Distributed projection-free algorithm for constrained aggregative optimization

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Abstract
In this paper, we focus on solving a distributed convex aggregative optimization problem in a network, where each agent has its own cost function which depends not only on its own decision variables but also on the aggregated function of all agents’ decision variables. The decision variable is constrained within a feasible set. In order to minimize the sum of the cost functions when each agent only knows its local cost function, we propose a distributed Frank–Wolfe algorithm based on gradient tracking for the aggregative optimization problem where each node maintains two estimates, namely an estimate of the sum of agents’ decision variable and an estimate of the gradient of global function. The algorithm is projection-free, but only involves solving a linear optimization to get a search direction at each step. We show the convergence of the proposed algorithm for convex and smooth objective functions over a time-varying network. Finally, we demonstrate the convergence and computational efficiency of the proposed algorithm via numerical simulations.

KEYWORDS
aggregative optimization, distributed algorithm, gradient projection-free, time-varying graph

1 | INTRODUCTION

In recent years, distributed optimization algorithms and their applications have received extensive attention in decision-making problems for multi-agent networks,¹-³ with applications in smart grids,⁴ resource allocation,⁵ and robot formations.⁶ In the framework of multi-agent networks, the agents in the network have a local interactive pattern, each of them can only access its own information and that of its neighboring agents, and the goal of the agents is to optimize the sum of the local objective functions in a cooperative manner.

In general, distributed optimization algorithms can be divided into unconstrained optimization,⁷,⁸ and constrained optimization,¹⁹ from the viewpoint of with or without constraints. For unconstrained optimization, various algorithms like consensus subgradient algorithm,⁷ dual averaging,¹⁰ EXTRA¹¹ and gradient tracking algorithm¹² were studied. For constrained optimization problems, methods based on projection dynamics and primal-dual dynamics have been studied. For example, Reference 13 studied the distributed projected subgradient algorithm. Reference 14 studied a projected
primal-dual algorithm for constrained optimization, and Reference 15 developed a distributed dual average push-sum algorithm with dual decomposition, while a constrained subproblem is solved at each step. Reference 16 investigated a distributed mirror-descent optimization algorithm based on the Bregman divergence and achieved an $O(\ln(T)/T)$ rate of convergence. Reference 17 proposed a consensus-based distributed regularized primal-dual gradient method. Compared to methods that required projection of estimates onto the constraint set at each iteration, the algorithm in Reference 17 only required one projection at the last iteration.

However, the projection-based distributed algorithm implies that the agent needs to solve a quadratic optimization problem at each iteration, to find the closest point within constraint set. When the constraints have complex structures (e.g., polyhedra), the computational cost of solving quadratic subproblem can prevent the agent from using projection-based dynamics, especially for high-dimensional optimization problems. Fortunately, the well-known Frank-Wolfe (FW)\textsuperscript{18} algorithm provides us with a way to derive an effective searching direction while maintaining decision feasibility. Each step of FW algorithm only needs to solve a constrained linear programming problem, which could have a closed form for specific problems or have effective solvers. Recently, FW methods have received renewed research attention due to its projection-free property and advantages for large-scale problems for online learning,\textsuperscript{19} machine learning,\textsuperscript{20} and traffic assignment.\textsuperscript{21} Briefly speaking, the FW method uses a linearized function to approximate the objective function and derives feasible descent directions by solving a linear objective optimization, by $s := \arg\min_{s \in D} \langle s, \nabla f(x^{(k)}) \rangle$, $x^{(k+1)} := (1 - \gamma)x^{(k)} + \gamma s$. There have been massive developments and applications for the FW method afterwards. For example, the primal-dual convergence rate has been analyzed in References 22, and Reference 23 has analyzed its convergence for nonconvex problems. In addition, Reference 24 developed communication efficient algorithms using the stochastic Frank-Wolfe (sFW) algorithm when there exist noises on the gradient computation. FW algorithms are also utilized and extended to distributed optimization. For example, Reference 25 studied the decentralized optimization problem by treating the algorithm as an inexact FW algorithm with consensus errors. By combing gradient tracking and FW algorithm, Reference 26 investigated the continuous-time algorithm based on FW dynamics, and Reference 27 studied FW algorithm for constrained stochastic optimization.

In the distributed optimization problems mentioned above, the agents decide the same variable while need to reach consensus on the optimal decision. However, in other scenarios, each agent only decides its own variable but the objective function of each agent is related to other agents’ decisions through an aggregated variable. For example, in multi-agents formation control problem,\textsuperscript{28} a group of networked agents wish to achieve a geometric pattern while surrounding a target, which can be regarded as a goal tracking problem. The dynamical tracking problem can be modeled as an online distributed optimization. In this case, each agent decides its location, while the objective function also depends on the centroid of all agents. Similar scenarios exist in resource allocation,\textsuperscript{29} smart grids,\textsuperscript{30} social networks\textsuperscript{31} as well. The above optimization problem is called distributed aggregative optimization in Reference 32. The aggregative optimization is also related to the well-studied aggregative games,\textsuperscript{33-35} but can be treated as a cooperative formulation in contrast to the noncooperative setting for multi-agent decision problem. Regarding the study of aggregative optimization, Reference 32 considered a static unconstrained framework, Reference 36 considered an online constrained framework, and Reference 37 considered a quantitative problem. To the best of our knowledge, no work has been done to improve the computational bottlenecks encountered by algorithms when dealing with complicated constraints that prohibiting projection algorithms.

Therefore, our interest is to design projection-free methods to solve distributed aggregative optimization problems with constraints. Motivated both by References 32 and 22, we propose a FW-based approach to solve the aggregative optimization in a distributed manner. The main contributions are as follows.

- Firstly, a novel distributed projection-free algorithm based on FW method with gradient tracking is designed to solve the aggregative optimization problem. Each agent’s local cost function depends both on its own variable and on aggregated variable, while the global information are not known by any single agent. The proposed algorithm uses the dynamical averaging tracking approach to estimate the global aggregation variable and the corresponding gradient term.
- Secondly, we prove that the algorithm converges to the optimal solution when the objective function is convex. Compared with the projected dynamics in Reference 36, the proposed algorithm is able to solve the aggregative optimization problem over time-varying communication graphs.
- Finally, we demonstrate the efficiency of the proposed algorithm with numerical studies.
The rest of the paper is organized as follows. Section 2 introduces notations, preliminary knowledge of graph theory, and illustrates the distributed aggregative optimization problem. Section 3 provides the proposed distributed algorithm and main convergence result. Section 4 provides the proof of convergence. Then, a numerical experiment is given in Section 5 and Section 6 concludes the paper.

1.1 | Notations

When referring to a vector $x$, it is assumed to be a column vector while $x^T$ denotes its transpose. $\langle x, y \rangle = x^T y$ denotes the inner product of vectors $x, y$. $\| x \|$ denotes the Euclidean vector norm, that is, $\| x \| = \sqrt{x^T x}$. Let $\otimes$ be the Kronecker product. Denote by $1_N$ and $0_N$ the column vectors. Denote by $\mathbb{R}^{n \times m}$ the set of real matrices with $n$-row and $m$-columns and $[A]_{ij}$ stands for the $ij$th element of matrix $A \in \mathbb{R}^{n \times m}$. A non-negative square matrix $A$ is called doubly stochastic if $A 1 = 1$ and $1^T A = 1^T$, where $1$ denotes the vector with each entry being $1$. $1_N \in \mathbb{R}^{N \times N}$ denotes the identity matrix. Let $\mathcal{G} = \{ \mathcal{N}, \mathcal{E} \}$ be a directed graph with $\mathcal{N} = \{1, \ldots, N\}$ denoting the set of players and $\mathcal{E}$ denoting the set of directed edges between players, where $(j, i) \in \mathcal{E}$ if player $j$ can obtain information from player $i$. The graph $\mathcal{G}$ is called strongly connected if for any $i, j \in \mathcal{N}$ there exists a directed path from $i$ to $j$. Moreover, there exists a sequence of edges $(i, l_1), (l_1, l_2), \ldots, (l_{p-1}, j)$ in the digraph with distinct nodes $l_m \in \mathcal{N}$, $\forall 1 \leq m \leq p - 1$. A differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is $\mu$-strongly convex if for all $\theta, \theta' \in \mathbb{R}^n$, $f(\theta) - f(\theta') \leq \langle \nabla f(\theta), \theta - \theta' \rangle - \frac{\mu}{2} \| \theta - \theta' \|_2^2$, moreover, $f$ is convex if the above is satisfied with $\mu = 0$.

2 | PROBLEM FORMULATION

In this section, we formulate the aggregative optimization problem over networks and introduce some basic assumptions.

2.1 | Problem statement

We define the aggregative optimization problem with $N$ agents:

$$
\min_{x \in \mathcal{X}, \delta(x) \in \mathcal{X}} \ f(x) = \sum_{i=1}^{N} f_i(x_i, \delta(x)) \quad \text{with} \quad \delta(x) = \frac{1}{N} \sum_{i=1}^{N} \phi_i(x_i),
$$

(1)

where $x = \text{col}(x_i)_{i \in \mathcal{N}}, \delta(x)$ is the global strategy variable with $x_i \in \mathbb{R}^{n_i}$ being the decision variable of agent $i$. The function $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is the local objective function of agent $i$ with $\phi_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}^d$ is only available to agent $i$. The goal is to design a distributed algorithm to cooperatively seek an optimal decision variable to the problem (1).

The gradient of $f(x)$ is defined by

$$
\nabla f(x) \triangleq \nabla f_i(x_i, \delta(x)) + \nabla \phi_i(x_i, \delta(x)) \otimes \frac{1}{N} \sum_{i=1}^{N} \nabla f_i(x_i, \delta(x)),
$$

where $\nabla f_i(x_i, \delta(x)) := \text{col}(\nabla \phi_i(x_i)_{x = \delta(x)})_{i \in \mathcal{N}}$, and

$$
\nabla \phi_i(x_i) := \begin{bmatrix}
    \nabla \phi_1(x_i) \\
    \vdots \\
    \nabla \phi_N(x_i)
\end{bmatrix}.
$$

In order to explicitly show the aggregation of the problem, let us specify the cost function as $f_i(x_i, \delta(x)) = \phi_i(x_i, z_i)|_{\delta = \delta(x)}$ with a compound function $g_i : \mathbb{R}^{n_i + d} \rightarrow \mathbb{R}, i \in \mathcal{N}$. To move forward, define $g(x, z) = \sum_{i=1}^{N} g_i(x_i, z_i) : \mathbb{R}^{n + Nd} \rightarrow \mathbb{R}$ for any $x \in \mathbb{R}^n$ and $z = \text{col}(z_i)_{i \in \mathcal{N}} \in \mathbb{R}^{Nd}$. The gradient of $g(x, z)$ is defined by $\nabla_1 g(x, z) := \text{col}(\nabla \phi_i(x_i, z_i)_{i \in \mathcal{N}}),$ $\nabla_2 g(x, z) := \text{col}(\nabla \phi_i(x_i, z_i)_{i \in \mathcal{N}}).$
Next, we impose some assumptions on the formulated problem. We require the agent-specific problem to be convex and continuously differentiable.

**Assumption 1.**

(a) For each agent \( i \in \mathcal{N} \), the strategy set \( X_i \) is closed, convex and compact. In addition, the diameter of \( X = \prod_{i=1}^{N} X_i \) is defined as \( \delta := \max_{\theta, \theta' \in X} \| \theta - \theta' \|_{\mathcal{L}}^2 \); (b) the global objective function \( f \) is convex and differentiable in \( x \in X \), and its gradient function is \( L \)-smooth in \( x \in X \), that is,

\[
    f(y) - f(x) \geq (y - x)^T \nabla f(x), \quad \| \nabla f(x) - \nabla f(y) \| \leq L \| x - y \|, \quad \forall x, y \in X.
\]

**Assumption 2.**

(a) \( \nabla_1 g(x, z) \) is \( l_1 \)-Lipschitz continuous in \( z \in \mathbb{R}^{Nd} \) for any \( x \in X \), i.e.,

\[
    \| \nabla_1 g(x, z_1) - \nabla_1 g(x, z_2) \| \leq l_1 \| z_1 - z_2 \|, \quad \forall z_1, z_2 \in \mathbb{R}^{Nd}.
\]

(b) \( \nabla_2 g(x, z) \) is \( l_2 \)-Lipschitz continuous in \( (x, z) \in X \times \mathbb{R}^{Nd} \), that is,

\[
    \| \nabla_2 g(x_1, z_1) - \nabla_2 g(x_2, z_2) \| \leq l_2 \| x_1 - x_2 \| + l_2 \| z_1 - z_2 \|, \quad \forall x_1, x_2 \in X, z_1, z_2 \in \mathbb{R}^{Nd}.
\]

(c) For each agent \( i \in \mathcal{N} \), \( \phi_i(x_i) \) is differentiable in \( x_i \in X_i \) and \( \nabla \phi_i(x_i) \in \mathbb{R}^{n_i} \) is uniformly bounded in \( x_i \in X_i \), that is, there exists a constant \( c_i > 0 \) such that \( \| \nabla \phi_i(x_i) \| \leq c_i \) for any \( x_i \in X_i \).

(d) For each agent \( i \in \mathcal{N} \), \( \phi_i(x_i) \) is \( l_3 \)-Lipschitz continuous in \( x_i \in X_i \).

### 2.2 Graph theory

We consider the information setting that each player \( i \in \mathcal{N} \) knows the information of its private functions \( f_i \), \( \phi \) and the local constraint \( X_i \), but have no access to the aggregate \( \delta(x) \). Instead, each player is able to communicate with its neighbors over a time-varying graph \( G_k = \{ \mathcal{N}, \mathcal{E}_k \} \). Define \( W_k = [w_{ij,k}]_{i,j=1}^{N} \) as the adjacency matrix of \( G_k \), where \( w_{ij,k} > 0 \) if and only if \( (j, i) \in \mathcal{E}_k \), and \( w_{ij,k} = 0 \), otherwise. Denote by \( N_{i,k} \triangleq \{ j \in \mathcal{N} : (j, i) \in \mathcal{E}_k \} \) the neighboring set of player \( i \) at time \( k \). We impose the following conditions on the time-varying communication graphs \( G_k = \{ \mathcal{N}, \mathcal{E}_k \} \).

**Assumption 3.**

(a) \( W_k \) is doubly stochastic for any \( k \geq 0 \); (b) There exists a constant \( 0 < \eta < 1 \) such that \( w_{ij,k} \geq \eta \), \( \forall j \in N_{i,k}, \forall i \in \mathcal{N}, \forall k \geq 0 \); (c) There exists a positive integer \( B \) such that the union graph \( \{ \mathcal{N}, \bigcup_{i=1}^{B} \mathcal{E}_{k+i} \} \) is strongly connected for all \( k \geq 0 \).

We define a transition matrix \( \Phi(k, s) = W_k W_{k-1} \cdots W_s \) for any \( k \geq s \geq 0 \) with \( \Phi(k, k+1) = I_N \), and state a result that will be used in the sequel.

**Lemma 1.** (Proposition 1)’ Let Assumption 3 hold. Then there exist \( \theta = (1 - \eta/(4N^2))^{-2} > 0 \) and \( \beta = (1 - \eta/(4N^2))^{1/B} \) such that for any \( k \geq s \geq 0 \),

\[
    \left| [\Phi(k, s)]_{ij} - 1/N \right| \leq \theta \beta^{k-s}, \quad \forall i, j \in \mathcal{N}.
\]

### 3 ALGORITHM DESIGN AND MAIN RESULTS

In this section, we design a distributed projection-free algorithm and provide its convergence performance.
3.1 Distributed projection-free algorithm

It is worth pointing out that the FW method for the optimization problem \( \min_{x \in C} f(x) \) requires minimizing a linear function over constraint sets,\(^{38}\) in contrast to the projected gradient methods which require the minimization of quadratic functions over constraint sets. Recall that the FW step is given by

\[
\begin{align*}
& (\text{FWA}) \quad \left\{ \begin{array}{l}
y_k = \arg\min_{y \in C} \langle \nabla f(x_k), y \rangle, \\
x_{k+1} = (1 - \alpha_k)x_k + \alpha_k y_k.
\end{array} \right.
\end{align*}
\]

where \( y \in C \) denote that \( y_k \) is definitely within the constraint set. That is, the FW method consists of a linear optimization oracle, followed by a convex averaging step of the current iterate and the oracle’s output.

The popular projected gradient descent (PGD) algorithms, in which a projection onto the constraint set is applied in each iteration of algorithms, are inefficient and perhaps even intractable in cases where the constraints are complex.\(^{23}\) Unlike projected gradient algorithms, FW algorithms are projection-free and solve linear minimization over constraint sets to make variables feasible.\(^{22}\) The cost of FW algorithms are usually much lower than that of PG algorithm, especially for unit-norm constraints and nuclear norm constraints in the machine learning, where linear minimization has a explicit solution.\(^{22}\)

Next we present the proposed distributed FW algorithm with gradient tracking (D-FWAGT).

**Algorithm 1.** DISTRIBUTED PROJECTION-FREE METHOD FOR AGGREGATIVE OPTIMIZATION

**Initialize:** Set \( k = 0, x_{i,0} \in X_i \) and \( v_{i,0} = \phi_i(x_{i,0}) \), and \( y_{i,0} = \nabla \phi_i(x_{i,0}, v_{i,0}) \) for each \( i \in \mathcal{N}. \)

**Iterate until convergence**

**Consensus.** Each player computes an intermediate estimate by

\[
\begin{align*}
\hat{v}_{i,k+1} &= \sum_{j \in \mathcal{N}_i} w_{i,j} v_{j,k}, \\
\hat{y}_{i,k+1} &= \sum_{j \in \mathcal{N}_i} w_{i,j} y_{j,k}.
\end{align*}
\]

**Strategy update.** Each agent \( i \in \mathcal{N} \) updates its decision variable and its estimate of the average aggregates by

\[
\begin{align*}
s_{i,k} &= \arg\min_{s_i \in X_i} \left( \langle \nabla g_i(x_{i,k}, \hat{v}_{i,k+1}) + \nabla \phi_i(x_{i,k}, \hat{y}_{i,k+1}, s_i) \rangle, \\
x_{i,k+1} &= (1 - \gamma_k)x_{i,k} + \gamma_k s_{i,k}, \\
v_{i,k+1} &= \hat{v}_{i,k+1} + \phi_i(x_{i,k+1}) - \phi_i(x_{i,k}), \\
y_{i,k+1} &= \hat{y}_{i,k+1} + \nabla g_i(x_{i,k+1}, v_{i,k+1}) - \nabla g_i(x_{i,k}, v_{i,k}).
\end{align*}
\]

where \( \gamma_k \in [0, 1) \), where \( s_i \in X_i \) denote that \( s_{i,k} \) is definitely within the constraint set.

Suppose that each agent \( i \) at stage \( k \) selects a strategy \( x_{i,k} \in X_i \) as an estimate of its optimal strategy, and holds the estimates \( v_{i,k} \) and \( y_{i,k} \) for the aggregate \( \delta(x) \) and the gradient term \( \frac{1}{N} \sum_{i=1}^{N} \nabla g_i(x_i, z) \|z=\delta(x)\|, \) respectively. At stage \( k + 1 \), player \( i \) observes or receives its neighbors’ information \( v_{j,k}, y_{j,k}, j \in \mathcal{N}_{i,k} \) and updates two intermediate estimates by the consensus step (5) and (6). Then it computes its gradient estimation, and updates its strategy \( x_{i,k+1} \) by a projection-free scheme (7) with a FW step. Set \( y_{i,0} = \nabla g_i(x_{i,0}, v_{i,0}) \) without loss of generality. Finally, player \( i \) updates the estimate for average aggregate \( v_{i,k+1} \) with the renewed strategy \( x_{i,k+1} \) by the dynamic average tracking scheme (8), and updates the estimate of gradient by the dynamic tracking scheme (9). The procedures are summarized in Algorithm 1.

Denote by \( x_k := \text{col}(x_{1,k}, \ldots, x_{N,k}) \) and similar notations for \( s_k, v_k, y_k, \hat{v}_k, \hat{y}_k \). We can write the Algorithm 1 in a compact form

\[
s_k = \arg\min_{s \in \mathcal{X}} \langle \nabla g_k(x_k, \hat{v}_{k+1}) + \nabla \phi_k(x_k, \hat{y}_{k+1}, s) \rangle.
\]
Lemma 2.33 we have the following result.

Wefirst establish boundson the consensus error of the aggregate and gradient tracking estimate measured by $\|\delta(x_k) - \hat{v}_{k+1}\|$ and $\|\hat{v}_{k+1} - 1 \otimes \hat{y}_k\|$, respectively, which will play an important role in the proof of Theorem 1. Similar to Lemma 2, we have the following result.

\[
x_{k+1} = (1 - \gamma_k)x_k + \gamma_k s_k,
\]

\[
v_{k+1} = \hat{v}_{k+1} + \phi(x_{k+1}) - \phi(x_k),
\]

\[
y_{k+1} = \hat{y}_{k+1} + \nabla_2 g(x_{k+1}, v_{k+1}) - \nabla_2 g(x_{k}, v_{k}),
\]

where $\hat{v}_{k+1} = W_{d,k}v_k, \hat{y}_{k+1} = W_{d,k}y_k, W_{d,k} = W_k \otimes I_d$ and $\phi(x_k) := \text{col}(\phi(x_{1,k}), \ldots, \phi(x_{N,k}))$.

Remark 1. Note that Algorithm 1 is a fully distributed algorithm. In the consensus step (5) and (6), each agent $i$ merely uses its neighbor’s information to update the intermediate estimate $\hat{v}_{i,k+1}$ and $\hat{y}_{i,k}$. For the strategy update step (7), each agent uses an estimate of the aggregate $\hat{v}_{i,k+1}$ and an estimate of the gradient term $\hat{g}_{i,k}$ instead of the true aggregate $\delta(x)$ and the true gradient term $\frac{1}{N} \sum_{i=1}^{N} \nabla f_i(x_i, \delta(x))$ at time $k$ to evaluate the gradient of its cost function $f_i$. Finally, in the update of the aggregate (8), agent $i$ uses its local function $\phi_i$ and its own previous estimate, (9) is the same. The design and analysis of a fully distributed algorithm differs from Reference 32 where the method is projection gradient method.

Remark 2. We highlight that contrary to other constrained optimization algorithms like PGD, the FW algorithm does not require access to a projection, hence why it is sometimes referred to as a projection-free algorithm. It instead relies on a routine that solves a linear problem over the domain $s_{i,k} = \text{argmin}_{s_i \in X_i} \left( \langle \nabla g_i(x_{i,k}, \hat{v}_{i,k+1}) + \nabla \phi_i(x_{i,k}) \hat{y}_{i,k+1}, s_i \rangle \right)$. This routine is commonly referred to as a linear minimization oracle. Moreover, we remark that each agent $i$ is only aware of its decision variable $x_{i,k}$, allowing the privacy preservation of the other agents estimate. In fact, each agent $i$ exchanges with its neighbors only the local aggregative variable estimate $v_{i,k}$ and the local gradient estimate $y_{i,k}$.

We impose the following conditions on the step-length sequence $(\gamma_k)_{k \in \mathbb{N}}$.

Assumption 4.

(i) (nonincreasing) $0 \leq \gamma_{k+1} \leq \gamma_k \leq 1$, for all $k \geq 0$;
(ii) (nonsummable) $\sum_{k=0}^{\infty} \gamma_k = \infty$;
(iii) (square-summable) $\sum_{k=0}^{\infty} (\gamma_k)^2 < \infty$.

Denote by $x^*$ the optimal solution to the aggregative optimization problem (1). We then present the main convergence result of this paper.

Theorem 1. Let Algorithm 1 be applied to the problem (1), where the step size $\gamma_k$ satisfies Assumption 4. Suppose that Assumptions 1–3 hold. Then

$$\lim_{k \to \infty} f(x_k) = f(x^*).$$

Remark 3. Theorem 1 provides sufficient conditions for distributed projection-free method, under which Algorithm 1 can find the optimal point of the aggregative optimization problem (1). Compared with the distributed algorithm proposed in Reference 32 for the aggregative optimization problem, we consider aggregative optimization problem with constraints, and Algorithm 1 replaces the costly projection step in PG based algorithms with a constrained linear optimization. We will further show the advantage of Algorithm 1 in terms of saving computation time through simulation.

4 PROOF OF MAIN RESULT

We first establish bounds on the consensus error of the aggregate and gradient tracking estimate measured by $\|\delta(x_k) - \hat{v}_{k+1}\|$ and $\|\hat{v}_{k+1} - 1 \otimes \hat{y}_k\|$, respectively, which will play an important role in the proof of Theorem 1. Similar to Lemma 2, we have the following result.
Lemma 2. Let Assumption 3 hold. Then there exist
\[
\tilde{v}_k = \frac{1}{N} \sum_{i=1}^{N} v_{i,k} = \frac{1}{N} \sum_{i=1}^{N} \phi_i(x_{i,k}),
\]  
(14)
\[
\tilde{y}_k = \frac{1}{N} \sum_{i=1}^{N} y_{i,k} = \frac{1}{N} \sum_{i=1}^{N} \nabla_v g_i(x_{i,k}, v_{i,k}).
\]  
(15)

Proof. In view of (12) and double-stochasticity in Assumption 3, multiplying \( \frac{1}{N} \mathbf{1}^T \) on both sides of (12) can lead to that
\[
\bar{v}_{k+1} = \bar{v}_k + \frac{1}{N} \sum_{i=1}^{N} \phi_i(x_{i,k+1}) - \frac{1}{N} \sum_{i=1}^{N} \phi_i(x_{i,k}),
\]
which further implies that
\[
\bar{v}_k - \frac{1}{N} \sum_{i=1}^{N} \phi_i(x_{i,k}) = \bar{v}_0 - \frac{1}{N} \sum_{i=1}^{N} \phi_i(x_{i,0}),
\]
combining the above equality and \( v_{i,0} = \phi_i(x_{i,0}) \) yields the first assertion of this lemma.

Secondly, we prove (15) by induction. Since the estimates are initialized as \( y_{i,0} = \nabla_v g_i(x_{i,0}, v_{i,0}) \) for all \( i \in \mathcal{N} \), we have \( \bar{y}_0 = \frac{1}{N} \sum_{i=1}^{N} \nabla_v g_i(x_{i,0}, v_{i,0}) \). At step \( k \), we assume that \( \bar{y}_k = \frac{1}{N} \sum_{i=1}^{N} \nabla_v g_i(x_{i,k}, v_{i,k}) \). We need to show that relation (15) holds at step \( k + 1 \):
\[
\bar{y}_{k+1} = \frac{1}{N} (\mathbf{1}^T \otimes I_d)(((W_k \otimes I_d) y_k) + \nabla_2 g(x_{i,k+1}, v_{i,k+1}) - \nabla_2 g(x_k, v_k))
\]
\[
= \frac{1}{N} (\mathbf{1}^T \otimes I_d)(((W_k \otimes I_d) y_k) + \frac{1}{N} \sum_{i=1}^{N} \nabla_v g_i(x_{i,k+1}, v_{i,k+1}) - \frac{1}{N} \sum_{i=1}^{N} \nabla_v g_i(x_{i,k}, v_{i,k}))
\]
\[
= \bar{y}_k + \frac{1}{N} \sum_{i=1}^{N} \nabla_v g_i(x_{i,k+1}, v_{i,k+1}) - \frac{1}{N} \sum_{i=1}^{N} \nabla_v g_i(x_{i,k}, v_{i,k})
\]
\[
= \frac{1}{N} \sum_{i=1}^{N} \nabla_v g_i(x_{i,k+1}, v_{i,k+1}),
\]
where the first equality follows by the update rule of \( y' \)'s in (8) of Algorithm 1, the second follows from definition of \( \bar{y}_k \), that is, \( \bar{y}_k = \frac{1}{N} (\mathbf{1}^T \otimes I_d) y_k \), the third follows by Assumption 3. While the last equality follows from the induction step \( k \).

Then, we establish a bound on the consensus error \( \|\delta(x_k) - \hat{v}_{i,k+1}\| \) of the aggregate. This proof is similar to Reference 33, we state the proof of this proposition based on Algorithm 1 to ensure the integrity of the work. For each \( i \in \mathcal{N} \), we define
\[
M_i \triangleq \max_{x_i \in X_i} \|x_i\|, \quad M_H \triangleq \sum_{j=1}^{N} M_j, \quad \rho_i \triangleq \max_{\theta, \theta' \in X_i} \|\theta - \theta'\|, \quad \rho \triangleq \max_{i \in \mathcal{N}} \rho_i.
\]  
(16)
(17)

Proposition 1. Consider Algorithm 1. Let Assumptions 1, and 3 hold. Then
\[
\|\delta(x_k) - \hat{v}_{i,k+1}\| \leq \theta M_H \rho^k + \theta N \rho \sum_{s=1}^{k} \rho^{k-s} \gamma_{s-1},
\]  
(18)
where the constants \( \theta, \beta \) are defined in (3).
Proof. From (14) it follows that
\[
\sum_{i=1}^{N} v_{i,k} = \sum_{i=1}^{N} \phi_i(x_{i,k}) = N \delta(x_k), \quad \forall k \geq 0.
\]  
(19)

Akin to Equation (16),\(^{33}\) we give an upper bound on \(\|\delta(x_k) - \hat{v}_{i,k+1}\|\). By combining (8) with (5), we have
\[
v_{i,k+1} = \sum_{j=1}^{N} [\Phi(k,0)]_{ij} v_{j,0} + \phi_i(x_{i,k+1}) - \phi_i(x_{i,k}) + \sum_{s=1}^{k} \sum_{j=1}^{N} [\Phi(k,s)]_{ij} (\phi_j(x_{j,s}) - \phi_j(x_{j,s-1})).
\]

Then by (8), we have
\[
\hat{v}_{i,k+1} = \sum_{j=1}^{N} [\Phi(k,0)]_{ij} v_{j,0} + \sum_{s=1}^{k} \sum_{j=1}^{N} [\Phi(k,s)]_{ij} (\phi_j(x_{j,s}) - \phi_j(x_{j,s-1})).
\]

By using (19), we have that
\[
\delta(x_k) = \frac{\sum_{j=1}^{N} v_{j,0}}{N} + \sum_{s=1}^{k} \sum_{j=1}^{N} \frac{1}{N} (\phi_j(x_{j,s}) - \phi_j(x_{j,s-1})).
\]

Therefore, we obtain the following bound.
\[
\|\delta(x_k) - \hat{v}_{i,k+1}\| \leq \sum_{j=1}^{N} \left| \frac{1}{N} - [\Phi(k,0)]_{ij} \right| \|v_{j,0}\|
\]
\[
+ \sum_{s=1}^{k} \sum_{j=1}^{N} \left| \frac{1}{N} - [\Phi(k,s)]_{ij} \right| \left\| \phi_j(x_{j,s}) - \phi_j(x_{j,s-1}) \right\|.
\]

Then by using (3), and \(v_{0,0} = x_{i,0}\), we obtain that
\[
\|\delta(x_k) - \hat{v}_{i,k+1}\| \leq \theta \rho^k \sum_{j=1}^{N} \|x_{j,0}\| + \theta \rho^k \sum_{s=1}^{k} \sum_{j=1}^{N} \left\| \phi_j(x_{j,s}) - \phi_j(x_{j,s-1}) \right\|.
\]  
(20)

Note that for any \(s \geq 1\) and each \(i \in \mathcal{N}\),
\[
\|\phi_i(x_{i,s}) - \phi_i(x_{i,s-1})\| \leq l_3 \|x_{i,s} - x_{i,s-1}\| = l_3 \|s_{i,s-1} - s_{i,s-1}\| \leq l_3 \gamma_{s-1} \rho.
\]  
(21)

where the first inequality follows by Assumption 2(d) (i.e., the \(l_1\)-Lipschitz continuous of \(\phi_i\)), and the second equality follows by (7), and the last inequality follows by the constraint set is convex and bounded as (17). By combining (21), (20) and (16), we prove (18). \( \blacksquare \)

Next, we establish a bound on the consensus error \(\|\hat{y}_{k+1} - 1 \otimes \bar{y}_k\|\) of the gradient tracking step, where \(\bar{y}_k\) is defined by Lemma 2.

**Proposition 2.** Consider Algorithm 1. Suppose that Assumptions 1–3 hold. Define \(C_k \triangleq \max_{i \in \mathcal{N}} \|\delta(x_k) - \hat{v}_{i,k+1}\|\). Then the following hold for all \(k \in \mathbb{N}\):
\[
\|\hat{y}_{k+1} - 1 \otimes \bar{y}_k\| \leq \theta \rho^k \|y_0\| + \sum_{s=1}^{k} \theta \rho^{k-s} (l_2 \bar{\rho} + l_2 l_3 \sqrt{N} \rho) \gamma_{s-1}
\]
\[
+ \sum_{s=1}^{k} l_2 l_3 (\sqrt{N} + 1) \rho \theta \rho^{k-s} \gamma_{s-1} + \sum_{s=1}^{k} l_2 \sqrt{N} \theta \rho^{k-s} (C_{s-1} + C_{s-2}),
\]  
(22)

where \(\theta, \beta\) are defined in (3), \(\bar{\rho}\) is the diameter of convex set \(X\) as in Assumption 1.
Proof. By telescoping (13), we obtain
\[ y_{k+1} = W_{d,k}(W_{d,k-1}y_{k-1} + \nabla g(x_k, v_k) - \nabla g(x_{k-1}, v_{k-1})) + \nabla g(x_{k+1}, v_{k+1}) - \nabla g(x_k, v_k) \]
\[ = \ldots \]
\[ = (\Phi(0, k) \otimes I_d)y_0 + \sum_{s=1}^k (\Phi(k, s) \otimes I_d)(\nabla g(x_s, v_s) - \nabla g(x_{s-1}, v_{s-1})) \]
\[ + \nabla g(x_{k+1}, v_{k+1}) - \nabla g(x_k, v_k). \quad (23) \]

By arranging (13), we can write \( W_{d,k}y_k = y_{k+1} - \nabla g(x_{k+1}, v_{k+1}) + \nabla g(x_k, v_k) \). Then by exploiting the equivalence in (23), we have
\[ W_{d,k}y_k = (\Phi(k, 0) \otimes I_d)y_0 + \sum_{s=1}^k (\Phi(k, s) \otimes I_d)(\nabla g(x_s, v_s) - \nabla g(x_{s-1}, v_{s-1})). \quad (24) \]

By Equation (15), we have \( \bar{y}_k = \frac{1}{N}(\mathbf{1}^T \otimes I_d)(\nabla g(x_s, v_s)), \forall s \geq 0 \), which leads to:
\[ \bar{y}_k = \frac{1}{N}(\mathbf{1}^T \otimes I_d)y_0 + \sum_{s=1}^N \frac{1}{N}(\mathbf{1}^T \otimes I_d)(\nabla g(x_s, v_s) - \nabla g(x_{s-1}, v_{s-1})). \quad (25) \]

From (24) and (25), we have the following:
\[
\begin{align*}
\| W_{d,k}y_k - 1 \otimes \bar{y}_k \| & = \| (\Phi(k, 0) - \frac{1}{N}\mathbf{1}^T)y_0 + \sum_{s=1}^k (\Phi(k, s) - \frac{1}{N}\mathbf{1}^T)(\nabla g(x_s, v_s) - \nabla g(x_{s-1}, v_{s-1})) \| \\
& \leq \left\| \Phi(k, 0) - \frac{1}{N}\mathbf{1}^T \right\| \| y_0 \| + \sum_{s=1}^k \left\| \Phi(k, s) - \frac{1}{N}\mathbf{1}^T \right\| \| \nabla g(x_s, v_s) - \nabla g(x_{s-1}, v_{s-1}) \| \\
& \leq \theta^k \| y_0 \| + \sum_{s=1}^k \theta^{k-s} \| \nabla g(x_s, v_s) - \nabla g(x_{s-1}, v_{s-1}) \|, \quad (26)
\end{align*}
\]
where the first inequality follows from the Kronecker and Cauchy–Schurz inequality, and the second inequality follows by Lemma 1.

Next, we find an upper bound for \( \| \nabla g(x_s, v_s) - \nabla g(x_{s-1}, v_{s-1}) \| \). Note by (21) that
\[
\| \phi(x_s) - \phi(x_{s-1}) \| = \sqrt{\sum_{i=1}^N \| \phi_i(x_{i,s}) - \phi_i(x_{i,s-1}) \|^2} \leq \sqrt{N}l_3\gamma_{s-1},
\]
\[
\| \delta(x_{s-1}) - \delta(x_{s-2}) \| = \frac{1}{N} \sum_{i=1}^N \left( \phi_i(x_{i,s-1}) - \phi_i(x_{i,s-2}) \right) \| \leq l_3\gamma_{s-2}.
\]

Then by recalling that \( \bar{\theta} := \max_{\theta, \theta' \in \mathbb{R}} \| \theta - \theta' \|_2^2 \), and using (11) and (12), we have:
\[
\begin{align*}
\| \nabla g(x_s, v_s) - \nabla g(x_{s-1}, v_{s-1}) \| & \leq l_2\| x_s - x_{s-1} \| + l_2\| v_s - v_{s-1} \| \\
& = l_2\| x_s - x_{s-1} \| + l_2\| v_s - v_{s-1} \| \\
& \quad + l_2\| \phi_s - \phi(x_s) - \Phi \| + 1 \otimes \delta(x_{s-1}) + 1 \otimes \delta(x_{s-1}) - 1 \otimes \delta(x_{s-2}) \\
& \quad + 1 \otimes \delta(x_{s-2}) - (v_{s-1} + \phi(x_{s-1}) - \Phi) \| \\
& \leq l_2\bar{\theta}\gamma_{s-1} + l_2\| \phi_s - 1 \otimes \delta(x_{s-1}) \| + l_2\| \phi(x_s) - \phi(x_{s-1}) \| + l_2\| \delta(x_{s-1}) - \delta(x_{s-2}) \| \\
& \quad + l_2\| 1 \otimes \delta(x_{s-2}) - v_{s-1} \| + l_2\| \phi(x_{s-1}) - \phi(x_{s-2}) \|
\end{align*}
\]
Now, we state a convergence property of Algorithm 1.

**Proposition 3.** Consider Algorithm 1. Let Assumptions 1–3 hold. Define \( h_k = f(x_k) - f(x^*) \) and \( c' = \max_{i \in A'} c_i \). Then

\[
 h_{k+1} \leq (1 - \gamma_k)h_k + \frac{L}{2}\gamma_k^2\rho^2 + \epsilon_{1,k}\gamma_k + \epsilon_{2,k}\gamma_k,
\]

where

\[
\epsilon_{1,k} = \bar{\rho} l_1 \sqrt{N} \left( \theta \beta^{k-1} M_{H} + \theta N \rho \sum_{i=1}^{k} \rho^{k-i} \gamma_{i-1} \right),
\]

\[
\epsilon_{2,k} = p c' \left( \theta \beta^k \|y_0\| + \sum_{i=1}^{k} \theta \rho^{k-i} (l_2 \bar{\rho} + l_2 l_3 \sqrt{N} \rho) \gamma_{i-1} + \sum_{i=1}^{k} l_2 l_3 \sqrt{N} \rho \gamma_{i-2} \\
+ \sum_{i=1}^{k} l_2 l_3 \sqrt{N} \beta^{k-i} (C_{k-1} + C_{k-2}) + \sqrt{N} l_2 \gamma_{k-1} \right) + \sqrt{N} l_2 l_3 \rho \gamma_{k-1} \right).
\]

**Proof.** Note by (11) that \( x_{k+1} - x_k = r_k (s_k - x_k) \). Then from the \( L \)-smoothness of \( f \) and the boundedness of \( X \), we have

\[
\begin{align*}
 f(x_{k+1}) &\leq f(x_k) + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L}{2} \| x_{k+1} - x_k \|^2 \\
 &\leq f(x_k) + \gamma_k \langle \nabla f(x_k), s_k - x_k \rangle + \frac{L}{2} \gamma_k^2 \rho^2,
\end{align*}
\]

where \( \bar{\rho} := \max_{\theta, \theta' \in X} \|\theta - \theta'\|^2_2 \).

Note that

\[
\nabla f(x_k) = \nabla_1 g(x_k, \delta(x_k)) + \nabla \phi(x_k) 1 \otimes \frac{1}{N} \sum_{i=1}^{N} \nabla_2 g_i(x_k, \delta(x_k)),
\]

where \( \delta(x_k) = 1 \otimes \delta(x_k) \). Thus, we see that for any \( s \in X \),

\[
\langle \nabla f(x_k), s \rangle = \langle \nabla_1 g(x_k, \bar{v}_{k+1}) + \nabla \phi(x_k) \bar{y}_{k+1}, s \rangle + \langle \nabla_1 g(x_k, \delta(x)) - \nabla_1 g(x_k, \bar{v}_{k+1}), s \rangle \\
+ \left( \nabla \phi(x_k) 1 \otimes \frac{1}{N} \sum_{i=1}^{N} \nabla_2 g_i(x_k, \delta(x)) - \nabla \phi(x_k) \bar{y}_{k+1}, s \right).
\]

Thus, we have

\[
\langle \nabla f(x_k), s_k - x^* \rangle = \langle \nabla_1 g(x_k, \bar{v}_{k+1}) + \nabla \phi(x_k) \bar{y}_{k+1}, s_k - x^* \rangle \\
+ \langle \nabla_1 g(x_k, \delta(x)) - \nabla_1 g(x_k, \bar{v}_{k+1}), s_k - x^* \rangle \\
+ \left( \nabla \phi(x_k) 1 \otimes \frac{1}{N} \sum_{i=1}^{N} \nabla_2 g_i(x_k, \delta(x_k)) - \nabla \phi(x_k) \bar{y}_{k+1}, s_k - x^* \right).
\]
\[
\leq (\nabla f(x_k), x_k - x^*) + \left( \nabla \phi(x_k) 1 \otimes \frac{1}{N} \sum_{i=1}^{N} \nabla g_i(x_{i,k}, \delta(x_k)) - \nabla \phi(x_k) \hat{y}_{k+1}, s_k - x^* \right)
\]
\[
\leq \bar{p} \| \nabla^2 g(x_{i,k}, \delta(x_k)) - \nabla^2 g(x_{i,k}, \hat{y}_{k+1}) \|
\]
\[
+ \bar{p} \left\| \frac{1}{N} \sum_{i=1}^{N} \nabla^2 g_i(x_{i,k}, \delta(x_k)) - \frac{1}{N} \sum_{i=1}^{N} \nabla^2 g_i(x_{i,k}, \hat{y}_{k+1}) \right\|.
\]
where the first inequality holds since \((\nabla^2 g(x_{i,k}, \delta(x_k)) + \nabla \phi(x_k) \hat{y}_{k+1}, s_k - x^*) \leq 0\) by (10), and the last inequality has utilized Assumption 1(a). Then by adding \((\nabla f(x_k), x^* - x_k)\) to both sides of (33) and using the triangle inequality, we have

\[
(\nabla f(x_k), s_k - x_k) \leq (\nabla f(x_k), x^* - x_k) + \bar{p} \| \nabla g(x_{i,k}, \delta(x_k)) - \nabla g(x_{i,k}, \hat{y}_{k+1}) \|
\]
\[
+ \bar{p} \left\| \frac{1}{N} \sum_{i=1}^{N} \nabla g_i(x_{i,k}, \delta(x_k)) - \hat{y}_{k+1} \right\|.
\]
where the last inequality has applied Assumption 2(a), Assumption 2(b), and \(c' = \max_{i \in A'} c_i\).

Next, we prove the bound of \(\| 1 \otimes \frac{1}{N} \sum_{i=1}^{N} \nabla g_i(x_{i,k}, \delta(x_k)) - \hat{y}_{k+1} \|\). By adding and subtracting \(\frac{1}{N} \sum_{i=1}^{N} \nabla g_i(x_{i,k}, v_{i,k})\), we have

\[
\left\| \hat{y}_{k+1} - 1 \otimes \frac{1}{N} \sum_{i=1}^{N} \nabla g_i(x_{i,k}, \delta(x_k)) \right\|
\]
\[
= \left\| \hat{y}_{k+1} - 1 \otimes \frac{1}{N} \sum_{i=1}^{N} \nabla g_i(x_{i,k}, v_{i,k}) + 1 \otimes \frac{1}{N} \sum_{i=1}^{N} \nabla g_i(x_{i,k}, v_{i,k}) - 1 \otimes \frac{1}{N} \sum_{i=1}^{N} \nabla g_i(x_{i,k}, \delta(x_k)) \right\|
\]
\[
= \left\| \hat{y}_{k+1} - 1 \otimes \bar{y}_k \right\| + \left\| 1 \otimes \frac{1}{N} \sum_{i=1}^{N} \nabla g_i(x_{i,k}, v_{i,k}) - 1 \otimes \frac{1}{N} \sum_{i=1}^{N} \nabla g_i(x_{i,k}, \delta(x_k)) \right\|
\]
\[
\leq \left\| \hat{y}_{k+1} - 1 \otimes \bar{y}_k \right\| + \left\| 1 \otimes \frac{1}{N} \sum_{i=1}^{N} \nabla g_i(x_{i,k}, v_{i,k}) - 1 \otimes \frac{1}{N} \sum_{i=1}^{N} \nabla g_i(x_{i,k}, \delta(x_{k-1})) \right\|
\]
\[
+ \left\| 1 \otimes \frac{1}{N} \sum_{i=1}^{N} \nabla g_i(x_{i,k}, \delta(x_{k-1})) - 1 \otimes \frac{1}{N} \sum_{i=1}^{N} \nabla g_i(x_{i,k}, \delta(x_k)) \right\|
\]

Then by using the triangle inequality and Assumption 2, we obtain that

\[
\left\| \hat{y}_{k+1} - 1 \otimes \frac{1}{N} \sum_{i=1}^{N} \nabla g_i(x_{i,k}, \delta(x_k)) \right\|
\]
\[
\leq \left\| \hat{y}_{k+1} - 1 \otimes \bar{y}_k \right\| + \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \| \nabla g_i(x_{i,k}, v_{i,k}) - \nabla g_i(x_{i,k}, \delta(x_{k-1})) \|
\]
\[
+ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \| \nabla g_i(x_{i,k}, \delta(x_k)) - \nabla g_i(x_{i,k}, \delta(x_{k-1})) \|.
\]
\[ \|\hat{y}_{k+1} - \mathbf{1} \otimes \bar{y}_k\| + \frac{1}{\sqrt{N}} \sum_{i=1}^{N} l_2 \| \delta(x_{k-1}) - v_{i,k}\| + \sqrt{N} l_2 \| \delta(x_k) - \delta(x_{k-1})\| \]

where the last inequality holds by the definition \( C_k = \max_{i \in [N]} \| \delta(x_k) - \hat{v}_{i,k+1} \| \) and (21). Then by applying Proposition 2 and recalling the definition of \( \epsilon_{2,k} \) in (30), we have

\[ \overline{p}_c \| \hat{y}_{k+1} - \mathbf{1} \otimes \bar{y}_k\| \leq \overline{p}_c \sqrt{N} \left( \theta \rho^{k-1} M_H + \theta N \rho \sum_{s=1}^{k} \rho^{k-s} r_{s-1} \right) = \epsilon_{1,k}. \]

Then by substituting (36) and (37) into (34), we derive

\[ \langle \nabla f(x_k), s_k - x_k \rangle \leq \langle \nabla f(x_k), x^* - x_k \rangle + \epsilon_{1,k} + \epsilon_{2,k}. \]

Therefore, by subtracting \( f(x^*) \) from both sides in the inequality (31), and using the above inequality, we obtain that

\[ h_{k+1} \leq h_k + \gamma_k \langle x^* - x_k, \nabla f(x_k) \rangle + \frac{L}{2} \gamma_k^2 \rho^2 + \epsilon_{1,k} \gamma_k + \epsilon_{2,k} \gamma_k. \]

Since \( f \) is convex, we observe \( \langle x^* - x_k, \nabla f(x_k) \rangle \leq -h_k \). This incorporating with (38) proves the proposition.

Next, we give the following convergence result of a recursive linear inequality.

**Proposition 4.** (Lemma 3)\(^\text{50}\) Let the nonnegative sequence \( \{ u_k \} \) be generated by \( u_{k+1} \leq q_k u_k + v_k \), where \( 0 \leq q_k \leq 1, v_k \geq 0 \). Suppose that

\[ \sum_{k=0}^{\infty} (1 - q_k) = \infty, \quad v_k / (1 - q_k) \to 0. \]

Then \( \lim_{k \to \infty} u_k = 0 \). In particular, if \( u_k \geq 0 \), then \( u_k \to 0 \)

**Lemma 3.** (Reference 40, Lemma 3) Let \( (\gamma_k)_{k \in [N]} \) be a scalar sequence.

(a) If \( \lim_{k \to \infty} \gamma_k = \gamma \) and \( 0 < r < 1 \), then \( \lim_{k \to \infty} \sum_{r=0}^{k} r^{k-r} \gamma_r = \gamma / (1 - r) \);

(b) If \( \gamma_k \geq 0 \) for all \( k \), \( \sum_{k=0}^{\infty} \gamma_k < \infty \) and \( 0 < r < 1 \), then \( \sum_{k=0}^{\infty} \sum_{r=0}^{k} r^{k-r} \gamma_r < \infty \).

Based on Propositions 1, 2, 3, and 4, we proceed to prove that the iterates generated by the proposed Algorithm 1 converge to the optimal point.

**Proof of Theorem 1.** Set \( q_k = 1 - \gamma_k \), and \( v_k = \frac{L}{2} \gamma_k^2 \rho^2 + \epsilon_{1,k} \gamma_k + \epsilon_{2,k} \gamma_k \). We will apply Proposition 4 to (28) to prove Theorem 1.
Firstly, we prove that
\[
\lim_{k \to \infty} \varepsilon_{1,k} + \varepsilon_{2,k} = 0.
\]
Since \(\lim_{k \to \infty} \gamma_k = 0\) by Assumption 4 and \(0 < \beta < 1\) by Lemma 1, we drive \(\lim_{k \to \infty} \sum_{s=1}^k \beta^{k-s} \gamma_s = 0\) and \(\lim_{k \to \infty} \sum_{s=1}^k \beta^{k-s} \gamma_s = 0\) by Lemma 3(a). Thus,
\[
\lim_{k \to \infty} \varepsilon_{1,k} = \lim_{k \to \infty} \bar{\rho} l_{i} \sqrt{N} \left( \theta \beta^{k-1} M_{H} + \theta N \rho \sum_{s=1}^k \beta^{k-s} \gamma_s \right) = 0.
\]
(40)

Then by definition \(C_k = \max_{i \in \mathcal{X}_k} \| \delta(x_i) - \hat{v}_{i,k} \|\) and using Proposition 1, we have \(\lim_{k \to \infty} C_k = 0\). Similarly, by using Lemma 3(a), we obtain that
\[
\lim_{k \to \infty} \sum_{s=1}^k \beta^{k-s} C_{s-1} = 0 \quad \text{and} \quad \lim_{k \to \infty} \sum_{s=1}^k \beta^{k-s} C_{s-2} = 0.
\]
Thus,
\[
\lim_{k \to \infty} \varepsilon_{2,k} = \lim_{k \to \infty} \rho \bar{c} \left( \theta \beta^k \| x_0 \| + \sum_{s=1}^k \theta \beta^{k-s} (l_2 \rho + l_3 \sqrt{N} \rho) \gamma_s \right) + \sum_{s=1}^k l_2 \sqrt{N} \theta \beta^{k-s} (C_{s-1} + C_{s-2}) + \sqrt{N} l_2 C_{k-1} + \left( \sqrt{N} + 1 \right) l_3 \rho \gamma_{k-1} = 0.
\]
(41)

From Assumption 4 it follows that \(\sum_{k=1}^k (1 - q_k) = k \sum_{k=1}^k \beta^k = \infty\) and \(\lim_{k \to \infty} \frac{\bar{c}}{\beta^k} \gamma_k = 0\). In summary, we have proved the condition (39) required by Proposition 4. Then by applying Proposition 4 to (28), we obtain \(\lim_{k \to \infty} f(x_k) - f(x^*) = 0\).

5 | NUMERICAL SIMULATION

In this section, we demonstrate the proposed algorithm by solving an example with \(N = 5\) agents for problem (1). Agent \(i\)'s local cost function is
\[
f_i(x_i, \delta(x)) = k_i (x_i - \chi_i)^2 + P(\delta(x)) x_i,
\]
(42)
where \(k_i\) is constant and \(\chi_i\) is the fixed entities for \(i = 1, \ldots, N\), and \(P(\sigma(x)) = a N \sigma(x) + p_0\) with \(\sigma(x) = \frac{1}{N} \sum_{i \in \mathcal{X}} x_i\). In simulation, the constraint set is set as an \(l_1\) norm ball constraint \(\Omega_i = \{x_i : \|x_i\|_1 \leq R_i\}\). Then, \(s_{i,k}\) in D-FWAGT admits a closed form solution
\[
s_{i,k} = \arg\min_{s \in \Omega_i} \langle s, d_{i,k} \rangle = R_i \cdot (-\text{sgn}[d_{i,k}]),
\]
with \(d_{i,k} \triangleq \nabla_{i} g_i(x_i, \hat{v}_{i,k+1}) + \nabla_{\phi}(x_i, \hat{v}_{i,k+1}) \hat{v}_{i,k+1}\) as in Algorithm 1. Let the feasible sets be \(\Omega_i = \{x_i \in \mathbb{R}^n : \|x_i\|_1 \leq 5, \|x_2\|_1 \leq 7, \|x_3\|_1 \leq 9, \|x_4\|_1 \leq 3, \|x_5\|_1 \leq 6\}\). And we have \(\text{col}(\chi_i) = [3, 5, 6, 1, 2]^T \otimes \mathbf{1}_n, \ a = 0.04, \ \text{and} \ p_0 = 5 \times \mathbf{1}_n\).

We set an undirected time-varying graph as the communication network. The graph at each iteration is randomly drawn from a set of three graphs, whose union graph is connected. Set the adjacency matrix \(W = [w_{ij}]\), where \(w_{ij} = 1\) for any \(i \neq j\) with \((i, j) \in E, w_{ii} = 1 - \sum_{j \neq i} w_{ij}\), and \(w_{ij} = 0\).

Figure 1 displays the convergence of the proposed algorithm, and it compares the effect of the step size of the FW-type algorithms on the convergence of the algorithm, by considering \(\gamma_k = 1/\sqrt{k}, \gamma_k = 1/k, \ \text{and} \ \gamma_k = 1/k^2\). Although the convergence rate of the algorithm is almost as fast for the three different decreasing step cases, too large decreasing step size could cause the result of the algorithm to be too far from the optimal solution.
To demonstrate the properties of our algorithm. We compare our Algorithm 1 with projection-based algorithm $(x_{i,k+1} = \Pi_{x}(x_{i,k} - \alpha V_{1}g_{i}(x_{i,k}, \hat{v}_{i,k+1}) + V\phi_{i}(x_{i,k}, \hat{y}_{i,k+1}))$, and select the dimensions of decision variables as the power of two. In Figure 2, the x-axis is for the real running time (CPU time) in seconds, while the y-axis is for the optimal solution errors in each algorithm. We learn from Figure 2 that as the dimension increases, the actual running time (CPU time) of the projection-based algorithm is significantly longer than that of the projection-free algorithm. The reason is that searching for poles on the boundary of a high-dimensional constraint set (solving a linear program) is faster than computing the projection of a high-dimensional constraint set (solving a quadratic program).

In addition, in Table 1, we list the average actual running time for solving the single-stage subproblem, that is, linear or quadratic programs. When the dimensional is low, the difference in time required to solve the linear program and the quadratic program on such constraint sets may not be too great. However, as the dimension explodes, solving the quadratic program becomes difficult in this case, but the time to solve the linear program hardly varies much. That is consistent with the advantages of the projection free approaches for large-scale problems.
6 | CONCLUSIONS

This paper proposes a distributed projection free gradient method for aggregative optimization problem based on FW method, and shows that the proposed method can achieve the convergence for the case of the cost function is convex. In addition, empirical results demonstrate that our method indeed brings speed-ups. It is of interest to explore how to give the convergence speed of the algorithm. And it is also of interest to explore the faster convergence rate projection-free algorithm for distributed aggregative optimization, and analysis the FW method to the other classes of network optimizations in distributed and stochastic settings.

CONFLICT OF INTEREST
The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

DATA AVAILABILITY STATEMENT
The data that support the findings of this study are available from the corresponding author upon reasonable request.

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TABLE 1 The average real running time of solving one-stage subproblems.

| Dimensions | \( n = 16 \) | \( n = 32 \) | \( n = 64 \) | \( n = 128 \) | \( n = 256 \) |
|------------|------------|------------|------------|------------|------------|
| D-FWAGT (msec) | 0.069 | 0.077 | 0.092 | 0.126 | 0.132 |
| PGA (msec) | 78.5 | 95.6 | 164.6 | 242.2 | 366.3 |
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