\infty^\times$, or more generally any smooth, summable function on $F_\infty^\times$ of moderate decay near zero which decays rapidly at infinity. There exists a smooth vector $\varphi \in V_\Pi$ whose corresponding archimedean local Whittaker coefficient $W_\varphi(y_\infty)$ satisfies

$$W_\varphi(y_\infty) := W_\varphi \left( \begin{pmatrix} y_\infty & \cdots & y_{n-1} \end{pmatrix} \right) = W(y_\infty),$$

i.e. as functions of $y_\infty \in F_\infty^\times$.

**Proof.** See Jacquet-Shalika [23, (3.8)]; cf. also [22, Lemma 5.1] and [6, § 2.5].

**Remark** Note that for applications, one can relax the conditions on the chosen function $W$ for Proposition 3.1 via a standard dyadic argument which we present in the subsequent section.

**Notations.** We identify any $F$-rational number $\alpha \in F^\times$ with its image in $A_F^\times$ under the diagonal embedding $\alpha \mapsto (\alpha, \alpha, \ldots) \in A_F^\times$, and write and $y = yf y_\infty \in A_F^\times$ to denote any idele with nonarchimedean component $y_\infty = (y_{n,j})_{j=1}^j \in F_\infty^\times \cong (R^\times)^d$. The notation $\alpha y_\infty$ then refers to the product $\alpha y_j y_\infty$ with $y_j = (1, 1, \ldots) \in A_F^\times$ trivial, so that $\alpha y_\infty = (\alpha, \alpha, \ldots)\alpha y_\infty$ has nonarchimedean component $(\alpha, \alpha, \ldots) \in A_F^\times$ and archimedean component $\alpha y_\infty = (\alpha y_{n,j})_{j=1}^d \in F_\infty^\times$. We shall use this notation without comment throughout the rest of the work. Let us remark however that the idele norm $|\alpha y_\infty|$ is then given by the archimedean norm on the component $(\alpha y_\infty) = (\alpha y_{n,j})_{j=1}^d \in F_\infty^\times \cong (R^\times)^d$.

### 3.1. Quadratic progressions via metaplectic theta series

We take for granted some basic background about automorphic forms on the metaplectic cover $\overline{G}$ of $\text{GL}_2$ following Gelbart [17], as well as the relevant discussions in [33] and [34]. Let us first consider metaplectic theta series $\theta_q$ corresponding to the $F$-rational quadratic form $q(x) = x^2$. In fact, we shall consider only the corresponding partial theta series associated to the principal class $1$ in the ideal class group $C(O_F)$ of $O_F$, as this is all we require for our applications. Viewed as an automorphic form on $\overline{G}(A_F)$, this partial theta series has the following Fourier-Whittaker expansion: Taking $x_\infty \in F_\infty$ with $y_\infty \in F_\infty^\times$ with norm $|y_\infty|$, we have the expansion

$$\theta_q \left( \begin{pmatrix} y_\infty & x_\infty \\ 1 & 1 \end{pmatrix} \right) = \theta_q,1 \left( \begin{pmatrix} y_\infty & x_\infty \\ 1 & 1 \end{pmatrix} \right) = |y_\infty|^\frac{d}{2} \sum_{a \in O_F} \psi(q(a)(x_\infty + iy_\infty)).$$

**Proposition 3.2.** Let $W$ be any smooth, summable function of $y_\infty \in F_\infty^\times$ of moderate decay near zero and rapid decay at infinity. Fix an $F$-integer $\alpha$, and let us use the same symbol $\alpha$ to denote its image under the diagonal embedding $\alpha = (\alpha, \alpha, \ldots) \in A_F^\times$ if $\alpha \neq 0$. Let $Y_\infty \in F_\infty^\times$ be any archimedean idele of idele norm $|Y_\infty| > |\alpha|$. Let $\varphi = \otimes_v \varphi_v \in V_\Pi$ be any pure tensor whose nonarchimedean local components are essential Whittaker vectors, and whose archimedean local component has corresponding local Whittaker function\(^2\)

$$W_\varphi(y_\infty) = |y_\infty|^{\frac{d}{2}} \psi(-iy_\infty) \psi \left( \frac{i \cdot \alpha}{Y_\infty} \right) W(y_\infty).$$

\(^2\)Note that the rapidly decaying function $W$ can be chosen with some flexibility after making a standard dyadic subdivision argument, restricting the chosen Whittaker function to dyadic intervals of the form $[2^k, 2^{k+1}]$, then checking that the sum over integers $k$ converges. Although we do not require such an argument here (taking $W$ to be sufficiently rapidly decaying), such a reduction is crucial in many applications.
given as a function of \( y_\infty \in F_\infty^\times \). Then, we have the integral presentation

\[
|Y_\infty|^\frac{1}{2} \int_{I\cong [0,1]^d \subset F_\infty} \mathbb{P}_1^n \varphi \cdot \overline{\mathcal{L}}_q \left( \begin{pmatrix} Y_\infty & x_\infty \\ 1 & 1 \end{pmatrix} \right) \psi(-\alpha x_\infty) dx_\infty
= \sum_{a \in \mathcal{O}_F \atop q(a) + \alpha \neq 0} \frac{q_1(q(a) + \alpha)}{|q(a) + \alpha|^2} W \left( \frac{q(a) + \alpha}{Y_\infty} \right).
\]

**Proof.** We open up Fourier-Whittaker expansions using Corollary 2.3 and the discussion above, switch the order of summation, and then use the orthogonality of additive characters on the compact abelian group \( I \cong [0,1]^d \cong (\mathbb{R}/\mathbb{Z})^d \subset F_\infty \) to compute

\[
\int_{I\cong [0,1]^d \subset F_\infty} \mathbb{P}_1^n \varphi \cdot \overline{\mathcal{L}}_q \left( \begin{pmatrix} Y_\infty & x_\infty \\ 1 & 1 \end{pmatrix} \right) \psi(-\alpha x_\infty) dx_\infty
= \int_{I\cong [0,1]^d \subset F_\infty} \mathbb{P}_1^n \varphi \cdot \overline{\mathcal{L}}_q \left( \begin{pmatrix} Y_\infty & x_\infty \\ 1 & 1 \end{pmatrix} \right) \psi(-\alpha x_\infty) dx_\infty
= \int_{I\cong [0,1]^d \subset F_\infty} |y_\infty|^\frac{1}{2} \sum_{\gamma \in F^\times} W_F \left( \begin{pmatrix} \gamma y_\infty & 1 \\ 1 & 1 \end{pmatrix} \right) \sum_{\alpha \in \mathcal{O}_F} \psi(iy_\infty q(a)) \psi(-q(a)x_\infty) \psi(-\alpha x_\infty) dx_\infty
= |y_\infty|^\frac{1}{2} \sum_{\alpha \in \mathcal{O}_F} \rho_\varphi((q(a) + \alpha)) W_F \left( \begin{pmatrix} (q(a) + \alpha) & 1 \\ 1 & 1 \end{pmatrix} \right) \psi(q(a)) dx_\infty.
\]

The claimed relation then follows after specialization to \( y_\infty = 1/Y_\infty \in F_\infty^\times \), after using the relation to \( L \)-function coefficients described in (4) above, and the explicit choice of \( \varphi \).

3.2. **Quadratic progressions via binary theta series.** Let us now consider the theta series associated to a positive definite \( F \)-rational binary quadratic form \( Q(a,b) \) having \( w = w_Q \) many automorphs. We shall only consider the theta series associated to the principal class \( 1 \) of such binary quadratic forms of a given discriminant (or equivalently of the ideal class group of the corresponding CM extension of \( F \)). Let \( \theta_Q = \theta_{Q,1} \) denote the corresponding theta series, viewed as a \( \text{GL}_2(A_F) \)-automorphic form. Taking \( x_\infty \in F_\infty \) and \( y_\infty \in F_\infty^\times \), this theta series has the expansion

\[
\theta_Q \left( \begin{pmatrix} Y_\infty & x_\infty \\ 1 & 1 \end{pmatrix} \right) = |y_\infty|^\frac{1}{4} \sum_{a,b \in \mathcal{O}_F} \psi(Q(a,b)(x_\infty + iy_\infty)).
\]

**Proposition 3.3.** Let \( W \) be any smooth, summable function of \( y_\infty \in F_\infty^\times \) of moderate decay near zero and rapid decay at infinity. Let \( \alpha \) be any \( F \)-integer, as well as its image under the diagonal embedding \( \alpha \rightarrow (\alpha, \alpha, \ldots) \in A_F^\times \) if \( \alpha \neq 0 \). Let \( Y_\infty \in F_\infty^\times \) be any archimedean idele of idele norm \( |Y_\infty| > |\alpha| \). Let \( \varphi = \otimes_v \psi_v \in V_H \) be a smooth vector whose nonarchimedean local components are essential Whittaker vectors, and whose archimedean local Whittaker function is given as a function of \( y_\infty \in F_\infty^\times \) by

\[
W_\varphi(y_\infty) = |y_\infty|^{\frac{n-2}{2}} \psi(-iy_\infty) \psi \left( \frac{i \cdot \alpha}{Y_\infty} \right) W(y_\infty)
\]

Keeping the same conventions from the discussion above, we derive the following

**Proposition 3.3.** Let \( W \) be any smooth, summable function of \( y_\infty \in F_\infty^\times \) of moderate decay near zero and rapid decay at infinity. Let \( \alpha \) be any \( F \)-integer, as well as its image under the diagonal embedding \( \alpha \rightarrow (\alpha, \alpha, \ldots) \in A_F^\times \) if \( \alpha \neq 0 \). Let \( Y_\infty \in F_\infty^\times \) be any archimedean idele of idele norm \( |Y_\infty| > |\alpha| \). Let \( \varphi = \otimes_v \psi_v \in V_H \) be a smooth vector whose nonarchimedean local components are essential Whittaker vectors, and whose archimedean local Whittaker function is given as a function of \( y_\infty \in F_\infty^\times \) by

\[
W_\varphi(y_\infty) = |y_\infty|^{\frac{n-2}{2}} \psi(-iy_\infty) \psi \left( \frac{i \cdot \alpha}{Y_\infty} \right) W(y_\infty)
\]

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for any archimedean idele \( y_\infty \in F_\infty^\times \). Then, we have the integral presentation

\[
|Y_\infty|^k \int_{I_{\equiv [0,1]^d} \subset F_\infty} \mathbb{P}_1^n \varphi \mathcal{P}_Q \left( \left( \begin{array}{c} y_\infty \\ x_\infty \\ 1 \end{array} \right) \right) \psi(-\alpha x_\infty) dx_\infty 
= \frac{1}{w} \sum_{a,b\in\mathcal{O}_F \mid Q(a,b) + \alpha \neq 0} \frac{\epsilon_1(Q(a,b) + \alpha)}{|Q(a,b) + \alpha|^2} W \left( \frac{Q(a,b) + \alpha}{Y_\infty} \right).
\]

Proof. The proof is derived in the same was as for Proposition 3.2 via the expansions of \( \mathbb{P}_1^n \varphi \) and \( \mathcal{P}_Q \), using the orthogonality of additive characters on the compact abelian group \( I \cong [0,1]^d \subset F_\infty \).

### 3.3. Shifts of arbitrary positive definite quadratic forms

Let us now record the following natural generalization of Propositions 3.3 and 3.2 to arbitrary positive definite quadratic forms, which does not seem to appear in literature on the shifted convolution problem, and which might be of independent interest.

Let \( f = f(a_1, \ldots, a_k) \) be an \( F \)-rational positive definite quadratic form in \( k \geq 1 \) many variables. Let \( p = (a_1, \ldots, a_k) \) be a homogeneous polynomial on \( \mathbb{R}^n \) which is harmonic with respect to \( f \). Hence \( \Delta_f p = 0 \), where \( \Delta_f \) is the unique homogeneous differential operator of order two which is invariant under the orthogonal group \( O(f) \). Let \( \theta_{f,p} \) denote the corresponding theta series, which determines an automorphic form on \( \mathrm{GL}_2(A_F) \) if \( k \) is even, and a genuine automorphic form on the metaplectic cover \( \overline{G}(A_F) \) if \( k \) is odd. Again, we shall really only consider the corresponding partial theta series associated to the principal class in \( C(\mathcal{O}_F) \) here, and not mention it again. Taking \( x_\infty \in F_\infty \) again to be a generic archimedean adele and \( y = y_\infty \in F_\infty^\times \) a generic archimedean idele, this (partial, metaplectic) theta series \( \theta_{f,p} \) has the Fourier series expansion

\[
\theta_{f,p} \left( \left( \begin{array}{c} y_\infty \\ x_\infty \\ 1 \end{array} \right) \right) = \theta_{f,p} \left( \left( \begin{array}{c} y_\infty \\ x_\infty \\ 1 \end{array} \right), 1 \right) 
= |y_\infty|^k \sum_{a_1, \ldots, a_k \in \mathcal{O}_F} p(a_1, \ldots, a_k) \psi(f(a_1, \ldots, a_k)(x_\infty + iy_\infty)).
\]

**Proposition 3.4.** Let \( W \) be any smooth, summable function of \( y_\infty \in F_\infty^\times \) of moderate decay near zero and rapid decay at infinity. Fix an \( F \)-integer \( \alpha \), and let us also write \( \alpha \) to denote the imagine under the diagonal embedding \( \alpha \to (\alpha, \alpha, \cdots) \in A_F^\times \) if \( \alpha \neq 0 \). Let \( Y_\infty \in F_\infty^\times \) be any archimedean idele of idele norm \( |Y_\infty| > |\alpha| \). Let \( \varphi = \otimes_v \varphi_v \in V_1 \) be a pure tensor whose nonarchimedean local components are essential Whittaker vectors, and whose archimedean component has the corresponding Whittaker coefficient

\[
W_\varphi(y_\infty) = |y_\infty|^{\frac{n-2}{2}} \psi(-iy_\infty) \psi \left( i \cdot \frac{\alpha}{Y_\infty} \right) W(y_\infty)
\]

for any \( y_\infty \in F_\infty^\times \). Writing \( a = (a_1, \cdots, a_k) \) to denote a \( k \)-tuple of \( F \)-integers, we have that

\[
|Y_\infty|^k \int_{I_{\equiv [0,1]^d} \subset F_\infty} \mathbb{P}_1^n \varphi \mathcal{P}_{f,p} \left( \left( \begin{array}{c} a_1 \\ x_\infty \\ 1 \end{array} \right) \right) \psi(-\alpha x_\infty) dx_\infty 
= \sum_{a = (a_1, \ldots, a_k) \in \mathcal{O}_F^k} p(a) \frac{\epsilon_1(f(a) + \alpha)}{|f(a) + \alpha|^2} W \left( \frac{f(a) + \alpha}{Y_\infty} \right).
\]
Proof. Let us again lighten notation by writing $a = (a_1, \ldots, a_k)$ to denote a $k$-tuple of $F$-integers $a_j \in \mathcal{O}_F$. Again, we open up Fourier-Whittaker expansions and switch the order of summation to compute

$$\int_{F^\times [0,1]^d \subset F_\infty} \mathbb{P}_1^n \phi \cdot \bar{\theta}_{f,p} \left( \begin{array}{c} y_\infty \\ x_\infty \\ 1 \end{array} \right) \psi(-\alpha x_\infty) dx_\infty$$

$$= |y_\infty|^\frac{k}{2} \left( \frac{y_\infty}{y_\infty^2} \right) \sum_{\gamma \in F^\times} W_{\bar{\phi}} \left( \begin{array}{c} \gamma y_\infty \\ 1 \end{array} \right) \sum_{a=(a_1, \ldots, a_k) \in \mathcal{O}^k_p} p(a) \psi(iy_\infty f(a))$$

$$\times \int_{F^\times [0,1]^d \subset F_\infty} \psi(\gamma x_\infty - f(a)x_\infty - \alpha x_\infty) dx_\infty$$

$$= |y_\infty|^\frac{k}{2} \left( \frac{y_\infty}{y_\infty^2} \right) \sum_{a=(a_1, \ldots, a_k) \in \mathcal{O}^k_p} \rho_{\bar{\phi}}(f(a) + \alpha) W_{\bar{\phi}}((f(a) + \alpha)y_\infty) p(a) \psi(iy_\infty f(a))$$

$$= |y_\infty|^\frac{k}{2} \left( \frac{y_\infty}{y_\infty^2} \right) \sum_{a=(a_1, \ldots, a_k) \in \mathcal{O}^k_p} c_{\Pi}(f(a) + \alpha) \frac{1}{|f(a) + \alpha|^{\frac{k}{2}}} W_{\bar{\phi}}((f(a) + \alpha)y_\infty) p(a) \psi(iy_\infty f(a)).$$

Specializing to $y_\infty = 1/Y_\infty \in F_\infty^\times$ as above then gives

$$\int_{F^\times [0,1]^d \subset F_\infty} \mathbb{P}_1^n \phi \cdot \bar{\theta}_{f,p} \left( \begin{array}{c} 1 \\ x_\infty \\ 1 \end{array} \right) \psi(-\alpha x_\infty) dx_\infty$$

$$= \frac{1}{Y_\infty} |y_\infty|^\frac{k}{2} \left( \frac{y_\infty}{y_\infty^2} \right) \sum_{a=(a_1, \ldots, a_k) \in \mathcal{O}^k_p} c_{\Pi}(f(a) + \alpha) \frac{1}{|f(a) + \alpha|^{\frac{k}{2}}} W_{\bar{\phi}}(\frac{f(a) + \alpha}{Y_\infty}) \psi \left( i \cdot \frac{f(a)}{Y_\infty} \right) p(a).$$

Choosing $\varphi = \otimes_v \varphi_v \in V_\Pi$ as above then gives the stated integral presentation. \hfill \Box

3.4. Linear progressions of two cuspidal forms. Let us now fix an integer $m \geq 2$, together with a cuspidal automorphic representation $\pi = \otimes_v \pi_v$ of $\text{GL}_m(A_F)$.

Proposition 3.5. Let $W_j$ for $j = 1, 2$ be any smooth, summable function of $y_\infty \in F_\infty^\times$ of moderate decay near zero and rapid decay at infinity. Fix a nonzero $F$-integer $\alpha$, and let us also write $\alpha$ to denote the image under the diagonal embedding $\alpha \to (\alpha, \alpha, \ldots) \in A_F^\times$. Fix an archimedean idele $Y_\infty \in F_\infty^\times$ of idele norm $|Y_\infty| > |\alpha|$. Let $\varphi = \otimes_v \varphi_v \in V_\Pi$ and $\varphi' = \otimes_v \varphi'_v \in V_\epsilon$ be pure tensors whose nonarchimedean local components are essential Whittaker vectors, and whose respective archimedean local Whittaker coefficients are specified as functions of $y_\infty \in F_\infty^\times$ by

$$W_{\varphi}(y_\infty) = |y_\infty|^{\frac{m-2}{2}} W_1(y_\infty)$$

and

$$W_{\varphi'}(y_\infty) = |y_\infty|^{\frac{m-2}{2}} W_2(y_\infty).$$

Then, we have the integral presentation

$$\int_{F^\times [0,1]^d \subset F_\infty} \mathbb{P}_1^n \phi \cdot \bar{\theta}_{f,p} \left( \begin{array}{c} 1 \\ x_\infty \\ 1 \end{array} \right) \psi(-\alpha x_\infty) dx_\infty$$

$$= \sum_{\gamma_1, \gamma_2 \in F_\infty^\times} c_{\Pi}(\gamma_1) c_{\Pi}(\gamma_2) |\gamma_1 \gamma_2|^{\frac{k}{2}} W_1(\gamma_1 Y_\infty) W_2(\gamma_2 Y_\infty).$$
Proof. Opening up Fourier-Whittaker expansions, switching the order of summation, and evaluating via orthogonality of additive characters again, we find that
\[
\int_{I \simeq [0,1]^{d} \subset F_{\infty}} \mathbb{P}_{n}^{m} \varphi \mathbb{P}_{1}^{m} \varphi' \left( \begin{pmatrix} y_{\infty} & x_{\infty} \\ 1 & 1 \end{pmatrix} \right) \psi(-\alpha x_{\infty}) dx_{\infty}
\]
\[
= |y_{\infty}|^{-\left(\frac{n-2}{2}\right)} \left(\varphi \right) \sum_{\gamma_{1}, \gamma_{2} \in F_{\infty}} c_{\Pi}(\gamma_{1}) c_{\pi}(\gamma_{2}) |\gamma_{1}|^{-\frac{n}{2}} |\gamma_{2}|^{-\frac{m}{2}} W_{\varphi}(\gamma_{1} y_{\infty}) W_{\varphi'}(\gamma_{2} y_{\infty}).
\]
Specializing to \(y_{\infty} = 1/Y_{\infty}\) as above then gives the identity
\[
\int_{I \simeq [0,1]^{d} \subset F_{\infty}} \mathbb{P}_{1}^{m} \varphi \mathbb{P}_{1}^{m} \varphi' \left( \begin{pmatrix} 1 & x_{\infty} \\ 1 & 1 \end{pmatrix} \right) \psi(-\alpha x_{\infty}) dx_{\infty}
\]
\[
= \left(\frac{1}{Y_{\infty}}\right)^{-\left(\frac{n-2}{2}\right)} \left(\varphi \right) \sum_{\gamma_{1}, \gamma_{2} \in F_{\infty}} c_{\Pi}(\gamma_{1}) c_{\pi}(\gamma_{2}) |\gamma_{1}|^{-\frac{n}{2}} |\gamma_{2}|^{-\frac{m}{2}} W_{\varphi}(\gamma_{1} Y_{\infty}) W_{\varphi'}(\gamma_{2} Y_{\infty}).
\]
Choosing the pure tensors \(\varphi = \otimes_{v} \varphi_{v} \in V_{\Pi}\) and \(\varphi' = \otimes_{v} \varphi'_{v} \in V_{\pi}\) as we do then gives the stated formula. □

4. Bounds for the shifted convolution problem

Fix a nonzero \(F\)-integer \(\alpha \in O_{F}\) and a totally positive archimedean idele \(Y_{\infty} = (Y_{\infty,j})_{j=1}^{d} \in F_{\infty}^{d}\) with idele norm \(|Y_{\infty}| > |\alpha|\). We can and do assume that each component \(Y_{\infty,j}\) is contained within the chosen fundamental domain \(\mathcal{F} = (F_{\infty,j})_{j=1}^{d} \subset \mathcal{F}^{d} \cong P_{2}(F_{\infty})\) of Theorem 2.5 above. We now derive bounds for the sums of \(L\)-function coefficients appearing in the integral presentations of Propositions 3.2, 3.3, 3.4 and 3.5, or rather the shifted convolution sums appearing in the statement of the main theorem. Here, we shall use spectral decompositions of automorphic forms on \(GL_{2}(A_{F})\) and its metaplectic cover \(\overline{G}(A_{F})\), building on ideas from [6], [33], and [34], and also developing the setting described above with the lift \(\mathbb{P}_{1}^{m} \varphi\) of the mirabolic form \(P_{1}^{m} \varphi\) to \(GL_{2}(A_{F})\). We shall also treat the special case of dimension \(n = 2\) separately from the generic case on all dimensions \(n \geq 2\), using Theorem 2.5 and Definition 2.6.

To begin, we first derive a variation of the integral presentations described above in terms of the lifted form \(\mathbb{P}_{1}^{m} \varphi\) (Proposition 4.2). The constraints coming from the choice of fundamental domain above lead us to fix a smooth partition of unity and dyadic subdivision, which allows us to make a suitable choice of archimedean local vectors in the pure tensor to derive integral presentations in terms of the lift. After reviewing some setup with Whittaker functions and Sobolev norms, we then argue that the \(L^{2}\)-automorphic form(s) \(\Phi\) we consider on \(GL_{2}(A)\) or its two-fold metaplectic cover \(\overline{G}(A)\) for the integral presentation of the corresponding shifted convolution problem have bounded spectral coefficients. This is automatic if the form \(\Phi\) is smooth (see Lemma 4.8), as is the case for the standard setup of dimension \(n = 2\). In the generic case of \(n \geq 2\) however, we give an additional argument with inner products of suitable Poincaré series (Proposition 4.9) or convolutions with smoothing kernels (described in the subsequent remark) to show that the spectral coefficients are bounded uniformly in a suitable sense. We then derive bounds in all cases after decomposing the form \(\Phi\) whose Fourier-Whittaker coefficient at \(\alpha\) describes the corresponding shifted convolution problem spectrally. The arguments after this point are relatively standard, and given in terms of the best uniform approximations to the generalized Ramanujan conjecture or the generalized Lindelöf hypothesis for \(GL_{2}(A_{F})\)-automorphic forms in the level aspect, i.e. uniform bounds for all such forms – as opposed to the individual forms we consider.

4.1. Integral presentations revisited. Let us start by returning to the integral presentations derived in Propositions 3.4 and 3.5 for the generic case of dimension \(n \geq 2\), replacing the mirabolic cusp form \(P_{1}^{m} \varphi\) by its corresponding lifted form \(\mathbb{P}_{1}^{m} \varphi \in L^{2}(GL_{2}(F) \setminus GL_{2}(A_{F}),1)^{K}\). In the setup of case (B) corresponding to linear shifts as in Proposition 3.5, we shall also make a similar modification, replacing \(P_{1}^{m} \varphi\) by its corresponding localized extended form \(\mathbb{P}_{1}^{m} \varphi' \in L^{2}(GL_{2}(F) \setminus GL_{2}(A_{F}),1)^{K}\). Here, we shall first introduce a smooth partition of unity and corresponding dyadic decompositions to reduce to the setting of compactly supported local weight functions. Although relatively standard, we spell out the details as we shall rely on this setup at some stages of the later argument. We shall also use the following observations about
the metaplectic group $\mathcal{G}(A_F)$ and $L^2$-automorphic forms on it, leading to some convenient modifications to the metaplectic theta series $\tilde{\theta}_{f,p}$ in the style of the liftings discussion of Theorem 2.5, Proposition 2.7, and Proposition 2.9 which we shall use to derive suitable integral presentations for our shifted convolution sums.

**Proposition 4.1.** The following assertions about $\mathcal{G}(A_F)$ and automorphic functions on it are true.

1. Each metaplectic element $g \in \mathcal{G}(A_F)$ can be expressed uniquely as $g = (g, \mu)$ with
   \[ g = \prod_{\zeta \in \Delta} \gamma \cdot \begin{pmatrix} y_\infty & x_\infty \\ r_\infty & 1 \end{pmatrix} \cdot \begin{pmatrix} \zeta & \zeta \end{pmatrix} \cdot k \in GL_2(A_F) \]
   decomposed uniquely according to Theorem 2.5 above (with the same conditions and constraints), and $\mu \in C_2 \cong \{\pm 1\}$ a square root of unity.

2. Let $\phi \in L^2(GL_2(F) \backslash \mathcal{G}(A_F), \omega)$ be any (genuine) $L^2$-automorphic form on $\mathcal{G}(A_F)$ with some central character $\omega$. Then, the function $\tilde{\phi}$ defined on an element $\tilde{g} = (g, \mu) \in \mathcal{G}(A_F)$ decomposed uniquely according to (1) above by the rule
   \[ \tilde{\phi}(\tilde{g}) = \frac{1}{\#(g, \mu)} \tilde{\phi}(\tilde{(g, \mu)}) := \phi\left(\begin{pmatrix} y_\infty & x_\infty \\ r_\infty & 1 \end{pmatrix}, \mu\right) \]
   determines a (genuine) automorphic form on $\mathcal{G}(A_F)$ in the $L^2$ sense having trivial central character and trivial right action by the maximal compact subgroup $K \subset GL_2(A_F)$. Moreover, for $y \in F_\infty^\times$ contained strictly within the chosen fundamental domain of Theorem 2.5, we have a matching of Fourier-Whittaker coefficients: For each $F$-integer $\alpha$, we have that
   \[ \int_{I \cong [0,1]^d \subset F_\infty} \tilde{\phi}\left(\begin{pmatrix} y_\infty & x_\infty \\ r_\infty & 1 \end{pmatrix}, \mu\right) \psi(-\alpha x_\infty) dx_\infty = \int_{I \cong [0,1]^d \subset F_\infty} \phi\left(\begin{pmatrix} y_\infty & x_\infty \\ r_\infty & 1 \end{pmatrix}, \mu\right) \psi(-\alpha x_\infty) dx_\infty. \]

**Proof.** The first claim (1) follows in a direct way from Theorem 2.5, i.e. applied to the matrix $g \in GL_2(A_F)$ in a generic metaplectic element $\tilde{g} = (g, \mu) \in \mathcal{G}(A_F)$. The second claim (2) can then be deduced by a minor variation of the arguments given above for Propositions 2.7 and 2.9, i.e. if not a more direct argument. □

4.1.1. **Choices of weight functions.** Recall that fix smooth functions $W$ and $W_j$ (for $j = 1, 2$) as in the statement of the main theorem. That is, we fix a smooth function $W$ on $y_\infty \in F_\infty^\times$, which decays as
   \[ W(y_\infty) = \begin{cases} O_C(|y_\infty|^{-C}) & \text{for any } C > 0 \text{ as } |y_\infty| \to \infty \\ O(|y_\infty|^\kappa) & \text{for some } 0 < \kappa < 1 \text{ as } |y_\infty| \to 0, \end{cases} \]
   and whose derivatives are bounded by $W^{(i)} \ll 1$ for all $i \geq 1$. Similarly, for each index $j = 1, 2$, we fix smooth weight functions $W_j$ on $y_\infty \in F_\infty^\times$, which decay as
   \[ W_j(y_\infty) = \begin{cases} O_C(|y_\infty|^{-C}) & \text{for any } C > 0 \text{ as } |y_\infty| \to \infty \\ O(|y_\infty|^\kappa) & \text{for some } 0 < \kappa < 1 \text{ as } |y_\infty| \to 0, \end{cases} \]
   and whose derivatives are bounded by $W_j^{(i)} \ll 1$ for all $i \geq 1$. Here, we also assume that $0 < \kappa_1 + \kappa_2 < 1$.

4.1.2. **A smooth partition of unity and dyadic decompositions.** Let us now introduce a smooth partition of unity and corresponding dyadic decompositions. That is, a standard result in analysis shows that we can fix a smooth partition of unity as follows: There exists a smooth function $U \in C^\infty_c(\mathbb{R}_{>0})$ supported on $[1, 2]$ with bounded derivatives $U^{(i)} \ll 1$ for each $i \geq 0$ together with a collection $\{\bar{R}\}$ of ranges $R \in \mathbb{R}_{>0}$ such that for any $y \in \mathbb{R}_{>0}$, we have the partition of unity
   \[ \sum_{\{\bar{R}\}} U\left(\frac{y}{R}\right) = 1. \]
Moreover, there exists for each integer \( l \in \mathbb{Z} \) a constant number (independent of \( l \)) of ranges \( R \) contained in the dyadic interval \( [2^{l}, 2^{l+1}] \), and so we can assume without loss of generality that this takes the form

\[
\sum_{l \in \mathbb{Z}} \sum_{2^{l} \in (R)} \sum_{R \cap [2^{l}, 2^{l+1}] \neq \emptyset} U \left( \frac{y}{R} \right) = \sum_{l \in \mathbb{Z}} \sum_{2^{l} \in (R)} \sum_{R \cap [2^{l}, 2^{l+1}] \neq \emptyset} U \left( \frac{y}{R} \right) = 1.
\]

Here, the notation means that we take a sum over all integers \( l \in \mathbb{Z} \), where for each corresponding dyadic interval \( [2^{l}, 2^{l+1}] \) there is a single positive real number \( R \in [2^{l}, 2^{l+1}] \in \mathbb{R}_{>0} \) which contributes to the partition of unity. We shall sometimes omit the reference to each single range \( \{R\} \) in the subsequent discussion to lighten the notation, as in the second sum of the latter expression, taking for granted that this preliminary discussion makes the context clear. Now, given any generic sum

\[
\sum_{r \geq 1} A(r),
\]

we can then consider the corresponding dyadic decomposition induced by (16)

\[
\sum_{l \in \mathbb{Z}} \sum_{2^{l} \in (R)} \sum_{R \cap [2^{l}, 2^{l+1}] \neq \emptyset} A(r) U \left( \frac{y}{R} \right) = \sum_{l \in \mathbb{Z}} A(r) \sum_{2^{l} \in (R)} \sum_{R \cap [2^{l}, 2^{l+1}] \neq \emptyset} U \left( \frac{y}{R} \right).
\]

4.1.3. Reduction to local weight functions. Recall that for a fixed nonzero \( F \)-integer \( \alpha \in \mathcal{O}_F \), and for weight functions \( W \) and \( W_j \) as above (and keeping the setup described in the introduction), we seek to estimate the shifted convolution sums

\[
\sum_{a=(a_1, \ldots, a_k) \in \mathcal{O}_F^k} p(a) c_n(f(a) + \alpha) \left| f(a) + \alpha \right|^\frac{1}{2} W \left( \frac{f(a) + \alpha}{Y} \right)
\]

and

\[
\sum_{\gamma_1, \gamma_2 \in \mathcal{F}} c_n(\gamma_1) c_\pi(\gamma_2) \left| \gamma_1 \gamma_2 \right|^\frac{1}{2} W_1 \left( \frac{\gamma_1}{Y} \right) W_2 \left( \frac{\gamma_2}{Y} \right).
\]

Using the partition of unity (16), each of these sums can be decomposed respectively as a sum over sums of local weight functions defined by

\[
\sum_{l \in \mathbb{Z}} \sum_{a=(a_1, \ldots, a_k) \in \mathcal{O}_F^k} p(a) c_n(f(a) + \alpha) \left| f(a) + \alpha \right|^\frac{1}{2} W \left( \frac{f(a) + \alpha}{Y} \right) \sum_{2^{l} \in (R)} \sum_{R \cap [2^{l}, 2^{l+1}] \neq \emptyset} U \left( \frac{\left| f(a) + \alpha \right|}{R} \right)
\]

and

\[
\sum_{l_1, l_2 \in \mathbb{Z}} \sum_{\gamma_1, \gamma_2 \in \mathcal{F}} c_n(\gamma_1) c_\pi(\gamma_2) \left| \gamma_1 \gamma_2 \right|^\frac{1}{2} W_1 \left( \frac{\gamma_1}{Y} \right) U \left( \frac{\left| \gamma_1 \right|}{R_1} \right) W_2 \left( \frac{\gamma_2}{Y} \right) \sum_{\gamma_2 \in \mathcal{F}} \sum_{R_2 \cap [2^{l_2}, 2^{l_2+1}] \neq \emptyset} U \left( \frac{\left| \gamma_2 \right|}{R_2} \right).
\]

Thus for each integer \( l \in \mathbb{Z} \), writing \( Y = |Y| \) as we do, and taking the local weight functions \( W_l \) and \( W_{j, l} \) (for \( j = 1, 2 \)) defined on \( y_\infty \in F_{\infty}^\times \) by

\[
W_l (y_\infty) = W(y_\infty) \sum_{2^{l} \in (R)} U \left( \frac{y_\infty |Y|}{R} \right)
\]

and

\[
W_{l,j} (y_\infty) = W_j (y_\infty) \sum_{2^{l} \in (R)} U \left( \frac{y_\infty |Y|}{R} \right),
\]

for \( j = 1, 2 \).
we have the respective decompositions into local sums
\[
\sum_{a=(a_1,\ldots,a_k)\in\mathbb{O}_F^k} p(a) \frac{c_H(f(a) + \alpha)}{|f(a) + \alpha|^2} W_i \left( \frac{f(a) + \alpha}{Y_\infty} \right)
\]
\[
= \sum_{l\in\mathbb{Z}} \sum_{a=(a_1,\ldots,a_k)\in\mathbb{O}_F^k} p(a) \frac{c_H(f(a) + \alpha)}{|f(a) + \alpha|^2} W_i \left( \frac{f(a) + \alpha}{Y_\infty} \right)
\]
and
\[
\sum_{\gamma_1,\gamma_2 \in \mathbb{F}_F^\times \gamma_1 \gamma_2 = \alpha} \frac{c_H(\gamma_1) c_\pi(\gamma_2)}{|\gamma_1 \gamma_2|^2} W_1 \left( \frac{\gamma_1}{Y_\infty} \right) W_2 \left( \frac{\gamma_2}{Y_\infty} \right)
\]
\[
= \sum_{l_1,l_2 \in \mathbb{Z}} \sum_{\gamma_1,\gamma_2 \in \mathbb{F}_F^\times \gamma_1 \gamma_2 = \alpha} \frac{c_H(\gamma_1) c_\pi(\gamma_2)}{|\gamma_1 \gamma_2|^2} W_{l_1,l_2} \left( \frac{\gamma_1}{Y_\infty} \right) W_{l_2} \left( \frac{\gamma_2}{Y_\infty} \right).
\]

4.1.4. Integral presentations for local weight functions. Recall that we want to derive bounds for the sums (19) and (20). Let us now observe that to derive upper bounds for these shifted convolution sums, it will suffice by the rapid decay properties of the weight functions \(W\) and \(W_j\) (for \(j = 1, 2\)) to bound the truncated sums in terms of the length \(Y = |Y_\infty|\) as
\[
\sum_{\gamma_1,\gamma_2 \in \mathbb{F}_F^\times \gamma_1 \gamma_2 = \alpha} \frac{c_H(\gamma_1) c_\pi(\gamma_2)}{|\gamma_1 \gamma_2|^2} W_1 \left( \frac{\gamma_1}{Y_\infty} \right) W_2 \left( \frac{\gamma_2}{Y_\infty} \right)
\]
\[
\sum_{\gamma_1,\gamma_2 \in \mathbb{F}_F^\times \gamma_1 \gamma_2 = \alpha} \frac{c_H(\gamma_1) c_\pi(\gamma_2)}{|\gamma_1 \gamma_2|^2} W_{l_1,l_2} \left( \frac{\gamma_1}{Y_\infty} \right) W_{l_2} \left( \frac{\gamma_2}{Y_\infty} \right).
\]

It is easy to deduce from the fact that the smooth function \(U\) used to define the partition of unity (16) is supported on \([1, 2]\) that for a given range \(R \in [2^j, 2^{j+1}]\), the function defined on \(y_\infty \in \mathbb{F}_F^{\times,+}\) by
\[
U \left( \frac{|y_\infty|}{R} \right)
\]

is supported on \(y_\infty \in \mathbb{F}_F^{\times,+}\) in the interval \(2^j \leq |y_\infty| \leq 2^{j+2} \).

Scaling out by \(Y\) in the definition, it is then easy to deduce that the truncated sums (21) and (22) can be reparametrized respectively as finite sums over local sums indexed by integers \(0 \leq l \leq \log Y/\log 2\) and \(0, l_1, l_2 \leq \log Y/\log 2\). In particular, we deduce that it will suffice to bound the finite sums over local sums
\[
\sum_{0 \leq l \leq \log Y} \sum_{\gamma_1,\gamma_2 \in \mathbb{F}_F^\times \gamma_1 \gamma_2 = \alpha} \frac{c_H(\gamma_1) c_\pi(\gamma_2)}{|\gamma_1 \gamma_2|^2} W_1 \left( \frac{\gamma_1}{Y_\infty} \right) W_{l_1} \left( \frac{\gamma_2}{Y_\infty} \right)
\]
and
\[
\sum_{0 \leq l_1,l_2 \leq \log Y} \sum_{\gamma_1,\gamma_2 \in \mathbb{F}_F^\times \gamma_1 \gamma_2 = \alpha} \frac{c_H(\gamma_1) c_\pi(\gamma_2)}{|\gamma_1 \gamma_2|^2} W_{l_1,l_2} \left( \frac{\gamma_1}{Y_\infty} \right) W_{l_2} \left( \frac{\gamma_2}{Y_\infty} \right).
\]

We shall now take this fact for granted, and in each case work to derive bounds for each of the inner sums indexed by an integer \(0 \leq l \leq \log Y\) or a pair of integers \(0 \leq l_1, l_2 \leq \log Y\). First, let us explain how we can derive integral presentations for any of the local sums corresponding to an integer \(l \in \mathbb{Z}\) in the dyadic decomposition (19) or a pair of integers \((l_1, l_2) \in \mathbb{Z}^2\) in the dyadic decomposition (20).

Taking the discussion above with Proposition 4.1 for granted, we can now derive the following integral presentations for the shifted convolution problems we consider in terms of the Fourier-Whittaker coefficients at a given nonzero \(F\)-integer \(\alpha\) of certain \(L^2\)-automorphic forms on \(GL(A_F)\) or its metaplectic cover \(\overline{G}(A_F)\).
Proposition 4.2. Fix a nonzero $F$-integer $\alpha \in \mathcal{O}_F$, and let $Y_\infty = (Y_\infty, j)_{j=1}^d \in F_\infty^x$ be any totally positive archimedean idele of norm $Y = |Y_\infty| > |\alpha|$ contained within our chosen fundamental domain of Theorem 2.5 above. Let $\Pi = \otimes_v \Pi_v$ be an irreducible cuspidal automorphic representation of $GL_m(A_F)$. Given an integer $2 \leq m \leq n$, we also consider a cuspidal automorphic representation $\pi = \otimes_v \pi_v$ of $GL_m(A_F)$. Let us also fix $0 \leq l \leq \log Y$ any integer in the subdivision (23) above, together with any pair of integers $0 \leq l_1, l_2 \leq \log Y$ in the subdivision (24). We have the following integral presentation for each of the corresponding inner sums.

(A) Let $W \in L^2(F_\infty^x)$ be any smooth function of moderate decay near zero which decays rapidly at infinity or is compactly supported, and which decays near zero as described above. Fixing a smooth partition of unity (16) and corresponding dyadic decomposition (19), let $W_l$ denote the local weight function (17) corresponding to some given integer $l \in \mathbb{Z}$ in this decomposition. Use then take $\varphi^{(l)} = \otimes_v \varphi^{(l)}_v \in V_\Pi$ to be a pure tensor whose nonarchimedean local components are each essential Whittaker vectors, and whose archimedean local component is chosen so that we have the identification of functions of $y_\infty \in F_\infty^x$

$$W_{\varphi^{(l)}}(y_\infty) := W_{\varphi^{(l)}} \left( \begin{pmatrix} y_\infty & 1_{n-1} \end{pmatrix} \right) = |y_\infty|^{\frac{m-2}{2}} \psi(-i|y_\infty|) \psi(i\alpha Y_\infty) W_l \left( \frac{1}{|Y_\infty|} \right).$$

Note that $W_{\varphi^{(l)}}(y_\infty) = W_{\varphi^{(l)}}(-y_\infty)$. Let $\theta_{f,p}$ be the theta series introduced above, with $\theta_{f,p} = T_{-1} \theta_{f,p}$ its image under the Hecke operator corresponding to $x \mapsto -x \in A_F$, and $\theta_{f,p}$ extension or descent according to Proposition 4.1 (ii). We have the integral presentation

$$\int_{x \in [0,1]^d \subset F_\infty} \mathcal{P}_{\Pi} \mathcal{P}_{\varphi^{(l)}} (Y_\infty x_\infty 1) \psi(-\alpha x_\infty) dx_\infty$$

$$= |Y_\infty|^\frac{1}{2} \sum_{a = (a_1, \ldots, a_n) \in \mathcal{O}_F^k} p(a) c_{\Pi}(a + \alpha) |f(a + \alpha)|^{\frac{1}{2}} W_l \left( \frac{1}{Y_\infty} \right).$$

(B) Let $W_1, W_2 \in L^2(F_\infty^x)$ by any smooth functions of moderate decay near zero which decay rapidly at infinity, or are compactly supported, and which decay near zero as described above. Fixing a smooth partition of unity (16) and corresponding dyadic decomposition (20) as above, let $W_j$ for $j = 1, 2$ denote the local weight functions corresponding to some given pair of integers $l_1, l_2 \in \mathbb{Z}$ as defined in (18). Let us then take $\varphi^{(l_1)} = \otimes_v \varphi^{(l_1)}_v \in V_\Pi$ be a pure tensor whose nonarchimedean local components are each essential Whittaker vectors, and whose archimedean local component is chosen so that we have the identification of functions of $y_\infty \in F_\infty^x$

$$W_{\varphi^{(l_1)}}(y_\infty) := W_{\varphi^{(l_1)}} \left( \begin{pmatrix} y_\infty & 1_{n-1} \end{pmatrix} \right) = |y_\infty|^{\frac{m-2}{2}} W_{1,l_1} \left( \frac{1}{|Y_\infty|} \right).$$

Note that $W_{\varphi^{(l_1)}}(y_\infty) = W_{\varphi^{(l_1)}}(-y_\infty)$. Let us also take $\varphi^{(l_2)} = \otimes_v \varphi^{(l_2)}_v \in V_\pi$ to be a pure tensor whose nonarchimedean local components are each essential Whittaker vectors, and whose archimedean local components are chosen so that as functions of $y_\infty \in F_\infty^x$

$$W_{\varphi^{(l_2)}}(y_\infty) := W_{\varphi^{(l_2)}} \left( \begin{pmatrix} y_\infty & 1_{m-1} \end{pmatrix} \right) = |y_\infty|^{\frac{m-2}{2}} W_{2,l_2} \left( \frac{1}{|Y_\infty|} \right).$$

Note again that $W_{\varphi^{(l_2)}}(y_\infty) = W_{\varphi^{(l_2)}}(-y_\infty)$. Then, we have the integral presentation

$$\int_{x \in [0,1]^d \subset F_\infty} \mathcal{P}_{\Pi} \mathcal{P}_{\varphi^{(l_1)}} (Y_\infty x_\infty 1) \psi(-\alpha x_\infty) dx_\infty$$

$$= \sum_{\gamma_1, \gamma_2 \in \mathcal{O}_F^x, \gamma_1 \notin \gamma_2 = 0} c_{\Pi}(\gamma_1) c_{\pi}(\gamma_2) W_{1,l_1} \left( \frac{\gamma_1}{Y_\infty} \right) W_{2,l_2} \left( \frac{\gamma_2}{Y_\infty} \right).$$

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The same argument works to make such a series of identifications for the modified form so that from which we derive the identification of unipotent integrals. We then use Definition 2.6 again to make the identification

For any \( x_\infty \in I_1 \cong [0, 1/2]^d \) in the first integral, we can use Definition 2.6 to make the easy identification

For any \( x_\infty \in I_2 \cong (1/2, 1]^d \) in the second integral, we first argue as in Proposition 2.9 that we have

We then use Definition 2.6 again to make the identification

whence we argue again following the proof of Proposition 2.9 that

so that

The same argument works to make such a series of identifications for the modified form \( \tilde{\theta}_{f,p} \). Thus, we have

from which we derive the identification of unipotent integrals

Proof. Let us start with (A), which we reduce to the calculation of Proposition 3.4 using Proposition 2.9. We split in the integral into two parts following the proof of Proposition 2.9, so that

Thus, we have
We can then evaluate the latter integral by a minor variation of the proof of Proposition 3.4. That is,

\[ \int_{f \geq 0} \mathbb{P}_{1 \over 2} \mathcal{P}(f) \cdot \mathcal{G}_{f,p} \left( \begin{array}{c} Y \infty \\ 1 \end{array} \right) \psi(-\alpha x_{\infty}) dx_{\infty} \]

\[ = \int_{f \geq 0} \mathbb{P}_{1 \over 2} \mathcal{P}(f) \left( \begin{array}{c} Y \infty \\ 1 \end{array} \right) \mathcal{G}_{f,p} \left( \begin{array}{c} Y \infty \\ 1 \end{array} \right) \psi(-\alpha x_{\infty}) dx_{\infty} \]

\[ = \left| Y_{\infty} \right| ^{\frac{1}{2}} \left( \frac{2\pi}{X} \right) \sum_{\gamma \in \mathcal{F}^\times} W_{\gamma} \left( \left( \begin{array}{c} Y \infty \\ 1 \end{array} \right) \mathcal{N}_{\gamma} \left( \begin{array}{c} \gamma Y \infty \\ 1 \end{array} \right) \right) \sum_{\alpha = (a_1, \ldots, a_k) \in \mathcal{O}^\times_p} p(a) \psi(i \cdot f(a)Y_{\infty}) \]

\[ \times \int_{f \geq 0} \psi(x_{\infty}(\gamma - f(a) - \alpha)) dx_{\infty} \]

\[ = \left| Y_{\infty} \right| ^{\frac{1}{2}} \left( \frac{2\pi}{X} \right) \sum_{\alpha = (a_1, \ldots, a_k) \in \mathcal{O}^\times_p} p(a) \rho_{\gamma}(f(a) + \alpha) W_{\gamma} ((f(a) + \alpha)Y_{\infty}) \psi(i \cdot f(a)Y_{\infty}) \]

\[ = \left| Y_{\infty} \right| ^{\frac{1}{2}} \left( \frac{2\pi}{X} \right) \sum_{\alpha = (a_1, \ldots, a_k) \in \mathcal{O}^\times_p} p(a) \epsilon_{\gamma}(f(a) + \alpha) |f(a) + \alpha|^{\frac{1}{2}} W_l \left( \frac{f(a) + \alpha}{Y_{\infty}} \right). \]

To prove (B) works in the same way, using (11) and the arguments of Proposition 2.9 to relate each of the integrals to the ones computed already in Proposition 3.5 above.

4.2. Whittaker functions and their decay properties. Let us first give a brief account of the normalized Whittaker functions we shall consider for our estimates below, and particularly the uniform bounds they satisfy, following [33, §7] and [6, §2]. Given complex numbers \( \kappa, \nu \in \mathbb{C} \), let \( W_{\kappa,\nu} \) denote the classical Whittaker function as described in [33, §7.1] (for instance), whose Mellin transform for \( \Re(s) > 1 \pm \nu \) can be described according to the calculation of [20, 7.621-11] as

\[ \int_0^\infty e^{-\frac{y}{2}} W_{\kappa,\nu}(y)y^s \frac{dy}{y} = \frac{\Gamma \left( \frac{1}{2} + s + \nu \right) \Gamma \left( \frac{1}{2} + s - \kappa \right)}{\Gamma (1 + s - \kappa)}. \]

(25)

Taking the inverse Mellin transform then gives us the more convenient integral presentation

\[ e^{-\frac{s}{2}} W_{\kappa,\nu}(y) = \int_{\Re(s) = \sigma} \frac{\Gamma \left( \frac{1}{2} + s + \nu \right) \Gamma \left( \frac{1}{2} + s - \kappa \right)}{\Gamma (1 + s - \kappa)} y^{-s} ds \]

\[ \frac{2\pi i}{2} \]

for any sufficiently large real number \( \sigma > 0 \).

4.2.1. Estimates. The latter integral presentation (26) can be used to derive the following estimates for the Whittaker function \( W_{\kappa,\nu}(y) \) as \( y \to 0 \) (and also as \( y \to \infty \)) according to the arguments given in [33, §7]. To describe these in more detail, let us first record the following auxiliary result.

**Proposition 4.3.** The following assertions about the gamma function \( \Gamma(s) \) are true:

(i) For any real parameter \( \sigma \in \mathbb{R} \), the function \( t \mapsto \Gamma(\sigma + it) \) is a decreasing function of \( |t| \).

(ii) (Stirling approximation). Fix \( \varepsilon > 0 \). For any \( s \in \mathbb{C} \) with \( |\arg(s)| > \pi - \varepsilon \), we have that

\[ \Gamma(s) = \left( \frac{2\pi}{s} \right)^{\frac{1}{2}} \left( \frac{s}{e} \right)^s \left( 1 + O \left( \frac{1}{|s|} \right) \right). \]

(iii) For any fixed \( \delta > 0 \), the value \( \Gamma(t + i|t|^{1+\delta}) \) is exponentially small as \( t \to \pm \infty \).
Proof. See [33, Lemma 7.1] for (i), [33, Lemma 7.2] for (ii), [33, Lemma 7.3] for (iii), and [33, Lemma 7.4] for (iv).

Proposition 4.4. We have the following uniform bounds for \( y \in R_{>0} \) as \( y \to 0 \).

(i) If \( \kappa, r \in R \), then for some constant \( A > 0 \) and any choice of \( \epsilon > 0 \),
\[
W_{\kappa,ir}(y) \ll \epsilon (|\kappa| + |r| + 1)^A y^{\frac{1}{2} - \epsilon}
\]

(ii) If \( \kappa \in R \) and \( 0 < \nu < \frac{1}{2} \), then for some constant \( A > 0 \) and any choice of \( \epsilon > 0 \),
\[
W_{\kappa,\nu}(y) \ll \epsilon (|\kappa| + 1)^A y^{\frac{1}{2} - \nu - \epsilon}.
\]

(iii) If \( \kappa, \nu \in R \) with \( \kappa - \nu - \frac{1}{2} \geq 0 \) and \( \nu > -\frac{1}{2} + \epsilon \), then for some \( A > 0 \) and any choice of \( \epsilon > 0 \),
\[
W_{\kappa,\nu}(y) \ll \epsilon (|\kappa| + |\nu| + 1)^A y^{\frac{1}{2} - \epsilon}.
\]

Proof. See [33, Proposition 3.1], which is proven in [33, §7]. Let us give a sketch of the argument for future reference. In each case, starting with the integral presentation (26), we shift the line of integration leftward to \( \Re(s) = \sigma = 0 \), crossing poles from the gamma functions.

For (i), we estimate the normalized contour integral presentation
\[
\frac{W_{\kappa,ir}(y)}{\Gamma\left(\frac{1}{2} + \kappa + ir\right)} = \frac{e^{\frac{\pi}{2}}}{\Gamma\left(\frac{1}{2} + \kappa + ir\right)} \int_{\Re(s) = \sigma} \frac{\Gamma\left(\frac{1}{2} + s + ir\right) \Gamma\left(\frac{1}{2} + s - ir\right)}{\Gamma(1 + s - \kappa)} y^{-s} ds \cdot 2\pi i.
\]
Shifting the line of integration to \( \Re(s) = \sigma = 0 \), we cross poles at \( s = -\frac{1}{2} - ir \) and \( s = -\frac{1}{2} + ir \). It is easy to see that the pole at \( s = -\frac{1}{2} - ir \) contributes
\[
\frac{e^{\frac{\pi}{2}} \cdot \Gamma(-2ir)}{\Gamma\left(\frac{1}{2} + \kappa + ir\right) \Gamma\left(\frac{1}{2} + \kappa - ir\right)} \cdot y^{\frac{1}{2} - ir} \ll \epsilon (|\kappa| + |r| + 1)^B y^{\frac{1}{2}}
\]
while the pole at \( s = -\frac{1}{2} + ir \) contributes
\[
\frac{e^{\frac{\pi}{2}} \cdot \Gamma(2ir)}{\Gamma\left(\frac{1}{2} + \kappa + ir\right) \Gamma\left(\frac{1}{2} + \kappa - ir\right)} \cdot y^{\frac{1}{2} - ir} \ll \epsilon (|\kappa| + |\nu| + 1)^B y^{\frac{1}{2}}
\]
Here, we use Proposition 4.3 (iv) and 4.3 (ii) (Stirling approximation) to derive the stated bounds. Note that in either case, we use the fact that for \( 0 < y < 1 \), we have \( 1 < e^{\frac{\pi}{2}} < 2 \), and hence the exponential term is absorbed into the larger quantity on the right. To estimate the remaining integral at \( \Re(s) = \sigma = 0 \), put \( s = it \) and \( \nu = ir \) so that we reduce to considering the ratio of gamma factors
\[
\frac{\Gamma\left(\frac{1}{2} + it + ir\right)}{\Gamma\left(\frac{1}{2} + it - ir\right)} \frac{\Gamma\left(\frac{1}{2} + it + ir\right)}{\Gamma\left(\frac{1}{2} + it - ir\right)}
\]
Using the exponential decay of the gamma function on vertical lines described in Proposition 4.3 (i), we may modify the denominator so that the two gamma functions are estimated along the same imaginary part \( \max(|r|, |t|) \). We can then apply the estimate of Proposition 4.3 (iv) to replace the real parameter \( \kappa \in R \) by
its fractional part \( \{ \kappa \} \), and apply Stirling approximation to estimate the remaining integral, which is then seen to have rapid decay as \( t \to \infty \) with polynomial control in \( \kappa \) and \( r \). This implies the stated bound.

The argument for (ii) is similar: Starting with the integral presentation

\[
e^{-\frac{y}{2} W_{\kappa, \nu}(y)} \frac{1}{\Gamma \left( \frac{1}{2} + \kappa \right)} \int_{\Re(s) = \sigma} \frac{\Gamma \left( s + \frac{1}{2} + \nu \right) \Gamma \left( s + \frac{1}{2} - \nu \right)}{\Gamma \left( s + 1 - \kappa \right)} y^{-s} ds \frac{2\pi i}{-s},
\]

we shift the line of integration to \( \Re(s) = \sigma = 0 \). The pole at \( s = -\frac{1}{2} - \nu \) contributes a residue of

\[
e^{-\frac{y}{2} \cdot \Gamma(-2\nu)} \frac{1}{\Gamma \left( \frac{1}{2} + \kappa \right) \Gamma \left( \frac{1}{2} - \nu - \kappa \right)} \cdot y^{\frac{1}{2} + \nu} \ll (1 + |\kappa|)^B \cdot y^{\frac{1}{2} + \nu}
\]

and the pole at \( s = -\frac{1}{2} + \nu \) contributes a residue of

\[
e^{-\frac{y}{2} \cdot \Gamma(2\nu)} \frac{1}{\Gamma \left( \frac{1}{2} + \kappa \right) \Gamma \left( \frac{1}{2} + \nu - \kappa \right)} \cdot y^{\frac{1}{2} - \nu} \ll (1 + |\kappa|)^B \cdot y^{\frac{1}{2} - \nu}
\]

To estimate the remaining integral over \( \sigma = 0 \), we use the same argument as given for (i) to estimate the corresponding ratio of gamma factors

\[
\frac{\Gamma \left( \frac{1}{2} + it + \nu \right) \Gamma \left( \frac{1}{2} + it - \nu \right)}{\Gamma \left( 1 + it - \kappa \right) \Gamma \left( \frac{1}{2} + \kappa \right)}.
\]

The argument for (iii) is also similar: Starting with the integral presentation

\[
e^{-\frac{y}{2} W_{\kappa, \nu}(y)} \frac{1}{\Gamma \left( \frac{1}{2} + \kappa - \nu \right) \Gamma \left( \frac{1}{2} + \kappa + \nu \right)} \int_{\Re(s) = \sigma} \frac{\Gamma \left( s + \frac{1}{2} + \nu \right) \Gamma \left( s + \frac{1}{2} - \nu \right)}{\Gamma \left( s + 1 - \kappa \right)} y^{-s} ds \frac{2\pi i}{-s},
\]

we shift the line of integration to \( \Re(s) = \sigma = 0 \). Now, observe that since \( \kappa - \nu - \frac{1}{2} \in \mathbb{Z} \), the function

\[
s \mapsto \frac{\Gamma \left( \frac{1}{2} + s - \nu \right)}{\Gamma \left( 1 + s - \kappa \right)}
\]

has no pole. There is a residue as \( s = -\frac{1}{2} - \nu \) which would contribute

\[
e^{\frac{y}{2} \cdot \Gamma(-2\nu)} \frac{1}{\Gamma \left( \frac{1}{2} + \kappa + \nu \right) \Gamma \left( \frac{1}{2} - \nu - \kappa \right)} \cdot y^{\frac{1}{2} + \nu},
\]

however we do not have to consider such a contribution as \( \nu > -\frac{1}{2} \). The remaining integral over \( \Re(s) = 0 \) is bounded using a similar argument as for (i) and (ii). That is, it will do to bound the ratio

\[
\frac{\Gamma \left( \frac{1}{2} + \nu + it \right) \Gamma \left( \frac{1}{2} - \nu + it \right)}{\sqrt{\left| \Gamma \left( \frac{1}{2} + \kappa - \nu \right) \Gamma \left( \frac{1}{2} + \kappa + \nu \right) \cdot \Gamma \left( 1 + it - \kappa \right) \right|}}.
\]

Observe that \( \sqrt{\Gamma \left( \frac{1}{2} + \kappa - \nu \right) \Gamma \left( \frac{1}{2} + \kappa + \nu \right)} \geq \sqrt{\Gamma(\kappa)} \), from which we deduce that it will suffice to bound

\[
\frac{\Gamma \left( it + \kappa \right) \Gamma \left( \frac{1}{2} + \nu + it \right) \Gamma \left( \frac{1}{2} - \nu + it \right)}{\Gamma \left( \kappa \right) \Gamma \left( 1 + it - \kappa \right) \Gamma \left( 1 + it + \kappa \right)}.
\]

Here, we can apply Proposition 4.3 (i) to derive the bound \( \Gamma(it + \kappa)/\Gamma(\kappa) \ll 1 \). Moreover, we can apply Proposition 4.3 (ii) and (iii) to deduce that this quantity \( \Gamma(it + \kappa)/\Gamma(\kappa) \) decays exponentially as \( |t| > \kappa^{1+\delta} \).

Now, since \( \nu \in \mathbb{R} \), we can apply Proposition 4.3 (iv) to the product of gamma factors in the remaining numerator. This allows us to replace the contribution of \( \nu \) by its fractional part \( \{ \nu \} \). Similarly, applying Proposition 4.3 (iv) to the product of gamma factors in the denominator allows us to replace the contribution of \( \kappa \) by its fractional part \( \{ \kappa \} \). We can then use Stirling approximation (Proposition 4.3 (ii)) to deduce that this remaining ratio is bounded uniformly by a polynomial in \( \kappa \) and \( \nu \), as required.

Let us now record how a variation of this argument allows us to derive similar quantitative uniform bounds as \( y \to \infty \). In fact, a variation of the same argument gives the same uniform, quantitative bounds as \( y \to \infty \):

**Corollary 4.5.** We have the following uniform bounds for \( y \in \mathbb{R}_{>0} \) as \( y \to \infty \):
(i) If $\kappa, \nu \in \mathbb{R}$, then for some constant $A > 0$ and any choice of $\varepsilon > 0$,
\[
\frac{W_{\kappa,ir}(y)}{\Gamma\left(\frac{1}{2} + ir + \kappa\right)} \ll_{\varepsilon} (|\kappa| + |r| + 1)^A y^{\frac{1}{2} + \varepsilon}.
\]

(ii) If $\kappa \in \mathbb{R}$ and $0 < \nu < \frac{1}{2}$, then for some constant $A > 0$ and any choice of $\varepsilon > 0$,
\[
\frac{W_{\kappa,\nu}(y)}{\Gamma\left(\frac{1}{2} + \kappa\right)} \ll_{\varepsilon} (|\kappa| + 1)^A y^{\frac{1}{2} + \nu + \varepsilon}.
\]

(iii) If $\kappa, \nu \in \mathbb{R}$ with $\kappa - \nu - \frac{1}{2} \geq 0$ and $\nu > -\frac{1}{2} + \varepsilon$, then for some $A > 0$ and any choice of $\varepsilon > 0$,
\[
\frac{W_{\kappa,\nu}(y)}{\Gamma\left(\frac{1}{2} + \kappa - \nu\right)\Gamma\left(\frac{1}{2} + \kappa + \nu\right)} \ll_{\varepsilon} (|\kappa| + |\nu| + 1)^A y^{\frac{1}{2} + \varepsilon}.
\]

**Proof.** We proceed in a similar way as for Proposition 4.4, shifting contours to the right. That is, fixing the real parameter $y > 1$, we assume the integral in (26) is defined for $\Re(s) = \sigma > 0$ sufficiently large, then shift the contour to the right beyond the line determined by $y^\sigma = e^{\frac{\pi}{2}}$. We then have to consider residues coming from the corresponding gamma factors at $-\frac{1}{2} \pm \nu + [\sigma]$, where $[\sigma]$ denotes the integer part of $\sigma > 0$, i.e. so that $\sigma = [\sigma] + \{\sigma\}$. The remaining contours are seen easily to be negligible, in contrast the case of $0 < y < 1$ shown above, though the final estimates (reflecting the contributions of the residual terms) are the same.

For (i), we estimate the contribution of
\[
\frac{W_{\kappa,ir}(y)}{\Gamma\left(\frac{1}{2} + ir + \kappa\right)} = \frac{e^y}{\Gamma\left(\frac{1}{2} + ir + \kappa\right)} \int_{\Re(s) = \sigma} \frac{\Gamma\left(s + \frac{1}{2} + ir\right)\Gamma\left(s + \frac{1}{2} - ir\right)}{\Gamma\left(s + 1 - \kappa\right)} y^{-s} ds \quad 2\pi i.
\]

Taking $\sigma > 0$ to be large enough so that $y^\sigma \geq e^{\frac{\pi}{2}}$, we shift the contour far enough to the right that we cross poles from each of the gamma factors in the numerator. The pole at $s = -\frac{1}{2} - ir + [\sigma]$ contributes
\[
\frac{e^y}{y^\sigma} \cdot \frac{\Gamma\left(-2ir + [\sigma]\right)}{\Gamma\left(\frac{1}{2} + ir + [\sigma] + \kappa\right)\Gamma\left(\frac{1}{2} + ir + [\sigma] - \kappa\right)} \ll (|\kappa| + |r| + 1)^B y^{\frac{3}{2}}.
\]

where again we use Proposition 4.3 (iv) to replace $[\sigma]$ by its fractional part $\{[\sigma]\} = 0$ followed by Proposition 4.3 (ii) (Stirling approximation). Similarly, we see that the pole at $s = -\frac{1}{2} + ir + [\sigma]$ contributes
\[
\frac{e^y}{y^\sigma} \cdot \frac{\Gamma\left(2ir + [\sigma]\right)}{\Gamma\left(\frac{1}{2} + ir + [\sigma] + \kappa\right)\Gamma\left(\frac{1}{2} + ir + [\sigma] - \kappa\right)} \ll (|\kappa| + |r| + 1)^B y^{\frac{3}{2}}.
\]

It is easy to see that the remaining integral is bounded above by a comparatively negligible quantity.

For (ii), we argue in a similar way to estimate the contribution of
\[
\frac{W_{\kappa,\nu}(y)}{\Gamma\left(\frac{1}{2} + \kappa\right)} = \frac{e^y}{\Gamma\left(\frac{1}{2} + \kappa\right)} \int_{\Re(s) = \sigma} \frac{\Gamma\left(s + \frac{1}{2} + \nu\right)\Gamma\left(s + \frac{1}{2} - \nu\right)}{\Gamma\left(s + 1 - \kappa\right)} y^{-s} ds \quad 2\pi i.
\]

Shifting the contour to the right in the same way, the pole at $s = -\frac{1}{2} - \nu - [\sigma]$ contributes
\[
\frac{e^y}{y^\sigma} \cdot \frac{\Gamma\left(-2\nu + [\sigma]\right)}{\Gamma\left(\frac{1}{2} + \kappa\right)\Gamma\left(\frac{1}{2} - \nu + [\sigma]\right)} \ll (1 + |\kappa|)^B y^{\frac{1}{2} + \nu},
\]

and the pole at $s = -\frac{1}{2} + \nu + [\sigma]$ contributes
\[
\frac{e^y}{y^\sigma} \cdot \frac{\Gamma\left(2\nu + [\sigma]\right)}{\Gamma\left(\frac{1}{2} + \kappa\right)\Gamma\left(\frac{1}{2} - \nu + [\sigma]\right)} \ll (1 + |\kappa|)^B y^{\frac{1}{2} - \nu}.
\]

Again, we argue that the remaining integral is bounded above by a comparatively negligible quantity.

Finally for (iii), we also argue in a similar way to estimate the contribution of
\[
\frac{W_{\kappa,\nu}(y)}{\Gamma\left(\frac{1}{2} + \kappa - \nu\right)\Gamma\left(\frac{1}{2} + \kappa + \nu\right)} = \frac{e^y}{\Gamma\left(\frac{1}{2} + \kappa - \nu\right)\Gamma\left(\frac{1}{2} + \kappa + \nu\right)} \int_{\Re(s) = \sigma} \frac{\Gamma\left(s + \frac{1}{2} + \nu\right)\Gamma\left(s + \frac{1}{2} - \nu\right)}{\Gamma\left(s + 1 - \kappa\right)} y^{-s} ds \quad 2\pi i.
\]
Shifting the contour to the right in the same way, we pick up a pole at $s = -\frac{1}{2} - \nu + [\sigma]$ which contributes
\[
\frac{e^{\frac{s}{2}}}{y^\sigma} \cdot \frac{\Gamma(-2\nu + [\sigma])}{\Gamma\left(\frac{1}{2} + \kappa + \nu\right) \Gamma\left(\frac{1}{2} + \kappa + \nu + [\sigma] - \kappa\right)} \ll (|\kappa| + |\nu| + 1)^B y^{\frac{1}{2} + \nu}
\]
and a pole at $s = -\frac{1}{2} + \nu + [\sigma]$ which contributes
\[
\frac{e^{\frac{s}{2}}}{y^\sigma} \cdot \frac{\Gamma(2\nu + [\sigma])}{\Gamma\left(\frac{1}{2} + \kappa + \nu\right) \Gamma\left(\frac{1}{2} + \kappa + \nu + [\sigma] - \kappa\right)} \ll (|\kappa| + |\nu| + 1)^B y^{\frac{1}{2} - \nu}
\]
Again, the remaining integral is seen to be bounded by a comparatively negligible quantity to derive the stated estimate. 

4.2.2. Remark on normalizations. Note that we could also consider the following normalized Whittaker functions here, in the style of Blomer-Harcos [6, §2.4]. To describe these normalized Whittaker functions, let us now take $k \in \mathbb{Z}$ any integer, and assume that the corresponding spectral parameter $\nu$ in constrained by
\[
\nu \in \begin{cases} 
\left(\frac{1}{2} + \mathbb{Z}\right) \cup i\mathbb{R} \cup (-\frac{1}{2}, \frac{1}{2}) & \text{if } k \equiv 0 \mod 2 \\
\mathbb{Z} \cup i\mathbb{R} & \text{if } k \equiv 1 \mod 2
\end{cases}
\]
For such parameters, we define the normalized Whittaker function on $y \in \mathbb{R}^\times$ by the formula
\[
W_{\frac{1}{2}, \nu}^*(y) = \frac{e^{\frac{s}{2}y^{\nu}} W_{\frac{1}{2}, \nu}(4\pi |y|)}{\Gamma\left(\frac{1}{2} - \nu + \text{sgn}(y)\frac{1}{2}\right) \Gamma\left(\frac{1}{2} + \nu + \text{sgn}(y)\frac{1}{2}\right)}.
\]
As explained in [6, §2.4] (cf. erratum), these normalized Whittaker functions are more natural to consider for spectral arguments, as they can be shown according to [9, §4] to form an orthonormal basis of the space of square integrable functions on $\mathbb{R}^\times$ with respect to the Haar measure $d\phi = dy/|y|$: For each choice of parity $l \in \{0, 1\}$, we have the Hilbert space decomposition
\[
L^2(\mathbb{R}^\times, d\phi) = \bigoplus_{k \equiv l \mod 2} C W_{\frac{1}{2}, \nu}^*; \quad \langle W_{\frac{1}{2}, \nu}^*, W_{\frac{1}{2}, \nu}^* \rangle = \begin{cases} 
1 & \text{if } k_1 = k_2 \\
0 & \text{if } k_1 \neq k_2
\end{cases}.
\]
It is easy to deduce that these normalized functions can be bounded uniformly as in Proposition 4.4 above.

4.2.3. Extension of estimates to totally real fields. Let us also note that since we work over a totally real number field $F$ of degree $d$, the explicit functions appearing in the Fourier-Whittaker expansions described above can be realized as $d$-fold products of these classical Whittaker functions, indexed by $d$-tuples of weights $\kappa = (\kappa_j)_{j=1}^d$ and spectral parameters $\nu = (\nu_j)_{j=1}^d$. Hence, we define the corresponding Whittaker functions on $y_\infty = (y_j)_{j=1}^d \in F_\infty^d$ via the $d$-fold products
\[
W_{\kappa, \nu}(y_\infty) = \prod_{j=1}^d W_{\kappa_j, \nu_j}(y_j) \quad \text{and} \quad W_{\frac{1}{2}, \nu}^*(y_\infty) = \prod_{j=1}^d W_{\frac{1}{2}, \nu_j}^*(y_j).
\]
Plainly, these functions can be estimated for $|y_\infty| \to 0, \infty$ via Proposition 4.4 and Corollary 4.5 above.

4.3. Sobolev norms. Let us now give some background on Sobolev norms for later arguments. Here, we refer to the general theory as described in [35, §2.9] and [29, §2.4], which apply to smooth or sufficiently smooth $L^2$-automorphic functions on $GL_2(\mathbb{A}_F)$. Note that we shall later supply additional justification to treat the $L^2$-automorphic functions we constructed from the automorphic functions $\mathbb{P}^n_{\kappa, \nu}$ on the mirabolic subgroup $P_2$ as considered above, which are not generally smooth.

The action of $GL_2(F_\infty)$ on the space of $L^2$-automorphic forms $L^2(GL_2(F) \setminus GL_2(\mathbb{A}_F), \omega)$ induces an action of the Lie algebra $gl_2(F_\infty)$ of $GL_2(F_\infty)$ on this space. The corresponding action of the Lie subalgebra $\mathfrak{g} = \mathfrak{sl}_2(F_\infty)$ induces an action of its universal enveloping algebra $U(\mathfrak{g})$ on $L^2(GL_2(F) \setminus GL_2(\mathbb{A}_F), \omega)$ via higher-order differential operators. To be more explicit, writing $e_j$ for a given index $1 \leq j \leq d$ to denote
the standard basis vector with $j$-th entry equal to one and all others zero, the action of $g$ is generated for $1 \leq j \leq d$ by the linearly independent vectors

$$H_j = \begin{pmatrix} e_j & 0 & 0 \\ 0 & e_j & 0 \\ 0 & 0 & e_j \end{pmatrix}, \quad R_j = \begin{pmatrix} 0 & e_j & 0 \\ e_j & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_j = \begin{pmatrix} 0 & 0 & 0 \\ e_j & 0 & 0 \\ 0 & e_j & 0 \end{pmatrix}.$$ 

Writing $k(\vartheta) \in \text{SO}_2(F_\infty)$ for $\vartheta = (\vartheta_j)_{j=1}^d \in (\mathbb{R}/\mathbb{Z})^d$ to denote a generic element, i.e. so that

$$k(\vartheta) = \begin{pmatrix} \cos \vartheta & \sin \vartheta \\ -\sin \vartheta & \cos \vartheta \end{pmatrix} \in \text{SO}_2(F_\infty),$$

the differential operators corresponding to these basis elements can be described explicitly as

$$dH_j = -2y_j \sin(2\vartheta_j) \partial_{x_j} + 2y_j \cos(2\vartheta_j) \partial_{y_j} + \sin(2\vartheta_j) \partial_{\vartheta_j},$$

$$dR_j = y_j \cos(2\vartheta_j) \partial_{x_j} + y_j \sin(2\vartheta_j) \partial_{y_j} + \sin(2\vartheta_j) \partial_{\vartheta_j},$$

$$dL_j = y_j \cos(2\vartheta_j) \partial_{x_j} + y_j \sin(2\vartheta_j) \partial_{y_j} - \cos(2\vartheta_j) \partial_{\vartheta_j}.$$ 

Note that this discussion extends directly to describe an action of the universal enveloping algebra $U(g)$ on the corresponding space of genuine $L^2$-automorphic forms on the metaplectic cover $L^2(GL_2(F) \backslash G(A_F), \omega)$, as there is an identification of the corresponding Lie subalgebras (cf. [33, §6.2], [29, §2.4]). Thus in either case, we can define Sobolev norms in a natural way as follows:

**Definition 4.6.** Given an integer $B \geq 1$ and a smooth function $\phi \in L^2(GL_2(F) \backslash GL_2(A_F), \omega)$ or more generally $\phi \in L^2(GL_2(F) \backslash \overline{G(A_F)}, \omega)$, we define the corresponding Sobolev norm $||\phi||_{S^B}$ by the rule

$$||\phi||_{S^B} = \sum_{\text{ord}(\mathcal{D}) \leq B} ||\mathcal{D}\phi||,$$

where the sum runs over all monomials $\mathcal{D} \in U(g)$ in $H_j, R_j,$ and $L_j$ of degrees less than or equal to $B$. The norm $|| \cdot ||$ on the right hand side of the definition denotes the norm coming from the inner product on $L^2(GL_2(F) \backslash GL_2(A_F), \omega)$ or $L^2(GL_2(F) \backslash \overline{G(A_F)}, \omega)$ respectively.

Now, observe that if we use the surjectivity of the archimedean local Kirillov map to choose an $L^2$-automorphic form $\phi \in L^2(GL_2(F) \backslash GL_2(A_F), \omega)$ in such a way that the archimedean local component of the corresponding Whittaker function $W_{\phi}(y)_{2x}$ is smooth as a function of $y_{2x} \in F_\infty$, then $||\phi||_{S^B}$ is convergent for each integer $B \geq 1$ (cf. [33, §2.4, Proposition 2.1]). The same assertion is true after multiplication by a metaplectic theta series $\theta$. That is, each norm $|| \cdot ||_{S^B}$ can be extended to the corresponding Hilbert space $L^2(GL_2(F) \backslash \overline{G(A_F)}, \omega)$ of $L^2$-automorphic forms on the metaplectic cover $\overline{G(A_F)}$. It is then relatively easy to deduce that $||\phi\theta||_{S^B} \ll 1$ for any choice of integer $B \geq 1$. Moreover, we have the following general result.

**Proposition 4.7.** Let $\phi$ be any smooth automorphic form on $GL_2(F) \backslash GL_2(A_F)$ or $GL_2(F) \backslash \overline{G(A_F)}$ with a given central character. Then, we have for each integer $B \geq 0$ the Sobolev norm bound $||\phi||_{S^B} \ll 1$, where the implied constant depends only on the choice of $B$.

**Proof.** See [33, Lemma 6.1] together with the more general discussion given in [29, §2.4] or [35, §2.9, 8]. That is, the result is standard for smooth cuspidal functions, and can be extended via height functions according to the argument of [33, Lemma 6.1] to treat the more general setting described in the statement. \hfill \Box

### 4.4. Spectral decompositions

We now begin the proof of Theorem 1.1. We first review some background on spectral decompositions for the Hilbert spaces of $L^2$-automorphic forms we consider. We then argue that the coefficients in the spectral expansion(s) of the form(s) $\Phi$ we consider are bounded uniformly in any case; see Lemma 4.8, Proposition 4.9, and the subsequent remark. We then proceed to bound the Fourier-Whittaker coefficient at $\alpha$ of $\Phi$ via spectral decomposition to prove Theorem 1.1.

#### 4.4.1. Spectral decompositions of (genuine and non-genuine) metaplectic forms

Let us first describe the space of $L^2$-automorphic forms on the metaplectic cover $\overline{G(A_F)}$ briefly. The two-fold cover $\overline{G}$ of the algebraic group $GL_2$ can be defined via cocycles, as described in Gelbart [17]. Writing $\mu_2 = \{\pm 1\}$ to denote the square roots of unity, the adelic points $G(A_F)$ fit into the exact sequence

$$1 \rightarrow \mu_2 \rightarrow \overline{G(A_F)} \rightarrow GL_2(A_F) \rightarrow 1.$$
This sequence splits over the group of $F$-rational points $GL_2(F)$, as well as the unipotent radical of its standard Borel subgroup. Recall that automorphic forms on $G(A_F)$ which transform nontrivially under $\mu_2$ are said to be genuine. These genuine forms correspond to modular forms of half-integral weight, the prototype example being the metaplectic theta series $\theta_q$ described above, and more generally the theta series $\theta_{f,p}$ associated to a positive definite quadratic form $f$ with an odd number of variables $k$. Note that these theta series can be described equivalently in terms of the Weil representation $r_f$ associated to $f$.

Let $\Phi$ be any genuine automorphic form in the space $L^2(GL_2(F) \backslash GL_2(A_F), \omega)$, where $\omega$ is some given idele class character of $F$. Hence, $\Phi$ transforms nontrivially under the group of square roots of unity $\mu_2$. We can decompose any $L^2$-autormorphic form $\Phi \in L^2(GL_2(F) \backslash GL_2(A_F), \omega)$ spectrally as follows, according to the description given in Gelbart [17]. That is, the space of genuine forms $L^2(GL_2(F) \backslash GL_2(A_F), \omega)$ decomposes into a Hilbert space direct sum of a discrete spectrum $L^2_{\text{disc}}(GL_2(F) \backslash GL_2(A_F), \omega)$ plus a continuous spectrum $L^2_{\text{cont}}(GL_2(F) \backslash GL_2(A_F), \omega)$ spanned by analytic continuations of metaplectic Eisenstein series. The discrete spectrum decomposes further into subspaces of cuspidal and residual forms

$$L^2_{\text{disc}}(GL_2(F) \backslash GL_2(A_F), \omega) = L^2_{\text{cusp}}(GL_2(F) \backslash GL_2(A_F), \omega) \oplus L^2_{\text{res}}(GL_2(F) \backslash GL_2(A_F), \omega).$$

Note that the residual forms (by definition) occur as residues of metaplectic Eisenstein series, and moreover $\theta_{f,p}$ can be described equivalently in terms of the Weil representation $\sigma_f$. Hence, $\Phi$ transforms nontrivially under the group of square roots of unity $\mu_2$. Furthermore, $\Phi$ can decompose any $L^2$-automorphic form $\Phi \in L^2(GL_2(F) \backslash GL_2(A_F), \omega)$ into a Hilbert space direct sum of a discrete spectrum $L^2_{\text{disc}}(GL_2(F) \backslash GL_2(A_F), \omega)$ plus a continuous spectrum $L^2_{\text{cont}}(GL_2(F) \backslash GL_2(A_F), \omega)$ spanned by analytic continuations of metaplectic Eisenstein series. The discrete spectrum decomposes further into subspaces of cuspidal and residual forms

$$L^2_{\text{disc}}(GL_2(F) \backslash GL_2(A_F), \omega) = L^2_{\text{cusp}}(GL_2(F) \backslash GL_2(A_F), \omega) \oplus L^2_{\text{res}}(GL_2(F) \backslash GL_2(A_F), \omega).$$

In the event that we evaluate at such an element $g \in GL_2(A_F)$, we shall simply write $\Phi = g \cdot \Phi$. Note as well that the spectral decomposition (28) is completely analogous to the well-known spectral decomposition of $L^2(GL_2(F) \backslash GL_2(A_F), \omega)$ into subspaces of cuspidal forms $L^2_{\text{cusp}}(GL_2(F) \backslash GL_2(A_F), \omega)$, residual forms $L^2_{\text{res}}(GL_2(F) \backslash GL_2(A_F), \omega)$, and forms spanned by analytic continuations of Eisenstein series $L^2_{\text{cont}}(GL_2(F) \backslash GL_2(A_F), \omega)$, which we write here as

$$L^2_{\text{cusp}}(GL_2(F) \backslash GL_2(A_F), \omega) \oplus L^2_{\text{res}}(GL_2(F) \backslash GL_2(A_F), \omega) \oplus L^2_{\text{cont}}(GL_2(F) \backslash GL_2(A_F), \omega).$$

Essentially, we shall use this latter decomposition (29) to treat the non-genuine cases of (A) and (B) considered above, i.e. for a quadratic form $f$ with an even number of variables $k \geq 2$ for (A) or any version of (B) in the setup with integral presentations described above (e.g. Proposition 4.2). The discussion of spectral decompositions of non-genuine forms factoring through $L^2(GL_2(F) \backslash GL_2(A_F), \omega)$ is formally identical to the
case of genuine forms in $L^2(\text{GL}_2(F)\backslash \mathcal{G}(A_F), \omega)$ for all of our subsequent arguments. We shall therefore only give full details for the genuine metaplectic case, the non-genuine case being easy to deduce as a consequence.

4.4.2. Convergence properties of the spectral coefficients. Let us now consider convergence properties of the spectral coefficients appearing in the decomposition (28), as well as the analogous coefficients appearing in the corresponding decomposition (29) with respect to a similar choice of orthonormal basis of non-genuine forms. Recall that if the form $\Phi \in L^2(\text{GL}_2(F)\backslash \mathcal{G}(A_F), \omega)$ is smooth (even if it is not $K$-finite), then we know by Proposition 4.7 that this decomposition is convergent in the Sobolev norm topology, i.e. that the Sobolev norm $||\Phi||_{S\omega}$ converges for any integer $B \geq 1$. We can then use the following well-known argument to bound the coefficients in the corresponding spectral decomposition (28) of $\Phi$.

Lemma 4.8. Suppose that the genuine form $\Phi \in L^2(\text{GL}_2(F)\backslash \mathcal{G}(A_F), \omega)$ is smooth, and hence that it has convergent Sobolev norm, or more generally that $\Phi$ is sufficiently smooth so as to have convergence Sobolev norm. Then, the spectral coefficients of $\Phi$ appearing in the decomposition (28) can be bounded as follows: For any choices for real numbers $C \in \mathbb{R}$, we have the bounds

$$\langle \Phi, \phi_i \rangle \ll_C \prod_{j=1}^{d} |\nu_{i,j}|^C, \quad \langle \Phi, \vartheta_\xi \rangle \ll_C \prod_{j=1}^{d} |\nu_{\xi,j}|^C \quad \langle \Phi, \mathcal{E}_{\omega}(*,s) \rangle \ll_C \prod_{j=1}^{d} |\nu_{\varpi,s,j}|^C$$

for all $\phi_i$, $\vartheta_\xi$, and $\mathcal{E}_{\omega}(*,s)$ in a fixed orthonormal basis $B$ as described above. Here, the implied constants depend on the choice of form $\Phi$, and in particular the representation $\Pi$.

Proof. See [33, 55-6], [6], or [29]. The idea is that for any arbitrary integer $A \geq 1$, we can apply the $A$-fold iterate $\Delta^{(A)}$ of the generalized Laplacian operator $\Delta$ to either side of (28) to derive the relation

$$\Delta^{(A)} \Phi = \sum_i \langle \Phi, \phi_i \rangle \cdot \prod_{j=1}^{d} \left( \frac{1}{4} + \nu_{i,j}^2 \right)^A \cdot \phi_i + \sum_\xi \langle \Phi, \vartheta_\xi \rangle \cdot \prod_{j=1}^{D} \left( \frac{1}{4} + \nu_{\xi,j}^2 \right)^A \cdot \vartheta_\xi$$

$$+ \sum_\omega \int_{\mathbb{R}(s) = 1/2} \langle \Phi, \mathcal{E}_{\omega}(*,s) \rangle \cdot \prod_{j=1}^{d} \left( \frac{1}{2} + \nu_{\omega,s,j}^2 \right)^A \cdot \mathcal{E}_{\omega}(*,s) \frac{ds}{2\pi i}.$$  

Note that this relation is also justified for non-$K$-finite forms $\Phi$: In classical terms, such a form $\Phi$ can be viewed as a convergent infinite linear combination of some fixed $K$-finite basis form under iterates of Maass weight raising operators, and this can be used to determine the weights of the corresponding basis forms which contribute to (28). Taking $L^2$-norms $|| \cdot || = || \cdot ||_{L^2} = \langle \cdot, \cdot \rangle$ then gives the corresponding relation

$$||\Delta^{(A)} \Phi|| = \sum_i \langle \Phi, \phi_i \rangle^2 \cdot \prod_{j=1}^{d} \left( \frac{1}{4} + \nu_{i,j}^2 \right)^{2A} ||\phi_i|| + \sum_\xi \langle \Phi, \vartheta_\xi \rangle^2 \cdot \prod_{j=1}^{D} \left( \frac{1}{4} + \nu_{\xi,j}^2 \right)^{2A} ||\vartheta_\xi||$$

$$+ \sum_\omega \left| \int_{\mathbb{R}(s) = 1/2} \langle \Phi, \mathcal{E}_{\omega}(*,s) \rangle^2 \cdot \prod_{j=1}^{d} \left( \frac{1}{2} + \nu_{\omega,s,j}^2 \right)^A \cdot \mathcal{E}_{\omega}(*,s) \frac{ds}{2\pi i} \right|.$$  

The key point here is that $||\Delta^{(A)} \Phi|| \ll ||\Phi||_{S\omega^{(A)}} \ll 1$ is bounded for any such choice of integer $A \geq 1$, i.e. where the exponent $B = B(A) \geq 1$ for the Sobolev norm is some integer depending only on the choice of $A$. Applying Parseval’s formula to the contribution of the discrete spectrum, and using adjointness properties of the inner product for the contribution of the continuous spectrum, the claimed bounds for the spectral coefficients $\langle \Phi, \phi_i \rangle$, $\langle \Phi, \vartheta_\xi \rangle$, and $\langle \Phi, \mathcal{E}_{\omega}(*,s) \rangle$ are then easy to deduce. Here, we use the fact that the integer $A \geq 1$ can be taken to be arbitrarily large.

Let us now suppose more generally that $\Phi \in L^2(\text{GL}_2(F)\backslash \mathcal{G}(A_F), \omega)$ is one of the forms appearing in the integral presentations of Proposition 4.2 above. In particular, we now consider the more general setting where the $L^2$-automorphic form $\Phi$ we consider is not generally smooth, specifically:

$$\Phi = \begin{cases} \frac{P_{\ell_3}^n(x)}{P_{\ell_2}^n(y)}, \Phi \in \tilde{P}_{\ell_3}^n \quad \text{corresponding to case (A) of Proposition 4.2} \\ \frac{P_{\ell_3}^n(y)}{P_{\ell_2}^n(x)}, \Phi \in \tilde{P}_{\ell_2}^n \quad \text{corresponding to case (B) of Proposition 4.2} \end{cases}$$
so that

\[
\Phi \in \begin{cases} 
L^2(\text{GL}_2(F) \setminus \overline{G}(A_F), 1)^K \\
L^2(\text{GL}_2(F) \setminus \text{GL}_2(A_F), 1)^K
\end{cases}
\]

corresponding to case (A) of Proposition 4.2

\[
L^2(\text{GL}_2(F) \setminus \text{GL}_2(A_F), 1)^K
\]

corresponding to case (B) of Proposition 4.2.

In any case (lifting to the metaplectic cover \(\overline{G}(A_F)\) if needed for (A)), we should like to consider the spectral decomposition of \(\Phi\) with respect to a fixed orthonormal basis, similar to the basis \(\mathcal{B}\) described above. To be clear, the form \(\Phi\) can be decomposed with respect to an orthonormal basis \(\mathcal{B}'\) of either of the corresponding Hilbert spaces: Writing \(\{f_i\}\) to denote an orthonormal basis of the cuspidal subspace, \(\{\theta_\xi\}\) an orthonormal basis for the subspace of residual forms, and \(\{E_{\varpi}(\ast, s)\}\) an orthonormal basis for the subspace of Eisenstein series spanning the continuous spectrum, we have

\[
(31) \quad \Phi = \sum_i \langle \Phi, f_i \rangle \cdot f_i + \sum_\xi \langle \Phi, \theta_\xi \rangle \cdot \theta_\xi + \sum_\varpi \int_{\mathbb{R}(s)=1/2} \langle \Phi, E_{\varpi}(\ast, s) \rangle \cdot E_{\varpi}(\ast, s) \frac{ds}{2\pi i}.
\]

Now, we are going to argue that this decomposition can be approximated by the corresponding linear combination of the smooth basis \(\mathcal{B}\),

\[
(32) \quad \Phi \approx \sum_i \langle \Phi, \phi_i \rangle \cdot \phi_i + \sum_\xi \langle \Phi, \theta_\xi \rangle \cdot \theta_\xi + \sum_\varpi \int_{\mathbb{R}(s)=1/2} \langle \Phi, E_{\varpi}(\ast, s) \rangle \cdot E_{\varpi}(\ast, s) \frac{ds}{2\pi i},
\]

and that the coefficients in this latter expression can be bounded using a variation of the standard argument given for Lemma 4.8 above, using inner products with Poincaré series.

**Proposition 4.9.** The spectral coefficients of \(\Phi \in L^2(\text{GL}_2(F) \setminus \overline{G}(A_F), 1)\) defined in (30) appearing in the decomposition (31) can be bounded as follows: We have for any choices for real numbers \(C \in \mathbb{R}\) the bounds

\[
\langle \Phi, f_i \rangle \ll_C \prod_{j=1}^d |\nu_{i', j}|^C, \quad \langle \Phi, \theta_\xi \rangle \ll_C \prod_{j=1}^d |\nu_{\xi', j}|^C, \quad \langle \Phi, E_{\varpi}(\ast, s) \rangle \ll_C \prod_{j=1}^d |\nu_{\varpi, s, j'}|^C.
\]

Here, the indices \(i', \xi', \varpi'\) on the right hand side corresponding to elements of the smooth basis \(\mathcal{B}\) can be chosen arbitrarily. That is, the bounds for the coefficients appearing in the spectral expansion (31) with respect to the fixed orthonormal basis \(\mathcal{B}'\) are given in terms of the spectral parameters of the basis \(\mathcal{B}\) of smooth forms described above, but without specification to a particular form within the respective cuspidal, residual, or continuous subspaces. The implied constants depend on \(\Phi\), and in particular the choices of archimedean local vectors determining the weight functions, the representation \(\Pi\), and the theta series \(\vartheta_{f,p}\) corresponding to case (A) of Proposition 4.2 or the representation \(\pi\) corresponding to case (B) of Proposition 4.2. They also depend on the choice of some auxiliary Poincaré series.

**Proof.** We argue in the style of [31, §2], replacing the polynomial \(P = P(\phi_1, \ldots, \phi_k)\) with a Poincaré series \(P_\phi \in L^2(\text{GL}_2(F) \setminus \overline{G}(A_F), 1)\) corresponding to a suitably chosen smooth function \(\phi\). That is, we shall first derive bounds for the linear combination appearing on the right hand side of (32) by taking the inner product with some suitable-chosen Poincaré series. We shall then argue that (32) gives a uniform approximation of the spectral decomposition (31), and in particular that the coefficients in (31) can be bounded similarly.

Recall that we write \(\psi = \otimes_v \psi_v\) to denote the standard additive character on \(A_F/F\). Consider the space

\[
C^\infty( N_2(A_F)Z_2(A_F) \setminus \text{GL}_2(A_F); \psi)
\]

deфиниции функции \(\phi : \text{GL}_2(A_F) \rightarrow \mathbb{C}\) которые являются лево инвариантными по центру \(Z_2(A_F)\), and which satisfy

\[
\phi(n\ast) = \psi(n)\phi(\ast) \quad \text{for all} \ n \in N_2(A_F) \text{ and} \ g \in \text{GL}_2(A_F).
\]

It is well-known (e.g. [11, §5]) that for a suitably chosen decomposable function \(\phi = \otimes_v \phi_v\) in this space, the corresponding Poincaré series defined on \(g \in \text{GL}_2(A_F)\) by the summation formula

\[
P_\phi(g) = \sum_{\gamma \in N_2(F)Z_2(F) \setminus \text{GL}_2(F)} \phi(\gamma g)
\]

converges absolutely, and uniformly on compact subsets. Moreover, for

\[
(33) \quad g = \prod_{\zeta \in \Delta} \gamma \cdot \left( y^\infty, x^\infty, 1 \right) \cdot \left( r^\infty, r^\infty \right) \cdot \left( \zeta \right) \cdot k \in \text{GL}_2(A_F)
\]

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decomposed uniquely according to Proposition 2.5 above, one can show that $P_\phi(g) \ll |y_\infty|^{-M+1}$ for some real parameter $M > 0$ depending on the choice of $\phi$ (e.g. [11, Proposition 5.2]). Note that with respect to the decomposition (33) as in Theorem 2.5, we can assume that

$$\phi \left( \begin{pmatrix} y_\infty & 0 \\ 0 & 1 \end{pmatrix} \right) = |y_\infty|^{-\sigma} \exp(-\epsilon|y_\infty|)$$

for some choices of real parameters $\sigma, \epsilon > 0$ (see e.g. [11, §5]). Let us henceforth choose such a function $\phi = \otimes_v \phi_v \in C^\infty(Z_2(A_F)Z_2(A_F) \backslash GL_2(A_F); \psi)$ as in (34) with $\epsilon > \sigma$. Granted the constant $M > 0$ is sufficiently large, we can view the corresponding Poincaré series $P_\phi$ as an $L^2$-automorphic form in the space $L^2(GL_2(F) \backslash GL_2(A_F), 1)$, or after lifting to the two-fold metaplectic cover $\overline{G}(A_F)$ in the natural way as an $L^2$-automorphic form in the space $L^2(GL_2(F) \backslash \overline{G}(A_F), 1)$. Let us also assume that this function $\phi = \otimes_v \phi_v$ is chosen more precisely in the following way. Fix $Y_\infty \in F_\infty^\times$ with $|Y_\infty| \gg 1$ as in the statement of the main theorem, and let $J(Y_\infty)$ be a compact neighbourhood around $Y_\infty$ which does not intersect the boundary of the chosen fundamental domain of Theorem 2.5. Put $I \cong [0, 1]^d \subset F_\infty$ as above, and let $\mathcal{J} = J(Y_\infty) \times I$ be the corresponding compact domain in $P_2(F_\infty) \cong \tilde{\mathcal{J}}^d$. We then choose $\phi = \otimes_v \phi_v$ so that:

- The component $\phi_\infty = \otimes_v \phi_v$, is compactly supported on $\mathcal{J} = J(Y_\infty) \times I \subset P_2(F_\infty) \subset GL_2(F_\infty)$.

- The function $\phi \in C^\infty_c(Z_2(A_F)N_2(A_F) \backslash GL_2(A_F); \psi)$ is right $K$-invariant.

Taking this choice of function $\phi = \otimes_v \phi_v \in C^\infty(Z_2(A_F)Z_2(A_F) \backslash GL_2(A_F); \psi)$ for granted, we now consider the corresponding Poincaré series $P_\phi$. Taking for the granted the analytic properties described above, it is well-known that we can decompose $P_\phi$ into a basis of smooth forms as described above (see e.g. [12, §1.1]). Hence, we decompose spectrally in terms of our fixed basis $\mathcal{B}$ described above as

$$P_\phi = \sum_i \langle P_\phi, \phi_i \rangle \cdot \phi_i + \sum_\xi \langle P_\phi, \vartheta_\xi \rangle \cdot \vartheta_\xi + \int_{\mathbb{R}(s)=1/2} \langle P_\phi, \mathcal{E}_\varpi^{(*)}, s \rangle \cdot \mathcal{E}_\varpi^{(*)}, s \rangle \frac{ds}{2\pi i}.$$ 

Notice that via the argument of Lemma 4.8, we can deduce that the spectral coefficients $\langle P_\phi, \phi_i \rangle$, $\langle P_\phi, \vartheta_\xi \rangle$, and $\langle P_\phi, \mathcal{E}_\varpi^{(*)}, s \rangle$ in this expansion are bounded in terms of the spectral parameters of the respective basis forms $\phi_i$, $\vartheta_\xi$, and $\mathcal{E}_\varpi^{(*)}, s \rangle$. That is, for any choices for real numbers $C \in \mathbb{R}$, we have the upper bounds

$$\langle P_\phi, \phi_i \rangle \ll \prod_{j=1}^d |\mu_{\xi,j}|^C, \quad \langle P_\phi, \vartheta_\xi \rangle \ll \prod_{j=1}^d |\mu_{\xi,j}|^C, \quad \langle P_\phi, \mathcal{E}_\varpi^{(*)}, s \rangle \ll \prod_{j=1}^d |\mu_{\varpi,s}|^C.$$ 

Notice as well that we can assume without loss of generality that each of these coefficients is nonvanishing, whence we have the strict lower bounds

$$|\langle P_\phi, \phi_i \rangle| > 0, \quad |\langle P_\phi, \vartheta_\xi \rangle| > 0, \quad \text{and} \quad |\langle P_\phi, \mathcal{E}_\varpi^{(*)}, s \rangle|_{\mathbb{R}(s)=1/2} > 0$$

for each index $i$, $\xi$, and $\varpi$ parametrizing our fixed basis $\mathcal{B}$. Let us now consider the inner product of $P_\phi$ against the form $\Phi$ defined in (30) above, i.e. where we lift to the metaplectic cover $\tilde{g} = (g, 1) \in \overline{G}(A_F)$ in any case, and consider inner product

$$\langle P_\phi, \Phi \rangle = \int_{Z_2(A_F) \backslash GL_2(F) \backslash GL_2(A_F)} P_\phi(g) \overline{\Phi(g)} dg.$$ 

To be clear, we take

$$\Phi(g) = \begin{cases} \Phi((g, 1)) & \text{for case (A) corresponding to Theorem 4.2 (i)} \\ \Phi(g) & \text{for case (B) corresponding to Theorem 4.2 (ii)} \end{cases}$$

in this definition, which is the same for both the genuine and non-genuine cases (cf. [17, p. 56]). Observe that after decomposing $P_\phi$ spectrally according to (35) above, this inner product is the same as

$$\langle P_\phi, \Phi \rangle = \sum_i \langle P_\phi, \phi_i \rangle \cdot \langle \phi_i, \Phi \rangle + \sum_\xi \langle P_\phi, \vartheta_\xi \rangle \cdot \langle \vartheta_\xi, \Phi \rangle + \sum_\varpi \int_{\mathbb{R}(s)=1/2} \langle P_\phi, \mathcal{E}_\varpi^{(*)}, s \rangle \cdot \langle \mathcal{E}_\varpi^{(*)}, s \rangle, \Phi \rangle \frac{ds}{2\pi i},$$

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In particular, it is simple to see by inspection of (38) with the lower bounds (37) of the spectral coefficients appearing in (35) that each of the spectral coefficients \( \langle \phi_i, \Phi \rangle \), \( \langle \psi_i, \Phi \rangle \), and \( \langle E_m(s, *) \Phi \rangle \) of \( \Phi \) can be bounded above by this inner product \( \langle P_\phi, \Phi \rangle \). On the other hand, unfolding with the Poincaré series \( P_\phi \), we find that

\[
\langle P_\phi, \Phi \rangle = \int_{Z_2(A_F) \backslash GL_2(F)} P_\phi(g) \Phi(g) dg
\]

\[
= \int_{Z_2(A_F) \backslash GL_2(F)} \sum_{\gamma \in Z_2(F) N_2(F) \backslash GL_2(F)} \phi(\gamma g) \Phi(g) dg
\]

\[
= \int_{Z_2(A_F) \backslash GL_2(F)} \phi(g) \Phi(g) dg.
\]

Here, we alter the appearance of the complex conjugation for simplicity (as we may). Using the definition (30) with choices of vectors and Definition 2.6 (via Theorem 2.5) with the compact domain \( J = J(Y_\infty) \times I \) described above for \( \phi \), it is easy to see via unfolding with the unique decomposition of Theorem 2.5 that

\[
\langle P_\phi, \Phi \rangle = \int_{J(Y_\infty)} \int_{I \subseteq [0, 1]^d \subseteq F_\infty} \phi \cdot \mathbb{P}_1 \mathbb{P}_1(t) \cdot \tilde{\theta}_{f, p} \left( \begin{pmatrix} y_\infty & x_\infty \\ 1 & 1 \end{pmatrix} \right) dx_\infty \frac{dy_\infty}{y_\infty^2}
\]

for the case corresponding to Proposition 4.2 (A), and by

\[
\langle P_\phi, \Phi \rangle = \int_{J(Y_\infty)} \int_{I \subseteq [0, 1]^d \subseteq F_\infty} \phi \cdot \mathbb{P}_1 \mathbb{P}_1(t) \cdot \mathbb{P}_1 \mathbb{P}_1(t) \left( \begin{pmatrix} y_\infty & x_\infty \\ 1 & 1 \end{pmatrix} \right) dx_\infty \frac{dy_\infty}{y_\infty^2}
\]

for the case corresponding to Proposition 4.2 (B). Let us first consider the constant coefficients determined by the inner unipotent integrals these expressions. Here, we assume (crucially) that the archimedean idele coordinate \( y_\infty \) is contained in our chosen fundamental domain, and hence that it is totally positive. Using the identification of Fourier-Whittaker coefficients from Proposition 2.9 for the extended mirabolic forms and theta series, we can open up Fourier-Whittaker expansions and evaluate via orthogonality of additive characters to calculate

\[
\mathcal{R}_A(y_\infty) := \int_{I \subseteq [0, 1]^d \subseteq F_\infty} \phi \cdot \mathbb{P}_1 \mathbb{P}_1(t) \cdot \tilde{\theta}_{f, p} \left( \begin{pmatrix} y_\infty & x_\infty \\ 1 & 1 \end{pmatrix} \right) dx_\infty
\]

\[
= \int_{I \subseteq [0, 1]^d \subseteq F_\infty} \phi \left( \begin{pmatrix} y_\infty & x_\infty \\ 1 & 1 \end{pmatrix} \right) \mathbb{P}_1 \mathbb{P}_1(t) \left( \begin{pmatrix} y_\infty & x_\infty \\ 1 & 1 \end{pmatrix} \right) \tilde{\theta}_{f, p} \left( \begin{pmatrix} y_\infty & x_\infty \\ 1 & 1 \end{pmatrix} \right) dx_\infty
\]

\[
= \int_{I \subseteq [0, 1]^d \subseteq F_\infty} \phi \left( \begin{pmatrix} y_\infty & x_\infty \\ 1 & 1 \end{pmatrix} \right) \mathbb{P}_1 \mathbb{P}_1(t) \left( \begin{pmatrix} y_\infty & x_\infty \\ 1 & 1 \end{pmatrix} \right) \tilde{\theta}_{f, p} \left( \begin{pmatrix} y_\infty & x_\infty \\ 1 & 1 \end{pmatrix} \right) dx_\infty
\]

\[
= \phi \left( \begin{pmatrix} y_\infty & 1 \\ 1 & 1 \end{pmatrix} \right) \sum_{\gamma \in F^\times} \mathbb{W}_{\varphi(i)} \left( \begin{pmatrix} \gamma y_\infty & 1 \\ 1 & 1 \end{pmatrix} \right) \sum_{a = (a_1, \ldots, a_k) \in O_F^k} p(a) \psi(i \cdot f(a) y_\infty)
\]

\[
\times \int_{I \subseteq [0, 1]^d \subseteq F_\infty} \psi(x_\infty + \gamma x_\infty - f(a)x_\infty) dx_\infty
\]

\[
= \phi \left( \begin{pmatrix} y_\infty & 1 \\ 1 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} (f(a) - 1) y_\infty & 1 \\ 1 & 1 \end{pmatrix} \right) \sum_{a = (a_1, \ldots, a_k) \in O_F^k} p(a) \psi(i \cdot f(a) y_\infty)
\]

\[
\times \psi(i \cdot a Y_\infty) \psi(i \cdot y_\infty) \left| y_\infty \right|^\frac{k}{2} \sum_{a = (a_1, \ldots, a_k) \in O_F^k} p(a) \frac{C_n(f(a) - 1)}{|f(a) - 1|^2} W_i \left( \frac{|f(a) - 1|}{|y_\infty|} \right)
\]

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for the first unipotent integral, and
\[
\tilde{A}_B(y_\infty) := \int_{I\in [0,1]} \phi \cdot \mathbb{P}_{m,\nu}^{(l_1)} \cdot \mathbb{P}_{m,\nu}^{(l_2)} \left( \left( \begin{array}{cc} y_\infty & x_\infty \\ 1 & 1 \end{array} \right) \right) dx_\infty
\]
\[
= \int_{I\in [0,1]} \phi \left( \left( \begin{array}{cc} y_\infty & x_\infty \\ 1 & 1 \end{array} \right) \right) \mathbb{P}_{m,\nu}^{(l_1)} \left( \left( \begin{array}{cc} y_\infty & x_\infty \\ 1 & 1 \end{array} \right) \right) \mathbb{P}_{m,\nu}^{(l_2)} \left( \left( \begin{array}{cc} y_\infty & x_\infty \\ 1 & 1 \end{array} \right) \right) dx_\infty
\]
\[
= \int_{I\in [0,1]} \phi \left( \left( \begin{array}{cc} y_\infty & x_\infty \\ 1 & 1 \end{array} \right) \right) \mathbb{P}_{m,\nu}^{(l_1)} \left( \left( \begin{array}{cc} y_\infty & x_\infty \\ 1 & 1 \end{array} \right) \right) \mathbb{P}_{m,\nu}^{(l_2)} \left( \left( \begin{array}{cc} y_\infty & x_\infty \\ 1 & 1 \end{array} \right) \right) dx_\infty
\]
\[
= \phi \left( \left( \begin{array}{cc} y_\infty & 1 \\ 1 & 1 \end{array} \right) \right) \int \sum_{\gamma_1 \in F^\times} W_{\psi^{(1)}} \left( \left( \begin{array}{cc} \gamma_1 y_\infty & y_\infty \\ 1 & 1 \end{array} \right) \right) \sum_{\gamma_2 \in F^\times} W_{\psi^{(2)}} \left( \left( \begin{array}{cc} \gamma_2 y_\infty & y_\infty \\ 1 & 1 \end{array} \right) \right) dx_\infty
\]
\[
= \phi \left( \left( \begin{array}{cc} y_\infty & 1 \\ 1 & 1 \end{array} \right) \right) \int \sum_{\gamma_1, \gamma_2 \in F^\times} \psi(x_\infty + \gamma_1 x_\infty - \gamma_2 x_\infty) dx_\infty
\]
\[
= \phi \left( \left( \begin{array}{cc} y_\infty & 1 \\ 1 & 1 \end{array} \right) \right) \int \sum_{\gamma_1, \gamma_2 \in F^\times} \psi(x_\infty + \gamma_1 x_\infty - \gamma_2 x_\infty) dx_\infty
\]

for the second. Note that we have used the explicit choice of pure tensors described in Proposition 4.2 to evaluate these constant coefficient kernels. We also use the choice of compact domain \(J(Y_\infty)\) to ensure that each archimedean idele \(y_\infty = (y_\infty,j)_{j=1}^d \in F^\times\) in this calculation is totally positive. Now, recall we assume that the chosen global weight functions \(W\) and \(W_j\) on \(y \in \mathbb{R}_{>0}\) (for \(j = 1,2\)) decay moderately as
\[
\left\{ \begin{array}{ll}
W(y) = O(y^\kappa) & \text{for some constant } 0 < \kappa < 1 \text{ when } 0 < y < 1 \\
W_j(y) = O(y^{\kappa_j}) & \text{for some constants } 0 < \kappa_j < 1 \text{ when } 0 < y < 1 \text{ for each } j = 1,2
\end{array} \right.
\]
for \(y \to 0\), and decay rapidly as
\[
\left\{ \begin{array}{ll}
W(y) = O_B(y^{-B}) & \text{for any choice of constant } B > 0 \text{ when } y > 1 \\
W_j(y) = O_B(y^{-B}) & \text{for any choice of constant } B > 0 \text{ when } y > 1 \text{ for each } j = 1,2
\end{array} \right.
\]
as \(y \to \infty\). Recall as well that for the respective fixed integers \(0 \leq l \leq \log Y\) and \(0 \leq l_j \leq \log Y\) the dyadic subdivisions coming from our fixed partition of unity (16) above, we consider the local weight functions defined (respectively) by (17) and (18). That is, as functions of \(y_\infty \in F^\times\), these are given by
\[
W_i(y_\infty) = W(y_\infty) \sum_{\gamma \in (R)} U \left( \frac{|y_\infty|}{R} \right)
\]
and (for each of \(j = 1,2\))
\[
W_{j,l}(y_\infty) = W_j(y_\infty) \sum_{\gamma \in (R)} U \left( \frac{|y_\infty|}{R} \right)
\]
Here again, the summation notation for \(\sum_{\gamma \in (R)}\) denotes the single range \(2^l \leq R \leq 2^{l+1}\) in our fixed partition of unity (16), i.e. so that the sum consists of this one element. Now, it is easy to deduce that these local weight functions are bounded in a similar way as for the global weight functions, uniformly in the range parameter \(R \in [2^l, 2^{l+1}]\). In particular, taking
\[
C_U := \max_{1 \leq l \leq 2} U(t)
\]
we can bound the corresponding partial Mellin transforms defining (39) and (40) respectively as

\[
W_l(y_\infty) = \begin{cases} O(|C_U \cdot y_\infty|^\kappa) & \text{for some } 0 < \kappa < 1 \text{ when } 0 < y < 1 \\ O_B \left(|C_U \cdot y_\infty|^{-B}\right) & \text{for any choice of constant } B > 0 \text{ when } y > 0 \end{cases}
\]

and

\[
W_{j,l}(y_\infty) = \begin{cases} O\left(|C_U \cdot y_\infty|^\kappa_j\right) & \text{for some } 0 < \kappa_j < 1 \text{ when } 0 < y < 1 \\ O_B \left(|C_U \cdot y_\infty|^{-B}\right) & \text{for any choice of constant } B > 0 \text{ when } y > 0 \end{cases}
\]

for each index \(j = 1, 2\). That is, these bounds (42) and (43) are uniform in the integer \(l \in \mathbb{Z}\) parametrizing the range \(R \in [2^l, 2^{l+1}]\) in the corresponding dyadic decomposition. We can then derive the following uniform bounds for the kernel functions \(K\), the range \(\varphi\) bounds (42) and (43). Namely, using the choice \(-\text{fold iterate } \Delta\)

Now, to use these estimates (44) and (45) for the inner product \(\langle \Delta, \varphi \rangle\) for the inner product \(\langle \varphi, \Phi \rangle\) to bound the spectral coefficients of our chosen \(L^2\)-automorphic form \(\Phi\) defined above, we first consider arbitrary \(D\)-fold iterates \(\Delta^{(D)}\) of the suitable Laplacian operator \(\Delta\) acting on the Poincaré series \(P_\varphi\). Thus for any integer \(D \geq 1\), we apply the \(D\)-fold iterate \(\Delta^{(D)}\) to each side of the decomposition (35) to derive the corresponding decomposition

\[
\Delta^{(D)}P_\varphi = \sum_i \langle P_\varphi, \phi_i \rangle \cdot \prod_{j=1}^d \left(1 + \nu_{\varphi,j}^2\right)^{D} \cdot \phi_i + \sum_\xi \langle P_\varphi, \vartheta_\xi \rangle \cdot \prod_{j=1}^d \left(1 + \nu_{\vartheta,j}^2\right)^{D} \cdot \vartheta_\xi + \sum_\nu \int_{\Re(s)=1/2} \langle P_\varphi, \mathcal{E}_\nu(*,s) \rangle \cdot \prod_{j=1}^d \left(1 + \frac{1}{2} \nu_{\nu,j}^2\right)^{D} ds \frac{d}{2\pi i}.
\]

Let us now consider the inner product of \(\Delta^{(D)}P_\varphi\) with our chosen \(L^2\)-automorphic form \(\Phi\),

\[
\langle \Delta^{(D)}P_\varphi, \Phi \rangle = \int_{Z_2(A_F) \backslash GL_2(F) \backslash GL_2(A_F)} \Delta^{(D)}P_\varphi(g) \Phi(g) dg.
\]

Observe that by using the decomposition (46) to describe the contribution of \(\Delta^{(D)} \cdot \Phi\) in this expression, we obtain the corresponding spectral decomposition

\[
\langle \Delta^{(D)}P_\varphi, \Phi \rangle = \sum_i \langle P_\varphi, \phi_i \rangle \cdot \langle \phi_i, \Phi \rangle \cdot \prod_{j=1}^d \left(1 + \nu_{\phi,j}^2\right)^{D} + \sum_\xi \langle P_\varphi, \vartheta_\xi \rangle \cdot \langle \vartheta_\xi, \Phi \rangle \cdot \prod_{j=1}^d \left(1 + \nu_{\vartheta,j}^2\right)^{D} + \sum_\nu \int_{\Re(s)=1/2} \langle P_\varphi, \mathcal{E}_\nu(*,s) \rangle \cdot \mathcal{E}_\nu(*,s,\Phi) \cdot \prod_{j=1}^d \left(1 + \nu_{\nu,j}^2\right)^{D} ds \frac{d}{2\pi i}.
\]
Note that the decompositions (46) and (47) hold for any choice of integer $D \geq 1$. On the other hand, observe as well that a simple variation of the unfolding calculations about allows us to derive the bounds

$$\langle \Delta^{(D)} P_\phi, \mathbf{f} \rangle$$

$$= \int_{J(Y_\infty)} \left( \Delta^{(D)} \phi \left( \begin{array}{c} y \infty \\ 1 \end{array} \right) \right) \psi(\text{i} a Y_\infty) \frac{1}{y_\infty} \sum_{a = \{a_1, \ldots, a_n\} \in \mathbb{D}_p^*} \frac{c_n(f(a) - 1)}{|f(a) - 1|^2} p(a) \psi(iy_\infty) W_i \left( \frac{|f(a) - 1|}{|y_\infty|} \right) \frac{dy_\infty}{y_\infty^2}$$

$$\ll \Delta^{(D)} \phi, \Pi, f, p \quad C_U \cdot Y^{-\sigma + \frac{5}{8} + (1 + \kappa) - 2} \cdot e^{-\alpha Y} \cdot e^{-2\pi \kappa a + 1}|Y$$

and

$$\langle \Delta^{(D)} P_\phi, \mathbf{f} \rangle$$

$$= \int_{J(Y_\infty)} \left( \Delta^{(D)} \phi \left( \begin{array}{c} y \infty \\ 1 \end{array} \right) \right) \sum_{\gamma_1 \gamma_2 \in \mathbb{F}^*} \frac{c_n(\gamma_1 \gamma_2)}{|\gamma_1 \gamma_2|^{\frac{5}{2}}} W_{1, l_1} \left( \frac{|\gamma_1|}{y_\infty} \right) W_{2, l_2} \left( \frac{|\gamma_2|}{y_\infty} \right) \frac{dy_\infty}{y_\infty^2}$$

$$\ll \Delta^{(D)} \phi, \Pi, \pi \cdot C_U \cdot Y^{-\sigma + (1 + \kappa_1 + \kappa_2) - 2} \cdot e^{-\alpha Y}$$

Since the Poincaré series $P_\phi$ is smooth and hence convergent in the Sobolev norm topology, we argue as in Lemma 4.8 above, after taking $L^2$ norms, that its spectral coefficients are bounded uniformly in terms of the spectral parameters of the basis forms as in (36). Putting these upper bounds together with the lower bounds (37) and upper bounds for the inner products (48) and (49) then allows to extend the standard argument (i.e. as given in Lemma 4.8) to derive the following uniform bounds: For each of the forms $\phi_i, \vartheta_\xi$, and $E_\mathbb{R}(s, s)$ in our fixed orthonormal basis $\mathcal{B}$, we have for any choice of real parameter $C \in \mathbb{R}$ the bounds

$$\langle \Phi, \phi_i \rangle \ll C \left( \prod_{j=1}^d |\nu_{i'}, j| \right)^C, \quad \langle \Phi, \vartheta_\xi \rangle \ll C \left( \prod_{j=1}^d |\nu_{\xi'}, j| \right)^C, \quad \langle \Phi, E_\mathbb{R}(s, s) \rangle |\Re(s) = 1/2 \ll C \left( \prod_{j=1}^d |\nu_{\vartheta', s, j} |\Re(s) = 1/2 \right)^C$$

Again, the indices $i', \xi'$, and $\vartheta'$ appearing on the right hand side, corresponding to elements of the smooth basis $\mathcal{B}$, can be chosen arbitrarily. Hence, the coefficients appearing on the right hand side of the expansion (32) are bounded according to the familiar setup described in Lemma 4.8. Now, to argue that this linear combination gives a uniform approximation to the spectral decomposition (31), we first observe that the linear combination defined by

$$\Phi_S := \sum_i \langle \Phi, \phi_i \rangle \cdot \phi_i + \sum_\xi \langle \Phi, \vartheta_\xi \rangle \cdot \vartheta_\xi + \sum_\vartheta \langle \Phi, E_\mathbb{R}(s, s) \rangle \cdot E_\mathbb{R}(s, s) \frac{ds}{2\pi i}$$

must in fact be the image of our chosen from $\Phi$ in the dense subspace of smooth forms. Indeed, it is easy to compare inner products with the Poincaré series $P_\phi$ (decomposing $P_\phi$ spectrally) to find that

$$\langle P_\phi, \Phi_S \rangle = \langle P_\phi, \Phi \rangle$$

We then argue that $\Phi_S$ must be an uniform approximation of $\Phi$ in the dense subspace of smooth forms, in which case it is easy to deduce that the spectral coefficients of $\Phi$ as described in (31) can be bounded in terms of those of $\Phi_S$. Another way to see this is to expand the Poincaré series $P_\phi$ with respect to the orthonormal basis $\mathcal{B}'$ described above, and then to compare spectral expansions. In this way, ordering the bases $\mathcal{B}'$ and $\mathcal{B}$ so that each cusp form $f_i$, residual form $\vartheta_\xi$, and Eisenstein series $E_\mathbb{R}(s, s)$ in $\mathcal{B}'$ has corresponding cusp form $f_i$, residual form $\vartheta_\xi$, or Eisenstein series $E_\mathbb{R}(s, s)$ respectively in the smooth basis $\mathcal{B}$, we derive the relations

$$P_\phi = \sum_i \langle P_\phi, f_i \rangle \cdot f_i + \sum_\xi \langle P_\phi, \vartheta_\xi \rangle \cdot \vartheta_\xi + \sum_\vartheta \langle P_\phi, E_\mathbb{R}(s, s) \rangle \cdot E_\mathbb{R}(s, s) \frac{ds}{2\pi i}$$

(51)
and

\[
(P_{\phi}, \Phi) = \sum_{i} \langle P_{\phi}, f_i \rangle \cdot \langle f_i, \Phi \rangle + \sum_{\xi} \langle P_{\phi}, \theta_{\xi} \rangle \cdot \langle \theta_{\xi}, \Phi \rangle + \sum_{\varphi} \int_{R(s)=1/2} \langle P_{\phi}, E_{\varphi}(s) \rangle \cdot \langle E_{\varphi}(s), \Phi \rangle \frac{ds}{2\pi i}
\]

\[= \sum_{i} \langle P_{\phi}, \phi_i \rangle \cdot \langle \phi_i, \Phi \rangle + \sum_{\xi} \langle P_{\phi}, \theta_{\xi} \rangle \cdot \langle \theta_{\xi}, \Phi \rangle + \sum_{\varphi} \int_{R(s)=1/2} \langle P_{\phi}, E_{\varphi}(s) \rangle \cdot \langle E_{\varphi}(s), \Phi \rangle \frac{ds}{2\pi i}
\]

\[= \langle P_{\phi}, \Phi \rangle.
\]

Since both bases \(\mathcal{B}\) and \(\mathcal{B}'\) are orthonormal, we derive from (51) and (36) the corresponding bounds

\[
\langle P_{\phi}, f_i \rangle \ll C \prod_{j=1}^{d} |\nu_{i,j}|^{C}, \quad \langle P_{\phi}, \theta_{\xi} \rangle \ll C \prod_{j=1}^{d} |\nu_{\xi,j}|^{C} \quad \langle P_{\phi}, E_{\varphi}(s) \rangle \ll C \prod_{j=1}^{d} |\nu_{\varphi,s}|^{C}
\]

for any choice(s) of real parameter \(C \in \mathbb{R}\). We can then derive the stated bounds from (52) and (50). \(\square\)

4.4.3. *Convergence of spectral coefficients via convolution with smoothing kernels.* Note that we could also consider the convolution of the \(L^2\)-automorphic form \(\Phi\) defined in (30) with a smoothing kernel \(\mathcal{R}\), e.g. of the form described in [8, §3]. The corresponding convolution \(\Phi \ast \mathcal{R}\) determines a smooth automorphic form, and hence it can be decomposed spectrally in terms of a basis of smooth forms \(\mathcal{B}\) as described above. In particular, the standard Sobolev norms argument presented in Lemma 4.8 applies to bound the spectral coefficients of this convolution \(\Phi \ast \mathcal{R}\) uniformly in the spectral parameters of the forms in the basis \(\mathcal{B}\). Let us now explain how we could also proceed to argue in this way, albeit at the expense of complicating our discussion of integral presentations in Proposition 4.2.

Let us now describe this in more detail, building on the setup of [8, §3]. Recall that we fix an identification \(P_2(F_{\infty}) \cong \mathfrak{h}^d\) of \(P_2(F_{\infty})\) with the \(d\)-fold upper-half plane \(\mathfrak{h}^d\). Let us write

\[z_\infty = (z_{\infty,j})_{j=1}^{d} = x_\infty + iy_\infty = (x_{\infty,j} + iy_{\infty,j})_{j=1}^{d} \in \mathfrak{h}^d\]

with \(x_\infty = (x_{\infty,j})_{j=1}^{d} \in F_{\infty} \cong \mathbb{R}^d\) and \(y_\infty = (y_{\infty,j})_{j=1}^{d} \in F_{\infty}^* \cong (\mathbb{R}^*)^d\) to denote a generic element, and also

\[d\mu(z_\infty) = \frac{dx_{\infty,1}dy_{\infty,1}}{y_{\infty,1}^\alpha} \cdots \frac{dx_{\infty,d}dy_{\infty,d}}{y_{\infty,d}^\alpha}\]

the corresponding measure on \(\mathfrak{h}^d \cong P_2(F_{\infty})\). As well, we shall put

\[\Phi(z_\infty) = \Phi(x_\infty + iy_\infty) := \Phi\left(\begin{array}{c} y_\infty \\ x_\infty \\ 1 \end{array}\right).
\]

Let \(\mathcal{R}(z_\infty, w_\infty) = \mathcal{R}(x_\infty + iy_\infty, u_\infty + it_\infty)\) be a point invariant on \(\mathfrak{h}^d \times \mathfrak{h}^d \cong P_2(F_{\infty}) \times P_2(F_{\infty})\). We consider the convolution \(\Phi \ast \mathcal{R}(z_\infty)\) defined by

\[\Phi \ast \mathcal{R}(z_\infty) = \int_{\mathfrak{h}^d} \mathcal{R}(z_\infty, w_\infty)\Phi(w_\infty) d\mu(w_\infty)
\]

\[= \int_{P_2(F_{\infty})} \mathcal{R}(x_\infty + iy_\infty, u_\infty + it_\infty) \Phi\left(\begin{array}{c} t_\infty \\ u_\infty \\ 1 \end{array}\right) d\mu(u_\infty + it_\infty).
\]

Now, it is easy to see that this \(\Phi \ast \mathcal{R}\) determines a smooth automorphic form on \(\text{SL}_2(O_F) \setminus \mathfrak{h}^d\). Indeed, smoothness is simple to check. To see the automorphy, observe that for any \(\gamma \in \text{SL}_2(O_F)\), we have that

\[\Phi \ast \mathcal{R}(\gamma \cdot z_\infty) = \int_{\mathfrak{h}^d} \mathcal{R}(\gamma \cdot z_\infty, w_\infty)\Phi(w_\infty) d\mu(w_\infty) = \int_{\mathfrak{h}^d} \mathcal{R}(z_\infty, \gamma^{-1} \cdot w_\infty)\Phi(\gamma^{-1} \cdot w_\infty) d\mu(w_\infty)
\]

\[= \int_{\mathfrak{h}^d} \mathcal{R}(z_\infty, w_\infty)\Phi(w_\infty) d\mu(w_\infty) = \Phi \ast \mathcal{R}(z_\infty).
\]

Thus, we can view the convolution \(\Phi \ast \mathcal{R}\) as a smooth automorphic form, and decompose it with respect to an orthonormal basis \(\mathcal{B}\) as described above, with a simple variation of Lemma 4.8 showing that the spectral coefficients are bounded uniformly in the spectral parameters of the basis forms.

Is it also easy to see that we can extract a given Fourier-Whittaker coefficient of \(\Phi\) from that of the corresponding coefficient at \(\alpha\) of the convolution \(\Phi \ast \mathcal{R}\). To justify this, let us first observe that making a
change of variables $r_\infty = x_\infty - u_\infty$ so that $x_\infty = r_\infty + u_\infty$ and $dr_\infty = dx_\infty$, and using the invariance of the point pair invariant $R$ under multiplication by the unipotent matrix

$$ \begin{pmatrix} 1 & -u_\infty \\ 0 & 1 \end{pmatrix} \in \text{SL}_2(F_\infty) \cong \text{Isom}(\mathbb{R}^d), $$

we find that for any nonzero $F$-integer $\alpha$ we have the identification(s)

$$ \int_{x_\infty \in I \simeq [0,1]^d} \Phi * \check{R}(x_\infty + iy_\infty) \psi(-\alpha x_\infty) dx_\infty $$

$$ = \int_{x_\infty \in I \simeq [0,1]^d} \int_{u_\infty + u_\infty \in \mathbb{R}^d} \check{R}(x_\infty + iy_\infty, u_\infty + it_\infty) \Phi(x_\infty + iy_\infty) d\mu(u_\infty + it_\infty) \psi(-\alpha x_\infty) dx_\infty $$

$$ = \int_{r_\infty \in I \simeq [0,1]^d} \int_{u_\infty \in I \simeq [0,1]^d} \int_{t_\infty \in F_\infty, + \simeq \mathbb{R}^d_{>0}} \check{R}(r_\infty + iy_\infty, u_\infty + it_\infty) \Phi(u_\infty + it_\infty) \frac{dt_\infty}{t_\infty^2} du_\infty \psi(-\alpha(r_\infty + u_\infty)) dr_\infty $$

$$ = \int_{r_\infty \in I \simeq [0,1]^d} \int_{u_\infty \in I \simeq [0,1]^d} \int_{t_\infty \in F_\infty, + \simeq \mathbb{R}^d_{>0}} \check{R}(r_\infty + iy_\infty, it_\infty) \Phi(u_\infty + it_\infty) \frac{dt_\infty}{t_\infty^2} du_\infty \psi(-\alpha(r_\infty + u_\infty)) dr_\infty $$

$$ = \int_{r_\infty \in I \simeq [0,1]^d} \int_{u_\infty \in I \simeq [0,1]^d} \int_{t_\infty \in F_\infty, + \simeq \mathbb{R}^d_{>0}} \check{R}(r_\infty + iy_\infty, it_\infty) \psi(-\alpha r_\infty) dr_\infty \int_{u_\infty \in I \simeq [0,1]^d} \Phi(u_\infty + it_\infty) \psi(-\alpha u_\infty) du_\infty \frac{dt_\infty}{t_\infty^2}. $$

Hence, writing the inner Fourier coefficients at $\alpha$ in this latter expression more simply as

$$ \rho_R(\alpha; iy_\infty; it_\infty) = \int_{r_\infty \in I \simeq [0,1]^d} \check{R}(r_\infty + iy_\infty, it_\infty) \psi(-\alpha r_\infty) dr_\infty $$

and

$$ \rho_\Phi(\alpha; it_\infty) = \int_{u_\infty \in I \simeq [0,1]^d} \Phi(u_\infty + it_\infty) \psi(-\alpha u_\infty) du_\infty, $$

we arrive at the expression

$$ \int_{x_\infty \in I \simeq [0,1]^d} \Phi * \check{R}(x_\infty + iy_\infty) \psi(-\alpha x_\infty) dx_\infty = \int_{r_\infty \in F_\infty, + \simeq \mathbb{R}^d_{>0}} \rho_R(\alpha; iy_\infty; it_\infty) \rho_\Phi(\alpha; it_\infty) \frac{dt_\infty}{t_\infty^2}. $$

Now, observe that this expression (53) implies that the Fourier-Whittaker coefficient of $\Phi * \check{R}(iy_\infty)$ at $\alpha$ on the left hand side can be viewed as the Mellin transform

$$ f^*(s) = \int_{t_\infty \in F_\infty, + \simeq \mathbb{R}^d_{>0}} f(t_\infty) t_\infty^s \frac{dt_\infty}{t_\infty} $$

at $s = -1$ of the function $f(t_\infty) := \rho_R(\alpha; iy_\infty; it_\infty) \rho_\Phi(\alpha; it_\infty)$. Hence by the inversion theorem, we have for some suitable choice of real parameter $\sigma > 0$ the relation

$$ \rho_R(\alpha; iy_\infty, it_\infty) \rho_\Phi(\alpha; it_\infty) = \int_{R(s) = \sigma} f^*(s) t_\infty^{-s} \frac{ds}{2\pi i}. $$

Applying the standard Sobolev norms arguments to bound the coefficient of $\Phi * \check{R}$ at $\alpha$ via spectral decomposition, we obtain a bound for the value $f^*(-1)$.

4.4.4. **Setup for the case of $n = 2$.** When $n = 2$, the projection operator $P_1^+$ is trivial, and we simply consider the form $\Phi = P_1^+ \varphi \cdot \check{\varphi}_{f, p} = \varphi \cdot \check{\varphi}_{f, p}$ for case (A) or $\Phi = P_1^+ \varphi \cdot \check{P}_n \varphi = \varphi \cdot \check{P}_n \varphi$ for case (B) of the main theorem. Corollary 4.7 implies in either case that the right hand side of the corresponding spectral decomposition (28) of $\Phi$ converges in the Sobolev norm topology, with $||\Phi||_B \ll B$ for each integer $B \geq 1$. That is, the argument of Lemma 4.8 justifies expanding the $\Phi$ spectrally with respect to the basis $B$ of smooth forms introduced above. We shall then consider this spectral decomposition in the integral presentation of Proposition 3.4.
That is, Proposition 4.2 shows that the left hand side of this latter decomposition can be viewed as an

\[ I = \int_{A_F/F} \Phi \left( \begin{array}{c} x \\ 1 \end{array} \right) \psi(-\alpha x)dx = \sum_{\tau \in \mathcal{B}} \langle \Phi, \varphi_\tau \rangle \int_{A_F/F} \varphi_\tau \left( \begin{array}{c} x \\ 1 \end{array} \right) \psi(-\alpha x)dx \]

in this case, where for each basis element \( \varphi_\tau \in \mathcal{B} \), writing \( \rho_\varphi \) to denote the nonarchimedean parts of the Fourier-Whittaker coefficients, with \( c_\tau \) the corresponding \( L \)-function coefficients, we have the relations

\[ \int_{A_F/F} \varphi_\tau \left( \begin{array}{c} x \\ 1 \end{array} \right) \psi(-\alpha x)dx = \rho_\varphi(\alpha)W_{\varphi_\tau}(\alpha \frac{Y_{\infty}}{Y_\infty}). \]

**Remark** Note that we could also express this decomposition in terms of the normalized Whittaker functions \( W_{\varphi_\tau} \) described above, at least for certain spectral parameters \( \nu_i \).

4.4.5. **Setup for the generic case of \( n \geq 2 \).** Suppose now that we are in the generic case of all dimensions \( n \geq 2 \). To be clear, recall that we shall give two proofs of the estimate for the special case of \( n = 2 \). In the generic case of all dimensions \( n \geq 2 \), we decompose the \( L^2 \)-automorphic form \( \Phi \) defined in (30) above spectrally, i.e. after lifting to the metaplectic cover \( \tilde{G}(A_F) \) when needed for case (A), so that we may treat cases (A) and (B) of Proposition 4.2 uniformly. Recall that the integral presentations derived in Proposition 4.2 work differently in this setting because of the definition of the lifting \( \tilde{\varphi}^m \), and in particular that we fix a smooth partition of unity (16) with corresponding dyadic subdivisions (19) and (20). Recall as well that we reduce via the choice of global weight functions \( W \) and \( W_j \) to the corresponding finite truncated sums over local weight functions (23) and (24). We shall take this reduction from Proposition 4.2 for granted in the discussion below, including the implicit choices of local weight functions \( W_j \) and \( W_j' \), for these estimates (as will do to prove the main theorem). Given a totally positive archimedean idele \( Y_\infty \in F_\infty^* \) of idele norm \( |Y_\infty| = Y \) contained in our chosen fundamental domain of Theorem 2.5, we consider the Fourier-Whittaker coefficient at a given nonzero \( F \)-integer \( \alpha \) of the function

\[ \Phi \left( \begin{array}{c} Y_\infty \\ x_{\infty} \\ 1 \end{array} \right). \]

Proposition 4.2 shows that these coefficients describe the shifted convolution sums we wish to bound. Now, we saw in (31) and Proposition 4.9 that we can decompose this form \( \Phi \) spectrally, and in particular that the coefficients in the spectral expansion of \( \Phi \) can be bounded in terms of the inner product \( \langle P_\phi, \Phi \rangle \) with some suitably chosen Poincaré series \( P_\phi \), i.e. which after unfolding can be calculated to extract an upper bound in terms of the chosen real parameter \( Y = |Y_\infty| > |\alpha| \). Moreover, we saw that these spectral coefficients can be bounded uniformly in terms of the spectral parameters of the automorphic forms in the smooth orthonormal basis \( \mathcal{B} \). Hence, we justify bounding the decomposition of \( \Phi \) in terms of the more general orthonormal basis \( \mathcal{B}' \) described above, and in particular for any \( x_\infty \in I \cong [0,1]^d \subset F_\infty \) the decomposition

\[ \Phi \left( \begin{array}{c} Y_\infty \\ x_{\infty} \\ 1 \end{array} \right) = \sum_i \langle \Phi, f_i \rangle \cdot f_i \left( \begin{array}{c} Y_\infty \\ x_{\infty} \\ 1 \end{array} \right) + \sum_{\xi} \langle \Phi, \theta_\xi \rangle \cdot \theta_\xi \left( \begin{array}{c} Y_\infty \\ x_{\infty} \\ 1 \end{array} \right) + \sum_{\varpi} \int_{\mathbb{R}(s)=1/2} \langle \Phi, E_{\varpi}(\ast, s) \rangle \cdot E_{\varpi} \left( \begin{array}{c} Y_\infty \\ x_{\infty} \\ 1 \end{array} \right), ds \frac{ds}{2\pi i}. \]

Taking unipotent integrals of each side (again, cf. Proposition 4.2) then gives us the relevant decomposition

\[ \int_{[0,1]^d} \Phi \left( \begin{array}{c} Y_\infty \\ x_{\infty} \\ 1 \end{array} \right) \psi(-\alpha x_{\infty})dx_{\infty} \]

\[ = \sum_i \langle \Phi, f_i \rangle \cdot \int_{[0,1]^d} f_i \left( \begin{array}{c} Y_\infty \\ x_{\infty} \\ 1 \end{array} \right) \psi(-\alpha x_{\infty})dx_{\infty} + \sum_{\xi} \langle \Phi, \theta_\xi \rangle \cdot \int_{[0,1]^d} \theta_\xi \left( \begin{array}{c} Y_\infty \\ x_{\infty} \\ 1 \end{array} \right) \psi(-\alpha x_{\infty})dx_{\infty} \]

\[ + \sum_{\varpi} \int_{\mathbb{R}(s)=1/2} \langle \Phi, E_{\varpi}(\ast, s) \rangle \cdot \int_{[0,1]^d} E_{\varpi} \left( \begin{array}{c} Y_\infty \\ x_{\infty} \\ 1 \end{array} \right), s \psi(-\alpha x_{\infty})dx_{\infty} \frac{ds}{2\pi i}. \]

That is, Proposition 4.2 shows that the left hand side of this latter decomposition can be viewed as an integral presentation for the shifted convolution problem for \( \text{GL}_n(A_F) \)-coefficients at shifts of the quadratic
form $f(a) = f(a_1, \ldots, a_k)$ in case (A), or as linear shifts coming from some other $GL_m(A_F)$-automorphic form in case (B). In particular, it will do to bound the Fourier-Whittaker coefficient at $\alpha$ of $\Phi$ in either case to derive estimates for the corresponding shifted convolution problem.

4.4.6. **Derivation of bounds.** We now derive bounds for the shifted convolution problem for both the standard case of $n = 2$ and the generic case of $n \geq 2$ in tandem. We shall supply separate arguments in each case owing to the distinct nature of the integral presentations, although we derive similar final bounds in all cases.

**Decompositions setup.** Recall that we write $\mathcal{B}$ to denote the fixed orthonormal basis of smooth forms (hence eigenvectors for the associated Laplacian operator) for the corresponding Hilbert space, with $\mathcal{B}'$ a fixed orthonormal basis for the more general space of $L^2$-automorphic forms which are not necessarily smooth (and hence not necessarily eigenvectors for the associated Laplacian operator). Before going on, let us recall that the result of Proposition 4.9 for the generic case of dimension $n \geq 2$ implies that the spectral coefficients of $\Phi$ decomposed with respect to $\mathcal{B}'$ can be bounded uniformly in terms of these spectral parameters of the basis of smooth forms $\mathcal{B}$. That is, recall that we consider on the one hand the expansion

$$\Phi = \sum_i \langle \Phi, f_i \rangle \cdot f_i + \sum_{\xi} \langle \Phi, \theta_\xi \rangle \cdot \theta_\xi + \sum_\varpi \int_{\mathbb{R}(s)=1/2} \langle \Phi, E_\varpi(\cdot, s) \rangle \cdot E_\varpi(\cdot, s) \frac{ds}{2\pi i}$$

of $\Phi$ in terms of the orthonormal basis $\mathcal{B}'$. On the other hand, the proof of Proposition 4.9 in effect allows us to approximate $\Phi$ by the smooth form defined by the corresponding linear combination

$$\Phi_S := \sum_i \langle \Phi, \phi_i \rangle \cdot \phi + \sum_{\xi} \langle \Phi, \theta_\xi \rangle \cdot \theta_\xi + \sum_\varpi \int_{\mathbb{R}(s)=1/2} \langle \Phi, E_\varpi(\cdot, s) \rangle \cdot E_\varpi(\cdot, s) \frac{ds}{2\pi i}$$

of forms the smooth orthonormal basis $\mathcal{B}$. To be clear, we can expand the (smooth) Poincaré series $P_\phi$ introduced in the proof of Proposition 4.9 above in terms of either orthonormal basis as in (51) to deduce that the corresponding spectral coefficients (including those for $\mathcal{B}'$) are bounded uniformly in terms of the spectral coefficients of the forms in $\mathcal{B}$. We can then compare Parseval-style spectral decompositions of the bounded inner product $\langle \Phi, P_\phi \rangle = \langle \Phi, P_\phi \rangle \ll 1$ as in (52) above to deduce the claim. This is a consequence of having bounds for the spectral coefficients of $P_\phi$ expanded with respect to $\mathcal{B}'$ together with the bounds derived above in the proof Proposition 4.9 for the inner product(s). Thus in the arguments that follow, we shall bound this smooth projection $\Phi_S$ of $\Phi$ in terms of the basis $\mathcal{B}$, as this is seen to suffice to derive bounds for the shifted convolution sums described in Proposition 4.2 above. Thus, we reduce to bounding the Fourier-Whittaker coefficient

$$I := \int_{I^d \subset F_\infty} \Phi \left( \begin{array}{c} \frac{1}{1} \\ x_\infty \\ 1 \end{array} \right) \psi(-\alpha x_\infty) dx_\infty$$

$$= \sum_i \langle \Phi, \phi_i \rangle \int_{I^d \subset F_\infty} \phi_i \left( \begin{array}{c} \frac{1}{1} \\ x_\infty \\ 1 \end{array} \right) \psi(-\alpha x_\infty) dx_\infty$$

$$+ \sum_{\xi} \langle \Phi, \theta_\xi \rangle \int_{I^d \subset F_\infty} \theta_\xi \left( \begin{array}{c} \frac{1}{1} \\ x_\infty \\ 1 \end{array} \right) \psi(-\alpha x_\infty) dx_\infty$$

$$+ \sum_\varpi \int_{\mathbb{R}(s)=1/2} \langle \Phi, E_\varpi(\cdot, s) \rangle \int_{I^d \subset F_\infty} E_\varpi \left( \begin{array}{c} \frac{1}{1} \\ x_\infty \\ 1 \end{array} \right) \psi(-\alpha x_\infty) dx_\infty \frac{ds}{2\pi i}$$
in the standard case of \( n = 2 \), and to bounding the coefficient (as described in Proposition 4.2)

\[
J := \int_{I \geq [0,1]^d \subset F_\infty} \Phi_S \left( \begin{array}{c}
Y_\infty \\
x_\infty \\
1
\end{array} \right) \psi(-\alpha x_\infty) dx_\infty
\]

\[
= \sum_i \langle \Phi, \phi_i \rangle \int_{I \geq [0,1]^d \subset F_\infty} \phi_i \left( \begin{array}{c}
Y_\infty \\
x_\infty \\
1
\end{array} \right) \psi(-\alpha x_\infty) dx_\infty
\]

\[
+ \sum_\xi \langle \Phi, \partial_\xi \rangle \int_{I \geq [0,1]^d \subset F_\infty} \partial_\xi \left( \begin{array}{c}
Y_\infty \\
x_\infty \\
1
\end{array} \right) \psi(-\alpha x_\infty) dx_\infty
\]

\[
+ \sum_\infty \int_{R(s)=1/2} \langle \Phi, \mathcal{E}_\psi(\cdot, s) \rangle \int_{I \geq [0,1]^d \subset F_\infty} \mathcal{E}_\psi \left( \begin{array}{c}
Y_\infty \\
x_\infty \\
1
\end{array} \right), s \right) \psi(-\alpha x_\infty) dx_\infty \frac{ds}{2\pi i}
\]

in the generic case of \( n \geq 2 \).

To bound these unipotent integrals \( I \) and \( J \), we separate out the contributions from residual forms \( \mathcal{B}_\text{RES} \) from our fixed smooth orthonormal basis \( \mathcal{B} \), and write \( \mathcal{B}_\text{NRES} \) to denote the complement given by nonresidual forms (i.e. cuspidal forms and Eisenstein series). We write \( \varphi_\tau \) again to denote a generic element of \( \mathcal{B} \). Note that this is only shorthand for the more precise setup described in Lemma 4.8 and Proposition 4.9 above.

Given a form \( \varphi_\tau \in \mathcal{B} \) in the smooth basis, viewing \( \varphi_\tau \) as a Laplacian eigenvector, let us write \( \kappa_\tau = (\kappa_{\tau,j})_j^{d=1} \) to denote the weight of \( \varphi_\tau \), and \( \nu_\tau = (\nu_{\tau,j})_j^{d=1} \) the spectral parameter of \( \varphi_\tau \).

**Bounds for the archimedean Whittaker functions.** Recall that the Whittaker functions \( W_{\varphi_\tau} \) appearing in our spectral expansions with respect to \( \mathcal{B} \) can be viewed as \( d \)-fold products over indices \( 1 \leq j \leq d \) of the classical Whittaker functions \( W_{\kappa_{\tau,j},\nu_{\tau,j}}(y) \) described above for Proposition 4.4 and Corollary 4.5. In particular (cf. [33, Lemma 6.2]), we deduce from Proposition 4.4 and Corollary 4.5 that we have the following bounds in \( y \in \mathbb{R}_{>0} \) with \( y \to 0 \): For some constant \( A > 0 \) and any choice of \( \varepsilon > 0 \), we have:

- If \( \varphi_\tau \) corresponds to a principal or complementary series, then for each \( j \),

\[
W_{\kappa_{\tau,j},\nu_{\tau,j}}(y) \ll_{\varepsilon} y^{\frac{1}{2} - \frac{\theta_0}{2} - \varepsilon} (1 + |\kappa_{\tau,j}| + |\nu_{\tau,j}|)^A ||\varphi_\tau||.
\]

- If \( \varphi_\tau \) corresponds to a holomorphic series, then for each \( j \),

\[
W_{\kappa_{\tau,j},\nu_{\tau,j}}(y) \ll_{\varepsilon} y^{\frac{1}{2} - \frac{\theta_0}{2} - \varepsilon} (1 + |\kappa_{\tau,j}| + |\nu_{\tau,j}|)^A ||\varphi_\tau||.
\]

Similarly, for \( y \to \infty \), we have for some choice of constant \( A > 0 \) and any choice of \( \varepsilon \) the bounds

- If \( \varphi_\tau \) corresponds to a principal or complementary series, then for each \( j \),

\[
W_{\kappa_{\tau,j},\nu_{\tau,j}}(y) \ll_{\varepsilon} y^{\frac{1}{2} + \frac{\theta_0}{2} + \varepsilon} (1 + |\kappa_{\tau,j}| + |\nu_{\tau,j}|)^A ||\varphi_\tau||.
\]

- If \( \varphi_\tau \) corresponds to a holomorphic series, then for each \( j \),

\[
W_{\kappa_{\tau,j},\nu_{\tau,j}}(y) \ll_{\varepsilon} y^{\frac{1}{2} + \frac{\theta_0}{2} + \varepsilon} (1 + |\kappa_{\tau,j}| + |\nu_{\tau,j}|)^A ||\varphi_\tau||.
\]

In both regimes, the exponent \( 0 \leq \theta_0 \leq \frac{1}{4} \) denotes the best uniform approximation towards the generalized Ramanujan conjecture for \( \text{GL}_2(\mathbb{A}_F) \)-automorphic forms. Note again that by taking the product over indices \( 1 \leq j \leq d \), we derive bounds for the Whittaker functions \( W_{\varphi_\tau} \) appearing in the expansions of \( \Phi \) and \( \Phi_S \).
Non-residual contributions. We first consider the contributions from the non-residual spectrum $\mathcal{B}_{NRES}$. That is, we now consider the sums defined by

$$I_{NRES} := \sum_{\varphi_\tau \in \mathcal{B}_{NRES}} \langle \Phi, \varphi_\tau \rangle \int_{I \geq [0,1] \subset F_\infty} \varphi_\tau \left( \left( \frac{1}{x_\infty} \ x_\infty \ 1 \right) \right) \psi(-\alpha x_\infty) \, dx_\infty$$

$$= \sum_{i=1}^{\infty} \langle \Phi, \phi_i \rangle \int_{I \geq [0,1] \subset F_\infty} \phi_i \left( \left( \frac{1}{x_\infty} \ x_\infty \ 1 \right) \right) \psi(-\alpha x_\infty) \, dx_\infty$$

$$+ \sum_{\varphi} \int_{|s| = 1/2} \langle \Phi, \mathcal{E}_\varphi (\ast, s) \rangle \int_{I \geq [0,1] \subset F_\infty} \mathcal{E}_\varphi \left( \left( \frac{1}{x_\infty} \ x_\infty \ 1 \right) \right), \psi(-\alpha x_\infty) \, dx_\infty \, ds \, 2\pi i$$

for the standard case of $n = 2$ and

$$J_{NRES} := \sum_{\varphi_\tau \in \mathcal{B}_{NRES}} \langle \Phi, \varphi_\tau \rangle \int_{I \geq [0,1] \subset F_\infty} \varphi_\tau \left( \left( \frac{Y_\infty}{x_\infty} \ x_\infty \ 1 \right) \right) \psi(-\alpha x_\infty) \, dx_\infty$$

$$= \sum_{i=1}^{\infty} \langle \Phi, \phi_i \rangle \int_{I \geq [0,1] \subset F_\infty} \phi_i \left( \left( \frac{Y_\infty}{x_\infty} \ x_\infty \ 1 \right) \right) \psi(-\alpha x_\infty) \, dx_\infty$$

$$+ \sum_{\varphi} \int_{|s| = 1/2} \langle \Phi, \mathcal{E}_\varphi (\ast, s) \rangle \int_{I \geq [0,1] \subset F_\infty} \mathcal{E}_\varphi \left( \left( \frac{Y_\infty}{x_\infty} \ x_\infty \ 1 \right) \right), \psi(-\alpha x_\infty) \, dx_\infty \, ds \, 2\pi i$$

for the generic case of $n \geq 2$.

Separating the nonarchimedean and archimedean components of the Whittaker functions appearing in these sums, and writing $0 \leq \delta_0 \leq \frac{1}{4}$ to denote the best exponent approximation for bounds for the $L$-functions coefficients of half-integral weight forms, we obtain the following bounds (cf. [33, (6.13)]). For any $\varepsilon > 0$,

$$I_{NRES} = \sum_{\varphi_\tau \in \mathcal{B}_{NRES}} \langle \Phi, \varphi_\tau \rangle \cdot \frac{c_{\varphi_\tau}(\alpha)}{\alpha^{\frac{1}{2}}} W_{\varphi_\tau} \left( \frac{\alpha}{Y_\infty} \right)$$

$$\ll_{\epsilon} \sum_{\varphi_\tau \in \mathcal{B}_{NRES}} \langle \Phi, \varphi_\tau \rangle \left( \left| \frac{\alpha}{Y_\infty} \right|^{\frac{1}{2} - \frac{\delta_0}{2} - \varepsilon} |\alpha|^{\delta_0 - \frac{1}{2}} \right) \left( \prod_{j=1}^{d} (1 + |\kappa_{\tau,j} + |\nu_{\tau,j}|-1) \right)^{A} ||\varphi_\tau||,$$

and similarly

$$J_{NRES} = \sum_{\varphi_\tau \in \mathcal{B}_{NRES}} \langle \Phi, \varphi_\tau \rangle \cdot \frac{c_{\varphi_\tau}(\alpha)}{\alpha^{\frac{1}{2}}} W_{\varphi_\tau} (\alpha Y_\infty)$$

$$\ll_{\epsilon} \sum_{\varphi_\tau \in \mathcal{B}_{NRES}} \langle \Phi, \varphi_\tau \rangle \left( |\alpha Y_\infty|^{\frac{1}{2} + \frac{\delta_0}{2} + \varepsilon} |\alpha|^{\delta_0 - \frac{1}{2}} \right) \left( \prod_{j=1}^{d} (1 + |\kappa_{\tau,j} + |\nu_{\tau,j}|) \right)^{A} ||\varphi_\tau||,$$

Note that for this latter bound (56), we use Corollary 4.5 (as described above).

Now recall by the theorem of Kohnen-Zagier [25] and more generally Baruch-Mao [1], the exponent $\delta_0$ is equivalent to the exponent in the best exponent approximation towards the generalized Lindelöf hypothesis for $GL_2(\mathbb{A}_F)$-automorphic forms in the level aspect.\(^3\) In the standard case of $n = 2$, taking $A \geq 1$ to be an arbitrary integer as in Lemma 4.8 above, we can deduce via the $A$-fold iterate $\Delta^{(A)}$ of a suitable Laplace operator $\Delta$ with the Plancherel formula (cf. [33, (6.4)]) that we also have the bound

$$\sum_{\varphi_\tau \in \mathcal{B}_{NRES}} \left( \prod_{j=1}^{d} (1 + |\kappa_{\tau,j} + |\nu_{\tau,j}|) \right)^{2A} ||\varphi_\tau||^2 \ll ||\Phi||_{L^2}^2 \ll 1$$

\(^3\)I.e. so that $L(1/2, \pi) \ll_{\epsilon} c(\pi)^{\delta_0 + \varepsilon}$ for any $GL_2(\mathbb{A}_F)$-automorphic representation $\pi$ of conductor $c(\pi)$ and any $\varepsilon > 0$. 47
Proposition 4.9 and the archimedean Whittaker bound (59) to derive the corresponding bound equality to the quantity on the right hand side of (56) with the corresponding spectral coefficient bounds of for the archimedean Whittaker functions in this expansion. We can then apply the Cauchy-Schwartz inequality to the quantity on the right hand side of (55) with the bound

$$\langle \Phi, \varphi \rangle \ll \epsilon, \Pi \left( \frac{\alpha}{Y_\infty} \right)^{\frac{1}{2} \frac{n_0 - \varepsilon}{2} - \varepsilon} |\alpha|^{\frac{1}{2} - \frac{1}{2}}.$$  

In the generic case of $n \geq 2$, we argue via Proposition 4.9 as that we can derive the similar bound

$$I_{\text{NRES}} = \sum_{\varphi \in \mathcal{B}_{\text{NRES}}} \langle \Phi, \varphi \rangle \cdot \frac{c_{\varphi}(\alpha)}{|\alpha|^\frac{1}{2}} \cdot W_{\varphi}(\frac{\alpha}{Y_\infty}) \ll \epsilon, \Pi \left( \frac{\alpha}{Y_\infty} \right)^{\frac{1}{2} \frac{n_0 - \varepsilon}{2} - \varepsilon} |\alpha|^{\frac{1}{2} - \frac{1}{2}}.$$

for the archimedean Whittaker functions in this expansion. We can then apply the Cauchy-Schwartz inequality to the quantity on the right hand side of (56) with the corresponding spectral coefficient bounds of Proposition 4.9 and the archimedean Whittaker bound (59) to derive the corresponding bound

$$J_{\text{NRES}} = \sum_{\varphi \in \mathcal{B}_{\text{NRES}}} \langle \Phi, \varphi \rangle \cdot \frac{c_{\varphi}(\alpha)}{|\alpha|^\frac{1}{2}} \cdot W_{\varphi}(\alpha Y_\infty) \ll \Pi, \epsilon |\alpha Y_\infty|^{\frac{1}{2} + \frac{n_0 - \varepsilon}{2} - \varepsilon} |\alpha|^{\frac{1}{2} - \frac{1}{2}}.$$

Note that each of the bounds (58) and (60) for the non-residual contribution give the same exponents for our final estimate via the integral presentations of Proposition 3.4 and Proposition 4.2 (A) respectively.

Residual contributions. We now consider the residual contributions

$$I_{\text{RES}} := \sum_{\varphi \in \mathcal{B}_{\text{RES}}} \langle \Phi, \varphi \rangle \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi \left( \begin{array}{c} \frac{1}{Y_\infty} x_\infty \\ 1 \end{array} \right) \psi(-\alpha x_\infty) dx_\infty$$

$$= \sum_{\xi} \langle \Phi, \varphi \rangle \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi \left( \begin{array}{c} \frac{1}{Y_\infty} x_\infty \\ 1 \end{array} \right) \psi(-\alpha x_\infty) dx_\infty$$

for the standard case of $n = 2$, and

$$J_{\text{RES}} := \sum_{\varphi \in \mathcal{B}_{\text{RES}}} \langle \Phi, \varphi \rangle \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi \left( \begin{array}{c} Y_\infty x_\infty \\ 1 \end{array} \right) \psi(-\alpha x_\infty) dx_\infty$$

$$= \sum_{\xi} \langle \Phi, \varphi \rangle \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi \left( \begin{array}{c} Y_\infty x_\infty \\ 1 \end{array} \right) \psi(-\alpha x_\infty) dx_\infty$$

for the generic case of $n \geq 2$. To estimate these contributions of the residual spectrum $\mathcal{B}_{\text{RES}}$ in either case, let us first recall that we fix a smooth weight function $W \in L^2(F_\infty)$ as in the statements of Propositions 3.4 and 4.2 which decays rapidly for $y_\infty \in F_\infty$ with $|y_\infty| \to \infty$. It is then easy to deduce that both $I_{\text{RES}}$ and $J_{\text{RES}}$ are linear in $W$. Now, we deduce by comparison with of Fourier-Whittaker expansions of metaplectic forms of the same weight $k/2$ (as determined by the number of variables $k \geq 1$ of the quadratic form $f$) that $I_{\text{RES}}$ is proportional to the scaling factor $|Y_\infty|^{-\frac{k}{2}}$, which $J_{\text{RES}}$ is proportional to the scaling factor $|Y_\infty|^{-\frac{k}{2}}$. Note that we have to multiply $J_{\text{RES}}$ out by the scaling factor $|Y_\infty|^{-\frac{k}{4}}$ according to the integral presentation described in Proposition 3.4 above. On the other hand, we multiply $J_{\text{RES}}$ by the scaling factor $|Y_\infty|^{-\frac{k}{4}}$ according to the
integral presentation of Proposition 4.2 (A) above. Thus in either case, we can interpret the contribution of
the residual spectrum to the shifted convolution sum for (A) as a constant times some linear functional $I(W)$
in the weight function $W$. We also deduce via inspection of the corresponding Fourier-Whittaker expansions
that the coefficients appearing in $I_{\mathrm{RES}}$ and $J_{\mathrm{RES}}$ are identically zero unless the $F$-integer $\alpha$ is totally positive.

Putting these observations together for the standard case with $n=2$, we can derive the crude estimate
\begin{equation}
I_{\mathrm{RES}} = \sum_{\varphi_r \in \mathcal{B}_{\mathcal{R}}} \langle \Phi, \varphi_r \rangle \cdot \frac{c_{\varphi_r}(\alpha)}{|\alpha|^{\frac{1}{2}}} \cdot W_{\varphi_r} \left( \frac{\alpha}{Y_{\infty}} \right) = |Y_{\infty}|^{-\frac{1}{2}} I(W) M_{\Pi, \alpha},
\end{equation}
where $I(W) = I_{\Phi}(W)$ is a linear functional in the weight function $W$ depending on $\Phi = \varphi \tilde{\theta}_{f,p}$, and $M_{\Pi, \alpha} \geq 0$ is
a constant which vanishes unless the $F$-integer $\alpha$ is totally positive (and as explained below) the symmetric
square $L$-function $L(s, \text{Sym}^2 \Pi)$ of $\Pi$ has a pole. Multiplying by the factor $|Y_{\infty}|^{-\frac{1}{2}}$ in our integral presentation
of Proposition 3.4, we can then describe the residual contribution of the shifted convolution sum as
\begin{equation}
|Y_{\infty}|^{-\frac{1}{2}} I_{\mathrm{RES}} = I(W) M_{\Pi, \alpha}.
\end{equation}
For the generic case of $n \geq 2$ according to Propositions 4.2 (A) and 4.9, we can also derive the crude estimate
\begin{equation}
|Y_{\infty}|^{-\frac{k}{2}} J_{\mathrm{RES}} = I(W) M_{\Pi, \alpha},
\end{equation}
where $I(W) = I_{\Phi}(W)$ is a linear functional in the chosen weight function $F$ depending on $\Phi = \varphi \tilde{\theta}_{f,p}$, and $M_{\Pi, \alpha} \geq 0$
more generally is some constant which vanishes unless the $F$-integer $\alpha$ is totally positive and the symmetric
square $L$-function $L(s, \text{Sym}^2 \Pi)$ has a pole.

Some remarks on the constant term $M_{\Pi, \alpha}$. Again, the constant $M_{\Pi, \alpha} \geq 0$ appearing in the residual
contributions vanishes unless the analytic continuation of the Dirichlet series corresponding to the Mellin
transform for the constant coefficient of $\Phi$ has a pole. In the special case where the quadratic form is given
by $f(x) = q(x) = x^2$, so that $\theta_q$ is the standard genuine metaplectic theta series, this analytic continuation is
none other than the symmetric square $L$-function $L(s, \text{Sym}^2 \Pi)$ of $\Pi$. In this special case, the constant term
vanishes unless $L(s, \text{Sym}^2 \Pi)$ has a pole and the $F$-integer $\alpha$ is totally positive, using Shimura’s well-known
theory of the integral presentation in this setting.

To give a more explicit description of the vanishing criterion for this constant term from the residual
spectrum in the generic setting of $n \geq 2$ with the metaplectic theta series $\tilde{\theta}_q$ corresponding to the quadratic
form $q(x) = x^2$, let us first note that the residual forms $\varphi_r = \varphi \tilde{\xi}$ are given as residues of Eisenstein series
$\varphi_r = \varphi \tilde{\xi} = \text{Res}_{s=0} \xi_r(x, s)$ (cf. [33, §4.7]), in addition to being translates of the metaplectic theta series
$\theta_q$. As explained in [33, §4.7] for the classical setting, this constant term can also be computed in terms of
residues at $s=1$ of the symmetric square $L$-function of $\Pi$. To explain this connection, let us first consider
the simplest case of the genuine metaplectic theta series $\tilde{\theta}_q$. We can first calculate the Mellin transform
$M(\rho_{\varphi_r \tilde{\xi}_q, 0}, s)$ of the constant coefficient $\rho_{\varphi_r \tilde{\xi}_q, 0}(y)$ in the expansion
\begin{equation}
\rho^n_{\varphi_r \tilde{\xi}_q} \left( \begin{array}{c} y \\ x \\ 1 \end{array} \right) = \rho_{\varphi_r \tilde{\xi}_q, 0}(y) + \sum_{\gamma \in F^x} \rho_{\varphi_r \tilde{\xi}_q} (\gamma y) W_{\varphi_r \tilde{\xi}_q}(\gamma y) \phi(y)
\end{equation}
of $\rho^n_{\varphi_r \tilde{\xi}_q}$ according to Proposition 5.1 below. To be more precise, taking $y = (y_j)_{j=1}^d \in F_{\infty,+}^d$ to be a totally
positive archimedean idele, we can compute
\begin{equation}
M(\rho_{\varphi_r \tilde{\xi}_q, 0}, s) = \int_{F_{\infty,+}^d} \rho_{\varphi_r \tilde{\xi}_q, 0}(y)|y|^sdy = \int_0^\infty \cdots \int_0^\infty \rho_{\varphi_r \tilde{\xi}_q, 0}(y_j y) dy_1 \cdots y_d dy_d,
\end{equation}
as
\begin{equation}
M(\rho_{\varphi_r \tilde{\xi}_q, 0}, s) = \sum_{a \neq 0} \frac{c_{\Pi}(a^2)}{|a|^s + \frac{1}{2} - (\frac{n-2}{2})^2} \int_{F_{\infty,+}^d} \psi(ia^2 y) W_{\varphi}(a^2 y) dy^x = 2 \cdot \frac{L_1(2 (s + \frac{1}{2} - (\frac{n-2}{2})^2), \text{Sym}^2 \Pi)}{L_1(4 (s + \frac{1}{2} - (\frac{n-2}{2})^2), \omega_{\Pi})} \int_{F_{\infty,+}^d} \psi(ia^2 y) W_{\varphi}(a^2 y) dy^x.
\end{equation}
Here, $L_1(s, \text{Sym}^2 \Pi)$ equals $L_1(s, \text{Sym}^2 \Pi)$ up to an Euler product which converges absolutely for $\Re(s) > \frac{1}{2}$,
where $L_1(s, \text{Sym}^2 \Pi)$ denotes the partial series expansion over principal ideals of $O_F$ of the symmetric square
L-function $L(s, \text{Sym}^2 \Pi)$ of $\Pi$. As well, $L_1(s, \omega_\Pi)$ denotes the corresponding partial Dirichlet series of the central character $\omega_\Pi$. The Mellin transforms appearing in these expressions can be computed explicitly in terms of the classical Whittaker functions $W_{k, \nu_j}(4\pi|y_j|)$ (Proposition 5.1). A convolution calculation then allows us to relate the Mellin transform $M(\rho_2^n, \nu, \sigma_0, s)$ to the inner product $\langle \varphi^\sigma_q, E(\sigma, s) \rangle$ of $\varphi^\sigma_q$ against a metaplectic Eisenstein series $E(\sigma, s) = E_q(\sigma, s)$ (Lemma 5.2). This in fact gives an analytic continuation for $M(\rho_2^n, \nu, \sigma_0, s)$, and hence for the partial Dirichlet series $L_1(s, \text{Sym}^2 \Pi)$. To be more precise, if we assume the archimedean local Whittaker function equals $W_\varphi(y) = \prod_{j=1}^d W_{k_j, \nu_j}(4\pi|y_j|)$, we obtain the relations (first for $\Re(s) > 1$)

$$M(\rho_2^n, \varphi_0, s, \nu_j) = 2 \cdot \frac{L_1^2(2 \left( s + \frac{1}{2} - \frac{n}{2} \right), \text{Sym}^2 \Pi)}{L_1(4 \left( s + \frac{1}{2} - \frac{n}{2} \right), \omega_\Pi)} \cdot \prod_{j=1}^d \frac{\Gamma \left( s + \frac{3}{4} - \frac{n}{2} - \nu_j \right)}{\Gamma \left( s + \frac{3}{4} - \frac{n}{2} - \nu_j \right)} \cdot \frac{\Gamma \left( s + \frac{3}{4} - \frac{n}{2} - \nu_j \right)}{\Gamma \left( s + \frac{3}{4} - \frac{n}{2} - \nu_j \right)}$$

and in the more general case of the metaplectic theta series $\theta_{f,p}$ described above the relations

$$M(\rho_2^n, \varphi_0, s, \nu_j) = \sum_{a=(a_1, \ldots, a_k) \in \mathcal{O}_p \setminus \{0\}} \frac{p(a) c_\Pi(f(a))}{|f(a)|^{s+\frac{1}{2}-\frac{n}{2}+\frac{k}{2}}} \cdot \prod_{j=1}^d \frac{\Gamma \left( s + \frac{1}{2} + \frac{k}{2} - \frac{n}{2} + \nu_j \right)}{\Gamma \left( s + \frac{1}{2} + \frac{k}{2} - \frac{n}{2} - \nu_j \right)}$$

Returning to the coefficients $\langle \Phi, \varphi_\tau \rangle = \langle \Phi, \text{Res}_{s=s_0} E_\tau \rangle = \langle \Phi, \theta_\xi \rangle$ appearing in (61) above, we deduce that a given $\varphi_\tau \in \mathcal{B}_{\text{RES}}$ contributes only if the corresponding $L$-series has a pole. Hence for $f = q$, we deduce from the theory of integral presentations of symmetric square $L$-functions that the residual contributions $I_{\text{RES}}$ and $J_{\text{RES}}$ vanishes unless $\Pi$ is orthogonal (and hence self-dual). In the general case on the quadratic form $f$, we argue that we can also deduce that the corresponding residual contribution will not vanish if $\Pi$ is orthogonal. Hence, we characterize the contribution of the residual spectrum $I_{\text{RES}}$ or $J_{\text{RES}}$ crudely in this way.

**Bounds in the genuine metaplectic case for (A).** Putting together (61) and (55) for the standard case of $n = 2$ with the scaling factor of Proposition 3.4, we derive the estimate

$$\sum_{a=(a_1, \ldots, a_k) \in \mathcal{O}_p \setminus \{0\}} p(a) \cdot \frac{c_\Pi(f(a) + \alpha)}{|f(a) + \alpha|^\frac{1}{2}} \cdot W \left( \frac{f(a) + \alpha}{Y_\infty} \right)$$

$$= I(W) M_{\Pi, \alpha} + O_{\Pi, \varepsilon} \left( |Y_\infty|^{\frac{1}{4} + \varepsilon} |\alpha|^{\frac{1}{2} - \varepsilon} |\alpha|^{-\frac{n}{2}} \right).$$

In the generic case $n \geq 2$, using the integral presentation of Proposition 4.2 (A), we can put together the bounds for the corresponding contributions (62) and (56) for to derive the more general estimate

$$\sum_{a=(a_1, \ldots, a_k) \in \mathcal{O}_p \setminus \{0\}} p(a) \cdot \frac{c_\Pi(f(a) + \alpha)}{|f(a) + \alpha|^\frac{1}{2}} \cdot W \left( \frac{f(a) + \alpha}{Y_\infty} \right)$$

$$= I(W) M_{\Pi, \alpha} + O_{\Pi, \varepsilon} \left( |Y_\infty|^{-\frac{1}{2}} |\alpha| Y_\infty^{\frac{1}{2} + \frac{n}{2} - \varepsilon} |\alpha|^{-\frac{n}{2}} \right)$$

$$= O_{\Pi, f, p, \alpha, W(1)} + O_{\Pi, \varepsilon} \left( |Y_\infty|^{\frac{1}{2} + \frac{n}{2} + \varepsilon} |\alpha|^\frac{1}{2} - \varepsilon \right).$$

This gives the corresponding estimate (for odd $k$) in Theorem 1.1 (A). Notice that we derive an improvement on the standard setup of dimension $n = 2$ by considering the modified theta series $\overline{\theta}_{f,p}$, i.e. to absorb the extra scaling factor of $|Y_\infty|^{\frac{1}{4}}$ that we would otherwise have to include (cf. e.g. [33, Theorem 1]).
Bounds for the non-genuine cases of (A) and (B). To deal with the remaining cases of Proposition 3.4 for even \( k \) and Proposition 3.5, we can use the same argument with the non-genuine forms \( \Phi = \phi \tilde{g}_{f,p} \), i.e. with \( f \) having an even number of variables \( k \geq 2 \), and also with \( \Phi = \phi \phi' \). Again, in the generic case \( n \geq 2 \), we consider the functions \( \Phi \) defined in (30) above for Proposition 4.2. Recall that these functions have convergent spectral coefficients by the argument of Proposition 4.9. We also saw in Proposition 4.2 (using Propositions 3.4, 3.5, and 2.9) that the Fourier-Whittaker coefficients of these restricted extended functions carry sufficient information about the shifted convolution problem via the underlying mirabolic form that it will suffice to bound the spectral decomposition of the unipotent integral describing the Fourier-Whittaker carry sufficient information about the shifted convolution problem via the underlying mirabolic form that it will suffice to bound the spectral decomposition of the unipotent integral describing the Fourier-Whittaker coefficient at \( \alpha \) of \( \Phi \). Here, we decompose the corresponding form \( \Phi \) as an \( L^2 \)-automorphic form on \( GL_2(A_F) \) in a completely analogous way, and repeat the same argument essentially verbatim. That is, we now consider the \( L^2 \)-spectral decomposition of \( \Phi \) in the Hilbert space \( L^2(GL_2(F) \setminus GL_2(A_F), \xi) \), as described for instance in [6, §2.2.(8)]:

\[
L^2(GL_2(F) \setminus GL_2(A_F), \omega) = \bigoplus_{\pi \in \mathfrak{c}_F \text{ cuspidal}} V_{\pi} \oplus \bigoplus_{\xi^2 = \omega \text{ residual}} V_{\xi} \oplus \int_{0}^{\infty} \bigoplus_{x_1 \chi_2 = \omega \text{ cont.}} V_{x_1, \chi_2} \, dy
\]

\[
= L^2_{\text{cusp}}(GL_2(F) \setminus GL_2(A_F), \omega) \oplus L^2_{\text{res}}(GL_2(F) \setminus GL_2(A_F), \omega) \oplus L^2_{\text{cont}}(GL_2(F) \setminus GL_2(A_F), \omega).
\]

Here, each \( V_{\pi} \) denotes the subspace generated by \( y \mapsto \chi(\det{g}) \), with \( \xi \) the idele class characters of \( F \) for which \( \xi^2 = \omega \). Repeating the arguments above for \( \Phi \) in this setting, the exponent \( \delta_0 \) is seen easily to be replaced by \( \theta_0 \) (as each form in the expansion is an \( L^2 \)-automorphic form on \( GL_2(A_F) \)). As well, it is not hard to deduce that special or residual spectral contributions only if the Rankin-Selberg \( L \)-function \( L(s, \Pi \times \pi) \) has a pole. This again comes from the fact that such forms \( \tilde{g}_{\xi} \) from the residual spectrum occur as residues of Eisenstein series \( \vartheta_{\xi} = \text{Res}_{\xi = \xi_0} E_\xi(*, s) \), so that the inner product \( \langle \Phi, \vartheta_{\xi} \rangle = \text{Res}_{\xi = \xi_0} \langle \Phi, E_\xi(*, s) \rangle \) vanishes unless \( \langle \Phi, E_\xi(*, s) \rangle \approx L(s, \Pi \times \pi) \) has a pole. Let us remark that the shifted convolution sum appearing in Propositions 4.2 (B) and 3.5 in particular can be estimated in the style of [6, Theorem 2]. As we explain below, this latter bound can be used to derive estimates for the corresponding subconvexity problem for central values of \( GL_n(A_F) \)-automorphic \( L \)-functions.

5. Remarks on analytic continuation of shifts by quadratic forms

We now explain how to derive the analytic continuation of the Dirichlet series (1), although this is not strictly necessary for the rest of the work. Let us now return to the setup of Proposition 3.4 above, but with the constant coefficient \( \alpha = 0 \). To be clear, let us fix a pure tensor \( \varphi = \otimes_v \varphi_v \in \mathfrak{V}_1 \) whose nonarchimedean local components \( \varphi_v \) are essential Whittaker vectors, but whose archimedean component is not yet specified. Let \( f(a_1, \ldots, a_k) \) be an \( F \)-rational positive definite quadratic form in \( k \) variables, with \( p(a_1, \ldots, a_k) \) an associated harmonic polynomial (possibly trivial), and \( \theta_{f,p} \) the corresponding theta series. We shall consider the automorphic form \( \mathbb{P}^n_{1, \varphi} \tilde{f}_{f,p} \) on \( GL_2(A_F) \), which for \( x \in A_F \) an adele and \( y = y_\infty y_f \in A_F^\infty \) an idele has the Fourier-Whittaker expansion

\[
\mathbb{P}^n_{1, \varphi} \tilde{f}_{f,p} \left( \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix} \right) = \rho_{\mathbb{P}^n_{1, \varphi} \tilde{f}_{f,p}, 0}(y) + \sum_{\gamma \in F_x} \rho_{\mathbb{P}^n_{1, \varphi} \tilde{f}_{f,p}, \gamma y_f}(\gamma y_f) W_{\mathbb{P}^n_{1, \varphi} \tilde{f}_{f,p}, \gamma y_f}(\gamma y_\infty).
\]

Let us now consider the constant coefficient \( \rho_{\mathbb{P}^n_{1, \varphi} \tilde{f}_{f,p}, 0}(y) \), and in particular its specialization to an archimedean idele \( y_\infty \in F_\infty^\times \). Writing \( F_\infty^\times = \mathbb{R}^d_{>0} \) as above to denote the totally positive plane, we compute the Mellin transform

\[
M(\rho_{\mathbb{P}^n_{1, \varphi} \tilde{f}_{f,p}, 0}, s) = \int_{F_\infty^\times} \rho_{\mathbb{P}^n_{1, \varphi} \tilde{f}_{f,p}, 0}(y_\infty) |y_\infty|^s \, dy_\infty^\times,
\]
which after expanding out the archimedean idele \( y_{\infty} = (y_j)_{j=1}^d = (y_{\infty,j})_{j=1}^d \in F_{\infty,+}^d \) with each \( y_j = y_{\infty,j} \in \mathbb{R}_{>0} \) is defined more explicitly as the \( d \)-fold Mellin transform

\[
M(\rho_{\mathbb{P}^n \varphi_{f,p},0}^n, s) = \int_0^\infty \cdots \int_0^\infty \rho_{\mathbb{P}^n \varphi_{f,p},0}^n ((y_j)_{j=1}^d) y_1^{\frac{s}{2}} \cdots y_d^{\frac{s}{2}} dy_1 \cdots dy_d.
\]

This Mellin transform can be computed explicitly as follows, using all of the same notations and conventions for (archimedean) Whittaker functions described above.

**Proposition 5.1.** Let \( \Pi = \otimes_v \Pi_v \) be a cuspidal automorphic representation of \( GL_n(A_F) \) with unitary central character for \( n \geq 2 \). Again, we write \( \phi \) to denote the image of \( \Phi \) under the projection operator \( \mathbb{P}^n_1 \), so that \( \mathbb{P}^n_1 \phi \) is a cuspidal \( L^2 \)-automorphic form on the mirabolic subgroup \( P_2(A_F) \subset GL_2(A_F) \). Let \( \varphi = \otimes_v \varphi_v \in V_\Pi \) be a pure tensor whose nonarchimedean local components are all essential Whittaker vectors. Let \( f(a_1, \ldots, a_k) \) be an \( F \)-rational positive definite quadratic form in \( k \) many variables, and \( p(a_1, \ldots, a_k) \) a harmonic polynomial for \( f \) (possibly trivial). Then,

\[
M(\rho_{\mathbb{P}^n_1 \varphi_{f,p},0}^n, s) = \sum_{a = (a_1, \ldots, a_k) \in O_F^k} p(a) \frac{c_{\Pi}(f(a))}{|f(a)|^{s + \frac{k}{2} - \left( \frac{n-2}{2} \right)}} \times \int_{F_{\infty,+}^k} \psi(if(a)y_{\infty}) W_{\phi}(f(a)y_{\infty}) |y_{\infty}|^{s + \frac{k}{2} - \left( \frac{n-2}{2} \right)} dy_{\infty}.
\]

In particular, if we assume that the archimedean Whittaker function

\[
W_{\varphi}(y_{\infty}) = W_{\varphi}(\begin{pmatrix} y_{\infty} \\ 1_{n-1} \end{pmatrix})
\]

with \( y_{\infty} = (y_j)_{j=1}^d \in F_{\infty,+}^d \cong \mathbb{R}_{>0}^d \) is given by the product

\[
W_{\varphi}(y_{\infty}) = \prod_{j=1}^d W_{\kappa_j, \nu_j}(4\pi|y_j|)
\]

for some \( d \)-tuples of complex numbers \( \kappa = (\kappa_j)_{j=1}^d \in \mathbb{C}^d \) and \( \nu = (\nu_j)_{j=1}^d \in \mathbb{C}^d \), where \( W_{\kappa_j, \nu_j}(y_j) \) denotes the standard Whittaker function described above, then

\[
M(\rho_{\mathbb{P}^n_1 \varphi_{f,p},0}^n, s) = \sum_{a = (a_1, \ldots, a_k) \in O_F^k} p(a) \frac{c_{\Pi}(f(a))}{|f(a)|^{s + \frac{k}{2} - \left( \frac{n-2}{2} \right)}} \times \prod_{j=1}^d \Gamma \left( s + \frac{1}{2} + \frac{k}{2} - \left( \frac{n-2}{2} \right) + \nu_j \right) \Gamma \left( s + 1 + \frac{k}{2} - \left( \frac{n-2}{2} \right) - \nu_j \right)
\]

for \( \Re(s) \) sufficiently large, i.e. for \( \Re(s) - \frac{k}{2} - \left( \frac{n-2}{2} \right) > -1 \).

**Proof.** The first claim follows as an easy consequence of the definitions. That is,

\[
M(\rho_{\mathbb{P}^n_1 \varphi_{f,p},0}^n, s) = \int_{F_{\infty,+}^d} \rho_{\mathbb{P}^n_1 \varphi_{f,p},0}(y_{\infty}) |y_{\infty}|^s dy_{\infty},
\]

which after expanding out the coefficient (as done in the proof of Proposition 3.4),

\[
\rho_{\mathbb{P}^n_1 \varphi_{f,p},0}(y_{\infty}) = |y_{\infty}|^{\frac{k}{2} - \left( \frac{n-2}{2} \right)} \sum_{a = (a_1, \ldots, a_k) \in O_F^k} p(a) \frac{c_{\Pi}(f(a))}{|f(a)|^{\frac{n-2}{2}}} \psi(if(a)y_{\infty}) W_{\phi}(f(a)y_{\infty}),
\]

then switching the order of summation gives the stated identity. Keep in mind that this is the same as

\[
\rho_{\mathbb{P}^n_1 \varphi_{f,p},0}(y_{\infty}) = |f(a)y_{\infty}|^{\frac{k}{2} - \left( \frac{n-2}{2} \right)} \sum_{a = (a_1, \ldots, a_k) \in O_F^k} p(a) c_{\Pi}(f(a)) |f(a)|^{\frac{n-2}{2}} \psi(if(a)y_{\infty}) W_{\phi}(f(a)y_{\infty}).
\]

To prove the second claim, we compute the remaining \( d \)-fold integral Mellin transform as follows. Let us write \((\tau_j)_{j=1}^d \) to denote the collection of real places of \( F \), i.e. the collection of embeddings \( \tau_j : F \to \mathbb{R} \).
Recall that we fix $\psi$ to be the standard additive character for which $\psi(y_\infty) = \prod_{j=1}^d \exp(2\pi iy_j)$. Hence $\psi(af(y_\infty)) = \prod_{j=1}^d \exp(-2\pi \tau_j(af(y_j)))$, whence unraveling definitions gives
\[
\int_{F_{\infty,+}^\times} \psi(af(y_\infty)) W_\varphi(f(af)(y_\infty)) |y_\infty|^{s+\frac{k}{2}-(\frac{n-2}{2})} dy_\infty^\times
= \prod_{j=1}^d \int_0^\infty e^{-2\pi \tau_j(af)(y)} W_{\kappa_j, \nu_j}(4\pi \tau_j(f(af))(y_j)) y_j^{s+\frac{k}{2}-(\frac{n-2}{2})} \frac{dy_j}{y_j}.
\]

Now, to evaluate each of the single-variable Mellin transforms
\[
\int_0^\infty e^{-2\pi \tau_j(f(af))(y)} W_{\kappa_j, \nu_j}(4\pi \tau_j(f(af))(y_j)) y_j^{s+\frac{k}{2}-(\frac{n-2}{2})} \frac{dy_j}{y_j}
\]
appearing in this latter expression, we use the formula (25) above. Hence for
\[
\Re(s) + k - \left(\frac{n-2}{2}\right) > -1,
\]
making a simple change of variables, we find that
\[
\int_0^\infty e^{-2\pi \tau_j(f(af))(y)} W_{\kappa_j, \nu_j}(4\pi \tau_j(f(af))(y_j)) y_j^{s+\frac{k}{2}-(\frac{n-2}{2})} dy_j
= (4\pi \tau_j(f(af)))^{-(s+\frac{k}{2}-(\frac{n-2}{2}))} \frac{\Gamma(s+\frac{k}{2}-(\frac{n-2}{2})+\nu_j)}{\Gamma(s+\frac{k}{2}-(\frac{n-2}{2})+\kappa_j)}
\]
Thus, we compute
\[
\int_{F_{\infty,+}^\times} \psi(af(y_\infty)) W_\varphi(f(af)(y_\infty)) |y_\infty|^{s+\frac{k}{2}-(\frac{n-2}{2})} dy_\infty^\times
= |f(af)|^{-s-\frac{k}{2}+(\frac{n-2}{2})} \prod_{j=1}^d \frac{\Gamma(s+\frac{k}{2}-(\frac{n-2}{2})+\nu_j)}{\Gamma(s+\frac{k}{2}-(\frac{n-2}{2})+\kappa_j)}
\]
from which the claim follows easily.

We now use this to derive the following semi-classical unfolding calculation. Let $\Gamma \subset P_2(F_\infty)$ be a discrete subgroup under which $\mathbb{P}_1 \varphi$ and $\theta_{f,p}$ are both invariant. Let us also consider
\[
\Gamma_\infty = \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} : \alpha \in \mathcal{O}_F \right\}.
\]
We then consider the Eisenstein series defined for $x \in \mathbb{A}_F$ and $y = y_\infty y_\infty \in \mathbb{A}_F^\times$ by
\[
E(y, s) = E\left( \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix}, s \right) = \left\{ \sum_{\Gamma \subset \Gamma_\infty} |\gamma y|^s \right\}^{k \equiv 0 \text{ mod } 2} \left\{ \sum_{\Gamma \subset \Gamma_\infty \setminus \Gamma} |\gamma y|^s \right\}^{k \equiv 1 \text{ mod } 2},
\]
where $j(y, \gamma)$ is the (metaplectic) automorphy factor defined by
\[
j(y, \gamma) = j\left( \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix}, \gamma \right) = \frac{\overline{\theta}_{f,p}}{\overline{\theta}_{f,p}} \left( \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix} \right),
\]
and $s \in \mathbb{C}$ is a complex variable with $\Re(s) > 1$. Let us also write $\mathcal{H}_F = \mathbb{A}_F \cup iF_{\infty,+}^\times$, which we identify with the mirabolic subgroup of $P_2(\mathbb{A}_F)$ of matrices $\begin{pmatrix} y_\infty & x \\ 1 & 1 \end{pmatrix}$ with $x \in \mathbb{A}_F$ and $y_\infty \in F_{\infty,+}^\times$ in the usual way.

Lemma 5.2. We have for $\Re(s) > 1$ that
\[
M(\rho_{\mathcal{H}_F}, s) = \int_{\mathcal{H}_F} \mathfrak{p}_\mathcal{H}_F^\times \mathfrak{p}_\mathcal{H}_F^\times \left( \begin{pmatrix} y_\infty & x \\ 1 & 1 \end{pmatrix} \right) \mathfrak{E} \left( \begin{pmatrix} y_\infty & x \\ 1 & 1 \end{pmatrix}, s \right) dx dy_\infty^\times,
\]
where $\mathfrak{F}_F$ denotes some fixed fundamental domain for the action of $\Gamma$ on $\mathcal{H}_F$. 

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Proof. We have that
\[
M(\rho_{F,0} \varphi_{f,p,0}, s) = \int_{F,0} \rho_{F} \varphi_{f,p,0}(y) |y|^{s} dxdy_{\infty}^{\times}
\]
\[
= \int_{F,0} \mathbb{A}_{F}/F \mathbb{P}_{1}^{n} \varphi \left( \left( \begin{array}{cc} y_{\infty} & x \\ 1 & 1 \end{array} \right) \right) \overline{\varphi}_{f,p} \left( \left( \begin{array}{cc} y_{\infty} & x \\ 1 & 1 \end{array} \right) \right) |y_{\infty}|^{s} dxdy_{\infty}^{\times},
\]
where \( \Gamma_{\infty} \) acts on \( \mathbb{H} \) by translation. Now, we argue that we can fix a set of representatives for this action of the form \( \cup \gamma \mathbb{S}_{F} \), where \( \gamma \) ranges over a set of representatives for the action of \( \Gamma \) modulo \( \Gamma_{\infty} \), and \( \mathbb{S}_{F} \) is a fundamental domain for the action of \( \Gamma \) on \( \mathbb{H} \). Hence, we can expand the latter integral expression as
\[
\sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} \int_{\gamma \mathbb{S}_{F}} \mathbb{P}_{1}^{n} \varphi \left( \left( \begin{array}{cc} y_{\infty} & x \\ 1 & 1 \end{array} \right) \right) \overline{\varphi}_{f,p} \left( \left( \begin{array}{cc} y_{\infty} & x \\ 1 & 1 \end{array} \right) \right) |y_{\infty}|^{s} dxdy_{\infty}^{\times},
\]
which after using the automorphy of \( \mathbb{P}_{1}^{n} \varphi \) and \( \overline{\varphi}_{f,p} \) with respect to \( \Gamma \) and switching the order of summation is the same as the stated integral
\[
\int_{\mathbb{S}_{F}} \mathbb{P}_{1}^{n} \varphi \left( \left( \begin{array}{cc} y_{\infty} & x \\ 1 & 1 \end{array} \right) \right) \overline{\varphi}_{f,p} \left( \left( \begin{array}{cc} y_{\infty} & x \\ 1 & 1 \end{array} \right) \right) E \left( \left( \begin{array}{cc} y_{\infty} & x \\ 1 & 1 \end{array} \right), s \right) dxdy_{\infty}^{\times}.
\]

This latter result allows us to derive the following from the well-known analytic properties of the Eisenstein series \( E(y,s) \) in either case on the parity of the number of variables \( k \) of the positive definite quadratic form \( f \) (see e.g. [30] or [18]).

Corollary 5.3. Fix \( n \geq 2 \), and let \( \Pi = \otimes \Pi_{v} \) be a cuspidal \( \text{GL}_{n}(\mathbb{A}_{F}) \)-automorphic representation. Again, we write \( c_{1} \) to denote the \( L \)-function coefficients of \( \Pi \), so that the Dirichlet series of the standard of \( L \)-function of \( \Pi \) can be written as \( \sum_{m \in \mathbb{O}_{F}} c_{1}(m)|Nm|^{-s} \) for \( \Re(s) > 1 \). Let \( f(a_{1}, \ldots, a_{k}) \) be a positive definite quadratic form in \( k \geq 1 \) many variables, and \( p(a_{1}, \ldots, a_{k}) \) a harmonic polynomial for \( f \) (possibly trivial). Then the Dirichlet series
\[
D(s, \Pi, f, p) = \sum_{a_{1}, \ldots, a_{k} \in \mathbb{O}_{F}} p(a_{1}, \ldots, a_{k}) c_{1}(f(a_{1}, \ldots, a_{k})) |f(a_{1}, \ldots, a_{k})|^{-s},
\]
defined a priori only for \( s \in \mathbb{C} \) with \( \Re(s) > 1 \), has an analytic continuation to all \( s \in \mathbb{C} \), and satisfies a functional equation relating values at \( s \) to \( 1 - s \).

6. Estimates for Twisted \( \text{GL}_{n}(\mathbb{A}_{F}) \)-Automorphic \( L \)-functions

We now give the proof of Theorem 1.2. That is, we now explain how to use the integral presentation of Proposition 3.5 (as adapted in Proposition 4.2) and the estimates of Theorem 1.1 (2) in the style of Blomer-Harcos [6, §3] and Cogdell [15], as well as an earlier work of Cogdell-Piatetski-Shapiro-Sarnak [11] (see [6, Remark 2]), to estimate central values of twisted \( \text{GL}_{n}(\mathbb{A}_{F}) \)-automorphic \( L \)-functions.

Recall that for \( \Re(s) > 1 \), we write \( L(s, \Pi) = \sum_{m \in \mathbb{O}_{F}} c_{1}(m)|Nm|^{-s} \) to denote the finite part of the standard \( L \)-function \( \Lambda(s, \Pi) = \prod_{v < \infty} L(s, \Pi_{v}) = L(s, \Pi_{\infty})L(s, \Pi) \) of \( \Pi \). Hence, the coefficients \( c_{1} \) are the \( L \)-function coefficients of \( \Pi \), and in the special case of dimension \( n = 2 \) described in [6, §2] have a direct relation to the Hecke eigenvalues \( \lambda_{\Pi} \). Assume that \( \Pi \) is irreducible. Fix a Hecke character \( \chi \) of \( F \) of conductor \( q \subset \mathbb{O}_{F} \). Hence, there exist characters \( \chi_{f} : (\mathcal{O}_{F}/q\mathcal{O}_{F})^{\times} \rightarrow \mathbb{S}^{1} \) and \( \chi_{\infty} : F_{\infty}^{\times} \rightarrow \mathbb{S}^{1} \) such that for any \( F \)-integer \( \alpha \in \mathcal{O}_{F} \) coprime to \( q \), we have \( \chi((\alpha)) = \chi_{f}(\alpha)\chi_{\infty}(\alpha) \). Note that we shall use the same notation \( \chi \) to denote the corresponding idele class character of \( F \), interchanging these notions freely when the context is clear.
6.1. **Esquisse.** Let us first sketch an outline of the main argument following the heuristic description of exponents given in Blomer-Harcos [6, §3.1], assuming for simplicity that $F = \mathbb{Q}$, and that $\Pi$ is everywhere unramified. We shall also ignore epsilons for now. Fixing a real parameter $L > 0$, and writing $l \sim L$ to denote the sum over $L \leq l \leq 2L$, the idea is to derive upper and lower bounds for the amplified second moment defined over Hecke characters $\xi$ of conductor $q$,

$$S := \sum_{\xi \mod q} \left| \sum_{l \sim L} \xi(l) \varphi_f(l) \right|^2 |L(1/2, \Pi \otimes \xi)|^2.$$

To begin, observe that we have the lower bound

$$S \gg L^2 |L(1/2, \Pi \otimes \chi)|^2.$$

On the other hand, using an approximate functional equation to describe each of the central values appearing in $S$, opening up squares gives an upper bound of

$$S \ll q \sum_{l_1, l_2 \sim L} \chi_f(l_1) \varphi_f(l_2) \sum_{m_1 m_2 \equiv 0 \mod q} \frac{c_n(m_1) c_n(m_2)}{(m_1 m_2)^2} V \left( \frac{m_1}{q^{2/3}} \right) V \left( \frac{m_2}{q^{2/3}} \right)$$

for a smooth and rapidly decaying function $V$ of $y \in \mathbb{R}_{>0}$ which depends only on the archimedean components $\Pi_\infty, \xi_\infty$, and $\chi_\infty$, and which by standard arguments can be replaced by a compactly supported function (see Corollary 6.2 (B)). Now, the diagonal term coming from the contribution $l_1 m_1 - l_2 m_2 = 0$ in this latter quantity is seen easily to be bounded above by $\ll_{\Pi} qL$ (see (73)). It therefore remains to estimate the contribution of the non-negligible part of the remaining off-diagonal sum

$$\left| \sum_{l_1, l_2 \sim L} \chi_f(l_1) \varphi_f(l_2) \right|^2 \sum_{1 \leq h \leq q} \sum_{L \leq m_1 m_2 \equiv 1 \mod q} \frac{c_n(m_1) c_n(m_2)}{(m_1 m_2)^2} V \left( \frac{m_1}{q^{2/3}} \right) V \left( \frac{m_2}{q^{2/3}} \right).$$

Using a variation of the argument of Propositions 3.5 and 4.2 (B) above (see Proposition 6.3), we can derive for each inner sum corresponding to the congruence pair $l_1 m_2 - l_2 m_1 = hq$ in this latter expression an integral presentation of the form

$$\int_{l \sim [0, 1] \subset \mathbb{R}} \Phi_{l_1, l_2} \left( \left( \begin{array}{c} Lq \frac{x}{2} \\ 1 \end{array} \right) \right) \psi(-hq x) dx,$$

and more intrinsically a presentation as a sum over Fourier-Whittaker coefficients

$$\int_{l \sim [0, 1] \subset \mathbb{R}} \Phi_{l_1, l_2} \left( \left( \begin{array}{c} Lq \frac{x}{2} \\ 1 \end{array} \right) \right) \psi(-hq x) dx \quad \text{for } \Phi_{l_1, l_2} := R_{(l_1)} \varphi \cdot R_{(l_2)} \varphi,$$

where $R_{(l_j)}$ for $j = 1, 2$ is the shift operator (cf. [6, (14) and also (15), (38)-(41)]) defined on an automorphic form $\phi$ of $p \in P_2(\mathbb{A}_F) \subset \text{GL}_2(\mathbb{A}_F)$ by the rule

$$(R_{(l_j)} \phi)(p) = \phi \left( p \left( \begin{array}{c} l_1^{-1} x \\ 1 \end{array} \right) \right).$$

Note that these operators are isometries (see [6, §2.9]). Hence, it remains to bound

$$\sum_{l_1, l_2 \sim L} \chi_f(l_1) \varphi_f(l_2) \int_{l \sim [0, 1] \subset \mathbb{R}} \Phi_{l_1, l_2} \left( \left( \begin{array}{c} Lq \frac{x}{2} \\ 1 \end{array} \right) \right) \psi(-hq x) dx.$$

Decomposing each automorphic form $\Phi = \Phi_{l_1, l_2}$ in the sum (64) spectrally according to the discussion above Proposition 4.9 for the generic case $n \geq 2$ as

$$\Phi = \sum_{\varphi \in \mathcal{B}} \langle \Phi, \varphi \rangle \cdot \varphi,$$

then allows us to derive bounds from Theorem 1.1 (B). Roughly speaking, we know that each archimedean local Whittaker coefficient $W_{\varphi, x}$ is a multiple of the standard Whittaker functions described above, and hence bounded in the spectral parameter. Using Plancharle’s formula, together with the fact that the Kirillov map and the operators $R_{(l_1)}$ for $i = 1, 2$ are isometries, we know that $\sum_{\varphi \in \mathcal{B}} |W_{\varphi, x}|^2 \approx |W_{\Pi, x}| \ll 1$. As well, we know that each $\Phi = \Phi_{l_1, l_2}$ has level of size $L^2$. By Weyl’s law, there should be approximately $L^2$ many
Theorem 1.1 (B) allow us to derive the estimate

Here, the second product on the right hand side runs over the real places of \( \Pi \). Using this latter bound to estimate the off-diagonal contribution for \( L \), we use the standard notation \( \Gamma_R(s) = \pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \), we view the Hecke character in terms of its corresponding idele class character \( \chi = \otimes_v \chi_v \), and we use the symbols \( \mu_i(\Pi_\infty) \) to denote the archimedean component \( L(s, \Pi_\infty \otimes \chi_\infty) \) is defined by the product

Using this latter bound to estimate the off-diagonal contribution for \( S \), we find that

\[
|L(1/2, \Pi \otimes \chi)|^2 \ll_{\Pi, \chi_\infty} qL^{-1} + q^{\frac{1}{2} + \theta_0} L^{\frac{1}{2} + \theta_0}.
\]

Making a suitable choice of parameter \( L \) of size \( q^n \) for some \( 0 \leq u \leq 1 \) then gives some corresponding bound for the modulus of the central value \( L(1/2, \Pi \otimes \chi) \). For instance, taking \( L = q^{1/4 - u/2} \) gives us the (surprising) uniform upper bound

\[
L(1/2, \Pi \otimes \chi) \ll_{\Pi, \chi_\infty} q^{\frac{1}{4} + \frac{u}{8}} + q^{\frac{1}{2} + \theta_0} + \frac{(1 - \theta_0)}{n}.
\]

We now give a more precise derivation of the bound (65), adapting the main line of argument of [6] to this setting, using a variation of Proposition 3.5 above (see Proposition 6.3) and the bounds of Theorem 1.1 (B) in lieu of [6, Theorem 2].

6.2. Reductions via approximate functional equations. To make the sketch above precise, we first review how to use the approximate functional equation to reduce the study of the value \( L(1/2, \Pi \otimes \xi) \) appearing in the definition of \( S \) to finite sums analogous to those described in [6, (75)]. Let us write \( \Lambda(s, \Pi \otimes \chi) = L(s, \Pi_\infty \otimes \chi_\infty) \) of \( L(s, \Pi \otimes \chi) \) to denote the standard \( L \)-function of the \( \text{GL}_n(A_F) \)-automorphic representation \( \Pi \otimes \chi \), where the archimedean component \( L(s, \Pi_\infty \otimes \chi_\infty) \) is defined by the product

Here, the second product on the right hand side runs over the real places of \( F \), we use the standard notation \( \Gamma_R(s) = \pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \), we view the Hecke character in terms of its corresponding idele class character \( \chi = \otimes_v \chi_v \), and we use the symbols \( \mu_i(\Pi_\infty) \) to denote the archimedean Satake parameters of \( \Pi_\infty \). Hence, the generalized Ramanujan-Petersson conjecture predicts that \( \Re(\mu_i(\Pi_\infty)) = 0 \) for each index \( i \), and we have the uniform upper bound

\[
\max_{1 \leq i \leq n} \Re(\mu_i(\Pi_\infty)) \leq \frac{1}{n^2 + 1}
\]

for any dimension \( n \geq 2 \) thanks to the theorem(s) of Luo-Rudnick-Sarnak [26], [27]. Note that if \( \chi \) is a wide ray class character, equivalently if \( \chi_\infty \) is the trivial character, then we have the useful identification \( L(s, \Pi_\infty) = L(s, \Pi_\infty \otimes \chi_\infty) \). In any case, \( \Lambda(s, \Pi \otimes \chi) \) has a well-known analytic continuation to all \( s \in \mathbb{C} \), and satisfies the function equation

\[
(\mathcal{N}(j(\Pi, \mathcal{D}_F^\infty)q^n)^s L(s, \Pi \otimes \chi) = \epsilon(1/2, \Pi \otimes \chi) (\mathcal{N}(j(\Pi, \mathcal{D}_F^\infty)q^n)^{1/2}) L(1 - s, \Pi \otimes \chi^{-1}),
\]

where \( j(\Pi) \subset \mathcal{O}_F \) denotes the conductor of \( \Pi \), \( \mathcal{D}_F \subset \mathcal{O}_F \) the discriminant of \( F \), \( \epsilon(1/2, \Pi \otimes \chi) \in \mathbb{S}^1 \) the root number of \( \Lambda(s, \Pi \otimes \chi) \), and \( \Pi = \otimes_v \tilde{\Pi}_v \) the contragredient representation. Following the discussions in [6] and [21], let us consider the corresponding analytic conductor \( C(\Pi \otimes \chi) = C(\Pi \otimes \chi \otimes |\det^{s - 1/2}|_{s = 1/2} \) of
We have the following estimates for the central value $L(1/2, \Pi \otimes \chi)$ in this setting thanks to the main result of Harcos [21].

**Theorem 6.1.** There exists a complex function $V : \mathbb{R}_{>0} \rightarrow \mathbb{C}$ and a complex number of modulus one $u \in S^1$ depending only on the archimedean local parameters $\chi_v(-1)\mu_v(\Pi_{\infty})$ for $v$ ranging over real places of $F$ and $1 \leq i \leq n$ such that

$$L(1/2, \Pi \otimes \chi) = \sum_{m \in \mathcal{O}_F} \frac{c_{\Pi}(m)\chi(m)}{Nm^2} V \left( \frac{Nm}{C(\Pi \otimes \chi)^{1/2}} \right) + u \cdot \epsilon(1/2, \Pi \otimes \chi) \sum_{m \in \mathcal{O}_F} \frac{\tau_{\Pi}(m)\chi(m)}{Nm^2} V \left( \frac{Nm}{C(\Pi \otimes \chi)^{1/2}} \right).$$

The function $V$ and its derivatives $V^{(j)}$ for each $j \geq 1$ satisfy the following uniform decay estimates:

$$V(y) = \begin{cases} 1 + O_\sigma(y^\sigma) & \text{as } y \rightarrow 0 \text{ for } 0 < \sigma < 1/(n^2 + 1) \\ O_\sigma(y^{-\sigma}) & \text{as } y \rightarrow \infty \text{ for any } \sigma > 0, \end{cases}$$

and

$$V^{(j)}(y) = O_{\sigma,j}(y^{-\sigma}) \text{ as } y \rightarrow \infty \text{ for } \sigma > j - 1/(n^2 + 1).$$

Here, the implied constants depend only on $\sigma$, $j$, $m$, and the degree $d = [F : \mathbb{Q}]$. As well, the region $0 < \sigma < 1/(n^2 + 1)$ in the first estimate can be widened to $0 < \sigma < 1/2$ if we known that $\Re(\mu_v(\Pi_{\infty})) = 0$ for any choice $1 \leq i \leq n$.

**Proof.** See [21, Theorem 1]. To be more precise about the definition of $V(y)$, let $H(s)$ be an entire function of $s \in \mathbb{C}$ which is bounded as $H(s) \ll_{\sigma,A} (1 + |s|)^{-A}$ for any choice $A > 0$, where $\sigma = \Re(s)$. Assume that $H(0) = 1$, and also that $H(s) = H(-s) = \overline{H(-s)}$. Note that such a function can be realized at the Mellin transform $H(s) = \int_0^{\infty} h(x)x^s\frac{dx}{x}$ of a smooth function $h : \mathbb{R}_{>0} \rightarrow \mathbb{R}$ having total mass one with respect to the measure $dx/x$ which satisfies the functional equation $h(x) = h(x^{-1})$, and whose derivatives decay faster than any negative power of $x$. Consider the function defined by

$$K(s, \Pi_{\infty} \otimes \chi_{\infty}) = \frac{1}{2} \left( \frac{N(\mathcal{D}_F\Pi)q^n}{C(\Pi \otimes \chi)^{1/2}} \right)^s \cdot \frac{L(1/2 + s, \Pi_{\infty} \otimes \chi_{\infty})}{L(1/2 - \Pi_{\infty} \otimes \chi_{\infty})} \cdot \frac{L(1/2, \Pi_{\infty} \otimes \chi_{\infty})}{L(1/2, \Pi_{\infty} \otimes \chi_{\infty})} + \frac{1}{2} C(\Pi \otimes \chi)^{1/2}.$$

The cutoff function $V(y)$ is then defined explicitly by the contour integral

$$V(y) = \int_{\Re(s)=\sigma} K(s, \Pi_{\infty} \otimes \chi_{\infty})H(s) \left( yC(\Pi \otimes \chi)^{1/2} \right)^{-s} ds \frac{2\pi i}{2\pi i}.$$

Moreover, $u \in S^1$ is given by the ratio $u = L(1/2, \Pi_{\infty} \otimes \chi_{\infty})/L(1/2, \Pi_{\infty} \otimes \chi_{\infty})$.

The result can be used to derive the following standard preliminary estimates.

**Corollary 6.2.** We have the following estimates for the central value $L(1/2, \Pi \otimes \chi)$.

(A) We have for any choice of $\varepsilon > 0$ the uniform convexity bound

$$L(1/2, \Pi \otimes \chi) \ll_{\varepsilon} C(\Pi \otimes \chi)^{1/2 + \varepsilon},$$

where the implied constant depends only on $\varepsilon$, $n$, and $d$.  

\[\square\]
(B) Writing $V^*(y)$ to denote the restriction of $V(y)$ to the compact interval $[2^{-1}, 2]$, we have for any choice of $\varepsilon > 0$ the estimate

$$L(1/2, \Pi \otimes \chi) \ll_{\Pi, \chi, \varepsilon} Nq^\varepsilon \max_{Y \leq C(\Pi \otimes \chi)^{2+\varepsilon}} \left| \sum_{m \in \mathcal{O}_F} \frac{c_H(m)}{Nm^{2}} V^* \left( \frac{Nm}{Y} \right) \right|.$$

**Proof.** See [21, Corollary 2] and [6, (75)] (cf. [7, § 5.1]).

Hence for some constant $c = c(\Pi, \chi, \varepsilon), \varepsilon > 0$, we can begin with the latter bound

$$L(1/2, \Pi \otimes \chi) \ll_{\Pi, \chi, \varepsilon} Nq^\varepsilon \max_{Y \leq Nq^{2+\varepsilon}} \left| \sum_{m \in \mathcal{O}_F} \frac{c_H(m)\chi(m)}{Nm^{2}} V^* \left( \frac{Nm}{Y} \right) \right|.$$

To estimate this latter quantity in the style outlined above (following [6, §3.3]), we must first split in the $\mathfrak{m}$-sum into its corresponding narrow ideal class components. Let us therefore fix a narrow ideal class of $\mathbb{F}$, together with an integer representative $\eta$ coprime to the conductor $q$ whose norm we can and do assume is bounded by $N\eta \ll_{\varepsilon} Nq^\varepsilon$. It will do to estimate the corresponding sum over totally positive nonzero ideals

$$\sum_{\gamma \in \mathfrak{C}_y \mod \mathcal{O}_F^*} \frac{c_H(\eta^{-1})\chi(\eta^{-1})}{N(\eta^{-1})^{2}} V^* \left( \frac{N\gamma}{Y} \right)$$

for $Y \ll_{\varepsilon} Nq^{2+\varepsilon}$, where $O^X_{\mathbb{F},+} \subset O^X_{\mathbb{F}}$ denotes the subgroup of totally positive units. Note that we shall always take sums over nonzero ideals in the subsequent discussion, but that we suppress this condition from the notations for simplicity.

Recall that we write $(\tau_j)_{j=1}^d$ to denote the collection of real embeddings of $\mathbb{F}$. Let us fix a fundamental domain $\mathcal{F}_0$ for the action of $O^X_{\mathbb{F},+}$ on the hyperboloid

$$\mathfrak{h} = \{ y \in F^X_{\infty, +} : N\eta = |y| = 1 \},$$

where again $F^X_{\infty, +} \subset F^X_{\infty}$ denotes the totally positive sub-plane. We assume that this action is fixed so that its image under the map

$$F^X_{\infty, +} \longrightarrow \mathbb{R}^d, \quad y \mapsto (\log \tau_1(y), \cdots, \log \tau_d(y))$$

is a fundamental parallotope for the image of the totally positive units $O^X_{\mathbb{F},+}$ under the same map. Note that the cone

$$\mathcal{F} = F^\text{diag}_{\infty, +} \mathcal{F}_0$$

is a fundamental domain for the action of the $O^X_{\mathbb{F},+}$ on $F^X_{\infty, +}$. To use these choices of fundamental domains to describe our sum (67) in a convenient way for estimates, let us first fix a smooth and compactly supported function $G_0 : \mathfrak{h} \longrightarrow \mathbb{C}$ satisfying

$$\sum_{u \in O^X_{\mathbb{F},+}} G_0(uy) = 1, \quad \text{for any } y \in \mathfrak{h}.$$

We extend this function $G_0$ to all $y \in F^X_{\infty, +}$ by defining

$$G(y) = G_0 \left( \frac{y}{Ny^2} \right).$$

Note that the support of the hyperboloid function $G_0$ is contained in some box $[c_1, c_2]^d \subset F^X_{\infty, +}$, and that the support of $G$ is contained in the cone $F^\text{diag}_{\infty, +} [c_1, c_2]^d$ of this box. Using these conventions, we can rewrite (67) as the finite sum

$$\sum_{0 \ll \gamma \in \eta} \frac{c_H(\eta^{-1})\chi(\eta^{-1})}{N(\eta^{-1})^{2}} G(\gamma)V^* \left( \frac{N\gamma}{Y} \right).$$

To be clear, since $\eta$ is a lattice in $F_{\infty}$, this latter sum is seen easily to be finite from the support of the chosen function $G$. More precisely, it vanishes outside of the box $[2c_1Y^{1/2}, 2c_2Y^{1/2}]^d$. Let us now fix a smooth
function $W : F_r^\infty \to \mathbf{C}$ with support contained in the box $[\frac{1}{2} c_1, 2 c_2]^d$, and such that $W(y) = 1$ on $[\frac{1}{2} c_1, 2 c_2]^d$. Making these choices, we can express the second moment for each of these sums $L$ so that

$$
(71)
$$

$$
W
$$
in the box $[1 \xi \mod q]$ in the interval $L$ covered by a finite number of $O_r^\infty$.

Before going on, we record that the functions $G(y)V^\ast(|y|) = G(y)V^\ast(Ny)$ and $W(y)$ appearing in this latter expression are smooth and compactly supported. Moreover, since $\chi_\infty(y) = \prod_{j=1}^d y_j^w$ for some fixed vector $s = (s_j)_{j=1}^d \in (i\mathbb{R})^d$, it is easy to see that we have the following bounds as functions of $w = (w_j)_{j=1}^d \in (i\mathbb{R})^d$, for any choices of constant $A > 0$ and integer $d$-tuple $\mu = (\mu_j)_{j=1}^d \in \mathbb{Z}_{\geq 0}^d$:

$$
(69)
$$
and

$$
(70)
$$

Let us now fix a vector $w = (w_j)_{j=1}^d \in (i\mathbb{R})^d$ in the integral on the right hand side of (68), deferring the integration until the last step in the argument. Given a character $\xi : (O_F / qO_F)^\times \to \mathbb{S}^1$, we then consider the sum defined by

$$
(71)
$$
so that $L_\xi(w)$ is the sum appearing in the integrand on the right hand side of (68). The next step is to form and estimate an amplified second moment for each of these sums $L_{\chi_j}(w)$, using a variation of Proposition 3.5 to describe the off-diagonal term.

6.3. Amplified second moments. Let us now focus on the sum (71), which observe has support contained in the box $[\frac{1}{2} c_1 Y^{\frac{1}{2}}, 3 c_2 Y^{\frac{1}{2}}]^d$. Note as well that the cone $C \subset F_r^\infty$ of this box is independent of $Y$, and covered by a finite number of $O_r^\infty$-translates of the fundamental domain $F$. Hence, fixing a parameter $L \geq \log(Nq)$, and writing $l \sim L$ to denote the sum over totally positive principal prime ideals $l \mid q \subset O_F$ in the interval $l \leq Nl \leq 2L$, we consider the amplified second moment defined by taking the sum over all characters $\xi : (O_F / qO_F)^\times$,}

$$
S = \sum_{\xi \mod q} \sum_{l \sim L} |\xi(l)\chi_f(l)|^2 |L_\xi(w)|^2 .
$$

It is easy to check that for any $\varepsilon > 0$, we have the lower bound

$$
\# \{ l \mid q \subset O_F \text{ principal, prime, and totally positive with } L \leq Nl \leq 2L \} \gg_{\varepsilon} Nq^{-\varepsilon}L.
$$
This implies the lower bound $S \gg \varepsilon Nq^{r}L^{2}|\mathcal{L}_{x}(w)|^{2}$, and hence the upper bound

$$|\mathcal{L}_{x}(w)|^{2} \ll \varepsilon \frac{Nq^{r}}{L^{2}} \sum_{\xi \mod q} \left| \mathcal{L}_{\xi}(w) \sum_{l \in P \cap L} \xi(l) \overline{\chi}_{f}(l) \right|^{2},$$

which by Plancherel’s formula for $(O_{F}/qO_{F})$ is the same as

$$|\mathcal{L}_{x}(w)|^{2} \ll \varepsilon \frac{Nq^{r}}{L^{2}} \sum_{x \in (O_{F}/qO_{F})^{\times}} \sum_{l \in P \cap L} \chi_{f}(l) \overline{\chi}_{f}(l) \sum_{\gamma \in \mathbb{G} \cap C_{\gamma}} \frac{c_{\Pi}(\gamma \eta^{-1})}{N(\gamma \eta^{-2})^{1/2}} W_{w}\left(\frac{\gamma}{Y^{1/2}}\right) W_{w}\left(\frac{\gamma}{Y^{1/2}}\right).$$

Here, we put $\varphi(q) = \#(O_{F}/qO_{F})^{\times}$. Now, extending the summation to all classes $x \mod q$ and opening up the square, we derive the more explicit upper bound

$$|\mathcal{L}_{x}(w)|^{2} \ll \varepsilon \frac{Nq^{1+r}}{L^{2}} \sum_{l_{1}, l_{2} \in P \cap L} \chi_{f}(l_{1}) \chi_{f}(l_{2}) \sum_{\gamma_{1}, \gamma_{2} \in \mathbb{G} \cap C_{\gamma}} \frac{c_{\Pi}(\gamma_{1} \eta^{-1})c_{\Pi}(\gamma_{2} \eta^{-1})}{N(\gamma_{1} \gamma_{2} \eta^{-2})^{1/2}} W_{w}\left(\frac{\gamma_{1}}{Y^{1/2}}\right) W_{w}\left(\frac{\gamma_{2}}{Y^{1/2}}\right).$$

(72)

Let us first consider the diagonal term in this latter expression (72) coming from the contribution of $l_{1} \gamma_{1} - l_{2} \gamma_{2} = 0$. This contribution is seen easily to be bounded above for any $\varepsilon > 0$ (uniformly in the choice of vector $w$ in $(iR)^{d}$) by the quantity

$$\ll \varepsilon \frac{Nq^{1+r}}{L^{2}} \sum_{l_{1}, l_{2} \in P \cap L} \sum_{\gamma_{1}, \gamma_{2} \in \mathbb{G} \cap C_{\gamma}} \frac{|c_{\Pi}(\gamma \eta^{-1})|^{2}}{N(\gamma \eta^{-2})^{1/2}} \# \{(l', \gamma') \in (O_{F} \cap \mathcal{F}) \times (\eta \cap \mathcal{C}) : l' \gamma' \equiv l \gamma\}.$$

Now, it is easy to see that

$$\sum_{l_{1}, l_{2} \in P \cap L} \sum_{\gamma_{1}, \gamma_{2} \in \mathbb{G} \cap C_{\gamma}} \# \{(l', \gamma') \in (O_{F} \cap \mathcal{F}) \times (\eta \cap \mathcal{C}) : l' \gamma' \equiv l \gamma\} \ll \varepsilon (LY)^{r}.$$

On the other hand, we have for any $Y \in \mathbb{R}_{>0}$ the well-known bound

$$\sum_{m \in \mathcal{O}_{F} \cap \mathcal{F} \cap \mathcal{C}} |c_{\Pi}(m)|^{2} \leq C_{\Pi} Y$$

as $Y \to \infty$ for some constant $C_{\Pi} > 0$ depending only on $\Pi$ (see e.g. [26, (14)]). Using these estimates, we see that the diagonal contribution is bounded above by

(73)

$$\ll \varepsilon \frac{Nq^{1+r}}{L^{2}} \# \{(l \subset O_{F} : l \sim L) \sum_{m \in \mathcal{O}_{F} \cap \mathcal{F} \cap \mathcal{C}} |c_{\Pi}(m)|^{2} \frac{N_{m}}{N_{m}} \ll \varepsilon, \frac{Nq^{1+r}}{L}.$$

Let us now consider the heart of the matter, which is the remaining off-diagonal contribution to the quantity in the right hand side of (72). Here again, we follow the reduction steps and setup of [6, §3.3] closely. Fix a box $[c_{3}, c_{4}]^{d} \in F_{\infty,+}^{\times}$ containing the fundamental domain $\mathcal{F}_{0}$. It is easy to see that the totally positive principal ideals $l_{1}, l_{2}$ in the sum are contained in the box $l_{1}, l_{2} \subset [c_{3} L^{1/2}, 2c_{4} L^{1/2}]^{d}$, and the totally positive $F$-integers $\gamma_{1}, \gamma_{2}$ in the box $\gamma_{1}, \gamma_{2} \subset [\alpha c_{1} Y^{1/2}, 3c_{2} Y^{1/2}]^{d}$, so that

$$l_{1} \gamma_{1} - l_{2} \gamma_{2} \in K := [-6c_{2} c_{4}(LY)^{1/2}, 6c_{2} c_{4}(LY)^{1/2}]^{d}.$$

Using this definition of box region $K$, we can describe the inner sum corresponding to a given pair $l_{1}, l_{2}$ in the off-diagonal sum of (72) more explicitly in terms of congruences modulo $F$-integers as

(74)

$$\sum_{\alpha \in \eta \cap K} \sum_{\gamma_{1}, \gamma_{2} \in \mathbb{G} \cap C_{\gamma_{1}} \cap \gamma_{2}} \frac{c_{\Pi}(\gamma_{1} \eta^{-1})c_{\Pi}(\gamma_{2} \eta^{-1})}{N(\gamma_{1} \gamma_{2} \eta^{-2})^{1/2}} W_{w,1}\left(\frac{l_{1} \gamma_{1}}{(LY)^{1/2}}\right) W_{w,2}\left(\frac{l_{2} \gamma_{2}}{(LY)^{1/2}}\right).$$

60
where for \( i = 1, 2 \) we use the functions \( W_{w,i} : \mathbb{F}_\infty^\times \to \mathbb{C} \) defined on \( y \in \mathbb{F}_\infty^\times \) by

\[
W_{w,i}(y) = \begin{cases} 
W_w \left( t^{-1} L^2 y \right) & \text{if } y \in \mathbb{F}_\infty^\times, \\
0 & \text{otherwise}.
\end{cases}
\]

Note that these functions \( W_{w,i} \) are smooth and supported on \([\frac{1}{3} c_1 c_3, 6 c_2 c_4]\) and moreover (as functions of the vector \( w = (w_j)_{j=1}^d \in (\mathbb{R})^d \)) bounded via (70) as

\[
\partial_{y_1}^{\mu_1} \cdots \partial_{y_d}^{\mu_d} W_{w,i} \ll_{\mu} \prod_{j=1}^d (1 + |w_j|)^{\mu_j}
\]

for any \( d \)-tuple \( \mu = (\mu_j)_{j=1}^d \in \mathbb{Z}_{\geq 0} \). We now derive an integral presentation for this sum (74) in the style of [6, Theorem 2, (115)], using a variation of Propositions 3.5 and 4.2 above with the projection operator \( \mathbb{P}_1^n \). Hence, we first revert to the adelic setup above, fixing a pure tensor \( \varphi = \otimes \varphi_v \in \mathcal{V}_1 \) as in Proposition 3.5. Recall that the projected form \( \mathbb{P}_1^n \varphi \) defines a cuspidal \( L^2 \)-automorphic form on the mirabolic subgroup \( P_2(\mathbb{A}_F) \subset GL_2(\mathbb{A}_F) \) having the Fourier-Whittaker described in Proposition 2.1 and Corollary 2.3, and also that we use the strong approximation and Iwasawa decomposition to lift to an \( L^2 \)-automorphic form in the space \( L^2(GL_2(F) \backslash GL_2(\mathbb{A}_F), \mathbf{1}) \) as in Definition 2.6, Proposition 2.7, and Proposition 2.7. Given a nonzero \( F \)-integer \( t \), and using the same notation to denote a fixed finite idele representative \( t \in \mathbb{A}_{F,f}^\times \), let us now consider the operator \( R_{(t)} \) defined on \( \mathbb{P}_1^n \varphi \) for a mirabolic matrix \( p \in P_2(\mathbb{A}_F) \) by

\[
R_{(t)} \mathbb{P}_1^n \varphi(p) = \mathbb{P}_1^n \varphi \left( p \left( \begin{array}{cc} t^{-1} & 1 \\ 0 & 1 \end{array} \right) \right)
\]

Hence, for \( x \in \mathbb{A}_F \) an adele, and \( y = y_f y_\infty \in \mathbb{A}_F^\times \) an idele, we deduce from Proposition 2.1 or more simply Proposition 2.3 that we have the Fourier expansion

\[
R_{(t)} \mathbb{P}_1^n \varphi(x) = \left| yt^{-1} \right|^{-\left( \frac{n-2}{2} \right)} \sum_{\gamma \in \mathbb{F}_\infty} c_{\mathbb{F}_\infty} (\gamma y f t^{-1}) \left| \gamma y f t^{-1} \right|^{-\frac{n-2}{2}} W_\varphi (\gamma y_\infty) \psi(\gamma x).
\]

This expansion allows us to derive the following key integral presentations.

**Proposition 6.3.** Let \( \Pi_i \) for \( i = 1, 2 \) be cuspidal automorphic representations of \( GL_n(\mathbb{A}_F) \). Fix pure tensors \( \varphi_i \in \mathcal{V}_i \), whose nonarchimedean local components are essential Whittaker vectors. Fix a nonzero integral ideal \( \eta \subset \mathcal{O}_F \), and let us use the same notation to denote a fixed finite idele representative \( \eta \in \mathbb{A}_{F,f}^\times \). Fix nonzero \( F \)-integers \( l_1, l_2, \) and \( \alpha \). Assume that \( l_i \) and \( \alpha \) are coprime for \( i = 1, 2 \). Let \( y = y_f y_\infty \) be an idele having some specified archimedean component \( y_\infty = Y_\infty^{-1} \), and nonarchimedean component \( y_f = \eta^{-1} \). Then, the Fourier-Whittaker coefficient at \( \alpha \) of the \( L^2 \)-automorphic form \( \Phi = R_{(l_1)} \mathbb{P}_1^n \varphi_1 \cdot R_{(l_2)} \mathbb{P}_1^n \varphi_2 \) on \( P_2(\mathbb{A}_F) \) has the expansion

\[
\int_{\mathbb{A}_F^\times / \mathbb{A}_F} R_{(l_1)} \mathbb{P}_1^n \varphi_1 \cdot R_{(l_2)} \mathbb{P}_1^n \varphi_2 \left( \left( \begin{array}{cc} y_f & x \\ 0 & 1 \end{array} \right) \right) \psi(-\alpha x) dx = \left| \eta Y_\infty \right|^{-n-2} \sum_{\gamma_1, \gamma_2 \in \mathcal{O}_F} c_{\mathcal{O}_F} (\gamma_1 \eta^{-1} \gamma_2 \eta^{-1}) \left| \gamma_1 \gamma_2^{-2} \right|^{-\frac{n-2}{2}} W_{\varphi_1} (\gamma_1 l_1 Y_\infty) W_{\varphi_2} (\gamma_2 l_2 Y_\infty^{-1})
\]
Proof. We open up Fourier-Whittaker expansions and evaluate via orthogonality of additive characters on $A_F/F$ as usual to find that

$$\int_{A_{F/F}} R_{(l_1)} \mathbb{P}_{n}^{1} \cdot R_{(l_2)} \mathbb{P}_{n}^{1} \varphi_1 \left( \begin{pmatrix} (\eta Y_{\infty})^{-1} & x \\ 1 & 1 \end{pmatrix} \right) \psi(-\alpha x) dx$$

$$= \int_{A_{F/F}} \mathbb{P}_{n}^{1} \varphi_1 \left( \begin{pmatrix} (l_1 Y_{\infty})^{-1} & x \\ 1 & 1 \end{pmatrix} \right) \mathbb{P}_{n}^{1} \varphi_2 \left( \begin{pmatrix} (l_2 Y_{\infty})^{-1} & x \\ 1 & 1 \end{pmatrix} \right) \psi(-\alpha x) dx$$

$$= \|l_1 Y_{\infty}\|^{n-2} \sum_{\gamma_1 \in F^\times} \frac{c_{n_1}(\gamma_1 l_1^{-1} \eta^{-1})}{|\gamma_1 l_1^{-1} \eta^{-1}|^{n-2}} W_{\varphi_1} \left( \frac{\gamma_1}{Y_{\infty}} \right) |l_2 Y_{\infty}|^{n-2} \sum_{\gamma_2 \in F^\times} \frac{c_{n_2}(\gamma_2 l_2^{-1} \eta^{-1})}{|\gamma_2 l_2^{-1} \eta^{-1}|^{n-2}} W_{\varphi_2} \left( \frac{\gamma_2}{Y_{\infty}} \right) \times \int_{A_{F/F}} \psi(y_{\infty} - \gamma_2 x - \alpha x) dx$$

$$= \|\eta Y_{\infty}\|^{n-2} \sum_{\gamma_1, \gamma_2 \in F^\times} \frac{c_{n_1}(\gamma_1 l_1^{-1} \eta^{-1})}{|\gamma_1 l_1^{-1} \eta^{-1}|^{n-2}} |\gamma_1 l_1^{-1} \eta^{-1}|^{n-2} W_{\varphi_1} \left( \frac{\gamma_1}{Y_{\infty}} \right) |\gamma_2 l_2^{-1} \eta^{-1}|^{n-2} W_{\varphi_2} \left( \frac{\gamma_2}{Y_{\infty}} \right)$$

Since the latter sum is supported only on $F$-integers in $\eta$, we derive the identity. 

\[ \Box \]

Corollary 6.4. Taking $Y_{\infty} \in F_{\infty}^\times$ of idele norm $|Y_{\infty}| = (YL)^{\frac{1}{2}}$ with $\Pi_1 = \Pi$ and $\Pi_2 = \widetilde{\Pi}$, let us choose pure tensors $\varphi_i \in V_{\Pi_i}$ in such a way that the corresponding archimedean local Whittaker functions $W_{\varphi_i} : F_{\infty}^\times \to \mathbb{C}$ for $i = 1, 2$ satisfy

$$W_{\varphi_i}(y_{\infty}) = \begin{cases} |l_i^{-1} y_{\infty}|^{n-2} W_{w_i}(y_{\infty}) / (l_i^{-1} y_{\infty} L_{\frac{1}{2}}^2) & \text{if } y_{\infty} \in F_{\infty,+}^\times, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we obtain the integral presentation

$$\int_{A_{F/F}} R_{(l_1)} \mathbb{P}_{n}^{1} \mathbb{P}_{n}^{1} \cdot R_{(l_2)} \mathbb{P}_{n}^{1} \mathbb{P}_{n}^{1} \varphi_1 \left( \begin{pmatrix} (\eta Y_{\infty} L)^{\frac{1}{2}} \right)^{-1} x \\ 1 \end{pmatrix} \right) \psi(-\alpha x) dx$$

$$= \|\eta\|^{n-2} \sum_{\gamma_1, \gamma_2 \in F^\times} \frac{c_{n}(\gamma_1 \eta^{-1})}{N(\gamma_1 \gamma_2 \eta^{-2})^{\frac{1}{2}}} W_{w_1} \left( \frac{l_1 \gamma_1}{(YL)^{\frac{1}{2}}} \right) W_{w_2} \left( \frac{l_2 \gamma_2}{(YL)^{\frac{1}{2}}} \right).$$

Proof. The stated identity is easy to check after making a direct substitution. 

\[ \Box \]

We also have the following less direct Corollary for the Fourier-Whittaker coefficient of the corresponding extended form $\Phi$ following Proposition 2.9 and Proposition 4.2 (B) above. Note that we fix a smooth partition of unity (16) and dyadic discussion as in the discussion above, and in particular reduce to working with compactly supported weight function as in Proposition 4.2 (B). Taking this standard reduction for granted, let us drop the additional notations from the discussion that follows for simplicity.

Corollary 6.5. Let $Y_{\infty} = (Y_{\infty,i})_{i=1}^{d} \in F_{\infty}^\times$ be a totally positive idele with idele norm $|Y_{\infty}| = (YL)^{\frac{d}{2}}$ exceeding that of the chosen nonzero $F$-integer $\alpha \in O_F$. Let us again consider $\Pi_1 = \Pi$ and $\Pi_2 = \widetilde{\Pi}$, and choose pure tensors $\varphi_i \in V_{\Pi_i}$ in such a way that the corresponding archimedean local Whittaker functions $W_{\varphi_i} : F_{\infty}^\times \to \mathbb{C}$ for $i = 1, 2$ satisfy

$$W_{\varphi_i}(y_{\infty}) = \begin{cases} |l_i^{-1} y_{\infty}|^{n-2} W_{w_i}(y_{\infty}) \left( \frac{1}{|l_i^{-1} y_{\infty}| L_{\frac{1}{2}}^2} \right) & \text{if } y_{\infty} \in F_{\infty,+}^\times, \\ 0 & \text{otherwise.} \end{cases}$$

Let us now write $\Phi_{l_1, l_2}$ to denote the $L^2$-automorphic form on the mirabolic subgroup $P_2(A_F)$ defined by $\Phi_{l_1, l_2} = R_{(l_1)} \mathbb{P}_{n}^{1} \mathbb{P}_{n}^{1} \cdot R_{(l_2)} \mathbb{P}_{n}^{1} \mathbb{P}_{n}^{1} \varphi_2$. However, let us also write $\Phi_{l_1, l_2} \in L^2(GL_2(F) \backslash GL_2(A_F), 1)^K$ to denote the
extension of $\Phi_{t_1,t_2}$ to an $L^2$-automorphic form on $\text{GL}_2(\mathbb{A}_F)$ via Theorem 2.5, Definition 2.6, and Proposition 2.7. Then, we obtain the integral presentation

$$\int_{I\subseteq [0,1]^d \subset F_\infty} \Phi_{t_1,t_2} \left( \left( \left( \eta(Y) L^\frac{1}{2} x_\infty \right) \frac{1}{1} \right) \psi(-\alpha x_\infty) dx_\infty \right)$$

$$= \int_{I\subseteq [0,1]^d \subset F_\infty} \Phi_{t_1,t_2} \left( \left( \left( \eta(Y) L^\frac{1}{2} x_\infty \right) \frac{1}{1} \right) \psi(-\alpha x_\infty) dx_\infty \right)$$

$$= |\eta|^{n-2} \sum_{\gamma_1,\gamma_2 \in \mathbb{N}} \frac{c_{\eta} (\gamma_1 \gamma_2^{-1}) \eta \gamma_2^{-1} - 1}{N(\gamma_1 \gamma_2^{-1})^\frac{1}{2}} W_{\Phi_{t_1,t_2}} \left( \left( \frac{l_1 \gamma_1}{(LY)^\frac{1}{2}} \right) W_{\Phi_{t_1,t_2}} \left( \left( \frac{l_2 \gamma_2}{(LY)^\frac{1}{2}} \right) \right) \right).$$

In particular, we may decompose this $L^2$-automorphic form $\Phi_{t_1,t_2}$ on $\text{GL}_2(\mathbb{A}_F)$, as its spectral coefficients are bounded by Proposition 4.9 (or the subsequent discussion of convolution with smoothing kernels) above.

Proof. We deduce the result from Proposition 6.3 and Corollary 6.4 using Proposition 2.9, using a minor variation of the argument given for Proposition 4.2 (B) above.

Now, Corollaries 6.4 and 6.5 allow us to express (74) as a sum over Fourier-Whittaker coefficients of the automorphic forms $\Phi_{t_1,t_2} = R(t_1)_{\frac{1}{2}} F_{1} \cdot R(t_2)_{\frac{1}{2}} F_{2}$ on $P_2(\mathbb{A}_F)$, and more generally (for the generic case of $n \geq 2$) of the $L^2$-automorphic form $\Phi_{t_1,t_2}$ on $\text{GL}_2(\mathbb{A}_F)$. In particular, for the generic case of $n \geq 2$ (distinct from the standard case of $n = 2$ as treated by Blomer-Harcos [6]), we have that

$$|\eta|^{2-n} \sum_{\gamma_1,\gamma_2 \in \mathbb{N}} \sum_{\alpha \in \mathbb{N}^{\mathbb{R}}} \int_{I\subseteq [0,1]^d \subset F_\infty} \Phi_{t_1,t_2} \left( \left( \left( \eta(Y) L^\frac{1}{2} x_\infty \right) \frac{1}{1} \right) \psi(-\alpha x_\infty) dx_\infty \right).$$

In particular, we may decompose this form spectrally via the argument of Proposition 4.9 above to derive bounds of the form described in Theorem 1.1 (B) to obtain a suitable bound this sum (75), i.e. so as to derive a suitable bound for the off-diagonal contribution in (72) above. In this way, we derive the following key estimate. Recall that we write $0 \leq \theta_0 < 1/2$ to denote the best known approximation towards the generalized Ramanujan-Petersson conjecture for $\text{GL}_2(\mathbb{A}_F)$-automorphic forms, with $\theta_0 = 0$ conjectured, and $\theta_0 = 7/64$ known thanks to the theorem of Blomer-Brunley [5].

**Theorem 6.6.** Assume that our initial cuspidal $\text{GL}_n$-automorphic form $\Pi = \otimes_v \Pi_v$ is irreducible. The sum $L_{\chi_f}(w)$ defined in (71) is bounded above for any $\varepsilon > 0$ as

$$|L_{\chi_f}(w)|^2 \ll_{\Pi_v} \frac{1}{n q^{1+\varepsilon}} L^{-1} + \frac{1}{n q^{2(\frac{1}{2}+\theta_0)+\varepsilon}} L^{\varepsilon+\theta_0+\varepsilon}.$$

Proof. Taking for granted the reductions leading up to the diagonal bound (73) and the sum defined by the box region $\mathcal{K}$ in (74), we reduce to bounding the sum

$$\sum_{\gamma_1,\gamma_2 \in \mathbb{N}} \sum_{\gamma_1,\gamma_2 \in \mathbb{N}} \frac{c_{\gamma_1 \gamma_2^{-1}} \eta \gamma_2^{-1} - 1}{N(\gamma_1 \gamma_2^{-1})^\frac{1}{2}} W_{\Phi_{t_1,t_2}} \left( \left( \frac{l_1 \gamma_1}{(LY)^\frac{1}{2}} \right) W_{\Phi_{t_1,t_2}} \left( \left( \frac{l_2 \gamma_2}{(LY)^\frac{1}{2}} \right) \right) \right),$$

which by (75) can be written equivalently as a sum of Fourier-Whittaker coefficients

$$|\eta|^{2-n} \sum_{\gamma_1,\gamma_2 \in \mathbb{N}} \sum_{\alpha \in \mathbb{N}^{\mathbb{R}}} \int_{I\subseteq [0,1]^d \subset F_\infty} \Phi_{t_1,t_2} \left( \left( \left( \eta(Y) L^\frac{1}{2} x_\infty \right) \frac{1}{1} \right) \psi(-\alpha x_\infty) dx_\infty \right).$$

Decomposing each of the smooth and compactly supported $L^2$-automorphic forms $\Phi_{t_1,t_2}$ on $\text{GL}_2(\mathbb{A}_F)$ spectrally in the style of the proofs of the shifted convolution sums estimates of Theorem 1.1 above, we thus reduce to bounding the corresponding sums in the decomposition

$$|\eta|^{2-n} \sum_{\gamma_1,\gamma_2 \in \mathbb{N}} \sum_{\alpha \in \mathbb{N}^{\mathbb{R}}} \int_{I\subseteq [0,1]^d \subset F_\infty} \Phi_{t_1,t_2} \left( \left( \left( \eta(Y) L^\frac{1}{2} x_\infty \right) \frac{1}{1} \right) \psi(-\alpha x_\infty) dx_\infty \right)$$

$$= |\eta|^{2-n} \sum_{\gamma_1,\gamma_2 \in \mathbb{N}} \sum_{\alpha \in \mathbb{N}^{\mathbb{R}}} \sum_{\gamma \in \mathbb{N}^{\mathbb{R}}} \frac{c_{\gamma \alpha^{-1}}}{N(\alpha^{-1})^\frac{1}{2}} W_{\Phi_{t_1,t_2}} \left( \left( \frac{\alpha(L)^\frac{1}{2}}{\alpha(L)^\frac{1}{2}} \right) \right).$$
Again, the spectral coefficients \( \langle \Phi_{l_1, l_2}, \varphi_r \rangle \) in this decomposition can be bounded via Proposition 4.9 above, and we use the shorthand notation introduced in the proof of Theorem 1.1 above for smooth basis elements \( \varphi_r \in \mathcal{B} \), i.e. for simplicity of exposition we do not write out the contribution from the continuous spectrum explicitly. Note again that we could also take the convolution of \( \Phi_{l_1, l_2} \) with a smoothing kernel, as indicated in the remark after Proposition 4.9, to reduce to estimating the sum on right hand side of this expression.

Now, recall that we choose the narrow class representative \( \eta \) so that \( Nq \ll_{\varepsilon} N\eta^c \). Using Weyl's law as described above (cf. [6, \S 3.1]), we argue that the basic sum over \( \varphi_r \in \mathcal{B} \) in this latter expression can be viewed as having approximately \( L^2 \) many terms. Putting these observations together, we then use the proof of Theorem 1.1 (B) to argue that the latter sum is bounded above for any choice of \( \varepsilon > 0 \) by the quantity

\[
\ll_{\Pi, \varepsilon} N\eta^c L^2 \cdot L^2 \sum_{\alpha \in q\mathbb{Z} \cap \mathcal{X}} N(\alpha\eta^{-1})^{\theta_0 + \varepsilon} \left( \frac{N\alpha}{YL} \right)^{\frac{1}{2} - \frac{\theta_0}{2} - \varepsilon}.
\]

To estimate the \( \alpha \)-sum in this latter expression, we observe (cf. [6, pp. 43-46]) that it can be expressed as a sum over integral ideals \( m \subset \mathcal{O}_F \) of absolute norm bounded by \( Nm \leq (LY)/N(\eta q) \leq L N\eta^{n-2} \). Hence, the sum (76) is bounded above by

\[
\ll_{\Pi, \varepsilon} N\eta^c L^4 \cdot (LY)^{\theta_0 - \frac{\varepsilon}{2} + \varepsilon} N\eta^{\theta_0} (LN\eta^{n-2})^{1 + \theta_0} = N\eta^c L^4 L^{\frac{1}{2} + \theta_0 - \varepsilon} N\eta^{\frac{3}{2} (\frac{1}{2} + \theta_0) - 1}.
\]

It follows that the corresponding off-diagonal term in (72) is bounded above by

\[
\ll_{\Pi, \varepsilon} \frac{N\eta^{1+\varepsilon}}{L^2} \cdot N\eta^c L^4 L^{\frac{1}{2} + \theta_0 - \varepsilon} N\eta^{\frac{3}{2} (\frac{1}{2} + \theta_0) - 1} = N\eta^{\frac{3}{2} (\frac{1}{2} + \theta_0) + \varepsilon} L^{\frac{3}{2} + \theta_0 + \varepsilon}.
\]

Putting this together with the diagonal bound (73) then implies the stated estimate. \( \square \)

Using this, we derive the following estimate for the central value \( L(1/2, \Pi \otimes \chi) \).

**Corollary 6.7.** Let \( \Pi = \otimes \Pi_v \) be an irreducible cuspidal \( \text{GL}_n(A_F) \)-automorphic representation, and \( \chi \) a Hecke character of \( F \) of conductor \( q \subset \mathcal{O}_F \). We have for any choice of parameters \( L = N\eta^u \) with \( 0 \leq u \leq 1 \) and \( \varepsilon > 0 \) the estimate

\[ |L(1/2, \Pi \otimes \chi)|^2 \ll_{\Pi, \chi, \varepsilon} N\eta^{1+\varepsilon} L^{-1} + N\eta^{\frac{3}{2} (\frac{1}{2} + \theta_0) + \varepsilon} L^{\frac{3}{2} + \theta_0 + \varepsilon}. \]

For instance, taking \( u = 1/4 - \theta_0/2 \) gives the estimate

\[ L(1/2, \Pi \otimes \chi) \ll_{\Pi, \chi, \varepsilon} N\eta^{\frac{3}{2} + \frac{\theta_0}{2}} + N\eta^{\frac{3}{2} (\frac{1}{2} + \theta_0) + \frac{1}{2} - \frac{\theta_0}{2} + \varepsilon}. \]

Taking \( u = (1 - 6\theta_0)/(14 - 4\theta_0) \) when \( n = 3 \) gives the estimate

\[ L(1/2, \Pi \otimes \chi) \ll_{\Pi, \chi, \varepsilon} N\eta^{\frac{3}{2} + \frac{2\theta_0}{3} + \varepsilon} + N\eta^{\frac{3}{2} (\frac{1}{2} + \theta_0) + \frac{(5 - 28\theta_0 - 12\theta_0^2)}{4(14 - 4\theta_0)}}, \]

and taking \( u = 0 \) when \( n \geq 4 \) gives the estimate

\[ L(1/2, \Pi \otimes \chi) \ll_{\Pi, \chi, \varepsilon} N\eta^{\frac{3}{2} (\frac{1}{2} + \theta_0) + \varepsilon}. \]

**Proof.** Following the argument of [6, \S 3.3], we use the bound of Theorem 6.6 above in the integral (68), together with the estimates (69) and (70) for the vector-valued functions \( \mathcal{V}(w) \) and \( \mathcal{W}_w(y) \) respectively, we derive the corresponding bound

\[ \left| \sum_{\gamma \mod \mathcal{O}_F^\times} \frac{c_n(\gamma \eta^{-1})\chi(\gamma \eta^{-1})}{N(\gamma \eta^{-1})} V^* \left( \frac{N\gamma}{Y} \right) \right|^2 \ll_{\varepsilon, \Pi, \chi} \frac{N\eta^{1+\varepsilon}}{L} + N\eta^{\frac{3}{2} (\frac{1}{2} + \theta_0) + \varepsilon} L^{\frac{3}{2} + \theta_0 + \varepsilon} \]

for the modulus squared of the sum (67), which by (66) suffices to deduce the corresponding bound for the modulus squared of the central value \( L(1/2, \Pi \otimes \chi) \). \( \square \)
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