HYPERSPACES OF COUNTABLE COMPACTA

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Abstract. Hyperspaces \( \mathcal{H}(X) \) of all countable compact subsets of a metric space \( X \) and \( \mathcal{A}_n(X) \) of infinite compact subsets which have at most \( n \) (\( n \in \mathbb{N} \)) or finitely many (\( n = \omega \)) or countably many (\( n = \omega + 1 \)) accumulation points are studied. By descriptive set-theoretical methods, we fully characterize them for 0-dimensional, dense-in-itself, Polish spaces and partially for \( \sigma \)-compact spaces \( X \). Using the theory of absorbing sets, we get characterizations of \( \mathcal{H}(X) \), \( \mathcal{A}_\omega(X) \) and \( \mathcal{A}_{\omega + 1}(X) \) for nondegenerate connected, locally connected Polish spaces \( X \) which are either locally compact or nowhere locally compact. For every \( n \in \mathbb{N} \), we show that if \( X \) is an interval or a simple closed curve, \( \mathcal{A}_n(X) \) is homeomorphic to the linear space \( c_0 = \{ (x_i) \in \mathbb{R}^\omega : \lim x_i = 0 \} \) with the product topology; if \( X \) is a Peano continuum and a point \( p \in X \) is of order \( \geq 2 \), then the hyperspace \( \mathcal{A}_1(X, \{p\}) \) of all compacta with exactly one accumulation point \( p \) also is homeomorphic to \( c_0 \).

1. Introduction

All spaces in the paper are metric.
Let \( \mathcal{K}(X) \) be the hyperspace of all nonempty compact subsets of \( X \) with the Vietoris topology. It is well known that \( \mathcal{K}(X) \) shares many basic topological properties of space \( X \) like, e.g., completeness, local compactness, compactness, connectedness, local connectedness, dimension 0. Recall also that \( \mathcal{K}(X) \) is an absolute neighborhood retract (ANR) if and only if \( X \) is locally continuum-connected and it is an absolute retract (AR) if, additionally, \( X \) is connected [11]; if \( X \) is nondegenerate noncompact, locally compact, locally connected (connected) then \( \mathcal{K}(X) \) is an \( I^\omega \)-manifold (\( \cong I^\omega \setminus \{ \text{point} \} \)) [11].

For a nondegenerate Peano continuum \( X \), \( \mathcal{K}(X) \cong I^\omega \) (the symbol \( \cong \) stands for “homeomorphic to”).

The hyperspace \( \mathcal{F}(X) \subset \mathcal{K}(X) \) of all finite subsets of \( X \) was also extensively studied for various spaces \( X \). Clearly, for the rationals \( \mathbb{Q} \), \( \mathcal{F}(\mathbb{Q}) \cong \mathbb{Q} \).

It follows from Lemma [3.9 (19 Lemma 3.1)] that \( \mathcal{F}(\mathbb{R} \setminus \mathbb{Q}) \cong \mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q}) \) and \( \mathcal{F}(\{0,1\}^\omega) \cong \mathbb{Q} \times \{0,1\}^\omega \). If \( X \) is locally path-connected (and connected) then \( \mathcal{F}(X) \) is an ANR (AR) which is homotopy dense in \( \mathcal{K}(X) \) [15]; for a nondegenerate Peano continuum \( X \), \( \mathcal{F}(X) \cong [0,1]^\omega \setminus (0,1)^\omega \) [13].

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Another interesting subspace of $\mathcal{K}(X)$ is the hyperspace $\mathcal{H}(X)$ of all nonempty at most countable compacta which seems to have been less recognized. In general, if $X$ is an uncountable Polish space, then $\mathcal{H}(X)$ is $\Pi^1_1$-complete [23, Theorem (27.5)] (in such case we will call it the Hurewicz set for $X$). The hyperspace $\mathcal{H}(\mathbb{Q})$ was characterized by H. Michalewski [29] as a first category, zero-dimensional, separable, metrizable space with the property that every nonempty clopen subset is $\Pi^1_1$-coanalytic.

Another interesting subspace of $\mathcal{K}(X)$ is $\mathcal{K}(\mathbb{Q})$. For any Polish space or a $\sigma$-compact metric space without isolated points, $\mathcal{K}(\mathbb{Q})$ is a $\Pi^1_1$-absorbing $\sigma\delta\sigma$-absorber $\Pi^1_1$. For any Polish space $X$ without isolated points, $\mathcal{K}(\mathbb{Q})$ is true absolute $F_{\sigma\delta}$ and $\mathcal{A}_\omega(X)$ is true absolute $F_{\sigma\delta\sigma}$. For $X = \mathbb{Q}$, $\mathcal{A}_n(\mathbb{Q})$ is in the small Borel class $D_{2n}(F_{\sigma\delta})$ in $\mathcal{K}(\mathbb{R})$ but we do not know if it is absolute $F_{\sigma\delta}$.

Next, we characterize $\mathcal{A}_n(X)$, $n \in \mathbb{N}$, for any 0-dimensional Polish space which is dense-in-itself (i.e., without isolated points) as the infinite product $\mathbb{Q}^\omega$. If $X$ is a 0-dimensional $\sigma$-compact metric space without isolated points then

$$\mathcal{A}_n(X, F) := \{ A \in \mathcal{A}_n(X) : A' \subset F \} \cong \mathbb{Q}^\omega$$

for any $F \in \mathcal{K}(X)$ of cardinality $|F| \geq n$. In particular, the hyperspace $\mathcal{A}_1(\mathbb{Q}, \{ q \}) = \{ A \in \mathcal{A}_1(\mathbb{Q}) : A' = \{ q \} \}$ also is homeomorphic to $\mathbb{Q}^\omega$. Thus, we get a partial answer to the question asked in [20] if $\mathcal{A}_1(\mathbb{R} \setminus \mathbb{Q})$ is homeomorphic to $\mathcal{A}_1(\mathbb{Q})$. The full positive answer is equivalent to the $F_{\sigma\delta\sigma}$-absoluteness of $\mathcal{A}_1(\mathbb{Q})$ which remains an open problem.

We show that for a dense-in-itself, 0-dimensional space $X$, the hyperspace $\mathcal{A}_\omega(X)$ is homeomorphic to the standard, everywhere $\Pi^1_1$-complete set $S_4 \subseteq \{0,1\}^\omega$ in two cases: $X$ a Polish space or a $\sigma$-compact metric space.

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If $X$ is a dense-in-itself, 0-dimensional Polish space, then the hyperspaces $\mathcal{A}_n(X)$ and $\mathcal{H}(X)$ are homeomorphic to $\mathcal{H}(\mathbb{Q})$.

Studying hyperspaces of compacta of reasonably nice spaces of positive dimensions, we unavoidably enter into infinite-dimensional topology. Here, we intensively employ the theory of absorbing sets. We describe (apparently new) an $F_{\sigma\delta\sigma}$-absorber $\Pi_3$ and an $F_{\sigma\delta\sigma}$-absorber $\Sigma_4$ in the Hilbert cube and
our main results in Section 8 are the following characterizations:
\[ A_\omega(X) \cong \Sigma_4, \quad A_{\omega+1}(X) \cong \mathcal{H}(X) \cong \mathcal{H}(I) \]
if \( X \) is nondegenerate, connected, locally connected and either (1) locally compact or (2) Polish, nowhere locally compact.

One of the simplest examples of type (2)-spaces is \( \mathbb{R}^2 \setminus \mathbb{Q}^2 \). Other natural examples of such spaces include the set of all “irrational points” of the Sierpiński carpet, infinite countable products of non-compact intervals, Nöbeling or Lipscomb universal spaces of dimension \( \geq 1 \). In particular, the characterization extends the Cauty’s characterization of \( \mathcal{H}(I) \) over all spaces \( X \) as in (1) and (2).

The hyperspace \( A_n(X) \) is more difficult to handle. In Sections 9 and 10 the following characterizations are obtained:
\[ A_n(I) \cong A_n([0,1]) \cong A_n(S^1) \cong \Pi_3 \cong c_0 : = \{ (x_i) \in \mathbb{R}^\omega : \lim x_i = 0 \}. \]
Incidentally, the characterizations answer [20 Question 2.17] and a question in [6] if \( A_n(S^1) \) is contractible.

Finally, in Section 11 we show that if \( X \) is a Peano continuum with a point \( p \) of order \( \geq 2 \) and \( p \in F \in \mathcal{F}(X) \), then
\[ \{ A \in A_n(X) : X \subset A \} \cong A_1(X, \{ p \}) \cong c_0. \]

2. Borel complexity of \( A_n(X) \) and \( A_\omega(X) \)

Let us recall the standard notations of absolute Borel classes of spaces:

- for a countable ordinal \( \alpha \geq 1 \), \( \Pi_\alpha^0 \) is the absolute \( \alpha \)-th multiplicative class (i.e., \( \Pi_1^0 \) is the class of compact metrizable spaces, \( \Pi_2^0 \) is the class of Polish spaces, etc.);
- for \( \alpha \geq 2 \), \( \Sigma_\alpha^0 \) is the absolute \( \alpha \)-th additive class (i.e., \( \Sigma_2^0 \) is the class of \( \sigma \)-compact spaces, \( \Sigma_3^0 \) is the class of absolute \( G^\delta_\sigma \)-spaces, etc.).

The class of absolute coanalytic spaces is denoted by \( \Pi^1_1 \).

Let \( \Gamma \) be a family of subsets of \( X \). For a natural number \( n \), let \( D_{2n}(\Gamma) \) be the family of sets of type \( \bigcup_{k=1}^n (A_{2k-1} \setminus A_{2k}) \) where \( (A_k)_{k=1}^n \) is an increasing sequence of sets from \( \Gamma \). For the class \( \Gamma \) of \( F^\delta_\sigma \)-sets, elements of the small Borel class \( D_{2n}(\Gamma) \) will be called sets of difference type \( D_{2n}(F^\delta_\sigma) \).

If \( \mathcal{P} \) is a topological property, then a subspace \( Y \subset Z \) is everywhere (nowhere) \( \mathcal{P} \) if every (no) nonempty, relatively open subset of \( Y \) has \( \mathcal{P} \).

For \( E, F \subset X \) and \( n \in \mathbb{N} \), denote:
- the closed subspace \( F_n(X) = \{ A \in \mathcal{K}(X) : |A| \leq n \} \) of \( \mathcal{K}(X) \),
- \( A_{=n}(X) = \{ A \in \mathcal{K}(X) : |A| = n \} \),
- \( A_n(E, F) = \{ A \in A_n(X) : A \setminus A' \subset E \setminus A' \subset F \} \),
- \( A_{=n}(E, F) = \{ A \in A_{=n}(X) : A \setminus A' \subset E \setminus A' \subset F \} \),
- \( \mathcal{K}(X)^E = \{ K \in \mathcal{K}(X) : F \subset K \} \), \( A_n(X)^E = A_n(X) \cap \mathcal{K}(X)^E \),
- \( \mathcal{K}(X)_F = \{ K \in \mathcal{K}(X) : F \cap K \neq \emptyset \} \), \( A_n(X)_F = A_n(X) \cap \mathcal{K}(X)_F \).
Theorem 2.1. For each \( n \in \mathbb{N} \),

1. \( A_n(X) \) and \( A_n(X, F) \), for any \( F \in \mathcal{K}(X) \), are \( F_{\sigma \delta} \)-sets in \( \mathcal{K}(X) \);
2. \( A_\omega(X) \) is \( F_{\sigma \delta} \) in \( \mathcal{K}(X) \);
3. if \( X \) is metric separable and \( F \) is an \( F_{\sigma} \)-subset of \( X \), then \( A_\omega(X) \) is \( F_{\sigma \delta} \) in \( \mathcal{K}(X) \);
4. \( A_n(X, F) \) is of type \( F_{\sigma \delta} \) in \( \mathcal{K}(X) \);
5. \( A_n(F, X) \) is of type \( G_{\delta \sigma} \) in \( \mathcal{K}(X) \);
6. \( A_n(F, X), A_n(X, F) \) and \( A_n(F) \) are of type \( D_{2n}(F_{\sigma \delta}) \) in \( \mathcal{K}(X) \).

Proof. K. Kuratowski proved in [26] that the derived set map

\[ D : \mathcal{K}(X) \setminus \mathcal{F}(X) \to \mathcal{K}(X), \quad D(A) = A' \]

is Borel of the second class. Actually, it is convenient to consider \( D \) as a map from the whole \( \mathcal{K}(X) \) to the space \( \mathcal{K}(X) \cup \{\emptyset\} \) with the isolated point \( \{\emptyset\} \) and then a direct proof in [10] of Kuratowski’s theorem shows that the preimage under \( D \) of each closed set in \( \mathcal{K}(X) \) is \( F_{\sigma \delta} \) in \( \mathcal{K}(X) \).

Observe that

\[ A_n(X) = D^{-1}(\mathcal{F}_n(X)) \quad \text{and} \quad A_n(X, F) = D^{-1}(F) \cap A_n(X) \]

which establishes (1) and yields (2).

Now, fix a metric \( d \) generating the topology of \( X \). For \( x \in X \) and \( \varepsilon > 0 \) denote by

\[ B(x, \varepsilon) = \{ y \in X : d(y, x) < \varepsilon \} \quad \text{and} \quad B[x, \varepsilon] = \{ y \in X : d(y, x) \leq \varepsilon \} \]

the open and closed \( \varepsilon \)-balls centered at \( x \). For \( A \subseteq X \), let

\[ B(A, \varepsilon) = \bigcup_{a \in A} B(a, \varepsilon) \quad \text{and} \quad B[A, \varepsilon] = \bigcup_{a \in A} B[a, \varepsilon] \]

Fix a countable dense set \( D \) in \( X \) and \( n \in \mathbb{N} \).

(3). The equality

\[ A_\equiv(X, F) \cap \bigcap_{m \in \mathbb{N}} \bigcup_{A \in [D] \leq n} \{ K \in A_\equiv(X) : K \setminus B[A, \frac{1}{m}] \in [F_k]^{\leq k} \} \]

witnesses that the sets \( A_\equiv(X, F) \) is of type \( F_{\sigma \delta} \) in \( A_\equiv(X) \).

(4). The equality

\[ A_\equiv(X) \setminus A_\equiv(X, F) = \bigcap_{k \in \mathbb{N}} \bigcup_{A \in [D] \leq n} \bigcup_{m \in \mathbb{N}} \{ K \in A_\equiv(X) : K \cap B(F_k, \frac{1}{m}) \setminus B[A, \frac{1}{m}] \in [X]^{\leq m} \} \]
Corollary 2.2. If $P$ will exploit the standard sets $F$ represented in slightly different from $[23]$ but equivalent forms: complete, respectively $[23, Exercise (23.1), Exercise (23.6)]$. They can be $(2), (6)$ of Theorem 2.1 are absolute.

(5). Since $A_n(F) = A_n(F, X) \cap A_n(X, F)$, the set $A_n(F)$ is of type $D_2(F_{\sigma\delta})$ in $A_n(X)$ according to the preceding two statements.

(6). For every $k \in \mathbb{N}$, choose an $F_{\sigma\delta}$-set $S_{2k}$ in $K(X)$ such that

$$S_{2k} \cap A_{=k}(X) = A_{=k}(F, X).$$

Since $A_k(X)$ and $A_{k-1}(X)$ are $F_{\sigma\delta}$-sets in $K(X)$, we can assume that $A_{k-1}(X) \subseteq S_{2k} \subseteq A_k(X)$. Put $S_1 = A_0(X) \setminus A_0(F)$ and $S_{2k-1} = A_{k-1}(X)$ for $k > 1$.

Since $A_n(F, X) = \bigcup_{k=1}^{n} (S_{2k} \setminus S_{2k-1})$, the set $A_n(F, X)$ is of type $D_{2n}(F_{\sigma\delta})$ in $K(X)$.

For every $k > 1$ put $T_{2k} = A_k(X)$ and choose an $F_{\sigma\delta}$-set $T_{2k-1}$ in $K(X)$ such that

$$T_{2k-1} \cap A_{=k}(X) = A_{=k}(X) \setminus A_{=k}(F, X).$$

Since $A_k(X)$ and $A_{k-1}(X)$ are $F_{\sigma\delta}$-sets in $K(X)$, we can assume that $A_{k-1}(X) \subseteq T_{2k-1} \subseteq A_k(X)$. Put also

$$T_2 = A_1(X), \quad T_1 = A_1(X) \setminus A_1(X, F) = A_{=1}(X) \setminus A_{=1}(X, F).$$

Since

$$A_n(X, F) = \bigcup_{k=1}^{n} (T_{2k} \setminus T_{2k-1}),$$

the set $A_n(X, F)$ is of type $D_{2n}(F_{\sigma\delta})$ in $K(X)$.

Since $A_{=1}(X) = A_1(X)$ is of type $F_{\sigma\delta}$ in $K(X)$, the set $A_{=1}(F, X) \setminus A_{=1}(X, F)$ is of type $F_{\sigma\delta}$ in $A_{=1}(X)$ (by statements (3) and (4)) and hence in $K(X)$. It means that the set

$$Q_1 = A_1(F, X) \setminus A_1(F) = A_{=1}(F, X) \setminus A_{=1}(X, F)$$

is of type $F_{\sigma\delta}$ in $K(X)$. For $k > 1$, let $Q_{2k-1} = S_{2k} \cap T_{2k-1}$.

Since

$$A_n(F) = \bigcup_{k=1}^{n} (S_{2k} \setminus Q_{2k-1}),$$

the set $A_n(F)$ is of type $D_{2n}(F_{\sigma\delta})$ in $K(X)$. □

**Corollary 2.2.** If $X$ is a Polish space then the Borel classes of sets in (1), (2), (6) of Theorem 2.1 are absolute.

In order to evaluate Borel classes of $A_n(X)$ and $A_\omega(X)$ from below, we will exploit the standard sets $P_3$ and $S_4$ which are $\Pi^0_4$-complete and $\Sigma^0_4$-complete, respectively [23, Exercise (23.1), Exercise (23.6)]. They can be represented in slightly different from [23] but equivalent forms:

- $P_3 = \{(x_i)_{i \in \mathbb{N}} : \forall j \in \omega \forall^\infty k \in \omega \ (x_{2j}(2k+1) = 0)\}$
- $S_4 = \{(x_i)_{i \in \mathbb{N}} : \forall^\infty j \in \omega \forall^\infty k \in \omega \ (x_{2j}(2k+1) = 0)\}$. 
Theorem 2.3. If \( X \) contains a compactum of the Cantor-Bendixson rank 3, then \( A_n(X) \) is not \( G_{\delta\sigma} \) in \( K(X) \) and \( A_\omega(X) \) is not \( G_{\delta\sigma\delta} \) in \( K(X) \). If \( X \) is dense-in-itself then

1. \( A_n(X) \) and \( A_\omega(X) \) are nowhere \( G_{\delta\sigma} \) and nowhere \( G_{\delta\sigma\delta} \) in \( K(X) \), respectively.
2. If \( F \in K(X) \) and \( |F| \geq n \) then \( A_n(X,F) \) is nowhere \( G_{\delta\sigma} \) in \( K(X)_F \)
3. If \( F \in K(X) \) is infinite then \( A_\omega(X,F) \) is nowhere \( G_{\delta\sigma\delta} \) in \( K(X)_F \).

Proof. Let the Cantor-Bendixson rank of some \( A \in K(X) \) equals 3. Without loss of generality we can assume that \( A = \text{cl}\{2^{-j} + 2^{-(j+k)} : j, k \in \omega\} \). For any \( n \in \mathbb{N} \), put

\[
\chi(n) = \begin{cases} 
\text{cl}\{2^{-j} + 2^{-(j+k)} : 1 \leq j \leq n-1, k \in \omega\}, & \text{if } n > 1; \\
\emptyset, & \text{if } n = 1.
\end{cases}
\]

Define a continuous map \( \psi_n : \{0,1\}^\mathbb{N} \to K(X) \) by

\[
\psi_n(x) = \chi(n) \cup \text{cl}\{2^{-j} + 2^{-(j+k)x_{2j-n(2k+1)}} : j \geq n, k \in \omega\} \subset A.
\]

One easily checks that

\[
\psi_n^{-1}(A_n(X)) = P_3 \quad \text{and} \quad \psi_1^{-1}(A_\omega(X)) = S_4.
\]

This guarantees that \( A_n(X) \) is not \( G_{\delta\sigma} \) and \( A_\omega(X) \) is not \( G_{\delta\sigma\delta} \).

Now, assume that \( X \) has no isolated points. Then every nonempty open subset of \( X \) contains a copy of \( A \). Consider a basic open set \( \mathcal{U} \) in the Vietoris topology in \( K(X) \):

\[
\mathcal{U} = \langle U_1, \ldots, U_k \rangle = \{K \in K(X) : K \subset \bigcup_{i=1}^k U_i, (\forall i) U_i \cap K \neq \emptyset\},
\]

where \( U_i \)’s are open subsets of \( X \) and pick points \( u_i \in U_i \) for \( i = 1, \ldots, k \).

(1). Assume, without loss of generality, that \( A \subset U_1 \). Then

\[
A \cup \{u_1, \ldots, u_k\} \in \mathcal{U}
\]

and the map

\[
\tilde{\psi}_n : \{0,1\}^\mathbb{N} \to \mathcal{U}, \quad \tilde{\psi}_n(x) = \psi_n(x) \cup \{u_1, \ldots, u_k\}
\]

satisfies

\[
(\tilde{\psi}_n)^{-1}(A_n(X) \cap \mathcal{U}) = P_3 \quad \text{and} \quad (\tilde{\psi}_1)^{-1}(A_\omega(X) \cap \mathcal{U}) = S_4
\]

which completes the proof of (1).

(2) and (3). Let \( \mathcal{U}_F = \langle U_1, \ldots, U_k \rangle \cap K(X)_F \). We can assume that \( A \subset \bigcup_{i=1}^k U_i \) and \( \{0\} \cup \{2^{-j} : j = 1, \ldots, n-1\} \subset F \) in case (2) and \( \{0\} \cup \{2^{-j} : j \in \mathbb{N}\} \subset F \) in case (3). Then

\[
(\tilde{\psi}_n)^{-1}(A_n(X,F) \cap \mathcal{U}_F) = P_3 \quad \text{and} \quad (\tilde{\psi}_1)^{-1}(A_\omega(X,F) \cap \mathcal{U}_F) = S_4
\]

in respective cases.
The following general fact can also be observed.

**Proposition 2.4.** Let $\Gamma$ be any absolute Borel class containing class $\Sigma_2^0$ or any projective class. If an uncountable metrizable separable space $X$ is not in $\Gamma$, then $A_n(X)$ is not in $\Gamma$ for each $n \in \mathbb{N} \cup \{\omega\}$.

**Proof.** Suppose $A_n(X)$ belongs to $\Gamma$. Since $X$ is uncountable, it contains infinitely many accumulation points. So, we can find $K \in A_n(X)$. Consider the continuous map $\delta : X \to A_n(X)$, $\delta(x) = \{x\} \cup K$ and observe that the image $\delta(X)$ is closed in $A_n(X)$, hence it is in $\Gamma$. Then $\delta(X) \setminus \{K\}$ belongs also to $\Gamma$. Since $\delta \upharpoonright \Omega \setminus K : X \setminus K \to \delta(X) \setminus \{K\}$ is a homeomorphism, the space $X \setminus K$ is in $\Gamma$ and so is the space $X = (X \setminus K) \cup K$, a contradiction.

□

3. **Hyperspaces $A_n(X)$ for 0-dimensional $X$**

In this section, we characterize hyperspaces $A_n(X)$, $n \leq \omega$, for 0-dimensional Polish or $\sigma$-compact spaces without isolated points.

**Lemma 3.1.** $P_3 \cong \mathbb{Q}^{\omega}$. In particular, $P_3$ is of the first category (in itself) and nowhere $G_{\delta\sigma}$.

**Proof.** Represent $\mathbb{N}$ as the countable disjoint union $\mathbb{N} = \bigcup_{n \in \omega} N_n$, where $N_n = \{2^p(2k + 1) : k \in \omega\}$ and let $\text{pr}_{N_n} : \{0,1\}^N \to \{0,1\}^{N_n}$ be the projection. Since the set $\{x \in \{0,1\}^N : \forall^\infty i (x_i = 0)\}$ is homeomorphic to $\mathbb{Q}$, the sets $Z_n = \text{pr}_{N_n}(P_3)$ are also homeomorphic to $\mathbb{Q}$. For each $n \in \omega$, fix a homeomorphism $h_n : Z_n \to \mathbb{Q}$ and let a homeomorphism $h : P_3 \to \mathbb{Q}^{\omega}$ be defined by $h(x) = (h_n(\text{pr}_{N_n}(x)))_{n \in \omega}$.

□

**Lemma 3.2.** The set $S_4$ is of the first category (in itself) and strongly homogeneous (i.e., every two nonempty clopen subsets are homeomorphic). In particular, $S_4$ is nowhere $G_{\delta\delta\sigma}$.

**Proof.** For each $m \in \omega$, put $T_m = \{x \in \{0,1\}^N : (\exists n \geq m) (\forall r \geq n) (\forall^\infty k) (x_{2^r(2k+1)} = 0)\}$.

Observe that
- $S_4 = \bigcup_m T_m$,
- $T_m \subset T_{m+1}$,
- $T_m$ is closed in $S_4$,
- $T_m$ is nowhere dense in $T_{m+1}$.
It follows that $S_4$ is of the first category.

The strong homogeneity follows directly from the definition of $S_4$. □

The following theorem is due to Steel [31, Theorem 2] and van Engelen [18, Theorem 4.6].

**Lemma 3.3.** If $\alpha \geq 3$, then any two 0-dimensional, metric separable, first category spaces from the class $\Pi^0_\alpha (\Sigma^0_\alpha)$ which are nowhere $\Sigma^0_\alpha (\Pi^0_\alpha)$ resp. are homeomorphic.

It was shown in [20] that $A_1(X)$ is of the first category for any second countable topological space $X$. We provide a quick argument for this fact valid for any $A_n(X)$ in the case of a dense-in-itself, metric, separable, 0-dimensional space $X$.

**Lemma 3.4.** Let $X$ be a dense-in-itself, metric, separable, 0-dimensional space, $n \leq \omega$ and $F \in K(X)$ be a set of cardinality $\geq n$. Then the hyperspaces $A_n(X)$, $A_n(X, F)$, $A_{\omega+1}(X)$ and $H(X)$ are of the first category.

**Proof.** Let $A$ denote any of the hyperspaces. Let $\{B_1, B_2, \ldots\}$ be a clopen base in $X$ which is closed under finite unions and $B_{i,k}$ be the family of all $A$ in $A$ such that $1 \leq |A \setminus B_i| \leq k$. Since $X$ has no isolated points, the sets $B_{i,k}$ are nowhere dense in $A$. Clearly, $A$ is the union $\bigcup_{i,k} B_{i,k}$. □

Now, we get the following characterizations.

**Theorem 3.5.** If $X$ is a dense-in-itself, 0-dimensional Polish space then

1. $A_n(X) \cong A_n(X, F) \cong P_\omega \cong \mathbb{Q}^\omega$ for each $n \in \mathbb{N}$ and $F \in K(X)$ of cardinality $\geq n$;
2. $A_\omega(X) \cong A_\omega(X, F) \cong S_4$ for each infinite $F \in K(X)$;
3. $A_{\omega+1}(X) \cong H(X) \cong H(\mathbb{Q})$.

**Proof.** Parts (1) and (2) follow from Corollary 2.2 Theorem 2.3 and Lemmas 3.2 3.4

For part (3), observe that nonempty clopen subsets of $A_{\omega+1}(X)$ and $H(X)$ are $\Pi^1_1$-complete. To see this, let $U = \langle U_1, \ldots, U_k \rangle$ be a basic clopen set in $K(X)$, where $U_1, \ldots, U_k$ are basic clopen subsets of $X$. Take a countable dense subset $Q$ in $\bigcup_{i=1}^k U_i$ and choose $A_1 \in A_1(U_1)$. The set $U$ is a Polish 0-dimensional space containing the Hurewicz set $H(Q) \cong H(\mathbb{Q})$. Now, the continuous map $f : U \to U$, $f(A) = A \cup A_1$ satisfies $f^{-1}(A_{\omega+1}(X) \cap U) = f^{-1}(H(X) \cap U) = H(Q)$. Thus, by Lemma 3.4 we can apply the Michalewski’s characterization 29. □
Theorem 3.6. If \( X \) is a 0-dimensional \( \sigma \)-compact metric space and \( F \in \mathcal{K}(X) \), then \( \mathcal{A}_n(X,F) \) is in \( \Pi_3^0 \) for each \( n \in \mathbb{N} \) and \( \mathcal{A}_\omega(X,F) \) is in \( \Sigma_3^0 \).

Proof. We can assume that \( X \) is contained in the Cantor set \( C \). Consider the derived set operator \( D \) on \( \mathcal{K}(C) \). The preimage \( D^{-1}(\mathcal{F}_n) \) is \( F_{\sigma\delta} \) in \( \mathcal{K}(C) \) for each \( n \in \mathbb{N} \) and \( F_{\sigma\delta} \) for \( n = \omega \). Let \( C = U_1 \supseteq U_2 \supseteq \ldots \). The sets \( X_j = X \cap (U_j \setminus U_{j+1}) \) are \( \sigma \)-compact. It follows that \( \mathcal{F}(X_j) \) are also \( \sigma \)-compact. Hence, each \( \mathcal{F}(X_j) \cup \{\emptyset\} \) as a subset of a compact space \( \mathcal{K}(C) \cup \{\emptyset\} \) is \( \sigma \)-compact.

Let \( \mathcal{B}_j = \{A \in \mathcal{K}(C) : A \cap (U_j \setminus U_{j+1}) \in \mathcal{F}(X_j) \cup \{\emptyset\}\} \).

The intersection map \( \Phi : \mathcal{K}(C) \to \mathcal{K}(C) \cup \{\emptyset\}, \quad \Phi(A) = A \cap (U_j \setminus U_{j+1}) \)

is continuous \([27, \text{Theorems 2 and 3, p.180}]\), consequently,

\[
\mathcal{B}_j = \Phi^{-1}(\mathcal{F}(X_j) \cup \{\emptyset\})
\]

is \( F_{\sigma} \) in \( \mathcal{K}(C) \).

Observe that

\[
\mathcal{A}_n(X,F) = D^{-1}(\mathcal{F}_n(K_i)) \cap \mathcal{A}_n(X) = D^{-1}(\mathcal{F}_n(K_i)) \cap \bigcap_{j=1}^{\infty} \mathcal{B}_j
\]

which shows that

\( \mathcal{A}_n(X,F) \) is \( F_{\sigma \delta} \) in \( \mathcal{K}(C) \) for \( n \in \mathbb{N} \), and is \( F_{\sigma \delta \sigma} \) in \( \mathcal{K}(C) \) for \( n = \omega \).

\( \square \)

Theorem 3.7. (1) If \( X \) is a dense-in-itself, 0-dimensional \( \sigma \)-compact metric space then \( \mathcal{A}_n(X,F) \cong \mathbb{Q}^\omega \) for any \( F \in \mathcal{K}(X) \) of cardinality \( |F| \geq n \);

(2) \( \mathcal{A}_w(X) \cong \mathcal{A}_w(X,F) \cong S_4 \) for any infinite \( F \in \mathcal{K}(X) \).

Proof. (1). \( \mathcal{A}_n(X,F) \) is in \( \Pi_3^0 \) by Theorem 3.6. Theorem 2.3 and Lemma 3.4 guarantee that the hyperspace is nowhere \( \Sigma_3^0 \) and of the first category, so Lemma 3.3 applies.

(2). The space \( X \) can be considered as an \( F_{\sigma} \)-subset of the Cantor set \( C \). By Theorem 2.1 (6), the hyperspaces \( \mathcal{A}_n(X) \), \( n \in \mathbb{N} \), are \( F_{\sigma \delta \sigma} \)-subsets of \( \mathcal{K}(C) \), so \( \mathcal{A}_w(X) \) also is \( F_{\sigma \delta \sigma} \) in \( \mathcal{K}(C) \), hence \( \mathcal{A}_w(X) \) is in \( \Sigma_1^3 \), \( \mathcal{A}_w(X,F) \) is in \( \Sigma_1^0 \) by Theorem 3.6. Both hyperspaces are nowhere \( \Pi_1^0 \) (Theorem 2.3) and of the first category (Lemma 3.4), hence they are homeomorphic to \( S_4 \) by Lemma 3.3. \( \square \)

Corollary 3.8. (1) \( \mathcal{A}_n(\mathbb{Q},F) \cong \mathcal{A}_n(\mathbb{R} \setminus \mathbb{Q}) \cong \mathcal{A}_n(\mathbb{R} \setminus \mathbb{Q},F') \cong \mathbb{Q}^\omega \) for any \( F \in \mathcal{K}(\mathbb{Q}) \) and \( F' \in \mathcal{K}(\mathbb{R} \setminus \mathbb{Q}) \) of cardinalities \( \geq n \).

(2) \( \mathcal{A}_w(\mathbb{Q}) \cong \mathcal{A}_w(\mathbb{Q},F) \cong \mathcal{A}_w(\mathbb{R} \setminus \mathbb{Q}) \cong \mathcal{A}_w(\mathbb{R} \setminus \mathbb{Q},F') \cong S_4 \) for any infinite \( F \in \mathcal{K}(\mathbb{Q}) \) and \( F' \in \mathcal{K}(\mathbb{R} \setminus \mathbb{Q}) \).
In view of Lemma 3.3 and by Theorem 3.5, Theorem 2.3 and Lemma 3.4, the question asked in [20] if \( A_1(\mathbb{R} \setminus \mathbb{Q}) \) is homeomorphic to \( A_1(\mathbb{Q}) \) reduces to the problem of the \( F_{\sigma \delta} \)-absoluteness of \( A_1(\mathbb{Q}) \) (more generally, of \( A_n(\mathbb{Q}) \)); equivalently, whether or not \( A_1(\mathbb{Q}) \) is \( F_{\sigma \delta} \) in \( K(C) \), where \( C \) is a Cantor set. Aiming at this direction, we observe several facts shedding some light on the structure of \( A_n(\mathbb{Q}) \).

We use the following lemma due to van Engelen [19, Lemma 3.1].

**Lemma 3.9.** Let \( X \) and \( Y \) be 0-dimensional metric separable spaces, \( X = \bigcup_{i=1}^{\infty} X_i \), \( Y = \bigcup_{i=1}^{\infty} Y_i \) with \( X_i \) (resp. \( Y_i \)) closed and nowhere dense in \( X \) (resp. \( Y \)) and let every nonempty clopen subset of \( X \) (resp. \( Y \)) contain a closed nowhere dense copy of each \( Y_i \) (resp \( X_i \)). Then \( X \sim Y \).

**Lemma 3.10.** Every nonempty clopen subset of \( A_n(\mathbb{Q}) \) contains a closed copy of \( A_n(\mathbb{Q}) \) for \( n \in \mathbb{N} \cup \{ \omega \} \).

**Proof.** We can assume that a clopen subset \( U \) of \( A_n(\mathbb{Q}) \) is of the form \( U = \langle U_1, \ldots, U_m \rangle \) for nonempty disjoint clopen subsets \( U_i \) of \( \mathbb{Q} \). The set \( A_n(U_1) \) is a closed copy of \( A_n(\mathbb{Q}) \).

Fix points \( u_i \in U_i, i = 2, \ldots, m \). Then

\[
\{ A \cup \{ u_2, \ldots, u_m \} : A \in A_n(U_1) \}
\]

is a closed copy of \( A_n(\mathbb{Q}) \) in \( U \). \( \square \)

By [19, Theorem 4.1] and Lemmas 3.4, 3.10 we get

**Proposition 3.11.** \( A_n(\mathbb{Q}) \) is strongly homogeneous and \( A_n(\mathbb{Q}) \cong A_n(\mathbb{Q}) \times \mathbb{Q} \) for \( n \in \mathbb{N} \cup \{ \omega \} \).

One can easily see

**Lemma 3.12.** \( A_1(\mathbb{Q})^{(q)} = \text{cl}(A_1(\mathbb{Q}), \{q\}) \) (the closure in \( A_1(\mathbb{Q}) \)).

**Proposition 3.13.** \( A_1(\mathbb{Q})^{(q)} \cong A_1(\mathbb{Q}) \cong A_1(\mathbb{Q})^{(q)} \setminus A_1(\mathbb{Q}, \{q\}) \) for every \( q \in \mathbb{Q} \).

**Proof.** We have:

\[
A_1(\mathbb{Q})^{(q)} = \bigcup_{p \in \mathbb{Q} \setminus \{q\}} A_1(\mathbb{Q})^{(p,q)} \quad \text{and} \quad A_1(\mathbb{Q}) = \bigcup_{p \in \mathbb{Q}} A_1(\mathbb{Q})^{(p)}.
\]

Each \( A_1(\mathbb{Q})^{(p,q)} \) is closed and nowhere dense in \( A_1(\mathbb{Q})^{(q)} \). Similarly, each \( A_1(\mathbb{Q})^{(p)} \) is closed and nowhere dense in \( A_1(\mathbb{Q}) \). If \( U = \langle U_1, \ldots, U_m \rangle \cap A_1(\mathbb{Q})^{(q)} \) is nonempty for nonempty disjoint clopen subsets \( U_i \) of \( \mathbb{Q} \) and \( q \in U_1 \), then, as in (3.1),

\[
\{ A \cup \{ u_2, \ldots, u_m \} : A \in A_1(U_1) \cap A_1(\mathbb{Q})^{(q)} \}
\]
is a closed and nowhere dense copy of \( A_1(\mathbb{Q})^{[p]} \) in \( U \). Analogously, each nonempty clopen set \((U_1, \ldots, U_m) \cap A_1(\mathbb{Q}) \) in \( A_1(\mathbb{Q}) \) contains a closed nowhere dense copy of \( A_1(\mathbb{Q})^{[p,q]} \). Now apply Lemma 3.9.

To prove the second equivalence, represent \( A_1(\mathbb{Q})^{[q]} \setminus A_1(\mathbb{Q}, \{q\}) \) as a union \( \bigcup_{k \in \mathbb{N}} B_k \), where

\[
B_k = \{ A \in A_1(\mathbb{Q})^{[q]} : A \subset \{q\} \cup (\mathbb{Q} \setminus (q - \frac{\sqrt{2}}{k}, q + \frac{\sqrt{2}}{k})) \}.
\]

One can easily check that each \( B_k \) is closed, nowhere dense in \( A_1(\mathbb{Q})^{[q]} \setminus A_1(\mathbb{Q}, \{q\}) \) as well as it can be embedded as a closed nowhere dense subset in each nonempty clopen subset of \( A_1(\mathbb{Q}) \). Conversely, each nonempty clopen subset of \( A_1(\mathbb{Q})^{[q]} \setminus A_1(\mathbb{Q}, \{q\}) \) contains a closed nowhere dense copy of \( A_1(\mathbb{Q})^{[q]} \).

\( \square \)

A map \( A_1(\mathbb{Q}, \{0\}) \times \mathbb{Q} \to A_1(\mathbb{Q}) \) given by the translation \((A, q) \mapsto A + q\) is a continuous bijection (it is not a homeomorphism, though). Hence, by Theorem 3.7, Lemma 3.12 and Proposition 3.13, we get

Corollary 3.14. \( A_1(\mathbb{Q}) \) is a one-to-one continuous image of \( \mathbb{Q}^{\omega} \). Equivalently, \( \text{cl}(A_1(\mathbb{Q}, \{q\})) \) is a one-to-one continuous image of \( A_1(\mathbb{Q}, \{q\}) \).

4. Preliminaries related to strongly universal and absorbing sets

From now on, all spaces are assumed to be metric separable and all maps continuous.

We recall a basic terminology and facts related to absorbing sets. The reader is referred to [1, 3, 4, 30] for more details.

The standard Hilbert cube \( I^\omega \) (\( I = [0, 1] \)) is considered with the metric

\[
d(x, y) = \sum_{k \in \omega} \frac{|x_k - y_k|}{2^k}.
\]

A map \( f : X \to Y \) is approximated arbitrarily closely by maps with property \( \mathcal{P} \) if for any open cover \( \mathcal{U} \) of \( Y \) there is a map \( g : X \to Y \) with property \( \mathcal{P} \) such that \( f \) is \( \mathcal{U} \)-close to \( g \), i.e., for each \( x \in X \) there is \( U \in \mathcal{U} \) containing \( \{f(x), g(x)\} \).

A closed subset \( B \subset X \) is a (strong) \( Z \)-set in \( X \) if the identity map of \( X \) can be approximated arbitrarily closely by maps \( f : X \to X \) such that \( B \cap f(X) = \emptyset \) (\( B \cap \text{cl}(f(X)) = \emptyset \)).

An embedding \( f : X \to Y \) is a \( Z \)-embedding if \( f(X) \) is a \( Z \)-set in \( Y \). A countable union of (compact) \( Z \)-sets in \( X \) will be called a (\( \sigma \)-compact) \( \sigma Z \)-set in \( X \).

A subset \( A \subset Y \) is homotopy dense in \( Y \) if there is a deformation \( H : Y \times [0, 1] \to Y \) such that \( H(Y \times (0, 1)) \subset A \) (a deformation through \( A \)).
Fact 4.1. [7, Lemma 2.6] If $M$ is an ANR, a subset $X \subset M$ is homotopy dense in $M$ and $Z$ is a strong $Z$-set in $M$, then $Z \cap X$ is a strong $Z$-set in $X$.

Let $M$ be an absolute neighborhood retract (ANR). It is known that

- $B$ is a $Z$-set in $M$ if and only if $M \setminus B$ is homotopy dense in $M$ (see [32, Corollary 3.3]),
- if $M$ is completely metrizable and $B$ is a $\sigma Z$-set in $M$, then $M \setminus B$ is homotopy dense in $M$ ([4, Exercise 3, p. 31]),
- if $A$ is homotopy dense in $M$ then $A$ is an ANR (an absolute retract (AR) if $M$ is an AR) (see [30, Theorem 4.1.6]).

A space $X$ has the strong discrete approximation property (SDAP) if any map $f : \bigoplus_{n \in \mathbb{N}} I^n \to X$ from the topological sum of finite-dimensional cubes can be approximated arbitrarily closely by maps $g : \bigoplus_{n \in \mathbb{N}} I^n \to X$ such that the family $\{g(I^n) : n \in \mathbb{N}\}$ is discrete.

Fact 4.2. [5, Proposition 1.7], [4, 1.4.1.] If $M$ is an ANR with SDAP or $M$ is locally compact then every $Z$-set in $M$ is a strong $Z$-set in $M$.

We will also need

Fact 4.3. If $X$ is a homotopy dense subset of a locally compact ANR $M$ and there are $Z$-sets $Z_i$ in $M$ such that $X \subset \bigcup_{i \in \mathbb{N}} Z_i$, then $X$ has SDAP.

The above fact can be easily derived from Facts 4.2, 4.1 and [4, Theorem 1.4.10] which says that each ANR $X$ that can be represented as a union of countably many strong $Z$-sets in $X$ has SDAP.

The famous Toruńczyk’s theorem [33] says that a space $X$ is an $\mathbb{R}^\omega$-manifold if and only if $X$ is a Polish ANR with SDAP.

The following theorem was proved by the first author [4].

Theorem 4.4. A space $X$ is an ANR with SDAP if and only if $X$ is homeomorphic to a homotopy dense subset of an $\mathbb{R}^\omega$-manifold.

Let $C$ be a topological class of spaces. A space $X$ is strongly $C$-universal if for each $C \in C$ and closed $B \subset C$, every map $f : C \to X$ which is a $Z$-embedding on $B$ can be approximated arbitrarily closely by $Z$-embeddings $g : C \to X$ such that $g \restriction B = f \restriction B$.

A space $X$ is called $C$-absorbing if

- $X$ is an ANR with SDAP,
- $X = \bigcup_{n \in \mathbb{N}} X_n$, where each $X_n$ is a $Z$-set in $X$ and $X_n \in C$,
- $X$ is strongly $C$-universal.

A fundamental theorem of M. Bestvina and J. Mogilski [5] says that a $C$-absorbing space is topologically unique up to a homotopy type. In particular,

Theorem 4.5. Any two $C$-absorbing AR’s are homeomorphic.
It is often more convenient to consider strongly universal pairs and absorbing pairs of spaces.

From now on, \( \tilde{C} \) will denote a class of pairs \((K, C)\) such that \( K \) is compact, \( C \subset K \) and \( C \in \tilde{C} \).

A pair of spaces \((M, X)\) \((X \subset M)\) is called

- **strongly \( \tilde{C} \)-universal** (some authors prefer to say \( X \) is strongly \( C \)-universal in \( M \)) if for each pair \((K, C)\) \(\in \tilde{C}\) and each closed \( B \subset K \) every map \( f : K \to M \) which is a \( Z \)-embedding on \( B \) and satisfies \((f \mid B)^{-1}(X) = B \cap C \) can be approximated arbitrarily closely by \( Z \)-embeddings \( g : K \to M \) such that \( g \mid B = f \mid B \) and \( g^{-1}(X) = C \).

**Remarks 4.6.** In the above definition,

1. if \( M \) is an ANR, then pairs \((K, C)\) can be replaced by pairs \((I_\omega J, C)\) \([1, Proposition 3.3]\); 
2. if \( M \) is an \( R_\omega \) or \( I_\omega \)-manifold, then map \( f \) can be replaced by an embedding \([4, 1.1.21, 1.1.26]\).

Proving strong universality of pairs is usually cumbersome. An easier property is the preuniversality which is verified as a first step.

A pair \((M, X)\) is

- **\( \tilde{C} \)-preuniversal** if for any pair \((K, C)\) \(\in \tilde{C}\) there exists a map \( f : K \to M \) such that \( f^{-1}(X) = C \);

- **everywhere \( \tilde{C} \)-preuniversal** if for any nonempty open set \( U \subset X \) and pair \((K, C)\) \(\in \tilde{C}\) there exists a map \( f : K \to M \) such that \( f^{-1}(X) = C \).

Henceforth, we restrict our attention to a Borel or projective class \( C \neq \Pi^0_2 \) containing all compacta.

We gather several general facts on strongly \( \tilde{C} \)-universal pairs.

**Fact 4.7.** \([1, Corollary 4.4]\) If \( M \) is an ANR (AR) and \((M, X)\) is strongly \( \tilde{C} \)-universal, then \( X \) and \( M \setminus X \) are homotopy dense in \( M \) ANR’s (AR’s).

**Fact 4.8.** \([1, Corollary 6.2]\) If \( M \) is an ANR, \( Y \subset M \) is homotopy dense in \( M \) and \((Y, X)\) is strongly \( \tilde{C} \)-universal, then \((M, X)\) is strongly \( \tilde{C} \)-universal.

**Fact 4.9.** \([1, Lemma 7.1]\) If \( M \) is an ANR, \((M, X)\) is strongly \( \tilde{C} \)-universal and \( U \) is an nonempty open subset of \( E \), then \((U, X \cap U)\) is strongly \( \tilde{C} \)-universal.

**Fact 4.10.** \([1, Proposition 7.2]\) If \( M \) is an ANR, \( U \) is an open cover of \( M \) and \((U, X \cap U)\) is strongly \( \tilde{C} \)-universal for every \( U \in U \), then \((M, X)\) is strongly \( \tilde{C} \)-universal.

**Fact 4.11.** \([1, Theorem 9.5]\) If \( M \) is an ANR, \((M, X)\) is strongly \( \tilde{C} \)-universal and \( A \) is a \( Z \)-set in \( M \), then \((M, X \cup A)\) is strongly \( \tilde{C} \)-universal for every subset \( B \subset A \).
Fact 4.12. [3] Theorem 3.1] Suppose \( M \) is an ANR, a subset \( X \subset M \) has SDAP, \( X \) is homotopy dense in \( M \) and the pair \((M,X)\) is strongly \( \mathcal{C} \)-absorbing. Then \( X \) is strongly \( \mathcal{C} \)-absorbing.

A pair \((M,X)\) is \( \mathcal{C} \)-absorbing (or \( X \) is a \( \mathcal{C} \)-absorber in \( M \)), if

- \( X \in \mathcal{C} \),
- \((M,X)\) is strongly \( \mathcal{C} \)-universal,
- \( X \) is contained in a \( \sigma \)-compact \( \sigma Z \)-set in \( M \).

A fundamental theorem on absorbing pairs is the following.

Theorem 4.13. [1 Corollary 10.8] If \( M_i \) is an \( \mathbb{R}^\omega \)- or \( \mathbb{I}^\omega \)-manifold and pairs \((M_i,X_i)\) are \( \mathcal{C} \)-absorbing, \( i = 1, 2 \), then \( X_1 \cong X_2 \) if and only if \( X_1 \) and \( X_2 \) are homotopically equivalent; in particular, if \( X_1 \) and \( X_2 \) are AR’s, then \( X_1 \cong X_2 \). If \( M_i \) is an AR for \( i = 1, 2 \), then \((M_1,X_1) \cong (M_2,X_2)\) under a homeomorphism \( h \) of pairs (i.e., \( h(M_1) = M_2 \) and \( h(X_1) = X_2 \)).

Standard \( \Pi^0_3 \)-absorbing pairs are \((\mathbb{R}^\omega, c_0)\) and \((\mathbb{I}^\omega, c_0)\), where \( c_0 = c_0 \cap \mathbb{I}^\omega \).

[16]. More examples of \( \Pi^0_3 \)-absorbing pairs can be found in [8 9 16 17 21 22 23 30].

The Hurewicz set \( \mathcal{H}(\mathbb{I}) \) is a \( \Pi^1_1 \)-absorber in \( K(\mathbb{I}) \) [7 1.4].

5. Strongly universal sets in Lawson semilattices

A topological semilattice is a topological space \( X \) endowed with a continuous commutative, associative operation \(* : X \times X \to X\) such that \( x * x = x \) for all \( x \in X \).

For subsets \( A, B \) of a semilattice \( X \) denote \( A * B := \{a * b : a \in A, b \in B\} \).

A subset \( A \) of \( X \) is a subsemilattice if \( A * A \subset A \). A topological semilattice \( X \) is called Lawson if it has a base of the topology consisting of subsemilattices. A subsemilattice \( A \) of \( X \) is a coideal if \((X \setminus A) * X \subset X \setminus A \).

Examples 5.1. Natural examples of Lawson semilattices are

1. Euclidean or Hilbert cubes with \((x_i) * (y_i) := (\max\{x_i, y_i\})\),
2. Vietoris hyperspaces \( K(X) \) with \( A * B := A \cup B \).
3. Hyperspaces \( F(X), A_\omega(X), A_{\omega+1}(X) \) and \( \mathcal{H}(X) \) are coideals in \( K(X) \).

A subset \( X \) of a space \( M \) is called locally path-connected in \( M \) if for any point \( x \in M \) and neighborhood \( U_x \subset M \) of \( x \) there exists a neighborhood \( V_x \subset M \) of \( x \) such that for any points \( y, z \in V_x \cap X \) there exists a continuous map \( \gamma : [0,1] \to U_x \cap X \) such that \( \gamma(0) = y \) and \( \gamma(1) = z \). Locally path-connected in \( M \) subsets \( X \) are also called \( LC^0 \) in \( M \).

A space \( X \) is locally path-connected (\( LC^0 \)) if \( X \) is \( LC^0 \) in \( X \). If \( X \) is locally path-connected in \( M \), then \( X \) is locally path-connected but not conversely.
The following useful result was proved by W. Kubiš, K. Sakai and M. Yaguchi in [21].

**Theorem 5.2.** If $X$ is a dense locally path-connected (and connected) sub-semilattice in a Lawson semilattice $M$, then $M$ and $X$ are ANR’s (AR’s) and $X$ is homotopy dense in $M$.

The next theorem is an important special case of a more general result recently proved by the first author [2, Theorem 9].

**Theorem 5.3.** Let $M$ be a Lawson semilattice and $X$ be a dense in $M$ coideal which is $LC^0$ in $M$. If our class $C$ is $\Pi^0_3$-hereditary (i.e. for each $C \in C$, any $G_\delta$-subset of $C$ belongs to $C$), then the following conditions are equivalent:

1. the pair $(M, X)$ is strongly $\tilde{C}$-universal;
2. $(M, X)$ is everywhere $\tilde{C}$-preuniversal.

If $M$ is a Polish space and $X$ has SDAP, then conditions (1) and (2) are equivalent to

3. $X$ is strongly $C$-universal.

### 6. TWO STANDARD BOREL ABSORBERS IN $\mathbb{I}^\omega$

Consider the following subsets of $\mathbb{I}^\omega$:

- $\Sigma_2 = \{(x_i)_{i \in \omega} \in \mathbb{I}^\omega : \exists n \forall m \geq n \ (x_m = 0)\}$;
- $\Pi_3 = \{(x_i)_{i \in \omega} \in \mathbb{I}^\omega : \forall n \exists k \forall m \geq k \ (x_{2^n(2m+1)} = 0)\}$;
- $\Sigma_4 = \{(x_i)_{i \in \omega} \in \mathbb{I}^\omega : \exists n_0 \forall n \geq n_0 \exists m_0 \forall m \geq m_0 \ (x_{2^n(2m+1)} = 0)\}$.

The sets $\Pi_3$ and $\Sigma_4$ belong to classes $\Pi_3$ and $\Sigma_4$, respectively, and are connected analogs of $p_3$ and $s_4$ used in the proof of Theorem 2.3.

As an application of Theorem 5.3, we are going to show that the pairs $(\mathbb{I}^\omega, \Pi_3)$, $(\mathbb{I}^\omega, \Sigma_4)$ are absorbing for Borel classes $\Pi_3^0$ and $\Sigma_4^0$, respectively.

**Lemma 6.1.**

1. The pair $(\mathbb{I}^\omega, \Sigma_2)$ is everywhere $\Sigma_2^0$-preuniversal;
2. The pair $(\mathbb{I}^\omega, \Pi_3)$ is everywhere $\Pi_3^0$-preuniversal;
3. The pair $(\mathbb{I}^\omega, \Sigma_4)$ is everywhere $\Sigma_4^0$-preuniversal.

**Proof.** We will first prove that the pairs are preuniversal. For every $m \in \omega$ let $pr_m : \mathbb{I}^\omega \to \mathbb{I}$, $pr_m : x \mapsto x(m)$, be the coordinate projection.

1. Given a compact metrizable space $K$ and an $F_\sigma$-set $C \subset K$, write $C$ as the union $C = \bigcup_{n \in \omega} C_n$ of an increasing sequence of closed sets $C_n$ in $K$. For every $n \in \omega$ choose a continuous function $f_n : K \to \mathbb{I}$ such that $f_n^{-1}(0) = C_n$. Consider the diagonal product $f = (f_n)_{n \in \omega} : K \to \mathbb{I}^\omega$ and observe that $f^{-1}(\Sigma_2) = \bigcup_{n \in \omega} C_n = C$.

2. Given a compact metrizable space $K$ and an $F_{\sigma\delta}$-set $C \subset K$, write $C$ as the intersection $C = \bigcap_{n \in \omega} C_n$ of a decreasing sequence $(C_n)_{n \in \omega}$ of $F_{\sigma\delta}$-sets in $K$. By the preceding item, for every $n \in \omega$ there exists a continuous
map \( f_n : K \to \mathbb{I}^\omega \) such that \( f_n^{-1}(\Sigma_2) = C_n \). Let \( g_0 : K \to \{0\} \subset \mathbb{I} \) be the constant function and, for every \( k \in \mathbb{N} \), let \( g_k = \text{pr}_m \circ f_n \) where \( n, m \in \omega \) are unique numbers such that \( k = 2^n(2m + 1) \). Consider the diagonal product \( g = (g_k)_{k \in \omega} : K \to \mathbb{I}^\omega \) and observe that \( g^{-1}(\Sigma_3) = \bigcap_{n \in \omega} C_n = C \).

3. Let \( C \) be any \( F_{\sigma\delta\sigma} \)-set in a compact metrizable space \( K \). By [23, 23.5(i)], there exists a sequence \((C_n)_{n \in \omega}\) of \( F_{\sigma} \)-sets \( C_n \) in \( K \) such that \( C = \bigcup_{m \in \omega} \bigcap_{n=m}^{\infty} C_n \). By the first item, for every \( n \in \omega \) there exists a continuous function \( f_n : K \to \mathbb{I}^\omega \) such that \( f_n^{-1}(\Sigma_2) = C_n \). Let \( g_0 : K \to \{0\} \subset \mathbb{I} \) be the constant function and, for every \( k \in \mathbb{N} \), let \( g_k = \text{pr}_m \circ f_n \) where \( n, m \in \omega \) are unique numbers such that \( k = 2^n(2m + 1) \). Consider the diagonal product \( g = (g_k)_{k \in \omega} : K \to \mathbb{I}^\omega \) and observe that \( g^{-1}(\Sigma_4) = \bigcup_{m \in \omega} \bigcap_{n=m}^{\infty} C_n = C \).

In order to see that the pairs are everywhere preuniversal, fix an open basic set \( U = U_0 \times \cdots \times U_n \times \mathbb{I} \times \mathbb{I} \times \cdots \) and apply an embedding \( h : \mathbb{I}^\omega \to U \) which is linear on each of the first \( n + 1 \) coordinates and the identity on the others. Observe that \( h \) sends each of the sets \( \Sigma_2, \Sigma_3, \Sigma_4 \) into itself and use their preuniversality in \( \mathbb{I}^\omega \).

Recall that \( \mathbb{I}^\omega \) is a Lawson semilattice (Examples 5.1).

**Lemma 6.2.** The sets \( \Sigma_2, \Sigma_3, \Sigma_4 \) are dense coideals in \( \mathbb{I}^\omega \) and are \( LC^0 \) in \( \mathbb{I}^\omega \).

**Proof.** The first two properties are evident. Let \( A \in \{ \Sigma_2, \Sigma_3, \Sigma_4 \} \). To see that \( A \) is \( LC^0 \) in \( \mathbb{I}^\omega \), consider an open basic set \( U = U_0 \times \cdots \times U_n \times \mathbb{I} \times \mathbb{I} \times \cdots \) in \( \mathbb{I}^\omega \), where \( U_i \) is connected open in \( \mathbb{I} \) for each \( i \leq n \), and choose arbitrary distinct points \((a_i)_{i \in \omega}, (b_i)_{i \in \omega} \in U \cap A \). Denote \( \Gamma = \{ i : a_i \neq b_i \} \). There is a segment \( \gamma(t) = (x_i(t))_{i \in \Gamma} \subset \mathbb{I}^\Gamma \) from \((a_i)_{i \in \Gamma} \) to \((b_i)_{i \in \Gamma} \), \( t \in \mathbb{I} \). Put \( \tilde{\gamma}(t) = (\tilde{x}_i(t))_{i \in \omega}, \) where

\[
\tilde{x}_i(t) = \begin{cases} 
  x_i(t), & \text{if } i \in \Gamma; \\
  a_i = b_i, & \text{otherwise.}
\end{cases}
\]

Then \( \tilde{\gamma}(t) \) is a segment from \((a_i)_{i \in \omega} \) to \((b_i)_{i \in \omega} \) in \( U \cap A \).

**Lemma 6.3.** The sets \( \Sigma_2, \Sigma_3, \Sigma_4 \) are contained in a \( \sigma \)-compact \( \sigma Z \)-set in \( \mathbb{I}^\omega \).

**Proof.** This follows from the inclusions

\[
\Sigma_2 \subset \Sigma_3 \subset \Sigma_4 \subset \bigcup_{i \in \omega} X_i,
\]

where \( X_i = \{(x_n) \in \mathbb{I}^\omega : x_n = 0 \text{ for } n \leq i \} \), and from the fact that the pseudo-interior \((0,1)^\omega \) is homotopy dense in the Hilbert cube \( \mathbb{I}^\omega \).

Now, Theorem 5.3 Lemmas 6.1 6.2 6.3 and Fact 4.3 imply the following corollary.

**Corollary 6.4.** (1) The pair \((\mathbb{I}^\omega, \Pi_3)\) is \( \Pi_3^3 \)-absorbing and \( \Pi_3 \) is \( \Pi_3^3 \)-absorbing.
Lemma 7.1. If a subspace $X \subset M$ is $LC^0$ in $M$, then $\mathcal{F}(X)$ is $LC^0$ in $K(M)$.

Proof. Let $U$ be an open in $M$ neighborhood of a point $x \in M$. There is an open in $M$ neighborhood $V \subset U$ of $x$ such that any two points $a,b \in V \cap X$ can be joined by a path in $U \cap X$.

Claim 7.1.1. For each finite sets $A,B \subset V \cap X$ there is a path

$$\gamma : I \to \langle U \rangle \cap \mathcal{F}(X) \quad \text{such that} \quad \gamma(0) = A \quad \text{and} \quad \gamma(1) = B.$$ 

Indeed, suppose $|A| \geq |B|$, choose a surjection $s : A \to B$ and paths $\gamma_{a,s(a)} : I \to \langle U \rangle \cap \mathcal{F}(X)$ such that $\gamma_{a,s(a)}(0) = a$ and $\gamma_{a,s(a)}(1) = s(a)$. Then $\gamma(t) := \bigcup_{a \in A} \gamma_{a,s(a)}(t)$ is the required path.

Now, let $\langle U_1, \ldots, U_k \rangle$ be a basic open set in the Vietoris topology in $K(M)$ and $K \in \langle U_1, \ldots, U_k \rangle$.

By the assumption and compactness of $K$, one can find open in $M$ sets $V_1, \ldots, V_m$ such that $K \in \langle V_1, \ldots, V_m \rangle \subset \langle U_1, \ldots, U_k \rangle$, each $V_i$ is contained in some $U_j$ and and any two points $a,b \in V_j \cap X$ can be joined by a path in $U_j \cap X$ for each $j$ such that $V_i \subset U_j$. Let $A,B \in \langle V_1, \ldots, V_m \rangle \cap \mathcal{F}(X)$. Using Claim 7.1.1 one constructs inductively a path from $A$ to $B$ in $\langle U_1, \ldots, U_k \rangle \cap \mathcal{F}(X)$.

Clearly, if $X$ is dense in $M$ then $\mathcal{F}(X)$ is dense in $K(M)$ and if $X$ is connected, then $\mathcal{F}(X)$ is connected either. Lemma 7.1 and Theorem 5.2 applied to $\mathcal{F}(X) \subset K(M)$ yield the following lemma.

Lemma 7.2. If $X \subset M$ is dense and $LC^0$ in $M$ (and connected), then $\mathcal{F}(X)$ and $K(X)$ are homotopy dense in $K(M)$ and the hyperspaces $\mathcal{F}(X)$, $K(X)$ and $K(M)$ are ANR’s (AR’s).

Theorem 7.3. If a subspace $X$ of a dense-in-itself space $M$ is dense and $LC^0$ in $M$ (and connected), then

1. $\mathcal{A}_n(X)$, $\mathcal{A}_{\omega+1}(X)$ and $\mathcal{H}(X)$ are ANR’s (AR’s) which are $LC^0$ in $K(M)$ and homotopy dense in $K(M)$;
2. $\mathcal{A}_n(X)$ is an ANR (AR) for each $n \in \mathbb{N}$.

Proof. First, let us notice that for each $n \in \mathbb{N} \cup \{\omega\}$ the hyperspace $\mathcal{A}_n(X)$ is dense in $K(M)$. This follows easily from the fact that $\mathcal{F}(X)$ is dense in $K(M)$, $M$ has no isolated points and there are nontrivial paths in $X$ in small neighborhoods of points of $M$. 

(2) The pair $(\mathbb{I}^\infty, \Sigma_4)$ is $\Sigma_4^0$-absorbing and $\Sigma_4$ is $\Sigma_4^0$-absorbing.
(1) Let $\mathcal{U} \subset \mathcal{K}(M)$ be an open neighborhood of $K \in \mathcal{K}(M)$. By Lemma 7.1 there is an open $\mathcal{V} \subset \mathcal{U}$ containing $K$ such that any two $F_1, F_2 \in \mathcal{F}(X) \cap \mathcal{V}$ can be joined by a path in $\mathcal{F}(X) \cap \mathcal{U}$. Let $A_1, A_2 \in \mathcal{A}_0(X) \cap \mathcal{V}$. By Lemma 7.2 there is a homotopy $H : \mathcal{K}(M) \times I \to \mathcal{K}(M)$, $H(Y, 0) = Y$, $H(Y, t) \in \mathcal{F}(X)$ for each $Y$ and $t > 0$. Choose sufficiently small $0 < t_0 < 1/2$ such that $H(A_i, [0, t_0]) \subset \mathcal{V}, i = 1, 2$. Put $\gamma_1(t) = A_1 \cup H(A_1, t)$ for $0 \leq t \leq t_0$ and let $\gamma : [0, 1 - t_0] \to \mathcal{F}(X) \cap \mathcal{U}$ be a path such that $\gamma(t_0) = H(A_1, t_0)$ and $\gamma(1 - t_0) = H(A_2, t_0)$. Then

$$
\forall = \begin{cases} 
\gamma_1(t), & 0 \leq t \leq t_0; \\
\gamma(t) \cup A_1, & t_0 \leq t \leq 1 - t_0; \\
H(A_2, t) \cup A_1, & 1 - t_0 \leq t \leq 1 
\end{cases}
$$

is a path in $\mathcal{A}_0(X) \cap \mathcal{U}$ from $A_1$ to $A_1 \cup A_2$. Similarly, there is a path in $\mathcal{A}_0(X) \cap \mathcal{U}$ from $A_2$ to $A_1 \cup A_2$. It means that the hyperspace $\mathcal{A}_0(X)$ is $LC^0$ in $\mathcal{K}(M)$. The proof for $\mathcal{A}_{n+1}(X)$ and $H(X)$ is the same.

Since $\mathcal{A}_0(X)$ is a dense subsemilattice of $\mathcal{K}(M)$, we conclude by Theorem 5.2 that $\mathcal{A}_0(X)$ is a homotopy dense in $\mathcal{K}(M)$ ANR, which implies that also $\mathcal{A}_{n+1}(X)$ and $H(X)$ are homotopy dense ANR’s. If $X$ is connected then $\mathcal{K}(M)$ is AR (Lemma 7.2), hence all the hyperspaces are AR’s, as homotopy dense subsets.

(2) Theorem 5.2 is not applicable to hyperspaces $\mathcal{A}_n(X), n \in \mathbb{N}$, for they are not subsemilattices of $\mathcal{K}(M)$. Therefore, we provide a more direct argument.

Consider basic open sets $\langle U_0, U_1, \ldots, U_k \rangle$ in the Vietoris topology in $\mathcal{K}(X)$, $k \geq 0$, where $U_i$’s are open path-connected subsets of $X$. The sets $\mathcal{A}_n(X) \cap \langle U_0, U_1, \ldots, U_k \rangle$ form an open base in $\mathcal{A}_n(X)$ which is closed under finite intersections. We are going to show that each of them is contractible in itself.

Choose points $x_0 \in U_0$ and $s_i \in U_i$ for each $0 < i \leq k$. Let $h : I \to h(I) \subset U_0$ be a homeomorphism such that $h(0) = x_0$.

For $r \in I$, let $C_r = h \left( \{0\} \cup \left\{ \frac{r}{j} : j \in \mathbb{N} \right\} \right)$ and $S = C_1 \cup \{s_1, \ldots, s_k\}$.

The set $\mathcal{F}(X) \cap \langle U_0, \ldots, U_k \rangle$ is dense in the Lawson semilattice $\langle U_0, \ldots, U_k \rangle$ and it is $LC^0$ in $\langle U_0, \ldots, U_k \rangle$, by Lemma 7.1 so it is an ANR homotopy dense in $\langle U_0, \ldots, U_k \rangle$, by Theorem 7.3. Hence, there is a homotopy $H : \langle U_0, \ldots, U_k \rangle \times I \to \langle U_0, \ldots, U_k \rangle$ through finite sets, i.e.

$$
H(K, 0) = K \quad \text{and} \quad H(K, t) \in \mathcal{F}(X) \quad \text{for} \quad t > 0.
$$
The subspace $U = \bigcup_{i=0}^{k} U_i$ is locally path-connected and
\[
\mathcal{E} = \mathcal{F}(X) \cap \langle U_0, U_1, \ldots, U_k \rangle
\]
is an expansion hyperspace in $U$ in the sense of [15]. Moreover, each element
of $\mathcal{E}$ intersects each component of $U$. Therefore $\mathcal{E}$ is an AR [15, Lemma
3.6], so it is contractible. Since $\{x_0, s_1, \ldots, s_k\} \in \mathcal{E}$, there is a homotopy
$F: \mathcal{E} \times \mathbb{I} \to \mathcal{E}$ such that
\[
F(Y, 0) = Y \quad \text{and} \quad F(Y, 1) = \{x_0, s_1, \ldots, s_k\}.
\]
Define a homotopy
\[
G: \mathcal{A}_n(X) \cap \langle U_0, U_1, \ldots, U_k \rangle \times \mathbb{I} \to \mathcal{A}_n(X) \cap \langle U_0, U_1, \ldots, U_k \rangle,
\]
by
\[
G(Y, t) = \begin{cases}
Y \cup H(Y, 4t), & \text{for } t \in [0, 1/4]; \\
Y \cup F(H(Y, 1), 4(t - 1/4)), & \text{for } t \in [1/4, 1/2]; \\
H(Y, 4(t - 1/2)) \cup C_{(t-1/2)} \cup \{s_1, \ldots, s_k\}, & \text{for } t \in [1/2, 3/4]; \\
F(H(Y, 1), 4(t - 3/4)) \cup \langle S \rangle, & \text{for } t \in [3/4, 1].
\end{cases}
\]
Homotopy $G$ is a deformation which contracts $\mathcal{A}_n(X) \cap \langle U_0, U_1, \ldots, U_k \rangle$
in itself to the point $S$. In particular, if $X$ is connected then $\mathcal{A}_n(X) = \mathcal{A}_n(X) \cap \langle X \rangle$ is contractible.

Summarizing: $\mathcal{A}_n(X)$ is a locally connected space with an open base
closed under finite intersections, each of whose elements is connected and
homotopically trivial. It means that $\mathcal{A}_n(X)$ is an ANR (see [30, Corollary
4.2.18]); if $X$ is connected then $\mathcal{A}_n(X)$ is contractible, hence an AR.

\[\square\]

8. Universality and absorbing properties of $\mathcal{A}_\omega(X)$, $\mathcal{A}_{\omega+1}(X)$
and $\mathcal{H}(X)$

**Lemma 8.1.** The pair $(\mathcal{K}(\mathbb{I}), \mathcal{A}_\omega(\mathbb{I}))$ is $\Sigma^0_4$-preuniversal. The pairs $(\mathcal{K}(\mathbb{I}), \mathcal{A}_{\omega+1}(\mathbb{I}))$
and $(\mathcal{K}(\mathbb{I}), \mathcal{H}(\mathbb{I}))$ are $\Pi^1_1$-preuniversal.

**Proof.** Consider the map
\[
\psi: \mathbb{I}^\omega \to \mathcal{K}(\mathbb{I}), \quad \psi((x_n)_{n \in \omega}) = \text{cl}(\{2^{-(n+1)} + 2^{-(n+m+1)}x_{2n(2m+1)} : n, m \in \omega\}).
\]
Observe that the preimage $\psi^{-1}(\mathcal{A}_\omega(\mathbb{I})) = \Sigma_4$. Now the strong $\Sigma^0_4$-universality
of the pair $(\mathbb{I}^\omega, \Sigma_4)$ (Corollary 5.4) implies the $\Sigma^0_4$-preuniversality of the pair
$(\mathcal{K}(\mathbb{I}), \mathcal{A}_\omega(\mathbb{I}))$.

The preuniversality of $(\mathcal{K}(\mathbb{I}), \mathcal{H}(\mathbb{I}))$ follows directly from the R. Cauty’s
result that the pair is $\Pi^1_1$-absorbing [14]. In fact, the construction in [14]
shows that $(\mathcal{K}(\mathbb{I}), \mathcal{A}_{\omega+1}(\mathbb{I}))$ is strongly $\Pi^1_1$-universal.

\[\square\]

Lemma 8.1 implies
Lemma 8.2. Let $X$ be a dense subspace of a space $M$ such that for any non-empty set $U \subset M$ the intersection $U \cap X$ contains a topological copy of the segment $I$. Then the pair $(\mathcal{K}(M), A_\omega(X))$ is everywhere $\Sigma^0_4$-preuniversal and pairs $(\mathcal{K}(M), A_{\omega+1}(X))$ and $(\mathcal{K}(M), \mathcal{H}(X))$ are everywhere $\Pi^1_3$-preuniversal.

Theorem 8.3. If $X$ is a dense subset of a dense-in-itself space $M$ and $X$ is $LC^0$ in $M$, then the pair $(\mathcal{K}(M), A_\omega(X))$ is strongly $\Sigma^0_4$-universal. The pairs $(\mathcal{K}(M), A_{\omega+1}(X))$ and $(\mathcal{K}(M), \mathcal{H}(X))$ are strongly $\Pi^1_3$-universal.

Proof. By Corollary 2.2, the hyperspace $A_\omega(X)$ is an $F_{\sigma\delta\sigma}$-subset of the Polish space $\mathcal{K}(M)$, hence $A_\omega(X) \in \Sigma^0_4$. By Theorem 5.3 $(\mathcal{K}(M), A_\omega(X))$ is strongly $\Sigma^0_4$-universal. The spaces $A_{\omega+1}(X)$ and $\mathcal{H}(X)$ belong to class $\Pi^1_3$ if $X$ is Polish; if not, then by [23] (33.5) $\mathcal{K}(X)$ is in $\Pi^1_3$ and the hyperspaces $A_{\omega+1}(X) = A_{\omega+1}(M) \cap \mathcal{K}(X)$ and $\mathcal{H}(X) = \mathcal{H}(M) \cap \mathcal{K}(X)$ also belong to $\Pi^1_3$ as intersections of a Polish space and $\Pi^1_3$-sets.

Notice that $A_\omega(X) \subset A_{\omega+1}(X) \subset \mathcal{H}(X)$. Therefore in case (1), it remains to find a $\sigma$-compact $\sigma Z$-subset of $\mathcal{K}(M)$ that covers $\mathcal{H}(X)$. We may use the following idea due to R. Cauty [7, Lemme 5.6]. We can assume that the metric $\rho$ in $M$ is bounded by 1. The locally path-connected space $X$ admits an equivalent metric

$$d(x, y) = \begin{cases} 
\inf \{\text{diam}_\rho(C) : C \text{ is a continuum in } X \text{ containing } x, y\}, \\
1, \text{otherwise.}
\end{cases}$$

Define

$$Z_k = \{K \in \mathcal{K}(M) : |K| \geq 2 \text{ and } (\exists x \in K) d(x, K \setminus \{x\}) \geq \frac{1}{k}\}.$$ 

Each $Z_k$ is a closed subset of $\mathcal{K}(M)$, thus it is $\sigma$-compact. In order to show that the sets are $Z$-sets in $\mathcal{K}(M)$, we apply to them a deformation
Theorem 8.5. \( H_t : \mathcal{K}(M) \to \mathcal{K}(M) \) through \( \mathcal{K}(X) \) (it exists by Lemma 7.2) followed by the, so called, expansion deformation

\[ E_t : \mathcal{K}(X) \to \mathcal{K}(X), \quad E_t(K) = \{ x \in X : d(x, K) \leq t \}. \]

More precisely, for \( t > 0 \) take a map \( f_t = E_t \circ H_t : \mathcal{K}(M) \to \mathcal{K}(X) \). Each \( Z_k \) contains an isolated point, while \( E_t(K) \) for \( t > 0 \) has no isolated points. Hence, for sufficiently small \( t > 0 \), \( f_t \) maps \( \mathcal{K}(M) \) into \( \mathcal{K}(M) \setminus \bigcup_{k \in \mathbb{N}} Z_k \) and approximates the identity map on \( \mathcal{K}(M) \). Moreover, since each \( A \in \mathcal{H}(X) \) contains an isolated point, we get the desired inclusion \( \mathcal{H}(X) \subset \bigcup_{k \in \mathbb{N}} Z_k \).

In case (2), \( \mathcal{K}(M) \) is an \( \mathbb{R}^\omega \)-manifold (see [12]) and \( A_\omega(X), A_{\omega+1}(X), \mathcal{H}(X) \) being homotopy dense in \( \mathcal{K}(M) \), they are ANR’s with SDAP by Theorem 4.3. Moreover, since the pairs \( (\mathcal{K}(M), A_\omega(X)), (\mathcal{K}(M), A_{\omega+1}(X)) \) and \( (\mathcal{K}(M), \mathcal{H}(X)) \) are strongly universal in respective classes \( 4.3 \) the spaces \( A_\omega(X), A_{\omega+1}(X) \) and \( \mathcal{H}(X) \) are universal in the classes by 4.12.

Since \( Z \)-sets in the \( \mathbb{R}^\omega \)-manifold \( \mathcal{K}(M) \) are strong \( Z \)-sets (Fact 4.2), the sets \( Z_k \cap A_\omega(X), Z_k \cap A_{\omega+1}(X) \) are \( Z \)-sets in \( A_\omega(X), A_{\omega+1}(X) \) and \( \mathcal{H}(X) \), respectively, by Theorem 7.3 and Fact 4.1. They also belong to the classes. Therefore all sufficient conditions for absorbing sets in the classes are satisfied.

Finally, we get the following characterizations.

**Theorem 8.5.** \( A_\omega(X) \cong \Sigma_4, A_{\omega+1}(X) \cong \mathcal{H}(X) \cong \mathcal{H}(I) \) in each of the following cases.

1. \( X \) is nondegenerate, connected, locally connected and locally compact (i.e. \( X \) is a nondegenerate generalized Peano continuum); if \( X \) is compact (i.e., \( X \) is a nondegenerate Peano continuum), then
   - \( (\mathcal{K}(M), A_\omega(X)) \cong (\mathbb{P}, \Sigma_4) \),
   - \( (\mathcal{K}(M), A_{\omega+1}(X)) \cong (\mathcal{K}(M), \mathcal{H}(X)) \cong (\mathcal{K}(I), \mathcal{H}(I)) \).
2. \( X \) is nondegenerate, Polish, connected, locally connected and nowhere locally compact.

**Proof.** (1) for noncompact \( X \) follows from Theorem 8.4 Corollary 6.4, Theorem 7.3 and Theorem 4.13. In case when \( X \) is a nondegenerate Peano continuum, we also use the Curtis-Schori characterization \( \mathcal{K}(M) \cong \mathbb{R}^\omega \) [14].

(2) follows from Theorem 8.4 Corollary 6.4 Theorem 7.3 and Theorem 4.5.

\[ \square \]

9. **Hyperspaces** \( A_n(I), n \in \mathbb{N} \)

Recall that \( \mathcal{K}(I) \cong \mathbb{P}^\omega \). It will be more convenient to work with the pair \( (\mathcal{K}(J), A_n(J)) \), where \( J = [-1, 1] \).

**Lemma 9.1.** The pair \( (\mathcal{K}(J), A_n(J)) \) is strongly \( \mathbb{P}^\omega \)-universal.
Proof. We apply an approach developed in [21].

Let $C$ be an $F_{\sigma\delta}$-subset of $\mathbb{I}^\omega$, $B$ a closed subset of $\mathbb{I}^\omega$, $f : \mathbb{I}^\omega \to \mathcal{K}(J)$ an embedding which is a $Z$-embedding on $B$ and $\epsilon > 0$. Our goal is to find a $Z$-embedding $g : \mathbb{I}^\omega \to \mathcal{K}(J)$ such that $g \upharpoonright B = f \upharpoonright B$, $g^{-1}(A_n(J)) \setminus B = C \setminus B$, and $\text{dist}(f(x), g(x)) < \epsilon$ for each $x \in \mathbb{I}^\omega$, where $\text{dist}$ denotes the Hausdorff distance in the hyperspace $\mathcal{K}(J)$ (see Remarks [4.6]).

The first ingredient in a construction of an approximation $g$ is the embedding $\phi_n : \mathbb{I}^\omega \to \mathcal{K}(J)$,

\begin{equation}
\phi_n((x_j)_{j \in \omega}) = \{-1\} \cup \{-2^{-(j+1)} + x_j2^{-2-j} : j \in \omega\} \cup \chi(n) \cup \\
\text{cl}\{2^{-(j+1)} + 2^{-(j+k+1)}x_{2j-n(2k+1)} : j \geq n, k \in \omega\},
\end{equation}

where $\chi(n)$ is defined in [21]. The positive part of $\phi_n((x_j)_{j \in \omega})$ is responsible for the property $\phi_n^{-1}(A_n(J)) = \Pi_3$, while the negative one exhibits 1-1 correspondence $\phi_n : \mathbb{I}^\omega \to \mathcal{K}(J)$.

The pair $(\mathbb{I}^\omega, \Pi_3)$ being strongly $\Pi^0_3$-universal, there is an embedding $\zeta : \mathbb{I}^\omega \to \mathbb{I}^\omega$ such that $\zeta^{-1}(\Pi_3) = C$. Put

\begin{equation}
\xi = \psi \zeta.
\end{equation}

We have

\begin{equation}
\xi^{-1}(A_n(J)) = C.
\end{equation}

Next, we need a deformation $H : \mathcal{K}(J) \times \mathbb{I} \to \mathcal{K}(J)$ through finite sets. Deformation $H$ can be easily modified to satisfy

$$\text{dist}(K, H(K, t)) \leq 2t \quad \text{and} \quad H(K, t) \subset [-1 + t, 1 - t]$$

(see [21] (1-4), p. 183).

We are going to verify that the embedding $g : \mathbb{I}^\omega \to \mathcal{K}(J)$ defined by

\begin{equation}
g(x) = H(f(x), \mu(x)) \cup \left(\min H(f(x), \mu(x)) + \mu(x)\xi(x)\right),
\end{equation}

where

\begin{equation}
\mu(x) = \frac{1}{4} \min\{\epsilon, \min\{\text{dist}(f(x), f(z)) : z \in B\}\}
\end{equation}

(we use standard operations $\alpha A := \{\alpha a : a \in A\}$ and $x + A := \{x + a : a \in A\}$), satisfies the definition of strong $\Pi^0_3$-universality of pair $(\mathcal{K}(J), A_n(J))$.

Clearly, $g$ is continuous and since $\mu(x) = 0$ for $x \in B$, it agrees with $f$ on $B$. It also $\epsilon$-approximates $f$, as

\begin{equation}
\text{dist}(f(x), g(x)) \leq \\
\text{dist}(f(x), H(f(x), \mu(x))) + \text{dist}(H(f(x), \mu(x)), g(x)) \leq \\
3\mu(x) = \frac{3}{4} \min\{\epsilon, \min\{\text{dist}(f(x), f(z)) : z \in B\}\}.
\end{equation}

Mapping $g$ is 1-1 on $B$. So, let $x, y \in \mathbb{I}^\omega \setminus B$. Then both numbers $\mu(x)$ and $\mu(y)$ are positive. Suppose $g(x) = g(y)$. It follows that

\begin{equation}
-\mu(x) = \min g(x) = \min g(y) = -\mu(y)
\end{equation}
hence $\mu(x) = \mu(y)$ and $\zeta(x)_j = \zeta(y)_j$ for each $j \in \omega$, so $x = y$. Suppose $x \in B$, $y \notin B$ and $g(x) = g(y)$. Then $g(y) = f(x)$. On the other hand,

$$\text{dist}(g(y), f(B)) \geq \text{dist}(f(y), f(B)) - \text{dist}(f(y), g(y)) \geq \frac{1}{4} \text{dist}(f(y), f(B)) > 0,$$

by (9.6) and since $f$ is 1-1, a contradiction. Thus, $g$ is 1-1.

It follows from (9.3) that $g^{-1}(A_n(\mathbb{I})) \setminus B = C \setminus B$.

The image $g(\mathbb{I}^\omega)$ is a $Z$-set in $K(\mathbb{J})$. Indeed, $g(\mathbb{I}^\omega \setminus B) = g(\mathbb{I}^\omega) \setminus g(B)$ is a $\sigma Z$-set in $K(\mathbb{J})$ because deformation $H$ through finite sets satisfies

$$\forall (t > 0) \ H(g(K(\mathbb{J}) \setminus B), t) \cap g(K(\mathbb{J}) \setminus B) = \emptyset$$

(since, for each $x \notin B$ and $t > 0$, the set $H(g(x), t)$ is finite whereas $g(x)$ is not). Now, $g(B)$ is a $Z$-set, so the union $g(B) \cup g(K(\mathbb{J}) \setminus B) = g(K(\mathbb{J}))$ is a compact $\sigma Z$-set, hence a $Z$-set in $K(\mathbb{J})$.

□

**Lemma 9.2.** $A_n(\mathbb{J})$ is contained in a $\sigma Z$-set in $K(\mathbb{J})$.

**Proof.** The $\sigma Z$-set we are looking for was constructed in a more general setting (for generalized Peano continua) in the proof of Case (1) of Theorem [8.1].

Since $A_n(\mathbb{I})$ is in class $\Pi^0_3$, Lemmas [9.1] and [9.2] imply

**Theorem 9.3.** The pair $(K(\mathbb{I}), A_n(\mathbb{I}))$ is $\Pi^3_3$-absorbing for each $n \in \mathbb{N}$. Consequently, $A_n(\mathbb{I}) \cong \Pi^3_3 \cong c_0 \cong c_0$.

**Corollary 9.4.** $A_n((0, 1))$, $A_n([0, 1))$ and $A_n((0, 1])$ are also $\Pi^3_3$-absorbers in $K(\mathbb{I})$ for each $n \in \mathbb{N}$. Hence they are all homeomorphic to $c_0$.

**Proof.** Observe that the set $B = \{A \in K(\mathbb{I}) : A \cap \{0, 1\} \neq \emptyset\}$ is a $Z$-set in $K(\mathbb{I})$ and $A_n((0, 1)) = A_n(\mathbb{I}) \setminus B$. It is known from [11, Corollary 9.4] that the difference of an $\Pi^3_3$-absorber and a $Z$-set in a Hilbert cube $K(\mathbb{I})$ is again an $\Pi^3_3$-absorber in $K(\mathbb{I})$. The argument for the remaining intervals is similar. □

**Remark 9.5.** Corollary [9.4] absorbs [20, Theorem 2.4, Corollary 2.5], [6, Theorem 5.1] and provides a positive answer to the question in [20, Question 2.17] of whether or not $S_c((0, 1])$ is homeomorphic to $S_c((0, 1))$.

**Remark 9.6.** The above method of showing the strong $\Pi^3_3$-universality is specific for $X$ an arc—we continuously select a point from a finite set (the point $\min H(f(x), \mu(x))$ from $H(f(x), \mu(x))$) and such selections are characteristic for arcs [28]. Using Corollary [9.4] and general facts about strongly $\mathcal{U}$-universal pairs, it is shown in Section [10] that the pair $(K(S^1), A_n(S^1))$ is $\Pi^3_3$-absorbing. One may ask for what other “nice” spaces $X$ the pair $(K(X), A_n(X))$ is $\Pi^3_3$-absorbing.
By $S^1$ we denote the unit circle in $\mathbb{R}^2$.

**Theorem 10.1.** The pair $(\mathcal{K}(S^1), A_n(S^1))$ is $\overrightarrow{\Pi}_3^0$-absorbing. Hence, $A_n(S^1) \cong c_0$.

**Proof.** The hyperspace $E = \mathcal{K}(S^m) \setminus \{S^1\}$ is an AR. Let $U$ be its open cover by sets $U_p = \mathcal{K}(S^1 \setminus \{p\})$, $p \in S^1$. The pair $(\mathcal{K}(I), A_n((0, 1)^m))$ is strongly $\overrightarrow{\Pi}_3^0$-universal by Corollary 9.3. By Fact 4.9 the pair $(\mathcal{K}((-1, 1)), A_n((0, 1)))$ is strongly $\overrightarrow{\Pi}_3^0$-universal. Let $h : (0, 1) \to S^1 \setminus \{p\}$ be a homeomorphism and $\overline{h} : \mathcal{K}((0, 1)) \to \mathcal{K}(S^1 \setminus \{p\})$ be the induced homeomorphism. Clearly, $\overline{h}(A_n((0, 1))) = A_n(S^1 \setminus \{p\})$ and the pair $(U_p, A_n(S^1 \setminus \{p\}))$ is strongly $\overrightarrow{\Pi}_3^0$-universal. Since $A_n(S^1 \setminus \{p\}) = U_p \cap A_n(S^1)$, we infer by Fact 4.10 that the pair $(E, A_n(S^m))$ is strongly $\overrightarrow{\Pi}_3^0$-universal. The singleton $\{S^1\}$ being a $Z$-set in $\mathcal{K}(S^1)$, $E$ is homotopy dense in $\mathcal{K}(S^1)$. Then, by Fact 4.8 the hyperspace $(\mathcal{K}(S^1), A_n(S^1))$ is also strongly $\overrightarrow{\Pi}_3^0$-universal.

The proof that $A_n(S^1)$ is contained in a $\sigma Z$-set in $\mathcal{K}(S^1)$ is the same as for Lemma 9.2.









11. Hyperspaces $A_n(X)^F$

As we have noticed in Remark 9.6 there is an essential obstacle in proving that $A_n(X)$ is an $\overrightarrow{\Pi}_3^0$-absorber in $\mathcal{K}(X)$ for nondegenerate Peano continua other than $I$ and $S^1$. The obstacle disappears for hyperspaces $A_n(X)^F \subseteq \mathcal{K}(X)^F$ and $A_1(X, \{p\}) \subseteq \mathcal{K}(X)^{\{p\}}$, where $F$ is a fixed finite subset of $X$ which contains a point of order $\geq 2$ and $p$ is a fixed point of order $\geq 2$ (a point $p \in X$ is of order $\geq 2$ if there is an arc $L \subseteq X$ containing $p$ in its combinatorial interior). The latter hyperspace is a natural counterpart of $c_0$ whose elements converge to the same number $0$.

**Theorem 11.1.** Suppose $X$ is a Peano continuum, $F \subseteq X$ is finite and contains a point $p$ of order $\geq 2$, $n \in \mathbb{N}$. Then the pairs $(\mathcal{K}(X)^F, A_n(X)^F)$ and $(\mathcal{K}(X)^{\{p\}}, A_1(X, \{p\})$ are $\overrightarrow{\Pi}_3^0$-absorbing.

Consequently, $A_n(X)^F \cong A_1(X, \{p\}) \cong c_0$.

**Proof.** Recall that $\mathcal{K}(X)^F$ is a Hilbert cube $[14]$. Clearly, $A_n(X)^F = A_n(X) \cap \mathcal{K}(X)^F$ is $F_{\sigma \delta}$ in $\mathcal{K}(X)^F$. Also $A_1(X, \{p\})$ is $F_{\sigma \delta}$ in $\mathcal{K}(X)^{\{p\}}$, since it equals the preimage $D^{-1}(\{p\})$, where $D$ is the derived set operator on $\mathcal{K}(X)^{\{p\}}$.

In order to prove the strong $\overrightarrow{\Pi}_3^0$-universal property, we proceed similarly to the proof of Lemma 9.1.

There is a deformation $H : \mathcal{K}(X) \times [0, 1] \to \mathcal{K}(X)$ through finite sets such that $\text{dist}(H(A,t), A) \leq 2t$. If we add $F$ to each $H(A,t)$ we get a
continuous deformation $K(X)^F \times [0,1] \to K(X)^F$ through finite sets satisfying $\text{dist}(H(A,t),A) \leq 2t$ for $A \in K(X)^F$. So, we can assume that $H : K(X)^F \times [0,1] \to K(X)^F$ is such. Choose an arc $L \subset X$ containing $p$ in its combinatorial interior. Note that each set $H(A,t)$ contains $p$. To simplify further description, assume without loss of generality that $L = \mathbb{I} = [-1,1]$ and $p = 0$. We modify the definition of embedding $\phi_n$ from (9.1) in its “negative” part in which the sequence $\{-2^{-(j+1)} + x_j2^{-(j+2)} : j \in \omega\}$ is now replaced with an increasing sequence $l(x)$ obtained in the following way. For any $x = (x_j) \in \mathbb{I}^\omega$, put 

$$a(x)_j = -(2^{2j} + \frac{x_j}{2^{2j}})^{-1}.$$  

Observe that the sequence $a(x) = (a(x)_j)$ satisfies

(1) $a(x)$ is strictly increasing and converging to 0,

(2) for each $x, y \in \mathbb{I}^\omega$ and $i < j$, vectors 

$$(a(x)_i, a(x)_{i+1}) \quad \text{and} \quad (a(y)_j, a(y)_{j+1})$$

are not parallel.

Let $x' \in \mathbb{I}^\omega$ be the sequence $1, x_1, 1, x_1, 2, 1, x_1, x_2, x_3, 1, \ldots$. Put $l(x) = a(x')$. Clearly, $l(x)$ also satisfies conditions (1-2).

Now, let

$$(11.1) \quad \psi_n(x) = l(x) \cup \chi(n) \cup \text{cl}(\{2^{-(j+1)} + 2^{-(j+k+1)}x_{2j-n(2k+1)} : j \geq n, k \in \omega\})$$

Note that, for each $n \in \mathbb{N}$, we have

$$(11.2) \quad \psi^{-1}_n(A_n(\mathbb{I})) = \Pi_3 = \psi^{-1}_1(A_1(\mathbb{I}, \{0\})).$$

Define

$$(11.3) \quad g(x) = H(f(x), \mu(x)) \cup \mu(x)\xi(x).$$

where $\mu(x)$ is defined in (9.5) and $\xi(x)$ is defined by (9.2) with $\psi_n$ modified as above. Now, $l(x)$ is responsible for $g(x)$ being 1-1. Indeed, suppose $x, y \in \mathbb{I}^\omega \setminus B$ and $g(x) = g(y)$. Then $\mu(x)$ and $\mu(y)$ are positive. If the set

$$W = (H(f(x), \mu(x)) \cup H(f(y), \mu(y))) \cap [-1,0)$$

is non-empty, let $\alpha = \max W$ and notice that, for sufficiently large $k$, say for $k \geq j$, numbers $\mu(x)l(\zeta(x))_k$ and $\mu(y)l(\zeta(y))_k$ are greater than $\alpha$; if $W = \emptyset$, then put $j = 0$. So, we can assume that $\mu(x)l(\zeta(x))_j = \mu(y)l(\zeta(y))_i$ for some $i \geq j$. Since sequences $\mu(x)l(\zeta(x))$ and $\mu(y)l(\zeta(y))$ are increasing, it follows that also $\mu(x)l(\zeta(x))_{j+1} = \mu(y)l(\zeta(y))_{i+1}$. Thus $i = j$ by property (2). But then $\mu(x)l(\zeta(x))_k = \mu(y)l(\zeta(y))_k$ for each $k \geq j$. Choose $m \geq j$
such that $(\zeta(x)')_m = 1 = (\zeta(y)')_m$. Then

$$-\mu(x)(2^{2m} + \frac{1}{2^{2m}})^{-1} = \mu(x)a(\zeta(x)')_m = \mu(x)l(\zeta(x))_m =$$

$$\mu(y)l(\zeta(y))_m = \mu(y)a(\zeta(y)')_m = -\mu(y)(2^{2m} + \frac{1}{2^{2m}})^{-1}$$

which implies $\mu(x) = \mu(y)$. Hence, for each $k \geq j$,

$$-(2^{2k} + \frac{\zeta(x')_k}{2^{2k}})^{-1} = a(\zeta(x)')_k = l(\zeta(x))_k =$$

$$l(\zeta(y))_k = a(\zeta(y)')_k = -(2^{2k} + \frac{\zeta(y')_k}{2^{2k}})^{-1},$$

so $(\zeta(x)')_k = (\zeta(y)')_k$. Consequently, $\zeta(x) = \zeta(y)$ and $x = y$.

The remaining arguments are exactly the same as in the proofs of Lemmas 9.1 and 9.2.

□

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