Sharp pointwise estimates for solutions of weakly coupled second-order parabolic system in a layer

Gershon Kresin and Vladimir Maz’ya

Department of Mathematics, Ariel University, Ariel, Israel; Department of Mathematical Sciences, University of Liverpool, Liverpool, UK; Department of Mathematics, Linköping University, Linköping, Sweden; RUDN University, Moscow, Russia

Abstract

We deal with $m$-component vector-valued solutions to the Cauchy problem for a linear both homogeneous and nonhomogeneous weakly coupled second-order parabolic system in the layer $\mathbb{R}_T^{n+1} = \mathbb{R}^n \times (0,T)$. We assume that coefficients of the system are real and depending only on $t$, $n \geq 1$ and $T < \infty$. The homogeneous system is considered with initial data in $[L^p(\mathbb{R}^n)]^m$, $1 \leq p \leq \infty$. For the nonhomogeneous system we suppose that the initial function is equal to zero and the right-hand side belongs to $[L^p(\mathbb{R}_T^{n+1})]^m \cap [C^\alpha(\mathbb{R}_T^{n+1})]^m$, $\alpha \in (0, 1)$. Explicit formulas for the sharp coefficients in pointwise estimates for solutions to these problems and their directional derivative are obtained.

1. Introduction

Parabolic equations and systems are classical subjects of mathematical physics (e.g. [1–4]). The present paper is a continuation of our recent work [5] on sharp pointwise estimates for the gradient of solutions to the Cauchy problem for the single parabolic equation of the second order with constant coefficients. We say that the estimate is sharp if the coefficient in front of the norm in the majorant part of the inequality cannot be diminished. Sharp pointwise estimates for solutions to the Laplace, modified Helmholtz, Lamé, Stokes, and heat equations, as well as for the analytic functions were obtained earlier in [5–10].

In this paper we study solutions to the Cauchy problem for parabolic weakly coupled system with real coefficients of the form

$$\frac{\partial u}{\partial t} = \sum_{j,k=1}^n a_{jk}(t) \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^n b_j(t) \frac{\partial u}{\partial x_j} + C(t)u + f(x, t)$$

(A)
in the layer $\mathbb{R}_T^{n+1} = \mathbb{R}^n \times (0, T)$ with the initial condition $u|_{t=0} = \varphi$, considering separately two cases, $f = 0$ and $\varphi = 0$. Here and henceforth $T < \infty$, $n \geq 1$, $u(x, t) = (u_1(x, t), \ldots, u_m(x, t))$ and $f(x, t) = (f_1(x, t), \ldots, f_m(x, t))$.

Throughout the article, we assume that $A(t) = ((a_{jk}(t)))$ is a symmetric positive definite $(n \times n)$-matrix-valued function on $[0, T]$, which elements satisfy the Hölder condition with exponent $\alpha/2$ ($0 < \alpha < 1$), $b_1(t), \ldots, b_n(t)$ are continuous functions on $[0, T]$, $C(t)$ is continuous on $[0, T]$ matrix-valued function of order $m$.

We obtain sharp pointwise estimates for $|u|$, $|\partial u/\partial \ell|$ and $\max_{|\ell|=1} |\partial u/\partial \ell|$, where $u$ solves the Cauchy problem for system (1) in the layer $\mathbb{R}_T^{n+1}$, $\cdot |$ denotes the Euclidean length of a vector and $\ell$ is a unit $n$-dimensional vector. By $\partial u/\partial \ell$ we mean the derivative of a vector-valued function $u(x, t)$ in the direction $\ell$:

$$\frac{\partial u}{\partial \ell} = \lim_{\lambda \to 0^+} \frac{u(x + \lambda \ell, t) - u(x, t)}{\lambda} = (\ell, \nabla_x)u = \sum_{j=1}^m \frac{\partial u_j}{\partial \ell} e_j,$$

where $\nabla_x = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$ and $e_j$ means the unit vector of the $j$-th coordinate axis.

The present paper consists of five sections, including Introduction. Section 2 is auxiliary. Section 3 is devoted to explicit formulas for solutions to the Cauchy problem in $\mathbb{R}_T^{n+1}$ for system (1).

In Section 4, we consider a solution to the Cauchy problem

$$\begin{cases}
\frac{\partial u}{\partial t} = \sum_{j,k=1}^n a_{jk}(t) \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^n b_j(t) \frac{\partial u}{\partial x_j} + C(t) u \\
u|_{t=0} = \varphi,
\end{cases}$$

where $\varphi \in [L^p(\mathbb{R}^n)]^m, p \in [1, \infty]$. The norm $\| \cdot \|_p$ in the space $[L^p(\mathbb{R}^n)]^m$ is defined by

$$\|\varphi\|_p = \left( \int_{\mathbb{R}^n} |\varphi(x)|^p \, dx \right)^{1/p}$$

for $1 \leq p < \infty$, and

$$\|\varphi\|_\infty = \text{ess sup}\{ |\varphi(x)| : x \in \mathbb{R}^n \}.$$

In this section we obtain two groups of sharp estimates. First of them concerns modulus to solution $u$ of problem (3). Namely, we derive the inequality

$$|u(x, t)| \leq \mathcal{H}_p(t) \|\varphi\|_p$$

with the sharp coefficient

$$\mathcal{H}_p(t) = \frac{e^{I_{C^0}(t)}}{(2\sqrt{\pi})^{n/p} \left( \det I_A^{1/2}(t) \right)^{1/p} (p')^{n/(2p')}}$$

where $(x, t)$ is an arbitrary point in the layer $\mathbb{R}_T^{n+1}$. Here and henceforth $I_F(t) = \int_0^t F(t) \, dt$, where $F$ can be a vector-valued or matrix-valued function, $p^{-1} + p'^{-1} = 1$, the symbol
* denotes passage to the transposed matrix and $|B| = \max_{|z|=1} |Bz|$ means the spectral norm of the $l \times l$ real-valued matrix $B$, $z \in \mathbb{R}^l$. It is known (e.g. [11, sect. 6.3]) that $|B| = \sqrt{\rho(B^*B)}$, where $\rho(M)$ is the spectral radius of the matrix $M$. If the matrix $B$ is a symmetric positive definite, then $|B^{-1/2}| = \sqrt{\rho(B^{-1})}$.

As a special case of (5) one has

$$H_\infty(t) = \|e^{IC^*t}\|.$$  \hfill (6)

For the single parabolic equation

$$\frac{\partial u}{\partial t} = \sum_{j,k=1}^{n} a_{jk}(t) \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{n} b_j(t) \frac{\partial u}{\partial x_j} + c(t)u + f(x,t)$$  \hfill (7)

with $f = 0$, formula (6) becomes

$$H_\infty(t) = \exp \left\{ \int_0^t c(t) \, dt \right\}.$$  \hfill (8)

The second group of sharp estimates concerns modulus of $\frac{\partial u}{\partial \ell}$, where $u$ is solution of problem (3). Namely, the formula for the sharp coefficient

$$K_{p,\ell}(t) = \frac{|I^{-1/2}_A(t)\ell| \|e^{IC^*t}\|}{2^n \pi^{(n+p-1)/2} \det I^{1/2}_A(t)^{1/p} \left\{ \Gamma \left( \frac{p'+1}{2} \right) \right\}^{1/p} \frac{\Gamma \left( \frac{p'(n+p')}{2} \right)}{p'(n+p')/2}}$$

in the inequality

$$\left| \frac{\partial u}{\partial \ell} (x,t) \right| \leq K_{p,\ell}(t) \|\varphi\|_p$$  \hfill (9)

is obtained, where $(x,t)$ is an arbitrary point in the layer $\mathbb{R}_{T}^{n+1}$.

As a consequence of (8), the sharp coefficient

$$K_p(t) = \max_{|\ell|=1} K_{p,\ell}(t)$$  \hfill (10)

in the inequality

$$\max_{|\ell|=1} \left| \frac{\partial u}{\partial \ell} (x,t) \right| \leq K_p(t) \|\varphi\|_p$$  \hfill (11)

is found. In particular,

$$K_\infty(t) = \frac{1}{\sqrt{\pi}} |I^{1/2}_A(t)| \|e^{IC^*t}\|.$$  \hfill (12)

For the single parabolic equation (7) with $f = 0$, the previous formula takes the form

$$K_\infty(t) = \frac{1}{\sqrt{\pi}} |I^{1/2}_A(t)| \exp \left\{ \int_0^t c(t) \, dt \right\}.$$  \hfill (13)
As a special case of (13) one has

$$\mathcal{K}_\infty(t) = \frac{1}{\sqrt{\pi}} \exp \left\{ \frac{\int_0^t c(t) \, dt}{\left( \int_0^t a(t) \, dt \right)^{1/2}} \right\},$$

for $A(t) = a(t)I$, where $I$ is the unit matrix of order $n$.

In Section 5 we consider a solution to the Cauchy problem

$$\begin{cases}
\frac{\partial u}{\partial t} = \sum_{j,k=1}^n a_{jk}(t) \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^n b_j(t) \frac{\partial u}{\partial x_j} + C(t)u + f(x,t) & \text{in } \mathbb{R}^{n+1},
\end{cases}$$

(14)

where $f \in [L^p(\mathbb{R}^{n+1})]^m \cap [C^\alpha(\mathbb{R}^{n+1})]^m$, $\alpha \in (0, 1)$. By $[C^\alpha(\mathbb{R}^{n+1})]^m$ we denote the space of $m$-component vector-valued functions $f(x,t)$ which are continuous and bounded in $\mathbb{R}^{n+1}$ and locally Hölder continuous with exponent $\alpha$ in $x \in \mathbb{R}^n$, uniformly with respect to $t \in [0, T]$. The space $[L^p(\mathbb{R}^{n+1})]^m$ is endowed with the norm

$$\|f\|_{p,T} = \left\{ \int_0^T \int_{\mathbb{R}^n} |f(x,\tau)|^p \, dx \, d\tau \right\}^{1/p},$$

for $1 \leq p < \infty$, and

$$\|f\|_{\infty,T} = \text{ess sup}\{ |f(x,\tau)| : x \in \mathbb{R}^n, \tau \in (0, T)\}.$$

In this section we also obtain two groups of sharp estimates. As before, first of them concerns modulus to solution $u$ of problem (14). Namely, we prove the inequality

$$|u(x,t)| \leq \mathcal{N}_p(t) \|f\|_{p,T}$$

(15)

with the sharp coefficient

$$\mathcal{N}_p(t) = \frac{1}{(2\sqrt{\pi})^{n/p} p^n/(2p')} \max_{|z|=1} \left\{ \int_0^t \frac{|e^{\mathcal{I}_c(t,\tau)} z|^{p'}}{(\det \mathcal{I}_A^{1/2}(t,\tau))^{p'-1}} \, d\tau \right\}^{1/p'},$$

(16)

where $(x,t)$ is an arbitrary point in the layer $\mathbb{R}_T^{n+1}$. Here and henceforth $\mathcal{I}_F(t,\tau) = \int_\tau^t F(s) \, ds$, where $F$ can be a vector-valued or matrix-valued function. Formula (16) is obtained under assumption that the integral in (16) converges. For instance, integral in (16) is convergent for $p > (n + 2)/2$ in the case of single parabolic equation with constant coefficients.
As a special case of (16) one has

$$\mathcal{N}_\infty(t) = \max_{|z|=1} \int_0^t \left| e^{\mathcal{C}_\ast(t,\tau)z} \right| d\tau. \quad (17)$$

For the single parabolic equation (7) the previous formula takes the form

$$\mathcal{N}_\infty(t) = \int_0^t \exp \left\{ \int_t^\tau c(t) \, d\tau \right\} d\tau. \quad (18)$$

The second group of sharp estimates concerns $\partial u / \partial \ell$. The explicit formula for the sharp coefficient

$$C_{p,\ell}(t) = \max_{|\ell|=1} \left\{ \int_0^t \left| \left| I_{A}^{-1/2}(t,\tau) \ell \right|^p \left| e^{\mathcal{C}_\ast(t,\tau)z} \right|^p \right\} \left( \frac{1}{p-1} \det I_A^{1/2}(t,\tau) \right)^{1/p} \frac{1}{p'} \right\} \quad (18)$$

in the inequality

$$\left| \frac{\partial u}{\partial \ell} (x, t) \right| \leq C_{p,\ell}(t) \| f \|_{p,t} \quad (19)$$

for solutions to the Cauchy problem (14) is found, where $(x, t) \in \mathbb{R}^{n+1}$.

Formula (18) is obtained under assumption that the integral in (18) converges. We note that this integral is convergent for $p > n + 2$ in the case of single parabolic equation with constant coefficients (see [5]).

As a consequence of (18), we arrive at the formula for the sharp coefficient

$$C_p(t) = \max_{|\ell|=1} C_{p,\ell}(t) \quad (20)$$

in the inequality

$$\max_{|\ell|=1} \left| \frac{\partial u}{\partial \ell} (x, t) \right| \leq C_p(t) \| f \|_{p,t} . \quad (21)$$

For instance,

$$C_{\infty}(t) = \frac{1}{\sqrt{\pi}} \max_{|\ell|=1} \max_{|z|=1} \int_0^t \left| I_{A}^{-1/2}(t,\tau) \ell \right| \left| e^{\mathcal{C}_\ast(t,\tau)z} \right| \, d\tau . \quad (22)$$

For the single parabolic equation (7) the previous formula takes the form

$$C_{\infty}(t) = \frac{1}{\sqrt{\pi}} \max_{|\ell|=1} \int_0^t \left| I_{A}^{-1/2}(t,\tau) \ell \right| \exp \left\{ \int_t^\tau c(t) \, d\tau \right\} \, d\tau . \quad (23)$$
In the particular case $A(t) = a(t)I$, formula (23) becomes

$$C_\infty(t) = \frac{1}{\sqrt{\pi}} \int_0^t \exp \left\{ \int_\tau^t c(t) \, dt \right\} \left[ \int_\tau^t a(t) \, dt \right]^{1/2} \, d\tau.$$ 

Note that the sharp coefficients (8) and (18) do not depend on the coefficient vector $b(t) = (b_1(t), \ldots, b_n(t))$.

Besides, worth to note that the coefficients in the sharp estimates for solutions to the homogeneous weakly coupled parabolic system in the layer do not depend on the antisymmetric component of the matrix-valued function $C(t)$.

2. The norm of a certain integral operator

Let $(\mathcal{X}, \mathcal{A}, \mu)$ be a measure space and let $1 \leq p \leq \infty$. We introduce the space $[L^p(\mathcal{X}, \mathcal{A}, \mu)]^n$ of real vector-valued functions endowed with the norm

$$\|f\|_p = \left\{ \int_\mathcal{X} |f(x)|^p \, d\mu(x) \right\}^{1/p}$$

for $1 \leq p < \infty$, and

$$\|f\|_\infty = \operatorname{ess \ sup}\{|f(x)| : x \in \mathcal{X}\}.$$ 

By $(\eta, \zeta)$ we denote the inner product of the vectors $\eta$ and $\zeta$ in a Euclidean space.

The following assertion is borrowed from our book [7] (Proposition 2.1) and presented here for the readers’ convenience.

**Proposition 2.1:** Let $G = (g_{ij})$ be an $(m \times n)$-matrix-valued function with the elements $g_{ij} \in L^p(\mathcal{X}, \mathcal{A}, \mu)$ whose values $g_{ij}$ are everywhere finite. The norm of the linear continuous operator $S : [L^p(\mathcal{X}, \mathcal{A}, \mu)]^n \to \mathbb{R}^m$ defined by

$$S(f) = \int_\mathcal{X} G(x)f(x) \, d\mu(x)$$

is equal to

$$\|S\|_p = \sup_{\|z\| = 1} \|G^*z\|_{p'},$$

where $G^*$ stands for the transposed matrix of $G$, $z \in \mathbb{R}^m$ and $p'$ is defined by $1/p + 1/p' = 1$.

**Proof:** 1. Upper estimate for $\|S\|_p$. For any vector $z \in \mathbb{R}^m$,

$$(S(f), z) = \int_\mathcal{X} (G(x)f(x), z) \, d\mu(x) = \int_\mathcal{X} (f(x), G^*(x)z) \, d\mu(x).$$

Hence by Hölder’s inequality

$$|(S(f), z)| \leq \int_\mathcal{X} |(f(x), G^*(x)z)| \, d\mu(x) \leq \int_\mathcal{X} |G^*(x)z||f(x)| \, d\mu(x) \leq \|G^*z\|_{p'} \|f\|_p.$$
Therefore, taking into account that \(|S(f)| = \sup \{|(S(f), z)| : |z| = 1\}\) we arrive at the estimate

\[
\|S\|_p \leq \sup_{|z|=1} \|G^* z\|_{p'}.
\] (28)

2. Lower estimate for \(\|S\|_p\). Let us fix \(z \in S^{m-1} = \{z \in \mathbb{R}^m : |z| = 1\}\). We introduce the vector-valued function with \(n\) components

\[
h_z(x) = g_z(x)h(x),
\] (29)

where \(h \in L_p(\mathcal{X}, \mathcal{A}, \mu), \|h\|_p \leq 1,\) and

\[
g_z(x) = \begin{cases} 
G^*(x)z|G^*(x)z|^{-1} & \text{for } |G^*(x)z| \neq 0, \\
0 & \text{for } |G^*(x)z| = 0.
\end{cases}
\]

Note that \(h_z \in [L^p(\mathcal{X}, \mathcal{A}, \mu)]^n\) and \(\|h\|_p \leq 1\). Setting (29) as \(f\) in (27) we find

\[
(S(h_z), z) = (S(g_z h), z) = \int_\mathcal{X} (g_z(x), G^*(x)z)h(x) \, d\mu(x) = \int_\mathcal{X} |G^*(x)z|h(x) \, d\mu(x).
\]

Hence

\[
\|S\|_p = \sup_{\|f\|_p \leq 1} \|S(f)\| \geq \sup_{\|h\|_p \leq 1} \|S(g_z h)\| \geq \sup_{\|h\|_p \leq 1} |(S(g_z h), z)| \\
= \sup_{\|h\|_p \leq 1} \left| \int_\mathcal{X} |G^*(x)z|h(x) \, d\mu(x) \right| = \|G^* z\|_{p'}.
\]

By the arbitrariness of \(z \in S^{m-1}\),

\[
\|S\|_p \geq \sup_{|z|=1} \|G^* z\|_{p'},
\] (30)

which together with (28) leads to (26).

\begin{remark} Let \(1 < p \leq \infty\). Estimate (30) can be derived with the help of the function

\[
h_z(x) = \begin{cases} 
G^*(x)z|G^*(x)z|^{p'-2} & \text{for } |G^*(x)z| \neq 0, \\
\|G^* z\|_{p'}^{-p'/p} & \text{for } |G^*(x)z| = 0,
\end{cases}
\] (31)

where \(z \in S^{m-1}\). Indeed, since \(p'/(p' - 1) = p\), it follows that

\[
\|h_z\|_p = \|G^* z\|_{p'}^{-p'/p} \left( \int_\mathcal{X} |G^*(x)z|^{(p'-1)p} \, d\mu(x) \right)^{1/p} = \|G^* z\|_{p'}^{-p'/p} \|G^* z\|_{p'}^{p'/p} = 1.
\]

Using (27) and (31), we obtain

\[
\|S\|_p = \sup_{\|f\|_p \leq 1} |S(f)| \geq (S(h_z), z) = \|G^* z\|_{p'}^{-p'/p} \int_\mathcal{X} |G^*(x)z|^{p'} \, d\mu(x)
\]

\[
= \|G^* z\|_{p'}^{-p'/p} \|G^* z\|_{p'}^{p'} = \|G^* z\|_{p'},
\]

which implies (30), because \(z \in S^{m-1}\) is arbitrary.

\end{remark}
3. Formulas for solutions of weakly coupled parabolic systems

By $(x, y)$ we mean the inner product of the vectors $x$ and $y$ in $\mathbb{R}^n$. The Schwartz class of $m$-component vector-valued rapidly decreasing $C^\infty$-functions on $\mathbb{R}^n$ will be denoted by $[S(\mathbb{R}^n)]^m$.

Since the matrix $A(t)$ is positive definite for any $t \in [0, T]$, the matrix $\mathcal{I}_A(t) = \int_0^t A(t)\,dt$ is positive definite for any $t \in (0, T)$. So, there exist positive definite matrices $\mathcal{I}_A^{1/2}(t)$ and $\mathcal{I}_A^{-1/2}(t)$ of order $n$ such that $(\mathcal{I}_A^{1/2}(t))^2 = \mathcal{I}_A(t)$ and $(\mathcal{I}_A^{-1/2}(t))^2 = \mathcal{I}_A^{-1}(t)$ for any $t \in (0, T]$ (e.g. [11], sect. 2.14).

We start with the Cauchy problem (3) for the homogeneous system. The assertion below can be proved analogously to the corresponding statement for the heat equation (e.g. [12, Th. 5.4], [13, Sect. 4.8], [14, Sect. 7.4]). We restrict ourselves to a formal argument.

Lemma 3.1: Let $\varphi \in [S(\mathbb{R}^n)]^m$. A solution $u$ of problem (3) is given by

$$u(x, t) = \int_{\mathbb{R}^n} G(x - y, t)\varphi(y)\,dy,$$

where

$$G(x, t) = \frac{e^{\mathcal{I}_C(t)}}{(2\sqrt{\pi})^n \det \mathcal{I}_A^{1/2}(t)} e^{-\left|\mathcal{I}_A^{-1/2}(t)(x + \mathcal{I}_b(t))\right|^2/4}.$$  

Proof: Let $u$ be solution to the Cauchy problem (3). We introduce the function

$$v = e^{-\mathcal{I}_C(t)} u.$$  

Then $v$ is solution of the problem

$$\begin{cases} \frac{\partial v}{\partial t} = \sum_{j,k=1}^n a_{jk}(t) \frac{\partial^2 v}{\partial x_j \partial x_k} + \sum_{j=1}^n b_j(t) \frac{\partial v}{\partial x_j} \quad \text{in } \mathbb{R}^{n+1}, \\
\left.v\right|_{t=0} = \varphi. \end{cases}$$

Applying the Fourier transform

$$\hat{v}(\xi, t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(x, \xi)} v(x, t)\,dx.$$  

to the Cauchy problem (35), we obtain

$$\frac{d\hat{v}}{dt} = \left\{ - (A(t)\xi, \xi) + i(b(t), \xi) \right\} \hat{v}, \quad \hat{v}(\xi, 0) = \hat{\varphi}(\xi).$$

The solution of problem (37) is

$$\hat{v}(\xi, t) = \hat{\varphi}(\xi) e^{\int_0^t \left\{ - (A(t)\xi, \xi) + i(b(t), \xi) \right\} \,dt} = \hat{\varphi}(\xi) e^{-\mathcal{I}_A(t)\xi, \xi + i\mathcal{I}_b(t), \xi}.$$  

By the inverse Fourier transform

$$v(x, t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(x, \xi)} \hat{v}(\xi, t)\,d\xi,$$
we deduce from (38)
\[ \nu(x, t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(x, \xi)} \phi(\xi) e^{-(\mathcal{I}_A(t)\xi + i\mathcal{I}_b(t)\xi)} \, d\xi \]
\[ = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x, \xi)} e^{-(\mathcal{I}_A(t)\xi + i\mathcal{I}_b(t)\xi)} \left\{ \int_{\mathbb{R}^n} e^{-i(y, \xi)} \varphi(y) \, dy \right\} \, d\xi \]
\[ = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} e^{i(x - y + \mathcal{I}_b(t)\xi)} e^{-(\mathcal{I}_A(t)\xi \cdot \xi)} \, d\xi \right\} \varphi(y) \, dy. \]  

(39)

Let us denote
\[ G_0(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-(\mathcal{I}_A(t)\xi + i(x + \mathcal{I}_b(t)\xi))} \, d\xi. \]  

(40)

The known formula (e.g. [15, Ch. 2, Sect. 9.7])
\[ \int_{\mathbb{R}^n} e^{-(M\xi + i\xi \cdot \xi)} \, d\xi = \frac{\pi^{n/2}}{\sqrt{\det M}} e^{-(M^{-1}\xi, \xi)/4}, \]  

(41)

where \( M \) is a symmetric positive definite matrix of order \( n \), together with (40) leads to
\[ G_0(x, t) = \frac{1}{(2\pi)^n \sqrt{\det \mathcal{I}_A(t)}} e^{-(\mathcal{I}_A^{-1}(t)(x + \mathcal{I}_b(t)) \cdot (x + \mathcal{I}_b(t)))/4}. \]  

(42)

Since \( (B^{-1}\xi, \xi) = (B^{-1/2}B^{-1/2}\xi, \xi) = (B^{-1/2}\xi, B^{-1/2}\xi) = |B^{-1/2}\xi|^2 \) as well as \( \det B = (\det B^{1/2})^2 \) for any symmetric positive definite matrix \( B \) and every vector \( \xi \in \mathbb{R}^n \), we can write (42) as
\[ G_0(x, t) = \frac{1}{(2\sqrt{\pi})^n \det \mathcal{I}_A(t)^{1/2}} e^{-|\mathcal{I}_A^{-1/2}(t)(x + \mathcal{I}_b(t))|^2/4}. \]  

(43)

It follows from (34), (39), (40) and (43) that the solution of problem (3) can be represented as (32), where \( G(x, t) \) is given by (33).

The next assertion can be proved in view of (34) and (35) on the base of Lemma 3.1 similarly to the analogous statement for the heat equation (e.g. [12, Th. 5.5], [14, Sect. 7.4]).

**Proposition 3.1:** Suppose that \( 1 \leq p \leq \infty \) and \( \varphi \in [L^p(\mathbb{R}^n)]^m \). Define \( u : \mathbb{R}^{n+1}_T \rightarrow \mathbb{R}^m \) by (32), where \( G \) is given by (33). Then \( u(x, t) \) is solution of the system
\[ \frac{\partial u}{\partial t} = \sum_{j,k=1}^n a_{jk}(t) \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^n b_j(t) \frac{\partial u}{\partial x_j} + C(t) u \]

in \( \mathbb{R}^{n+1}_T \). If \( 1 \leq p \leq \infty \), then \( u(\cdot, t) \rightarrow \varphi \) in \( [L^p]^m \) as \( t \rightarrow 0^+ \).

**Remark 3.1:** It is known (e.g. [2, Ch.1, Sect. 6, 7 and 9]) that
\[ \nu(x, t) = \int_{\mathbb{R}^n} G_0(x - y, t) \varphi(y) \, dy, \]

where \( G_0 \) is given by (42), represents a unique bounded solution of the Cauchy problem (35) and \( \nu(x, t) \rightarrow \varphi(x) \) as \( t \rightarrow 0^+ \) at any \( x \in \mathbb{R}^n \) under assumption that \( \varphi \) belongs
to \([C(\mathbb{R}^n)]^m \cap [L^\infty(\mathbb{R}^n)]^m\). This fact together with (34) and the Lusin’s theorem (see [16, Ch. VI, Sect. 6]) implies that \(u(\cdot, t) \to \varphi\) almost everywhere in \(\mathbb{R}^n\) as \(t \to 0^+\), where \(u\) is a solution of problem (3) with \(\varphi \in [L^\infty(\mathbb{R}^n)]^m\).

Further, let us consider solution \(u\) to the Cauchy problem (14) for the nonhomogeneous system. We introduce the function (34). Then \(v\) is solution of the problem

\[
\begin{align*}
\frac{\partial v}{\partial t} &= \sum_{j,k=1}^n d_{jk}(t) \frac{\partial^2 v}{\partial x_j \partial x_k} + \sum_{j=1}^n b_j(t) \frac{\partial v}{\partial x_j} + e^{-\mathcal{I}_c(t)} f(x, t) \quad \text{in } \mathbb{R}^{n+1}, \\
|v|_{t=0} &= 0.
\end{align*}
\]

In view of (34) and (44), the next statement can be proved analogously to the similar assertion for the heat equation (e.g. [13, Sect. 4.8]). We restrict ourselves to a formal argument.

**Lemma 3.2:** Let \(f(\cdot, t) \in [\mathcal{S}(\mathbb{R}^n)]^m\) for any \(t \in [0, T]\) and let the quantities \(C_{\alpha,m}\) in the estimates

\[
(1 + |x|^m) |\partial^\alpha_x f(x, t)| \leq C_{\alpha,m}
\]

are independent of \(t\) for any integer \(m \geq 0\) and multiindex \(\alpha\).

The solution of problem (14) is given by

\[
u(x, t) = \int_0^t \int_{\mathbb{R}^n} P(x - y, t, \tau) f(y, \tau) \, dy \, d\tau,
\]

where \(P\) is defined by

\[
P(x, t, \tau) = \frac{e^{\mathcal{I}_c(t, \tau)}}{(2\sqrt{\pi})^n \det \mathcal{I}_A^{1/2}(t, \tau)} e^{-\left|\mathcal{I}_A^{-1/2}(t, \tau)(x - \mathcal{I}_b(t, \tau))\right|^2 / 4}.
\]

**Proof:** Applying the Fourier transform to the Cauchy problem (44), we obtain

\[
\frac{d\hat{v}}{dt} = \{ - (A(t)\xi, \xi) + i(b(t), \xi) \} \hat{v} + e^{-\mathcal{I}_c(t)} \hat{f}(\xi, t), \quad \hat{v}(\xi, 0) = 0.
\]

The solution of problem (47) is

\[
\hat{v}(\xi, t) = e^{-\left(\mathcal{I}_A(t, \xi, \xi) + i(B(t), \xi)\right)} \int_0^t e^{-\mathcal{I}_c(\tau)} \hat{f}(\xi, \tau) e^{\left(\mathcal{I}_A(\tau, \xi, \xi) + i(B(\tau), \xi)\right)} \, d\tau
\]

\[
= \int_0^t e^{-\mathcal{I}_c(\tau)} \hat{f}(\xi, \tau) e^{\int_0^t \left(- (A(\xi), \xi) + i(b(\xi), \xi)\right) \, ds} \, d\tau.
\]

By the inverse Fourier transform in (48), we have

\[
v(x, t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left\{ \int_0^t e^{-\mathcal{I}_c(\tau)} \hat{f}(\xi, \tau) e^{\int_0^t \left(- (A(\xi), \xi) + i(b(\xi), \xi)\right) \, ds} \, d\tau \right\} e^{i(x, \xi)} \, d\xi
\]

\[
= \frac{1}{(2\pi)^{n/2}} \int_0^t \left\{ \int_{\mathbb{R}^n} e^{i(x, \xi)} e^{-\mathcal{I}_c(\tau)} \hat{f}(\xi, \tau) e^{\int_0^t \left(- (A(\xi), \xi) + i(b(\xi), \xi)\right) \, ds} \, d\xi \right\} \, d\tau
\]
in view of (34), we arrive at

$$u(x, t) = \int_0^t \int_{\mathbb{R}^n} \left\{ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-y+j_t^i b(s, \xi) d\xi)} e^{-j_t^i (A(s) \xi, \xi)} d\xi \right\} e^{-I_C(\tau)} f(y, \tau) dy d\tau. \tag{49}$$

By (41), expression inside of the braces in the right-hand side of (49) is equal to

$$\frac{e^{I_C(t, \tau)}}{(2\sqrt{\pi})^n \sqrt{\det I_A(t, \tau)}} e^{-\left( I_{-1}^{-1}(t, \tau)(x-y+I_b(t, \tau)) - y - y + I_b(t, \tau) \right)/4}.$$ 

Applying the same arguments as in the proof of Lemma 3.1, we can rewrite (49) as (45), where $P(x, t, \tau)$ is given by (46).

### 4. Sharp estimates for solutions to the homogeneous weakly coupled parabolic system

In this section we obtain estimates for solution to the Cauchy problem (3). First, we prove the sharp pointwise estimate for $|u|$ with $\varphi \in L^p(\mathbb{R}^n)$, where $p \in [1, \infty]$.

**Theorem 4.1:** Let $(x, t)$ be an arbitrary point in $\mathbb{R}^{n+1}$ and $u$ be solution of problem (3). The sharp coefficient $H_p(t)$ in inequality (4) is given by (5). As a special case of (5) with $p = \infty$ one has (6).

**Proof:** By Proposition 2.1 and (32), (33), the sharp coefficient in inequality (4) is given by

$$H_p(t) = \max_{|z|=1} \left| \left( e^{I_C(t)} \right)^* z \right| \left( \int_{\mathbb{R}^n} e^{-p' \left| I_{-1}^{-1/2}(t)(x-y+I_b(t)) \right|^2 / 4} dy \right)^{1/p'} \tag{50}$$

Since

$$\left( e^{I_C(t)} \right)^* = e^{I_{C^*}(t)}, \tag{51}$$

it follows from (50) that

$$H_p(t) = \left| \frac{e^{I_{C^*}(t)}}{(2\sqrt{\pi})^n \det I_A^{1/2}(t)} \right| \left( \int_{\mathbb{R}^n} e^{-p' \left| I_{-1}^{-1/2}(t)(x-y+I_b(t)) \right|^2 / 4} dy \right)^{1/p'} \tag{52}$$

Further, we introduce the new variable $\xi = I_{-1}^{1/2}(t)(x-y+I_b(t))$. Since $y = -I_A^{1/2}(t)\xi + x + I_b(t)$, we have $dy = \det I_A^{1/2}(t) d\xi$, which together with (52) leads to the...
following representation

\[ \mathcal{H}_p(t) = \frac{\left| e^{z(t)} \right|}{(2\sqrt{\pi})^n \left( \det \mathcal{I}^{-1/2}_A(t) \right)^{1/p}} \left\{ \int_{\mathbb{R}^n} e^{-p'|\xi|^2/4} \, d\xi \right\}^{1/p'}, \tag{53} \]

Passing to the spherical coordinates in (53), we obtain

\[ \mathcal{H}_p(t) = \frac{\left| e^{z(t)} \right|}{(2\sqrt{\pi})^n \left( \det \mathcal{I}^{-1/2}_A(t) \right)^{1/p}} \left\{ \int_{\mathbb{S}^{n-1}} \, d\sigma \int_0^{\infty} \rho^{n-1} e^{-p'\rho^2/4} \, d\rho \right\}^{1/p'}, \tag{54} \]

where \( \omega_n = 2\pi^{n/2}/\Gamma(n/2) \) is the area of the unit sphere \( \mathbb{S}^{n-1} \) in \( \mathbb{R}^n \). Further, making the change of variable \( \rho = \sqrt{u} \) in the integral

\[ \int_0^{\infty} \rho^{n-1} e^{-p'\rho^2/4} \, d\rho \]

and using the formula (e.g. [17], 3.381, item 4)

\[ \int_0^{\infty} x^{\alpha-1} e^{-\beta x} \, dx = \beta^{-\alpha} \Gamma(\alpha) \tag{55} \]

with positive \( \alpha \) and \( \beta \), we obtain

\[ \int_0^{\infty} \rho^{n-1} e^{-p'\rho^2/4} \, d\rho = \frac{1}{2} \int_0^{\infty} u^{((n/2)-1)} e^{-p'u/4} \, du = \frac{1}{2} \left( \frac{4}{p'} \right)^{n/2} \Gamma \left( \frac{n}{2} \right), \]

which leads to

\[ \omega_n \int_0^{\infty} \rho^{n-1} e^{-p'\rho^2/4} \, d\rho = \frac{2\pi^{n/2}}{\Gamma \left( \frac{n}{2} \right)} \frac{1}{2} \left( \frac{4}{p'} \right)^{n/2} \Gamma \left( \frac{n}{2} \right) = \frac{2\pi \pi^{n/2}}{p'^{n/2}}. \tag{56} \]

Substituting (56) into (54), we arrive at (5). As a particular case of (5) with \( p = \infty \), we obtain (6).

In the next assertion we prove the sharp pointwise estimate for \( |\partial u/\partial \ell| \) with \( \varphi \in [L^p(\mathbb{R}^n)]^m, p \in [1, \infty] \).

**Theorem 4.2:** Let \( (x, t) \) be an arbitrary point in \( \mathbb{R}^n_{\ell+1} \) and \( u \) be solution of problem (3). The sharp coefficient \( \mathcal{K}_{p,\ell}(t) \) in inequality (9) is given by (8).

As a consequence, the sharp coefficient \( \mathcal{K}_p(t) \) in inequality (11) is given by

\[ \mathcal{K}_p(t) = \frac{\left| e^{z(t)} \right|}{2^n \pi^{(n+p-1)/2} \left( \det \mathcal{I}^{-1/2}_A(t) \right)^{1/p} \left( \frac{\Gamma \left( \frac{p'+1}{2} \right)}{p'(n+p')/2} \right)^{1/p'}}. \tag{57} \]

As a special case of (57) with \( p = \infty \) one has (12).
**Proof:** Differentiating in (33) with respect to $x_j$, $j = 1, \ldots, n$, we obtain

\[
\frac{\partial}{\partial x_j} G(x, t) = -\frac{e^{I_C(t)}}{2(2\sqrt{\pi})^n \det \mathcal{I}_A^{1/2}(t)} \left\{ \mathcal{I}_A^{-1}(t)(x + \mathcal{I}_b(t)) \right\} \frac{1}{2} e^{-\left| \mathcal{I}_A^{-1/2}(t)(x + \mathcal{I}_b(t)) \right|^2 / 4}, \tag{58}
\]

which together with (2) and (32), leads to

\[
\frac{\partial u}{\partial \ell} = (\ell, \nabla_x) u = \sum_{j=1}^n \int_{\mathbb{R}^n} \ell_j \frac{\partial}{\partial x_j} G(x - y, t) \varphi(y) \, dy = -\frac{1}{2(2\sqrt{\pi})^n \det \mathcal{I}_A^{1/2}(t)} \times \int_{\mathbb{R}^n} \left( \mathcal{I}_A^{-1}(t)(x - y + \mathcal{I}_b(t)), \ell \right) e^{-\left| \mathcal{I}_A^{-1/2}(t)(x - y + \mathcal{I}_b(t)) \right|^2 / 4} e^{I_C(t)} \varphi(y) \, dy. \tag{59}
\]

Applying Proposition 2.1 to (59), we conclude that the sharp coefficient in estimate (9) is given by

\[
K_{p, \ell}(t) = \max_{|z|=1} \frac{\left| (e^{I_C(t)})^* z \right|}{2(2\sqrt{\pi})^n \det \mathcal{I}_A^{1/2}(t)} \times \left\{ \int_{\mathbb{R}^n} \left| (\mathcal{I}_A^{-1}(t)(x - y + \mathcal{I}_b(t)), \ell) \right|^p' e^{-p' \left| \mathcal{I}_A^{-1/2}(t)(x - y + \mathcal{I}_b(t)) \right|^2 / 4} \, dy \right\}^{1/p'},
\]

which, in view of (51), implies

\[
K_{p, \ell}(t) = \frac{\left| e^{I_C(t)} \right|}{2(2\sqrt{\pi})^n \det \mathcal{I}_A^{1/2}(t)} \times \left\{ \int_{\mathbb{R}^n} \left| (\mathcal{I}_A^{-1}(t)(x - y + \mathcal{I}_b(t)), \ell) \right|^p' e^{-p' \left| \mathcal{I}_A^{-1/2}(t)(x - y + \mathcal{I}_b(t)) \right|^2 / 4} \, dy \right\}^{1/p'}.
\]

Changing the variable $\xi = \mathcal{I}_A^{-1/2}(t)(x - y + \mathcal{I}_b(t))$ in the last integral in view of $dy = (\det \mathcal{I}_A^{1/2}(t)) \, d\xi$, we arrive at the following representation

\[
K_{p, \ell}(t) = \frac{\left| e^{I_C(t)} \right| (\det \mathcal{I}_A^{1/2}(t))^{1/p'}}{2(2\sqrt{\pi})^n \det \mathcal{I}_A^{1/2}(t)} \left\{ \int_{\mathbb{R}^n} \left| (\mathcal{I}_A^{-1/2}(t)\xi, \ell) \right|^p' e^{-p'|\xi|^2 / 4} \, d\xi \right\}^{1/p'}.
\]

By the symmetricity of $\mathcal{I}_A^{-1/2}(t)$, we have

\[
K_{p, \ell}(t) = \frac{\left| e^{I_C(t)} \right|}{2(2\sqrt{\pi})^n (\det \mathcal{I}_A^{1/2}(t))^{1/p}} \left\{ \int_{\mathbb{R}^n} \left| (\xi, \mathcal{I}_A^{-1/2}(t)\ell) \right|^p' e^{-p'|\xi|^2 / 4} \, d\xi \right\}^{1/p'}. \tag{60}
\]
Passing to the spherical coordinates in (60), we obtain

\[
K_{p, \ell}(t) = \frac{|e^{\mathcal{I}C^s(t)}|}{2(2\sqrt{\pi})^n (\det \mathcal{I}_A^{1/2}(t))^{1/p}} \times \left\{ \int_0^\infty \rho^{p' + n - 1} e^{-\rho^2/4} \, d\rho \int_{S^{n-1}} |(e_\sigma, \mathcal{I}_A^{-1/2}(t) \ell)|^{p'} \, d\sigma \right\}^{1/p'},
\]

(61)

where \(e_\sigma\) is the \(n\)-dimensional unit vector joining the origin to a point \(\sigma\) of the sphere \(S^{n-1}\). Let \(\vartheta\) be the angle between \(e_\sigma\) and \(\mathcal{I}_A^{-1/2}(t) \ell\). We have

\[
\int_{S^{n-1}} |(e_\sigma, \mathcal{I}_A^{-1/2}(t) \ell)|^{p'} \, d\sigma = 2 \omega_{n-1} |\mathcal{I}_A^{-1/2}(t) \ell|^{p'} \int_0^{\pi/2} \cos^{p'} \vartheta \sin^{n-2} \vartheta \, d\vartheta
\]

(62)

Further, making the change of variable \(\rho = \sqrt{u}\) in the integral

\[
\int_0^\infty \rho^{p' + n - 1} e^{-\rho^2/4} \, d\rho
\]

and applying (55), we obtain

\[
\int_0^\infty \rho^{p' + n - 1} e^{-\rho^2/4} \, d\rho = \frac{1}{2} \int_0^\infty u^{(p' + n)/2 - 1} e^{-u/4} \, du
\]

\[
= \frac{1}{2} \left( \frac{4}{p'} \right)^{(p' + n)/2} \Gamma \left( \frac{n + p'}{2} \right).
\]

(63)

Combining (62) and (63) with (61), we arrive at (8).

Formula (57) follows from (8) and (10). As a particular case of (57) with \(p = \infty\), we obtain (12).

Remark 4.1: We note that

\[
|e^{\mathcal{I}C^s(t)}| = |e^{\int_0^t C^s(t) \, dt}| = \left\{ \rho \left( e^{\int_0^t C(t) \, dt} e^{\int_0^t C^s(t) \, dt} \right) \right\}^{1/2} = \left\{ \rho \left( e^{2 \int_0^t C_s(t) \, dt} \right) \right\}^{1/2},
\]

where \(C_s(t) = (C^s(t) + C(t))/2\) is the symmetric component of the matrix-valued function \(C(t)\). This means that the coefficients in all sharp estimates for solutions to the homogeneous weakly coupled parabolic system in the layer do not depend on the antisymmetric component of the matrix-valued function \(C(t)\).
5. Sharp estimates for solutions to the nonhomogeneous weakly coupled parabolic system

In this section we derive estimates for solution to the Cauchy problem (14). Here we suppose that \( f \in L^p(\mathbb{R}^{n+1}_T)^m \cap C^\alpha(\mathbb{R}^{n+1}_T)^m, \alpha \in (0,1). \) It follows from the known assertions for the single parabolic equation (e.g. [2, Ch. 1, Sect. 7 and 9]) and (34), (44) that formula (45), where \( P \) is defined by (46), solves problem (14) with \( f \in L^p(\mathbb{R}^{n+1}_T)^m \cap C^\alpha(\mathbb{R}^{n+1}_T)^m. \)

First, we prove the sharp pointwise estimate for \(|u|\), where \( u \) is solution of problem (14).

**Theorem 5.1:** Let \((x, t)\) be an arbitrary point in \( \mathbb{R}^{n+1}_T \) and \( u \) be solution of problem (14). Let us suppose that the integral

\[
\int_0^t \frac{|e^{I_C^*(t, \tau)}z|^{p'}}{(\det I_A^{1/2}(t, \tau))^{p'-1}} d\tau
\]

is convergent. The sharp coefficient \( N_p(t) \) in inequality (15) is given by (16). As a special case of (16) with \( p = \infty \) one has (17).

**Proof:** By Proposition 2.1 and (45), (46), the sharp coefficient in inequality (15) is given by

\[
N_p(t) = \frac{1}{(2\sqrt{\pi})^n} \max_{|z|=1} \left\{ \int_0^t \int_{\mathbb{R}^n} \frac{|e^{I_C^*(t, \tau)}z|^{p'}}{(\det I_A^{1/2}(t, \tau))^{p'-1}} e^{-p|I_A^{-1/2}(t, \tau)(x-y+I_b(t, \tau))|^2/4} dy d\tau \right\}^{1/p'}
\]

which, in view of

\[
\left(e^{I_C^*(t, \tau)}\right)^* = e^{I_C^*(t, \tau)},
\]

implies

\[
N_p(t) = \frac{1}{(2\sqrt{\pi})^n} \max_{|z|=1} \left\{ \int_0^t \int_{\mathbb{R}^n} \frac{|e^{I_C^*(t, \tau)}z|^{p'}}{(\det I_A^{1/2}(t, \tau))^{p'-1}} e^{-p|I_A^{-1/2}(t, \tau)(x-y+I_b(t, \tau))|^2/4} dy d\tau \right\}^{1/p'}
\]

(65)

Now, we introduce the new variable \( \xi = I_A^{-1/2}(t, \tau)(x-y+I_b(t, \tau)) \). Since \( y = -I_A^{-1/2}(t, \tau)\xi + x + I_b(t, \tau) \), we have \( dy = \det I_A^{1/2}(t, \tau) d\xi \), which together with (65)
leads to the following representation

\[ N_p(t) = \frac{1}{(2\sqrt{\pi})^n} \max_{|z|=1} \left\{ \int_0^t \int_{\mathbb{R}^n} \frac{|e^{\mathcal{I}_C^+(t,\tau)} z|^{p'}}{\left( \det \mathcal{I}_A^{1/2}(t, \tau) \right)^{p'-1}} e^{-p'|z|^2/4} \, d\xi \, d\tau \right\}^{1/p'} \]. \tag{66}

Passing to the spherical coordinates in (66), we obtain

\[ N_p(t) = \frac{1}{(2\sqrt{\pi})^n} \max_{|z|=1} \left\{ \int_0^t \frac{|e^{\mathcal{I}_C^+(t,\tau)} z|^{p'}}{\left( \det \mathcal{I}_A^{1/2}(t, \tau) \right)^{p'-1}} \, d\tau \int_{S^{n-1}} \int_0^\infty \rho^{n-1} e^{-p'\rho^2/4} \, d\rho \right\}^{1/p'} \]

\[ = \frac{1}{(2\sqrt{\pi})^n} \max_{|z|=1} \left\{ \omega_n \int_0^t \frac{|e^{\mathcal{I}_C^+(t,\tau)} z|^{p'}}{\left( \det \mathcal{I}_A^{1/2}(t, \tau) \right)^{p'-1}} \, d\tau \int_0^\infty \rho^{n-1} e^{-p'\rho^2/4} \, d\rho \right\}^{1/p'} \]. \tag{67}

Substituting (56) into (67), we arrive at (16). As a particular case of (16) with \( p = \infty \), we obtain (17).

In the next assertion we prove the sharp pointwise estimate for \( |\partial u/\partial \ell| \), where \( u \) is solution of problem (14).

**Theorem 5.2:** Let \( (x, t) \) be an arbitrary point in \( \mathbb{R}_T^{n+1} \) and let \( u \) solves problem (14). Suppose that the integral

\[ \int_0^t \left\{ \mathcal{I}_A^{-1/2}(t, \tau) \ell \right\}^{p'} \left\| e^{\mathcal{I}_C^+(t,\tau)} z \right\|^{p'} \left( \det \mathcal{I}_A^{1/2}(t, \tau) \right)^{p'-1} \, d\tau \]

is convergent for every unit \( n \)-dimensional vector \( \ell \). Then the sharp coefficient \( C_{p,\ell}(t) \) in inequality (19) is given by (18).

As a consequence of (18), the sharp coefficient \( C_p(t) \) in inequality (21) is given by

\[ C_p(t) = \frac{1}{\left\{ 2^n \pi (n+p-1)/2 \right\}^{1/p}} \left\{ \Gamma \left( \frac{p'+1}{2} \right) \right\}^{1/p'} \left\{ \frac{p'(n+p)/2}{p'(n+p)/2} \right\}^{1/p'} \]

\[ \times \max \max_{|\ell|=1} \max_{|z|=1} \left\{ \int_0^t \left\| \mathcal{I}_A^{-1/2}(t, \tau) \ell \right\|^{p'} \left\| e^{\mathcal{I}_C^+(t,\tau)} z \right\|^{p'} \left( \det \mathcal{I}_A^{1/2}(t, \tau) \right)^{p'-1} \, d\tau \right\}^{1/p'} \]. \tag{68}

As a special case of (68) with \( p = \infty \) one has (22).
Proof: Differentiating in (46) with respect to \( x_j, j = 1, \ldots, n \), we obtain

\[
\frac{\partial}{\partial x_j} P(x, t, \tau) = -\frac{e^{IC(t, \tau)}}{2(2\sqrt{\pi})^n \det I_A^{1/2}(t, \tau)} \times \{ I_A^{-1}(t, \tau)(x + I_b(t, \tau)) \} j e^{-\frac{1}{4} (I_A^{-1/2}(t, \tau)(x + I_b(t, \tau)))^2}
\]

which together with (2) and (45), leads to

\[
\frac{\partial u}{\partial \ell} = (\ell, \nabla_x) u = \sum_{j=1}^{n} \int_0^t \int_{\mathbb{R}^n} \ell_j \frac{\partial}{\partial x_j} P(x - y, t, \tau) f(y, \tau) \, dy \, d\tau
\]

\[
= -\frac{1}{2(2\sqrt{\pi})^n} \times \int_0^t \int_{\mathbb{R}^n} \frac{e^{IC(t, \tau)}}{\det I_A^{1/2}(t, \tau)} (I_A^{-1}(t, \tau)(x - y + I_b(t, \tau)), \ell)
\]

\[
\times e^{-\frac{1}{4} (I_A^{-1/2}(t, \tau)(x - y + I_b(t, \tau)))^2} f(y, \tau) \, dy \, d\tau.
\]

Applying Proposition 2.1 to the last representation, we conclude that the sharp coefficient in estimate (19) is given by

\[
C_{p,\ell}(t) = \frac{1}{2(2\sqrt{\pi})^n}
\]

\[
\times \max_{|z|=1} \left\{ \int_0^t \int_{\mathbb{R}^n} \left| (e^{IC(t, \tau)^*} z)^{p'} \right|^{p'} \left| (I_A^{-1}(t, \tau)(x - y + I_b(t, \tau)), \ell) \right|^{p'}
\]

\[
\times e^{-\frac{p'}{4} (I_A^{-1/2}(t, \tau)(x - y + I_b(t, \tau)))^2} \, dy \, d\tau \right\}^{1/p'}
\]

which in view of (64), implies

\[
C_{p,\ell}(t) = \frac{1}{2(2\sqrt{\pi})^n}
\]

\[
\times \max_{|z|=1} \left\{ \int_0^t \int_{\mathbb{R}^n} \left| e^{IC(t, \tau)} z \right|^{p'} \left| (I_A^{-1}(t, \tau)(x - y + I_b(t, \tau)), \ell) \right|^{p'}
\]

\[
\times e^{-\frac{p'}{4} (I_A^{-1/2}(t, \tau)(x - y + I_b(t, \tau)))^2} \, dy \, d\tau \right\}^{1/p'}.
\]
Changing the variable \( \xi = I_A^{-1/2}(t, \tau)(x - y + I_b(t, \tau)) \) in the last integral in view of 
\( \mathrm{d}y = (\det I_A^{-1/2}(t, \tau)) \mathrm{d}\xi \), we arrive at the following representation

\[
C_{p, \ell}(t) = \frac{1}{2(2\sqrt{\pi})^n} \times \max_{|z|=1} \left\{ \int_0^t \int_{\mathbb{R}^n} \left| e^{I_C^*(t, \tau) z} \right|^{p'} |(I_A^{-1/2}(t, \tau) \xi, \ell)|^{p'} e^{-p'|\xi|^{2/4}} \mathrm{d}\xi \mathrm{d}\tau \right\}^{1/p'}.
\]

By the symmetricity of \( I_A^{-1/2}(t, \tau) \), we have

\[
C_{p, \ell}(t) = \frac{1}{2(2\sqrt{\pi})^n} \times \max_{|z|=1} \left\{ \int_0^t \int_{\mathbb{R}^n} \left| e^{I_C^*(t, \tau) z} \right|^{p'} |(I_A^{-1/2}(t, \tau) \xi, \ell)|^{p'} e^{-p'|\xi|^{2/4}} \mathrm{d}\xi \mathrm{d}\tau \right\}^{1/p'}.
\]

Passing to the spherical coordinates in the inner integral, we obtain

\[
C_{p, \ell}(t) = \frac{1}{2(2\sqrt{\pi})^n} \times \max_{|z|=1} \left\{ \int_0^t \int_{\mathbb{R}^n} \left| e^{I_C^*(t, \tau) z} \right|^{p'} |(I_A^{-1/2}(t, \tau) \xi, \ell)|^{p'} e^{-p'|\xi|^{2/4}} \mathrm{d}\xi \mathrm{d}\tau \right\}^{1/p'}
\]

where \( e_\sigma \) is as before, the \( n \)-dimensional unit vector joining the origin to a point \( \sigma \) of the sphere \( S^{n-1} \).

Let \( \vartheta \) be the angle between \( e_\sigma \) and \( I_A^{-1/2}(t, \tau) \ell \). Similarly to (62), we have

\[
\int_{S^{n-1}} |(e_\sigma, I_A^{-1/2}(t, \tau) \ell)|^{p'} \mathrm{d}\sigma = |I_A^{-1/2}(t, \tau) \ell|^{p'} 2\pi^{(n-1)/2} \Gamma\left(\frac{p'+1}{2}\right) \Gamma\left(\frac{n+p'}{2}\right).
\]

Combining (63) and (71) with (70), we arrive at (18). Equality (68) follows from (18) and (20). Putting \( p = \infty \) in (68), we arrive at (22).

\[\blacksquare\]

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References

[1] Eidel’man SD. Parabolic systems. Amsterdam: North-Holland and Noordhoff; 1969.
[2] Friedman A. Partial differential equations of parabolic type. Malabar (FL): R.E. Krieger Publ. Comp.; 1983.
[3] Sobolev SL. Partial differential equations of mathematical physics. New York: Dover Publications; 1989.
[4] Tikhonov AN, Samarskii AA. Equations of mathematical physics. New York: Dover Publications; 1990.
[5] Kresin G, Maz’ya V. Sharp estimates for the gradients of solutions to linear parabolic second order equation in the layer. Publ. online in Appl. Anal.; 1990. (doi: 10.1080/00036811.2020.1732356).
[6] Kresin G, Maz’ya V. Sharp real-part theorems. A unified approach. Berlin: Springer; 2007. (Lect. Notes in Math.; 1903).
[7] Kresin G, Maz’ya V. Maximum principles and sharp constants for solutions of elliptic and parabolic systems. Providence (RI): Amer. Math. Soc.; 2012. (Math. Surveys and Monographs; 183).
[8] Kresin G, Maz’ya V. Optimal estimates for derivatives of solutions to Laplace, Lamé and Stokes equations. J Math Sci, New York. 2014;196(3):300–321.
[9] Kresin G, Maz’ya V. Sharp estimates for the gradient of solutions to the heat equation. Algebra i Analiz. 2019;31(3):136–153. (transl. in St. Petersburg Mathematical Journal).
[10] Kresin G, Ben Yaakov T. Sharp pointwise estimates for solutions of the modified Helmholtz equation. Pure Appl Funct Anal. 2020;5(2):349–367.
[11] Lancaster P. Theory of matrices. New York: Academic Press; 1969.
[12] Hunter JK. Notes on partial differential equations. University of California, Davis; 2014. (el. edition).
[13] Oleinik OA. Lectures on partial differential equations. 2nd ed. BINOM: Moscow; 2005. (in Russian).
[14] Shubin MA. Lectures on Equations of Mathematical Physics, 2nd ed., Moscow: MCNMO; 2003. (in Russian) (Engl. transl. Invitation to Partial Differential Equations, el. edition).
[15] Vladimirov VS. Equations of mathematical physics. 2nd English ed. Moscow: Mir Publ.; 1984.
[16] Vulich BZ. A short course in the theory of functions of the real variable. 2nd ed. translator; Jeffrey A, editor. Moscow: Nauka; 1973. (in Russian).
[17] Gradshteyn IS, Ryzhik IM, Jeffrey A. Table of integrals, series and products. 5th ed. New York: Academic Press; 1994.