Commutative Rings of Differential Operators Corresponding to Multidimensional Algebraic Varieties

A.E. Mironov

1 Introduction

In this article we construct commutative rings of multidimensional $(N \times N)$-matrix differential operators whose common eigenfunctions and eigenvalues are parametrized by points of a spectral variety $Y^k$, the intersection of some smooth hypersurfaces $Y_{a_1} \cap \ldots \cap Y_{a_k}$ in a principally polarized abelian variety $X^g$ of dimension $g$, $k < g - 1$. The hypersurface $Y_{a_j}$ is the translate of a theta-divisor $Y \subset X^g$ by an element $a_j \in X^g$. The number $N$ equals $rd_g$, where $d_g$ is the $g$-fold self-intersection index of the hypersurface $Y$. Denote by $Q^j$ the variety $Y^j \cap Y$. Below we suppose that the variety $Y^j$ intersects $Y_{a_{j+s}}$ and $Y$, $j + s \leq k$, transversally. We assume that $Y^j$ and $Q^j$ are smooth and irreducible and the collection $a_1, \ldots, a_k$ is in general position (i.e., belongs to some open everywhere dense set in $X^g \times \ldots \times X^g$).

These commutative rings relate to some analog of the Kadomtsev–Petviashvili hierarchy to be indicated in this article.

Our main result is the following

**Theorem 1.** There is an embedding $L_k$ of the ring of meromorphic functions on the variety $Y^k$ with a pole on $Q^k$ into the ring of $(N \times N)$-matrix differential operators in $g - k$ variables whose coefficients are analytic in a neighborhood of 0:

$$L_k : \mathcal{A}_k \rightarrow \text{Mat}(N, g - k).$$

The range of the embedding is a commutative ring of $(g - k)$-dimensional matrix differential operators.

Using the Riemann–Roch–Hirzebruch theorem, we can demonstrate that the number $d_g$, and in consequence $N$, is a multiple of $g!$.

The operators $L_k(\mathcal{A}_k)$ have rank $r$. This means that to each point of $Y^k$ there corresponds $r$ linearly independent eigenfunctions.

The two-dimensional operators $L_k(\mathcal{A}_k)$ with doubly periodic coefficients are finite-gap at every energy level $E$; i.e., the Blöch vector-functions (eigenfunctions of both the operators $L_k(\lambda)$, $\lambda \in \mathcal{A}_k$, and the translation operators
by periods) are parametrized by a Riemann surface of finite genus defined in the spectral surface by the equation $\lambda = E$.

If the dimension of $\Lambda_2^k$ equals 2 then, using the adjunction formula and the Lefschetz embedding theorem, we can demonstrate that the Kodaira dimension of the spectral surface $\Lambda_2^k$ equals 2; i.e., this surface is a surface of general type.

We prove Theorem 1, using Nakayashiki’s results [1] (see also [2]) who constructed an embedding of the ring of meromorphic functions on $X^g$ with a pole on $Y$ into the ring of $g$-dimensional $(N \times N)$-matrix differential operators. The $(2 \times 2)$-matrix operators of this kind in two variables (the Nakayashiki operators) were studied in the author’s articles [3, 4]. In particular, it was proven in [4] that there are no two-dimensional real Nakayashiki operators that are finite-gap at every energy level and have doubly periodic coefficients, but there exist two-dimensional real Nakayashiki operators with singular doubly periodic coefficients which are finite-gap at every energy level. In [4] we also indicated smooth real Nakayashiki operators, including a second-order operator $H$ whose diagonal is constituted by Schrödinger operators in doubly periodic magnetic fields with doubly periodic potentials of the form

$$(\partial_{y_1} - A_1)^2 + (\partial_{y_2} - A_2)^2 + u(y), \quad y = (y_1, y_2).$$

The magnetic Bloch vector-functions of the operator $H$ (the common eigenfunctions of $H$ and the magnetic translation operators $T_j^*, T_j\varphi(y) = \varphi(y + e_j) \exp(2\pi y_j)$, $j = 1, 2$, where $e_j$ are the periods) are parametrized by a Riemann surface of finite genus at each energy level. This property is an analog of the finite-gap property at each energy level for operators with doubly periodic coefficients.

In the particular case when $g = 3$, $r = 1$, and the spectral surface is a theta-divisor Theorem 1 was proven by Nakayashiki [1].

Rothstein [5] constructed another example of commuting matrix differential operators. In this example $g = 5$, $r = 1$, the size $N$ of the matrices equals 5, and the spectral surface is the Fano surface.

Let us recall Krichever’s construction [6] of commuting ordinary differential operators of rank 1. Suppose that $\Gamma$ is a Riemann surface of genus $g$, $P = p_1 + \ldots + p_g$ is a non-special positive divisor on $\Gamma$, $\infty$ is a point on $\Gamma$ other than the points of $P$, $k^{-1}$ is a local parameter at $\infty$, and $k^{-1}(\infty) = 0$. There is a Baker–Akhiezer function $\psi(p, x)$, $p \in \Gamma$, meromorphic on $\Gamma \setminus \infty$ and whose set of poles coincides with $P$ and is independent of $x$; moreover,
the function $\psi \exp(-kx)$ is analytic in a neighborhood of $\infty$. For every meromorphic function $f(p)$ on $\Gamma$ with a sole pole at $\infty$ there is a unique differential operator $L(f)$ such that

$$L(f)\psi = f\psi.$$ 

The operators $L(f)$ commute pairwise for different $f$. Hence, we obtain a relation between the spectral data of the commuting Burchnall–Chaundy–Krichever operators and the spectral data of the operators $L_k(\mathcal{A}_k)$:

$$\{\Gamma, \infty, P, f\} \leftrightarrow \{Y^k, Q^k, Q^k_c, \lambda\},$$

where $Q^k_c = Y^k \cap Y_c$ and $c \in X^g$ is some nonzero element.

As in the one-dimensional case we can construct operators $L_\alpha$ whose coefficients depend on time and satisfy some evolution equations.

**Theorem 2.** There is a multidimensional analog of the Kadomtsev–Petviashvili hierarchy

$$[\partial_{t_\alpha} - L_\alpha, \partial_{t_\beta} - L_\beta] = 0,$$

where $L_\alpha$ and $L_\beta$ are $(N \times N)$-matrix differential operators in $g - k$ variables whose coefficients depend on $t_\alpha$ and $t_\beta$ with $\alpha$ and $\beta$ varying in some countable set of indices.

As was already mentioned in [4], the coefficients of the Nakayashiki operators cannot satisfy evolution equations of the Kadomtsev–Petviashvili hierarchy type.

In Section 2 we introduce vector theta-functions which determine sections of holomorphic vector bundles of rank $r$ over an abelian variety $X^g$. For $r = 1$ the vector theta-functions coincide with the classical Riemann theta-functions. Using vector theta-functions, we can write down explicitly sections of holomorphic vector bundles over Riemann surfaces. If $X^g$ is the Jacobi variety of a Riemann surface $\Gamma \subset X^g$ then the restriction of a vector theta-function to $\Gamma$ is a section of the vector bundle over $\Gamma$ of rank $r$ and degree $rsg$, where $s$ is some natural number and $g$ is the genus of $\Gamma$.

In Section 3, using the Fourier–Mukai transform [7], we introduce the Baker–Akhiezer module over the ring of differential operators whose elements are expressed in terms of vector theta-functions. Theorems 1 and 2 ensue from Theorem 3 which claims that the Baker–Akhiezer module is free.

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2  Vector Theta-Functions

In this section we indicate the coefficients of the Fourier series expansion of vector theta-functions. In Lemma 1 we find the dimension of the space of vector theta-functions.

Denote by \( X^g = \mathbb{C}^g / \{\mathbb{Z}^g + \Omega \mathbb{Z}^g\} \) a principally polarized complex abelian variety, where \( \Omega \) is a symmetric \((g \times g)\)-matrix with \( \text{Im} \Omega > 0 \). Given non-degenerate pairwise commuting \((r \times r)\)-matrices \( A_j, j = 1, \ldots, g \), introduce the set of matrix functions (multipliers) on \( \mathbb{C}^g \):

\[
e_{n+\Omega m}(z) = \exp(-s\pi i \langle m, \Omega m \rangle - 2s\pi i \langle m, z \rangle) A_1^{m_1} \cdots A_g^{m_g},
\]

where \( m, n \in \mathbb{Z}^g, \langle m, z \rangle = m_1z_1 + \ldots + m_gz_g \), and \( s \) is some natural number. It is easy to check that these functions satisfy the equalities

\[
e_\lambda(z + \lambda')e_{\lambda'}(z) = e_{\lambda'}(z + \lambda)e_\lambda(z), \quad \lambda, \lambda' \in \mathbb{Z}^g + \Omega \mathbb{Z}^g.
\]

An arbitrary collection of matrix functions satisfying these equalities determines a vector bundle of rank \( r \) over \( X^g \) which is obtained by factoring \( \mathbb{C}^g \times \mathbb{C}^r \) by the action of the lattice \( \mathbb{Z}^g + \Omega \mathbb{Z}^g \):

\[
(z, v) \sim (z + \lambda, e_\lambda(z)v), \quad v \in \mathbb{C}^r.
\]

Global sections are given by vector-functions on \( \mathbb{C}^g \) with the periodicity properties

\[
f(z + \lambda) = e_\lambda f(z).
\]

A vector theta-function of rank \( r \) and degree \( s \) is a vector-function

\[
\theta^{r,s}(z) = (\theta_1^s(z), \ldots, \theta_r^s(z))^\top, \quad z \in \mathbb{C}^g,
\]

with entire components which possesses the property

\[
\theta^{r,s}(z + \Omega m + n) = \exp(-s\pi i \langle m, \Omega m \rangle - 2s\pi i \langle m, z \rangle) A_1^{m_1} \cdots A_g^{m_g} \theta^{r,s}(z).
\]

By periodicity, \( \theta^{r,s} \) expands in the series

\[
\theta^{r,s} = \sum_{l \in \mathbb{Z}^g} \exp(2\pi i \langle l, z \rangle) a_l, \quad a_l = (a_l^1, \ldots, a_l^r)^\top \in \mathbb{C}^r.
\]

Find the recurrent relations for the coefficients \( a_l \):

\[
\theta^{r,s}(z + \Omega e_j) = \sum_{l \in \mathbb{Z}^g} \exp(2\pi i \langle l, \Omega e_j \rangle) \exp(2\pi i \langle l, z \rangle) a_l
\]
\[ \sum_{l \in \mathbb{Z}^g} \exp(-s \pi i \Omega_{jj}) \exp(2 \pi i \langle l - s e_j, z \rangle) A_j a_l; \]

consequently,

\[ a_{l+se_j} = \exp(s \pi i \Omega_{jj} + 2 \pi i \langle l, \Omega e_j \rangle) A_j^{-1} a_l, \]

where \( e_j = (0, \ldots, 1, \ldots, 0)^\top \) (with 1 at the \( j \)th place). It follows from the last formula that \( \theta^{r,s} \) is determined by the coefficients \( a_l \), where the components of \( l \) vary within \( 0 \leq l_\alpha \leq s - 1 \); therefore, the dimension of the space of vector theta-functions does not exceed \( rs^g \). Show that for every choice of \( a_l \), \( 0 \leq l_\alpha \leq s - 1 \), the series for the vector theta-function \( \theta^{r,s} \) converges. To this end, rewrite it as follows:

\[ \theta^{r,s}(z) = \sum_{l_0} \sum_{l \in \mathbb{Z}^g} \exp(2 \pi i \langle l_0 + sl, z \rangle) a_{l_0+sl}, \]

where the components of \( l_0 \) vary between 0 and \( s - 1 \). The above recurrent relations can be resolved explicitly:

\[ a_{l_0+sl} = \exp(s \pi i \langle l, \Omega l \rangle + 2 \pi i \langle l_0, \Omega l \rangle) A^{-l_1}_1 \ldots A^{-l_g}_g a_{l_0}. \]

Put

\[ \theta^{r,s}_{a_{l_0}} = \sum_{l \in \mathbb{Z}^g} \exp(s \pi i \langle l, \Omega l \rangle + 2 \pi i \langle l_0, \Omega l \rangle + 2 \pi i \langle l_0 + sl, z \rangle) A^{-l_1}_1 \ldots A^{-l_g}_g a_{l_0}. \]

Then

\[ \theta^{r,s} = \sum_{l_0} \theta^{r,s}_{a_{l_0}}. \]

Denote by \( C_j \) the greatest of the two numbers \( \|A_j^{-1}\| \) and \( \|A_j\| \). Then the norm of each summand in the series for \( \theta^{r,s}_{a_{l_0}} \) does not exceed

\[ |\exp(s \pi i \langle l, \Omega l \rangle + 2 \pi i \langle l_0, \Omega l \rangle + 2 \pi i \langle l_0 + sl, z \rangle)| C_1^{l_1} \ldots C_g^{l_g} \|a_{l_0}\|; \]

consequently, by positive definiteness of \( \text{Im} \, \Omega \) this series converges absolutely. We obtain the following

**Lemma 1.** The dimension of the space of vector theta-functions of degree \( s \) and rank \( r \) equals \( rs^g \).

We give an example of the matrices \( A_j \). We take \( A_1 \) to be some matrix with nondiagonal Jordan form and the remaining \( A_j \) to be polynomials in \( A_1 \). If the matrices \( A_j \) have diagonal Jordan forms then the bundle corresponding to the collection \( A_j \) is the direct sum of line bundles.
3  Commuting Operators

In this section we state Nakayashiki’s theorem [1] in the particular case of holomorphic vector bundles of rank \( r \geq 1 \) invariant under translations by elements of \( X^g \) which we need below. Using the Fourier–Mukai transform of these bundles, we introduce the Baker–Akhiezer modules \( M^j_c \) over the ring \( \mathcal{D}_j \) of differential operators. We show in Corollary 2 that the restriction map of functions in \( M^j_c \) to the variety \( Y^{j+1} \subset Y^j \) determines an epimorphism \( M^j_c \rightarrow M^{j+1}_c \). In Theorem 3 we prove that the \( \mathcal{D}_j \)-module \( M^j_c \) is free. In Corollary 3 we show that the coefficients of the operators \( L_k(A_k) \) satisfy some evolution equations.

Denote by \( \text{Pic}^0(X^g) \) the Picard variety of \( X^g \). In our case \( X^g \) and \( \text{Pic}^0(X^g) \) are isomorphic. Denote by \( P \) the Poincaré bundle over \( X^g \times \text{Pic}^0(X^g) \). The sections of \( P \) under the lift to \( \mathbb{C}^g \times \mathbb{C}^g \) are determined by functions \( f(z,x) \) such that

\[
 f(z + \Omega m_1 + n_1, x + \Omega m_2 + n_2) = \exp(-2\pi i((m_1, x) + (m_2, z))) f(z,x),
\]

where \( m_j, n_j \in \mathbb{Z}^g \).

Let \( Y \) represent the zeros of some theta-function \( \vartheta \) (of rank 1) of degree \( s \):

\[
 \vartheta(z + \Omega m + n) = \exp(-s\pi i(m, \Omega m) - 2s\pi i(m, z)) \vartheta(z).
\]

Denote by \( L_c \) the holomorphic vector bundle over \( X^g \) whose sections are given by the vector-functions \( f(z) \) of rank \( r \) on \( \mathbb{C}^g \) with the property

\[
 f(z + \Omega m + n) = \exp(-2\pi i(m, c)) A_1^{m_1} \cdots A_g^{m_g} f(z), \quad m, n \in \mathbb{Z}^g, \ c \in \mathbb{C}^g. \tag{1}
\]

Observe that the bundle \( L_c \) is invariant under translations by the elements of \( X^g \). Let \( \mathcal{L} \) be the space of global sections of the bundle \( \mathcal{L}_0 \) with a pole on \( Y \) and let \( \pi \) be the projection \( X^g \times \text{Pic}^0(X^g) \rightarrow X^g \). Denote by \( F(Y, \mathcal{L}_0)(U) \) the space of meromorphic sections of the bundle \( \pi^* \mathcal{L}_0 \otimes \mathcal{P} \) over \( X^g \times U \) with a pole on \( Y \times U \), where \( U \) is an open subset in \( \text{Pic}^0(X^g) \). For a given \( x \in U \) the space \( F(Y, \mathcal{L}_0)(U) \) coincides with the space \( \bigcup_{j=1}^{\infty} H^0(X^g, \mathcal{L}_x(jY)) \). We sometimes denote by \( \mathcal{L}_x(jY) \) the bundle \( \mathcal{L}_x \otimes [jY] \), where \([jY]\) is the line bundle associated with the divisor \( jY \). For simplicity we denote vector bundles and the corresponding bundles of analytic sections by the same symbol.

We identify the space \( H^0(X^g, \mathcal{L}_x(jY)) \) with the space of global sections of the bundle \( \mathcal{L}_x \) with a pole on \( Y \); moreover, the order of the pole does not exceed \( j \).
The space $F(Y, \mathcal{L}_0)(U)$ is the *Fourier–Mukai transform* over $U$ of the space $\mathcal{L}$.

The covariant differentiation operators act on $F(Y, \mathcal{L}_0)(U)$:

$$\nabla_j = \partial_{x_j} - \frac{1}{s} \partial_{z_j} \log \vartheta(z) : F(Y, \mathcal{L}_0)(U) \to F(Y, \mathcal{L}_0)(U),$$

$$\nabla_k \nabla_j = \nabla_j \nabla_k, \quad k, j = 1, \ldots, g,$$

which furnish $F(Y, \mathcal{L}_0)(U)$ with the structure of a module over the ring $\mathcal{O}_U[\nabla_1, \ldots, \nabla_g]$, where $\mathcal{O}_U$ is the ring of analytic functions on $U$. It follows from the construction that $F(Y, \mathcal{L}_0)(U)$ is also a module over the ring $\mathcal{A}_0$ of meromorphic functions on $X^g$ with a pole on $Y$.

Denote by $D_g$ the ring of differential operators $\mathcal{O}_g[\partial_1, \ldots, \partial_g]$, where $\mathcal{O}_g$ is the ring of analytic functions in the variables $x_1, \ldots, x_g$ which are defined in a neighborhood of $0 \in \mathbb{C}^g$. In [1] Nakayashiki introduced the Baker–Akhiezer module $M_c = \bigcup_{n=1}^{\infty} M_c(n)$ over the ring $\mathcal{D}_g$ of differential operators, where

$$M_c(n) = \left\{ f(z, x) \exp \left( -\sum_{j=1}^{g} \frac{x_j}{s} \partial_{x_j} \log \vartheta(z) \right), \ f(z, x) \in H^0(X, \mathcal{L}_{c+} (nY)) \right\}.$$

We need one more $\mathcal{D}_g$-module

$$\mathcal{D}_g M_c(n) = \left\{ \sum d \varphi, \ d \in \mathcal{D}_g, \ \varphi \in M_c(n) \right\}.$$

We can express the elements of $M_c$ in terms of vector theta-functions. Every vector-function in $M_c$ is representable as the sum of vector-functions of the form

$$g(x) \frac{\theta^{r,sn}(z + \frac{x+c}{sn})}{\theta^m(z)} \exp \left( -\sum_{j=1}^{g} \frac{x_j}{s} \partial_{x_j} \log \vartheta(z) \right),$$

where $g(x) \in \mathcal{O}_g$ and $\theta^{r,sn}$ is some vector theta-function.

The following theorem is proven in [1]:

**Nakayashiki’s Theorem.** For $c$ in general position, $M_c$ is a free $\mathcal{D}_g$-module of rank $N$. The equality $M_c = \mathcal{D}_g M_c(g)$ is valid.

The equality $M_c = \mathcal{D}_g M_c(g)$ means that the $\mathcal{D}_g$-module $M_c$ is generated by the elements of $M_c(g)$.

Fix a basis $\Phi_c = (\phi_{1,c}(z, x), \ldots, \phi_{N,c}(z, x))^\top$ for the $\mathcal{D}_g$-module $M_c$. Sometimes, like in the following corollary, by $\Phi_c$ we mean the matrix function with
$N$ rows and $r$ columns, since each component $\phi_{j,c}(z,x)$ is itself a vector-function of size $r$.

**Corollary 1** [1]. There is a ring embedding

$$L_0 : \mathcal{A}_0 \to \text{Mat}(N,g)$$

defined by the equality

$$L_0(\lambda)\Phi_c = \lambda \Phi_c, \quad \lambda \in \mathcal{A}_0.$$  

The range of the embedding is a commutative ring of $g$-dimensional matrix differential operators.

We turn to the construction. In fact, we prove a stronger version of Theorem 1. We assume that the hypersurface $Y_{aj}$ can be not only a translate of $Y$ but also a translate of some smooth hypersurface linearly equivalent to $Y$; i.e., $Y_{aj}$ is the set of zeros of some theta-function of degree $s$ with a translation:

$$Y_{aj} = \{ z \in X^g, \vartheta_j(z - a_j) = 0 \}.$$  

Denote by $\mathcal{L}^k_c$ the line bundle over $Y^k$ whose sections are given by the functions $f(z)$ on $Y^k \subset \mathbb{C}^g$ with property (1).

Introduce the Baker–Akhiezer module $M^k_c = \bigcup_{n=1}^{\infty} M^k_c(n)$ over $\mathcal{D}_{g-k}$, where

$$M^k_c(n) = \left\{ f(z, x) \exp \left( -\sum_{j=1}^{g} \frac{x_j}{s} \partial_{x_j} \log \vartheta(z) \right), f(z, x) \in H^0(Y^k, \mathcal{L}^k_{c+x}(nQ^k)) \right\}.$$  

**Theorem 3.** For $c$ in general position, $M^k_c$ is a free $\mathcal{D}_{g-k}$-module of rank $N$.

To prove this theorem, we need

**Lemma 2.** The restriction map

$$\pi_j : H^0(Y^{j}, \mathcal{L}^j_{c+x}(nQ^j)) \to H^0(Y^{j+1}, \mathcal{L}^{j+1}_{c+x}(nQ^{j+1})), \quad \pi_j(\varphi) = \varphi|_{Y^{j+1}}, \quad n \geq 1, \ j \geq 0,$$

is an epimorphism for $x$ in general position.

We let $Y^0$, $\mathcal{L}^0_c$, and $Q^0$ denote $X^g$, $\mathcal{L}_c$, and $Y$.

**Proof.** Let $F$ be a bundle of rank $r$ over $X^g$ invariant under translations. Denote by $F_c$ the bundle $F \otimes \mathcal{P}_c$, where $\mathcal{P}_c$ is the restriction of the Poincaré
bundle to \( X^g \times \{c\} \). In [1] (see Example 5.8 and Proposition 5.10) it was proven that

\[
H^i(X^g, F_c(nY)) = 0, \quad i \geq 1, \; n \geq 1, \quad (2)
\]

\[
H^i(X^g, F_c(nY)) = 0, \quad i \neq g, \; n \leq -1, \quad (3)
\]

and the equality

\[
H^i(X^g, F_c) = 0 \quad (4)
\]

is valid for a point \( c \) in general position for \( i \geq 0 \). Observe that the bundle

\[
L_c \otimes [sY] \otimes [-Y_{a_1}] \otimes \ldots \otimes [-Y_{a_s}]
\]

is invariant under translations, where \( 1 \leq s \leq k \), since so are \( L_c \) and \([Y] \otimes [-Y_{a_s}]\). Hence,

\[
H^i(X^g, L_c \otimes [nY] \otimes [-Y_{a_1}] \otimes \ldots \otimes [-Y_{a_s}]) = 0, \quad (5)
\]

where \( 1 \leq i < g \) and \( n \in \mathbb{Z} \). We have the exact sequence of bundles

\[
0 \to L^j_c \otimes [nQ^j] \otimes [-Y_{j+1}] \to L^j_c \otimes [nQ^j] \to L^{j+1}_c \otimes [nQ^{j+1}] \to 0. \quad (6)
\]

It follows from the long exact cohomology sequence corresponding to this sequence that, for proving surjectivity of \( \pi_j \), it suffices to establish the equality

\[
H^i(Y^j, L^j_c \otimes [nQ^j] \otimes [-Y_{j+1}]) = 0. \quad (7)
\]

From (5) we immediately obtain surjectivity of \( \pi_0 \). To prove (7), consider the following exact sequence:

\[
0 \to L^j_c \otimes [nQ^j] \otimes [-(Y^j \cap Y_{a_{j+1}})] \otimes \ldots \otimes [-(Y^j \cap Y_{a_{j+s}})]
\]

\[
\to L^j_c \otimes [nQ^j] \otimes [-(Y^j \cap Y_{a_{j+2}})] \otimes \ldots \otimes [-(Y^j \cap Y_{a_{j+s}})]
\]

\[
\to L^{j+1}_c \otimes [nQ^{j+1}] \otimes [-(Y^{j+1} \cap Y_{a_{j+2}})] \otimes \ldots \otimes [-(Y^{j+1} \cap Y_{a_{j+s}})] \to 0, \quad (8)
\]

where \( j + s \leq k \). From the long exact cohomology sequences corresponding to (6) and (8), using (2)–(4) and inducting on \( j \), we obtain

\[
H^i(Y^j, L^j_c \otimes [nQ^j] \otimes [-(Y^j \cap Y_{a_{j+1}})] \otimes \ldots \otimes [-(Y^j \cap Y_{a_{j+s}})]) = 0, \quad (9)
\]

where \( 1 \leq i < g - j \) and \( j + s \leq k \). Consequently, \( \pi_j \) is surjective. The lemma is proven.

Observe also that if \( n > g \) then (9) is valid for \( i \geq 1 \).
From Lemma 2 we derive

**Corollary 2.** The restriction map

\[ \pi_j : M^j_c \to M^j+1_c, \quad \pi_j(\varphi) = \varphi|_{Y^{j+1}}, \]

is an epimorphism for c in general position.

We also need

**Lemma 3.** The linear span of the set

\[ \bigcup_{b, \varphi} \frac{\partial}{\partial z} (z - b) \varphi, \quad \varphi \in H^0(X^g, L_{c+sb}((n - 1)Y), \ b \in \mathbb{C}^g) \}

where \( n > g \) and the union is taken over all \( b \) and \( \varphi \), coincides with \( H^0(X^g, L_c(nY)) \).

**Proof.** Consider the sequence of mappings

\[ 0 \longrightarrow H^0(X^g, L_c(nY)) \xrightarrow{\pi_0} H^0(Y^1, L^1_c(nQ^1)) \xrightarrow{\pi_1} \]

\[ \ldots \xrightarrow{\pi_{g-2}} H^0(Y^{g-1}, L^{g-1}_c(nQ^{g-1})) \longrightarrow 0. \]

Here \( Y^{g-1} \) is the Riemann surface \( Y^{g-2} \cap Y_{a_{g-1}} \), where \( a_{g-1} \) is some element of \( X^g \). The mappings \( \pi_0, \ldots, \pi_{g-3} \) are surjective by Lemma 2. Since (9) holds for \( n > g \); therefore, \( \pi_{g-2} \) is also surjective. Consequently, to prove the lemma, it suffices to demonstrate that the linear span of the restrictions of the vector-functions listed in the lemma to \( Y^{g-1} \) coincides with \( H^0(Y^{g-1}, L^{g-1}_c(nQ^{g-1})) \). Take \( b_1, b_2 \in \mathbb{C}^g \) so that the divisors \( B_1 = Y_{b_1} \cap Y^{g-1} \) and \( B_1 = Y_{b_1} \cap Y^{g-1} \) be disjoint. Note that from (6), (8), and (9) we obtain the equalities

\[ H^1(L^{g-1}_c \otimes [nQ^{g-1}]) = 0, \]

\[ H^1(Y^{g-1}, L^{g-1}_c \otimes [nQ^{g-1}] \otimes [-B_1]) = 0, \]

\[ H^1(Y^{g-1}, L^{g-1}_c \otimes [nQ^{g-1}] \otimes [-B_1] \otimes [-B_2]) = 0. \]

Then by the Riemann–Roch theorem

\[ h^0(L^{g-1}_c \otimes [nQ^{g-1}]) = \deg(L^{g-1}_c \otimes [nQ^{g-1}]) - (g(Y^{g-1}) - 1)r, \]

\[ h^0(L^{g-1}_c \otimes [nQ^{g-1}] \otimes [-B_1]) = \deg(L^{g-1}_c \otimes [nQ^{g-1}] \otimes [-B_1]) - (g(Y^{g-1}) - 1)r, \]

\[ h^0(L^{g-1}_c \otimes [nQ^{g-1}] \otimes [-B_1] \otimes [-B_2]) = \deg(L^{g-1}_c \otimes [nQ^{g-1}] \otimes [-B_1] \otimes [-B_2]) - (g(Y^{g-1}) - 1)r, \]
where $h^0$ is the dimension of $H^0$ and $g(Y^{g-1})$ is the genus of $Y^{g-1}$. Since the divisors $B_1$ and $B_2$ are disjoint, we have

$$
\dim(H^0(Y^{g-1}, \mathcal{L}_c^{g-1}(nQ^{g-1}) \otimes [-B_1]) \cap H^0(Y^{g-1}, \mathcal{L}_c^{g-1}(nQ^{g-1}) \otimes [-B_2]))
$$

$$
= h^0(\mathcal{L}_c^{g-1}(nQ^{g-1}) \otimes [-B_1] \otimes [-B_2]).
$$

Recall that we identify $H^0(Y^{g-1}, \mathcal{L}_c^{g-1}(nQ^{g-1}) \otimes [-B_j])$ with the space of global sections of $\mathcal{L}_c^{g-1}(nQ^{g-1})$ having zeros at the points of the divisor $B_j$. Hence, we obtain the equality

$$
h^0(\mathcal{L}_c^{g-1}(nQ^{g-1}) \otimes [nQ^{g-1}]) = h^0(\mathcal{L}_c^{g-1}(nQ^{g-1}) \otimes [-B_1])
$$

$$
+ h^0(\mathcal{L}_c^{g-1}(nQ^{g-1}) \otimes [-B_2]) - h^0(\mathcal{L}_c^{g-1}(nQ^{g-1}) \otimes [-B_1] \otimes [-B_2]),
$$

which means that the linear span of the restriction of $W(b_1) \cup W(b_2)$ to $Y^{g-1}$ coincides with $H^0(\mathcal{L}_c^{g-1}(nQ^{g-1}) \otimes [nQ^{g-1}])$, where

$$
W(b) = \left\{ \frac{\partial_j + 1(z - b)}{\partial(z)} \varphi, \ \varphi \in H^0(X^g, \mathcal{L}_{c+sb}((n-1)Y) \right\}.
$$

This completes the proof of the lemma.

Note that we have proven even more. In the condition of Lemma 3 we need not take the union over all $b$; it suffices to take the set

$$
W(a_1) \cup \ldots \cup W(a_{g-2}) \cup W(b_1) \cup W(b_2).
$$

Denote by $S^g_n$ the dimension of the space of differential operators in $g$ variables with constant coefficients whose degree does not exceed $n - 1$. It is easy to verify that

$$
S^g_n = C_{n+g-1}^{m-1} = \frac{n(n+1) \ldots (n+g-1)}{g!}.
$$

Introduce one more notation

$$
F_j(n) = \dim H^0(Y^j, \mathcal{L}_c^j(nQ^j)), \quad 0 \leq j < g - 1.
$$

**Proof of Theorem 2.** Take a homogeneous basis $\Phi_c$ for the $\mathcal{D}_g$-module $M_c$ so that its restriction to the variety $Y^j$ generates the $\mathcal{D}_j$-module $M_c^j$; i.e.,

$$
M_c^j = \{ d_1 \phi_{1,c}|_{Y^j} + \ldots + d_N \phi_{N,c}|_{Y^j}, \ d_i \in \mathcal{D}_g \}.
$$
This is possible by Corollary 2. By homogeneity of the basis we mean the following. First, all elements of the basis $\Phi_c$ are contained in $M_c(g)$ (this requirement is fulfilled by Nakayashiki’s theorem). And, second, if $\phi_1, \ldots, \phi_{1,K}$ are the elements of the basis that belong to $M_c(n)$, $n \leq g$, then they generate $M_c(n)$. In other words,

$$\{d_1\phi_1 + \ldots + d_K\phi_K, \ d_j \in D_g\} \cap M_c(n) = M_c(n).$$

Denote by $a^g_k$ the number of elements of the basis $\Phi_c$ belonging to $M_c(n)$ but not to $M_c(n-1)$. Since the basis is homogeneous and the $D_g$-module $M_c$ is free, we have

$$a^g_1S^g_{n} + \ldots + a^g_gS^g_{n-g+1} = F_0(n), \ n > g.$$ 

Denote by $D_{g-j} \Phi^j_c \subset M^j_c$ the $D_{g-j}$-module

$$\{\varphi|_{Y^j}, \ \varphi = d_1\phi_1 + \ldots + d_N\phi_N, \ d_s \in D_{g-j}\}.$$

Inducting on $k$, prove that $D_{g-k} \Phi^k_c$ is a free $D_{g-k}$-module of rank $N$. Then, computing the dimensions of the spaces $M^j_c(n)$ and $D_{g-j} \Phi^j_c \cap M^j_c(n)$ (for a fixed $x$), we establish that these $D_{g-k}$-modules coincide.

The initial induction step is Nakayashiki’s theorem. Suppose that the assertion is proven for $k = j$. Since the $D_{g-j}$-module $M^j_c$ is free, we derive the equality

$$a^g_1S^g_{n-j} + \ldots + a^g_gS^g_{n-g+1} = F_j(n), \ n > g. \quad (10)$$

Suppose that the $D_{g-j-1}$-module $D_{g-j-1} \Phi^{j+1}_c$ is not free for $k = j + 1 < g$. Then there exist operators $\tilde{d}_i \in D_{g-j-1}$ such that

$$\phi = \tilde{d}_1\phi_1 + \ldots + \tilde{d}_N\phi_N, \ \phi|_{Y^{j+1}} = 0.$$

This is equivalent to the fact that $M^j_c(n)$ (we may assume that $n > g$) contains an element of the form $\frac{\varphi_j}{\varphi_0} (z-a_{j+1}) \varphi, \ \varphi \in M^j_{c-a_{j+1}}(n-1)$, for which

$$\tilde{d}_1\phi_1 + \ldots + \tilde{d}_N\phi_N = \frac{\varphi_{j+1}(z-a_{j+1})}{\varphi(z)}, \ z \in Y^j. \quad (11)$$

Consider the following subspace in $H^0(Y^j, L^j_{c+x}(nQ^j))$:

$$V^j_{c+x}(n) = \left\{ \frac{\psi}{\epsilon|_{Y^j}}, \ \psi = d_1\phi_1 + \ldots + d_N\phi_N, \ d_i \in D_{g-j-1}\right\} \cap H^0(Y^j, L^j_{c+x}(nQ^j)),$$
where
\[ e = \exp \left( - \sum_{j=1}^{g} \frac{x_j}{s} \partial_{x_j} \log \vartheta(z) \right). \]

Find the dimension of \( V_{c+x}^{j}(n) \). From (10) we obtain the equality
\[ a_{1}^{g}(S_{n}^{g-j} - S_{n-1}^{g-j}) + \ldots + a_{g}^{g}(S_{n-g+1}^{g-j} - S_{n-g}^{g-j}) = a_{1}^{g}S_{n}^{g-j-1} + \ldots + a_{g}^{g}S_{n-g+1}^{g-j-1} \]
\[ = F_{j}(n) - F_{j}(n-1) = F_{j+1}(n); \]
consequently,
\[ \dim V_{c+x}^{j}(n) = F_{j}(n) - F_{j}(n-1). \] (12)

Introduce one more subspace in \( H^{0}(Y^{j}, L_{c+x}^{j}(nQ^{j})) \) which depends on the element \( a_{j+1} \):
\[ W_{c+x}^{j}(n) = \left\{ \frac{\vartheta_{j+1}(z - a_{j+1})}{\vartheta(z)} \varphi, \varphi \in H^{0}(Y^{j}, L_{c+x+a_{j+1}}^{j}((n-1)Q^{j}), z \in Y^{j} \right\}. \]

It is clear that
\[ \dim W_{c+x}^{j}(n) = F_{j}(n-1), \]
since \( \dim H^{0}(Y^{j}, L_{c+x+a_{j+1}}^{j}((n-1)Q^{j})) = F_{j}(n-1). \)

It follows from Lemmas 2 and 3 that there is an element \( a_{j+1} \) for which an equality like (11) is impossible. Since
\[ \dim V_{c+x}^{j}(n) + \dim W_{c+x}^{j}(n) = \dim H^{0}(Y^{j}, L_{c+x}^{j}(nQ^{j})), \]
an equality like (11) is impossible for elements of some small neighborhood of \( a_{j+1} \). By analytic dependence of the space \( W_{c+x}^{j}(n) \) on \( a_{j+1} \), an equality like (11) does not hold for an open everywhere dense set of such \( a_{j+1} \). Consequently, since the set \( a_{1}, \ldots, a_{k} \) is in general position by assumption, \( D_{g-j-1}^{\phi^{j+1}} \) is a free \( D_{g-j-1} \)-module of rank \( N \).

Prove that the \( D_{g-j-1} \)-modules \( D_{g-j-1}^{\phi^{j+1}} \) and \( M_{c}^{j+1} \) coincide.

Since \( \frac{\vartheta_{j}(z)}{\vartheta(z)} \) is a meromorphic function and sequence (6) is exact, we obtain the equality
\[ \dim H^{0}(Y^{j+1}, L_{c+x}^{j+1}(nQ^{j+1})) \]
\[ = \dim H^{0}(Y^{j}, L_{c+x}^{j}(nQ^{j})) - \dim H^{0}(Y^{j}, L_{c+x}^{j}((n)Q^{j}) \otimes [-Y^{j+1}]) \]
\[ = \dim H^{0}(Y^{j}, L_{c+x}^{j}(nQ^{j})) - \dim H^{0}(Y^{j}, L_{c+x}^{j}((n-1)Q^{j})) = F_{j}(n) - F_{j}(n-1). \]
Observing the inclusion $\mathcal{D}_{g-j-1} \Phi_{c}^{j+1} \subset M_{c}^{j+1}$, from (12) we find
\[ \dim \mathcal{H}^0(Y^{j+1}, \mathcal{L}_{c+x}^{j+1}(nQ_{j+1})) = \dim \mathcal{V}^{j}_{c+x}(n) = \mathcal{F}_{j+1}(n). \]
So the $\mathcal{D}_{g-j-1}$-modules $\mathcal{D}_{g-j-1} \Phi_{c}^{j+1}$ and $M_{c}^{j+1}$ coincide. Theorem 3 is proven.

We now demonstrate how to derive Theorems 1 and 2 from Theorem 3. Denote by
\[
\Phi_c = (\phi_{1,c}(z,x), \ldots, \phi_{N,c}(z,x))^\top
\]
a basis for the $\mathcal{D}_{g-k}$-module $M_{c}^{k}$. Then, by Theorem 3, for $\lambda \in \mathcal{A}_k$, there is a unique operator $L_k(\lambda) \in \text{Mat}(N, g-k)$ such that
\[
L_k(\lambda) \Phi_c = \lambda \Phi_c.
\]
The operators $L_k(\lambda)$ with different $\lambda$’s commute pairwise. Theorem 1 is proven.

Denote by $T_j \in \text{Mat}(N, g-k)$ the operator of order $g$ defined by the equality
\[
T_j \Phi_c = \partial_{t_j} \Phi_c;
\]
here we identify time $t_j, 1 \leq j \leq k$, with the variable $x_{g-k}$. The equalities
\[
[L_k(\lambda), T_j - \partial_{t_j}] \Phi = 0, \quad [T_m - \partial_{t_m}, T_n - \partial_{t_n}] \Phi = 0
\]
hold. Then Theorem 3 yields

**Corollary 3.** The following evolution equations are valid:

\[
\frac{\partial L_k(\lambda)}{\partial t_j} = [L_k(\lambda), T_j], \quad \lambda \in \mathcal{A}_k,
\]
\[
\frac{\partial T_m}{\partial t_n} - \frac{\partial T_n}{\partial t_m} = [T_n, T_m].
\]

We turn to proving Theorem 2. We divide each vector-function $\phi_{j,c}(z,x)$ by
\[
\exp \left(- \sum_{j=g-k+1}^{g} \frac{x_j}{s} \partial_{z_j} \log \vartheta(z) \right),
\]
and then replace $x = (x_1, \ldots, x_g)$ with
\[
(x,t) = (x_1, \ldots, x_{g-k}, \sum_{m} t_{1,m}, \ldots, \sum_{m} t_{k,m}),
\]

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where $m = (m_1, \ldots, m_g) \in \mathbb{Z}^g$, $m_1 + \ldots + m_g \geq 2$, $m_i \geq 0$, and, finally, multiply by

$$\exp\left( - \sum_{j=1}^{k} \sum_{m} \frac{t_{j,m}}{s} \left( \partial_{z_{g-k+j}} \log \vartheta(z) + \partial_z^{m} \log \vartheta(z) \right) \right).$$

where $\partial_z^{m} \log \vartheta(z) = \partial_{z_1}^{m_1} \ldots \partial_{z_g}^{m_g} \log \vartheta(z)$. We obtain the vector-function $\psi_{j,c}(z, x, t)$ which is representable as the sum of vector-functions:

$$g(x, t) \frac{\partial^{r,sn}(z + \frac{(x,t) + c}{sn})}{\partial^{n}(z)} \exp\left( - \sum_{j=1}^{g-k} \frac{x_j}{s} \partial_{z_j} \log \vartheta(z) \right.$$

$$- \sum_{j=1}^{k} \sum_{m} \frac{t_{j,m}}{s} \left( \partial_{z_{g-k+j}} \log \vartheta(z) + \partial_z^{m} \log \vartheta(z) \right).$$

Then for

$$\Psi = (\psi_{1,c}(z, x, t), \ldots, \psi_{N,c}(z, x, t))^\top$$

we have the equality

$$L_{j,m} \Psi = \partial_{t_{j,m}} \Psi.$$

By Theorem 3,

$$[\partial_{t_{j,m}} - L_{j,m}, \partial_{t_{i,n}} - L_{i,n}] = 0.$$

Theorem 2 is proven.

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Sobolev Institute of Mathematics, Novosibirsk
mironov@math.nsc.ru