ON ALEXANDER POLYNOMIALS OF CERTAIN \((2, 5)\) TORUS CURVES

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ABSTRACT. In this paper, we compute Alexander polynomials of a torus curve \(C\) of type \((2, 5)\), \(C : f(x, y) = f_2(x, y)^5 + f_5(x, y)^2 = 0\), under the assumption that the origin \(O\) is the unique inner singularity and \(f_2 = 0\) is an irreducible conic. We show that the Alexander polynomial remains the same with that of a generic torus curve as long as \(C\) is irreducible.

1. INTRODUCTION

A plane curve \(C \subset \mathbb{P}^2\) of degree \(pq\) is called a curve of torus type \((p, q)\) with \(p > q \geq 2\), if there is a defining polynomial \(F\) of \(C\) of the form \(F = F_p + F_q\), where \(F_p, F_q\) are homogeneous polynomials of \(X, Y, Z\) of degree \(p\) and \(q\) respectively. A singularity \(P \in C\) is called inner if \(F_p(P) = F_q(P) = 0\). Otherwise, \(P\) is called an outer singularity. A torus curve \(C\) is called tame if it has no outer singularity. We assume \(O = (0, 0)\) hereafter. In [5], the first author classified the topological types of the germs of inner singularity of curves of \((2, 5)\) torus type. In this paper, we are interested in the Alexander polynomial of \(C\) which is an important topological invariant. In the case of irreducible sextics of torus type \((2, 3)\), there are only 3 possible Alexander polynomials: \(\Delta_{4, 2}(t) = (t^2 - t + 1)^j, j = 1, 2, 3\) ([11]).

A tame torus curve \(C\) of type \((p, q)\) is said to be generic if the associated curves \(C_p = \{F_p = 0\}\) and \(C_q = \{F_q = 0\}\) intersect transversely at \(pq\) distinct points. It is known that the Alexander polynomial of a generic \(C\) is equal to \(\Delta_{p,q}(t)\) ([12]) where

\[
\Delta_{p,q}(t) := \frac{(pq/r - 1)^r(t - 1)}{(tp - 1)(tq - 1)}, \quad r = \gcd(p, q).
\]

Moreover it is also known that the Alexander polynomial of \(C\) is still equal to \(\Delta_{p,q}(t)\), if \(C\) is tame and \(C_p, C_q\) intersect at \(O\) with intersection multiplicity \(pq\) and \(C_p\) is smooth ([2, 3]).

Let \(C\) be a torus curve of type \((2, 5)\) such that \(C\) has a unique inner singularity, say \(O \in C\) (thus \(I(C_2, C_5; O) = 10\)) and we assume that \(C\) has no outer singularity. Then we have shown that there are 22 possible singularities for \((C, O)\) under the assumption that \(C_2\) is irreducible ([5]). For 8 classes among 22 type of singularities, \(C\) can be either irreducible or reducible. We list those 22-singularities below. Throughout this paper, we use the same notations of singularities as in [5, 10].

(I) Assume that \(C\) is irreducible, the possibilities are:

\[
\begin{align*}
B_{50,2}, \quad B_{43,2} \circ B_{2,3}, \quad B_{36,2} \circ B_{4,3}, \quad B_{29,2} \circ B_{6,3}, \quad B_{22,2} \circ B_{8,3}, \quad B_{15,2} \circ B_{10,3}, \quad B_{25,4}, \\
(B^2_{4,2})_{B_{32,2}+B_{2,2}}, \quad (B^2_{4,2})_{B_{32,2}+B_{2,2}}, \quad (B^2_{6,2})_{B_{23,2}+B_{4,2}}, \quad (B^2_{6,2})_{B_{14,2}+B_{4,2}}, \quad (B^2_{10,2})_{2B_{5,2}}, \\
(B^2_{11,2})_{B_{6,2}}, \quad (B^2_{12,2})_{2B_{1,1}}, \quad (B^2_{6,2})_{B_{16,2}+B_{1,2} \circ B_{2,1}}, \quad (B^2_{6,2})_{2B_{6,2}+B_{1,2} \circ B_{2,1}}, \\
(B^2_{9,2})_{B_{6,2} \circ B_{2,1}}, \quad B_{29,2} \circ B_{2,1} \circ (B^2_{2,1})_{B_{k,2}} (k = 1, 2, 3, 5).
\end{align*}
\]

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Let us consider the affine coordinate $\mathbb{C}^2 = \mathbb{P}^2 \setminus \{Z = 0\}$ and let $x = X/Z, y = Y/Z$. Let $C$ be a given plane curve of degree $d$ defined by $f(x, y) = 0$ and let $O \in C$ be a singular point of $C$ where $O = (0, 0)$. We assume that the line at infinity $\{Z = 0\}$ is generic with respect to $C$.
2.1. Loeser-Vaquie formula. Consider an embedded resolution of \((C,O) \subset (\mathbb{C}^2, O)\), \(\pi : \tilde{U} \rightarrow U\) where \(U\) is an open neighborhood of \(O\) and let \(E_1, \ldots, E_n\) be the exceptional divisors. Let \((u, v)\) be a local coordinate system centered at \(O\) and \(k_i\) and \(m_i\) be respective order of zero of the canonical two form \(\pi^*(du \wedge dv)\) and \(\pi^* f\) along the divisor \(E_i\). The adjunction ideal \(\mathcal{J}_{O,k,d}\) of \(O\) is defined by

\[
\mathcal{J}_{O,k,d} = \{ \phi \in O_O \mid (\pi^* \phi) \geq \sum_i ([km_i/d] - k_i)E_i \}, \quad k = 1, \ldots, d - 1
\]

where \(\lfloor r \rfloor\) is the largest integer \(n\) such that \(n \leq r\) for \(r \in \mathbb{Q}\) (\cite{12}). Let \(O(j)\) be the set of polynomials in \(x, y\) whose degree is less than or equal to \(j\). We consider the canonical mapping \(\sigma : \mathbb{C}[x, y] \rightarrow O_O\) and its restriction:

\[
\sigma_k : O(k - 3) \rightarrow O_O.
\]

Put \(V_k(O) = O_O/\mathcal{J}_{O,k,d}\) and we denote the composition \(O(k - 3) \rightarrow O_O \rightarrow V_k(O)\) by \(\tilde{\sigma}_k\). Then the Alexander polynomial is given as follows.

**Lemma 1.** (\cite{6, 7, 1, 4}) The reduced Alexander polynomial \(\tilde{\Delta}_C(t)\) is given by the product

\[
\tilde{\Delta}_C(t) = \prod_{k=1}^{d-1} \Delta_k(t)^{\ell_k}
\]

where \(d\) is the degree of \(f\), \(\ell_k\) is the dimension of \(\text{Coker} \tilde{\sigma}_k\) and

\[
\Delta_k(t) = \left(t - \exp\left(\frac{2k\pi i}{d}\right)\right)\left(t - \exp\left(-\frac{2k\pi i}{d}\right)\right).
\]

We use the method of Esnault-Artal (\cite{11}) to compute \(\ell_k\).

**Remark 1.** The Alexander polynomial \(\Delta_C(t)\) is given as

\[
\Delta_C(t) = (t - 1)^{r-1}\tilde{\Delta}_C(t)
\]

where \(r\) is the number of irreducible components of \(C\) (\cite{12}). Note that for the case of curve of degree 10.

\[
\Delta_5(t) = (t + 1)^2, \quad \Delta_6(t)\Delta_8(t) = t^4 + t^3 + t^2 + t + 1, \quad \Delta_7(t)\Delta_9(t) = t^4 - t^3 + t^2 - t + 1.
\]

2.2. Plücker’s formula. We denote the Milnor number of the singularity of \((C, P)\) by \(\mu(C, P)\) and the number of locally irreducible components of \((C, P)\) by \(r(C, P)\). We recall the generalized Plücker’s formula. Let \(C_1, \ldots, C_r\) be irreducible components of \(C\) and let \(\tilde{C}_1, \ldots, \tilde{C}_r\) be their normalizations, let \(g(\tilde{C}_i)\) be the genus of \(\tilde{C}_i\) and let \(\Sigma(C)\) be the singular locus of \(C\). Then

\[
\chi(\tilde{C}) = \sum_{i=1}^{r} (2 - 2g(\tilde{C}_i)) = d(3 - d) + \sum_{p \in \Sigma(C)} (\mu(C, P) + r(C, P) - 1) \leq 2r
\]

For further details, we refer to \cite{8, 9, 13}.

3. Outline of the proof of Theorem \cite{1}

We have to consider the following 22-singularities. We denote a class of a singularity \((C, O)\) which can appear both as an irreducible curve and a reducible curve by \(\sharp(C, O)\). In the section
3.2, we will use notation $\text{irr}(C, O)$, $\text{red}(C, O)$ to distinguish the case of $C$ being irreducible and reducible.

$$B_{50,2}, B_{43,2} \circ B_{2,3}, B_{36,2} \circ B_{4,3}, \frac{t}{B}_{29,2} \circ B_{6,3}, B_{22,2} \circ B_{8,3}, B_{15,2} \circ B_{10,3}, B_{20,5}, B_{25,4}, (B_{4,2})^{B_{32,2}+B_{2,2}} , (B_{4,2})^{B_{32,2}+B_{2,2}} , (B_{6,2})^{B_{32,2}+B_{2,2}} , (B_{8,2})^{B_{32,2}+B_{2,2}} , (B_{10,2})^{B_{32,2}+B_{2,2}}, (B_{11,2})^{B_{32,2}+B_{2,2}} , (B_{12,2})^{B_{32,2}+B_{2,2}} , (B_{6,2})^{B_{32,2}+B_{2,2}} , (B_{8,2})^{B_{32,2}+B_{2,2}} , (B_{12,2})^{B_{32,2}+B_{2,2}} , (B_{6,2})^{B_{32,2}+B_{2,2}} , (B_{8,2})^{B_{32,2}+B_{2,2}} , (B_{12,2})^{B_{32,2}+B_{2,2}} , (B_{6,2})^{B_{32,2}+B_{2,2}}$$

3.1. Divisibility principle and Sandwich principle. Suppose we have a degeneration family $C_s$, $s \in W$ of reducible curves such that $C_s$, $s \neq 0$ are equisingular family of plane curves. Here $W$ is an open neighbourhood of the origin in $C$. We denote this situation as $C_s \xrightarrow{s \rightarrow 0} C_0$. Then we have the divisibility $\Delta_{C_s}(t) | \Delta_{C_0}(t)$ (Theorem 26 of [12]). Suppose that we have two degeneration series $C_s \xrightarrow{s \rightarrow 0} C_0$ and $D_r \xrightarrow{r \rightarrow 0} D_0$ such that $C_0 \cong D_r$ ($r \neq 0$) and assume that $\Delta_{C_s}(t) = \Delta_{D_0}(t)$. Then the divisibility implies that $\Delta_{C_s}(t) = \Delta_{C_0}(t)$ (the Sandwich principle).

3.2. Degeneration series. Recall that we have the following degeneration series among the above singularities ([5]):

(1) Main sequence:

$$B_{50,2} \rightarrow B_{43,2} \circ B_{2,3} \rightarrow (B_{4,2})^{B_{32,2}+B_{2,2}} \rightarrow B_{36,2} \circ B_{4,3}$$

$$\rightarrow \frac{t}{B}_{29,2} \circ B_{6,3} \rightarrow B_{22,2} \circ B_{8,3} \rightarrow B_{15,2} \circ B_{10,3} \rightarrow B_{20,5}$$

where the branched sequences (a) from $(B_{4,2})^{B_{32,2}+B_{2,2}}$ and (b), (c) from $B_{29,2} \circ B_{6,3}$ in the main sequence are as follows.

(a) $(B_{4,2})^{B_{32,2}+B_{2,2}} \rightarrow (B_{6,2})^{B_{32,2}+B_{2,2}} \rightarrow (B_{8,2})^{B_{32,2}+B_{2,2}} \rightarrow (B_{10,2})^{B_{32,2}+B_{2,2}} \rightarrow (B_{12,2})^{B_{32,2}+B_{2,2}} \rightarrow B_{25,4}$

(b) (i) $\text{irr} B_{29,2} \circ B_{6,3} \rightarrow \text{irr} (B_{6,2})^{B_{16,2}+B_{2,2}} \circ B_{2,1} \rightarrow \text{irr} (B_{8,2})^{B_{16,2}+B_{2,2}} \circ B_{2,1} \rightarrow \text{irr} (B_{9,2})^{B_{5,2}} \circ B_{2,1}$.

(ii) $\text{irr} B_{29,2} \circ B_{6,3} \rightarrow \text{irr} (B_{6,2})^{B_{16,2}+B_{2,2}} \circ B_{2,1} \rightarrow \text{irr} (B_{8,2})^{B_{16,2}+B_{2,2}} \circ B_{2,1} \rightarrow \text{irr} (B_{9,2})^{B_{5,2}} \circ B_{2,1}$.

(c) (i) $\text{red} B_{29,2} \circ B_{6,3} \rightarrow \text{red} (B_{6,2})^{B_{16,2}+B_{2,2}} \circ B_{2,1} \rightarrow \text{red} (B_{8,2})^{B_{16,2}+B_{2,2}} \circ B_{2,1}$

(ii) $\text{red} B_{29,2} \circ B_{6,3} \rightarrow \text{red} (B_{6,2})^{B_{16,2}+B_{2,2}} \circ B_{2,1} \rightarrow \text{red} (B_{8,2})^{B_{16,2}+B_{2,2}} \circ B_{2,1}$

The main sequence is obtained through the degenerations of the tangent cone of $C_5$ at $O$, keeping the irreducibility of $C_2$. In the last degeneration $B_{15,2} \circ B_{10,3} \rightarrow B_{20,5}$ of the main sequence, $C$ degenerates into a reducible curve.

The branched sequence (a) from $(B_{4,2})^{B_{32,2}+B_{2,2}}$ is obtained by degenerating $(C_5, O)$, fixing the tangent cone of $C_5$ at $O$. More precisely, the tangent cone of $(C_5, O)$ is a line with
multiplicity 2 and the generic singularity of \((C_5, O)\) is \(A_3\) and the corresponding degenerations of \((C_5, O)\) are:
\[
(C_5, O): \quad B_{4,2} \to B_{6,2} \to B_{8,2} \to B_{10,2} \to B_{11,2} \to B_{12,2} \to B_{13,2}.
\]

The branched sequence \((b)\) (respectively, \((c)\)) from \(\text{Irr} B_{29,2} \circ B_{6,3}\) (resp. \(\text{Red} B_{29,2} \circ B_{6,3}\)) is also obtained by degenerating \((C_5, O)\) fixing the tangent cone of \(C_5\) at \(O\) (See §3.4).

3.3. **Strategy.** Our strategy is the following. The singularity \(B_{50,2}\) is obtained when \(C_2\) and \(C_5\) has a maximal contact at \(O\) and \((C_5, O)\) is smooth. In this case, it is known that \(\Delta_C(t) = t^4 - t^3 + t^2 - t + 1\) by Theorem 2 of [2]. Hence by virtue of the Sandwich principle, it is enough to show

1. the irreducibility of \(C\) and
2. \(\tilde{\Delta}_C(t) = \Delta_{5,2}(t)\) for the case \((C, O)\) being one of the following singularities which are the end of the degenerations.

\[B_{15,2} \circ B_{10,3}, \quad B_{25,4}, \quad \sharp(B_{9,2}) B_{5,2} \circ B_{2,1}, \quad \sharp B_{29,2} \circ B_{2,1} \circ (B_{2,1}^2 B_{5,2}).\]

By virtue of Lemma 1 to show \(\tilde{\Delta}_C(t) = \Delta_{5,2}(t)\) is equivalent to show that

1. \(\tilde{\sigma}_k : O(k - 3) \to V_k(O)\) has one-dimensional cokernel for \(k = 7, 9\) and surjective for other cases.

So for the proof of the assertions (1) and (2) of Theorem 1, we will actually show the above property (2).

The last singularity \(B_{20,5}\) of the main sequence appears when \(C\) consists of five conics. We treat this case separately in the later section.

3.4. **Irreducibility of \(C\).** Now we will discuss the irreducibility of \(C\) using the generalized Plücker’s formula. First we show that \(C\) is irreducible if \((C, O)\) is one of 2 singularities \(B_{15,2} \circ B_{10,3}\) and \(B_{25,4}\).

**Case** \((C, O) \sim B_{15,2} \circ B_{10,3}\): Note that the singularities \(B_{15,2}\) and \(B_{10,3}\) are locally irreducible singularities. As \(\mu(B_{15,2}) = 14\), \(\mu(B_{10,3}) = 18\) and each singularity appears for sextics or higher degree curves. Thus \(C\) must be irreducible, as the degree of \(C\) is 10.

**Case** \((C, O) \sim B_{25,4}\): The singularity \(B_{25,4}\) is a locally irreducible singularity and thus \(C\) is irreducible.

**Case** \((C, O) \sim B_{29,2} \circ B_{6,3}\): Next we consider the case \((C, O) \sim B_{29,2} \circ B_{6,3}\) and we will show that \(C\) can be either irreducible or reducible. Recall that the singularity \(B_{29,2} \circ B_{6,3}\) appears in the case that \(C_2\) and \(C_5\) satisfies following three conditions (5):

1. \(C_2\) is irreducible and \(I(C_2, C_5; O) = 10\).
2. \((C_5, O)\) has the multiplicity 3 and the tangent cone consists of a multiple line \(L_1\) of the multiplicity 2 and a single line \(L_2\).
3. The conic \(C_2\) is tangent to the line \(L_1\) at \(O\).

Under the condition \(I(C_2, C_5; O) = 10\), we have generically \((C, O) \sim B_{29,2} \circ B_{6,3}\). The singularity \(B_{29,2}\) is locally irreducible and \(B_{29,2}\) appears for curves of degree \(d \geq 7\) as \(\mu(B_{29,2}) = 28\). Hence we have four possibilities:

1. \(C\): irreducible,
2. \(C = D_9 \cup D_1\),
3. \(C = D_8 \cup D_2\),
4. \(C = D_7 \cup D_3\)

where \(D_d\) is a curve of degree \(d\). But the cases (3) and (4) are impossible. Indeed, if \(C = D_7 \cup D_3\), then either (a) \((D_7, O) \sim B_{29,2}, (D_3, O) \sim B_{6,3}\) or (b) \((D_7, O) \sim B_{29,2} \circ B_{2,1}, (D_3, O) \sim B_{4,2}\). We observe that \(\mu(D_3, O) = 10\) in the case (a) and \(\mu(D_7, O) = 35\) in the case (b) and neither case is possible by the generalized Plücker’s formula. By the same argument, we see that the case (3) is impossible. Hence we have two possibilities:
(i) $C$ is irreducible or
(ii) $C$ consists of a line and a curve of degree 9.
If $C$ has a line component, this line must be defined by \{y = 0\}. In fact, this case is given by the normal forms of $f_2$, $f_5$:

\[
\begin{align*}
f_2(x, y) &= a_{02} y^2 + (a_{11} x + 1) y - k^2 x^2, \\
f_5(x, y) &= (t + a_{02} b_{04}) y^5 + \phi_4(x) y^4 + \phi_3(x) y^3 + \phi_2(x) y^2 + \phi_1(x) y - k^5 x^5,
\end{align*}
\]

where $\phi_1, \phi_2, \phi_3, \phi_4$ take the forms:

\[
\begin{align*}
\phi_4(x) &= (a_{02} b_{13} - a_{02}^2 b_{12} + a_{11} b_{04}) x + b_{04}, \\
\phi_3(x) &= (b_{13} a_{11} - k^2 b_{04} - 2b_{12} a_{02} a_{11} + b_{22} a_{02}) x^2 + b_{13} x, \\
\phi_2(x) &= (a_{02} k^3 + k^2 a_{02} b_{12} - k^2 b_{13} - b_{12} a_{11}^2 + b_{22} a_{11}) x^3 + b_{22} x^2 + b_{12} x, \\
\phi_1(x) &= (a_{11} k^3 + b_{12} k^2 a_{11} - b_{22} k^2) x^4 + (k^3 - k^2 b_{12}) x^3.
\end{align*}
\]

The branched sequence (b), (c) in §3.2 are obtained by degenerating $(C, O)$, fixing the tangent cone of $(C, O)$ and keeping irreducibility of $C$.

**Case** $(C, O) \sim B_{20,5}$: This is the last singularity in the main sequence. We will show that $C$ cannot be irreducible in this case. As $\mu(B_{20,5}) = 76$, the number of irreducible components $r$ of $C$ must be at least 5 by the generalized Plücker’s formula. On the other hand, the singularity $B_{20,5}$ consists of 5 smooth local components. Any two components intersect with intersection multiplicity 4. Thus each local component corresponds to a global component and its degree must be 2, namely a conic.

4. Calculation of $\Delta_C(t)$ I: Non-degenerate case

We divide the calculation of the Alexander polynomial $\Delta_C(t)$ in two cases, according to $(C, O)$ being non-degenerate or not. In this section, we treat the first case.

4.1. Characterization of the adjunction ideal for non-degenerate singularities. In general, the computation of the ideal $J_{O,k,d}$ requires an explicit computation of the resolution of the singularity $(C, O)$. However for the case of non-degenerate singularities, the ideal $J_{O,k,d}$ can be obtained combinatorially by a toric modification. Let $(u, v)$ be a local coordinate system centered at $O$ such that $(C, O)$ is defined by a function germ $f(u, v)$ and the Newton boundary $\Gamma(f; u, v)$ is non-degenerate. Let $Q_1, \ldots, Q_s$ be the primitive weight vectors which correspond to the faces $\Delta_1, \ldots, \Delta_s$ of $\Gamma(f; u, v)$. Let $\pi : \hat{U} \rightarrow U$ be the canonical toric modification and let $\hat{E}(Q_i)$ be the exceptional divisor corresponding to $Q_i$. Recall that the order of zeros of the canonical two form $\pi^*(du \wedge dv)$ along the divisor $\hat{E}(Q_i)$ is simply given by $|Q_i| - 1$ where $|Q_i| = p + q$ for a weight vector $Q_i = t(p_i, q_i)$ (see [11]). For a function germ $g(u, v)$, let $m(g, Q_i)$ be the multiplicity of the pull-back $(\pi^*g)$ on $\hat{E}(Q_i)$. Then

**Lemma 2 ([11])**. A function germ $g \in O_O$ is contained in the ideal $J_{O,k,d}$ if and only if $g$ satisfies following condition:

\[ m(g, Q_i) \geq \frac{k}{d} m(f, Q_i) - |Q_i| + 1, \quad i = 1, \ldots, s. \]

The ideal $J_{O,k,d}$ is generated by the monomials satisfying the above conditions.

We consider the following integers for each singular point $P \in \Sigma(C)$:

\[ \rho_k(P) := \dim V_k(P), \quad \bar{\rho}(k) := \sum_{P \in \Sigma(C)} \rho_k(P) - \dim O(k - 3), \quad \iota_k(P) := \min_{g \in J_{P,k,d}} I(g, f; P), \]

where $\rho_k(P)$ is the dimension of the space of global sections of $O(P)$, $\rho_k(P)$ is the dimension of the local space of sections, and $\iota_k(P)$ is the minimum of the intersection number of $g$ with $f$ at $P$. The numbers $\rho_k(P)$ and $\bar{\rho}(k)$ are related to the multiplicity of the pull-back and the slope of the pull-back, respectively. The number $\iota_k(P)$ is a measure of how well a function germ $g$ avoids the singularity $P$.
where $V_k(P) = \mathcal{O}_P/J_{P,k,d}$. Then the multiplicity $\ell_k$ in the formula (1) of Loeser-Vaquíé is given as

$$\ell_k = \dim \text{Coker} \sigma_k = \bar{\rho}(k) + \dim \text{Ker} \sigma_k,$$

where $\sigma_k$ is defined in §2.1. We consider the integer $\sum_{P \in \Sigma(C)} \iota_k(P)$.

**Proposition 1.** If $\sum_{P \in \Sigma(C)} \iota_k(P) > d(k - 3)$, then

(a) $C$ is irreducible and $\sigma_k$ is injective and $\ell_k = \bar{\rho}(k)$ or

(b) $C$ is reducible.

**Proof.** Suppose $0 \neq g \in \text{Ker} \sigma_k \subset O(k - 3)$. Then by Bézout theorem, we have

$$d(k - 3) \geq I(G, C) \geq \sum_{P \in \Sigma(C)} I(G, C; P) \geq \sum_{P \in \Sigma(C)} \iota_k(P) > d(k - 3)$$

where $G = \{g = 0\}$. This is an obvious contradiction unless $g \mid f$. Thus this implies either $f$ is irreducible and $\sigma_k$ is injective or $f$ is reducible (and $g \mid f$).

4.2. **The singularities** $B_{15,2} \circ B_{10,3}$ and $B_{25,4}$. Now we consider the following two non-degenerate singularities $B_{15,2} \circ B_{10,3}$, and $B_{25,4}$ which appear as the last singularities of the respective degenerations with $C$ being irreducible. We assume that we have chosen local analytic coordinates $(u, v)$ so that

$B_{15,2} \circ B_{10,3}: f(u, v) = u^{25} + u^{10}v^2 + v^5 + (\text{higher terms}),$

$B_{25,4}: f(u, v) = u^{25} + u^4 + (\text{higher terms}).$

The local data are given by the following tables.

### $B_{15,2} \circ B_{10,3}$

| $k$ | $\mathcal{J}_{O,k,10}$ | $\rho_k(O)$ | $\iota_k(O)$ |
|-----|------------------------|--------------|--------------|
| 3   | $\langle u, v \rangle$ | 1            | 5            |
| 4   | $\langle u^3, v \rangle$ | 3            | 15           |
| 5   | $\langle u^5, uv, v^2 \rangle$ | 6          | 23           |
| 6   | $\langle u^7, u^3v, v^2 \rangle$ | 10         | 33           |
| 7   | $\langle u^{10}, u^5v, uv^2, v^3 \rangle$ | 16        | 43           |
| 8   | $\langle u^{12}, u^6v, u^3v^2, v^3 \rangle$ | 21        | 52           |
| 9   | $\langle u^{15}, u^8v, u^5v^2, uv^3, v^4 \rangle$ | 29        | 63           |

### $B_{25,4}$

| $k$ | $\mathcal{J}_{O,k,10}$ | $\rho_k(O)$ | $\iota_k(O)$ |
|-----|------------------------|--------------|--------------|
| 3   | $\langle u, v \rangle$ | 1            | 4            |
| 4   | $\langle u^3, v \rangle$ | 3            | 12           |
| 5   | $\langle u^6, v \rangle$ | 6            | 24           |
| 6   | $\langle u^8, u^2v, v^2 \rangle$ | 10        | 32           |
| 7   | $\langle u^{11}, u^5v, v^2 \rangle$ | 16        | 44           |
| 8   | $\langle u^{13}, u^7v, uv^2, v^3 \rangle$ | 21        | 52           |
| 9   | $\langle u^{16}, u^{10}v, u^3v^2, v^3 \rangle$ | 29        | 62           |
Case \((C, O) \sim B_{15.2} \circ B_{10.3} \text{ and } B_{25.4}\). In this case, we have the inequalities \(\iota_k(O) > 10(k - 3)\) for all \(k = 3, \ldots, 9\) by the local data. Hence \(\tilde{\sigma}_k\) is injective for all \(k\) by Proposition 1 and we obtain the property \((\#)\):
\[
\ell_k = \tilde{\rho}(k) = \begin{cases} 
1 & k = 7, 9, \\
0 & k \neq 7, 9.
\end{cases}
\]
Therefore \(\Delta_C(t) = \Delta_{5.2}(t) = t^4 - t^3 + t^2 - t + 1\).

4.3. Exceptional case: \((C, O) \sim B_{20.5}\). In this section, we consider the last singularity \(B_{20.5}\) which takes place for reducible \(C\). Recall that \(C\) is a union of five conics. We assume that we have chosen local coordinates \((u, v)\) so that \((C, O)\) is defined by
\[
B_{20.5} : f(u, v) = u^{20} + v^5 + (\text{higher terms}),
\]
where we ignore the coefficients of the monomials and other monomials corresponding to other integral points on the Newton boundary.

| \(k\) | \(\mathcal{J}_{O,k,10}\) | \(\rho_k(O)\) | \(\iota_k(O)\) |
|---|---|---|---|
| 3 | \(\langle u^2, v\rangle\) | 2 | 10 |
| 4 | \(\langle u^4, v\rangle\) | 4 | 20 |
| 5 | \(\langle u^6, u^2v, v^2\rangle\) | 8 | 30 |
| 6 | \(\langle u^8, u^4v, v^2\rangle\) | 12 | 40 |
| 7 | \(\langle u^{10}, u^6v, u^2v^2, v^3\rangle\) | 18 | 50 |
| 8 | \(\langle u^{12}, u^8v, u^4v^2, v^3\rangle\) | 24 | 60 |
| 9 | \(\langle u^{14}, u^{10}v, u^6v^2, u^2v^3, v^4\rangle\) | 32 | 70 |

Again we have the inequalities \(\iota_k(O) - 10(k - 3) > 0\) for all \(k = 3, \ldots, 9\). We claim that \(\tilde{\sigma}_k\) is injective for all \(k\). In fact, assuming \(0 \neq g \in \text{Ker } \tilde{\sigma}_k\), we have \(g | f\) by the proof of Proposition 1 and this means \(g\) is a union of conics which are components of \(f\). Consider the factorization \(f = h_1h_2h_3h_4h_5\) where \(\{h_i = 0\}\) is a smooth conic component of \(C\). Then we may assume that
\[
f \overset{\sigma}{\longrightarrow} u^{20} + v^5 + (\text{higher terms}), \quad h_i \overset{\sigma}{\longrightarrow} u^4 + \zeta^iv + (\text{higher terms}), \quad i = 1, \ldots, 5
\]
where \(\zeta = \exp(\pi i / 5)\). Thus suppose that \(g = h_i_1 \cdots h_i_j\). Then \(2j \leq k - 3\) or \(j \leq \lfloor \frac{k-3}{2}\rfloor\) and \(\sigma_k(g)\) must contain \(v^j\) with a non-zero coefficient. This implies that \(j \leq 0, 1, 1, 2, 3, 4\) for \(k = 3, 4, \ldots, 9\) respectively. On the other hand, \(v^j \in \mathcal{J}_{O,k,10}\) implies from the table of \(B_{20.5}\) that \(j \geq 1, 1, 2, 2, 3, 3, 4\) for \(k = 3, \ldots, 9\) respectively. This gives an obvious contradiction. Hence we have
\[
\ell_k = \tilde{\rho}(k) = \begin{cases} 
1 & k = 3, 4, \\
2 & k = 5, 6, \\
3 & k = 7, 8, \\
4 & k = 9.
\end{cases}
\]
Therefore by the formula \((\#)\) in Lemma \(\#\) we obtain the equality:
\[
\Delta_C(t) = (t - 1)^4(t + 1)^4(t^4 - t^3 + t^2 - t + 1)^4(t^4 + t^3 + t^2 + t + 1)^3.
\]
5. Calculation of $\Delta_C(t)$, II: Degenerate cases

Next we calculate the Alexander polynomial of following two degenerate singularities:
- $(B_{2,2}^h)^{B_{2,2}} \circ B_{2,1}$: this is the last singularity of the sequence of (b-i) or (c-i).
- $B_{2,2} \circ B_{2,1} \circ (B_{2,1}^h)^{B_{2,2}}$: this is the last singularity of the sequence of (b-ii) or (c-ii).

5.1. Characterization of the adjunction ideal for degenerate cases. For degenerate singularities, we proceed several toric modifications to obtain their resolutions. Consider an embedded resolution of $(C,O) \subset (\mathbb{C}^2, O)$, $\pi : \tilde{U} \to U$ where $U$ is an open neighborhood of $O$ and let $E_1, \ldots, E_s$ be the exceptional divisors. We put the ideal $\tilde{I}_{O,k,d}$ of $O_O$

$$\tilde{I}_{O,k,d} := \langle M \in O_O \mid M : \text{monomial, } (\pi^* M) \geq \sum_i ([km_i/d] - k_i)E_i \rangle, \quad 1 \leq k \leq d - 1.$$ 

In general, $\tilde{I}_{O,k,d} \subset J_{O,k,d}$ and $\tilde{I}_{O,k,d} = J_{O,k,d}$ if $(C,O)$ is non-degenerate from Lemma 2. If $(C,O)$ is degenerate singularity, there exist several other (non-monomial) polynomials $h_i, i = 1, \ldots, r$ such that $h_i \in \tilde{I}_{O,k,d} \setminus \tilde{I}_{O,k,d}$ and

$$\tilde{I}_{O,k,d} = \langle M, h_i \mid M \in \tilde{I}_{O,k,d}, i = 1, \ldots, r \rangle.$$ 

5.1.1. Formulation of the multiplicities. We recall how the multiplicities of the pull-back of a function after toric modifications along the exceptional divisors can be computed.

Let $D = \{g = 0\}$ be a plane curve and let $P \in D$ be a singular point. Suppose that its Newton boundary $\Gamma(g; u, v)$ consists of $m$-faces $\Delta_1, \ldots, \Delta_m$ where $(u, v)$ is a local coordinates centered at $P$. Then the face function of $g$ with respect to a face $\Delta_i$ takes the form:

$$g_{\Delta_i}(u, v) = cu^{w_i}v^{b_i} \prod_{j=1}^{k_i} (v^{a_i} - \gamma_{i,j} u^{b_i})^{\nu_{i,j}}, \quad c \neq 0$$

where $P_i = t(a_i, b_i)$ is the weight vector corresponding to $\Delta_i$. Let $\{E_0, P_1, \ldots, P_m, E_2\}$ be the vertices of the dual Newton diagram $\Gamma^*(g; u, v)$ where $E_1 = t(1,0)$ and $E_2 = t(0,1)$. Let $\pi_1 : X_1 \to \mathbb{C}^2$ be the toric modification associated with $\{\Sigma_1^*, (u,v), P\}$ where $\Sigma_1^* = \{E_1, Q_1, \ldots, Q_{m'}, E_2\}$ is the canonical regular simplicial cone subdivision of $\{E_1, P_1, \ldots, P_m, E_2\}$ ([10]). Then we can write the divisor $(\pi^*_1 g)$ as

$$\pi^*_1 g = \tilde{D} + \sum_{s=1}^{m'} m(g, Q_s) \tilde{E}(Q_s)$$

where $\tilde{D}$ is the strict transform of $D$ and $\tilde{E}(Q_j)$ is the exceptional divisor corresponding to the vertex $Q_j$. We assume that $P_i = Q_{\nu_i}$ for $i = 1, \ldots, m$. Then the exceptional divisors $\tilde{E}(Q_{\nu_i}) = \tilde{E}(P_i)$ intersects with the strict transform $\tilde{D}$. We take the toric coordinates $(\mathbb{C}^2_{\sigma_{\nu_i}}(u_i, v_i))$ where $\sigma_{\nu_i} = \text{Cone}(Q_{\nu_i}, Q_{\nu_i+1})$ so that $\{u_i = 0\}$ defines $\tilde{E}(Q_{\nu_i}) \cap \mathbb{C}^2_{\sigma_{\nu_i}}$. Then $\tilde{D}$ and the total transform $\pi^*_1 D$ are defined in this coordinate as

$$\tilde{D} : \quad \bar{g}(u_i, v_i) = c_i (v_i - \gamma_{i,j} u_i)^{\nu_{i,j}} + R(u_i, v_i), \quad c_i \neq 0$$

$$\pi^*_1 D : \quad \pi^* g(u_i, v_i) = u_i^{d(P_i; g)} v_i^{d(Q_{\nu_i+1}; g)} \bar{g}(u_i, v_i)$$

where $R \equiv 0$ modulo $(u_i)$. Thus $\xi_{i,j} := (0, \gamma_{i,j})$ is the intersection points of $\tilde{D}$ and $\tilde{E}(Q_{\nu_i})$ for $j = 1, \ldots, k_i$. We take an admissible translated coordinates $(u_i, v'_i)$ with $v'_i = v_i - \gamma_{i,j} + h(u_i)$ in an open neighbourhood of $\xi_{i,j}$ where $h$ is a suitable polynomial with $h(0) = 0$. Suppose that $(\tilde{D}, \xi_{i,j})$ has a non-degenerate singularity with respect to the coordinates $(u_i, v'_i)$ and suppose
that the Newton boundary has a unique face $\Delta_{ij}$ for $j = 1, \ldots, k_i$. (For our purpose, this case is enough to be considered.) Let $S_{ij} = t^i(s_{ij}, t_{ij})$ be the primitive dual vector which corresponds to the face $\Delta_{ij}$ and assume the germ $(\tilde{D}, \xi_{ij})$ is equivalent to the Brieskorn singularity $B_{c_{ij}, d_{ij}}$ with $t_{ij}c_{ij} = s_{ij}d_{ij}$. This means the dual Newton diagram $\Gamma^*(\tilde{g}; u_i, v'_i)$ is given by $\{E_1, S_{ij}, E_2\}$.

We may assume $S_{ij} = T_{ij,k_0}$ for some $k_0 \in \{1, \ldots, m_j\}$. At each point $\xi_{ij}$, we take the toric modification $\pi_{ij} : X_{ij} \to X_1$ with respect to $\{\Sigma^*_t, (u_i, v'_i), \xi_{ij}\}$. These modifications are compatible each other and let $\pi_2 : X_2 \to X_1$ be the composition of these modifications for every $i, j$ so that the exceptional divisors of $\pi_2$ are bijectively corresponding to the vertices of $\Sigma^*_t$, $i = 1, \ldots, m, \ j = 1, \ldots, k_i$. What is necessary to be checked are the multiplicities of $\pi^*g$ and $\pi^*(du \wedge dv)$ along the exceptional divisors $\tilde{E}(T_{ij,k})$ where $\pi : X_2 \to \mathbb{C}^2$ is the composition of $\pi_2 : X_2 \to X_1$ and $\pi_1 : X_1 \to \mathbb{C}^2$. Then we can write:

\[
\begin{align*}
(\pi^*g) &= \tilde{D} + \sum_{s=1}^{m'} m(g, Q_s)\tilde{E}(Q_s) + \sum_{i=1}^m \sum_{j=1}^{k_i} \sum_{k=1}^{m_j} m(g, T_{ij,k})\tilde{E}(T_{ij,k}), \\
(\pi^*K) &= \sum_{s=1}^{m'} k(Q_s)\tilde{E}(Q_s) + \sum_{i=1}^m \sum_{j=1}^{k_i} \sum_{k=1}^{m_j} k(T_{ij,k})\tilde{E}(T_{ij,k})
\end{align*}
\]

where $K = du \wedge dv$ is the canonical two form in the base space.

**Lemma 3.** Under the above situations, the multiplicities are given as follows. Put $T_{ij,k} = t^i(\varepsilon_{ij,k}, \eta_{ij,k})$.

1. The multiplicities $m(g, P_i), m(g, T_{ij,k})$ of $\pi^*g$ along the divisors $\tilde{E}(P_i)$ and $\tilde{E}(T_{ij,k})$ are given by

\[
m(g, P_i) = d(P_i, g), \quad m(g, T_{ij,k}) = \varepsilon_{ij,k}m(g, P_i) + d(T_{ij,k}, \tilde{g}).
\]

2. The multiplicities $k(Q_s), k(T_{ij,k})$ of the pull-back of the canonical two form $K = du \wedge dv$ along the divisors $\tilde{E}(Q_s)$ and $\tilde{E}(T_{ij,k})$ are given by

\[
k(Q_s) = |Q_s| - 1, \quad k(T_{ij,k}) = |T_{ij,k}| - 1 + \varepsilon_{ij,k}k(P_i)
\]

where $|t^i(a, b)| = a + b$.

The proof follows easily from Theorem 3.8 and Proposition 7.2, Chapter III of [10].

### 5.2. Generalization of Lemma 2

**Lemma 4.** Under the above assumptions, a germ $\varphi \in \mathcal{O}_P$ is contained in the ideal $J_{P,k,d}$ if and only if $\varphi$ satisfies:

1. $m(\varphi, P_i) \geq \lfloor \frac{k}{d} m(g, P_i) \rfloor - k(P_i)$ for $i = 1, \ldots, m$, and

2. $m(\varphi, S_{ij}) \geq \lfloor \frac{k}{d} m(g, S_{ij}) \rfloor - k(S_{ij})$ for $j = 1, \ldots, k_i$.

Note that there are no conditions on other exceptional divisors $\tilde{E}(T_{ij,k})$. 

Proof. The proof is almost parallel to that of Lemma 2 of [11]. Assume that \( \varphi \) satisfies the conditions (1) and (2). It is enough to show that

\[
(2 - \text{bis}) \quad m(\varphi, T_{i,j,k}) \geq \left( \frac{k}{d} m(g, T_{i,j,k}) \right) - k(T_{i,j,k}), \quad j = 1, \ldots, k_i, \quad k = 1, \ldots, m_j.
\]

Note that the condition (2) is equivalent to

\[
(2) \quad m(\varphi, S_{i,j}) > \frac{k}{d} m(g, S_{i,j}) - (|S_{i,j}| + s_{i,j} k(P_i)) \quad \text{for} \quad j = 1, \ldots, k_i.
\]

First we observe that \( m(g, T_{i,j,0}) = m(g, P_i) \) and \( m(g, T_{i,j,m_{j+1}}) = 0 \). Take \( T_{i,j,k} \) for \( k < k_0 \) for example. We can write \( T_{i,j,k} = \alpha_k S_{i,j} + \beta_k T_{i,j,0} \) for some positive rational numbers \( \alpha_k, \beta_k \).

Note that

\[
|T_{i,j,k}| = \alpha_k |S_{i,j}| + \beta_k |T_{i,j,0}| = \alpha_k |S_{i,j}| + \beta_k,
\]

\[
m(g, T_{i,j,k}) = \alpha_k m(g, S_{i,j}) + \beta_k m(g, T_{i,j,0}).
\]

Here the second equality follows as \( \Delta(S_{i,j}, \pi_1^i g) \cap \Delta(T_{i,j,0}, \pi_1^i g) \neq \emptyset \) by the admissibility of the canonical subdivision \( \Sigma_{i,j}^* \). Thus we have

\[
m(\varphi, T_{i,j,k}) \geq \alpha_k m(\varphi, S_{i,j}) + \beta_k m(\varphi, T_{i,j,0})
\]

\[
> \alpha_k \left( \frac{k}{d} m(g, S_{i,j}) - (|S_{i,j}| + s_{i,j} k(P_i)) \right) + \beta_k \left( \frac{k}{d} m(g, T_{i,j,0}) - (1 + k(P_i)) \right)
\]

\[
= \frac{k}{d} m(g, T_{i,j,k}) - (|T_{i,j,k}| + \varepsilon_{i,j,k} k(P_i))
\]

as \( \varepsilon_{i,j,k} = \alpha_k s_{i,j} + \beta_k \) by the equality \( T_{i,j,k} = \alpha_k S_{i,j} + \beta_k T_{i,j,0} \). This inequality is equivalent:

\[
m(\varphi, T_{i,j,k}) \geq \left( \frac{k}{d} m(g, T_{i,j,k}) \right) - k(T_{i,j,k}).
\]

For \( T_{i,j,k} \) with \( k > k_0 \), the argument is similar. Hence we have \( \varphi \in J_{P,k,d} \). \( \square \)

Now we consider the ideal \( J_{P,k,d} \) in more detail. Take \( \varphi \in \mathcal{O}_P \). We compute the multiplicity of \( \varphi \) along the divisors \( E(P_i) \) and \( E(S_{i,j}) \). We divide our consideration into the two cases:

1. \( \varphi \) is a monomial,
2. \( \varphi \) is a polynomial (non-monomial).

First we see the case (1) and we put \( \varphi(u, v) = u^\alpha v^\beta \). As \( \pi_1^i \varphi \) is also a monomial in \( u_i, v_i \), we can check easily following

\[
m(\varphi, P_i) = d(P_i, \varphi) = a_i \alpha + b_i \beta, \quad m(\varphi, S_{i,j}) = s_{i,j} m(\varphi, P_i).
\]

Next we consider the case (2). We can write \( \varphi(u, v) = \varphi_{P_i}(u, v) + R(u, v) \) where \( R(u, v) \) consist of monomials of degree strictly greater than \( d(P_i, \varphi) \). If \( \Delta(\varphi, P_i) \) is zero dimensional, then the multiplicities \( m(\varphi, P_i) \) and \( m(\varphi, S_{i,j}) \) are equal to that of the monomial \( \varphi_{P_i}(u, v) \). If \( \Delta(\varphi, P_i) \) is one dimensional, then the face function \( \varphi_{P_i}(u, v) \) can be written by

\[
\varphi_{P_i}(u, v) = c_i u^\alpha v^\beta \prod_{j=1}^{\kappa_i} (v^{\delta_{i,j}} - d_{i,j} u^{b_i})^{\mu_{i,j}}, \quad c_i, \delta_{i,j} \neq 0.
\]

Then the multiplicities \( m(\varphi, P_i) \) is given by

\[
m(\varphi, P_i) = a_i \alpha + b_i \beta + a_i b_i \sum_{j=1}^{\kappa_i} \mu_{i,j}.
\]
In the admissible translated coordinates \((u_i, v'_i)\), the function \(\pi_1^* \varphi\) is written by

\[
\pi_1^* \varphi(u_i, v'_i) = c_i u_i^{m(\varphi, P_i)} \tilde{\varphi}(u_i, v'_i),
\]

\[
\tilde{\varphi}(u_i, v'_i) = \prod_{j=1}^{\kappa_i} (v'_i + (\gamma_{i,j} - \delta_{i,j}) - h(u_i))^{\mu_{i,j}} + \tilde{R}(u_i, v'_i)
\]

where \(\tilde{R}(u_i, v'_i) \equiv 0 \mod (u_i)\). Thus we obtain

\[
m(\varphi, S_{i,j}) = \begin{cases} s_{i,j} m(\varphi, P_i) & \text{if } \delta_{i,j} \neq \gamma_{i,j} \text{ for all } j, \\ s_{i,j} m(\varphi, P_i) + d(S_{i,j}, \tilde{\varphi}) & \text{if } \delta_{i,j} = \gamma_{i,j} \text{ for some } j. \end{cases}
\]

Note that the multiplicity \(d(S_{i,j}, \tilde{\varphi})\) depends on the form \(h, R\) and \(S_{i,j}\).

5.3. The case of \((B^2_0)^{B_{5,2}} \circ B_{2,1}\). By the local classification in [5], this singularity \((B^2_0)^{B_{5,2}} \circ B_{2,1}\) appears when the associated curves \(C_2\) and \(C_5\) satisfies following conditions:

1. \(C_2\) is irreducible and \(I(C_2, C_5; O) = 10\).
2. The multiplicity of \((C_5, O)\) is 3 and the tangent cone of \(C_5\) consists of a line \(L_1\) with multiplicity 2 and a single line \(L_2\).
3. The conic \(C_2\) is tangent to the line \(L_1\) at \(O\).

Suppose that \(C_2\) and \(C_5\) satisfies the above conditions. Then we may assume that the defining polynomials of \(C_2\) and \(C_5\) are the following forms:

\[
f_2(x, y) = y + a_{20} x^2 + a_{11} x y + a_{02} y^2, \quad a_{20} \neq 0,
\]

\[
f_5(x, y) = b_{05} y^5 + ((a_{02} b_{12} + a_{11} b_{04}) x + b_{04}) y^4 + ((2 b_{12} a_{02} a_{11} + a_{20} b_{04}) x^2 + 2 a_{02} b_{12} x) y^3 \\
+ ((2 a_{20} a_{02} b_{12} + b_{12} a_{11}^2) x^3 + 2 b_{12} a_{11} x^2 + b_{12} x) y^2 \\
+ (2 a_{11} b_{12} a_{20} x^4 + 2 b_{12} a_{02} x^3) y + a_{02}^2 b_{12} x^5
\]

where \(b_{12} \neq 0\) and \(a_{20} + b_{12}^2 \neq 0\) in general. If \(a_{20} + b_{12}^2 = 0\), \((C, O)\) has the same type of singularity but \(C\) is not irreducible and has a line component which is defined by \(\{y = 0\}\). Now we take a local coordinates \((u, v)\) of the following type so that

\[
x = u, \quad y = v + \varphi(u), \quad \varphi(u) = -a_{20} u^2 + \cdots, \quad a_{20} \neq 0,
\]

\[
f_2(u, v + \varphi(u)) = v + c_1 v^5 + \cdots,
\]

\[
f_5(u, v + \varphi(u)) = b_{04} v^4 + b_{12} u v^2 + c_2 u^{10} + \text{(higher terms)}, \quad b_{12}, c_2 \neq 0,
\]

\[
f(u, v + \varphi(u)) = v^5 + u^2 (b_{12} v^2 + c_2 u^9) + \text{(higher terms)}.
\]

Then the Newton boundary \(\Gamma(f; u, v)\) consists of two faces \(\Delta_i\) \((i = 1, 2)\) so that the respective face functions are given by

\[
f_{\Delta_1}(u, v) = v^4 (v + b_{12} u^2), \quad f_{\Delta_2}(u, v) = u^2 (b_{12} v^2 + c_2 u^9)^2.
\]

Note that \(f(u, v)\) is degenerate on \(\Delta_2\). We take the canonical toric modification \(\pi_1 : X_1 \to \mathbb{C}^2\) with respect to \(\{\Sigma_1^*, (u, v), O\}\) where \(\Sigma_1^*\) is the canonical regular simplicial cone subdivision with vertices \(\{E_1, Q_1, \ldots, Q_6, E_2\}\) where

\[
Q_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad Q_4 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad Q_5 = \begin{pmatrix} 2 \\ 9 \end{pmatrix}, \quad Q_6 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}
\]
and the weight vectors $Q_2$ and $Q_5$ correspond to the faces $\Delta_1$ and $\Delta_2$ respectively. Then we can write the divisor $(\pi_1^*f)$ as

$$(\pi_1^*f) = \tilde{C} + \sum_{i=1}^{6} m(f,Q_i) \hat{E}(Q_i),$$

where $\tilde{C}$ is the strict transform of $C$ and intersects only with the exceptional divisors $\hat{E}(Q_2)$ and $\hat{E}(Q_5)$. We can see that $\tilde{C}$ is smooth and intersects transversely at $\tilde{C} \cap \hat{E}(Q_2)$ but $\tilde{C}$ has the singularity at the intersection $\tilde{C} \cap \hat{E}(Q_5)$. Let $\xi = \tilde{C} \cap \hat{E}(Q_5)$. In the toric coordinates $(u_1,v_1)$ of $\mathbb{C}^2$ with $\tau = \text{Cone}(Q_5,Q_6)$ (see [10] for the notations), $\xi = (0, -c_2/b_1)$. To see the singularity $(\tilde{C}, \xi)$, we take the admissible translated toric coordinates $(u_1,v'_1)$ with $v'_1 = v_1 + c_2/b_1 + h(u_1)$ where $h$ take the form $h(u_1) = q_1u_1 + q_2u_1^2$. Then we can see that

$$\pi_1^*f(u_1,v'_1) = c_1u_1^2(u'_1 + \beta u_1^3 + \text{(higher terms)})$$

and $(\tilde{C}, \xi) \sim B_{5,2}$. Now we take the second toric modification $\pi_2 : X_2 \rightarrow X_1$ with respect to $\{\Sigma_2, (u_1, v'_1), \xi\}$ where $\Sigma_2$ is the canonical regular simplicial cone subdivision with vertices $\{E_1, T_1, \ldots, T_4, E_2\}$ where

$$T_1 = \left( \frac{1}{1} \right), \quad T_2 = \left( \frac{2}{1} \right), \quad T_3 = \left( \frac{2}{3} \right), \quad T_4 = \left( \frac{1}{3} \right),$$

and the weight vector $T_3$ corresponds to the unique face of $\pi_1^*f(u_1,v'_1)$. Note also the exceptional divisor which corresponds to $E_1$ is nothing but the exceptional divisor $\hat{E}(Q_5)$ in the previous modification $\pi_1$. Then we have

$$(\pi^*f) = 5\hat{E}(Q_1) + 10\hat{E}(Q_2) + 14\hat{E}(Q_3) + 18\hat{E}(Q_4) + 40\hat{E}(Q_5) + 20\hat{E}(Q_6)$$

$$+ 42\hat{E}(T_1) + 44\hat{E}(T_2) + 90\hat{E}(T_3) + 45\hat{E}(T_4)$$

and we consider two polynomials $h_2(u,v)$ and $r_2(u,v)$ which are defined by $h_2(u,v) = b_{12}v^2 + c_2u^9$ and $r_2(u,v) = h_2(u,v) - \frac{b_{12}v^2}{c_2}uv^3$. Then we can see by a direct computation

$$\pi_1^*h_2(u_1,v'_1) = u_1^{14}(d_3v'_1 + d_4u_1 + \text{(higher terms)}),$$

$$\pi_1^*r_2(u_1,v'_1) = u_1^{18}(d'_3v'_1 + d'_4u_1^2 + \text{(higher terms)}),$$

$$m(h_2,Q_2) = 4, \quad m(h_2,Q_5) = 18, \quad m(h_2,T_3) = 38,$$

$$m(r_2,Q_2) = 4, \quad m(r_2,Q_5) = 18, \quad m(h_2,T_3) = 40.$$

**Assertion 1.** The adjunction ideals $\mathcal{J}_{O,k,10}$ are given by

$$\mathcal{J}_{O,3,10} = \langle u,v \rangle, \quad \mathcal{J}_{O,4,10} = \langle u^3, v \rangle, \quad \mathcal{J}_{O,5,10} = \langle u^3, uv, v^2 \rangle, \quad \mathcal{J}_{O,6,10} = \langle u^7, u^3v, v^2 \rangle,$$

$$\mathcal{J}_{O,7,10} = \langle u^{10}, u^5v, uv^2, v^3 \rangle, \quad \mathcal{J}_{O,8,10} = \langle u^{12}, u^7v, u^3v^2, v^3, h_2^{(2,0)} \rangle,$$

$$\mathcal{J}_{O,9,10} = \langle u^{14}, u^{10}v, u^5v^2, uv^3, v^4, r_2^{(4,0)} \rangle$$

where $h_2^{(2,0)}(u,v) := u^2h_2(u,v)$ and $r_2^{(4,0)}(u,v) := u^4r_2(u,v)$.

The proof follows from Lemma 3 and Lemma 4 and by an easy computation.

Thus we have $\rho_8(O) = 21$, $\rho_9(O) = 29$ and

$$\hat{\rho}(k) = \begin{cases} 1 & k = 7, 9, \\ 0 & k \neq 7, 9, \end{cases} \quad \nu_k(O) > 10(k - 3), \quad 3 \leq k \leq 9.$$
Assertion 2. The map $\sigma_k$ is injective for all $k = 3, \ldots, 9$.

Proof. Recall that $C$ can be either irreducible or reducible in this case. As $\iota_k(O) > 10(k - 3)$, if $C$ is irreducible, then the assertion follows from Proposition 1.

Assume $C$ is not irreducible. We have seen in the previous argument in §3.4, $C$ has two irreducible components of respective degree 1 and 9. Namely we can write $C = C_1 \cup C_9$ where $C_1 = \{y = 0\}$. Suppose that there exists a non-zero $g \in \text{Ker } \sigma_k \subset O(k - 3)$. As $\iota_k(O) > 10(k - 3)$, $g$ divides $f$ by the proof of Proposition 1. This is possible only if $k \geq 4$ and $\deg g = 1$. By the assumption, we have $g = cy$ with $c \neq 0$. As $y = v + \varphi(u)$, we see that $g$ can not be in the ideal $\mathcal{J}_{O,k,10}$ for $k \geq 5$, as $v \notin \mathcal{J}_{O,k,10}$ by Assertion 1. This implies that $\sigma_k$ is injective for $k \neq 4$. Assume $k = 4$. As $a_{20} \neq 0$, $\text{ord}_u \varphi(u) = 2$ and $\mathcal{J}_{O,4,10} = \langle u^3, v \rangle$, again we see that $v + \varphi(u) \notin \mathcal{J}_{O,4,10}$. This is a contradiction for $g \in \text{Ker } \sigma_4$ and the proof is completed.

Therefore we obtain the property $(\natural)$: $\iota_k = 1$ for $k = 7, 9$ and $\iota_k = 0$ otherwise. Thus the reduced Alexander polynomial is given by $\Delta_C(t) = t^4 - t^3 + t^2 - t + 1$ for the case $(C, O) \sim (B_{9,2}^2 B_{8,2} \circ B_{2,1})$.

5.4. The case of $B_{20,2} \circ B_{2,1} \circ (B_{2,1}^2).$ By local classification [5], this singularity appears in the case that the associated curves $C_2$ and $C_5$ satisfies following conditions:

1. $C_2$ is irreducible and $I(C_2, C_5; O) = 10$.
2. The multiplicity of $(C_5, O)$ is 3 and the tangent cone of $C_5$ at $O$ consists of a line $L_1$ with multiplicity 2 and a single line $L_2$.
3. The conic $C_2$ is tangent to the line $L_1$ at $O$.

Suppose that $C_2$ and $C_5$ satisfies the above conditions. Then we may assume that the defining polynomials of $C_2$ and $C_5$ are following forms:

$$f_2(x, y) = y + a_{20}x^2 + a_{11}xy + a_{02}y^2, \quad a_{20} \neq 0$$

$$f_5(x, y) = b_{05}y^5 + a_{12}b_{12}xy^4 + 2a_{02}b_{12}x^2 + 2a_{11}b_{12}^2 + (a_{11}x + 1)y^3$$

$$+ \left( \frac{1}{27} b_{12} + \frac{54a_{02}b_{20} + 27a_{11}^2}{27} x^3 + 2a_{11}b_{12} x^2 + b_{12} x \right) y^2$$

$$+ \left( \frac{2}{27} b_{12}x^3 + \frac{27a_{20}}{27} (a_{11}x + 1) \right) y + \frac{1}{27} b_{12}^2 a_{20} + 4b_{12}^2 x^5$$

where $b_{12} \neq 0$ and $b_{12}^2 + 9a_{20} \neq 0$ in general. If $b_{12}^2 + 9a_{20} = 0$, $C$ has the line component which is defined by $\{y = 0\}$. Now we take a local coordinates $(u, v)$ of the following type so that

$$x = u, \quad y = v + \varphi(u), \quad \varphi(u) = -a_{20}u^2 + \cdots, \quad a_{20} \neq 0,$$

$$f_2(u, v + \varphi(u)) = v + \psi(u) = v + \beta_7 u^7 + (\text{higher terms}), \quad \beta_7 \neq 0,$$

$$f_5(u, v + \varphi(u)) = b_{05} v^5 + b_{12}uv(v + \frac{4}{27} b_{12} u^2) + c_4 u^{18} + (\text{higher terms}), \quad b_{12} \neq 0,$$

$$f(u, v + \varphi(u)) = v^2 + d_1 u^2(v + d_2 u^2)^2 + \beta_5 u^{35} + (\text{higher terms}), \quad d_1, d_2 \neq 0.$$

By an explicit calculation, we have $d_2 = \frac{4}{9} b_{12}$ and $d_2 + a_{20} \neq 0$. (If $d_2 + a_{20} = 0$, $f$ becomes a non-reduced polynomial.) Then the Newton boundary $\Gamma(f; u, v)$ consists of two faces $\Delta_1$ and $\Delta_2$ so that their face functions are given by

$$f_{\Delta_1}(u, v) = v^2(v + d_1 u^2)(v + d_2 u^2)^2, \quad f_{\Delta_2}(u, v) = u^6(d_1 d_2 u^2 + \beta_5 u^{29}).$$
Note that \( f(u, v) \) is degenerate on \( \Delta_1 \). We take the canonical toric modification \( \pi_1 : X_1 \to \mathbb{C}^2 \) with respect to \( \{ \Sigma_1^*, (u, v), O \} \) where \( \Sigma_1^* \) is the canonical regular simplicial cone subdivision with vertices

\[
E_1, \quad Q_k = \left( \frac{1}{k} \right) \quad (1 \leq k \leq 14), \quad Q_{15} = \left( \frac{2}{29} \right), \quad Q_{16} = \left( \frac{1}{15} \right), \quad E_2
\]

where \( Q_2 \) and \( Q_{15} \) are the weight vectors of the faces \( \Delta_1 \) and \( \Delta_2 \) respectively. Then the divisor \( (\pi_1^* f) \) is given by

\[
(\pi_1^* f) = \tilde{C} + \sum_{i=1}^{16} m(f, Q_i) \tilde{E}(Q_i),
\]

where \( \tilde{C} \) is the strict transform of \( C \) and intersects only with the exceptional divisors \( \tilde{E}(Q_2) \) and \( \tilde{E}(Q_{15}) \). We can see that \( \tilde{C} \) is smooth at \( \tilde{C} \cap \tilde{E}(Q_{15}) \) and the intersection is transverse. On the other hand, \( \tilde{C} \) intersects with \( \tilde{E}(Q_2) \) at two points \( \xi_{1,1}, \xi_{1,2} \) where \( \xi_{1,1} = (0, -d_1), \xi_{1,2} = (0, -d_2) \) in the toric coordinates \( (u_1, v_1) \) of the chart \( \mathbb{C}^2_x \) with \( \tau = \text{Cone}(Q_2, Q_3) \). Note that \( (\tilde{C}, \xi_{1,1}) \) is smooth and the intersection with \( \tilde{E}(Q_2) \) is transverse at \( \xi_{1,1} = (0, -d_1) \).

On the other hand, \( (\tilde{C}, \xi_{1,2}) \) has singularity. To see the singularity \( (\tilde{C}, \xi_{1,2}) \), we take the admissible translated coordinates \( (u_1, v_1') \) with \( v_1' = v_1 + d_2 + h(u_1) \) where \( h \) takes the form \( h(u_1) = q_1 u + q_2 u^2 \). Then we see that \( \pi_1^* f(u_1, v_1') = cu_1^1(v_1'^2 + \beta u_1^2) + (\text{higher terms}) \) and \( (\tilde{C}, \xi_{1,2}) \sim B_{5,2} \). Now we take the second toric modification \( \pi_2 : X_2 \to X_1 \) with respect to \( \{ \Sigma_2^*, (u_1, v_1'), \xi_{1,2} \} \) where \( \Sigma_2^* \) is the canonical regular simplicial cone subdivision with vertices

\[
E_1, \quad T_1 = \left( \frac{1}{1} \right), \quad T_2 = \left( \frac{1}{2} \right), \quad T_3 = \left( \frac{2}{5} \right), \quad T_4 = \left( \frac{1}{3} \right), \quad E_2
\]

where the weight vector \( T_3 \) corresponds to the unique face of \( \Gamma(\pi_1^* f; u_1, v_1') \). Then we have

\[
(\pi^* f) = 5\tilde{E}(Q_1) + \sum_{i=2}^{14} 2(i + 3)\tilde{E}(Q_i) + 70\tilde{E}(Q_{15}) + 35\tilde{E}(Q_{16})
\]

\[
+ 12\tilde{E}(T_1) + 14\tilde{E}(T_2) + 30\tilde{E}(T_3) + 15\tilde{E}(T_4).
\]

\[
(\pi^* K) = \tilde{E}(Q_1) + \sum_{i=2}^{14} i\tilde{E}(Q_i) + 30\tilde{E}(Q_{15}) + 15\tilde{E}(Q_{16}) + 3\tilde{E}(T_1) + 4\tilde{E}(T_2) + 10\tilde{E}(T_3) + 5\tilde{E}(T_4)
\]

and we consider two polynomials \( h_1(u, v) \) and \( r_1(u, v) \) which are defined by \( h_1(u, v) = v + d_2 u^2 \) and \( r_1(u, v) = h_1(u, v) = \frac{a_{12}}{d_2} u^3 \). Then

\[
\pi_1^* h_1(u_1, v_1') = u_1^2(e_3 v_1' + e_4 u_1) + (\text{higher terms}),
\]

\[
\pi_1^* r_1(u_1, v_1) = u_1^2(e_3 v_1' + e_4 u_1^2) + (\text{higher terms}),
\]

\[
m(h_1, Q_2) = 2, \quad m(h_1, Q_{15}) = 4, \quad m(h_1, T_3) = 6,
\]

\[
m(r_1, Q_2) = 2, \quad m(r_1, Q_{15}) = 4, \quad m(r_1, T_3) = 8.
\]

**Remark 2.** The Alexander polynomial does not change on the irreducible component of the configuration space of the fixed topological type of the singularity. Therefore for the practical computation, it is easier to choose some explicit values. We take \( b_{05} = b_{12} = a_{02} = \)
$a_{20} = a_{11} = 1$. Then we have
\[ f_2(x, y) = y^2 + (x + 1)y + x^2 \]
\[ f_5(x, y) = y^5 + xy^4 + 2(x^2 + x)y^3 + \left(\frac{85}{27}x^3 + 2x^2 + x\right)y^2 + \left(\frac{58}{27}x^4 + \frac{58}{27}x^3\right)y + \frac{31}{27}x^5 \]
\[ \varphi(u) = -u^2 + u^3 - 2u^4 + 4u^5 - 9u^6 + 11u^7 - \frac{183}{2}u^8 + 316u^9 - 1079u^{10} + \frac{7259}{2}u^{11} \]
\[- \frac{801559}{64}u^{12} + \frac{2872109}{64}u^{13} - \frac{5348333}{32}u^{14}. \]

Assertion 3. Under the above situation,
(a) The ideals $\mathcal{J}_{O,k,10}$ are given by
\[ \mathcal{J}_{O,3,10} = \langle u, v \rangle, \quad \mathcal{J}_{O,4,10} = \langle u^2, v \rangle, \quad \mathcal{J}_{O,5,10} = \langle u^3, uv, v^2 \rangle, \quad \mathcal{J}_{O,6,10} = \langle u^6, u^2v, v^2 \rangle, \]
\[ \mathcal{J}_{O,7,10} = \langle u^{10}, u^4v, u^2v^2, v^3, h_1^{(1,1)} \rangle, \quad \mathcal{J}_{O,8,10} = \langle u^{13}, u^5v, u^3v^2, uv^3, v^4, h_1^{(2,1)}, h_1^{(0,2)} \rangle, \]
\[ \mathcal{J}_{O,9,10} = \langle u^{17}, u^7v, u^5v^2, u^3v^3, uv^4, v^5, r_1^{(3,1)}, r_1^{(1,2)}, h_1^{(0,3)} \rangle, \]
where $h_1^{(r,s)}(u,v) := u^rv^sh_1(u,v)$ and $r_1^{(r,s)}(u,v) := u^rv^sr_1(u,v)$.
(b) The kernel of $\sigma_k$ are given by
\[ \text{Ker} \sigma_3 = \langle 0 \rangle, \quad \text{Ker} \sigma_4 = \langle y \rangle, \quad \text{Ker} \sigma_5 = \langle y^2, xy \rangle, \quad \text{Ker} \sigma_6 = \langle y^2, y^3 \rangle, \]
\[ \text{Ker} \sigma_7 = \langle y^2f_2 \rangle, \quad \text{Ker} \sigma_8 = \langle y^3f_2 \rangle, \quad \text{Ker} \sigma_9 = \langle 3yf_5 - 2b_{12}xyf_2 - c_0y^2f_2^2 \rangle. \]
with $c_0 = \frac{81a_{11}a_{20}}{4b_{12}(9a_{20} + 4b_{12}^2)}$.

Proof. The assertion (a) follows from Lemma 3 and Lemma 4. We consider the assertion (b).
By the choice of the local coordinates $(u,v)$, we have relations:
\[ x = u, \quad y = v + \varphi(u) = v - a_{20}u^2 + \cdots, \]
\[ f_2(u, v + \varphi(u)) = v + \psi(u) = v + \beta_7u^7 + \text{(higher terms)}, \quad \beta_7 \neq 0, \]
\[ f_5(u, v + \varphi(u)) = b_{05}u^5 + b_{12}uv(v + \frac{4}{27}b_{12}^2u^2) + c_4u^{18} + \text{(higher terms)}. \]
Put $\varphi(u) = \sum_{j=2}^{\infty} \alpha_j u^j$ with $\alpha_2 = -a_{20}$. We define
\[ \text{ord} \mathcal{J}_{O,k,10} := \min \{ \text{ord}_{(u,v)} h \mid h \in \mathcal{J}_{O,k,10} \}. \]
Thus for any $g \in \text{Ker} \sigma_k$, we have
\[ \text{ord}_{(x,y)} g = \text{ord}(u,v)\sigma_k(g) \geq \text{ord} \mathcal{J}_{O,k,10}. \]

Case $k = 4$: Ker $\sigma_4 = \langle y \rangle$.

Proof. The inclusion $\langle y \rangle \subset \text{Ker} \sigma_4$ holds by the definition of $\sigma_4$. For any $g \in \text{Ker} \sigma_4 \subset O(1)$, writing $g(x, y) = c_1 + c_2x + c_3y,$
\[ \sigma_4(g)(u, v) = c_1 + c_2u + c_3(v + \varphi(u)) \in \mathcal{J}_{O,4,10} = \langle u^2, v \rangle. \]
Hence we have $c_1 = c_2 = 0$ and Ker $\sigma_4 \subset \langle y \rangle$.

Case $k = 5$: Ker $\sigma_5 = \langle y^2, xy \rangle$.

Proof. First we show that $y^2, xy \in \text{Ker} \sigma_5$. By the definition of $\sigma_5$, we have
\[ \sigma_5(y^2) = (v + \varphi(u))^2 = v^2 - 2a_{20}u^2v + a_{20}^2u^4 + \text{(higher terms)} \in \mathcal{J}_{O,5,10} \]
\[ \sigma_5(xy) = u(v + \varphi(u)) = uv - a_{20}u^3 + \text{(higher terms)} \in \mathcal{J}_{O,5,10} \]
as $\mathcal{J}_{O,5,10} = \langle u^3, uv, v^2 \rangle$. Next we show that $\text{Ker } \sigma_5 \subset \langle y^2, xy \rangle$. Take $g \in \text{Ker } \sigma_5 \subset O(2)$. As $\text{ord} \mathcal{J}_{O,5,10} = 2$, we can write $g(x,y) = c_1 x^2 + c_2 xy + c_3 y^2$ by \[2\] and

$$\sigma_5(g)(u,v) = c_1 u^2 + c_2 uv + c_3 v^2 + \text{(higher terms)} \in \mathcal{J}_{O,5,10} = \langle u^3, uv, v^2 \rangle.$$ 
Hence we have $c_1 = 0$ and $\text{Ker } \sigma_5 = \langle y^2, xy \rangle$. 

\[\square\]

**Case** $k = 6$: $\text{Ker } \sigma_6 = \langle yf_2, y^3 \rangle$.

**Proof.** First we show that $yf_2, y^3 \in \text{Ker } \sigma_6$. By the definition of $\sigma_6$, we have

$$\sigma_6(yf_2) = (v + \varphi(u))(v + \psi(u))$$
$$= v^2 - a_20 u^2 v - a_20 \beta_7 u^9 + \text{(higher terms)} \in \mathcal{J}_{O,6,10}$$

$$\sigma_6(y^3) = (v + \varphi(u))^3 = v^3 - 3a_20 u^2 v^2 + 3a_20^2 u^4 v - a_3^2 u^6 + \text{(higher terms)} \in \mathcal{J}_{O,6,10}$$
as $\mathcal{J}_{O,6,10} = \langle u^6, u^2 v, v^2 \rangle$.

Next we show that $\text{Ker } \sigma_6 \subset \langle yf_2, y^3 \rangle$. Take $g \in \text{Ker } \sigma_6 \subset O(3)$. As $\sigma_6(g) \in \mathcal{J}_{O,6,10}$, we can write

$$\sigma_6(g)(u,v) = g(u,v + \varphi(u)) = a_1(u,v) u^6 + a_2(u,v) u^2 v + a_3(u,v) v^2$$
where $a_i \in O_O$ ($i = 1, 2, 3$). Define $g'/(u,v)$ by the above right side polynomial. Then we see that

$$I(y,g;O) = \text{ord}_u g'(u,-\varphi(u)) \geq 4.$$ 
On the other hand, if $y$ does not divide $g$, $I(y,g;O) \leq 3$ by Bézout’s theorem which is an obvious contradiction. Therefore $y$ divides $g$. Thus we can write $g(x,y) = yg_2(x,y)$ where $g_2 \in O(2)$. Dividing $g_2$ by $f_2$ as a polynomial of $x$, we can write $g_2$ as $g_2 = c_0 f_2 + (c_1 + c_2 y)x + c_3 y^2 + c_4 y + c_5$ for some constants $c_0, \ldots, c_5$. As $yf_2, y^3 \in \text{Ker } \sigma_6$, we need to have $g((c_1 + c_2 y)x + c_4 y + c_5) \in \text{Ker } \sigma_6$. By a simple computation, we conclude $c_1 = c_2 = c_4 = c_5 = 0$ and

$$g(x,y) = c_0 yf_2(x,y) + c_3 y^3 \in \langle yf_2, y^3 \rangle.$$ 

\[\square\]

**Case** $k = 7$: $\text{Ker } \sigma_7 = \langle y^2 f_2 \rangle$.

**Proof.** First we show that $y^2 f_2 \in \text{Ker } \sigma_7$. By the definition of $\sigma_7$, we have

$$\sigma_7(y^2 f_2)(u,v) = (v + \varphi(u))^2(v + \psi(u))$$
$$= v^3 - 2a_20 u^2 v^2 + a_20^2 u^4 v + a_20^2 \beta_7 u^{11} + \text{(higher terms)} \in \mathcal{J}_{O,7,10}$$
as

$$\mathcal{J}_{O,7,10} = \langle u^{10}, u^4 v, u^2 v^2, v^3 \rangle, \quad \mathcal{J}_{O,7,10} = \langle u^{10}, u^4 v, u^2 v^2, v^3, h_1^{(1,1)} \rangle$$

where $h_1^{(1,1)}(u,v) := uv + (d_2 v^2)$. Next we show that $\text{Ker } \sigma_7 \subset \langle y^2 f_2 \rangle$. Take $g \in \text{Ker } \sigma_7 \subset O(4)$ and we can write $\sigma_7(g)$ as

$$\sigma_7(g)(u,v) = \sum_{i \geq 0} g_i(u)v^i, \quad \text{ord}_u g_0(u) \geq 10, \quad \text{ord}_u g_1(u) \geq 3, \quad \text{ord}_u g_2(u) \geq 1.$$ 
Then we see that $I(g,y;O) = \text{ord}_u \sigma_7(g)(u,-\varphi(u)) \geq 5$ and by Bézout’s theorem, $y$ divides $g$. Similarly we can see that we have $I(g,f_2;O) = \text{ord}_u \sigma_7(g)(u,-\psi(u)) \geq 10$ and again by Bézout’s theorem, we conclude $f_2$ divides $g$. Thus we can write $g(x,y) = yf_2(c_0 + c_1 x + c_2 y)$ for some $c_0, c_1, c_2 \in \mathbb{C}$. The assumption $g, y^2 f_2 \in \text{Ker } \sigma_7$ implies that $g(x,y) - c_2 y^2 f_2 =
\((c_0 + c_1x)\phi_2 \in \text{Ker} \sigma_7\). Thus we have \(\sigma_7(c_0\phi_2)(u,0) = -c_0a_{20}\beta_7u^8 + \cdots \in \mathcal{J}_{O,7,10}\). Therefore \(c_0 = 0\) as \(\text{ord}_u \sigma_7(c_0\phi_2)(u,0) \geq 10\). Moreover we have

\[
\sigma_7(\phi) \equiv \sigma_7(c_1\phi_2) \equiv c_1 uv(v - a_{20}u^2) \mod \mathcal{J}_{O,7,10}.
\]

As \(d_2 + a_{20} \neq 0\), we see that \(uv(v - a_{20}u^2) \notin \mathcal{J}_{O,7,10}\). Hence we have \(c_1 = 0\) and we conclude \(g(x,y) = c_2 y^2 f_2\).

**Case** \(k = 8\): Ker \(\sigma_8 = \langle y^3 f_2 \rangle\).

**Proof.** First we show that \(y^3 f_2 \in \text{Ker} \sigma_8\). By the definition of \(\sigma_8\), we have

\[
\sigma_8(y^3 f_2)(u,v) = (v + \omega(u))^3(v + \omega(u))
\]

\[
= v^4 - 3a_{20}u^2v^2 + 3a_{20}^2u^4 - a_{20}^3u^8 + (\text{higher terms}).
\]

As \(\mathcal{J}_{O,8,10} = \langle u^{13}, u^5v, u^3v^2, u^2, v^4 \rangle\), we see that \(\sigma_8(y^3 f_2) \in \mathcal{J}_{O,8,10}\).

Next we show that \(\text{Ker} \sigma_8 \subset \langle y^3 f_2 \rangle\). Take \(g \in \text{Ker} \sigma_8 \subset O(3)\) and write \(\sigma_8(g)\) as

\[
\sigma_8(g)(u,v) = \sum_{i \geq 0} g_i(u)v^i, \quad \text{ord}_u g_0(u) \geq 13, \quad \text{ord}_u g_1(u) \geq 4, \quad \text{ord}_u g_2(u) \geq 2.
\]

As \(I(g;y;O) = \text{ord}_u \sigma_8(g)(u,\omega(u)) \geq 6\), we see that \(y\) divides \(g\) by Bézout’s theorem. Similarly we can see that \(I(g, f_2; O) = \text{ord}_u \sigma_8(g)(u, -\omega(u)) \geq 11\), we see that \(f_2\) divides \(g\). Hence we have \(g(x,y) = yf_2g'(x,y)\) for some \(g' \in O(2)\). We put \(g'(x,y) = c_02y^2 + r(x,y)\) where \(r(x,y) = c_{11}x^2 + c_{10}y + c_{20}x^2 + c_{10}x + c_{00}\). As \(y^3 f_2 \in \text{Ker} \sigma_8\), we have \(yf_2r(x,y) \in \text{Ker} \sigma_8\).

Consider the expression

\[
\sigma_8(yf_2r)(u,v) = \sum_{i \geq 0} \psi_i(u)v^i
\]

We can see that \(\psi_2(u) = \psi_0(u) = (c_{00}a_{20} - c_{10})u + u^2 \tilde{\psi}_2(u)\). Thus \(\psi_0 = 0\) and \(c_{10} = 0\) as \(\text{ord}_u \psi_0(u) \geq 2\). Now we have

\[
\psi_0(u) = -a_{20}\beta_7(a_{20}c_{01} - c_{20})u^{11} + a_{20}^2(d(a_{20}c_{01} - c_{20}) + \beta_7(a_{11}c_{01} - c_{11}))u^{12} + u^{13}\tilde{\psi}_0(u).
\]

As \(\text{ord}_u \psi_0(u) \geq 13\) by the assumption \(\sigma_8(yf_2r) \in \mathcal{J}_{O,8,10}\), the coefficients of \(u^{11}, u^{12}\) in \(\psi_0(u)\) must vanish. Therefore \(a_{20}c_{01} - c_{20} = 0\) and \(a_{11}c_{01} - c_{11} = 0\). Thus we conclude \(r(x,y) = c_{01}(y + a_{20}x^2 + a_{11}xy)\). Consider the weight vector \(P = \ell(1,2)\) for the variables \(u, v\). Then we compute the leading term of \(\sigma_8(yf_2r)(u,v)\) with respect to \(P\):

\[
\sigma_8(yf_2r)(u,v) = c_{01}v^2(v - a_{20}u^2)
\]

As the lowest degree of elements in \(\mathcal{J}_{O,8,10}\) is 6 and they are generated by \(h_1^{(2,1)} = u^2(v + d_2u^2)\) and \(h_1^{(0,2)} = v^2(v + d_2u^2)\). Thus we must have the equality \(\sigma_8(yf_2r)(u,v)|_{v = -d_2u^2} = 0\). This implies that \(c_{01} = 0\) as \(d_2 + a_{20} \neq 0\).

**Case** \(k = 9\): Ker \(\sigma_9 = \langle 3y_5 - 2b_{12}xyf_2 - c_0y_2 f_2 \rangle\) where \(c_0 = \frac{81a_{11}a_{20}}{4b_{12}(9a_{20} + 4b_{12}^2)}\).

**Proof.** The proof of this case is most computational. As the Alexander polynomial is a topological invariant, we can choose any polynomial in the connected component of the moduli
space. Thus we use the polynomial in Remark 2. (We take \( b_{05} = b_{12} = a_{02} = a_{20} = a_{11} = 1 \).)

\[
f_2(x, y) = y^2 + (x + 1)y + x^2
\]

\[
f_5(x, y) = y^5 + xy^4 + 2(x^2 + x)y^3 + \left( \frac{85}{27} x^3 + 2x^2 + x \right) y^2 + \left( \frac{58}{27} x^4 + \frac{58}{27} x^3 \right) y + \frac{31}{27} x^5
\]

\[
\varphi(u) = -u^2 + u^3 - 2u^4 + 4u^5 - 9u^6 + \frac{111}{4} u^7 - \frac{183}{2} u^8 + 316u^9 - 1079u^{10} + \frac{7259}{2} u^{11}
\]

\[
- \frac{801559}{64} u^{12} + \frac{2872109}{64} u^{13} - \frac{5348333}{32} u^{14}.
\]

and then \( h_1(u, v) = v + \frac{4}{9} u^2, r_1(u, v) = (v + \frac{4}{9} u^2) - \frac{13}{9} u^3 \) and \( c_0 = \frac{81}{92} \).

First we show that \( p(x, y) := 3yf_3 - 2xyf_2 - \frac{81}{12} y^2f_2^2 \) is in the kernel of \( \sigma_9 \). Put \( p_1(x, y) = 3yf_5 - 2xyf_2^2 \). We observe that

\[
\sigma_9(p_1)(u, v) \equiv uv(v - u^2)(v + \frac{4}{9} u^2) + u^2v^2 \left( 2v - \frac{5}{9} u^2 \right) \mod \mathcal{J}_{O,9,10}.
\]

\[
\sigma_9\left( \frac{81}{52} y^2 f_2^2 \right) = \frac{81}{52} v^2 (v - u^2)^2 \mod \mathcal{J}_{O,9,10}.
\]

\[
\sigma_9(p)(u, v) \equiv - \left( 1 + \frac{13}{4} u \right) r_1(3,1) + \left( 1 + \frac{151}{26} u \right) r_1(1,2) - \frac{81}{52} h_{10}^{(0,3)} \mod \mathcal{J}_{O,9,10}.
\]

Thus we conclude \( \sigma_9(p) \in \mathcal{J}_{O,9,10} \), as \( \mathcal{J}_{O,9,10} = \langle u^{17}, v^7, u^5v^2, u^3v^3, uv^4, v^5, r_1^{(3,1)}, r_1^{(1,2)}, h_{10}^{(0,3)} \rangle \).

Next we will show that \( \ker \sigma_9 \) is generated by \( p \). Take \( g \in \ker \sigma_9 \subset O(6) \) with \( g \neq 0 \). As \( \text{ord} \mathcal{J}_{O,9,10} = 4 \), we have \( \text{ord} (x,y) g = 4 \). Hence we can put

\[
g(x, y) = \sum_{4 \leq r + s \leq 6} c_{rs} x^r y^s, \quad \sigma_9(g)(u, v) = \sum_{i \geq 0} \psi_1(u) u^i,
\]

where

\[
\psi_0(u) = \sum_{i \geq 4} a_i u^i, \quad \psi_1(u) = \sum_{i \geq 3} b_i u^i, \quad a_i, b_i \text{ are linear polynomials in } c_{rs}.
\]

By the assumption \( \sigma_9(g) \in \mathcal{J}_{O,9,10} \), we have \( \text{ord}_u \psi_0(u) \geq 17 \) and \( \text{ord}_u \psi_1(u) \geq 5 \). Hence we solve the 15-equations \( a_i = 0, \, 4 \leq i \leq 16 \) and \( b_i = 0, \, i = 3, 4 \) in \( c_{rs} \) for the lexicographical order. After solving these equations, \( g \) takes the form:

\[
g(x, y) = c_{06} y^6 + (c_{15} x + c_{05}) y^5 + \left( (2c_{15} - \frac{1}{2} c_{05}) x^2 + (2c_{15} - c_{05}) x + \frac{1}{2} c_{05} \right) y^4
\]

\[
+ \left( \frac{4}{27} c_{06} - \frac{56}{27} c_{05} + 3c_{15} \right) x^3 + (2c_{15} - c_{05}) x^2 + (c_{15} - c_{05}) x \right) y^3
\]

\[
+ \left( \frac{4}{27} c_{06} - \frac{85}{54} c_{05} + 2c_{15} \right) x^4 + (2c_{15} + \frac{4}{27} c_{06} - \frac{56}{27} c_{05}) x^3 \right) y^2
\]

\[
+ \left( c_{15} - \frac{29}{27} c_{05} + \frac{4}{27} c_{06} \right) x^5 y
\]

where \( g \) has still 3-parameters \( c_{05}, c_{15}, c_{06} \). We get the equality

\[
\sigma_9(g)(u, v) \equiv (c_{15} - c_{05}) u v^3 + \left( \frac{4}{27} c_{06} - c_{15} + \frac{25}{27} c_{05} \right) u^3 v^2 + (c_{05} - \frac{4}{27} c_{06}) u v^5
\]

\[
+ \frac{1}{2} c_{05} v^4 + (2c_{15} - 3c_{05}) u^2 v^3 + \left( \frac{4}{27} c_{06} - c_{15} + \frac{77}{54} c_{05} \right) u^4 v^2 \mod \mathcal{J}_{O,9,10}.
\]
Now we consider the weight vector \( P = ^t(1, 2) \) for variables \( u, v \) as in the previous case. Then 
\[
\deg_P \sigma_9(g)(u, v) = 7 \quad \text{and} \quad 
\sigma_9(g)_P(u, v) = (c_{15} - c_{05})uv^3 + \left( \frac{4}{27}c_{06} - c_{15} + \frac{25}{27}c_{05}\right)u^3v^2 + \left( \frac{2}{27}c_{05} - \frac{4}{27}c_{06}\right)u^5v,
\]
As the lowest degree of the generators of \( J_{O,9,10} \) is also 7 and they are \( r_1^{(3,1)} \) and \( r_1^{(1,2)} \). Thus we must have 
\[
\sigma_9(g)_P(u, v) = (a \cdot r_1^{(3,1)} + b \cdot r_1^{(1,2)})_P, \quad \text{for some } a, b \in \mathbb{C}.
\]
Thus we see that \( \sigma_9(g)_P(u, -\frac{4}{9}u^2) = 0 \). This gives the equality 
\[
5c_{05} - 6c_{15} + 2c_{06} = 0
\]
and we eliminate \( c_{05} \) using the above equality and then \( a, b \) are solved as follows.
\[
a = \frac{1}{5}c_{15} - \frac{2}{5}c_{06}, \quad b = -\frac{1}{5}c_{15} + \frac{2}{5}c_{06}.
\]
We put \( g_1 = \sigma_9(g) - (a \cdot r_1^{(3,1)} + b \cdot r_1^{(1,2)}) \). Then we see that \( \deg_P g_1 = 8 \). Thus we can write 
\[
g_1(u, v) = a' \cdot r_1^{(3,1)} + b' \cdot r_1^{(1,2)} + c' \cdot h_1^{(0,3)}, \quad \text{for some } a', b', c' \in \mathbb{C}.
\]
Again we need to have \( g_1(u, -\frac{4}{9}u^2) = 0 \) which gives the equality 
\[
15c_{15} + 22c_{06} = 0.
\]
Eliminating the parameter \( c_{15} \), we finally obtain the expression 
\[
g(x, y) = \frac{c_{06}}{675}y \left( 675y^5 - (990x + 1458)y^4 - (1251x^2 + 522x + 729)y^3 + (154x^2 - 522x + 468)xy^2 + (415x^4 + 1144x^3)y + 676x^5 \right)
\]
and we conclude that 
\[
g(x, y) = \frac{52}{t^5}c_{06} p(x, y).
\]

The proof of Assertion 3 is now completed.

Now we are ready to compute the Alexander polynomial for the case \( (C, O) \sim B_{29,2} \circ B_{2,1} \circ (B_{2,1}^2)^B5,2 \). By above assertions, we have \( \rho_7(O) = 15 \), \( \rho_8(O) = 20 \) and \( \rho_9(O) = 28 \) hence we obtain the property (\( \sharp \)): \( \ell_k = 1 \) for \( k = 7, 9 \) and \( \ell_k = 0 \) otherwise. Therefore we have \( \Delta_C(t) = t^4 - t^3 + t^2 - t + 1 \) by Lemma 1. Thus the proof of Theorem 1 is completed.

5.5. Linear torus curve. The singularity \( B_{50,2} \) appears also as a linear torus curve of type \((5,2)\):

\[
C: \quad f_5(x, y)^2 - y^{10} = 0
\]
with \( I(f_5, y; O) = 5 \) ([2]). In this case, \( C \) consists of two smooth quintics and the Alexander polynomial is given by following ([2]):

\[
\Delta_C(t) = \frac{(t^{10} - 1)}{t + 1}.
\]
5.6. **Proofs of Corollary 1 and Corollary 2.** The assertion of Corollary 1 is an immediate consequence of the Sandwich principle. The assertion of Corollary 2 is a result of [2]. In fact, we only need to observe that the equivalence class of such torus curves correspond bijectively to the partitions of 10 by locally intersection numbers of $C_2$ and $C_5$. In particular, such a curve degenerates into an irreducible torus curve with a unique singularity $B_{50,2}$ which corresponds to the partition $10 = 10$.

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