Enumeration of singular hypersurfaces on arbitrary complex manifolds

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Abstract

In this paper we obtain an explicit formula for the number of hypersurfaces in a compact complex manifold $X$ (passing through the right number of points), that has a simple node, a cusp or a tacnode. The hypersurfaces belong to a linear system, which is obtained by considering a holomorphic line bundle $L$ over $X$. Our main tool is a classical fact from differential topology: the number of zeros of a generic smooth section of a vector bundle $V$ over $M$, counted with a sign, is the Euler class of $V$ evaluated on the fundamental class of $M$.

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1 Introduction

Enumeration of singular curves in $\mathbb{P}^2$ (complex projective space) is a classical problem in algebraic geometry. For certain singularities $\mathcal{X}$, the authors in [6] and [1] use a purely topological method to obtain an explicit answer for the following question:

**Question 1.1.** Let $\mathcal{X}$ be a codimension $k$-singularity.\(^a\) How many degree $d$-curves are there in $\mathbb{P}^2$, passing through $(d(d + 3)/2 - k)$ generic points and having a singularity of type $\mathcal{X}$?

However, one of the power of that topological method is that it generalizes to enumerating curves on *any* complex surface. More generally, the topological method generalizes to enumerating singular hypersurfaces on an arbitrary compact complex manifold of a given dimension.

Let us first make a definition.

**Definition 1.2.** Let $L \to X$ be a holomorphic line bundle over an $m$-dimensional complex manifold $X$ and $f : X \to L$ a holomorphic section. A point $q \in f^{-1}(0)$ is of **singularity type** $A_k$ if there exists

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\(^a\)By codimension we mean the number of conditions having that particular singularity imposes on the space of curves. For example a node is a codimension one singularity, a cusp is a codimension two singularity, a triple point is a codimension four singularity and so on.
a coordinate system \((z_1, z_2, \ldots, z_m) : (U, q) \to (\mathbb{C}^m, 0)\) such that \(f^{-1}(0) \cap U\) is given by

\[
z_1^{k+1} + \sum_{i=2}^{m} z_i^2 = 0,
\]

where \(m\) is the complex dimension of the manifold \(X\).

Our main theorem can be summarized as follows:

**Theorem 1.3.** Let \(X\) be an \(m\)-dimensional compact complex manifold and \(L \to X\) a holomorphic line bundle. Let

\[
\mathcal{D} := \mathbb{P} H^0(X, L) \cong \mathbb{P} \mathcal{C}, \quad c_1 := c_1(L) \quad \text{and} \quad x_i := c_i(T^*X)
\]

where \(c_i\) denotes the \(i\)th Chern class. Denote \(\mathcal{N}(X)\) to be the number of hypersurfaces in \(X\), that belong to the linear system \(H^0(X, L)\), passing through \(\delta_L - k\) generic points and having a singularity of type \(X\), where \(k\) is the codimension of the singularity \(X\). Then

\[
\mathcal{N}(A_1) = \sum_{i=0}^{m} (m + 1 - i)x_i c_1^{m-i},
\]

\[
\mathcal{N}(A_2) = \sum_{i=0}^{m} \left( m + 2 - i \right) x_i c_1^{m-i} + \sum_{i=0}^{m-1} 2 \left( m + 1 - i \right) x_1 x_i c_1^{m-i-1},
\]

\[
\mathcal{N}(A_3) = \sum_{i=0}^{m-2} t_2 c_1^{m-i-2} x_i + \sum_{i=0}^{m-1} t_1 c_1^{m-i-1} x_i + \sum_{i=0}^{m} t_0 c_1^{m-i} x_i,
\]

where \(t_2 := \left( m + 1 - i \right) \left( \frac{c_1^2}{2} (3 m^2 - m) + c_1 x_1 (6 m - 1) + 6 x_2 \right)\),

\[
t_1 := \left( m + 1 - i \right) \left( c_1 (3 m^2 - m) + x_1 (6 m - 1) \right) \quad \text{and}
\]

\[
t_0 := \left( m + 1 - i \right) \left( \frac{3 m^2}{2} - \frac{m}{2} \right)
\]

provided the sections \(\psi_{A_0}, \psi_{A_1}, \psi_{A_2}\) and \(\Psi_{P, A_3}\) defined in (5), (6), (10) and (12) are transverse to the zero set, respectively.

**Remark 1.4.** In equations (1) to (3) we are making an obvious abuse of notation; the lhs is an integer, while the rhs is a cohomology class. Our intended meaning is the rhs evaluated on the fundamental class \([X]\).

## 2 Topological computations

The main tool that we will use is the following well known fact from differential topology (cf. [3], Proposition 12.8).

**Theorem 2.1.** Let \(V \to X\) be an oriented vector bundle over a compact manifold \(X\) and \(s : X \to V\) a smooth section that is transverse to the zero set. Then the zero set of \(s\) defines an integer homology class in \(X\), whose Poincaré dual is the Euler class of \(V\). In particular, if the rank of \(V\) is
same as the dimension of $X$, then the signed cardinality of $s^{-1}(0)$ is the Euler class of $V$, evaluated on the fundamental class of $X$, i.e.,

$$|\pm s^{-1}(0)| = \langle e(V), [X] \rangle.$$ 

An immediate corollary is:

**Corollary 2.2.** Let $X$ be a compact, complex manifold, $V$ a holomorphic vector bundle (with their natural orientations) and $s$ a holomorphic section that is transverse to the zero set. If the rank of $V$ is same as the dimension of $X$, then the signed cardinality of $s^{-1}(0)$ is same as its actual cardinality. In particular

$$|s^{-1}(0)| = \langle e(V), [X] \rangle.$$ 

We are now ready to give a proof of formulas (1), (2) and (3). Let $\mathcal{D}_n \cong \mathbb{P}^n \subset \mathcal{D}$ be the space of hypersurfaces through $\delta_L - n$ generic points. Then $\mathcal{N}(A_1)$ is the cardinality of the set

$$\{([f], q) \in D_1 \times X : f(q) = 0, \nabla f|_q = 0\}. \quad (4)$$

Let us now define sections of the following bundles:

$$\psi_{A_0} : \mathcal{D} \times X \rightarrow \mathcal{L}_{A_0} := \gamma_0^* \otimes L, \quad \{\psi_{A_0}([f], q)\}(f) := f(q), \quad (5)$$

$$\psi_{A_1} : \psi_{A_0}^{-1}(0) \rightarrow \mathcal{V}_{A_1} := \gamma_0^* \otimes T^* X \otimes L, \quad \{\psi_{A_1}([f], q)\}(f) := \nabla f|_q. \quad (6)$$

Here $\gamma_{\mathcal{D}}$ is the tautological line bundle over $\mathcal{D}$. In equation (5), the rhs is an element of the vector space $L_q$, the fiber of the line bundle $L$ over $q$. Hence $\psi_{A_0}$ is a section of $\mathcal{L}_{A_0}$. Similarly, the rhs of (6) is an element of $T^*_q X \otimes L_q$. Hence $\psi_{A_1}$ is a section of $\mathcal{V}_{A_1}$.

Next, let us assume that $\psi_{A_0}$ and $\psi_{A_1}$ are transverse to the zero set. Since these two sections are holomorphic, we conclude that

$$\mathcal{N}(A_1) = \langle e(\mathcal{V}_{A_1}), \psi_{A_0}^{-1}(0) \rangle = \langle e(\mathcal{L}_{A_0})e(\mathcal{V}_{A_1}), [\mathcal{D}_1 \times X] \rangle. \quad (7)$$

The second equality follows from the fact that the Poincaré Dual of $[\psi_{A_0}^{-1}(0)]$ in $\mathcal{D}_1 \times X$ is $e(\mathcal{L}_{A_0})$. Using the splitting principal, (7) simplifies to (1).

Next, let us prove (2). Note that $\mathcal{N}(A_2)$ is the cardinality of the following set

$$\{([f], q) \in \mathcal{D}_2 \times X : f(q) = 0, \nabla f|_q = 0, \det \nabla^2 f|_q = 0\}. \quad (8)$$

Continuing with the setup of the proof of (1), we define a section of the following line bundle

$$\psi_{A_2} : \psi_{A_1}^{-1}(0) \rightarrow \mathcal{L}_{A_2} := \gamma_D^{\star m} \otimes (\Lambda^m T^* X)^{\otimes 2} \otimes L^{\otimes m}, \quad (9)$$

given by

$$\{\psi_{A_2}([f], q)\}(f^{\otimes m} \otimes (v_{i_1} \wedge \ldots v_{i_m})^{\otimes 2}) := \det \begin{pmatrix}
\nabla^2 f|_q(v_{i_1}, v_{i_1}) & \ldots & \nabla^2 f|_q(v_{i_1}, v_{i_m}) \\
\nabla^2 f|_q(v_{i_2}, v_{i_1}) & \ldots & \nabla^2 f|_q(v_{i_2}, v_{i_m}) \\
\vdots & \ddots & \vdots \\
\nabla^2 f|_q(v_{i_m}, v_{i_1}) & \ldots & \nabla^2 f|_q(v_{i_m}, v_{i_m})
\end{pmatrix}. \quad (10)$$

\[\text{bWhether or not this assumption actually holds, will depend on the specific example of } L \text{ and } X.\]
Note that in (10), the rhs is an element of $L_q^\otimes m$. Hence $\psi_{A_2}$ is a section of $L_{A_2}$. Assume that this section is transverse to the zero set. Since it is holomorphic, we conclude that

$$\mathcal{N}(A_2) = \langle e(L_{A_2}), [\psi_{A_2}^{-1}(0)] \rangle = \langle e(L_{A_0})e(V_{A_1})e(L_{A_2}), [D_2 \times S] \rangle. \quad (11)$$

The second equality follows from the fact that the Poincaré Dual of $[\psi_{A_2}^{-1}(0)]$ in $D_2 \times X$ is $e(L_{A_0})e(V_{A_1})$. Using the splitting principal, (11) simplifies to (2). \hfill \square

Next, we will give a proof of (3). This requires a bit more ingenuity. First, let us compute $\mathcal{N}(A_2)$ in an alternate way. Let $\mathbb{P}TX$ denote the projectivization of $TX$ and $\hat{\gamma} \rightarrow \mathbb{P}TX$ the tautological line bundle over $\mathbb{P}TX$. Then $\mathcal{N}(A_2)$ is also the cardinality of the set

$$\{( [f], l_q) \in D_2 \times \mathbb{P}TX : (f, q) \in \psi_{A_1}^{-1}(0), \nabla^2 f|_q(v, \cdot) = 0 \ \forall \ v \in l_q \}. \quad (12)$$

Let $\pi : D \times \mathbb{P}TX \rightarrow D \times X$ be the projection map. We now define a section of the following bundle

$$\Psi_{P,A_2} : \pi^* \psi_{A_1}^{-1}(0) \rightarrow V_{P,A_2} := \hat{\gamma}^* \otimes \gamma_D^* \otimes T^* X \otimes L, \quad \text{given by}$$

$$\{ \Psi_{P,A_2}([f], l_q) \} (v \otimes f) := \nabla^2 f|_q(v, \cdot) \quad \forall \ v \in l_q. \quad (13)$$

Note that the rhs of (13) belongs to $T^*_q X \otimes L_q$. Hence $\Psi_{P,A_2}$ is a section of $V_{P,A_2}$. Assume that this section is transverse to the zero set. Since it is holomorphic, we conclude that

$$\mathcal{N}(A_2) = \langle e(V_{P,A_2}), [\pi^* \psi_{A_1}^{-1}(0)] \rangle = \langle e(\pi^* L_{A_0})e(\pi^* V_{A_1})e(V_{P,A_2}), [D_2 \times \mathbb{P}TX] \rangle. \quad (14)$$

We now use the ring structure of $H^*(\mathbb{P}TX, \mathbb{Z})$ (3), pp. 270) to conclude that

$$\lambda^m - c_1 \lambda^{m-1} + c_2 \lambda^{m-2} + \ldots + (-1)^m c_m = 0, \quad (15)$$

where $\lambda := c_1(\hat{\gamma}^*)$. Using the splitting principal, equations (14) combined with (15) simplifies to (2).

We are now ready to prove (3). First, we note that $\mathcal{N}(A_2)$ is also the cardinality of the set

$$\{( [f], l_q) \in D_3 \times \mathbb{P}TX : (f, q) \in \Psi_{P,A_2}^{-1}(0), \nabla^3 f|_q(v, v, v) = 0 \ \forall \ v \in l_q \}. \quad (16)$$

Let us now define a section of the following bundle

$$\Psi_{P,A_3} : \Psi_{P,A_2}^{-1}(0) \rightarrow \mathbb{L}_{P,A_3} := \hat{\gamma}^{*3} \otimes \gamma_D^* \otimes L, \quad \text{given by}$$

$$\{ \Psi_{P,A_3}([f], l_q) \} (v^{*3} \otimes f) := \nabla^3 f|_q(v, v, v) \quad \forall \ v \in l_q. \quad (16)$$

Note that the rhs of (16) is an element of $L_q$. Hence $\Psi_{P,A_3}$ is a section of $\mathbb{L}_{P,A_3}$. Assume that this section is transverse to the zero set. Since it is holomorphic, we conclude that

$$\mathcal{N}(A_3) = \langle e(\mathbb{L}_{P,A_3}), [\Psi_{P,A_2}^{-1}(0)] \rangle = \langle e(\pi^* L_{A_0})e(\pi^* V_{A_1})e(V_{P,A_2})e(\mathbb{L}_{P,A_3}), [D_3 \times \mathbb{P}TS] \rangle. \quad (17)$$

Using the splitting principal, equations (17) combined with (15) simplifies to (3). \hfill \square
3 Examples

Example 3.1. Let $X$ be a surface. Then (1) simplifies to

$$\mathcal{N}(A_1) = 3c_1^2 + 2c_1 x_1 + x_2,$$

which agrees with the result of Vainsencher in [5].

Example 3.2. Let $X := \mathbb{P}^m$ and $L := \gamma_{\mathbb{P}^m}^d$. Then

$$c_1^{m-i}x_i = (-1)^i \binom{m+1}{i} d^{m-i}.$$  \hfill (19)

Equations (1), (2) and (3) combined with (19) imply that

$$\mathcal{N}(A_1) = (m+1)(d-1)^m,$$

$$\mathcal{N}(A_2) = \frac{m(m+1)(m+2)}{2}(d-1)^{m-1}(d-2),$$

$$\mathcal{N}(A_3) = \frac{m(m+1)(m+2)}{12}(d-1)^{m-2}(m_2 d^2 + m_1 d + m_0),$$

where $m_2 := m^2 + 2m - 1$, $m_1 := -12m^2 - 28m + 8$ and $m_0 := 3m^2 + 8m - 3$.

For $m = 2$ this gives us

$$\mathcal{N}(A_1) = 3(d-1)^2, \quad \mathcal{N}(A_2) = 12(d-1)(d-2), \quad \mathcal{N}(A_3) = 50d^2 - 192d + 168,$$

which recovers the formulas obtained in [6] and [1]. For general $m$, the numbers $\mathcal{N}(A_2)$ and $\mathcal{N}(A_3)$ agree with the results of Kerner in [4].

Example 3.3. Let $X := \mathbb{P}^1 \times \mathbb{P}^1$ and $L := \pi_1^* \gamma_{\mathbb{P}^1}^d \otimes \pi_2^* \gamma_{\mathbb{P}^1}^d$. Then

$$c_1^2 = 2d_1 d_2, \quad c_1 x_1 = -2(d_1 + d_2), \quad x_1^2 = 8, \quad x_2 = 4.$$  \hfill (24)

Equation (24), combined with (1), (2) and (3) gives

$$\mathcal{N}(A_1) = 6d_1 d_2 - 4(d_1 + d_2) + 4,$$

$$\mathcal{N}(A_2) = 24(d_1 - 1)(d_2 - 1)$$

and

$$\mathcal{N}(A_3) = 100d_1 d_2 - 128(d_1 + d_2) + 136.$$  \hfill (25)

Let us take $d_1 = d_2 = 1$. The formulas for $\mathcal{N}(A_2)$ and $\mathcal{N}(A_3)$ confirm the fact that there are no curves of type $(1,1)$ with either a cusp or a tacnode. The formula for $\mathcal{N}(A_1)$ confirms the fact that there are two curves of type $(1,1)$ through two generic points that have a node.

Remark 3.4. In [2], we show why the relevant sections in example 3.2 are transverse to the zero set (for $m = 2$). The proof follows by unwinding definitions and writing the section in a local coordinate chart and trivialization. The proof of why the relevant sections in example 3.3 are transverse to the zero set is similar (provided $d_1$ and $d_2$ are sufficiently large).

Acknowledgements. The author thanks Indranil Biswas for suggesting example 3.3. Furthermore, the author is also grateful to Somnath Basu, Shane D’Mello and Vamsi Pingali for relevant discussions and comments about this paper.

The number $\mathcal{N}(A_1)$ is not explicitly stated in [4].


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