DEM AzURE MOD ULES, FUSION PRODUCTS AND Q–SYSTEMS

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Abstract. In this paper, we introduce a family of indecomposable finite–dimensional graded modules for the current algebra associated to a simple Lie algebra. These modules are indexed by an \(| R^+ | \)–tuple of partitions \( \xi = (\xi^\alpha) \), where \( \alpha \) varies over a set \( R^+ \) of positive roots of \( g \) and we assume that they satisfy a natural compatibility condition. In the case when the \( \xi^\alpha \) are all rectangular, for instance, we prove that these modules are Demazure modules in various levels. As a consequence we see that the defining relations of Demazure modules can be greatly simplified. We use this simplified presentation to relate our results to the fusion products, defined in [15], of representations of the current algebra. We prove that the \( Q \)–system of [22] can be actually thought of as a canonical short exact of the fusion products of representations associated to certain special partitions \( \xi \). Finally, in the last section we deal with the case of \( \mathfrak{sl}_2 \) and prove that the modules we define are just fusion products of irreducible representations of the associated current algebra and give monomial bases for these modules.

Introduction

The current algebra associated to a simple Lie algebra \( g \) is just the Lie algebra of polynomial maps \( \mathbb{C} \rightarrow g \). As a vector space it is isomorphic to \( g \otimes \mathbb{C}[t] \), where \( \mathbb{C}[t] \) is the polynomial ring in the indeterminate \( t \) and the Lie bracket is given in the obvious way. Both \( \mathfrak{g}[t] \) and its universal enveloping algebra inherit a grading coming from the natural grading on \( \mathbb{C}[t] \). The study of the category of graded finite–dimensional representations of the current algebra has been of interest in recent years for a variety of reasons. It has connections with with problems arising in mathematical physics, for instance the \( X = M \) conjectures, see [1], [12], [32]. The work of [26] relates graded characters of certain representations to the Poincare polynomials of quiver varieties. The homological properties of the category are similar to those of the BGG category \( \mathcal{O} \) for the simple Lie algebra. This similarity has been explored to some extent in [3] and [4] and leads to connections with symmetric functions and Macdonald polynomials.

One of the original motivations for the study of this category was that it is closely related to the representation theory of the corresponding quantum affine algebra. The results of [5] and [11] showed that many interesting families of irreducible representations of the quantum affine algebra (associated to \( g \)), when specialized to \( q = 1 \) give indecomposable representations of the current algebra. The Kirillov–Reshetikhin modules which are indexed by pairs \((i, m)\) where \( i \) varies over the index set of the vertices of the Dynkin diagram of \( \mathfrak{g} \) and \( m \) over the positive integers are one such family. The definition of these modules for the current algebra was given in [10]. It was also shown in that paper and in [20], that the Kirillov–Reshetikhin
modules are isomorphic to certain Demazure modules in positive level representations of the affine Lie algebra.

Our paper is motivated by two questions. It is not hard to see that if \( V_1 \) and \( V_2 \) are representations of the quantum affine algebra, then the specialization of \( V_1 \otimes V_2 \) to \( q = 1 \) is not isomorphic to the tensor product of the specializations. Indeed, this problem was studied in [11] and led to the definition of the local Weyl modules for the affine Lie algebra and hence also the current algebra associated to \( \mathfrak{g} \). At about the same time, Feigin and Loktev introduced the notion of a fusion product of graded representations of the current algebra. It was later proved in [9], [20], [31] that a local Weyl module is a fusion product of fundamental local Weyl modules. Together with results established in [2], [6], it follows that the local Weyl module is the \( q = 1 \) limit of a tensor product of irreducible representations of quantum affine algebra. In other words, at least in this very special case, it is true, that the specialization of a tensor product of representation of the quantum affine algebras is isomorphic to the fusion product of representations of the current algebra. Sections 4 and 5 show that this remains true for certain kinds of Kirillov–Reshetikhin modules. In fact, the results of Section 5 can be interpreted as follows: they show that the short exact sequences (T–systems) studied in [23], [29], [30] of tensor products of Kirillov–Reshetikhin modules, give rise by specializing, to short exact sequences of fusion products of the Kirillov–Reshetikhin modules of the current algebra.

The second motivation for our paper was to understand the presentation given in [13] of the fusion product of irreducible finite–dimensional representations of the current algebra associated to \( \mathfrak{sl}_2 \). Although the definition of fusion products are given in a purely representation theoretic way, the proofs in [13] do not use this definition in any direct way. Moreover, their relations do not include (at least in any straightforward way), certain obvious ones which hold in the fusion product. In fact it is exactly these obvious relations which are critical for our study of Demazure modules and \( Q \)–systems for arbitrary simple Lie algebras. In Section 6 of our paper, we recover the presentation of [13] and more. We give three presentations of the fusion product and in one of them (see Proposition 2.7) the Demazure type relations show up in a natural way. Finally, we give a monomial bases for fusion products which are consistent with certain canonical short exact sequences of modules for \( \mathfrak{sl}_2[t] \).

The paper is organized as follows. Section one establishes the basic notation and elementary results needed in the rest of the paper. In Section two, we define the modules \( V(\xi) \), where \( \xi \) is a tuple of partitions indexed by the positive roots. We give three equivalent presentations of these modules. In Section 3, we consider the case when \( \xi \) consists of partitions which are either rectangular or certain types of fat hooks. We show that in this case, the module \( V(\xi) \) is isomorphic to a Demazure module of level \( \ell \) and that the defining relations of these modules can be highly simplified. These are generalizations of results which were previously known only for the level one Demazure modules (see [20], [31]). Even in the level one case, our methods are very different and our approach is more uniform. In Section 4, we recall the definition of fusion products and prove a “Schur positivity” type result for the fusion product of Demazure modules (see Proposition 4.5). In Section 5, we show that there exists a short exact sequence of graded \( \mathfrak{g}[t] \)–modules corresponding to the \( Q \)–system defined in [22]. Finally, in Section 6, we study the case of \( \mathfrak{sl}_2 \) and compute the dimension and give a monomial basis of the modules.
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1. Preliminaries

Throughout this paper, $\mathbb{C}$ will denote the field of complex numbers and $\mathbb{Z}$ (resp. $\mathbb{Z}_+$, $\mathbb{N}$) the set of integers (resp. non–negative, positive integers). We let $\mathbb{C}[t]$ be the polynomial ring in an indeterminate $t$.

1.1. Given a complex Lie algebra $\mathfrak{a}$, we let $U(\mathfrak{a})$ be the corresponding universal enveloping algebra. The associated current Lie algebra is denoted by $\mathfrak{a}[t]$: as a vector space it is just $\mathfrak{a} \otimes \mathbb{C}[t]$ and the Lie bracket is given in the natural way: $[a \otimes f, b \otimes g] = [a, b] \otimes fg$, for all $a, b \in \mathfrak{a}$ and $f, g \in \mathbb{C}[t]$. Let $\mathfrak{a}[t]_+$ be the ideal $\mathfrak{a} \otimes t \mathbb{C}[t]$. We shall freely identify $\mathfrak{a}$ with the Lie subalgebra $\mathfrak{a} \otimes 1$ of $\mathfrak{a}[t]$ and we clearly have an isomorphism of vector spaces

$$\mathfrak{a}[t] = \mathfrak{a}[t]_+ \bigoplus \mathfrak{a}, \quad U(\mathfrak{a}) \cong U(\mathfrak{a}[t]_+) \otimes U(\mathfrak{a}).$$

The degree grading on $\mathbb{C}[t]$ defines a natural $\mathbb{Z}_+$–grading on $\mathfrak{a}[t]$ and hence also $U(\mathfrak{a}[t])$: an element of the form $(a_1 \otimes t^{s_1}) \cdots (a_s \otimes t^{s_s})$ has grade $r_1 + \cdots + r_s$. Denote by $U(\mathfrak{a}[t])[r]$ the subspace of grade $r$.

1.2. A graded representation of $\mathfrak{a}[t]$ is a $\mathbb{Z}$–graded vector space which admits a compatible Lie algebra action of $\mathfrak{a}[t]$, i.e.,

$$V = \bigoplus_{r \in \mathbb{Z}} V[r], \quad (\mathfrak{a} \otimes t^s)V[r] \subset V[r+s], \quad r \in \mathbb{Z}, \quad s \in \mathbb{Z}_+.$$

If $V$ and $V'$ are graded $\mathfrak{a}[t]$–modules, we say that $\pi : V \to V'$ is a morphism of graded $\mathfrak{a}[t]$–modules if $\pi$ is a degree zero morphism of $\mathfrak{a}[t]$–modules. For $r \in \mathbb{Z}$, let $\tau_r$ be the grading shift operator: if $V$ is a graded $\mathfrak{a}[t]$–module then $\tau_r V$ is the graded $\mathfrak{a}[t]$–module with the graded pieces shifted uniformly by $r$ and the action of $\mathfrak{a}[t]$ unchanged. If $M$ is an $\mathfrak{a}$–module and $z \in \mathbb{C}$, define a $\mathfrak{a}[t]$–module structure on $M$ by: $(a \otimes t^s) m = z^s am$. We denote this module as $ev_z M$. Clearly $ev_z M$ is an irreducible $\mathfrak{a}[t]$–module iff $M$ is an irreducible $\mathfrak{a}$–module. Moreover the module $ev_0 M$ is a graded $\mathfrak{a}[t]$–module and $(ev_0 M) [0] = M$. In particular, $\mathfrak{a}[t]_+ (ev_0 M) = 0$.

1.3. From now on, $\mathfrak{g}$ will be an arbitrary simple finite–dimensional complex Lie algebra and $\mathfrak{h}$ will be a fixed Cartan subalgebra of $\mathfrak{g}$ and we assume that $\dim \mathfrak{h} = n$ or equivalently that the rank of $\mathfrak{g}$ is $n$. Let $R$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. The restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$ induces an isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$ and hence also a symmetric non–degenerate form $(\cdot, \cdot)$ on $\mathfrak{h}^*$. We shall assume that this form on $\mathfrak{h}^*$ is normalized so that the square length of a long root is two. For $\alpha \in R$, let $t_\alpha \in \mathfrak{h}$ be the element that maps to $\alpha$ and set,

$$d_\alpha = \frac{2}{(\alpha, \alpha)}, \quad h_\alpha = d_\alpha t_\alpha.$$
For $\alpha \in R$, let $\mathfrak{g}_{\alpha}$ be the corresponding root space of $\mathfrak{g}$ and set $\mathfrak{n}^\pm = \bigoplus_{\alpha \in R^\pm} \mathfrak{g}_{\pm \alpha}$. Fix non-zero elements $x^\pm_\alpha \in \mathfrak{g}_{\pm \alpha}$, such that

$$[h_\alpha, x^\pm_\alpha] = \pm 2x^\pm_\alpha, \quad [x^+_\alpha, x^-_\alpha] = h_\alpha,$$

and denote the corresponding subalgebra as $\mathfrak{sl}_2(\alpha)$. Let $I = \{1, \ldots, n\}$ and fix a set $\{\alpha_i : i \in I\}$ of simple roots for $R$ and a set $\{\omega_i : i \in I\} \subset \mathfrak{h}^*$ of fundamental weights, i.e., $(\omega_i, d_j \alpha_j) = \delta_{i,j}$.

Let $Q$ (resp. $Q^+$) be the integer span (resp. the nonnegative integer span) of $\{\alpha_i : i \in I\}$ and similarly define $P$ (resp. $P^+$) to be the $\mathbb{Z}$ (resp. $\mathbb{Z}_+$) span of $\{\omega_i : i \in I\}$. Set $R^+ = R \cap Q^+$ and for $\alpha = \sum_{j=1}^n m_j \alpha_j \in R^+$, let $ht \alpha = \sum_{j=1}^n m_j$. Denote by $\theta \in R^+$ be the highest root in $R$ and recall that $[x_\theta^+, \mathfrak{n}^-] = 0$. For $i \in I$, we write $x_i^+, h_i, d_i$ for $x^+_\alpha, h_\alpha, d_\alpha$.

The next result is elementary, but we record it since it is important for Sections 3 and 4.

**Lemma.** Suppose that $\alpha, \beta \in R$ are such that $\alpha + \beta \in R$. Then $\alpha + \beta$ is long if $\alpha$ and $\beta$ are long.

**1.4.** We shall also need,

**Lemma.** Suppose that $\lambda \in P^+$ is such that $\lambda = \sum_{i \in I} d_i s_i \omega_i$ for some $s_i \in \mathbb{Z}_+$. Then for $\alpha \in R^+$, there exists $s_\alpha \in \mathbb{Z}_+$ such that

$$\lambda(h_\alpha) = d_\alpha s_\alpha.$$

**Proof.** Proceed by induction on $ht \alpha$. If $ht \alpha = 1$, then $\alpha = \alpha_i$ for some $i \in I$ and $\lambda(h_i) = s_i d_i$ showing that induction begins. If $ht \alpha > 1$, we can write $\alpha = \beta + \gamma$ with $\beta, \gamma \in R^+$ so that the inductive hypothesis applies to $\beta$ and $\gamma$. If all three roots have the same length, then $h_\alpha = h_\beta + h_\gamma$ and the inductive step is immediate. If $\alpha$ is long then $d_\alpha = 1$ and there is nothing to prove. If $\alpha$ is short, then by Lemma 1.3 we may assume without loss of generality that $\beta$ is short and $\gamma$ long. This gives,

$$h_\alpha = h_\beta + d_\alpha h_\gamma.$$

It follows from the inductive hypothesis applied to $\beta$ that $\lambda(h_\beta) = s_\beta d_\beta$ for some $s_\beta \in \mathbb{Z}_+$ and since $d_\alpha = d_\beta$, we get $\lambda(h_\alpha) = d_\alpha (s_\beta + s_\gamma)$.

**1.5.** For $\mu \in P^+$, let $V(\mu)$ be the irreducible finite-dimensional $\mathfrak{g}$-module generated by an element $v_\mu$ with defining relations

$$x_i^+ v_\mu = 0, \quad h_i v_\mu = \mu(h_i) v_\mu, \quad (x_i^- \otimes 1)^{\mu(h_i)+1} v_\mu = 0, \quad i \in I.$$

It is well-known that any finite-dimensional $\mathfrak{g}$-module $V$ is isomorphic to a direct sum of irreducible modules $V(\mu), \mu \in P^+$. Further, we may write,

$$V = \bigoplus_{\nu \in P} V_\nu, \quad V_\nu = \{v \in V : hv = \nu(h)v, \quad h \in \mathfrak{h}\},$$

and we set $\text{wt} V = \{\nu \in P : V_\nu \neq 0\}$. 

1.6. We conclude this section with the following Lemma which is needed in later sections.

**Lemma.** Suppose that \( V \) is a \( \mathfrak{g}[t] \)-module and \( v \in V \) is such that
\[
(x_i^- \otimes t^{s_i})v = 0,
\]
for all \( i \in I \) and some \( s_i \in \mathbb{Z}_+ \). Set \( \lambda = \sum_{i \in I} d_i s_i \omega_i \). For all \( \alpha \in R^+ \), we have, \( (x^-_\alpha \otimes t^{s_\alpha})v = 0 \), \( s_\alpha \in \mathbb{Z}_+ \) such that \( \lambda(h_\alpha) = d_\alpha s_\alpha \).

**Proof.** We prove this by induction on \( h\alpha \), with induction beginning for the simple roots by assumption. For \( h\alpha > 1 \), choose \( \beta, \gamma \in R^+ \) such that \( \alpha = \beta + \gamma \), in which case using Lemma 1.3 we see that \( s_\alpha = s_\beta + s_\gamma \). Since
\[
(x^-_\alpha \otimes t^{s_\alpha}) = a(x^-_\beta \otimes t^{s_\beta}, x^-_\gamma \otimes t^{s_\gamma})
\]
for some non–zero complex number \( a \), the inductive step follows and proves the Lemma. \( \square \)

2. The modules \( V((\xi^\alpha)_{\alpha \in R^+}) \)

In this section, we first recall the definition of a local Weyl module \( W_{\text{loc}}(\lambda) \), \( \lambda \in P^+ \) and introduce non–zero gruch thataded quotient of these modules which are indexed by an \(|R^+|\)-tuple of partitions \( \xi = (\xi^\alpha)_{\alpha \in R^+} \) satisfying \( |\xi^\alpha| = \lambda(h_\alpha) \) for all \( \alpha \in R^+ \). We then give two further equivalent presentations of these modules.

2.1. The definition of the local Weyl modules was given originally in [11] and later in [7] and [15]. For \( \lambda \in P^+ \), the local Weyl module \( W_{\text{loc}}(\lambda) \), is the \( \mathfrak{g}[t] \)-module generated by an element \( w_\lambda \) with defining relations: for \( i \in I \) and \( s \in \mathbb{Z}_+ \),
\[
(x_i^+ \otimes C[t])w_\lambda = 0, \quad (h_i \otimes t^s)w_\lambda = \lambda(h_i)\delta_{s,0}w_\lambda, \quad (x_i^- \otimes 1)^{\lambda(h_i)+1}w_\lambda = 0,
\]
Here \( \delta_{s,0} \) is the Kronecker delta symbol. It is simple to see that \( \text{wt} W_{\text{loc}}(\lambda) \subset \lambda - Q^+ \) and that \( \dim W_{\text{loc}}(\lambda) = 1 \). It follows that \( W_{\text{loc}}(\lambda) \) is an indecomposable module and that \( W_{\text{loc}}(0) \) is isomorphic to the trivial \( \mathfrak{g}[t] \)-module. It was proved in [11] (see also [7]) that the local Weyl modules are finite–dimensional and so, in particular, we have
\[
(x^-_\alpha \otimes 1)^{\lambda(h_\alpha)+1}w_\lambda = 0, \quad \alpha \in R^+.
\]
The local Weyl module is clearly graded by \( \mathbb{Z}_+ \), once we declare the grade of \( w_\lambda \) to be zero, and
\[
W_{\text{loc}}(\lambda)[0] \cong_\mathfrak{g} V(\lambda).
\]
Moreover, \( e\nu_0 V(\lambda) \) is the unique graded irreducible quotient of \( W_{\text{loc}}(\lambda) \).
2.2. Given a non–zero element \( \lambda \in P^+ \), we say that \( \xi = (\xi^\alpha)_{\alpha \in R^+} \) is a \( \lambda \)-compatible \(|R^+|\)-tuple of partitions, if
\[
\xi^\alpha = (\xi^\alpha_1 \geq \cdots \geq \xi^\alpha_s \geq \cdots \geq 0), \quad |\xi| = \sum_{j \geq 1} \xi_j^\alpha = \lambda(h_\alpha).
\]
Define \( V(\xi) \) be the graded quotient of \( W_{\text{loc}}(\lambda) \) by the submodule generated by the graded elements
\[
\{(x^+_\alpha \otimes t)^s(x^-_\alpha \otimes 1)^{s+r} w_\lambda : \alpha \in R^+, \ s, r, k \in \mathbb{N}, \ s + r \geq 1 + rk + \sum_{j \geq k+1} \xi_j^\alpha \}.
\]
Denoting by \( v_\xi \) the image of \( w_\lambda \) in \( V(\xi) \), it is clear that \( V(\xi) \) is the \( g[t] \)-module generated by \( v_\xi \) with defining relations :
\[
\begin{align*}
n^+t[v_\xi] &= 0, \quad (h \otimes t^s)v_\xi = \delta_{s,0} \lambda(h)v_\xi, \quad h \in \mathfrak{h}, \ s \in \mathbb{Z}_+ \quad (2.4) \\
(x^-_i \otimes 1)^{\lambda(h_i)+1}v_\xi &= 0, \quad i \in I \quad (2.5) \\
(x^+_\alpha \otimes t)^s(x^-_\alpha \otimes 1)^{s+r}v_\xi &= 0, \quad \alpha \in R^+, \ s, r, k \in \mathbb{N}, \ s + r \geq 1 + rk + \sum_{j \geq k+1} \xi_j^\alpha. \quad (2.6)
\end{align*}
\]

For the rest of this section we shall work with a fixed non–zero element \( \lambda \in P^+ \) and a fixed \( \lambda \)-compatible \( \xi = (\xi^\alpha)_{\alpha \in R^+} \) tuple of partitions.

2.3. To prove that \( V(\xi) \) is non–zero and to give alternate formulations of (2.6) we need some more information. For \( r, s \in \mathbb{Z}_+ \), let
\[
S(r, s) = \left\{ (b_p)_{p \geq 0} : b_p \in \mathbb{Z}_+, \ \sum_{p \geq 0} b_p = r, \ \sum_{p \geq 0} pb_p = s \right\}. \quad (2.7)
\]
Notice that \( S(0, s) \) is the empty set if \( s > 0 \) and that
\[
(b_p)_{p \geq 0} \in S(r, s) \quad \Rightarrow \quad b_p = 0 \quad \text{if} \quad p > s.
\]
In particular, \( S(r, s) \) is finite.

Given \( x \in \mathfrak{g} \) and \( r, s \in \mathbb{Z}_+ \), define elements \( x(r, s) \in \text{U}(\mathfrak{g}[t]) \) by,
\[
x(r, s) = \sum_{(b_p)_{p \geq 0} \in S(r, s)} (x \otimes 1)^{(b_0)}(x \otimes t)^{(b_1)} \cdots (x \otimes t^s)^{(b_s)}, \quad (2.8)
\]
where for any integer \( p \) and any \( x \in \mathfrak{g}[t] \), we set \( x^{(p)} = x^p/p! \). Here we understand that \( x(r, s) = 0 \) if \( S(r, s) \) is the empty set. In particular,
\[
x(0, s) = \delta_{s,0}, \quad x(1, s) = x \otimes t^s. \quad (2.9)
\]

The following result was proved in [21] (see [11, Lemma 1.3] for the current formulation).
Lemma. Given \( s \in \mathbb{N}, r \in \mathbb{Z}_+ \) and \( \alpha \in R^+ \) we have,
\[
(x^+_\alpha \otimes t)^s(x^-_\alpha \otimes 1)^{s+r} + (-1)^s x^-_\alpha (r, s) \in U(g[t])n^+[t] \bigoplus U(n^-[t])h[t]_+.
\]
\( \square \)

Using the Lemma we see immediately that,
\[
((x^+_\alpha \otimes t)^s(x^-_\alpha \otimes 1)^{s+r} + (-1)^s x^-_\alpha (r, s)) v_{\xi} = 0,
\]
and hence (2.6) is equivalent to: for \( \alpha \in R^+ \), \( s, r, k \in \mathbb{N} \),
\[
x^-_\alpha (r, s) v_{\xi} = 0, \quad \text{if } s + r \geq 1 + rk + \sum_{j \geq k+1} \xi^\alpha_j.
\]

(2.10)

From now on, we shall use freely both presentations of \( V(\xi) \).

2.4. Given \( \lambda \in P^+ \), let \( \{\lambda\} = \{(\lambda(h_\alpha)_{\alpha \in R^+}\} \) be the \(|R^+|\)-tuple of partitions where each partition has at most one part. We now prove,

Proposition. (i) The module \( ev_0 V(\lambda) \) is the unique irreducible quotient of \( V(\xi) \) and hence \( V(\xi) \) is a non–zero, indecomposable \( g[t] \)-module.

(ii) We have an isomorphism,
\[
ev_0 V(\lambda) \cong g[t] V(\{\lambda\}).
\]

Proof. Part (i) follows if we prove that \( v_\lambda \) satisfies (2.10). If \( r, s \in \mathbb{N} \), then
\[
(b_p)_{p \geq 0} \in S(r, s) \implies b_p > 0 \quad \text{for some } \; p > 0,
\]
and hence
\[
x^-_\alpha (r, s) \in U(n^-[t])n^-[t]_+, \; r, s \in \mathbb{N}.
\]

Since \( g[t]_+ ev_0 V(\lambda) = 0 \) we get that \( x^-_\alpha (r, s) v_\lambda = 0 \) for \( r, s \in \mathbb{N} \) as required. To prove part (ii) recall that \( \theta \in R^+ \) is the highest root and notice that the relation
\[
x^-_\theta (1, 1)v(\lambda) = 0 = (x^-_\theta \otimes t)v(\lambda),
\]
holds in \( V(\{\lambda\}) \) by taking \( s = r = k = 1 \) in the defining relations. Since \([x^-_\theta, n^-] = 0\) and \( V(\{\lambda\}) = U(n^-[t])v(\lambda) \) it now follows that
\[
(x^-_\theta \otimes t)V(\{\lambda\}) = 0.
\]
Applying elements of \( n^+[t] \oplus h[t]_+ \) repeatedly, it is now straightforward to see that
\[
g[t]_+ V(\{\lambda\}) = 0, \quad \text{i.e.}, \quad ev_0 V(\lambda) \cong g[t] V(\{\lambda\}).
\]
\( \square \)
2.5. The third presentation requires an alternate description of the set $S(r, s)$. For $k \in \mathbb{Z}_+$, let $S(r, s)_k$ (resp. $kS(r, s)$) be the subset of $S(r, s)$ consisting of elements $(b_p)_{p \geq 0}$, satisfying

$$b_p = 0, \ p \geq k, \ \text{(resp. } b_p = 0 \ p < k).$$

For $0 \leq \ell \leq r$ and $0 \leq m \leq s$, we have canonical one–to–one maps

$$S(r - \ell, s - m)_k \times kS(\ell, m) \rightarrow S(r, s),$$

$$(b_p)_{p \geq 0}, \ (c_p)_{p \geq 0} \rightarrow (b_0, \cdots, b_{k-1}, c_k, \cdots).$$

By abuse of notation, we denote the image of this map by $S(r - \ell, s - m)_k \times kS(\ell, m)$, and we clearly have,

$$S(r, s) = kS(r, s) \bigcup_{r' \leq s} (S(r - r', s - s')_k \times kS(r', s')). \quad (2.11)$$

For $x \in \mathfrak{g}$, define elements $x(\ell, m)_k$ and $kx(\ell, m)$ of $U(\mathfrak{g}[t])$,

$$x(\ell, m)_k = \sum_{\{b_p\} \in S(\ell, m)_k} (x \otimes 1)^{(b_0)} \cdots (x \otimes t^{k-1})^{(b_{k-1})},$$

$$kx(\ell, m) = \sum_{\{b_p\} \in kS(\ell, m)} (x \otimes t^k)^{(b_k)} \cdots (x \otimes t^m)^{(b_m)}.$$

As always, we understand that the element $x(\ell, m)_k$ etc. is zero if $S(\ell, m)_k$ etc. is the empty set. In particular, it follows that

$$x(\ell, m)_k \neq 0 \implies m \leq (k-1)\ell, \quad kx(\ell, k\ell) = (x \otimes t^k)^\ell. \quad (2.12)$$

Lemma. Suppose that $s, r, k \in \mathbb{N}$ are such that $s + r \geq kr + K$ for some $K \in \mathbb{Z}_+$. Then,

$$x(r, s) = kx(r, s) + \sum x(r - r', s - s')_k \ kx(r', s'),$$

where the sum is over all pairs $r', s' \in \mathbb{Z}_+$ satisfying $r' < r$, $s' \leq s$ and $s' + r' \geq r'k + K$.

Proof. It is trivial to see from the alternative description of the set $S(r, s)$ given in (2.11), that

$$x(r, s) = kx(r, s) + \sum x(r - r', s - s')_k \ kx(r', s'),$$

where the sum is over all $r' < r$ and $s' \leq s$. Using (2.12) we see that we can also assume that $s - s' \leq (k-1)(r - r')$. Since

$$s + r = (s - s') + (r - r') + s' + r' \geq k(r - r') + kr' + K,$$

we get $s' + r' \geq kr' + K$ as required. \hfill \square

2.6. We now prove,

Proposition. Let $V$ be any representation of $\mathfrak{g}[t]$ and let $v \in V$, $x \in \mathfrak{g}$ and $K \in \mathbb{Z}_+$. Then,

$$x(r, s)v = 0 \text{ for all } s, r, k \in \mathbb{N} \text{ with } s + r \geq 1 + kr + K \iff kx(r, s)v = 0 \text{ for all } r, s, k \in \mathbb{N} \text{ with } s + r \geq 1 + kr + K.$$
Proof. Suppose first that \( x(r,s)v = 0 \) for all \( r, s, k \in \mathbb{N} \) with \( s + r \geq 1 + kr + K \). We shall prove by induction on \( r \) that \( kx(r,s)v = 0 \) for all \( r, s, k \in \mathbb{N} \) with \( s + r \geq 1 + kr + K \). Suppose that \( r = 1 \) and let \( s \in \mathbb{N} \) be such that \( s + 1 \geq 1 + k + K \). Then we have \( s \geq k + 1 \) and so

\[
x(1,s) = kx(1,s) = x \otimes t^s,
\]

and hence we have that \( kx(1,s)v = 0 \). Assume now that we have proved the statement for all \( r' < r \) and all \( s' \leq s \) with \( s' + r' \geq 1 + kr' + K \). Using Lemma 2.5, we get

\[
0 = x(r,s)v = kx(r,s)v + \left( \sum_{r' < r, s' \leq s} x(r' - r', s - s') kx(r', s') \right) v,
\]

where the sum is over all \( r' < r \) and \( s' \leq s \) with \( r' + s' \geq 1 + kr' + K \). The inductive hypothesis applies to the second term on the right hand side and hence we get \( kx(r,s)v = 0 \). The converse statement is obvious by using Lemma 2.5. \( \square \)

2.7. We can now give the third presentation of \( V(\xi) \).

Proposition. The module \( V(\xi) \) is generated by the element \( v_\xi \) with defining relations (2.4), (2.5) and,

\[
kx_\alpha^{-}(r,s)v_\xi = 0, \quad \alpha \in R^+, \quad r, s, k \in \mathbb{N}, \quad s + r \geq 1 + kr + \sum_{j \geq k+1} \xi_j^\alpha. \tag{2.13}
\]

In particular, for all \( \alpha \in R^+, r, k \in \mathbb{N} \) with \( r \geq 1 + \sum_{j \geq k+1} \xi_j^\alpha \), we have

\[
(x_\alpha^- \otimes t^k)^r v_\xi = 0. \tag{2.14}
\]

Proof. The first statement is immediate from Proposition 2.6. For the second, take \( s = kr \). Then we have \( s + r \geq 1 + kr + \sum_{j \geq k+1} \xi_j^\alpha \) and using the second equation in (2.12), we get

\[
kx_\alpha^{-}(r,kr)v_\xi = (x_\alpha^- \otimes t^k)^r v_\xi = 0.
\]

Corollary. If \( |\xi^\alpha| > 0 \) and \( s_\alpha \) is the number of parts of \( \xi^\alpha \), then

\[
(x_\alpha^- \otimes t^p)^r v_\xi = 0, \quad p \geq s_\alpha.
\]

\( \square \)

3. The connection with Demazure modules.

In this section, we study very special kinds of \( \lambda \)-compatible partitions and prove that in this case, the defining relations of \( V(\xi) \) can be greatly simplified. This allows us to make connections with well–known \( g[t] \)-modules, such as the local Weyl modules, Kirillov–Reshetikhin modules and the Demazure modules of level \( \ell \). Our results are new when \( \ell > 1 \) and are much simpler than the original proofs (see [9],[11],[20],[31]) even when \( \ell = 1 \). We shall use freely and without comment, the notation established in the earlier sections.
3.1. It will be convenient to use another standard notation for partitions. Namely, if \( i_1 > \cdots > i_k \) are the distinct non–zero parts of the partition and \( i_k \) occurs \( s_k \) times then we shall denote this partition by \((i_1^{s_1}, \cdots, i_k^{s_k})\). A partition \( \xi \) is said to be rectangular if it is either the empty partition or of the form \((k^m)\) for some \( k, m \in \mathbb{N} \). A partition is said to be a fat hook if it is of the form \((k_1^{s_1}, k_2^{s_2})\) with \( k_j, s_j \in \mathbb{N}, j = 1, 2 \). We shall say that the fat hook is special if \( s_2 = 1 \).

**Theorem 1.** Let \( \xi = (\xi^\alpha)_{\alpha \in R^+} \) be a \( \lambda \)-compatible \(|R^+|\)-tuple of partitions. Assume that \( \xi^\alpha \) is either rectangular or a special fat hook for \( \alpha \in R^+ \). Then, \( V(\xi) \) is isomorphic to the quotient of \( W_{\text{loc}}(\lambda) \) by the submodule generated by the elements,

\[
\{(x^-_{\alpha} \otimes t^{s_{\alpha}})w_\lambda : \alpha \in R^+\} \cup \{(x^-_{\alpha} \otimes t^{s_{\alpha}-1})^{\xi^\alpha+1}w_\lambda : \alpha \in R^+, \ \xi^\alpha \text{ a special fat hook}\}, \tag{3.1}
\]

where \( s_{\alpha} \) is zero if \( \xi^\alpha \) is the empty partition and is the number of non–zero parts of \( \xi^\alpha \) otherwise.

**Proof.** Let \( U \) be the submodule of \( W_{\text{loc}}(\lambda) \) generated by the elements in \((3.1)\) and let \( \tilde{V}(\xi) \) be the corresponding quotient of \( W_{\text{loc}}(\lambda) \). Using Proposition \((2.7)\) and taking \( k \in \{s_\alpha - 1, s_\alpha\} \) in equation \((2.14)\), we see that that \( V(\xi) \) is a quotient of \( \tilde{V}(\xi) \). To prove that they are isomorphic, we must show that: for \( \alpha \in R^+ \) and \( k, r, s \in \mathbb{N} \), either

\[
s + r \geq 1 + rk + \sum_{j \geq k+1} \xi^\alpha_j \implies (x^-_{\alpha} \otimes t^s)(x^-_{\alpha} \otimes 1)^{s+r}w_\lambda \in U, \tag{3.2}
\]

or

\[
s + r \geq 1 + rk + \sum_{j \geq k+1} \xi^\alpha_j \implies x^-_{\alpha}(r, s)w_\lambda \in U. \tag{3.3}
\]

If \( r \geq \xi^\alpha_1 \), then

\[
s + r \geq 1 + k\xi^\alpha_1 + \sum_{j \geq k+1} \xi^\alpha_j \geq 1 + \sum_{j \geq 1} \xi^\alpha_j = |\xi^\alpha| + 1.
\]

By \((3.3)\), we know \((x^-_{\alpha} \otimes 1)^{s+r}w_\lambda = 0 \) and so equation \((3.2)\) is proved in this case.

If \( r < \xi^\alpha_1 \), then by our assumptions on \( \xi \), we have \( r \leq \xi^\alpha_{s_\alpha} \) and we shall prove that \((3.3)\) is satisfied. Observe that if \( (b_p)_{p \geq 0} \in S(r, s) \) is such that \( b_m > 0 \) for some \( m \geq s_\alpha \), then

\[
(x^-_{\alpha} \otimes 1)^{(b_0)} \cdots (x^-_{\alpha} \otimes t^m)^{(b_m)} \cdots)w_\lambda \in U,
\]

and hence, we get

\[
(x^-_{\alpha}(r, s) - x^-_{\alpha}(r, s)_{s_\alpha})w_\lambda \in U. \tag{3.4}
\]

We claim that,

\[
s + r \geq 1 + kr + \sum_{j \geq k+1} \xi^\alpha_j \implies s + r \geq 1 + s_\alpha r. \tag{3.5}
\]

For the claim, notice there is nothing to prove if \( k \geq s_\alpha \), and if \( k < s_\alpha \), then

\[
s + r \geq 1 + kr + \sum_{j \geq k+1} \xi^\alpha_j \geq 1 + kr + (s_\alpha - k)\xi^\alpha_{s_\alpha} \geq 1 + kr + (s_\alpha - k)r.
\]
This means that if \((b_p)_{p \geq 0} \in S(r, s)\), then we must have \(b_m > 0\) for some \(m \geq s_{\alpha}\), since otherwise we would have \(s = \sum_{p < s_{\alpha}} pb_p \leq r(s_{\alpha} - 1)\). In particular, we get \(x_{-\alpha}^{-1}(r, s)s_{\alpha} = 0\) and equation (3.4) now proves that \(x_{-\alpha}^{-1}(r, s)w_\lambda \in U\). This also completes the proof when \(\xi^\alpha\) is rectangular. If \(\xi^\alpha\) is a special fat hook we have to also consider the case when \(\xi_1^\alpha > r > \xi_{s_{\alpha}}^\alpha\). If \((b_p)_{p \geq 0} \in S(r, s)s_{\alpha}\) then \(s = (s_{\alpha} - 1)b_{s_{\alpha} - 1} + \cdots + b_1\). If we prove that

\[
\tag{3.6}
 b_{s_{\alpha} - 1} \geq \xi^\alpha_{s_{\alpha}} + 1,
\]

then it obviously follows that \(x_{-\alpha}^{-1}(r, s)s_{\alpha}w_\lambda \in U\). Using (3.4) again, we will have \(x_{-\alpha}^{-1}(r, s)w_\lambda \in U\) which would complete the proof of the theorem.

To prove equation (3.6) observe that

\[
(s_{\alpha} - 1)b_{s_{\alpha} - 1} + (r - b_{s_{\alpha} - 1})(s_{\alpha} - 2) \geq s \geq 1 + r(k - 1) + \sum_{j \geq k + 1} \xi^\alpha_j,
\]

and hence

\[
b_{s_{\alpha} - 1} \geq 1 + r(k - s_{\alpha} + 1) + \sum_{j \geq k + 1} \xi^\alpha_j.
\]

Since \(r > \xi^\alpha_{s_{\alpha}}\), we see that equation (3.6) is immediate if \(k \geq s_{\alpha}\). If \(k < s_{\alpha}\), then

\[
b_{s_{\alpha} - 1} \geq 1 + \sum_{s_{\alpha} > j \geq k + 1} (\xi^\alpha_j - r) + \xi^\alpha_{s_{\alpha}} \geq 1 + \xi^\alpha_{s_{\alpha}}
\]

where the last inequality is because \(\xi^\alpha\) is a special fat hook, which means \(\xi^\alpha_1 = \xi^\alpha > r\) for all \(1 \leq j \leq s_{\alpha} - 1\).

\[\square\]

**3.2.** Given \(\ell \in \mathbb{N}\) and a non–zero element \(\lambda \in P^+\), we now define in a canonical way, an \(|R^+|\)-tuple of \(\lambda\)-compatible partitions. For \(\alpha \in R^+\), with \(\lambda(h_\alpha) > 0\), let \(s_{\alpha}, m_\alpha \in \mathbb{N}\) be the unique positive integers so that

\[
\lambda(h_\alpha) = (s_{\alpha} - 1)d_\alpha \ell + m_\alpha, \quad 0 < m_\alpha \leq d_\alpha \ell,
\]

where we recall that \(d_\alpha = 2/(\alpha, \alpha)\). If \(\lambda(h_\alpha) = 0\) set \(s_{\alpha} = 0 = m_\alpha\). Let \(\xi(\ell, \lambda) = (\xi^\alpha)_{\alpha \in R^+}\) be the \(|R^+|\)-tuple of partitions given by: \(\xi^\alpha\) is the empty partition if \(\lambda(h_\alpha) = 0\) and otherwise, is the partition \((((d_\alpha \ell)^{s_{\alpha} - 1}, m_\alpha)\). In particular, \(\xi^\alpha\) is either rectangular or a special fat hook. In the rest of this section we use Theorem I to analyze the modules \(V(\xi(\ell, \lambda))\).

**3.3.** The first result is,

**Lemma.** Let \(\ell \in \mathbb{Z}_+\) and suppose that \(\lambda = \ell \left(\sum_{i \in I} d_is_i\omega_i\right)\). Then, \(V(\xi(\ell, \lambda))\) is the quotient of \(W_{\text{loc}}(\lambda)\) by the submodule generated by the elements

\[
\{(x_i^{-1} \otimes t^{s_i})w_\lambda : i \in I, \ s_i > 0\}.
\]

**Proof.** Applying Lemma 1.4 to \((1/\ell)\lambda\) proves that \(\lambda(h_\alpha) = d_\alpha \ell s_{\alpha}\) for some \(s_{\alpha} \in R^+\), i.e., that \(\xi^\alpha\) is rectangular for all \(\alpha \in R^+\). Hence by Theorem I it suffices to prove that

\[
(x_i^{-1} \otimes t^{s_i})\xi(\ell, \lambda) = 0, \ i \in I \implies (x_\alpha^{-1} \otimes t^{s_\alpha})\xi(\ell, \lambda) = 0, \ \alpha \in R^+.
\]

Recalling that \((x_i^{-1} \otimes 1)w_\lambda = 0\) if \(s_i = 0\) equation (3.7) follows by applying Lemma 1.6 to \((1/\ell)\lambda\).
3.4. Our next result considers the case when $\lambda \in P^+$ and $\ell = 1$.

**Proposition.** For $\lambda \in P^+$, the module $V(\xi(1, \lambda))$ is the quotient of the local Weyl module by the submodule generated by the elements in the following two sets:

$$\{(x^{-}_\alpha \otimes t^{s_\alpha})w_\lambda : \alpha \in R^+ \text{ with } d_\alpha > 1\}, \quad (3.8)$$

$$\{(x^{-}_\alpha \otimes t^{s_\alpha-1})^2w_\lambda : \alpha \in R^+ \text{ with } d_\alpha = 3 \text{ and } m_\alpha = 1\}. \quad (3.9)$$

In particular, if $g$ is simply laced, then $W_{\text{loc}}(\lambda) \cong V(\xi(1, \lambda))$.

**Proof.** Let $U$ be the submodule generated by the elements in (3.8) and (3.9). Since $\ell = 1$, we have $0 < m_\alpha \leq d_\alpha$ and $\lambda(h_\alpha) = d_\alpha(s_\alpha - 1) + m_\alpha$. In particular if $d_\alpha = 1$ then $\xi^\alpha$ is rectangular and $s_\alpha = \lambda(h_\alpha)$. Applying Theorem 1 to $V(\xi(1, \lambda))$, we see that the result follows if we prove that:

$$d_\alpha = 1 \implies (x^{-}_\alpha \otimes t^{s_\alpha})w_\lambda = 0,$$

$$d_\alpha > 1 \text{ and } m_\alpha = d_\alpha - 1 \implies (x^{-}_\alpha \otimes t^{s_\alpha-1})^{d_\alpha}w_\lambda \in U.$$

The first statement follows by using (2.3) and Lemma 2.3 again, that $0 = (x^{-}_\alpha \otimes t)\xi^\alpha(x^{-}_\alpha \otimes 1)^{\xi^\alpha+1}w_\lambda = (x^{-}_\alpha \otimes t^{s_\alpha})w_\lambda = x^{-}_\alpha(1, s_\alpha)w_\lambda$.

Suppose that $d_\alpha > 1$ and $m_\alpha = d_\alpha - 1$, in which case $\lambda(h_\alpha) = s_\alpha d_\alpha - 1$ and we get by Using (2.3) and Lemma 2.3 again, that

$$0 = (x^{-}_\alpha \otimes t)^{d_\alpha(s_\alpha - 1)}(x^{-}_\alpha \otimes 1)^{d_\alpha s_\alpha}w_\lambda = x^{-}_\alpha(d_\alpha, d_\alpha(s_\alpha - 1))w_\lambda$$

$$= (x^{-}_\alpha \otimes t^{s_\alpha-1})^{d_\alpha}w_\lambda + Xw_\lambda,$$

where $X$ is in the left ideal of $U(n^-[t])$ generated by the elements $(x^{-}_\alpha \otimes t^p)$ with $p \geq s_\alpha$. Hence $Xw_\lambda \in U$ which give

$$(x^{-}_\alpha \otimes t^{s_\alpha-1})^{d_\alpha}w_\lambda \in U.$$  

3.5. We now recall from [20] and [31] the definition of the Demazure modules which are appropriate to our study. For $\ell \in Z_+$ and $\lambda \in P^+$ the Demazure module $D(\ell, \lambda)$ is the graded quotient of $W_{\text{loc}}(\lambda)$ generated by the elements,

$$\{(x^{-}_\alpha \otimes t^p)^{r+1}w_\lambda : p \in Z_+, \ r \geq \max\{0, \lambda(h_\alpha) - d_\alpha \ell p\}, \text{ for } \alpha \in R^+\}. \quad (3.10)$$

**Theorem 2.** Let $\ell \in Z_+$ and $\lambda \in P^+$. We have an isomorphism of graded $g[t]$-modules $V(\xi(\ell, \lambda)) \cong D(\ell, \lambda)$. Equivalently, $D(\ell, \lambda)$ is the quotient of $W_{\text{loc}}(\lambda)$ by the submodule generated by the elements

$$\{(x^{-}_\alpha \otimes t^{s_\alpha})w_\lambda : \alpha \in R^+\} \cup \{(x^{-}_\alpha \otimes t^{s_\alpha-1})^{m_\alpha+1}w_\lambda : \alpha \in R^+, \ m_\alpha < d_\alpha \ell \}$$. \quad (3.11)

**Proof.** Let $U'$ be the submodule of $W_{\text{loc}}(\lambda)$ by the elements in (3.10). Taking $p \in \{s_\alpha, s_\alpha - 1\}$ in (3.10) and recalling that $\lambda(h_\alpha) = d_\alpha \ell(s_\alpha - 1) + m_\alpha$, we see that the elements $(x^{-}_\alpha \otimes t^{s_\alpha})w_\lambda$ and $(x^{-}_\alpha \otimes t^{s_\alpha-1})^{m_\alpha+1}$ are in $U'$. Theorem 1 implies that the canonical map $W_{\text{loc}}(\lambda) \to D(\ell, \lambda)$ factors through to a map of $g[t]$-modules

$$V(\xi(\ell, \lambda)) \to D(\ell, \lambda).$$
To prove that it is an isomorphism we must prove that the additional defining relations of 
\( D(\ell, \lambda) \) hold in \( V(\xi(\ell, \lambda)) \). Namely that for all \( \alpha \in R^+ \) and \( p \in \mathbb{Z}_+ \), we have
\[
(x^-_\alpha \otimes t^p)^{r+1}v_{\xi(\ell, \lambda)} = 0, \quad r \geq \max\{0, \lambda(h_\alpha) - d_\alpha \ell p\}. \tag{3.12}
\]
If \( p \geq s_\alpha \), we have \( r \geq 0 \) while, if \( p < s_\alpha \), then
\[
r \geq \lambda(h_\alpha) - d_\alpha \ell p = d_\alpha \ell (s_\alpha - p - 1) + m_\alpha \geq \sum_{j \geq p+1} \xi_j^\alpha.
\]
In either case, (3.12) follows by taking \( k = p \) in (2.14) and the proof of the Theorem is complete. \( \Box \)

Together with Proposition 3.4 we have,

**Corollary.** The module \( D(1, \lambda) \) is the quotient of the local Weyl module by the submodule generated by the elements in the following two sets:
\[
\begin{align*}
\{(x^-_\alpha \otimes t^s_\alpha)w_\lambda : \alpha \in R^+ \text{ such that } d_\alpha > 1\} \\
\{(x^-_\alpha \otimes t^{s_\alpha - 1})^2w_\lambda : \alpha \in R^+ \text{ such that } d_\alpha = 3 \text{ and } m_\alpha = 1\}.
\end{align*}
\]
In particular, if g is simply laced, then \( W_{\text{loc}}(\lambda) \cong D(1, \lambda) \). \( \Box \)

### 3.6. We end this section with some remarks. Corollary 2 was proved earlier by very different methods. In [11, Section 6] it was shown that \( W_{\text{loc}}(\ell \omega) \) is of dimension \( 2^\ell \) when g is isomorphic to \( \mathfrak{sl}_2 \). On the other hand it was known that \( D(1, \ell \omega) \) is also of dimension \( 2^\ell \) ([19],[27],[28]) which shows that \( W_{\text{loc}}(\ell \omega) \cong D(1, \ell \omega) \). This was then used in [9, Section 3] and later in [20, Section 3.4] to prove the result for \( \ell = 1 \) and \( \mathfrak{sl}_{r+1} \) and for an arbitrary simply laced Lie algebra respectively. The case of the non–simply laced algebras the result was proved in [31, Section 4] using a case by case approach. The methods used in these papers are more complicated and do not appear to generalize to the case of \( \ell > 1 \) and arbitrary \( \lambda \).

### 4. Applications to Fusion Products of Demazure modules

We begin by recalling the definition of fusion products of \( g[t] \)-modules given in [15]. We then prove some elementary results which are used repeatedly in the rest of the paper. The main result of this section is Proposition 4.5 on the fusion product of Demazure modules.

**4.1.** Suppose that \( V \) is a cyclic \( g[t] \)-module generated by an element \( v \). Define a filtration on \( F^rV, r \in \mathbb{Z}_+ \) by
\[
F^rV = \sum_{0 \leq s \leq r} U(g[t])[s]v.
\]
Clearly each \( F^rV \) is a \( g \)-module and the associated graded space \( grV \) acquires a natural structure of a cyclic graded \( g[t] \)-module with action given by
\[
(x \otimes t^s)(w) = (x \otimes t^s)w, \quad w \in F^rV/F^{r-1}V.
\]
Moreover, \( grV \cong V \) as \( g \)-modules and \( grV \) is the cyclic \( g[t] \)-module generated by the image \( \bar{v} \) of \( v \) in \( grV \). The following is trivial but very useful.
Lemma. Let $V$ be a cyclic $\mathfrak{g}[t]$–module generated by $v \in V$. For all $u \in V$, $x \in \mathfrak{g}$, $r \in \mathbb{N}$, $a_1, \ldots, a_r \in C$, we have

$$(x \otimes t^r)\overline{u} = (x \otimes (t - a_1) \cdots (t - a_r))\overline{u},$$

where $\overline{u}$ is the image of $u$ in $\text{gr} V$.

4.2. Given any $\mathfrak{g}[t]$–module $V$ and $z \in C$ let $V^z$ be the $\mathfrak{g}[t]$–module defined by

$$(x \otimes t^r)v = (x \otimes (t + z)^r)v, \quad x \in \mathfrak{g}, \quad r \in \mathbb{Z}_+, \quad v \in V.$$

Let $V_1, \ldots, V_m$ be finite–dimensional graded $\mathfrak{g}[t]$–modules generated by elements $v_j$, $1 \leq j \leq m$ and let $z_1, \ldots, z_m$ be distinct complex numbers. Let

$$V = V_1^{z_1} \otimes \cdots \otimes V_m^{z_m},$$

be the corresponding product of $\mathfrak{g}[t]$–modules. It is easily checked (see [15, Proposition 1.4]) that

$$V = U(\mathfrak{g}[t])(v_1 \otimes \cdots \otimes v_m).$$

The corresponding associated graded $\mathfrak{g}[t]$–module $\text{gr} V$ is called the fusion product of $V_1, \ldots, V_m$ with parameters $z_1, \ldots, z_m$. It is denoted as $V_1^{z_1} \ast \cdots \ast V_m^{z_m}$ and is generated by the image of $v_1 \otimes \cdots \otimes v_m$.

In the rest of the paper, we shall frequently, for ease of notation, suppress the dependence of the fusion product on the parameters and just write $V_1 \ast \cdots \ast V_m$ for $V_1^{z_1} \ast \cdots \ast V_m^{z_m}$. But unless explicitly stated, it should be assumed that the fusion product does depend on these parameters.

Given elements $u_s \in V_s$, $1 \leq s \leq m$, we shall denote by $u_1 \ast \cdots \ast u_m \in V_1 \ast \cdots \ast V_m$ the image of the element $u_1 \otimes \cdots \otimes u_m \in V_1^{z_1} \otimes \cdots \otimes V_m^{z_m}$.

4.3.

Lemma. Let $\lambda_s \in P^+$ and $V_s$ be a $\mathfrak{g}[t]$–module quotient of $W_{\text{loc}}(\lambda_s)$ for $1 \leq s \leq m$. Then $V_1 \ast \cdots \ast V_m$ is a graded $\mathfrak{g}[t]$–module quotient of $W_{\text{loc}}(\lambda)$, where $\lambda = \sum_{s=1}^m \lambda_s$.

Proof. Let $v_s$ be the image of $w_{\lambda_s}$ in $V_s$. It suffices to prove that $v_1 \ast \cdots \ast v_m$ satisfies the defining relations of $w_\lambda$. Note that for $r > 0$ and $h \in \mathfrak{h}$, we have

$$(h \otimes t^r)(v_1 \otimes \cdots \otimes v_m) = \left( \sum_{s=1}^m z_s^r \lambda_s(h) \right) (v_1 \otimes \cdots \otimes v_m) \in U(\mathfrak{g})(v_1 \otimes \cdots \otimes v_m).$$

Hence if $r > 0$, we see that the image of $(h \otimes t^r)(v_1 \otimes \cdots \otimes v_m)$ in the fusion product is zero. The other defining relations of $W_{\text{loc}}(\lambda)$ are trivially satisfied and the proof is complete.

4.4. We shall need some results on Demazure modules which were proved in [19], [20]. We note here, that these papers work with a special family of Demazure modules $D(\ell, \lambda)$. In the notation of the current paper, they only work with the modules of the form $D(\ell, \ell \mu)$, where $\mu \in P^+$ is of the form $\mu = \ell \left( \sum_{i \in I} d_i \omega_i \right) \in P^+$. In view of this, it is convenient to define a subset $\Gamma$ of $\mathbb{Z}_+ \times P^+$ as follows:

$$\Gamma = \left\{ (\ell, \lambda) \in \mathbb{Z}_+ \times P^+ : \lambda = \ell \sum_{i \in I} d_is_i\omega_i \right\}.$$
In particular, if \((\ell, \lambda) \in \Gamma\) it follows from Theorem 2(ii) that the Demazure module \(D(\ell, \lambda)\), where \(\lambda = \ell \left( \sum_{i \in I} d_i s_i \omega_i \right)\) is the quotient of the local Weyl module by the relations \((x_i^- \otimes t^{s_i}) w_\lambda, \quad i \in I\) with \(s_i > 0\).

The following was established in [20, section 4], [19, section 3] Theorem 3. Let \((\ell, \lambda), (\ell, \mu) \in \Gamma\).

(i) For all \(s \in \mathbb{Z}\), we have \(\dim \text{Hom}_{\mathfrak{g}[t]}(\tau_s D(\ell, \lambda), D(\ell, \mu)) \leq 1\) and moreover, any non–zero map is injective.

(ii) \(\dim D(\ell, (\lambda + \mu)) = \dim D(\ell, \lambda) \dim D(\ell, \mu)\). \(\square\)

4.5. We can now prove,

**Proposition.** Let \((\ell, \lambda) \in \Gamma\) and suppose that there exists \((p_j, \mu_j) \in \Gamma, 1 \leq j \leq m\) such that

\[ \lambda = \mu_1 + \cdots + \mu_m, \quad \frac{1}{\ell} \lambda(h_i) \geq \sum_{j=1}^{m} \frac{1}{p_j} \mu_j(h_i), \quad i \in I. \]

There exists a non–zero surjective map of graded \(\mathfrak{g}[t]\)-modules,

\[ D(\ell, \lambda) \rightarrow D(p_1, \mu_1) \ast \cdots \ast D(p_m, \mu_m) \rightarrow 0, \]

and in the special case when \(p_1 = \cdots = p_m = \ell\) we have an isomorphism

\[ D(\ell, \lambda) \cong D(\ell, \mu_1) \ast \cdots \ast D(\ell, \mu_m). \]

The following is immediate.

**Corollary.** The fusion product of a finite number of modules of the form \(D(\ell, \mu), (\ell, \mu) \in \Gamma\) for a fixed \(\ell\) is independent of the choice of parameters.

**Remark.** The second statement of the proposition and the Corollary was proved earlier in [20, Section 3.5] using a result in [16].

**Proof of Proposition 4.5.** Let \(v_s \in D(p_s, \mu_s)\) be the image of the generator \(w_{\mu_s}\) of \(W_{\text{loc}}(\mu_s)\) for \(1 \leq s \leq m\). Using Lemma 4.3, we see that there exists a surjective map of graded \(\mathfrak{g}[t]\)-modules,

\[ W_{\text{loc}}(\lambda) \rightarrow D(p_1, \mu_1) \ast \cdots \ast D(p_m, \mu_m) \rightarrow 0. \]

Writing \(\lambda = \ell \sum_{i \in I} d_i s_i \omega_i\), and using Theorem 2(ii) (see the comments preceding the statement of Theorem 3), it suffices to show that

\[ (x_i^- \otimes t^{s_i})(v_1 \ast \cdots \ast v_m) = 0, \quad i \in I, s_i > 0. \]

Write \(\mu_k = p_k \sum_{i \in I} d_i s_{i,k} \omega_i, \quad 1 \leq k \leq m\), and note that we are given that

\[ s_i \geq \sum_{k=1}^{m} s_{i,k}, \quad i \in I. \]
Setting $b_i = s_i - \sum_k s_{i,k}$ and taking $z_1, \ldots, z_m$ be the parameters involved in the fusion product, we see that
\[(x_i^- \otimes t^b_i (t - z_1)^{s_{i,1}} \cdots (t - z_m)^{s_{i,m}})(v_1 \otimes \cdots \otimes v_m)\]
\[= \sum_{r=1}^m v_1 \otimes \cdots \otimes (x_i^- \otimes (t + z_r)^{b_i} (t - z_1 + z_r)^{s_{i,1}} \cdots t^{s_{i,r}} \cdots (t + z_r - z_m)^{s_{i,m}}) v_r \otimes \cdots \otimes v_m.\]

For $1 \leq r \leq m$, we know that the relation
\[(x_i^- \otimes t^b_i)v_r = 0, \quad b \geq s_{i,r},\]
holds in $D(p_r, \mu_r)$ and hence we have shown that
\[(x_i^- \otimes t^b_i (t - z_1)^{s_{i,1}} \cdots (t - z_m)^{s_{i,m}})(v_1 \otimes \cdots \otimes v_m) = 0.\]

It now follows by using Lemma 4.1 that
\[(x_i^- \otimes t^b_i (t - z_1)^{s_{i,1}} \cdots (t - z_m)^{s_{i,m}})(v_1 \cdots \otimes v_m) = 0 = (x_i^- \otimes t^s_i)(v_1 \cdots \otimes v_m) = 0,
\]
proving the existence of the map
\[D(\ell, \lambda) \to D(p_1, \mu_1) \cdots \ast D(p_m, \mu_m) \to 0.\]

The second statement of the proposition is now immediate by using Theorem 3(ii). \[\square\]

5. Kirillov–Reshetikhin modules and $Q$–systems

In this section, we discuss further consequences of our study and establish the connections with the $Q$–systems introduced in [22]. We will use freely the notation established in the previous sections.

5.1. We recall from [10] Section 2] the definition of the Kirillov–Reshetikhin modules. Thus given $i \in I$ and $m \in \mathbb{Z}_+$, the Kirillov–Reshetikhin module $KR(m\omega_i)$ is the quotient of the $W_{\text{loc}}(m\omega_i)$ by the submodule generated by the element $(x_i^- \otimes t)(m\omega_i)$. We will denote the image of $w_{m\omega_i}$ in $KR(m\omega_i)$ by $w_{i,m}$. It is trivially checked that
\[(x_i^- \otimes t^s)(w_{i,m}) = 0, \quad s \geq 1. \quad (5.1)\]

The following is a consequence of Lemma 3.3, Theorem 2 and Proposition 4.5. We remark that the isomorphism between the Kirillov–Reshetikhin modules and the Demazure modules was proved earlier in [10] Section 5 and [20] Section 3.2.

**Proposition.** For $i \in I$ and $\ell \in \mathbb{Z}_+$, we have an isomorphism of $\mathfrak{g}[t]$–modules,
\[KR(d_i \ell \omega_i) \cong V(\xi(\ell, d_i \ell \omega_i)) \cong D(\ell, d_i \ell \omega_i).\]

Moreover if $\lambda = \ell(\sum_{i \in I} d_i s_i \omega_i)$ then,
\[D(\ell, \lambda) \cong KR(d_1 \ell \omega_1)^{ss_1} \ast \cdots \ast KR(d_n \ell \omega_n)^{ss_n} \cong V(\xi(\ell, \lambda)). \square\]
5.2. We recall the definition of the $Q$–system given in [22, Section 7]. Consider the ring $\mathbb{Z}[x_1^\pm, \ldots, x_n^\pm]$ in the indeterminates $x_1, \ldots, x_n$, where we recall that $n$ is the rank of $g$. A $Q$–system for $g$ is a family of functions $\{Q_m(i) : i \in I, m \in \mathbb{Z}_+\}$ satisfying $Q_0^{(i)} = 1$, and

$$ Q_m^{(i)} Q_m^{(j)} = Q_{m+1}^{(i)} Q_{m-1}^{(i)} + \prod_{j \sim i} \prod_{k=0}^{-C_{ij} - 1} Q_{mC_{ij} - k}^{(j)} . $$

Here, $C_{ij} = d_i(\alpha_i, \alpha_j)$ and we say that $j \sim i$ if $C_{i,j} < 0$. Theorem 2.2 of [10] and Theorem 7.1 of [22] together, prove the following:

**Proposition.** Assume that $g$ is of type $A, B, C$ or $D$. The $g$–characters of the Kirillov-Reshhetikhin modules satisfy the $Q$–system. More precisely, for $i \in I$ and $m \in \mathbb{Z}_+$, we have a (non–canonical) short exact sequence of $g$–modules,

$$ 0 \to K_{i,m} \to KR(m\omega_i) \otimes KR(m\omega_i) \to KR((m + 1)\omega_i) \otimes KR((m - 1)\omega_i) \to 0, $$

where

$$ K_{i,m} \cong \bigotimes_{j \sim i} K_{(mC_{ji} - k)/C_{ij}}(\omega_j). $$

5.3. We shall prove the stronger statement:

**Theorem 4.** Assume that $g$ is of type $A, B, C$ or $D$. Given $i \in I$, there exists a canonical, non–split short exact sequence of graded $g[t]$–modules:

$$ 0 \to \tau_{d,m}K_{i,d,m}^* \to KR(d_i m\omega_i) \otimes KR(d_i m\omega_i) \to KR((d_i m + 1)\omega_i) \otimes KR((d_i m - 1)\omega_i) \to 0, $$

where $K_{i,d,m}^*$ is obtained from $K_{i,d,m}$ by replacing the tensor product by fusion product and $\tau_{d,m}$ is the grading shift operator.

The proof of the theorem occupies the rest of this section. Moreover $g$ will continue to be an arbitrary complex simple Lie algebra, unless otherwise stated.

5.4. Let us first make the tensor product expression for $K_{i,d,m}$ more explicitly. If $d_i = 1$ and $j \sim i$, then $C_{i,j} = -1$, $k = 0$, $C_{j,i} = -d_j$ and we have

$$ K_{i,d,m} \cong \bigotimes_{j \sim i} KR(d_j m), \quad d_i = 1. $$

If $d_i = 2$ and $j \sim i$ then we have two possibilities: either $d_j = d_i = 2$ in which case $C_{i,j} = -1 = C_{j,i}$ and $k = 0$ or $d_j = 1$ in which case $C_{i,j} = -2, C_{j,i} = -1$ and $k = 0,1$. Hence we get,

$$ K_{i,d,m} = \left( \bigotimes_{j \sim i : d_j = 2} KR(d_j m\omega_j) \right) \bigotimes_{j \sim i : d_j = 1} KR(d_j m\omega_j)^{\otimes 2}. $$

If $d_i = 3$ then $d_j = 1$ and $C_{i,j} = -3, C_{j,i} = -1$ and $k \in \{0,1,2\}$. Hence we get

$$ K_{i,d,m} = KR(m\omega_j)^{\otimes 3}. $$
5.5. Set
\[
\lambda = \begin{cases} 
  m \left( \sum_{j \sim i} d_j \omega_j \right), & d_i = 1, \\
  m \left( \sum_{j \sim i; d_j = 2} d_j \omega_j + \sum_{j \sim i; d_j = 1} 2d_j \omega_j \right), & d_i = 2, \\
  3m \omega_j, & d_i = 3.
\end{cases}
\]

Using Proposition 5.1, we see that
\[ K_{i,d,m}^* \cong g[t] D(m, \lambda) \cong V(\xi(m, \lambda)), \quad KR(d_i \omega_i) * KR(d_i \omega_i) \cong V(\xi(m, 2d_i \omega_i)), \]
and hence an equivalent form of Theorem 4 is the following:

Theorem 4. Assume that $g$ is of type $A$, $B$, $C$ or $D$. Given $i \in I$, there exists a canonical, non-split short exact sequence of graded $g[t]$-modules:
\[ 0 \to \tau_{d_i} V(\xi(m, \lambda)) \xrightarrow{i} V(\xi(m, 2d_i \omega_i)) \xrightarrow{\pi} KR((d_i m + 1) \omega_i) * KR((d_i m - 1) \omega_i) \to 0. \]

We now prove Theorem 4.

5.6. The following Lemma proves the existence of $\tilde{\pi}$.

Lemma. There exists a surjective map of $g[t]$-modules
\[ \tilde{\pi} : V(\xi(m, 2d_i \omega_i)) \to KR((d_i m + 1) \omega_i) * KR((d_i m - 1) \omega_i), \]
such that
\[ \tilde{\pi}(v_{\xi(m, 2d_i \omega_i)}) = w_{i,d,m+1} * w_{i,d,m-1}. \]

Moreover,
\[ 0 \neq (x_{i}^- \otimes t)^{d_i} v_{\xi(m, 2d_i \omega_i)} \in \ker \tilde{\pi}. \]

Proof. By Lemma 4.3, we know that $KR((d_i m + 1) \omega_i) * KR((d_i m - 1) \omega_i)$ is a quotient of $W_{\text{loc}}(2d_i \omega_i)$. Hence by Lemma 5.5, it is enough to prove that
\[ (x_{i}^- \otimes t^2)(w_{i,d,m+1} * w_{i,d,m-1}) = 0. \]

Let $z_1, z_2$ be the distinct complex numbers involved in the fusion product. By Lemma 4.1, it is enough to prove that
\[ (x_{i}^- \otimes (t - z_1)(t - z_2))(w_{i,d,m+1} * w_{i,d,m-1}) = 0. \]

This follows from,
\[ (x_{i}^- \otimes (t - z_1)(t - z_2))(w_{i,d,m+1} \otimes w_{i,d,m-1}) = \]
\[ (x_{i}^- \otimes (t(t - z_2 + z_1))w_{i,d,m+1} \otimes w_{i,d,m-1} + w_{i,d,m+1} \otimes (x_{i}^- \otimes (t - z_1 + z_2))w_{i,d,m-1} = 0, \]
where the last equality is a consequence of equation 5.1. This proves the existence of $\tilde{\pi}$.

We now prove that
\[ (x_{i}^- \otimes t)^{d_i} (w_{i,d,m+1} * w_{i,d,m-1}) = 0. \]
Again, using Lemma 4.1 we have,

\[(x^-_i \otimes t)^{d,m}(w_{i,d,m+1} * w_{i,d,m-1}) = (x^-_i \otimes (t - z_1)^{d,m})(w_{i,d,m+1} * w_{i,d,m-1}),\]

and this time we get,

\[\left((x^-_i \otimes t)w_{i,d,m+1}\right) \otimes w_{i,d,m-1} + w_{i,d,m+1} \otimes \left((x^-_i \otimes (t - z_1 + z_2)^{d,m})w_{i,d,m-1}\right)\]

The first term on the right hand side is zero by equation (5.1) while the second term is zero because

\[(d_i m - 1)\omega_i - (d_i m \alpha_i) \notin \text{wt}(KR((d_i m - 1)\omega_i)).\]

This proves, that

\[(x^-_i \otimes t)^{d,m}v_{\xi(m,2d_i m\omega_i)} \in \ker \tilde{\pi}.\]

To complete the proof of the Lemma, we must show that

\[(x^-_i \otimes t)^{d,m}v_{\xi(m,2d_i m\omega_i)} \neq 0,\]

and using the isomorphism

\[V(\xi(m, 2md_i \omega_i)) \cong KR(m\omega_i) \ast KR(m\omega_i),\]

it is enough to show that \((x^-_i \otimes t)^{d,m}(w_{i,d,m} * w_{i,d,m})\) is a non-zero element of the fusion product \(KR(d_i m\omega_i) * KR(d_i m\omega_i)\). We proceed by contradiction. A familiar argument by now, shows that we would have

\[0 = (x^-_i \otimes t)^{d,m}(w_{i,d,m} * w_{i,d,m}) = w_{i,d,m} * (x^-_i \otimes (t - z_1 + z_2)^{d,m})w_{i,d,m} = (z_2 - z_1)^{d,m}w_{i,d,m} * (x^-_i)^{d,m}w_{i,d,m} \]

Consider now the \(g\)-module isomorphism

\[KR(d_i m\omega_i) \otimes KR(d_i m\omega_i) \rightarrow KR(d_i m\omega_i) * KR(d_i m\omega_i), \quad v \otimes w \rightarrow v * w.\]

We would now have that the element

\[w_{i,d,m} \otimes (x^-_i)^{d,m}w_{i,d,m} \mapsto 0.\]

Applying \((x^+_i \otimes 1)^{d,m}\) gives

\[w_{i,d,m} \otimes w_{i,d,m} \mapsto 0,\]

which gives the required contradiction.

\[\square\]

5.7. The next result establishes the existence of \(i\).

Lemma. There exists an injective non-zero map of graded \(\mathfrak{g}[t]\)-modules,

\[\tau : \tau_{d,m}K^*_i d_m \rightarrow \ker \pi.\]
Proof. Since $K_{i,d,m}^* \cong V(\xi(m, \lambda))$, the Lemma will follow if we prove that the non–zero element $(x_i^- \otimes t)^{d,m} v_{\xi(m,2d,m\omega_i)} \in \ker \tilde{\pi}$ satisfies the relations of $\tau_{d,m} V(\xi(m, \lambda))$. It is trivial to see that the element satisfies the relations of the grade shifted local Weyl module $\tau_{d,m} W_{\text{loc}}(\lambda)$. By Lemma 3.3 we only need to check that the following relations hold for all $j \sim i$:

$$(x_j^- \otimes t)(x_i^- \otimes t)^{d,m} v_{\xi(m,2d,m\omega_i)} = 0, \quad d_i = 1 \text{ or } d_i = d_j = 2,$$

$$(x_j^- \otimes t^2)(x_i^- \otimes t)^{d,m} v_{\xi(m,2d,m\omega_i)} = 0, \quad d_i = 2, \quad d_j = 1.$$

If $i = j$, the result is immediate from (5.10). If $i \neq j$, then $\alpha_i, \alpha_j$ span a root system of type $A_2$ or $C_2$, we see, by using the commutation relations in these Lie algebras, that the equalities follow if we prove that

$$(x_{\alpha_i + \alpha_j}^- \otimes t^2)v_{\xi(m,2d,m\omega_i)} = 0,$$

and also if $d_i = 2$, $d_j = 1$, then

$$(x_{2\alpha_i + \alpha_j}^- \otimes t^4)v_{\xi(m,2d,m\omega_i)} = 0.$$

These are proved by a straightforward case by case analysis using the relations (3.10). Thus we have,

$$d_i = d_j \implies d_{\alpha_i + \alpha_j} = d_i, \quad \omega_i(h_{\alpha_i + \alpha_j}) = 1,$$

$$d_i = 1, \quad d_j = 2 \implies d_{\alpha_i + \alpha_j} = 2, \quad \omega_i(h_{\alpha_i + \alpha_j}) = 2,$$

$$d_i = 2, \quad d_j = 1 \implies d_{\alpha_i + \alpha_j} = 2, \quad \omega_i(h_{\alpha_i + \alpha_j}) = 1,$$

$$d_i = 2, \quad d_j = 1 \implies d_{2\alpha_i + \alpha_j} = 1, \quad \omega(h_{2\alpha_i + \alpha_j}) = 1.$$

In the first three cases, this means that $2d_i m \omega_i(h_{\alpha_i + \alpha_j}) = 2d_{\alpha_i + \alpha_j} m$ and in the last case, we have $2d_i m \omega_i(h_{2\alpha_i + \alpha_j}) = 4d_{2\alpha_i + \alpha_j} m$. Equations (5.3) and (5.4) are now immediate from (3.10). The case when $d_i = 3$ is similar and we omit the details. Since $\tilde{i}$ is a non–zero map between Demazure modules, it follows from Theorem 3(i) that $\tilde{i}$ is injective and the proof of the Lemma is complete.

\[ \square \]

5.8. Note that Lemma 5.6 and Lemma 5.7 prove that

$$\dim(KR(d_i m \omega_i) * KR(d_i m \omega_i)) \geq \dim K_{d,m}^* + \dim KR((d_i m + 1) \omega_i) * KR((d_i m - 1) \omega_i)).$$

Hence Theorem 4 obviously follows if we establish the reverse inequality. Since the dimension is unchanged when we replace tensor products by fusion products, we must prove that

$$\dim(KR(d_i m \omega_i) \otimes KR(d_i m \omega_i)) = \dim K_{d,m} + \dim KR((d_i m + 1) \omega_i) \otimes KR((d_i m - 1) \omega_i)).$$

But this is immediate from Proposition 5.2 if $\mathfrak{g}$ is of classical type. Finally, since $V(\xi(m, 2d_i m \omega_i))$ is an indecomposable module, it follows that the sequence is non–split and the theorem is proved.
Theorem 4 can also be deduced for arbitrary simple Lie algebras from Lemma 5.6 and Lemma 5.7 in a different way and we explain this very briefly. The Kirillov–Reshetikhin modules for the current algebra were proved in [5], [25] to be the specializations to $q = 1$ of Kirillov–Reshetikhin modules for the quantized enveloping algebra of the loop algebra. An obvious consequence of results proved in [23], [29], [30] is that the quantum Kirillov–Reshetikhin modules satisfy the required conditions on the dimensions in (5.5). Since the dimension is unchanged on specializing we have the necessary equality of dimensions for the current algebra. However all these results are very difficult, use many different ideas and deep results in the representation theory of quantum and classical affine Lie algebras. Our approach keeps us in the purely classical realm of Lie algebras and shows that the only missing piece of information for proving Theorem 4 for exceptional Lie algebras is (5.5). Since the Kirillov–Reshetikhin modules are actually Demazure modules, it might be possible to use the literature on Demazure modules to prove (5.5) directly, although this too seems hard at the moment.

6. THE $\mathfrak{sl}_2$ CASE

In this section, we focus on the Lie algebra $\mathfrak{sl}_2$ and prove that the modules $V(\xi)$ are fusion products of evaluation modules $ev_0 V(r)$, $r \in \mathbb{Z}_+$ and vice-versa. In particular, this gives generators and relations for such fusion products and recovers in a very different and self-contained way the results of [13]. We also give an explicit monomial basis for the modules $V(\xi)$.

6.1. In the case of $\mathfrak{sl}_2$, the set $I$ is the singleton set $\{1\}$ and so we just denote the elements $x_1^\pm$ by $x^\pm$ etc. We identify $P$ with the integers and given $r \in \mathbb{Z}_+$, a $r$–compatible partition $\xi$ is just a partition of $r$, i.e., $\xi = (\xi_1 \geq \xi_2 \geq \cdots)$ and $|\xi| = r$.

6.2. Given a partition $\xi = (\xi_1 \geq \cdots \geq \xi_\ell > 0)$ with $\ell$ parts we define partitions $\xi^\pm$ as follows. If $\ell = 1$, then $\xi^+ = \xi$ and $\xi^-$ is the empty partition. If $\ell > 1$, then $\xi^- = (\xi_1^+ \geq \cdots \geq \xi_{\ell-2}^- \geq \xi_{\ell-1}^- \geq 0)$ is given by

$$\xi^-_r = \begin{cases} \xi_r, & r < \ell - 1, \\ \xi_{\ell-1} - \xi_\ell, & r = \ell - 1, \\ 0, & r \geq \ell. \end{cases}$$

In particular $|\xi^-| = |\xi| - 2\xi_\ell$. We take $\xi^+ = (\xi_1^+ \geq \cdots \geq \xi_{\ell-1}^+ \geq \xi_\ell^+ \geq 0)$ is the unique partition associated to the $n$–tuple $(\xi_1, \cdots, \xi_{\ell-2}, \xi_{\ell-1} + 1, \xi_\ell - 1)$. It can be described explicitly as follows: let $0 \leq \ell(\xi) \leq \ell - 2$ be minimal such that $\xi_{\ell(\xi) + 1} = \xi_{\ell-1}$.

Then,

$$\xi^+_j = \begin{cases} \xi_j, & 1 \leq j \leq \ell(\xi) \text{ or } j > \ell, \\ \xi_{\ell-1} + 1, & j = \ell(\xi) + 1, \\ \xi_{\ell-1} - 1, & \ell(\xi) + 2 \leq j \leq \ell - 1, \\ \xi_{\ell - 1}, & j = \ell. \end{cases}$$
and hence,  

\[
\sum_{j \geq k + 1} \xi^+_j = \begin{cases} 
\sum_{j \geq k + 1} \xi_j, & k \leq \ell(\xi) \text{ or } k \geq \ell, \\
-1 + \sum_{j \geq k + 1} \xi_j, & \ell(\xi) < k \leq \ell - 1.
\end{cases}
\]  

(6.1)

6.3. Define a subset $\mathcal{I}(\xi)$ of $\mathbb{Z}^{\ell}_{+}$ by:  

\[i = (i_1, \ldots, i_\ell) \in \mathcal{I}(\xi), \text{ iff for all } 2 \leq k \leq \ell + 1 \text{ and } 1 \leq j \leq k - 1, \text{ we have}, \]

\[(ji_{k-1} + (j + 1)i_k) + 2 \sum_{p=k+1}^{\ell} i_p \leq \sum_{p=k-j}^{\ell} \xi_p, \]

where we understand that $i_{\ell+1} = 0$. We shall prove,

**Theorem 5.** Assume that $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_2$ and that $\xi = (\xi_1 \geq \cdots \geq \xi_\ell > 0)$.

(i) For $\ell > 1$, there exists a short exact sequence of $\mathfrak{g}[t]$–modules,  

\[0 \to \tau(\ell-1)\xi V(\xi^-) \xrightarrow{\varphi^-} V(\xi) \xrightarrow{\varphi^+} V(\xi^+) \to 0.\]

(ii) For all $\ell \geq 1$, we have an isomorphism of $\mathfrak{g}[t]$–modules,  

\[V(\xi) \cong V^{z_1}(\xi_1) \ast \cdots \ast V^{z_\ell}(\xi_\ell),\]

for any set $z_1, \ldots, z_\ell$ of distinct scalars.

(iii) The elements  

\[\{(x^- \otimes 1)^{i_1} \cdots (x^- \otimes t^{\ell-1})^{i_\ell} v_\xi : (i_1, \ldots, i_\ell) \in \mathcal{I}(\xi)\},\]

form a basis of $V(\xi)$.

The proof of the Theorem occupies the rest of this section. The first two parts of the theorem are proved simultaneously by an induction on $\ell$, namely we shall prove that if (ii) holds for a partition with $\ell$ parts, then (i) and (ii) hold for a partition with $\ell + 1$ parts. Proposition 2.4 shows that (ii) holds when $\ell = 1$ and so induction begins. *In the rest of the section, given $r, s \in \mathbb{Z}_{+}$, we set,*  

\[X(r, s) = (x^+ \otimes t)^{(s)}(x^- \otimes 1)^{(s+r)}.\]

6.4. The following result establishes the existence of $\varphi^+$.

**Proposition.** There exists a surjective morphism of $\mathfrak{g}[t]$–modules $\varphi^+ : V(\xi) \to V(\xi^+)$ such that  

\[\varphi(v_\xi) = v_{\xi^+}, \quad \ker \varphi^+ = U(\mathfrak{g}[t])(x^- \otimes t^{\ell-1})^{\xi_1} v_\xi.\]

**Proof.** To prove that $\varphi^+$ exists, it suffices to show that  

\[X(r, s)v_{\xi^+} = 0, \text{ for all } s, r, k \in \mathbb{N}, \text{ with } s + r \geq 1 + rk + \sum_{j \geq k+1} \xi_j.\]

But this is immediate by noting that $\sum_{j \geq k+1} \xi_j \geq \sum_{j \geq k+1} \xi_j^+$. 
Using (2.14) we see that
\[(x^- \otimes t^{\ell-1})^{\xi^v}v^+ = 0, \quad \text{i.e.,} \quad (x^- \otimes t^{\ell-1})^{\xi^v}v^+ \in \ker \phi^+.
\]

To complete the proof we must show that \((x^- \otimes t^{\ell-1})^{\xi^v}v^+\) generates \(\ker \phi\). Suppose that \(Xv^+ = 0\) for some \(X \in \mathbf{U}(\mathfrak{g}[t])\). Then we may write \(X = Y + Z\) where \(Y\) is in the left ideal of \(\mathbf{U}(\mathfrak{g}[t])\) generated by the set
\[
\{x^+ \otimes t^p, (h \otimes t^p) - \delta_{p,0}|\xi|, (x^- \otimes 1)|\xi|+1 : p \in \mathbb{Z}_+\},
\]
and \(Z\) is in the left ideal of \(\mathbf{U}(\mathfrak{g}[t])\) generated by
\[
\{X(r, s) : r, s, k \in \mathbb{N}, s + r \geq 1 + rk + \sum_{j \geq k+1} \xi^+_j\}.
\]

Since \(Yv^+ = 0\) as well, we need to consider only the case when \(X = X(r, s)\) for \(r, s \in \mathbb{N}\) with \(s + r \geq 1 + rk + \sum_{j \geq k+1} \xi^+_j\) for some \(k \in \mathbb{N}\). Moreover, we see by using equation (6.1), that the only time \(X(r, s)v^+ = 0\) is not obviously a defining relation of \(v^+\) is when \(s + r = rk + \sum_{j \geq k+1} \xi^+_j\), for some \(\ell(\xi) < k \leq \ell - 1\). Since \(\xi^+_j = \xi^t - \ell\) if \(\ell(\xi) + 2 \geq j \leq \ell - 1\) we have
\[
\ell(\xi) < k \leq \ell - 1 \implies s + r = rk + (\ell - k - 1)\xi^t - \ell + \xi^t.
\]

If \(\xi^t < r\), then we have
\[
s + r = (k - 1)r + (r - \xi^t) + \xi^t - \ell + \xi^t - 1 + (k - 1)r + \sum_{j \geq k} \xi^+_j,
\]
(since \(\xi^+_j = \xi^t - \ell\) under the hypothesis that \(\ell - 1 \geq k > \ell(\xi)\)). This means that \(X(r, s)v^+ = 0\) and we are done in this case.

Assume that \(\xi^t \geq r\) and let
\[
(b_p)p \geq 0 \in \mathbf{S}(r, s), \quad s + r = rk + (\ell - k - 1)\xi^t + \xi^t
\]
for some \(\ell(\xi) < k \leq \ell - 1\). Recalling from Corollary 2.7 that \((x^- \otimes t^\ell)v^+ = 0\), we see that
\[
(x^- \otimes 1)^{(b_0)}(x^- \otimes t)^{(b_1)} \cdots (x \otimes t^\ell)^{(b_\ell)}v^+ \neq 0 \implies s \leq (\ell - 1)b^t + (r - b^t)(\ell - 2).
\]

This gives,
\[
r(k - 1) + (\ell - k - 1)\xi^t - 1 + \xi^t \leq b^t + r(\ell - 2),
\]
i.e.,
\[
0 \leq (\ell - k - 1)(\xi^t - r) \leq b^t - \xi^t.
\]
Hence \(b^t \geq \xi^t\) and so,
\[
X(r, s)v^+ = x^t(r, s)v^+ \in \mathbf{U}(\mathfrak{n}^-[t])(x^- \otimes t^{\ell-1})^{\xi^v}v^+,
\]
which completes the proof. \(\square\)
6.5. We need some additional results to prove the existence of $\varphi^-$. The following Lemma is checked by a straightforward induction on $p$.

**Lemma.** Given $a, b, p \in \mathbb{Z}_+$, we have,

\[
[x^+ \otimes t^a, (x^- \otimes t^b)^{(p)}] = (x^- \otimes t^b)^{(p-1)}(h \otimes t^{a+b}) - (x^- \otimes t^{a+2b})(x^- \otimes t^b)^{(p-2)},
\]

\[
[h \otimes t^a, (x^- \otimes t^b)^{(p)}] = -2(x^- \otimes t^{a+b})(x^- \otimes t^b)^{(p-1)},
\]

\[
[(x^- \otimes t^a), (x^+ \otimes t^b)^{(p)}] = -(x^+ \otimes t^b)^{(p-1)}(h \otimes t^{a+b}) - (x^+ \otimes t^b)^{(p-2)}(x^+ \otimes t^{a+2b}),
\]

where we understand as usual that $(x \otimes t^b)^p = 0$ if $p < 0$.

\[\square\]

6.6. The next result we need is the following.

**Lemma.** Suppose that $V$ is a $\mathfrak{g}[t]$–module and $v \in V$ satisfies,

\[n[t]v = 0, \quad b[t]v = 0, \quad X(r, s)v = 0, \quad s, r \in \mathbb{N} \quad s + r \geq N,\]

for some $N \in \mathbb{N}$. Then for all $j \in \mathbb{N}$, we have

\[X(r, s)(x^- \otimes t^j)v = 0 \text{ for all } s, r \in \mathbb{Z}_+ \text{ with } s + r \geq N - 2.\]

**Proof.** For $j \in \mathbb{N}$, set $v_j = (x^- \otimes t^j)v$ and note that $n^+[t]v_j = 0$. Write

\[(r + 1)X(r + 1, s + 1) = (x^+ \otimes t)^{(r)}(x^+ \otimes t)(x^- \otimes t)^{(r+s+2)}.\]

Using the first equation in Lemma 6.5 and the relations satisfied by $v$, we find that

\[(r + 1)X(r + 1, s + 1)v = -X(r, s)v_1.\]

Since the left hand side is zero if $s + r + 2 \geq N$, it follows that

\[X(r, s)v_1 = 0, \quad s + r \geq N - 2. \tag{6.2}\]

Using the second and third equations in Lemma 6.5 we see that, for $i \geq 0$, we have

\[(x^- \otimes t^i)X(r, s)v = X(r, s)(x^- \otimes t^i)v + 2X(r, s - 1)v_{i+1} + X(r, s - 2)v_{i+2}.\]

If we take $s, r$ such that $s + r \geq N$, then the left hand side of the equation is zero, i.e.,

\[X(r, s)(x^- \otimes t^i)v + 2X(r, s - 1)v_{i+1} + X(r, s - 2)v_{i+2} = 0, \quad s + r \geq N.\]

If $i = 0$, then the first term on the left hand side is a multiple of $X(r + 1, s)v$ and hence is zero. Using (6.2) we see that since $r + s - 1 \geq N - 2$ the second term is zero which proves finally that

\[X(r, s - 2)v_2 = 0, \quad r + s \geq N, \quad \text{i.e.,} \quad X(r, s)v_2 = 0, \quad r + s \geq N - 2.\]

An obvious induction on $i$ now gives the proposition.

\[\square\]

**Corollary.** Let $v \in V$ be as in the proposition and assume also that $(x^- \otimes t^p)^{(p)} = 0$. Then

\[X(r, s)(x^- \otimes t^{p-1})^kv = 0, \quad s, r \in \mathbb{Z}_+, \quad s + r \geq N - 2k.\]

**Proof.** The proof is immediate once we observe that the element $(x^- \otimes t^{p-1})^kv$ satisfies the same relations as $v$ with $N$ replaced by $N - 2k$. 

\[\square\]
6.7. We now establish the existence of $\varphi^-$.  

**Proposition.** There exists a surjective morphism of $\mathfrak{g}[t]$–modules satisfying  
\[
\varphi^- : V(\xi^-) \to \ker \varphi^+, \quad \varphi^-(v_{\xi^-}) = (x^- \otimes t^{\ell-1})^{\xi}\xi v_{\xi}.
\]

**Proof.** It is trivial to check that there exists a morphism $W_{\text{loc}}(\xi^-) \to V(\xi)$ which maps  
\[w_{\xi_-} \to (x^- \otimes t^{\ell-1})^{\xi}\xi v_{\xi}.
\]
To prove that the map factors through to $V(\xi^-)$ we must show that  
\[X(r,s)(x^- \otimes t^{\ell-1})^{\xi}\xi v_{\xi} = 0, \quad s, r, k \in \mathbb{N}, \text{ with } s + r \geq 1 + r k + \sum_{j \geq k+1} \xi_j^-.
\]
If $k \leq \ell - 2$, then we have  
\[1 + r k + \sum_{j \geq k+1} \xi_j^- = 1 + r k + \left( \sum_{j \geq k+1} \xi_j \right) - 2\xi_\ell.
\]
Since $X(r,s)v_{\xi} = 0$ if $s + r \geq 1 + \sum_{j \geq k+1} \xi_j$ and  
\[(x^- \otimes t^{\ell})v_{\xi} = 0 \implies (x^- \otimes t^{\ell})(x^- \otimes t^{\ell-1})^{\xi}\xi v_{\xi} = 0,
\]
the result is immediate from Corollary 6.6.  

If $k \geq \ell - 1$, then $s \geq 1 + r(\ell - 1)$ and hence  
\[(b_p)_{p \geq 0} \in S(r,s) \implies b_p > 0, \quad \text{for some } p \geq \ell.
\]
This means that  
\[x^-(r,s) = \sum_{p \geq \ell} X_p(x^- \otimes t^p)
\]
for some $X_p \in U(n^-[t])$ which gives,  
\[x^-(r,s)(x^- \otimes t^{\ell-1})^{\xi}\xi v_{\xi} = (x^- \otimes t^{\ell-1})^{\xi}\xi \left( \sum_{p \geq \ell} X_p(x^- \otimes t^p) \right) v_{\xi} = 0,
\]
or equivalently $X(r,s)(x^- \otimes t^{\ell-1})^{\xi}\xi = 0$ as required. \[\square\]

6.8. The existence of the surjective map $\varphi^+$ and the map $\varphi^-$ implies that we have  
\[\dim V(\xi) \leq \dim V(\xi^+) + \dim V(\xi^-). \quad (6.3)
\]
To prove the reverse inequality we need the following.  

**Proposition.** The assignment $v_{\xi} \to v_{\xi_1} \cdots v_{\xi_\ell}$ defines a surjective homomorphism  
\[V(\xi) \to V(\xi_1) \cdots V(\xi_\ell) \to 0.
\]
In particular,  
\[\dim V(\xi) \geq \dim V(\xi_1) \cdots \dim V(\xi_\ell) = \prod_{s=1}^{\ell} (\xi_s + 1).
\]
Assuming the proposition, the proof of parts (i) and (ii) of the Theorem is completed as follows. Assume that (i) and (ii) hold for all $1 \leq m \leq \ell - 1$. We must prove that (i) and (ii) hold for $\ell$. If $\xi$ has $\ell$ parts and $\xi_\ell = 1$, then $\xi^{\pm}$ have at most $(\ell - 1)$ parts and hence by induction we see that

$$\dim V(\xi) \geq \prod_{s=1}^{\ell}(\xi_s + 1) = \prod_{s=1}^{\ell-1}(\xi_s^{+} + 1) + \prod_{s=1}^{\ell-1}(\xi_s^{-} + 1) = \dim V(\xi^{+}) + \dim V(\xi^{-}).$$

Together with (6.3) we get

$$\dim V(\xi) = \prod_{s=1}^{\ell}(\xi_s + 1) = \dim V(\xi^{+}) + \dim V(\xi^{-}),$$

and parts (i) and (ii) are proved in this case. An obvious further induction on $\xi_\ell$ now completes the proof in the case when $\xi$ has $\ell$ parts and proves the inductive step.

6.9. We need some additional comments to prove Proposition 6.8. The following Lemma is trivially checked.

**Lemma.** Given any $f \in \mathbb{C}[t]$, we have an injective map $\psi_f : \mathfrak{sl}_2 \otimes \mathbb{C}[t] \to \mathfrak{sl}_2 \otimes \mathbb{C}[t, 1/f]$ of Lie algebras

$$\psi_f(x^+ \otimes t^p) = x^+ \otimes t^p / f, \quad \psi_f(x^- \otimes t^p) = x^- \otimes t^p f, \quad \psi(h \otimes t^p) = h \otimes t^p.$$

In particular, if $a \in \mathbb{C}$ is such that $f(a) \neq 0$, and $V$ is any representation of $\mathfrak{g}$, then $\text{ev}_a V$ is a representation of $\mathfrak{sl}_2 \otimes \mathbb{C}[t, 1/f]$, given by

$$(x \otimes g)v = g(a)xv, \quad g \in \mathbb{C}[t, 1/f], \quad x \in \mathfrak{g}.$$  

We note some consequences of the Lemma. For $r, s, k \in \mathbb{N}$, we have

$$\psi_{rk}(x^- (r, s)) = kx^- (r, s + rk).$$  

For $r, s \in \mathbb{N}$ and $f \in \mathbb{C}[t]$, we have, by using Lemma 2.3,

$$\psi_f(X(r, s)) = (-1)^s \psi_f(x^- (r, s)) + \sum_{p \geq 1} \psi_f(Y_p) H_p + \psi_f(X),$$  

for some $Y_p \in \mathbb{U}(n^- [t])[s - p], H_p \in \mathbb{U}(h[t])[p]$ and $X \in \mathbb{U}(g[t])(n^+ [t])$.

Suppose that $f \in \mathbb{C}[t]$ and $a_1, \ldots, a_m \in \mathbb{C}$, $m \in \mathbb{N}$ are such that $f(a_s) \neq 0$ for $1 \leq s \leq m$. If $V_1, \ldots, V_m$ are arbitrary modules for $\mathfrak{sl}_2$, then

$$V = \text{ev}_{a_1} V_1 \otimes \cdots \otimes \text{ev}_{a_m} V_m$$

is a module for $\mathfrak{sl}_2[t, 1/f]$. 

Proof of Proposition 6.3. Using Lemma 4.3 and Proposition 2.7 it is enough to prove that
\[ kx^-(r, s)(v_{\xi_1} \cdots v_{\xi_\ell}) = 0, \quad s, r, k \in \mathbb{N}, s + r \geq 1 + kr + \sum_{j \geq k+1} \xi_j. \]

Using (6.4) this is equivalent to proving that
\[ \psi_{tk}(x^-(r, s-rk))(v_{\xi_1} \cdots v_{\xi_\ell}) = 0, \quad s, r, k \in \mathbb{N}, s + r \geq 1 + kr + \sum_{j \geq k+1} \xi_j. \]

Let \( z_1, \ldots, z_\ell \) be the distinct parameters used to define the fusion product. For \( 1 \leq k \leq \ell \), set \( f_k = (t - z_1) \cdots (t - z_k) \). Using Lemma 4.3 and Proposition 2.7, it is enough to show that
\[
\psi_{tk}(x^-(r, s-rk))(v_{\xi_1} \cdots v_{\xi_\ell}) = \psi_{f_k}(x^-(r, s-rk))(v_{\xi_1} \cdots v_{\xi_\ell}).
\]

Since
\[
(x \otimes f_k)(V(\xi_1) \otimes \cdots \otimes V(\xi_k)) = 0, \quad x \in \mathfrak{g},
\]
and \( \psi_{f_k}(x^- \otimes t') = (x^- \otimes t') f_k \), we see that,
\[
\psi_{f_k}(x^-(r, s-rk))(v_{\xi_1} \otimes \cdots \otimes v_{\xi_\ell}) = (v_{\xi_1} \otimes \cdots \otimes v_{\xi_k}) \otimes \psi_{f_k}(x^-(r, s-rk))(v_{\xi_{k+1}} \otimes \cdots \otimes v_{\xi_\ell}).
\]

Using (6.5) and the fact that
\[
(n^+/t, 1/f_k)(v_{\xi_{k+1}} \otimes \cdots \otimes v_{\xi_\ell}) = 0,
\]
we get,
\[
((-1)^s \psi_{f_k}(x^-(r, s-rk)) + \sum_{p \geq 1} \psi_{f_k}(Y_p) H_p - \psi_{f_k}(X(r, s-rk)))(v_{\xi_{k+1}} \otimes \cdots \otimes v_{\xi_\ell}) = 0,
\]
where \( Y_p \in U(n^-[t])[s-rk-p] \) and \( H_p \in U(b[t]_+)[p] \). Now,
\[
\psi_{f_k}((x^+ \otimes t)^{s-rk}(x^- \otimes 1)^{s-r(k-1)})(v_{\xi_{k+1}} \otimes \cdots \otimes v_{\xi_\ell}) = 0,
\]
since \( s - r(k-1) \geq 1 + \sum_{j \geq k+1} \xi_j \) and so cannot be a weight of \( V(\xi_{k+1}) \otimes \cdots \otimes V(\xi_\ell) \), i.e.
\[
\psi_{f_k}(X(r, s-rk))(v_{\xi_{k+1}} \otimes \cdots \otimes v_{\xi_\ell}) = 0, \quad s + r \geq 1 + kr + \sum_{j \geq k+1} \xi_j.
\]

Further
\[
H_p(v_{\xi_{k+1}} \otimes \cdots \otimes v_{\xi_\ell}) = A_p(v_{\xi_{k+1}} \otimes \cdots \otimes v_{\xi_\ell}),
\]
for some \( A_p \in \mathbb{C} \) and we have,
\[
(-1)^s \psi_{f_k}(x^-(r, s-rk))(v_{\xi_{k+1}} \otimes \cdots \otimes v_{\xi_\ell}) = \sum_{p \geq 1} A_p \psi_{f_k}(Y_p)(v_{\xi_{k+1}} \otimes \cdots \otimes v_{\xi_\ell}).
\]

Putting all this together we find finally that
\[
\psi_{f_k}(x^-(r, s-rk))(v_{\xi_1} \otimes \cdots \otimes v_{\xi_\ell}) = (-1)^s \sum_{p \geq 1} A_p \psi_{f_k}(Y_p)(v_{\xi_1} \otimes \cdots \otimes v_{\xi_k} \otimes v_{\xi_{k+1}} \otimes \cdots \otimes v_{\xi_\ell}).
\]

Since \( \psi_{f_k}(Y_p) \in U(n^-[t])[s-p] \) and \( p \geq 1 \), we get that
\[
\psi_{f_k}(x^-(r, s-rk))(v_{\xi_1} \cdots v_{\xi_\ell}) = 0,
\]
as needed.
It remains to consider the case when \( k \geq \ell + 1 \). But here we have
\[
\psi_{f_t}(x^-(r,s))(v_{\xi_1} \otimes \cdots \otimes v_{\xi_\ell}) = 0,
\]
and the proof is complete.

6.11. We shall deduce Theorem 5(iii) from the following proposition.

**Proposition.** We have,
\[
I(\xi) = I(\xi^-; \xi_\ell) \bigcup I(\xi^+)
\]
where
\[
I(\xi^-; \xi_\ell) = \{(i_1, \cdots, i_{\ell-1}, \xi_\ell) : (i_1, \cdots, i_{\ell-1}) \in I(\xi^-)\}.
\]

**Proof.** Using the explicit formulae for \( \xi^+ \) given in Section 6.2, it is trivial to see that \( I(\xi^+) \subset I(\xi) \).

Note also that
\[
(i_1, \cdots, i_\ell) \in I(\xi^+) \implies i_\ell < \xi_\ell.
\]
To see this, recall from the definition of \( I(\xi^+) \) that the inequality
\[
i_\ell \leq \xi_\ell^- = \xi_\ell - 1,
\]
must be satisfied.

Suppose that \( \mathbf{i} = (i_1, \cdots, i_{\ell-1}, \xi_\ell) \in I(\xi^-; \xi_\ell) \), in which case, for all \( 2 \leq k \leq \ell \) and \( 1 \leq j \leq k - 1 \), we have,
\[
ji_{k-1} + (j + 1)i_k + 2(i_{k+1} + \cdots + i_{\ell-1}) \leq \xi_{\ell-1} - \xi_\ell + \sum_{p=k-j}^{\ell-2} \xi_p
\]
which is clearly equivalent to
\[
ji_{k-1} + (j + 1)i_k + 2(i_{k+1} + \cdots + i_{\ell-1} + \xi_\ell) \leq \sum_{p=k-j}^{\ell} \xi_p.
\]
If \( k = \ell + 1 \) and \( 1 \leq j \leq \ell \), then we have
\[
ji_\ell \leq \sum_{p=\ell+1-j}^{\ell} \xi_p,
\]
and hence we have proved that
\[
I(\xi^-; \xi_\ell) \subset I(\xi), \quad I(\xi^+) \cap I(\xi^-; \xi_\ell) = \emptyset.
\]
Notice also that the preceding argument also proves that
\[
\mathbf{i} \in I(\xi), \quad i_\ell = \xi_\ell \implies (i_1, \cdots, i_{\ell-1}) \in I(\xi^-),
\]
i.e.
\[
I(\xi^-; \xi_\ell) = \{(i_1, \cdots, i_\ell) \in I(\xi) : i_\ell = \xi_\ell\}.
\]
Thus, to complete the proof of the proposition we must prove,
\[
(i_1, \cdots, i_\ell) \in I(\xi), \quad i_\ell < \xi_\ell \implies \mathbf{i} \in I(\xi^+).
\] (6.6)
If $2 \leq k \leq \ell + 1$ and $1 \leq j \leq k - 1$ are such that $k - j \leq \ell(\xi) + 1$, then the inequalities in $\mathbb{I}(\xi)$ and $\mathbb{I}(\xi^+)$ are the same and there is nothing to prove. If $k - j > \ell(\xi) + 1$, then we must prove that

$$j\ell_{k-1} + (j + 1)i_k + 2(i_{k+1} + \cdots + i_\ell) < (\ell - k + j)\xi_{\ell-1} + \xi_\ell.$$  

(6.7)

In other words, we must prove that if $\ell(\xi) + 3 \leq k \leq \ell + 1$ and $1 \leq j < k - \ell(\xi) - 1$, then

$$j\ell_{k-1} + (j + 1)i_k + 2(i_{k+1} + \cdots + i_\ell) < (\ell - k + j)\xi_{\ell-1} + \xi_\ell.$$  

(6.8)

We prove this by a downward induction on $k$. If $k = \ell + 1$ and $1 \leq j < \ell - \ell(\xi)$, then we have

$$ji_\ell < j\xi_\ell \leq (j - 1)\xi_{\ell-1} + \xi_\ell,$$

and hence induction begins.

Assume now that we have proved the result for all $r > k$. To prove the result for $k$, we proceed by an induction on $j$. Taking $j = 1$ and $r = k + 1$, we have,

$$i_k + 2(i_{k+1} + \cdots + i_\ell) < (\ell - k)\xi_{\ell-1} + \xi_\ell.$$  

(6.9)

Taking $j = 2$ and $r = k$, (notice that this is allowed since $k - \ell(\xi) \geq 3$, i.e., $k \geq 3$), we have,

$$2i_{k-1} + 3i_k + 2(i_{k+1} + \cdots + i_\ell) \leq \sum_{p=k-2}^\ell \xi_p = (\ell - k + 2)\xi_{\ell-1} + \xi_\ell.$$  

(6.10)

Adding (6.9) and (6.10) gives,

$$2(i_{k-1} + 2i_k + 2(i_{k+1} + \cdots + i_\ell)) < 2(\ell - k + 1)\xi_{\ell-1} + 2\xi_\ell,$$

i.e.,

$$i_{k-1} + 2i_k + 2(i_{k+1} + \cdots + i_\ell) < (\ell - k + 1)\xi_{\ell-1} + \xi_\ell.$$  

This shows that (6.8) holds for $j = 1$ and $k$. Assume that we have proved the result for all $1 \leq j' < j < k - \ell(\xi) - 1$. Then taking $j' = j - 1$ we have,

$$(j - 1)i_{k-1} + j i_k + 2(i_{k+1} + \cdots + i_\ell) < (\ell - k + j - 1)\xi_{\ell-1} + \xi_\ell.$$  

(6.11)

On the other hand, since $2 \leq j < k - \ell(\xi) - 1 \leq k - 1$, we have $j + 1 \leq k - 1$ and so, we have the inequality

$$(j + 1)i_{k-1} + (j + 2)i_k + 2(i_{k+1} + \cdots + i_\ell) \leq (\ell - k + j + 1)\xi_{\ell-1} + \xi_\ell.$$  

(6.12)

Adding equations (6.11) and (6.12) gives,

$$2(ji_{k-1} + (j + 1)i_k + 2(i_{k+1} + \cdots + i_\ell)) < 2(\ell - k + j)\xi_{\ell-1} + 2\xi_\ell,$$

which proves the inductive step for $j$ and completes the proof. □
6.12. The proof of Theorem 5(iii) is completed as follows. We proceed by induction on $\ell$. If $\ell = 1$ then $V(\xi) \cong ev_0 V(\xi_1)$. It is an elementary result in the representation theory of $\mathfrak{sl}_2$ that $\{ (x^{-})^{i_\xi} : 0 \leq i \leq \xi_1 \}$ is a basis for $ev_0 V(\xi_1)$. For the inductive step, note that if $(i_1, \ldots, i_{\ell-1}) \in I(\xi^-)$,
\[ \varphi^-( (x^- \otimes 1)^{i_1} \cdots (x^- \otimes t^{\ell-2})^{i_{\ell-1}} v_{\xi^-} ) = (x^- \otimes 1)^{i_1} \cdots (x^- \otimes t^{\ell-2})^{i_{\ell-1}} v_{\xi^-}. \]
Since $\varphi^-$ is injective, it follows from the inductive hypothesis that the elements
\[ \left\{ (x^- \otimes 1)^{i_1} \cdots (x^- \otimes t^{\ell-2})^{i_{\ell-1}} v_{\xi^-} : (i_1, \ldots, i_{\ell-1}, \xi_{\ell}) \in I(\xi^-; \xi_{\ell}) \right\}, \]
are a basis of the image of $\varphi^-$. If $(i_1, \ldots, i_{\ell}) \in I(\xi)$, with $i_{\ell} < \xi_{\ell}$, then by the inductive hypothesis, the image under $\varphi^+$ of the elements
\[ \left\{ (x^- \otimes 1)^{i_1} \cdots (x^- \otimes t^{\ell-2})^{i_{\ell-1}} v_{\xi} : (i_1, \ldots, i_{\ell-1}, i_{\ell}) \in I(\xi), \ i_{\ell} < \xi_{\ell} \right\}, \]
is a basis for $V(\xi^+)$. Theorem 5(iii) is now immediate from Theorem 5(i) and Proposition 6.11.

References

[1] E. Ardonne and R. Kedem, Fusion products of Kirillov-Reshetikhin modules and fermionic multiplicity formulas, J. Algebra, 308, (2007), 270-294.
[2] J. Beck and H. Nakajima, Crystal bases and two-sided cells of quantum affine algebras, Duke Math. J. 123 (2004), no. 2, 335-402.
[3] M. Bennett, A. Berenstein, V. Chari, A. Khoroshkin and S. Loktev, BGG reciprocity for the current algebra of $\mathfrak{sl}_{n+1}$, in preparation.
[4] M. Bennett, V. Chari, and N. Manning, BGG reciprocity for current algebras, Advances in Mathematics, 231, no. 1, 276-305.
[5] V. Chari, On the fermionic formula and the Kirillov-Reshetikhin conjecture, Internat. Math. Res. Notices (2001), no. 12, 629–654.
[6] V. Chari, Braid group actions and tensor products, Int. Math. Res. Not. 2003 (2002), no. 7, 357–382.
[7] V. Chari, G. Fourier and T. Khandai, A categorical approach to Weyl modules, Transform. Groups 15 (2010), no. 3, 517–549.
[8] V. Chari and D. Hernandez, Beyond Kirillov-Reshetikhin modules, Contemp. Math. 506 (2010), 49–81.
[9] V. Chari and S. Loktev, Weyl, Demazure and fusion modules for the current algebra of $\mathfrak{sl}_{n+1}$, Adv. Math. 207 (2006), 928–960.
[10] V. Chari and A. Moura, The restricted Kirillov-Reshetikhin modules for the current and twisted current algebras, Comm. Math. Phys. 266 (2006), no. 2, 431–454.
[11] V. Chari and A. Pressley, Weyl modules for classical and quantum affine algebras, Represent. Theory 5 (2001), 191–223.
[12] P. Di Francesco, and R. Kedem, Proof of the combinatorial Kirillov-Reshetikhin conjecture, Int. Math. Res. Not. IMRN 2008, no. 7, Art. ID rnm006, 57 pp.
[13] B. L. Feigin and E. Feigin, $q$-characters of the tensor products in $\mathfrak{sl}_2$-case, Mosc. Math. J. 2 (2002), no. 3, 567–588, math.QA/0201111.
[14] B. Feigin and E. Feigin, Schubert varieties and the fusion products. Publ. Res. Inst. Math. Sci. 40 (2004), no. 3, 625–668.
[15] B. Feigin and S. Loktev, On Generalized Kostka Polynomials and the Quantum Verlinde Rule, Differential topology, infinite-dimensional Lie algebras, and applications, Amer. Math. Soc. Transl. Ser. 2, Vol. 194 (1999), p. 61–79, math.QA/9812093.
[16] B. Feigin, A.N. Kirillov and S. Loktev, Combinatorics and geometry of higher level Weyl modules. Moscow Seminar on Mathematical Physics. II, 3347, Amer. Math. Soc. Transl. Ser. 2, 221, Amer. Math. Soc., Providence, RI, 2007.

[17] E. Feigin, Schubert varieties and the fusion products: the general case, Int. Math. Res. Not. 2004, no. 59, 3153–3175.

[18] E. Feigin, Infinite fusion products and $\widehat{s}_1\hat{s}_2$ cosets, J. Lie Theory 17 (2007), no. 1, 145–161.

[19] G. Fourier and P. Littelmann, Tensor product structure of affine Demazure modules and limit constructions, Nagoya Math. J. 182 (2006), 171–198.

[20] G. Fourier and P. Littelmann, Weyl modules, Demazure modules, KR-modules, crystals, fusion products and limit constructions, Adv. Math. 211 (2007), no. 2, 566–593.

[21] H. Garland, The arithmetic theory of loop algebras, J. Algebra 53 (1978), 480–551.

[22] G. Hatayama, A. Kuniba, M. Okado, T. Takagi and Y. Yamada, Remarks on fermionic formula, in Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998), 243–291, Contemp. Math., 248, Amer. Math. Soc., Providence, RI (1999).

[23] D. Hernandez, The Kirillov-Reshetikhin conjecture and solutions of T-systems, J. Reine Angew. Math. 596, (2006), 63–87.

[24] R. Kedem, Q-systems as cluster algebras, J. Phys. A 41 (2008), no. 19, 194011, 14 pp.

[25] R. Kedem, A pentagon of identities, graded tensor products, and the Kirillov-Reshetikhin conjecture, New trends in quantum integrable systems, 173–193, World Sci. Publ., Hackensack, NJ, 2011.

[26] R. Kodera and K. Naoi, Locuy series of Weyl modules and the Poincare polynomials of quiver varieties, Publ. Res. Inst. Math. Sci. 48 (2012), no. 3, 477–500.

[27] A. Kuniba, K. C. Misra, M. Okado, T. Takagi and J. Uchiyama, Paths, Demazure Crystals and Symmetric Functions, J. Math. Phys. 41 (2000), no. 9, 6477–6486.

[28] P. Magyar, Littelmann Paths for the Basic Representations of an Affine Lie Algebra, J. Algebra 305 (2006), no. 2, 1037–1054.

[29] H. Nakajima, t-analogs of q-characters of Kirillov-Reshetikhin modules of quantum affine algebras, Represent. Theory 7, (electronic) (2003), 259274.

[30] H. Nakajima, Quiver Varieties and t-Analogs of q-Characters of Quantum Affine Algebras, Ann. of Math. 160, (2004), 1057 - 1097.

[31] K. Naoi, Weyl modules, Demazure modules and finite crystals for non-simply laced type, Adv. Math. 229 (2012), no. 2, 875–934.

[32] K. Naoi, Fusion products of Kirillov-Reshetikhin modules and the $X = M$ conjecture, Adv. Math. 231 (2012), no. 3–4, 1546–1571.

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