Leading and next to leading large $n_f$ terms in the cusp anomalous dimension and the quark–antiquark potential

Andrey Grozin

Mainz Institute of Theoretical Physics, Mainz University, Germany
Budker Institute of Nuclear Physics, Novosibirsk, Russia
E-mail: A.G.Grozin@inp.nsk.su

I discuss 3 related quantities: the cusp anomalous dimension, the HQET heavy-quark field anomalous dimension, and the quark–antiquark potential. Leading large $n_f$ terms can be calculated to all orders in $\alpha_s$. Next to leading terms with the abelian color structure $C_F^2$ also can be found to all orders (but not non-abelian $C_F C_A$ terms). This talk is based on Appendices C and D in [1].
1. Introduction

The one-loop cusp anomalous dimension

$$\Gamma(\alpha_s, \varphi) = C_F \frac{\alpha_s}{\pi} (\varphi \coth \varphi - 1)$$

(1.1)

follows from the soft radiation function in classical electrodynamics: when a charge suddenly changes its velocity, it emits electromagnetic waves; integrating the intensity over directions, one obtains \(\varphi \coth \varphi - 1\). This result is probably known for more than 100 years, and should be included in The Guinness Book of Records as the anomalous dimension known for a longest time. The two-loop term has been calculated 30 years ago [3] (and rewritten via \(\text{Li}_2, \text{Li}_3\) in [4]). The three-loop term has been calculated recently [5, 6, 1].

The HQET heavy-quark field anomalous dimension (or the anomalous dimension of a straight Wilson line) is known up to 3 loops. At 2 loops, after a wrong calculation [7], the correct result has been obtained in [8], and later in [9, 10, 11, 12]. The three-loop result has been obtained in [13, 14] (in the first paper [13] it has been found as a by-product of the calculation of the QCD on-shell heavy-quark field renormalization constant, from the requirement that the QCD/HQET matching coefficient for the heavy-quark field [15] is finite; at 2 loops this has been done in [11]).

The quark–antiquark potential is known at two [16, 17] and three [18, 19, 20] loops.

Some terms in perturbative series for these quantities can be obtained to all orders in \(\alpha_s\).

2. Large \(n_f\) terms

The terms with the highest power of \(n_f\) at each order of perturbation theory for the cusp anomalous dimension \(\Gamma\) have the structures \(C_F (T_F n_f)^{L-1} \alpha_s^L (L \geq 1)\). They are known to all orders in \(\alpha_s\). The terms with next to highest power of \(n_f\) have the structures \(C_F^2 (T_F n_f)^{L-2} \alpha_s^L\) and \(C_F C_A (T_F n_f)^{L-2} \alpha_s^L (L \geq 3)\). The abelian ones (without \(C_A\)) can be also found to all orders in \(\alpha_s\). For this purpose it is sufficient to consider QED with \(n_f\) massless lepton flavors: \(C_F = T_F = 1, C_A = 0, \beta_0 = -\frac{4}{3} n_f\). Let’s introduce

$$b = \beta_0 \frac{\alpha_s}{4\pi}$$

(2.1)

We assume \(b \sim 1\) and take into account all powers of \(b\); \(1/\beta_0 \ll 1\) is our small parameter, and we consider only a few terms in expansions in \(1/\beta_0\).

At the leading and next-to-leading large-\(\beta_0\) orders (\(L\beta_0\) and \(NL\beta_0\), the coordinate-space Wilson line of any shape is equal to

$$\log W =$$

(2.2)

where the thick photon line is the full photon propagator with the \(NL\beta_0\) accuracy. This simple exponentiation formula is first broken at \(\text{NNL}\beta_0\) order by the light-by-light diagram (figure 1).
Figure 1: The light-by-light diagram is \( n_f \alpha^4 \), and hence NNL\( \beta_0 \).

With the NNL\( \beta_0 \) accuracy the renormalization constant \( Z \) of the heavy-to-heavy current (the cusp) is given by

\[
\log W(t,t';\varphi) - \log W(t,t';0) = \quad = \log Z + \text{finite} \quad (2.3)
\]

(diagrams where both photon-interaction vertices are before the cusp, or after the cusp, cancel in this difference). Going to momentum space, we can express it via the vertex function \( V(\omega, \omega'; \varphi) \) (it is convenient to set \( \omega' = \omega \), in order to have a single-scale problem):

\[
V(\omega, \omega; \varphi) - V(\omega, \omega;0) = \quad = \log Z + \text{finite}. \quad (2.4)
\]

The HQET field renormalization can be obtained from \( V(\omega, \omega;0) \).

The static quark–antiquark potential can be considered similarly. The terms with the highest power of \( n_f \) in each order of perturbation theory have the structures \( C_F (T_F n_f)^L \alpha_s^{L+1} (L \geq 0) \). The terms with next to highest power of \( n_f \) have the structures \( C_F^2 (T_F n_f)^{L-1} \alpha_s^{L+1} \) and \( C_F C_A (T_F n_f)^{L-1} \alpha_s^{L+1} \) \( (L \geq 2); \) we’ll consider only the abelian ones. In the Coulomb gauge, up to NNL\( \beta_0 \) the potential is given by the full Coulomb photon propagator

\[
V(\vec{q}) = \quad = -\frac{e^2}{q^2} \frac{1}{1 - \Pi(-\vec{q}^2)} \quad (2.5)
\]

(\( \Pi(q^2) \) is gauge invariant in QED, and can be taken from covariant-gauge calculations). This simple equality is first broken at NNL\( \beta_0 \) order by the light-by-light diagram (figure 2).

Figure 2: The light-by-light diagram is \( n_f \alpha^4 \), and hence NNL\( \beta_0 \).

As discussed in [1], conformal symmetry leads to the relation between \( \Gamma(\pi - \delta) \) at \( \delta \to 0 \) and \( V(\vec{q}) \):

\[
\Delta \equiv [\delta \Gamma(\pi - \delta; \alpha_s)]_{\delta \to 0} - \frac{\vec{q}^2 V(\vec{q}; \alpha_s)}{4\pi} = 0 \quad (2.6)
\]
(this relation has been observed in [21] at 2 loops). In QCD (and QED) conformal symmetry is anomalous (thus leading to non-zero $\beta$ function), and [1]

$$\Delta = \frac{\pi}{108} \beta_0 C_F \left( \frac{\alpha_s}{\pi} \right)^3 \left( 47C_A - 28T_F n_f \right) + O'(\alpha_s^4).$$

(2.7)

3. Leading $\beta_0$ order

The photon self energy at the $L\beta_0$ order is $\sim 1$:

$$\Pi_0(k^2) = \beta_0 \frac{e_0^2}{(4\pi)^{d/2}} e^{-\epsilon} D(\epsilon) (-k^2)^{-\epsilon},$$

$$D(\epsilon) = e^{\epsilon} \frac{(1-\epsilon)\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{(1-2\epsilon)(1-\frac{\epsilon}{2})\Gamma(1-2\epsilon)} = 1 + \frac{5}{3} \epsilon + \cdots$$

(3.1)

The charge renormalization in the $\overline{\text{MS}}$ scheme is

$$\beta_0 \frac{e_0^2}{(4\pi)^{d/2}} e^{-\epsilon} = bZ_\alpha(b) \mu^{2\epsilon}.$$ (3.2)

At the $L\beta_0$ order we can solve the RG equation

$$\frac{d \log Z_\alpha}{d \log b} = - \frac{b}{\epsilon + b}$$

and obtain

$$Z_\alpha = \frac{1}{1 + b/\epsilon}.$$ (3.3)

The vertex $V(\omega, \omega; \phi)$ is given by the one-loop diagram with the factor $1/(1 - \Pi(k^2))$ inserted in the integrand. At the $L\beta_0$ order (figure 3) the result can be written in the form

$$V(\omega, \omega; \phi) = \frac{1}{\beta_0} \sum_{L=1}^{\infty} \frac{f(\epsilon, L\epsilon; \phi)}{L} \Pi_0^L + O\left( \frac{1}{\beta_0^2} \right),$$ (3.4)

where $L$ is the number of loops and $\Pi_0$ (3.1) is taken at $-k^2 = (-2\omega)^2$. Reduction of such integrals to master ones, as well as evaluation of these master integrals, has been considered in [22].

![Figure 3](image)

**Figure 3:** The $L$-loop vertex diagram at the $L\beta_0$ order contains $L - 1 \Pi_0$ insertions.
Landau gauge we obtain
\[
f(\epsilon, u; \varphi) = \frac{(1 - \frac{2}{\epsilon})\Gamma(2 - 2\epsilon)\Gamma(1 - u)\Gamma(1 + 2u)}{(1 - \epsilon)\Gamma(1 - \epsilon)\Gamma(1 + \epsilon)\Gamma(2 + u - \epsilon)}
\times \left[ (2 + u - 2\epsilon) \cos \varphi - u \right]_{2F1} \left( 1, 1 - u; \frac{1 - \cos \varphi}{2} \right) + 1 \tag{3.5}
\]
(in an arbitrary covariant gauge, a one-loop gauge-dependent contribution should be added). The function \(f(\epsilon, u; \varphi)\) is regular at the origin:
\[
f(\epsilon, u; \varphi) = \sum_{n,m=0}^{\infty} f_{nm}(\varphi) \epsilon^n u^m. \tag{3.6}
\]

The renormalization constant \(Z\) can be written as
\[
\log Z = \frac{Z_1}{\epsilon} + \frac{Z_2}{\epsilon^2} + \cdots, \quad Z_n = \mathcal{O}(b^n).
\]
Only \(Z_1\) is needed in order to obtain
\[
\Gamma(b; \varphi) = -2\frac{dZ_1(b; \varphi)}{d\log b};
\]
higher \(Z_n\) contain no new information, and are uniquely reconstructed from \(Z_1\) using self-consistency conditions. Choosing
\[
\mu^2 = D(\epsilon)^{-1/\epsilon}(-2\omega)^2 \rightarrow e^{-\frac{2}{\epsilon}}(-2\omega)^2
\]
we have
\[
V(\omega, \omega; \varphi) - V(\omega, 0; 0) = \frac{1}{\beta_0} \sum_{L=1}^{\infty} \frac{\tilde{f}(\epsilon, L\epsilon; \varphi)}{L} \left( \frac{b}{\epsilon + b} \right)^L + \mathcal{O}\left( \frac{1}{\beta_0^2} \right), \tag{3.7}
\]
where \(\tilde{f}(\epsilon, u; \varphi) = f(\epsilon, u; \varphi) - f(\epsilon, u; 0)\). We expand in \(b\), expand \(\tilde{f}(\epsilon, u; \varphi)\) in \(\epsilon\) and \(u\) and select only \(\epsilon^{-1}\) terms in order to obtain \(Z_1\). All coefficients but \(f_{n0}\) cancel:
\[
Z_1(b; \varphi) = 2\frac{\varphi \cot \varphi - 1}{\beta_0} \sum_{n=0}^{\infty} \frac{\hat{f}_n}{n + 1} (-b)^{n+1},
\]
where
\[
\hat{f}_n(\epsilon; 0; \varphi) = -2\hat{f}(\epsilon)(\varphi \cot \varphi - 1), \quad \hat{f}(\epsilon) = \sum_{n=0}^{\infty} \hat{f}_n \epsilon^n.
\]
Therefore at the \(L\beta_0\) we obtain [23]
\[
\Gamma(b; \varphi) = 4\frac{b}{\beta_0} \gamma_0(b)(\varphi \cot \varphi - 1) + \mathcal{O}\left( \frac{1}{\beta_0^2} \right),
\]
\[
\gamma_0(b) = \hat{f}(-b) = \frac{(1 + \frac{2}{\epsilon}b)\Gamma(2 + 2b)}{(1 + b)^3(1 + b)\Gamma(1 - b)}
\]
\[
= 1 + \frac{5}{3}b - \frac{1}{3}b^2 - \left( 2\zeta_3 - \frac{1}{3} \right) b^3 + \left( \frac{\pi^4}{30} - \frac{10}{3}\zeta_3 - \frac{1}{3} \right) b^4 + \cdots \tag{3.8}
\]
As a free bonus, we can obtain the HQET field anomalous dimension. The vertex function \( V \) at \( \varphi = 0 \) is related to the HQET propagator \( S \) by the Ward identity

\[
V(\omega, \omega'; 0) = \frac{S^{-1}(\omega') - S^{-1}(\omega)}{\omega' - \omega}, \quad V(\omega, \omega; 0) = \frac{dS^{-1}(\omega)}{d\omega}.
\]

Therefore the renormalization constant of the HQET quark field \( Z_h \) is given by

\[
\log V(\omega, \omega'; 0) = -\log Z_h + \text{finite}.
\]

Using

\[
f(\varepsilon, u; 0) = -\frac{(1 + \frac{2}{3} \varepsilon)^2 \Gamma(2 - 2 \varepsilon) \Gamma(1 + 2 \varepsilon)}{(1 - \varepsilon)^2 \Gamma(1 - \varepsilon) \Gamma(1 + \varepsilon) \Gamma(2 + u - \varepsilon)},
\]

we obtain in the Landau gauge [24]

\[
\gamma_h(b) = 2 \frac{b}{\beta_0} \gamma_0(b) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right),
\]

\[
\gamma_0(b) = f(-b, 0; 0) = \frac{(1 + \frac{2}{3} b)^2 \Gamma(2 + 2b)}{(1 + b)^2 \Gamma(1 + b) \Gamma(1 - b)}
\]

\[
= 1 + \frac{4}{3} b - \frac{5}{9} b^2 - \left(2 \zeta_3 - \frac{2}{3}\right) b^3 + \left(\frac{\pi^4}{30} - \frac{8}{3} \zeta_3 - \frac{7}{9}\right) b^4 + \cdots
\]

(in an arbitrary covariant gauge, a one-loop gauge-dependent contribution should be added).

Now we consider the potential \( V(\bar{q}) \) at the L\( \beta_0 \) order. Choosing \( \mu^2 = \bar{q}^2 \) we have

\[
V(\bar{q}) = \frac{(4\pi)^{D/2} e^{2\mu} \varepsilon}{\beta_0 \Gamma(\varepsilon)} \sum_{L=1}^{\infty} \left(D(\varepsilon) \frac{b}{\varepsilon + b}\right)^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right).
\]

The sum here can be written as

\[
\sum_{L=1}^{\infty} \left(\frac{b}{\varepsilon + b}\right)^L, \quad g(\varepsilon, u) = D(\varepsilon)^{n/\varepsilon} = \sum_{n,m=0}^{\infty} g_{nm} \varepsilon^n u^m.
\]

This sum is equal to

\[
\frac{b}{\varepsilon} \sum_{n=0}^{\infty} n! g_{0n} b^n + \mathcal{O}(\varepsilon^0)
\]

(1/\varepsilon^n terms with \( n > 1 \) vanish, so that \( V(\bar{q}) \) is automatically finite), where

\[
g(0, u) = e^{\frac{2}{3} u}, \quad g_{0n} = \frac{n!}{3^n} \left(\frac{5}{3}\right)^n.
\]

Therefore

\[
V(\bar{q}) = -\frac{(4\pi)^2}{\bar{q}^2} \frac{b}{\beta_0} V_0(b) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right), \quad V_0(b) = \frac{1}{1 - \frac{5}{3} b}.
\]

The conformal anomaly \((2.6)\) at the L\( \beta_0 \) order is

\[
\Delta = 4\pi \frac{b^3}{\beta_0} \delta_0(b) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right),
\]

\[
\delta_0(b) = \frac{V_0(b) - \gamma_0(b)}{b^2} = \frac{28}{9} + 2 \left(\frac{28}{58} + \frac{58}{27}\right) b - \frac{1}{3} \left(\frac{\pi^4}{10} - 10 \zeta_3 - \frac{652}{27}\right) b^2 + \cdots
\]

The first term here reproduces the \( T_\gamma n_f \) term in \((2.7)\).
4. Next to leading $\beta_0$ order

To obtain the photon propagator with the NL$\beta_0$ accuracy, we need the photon self-energy up to $1/\beta_0$:

$$
\Pi(k^2) = \frac{\Pi_1(k^2)}{\beta_0} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right),
$$

(4.1)

where the photon propagators in $\Pi_1$ are taken at the L$\beta_0$ order. The NL$\beta_0$ contribution can be written in the form [25, 26]

$$
\Pi_1(k^2) = 3\varepsilon \sum_{L=2}^{\infty} \frac{F(\varepsilon, L\varepsilon)}{L} \Pi_0(k^2)^L.
$$

(4.2)

Using integration by parts, one can reduce it to

$$
F(\varepsilon, u) = \frac{2(1 - 2\varepsilon)^2(3 - 2\varepsilon)\Gamma^2(1 - 2\varepsilon)}{9(1 - \varepsilon)(1 - u)(2 - \varepsilon)\Gamma^2(1 - \varepsilon)\Gamma^2(1 + \varepsilon)}
\times \left[ -u \frac{2 - 3\varepsilon - \varepsilon^2 + \varepsilon(2 + \varepsilon)u - \varepsilon u^2}{\Gamma^2(1 - \varepsilon)} I(1 + u - 2\varepsilon)
+ 2 \frac{(1 + \varepsilon)(2 - \varepsilon)^2 - (4 + 11\varepsilon - 7\varepsilon^2)u + (8 - 3\varepsilon)u^2 - \varepsilon u^3}{(1 - \varepsilon)(2 - \varepsilon)(1 - u - \varepsilon)(2 - u - \varepsilon)} \Gamma(1 + u)\Gamma(1 - u + \varepsilon) \right]
\sum_{n,m=0}^{\infty} F_{nm}\zeta^n u^m,
$$

(4.3)

where the integral

$$
I(n) = \frac{1}{\pi^d} \int \frac{d^d k_1 d^d k_2}{k_1^2 k_2^2 (k_1 + p)^2 (k_2 + p)^2 [(k_1 - k_2)^2]^n}
$$

(euclidean, $p^2 = 1$) can be expressed via a $3 F_2$ function of unit argument [27, 28] (see the review [29] for more references). The $3 F_2$ function can be expanded up to any desired order using known algorithms, the coefficients are expressed via multiple $\zeta$ values; therefore, the coefficients $F_{nm}$ can be calculated to any desired order.

The function $F(\varepsilon, u)$ simplifies in some cases. In particular [25],

$$
F(\varepsilon, 0) = \frac{(1 + \varepsilon)(1 - 2\varepsilon)^2(1 - \frac{3}{2}\varepsilon)\Gamma(1 - 2\varepsilon)}{(1 - \varepsilon)^2(1 - \frac{1}{2}\varepsilon)\Gamma(1 + \varepsilon)\Gamma^3(1 - \varepsilon)}
$$

(4.4)

so that $F_{n0}$ contain no multiple $\zeta$ values, only $\zeta^n$. Also [26]

$$
F(0, u) = \frac{2}{3} \psi'\left(\frac{2 - u}{2}\right) - \psi'\left(\frac{1 + u}{2}\right) - \psi'\left(\frac{3 - u}{2}\right) + \psi'\left(\frac{1 + u}{2}\right)
\frac{1}{(1 - u)(2 - u)}
$$

(4.5)

so that $F_{0m}$ contains only $\zeta_{2n+1}$ [26]:

$$
F_{0m} = -\frac{32}{3} \sum_{s=1}^{[(m+1)/2]} s \left(1 - 2^{-2s}\right) \left(1 - 2^{2s-m-2}\right) \zeta_{2s+1} + \frac{4}{3}(m+1)(m+(m+6)2^{-m-3})
$$

(4.6)
It is convenient to choose

\[ F(\epsilon, 2\epsilon) = \frac{2 - 3 - 2\epsilon}{9\epsilon^2} \left[ 2 \left( 1 - 2\epsilon \right)^2 \left( 2 - 2\epsilon + \epsilon^2 \right) \Gamma(1 + 2\epsilon) \Gamma^2(1 - 2\epsilon) \right] \]

Let’s write the charge renormalization constant \( Z_\alpha \) with the NL\( \beta_0 \) accuracy as

\[ Z_\alpha(b) = \frac{1}{1 + b/\epsilon} \left[ 1 + \frac{Z_{\alpha1}(b)}{\beta_0} + \mathcal{O} \left( \frac{1}{\beta_0^2} \right) \right], \]

\[ Z_{\alpha1}(b) = \frac{Z_{\alpha11}(b)}{\epsilon} + \frac{Z_{\alpha12}(b)}{\epsilon^2} + \cdots, \]

\[ Z_{\alpha1n} = \mathcal{O}(b^{n+1}). \] (4.7)

In the abelian theory, \( \log(1 - \Pi) \) expressed (3.2) via renormalized \( b \) should be equal to \( \log Z_\alpha + \) finite. Equating the coefficients of \( \epsilon^{-1} \) in the \( 1/\beta_0 \) terms in this relation, we see that \( Z_{\alpha11} \) (4.7) is given by the coefficient of \( \epsilon^{-1} \) in

\[ - \left( 1 + \frac{b}{\epsilon} \right) \Pi_1. \]

It is convenient to choose

\[ \mu^2 = D(\epsilon)^{-1/\epsilon}(-k^2) \to e^{-\epsilon/2\epsilon}(-k^2), \]

then

\[ \Pi_1 = 3\epsilon \sum_{L=2}^{\infty} \frac{F(\epsilon, L\epsilon)}{L} \left( \frac{b}{\epsilon + b} \right)^L. \]

We expand in \( b \) and expand \( F(\epsilon, u) \) in \( \epsilon \) and \( u \); selecting \( \epsilon^{-1} \) terms, we find that all coefficients but \( F_{n0} \) cancel:

\[ Z_{\alpha11} = -3 \sum_{n=0}^{\infty} \frac{F_{n0}(-b)^{n+2}}{(n+1)(n+2)}. \] (4.8)

The \( \beta \) function with NL\( \beta_0 \) accuracy is

\[ \beta(b) = b + \frac{\beta_1(b)}{\beta_0} + \mathcal{O} \left( \frac{1}{\beta_0^2} \right), \] (4.9)

where [25, 26]

\[ \beta_1(b) = -\frac{dZ_{\alpha11}(b)}{d\log b} = 3 \sum_{n=0}^{\infty} \frac{F_{n0}(-b)^{n+2}}{n+1} \]

\[ = 3b^2 + \frac{11}{4} b^3 - \frac{77}{36} b^4 - \frac{1}{2} \left( 3\zeta_3 + \frac{107}{48} \right) b^5 + \frac{1}{5} \left( \frac{\pi^4}{10} - 11\zeta_3 + \frac{251}{48} \right) b^6 + \cdots \] (4.10)

(the coefficients \( F_{n0} \) follow from \( F(\epsilon, 0) \) (4.4)). The corresponding terms in the 5-loop QED \( \beta \) function [30] are reproduced. We shall need the full \( Z_{\alpha1} \), not just \( Z_{\alpha11} \); integrating the RG equation with the \( 1/\beta_0 \) accuracy we obtain

\[ Z_{\alpha1}(b) = -\epsilon \int_0^b \frac{\beta_1(b) \, db}{b(\epsilon + b)^2} = \frac{3b^2}{2\epsilon} + \frac{1}{2} \left( 4 + F_{10}\epsilon \right) \frac{b^3}{\epsilon^2} - \frac{1}{4} \left( 9 + 3F_{10}\epsilon + F_{20}\epsilon^2 \right) b^4 + \cdots \]
At the NL$\beta_0$ order we should expand the photon propagator $(1 - \Pi_0 - \Pi_1/\beta_0)^{-1}$ up to $1/\beta_0$ (Fig. 4). The vertex function (3.7) becomes

$$
V(\omega, \omega; \varphi) - V(\omega, \omega; 0) = \frac{1}{\beta_0} \sum_{L=1}^{\infty} \frac{f(\varepsilon, L\varepsilon; \varphi)}{L} \left( \frac{b}{\varepsilon + b} \right)^L
$$

$$
\times \left[ 1 + LZ_{a1} + \frac{3\varepsilon}{\beta_0} \sum_{L'=2}^{L-1} L' \frac{L - L'}{L'F(\varepsilon, L'\varepsilon)} \right] + O\left( \frac{1}{\beta_0^3} \right),
$$

where $L'$ is the number of loops in the $\Pi_1$ insertion, and the $1/\beta_0$ correction $Z_{a1}$ to the charge renormalization (4.7) is taken into account. We expand in $b$ and substitute the expansions (4.3) and (3.6); in $Z_1$, the coefficient of $\varepsilon^{-1}$, all $\tilde{f}_{am}$ except $\tilde{f}_{a0}$ cancel. At the NL$\beta_0$ order the cusp anomalous dimension is determined by the same $\tilde{f}_n$ coefficients as at the L$\beta_0$ order:

$$
\Gamma(b; \varphi) = 4 \left[ \frac{b}{\beta_0} \gamma_0(b) - \frac{b^3}{\beta_0^2} \gamma_1(b) \right] (\varphi \cot \varphi - 1) + O\left( \frac{1}{\beta_0^3} \right),
$$

where

$$
\gamma_1(b) = -\frac{3}{2} \left[ F_{10} + 2F_{01} - 2\tilde{f}_1 \right] + \left[ 2F_{20} + 3(F_{11} + F_{02}) + 3F_{01}\tilde{f}_1 - 6\tilde{f}_2 \right] b
$$

$$
- \left[ \frac{3}{4} (3F_{30} + 4(F_{21} + F_{12} + F_{03})) + (F_{20} + 3(F_{11} + F_{02}))\tilde{f}_1 - \frac{3}{2} (F_{10} - 2F_{01})\tilde{f}_2 - 9\tilde{f}_3 \right] b^2 + \cdots
$$

Figure 4: NL$\beta_0$ order diagrams contain one $\Pi_1$ insertion (with any number of $\Pi_0$ insertions inside) and any number of $\Pi_0$ insertions to the left and to the right of it.
Substituting $F_{nm}$ we obtain
\begin{align*}
\gamma(b) &= 12\zeta_3 - \frac{55}{4} + \left( -\frac{\pi^4}{5} + 40\zeta_3 - \frac{299}{18} \right) b \\
+ &\left( 24\zeta_5 - \frac{2}{3}\pi^4 + \frac{233}{6}\zeta_3 + \frac{15211}{864} \right) b^2 \\
+ &\left( -48\zeta_3^2 - \frac{2}{63}\pi^6 + 80\zeta_5 - \frac{167}{225}\pi^4 + \frac{1168}{15}\zeta_3 - \frac{971}{240} \right) b^3 \\
+ &\left( 36\zeta_7 + \frac{8}{5}\pi^4 \zeta_3 - 160\zeta_7^2 - \frac{20}{189}\pi^6 + \frac{377}{3}\zeta_5 - \frac{23}{15}\pi^4 + \frac{929}{12\zeta_3} - \frac{8017}{1728} \right) b^4 \\
+ &\left( -240\zeta_5\zeta_3 - \frac{4}{225}\pi^8 + 120\zeta_7 + \frac{16}{3}\pi^4 \zeta_3 - \frac{2776}{21}\zeta_5^2 - \frac{914}{3969}\pi^6 \\
+ &\frac{6826}{21}\zeta_5 - \frac{1793}{1350}\pi^4 - \frac{31693}{315}\zeta_3 + \frac{79433}{4320} \right) b^5 + \cdots
\end{align*}
(4.13)

This expansion can be extended to any number of loops. The first term in (4.13) agrees with the $C_{T'}^2(T_{F n_f})^2\alpha_s^4$ term in the three-loop result [5, 6, 1]. The next term coincides with the $C_{T'}^2(T_{F n_f})^2\alpha_s^4$ term in $\Gamma$ recently calculated in [31]. Note that the last (8-loop) term here contains $F_{nm}$ with $n + m = 6$, $n > 0$, $m > 0$, which contain $\zeta_{5,3}$; but they enter as the combination $F_{51} + F_{42} + F_{33} + F_{24} + F_{15}$ in which this $\zeta_{5,3}$ cancels.

Similarly, the field anomalous dimension in Landau gauge at the NL$\beta_0$ order is
\begin{align*}
\gamma(b) &= -6 \left( \frac{b}{\beta_0} \gamma_0(b) - \frac{b^3}{\beta_0^2} \gamma_1(b) \right) + O\left( \frac{1}{\beta_0^3} \right), \\
\gamma_1(b) &= 3 \left( 4\zeta_5 - \frac{17}{4} \right) + \left( -\frac{\pi^4}{5} + 36\zeta_3 - \frac{103}{9} \right) b \\
+ &\left( 24\zeta_5 - \frac{3}{5}\pi^4 + \frac{59}{2}\zeta_3 + \frac{14579}{864} \right) b^2 \\
+ &\left( -48\zeta_3^2 - \frac{2}{63}\pi^6 + 72\zeta_5 - \frac{44}{75}\pi^4 + \frac{3229}{45}\zeta_3 - \frac{5191}{540} \right) b^3 \\
+ &\left( 36\zeta_7 + \frac{8}{5}\pi^4 \zeta_3 - 144\zeta_7^2 - \frac{2}{21}\pi^6 + 107\zeta_5 - \frac{946}{675}\pi^4 + \frac{9601}{180}\zeta_3 + \frac{22859}{8640} \right) b^4 \\
+ &\left( -240\zeta_5\zeta_3 - \frac{4}{225}\pi^8 + 108\zeta_7 + \frac{24}{5}\pi^4 \zeta_3 - \frac{664}{7}\zeta_5^2 - \frac{272}{1323}\pi^6 \\
+ &\frac{18574}{63}\zeta_5 - \frac{119}{135}\pi^4 - \frac{6263}{63}\zeta_3 + \frac{16103}{1296} \right) b^5 + \cdots
\end{align*}
(4.14)

The first term here coincides with the $C_{T'}^2(T_{F n_f})^2\alpha_s^4$ term in the three-loop result obtained by a direct calculation [13, 14]. The last term contains the same combination of $F_{nm}$ with $n + m = 6$, so that $\zeta_{5,3}$ cancels.

The static potential at the NL$\beta_0$ level is
\begin{align*}
V(q) &= -\left( \frac{4\pi^2}{\beta_0^2} \right) \sum_{L=1}^{\infty} g(\epsilon, L\epsilon) \left( \frac{b}{\epsilon + b} \right)^L \left[ 1 + L\frac{Z_{a\epsilon}}{\beta_0} + \frac{3\epsilon}{\beta_0} \sum_{L=2}^{L-1} \frac{L-L'}{L'} F(\epsilon, L'\epsilon) \right] + O\left( \frac{1}{\beta_0^3} \right) \\
&= -\left( \frac{4\pi^2}{\beta_0^2} \right) \left[ \frac{b}{\beta_0} V_0(b) - \frac{b^3}{\beta_0^3} V_1(b) \right] + O\left( \frac{1}{\beta_0^3} \right)
\end{align*}
(4.15)
where
\[ V_1(b) = -\frac{3}{2} [F_{10} + 2F_{01} + 2g_{01}] + \frac{1}{2} [F_{20} - 6F_{02} - 6(F_{10} + 3F_{01})g_{01} - 30g_{02}] b \]
\[ - \frac{1}{4} [F_{30} + 24F_{03} - 4(F_{20} + 12F_{02})g_{01} + 36(F_{10} + 4F_{01})g_{02} + 312g_{03}] b^2 + \cdots \]
contains only the same coefficients $g_{0n}$ (3.11) as the $L\beta_0$ result, and only $F_{n0}$ and $F_{0n}$ are involved (see (4.4–4.6)). We obtain
\[ V_1(b) = 12\zeta_3 - \frac{55}{4} + \left( \frac{78\zeta_3 - 7001}{72} \right) b + \left( \frac{60\zeta_5 + 723}{2} - \frac{147851}{288} \right) b^2 \]
\[ + \left( \frac{770\zeta_3 + \pi^4}{200} + \frac{276901}{180} - \frac{70418923}{25920} \right) b^3 \]
\[ + \left( \frac{1134\zeta_7 + 32297}{5} \zeta_5 + \frac{41}{1800} \pi^4 + \frac{402479}{60} - \frac{1249510621}{77760} \right) b^4 \]
\[ + \left( \frac{21735\zeta_7}{5} + \frac{\pi^6}{1323} + \frac{5911849}{126} \zeta_5 + \frac{41}{720} \pi^4 + \frac{48558187}{1512} - \frac{10255708489}{93312} \right) b^5 + \cdots \]
(4.16)

Thus we have reproduced the $C_F(T_Fn_f)\alpha_s^3$ and $C_F^2T_Fn_f\alpha_s^2$ terms in the two-loop potential [17], as well as the $C_F(T_Fn_f)^3\alpha_s^4$ and $C_F^2(T_Fn_f)^2\alpha_s^3$ terms in the three-loop one [18]. This expansion can be extended to any order; it contains only $\zeta_n$ because only $F_{n0}$ and $F_{0n}$ are present. Note the pattern of the highest weights in (4.16): 3, 3, 5, 5, 7, 7, whereas one would expect 3, 4, 5, 6, 7, 8, as in (4.13), (4.14). The conformal anomaly (2.6) at the NL$\beta_0$ order is
\[ \Delta = 4\pi \left[ \frac{b^4}{\beta_0^2} \delta_0(b) - \frac{b^4}{\beta_0^2} \delta_1(b) \right] + O \left( \frac{1}{\beta_0^2} \right), \]
\[ \delta_1(b) = \frac{\pi^4}{5} + 38\zeta_3 - \frac{645}{8} + \left( \frac{36\zeta_5 + \frac{2}{3} \pi^4 + \frac{968}{3} \zeta_3 - 114691}{216} \right) b \]
\[ + \left( \frac{48\zeta_3 + \frac{2}{63} \pi^6 + 690\zeta_5 + \frac{269}{360} \pi^4 + \frac{52577}{36} \zeta_3 - 14062811}{5184} \right) b^2 \]
\[ + \left( \frac{1098\zeta_7 - \frac{8}{5} \pi^4 \zeta_3 + 160\zeta_5^2 + \frac{20}{189} \pi^6 + \frac{9506}{15} \zeta_5 + \frac{2801}{1800} \pi^4 + \frac{198917}{30} \zeta_3 - 39035933}{2430} \right) b^3 \]
\[ + \left( \frac{240\zeta_7 + \frac{4}{225} \pi^6 + 21615\zeta_5 - \frac{16}{3} \pi^4 \zeta_3 + \frac{397}{3} \zeta_5^2 + 131}{567} \pi^6 \right) \]
\[ + \left( \frac{838699}{18} \zeta_5 + \frac{14959}{10800} \pi^4 + \frac{34793081}{1080} \zeta_3 - \frac{51287121209}{466560} \right) b^4 + \cdots \]  
(4.17)
The $b^3/\beta_0^2$ term has canceled, so that the coefficient of $C_F$ in the bracket in (2.7) is 0.

5. Conclusion

The terms with the highest powers of $n_f$ at each order of perturbation theory ($C_F(T_Fn_f)^{l-1}\alpha_s^l$ in $\Gamma$, $\gamma_l$; $C_F(T_Fn_f)^{l}\alpha_s^{l+1}$ in $V(\bar{q})$) are known, and given by explicit formulas (3.8), (3.10), (3.12). The terms with the next to highest power of $n_f$ can have abelian ($C_F^2$) or non-abelian ($C_FC_A$) color
structure. The abelian terms \((C_F^2 T_F n_f)^{L-2} \alpha_s^L (L \geq 3)\) in \(\Gamma, \gamma_h; C_F^2 (T_F n_f)^{L-1} \alpha_s^{L+1} (L \geq 2)\) in \(V(\vec{q})\) are also known to all orders in \(\alpha_s\), but only as algorithms which allow one to obtain (in principle) any number of terms, see (4.13), (4.14), (4.16). The simple method used here is not applicable to non-abelian terms.

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