Ultrametric Smale’s $\alpha$-theory
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Abstract

We present a version of Smale’s $\alpha$-theory for ultrametric fields, such as the $p$-adics and their extensions, which gives us a multivariate version of Hensel’s lemma.

Hensel’s lemma [4, §3.4] gives us sufficient condition for lifting roots mod $p^k$ to roots in $\mathbb{Z}_p$. Alternatively, Hensel’s lemma gives us sufficient conditions for Newton’s method convergence towards an approximate root. Unfortunately, in the multivariate setting, versions of Hensel’s lemma are scarce [2]. However, in the real/complex world, Smale’s $\alpha$-theory [3] gives us a clean sufficient criterion for deciding if Newton’s method will converge quadratically. In the $p$-adic setting, Breiding [1] proved a version of the $\gamma$-theorem, but he didn’t provide a full $\alpha$-theory. In this short communication, we provide an ultrametric version of Smale’s $\alpha$-theory for square systems—initially presented as an appendix in [5]—, together with an easy proof.

In what follows, and for simplicity\(^1\), $\mathbb{F}$ is a non-archimedean complete field of characteristic zero with (ultrametric) absolute value $| |$ and $\mathcal{P}_{n,d}[n]$ the set of polynomial maps $f : \mathbb{F}^n \to \mathbb{F}^n$ where $f_1$ is of degree $d_i$. In this setting, we will consider on $\mathbb{F}^n$ the ultranorm given by $\|x\| := \max\{x_1, \ldots, x_n\}$, its associated distance $\text{dist}(x, y) := \|x - y\|$, and on $k$-multilinear maps $A : (\mathbb{F}^n)^k \to \mathbb{F}^q$ the induced ultranorm, which is given by

$$\|A\| := \sup_{v_1, \ldots, v_k \neq 0} \frac{\|A(v_1, \ldots, v_k)\|}{\|v_1\| \cdots \|v_k\|}.$$  

In this context, we can define Smale’s parameters as follows. Below $D_x f$ denotes the differential map of $f$ at $x$ and $D^k_x f$ the $k$-linear map induced by the $k$th order partial derivatives of $f$ at $x$.

Definition 1 (Smale’s parameters). Let $f \in \mathcal{P}_{n,d}[n]$ and $x \in \mathbb{F}^n$. We define the following:

(a) Smale’s $\alpha$: $\alpha(f, x) := \beta(f, x) \gamma(f, x)$, if $D_x f$ is non-singular, and $\alpha(f, x) := \infty$, otherwise.

(b) Smale’s $\beta$: $\beta(f, x) := \|D_x f^{-1} f(x)\|$, if $D_x f$ is non-singular, and $\alpha(f, x) := \infty$, otherwise.

(c) Smale’s $\gamma$: $\gamma(f, x) := \sup_{k \geq 2} \left\|D_x f^{-1} \frac{D^k_x f}{k!}\right\|^{\frac{1}{k-1}}$, if $D_x f$ is invertible, and $\gamma(f, x) := \infty$, otherwise.

\(^1\)We focus on characteristic zero and polynomials to avoid technical details related to Taylor series.
If $D_x f$ is invertible, then the Newton operator,

$$N_f : x \mapsto x - D_x f^{-1} f(x),$$

is well-defined at $x$. For a point $x$, the Newton sequence is the sequence $\{N_f^k(x)\}$. Note that this sequence is well-defined (i.e., $N_f^k(x)$ makes sense for all $k$) if and only if $D_{N_f^k(x)} f$ is invertible at every $k$. Also note that

$$\beta(f, x) = \|x - N_f(x)\|,$$

so $\beta$ measures the length of a Newton step.

**Theorem 1** (Ultrametric $\alpha/\gamma$-theorem). Let $f \in \mathcal{P}_{n,d}[n]$ and $x \in \mathbb{F}^n$. Then the following are equivalent:

- $(\alpha)$ $\alpha(f, x) < 1$ and $(\gamma)$ $\text{dist}(x, f^{-1}(0)) < 1/\gamma(f, x)$

Moreover, if any of the above equivalent conditions holds, then the Newton sequence, $\{N_f^k(x)\}$, is well-defined and it converges quadratically to a non-singular zero $\zeta$ of $f$. More specifically, for all $k$, the following holds:

- $(a)$ $\alpha(f, N_f^k(x)) \leq \alpha(f, x)^{2^k}$. 
- $(b)$ $\beta(f, N_f^k(x)) \leq \beta(f, x) \alpha(f, x)^{2^k-1}$. 
- $(c)$ $\gamma(f, N_f^k(x)) \leq \gamma(f, x)\alpha(f, x)^{2^k-1} \beta(f, x) < \alpha(f, x)^{2^k} / \gamma(f, x)$.

In the univariate $p$-adic setting, we have that for $f \in \mathbb{Z}_p[X]$ and $x \in \mathbb{Z}_p$,

$$\gamma(f, x) \leq 1/|f'(x)|,$$

since $|1/k! f^{(k)}(x)| < 1$ and $|f'(x)| \leq 1$. Therefore we can see that the condition $|f(x)| < |f'(x)|^2$ of Hensel’s lemma implies $\alpha(f, x) < 1$ for a $p$-adic integer polynomial. In this way, we can see that Theorem 1 generalizes Hensel’s lemma to the multivariate case.

Moreover, in the univariate setting, we can show the following proposition which gives a precise characterization of Smale’s $\gamma$ in the ultrametric setting as the separation between ‘complex’ roots— not only a bound as it happens in the complex/real setting.

**Proposition 2** (Ultrametric separation theorem for $\gamma$). [5, Theorem 3.15] Fix an algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$ with the corresponding extension of the ultranorm. Let $f \in \mathbb{F}[X]$ and $\zeta \in \overline{\mathbb{F}}$ a simple root, then

$$\frac{1}{\gamma(f, \zeta)} = \text{dist}(\zeta, f^{-1}(0) \setminus \{\zeta\}).$$

□

**Proof of Theorem 1**

The proof of the theorem relies in the following three lemmas, stated for $f \in \mathcal{P}_{n,d}[n]$ and $x, y \in \mathbb{F}^n$.

**Lemma 3.** If $\gamma(f, x) \|x - y\| < 1$, then $D_y f$ is invertible and $\|D_y f^{-1} D_x f\| = 1$.

**Lemma 4** (Variations of Smale’s parameters). If $\rho := \gamma(f, x) \|x - y\| < 1$, then:

- $(a)$ $\alpha(f, y) \leq \max\{\alpha(f, x), \rho\}$. 
- $(b)$ $\beta(f, y) \leq \max\{\beta(f, x), \|y - x\|\}$. 
- $(c)$ $\gamma(f, y) = \gamma(f, x)$.

Moreover, if $\|y - x\| < \beta(f, x)$, all are equalities.
Lemma 5 (Variations along Newton step). If \( \alpha(f, x) < 1 \), then:

(a) \( \alpha(f, N_f(x)) \leq \alpha(f, x)^2 \).

(b) \( \beta(f, N_f(x)) \leq \alpha(f, x) \beta(f, x) \).

(c) \( \gamma(f, N_f(x)) = \gamma(f, x) \).

In particular, \( N_f(N_f(x)) \) is well-defined.

Proof of Theorem 1. If \( \alpha(f, x) < 1 \), then, using induction and Lemma 5, we obtain that (a), (b) and (c) hold. But then the sequence \( \{N_f^k(x)\} \) converges since \( \lim_{k \to \infty} \|N_f^{k+1}(x) - N_f^k(x)\| = 0 \) and so it is a Cauchy sequence. Finally, (Q) follows from noting that for \( l \geq k \)

\[
\|N_f^l(x) - N_f^k(x)\| \leq \alpha(f, x)^{2^{l-k}} \beta(f, N_f^k(x))
\]

and taking infinite sum together with the equality case of the ultrametric inequality. In particular, we have \( \text{dist}(x, f^{-1}(0)) = \|x - \zeta\| = \beta(f, x) < 1/\gamma(f, x) \). This shows that (\( \alpha \)) implies (\( \gamma \)).

For the other direction, assume that \( \text{dist}(x, f^{-1}(0)) < 1/\gamma(f, x) \). Then \( \gamma(f, x) \) is finite, since otherwise \( \text{dist}(x, f^{-1}(0)) \leq 0 \), which is impossible. Let \( \zeta \in \mathbb{F}^n \) be a zero of \( f \) such that \( \text{dist}(x, \zeta) < 1/\gamma(f, x) \). Then \( 0 = f(\zeta) = f(x) + D_x f(\zeta - x) + \sum_{k=2}^{\infty} \frac{D_x f^k}{k!} (\zeta - x, \ldots, \zeta - x) \), and so

\[
-D_x f^{-1} f(x) = \zeta - x + \sum_{k=2}^{\infty} D_x f^{-1} \frac{D_x f^k}{k!} (\zeta - x, \ldots, \zeta - x).
\]

Now, the higher order terms satisfy that

\[
\|D_x f^{-1} \frac{D_x f^k}{k!} (\zeta - x, \ldots, \zeta - x)\| \leq (\gamma(f, x))^{k-1} \|\zeta - x\| < (\|\zeta - x\| < 1/\gamma(f, x)) ,
\]

as desired. \( \Box \)

Now, we prove the auxiliary lemmas 3, 4 and 5.

Proof of Lemma 3. We have that \( D_x f^{-1} D_y f = I + \sum_{k=1}^{\infty} D_x f^{-1} \frac{D_x f^k}{k!} (y - x, \ldots, y - x) \). Now, under the given assumption, \( \|D_x f^{-1} \frac{D_x f^k}{k!} (y - x, \ldots, y - x)\| \leq (\gamma(f, x))^{k-1} \|y - x\| < 1 \) for \( k \geq 2 \), and so, by the ultrametric inequality, \( \|D_x f^{-1} D_y f - I\| < 1 \). Therefore \( \sum_{k=0}^{\infty} (I - D_x f^{-1} D_y f)^k \) converges, and it does so to the inverse of \( D_x f^{-1} D_y f \). Since, by assumption \( D_x f \) is invertible, so it is \( D_y f \).

Finally, by the invertibility of \( D_y f \), we have that \( D_y f^{-1} D_x f = \sum_{k=0}^{\infty} (I - D_x f^{-1} D_y f)^k \), and so, by the equality case of the ultrametric inequality, \( \|D_y f^{-1} D_x f\| = 1 \), as desired. \( \Box \)

Proof of Lemma 4. We first prove (c) and then (b). (a) follows from (b) and (c) immediately.

(c) We note that under the given assumption, for \( k \geq 2 \),

\[
\left\| D_x f^{-1} \frac{D_x f^k}{k!} \right\| \leq \gamma(f, x)^{k-1} . \tag{2}
\]

For this, we expand the Taylor series of \( \frac{D_x f^k}{k!} \) (with respect \( y \)) and note that its \( l \)th term is dominated by

\[
\gamma(f, x)^{k+l-1} \|y - x\|^l ,
\]

which, by the ultrametric inequality, gives the above inequality. In this way, for \( k \geq 2 \),

\[
\left\| D_y f^{-1} \frac{D_y f^k}{k!} \right\| \leq \left\| D_y f^{-1} D_x f \right\| \left\| D_x f^{-1} \frac{D_x f^k}{k!} \right\| \leq \gamma(f, x)^{k-1} .
\]
by Lemma 3 and (2). Thus $\gamma(f, y) \leq \gamma(f, x)$. Now, due to this, the hypothesis $\gamma(f, y) \|x - y\| < 1$ holds, and so, by the same argument, $\gamma(f, x) \leq \gamma(f, y)$, which is the desired equality.

(b) Arguing as in (c), we can show that

$$
\|D_x f^{-1} f(y)\| \leq \max\{\|D_x f^{-1} f(x) + y - x\|, \gamma(f, x)\|y - x\|^2\}
$$

by noting that the general term (of the Taylor series of $D_x f^{-1} f(y)$ with respect $y$) is dominated by $\gamma(f, x)^{k-1}\|y - x\|^k < \gamma(f, x)\|y - x\|^2$. Now, $\beta(f, y) \leq \|D_y f^{-1} D_x f\|\|D_x f^{-1} f(y)\|$, and so, by Lemma 3 and (3),

$$
\beta(f, y) \leq \max\{\|D_x f^{-1} f(x) + y - x\|, \gamma(f, x)\|y - x\|^2\} \leq \max\{\beta(f, x), \|y - x\|\}.
$$

For the equality case, note that, by the same argument, we have $\beta(f, x) \leq \max\{\beta(f, y), \|y - x\|\} = \beta(f, y)$ where the equality on the right-hand side follows from $\beta(f, x) > \|y - x\|$.

Proof of Lemma 5. (a) follows from combining (b) and (c), and (c) from Lemma 4 (c). We only need to show (b). We use equation (3) in the proof of Lemma 4 with $y = N_f(x)$. By (3) and Lemma 3,

$$
\beta(f, N_f(x)) \leq \max\{\|D_x f^{-1} f(x) + N_f(x) - x\|, \gamma(f, x)\|N_f(x) - x\|^2\}.
$$

Now, $N_f(x) - x = -D_x f^{-1} f(x)$, so the above becomes $\beta(f, N_f(x)) \leq \max\{0, \gamma(f, x)\beta(f, x)^2\}$, which gives the desired claim.

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