THE LIMIT SET FOR DISCRETE COMPLEX HYPERBOLIC GROUPS

ANGEL CANO, BINGYUAN LIU, AND MARLON M. LÓPEZ

ABSTRACT. Given a discrete subgroup $\Gamma$ of $PU(1, n)$ that acts by isometries on the unit complex ball $\mathbb{H}^n_C$. In this setting a lot of work has been done in order to understand the action of the group. However, when we look at the action of $\Gamma$ on all of $\mathbb{P}^n_C$ little or nothing is known. In this paper, we study the action in the whole projective space and we are able to show that its equicontinuity agrees with its Kulkarni discontinuity set. Moreover, in the non-elementary case, this set turns out to be the largest open set on which the group acts properly and discontinuously. It can be described as the complement of the union of all complex projective hyperplanes in $\mathbb{P}^n_C$ which are tangent to $\partial H^n_C$ at points in the Chen-Greenberg limit set $\Lambda_{CG}(\Gamma)$.

INTRODUCTION

The theory of Complex Kleinian groups is still in its early developments. One unsolved problem is about the existence of largest open sets where a given group acts properly discontinuously. Little is known about this problem, see for instance [CNS13, CS10, Fra05, Men15, SV02, SV03]. In this article, we answer this question in a special case. We do it for complex hyperbolic groups. In this case we prove:

Theorem 0.1. Let $\Gamma \subset PU(1, n)$ be discrete subgroup, then the Kulkarni limit set of $\Gamma$ can be described as the hyperplanes tangent to $\partial H^n_C$ at points in the Chen-Greenberg limit set of $\Gamma$, i.e.

$$\Lambda_{Kul}(\Gamma) = \bigcup_{p \in \Lambda_{CG}(\Gamma)} p^\perp.$$ 

Moreover, if $\Gamma$ is non-elementary, then $\Omega_{Kul}(\Gamma)$ agrees with the equicontinuity set of $\Gamma$ and is the largest open set on which the group acts properly and discontinuously.

This result was proven (essentially) by J. P. Navarrete in [Nav06] for $n=2$. In [CS10], J. Seade and one of the authors studied the higher dimensional case. They proved that, for $\Gamma$ as above, the region of equicontinuity is as stated in the theorem, and they asked whether the full statement of the theorem hold. In this article we answer the question affirmatively.

In the 2-dimensional scenario, the proof of Theorem 0.1 is a key step to show that in the “generic case” there is a well defined notion of limit set, see [CNS13], and we expect that in the higher dimensional setting we get a similar result. In a

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series of forthcoming articles, we will clarify this assertion, see [ACCML5, Uca].

The paper is organized as follows: in Section 1, we review some general facts and introduce the notation used along the text. In Section 2, we provide an example which depicts the problem of finding a maximum region where a group acts properly and discontinuously. In Section 3, we provide a lemma that helps us to describe the dynamic of compact sets. In Section 4, we construct a group of transformations, called the control group, which helps us to describe the induced dynamic in the Grassmanian $Gr(1, n)$. In Section 5, we provide a proof of the main theorem. Finally, in Section 6, we provide a relation between the control group and the Cartan angular invariant of triplets contained in the Chen-Greenberg limit set.

1. Preliminaries

1.1. Projective Geometry. The complex projective space $\mathbb{P}_n^\mathbb{C}$ is defined as:
$$\mathbb{P}_n^\mathbb{C} = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*,$$
where $\mathbb{C}^*$ acts by the usual scalar multiplication. This is a compact connected complex $n$-dimensional manifold equipped with the Fubini-Study metric $d_n$.

If $[\cdot] : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}_n^\mathbb{C}$ is the quotient map, then a non-empty set $H \subset \mathbb{P}_n^\mathbb{C}$ is said to be a projective subspace of dimension $k$ if there is a $\mathbb{C}$-linear subspace $\tilde{H}$ of dimension $k + 1$ such that $[\tilde{H} \setminus \{0\}] = H$. The projective subspaces of dimension $(n - 1)$ are called hyperplanes and the complex projective subspaces of dimension 1 are called lines. In this article, $e_1, \ldots, e_{n+1}$ will denote the standard basis for $\mathbb{C}^{n+1}$.

Given a set of points $P$ in $\mathbb{P}_n^\mathbb{C}$, we define:
$$\text{Span}(P) = \bigcap \{ l \subset \mathbb{P}_n^\mathbb{C} \mid l \text{ is a projective subspace containing } P \}.$$
Clearly, $\text{Span}(P)$ is a projective subspace of $\mathbb{P}_n^\mathbb{C}$. If $p, q$ are distinct points then $\text{Span}(\{p, q\})$ is the unique complex line passing through them. In this case, we will write $\langle p, q \rangle$ instead of $\text{Span}(\{p, q\})$.

1.2. Projective and Pseudo-projective Transformations. It is clear that every linear isomorphism of $\mathbb{C}^{n+1}$ defines a holomorphic automorphism of $\mathbb{P}_n^\mathbb{C}$. Also, it is well-known that every holomorphic automorphism of $\mathbb{P}_n^\mathbb{C}$ arises in this way. The group of projective automorphisms of $\mathbb{P}_n^\mathbb{C}$ is defined:
$$\text{PSL}(n + 1, \mathbb{C}) := GL(n + 1, \mathbb{C})/\mathbb{C}^*,$$
where $\mathbb{C}^*$ acts by the usual scalar multiplication. Then $\text{PSL}(n + 1, \mathbb{C})$ is a Lie group whose elements are called projective transformations. We denote by $[[\cdot]] : GL(n + 1, \mathbb{C}) \to \text{PSL}(n + 1, \mathbb{C})$ the quotient map. Given $\gamma \in \text{PSL}(n + 1, \mathbb{C})$, we say that $\tilde{\gamma} \in GL(n + 1, \mathbb{C})$ is a lift of $\gamma$ if $[[\tilde{\gamma}]] = \gamma$.

1.3. Complex Hyperbolic Groups. In the rest of the paper, we will be interested in studying those subgroups of $\text{PSL}(n + 1, \mathbb{C})$ preserving the unitary complex ball.
We start by considering the following Hermitian matrix:

\[
H = \begin{pmatrix} I_{n-1} & 1 \\ 1 & \end{pmatrix},
\]

where \(I_n\) denotes the identity matrix of size \((n - 1) \times (n - 1)\). We will set

\[
U(1, n) = \{ g \in GL(n + 1, \mathbb{C}) : gHg^* = H \}.
\]

and \(\langle , \rangle : \mathbb{C}^{n+1} \to \mathbb{C}\) the hermitian form induced by \(H\). Clearly, \(\langle , \rangle\) has signature \((1, n), U(1, n)\) is the the group preserving \(\langle , \rangle\), see [Ner11]. And the corresponding projectivization \(PU(1, n)\) preserves the unitary complex ball:

\[
\mathbb{H}_C^n = \{ [w] \in P^n_C \mid \langle w, w \rangle < 0 \}
\]

Given a group \(\Gamma \subset PU(1, n)\), we define the following notion of limit set due to Chen and Greenberg, see [CG74].

**Definition 1.1.** Let \(\Gamma \subset PU(1, n)\), then \(\Lambda_{CG}(\Gamma)\) is to be defined as as the set of accumulation points in \(\partial \mathbb{H}_C^n\) of the orbit of any point in \(\mathbb{H}_C^n\).

As in the Fuchsian groups case, it is not hard to show that \(\Lambda_{CG}(\Gamma)\) does not depend on the choice of \(x\) and \(\Lambda_{CG}(\Gamma)\) has either 1,2 or infinite points. A group is said to be non-elementary if \(\Lambda_{CG}(\Gamma)\) has infinite points.

In the following, given a projective subspace \(P \subset P^n_C\) we will define

\[
P^1 = \{ [w] \in \mathbb{C}^{n+1} \mid \langle w, v \rangle = 0 \text{ for all } v \in [P]^{-1} \} \setminus \{0\}.
\]

Also, we will say that \(P\) is a Lagrangian plane if there is a \(\mathbb{R}\)-vectorial subspace \(R \subset \mathbb{C}^{n+1}\) of dimension 3, such that \([R \setminus \{0\}] = \mathbb{P}\) and \(\langle w, v \rangle \in \mathbb{R}\) for each \(v, w \in R\). A non elementary group \(\Gamma \subset PU(1, n)\) is said to be \(\mathbb{C}\)-Fuchsian (resp. \(\mathbb{R}\)-Fuchsian) if there is a complex line (resp. a Lagrangian plane) \(\ell\) invariant under \(\Gamma\).

Let us define the Cartan angular invariant. Given \(((x_1, y_1), (x_2, y_2), (x_3, y_3)) \in \partial(\mathbb{H}_C^n)^3\) a triplet of distinct points, the Cartan angular invariant of this triplet is defined as

\[
\mathcal{A}((x_1, y_1), (x_2, y_2), (x_3, y_3)) = \arg(-\langle x, y, z \rangle \langle z, x \rangle).
\]

It can be shown, see [Go99], that \(2|\mathcal{A}((x_1, y_1), (x_2, y_2), (x_3, y_3))| \leq \pi\) and

1. we have \(2\mathcal{A}((x_1, y_1), (x_2, y_2), (x_3, y_3)) = \pm \pi\) if and only if the points \([x_1, y_1, z_1]\) lie in a complex line,
2. we have \(\mathcal{A}((x_1, y_1), (x_2, y_2), (x_3, y_3)) = 0\) if and only if the points \([x_1, y_1, z_1]\) lie in a Lagrangian plane.

We also, have the following result:

**Theorem 1.2.** Given \(((x_1, y_1), (x_2, y_2), (x_3, y_3)) \in \partial(\mathbb{H}_C^n)^3\) be triplets of distinct points we have \(\mathcal{A}((\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)) = \mathcal{A}((x_1, x_2, x_3))\) if and only if there is a transformation \(\gamma \in PU(1, n)\) such that \(\gamma([x_i]) = [\tilde{x}_i]\).

The following result will be useful, see [Kna02][Ner11].

**Theorem 1.3** (Cartan’s Decomposition). For every \(\gamma \in PU(1, n)\) there are elements \(k_1, k_2 \in K := PU(n + 1) \cap PU(1, n)\) and a unique \(\mu(\gamma) \in PU(1, n)\), such
that $\gamma = k_1 \mu(\gamma) k_2$ and $\mu(\gamma)$ has a lift in $SL(n+1, \mathbb{C})$ given by

$$
\begin{pmatrix}
1 \\
\vdots \\
1 \\
e^{\lambda(\gamma)}
\end{pmatrix},
$$

where $\lambda(\gamma) \geq 0$.

1.4. **Pseudo-projective Transformation.** The space of linear transformations from $\mathbb{C}^{n+1}$ to $\mathbb{C}^{n+1}$, denoted by $M(n+1, \mathbb{C})$, is a linear complex space of dimension $(n+1)^2$, where $GL(n+1, \mathbb{C})$ is an open dense set in $M(n+1, \mathbb{C})$. Then $PSL(n+1, \mathbb{C})$ is an open dense set in $QP(n+1, \mathbb{C}) = (M(n+1, \mathbb{C}) \setminus \{0\})/\mathbb{C}^*$ called the space of pseudo-projective maps. Let $\tilde{M} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ be a non-zero linear transformation. Let $Ker(M)$ be its kernel and $Ker([\tilde{M}])$ denote the respective projectivization, then $\tilde{M}$ induces a well defined map $[\tilde{M}] : \mathbb{P}_\mathbb{C}^n \setminus Ker([\tilde{M}]) \to \mathbb{P}_\mathbb{C}^n$ by:

$$
[[\tilde{M}]]([v]) = [\tilde{M}(v)].
$$

The following result provides a link between pointwise convergence in $QP(n+1, \mathbb{C})$ and uniform convergence in a projective space.

**Proposition 1.4** (See [CS10]). Let $(\gamma_m)_{m \in \mathbb{N}} \subset PSL(n+1, \mathbb{C})$ be a sequence of distinct elements, then

1. There is a subsequence $(\tau_m)_{m \in \mathbb{N}} \subset (\gamma_m)_{m \in \mathbb{N}}$ and $\tau_0 \in M(n+1, \mathbb{C}) \setminus \{0\}$ such that $\tau_m \xrightarrow{m \to \infty} \tau_0$ as points in $QP(n+1, \mathbb{C})$.

2. If $(\tau_m)_{m \in \mathbb{N}}$ is the sequence given by the previous part of this lemma, then $\tau_m \xrightarrow{m \to \infty} \tau_0$, as functions, uniformly on compact sets of $\mathbb{P}_\mathbb{C}^n \setminus Ker(\tau_0)$.

1.5. **The Grassmanians.** Let $0 \leq k < n$, we define the Grassmanian $Gr(k, n)$ as the space of all $k$-dimensional projective subspaces of $\mathbb{P}_\mathbb{C}^n$ endowed with the Hausdorff topology. One has that $Gr(k, n)$ is a compact, connected complex manifold of dimension $k(n-k)$. A method to realize the Grassmanian $Gr(k, n)$ as a subvariety of the projective space of the $(k+1)$-th exterior power of $\mathbb{C}^{n+1}$, in symbols $P(\wedge^{k+1} \mathbb{C}^{n+1})$ is done by the so called Plücker embedding which is given by:

$$
\iota : Gr(k, n) \to P(\wedge^{k+1} \mathbb{C}^{n+1})
$$

$$
\iota(V) \mapsto [v_1 \wedge \cdots \wedge v_{k+1}]
$$

where $Span(\{v_1, \cdots, v_{k+1}\}) = V$. We can make $PSL(n+1, \mathbb{C})$ act on $Gr(k, n)$ and $P(\wedge^{k+1} \mathbb{C}^{n+1})$, which makes $\iota$ a $PSL(n+1, \mathbb{C})$-equivalent embedding.

1.6. **Kulkarni Limit Set.** When we look at the action of a group acting on a general topological space, in general there are no natural notions of limit set. Hence, we study the notion introduced by Kulkarni.

**Definition 1.5** (see [Kul78]). Let $\Gamma \subset PSL(n+1, \mathbb{C})$ be a subgroup. We define

1. The set $L_0(\Gamma)$ as the closure of the set of points in $\mathbb{P}_\mathbb{C}^n$ with infinite isotropy group.
(2) The set \( L_1(\Gamma) \) as the closure of the set of cluster points of \( \{g(z) : g \in \Gamma \} \) where \( z \) runs over \( \mathbb{P}_n^\mathbb{C} \setminus L_0(\Gamma) \).
(3) The set \( \Lambda(\Gamma) = L_0(\Gamma) \cup L_1(\Gamma) \).
(4) The set \( L_2(\Gamma) \) as the closure of cluster points of \( \Gamma K \) where \( K \) runs over all the compact sets in \( \mathbb{P}_n^\mathbb{C} \setminus \Lambda(\Gamma) \).
(5) The Kulkarni’s limit set of \( \Gamma \) as:
\[ \Lambda_{\text{Kul}}(\Gamma) = \Lambda(\Gamma) \cup L_2(\Gamma) \]
(6) The Kulkarni’s discontinuity region of \( \Gamma \) as:
\[ \Omega_{\text{Kul}}(\Gamma) = \mathbb{P}_n^\mathbb{C} \setminus \Lambda_{\text{Kul}}(\Gamma) \]

The Kulkarni’s limit set has the following properties, for a more detailed discussion on this topic in the 2 dimensional setting see [CNS13].

Proposition 1.6 (See [CNS13, CS10, Kul78]). Let \( \Gamma \) be a complex Kleinian group. Then:
(1) The sets \( \Lambda_{\text{Kul}}(\Gamma), \Lambda(\Gamma), L_2(\Gamma) \) are \( \Gamma \)-invariant closed sets.
(2) The group \( \Gamma \) acts properly discontinuously on \( \Omega_{\text{Kul}}(\Gamma) \).
(3) Let \( \mathcal{C} \subset \mathbb{P}_n^\mathbb{C} \) be a closed \( \Gamma \)-invariant set such that for every compact set \( K \subset \mathbb{P}_n^\mathbb{C} \setminus \mathcal{C} \), the set of cluster points of \( \Gamma K \) is contained in \( \Lambda(\Gamma) \cap \mathcal{C} \), then \( \Lambda_{\text{Kul}}(\Gamma) \subset \mathcal{C} \).
(4) The equicontinuity set of \( \Gamma \) is contained in \( \Omega_{\text{Kul}}(\Gamma) \).

2. A starting example

Let us consider the element \( \gamma \in PU(1, n) \) induced by the following matrix
\[ \widetilde{\gamma} = \begin{pmatrix} 2 & I_{n-1} & \frac{1}{2} & -1 \\ \end{pmatrix} \]
and \( \Gamma = \langle \langle \gamma \rangle \rangle \). It is not hard to show that
\[ Eq(\Gamma) = \mathbb{P}_n^\mathbb{C} \setminus (\{e_1\}^\perp \cup \{e_{n+1}\}^\perp) = \Omega_{\text{Kul}}(\Gamma) \].

On the other hand, when one tries to determine if the previous sets are maximal open sets on which \( \Gamma \) acts properly discontinuously, we find the following phenomena: given \( U, W \subset \text{Span}(\{e_2, \ldots, e_n\}) = \mathcal{L} \) disjoint open sets such that \( \overline{U} \cup \overline{W} = \mathcal{L} \), define
\[ U = \{\frac{e_1+v}{\|v\|} : v \in \overline{U}\} \]
\[ V = \{\frac{e_{n+1}+v}{\|v\|} : v \in \overline{V}\} \].

Is not hard to show \( \mathbb{P}_n^\mathbb{C} \setminus (U \cup V) \) is a maximal open set on which \( \Gamma \) acts properly discontinuously and every maximal open set for the action of \( \Gamma \) arises in this way.

3. The lambda lemma

In this section, we develop a tool that will enable us to determine the accumulation points of the orbit of compact sets, compare results with those in [Fra05, Men15].

Definition 3.1. Let \( (\gamma_m) \subset PU(1, n) \) be a sequence of distinct elements, we will say that \( (\gamma_m) \) tends simply to the infinite if:
(1) The sequences of compact factors in the Cartan decomposition of \((\gamma_m)\) converge in \(U(n + 1) \cap U(1, n)\).

(2) The sequence \(\lambda(\gamma_m)\) converges to infinity.

Given a discrete group \(\Gamma \subset PU(1, n)\) and a sequence \((\gamma_m) \subset \Gamma\) of distinct elements, there is a subsequence \((\tau_m) \subset (\gamma_m)\) tending simply to infinity.

**Definition 3.2.** Let \((\gamma_m) \subset PU(1, n)\) be a sequence tending simply to infinity and \(x \in \mathbb{P}_C^n\), we define:

\[
D_{(\gamma_m)}(x) = \bigcup \{\text{accumulation points of } (\gamma_m(x_m))\}.
\]

The union is taken over all sequences converging to \(x\).

Clearly, if \(\Gamma \subset PU(1, n)\) is a discrete group, \(\Omega \subset \mathbb{P}_C^n\) is an open set on which \(\Gamma\) acts properly discontinuously, \(x \in \Omega\) and \((\gamma_m) \subset \Gamma\) is any sequence tending simply to infinity, then \(D_{(\gamma_m)}(x) \subset \mathbb{P}_C^n\).

Before we provide the main result of this section, let us make a brief digression to some projective geometry.

**Definition 3.3.** Given a hyperplane \(H \subset \mathbb{P}_C^n\) and a point \(p \in H\) we define

\[
H(p) = \{\ell \in Gr_1(\mathbb{P}_C^n) : p \in \ell \subset H\}.
\]

The following proposition is straightforward.

**Lemma 3.4.** Given \(p \in \partial \mathbb{H}_C^n\), define the following function in \(p^+(p) \times p^+(p)\):

\[
d_p(\ell_1, \ell_2) = \arccos \left( \frac{\langle q_1, q_2 \rangle \langle q_1, q_2 \rangle}{\langle q_1, q_1 \rangle \langle q_2, q_2 \rangle} \right)
\]

where \(q_1, q_2 \in \mathbb{C}^{n+1} \setminus \{0\}\) are points satisfying \(\ell_i = \frac{1}{\langle q_i, q_i \rangle} q_i\). Then \((p^+(p), d_p)\) is a metric space isometric to \((\mathbb{P}_C^{n-1}, d_{n-1})\).

**Definition 3.5.** Given \(p, q \in \partial \mathbb{H}_C^n\), we will denote by \(Isom(p, q)\) the set of isometries from \((p^+(p), d_p)\) to the space \((q^+(q), d_q)\)

The following proposition will be crucial along the paper, see also [CS10, Fra05, Men15, Nav06].

**Proposition 3.6.** Let \(\Gamma \subset PU(1, n)\) be a discrete group and \((\gamma_m) \subset \Gamma\) a sequence tending simply to infinity, then there are two pseudo projective transformations \(\tau, \vartheta\) such that

1. We have \(\gamma_m \xrightarrow{m \to \infty} \tau\) and \(\gamma_m^{-1} \xrightarrow{m \to \infty} \vartheta\).

2. The sets \(\text{Im}(\tau)\) and \(\text{Im}(\vartheta)\) are points in the Chen-Greenberg limit set satisfying \(\text{Im}(\tau)^{\perp} = \text{Ker}(\vartheta)\) and \(\text{Im}(\vartheta)^{\perp} = \text{Ker}(\tau)\).

Moreover, there is a projective equivalence \(\phi : \text{Im}(\vartheta)^{\perp} \circ (\text{Im}(\vartheta)) \to \text{Im}(\tau)^{\perp} \circ (\text{Im}(\tau))\) satisfying:

1. The equivalence \(\phi\) belongs to \(Isom(\text{Im}(\vartheta), \text{Im}(\tau))\).
2. Given \(\ell \in \text{Im}(\vartheta)^{\perp} \circ (\text{Im}(\vartheta))\), and \(y \in \ell \setminus \text{Im}(\vartheta)\), we know \(D_{(\gamma_m)}(x) = \phi(\ell)\).
(c) Given $\ell \in \text{Im}(\tau)^{\perp}(\text{Im}(\tau))$, and $y \in \ell \setminus \text{Im}(\tau)$, we know
\[ D_{(\gamma^{-1}_m)}(x) = \phi^{-1}(\ell). \]

Proof. Let us show part (1). By the Cartan’s decomposition theorem there are sequences $(\alpha_m) \in \mathbb{R}^+$, $(\kappa_m), (\tilde{\kappa}_m) \in K = U(n+1) \cap U(1, n)$, such that
\[ \gamma = \left[ \begin{array}{cc} e^{\alpha_m} & 1 \\ \vdots & \ddots \\ 1 & e^{-\alpha_m} \end{array} \right] \tilde{\kappa}_m. \]
Equation (3.1) shows that
\[ \gamma \gamma_m \xrightarrow{m \to \infty} \kappa_1 \kappa_2 \in K \text{ and } \alpha_m \xrightarrow{m \to \infty} \infty. \]
Equation (5.1) shows that
\[ \gamma_m \xrightarrow{m \to \infty} \tau = \left[ \begin{array}{cc} \kappa_1 & 0 \\ \vdots & \ddots \\ 0 & \kappa_2 \end{array} \right] \]
Equation (3.1) shows that
\[ \gamma^{-1}_m \xrightarrow{m \to \infty} \vartheta = \left[ \begin{array}{cc} \kappa_2^{-1} & 0 \\ \vdots & \ddots \\ 0 & \kappa_1^{-1} \end{array} \right], \]
and it shows part (1).

Let us show part (2). Observe that equation (5.2) yields
\[ \text{Im}(\tau) = \kappa_1(e_1) \]
\[ \text{Im}(\vartheta) = \kappa_2^{-1}(e_{n+1}) \]
\[ \text{Ker}(\tau) = \kappa_2^{-1}((\text{Span}(e_2, \ldots, e_{n+1})) \]
\[ \text{Ker}(\vartheta) = \kappa_1(\text{Span}(e_1, \ldots, e_n)), \]
which shows part (2).

Now let us show part (a). Let us define
\[ \phi : \text{Im}(\vartheta)^{\perp}(\text{Im}(\vartheta)) \to \text{Im}(\tau)^{\perp}(\text{Im}(\tau)) \]
\[ \phi(\ell) = \kappa_1(H(\kappa_2(\ell))). \]
it is not hard to see that $\phi$ is an isometry from $(\text{Im}(\vartheta)^{\perp}(\text{Im}(\vartheta)), d_{\text{Im}(\vartheta)})$ to the space $(\text{Im}(\tau)^{\perp}(\text{Im}(\tau)), d_{\text{Im}(\tau)}).$

Now let us show part (b). Let $\ell \in \text{Im}(\vartheta)^{\perp}(\text{Im}(\vartheta))$, $w \in \text{Im}(\vartheta)^{\perp} \setminus \text{Im}(\vartheta)$, and $(x_m) \subset \mathbb{C}$ such that $x_m \xrightarrow{m \to \infty} w$, then $\kappa_2(x_m) \xrightarrow{m \to \infty} \kappa_2(w)$, where $\kappa_2(w) \in (\text{Span}(e_2, \ldots, e_{n+1}) \setminus \{e_{n+1}\}, \kappa_2(x_m) = [x_{1m}, \ldots, x_{n+1,m}], \kappa_2(w) = [0, x_2, \ldots, x_{n+1}], \sum_{j=2}^n |x_j| \neq 0$ and $x_{jm} \xrightarrow{m \to \infty} x_j$. Then
\[
\begin{bmatrix}
1 & \cdots & 1 \\
0 & \ddots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
x_{1m} \\
x_{2m} \\
\vdots \\
x_{n,m}
\end{bmatrix}
= \begin{bmatrix}
e^{\alpha_m} x_{1m} \\
e^{\alpha_m} x_{2m} \\
\vdots \\
e^{\alpha_m} x_{n,m}
\end{bmatrix}
= w_m,
\]

thus the accumulation points of \((w_m)\) lie on the line \(e_1, [0, x_2, \ldots, x_n, 0]\), thus the accumulation points of \((\gamma_m(x_m))\) lie on the line \(\tilde{\imath} m(\tau), \kappa_1([0, x_2, \ldots, x_n, 0]) = \phi(\ell)\).

Now, the proof of this part follows from Equation 3.3.

In order to conclude the proof, one must observe that the proof of part (c) is similar to the proof of part (a). □

The proofs of the next two corollaries follow straightforwardly.

**Corollary 3.7 (See [Nav06]).** Let \(\Gamma \subset PU(2,1)\) be a discrete group non-elementary group, then

1. The Kulkarni discontinuity region \(\Omega_{\text{Kul}}(\Gamma)\) is the largest open set on which \(\Gamma\) acts properly and discontinuously and agrees with \(\text{Eq}(\Gamma)\).
2. The Kulkarni limit set of \(\Gamma\) can be described as the hyperplanes tangent to \(\partial H^2\) at points in the Chen-Greenberg limit set of \(\Gamma\), i.e.

\[
\Lambda_{\text{Kul}}(\Gamma) = \bigcup_{p \in \Lambda_{\text{CG}}(\Gamma)} p^\perp.
\]

**Corollary 3.8.** [See [GST03]] Let \(\Gamma \subset PU(1, n)\) be a discrete group, then the complement of the equicontinuity region of \(\Gamma\) are the hyperplanes tangent to \(\partial H^n\) at points in the Chen-Greenberg limit set of \(\Gamma\), i.e.

\[
\mathbb{P}_C^n \setminus \text{Eq}(\Gamma) = \bigcup_{p \in \Lambda_{\text{CG}}(\Gamma)} p^\perp.
\]

4. THE CONTROL GROUP

As we will see, in order to understand the dynamic of \(\Gamma\) it will suffice to understand the dynamic in \(Gr(1, n)\). Such thing can be done by introducing the action of an adequate group. Let us consider the following definition.

**Definition 4.1.** Let \(\Gamma \subset PU(1, n)\) be a discrete non-elementary group and \(x, y \in \Lambda_{\text{CG}}(\Gamma)\). A transformation \(\gamma : y^+(y) \to x^+(x)\) is called a Cartan map for \((y, x, \Gamma)\) if there is a sequence \((\gamma_m) \subset \Gamma\) tending simply to infinity such that

1. We have \(\gamma_m \xrightarrow{m \to \infty} x\) uniformly on compact sets of \(\mathbb{P}_C^n \setminus y^\perp\).
2. Also \(\gamma_m^{-1} \xrightarrow{m \to \infty} y\) uniformly on compact sets of \(\mathbb{P}_C^n \setminus x^+\).
3. For each \(\ell \in y^+(y)\) and each \(w \in \ell\) we have:

\[
D_{(\gamma_m)}(w) = \gamma(\ell)
\]
(4) For each \( \ell \in x^+(x) \) and each \( w \in \ell \) we have:
\[
D_{(\gamma_m^{-1})}(w) = \gamma^{-1}(\ell)
\]
The sequence \( (\gamma_m) \) is called the Cartan sequence associated to \( \gamma \).

The following lemma will be useful.

**Lemma 4.2** (See [Kam91].) Let \( \Gamma \subset PU(1, n) \) be a discrete non-elementary group and points \( x_+, x_- \in \Lambda_{CG}(\Gamma) \), then there is a sequence \( (\gamma_m) \subset \Gamma \) of distinct elements such that
\[
\gamma_m^{\pm 1} \xrightarrow{m \to \infty} x_\pm
\]
uniformly on compact set of \( \mathbb{H}^n_{\mathbb{C}} \).

**Lemma 4.3.** Let \( \Gamma \subset PU(1, n) \) be a discrete and non-elementary group and \( x, y \in \Lambda_{CG}(\Gamma) \) be distinct points, then
\[
\Gamma(y, x) = \{ \gamma : y^+(y) \to x^+(x) | \gamma \text{ is a cartan map for } (y, x, \Gamma) \}
\]
is a non-empty subset of \( Isom(y, x) \).

**Proof.** Let us show that \( \Gamma(y, x) \) is non-empty. Let \( (\gamma_m) \subset \Gamma \) be the sequence given by Lemma 4.2. By Lemma 3.6 we can assume that \( (\gamma_m) \) tends simply to infinity, \( \gamma_m \xrightarrow{m \to \infty} x \) uniformly on \( \mathbb{P}^n_{\mathbb{C}} \setminus y^+ \), \( \gamma_m^{-1} \xrightarrow{m \to \infty} y \) uniformly on \( \mathbb{P}^n_{\mathbb{C}} \setminus x^- \) and there is \( \mu : y^+(y) \to x^+(x) \) such that for each \( \ell \in y^+(y) \) (resp. \( \ell \in x^+(x) \)) and \( w \in \ell \setminus \{y\} \) (resp \( w \in \ell \setminus \{x\} \)). We have \( D(\gamma_m)(w) = \mu(\ell) \) (resp. \( D(\gamma_m^{-1})(w) = \mu^{-1}(\ell) \)). Therefore, \( \Gamma(y, x) \) is non-empty. \( \square \)

By means of the arguments used in the proof of the previous lemma, it is not hard to show that \( \Gamma(y, y) \) is a non-empty set of \( Isom(y, y) \).

**Definition 4.4.** Let \( \Gamma \subset PU(1, n) \) be a discrete group, \( x, y \in \Lambda_{CG}(\Gamma) \), \( \mu \in \Gamma(y, x) \), \( (\gamma_m) \subset \Gamma \) a Cartan sequence associated to \( \mu \) and \( T \) a projective transformation of \( \mathbb{P}^2_{\mathbb{C}} \) such that \( \mu = T(\gamma_m) \). If
\begin{enumerate}
    \item The Cartan decomposition of \( \gamma_m \) is \( k_m^+A_mk_m^- \).
    \item We have \( k_m^\pm \xrightarrow{m \to \infty} k^\pm \), where \( k^\pm, k^- \in K = PU(n + 1) \cap PU(1, n) \).
    \item Also \( T = k^+Hk^- \).
\end{enumerate}
Moreover, we will say that \( T \) is a Cartan extension for \( \mu \) if there is a Cartan sequence \( (\gamma_m) \) associated to \( \mu \) such that \( T \) is a Cartan extension for \( \mu \) with respect \( (\gamma_m) \).

The proof of the next lemma follows straightforwardly and we omit it here.

**Lemma 4.5.** Let \( \Gamma \subset PU(1, n) \) be a discrete group, \( x, y \in \Lambda_{CG}(\Gamma) \) and \( \mu \in \Gamma(y, x) \). If \( T \) is a Cartan extension for \( \mu \), then \( T(\ell) = \mu(\ell) \) for each \( \ell \in y^+(y) \).

**Lemma 4.6.** Let \( \Gamma \subset PU(1, n) \) be a discrete group and \( x, y \in \Lambda_{CG}(\Gamma) \) be distinct points, then \( \Gamma(y, x) \) is a closed subset of \( Isom(y, x) \).

**Proof.** Let \( (\mu_m) \subset \Gamma(y, x) \) and \( \mu \in Isom(y, x) \) such that \( \mu_m \xrightarrow{m \to \infty} \mu \). For each \( m \in \mathbb{N} \) let \( (\gamma_m) \subset \Gamma \) be a Cartan sequence associated to \( \mu_m \) satisfying:
\begin{enumerate}
    \item The Cartan decomposition of \( \gamma_{jm} \) is given by \( \gamma_{jm} = k_{jm}^+A_{jm}k_{jm}^- \).
    \item There are \( k_{jm}^+, k_{jm}^- \subset U(n + 1) \cap U(1, n) \) such that \( d(k_{jm}^+, k_{jm}^-) < 2^{-j} \). 
\end{enumerate}
Therefore, \( T_n = k_n^+ H k_n^- \) is a Cartan extension for \( \mu_n \) with respect \( (\gamma_{jm}) \). Since \( K \) is compact, there is a strictly increasing sequence \( (n_m) \subset \mathbb{N} \) and \( k_+, k_- \in K = PU(n + 1) \cap PU(1, n) \), such that
\[
d(k_n^+, k_n^-) < 2^{-i}.
\]
Therefore, if \( \ell \in y^+(y) \) then
\[
\mu_{n_m}(\ell) = T_{n_m}(\ell) \xrightarrow{m \to \infty} k_+(H(k_-(\ell))).
\]
Taking \( T = k_+ H k_- \), we have \( T(\ell) = \mu(\ell) \) for each \( \ell \in y^+(y) \). In order to conclude the proof, observe there is a strictly increasing sequence \( l_m \) such that \( (\gamma_{l_m}) \) is a Cartan sequence associated to \( \mu \).

\[\square\]

**Remark 4.7.** Given \( \Gamma \subset PU(1, n) \) a discrete subgroup and \( y \in \Lambda_{cul}(\Gamma) \), by means of the arguments used in the proof of the previous lemma, it is not hard to show that \( \Gamma(y, y) \) is a non-empty closed set of \( Isom(y, y) \).

Let us define the following binary operation in \( Bihol(p, q) \)

**Definition 4.8.** Let \( p, q \in \partial \mathbb{H}_\mathbb{C} \), let
\[
Bihol(p, q) = \{ \mu : p^+(p) \to q^+(q) | \mu \text{ is biholomorphic} \}
\]

We define the binary operation \( \ast \) on \( Bihol(p, q) \)
\[
\mu \ast \nu(\ell) = \begin{cases} 
\mu(\nu(\ell)) & \text{if } p = q \\
\mu(Span(\{\nu(\ell) \cap p^+(p), p\}) & \text{otherwise}
\end{cases}
\]

The following result will show that the operation \( \ast \) is quite usual.

**Lemma 4.9.** Given \( p, q \in \partial \mathbb{H}_\mathbb{C} \), the algebraic structure \( (Bihol(p, q), \ast) \) is a group isomorphic to \( (PSL(n-1, \mathbb{C}), \circ) \). Moreover, \( (Isom(p, q), \ast) \) is an isomorphic group to \( (PU(n-1), \circ) \).

**Lemma 4.10.** Let \( \Gamma \subset PU(1, n) \) be a non-elementary discrete group and \( x, y \in \Lambda_{cul}(\Gamma) \) be distinct points, then \( \Gamma(y, x) \) is closed under the operation \( \ast \). In particular, \( \Gamma(y, x) \) is a closed monoid.

**Proof.** Let \( \gamma_1, \gamma_2 \in \Gamma(y, x) \) and \( (\gamma_1) \), \( (\gamma_2) \) be the Cartan sequences associated to \( \gamma_1 \) and \( \gamma_2 \), respectively. A straightforward calculation shows \( \gamma_{1m} \gamma_{2m} \xrightarrow{m \to \infty} x \) uniformly on compact sets of \( \mathbb{P}^n_\mathbb{C} \setminus y^+ \) and \( \gamma_{2m}^{-1} \gamma_{1m}^{-1} \xrightarrow{m \to \infty} y \) uniformly on compact sets of \( \mathbb{P}^n_\mathbb{C} \setminus x^+ \). By Proposition 3.3 there is a strictly increasing sequence \( (n_m) \subset \mathbb{N} \) and \( \nu \in \Gamma(y, x) \) such that for every \( \ell \in y^+(y) \) and \( w \in \ell \setminus \{y\} \) we have
\[
D_{(\gamma_{1m} \gamma_{2m})(w)}(w) = \nu(\ell)
\]

Let \( \ell \in y^+(y) \), \( w \in \ell \setminus \{y\} \) and \( (w_m) \subset \mathbb{P}^n_\mathbb{C} \) such that \( w_m \xrightarrow{m \to \infty} w \). Also, let us assume that there are \( w_1, w_2 \in \mathbb{P}^n_\mathbb{C} \) such that
\[
\gamma_{2m}(w_m) \xrightarrow{m \to \infty} w_1, \quad \gamma_{1m}(\gamma_{2m}(w_m)) \xrightarrow{m \to \infty} w_2.
\]

Observe that in case \( w_1 \notin y^+ \), then \( w_2 = x \). On the other hand, when \( w_1 \in y^+ \), we have \( w_1 = \gamma_1(\ell) \cap y^+ \). Consequently, \( w_2 \in \gamma_1(Span(\{\gamma_2(\ell) \cap y^+, y\})) \). Therefore, \( \nu = \gamma_1 \ast \gamma_2 \). \[\square\]
Theorem 4.11. Let $\Gamma \subset PU(1, n)$ be a non-elementary discrete group and $x, y \in \Lambda_{CG}(\Gamma)$ be distinct points, then $\Gamma(y, x)$ is a compact Lie group.

Proof. We will show $\Gamma(y, x)$ is indeed a group. Since $(Isom(p, q), *)$ is isomorphic to $(PU(n-1), \circ)$. We will also show that $\Gamma(y, x)$ is a subgroup of $(PU(n-1), \circ)$ after identifying $(Isom(p, q), *)$ with $(PU(n-1), \circ)$. Please note $(Isom(p, q), *)$ is also compact.

Let $a \in \Gamma(y, x)$ be an arbitrary element, and consider the set

$$A := \{a, a^2, a^3, \ldots, a^n, \ldots\}.$$ 

One can easily see $A \subset \Gamma(y, x) \subset PU(n-1)$ is a closed Abelian subsemigroup. Since $PU(n-1)$ is compact, $A$ is also compact which implies $A$ is an Abelian abundant semigroup. That is, $A$ has minimal ideals and each ideal contains an idempotent element. Notice that $s = id$ is the only element in $PU(n-1)$ satisfying $s^2 = s$.

Now, let $I \subset A$ be a minimal ideal of $A$, and thus $id \in I$. It is clear that $I$ is a group because for any element $t \in I$, we have $tI = I = It$, since $I$ is a two-sided minimal ideal and cannot be smaller. By Green’s theorem (in context of Green’s relations), one can see $I$ is a subgroup of $A$. Since $I$ is an ideal with $id$, $I = A$. Now, $A$ is a group and $a^{-1} \in A \subset \Gamma(y, x)$ which completes the proof. \qed

Definition 4.12. Let $\Gamma \subset PU(1, n)$ be a non-elementary discrete subgroup, a point $y \in \Lambda_{Kw}(\Gamma)$ and $\mu_0 \in \Gamma(y, y)$. We will say that $\mu_0$ is a $y$-Cartan limit if there are:

1. A sequence of distinct elements $(y_m) \subset \Lambda_{Kw}(\Gamma)$ converging to $y$.
2. For each $m \in \mathbb{N}$ a Cartan map $\mu_m \in \Gamma(y_m, y)$.
3. For each $m \in \mathbb{N} \cup \{0\}$ a Cartan extension $T_m$ for $\mu_m$.

Such that $T_m$ converges to $T_0$ as projective transformations. Also, we will say that $(\mu_m) \subset \bigcup_{x, y \in \Lambda_{CG}(\Gamma)} \Gamma(x, y) = \mathcal{C}(\Gamma)$ converges to $\mu_0 \in \mathcal{C}(\Gamma)$, if for each $m \in \mathbb{N} \cup \{0\}$ there is a Cartan extension $T$ of $\mu_m$ such that $T_m \xrightarrow{m \to \infty} T_0$, as projective transformations.

Lemma 4.13. Let $\Gamma \subset PU(1, n)$ be a non-elementary discrete subgroup and $y \in \Lambda_{CG}(\Gamma)$, then

$$\Gamma(y) = \{\mu : y^\pm(y) \to y^\pm(y) \mid \mu \text{ is a finite composition of } y\text{-Cartan limits}\}$$

is a closed monoid contained in $\Gamma(y, x)$.

Proof. First, let us show that $\Gamma(y)$ is non-empty. Let $(y_m) \subset \Lambda_{CG}(\Gamma)$ be a sequence of distinct elements such that $y_m \xrightarrow{m \to \infty} y$. For each $m \in \mathbb{N}$ let $\mu_m \in \Gamma(y_m, x)$ and $(\gamma_{jm}) \subset \Gamma$ a Cartan sequence associated to $\mu_m$ such that:

1. The Cartan decomposition of $\gamma_{jm}$ is given by $\gamma_{jm} = k_{jm}^+A_{jm}k_{jm}^-$. 
2. There are $k_{jm}^+, k_{jm}^- \in K$ such that

$$d(k_{jm}^+, k_{jm}^-) < 2^{-j}.$$

Thus $T_m = k_{jm}^+Hk_{jm}^-$ is a Cartan extension of $\mu_m$. Since $K$ is compact there are $k_+, k_- \in U(n+1) \cap U(1, n)$ and a strictly increasing sequence $(n_m) \subset \mathbb{N}$ such that

$$d(k_{nm}^+, k_{nm}^-) < 2^{-n_m}.$$ 

Therefore,

$$T_{nm} \xrightarrow{m \to \infty} k_+Hk_- = T.$$
Corollary 4.14. The space $\mathcal{E}(\Gamma)$ is sequentially compact.

Clearly, $T(y^+(y)) = y^+(y)$. Now, let $(l_m) \subset \mathbb{N}$ be a strictly increasing sequence such that $(\gamma_{l_m})$ is a sequence tending simply to infinity. Then is straightforward that $(\gamma_{l_m})$ is a Cartan sequence associated to $T_{|y^+(y)}$, which concludes this part of the proof.

In order to conclude the proof, it will suffice to show that the product of two $y$-Cartan limits is in $\Gamma(y, y)$. Let $\mu_1, \mu_2 \in Isom(y, y)$ be $y$-Cartan limit maps, then for each $i \in \{1, 2\}$ there are sequences $(y_{mi}) \subset \mathcal{ACC}(\Gamma)$ and $(\mu_{mi}) \in \mathcal{E}(\Gamma)$ such that $(y_{mi})$ is a sequence of distinct elements converging to $y$, $\mu_{mi} \in \Gamma(y_{mi}, y)$ and $\mu_{mi} \xrightarrow{m \to \infty} \mu_i$. For each $m \in \mathbb{N}$ and $i \in \{1, 2\}$ let $(\gamma_{jmi})_{j \in \mathbb{N}}$ be a Cartan sequence associated to $\mu_{mi}$ satisfying:

1. The Cartan decomposition of $\gamma_{jmi}$ is $k_{jmi}^+ A_{jmi} k_{jmi}^-$.  
2. There are $k_{mi}, k_{mi}^- \in K$ such that
   \[d(k_{jmi}^+, k_{jmi}^-) < 2^{-j} \] 
3. There are $k_i^+, k_i^- \in K$ such that
   \[d(k_{mi}^+, k_{mi}^-) < 2^{-m} \] 
4. The projective transformation, $T_{mi} = k_{mi}^+ H k_{mi}^-$, is a Cartan extension of $\mu_{mi}. \]
5. The projective transformation, $T_\ell = k_\ell^+ H k_\ell^-$, is a Cartan extension of $\mu_\ell$.

For each $m \in \mathbb{N}$, let us consider the sequence $(\tau_{jm})_{j \to \infty}$ uniformly on $P^n \setminus y_{m2}$ and $(\tau_{jm}^{-1})_{j \to \infty}$ uniformly on $P^n \setminus y^+$. Let $n_m \subset \mathbb{N}$ be a strictly increasing sequence such that $(\tau_{n_m})$ is a sequence tending simply to infinite, then $(\gamma_{n_m})$ is a Cartan sequence associated to $\mu$ and

\[
\begin{align*}
\tau_{n_m} & \xrightarrow{m \to \infty} y \text{ uniformly on } P^n \setminus y^+ \\
\tau_{n_m}^{-1} & \xrightarrow{m \to \infty} y \text{ uniformly on } P^n \setminus y^+.
\end{align*}
\]

By Proposition 3.3, there is $\nu \in \Gamma(y, y)$ such that for every $\ell \in y^+(y)$ and $w \in \ell \setminus \{y\}$ we have

\[
\begin{align*}
D_{(\gamma_{n_m} 1 \gamma_{n_m} 2)}(w) & = \nu(\ell) \\
D_{(\gamma_{n_m} 1 \gamma_{n_m} 2)}^{-1}(w) & = \nu^{-1}(\ell).
\end{align*}
\]

Let $\ell \in y^+(y)$, $w \in \ell \setminus \{y\}$ and $(w_m) \subset P^n_\mathbb{C}$ such that $w_m \xrightarrow{m \to \infty} w$, also let us assume that there are $w_1, w_2 \in P^n_\mathbb{C}$ such that

\[
\begin{align*}
\gamma_{n_m 2}(w_m) & \xrightarrow{m \to \infty} w_1 \\
\gamma_{n_m 1}(\gamma_{n_m 2}(w_m)) & \xrightarrow{m \to \infty} w_2.
\end{align*}
\]

Observe that in case $w_1 \notin y^+$, then $w_2 = y$. On the other hand, when $w_1 \in y^+$, we have $w_1 \in \mu_2(\ell)$. Consequently, $w_2 \in \mu_1(\mu_2(\ell))$, therefore $\nu = \mu_1 \circ \mu_2$. 

We have the following straightforward results.

Corollary 4.14. The space $\mathcal{E}(\Gamma)$ is sequentially compact.
Corollary 4.15. Let \((y_m) \subset \Lambda_{CG}(\Gamma)\) be a sequence of distinct elements converging to \(y\), the there is a subsequence \((z_m) \subset (y_m)\), such that the identity \(I_m \in \Gamma(z_m)\) converges to the identity \(I \in \Gamma(y, y)\).

Before we end this section, let us show that indeed \(\Gamma(y)\) is a group.

**Theorem 4.16.** Let \(\Gamma \subset PU(1, n)\) be a non-elementary discrete subgroup and \(y \in \Lambda_{CG}(\Gamma)\), then \(\Gamma(y) = \Gamma(y, y)\). In particular \(\Gamma(y)\) is a compact Lie subgroup naturally embedded in \(PU(n - 1)\).

**Proof.** In order to conclude the proof, it will suffice to show that every \(\mu_{-1} \in \Gamma(y, y)\) is indeed a \(y\)-Cartan limit. Let \((y_m) \subset \Lambda_{CG}(\Gamma)\) be a sequence of distinct elements converging to \(y\), such that if \(\mu_m \in \Gamma(y_m, y_m)\) is the identity of \((Bihol(y_m, y_m), \ast)\). Then \(\mu_m \xrightarrow{m \to \infty} \mu_0\), where \(\mu_0\) is the identity of \((Bihol(y, y), \ast)\). For each \(m \in \mathbb{N}\) let us define

\[
\mu_{-1} \ast \mu_m : y_m^\perp \to y^\perp(y)
\mu_{-1}(\text{Span}(\{\mu_m(\ell) \cap y^\perp(\ell), \ell\}), \mu_m)
\]

as in the proof of Lemma \[4.10\] one can easily show that \(\mu \ast \mu_m \in \Gamma(y_m, y)\). For each \(m \in \mathbb{N}\), let \(S_m\) be a Cartan extension of \(\nu_m = \mu \ast \mu_m\). By Corollary \[4.15\] we can assume that there is \(S_0 \in \mathcal{E}(\Gamma)\) such that \(S_m \xrightarrow{m \to \infty} S_0\) as projective transformations. Moreover, we can say that there is \(\nu_0 \in \Gamma(y)\) such that \(S_0\) is the Cartan extension of \(\nu_0\). Consequently, \(\nu_m \xrightarrow{m \to \infty} \nu_0\). For each element \(m \in \mathbb{N} \cup \{0, -1\}\), let \((\gamma_{jm})_{j \in \mathbb{N}}\) be a Cartan sequence associated to \(\mu_m\) satisfying:

1. The Cartan decompositions of \(\gamma_{jm}\) and \(\gamma_{j-1} \gamma_{jm}\), respectively, are \(k_{jm}^+ A_{jm} k_{jm}^-\) and \(r_{jm}^+ B_{jm} r_{jm}^-\).
2. There are \(k_{m}^+, k_{m}^-, r_{m}^+, r_{m}^- \in K\) such that
   \[\max\{d(k_{jm}^+, k_{jm}^-), d(r_{jm}^+, r_{jm}^-)\} < 2^{-j}\]
3. There are \(k_{0}^+, k_{0}^-, r_{0}^+, r_{0}^- \in K\) such that
   \[\max\{d(k_{m}^+, k_{m}^-), d(r_{m}^+, r_{m}^-)\} < 2^{-m}\]
4. We have \(T_m = k_m^+ H k_m^-\) and \(S_m = r_m^+ H r_m^-\).

Let us consider \(\sigma_m = \gamma_{m-1} \gamma_{mm}\), then it is not hard to show that \(\sigma_m \xrightarrow{m \to \infty} y\), uniformly on compact sets of \(\mathbb{P}_C^n \setminus y^\perp\). Let \(\ell \in y^\perp(y), w \in \ell \setminus \{y\}\) and \((w_m) \subset \mathbb{P}_C^n\) such that \(w_m \xrightarrow{m \to \infty} w\). Also, let us assume that there are \(w_1, w_2 \in \mathbb{P}_C^n\) such that

\[
\gamma_{mm}(w_m) \xrightarrow{m \to \infty} w_1
\gamma_{m-1} \gamma_{mm}(w_m) \xrightarrow{m \to \infty} w_2.
\]

In case \(w_1 \in \ell\), then \(w_2 \in \mu_{-1}(\ell)\). Therefore, \(\nu_0 = \mu_{-1} \ast \mu_0\). \(\square\)

5. **Proof of the main theorem**

Let us consider the following result:

**Theorem 5.1.** Let \(\Gamma \subset PU(1, n)\) be a non-elementary discrete subgroup, then

1. The Kulkarni discontinuity region \(\Omega_{Kul}(\Gamma)\) is the largest open set on which \(\Gamma\) acts properly and discontinuously and agrees with \(Eq(\Gamma)\).
The Kulkarni limit set of $\Gamma$ can be described as the union of hyperplanes tangent to $\partial \mathbb{H}^n_c$ at points in the Chen-Greenberg limit set of $\Gamma$, i.e.,

$$\Lambda_{Kul}(\Gamma) = \bigcup_{p \in \Lambda_{CG}(\Gamma)} p^\perp.$$

**Proof.** On the contrary, let us assume that $Eq(\Gamma) \neq \Omega_{Kul}(\Gamma)$. By Proposition 1.6 and Corollary 3.8, we conclude that there is $p \in \Lambda_{CG}(\Gamma)$ such that $p^\perp \not\subset \Lambda_{Kul}(\Gamma)$. Then there is $w_0 \in p^\perp \setminus \{p\}$ such that $w_0 \in \Omega_{Kul}(\Gamma)$. From Lemma 4.13, we have $\text{Id}_{p^\perp}(p) \in \Gamma(p, p)$, then there is a Cartan sequence $(\gamma_m)$ associated to $\text{Id}_{p^\perp}(p)$. Consequently, $\overrightarrow{w_0, p} = \mathcal{D}(\gamma_m)(w_0) \subset \Lambda_{kul}(\Gamma)$, which is a contradiction. Therefore, $\Omega_{Kul}(\Gamma) = Eq(\Gamma)$ and $\Lambda_{Kul}(\Gamma) = \bigcup_{p \in \Lambda_{CG}(\Gamma)} p^\perp$.

Through similar arguments, we can show that $\Omega_{Kul}(\Gamma)$ is the largest open set on which $\Gamma$ acts properly discontinuously. $\Box$

**Proof of Theorem 0.1.** In virtue of Theorem 5.1, we only need to show the theorem when $\Gamma$ is elementary. Consider the following cases:

**Case 1.** The Chen-Greenberg limit set of $\Gamma$ is a single point. Let $p \in \Lambda_{CG}(\Gamma)$ be the unique point, by Proposition 3.6 and arguments as in Theorem 4.11, we can ensure that $\text{Id}_{p^\perp}(p) \in \Gamma(p, p)$. If there is $w_0 \in p^\perp \setminus \Lambda(\Gamma)$, let $(\gamma_m) \subset \Gamma$ be a Cartan sequence associated to $\text{Id}_{p^\perp}(p)$. Thus $\overrightarrow{w_0, p} = \mathcal{D}(\gamma_m)(w_0) \subset p^\perp$, which proves the theorem in this case.

**Case 2.** The Chen-Greenberg limit set of $\Gamma$ has exactly two points. After conjugating with an element in $\text{PU}(1, n)$, if it is necessary, we can assume that $\{[e_1], [e_{n+1}]\} = \Lambda_{CG}(\Gamma)$. Let $\Gamma_0 = \text{Isot}(\Gamma, [e_1]) \cap \text{Isot}(\Gamma, [e_{n+1}])$, thus $\Gamma_0$ is a subgroup of $\Gamma$ with finite index, therefore $\Lambda_{Kul}(\Gamma_0) = \Lambda_{Kul}(\Gamma)$. Let $(\gamma_m)_{m \in \mathbb{N}} \subset \Gamma$ be a sequence of distinct elements, then

$$\gamma_m = \begin{bmatrix} r_m c_m & u_m \\ r_m^{-1} c_m & r_m \end{bmatrix}$$

where $c_m^2 = \det(U_m)$, $r_m \in \mathbb{R}^+$ and $U_m \in U(n-1)$. From Equation 5.1 we conclude $L_0(\Gamma_0) = L_1(\Gamma_0)$ and $L_2(\Gamma) = [e_1]^\perp \cup [e_{n+1}]^\perp$, which concludes the proof. $\Box$

As a corollary of the main theorem, we have the following result, compare with results in [BN09, CS10].

**Corollary 5.2.** Let $\Gamma \subset \text{PU}(1, n)$ be a discrete group such that $\Gamma$ is irreducible, then each connected component of $\Omega_{Kul}(\Gamma)$ is a complete Kobayashi hyperbolic space, pseudoconvex, a domain of holomorphy and a Stein Manifold. Moreover, $\Lambda_{CG}(\Gamma) \subset \Lambda_{Kul}(\Gamma)$ is the unique minimal closed set for the action of $\Gamma$ on $\mathbb{P}_n^\mathbb{C}$.

### 6. The Cartan invariant and the control group

In this section, we will show how the control group “encodes the geometry” of the Chen-Greenberg limit set trough the Cartan angular invariant.
**Definition 6.1.** Given \( z = [z_1, z_2, \ldots, z_{n+1}] \in \partial \mathbb{H}^n \setminus \{e_1, e_{n+1}\} \) we define:

\[
\phi_z : \text{Span}\{e_2, \ldots, e_n\} \rightarrow \text{Span}\{e_2, \ldots, e_n\}
\]

\[
\phi_z(w) = \text{Span}(\text{Span}\{(w, e_{n+1})\} \cap z^+) \cup \{z\} \cap e_1^+) \cup \{e_1\} \cap e_{n+1}^+
\]

The following lemma is the computation of the transformation \( \phi_z \).

**Lemma 6.2.** Let \( z = [z_1, z_2, \ldots, z_{n+1}] \in \partial \mathbb{H}^n \setminus \{e_1, e_{n+1}\} \), then \( \phi_z(w) = [\{M_z\}]\{w\} \), where

\[
M_z = \begin{pmatrix}
(z_{n+1} \bar{z}_1 + |z_2|^2 & z_2 \bar{z}_3 & \cdots & z_2 \bar{z}_n \\
z_3 \bar{z}_2 & (z_{n+1} \bar{z}_1 + |z_3|^2 & \cdots & z_3 \bar{z}_n \\
\vdots & \vdots & \ddots & \vdots \\
z_n \bar{z}_{n-1} & \bar{z}_n \bar{z}_1 & \cdots & z_{n+1} \bar{z}_1 + |z_n|^2
\end{pmatrix}
\]

**Proof.** Let \( w \in \text{Span}\{e_2, \ldots, e_n\} \), then we can assume \( w = [0, w_2, \ldots, w_n, 0] \), thus

\[
\text{Span}\{(w, e_{n+1})\} = \{0, (1 - \lambda)w_2, \ldots, (1 - \lambda)w_n, \lambda|\lambda \in \hat{C}\}.
\]

To obtain \( w_1 = \text{Span}\{(w, e_{n+1})\} \cap z^+ \), we use the Hermitian form \( \langle \cdot, \cdot \rangle \), so \( w_1 \) is obtained through the solution of

\[
\lambda \bar{z}_1 + \sum_{j=2}^n (1 - \lambda)w_j \bar{z}_j = 0.
\]

And the solution is

\[
\sigma = \frac{\sum_{j=2}^n w_j \bar{z}_j}{-\bar{z}_1 + \sum_{j=2}^n w_j \bar{z}_j}.
\]

Therefore,

\[
w_1 = \sum_{j=2} (1 - \sigma)w_j, \ldots, (1 - \sigma)w_n, \sigma].
\]

If \( x \in \text{Span}\{(w_1, z)\} \), we have

\[
x = [(1 - \lambda)z_1, \lambda(1 - \sigma)w_2 + (1 - \lambda)z_2, \ldots, \lambda(1 - \sigma)w_n + (1 - \lambda)z_n, \lambda \sigma + (1 - \lambda)z_{n+1}],
\]

where \( \lambda \in \hat{C} \). Thus, \( w_2 = \text{Span}\{(w_1, z)\} \cap e_1^+ \) is given by the solution of

\[
\lambda \sigma + (1 - \lambda)z_{n+1} = 0.
\]

Such solution is \( \eta = \frac{z_{n+1}}{\lambda \sigma + (1 - \lambda)z_{n+1}} \). Now,

\[
w_2 = [(1 - \eta)z_1, \eta(1 - \sigma)w_2 + (1 - \eta)z_2, \ldots, \eta(1 - \sigma)w_n + (1 - \eta)z_n, \eta \sigma + (1 - \eta)z_{n+1}].
\]

Next, if \( y \in \text{Span}\{(w_2, e_1)\} \), then

\[
y = [\lambda(1 - \eta)z_1 + (1 - \lambda), \lambda(1 - \sigma)w_2 + (1 - \eta)z_2, \ldots, \lambda(1 - \sigma)w_n + (1 - \eta)z_n, \lambda \eta \sigma + (1 - \eta)z_{n+1}]
\]

and \( \phi_z(w) = \text{Span}\{(w_2, e_1)\} \cap e_{n+1}^+ \) is induced by the solution of

\[
\lambda(1 - \eta)z_1 + (1 - \lambda) = 0,
\]

which is \( \lambda = \frac{1}{1 - \eta} \).

A straightforward calculation shows that

\[
\phi_z(w) = \left[ \sum_{i=2}^n (|z_{n+1} \bar{z}_1 + |z_i|^2)w_i + z_i \sum_{j \neq 1, i} w_j \bar{z}_j e_i \right],
\]
which concludes the proof. \qed

We have the following straightforward corollary.

**Corollary 6.3.** If \( z \in \mathbb{H}^n_C \setminus \{ e_1, e_{n+1} \} \). Then,

1. We have \( \phi_z = \text{Id} \) if and only if \( z \in \text{Span}(\{ e_1, e_{n+1} \}) \).
2. Also, \( \det(M_z - z_{n+1} \bar{z}_1 \text{Id}) = 0 \).

Now, we get the following properties of \( M_z \).

**Lemma 6.4.** Let \( z \in \mathbb{H}^n_C \setminus \{ e_1, e_{n+1} \} \). Then,

1. We have \([M_z] \in PU(n-1)\).
2. The projective subspace \( W_z = e_+^1 \cap e_+^{n+1} \cap z^\perp \), satisfies \( W_z \subset \text{Fix}[M_z] \).

Moreover, if \( z = [z_1, \ldots, z_{n+1}] \notin \text{Span}(\{ e_1, e_{n+1} \}) \) and \( z_0 = [0, z_2, \ldots, z_n, 0] \), then

1. The vector \( \bar{z}_0 = (z_2, \ldots, z_n) \) is an eigenvector \( M_z \) with eigenvalue \( \lambda_z = z_{n+1} \bar{z}_1 + \sum_{j=2}^{n} |z_j|^2 = -z_{n+1} \bar{z}_1 + |z_{n+1}^*| e^{iA(e_1, z, e_{n+1})} \).
2. We have \( \text{Fix}(\phi_z) = W_z \cup \{ z_0 \} \).

**Proof.** Let us show part (1). Taking \( z = [z_1, z_2, \ldots, z_{n+1}] \), by Lemma 6.2 and a straightforward calculation, we have

\[
M_z M_z^* = \begin{pmatrix}
|z_{n+1}|^2 |z_1| & 0 & \cdots & 0 \\
0 & |z_{n+1}|^2 |z_1| & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & |z_{n+1}|^2 |z_1|^2
\end{pmatrix}
\]

which shows the claim.

Let us show part (2). Let \( w \in W_z \), then

\[
\text{Span}(\{ w, e_{n+1} \}) \cap z^\perp = \{ w \}
\]

\[
\text{Span}(\{ w, z \}) \cap e_+^1 = \{ w \}
\]

\[
\text{Span}(\{ w, e_1 \}) \cap e_+^{n+1} = \{ w \}
\]

which shows \( \phi_z(w) = w \), as required.

The proof of part (a) is a straightforward calculation, so we omit it here. To conclude the proof, let us show part (b). Since \( \phi_z \) is not the identity, we conclude that \( W_z \) is a projective space of dimension \( n - 3 \) contained in \( \text{Span}\{e_2, \ldots, e_n\} \). On the other hand, since \( \phi_z \in PU(n-1) \), it suffices to show \( z_0 \notin W_z \). If this is not true, we must have \( z_0 \in z^\perp \), which is equivalent to the fact \( \sum_{j=2}^{n} |z_j|^2 = 0 \), which is not possible. Therefore, \( z_0 \notin W_z \). \qed

As an easy corollary, we get

**Corollary 6.5.** Let \( z \in \mathbb{H}^n_C \setminus \{ e_1, e_{n+1} \} \), then there is \( H : e_+^1 \cap e_+^{n+1} \rightarrow e_+^1 \cap e_+^{n+1} \) a projective transformation such that:

\[
M \phi_z M^{-1} = \begin{pmatrix}
-e^{iA(z, e_1, e_{n+1})} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
\cdots & \cdots & -e^{iA(z, e_1, e_{n+1})} & \cdots \\
e^{iA(e_1, z, e_{n+1})} & \cdots & \cdots & \cdots
\end{pmatrix}
\]
Theorem 6.6. Let $\Gamma \subset PU(1,n)$ be a non-elementary discrete subgroup and $x,y,z \in \Lambda_{CG}(\Gamma)$ be three distinct elements, set

$$\phi_{xyz} : x^\perp \cap z^\perp \rightarrow x^\perp \cap z^\perp$$

$$\phi_{xyz}(w) = \text{Span}((\text{Span}((w,z) \cap y^\perp) \cup \{y\}) \cap x^\perp) \cup \{x\} \cap z^\perp,$$

and

$$\Phi_{xyz} : z^\perp(z) \rightarrow z^\perp(z)$$

$$\Phi_{xyz}(\ell) = \text{Span}(\phi_{xyz}(\ell \cap x^\perp) \cup \{z\}).$$

Then

1. There is an ordered base $\beta = \{v_1, \ldots, v_{n-1}\}$ such that

$$\phi_{xyz} \left( \sum_{j=1}^{n-1} \alpha_j v_j \right) = \left[ e^{i\delta(x,y,z)} \alpha_{n-1} v_j - \sum_{j=1}^{n-2} e^{i\delta(y,x,z)} \alpha_j v_j \right]$$

2. The transformation $\Phi_{xyz}$ belongs to $\Gamma(z,z)$.

Proof. The proof of part (1) is straightforward, so we will omit it here. Let us show part (2). Let $I_1 \in \Gamma(z,y)$ be the identity of $(\text{Bihol}(y,z), \star)$ and $I_2 \in \Gamma(x,z)$ be the identity of $(\text{Bihol}(x,z), \star)$. Also, let $(\gamma_{jm})$ be a Cartan sequence associated to $I_j$. It is not hard to show that $\gamma_{2m} \gamma_{1m} \xrightarrow{m \to \infty} z$ uniformly on compact sets of $\mathbb{P}^n_\mathbb{C} \setminus z^\perp$. By Proposition 3.6 there is a strictly increasing sequence $(n_m) \subset \mathbb{N}$ and $\nu \in \Gamma(z,z)$ such that for every $\ell \in z^\perp(z)$ and $w \in \ell \setminus \{z\}$ we have

$$\mathcal{D}(\gamma_{2n_m} \gamma_{1n_m})(w) = \nu(\ell).$$

Let $\ell \in z^\perp(z)$, $w \in \ell \setminus \{z\}$ and $(w_m) \subset \mathbb{P}^n_\mathbb{C}$ such that $w_m \xrightarrow{m \to \infty} w$. Also, let us assume that there are $w_1, w_2, w_3 \in \mathbb{P}^n_\mathbb{C}$ such that

$$\gamma_{1n_m}(w_m) \xrightarrow{m \to \infty} w_1$$

$$\gamma_{2n_m}(\gamma_{1n_m}(w_m)) \xrightarrow{m \to \infty} w_2$$

As in the proof of Lemma 4.10 we deduce

$$w_1 \in I_1(\ell) = \text{Span}(\{\ell \cap y^\perp \cup \{y\}\}).$$

Consequently,

$$w_2 \in I_2(\text{Span}((I_1(\ell) \cap x^\perp) \cup \{x\})) = \text{Span}((\text{Span}((I_1(\ell) \cap x^\perp) \cup \{x\}) \cap z^\perp) \cup \{z\})) = \Phi_{xyz}(\ell).$$

Therefore, $\nu = \Phi_{xyz}$, which concludes the proof. \qed

We get the straightforward corollary.

Corollary 6.7. Let $\Gamma \subset PU(1,n)$ be a non-elementary discrete subgroup, set $\Gamma(x,y,z) = \{\Phi_{xyz}, x,y,z \in \partial \mathbb{H}^n_\mathbb{C} \text{ are distinct points}\}$ then

1. We have $\Gamma(x,y,z) = \{Id_{z^\perp(z)} \} \in \Lambda_{CG}(\Gamma)$ if and only if $\Gamma$ is a $\mathbb{C}$-Fuchsian group.

2. Every element in $\Gamma(x,y,z)$ has order 2 if and only if $\Gamma$ is a $\mathbb{R}$-Fuchsian group.

Definition 6.8. Let $y \in \Lambda_{CG}(\Gamma)$, we define the Cartan group at $y$ as

$$\mathcal{C}(y) = \{\Phi_{x_1y_1} \cdots \Phi_{x_my_n} | x_i, y_i \in \Lambda_{CG}(\Gamma) \text{ and } x_i \neq y_i\}$$

We will illustrate the Cartan group in a forthcoming article.
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