Ricci Flow and Nonlinear Reaction–Diffusion Systems in Biology, Chemistry, and Physics

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Abstract

This paper proposes the Ricci–flow equation from Riemannian geometry as a general geometric framework for various nonlinear reaction–diffusion systems (and related dissipative solitons) in mathematical biology. More precisely, we propose a conjecture that any kind of reaction–diffusion processes in biology, chemistry and physics can be modelled by the combined geometric–diffusion system. In order to demonstrate the validity of this hypothesis, we review a number of popular nonlinear reaction–diffusion systems and try to show that they can all be subsumed by the presented geometric framework of the Ricci flow.

Keywords: geometrical Ricci flow, nonlinear bio–reaction–diffusion, dissipative solitons and breathers

1 Introduction

Parabolic reaction–diffusion systems are abundant in mathematical biology. They are mathematical models that describe how the concentration of one or more substances distributed in space changes under the influence of two processes: local chemical reactions in which the substances are converted into each other, and diffusion which causes the substances to spread out in space. More formally, they are expressed as semi–linear parabolic partial differential equations (PDEs, see e.g., [55]). The evolution of the state vector \( u(x, t) \) describing the concentration of the different reagents is determined by anisotropic diffusion as well as local reactions:

\[
\partial_t u = D \Delta u + R(u), \quad (\partial_t = \partial/\partial t),
\]

where each component of the state vector \( u(x, t) \) represents the concentration of one substance, \( \Delta \) is the standard Laplacian operator, \( D \) is a symmetric positive–definite matrix of diffusion coefficients (which are proportional to the velocity of the diffusing particles) and \( R(u) \) accounts for all local reactions. The solutions of reaction–diffusion equations display a wide range of behaviors, including the formation of travelling waves and other self–organized patterns like dissipative solitons (DSs).
On the other hand, the Ricci flow equation (or, the parabolic Einstein equation), introduced by R. Hamilton in 1982 \[16\], is the nonlinear heat–like evolution equation\[2\]

\[\partial_t g_{ij} = -2R_{ij},\]  

(2)

for a time–dependent Riemannian metric \( g = g_{ij}(t) \) on a smooth real\[2\] \( n \)–manifold \( M \) with the Ricci curvature tensor \( R_{ij} \). This equation roughly says that we can deform any metric on a 2–surface or \( n \)–manifold by the negative of its curvature; after normalization (see Figure ??), the final state of such deformation will be a metric with constant curvature. However, this is not true in general since, in addition to the presence of singularities, the limits could be Ricci solitons (see below). The factor of 2 in (2) is more or less arbitrary, but the negative sign is essential to insure a kind of global volume exponential decay\[4\] since the Ricci flow equation (2) is a kind of nonlinear geometric generalization of the standard linear heat equation\[5\]

\[\partial_t u = \Delta u.\]  

(3)

Like the heat equation (3), the Ricci flow equation (2) is well behaved in forward time and acts as a kind of smoothing operator (but is usually impossible to solve in backward time). If some parts of a solid object are hot and others are cold, then, under the heat equation, heat will flow from hot to cold, so that the object gradually attains a uniform temperature. To some extent the Ricci flow behaves similarly, so that the Ricci curvature ‘tries’ to become more uniform \[43\], thus

\[A \] hot topic in geometric topology is the Ricci flow, a Riemannian evolution machinery that recently allowed G. Perelman to prove the celebrated Poincaré Conjecture, a century–old mathematics problem (and one of the seven Millennium Prize Problems of the Clay Mathematics Institute) – and won him the 2006 Fields Medal (which he declined in a public controversy) \[41\]. The Poincaré Conjecture can roughly be put as a question: Is a closed 3–manifold \( M \) topologically a sphere if every closed curve in \( M \) can be shrunk continuously to a point? In other words, Poincaré conjectured: A simply-connected compact 3–manifold is diffeomorphic to the 3–sphere \( S^3 \) (see e.g., \[63\]).

\[2\] For the related Kähler–Ricci flow on complex manifolds, see e.g., \[27, 28\].

\[3\] This particular PDE (2) was chosen by Hamilton for much the same reason that A. Einstein introduced the Ricci tensor into his gravitation field equation,

\[R_{ij} - \frac{1}{2}g_{ij}R = 8\pi T_{ij},\]

where \( T_{ij} \) is the energy–momentum tensor. Einstein needed a symmetric 2–index tensor which arises naturally from the metric tensor \( g_{ij} \) and its first and second partial derivatives. The Ricci tensor \( R_{ij} \) is essentially the only possibility. In gravitation theory and cosmology, the Ricci tensor has the volume–decreasing effect (i.e., convergence of neighboring geodesics, see \[21\]).

\[4\] This complex geometric process is globally similar to a generic exponential decay ODE:

\[\dot{x} = -\lambda f(x),\]

for a positive function \( f(x) \). We can get some insight into its solution from the simple exponential decay ODE,

\[\dot{x} = -\lambda x \quad \text{with the solution} \quad x(t) = x_0 e^{-\lambda t},\]

(where \( x = x(t) \) is the observed quantity with its initial value \( x_0 \) and \( \lambda \) is a positive decay constant), as well as the corresponding nth order rate equation (where \( n > 1 \) is an integer),

\[\dot{x} = -\lambda x^n \quad \text{with the solution} \quad \frac{1}{x^{n-1}} = \frac{1}{x_0^{n-1}} + (n - 1)\lambda t.\]

\[5\] More precisely, the negative sign is to make the equation parabolic so that there is a theory of existence and uniqueness. Otherwise the equation would be backwards parabolic and not have any theory of existence, uniqueness, etc.
resembling a monotonic entropy growth \[ \partial_t S \geq 0, \] which is due to the positive definiteness of the metric \( g_{ij} \geq 0, \) and naturally implying the arrow of time \[ \text{[50, 28, 27]}. \]

In a suitable local coordinate system, the Ricci flow equation \[ (2) \] has a nonlinear heat–type form, as follows. At any time \( t, \) we can choose local harmonic coordinates so that the coordinate functions are locally defined harmonic functions in the metric \( g(t). \) Then the Ricci flow takes the general form (see e.g., \[ [4] \])

\[
\partial_t g_{ij} = \Delta_M g_{ij} + Q_{ij}(g, \partial g),
\]

where \( \Delta_M \) is the Laplace–Beltrami operator \[ [5] \] and \( Q = Q_{ij}(g, \partial g) \) is a lower–order term quadratic in \( g \) and its first order partial derivatives \( \partial g. \) From the analysis of nonlinear heat PDEs, one obtains existence and uniqueness of forward–time solutions to the Ricci flow on some time interval, starting at any smooth initial metric \( g_0. \)

The quadratic Ricci flow equation \[ (4) \] is our geometric framework for general bio–reaction–diffusion systems, so that the spatio–temporal PDE \[ (1) \] corresponds to the quadratic Ricci flow PDE

\[
\begin{align*}
\partial_t u &= D \Delta u + R(u), \\
\partial_t g_{ij} &= \Delta_M g_{ij} + Q_{ij}(g, \partial g)
\end{align*}
\]

with:

- the metric \( g = g_{ij} \) on an \( n \)-manifold \( M \) corresponding to the \( n \)-dimensional (or \( n \)-component, or \( n \)-phase) concentration \( u(x, t); \)

- the Laplace–Beltrami differential operator \( \Delta_M, \) as defined on \( C^2 \)-functions on an \( n \)-manifold \( M, \) with respect to the Riemannian metric \( g_{ij}, \) by

\[
\Delta_M \equiv \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^i} \left( \sqrt{\det(g)} g^{ij} \frac{\partial}{\partial x^j} \right)
\]

- corresponding to the \( n \)-dimensional bio–diffusion term \( D \Delta u; \) and

- the quadratic \( n \)-dimensional Ricci–term, \( Q = Q_{ij}(g, \partial g), \) corresponding to the \( n \)-dimensional bio–reaction term, \( R(u). \)

As a simple example of the Ricci flow equations \[ (2)–(4) \], consider a round spherical boundary \( S^2 \) of the 3-ball radius \( r. \) The metric tensor on \( S^2 \) takes the form

\[
g_{ij} = r^2 \hat{g}_{ij},
\]

where \( \hat{g}_{ij} \) is the metric for a unit sphere, while the Ricci tensor

\[
R_{ij} = (n - 1) \hat{g}_{ij}
\]

Note that two different kinds of entropy functional have been introduced into the theory of the Ricci flow, both motivated by concepts of entropy in thermodynamics, statistical mechanics and information theory. One is Hamilton’s entropy, the other is Perelman’s entropy. While in Hamilton’s entropy, the scalar curvature \( R \) of the metric \( g_{ij} \) is viewed as the leading quantity of the system and plays the role of a probability density, in Perelman’s entropy the leading quantity describing the system is the metric \( g_{ij} \) itself. Hamilton established the monotonicity of his entropy along the volume-normalized Ricci flow on the 2–sphere \( S^2 \) [18]. Perelman established the monotonicity of his entropy along the Ricci flow in all dimensions [51].
is independent of $r$. The Ricci flow equation on $S^2$ reduces to

$$
\dot{r}^2 = -2(n - 1), \quad \text{with the solution} \quad r^2(t) = r^2(0) - 2(n - 1)t.
$$

Thus the boundary sphere $S^2$ collapses to a point in finite time (see [43]).

More generally, the geometrization conjecture [60] holds for any 3–manifold $M$ (see below). Suppose that we start with a compact initial 3–manifold $M_0$ whose Ricci tensor $R_{ij}$ is everywhere positive definite. Then, as $M_0$ shrinks to a point under the Ricci flow (2), it becomes rounder and rounder. If we rescale the metric $g_{ij}$ on $M_0$ so that the volume of $M_0$ remains constant, then $M_0$ converges towards another compact 3–manifold $M_1$ of constant positive curvature (see [16]).

In case of even more general 3–manifolds (outside the class of positive Ricci curvature metrics), the situation is much more complicated, as various singularities may arise. One way in which singularities may arise during the Ricci flow is that a spherical boundary $S^2 = \partial M$ of an 3–manifold $M$ may collapse to a point in finite time. Such collapses can be eliminated by performing a kind of ‘geometric surgery’ on the 3–manifold $M$, that is a sophisticated sequence of cutting and pasting without accumulation of time error$^7$ (see [52]). After a finite number of such surgeries, each component either: (i) converges towards a 3–manifold of constant positive Ricci curvature which shrinks to a point in finite time, or possibly (ii) converges towards an $S^2 \times S^1$ which shrinks to a circle $S^1$ in finite time, or (iii) admits a ‘thin–thick’ decomposition of [60]. Therefore, one can choose the surgery parameters so that there is a well defined Ricci flow with surgery, that exists for all time [52].

In this paper we use the evolving $n$–dimensional geometric machinery of the volume–decaying and entropy–growing Ricci flow $g(t)$, given by equations (2)–(4), for modelling various biological reaction–diffusion systems and dissipative solitons, defined by special cases of the general spatio–temporal model (1).

## 2 Bio–reaction–diffusion systems

In case of ideal mixtures, the driving force for the general diffusion $D\Delta u$ (1) is the concentration gradient $-\nabla u$, or the gradient of the chemical potential $-\nabla u_i$ of each species $u_i$, ($i = 1, ..., n$), giving the diffusion flux by the First Fick’s law,

$$
J = -D\nabla u.
$$

Assuming the diffusion coefficients $D$ to be a constant, the Second Fick’s law gives the linear parabolic heat equation,

$$
\partial_t u = D\Delta u,
$$

while, in case of variable diffusion coefficients $D$, we get (slightly) more general parabolic diffusion equation,

$$
\partial_t u = \nabla \cdot (D \nabla u),
$$

$^7$Hamilton’s idea was to perform surgery to cut off the singularities and continue his flow after the surgery. If the flow develops singularities again, one repeats the process of performing surgery and continuing the flow. If one can prove there are only a finite number of surgeries in any finite time interval, and if the long-time behavior of solutions of the Ricci flow (with surgery) is well understood, then one would be able to recognize the topological structure of the initial manifold. Thus Hamilton’s program, when carried out successfully, would lead to a proof of the Poincaré Conjecture and Thurston’s Geometrization Conjecture [63].
which is still analogous to the ‘linear’ part of the quadratic Ricci flow equation \( \text{(4)} \),

\[
\partial_t g_{ij} = \Delta_M g_{ij},
\]
due to general ‘diffusion properties’ of the Laplace–Beltrami operator \( \Delta_M \).

The \( n \)-dimensional diffusion coefficient \( D = D(T) \) at different temperatures \( T \) can be approximated by the Arrhenius exponential–decay relation,

\[
D(t) = D_0 e^{-\frac{E_A}{rT}},
\]
where \( D_0 \) is the maximum possible diffusion coefficient (at infinite temperature \( T \)), \( E_A \) is the activation energy for diffusion (i.e., the energy that must be overcome in order for a chemical reaction to occur) and \( r \) is the gas constant.

Using the First Fick’s first law \( \text{(6)} \), the diffusion equation \( \text{(8)} \) can be derived in a straightforward way from the continuity equation, which states that a change in density in any part of the system is due to inflow and outflow of material into and out of that part of the system (effectively, no material is created or destroyed),

\[
\partial_t u + \nabla \cdot j = 0,
\]
where \( j \) is the flux of the diffusing material.

The most important special case of \( \text{(7)} \) is at a steady state, when the concentrations \( u \) do not change in time, giving the Laplace’s equation,

\[
\Delta u = 0, \quad \text{or} \quad \Delta u_i = 0,
\]
for harmonic functions \( u = \{u_i\} \).

The stochastic version of the deterministic heat equation \( \text{(7)} \), connected with the study of Brownian motion\(^8\) is the Fokker–Planck equation (see e.g., \( \text{(31)} \)),

\[
\partial_t f = -\partial_{x^i} \left[ D_1^i(x^i,f) \right] + \partial_{x^i x^j} \left[ D_2^{ij}(x^i,f) \right],
\]
\((\partial_{x^i} = \frac{\partial}{\partial x^i}, \partial_{x^i x^j} = \frac{\partial^2}{\partial x^i \partial x^j})\), where \( D_1^i \) is the drift vector and \( D_2^{ij} \) the diffusion tensor (which results from the presence of the stochastic force). The Fokker–Planck equation \( \text{(10)} \) is used for computing the probability densities of stochastic differential equations\(^9\).

\(^8\)Brownian motion is the random movement of particles suspended in a liquid or gas or the mathematical model used to describe such random movements, often called a particle theory. The infinitesimal generator (and hence characteristic operator) of a Brownian motion on \( \mathbb{R}^n \) is \( \frac{1}{2} \Delta \), where \( \Delta \) is the Laplacian on \( \mathbb{R}^n \). More generally, a Brownian motion on an \( n \)-manifold \( M \) is given by one-half of the Laplace–Beltrami operator \( \Delta_M \).\(^5\)

\(^9\)Consider the Itô stochastic differential equation,

\[
dX_t = \mu(X_t, t) dt + \sigma(X_t, t) dW_t,
\]
where \( X_t \in \mathbb{R}^n \) is the state of an \( n \)-dimensional stochastic system at time \( t \) and \( W_t \in \mathbb{R}^m \) is the standard \( m \)D Wiener process. If the initial distribution is \( X_0 \sim f(x, 0) \), then the probability density of the state is given by the Fokker–Planck equation \( \text{(10)} \) with the drift and diffusion terms,

\[
D_1^i(x, t) = \mu_i(x, t) \quad \text{and} \quad D_2^{ij}(x, t) = \frac{1}{2} \sum_k \sigma_{ik}(x, t) \sigma_{kj}^T(x, t).
\]
Also, notice that the real–valued heat equation (7) is formally similar to the complex–valued Schrödinger equation (see e.g., [32]),

$$\partial_t \psi = \frac{i \hbar}{2m} \Delta \psi,$$

where $\psi = \psi(x, t)$ is the wave–function of the particle, $i = \sqrt{-1}$, and $\hbar$ is Planck’s constant divided by $2\pi$.

In the remainder of this section, we will review a number of particular bio–reaction–diffusion systems, which are likely to be subsumed by the quadratic Ricci flow model [4].

2.1 1–component systems

2.1.1 Kolmogorov–Petrovsky–Piscounov equation

The simplest bio–reaction–diffusion PDE concerning the concentration $u = u(x, t)$ of a single substance in one spatial dimension,

$$\partial_t u = D \partial_x^2 u + R(u),$$

is also referred to as the Kolmogorov–Petrovsky–Piscounov (KPP) equation. If the reaction term vanishes, then the equation represents a pure diffusion process described by the heat equation. In particular, the choice

$$R(u) = u(1 - u)$$

yields Fisher’s equation that was originally used to describe the spreading of biological populations [3].

The one–component KPP equation (12) can also be written in the variational (gradient) form

$$\partial_t u = -\frac{\delta F}{\delta u},$$

and therefore describes a permanent decrease (a kind of exponential decay) of the system’s free energy functional

$$F = \int_{-\infty}^{\infty} \left[ \frac{D}{2} (\partial_x u)^2 + V(u) \right] dx,$$

where $V(u)$ is the potential such that

$$R(u) = -\frac{dV(u)}{du}.$$
2.1.2 Swift–Hohenberg equation

The Swift–Hohenberg (SH) equation, noted for its pattern–forming behavior, is the decaying reaction–diffusion PDE,

\[ \partial_t u = -(1 + \Delta)^2 u + R(u), \quad (15) \]

given by the variational (gradient) equation (13) with the free energy functional

\[ F = \int_{\Omega} \left[ V(u) + \frac{1}{2} ((1 + \Delta)u)^2 \right] dxdy, \]

where \( R(u) \) is given by (14), while \( \Omega \) is a 2–dimensional region in which (bio)chemical pattern formation occurs.

The time derivative of the free energy \( F \) is given by

\[ \partial_t F = \int_{\Omega} \left[ \frac{dV(u)}{du} + (1 + \Delta)u \right] \partial_t u dxdy, \]

and, since the expression in square brackets is equal to the negative right–hand side of (15), we have

\[ \dot{F} = -\int_{\Omega} (\partial_t u)^2 dxdy \leq 0. \]

Therefore, the free energy \( F \) is the Lyapunov functional that may only decrease as it evolves along its trajectory in some phase space. If \( F \) has no minima, then when the horizontal scale of the liquid container is large compared to the instability wavelength, a propagating front will be observed (e.g., in chemically reacting flames). In this case, \( F \) will decrease continuously until the front approaches the boundary of the medium. An alternative possibility is realized when \( F \) has one or several minima, each corresponding to a local equilibrium state in time. In this case the so–called multi–stability is possible. Therefore, the limit behavior of gradient systems of the form of (13) is characterized by either a steady attractor or propagating fronts [56].

2.1.3 Ginzburg–Landau equation

One of the most popular models in the pattern–formation theory is the complex Ginzburg–Landau equation (see e.g., [56]),

\[ \partial_t A = \varepsilon A + (1 + i\alpha) \Delta A - (1 + i\beta)|A|^2 A, \quad (16) \]

where \( A \) is the complex wave amplitude, \( i = \sqrt{-1}, \varepsilon \) is the super-criticality parameter, while \( \alpha \) and \( \beta \) measure linear and nonlinear dispersion (the dependence of the frequency of the waves on the wave-number), respectively. The equation (16) describes a vast array of phenomena including nonlinear waves, second-order phase transitions, Rayleigh–Bénard convection and superconductivity. The equation describes the evolution of amplitudes of unstable modes for any process exhibiting a Hopf bifurcation, for which a continuous spectrum of unstable wave–numbers is taken into account. It can be viewed as a highly general normal form for a large class of bifurcations and nonlinear wave phenomena in spatially extended systems [57].

\[ \text{The extension of the complex Ginzburg–Landau equation (16), which describes strongly resonant multi-frequency forcing of the form} \]

\[ F = f_1 e^{i\omega t} + f_2 e^{2i\omega t} + f_3 e^{3i\omega t} + \text{c.c.} \]
In particular, if we put $\alpha = \beta = 0$ in (16), we get the real, or dissipative, Ginzburg–Landau equation,

$$\partial_t A = \varepsilon A + \Delta A - |A|^2 A,$$

(18)

which is a gradient equation: $\partial_t A = -\delta F/\delta A$, with the free energy functional

$$F = -\int_{\Omega} \left[ \varepsilon |A|^2 - \frac{1}{2} |A|^4 + (\nabla A)^2 \right] dx dy.$$

Using the fact that

$$\dot{F} = -\int_{\Omega} |\partial_t A|^2 dx dy \leq 0,$$

solutions of (18) at $t \to \infty$ are either stationary field–distributions satisfying, for $\varepsilon = 1$, the equation

$$\Delta A + A - |A|^2 A = 0,$$

(19)

of fronts whose propagation is accompanied by a decrease of the functional $F$. The functional must reach its minimum at stable stationary solutions of (19).

### 2.1.4 Neural field theory

The dynamical system from which the temporal evolution of neural activation fields is generated is constrained by the postulate that localized peaks of activation are stable objects, or formally, fixed–point attractors. Such a field dynamics has the generic form

$$\tau \partial_t u = -u + \text{resting level} + \text{input} + \text{interaction},$$

(20)

where $u = u(x, t)$ is the activation field defined over the metric dimension $x$ and time $t$. The first three terms define an input driven regime, in which attractor solutions have the form

$$u(x, t) = \text{resting level} + \text{input}.$$

The rate of relaxation is determined by the time scale parameter $\tau$. The interaction stabilizes localized peaks of activation against decay by local excitatory interaction and against diffusion by global inhibitory interaction. In Amari’s formulation [3], the conceptual model (20) is specified as a continuous model for neural activity in cortical structures,

$$\tau \partial_t u(x, t) = -u(x, t) + h + S(x, t) + \int dx' w(x - x')\sigma(u(x', t)),$$

(21)

was recently proposed in [9] by considering the analogous center–manifold reduction of the extended dynamical system in which the forcing amplitudes $f_1$, $f_2$, and $f_3$ are considered as dynamical variables that vary on the slow time scale $t$. Under time translations $T_\nu : A \to Ae^{i\nu t}$, they transform as

$$f_1 \to f_1 e^{i\nu t}, \quad f_2 \to f_2 e^{2i\nu t}, \quad f_3 \to f_3 e^{3i\nu t}.$$

To cubic order in $A$ the most general equation that is equivariant under $T_\nu$ is then given by

$$\partial_t A = a_1 + a_2 A + a_3 \Delta A + a_4 |A|^2 A + a_5 A^2 + a_6 A^3,$$

(17)

where $a_1 = b_{11} f_3 + b_{12} f_2 f_3$, $a_2 = b_{21} + b_{22} |f_3|^2$, $a_3 = b_{31} f_2$, $a_5 = b_{51} f_3$, $a_6 = b_{61} f_3$. The forcing terms $f_j$ satisfy decoupled evolution equations on their own. In the simplest case this evolution expresses a de-tuning $\nu_j$ of the forcing $f_j$ from the respective resonance and the $f_j$ satisfy

$$f_j = i\nu_j f_j, \quad (j = 1 \ldots 3).$$

In general, the de-tuning introduces time dependence into (17).
where $h < 0$ is a constant resting level, $S(x, t)$ is spatially and temporally variable input function, $w(x)$ is an interaction kernel and $\sigma(u)$ is a sigmoidal nonlinear threshold function. The interaction term collects input from all those field sites $x'$ at which activation is sufficiently large. The interaction kernel determines if inputs from those sites are positive, driving up activation (excitatory), or negative, driving down activation (inhibitory). Excitatory input from nearby location and inhibitory input from all field locations generically stabilizes localized peaks of activation. For this class of dynamics, detailed analytical results provide a framework for the inverse dynamics task facing the modeler, determining a dynamical system that has the appropriate attractor solutions [26] [33] [12].

### 2.2 2–component systems

Two–component systems allow for a much larger range of possible phenomena than their one–component counterparts. An important idea that was first proposed by A. Turing is that a state that is stable in the local system should become unstable in the presence of diffusion [61]. This idea seems counter–intuitive at first glance as diffusion is commonly associated with a stabilizing effect. However, the linear stability analysis shows that when linearizing the general two–component

$$
\tau \dot{u}_i = -u_i + h + s_{beh}^{beh} + c_{mot} \cdot \sigma(m_i) + \alpha_{self}^{beh} + \alpha_{self}^{beh} - \alpha_{self}^{beh},
$$

where the system parameters have the following meaning:

- $\tau$, the constant relaxation rate, i.e., the time scale on which the dynamics reacts to changes;
- $h$, the constant negative resting level of neural activation;
- $\sigma(.)$, a sigmoidal function, which maps the value of neural activity onto $[0, 1]$, given by $\sigma(u) = \frac{1}{1 + e^{-\beta u}}$, where $\beta (=100)$ parameterizes the slope of the resulting function;
- $s_{beh}^{beh}$, the adequate stimulus provided by sensory input of a certain duration;
- $u_i$, activity of behavioral neuron $i$, i.e., activity of behavior $i$;
- $c_{mot}$, a constant for weighting the motivational contribution, $c_{mot} < |h|$;
- $\alpha_{self}^{beh}$ excitatory contribution of neuron $i$'s own activity $u_i$;
- $\alpha_{self}^{beh}$ all excitatory contribution of active neurons connected to neuron $i$;
- $\alpha_{self}^{beh}$ all inhibitory contribution of active neurons connected to neuron $i$;
- $m_i$, activity of motivational neuron $i$, i.e., motivation of behavior $i$ is in [58] defined by the following NAD–equation, similar to (22):

$$
\tau \dot{m}_i = -m_i + h + s_{mot}^{mot} + \alpha_{self}^{mot} + \alpha_{self}^{mot} - \alpha_{self}^{mot},
$$

where

- $\alpha_{self}^{mot}$ excitatory contribution of neuron $i$'s own motivation $m_i$;
- $\alpha_{self}^{mot}$ all excitatory contribution of motivation neurons connected to neuron $i$;
- $\alpha_{self}^{mot}$ all inhibitory contribution of motivation neurons connected to neuron $i$.

In this framework, a nonlinear neural dynamical and control system generates the temporal evolution of behavioral variables, such that desired behaviors are fixed-point attractor solutions while un-desired behaviors are repellors. This kind of attractor & repeller dynamics [26] provides the basis for understanding cognition, both natural and artificial [35] [29] [30].
system
\[
\begin{pmatrix}
\frac{\partial u}{\partial t} \\
\frac{\partial v}{\partial t}
\end{pmatrix}
= \begin{pmatrix} D_u & 0 \\
0 & D_v \end{pmatrix}
\begin{pmatrix}
\frac{\partial_{xx} u}{\partial_{xx} v}
\end{pmatrix}
+ \begin{pmatrix} F(u,v) \\
G(u,v) \end{pmatrix}
\]
and perturbing the system against plane waves
\[
\tilde{u}_k(x,t) = \begin{pmatrix} \tilde{u}(t) \\
\tilde{v}(t) \end{pmatrix} e^{ikx}
\]
close to a stationary homogeneous solution one finds [30]
\[
\begin{pmatrix}
\frac{\partial \tilde{u}_k(t)}{\partial t} \\
\frac{\partial \tilde{v}_k(t)}{\partial t}
\end{pmatrix}
= -k^2 \begin{pmatrix} D_u \tilde{u}_k(t) \\
D_v \tilde{v}_k(t) \end{pmatrix}
+ \mathbf{R}' \begin{pmatrix} \tilde{u}_k(t) \\
\tilde{v}_k(t) \end{pmatrix}
\].

Turing’s idea can only be realized in four equivalence classes of systems characterized by the signs of the Jacobian \( \mathbf{R}' \) of the reaction function. In particular, if a finite wave vector \( k \) is supposed to be the most unstable one, the Jacobian must have the signs
\[
\left( \begin{array}{cc}
+ & - \\
+ & -
\end{array} \right),
\left( \begin{array}{cc}
+ & + \\
+ & -
\end{array} \right),
\left( \begin{array}{cc}
- & + \\
- & +
\end{array} \right),
\left( \begin{array}{cc}
- & - \\
+ & +
\end{array} \right).
\]
This class of systems is named activator–inhibitor system after its first representative: close to the ground state, one component stimulates the production of both components while the other one inhibits their growth. Its most prominent representative is the FitzHugh–Nagumo equation (25).

2.2.1 Brusselator

Classical model of an autocatalytic chemical reaction is Prigogine’s Brusselator (see e.g., [54])
\[
\begin{align*}
\partial_t u &= D_u \Delta u + \alpha + u^2 v - (1 + \beta)u, \\
\partial_t v &= D_v \Delta v - u^2 v + \beta u,
\end{align*}
\]
which describe the spatio–temporal evolution of the intermediate components \( u \) and \( v \), with diffusion coefficients \( D_u \) and \( D_v \), while reactions
\[
\alpha \xrightarrow{r_1} u, \quad 2u + v \xrightarrow{r_2} 3u, \quad \beta + u \xrightarrow{r_3} v + d, \quad u \xrightarrow{r_4} c
\]
describe the concentration of the original substances \( \alpha \) and \( \beta \), for which the final products \( c \) and \( d \) are constant when all four reaction rates \( r_i \) equal unity.

A discretized (temporal only) version of the Brusselator PDE (23) reads
\[
\dot{u} = \alpha + u^2 v - (1 + \beta)u, \quad \dot{v} = \beta u - u^2 v.
\]

The Brusselator displays oscillatory behavior in the species \( u \) and \( v \) when reverse reactions are neglected and the concentrations of \( \alpha \) and \( \beta \) are kept constant.

2.2.2 2–component model of excitable media

Turbulence of scroll waves is a kind of spatio–temporal chaos that exists in 3–dimensional excitable media. Cardiac tissue and the Belousov–Zhabotinsky reaction are examples of such media. In cardiac tissue, chaotic behavior is believed to underlie fibrillation which, without intervention,
precedes cardiac death. Fast computer–simulation of waves in excitable media have been often performed using the 2–component Barkley model of excitable media [5],

\[
\partial_t u = \frac{1}{\epsilon} u (1 - u) \left( u - \frac{v + b(t)}{a} \right) + \nabla^2 u + h(t), \quad \partial_t v = u - v,
\]

(24)

where \( \epsilon \) is a small parameter \( \epsilon \ll 1 \) characterising mutual time scales of the fast \( u \) and slow \( v \) variables, and \( a \) and \( b \) specify the kinetic properties of the system. Parameter \( b \) determines the excitation threshold and thus controls the excitability of the medium. The term \( h(t) \) represents an ‘extra transmembrane current’.

Suppression of the turbulence using stimulation of two different types, ‘modulation of excitability’ and ‘extra transmembrane current’ was performed in [46], using the Barkley model (24). With cardiac defibrillation in mind, the authors used a single pulse as well as repetitive extra current with both constant and feedback controlled frequency. They show that turbulence can be terminated using either a resonant modulation of excitability or a resonant extra current. The turbulence is terminated with much higher probability using a resonant frequency perturbation than a non-resonant one. Suppression of the turbulence using a resonant frequency is up to fifty times faster than using a non-resonant frequency, in both the modulation of excitability and the extra current modes. They also demonstrate that resonant perturbation requires strength one order of magnitude lower than that of a single pulse, which is currently used in clinical practice to terminate cardiac fibrillation.

2.2.3 Gierer–Meinhardt activator–inhibitor system

Spontaneous pattern formation in initially almost homogeneous systems is common in both organic and inorganic systems. The Gierer–Meinhardt model [14] is a reaction–diffusion system of the activator–inhibitor type that appears to account for many important types of pattern formation and morphogenesis observed in biology, chemistry and physics. The model describes the concentration of a short–range autocatalytic substance, the activator, that regulates the production of its long–range antagonist, the inhibitor. It is given as a 2–component nonlinear PDE system,

\[
\partial_t a = -\mu_a a + \rho a^2/h + D_a \partial_x^2 a + \rho_a, \quad \partial_t h = -\mu_h h + \rho a^2 + D_h \partial_x^2 h + \rho_h,
\]

where \( a \) is a short–range autocatalytic substance, i.e., activator, and \( h \) is its long–range antagonist, i.e., inhibitor. \( \partial_t a \) and \( \partial_t h \) describe respectively the changes of activator and inhibitor concentrations per second, \( \mu_a \) and \( \mu_h \) are the corresponding decay rates, while \( D_a \) and \( D_h \) are the corresponding diffusion coefficients. \( \rho \) is a positive constant. \( \rho_a \) is a small activator–independent production rate of the activator and is required to initiate the activator autocatalysis at very low activator concentration, e.g., in the case of regeneration. A low baseline production of the inhibitor, \( \rho_h \), leads to a stable non–patterned steady state; the system can be asleep until an external trigger occurs by an elevation of the activator concentration above a threshold [32].

2.2.4 Fitzhugh–Nagumo activator–inhibitor system

An important example of bio–reaction–diffusion systems, frequently used in neurodynamics, is the 2–component Fitzhugh–Nagumo activator–inhibitor system [12, 17] (see also [24, 29])

\[
\tau_u \partial_t u = D_u^2 \Delta u + f(u) - \sigma v, \quad \tau_v \partial_t v = D_v^2 \Delta v + u - v,
\]

(25)

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with \( f(u) = \lambda u - u^3 - \kappa \), which describes how an action potential travels through a nerve. \( D_u \) and \( D_v \) are diffusion coefficients, \( \tau_u \) and \( \tau_v \) are time characteristics, while \( \kappa, \sigma \) and \( \lambda \) are positive constants. In matrix form, system (25) reads

\[
\begin{pmatrix}
\tau_u \partial_t u \\
\tau_v \partial_t v
\end{pmatrix} = \begin{pmatrix} D_u^2 & 0 \\
0 & D_v^2 \end{pmatrix} \begin{pmatrix} \Delta u \\
\Delta v
\end{pmatrix} + \begin{pmatrix} \lambda u - u^3 - \kappa - \sigma v \\
u - u
\end{pmatrix}.
\]

When an activator–inhibitor system undergoes a change of parameters, one may pass from conditions under which a homogeneous ground state is stable to conditions under which it is linearly unstable. The corresponding bifurcation may be either a Hopf bifurcation to a globally oscillating homogeneous state with a dominant wave number \( k = 0 \) or a Turing bifurcation to a globally patterned state with a dominant finite wave number. The latter in two spatial dimensions typically leads to stripe or hexagonal patterns.

In particular, for the Fitzhugh–Nagumo system (25), the neutral stability curves marking the boundary of the linearly stable region for the Turing and Hopf bifurcation are given by

\[
q^H_T(n)(k) : \frac{1}{\tau} + \frac{d^2 u + \frac{1}{\tau} d^2 v}{k^2} = f'(u_h),
\]

\[
q^T_H(n)(k) : \frac{\kappa}{1 + \sigma \gamma k} + \frac{d^2 u}{k^2} = f'(u_h).
\]

If the bifurcation is subcritical, often localized structures (i.e., dissipative solitons) can be observed in the hysteretic region where the pattern coexists with the ground state. Other frequently encountered structures comprise pulse trains, spiral waves and target patterns.

The reduced (temporal) non–dimensional Fitzhugh–Nagumo equations read:

\[
\dot{v} = v(a - v)(v - 1) - w + I_a, \quad (26)
\]

\[
\dot{w} = bv - \gamma w, \quad (27)
\]

where \( 0 < a < 1 \) is essentially the threshold value, \( b \) and \( \gamma \) are positive constants and \( I_a \) is the applied current. The drift field for this model is given by

\[
u_1(v, w) = v(a - v)(v - 1) - w, \quad u_2(v, w) = bv - \gamma w.
\]

As can be seen from (27) the null cline of the deterministic dynamics of this equations is the line \( v = \frac{\gamma}{b} w \). By substitution on the r.h.s of equation (26) we find the following equation for steady states: \( v(a - v)(v - 1) - \frac{b}{\gamma} v = 0 \).

When this system is in a noisy environment, in the limit of weak noise, we can approximate the dynamics of the fluctuations by the Langevin equation \(24, 31\)

\[
\dot{v} = v(a - v)(v - 1) - \frac{b}{\gamma} v + \xi(t),
\]

that is, the fluctuations run along the line \( v = \frac{\gamma}{b} w \).

In particular, parameters in the FitzHugh–Nagumo neuron model \(33, 30\)

\[
\dot{v} = a + bv + cv^2 + dw^3 - u, \quad \dot{u} = \varepsilon(v - u),
\]

can be tuned so that the model describes spiking dynamics of many resonator neurons. Since one needs to simulate the shape of each spike, the time step in the model must be relatively small, e.g., \( \tau = 0.25 \text{ ms} \). Since the model is a 2–dimensional system of ODEs, without a reset, it cannot exhibit autonomous chaotic dynamics or bursting. Adding noise to this, or some other 2–dimensional models, allows for stochastic bursting.
2.2.5 2–component Belousov–Zhabotinsky reaction

Classical Belousov–Zhabotinsky (BZ) reaction is a family of oscillating chemical reactions. During these reactions, transition–metal ions catalyze oxidation of various, usually organic, reductants by bromic acid in acidic water solution. Most BZ reactions are homogeneous. The BZ reaction makes it possible to observe development of complex patterns in time and space by naked eye on a very convenient human time scale of dozens of seconds and space scale of several millimeters. The BZ reaction can generate up to several thousand oscillatory cycles in a closed system, which permits studying chemical waves and patterns without constant replenishment of reactants [65].

Consider the water–in–oil micro–emulsion BZ reaction [35, 34]

\[
\begin{align*}
\partial_t v &= D_v \Delta v + \frac{1}{\varepsilon_0} \left[ f_0 z + i_0 (1 - mz) \right] \frac{v - q_0}{v + q_0} + \frac{1}{\varepsilon_0} \left[ 1 - \frac{1 - mz}{1 - mz + \varepsilon_1} \right] v - v^2, \\
\partial_t z &= D_z \Delta z - z + v \left[ \frac{1 - mz}{1 - mz + \varepsilon_1} \right],
\end{align*}
\]

(28)

where \(v, z\) are dimensionless concentrations of activator HBrO\(_2\) and oxidized catalyst \([\text{Ru}(bpy)_3]^{3+}\) respectively; \(D_v\) and \(D_z\) are dimensionless diffusion coefficients of activator and catalyst; \(f, \varepsilon_0, \varepsilon_1\) and \(q\) are parameters of the standard Tyson model [62]; \(i_0\) represents the photoinduced production of inhibitor, and \(m\) represents the strength of oxidized state of the catalyst with \(0 < mz < 1\). This reaction was shown experimentally and numerically to admit localized spot patterns that persist for long time [35, 34].

We can rescale the variables in (28) as [37]

\[
z = 1/m - m^{-3/2}w\varepsilon_1, \quad v = m^{-1/2}\hat{v}, \quad t = \varepsilon_0 m^{1/2}\hat{t}.
\]

In the new variables, after dropping the hats, we obtain the non–dimensional 2–component BZ reaction

\[
\partial_t \hat{v} = \varepsilon_2 \Delta \hat{v} + f(\hat{v}, w), \quad \tau \partial_t \hat{w} = D \Delta \hat{w} + g(\hat{v}, w),
\]

where

\[
f(v, z) = -\left[ f_0 + f_1 w \right] \frac{v - q}{v + q} + \left[ \frac{w}{1 + \alpha w} \right] v - v^2, \quad g(v, w) = 1 - \left[ \frac{w}{1 + \alpha w} \right] v,
\]

with the non–dimensional constants given by

\[
\alpha = m^{-1/2}, \quad f_1 = \varepsilon_1 m^{1/2} \left( i_0 - \frac{f_0}{m} \right), \quad q = q_0 m^{1/2},
\]

\[
\varepsilon^2 = \varepsilon_0 D_v m^{1/2}, \quad D = D_z \varepsilon_1 m^{-1/2}, \quad \tau = \frac{1}{m \varepsilon_0}.
\]

2.3 3–component and multi–component systems

2.3.1 Oregonator

The Oregonator model is based on the so–called FKN–mechanism [11], which provided the first successful explanation of the chemical oscillations that occur in the experimental Belousov–Zhabotinsky reaction. It is is composed of five coupled elementary chemical stoichiometries. During the last
two decades, the Oregonator model has been modified in many ways by inclusion of additional chemical reaction steps or by changing the rate constants. If we denote the concentration of the species \( S \) by \([S]\), then we define:

\[
A = [\text{BrO}_3^-], \quad H = [H^+], \quad X = [\text{HBrO}_2], \quad Y = [\text{Br}^-], \quad Z = [\text{Ce}^{4+}].
\]

The original Oregonator model was described by the following three coupled nonlinear PDEs,

\[
\begin{align*}
\frac{\partial t}{X} & = k_1 AH^2 Y - k_2 HXY - 2k_3 X^2 + k_4 AHX + D_X \nabla^2 X, \\
\frac{\partial t}{Y} & = -k_1 AH^2 Y - k_2 HXY + k_5 f Z + D_Y \nabla^2 Y, \\
\frac{\partial t}{Z} & = 2k_4 AHX - k_5 Z + D_Z \nabla^2 Z,
\end{align*}
\]

where \( f \) is a stoichiometric factor [49], \( k_i (i = 1, ..., 5) \) are rate constants, while \( D_X, D_Y, \) and \( D_Z \) are the diffusion constants of the species \( \text{HBrO}_2, \) \( \text{Br}^- \), and \( \text{Ce}^{4+} \) respectively (for dilute solutions, the diffusion matrix is diagonal). For a thorough discussion of the chemistry on which the Oregonator is based, the reader is referred to [62].

The Oregonator temporal mass–action dynamics is a well–stirred, homogeneous system of ODEs given by

\[
\begin{align*}
\dot{X} & = k_1 AY - k_2 XY + k_3 AX - 2k_4 X^2, \\
\dot{Y} & = -k_1 AH^2 Y - k_2 HXY + 1/2k_c f BZ, \\
\dot{Z} & = 2k_4 AHX - k_5 Z + k_b BZ,
\end{align*}
\]

which are typically scaled as [62]

\[
\epsilon (dx/d\tau) = qy - xy + x(1 - x), \quad \epsilon' (dy/d\tau) = -qy - xy + fz, \quad dz/d\tau = x - z.
\]

The basic chemistry of the BZ–oscillations involves jumps between high and low \( \text{HBrO}_2 (X) \) states, which is reflected in the relaxation oscillator nature of the Oregonator. This fundamental bistability may be stabilized in a flow reactor (CSTR) with reactants and \( \text{Br}^- \) in the feed stream. Hysteresis between the two states is observed both experimentally and in the Oregonator. Quasiperiodicity and chaos also are observed in CSTR and can be modeled by the Oregonator [10].

### 2.3.2 Multi–phase tumor growth equations

Our last reaction–diffusion system is a general model of multi–phase tumor growth, in the form of nonlinear parabolic PDE, as reviewed recently in [57]

\[
\begin{align*}
\partial_t \Phi_i &= \nabla \cdot (D_i \Phi_i) - \nabla \cdot (v_i \Phi_i) + \lambda_i(\Phi_i, C_i) - \mu_i(\Phi_i, C_i) \quad (30)
\end{align*}
\]

\( (\partial_t \equiv \partial/\partial t) \), where for phase \( i \), \( \Phi_i \) is the volume fraction \( (\sum_i \Phi_i = 1) \), \( D_i \) is the random motility or diffusion, \( \lambda_i(\Phi_i, C_i) \) is the chemical and phase dependent production, and \( \mu_i(\Phi_i, C_i) \) is the chemical and phase dependent degradation/death, and \( v_i \) is the cell velocity defined by the constitutive equation

\[
v_i = -\mu \nabla p, \quad (31)
\]

where \( \mu \) is a positive constant describing the viscous–like properties of tumor cells and \( p \) is the spheroid internal pressure.

In particular, the multi–phase equation (30) splits into two heat–like mass–conservation PDEs [57].

\[
\begin{align*}
\partial_t \Phi^C &= S^C - \nabla \cdot (v^C \Phi^C), \\
\partial_t \Phi^F &= S^F - \nabla \cdot (v^F \Phi^F),
\end{align*}
\]

(32)
where $\Phi^C$ and $\Phi^F$ are the tissue cell/matrix and fluid volume fractions, respectively, $v^C$ and $v^F$ are the cell/matrix and the fluid velocities (both defined by their constitutive equations of the form of (31)), $S^C$ is the rate of production of solid phase tumor tissue and $S^F$ is the creation/degradation of the fluid phase. Conservation of matter in the tissue, $\Phi^C + \Phi^F = 1$, implies that $\nabla \cdot (v^C \Phi^C + v^F \Phi^F) = \Phi^C + \Phi^F$. The assumption that the tumor may be described by two phases only implies that the new cell/matrix phase is formed from the fluid phase and vice versa, so that $S^C + S^F = 0$. The detailed biochemistry of tumor growth can be coupled into the model above through the growth term $S^C$, with equations added for nutrient diffusion, see [57] and references therein.

The multi-phase tumor growth model (30) has been derived from the classical transport/mass conservation equations for different chemical species [57].

$$\partial_t u_i = P_i - \nabla \cdot N_i,$$

(33)

Here $C_i$ are the concentrations of the chemical species, subindex $a$ for oxygen, $b$ for glucose, $c$ for lactate ion, $d$ for carbon dioxide, $e$ for bicarbonate ion, $f$ for chloride ion, and $g$ for hydrogen ion concentration; $P_i$ is the net rate of consumption/production of the chemical species both by tumor cells and due to the chemical reactions with other species; and $N_i$ is the flux of each of the chemical species inside the tumor spheroid, given (in the simplest case of uncharged molecules of glucose, $O_2$ and $CO_2$) by Fick’s law,

$$N_i = -D_i \nabla u_i,$$

where $D_i$ are (positive) constant diffusion coefficients. In case of charged molecules of ionic species, the flux $N_i$ contains also the (negative) gradient of the volume fractions $\Phi_i$.

There are three distinct stages to cancer development: avascular, vascular, and metastatic – researchers often concentrate their efforts on answering specific OUPC–related questions on each of these stages [57]. In particular, as some tumor cell lines grown in vitro form spherical aggregates, the relative cheapness and ease of in vitro experiments in comparison to animal experiments has made 3D multicellular tumor spheroids (MTS, see Figure 6 in [57]) very popular in vitro model system of avascular tumors [38]. They are used to study how local micro-environments affect cellular growth/decay, viability, and therapeutic response [59]. MTS provide, allowing strictly controlled nutritional and mechanical conditions, excellent experimental patterns to test the validity of the proposed mathematical models of tumor growth/decay [53].

### 3 Dissipative evolution under the Ricci flow

In this section we will derive the geometric formalism associated with the quadratic Ricci–flow equation (4), as a general framework for all presented bio–reaction–diffusion systems.

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13 In vitro cultivation of tumor cells as multicellular tumor spheroids (MTS) has greatly contributed to the understanding of the role of the cellular micro-environment in tumor biology (for review see [29, 25]). These spherical cell aggregates mimic avascular tumor stages or micro-metastases in many aspects and have been studied intensively as an experimental model reflecting an in vivo-like micro-milieu with 3D metabolic gradients. With increasing size, most MCTS not only exhibit proliferation gradients from the periphery towards the center but they also develop a spheroid type-specific nutrient supply pattern, such as radial oxygen partial pressure gradients. Similarly, MCTS of a variety of tumor cell lines exhibit a concentric histo-morphology, with a necrotic core surrounded by a viable cell rim. The spherical symmetry is an important prerequisite for investigating the effect of environmental factors on cell proliferation and viability in a 3D environment on a quantitative basis.
3.1 Geometrization Conjecture

Geometry and topology of smooth surfaces are related by the Gauss–Bonnet formula for a closed surface $\Sigma$ (see, e.g., [25, 28])

$$\frac{1}{2\pi} \int \int_{\Sigma} K \, dA = \chi(\Sigma) = 2 - 2 \text{gen}(\Sigma),$$  \hspace{1cm} (34)

where $dA$ is the area element of a metric $g$ on $\Sigma$, $K$ is the Gaussian curvature, $\chi(\Sigma)$ is the Euler characteristic of $\Sigma$ and gen($\Sigma$) is its genus, or number of handles, of $\Sigma$. Every closed surface $\Sigma$ admits a metric of constant Gaussian curvature $K = +1$, 0, or $-1$ and so is uniformized by elliptic, Euclidean, or hyperbolic geometry, which respectively have gen($S^2$) = 0 (sphere), gen($T^2$) = 1 (torus) and gen($\Sigma$) > 1 (torus with several holes). The integral (34) is a topological invariant of the surface $\Sigma$, always equal to 2 for all topological spheres $S^2$ (that is, for all closed surfaces without holes that can be continuously deformed from the geometric sphere) and always equal to 0 for the topological torus $T^2$ (i.e., for all closed surfaces with one hole or handle).

Topological framework for the Ricci flow (2) is Thurston’s Geometrization Conjecture [60], which states that the interior of any compact 3–manifold can be split in an essentially unique way by disjoint embedded 2–spheres $S^2$ and tori $T^2$ into pieces and each piece admits one of 8 geometric structures (including (i) the 3–sphere $S^3$ with constant curvature +1; (ii) the Euclidean 3–space $\mathbb{R}^3$ with constant curvature 0 and (iii) the hyperbolic 3–space $H^3$ with constant curvature $-1$). The geometrization conjecture (which has the Poincaré Conjecture as a special case) would give us a link between the geometry and topology of 3–manifolds, analogous in spirit to the case of 2–surfaces.

In higher dimensions, the Gaussian curvature $K$ corresponds to the Riemann curvature tensor $\mathcal{R}$ on a smooth $n$–manifold $M$, which is in local coordinates on $M$ denoted by its $(4, 0)$–components $R_{ijkl}$, or its $(3, 1)$–components $R_{ij}^{kl}$ (see Appendix, as well as e.g., [25, 28]). The trace (or, contraction) of $\mathcal{R}$, in $(4, 0)$–case using the inverse metric tensor $g^{ij} = (g_{ij})^{-1}$, is the Ricci tensor $\mathcal{R}_c$, the contracted curvature tensor, which is in a local coordinate system $\{x^i\}_{i=1}^n$ defined in an open set $U \subset M$, given by

$$R_{ij} = \text{tr}(\mathcal{R}) = g^{kl} R_{ijkl}$$

(using Einstein’s summation convention), while the scalar curvature is now given by the second contraction of $\mathcal{R}$ as

$$R = \text{tr}(\mathcal{R}_c) = g^{ij} R_{ij}.$$ 

In general, the Ricci flow $g_{ij}(t)$ is a one–parameter family of Riemannian metrics on a compact $n$–manifold $M$ governed by the equation (2), which has a unique solution for a short time for an arbitrary smooth metric $g_{ij}$ on $M$ [16]. If $\mathcal{R}_c > 0$ at any local point $x = \{x^i\}$ on $M$, then the Ricci flow (2) contracts the metric $g_{ij}(t)$ near $x$, to the future, while if $\mathcal{R}_c < 0$, then the flow (2)

\begin{footnote}
Another five allowed geometric structures are represented by the following examples: (iv) the product $S^2 \times S^1$; (v) the product $\mathbb{H}^2 \times S^1$ of hyperbolic plane and circle; (vi) a left invariant Riemannian metric on the special linear group $SL(2, \mathbb{R})$; (vii) a left invariant Riemannian metric on the solvable Poincaré-Lorentz group $E(1, 1)$, which consists of rigid motions of a $(1 + 1)$–dimensional space-time provided with the flat metric $dt^2 - dx^2$; (viii) a left invariant metric on the nilpotent Heisenberg group, consisting of $3 \times 3$ matrices of the form

$$\begin{pmatrix}
1 & * & * \\
0 & 1 & * \\
0 & 0 & 1
\end{pmatrix}.$$ 

In each case, the universal covering of the indicated manifold provides a canonical model for the corresponding geometry [43].
\end{footnote}
expands \( g_{ij}(t) \) near \( x \). The solution metric \( g_{ij}(t) \) of the Ricci flow equation (2) shrinks in positive Ricci curvature direction while it expands in the negative Ricci curvature direction, because of the minus sign in the front of the Ricci tensor \( R_{ij} \). In particular, on a 2-sphere \( S^2 \), any metric of positive Gaussian curvature will shrink to a point in finite time. At a general point, there will be directions of positive and negative Ricci curvature along which the metric will locally contract or expand (see [3]). Also, if a simply-connected compact 3-manifold \( M \) has a Riemannian metric \( g_{ij} \) with positive Ricci curvature then it is diffeomorphic to the 3-sphere \( S^3 \) [10]. More generally speaking, the Ricci flow deforms manifolds with positive Ricci curvature to a point which can be renormalized to the 3-sphere.

3.2 Reaction–diffusion–type evolution of curvatures and volumes

All three Riemannian curvatures \((R, R_c \text{ and } R_m)\), as well as the associated volume forms, evolve during the Ricci flow (2). In general, the Ricci–flow evolution equation (2) for the metric tensor \( g_{ij} \) implies the reaction–diffusion–type evolution equation for the Riemann curvature tensor \( R_m \) on an \( n \)-manifold \( M \),

\[
\partial_t R_m = \Delta R_m + Q_n,
\]

(35)

where \( Q_n \) is a quadratic expression of the Riemann \( n \)-curvatures, corresponding to the \( n \)-component bio–chemical reaction, while the term \( \Delta R_m \) corresponds to the \( n \)-component diffusion. From the general \( n \)-curvature evolution (35) we have two important particular cases:

1. The evolution equation for the Ricci curvature tensor \( R_c \) on a 3–manifold \( M \),

\[
\partial_t R_c = \Delta R_c + Q_3,
\]

(36)

where \( Q_3 \) is a quadratic expression of the Ricci 3-curvatures, corresponding to the 3-component bio–chemical reaction, while the term \( \Delta R_c \) corresponds to the 3-diffusion.

2. The evolution equation for the scalar curvature \( R \),

\[
\partial_t R = \Delta R + 2|R_c|^2,
\]

(37)

in which the term \( 2|R_c|^2 \) corresponds to the 2-component bio–chemical reaction, while the term \( \Delta R \) corresponds to the 2-component diffusion. By the maximum principle (see subsection 3.4), the minimum of the scalar curvature \( R \) is non–decreasing along the flow \( g(t) \), both on \( M \) and on its boundary \( \partial M \) (see [51]).

Let us now see in detail how various geometric quantities evolve given the short–time solution of the Ricci flow equation (2) on an arbitrary \( n \)-manifold \( M \). For this, we need first to calculate the

\[\text{By expanding the maximum principle for tensors, Hamilton proved that Ricci flow } g(t) \text{ given by (2) preserves the positivity of the Ricci tensor } R_c \text{ on 3-manifolds (as well as of the Riemann curvature tensor } R_m \text{ in all dimensions); moreover, the eigenvalues of the Ricci tensor on 3-manifolds (and of the curvature operator } R_m \text{ on 4-manifolds) are getting pinched point-wisely as the curvature is getting large [16, 17]. This observation allowed him to prove the convergence results: the evolving metrics (on a compact manifold) of positive Ricci curvature in dimension 3 (or positive Riemann curvature in dimension 4) converge, modulo scaling, to metrics of constant positive curvature. However, without assumptions on curvature, the long time behavior of the metric evolving by Ricci flow may be more complicated [52]. In particular, as } t \text{ approaches some finite time } T, \text{ the curvatures may become arbitrarily large in some region while staying bounded in its complement. On the other hand, Hamilton [19] discovered a remarkable property of solutions with nonnegative curvature tensor } R_m \text{ in arbitrary dimension, called the differential Harnack inequality, which allows, in particular, to compare the curvatures of the solution of (2) at different points and different times.}\]
variation formulas for the Christoffel symbols and curvature tensors on \( M \); then the corresponding evolution equations will naturally follow (see [16, 7, 8]). If \( g(s) \) is a 1–parameter family of metrics on \( M \) with \( \partial_s g_{ij} = v_{ij} \), then the variation of the Christoffel symbols \( \Gamma^k_{ij} \) on \( M \) is given by

\[
\partial_s \Gamma^k_{ij} = \frac{1}{2} g^{kl} (\nabla_i v_{jl} + \nabla_j v_{il} - \nabla_l v_{ij}) ,
\]

(38)

(where \( \nabla \) is the covariant derivative with respect to the Riemannian connection) from which follows the evolution of the Christoffel symbols \( \Gamma^k_{ij} \) under the Ricci flow \( g(t) \) on \( M \) given by \([2]\),

\[
\partial_t \Gamma^k_{ij} = -g^{kl} (\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}) .
\]

From \([38]\) we calculate the variation of the Ricci tensor \( R_{ij} \) on \( M \) as

\[
\partial_s R_{ij} = \nabla_m (\partial_s \Gamma^m_{ij}) - \nabla_i (\partial_s \Gamma^m_{mj}) ,
\]

(39)

and the variation of the scalar curvature \( R \) on \( M \) by

\[
\partial_s R = -\Delta V + \text{div}(\text{div} v) - \langle v, \text{Rc} \rangle ,
\]

(40)

where \( V = g^{ij} v_{ij} = \text{tr}(v) \) is the trace of \( v = (v_{ij}) \).

If an \( n \)--manifold \( M \) is oriented, then the volume \( n \)--form on \( M \) is given, in a positively–oriented local coordinate system \( \{x^i\} \in U \subset M \), by \([23]\)

\[
d\mu = \sqrt{\det(g_{ij})} \, dx^1 \land dx^2 \land ... \land dx^n .
\]

(41)

If \( \partial_s g_{ij} = v_{ij} \), then \( \partial_s d\mu = \frac{1}{2} V d\mu \). The evolution of the volume \( n \)--form \( d\mu \) under the Ricci flow \( g(t) \) on \( M \) is given by the exponential decay/growth relation with the scalar curvature \( R \) as the (variable) rate parameter,

\[
\partial_t d\mu = -Rd\mu ,
\]

(42)

which gives an exponential decay for \( R > a > 0 \) (elliptic geometry) and exponential growth for \( R <= a < 0 \) (hyperbolic geometry) – for any small constant \( a \) (scalar curvature must be bounded away from zero). The elementary volume evolution \([12]\) implies the integral form of the exponential relation for the total \( n \)--volume

\[
\text{vol}(g) = \int_M d\mu , \quad \text{as} \quad \partial_t \text{vol}(g(t)) = -\int_M Rd\mu ,
\]

which again gives an exponential decay for elliptic \( R > 0 \) and exponential growth for hyperbolic \( R < 0 \).

This is a crucial point for the tumor suppression by the body: the immune system needs to keep the elliptic geometry of the MTS, evolving by \([30]\) – by all possible means\(^{16}\). In the healthy

\[^{16}\text{As a tumor decay control tool, a monoclonal antibody therapy is usually proposed. Monoclonal antibodies (mAb) are mono-specific antibodies that are identical because they are produced by one type of immune cell that are all clones of a single parent cell. Given (almost) any substance, it is possible to create monoclonal antibodies that specifically bind to that substance; they can then serve to detect or purify that substance. The invention of monoclonal antibodies is generally accredited to Georges Köhler, César Milstein, and Niels Kaj Jerne in 1975 [36], who shared the Nobel Prize in Physiology or Medicine in 1984 for the discovery. The key idea was to use a line of myeloma cells that had lost their ability to secrete antibodies, come up with a technique to fuse these cells with healthy antibody producing B–cells, and be able to select for the successfully fused cells.}
organism this normally happens, because the initial MTS started as a spherical shape with \( R > 0 \).
The immune system just needs to keep the MTS in the spherical/elliptic shape and prevent any hyperbolic distortions of \( R < 0 \). Thus, it will naturally have an exponential decay and vanish.

Since the \( n \)--volume is not constant and sometimes we would like to prevent the solution from shrinking to an \( n \)--point on \( M \) (elliptic case) or expanding to an infinity (hyperbolic case), we can also consider the **normalized Ricci flow** on \( M \) (see [7]):

\[
\partial_t \hat{g}_{ij} = -2 \hat{R}_{ij} + \frac{2}{n} \hat{r} \hat{g}_{ij}, \quad \text{where} \quad \hat{r} = \text{vol}(\hat{g})^{-1} \int_M \hat{R} d\mu
\]

(43)
is the average scalar curvature on \( M \). We then have the \( n \)--volume conservation law:

\[
\partial_t \text{vol}(\hat{g}(t)) = 0.
\]

To study the **long–time existence** of the normalized Ricci flow (43) on an arbitrary \( n \)--manifold \( M \), it is important to know what kind of curvature conditions are preserved under the equation. In general, the Ricci flow \( g(t) \) on \( M \), as defined by the fundamental relation (2), tends to preserve some kind of positivity of curvatures. For example, positive scalar curvature \( R \) (i.e., elliptic geometry) is preserved both on \( M \) and on its boundary \( \partial M \) in any dimension. This follows from applying the maximum principle to the evolution equation (37) for scalar curvature \( R \) both on \( M \) and on \( \partial M \). Similarly, positive Ricci curvature is preserved under the Ricci flow on a 3–manifold \( M \). This is a special feature of dimension 3 and is related to the fact that the Riemann curvature tensor may be recovered algebraically from the Ricci tensor and the metric on 3–manifolds [7].

In particular, we have the following result for \( 2 \)--surfaces (see [18]): Let \( S = \partial M \) be a closed \( 2 \)--surface, which is a boundary of a compact 3–manifold \( M \). Then for any initial \( 2 \)--metric \( g_0 \) on \( \partial M \), the solution to the normalized Ricci flow (43) on \( \partial M \) exists for all time. In other words, the normalized Ricci flow in 2D always converges. Moreover, (i) if the Euler characteristic of \( \partial M \) is non–positive, then the solution metric \( g(t) \) on \( \partial M \) converges to a constant curvature metric as \( t \to \infty \); and (ii) if the scalar curvature \( R \) of the initial metric \( g_0 \) is positive, then the solution metric \( g(t) \) on \( \partial M \) converges to a positive constant curvature metric as \( t \to \infty \). (For surfaces with non–positive Euler characteristic, the proof is based primarily on maximum principle estimates for the scalar curvature.)

Applying to the tumor evolution (30), the normalized Ricci flow (43) of the MTS will make it completely round with a geometric sphere shell, which is ideal for surgical removal. This is our second option for the MTS growth/decay control. If we cannot force it to exponential decay, then we must try to normalize into a round spherical shell – which is suitable for surgical removal.

The negative flow of the total \( n \)--volume \( \text{vol}(g(t)) \) represents the **Einstein–Hilbert functional** (see [44, 7, 4])

\[
E(g) = \int_M R d\mu = -\partial_t \text{vol}(g(t)).
\]

If we put \( \partial_s g_{ij} = v_{ij} \), we have

\[
\partial_t E(g) = \int_M \left( -\Delta V + \text{div}(\text{div} v) - \langle v, \mathfrak{R} \rangle + \frac{1}{2} RV \right) d\mu = \int_M \left( v, \frac{1}{2} R g_{ij} - R_{ij} \right) d\mu,
\]

so, the critical points of \( E(g) \) satisfy **Einstein’s equation** \( \frac{1}{2} R g_{ij} = R_{ij} \) in the vacuum. The gradient flow of \( E(g) \) on an \( n \)--manifold \( M \),

\[
\partial_t g_{ij} = 2 (\nabla E(g))_{ij} = R_{ij} - 2 R_{ij},
\]

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is the Ricci flow (2) plus $R_{ij}$. Thus, Einstein metrics are the fixed points of the normalized Ricci flow\[17\]

Let $\Delta$ denote the Laplacian acting on functions on a closed $n$–manifold $M$, which is in local coordinates $\{x^i\} \in U \subset M$ given by

$$\Delta = g^{ij} \nabla_i \nabla_j = g^{ij} \left( \partial_{ij} - \Gamma^k_{ij} \partial_k \right).$$

For any smooth function $f$ on $M$ we have \[16, 8\]

$$\Delta \nabla_i f = \nabla_i \Delta f + R_{ij} \nabla_j f \quad \text{and} \quad \Delta |\nabla f|^2 = 2 |\nabla_i \nabla_j f|^2 + 2 R_{ij} \nabla_i f \nabla_j f + 2 \nabla_i f \nabla_j \Delta f,$$

from which it follows that if we have

$$\Re c \geq 0, \quad \Delta f \equiv 0, \quad |\nabla f| \equiv 1,$$

then

$$\nabla \nabla f \equiv 0 \quad \text{and} \quad \Re c (\nabla f, \nabla f) \equiv 0.$$

Using this Laplacian $\Delta$, we can write the linear heat equation on $M$ as $\partial_t u = \Delta u$, where $u$ is the temperature. In particular, the Laplacian acting on functions with respect to $g(t)$ will be denoted by $\Delta_{g(t)}$. If $(M, g(t))$ is a solution to the Ricci flow equation (2), then we have

$$\partial_t \Delta_{g(t)} = 2 R_{ij} \nabla_i \nabla_j.$$

The evolution equation [37] for the scalar curvature $R$ under the Ricci flow (2) follows from [10]. Using equation [51] from Appendix, we have:

$$\text{div}(\Re c) = \frac{1}{2} \nabla R, \quad \text{so that} \quad \text{div}(\text{div}(\Re c)) = \frac{1}{2} \Delta R,$$

showing again that the scalar curvature $R$ satisfies a heat–type equation with a quadratic nonlinearity both on a 3–manifold $M$ and on its boundary 2–surface $\partial M$.

Next we will find the exact form of the evolution equation [39] for the Ricci tensor $\Re c$ under the Ricci flow $g(t)$ given by (2) on any 3–manifold $M$. (Note that in higher dimensions, the appropriate formula of huge complexity would involve the whole Riemann curvature tensor $\Re m$.) Given a variation $\partial_s g_{ij} = v_{ij}$, from [39] we get

$$\partial_s R_{ij} = \frac{1}{2} \left( \Delta_L v_{ij} + \nabla_i \nabla_j V - \nabla_i (\text{div} v)_j - \nabla_j (\text{div} v)_i \right),$$

where $\Delta_L$ denotes the so–called Lichnerowicz Laplacian (which depends on $\Re m$, see [16, 8]). Since

$$\nabla_i \nabla_j R - \nabla_i (\text{div}(\Re c))_j - \nabla_j (\text{div}(\Re c))_i = 0,$$

by [51] (after some algebra) we get that under the Ricci flow (2) the evolution equation for the Ricci tensor $\Re c$ on a 3–manifold $M$ is

$$\partial_t R_{ij} = \Delta R_{ij} + 3 R R_{ij} - 6 R_{im} R_{jm} + \left( 2 |\Re c|^2 - R^2 \right) g_{ij}.$$
So, just as in case of the evolution of the scalar curvature \( \partial_t R \) (both on a 3–manifold \( M \) and on its 2–boundary \( \partial M \)), we get a heat–type evolution equation with a quadratic nonlinearity for \( \partial_t R_{ij} \), which means that positive Ricci curvature \( (\mathcal{R}_c > 0) \) of elliptic 3–geometry is preserved under the Ricci flow \( g(t) \) on \( M \).

More generally, we have the following result for 3–manifolds (see [16]): Let \( (M, g_0) \) be a compact Riemannian 3–manifold with positive Ricci curvature \( \mathcal{R}_c \). Then there exists a unique solution to the normalized Ricci flow \( g(t) \) on \( M \) with \( g(0) = g_0 \) for all time and the metrics \( g(t) \) converge exponentially fast to a constant positive sectional curvature metric \( g_\infty \) on \( M \). In particular, \( M \) is diffeomorphic to a 3–sphere \( S^3 \). (As a consequence, such a 3–manifold \( M \) is necessarily diffeomorphic to a quotient of the 3–sphere by a finite group of isometries. It follows that given any homotopy 3–sphere, if one can show that it admits a metric with positive Ricci curvature, then the Poincaré Conjecture would follow [7].) In addition, compact and closed 3–manifolds which admit a non-singular solution can also be decomposed into geometric pieces [20].

From the geometric evolution equations reviewed in this subsection, we see that both short–time and long–time geometric solution can always be found for 2–component bio–reaction–diffusion equations, as they correspond to evolution of the scalar 2–curvature \( \mathcal{R}_c \). Regarding the 3–component bio–reaction–diffusion equations, corresponding to evolution of the Ricci 3–curvature \( \mathcal{R}_c \), we can always find the short–time geometric solution, while the long–time solution exists only under some additional (compactness and/or closure) conditions. Finally, in case of \( n \)-component bio–reaction–diffusion equations, corresponding to evolution of the Riemann \( n \)-curvature \( \mathcal{R}_m \), only short–time geometric solution is possible.

### 3.3 Dissipative solitons and Ricci breathers

An important class of bio–reaction–diffusion systems are dissipative solitons (DSs), which are stable solitary localized structures that arise in nonlinear spatially extended dissipative systems due to mechanisms of self–organization. They can be considered as an extension of the classical soliton concept in conservative systems. Apart from aspects similar to the behavior of classical particles like the formation of bound states, DSs exhibit entirely nonclassical behavior – e.g., scattering, generation and annihilation – all without the constraints of energy or momentum conservation. The excitation of internal degrees of freedom may result in a dynamically stabilized intrinsic speed, or periodic oscillations of the shape.

In particular, stationary DSs are generated by production of material in the center of the DSs, diffusive transport into the tails and depletion of material in the tails. A propagating pulse arises from production in the leading and depletion in the trailing end. Among other effects, one finds periodic oscillations of DSs, the so–called ‘breathing’ dissipative solitons [15].

DSs in many different systems show universal particle–like properties. To understand and describe the latter, one may try to derive ‘particle equations’ for slowly varying order parameters like position, velocity or amplitude of the DSs by adiabatically eliminating all fast variables in the field description. This technique is known from linear systems, however mathematical problems arise from the nonlinear models due to a coupling of fast and slow modes [13].

Similar to low–dimensional dynamic systems, for supercritical bifurcations of stationary DSs one finds characteristic normal forms essentially depending on the symmetries of the system; e.g., for a transition from a symmetric stationary to an intrinsically propagating DS one finds the
Pitchfork normal form for the DS–velocity $\mathbf{v}$ \[6\],

$$\dot{\mathbf{v}} = (\sigma - \sigma_0)\mathbf{v} - |\mathbf{v}|^2\mathbf{v}$$

where $\sigma$ represents the bifurcation parameter and $\sigma_0$ the bifurcation point. For a bifurcation to a ‘breathing’ DS, one finds the Hopf normal form \[26\]

$$\dot{A} = (\sigma - \sigma_0)A - |A|^2A$$

for the amplitude $A$ of the oscillation. Note that the above problems do not arise for classical solitons as inverse scattering theory yields complete analytical solutions \[15\].

3.3.1 Ricci breathers and Ricci solitons

Closely related to dissipative solitons are the so–called breathers, solitonic structures given by localized periodic solutions of some nonlinear soliton PDEs, including the exactly–solvable sine–Gordon equation \[18\] and the focusing nonlinear Schrödinger equation \[19\].

A metric $g_{ij}(t)$ evolving by the Ricci flow $g(t)$ given by \[2\] on any 3–manifold $M$ is called a Ricci breather, if for some $t_1 < t_2$ and $\alpha > 0$ the metrics $\alpha g_{ij}(t_1)$ and $g_{ij}(t_2)$ differ only by a diffeomorphism; the cases $\alpha = 1, \alpha < 1, \alpha > 1$ correspond to steady, shrinking and expanding breathers, respectively. Trivial breathers on $M$, for which the metrics $g_{ij}(t_1)$ and $g_{ij}(t_2)$ differ only by diffeomorphism and scaling for each pair of $t_1$ and $t_2$, are called Ricci solitons. Thus, if one considers Ricci flow as a dynamical system on the space of Riemannian metrics modulo diffeomorphism and scaling, then breathers and solitons correspond to periodic orbits and fixed points respectively. At each time the Ricci soliton metric satisfies on $M$ an equation of the form \[51\]

$$R_{ij} + cg_{ij} + \nabla_i b_j + \nabla_j b_i = 0,$$

where $c$ is a number and $b_i$ is a 1–form; in particular, when $b_i = \frac{1}{2} \nabla_i a$ for some function $a$ on $M$, we get a gradient Ricci soliton. An important example of a gradient shrinking soliton is the Gaussian soliton, for which the metric $g_{ij}$ is just the Euclidean metric on $\mathbb{R}^3$, $c = 1$ and $a = -|x|^2/2$.

\[18\] An exact solution $u = u(x,t)$ of the (1+1)D sine–Gordon equation

$$\partial_t^2 u = \partial_x^2 u - \sin u,$$

is \[1\]

$$u = 4 \arctan \left( \frac{\sqrt{1 - \omega^2} \cos(\omega t)}{\omega \cosh(\sqrt{1 - \omega^2} x)} \right),$$

which, for $\omega < 1$, is periodic in time $t$ and decays exponentially when moving away from $x = 0$.

\[19\] The focusing nonlinear Schrödinger equation is the dispersive complex–valued (1+1)D PDE \[2\],

$$i \partial_t u + \partial_x^2 u + |u|^2 u = 0,$$

with a breather solution of the form:

$$u = \left( \frac{2b^2 \cosh(\theta) + 2ib \sqrt{2 - b^2} \sinh(\theta)}{2 \cosh(\theta) - \sqrt{2} \sqrt{2 - b^2} \cos(\omega a b x)} - 1 \right) a \exp(i a^2 t) \quad \text{with} \quad \theta = a^2 b \sqrt{2 - b^2} t,$$

which gives breathers periodic in space $x$ and approaching the uniform value $a$ when moving away from the focus time $t = 0$. 

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3.4 Smoothing/Averaging heat equation and Ricci entropy

Given a $C^2$ function $u : M \to \mathbb{R}$ on a Riemannian 3–manifold $M$, its Laplacian is defined in local coordinates $\{x^i\} \in U \subset M$ to be

$$\Delta u = \text{tr}_g (\nabla^2 u) = g^{ij} \nabla_i \nabla_j u,$$

where $\nabla_i$ is the covariant derivative (Levi–Civita connection, see Appendix). We say that a $C^2$ function $u : M \times [0, T) \to \mathbb{R}$, where $T \in (0, \infty]$, is a solution to the heat equation if (3) holds. One of the most important properties satisfied by the heat equation is the maximum principle, which says that for any smooth solution to the heat equation, whatever point-wise bounds hold at $t = 0$ also hold for $t > 0$. More precisely, we can state:

Let $u : M \times [0, T) \to \mathbb{R}$ be a $C^2$ solution to the heat equation (3) on a complete Riemannian 3–manifold $M$. If $C_1 \leq u(x, 0) \leq C_2$ for all $x \in M$, for some constants $C_1, C_2 \in \mathbb{R}$, then $C_1 \leq u(x, t) \leq C_2$ for all $x \in M$ and $t \in [0, T)$. This property exhibits the averaging behavior of the heat equation (3) on $M$.

Now, consider Perelman’s entropy functional $\mathcal{F}$ on a 3–manifold $M$:

$$\mathcal{F} = \int_M (R + |\nabla f|^2)e^{-f}d\mu$$

(44)

for a Riemannian metric $g_{ij}$ and a (temperature-like) scalar function $f$ (which satisfies the backward heat equation) on a closed 3–manifold $M$, where $d\mu$ is the volume 3–form $\delta$. During the Ricci flow (2), $\mathcal{F}$ evolves on $M$ as

$$\partial_t \mathcal{F} = 2 \int |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f}d\mu.$$  (45)

Now, define $\lambda(g_{ij}) = \inf \mathcal{F}(g_{ij}, f)$, where infimum is taken over all smooth $f$, satisfying

$$\int_M e^{-f}d\mu = 1. $$

(46)

$\lambda(g_{ij})$ is the lowest eigenvalue of the operator $-4\Delta + R$. Then the entropy evolution formula (45) implies that $\lambda(g_{ij}(t))$ is nondecreasing in $t$, and moreover, if $\lambda(t_1) = \lambda(t_2)$, then for $t \in [t_1, t_2]$ we have $R_{ij} + \nabla_i \nabla_j f = 0$ for $f$ which minimizes $\mathcal{F}$ on $M$. Thus a steady breather on $M$ is necessarily a steady soliton.

If we define the conjugate heat operator on $M$ as

$$\Box^* = -\partial/\partial t - \Delta + R$$

we have $f(R_{ij}g_{ij})$ is a certain linear function of $R_{ij}$ and $g_{ij}$. According to Penrose’s Weyl curvature hypothesis, the entire history of a closed universe starts from a uniform low–entropy Big Bang with zero Weyl curvature tensor $\mathcal{W}$ (i.e., $\mathcal{W}$ with the Ricci tensor $\mathcal{R}$ removed),

$$\mathcal{W} = \mathcal{R} - f(R_{ij}g_{ij}),$$

where $f(R_{ij}g_{ij})$ is a certain linear function of $R_{ij}$ and $g_{ij}$. According to Penrose’s Weyl curvature hypothesis, the entire history of a closed universe starts from a uniform low–entropy Big Bang with zero Weyl curvature tensor of the cosmological gravitational field and ends with a high–entropy Big Crunch, representing the congealing of may black holes, with Weyl tensor approaching infinity (see [50, 21]).

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then we have the conjugate heat equation\(^{21}\) \([51]\)
\[
\Box^* u = 0.
\] (48)

The entropy functional \((44)\) is nondecreasing under the coupled Ricci–diffusion flow on \(M\) (see \([64, 40]\))
\[
\partial_t g_{ij} = -2R_{ij}, \quad \partial_t u = -\Delta u + \frac{R}{2} u - \frac{\nabla u}{u}, \tag{49}
\]
where the second equation ensures \(\int_M u^2 d\mu = 1\), to be preserved by the Ricci flow \(g(t)\) on \(M\). If we define \(u = e^{-\frac{t}{4}}\), then the right–hand equation in \((49)\) is equivalent to the generic scalar–field \(f\)–evolution equation on \(M\),
\[
\partial_t f = -\Delta f - R + |\nabla f|^2,
\]
which instead preserves \((46)\).

The coupled Ricci–diffusion flow \((49)\), or equivalently, the dual system
\[
\partial_t g_{ij} = \Delta_M g_{ij} + Q_{ij}(g, \partial g), \quad \partial_t f = -\Delta f - R + |\nabla f|^2,
\] (50)
is our global decay model for a general \(n\)–dimensional bio–reaction–diffusion process, including both geometric and bio–chemical multi–phase evolution.

The sole hypothesis of this paper is that any kind of reaction–diffusion processes in biology, chemistry and physics is subsumed by the geometric–diffusion system \((49)\), or the dual system \((50)\).

### 4 Conclusion

In this paper we have conjectured that the Ricci–flow equation from Riemannian geometry:
\[
\partial_t g_{ij} = \Delta_M g_{ij} + Q_{ij}(g, \partial g),
\]
\(^{21}\)In \([51]\) Perelman stated a differential Li–Yau–Hamilton (LYH) type inequality \([22]\) for the fundamental solution \(u = u(x, t)\) of the conjugate heat equation \((45)\) on a closed \(n\)–manifold \(M\) evolving by the Ricci flow \((2)\). Let \(p \in M\) and
\[
u = \frac{1}{4\pi t^2} e^{-f} u,
\]
be the fundamental solution of the conjugate heat equation in \(M \times (0, T)\),
\[
\Box^* u = 0, \quad or \quad \partial_t u + \Delta u = Ru,
\]
where \(\tau = T - t\) and \(R = R(\cdot, t)\) is the scalar curvature of \(M\) with respect to the metric \(g(t)\) with \(\lim_{t \to T} u = \delta_p\) (in the distribution sense), where \(\delta_p\) is the delta–mass at \(p\). Let
\[
v = |\tau(2\Delta f - |\nabla f|^2 + R) + f - n|u,
\]
where \(\tau = T - t\). Then we have a differential LYH–type inequality
\[
v(x, t) \leq 0 \quad in \quad M \times (0, T).
\] (47)
This result was used by Perelman to give a proof of the pseudolocality theorem \([51]\) which roughly said that almost Euclidean regions of large curvature in closed manifold with metric evolving by Ricci flow \(g(t)\) given by \((2)\) remain localized.

In particular, let \((M, g(t)), 0 \leq t \leq T, \partial M \neq \phi\), be a compact 3–manifold with metric \(g(t)\) evolving by the Ricci flow \(g(t)\) given by \((2)\) such that the second fundamental form of the surface \(\partial M\) with respect to the unit outward normal \(\partial/\partial v\) of \(\partial M\) is uniformly bounded below on \(\partial M \times [0, T]\). A global Li–Yau gradient estimate \([39]\) for the solution of the generalized conjugate heat equation was proved in \([22]\) (using a a variation of the method of P. Li and S.T. Yau, \([39]\)) on such a manifold with Neumann boundary condition.
can be considered as a general geometric framework for various nonlinear reaction-diffusion systems (and related dissipative solitons) in mathematical biology. More precisely, we proposed a hypothesis that any kind of reaction-diffusion processes in biology, chemistry and physics can be modelled by the combined geometric-diffusion system represented by the Ricci-flow equation. The validity of this hypothesis was demonstrated by reviewing a number of popular nonlinear reaction-diffusion systems from biology, chemistry and physics and showed that they could all be subsumed by the geometric framework of the Ricci flow.

5 Appendix: Riemann and Ricci curvatures on a smooth $n$–manifold

Recall that proper differentiation of vector and tensor fields on a smooth Riemannian $n$–manifold is performed using the Levi–Civita covariant derivative (see, e.g., [25 28]). Formally, let $M$ be a Riemannian $n$–manifold with the tangent bundle $TM$ and a local coordinate system $\{x^i\}_{i=1}^n$ defined in an open set $U \subset M$. The covariant derivative operator, $\nabla_X : C^\infty(TM) \rightarrow C^\infty(TM)$, is the unique linear map such that for any vector fields $X, Y, Z$, constant $c$, and function $f$ the following properties are valid:

$$\nabla_X (Y + fZ) = \nabla_X Y + (Xf)Z + f\nabla_X Z,$$

$$\nabla_X Y - \nabla_Y X = [X, Y], \quad \text{(torsion free property)}$$

where $[X, Y]$ is the Lie bracket of $X$ and $Y$ (see, e.g., [23]). In local coordinates, the metric $g$ is defined for any orthonormal basis $(\partial_i = \partial_x^i)$ in $U \subset M$ by

$$g_{ij} = g(\partial_i, \partial_j) = \delta_{ij}, \quad \partial_k g_{ij} = 0.$$

Then the affine Levi–Civita connection is defined on $M$ by

$$\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k,$$

where

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left( \partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right)$$

are the (second-order) Christoffel symbols.

Now, using the covariant derivative operator $\nabla_X$ we can define the Riemann curvature $(3, 1)$–tensor $\mathfrak{R}m$ by (see, e.g., [25 28])

$$\mathfrak{R}m(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

$\mathfrak{R}m$ measures the curvature of the manifold by expressing how noncommutative covariant differentiation is. The $(3, 1)$–components $R^l_{ijk}$ of $\mathfrak{R}m$ are defined in $U \subset M$ by

$$\mathfrak{R}m(\partial_i, \partial_j) \partial_k = R^l_{ijk} \partial_l, \quad \text{which expands (see [44]) as}$$

$$R^l_{ijk} = \partial_l \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^m_{jk} \Gamma^l_{im} - \Gamma^m_{ik} \Gamma^l_{jm}.$$
The first and second Bianchi identities for the Riemann $(4,0)$–tensor $R_{ijkl}$ hold,
\[ R_{ijkl} + R_{jkil} + R_{kijl} = 0, \quad \nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0, \]
while the twice contracted second Bianchi identity reads
\[ 2 \nabla_j R_{ij} = \nabla_i R. \quad (51) \]

The $(0,2)$ Ricci tensor $\mathfrak{R}$ is the trace of the Riemann $(3,1)$–tensor $\mathfrak{R}^m$,
\[ \mathfrak{R}(Y,Z) + \text{tr}(X \rightarrow \mathfrak{R}^m(X,Y)Z), \] so that \[ \mathfrak{R}(X,Y) = g(\mathfrak{R}^m(\partial_i X)\partial_i Y), \]
Its components $R_{jk} = \mathfrak{R}(\partial_j, \partial_k)$ are given in $U \subset M$ by the contraction $[44]$
\begin{align*}
R_{jk} &= R^i_{jk}, \quad \text{or, in terms of Christoffel symbols,} \\
R_{jk} &= \partial_i \Gamma^i_{jk} - \partial_k \Gamma^i_{ji} + \Gamma^i_{mi} \Gamma^m_{jk} - \Gamma^i_{mk} \Gamma^m_{ji}.
\end{align*}
Being a symmetric second–order tensor, $\mathfrak{R}$ has $n + 12$ independent components on an $n$–manifold $M$. In particular, on a 3–manifold, it has 6 components, and on a 2–surface it has only the following 3 components:
\begin{align*}
R_{11} &= g^{22} R_{2112}, & R_{12} &= g^{12} R_{2121}, & R_{22} &= g^{11} R_{1221},
\end{align*}
which are all proportional to the corresponding coordinates of the metric tensor,
\[ \frac{R_{11}}{g_{11}} = \frac{R_{12}}{g_{12}} = \frac{R_{22}}{g_{22}} = \frac{R_{1212}}{\det(g)}. \quad (52) \]

Finally, the scalar curvature $R$ is the trace of the Ricci tensor $\mathfrak{R}$, given in $U \subset M$ by: \[ R = g^{ij} R_{ij}. \]

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