Classification of the phases of 1D spin chains with commuting Hamiltonians

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Abstract
We consider the class of spin Hamiltonians on a 1D chain with periodic boundary conditions that are (i) translational invariant, (ii) commuting and (iii) scale invariant, where by the latter we mean that the ground state degeneracy is independent of the system size. We correspond a directed graph to a Hamiltonian of this form and show that the structure of its ground space can be read from the cycles of the graph. We show that the ground state degeneracy is the only parameter that distinguishes the phases of these Hamiltonians. Our main tool in this paper is the idea of Bravyi and Vyalyi (2005 Quantum Inform. Comput. 5 187–215) in using the representation theory of finite-dimensional C*-algebras to study commuting Hamiltonians.

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1. Introduction
Classification of the phases of matter is a major problem at the heart of recent activities in quantum many-body physics research, especially after the discovery of topologically ordered phases. The quantum double model of Kitaev [1] and the string-net condensation model of Levin and Wen [2] show that the phase diagram of 2D systems can be very complicated. These two models of commuting Hamiltonians are exactly solvable and exhibit a rich structure in their ground states as well as elementary excitations. In 1D, however, we are not aware of any essential example of a commuting Hamiltonian besides the Ising model. Moreover, simulation of the ground states based on tensor networks has been proven to be efficient in 1D gapped systems [3]. So we expect to have an easier theory in 1D. Before reviewing known results in this regard, let us first describe what we mean exactly by a phase.

Hamiltonians of interest are translational-invariant spin Hamiltonians defined on a lattice with periodic boundary conditions. We assume that Hamiltonians are scale invariant, meaning that the ground state degeneracy is independent of the system size, and also have a constant gap above the ground state energy, which again is independent of the size of the system.
say that two such Hamiltonians (their associated ground spaces) belong to the same phase if there exists a continuous path of Hamiltonians satisfying the above conditions starting with one of them and ending with the other.

Chen et al [4] and Schuch et al [5] have recently shown that assuming the ground space of such a 1D Hamiltonian is exactly describable by matrix product states (MPSs) [6], the only parameter that distinguishes phases is the ground state degeneracy\(^1\). Although it seems that we can remove the assumption in this result by using the proven area law in 1D [3], there are some technical obstacles. First, we are able to approximate the ground state of any gapped 1D Hamiltonian by an MPS, whose bond dimension scales (polynomially) with the system size. In [4] and [5] however, the bond dimension is assumed to be fixed independent of the system size. Second, we need to show that the true ground state and the approximate MPS are in the same phase, which again should be proved with a gap that is independent of the system size.

Yoshida [7] considered the problem of classification of phases in the special case of stabilizer Hamiltonians. He showed that any translational-invariant and scale-invariant stabilizer Hamiltonian in 1D is equivalent to some independent copies of the Ising model, and in 2D we essentially have the 2D Ising model and the toric code. He proved a similar result in 3D assuming some extra conditions [8]. Although stabilizer Hamiltonians seem very restrictive, they have been extensively studied in the theory of quantum error correcting codes and quantum memories because of their simple structure of syndrome measurements. The advantage of these results, comparing to [4, 5], is that Yoshida’s classification gives a characterization of logical operators, and consequently low-energy excitations as well. We emphasize on the latter because, for instance, in the models of quantum double and string-net condensation, the structures of elementary excitations are much richer than those of ground states.

In this paper, we generalize Yoshida’s results in 1D. We consider frustration-free commuting Hamiltonians that are both translational and scale invariant. Due to the commutativity assumption, these Hamiltonians are automatically gapped. We associate a graph with a Hamiltonian of this form and show that cycles of this graph encode the ground space of the Hamiltonian. We also comment that the low-energy excited states are described by paths of the graph. Our main result is that the phases of the ground states of these systems are determined only by their degeneracy.

Although our assumption of commutativity, comparing to that of [4, 5], seems very restrictive, this at least can be easily verified. Even the scale invariance of Hamiltonians is hard to check in general, but for commuting Hamiltonians we provide a simple criterion to verify that. In the last section, we show that our results can be applied to a wider class of Hamiltonians than commuting ones.

### 2. Some key observations

Let us first fix some notations. Hilbert spaces are denoted by \(\mathcal{H}\) and \(\mathcal{H}_A\) is the Hilbert space corresponding to register \(A\). \(I_A\) is the identity operator of \(\mathcal{H}_A\) and \(\mathcal{L}(\mathcal{H}_A)\) is the space of linear operators acting on \(\mathcal{H}_A\). We may consider \(\mathcal{L}(\mathcal{H}_A)\) as a Hilbert space equipped with the Hilbert–Schmidt inner product.

In this paper, we are interested in the phases of ground spaces of Hamiltonians defined on a 1D chain with periodic boundary conditions. We say that the ground spaces of two Hamiltonians belong to the same phase if there exists a continuous path of gapped Hamiltonians starting with one of them and ending with the other. Here we often consider translational- and

\(^1\) 1D systems with symmetries have also been studied in these papers.
Local unitaries also do not change the phase of a ground space. That is, the Hamiltonian's energy is zero. Let assume that the local terms of the Hamiltonian are positive semidefinite and the ground space acts nontrivially only on for simplification. By this definition when two Hamiltonians $H$ and $H'$ have the same ground spaces, they belong to the same phase. In fact, the linear interpolation $tH + (1-t)H'$, $0 \leq t \leq 1$, is the path between them and one can easily show that these Hamiltonians for all $t$ are gapped. Local unitaries also do not change the phase of a ground space. That is, the Hamiltonians $H = \sum_j h_{j,j+1}$ and $H' = \sum_j U h_{j,j+1} U^\dagger$, for unitary $U$, belong to the same phase and the path $H(t) = \sum_j e^{itX} h_{j,j+1} e^{-itX}, 0 \leq t \leq 1$, where $U = e^{itX}$, connects the two Hamiltonians. Note that translational and scale invariance are preserved in these two interpolations. We may also embed the Hilbert space of local spins into larger Hilbert spaces (and modify the local terms if and only if $H'$ is gapped. Moreover, $H$ and $H'$ have exactly the same ground spaces and by the above observation belong to the same phase. This means that without loss of generality we may assume that the local terms of the Hamiltonians are projections. Note that if $H$ is commuting, then $H'$ is commuting as well. This is because $P_{j,j+1}$ belongs to the algebra generated by $h_{j,j+1}$ and the identity operator.

The following lemma due to Bravyi and Vyalyi [9] is a key step toward understanding the ground space of commuting Hamiltonians. For the sake of completeness, we also include a sketch of the proof.

**Lemma 2.1.** Suppose that $X_{AB}$ and $Y_{BC}$ are two Hermitian operators acting on the finite-dimensional Hilbert spaces $\mathcal{H}_A \otimes \mathcal{H}_B$ and $\mathcal{H}_B \otimes \mathcal{H}_C$, respectively. Assume that $X_{AB} \otimes I_C$ and $I_A \otimes Y_{BC}$ commute. Then there exist an index set $V$ and decomposition

$$\mathcal{H}_B \cong \bigoplus_{\beta \in V} \mathcal{H}_{\beta_A} \otimes \mathcal{H}_{\beta_B},$$

such that $X_{AB}$ acts nontrivially only on $\mathcal{H}_{\beta_A}$ and not on $\mathcal{H}_{\beta_B}$ (for all $\beta \in V$), and similarly $Y_{BC}$ acts nontrivially only on $\mathcal{H}_{\beta_B}$. More precisely, if we let $\Pi_{\beta}$ be the orthogonal projection onto the subspace $\mathcal{H}_{\beta_A} \otimes \mathcal{H}_{\beta_B} \subset \mathcal{H}_B$, then

$$X_{AB} = \sum_{\beta} (I_A \otimes \Pi_{\beta}) X_{AB} (I_A \otimes \Pi_{\beta}) = \sum_{\beta} R_{\beta_A} \otimes I_{\beta_B},$$

$$Y_{BC} = \sum_{\beta} (\Pi_{\beta} \otimes I_C) Y_{BC} (\Pi_{\beta} \otimes I_C) = \sum_{\beta} I_{\beta_A} \otimes S_{\beta_B C},$$

for some Hermitian operators $R_{\beta_A}$ and $S_{\beta_B C}$ acting on $\mathcal{H}_A \otimes \mathcal{H}_{\beta_A}$ and $\mathcal{H}_{\beta_B} \otimes \mathcal{H}_{C}$, respectively.

**Sketch of proof.** Using the Schmidt decomposition of $X_{AB}$ and $Y_{BC}$ as vectors in the bipartite Hilbert spaces $L(\mathcal{H}_A) \otimes L(\mathcal{H}_B)$ and $L(\mathcal{H}_B) \otimes L(\mathcal{H}_C)$, and writing down the constraint $[X_{AB} \otimes I_C, I_A \otimes Y_{BC}] = 0$, we find that it suffices to prove the following statement.

\[\rho = \alpha \gamma, \quad \alpha, \gamma \neq 0, \quad \alpha, \gamma \in \mathbb{C}, \quad \alpha \cdot \gamma = \alpha^\dagger \cdot \gamma^\dagger.\]
Let $S$ and $S'$ be two subsets of Hermitian matrices acting on $\mathcal{H}_S$ such that all operators in $S$ commute with elements of $S'$. Then there exists a decomposition (1) such that elements of $S$ ($S'$) act trivially on $\mathcal{H}_{S'}$ ($\mathcal{H}_S$).

We need to prove the above statement. Let $\mathcal{Z}(S)$ be the space of all matrices that commute with elements of $S$. Since elements of $S$ are Hermitian, $\mathcal{Z}(S)$ is a $C^*$-algebra, and it contains $S'$. Using the representation theory of finite-dimensional $C^*$-algebras [10], we find that there exists a decomposition (1) such that

$$\mathcal{Z}(S) \cong \bigoplus_{\beta \in V} \mathbb{1}_\beta \otimes \mathcal{L}(\mathcal{H}_{\beta}).$$

Thus, elements of $S'$ have the desired form. Furthermore, it is not hard to see that a matrix commuting with such $\mathcal{Z}(S)$ has to act trivially on $\mathcal{H}_{\beta}$.

3. Graphs encode ground states of 1D commuting Hamiltonians

Consider a 1D chain of $N$ spins of finite dimension $d$ (qudits) with periodic boundary conditions. We let the Hamiltonian be $H_N = \sum_{j=1}^N P_{j,j+1}$, where $P_{j,j+1}$ acts only on sites $j, j+1$ (hereafter we adopt indices to be modulo $N$). By the observation of the previous section, since we let $H_N$ to be frustration free, we further assume that $P_{j,j+1}$s are projections. The extra assumptions that we impose in this section are that $H_N$ is translational invariant and $H_N$ is commuting.

According to lemma 2.1 since $P_{j-1,j}$ and $P_{j,j+1}$ commute, there exists a decomposition

$$C^d \cong \bigoplus_{\alpha \in V} \mathcal{H}_{\alpha,j} \otimes \mathcal{H}_{\alpha,j+1}$$

(2)

of the Hilbert space of the $j$th qudit such that $P_{j-1,j}$ ($P_{j,j+1}$) acts trivially on $\mathcal{H}_{\alpha,j}$ ($\mathcal{H}_{\alpha,j+1}$). Using the fact that $P_{j,j+1}$s are translations of each other, we may consider the same decomposition at site $j+1$. As a result, $P_{j,j+1}$ acts trivially on $\mathcal{H}_{\alpha,j+1}$ as well. Therefore, $P_{j,j+1}$ can be written as

$$P_{j,j+1} = \sum_{\alpha, \beta \in V} \frac{1}{\mathcal{H}_{\alpha,j} \otimes \mathcal{H}_{\beta,j+1}},$$

(3)

where $\mathcal{H}_{\alpha,j} \otimes \mathcal{H}_{\beta,j+1}$ acts on $\mathcal{Z}(S)$ and is a projection. Since $P_{j,j+1}$s are shifts of each other we may ignore the index $j$ and represent $Q_{\alpha,j+1}$ by $Q_{\alpha,\beta}$. The combined Hilbert space of all $N$ qudits then can be written as

$$C^d = \bigoplus_{\alpha \in V} \mathcal{H}_{\alpha,i} \otimes \mathcal{H}_{\alpha,i+1}.$$

(4)

We correspond a directed graph $G$ to our Hamiltonian. The vertex set of $G$ is the index set $V$ of decomposition (2). A directed edge is drawn from $\alpha \in V$ to $\beta \in V$ if $\ker Q_{\alpha,\beta} \subseteq \mathcal{H}_{\alpha} \otimes \mathcal{H}_{\beta}$ is non-zero. Note that this graph may contain loops, i.e. edges from a vertex to itself. Moreover, $G$ is defined based on the local terms $P_{i,i+1}$ and is independent of the system size $N$.

$G$ encodes the ground space of the Hamiltonian as follows. Let $(\alpha^1, \ldots, \alpha^N)$ be a directed cycle of length $N$ in $G$, which may contain a vertex or edge several times. Since $\alpha^i \to \alpha^{i+1}$ is an edge of $G$, there is a non-zero vector $|\psi_j\rangle$ in $\ker Q_{\alpha^j,\alpha^{j+1}}$ for $j = 1, \ldots, N$. By decomposition (4), $|\psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_N\rangle$ is a state in the Hilbert space of $N$ spins. Then using (3), we have
are directed edges from vertices in
represent each site with two spin-half particles. Define corresponds to the
entangled state to
Here we must consider ordered cycles because we have the freedom of choosing which vertex
translational invariant, instead of a single index set
Note that the translational invariance assumption in this result is not crucial. If
As a summary we conclude that the ground space of \( H_N \), for all \( N \), is completely
Similarly to the ground states that correspond to cycles of \( G \), it is easy to see that excited
Note that the translational invariance assumption in this result is not crucial. If \( H_N \) is not
An implication of our result is that the ground states of commuting Hamiltonians can be
represented by MPSs. An MPS representation consists of a chain of maximally entangled
states on which we apply some local linear transformations [11] (see figure 1). To write the
state \( |\psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_N\rangle \) defined above as an MPS, we need only to map the
th maximally entangled state to \( |\psi_j\rangle \). Thus, \( |\psi\rangle \) is an MPS whose bond dimension is a constant independent
of \( N \).
Let us work out an example to clarify the ideas. Let \( d = 4 \), and using \( \mathbb{C}^4 \cong \mathbb{C}^2 \otimes \mathbb{C}^2 \) represent each site with two spin-half particles. Define
\[
P_{j,j+1} = (|1\rangle \otimes |j+1\rangle - (\sigma_x \otimes \sigma_x)_j \otimes (\sigma_z \otimes \sigma_z)_{j+1})/2,
\]
where \( \sigma_x \) and \( \sigma_z \) are Pauli matrices. Then \( H_N = \sum_j P_{j,j+1} \) is commuting and translational
invariant. The index set \( V \) associated with \( H_N \) has four elements and components of decomposition (2) are
\[
\mathcal{H}_{\alpha_0} \otimes \mathcal{H}_{\alpha_0} = \mathbb{C} (|00\rangle + |11\rangle), \\
\mathcal{H}_{\beta_0} \otimes \mathcal{H}_{\beta_0} = \mathbb{C} (|01\rangle - |10\rangle), \\
\mathcal{H}_N \otimes \mathcal{H}_N = \mathbb{C} (|00\rangle - |11\rangle), \\
\mathcal{H}_{\beta_0} \otimes \mathcal{H}_{\beta_0} = \mathbb{C} (|01\rangle + |10\rangle).
\]
Figure 1. An $N$-qudit MPS (with periodic boundary conditions and) with bond dimension $\chi$ is obtained by a chain of $N$ maximally entangled states $|\Phi\rangle$ of local dimension $\chi$ followed by the local maps $S_j: C^\chi \otimes C^\chi \rightarrow C^d$. Then the final state equals $(S_1 \otimes \cdots \otimes S_N)|\Phi\rangle \otimes \cdots \otimes |\Phi\rangle$. For translational-invariant MPSs, we have $S_1 = S_2 = \cdots = S_N$. From this representation, it is clear that if $|\varphi\rangle$ is an MPS, then $(X_1 \otimes \cdots \otimes X_N)|\varphi\rangle$ is also an MPS with the same bond dimension. Moreover, the maximally entangled states $|\Phi\rangle$ can be replaced with any bipartite state with local dimension $\chi$ and we still obtain an MPS with bond dimension $\chi$.

Figure 2. The graph associated with the local projection $P_{j,j+1} = (1_j \otimes 1_{j+1} - (\sigma_x \otimes \sigma_x)_{j} \otimes (\sigma_z \otimes \sigma_z)_{j+1})/2$ and Hamiltonian $H_N = \sum_{j} P_{j,j+1}$. This Hamiltonian is commuting and translational invariant but not scale invariant.

Note that in this case since these subspaces are all one dimensional, we do not need to specify the subsystems $H_{\alpha}$, $H_{\alpha_1}$, $\ldots$ individually. The corresponding graph $G$ has four vertices and is depicted in figure 2. For $N = 3$, for instance, $(\alpha, \gamma, \theta)$ is a cycle of length 3 and then $((00) \oplus (11)) \otimes ((00) - (11)) \otimes ((01) + (10))$ is a ground state. However, $(\alpha, \gamma, \beta)$ is a path which cannot be completed to a cycle and then $((00) + (11)) \otimes ((00) - (11)) \otimes ((01) - (10))$ is an excited state with energy 1.

4. Scale invariance

We now apply the assumption that the system is scale invariant, meaning that for sufficiently large $N$, the ground state degeneracy is a constant independent of $N$. So we let $H_N$ be commuting, and translational and scale invariant. Our goal in this section is to show that for every $N$,

$$\dim \ker H_N = |C_1|,$$

where $C_1$ is the set of loops of $G$.

Let $D_1 \in C_N$ be a cycle of $G$. By going around $D_1$ many times, we may assume with no loss of generality that $N$, the length of this cycle, is sufficiently large. Let $C_N = \{D_1, \ldots, D_k\}$ and $C_{N+1} = \{E_1, \ldots, E_k\}$. We have

$$\dim \ker H_N = \sum_{i=1}^{k} \prod_{a \rightarrow \beta \in D_i} \dim \ker Q_{a,\beta} = \dim \ker H_N.$$
The scale-invariance assumption implies that all cycles of immediate consequence of (10).

Now applying the scale-invariance assumption, we have \( \dim \ker H_{N+1} = \dim \ker H_N = \dim \ker H_{N+1} \). So by comparing equations (6)–(9), we obtain

\[
\dim \ker H_{N+1} = \sum_{i=1}^{k'} \prod_{a \to \beta \in E_i} \left( \dim \ker Q_{a, \beta} \right)^{N+1},
\]

(8)

\[
\dim \ker H_{N+1} = \sum_{i=1}^{k'} \prod_{a \to \beta \in E_i} \left( \dim \ker Q_{a, \beta} \right)^{N}.
\]

(9)

for every edge \( a \to \beta \) in one of the cycles \( D_i \) or \( E_i \). (Note that when \( a \to \beta \) is an edge of \( G \), \( \dim \ker Q_{a, \beta} \) is non-zero.) We further conclude that \( k = k' = \dim \ker H_{N+1} \) and indeed

\[
C_{N+1} = \{ D_1^{N+1}, \ldots, D_k^{N+1} \} = \{ E_1^N, \ldots, E_k^N \}.
\]

Thus, for some \( i, D_i^{N+1} = E_i^N \). By the fact that \( N \) and \( N + 1 \) are relatively prime, we find that the arbitrarily chosen cycle \( D_i \) consists of the repetition of a single loop. We conclude that the scale-invariance assumption implies that all cycles of \( G \) are essentially loops. Then (5) is an immediate consequence of (10).

The structure of the ground space of \( H_N \) is even simpler with the scale-invariance assumption. Let \( a \to a \) be a loop of \( G \), and let \( |\psi_a\rangle \in \mathcal{H}_a \otimes \mathcal{H}_a \) be a vector that spans \( \ker Q_{a,a} \). Then \( |\psi_a\rangle^{\otimes N} \) is a ground state, and the ground space of \( H_N \) is spanned by these vectors for all loops \( a \to a \) of \( G \). The vectors \( |\psi_a\rangle^{\otimes N} \) are still MPSs and in fact translational-invariant MPSs.

The example of figure 2 is not scale invariant because the graph contains cycles that do not come from loops. For the special case of the 1D Ising model, the graph \( G \) consists of two vertices with two loops and no other edges. From these two loops and the above construction, we obtain all spin-up and all spin-down states as the ground states of the Ising model.

Elementary excitations are also easier to characterize with the scale-invariance assumption. As mentioned in the previous section, elementary excitations correspond to paths of \( G \), and excitations with energy 2 come from unions of two paths. A sufficiently long path in a finite graph must contain a cycle. Since loops are the only cycles of such \( G \), any elementary excitation of \( H_N \), for sufficiently large \( N \), contains several copies of states \( |\psi_a\rangle \), for loops \( a \to a \), in their subsystems. In fact an energy 1 eigenstate corresponds to a path of the form either \( a \to a \cdots \to a \to \beta \) or \( \beta \to a \to a \cdots a \). For example, the Ising model does not have any energy 1 eigenstate because its corresponding graph is a union of two loops and there is no edge of the form \( a \to \beta \) for \( a \neq \beta \).

5. Phases are distinguished by the ground state degeneracy

We are now ready to study the phases of 1D spin chains with commuting Hamiltonians. Objects of interest are ground spaces of 1D commuting Hamiltonians that are both translational and
scale invariant. By the discussion of section 2 without loss of generality we can also assume that the Hamiltonian is a summation of local projections. Thus, we may use results of the previous two sections, which are summarized as

\[ \ker H_N = \bigoplus_{a \rightarrow r \in C_1} (\ker Q_a, a)^\otimes \overline{N}, \]

(11)

where \( \ker Q_a, a \) is one dimensional and is spanned by \( |\psi_a\rangle \).

Here since we are interested only in the ground spaces, we can replace the other \( Q_a, \beta \)'s that do not appear in the above expression with the identity operator. More precisely, for every \( \alpha \neq \beta \), or \( \alpha = \beta \) where \( \alpha \rightarrow \alpha \) is not a loop, define \( \tilde{Q}_a, \beta = I_{a} \otimes I_{\beta} \), and for loops \( \alpha \rightarrow \alpha \in C_1 \) let \( \tilde{Q}_a, a = Q_{a, a} \). Furthermore, define

\[ \tilde{P}_{j, j+1} = \sum_{a, \beta \in V} 1_a \otimes \tilde{Q}_a, \beta \otimes I_{\beta^{j+1}}, \]

and \( \tilde{H}_N = \sum_{j} \tilde{P}_{j, j+1} \). Then \( \tilde{H}_N \) is still commuting and translational invariant. Moreover, its corresponding graph consists of the same loops as \( G \), but no other edges. In fact, using (11), \( \tilde{H}_N \) has the same ground space as \( H_N \) and they are in the same phase. So we may replace \( H_N \) with \( \tilde{H}_N \), i.e. we assume that \( \tilde{P}_{j, j+1} = P_{j, j+1} \) and \( H_N = \tilde{H}_N \).

For every loop \( \alpha \rightarrow \alpha \), fix the arbitrary states \( |\xi_\alpha\rangle \in \mathcal{H}_{a_\alpha} \) and \( |\xi_{\gamma}\rangle \in \mathcal{H}_{a_\gamma} \). Let \( U_\alpha \) be a unitary operator that acts on \( \mathcal{H}_{a_\alpha} \otimes \mathcal{H}_{a_\gamma} \) in such a way that \( U_\alpha |\psi_\alpha\rangle = |\xi_\alpha\rangle \otimes |\xi_{\gamma}\rangle \). Recall that the Hilbert space of two qudits \( j \) and \( j+1 \) can be decomposed as

\[ \bigoplus_{|\beta, y\rangle \in V} \mathcal{H}_{a_{\beta}^{j}} \otimes \mathcal{H}_{a_{\gamma}^{j+1}} \otimes \mathcal{H}_{y^{j+1}}. \]

(12)

Thus, we can define the two-qudit unitary operator \( U \) that acts as \( U_\alpha \) on the subsystem/subspace \( \mathcal{H}_{a_{\alpha}^{j}} \otimes \mathcal{H}_{a_{\gamma}^{j+1}} \) when \( \alpha \rightarrow \alpha \) is a loop, and acts as the identity operator elsewhere.

Define \( P_{j, j+1} = U P_{j, j+1} U^\dagger \) and \( H'_N = \sum_{j} P_{j, j+1} \). Since \( U \) is block-diagonal with respect to decomposition (12) and acts trivially on \( \mathcal{H}_{a_{\beta}^{j}} \) and \( \mathcal{H}_{y^{j+1}} \), the new Hamiltonian \( H'_N \) is commuting. The corresponding graph of \( H'_N \) is the same as that of \( H_N \) and consists of a union of loops. The only difference is that the state corresponding to the loop \( \alpha \rightarrow \alpha \) is equal to \( U |\psi_\alpha\rangle = |\xi_\alpha\rangle \otimes |\xi_{\gamma}\rangle \).

The two Hamiltonians \( H_N \) and \( H'_N \) belong to the same phase because they differ only by local unitaries (see section 2). So again without loss of generality, we assume \( P_{j, j+1} = P_{j, j+1} \) and \( H_N = H'_N \). In this step we turned \( |\psi_\alpha\rangle \), which belongs to \( \mathcal{H}_{a_\alpha} \otimes \mathcal{H}_{a_\gamma} \), into a product state.

By (11), the ground space of \( H_N \) is spanned by vectors,

\[ \bigotimes_{j=1}^N |\xi_{a_j}\rangle \otimes |\xi_{a^{j+1}}\rangle = \bigotimes_{j=1}^N |\xi_{a_j}\rangle \otimes |\xi_{a_j}\rangle = |\alpha\rangle_1 \otimes \cdots \otimes |\alpha\rangle_N, \]

for loops \( \alpha \rightarrow \alpha \), where we set \( |\alpha\rangle_j = |\xi_{a_j}\rangle \otimes |\xi_{a_j}\rangle \). If we let \( \hat{H}_N = \sum_{j} \tilde{P}_{j, j+1} \) such that

\[ \hat{P}_{j, j+1} = \mathbb{1} \otimes \mathbb{1} \otimes |0\rangle_0 \otimes |0\rangle_0 + |1\rangle_1 \otimes |1\rangle_1. \]

then \( \hat{H}_N \) is commuting and translational invariant, and has the same ground space as \( H_N \). Then they belong to the same phase. Moreover, since \( |\xi_{a_\alpha}\rangle \) and \( |\xi_{a_\gamma}\rangle \) were arbitrarily chosen, the only parameter that determines \( \hat{H}_N \) is the size of \( C_1 \), i.e. the ground space degeneracy. For example, for two loops we obtain the Ising model whose local projections are given by

\[ P = \mathbb{1} \otimes \mathbb{1} - (|0\rangle_0 \otimes |0\rangle_0 + |1\rangle_1 \otimes |1\rangle_1). \]

We conclude that the phases of ground spaces of translational- and scale-invariant commuting Hamiltonians in 1D are characterized by their degeneracy.
6. Summary and outlook

In this paper, we described the structure of the ground states of translational- and scale-invariant, 1D commuting Hamiltonians. We associated a graph with a commuting Hamiltonian which encodes the ground space in its cycles and the low energy states in its paths. Our results generalize Yoshida’s work in 1D who considers stabilizer Hamiltonians [7]. Comparing to [4, 5] instead of assuming that the ground states are described by MPSs, we imposed the assumption that the Hamiltonian is commuting.

In section 4, we argued that the ground states of the Hamiltonians under consideration can be exactly written as translational-invariant MPSs. Thus, we could have skipped the previous section and directly used the result of [4, 5] to conclude that the ground state degeneracy is the only parameter that distinguishes phases. Our arguments, however, are based on much simpler techniques and we preferred not to refer to [4, 5].

Our results can be applied on a larger class of Hamiltonians than the commuting ones. Let $H_N = \sum_j h_{j,j+1}$ be an arbitrary translational-invariant (frustration-free) Hamiltonian. As before without loss of generality, we may assume that $h_{j,j+1}$ is positive semidefinite and the ground state energy of $H_N$ is zero. Suppose that there exists a positive definite matrix $X$ such that

\[(h_{j-1,j} \otimes 1_{j+1}) \cdot (1_{j-1} \otimes X_j \otimes 1_{j+1}) \cdot (1_{j-1} \otimes h_{j,j+1}) = (1_{j-1} \otimes h_{j+1,j}) \cdot (1_{j-1} \otimes X_j \otimes 1_{j+1}) \cdot (h_{j-1,j} \otimes 1_{j+1}). \tag{13}\]

Then the Hamiltonian $H'_N = \sum_j h'_{j,j+1}$ with the local term

\[h'_{j,j+1} = (X_j^{1/2} \otimes X_{j+1}^{1/2}) h_{j,j+1} (X_j^{1/2} \otimes X_{j+1}^{1/2}) \] \tag{14}

is commuting. These local terms are still positive semidefinite, and one can easily observe that $|\psi\rangle$ is a zero-energy state of $H'_N$ if and only if $X_j^{1/2} \otimes \cdots \otimes X_{N-1}^{1/2} |\psi\rangle$ is the ground state of $H_N$. Thus, the new Hamiltonian is frustration free as well. Furthermore, since $X$ is assumed to be positive definite and then invertible, the correspondence between ground states is one to one. Therefore, $H'_N$ is scale invariant if and only if $H_N$ is scale invariant, and in this case results of our paper are applied. For instance, we showed that ground states of commuting Hamiltonian have MPS representations. Moreover, if $|\psi\rangle$ is an MPS, then $X_1^{1/2} \otimes \cdots \otimes X_N^{1/2} |\psi\rangle$ is an MPS as well (see figure 1). As a summary, the existence of a positive definite matrix $X$ satisfying (13) implies that the ground states of $H_N = \sum_j h_{j,j+1}$ have MPS representations, and results of [4, 5] are applied.

With the above technique, sometimes we can turn a Hamiltonian whose ground space has MPS representation into a commuting one. In particular, the converse of this observation holds for Hamiltonians with a unique injective MPS ground state in the following sense. Consider a translational-invariant Hamiltonian with a unique translational-invariant MPS ground state $|\psi\rangle$. We assume that the bond dimension of this MPS representation is $\chi$ and the corresponding map is $S : \mathbb{C}^\chi \otimes \mathbb{C}^\chi \to \mathbb{C}^d$. Then we have

\[|\psi\rangle = (S \otimes \cdots \otimes S) |\Phi\rangle \otimes \cdots \otimes |\Phi\rangle,\]

where $|\Phi\rangle$ is the maximally entangled state with local dimension $\chi$. We may also replace $S$ with $US$, where $U$ is a unitary map. In this case, $|\psi\rangle$ is replaced with $U \otimes \cdots \otimes U |\psi\rangle$, which by the discussion of section 2 belongs to the same phase as $|\psi\rangle$.

$S$ is injective because we assume that $|\psi\rangle$ is an injective MPS. Then the map $S^{-1}$ is well defined on the support of $S$. In fact, we may identify the domain and support of $S$, and by applying an appropriate $U$ (replacing $S$ with $US$) assume that $S$ is Hermitian and positive definite.
We now introduce a commuting Hamiltonian as follows. Consider $2N$ spins of dimension $\chi$ sitting on $N$ sites of a chain. So there are two spins on each site which are denoted by $l$ and $r$, and the Hilbert space corresponding to the sites $j$ and $j + 1$ is

$$(C^l_{ij} \otimes C^r_{ij}) \otimes (C^l_{(i+1)j} \otimes C^r_{(i+1)j}).$$

Define the local projection

$$P_{j,j+1} = \mathbb{1}_l \otimes \mathbb{1}_r \otimes \mathbb{1}_{l(j+1)} \otimes \mathbb{1}_{r(j+1)} - \mathbb{1}_l \otimes |\Phi\rangle \langle \Phi| \mathbb{1}_{l(j+1)} \otimes \mathbb{1}_{r(j+1)},$$

and let $\tilde{H} = \sum_j P_{j,j+1}$. $\tilde{H}$ is commuting, translational invariant and frustration free with the unique ground state $|\Phi\rangle \otimes \cdots \otimes |\Phi\rangle$. Moreover, if we define

$$h_{j,j+1} = (S^{-1}_j \otimes S^{-1}_{j+1}) P_{j,j+1} (S^{-1}_j \otimes S^{-1}_{j+1}),$$

then $H = \sum_j h_{j,j+1}$ has a unique ground state which is $|\psi\rangle$. As a result, the Hamiltonian $H$ with the ground state $|\psi\rangle$ can be turned into a commuting one using equation (14) with $X^{1/2} = S$.

We conclude that MPS ground states and commuting Hamiltonians are in close relation via (14). The advantage of working with commuting Hamiltonians, however, is that verifying commutativity in general is much easier than checking whether the ground space is describable by MPSs.

We leave the problem of classification of the phases of 1D systems for general Hamiltonians, without the commutativity assumption and that of [4, 5], for future works.

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