Stabilization of polynomial dynamical systems using linear programming based on Bernstein polynomials

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Abstract—In this paper, we deal with the problem of synthesizing static output feedback controllers for stabilizing polynomial systems. Our approach jointly synthesizes a Lyapunov function and a static output feedback controller that stabilizes the system over a given subset of the state-space. Specifically, our approach is simultaneously targeted towards two goals: (a) asymptotic Lyapunov stability of the system, and (b) invariance of a box containing the equilibrium. Our approach uses Bernstein polynomials to build a linear relaxation of polynomial optimization problems, and the use of a so-called “policy iteration” approach to deal with bilinear optimization problems. Our approach can be naturally extended to synthesizing hybrid feedback control laws through a combination of state-space decomposition and Bernstein polynomials. We demonstrate the effectiveness of our approach on a series of numerical benchmark examples.

I. INTRODUCTION

The problem of designing stabilizing controllers for nonlinear dynamical systems is of great importance. In this paper, we study the problem of synthesizing static output feedback controllers for polynomial systems by solving a polynomial optimization problem to directly obtain the controller along with the associated Lyapunov functions that yields the proof of stability.

Our approach inputs the description of a polynomial system and a desired region $R$ to be stabilized. It then proceeds to find a static output feedback control law and an associated Lyapunov function to ensure local stability in $R$. Simultaneously, we ensure that the region $R$ is an invariant of the resulting closed loop system. Our approach assumes a given structure for the feedback as a polynomial function of the outputs of the system. Furthermore, we assume a polynomial template form for the unknown Lyapunov function. We proceed to encode the conditions for the Lyapunov function, obtaining a hard polynomial optimization problem that involves the coefficients of the Lyapunov functions and those of the feedback.

The second part of the paper iteratively solves this optimization problem through an iterative method variously called “V-K” iteration [19] or policy iteration [18]. The $i$th iteration of the approach selects a positive definite polynomial $V_i$ and a feedback law $u_i$. Ideally, we require $V_i$ to be negative definite inside the region $R$ for $V_i$ to be a Lyapunov function guaranteeing asymptotic stability. Failing this, we first search for a new positive definite polynomial $V_{i+1}$ whose Lie derivative $V_{i+1}'$ has a larger maxima inside $R$ fixing $u_i$, and adjust to a new feedback law $u_{i+1}$ that improves the maximal value of $V_{i+1}'$ inside $R$. Each iteration is reduced to solving a Linear Programming (LP) problem using Bernstein polynomials combined with a reformulation linearization technique [5]. It is well-known that policy iteration does not necessarily converge to a global minimum, in general. However, our evaluation over a wide variety of benchmark examples shows that our approach is effective at converging to a global minimum by discovering an appropriate feedback law $u^*$ and an associated Lyapunov function $V^*$.

Automatic static output feedback design, or more generally, finding feedback that satisfies given structural constraints is well-known to be a hard problem in general. In fact, static output feedback stabilization of linear systems yields bilinear matrix inequalities (BMIs) rather than LMIs. A direct approach given by Henrion et al. [6] uses the characteristic polynomial of the transfer function matrix, and derives constraints that ensure the Hermite stability criterion for this matrix. As a result, they obtain a system of PMI (polynomial matrix inequalities), that is solved using a local optimization solver (PENBMI). In contrast, an indirect approach reduces the non convex BMIs to a series of convex LMIs. This was proposed as the so-called $V-K$ iteration was proposed by El Ghaoui and Balakrishnan [19]. The approach iteratively solves a bilinear problem by fixing one set of variables while modifying the other to result in a decrease in the objective values. The iteration alternates between the two sets of variables, until reaching a feasible solution. Our goal is to use this technique for polynomial systems while replacing BMI and LMI with linear and bilinear programs that can be solved more efficiently. A similar idea for solving bilinear problems appears in the work of Gaubert et al. [18], for finding invariants for discrete-time systems. Therein, the idea is called policy iteration. In this work, we will call our approach policy iteration, as well. The main differences between our work and that of El Gahoui et al. lie in our focus on polynomial systems, yielding more general polynomial optimization problems that involve the “$V$” variables relating to the Lyapunov function and the “$K$” variables relating to the feedback. Yet, by using policy iteration, we can separately focus on problems with a single set of variables at a time and use linear programming relaxations through a combination of Bernstein polynomials and reformulation linearization, discussed in our earlier work [5].

Existing approaches to stabilizing polynomial systems rely on linearization around the equilibrium. However, linearization can sometimes fail to be controllable, or yield region of stability that is much smaller than desired. Furthermore,
the output feedback stabilization for a linear system (or finding a feedback law satisfying a given structure) yields non-convex problems that are no easier to solve. Another class of methods (more related to our work) consists on reducing the problem to a set of LMIs or Sum-Of-Squares (SOS) formulations (see [20], [21] and references therein). In [21], an iterative SOS approach is proposed. This approach uses the Schur complement to produce a set of BMIs relaxed to an SOS problem. More precisely, an additional design nonlinear term $\epsilon(x)$ is introduced, and causes bilinearity. An iterative approach is then obtained by fixing a guess for $\epsilon(x)$ and iteratively updating it until feasibility is obtained. Once again, the major problem arises from the fact that the Lyapunov function and a static output feedback are needed simultaneously. Other approaches to controlling polynomial systems include the use of nonlinear optimal control techniques, feedback linearization, backstepping, and exact linearization. However, these techniques rely on the system being of a certain form and mostly involve state-feedback. A detailed comparison of the relative advantages of the direct approach presented here with other approaches to nonlinear stabilization will form an important part of our future work.

II. PROBLEMS FORMULATION AND POLYNOMIAL OPTIMIZATION PROBLEMS

A. Problem formulation

In this work, we consider a nonlinear control-affine system subject to input constraints:

$$\begin{align*}
\dot{x}(t) &= f(x(t)) + g(x(t))u(y(t)), & u \in U. \\
y(t) &= h(x(t)).
\end{align*}$$

wherein $x \in \mathbb{R}^n$ represents the state variables, $u \in U$ represents the control inputs ranging over a compact set $U \subseteq \mathbb{R}^p$, and $y \in \mathbb{R}^q$ are the outputs.

We assume that the functions $f : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}^q$ and the control matrix $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{(n \times p)}$ defining the dynamics of the system are multivariate polynomial maps. The set of inputs $U$ is a convex compact polytope:

$$U = \{u \in \mathbb{R}^p | \alpha_{U,k} \cdot u \leq \beta_{U,k}, \forall k \in K_U\} \quad \text{where} \quad \alpha_{U,k} \in \mathbb{R}^p, \beta_{U,k} \in \mathbb{R} \text{ and } K_U \text{ is a finite set of indices.}$$

Finally, we assume that $x^* = 0_n$ is an equilibrium for the system, i.e. $f(0_n) + g(0_n)u(h(0_n)) = 0_n$.

We define a region of interest $R$ as a hyper-rectangle, $R : [x_1, x_1] \times \cdots \times [x_n, x_n]$ with $x_k < x_k$ for all $k \in \{1, \ldots, n\}$.

Stabilizing Feedback: In this work, we assume that the desired feedback is given by a function $u : \mathbb{R}^q \to U$ mapping outputs $y$ to control inputs $u$ to yield a closed-loop system

$$\dot{x} = f(x) + g(x)u(y), \quad y = h(x) \quad (2)$$

We require that the closed loop system be asymptotically stable in $R$. This is achieved by ensuring two important properties.

Problem 1 (Existence of Local Lyapunov Function): The system has a local Lyapunov function $V(x)$ in the region $R$ such that

1) $V(x)$ is positive definite over $R$, i.e. $V(x) > 0$ for all $x \in R \setminus \{0_n\} \text{ and } V(0_n) = 0$.

2) $\frac{dV}{dt} = \nabla V \cdot (f(x) + g(x)u(h(x)))$ is negative definite over $R$.

As such, a local Lyapunov function inside $R$ guarantees that the system is asymptotically stable in some neighborhood $N$ of $0_n$, where $N \subseteq R$. Specifically, $N$ contains the largest sublevel set of $V$ inside $R$ as the stability region, but does not have to include $R$. To ensure that the system is stable inside all of $R$, we additionally require positive invariance of $R$.

Problem 2 (Positive Invariance of $R$): The system is $R$-invariant, iff all trajectories with $x(0) \in R$ satisfy $x(t) \in R$ for all $t \geq 0$.

Finding a feedback $u(y)$ that solves problems 1 and 2 ensures asymptotic stability in the whole region $R$.

Feedback Structure Finally, we consider feedback functions that conform to a given fixed structure. In other words, we consider feedback functions of the following form

$$u(y) = H(y) \cdot \theta = \mathcal{H}(x) \cdot \theta$$

where $\theta \in \mathbb{R}^q$ is a set of gain parameters to be determined by the synthesis procedure, the matrix $H : \mathbb{R}^n \to \mathbb{R}^{(p \times q)}$ is a given multivariate polynomial map that specifies the controller structure. Often, $H$ is specified to include all monomial terms up to a given degree. However, more complex situations such as decentralized control may involve choosing specific structure for $H$. Figure 1 depicts the structure of the controller schematically.

Let $\mathcal{H}(x) : H(h(x))$ be the equivalent map as a function of the state variables. The input constraints (i.e. for all $x \in R$, $u \in U$) is then equivalent to

$$\forall k \in K_U, \forall x \in R, \quad \alpha_{U,k} \cdot \mathcal{H}(x) \leq \beta_{U,k}. \quad (3)$$

Let $O$ represent the values of $\theta$ that satisfy Eq. (3). Under these assumptions, the dynamics of the controlled system can be rewritten under the form

$$\dot{x}(t) = f(x(t)) + G(x(t))\theta,$$

where the matrix of polynomials $G(x) = g(x)\mathcal{H}(x)$, and $\theta \in O$.

B. Reduction to polynomial optimization problems (POP)

The first step is to fix a template form for the Lyapunov function $V$. We assume a polynomial form:

$$V = V_c(x) = \sum_{|\alpha| \leq D} c_\alpha x^\alpha,$$
where $\alpha \in \mathbb{N}^n$, $|\alpha| = \sum \alpha_i$, $c : (c_\alpha)_{|\alpha| \leq D}$ are the unknown coefficients of the Lyapunov function and $D \in \mathbb{N}$ is the maximal degree.

We now focus on solving Problem 1. For a relatively small $\epsilon > 0$, this problem can be formulated as follows:

1) Find a feasible set $C$ s.t
\[
C : \{ c : \min_{x \in R} V_c(x) - \epsilon ||x||^2 \geq 0 \}
\]

2) Find feasible sets $c \in C'$ and $\theta \in O'$ s.t for all $c \in C'$ and $\theta \in O'$,
\[
\min_{x \in R} -\nabla V_c \cdot (f + G\theta) - \epsilon ||x||^2 \geq 0
\]

Recall the set $O$ from Eq. (3).

Theorem 1: If $C \cap C' \neq \emptyset$ and $O \cap O' \neq \emptyset$, then each $c^* \in C \cap C'$ and $\theta^* \in O \cap O'$ solves the local Lyapunov function existence problem (Problem 1).

Proof: It is easy to see that the first condition will imply that $V_c$ will be positive definite, the second one implies that its derivatives $\frac{dV}{dx}$ is negative definite. The last condition implies that the controller is admissible i.e $u \in U$.

To solve the invariance problem (Problem 2), we should find a controller (i.e a coefficient vector $\theta$) ensuring that all the facets of the rectangle $R$ are blocked.

Definition 1 (Blocked Facets): A facet $F$ of the hyper-rectangle $R$ is said to be blocked for the system [2] if and only if
\[
\forall x \in F, n_F \cdot (f(x) + G(x)\theta) < 0,
\]
where $n_F$ is its outer normal of the facet $F$.

Let $F$ denote the set of facets of the rectangle $R$, then solving Problem 2 can be formulated as follows :

- Find feasible set $O_F$ such that for all $\theta \in O_F$ s.t
\[
\min_{x \in F} n_F \cdot (f(x) + G(x)\theta) < 0,
\]

for all facet $F \in F$.

Recall that $O$ represents the feasible set from [3].

Theorem 2: If $O_F \cap O \neq \emptyset$, then each $\theta^* \in O_F \cap O$ ensure the invariance of the rectangle $R$ and solve Problem 2 where $O_F = \bigcap_{F \in F} O_F$.

Proof: Since $\theta^* \in O_F$ then all the facets of $R$ are blocked implying its invariance. The fact that $\theta^* \in O'$ proves that the controller is admissible.

III. REDUCTION TO LINEAR AND BILINEAR FEASIBILITY PROBLEMS

In this section, we are going to relax the previous polynomial optimization problems to a set of linear and bilinear feasibility problems. For doing so, we will briefly recall a relevant result showing how a general POP can be relaxed to a linear program using Bernstein polynomials [5], then we will use this relaxation to build our linear and bilinear feasibility problems in order to solve our two given problems.

A. Linear relaxation of a POP using Bernstein polynomials

In this section, we are going to use Bernstein polynomials to establish lower bounds for our polynomial optimization problems (POP). More precisely, we seek tight lower bound for the optimal solution of the following POP:

\[
\text{minimize } p(x) \text{ s.t. } x \in R.
\]

We build a linear relaxation for problem [4], as follows:

1) Change of variable $q_U$ mapping $R$ to the unit box $U = [0,1]^n$. Let $p_U = p \circ q_U$.

2) Write $p_U$ in the Bernstein basis.

3) Write an equivalent POP in the Bernstein basis.

4) Exploit properties of Bernstein polynomials to formulate a linear programming problem whose optimum is guaranteed to lower bound the POP in Eq. (4).

We now explain the procedure in further detail. First of all, the mapping $q_U$ from any rectangle $R$ to the unit box $[0,1]^n$ is an affine transformation. Therefore, the multi-variate polynomial $p_U$ is also of degree $\delta$ and we can write:

\[
p_U(y) = \sum_{\alpha \leq \delta} b_\alpha y^\alpha \text{ for all } y \in U,
\]

where Bernstein coefficients $(b_\alpha)_{\alpha \leq \delta}$ are given as follows:

\[
b_\alpha = \sum_{J \leq \delta} \left( \begin{array}{c} i_1 \\ \vdots \\ i_n \end{array} \right) \delta^J \left( \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} \right) p_J = \sum_{J \leq \delta} \left( \begin{array}{c} I \\ J \end{array} \right) p_J.
\]

\[
B_{I,\delta}(y) = \begin{cases} \delta - I \quad \text{if } \delta - I \leq 0 \\ I \quad \text{if } \delta - I \geq 0 \end{cases}.
\]

For the third step it is sufficient to replace the canonical form by the Bernstein form in the optimization problem, we then get the following optimization problem:

\[
\text{minimize } \sum_{I \leq \delta} b_{I,\delta} B_{I,\delta}(y) \text{ s.t. } y \in U, \quad z_I = B_{I,\delta}(y).
\]

The final step is now to remove the nonlinearity caused by the Bernstein polynomials by replacing each Bernstein polynomial $B_{I,\delta}$ by a fresh variable $z_I$. In effect, we drop
the maximal degrees of \( U \) in (3). Let \( \hat{\beta} \) result in linear programming, to dualize eq. (10) and obtain

\[
\min \{ b_i \} \quad \text{s.t.} \quad 0 \leq B_i(y) \leq B_i(\frac{1}{2}), \quad \forall i \leq \delta.
\]

By injecting these properties in (7), we obtain the following linear relaxation:

\[
\begin{align*}
\text{minimize} & \quad \sum_{i \leq \delta} b_i \hat{\beta} z_i, \\
\text{s.t.} & \quad z_i, \hat{\beta} \in \mathbb{R}, \quad i \leq \delta, \\
& \quad 0 \leq z_i, \hat{\beta} \leq B_i(\frac{1}{2}), \quad i \leq \delta, \\
& \quad \sum_{i \in \delta} z_i = 1.
\end{align*}
\]

**Lemma 1**: The optimal value of (8) gives a lower bound for the POP (4).

**B. Linear and bilinear feasibility programs for existence of Lyapunov function (Problem 7)**

Let \( V(x,c) \) be the assumed polynomial form for the Lyapunov function with unknowns \( c \). We first focus on encoding the positive definiteness of \( V \) inside \( R \). We recall the sets \( C, C', O, O' \) from section II-B.

First, we consider the set

\[
C : \left\{ c \mid \min_{x \in R} \left( \epsilon \|x\|^2 - V(x,c) \right) \leq 0 \right\}.
\]

Let \( m(x) \) represent a vector of monomials involved in \( \epsilon \|x\|^2 - V(x,c) \) so that we may write \( \epsilon \|x\|^2 - V(x,c) = \hat{c}^L m_\epsilon \), where \( \hat{c} = \begin{pmatrix} 1 & c \end{pmatrix} \) for a suitable matrix \( L \). Writing \( m \) in the Bernstein basis, we obtain \( m : B z \) where \( z \) represents a vector of polynomials in the Bernstein basis and \( B \) is a linear transformation. Therefore, the problem (8) is written equivalently as

\[
\begin{align*}
\min & \quad -\hat{c}^L B z \\
\text{s.t.} & \quad A z \leq b
\end{align*}
\]

Let \( \tilde{C} \) be the set of all values of \( c \) such that problem (9) with \( c \in \tilde{C} \) yields a non-positive optimal value. In other words,

\[
\tilde{C} : \left\{ c \mid (\forall z) A z \leq b \Rightarrow -\hat{c}^L B z \leq 0 \right\}.
\]

**Lemma 2**: \( \tilde{C} \subseteq C \).

To represent the set \( C \), we use Farkas lemma, a well known result in linear programming, to dualize eq. (10) and obtain our first linear feasibility problem for computing \( \tilde{C} \subseteq C \).

**Lemma 3**: The vector \( c \) is a solution to the problem in eq. (10) if and only if there exist multipliers \( c \) and \( \lambda \) such that

\[
A^t \lambda = -B^t L \hat{c}, \quad b^t \lambda \leq 0, \quad \text{and} \quad \lambda \geq 0
\]

Next, we consider the set \( O \) encoding the input constraints in (3). Let \( \mathcal{H} \) denotes the Bernstein matrix associated to the \( i \)-th row of the the polynomial matrix \( \mathcal{H} \) after mapping it to the unit box \( U \) (with respect to the degree \( \delta \in \mathbb{N}^n \) equal to the maximal degrees of \( \mathcal{H} \)). Consider the set \( \hat{O} \) defined as the feasible values of \( \theta \) that satisfy the following constraints

\[
\alpha_{U,k} \cdot \mathcal{H} \theta \leq \beta_{U,k}, \quad k \in K_U, \quad \forall i = 1, \ldots, m.
\]

**Lemma 4**: \( \hat{O} \subseteq O \).

Now we will show that finding the feasible sets \( C' \) and \( O' \) leads to a bilinear program. First, we can find a polynomial matrix \( B(x) \) to allow us to write

\[
-\nabla V_c(x) \cdot (f(x) + G(x) \theta) = c^t B(x) \hat{\theta},
\]

where \( \hat{\theta} = \begin{pmatrix} 1 & \theta \end{pmatrix} \) and \( B(x) = (\nabla V_m(x))^t \cdot (f(x) + G(x)) \).

Here \( (\nabla V_m(x))^t \) denotes the matrix where each column corresponds to the Jacobian of one of the monomials of the Lyapunov function.

The main difference with the previous case is that instead of the vector of monomials \( m \) we have \( B(x) \hat{\theta} \). The degree \( \delta \) will be chosen as the maximal degrees of the polynomials in \( B(x) \). By consequence, the Bernstein conversion matrix will be a set of \( n \) matrices \( B_{\theta,i} = B_i \hat{\theta} \) where \( B_i \) is the Bernstein conversion matrix corresponding to the polynomial row \( B_i(x) \) of the polynomial matrix \( B(x) \) after mapping it to the unit box \( U \). Now using the same ideas as previously we will get by applying farkas lemma a set of linear program:

**Lemma 5**: \( c \) is a solution to the problem in eq. (10) if and only if there exist multipliers \( c \) and \( \lambda \) such that

\[
A^t \lambda = -B_{\theta,i} c, \quad b^t \lambda \leq 0, \quad \text{and} \quad \lambda \geq 0,
\]

for all \( i = 1, \ldots, n \).

**C. Linear feasibility programs for positive invariance (Problem 2)**

We now turn to the problem of encoding the invariance of the region \( R \). Our approach reuses ideas from earlier work by Ben Sassi and Girard using the blossoming principle to enforce the invariance of a polytope for a polynomial system [4]. We obtain linear constraints over \( \theta \) that define a feasible region \( \hat{O} \subseteq \mathbb{R}^n \) such that choosing any \( \theta \in \hat{O} \) guarantees that the region \( R \) will be maintained invariant.

First, we will need to define a facet and its outer normal [1] for a general rectangle \( R_n = \prod_{k=1}^{\infty} [a_k,b_k] \):

\[
\begin{align*}
\xi_k : \{ a_k, b_k \} & \rightarrow \{ 0, 1 \} \quad \text{when for all} \ k \in \{ 1, \ldots, n \}, \\
\xi_k(a_k) & = 0 \quad \text{and} \ \xi_k(b_k) = 1, \\
F_j,\xi_j(w_j) & = \{ x \in R_n \mid x_j = w_j \} \quad \text{the set of facets of} \ R_n \ \text{where for all} \ j \in \{ 1, \ldots, n \}, \ w_j \in \{ a_j, b_j \}, \\
\eta_j,\xi_j(w_j) & = (-1)(\xi_j(w_j)+1)e_j \quad \text{the outer normal of the facet} \ F_j,\xi_j(w_j) \ \text{where the vectors} \ e_j \ \text{form the canonical basis of} \ R^n.
\end{align*}
\]

For the invariance context, all the results are derived from [4] so they are given without demonstration. We simply adapt the main result (Theorem 6 in [4]) to the specific form of
the controller required in this work. For doing so we define for a fixed degree \( \delta = (\delta_1, \ldots, \delta_n) \), for all \( j \in \{1, \ldots, n\} \) and all \( I \in \{1, \ldots, \delta_j\} \):

\[
I_{j,I} = \{ I = (i_1, \ldots, i_n) \in \mathbb{N}^n, \text{ such that } I \leq \delta \text{ and } i_j = l \}.
\]

More precisely, we need to replace in [4] the vector field \( f \) by \( f + G \theta \) and the blossom values by the Bernstein coefficients. Let \( f_U \) and \( G_U \) denote the polynomial vector field \( f \) and the polynomial matrix \( G \) after mapping them to the unit box \( U \) and let \( f_{U,I} \) and \( G_{U,I} \) the associated Bernstein coefficient vector and matrix for all multi-index \( I \leq \delta \). We will obtain the following result:

**Corollary 1:** For all \( j \in \{1, \ldots, n\} \), we have:

1. The facet \( F_{j,i}(a_j) \) of the rectangle \( R_n \) is blocked for the controlled system \( \dot{x} = f + G \theta \) if \( f_{U,I,j} + G_{U,I,j} \theta \geq 0 \) for all \( I \in I_{j,0} \).
2. The facet \( F_{j,i}(b_j) \) of the rectangle \( R_n \) is blocked for the controlled system \( \dot{x} = f + G \theta \) if \( f_{U,I,j} + G_{U,I,j} \theta \leq 0 \) for all \( I \in I_{j,0} \).

where \( f_{U,I,j} \) and \( G_{U,I,j} \) are respectively the \( j \) component (row) of the vector \( f_{U,I} \) (matrix \( G_{U,I} \)).

The corollary gives us a linear program allowing to compute the feasible sets \( \hat{O}_F \) for all facets \( F \in \mathcal{F} \).

**IV. J\textsc{o}INT SY\textsc{N}TH\textsc{ESIS OF POLYNOMIAL L\textsc{Y}APUNOV FUNCTIONS AND CONTROLLERS**

First of all, we are going to present an algorithm to solve our stability problem, then we will show how the results can be improved by using a decomposition criterion and extend the results using this decomposition to a particular class of hybrid systems.

**A. Algorithm**

In this section, we give an algorithm allowing to summarize the previous results in order to solve our stabilization problems by synthesizing jointly the controller that stabilize the system and the Lyapunov function for the controlled system. In fact, the main problem when regrouping the feasibility problems of the previous section is that we have to deal with a bilinear program for which there is no practical way to solve it. We will define an iterative approach where for each step one of the parameters (\( \theta \) for the controller or \( c \) for the Lyapunov function) is fixed and the other is computed by solving a linear program. The overall approach is given as follows:

1. Initialize \( \theta^* = 0 \).
2. Compute feasible set \( C \) using feasibility problem (11).
3. Find a "maximal" coefficient vector \( c \in C \) for the Lyapunov function:
   - We fix \( \theta = \theta^* \) and we solve the feasibility problems by relaxing " \( \leq 0\)" by " \( \leq t\)" where \( t \) will be a positive decision variable to be minimized. The outputs of the linear program are \( (c^*, t^*) \).
4. Find a "maximal" coefficient vector \( \theta \) for the controller:
   - We fix \( c = c^* \) and we solve the feasibility problems given by the (RHS) of (12) and the ones of Corollary 1.

By using the same idea of relaxing " \( \leq 0\)" by " \( \leq t\)" for a positive decision variable \( t \) and minimize over \( t \), we get outputs \( (\theta^*, t^*) \). If \( t^* \approx 0 \) STOP ; else Go back to the previous step.

When the algorithm terminates, the outputs \( (c^*, \theta^*) \) will give us the admissible controller and the Lyapunov function proving the asymptotic stability of the controlled system. The invariance problem of the rectangular domain will be ensured.

**B. Decomposition and generalization for a particular class of hybrid systems**

As mentioned in [5], the Bernstein relaxation (8) can be much more efficient once a good decomposition is provided. By "good" we mean a box decomposition where local minima will belong to the edge of the box. Since the global minimum of the Lyapunov function is known in advance (0 in our case), a decomposition of the rectangle \( R \) around zero (by putting zeros on the edges of the resulting rectangles) will significantly improve the precision of the approach.

The drawback is that \( 2^n \) decompositions are needed. In fact by using this decomposition, each feasibility problem in the previous algorithm (except the invariance ones) will be replaced by \( 2^n \) feasibility problems.

Now, since the approach deals with a box partition of the state space, one can easily extend the dynamical system (1) to the following class of hybrid system where the state space is decomposed to boxes and each box has its own polynomial dynamic. More precisely, for all \( i \leq 2^n \), let \( R_i \) be the set of boxes of our 'zero' decomposition and the hybrid system will be following :

\[
\begin{align*}
\dot{x}_i(t) &= f_i(x(t)) + G(x)\theta_i, \quad \theta_i \in O \times R_i.
\end{align*}
\]

The difference here is that each of the \( 2^n \) feasibility problems (Step 4) will provide an admissible controller \( \theta_i \) trying to make the Lyapunov function decreasing in the corresponding box. So we will get a common Lyapunov function having multiple derivatives (one for each box). Also we should remark that when dealing with the invariance problem, linear feasibility problems of Corollary 1 should be adapted. In fact, for each box one should ensure the feasibility problems with respect to the facets that should be blocked.

**Remark 1:** The previous result will hold for each other box decomposition. In fact we can always be reduced to the previous case by decomposing each sub box containing \( 0_n \) into sub boxes where \( 0_n \) will belong to the edges.

**V. NUMERICAL RESULTS**

**A. Illustrative example**

To illustrate the approach, we consider the following 2-dimensional polynomial system and a box \( R = [-1, 1]^2 \):

\[
\begin{align*}
\dot{x}_1 &= f_1(x) = x_2 - x_1^2 + 3x_2^2 - 2x_1x_2, \\
\dot{x}_2 &= f_2(x) = -x_1 - 3x_1^2 + x_2^2 + 2x_1x_2.
\end{align*}
\]

By simulation, one can see that the origin is not asymptotically stable and that the box \([-1, 1]^2\) is not invariant for the
For the Lyapunov function, we fix the following form:

\[
\dot{x}(t) = f(x(t)) + g(x(t))u(x(t)),
\]

where \( g(x) = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( u(x) = Ax \) where

\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.
\]

Since we look for a linear state feedback controller, we can write \( u(x) = H(x)\theta \) where

\[
H(x) = \begin{pmatrix} x_1 & x_2 & 0 & 0 \\ 0 & 0 & x_1 & x_2 \end{pmatrix}
\]

and \( \theta = (a_{11}, a_{12}, a_{21}, a_{22})^T \).

For the Lyapunov function, we fix the following form:

\[
V_c(x) = c_1 x_1 + c_2 x_2 + c_3 x_1^2 + c_4 x_2^2 + c_5 x_1 x_2 + c_6 x_1^4 + c_7 x_2^4.
\]

We impose that \(-5 \leq c_i \leq 5\) for all \(i \in \{1, \ldots, 7\}\) and add the fact that \(c_i \geq 0.01\) for all \(i \in \{3, 4\}\) in order to ensure that \(V\) is positive definite. Also the linear coefficients of the controller are bounded by \(-5\) and \(5\).

The iterative approach needs two iterations to globally stabilize \(R\). Outputs are:

- \(A = \begin{pmatrix} -4.5471 & 0.7000 \\ 3.9290 & -4.6218 \end{pmatrix}\).
- \(V(x) = 0.01(x_1^2 + x_2^2) + 0.009x_1x_2 + 0.036x_1^4 + 0.023x_2^4\).

One can simulate the obtained system and verify the asymptotic stability and the invariance of \(R\) (see Figure 3). Now, we will use the approach to deal with the Hybrid case. More precisely, we decompose \(R\) around zero \((R_1 = [-1, 0]^2, R_2 = [-1, 0] \times [0, 1], R_3 = [0, 1] \times [-1, 0], R_4 = [0, 1]^2)\) and try to find for each sub-box \(R_i\) a linear controller \(u_i\) such that the following Hybrid system

\[
\dot{x}(t) = f(x(t)) + g(x(t))u_i(x),
\]

is globally stable with respect to \(R\) where \(u_i(x) = A_ix\) for all \(x \in R_i\) and all \(i \in \{1, \ldots, 4\}\).

In this case, only one iteration is needed to stabilize the system inside \(R\) since we have more freedom in the choice of the controller. Outputs are:

- \(A_1 = \begin{pmatrix} -4.4721 & -3.4219 \\ -2.9376 & -4.0957 \end{pmatrix}\).
- \(A_2 = \begin{pmatrix} -4.3795 & 0.3130 \\ 1.1904 & -4.3770 \end{pmatrix}\).
- \(A_3 = \begin{pmatrix} -4.3331 & 2.6016 \\ 3.3924 & -4.2926 \end{pmatrix}\).
- \(A_4 = \begin{pmatrix} -4.1427 & -3.0418 \\ -3.3052 & -4.4195 \end{pmatrix}\).
- \(V(x) = 4.7737x_1^2 + 4.7743x_2^2 + 4.8172x_1^4 + 4.8175x_2^4\).

By simulating trajectories in those boxes, we can verify that the stability property and the box invariance hold (see Figure 4 for \(R_1\) and \(R_2\)).
B. Benchmarks

We discuss the results obtained for a set of benchmarks borrowed from the literature. We run the algorithm until a good precision \( \varepsilon \) is reached or a fixed number of iterations (the approach fails to make progress). In the latter case one can add more flexibility in the templates by adding terms of higher degrees. In failure cases, we remove the invariance constraints in order to achieve just the asymptotic stability property. We report separately stability (Stab column) and invariance (Inv column). A threshold of precision around \( 10^{-6} \) is considered to confirm that the property holds. We report also the number of iteration needed to achieve the given precision. A detailed description of the systems, explicit expression of Lyapunov functions and controllers are given in the Appendix.

| Table 1 | Table showing performance of our method on a set of benchmarks. |
|---|---|
| Id | \( R \) | \( U \) | \( \varepsilon \) | Stab | Inv | Iter |
| 1 | \([-0.5, 0.5]^2 \) | \([-1, 1] \) | \( 4 \times 10^{-12} \) | ✓ | ✓ | 1 |
| 2 | \([-1, 1]^2 \) | \([-2, 2] \) | \( 4 \times 10^{-9} \) | ✓ | ✓ | 2 |
| 3 | \([-1, 1]^2 \) | \([-4, 4] \) | \( 2 \times 10^{-17} \) | ✓ | ✓ | 1 |
| 4 | \([-1, 1]^2 \) | \([-1, 1] \) | \( 4 \times 10^{-7} \) | ✓ | ✓ | 3 |
| 5 | \([-1, 1]^3 \) | \([-10, 10] \) | \( 2 \times 10^{-6} \) | ✓ | ✓ | 4 |
| 6 | \([-0.5, 0.5]^3 \) | \([-5, 5] \) | \( 2 \times 10^{-7} \) | ✓ | ✓ | 6 |
| 7 | \([-0.5, 0.5]^3 \) | \([-3, 3] \) | \( 9 \times 10^{-7} \) | ✓ | ✓ | 3 |
| 8 | \([-0.5, 0.5]^3 \) | \([-1, 1] \) | \( 4 \times 10^{-5} \) | ✓ | ✓ | 9 |
| 9 | \([-0.1, 0.1]^3 \) | \([-5, 5] \) | \( 5 \times 10^{-5} \) | ✓ | ✓ | 3 |
| 10 | \([-0.1, 0.1]^3 \) | \([-10, 10] \) | \( 8 \times 10^{-5} \) | ✓ | ✓ | 4 |
| 11 | \([-0.05, 0.05]^3 \) | \([-1, 1] \) | \( 6 \times 10^{-5} \) | ✓ | ✓ | 2 |

Note that that invariance conditions usually make the feasibility of the approach very restricted since it needs to hold simultaneously with the stability conditions. This explains the fact that only few stabilizable systems can only have the invariance box property. The computation time is roughly in size of the problem and the templates: roughly each iteration of two dimensional systems (systems 1, 2, 3, 4) required almost one second, for three dimensional systems it required between two and three seconds (systems 4, 5, 6, 7).

VI. APPENDIX

Example 1: (see [22])

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -x + u(y).
\end{align*}
\]
- \( u(y) = -2y \).
- \( V(x, y) = 0.01(x^2 + y^2) \)

Example 2: (see Lectures on back-stepping\( ^3 \))

\[
\begin{align*}
\dot{x} &= y - x^3, \\
\dot{y} &= u(x, y).
\end{align*}
\]
- \( u(x, y) = -x \) \( \frac{3}{2} y + \frac{1}{4} x^3 \).
- \( V(x, y) = 0.01(y^2 + x^2 y^2) + 0.0102x^2 + 0.0007xy. \)

Example 3:

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= u(y)y^2 - x.
\end{align*}
\]
- \( u(y) = 4(y^2 - y). \)
- \( V(x, y) = 0.01(x^2 + y^2 + x^2 y^2) + 0.005(x^4 + y^4). \)

Example 4: (See [23])

\[
\begin{align*}
\dot{x} &= -x(0.1 + (x + y)^2) \\
\dot{y} &= (u(x) + x)(0.1 + (x + y)^2).
\end{align*}
\]
- \( u(x) = -x. \)
- \( V(x, y) = 0.01(y^2 + x^2 y^2) + 0.0657x^2 + 0.0022xy + 0.0019y^4. \)

Example 5:

\[
\begin{align*}
\dot{x} &= y + 0.5z^2, \\
\dot{y} &= z, \\
\dot{z} &= u(x, y, z).
\end{align*}
\]
- \( u(x, y, z) = -0.59185x - 5.9217y - 0.51825z + 0.061785x^2 + 0.12415xy - 0.4642xz + 0.048453x^3 - 0.57345y^3. \)
- \( V(x, y, z) = 0.01x^2 + 0.0583y^2 + 0.0099z^2 + 0.0134xy + 0.0033xz + 0.0049y^2 + 0.0024yz + 0.0003z^4. \)

Example 6: (See [24])

\[
\begin{align*}
\dot{x} &= -x + y - z, \\
\dot{y} &= -x(z + 1) - y, \\
\dot{z} &= -x + u(x, z).
\end{align*}
\]
- \( u(x, z) = 1.7652xz - 4.7037z. \)
- \( V(x, y, z) = 0.01(x^2 + y^2) + 0.013z^2. \)

Example 7: (see Lectures on back-stepping)

\[
\begin{align*}
\dot{x} &= -x^3 + y, \\
\dot{y} &= y^3 + z, \\
\dot{z} &= u(x, y, z).
\end{align*}
\]
- \( u(x, y, z) = -0.083339x - 3.5413y - 0.33868z - 0.4325x^3. \)
- \( V(x, y, z) = 0.01(x^2 + z^2) + 0.0333z^2 + 0.0179xy + 0.0048xz + 0.0061yz. \)

Example 8: (See [24])

\[
\begin{align*}
\dot{x} &= z^3 - y, \\
\dot{y} &= z, \\
\dot{z} &= u(x, y, z).
\end{align*}
\]
- \( u(x, y, z) = -0.86597x - 0.16208y - 0.61597z. \)
- \( V(x, y, z) = 0.01(x^2 + z^2) + 0.0333z^2 + 0.0179xy + 0.0129xz + 0.0127y^2. \)

Example 9:

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -0.1y - 10z + xv^2, \\
\dot{z} &= v, \\
v &= -z - v + u(x, y, z, v).
\end{align*}
\]
- \( u(x, y, z, v) = -12.0271x - 8.1243y - 10.2755z - 10.047v. \)
- \( V(x, y, z, v) = 0.1202x^2 + 0.01(y^2 + v^2) + 0.2201z^2 + 0.2556xz + 0.0101xy + 0.01578yz + 0.0115vy. \)

\(^1\) \( \varepsilon \) denotes the precision \( \varepsilon^* \) of the algorithm.

[^3]: [http://control.ee.ethz.ch/~apnco/Lectures2014](http://control.ee.ethz.ch/~apnco/Lectures2014)
Example 10: (Ball and Beam example [25])

\[
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= -9.8z + 1.6z^3 + xv^2, \\
\dot{z} &= v, \\
\dot{v} &= u(x).
\end{aligned}
\]

- \( u(x) = -6x \).
- \( V(x, y, z, v) = 0.0672x^2 + 0.01y^2 + 0.1074z^2 + 0.0136y^2 + 0.0043xyz + 0.149xz + 0.0023zv + 0.008yz + 0.0189yv - 0.003zv. \)

Example 11:

\[
\begin{aligned}
\dot{x} &= -0.1x^2 - 0.4xv - x + y + 3z + 0.5v, \\
\dot{y} &= y^2 - 0.5yw + x + z, \\
\dot{z} &= 0.5z^2 + x - y + 2z + 0.1v - 0.5w, \\
\dot{v} &= y + 2z + 0.1v - 0.2w + u(x, y, z, v, w), \\
\dot{w} &= z - 0.1v + u(x, y, z, v, w).
\end{aligned}
\]

- \( u(x, y, z, v, w) = -1.5x - 1.5y - 1.5z - 1.5v - 1.5w. \)
- \( V(x, y, z, v, w) = 0.01(x^2 + y^2 + z^2 + w^2) - 0.0066xy - 0.0252xz - 0.008(x + y + z + w) + 0.005yv + 0.001yz + 0.0167yw - 0.0023zw - 0.001vw. \)

VII. CONCLUSION

In this paper a linear programming approach is presented allowing to deal with the stabilization problem of polynomial systems. The approach is based on Bernstein polynomials and propose a policy iteration technique allowing to avoid bilinear programs by having an iterative approach of linear programs instead. The benchmarks results show that the method can be efficient in practice. The drawback of this technique is that no convergence result is guaranteed and even in case of convergence there is no guaranty that it will be to a local minima. A future work will be a deeper study of the failure case or the fix point (once the algorithm result does not improve): an idea is to fix small variation for each variable of the bilinear program and try to find a descent direction helping the algorithm to improve.

REFERENCES

[1] C. Beta and L.C.G.J.M Habets, Controlling a class of non-linear systems on rectangles, IEEE Transactions on Automatic Control, vol. 51, no. 11, 2006, pp. 1749-1759.
[2] M.A.Ben Sassi and A. Girard, Computation of polytopic invariants for polynomial dynamical systems using linear programming, Automatica, 2012.
[3] M.A.Ben Sassi and A. Girard, Controller synthesis for robust invariance of polynomial dynamical systems using linear programming, Systems and Control Letters, vol. 61, no. 4, 2012, pp. 506-512.
[4] M.A.Ben Sassi and A. Girard, Control of polynomial dynamical systems on rectangles, European Control Conference, Zurich, 2013.
[5] M.A.Ben Sassi and S. Sankaranarayanan and X. Chen and E. Abraham, Linear Relaxations of Polynomial Positivity for Polynomial Lyapunov Function Synthesis, IMA Journal of Mathematical Control and Information.
[6] D. Heurton and J. Lobberg and M. Kocvara and M. Stingl, Solving polynomial static output feedback problems with PENBMI, Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference, Seville, 2005.
[7] L. El Ghaoui and V. Balakrishnan, Synthesis of fixed-structure controllers via numerical optimization, Proceedings of the 33rd Conference on Decision and Control, 1994.
[8] E. Sontag, A "universal" construction of Artstein's theorem on non-linear stabilization, Systems, Control Letters, vol. 13, no. 2, 1989, pp. 117–123.
[9] I. Karafyllis, Global Stabilization of Nonlinear Systems Based on Vector Control Lyapunov Functions, Automatic control, IEEE transactions, vol. 58, no. 10, 2013, pp. 2550–2562.
[10] Z. H. Li and M. Krstic, Maximizing Regions of Attraction via Backstepping and CLFs with Singularities, Syst., Control Lett., vol. 30, no. 4, pp. 195-207, May 1997.
[11] A. Astolfi and P.A Colaneri, Static output feedback stabilization of linear and nonlinear systems, Proceedings of the 39th IEEE Conference on Decision and Control, Sydney, Australia, 2000, pp. 2920-2925.
[12] A. Astolfi and P.A Colaneri, Hamilton?Jacobi setup for the static output feedback stabilization of nonlinear systems, IEEE Transactions on Automatic Control 2002, vol. 47, no. 12, pp. 2038–2041, 2002.
[13] R. M. Hirschorn, Output Tracking through Singularities, Proc. 41st IEEE CDC, pp. 3843-3848, Dec. 2002.
[14] W. Tan. Nonlinear control analysis and synthesis using sumof- squares programming. Ph.D. Thesis, University of California, Berkeley, 2006.
[15] G. Chesi and Y.S. Hung, Analysis and synthesis of nonlinear systems with uncertain initial conditions, IEEE Transactions on Automatic Control, vol. 53, no. 5, pp.1262/1267, 2008.
[16] S. Gaubert and E. Goubault and A. Taly and S. Zennou, Static Analysis by Policy Iteration on Relational Domains, European Symposium on Programming (ESOP), Volume 4421 of Lecture Notes in Computer Science, Springer, 2007.
[17] M.R. Garey and D.S. Johnson, Computers and Intractability: Guide to the Theory of NP-Completeness, Macmillan, 1979.
[18] S. Gaubert and E. Goubault and A. Taly and S. Zennou, Static Analysis by Policy Iteration on Relational Domains, European Symposium on Programming (ESOP), Volume 4421 of Lecture Notes in Computer Science, Springer, 2007.
[19] L. El Ghaoui and V. Balakrishnan, Synthesis of fixed-structure controllers via numerical optimization, Proceedings of the 33rd Conference on Decision and Control, 1994.
[20] D. Zhao and J. Wang, Robust static output feedback design for polynomial nonlinear systems, International Journal of Robust and Nonlinear Control, vol. 20, no. 14, 2010, pp. 1637-1654.
[21] S. Nguang and M. Krug and S. Saat, Nonlinear Static Output Feedback Controller Design for Uncertain Polynomial Systems: An Iterative Sums of Squares Approach, 6th IEEE Conference on Industrial Electronics and Applications, 2011.
[22] D. Liberzon and S. Morse, Basic problems in stability and design of switched systems, Control Systems, IEEE, vol. 19, no. 5, 1999, pp. 59–70.
[23] W. Perruquetti and J.P. Richard and P. Borne, Lyapunov analysis of sliding motions: Applications to bounded control, Mathematical problems in engineering, vol. 3, 1995, pp. 1–25.
[24] D.H. Yeom and Y.H. Joo, Control Lyapunov Function Design by Cancelling Input Singularity, International Journal of Fuzzy Logic and Intelligent Systems, vol. 12, no. 2, 2012, pp. 131–136.
[25] R. Kadiyala, A Tool Box for Approximate Linearization of Nonlinear Systems.