Solution of the Linearly Structured Partial Polynomial Inverse Eigenvalue Problem

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Abstract

In this paper, linearly structured partial polynomial inverse eigenvalue problem is considered for the $n \times n$ matrix polynomial of arbitrary degree $k$. Given a set of $m$ eigenpairs (1 $\leq$ $m$ $\leq$ $kn$), this problem concerns with computing the matrices $A_i \in \mathbb{R}^{n \times n}$ for $i = 0, 1, 2, \ldots, (k - 1)$ of specified linear structure such that the matrix polynomial $P(\lambda) = \lambda^k I_n + \sum_{i=0}^{k-1} \lambda^i A_i$ has the given eigenpairs as its eigenvalues and eigenvectors. Many practical applications give rise to the linearly structured structured matrix polynomial. Therefore, construction of the linearly structured structured matrix polynomial is the most important aspect of the polynomial inverse eigenvalue problem (PIEP). In this paper, a necessary and sufficient condition for the existence of the solution of this problem is derived. Additionally, we characterize the class of all solutions to this problem by giving the explicit expressions of solutions. The results presented in this paper address some important open problems in the area of PIEP raised in De Teran, Dopico and Van Dooren [SIAM Journal on Matrix Analysis and Applications, 36(1) (2015), pp 302 – 328]. An attractive feature of our solution approach is that it does not impose any restriction on the number of eigendata for computing the solution of PIEP. The proposed method is validated with various numerical examples on a spring mass problem.

Keywords: Matrix polynomial, linearly structured matrix, polynomial inverse eigenvalue problem, polynomial eigenvalue problem.

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1. Introduction

Consider the higher order system of ordinary differential equations of the form
\[ A_k \frac{d^k v(t)}{dt^k} + A_{k-1} \frac{d^{k-1} v(t)}{dt^{k-1}} + \cdots + A_1 \frac{dv(t)}{dt} + A_0 v(t) = 0 \] (1)
where \( A_i \in \mathbb{R}^{n \times n} \) for \( i = 0, 1, 2, \ldots, k \) and \( A_k \) is a nonsingular matrix.

Assuming the solution of (1) is of the form \( v(t) = xe^{\lambda t} \), using separation of variables, (1) leads to the higher order polynomial eigenvalue problem
\[ P(\lambda)x = 0 \] (2)
where \( P(\lambda) = \lambda^k A_k + \lambda^{k-1} A_{k-1} + \cdots + \lambda A_1 + A_0 \in \mathbb{R}^{n \times n}[\lambda] \) is known as matrix polynomial of degree \( k \). The comprehensive theory and application of the matrix polynomial is discussed in the classic reference [24].

A matrix polynomial \( P(\lambda) \) is regular when \( P(\lambda) \) is square and the scalar polynomial \( \det(P(\lambda)) \) has at least one nonzero coefficient. Otherwise, \( P(\lambda) \) is said to be singular. We assume the matrix polynomial \( P(\lambda) \) is regular throughout this paper. The roots of \( \det(P(\lambda)) = 0 \) are the eigenvalues of the matrix polynomial \( P(\lambda) \). The vectors \( y \neq 0 \) and \( z \neq 0 \) are corresponding left and right eigenvectors satisfying \( z^H P(\lambda) = 0 \) and \( P(\lambda)y = 0 \) where \( z^H \) denotes the conjugate transpose of \( z \). If the matrix \( A_k \) is nonsingular, then the matrix polynomial \( P(\lambda) \) has \( kn \) finite eigenvalues and eigenvectors. The \( kn \) eigenvalues of \( P(\lambda) \) are either real numbers or if not, are complex conjugate pairs. The polynomial eigenvalue problem concerns with determining the eigenvalues and corresponding eigenvectors of the matrix polynomial \( P(\lambda) \). This problem arises in many practical situations, for instance, vibration analysis of structural mechanical and acoustic system, electrical circuit simulation, fluid mechanics, etc [13, 21]. This problem is well studied in the literature and a lot of literature exists addressing the ways to solve the polynomial problem (see [7, 13, 30, 46] and the references therein).

Mostly, matrix polynomial arising from practical applications are often inherently structured. For example, they are all symmetric [8], skew-symmetric [19], they alternate between symmetric and skew-symmetric [33], symmetric tridiagonal [2], etc. Also, pentadiagonal matrices occur in the discretization of the fourth-order differential systems [22]. Generally, these matrices \( A_i \) for \( i = 0, 1, 2, \ldots, k \) are linearly structured matrices [27]. A matrix polynomial \( P(\lambda) \) in which the coefficient matrices are linearly structured, is known as linearly structured matrix polynomial. Since the matrix \( A_k \) is often diagonal
and positive definite in various applications, we assume, without loss of generality, that the leading coefficient $A_k$ is an identity matrix. In this case, the matrix polynomial is referred to as a monic matrix polynomial of degree $k$.

The polynomial inverse eigenvalue problem (PIEP) addresses the construction of a matrix polynomial $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i \in \mathbb{R}^{n \times n}[\lambda]$ from the given eigenvalues and associated eigenvectors. PIEP arises in many applications where parameters of a certain physical system are to be determined from the knowledge of its dynamical behavior. It has applications in the mechanical vibrations, aerospace engineering, molecular spectroscopy, particle physics, geophysical applications, numerical analysis, differential equations etc (see for instance \[3, 4, 10, 11, 36, 38\]).

Generally, a small number of eigenvalues and eigenvectors of the associated eigenvalue problem are available from the computation or measurement. Unfortunately there is no analytical tool available to evaluate the entire eigendata of a large physical system. It should be mentioned that when the problem is large, as in the case with the most engineering applications, state of art computational methods are capable of computing a very few eigenvalues and associated eigenvectors. Therefore, it might be more sensible to solve the polynomial inverse eigenvalue problem when only a few measured eigenvalues and associated eigenvectors are available.

The construction of the matrix polynomial $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i \in \mathbb{R}^{n \times n}[\lambda]$ using the partially described eigendata is known as the partial polynomial inverse eigenvalue problem (PPIEP). In view of practical applications, it might be more realistic to solve PPIEP with these structure constraints on the coefficient matrices. This problem is termed as the structured partial polynomial inverse eigenvalue problem (LPPIEP). The structure constraint imposes a great challenge for solving this problem.

The inverse eigenvalue problem (IEP) for linear and quadratic matrix polynomial have been well studied in the literature since the 1970s (see \[17\] the references therein). Some previous attempts at solving the inverse eigenvalue problem are listed in \[1, 23, 25, 39, 40\]. A large number of papers have been published on the linear inverse eigenvalue problem \[20, 41, 44\]. An excellent review of this area can be found in the classic reference \[10\]. Special attention is paid to the quadratic inverse eigenvalue problem (QIEP) (see \[2, 8, 12, 16, 26, 28, 29, 43, 48\]). Most of the papers solve QIEP for the symmetric structure (see \[48, 8, 26\]) and symmetric tridiagonal structure (see \[2, 43\]). The quadratic inverse eigenvalue problem is considered in the context of solving the finite element model updating problem \[21, 34, 35, 42\].
and eigenstructure assignment problem [14, 37].

Some earlier attempts at solving the higher order PIEP are listed in [3, 5, 18, 32]. Also, IEP for the matrix polynomial of degree \( k \) is considered in the context of solutions of active vibration control (see [9, 31, 45, 47]). Most significant contributions to the solution of the higher order PIEP have been made in [3, 18]. In [3], higher order PIEP for the \( T \)-Alternating and \( T \)-Palindromic matrix polynomials of degree \( k \) are considered. These results are most phenomenal so far on the solution of higher order structured PIEP. In [18], authors mention an important open problem in this area, namely, the inverse eigenvalue problems for structured matrix polynomials such as symmetric, skew-symmetric matrix polynomials, etc. In this paper, we attempt at addressing this open problem providing the solution of PIEP.

Throughout this paper, we shall adopt the following notations. \( A \otimes B \) denotes the Kronecker product of the two matrices \( A \) and \( B \). Also, \( \text{Vec}(A) \) denotes the vectorization of the matrix \( A \). \( \|A\|_F \) and \( \|A\|_2 \) denote the Frobenius norm and 2-norm of the matrix \( A \) respectively. \( \mathcal{L} \) denotes the real linear subspaces of \( \mathbb{R}^{n \times n} \) representing the linearly structured matrices. \( A^\dagger \) is the Moore Penrose pseudoinverse of \( A \). \( I_n \) denotes the identity matrix of size \( n \times n \). Also, \( e_i \) is the \( i \)th row of \( I_k \) for \( 1 \leq i \leq k \).

**Problem Statement 1.1. LPPIEP:** Given two positive integers \( k \) and \( n \), a set of partial eigenpairs \((\lambda_j, \phi_j)_{j=1}^m\) (where \( 1 \leq m \leq kn \)), construct a monic matrix polynomial \( P(\lambda) = \lambda^k I_n + \sum_{i=0}^{k-1} \lambda^i A_i \in \mathbb{R}^{n \times n}[\lambda] \) of degree \( k \) in such a way that matrices \( A_i \in \mathcal{L} \) are symmetric for \( i = 0, 1, 2, \ldots, (k-1) \) and \( P(\lambda) \) has the specified set \((\lambda_j, \phi_j)_{j=1}^m\) as its eigenpairs.

**Contributions**

In this paper, we consider the **linearly structured partial polynomial inverse eigenvalue problem for the monic matrix polynomial of arbitrary degree** \( k \). The authors believe that this problem, in its full generality, has not been addressed earlier in the literature. Our results solve some open problems in the theory of polynomial inverse eigenvalue problem (see [18]).

In particular, key contributions made in this paper are listed below:

- The proposed method is capable to solve LPPIEP using a set of \( m \) (\( 1 \leq m \leq kn \)) eigenpairs without imposing any restrictions on it, unlike some instances in the past where certain restrictions on \( m \) are imposed (see [2, 8, 48]) for computing the solution of inverse eigenvalue problem in the case of quadratic matrix polynomial.
• The proposed method is capable to solve LPPIEP for a monic matrix polynomial of arbitrary degree $k$.

• We derive some necessary and sufficient conditions on the eigendata for the existence of solution of this problem.

• We completely characterize the class of solutions of this problem and present the explicit expression of the solution.

Real-Form Representations of Eigenvalues and Eigenvectors

We assume that the $m$ eigenvalues of a matrix polynomial are given of which $t$ are complex conjugate pairs and remaining $m - 2t$ are real. Also, complex eigenvalues are $\alpha_j \pm i\beta_j$ for $j = 1, 2, \ldots, t$ and real eigenvalues are $e_{2t+1}, e_{2t+2}, \ldots, e_m$. Eigenvectors corresponding to the complex eigenvalues are $u_j \pm iv_j$ and eigenvectors corresponding to the real eigenvalues are $\phi_{2t+1}, \phi_{2t+2}, \ldots, \phi_m$.

We relate this pair of complex eigenvalues with a matrix $E_j \in \mathbb{R}^{2 \times 2}$ given by

$$ E_j = \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix}. $$

Thus given a set of $m$ eigenvalues, we relate these numbers with a real block-diagonal matrix $E \in \mathbb{R}^{m \times m}$ of the following form

$$ E = \text{diag}(E_1, E_2, E_3, \ldots, E_t, e_{2t+1}, \ldots, e_m). \quad (3) $$

Then $E$ is the real-form matrix representation of these $m$ eigenvalues in real form. Similarly, for a set of $m$ eigenvectors a real-form matrix representation is given by

$$ X = \begin{bmatrix} u_1 & v_1 & \ldots & u_t & v_t & \phi_{2t+1} & \ldots & \phi_m \end{bmatrix} \in \mathbb{R}^{n \times m}. \quad (4) $$

Thus the pair $(X, E)$ is a real matrix eigenpair of the matrix polynomial of degree $k$, then it satisfies

$$ \sum_{i=0}^{k} A_i X E^i = 0. \quad (5) $$

This relation is known as eigenvalue eigenvector relation for the matrix polynomial of degree $k$. 

5
1.1. Linearly structured matrices and its structure specifications

Linearly structured matrix is a linear combinations of sub structured matrices. Let $A \in \mathcal{L}$ be a linearly structured matrix of the form

$$A = \sum_{\ell=1}^{r} S_{\ell} \alpha_{\ell}$$

(6)

where $\alpha_{1}, \alpha_{2}, \ldots \alpha_{r}$ are the structure parameters, $r$ is the dimension and $\{S_{\ell} \in \mathbb{R}^{n \times n} : \ell = 1, 2, \ldots r\}$ is a standard basis of $\mathcal{L}$. Here $[\alpha_{1} \alpha_{2} \alpha_{3} \cdots \alpha_{r-1} \alpha_{r}]^{T}$ is the coordinate vector of $A$ w.r.t the above standard basis.

Matrix $A$ is the linear combinations of the sub structured matrices $S_{\ell}$ for $\ell = 1, 2, \ldots r$.

We give some examples of linearly structured matrices in the table given below.
2. Solution of LPPIEP

In this section, we obtain the solution of LPPIEP from the eigenvalue-eigenvector relation for monic matrix polynomial of degree $k$ which is given by

$$\sum_{i=0}^{k-1} A_i X E^i = -X E^k$$

(7)

where $X \in \mathbb{R}^{n \times m}$ and $E \in \mathbb{R}^{m \times m}$.

It is clear that (7) is a nonhomogenous linear system of $nm$ equations. Therefore, the solution of LPPIEP is obtained by computing the linearly structured solution $A_i$ of (7).

We now discuss an important concept of vectorization of a matrix which will be used to derive the solution of LPPIEP.

**Vectorization of a linearly structured matrix**

Vectorization of a matrix $A \in \mathcal{L}$, is denoted by $\text{Vec}(A)$ and is defined as a vector in $\mathbb{R}^{n^2 \times 1}$ obtained by stacking the columns of the matrix $A$ on top of one another.

Define the vector $\text{Vec}_1(A)$ as

$$\text{Vec}_1(A) = [\alpha_1 \alpha_2 \alpha_3 \ldots \alpha_{r-1} \alpha_r]^T.$$

We define the matrix $P \in \mathbb{R}^{n^2 \times r}$ as

$$P = [\text{Vec}(S_1) \text{Vec}(S_2) \cdots \text{Vec}(S_r)]$$

(8)
where \( \{ S_\ell \in \mathbb{R}^{n \times n} : \ell = 1, 2, \ldots, r \} \) is a standard basis of \( \mathcal{L} \) such that \( \text{Vec}_1(A) \) is the coordinate vector of \( A \in \mathcal{L} \) w.r.t the above basis.

It is easy to see that \( \text{Vec}(A) \) and \( \text{Vec}_1(A) \) are related through the matrix \( P \) as:

\[
\text{Vec}(A) = P \text{Vec}_1(A) \quad (9)
\]

**Example 2.1.** Consider the symmetric matrix (linearly structured) \( A \in \mathbb{R}^{3 \times 3} \) as

\[
A = \begin{bmatrix}
4 & 2 & 8 \\
2 & 7 & 9 \\
8 & 9 & 5
\end{bmatrix}
\]

Then \( \text{Vec}(A) \in \mathbb{R}^{9 \times 1} \) and \( \text{Vec}_1(A) \in \mathbb{R}^{5 \times 1} \) are given by

\[
\text{Vec}(A) = \begin{bmatrix}
4 & 2 & 8 & 2 & 7 & 9 & 8 & 9 & 5
\end{bmatrix}^T \\
\text{Vec}_1(A) = \begin{bmatrix}
4 & 2 & 8 & 7 & 9 & 5
\end{bmatrix}^T
\]

Let, \( \{ S_\ell \in \mathbb{R}^{n \times n} : \ell = 1, 2, \ldots, 5 \} \) be the standard basis of the space of all symmetric matrices where

\[
S_1 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad S_2 = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad S_3 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}, \quad S_4 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad S_5 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}, \quad S_6 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

The matrix \( P \in \mathbb{R}^{9 \times 6} \) is given by

\[
P = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

For the symmetric matrix \( A \), it is straightforward to verify that (9) holds.

**Existence of a solution of LPPIEP**

In this subsection, we derive a necessary and sufficient condition on the eigendata for the existence of a solution of LPPIEP. Applying vectorization
operation on (7), we get,

\[
\text{Vec} \left( \sum_{i=0}^{k-1} A_i X^i \right) = -\text{Vec} \left( X^k \right)
\]

\[
\Rightarrow \sum_{i=0}^{k-1} \left( (X^i)^T \otimes I \right) \text{Vec} (A_i) = -\text{Vec} \left( X^k \right)
\]

\[
\Rightarrow \sum_{i=0}^{k-1} \left( (X^i)^T \otimes I \right) P\text{Vec}_1 (A_i) = -\text{Vec} \left( X^k \right) \quad \text{... using (9)}
\]

\[
\Rightarrow \left[ ((X^{k-1})^T \otimes I)P ((X^{k-2})^T \otimes I)P \cdots (X^T \otimes I)P \right] \begin{bmatrix}
\text{Vec}_1 (A_{k-1}) \\
\text{Vec}_1 (A_{k-2}) \\
\vdots \\
\text{Vec}_1 (A_0)
\end{bmatrix} = -\text{Vec}(X^k)
\]

\[
\Rightarrow Ux = b \quad (10)
\]

where

\[
U = \left[ ((X^{k-1})^T \otimes I_n)P ((X^{k-2})^T \otimes I_n)P \cdots (X^T \otimes I_n)P \right] \in \mathbb{R}^{mn \times kr},
\]

\[
x = \begin{bmatrix}
\text{Vec}_1 (A_{k-1}) \\
\text{Vec}_1 (A_{k-2}) \\
\vdots \\
\text{Vec}_1 (A_0)
\end{bmatrix} \in \mathbb{R}^{kr \times 1},
\]

\[
b = \text{Vec}(-X^k) \in \mathbb{R}^{mn \times 1}.
\]

Above system of linear equations (10) has \( mn \) equations and \( kr \) unknowns. We now state a necessary and sufficient condition for the existence of the solution of a system of linear equations in the following theorem.

**Theorem 2.2.** [6] Let \( \Psi \zeta = \eta \) be a system of linear equations where \( \Psi \in \mathbb{R}^{p \times q} \) and \( \eta \in \mathbb{R}^p \). Then \( \Psi \zeta = \eta \) is consistent if and only if \( \Psi \Psi^\dagger \eta = \eta \) where \( \Psi^\dagger \) is the generalized inverse of \( \Psi \in \mathbb{R}^{p \times q} \). General solution of \( \Psi \zeta = \eta \) is given by

\[
\zeta = \Psi^\dagger \eta + (I_q - \Psi^\dagger \Psi)y
\]

where \( y \in \mathbb{R}^{q \times 1} \) is an arbitrary vector. Moreover, \( \Psi \zeta = \eta \) has a unique solution if and only if \( \Psi^\dagger \Psi = I_q \), \( \Psi \Psi^\dagger \eta = \eta \) and the unique solution is given by

\[
\zeta = \Psi^\dagger \eta
\]
First, we transform the eigenvalue eigenvector relation (7) to a system of linear equations \( Ux = v \). Therefore, determination of solution of LPPIEP is equivalent to finding the solution of the system of linear equations in (10). Thus, necessary and sufficient conditions for the existence of solution of LPPIEP is same as the system of linear equation \( Ux = v \). We now present the main theorem to find a necessary and sufficient condition for the existence of the solution of LPPIEP.

**Theorem 2.3.** Let an arbitrary matrix eigenpair \((E, X) \in \mathbb{R}^{m \times m} \times \mathbb{R}^{n \times m}\) be given as in Equations (3) and (4). Then LPPIEP has a solution if and only if \( UU^\dagger b = b \) where \( U \) and \( b \) are defined by (11) and (13). In that case expression of \( A_i \in \mathcal{L} \) for \( i = 0, 1, 2, \ldots, (k - 1) \) are given by

\[
Vec(A_i) = P \left( (e_{k-i} \otimes I_r) \left( U^\dagger b + (I_{kr} - U^\dagger U)y \right) \right),
\]

where \( y \in \mathbb{R}^{kr \times 1} \) is an arbitrary vector. Moreover, LPPIEP has a unique solution if and only if \( UU^\dagger b = b, U^\dagger U = I_{kr} \). Explicit expressions of \( A_i \in \mathcal{L} \) are given below as

\[
Vec(A_i) = P \left( (e_{k-i} \otimes I_r)U^\dagger b \right).
\]

**Proof.** Computing the solution of LPPIEP is equivalent to solving the system of linear equations \( Ux = b \) where \( U \) and \( b \) are defined by (11) and (13). Necessary and sufficient condition for the existence of the solution of \( Ux = b \) is \( UU^\dagger b = b \) and general solution is given by

\[
x = U^\dagger b + (I_{kr} - U^\dagger U)y.
\]

where \( y \in \mathbb{R}^{kr \times 1} \) is an arbitrary vector. Note that, \( x \) is of the form as in (12) and \( Vec_1(A_{k-1}) \) can be obtained from \( x \) as follows.

\[
[I_r \ \Theta \ \Theta \ \Theta \ \ldots \Theta] x = \begin{bmatrix} Vec_1(A_{k-1}) \\ Vec_1(A_{k-2}) \\ \vdots \\ Vec_1(A_0) \end{bmatrix} = Vec_1(A_{k-1})
\]

where \( \Theta \in \mathbb{R}^{r \times r} \) be a zero matrix.

\[
\Rightarrow Vec_1(A_{k-1}) = (e_1 \otimes I_r)x
\]
Similarly, $\text{Vec}_1(A_i)$ are given by

$$\text{Vec}_1(A_i) = (e_{k-i} \otimes I_r)x \quad \text{for } i = 0, 1, 2, 3, \ldots, (k-1) \quad (18)$$

Substituting the expression of $x$ in $[18]$, $\text{Vec}_1(A_i)$ can be obtained as in the following as:

$$\text{Vec}_1(A_i) = (e_{k-i} \otimes I_r) (U^\dagger b + (I_{kr} - U^\dagger U)y) \quad (19)$$

General solution $A_i$ is obtained from the vector $\text{Vec}_1(A_i)$ using the relation $[9]$ as

$$\text{Vec}(A_i) = P \text{Vec}_1(A_i). \quad (20)$$

Substituting the expressions of $\text{Vec}_1(A_i)$ in the above equations, we get

$$\text{Vec}(A_i) = P \left( (e_{k-i} \otimes I_r) (U^\dagger b + (I_{kr} - U^\dagger U)y) \right).$$

Further, $Ux = b$ has a unique solution if and only if $UU^\dagger b = b$ and $U^\dagger U = I_{kr}$. Explicitly, the unique solution $x$ is given by $x = U^\dagger b$ (see Theorem $[2.2]$). If $Ux = b$ has a unique solution then LPPIEP has a unique solution $A_i$. In that case, matrices $A_i$ are given by uniquely as

$$\text{Vec}(A_i) = P \left( (e_{k-i} \otimes I_r)U^\dagger b \right). \quad (21)$$

**Remark 2.4.** We considered the standard ordered basis of $\mathcal{L}$ to represent any linearly structured matrix and we construct the matrix $P$ using this basis. However, any other ordered basis can be chosen to construct the matrix $P$. Result of Theorem $[2.3]$ is also true if we choose any other ordered basis.

**Construction of symmetric non-monic matrix polynomials**

In Theorem $[2.3]$ we construct the monic linearly structured matrix polynomial using partial eigendata. Now we generalize this result to find the symmetric non-monic polynomials with positive definite leading coefficients using similarity transformation.

Consider the matrix polynomial $P(\lambda) = \lambda^k A_k + \lambda^{k-1} A_{k-1} + \cdots + \lambda A_1 + A_0 \in \mathbb{R}^{n \times n}[\lambda]$ where $A_i$ are symmetric and $A_k$ is positive definite matrix. Let, $A_k^{1/2}$ be the positive definite square-root of $A_k$ and modify the problem
by writing $\xi = A_k^{1/2}x$ and observe that Eq. (2) reduces to the monic problem as

$$
(\lambda^k A_k^{-1/2} A_k^{-1/2} + \lambda^{k-1} A_k^{-1/2} A_{k-1}^{-1/2} + \cdots + A^{-1/2} A_0 A^{-1/2}) A_k^{1/2} x = 0
\Rightarrow (\lambda^k I + \lambda^{k-1} \hat{A}_{k-1} + \cdots + \lambda \hat{A}_1 + \hat{A}_0) \xi = 0
$$

where $\hat{A}_i = A_k^{-1/2} A_i A_k^{-1/2}$ are symmetric matrices.

3. Numerical Example

In this section, we give three numerical examples to illustrate the validity of our proposed approach.

**Example 3.1.** Consider the mass-spring system having three degrees of freedom with the following target set of eigenvalues $-1.3064 \pm 0.5436i$, $-0.2582$.

The eigenvalue and the eigenvector matrices are given by

$$
X = \begin{bmatrix}
-0.0406 & -0.4699 & 0.4231 \\
-0.4504 & -0.2542 & 0.3510 \\
0.7128 & -0.0438 & -0.8353 \\
\end{bmatrix}
$$

$$
E = \begin{bmatrix}
-1.3064 & 0.5436 & 0 \\
-0.5436 & -1.3064 & 0 \\
0 & 0 & -0.2582 \\
\end{bmatrix}
$$

For this mass-spring system $m = 3$ and $n = 3$. Now we construct the monic symmetric matrix polynomial $P(\lambda) = \lambda^2 I_3 + \lambda A_1 + A_0$ of degree 2.

We take the standard basis of the space of all symmetric matrices $\{S_\ell \in \mathbb{R}^{n \times n} : \ell = 1, 2, \ldots, 6\}$ where $S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $S_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $S_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $S_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $S_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $S_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Now, we construct symmetric matrices $A_0$ and $A_1$ from the above partial eigendata. Here, $UU^\dagger b = b$ and $UU \neq I_{10}$ where $U \in \mathbb{R}^{12 \times 10}$ and $b \in \mathbb{R}^{12 \times 1}$. Equation (10) has an infinite number of solutions. Therefore, LPPI EP has an infinite number of solutions. Using Theorem 2.3, symmetric matrices $A_0$ and $A_1$ are given by

$$
A_0 = \begin{bmatrix}
4.2248 & -0.0174 & 2.4278 \\
-0.0174 & 1.8133 & 0.2806 \\
2.4278 & 0.2806 & 1.5618 \\
\end{bmatrix}
$$

$$
A_1 = \begin{bmatrix}
2.3283 & 1.2405 & 2.7130 \\
1.2405 & 0.1189 & -1.2603 \\
2.7130 & -1.2603 & 1.9321 \\
\end{bmatrix}
$$
Next, we study the effect of choosing different ordered basis of the space of all symmetric matrices to the solution. Using the ordered basis \( \{ S_\ell \in \mathbb{R}^{n \times n} : \ell = 1, 2, \ldots, 6 \} \) where \( S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, S_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, S_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \)

\( S_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, S_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, S_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \)

we construct the matrices \( P \) and \( D \). Here, \( UU^\dagger b = b \) and \( U^\dagger U \neq I_{10} \) where \( U \in \mathbb{R}^{12 \times 10} \) and \( b \in \mathbb{R}^{12 \times 1} \). We also get the same symmetric matrices \( A_0 \) and \( A_1 \) as above.

Therefore, if we take two different basis of the space of all symmetric matrices for this example, we get same result.

**Example 3.2.** Consider the mass-spring system having four degrees of freedom with the following target set of eigenvalues \( 0.5950 + 9.5092i, 0.5950 - 9.5092i \). The eigenvalue and the eigenvector matrices are given by

\[
X = \begin{bmatrix}
-0.2164 & -0.6066 \\
-0.5435 & -0.0169 \\
-0.3518 & 0.2746 \\
-0.1845 & 0.2374
\end{bmatrix}
\]

\[
E = \begin{bmatrix}
0.5950 & 9.5092 \\
-9.5092 & 0.5950
\end{bmatrix}
\]

For this mass-spring system \( m = 2 \) and \( n = 4 \).

Now we construct the monic skew symmetric matrix polynomial \( P(\lambda) = \lambda^2 I_4 + \lambda A_1 + A_0 \) of degree 2.

We take the standard basis of the space of all skew symmetric matrices \( \{ S_\ell \in \mathbb{R}^{n \times n} : \ell = 1, 2, \ldots, 6 \} \) where \( S_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, S_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, S_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \)

\( S_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, S_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, S_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \)

Here, \( UU^\dagger b = b \) and \( U^\dagger U \neq I_{12} \) where \( U \in \mathbb{R}^{8 \times 12} \) and \( b \in \mathbb{R}^{8 \times 1} \). Equation (10) has an infinite number of solutions. Therefore, LPPIEP has an infinite number of solutions. One of the solution \( x \) of Equation (10) is given by \( x = [6.1761 \ 5.1682 \ 3.0933 \ 2.9398 \ 2.5033 \ 0.6224 \ 3.7036 \ 3.0992 \ 1.8550 \ 1.7629 \ 1.5011 \ 0.3732]^T \).

Using Theorem 2.3 matrices \( A_0 \) and \( A_1 \) are given by

\[
A_0 = \begin{bmatrix}
0 & 3.7036 & 3.0992 & 1.8550 \\
-3.7036 & 0 & 1.7629 & 1.5011 \\
-3.0992 & -1.7629 & 0 & 0.3732 \\
-1.8550 & -1.5011 & -0.3732 & 0
\end{bmatrix}
\]
Constructed matrices \( A_0 \) and \( A_1 \) are skew symmetric and they satisfy the eigenvalue and eigenvector relation \( XE^2 + A_1XE + A_0X = 0 \) as \( \|XE^2 + A_1XE + A_0X\|_F^2 = 8.0185 \times 10^{-6} \). Total computational time for running this program in a system with 4Gb ram is 0.078 seconds. Therefore, we successfully reproduced the eigenvalues and eigenvectors from the constructed monic skew symmetric quadratic matrix polynomial.

Next, we study the effect of choosing different ordered basis of the space of all skew symmetric matrices to the solution.

Using the ordered basis \( \{ S_\ell \in \mathbb{R}^{n\times n} : \ell = 1, 2, \ldots, 6 \} \) where \( S_1 = \begin{bmatrix} 0 & 1 & -2 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, S_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, S_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, S_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, S_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \)

\( S_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \), matrices \( A_0 \) and \( A_1 \) are given by

\[
A_0 = \begin{bmatrix} 0 & -1.2396 & 6.4982 & 2.0008 \\ 1.2396 & 0 & 4.0440 & 3.6581 \\ -6.4982 & -4.0440 & 0 & 0.3732 \\ -2.0008 & -3.6581 & -0.3732 & 0 \end{bmatrix}
\]

\[
A_1 = \begin{bmatrix} 0 & 6.1815 & 5.1892 & 3.6862 \\ -6.1815 & 0 & 2.7181 & 1.7956 \\ -5.1892 & -2.7181 & 0 & 1.3404 \\ -3.6862 & -1.7956 & -1.3404 & 0 \end{bmatrix}
\]

If we take two different basis of the space of all skew symmetric matrices for this example, we get different skew symmetric matrices.

**Example 3.3.** Consider a 50 \( \times \) 50 triplet \((I_{50}, A_1, A_0)\) where symmetric tridiagonal matrices \( A_0, A_1 \) are generated using the MATLAB as

\[
A_1 = \text{diag}(a_1) + \text{diag}(b_1, -1) + \text{diag}(b_1, 1)
\]

\[
A_0 = \text{diag}(a_2) + \text{diag}(b_2, -1) + \text{diag}(b_2, 1)
\]

where \( a_1 = [10 20 6 8 40 10 50 60 3 70 30 7 9 4 80 4 2 6 5 8 1 1.2 6.2 2.7 4.3 3.2 2.6 14 2.9 13 12.4 4.6 14.2 8 1.9 2.4 1.6 25 10.84 22.3 42.62 54.24 26.24 1 4 0 5 0.3 7 3 8 0.9 5 0.2],
\]

\( b_1 = [2.8 1.2 36 8 4 16 2 1.2 28 12 32 3.6 20 0.8 1.8 0.96 3.92 3.24 1.04 6 0.9 3 0.4 4 0.2 2 0.5 0.6 0.8 0.3 2 1 0.9 3 0.4 4 0.2 2 5 2 1.07 8 0.2 0.6 7 0.4 7],
\]

\( a_2 = [5.6 2.4 16 8 48 7.2 24 3.2 32 1.6 16 4 4.8 6.4 72 80168 328 432 200 17.6 26.4 23.2 17.6 96 19.2 84 75.2 35.6 85.6 52 12.4 15.6 11.2 168 85.04 175.8 337.72 433.44 207.44 0.4 4 0.2 2 0.5 0.6 0.8 9 10 21],
\]
Infinite \|X E^2 + A_1 X E + A_0 X\|_F = 7.1 \times 10^{-8}.

Total computational time for running this program in a system with 4 GB ram is 1.158 seconds. Therefore, we successfully reproduced the eigenvalues and eigenvectors from the constructed monic symmetric tridiagonal quadratic matrix polynomial.

Similarly for various cases of partial eigendata where \( m = 2, 6 \) and \( 10 \), we construct the matrix eigenpairs \((E, X) \in \mathbb{R}^{m \times m} \times \mathbb{R}^{50 \times m}\). We construct the symmetric tridiagonal matrices \( A_0 \in \mathbb{R}^{50 \times 50} \) and \( A_1 \in \mathbb{R}^{50 \times 50} \) using Theorem 2.3 and they satisfy the eigenvalue and eigenvector relation \( X E^2 + A_1 X E + A_0 X = 0 \) as \( \|X E^2 + A_1 X E + A_0 X\|_F = 7.1 \times 10^{-8} \).

| n | m | Conditions Satisfied | \(\|X E^2 + A_1 X E + A_0 X\|_F\) | Solution | Time(s) |
|---|---|----------------------|-------------------------------|----------|---------|
| 50 | 2 | \(UU^t b = b, U^t U \neq I_{198}\) | \(2.5 \times 10^{-11}\) | Infinite | 1.12 s |
| 50 | 4 | \(UU^t b = b, U^t U = I_{198}\) | \(7.1 \times 10^{-8}\) | Unique | 1.15 s |
| 50 | 6 | \(UU^t b = b, U^t U = I_{198}\) | \(3.6 \times 10^{-8}\) | Unique | 1.22 s |
| 50 | 10 | \(UU^t b = b, U^t U = I_{198}\) | \(5.74 \times 10^{-6}\) | Unique | 1.29 s |

4. Conclusions

In this paper, we have studied the linearly structured partial polynomial inverse eigenvalue problem. A necessary and sufficient condition for the existence of solution to this problem is derived in this paper. Additionally, we present an analytical expression of the solution. Further, we discuss the sensitivity of the solution when the eigendata is not exactly known. Thus, this
paper presents a complete theory on the structured solution of the inverse eigenvalue problem for a monic matrix polynomial of arbitrary degree.

References

References

[1] Q. Al-Hassan. An inverse eigenvalue problem for general tridiagonal matrices. *International Journal of Contemporary Mathematical Sciences*, 4(13):625–634, 2009.

[2] Z. J. Bai. Symmetric tridiagonal inverse quadratic eigenvalue problems with partial eigendata. *Inverse Problems*, 24(1):015005, 2007.

[3] V. Barcilon. On the solution of inverse eigenvalue problems of high orders. *Geophysical Journal International*, 39(1):143–154, 1974.

[4] D. C. Barnes and R. Knobel. The inverse eigenvalue problem with finite data for partial differential equations. *SIAM Journal on Mathematical Analysis*, 26(3):616–632, 1995.

[5] L. Batzke and C. Mehl. On the inverse eigenvalue problem for T-alternating and T-palindromic matrix polynomials. *Linear Algebra and its Applications*, 452:172–191, 2014.

[6] A. Ben-Israel and T. NE. Greville. *Generalized Inverses: Theory and Applications*, volume 15. Springer Science & Business Media, 2003.

[7] M. Berhanu. *The polynomial eigenvalue problem*. PhD thesis, University of Manchester, 2005.

[8] Y. F. Cai, Y. C. Kuo, W. W. Lin, and S. F. Xu. Solutions to a quadratic inverse eigenvalue problem. *Linear Algebra and its Applications*, 430(5):1590–1606, 2009.

[9] Y. F. Cai, J. Qian, and S. F. Xu. Robust partial pole assignment problem for high order control systems. *Automatica*, 48(7):1462–1466, 2012.

[10] M. T. Chu. Inverse eigenvalue problems. *SIAM review*, 40(1):1–39, 1998.

[11] M. T. Chu. Inverse eigenvalue problems: theory and applications. *A series of lectures presented at IRMA, CRN, Bari, Italy*, 2001.
[12] M. T. Chu, Y. C. Kuo, and W. W. Lin. On inverse quadratic eigenvalue problems with partially prescribed eigenstructure. *SIAM Journal on Matrix Analysis and Applications*, 25(4):995–1020, 2004.

[13] B. N. Datta. *Numerical Linear Algebra and Applications*. SIAM, 2010.

[14] B. N. Datta, S. Elhay, Y. Ram, and D. Sarkissian. Partial eigenstructure assignment for the quadratic pencil. *Journal of Sound and Vibration*, 230(1):101–110, 2000.

[15] B. N. Datta and D. Sarkissian. Theory and computations of some inverse eigenvalue problems for the quadratic pencil. *Contemporary Mathematics*, 280:221–240, 2001.

[16] B. N. Datta and V. Sokolov. A solution of the affine quadratic inverse eigenvalue problem. *Linear Algebra and its Applications*, 434(7):1745–1760, 2011.

[17] E. M. De Sa. Imbedding conditions for $\lambda$-matrices. *Linear Algebra and its Applications*, 24:33–50, 1979.

[18] F. De Terán, F. M. Dopico, and P. Van Dooren. Matrix polynomials with completely prescribed eigenstructure. *SIAM Journal on Matrix Analysis and Applications*, 36(1):302–328, 2015.

[19] A. Dmytryshyn. *Skew-Symmetric Matrix Pencils: Stratification Theory and Tools*. PhD thesis, Umeå universitet, 2014.

[20] S. Elhay and Y. M. Ram. An affine inverse eigenvalue problem. *Inverse Problems*, 18(2):455, 2002.

[21] M. Friswell and J. Mottershead. *Finite Element Model Updating in Structural Dynamics*, volume 38. Springer Science & Business Media, 1995.

[22] G. M. L. Gladwell. Inverse problems in vibration. *Applied Mechanics Reviews*, 39(7):1013–1018, 1986.

[23] G. M. L. Gladwell, T. H. Jones, and N. B. Willms. A test matrix for an inverse eigenvalue problem. *Journal of Applied Mathematics*, 2014, 2014.
[24] I. Gohberg, P Lancaster, and L. Rodman. Matrix Polynomials. Springer, 2005.

[25] O. H. Hald. Inverse eigenvalue problems for jordi matrices. Linear Algebra and Its Applications, 14(1):63–85, 1976.

[26] Y. C. Kuo, W.W. Lin, and S. F. Xu. Solutions of the partially described inverse quadratic eigenvalue problem. SIAM Journal on Matrix Analysis and Applications, 29(1):33–53, 2006.

[27] P. Lancaster. Lambda Matrices and Vibrating Systems. Courier Corporation, 2002.

[28] P. Lancaster. Inverse spectral problems for semisimple damped vibrating systems. SIAM Journal on Matrix Analysis and Applications, 29(1):279–301, 2007.

[29] P. Lancaster and I. Zaballa. On the inverse symmetric quadratic eigenvalue problem. SIAM Journal on Matrix Analysis and Applications, 35(1):254–278, 2014.

[30] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Structured polynomial eigenvalue problems: Good vibrations from good linearizations. SIAM Journal on Matrix Analysis and Applications, 28(4):1029–1051, 2006.

[31] X. Mao and H. Dai. Minimum norm partial eigenvalue assignment of high order linear system with no spill-over. Linear Algebra and its Applications, 438(5):2136–2154, 2013.

[32] J. R. McLaughlin. An inverse eigenvalue problem of order four. SIAM Journal on Mathematical Analysis, 7(5):646–661, 1976.

[33] V. Mehrmann and D. Watkins. Polynomial eigenvalue problems with hamiltonian structure. Electronic Transactions on Numerical Analysis, 13:106–118, 2002.

[34] J. Moreno, B. N. Datta, and M. Raydan. A symmetry preserving alternating projection method for matrix model updating. Mechanical Systems and Signal Processing, 23(6):1784–1791, 2009.
[35] J. Mottershead and M. Friswell. Model updating in structural dynamics: a survey. *Journal of Sound and Vibration*, 167(2):347–375, 1993.

[36] M. Müller. An inverse eigenvalue problem: Computing B-stable Runge-Kutta methods having real poles. *BIT Numerical Mathematics*, 32(4):676–688, 1992.

[37] N. K. Nichols and J. Kautsky. Robust eigenstructure assignment in quadratic matrix polynomials: Nonsingular case. *SIAM Journal on Matrix Analysis and Applications*, 23(1):77–102, 2001.

[38] R. L. Parker and K. A. Whaler. Numerical methods for establishing solutions to the inverse problem of electromagnetic induction. *Journal of Geophysical Research: Solid Earth*, 86(B10):9574–9584, 1981.

[39] B. Parlett, F. M. Dopico, and C. Ferreira. The inverse eigenvector problem for real tridiagonal matrices. *SIAM Journal on Matrix Analysis and Applications*, 37(2):577–597, 2016.

[40] H. Pickmann, R. L. Soto, J. Egana, and M. Salas. An inverse eigenvalue problem for symmetrical tridiagonal matrices. *Computers & Mathematics with Applications*, 54(5):699–708, 2007.

[41] S. Rakshit and S. R. Khare. Symmetric band structure preserving finite element model updating with no spillover. *Mechanical Systems and Signal Processing*, 116:415–431, 2019.

[42] S. Rakshit, S. R. Khare, and B. N. Datta. Symmetric tridiagonal structure preserving finite element model updating problem for the quadratic model. *Mechanical Systems and Signal Processing*, 107:278–290, 2018.

[43] Y. Ram and S. Elba. An inverse eigenvalue problem for the symmetric tridiagonal quadratic pencil with application to damped oscillatory systems. *SIAM Journal on Applied Mathematics*, 56(1):232–244, 1996.

[44] Y. M. Ram. Inverse eigenvalue problem for a modified vibrating system. *SIAM Journal on Applied Mathematics*, 53(6):1762–1775, 1993.

[45] M. A. Ramadan and E. A. El-Sayed. Partial eigenvalue assignment problem of high order control systems using orthogonality relations. *Computers & Mathematics with Applications*, 59(6):1918–1928, 2010.
[46] F. Tisseur and K. Meerbergen. The quadratic eigenvalue problem. *SIAM Review*, 43(2):235–286, 2001.

[47] X. T. Wang and L. Zhang. Partial eigenvalue assignment of high order systems with time delay. *Linear Algebra and its Applications*, 438(5):2174–2187, 2013.

[48] Y. Yuan and H. Dai. On a class of inverse quadratic eigenvalue problem. *Journal of Computational and Applied Mathematics*, 235(8):2662–2669, 2011.