Good filtrations and strong $F$-regularity of the ring of $U_P$-invariants

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Abstract

Let $k$ be an algebraically closed field of positive characteristic, $G$ a reductive group over $k$, and $V$ a finite dimensional $G$-module. Let $P$ be a parabolic subgroup of $G$, and $U_P$ its unipotent radical. We prove that if $S = \text{Sym} V$ has a good filtration, then $S^{U_P}$ is strongly $F$-regular.

1. Introduction

Throughout this paper, $k$ denotes an algebraically closed field, and $G$ a reductive group over $k$. We fix a maximal torus $T$ and a Borel subgroup $B$ which contains $T$. We fix a base $\Delta$ of the root system $\Sigma$ of $G$ so that $B$ is negative. For any weight $\lambda \in X(T)$, we denote the induced module $\text{ind}^G_B(\lambda)$ by $\nabla_G(\lambda)$. We denote the set of dominant weights by $X^+$. For $\lambda \in X^+$, we call $\nabla_G(\lambda)$ the dual Weyl module of highest weight $\lambda$. Note that for $\lambda \in X(T)$, $\text{ind}^G_B(\lambda) \neq 0$ if and only if $\lambda \in X^+$ [Jan. (II.2.6)], and if this is the case, $\nabla_G(\lambda) = \text{ind}^G_B(\lambda)$ is finite dimensional [Jan. (II.2.1)]. We denote $\nabla_G(-w_0\lambda)^*$ by $\Delta_G(\lambda)$, and call it the Weyl module of highest weight $\lambda$, where $w_0$ is the longest element of the Weyl group of $G$.

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We say that a $G$-module $W$ is good if $\Ext^1_G(\Delta_G(\lambda), W) = 0$ for any $\lambda \in X^+$. A filtration $0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_r$ or $0 = W_0 \subset W_1 \subset W_2 \subset \cdots$ of $W$ is called a good filtration of $W$ if $\bigcup_i W_i = W$, and for any $i \geq 1$, $W_i/W_{i-1} \cong \nabla_G(\lambda(i))$ for some $\lambda(i) \in X^+$. A $G$-module $W$ has a good filtration if and only if $W$ is good and of countable dimension [Don1]. See also [Fr] and [Has1, (III.1.3.2)].

Let $V$ be a finite dimensional $G$-module. Let $P$ be a parabolic subgroup of $G$ containing $B$, and $U_P$ its unipotent radical. The objective of this paper is to prove the following.

**Corollary 5.5** Let $k$ be of positive characteristic. Let $V$ be a finite dimensional $G$-module, and assume that $S = \Sym V$ is good as a $G$-module. Then $S^{U_P}$ is a finitely generated strongly $F$-regular Gorenstein UFD.

An $F$-finite Noetherian ring $R$ of characteristic $p$ is said to be strongly $F$-regular if for any nonzerodivisor $a$ of $R$, there exists some $r > 0$ such that the $R^{(r)}$-linear map $aF^r : R^{(r)} \to R$ ($x^{(r)} \mapsto ax^{(r)}$) is $R^{(r)}$-split [HH1]. See [2.1] for the notation. A strongly $F$-regular $F$-finite ring is $F$-regular in the sense of Hochster–Huneke [HH2], and hence it is Cohen–Macaulay normal ([HH3 (4.2)], [Kun], and [Vel (0.10)]).

Under the same assumption as in Corollary 5.5, it has been known that $S^G$ is strongly $F$-regular [Has2]. This old result is a corollary to our Corollary 5.5 since $T$ is linearly reductive and $S^G = S^B = (S^U)^T$ is a direct summand subring of $S^U$. Under the same assumption as in Corollary 5.5, it has been proved that $S^U$ is $F$-pure [Has6]. An $F$-finite Noetherian ring $R$ of characteristic $p$ is said to be $F$-pure if the Frobenius map $F : R^{(1)} \to R$ splits as an $R^{(1)}$-linear map. Almost by definition, an $F$-finite strongly $F$-regular ring is $F$-pure, and hence Corollary 5.5 (or Corollary 4.14) yields this old result, too.

Popov [Pop3] proved that if the characteristic of $k$ is zero, $G$ is a reductive group over $k$, and $A$ is a finitely generated $G$-algebra, then $A$ has rational singularities if and only if $A^{U}$ does so. Corollary 5.5 (or Corollary 4.14) can be seen as a weak characteristic $p$ version of one direction of this result. For a characteristic $p$ result related to the other direction, see Corollary 3.9.

Section 2 is preliminaries. We review the Frobenius twisting of rings, modules, and representations. We also review the basics of $F$-singularities such as $F$-rationality and $F$-regularity.
In Section 3, we study the ring theoretic properties of the invariant subring $k[G]^U$ of the coordinate ring $k[G]$. The main results of this section are Lemma 3.8 and Corollary 3.9.

In Section 4, we state and prove our main result for $P = B$. In order to do so, we introduce the notion of $G$-strong $F$-regularity and $G$-$F$-purity. These notions have already appeared in [Has2] essentially. Our main theorem in the most general form can be stated using these words (Theorem 4.12). As in [Has2], Steinberg modules play important roles.

In Section 5, we generalize the main results in Section 4 to the case of general $P$. Donkin’s results on $U_P$-invariants of good $G$-modules play an important role here.

In Section 6, we give some examples. The first one is the action associated with a finite quiver. The second one is a special case of the first, and is a determinantal variety studied by De Concini and Procesi [DP]. The third one is also an example of the first. It gives some new understandings on the study of Goto–Hayasaka–Kurano–Nakamura [GHKN]. It also has some relationships with Miyazaki’s study [Miy].

In Section 7, we prove the following.

**Theorem 7.11** Let $S$ be a scheme, $G$ a reductive $S$-group acting trivially on a Noetherian $S$-scheme $X$. Let $M$ be a locally free coherent $(G, O_X)$-module. Then

$$\text{Good}(\text{Sym} M) = \{ x \in X \mid \text{Sym}(\kappa(x) \otimes_{O_X} M_x) \text{ is a good } (\text{Spec} \kappa(x) \times_S G) \text{-module} \},$$

and $\text{Good}(\text{Sym} M)$ is Zariski open in $X$.

For a reductive group $G$ over a field which is not linearly reductive, there is a finite dimensional $G$-module $V$ such that $(\text{Sym } V)^G$ is not Cohen–Macaulay [Kem]. On the other hand, in characteristic zero, a reductive group $G$ is linearly reductive, and Hochster and Roberts [HR] proved that $(\text{Sym } V)^G$ is Cohen–Macaulay for any finite dimensional $G$-module $V$. Later, Boutot proved that $(\text{Sym } V)^G$ has rational singularities [Bt]. In view of Corollary 5.5 and Theorem 7.11 it seems that the condition Sym $V$ being good is an appropriate condition to ensure that the good results in characteristic zero still holds.
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2. Preliminaries

(2.1) Throughout this paper, \( p \) denotes a prime number. Let \( K \) be a perfect field of characteristic \( p \).

For a \( K \)-space \( V \) and \( e \in \mathbb{Z} \), we denote the abelian group \( V \) with the new \( K \)-space structure \( \alpha \cdot v = \alpha^{p^{-e}} v \) by \( V^{(e)} \), where the product of \( \alpha^{p^{-e}} \) and \( v \) in the right hand side is given by the original \( K \)-space structure of \( V \). An element of \( V \), viewed as an element of \( V^{(e)} \) is sometimes denoted by \( v^{(e)} \) to avoid confusion. Thus we have \( v^{(e)} + w^{(e)} = (v + w)^{(e)} \) and \( \alpha v^{(e)} = (\alpha^{p^{-e}}) v^{(e)} \).

If \( f : V \to W \) is a \( K \)-linear map, then \( f^{(e)} : V^{(e)} \to W^{(e)} \) given by \( f^{(e)}(v^{(e)}) = w^{(e)} \) is a \( K \)-linear map again. Note that \( (?)^{(e)} \) is an autoequivalence of the category of \( K \)-vector spaces.

If \( A \) is a \( K \)-algebra, then \( A^{(e)} \) with the multiplicative structure of \( A \) is a \( K \)-algebra. So \( a^{(e)} b^{(e)} = (ab)^{(e)} \) for \( a, b \in A \). If \( M \) is an \( A \)-module, then \( M^{(e)} \) is an \( A^{(e)} \)-module by \( a^{(e)} m^{(e)} = (am)^{(e)} \). For a \( K \)-algebra \( A \) and \( r \geq 0 \), the \( r \)th Frobenius map \( F^r = F_A^r : A \to A \) is defined by \( F^r(a) = a^{p^r} \). Then \( F^r : A^{(r+e)} \to A^{(e)} \) is a \( K \)-algebra map for \( e \in \mathbb{Z} \). Note that \( F^r(a^{(r+e)}) = (a^{p^r})^{(e)} \).

In commutative ring theory, \( A^{(e)} \) is sometimes denoted by \( ^eA \) or \( A^{[p^e]} \).

(2.2) For a \( K \)-scheme \( X \), the scheme \( X \) with the new \( K \)-scheme structure \( X \xrightarrow{f} \text{Spec } K \xrightarrow{a(F_{K_{(e)}})} \text{Spec } K \) is denoted by \( X^{(e)} \), where \( f \) is the original structure map of \( X \) as a \( K \)-scheme. So for a \( K \)-algebra \( A \), \( \text{Spec } A^{(e)} \) is identified with \( (\text{Spec } A)^{(e)} \). The Frobenius map \( F^r : X \to X^{(r)} \) is a \( K \)-morphism. Note that \( (?)^{(e)} \) is an autoequivalence of the category of \( K \)-schemes with the quasi-inverse \( (?)^{(-e)} \), and it preserves the product. So the canonical map \( (X \times Y)^{(e)} \to X^{(e)} \times Y^{(e)} \) is an isomorphism. If \( G \) is a \( K \)-group scheme, then with the product \( G^{(e)} \times G^{(e)} \cong (G \times G)^{(e)} \xrightarrow{\mu^{(e)}} G^{(e)} \), \( G^{(e)} \) is a \( K \)-group scheme, and \( F^r : G^{(e)} \to G^{(e+r)} \) is a homomorphism of \( K \)-group schemes. If \( V \) is a \( G \)-module, then \( V^{(e)} \) is a \( G^{(e)} \)-module in a natural way. Thus \( V^{(r)} \) is a \( G \)-module again for \( r \geq 0 \) via \( F^r : G \to G^{(r)} \). If \( V \) has a basis \( v_1, \ldots, v_n \), \( g \in G(K) \), and \( g v_j = \sum_i c_{ij} v_i \), then \( g v_j^{(r)} = \sum_i c_{ij} v_i^{(r)} \). If \( A \) is a \( G \)-algebra, then \( A^{(r)} \) is a \( G \)-algebra again. If \( M \) is a \( (G,A) \)-module, then
Let $A$ be an $F$-finite Noetherian $K$-algebra. We say that $A$ is $F$-regular if $A$ is $F$-finite and $A$ is a finite $A$-module. An $F$-regular ring is weakly $F$-regular. A weakly $F$-regular ring is $F$-rational. We say that $A$ is $F$-rational if $I = I^*$ for any ideal $I$ of $A$ [HH2]. We say that $A$ is $F$-rational if $I = I^*$ for any ideal $I$ generated by ht $I$ elements, where ht $I$ denotes the height of $I$.

Lemma 2.4. Let $A$ be a Noetherian $F$-algebra.

(i) If $A$ is very strongly $F$-regular, then it is strongly $F$-regular. The converse is true, if $A$ is either local, $F$-finite, or essentially of finite type over an excellent local ring.

(ii) If $A$ is strongly $F$-regular, then it is $F$-regular. An $F$-regular ring is weakly $F$-regular. A weakly $F$-regular ring is $F$-rational.

(iii) A pure subring of a strongly $F$-regular ring is strongly $F$-regular.

(iv) An $F$-rational ring is normal.

(v) An $F$-rational ring which is a homomorphic image of a Cohen–Macaulay ring is Cohen–Macaulay.
A locally excellent $F$-rational ring is Cohen–Macaulay.

If $A = \bigoplus_{i \geq 0} A_i$ is graded and $A_0$ is a field, and if $A$ is weakly $F$-regular, then $A$ is very strongly $F$-regular.

A Gorenstein $F$-rational ring is strongly $F$-regular.

Proof. (i) is [Has5, (3.6), (3.9), (3.35)]. (ii) is [Has5, (3.7)], [HH2, (4.15)], and [HH3, (4.2)]. (iii) is [Has5, (3.17)]. (iv) and (v) are [HH3, (4.2)]. (vi) is [Vel, (0.10)].

(vii) is [LS, (4.3)], if the field $A_0$ is $F$-finite. We prove the general case. By [HH2, (4.15)], $A_m$ is weakly $F$-regular, where $m = \bigoplus_{i > 0} A_i$ is the irrelevant ideal. Let $K$ be the perfect closure (the largest purely inseparable extension) of $A_0$, and set $B := K \otimes_{A_0} A$. Then $B$ is purely inseparable over $A$. It is easy to see that $B_m := B \otimes_A A_m$ is a local ring whose maximal ideal is $mB_m$. By [HH3, (6.17)], $B_m$ is weakly $F$-regular. By the proof of [LS, (4.3)], $B$ and $B_m$ are strongly $F$-regular. By [Has5, (3.17)], $A$ is strongly $F$-regular. As $A$ is finitely generated over the field $A_0$, $A$ is very strongly $F$-regular by (i).

(viii) Let $A$ be a Gorenstein $F$-rational ring. By [HH3, (4.2)], $A_m$ is Gorenstein $F$-rational for any maximal ideal $m$ of $A$. If $A_m$ is strongly $F$-regular for any maximal ideal $m$ of $A$, then $A$ is strongly $F$-regular by [Has5, (3.6)]. Thus we may assume that $(A, m)$ is local. Let $(x_1, \ldots, x_d)$ be a system of parameters of $A$. Then an element of $H^d_m(A)$ as the $d$th cohomology group of the modified Čech complex [BH, (3.5)] is of the form $a/(x_1 \cdots x_d)^t$ for some $t \geq 0$ and $a \in A$. This element is zero if and only if $a \in (x_1^t, \ldots, x_d^t)$, by [BH, (10.3.20)]. So this element is in the tight closure $(0)^*_m H^d_m(A)$ of $0$ if and only if $a \in (x_1^t, \ldots, x_d^t)^* = (x_1^t, \ldots, x_d^t)$, and hence $(0)^*_m H^d_m(A) = 0$. As $A$ is Gorenstein, $H^d_m(A)$ is isomorphic to the injective hull $E_A(A/m)$ of the residue field $A/m$. By [Has5, (3.6)], $A$ is strongly $F$-regular.

(2.5) Let $K$ be a field of characteristic zero, and $A$ a $K$-algebra of finite type. We say that $A$ is of strongly $F$-regular type if there is a finitely generated $\mathbb{Z}$-subalgebra $R$ of $A$ and a finitely generated flat $R$-algebra $A_R$ such that $A \cong K \otimes_R A_R$, and for any maximal ideal $m$ of $R$, $R/m \otimes_R A_R$ is strongly $F$-regular. See [Har, (2.5.1)].
3. The invariant subring $k[G]^U$

(3.1) Let the notation be as in the introduction. Let $\Lambda$ be an abelian group. We say that $A = \bigoplus_{\lambda \in \Lambda} A_{\lambda}$ is a $\Lambda$-graded $G$-algebra if $A$ is both a $G$-algebra and a $\Lambda$-graded $k$-algebra, and each $A_{\lambda}$ is a $G$-submodule of $A$ for $\lambda \in \Lambda$. This is the same as to say that $A$ is a $G \times \text{Spec } k\Lambda$-algebra, where $k\Lambda$ is the group algebra of $\Lambda$ over $k$. It is a commutative cocommutative Hopf algebra with each $\lambda \in \Lambda$ group-like.

We say that a $\mathbb{Z}$-graded $k$-algebra $A = \bigoplus_{i \in \mathbb{Z}} A_i$ is positively graded if $A_i = 0$ for $i < 0$ and $k \cong A_0$.

(3.2) Let the notation be as in the introduction.

We need to review Popov–Grosshans filtration [Pop2], [Grs2].

Let us fix (until the end of this section) a function $h : X(T) \rightarrow \mathbb{Z}$ such that (i) $h(X^+) \subset \mathbb{N} = \{0, 1, \ldots\}$; (ii) $h(\lambda) > h(\mu)$ whenever $\lambda > \mu$; (iii) $h(\chi) = 0$ for $\chi \in X(G)$. Such a function $h$ exists [Grs2 Lemma 6].

Let $V$ be a $G$-module. For a poset ideal $\pi$ of $X^+$, we define $O_{\pi}(V)$ to be the sum of all the $G$-submodules $W$ of $V$ such that $W$ belongs to $\pi$, that is, if $\lambda \in X^+$ and $W_\lambda \neq 0$, then $\lambda \in \pi$. $O_{\pi}(V)$ is the biggest $G$-submodule of $V$ belonging to $\pi$. We set $\pi(n) := h^{-1}\{0, 1, \ldots, n\}$ for $n \geq 0$ and $\pi(n) = \emptyset$ for $n < 0$. We also define $V \langle n \rangle := O_{\pi(n)}(V)$.

For a $G$-algebra $A$, $(A\langle n \rangle)$ is a filtration of $A$. That is, $1 \in A\langle 0 \rangle \subset A\langle 1 \rangle \subset \cdots, \bigcup_n A\langle n \rangle = A$, and $A\langle n \rangle \cdot A\langle m \rangle \subset A\langle n + m \rangle$. The Rees ring $R(A)$ of $A$ is the subring $\bigoplus_n A\langle n \rangle t^n$ of $A[t]$. Letting $G$ act on $t$ trivially, $A[t]$ is a $G$-algebra, and $R(A)$ is a $G$-subalgebra of $A[t]$. So the associated graded ring $G(A) := R(A)/tR(A)$ is also a $G$-algebra.

We denote the opposite of $U$ by $U^+$.

Theorem 3.3 (Grosshans [Grs1, Theorem 16]). Let $A$ be a $G$-algebra which is good as a $G$-module. There is a $G$-algebra isomorphism $\Phi : G(A) \rightarrow (A^{U^+} \otimes k[G]^U)^T$, where $U$ acts right regularly on $k[G]$, $T$ acts right regularly on $k[G]^U$ (because $T$ normalizes $U$), and $G$ acts left regularly on $k[G]^U$ and trivially on $A^{U^+}$.

The direct product $G \times G$ acts on the coordinate ring $k[G]$ by

$((g_1, g_2)f)(g) = f(g_1^{-1}gg_2) \quad (f \in k[G], g, g_1, g_2 \in G(k)).$

In particular, $k[G]$ is a $G \times B$-algebra. Taking the invariant subring by the subgroup $U = \{e\} \times U \subset G \times B$, $k[G]^U$ is a $G \times T$-algebra, since $T$
normalizes $U$. Thus $k[G]^U = \bigoplus_{\lambda \in X(T)} k[G]_{\lambda}^U$ is an $X(T)$-graded $G$-algebra. As a $G$-module,

$$k[G]_{\lambda}^U = \{ f \in k[G] \mid f(gb) = \lambda(b)f(g) \} \cong ((-\lambda) \otimes k[G])^B = \text{ind}_G^B(-\lambda)$$

for $\lambda \in X(B) = X(T)$ by the definition of induction, see [Jan, (I.3.3)]. Thus we have:

**Lemma 3.4.** $k[G]^U \cong \bigoplus_{\lambda \in X^+} \nabla G(\lambda) \boxtimes (-\lambda)$ as a $G \times T$-module. It is an integral domain.

The converse is also true.

**Lemma 3.5.** Let $A$ be a $G \times T$-algebra such that $A \cong \bigoplus_{\lambda \in X^+} \nabla G(\lambda) \boxtimes (-\lambda)$ as a $G \times T$-module and that $A^{U^+}$ is a domain, where $U^+ = U^+ \times \{ e \} \subset G \times T$. Then $A \cong k[G]^U$ as a $G \times T$-algebra.

**Proof.** Let $\varphi : X^+ \to X(T) \times X(T)$ be the semigroup homomorphism given by $\varphi(\lambda) = (\lambda, -\lambda)$. $A^{U^+}$ is a $\varphi(X^+)$-graded domain, and each homogeneous component $A^{U^+}_{\varphi^{\lambda}} = \nabla G(\lambda)^{U^+} \boxtimes (-\lambda)$ is one-dimensional. So by [Has3, Lemma 5.5], $A^{U^+} \cong k\varphi(X^+)$ as an $X(T) \times X(T)$-graded $k$-algebra.

Set $G' = G \times T$, and $T' = T \times T$. Define $h' : X(T') \cong X(T) \times X(T) \to \mathbb{Z}$ by $h'(\lambda, \mu) = h(\lambda)$. For a $G'$-algebra $A'$, we have a filtration of $A'$ from $h'$ as in [8.2]. We denote the associated graded algebra by $G'(A')$.

It is easy to see that $A \cong G'(A)$, and this is isomorphic to $B := (k\varphi(X^+) \otimes (k[G]^U \boxtimes k[T]))^{T'}$ by Theorem 3.3 (applied to $G'$). We define $\psi : k[G]^U \to B$ by

$$\psi(a \otimes t_{-\lambda}) = (t_\lambda \otimes t_{-\lambda}) \otimes (a \otimes t_{-\lambda}) \otimes t_\lambda,$$

where $a \in \nabla G(\lambda)$ and for $\mu \in X(T)$, $t_\mu$ is the element $\mu$ considered as a basis element of $kX(T) = k[T]$. We consider that $\nabla G(\lambda) \boxtimes (-\lambda) \subset k[G]^U$. With respect to the left regular action, $t_\lambda$ is of weight $-\lambda$. So $\psi$ is a $G \times T$-algebra isomorphism. Thus $A \cong k[G]^U$ as a $G \times T$-algebra, as desired.

Assume that $G$ is semisimple simply connected. Then by [Pop2], $X(B) \to \text{Pic}(G/B) (\lambda \mapsto \mathcal{L}(\lambda))$ is an isomorphism, where $\mathcal{L}(\lambda) = \mathcal{L}_{G/B}(\lambda)$ is the $G$-linearized invertible sheaf on the flag variety $G/B$, associated to the $B$-module $\lambda$, see [Jan, (I.5.8)]. Thus we have
Lemma 3.6. If $G$ is semisimple and simply connected, then the Cox ring (the total coordinate ring, see [Cox], [EKW]) $\text{Cox}(G/B)$ is isomorphic to $\bigoplus_{\lambda \in X^+} \nabla_G(\lambda)$, as an $X(T)$-graded $G$-module (that is, $G \times T$-module), where both $H^0(G/B, \mathcal{L}(\lambda)) \subseteq \text{Cox}(G/B)$ and $\nabla_G(\lambda)$ are assigned degree $-\lambda$. The Cox ring $\text{Cox}(G/B)$ is also an integral domain, and hence isomorphic to $k[G]^U$ as an $X(T)$-graded $G$-module.

Proof. The first assertion follows from the fact that $H^0(G/B, \mathcal{L}(\lambda)) = \nabla_G(\lambda)$ for $\lambda \in X^+$ and $H^0(G/B, \mathcal{L}(\lambda)) = 0$ for $\lambda \in X(B) \setminus X^+$ [Jan, (II.2.6)].

Consider the $\mathcal{O}_{G/B}$-algebra $S := \text{Sym}(\mathcal{L}(\lambda_1) \oplus \cdots \oplus \mathcal{L}(\lambda_l))$, where $\lambda_1, \ldots, \lambda_l$ are the fundamental dominant weights. Being a vector bundle over $G/B$, $\text{Spec} S$ is integral. Hence $\text{Cox}(G/B) \cong \Gamma(G/B, S)$ is a domain.

The last assertion follows from Lemma 3.5. □

Lemma 3.7. If $G$ is semisimple and simply connected, then $k[G]^U$ is a UFD.

Proof. This is a consequence of Lemma 3.6 and [EKW, Corollary 1.2].

There is another proof. Popov [Pop2] proved that $k[G]$ is a UFD. Moreover, $G$ does not have a nontrivial character, since $G = [G,G]$, see [Hum, (29.5)]. It follows easily that $k[G]^\times = k^\times$ by [Ros, Theorem 3]. As $U$ is unipotent, $U$ does not have a nontrivial character. The lemma follows from Remark 3 after Proposition 2 of [Pop1]. See also [Has7, (4.31)]. □

By [Grs1, (2.1)], $k[G]^U$ is finitely generated. See also [RR] and [Grs2, Theorem 9].

By [Has3, Lemma 5.6] and Lemma 3.4, $k[G]^U$ is strongly $F$-regular in positive characteristic, and strongly $F$-regular type in characteristic zero. In any characteristic, $k[G]^U$ is Cohen–Macaulay normal.

In any characteristic, if $G$ is semisimple simply connected, being a finitely generated Cohen–Macaulay UFD, $k[G]^U$ is Gorenstein [Mur].

Combining the observations above, we have:

Lemma 3.8. $k[G]^U$ is finitely generated. It is strongly $F$-regular in positive characteristic, and strongly $F$-regular type in characteristic zero. If $G$ is semisimple and simply connected, then $k[G]^U$ is a Gorenstein UFD.

Corollary 3.9. Let $k$ be of positive characteristic, and $A$ be a $G$-algebra which is good as a $G$-module. If $A^U$ is finitely generated and strongly $F$-regular, then $A$ is finitely generated and $F$-rational.
Proof. This is proved similarly to [Pop3, Proposition 10] and [Grs2, Theorem 17].

As $U$ and $U^+$ are conjugate, $A^{U^+} \cong A^U$, and it is finitely generated and strongly $F$-regular by assumption. Note that $A$ is finitely generated [Grs2, Theorem 9].

Note that $k[G]^U$ is finitely generated and strongly $F$-regular by Lemma 3.8.

So the tensor product $A^{U^+} \otimes k[G]^U$ is finitely generated and strongly $F$-regular by [Has3, (5.2)]. Thus its direct summand subring $(A^{U^+} \otimes k[G]^U)^T$ is also finitely generated and strongly $F$-regular [HH1 (3.1)]. By Theorem 3.3, $G(A)$ is finitely generated and strongly $F$-regular, hence is $F$-rational ([HH1 (3.1)] and [HH3 (4.2)]). By [HM, (7.14)], $A$ is $F$-rational. \hfill \Box

4. The main result

Let the notation be as in the introduction. In this section, the characteristic of $k$ is $p > 0$.

(4.1) For a $G$-module $W$ and $r \geq 0$, $W^{(r)}$ denotes the $r$th Frobenius twist of $W$, see Section 2 and [Jan (I.9.10)].

Let $ρ$ denote the half sum of positive roots. For $r \geq 0$, let $St_r$ denote the $r$th Steinberg module $∇_G((p^r - 1)ρ)$, if $(p^r - 1)ρ$ is a weight of $G$. Note that $(p^r - 1)ρ$ is a weight of $G$ if $p$ is odd or $[G, G]$ is simply connected.

The following lemma, which is the dual assertion of [Has2, Theorem 3], follows immediately from [Jan (10.6)].

Lemma 4.2. Let $(p^r - 1)ρ$ be a weight of $G$ for any $r \geq 0$. Let $V$ be a finite dimensional $G$-module. Then there exists some $r_0 \geq 1$ such that for any $r \geq r_0$ and any subquotient $W$ of $V$, any nonzero (or equivalently, surjective) $G$-linear map $ϕ : St_r \otimes W \to St_r$ admits a $G$-linear map $ψ : St_r \to St_r \otimes W$ such that $ϕψ = id_{St_r}$.

We set $\tilde{G} = rad G \times Γ$, where $rad G$ is the radical of $G$, and $Γ \to [G, G]$ is the universal covering of the derived subgroup $[G, G]$ of $G$. Note that there is a canonical surjective map $\tilde{G} \to G$, and hence any $G$-module (resp. $G$-algebra) is a $\tilde{G}$-module (resp. $\tilde{G}$-algebra) in a natural way. The restriction functor $Res^G_{\tilde{G}}$ is full and faithful.

Let $S = \bigoplus_{i \geq 0} S_i$ be a positively graded finitely generated $G$-algebra which is an integral domain.
Assume first that \((p^r - 1)\rho\) is a weight of \(G\) for \(r \geq 0\). We say that \(S\) is \(G\)-strongly \(F\)-regular if for any nonzero homogeneous element \(a\) of \(S^G\), there exists some \(r \geq 1\) such that the \((G, S^{(r)})\)-linear map

\[
\text{id} \otimes aF^r : St_r \otimes S^{(r)} \rightarrow St_r \otimes S \quad (x \otimes s^{(r)} \mapsto x \otimes as^{p^r})
\]

is a split mono. In general, we say that \(S\) is \(G\)-strongly \(F\)-regular if it is so as a \(\tilde{G}\)-algebra.

The following is essentially proved in [Has2]. We give a proof for completeness.

**Lemma 4.3.** If \(S\) is a \(G\)-strongly \(F\)-regular positively graded finitely generated \(G\)-algebra domain, then \(S^G\) is strongly \(F\)-regular.

**Proof.** We may assume that \(G = \tilde{G}\). Let \(A := S^G\).

As we assume that \(S\) is a finitely generated positively graded domain, \(A\) is a finitely generated positively graded domain, see [MFK, Appendix to Chapter 1, A]. Let \(a\) be a nonzero homogeneous element of \(A\) such that \(A[1/a]\) is regular. Take \(r \geq 1\) so that \(\text{id} \otimes aF^r : St_r \otimes S^{(r)} \rightarrow St_r \otimes S\) is a split mono. Let \(\Phi : St_r \otimes S \rightarrow St_r \otimes S^{(r)} \) be a \((G, S^{(r)})\)-linear map such that \(\Phi \circ (\text{id} \otimes aF^r) = \text{id}\). Then consider the commutative diagram of \((G, A^{(r)})\)-modules

\[
\begin{array}{ccc}
St_r \otimes A^{(r)} & \longrightarrow & St_r \otimes S^{(r)} \\
\downarrow \downarrow & & \downarrow \downarrow \\
St_r \otimes A & \longrightarrow & St_r \otimes S.
\end{array}
\]

Then applying the functor \(\text{Hom}_G(St_r, ?)\) to this diagram, we get the commutative diagram of \(A^{(r)}\)-modules

\[
\begin{array}{ccc}
A^{(r)} & \longrightarrow & A^{(r)} \\
\downarrow \downarrow & & \downarrow \downarrow \\
A & \longrightarrow & \text{Hom}_G(St_r, St_r \otimes S),
\end{array}
\]

see [Has2, Proposition 1, 5]. This shows that the \(A^{(r)}\)-linear map \(aF^r : A^{(r)} \rightarrow A\) splits. By [HH1, (3.3)], \(A\) is strongly \(F\)-regular. \(\square\)

The following is also proved in [Has2] (see the proof of [Has2, Theorem 6]).
Theorem 4.4. Let $V$ be a finite dimensional $G$-module. If $S = \text{Sym} V$ has a good filtration (see the introduction for definition), then $S$ is $G$-strongly $F$-regular.

Lemma 4.5. Let $S$ be a $G$-strongly $F$-regular positively graded finitely generated $G$-algebra domain, and assume that there exists some $a \in S^G \setminus \{0\}$ such that $S[1/a]$ is strongly $F$-regular. Then $S$ is strongly $F$-regular.

Proof. We may assume that $G = \tilde{G}$. Let $I$ be the radical ideal of $S$ which defines the non-strongly $F$-regular locus of $S$. Such an ideal exists, see [HH1, (3.3)]. Then $I$ is $G \times \mathbb{G}_m$-stable, and hence $I \cap S^G$ is $\mathbb{G}_m$-stable. In other words, $I \cap S^G$ is a homogeneous ideal of $S^G$. By assumption, $0 \neq a \in I \cap S^G$ contains a nonzero homogeneous element $b$. Take $r \geq 1$ so that $1 \otimes b F^r : S^r \otimes S \to S^r \otimes S$ has a splitting. Let $x$ be any nonzero element of $S^r$. Then $x \otimes \text{id} : S^r \cong k \otimes S^r \to S^r \otimes S^r$ given by $s^r \mapsto x \otimes s^r$ is a split mono as an $S^r$-linear map. Thus $(x \otimes \text{id}) (b F^r) = (\text{id} \otimes b F^r)(x \otimes \text{id})$ is a split mono as an $S^r$-linear map, and hence so is $b F^r : S^r \to S$. By [HH1, (3.3)], $S$ is strongly $F$-regular.\hfill \Box

Let $S$ be a finitely generated $G$-algebra. We say that $S$ is $G$-$F$-pure if there exists some $r \geq 1$ such that $\text{id} \otimes F^r : S^r \otimes S^r \to S^r \otimes S$ splits as a $(G, S^r)$-linear map. Obviously, a $G$-strongly $F$-regular finitely generated positively graded $G$-algebra domain is $G$-$F$-pure. The following is essentially proved in [Has6].

Lemma 4.6. Let $S$ be a $G$-$F$-pure finitely generated $G$-algebra. Then $S^G$ is $F$-pure.

Proof. This is proved similarly to Lemma 4.3. See also [Has6].\hfill \Box

Lemma 4.7. Let $S$ and $S'$ be a $G$-$F$-pure finitely generated $G$-algebras. Then the tensor product $S \otimes S'$ is $G$-$F$-pure.

Proof. This is easy, and we omit the proof.\hfill \Box

Lemma 4.8. Let $S$ be a $G$-$F$-pure finitely generated $G$-algebra, and assume that the $(G, S^r)$-linear map

$$\text{id} \otimes F^r : S^r \otimes S \to S^r \otimes S$$

splits. Then the $(G, S^{nr})$-linear map

$$\text{id} \otimes F^{nr} : S_{nr} \otimes S^{nr} \to S_{nr} \otimes S$$

splits for any $n \geq 0$.\hfill 12
Proof. Induction on $n$. The case that $n = 0$ is trivial. Assume that $n > 0$. Note that $St_{nr} \cong St_r \otimes St_{(n-1)r}$. So

$$id \otimes (F^{(n-1)r}) (r) : St_{nr} \otimes S^{(nr)} \rightarrow St_{nr} \otimes S^{(r)}$$

is identified with the map

$$id \otimes (id \otimes F^{(n-1)r}) (r) : St_r \otimes (St_{(n-1)r} \otimes S^{(nr)}) (r) \rightarrow St_r \otimes (St_{(n-1)r} \otimes S^{(r)}) ,$$

and it has an $(G, S^{(nr)})$-linear splitting by the induction assumption. On the other hand, $id \otimes F^r : St_{nr} \otimes S^{(r)} \rightarrow St_{nr} \otimes S$ splits by assumption, as $St_{nr} \cong St_r \otimes St_{(n-1)r}$. Thus the composite

$$St_{nr} \otimes S^{(nr)} \rightarrow St_{nr} \otimes S^{(r)} \rightarrow St_{nr} \otimes S,$$

which agrees with $id \otimes F^{nr}$, has a splitting, as desired.

Lemma 4.9. Let $S = \bigoplus_{n \geq 0} S_n$ be a finitely generated positively graded $G$-algebra which is an integral domain. Then the following are equivalent.

1 $S$ is $G$-strongly $F$-regular.

2 $S$ is $[G, G]$-strongly $F$-regular.

3 $S$ is $\Gamma$-strongly $F$-regular, where $\Gamma \rightarrow [G, G]$ is the universal covering.

Proof. The implications 1$\Rightarrow$2$\iff$3 is trivial. We prove the direction 3$\Rightarrow$1. Replacing $G$ by $\tilde{G}$ if necessary, we may assume that $G = R \times \Gamma$, where $R$ is a torus, and $\Gamma$ is a semisimple and simply connected algebraic group. Let $a \in S^G$ be any nonzero homogeneous element. Then by assumption, the $(R, (S^{(r)})^*)$-linear map

$$(aF^r)^* : \text{Hom}_{R,S^{(r)}}(St_r \otimes S, St_r \otimes S^{(r)}) \rightarrow \text{Hom}_{R,S^{(r)}}(St_r \otimes S^{(r)}, St_r \otimes S^{(r)})$$

is surjective. Taking the $R$-invariant,

$$(aF^r)^* : \text{Hom}_{G,S^{(r)}}(St_r \otimes S, St_r \otimes S^{(r)}) \rightarrow \text{Hom}_{G,S^{(r)}}(St_r \otimes S^{(r)}, St_r \otimes S^{(r)})$$

is still surjective, since $R$ is linearly reductive. This is what we wanted to prove.
The following is proved similarly.

**Lemma 4.10.** Let $S = \bigoplus_{n \geq 0} S_n$ be a finitely generated $G$-algebra. Then the following are equivalent.

1. $S$ is $G$-$F$-pure.
2. $S$ is $[G,G]$-$F$-pure.
3. $S$ is $\Gamma$-$F$-pure, where $\Gamma \to [G,G]$ is the universal covering. \hfill $\square$

**Lemma 4.11.** Let $G$ be semisimple and simply connected. Then $k[G]^U$ is $G$-$F$-pure.

*Proof.* This is [Has6, Lemma 3]. \hfill $\square$

The following is the main theorem of this paper.

**Theorem 4.12.** Let $S = \bigoplus_{n \geq 0} S_n$ be a finitely generated positively graded $G$-algebra. Assume that

1. $S$ is $F$-rational and Gorenstein.
2. $S$ is $G$-$F$-pure.

Then $S$ is a $G$-strongly $F$-regular integral domain.

*Proof.* Note that $S$ is normal [HH3, (4.2)]. As $S$ is positively graded, $S$ is an integral domain.

Replacing $G$ by $\Gamma$, where $\Gamma \to [G,G]$ is the universal covering, we may assume that $G$ is semisimple and simply connected, by Lemma 4.9 and Lemma 4.10.

As $S$ is $G$-$F$-pure, there exists some $l \geq 1$ such that $\text{id} \otimes F^l : St_1 \otimes S^{(l)} \to St_1 \otimes S$ has a $(G,S^{(l)})$-linear splitting $\psi : St_1 \otimes S \to St_1 \otimes S^{(l)}$.

Note that any graded $(G,S)$-module which is rank one free as an $S$-module is of the form $S(n)$, where $S(n)$ is $S$ as a $(G,S)$-module, but the grading is given by $S(n)_i = S_{n+i}$. In fact, let $-n$ be the generating degree of the rank one free graded $(G,S)$-module, say $M$, then $M_{-n} \otimes S \to M$ is a $(G,S)$-isomorphism. As $M_{-n}$ is trivial as a $G$-module (since $G$ is semisimple), $M_{-n} \cong k(n)$ as a graded $G$-module. Thus $M \cong k(n) \otimes S \cong S(n)$. 

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Let $a$ be the $a$-invariant of the Gorenstein positively graded ring $S$. Namely, $\omega_S \cong S(a)$ (as a graded $(G,S)$-module, see the last paragraph). Then

$$\text{Hom}_{S(r)}(S, S^{(r)}) \cong \text{Hom}_{S(r)}(S, (\omega_S)^{(r)}(-p^r a)) \cong \omega_S(-p^r a) \cong S((1 - p^r)a)$$

for $r \geq 0$.

Let $\sigma$ be any nonzero element of $\text{Hom}_{S(r)}(S, S^{(1)})_{(p-1)a} \cong S_0$. As $S_0 = k$ is $G$-trivial, $\sigma : S \to S^{(1)}$ is $(G, S^{(1)})$-linear of degree $(p-1)a$.

For $r \geq 0$, let $\sigma_r$ be the composite

$$S \xrightarrow{\sigma} S^{(1)} \xrightarrow{\sigma^{(1)}} S^{(2)} \xrightarrow{\sigma^{(2)}} \cdots \xrightarrow{\sigma^{(r-1)}} S^{(r)}.$$  

It is $(G, S^{(r)})$-linear of degree $(p^r - 1)a$. Note that $\sigma_u = \sigma_{u-r} \sigma_r$ for $u \geq r$.

Hence by the composite map

$$(1) \quad Q_{r,u} : \text{Hom}_{S(r)}(S, S^{(r)}) \xrightarrow{\sigma_u^{(r)}} \text{Hom}_{S(r)}(S, \text{Hom}_{S(u)}(S^{(r)}, S^{(u)})) \cong \text{Hom}_{S(u)}(S \otimes_{S(r)} S, S^{(u)}) \cong \text{Hom}_{S(u)}(S, S^{(u)}),$$

the element $\sigma_r$ is mapped to $\sigma_u$, where the first map $\sigma_u^{(r)}$ maps $f \in \text{Hom}_{S(r)}(S, S^{(r)})$ to the map $x \mapsto f(x) \cdot \sigma_u^{(r)}$. More precisely, we have $Q_{r,u}(f) = \sigma_u^{(r)} \circ f$.

By the induction on $u$, it is easy to see that $Q_{1,u}$ is an isomorphism, and $\sigma_u$ is a generator of the rank one $S$-free module $\text{Hom}_{S(u)}(S, S^{(u)})$. It follows that $Q_{r,u}$ is an isomorphism for any $u \geq r$.

We continue the proof of Theorem 4.12. Take a nonzero homogeneous element $b$ of $A = S^G$. It suffices to show that there exists some $u \geq 1$ such that $\text{id} \otimes bF^u : St_u \otimes S^{(u)} \to St_u \otimes S$ splits as a $(G, S^{(u)})$-linear map.

As $S$ is $F$-rational Gorenstein, it is strongly $F$-regular by Lemma 2.4 (viii). So there exists some $r \geq 1$ such that

$$(bF_S^r)^* : \text{Hom}_{S(r)}(S, S^{(r)}) \to \text{Hom}_{S(r)}(S^{(r)}, S^{(r)})$$

given by $(bF_S^r)^*(x) = \varphi bF_S^r$ is surjective.

Let $V$ be the degree $-(p^r - 1)a - d$ component of $S$, where $d$ is the degree of $b$. Note that $V \cong \text{Hom}_{S(r)}(S, S^{(r)})_{-d}$ is mapped onto $k \cong \text{Hom}_{S(r)}(S^{(r)}, S^{(r)})_0$ by $(bF_S^r)^*$. In particular, $-(p^r - 1)a - d \geq 0$. So $a \leq 0$. If $S \neq k$, then it is easy to see that $a < 0$. 

By Lemma 4.2, there exists some \( u_0 \geq 1 \) such that for any \( u \geq u_0 \), for any subquotient \( W \) of \( V \), and any \( G \)-linear nonzero map \( f : St_u \otimes W \rightarrow St_u \), there exists some \( G \)-linear map \( g : St_u \rightarrow St_u \otimes W \) such that \( fg = \text{id} \). Take \( u \geq u_0 \) such that \( u - r \) is divisible by \( l \).

Now the diagram

\[
\begin{array}{ccc}
\Hom_{S(r)}(S, S(r)) & \xrightarrow{(bF^r)^*} & \Hom_{S(r)}(S(r), S(r)) \\
\approx & Q_{r,u} & \approx Q_{0,u-r} \\
\Hom_{S(u)}(S, S(u)) & \xrightarrow{(bF^r)^*} & \Hom_{S(u)}(S(r), S(u))
\end{array}
\]

is commutative. So the bottom \((bF^r)^*\) is surjective. Let us consider the surjection

\[
(bF^r)^* : W = ((bF^r)^*)^{-1}(k \cdot \sigma_{u-r}) \cap \Hom_{S(u)}(S, S(u))_{ap^r(p^{u-r}-1)-d} \rightarrow k \cdot \sigma_{u-r}.
\]

By definition, \( W \) is contained in the degree \( ap^r(p^{u-r}-1) - d \) component of \( \Hom_{S(u)}(S, S(u)) = S(-a(p^u - 1)) \), which is isomorphic to \( V \) as a \( G \)-module. So \( W \) is isomorphic to a \( G \)-submodule of \( V \).

Let \( E := \Hom(St_u, St_u) \). Then by the choice of \( u_0 \) and \( u \), there exists some \( G \)-linear map \( g_1 : E \rightarrow E \otimes W \) such that the composite

\[
E \xrightarrow{g_1} E \otimes W \xrightarrow{1 \otimes (bF^r)^*} E \otimes (k \cdot \sigma_{u-r}^r)
\]

maps \( \varphi \) to \( \varphi \otimes \sigma_{u-r}^r \).

We identify \( E \otimes \Hom_{S(u)}(S, S(u)) \) by \( \Hom_{S(u)}(St_u \otimes S, St_u \otimes S(u)) \) in a natural way. Similarly, \( E \otimes \Hom_{S(r)}(S(r), S) \) is identified with \( \Hom_{S(r)}(St_u \otimes S(r), St_u \otimes S) \), and so on.

Then letting \( \nu := g_1(\text{id}_{St_u}) \), the composite

\[
St_u \otimes S(r) \xrightarrow{\text{id} \otimes bF^r} St_u \otimes S \xrightarrow{\nu} St_u \otimes S(u)
\]

agrees with \( \text{id} \otimes \sigma_{u-r}^r \).

Since \( S \) is \( G \)-\( F \)-pure, \( u - r \) is a multiple of \( l \), and \( St_u \cong St_r \otimes St_{u-r} \), there exists some \((G, S(u))\)-linear map \( \Phi : St_u \otimes S(r) \rightarrow St_u \otimes S(u) \) such that \( \Phi \circ (\text{id}_{St_u} \otimes F_{S}^{u-r}) = \text{id} \) by Lemma 4.8.

Viewing \( \Phi \) as an element of \( E \otimes \Hom_{S(u)}(S(r), S(u)) \), let

\[
\beta \in E \otimes \Hom_{S(r)}(S(r), S(r))
\]
be the element \((\text{id}_E \otimes (Q^{(r)}_{u})^{-1}) (\Phi)\). In other words, \(\beta : St_u \otimes S^{(r)} \to St_u \otimes S^{(r)}\) is the unique map such that the composite

\[
St_u \otimes S^{(r)} \xrightarrow{\beta} St_u \otimes S^{(r)} \xrightarrow{\text{id} \otimes \sigma^{(r)}_{u-r}} St_u \otimes S^{(u)}
\]

is \(\Phi\).

Write \(\beta = \sum_i \varphi_i \otimes a_i^{(r)}\), where \(\varphi_i \in E\) and \(a_i \in S\). Define \(\beta' \in E \otimes \text{Hom}_S(S, S)\) by \(\beta' = \sum_i \varphi_i \otimes a_i^{(r)}\). Then it is easy to check that \((\text{id} \otimes bF^r) \circ \beta = \beta' \circ (\text{id} \otimes bF^r)\) as maps \(St_u \otimes S^{(r)} \to St_u \otimes S\).

Combining the observations above, the whole diagram of \((G, S^{(u)})\)-modules

\[
\begin{array}{ccc}
St_u \otimes S^{(u)} & \xrightarrow{\text{id} \otimes bF^{u-r}} & St_u \otimes S^{(r)} \\
\downarrow \Phi & & \downarrow \beta \\
St_u \otimes S^{(u)} & \xrightarrow{\nu} & St_u \otimes S^{(r)} & \xrightarrow{\beta} & St_u \otimes S \\
\downarrow \nu & & & \downarrow \beta' & \downarrow \beta' \\
St_u \otimes S & & & & St_u \otimes S
\end{array}
\]

is commutative. So \(\text{id} \otimes bF^u : St_u \otimes S^{(u)} \to St_u \otimes S\) has a \((G, S^{(u)})\)-linear splitting \(\nu \beta'\).

**Corollary 4.13.** Let \(S\) be as in Theorem 4.12. Then \(S^U\) is finitely generated and strongly \(F\)-regular.

**Proof.** Finite generation is by [Gr2, Theorem 9].

We prove the strong \(F\)-regularity. We may assume that \(G\) is semisimple and simply connected. Then \(k[G]^U\) is a strongly \(F\)-regular Gorenstein domain by Lemma 3.8. Hence the tensor product \(S \otimes k[G]^U\) is also a strongly \(F\)-regular Gorenstein domain, see [Has3, Theorem 5.2]. As \(S\) is assumed to be \(G\)-\(F\)-pure and \(k[G]^U\) is \(G\)-\(F\)-pure by Lemma 4.11, the tensor product \(S \otimes k[G]^U\) is also \(G\)-\(F\)-pure by Lemma 4.7. Hence by the theorem, \(S \otimes k[G]^U\) is \(G\)-strongly \(F\)-regular. It follows that \((S \otimes k[G]^U)^G\) is strongly \(F\)-regular by Lemma 4.3. As \(S^U \cong (S \otimes k[G]^U)^G\) (see the proof of [Gr1, (1.2)]. See also [Dol, Lemma 4.1]), we are done.

**Corollary 4.14.** Let \(V\) be a finite dimensional \(G\)-module, and assume that \(S = \text{Sym} V\) has a good filtration as a \(G\)-module. Then \(S^U\) is finitely generated and strongly \(F\)-regular.

**Proof.** Follows immediately from Corollary 4.13 and Theorem 4.4. 

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5. The unipotent radicals of parabolic subgroups

Let the notation be as in the introduction. Let $I$ be a subset of $\Delta$. Let $L = L_I$ be the corresponding Levi subgroup $C_G(\bigcap_{\alpha \in I} \text{Ker} \alpha)^\circ$, where $(\cdot)^\circ$ denotes the identity component, and $C_G$ denotes the centralizer. Let $P = P_I$ be the parabolic subgroup generated by $B$ and $L$. Let $U_P$ be the unipotent radical of $P$. Let $B_L := B \cap L$, and $U_L$ the unipotent radical of $B_L$.

Here are two theorems due to Donkin.

**Theorem 5.1** (Donkin [Don3 (1.2)]). Let $w_0$ and $w_L$ denote the longest elements of the Weyl groups of $G$ and $L$, respectively. For $\lambda \in X^+$, we have $\nabla_G(\lambda)^{U_P} \cong \nabla_L(w_L w_0 \lambda)$ as $L$-modules.

**Theorem 5.2** (Donkin [Don3 (1.4)], [Don4 (3.9)]). Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of $G$-modules. If $M_1$ is good, then $0 \to M_1^{U_P} \to M_2^{U_P} \to M_3^{U_P} \to 0$ is exact. In other words, if $M$ is a good $G$-module, then $R^i(H^0(U_P, ?) \circ \text{res}^{G}_{U_P})(M) = 0$ for $i > 0$.

From these two theorems, it follows immediately:

**Lemma 5.3.** Let $M$ be a good $G$-module. Then $M^{U_P}$ is a good $L$-module.

So we have:

**Proposition 5.4.** Let $k$ be of positive characteristic. Let $S$ be a finitely generated positively graded $G$-algebra. Assume that $S$ is Gorenstein $F$-rational, and $G$-$F$-pure. Then $S^{U_P}$ is finitely generated and $F$-rational.

**Proof.** By Lemma 5.3, $S^{U_P}$ is good as an $L$-module. By Corollary 4.13, $(S^{U_P})^{U_L} \cong S^U$ is finitely generated and strongly $F$-regular. By Corollary 3.9, applied to the action of $L$ on $S^{U_P}$, we have that $S^{U_P}$ is finitely generated and $F$-rational.

**Corollary 5.5.** Let $k$ be of positive characteristic. Let $V$ be a finite dimensional $G$-module, and assume that $S = \text{Sym} V$ is good as a $G$-module. Then $S^{U_P}$ is a finitely generated strongly $F$-regular Gorenstein UFD.
Proof. As \( U_P \) is unipotent, \( S^U_P \) is a UFD by Remark 3 after Proposition 2 of [Pop1].

On the other hand, \( S \) satisfies the assumption of Proposition 5.4 by Theorem 4.4. So by Proposition 5.4, \( S^U_P \) is finitely generated and \( F \)-rational. Being a finitely generated Cohen–Macaulay UFD, it is Gorenstein [Mur], and hence is strongly \( F \)-regular by Lemma 2.4 (viii).

Remark 5.6. Let \( k \) be of characteristic zero. The characteristic-zero counterpart of Proposition 5.4 is stated as follows: If \( S \) is a finitely generated \( G \)-algebra with rational singularities, then \( S^U_P \) is finitely generated with rational singularities. This is proved in the same line as Proposition 5.4. Note that \( S^U \) is finitely generated with rational singularities by [Pop3, Corollary 4, Theorem 6]. Then applying [Pop3, Corollary 4, Theorem 6] again to the action of \( L \) on \( S^U_P \), \( S^U_P \) is finitely generated and has rational singularities, since \( (S^U_P)^{U_L} \sim S^U \) is so.

The characteristic-zero counterpart of Corollary 5.5 is stated as follows: If \( S \) is a finitely generated \( G \)-algebra with rational singularities and is a UFD, then \( S^U_P \) is a Gorenstein finitely generated UFD which is of strongly \( F \)-regular type. As we have already seen, \( S^U_P \) is finitely generated with rational singularities. \( S^U_P \) is a UFD by Remark 3 after Proposition 2 of [Pop1]. As \( S^U_P \) is also Cohen–Macaulay [KKMS, p. 50, Proposition], \( S^U_P \) is Gorenstein [Mur]. A Gorenstein finitely generated algebra with rational singularities is of strongly \( F \)-regular type, see [Har, (1.1), (5.2)].

6. Applications

The following is pointed out in the proof of [SvdB, (5.2.3)].

Lemma 6.1. Let \( K \) be a field of characteristic zero, \( H \) be an extension of a finite group scheme by a torus over \( K \), and \( A \) a finitely generated \( H \)-algebra of strongly \( F \)-regular type. Then \( A^H \) is of strongly \( F \)-regular type.

Proof. Set \( B = A^H \). Let \( \bar{K} \) be the algebraic closure of \( K \). As can be seen easily, if \( \bar{K} \otimes_K B \) is of strongly \( F \)-regular type, then so is \( B \). Since \( \bar{K} \otimes_K B \cong (\bar{K} \otimes_K A)^{K \otimes_K H} \), replacing \( K \) by \( \bar{K} \), we may assume that \( K \) is algebraically closed. Then \( H \) is an extension of a finite group \( \Gamma \) by a split torus \( \mathbb{G}_m^r \) for some \( r \). As \( A^H \cong (A^{\mathbb{G}_m^r})^\Gamma \), we may assume that \( H \) is either a split torus \( \mathbb{G}_m^r \) or a finite group \( \Gamma \).
Now we can take a finitely generated \( \mathbb{Z} \)-subalgebra \( R \) of \( K \) and a finitely generated flat \( R \)-algebra \( A_R \) such that \( K \otimes_R A_R \cong A \), and for any closed point \( x \) of \( \text{Spec} \, R \), \( \kappa(x) \otimes R A_R \) is strongly \( F \)-regular. Extending \( R \) if necessary, we have an action of \( H_R \) on \( A_R \) which is extended to the action of \( H \) on \( A \), where \( H_R = (\mathbb{G}_m)^r_R \) or \( H_R = \Gamma \). Extending \( R \) if necessary, we may assume that \( n \in R^\times \), where \( n \) is the order of \( \Gamma \), if \( H = \Gamma \).

Now set \( B_R := A_{H_R}^H \).

If \( H = (\mathbb{G}_m)^r_R \), then \( B_R \) is the degree zero component of the \( \mathbb{Z}^r \)-graded finitely generated \( R \)-algebra \( A_R \), and it is finitely generated, and is a direct summand subring of \( A_R \).

If \( H = \Gamma \), then \( B_R \) is the degree zero component of the \( \mathbb{Z}^r \)-graded finitely generated \( R \)-algebra \( A_R \), and it is finitely generated, and is a direct summand subring of \( A_R \).

In either case, \( B_R \) is finitely generated over \( R \), so extending \( R \) if necessary, we may assume that \( B_R \) is \( R \)-flat. Note that \( B \cong K \otimes_R B_R \), since \( K \) is \( R \)-flat, and the invariance is compatible with a flat base change. Note also that \( \kappa(x) \otimes R B_R \) is a direct summand subring of \( \kappa(x) \otimes R A_R \), and \( \kappa(x) \otimes R A_R \) is strongly \( F \)-regular. Hence \( \kappa(x) \otimes R B_R \) is strongly \( F \)-regular by Lemma 2.4 (iii). This shows that \( A^H = B \) is of strongly \( F \)-regular type.

The following is a refinement of [SvdB, (5.2.3)].

**Corollary 6.2.** Let \( K \) be a field of characteristic zero, \( H \) an affine algebraic group scheme over \( K \) such that \( H^o \) is reductive. Let \( S \) be a finitely generated \( H \)-algebra which has rational singularities and is a UFD. Then \( S^H \) is of strongly \( F \)-regular type.

**Proof.** Let \( H' := [H^o, H^o] \). Then \( K \otimes_K H' \) is semisimple, and does not have a nontrivial character. Thus \( S^H \) has rational singularities by Boutot’s theorem [Bt] and is a UFD by [Has7] (4.28)]. So it is of strongly \( F \)-regular type by Hara [Har] (1.1), (5.2)]. As \( (H/H^o)^c \) is a torus, \( S^H = (S^{H'})^{H/H^o} \) is of strongly \( F \)-regular type by Lemma 6.1.

**Theorem 6.3.** Let \( k \) be an algebraically closed field, and \( Q = (Q_0, Q_1, s, t) \) a finite quiver, where \( Q_0 \) is the set of vertices, \( Q_1 \) is the set of arrows, and \( s \) and \( t \) are the source and the target maps \( Q_1 \to Q_0 \), respectively. Let \( d : Q_0 \to \mathbb{N} \) be a map. For \( i \in Q_0 \), set \( M_i := k^{d(i)} \), and let \( H_i \) be any closed subgroup scheme of \( \text{GL}(M_i) \) of the following:

(1) \( \text{GL}(M_i), \text{SL}(M_i) \);
(2) \(Sp_{d(i)}\) (in this case, \(d(i)\) is required to be even);

(3) \(SO_{d(i)}\) (in this case, the characteristic of \(k\) must not be two);

(4) Levi subgroup of any of \((1)-(3)\);

(5) Derived subgroup of any of \((1)-(4)\);

(6) Unipotent radical of a parabolic subgroup of any of \((1)-(5)\);

(7) Any subgroup \(H_i\) of \(GL(M_i)\) with a closed normal subgroup \(N_i\) of \(H_i\) such that \(N_i\) is any of \((1)-(6)\), and \(H_i/N_i\) is a linearly reductive group scheme. In characteristic zero, we require that \((H_i/N_i)^{\circ}\) is a torus.

Set \(H := \prod_{i \in Q_0} H_i\) and \(M := \prod_{\alpha \in Q_1} \text{Hom}(M_{s(\alpha)}, M_{t(\alpha)})\). Then \((\text{Sym} M^*)^H\) is finitely generated, and strongly \(F\)-regular if the characteristic of \(k\) is positive, and strongly \(F\)-regular type if the characteristic of \(k\) is zero.

Proof. If \(H_i\) satisfies \((7)\) and the corresponding \(N_i\) satisfies \((x)\), where \(1 \leq x \leq 6\), then we say that \(H_i\) is of type \((7,x)\).

Note that \(\text{Sym} M^* \cong \bigotimes_{\alpha \in Q_1} \text{Sym}(M_{s(\alpha)} \otimes M_{t(\alpha)}^*)\).

First we prove that \(\text{Sym} M^*\) has a good filtration as an \(H\)-module if each \(H_i\) is as in \((1)-(5)\). To verify this, we only have to show that \(\text{Sym}(M_{s(\alpha)} \otimes M_{t(\alpha)}^*)\) is a good \(H\)-module for each \(\alpha\), by Mathieu’s tensor product theorem [Mat]. This module is trivial as an \(H_i\)-module if \(i \neq s(\alpha)\), \(t(\alpha)\). Thus it suffices to show that this is good as an \(H_{s(\alpha)} \times H_{t(\alpha)}\)-module if \(s(\alpha) \neq t(\alpha)\), and as an \(H_{s(\alpha)}\)-module if \(s(\alpha) = t(\alpha)\), see [Has2, Lemma 4]. By [Has2, Lemma 3, 3, 5, 6] and [Don2, (3.2.7), (3.4.3)], the assertion is true for \(H_{s(\alpha)}\), \(H_{t(\alpha)}\) of \((1)-(3)\). By Mathieu’s theorem [Mat, Theorem 1], the groups of type \((4)\) is also allowed. By [Don2, (3.2.7)] again, the groups of type \((5)\) is also allowed. By [Has2, Theorem 6], the conclusion of the theorem holds this case.

We consider the general case. If \(H_i\) is of the form \((1)-(5)\), then considering \(N_i = U_i \subset B_i \subset H_i\), where \(B_i\) is a Borel subgroup of \(H_i\) and \(U_i\) its unipotent radical, \(B_i\) is a group of the form \((7)\), and as the \(H_i\)-invariant and the \(B_i\)-invariant are the same thing for an \(H_i\)-module, we may replace \(H_i\) by \(B_i\) without changing the invariant subring. Hence in this case, we may assume that \(H_i\) is of the form \((7, 6)\). Clearly, a group of the form \((6)\) is also of the form \((7, 6)\), letting \(N_i = H_i\). So we may assume that each \(H_i\) is of type \((7)\). If \((\text{Sym} M^*)^N\) is strongly \(F\)-regular (type), where \(N = \prod_{i \in Q_0} N_i\), then
(Sym $M^*)^H \cong ((\text{Sym } M^*)^N)^{H/N}$ is also strongly $F$-regular in positive characteristic, since $H/N \cong \prod_{i \in Q_0} H_i/N_i$ is linearly reductive and $(\text{Sym } M^*)^H$ is a direct summand subring of $(\text{Sym } M^*)^N$. In characteristic zero, $(H/N)^o$ is a torus, and we can invoke Lemma 6.1. Thus we may assume that each $H_i$ is of the form (1)–(6). Then again by the argument above, we may assume that each $H_i$ is of the form (7,6). Again by the argument above, we may assume that each $H_i$ is of the form (6). Now suppose that $H_i \subset G_i \subset GL(M_i)$, and each $G_i$ if of the form (1)–(5), and $H_i$ is the unipotent radical of the parabolic subgroup $P_i$ of $G_i$. Then letting $G := \prod G_i$ and $P := \prod P_i$, $H = \prod H_i$ is the unipotent radical of the parabolic subgroup $P$ of $G$. As Sym $M^*$ has a good filtration as a $G$-module by the first paragraph, $(\text{Sym } M^*)^H$ is finitely generated and strongly $F$-regular (type) by Corollary 5.5 and Remark 5.6.

This covers Example 1 and Example 2 of [Has2], except that we do not consider the case $p = 2$ here, if $O_n$ or $SO_n$ is involved. For example,

**Example 6.4.** Let $Q = 1 \to 2 \to 3$, $(d(1), d(2), d(3)) = (m, t, n)$, $H_1 = H_3 = \{e\}$, and $H_2 = GL_t$. Then $M = \text{Hom}(M_2, M_3) \times \text{Hom}(M_1, M_2)$, and $M \to M//H$ is identified with

$$\pi : M \to Y_t = \{ f \in \text{Hom}(M_1, M_3) \mid \text{rank } f \leq t \},$$

where $\pi(\varphi, \psi) = \varphi \psi$ (De Concini–Procesi [DP]). Thus (the coordinate ring of) $Y_t$ is strongly $F$-regular (type), as was proved by Hochster–Huneke [HH4, (7.14)] ($F$-regularity and strong $F$-regularity are equivalent for positively graded rings, see Lemma 2.4).

Next we consider an example which really requires a group of type (7) in Theorem 6.3.

Let $K$ be a field, and $M = K^m$, $N = K^n$. Let $1 \leq s \leq n$, and $a = (0 = a_0 < a_1 < \cdots < a_s = n)$ be an increasing sequence of integers. Let $\mathfrak{G}$, $\mathfrak{S}$, and $\mathfrak{T}$ be disjoint subsets of $\{1, \ldots, s\}$ such that $\mathfrak{G} \bigsqcup \mathfrak{S} \bigsqcup \mathfrak{T} = \{1, \ldots, s\}$. Let

$$H = H(a; \mathfrak{G}, \mathfrak{S}, \mathfrak{T}) := \begin{pmatrix}
H_1 & & & \\
& H_2 & & \\
& & \ddots & \\
& & & H_s \\
o & & & H_s
\end{pmatrix} \subset GL_m(K) \cong GL(M),$$

where $H_l$ is $GL_{a_l-a_{l-1}}$ if $l \in \mathfrak{G}$, $SL_{a_l-a_{l-1}}$ if $l \in \mathfrak{S}$, and $\{E_{a_l-a_{l-1}}\}$ if $l \in \mathfrak{T}$. 

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Let us consider the symmetric algebra $S = \text{Sym}(M \otimes N)$. It is a graded polynomial algebra over $K$ with each variable degree one. Let $e_1, \ldots, e_m$ and $f_1, \ldots, f_n$ be the standard bases of $M = K^m$ and $N = K^n$, respectively. For sequences $1 \leq c_1, \ldots, c_u \leq m$ and $1 \leq d_1, \ldots, d_u \leq n$, we define $[c_1, \ldots, c_u \mid d_1, \ldots, d_u]$ to be the determinant $\det(e_{c_i} \otimes f_{d_j})_{1 \leq i,j \leq u}$. It is a minor of the matrix $(e_i \otimes f_j)$ up to sign, or zero. Let $\Sigma$ be the set of minors

$$\{[c_1, \ldots, c_u \mid d_1, \ldots, d_u] \mid 1 \leq u \leq \min(m, n), 1 \leq c_1 < \cdots < c_u \leq m, 1 \leq d_1 < \cdots < d_u \leq n\}.$$  

We say that $[c_1, \ldots, c_u \mid d_1, \ldots, d_u] \leq [c'_1, \ldots, c'_v \mid d'_1, \ldots, d'_u]$ if $u \geq v$, and $c'_i \geq c_i$ and $d'_i \geq d_i$ for $1 \leq i \leq v$. It is easy to see that $\Sigma$ is a distributive lattice.

Set $\epsilon := \min \mathcal{G}$. For $1 \leq l < \epsilon$, set

$$\Gamma_l := \{[1, \ldots, a_l \mid d_1, \ldots, d_{a_l}] \mid 1 \leq d_1 < \cdots < d_{a_l} \leq n\}$$

if $l \in \mathcal{G}$, and

$$\Gamma_l := \{[c_1, \ldots, c_u \mid d_1, \ldots, d_u] \mid a_{l-1} < u \leq a_l, 1 \leq c_1 < \cdots < c_u \leq a_l, c_t = t (t \leq a_{l-1}), 1 \leq d_1 < \cdots < d_u \leq n\}$$

if $l \in \mathcal{S}$. Set $\Gamma = \bigcup_{l \leq \epsilon} \Gamma_l$. Note that $\Gamma$ is a sublattice of $\Sigma$.

It is well-known that $S$ is an ASL on $\Sigma$ over $K$ \cite{BH} (7.2.7). For the definition of ASL, see \cite{BH} (7.1).

**Lemma 6.5.** Let $B$ be a graded ASL on a poset $\Omega$ over a field $K$. Let $\Xi$ be a subset of $\Omega$ such that for any two incomparable elements $\xi, \eta \in \Xi$,

$$(2) \quad \xi \eta = \sum c_i m_i$$

in $S$ with each $m_i$ in the right hand side being a monomial of $\Xi$ divisible by an element $\xi_i$ in $\Xi$ smaller than both $\xi$ and $\eta$. Then the subalgebra $K[\Xi]$ of $B$ is a graded ASL on $\Xi$.

**Proof.** We may assume that $m_i$ in the right hand side of (2) has the same degree as that of $\xi \eta$ for each $\xi, \eta$, and $m_i$. For a monomial $m = \prod_{\omega \in \Omega} \omega^{e(\omega)}$, the weight $w(m)$ of $m$ is defined to be $\sum_{\omega} e(\omega)3^{\text{coht}(\omega)}$, where $\text{coht}(\omega)$ is the maximum of the lengths of chains $\omega = \omega_0 < \omega_1 < \cdots$ in $\Omega$. Then
\[ w(mm') = w(m) + w(m') \], and for each \( i \), \( w(m_i) > w(\xi \eta) \) in (2). So each time we use (2) to rewrite a monomial, the weight goes up. On the other hand, there are only finitely many monomials of a given degree, this rewriting procedure will stop eventually, and we get a linear combination of standard monomials in \( \Xi \). Now \( (H_2) \) condition in [BH (7.1)] is clear, while \( (H_0) \) and \( (H_1) \) are trivial.

We call \( K[\Xi] \) a subASL of \( B \) generated by \( \Xi \) if the assumption of the lemma is satisfied.

**Theorem 6.6.** Let the notation be as above. Let \( H \) act on \( S \) via \( h(m \otimes n) = h(m) \otimes n \). Set \( A := S^H \). Then

1. \( A = K[\Gamma] \).
2. \( K[\Gamma] \) is a subASL of \( S = K[\Sigma] \) generated by \( \Gamma \).
3. \( A \) is a Gorenstein UFD. It is strongly F-regular if the characteristic of \( K \) is positive, and is of strongly F-regular type if the characteristic of \( K \) is zero.

**Proof.** First we prove that \( A \) is strongly F-regular (type). To do so, we may assume that \( K = k \) is algebraically closed. Let \( B^+ \) be the subgroup of upper triangular matrices in \( GL_m \), and set \( B^+_H := B^+ \cap H \). Then it is easy to see that \( A = S^{B^+_H} \).

Now let \( Q \) be the quiver \( 1 \to 2 \), \( d = (d(1), d(2)) = (m, n) \), \( G_1 = B^+_H \subset GL_m \), and \( G_2 = \{e\} \). Let \( U^+_H \) be the unipotent radical of \( B^+_H \). Then \( U^+_H \) is the unipotent radical of an appropriate parabolic subgroup of \( GL_m \), \( U^+_H \) is normal in \( B^+_H \), and \( B^+_H/U^+_H \) is a torus. Thus the assumption (7) of Theorem 6.3 is satisfied, and thus \( A = S^{B^+_H} \) is strongly F-regular (type).

The assertion (2) is a consequence of the straightening relation of the ASL \( S \). See [ABW] for details.

Assume that (1) is proved. Then by the definition of \( \Gamma \), letting \( M' \) be the subspace of \( M \) spanned by \( e_1, \ldots, e_{a-1} \), \( A = \text{Sym}(M' \otimes N)^{H'} \), where \( H' = H \cap GL(M') \), by (1) again \( (GL(M') \) is viewed as a subgroup of \( GL(M) \) via \( g'(e_i) = e_i \) for \( i > a_{a-1} \). As \( H' \) is connected and \( \bar{K} \otimes_K H' \) does not have a nontrivial character, \( A \) is a UFD by [Has7 (4.28)], where \( \bar{K} \) is the algebraic closure of \( K \). So assuming (1), the assertion (3) is proved.
It remains to prove (1). It is easy to see that $\Gamma \subset A$. So it suffices to prove that $\dim_K A_d = \dim_K K[\Gamma]_d$ for each degree $d \geq 0$. To do so, we may assume that $K$ is algebraically closed.

Let $P^+$ be the parabolic subgroup $H(\omega; \{1, \ldots, s\}, \emptyset, \emptyset)$ of $GL_m$, and $U_{P^+}$ the unipotent radical of $P^+$. If

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence of good $GL(M) \times GL(N)$-modules, then

(3) $$0 \to M_1^{U_{P^+}} \to M_2^{U_{P^+}} \to M_3^{U_{P^+}} \to 0$$

is an exact sequence of good $P^+/U_{P^+}$-modules by Lemma 5.3 and Theorem 5.2. Note that $P^+/U_{P^+}$ is identified with $\prod_{i=1}^s GL_{a_i-a_{i-1}}$, and $H/U_{P^+}$ is identified with its subgroup $\prod_{i=1}^s H_i$. As each $H_i$ is either $GL_{a_i-a_{i-1}}$, $SL_{a_i-a_{i-1}}$, or trivial, it follows that a good $P^+/U_{P^+}$-module is also good as an $H/U_{P^+}$-module. Applying the invariance functor $(?)^{H/U_{P^+}}$ to (3),

$$0 \to M_1^H \to M_2^H \to M_3^H \to 0$$

is exact.

Now we employ the standard convention for $GL(M)$. Let $T$ be the set of diagonal matrices in $G := GL(M) = GL_m$, and we identify $X(T)$ with $\mathbb{Z}^m$ by the isomorphism

$$\mathbb{Z}^m \ni (\lambda_1, \lambda_2, \ldots, \lambda_m) \mapsto \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & \ddots \\ & & & t_m \end{pmatrix} \mapsto t^\lambda = t_1^{\lambda_1} t_2^{\lambda_2} \cdots t_m^{\lambda_m} \in X(T).$$

We fix the base of the root system of $GL(M)$ so that the set of lower triangular matrices in $GL(M)$ is negative. Then the set of dominant weights $X^+_{GL(M)}$ is the set

$$\{ \lambda = (\lambda_1, \ldots, \lambda_m) \in X(T) \mid \lambda_1 \geq \cdots \geq \lambda_m \}.$$  

We use a similar convention for $GL(N)$. See [Jan, (II.1.21)] for more information on this convention.

For $\lambda \in X^+_{GL(M)}$, $\nabla_{GL(M)}(\lambda)^{U_{P^+}}$ is a single dual Weyl module by Theorem 5.1. But obviously, the highest weight of $\nabla_{GL(M)}(\lambda)^{U_{P^+}}$ is $\lambda$. Thus $\nabla_{GL(M)}(\lambda)^{U_{P^+}} \cong \nabla_{P^+/U_{P^+}}(\lambda)$. Now the following is easy to verify:
Lemma 6.7. For $\lambda = (\lambda_1, \ldots, \lambda_m) \in X_{GL(M)}^+$,

$$\nabla_{GL(M)}(\lambda)^H \cong \begin{cases} 
\nabla_{GL_{a_1}}(\lambda(1)) \otimes \cdots \otimes \nabla_{GL_{a_s-a_{s-1}}}(\lambda(s)) & (\lambda \in \Theta) \\
0 & \text{(otherwise)}
\end{cases}$$

as $P^+/H$-modules, where $\lambda(l) := (\lambda_{a_l+1}, \ldots, \lambda_{a_l})$ for each $l$, and $\Theta$ is the subset of $X_{GL(M)}^+$ consisting of sequences $\lambda = (\lambda_1, \ldots, \lambda_m)$ such that $\lambda(l) = (0, 0, \ldots, 0)$ for each $l \in \mathfrak{g}$, and $\lambda(l) = (t, t, \ldots, t)$ for some $t \in \mathbb{Z}$ for each $l \in \mathfrak{g}$.

Let $r := \min(m, n)$, and set

$$\mathcal{P}(d) = \{ \lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r \mid \lambda_1 \geq \cdots \geq \lambda_r \geq 0, |\lambda| = d \},$$

where $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_r$. We consider that

$$(\lambda_1, \ldots, \lambda_r) = (\lambda_1, \ldots, \lambda_r, 0, \ldots, 0),$$

and $\mathcal{P}(d) \subset X_{GL(M)}^+$. Similarly, we also consider that $\mathcal{P}(d) \subset X_{GL(N)}^+$. By the Cauchy formula [ABW, (III.1.4)], $S_d$ has a good filtration as a $GL(M) \times GL(N)$-module whose associated graded object is

$$\bigoplus_{\lambda \in \mathcal{P}(d) \cap \Theta} \nabla_{GL(M)}(\lambda) \boxtimes \nabla_{GL(N)}(\lambda).$$

Note that $\nabla_{GL(M)}(\lambda)$ is isomorphic to the Schur module $L_{\tilde{\lambda}}M$ in [ABW], where $\tilde{\lambda}$ is the transpose of $\lambda$. That is, $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots)$ is given by $\tilde{\lambda}_i = \# \{ j \geq 1 \mid \lambda_j \geq i \}$.

By Lemma 6.7, $S_d^H$ has a filtration whose associated graded object is

$$\bigoplus_{\lambda \in \mathcal{P}(d) \cap \Theta} \nabla_{GL_{a_1}}(\lambda(1)) \otimes \cdots \otimes \nabla_{GL_{a_s-a_{s-1}}}(\lambda(s)) \boxtimes \nabla_{GL(N)}(\lambda).$$

In particular,

$$\dim S_d^H = \sum_{\lambda \in \mathcal{P}(d) \cap \Theta} \dim \nabla_{GL(N)}(\lambda) \prod_l \dim \nabla_{GL_{a_l-a_{l-1}}}(\lambda(l)).$$

Next we count the dimension of $K[\Gamma]_d$. This is the number of standard monomials of degree $d$ in $K[\Gamma]$. For a standard monomial

$$v = \prod_{b=1}^{\alpha} [c_{b,1}, \ldots, c_{b,\mu_b} \mid d_{b,1}, \ldots, d_{b,\mu_b}]$$

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where \([c_{b,1}, \ldots, c_{b,\mu_b}]\) increses when \(b\) increases) in \(\Sigma\), we define \(\mu(v) = (\mu_1, \ldots, \mu_\alpha)\), and \(\lambda(v)\) its transpose. Such a standard monomial \(v\) of \(\Gamma\) of degree \(d\) exists if and only if \(\lambda(v) \in \Theta \cap P(d)\).

For a standard monomial \(v\) of \(\Sigma\) such that \(\lambda(v) = \lambda \in P(d) \cap \Theta\), \(v\) is a monomial of \(\Gamma\) if and only if the following condition holds. For each \(1 \leq b \leq \lambda_1, 1 \leq l \leq s\), and each \(a_{l-1} < i \leq a_l\), it holds \(a_{l-1} < c_{s,i} \leq a_l\). The number of such monomials agrees with \(\dim \nabla_{GL(N)}(\lambda) \prod_l \dim \nabla_{GL(a_{l-1})}(\lambda(l))\), as can be seen easily from the standard basis theorem [ABW, (II.2.16)]. So \(\dim_K K[\Gamma]\) agrees with the right hand side of (4), and we have \(\dim_K A_d = \dim_K S_d^{H} = \dim_K K[\Gamma_d]\), as desired.

**Remark 6.8.** The case that \(s = 2, a_1 = l, \mathcal{G} = \emptyset, \mathcal{T} = \{2\}\) and \(\mathcal{S} = \{1\}\) is studied by Goto–Hayasaka–Kurano–Nakamura [GHKN]. Gorenstein property and factoriality are proved there for this case. The case that \(s = m, a_l = l (l = 1, \ldots, m), \mathcal{G} = \mathcal{S} = \emptyset, \mathcal{T} = \{1, \ldots, m\}\) is a very special case of the study of Miyazaki [Miy].

### 7. Openness of good locus

(7.1) Let \(R\) be a Noetherian commutative ring, and \(G\) a split reductive group over \(R\). We fix a split maximal torus \(T\) of \(G\) whose embedding into \(G\) is defined over \(\mathbb{Z}\). We fix a base \(\Delta\) of the root system, and let \(B\) be the negative Borel subgroup. For a dominant weight \(\lambda\), the dual Weyl module \(\nabla_G(\lambda)\) is defined to be \(\text{ind}_B^G(\lambda)\), and the Weyl module \(\Delta_G(\lambda)\) is defined to be \(\nabla_G(-w_0\lambda)^*\).

A \(G\)-module \(M\) is said to be good if \(\text{Ext}_G^1(\Delta_G(\lambda), M) = 0\) for any \(\lambda \in X^+\), where \(X^+\) is the set of dominant weights, see [HasI, (III.2.3.8)].

**Lemma 7.2.** The notion of goodness of a \(G\)-module \(M\) is independent of the choice of \(T\) or \(\Delta\), and depends only on \(M\).

**Proof.** Let \(T'\) and \(\Delta'\) be another choice of a split maximal torus defined over \(\mathbb{Z}\) and a base of the root system (with respect to \(T'\)). Let \(B'\) be the corresponding negative Borel subgroup.

Assume that \(R\) is an algebraically closed field. Then there exists some \(g \in G(R)\) such that \(gBg^{-1} = B'\). So \(\text{ind}_B^G \lambda \cong \text{ind}_{B'}^G(\lambda')\) for any \(\lambda \in X(B)\), where \(\lambda'\) is the composite

\[
B' \xrightarrow{B' \to g^{-1}g} B \xrightarrow{\lambda} \mathbb{G}_m.
\]
So this case is clear.

When $R$ is a field, then a $G$-module $M$ is good if and only if $\bar{R} \otimes_R M$ is so as an $\bar{R} \otimes_R G$-module, and this notion is independent of the choice of $B$, where $\bar{R}$ is the algebraic closure of $R$.

Now consider the general case. If $M$ is $R$-finite $R$-projective, then the assertion follows from [Has1, (III.4.1.8)] and the discussion above. If $M$ is general, then $M$ is good if and only if there exists some filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots$$

of $M$ such that $\bigcup_i M_i = M$, and for each $i \geq 1$, $M_i/M_{i-1} \cong N_i \otimes V_i$ for some $R$-finite $R$-projective good $G$-module $N_i$ and an $R$-module $V_i$. Indeed, the only if part is [Has1, (III.2.3.8)], while the if part is a consequence of the goodness of $N_i \otimes V_i$, see [Has1, (III.4.1.8)]. This notion is independent of the choice of $T$ or $\Delta$, and we are done.

Note that if $R \to R'$ is a Noetherian $R$-algebra, then an $R' \otimes_R G$-module $M'$ is good if and only if it is so as a $G$-module. This comes from the isomorphism

$$\text{Ext}_G^i(\Delta_G(\lambda), M') \cong \text{Ext}_{R' \otimes G}^i(\Delta_{R' \otimes G}(\lambda), M').$$

If $M$ is a good $G$-module, and $R'$ is $R$-flat or $M$ is $R$-finite $R$-projective, then $R' \otimes_R M$ is a good $R' \otimes_R G$-module by [Has1, (I.3.6.20)] and [Has1, (III.4.8)], see [Has1, Chapter 29]. If $M$ is good and $V$ is a flat $R$-module, then $M \otimes V$ is good. This follows from the canonical isomorphism

$$\text{Ext}_G^i(\Delta_G(\lambda), M \otimes V) \cong \text{Ext}_G^i(\Delta_G(\lambda), M) \otimes V,$$

see [Has1, (I.3.6.16)].

If $R'$ is faithfully flat over $R$ and $R' \otimes_R M$ is good, then $M$ is good by [Has1, (I.3.6.20)].

(7.3) Let $S$ be a scheme, and $G$ a reductive group scheme over $S$, and $X$ a Noetherian $S$-scheme on which $G$ acts trivially. Let $M$ be a quasi-coherent $(G, O_X)$-module. For $(G, O_X)$-modules, see [Has1, Chapter 29]. Almost by definition, a $(G, O_X)$-module and a $(G \times_S X, O_X)$-module (note that $G \times_S X$ is an $X$-group scheme) are the same thing.

We say that $M$ is good if there is a Noetherian commutative ring $R$ and a faithfully flat morphism of finite type $f : \text{Spec} R \to X$ such that $G_R :=$
Spec $R \times_S G$ is a split reductive group scheme over $R$, and $\Gamma(\text{Spec } R, f^*M)$ is a good $G_R$-module. This notion is independent of the choice of $f$ such that $G_R$ is split reductive. When $X = \text{Spec } B$ is affine, then we also say that $\Gamma(X, M)$ is good, if $M$ is good. If $g : X' \to X$ is a flat morphism of Noetherian schemes and $M$ is a good quasi-coherent $(G, \mathcal{O}_X)$-module, then $g^*M$ is good. If $M$ is a quasi-coherent $(G, \mathcal{O}_X)$-module, $g$ is faithfully flat, and $g^*M$ is good, then $M$ is good.

For a quasi-coherent $(G, \mathcal{O}_X)$-module $M$, we define the good locus of $M$ to be

$$\text{Good}(M) = \{ x \in X \mid M_x \text{ is a good (Spec } \mathcal{O}_{X,x} \times_S G)\text{-module} \}.$$ 

If $g : X' \to X$ is a flat morphism of Noetherian schemes, then $g^{-1}(\text{Good}(M)) = \text{Good}(g^*M)$. If $X = \text{Spec } R$ is affine, then for a $(G, R)$-module $N$, $\text{Good}(N)$ stands for $\text{Good}(\tilde{N})$, where $\tilde{N}$ is the sheaf associated with $N$.

(7.4) Let the notation be as in (7.1).

For a poset ideal $\pi$ of $X^+$ and a $G$-module $M$, we say that $M$ belongs to $\pi$ if $M_\lambda = 0$ for $\lambda \in X^+ \setminus \pi$.

**Proposition 7.5.** Let $\pi$ be a poset ideal of $X^+$ and $M$ a $G$-module. Then the following are equivalent.

1. $M$ belongs to $\pi$.
2. For any $R$-finite subquotient $N$ of $M$ and any $R$-algebra $K$ that is a field, $K \otimes_R N$ belongs to $\pi$.
3. For any $R$-finite subquotient $N$ of $M$, any $R$-algebra $K$ that is a field, and $\lambda \in X^+ \setminus \pi$, $\text{Hom}_G(\Delta_G(\lambda), K \otimes_R N) = 0$.
4. For any $\lambda \in X^+ \setminus \pi$, $\text{Hom}_G(\Delta_G(\lambda), M) = 0$.
5. $M$ is a $C_\pi$-comodule, where $C_\pi$ is the Donkin subcoalgebra of $C$ with respect to $\pi$, see [Has1, (III.2.3.13)].

**Proof.** (1)$\Rightarrow$(2) is obvious.

(2)$\Rightarrow$(3) We may assume that $R = K$ and $N = M$. Then

$$\text{Hom}_G(\Delta_G(\lambda), M) \cong \text{Hom}_G(M^*, \text{ind}_B^G(-w_0\lambda))$$

$$\cong \text{Hom}_B(M^*, -w_0\lambda) \cong \text{Hom}_B(w_0\lambda, M) \subset M_{w_0\lambda} = 0.$$
(3)⇒(4) As $M$ is the inductive limit of $R$-finite $G$-submodules of $M$, we may assume that $M$ is $R$-finite. We use the Noetherian induction, and we may assume that the implication is true for $R/I$ for any nonzero ideal $I$ of $R$. If $R$ is not a domain, then there is a nonzero ideal $I$ of $R$ such that the annihilator $0 : I$ of $R$ is also nonzero. As $\text{Hom}_G(\Delta_G(\lambda), M/IM) = 0$ and $\text{Hom}_G(\Delta_G(\lambda), IM) = 0$, we have that $\text{Hom}_G(\Delta_G(\lambda), M) = 0$. So we may assume that $R$ is a domain. Let $N$ be the torsion part of $M$. Note that $0 \to N \to M \to K \otimes_R M$

is exact, where $K$ is the field of fractions of $R$. Hence $N$ is a $G$-submodule of $M$. The annihilator of $N$ is nontrivial, and hence $\text{Hom}_G(\Delta_G(\lambda), N) = 0$. On the other hand, by assumption, $\text{Hom}_G(\Delta_G(\lambda), K \otimes_R M) = 0$. So $\text{Hom}_G(\Delta_G(\lambda), M) = 0$, and we are done.

(4)⇒(5) is [Has1, (III.2.3.5)].

(5)⇒(1) As the coaction $\omega_M : M \to M' \otimes_R C_\pi$ is injective, it suffices to show that $M' \otimes_R C_\pi$ belongs to $\pi$, where $M'$ is the $R$-module $M$ with the trivial $G$-action. For this, it suffices to show that $C_\pi$ belongs to $\pi$. This is proved easily by induction on the number of elements of $\pi$, if $\pi$ is finite, almost by the definition of the Donkin system [Has1, (III.2.2)], and the fact that $\nabla_G(\lambda)$ belongs to $\pi$. Then the general case follows easily from the definition of $C_\pi$, see [Has1, (III.2.3.13)].

Corollary 7.6. Let $M$ be a $G$-module, and $\pi$ a poset ideal of the set of dominant weights $X^+$. If $M$ belongs to $\pi$, then $\text{Ext}_G^i(\Delta_G(\lambda), M) = 0$ for $i \geq 0$ and $\lambda \in X^+ \setminus \pi$.

Proof. We use the induction on $i$. The case $i = 0$ is already proved in Proposition 7.5.

Let $i > 0$. Let $C_\pi$ denote the Donkin subcoalgebra of $k[G]$. Consider the exact sequence

$$0 \to M \xrightarrow{\omega_M} M' \otimes_R C_\pi \to N \to 0.$$ 

Then $N$ belongs to $\pi$, and $\text{Ext}_G^{i-1}(\Delta_G(\lambda), N) = 0$ by induction assumption. On the other hand, as $C_\pi$ is good and $R$-finite $R$-projective by construction, $M' \otimes_R C_\pi$ is also good by [Has1, (III.4.1.8)]. Hence $\text{Ext}_G^i(\Delta_G(\lambda), M' \otimes_R C_\pi) = 0$. By the long exact sequence of the Ext-modules, we have that $\text{Ext}_G^i(\Delta_G(\lambda), M) = 0$. \hfill \Box

Lemma 7.7. Let the notation be as in (7.3). Let $M$ be a coherent $(G, \mathcal{O}_X)$-module. Then $\text{Good}(M)$ is Zariski open in $X$. 30
Proof. Let $f : \text{Spec } R \to X$ be a faithfully flat morphism of finite type such that $G_R$ is split reductive. Let $M_R := \Gamma(\text{Spec } R, f^*M)$. Then $\text{Good}(M_R) = f^{-1}(\text{Good}(M))$. As $f$ is a surjective open map, it suffices to show that $\text{Good}(M_R)$ is open in $\text{Spec } R$. So we may assume that $S = X = \text{Spec } R$ is affine and $G$ is split, and we are to prove that $\text{Good}(N)$ is open for an $R$-finite $G$-module $N$.

As $N$ is $R$-finite, there exists some finite poset ideal $\pi$ of $X^+$ to which $N$ belongs. Then $\text{Ext}^i_G(\Delta_G(\lambda), N) = 0$ for $\lambda \in X^+ \setminus \pi$ and $i \geq 0$. Set $L := \bigoplus_{\lambda \in \pi} \Delta_G(\lambda)$. Then $\text{Good}(N)$ is nothing but the complement of the support of the $R$-module $\text{Ext}^1_G(L, N)$ by \cite[(III.2.3.8)]{Has1}. As $\text{Ext}^1_G(L, N)$ is $R$-finite by \cite[(III.2.3.19)]{Has1}, the support of $\text{Ext}^1_G(L, N)$ is closed, and we are done. \hfill \Box

(7.8) Let the notation be as in (7.3). For a quasi-coherent $(G, \mathcal{O}_X)$-module $M$, the good dimension $\text{GD}(M)$ is defined to be $-\infty$ if $M = 0$. If $M \neq 0$ and there is an exact sequence

\begin{align}
0 \to M \to N_0 \to \cdots \to N_s \to 0
\end{align}

such that each $N_i$ is good, then $\text{GD}(M)$ is defined to be the smallest $s$ such that such an exact sequence exists. If there is no such an exact sequence, $\text{GD}(M)$ is defined to be $\infty$.

(7.9) Assume that $X = \text{Spec } R$ is affine and $G$ is split reductive. For a $G$-module $M$,

$$\text{GD}(M) = \sup\{i \mid \bigoplus_{\lambda \in X^+} \text{Ext}^i_G(\Delta_G(\lambda), M) \neq 0\}.$$ 

Note that $M$ is good if and only if $\text{GD}(M) \leq 0$. If $r \geq 0, s \geq -1$, and

$$0 \to M \to M_s \to \cdots M_0 \to N \to 0$$

is an exact sequence of $G$-modules with $\text{GD}(M_i) \leq i + r$, then $\text{GD}(M) \leq s + r + 1$ if and only if $\text{GD}(N) \leq r$.

If $M$ and $N$ are good and $M$ is $R$-finite $R$-projective, then $M \otimes N$ is good, see \cite[(III.4.5.10)]{Has1}. Moreover, if $M$ is $R$-finite $R$-projective with $\text{GD}(M) \leq s$, then $M$ has an exact sequence of the form (5) such that each $N_i$ is $R$-finite $R$-projective and good. Indeed, $M$ belongs to some finite poset ideal $\pi$ of $X^+$, and when we truncate the cobar resolution of $M$ as a $C_\pi$-comodule, then we obtain such a sequence.

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It follows that for an $R$-finite $R$-projective $G$-module $M$, $\text{GD}(M) \leq s$ if and only if $\text{GD}(\kappa(m) \otimes_R M) \leq s$ for any maximal ideal $m$ of $R$ by [Has1, (III.4.1.8)].

It also follows that if $\text{GD}(M) \leq s$ and $\text{GD}(N) \leq t$ with $M$ being $R$-finite $R$-projective, then $\text{GD}(M \otimes N) \leq s + t$.

**Lemma 7.10.** Let $V$ be an $R$-finite $R$-projective $G$-module with rank $V \leq n < \infty$. Then the following are equivalent.

1. $\text{Sym} V$ is good.
2. $\bigoplus_{i=1}^{n-1} \text{Sym}_i V$ is good.
3. For $i = 1, \ldots, n - 1$, $\text{GD}(\bigwedge^i V) \leq i - 1$.
4. For $i \geq 1$, $\text{GD}(\bigwedge^i V) \leq i - 1$.

**Proof.** We may assume that $R$ is a field.

(1)$\Rightarrow$(2) is trivial.

(2)$\Rightarrow$(3) We use the induction on $i$.

By assumption and the induction assumption, $\text{GD}(\text{Sym}_{i-j} V \otimes \bigwedge^j V) \leq j - 1$ for $j = 1, \ldots, i - 1$. On the other hand, $\text{Sym}_i V$ is good. So by the exact sequence

$$0 \to \bigwedge^i V \to \text{Sym}_1 V \otimes \bigwedge^{i-1} V \to \cdots \to \text{Sym}_{i-1} V \otimes \bigwedge^1 V \to \text{Sym}_i V \to 0,$$

$\text{GD}(\bigwedge^i V) \leq i - 1$.

(3)$\Rightarrow$(4) is trivial, as $\text{dim} \bigwedge^i V \leq 1$ for $i \geq n$.

(4)$\Rightarrow$(1) Note that $\text{Sym}_0 V = R$ is good. Now use induction on $i \geq 1$ to prove that $\text{Sym}_i V$ is good (use the exact sequence (6) again). \hfill $\square$

**Theorem 7.11.** Let $S$ be a scheme, $G$ a reductive $S$-group acting trivially on a Noetherian $S$-scheme $X$. Let $M$ be a locally free coherent $(G, \mathcal{O}_X)$-module. Then

(7) $\text{Good}(\text{Sym} \ M) = \{ x \in X \mid \text{Sym}(\kappa(x) \otimes_{\mathcal{O}_{X,x}} M_x) \\
is a good $(\text{Spec} \kappa(x) \times_S G)$-module\}$,

and $\text{Good}(\text{Sym} \ M)$ is Zariski open in $X$. 

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Proof. Take a faithfully flat morphism of finite type $f : \text{Spec} R \to X$ such that $\text{Spec} R \times_S G$ is split reductive. Note that $f$ is a surjective open map, and $f^{-1}(\text{Good}(\text{Sym} M)) = \text{Good}(\text{Sym} f^* M)$.

First we prove that $\text{Good}(\text{Sym} M)$ is open. We may assume that $S = X = \text{Spec} R$ is affine, and $G$ is split reductive.

Then by Lemma 7.10 and Lemma 7.7,

$$\text{Good}(\text{Sym} M) = \text{Good}(\bigoplus_{i=1}^n \text{Sym}_i M)$$

is open, where the rank of $M$ is less than or equal to $n$.

Next we prove that the equality (7) holds. Let $P \in \text{Spec} R$, and $x = f(P)$. Then $\text{Sym}(\kappa(x) \otimes_{\mathcal{O}_{X,x}} M_x)$ is good if and only if $\text{Sym}(\kappa(P) \otimes_{R_P} \Gamma(\text{Spec} R, f^* M)_P)$ is good. So we may assume that $S = X = \text{Spec} R$ is affine, and $G$ is split reductive. Let $N$ be an $R$-finite $R$-projective $G$-module of rank at most $n$. Then $(\text{Sym} N)_P$ is good if and only if $(\bigoplus_{i=1}^n \text{Sym}_i N)_P$ is good by Lemma 7.10. By [Has1, (III.4.1.8)], $(\bigoplus_{i=1}^n \text{Sym}_i N)_P$ is good if and only if $\kappa(P) \otimes_{R_P} (\bigoplus_{i=1}^n \text{Sym}_i N)_P$ is good. By Lemma 7.10 again, it is good if and only if $\kappa(P) \otimes_{R_P} (\text{Sym} N)_P$ is so. Thus the equality (7) was proved.

Corollary 7.12. Let $R$ be a Noetherian domain of characteristic zero, and $G$ a reductive group over $R$. If $M$ is an $R$-finite $R$-projective $G$-module, then \{ $P \in \text{Spec} R \mid \text{Sym}(\kappa(P) \otimes_R M)$ is good \} is a dense open subset of $\text{Spec} R$.

Proof. By Theorem 7.11, it suffices to show that $\text{Good}(\text{Sym} M)$ is non-empty. But the generic point $\eta$ of $\text{Spec} R$ is in $\text{Good}(\text{Sym} M)$. Indeed, $\kappa(\eta)$ is a field of characteristic zero, and any $\kappa(\eta) \otimes_R G$-module is good.

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