SPLICING KNOT COMPLEMENTS AND BORDERED FLOER HOMOLOGY

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Abstract. We show that the integer homology sphere obtained by splicing two nontrivial knot complements in integer homology sphere L-spaces has Heegaard Floer homology rank strictly greater than one. In particular, splicing the complements of nontrivial knots in the 3-sphere never produces an L-space. The proof uses bordered Floer homology.

1. Introduction

A rational homology 3-sphere $Y$ is called an L-space if the rank of its Heegaard Floer homology group $\hat{HF}(Y)$ equals the order of $H_1(Y;\mathbb{Z})$. Examples of L-spaces include $S^3$, lens spaces, all manifolds with finite fundamental group [17, Proposition 2.3], and the branched double covers of alternating (or, more generally, quasi-alternating) links in $S^3$ [23, Proposition 3.3]. Since the rank of $\hat{HF}(Y)$ is always greater than or equal to $|H_1(Y;\mathbb{Z})|$ [22, Proposition 5.1], L-spaces are the manifolds with the smallest possible Heegaard Floer homology, and it is natural to ask for a complete classification of L-spaces or a more topological characterization [18, Question 11]. The following conjecture is of central importance to Heegaard Floer theory:

Conjecture 1. If $Y$ is an irreducible homology sphere that is an L-space, then $Y$ is homeomorphic to either $S^3$ or the Poincaré homology sphere.

Thus, the conjecture asserts that the classification of L-spaces with the singular homology of the 3-sphere is extremely simple: they are simply the connected sums of zero or more copies of the Poincaré sphere (with either orientation). Conjecture 1 is known to hold for manifolds obtained by Dehn surgery on knots in $S^3$ [21, Proof of Corollary 1.3], [6, Proof of Corollary 1.5] and for all Seifert fibered spaces [27]. In light of the Geometrization Theorem [24, 25, 14], one should consider how Heegaard Floer homology behaves under the operation of gluing along incompressible tori. The following conjecture would reduce Conjecture 1 to the case of hyperbolic 3-manifolds:

Conjecture 2. If $Y$ is an irreducible homology sphere that contains an incompressible torus, then $Y$ is not an L-space.

The purpose of this paper is to prove a special case of Conjecture 2.

To describe our result, let the exterior of a knot $K$ in a homology sphere $Y$ be denoted by $X_K$. The meridian and Seifert longitude of $K$, viewed as curves in $\partial X_K$, are respectively denoted $\mu_K$ and $\lambda_K$. Given knots $K_1 \subset Y_1$ and $K_2 \subset Y_2$, let $Y(K_1, K_2)$ denote the manifold obtained by gluing $X_{K_1}$ and $X_{K_2}$ via an orientation-reversing diffeomorphism $\phi: \partial X_{K_1} \to \partial X_{K_2}$ taking $\lambda_{K_1}$ to $\mu_{K_2}$ and $\lambda_{K_2}$ to $\mu_{K_1}$. We say that $Y(K_1, K_2)$ is obtained by splicing the knot complements $X_{K_1}$ and $X_{K_2}$. The Mayer–Vietoris sequence shows that $Y(K_1, K_2)$ is a
homology sphere. The image of $\partial X_{K_1}$ is incompressible in $Y(K_1, K_2)$ if and only if the knots $K_1$ and $K_2$ are both nontrivial. Furthermore, a separating torus $T$ in a homology sphere $Y$ canonically determines a decomposition $Y = Y(K_1, K_2)$: if $Y = X_1 \cup_T X_2$, we obtain $Y_1$ (resp. $Y_2$) by Dehn filling $X_1$ (resp. $X_2$) along the unique slope in $T$ that bounds a surface in $Y_2$ (resp. $Y_1$), and we let $K_1$ (resp. $K_2$) be the core of the glued-in solid torus.

The main result of this paper is the following:

**Theorem 1.** Let $Y_1$ and $Y_2$ be $L$-space homology spheres, and let $K_1 \subset Y_1$ and $K_2 \subset Y_2$ be nontrivial knots. Then $\dim \widehat{HF}(Y(K_1, K_2)) > 1$.

Removing the hypothesis that $Y_1$ and $Y_2$ are themselves $L$-spaces would complete the proof of Conjecture [2]. Of course we have the immediate corollary:

**Corollary 2.** Splicing the complements of nontrivial knots in the 3-sphere never produces an $L$-space.

Our strategy for studying $\widehat{HF}(Y(K_1, K_2))$ is to relate it to the knot Floer homology of $K_1$ and $K_2$. For a knot $K \subset Y$ in an integral homology sphere, $\widehat{HF}(Y, K)$ is a bigraded vector space over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$

$$\widehat{HF}(Y, K) = \bigoplus_{a, m \in \mathbb{Z}} \widehat{HF}_m(Y, K, a),$$

whose graded Euler characteristic is the Alexander polynomial of $K$ [16, 26]. These groups detect the Seifert genus of $K$ [21, Theorem 1.2], in the sense that

$$g(K) = \max \{a \mid \widehat{HF}_a(Y, K, a) \neq 0\} = -\min \{a \mid \widehat{HF}_a(Y, K, a) \neq 0\}.$$

If $K_1$ and $K_2$ are nontrivial knots, we show that $\widehat{HF}(Y(K_1, K_2))$ contains a subspace of dimension

$$2 \cdot \dim \widehat{HF}_*(Y_1, K_1, -g(K_1)) \cdot \dim \widehat{HF}_*(Y_2, K_2, -g(K_2)) \geq 2,$$

which implies Theorem [1]. Indeed, since $\dim \widehat{HF}_*(Y, K, -g(K)) = 1$ if and only if $K$ is a fibered knot [6, 15], we obtain a stronger lower bound on $\dim \widehat{HF}(Y(K_1, K_2))$ if either $K_1$ or $K_2$ is non-fibered.

Our basic tool for proving Theorem [1] is bordered Floer homology [12], which can be used to compute the Heegaard Floer homology of a closed 3-manifold obtained by gluing two pieces along a common boundary as the homology of the derived tensor product of algebraic invariants associated to the pieces. We review some of the basics of this theory in Section [2].

In the present setting, we have

$$\widehat{HF}(Y(K_1, K_2)) \cong H_*(\widehat{CFA}(X_{K_1}) \otimes \widehat{CFD}(X_{K_2})), $$

where $\widehat{CFA}(X_{K_1})$ and $\widehat{CFD}(X_{K_2})$ are the bordered invariants of $X_{K_1}$ and $X_{K_2}$ with suitable boundary parameterizations. Lipshitz, Ozsváth, and Thurston give a formula describing $\widehat{CFD}$ of the complement of a knot in an $L$-space homology sphere in terms of the knot Floer complex of the knot [12, and a simple algorithm (given below as Theorem [2.2]) yields a similar description of $\widehat{CFA}$. Using an Alexander grading on the bordered invariants, we can identify subspaces of $\widehat{CFA}(X_{K_1})$ and $\widehat{CFD}(X_{K_2})$ that are isomorphic to the corresponding knot Floer homology groups in extremal Alexander grading and whose algebraic structure.
can be understood quite explicitly. These subspaces combine in the tensor product to produce the subgroup of \( \hat{HF}(Y(K_1, K_2)) \) described above.

In a sequence of papers, Eftekhary \cite{3, 1, 4, 2} proposes a proof of Conjecture \cite{2}. The main theorem of \cite{3} provides a chain complex that ostensibly computes \( \hat{HF}(Y(K_1, K_2)) \), and which plays a central role in the subsequent papers. We have observed that when \( K_1 \) and \( K_2 \) are both the right-handed trefoil in \( S^3 \), the homology of Eftekhary’s complex has rank 13. However, a computation using bordered Floer homology, given in Section 4, shows that the rank of \( \hat{HF}(Y(K_1, K_2)) \) is only 7. Our calculation agrees with an independent calculation provided by results of \cite{7}. In particular, results of \cite{7} allow for an easy computation of the Floer homology of +1 surgery on the untwisted Whitehead double of the right-handed trefoil, using a surgery formula from \cite{16}. This latter manifold, however, can be identified as the splice of two right-handed trefoil complements.

Acknowledgments. The authors are grateful to Robert Lipshitz, Peter Ozsváth, and Dylan Thurston for many helpful conversations.

2. Bordered Heegaard Floer homology

We begin by reviewing a few basic definitions and facts regarding bordered Heegaard Floer homology \cite{12}, focusing on the case of manifolds with torus boundary. Some of this material is adapted from the second author’s exposition in \cite[Section 2]{10}.

2.1. Algebraic preliminaries. In this section, we recall the key algebraic structures that occur in bordered Floer homology, known as \( A_\infty \)-modules and type D structures. While these objects can be defined in general over an underlying \( A_\infty \)-algebra \( \mathcal{A} \), the relevant algebra for our purposes is merely differential graded, so it will be convenient to give the definitions in this simplified setting.

Let \((\mathcal{A}, d)\) be a unital differential algebra over \( \mathbb{F} = \mathbb{Z}/2\mathbb{Z} \), and assume that the set \( \mathcal{I} \) of idempotents in \( \mathcal{A} \) is a commutative subring of \( \mathcal{A} \) and possesses a basis \( \{i_1\} \) over \( \mathbb{F} \) such that \( i_1i_2 = \delta_{i_1i_2}i_1 \) and \( \sum_i i_1 = 1 \), the identity element of \( \mathcal{A} \). A (right) \( A_\infty \)-module or (right) type A module over \( \mathcal{A} \) is a vector space \( M \) equipped with a right action of \( \mathcal{I} \) such that \( M = Mt_1 \oplus ... \oplus Mt_n \) as a vector space, and multiplication maps

\[
m_{k+1} : M \otimes_I \underbrace{\mathcal{A} \otimes_I \cdots \otimes_I \mathcal{A}}_{k \text{ times}} \to M
\]

satisfying the \( A_\infty \) relations: for any \( x \in M \) and \( a_1, \ldots, a_n \in \mathcal{A} \),

\[
0 = \sum_{i=0}^{n} m_{n-i+1}(m_{i+1}(x \otimes a_1 \otimes \cdots \otimes a_i) \otimes a_{i+1} \otimes \cdots \otimes a_n) + \sum_{i=1}^{n} m_{n+1}(x \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes d(a_i) \otimes a_{i+1} \otimes \cdots \otimes a_n) + \sum_{i=1}^{n-1} m_{n}(x \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes a_ia_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_n).
\]

We also require that \( m_2(x \otimes 1) = x \) and \( m_k(x \otimes \cdots \otimes 1 \otimes \cdots) = 0 \) for \( k > 2 \). We say that \( M \) is bounded if \( m_k = 0 \) for all \( k \) sufficiently large.

A \textit{(left) type D structure} over $\mathcal{A}$ is an $F$-vector space $N$, equipped with a left action of $\mathcal{I}$ such that $N = \iota_0 N \oplus \iota_1 N$, and a map

$$\delta_1 : N \to \mathcal{A} \otimes_{\mathcal{I}} N$$

satisfying the relation

$$\delta_1 = (\mu \otimes \text{id}_N) \circ (\text{id}_{\mathcal{A}} \otimes \delta_1) \circ \delta_1 + (d \otimes \text{id}_N) \circ \delta_1 = 0,$$

where $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ denotes the multiplication on $\mathcal{A}$. If $N$ is a type $D$ module, the tensor product $\mathcal{A} \otimes_{\mathcal{I}} N$ is naturally a left differential module over $\mathcal{A}$, with module structure given by $a \cdot (b \otimes x) = \mu(ab) \otimes x$, and differential $\partial(a \otimes x) = a \cdot \delta_1(x) + d(a) \otimes x$. Condition (2) translates to $\partial^2 = 0$. We inductively define maps

$$\delta_k : N \to \mathcal{A} \otimes_{\mathcal{I}} \cdots \otimes_{\mathcal{I}} \mathcal{A} \otimes_{\mathcal{I}} N$$

by $\delta_0 = \text{id}_N$ and $\delta_k = (\text{id}_{\mathcal{A}^k} \otimes \delta_1) \circ \delta_{k-1}$. We say $N$ is \textit{bounded} if $\delta_k = 0$ for all $k$ sufficiently large.

If $M$ is a type $A$ module and $N$ is a type $D$ module, the $A_{\infty}$-tensor product $M \hat{\otimes} N$ [12, Definition 2.12] is a chain complex whose chain homotopy type depends only on the chain homotopy types of $M$ and $N$ (using suitable notions of chain homotopy equivalence for type $A$ and $D$ modules).

We say that the pair $(M, N)$ is \textit{relatively bounded} if there exists a constant $K$ such that for all $x \in M$ and $y \in N$ and all $k \geq K$,

$$\delta_{k+1} \otimes \text{id}_N)(x \otimes \delta_k(y)) = 0.$$

For instance, this will be true if either $M$ or $N$ is bounded. If $(M, N)$ is relatively bounded, the \textit{box tensor product} $M \boxtimes N$ is the vector space $M \otimes_{\mathcal{I}} N$, equipped with the differential

$$\partial^\boxtimes(x \otimes y) = \sum_{k=0}^{\infty} (m_{k+1} \otimes \text{id}_N)(x \otimes \delta_k(y)).$$

This is a finite sum by (3), and (1) and (2) imply that $\partial^\boxtimes \circ \partial^\boxtimes = 0$. Lipshitz, Oszváth, and Thurston [12, Proposition 2.34] show that when $M$ or $N$ is bounded, $M \boxtimes N$ is chain homotopy equivalent to $M \hat{\otimes} N$, and it is not hard to see that their proof extends to the case where the pair $(M, N)$ is relatively bounded.

In addition to type $A$ modules and $D$ structures over $\mathcal{A}$, we can also talk about bimodules (or trimodules, et cetera). These come in several flavors, known as type $AA$, $AD$, $DA$, $DD$. For instance, for differential graded algebras $\mathcal{A}$ and $\mathcal{B}$ a left-left type $DD$ bimodule over $(\mathcal{A}, \mathcal{B})$ is simply a left type $D$ module over $\mathcal{A} \otimes \mathcal{B}$; the other types are slightly more complicated. The $\mathcal{A}_{\infty}$ tensor products of bimodules behave as expected: for instance, given a right type $A$ module $M$ over $\mathcal{A}$ and a type $DD$ bimodule $N$ over $(\mathcal{A}, \mathcal{B})$, $M \hat{\otimes}_{\mathcal{A}} N$ is a type $D$ module over $\mathcal{B}$. The box tensor product $\boxtimes$ may be used in place of $\hat{\otimes}$ under suitable conditions. See [11, Section 2] for the complete definitions.

2.2. \textbf{Invariants of bordered manifolds.} We will focus solely on the case of torus boundary. We consider $T^2 = S^1 \times S^1$, oriented by choosing the same orientation on both $S^1$ factors and taking the product orientation.
The torus algebra \( A = \mathcal{A}(T^2) \) is freely generated as a vector space over \( \mathbb{F} \) by mutually orthogonal idempotents \( \iota_0 \) and \( \iota_1 \) and additional elements \( \rho_1, \rho_2, \rho_3, \rho_{12}, \rho_{23}, \) and \( \rho_{123} \), with the following nonzero multiplications:

\[
\begin{align*}
\iota_0 \rho_1 &= \rho_1 \iota_1 = \rho_1 & \iota_1 \rho_2 &= \rho_2 \iota_0 = \rho_2 & \iota_0 \rho_3 &= \rho_3 \iota_1 = \rho_3 \\
\iota_0 \rho_{12} &= \rho_{12} \iota_0 = \rho_{12} & \iota_1 \rho_{23} &= \rho_{23} \iota_1 = \rho_{23} & \iota_0 \rho_{123} &= \rho_{123} \iota_1 = \rho_{123} \\
\rho_1 \rho_2 &= \rho_{12} & \rho_2 \rho_3 &= \rho_{23} & \rho_{12} \rho_3 &= \rho_{123} \rho_{23} &= \rho_{123}
\end{align*}
\]

(All other multiplications among the generators zero.) The multiplicative identity in \( A \) is \( 1 = \iota_0 + \iota_1 \). The differential on \( A \) is defined to be zero; note that this eliminates the second sum in (1) and the third term in (2).

A bordered manifold (with torus boundary) is an oriented 3-manifold \( Y \) along with a diffeomorphism \( \phi : T^2 \to \partial Y \), which we consider up to isotopy fixing a neighborhood of a point. We call \((Y, \phi)\) type \( A \) if \( \phi \) is orientation-preserving and type \( D \) if \( \phi \) is orientation-reversing. Lipshitz, Ozsváth, and Thurston associate to a type \( A \) bordered manifold \((Y_1, \phi_1)\) a type \( A \) module \( \widehat{\text{CF}A}(Y_1, \phi_1) \) over \( A \), and to a type-\( D \) bordered manifold \((Y_2, \phi_2)\) a type-\( D \) module \( \widehat{\text{CF}A}(Y_2, \phi_2) \) over \( A \). (The maps \( \phi_1 \) and \( \phi_2 \) are often suppressed from the notation if they are understood from the context.) Up to the appropriate notion of chain homotopy equivalence, each of these modules is a diffeomorphism invariant of the manifold with parametrized boundary. These invariants are defined in terms of counts of pseudo-holomorphic curves in \( \Sigma \times [0, 1] \times \mathbb{R} \), where \( \Sigma \) is a bordered Heegaard diagram; we shall say nothing more about the definition. The pairing theorem states that the Heegaard Floer homology of the closed, oriented 3-manifold gotten by gluing \( Y_1 \) to \( Y_2 \) along their boundaries via the diffeomorphism \( \phi_2 \circ \phi_1^{-1} \) is determined by the bordered invariants of \( Y_1 \) and \( Y_2 \):

\[
\widehat{\text{HF}}(Y_1 \cup_{\phi_2 \circ \phi_1^{-1}} Y_2) \cong H_* \left( \widehat{\text{CF}A}(Y_1, \phi_1) \otimes \widehat{\text{CF}A}(Y_2, \phi_2) \right).
\]

There are also various bimodules associated to manifolds with two boundary components, denoted \( \widehat{\text{CF}A}, \widehat{\text{CF}D}, \widehat{\text{CF}A}, \) and \( \widehat{\text{CF}D} \) according to whether the parameterizations of the boundary components are orientation-preserving or orientation-reversing, and similar gluing theorems apply. See [12, 11] for further details.

Lipshitz, Ozsváth, and Thurston provide a convenient notation for type \( D \) modules over \( \mathcal{A}(T^2) \) [12 Section 11.1]. For any finite sequence of integers \( I \), let first(\( I \)) and last(\( I \)) denote the first and last elements of \( I \), respectively. A finite sequence \( I \) is called alternating if its entries alternate in parity. If \( I_1, \ldots, I_k \) are finite sequences of integers, let \( I_1 \cdots I_k \) denote their concatenation. Let \( \mathcal{R} \) denote the set of nonempty, strictly increasing sequences of consecutive integers in \( \{1, 2, 3\} \), and let \( \mathcal{R}' = \mathcal{R} \cup \{\emptyset\} \). Thus, the non-idempotent generators of \( \mathcal{A}(T^2) \) correspond to elements of \( \mathcal{R} \); for convenience, we define \( \rho_{\emptyset} = 1 \).

Let \( V = V^0 \oplus V^1 \) be a \( \mathbb{Z}/2 \)-graded vector space over \( \mathbb{F} = \mathbb{Z}/2\mathbb{Z} \). A collection of coefficient maps consists of a linear map \( D = D_0 : V \to V \) taking \( V^0 \) to \( V^0 \) and \( V^1 \) to \( V^1 \), and, for each \( I = (i_1, \ldots, i_n) \in \mathcal{R} \), a linear map \( D_I : V^{[i_1]} \to V^{[i_n]} \) (where for \( i \in \mathbb{Z} \), \( [i] \in \{0, 1\} \) denotes the mod-2 reduction of \( i \)), satisfying the condition that for each \( I \in \mathcal{R}' \),

\[
\sum_{J, K \in \mathcal{R}' \mid JK = I} D_K \circ D_J = 0,
\]

where the sum is taken over all pairs of elements in \( \mathcal{R}' \) whose concatenation is \( I \). In other words, \( D_0 \) is a differential; \( D_1, D_2, \) and \( D_3 \) are chain maps; \( D_{12} \) and \( D_{23} \) are nullhomotopies.
of \( D_2 \circ D_1 \) and \( D_3 \circ D_2 \), respectively; and \( D_{123} \) is a homotopy between \( D_{23} \circ D_1 \) and \( D_3 \circ D_{12} \). For convenience, we may trivially extend each \( D_I \) over all of \( V^0 \oplus V^1 \). A collection of coefficient maps determines a type \( D \) structure on \( V \): define multiplication by \( \iota_0 \) and \( \iota_1 \) by projection onto \( V^0 \) and \( V^1 \) respectively, and for each \( v \in V \), define
\[
\delta_1(v) = \sum_{I \in \mathcal{I}^V} \rho_I \otimes D_I(v).
\]
The higher maps \( \delta_k \) are then given by compositions of the maps \( D_I \):
\[
\delta_k(v) = \sum_{I_1, \ldots, I_k \in \mathcal{I}^V} \rho_{I_1} \otimes \cdots \otimes \rho_{I_k} \otimes (D_{I_k} \circ \cdots \circ D_{I_1})(v).
\]
Furthermore, any type \( D \) structure over \( A \) can be obtained in this manner \([12\text{ Lemma 11.5}]\).

We say that \((V, \{D_I\})\) is reduced if \( D_0 = 0 \), in which case the relations in \((1)\) simplify to:
\[
(5) \quad D_2 \circ D_1 = 0 \quad D_3 \circ D_2 = 0 \quad D_3 \circ D_{12} = D_{23} \circ D_1.
\]

It is not hard to see that any type \( D \) structure is homotopy equivalent to a reduced one. See \([10\text{ Section 2.6}]\) for more details.

Finally, if \( M \) is a type \( A \) module and the pair \((M, V)\) is relatively bounded, then the differential on the box tensor product \( M \boxtimes V \) is given explicitly by
\[
\partial_\mathcal{I}^V(x \otimes y) = \sum_{I_1, \ldots, I_r \in \mathcal{I}} m_{r+1}(x \otimes \rho_{I_1} \otimes \cdots \otimes \rho_{I_r}) \otimes (D_{I_r} \circ \cdots \circ D_{I_1})(y).
\]
for each \( x \in M \) and \( y \in V \). (The sum includes an \( r = 0 \) term, where the composition of zero coefficient maps is the identity on \( V \).)

### 2.3. Computing \( \widehat{\mathrm{CFA}} \) from \( \widehat{\mathrm{CFD}} \)
Let \( r : T^2 \to T^2 \) be the orientation-reversing involution that interchanges the two coordinates of \( S^1 \times S^1 \). This involution gives a one-to-one correspondence between type \( A \) and type \( D \) bordered manifolds, given by \((Y, \phi) \mapsto (Y, \phi \circ r)\). The bordered invariants of \((Y, \phi)\) and \((Y, \phi \circ r)\) are related by taking tensor products with the appropriate identity bimodules, \( \widehat{\mathrm{CFAA}}(\mathbb{I}) \) and \( \widehat{\mathrm{CFDD}}(\mathbb{I}) \). Here \( \mathbb{I} \) denotes the manifold \( T^2 \times \mathbb{I} \), with boundary parametrized appropriately. If \((Y, \phi)\) is a type \( A \) bordered 3-manifold, \([11\text{ Corollary 1.1}]\) says that
\[
(6) \quad \widehat{\mathrm{CFD}}(Y, \phi \circ r) \simeq \widehat{\mathrm{CFA}}(Y, \phi) \otimes \widehat{\mathrm{CFDD}}(\mathbb{I}) \quad \text{and} \quad \widehat{\mathrm{CFA}}(Y, \phi) \simeq \widehat{\mathrm{CFAA}}(\mathbb{I}) \otimes \widehat{\mathrm{CFD}}(Y, \phi \circ r).
\]
Here, we view \( \widehat{\mathrm{CFAA}}(\mathbb{I}) \) as a right-left \( AA \) bimodule and \( \widehat{\mathrm{CFDD}}(\mathbb{I}) \) as a left-left \( DD \) bimodule, each over two copies of \( \mathcal{A}(\mathbb{I}) \). Thus, if a parametrization \( \phi \) (either orientation-preserving or orientation-reversing) is understood from context, we will simply speak of \( \widehat{\mathrm{CFA}}(Y) \) and

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1Our perspective here is slightly different from that of Lipshitz, Ozsváth, and Thurston, who use two distinct algebras associated to \( T^2 \) and \(-T^2\), denoted \( \mathcal{A}(T^2) \) and \( \mathcal{A}(-T^2) \), where \( \mathcal{A}(-T^2) = \mathcal{A}(T^2)^{\text{op}} \). If \((Y, \phi)\) is a type \( A \) bordered manifold (in the sense used above), then one can define \( \widehat{\mathrm{CFD}}(Y, \phi) \) as a type \( D \) structure over \( \mathcal{A}(-T^2) \), and one views \( \widehat{\mathrm{CFAA}}(\mathbb{I}) \) and \( \widehat{\mathrm{CFDD}}(\mathbb{I}) \) as \( (\mathcal{A}(T^2), \mathcal{A}(-T^2))-\)bimodules. To see how \([11]\) follows from \([11\text{ Corollary 1.1}]\), note that the map \( r \) (which can be realized as the symmetry of the pointed matched circle associated to the torus) induces an isomorphism between \( \mathcal{A}(T^2) \) and \( \mathcal{A}(-T^2) \), which gives the identification between \( \widehat{\mathrm{CFD}}(Y, \phi) \) (as a type \( D \) module over \( \mathcal{A}(-T^2) \)) and \( \widehat{\mathrm{CFD}}(Y, \phi \circ r) \) (as a type \( D \) module over \( \mathcal{A}(T^2) \)). We find it conceptually simpler to work with a single algebra, at the cost of being more explicit about the role of \( r \).
Note that the two halves of (6) are equivalent statements, since by [13, Theorem 2.3] the operations of tensoring with $\tilde{CFA}(\mathbb{I})$ and $\tilde{CFDD}(\mathbb{I})$ are inverses up to homotopy equivalence. That is, if $M \simeq \mathbb{I}$ is a type $A$ module and $N$ is a type $D$ module, then $M \simeq \tilde{CFA}(\mathbb{I}) \otimes N$ if and only if $M \otimes \tilde{CFDD}(\mathbb{I}) \simeq N$.

We now describe an algorithm for computing $\tilde{CFA}(Y)$ from $\tilde{CFD}(Y)$, based on an idea described to us by Peter Ozsváth. The basic idea is as follows. We begin by taking a basis for $\hat{C}$. Described to us by Peter Ozsváth. The basic idea is as follows. We begin by taking a basis for $\hat{C}$, which we shall write elements of $\mathbb{R}$ as strings of digits, without parentheses or commas.

For example, suppose in $\tilde{CFD}(Y)$ we have $D_{23} \circ D_{23} \circ D_{123}(v) = w$. We first take the string $123232$ and replace it with $321212$, which we then parse as $3, 2, 1, 12, 12, 1$. This tells us that in $\tilde{CFA}(Y)$ we have a multiplication $m_6(v \otimes \rho_3 \otimes \rho_2 \otimes \rho_1 \otimes \rho_1) = w$. (See Section 4 for an example of this procedure applied to $\tilde{CFA}$ of the trefoil complement.)

To be more precise, let $\mathcal{S}$ denote the set of strictly decreasing, nonempty sequences of consecutive elements of $\{1, 2, 3\}$ and let $\phi: \mathcal{S} \to \mathcal{R}$ denote the bijection defined by interchanging the roles of 1 and 3:

$$\phi(1) = 3 \quad \phi(2) = 2 \quad \phi(3) = 1$$

$$\phi(21) = 23 \quad \phi(32) = 12 \quad \phi(321) = 123.$$  

Strong induction on length shows that for any alternating sequence $I$ of elements of $\{1, 2, 3\}$, there is a unique decomposition $I = J_1 \cdots J_j$ such that $J_1, \ldots, J_j \in \mathcal{S}$ and for each $i = 1, \ldots, j - 1$, last($J_i$) < first($J_{i+1}$). In this case, we define $\Psi(I) = (J_1, \ldots, J_j)$. The following lemma is an immediate consequence of the definition of $\Psi$:

**Lemma 2.1.** Let $I$ and $I'$ be alternating sequences whose concatenation $II'$ is alternating. Suppose that $\Psi(I) = (J_1, \ldots, J_j)$ and $\Psi(I') = (K_1, \ldots, K_k)$. Then

$$\Psi(II') = \begin{cases} (J_1, \ldots, J_j, K_1, \ldots, K_k) & \text{if } \text{last}(I) < \text{first}(I') \\ (J_1, \ldots, J_{j-1}, J_j, K_1, K_2, \ldots, K_k) & \text{if } \text{last}(I) > \text{first}(I'). \end{cases}$$

The algorithm is given by the following theorem:

**Theorem 2.2.** Let $(V, \delta_1)$ be a reduced type-$D$ module over $\mathbb{A}$, seen as a finite-dimensional vector space $V = V^0 \oplus V^1$, with coefficient maps $D_1, D_2, D_3, D_{12}, D_{23}, D_{123}$ satisfying (5). For $k \geq 0$, define maps

$$m_{k+1}: V \otimes \mathbb{A}^\otimes k \to V$$

as follows. Set $m_1 = 0$. For $k > 1$ and any $I_1, \ldots, I_k \in \mathcal{R}$ whose concatenation $I_1 \cdots I_k$ is alternating and for which $\text{last}(I_i) > \text{first}(I_{i+1})$ for all $i = 1, \ldots, k - 1$, write $\Psi(I_1 \cdots I_k) = \phi$.

---

1. We shall write elements of $\mathcal{R}$ and $\mathcal{S}$ as strings of digits, without parentheses or commas.
(J_1, \ldots, J_j) and define

$$m_{k+1}(v \otimes \rho_{I_1} \otimes \cdots \otimes \rho_{I_k}) = (D_{\phi(J_j)} \circ \cdots \circ D_{\phi(J_1)})(v)$$

for all \( v \in V \). For any other \( I_1, \ldots, I_k \), define

$$m_{k+1}(v \otimes \rho_{I_1} \otimes \cdots \otimes \rho_{I_k}) = 0.$$  

Then the maps \( m_k \) satisfy the \( \mathcal{A}_\infty \) relations. Furthermore, we have

$$(V, \delta_1) \simeq (V, \{ m_k \}) \cong \text{CFDD}(I) \quad \text{and} \quad (V, \{ m_k \}) \simeq \text{CFAA}(I) \cong (V, \delta_1).$$

Proof of Theorem 2.2. To see that the maps \( m_k \) satisfy the \( \mathcal{A}_\infty \) relations, we must show that for any \( I_1, \ldots, I_k \in \mathcal{R} \) and any \( v \in V \),

$$\sum_{i=1}^{k-1} m_{k-i+1}(m_{i+1}(v \otimes \rho_{I_1} \otimes \cdots \otimes \rho_{I_k}) \otimes \rho_{I_{i+1}} \otimes \cdots \otimes \rho_{I_k}) + \sum_{i=1}^{k-1} m_{k-1}(v \otimes \rho_{I_1} \otimes \cdots \otimes \rho_{I_{i+1}} \otimes \rho_{I_i} \rho_{I_{i+1}} \otimes \rho_{I_{i+2}} \otimes \cdots \otimes \rho_{I_k}) = 0. $$

We may assume that \( I_1 \ldots I_k \) is alternating, since otherwise all the terms in (7) would vanish because the tensor products are taken over the ring of idempotents. Indeed, if we had a term such as \( \rho_1 \otimes \rho_3 \) we could write it as \( \rho_1 \otimes \iota_1 \rho_3 = \rho_1 \otimes 0 \), with similar expressions for any other non-alternating occurrence.

If \( \text{last}(I_i) < \text{first}(I_{i+1}) \), then for any \( i' \neq i \), the \( i' \)th terms of both sums in (7) must both vanish by definition. Thus, we may assume that there is at most one value of \( i \) for which \( \text{last}(I_i) < \text{first}(I_{i+1}) \). For this \( i \), if it exists, Lemma 2.1 implies that if

$$\Psi(I_1 \cdots I_i) = (L^i_1, \ldots, L^i_k), \quad \Psi(I_{i+1} \cdots I_k) = (M^i_1, \ldots, M^i_{m_i}),$$

then \( \ell_i + m_i = j \) and

$$(L^i_1, \ldots, L^i_{\ell_i}, M^i_1, \ldots, M^i_{m_i}) = (J_1, \ldots, J_j).$$

Thus the \( i \)th term of the first sum in (7) equals

$$(D_{\phi(J_j)} \circ \cdots \circ D_{\phi(J_1)})(v).$$

Since \( \rho_{I_i} \rho_{I_{i+1}} = \rho_{I_i L_{i+1}} \), the \( i \)th term of the second sum in (7) equals (8) as well. Thus, the \( i \)th terms of the two sums in (7) cancel each other, and all other terms in both sums vanish.

Thus, we may assume that for all \( i = 1, \ldots, k - 1 \), we have \( \text{last}(I_i) > \text{first}(I_{i+1}) \). Since \( \rho_{I_i} \rho_{I_{i+1}} = 0 \), the entire second sum in (7) vanishes. Suppose that \( \Psi(I_1 \cdots I_k) = (J_1, \ldots, J_j) \). For each \( i = 1, \ldots, k - 1 \), suppose that \( \Psi(I_1 \cdots I_i) = (L^i_1, \ldots, L^i_{\ell_i}) \) and \( \Psi(I_{i+1} \cdots I_k) = (M^i_1, \ldots, M^i_{m_i}) \). Thus, the first sum in (7) equals

$$(9) \quad \sum_{i=1}^{k-1} (D_{\phi(L^i_{\ell_i})} \circ \cdots \circ D_{\phi(L^i_1)})(v).$$

Furthermore, Lemma 2.1 implies that \( \ell_i + m_i + 1 = j \) and

$$(J_1, \ldots, J_j) = (L^i_1, \ldots, L^i_{\ell_i-1}, L^i_{\ell_i} M^i_1, M^i_2, \ldots, M^i_{m_i}).$$

In particular, the concatenation \( L^i_{\ell_i} M^i_1 \) equals either 21, 32, or 321.
If \( L_{i_1} = 3 \) and \( M_i^1 = 2 \), then \( D_{\phi(M_1^1)} \circ D_{\phi(L_{i_1}^1)} = D_2 \circ D_1 = 0 \) by (5), and therefore the \( i \)th term of (9) vanishes. The same argument holds when \( L_{i_1} = 2 \) and \( M_i^1 = 1 \) using the fact that \( D_3 \circ D_2 = 0 \).

If \( L_{i_1} = 32 \) and \( M_i^1 = 1 \), the concatenation \( I_1 \cdots I_i \) ends in 32, and we must have \( i > 1 \), last(\( I_{i-1} \)) = 3, \( I_i = 2 \), and first(\( I_{i+1} \)) = 1. Therefore,

\[
\Psi(I_1 \cdots I_{i-1}) = (L_{i_1}, \ldots, L_{i_{\ell_i-1}}, 3) \quad \text{and} \quad \Psi(I_i \cdots I_k) = (21, M_{i_2}, \ldots, M_{m}^i).
\]

The sum of the \((i-1)\)th and \(i\)th terms of (9) then equals

\[
\sum_{i=1}^{k-1} (D_{\phi(M_{m_1}^i)} \circ \cdots \circ D_{\phi(M_{i_1}^i)} \circ (D_3 \circ D_{12} + D_{23} \circ D_1) \circ D_{\phi(L_{i_{\ell_i-1}}^i)} \circ \cdots \circ D_{\phi(L_{i_1}^i)})(v),
\]

which vanishes by (5).

If \( L_{i_1} = 3 \) and \( M_i^1 = 21 \), a similar argument shows that the \( i \)th and \((i+1)\)th terms of (9) cancel. This completes the proof of (7). Thus, the maps \( m_k \) satisfy the \( A_\infty \) relations.

For the second part of the theorem, as noted in the discussion following (6), it suffices to show that

\[
(V, \delta_1) \simeq (V, \{m_k\}) \boxtimes \widehat{\text{CFDD}(I)}.
\]

According to [11, Proposition 10.1], the left-left \( DD \) bimodule \( \widehat{\text{CFDD}(I)} \) has generators \( p, q \), with idempotent action given by

\[
(\iota_0 \otimes \iota_0) \cdot p = p \quad \text{and} \quad (\iota_1 \otimes \iota_1) \cdot q = q
\]

and structure map given by

\[
\delta_1(p) = (\rho_1 \otimes \rho_3 + \rho_3 \otimes \rho_1 + \rho_{123} \otimes \rho_{123}) \otimes q \quad \text{and} \quad \delta_1(q) = \rho_2 \otimes \rho_2 \otimes p.
\]

Thus, \( (V, \{m_k\}) \boxtimes \widehat{\text{CFDD}(I)} \) is isomorphic to \( V \) as a vector space. According to the definition of the box tensor product of a type \( A \) module and a type \( DD \) bimodule, for \( v \in V^0 \), we have:

\[
\delta_1(v \otimes p) = (\rho_1 \otimes m_2(v, \rho_3) + \rho_3 \otimes m_2(v, \rho_1) + \rho_{123} \otimes (m_2(v, \rho_{123}) + m_4(v, \rho_3, \rho_2, \rho_1))) \otimes q
\]

\[
+ \rho_{12} \otimes m_3(v, \rho_3, \rho_2) \otimes p
\]

\[
= (\rho_1 \otimes D_1(v) + \rho_3 \otimes D_3(v) + \rho_{123} \otimes (D_3 \circ D_2 \circ D_1(v) + D_{123}(v))) \otimes q
\]

\[
+ \rho_{12} \otimes D_{12}(v) \otimes p
\]

\[
= (\rho_1 \otimes D_1(v) + \rho_3 \otimes D_3(v) + \rho_{123} \otimes D_{123}(v)) \otimes q + \rho_{12} \otimes D_{12}(v) \otimes p,
\]

where the final line follows from the fact that \( D_2 \circ D_1 = D_3 \circ D_2 = 0 \). Likewise, for \( w \in V^1 \),

\[
\delta_1(w \otimes q) = \rho_2 \otimes m_2(w, \rho_3) \otimes p + \rho_{23} \otimes m_3(w, \rho_2, \rho_1) \otimes q
\]

\[
= \rho_2 \otimes D_2(w) \otimes p + \rho_{23} \otimes D_{23}(w) \otimes q.
\]

Thus, the differential on \( (V, \{m_k\}) \boxtimes \widehat{\text{CFDD}(I)} \) is equal to the original differential on \( V \).
2.4. Bordered invariants of knot complements. If $K$ is a knot in a homology sphere $Y$, and $X_K$ denotes the exterior of $K$, let $\phi_K: T^2 \to \partial X_K$ be the orientation-reversing parametrization taking $S^1 \times \{pt\}$ to a 0-framed longitude of $K$. When $Y$ is an L-space, Lipshitz, Ozsváth, and Thurston give a formula for $\widehat{CFD}(X_K, \phi_K)$ in terms of the knot Floer complex of $(Y, K)$, which we now describe (adding a few details).

We begin by reviewing some facts about knot Floer homology, as defined in \cite{16, 26}. Let $C^- = \text{CFK}^-(Y, K)$ denote the knot Floer complex of $K$, a finitely generated free chain complex over $\mathbb{F}[U]$ with a bounded-above filtration

$$\cdots \subset \mathcal{F}_i \subset \mathcal{F}_{i+1} \subset \cdots \subset C^-$$

such that $U \cdot \mathcal{F}_i \subset \mathcal{F}_{i-1}$ for all $i$. The filtered chain homotopy type of this complex is an invariant of the knot.

For any nonzero $x \in C^-$, let $A(x) = \min\{i \mid x \in \mathcal{F}_i\}$; we call this the filtration level or Alexander grading of $x$. Multiplication by $U$ decreases the Alexander grading by one: $A(U \cdot x) = A(x) - 1$. By convention, $A(0) = -\infty$. We may assume that $C^-$ is reduced, in the sense that for any $x \in C^-$, $\partial x = U \cdot y + z$, where $A(z) < A(x)$. The manner by which knot Floer homology detects the genus \cite{21} Theorem 1.2 implies $F_{g(K)-1} \subseteq F_{g(K)} = C^-$, $F_{-g(K)-1} \subset UC^-$, and $F_{-g(K)} \not\subseteq UC^-$. We assume that $\text{rank}_{\mathbb{F}[U]} C^- = 2n + 1$.

Let $C^\infty = C^- \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}]$, and extend the filtration to $C^\infty$ accordingly. If $\{x_0, \ldots, x_{2n}\}$ is a basis for $C^-$ over $\mathbb{F}[U]$, then $\{U^ix_k \mid k = 0, \ldots, 2n, i \in \mathbb{Z}\}$ is a basis for $C^\infty$ over $\mathbb{F}$. We may picture these basis elements living on the integer lattice in $\mathbb{R}^2$, with the element $U^ix_k$ at the point $(-k, A(x_k) - k)$. We refer to the coordinates in the plane as $i$ and $j$, and we identify $C^-$ with the subcomplex $C\{i \leq 0\} \subset C^\infty$ generated by the basis elements at lattice points with $i \leq 0$. The complexes $C\{i \leq s\} (s \in \mathbb{Z})$ provide a second filtration on $C^-$ and $C^\infty$.

Let $C^v = C^-/UC^-$, and let $\partial^v$ denote the induced differential. Let $C^h = \mathcal{F}_0(C^\infty)/\mathcal{F}_{-1}(C^\infty)$, and let $\partial^h$ denote the induced differential. We refer to $(C^v, \partial^v)$ and $(C^h, \partial^h)$ as the vertical and horizontal complexes, respectively.

The associated graded object of $C^-$ (with respect to the original filtration) is the free $\mathbb{F}[U]$-module

$$\text{gr}(C^-) = \bigoplus_{i \in \mathbb{Z}} \mathcal{F}_i/\mathcal{F}_{i-1},$$

with the induced multiplication by $U$. Note that the direct sum is as an $\mathbb{F}$-vector space, and not as an $\mathbb{F}[U]$-module since multiplication by $U$ decreases the filtration by one. For $x \in C^-$, let $[x] \in \text{gr}(C^-)$ denote the image of $x$ in $\mathcal{F}_{A(x)}/\mathcal{F}_{A(x)-1}$. Note that $[Ux] = U[x]$. A basis $\{x_0, \ldots, x_{2n}\}$ for $C^-$ is called a filtered basis if $\{[x_0], \ldots, [x_{2n}]\}$ is a basis for $\text{gr}(C^-)$ over $\mathbb{F}[U]$. Any two filtered bases $\{x_0, \ldots, x_{2n}\}$ and $\{x_0', \ldots, x_{2n}'\}$ are related by a filtered change of basis: if $x_i = \sum_j a_{ij}x'_j$ and $x'_i = \sum_j b_{ij}x_j$, where $a_{ij}, b_{ij} \in \mathbb{F}[U]$, then $A(a_{ij}x'_j) \leq A(x_i)$ and $A(b_{ij}x_j) \leq A(x'_i)$ for all $i, j$. In particular, if $a_{ij} \neq 0 \mod UC^-$, then $A(x'_j) \leq A(x_i)$, and similarly for $b_{ij}$.

A key tool for our main theorem is a formula which expresses $\widehat{CFD}(X_K)$ in terms of $\text{CFK}^-(Y, K)$. The most useful way to express this formula is by picking a basis for $\text{CFK}^-(Y, K)$ and describing $\widehat{CFD}(X_K)$ in terms of this basis. To do this it will be useful to have particularly nice bases for $\text{CFK}^-$, whose definitions we now recall.
Definition 2.3. A filtered basis \( \{\xi_0, \ldots, \xi_{2n}\} \) for \( C^- \) over \( F[U] \) is called \textit{vertically simplified} if, for \( j = 1, \ldots, n \),
\[
A(\xi_{2j-1}) - A(\xi_{2j}) = k_j > 0 \quad \text{and} \quad \partial \xi_{2j-1} \cong \xi_{2j} \pmod{UC^-},
\]
while for \( p = 0, 1, \ldots, n \), we have \( \partial \xi_{2p} \in UC^- \). We say that there is a \textit{vertical arrow of length} \( k_j \) from \( \xi_{2j-1} \) to \( \xi_{2j} \) and that \( \xi_0 \) is the \textit{generator of vertical homology}.

The name is motivated by the fact that in a vertically simplified basis the differential on the vertical complex \( (C^v, \partial^v) \) is particularly simple; indeed, in a vertically simplified basis \( \partial^v \) can be represented by a collection of vertical arrows which pair up the even and odd basis elements, and where \( \xi_0 \) has no incoming or outgoing arrows. Similarly, for the horizontal complex we have

Definition 2.4. A filtered basis \( \{\eta_0, \ldots, \eta_{2n}\} \) for \( C^- \) over \( F[U] \) is called \textit{horizontally simplified} if, for \( j = 1, \ldots, n \),
\[
A(\eta_{2j}) - A(\eta_{2j-1}) = \ell_j > 0 \quad \text{and} \quad \partial \eta_{2j-1} \cong U^{\ell_j} \eta_{2j} \pmod{F_{A(\eta_{2j-1})-1}},
\]
while for \( p = 0, 1, \ldots, n \), we have \( A(\partial \eta_{2p}) < A(\eta_{2p}) \). We say that there is a \textit{horizontal arrow of length} \( \ell_j \) from \( \eta_{2j-1} \) to \( \eta_{2j} \) and that \( \eta_0 \) is the \textit{generator of horizontal homology}.

Lipshitz, Ozsváth, and Thurston showed that \( C^- \) always admits both horizontal and vertically simplified bases \cite[Proposition 11.52]{Def}. Furthermore, for any vertically simplified basis \( \{\xi_0, \ldots, \xi_{2n}\} \) and horizontally simplified basis \( \{\eta_0, \ldots, \eta_{2n}\} \), the unordered tuples \( \{k_1, \ldots, k_n\} \) and \( \{\ell_1, \ldots, \ell_n\} \) are equal; this follows from the symmetry of knot Floer homology under reversing the knot orientation \cite[Section 3.5]{Def}.

Two particularly useful derivatives of the filtered chain homotopy type of \( C^\infty \) can be expressed easily in terms of a vertical or horizontally simplified basis. The first is the Ozsváth-Szabó concordance invariant \cite{Def, Def}. Denoted \( \tau(K) \), this invariant is a homomorphism from the smooth concordance group to the integers which bounds the smooth 4-genus \( |\tau(K)| \leq g_4(K) \). In terms of a vertically simplified basis, we have
\[
\tau(K) = A(\xi_0),
\]
while in terms a horizontally simplified basis we have
\[
\tau(K) = -A(\eta_0).
\]
The latter equality again follows from the orientation reversal symmetry.

The second invariant we derive from \( C^\infty \) is Hom’s invariant \( \epsilon(K) \in \{-1, 0, 1\} \), which captures when the four-dimensional cobordisms obtained by attaching two-handles to \( Y \) along \( K \) induce nontrivial maps on Floer homology in certain \( \text{Spin}^c \)-structures \cite[Definition 3.1]{Def}. This invariant can also be expressed in terms of vertically and horizontally simplified bases. Let \( [\eta_0] \) denote the image of \( \eta_0 \) in the vertical complex \( C^v \). Also, viewing \( \xi_0 \) as an element of \( C^\infty \), the chain \( \xi'_0 = U^{A(\xi_0)} \xi_0 \) is in \( F_0 \), so we may consider its image \( [\xi'_0] \) in the horizontal complex \( C^h \). Then:

- If \( \epsilon(K) = -1 \), then \( \partial^v[\eta_0] \neq 0 \) and \( \partial^h[\xi'_0] \neq 0 \).
- If \( \epsilon(K) = 0 \), then \( [\eta_0] \in \ker \partial^v \cap \im \partial^v \) and \( [\xi'_0] \in \ker \partial^h \cap \im \partial^h \).
- If \( \epsilon(K) = 1 \), then \( [\eta_0] \in \im \partial^v \) and \( [\xi'_0] \in \im \partial^h \).

The following proposition tells us that the change of basis passing between horizontally and vertically simplified bases can be assumed to be relatively well-behaved.
Proposition 2.5. There exist filtered bases \( \{\xi_0, \ldots, \xi_{2n}\} \) and \( \{\eta_0, \ldots, \eta_{2n}\} \) for \( C^- \) over \( \mathbb{F}[U] \) with the following properties:

(1) \( \{\xi_0, \ldots, \xi_{2n}\} \) is a vertically simplified basis.

(2) \( \{\eta_0, \ldots, \eta_{2n}\} \) is a horizontally simplified basis.

(3) If \( \epsilon(K) = -1 \), then \( \xi_0 = \eta_1 \). If \( \epsilon(K) = 0 \), then \( \xi_0 = \eta_0 \). If \( \epsilon(K) = 1 \), then \( \xi_0 = \eta_2 \).

(4) If

\[
\xi_p = \sum_{q=0}^{2n} a_{p,q} \eta_q \quad \text{and} \quad \eta_p = \sum_{q=0}^{2n} b_{p,q} \xi_q,
\]

where \( a_{p,q}, b_{p,q} \in \mathbb{F}[U] \), then \( a_{p,q} = 0 \) whenever \( A(\xi_p) \neq A(a_{p,q} \eta_q) \), and \( b_{p,q} = 0 \) whenever \( A(\eta_p) \neq A(b_{p,q} \xi_q) \). In other words, each \( \xi_p \) is an \( \mathbb{F}[U] \)-linear combination of the elements \( \eta_q \) that are the same filtration level as \( \xi_p \), and vice versa.

Proof. According to Hom \( \mathbb{F}[U] \), Lemmas 3.2, 3.3], we may find vertically and horizontally simplified bases \( \{\xi_0, \ldots, \xi_{2n}\} \) and \( \{\eta_0, \ldots, \eta_{2n}\} \) satisfying \( \Box \) (with \( \eta_i \) replaced by \( \eta'_i \)). We shall modify the latter basis to produce a new basis \( \{\eta_0, \ldots, \eta_{2n}\} \) satisfying the conclusions of the proposition.

As above, any two filtered bases are related by a filtered change of bases, so let

\[
\xi_p = \sum_{q=0}^{2n} a'_{p,q} \eta'_q \quad \text{and} \quad \eta_p' = \sum_{q=0}^{2n} b'_{p,q} \xi_q,
\]

be filtered change of bases. That is, for all \( p, q \in \{0, \ldots, 2n\} \), we have

\[
A(a'_{p,q} \eta'_q) \leq A(\xi_p) \quad \text{and} \quad A(b'_{p,q} \xi_q) \leq A(\eta'_p).
\]

Let

\[
b_{p,q} = \begin{cases} 
  b'_{p,q} & \text{if } A(b'_{p,q} \xi_q) = A(\eta'_p) \\
  0 & \text{if } A(b_{p,q} \xi_q) < A(\eta'_p),
\end{cases}
\]

and define

\[
\eta_p = \sum_{q=0}^{2n} b_{p,q} \xi_q \quad \text{and} \quad \Delta_p = \eta'_p - \eta_p.
\]

Note that \( A(\eta_p) = A(\eta'_p) \), while \( A(\Delta_p) < A(\eta'_p) \). The change-of-basis matrix \( (b_{p,q}) \) is in block-diagonal form (after reordering rows and columns according to filtration level), so its inverse is as well. Thus the bases \( \{\xi_0, \ldots, \xi_{2n}\} \) and \( \{\eta_0, \ldots, \eta_{2n}\} \) satisfy \( \Box \). Furthermore, if \( i \in \{0, 1, 2\} \) is the index for which \( \xi_0 = \eta'_i \), then \( \eta_0 = \eta'_i \) by construction, so \( \Box \) also holds.

It remains to show that the basis \( \{\eta_0, \ldots, \eta_{2n}\} \) is horizontally simplified. For any \( j = 1, \ldots, n \), we have:

\[
\partial \eta_{2j-1} = \partial \eta'_{2j-1} - \partial \Delta_{2j-1}
\]

\[
\equiv \partial \eta'_{2j-1} \pmod{\mathcal{F}_A(\eta'_{2j-1})-1}
\]

\[
\equiv U^{\ell_j} \eta'_j \pmod{\mathcal{F}_A(\eta'_{2j-1})-1}
\]

\[
= U^{\ell_j} \eta_{2j} + U^{\ell_j} \Delta_{2j}
\]

\[
\equiv U^{\ell_j} \eta_{2j} \pmod{\mathcal{F}_A(\eta_{2j-1})-1},
\]

where \( \ell_j \) is the index for which \( \eta_{2j-1} = \eta'_j \).
where the last line follows from the fact that
\[ A(U^\ell_j \Delta_{2j}) = A(\Delta_{2j}) - \ell_j < A(\eta_{2j}') - \ell_j = A(\eta_{2j-1}') = A(\eta_{2j-1}). \]
Likewise, for \( j = 0, 1, \ldots, n \), we have
\[ \partial \eta_{2j} = \partial \eta_{2j}' + \partial \Delta_{2j} \in F_{A(\eta_{2j}')}, \]
as required. \[ \square \]

For the remainder of this section, choose vertically and horizontally simplified bases \( \{\tilde{\xi}_0, \ldots, \tilde{\xi}_{2n}\} \) and \( \{\tilde{\eta}_0, \ldots, \tilde{\eta}_{2n}\} \) for \( \text{CFK}^- (Y, K) \) satisfying the conclusions of Proposition 2.5. Assume that
\[ \tilde{\xi}_p = \sum_{q=0}^{2n} a_{p,q} \tilde{\eta}_q \quad \text{and} \quad \tilde{\eta}_p = \sum_{q=0}^{2n} b_{p,q} \tilde{\xi}_q, \]
where \( a_{p,q}, b_{p,q} \in \mathbb{F}[U] \), and let \( a_{p,q} = a_{p,q}|_{U=0} \) and \( b_{p,q} = b_{p,q}|_{U=0} \). According to Lipshitz, Ozsváth, and Thurston [12, Theorem 11.27 and Theorem A.11], \( \widehat{\text{CFD}}(X_K, \phi_K) \) is completely determined by the lengths of the arrows (i.e., \( k_j \) and \( \ell_j \)), \( \tau(K) \), and the change-of-basis matrix \( (a_{p,q}) \), as follows.

**Theorem 2.6.** With notation as above, \( \widehat{\text{CFD}}(X_K) \) satisfies the following properties:

- The summand \( \iota_0 \widehat{\text{CFD}}(X_K) \) has dimension \( 2n + 1 \), with designated bases \( \{\xi_0, \ldots, \xi_{2n}\} \) and \( \{\eta_0, \ldots, \eta_{2n}\} \) related by
  \[ \xi_p = \sum_{q=0}^{2n} a_{p,q} \eta_q \quad \text{and} \quad \eta_p = \sum_{q=0}^{2n} b_{p,q} \xi_q. \]

- The summand \( \iota_1 \widehat{\text{CFD}}(X_K) \) has dimension \( \sum_{j=1}^{n} (k_j + l_j) + t \), where \( t = 2 |\tau(K)| \), with basis
  \[ \bigcup_{j=1}^{n} \{k_1^j, \ldots, k_{k_j}^j\} \cup \bigcup_{j=1}^{n} \{\lambda_1^j, \ldots, \lambda_{l_j}^j\} \cup \{\mu_1, \ldots, \mu_t\}. \]

- For \( j = 1, \ldots, n \), corresponding to the vertical arrow \( \tilde{\xi}_{2j-1} \to \tilde{\xi}_{2j} \) of length \( k_j \), there are coefficient maps
  \[ \xi_{2j} \xrightarrow{D_{123}} k_1^j \xrightarrow{D_{23}} \cdots \xrightarrow{D_{23}} k_{k_j}^j \xleftarrow{D_1} \xi_{2j-1}. \]

- For \( j = 1, \ldots, n \), corresponding to the horizontal arrow \( \tilde{\eta}_{2j-1} \to \tilde{\eta}_{2j} \) of length \( l_j \), there are coefficient maps
  \[ \eta_{2j-1} \xrightarrow{D_3} \lambda_1^j \xrightarrow{D_{23}} \cdots \xrightarrow{D_{23}} \lambda_{l_j}^j \xrightarrow{D_2} \eta_{2j}, \]

- Depending on \( \tau(K) \), there are additional coefficient maps
  \[ \begin{cases} 
  \eta_0 \xrightarrow{D_3} \mu_1 \xrightarrow{D_{23}} \cdots \xrightarrow{D_{23}} \mu_t \xleftarrow{D_1} \xi_0 & \tau(K) > 0 \\
  \xi_0 \xrightarrow{D_{23}} \eta_0 & \tau(K) = 0 \\
  \xi_0 \xrightarrow{D_{23}} \mu_1 \xrightarrow{D_{23}} \cdots \xrightarrow{D_{23}} \mu_t \xrightarrow{D_2} \eta_0 & \tau(K) < 0.
  \end{cases} \]

\(^3\text{We use tildes for the generators of } \text{CFK}^- (Y, K) \text{ in order to distinguish them from the corresponding elements of } \widehat{\text{CFD}}(X_K).\)
We refer to the subspaces of \( \hat{\text{CFD}}(X_K) \) spanned by the generators in (10), (11), and (12) as the vertical chains, horizontal chains, and unstable chain, respectively.\(^4\)

As described in [12, Lemma 11.40], \( \hat{\text{CFD}}(X_K) \) admits a grading by half-integers, taking integer values on \( \iota_0 \hat{\text{CFD}}(X_K) \) and non-integer values on \( \iota_1 \hat{\text{CFD}}(X_K) \):

\[
\iota_0 \hat{\text{CFD}}(X_K) = \bigoplus_{s \in \mathbb{Z}} \hat{\text{CFD}}(X_K, s) \quad \text{and} \quad \iota_1 \hat{\text{CFD}}(X_K) = \bigoplus_{s \in \mathbb{Z} + \frac{1}{2}} \hat{\text{CFD}}(X_K, s).
\]

We refer to this grading as the Alexander grading. This is justified since there are canonical identifications:

\[
\iota_0 \hat{\text{CFD}}(X_K) \cong \hat{\text{HFK}}(K) \quad \text{and} \quad \iota_1 \hat{\text{CFD}}(X_K) \cong \hat{\text{HFL}}(K),
\]

where the latter invariant is the longitude Floer homology [5]. Under these identifications the grading on the summands of \( \hat{\text{CFD}}(X_K) \) agrees with the Alexander gradings on each of these groups. Proposition 2.5 implies that the Alexander gradings (in \( \hat{\text{CFD}}(X_K) \)) of \( \xi_0, \ldots, \xi_{2n} \) and \( \eta_0, \ldots, \eta_{2n} \) are equal to the filtration levels (in \( \text{CFK}^{-}(Y, K) \)) of \( \xi_0, \ldots, \xi_{2n} \) and \( \tilde{\eta}_0, \ldots, \tilde{\eta}_{2n} \), respectively, and that the change of basis is homogeneous. We denote the grading of a homogeneous element \( x \) by \( A(x) \), and for each \( s \in \frac{1}{2}\mathbb{Z} \), let \( \pi_s: \hat{\text{CFD}}(X_K) \to \hat{\text{CFD}}(X_K, s) \) be the projection map coming from (13). The Alexander grading on \( \hat{\text{CFD}}(X_K) \) will eventually enable us to isolate certain pieces of the chain complex for a spliced manifold. As seen in [12], the coefficient maps on \( \hat{\text{CFD}}(X_K) \) are all homogeneous with respect to \( A \), with the following degrees:

| Coefficient map | \( D_1 \) | \( D_2 \) | \( D_3 \) | \( D_{12} \) | \( D_{23} \) | \( D_{123} \) |
|----------------|---------|---------|---------|---------|---------|---------|
| \( A \)-degree | \(-\frac{1}{2}\) | \( \frac{1}{2}\) | \( \frac{1}{2}\) | \( 0\) | \( 1\) | \( \frac{1}{2}\) |

Note that \( \hat{\text{CFD}}(X_K) \) need not be bounded; for instance, if \( K \) is the unknot, then \( \hat{\text{CFD}}(X_K) \) has a single generator \( \xi \), with \( \delta_1(\xi) = \rho_{12} \otimes \xi \), and therefore \( \delta_k(\xi) = \rho_{12} \otimes \cdots \otimes \rho_{12} \otimes \xi \) for all \( k \geq 1 \).

3. Splicing knot complements

For any knots \( K_1 \subset Y_1 \) and \( K_2 \subset Y_2 \), note that the composition

\[
\phi_{K_2} \circ r \circ \phi_{K_1}^{-1}: \partial X_{K_1} \to \partial X_{K_2}
\]

is orientation-reversing and takes a 0-framed longitude of \( K_1 \) to a meridian of \( K_2 \) and a meridian of \( K_1 \) to a 0-framed longitude of \( K_2 \). The manifold gotten by gluing \( X_{K_1} \) and \( X_{K_2} \) via \( \phi_{K_2} \circ r \circ \phi_{K_1}^{-1} \) is thus precisely \( Y(K_1, K_2) \), as defined in the introduction. Therefore, we have

\[
\hat{\text{HF}}(Y(K_1, K_2)) = H_*(\hat{\text{CFA}}(X_{K_1}, \phi_{K_1} \circ r) \otimes \hat{\text{CFD}}(X_{K_2}, \phi_{K_2})).
\]

Our strategy will be to describe, for any knot \( K \) in an L-space homology sphere \( Y \), the behavior of those elements of \( \hat{\text{CFD}}(X_K) \) which come from the part of knot Floer homology in lowest Alexander grading, \( \hat{\text{HFK}}(Y, K, -g(K)) \). We will then use Theorem 2.2 to describe

\(^4\)Note that our notation differs slightly from that of [12]: the generators \( \kappa_1^1, \ldots, \kappa_{l_1}^1 \) are indexed in the reverse order, as are \( \mu_1, \ldots, \mu_t \) in the case where \( \tau(K) > 0 \).
the corresponding elements of \( \widehat{\mathrm{CFA}}(X_K, \phi_K \circ r) \). The tensor products of these elements will give rise to the homology classes needed for Theorem \( \blacksquare \).

First, we show that we may work with the box tensor product.

**Proposition 3.1.** If \( K_1 \) and \( K_2 \) are knots in L-space homology spheres, then the pair \((\widehat{\mathrm{CFA}}(X_{K_1}), \widehat{\mathrm{CFD}}(X_{K_2}))\) is relatively bounded.

This follows from a more technical lemma:

**Lemma 3.2.** Let \( K \) be a knot with genus \( g \) in an L-space homology sphere \( Y \), and consider \( \widehat{\mathrm{CFD}}(X_K, \phi_K) \) as described by Theorem \( \red{2.6} \), and \( \widehat{\mathrm{CFA}}(X_K, \phi_K \circ r) \) as obtained by applying Theorem \( \red{2.5} \) to \( \widehat{\mathrm{CFD}}(X_K, \phi_K) \). Then there is a constant \( N(K) \) such that

1. If \( I_1, \ldots, I_r \in \mathfrak{A} \) are such that \( D_{I_r} \circ \cdots \circ D_{I_1} \neq 0 \), then at most \( N(K) \) of the tuples \( I_1, \ldots, I_r \) are not equal to 12.
2. If \( I_1, \ldots, I_r \in \mathfrak{A} \) are such that \( m_{r+1}(\cdot \otimes \rho_{I_1} \otimes \cdots \otimes \rho_{I_r}) \neq 0 \), then at most \( N(K) \) of the tuples \( I_1, \ldots, I_r \) are not equal to 23.

**Proof.** For the first statement, note for any \( I \in \mathfrak{A}, \) \( D_I \circ D_1 = 0 \), which implies that \( I_i \neq 1 \) for each \( i < r \). Therefore, the maps \( D_{I_r}, \ldots, D_{I_2} \) do not decrease the Alexander grading; if \( I_i \neq 12 \), then \( D_{I_i} \) increases the Alexander grading by at least \( \frac{1}{2} \). Since the Alexander gradings of all elements of \( \widehat{\mathrm{CFD}}(X_K) \) are between \(-g\) and \( g \), all but at most 4\( g \) of the tuples \( I_1, \ldots, I_{r-1} \) must therefore be equal to 12.

For the second statement, if \( \Psi(I_1 \cdots I_r) = (J_1, \ldots, J_s) \), then the first part of the lemma implies that at most \( 4g + 1 \) of the tuples \( \phi(J_1), \ldots, \phi(J_s) \) are not equal to 12, so at most \( 4g + 1 \) of the tuples \( J_1, \ldots, J_s \) are not equal to 32. Thus, at most \( 4g + 1 \) of the digits in the concatenation \( I_1 \cdots I_r \) are 1’s, so at most \( 4g + 1 \) of the tuples \( I_1, \ldots, I_r \) are equal to 1, 12, or 123. Furthermore, since we require that \( \text{last}(I_i) > \text{first}(I_{i+1}) \) for each \( i = 1, \ldots, r - 1 \), \( I_i = 2 \) only if \( i = r \) or \( \text{first}(I_{i+1}) = 1 \), so at most \( 4g + 2 \) of \( I_1, \ldots, I_r \) can equal 2. Likewise, only \( I_1 \) can be 3. In total, all but at most \( 8g + 4 \) of \( I_1, \ldots, I_r \) are not 23, so we define \( N(K) = 8g + 4 \).

**Proof of Proposition \( \blacksquare \).** By Lemma \( \red{3.2} \) if \( r > 2 \max(N(K_1), N(K_2)) \), then for any \( I_1, \ldots, I_r \in \mathfrak{A} \), at least one of the maps

\[
m_{r+1}(\cdot \otimes \rho_{I_1} \otimes \cdots \otimes \rho_{I_r}) \colon \widehat{\mathrm{CFA}}(X_{K_1}) \to \widehat{\mathrm{CFA}}(X_{K_1})
\]

\[
D_{I_r} \circ \cdots \circ D_{I_1} \colon \widehat{\mathrm{CFD}}(X_{K_2}) \to \widehat{\mathrm{CFD}}(X_{K_2})
\]

vanishes, which implies that \( (\tau \otimes \text{id}_{\widehat{\mathrm{CFD}}} \otimes \text{id}_{\widehat{\mathrm{CFA}}} \otimes \delta_r) = 0 \) for all \( r \) sufficiently large. \( \blacksquare \)

Assume, for the duration of this section, that we have bases \( \{\xi_0, \ldots, \xi_{2n}\} \) and \( \{\tilde{\eta}_0, \ldots, \tilde{\eta}_{2n}\} \) for \( \mathrm{CFK}^-(Y, K) \) just as in Theorem \( \red{2.6} \) We begin by considering the basis elements that have Alexander grading equal to \(-g(K)\).

By the definitions of vertically and horizontally reduced bases, for any \( j \in \{1, \ldots, n\} \), neither \( \xi_{2j-1} \) nor \( \tilde{\eta}_{2j-1} \) can have Alexander grading equal to \(-g(K)\), since that would require \( \xi_{2j} \) or \( \tilde{\eta}_{2j} \) to have Alexander grading less than \(-g(K)\). Furthermore, if \( K \) is a nontrivial knot and \( A(\xi_0) = -g(K) \), then \( \tau(K) = -g(K) \) and \( \epsilon(K) = -1 \), since \( \xi_0 \) is congruent modulo \( UC^-(Y, K) \) to a linear combination of \( \tilde{\eta}_1, \ldots, \tilde{\eta}_{2j-1} \). Likewise, if \( A(\tilde{\eta}_0) = -g(K) < 0 \), then \( \tau(K) = g(K) \) and \( \epsilon(K) = 1 \).
Proof. Suppose, toward a contradiction, that reduced bases \( \{ \tilde{\tau}_i \} \) is congruent modulo \( UC^- \) to a linear combination of \( \tilde{\xi}_0, \tilde{\xi}_2, \tilde{\xi}_4, \ldots, \tilde{\xi}_{2n} \). Furthermore, the coefficient of \( \tilde{\xi}_0 \) is zero unless \( \tau(K) = -g(K) + 1 \) and \( g(K) > 1 \).

Lemma 3.3. If \( A(\tilde{\eta}_{2j-1}) = -g(K) \) and \( \ell_j = 1 \), then \( \tilde{\eta}_{2j} \) is congruent modulo \( UC^- \) to a linear combination of \( \tilde{\xi}_0, \tilde{\xi}_2, \tilde{\xi}_4, \ldots, \tilde{\xi}_{2n} \). Furthermore, the coefficient of \( \tilde{\xi}_0 \) is zero unless \( \tau(K) = -g(K) + 1 \) and \( g(K) > 1 \).

Lemma 3.4. If \( Y \) is an L-space homology sphere, and \( K \subset Y \) is a knot with \( g(K) = 1 \) and \( \tau(K) = 0 \), then \( \epsilon(K) = 0 \).

Proof. Suppose, toward a contradiction, that \( \epsilon(K) = 1 \). We may find horizontally and reduced bases \( \{ \eta_0, \ldots, \eta_{2n} \} \) and \( \{ \xi_0, \ldots, \xi_{2n} \} \) satisfying the conclusions of Proposition 2.5 in particular, \( \xi_0 = \eta_2 \). Since \( g(K) = 1 \), the horizontal arrow from \( \eta_1 \) to \( \eta_2 \) has length 1, which means that \( A(\eta_1) = -1 \) and \( \partial \eta_1 = U\xi_0 + \gamma \), where \( A(\gamma) < -1 \). As above, \( \gamma = U\delta \) for some \( \delta \) with \( A(\delta) \leq -1 \) since there are no chains with \( U \) power zero having Alexander grading less than \( -g = -1 \). Now the filtration levels of \( \xi_0 \) and each of the \( \xi_{2j-1} \) are strictly greater than \( -1 \), because the vertical differential decreases the Alexander grading and \( A(\xi_0) = \tau(K) = 0 \). It follows that \( \eta_1 \) is in the span of \( \{ \xi_2, \ldots, \xi_{2n} \} \), so there exist elements \( \xi, \alpha \) such that \( \partial \xi = \eta_1 + U\alpha \). Hence,
\[
0 = \partial^2 \xi = \partial \eta_1 + U \partial \alpha = U(\xi_0 + \delta + \partial \alpha),
\]
so, by the injectivity of multiplication by \( U \)
\[
\partial \alpha = \xi_0 + \delta.
\]

If we write \( \delta = a_0\xi_0 + \cdots + a_{2n}\xi_{2n} \), where \( a_i \in \mathbb{F}[U] \), the fact that \( A(\delta) \leq -1 \) implies that \( a_0 \) and \( a_1, a_3, \ldots, a_{2n-1} \) must be divisible by \( U \). Setting
\[
\alpha' = \alpha + \sum_{j=1}^{n} a_{2j}\xi_{2j-1},
\]
we see that
\[
\partial \alpha' \equiv \xi_0 \quad (\text{mod } UC^-),
\]
which means that \( \xi_0 \) is in the image of the vertical differential, a contradiction.

If \( \epsilon(K) = -1 \), we reduce to the previous case by considering the mirror \( \overline{K} \) in place of \( K \). \( \square \)
We now return to the bordered invariants. Let
\[ B_K = \widehat{\text{CFD}}(X_K, -g(K)). \]
Note that \( B_K \cong \text{HFK}(Y_K, -g(K)) \), and is generated by some subset of \( \{ \xi_{2j} \mid j = 1, \ldots, n \} \), along with \( \xi_0 \) if \( \tau(K) = -g(K) \); it is also generated by some subset of \( \{ \eta_{2j-1} \mid j = 1, \ldots, n \} \), along with \( \eta_0 \) if \( \tau(K) = g(K) \). Let \( \pi_B = \pi_{-g} \) denote the projection onto \( B_K \).

Additionally, note that \( \widehat{\text{CFD}}(X_K, -g(K) + \frac{1}{2}) \) is generated by the elements \( \kappa_1^j, \lambda_1^j \), and \( \mu_1 \) that are “adjacent” to the generators of \( \widehat{\text{CFD}}(X_K, -g(K)) \) in the vertical, horizontal, and unstable chains. To be precise, let
\[ V_K = \text{subspace generated by } \{ \kappa_1^j \mid A(\xi_{2j}) = -g(K) \}, \text{ and } \mu_1 \text{ if } \tau(K) = -g(K), \]
\[ H_K = \text{subspace generated by } \{ \lambda_1^j \mid A(\eta_{2j-1}) = -g(K) \}, \text{ and } \mu_1 \text{ if } \tau(K) = g(K). \]

Clearly, \( \widehat{\text{CFD}}(X_K, -g(K) + \frac{1}{2}) = V_K \oplus H_K \). Furthermore, \( V_K \) and \( H_K \) each have the same rank as \( B_K \); indeed, the restriction of \( D_{123} + D_3 \) to \( B_K \) gives isomorphisms from \( B_K \) to \( V_K \) and \( H_K \), respectively. Let \( \pi_V : \widehat{\text{CFD}}(X_K) \to V_K \) and \( \pi_H : \widehat{\text{CFD}}(X_K) \to H_K \) be the composition of \( \pi_{-g+1/2} \) with projection onto the appropriate factors.

The next two propositions describe all of the differentials into and out of \( B_K \) and \( V_K \). \( (H_K \) turns out not to be as useful for the present purposes.)

**Proposition 3.5.** Let \( K \) be a nontrivial knot with genus \( g > 0 \) in an L-space homology sphere, and consider the subspace \( B_K \subset \widehat{\text{CFD}}(X_K) \) as described above.

1. Elements of \( B_K \) have no incoming coefficient maps of any type. More precisely, for each \( I \in \mathcal{R} \), we have \( \pi_B \circ D_I = 0 \).
2. If \( I_1, \ldots, I_r \) are elements of \( \mathcal{R} \) such that the restriction of \( D_{I_1} \circ \cdots \circ D_{I_r} \) to \( B_K \) is nontrivial, then:
   - (a) \( I_1 = 3 \text{ or } 123 \).
   - (b) If \( I_1 = 123 \) and \( r > 1 \), then \( I_2 = 23 \).
   - (c) If \( I_1 = 3 \) and \( r > 1 \), then \( I_2 = 2 \text{ or } 23 \); if \( I_2 = 2 \) and \( r > 2 \), then \( I_3 = 123 \).

**Proof.** The first statement follows immediately from Theorem 2.6 and the fact that \( \widehat{\text{CFD}}(X_K, -g) \) does not contain elements of the form \( \eta_{2j} \) for \( j = 1, \ldots, n \), and does not contain \( \eta_0 \) if \( \tau(K) \leq 0 \) (the only cases where \( \eta_0 \) has an incoming coefficient map).

For the second statement, note that \( D_1 \) and \( D_{12} \) restricted to \( B_K \) are both zero, so we may reduce to the two cases where \( I_1 = 3 \) or \( 123 \), which we treat separately.

In the case where \( I_1 = 123 \), we consider the vertical basis for \( B_K \). If \( \xi_{2j} \in \widehat{\text{CFD}}(X_K, -g) \), where \( j \in \{ 1, \ldots, n \} \), then the only nonzero sequence of coefficient maps coming from \( \xi_{2j} \) and starting with \( D_{123} \) is the vertical chain
\[ \xi_{2j} \xrightarrow{D_{123}} \kappa_1^j \xrightarrow{D_{23}} \cdots \xrightarrow{D_{k_j^j}} \kappa_{k_j^j}. \]

If \( \xi_0 \in \widehat{\text{CFD}}(X_K, -g) \), then \( \tau(K) = -g(K) < 0 \), so the unstable chain provides the sequence
\[ \xi_0 \xrightarrow{D_{123}} \mu_1 \xrightarrow{D_{23}} \cdots \xrightarrow{D_{23}} \mu_{2g} \xrightarrow{D_2} \eta_0, \]
with at least one \( D_{23} \). Thus, the only \( I \) such that \( D_I \circ D_{123} \mid_{B_K} \) can be nonzero is \( I = 23 \).
In the case where $I_1 = 3$, we use the horizontal basis. If $\eta_0 \in \widehat{\text{CFD}}(X_K, -g)$, then $\tau(K) = g(K) > 0$, so the unstable chain provides the sequence

$$\eta_0 \overset{D_{23}}{\longrightarrow} \mu_1 \overset{D_{23}}{\longrightarrow} \cdots \overset{D_{23}}{\longrightarrow} \mu_{2g}.$$

If $\eta_{2j-1} \in \widehat{\text{CFD}}(X_K, -g)$, the horizontal chain from $\eta_{2j-1}$ to $\eta_{2j}$ provides the sequence

$$\eta_{2j-1} \overset{D_{23}}{\longrightarrow} \lambda_1^{j} \overset{D_{23}}{\longrightarrow} \cdots \overset{D_{23}}{\longrightarrow} \lambda_{\ell_j} \overset{D_{2}}{\longrightarrow} \eta_{2j}.$$  

Thus, it suffices to consider the case where $\ell_j = 1$. Lemma 3.3 says that $\eta_{2j}$ is a linear combination of $\xi_2, \xi_4, \ldots, \xi_{2n}$, along with $\xi_0$ provided that $\tau(K) = -g(K) + 1$ and $g(K) > 1$, and only $D_{123}$ is nonzero on these elements (via corresponding vertical or unstable chains). Hence, the only $I$ such that $D_I \circ D_2 \circ D_3|_{B_K}$ can be nonzero is $I = 123$, as required. 

**Proposition 3.6.** Let $K$ be a nontrivial knot with genus $g > 0$ in an L-space homology sphere, and consider the subspace $V_K \subseteq \widehat{\text{CFD}}(X_K)$ as described above.

1. The only possible nonzero sequences of coefficient maps into $V_K$ are $D_{123}$ and $D_I$.
   More precisely, if $\pi_V \circ D_{1r} \circ \cdots \circ D_{I_1} \neq 0$, then $r = 1$ and $I_1 = 123$ or 1.

2. If the restriction of $D_{1r} \circ \cdots \circ D_{I_1}$ to $V_K$ is nontrivial, then $I_1 = 23$.

**Proof.** By Theorem 2.6, the only coefficient maps whose image have nonzero projection to $V_K$ are $D_1$ and $D_{123}$. Furthermore, the only nonzero contribution to $\pi_V \circ D_1$ comes when $A(\xi_2) = -g(K)$ and $k_j = 1$, in which case $D_1(\xi_{2j-1}) = \kappa_1^{j}$. It remains to verify that $\xi_{2j-1}$ has no incoming coefficient maps coming from the horizontal or unstable chains. If $\eta_{2i}$ has a nonzero $\xi_{2j-1}$ component, then $A(\eta_{2i}) = A(\xi_{2j-1}) = -g(K) + 1$ and $A(\eta_{2i-1}) = -g(K)$, so by Lemma 3.3, $\eta_{2i}$ is in the span of $\xi_0, \xi_2, \ldots, \xi_{2n}$, a contradiction. Likewise, if $\eta_0$ has a nonzero $\xi_{2j-1}$ component, then $\tau(K) = -A(\eta_0) = g(K) - 1$ and $\epsilon(K) = -1$, so $g(K) > 1$ by Lemma 3.3. Hence, $\tau(K) > 0$. The unstable chain then gives $\eta_0$ an outgoing differential $(\eta_0 \overset{D_{3}}{\longrightarrow} \mu_1)$, not an incoming one. This concludes the proof of the first statement.

The second statement follows Proposition 3.5 and the fact that $D_{123}|_{B_K} : B_K \rightarrow V_K$ is an isomorphism.  

Next, we use the algorithm of Theorem 2.2 to give analogous results for $\widehat{\text{CFA}}(X_K)$. We view $\widehat{\text{CFA}}(X_K)$ as having the same underlying vector space as $\widehat{\text{CFD}}(X_K)$, with $\mathcal{A}_\infty$ multiplications given by Theorem 2.2. We may then think of $B_K$, $V_K$, and $H_K$ as subspaces of $\text{CFA}(X_K)$.

**Proposition 3.7.** Let $K$ be a nontrivial knot with genus $g > 0$ in an L-space homology sphere, and consider the subspace $B_K \subseteq \widehat{\text{CFA}}(X_K)$ as described above.

1. Elements of $B_K$ have no incoming multiplications of any type. More precisely, for any $a_1, \ldots, a_k \in A$, the composition $\pi_{-g} \circ m_{k+1}(\cdot \otimes a_1 \otimes \cdots \otimes a_k)$ is trivial.

2. If $I_1, \ldots, I_r$ are elements of $\mathfrak{R}$ such that the restriction of $m_{r+1}(\cdot \otimes \rho_{I_1} \otimes \cdots \otimes \rho_{I_r})$ to $B_K$ is nonzero, then:
   (a) If $I_1 = 3$, then $r \geq 3$, $I_2 = 2$, and $I_3 = 1$ or 12.
   (b) If $I_1 = 123$, then $r \geq 2$ and $I_2 = 2$.

**Proof.** This proposition follows by applying Theorem 2.2 to the results of Proposition 3.5. For any $I_1, \ldots, I_r \in \mathfrak{R}$ with $I_1 \cdots I_r$ alternating and last($I_i$) > first($I_{i+1}$) for all $i$, we have

$$m_{r+1}(\cdot \otimes \rho_{I_1} \otimes \cdots \otimes \rho_{I_r}) = D_{\phi(J)} \circ \cdots \circ D_{\phi(J_r)}.$$
where \((J_1, \ldots, J_r) = \Psi(I_1 \cdots I_r)\). If the restriction of \(m_{r+1}(\cdot \otimes \rho_{I_1} \otimes \cdots \otimes \rho_{I_r})\) to \(B_K\) or \(V_K\) is nonzero, the sequence \((\phi(J_1), \ldots, \phi(J_r))\) must satisfy the conclusions of Proposition 3.8. Specifically:

- If \(I_1 = 3\), then \(\phi(J_1)\) begins with 1, so Proposition 3.8 says that \(\phi(J_1) = 123\) and \(\phi(J_2) = 23\) if \(j > 1\). Hence \(I_1 \cdots I_r = 321\) or \(32121\), so \(I_2 = 2\) and \(I_3 = 1\) or 12.
- If \(I_1 = 123\), then \(\phi(J_1) = 3\) and \(\phi(J_2) = 2\), so Proposition 3.8 says that \(j > 2\) and \(J_3 = 123\). Hence \(I_1 \cdots I_r = 12321\), so \(I_2 = 2\).

A similar argument shows:

**Proposition 3.8.** Let \(K\) be a nontrivial knot with genus \(g > 0\) in an \(L\)-space homology sphere, and consider the subspace \(V_K \subset \widehat{\text{CFA}}(X_K)\) as described above.

1. The only possible nonzero \(\mathcal{A}_\infty\) multiplications into \(V_K\) are \(m_2(\cdot \otimes \rho_3)\) and \(m_4(\cdot \otimes \rho_3 \otimes \rho_2 \otimes \rho_1)\). More precisely, if \(\pi_V \circ m_{r+1}(\cdot \otimes \rho_1 \otimes \cdots \otimes \rho_{I_r}) \neq 0\), then either \(r = 1\) and \(I_1 = 3\), or \(r = 3\) and \((I_1, I_2, I_3) = (3, 2, 1)\).
2. If the restriction of \(m_{r+1}(\cdot \otimes \rho_1 \otimes \cdots \otimes \rho_{I_r})\) to \(B_K\) is nonzero, then \(I_1 = 2\).

**Proof of Theorem\[.]** Let \(K_1 \subset Y_1\) and \(K_2 \subset Y_2\) be nontrivial knots in \(L\)-space homology spheres. The Alexander gradings on \(\widehat{\text{CFA}}(K_1)\) and \(\widehat{\text{CFD}}(K_2)\) give a direct sum decomposition

\[
\widehat{\text{CFA}}(X_{K_1}) \otimes \widehat{\text{CFD}}(X_{K_2}) = \bigoplus_{s \in \mathbb{Z}} C_s,
\]

where

\[
C_s = \bigoplus_{t \in \frac{1}{2} \mathbb{Z}} \widehat{\text{CFA}}(X_{K_1}, t) \otimes \widehat{\text{CFD}}(X_{K_2}, s - t).
\]

Note that \(C_{-g(K_1) - g(K_2)} = B_{K_1} \otimes B_{K_2}\) and \(C_{-g(K_1) - g(K_2) + 1} = (V_{K_1} \otimes V_{K_2}) \oplus (V_{K_1} \otimes H_{K_2}) \oplus (H_{K_1} \otimes V_{K_2}) \oplus (H_{K_1} \otimes H_{K_2})\). We claim that the direct summands \(B = B_{K_1} \otimes B_{K_2}\) and \(V = V_{K_1} \otimes V_{K_2}\), each of dimension \(\dim \widehat{\text{HF}}(Y_1, K_1, -g(K_1)) \cdot \dim \widehat{\text{HF}}(H_2, K_2, -g(K_2))\), both survive in the homology of \(\widehat{\text{CFA}}(X_{K_1}) \otimes \widehat{\text{CFD}}(X_{K_2})\), which will prove that

\[
\dim \widehat{\text{HF}}(Y(K_1, K_2)) \geq 2 \dim \widehat{\text{HF}}(Y_1, K_1, -g(K_1)) \cdot \dim \widehat{\text{HF}}(H_2, K_2, -g(K_2)) \geq 2,
\]

as required.

To see that the differential on \(\widehat{\text{CFA}}(X_{K_1}) \otimes \widehat{\text{CFD}}(X_{K_2})\) is identically zero on \(B\), we simply note that there do not exist \(I_1, \ldots, I_r \in \mathfrak{R}\) satisfying the conclusions of the second parts Propositions 3.3 and 3.7 simultaneously. Thus, for any \(x \in B_{K_1}\) and \(y \in B_{K_2}\),

\[
\varphi^B(x \otimes y) = \sum_{I_1, \ldots, I_r \in \mathfrak{R}} m_{r+1}(x \otimes \rho_{I_1} \otimes \cdots \otimes \rho_{I_r}) \otimes (D_{I_1} \circ \cdots \circ D_{I_r})(y) = 0.
\]

(Here \(m_{r+1}\) denotes an \(\mathcal{A}_\infty\) multiplication on \(\widehat{\text{CFA}}(X_{K_1})\), while \(D_{I_1}, \ldots, D_{I_r}\) denote coefficient maps on \(\widehat{\text{CFD}}(X_{K_2})\).) Furthermore, the first parts of Propositions 3.3 and 3.7 imply that the composition of \(\varphi^B\) with the projection onto \(B\) coming from the direct sum decomposition is zero. Thus, \(B\) survives in homology.
The proof for $V$ is similar, using Propositions 3.6 and 3.8. Just as above, we see that so the restriction of $\partial^{33}$ to $V$ vanishes. Furthermore, if $x \in \text{CFA}(X_{K_1})$ and $y \in \text{CFD}(X_{K_2})$ are such that $\partial^{33}(x \otimes y)$ has nontrivial projection to $V$, there must be $I_1, \ldots, I_r$ that simultaneously satisfy the first parts of Propositions 3.6 and 3.8, but clearly this is impossible. □

4. Examples

Let $L$ and $R$ denote the left- and right-handed trefoils in $S^3$, respectively. $\text{CFD}(X_L)$ and $\text{CFD}(X_R)$ are as follows:

According to Theorem 2.2, $\text{CFA}(X_R)$ is as follows (using capital Greek letters to avoid confusion when we take tensor products below):

We may use these results to compute the tensor product complexes $\text{CFA}(X_R) \boxtimes \text{CFD}(X_L)$ and $\text{CFA}(X_R) \boxtimes \text{CFD}(X_R)$, illustrated in Figures 1 and 2. In each of these figures, the two homology classes provided by the proof of Theorem 1 are indicated in boldface.

From these complexes, it is easy to verify that

$$\text{dim } \hat{\text{HF}}(Y(R, L)) = \text{dim } H_*(\text{CFA}(X_R) \boxtimes \text{CFD}(X_L)) = 9$$

and

$$\text{dim } \hat{\text{HF}}(Y(R, R)) = \text{dim } H_*(\text{CFA}(X_R) \boxtimes \text{CFD}(X_R)) = 7.$$
The reader is encouraged verify these results in another way by computing $\widehat{\text{CFA}}(X_L)$ and evaluating its box tensor product with $\widehat{\text{CFD}}(X_L)$ and $\widehat{\text{CFD}}(X_R)$.

5. Future Directions

We conclude by discussing the prospects for generalizing Theorem 1 to manifolds obtained by splicing knots in arbitrary homology spheres, which would prove Conjecture 2. If $K$ is a knot in a homology sphere $Y$, the proof of Theorem 2.6 given in [12] can be adapted to give a description of $\widehat{\text{CFD}}(X_K)$ in terms of $\text{CFK}^-(Y, K)$, with multiple unstable chains when $Y$ is not an L-space. However, the structure of the unstable chains depends on the isomorphism induced on homology by a certain chain homotopy equivalence $J: (Ch, \partial_h) \to (Cv, \partial^v)$ that arises in the course of the proof, and this isomorphism is not a priori determined merely by $\text{CFK}^-(X, K)$. Furthermore, even though $(Ch, \partial_h)$ and $(Cv, \partial^v)$ are filtered chain homotopy equivalent, the map $J$ need not be a filtered chain homotopy equivalence. In particular, an unstable chain may connect a horizontal generator $\eta_0$ and a vertical generator $\xi_0$ with $A(\eta_0) \neq -A(\xi_0)$.

As a result, Propositions 3.5 through 3.8 no longer hold when $Y$ is not an L-space. For example, let $Y$ be the manifold obtained by +1 surgery on $L$ (i.e., the Brieskorn sphere $-\Sigma(2, 3, 7)$), and let $K$ be the core of the surgery torus. Note that $X_K = X_L$, but the
parametrization $\phi_K$ differs from $\phi_L$ by a longitudinal Dehn twist. Thus,

$$\hat{\text{CFD}}(X_K, \phi_K) \cong \hat{\text{CFDA}}(\tau^{-1}) \boxtimes \hat{\text{CFD}}(X_L, \phi_L),$$

where $\hat{\text{CFDA}}(\tau^{-1})$ is one of the Dehn twist bimodules computed in [11, Section 10.2]. By evaluating this tensor product and simplifying, the reader may verify that $\hat{\text{CFD}}(X_K, \phi_K)$ has the following form:

Here, $\eta_0$, $\eta_2$, and $\xi_0$ are the generators of vertical homology, and $\xi_0$, $\xi_1$, and $\xi_2$ are the generators of horizontal homology. The only generator in Alexander grading $-1$ is $\eta_1$. Notice that $D_2 \circ D_{123}(\eta_1)$ and $D_{12} \circ D_2 \circ D_3(\eta_1)$ are both nonzero (and distinct), contrary to Proposition [3.5]. Furthermore, by Theorem [2.2] the corresponding generator in $\hat{\text{CFA}}(X_K)$
has outgoing $m_4(\cdot \otimes \rho_3 \otimes \rho_2 \otimes \rho_1)$ and $m_3(\cdot \otimes \rho_{123} \otimes \rho_2)$ multiplications, contrary to Proposition 3.7. Therefore, when $K_1$ and $K_2$ are knots in arbitrary homology spheres, the subgroup $B_{K_1} \otimes B_{K_2} \subset \hat{\mathcal{CFA}}(X_{K_1}) \otimes \hat{\mathcal{CFD}}(X_{K_2})$ does not necessarily survive in homology, unlike in our proof of Theorem 1. A different strategy will thus be required for a proof of Conjecture 2.

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