General Integral Inequalities Including Weight Functions

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Abstract

In this note, we present two general classes of integral inequalities motivated by their applications to infinite dimensional systems. The inequalities possess general structures in terms of weight functions and lower quadratic bounds. Many existing inequalities in the published literature, including those with free matrix variables, are the special cases of our inequalities. An relation on the lower bounds of the proposed inequalities is also established. For specific applications, our inequalities are applied to construct a Liapunov-Krasovskii functional for the stability analysis of a linear coupled differential-difference system with a distributed delay, which gives to equivalent stability conditions based on the properties of the proposed inequalities. Finally, it is worthy to note that the inequalities in this note can be applied in general contexts such as the stability analysis of PDE-related systems or sampled-data systems.

Index Terms

Integral Inequalities, Free Matrix Type Integral Inequalities, Coupled Differential-Difference Systems.

I. INTRODUCTION

Many control and optimization problems involve the applications of integral inequalities. Notable examples can be found in the stability analysis and stabilization of linear delay systems [1], [2] and PDE-related systems [3], [4], [5], [6] based on the application of the direct Liapunov method. Unlike analyzing the stability of an LTI system, functionals with integral structures are required to be constructed for the stability analysis of infinite dimensional systems, and the existing approaches may only lead to sufficient stability conditions due to the intrinsic limitations of their underlying mathematical structures.

In this note, we present two classes of integral inequalities which could be applied to a variety of applications such as delay (time-varying) related systems, PDE-related systems, and sampled-data systems, etc. In Section II, we present the first class of integral inequalities which contain no extra matrix variables other than the original matrix term in the quadratic lower bound. This class of inequalities possesses general structures which can be reflected by the fact that it can generalize many existing inequalities in [7], [1], [8], [2], [9], [10], [11], [12]. On the other hand, the second class of integral inequalities, which is of the free matrix type, is derived in Section III which can generalize the existing inequalities in [13], [14], [15]. We then prove an important conclusion with respect to the relation between the lower bounds of the two classes of inequalities in this note, through which relations with respect to the lower bounds of many existing inequalities could be established. To show a specific application of our inequalities, we apply them in Section IV to derive stability conditions for a linear coupled differential-difference system (CDDS) [16] with a distributed delay by constructing a parameterized complete Liapunov-Krasovskii functional. We show

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that equivalent stability conditions, whose solvability is invariant with respect to a parameter of the Liapunov-Krasovskii functional, can be obtained by the application of our inequalities. The core contributions in this note are rooted in the generality of the proposed inequalities supported by their nice properties. This provides great potential to apply them to tackle problems in the contexts of control and optimizations.

**NOTATION**

Let $S^n := \{ X \in \mathbb{R}^{n \times n} : X = X^\top \}$ and $\mathbb{R}_{\geq a} := \{ x \in \mathbb{R} : x \geq a \}$ and $\mathbb{R}^{n \times m} = \{ X \in \mathbb{R}^{n \times m} : \text{rank}(X) = r \}$. $I_f(\mathcal{X}; \mathcal{Y})$ denotes the space of all functions which are Lebesgue integrable from the set $\mathcal{X}$ onto $\mathcal{Y}$. We frequently utilize the notations of universal quantifier $\forall$ and the existential quantifier $\exists$ in this paper. $\text{Sy}(X) := X + X^\top$ stands for the sum of a matrix with its transpose. $\text{Col}_{i=1}^n x_i := [\text{Row}_{i=1}^n x_i]^\top = [x_1^\top \cdots x_i^\top \cdots x_n^\top]^\top$ stands for a column vector containing a sequence of mathematical objects (scalars, vectors, matrices etc.). The symbol $\ast$ is used to indicate $[\ast]YX = X^\top YX$ or $X^\top Y[\ast] = X^\top YX$ or $[A \ast B] = [A \ast B]^\top$. $O_{n \times n}$ denotes a $n \times n$ zero matrix which can be abbreviated into $O_n$, while $0_n$ represents a $n \times 1$ column vector. We frequently use $X \otimes Y = [X \ast Y]$ to denote the diagonal sum of two matrices, respectively. $\otimes$ stands for the Kronecker product. Finally, we assume the order of matrix operations to be matrix (scalars) multiplications $> \otimes > \oplus > +$.

**II. FIRST INTEGRAL INEQUALITIES**

In this section, we present the first general class of inequalities whose generality can be demonstrated by its structures and the fact that it generalizes many existing inequalities.

We will frequently apply the following lemma concerning the property of Kronecker product throughout the entire paper.

**Lemma 1.** $\forall X \in \mathbb{R}^{n \times m}, \forall Y \in \mathbb{R}^{m \times p}, \forall Z \in \mathbb{R}^{q \times r}$,

$$(X \otimes I_q)(Y \otimes Z) = (XY) \otimes (I_qZ) = (XY) \otimes Z = (XY) \otimes (ZI_r) = (X \otimes Z)(Y \otimes I_r).$$ (1)

The following lemma is partially taken from Lemma 4.1 in [17] which is crucial for the derivation of the results in this note.

**Lemma 2.** Given matrices $C \in \mathbb{S}^m_{>0}$, $B \in \mathbb{R}^{m \times n}$, then

$$\forall M \in \mathbb{R}^{m \times n}, \quad B^\top C^{-1}B \succeq M^\top B + B^\top M - M^\top CM$$ (2)

where the inequality in (2) becomes an equality with $M = C^{-1}B$.

To present the first class of inequalities in the theorem in this section, we define the weighted Lebesgue function space

$$\mathcal{L}_2^\omega (\mathcal{K}; \mathbb{R}^d) := \left\{ \phi(\cdot) \in L_f(\mathcal{K}; \mathbb{R}^d) : \| \phi(\cdot) \|_{2, \omega} < \infty \right\}$$ (3)

with $d \in \mathbb{N}$ and the semi-norm $\| \phi(\cdot) \|_{2, \omega} := \int_{\mathcal{K}} \omega(\tau) \phi(\tau)^2 d\tau$ where $\omega(\cdot) \in L_f(\mathcal{K}; \mathbb{R}_{\geq 0})$ and $\omega(\cdot)$ has only countably infinite or finite numbers of zero values. Furthermore, $\mathcal{K} \subseteq \mathbb{R} \cup \{ \pm \infty \}$ and the Lebesgue measure of $\mathcal{K}$ is non-zero.

**Theorem 1.** Given $\omega(\cdot)$ in (3) and $f(\cdot) \in \mathcal{L}_2^\omega (\mathcal{K}; \mathbb{R}^d)$ which satisfies

$$\int_{\mathcal{K}} \omega(\tau) f(\tau) f^\top(\tau) d\tau > 0,$$ (4)
then the inequality
\[
\int_{\mathcal{K}} \varpi(\tau) x^\top(\tau) U x(\tau) d\tau \geq \left( \int_{\mathcal{K}} \varpi(\tau) F(\tau) x(\tau) d\tau \right)^\top (F \otimes U) \int_{\mathcal{K}} \varpi(\tau) F(\tau) x(\tau) d\tau,
\]
holds for all \( x(\cdot) \in L^2(\mathcal{K}; \mathbb{R}^n) \) with \( U \in \mathbb{S}_{+}^n \). Moreover, we have for all \( x(\cdot) \in L^2(\mathcal{K}; \mathbb{R}^n) \) and for all \( \omega \in \mathbb{R}^d 
\]
\[
\int_{\mathcal{K}} \varpi(\tau) x^\top(\tau) U x(\tau) d\tau \geq \vartheta^\top (I_d \otimes U) \vartheta \geq \text{Sy} \left( \vartheta^\top (F^{-1} \otimes U) \omega \right) - \omega^\top (F \otimes U) \omega
\]
with \( U \in \mathbb{S}_{+}^n \), where \( \vartheta = \int_{\mathcal{K}} \varpi(\tau) F(\tau) x(\tau) d\tau \) and \( F(\tau) = f(\tau) \otimes I_n \) and \( F^{-1} = \int_{\mathcal{K}} \varpi(\tau) f(\tau) f^\top(\tau) d\tau \). Furthermore, with \( \omega = (F \otimes I_n) \vartheta \) in (6), the second inequality in (6) becomes an equality where (6) can be written as (5).

**Proof:** Let \( \varepsilon(\tau) = x(\tau) - F^\top(\tau) \omega \) with \( F(\tau) = f(\tau) \otimes I_n \). By the expression of \( \varepsilon(\tau) \) with \( \int_{\mathcal{K}} \varpi(\tau) e^\top(\tau) U e(\tau) d\tau \), we have
\[
\int_{\mathcal{K}} \varpi(\tau) e^\top(\tau) U e(\tau) d\tau = \int_{\mathcal{K}} \varpi(\tau) x^\top(\tau) U x(\tau) d\tau - 2 \int_{\mathcal{K}} \varpi(\tau) x^\top(\tau) U F^\top(\tau) d\tau \omega + \omega^\top \int_{\mathcal{K}} \varpi(\tau) F(\tau) U F^\top(\tau) d\tau \omega.
\]
Now apply (1) to \( UF^\top(\tau) \) in (7) with \( F(\tau) = f(\tau) \otimes I_n \), we have
\[
UF^\top(\tau) = U(f^\top(\tau) \otimes I_n) = f^\top(\tau) \otimes U = F^\top(\tau)(I_d \otimes U).
\]
Moreover, applying (8) to \( \int_{\mathcal{K}} \varpi(\tau) x^\top(\tau) UF^\top(\tau) d\tau \omega \) in (7) yields
\[
\int_{\mathcal{K}} \varpi(\tau) x^\top(\tau) UF^\top(\tau) d\tau = \int_{\mathcal{K}} \varpi(\tau) x^\top(\tau) F^\top(\tau)(I_d \otimes U) \omega = \vartheta^\top(I_d \otimes U) \omega
\]
with \( \vartheta = \int_{\mathcal{K}} \varpi(\tau) F(\tau) x(\tau) d\tau \). By (8) and (1), we have
\[
\int_{\mathcal{K}} \varpi(\tau) F(\tau) UF^\top(\tau) d\tau = \int_{\mathcal{K}} \varpi(\tau) F(\tau) F^\top(\tau)(I_d \otimes U) =
\]
\[
\left[ \int_{\mathcal{K}} \varpi(\tau) f(\tau) f^\top(\tau) d\tau \otimes I_n \right] (I_d \otimes U) = F^{-1} \otimes U
\]
where \( F^{-1} = \int_{\mathcal{K}} \varpi(\tau) f(\tau) f^\top(\tau) d\tau \succ 0 \) given (4). By (9) and (10), the expressions in (7) can be simplified as
\[
\int_{\mathcal{K}} \varpi(\tau) e^\top(\tau) U e(\tau) d\tau = \int_{\mathcal{K}} \varpi(\tau) x^\top(\tau) U x(\tau) d\tau - \text{Sy} \left( \vartheta^\top (I_d \otimes U) \omega \right) + \omega^\top (F^{-1} \otimes U) \omega,
\]
Now given \( U \succ 0 \) with \( \omega = (F \otimes I_n) \vartheta \), (11) gives (5). Furthermore, assume \( U \succ 0 \), we have that \( \forall x(\cdot) \in L^2(\mathcal{K}; \mathbb{R}^n) \), \( \forall \omega \in \mathbb{R}^d \)
\[
\text{Sy} \left( \vartheta^\top (I_d \otimes U) \omega \right) - \omega^\top (F^{-1} \otimes U) \omega \leq \vartheta^\top (I_d \otimes U)(F \otimes U^{-1})(I_d \otimes U) \vartheta = \vartheta^\top(F \otimes U) \vartheta
\]
by the application of Lemma 2, where the inequality in (12) becomes an equality with \( \omega = (F \otimes I_n) \vartheta \). By both (12) and (5), then (6) and the rest of the statements in Theorem 1 is proved.

**Remark 1.** The constraint in (4) indicates that the functions \( \{f_i(\cdot)\}_{i=1}^d \) in \( f(\cdot) = \text{Col}_{i=1}^d f_i(\tau) \) are linear independently in a Lebesgue sense. (See the Theorem 7.2.10 [18]) Since \( f(\cdot) \in L^2(\mathcal{K}; \mathbb{R}^d) \) with general options for \( \varpi(\cdot) \), hence the generality of the structure of \( f(\cdot) \) is clearly evident, which provides a tremendous degree of freedom for the structures of (5). For instance, one may assume that \( f(\cdot) \) contains orthogonal functions [8], elementary functions or other types of functions as long as \( \{f_i(\cdot)\}_{i=1}^d \) in \( f(\cdot) \) are linearly independent in a Lebesgue sense.
Remark 2. Note that $\varepsilon(\tau) = x(\tau) - F^T(\tau)\omega$ in the proof of Theorem 1 can be interpreted as using all the functions in $f(\cdot)$ to approximate the functions in each row of $x(\cdot)$ individually, where $\varepsilon(\cdot)$ here measure the error of approximation. Specifically, we have $\varepsilon(\tau) = x(\tau) - F^T(\tau)\omega = x(\tau) - \sum_{i=1}^{d} f_i(\tau)\omega_i = x(t) - \left[\text{Row}_{i=1}^{d} \omega_i\right] f(\tau)$ where $\omega = \text{Col}_{i=1}^{d} \omega_i$ with $\omega_i \in \mathbb{R}^n$, by which it is clear that $\omega$ can be interpreted as an approximation coefficient. Indeed, one can regard the lower bound of (6) as an optimization problem

$$\max_{\omega \in \mathbb{R}^d} \left[Sy \left(\theta^T (I_n \otimes U) \omega \right) - \omega^T (F^{-1} \otimes U) \omega \right].$$

with given $f(\cdot), U > 0$ and $x(\cdot)$ and by treating $\omega$ as a decision variable. Based on the proof of Theorem 1, we have shown that $\omega = (F \otimes I_n)\theta$ is the ‘best’ coefficient one can construct to obtain the largest lower bound for (13) with given $f(\cdot), U > 0$ and $x(\cdot)$. This shows the optimality of the inequality in (5) with $U > 0$.

Remark 3. The optimal value of $\omega$ for (13) may be determined by differentiating $\omega$ in the expression of (13). This kind of idea has been considered in page 2 of [19] and the proof of Lemma 3 in [20], which may also require the application of Hessian matrix [20]. On the other hand, the use of Lemma 2 in the proof of Theorem 1 also provides a way to show the optimality of (5) with $U > 0$.

Remark 4. With appropriate $x(\cdot), K = [-r, 0]$ and mathematical manipulations, the inequalities in [1, eq.(5)–(6)] can be obtained by (5) if $f(\cdot)$ contains the Legendre polynomials over $[-r, 0]$. Meanwhile, if $f(\cdot)$ contains only orthogonal functions, then [8, eq.(5)] can be obtained by (5) via the use of commutation matrices [21]. Let $K = [0, +\infty)$, then the inequality in [12, eq.(9)] is the special case of (5) with appropriate $\varpi(\cdot)$ and $f(\cdot)$. Finally, with $\varpi(\tau) = 1$, (5) becomes [2, eq.(16)].

Remark 5. One can conclude that the polynomials in [9, eq.(13)–(14)]; [11, eq.(3)–(4)]; [10, eq.(2)] and [20, eq.(5)–(7)] are the special cases of the Jacobi polynomials [22, 22.3.2] over $[a, b]$ with appropriate weight functions. Note that the above conclusions on [20, eq.(5)–(7)] and [11, eq.(3)–(4)] are established based on the use of the Cauchy formula for repeated integrations [9, See eq.(5)–(6) and eq.(25)–(26)]. As a result, the inequalities in [9, eq.(27), (34)]; [10, eq.(2)]; [11, eq.(1)–(2)]; and the inequalities in [20, (8)–(9)] with finite terms of summations, are covered by (5) by choosing appropriate $x(\cdot)$ and letting $f(\cdot)$ be the Jacobi polynomials over $[a, b]$ with the corresponding $\varpi(\tau) = (\tau - a)^p$ or $\varpi(\tau) = (b - \tau)^p$.

Remark 6. Note that an substitution $Gf(\tau) \rightarrow f(\tau)$ for (5) with an invertible $G \in \mathbb{R}^{n \times n}$ gives a lower bound which is equivalent to the lower bound of (5).

### III. Second Class of Integral Inequalities

This section is devoted to presenting another general class of inequalities named as the free matrix type. This type of inequalities has been previously researched in [13], [14], [15], which can be useful in dealing with the stability analysis of systems with time-varying delays. Finally, a relation between the proposed inequalities in Sections II and III is established in a theorem.

The following lemma, which will be applied for the derivations of Theorem 2, can be obtained via the definition of matrix multiplications and Kronecker products.

**Lemma 3.** Given a matrix $X := \text{Row}_{i=1}^{d} X_i \in \mathbb{R}^{n \times d \rho n}$ with $n; d; \rho \in \mathbb{N}$ and a function $f(\tau) = \text{Col}_{i=1}^{d} f_i(\tau) \in \mathbb{R}^{d}$. Let $\tilde{X} := \text{Col}_{i=1}^{d} X_i \in \mathbb{R}^{dn \times \rho n}$ we have

$$X(f(\tau) \otimes I_{\rho n}) = \sum_{i=1}^{d} f_i(\tau) X_i = \left(f^T(\tau) \otimes I_n\right) \tilde{X}.$$

(14)
Theorem 2. Given \( \rho \in \mathbb{N} \) and the same \( \varpi(\cdot) \), and \( f(\cdot) \) defined in Theorem 1 and \( U \in \mathbb{S}_{>0}^n \), if there exist \( Y \in \mathbb{S}^{\rho n} \) and \( X = \text{Row}_{i=1}^d X_i \in \mathbb{R}^{n \times \rho n} \) such that
\[
\begin{bmatrix}
U & -X \\
* & Y
\end{bmatrix} \succeq 0,
\]
then we have for all \( x(\cdot) \in \mathbb{L}^2_\varpi(K; \mathbb{R}^n) \) and for all \( z \in \mathbb{R}^{\rho n} \)
\[
\int_K \varpi(\tau) x^\top(\tau) U x(\tau) d\tau \geq \text{Sy} \left( \vartheta^\top \tilde{X} z \right) - z^\top \mathbf{W} z
\]
(16)
where \( \vartheta = \int_K \varpi(\tau) \mathbf{F}(\tau) x(\tau) d\tau \) and \( \tilde{X} = \text{Col}_{i=1}^d X_i \in \mathbb{R}^{dn \times \rho n} \) and \( \mathbf{W} = \int_K \varpi(\tau)(f^\top(\tau) \otimes I_{\rho n}) Y (f(\tau) \otimes I_{\rho n}) d\tau \in \mathbb{S}^{\rho n} \). Furthermore, let \( \Upsilon \in \mathbb{R}^{dn \times \rho n} \) and \( z \in \mathbb{R}^{\rho n} \) be any matrix and vector satisfy
\[
\Upsilon z = \int_K \varpi(\tau) \mathbf{F}(\tau) x(\tau) d\tau
\]
(17)
for all \( x(\cdot) \in \mathbb{L}^2_\varpi(K; \mathbb{R}^n) \), then we have for all \( x(\cdot) \in \mathbb{L}^2_\varpi(K; \mathbb{R}^n) \)
\[
\int_K \varpi(\tau) x^\top(\tau) U x(\tau) d\tau \geq z^\top \left[ \text{Sy} \left( \Upsilon^\top \tilde{X} \right) - \mathbf{W} \right] z.
\]
(18)

Proof: Given (15), we have
\[
\int_K \varpi(\tau) \begin{bmatrix}
x(\tau) \\
f(\tau) \otimes z
\end{bmatrix}^\top \begin{bmatrix}
U & -X \\
* & Y
\end{bmatrix} \begin{bmatrix}
x(\tau) \\
f(\tau) \otimes z
\end{bmatrix} d\tau
= \int_K \varpi(\tau) x^\top(\tau) U x(\tau) d\tau - \text{Sy} \left[ \int_K \varpi(\tau) x^\top(\tau) X (f(\tau) \otimes z) d\tau \right]
+ \int_K \varpi(\tau) (f(\tau) \otimes z)^\top Y (f(\tau) \otimes z) d\tau \geq 0.
\]
(19)
Now using (1) and (14) to the terms in (19) yields
\[
\int_K \varpi(\tau) x^\top(\tau) X (f(\tau) \otimes z) d\tau = \int_K \varpi(\tau) x^\top(\tau) X (f(\tau) \otimes I_{\rho n}) d\tau z
= \int_K \varpi(\tau) x^\top(\tau) (f^\top(\tau) \otimes I_{\rho n}) \vartheta \tilde{X} z = \vartheta^\top \tilde{X} z,
\]
(20)
\[
\int_K \varpi(\tau) (f(\tau) \otimes z)^\top Y (f(\tau) \otimes z) d\tau = z^\top \int_K \varpi(\tau) \left( f^\top(\tau) \otimes I_{\rho n} \right) Y (f(\tau) \otimes I_{\rho n}) d\tau z = z^\top \mathbf{W} z
\]
(21)
where \( X = \text{Row}_{i=1}^d X_i \in \mathbb{R}^{n \times \rho n} \) and \( \tilde{X} = \text{Col}_{i=1}^d X_i \). Substituting (20)–(21) into (19) yields (16). With the equality \( \Upsilon z = \int_K \varpi(\tau) \mathbf{F}(\tau) x(\tau) d\tau \) with \( \Upsilon \in \mathbb{R}^{dn \times \rho n} \), then (16) becomes (18).

Since \( f(\cdot) \) in Theorem 2 is subject to the same constraints (4) as in Theorem 1, hence (18) gives a free matrix type inequality with more general structures compared to existing results in the literature.

Remark 7. Let \( K = [a, b] \), \( \varpi(\cdot) = 1 \) and \( f(\tau) \) consists of the Legendre polynomials over \( K \), then Lemma 3 in [14] can be obtained by Theorem 2 with appropriate \( \Upsilon \) and \( z \) and the substitution \( \dot{x}(\cdot) \rightarrow x(\cdot) \). Furthermore, let \( K = [a, b] \), \( \varpi(\cdot) = 1 \) and \( f(\cdot) = \text{Col}_{i=0}^m \frac{(r-a)^i}{(b-a)^i} \), then by (16) with the substitution \( x(t) \rightarrow \dot{x}(t) \) and considering the properties in [15, eq.(4)–(5)], one can derive the result of Lemma 3 in [15].

The following theorem shows an “equivalent” relation between the lower bounds of (5) and (18).

Theorem 3. By choosing the same \( \varpi(\cdot) \), \( U \) and \( f(\cdot) \) for Theorems 1 and 2 with \( U \succ 0 \), one can always find \( X \) and \( Y \) for (15) to render the inequalities in (18) and (5) to be identical. Moreover, (5) corresponds to the largest lower bound of (18).
Proof. Let $U > 0$ and $\varpi(\cdot)$ in (3) and $f(\cdot)$ satisfying (4) be given. Using the Schur complement [23, Theorem 1.12] to (15) concludes that $Y \succeq X^\top U^{-1}X$ for any $Y$, $X = \text{Row}_{i=1}^d X_i \in \mathbb{R}^{n \times \rho_{dn}}$ satisfying (15). Now consider $W$ in Theorem 2 with $Y \succeq X^\top U^{-1}X$ and (14), we have

$$W \geq \int_K \varpi(\tau)(f^\top(\tau) \otimes I_{nd})X^\top U^{-1}X(f(\tau) \otimes I_{nd})d\tau = \hat{X}^\top \int_K \varpi(\tau)(f(\tau) \otimes I_{n})U^{-1}(f^\top(\tau) \otimes I_{n})d\tau \hat{X}$$

$$= \hat{X}^\top \int_K \varpi(\tau)(I_{d} \otimes U^{-1})(f(\tau)f^\top(\tau) \otimes I_{n})d\tau \hat{X} = \hat{X}^\top (F^{-1} \otimes U^{-1}) \hat{X}$$

(22)

with $F^{-1} = \int_K \varpi(\tau)f(\tau)f^\top(\tau)d\tau \in S_d^d$ and $\hat{X} = \text{Col}_{i=1}^d X_i \in \mathbb{R}^{dn \times \rho_{n}}$. It is clear that the inequality in (22) becomes an equality with $Y = X^\top U^{-1}X$ for any $U > 0$ and $X \in \mathbb{R}^{n \times \rho_{dn}}$.

By (22) and (2), one can conclude that

$$S_Y(\Upsilon^\top \hat{X}) - W \preceq S_Y(\Upsilon^\top X) - \hat{X}^\top (F^{-1} \otimes U^{-1}) \hat{X} \preceq \Upsilon^\top (F \otimes U) \Upsilon$$

(23)

holds for any $Y$ and $X$ satisfying (15) with $\Upsilon \in \mathbb{R}^{dn \times \rho_{m}}$ in (17). Moreover, by Lemma 1, one can conclude that the two inequalities in (23) become equalities with $\hat{X} = (F \otimes U) \Upsilon$ and $Y = X^\top U^{-1}X$ where the value of $X$ here can be uniquely determined by $\hat{X} = (F \otimes U) \Upsilon$ with given $U$ and $\Upsilon$. By (23) with its aforementioned property and considering the assumption in (17), we have

$$z^\top [S_Y(\Upsilon^\top \hat{X}) - W] z \leq z^\top \Upsilon^\top (F \otimes U) \Upsilon z = \int_K \varpi(\tau)x^\top(\tau)F^\top(\tau)(F \otimes U) \int_K \varpi(\tau)x(\tau)d\tau$$

(24)

holds for any $Y$ and $X$ satisfying (15) with $U > 0$, and the inequality in (24) becomes an equality with $\hat{X} = (F \otimes U) \Upsilon$ and $Y = X^\top U^{-1}X$. As a result, the above arguments show that under the same $\varpi(\cdot)$, $U > 0$ and $f(\cdot)$, one can always find $X$ and $Y$ for (15) to render (18) to become identical to (5) which corresponds to the largest lower bound of (18).

A particular case of (18) with less matrix variables is presented in the following corollary, where the largest lower bound of this inequality is also identical to (5).

Corollary 1. Choosing the same $\varpi(\cdot)$, $U > 0$ and $f(\cdot)$ as in Theorems 1 and 2 with $\rho \in \mathbb{N}$, we have

$$\int_K \varpi(\tau)x^\top(\tau)Ux(\tau)d\tau \geq z^\top [S_Y(\Upsilon^\top \hat{X}) - \hat{X}^\top (F^{-1} \otimes U^{-1}) \hat{X}] z$$

(25)

with $\hat{X} = \text{Col}_{i=1}^d X_i \in \mathbb{R}^{dn \times \rho_{m}}$ and $F^{-1} = \int_K \varpi(\tau)f(\tau)f^\top(\tau)d\tau$, where $\Upsilon$ and $z$ satisfy (17). Moreover, (25) is a special case of (18) and the largest lower bound of (25) is identical to (5) with $\hat{X} = (F \otimes U)$ for (25).

Proof. Let the same $\varpi(\cdot)$, $U > 0$ and $f(\cdot)$ in Theorems 1 and 2 to be given. Now let $Y = X^\top U^{-1}X$, we have $W = \hat{X}^\top (F^{-1} \otimes U^{-1}) \hat{X}$ considering (22), with which the inequality in (18) becomes (25). Moreover, one can conclude by the application of Lemma 2 that the largest lower bound of (25) is attained with $\hat{X} = (F \otimes U) \Upsilon$ where (25) becomes (5).

Remark 8. Let $K = [a, b]$, $\varpi(\cdot) = 1$ and $f(\tau)$ to contain the Legendre polynomials over $[a, b]$, then the Lemma 1 in [24] can be obtained from the corresponding (25) with appropriate $z$ and $\Upsilon$ using the substitution $\hat{x}(\cdot) \rightarrow x(\cdot)$. Now consider the fact that the left hand of the inequality (9) in [24] can be rewritten as a one fold integral with a weight function by using the Cauchy formula for repeated integrations. Let $K = [a, b]$, $\varpi(\tau) = (\tau - a)^m$ and $f(\tau)$ to contain Jacobi polynomials associated with $\varpi(\tau)$ over $[a, b]$, then [24, eq.(9)] can be obtained by the corresponding (25) with appropriate $z$ and $\Upsilon$ using the substitution $\hat{x}(\cdot) \rightarrow x(\cdot)$. Finally, since the largest lower
bound of (25) is identical to the lower bound of (5) under the same \( \varpi(\cdot), f(\cdot) \) and \( U > 0 \), it also indicates that relations in terms of lower bounds can be established among the inequalities in [11], [24].

Remark 9. Let \( \mathcal{K} = [a, b] \), \( \varpi(\cdot) = 1 \) and \( f(\tau) \) to contain the Legendre polynomials over \([a, b]\), then the conclusion of Theorem 1 in [14] can be obtained from Theorem 3 with appropriate \( \Upsilon \) and \( z \) considering the substitution \( \dot{x}(\cdot) \to x(\cdot) \). As we have proved that (25) is equivalent to (5), thus (18) is equivalent to (25). Consequently, it is possible to show that equivalent relations\(^1\) in terms of lower bounds can be established between the inequalities in [14], [11], [24] given what we have presented in Remark 8.

The conclusion in Theorem 3 is very important to understand the relationship among (5), (18) and (25). Since all these three inequalities are essentially equivalent in terms of their lower bounds under the same \( \varpi(\cdot), f(\cdot) \) and \( U > 0 \), hence if one finds a special example of one of these three inequalities then it corresponds to two 'equivalent' inequalities.

Remark 10. Similar to Remark 6 and also consider Theorem 3, an invertible linear transformation \( G \) acting on \( f(\cdot) \), namely, \( Gf(\cdot) \to f(\cdot) \), gives an equivalent lower bound as the one in (18).

IV. APPLICATIONS OF INTEGRAL INEQUALITIES TO THE STABILITY ANALYSIS OF A SYSTEM WITH DELAYS

The proposed inequalities could be applied to various contexts such as the stability analysis and control of infinite dimensional systems such as delay [1], [2] and ODE-PDE coupled system [5], [6]. To show an application in this paper, we derive two stability conditions in this section for a linear CDDS with a distributed delay via the application of (5) and (25). We also show that the resulting stability conditions are equivalent whose solvability is invariant with respect to a matrix parameter in the Liapunov Krasovskii functional.

Consider a linear CDDS of the form

\[
\begin{align*}
\dot{x}(t) &= A_1 x(t) + A_2 y(t - r) + \int_{-r}^{0} \tilde{A}_3(\tau) y(t + \tau) d\tau \\
y(t) &= A_4 x(t) + A_5 y(t - r), \quad t \geq t_0 \\
x(t_0) &= \xi, \quad \forall t \in [-r, 0], \quad y(t_0 + \theta) = \phi(\theta)
\end{align*}
\]

(26)

with a distributed delay, where \( t_0 \in \mathbb{R} \) and \( \xi \in \mathbb{R}^n \) and \( \phi(\cdot) \in \mathcal{C}([-r, 0]; \mathbb{R}^n) \). The notation \( \mathcal{C}([-r, 0]; \mathbb{R}^n) \) stands for the space of bounded right piecewise continuous functions endowed with the norm \( \|\phi(\cdot)\|_{\infty} = \sup_{\tau \in \mathbb{R}} \|\phi(\tau)\|_2 \).

We also assume that \( \rho(A_3) < 1 \) which ensures [16] the input to state stability of \( y(t) = A_4 x(t) + A_5 y(t - r) \), where \( \rho(A_3) \) is the spectral radius of \( A_3 \). Since \( \rho(A_3) < 1 \) is independent from \( r \), thus this condition ensures the input to state stability of \( y(t) = A_4 x(t) + A_5 y(t - r) \) for all \( r > 0 \). Finally, \( \tilde{A}_3(\tau) \) in (26) satisfies the following assumption.

Assumption 1. There exist \( \text{Col}_{i=1}^d f_i(\tau) = f(\cdot) \in \mathcal{C}^1(\mathbb{R} \cap \mathbb{R}^d) \) with \( d \in \mathbb{N} \), and \( A_3 \in \mathbb{R}^{n \times d} \) such that for all \( \tau \in [-r, 0] \) we have \( \tilde{A}_3(\tau) = A_3 F(\tau) \in \mathbb{R}^{n \times d} \) where \( F(\tau) = f(\tau) \otimes I_{\nu} \in \mathbb{R}^{d \times d} \). In addition, we assume that \( f(\cdot) \) here satisfies the following constraints

\[
\begin{align*}
\int_{-r}^{0} f(\tau) f^T(\tau) d\tau > 0 \quad \text{and} \\
\exists M \in \mathbb{R}^{d \times d}, \quad \frac{df(\tau)}{d\tau} = M f(\tau).
\end{align*}
\]

(27)

(28)

\(^1\)The equivalence relations here are understood by considering the structure of inequalities irrespective of using \( x(\cdot) \) or \( \dot{x}(\cdot) \).
Theorem 4. For any value of the complete Liapunov-Krasovskii functional proposed in \( G \) in Assumption 1, \( G \) is defined by the equality 

\[
\frac{d^+}{dt} v(x(t), y_t(\cdot)) \bigg|_{t=t_0, x(t_0)=\xi, y_{t_0}(\cdot)=\phi(\cdot)} \leq -\epsilon_3 \|\xi\|^2_2
\]

for any \( \xi \in \mathbb{R}^n \) and \( \phi(\cdot) \in \mathcal{C}([-r, 0); \mathbb{R}^\nu) \) in (26), where \( t_0 \in \mathbb{R} \) and \( \frac{d^+}{dt} f(x) = \limsup_{\eta \downarrow 0} \frac{f(x+\eta)-f(x)}{\eta} \). Furthermore, \( y_t(\cdot) \) in (30) is defined by the equality \( \forall t \geq t_0, \forall \theta \in [-r, 0), y_t(\theta) = y(t+\theta) \) where \( x(t) \) and \( y(t) \) satisfying (26).

Proof. Let \( u(\cdot), v(\cdot), w(\cdot) \) in Theorem 3 of [16] be quadratic functions with the multiplier factors \( \epsilon_1; \epsilon_2; \epsilon_3 > 0 \). Since (26) is a particular case of the general system considered in Theorem 3 of [16], then Lemma 4 is obtained.

To analyze the stability of the origin (26), consider the following parameterized Krasovskii functional

\[
v(\xi, \phi(\cdot)) := \left[ \int_{t_0}^0 \xi^T \hat{G}(\tau) \phi(\tau) d\tau \right]^T \hat{P} \int_{t_0}^0 \xi^T \hat{G}(\tau) \phi(\tau) d\tau + \int_{t_0}^0 \phi^T(\tau) [S + (\tau + r)U] \phi(\tau) d\tau
\]

with \( \hat{G}(\tau) = g(\tau) \otimes I_\nu, g(\tau) = Gf(\tau), G \in \mathbb{R}^{d \times d} \) and \( f(\cdot) \) in Assumption 1, where \( \xi \in \mathbb{R}^n, \phi(\cdot) \in \mathcal{C}([-r, 0); \mathbb{R}^\nu) \) in (31) are the initial conditions in (26), and \( \hat{P} \in \mathbb{S}^{n+\nu} \) and \( S; U \in \mathbb{S}^\nu \) are unknown parameters to be determined. Note that \( \hat{G}(\tau) \) can be rewritten as \( \hat{G}(\tau) = g(\tau) \otimes I_\nu = Gf(\tau) \otimes I_\nu = (G \otimes I_\nu)F(\tau) \) with \( F(\tau) := f(\tau) \otimes I_\nu \) based on the property of Kronecker product in (1). Note that also (31) can be regarded as a parameterized version of the complete Liapunov-Krasovskii functional proposed in [16].

We will show in the following theorem that the solvability of the resulting stability conditions remain unchanged for any value of \( G \in \mathbb{R}^{d \times d} \) in (31) when (5) or (25) is applied for the derivation.

Theorem 4. Given \( G \in \mathbb{R}^{d \times d} \) and \( f(\cdot), M \) in (28), then the origin of (26) under Assumption 1 is globally uniformly asymptotically stable if there exists \( \hat{P} \in \mathbb{S}^{n+\nu} \) and \( S; U \in \mathbb{S}^\nu \) such that

\[
\hat{P} + \left[ \mathcal{O}_n \oplus ([*]FG^{-1} \otimes S) \right] > 0
\]

\[
S > 0, \quad U > 0, \quad \Phi_1 < 0
\]

hold, or equivalently if there exist \( \hat{P} \in \mathbb{S}^{n+\nu} \), \( S; U \in \mathbb{S}^\nu \), and \( X_1; X_2 \in \mathbb{S}^{d\nu} \) such that

\[
\begin{bmatrix}
\hat{P} + \left[ \mathcal{O}_n \oplus (2X_1) \right] & \left[ \mathcal{O}_{d\nu \times n} \text{ } X_1 \right]^T \\
\ast & \left[ [*]FG^{-1} \otimes S \right]
\end{bmatrix} > 0
\]

\[
S > 0, \quad U > 0, \quad \Phi_2 \begin{bmatrix}
\mathcal{O}_{d\nu \times (\nu+n)} \text{ } X_2 \end{bmatrix}^T < 0
\]

(34) (35)
hold, where $F^{-1} = \int_{-r}^{r} f(\tau) f^\top(\tau) d\tau$ and
\[
\Phi_1 = \mathbb{S} \mathbb{Y} \left( H \hat{P} \left[ A^\top \ G^\top \right]^\top \right) + \Gamma^\top (S + rU) \Gamma - \left( O_n \oplus S \oplus ([*_FG^{-1} \oplus U] \right)
\]
\[
\Phi_2 = \mathbb{S} \mathbb{Y} \left( H \hat{P} \left[ A^\top \ G^\top \right]^\top \right) + \Gamma^\top (S + rU) \Gamma - \left( O_n \oplus S \oplus 2X_2 \right)
\]
with
\[
H = \begin{bmatrix}
I_n & O_{n \times d
}
O_{n \times d} & O_{d \times d}
\end{bmatrix}, \quad \Gamma := \begin{bmatrix}
A_4 & A_5 & O_{d \times d}
\end{bmatrix}
\]
\[
A = \begin{bmatrix}
A_1 & A_2 & A_3(G^{-1} \otimes I_\nu)
\end{bmatrix}
\]
\[
G = \begin{bmatrix}
\hat{G}(0)A_4 & \hat{G}(0)A_5 & 0 \end{bmatrix}
\]
in which $\hat{G}(0) = (G \otimes I_\nu)F(0) = Gf(0) \otimes I_\nu$ and $\hat{G}(r) = (G \otimes I_\nu)F(-r) = Gf(-r) \otimes I_\nu$ and $\hat{M} = (G \otimes I_\nu)(M \otimes I_\nu)(G^{-1} \otimes I_\nu) = GMG^{-1} \otimes I_\nu$. Finally, the solvability of the matrix inequalities in (32)–(35) is invariant for any value of $G \in \mathbb{R}^{d \times d}$.

Proof. Let $G \in \mathbb{R}^{d \times d}$ and $f(\cdot)$ with $M$ in Assumption 1 be given. Given the fact that the eigenvalues of $S + (\tau + r)U$, $\tau \in [-r, 0]$ are bounded and $\hat{G}(\tau) = (G \otimes I_\nu)F(\tau)$, it is obvious to see that (31) satisfies the following property that there exist $\lambda; \eta > 0$ such that

\[
v(\xi, \phi(\cdot)) \leq \left[ \int_{-r}^{0} F(\tau)\phi(\tau) d\tau \right]^\top \lambda \left[ \int_{-r}^{0} F(\tau)\phi(\tau) d\tau \right] + \int_{-r}^{0} \phi^\top(\tau) \lambda \phi(\tau) d\tau
\]

\[
\leq \lambda \|\xi\|^2_2 + \int_{-r}^{0} \phi^\top(\tau) F^\top(\tau) \phi(\tau) d\tau \lambda \int_{-r}^{0} F(\tau) \phi(\tau) d\tau + \lambda \|\phi(\cdot)\|^2_\infty
\]

\[
\leq \lambda \|\xi\|^2_2 + \lambda r \|\phi(\cdot)\|^2_\infty + \left[ \int_{-r}^{0} F(\tau) \phi(\tau) d\tau \right]^\top \left( \eta F \otimes I_\nu \right) \left[ \int_{-r}^{0} F(\tau) \phi(\tau) d\tau \right]
\]

\[
\leq \lambda \|\xi\|^2_2 + \lambda r \|\phi(\cdot)\|^2_\infty + \int_{-r}^{0} \phi^\top(\tau) \eta \phi(\tau) d\tau
\]

\[
\leq \lambda \|\xi\|^2_2 + (\lambda r + \eta r) \|\phi(\cdot)\|^2_\infty \leq (\lambda r + \eta r)(\|\xi\|^2_2 + \|\phi(\cdot)\|^2_\infty)
\]

\[
\leq 2 (\lambda r + \eta r) \left[ \max (\|\xi\|_2, \|\phi(\cdot)\|_\infty) \right]^2
\]

for any $\xi \in \mathbb{R}^n$ and $\phi(\cdot) \in \hat{C}([-r_2, 0]; \mathbb{R}^\nu)$ in (26), where (41) is derived via the property of quadratic forms: $\forall X \in \mathbb{R}^n, \exists \lambda > 0 : \forall x \in \mathbb{R}^n \setminus \{0\}, x^\top (\lambda I_n - X)x > 0$ together with the application of (5) with $f(\cdot)$ in (1). Consequently, (41) shows that (31) satisfies the upper bound property in (29).

Now apply (5) with $\varphi(\tau) = 1$ to $\int_{-r}^{0} \phi^\top(\tau) S \phi(\tau) d\tau$ in (31) given $S > 0$ and $f(\cdot)$ in Assumption 1 and the fact that $\phi(\cdot) \in \hat{C}([-r_2, 0]; \mathbb{R}^\nu) \subset L^2([-r_2, 0]; \mathbb{R}^\nu)$. It yields

\[
\int_{-r}^{0} \phi^\top(\tau) S \phi(\tau) d\tau \geq \left( \int_{-r}^{0} \hat{G}(\tau) \phi(\tau) d\tau \right)^\top \left( [*_FG^{-1} \otimes S] \int_{-r}^{0} \hat{G}(\tau) \phi(\tau) d\tau \right)
\]

for any $\xi \in \mathbb{R}^n$ and $\phi(\cdot) \in \hat{C}([-r_2, 0]; \mathbb{R}^\nu)$ in (26).

Now by (42) and (31), we can conclude that if (32) is feasible, then it infers the existence of $\epsilon_1; \epsilon_2 > 0$ and (31) satisfying (29) given what we have shown in (41). On the other hand, given the property of congruence transformations with the fact that $G \in \mathbb{R}^{d \times d}$, one can conclude that (32) holds if and only if
\[
[I_n \oplus (G^\top \otimes I_\nu)] \tilde{P} [I_n \oplus (G \otimes I_\nu)] + [I_n \oplus (G^\top \otimes I_\nu)] [O_n \oplus ([\ast]FG^{-1} \otimes S)] [I_n \oplus (G \otimes I_\nu)]
= P + [O_n \oplus (F \otimes S)] > 0 \quad (43)
\]

with \( P = [I_n \oplus (G^\top \otimes I_\nu)] \tilde{P} [I_n \oplus (G \otimes I_\nu)] \). By viewing \( P \) as a new variable, it shows that the solvability of the last matrix inequality in (43), namely (32), is invariant with respect to the values of \( G \).

Now we start to construct stability conditions inferring (30) via (31). Differentiate \( v(x(t), y_t(\cdot)) \) along the trajectory of (26) at \( t = t_0 \) and consider the relation

\[
\frac{d}{dt} \int_{-r}^{0} \hat{G}(\tau)y(t + \tau)d\tau \bigg|_{t=t_0} = (G \otimes I_\nu)F(0)\phi(0) - (G \otimes I_\nu)F(-r)\phi(-r) - \hat{M} \int_{-r}^{0} (Gf(\tau) \otimes I_\nu) \phi(\tau)d\tau
= \hat{G}(0)A_4 \xi + \left[ \hat{G}(0)A_5 - \hat{G}(-r) \right] \phi(-r) - \hat{M} \int_{-r}^{0} \hat{G}(\tau) \phi(\tau)d\tau \quad (44)
\]

where \( \hat{M} = (G \otimes I_\nu)(M \otimes I_\nu)(G^{-1} \otimes I_\nu) \) and (44) is obtained by considering (28). Then we have

\[
\frac{d^+}{dt} \left|_{t=t_0, x(t_0)=\xi, y_{t_0}(\cdot)=\phi(\cdot)} \right. \frac{d}{dt} v(x(t), y_t(\cdot)) = \chi^\top Sy \left( H \tilde{P} \begin{bmatrix} A \\ G \end{bmatrix} \right) \chi - \int_{-r}^{0} \phi^\top(\tau)U\phi(\tau)d\tau + \chi^\top \left[ \Gamma^\top(S + rU) \Omega - \left( O_n \oplus S \oplus O_{d\nu} \right) \right] \chi \quad (45)
\]

where \( H, A, G \) and \( \Gamma \) have been given in (38)-(40) and

\[
\chi := \text{Col} (\xi, \phi(-r), \int_{-r}^{0} \hat{G}(\tau) \phi(\tau)d\tau) \quad (46)
\]

Given \( U > 0 \) in (32) and apply (5) with \( c(\tau) = 1 \) to \( \int_{-r}^{0} \phi^\top(\tau)U\phi(\tau)d\tau \) in (45) similar to the procedure in (42). It yields

\[
\int_{-r}^{0} \phi^\top(\tau)U\phi(\tau)d\tau \geq \left[ \int_{-r}^{0} \hat{G}(\tau) \phi(\tau)d\tau \right]^\top \left[ [\ast]FG^{-1} \otimes U \right] \int_{-r}^{0} \hat{G}(\tau) \phi(\tau)d\tau \quad (47)
\]

for any \( \xi \in \mathbb{R}^n \) and \( \phi(\cdot) \in \mathcal{C}([-r_2, 0); \mathbb{R}^\nu) \) in (26). By using (47) to (45) with \( U > 0 \), we have

\[
\frac{d^+}{dt} v(x(t), y_{t}(\cdot)) \bigg|_{t=t_0, x(t_0)=\xi, y_{t_0}(\cdot)=\phi(\cdot)} \leq \chi^\top \Phi_1 \chi \quad (48)
\]

in (32), where \( \Phi_1 \) is given in (36). By (48) and (46), it is easy to see that the feasible solutions of (33) infer the existence of \( \epsilon_3 > 0 \) and (31) satisfying (30).

Now considering the property of congruence transformations with the fact that \( G \in \mathbb{R}^{d \times d} \), it is true that

\[
\Phi_1 < 0 \iff [\ast] \Phi_1 [I_{n+d\nu} \oplus (G \otimes I_\nu)] = \Theta := Sy (H\tilde{P}\Psi) + \Gamma^\top(S + rU) \Omega - \left( O_n \oplus S \oplus (F \otimes U) \right) < 0 \quad (49)
\]

with \( P = [I_n \oplus (G^\top \otimes I_\nu)] \tilde{P} [I_n \oplus (G \otimes I_\nu)] \) and \( H \) in (38) and

\[
\Psi = \begin{bmatrix} A_1 & A_2 & A_3 \\ F(0)A_4 & F(0)A_5 - F(-r) & -M \otimes I_\nu \end{bmatrix} \quad (50)
\]

which can be derived via (1). By treating \( P \) as the same in (43) and as a new variable, it is clear to see that the solvability of \( \Phi_1 < 0 \) in (49) is invariant with respect to the values of \( G \in \mathbb{R}^{n \times n} \), which indicates the feasibility of (33) remains unchanged for any invertible \( G \).

Finally, let \( Y = I_{d\nu} \) and \( \bar{X} = X_1 \in S^{d\nu} ; \bar{X} = X_2 \in S^{d\nu} \) in (25), respectively, then one can apply (25) with \( f(\cdot) \) in Assumption 1 for the steps at (42) and (47) to derive (34)-(35) via the Schur complement. By Theorem 3, one can conclude that (34)-(35) is equivalent to (32)-(33). Since the feasibility of (32)-(33) is invariant with respect to the value of \( G \in \mathbb{R}^{n \times n} \), thus the feasibility of (32)-(33) is also invariant with respect to the value of \( G \).
Remark 12. Although the values of \( G \in \mathbb{R}^{n \times n} \) do not affect the solvability of (32)–(35), it can be still beneficial to use orthonormal functions \( Gf(\tau) \) with \( G^{-2} = \int_{-r}^{0} \varpi(\tau)f(\tau)f^T(\tau)d\tau \) in Theorem 4 as it gives \( *FG^{-1} = I_d \) which makes the relevant diagonal blocks in (32)–(35) more regular towards numerical calculations.

V. NUMERICAL EXAMPLE

The numerical example in this section is tested in Matlab with Yalmip [25] and Mosek 8 [26]. Consider the following linear CDDS

\[
\begin{align*}
\dot{x}(t) &= 0.35x(t) + 0.035y(t - r) - \int_{-r}^{0} 5\cos(12\tau)y(t + \tau)d\tau \\
y(t) &= x(t) + 0.1y(t - r).
\end{align*}
\]  

(51)

which corresponds to \( A_1 = 0.35, \ A_2 = 0.035, \tilde{A}_3(\tau) = -5\cos(12\tau), \ A_4 = 1 \) and \( A_5 = 0.1 \) with \( n = 1 \) in (26).

Now let \( f(\cdot) \) and \( M \) in (28) be

\[
f(\tau) = \begin{bmatrix} 1 \\ \sin(12\tau) \\ \cos(12\tau) \end{bmatrix} \quad \text{with} \quad M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 12 \\ 0 & -12 & 0 \end{bmatrix}
\]  

(52)

with \( A_3 = \begin{bmatrix} 0 & 0 & -5 \end{bmatrix} \) which satisfies Assumption 1.

We will calculate the delay margins of (51) via (32)–(33) and (34)–(35) with (52) and different values of \( G \) to show that the numerical results are not affected by using different values of invertible \( G \).

Now apply (32)–(33) with \( G = I_3 \) and (52) to the system in (51), we obtain the boundaries of the detectable stable points as

\[
[0.084, 0.178], \quad [0.607, 0.702], \quad [1.131, 1.225], \quad [1.654, 1.749], \quad [2.178, 2.273]
\]  

(53)

by using a testing vector \( r = (1 : 2500)/1000 \) where at each value of \( r \) the corresponding optimization program has 12 decision variables. By using (32)–(33) with \( G = \begin{bmatrix} 1 & 0.5 & 0.2 \\ 0 & 2 & -1 \end{bmatrix} \) and (52) to the system in (51), the same results in (53) can be obtained where for each value of \( r \) the same number of decision variables is required as for \( G = I_3 \). Moreover, the same boundaries of (53) can be obtained by using the stability condition (34)–(35) with (52) and \( G = I_3 \) or \( G = \begin{bmatrix} 1 & 0.5 & 0.2 \\ 0 & 2 & -1 \end{bmatrix} \) to the system in (51), which requires 18 decision variables for each value of \( r = (1 : 2500)/1000 \). The above results on the stability of (52) are consistent with what have been proved in Theorem 4 that (32)–(33) and (34)–(35) are equivalent and their solvability is not affected by having different values of \( G \).

Now we want to show the impact of choosing different \( f(\cdot) \), here we modify the conditions (32)–(33) with \( G = I_d \) and \( f(\tau) = \ell_d(\tau) \) in accordance to the polynomials approximation approach in [27], where \( \ell_d(\tau) \) contains Legendre polynomials over \([-r, 0]\) up to degree \( d \in \mathbb{N}_0 \). It occurs that the modified condition requires \( d = 22 \) with 302 decision variables for each value of \( r = (1 : 2500)/1000 \) to detect the boundaries in (53), which is in a sharp contrast with the numbers of variables required by (32)–(33) and (34)–(35) with (52). The above results show the advantage of using (52) with our methods over the polynomials approximation approach in [27].

VI. CONCLUSION

In this note, two general classes of integral inequalities have been proposed which generalize many existing integral inequalities in the existing literature. Moreover, the relation between (5) and (18) is established in Theorem
3 by which one can conclude that the lower bounds in many existing quadratic integral inequalities are essentially equivalent. For a specific application, the inequalities presented in this note have been utilized to derive equivalent stability conditions for a linear CDDS with a distributed delay. Finally, the proposed inequalities have great potential to be applied in wider contexts such as the stability analysis of PDE-related systems or sampled-data systems or other types of infinite dimensional systems whenever the contexts are suitable.

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