Induced Matter Theory of gravity from a Weitzenböck 5D vacuum and pre-big bang collapse of the universe

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Abstract

We extend the Induced Matter Theory of gravity (IMT) to 5D curved spacetimes by using the Weitzenböck representation of connections on a 5D curved spacetime. In this representation the 5D curvature tensor becomes null, so that we can make a static foliation on the extra noncompact coordinate to induce in the Weitzenböck representation the Einstein equations. Once we have done it, we can rewrite the effective 4D Einstein equations in the Levi-Civita representation. This generalization of IMT opens a huge window of possible applications for this theory. A pre-big bang collapsing scenario is explored as an example.

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I. INTRODUCTION AND MOTIVATION

Pre-big bang cosmology is a new and fascinating cosmological paradigm\[1\] that emerges from the possibility that the our universe could be born from a unstable compact object (could be a black-hole), with a low state of entropy. Under this point of view the universe is just one of many in a cyclical chain, with each Big Bang starting up a new universe in place of the one before, and the end of each universe will involve a return to low entropy. This is because black holes suck in all the matter, energy, and information they encounter, which works to remove entropy from our universe. However, the investigation of the black-holes formation is not closed. The problem with this issue consists of explaining how the gravitational collapse does not end with a singularity with infinity energy density. This topic has been studied earlier by Liu and Wesson\[4\] in the framework of IMT. In particular, the Induced Matter Theory (IMT) is based on the assumption that ordinary matter and physical fields that we can observe in our 4D universe can be geometrically induced from a 5D Ricci-flat metric with a space-like noncompact extra dimension on which we define a physical vacuum\[5, 6\]. In order to study the gravitational collapse resulting in the formation of compact objects with finite size and energy densities from a 5D vacuum state, we shall incorporate the Weitzenböck connections to extend the Induced Matter Theory (IMT) formalism\[5\], which is a very active extra dimensional theory. The induced matter theory is regarded a non-compact KK theory in 5D, since the fifth extra dimension is assumed extended. This theory is mathematically supported by the embedding Campbell-Magaard theorem \[2, 3\]. The main idea in this theory is that matter in 4D can be geometrically induced from a 5D Ricci-flat metric. Thus, the theory considers a 5D apparent vacuum defined by \( R_{AB} = 0 \), which are the field equations of the theory.

In this work we study an extension of the IMT to using the Weitzenböck\[7\] representation of connections on a 5D curved spacetime. In this representation the 5D curvature tensor (which could be nonzero in a Levi-Civita representation) becomes null in a Weitzenböck one, so that we can make static foliation on the extra noncompact coordinate to induce in the Weitzenböck representation the effective 4D Einstein-Cartan equations. Once we have done it, we can rewrite the effective 4D Einstein equations in the Levi-Civita representation.

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1 In our conventions indices "a,b,c,...,h" run from 1 to 5, "A,B,C,...,H" run from 1 to 5, Greek indices \( \alpha, \beta, \gamma \) run from 1 to 4 and indices "i,j,k,..." run from 2 to 4.
There are multiple possible applications for this formalism. One of these applications is the treatment of a pre-big bang collapsing universe. This issue is studied in the Sect. III.

II. WEITZENBÖCK CONNECTIONS AND THE EXTENDED IMT

In this section we shall introduce the Weitzenböck connections to develop the IMT in the framework of the Weitzenböck treatment.

A. Weitzenböck treatment in 5D

In order to extend the IMT formalism we shall use the Weitzenböck connections, which are defined when we make a transformation

$$e_a = e^A_a E_A,$$  

from a coordinated basis $E_A$ to another $e_a$. Hence, all tensors can be written using the vielbein $e^A_a$ and its inverse $e^a_A$, such that, if $e^A_a e^b_A = \delta^b_a$ and

$$g_{ab} = e^A_a e^B_b g^{AB}.$$  

When the covariant derivative of the vielbein becomes zero: $\nabla_b e^A_a = 0$, the connection with respect to which we make the covariant derivative, is a Weitzenböck connection, $\bar{\Gamma}^c_{ba}$:

$$\nabla_b e^A_a = \partial_b e^A_a - \bar{\Gamma}^c_{ba} e^A_c = 0.$$  

We shall consider that the curvature tensor of the 5D space represented by the metric tensor $g_{AB}$ is zero from the point of view of the Levi-Civita and Weitzenböck connections. However, the transformed space represented by tensor metric $g_{ab}$ is Weitzenböck-flat: $\bar{R}^a_{bcd} = 0$, but not Riemann-flat: $R^a_{bcd} \neq 0$. The curvature in the Riemann and Weitzenböck representations, is given (on the space $g_{ab}$), by

$$\bar{R}^a_{bcd} = \bar{\Gamma}^a_{bd,c} - \bar{\Gamma}^a_{bc,d} + \bar{\Gamma}^e_{bd} \bar{\Gamma}^a_{ec} - \bar{\Gamma}^e_{bc} \bar{\Gamma}^a_{ed} = 0,$$  

$$R^a_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^e_{bd} \Gamma^a_{ec} - \Gamma^e_{bc} \Gamma^a_{ed} \neq 0.$$  

Here, the Weitzenböck and Levi-Civita connections on $g_{ab}$, are, respectively, given by

$$\Gamma^a_{bc} = e^A_a \partial_e e^A_b,$$  

$$\Gamma^a_{bc} = \frac{1}{2} g^{as} \left( \partial_b g_{cs} + \partial_c g_{es} - \partial_s g_{bc} \right).$$
such that both connections are related by

$$\bar{\Gamma}^a_{bc} = \Gamma^a_{bc} - \bar{K}^a_{bc},$$

(8)

$$\bar{K}^a_{bc}$$ being the contortion in the Weitzenb"{o}ck space, which is defined from the torsion

$$T^a_{bc} = \Gamma^a_{bc} - \Gamma^{a}_{cb}$$

(9)

By contracting the tensor $\bar{R}^a_{bcd}$, it is possible to obtain two tensors, one symmetric $\bar{R}^a_{bc} = \bar{R}^a_{cb}$, and the other antisymmetric, $\bar{S}^a_{cd} = \bar{S}^a_{dc}$, which are

$$\bar{R}^a_{bc} = \bar{\Gamma}^a_{ba,c} - \bar{\Gamma}^a_{bc,a} + \bar{\Gamma}^e_{ba} \bar{\Gamma}^a_{ec} - \bar{\Gamma}^e_{bc} \bar{\Gamma}^a_{ea} = 0,$$

(10)

$$\bar{S}^a_{cd} = \bar{\Gamma}^a_{ad,c} - \bar{\Gamma}^a_{ac,d} = 0.$$

(11)

B. IMT with the Weitzenb"{o}ck representation

Now we shall study the effective 4D Einstein equations after we make a static foliation on the 5D Einstein equations using the Weitzenb"{o}ck representation. We consider Eq. (10). If we take into account only the 4D components of the tensor $\bar{R}^a_{ab}$, we obtain, after separately writting the sum over the 4D and extra dimensional components

$$\underbrace{\bar{R}^{\gamma}_{\beta\gamma}}_{l=l_0} = \underbrace{\bar{\Gamma}^\alpha_{\beta\alpha,\gamma} - \bar{\Gamma}^\alpha_{\beta\gamma,\alpha}}_{l=l_0} + \underbrace{\bar{\Gamma}^\alpha_{\beta\alpha} \bar{\Gamma}^\alpha_{\epsilon\gamma} - \bar{\Gamma}^\epsilon_{\beta\gamma} \bar{\Gamma}^\alpha_{\epsilon\alpha}}_{l=l_0}$$

$$+ \left[ \left( \bar{\Gamma}^5_{\beta5,\gamma} - \bar{\Gamma}^5_{\beta\gamma,5} \right) + \left( \bar{\Gamma}^5_{5\gamma} \bar{\Gamma}^5_{\beta5} + \bar{\Gamma}^\mu_{\beta\gamma} \bar{\Gamma}^5_{\mu5} + \bar{\Gamma}^\mu_{5\gamma} \bar{\Gamma}^5_{\mu5} - \bar{\Gamma}^\mu_{\beta\gamma} \bar{\Gamma}^\mu_{\mu5} \right) \right]_{l=l_0} = 0.$$

(12)

We shall restrict our study to consider canonical metrics: $dS^2 = \left( \frac{l}{l_0} \right)^2 h_{\alpha\beta}(y^\gamma) dy^\alpha dy^\beta - dl^2$, where $l$ is the noncompact extra dimension and $l_0$ is a constant introduced by consistency of physical dimensions. It is possible to demonstrate that $\Gamma^{e}_{\beta\alpha} |_{l=l_0} = \bar{\Gamma}^{e}_{\beta\alpha}$, where $\bar{\Gamma}^{e}_{\beta\alpha}$ is the effective 4D connection of 4D hypersurface $h_{\alpha\beta}$. The induced 4D Weitzenb"{o}ck tensor
\( \overleftarrow{R}_{\beta\gamma} \equiv \overleftarrow{R}^a_{\beta\gamma a} \) is given by

\[
\overleftarrow{R}_{\beta\gamma} = - \left[ (\overleftarrow{\Gamma}^{\alpha}_{\beta5,\gamma} - \overleftarrow{\Gamma}^{\alpha}_{\beta\gamma,5}) + (\overleftarrow{\Gamma}^{\alpha}_{5\gamma,\beta5} + \overleftarrow{\Gamma}^{\alpha}_{\mu\gamma} \overleftarrow{\Gamma}^{\mu}_{\beta5} + \overleftarrow{\Gamma}^{\alpha}_{\beta5} \overleftarrow{\Gamma}^{\beta}_{\gamma\alpha} - \overleftarrow{\Gamma}^{\alpha}_{5\gamma} \overleftarrow{\Gamma}^{\beta}_{\gamma\alpha} - \overleftarrow{\Gamma}^{\alpha}_{\mu5} \overleftarrow{\Gamma}^{\mu}_{\beta\gamma} - \overleftarrow{\Gamma}^{\alpha}_{55} \overleftarrow{\Gamma}^{\beta}_{\gamma\alpha} - \overleftarrow{\Gamma}^{\alpha}_{\mu5} \overleftarrow{\Gamma}^{\mu}_{\beta\gamma} - \overleftarrow{\Gamma}^{\alpha}_{55} \overleftarrow{\Gamma}^{\beta}_{\gamma\alpha}) \right] \bigg|_{l=l_0} \tag{13}
\]

which is symmetric with respect to \( \beta\gamma \). The induced antisymmetric tensor that comes from the contraction of the 5D Weitzenböck curvature tensor is

\[
\overleftarrow{S}_{\beta\gamma} \equiv \overleftarrow{R}^a_{a\beta\gamma} \bigg|_{l=l_0} .
\]

This antisymmetric tensor is given by

\[
\overleftarrow{S}_{\beta\gamma} = - \left[ (\overleftarrow{\Gamma}^{\alpha}_{5\delta,\gamma} - \overleftarrow{\Gamma}^{\alpha}_{5\gamma,\delta}) \right] \bigg|_{l=l_0=1/H} . \tag{14}
\]

To induce the effective 4D scalar Weitzenböck curvature, we must notice that

\[
\overleftarrow{\Gamma}^{5}_{\beta5,\gamma} \equiv \overleftarrow{\Gamma}^{5}_{\beta\gamma,5} = 0 .
\]

Hence, if we identify in (13)

\[
\left. \overleftarrow{R}_{\beta\gamma} \right|_{l=l_0} = h^{\beta\gamma} \overleftarrow{R}_{\beta\gamma} \text{ as the effective 4D induced Weitzenböck scalar curvature}.
\]

We obtain that this scalar curvature is

\[
\left. \overleftarrow{R}_{\beta\gamma} \right|_{l=l_0} = - \left. \overleftarrow{\Gamma}^{5}_{\beta\gamma,5} \right|_{l=l_0} = 0.
\]

\[
\frac{4D}{5D} \overleftarrow{G}_{\beta\gamma} \left. = \overleftarrow{R}_{\beta\gamma} - \frac{1}{2} h_{\beta\gamma} \overleftarrow{R} \right. \left( \frac{4D}{5D} \overleftarrow{\Gamma}_{\beta\gamma} \right). \tag{16}
\]

\[
\frac{4D}{5D} \overleftarrow{S}_{\beta\gamma} = -8\pi G \left. \frac{4D}{5D} \overleftarrow{T}_{\beta\gamma} \right( \text{sym} ) , \tag{17}
\]

\[
\frac{4D}{5D} \overleftarrow{R}_{a5} = 0, \tag{18}
\]

\[
\frac{4D}{5D} \overleftarrow{S}_{a5} = 0, \tag{19}
\]

where \( \frac{4D}{5D} \overleftarrow{T}_{\beta\gamma} \left( \text{sym} \right) = \frac{1}{2} \left( \frac{4D}{5D} \overleftarrow{T}_{\beta\gamma} + \frac{4D}{5D} \overleftarrow{T}_{\gamma\beta} \right) \) is the symmetrized energy-momentum tensor in the

\[
\frac{4D}{5D} \overleftarrow{T}_{\beta\gamma} \left( \text{ant} \right) = \frac{1}{2} \left( \frac{4D}{5D} \overleftarrow{T}_{\beta\gamma} - \frac{4D}{5D} \overleftarrow{T}_{\gamma\beta} \right) \] the antisymmetrized contribu-
tion. These equations provide the effective 4D relativistic dynamics. In particular, Eq. (16) describes the dynamics and Eqs. (18)-(19) provide us with the constraint conditions.

C. Effective 4D Levi-Civita representation from the induced 4D Weitzenböck representation

In order to obtain the effective 4D Einstein equations in the Levi-Civita representation, we shall use the expression for the connections (8) in Eq. (12). Then we obtain the effective 4D Ricci tensor

\[
4D \tilde{R}_{\beta\gamma} \equiv \Gamma^\alpha_{\beta\alpha,\gamma} - \Gamma^\alpha_{\beta\gamma,\alpha} + \Gamma^\epsilon_{\beta\alpha} \Gamma^\alpha_{\epsilon\gamma} - \Gamma^\epsilon_{\beta\gamma} \Gamma^\alpha_{\epsilon\alpha} \bigg|_{l=l_0} \]

\[
= \tilde{R}_{\beta\gamma} + \tilde{K}^\alpha_{\beta\alpha,\gamma} - \tilde{K}^\alpha_{\beta\gamma,\alpha} + \tilde{K}^\epsilon_{\beta\alpha} \tilde{K}^\alpha_{\epsilon\gamma} - \tilde{K}^\epsilon_{\beta\gamma} \tilde{K}^\alpha_{\epsilon\alpha} \bigg|_{l=l_0},
\]

which depends on the effective 4D Ricci tensor in the Weitzenböck representation given by Eq. (13) and the contortion \(\tilde{K}^\alpha_{\beta\gamma}\). Finally, the Einstein equations for a scalar field in the Levi-Civita representation are

\[
4D \tilde{G}_{\beta\gamma} = 4D \tilde{R}_{\beta\gamma} - \frac{1}{2} h_{\beta\gamma} 4D \tilde{R} \equiv -8\pi G 4D \tilde{T}^{(sym)}_{\beta\gamma},
\]

\[
4D \tilde{S}_{\beta\gamma} \equiv \left[ \tilde{K}^a_{\alpha\gamma,\beta} - \tilde{K}^a_{\beta\alpha,\gamma} \right] \bigg|_{l=l_0=1/H} = -8\pi G 4D \tilde{T}^{(ant)}_{\beta\gamma},
\]

with \(\tilde{R}_{a5} = 0\) and \(\tilde{S}_{a5} = 0\). On the other hand \(4D \tilde{T}^{(ant)}_{\beta\gamma}\) is the effective 4D induced antisymmetric energy-momentum tensor in the Levi-Civita representation. On the other hand \(4D \tilde{T}^{(sym)}_{\beta\gamma}\) is the effective 4D induced symmetric energy-momentum tensor in the Levi-Civita representation, which invariant of form with respect to \(4D \tilde{T}^{(sym)}_{\beta\gamma}\) in terms of the scalar fields and their covariant
derivatives\textsuperscript{2}. Equations (21,22) are the same than obtained by Cartan\textsuperscript{10} and describe the well-known Cartan-Einstein formalism. A perfect fluid representation of the stress tensor components should be

\[
\begin{align*}
\left(\overset{4D}{T}\right)_{\beta\gamma}^{(sym)} & = \frac{1}{2}(P + \rho)\{u_\beta, u_\gamma\} - P g_{\beta\gamma}, \\
\left(\overset{4D}{T}\right)_{\beta\gamma}^{(ant)} & = \frac{1}{2}(P + \rho)[u_\beta, u_\gamma],
\end{align*}
\]  

(23)  

(24)

where \(\{u_\beta, u_\gamma\}\) and \([u_\beta, u_\gamma]\) denotes, respectively, the anti-commutator and commutator between the effective tetra-velocities \(u_\beta\) and \(u_\gamma\). Here, we denote the radiation energy density and pressure by \(\rho\) and \(P\), respectively. Of course, this representation excludes the fermion spinor contributions, which should be included in a high density fermion system\textsuperscript{11}.

The cases with \(\overset{4D}{R}_{\beta\gamma} = 0\) and \(\overset{4D}{R}_{\beta\gamma} \neq 0\) describe a purely Einstein formalism, but cases with zero curvature \(\overset{4D}{R}_{\beta\gamma} = 0\) and nonzero torsion \(\overset{4D}{S}_{\beta\gamma} \neq 0\), can be related with the \(f(T)\) theories\textsuperscript{12}.

\section*{III. AN EXAMPLE: PRE-BIG BANG COLLAPSE}

We consider the 5D Riemann-flat spacetime \(dS^2 = g_{AB} \, dx^A dx^B\)

\[
dS^2 = \left(\frac{l}{l_0}\right)^2 [dt^2 - e^{-2t/l_0} \, dr^2] - dl^2, \tag{25}
\]

such that \(l\) is the non-compact extra coordinate and the 3D spatial coordinates (we shall consider units \(c = \hbar = 1\)) are considered as cartesian \(dr^2 = dx^2 + dy^2 + dz^2\). The metric (25) describes 5D extended universe with constant (and negative) cosmological parameter, which is contracting with the time \(t\). Furthermore, \(dr^2 = dx^2 + dy^2 + dz^2\) is the 3D Euclidean

\textsuperscript{2} Notice that for fermionic fields (for instance with spin 1/2), the Cartan equation (22) should be

\[
\left(\overset{4D}{S}\right)_{\beta\gamma} - \frac{1}{2}\sigma_{\beta\gamma} S = -8\pi G \left(\overset{4D}{T}\right)_{\beta\gamma}^{(ant)},
\]

where \(S = \sigma^{\mu\nu} S_{\mu\nu}, \sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]\) and \(\gamma^\mu\) are the Dirac matrices. However, this issue is beyond the scope of this work.
metric. Since the metric (25) is Riemann-flat (and therefore Ricci-flat), hence it is suitable to describe a 5D vacuum \( G_{AB} = 0 \) in the framework of Space-Time-Matter (STM) theory of gravity. When we make a constant foliation \( l = l_0 \) on this metric we obtain an effective 4D collapsing universe with an asymptotic singular size and an energy density which tends to infinity. In order to avoid this problem, we shall consider a different metric \( dS^2 = g_{ab} dy^a dy^b \):

\[
dS^2 = \left( \frac{l}{l_0} \right)^2 \left[ dt^2 - \frac{\psi^2(t)}{\psi_0^2} e^{-2t/l_0} dr^2 \right] - dl^2,
\]

where \( \psi(t) = \psi_0 \cosh (t/\psi_0) \) is a function of the time \( t \). The metrics (25) and (26) are related by the transformations

\[
g_{ab} = e^A_b e^B_a g_{AB},
\]

such that \( e^A_b \) and its inverse \( \bar{e}^a_B \), are given by

\[
e^A_b = \text{diag} \left[ \left( 1, \frac{\psi(t)}{\psi_0}, \frac{\psi(t)}{\psi_0}, \psi_0, 1 \right) \right], \tag{28}
\]

\[
\bar{e}^a_B = \text{diag} \left[ \left( 1, \frac{\psi_0}{\psi(t)}, \frac{\psi_0}{\psi(t)}, \frac{\psi(t)}{\psi_0}, 1 \right) \right]. \tag{29}
\]

Notice that the metric (25) is also Weitzenböck-flat. The Weitzenböck torsion is zero, so that the Levi-Civita connection \( \Gamma^A_{BC} \) is coincident with the Weitzenböck connection \( (W) \Gamma^A_{BC} = \bar{\Gamma}^A_{BC} \). On the other hand, the metric (26) is not Riemann-flat \( R^a_{bcd} \neq 0 \), so that it is not a good candidate to realize standard IMT. However, from the point of view of the Weitzenböck geometry, it is Weitzenböck-flat and also has nonzero torsion:

\[
\bar{R}^a_{bcd} = 0, \bar{T}^a_{bc} \neq 0.
\]

In the spacetime (26) the connections are related by the contortion

\[
K^a_{bc} = \bar{c}^a_{bc} - \bar{c}^a_{bc}, \text{ with } \bar{c}^a_{bc} = \bar{e}^a_A \partial_c e^A_b.
\]

In order to describe the collapse, we shall consider the action on a Riemann-flat 5D metric (25), for a Levi-Civita scalar field \( \varphi(x^A) \), which is free of interactions

\[
I = \int d^4 x \sqrt{|g_1|} \left( \frac{\mathcal{R}}{16 \pi G} + \frac{1}{2} g^{AB} \nabla_A \varphi \nabla_B \varphi \right), \tag{30}
\]

where \( g_1 \) is the determinant of the tensor metric \( g_{AB} \). The equation of motion for \( \varphi(x^A) \) on the Riemann-flat and for \( \bar{\varphi}(x^A) \) on the Weitzenböck-flat spacetime (25), are \( \nabla^A \varphi_A = \nabla^A \bar{\varphi}_A = 0 \), such that

\[
\nabla^A \varphi_A \equiv g^{AB} \left( \partial_A \varphi_B - \Gamma^C_{AB} \bar{\varphi}_C \right), \tag{31}
\]

\[
\bar{\nabla}^A \bar{\varphi}_A \equiv g^{AB} \left( \partial_A \bar{\varphi}_B - \bar{\Gamma}^C_{AB} \bar{\varphi}_C \right). \tag{32}
\]
Since \( g^{AB} \tilde{\varphi}_{A} \tilde{\varphi}_{B} = g^{ab} \tilde{\varphi}_{a} \tilde{\varphi}_{b} \), we can consider that the action \( I \) on the metric (26) is an invariant after making the transformation \( x^A \rightarrow y^a \), so that one can describe the Weitzenböck field \( \tilde{\varphi}(y^a) \) on (26), as

\[
I = \int d^4y d\ell \sqrt{|g_2|} \left( \frac{\mathcal{R}}{16\pi G} + \frac{1}{2} g^{ab} \tilde{\varphi}_{a} \tilde{\varphi}_{b} \right),
\]

where \( g_2 \) is the determinant of the metric tensor \( g_{ab} \) and \( \mathcal{R} = 0 \) is a Weitzenböck invariant: \( g^{ab} \mathcal{R}_{abc} \). The equation of motion for \( \tilde{\varphi}(y^a) \) is given by the null D’Alembertian of Weitzenböck: \( \tilde{\nabla}_{a} \tilde{\varphi}_{a} = 0 \), or

\[
\ddot{\tilde{\varphi}} - \frac{e^{2t/l_0}}{\cosh^2(t/l_0)} \nabla^2_{\ell} \tilde{\varphi} - \left( \frac{l}{l_0} \right)^2 \frac{\partial^2 \tilde{\varphi}}{\partial \ell^2} = 0.
\]

This means that the scalar field \( \tilde{\varphi}(y^a) \) is a free scalar field in a Weitzenböck representation. In other words, its origin is not geometric, but their physical properties (describing a 5D physical vacuum in absence of interactions in a Weitzenböck representation) have a geometric dependence. On the other hand, from the point of view of the Levi-Civita representation, this means that

\[
\nabla_{a} \varphi_{a} = \tilde{\nabla}_{a} \varphi_{a} - g^{ab} K_{c}{}_{ab} \varphi_{c} = 0,
\]

where we have made use of the fact that \( \tilde{\nabla}_{a} \varphi_{a} = \nabla_{a} \varphi_{a} = 0 \). The relevant contortion components are

\[
K_{122} = \frac{e^{-2t/l_0}}{l_0} \cosh(t/l_0) \left[ \sinh(t/l_0) - \cosh(t/l_0) \right], \quad K_{155} = 0,
\]

\[
K_{511} = \frac{l}{l_0^2}, \quad K_{522} = -\frac{l}{l_0^2} e^{-2t/l_0} \cosh^2(t/l_0).
\]

Hence, the equation of motion for the Levi-Civita scalar field \( \varphi(y^a) \), finally, as a result is found to be

\[
\ddot{\varphi} + \frac{3}{l_0} \left[ \tanh(t/l_0) - 1 \right] \dot{\varphi} - \frac{e^{2t/l_0}}{\cosh^2(t/l_0)} \nabla^2_{\ell} \varphi - \left( \frac{l}{l_0} \right)^2 \left[ \frac{\partial^2 \varphi}{\partial \ell^2} + \frac{4 \partial \varphi}{\partial \ell} \right] = 0.
\]

This field can be written as a Fourier expansion in terms of the modes \( \varphi_{k}(y^a) = A_{k} \xi_{k}(t) e^{i \vec{k} \cdot \vec{y}} \Lambda(l) \), where \( \Lambda(l) \) and \( \xi_{k}(t) \) are given, respectively, by the solutions of the equations

\[
\left( \frac{l}{l_0} \right)^2 \left[ \frac{d^2 \Lambda}{dl^2} + \frac{4 d \Lambda}{l dl} \right] = M^2 \Lambda(l),
\]

\[
\ddot{\xi}_{k} + \frac{3}{l_0} \left[ \tanh(t/l_0) - 1 \right] \dot{\xi}_{k} + \left[ \frac{e^{2t/l_0}}{\cosh^2(t/l_0)} k^2 - M^2 \right] \xi_{k}(t) = 0.
\]
where $M^2$ is a separation of variables constant and de dot denotes the derivative with respect to $t$. The solutions for Eqs. (39) and (40) are

$$
\Lambda(l) = \left(\frac{l}{l_0}\right)^{-3/2} \left[ \Lambda_1 \left(\frac{l}{l_0}\right) \frac{\sqrt{9+4M^2l_0^2}}{2} + \Lambda_2 \left(\frac{l}{l_0}\right) \frac{\sqrt{9+4M^2l_0^2}}{2} \right],
$$

(41)

$$
\xi_k(t) = [1 + \tanh (t/l_0)] [\tanh (t/l_0) - 1] \frac{\sqrt{M^2 - 4k^2}}{2} \times \left\{ A_{12} F_1 \left[ \left[ a_1, b_1 \right], c_1, 2 \frac{\sqrt{9+4M^2l_0^2}}{2} \left[1 + \tanh (t/l_0)\right] \right] \right.

\left. \left. + B_{12} F_1 \left[ \left[ a_2, b_2 \right], c_2, 2 \frac{\sqrt{9+4M^2l_0^2}}{2} \left[1 + \tanh (t/l_0)\right] \right] \right\},
$$

(42)

where ($A_1, A_2$) are constants and $F_1$ denotes the Gaussian hypergeometric function with parameters

$$
a_{1,2} = \frac{l_0}{2} \sqrt{M^2 - 4k^2} + \sqrt{1 - l_0^2 k^2} \mp \frac{\sqrt{l_0^2 M^2 + 9 + 1}}{2},
$$

(43)

$$
b_{1,2} = \frac{l_0}{2} \sqrt{M^2 - 4k^2} - \sqrt{1 - l_0^2 k^2} \mp \frac{\sqrt{l_0^2 M^2 + 9 + 1}}{2},
$$

(44)

$$
c_{1,2} = \left[ 1 \mp \sqrt{9 + l_0^2 M^2} \right].
$$

(45)

In the ultraviolet (UV) limit of the spectrum, for which $k \gg M$ and $kl_0 \gg 1$, these parameters take the asymptotic form

$$
a_{1,2} \simeq 2i l_0 k \mp \frac{\sqrt{l_0^2 M^2 + 9 + 1}}{2},
$$

(46)

$$
b_{1,2} \simeq \mp \frac{\sqrt{l_0^2 M^2 + 9 + 1}}{2},
$$

(47)

$$
c_{1,2} = \left[ 1 \mp \sqrt{9 + l_0^2 M^2} \right].
$$

(48)

When $t \to \infty$, one can see that $\tanh (t/l_0) \to 1$, so that the equation of motion (40) tends asymptotically to

$$
\ddot{\xi}_k + [4k^2 - M^2] \xi_k(t) = 0,
$$

(49)

which has the asymptotic solution

$$
\xi_k(t) \big|_{t \to \infty} \to A(k) e^{i \sqrt{4k^2 - M^2} t} + B(k) e^{-i \sqrt{4k^2 - M^2} t}.
$$

(50)

This means that the amplitude of the fluctuations remains asymptotically constant as $t \to \infty$. 


A. Effective 4D dynamics

Now we consider a constant foliation \( l = l_0 = 1/H \) on the fifth coordinate. The effective 4D metric for observers that move with Weitzenböck velocities \( \bar{u}^a = (1, 0, 0, 0, 0) \), will be

\[
\begin{align*}
    dS^2 &= h_{\alpha\beta}dy^\alpha dy^\beta, \\
    dS^2 &= (l/l_0)^2 \left[ dt^2 - \left( \frac{\psi(t)}{\psi_0^2} e^{-2t/l_0} \right)^2 dr^2 \right] - d^2 l \
    &\to dS^2 = dt^2 - \cosh^2 (H_0 t) e^{-2H_0 t} dr^2.
\end{align*}
\]

Here, the scale factor of the universe which collapses is \( a(t) = \cosh (t/l_0) e^{-l_0^{-1}t} \), the Hubble parameter \( H(t) = \frac{\dot{a}}{a} \) and the deceleration parameter \( q = -\frac{\ddot{a}}{a^2} \). This is a very interesting behavior because the model describes a contracting universe which has an asymptotic finite size \( a_{\text{min}} = 1/(2H_0) \). This effect is due to the fact that in General Relativity the action is invariant under time reflections. Thus, to any standard cosmological solution \( H(t) \), describing decelerated expansion and decreasing curvature \( (H > 0, \dot{H} < 0) \), time reversal associates a reflected solution, \( H(-t) \), describing a contracting Universe. In a string cosmology context, these solutions are called dual [8]. In this work we are dealing with an extra dimensional cosmological model where the extra dimension is non-compact. However, this duality is preserved and the interpretation of the results obtained by Gasperini and Veneziano in [8] are preserved. A possible cosmological application of such a cosmological scenario is bouncing cosmology, in which the cosmic singularity \( (a = 0) \) is avoided due to the repulsive effects produced by fermions during the collapse, which are more significant at very short distances [9].

In our example \( S_{\beta\gamma} = 0 \), so that the relativistic dynamics on the effective 4D metric (51), being given by Eqs. (21) and (22). The physical information is provided by the effective 4D energy-momentum tensor in the Levi-Civita representation:

\[
\begin{align*}
    T_{\beta\gamma}^{4D} &= \phi_{,\beta} \phi_{,\gamma} - g_{\beta\gamma} \left[ \frac{1}{2} g^{\sigma\mu} \phi_{,\sigma} \phi_{,\mu} + \frac{1}{2} g^{55} (\phi_{,5})^2 \right] \bigg|_{l = l_0 = 1/H_0}.
\end{align*}
\]

Notice that \( g^{\alpha\beta} \big|_{l = l_0 = 1/H_0} = h^{\alpha\beta} \). Here, the last term in the brackets can be identified with the effective 4D scalar potential in the Levi-Civita representation: \( V(\phi) \equiv -\frac{1}{2} g^{55} (\phi_{,5})^2 \big|_{l = l_0 = 1/H_0} \). Using Eqs. (16), (22), (18), (19), we obtain the effective 4D relativistic dynamics. In our example \( S_{\beta\gamma} = 0 \), which agrees with what one expects for a
scalar field because the antisymmetric contribution of the effective 4D stress tensor becomes null: $\langle T_{\beta\gamma}^{(ant)} \rangle = 0$. The effective 4D symmetric contribution of the energy-momentum tensor components are

\[
\langle T_0^0 \rangle = \frac{\dot{\varphi}^2}{2} + \frac{1}{2a^2} \left( \vec{\nabla} \varphi \right)^2 + V(\varphi),
\]

\[
\langle T_i^j \rangle = - \left[ \frac{\dot{\varphi}^2}{2} - \frac{1}{6a^2} \left( \vec{\nabla} \varphi \right)^2 - V(\varphi) \right] \delta^i_j,
\]

where

\[
V(\varphi) = \left. \frac{1}{2} \left[ \frac{d}{dl} \ln |\Lambda| \right]^2 \varphi^2 \right|_{l_0}.
\]

In other words, the effective 4D squared mass of the scalar field is related to the extra dimensional static solution of $\varphi$ by the expression

\[
M_{eff}^2 = \left. \left[ \frac{d}{dl} \ln |\Lambda| \right]^2 \right|_{l=l_0}.
\]

Finally, the effective 4D Einstein equations in the Levi-Civita representation, are (we use 3D cartesian coordinates and the foliation $l = l_0 = 1/H_0$)

\[
G_0^0 = - \frac{3H_0^2}{\cosh^2(H_0t)} \left[ \cosh (H_0t) - \sinh (H_0t) \right]^2,
\]

\[
G_i^j = - \frac{H_0^2}{\cosh^2(H_0t)} \left[ \cosh (H_0t) - \sinh (H_0t) \right] \left[ 5 \cosh (H_0t) - \sinh (H_0t) \right] \delta^i_j,
\]

so that, using the fact that the Einstein equations are, respectively, $G_0^0 = -8\pi G \rho$ and $G^x_y = G^y_x = 8\pi G P$, we obtain the equation of state for the universe

\[
\frac{P}{\rho} = \omega(t) = - \frac{1}{3} \frac{\left[ 5 \cosh (H_0t) - \sinh (H_0t) \right]}{\left[ \cosh (H_0t) - \sinh (H_0t) \right]}.
\]

Notice that $\omega$ always remains with negative values $\omega(t) < -1$, and evolves from $-5/3$ to $-\infty$, for large asymptotic times. The effective 4D scalar curvature

\[
R = \frac{6H_0^2}{\cosh^2(H_0t)} \left[ \cosh (H_0t) - \sinh (H_0t) \right] \left[ 3 \cosh (H_0t) - \sinh (H_0t) \right],
\]

3 In the cases where $M_{eff}^2 > 0$ and the universe expands with a nearly constant energy density given by $\rho \simeq \langle V(\varphi) \rangle$ and a pressure $P \simeq - \langle V(\varphi) \rangle$, the potential (55) is a good candidate to describe inflation.
decreases with the time and has a null asymptotic value \( R|_{t \to \infty} \to 0 \). We are considering a spatially isotropic and homogeneous background, so that we shall consider an averaging value with respect to a Gaussian distribution on a Euclidean 3D volume. The late time expectation values for both, the energy density and the pressure, are zero [see Eqs. (57) and (58)]

\[
\rho|_{t \to \infty} = \langle 0|T^0_0|0 \rangle = \left. \left[ \frac{1}{2} \dot{\varphi}^2 + \frac{1}{2a^2(t)} \left( \nabla \varphi \right)^2 + \frac{1}{2} M_{eff}^2 \varphi^2 \right] \right|_{l=1/H_0} = 0, \quad (61)
\]

\[
P|_{t \to \infty} = \langle 0|T^i_i|0 \rangle = \delta^i_j \left. \left[ \frac{1}{2} \dot{\varphi}^2 - \frac{1}{6a^2(t)} \left( \nabla \varphi \right)^2 - \frac{1}{2} M_{eff}^2 \varphi^2 \right] \right|_{l=1/H_0} = 0, \quad (62)
\]

where we denote by \( \langle 0|...|0 \rangle \) as the quantum expectation value calculated on a 4D vacuum state.

**B. Particular solution: asymptotic collapsing state**

In order to obtain the effective 4D squared mass \( M_{eff}^2 \) we must calculate the asymptotic energy density, which is zero because \( G^0_0 \to 0 \) for late times (see Eq. (57). The effective 4D scalar field \( \varphi(y^a, l = l_0) \) can be written as a Fourier expansion on the 4D hypersurface

\[
\varphi(y^a, l_0) = \frac{\Lambda(l_0)}{(2\pi)^{3/2}} \int d^3k \left[ A_k e^{i\vec{k} \cdot \vec{y}} \xi_k(t) + h.c. \right], \quad (63)
\]

such that the operators of creation and destruction comply with the algebra

\[
\left[ A_k, A_{k'}^\dagger \right] = \delta^{(3)}(\vec{k} - \vec{k}'), \quad \left[ A_{k}^\dagger, A_{k'} \right] = \left[ A_{k}, A_{k'}^\dagger \right] = 0. \quad (64)
\]

Hence the canonical structure \( \left[ \varphi(t, \vec{y}, l_0), \dot{\varphi}(t, \vec{y'}, l_0) \right] = i a^{-3} \delta^{(3)}(\vec{y} - \vec{y'}) \), will be ensured when the condition of normalization

\[
\xi_k \dot{\xi}_k^* - \dot{\xi}_k \xi_k^* = \frac{i}{a^3(t)}, \quad (65)
\]

is fulfilled. To make this happen on the asymptotic late-time state, we must require that

\[
\xi_k \dot{\xi}_k^* - \dot{\xi}_k \xi_k^* \bigg|_{t \to \infty} = \frac{i}{a^3_{\text{min}}}, \quad (66)
\]

in Eq. (50). A particular solution of the condition (66) is given by \( A(k) = 0 \) and \( B(k) = \frac{2H_0^3}{[4k^2 - M^2]^{1/4}} \). With this solution we find that the asymptotic energy density is

\[
\rho|_{t \to \infty} = \frac{H_0^3}{\pi^2} \int_0^{4\pi H_0} dk \frac{k^2}{[4k^2 - M^2]^{1/2}} \left[ \frac{(4k^2 - M^2)}{2} + 2k^2 + \frac{M_{eff}^2}{2} \right] = 0, \quad (67)
\]
where the limits of integration are \((k_{\text{min}} = 0, k_{\text{max}} = 2\pi/a_{\text{min}} = 4\pi H_0)\), such that the minimum scale factor is given by the asymptotic scale factor during the collapse: \(a_{\text{min}} = 1/(2H_0)\). On the other hand, from Eq. (58) we know that the asymptotic pressure is zero, so that from the expression (62), we obtain

\[
P\big|_{t\to\infty} = \frac{H_0^3}{\pi^2} \int_0^{4\pi H_0} dk \frac{k^2}{\left[4k^2 - M^2 \right]^{1/2}} \left[ \frac{(4k^2 - M^2)}{2} - \frac{2}{3} k^2 - \frac{M_{\text{eff}}^2}{2} \right] = 0. \tag{68}
\]

From Eqs. (67) and (68), we obtain the effective 4D mass of the potential \(V(\varphi)\) and the parameter \(M^2\)

\[
M_{\text{eff}}^2 = -32\pi^2 H_0^2, \quad M^2 = 64\pi^2 H_0^2. \tag{69}
\]

This means that the effective 4D potential \(\langle 0|V(\varphi)|0 \rangle < 0\) is negative and the asymptotic system is unstable. The system which we are describing is very similar to a gravitational collapse suffered by an observer in a Schwarzschild black-hole, but in our case there is no cosmic singularity, because \(a_{\text{min}} \neq 0\). Notice that the asymptotic values of radiation energy density and pressure are zero, because all the matter is eaten by the black-hole. To make a more realistic model one would include the contribution of fermions as a condensate, in order to describe the physical origin of repulsion on very small scales. Of course the study of such a physical problem deserves a more intense study, but it goes beyond the scope of this paper.

### IV. final remarks

We have extended the theoretical background for the IMT of gravity. The fact that one can define a 5D vacuum on a Levi-Civita 5D curved spacetime using the Weitzenböck representation opens a huge window of possible applications for this theory. In this framework we study a pre-big bang collapsing universe that ends with finite size and energy density. Notice that we have restricted our analysis to a 5D coordinate basis, but the formalism can be extended to a non-coordinate basis. As an example we have studied a pre-big bang collapsing universe. The difference with respect to earlier studies\(^1\) lies in the fact that the scalar field induced in the Levi-Civita representation is massive, its squared mass being given by the expression (56). We have found that the effective 4D expectation value for the potential is quadratic and negative, so that the asymptotic late time collapse is an unstable
system with an equation of state \( \omega = -\infty \). This suggests a good initial state for a new big-bang.

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