HIGH–ENERGY SCATTERING AND THE EUCLIDEAN–MINKOWSKIAN DUALITY

(Can the *pomeron* be derived from a Euclidean–Minkowskian duality?)

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**OUTLINE**

- After a brief review of some relevant *analyticity* and *crossing–symmetry* properties of the correlation functions of two Wilson lines or two Wilson loops in QCD, when going *from Euclidean to Minkowskian theory*, ¹ . . .

- . . . we shall see how these properties can be related to the still unsolved problem of the *asymptotic s–dependence* of the *hadron–hadron total cross sections*. In particular, we critically discuss the question if (and how) a *pomeron*–like behaviour can be derived from this *Euclidean–Minkowskian duality*. ²

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¹ M. Giordano and E. Meggiolaro, Phys. Rev. D 74 (2006) 016003; hep-ph/0602143.
² E. Meggiolaro, hep-ph/0612307, revised version, May 2007.
The quark–quark scattering amplitude, at high squared energies $s$ in the center of mass and small squared transferred momentum $t$ (that is to say: $|t| \leq 1 \text{ GeV}^2 \ll s$), can be described by the expectation value of two infinite lightlike Wilson lines, running along the classical trajectories of the colliding particles [Nachtmann, 1991; EM1996; EM2001].

$$\Longrightarrow$$ there are infrared (IR) divergences, which are typical of 3+1 dimensional gauge theories!

→ One can regularize this IR problem by letting the Wilson lines coincide with the classical trajectories for quarks with a non-zero mass $m$ (so forming a certain finite hyperbolic angle $\chi$ in Minkowskian space–time: of course, $\chi \to +\infty$ when $s \to \infty$), and, in addition, by considering finite Wilson lines, extending in proper time from $-T$ to $T$ (and eventually letting $T \to +\infty$) [Verlinde, 1993; EM2002]:

$$\mathcal{M}^{qq}(s; t)_{\alpha'i;i'j'}^{\alpha\beta} \sim -i 2s \delta_{\alpha'\alpha} \delta_{\beta'\beta} g_{M}^{qq}(\chi \to +\infty; T \to \infty; t)_{i'i;j'j'},$$

$$g_{M}^{qq}(\chi; T; t) \equiv \frac{1}{[Z_M(T)]^2} \int d^2 \mathbf{z}_\perp e^{i\vec{q}_\perp \cdot \vec{z}_\perp} \langle [W_{p_1}^{(T)}(\vec{z}_\perp) - \mathbb{I}_{i'i}] [W_{p_2}^{(T)}(\vec{0}_\perp) - \mathbb{I}_{j'j}] \rangle,$$

$$Z_M(T) \equiv \frac{1}{N_c} \langle \text{Tr}[W_{p_1}^{(T)}(\vec{0}_\perp)] \rangle = \frac{1}{N_c} \langle \text{Tr}[W_{p_2}^{(T)}(\vec{0}_\perp)] \rangle;$$

where $t = -|\vec{q}_\perp|^2$, $\vec{q}_\perp$ being the transferred momentum, and $\vec{z}_\perp = (z^2, z^3)$ is impact parameter.
The two IR-regularized Wilson lines are defined as \( z = (0, 0, \vec{z}_\perp) \):

\[
W^{(T)}_{p_1}(\vec{z}_\perp) \equiv T \exp \left[ -ig \int_{-T}^{+T} A_\mu \left( z + \frac{p_1}{m} \tau \right) \frac{p_1^\mu}{m} d\tau \right],
\]

\[
W^{(T)}_{p_2}(\vec{0}_\perp) \equiv T \exp \left[ -ig \int_{-T}^{+T} A_\mu \left( \frac{p_2}{m} \tau \right) \frac{p_2^\mu}{m} d\tau \right],
\]

along quark trajectories (speed \( \pm V = \pm \tanh \frac{x}{2} \) along \( x^1 \)):

\[
p_{1,2} = m \left( \cosh \frac{\chi}{2}, \pm \sinh \frac{\chi}{2}, \vec{0}_\perp \right).
\]

Therefore \((p_1 \cdot p_2 = m^2 \cosh \chi)\):

\[
s \equiv (p_1 + p_2)^2 = 2m^2 (\cosh \chi + 1), \quad \text{i.e.:} \quad \chi \sim \log \left( \frac{s}{m^2} \right).
\]
The expectation values $\langle W_{p_1} W_{p_2} \rangle$, $\langle W_{p_1} \rangle$ and $\langle W_{p_2} \rangle$ are averages in the sense of the QCD functional integrals:

$$\langle \mathcal{O}[A] \rangle = \frac{1}{Z} \int [dA] \det(Q[A]) e^{iS_A} \mathcal{O}[A],$$

$$Z = \int [dA] \det(Q[A]) e^{iS_A},$$

where $S_A$ is the pure-gauge (Yang–Mills) action and $Q[A]$ is the quark matrix, coming from the functional integration over the fermion degrees of freedom.

By virtue of the invariance under parity transformations and $O(3)$ spatial rotations, the domain of the function $g_M$ in the variable $\chi$ can be restricted to the real positive axis, $\chi \in \mathbb{R}^+$. In fact, a parity transformation together with a $180^\circ$ rotation around the $x^1$ axis, i.e., a transformation

$$x \rightarrow x' = \Lambda x, \quad \Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

brings $\chi$ into $-\chi$ without modifying the functional integral:

$$g_M^{qq}(-\chi; T; t)_{i'; j'; j} = g_M^{qq}(\chi; T; t)_{i'; j'; j}, \quad \forall \chi \in \mathbb{R}.$$
The Minkowskian correlator \( g^{qq}_M(\chi; T; t) \) can be reconstructed from the corresponding Euclidean correlator \( g^{qq}_E(\theta; T; t) \), defined as:

\[
g^{qq}_E(\theta; T; t) \equiv \frac{1}{[Z_E(T)]^2} \int d^2 \vec{z}_\perp e^{i \vec{q}_\perp \cdot \vec{z}_\perp} \langle [\tilde{W}^{(T)}_{p1E}(\vec{0}_\perp) - \mathbb{I}]_{i'i} [\tilde{W}^{(T)}_{p2E}(\vec{0}_\perp) - \mathbb{I}]_{j'j} \rangle_E,
\]

\[
Z_E(T) \equiv \frac{1}{N_c} \langle \text{Tr}[\tilde{W}^{(T)}_{p1E}(\vec{0}_\perp)] \rangle_E = \frac{1}{N_c} \langle \text{Tr}[\tilde{W}^{(T)}_{p2E}(\vec{0}_\perp)] \rangle_E,
\]

where \([z_E = (0, \vec{z}_\perp, 0)]\):

\[
\tilde{W}^{(T)}_{p1E}(\vec{z}_\perp) \equiv T \exp \left[ -ig \int_{-T}^{+T} A^{(E)}_\mu \left( z_E + \frac{p_{1E}}{m} \tau \right) \frac{p_{1E}\mu}{m} d\tau \right],
\]

\[
\tilde{W}^{(T)}_{p2E}(\vec{0}_\perp) \equiv T \exp \left[ -ig \int_{-T}^{+T} A^{(E)}_\mu \left( \frac{p_{2E}}{m} \tau \right) \frac{p_{2E}\mu}{m} d\tau \right],
\]

and:

\[
\langle \mathcal{O}[A^{(E)}] \rangle_E = \frac{1}{Z^{(E)}} \int [dA^{(E)}] \det(Q^{(E)}[A^{(E)}]) e^{-S^{(E)}_A} \mathcal{O}[A^{(E)}],
\]

\[
Z^{(E)} = \int [dA^{(E)}] \det(Q^{(E)}[A^{(E)}]) e^{-S^{(E)}_A}.
\]

The Euclidean four-vectors \( p_{1E} \) and \( p_{2E} \) are chosen to be:

\[
p_{1E,2E} = m \left( \pm \sin \frac{\theta}{2}, \vec{0}_\perp, \cos \frac{\theta}{2} \right),
\]

(i.e., \( p_{1E} \cdot p_{2E} = m^2 \cos \theta \), where \( \theta \) = angle formed by the two trajectories in the Euclidean four-space.)
By virtue of the $O(4)$ symmetry of the Euclidean theory, the domain of the function $g_E$ in the variable $\theta$ can be restricted to the interval $(0, \pi)$. In fact, the invariance of the functional integral under the following $O(4)$ transformation:

$$x_E \rightarrow x'_E = \mathcal{R}_1 x_E,$$

$$\mathcal{R}_1 = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},$$

leads to the following relation:

$$g^{qq}_{E}(-\theta; T; t)_{i'j';i'j} = g^{qq}_{E}(\theta; T; t)_{i'j';i'j}, \quad \forall \theta \in \mathbb{R}. \tag{1}$$

Similarly, the invariance of the functional integral under the following $O(4)$ transformation:

$$x_E \rightarrow x'_E = \mathcal{R}_2 x_E,$$

$$\mathcal{R}_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},$$

leads to the following relation:

$$g^{qq}_{E}(2\pi - \theta; T; t)_{i'j';i'j} = g^{qq}_{E}(\theta; T; t)_{i'j';i'j}, \quad \forall \theta \in \mathbb{R}. \tag{2}$$

These two relations imply the possibility of restricting the domain in the angular variable $\theta$ to the interval $(0, \pi)$, as said above.
The correlator \( g_{qq}^M(\chi; T; t) \) with \( \chi \in \mathbb{R}^+ \) can be reconstructed from the corresponding Euclidean correlator \( g_{qq}^E(\theta; T; t) \), with \( \theta \in (0, \pi) \), by an analytic continuation in the angular variables and in the IR cutoff [EM1997, EM1998, EM2002]:

\[
\overline{g}_{qq}^E(\theta; T; t) = \overline{g}_{M}^{qq}(i\theta; -iT; t), \quad \forall \theta \in \mathcal{D}_E;
\]
\[
\overline{g}_{qq}^M(\chi; T; t) = \overline{g}_{E}^{qq}(-i\chi; iT; t), \quad \forall \chi \in \mathcal{D}_M,
\]

where \( \overline{g}_E \), as a function of the complex variable \( \theta \), is the analytic extension of \( g_E \) from the real segment \((0 < \text{Re}\theta < \pi, \text{Im}\theta = 0)\) to a domain \( \mathcal{D}_E \) which also includes the negative imaginary axis \((\text{Re}\theta = 0+, \text{Im}\theta < 0)\).

Of course one is assuming that such an analytic extension exists!
Similarly \( \bar{g}_M \), as a function of the complex variable \( \chi \), is the analytic extension of \( g_M \) from the positive real axis \((\text{Re}\chi > 0, \text{Im}\chi = 0+)\) to a domain \( D_M = \{\chi \in \mathbb{C} \mid -i\chi \in D_E\} \) which also includes the imaginary segment \((\text{Re}\chi = 0, 0 < \text{Im}\chi < \pi)\).

This analytic continuation (assuming certain analyticity hypotheses) is an exact, i.e., nonperturbative result, valid both for the Abelian and the non–Abelian case.
From Wilson lines to Wilson loops

→ The quantities $g_M(\chi; T; t)$ and $g_E(\theta; T; t)$, while being finite at any given value of $T$, are divergent in the limit $T \to \infty$ (even if in some cases this IR divergence can be factorized out . . .).

→ A way to get rid of the problem of the IR–cutoff dependence is to consider an IR–finite physical quantity, like the elastic scattering amplitude of two colourless states in gauge theories, e.g., two $q\bar{q}$ meson states [Balitsky & Lipatov, 1978; 1979].

→ The high–energy meson–meson elastic scattering amplitude can be approximately reconstructed by first evaluating, in the functional–integral approach, the high–energy elastic scattering amplitude of two $q\bar{q}$ pairs (usually called dipoles), of given transverse sizes $\vec{R}_{1\perp}$ and $\vec{R}_{2\perp}$ and given longitudinal–momentum fractions $f_1$ and $f_2$ of the two quarks in the two dipoles respectively [Nachtmann, 1997; Dosch et al., 1994]:

$$M_{\mu\mu}(s, t; \vec{R}_{1\perp}, f_1, \vec{R}_{2\perp}, f_2) \equiv -i \ 2s \int d^2\vec{z}_1 e^{i\vec{q}_1 \cdot \vec{z}_1} \left[ \frac{\langle W_1 W_2 \rangle}{\langle W_1 \rangle \langle W_2 \rangle} - 1 \right],$$

→ $W_1$ and $W_2$ are two Wilson loops, defined as:

$$W^{(T)}_{1,2} \equiv \frac{1}{N_c} \ Tr \left\{ \mathcal{P} \exp \left[ -ig \int_{C_{1,2}} A_\mu(x) dx^\mu \right] \right\},$$
\[ \mathcal{C}_1 : X_{1q}^\mu(\tau) = z^\mu + \frac{p_1^\mu}{m} \tau + (1 - f_1)R_1^\mu, \quad X_{1\bar{q}}^\mu(\tau) = z^\mu + \frac{p_1^\mu}{m} \tau - f_1 R_1^\mu, \]

\[ \mathcal{C}_2 : X_{2q}^\mu(\tau) = \frac{p_2^\mu}{m} \tau + (1 - f_2)R_2^\mu, \quad X_{2\bar{q}}^\mu(\tau) = \frac{p_2^\mu}{m} \tau - f_2 R_2^\mu, \]

and are closed by straight–line paths at proper times \( \tau = \pm T \), where \( T \) plays the role of an IR cutoff, which can and must be removed in the end (\( T \rightarrow \infty \)).
Let us introduce the following notations for the normalized correlators $\langle \mathcal{W}_1 \mathcal{W}_2 \rangle / \langle \mathcal{W}_1 \rangle \langle \mathcal{W}_2 \rangle$ in the Minkowskian and in the Euclidean theory, in the presence of a finite IR cutoff $T$:

\[
G_M(\chi, T, \bar{z}_\perp; 1, 2) \equiv \frac{\langle \mathcal{W}_1^{(T)} \mathcal{W}_2^{(T)} \rangle}{\langle \mathcal{W}_1^{(T)} \rangle \langle \mathcal{W}_2^{(T)} \rangle},
\]

\[
G_E(\theta, T, \bar{z}_\perp; 1, 2) \equiv \frac{\langle \tilde{\mathcal{W}}_1^{(T)} \tilde{\mathcal{W}}_2^{(T)} \rangle_E}{\langle \tilde{\mathcal{W}}_1^{(T)} \rangle_E \langle \tilde{\mathcal{W}}_2^{(T)} \rangle_E}.
\]

with “1 $\equiv \tilde{R}_{1\perp}, f_1$” and “2 $\equiv \tilde{R}_{2\perp}, f_2$” in the left–hand sides.

$\leftrightarrow$ As in the case of Wilson lines, the Minkowskian quantity $G_M$ with $\chi \in \mathbb{R}^+$ can be reconstructed from the corresponding Euclidean quantity $G_E$, with $\theta \in (0, \pi)$, by an analytic continuation in the angular variables $\theta \to -i\chi$ and in the IR cutoff $T \to iT$ [EM2002,EM2005]:

\[
\overline{G}_E(\theta, T, \bar{z}_\perp; 1, 2) = \overline{G}_M(i\theta, -iT, \bar{z}_\perp; 1, 2), \quad \forall \theta \in \mathcal{D}_E;
\]

\[
\overline{G}_M(\chi, T, \bar{z}_\perp; 1, 2) = \overline{G}_E(-i\chi, iT, \bar{z}_\perp; 1, 2), \quad \forall \chi \in \mathcal{D}_M.
\]

$\leftrightarrow$ This analytic continuation (as the corresponding result for the line–line case) is an exact, i.e., nonperturbative result, valid both for the Abelian and the non–Abelian case.
Analytic continuation and IR finiteness

As we have said above, the loop–loop correlation functions, both in the Minkowskian and in the Euclidean theory, are expected to be IR–finite quantities, i.e., to have finite limits when $T \to \infty$, differently from what happens in the case of Wilson lines. One can then define the following loop–loop correlation functions with the IR cutoff removed:

$$
\mathcal{C}_M(\chi, z_\perp; 1, 2) \equiv \lim_{T \to \infty} [\mathcal{G}_M(\chi, T, z_\perp; 1, 2) - 1],
$$

$$
\mathcal{C}_E(\theta, z_\perp; 1, 2) \equiv \lim_{T \to \infty} [\mathcal{G}_E(\theta, T, z_\perp; 1, 2) - 1].
$$

It has been proved in [EM2005] that, under certain analyticity conditions in the complex variable $T$ [conditions which are also sufficient to make the previous analytic–continuation relations meaningful], these two quantities, obtained after the removal of the IR cutoff ($T \to \infty$), are still connected by the usual analytic continuation in the angular variables only:

$$
\overline{\mathcal{C}}_E(\theta, z_\perp; 1, 2) = \overline{\mathcal{C}}_M(i\theta, z_\perp; 1, 2), \quad \forall \theta \in \mathcal{D}_E;
$$

$$
\overline{\mathcal{C}}_M(\chi, z_\perp; 1, 2) = \overline{\mathcal{C}}_E(-i\chi, z_\perp; 1, 2), \quad \forall \chi \in \mathcal{D}_M.
$$

The validity of this relation has been also verified by an explicit calculation in QCD perturbation theory up to the order $\mathcal{O}(g^6)$ [Babansky & Balitsky, 2003].
Analyticity and crossing symmetry

We can derive a nice geometrical interpretation of the so-called crossing symmetry between the quark–quark and quark–antiquark scattering amplitudes (and also between dipole–dipole scattering amplitudes) using the functional integral approach.

Changing from a quark to an antiquark just corresponds, in our formalism, to substitute the corresponding Wilson line (in the fundamental representation $T_a$) with its complex conjugate (the Wilson line in the complex conjugate representation $T_a' = -T_a^*$):

$$ q \to \bar{q} \iff T_a \to T_a' = -T_a^* \iff W_p \to W_p^*. $$

Therefore, the high–energy scattering amplitude for:

$$ q(p_1, \alpha, i) + \bar{q}(p_2, \beta, j) \to q(p_1' \simeq p_1, \alpha', i') + \bar{q}(p_2' \simeq p_2, \beta', j'), $$

is given by the formula:

$$ \mathcal{M}^{q\bar{q}}(s; t)^{\alpha'\alpha;\beta'\beta}_{i'j;i} \sim -i 2s \delta_{\alpha'\alpha} \delta_{\beta'\beta} g_M^{q\bar{q}}(\chi \to +\infty; T \to \infty; t)_{i'j;i}^j, $$

where the quark–antiquark correlator $g_M^{q\bar{q}}(\chi; T; t)_{i'j;i}^j$ is defined as:

$$ g_M^{q\bar{q}}(\chi; T; t) = \frac{1}{[Z_M(T)]^2} \int d^2\vec{z}_\perp e^{i\vec{q}_\perp \cdot \vec{z}_\perp} \langle [W_{p_1}^{(T)}(\vec{z}_\perp) - \mathbb{I}]_{i'i} [W_{p_2}'^{(T)*}(\vec{0}_\perp) - \mathbb{I}]_{j'j} \rangle. $$
Crossing symmetry relates the amplitude of this process to the amplitude of the "crossed process", defined as:

\[ q(p_1, \alpha, i) + q(-p_2', \beta', j') \rightarrow q(p_1', \alpha', i') + q(-p_2, \beta, j). \]

Using the fact that the generators \( T_a \) are hermitian and the variables \( A^a_{\mu} \) are real, we can also write:

\[
\left[ W_{p_2}(T)^*(\vec{0}_\perp) \right]_{j'j} = \left[ W_{p_2}^{(T)}(\vec{0}_\perp) \right]_{j'j} = \left[ W_{-p_2}^{(T)}(\vec{0}_\perp) \right]_{j'j}.
\]

And, therefore:

\[
g_{qq}^M(p_1, p_2; T; t)_{i'i';jj'} = \frac{1}{[Z_M(T)]^2} \int d^2z_\perp e^{i\vec{q}_1 \cdot \vec{z}_\perp} \langle [W_{p_1}^{(T)}(\vec{z}_\perp) - \mathbb{I}]_{i'i'} [W_{-p_2}^{(T)}(\vec{0}_\perp) - \mathbb{I}]_{jj'} \rangle.
\]

That is, reminding the definition of the quark–quark correlator:

\[
g_{qq}^M(p_1, p_2; T; t)_{i'i';jj'} = g_{qq}^M(p_1, -p_2; T; t)_{i'i';jj'}.
\]

But \( \tilde{p}_2 = -p_2 \) is unphysical! Observe that:

\[
(p_1, p_2) \rightarrow (p_1, \tilde{p}_2 = -p_2) \implies \cosh \chi \rightarrow -\cosh \chi
\]

\[\implies g_{qq}^M(\chi; T; t)_{i'i';jj'} = g_{qq}^M(\chi; T; t)_{i'i';jj'}.
\]

\[\rightarrow \text{To determine unambiguously which complex values of } \chi \text{ this substitution corresponds to, we will make use of the analytic–continuation relation between the Minkowskian and the Euclidean theory and of the } O(4) \text{ symmetry of the latter.}\]
Similarly, for Euclidean Wilson lines:

\[
\begin{bmatrix}
\tilde{W}^{(T)*}_{p2E}(0_{\perp})
\end{bmatrix}_{j'j} = \begin{bmatrix}
\tilde{W}^{(T)}_{p2E}(0_{\perp})
\end{bmatrix}^{\dagger}_{jj'} = \begin{bmatrix}
\tilde{W}^{(T)}_{-p2E}(0_{\perp})
\end{bmatrix}_{jj'},
\]

and so:

\[
g_{E}^{q\bar{q}}(p_{1E}, p_{2E}; T; t)_{i'ij'j} = g_{E}^{q\bar{q}}(p_{1E}, -p_{2E}; T; t)_{i'ij'j}.
\]

That is to say:

\[
g_{E}^{q\bar{q}}(\theta; T; t)_{i'ij'j} = g_{E}^{q\bar{q}}(\pi + \theta; T; t)_{i'ij'j} = g_{E}^{q\bar{q}}(\pi - \theta; T; t)_{i'ij'j},
\]

using the invariance under the \(O(4)\) 90° “rotation” \(R_{3}\) in the \((x_{E1}, x_{E4})\) plane and under the \(O(4)\) “time–reversal” transf. \(R_{2}:
\]
Therefore we have obtained:

\[ g^{q}q_{E}(\theta; T; t)_{ii'jj} = g^{q}q_{E}(\pi - \theta; T; t)_{ii'jj'}, \quad \forall \theta \in \mathbb{R}. \]

Suppose now that this relation can be analytically extended to values of \( \theta \) in a common analyticity domain \( \mathcal{D}_E \) for \( g^{q}q_{E} \) and \( g^{q}q_{E} \):

\[ \overline{g}^{q}q_{E}(\theta; T; t)_{ii'jj} = \overline{g}^{q}q_{E}(\pi - \theta; T; t)_{ii'jj'}, \quad \forall \theta \in \mathcal{D}_E. \]

\[ (\theta \in \mathcal{D}_E \iff \pi - \theta \in \mathcal{D}_E) \]
Using the analytic-continuation relations, we thus obtain:

\[
g_{M}(\chi; T; t)_{i' i; j' j} = g_{E}(i\pi - \chi; T; t)_{i' i; j' j} = g_{M}(\pi + i\chi; i T; t)_{i' i; j' j'}, \quad \forall \chi \in \mathcal{D}_{M},
\]

where \(\mathcal{D}_{M} = \{\chi \in \mathbb{C} | -i\chi \in \mathcal{D}_{E}\}\) is the common analyticity domain of \(g_{M}\) and \(g_{E}\):

\[
g_{M}(\chi; T; t)_{i' i; j' j} = g_{M}(i\pi - \chi; T; t)_{i' i; j' j'}, \quad \forall \chi \in \mathbb{R}^{+}.
\]

\[
(\chi \to i\pi - \chi \quad \Rightarrow \quad s \to e^{-i\pi} s)
\]

\[
(p_2 \leftrightarrow -p_2 \quad \Rightarrow \quad s = (p_1 + p_2)^2 \leftrightarrow u = (p_1 - p_2)^2 \simeq -s)
\]
Crossing relations for loop–loop correlators

Let us consider a certain Wilson loop:

\[ \mathcal{W}_p^{(T)}(\vec{b}_\perp, \vec{R}_\perp, f) = \frac{1}{N_c} \text{Tr} \left\{ \mathcal{P} \exp \left[ -ig \oint_{\mathcal{C}(p, b, R, f)} A_\mu(x) dx^\mu \right] \right\}, \]

defined on the rectangular path \( \mathcal{C}(p, b, R, f) \), consisting of the straight–line quark–antiquark trajectories \([b = (0, 0, \vec{b}_\perp), R = (0, 0, \vec{R}_\perp)]\):

\[ \mathcal{C} : X_q^\mu(\tau) = b^\mu + \frac{p^\mu}{m} \tau + (1 - f) R^\mu, \quad X_{\bar{q}}^\mu(\tau) = b^\mu + \frac{p^\mu}{m} \tau - f R^\mu. \]

Let us define the corresponding antiloop \( \overline{\mathcal{W}} \) by exchanging the quark and the antiquark trajectories:

\[ \overline{\mathcal{W}}_p^{(T)}(\vec{b}_\perp, \vec{R}_\perp, f) = \frac{1}{N_c} \text{Tr} \left\{ \mathcal{P} \exp \left[ -ig \oint_{\overline{\mathcal{C}}(p, b, R, f)} A_\mu(x) dx^\mu \right] \right\}, \]

\[ \overline{\mathcal{C}}(p, b, R, f) = \mathcal{C}(-p, b, R, f) = \mathcal{C}(p, b, -R, 1 - f). \]

Consequently:

\[ \overline{\mathcal{W}}_p^{(T)}(\vec{b}_\perp, \vec{R}_\perp, f) = \mathcal{W}_p^{(T)}(\vec{b}_\perp, \vec{R}_\perp, f) = \mathcal{W}_p^{(T)}(\vec{b}_\perp, -\vec{R}_\perp, 1 - f). \]

Let us define the loop–antiloop correlator \( \mathcal{G}_M^{(II)} \) by substituting \( \mathcal{W}_2 \to \overline{\mathcal{W}}_2 \) in the loop–loop correlator \( \mathcal{G}_M \):

\[ \mathcal{G}_M^{(II)}(\chi, T, \vec{z}_\perp; \vec{R}_{1\perp}, f_1, \vec{R}_{2\perp}, f_2) = \frac{\langle \mathcal{W}_1^{(T)} \overline{\mathcal{W}}_2^{(T)} \rangle}{\langle \mathcal{W}_1^{(T)} \rangle \langle \overline{\mathcal{W}}_2^{(T)} \rangle}. \]
Going on as for the line–line case, from \( \overline{\mathcal{C}}(p, b, R, f) = \mathcal{C}(-p, b, R, f) \) we immediately obtain:

\[
\mathcal{G}_E^{(II)}(\theta, T, \tilde{z}_\perp; \tilde{R}_{1\perp}, f_1, \tilde{R}_{2\perp}, f_2) = \mathcal{G}_E(\pi - \theta, T, \tilde{z}_\perp; \tilde{R}_{1\perp}, f_1, \tilde{R}_{2\perp}, f_2),
\]

\[
\mathcal{G}_M^{(II)}(\chi, T, \tilde{z}_\perp; \tilde{R}_{1\perp}, f_1, \tilde{R}_{2\perp}, f_2) = \overline{\mathcal{G}}_M(i\pi - \chi, T, \tilde{z}_\perp; \tilde{R}_{1\perp}, f_1, \tilde{R}_{2\perp}, f_2),
\]

But from \( \overline{\mathcal{C}}(p, b, R, f) = \mathcal{C}(p, b, -R, 1 - f) \) we also get:

\[
\mathcal{G}_E^{(II)}(\theta, T, \tilde{z}_\perp; \tilde{R}_{1\perp}, f_1, \tilde{R}_{2\perp}, f_2) = \mathcal{G}_E(\theta, T, \tilde{z}_\perp; \tilde{R}_{1\perp}, f_1, -\tilde{R}_{2\perp}, 1 - f_2),
\]

\[
\mathcal{G}_M^{(II)}(\chi, T, \tilde{z}_\perp; \tilde{R}_{1\perp}, f_1, \tilde{R}_{2\perp}, f_2) = \mathcal{G}_M(\chi, T, \tilde{z}_\perp; \tilde{R}_{1\perp}, f_1, -\tilde{R}_{2\perp}, 1 - f_2).
\]

Therefore we find the following \textit{crossing–symmetry relations}:

\[
\mathcal{G}_E(\pi - \theta, T, \tilde{z}_\perp; \tilde{R}_{1\perp}, f_1, \tilde{R}_{2\perp}, f_2) = \mathcal{G}_E(\theta, T, \tilde{z}_\perp; \tilde{R}_{1\perp}, f_1, -\tilde{R}_{2\perp}, 1 - f_2),
\]

\[
\overline{\mathcal{G}}_M(i\pi - \chi, T, \tilde{z}_\perp; \tilde{R}_{1\perp}, f_1, \tilde{R}_{2\perp}, f_2) = \mathcal{G}_M(\chi, T, \tilde{z}_\perp; \tilde{R}_{1\perp}, f_1, -\tilde{R}_{2\perp}, 1 - f_2).
\]

And also, removing the IR cutoff \( T \) \( (T \to \infty) \):

\[
\mathcal{C}_E(\pi - \theta, \tilde{z}_\perp; \tilde{R}_{1\perp}, f_1, \tilde{R}_{2\perp}, f_2) = \mathcal{C}_E(\theta, \tilde{z}_\perp; \tilde{R}_{1\perp}, f_1, -\tilde{R}_{2\perp}, 1 - f_2),
\]

\[
\overline{\mathcal{C}}_M(i\pi - \chi, \tilde{z}_\perp; \tilde{R}_{1\perp}, f_1, \tilde{R}_{2\perp}, f_2) = \mathcal{C}_M(\chi, \tilde{z}_\perp; \tilde{R}_{1\perp}, f_1, -\tilde{R}_{2\perp}, 1 - f_2).
\]
Perturbative expansion of the loop–loop correlators

Perturbation theory is the only calculation technique from first principles available both in the Minkowskian and in the Euclidean theory, and it can surely give us some useful insights about the analytic structure of the real (nonperturbative) correlation functions.

As a pedagogic example to illustrate these considerations, we first consider the simple case of QED, in the so–called quenched approximation, where fermion loops are neglected, i.e.:

$$\det(Q[A]) = 1, \quad \det(Q^E[A^E]) = 1.$$  

In such an approximation the functional integrals become simple Gaussian integrals and one finds (taking $f_1 = f_2 = \frac{1}{2}$) [EM2005]:

$$C_M(\chi, \vec{z}_\perp; 1, 2) = \exp \left[-i4e^2|\coth \chi| \, t(\vec{z}_\perp, \vec{R}_1, \vec{R}_2)\right] - 1,$$

$$C_E(\theta, \vec{z}_\perp; 1, 2) = \exp \left[-4e^2 \frac{\cos \theta}{|\sin \theta|} \, t(\vec{z}_\perp, \vec{R}_1, \vec{R}_2)\right] - 1,$$

where the coupling constant is now the electric charge $e$ and:

$$t(\vec{z}_\perp, \vec{R}_1, \vec{R}_2) \equiv \int \frac{d^2k_\perp}{(2\pi)^2} \frac{e^{-i\vec{k}_\perp \cdot \vec{z}_\perp}}{k_\perp^2} \sin \left(\frac{k_\perp \cdot \vec{R}_1}{2}\right) \sin \left(\frac{k_\perp \cdot \vec{R}_2}{2}\right)$$

$$= \frac{1}{8\pi} \log \left(\frac{|\vec{z}_\perp + \frac{\vec{R}_1}{2} + \frac{\vec{R}_2}{2}|}{|\vec{z}_\perp + \frac{\vec{R}_1}{2} - \frac{\vec{R}_2}{2}|} \left|\vec{z}_\perp - \frac{\vec{R}_1}{2} - \frac{\vec{R}_2}{2}\right| \left|\vec{z}_\perp - \frac{\vec{R}_1}{2} + \frac{\vec{R}_2}{2}\right|\right).$$
The analytic extension $\overline{C}_M$ of the Minkowskian correlator from the positive real axis $\chi \in \mathbb{R}^+$ and the analytic extension $\overline{C}_E$ of the Euclidean correlator from the real segment $\theta \in (0, \pi)$ are given by:

$$\overline{C}_M(\chi, \vec{z}_\perp; 1, 2) = \exp \left[ -i4e^2 \coth \chi \, t(\vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) \right] - 1,$$

$$\overline{C}_E(\theta, \vec{z}_\perp; 1, 2) = \exp \left[ -4e^2 \cot \theta \, t(\vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) \right] - 1,$$

with the following analyticity domains:

$$\mathcal{D}_M = \{ \chi \in \mathbb{C} | \chi \neq ik\pi, \ k \in \mathbb{Z} \};$$

$$\mathcal{D}_E = \{ \theta \in \mathbb{C} | i\theta \in \mathcal{D}_M \} = \{ \theta \in \mathbb{C} | \theta \neq k\pi, \ k \in \mathbb{Z} \}.$$

As shown in [EM2005], the Abelian results can be used to derive the corresponding results in the case of a non–Abelian gauge theory with $N_c$ colours, up to the order $\mathcal{O}(g^4)$ in perturbation theory [Shoshi, Steffen, Dosch & Pirner, 2003; Babansky & Balitsky, 2003]:

$$\overline{C}_M(\chi, \vec{z}_\perp; 1, 2)|_{g^4} = -2g^4 \left( \frac{N_c^2 - 1}{N_c^2} \right) \coth^2 \chi \ [t(\vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp})]^2,$$

$$\overline{C}_E(\theta, \vec{z}_\perp; 1, 2)|_{g^4} = 2g^4 \left( \frac{N_c^2 - 1}{N_c^2} \right) \cot^2 \theta \ [t(\vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp})]^2.$$

Both the analytic–continuation relations and the crossing–symmetry relations are trivially satisfied!

The loop–loop correlators (both in the Minkowskian and in the Euclidean theory) have been also computed up to the order $\mathcal{O}(g^6)$ of QCD perturbation theory [Babansky & Balitsky, 2003].
Angular singularities vs. bound states

It is well known that the loop–loop Euclidean correlation function $G_E(\theta, T, \bar{z}_\perp; 1, 2)$ in the case $\theta = 0$ and $T \to \infty$ is related to the van der Waals potential $V_{dd}(\bar{z}_\perp; 1, 2)$ between two static fermion–antifermion dipoles:

$$G_E(\theta = 0, T, \bar{z}_\perp; 1, 2) \simeq \exp \left[-2T \ V_{dd}(\bar{z}_\perp; 1, 2)\right].$$

$\leftrightarrow$ As a pedagogic example, in quenched QED this quantity can be easily calculated from the expressions reported in [EM2005]:

$$V_{dd}(\bar{z}_\perp; 1, 2) = \frac{e^2}{4\pi} \left( \frac{1}{|\bar{z}_\perp + \bar{R}_{1\perp} - \bar{R}_{2\perp}|} + \frac{1}{|\bar{z}_\perp - \bar{R}_{1\perp} + \bar{R}_{2\perp}|} - \frac{1}{|\bar{z}_\perp + \bar{R}_{1\perp} + \bar{R}_{2\perp}|} - \frac{1}{|\bar{z}_\perp - \bar{R}_{1\perp} - \bar{R}_{2\perp}|} \right).$$

$\leftrightarrow$ The above–written relation tells us that the correlator $G_E$ when $T \to \infty$ has a singularity in $\theta = 0$. The use of the crossing–symmetry relation then immediately tells us that $G_E$ when $T \to \infty$ has also a singularity in $\theta = \pi$ and, by virtue of the periodicity in $\theta$, a singularity is expected in each point $\theta = k\pi$, $k \in \mathbb{Z}$.

$\leftrightarrow$ A similar result is expected to hold also for the line–line Euclidean correlation functions . . .
From Wilson loops to hadrons

From the \textit{loop–loop scattering amplitude} ...

\[ \mathcal{M}(ll)(s,t; \vec{R}_{1\perp}, f_1, \vec{R}_{2\perp}, f_2) = -i \, 2s \, \widetilde{C}_M \left( \chi \sim \log \left( \frac{s}{m^2} \right), t; 1, 2 \right), \]

\[ \widetilde{C}_M(\chi, t; 1, 2) \equiv \int d^2 \vec{z}_\perp e^{i \vec{q}_\perp \cdot \vec{z}_\perp} C_M(\chi, \vec{z}_\perp; 1, 2), \]

\[ \rightarrow \ldots \text{to the \textit{hadron–hadron elastic scattering amplitude}:} \]

\[ \mathcal{M}_{(hh)}(s, t) = -i \, 2s \, \widetilde{C}_{hh}^{(hh)} \left( \chi \sim \log \left( \frac{s}{m^2} \right), t \right), \]

\[ \widetilde{C}_{hh}^{(hh)}(\chi, t) \equiv \int d^2 \vec{R}_{1\perp} \int_0^1 df_1 \, |\psi_1(\vec{R}_{1\perp}, f_1)|^2 \times \int d^2 \vec{R}_{2\perp} \int_0^1 df_2 \, |\psi_2(\vec{R}_{2\perp}, f_2)|^2 \widetilde{C}_M(\chi, t; 1, 2). \]

\[ \rightarrow \text{It can be derived from the \textit{analytic continuation} } \theta \rightarrow -i \chi \text{ of the corresponding Euclidean quantity:} \]

\[ \overline{\widetilde{C}_{hh}^{(hh)}(\chi, t)} = \overline{\widetilde{C}_{hh}^{(hh)}(-i \chi, t)}, \quad \forall \chi \in \mathcal{D}_M, \]

\[ \widetilde{C}_{E}^{(hh)}(\theta, t) \equiv \int d^2 \vec{R}_{1\perp} \int_0^1 df_1 \, |\psi_1(\vec{R}_{1\perp}, f_1)|^2 \times \int d^2 \vec{R}_{2\perp} \int_0^1 df_2 \, |\psi_2(\vec{R}_{2\perp}, f_2)|^2 \widetilde{C}_E(\theta, t; 1, 2), \]

which can be evaluated non-perturbatively by well-known and well-established techniques available in the \textit{Euclidean theory}. 
There exist in the literature some nonperturbative estimates of the Euclidean correlation functions:

- **loop–loop correlation model** (QCD vacuum = perturbative gluon exchange + nonperturbative *Stochastic Vacuum Model* [Shoshi, Steffen, Dosch & Pirner, 2003]:
  
  \[ C_E(\theta; \ldots) \text{ is an analytic function of } \cot \theta \]
  with analyticity domain \( D_E = \{ \theta \in \mathbb{C} | \theta \neq k\pi, \ k \in \mathbb{Z} \} \).

- **one–instanton contribution** [Shuryak & Zahed, 2000; Dorokhov & Cherednikov, 2004]:
  
  - the colour–elastic \( g_E(\theta; \ldots) \) and \( C_E(\theta; \ldots) \) scale as \( 1/ \sin \theta \)
    with analyticity domain \( D_E = \{ \theta \in \mathbb{C} | \theta \neq k\pi, \ k \in \mathbb{Z} \} \);
  
  - the colour–changing inelastic \( g_E(\theta; \ldots) \) scales as \( \cot \theta \)
    with analyticity domain \( D_E = \{ \theta \in \mathbb{C} | \theta \neq k\pi, \ k \in \mathbb{Z} \} \).

Using the AdS/CFT correspondence, in the \( \mathcal{N} = 4 \) SYM theory in the limit of large number of colours (\( N_c \to \infty \)) and strong coupling [Janik & Peschanski, 2000 (I)]:

- \( C_E(\theta; \ldots) \) is a combination of \( \{ 1/ \sin \theta, \ \cot \theta, \ \cos^2 \theta/ \sin \theta \} \)
  with analyticity domain \( D_E = \{ \theta \in \mathbb{C} | \theta \neq k\pi, \ k \in \mathbb{Z} \} \).

By virtue of the optical theorem:

\[
\sigma_{tot}^{(hh)}(s) \sim \frac{1}{s} \Im [\mathcal{M}_{(hh)}(s, t = 0)] \sim -2 \Re \left[ \tilde{C}_M^{(hh)} \left( \chi \sim \log \left( \frac{s}{m^2} \right), t = 0 \right) \right],
\]

all these results seem to imply (apart from possible \( s \)-dependences in the hadron wave functions!) \( s \)-independent hadron–hadron total cross sections in the asymptotic high–energy limit . . .
in apparent contradiction to the experimental observations, which seem to be well described by a \textit{pomeron}-like high-energy behaviour:

\[
\sigma_{\text{tot}}^{(hh)}(s) \underset{s \to \infty}{\approx} \sigma_0^{(hh)} \left( \frac{s}{s_0} \right)^{\epsilon_P}, \quad \text{with: } \epsilon_P \approx 0.08.
\]

A behaviour like this seems to emerge directly when applying the Euclidean–to–Minkowskian analytic–continuation approach in these two cases:

Using the \textbf{AdS/CFT correspondence}, in strongly coupled non-conformal, i.e., \textit{confining gauge theories} [Janik & Peschanski, 2000 (II); Janik, 2001]:

- branch cuts in the complex $\theta$ and $T$ planes appear, coming from logarithms and square roots, in the \textit{line–line correlator} (\equiv an ambiguity in the Euclidean–to–Minkowskian analytic continuation !?)

By an exact computation of the loop–loop correlation functions (both in the Minkowskian and in the Euclidean theory) in the first two orders $\mathcal{O}(g^4)$ and $\mathcal{O}(g^6)$ of \textit{QCD perturbation theory} [Babansky & Balitsky, 2003]:

- Analyticity of the loop–loop correlation function in the angle.
- The first iteration of the BFKL \textit{kernel} in the leading–log approximation, the so–called \textbf{BFKL–pomeron} behaviour, is reproduced [BFKL, 1975–1978]:

\[
\sigma_{(dd)} \sim s^{\frac{12\alpha_s}{\pi} \log^2}, \quad \text{with } \alpha_s = g^2/4\pi.
\]
How a pomeron–like behaviour can be derived

We start by writing the Euclidean hadronic correlation function in a partial–wave expansion:

\[ \tilde{C}_E^{(hh)}(\theta, t) = \sum_{l=0}^{\infty} (2l + 1)A_l(t)P_l(\cos \theta). \]

Because of the crossing–symmetry relations, it is natural to decompose our hadronic correlation function \( \tilde{C}_E^{(hh)}(\theta, t) \) as a sum of a crossing–symmetric function \( \tilde{C}_E^+(\theta, t) \) and of a crossing–antisymmetric function \( \tilde{C}_E^-(\theta, t) \):

\[ \tilde{C}_E^{(hh)}(\theta, t) = \tilde{C}_E^+(\theta, t) + \tilde{C}_E^-(\theta, t), \]

\[ \tilde{C}_E^\pm(\theta, t) \equiv \frac{\tilde{C}_E^{(hh)}(\theta, t) \pm \tilde{C}_E^{(hh)}(\pi - \theta, t)}{2}. \]

Because of the relation \( P_l(\cos \theta) = (-1)^l P_l(\cos \theta), \ \forall l \in \mathbb{N} \), we can replace \( A_l(t) \) respectively with \( A_l^\pm(t) \equiv \frac{1}{2}[1 \pm (-1)^l]A_l(t) \):

\[ A_l^+(t) = \begin{cases} A_l(t) & \text{for even } l \\ 0 & \text{for odd } l \end{cases}; \quad A_l^-(t) = \begin{cases} 0 & \text{for even } l \\ A_l(t) & \text{for odd } l \end{cases}. \]

The functions \( \tilde{C}_E^\pm(\theta, t) \) can also be called even–signatured and odd–signatured correlation functions respectively.
If we remember that:

\[ \tilde{C}_E^{(hh)}(\theta, t) \equiv \int d^2 \vec{R}_{1\perp} \int_0^1 df_1 |\psi_1(\vec{R}_{1\perp}, f_1)|^2 \times \int d^2 \vec{R}_{2\perp} \int_0^1 df_2 |\psi_2(\vec{R}_{2\perp}, f_2)|^2 \tilde{C}_E(\theta, t; 1, 2), \]

and we make use i) of the crossing-symmetry relations:

\[ \tilde{C}_E(\pi - \theta, t; \vec{R}_{1\perp}, f_1, \vec{R}_{2\perp}, f_2) = \tilde{C}_E(\theta, t; \vec{R}_{1\perp}, f_1, -\vec{R}_{2\perp}, 1 - f_2), \]

and ii) of the rotational- and C-invariance of the squared hadron wave functions:

\[ |\psi_i(\vec{R}_{i\perp}, f_i)|^2 = |\psi_i(-\vec{R}_{i\perp}, f_i)|^2 = |\psi_i(\vec{R}_{i\perp}, 1 - f_i)|^2 = |\psi_i(-\vec{R}_{i\perp}, 1 - f_i)|^2, \]

then we immediately conclude that the hadronic correlation function is automatically crossing symmetric (\( \theta \leftrightarrow \pi - \theta \)):

\[ \tilde{C}_E^{(hh)}(\theta, t) = \tilde{C}_E^+(\theta, t), \quad \tilde{C}_E^-(\theta, t) = 0. \]

Upon Euclidean-to-Minkowskian analytic continuation:

\[ \tilde{C}_M^{(hh)}(\chi, t) = \tilde{C}_M^+(\chi, t), \quad \tilde{C}_M^-(\chi, t) = 0, \]

and therefore also the high-energy meson-meson elastic scattering amplitude \( \mathcal{M}_{(hh)} \) turns out to be automatically crossing symmetric (\( \chi \leftrightarrow i\pi - \chi \)) \( \rightarrow \) Pomeranchuk theorem.

- **NO odderon** (i.e., \( C = -1 \)) exchange is possible for high-energy meson-meson scattering [Dosch & Rueter, 1996], . . .

- . . . while a **pomeron** (i.e., \( C = +1 \)) exchange is possible.
Let us therefore proceed by considering our \textit{crossing-symmetric} Euclidean correlation function:

\[
\widetilde{C}_E^{(hh)}(\theta, t) = \widetilde{C}_E^+(\theta, t) = \frac{1}{2} \sum_{l=0}^{\infty} (2l + 1) A_l^+(t) [P_l(\cos \theta) + P_l(- \cos \theta)].
\]

We can now use Cauchy’s theorem to rewrite this partial-wave expansion as an integral over \( l \), the so-called \textit{Sommerfeld–Watson} transform:

\[
\widetilde{C}_E^{(hh)}(\theta, t) = \widetilde{C}_E^+(\theta, t) = -\frac{1}{4i} \int_C \frac{(2l + 1) A_l^+(t) [P_l(- \cos \theta) + P_l(\cos \theta)]}{\sin(\pi l)} \, dl
\]

[see, e.g., \textit{“Pomeron Physics and QCD”}, by Donnachie, Dosch, Landshoff & Nachtmann, 2002]. As in the original derivation ...
Then we can use again Cauchy’s theorem to reshape the contour \( C \) into the straight line \( \Re(l) = -\frac{1}{2} \) and rewrite the integral as:

\[
\tilde{C}^{(hh)}_E(\theta, t) = \tilde{C}^+_E(\theta, t) = \\
-\frac{\pi}{2} \sum_{\Re(\sigma^+_n) > -\frac{1}{2}} \frac{(2\sigma^+_n(t) + 1)r^+_n(t)[P_{\sigma^+_n(t)}(-\cos \theta) + P_{\sigma^+_n(t)}(\cos \theta)]}{\sin(\pi\sigma^+_n(t))} \\
-\frac{1}{4i} \int_{-\frac{1}{2} + i\infty}^{\frac{1}{2} + i\infty} (2l + 1)A^+_l(t)[P_l(-\cos \theta) + P_l(\cos \theta)] \frac{dl}{\sin(\pi l)},
\]

where \( \sigma^+_n(t) \) is a pole of \( A^+_l(t) \) in the complex \( l \)-plane and \( r^+_n(t) \) is the corresponding residue. We have also assumed that the large-

\( l \) behaviour of \( A^+_l(t) \) is such that the integrand function vanishes enough rapidly (faster than \( 1/l \)) as \( |l| \to \infty \) in the right half-plane, so that the contribution from the infinite contour is zero. (Then \( A^+_l \) is unique, by virtue of Carlson’s theorem!)
Making use of the Euclidean–to–Minkowskian analytic continuation $\theta \rightarrow -i\chi$ in the angular variable, we derive that, $\forall \chi \in \mathbb{R}^+$:

$$\tilde{C}_M^{(hh)}(\chi, t) = \tilde{C}_E^{(hh)}(-i\chi, t) =$$

$$-\frac{\pi}{2} \sum_{\text{Re}(\sigma_n^+) > -\frac{1}{2}} \frac{(2\sigma_n^+(t) + 1)r_n^+(t)[P_{\sigma_n^+}(t)(-\cosh \chi) + P_{\sigma_n^+}(t)\cosh \chi]}{\sin(\pi\sigma_n^+(t))}$$

$$-\frac{1}{4i} \int_{-\frac{1}{2} + i\infty}^{-\frac{1}{2} - i\infty} (2l + 1)A_l^+(t)[P_l(-\cosh \chi) + P_l(\cosh \chi)] dl.$$

Now we take the large–$\chi$ (large–$s$) limit of this expression, with:

$$\cosh \chi = \frac{s}{2m^2} - 1, \quad \text{i.e.:} \quad \chi \sim \log \left(\frac{s}{m^2}\right),$$

- Indeed, by virtue of the usual $i\varepsilon$–prescription $m^2 \rightarrow m^2 - i\varepsilon$, with $\varepsilon \rightarrow 0+$, (i.e., $s \rightarrow s + i\varepsilon$, with $\varepsilon \rightarrow 0+$, as is well known), we have that: $\chi \rightarrow \chi + i\varepsilon$, with $\chi \in \mathbb{R}^+, \varepsilon \rightarrow 0+$; that is to say: $\text{Im}(z \equiv \cosh \chi) > 0$.

- The asymptotic form of $P_{\nu}(z)$ when $z \rightarrow \infty$ is known to be:

$$P_{\nu}(z) \sim \frac{1}{\sqrt{\pi}} \left[\frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu + 1)}(2z)^\nu + \frac{\Gamma(-\nu - \frac{1}{2})}{\Gamma(-\nu)}(2z)^{-\nu - 1}\right].$$

- We thus obtain, for each term in the sum [Re($\sigma_n^+$) > $-\frac{1}{2}$]:

$$P_{\sigma_n^+}(\cosh \chi) + P_{\sigma_n^+}(-\cosh \chi) \sim \left[1 + e^{-i\pi\sigma_n^+}\right] \frac{1}{\sqrt{\pi}} \frac{\Gamma(\sigma_n^+ + \frac{1}{2})}{\Gamma(\sigma_n^+ + 1)} \left(\frac{s}{m^2}\right)^{\sigma_n^+}. $$

- The integral, instead, usually called the background term, vanishes at least as $1/\sqrt{s}$ and thus can be neglected.
Therefore, in the limit \( s \to \infty \), with a fixed \( t \) (\(|t| \ll s\)), we are left with the following expression:

\[
\mathcal{C}^{(hh)}_{M} \left( \chi_{s \to \infty} \log \left( \frac{s}{m^2} \right), t \right) \sim \sum_{\text{Re}(\sigma_n^+) > -\frac{1}{2}} \beta_n^+(t)s^{\sigma_n^+(t)},
\]

\( \Rightarrow \) ... and, for the scattering amplitude:

\[
\mathcal{M}_{(hh)}(s, t) \sim -2i \sum_{\text{Re}(\sigma_n^+) > -\frac{1}{2}} \beta_n^+(t)s^{1+\sigma_n^+(t)}.
\]

\( \Rightarrow \) This sort of behaviour for the scattering amplitude is known in the literature as a Regge behaviour, ... 

\( \Rightarrow \) Denoting with \( \sigma_P(t) \) the pole with the largest real part (at that given \( t \)) and with \( \beta_P(t) \) the corresponding coefficient \( \beta_n^+(t) \):

\[
\mathcal{M}_{(hh)}(s, t) \sim -2i \beta_P(t) s^{\alpha_P(t)},
\]

where \( \alpha_P(t) \equiv 1 + \sigma_P(t) \) is the pomeron trajectory.

\( \Rightarrow \) Therefore, by virtue of the optical theorem:

\[
\sigma_{\text{tot}}^{(hh)}(s) \sim \sigma_0^{(hh)} \left( \frac{s}{s_0} \right)^{\epsilon_P}, \quad \text{with: } \epsilon_P = \text{Re}[\alpha_P(0)] - 1.
\]

\( \Rightarrow \) In the original derivation the asymptotic behaviour is recovered by analytically continuing the \( t \)-channel scattering amplitude to very large imaginary values of the angle between the trajectories of the two exiting particles in the \( t \)-channel scattering process.
Two important issues:

- **NO** energy dependence of hadron wave functions
  \[ A^+_l, \beta^+_n \text{ and } \sigma^+_n \text{ do NOT depend on } s, \text{ but only on } t! \]

- However, this is not enough to guarantee the experimentally-observed *universality* of the *pomeron trajectory* \( \alpha_P(t) \! \)!

We may start from the partial-wave expansion of the *fundamental* loop-loop Euclidean correlation function:

\[
\tilde{\mathcal{C}}_E(\theta, t; 1, 2) = \sum_{l=0}^{\infty} (2l + 1) A_l(t; 1, 2) P_l(\cos \theta),
\]

... and we arrive at the following Regge expansion for the *(even-signatured)* loop-loop Minkowskian correlator:

\[
\tilde{\mathcal{C}}_M^+(\chi, s \to \infty \log \left( \frac{s}{m^2} \right), t; 1, 2) \sim \sum_{\text{Re}(a_n^+)>-\frac{1}{2}} b_n^+(t; 1, 2) s^{a_n^+(t; 1, 2)}.
\]

If we assume that (at least) the location of the pole \( \sigma_P(t) \equiv \alpha_P(t) \equiv a_n^+(t; 1, 2) \) with the largest real part does **NOT** depend on \( R_{1\perp}, f_1 \) and \( R_{2\perp}, f_2 \), but only depends on \( t \), we then find:

\[
\mathcal{M}_{(hh)}(s, t) \sim_{s \to \infty} -2i \beta_P(t) s^{\alpha_P(t)},
\]

with a *universal pomeron trajectory* \( \alpha_P(t) = 1 + \sigma_P(t), \ldots \)

... while the coefficient \( \beta_P(t) \) in front explicitly depends on the specific type of hadrons involved in the process.
Concluding remarks and prospects

→ Certain apparently reasonable analyticity hypotheses of the line–line and loop–loop correlation functions imply both the Euclidean–to–Minkowskian analytic–continuation relations and (directly from this) also the crossing–symmetry relations.

→ The reasonableness of the above–mentioned analyticity hypotheses comes essentially from perturbation theory.

→ A real nonperturbative foundation of these properties is at the moment out of our reach.

→ One is immediately faced with the following series of questions:

- Are the analyticity hypotheses considered above maybe too strong and to what degree can they be relaxed?

- What about the Euclidean–to–Minkowskian analytic continuation if weaker analyticity hypotheses are kept in place of those discussed above?

- What about, finally, the crossing symmetry relation in the presence of a more complicated analytic structure, when the Euclidean–to–Minkowskian analytic continuation cannot be trusted, at least in the form presented above?

⟵ We have not the answers, at the moment . . .
The Euclidean–to–Minkowskian analytic–continuation approach can, with the inclusion of some extra assumptions, easily reproduce a pomeron–like behaviour for the high–energy total cross sections, in apparent agreement with the present–day experimental observations.

The pomeron–like behaviour is, strictly speaking, forbidden (at least if considered as a true asymptotic behaviour) by the well–known Froissart–Lukaszuk–Martin (FLM) theorem, according to which, for \( s \to \infty \):

\[
\sigma_{\text{tot}}(s) \leq \frac{\pi}{m_{\pi}^2} \log^2 \left( \frac{s}{s_0} \right).
\]

The pomeron–like behaviour can at most be regarded as a sort of pre–asymptotic (but not really asymptotic !) behaviour of the high–energy total cross sections.

Immediately the following question arises:

why our approach, which was formulated so to give the really asymptotic large–s behaviour of scattering amplitudes and total cross sections, is also able to reproduce pre–asymptotic behaviours [violating the FLM bound] like the pomeron?

extra assumptions, models !

A direct LATTICE calculation of the loop–loop Euclidean correlation functions could provide a criterion to investigate the goodness of a given existing analytic model (Instantons, SVM, AdS/CFT, BFKL \ldots) or even to open the way to some new model.

Work is in progress along this line of research.
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The line–line case

In the case of the line–line correlators we cannot simply remove the IR cutoff $T$, as we have done for the loop–loop case, since the limits $T \to \infty$ are divergent!

Nevertheless, we can remove the IR cutoff $T$ ($T \to \infty$) provided that another IR cutoff $\lambda$ has been introduced to regularize the line–line correlators, …

… e.g., by giving a small mass $\lambda$ to the gluons (or photons) exchanged in each graph in perturbation theory (we can call this a perturbative IR cutoff, while $T$ is a nonperturbative IR cutoff):

$$g_M^{(\lambda)}(\chi; t) \equiv \lim_{T \to \infty} g_M^{(\lambda)}(\chi; T; t),$$

$$g_E^{(\lambda)}(\theta; t) \equiv \lim_{T \to \infty} g_E^{(\lambda)}(\theta; T; t).$$

Under certain analyticity conditions in the complex variable $T$ for the two IR–regularized line–line correlators $g_M^{(\lambda)}(\chi; T; t)$ and $g_E^{(\lambda)}(\theta; T; t)$, the two quantities obtained after the removal of the nonperturbative IR cutoff $T$ are still connected by the usual analytic continuation in the angular variables only:

$$\overline{g}_E^{(\lambda)}(\theta; t) = \overline{g}_M^{(\lambda)}(i\theta; t), \quad \forall \theta \in \mathcal{D}_E;$$

$$\overline{g}_M^{(\lambda)}(\chi; t) = \overline{g}_E^{(\lambda)}(-i\chi; t), \quad \forall \chi \in \mathcal{D}_M.$$
For example, in *quenched* QED the calculation gives the following results for the fermion–fermion correlation functions in the Feynman gauge (gauge–fixing parameter $\alpha = 1$) [EM1997]:

$$g_{fF}^f(\chi; t)^{(\lambda)} = \int d^2z_\perp e^{iq_\perp \cdot z_\perp} \exp \left[ -ie^2|\coth \chi| \int \frac{d^2k_\perp}{(2\pi)^2} e^{ik_\perp \cdot z_\perp} \frac{1}{k_\perp^2 + \lambda^2} \right],$$

$$g_{fF}^f(\theta; t)^{(\lambda)} = \int d^2z_\perp e^{iq_\perp \cdot z_\perp} \exp \left[ -e^2 \frac{\cos \theta}{|\sin \theta|} \int \frac{d^2k_\perp}{(2\pi)^2} e^{ik_\perp \cdot z_\perp} \frac{1}{k_\perp^2 + \lambda^2} \right].$$

For obtaining the fermion–antifermion correlation function it is clearly sufficient to exchange $e^2 \rightarrow -e^2$:

$$g_{f\bar{f}}^f(\chi; t)^{(\lambda)} = \int d^2z_\perp e^{iq_\perp \cdot z_\perp} \exp \left[ ie^2|\coth \chi| \int \frac{d^2k_\perp}{(2\pi)^2} e^{ik_\perp \cdot z_\perp} \frac{1}{k_\perp^2 + \lambda^2} \right],$$

$$g_{f\bar{f}}^f(\theta; t)^{(\lambda)} = \int d^2z_\perp e^{iq_\perp \cdot z_\perp} \exp \left[ e^2 \frac{\cos \theta}{|\sin \theta|} \int \frac{d^2k_\perp}{(2\pi)^2} e^{ik_\perp \cdot z_\perp} \frac{1}{k_\perp^2 + \lambda^2} \right].$$

The analytic extensions $\overline{g}_{fF}^f, \overline{g}_{f\bar{f}}^f, \overline{g}_{fF}^f$ and $\overline{g}_{f\bar{f}}^f$ to the usual domains $\mathcal{D}_M$ and $\mathcal{D}_E$ are obtained from these expressions by the simple substitutions:

$$|\coth \chi| \rightarrow \coth \chi, \quad \cos \theta/|\sin \theta| \rightarrow \cot \theta.$$

The analytic–continuation relations are trivially satisfied and so is the crossing–symmetry relation:

$$g_{M}^{ff}(\chi; t)^{(\lambda)} = \overline{g}_{M}^{ff}(i\pi - \chi; t)^{(\lambda)}, \quad \forall \chi \in \mathbb{R}^+.$$
We observe that:
\[
\int \frac{d^2 \vec{k}_\perp}{(2\pi)^2} \frac{e^{-i\vec{k}_\perp \cdot \vec{z}_\perp}}{k_\perp^2 + \lambda^2} \equiv \frac{1}{2\pi} K_0(\lambda |\vec{z}_\perp|),
\]
where \( K_0 \) is the modified Bessel function.
In the limit of small \( \lambda \) this last expression can be replaced by:
\[
\frac{1}{2\pi} K_0(\lambda |\vec{z}_\perp|) \sim -\frac{1}{2\pi} \log \left( \frac{1}{2} e^{i\gamma \lambda} |\vec{z}_\perp| \right) = \Lambda_{\text{IR}} - \frac{1}{2\pi} \log |\vec{z}_\perp|,
\]
where \( \Lambda_{\text{IR}} \equiv -(1/2\pi) \log(\frac{1}{2} e^{\gamma \lambda}) \) is an infinite constant phase (\( \Lambda_{\text{IR}} \to \infty \) when \( \lambda \to 0 \)) and is therefore physically unobservable.

Thus we can rewrite the previous expressions as:
\[
g_M^{ff}(\chi; t)^{\lambda} = e^{-i e^2 \Lambda_{\text{IR}} |\coth \chi|} \int d^2 \vec{z}_\perp e^{i \vec{q}_\perp \cdot \vec{z}_\perp} \exp \left( i \frac{e^2}{2\pi} |\coth \chi| \log |\vec{z}_\perp| \right),
\]
\[
g_E^{ff}(\theta; t)^{\lambda} = e^{-i e^2 \Lambda_{\text{IR}} \cos \theta / \sin \theta} \int d^2 \vec{z}_\perp e^{i \vec{q}_\perp \cdot \vec{z}_\perp} \exp \left( \frac{e^2 \cos \theta}{2\pi |\sin \theta|} \log |\vec{z}_\perp| \right).
\]

One thus obtains the “standard” eikonal formula for the high-energy fermion–fermion e.m. elastic scattering amplitude [Cheng & Wu, 1969; Abarbanel & Itzykson, 1969; Jackiw et al., 1992]:
\[
M^{ff}(s; t)_{\alpha'; \beta' \beta}^{\alpha; \alpha'} \sim \delta_{\alpha' \alpha} \delta_{\beta' \beta} 2s \frac{\Gamma(1 + i\alpha)}{4\pi i\lambda^2 \Gamma(-i\alpha)} \left( \frac{4\lambda^2}{-t} \right)^{1+i\alpha},
\]
where \( \alpha = e^2 / 4\pi \).
In QCD perturbation theory:

\[
g^{qg}_{M}(x; T; t)_{i'j'; j'} = \frac{1}{[Z_{M}(T)]^2} \sum_{r,s=1}^{\infty} (-ig)^{r+s} \frac{p_1^{\mu_1}}{m} \ldots \frac{p_r^{\mu_r}}{m} \frac{p_1^{\nu_1}}{m} \ldots \frac{p_s^{\nu_s}}{m}
\times (T_{a_1} \ldots T_{a_r})_{i'i}(T_{b_1} \ldots T_{b_s})_{j'j}
\times \int d^2 \bar{z}_L^{\perp} e^{i\bar{q}_L^{\perp} \cdot \bar{z}_L} \int_{-T}^{+T} d\tau_1 \ldots \int_{-T}^{+T} d\tau_r \int_{-T}^{+T} d\sigma_1 \ldots \int_{-T}^{+T} d\sigma_s
\times \theta(\tau_1 - \tau_2) \ldots \theta(\tau_{r-1} - \tau_r) \theta(\sigma_1 - \sigma_2) \ldots \theta(\sigma_{s-1} - \sigma_s)
\times G_{\mu_1 \ldots \mu_r \nu_1 \ldots \nu_s}^{a_1 \ldots a_r b_1 \ldots b_s}(z + \frac{p_1^1}{m} \tau_1, \ldots, z + \frac{p_1^r}{m} \tau_r, \frac{p_2^1}{m} \sigma_1, \ldots, \frac{p_2^s}{m} \sigma_s),
\]

\[
g^{qg}_{M}(x; T; t)_{i'j'; j'} = \frac{1}{[Z_{M}(T)]^2} \sum_{r,s=1}^{\infty} (-ig)^{r+s} \frac{p_1^{\mu_1}}{m} \ldots \frac{p_r^{\mu_r}}{m} \frac{p_1^{\nu_1}}{m} \ldots \frac{p_s^{\nu_s}}{m}
\times (T_{a_1} \ldots T_{a_r})_{i'i}(T_{b_1} \ldots T_{b_s})_{j'j}
\times \int d^2 \bar{z}_L^{\perp} e^{i\bar{q}_L^{\perp} \cdot \bar{z}_L} \int_{-T}^{+T} d\tau_1 \ldots \int_{-T}^{+T} d\tau_r \int_{-T}^{+T} d\sigma_1 \ldots \int_{-T}^{+T} d\sigma_s
\times \theta(\tau_1 - \tau_2) \ldots \theta(\tau_{r-1} - \tau_r) \theta(\sigma_1 - \sigma_2) \ldots \theta(\sigma_{s-1} - \sigma_s)
\times G_{\mu_1 \ldots \mu_r \nu_1 \ldots \nu_s}^{a_1 \ldots a_r b_1 \ldots b_s}(z + \frac{p_1^1}{m} \tau_1, \ldots, z + \frac{p_1^r}{m} \tau_r, \frac{p_2^1}{m} \sigma_1, \ldots, \frac{p_2^s}{m} \sigma_s),
\]

where: \( G_{\mu_1 \ldots \mu_p}^{a_1 \ldots a_p}(x_1, \ldots, x_p) \equiv \langle A_{\mu_1}^{a_1}(x_1) \ldots A_{\mu_p}^{a_p}(x_p) \rangle \)

\( \rightarrow \) We get a “crossing relation” for Feynman graphs:

\( (T_{b_1} \ldots T_{b_s})_{j'j} \rightarrow (-1)^s (T_{b_s} \ldots T_{b_1})_{j'j}. \)
Let us see, in particular, how this works in the case of correlators evaluated in QCD perturbation theory up to the order $\mathcal{O}(g_R^4)$. One finds (always in the Feynman gauge $\alpha = 1$) [EM1997]:

\[
\begin{align*}
g_M^{qq}(\chi; t)_{i'j',j}^{(\lambda)}|g_R^4 &\quad = ig_R^2 \frac{1}{t} \coth \chi \cdot (G_1)_{i'j; j} \\
&\quad \cdot \left[ 1 - g_R^2 \left( F^{(2)}(t) + \frac{2N_c B}{4\pi^2} + \frac{N_c t I(t)}{4\pi} \right) \right] \\
&\quad - \frac{1}{2} g_R^4 I(t) \coth^2 \chi \cdot (G_2)_{i'j; j} \end{align*}
\]

\[
\begin{align*}
g_E^{qq}(\theta; t)_{i'j',j}^{(\lambda)}|g_R^4 &\quad = g_R^2 \frac{1}{t} \cos \frac{\theta}{\sin \theta} \cdot (G_1)_{i'j; j} \\
&\quad \cdot \left[ 1 - g_R^2 \left( F^{(2)}(t) + \frac{2N_c B}{4\pi^2} + \frac{N_c t I(t)}{4\pi} \right) \right] \\
&\quad + \frac{1}{2} g_R^4 I(t) \cot^2 \theta \cdot (G_2)_{i'j; j}, \quad \text{with : } \{\theta\} \equiv 2 \int_0^{\frac{\tan \theta}{2}} \frac{dx}{1 + x^2},
\end{align*}
\]

where:

\[
\begin{align*}
(G_1)_{i'j; j} &\equiv (T_a)_{i} (T_a)_{j} \\
(G_2)_{i'j; j} &\equiv (T_a T_b)_{i} (T_a T_b)_{j}
\end{align*}
\]

\[
I(t) \equiv \int \frac{d^2 \vec{k}}{(2\pi)^2} \frac{1}{\vec{k}^2 + \lambda^2} \frac{1}{\vec{q}^2 + \lambda^2},
\]

\[
\begin{align*}
\{(-\theta)\} &\equiv \{\theta\}, \\
\{(2\pi - \theta)\} &\equiv \{\theta\}.
\end{align*}
\]
Feynman diagrams contributing to the quark–quark and quark–antiquark correlators up to the order $\mathcal{O}(g^4_R)$; ...
... other Feynman diagrams contributing to the quark–quark and quark–antiquark correlators up to the order $\mathcal{O}(g_R^4)$. 
Moreover, since \( \{ \theta \} = \theta \) for \( \theta \in (0, \pi) \), one immediately finds that the analytic extensions \( \bar{g}_{M}^{q q} \) and \( \bar{g}_{E}^{q q} \) are given by:

\[
\bar{g}_{M}^{q q}(\chi; t)_{i' i ; j' j}^{(\lambda)} g_{R}^{4} = i g_{R}^{2} \frac{1}{t} \coth \chi \cdot (G_{1})_{i' i ; j' j} \\
\quad \cdot \left[ 1 - g_{R}^{2} \left( F^{(2)}(t) + \frac{2 N_{c} B}{(4 \pi)^{2}} + \frac{N_{c}}{4 \pi} t I(t) \chi \coth \chi \right) \right] \\
\quad - \frac{1}{2} g_{R}^{4} I(t) \cot^{2} \chi \cdot (G_{2})_{i' i ; j' j}, \quad \forall \chi \in D_{M},
\]

\[
\bar{g}_{E}^{q q}(\theta; t)_{i' i ; j' j}^{(\lambda)} g_{R}^{4} = g_{R}^{2} \frac{1}{t} \cot \theta \cdot (G_{1})_{i' i ; j' j} \\
\quad \cdot \left[ 1 - g_{R}^{2} \left( F^{(2)}(t) + \frac{2 N_{c} B}{(4 \pi)^{2}} + \frac{N_{c}}{4 \pi} t I(t) \theta \cot \theta \right) \right] \\
\quad + \frac{1}{2} g_{R}^{4} I(t) \cot^{2} \theta \cdot (G_{2})_{i' i ; j' j}, \quad \forall \theta \in D_{E},
\]

with the usual analyticity domains \( D_{M} \) and \( D_{E} \) defined above.

From these explicit expressions one immediately verifies that the analytic–continuation relations are verified.

Let us consider now the quark–antiquark correlator. We can simply use the \underline{crossing relation} derived above:

\[
(G_{1})_{i' i ; j' j} = (T_{a})_{i' i} (T_{a})_{j' j} \rightarrow -(T_{a})_{i' i} (T_{a})_{j' j} = -(G_{1})_{i' i ; j' j}.
\]

\[
M(b) \cdot (T_{a} T_{b})_{i' i} (T_{a} T_{b})_{j' j} \rightarrow M(b) \cdot (T_{a} T_{b})_{i' i} (T_{b} T_{a})_{j' j},
\]

\[
M(c) \cdot (T_{a} T_{b})_{i' i} (T_{b} T_{a})_{j' j} \rightarrow M(c) \cdot (T_{a} T_{b})_{i' i} (T_{a} T_{b})_{j' j}.
\]
In [EM1997] it was found that:

\[ M(b) = i \frac{g_R^4}{2\pi} I(t)(i\pi - |\chi|) \coth^2 \chi, \]
\[ M(c) = i \frac{g_R^4}{2\pi} I(t)|\chi| \coth^2 \chi, \]

so that: \( M(b) + M(c) = -\frac{1}{2} g_R^4 I(t) \coth^2 \chi. \)

Making use of the following relation for the colours factors:

\[
(T_a T_b)_{ij}(T_b T_a)_{kl} = (T_a T_b)_{ij}(T_a T_b)_{kl} + \frac{N_c}{2} (T_c)_{ij}(T_c)_{kl} \\
\equiv (G_2)_{ij:kl} + \frac{N_c}{2} (G_1)_{ij:kl},
\]

the following result is found for the quark–antiquark correlator at order \( \mathcal{O}(g_R^4), \) for positive hyperbolic angle \( \chi > 0: \)

\[
g_{M}(\chi; t)^{(\lambda)}_{i'i';j'j} |_{g_R^4} = -i g_R^2 \frac{1}{t} \coth \chi \cdot (G_1)_{i'i;j'j'} \\
\cdot \left[ 1 - g_R^2 \left( F^{(2)}(t) + \frac{2N_c B}{(4\pi)^2} - \frac{N_c}{4\pi} t I(t)(i\pi - \chi) \coth \chi \right) \right] \\
- \frac{1}{2} g_R^4 I(t) \coth^2 \chi \cdot (G_2)_{i'i;j'j'}, \quad \forall \chi \in \mathbb{R}^+.
\]

This is also the expression of the analytic extension \( \overline{g}_{M}^{qq} \) from the real positive \( \chi \)-axis to the same analyticity domain \( \mathcal{D}_M = \{ \chi \in \mathbb{C} | \chi \neq ik\pi, \ k \in \mathbb{Z} \} \) introduced above for the quark–quark correlator \( g_{M}^{qq}. \)

The crossing–symmetry relation is verified:

\[
g_{M}(\chi; t)^{(\lambda)}_{i'i;j'j} |_{g_R^4} = \overline{g}_{M}^{qq}(i\pi - \chi; t)^{(\lambda)}_{i'i;j'j} |_{g_R^4}, \quad \forall \chi \in \mathbb{R}^+.
\]
The Feynman diagrams with exchange of one, two and three gluons which contribute to the quark–quark and quark–antiquark correlators.
Three–gluon–exchange diagrams contributing to $qq$ and $q\bar{q}$ correlators

For simplicity we limit ourselves to the study of the *diffractive* part of the three–gluon–exchange graphs:

$$g_M(\chi; t)^{(D)} \equiv \sum_{i,j=1}^{N_c} g_M(\chi; t)_{ii:jj}$$

($\rightarrow$ process without exchange of colour).

$\rightarrow$ The diffractive contributions of the six diagrams to the correlators $g_M^{qq}$ and $g_M^{q\bar{q}}$ are:

$$\Delta g_M^{qq}(\chi, t)^{(D)}_{3\text{gluon}} = S_3 L(\chi, t) + S'_3 X(\chi, t)$$

$$= S_3 [L(\chi, t) + X(\chi, t)] + \frac{N_c}{2} S_2 X(\chi, t);$$

$$\Delta g_M^{q\bar{q}}(\chi, t)^{(D)}_{3\text{gluon}} = (-1)^3 [S'_3 L(\chi, t) + S_3 X(\chi, t)]$$

$$= -S_3 [L(\chi, t) + X(\chi, t)] - \frac{N_c}{2} S_2 L(\chi, t),$$

where:

$$S_3 \equiv \text{Tr} [T_a T_b T_c] \text{Tr} [T_a T_b T_c] = -\frac{N_c^2 - 1}{4 N_c},$$

$$S'_3 \equiv \text{Tr} [T_a T_b T_c] \text{Tr} [T_b T_a T_c] = S_3 + \frac{N_c}{2} S_2,$$

$$S_2 \equiv \text{Tr} [T_a T_b] \text{Tr} [T_a T_b] = \frac{N_c^2 - 1}{4}. $$
The three-gluon-exchange contributions are (for $\chi \in \mathbb{R}^+$):

\[
\Delta g_M^{qq}(\chi; t)^{(D)}_{3\text{gluon}} = i g_R^6 \coth^3 \chi \\
\times \left\{ S_3 \left[ \frac{1}{6} I_1(t) \right] + \frac{N_c}{2} S_2 \left[ \frac{i}{2\pi} \left( \chi - i \frac{2\pi}{3} \right) I_1(t) + \frac{1}{2\pi^2} H(\chi) \right] \right\},
\]

\[
\Delta g_M^{q\bar{q}}(\chi; t)^{(D)}_{3\text{gluon}} = -i g_R^6 \coth^3 \chi \\
\times \left\{ S_3 \left[ \frac{1}{6} I_1(t) \right] - \frac{N_c}{2} S_2 \left[ \frac{i}{2\pi} \left( \chi - i \frac{\pi}{3} \right) I_1(t) + \frac{1}{2\pi^2} H(\chi) \right] \right\},
\]

where:

\[
I_1(t) \equiv \int \frac{d^2 k_{1\perp}^- d^2 k_{2\perp}^-}{(2\pi)^2} \frac{1}{(k_{1\perp}^-)^2 + \lambda^2 (k_{2\perp}^-)^2 + \lambda^2 (\vec{q}_\perp - \vec{k}_{1\perp}^- - \vec{k}_{2\perp}^-)^2 + \lambda^2},
\]

and the function $H(\chi)$ is defined by the integral:

\[
H(\chi) = \int \frac{d^2 \vec{k}_{1\perp}^- d^2 \vec{k}_{2\perp}^- d^2 \vec{k}_{3\perp}^-}{(2\pi)^2} (2\pi)^2 \delta^{(2)} \left( \vec{q}_\perp - \vec{k}_{1\perp}^- - \vec{k}_{2\perp}^- - \vec{k}_{3\perp}^- \right) \\
\times h \left( \chi; \vec{k}_{1\perp}^-, \vec{k}_{2\perp}^-, \vec{k}_{3\perp}^- \right),
\]

\[
h \left( \chi; \vec{k}_{1\perp}^-, \vec{k}_{2\perp}^-, \vec{k}_{3\perp}^- \right) = \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} d\eta P_\xi^{-1} P_\eta^{-1} \\
\times \left( \frac{1}{\xi^2 + \vec{k}_{1\perp}^-^2 + \lambda^2 \eta^2 + \vec{k}_{2\perp}^-^2 + \lambda^2} - \frac{1}{\vec{k}_{1\perp}^-^2 + \lambda^2 \vec{k}_{2\perp}^-^2 + \lambda^2} \right) \\
\times \frac{1}{\xi^2 + \eta^2 - 2\xi\eta \cosh \chi + \vec{k}_{3\perp}^- + \lambda^2 - i\epsilon},
\]

where:

\[
P_\xi^{-1} \equiv \frac{1}{2} \left( \frac{1}{\xi - i\epsilon} + \frac{1}{\xi + i\epsilon} \right).
\]
The explicit form of $H(\chi)$ is not too enlightening; anyway, it can be continued analytically from the positive real axis $\chi \in \mathbb{R}^+$ into a domain including also the imaginary segment $(\text{Re}\chi = 0, 0 < \text{Im}\chi < \pi)$ and the semiaxis $(\text{Re}\chi < 0, \text{Im}\chi = \pi)$. Using the notation previously introduced, we denote such an extension as $\overline{H}(\chi)$; as one immediately sees, it has the property:

$$\overline{H}(i\pi - \chi) = -\overline{H}(\chi).$$

Repeating the calculation in the Euclidean case, we obtain the result, for $\theta \in (0, \pi)$:

$$\Delta g_E^{qq}(\theta; t)^{(D)}_{3\text{gluon}} = -g_R^6 \cot^3 \theta \times \left\{ S_3 \left[ \frac{1}{6} I_1(t) \right] + \frac{N_c}{2} S_2 \left[ \frac{i}{2\pi} \left( i\theta - i \frac{2\pi}{3} \right) I_1(t) + \frac{1}{2\pi^2} \overline{H}(i\theta) \right] \right\},$$

immediately verifying the analytic-continuation relation:

$$\Delta g_E^{qq}(\theta; t)^{(D)}_{3\text{gluon}} = \overline{\Delta g_M^{qq}(i\theta; t)^{(D)}_{3\text{gluon}}},$$

The contributions of the three-gluon-exchange diagrams to the $qq$ and $q\bar{q}$ correlators satisfy the crossing-symmetry relation:

$$\Delta g_M^{q\bar{q}}(\chi; t)^{(D)}_{3\text{gluon}} = \overline{\Delta g_M^{qq}(i\pi - \chi; t)^{(D)}_{3\text{gluon}}}. $$
Let us discuss the high-energy behaviour of the three-gluon-exchange contribution to the $qq$ and $q\bar{q}$ scattering amplitudes:

$$\lim_{\chi \to +\infty} \frac{dH(\chi)}{d\chi} = 0 \implies \lim_{\chi \to +\infty} H(\chi) = H_0.$$  

The high-$\chi$ contributions of the three-gluon diagrams to the $qq$ and $q\bar{q}$ correlators are then:

$$\Delta g^q=q_M(\chi \to +\infty; t)^{(D)}_{3\text{gluon}} \simeq -g_R^6 \frac{N_c S_2}{4\pi} \chi I_1(t) + \text{const. imaginary part},$$

$$\Delta g^{q\bar{q}}_M(\chi \to +\infty; t)^{(D)}_{3\text{gluon}} \simeq -g_R^6 \frac{N_c S_2}{4\pi} \chi I_1(t) + \text{const. imaginary part}.$$  

In the limit $\chi \simeq \log(s/m^2) \to +\infty$, the optical theorem relates them to the colour-averaged quark-quark and quark-antiquark total cross sections:

$$\bar{\sigma}^{qq/q\bar{q}}_{tot} \equiv \frac{1}{N_c^2} \sum_{i,j=1}^{N_c} \sigma^{qq/q\bar{q}}_{tot ij} \simeq \frac{\text{Im}[-i2sg_M(\chi \to +\infty; 0)^{(D)}]}{s \cdot N_c^2} = -\frac{2}{N_c^2} \text{Re} \left[ g_M(\chi \to +\infty; 0)^{(D)} \right].$$

We thus find the following result for the high-energy colour-averaged $qq$ and $q\bar{q}$ total cross sections up to the order $O(g_R^6)$:

$$\bar{\sigma}^{qq}_{tot}|_{g_R^6} \simeq \frac{g_R^4 N_c^2 - 1}{4 N_c^2} I(0) + g_R^6 \frac{N_c^2 - 1}{8\pi N_c} I_1(0) \log \left( \frac{s}{m^2} \right).$$

According to [Cheng & Wu] this is the entire contribution of order $O(g_R^6)$ to the color-averaged total cross sections at high energies.