Note on the Absolutely Continuous Spectrum for the Anderson Model on Cayley Trees of Arbitrary Degree

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Abstract

We provide a simplified version of the geometric method given by Froese, Hasler and Spitzer in [4] and use it to prove the existence of absolutely continuous spectrum for a Cayley tree of arbitrary degree $k$.

1 Introduction

One of the most important open problems in the field of random Schrödinger operators is to prove the existence of absolutely continuous spectrum for weak disorder in the Anderson model [2] in three and higher dimensions. The first result in this direction is Abel Klein’s, for random Schrödinger operators acting on a tree, or Cayley tree, or Bethe lattice, of any constant degree larger than 2. Klein [7] proves that for weak disorder, almost all potentials will produce absolutely continuous spectrum. This means that there must be many potentials on a tree for which the corresponding Schrödinger operator has absolutely continuous spectrum without there being an obvious reason, such as periodicity or decrease at infinity. Later on, different other proofs were given to the same result (see [4] and [11]). This paper simplifies the geometric method in [4]. The simplifications make possible the generalization from a Bethe lattice of degree 3 to one of any degree $M + 1$, with $M \geq 2$.

2 The Model and the Results

A Bethe lattice (or Cayley tree), $\mathcal{B}$, is a connected infinite graph with no closed loops and a fixed degree (number of nearest neighbors) at each vertex (site or

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point), $x$. The distance between two sites $x$ and $y$ will be denoted by $d(x, y)$ and is equal to the length of the shortest (only) path connecting $x$ and $y$.

The Anderson model on the Bethe lattice is given by the random Hamiltonian

$$
H = \Delta + k q
$$
on the Hilbert space $\ell^2(\mathbb{B}) = \{ \varphi : \mathbb{B} \to \mathbb{C} ; \sum_{x \in \mathbb{B}} |\varphi(x)|^2 < \infty \}$. The (centered) Laplacian $\Delta$ is defined by

$$(\Delta \varphi)(x) = \sum_{y : d(x, y) = 1} \varphi(y)$$

and has spectrum $\sigma(\Delta) = [-2 \sqrt{M}, 2 \sqrt{M}]$. The operator $q$ is a random potential, with $q(x), x \in \mathbb{B}$, being independent, identically distributed real random variables with common probability distribution $\nu$. We assume the $(2(1 + p)$ moment,

$$
\int |q|^{2(1+p)} \, d\nu, \text{is finite for some } p > 0.
$$
The coupling constant $k$ measures the disorder.

As mentioned above, the existence of purely absolutely continuous spectrum for the Anderson model on the Bethe lattice was first proved, in a different manner, by Klein in 1998. Given any closed interval $E$ contained in the interior of the spectrum of $\Delta$ on the Bethe lattice, he proved that for small disorder, $H$ has purely absolutely continuous spectrum in some interval $E$ with probability one, and its integrated density of states is continuously differentiable on the interval (he only needed a finite second moment, whereas we have a finite $2(1+p)$ moment in our model). We prove a similar result in this chapter. A key point is the definition of a weight function appearing in the proofs. This definition is motivated by hyperbolic geometry.

**Theorem 1.** For any $E$, with $0 < E < 2 \sqrt{M}$ and $H$ defined above, there exists $k(E) > 0$ such that for all $0 < |k| < k(E)$ the spectrum of $H$ is purely absolutely continuous in $[-E, E]$ with probability one, i.e., we have almost surely

$$
\sigma_{ac} \cap [-E, E] = [-E, E], \quad \sigma_{pp} \cap [-E, E] = \emptyset, \quad \sigma_{sc} \cap [-E, E] = \emptyset.
$$

Let $\mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$ denote the complex upper half plane. For convenience, we fix an arbitrary site in $\mathbb{B}$ to be the origin and denote it by $0$. For each $x \in \mathbb{B}$ we have at most one neighbour towards the root and two or more in what we refer to as the forward direction. We say that $y \in \mathbb{B}$ is in the future of $x \in \mathbb{B}$ if the path connecting $y$ and the root runs through $x$. Let $x \in \mathbb{B}$ be an arbitrary vertex, the subtree consisting of all the vertices in the future of $x$, with $x$ regarded as its
root, is denoted by $B^x$. We will write $H^x$ for $H$ when restricted to $B^x$ and set $G^x(\lambda) = \langle \delta_x, (H^x - \lambda)^{-1} \delta_x \rangle$ the Green function for the truncated graph. $G^x$ is called the forward Green function.

**Proposition 2.** For any $\lambda \in \mathbb{H}$ we have

$$G(\lambda) = \langle \delta_0, (H - \lambda)^{-1} \delta_0 \rangle = -\left( \sum_{x: d(x, 0) = 1} G^x(\lambda) + \lambda - k q(0) \right)^{-1} \tag{1}$$

and, for any site $x \in \mathbb{B}$,

$$G^x(\lambda) = -\left( \sum_{y: d(y, x) = 1, y \in \mathbb{B}} G^y(\lambda) + \lambda - k q(x) \right)^{-1} \tag{2}$$

**Proof.** We will prove (1); (2) is proven in exactly the same way. Let us write $H = \tilde{H} + \Gamma$, where

$$\tilde{H} = k q(0) \oplus \left( \bigoplus_{x: d(x, 0) = 1} H^x \right)$$

is the direct sum corresponding to the decomposition $\mathbb{B} = \{0\} \cup \left( \bigcup_{x: d(x, 0) = 1} \mathbb{B}^x \right)$. The operator $\Gamma$ has matrix elements $\langle \delta_x, \Gamma \delta_y \rangle = \delta_y$, $\langle \delta_0, \Gamma \delta_x \rangle = 1$ if $d(x, 0) = 1$, with all other matrix elements being 0. The resolvent identity gives

$$(\tilde{H} - \lambda)^{-1} = (H - \lambda)^{-1} + (\tilde{H} - \lambda)^{-1} \Gamma (H - \lambda)^{-1}.$$ 

Also,

$$(\tilde{H} - \lambda)^{-1} = (k q(0) - \lambda)^{-1} \oplus \left( \bigoplus_{x: d(x, 0) = 1} (H^x - \lambda)^{-1} \right).$$

Thus

$$\langle \delta_0, (\tilde{H} - \lambda)^{-1} \delta_0 \rangle = \langle \delta_0, (H - \lambda)^{-1} \delta_0 \rangle + \langle \delta_0, (\tilde{H} - \lambda)^{-1} \Gamma (H - \lambda)^{-1} \delta_0 \rangle.$$ 

Hence

$$G(\lambda) = (q(0) - \lambda)^{-1} - (k q(0) - \lambda)^{-1} \sum_{x: d(x, 0) = 1} \langle \delta_x, (H - \lambda)^{-1} \delta_0 \rangle \tag{3}.$$ 

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The resolvent formula also implies that for each \( x \) with \( d(x, 0) = 1 \),
\[
\left\langle \delta_x, (H - \lambda)^{-1} \delta_0 \right\rangle = -G^*(\lambda) G(\lambda).
\] (4)

(2) follows from (3) and (4). \( \square \)

The recursion relation for \( G^*(\lambda) \) that we just proved leads us to the following transformation \( \phi: \mathbb{H}^M \times \mathbb{R} \times \mathbb{H} \rightarrow \mathbb{H} \) defined by
\[
\phi(z_1, \ldots, z_M, q, \lambda) = -\frac{1}{z_1 + \ldots + z_M + \lambda - q}.
\] (5)

It is easy to see the equivalence between (1) and (5). Let \( q \equiv 0 \). If \( \Im(\lambda) > 0 \), the transformation \( z \mapsto \phi(z, \ldots, z, 0, \lambda) \) has a unique fixed point, \( z_\lambda \), in the upper half plane, i.e. \( \Im(z_\lambda) > 0 \) (for details see Proposition 2.1, in [3]). Explicitly,
\[
z_\lambda = -\frac{\lambda}{2M} + \frac{1}{M} \sqrt{\lambda/2 - M},
\]
where we will always make the choice \( \Im\sqrt{\cdot} \geq 0 \) (and \( \sqrt{a} > 0 \) for \( a > 0 \)). This fixed point as a function of \( \lambda \in \mathbb{H} \) extends continuously onto the real axis. This extension yields, for \( \Im(\lambda) = 0 \) and \( |\lambda| < 2 \sqrt{M} \), the fixed point
\[
z_\lambda = -\frac{\lambda}{2M} + \frac{i}{2M} \sqrt{4M - \lambda^2},
\]
lying on an arc of the circle \( |z| = 1/\sqrt{M} \). When \( \Im(\lambda) = 0 \) and \( |\lambda| \leq E < 2 \sqrt{M} \), the arc is strictly contained in the upper half plane. Thus, when \( \lambda \) lies in the strip
\[
R(E, \epsilon) = \{ z \in \mathbb{H} : \Re(z) \in [-E, E], 0 < \Im(z) \leq \epsilon \}
\]
with \( 0 < E < 2 \sqrt{M} \) and \( \epsilon \) sufficiently small, \( \Im(z_\lambda) \) is bounded below and \( |z_\lambda| \) is bounded above by a positive constant.

In order to prove that the spectral measures are absolutely continuous we need to establish bounds for \( \mathbb{E}(|G^*(\lambda)|^{1+p}) \). Since \( z_\lambda \) equals \( G^*(\lambda) \) for the case \( q \equiv 0 \) and any \( x \in \mathbb{B} \), in order to prove the desired bounds we will use the weight function \( w(z) \) defined by
\[
w(z) = 2 \left( \cosh(\text{dist}_{\mathbb{H}}(z, z_\lambda)) - 1 \right) = \frac{|z - z_\lambda|^2}{\Im(z) \Im(z_\lambda)}.
\] (6)

Up to constants, \( w(z) \) is the hyperbolic cosine of the hyperbolic distance from \( z \) to \( z_\lambda \), provided \( \lambda \in R(E, \epsilon) \) with \( 0 < E < 2 \sqrt{M} \) and \( \epsilon \) sufficiently small. This notation
suppresses the $\lambda$ dependence. In essence, we are looking at the hyperbolic cosine of the distance between $G^\lambda(\lambda)$ for the free Laplacian and the one for the perturbed one, $H$. The goal is to prove that this quantity, which blows up on the boundary, stays mostly finite.

To prove a bound for $\mathbb{E}(w^{1+p}(G^\lambda(\lambda)))$ we will need to use (5), more than once, to express the forward Green function as a function of the forward Green functions at future nodes. As a result, the study of the following quantity becomes needed:

$$\mu_{3,p}(z_1 \ldots z_{2M-1}, q_1, q_2, \lambda) = \sum_{\sigma} w^{1+p}(\phi(z_{\sigma_1} \ldots z_{\sigma_M}, q_1, \lambda), z_{\sigma_{M+1}} \ldots z_{\sigma_{2M-1}}, q_2, \lambda) \frac{w^{1+p}(\phi(z_1) + \ldots + w^{1+p}(z_{2M-1}))}{w^{1+p}(z_1) + \ldots + w^{1+p}(z_{2M-1})}$$

where $\sigma$ are all cyclic permutations. We will state here the needed lemmas, but we will give the proofs later.

**Lemma 3.** For any $E, 0 < E < 2\sqrt{M}$ and any $0 < p < 1$, there exist positive constants $\epsilon, \eta_1, \epsilon_0$ and a compact set $\mathcal{M} \in H^{2M-1}$ such that

$$\mu_{3,p}(\mathcal{M}^c \times [-\eta_1, \eta_1]^{2} \times R(E, \epsilon_0)) \leq 1 - \epsilon. \quad (7)$$

Here $\mathcal{M}^c$ denotes the complement $H^{2M-1} \setminus \mathcal{M}$.

**Lemma 4.** For any $E, 0 < E < 2\sqrt{M}$ and any $0 < p < 1$, there exist positive constants $\epsilon_0, C$ and a compact set $\mathcal{M} \in H^{2M-1}$ such that

$$\mu_{3,p}(\mathcal{M}^c \times \mathbb{R}^2 \times R(E, \epsilon_0)) \leq C(1 + \sum_{i=1}^{2} |q_i|^{2(1+p)}) \quad (8)$$

Similarly, if we define

$$\mu_{3,p}'(z_1, \ldots, z_{M+1}) = \frac{w(- (\sum_{i=1}^{M+1} z_i + \lambda - q)^{-1})^{1+p}}{w(z_1)^{1+p} + \ldots + w(z_{M+1})^{1+p}},$$

then

$$\mu_{3,p}'(\mathcal{M}^c \times \mathbb{R}^2 \times R(E, \epsilon_0)) \leq C(1 + |q|^{2(1+p)}).$$

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**Theorem 5.** Let \( x \) be a nearest neighbour of 0. For any \( E, 0 < E < 2\sqrt{M} \) and all \( 0 < p < 1 \), there exists \( k(E) > 0 \) such that for all \( 0 < |k| < k(E) \) we have

\[
\sup_{\lambda \in \mathbb{R}(E, \epsilon)} \mathbb{E}\left( w^{1+p}(G^x(\lambda)) \right) < \infty.
\]

**Proof.** In order to prove that the above quantity is bounded we need a couple of preparatory steps.

Let \( \eta_1 \) and \( p \) be given by Lemma 3, and choose \( \epsilon_0 \) and \( M \) that work in both Lemma 3 and Lemma 4. For \((z_1, \ldots, z_{2M-1}) \in M^c\), we estimate

\[
\int_{\mathbb{R}^2} \mu_{3,p}(z_1, \ldots, z_{2M-1}, k q_1, k q_2, \lambda) d\nu(q_1) d\nu(q_2)
\]

\[
\leq (1 - \epsilon) \int_{\left[\frac{-\eta_1}{\sqrt{M}}, \frac{\eta_1}{\sqrt{M}}\right]^2} d\nu(q_1) d\nu(q_2) + C \int_{\mathbb{R}^2} \left(1 + \sum_{i=1}^{2} |k q_i|^{2(1+p)}\right) d\nu(q_1) d\nu(q_2)
\]

\[
\leq (1 - \epsilon) + C \int_{\mathbb{R}^2} d\nu(q_1) d\nu(q_2) + 2C|k|^{2(1+p)} M_{2(1+p)} \leq 1 - \epsilon/2
\]

provided \( k \) is sufficiently small. Here \( M_{2(1+p)} \) denotes the moment \( \int |q|^{2(1+p)} d\nu(q) \).

The probability distributions for \( G \) and \( G^x \) on the hyperbolic plane are defined by \( \rho_G(A) = \text{Prob}\{ G(\lambda) \in A \} \) and \( \rho(A) = \text{Prob}\{ G^x(\lambda) \in A \} \). This implies

\[
\rho(A) = \text{Prob}\{ \phi(z_1, \ldots, z_M, k q, \lambda) \in A \} = \text{Prob}\{ (z_1, \ldots, z_M, k q, \lambda) \in \phi^{-1}(A) \}
\]

\[
= \int_{\phi^{-1}(A)} d\rho(z_1) \ldots d\rho(z_M) d\nu(q) = \int_{\mathbb{H} \times \mathbb{R}} \chi_A(\phi(z_1, \ldots, z_M, k q, \lambda)) d\rho(z_1) \ldots d\rho(z_M) d\nu(q)
\]

which gives us that for any bounded continuous function \( w(z) \),

\[
\int_{\mathbb{H}} w(z) d\rho(z) = \int_{\mathbb{H} \times \mathbb{R}} w(\phi(z_1, \ldots, z_M, k q, \lambda)) d\rho(z_1) \ldots d\rho(z_M) d\nu(q).
\]

Now we have all the ingredients needed to prove our theorem. Using the previous relation twice, for \( \lambda \in R(E, \epsilon_0) \), we obtain:
\[ \mathbb{E}(w^{1+p}(G^\lambda)) = \int_{\mathbb{H}} w^{1+p}(z) \, d\rho(z) \]

\[ = \int_{\mathbb{H}^M \times \mathbb{R}} w^{1+p}(\phi(z_1, \ldots, z_M, k q_1, \lambda)) \, d\rho(z_1) \ldots d\rho(z_M) \, d\nu(q_1) \]

\[ = \int_{\mathbb{H}^{2M} \times \mathbb{R}^2} w^{1+p}(\phi(z_1, \ldots, z_M, k q_1, \lambda), z_{M+1} \ldots z_{2M-1}, k q_2, \lambda)) \]

\[ = \int_{\mathbb{H}^{2M} \times \mathbb{R}^2} \frac{1}{2M-1} \sum_{\sigma} w^{1+p}(\phi(z_{\sigma_1}, \ldots, z_{\sigma_M}, k q_1, \lambda), z_{\sigma_{M+1}} \ldots z_{\sigma_{2M-1}}, k q_2, \lambda)) \]

\[ \leq (1 - \epsilon/2) \int_{\mathbb{H}} w^{1+p}(z) \, d\rho(z) + C = (1 - \epsilon/2) \mathbb{E}(w^{1+p}(G^\lambda)) + C . \]

where \( C \) is some finite constant, only depending on the choice of \( M \).

Note: We used the fact that
\[ \int_{\mathbb{H}} w^{1+p}(z) \, d\rho(z) = \frac{1}{2M-1} \int_{\mathbb{H}^{2M-1}} (w^{1+p}(z_1) + \ldots + w^{1+p}(z_{2M-1})) \, d\rho(z_1) \ldots d\rho(z_{2M-1}) \]

This implies that for all \( \lambda \in R(E, e_0) \),
\[ \mathbb{E}(w^{1+p}(G^\lambda)) \leq \frac{2C}{\epsilon} . \]

\[ \square \]

**Theorem 6.** Let \( x \in \mathbb{B} \). Under the hypotheses of Theorem 5

\[ \sup_{\lambda \in R(E, e)} \mathbb{E}\left( |\langle \delta_x, (H - \lambda)^{-1} \delta_x \rangle|^{1+p} \right) < \infty \]

for some \( \epsilon > 0 \).
Proof. It is an immediate consequence of Theorem 5 and the following inequality:

$$|z| \leq 4\text{Im}(s) \frac{|z - s|^2}{\text{Im}(z)\text{Im}(s)} + 2|s|.$$ (9)

The inequality clearly holds for $|z| \leq 2|s|$. In the complementary case, we have $|z| > 2|s|$ and thus $|z - s| \geq ||z| - |s|| \geq |s|$, implying $|z|\text{Im}(z) \leq |z|^2 \leq 2|z - s|^2 + 2|s|^2 \leq 4|z - s|^2$. This proves (9).

Using (9) with $s = z_0$ yields that for $\lambda \in R(E, \epsilon)$, $|z| \leq 4w(z) + C$, where $C$ depends only on $E$ and $\epsilon$.

To finish the proof we need to transfer the estimate from $\rho$ to $\rho_G$ and therefore prove the inequality for $x = 0$. By symmetry it extends to any vertex $x \in \mathcal{B}$. In the proof of the following estimate we need the elementary fact that for $z_1, \ldots, z_{M+1} \in \mathcal{M}$, $w^{1+p}\left(\sum_{i=1}^{M+1} z_i - \lambda - q\right)^{-1}\leq C \left(1 + |q|^{2(1 + p)}\right)$. Let $R$ denote $R(E, \epsilon)$, then

$$\sup_{x \in \mathcal{R}} \left|\left(\delta_0, (H - \lambda)^{-1}\delta_0\right)\right|^{1+p} = \sup_{x \in \mathcal{R}} \int |z|^{1+p} d\rho_G(z)$$

$$\leq C_1 \sup_{x \in \mathcal{R}} \int_{\mathbb{H}} w^{1+p}(z) d\rho_G(z) + C_2$$

$$= C_1 \sup_{x \in \mathcal{R}} \int_{\mathbb{H}^{M+1} \times \mathbb{R}} w^{1+p}\left(\sum_{i=1}^{M+1} z_i - \lambda - k q\right)^{-1}\text{d}p(z_1)\ldots\text{d}p(z_{M+1}) \text{d}v(q) + C_2$$

$$\leq C_1 \sup_{x \in \mathcal{R}} \int_{\mathbb{H}^{M+1} \times \mathbb{R}} \mu_3'(\sigma_1, \ldots, \sigma_{M+1}, k q, \lambda) \times (w^{1+p}(z_1) + \ldots + w^{1+p}(z_{M+1})) \text{d}p(z_1)\ldots\text{d}p(z_{M+1}) \text{d}v(q) + C_2'$$

$$\leq C \int_{\mathbb{H} \times \mathbb{R}} (1 + |k|^{2(1 + p)})w^{1+p}(z) d\rho(p(z) d\nu(q) + C_2 \leq C \int_{\mathbb{H}} w^{1+p}(z) d\rho(z) + C_3$$

$$= C E (w^{1+p}(G^4(\lambda))) + C_3 \leq C_4 ,$$

where $C$, $C_1$, $C_2$, $C_3$ and $C_4$ are positive constants. □

As it was proven in [6] (or in the next chapter), this theorem implies the main result of this chapter: Theorem 1 For any $E$, with $0 < E < 2 \sqrt{M}$, there exists $k(E) > 0$ such that for all $0 < |k| < k(E)$ the spectrum of $H$ is purely absolutely continuous in $[-E, E]$ with probability one, i.e., we have almost surely

$$\sigma_{ac} \cap [-E, E] = [-E, E], \quad \sigma_{pp} \cap [-E, E] = \emptyset, \quad \sigma_{sc} \cap [-E, E] = \emptyset .$$
3 Analysis of $\mu_2$ and Proofs of Lemmas

For the proofs of our technical lemmas we need to analyse a quantity, $\mu_2$, which will prove to play a significant role in the expression for $\mu_{3,p}$. We define $\mu_2$ by

$$
\mu_2(z_1 \ldots z_M, q, \lambda) = \frac{M w(\phi(z_1 \ldots z_M, q, \lambda))}{w(z_1) + \ldots + w(z_M)}
$$

as a function from $\mathbb{H}^M \setminus \{(z_1, \ldots, z_i)\} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. In this section $R = R(E, \epsilon)$, for some $0 < E < 2 \sqrt{M}$ and $\epsilon > 0$.

**Proposition 7.** For all $z_1, \ldots, z_M \in \mathbb{H}^M \setminus \{(z_1, \ldots, z_i)\}$ and $\lambda \in \mathbb{R}$,

$$
\mu_2(z_1, \ldots, z_M, 0, \lambda) < 1.
$$

**Proof.** For $z, s \in \mathbb{H}$ set

$$
c(s, z) = 2(\cosh(\text{dist}_{\mathbb{H}}(s, z)) - 1) = \frac{|s - z|^2}{\text{Im}(s)\text{Im}(z)}.
$$

Note that $z \mapsto c(s, z)$ is strictly convex. This can be seen for example by noting that its Hessian has strictly positive eigenvalues. Also, for $s = z_{\lambda}$, $c(z_{\lambda}, z) = w(z)$. The transformation $\phi'(z) = -1/(z + \lambda)$ is a hyperbolic contraction (see [3], Proposition 2.1) and since $\phi'(z_1 + \ldots + z_M) = \phi(z_1 \ldots z_M, 0, \lambda)$ we have $\phi'(Mz_{\lambda}) = z_{\lambda}$. This implies

$$
\text{dist}_{\mathbb{H}}(\phi'(Mz_{\lambda}), \phi'(z_1 + \ldots + z_M)) < \text{dist}_{\mathbb{H}}(Mz_{\lambda}, z_1 + \ldots + z_M) \iff \\
\cosh(\text{dist}_{\mathbb{H}}(\phi'(Mz_{\lambda}), \phi'(z_1 + \ldots + z_M))) < \cosh(\text{dist}_{\mathbb{H}}(Mz_{\lambda}, z_1 + \ldots + z_M)) \iff \\
c(z_{\lambda}, \phi(z_1, \ldots, z_M, 0, \lambda)) < c(Mz_{\lambda}, z_1 + \ldots + z_M) = c\left(z_{\lambda}, \frac{z_1 + \ldots + z_M}{M}\right) \\
\leq \frac{1}{M} \sum_{i=1}^{M} c(z_{\lambda}, z_i),
$$

hence

$$
\frac{Mc(z_{\lambda}, \phi(z_1, \ldots, z_M, 0, \lambda))}{\sum_{i=1}^{M} c(z_{\lambda}, z_i)} < 1
$$

Also, from Proposition 2.1 [3], if $\text{Im}(\lambda) = 0$ then $\phi'$ is a hyperbolic isometry. Therefore

$$
c(\phi'(Mz_{\lambda}), \phi'(z_1 + \ldots + z_M)) = c(Mz_{\lambda}, z_1 + \ldots + z_M) \\
= c\left(z_{\lambda}, \frac{z_1 + \ldots + z_M}{M}\right) \leq \frac{1}{M} \sum_{i=1}^{M} c(z_{\lambda}, z_i)$$

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If \( \text{Im}(\lambda) = 0 \), then \( \mu_2(z, \ldots, z_M, q, \lambda) = 1 \). If \( \text{Im}(\lambda) > 0 \), since \( \phi' \) is a hyperbolic contraction, \( \mu_2(z, \ldots, z_M, q, \lambda) = 1 \) iff \( z_1 = \ldots = z_M = z_1 \).

Since in our lemmas we will use a compactification argument, we need to understand the behavior of \( \mu_2(z_1, \ldots, z_M, q, \lambda) \) as \( z_1, \ldots, z_M \) approach the boundary of \( \mathbb{H} \) and \( \lambda \) approaches the real axis. Thus, it is natural to introduce the compactification \( \overline{\mathbb{H}}^M \times \mathbb{R} \times \overline{\mathbb{R}} \). Here \( \overline{\mathbb{R}} \) denotes the closure and \( \overline{\mathbb{H}} \) is the compactification of \( \mathbb{H} \) obtained by adjoining the boundary at infinity. (The word compactification is not quite accurate here because of the factor \( \mathbb{R} \), but we will use the term nevertheless.)

The boundary at infinity is defined as follows. We cover the upper half plane model of the hyperbolic plane \( \mathbb{H} \) with the atlas \( \mathcal{A} = \{(U_i, \psi_i)_{i=1,2}\} \). We have \( U_1 = \{z \in \mathbb{C} : \text{Im}(z) > 0, |z| < C\} \), \( \psi_1(z) = z \), \( U_2 = \{z \in \mathbb{C} : \text{Im}(z) > 0, |z| > C\} \) and \( \psi_2(z) = -1/z = w \). The boundary at infinity consists of the sets \( \{\text{Im}(z) = 0\} \) and \( \{\text{Im}(w) = 0\} \) in the respective charts. The compactification \( \overline{\mathbb{H}} \) is the upper half plane with the boundary at infinity adjoined. We will use \( i\infty \) to denote the point where \( w = 0 \).

With this convention, \( \mu_2 \) is defined in the interior of the compactification \( \overline{\mathbb{H}}^M \times \mathbb{R} \times \overline{\mathbb{R}} \) and we want to know how it behaves near the boundary. It turns out that in the coordinates introduced above, \( \mu_2 \) is a rational function. For the majority of points on the boundary the denominator does not vanish in the limit and \( \mu_2 \) has a continuous extension. There are, however, points where both numerator and denominator vanish and at these singular points the limiting value of \( \mu_2 \) depends on the direction of approach. By blowing up the singular points, it would be possible to define a compactification to which \( \mu_2 \) extends continuously. However, this is more than we need for our analysis. We will do a partial resolution of the singularities of \( \mu_2 \) and then extend \( \mu_2 \) to an upper semi-continuous function on the resulting compactification.

The reciprocal of the function \( w(z), \chi(z) = 1/w(z) = \text{Im}(z)/|z-z_d|^2 \) is a boundary defining function for \( \mathbb{H} \). This means that in each of the two charts above, \( \chi \) is positive near infinity and vanishes exactly to first order on the boundary at infinity. Further more, we can express \( \mu_2 \) as follows:

\[
\mu_2(z_1 \ldots z_M, q, \lambda) = \frac{M}{\chi(\phi(z_1 \ldots z_M, q, \lambda)) \left[ \frac{1}{\chi(z_1)} + \ldots + \frac{1}{\chi(z_M)} \right]}
\]

or

\[
\mu_2(z_1 \ldots z_M, q, \lambda) = \frac{M \chi(z_1) \ldots \chi(z_M)}{\chi(\phi(z_1 \ldots z_M, q, \lambda)) \left[ \chi(z_1) \ldots \chi(z_{M-1}) + \ldots + \chi(z_2) + \chi(z_M) \right]}
\]

Since

\[
\chi(\phi(z_1 \ldots z_M, q, \lambda)) = \frac{\text{Im}(\phi(z_1 \ldots z_M, q, \lambda))}{|z_1 - \phi(z_1 \ldots z_M, q, \lambda)|^2} = \frac{\text{Im}(z_1 + \ldots + z_M + \lambda)}{|z_d(z_1 + \ldots + z_M) + \lambda z_d - q z_d + 1|^2}
\]
we obtain
\[
\mu_2(z_1 \ldots z_M, q, \lambda) = \frac{M \prod_{i=1}^{M} \chi(z_i) \left| \sum_{i=1}^{M} z_i + \lambda z_{i'} - q z_{i'} + 1 \right|^2}{\left[ \sum_{j=1}^{M} \prod_{i=1}^{M} \chi(z_i) \left| \sum_{i=1}^{M} \chi(z_i) z_i - z_{i'} \right|^2 + \text{Im}(\lambda)/r_1 \right]}
\] (10)

We will now describe our compactification of \( \mathbb{H}^M \times \mathbb{R} \times \mathbb{R} \). Start with \( \mathbb{H}^M \times \mathbb{R} \times \mathbb{R} \). Our blow-up consists of writing \( \chi(z_1), \ldots, \chi(z_M) \) in polar co-ordinates. Thus we introduce new variables \( r_1, \beta_1, \ldots, \beta_M \) and impose the equations
\[
\chi(z_1) = r_1 \beta_1, \ldots, \chi(z_M) = r_1 \beta_M \quad \text{and} \quad \beta_1 + \ldots + \beta_M = 1.
\]
The blown up space, \( \mathcal{K} \), is the variety in \( \mathbb{H}^M \times \mathbb{R} \times \mathbb{R}^{M+1} \) containing all points \((z_1, \ldots, z_M, q, \lambda, r_1, \beta_1, \ldots, \beta_M)\) that verify the blow-up constraints. The topology is the one given by the local description as a closed subset of Euclidean space. The set \( \mathcal{K} \setminus \partial_{\infty} \mathcal{K} \) can be identified with \( \mathbb{H}^M \times \mathbb{R} \times \mathbb{R} \). After the first blow-up, \( \mu_2 \) becomes
\[
\mu_2 = \frac{M \prod_{i=1}^{M} \beta_i \left| \sum_{i=1}^{M} z_i + \lambda z_{i'} - q z_{i'} + 1 \right|^2}{\left[ \sum_{j=1}^{M} \prod_{i=1}^{M} \beta_i \left| \sum_{i=1}^{M} \beta_i z_i - z_{i'} \right|^2 + \text{Im}(\lambda)/r_1 \right]}\] (11)

We can extend \( \mu_2 \) to an upper semi-continuous function on \( \mathcal{K} \) by defining, for points \( k \in \partial_{\infty} \mathcal{K} \),
\[
\mu_2(k) = \lim_{k_n \to k} \sup_{k_n \in \mathcal{K} \setminus \partial_{\infty} \mathcal{K}} \mu_2(k_n).
\]
Here \( k_n = (z_1,n, z_2,n, \ldots, z_M,n, q_1,n, \lambda,n) \) and it converges to \( k \) in \( \mathcal{K} \).

Let us define \( \Sigma \) to be the subset of \( \mathcal{K} \) where \( \mu_2 = 1 \) and let \( \mathcal{K}_0 \) denote the subset of \( \partial_{\infty} \mathcal{K} \) where \( \lambda \in (-2 \sqrt{M}, 2 \sqrt{M}) \) and \( q = 0 \). For the analysis of \( \mu_3 \) we need the following lemma:

**Lemma 8.** Let \( \Gamma = \{ k \in \mathcal{K} : k = (z_1, \ldots, z_M, 0, 0, 0, \lambda, \beta, \ldots, \beta) \} \subset \mathcal{K} \), it contains points in \( \mathcal{K} \) with \( \beta_1 = \ldots = \beta_M = \beta \). Then,
\[
\Gamma \cap \Sigma \cap \mathcal{K}_0 = \{ k \in \mathcal{K}_0 : k = (z, \ldots, z, 0, \lambda, 0, \beta, \ldots, \beta) \}.
\]

**Proof.** Let us first derive an upper bound \( \mu_2^* \) for \( \mu_2 \).
For $k = (z_1, \ldots, z_M, 0, \lambda, r_1, \beta_1, \ldots, \beta_M) \in \mathcal{K} \setminus \partial \omega \mathcal{K}$ we have
\[
\mu_2(k) = \frac{M w(\phi(z_1, \ldots, z_M, q, \lambda))}{\sum_{i=1}^{M} w(z_i)} = \frac{M c(z_1, \phi(z_1, \ldots, z_M, q, \lambda))}{\sum_{i=1}^{M} c(z_1, z_i)} \\
\leq \frac{M c(Mz_1, z_1 + \ldots + z_M)}{\sum_{i=1}^{M} c(z_1, z_i)} = \frac{M w(\frac{1}{M} \sum_{i=1}^{M} z_i)}{\sum_{i=1}^{M} w(z_i)}.
\]

Therefore we can define
\[
\mu_2^*(k) = \max \left( \frac{M w(\frac{1}{M} \sum_{i=1}^{M} z_i)}{\sum_{i=1}^{M} w(z_i)} \right) = \max \left( \frac{M \prod_{i=1}^{M} |z_i - z_1|^2}{\sum_{i=1}^{M} \prod_{i \neq j=1}^{M} |z_i - z_1|^2} \right)
\]

Clearly $\mu_2 \leq \mu_2^*$, with equality when $\lambda$ is real.

Let $k = \mu \in \Gamma \cap \Sigma \cap \mathcal{K}_0$. If $k$ is a point of continuity for $\mu_2^*$ then $\mu_2^*(k) = 1$. At a point of continuity $k$,
\[
1 = \mu_2(k) = \lim_{k \to k_0} \sup \mu_2(k_n) \leq \lim_{k \to k_0} \sup \mu_2^*(k_n) \leq 1.
\]
The last inequality holds because at a point of continuity, the lim sup is actually a limit which can be evaluated in any order. If we take the limit in $\lambda$ and $q$ first, we may use the fact that for $\lambda \in (-2 \sqrt{M}, 2 \sqrt{M})$, $\mu_2 = \mu_2^*$. Proposition [7] proves that the limit in $z_i$ is at most 1, which implies $\mu_2^*(k) = 1$ at the points of continuity.

Since we do not need to know the entire behavior of $\mu_2$ at the boundary, we will concentrate only on the situations needed in the analysis of $\mu_3$. Therefore we need two cases to consider:

Case 1: Let $k = \mu \in \Gamma \cap \Sigma \cap \mathcal{K}_0$ such that $z_1, \ldots, z_M \in \partial \omega \mathcal{H}$ and $z_i \neq \infty$ for all $i = 1, \ldots, M$. This is a point of continuity and we have:
\[
\mu_2^*(k) = \frac{\sum_{i=1}^{M} |z_i - z_1|^2}{M \sum_{i=1}^{M} |z_i - z_1|^2}.
\]

By the triangle inequality and the Cauchy Schwwarz inequality,
\[
\sum_{i=1}^{M} |z_i - z_1|^2 \leq \left[ \sum_{i=1}^{M} |z_i - z_1| \right]^2 \leq M \left[ \sum_{i=1}^{M} |z_i - z_1| \right]^2.
\]
The first inequality turns into equality if $z_i - z_\lambda$ have the same argument for all $i$ and the second one if $z_i - z_\lambda$ are equal in absolute values. Therefore, $\mu_2^* = 1$ iff all $z_i$ are equal.

Case II: Let $k \in \Gamma \cap \Sigma \cap \mathcal{K}_0$, $z_1 = \ldots = z_{a} = i\infty$, and $z_{a+1}, \ldots, z_M$ are real, for some $a$, $1 < a < M$. Suppose $(k_n)$ is a sequence that realizes the lim sup in the definition of $\mu_2(k)$.

$$
\mu_2^*(k_n) = \frac{| \sum_{i=1}^a (z_i - z_\lambda) + \sum_{i=a+1}^M (z_i - z_\lambda)|^2}{M \sum_{i=1}^M |z_i - z_\lambda|^2} \leq \frac{a | \sum_{i=1}^a (z_i - z_\lambda) + \sum_{i=a+1}^M (z_i - z_\lambda)|^2}{M \sum_{i=1}^M |z_i - z_\lambda|^2}.
$$

The second term in the numerator stays finite in the limit and therefore, obviously

$$
\mu_2^*(k_n) \leq \frac{a}{M}.
$$

We end this section with the proofs of our previous lemmas, Lemma 3 and Lemma 4.

Proof of Lemma 3: In order to simplify the notation, let us define $Z = (z_1, \ldots, z_{2M-1})$, $Q = (q_1, q_2)$, $\xi_\sigma(Z, Q, \lambda) = (z_{\sigma_1}, \ldots, z_{\sigma_{M}}, q_1, \lambda)$, $\tau_\sigma(Z, Q, \lambda) = (\phi(\xi_\sigma(Z, Q, \lambda)), z_{\sigma_{M+1}}, \ldots, z_{\sigma_{2M-1}}, q_2, \lambda)$ and

$$
\nu_i = \frac{w(z_i)}{w(z_1) + \ldots + w(z_{2M-1})}.
$$

Extend $\mu_{3,p}$ to an upper semi-continuous function on $\mathbb{H}^{2M-1} \times \mathbb{R}^2 \times \mathbb{R}$ by setting, at points $Z_0$, $Q_0$, $\lambda_0$ where it is not already defined,

$$
\mu_{3,p}(Z_0, Q_0, \lambda_0) = \limsup_{Z \to Z_0, Q \to Q_0, \lambda \to \lambda_0} \mu_{3,p}(Z, Q, \lambda, \lambda_0).
$$

The points $Z$, $Q$ and $\lambda$ are approaching their limits in the topology of $\mathbb{H}^{2M-1} \times \mathbb{R}^2 \times \mathbb{R}$. To prove the lemma it is enough to show that

$$
\mu_{3,p}(Z, Q, \lambda) < 1
$$

for $(Z, Q, \lambda)$ in the compact set $\partial_{M} \mathbb{H}^{2M-1} \times [0]^2 \times [-E, E]$, since this implies that for some $\epsilon > 0$, the upper semi-continuous function $\mu_{3,p}(Z, Q, \lambda)$ is bounded by $1 - 2\epsilon$ on the set, and by $1 - \epsilon$ in some neighborhood. We have
\[
\mu_{3,p}(Z, Q, \lambda) = \sum_{\sigma} \frac{w^{1+p}(\tau_\sigma(Z, Q, \lambda))}{w^{1+p}(z_1) + \ldots + w^{1+p}(z_{2M-1})}
\]
\[
= \sum_{\sigma} \left( \frac{w(\tau_\sigma(Z, Q, \lambda))}{w(z_1) + \ldots + w(z_{2M-1})} \right)^{1+p} \frac{1}{\nu_1^{1+p} + \ldots + \nu_{2M-1}^{1+p}} = \sum_{\sigma} \left[ \mu_2(\tau_\sigma) \left( \frac{1}{M^2} \mu_2(\xi_\sigma)(\nu_{\sigma_1} + \ldots + \nu_{\sigma_M}) + \frac{1}{M} (\nu_{M+1} + \ldots + \nu_{2M-1}) \right) \right]^{1+p}.
\]

Define \( \chi(z_1) = \frac{1}{w(z_1)} = R_1 \Omega_1, \ldots, \chi(z_{2M-1}) = \frac{1}{w(z_{2M-1})} = R_1 \Omega_{2M-1} \), where \( R_1, \Omega_1, \Omega_2, \ldots, \Omega_{2M-1} \) are defined functions of \( Z \) with the property \( \Omega_1^2 + \ldots + \Omega_{2M-1}^2 = 1 \). Notice that for any cyclic permutation \( \sigma \),

\[
\nu_{\sigma_j} = \frac{2M-1}{\sum_{j=1}^{2M-1} \prod_{\substack{k=1 \atop k \neq j}}^{2M-1} \Omega_k} \prod_{j=1}^{2M-1} \Omega_j
\]

In the analysis of \( \mu_2(\xi_\sigma) \) we use the blow-up with coordinates \( r_{1\sigma}(\xi_\sigma) \) and \( \beta_{\sigma_j}(\xi_\sigma) \) where \( j = 1, \ldots, M \) and in the analysis of \( \mu_2(\tau_\sigma) \) we use the blow-up with coordinates \( r_{2\sigma}(\tau_\sigma) \) and \( \beta_{\sigma_j}(\tau_\sigma) \) where \( j = M, \ldots, 2M-1 \). Therefore we have the following relations:

- \( R_1 \Omega_{\sigma_j} = r_{1\sigma}(\xi_\sigma) \) when \( j = 1, \ldots, M \)
- \( R_1 F = \chi(\phi(\xi_\sigma)) = r_{2\sigma}(\tau_\sigma) \)
- \( R_1 \Omega_{\sigma_j} = r_{2\sigma}(\tau_\sigma) \) when \( j = M + 1, \ldots, 2M-1 \)

where

\[
F = \frac{\chi(\phi(\xi_\sigma))}{R_1} = \frac{r_{1\sigma} M \prod_{i=1}^{M} \beta_{\sigma_i}}{R_1 \mu_2(\xi_\sigma(Z, Q, \lambda)) \sum_{j=1}^{M} \prod_{i \neq j} \beta_{\sigma_i}} = \frac{M \Omega_{\sigma_j} M \prod_{i=2}^{M} \beta_{\sigma_i}}{M \mu_2(\xi_\sigma(Z, Q, \lambda)) \sum_{j=1}^{M} \prod_{i \neq j} \beta_{\sigma_i}}.
\]

Consequently

\[
\Omega_{\sigma_j}^2 = \beta_{\sigma_j}^2(\xi_\sigma) (\Omega_{\sigma_1}^2 + \ldots + \Omega_{\sigma_M}^2) \text{ for } j = 1, \ldots, M
\]
\[\Omega^2_{\sigma_j} = \beta_2^2(\tau_\sigma)(F + \Omega^2_{\sigma_{M+1}} + \ldots + \Omega^2_{\sigma_{2M-1}})\text{ for } j = M, \ldots, 2M - 1.\]

Suppose that \(\mu_{3,p}(Z, Q, \lambda) = 1\) for some \((Z, Q, \lambda) \in \partial_{\infty} \mathbb{H}^{2M-1} \times \{0\}^2 \times [-E, E]\). Then there must exist a sequence \((Z_n, Q_n, \lambda_n)\) with \(Z_n \rightarrow Z\) in \(\mathbb{H}^{2M-1}\), \(Q_n \rightarrow (0, 0)\) and \(\lambda_n \rightarrow \lambda \in [-E, E]\) such that

\[
\lim \mu_{3,p}(Z_n, Q_n, \lambda_n) = 1.
\]

From now on \(Z\) and \(\lambda\) will denote the limiting values of the sequences \(Z_n\) and \(\lambda_n\). Similarly, we will denote by \(\nu_i\) and \(\Omega_i\) the limits of \(\nu_i(Z_n)\) and \(\Omega_i(Z_n)\).

We claim that

\[
\nu_1 = \ldots = \nu_{2M-1} = \frac{1}{2M - 1}.
\]

This follows from the expression for \(\mu_{3,p}(Z, Q, \lambda)\), the bound for \(\mu_2\) and the convexity of \(x \mapsto x^{1+p}\):

\[
1 = \mu_{3,p}(Z, Q, \lambda) = \sum_{\sigma} \left[ \mu_2(\tau_\sigma) \left( \frac{1}{M^2} \mu_2(E_\sigma) (\nu_{\sigma_1} + \ldots + \nu_{\sigma_M}) + \frac{1}{M} (\nu_{\sigma_{M+1}} + \ldots + \nu_{\sigma_{2M-1}}) \right) \right]^{1+p} \cdot \frac{1}{\nu_1^{1+p} + \ldots + \nu_{2M-1}^{1+p}} \leq \sum_{\sigma} \left[ \left( \frac{1}{M^2} (\nu_{\sigma_1} + \ldots + \nu_{\sigma_M}) + \frac{1}{M} (\nu_{\sigma_{M+1}} + \ldots + \nu_{\sigma_{2M-1}}) \right) \right]^{1+p} \frac{1}{\nu_1^{1+p} + \ldots + \nu_{2M-1}^{1+p}} \leq \sum_{\sigma} \left[ \left( \frac{1}{M^2} (\nu_{\sigma_1}^{1+p} + \ldots + \nu_{\sigma_M}^{1+p}) + \frac{1}{M} (\nu_{\sigma_{M+1}}^{1+p} + \ldots + \nu_{\sigma_{2M-1}}^{1+p}) \right) \right]^{1+p} \frac{1}{\nu_1^{1+p} + \ldots + \nu_{2M-1}^{1+p}} = 1,
\]

so the inequalities must actually be equalities. Since \(p > 0\), strict convexity implies that equality only holds if \(\nu_1 = \ldots = \nu_{2M-1}\). Since their sum is 1, their common value must be \(\frac{1}{2M - 1}\).

By going to a subsequence, we may assume that \(\Omega_i(Z_n)\) converge. Then (13) and (14) imply that their limiting values along the sequence must be

\[
\Omega_1 = \ldots = \Omega_{2M-1} = \frac{1}{\sqrt{2M - 1}}.
\]

One consequence is that

\[
z_i \in \partial_{\infty} \mathbb{H}
\]

for \(i = 1, \ldots, 2M - 1\).
Now consider the values of $\xi_{\sigma}(Z_n, Q_n, \lambda_n)$ and $\tau_{\sigma}(Z_n, Q_n, \lambda_n)$. Since these values vary in a compact region in $\mathcal{M}$ we may, again by going to a subsequence, assume that they converge in $\mathcal{M}$ to values which we will denote $\xi_{\sigma}$ and $\tau_{\sigma}$. Using (14) and the bound $\mu_2 \leq 1$, we find that

$$1 = \lim_{n \to \infty} \sum_{\sigma} \left[ \mu_2(\tau_{\sigma}(Z_n, Q_n, \lambda_n)) \left( \frac{\mu_2(\xi_{\sigma}(Z_n, Q_n, \lambda_n)) + M - 1}{M(2M - 1)} \right) \right]^{1+\rho} (2M - 1)^{\rho}$$

$$\leq \frac{1}{2M - 1} \sum_{\sigma} \left[ \frac{1}{M} \mu_2(\tau_{\sigma})(\mu_2(\xi_{\sigma}) + M - 1) \right]^{1+\rho} \leq 1.$$  

This implies that for every $\sigma$ occurring in the sum we have $\mu_2(\xi_{\sigma}) = \mu_2(\tau_{\sigma}) = 1$. Therefore, using (16) we conclude that for each $\sigma$, $\xi_{\sigma}$ and $\tau_{\sigma}$ lie in the set $\Sigma$ given by Lemma 8.

Now consider the coordinates $\beta_{\sigma}$, $i = 1, \ldots, M$ for the point $\xi_{\sigma}$. These are the limiting values of $\beta_{\sigma}(z_{\sigma_1}, \ldots, z_{\sigma_M})$ along our sequence. Since $\Omega_{\sigma_{i,j}} = \beta_{\sigma_{i,j}}(\Omega_1^2 + \cdots + \Omega_M^2)$ and $\Omega_i = \frac{1}{\sqrt{2M - 1}}$, we have $\beta_{\sigma_{i,j}} = \frac{1}{\sqrt{M}}$ for $i = 1, \ldots, M$. Going back to the analysis of $\mu_2$, Lemma 8 we conclude that the $\mathbb{R}$ coordinates of $\xi_{\sigma}$, namely the limiting values of $z_{\sigma_1}, \ldots, z_{\sigma_M}$ must be equal. Since this is true for every cyclic permutation, we conclude that

$$z = z_1 = z_2 = \ldots = z_{2M-1} \in \partial_{\infty} \mathbb{W}.$$  

We have two distinct cases:

- If $z \in \mathbb{R}$ then $\phi(z_{\sigma_1}, \ldots, z_{\sigma_M}, q, \lambda) \to \phi(z, \ldots, z, 0, \lambda) = \frac{1}{M \in z_{\lambda}}$. From the analysis of $\mu_2$, Case I, the only way $\tau_{\sigma} = (\phi(z, \ldots, z, 0, \lambda), z, \ldots, z)$ can lie in $\Sigma$ is if $\phi(z, \ldots, z, 0, \lambda) = z$ which would imply $z = z_{\lambda}$ and this cannot happen since $z_{\lambda} \notin \partial_{\infty} \mathbb{W}$.

- If $z = i \infty$ then $\phi(z_{\sigma_1}, \ldots, z_{\sigma_M}, q, \lambda) \to 0$ therefore $\tau_{\sigma} \to (0, i \infty, \ldots, i \infty)$. Since $\Omega_{\sigma_{i,j}} = \beta_{\sigma_{i,j}}(F + \Omega_{\sigma_{M+1}}^2 + \cdots + \Omega_{2M-1}^2)$ for $j = M, \ldots, 2M-1$ and $F = \frac{1}{\sqrt{2M-3}}$ in the limiting case, $\beta_{\sigma_{i,j}}(\tau_{\sigma})$ are equal. Going back to the analysis of $\mu_2$, Case II, we conclude that $\mu_2(\tau_{\sigma}) < 1$.

Therefore, $\mu_{3,p}(Z, Q, \lambda) < 1$.  

**Proof of Lemma 4** Each term in the sum appearing in $\mu_{3,p}$ can be estimated.
where \( \phi(\cdot, \cdot, \cdot) \) denotes \( \phi(z_{\sigma_1}, \ldots, z_{\sigma_M}, q_1, \lambda), z_{\sigma_{M+1}}, \ldots, z_{\sigma_{2M-1}}, q_2, \lambda \). Therefore it is enough to prove

\[
\frac{w(\phi(\cdot, \cdot, \cdot))}{w(z_1 + \ldots + w(z_{2M-1}))} \leq C(1 + \sum_{i=1}^2 |q_i|^2).
\]

Let \( \phi(\cdot) \) denote \( \phi(z_{\sigma_1}, \ldots, z_{\sigma_M}, q_1, \lambda) \). We have

\[
\frac{w(\phi(\cdot, \cdot, \cdot))}{w(z_1 + \ldots + w(z_{2M-1}))} = \left| 1 + \frac{\sum_{i=M+1}^{2M-1} z_{\sigma_i} + \lambda - q_2}{\text{Im} \left[ \sum_{i=M+1}^{2M-1} z_{\sigma_i} + \lambda \right]} \right|^2 \cdot \frac{1}{1 + \sum_{i=1}^{2M-1} |z_i - z_{\sigma_i}|^2 / \text{Im}(z)}.
\]

\[
\leq C \left( \frac{1}{\text{Im}(\sum_{i=M+1}^{2M-1} z_{\sigma_i})} + \frac{|1 + \left( \sum_{i=M+1}^{2M-1} z_{\sigma_i} + \lambda - q_2 \right) \left( \sum_{i=M+1}^{2M-1} z_{\sigma_i} + \lambda - q_2 \right) |^2}{\sum_{i=1}^{2M-1} |z_i - z_{\sigma_i}|^2 / \text{Im}(z)} \right) \cdot \frac{1}{\sum_{i=1}^{2M-1} |z_{\sigma_i} - z_i|^2 / \text{Im}(z)}.
\]
\[
\leq C \left( \frac{1}{\text{Im}(\sum_{i=M+1}^{2M-1} z_{c_i})} + 2 \left( \frac{1}{\text{Im}(\sum_{i=1}^{M} z_{c_i})} + \frac{1}{\text{Im}(\sum_{i=1}^{M} z_{c_i}) + \text{Im}(\sum_{i=M+1}^{2M-1} z_{c_i})} \right) \right) \frac{2M-1}{\sum_{i=1}^{2M-1} |z_{c_i} - z_d|^2/\text{Im}(z_i)}.
\]

Choose the compact set \( \mathcal{M} \) so that \( \sum_{i=1}^{2M-1} |z_{c_i} - z_d|^2/\text{Im}(z_i) \geq C > 0 \) for some constant \( C \) and \( (z_1, \ldots, z_{2M-1}) \in \mathcal{M}^c \). Then we can estimate each term depending on whether \( z_{c_i} \) is close to \( z_d \).

If all \( z_{c_i} \) are sufficiently close to \( z_d \), then \( \text{Im}(z_{c_i}) \) is bounded below and \( |z_{c_i}| \) is bounded above by a constant. Thus

\[
\text{Im}\left( \sum_{i=M+1}^{2M-1} z_{c_i} \right) \sum_{i=1}^{2M-1} |z_{c_i} - z_d|^2/\text{Im}(z_i) \geq \text{Im}\left( \sum_{i=M+1}^{2M-1} z_{c_i} \right) C \geq C' > 0,
\]

\[
\text{Im}\left( \sum_{i=1}^{M} z_{c_i} \right) \sum_{i=1}^{2M-1} |z_{c_i} - z_d|^2/\text{Im}(z_i) \geq \text{Im}\left( \sum_{i=1}^{M} z_{c_i} \right) C \geq C' > 0
\]

and

\[
\left| \sum_{i=M+1}^{2M-1} z_{c_i} + \lambda - q_{c_2} \right|^2 \leq \left( \left| \sum_{i=M+1}^{2M-1} z_{c_i} \right| + |q_{c_2}| \right)^2 \leq \left( \left| \sum_{i=M+1}^{2M-1} z_{c_i} + \lambda \right|^2 + 1 \right) \left| q_{c_2} \right|^2 + 1
\]

\[
\leq \left( \left( \left| \sum_{i=M+1}^{2M-1} |z_{c_i}| + |\lambda| \right|^2 + 1 \right) \left| q_{c_2} \right|^2 + 1 \right) \leq \left( \left| \sum_{i=M+1}^{2M-1} \left| z_{c_i} \right|^2 (|\lambda|^2 + 1) + 1 \right) \left| q_{c_2} \right|^2 + 1 \right)
\]

\[
\leq C \left( \left| \sum_{i=M+1}^{2M-1} \left| z_{c_i} \right|^2 + 1 \right) \left| q_{c_2} \right|^2 + 1 \right) \leq C \left( 1 + |q_{c_2}|^2 \right), \text{ so we are done.}
\]
If all $z_{\sigma_j}$ are far from $z_1$, $\Im\left(\sum_{i=M+1}^{2M-1} z_{\sigma_j} \sum_{i=1}^{2M-1} |z_i - z_1|^2/\Im(z_i)\right)\geq \sum_{i=M+1}^{2M-1} |z_{\sigma_j} - z_1|^2 \geq \frac{1}{M-2} \left| \sum_{i=M+1}^{2M-1} (z_{\sigma_j} - z_1) \right|^2 \geq C(1 + \left| \sum_{i=M+1}^{2M-1} z_{\sigma_j} \right|^2)$ so that $\left| \sum_{i=M+1}^{2M-1} z_{\sigma_j} + \lambda - q_{\sigma_2} \right|^2 / \left( \Im\left(\sum_{i=M+1}^{2M-1} z_{\sigma_j} \sum_{i=1}^{2M-1} |z_i - z_1|^2/\Im(z_i)\right) \right) \leq C(1 + |q_{\sigma_2}|^2)$ in this case too. Also,

$$\Im\left(\sum_{i=1}^{M} \sum_{i=M+1}^{2M-1} |z_i - z_1|^2/\Im(z_i)\right) \geq \sum_{i=1}^{M} |z_{\sigma_j} - z_1|^2 \geq C(1 + \left| \sum_{i=M+1}^{2M-1} z_{\sigma_j} \right|^2).$$

If at least one $z_{\sigma_j}$ is not close to $z_1$ for $j = 1, \ldots, M$, the first term is still bounded. If at least one $z_{\sigma_j}$ is close to $z_1$ for $j = M + 1, \ldots, 2M - 1$, then the second term is finite and

$$\Im\left(\sum_{i=M+1}^{2M-1} z_{\sigma_j} \sum_{i=1}^{2M-1} |z_i - z_1|^2/\Im(z_i)\right) \geq \sum_{i=M+1}^{2M-1} |z_{\sigma_j} - z_1|^2 \geq C(1 + |z_{\sigma_j}|^2).$$

Therefore

$$\left| \sum_{i=M+1}^{2M-1} z_{\sigma_j} + \lambda - q_{\sigma_2} \right|^2 / \Im\left(\sum_{i=M+1}^{2M-1} z_{\sigma_j} \sum_{i=1}^{2M-1} |z_i - z_1|^2/\Im(z_i)\right) \leq C \frac{(C_1 + |z_{\sigma_j}|^2)(1 + |q_{\sigma_2}|^2)}{C_2 + |z_{\sigma_j}|^2} \leq C(1 + |q_{\sigma_2}|^2).$$

The estimates for $\mu_{3,\rho}^j$ are very similar. We omit the details. \qed

References

[1] M. Aizenman, R. Sims and S. Warzel. Stability of the Absolutely Continuous Spectrum of Random Schrödinger Operators on Tree Graphs. *Prob. Theor. Rel. Fields*, (136):363-394, 2006.

[2] P. W. Anderson. Absence of Diffusion in Certain Random Lattices *Phys. Rev.*, (109):1492-1505, 1958.

[3] R. Froese, D. Hasler and W. Spitzer. Transfer matrices, hyperbolic geometry and absolutely continuous spectrum for some discrete Schrödinger operators on graphs. *J. Func. Anal.*, (230):184-221, 2006.
[4] R. Froese, D. Hasler and W. Spitzer. Absolutely Continuous Spectrum for the Anderson Model on a Tree: A Geometric Proof of Klein’s Theorem. *Commun. Math. Phys.*, (269):239-257, 2007.

[5] R. Froese, D. Hasler and W. Spitzer. Absolutely continuous spectrum for a random potential on a tree with strong transverse correlations and large weighted loops. Rev. Math. Phys., Vol. 21, no. 6. 709-733 2009.

[6] F. Halasan. Absolutely Continuous Spectrum for the Anderson Model on a Tree-like Graph [arXiv:0810.2516] 2008

[7] A. Klein. Extended States in the Anderson Model on the Bethe Lattice. *Advances in Math.*, (133):163-184, 1998.

[8] B. Simon. Spectral analysis of rank one perturbations and applications. *Mathematical quantum theory. II. Schrödinger operators (Vancouver, BC, 1993) C.R.M. Proc. Lecture Notes, Amer. Math. Soc., Providence, RI*, (8):109-149, 1995.

[9] B. Simon. $L^p$ norms of the Borel transform and the decomposition of measures. *Proceedings of the American Mathematical Society*, 123(12):3749-3755, Dec. 1995.