Network synchronization with an adaptive strength

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Abstract

In this paper, new schemes to synchronize linearly or nonlinearly coupled chaotic systems with an adaptive coupling strength are proposed. Unlike other adaptive schemes, which synchronize coupled chaotic systems to a special trajectory (or an equilibrium point) of the uncoupled node by adding negative feedbacks adaptively; here, adaptive schemes for the coupling strength are used to synchronize coupled chaotic systems without knowing the synchronization trajectory. Moreover, in many applications, the state variables are not observable; instead, some functions of the states can be observed. How to synchronize coupled systems with the observed data is of great significance. In this paper, synchronization of nonlinearly coupled chaotic systems with an adaptive coupling strength is also discussed. The validity of those schemes are proved rigorously. Moreover, simulations show that by choosing some proper parameter $\alpha$, the coupling strength obtained by adaptation could be much smaller. It means that chaos oscillators can be easily synchronized with a very weak coupling.

Key words: Synchronization, Adaptive Coupling Strength, Nonlinearly Coupling, Time-varying Coupling Matrix.

1 Introduction

Recently, an increasing interest has been devoted to the study of complex networks (see Strogaze 2001 - Wang et.al 2002). Among them, synchronization of coupled complex networks has received more attentions, because synchronization not only can explain many
natural phenomena (see Mirollo and Strogatz 1990), but also have many applications, such as image processing, secure communication (see Wei and Jia 2002 - Lu and Chen 2004) and so on.

Generally, linearly coupled systems can be described as:

\[
\dot{x}_i(t) = f(x_i(t), t) + c \sum_{j \neq i} a_{ij} \Gamma \left[ x_j(t) - x_i(t) \right] \quad i = 1, 2, \cdots, N
\]

where \( x_i(t) = (x_1^i(t), \cdots, x_n^i(t))^T \in \mathbb{R}^n \); \( f(\cdot, t) : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n \) is continuous. The outer coupling matrix \( A \) satisfies: \( a_{ij} \geq 0 \), for \( i \neq j \), and let \( a_{ii} = -\sum_{j=1, j \neq i}^{N} a_{ij} \); while the inner coupling matrix \( \Gamma = \text{diag}(\gamma_1, \cdots, \gamma_n) \) is positive definite. \( c \) denotes the coupling strength.

If \( \lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0 \) for all \( i, j = 1, 2, \cdots, N \), where \( \| \cdot \| \) denotes some norm, then coupled systems (1) are said to be synchronized completely. In the following, we will investigate complete synchronization. Hitherto, many approaches and criteria to ensure synchronization have been derived (Lu et. al 2004 - Pikovsky, Rosenblum, and Kurths 2001).

Moreover, in practice, the state variables \( x_i(t) \) may be unobservable; instead, we can observe \( g(x_i(t)) \), where \( g(\cdot) \) is a monotone increasing function. How to synchronize coupled systems with the observed data is of great significance. In (Chen, Zhu 2007), following nonlinearly coupled systems

\[
\dot{x}_i(t) = f(x_i(t), t) + c \sum_{j \neq i} a_{ij} \left[ g(x_j(t)) - g(x_i(t)) \right] \quad i = 1, 2, \cdots, N
\]

were proposed, where \( g(x_j(t)) = (g_1(x_j^1(t)), \cdots, g_n(x_j^n(t)))^T \) and every \( g_i(\cdot) \) is a nonlinear monotone increasing function.

It is pointed out in Pikovsky, Rosenblum, and Kurths 2001 that if the coupling strength is larger than a critical value \( c^* \), then coupled systems (1) can be synchronized. In fact, \( c^* \) depends not only on the coupling matrix but also depends on the dynamical behavior of function \( f(x(t), t) \) in the uncoupled system \( \dot{x}(t) = f(x(t), t) \). Thus, the critical value \( c^* \)
(suitable for all systems with various $f(x(t), t)$) is much larger than the coupling strength $c$ needed for specified coupled systems (see Lu and Chen 2004).

On the other hand, it is well known that a chaotic attractor typically has embedded within it an infinite number of unstable periodic orbits. In (Ott, Grebogi and Yorke 1990), it is shown that one can convert a chaotic attractor to any one of a large number of possible attracting time-periodic motions by making only small time-dependent perturbations of an available system parameter. On the other hand, if the attractor is not chaotic but is, say, periodic, then small parameter perturbations can only change the orbit slightly.

Therefore, how to synchronize a large number of specified chaotic oscillators (for example, Lorenz oscillator or other oscillators) with a relatively small coupling strength is an interesting problem.

For this purpose, we replace coupled systems (1) and (2) by

$$\dot{x}_i(t) = f(x_i(t), t) + c(t) \sum_{j \neq i} a_{ij} \Gamma \left[ x_j(t) - x_i(t) \right] \quad i = 1, 2, \ldots, N \quad (3)$$

$$\dot{x}_i(t) = f(x_i(t), t) + c(t) \sum_{j \neq i} a_{ij} \left[ g(x_j(t)) - g(x_i(t)) \right] \quad i = 1, 2, \ldots, N \quad (4)$$

with an adaptive coupling strength $c(t)$.

The adaptation of parameters is widely used in the signal processing and other research fields. Recently, the adaptive approach is used to synchronize master-salve systems or mutually coupled systems to a specified trajectory of the uncoupled node by adding negative feedbacks (see Chen and Zhou 2006 - Boccaletti 2006). For example, in (Chen and Zhou 2006 and Zhou et.al 2006), the authors investigated how to synchronize coupled systems:

$$\dot{x}_i(t) = f(x_i(t), t) + h_i(x_1(t), x_2(t), \ldots, x_N(t)) \quad 1 \leq i \leq N \quad (5)$$

to a specified trajectory $\dot{s}(t) = f(s(t), t)$ by adding controls $-d_i(t) \left[ x_i(t) - s(t) \right]$ with adaptation rule $\dot{d}_i = k_i \|x_i(t) - s(t)\|_2^2$.  

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However, this adaptation rule does not suit complete synchronization. Because, the synchronization trajectory, which generally is not a trajectory of the uncoupled system, is unknown.

In this paper, new schemes to synchronize chaotic oscillators with an adaptive coupling strength are proposed. It is revealed that a large number of chaotic oscillators (including Lorenz oscillators, Chen’s oscillators, Rössler oscillators and Chua’s circuits) can be synchronized with a very small coupling strength. It means that one can synchronize a large number of chaotic oscillators with very weak coupling.

This paper is organized as follows: In section 2, some necessary definitions, lemmas, and hypotheses are given; in section 3, adaptive schemes for linearly and nonlinearly coupled system are given; in section 4 and section 5, some simulations are given to verify our theoretical results; while in section 6, an example is given to illustrate that: for coupled periodic systems the coupling strength should be very large; and the paper is concluded in section 7.

2 Preliminaries

In this section, some definitions, denotations and lemmas throughout the paper are presented.

**Definition 1**  
The set \( S = \{ (x_1^T, x_2^T, \cdots, x_N^T) | x_i = x_j; \ i, j = 1, 2, \cdots, N \} \) is called the synchronization manifold.

**Definition 2**  
Matrix \( A = (a_{ij})_{i,j=1}^N \) of order \( N \) is said to satisfy Condition A1, if

1. \( a_{ij} \geq 0, i \neq j \), \( a_{ii} = -\sum_{j=1,j\neq i}^N a_{ij}, i = 1, 2, \cdots, N \)

2. Eigenvalues of \( A \) are all negative except an eigenvalue 0 with multiplicity 1.

Furthermore, if \( A \in A1 \), and \( a_{ij} = a_{ji}, i \neq j \), then we say \( A \in A2 \).
Definition 3   Suppose that $\Delta = \text{diag}\{\delta_1, \delta_2, \cdots, \delta_n\}$ is a diagonal matrix, and $\varpi > 0$. $f : R^n \times R^+ \rightarrow R^n$ is continuous. we say $f \in \text{QUAD}(\Delta, \varpi)$, if and only if

$$(x - y)^T[f(x, t) - f(y, t)] - (x - y)^T \Delta (x - y) \leq -\varpi (x - y)^T(x - y) \quad (6)$$

holds for any $x, y \in R^n$.

Remark 1   It is some control for the case $(x - y)^T[f(x, t) - f(y, t)] > 0$ (roughly speaking, $f(x, t) - f(y, t)$ and $x - y$ have same sign), $||f(x, t) - f(y, t)||$ should be less than $(x - y)^T(\Delta - \varpi I_n)(x - y)$. Instead, if $(x - y)^T[f(x, t) - f(y, t)] \leq 0$, then inequality (6) is satisfied automatically.

Lemma 1   If a matrix $A_{N \times N} \in A1$. Then (see Wu 2005, Lu Chen 2006):

1. $[1, 1, \cdots, 1]^T$ is the right eigenvector of $A$. corresponding to eigenvalue 0 with multiplicity 1;

2. The left eigenvector of $A$: $\xi = [\xi_1, \xi_2, \cdots, \xi_N]^T \in R^N$ corresponding to eigenvalue 0 has following properties: it is non-zero and its multiplicity is 1; all $\xi_i \geq 0$, $i = 1, 2, \cdots, N$. More precisely, $A$ is irreducible if and only if all $\xi_i > 0$, $i = 1, 2, \cdots, N$.

In the following, we always assume that $A$ is irreducible and $\sum_{i=1}^{N} \xi_i = 1$, $\xi_i > 0$, which is called the normalized left eigenvector. It is clear that if $A_{N \times N} \in A2$, then $\xi_i = \frac{1}{N}$ for $i = 1, \cdots, N$.

3 Main Results

In this section, we give the main results about how to design synchronization algorithms with an adaptive coupling strength for linearly and nonlinearly coupled systems.
3.1 Linearly coupled systems

In this part, we discuss synchronization of linearly coupled systems with the coupling strength adaptively.

3.1.1 Constant coupling matrix

In this part, we investigate the case that the coupling matrix is time independent. Consider following linearly coupled systems with an adaptive coupling strength:

\[ \dot{x}_i(t) = f(x_i(t), t) + c(t) \sum_{j \neq i} a_{ij} \Gamma \left[ x_j(t) - x_i(t) \right] \]  

(7)

the coupling matrix \( A = (a_{ij}) \) is not assumed to be symmetric.

For any irreducible (asymmetric) matrix \( A_{N \times N} \in \mathbf{A1} \), Denote \( \Xi = \text{diag}\{\xi_1, \cdots, \xi_N\} \), \( U = \Xi - \xi \xi^T \), where \( \xi = (\xi_1, \cdots, \xi_N)^T \in \mathbb{R}^N \) is the normalized left eigenvector corresponding to eigenvalue 0. It is easy to check that \(-U \in \mathbf{A2}\).

Furthermore, denote \( X(t) = (x_1^T(t), \cdots, x_N^T(t))^T \), \( F(X(t)) = (f(x_1(t), t)^T, \cdots, f(x_N(t), t)^T)^T \), \( \Delta = I_N \otimes \Delta \), \( A = A \otimes \Gamma \), and \( U = U \otimes I_n \), \( \Xi = \Xi \otimes I_n \), where \( \otimes \) is the Kronecker product. Then, (7) can be rewritten in a compact form:

\[ \dot{X}(t) = F(X(t)) + c(t)AX(t) \]  

(8)

The following Lemma gives a necessary and sufficient condition whether a coupled system is synchronized. It is clear that the left eigenvector \( \xi \) plays a key role to prove synchronization.

**Lemma 2** \( X(t) \in \mathbf{S} \) if and only if 

\[ X^T(t)UX(t) = 0 \]  

(9)
In fact, for any \( X(t) = (x_1^T(t), \ldots, x_N^T(t))^T, Y(t) = (y_1^T(t), \ldots, y_N^T(t))^T, \) we have (also see Wu 2005)

\[
X^T(t)U Y(t) = \frac{1}{2} \sum_{i,j=1}^{N} \xi_i \xi_j (x_i(t) - x_j(t))^T (y_i(t) - y_j(t))
\]

(10)

Now, we propose a new scheme to synchronize linearly coupled systems with an adaptive coupling strength and prove the following theorem.

**Theorem 1**  Suppose \( A \in A1 \) is irreducible and \( f \in QUAD(\Delta, \varpi) \). Then, following coupled systems with an adaptive coupling strength \( c(t) \):

\[
\begin{align*}
\dot{X}(t) &= F(X(t)) + c(t)AX(t), \\
\dot{c}(t) &= -\frac{\alpha}{2}X^T(t)\Xi A X(t)
\end{align*}
\]

(11)

can finally achieve synchronization with a relatively small coupling strength, where \( c(0) = 0 \) and \( \alpha > 0 \).

**Proof:** Pick a sufficiently large constant \( c > 0 \) and define a Lyapunov function as:

\[
V(X(t)) = \frac{1}{2}X^T(t)UX(t) + \frac{1}{\alpha}(c - c(t))^2
\]

(12)

Then, noting \( UA = \Xi A \), we have

\[
\frac{dV(X(t))}{dt} = X(t)^T U [F(X(t)) + c(t)AX(t)] + (c - c(t))X^T(t)\Xi A X(t)
\]

\[= X(t)^T U[F(X(t)) - \Delta X(t)] + X^T(t)U\Delta X(t) + cX^T(t)\Xi A X(t)
\]

By the identity (10) and the assumption \( f \in QUAD(\Delta, \varpi) \), we have

\[
X^T(t)U[F(X(t)) - \Delta X(t)]
\]

\[= \sum_{i,j=1; i \neq j}^{N} \xi_i \xi_j [x_i(t) - x_j(t)]^T [f(x_i(t), t) - f(x_j(t), t)] - \Delta (x_i(t) - x_j(t))
\]

\[\leq -\varpi \sum_{i,j=1; i \neq j}^{N} \xi_i \xi_j [x_i(t) - x_j(t)]^T [x_i(t) - x_j(t)] = -\varpi X^T(t)UX(t)
\]
Therefore,

\[
\frac{dV(t)}{dt} \leq -\varpi X(t)^T UX(t) + X^T(t)U \Delta X(t) + cX^T(t)\Xi A X(t) \tag{13}
\]

Write \( \tilde{x}_j(t) = (x_1^j(t), \cdots, x_N^j(t))^T \) for \( j = 1, 2, \cdots, n \), we have

\[
X^T(t)U(\Delta + cA)X(t) = \sum_{j=1}^{n} \tilde{x}_j^T(t)\delta_j U \tilde{x}_j(t) + c \sum_{j=1}^{n} \gamma_j \tilde{x}_j^T(t) \Xi A \tilde{x}_j(t) \tag{14}
\]

It can be seen that \( \Xi A + A^T \Xi \) is a symmetric matrix with negative diagonal and row-sum zero. Now, let \( v_1, \cdots, v_N \) be the normalized eigenvectors of the matrix \( \frac{1}{2}(\Xi A + A^T \Xi) \) with corresponding eigenvalues \( \lambda_1 = 0 > \lambda_2 \geq \cdots \geq \lambda_N \). Moreover, by \( \tilde{x}_j^*(t) \) denote the projection of \( \tilde{x}_j(t) \) on the subspace \( L \) spanned by \( v_2, \cdots, v_N \). Then,

\[
c \sum_{j=1}^{n} \tilde{x}_j^T(t)\gamma_j \Xi A \tilde{x}_j(t) = \frac{c}{2} \sum_{j=1}^{n} \tilde{x}_j^T(t)\gamma_j (\Xi A + A^T \Xi) \tilde{x}_j(t) \\
= \frac{c}{2} \sum_{j=1}^{n} \tilde{x}_j^*(t)\gamma_j (\Xi A + A^T \Xi) \tilde{x}_j^*(t) \leq c\lambda_2 \sum_{j=1}^{n} \gamma_j \tilde{x}_j^*T(t) \tilde{x}_j^*(t) \tag{15}
\]

\[
\sum_{j=1}^{n} \tilde{x}_j^T(t)\delta_j U \tilde{x}_j(t) = \sum_{j=1}^{n} \tilde{x}_j^*T(t) \delta_j U \tilde{x}_j^*(t) \tag{16}
\]

Substituting into (13), we have

\[
\frac{dV(t)}{dt} \leq -\varpi X(t)^T UX(t) + \sum_{j=1}^{n} \tilde{x}_j^*T(t)\delta_j U \tilde{x}_j^*(t) + c\lambda_2 \sum_{j=1}^{n} \gamma_j \tilde{x}_j^*T(t) \tilde{x}_j^*(t) \tag{17}
\]

It is clear that if \( c \) is sufficient large, then

\[
\sum_{j=1}^{n} \tilde{x}_j^*T(t)\delta_j U \tilde{x}_j^*(t) + c\lambda_2 \sum_{j=1}^{n} \gamma_j \tilde{x}_j^*T(t) \tilde{x}_j^*(t) < 0 \tag{18}
\]

In summary, we have \( \frac{dV(t)}{dt} \leq -\varpi X(t)^T UX(t) \leq 0 \).

Similarly, it can be seen that \( \dot{c}(t) \geq 0 \) and \( \dot{c}(t) = 0 \) if and only if \( X \in S \), so \( c(t) > 0 \) for \( t > 0 \).
It is obvious that $\dot{V} = 0$ if and only if $X \in S$. According to the well-known Lyapunov-LaSall type theorem for functional differential equations (see Kuang 1993), the trajectory of coupled systems, starting with arbitrary initial value, converges asymptotically to the largest invariant set $H_1$ contained in $H_2 = \{ \dot{V}(t) = 0 \}$ as $t \to +\infty$, where $H_1 = \{ [X, c]^T : X^T U X = 0, c = c_0 \in \mathbb{R}^+ \}$. Therefore, $X^T(t) U X(t) \to 0$ and $c(t) \to c_0$ for some constant $c_0$.

Theorem 1 is proved completely.

**Remark 2** Adaptive algorithm is often used in many research fields. Many authors use $-d_i \left[ x_i(t) - s(t) \right]$ as the negative feedback adaptation, where $\dot{d}_i = k_i \| x_i(t) - s(t) \|^2_2$, $k_i$ are positive constants and $s(t)$ is a special solution (or an equilibrium point) of the uncoupled system. Here, the synchronization state is unknown. Therefore, previous negative feedback adaptation methods are invalid. Theorem 1 provides a new adaptive algorithm, which succeeds in the synchronization of complex networks with an adaptive coupling strength.

### 3.1.2 Unknown constant coupling matrix

In some physical coupled systems, the coupling matrix may be unknown, though we know $A \in A_1$. Can we design an adaptive algorithm to synchronize coupled systems with an adaptive coupling strength for a unknown coupling matrix? In this subsection, we will give an affirmative answer by proving the following theorem.

**Theorem 2** Suppose that the unknown coupling matrix $A \in A_1$ is irreducible and $f \in QUAD(\Delta, \varpi)$. Then, following coupled systems with an adaptive coupling strength $c(t)$:

\[
\begin{align*}
\dot{X}(t) &= F(X(t)) + c(t) A X(t), \\
\dot{c}(t) &= -\frac{\alpha}{2} X^T(t) \tilde{A} X(t)
\end{align*}
\]  

(19)

can achieve synchronization with a relatively small coupling strength, where $\tilde{A} = \tilde{A} \otimes I_n$, $\tilde{A} \in A_2$ is any irreducible matrix, $c(0) = 0$ and $\alpha > 0$. 

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**Proof:** Define a slightly different Lyapunov function

\[
V_1(X(t)) = \frac{1}{2} X^T(t) U X(t) + \frac{\eta}{\alpha}(c - c(t))^2
\]  

(20)

where \(\eta > 0\) is a constant. Then, noting \(UA = \Xi A\), we have

\[
\frac{dV_1(X(t))}{dt} = X(t)^T U [F(X(t)) + c(t) A X(t)] + \eta(c - c(t)) X^T(t) \tilde{A} X(t)
\]

\[
= X(t)^T U [F(X(t)) - \Delta X(t)] + X^T(t) U \Delta X(t)
\]

\[
+ c(t) X^T(t) \Xi A X(t) + \eta(c - c(t)) X^T(t) \tilde{A} X(t)
\]

\[
\leq -\omega X(t)^T U X(t) + X^T(t) \left( U \Delta + c \eta \tilde{A} \right) X(t)
\]

\[
+ c(t) X^T(t) \left( \Xi A - \eta \tilde{A} \right) X(t)
\]

Now, pick a sufficiently large constant \(c\) and a sufficiently small \(\eta\) such that

\[
\begin{align*}
& U \Delta + c \eta \tilde{A} \leq 0 \\
& \{\Xi A\}^* - \eta \tilde{A} \leq 0
\end{align*}
\]

(21)

Therefore, \(\frac{dV_1(X(t))}{dt} \leq -\omega X(t)^T U X(t) \leq 0\). By similar arguments used in Theorem 1, we can prove Theorem 2.

### 3.1.3 Time-varying coupling matrix

In practice, the coupling matrix is time-dependent, which means that the coupling matrix changes along with time. Therefore, it is natural to investigate following linearly coupled systems with a time-varying coupling matrix:

\[
\dot{X}(t) = F(X(t)) + c(t) A(t) X(t)
\]

(22)

The following theorem for time-varying coupling can be proved with the same arguments used in the proof of Theorem 1.
Theorem 3  Suppose that $f \in QUAD(\Delta, \varpi)$, the coupling matrix $A(t) \in A1$ is irreducible and has the same left eigenvector $\xi$ corresponding to eigenvalue 0 for all $t$. If the largest non-zero eigenvalue of the matrix $\Xi A(t) + A^T(t) \Xi$ satisfies $\lambda_2(t) \leq \lambda < 0$ for all $t$. Then, following coupled systems with an adaptive coupling strength $c(t)$:

$$
\begin{align*}
\dot{X}(t) &= F(X(t)) + c(t)A(t)X(t), \\
\dot{c}(t) &= -\frac{\alpha}{2}X^T(t)\Xi A(t)X(t)
\end{align*}
$$

(23)

can finally achieve synchronization with a relatively small coupling strength, where $c(0) = 0$ and $\alpha > 0$.

Remark 3  The condition that $A(t) \in A1$ is irreducible and has the same left eigenvector $\xi$ corresponding to eigenvalue 0 for all $t$ looks quite strong. However, it just suits the node-balanced coupling networks discussed in (Belykh, V. et.al 2006), where the coupling matrix $A(t)$ is assumed to be row-sum zero as well as column-sum zero for each $t$. In this case, $A(t) \in A1$ has the same left eigenvector $[1, \cdots, 1]^T$ corresponding to eigenvalue 0 for all $t$. Therefore, Theorem 3 applies to the node-balanced coupling networks.

As a direct consequence of Theorem 3, we can obtain the following simple adaptive scheme, if every $A(t)$ is symmetric.

Corollary 1  Suppose that $f \in QUAD(\Delta, \varpi)$, the coupling matrix $A(t) \in A2$ is irreducible. If the largest non-zero eigenvalue of the matrix $A(t)$ satisfies $\lambda_2(t) \leq \lambda < 0$, for all $t$. Then, following coupled systems with an adaptive coupling strength $c(t)$:

$$
\begin{align*}
\dot{X}(t) &= F(X(t)) + c(t)A(t)X(t), \\
\dot{c}(t) &= -\frac{\alpha}{2}X^T(t)A(t)X(t)
\end{align*}
$$

(24)

can finally achieve synchronization with a relatively small coupling strength, where $c(0) = 0$ and $\alpha > 0$. 

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Remark 4  In Theorem 3 and Corollary 1, condition $\lambda_2(t) \leq \lambda < 0$ plays a key role. However, calculating $\lambda_2(t)$ for all $t$ is impossible numerically. If all $A(t)$ can be dominated by a constant matrix $\hat{A}$, then we can synchronize chaotic oscillators directly using (11), only replacing $A$ in the adaptive algorithm by $\hat{A}$. Following corollary explains it.

Corollary 2  Suppose that $f \in QUAD(\Delta, \varpi)$, the coupling matrix $A(t) \in A2$ is irreducible. If there exists a constant matrix $\hat{A} = (\hat{a}_{ij}) \in A2$ such that $\hat{a}_{ij} \geq a_{ij}(t)$, $i \neq j$ hold for all $t$. Then, following coupled systems with an adaptive coupling strength $c(t)$

$$
\begin{align*}
\dot{X}(t) &= F(X(t)) + c(t)A(t)X(t), \\
\dot{c}(t) &= -\frac{\alpha}{2}X^T(t)\hat{A}X(t) \\
\end{align*}
$$

(25)

can finally achieve synchronization with a relatively small coupling strength, where $c(0) = 0$ and $\alpha > 0$.

Proof  In this case, $\xi_i = \frac{1}{N}$, which implies $\Xi A(t) = \frac{1}{N} A(t)$, and

$$
X^T(t)A(t)X(t) = \sum_{i,j=1;i\neq j}^{N} a_{ij}(t)(x_i(t) - x_j(t))^T(x_i(t) - x_j(t)) \\
\leq \sum_{i,j=1;i\neq j}^{N} \hat{a}_{ij}(x_i(t) - x_j(t))^T(x_i(t) - x_j(t)) = X^T(t)\hat{A}X(t) \\
$$

(26)

Using the Lyapunov function

$$
V(X(t)) = \frac{1}{2}X^T(t)UX(t) + \frac{1}{\alpha}(c - c(t))^2 \\
$$

(27)

we have

$$
\frac{dV(t)}{dt} \leq X(t)^T[U[F(X(t)) - \Delta X(t)] + X^T(t)U(\Delta + cN\hat{A})]X(t) \\
$$

By the same arguments used in the proof of Theorem 1, we can obtain Corollary 2.
3.2 Nonlinearly coupled systems

We consider following nonlinearly coupled systems with an adaptive coupling strength:

$$\dot{x}_i(t) = f(x_i(t), t) + c(t) \sum_{j \neq i} a_{ij} \left[ g(x_j(t)) - g(x_i(t)) \right]$$ \hspace{1cm} (28)

where $g(x_i(t)) = [g_1(x_1^i(t)), \ldots, g_n(x_n^i(t))]^T$.

Denote $G(X(t)) = (g^T(x_1(t)), \ldots, g^T(x_N(t)))^T$ and $A = A \otimes I_n$, then we have

$$\frac{dX(t)}{dt} = F(X(t)) + c(t)AG(X(t))$$ \hspace{1cm} (29)

**Theorem 4** Suppose $A \in A2$ is irreducible, $f \in QUAD(\Delta, \varpi)$, $\frac{g_k(u) - g_k(v)}{u - v} \geq \beta$ for $\forall u \neq v, k = 1, \ldots, n$, where $\beta$ is a positive constant. Then, following nonlinearly coupled systems with an adaptive coupling strength $c(t)$:

$$\left\{\begin{array}{l}
\dot{X}(t) = F(X(t)) + c(t)AG(X(t)), \\
\dot{c}(t) = -\frac{\alpha}{2}X^T(t)AG(X(t))
\end{array}\right.$$ \hspace{1cm} (30)

will be synchronized with a relatively small coupling strength, where $c(0) = 0$ and $\alpha > 0$.

**Proof:** Because $A \in A2$, then $\xi_i = \frac{1}{N}$ and $UA = \frac{1}{N}A$. Using the same Lyapunov function as in the Theorem 1, we have

$$\frac{dV(t)}{dt} = X(t)^T U [F(X(t)) + c(t)AG(X(t))] + (c - c(t))X^T(t)UAG(X(t))$$

$$= X(t)^T U [F(X(t)) - \Delta X(t)] + X^T(t)U\Delta X(t) + cNX^T(t)UAG(X(t))$$

Furthermore,

$$X^T(t)UAG(X(t)) = \sum_{i=1}^{n} \tilde{x}_i^T(t)UA\tilde{g}_i(\tilde{x}_i(t)) = \sum_{i=1}^{n} \frac{1}{N} \tilde{x}_i(t)^T A\tilde{g}_i(\tilde{x}_i(t))$$

$$= -\sum_{i=1}^{n} \frac{1}{N} \sum_{j > k} a_{jk} [x_j^i(t) - x_k^i(t)]^T [g_i(x_j^i(t)) - g_k(x_k^i(t))]$$

$$\leq -\beta \sum_{i=1}^{n} \frac{1}{N} \sum_{j > k} a_{jk} [x_j^i(t) - x_k^i(t)]^T [x_j^i(t) - x_k^i(t)]$$

$$= \beta \sum_{i=1}^{n} \frac{1}{N} \tilde{x}_i(t)^T A\tilde{x}_i(t) = \beta \sum_{i=1}^{n} \tilde{x}_i(t)^T U A\tilde{x}_i(t)$$

13
where \( \tilde{g}_i(x_i(t)) = (g_i(x_i^1(t)), \ldots, g_i(x_i^N(t)))^T \), for \( i = 1, 2, \ldots, n \). Therefore,

\[
\frac{dV(t)}{dt} \leq -\omega X(t)^T U X(t) + \sum_{j=1}^{n} \tilde{x}_j^T(t) [\delta_j U + c\beta NU A] \tilde{x}_j(t)
\]

(31)

By similar arguments used in Theorem 1, we complete the proof of Theorem 4.

Similar to linearly coupled systems, for nonlinearly coupled systems with a time-varying coupling matrix:

\[
\frac{dX(t)}{dt} = F(X(t)) + c(t)A(t)G(X(t))
\]

(32)

we have

**Corollary 3**  **Suppose** \( A(t) \in \mathbb{A}^2 \) **is irreducible,** \( f \in QUAD(\Delta, \omega) \), \( \frac{\phi_k(u) - \phi_k(v)}{u - v} \geq \beta \) **for** \( \forall u \neq v, k = 1, \ldots, n \), **where** \( \beta \) **is a positive constant.** **If the largest non-zero eigenvalue of** \( A(t) \) **satisfies** \( \lambda_2(t) \leq \lambda < 0 \), **for all** \( t \). **Then, following nonlinearly coupled systems with an adaptive coupling strength**

\[
\begin{align*}
\dot{X}(t) & = F(X(t)) + c(t)A(t)G(X(t)), \\
\dot{c}(t) & = -\frac{\alpha}{2}X^T(t)A(t)G(X(t))
\end{align*}
\]

(33)

**will be synchronized with a relatively small coupling strength, where** \( c(0) = 0 \) **and** \( \alpha > 0 \).

**4 Numerical Simulation**

To validate the effectiveness of the proposed synchronization algorithms with an adaptive coupling strength, in the following, we couple 100 chaotic oscillators by following linearly coupled systems with adaptive strength

\[
\begin{align*}
\dot{X}(t) & = F(X(t)) + c(t)A X(t), \\
\dot{c}(t) & = -\frac{\alpha}{2}X^T(t)\Xi A X(t)
\end{align*}
\]

(34)
and nonlinearly coupled systems with adaptive strength

\[
\begin{align*}
\dot{X}(t) &= F(X(t)) + c(t)AG(X(t)), \\
\dot{c}(t) &= -\frac{\alpha}{2}X^T(t)BG(X(t))
\end{align*}
\tag{35}
\]

where \( X(t) = (x_1^T(t), \ldots, x_{100}^T(t))^T \), \( F(X(t)) = (f(x_1(t), t)^T, \ldots, f(x_{100}(t), t)^T)^T \), \( A = A \otimes I_3, A \in R^{100 \times 100} \), \( B = B \otimes I_3, \) and \( B = \frac{1}{2}(A+A^T) \). \( G(X(t)) = (g(x_1(t)), g(x_2(t)), \ldots, g(x_{100}(t))^T, g(x_i(t)) = (x_1^i(t) + \tanh(x_1^i(t)), x_2^i(t) + \tanh(x_2^i(t)), x_3^i(t) + \tanh(x_3^i(t))^T \), \( i = 1, \ldots, 100 \). The prototypes of \( f(\cdot) \) are Chua’s circuit, Chen’s oscillator, Lorenz’s oscillator and Rössler’s oscillator, respectively. For the coupling matrix \( A \), first, we construct a coupling matrix of a small-world network generated by the method proposed in (Watts et.al 1998). Then, replace every non-zero element \( a_{ij}, i \neq j \) with a positive random scalar, which is equally distributed in \([0, 1]\), and choose the diagonal elements to ensure \( A \in A_1 \).

We also couple 100 chaotic oscillators by following linearly coupled systems with an adaptive strength

\[
\begin{align*}
\dot{X}(t) &= F(X(t)) + c(t)AX(t), \\
\dot{c}(t) &= -\frac{\alpha}{2}X^T(t)\tilde{A}X(t)
\end{align*}
\tag{36}
\]

where \( \tilde{A} = \tilde{A} \otimes I_3, \tilde{A} \in R^{100 \times 100} \) are globally connected matrix or randomly connected matrix respectively.

In all simulations, we use the quantity \( E(t) = \sqrt{\frac{1}{99} \sum_{j=2}^{100} \| x_i(t) - x_1(t) \|^2 / 100} \) as a measure of synchronization error.

### 4.1 Chua’s Circuits

The uncoupled equation is:

\[
\begin{align*}
\frac{dx}{dt} &= \bar{m}[y - \bar{h}(x)] \\
\frac{dy}{dt} &= x - y + z \\
\frac{dz}{dt} &= -\bar{n}y
\end{align*}
\]
where $\tilde{h}(x) = \frac{2}{7}x - \frac{3}{14}|x+1| - |x-1|$, $\bar{m} = 9$ and $\bar{n} = 14\frac{5}{7}$. Figure 1. (a) and (b) show the dynamics of $c(t)$ and $E(t)$ for linearly coupled systems (34); while (c) and (d) show the dynamics of $c(t)$ and $E(t)$ for nonlinearly coupled systems (35).

Figure 1. The dynamics of $c(t)$ and $E(t)$ for 100 linearly and nonlinearly coupled chua’s circuits with an adaptive coupling strength

Figure 2. (a) and (b) show the dynamics of $c(t)$ and $E(t)$ for linearly coupling systems (36) when $\tilde{A}$ is the globally coupled matrix; while (c) and (d) show the dynamics of $c(t)$ and $E(t)$ for linearly coupled systems (36) when $\tilde{A} \in \mathbf{A}_2$ is a random matrix.
Figure 2. The dynamics of $c(t)$ and $E(t)$ for (36) with globally and randomly connected $\tilde{A}$

4.2 Chen’s Oscillator

The uncoupled equation is:

\[
\begin{align*}
\dot{x} &= a(y - x) \\
\dot{y} &= (c - a)x - xz + cy \\
\dot{z} &= xy - bz
\end{align*}
\]

where $a = 35$, $b = 3$ and $c = 28$. Figure 3. (a) and (b) show the dynamics of $c(t)$ and $E(t)$ for linearly coupled systems (34); while (c) and (d) show the dynamics of $c(t)$ and $E(t)$ for nonlinearly coupled systems (35).
Figure 3. The dynamics of $c(t)$ and $E(t)$ for 100 linearly and nonlinearly coupled Chen’s chaotic oscillators with an adaptive coupling strength.

Figure 4. (a) and (b) show the dynamics of $c(t)$ and $E(t)$ for linearly coupling systems (36) when $\tilde{A}$ is the globally coupled matrix; while (c) and (d) show the dynamics of $c(t)$ and $E(t)$ for linearly coupled systems (36) when $\tilde{A} \in \mathbf{A}_2$ is a random matrix.
Figure 4. The dynamics of $c(t)$ and $E(t)$ for (36) with globally and randomly connected $\tilde{A}$

4.3 Lorenz’s Oscillator

The uncoupled equation is:

$$
\begin{align*}
\dot{x}_1 &= \beta(x_2 - x_1) \\
\dot{x}_2 &= \alpha x_1 - x_1 x_3 - x_2 \\
\dot{x}_3 &= x_1 x_2 - bx_3
\end{align*}
$$

where $\beta = 10$, $\alpha = 28$, and $b = \frac{8}{3}$. Figure 5. (a) and (b) show the dynamics of $c(t)$ and $E(t)$ for linearly coupled systems (34); while (c) and (d) show the dynamics of $c(t)$ and $E(t)$ for nonlinearly coupled systems (35).
Figure 5. The dynamics of $c(t)$ and $E(t)$ for 100 linearly and nonlinearly coupled Lorenz’s chaotic oscillators with an adaptive coupling strength.

Figure 6. (a) and (b) show the dynamics of $c(t)$ and $E(t)$ for linearly coupling systems (36) when $\tilde{A}$ is the globally coupled matrix; while (c) and (d) show the dynamics of $c(t)$ and $E(t)$ for linearly coupled systems (36) when $\tilde{A} \in \mathbf{A}2$ is a random matrix.
Figure 6. The dynamics of $c(t)$ and $E(t)$ for (36) with globally and randomly connected $\tilde{A}$

4.4 Rössler’s Oscillator

The uncoupled equation is:

\[
\begin{align*}
\dot{x}_1 &= -x_2 - x_3 \\
\dot{x}_2 &= x_1 + 0.2x_2 \\
\dot{x}_3 &= 0.2 + x_3(x_1 - \mu)
\end{align*}
\]

where $\mu = 5.7$. Figure 7. (a) and (b) show the dynamics of $c(t)$ and $E(t)$ for linearly coupled systems (34); while (c) and (d) show the dynamics of $c(t)$ and $E(t)$ for nonlinearly coupled systems (35).
Figure 7. The dynamics of $c(t)$ and $E(t)$ for 100 linearly and nonlinearly coupled Rössler’s chaotic oscillators with an adaptive coupling strength.

Figure 8. (a) and (b) show the dynamics of $c(t)$ and $E(t)$ for linearly coupling systems (36) when $\tilde{A}$ is the globally coupled matrix; while (c) and (d) show the dynamics of $c(t)$ and $E(t)$ for linearly coupled systems (36) when $\tilde{A} \in A2$ is a random matrix.
Figure 8. The dynamics of $c(t)$ and $E(t)$ for (36) with globally and randomly connected $\tilde{A}$

5 Numerical Example 2

In this section, we give some simulations for coupled systems with a time-varying coupling matrix. As a special case, it includes node-balanced coupled systems.

We pick

$$A(t) = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \begin{pmatrix} -5 - \sin t - \cos t & 3 + \sin t & 2 + \cos t \\ 2 + \cos t & -5 - \sin t - \cos t & 3 + \sin t \\ 3 + \sin t & 2 + \cos t & -5 - \sin t - \cos t \end{pmatrix}$$ (37)
as the time-varying coupling matrix, where $p_1, p_2, p_3$ are positive constants.

It is clear that for all $t$, $A(t)$ has the same left eigenvector $\xi = \left(\frac{1}{3p_1}, \frac{1}{3p_2}, \frac{1}{3p_3}\right)^T$ corresponding to eigenvalue 0. In particular, if $p_1 = p_2 = p_3$, then the coupling matrix $A(t)$ is node-balanced for each $t$.

Moreover, it is easy to check that

$$
\lambda_2(t) = -(5 + \sin t + \cos t) \leq -(5 - \sqrt{2}) < 0
$$

is the largest non-zero eigenvalue of the matrix

$$
\Xi A(t) + A(t)^T \Xi = \frac{5 + \sin t + \cos t}{3} \begin{pmatrix}
-2 & 1 & 1 \\
1 & -2 & 1 \\
1 & 1 & -2
\end{pmatrix}
$$

We couple three Chua’s circuits by

$$
\begin{cases}
\dot{X}(t) = F(X(t)) + c(t)A(t)X(t), \\
\dot{c}(t) = -\frac{\alpha}{2}X^T(t)\Xi A(t)X(t)
\end{cases}
$$

(38)

We also use $E(t) = \sqrt{\left(\|x_2(t) - x_1(t)\|^2 + \|x_3(t) - x_1(t)\|^2\right)/2}$ to measure the error of synchronization.

Figure 9. shows the dynamics of the coupling strength $c(t)$ and error for different $p_1, p_2, p_3$, Where sub-figures (a) and (b) denote the dynamics of $c(t)$ and $E(t)$ for $p_1 = p_2 = 1, p_3 = 2$; and sub-figures (c) and (d) denote the dynamics of $c(t)$ and $E(t)$ for $p_1 = p_2 = p_3 = 1$. 
6 Conclusions

In this paper, we propose new algorithms to synchronize linearly or nonlinearly coupled systems with an adaptive coupling strength. Unlike those adaptive algorithms existing in the literature, where coupled systems are synchronized to a special trajectory $s(t)$ or an equilibrium of the uncoupled system by adding a negative feedback controller; in this paper, we synchronize linearly and nonlinearly coupled systems with an adaptive coupling strength without knowing the synchronization trajectory. By adapting the coupling strength, we
reveal that a large scale of chaotic oscillators can be synchronized even with a very small coupling strength. It indicates that chaotic oscillators are very easy to be synchronized.

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(a) $X_1^1(t)$

(b) $X_2^1(t)$

(c) $X_1^2(t)$

(d) $X_1^1(t)$
(a) $X_1^1(t)$
(b) $X_2^1(t)$
(c) $X_1^1(t)$
(d) $X_1^1(t)$