Approximation by Sampling-Type Nonlinear Discrete Operators in $\varphi$-Variation

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Abstract. In the present paper, our purpose is to obtain a nonlinear approximation by using convergence in $\varphi$-variation. Angeloni and Vinti prove some approximation results considering linear sampling-type discrete operators. These types of operators have close relationships with generalized sampling series. By improving Angeloni and Vinti's one, we aim to get a nonlinear approximation which is also generalized by means of summability process. We also evaluate the rate of approximation under appropriate Lipschitz classes of $\varphi$-absolutely continuous functions. Finally, we give some examples of kernels, which fulfill our kernel assumptions.

1. Introduction

Sampling-type operators have numerous applications in speech processing, geophysics, medicine and etc (see [4, 9, 20–28, 42]). These operators are dealing with the generalized sampling series. In this study, we concentrate on the paper [2], where Angeloni and Vinti have some convergence results concerning sampling-type discrete operators. Our goal is to obtain more general approximations than their studies. To this end, we construct a nonlinear form of the operators

$$T_w(f;x) = \sum_{k \in \mathbb{Z}} f(x - \frac{k}{w}) l_{k,w} \quad (x \in \mathbb{R} \text{ and } w \in \mathbb{N})$$

(1)

given in [7, 8] and we improve them via Bell-type summability method [18, 19]. Note that, Bell’s method is considerably general and beside the classical convergence, it includes Cesaro convergence, almost convergence and so on (see [30, 32, 33, 36]). Although there are many works about usages of Bell’s methods on positive linear operators [10, 29, 34, 35, 40, 44, 46], there are only a few works on nonlinear cases [11–14] in approximation theory.

Assume that $\mathcal{A} = \{A^n\} = \{(a_{nm}^n)\} (w, n, v \in \mathbb{N})$ is a family of infinite matrices of real or complex numbers. Then for a given sequence $(x_w)_{w \in \mathbb{N}}$, the double sequence $t^n_m := \sum_{n=1}^{\infty} a_{nm}^n x_w$ is called $\mathcal{A} -$ transform of $(x_w)$ provided that it is convergent for all $n, v \in \mathbb{N}$. In addition, it is called “$(x_w)$ is $\mathcal{A} -$ summable to $L$” if

$$\lim_{n \to \infty} \sum_{w=1}^{\infty} a_{nm}^n x_w = L \quad (\text{uniformly in } v) \ [18].$$

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This approximation is denoted by $\mathcal{A} \sim \lim x = L$. $\mathcal{A}$ is called regular if $\lim_{x \to a} x = L$ implies $\mathcal{A} \sim \lim x = L$. A characterization of regularity of $\mathcal{A}$ is also given by Bell in [19]: $\mathcal{A}$ is regular if and only if

(a) for every $w \in \mathbb{N}$, $\lim_{n \to \infty} a_{nw}^\mu = 0$ (uniformly in $v$)

(b) $\lim_{n \to \infty} \sum_{n=1}^{\infty} d_{nw}^\mu = 1$ (uniformly in $v$)

(c) for each $n, v \in \mathbb{N}$, $\sum_{n=1}^{\infty} a_{nv}^\mu = a_{n, v}$ is finite and there exist positive integers $N, M$ satisfying that $\sup_{n \geq N, v \in \mathbb{N}} \sum_{n=1}^{\infty} [a_{nv}^\mu] = M$.

The variation of a function was first given by Jordan in [31] and then it was developed, e.g., in [37, 45, 47, 48]. Afterwards, taking these generalizations into account, Musielak and Orlicz introduced $\varphi$-variation [41], which is known as the Musielak Orlicz $\varphi$-variation. This concept is a strict generalization of classical Jordan variation and retains many properties of it. For other applications about $\varphi$-variation, see [1, 4, 6–8, 17, 39]. We also refer to [5, 15], which are related to the topic of this paper.

Let $\varphi: \mathbb{R}_+^\ast \to \mathbb{R}_+^\ast$ be a $\varphi$-function, that is, $\varphi$ is continuous, nondecreasing such that $\varphi(0) = 0$, $\varphi(x) > 0$ for all $x > 0$ and $\lim_{x \to 0^+} \varphi(x) = +\infty$.

Throughout the paper, we assume that $\mathcal{A}$ is regular with nonnegative real entries and $\varphi$ is a convex $\varphi$-function together with the following limit condition

$$\lim_{x \to 0^+} \frac{\varphi(x)}{x} = 0. \quad (+)$$

Note that, this limit condition is needed to have the following inclusion $BV(\mathbb{R}) \subset BV_\varphi(\mathbb{R})$, i.e., the inclusion is strict in general (for further information, see Remark 4.5. in [1]).

Suppose that $\mathcal{P} = \{x_i\}_{i=0}^{m}$ is an increasing sequence in $\mathbb{R}$. Then $\varphi$-variation of a given measurable function $f: \mathbb{R} \to \mathbb{R}$ is defined by

$$V_\varphi[f] = \sup_{\mathcal{P}} \sum_{i=1}^{m} \varphi\left(|f(x_i) - f(x_{i-1})|\right). \quad [41].$$

In addition, $f$ is called bounded $\varphi$-variation, if there exists a $\lambda > 0$ such that $V_\varphi[\lambda f] < \infty$. By $BV_\varphi(\mathbb{R})$, we denote the space of all functions of bounded $\varphi$-variation.

One significant property of $\varphi$-variation is that,

$$V_\varphi[\sum_{i=1}^{n} f_i] \leq \frac{1}{n} \sum_{i=1}^{n} V_\varphi[f_i] \quad (2)$$

holds for every measurable function $f_i: \mathbb{R} \to \mathbb{R}$ ($i = 1, \ldots, n$) (see [41]).

By $AC_\varphi(\mathbb{R})$, we denote the space of all $\varphi$-absolutely continuous functions on $\mathbb{R}$, namely, the space of all functions of bounded $\varphi$-variation such that there exists a $\lambda > 0$ for which for all $\epsilon > 0$ and for all bounded interval $I = [a, b] \subset \mathbb{R}$, there exists a $\delta > 0$ satisfying that

$$\sum_{i=1}^{n} \varphi(\lambda |f(\beta_i) - f(\alpha_i)|) < \epsilon$$

holds for any collections of non-overlapping intervals $[\alpha_i, \beta_i] \subset I$, whenever

$$\sum_{i=1}^{n} \varphi(\beta_i - \alpha_i) < \delta.$$

Now that we have given some basic concepts, we can define our operator as follows.

Let $f: \mathbb{R} \to \mathbb{R}$ be a bounded function. Then consider the following operator

$$T_{n, v}[f; x] = \sum_{i=1}^{\infty} a_{i, n}^\mu \sum_{k \in \mathbb{Z}} H_{nw}(f(x - \frac{k}{v})) l_{i, w} \quad (x \in \mathbb{R} \text{ and } n, v \in \mathbb{N}), \quad (3)$$
where $H_w : \mathbb{R} \to \mathbb{R}, H_w(0) = 0$ and $H_w$ is a $\psi$-Lipschitz kernel ($|H_w(x) - H_w(y)| \leq K\psi(|x - y|)$ for all $x, y \in \mathbb{R}$).

Here, $\psi$ is a $\varphi$-function and $l_w \in l^1(\mathbb{Z})$ is a family of discrete kernels for every $w \in \mathbb{N}$. Then, it is not hard to see that (3) is well-defined for all real-valued bounded functions $f$.

In this work, by using $\varphi$-absolutely continuous functions, we investigate the existence of $\mu > 0$ such that the following limit holds

$$\lim_{n \to \infty} V_\varphi[\mu (T_{n,w}(f) - f)] = 0 \text{ (uniformly in } v \in \mathbb{N}),$$

where $T_{n,w}(f)$ is defined above.

Then, we will check the rate of approximation under some Lipschitz classes of $\varphi$-absolutely continuous functions. By using the relation between them, we also get the following result

$$\lim_{n \to \infty} V_\varphi[\mu (S_{n,w}(f) - f)] = 0 \text{ (uniformly in } v \in \mathbb{N}),$$

where

$$S_{n,w}(f;x) = \sum_{w=1}^{\infty} a_{w}^{\nu} \sum_{k \in \mathbb{Z}} H_w\left(f\left(\frac{k}{w}\right)\right)\chi(\omega x - k),$$

(4)

namely, $S_{n,w}(f)$ is $\mathcal{A}$-transform of nonlinear generalized sampling series. Furthermore, we give an application of Theorem 2.4 and Theorem 3.1 at the end of the paper.

2. Convergence in $\varphi$-Variation

In this section, we prove our main approximation theorem using convergence in $\varphi$-variation.

We require the following conditions:

(1) $\frac{1}{A} \leq \sum_{k \in \mathbb{Z}} |l_{k,w}|^2 \leq A$, for some constant $A > 0$,

(2) $\mathcal{A} - \lim \left(\sum_{k \in \mathbb{Z}} |l_{k,w}|\right) = 1$,

(3) $\exists r > 0$ such that $\mathcal{A} - \lim \left(\sum_{k \in \mathbb{Z}} |l_{k,w}|\right) = 0$,

(4) For every $\gamma > 0$, there exists a $\lambda > 0$ such that, for every (proper) bounded interval $I \subset \mathbb{R}$, $\mathcal{A} - \lim \forall \varphi \left[\lambda G_w(f)\right] = 0$ uniformly in $I \subset \mathbb{R}$, where $G_w(u) = H_w(u) - u$ and $\forall \varphi \left[\lambda G_w(f)\right]$ denotes the $\varphi$-variation of $\lambda G_w$ on the interval $I$.

It can be easily seen that taking $\mathcal{A} = \{I\}$, the identity matrix, then (1) - (3) turn into (A1)-(A2) given in [2]. Here, condition (h) is a natural condition due to the nonlinearity of the kernel. For the examples of $H_w$ in case of $\mathcal{A} = \{I\}$, see [1, 8]. At the end of the paper, we give a specific kernel satisfying (l1) - (l3) and (h).

The following growth condition on $\psi$ corresponding to $\psi$-Lipschitz condition of $H_w$ is also needed.

**Definition 2.1.** Let $\varphi, \eta, \psi$ be a $\varphi$-function. If for all $\gamma \in (0, 1)$, there exists a constant $C_\gamma$, such that

$$\varphi(C_\gamma \psi(|g|)) \leq \eta(\gamma |g|)$$

for every measurable function $g : \mathbb{R} \to \mathbb{R}$, then $(\varphi, \eta, \psi)$ is called properly directed.

Throughout the paper, we will assume that $(\varphi, \eta, \psi)$ is properly directed. In the nonlinear setting, this condition is common (see [1, 7, 16, 17, 38, 43]) and some examples of the triple $(\varphi, \eta, \psi)$ can be found in [1].
Lemma 2.2. Let $f \in BV_\eta(\mathbb{R})$. If (l1) is satisfied, then $T_{n,\nu}$ maps from $BV_\eta(\mathbb{R})$ into $BV_\psi(\mathbb{R})$, namely, there exists a $\mu > 0$ such that

$$V_\psi(\mu T_{n,\nu}f) \leq V_\eta(\lambda f)$$

holds, where $\lambda > 0$ is sufficiently small for which $V_\eta(\lambda f) < \infty$.

Proof. Let $\{x_i\}_{i=1}^m$ be an increasing sequence in $\mathbb{R}$. For all $\mu > 0$, it is not hard to see from Jensen’s inequality that

$$\sum_{i=1}^m \varphi(\mu |T_{n,\nu}(f; x_i) - T_{n,\nu}(f; x_{i-1})|) \leq \sum_{i=1}^m \varphi\left(\mu \sum_{w=1}^\infty a_{nw}^{\nu} \sum_{k \in \mathbb{Z}} |h_{k,w}| |H_w(f(x_i - \frac{k}{\nu}) - H_w(f(x_{i-1} - \frac{k}{\nu}))|\right) \leq \frac{1}{\alpha_{n,\psi}} \sum_{w=1}^\infty \sum_{k \in \mathbb{Z}} a_{nw}^{\nu} \varphi\left(\mu a_{n,\psi} A H_w\left(\frac{|f(x_i) - f(x_{i-1})|}{\nu}\right)\right)$$

where $a_{n,\psi} \equiv \sum_{w=1}^\infty \alpha_{nw}^{\nu} < \infty$ by (a3). Then, using Jensen’s inequality one more time and taking supremum, we get the following inequality,

$$\sum_{i=1}^m \varphi(\mu |T_{n,\nu}(f; x_i) - T_{n,\nu}(f; x_{i-1})|) \leq \frac{1}{\alpha_{n,\psi} A} \sum_{w=1}^\infty \sum_{k \in \mathbb{Z}} |h_{k,w}| \sum_{i=1}^m \varphi\left(\mu a_{n,\psi} A K \varphi\left(\frac{|f(x_i) - f(x_{i-1})|}{\nu}\right)\right)$$

Since $H_w$ is $\psi$-Lipschitz, then there holds

$$\sum_{i=1}^m \varphi(\mu |T_{n,\nu}(f; x_i) - T_{n,\nu}(f; x_{i-1})|) \leq \frac{1}{\alpha_{n,\psi} A} \sum_{w=1}^\infty \sum_{k \in \mathbb{Z}} |h_{k,w}| \sum_{i=1}^m \varphi\left(\mu a_{n,\psi} A K \varphi\left(\frac{|f(x_i) - f(x_{i-1})|}{\nu}\right)\right)$$

where $K$ is $\psi$-Lipschitz constant of $H_w$. Now, from (5) for every $\lambda \in (0, 1)$ for which $V_\eta(\lambda f) < \infty$, there exists a constant $C_3 \in (0, 1)$ such that

$$\sum_{i=1}^m \varphi(\mu |T_{n,\nu}(f; x_i) - T_{n,\nu}(f; x_{i-1})|) \leq \frac{1}{\alpha_{n,\psi} A} \sum_{w=1}^\infty \sum_{k \in \mathbb{Z}} |h_{k,w}| \sum_{i=1}^m \eta(\lambda |f(x_i) - f(x_{i-1})|) \leq \frac{1}{\alpha_{n,\psi} A} \sum_{w=1}^\infty \sum_{k \in \mathbb{Z}} |h_{k,w}| V_\eta(\lambda f)$$

holds for all $0 < \mu \leq C_3/(a_{n,\psi} AK)$. Since

$$V_\eta(\lambda f) = V_{\psi}(\frac{-\lambda f}{\nu})$$

we derive from (l1) that

$$\sum_{i=1}^m \varphi(\mu |T_{n,\nu}(f; x_i) - T_{n,\nu}(f; x_{i-1})|) \leq \frac{V_{\psi}(\lambda f)}{\alpha_{n,\psi} A} \sum_{w=1}^\infty \sum_{k \in \mathbb{Z}} |h_{k,w}| \leq V_\eta(\lambda f)$$

Consequently, if we take supremum over $\{x_i\}_{i=1}^m$, the proof is done. \qed
Lemma 2.3. Let \( f \in AC_\eta (\mathbb{R}) \). If \((l_i)\) is satisfied, then \( T_{n,v}(f) \in AC_\psi (\mathbb{R}) \) for all \( n, v \in \mathbb{N} \).

Proof. Assume that \( \varepsilon > 0 \) be given and let \( \delta := \delta(\varepsilon) > 0 \) corresponds to \( \eta \)-absolute continuity of \( f \) where \( [[\alpha_i, \beta_i]]_{i=1}^m \) be a finite nonoverlapping intervals of \( I = [a, b] \subset \mathbb{R} \) such that \( \sum_{i=1}^{m} \eta(\beta_i - \alpha_i) < \delta \). Then, applying Jensen’s inequality we may clearly see that

\[
\sum_{i=1}^{m} \varphi \left( \mu \left| T_{n,v}(f; \beta_i) - T_{n,v}(f; \alpha_i) \right| \right) \\
\leq \sum_{i=1}^{m} \varphi \left( \mu \sum_{w=1}^{\infty} \sum_{k \in \mathbb{Z}} |h_{k,w}| \left| H_w \left( f (\beta_i - \frac{k}{w}) \right) - H_w \left( f (\alpha_i - \frac{k}{w}) \right) \right| \right) \\
\leq \frac{1}{a_{n,v}A} \sum_{w=1}^{\infty} \sum_{k \in \mathbb{Z}} |h_{k,w}| \varphi \left( \mu a_{n,v}A \left| H_w \left( f (\beta_i - \frac{k}{w}) \right) - H_w \left( f (\alpha_i - \frac{k}{w}) \right) \right| \right) \\
\leq \frac{1}{a_{n,v}A} \sum_{w=1}^{\infty} \sum_{k \in \mathbb{Z}} \varphi \left( \mu a_{n,v}AK\psi \left( f (\beta_i - \frac{k}{w}) - f (\alpha_i - \frac{k}{w}) \right) \right).
\]

Since \( (\varphi, \eta, \psi) \) is properly directed, then for every \( \lambda \in (0, 1) \) there exists a \( C_\lambda > 0 \) such that

\[
\sum_{i=1}^{m} \varphi \left( \mu \left| T_{n,v}(f; \beta_i) - T_{n,v}(f; \alpha_i) \right| \right) \\
\leq \frac{1}{a_{n,v}A} \sum_{w=1}^{\infty} \sum_{k \in \mathbb{Z}} \varphi \left( \sum_{i=1}^{m} \eta \left( \lambda \left| f (\beta_i - \frac{k}{w}) - f (\alpha_i - \frac{k}{w}) \right| \right) \right)
\]

holds for all \( 0 < \mu \leq C_\lambda/(a_{n,v}AK) \). Moreover, seeing that \( f \) is \( \eta \)-absolutely continuous, then there exists a \( \gamma > 0 \) such that

\[
\sum_{i=1}^{m} \eta \left( \left| f (\beta_i - \frac{k}{w}) - f (\alpha_i - \frac{k}{w}) \right| \right) < \varepsilon
\]

whenever

\[
\sum_{i=1}^{m} \eta \left( \beta_i - \frac{k}{w} \right) - \left( \alpha_i - \frac{k}{w} \right) = \sum_{i=1}^{m} \eta (\beta_i - \alpha_i) < \delta.
\]

Using the previous expression together with \((l_1)\) and \((a_3)\), we finally get

\[
\sum_{i=1}^{m} \varphi \left( \mu \left| T_{n,v}(f; \beta_i) - T_{n,v}(f; \alpha_i) \right| \right) \leq \frac{1}{a_{n,v}A} \sum_{w=1}^{\infty} \sum_{k \in \mathbb{Z}} |h_{k,w}| \varepsilon
\]

for all \( 0 < \lambda \leq \gamma \). \( \square \)

Now, we state our main approximation theorem.

Theorem 2.4. Assume that \((l_1) - (l_3)\) and \((h)\) hold. Then, there exists a \( \mu > 0 \) such that for a given \( f \in AC_\psi (\mathbb{R}) \cap BV_\psi (\mathbb{R}) \), we have

\[
\lim_{n \to \infty} V_\psi \left[ \mu \left( T_{n,v}(f) - f \right) \right] = 0 \text{ (uniformly in } v \in \mathbb{N} \).
\] (6)
Proof. Let \( \{x_n\}_{n=1}^{\infty} \) be an increasing sequence in \( \mathbb{R} \). Then, for all \( \mu > 0 \)

\[
I = \sum_{i=1}^{m} \varphi \left( \mu \left| T_{n,\nu} (f; x_i) - f (x_i) - T_{n,\nu} (f; x_{i-1}) + f (x_{i-1}) \right| \right)
\]

\[
= \sum_{i=1}^{m} \varphi \left( \mu \left| \sum_{n=1}^{\infty} a_{n,\nu} \sum_{k \in \mathbb{Z}} l_{k,\nu} \left| H_w \left( f \left( x_i - \frac{x}{w} \right) \right) - f (x_i) \right| - H_w \left( f \left( x_{i-1} - \frac{x}{w} \right) \right) \right| + f \left( x_{i-1} - \frac{x}{w} \right) \right) \right) + \sum_{n=1}^{\infty} a_{n,\nu} \sum_{k \in \mathbb{Z}} l_{k,\nu} \left| f \left( x_i - \frac{x}{w} \right) - f (x_i) \right| + f \left( x_{i-1} - \frac{x}{w} \right) + f (x_{i-1}) \right| + f (x_{i-1}) \right| \right) \right)
\]

holds. Now, using the convexity of \( \varphi \), one can observe the following,

\[
I \leq \frac{1}{3} \sum_{i=1}^{m} \varphi \left( 3 \mu a_{n,\nu} \sum_{k \in \mathbb{Z}} l_{k,\nu} \left| H_w \left( f \left( x_i - \frac{x}{w} \right) \right) - f (x_i) \right| - f \left( x_{i-1} - \frac{x}{w} \right) \right) \right) + \frac{1}{3} \sum_{i=1}^{m} \varphi \left( \left| f \left( x_i - \frac{x}{w} \right) - f (x_i) \right| - f \left( x_{i-1} - \frac{x}{w} \right) + f (x_{i-1}) \right| \right) \right) \right) \right)
\]

In \( I_1 \), using two times Jensen’s inequality we immediately get

\[
I_1 \leq \frac{1}{3 a_{n,\nu} A} \sum_{n=1}^{\infty} a_{n,\nu} \sum_{k \in \mathbb{Z}} l_{k,\nu} \left| \sum_{i=1}^{m} \varphi \left( 3 \mu a_{n,\nu} A \left| H_w \left( f \left( x_i - \frac{x}{w} \right) \right) - f (x_i) \right| - f \left( x_{i-1} - \frac{x}{w} \right) \right) \right) \right) + \sum_{i=1}^{m} \varphi \left( 3 \mu a_{n,\nu} \sum_{k \in \mathbb{Z}} l_{k,\nu} \left| H_w \left( f \left( x_i - \frac{x}{w} \right) \right) - f (x_i) \right| - f \left( x_{i-1} - \frac{x}{w} \right) \right) \right) \right) \right) \right)
\]

It is known from (as) that \( a_{n,\nu} := \sum_{n=1}^{\infty} a_{n,\nu} \leq M \) for sufficiently large \( n \in \mathbb{N} \). Then, from the convexity of \( \varphi \)

\[
I_1 \leq \frac{1}{3 M A} \sum_{n=1}^{\infty} a_{n,\nu} \sum_{k \in \mathbb{Z}} l_{k,\nu} \left| \sum_{i=1}^{m} \varphi \left( 3 \mu M A \left| H_w \left( f \left( x_i - \frac{x}{w} \right) \right) - f (x_i) \right| - f \left( x_{i-1} - \frac{x}{w} \right) \right) \right) \right) + \sum_{i=1}^{m} \varphi \left( 3 \mu M A \left| H_w \left( f \left( x_i - \frac{x}{w} \right) \right) - f (x_i) \right| - f \left( x_{i-1} - \frac{x}{w} \right) \right) \right) \right) \right) \right)
\]

yields. Now, using the fact that

\[
V_{\psi} \left[ 3 \mu M A \left( H_w \left( f \left( x_i - \frac{x}{w} \right) \right) - f (x_i) \right) \right] = V_{\psi} \left[ 3 \mu M A \left( H_w (f) - f \right) \right],
\]

then holds

\[
I_1 \leq \frac{1}{3 M} \sum_{i=1}^{m} a_{i,\nu} V_{\psi} \left[ 3 \mu M A \left( H_w (f) - f \right) \right].
\]

Considering (ii) together with Lemma 1 in [8], we observe that for all \( \gamma > 0 \), there exists a \( \lambda > 0 \) such that \( \forall \varepsilon > 0 \), there exists a number \( n_0 \) satisfying that

\[
I_1 < \frac{V_{\psi} \left[ y \right]}{3 M} \varepsilon
\]
for all \( n > n_0 \) and \( 0 < \mu \leq \frac{1}{3M} \).

About \( I_2 \), using the convexity of \( \varphi \), Jensen’s inequality and \((a_3)\), there holds

\[
I_2 \leq \frac{1}{3MA} \sum_{w=1}^{\infty} \sum_{k \in \mathbb{Z}} \left| \mu w \right| \left[ 3\mu MA \left( f \left( \cdot - \frac{k}{w} \right) - f \left( \cdot \right) \right) \right]
\]

for sufficiently large \( n \in \mathbb{N} \). Here, one can observe the \( \varphi \)-modulus of smoothness of \( f \in AC_{\varphi}(\mathbb{R}) \) by Subsection 2.4. in [41], that is, if \( \varphi \) satisfies \((+)\), then \( \lim_{b \to 0} \sup_{|t| \leq b} \varphi \left[ \lambda \left( f \left( \cdot - t \right) - f \left( \cdot \right) \right) \right] = 0 \) for some \( \lambda > 0 \) if and only if \( f \in AC_{\varphi}(\mathbb{R}) \). So, one can find a \( \lambda_1 > 0 \) such that for all \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that

\[
\varphi \left[ \lambda_1 \left( f \left( \cdot - t \right) - f \left( \cdot \right) \right) \right] < \varepsilon
\]

whenever \(|t| < \delta\). Now, from (2) we can divide the sum in (7) as follows

\[
I_2 \leq \frac{1}{3MA} \sum_{w=1}^{\infty} \sum_{k \in \mathbb{Z}} \left| \mu w \right| \left[ 3\mu MA \left( f \left( \cdot - \frac{k}{w} \right) - f \left( \cdot \right) \right) \right]
\]

for all \( w > w_1 \).

In \( I_2 \), since \( \varphi \left[ 6\mu MA f \left( \cdot - \frac{k}{w} \right) \right] = \varphi \left[ 6\mu MA f \right] \), it can easily be observed from \((l_1)\) that

\[
I_2^{\prime} \leq \frac{1}{3MA} \sum_{w=1}^{\infty} \mu w \sum_{k \in \mathbb{Z}} \left| \mu w \right| \left[ 3\mu MA \left( f \left( \cdot - \frac{k}{w} \right) - f \left( \cdot \right) \right) \right]
\]

where \( r > 0 \) is given in \((l_3)\) and \( w_1 \) is such that

\[
\frac{k}{w} < \frac{r}{w_1} \leq \delta
\]

for all \( w > w_1 \).

From \((8)\), \((l_1)\) and \((a_3)\) we obtain

\[
I_2^{\prime} \leq \frac{\varepsilon}{3}
\]

for all \( 0 < \mu \leq \frac{\tilde{\mu}}{6MA} \) and for sufficiently large \( n \in \mathbb{N} \).

From \((8)\), \((l_1)\) and \((a_3)\) we obtain

\[
I_2^{\prime} \leq \frac{\varepsilon}{3}
\]

for all \( 0 < \mu \leq \lambda_1 / (6MA) \).

From \((l_3)\), we get

\[
I_2^{\prime} \leq \frac{\varepsilon}{3}
\]
for sufficiently large $n \in \mathbb{N}$.

On the other hand, since 
\[ \left| \sum_{k=1}^{\infty} a_{nw}^\nu \sum_{k \in \mathbb{Z}} l_{k,w} - 1 \right| < 1 \]
for sufficiently large $n \in \mathbb{N}$, by the convexity of $\varphi$

\[ l_3 \leq \frac{1}{3} \sum_{i=1}^{m} \varphi \left( 3 \mu \left| f(x_i) - f(x_{i-1}) \right| \right) \left| \sum_{n=1}^{\infty} a_{nw}^\nu \sum_{k \in \mathbb{Z}} l_{k,w} - 1 \right| \]
\[ \leq \frac{1}{3} \varphi \left[ 3 \mu f \right] \left| \sum_{n=1}^{\infty} a_{nw}^\nu \sum_{k \in \mathbb{Z}} l_{k,w} - 1 \right| \]
holds. Then from $(l_2)$, we get

\[ l_3 < \frac{\varphi \left[ 3 \mu f \right]}{3} \epsilon \]

for sufficiently large $n \in \mathbb{N}$. Finally, taking supremum over $\{x_i\}_{i \in \{1, \ldots, m\}}$ in the first inequality, we complete the proof. $\square$

3. Order of Approximation

In this section, we examine the order of approximation. For this reason, we first consider the following Lipschitz class

\[ V_{\varphi}Lip(\alpha) = \{ f \in AC_{\varphi}(\mathbb{R}) : \exists \rho > 0 \text{ s.t. } V_{\varphi} \left[ \rho \left| f(\cdot - t) - f(\cdot) \right| \right] = O(\|t\|^\alpha) \text{ as } t \to 0 \} \]

for any $\alpha > 0$ (see also [3]).

For a given nonnegative regular method $A = \{ (a_{nw}) \}_{n \in \mathbb{N}}$ and $\alpha > 0$, we take into account the following orders of approximations:

\[ \left( \sum_{w=1}^{\infty} a_{nw}^\nu \sum_{k \in \mathbb{Z}} l_{k,w} - 1 \right) = O(n^{-\alpha}) \text{ as } n \to \infty \text{ (uniformly in } \nu), \]  
(9)

there exists a number $\bar{r} > 0$ such that

\[ \sum_{w=1}^{\infty} a_{nw}^\nu \sum_{|t| < \bar{r}} = O(n^{-\alpha}) \text{ as } n \to \infty \text{ (uniformly in } \nu), \]  
(10)

\[ \sum_{w=1}^{\infty} a_{nw}^\nu \sum_{|t| \geq \bar{r}} |l_{k,w}| = O(n^{-\alpha}) \text{ as } n \to \infty \text{ (uniformly in } \nu) \]  
(11)

and for each $w \in \mathbb{N},$

\[ a_{nw}^\nu = O(n^{-\alpha}) \text{ as } n \to \infty \text{ (uniformly in } \nu). \]  
(12)

**Theorem 3.1.** Assume that (9)-(12) and $(l_1)$ hold. Assume further that for every $\gamma > 0$, there exists a $\lambda > 0$ such that

\[ \sum_{w=1}^{\infty} a_{nw}^\nu \varphi \left[ \lambda G_{nw} \right] = O(n^{-\alpha}) \text{ as } n \to \infty \text{ (uniformly in } \nu and uniformly in every proper bounded interval } J \subset \mathbb{R}. \]  
(13)

Then, there exists a $\mu > 0$ such that

\[ V_{\varphi} \left[ \mu (T_{\nu,w} (f) - f) \right] = O(n^{-\alpha}) \text{ as } n \to \infty \text{ (uniformly in } \nu) \]

for all $f \in V_{\varphi}Lip(\alpha) \cap BV_{\varphi}(\mathbb{R})$. 

Proof. By the proof of Theorem 2.4, we may easily obtain the following inequality

\[
V_p \left[ \mu \left( T_{n,w} (f) - f \right) \right] \leq \frac{1}{3MA} \sum_{w=1}^{\infty} a_{nw}^* \sum_{k \in \mathbb{Z}} \left| l_{k,w} \right| V_p \left[ 3\mu AM (H_w (f) - f) \right] \\
+ \frac{1}{3MA} \sum_{w=1}^{\infty} a_{nw}^* \sum_{k \in \mathbb{Z}} \left| l_{k,w} \right| V_p \left[ 3\mu AM (f (\cdot - \frac{1}{w}) - f (\cdot)) \right] \\
+ \frac{V_p \left[ 3\mu f \right]}{3} \sum_{w=1}^{\infty} a_{nw}^* \sum_{k \in \mathbb{Z}} l_{k,w} - 1 \\
=: J_1 + J_2 + J_3
\]

for sufficiently large \( n \in \mathbb{N} \). Considering (13) in [8], there exists a constant \( L > 0 \) such that

\[
J_1 = \frac{1}{3MA} \sum_{w=1}^{\infty} a_{nw}^* V_p \left[ 3\mu AM (H_w (f) - f) \right] \sum_{k \in \mathbb{Z}} \left| l_{k,w} \right| \\
\leq \frac{L}{3M} V_p \left[ yf \right] n^{-\alpha} \\
= O \left( n^{-\alpha} \right) \text{ as } n \to \infty \text{ (uniformly in } v \text{)}
\]

for sufficiently small \( \mu > 0 \).

In \( J_2 \), since \( f \in V_p \text{Lip}(\alpha) \), there exist \( \rho, N, \delta > 0 \) s.t. \( V_p \left[ \rho |f (\cdot - t) - f (\cdot)| \right] \leq N |t|^\alpha \) if \( |t| < \delta \). Moreover, for a given \( \bar{r} > 0 \), we can find a number \( w' \) such that

\[
\frac{k}{w} < \frac{\bar{r}}{w} < \delta
\]

for every \( w > w' \). Taking these arguments into account, we divide \( J_2 \) as follows,

\[
J_2 = \frac{1}{3MA} \sum_{w=w'+1}^{\infty} a_{nw}^* \sum_{|k| < r} |l_{k,w}| V_p \left[ 3\mu AM \left( f (\cdot - \frac{1}{w}) - f (\cdot) \right) \right] \\
+ \frac{1}{3MA} \sum_{w=w'+1}^{\infty} a_{nw}^* \sum_{|k| < r} |l_{k,w}| V_p \left[ 3\mu AM \left( f (\cdot - \frac{1}{w}) - f (\cdot) \right) \right] \\
+ \frac{1}{3MA} \sum_{w=w'+1}^{\infty} a_{nw}^* \sum_{|k| \geq r} |l_{k,w}| V_p \left[ 3\mu AM \left( f (\cdot - \frac{1}{w}) - f (\cdot) \right) \right] \\
=: J_2^1 + J_2^2 + J_2^3.
\]

Then, it follows from (10) that

\[
J_2^2 \leq \frac{N}{3MA} \sum_{w=w'+1}^{\infty} a_{nw}^* \sum_{|k| < r} |l_{k,w}| \left| \frac{k}{w} \right|^\alpha \\
\leq \frac{N\rho^\alpha}{3M} \sum_{w=w'+1}^{\infty} a_{nw}^* \sum_{|k| < r} \frac{1}{|w|^\rho} \\
= O \left( n^{-\alpha} \right) \text{ as } n \to \infty \text{ (uniformly in } v \text{)}
\]

for all \( 0 < \mu \leq \frac{\rho}{3MA} \). On the other hand, for \( J_2^1 \) it is not hard to see from (2) that

\[
J_2^1 \leq \frac{1}{3MA} \sum_{w=w'+1}^{\infty} a_{nw}^* V_p \left[ 6\mu AM f \right]
\]

and therefore, from (12)

\[
J_2^1 = O \left( n^{-\alpha} \right) \text{ as } n \to \infty \text{ (uniformly in } v \text{)}
\]
holds. About $J_2^3$, from (2) and (11), we observe the following

$$J_2^3 \leq \frac{\psi_{3\mu AM}}{3MA} \sum_{w=1}^{\infty} \sum_{|k| \geq r} |l_{k,w}|$$

$$= O(n^{-\alpha}) \text{ as } n \to \infty \text{ (uniformly in } v).$$

Finally, directly from (9) we get

$$J_3 = O(n^{-\alpha}) \text{ as } n \to \infty \text{ (uniformly in } v).$$

Now, we investigate a special case of the operator (3), where $l_{k,w} \equiv \chi(k)$ and $\chi : \mathbb{R} \to \mathbb{R}$, namely, $l_{k,w}$ is not depending on $w$. Then, (3) reduces to

$$\mathcal{T}_{n,v} (f; x) = \sum_{w=1}^{\infty} \sum_{k \in \mathbb{Z}} H_w \left( f \left( x - \frac{k}{w} \right) \right) \chi(k),$$

which is (in some cases) equivalent to $\mathcal{A}$-transform of nonlinear generalized sampling series given in (4).

Under these considerations, $(l_1)$ and $(l_2)$ turn into the following assumptions

$$(l'_1) \chi \in l^1(\mathbb{Z})$$

$$(l'_2) \sum_{k \in \mathbb{Z}} |\chi(k)| = 1$$

where on the other hand $(l_3)$ is clearly not satisfied. But these two conditions are still enough to verify the following theorem.

**Theorem 3.2.** Let $f \in AC_{\psi} (\mathbb{R}) \cap BV_{\eta} (\mathbb{R})$. If $(l'_1)$, $(l'_2)$ and $(h)$ hold, then there exists a $\mu > 0$ such that

$$\lim_{n \to \infty} V_{\psi} \left[ \mu \left( \mathcal{T}_{n,v} (f) - f \right) \right] = 0 \text{ (uniformly in } v \in \mathbb{N}).$$

**Proof.** Considering $(l'_2)$ in the proof of Theorem 2.4, then for every $\mu > 0$

$$V_{\psi} \left[ \mu \left( \mathcal{T}_{n,v} (f) - f \right) \right] \leq \frac{1}{3MA} \sum_{w=1}^{\infty} \sum_{k \in \mathbb{Z}} |\chi(k)| V_{\psi} \left[ 3\mu MA \left( H_w \circ f - f \right) \right]$$

$$+ \frac{1}{3MA} \sum_{w=1}^{\infty} \sum_{k \in \mathbb{Z}} |\chi(k)| V_{\psi} \left[ 3\mu MA \left( f \left( \cdot - \frac{k}{w} \right) - f \left( \cdot \right) \right) \right]$$

$$+ \frac{1}{3} V_{\psi} \left[ 3\mu f \right] \left| \sum_{l=1}^{\infty} a_{lw} - 1 \right|$$

$$=: L_1 + L_2 + L_3$$

holds, where $A = \|\chi\|_1$. From $(h)$, $(l'_1)$, and Lemma 1 in [8], one can clearly see that

$$L_1 < \frac{V_{\psi} \left[ yf \right]}{3M} \epsilon$$

for sufficiently large $n \in \mathbb{N}$ and for all $0 < \mu \leq \lambda/(3MA)$ where $\lambda$ and $\gamma$ correspond to Lemma 1 in [8]. On the other hand, since $\chi \in l^1(\mathbb{Z})$, for all $\epsilon > 0$ there exists a $r > 0$ such that

$$\sum_{|k| \geq r} |\chi(k)| < \epsilon.$$
Hence, if we divide $L_2$ into two parts as follows,

$$L_2 = \frac{1}{3\lambda A} \sum_{m=1}^{\infty} a_{m}^n \sum_{|k|\leq r} |\chi(k)| V_{\chi} [3\mu MA \left(f\left(-\frac{k}{w}\right) - f()\right)]$$

$$+ \frac{1}{3\lambda A} \sum_{m=r+1}^{\infty} a_{m}^n \sum_{|k|<r} |\chi(k)| V_{\chi} [3\mu MA \left(f\left(-\frac{k}{w}\right) - f()\right)]$$

$$= L_2^1 + L_2^2$$

then, there holds

$$L_2^1 < \frac{V_{\chi} [6\mu MA f]}{3\lambda} \epsilon$$

For $L_2^2$, using $\varphi$-modulus of smoothness of the function $f \in AC_{\varphi}(\mathbb{R})$, we obviously see that for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\frac{\delta}{w} < \frac{\delta}{w} < \delta$$

for all $w > \varphi$, which implies

$$V_{\chi} [3\mu MA \left(f\left(-l-t\right) - f\left(t\right)\right)] < \epsilon.$$  

Then, dividing $L_2^2$ as follows,

$$L_2^2 = \frac{\varphi}{3\lambda A} \sum_{m=1}^{\varphi} a_{m}^n \sum_{|k|\leq r} |\chi(k)| V_{\chi} [3\mu MA \left(f\left(-\frac{k}{w}\right) - f()\right)]$$

$$+ \frac{1}{3\lambda A} \sum_{m=r+1}^{\infty} a_{m}^n \sum_{|k|<r} |\chi(k)| V_{\chi} [3\mu MA \left(f\left(-\frac{k}{w}\right) - f()\right)]$$

we may easily obtain

$$L_2^2 < \left(\frac{\varphi V_{\chi} [6\mu MA f]}{3\lambda A} + \frac{1}{3}\right) \epsilon.$$  

Finally, using (a2) we conclude

$$L_3 < \frac{V_{\chi} [3\mu f]}{3} \epsilon$$

for sufficiently large $n \in \mathbb{N}$, which completes the proof.  

\[\square\]

**Remark 3.3.** Note that, the operators $T$ and $S$ are different in general but, in some cases, they coincide.

**Corollary 3.4.** Assume that $f \in B_{1\nu}^{1} (\mathbb{R}) \cap BV_{\varphi} (\mathbb{R})$ and $\psi \left(\int f \right) \in B_{1\nu}^{1} (\mathbb{R})$ (the Paley-Wiener Space $B_{1\nu}^{1} (\mathbb{R})$ = \{ $f \in L^r (\mathbb{R}) : f$ has an extension to whole $\mathbb{C}$ s.t. $\int f \leq \exp (\pi \nu |z|) \int f$ for every $z \in \mathbb{C}$\}) for some $\varphi > 0$, where $\|\cdot\|$ denotes supremum norm. If $\chi \in B_{1\nu}^{\infty} (\mathbb{R})$ and $(l_1^2)$, $(l_1^2)$, $(h)$ are satisfied, then there exists a $\mu > 0$ such that

$$\lim_{n \to \infty} V_{\chi} [\mu (S_{n\nu} f - f)] = 0$$

(uniformly in $v \in \mathbb{N}$).

**Proof.** First of all, we should say that since $\|H_{w} f\| \leq K_{\psi} \left(\int f \right)$ and $\psi \left(\int f \right) \in B_{1\nu}^{1} (\mathbb{R})$, then $H_{w} f \in B_{1\nu}^{1} (\mathbb{R})$. From Proposition 4.3. in [2] and (\text{+}), we may easily see that $B_{1\nu}^{1} (\mathbb{R}) \subset AC_{\varphi} (\mathbb{R})$. Therefore, using the similar arguments on Lemma 4.2. in [2], we deduce that

$S_{n\nu} (f) = T_{n\nu} (f)$

for all $n, \nu \in \mathbb{N}$. Consequently, by the Theorem 3.2 the proof completes.  

\[\square\]

An example of $\chi \in B_{1\nu}^{\infty} (\mathbb{R})$ satisfying $(l_1^2)$ and $(l_2^2)$ can be found in Example 4.5. in [2].
4. Conclusions and Applications

We remark that operator (3) can be written as

\[ T_{n, \nu} (f; x) = \sum_{w=1}^{\infty} a_{\nu w} T_w (f; x) \]

where \( T_w (f; x) \) is introduced by

\[ T_w (f; x) = \sum_{k \in \mathbb{Z}} H_w (f (x - \frac{k}{w})) l_{k,w}. \]

Using certain methods, some significant results of Theorem 2.4 are given below:

- If we take \( \mathcal{A} = \{C_1\} \), Cesàro matrix \([30]\), where \( C_1 = [c_{nw}] \) is such that
  \[ c_{nw} = \begin{cases} 1/n; & \text{if } 1 \leq w \leq n \\ 0; & \text{otherwise}, \end{cases} \]
  then we get
  \[ \lim_{n \to \infty} V_{\phi} \left[ \frac{T_1 (f) + T_2 (f) + \cdots + T_n (f)}{n} - f \right] = 0 \]
  for all \( f \in AC_{\phi} (\mathbb{R}) \).

- Putting \( \mathcal{A} = \mathcal{F} \), the almost convergence matrix \([36]\), where \( \mathcal{F} = \{[c_{\nu w}]\} \) is such that
  \[ c_{\nu w} = \begin{cases} 1/\nu; & \text{if } \nu \leq w \leq n + \nu - 1 \\ 0; & \text{otherwise}, \end{cases} \]
  then we get
  \[ \lim_{n \to \infty} V_{\phi} \left[ \frac{T_{\nu} (f) + T_{\nu+1} (f) + \cdots + T_{n+\nu-1} (f)}{n} - f \right] = 0 \text{ uniformly in } \nu \]
  for all \( f \in AC_{\phi} (\mathbb{R}) \).

- If \( \mathcal{A} = \{I\} \), the identity matrix, then we get
  \[ \lim_{n \to \infty} V_{\phi} [T_n (f) - f] = 0, \]
  where \( T_n \) is nonlinear form of (1).

- If one take \( H_w (u) = u \), then \( T_n \) reduces to linear case given in (1) and the previous estimations hold for the operator (1).

- On the other hand, all the previous results are still valid for the generalized sampling series \( S_{n, \nu} (f) \) given in (4).

Now, we will investigate the existence of kernels which satisfy \((l_1) - (l_3), (h)\) and conditions \((9) - (13)\). Let \( \mathcal{A} = \mathcal{F} = \{F\} \), \( \alpha = 1/2 \) and \( l_{k,w}, H_w \) and \( \psi \) are defined by

\[ l_{k,w} := \begin{cases} 1 \quad & w = m^2 \ (m \in \mathbb{N}) \\ \frac{2^w - 1}{2^w}; & w \neq m^2 \ (m \in \mathbb{N}), \end{cases} \]
\( H_w(u) := u + \tanh \left( \frac{u}{w} \right) \) and \( \psi(|u|) := |u| \). Then, if \( w = m^2 (m \in \mathbb{N}) \), we have

\[
\sum_{k \in \mathbb{Z}} |l_{k,w}| = 2 \left( \frac{2^w + 1}{2^w - 1} \right) \leq 6
\]

and if \( w \neq m^2 \), we have

\[
\sum_{k \in \mathbb{Z}} |l_{k,w}| = 1,
\]

which implies \((l_1)\) for \( A = 6 \).

For \((l_2)\) and \((9)\), consider the following inequality

\[
\left| \sum_{w=v}^{n+\nu-1} \frac{1}{n} \sum_{k \in \mathbb{Z}} |l_{k,w} - 1| \right| \leq \sum_{w=v}^{n+\nu-1} \frac{1}{n} \left| \sum_{k \in \mathbb{Z}} l_{k,w} - 1 \right|
\]

\[
\leq \frac{5}{n} \left( \sqrt{n + v - 1} - \sqrt{v} + 1 \right)
\]

\[
= \frac{5(n-1)}{n \left( \sqrt{n + v - 1} + \sqrt{v} \right)} + \frac{5}{n}
\]

\[
\leq \frac{5}{\sqrt{n + v - 1} + \sqrt{v}} + \frac{5}{n}
\]

\[
\leq \frac{10}{\sqrt{n}} = O \left( \frac{1}{\sqrt{n}} \right) \quad \text{(uniformly in } v)\]

which proves \((l_2)\) and \((9)\).

For \((l_3)\) and \((11)\), if \( w = m^2 \), then

\[
\sum_{k \in \mathbb{Z}} |l_{k,w}| = 4 \left( \frac{2^w}{2^w - 1} \right) \frac{1}{2^{w^2}}
\]

and if \( w \neq m^2 \),

\[
\sum_{k \in \mathbb{Z}} |l_{k,w}| = 2 \left( \frac{2^w - 1}{2^w + 1} \right) \left( \frac{2^w}{2^w - 1} \right) \frac{1}{2^{w^2}}
\]

hold. Therefore, we get the following expression

\[
\sum_{w=v}^{n+\nu-1} \frac{1}{n} \sum_{k \in \mathbb{Z}} |l_{k,w}| \leq \frac{4}{n} \sum_{w=v}^{n+\nu-1} \left( \frac{2^w}{2^w - 1} \right) \frac{1}{2^{w^2}}
\]

\[
\leq \frac{8}{n} \sum_{w=v}^{n+\nu-1} \frac{1}{2^{w^2}}
\]

\[
\leq \frac{8}{n} \sum_{w=0}^{\infty} \frac{1}{2^{w^2}}
\]

\[
= \frac{8}{n} \left( \frac{2'}{2'} - 1 \right),
\]
which shows \((l_3)\) is satisfied for \(r = 1\). Furthermore, by the fact that for all \(r \geq 1\)
\[
\left(\frac{2^r}{2^r - 1}\right) \leq 2
\]
and so, we conclude
\[
\sum_{w=0}^{n+\nu-1} \frac{1}{n} \sum_{H \in \mathcal{G}} |h_{w}| \leq \frac{16}{n} \leq \frac{16}{\sqrt{n}}
= O\left(\frac{1}{\sqrt{n}}\right) \text{ (uniformly in } \nu\).
\]

For the condition (10), we may clearly get
\[
\frac{1}{n} \sum_{w=0}^{n+\nu-1} \frac{1}{w} \leq \frac{2(\sqrt{n+\nu} - 1)}{n}
\leq \frac{2n(\nu+1)}{n}\sqrt{n+\nu+1} + \sqrt{n}
\leq \frac{2}{\nu^n}
= O\left(\frac{1}{\nu^n}\right) \text{ (uniformly in } \nu\).
\]

Moreover, by the definition of \(F\), we obtain the following
\[
c_{\nu} \leq \frac{1}{n} \leq \frac{1}{\sqrt{\nu}}
= O\left(\frac{1}{\sqrt{n}}\right) \text{ (uniformly in } \nu\).
\]

On the other hand, by the definition of \(H_{w}\), it is clear that \(H_{w}(0) = 0\) and \(H_{w}\) is 1-Lipschitz (see also Figure 1). In addition, \(G_{w}(u) = H_{w}(u) - u = \tanh\left(\frac{u}{w}\right)\) is an increasing function and hence choosing \(\lambda = \gamma\) and \(J = [a, b]\) we have the following equality
\[
\frac{V_{\phi} [\gamma G_{w}, J]}{\phi (y m (J))} = \frac{\phi (\gamma (G_{w}(b) - G_{w}(a)))}{\phi (y m (J))}.
\]

Furthermore, by the convexity of \(\psi\)
\[
\frac{V_{\phi} [\gamma G_{w}, J]}{\phi (y m (J))} \leq \frac{\phi (\gamma (\frac{b}{w} - \frac{a}{w}))}{\phi (y m (J))}
\leq \frac{1}{w} \frac{\phi (\gamma (b - a))}{\phi (y m (J))}
= \frac{1}{w}
\]
holds, where \(1/w \to 0\) as \(w \to \infty\). Then we obtain from (14) that
\[
\frac{1}{n} \sum_{w=0}^{n+\nu-1} \frac{1}{w} \leq \frac{1}{n} \sum_{w=0}^{n+\nu-1} \frac{1}{\sqrt{w}}
= O\left(\frac{1}{\sqrt{n}}\right) \text{ as } n \to \infty \text{ (uniformly in } \nu\)
\]
which verifies (13) and (14).
I. Aslan, Filomat 35:8 (2021), 2731–2746

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