Recency-based preferential attachment models

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Abstract

Preferential attachment models were shown to be very effective in predicting such important properties of real-world networks as the power-law degree distribution, small diameter, etc. Many different models are based on the idea of preferential attachment: LCD, Buckley-Osthus, Holme-Kim, fitness, random Apollonian network, and many others.

Although preferential attachment models reflect some important properties of real-world networks, they do not allow to model the so-called recency property. Recency property reflects the fact that in many real networks vertices tend to connect to other vertices of similar age. This fact motivated us to introduce a new class of models – recency-based models. This class is a generalization of fitness models, which were suggested by Bianconi and Barabási. Bianconi and Barabási extended preferential attachment models with pages’ inherent quality or fitness of vertices. When a new vertex is added to the graph, it is joined to some already existing vertices that are chosen with probabilities proportional to the product of their fitness and incoming degree.

We generalize fitness models by adding a recency factor to the attractiveness function. This means that pages are gaining incoming links according to their attractiveness, which is determined by the incoming degree of the page (current popularity), its inherent quality (some page-specific constant) and age (new pages are gaining new links more rapidly).

We analyze different properties of recency-based models. For example, we show that some distributions of inherent quality lead to the power-law degree distribution.
1 Introduction

Numerous models have been suggested to reflect and predict the growth of the Web [4, 6, 10], the most well-known ones are preferential attachment models. One of the first attempts to propose a realistic mathematical model of the Web growth was made in [2]. The main idea is to start with the assumption that new pages often link to old popular pages. Barabási and Albert defined a graph construction stochastic process, which is a Markov chain of graphs, governed by the \textit{preferential attachment}. At each step in the process, a new node is added to the graph and is joined to \( m \) different nodes already existing in the graph that are chosen with probabilities proportional to their incoming degree (the measure of popularity). This model successfully explained some properties of the Web graph like its small diameter and power law distribution of incoming degrees. Later, many modifications to the Barabási–Albert model have been proposed, e.g., [7, 8, 9], in order to more accurately depict these but also other properties (see [1, 5] for details).

It was noted by Bianconi and Barabási in [3] that in real networks some nodes are gaining new incoming links not only because of their incoming degree (popularity), but also because of their own intrinsic properties. Motivated by this observation, Bianconi and Barabási extended preferential attachment models with pages’ inherent quality or \textit{fitness} of nodes. When a new node is added to the graph, it is joined to some already existing nodes that are chosen with probabilities proportional to the product of their fitness and incoming degree.

One of the main drawbacks of these models is that they pay too much attention to old pages and do not realistically explain how links pointing to newly-created pages appear. For example, most new media pages like news and blog posts are popular only for a short period of time, i.e., such pages are mostly cited and visited for several days after they appeared. In [11] a \textit{recency property} was introduced, which reflects the fact that new media pages tend to connect to other media pages of similar age. Namely, for the media related part of the Web it was shown that \( e(T) \) the fraction of edges connecting nodes whose age difference is greater than \( T \) decreases exponentially fast.

Although preferential attachment models reflect some important properties of real-world networks, they do not allow to model the \textit{recency property}. Here we discuss recency-based models – a generalization of fitness models, where a recency factor is added to the attractiveness function. This means that pages are gaining incoming links according to their \textit{attractiveness}, which
is determined by the incoming degree of the page (current popularity), its inherent quality (some page-specific constant) and age (new pages are gaining new links more rapidly).

Recency-based models have been already studied in the mean field approximation and compared with the previous models by estimating the likelihood of the real-world link graph given each model in [11]. One of the most surprising results is that in the Media Web the probability for a post to be cited is determined, most likely, by its inherent quality rather than by its current popularity.

In this paper, we theoretically analyze different properties of some recency-based models more thoroughly using combinatorial approach. Our analysis shows that for the considered models the power law distribution of inherent quality leads to the power-law degree distribution. We also analyze recency property, i.e., the behavior of $e(T)$.

2 Model

In this section we formalize the models introduced in [11]. We construct a sequence of random graphs $\{G_n\}$. This sequence has the following parameters: a positive integer constant $m$ (vertex outdegree) and an integer function $N(n)$. We also need a sequence of mutually independent random variables $\zeta_1, \zeta_2, \ldots$ with some given distribution taking positive values. Each graph $G_n$ is defined according to its own constructing procedure which is based on the idea of preferential attachment.

Let us now define the random graph $G_n$. At the beginning of the constructing process we have two vertices and one edge between them (graph $\tilde{G}^0_n$). The first two vertices have inherent qualities $q(1) := \zeta_1$ and $q(2) := \zeta_2$. At the $t + 1$-th step ($2 \leq t \leq n - 1$) one vertex and $m$ edges are added to $\tilde{G}^t_n$. New vertex $t + 1$ has an inherent quality $q(t + 1) := \zeta_{t+1}$. New edges are drawn independently and they go from the new vertex to previous vertices. For each edge the probability that it goes to a vertex $i$ ($1 \leq i \leq t$) is equal to

$$\frac{\text{attr}_t(i)}{\sum_{j=1}^{t} \text{attr}_t(j)},$$

where

$$\text{attr}_t(i) = (1 \text{ or } q(i)) \cdot (1 \text{ or } d_t(i)) \cdot \left(1 \text{ or } I[i > t - N(n)] \text{ or } e^{-\frac{t-i}{N(n)}}\right)$$

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and $d_t(i)$ is the degree of the vertex $i$ in $\tilde{G}^n_t$. Further we omit $n$ in the notation $N(n)$. According to the definition loops are not allowed, although multiple edges may appear.

It is important to note that in contrast to standard definitions of preferential attachment models, in our case a graph $G_n$ cannot be obtained from a graph $G_{n-1}$. Each graph has its own constructing procedure which is based on preferential attachment. This unusual definition allows us to analyze both the power law degree distribution and the behavior of $e(T)$, which is the fraction of edges connecting vertices $i$ and $j$ with $|i-j| > T$.

According to the definition, 12 different models are possible. If $\text{attr}_t(i) = d_t(i)$, then we get the preferential attachment model, and if $\text{attr}_t(i) = q(i) d_t(i)$, we get the fitness model. Since models without a recency factor (i.e., without $I[i > t - N]$, or $e^{-\frac{i}{N}}$, or other functions decreasing with age) in the attractiveness function were previously studied, we are interested here in models with the recency factor. Therefore the following attractiveness functions are interesting:

1. $q(i) I[i > t - N]$,
2. $q(i) d_t(i) I[i > t - N]$,
3. $q(i) e^{-\frac{i}{N}}$,
4. $q(i) d_t(i) e^{-\frac{i}{N}}$.

It was shown empirically in [11] that the attractiveness based on quality and recency better reflects the behavior of the Media Web than the attractiveness based on degree, quality, and recency. Therefore, in this paper we focus on the attractiveness functions (1) and (3).

Note that in [11] only the recency factor $e^{-\frac{i}{N}}$ was considered. In this paper we also introduce the recency factor $I[i > t - N]$. We do this because of two reasons. First, as we show in this paper, both recency factors are similar in terms of the degree distribution, but the theoretical analysis of the recency factor $I[i > t - N]$ is less complicated, therefore it can be considered as the natural first step. Second, the recency factor $I[i > t - N]$ has the following natural interpretation. Links to a lot of media pages can usually be found on some pages which are content sources. And new pages are popular while they can be found on such content sources. After some period of time other new pages appear on a content source and they replace old ones. Therefore
it is natural to assume that after some period of time old pages become unpopular.

3 Attractiveness function \( q(i)I[i > t - N] \)

In this section, we assume that the attractiveness function of a vertex \( i \) is \( \text{attr}_t(i) = q(i)I[i > t - N] \). Once again, indicator function means that a vertex \( i \) accumulates incoming edges only during the next \( N \) steps after its appearance and we call this period a lifespan of a vertex. We say that during this lifespan a vertex is alive, after this period a vertex dies.

According to [11], we also assume that the random variables \( \zeta_1, \zeta_2, \ldots \) have the Pareto distribution with the density function \( f(x) = \frac{\gamma a^\gamma}{x^{\gamma+1}}I[x>a] \), where \( \gamma > 1, a > 0 \). Further we denote by \( \zeta \) a random variable with the Pareto distribution defined above.

Finally, our random graph has the following parameters: 1) number of vertices \( n \), 2) vertex outdegree \( m \), 3) lifespan length \( N \), 4) quality exponent \( \gamma \), and 5) minimal quality \( a \).

3.1 Degree distribution

3.1.1 Results

Let \( \#(d) \) be the number of vertices with degree \( d \) in \( G_n \). We prove the following theorem.

**Theorem 1.** Assume that \( d = d(n) \) increases with \( n \) and \( d = o\left(\left(\frac{N}{n}\right)^{\frac{1}{\gamma+1}}\right) \).

If \( \gamma > 2 \) and \( d = o\left(N^{\frac{\alpha}{\gamma+1}}\right) \) or \( 1 < \gamma \leq 2 \) and \( d = o\left(N^{\frac{\alpha}{\gamma+\alpha+1}}\right) \) for any \( 1 < \alpha < \gamma \), then

\[
\frac{\text{E}\#(d)}{n} = \frac{\gamma}{d^{\gamma+1}} \left(\frac{(\gamma - 1)m}{\gamma}\right)^\gamma (1 + o(1)).
\]

Theorem 1 shows that the expectation of the number of vertices with degree \( d \) decreases as \( d^{-\gamma-1} \). In order to get the power law degree distribution we also need to prove the concentration of the number of vertices with degree \( d \) near its expectation.
Theorem 2. For every $d$ the following inequality holds:

$$P\left( |\#(d) - E\#(d)| \geq \sqrt{Nn \log n} \right) \leq \frac{2}{\log n}.$$ 

Note that for $d = o\left( \left( \frac{n}{N \log n} \right)^{1/2(\gamma + 1)} \right)$ we have $\sqrt{Nn \log n} = o\left( \frac{n}{d^{\gamma + 1}} \right)$, so Theorem 2 gives the concentration.

We prove Theorem 1 in Sections 3.1.2 and 3.1.3. Theorem 2 is proved in Section 3.1.4.

3.1.2 Concentration of the weight

Let us now fix some $n$ and $N = N(n)$. In this section we consider only the vertices $N \leq p \leq n - N + 1$.

Let us denote by $\bar{d}(p)$ the degree of a vertex $p$ after its death and by $\bar{d}_{in}(p)$ the incoming degree of a vertex $p$ after its death, i.e., $\bar{d}_{in}(p) = \bar{d}(p) - m$. By $Q(t)$ we denote the sum of qualities of the alive vertices at the $t$-th step, i.e.,

$$Q(t) = \sum_{k=t-N}^{t-1} q(k).$$

We also say that $Q(t)$ is the weight of vertices at $t$-th step. Note that

$$E\left( \bar{d}_{in}(p) \mid q(p - N + 1), \ldots, q(p + N - 1) \right) = \sum_{i=1}^{N} \frac{mq(p)}{Q(p+i)}.$$

Indeed, for each $1 \leq i \leq N$ the probability of an edge $(p + i, p)$ is equal to $\frac{mq(p)}{Q(p+i)}$ according to the definition of the model, since $Q(p + i)$ is the overall attractiveness of all vertices at $(p + i)$-th step.

Consider the lifespan of a vertex $p$ with a quality $q(p)$. We have $E(Q(p + i) \mid q(p)) = q(p) + (N - 1)E\xi$ for $1 \leq i \leq N$. We want to estimate the probability of this weight $E(Q(p + i) \mid q(p))$ to deviate from the value $NE\xi$.

Let $\xi_1, \ldots, \xi_{N-1}$ be the weights of vertices $p - N + 1, \ldots, p - 1$ and $\eta_1, \ldots, \eta_{N-1}$ be the weights of vertices $p + 1, \ldots, p + N - 1$. Let $W_p^q(i)$ be the overall weight of all living vertices when the age of $p$ equals $i$ given that $p$ has the quality $q$, i.e.,

$$W_p^q(i) = \sum_{k=1}^{i-1} \eta_k + q + \sum_{k=i}^{N-1} \xi_k.$$
We will need the following lemma.

**Lemma 1.** Let \( \xi_1, \ldots, \xi_n \) be mutually independent random variables, \( E\xi_i = 0, E|\xi_i|^\alpha < \infty, 1 \leq \alpha \leq 2 \), then

\[
E|\xi_1 + \ldots + \xi_n|^\alpha \leq 2^\alpha (E|\xi_1|^\alpha + \ldots + E|\xi_n|^\alpha).
\]

**Proof.**

We use the following two facts.

**Fact 1.** If \( \xi \) and \( \eta \) are independent random variables and \( \eta \) is symmetrically distributed, then for any \( 1 \leq \alpha \leq 2 \)

\[
E|\xi + \eta|^\alpha \leq E|\xi|^\alpha + E|\eta|^\alpha.
\]

**Proof.**

\[
E|\xi + \eta|^\alpha = \frac{1}{2} (E|\xi + \eta|^\alpha + E|\xi - \eta|^\alpha)
\]

and it remains to show that for any \( x, y, \) and \( 1 \leq \alpha \leq 2 \) we have

\[
\frac{1}{2} (|x + y|^\alpha + |x - y|^\alpha) \leq |x|^\alpha + |y|^\alpha.
\]

Without loss of generality we assume that \( x \geq y \geq 0 \) and consider the function \( f(x, y) = \frac{1}{2} ((x + y)^\alpha + (x - y)^\alpha) - x^\alpha - y^\alpha \). In order to show that \( f(x, y) \leq 0 \) we note that \( f(x, 0) = 0 \) and \( \frac{\partial f(x, y)}{\partial y} \leq 0 \). In turn, \( \frac{\partial^2 f(x, y)}{\partial x \partial y} \leq 0 \).

**Fact 2.** If \( \alpha \geq 1, \xi \) and \( \eta \) are independent random variables, \( E\eta = 0, E|\xi|^\alpha < \infty, E|\eta|^\alpha < \infty \), then

\[
E|\xi + \eta|^\alpha \geq E|\xi|^\alpha.
\]

**Proof.** Fact 2 follows directly from Jensen’s inequality.

Now, let us prove Lemma 1. Consider random variables \( \xi'_1, \ldots, \xi'_n \), such that \( \xi'_i \) has the same distribution as \( \xi_i \) and \( \xi_1, \ldots, \xi_n, \xi'_1, \ldots, \xi'_n \) are mutually independent. Then from Facts 1 and 2 it follows that

\[
E|\xi_1 + \ldots + \xi_n|^\alpha \leq E|\xi_1 - \xi'_1 + \ldots + \xi_n - \xi'_n|\alpha \leq E|\xi_1 - \xi'_1|^\alpha + \ldots + |\xi_n - \xi'_n|^\alpha.
\]

Finally, it remains to note that \( E|\xi_i - \xi'_i|^\alpha \leq 2^\alpha E|\xi_i|^\alpha \).

\[\square\]
Theorem 3. Consider a vertex $p$ such that $N \leq p \leq n - N + 1$ and some constant $c > 0$ such that $|q(p) - E\zeta| \leq Nc/2$. There are two possibilities:

1) if $\gamma > 2$, then

$$P \left( \max_{1 \leq i \leq N} |W^q_p(i) - NE\zeta| \geq Nc \right) \leq \frac{48 Var(\zeta)}{N^{2c - 1}};$$

2) if $1 < \gamma \leq 2$ and $1 < \alpha < \gamma$, then

$$P \left( \max_{1 \leq i \leq N} |W^q_p(i) - NE\zeta| \geq Nc \right) \leq \frac{320 E|\zeta - E\zeta|^\alpha}{N^{\alpha c - 1}}.$$

Proof.

Note that

$$P \left( \max_{1 \leq i \leq N} |W^q_p(i) - EW^q_p(1)| \geq x \right) \leq P (|W^q_p(1) - EW^q_p(1)| \geq x/2) + P \left( \max_{2 \leq i \leq N} |W^q_p(i) - W^q_p(1)| \geq x/2 \right).$$

Indeed, $\max_{1 \leq i \leq N} |W^q_p(i) - EW^q_p(1)| \leq |W^q_p(1) - EW^q_p(1)| + \max_{2 \leq i \leq N} |W^q_p(i) - W^q_p(1)|$ and if $\max_{1 \leq i \leq N} |W^q_p(i) - EW^q_p(1)| \geq x$ then either $|W^q_p(1) - EW^q_p(1)| \geq x/2$ or $\max_{2 \leq i \leq N} |W^q_p(i) - W^q_p(1)| \geq x/2$.

In the case $\gamma > 2$ the random variables have finite variances and we can apply Chebyshev’s and Kolmogorov’s inequalities.

Chebyshev’s inequality gives

$$P (|W^q_p(1) - EW^q_p(1)| \geq x/2) \leq \frac{4NVar(\zeta)}{x^2}.$$

Kolmogorov’s inequality gives

$$P \left( \max_{2 \leq i \leq N} |W^q_p(i) - W^q_p(1)| \geq x/2 \right) =$$

$$= P \left( \max_{1 \leq i \leq N - 1} \sum_{k=1}^i (\eta_k - \xi_k) \geq x/2 \right) \leq \frac{8NVar(\zeta)}{x^2}.$$  

So, finally we get

$$P \left( \max_{1 \leq i \leq N} |W^q_p(i) - EW^q_p(1)| \geq x \right) \leq \frac{12NVar(\zeta)}{x^2}.$$
Take \( x = N^c/2 \) and note that \( |EW_p^q(1) - NE\zeta| = |q(p) - E\zeta| \leq N^c/2 \). Therefore we get

\[
P\left( \max_{1 \leq i \leq N} |W_p^q(i) - NE\zeta| \geq N^c \right) \leq \frac{48 \Var(\zeta)}{N^{2c-1}}.
\]

Now consider the case \( 1 < \gamma \leq 2 \). Fix some constant \( \alpha \) with \( 1 < \alpha < \gamma \).

Instead of Chebyshev’s inequality, we can apply Markov’s inequality and Lemma 1:

\[
P(|W_p^q(1) - EW_p^q(1)| \geq x/2) = P\left( |W_p^q(1) - EW_p^q(1)|^\alpha \geq (x/2)^\alpha \right) \leq \frac{\E|W_p^q(1) - EW_p^q(1)|^\alpha}{(x/2)^\alpha} \leq \frac{4^\alpha N \E|\zeta - E\zeta|^\alpha}{x^\alpha}.
\]

Instead of Kolmogorov’s inequality, we use Doob’s martingale inequality and Lemma 1. Note that \( S_i = \sum_{j=1}^i (\eta_j - \xi_j) \) is a submartingale as a convex function of a martingale. Thus,

\[
P\left( \max_{1 \leq i \leq N-1} \left| \sum_{j=1}^i (\eta_j - \xi_j) \right| \geq x/2 \right) \leq \frac{\E\left| \sum_{j=1}^{N-1} (\eta_j - \xi_j) \right|^\alpha}{(x/2)^\alpha} \leq \frac{4^\alpha N \E|\eta_1 - \xi_1|^\alpha}{x^\alpha} \leq \frac{8^\alpha N \E|\zeta - E\zeta|^\alpha}{x^\alpha}.
\]

So, finally we get

\[
P\left( \max_{1 \leq i \leq N} |W_p^q(i) - EW_p^q(1)| \geq x \right) \leq \frac{4^\alpha (2^\alpha + 1) \N \E|\zeta - E\zeta|^\alpha}{x^\alpha}.
\]

Now take \( x = N^c/2 \) and note that \( |EW_p^q(1) - NE\zeta| \leq N^c/2 \). As before, we can estimate

\[
P\left( \max_{1 \leq i \leq N} |W_p^q(i) - NE\zeta| \geq N^c \right) \leq \frac{8^\alpha (2^\alpha + 1) \N \E|\zeta - E\zeta|^\alpha}{N^{\alpha c}} \leq \frac{320 \E|\zeta - E\zeta|^\alpha}{N^{\alpha c - 1}}.
\]
3.1.3 Expectation

Let \( \rho(d, q) \) be the conditional probability that a vertex \( p \) such that \( N \leq p \leq n - N + 1 \) with a quality \( q \) has an in-degree \( d \), i.e., \( \rho(d, q) = P(\bar{d}_\text{in}(p) = d | q(p) = q) \). Note that \( \rho(d, q) \) does not depend on \( p \). By \( \#_\text{in}(d) \) we denote the number of vertices with in-degree \( d \), so \( \#_\text{in}(d) = \#(d + m) \). The expectation of \( \#_\text{in}(d) \) is

\[
E\#_\text{in}(d) = (n - 2N) \int_a^\infty f(q) \rho(d, q) dq + r(N),
\]

where \( f(q) \) is the density function of Pareto distribution defined above and \( r(N), 0 \leq r(N) \leq 2N, \) is the error term. We have this error term since the first and the last \( N \) vertices behave differently.

Let \( c \) be some positive constant. We estimate the integral

\[
I = \int_a^\infty f(q) \rho(d, q) dq = \int_a^{N_c/2} f(q) \rho(d, q) dq + \int_{N_c/2}^{\infty} f(q) \rho(d, q) dq = I_1 + I_2.
\]

Note that

\[
I_2 = \int_{N_c/2}^{\infty} f(q) \rho(d, q) dq \leq \int_{N_c/2}^{\infty} f(q) dq = \int_{N_c/2}^{\infty} \frac{\gamma a^\gamma}{q^{\gamma+1}} dq = \frac{(2a)^\gamma}{N_c^\gamma}.
\]

Consider the event

\[
A = \left\{ \max_{1 \leq i \leq N} |Q(p + i) - NE\zeta| \leq N_c \right\}
\]

and the following conditional probabilities:

\[
\rho_A(d, q) = P(\bar{d}_\text{in}(p) = d | q(p) = q, A),
\]

\[
\rho_{\bar{A}}(d, q) = P(\bar{d}_\text{in}(p) = d | q(p) = q, \bar{A}).
\]

Then we have

\[
\rho(d, q) = \rho_A(d, q) P(A | q(p) = q) + \rho_{\bar{A}}(d, q) P(\bar{A} | q(p) = q).
\]

Let us use this representation to split \( I_1 \) into two integrals using (3):

\[
I_1 = \int_a^{N_c/2} f(q) \rho_A(d, q) P(A_q) dq + \int_{N_c/2}^{N_c} f(q) \rho_{\bar{A}}(d, q) P(\bar{A}_q) dq = I_1^1 + I_1^2,
\]

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where we use the following notation:

\[ A_q = [\{ A|q(p) = q \} = \left\{ \max_{1 \leq i \leq N} |W_p^q(i) - NE\zeta| \leq N^c \right\} \),
\]

\[ \tilde{A}_q = [\{ \tilde{A}|q(p) = q \} = \left\{ \max_{1 \leq i \leq N} |W_p^q(i) - NE\zeta| > N^c \right\} \].

Let us assume that \( N^c/2 > E\zeta \), this holds if \( N \) is large enough (the fact that \( N \) grows follows from the statement of Theorem 1, while \( E\zeta \) is constant). Note that

\[ I_1^2 \leq \max_{q \in N^c/2} P(\tilde{A}_q) \] (4)

and since \( q \leq N^c/2 \) Theorem 3 gives us an upper bound for it, i.e.,

\[ \max_{q \in N^c/2} P(\tilde{A}_q) = O \left( N^{1-2\gamma} \right) \text{ if } \gamma > 2 , \] (5)

\[ \max_{q \in N^c/2} P(\tilde{A}_q) = O \left( N^{1-\alpha c} \right) \text{ if } 1 < \gamma \leq 2 , \] (6)

where \( 1 < \alpha < \gamma \).

So, let us now focus on \( I_1^1 \). First we estimate \( \rho_A(d, q) \). Recall that \( \rho_A(d, q) = P(\tilde{d}_{in}(p) = d|q(p) = q, A) \). Note that during the lifespan of a vertex \( p \) there are \( mN \) mutually independent edges which may lead to \( p \). For an edge from a vertex \( p + i \) the probability to choose \( p \) is \( qW_p^q(i) \). Given the event \( A_q \) we have

\[ NE\zeta - N^c \leq W_p^q(i) \leq NE\zeta + N^c \].

Therefore we have the following bounds for \( \rho_A(q, d) \):

\[ \left( \frac{mN}{d} \right) \left( \frac{q}{NE\zeta + N^c} \right)^d \left( 1 - \frac{q}{NE\zeta - N^c} \right)^{mN-d} \leq \rho_A(d, q) \leq \left( \frac{mN}{d} \right) \left( \frac{q}{NE\zeta - N^c} \right)^d \left( 1 - \frac{d}{NE\zeta + N^c} \right)^{mN-d} \].

Thus, \( (1 - \max_{q \in N^c/2} P(\tilde{A}_q)) S_- \leq I_1^1 \leq S_+ \) where

\[ S_\pm = \int_a^{N^c/2} f(q) \left( \frac{mN}{d} \right) \left( \frac{q}{NE\zeta \pm N^c} \right)^d \left( 1 - \frac{q}{NE\zeta \mp N^c} \right)^{mN-d} dq . \]

We will use the following lemma.
Lemma 2. Assume that both $N$ and $d$ grow but $d = o(N^{1-c})$. If $1/2 \leq c \leq 1$, then

$$S_\pm = \frac{\gamma}{d^{\gamma+1}} \left( \frac{(\gamma - 1)m}{\gamma} \right)^\gamma (1 + o(1)).$$

We placed the proof of this technical lemma to the appendix. Now we will use this lemma to prove the theorem. Using Equations (1), (2), and (4) we get the following bounds for $\frac{E(#(d+m))}{n}$:

$$\left( 1 - \frac{2N}{n} \right) \left( 1 - \max_{q \leq N^{\gamma/2}} P(\bar{A}_q) \right) S_- \leq \frac{E(#(d+m))}{n} \leq \begin{cases} S_\pm + \max_{q \leq N^{\gamma/2}} P(\bar{A}_q) + \left( 2a \right)^\gamma \frac{2N}{n} & \text{if } d = o \left( \frac{(2a)^\gamma}{N^{\gamma/2}} \right) \\ \geq I_1 \end{cases}, \quad \begin{cases} \geq I_2 \end{cases}, \quad \begin{cases} \geq I_2 \\ \geq I_2 \end{cases} . \quad (7)$$

Now we show that for some parameter $c$ all error terms in Equation (7) are negligible in comparison with the main term $d^{-\gamma-1}$ from Lemma 2.

First, consider the case $\gamma > 2$. Take $c = \frac{\alpha + 2}{\gamma + 3}$. Note that we can apply Lemma 2 since $d = o \left( \frac{N^{c\gamma}}{(\gamma+1)} \right)$ due to the statement of Theorem 1.

1. $\frac{(2a)^\gamma}{N^{\gamma/2}} = o(d^{-\gamma-1})$ if $d = o \left( \frac{N^{c\gamma}}{(\gamma+1)} \right)$. This holds for $c = \frac{\gamma + 2}{\gamma + 3}$ and $d = o \left( \frac{1}{N^{\gamma+1}} \right)$.

2. For $\gamma > 2$, $P(\bar{A}_q) = O \left( N^{1-2c} \right) = o(d^{-\gamma-1})$ if $d = o \left( \frac{N^{(2c-1)/(\gamma+1)}}{(\gamma+1)} \right)$, i.e., $d = o \left( \frac{1}{N^{1/(\gamma+3)}} \right)$. Here we used Equation (5).

3. $N = o \left( n d^{-\gamma-1} \right)$, since $d = o \left( \left( \frac{n}{N} \right)^{\frac{1}{\gamma+1}} \right)$.

Consider the case $\gamma \leq 2$. Take $c = \frac{\gamma + 2}{\gamma + \alpha + 1}$. Note that we can apply Lemma 2 since $d = o \left( N^{1-c} \right)$ due to the statement of Theorem 1.

1. $\frac{(2a)^\gamma}{N^{\gamma/2}} = o(d^{-\gamma-1})$ if $d = o \left( \frac{N^{c\gamma}}{(\gamma+1)} \right)$. This holds for $c = \frac{\gamma + 2}{\gamma + \alpha + 1}$ and $d = o \left( \frac{1}{N^{\gamma+1}} \right)$.

2. For $\gamma \leq 2$, $P(\bar{A}_q) = O \left( N^{1-c\alpha} \right) = o(d^{-\gamma-1})$ if $d = o \left( \frac{N^{(c\alpha-1)/(\gamma+1)}}{(\gamma+1)} \right)$, i.e., $d = o \left( \frac{1}{N^{\gamma+1}} \right)$. Here we used Equation (6).
3. \( N = o(nd^{-\gamma-1}) \), since \( d = o\left(\left(\frac{n}{N}\right)^{-\frac{1}{\gamma+1}}\right) \).

It remains to note that the asymptotic for \( \#(d) \) is the same as for \( \#(d + m) \). This concludes the proof of Theorem 1.

### 3.1.4 Concentration

We use Chebyshev’s inequality to prove concentration. In order to do this we first estimate \( \text{Var}(\#(d)) \). Note that if \( |i - j| \geq N \) then the degrees of \( i \) and \( j \) are independent. Therefore

\[
\text{Var}(\#(d)) = \sum_{i,j=1}^{n} (\mathbb{P}(d_n(i) = d, d_n(j) = d) - \mathbb{P}(d_n(i) = d) \mathbb{P}(d_n(j) = d)) \leq 2nN.
\]

Applying Chebyshev’s inequality we get

\[
\mathbb{P}\left(\left|\#(d) - \mathbb{E}(\#(d))\right| > \sqrt{Nn \log n}\right) \leq \frac{\text{Var}(\#(d))}{Nn \log n} \leq \frac{2}{\log n}.
\]

**Remark.** Note that instead we could use Azuma–Hoeffding inequality, since \( |\mathbb{E}(\#(d)|G_i) - \mathbb{E}(\#(d)|G_{i-1})| \leq (N + 1)m \). In this case we get

\[
\mathbb{P}\left(\left|\#(d) - \mathbb{E}(\#(d))\right| \geq \sqrt{n \log n (N + 1)}\right) \leq 2n^{-1/2m^2}.
\]

So, on the one hand the range of degrees for which we get concentration is smaller in this case. We get concentration for \( d = o\left(\left(\frac{\sqrt{n}}{N \log n}\right)^{1/(\gamma+1)}\right) \). On the other hand, the concentration is tighter, so we can say that for all \( d \) in this range the number of vertices of degree \( d \) is near its expectation.

### 3.2 Recency property

Let \( e(T) \) be the fraction of edges in a graph which connect vertices with age difference greater than \( T \), i.e., vertices \( i \) and \( j \) with \( |i - j| > T \). In [11] a **recency property** was introduced, which reflects the fact that new media pages tend to connect to other media pages of similar age. Namely, for the media related part of the Web it was shown that \( e(T) \) decreases exponentially fast. In this section we show that we have linear decay of \( e(T) \) for the model under consideration.
Theorem 4. For any integer $T$

$$Ee(T) = \begin{cases} 
1 - \frac{T}{N} + O\left(\frac{N}{n}\right), & \text{if } T \leq N; \\
0, & \text{if } T > N.
\end{cases}$$

Proof. Consider any vertex $n > N$ and any edge $ni$, $i < n$, drawn from this vertex. The probability that $n - i > T$ is the probability to choose one vertex from $n - N, \ldots, n - T - 1$. Since qualities of vertices are i.i.d. random variables, this probability equals $\frac{N - T}{N}$. From this the theorem follows.

\[\square\]

Theorem 5.

$$P\left(|e(T) - Ee(T)| \geq \sqrt{\frac{\log n}{n}}\right) \leq \frac{1}{m \log n}.$$

Proof. Here we again use Chebyshev’s inequality. Let $e_1$ and $e_2$ be any two different edges in our graph. Let $l(e_i)$ be the age difference between endpoints of the edge $e_i$. It is easy to see that

$$P(l(e_1) > T, l(e_2) > T) - P(l(e_1) > T)P(l(e_2) > T) = 0.$$

Indeed, if all qualities are fixed, then edges are independent. Then we just integrate over all qualities.

From this we get $Var(mn e(T)) \leq mn$, since we take into account only the summands corresponding to coinciding edges and $P(l(e_1) > T, l(e_1) > T) - P(l(e_1) > T)P(l(e_1) > T) \leq 1$.

Therefore

$$P\left(mn |e(T) - Ee(T)| \geq m \sqrt{n \log n}\right) \leq \frac{Var(mn e(T))}{m^2 n \log n} \leq \frac{1}{m \log n}.$$

\[\square\]

4 Attractiveness function $q(i)e^{-\frac{t-i}{N}}$

Now we switch to the attractiveness function $q(i)e^{-\frac{t-i}{N}}$. In this case, the popularity of a vertex decreases exponentially with the age of the vertex. Again, we assume that the random variables $\zeta_1, \zeta_2, \ldots$ have the Pareto distribution with the density function $f(x) = \frac{\gamma a^{\gamma}}{x^{\gamma+1}}$, where $\gamma > 1$, $a > 0$. And $\zeta$ again is a random variable with the Pareto distribution defined above.
4.1 Degree distribution

4.1.1 Results

For the model with exponential recency factor we get the results similar to ones for the model with indicator recency factor (see Section 3.1.1).

**Theorem 6.** Assume that \( d = d(n) \) increases with \( n \) and \( d = o \left( \left( \frac{n}{N \log N} \right)^{\frac{1}{1+\gamma}} \right) \). If \( \gamma > 2 \) and \( d = o \left( \left( \frac{n}{N \log n} \right)^{\frac{1}{\gamma+1}} \right) \) or \( 1 < \gamma \leq 2 \) and \( d = o \left( \left( \frac{n}{\alpha N^{\gamma+1}} \right)^{\frac{1}{\gamma+1}} \right) \) for any \( 1 < \alpha < \gamma \), then

\[
\frac{E\#(d)}{n} = \frac{\gamma}{d^{\gamma+1}} \left( \frac{(\gamma-1)m}{\gamma} \right)^{\gamma} \left( 1 + o(1) \right).
\]

Again, the expectation of the number of vertices with degree \( d \) decreases as \( d^{-\gamma-1} \). The next theorem shows that the number of vertices of degree \( d \) is concentrated near its expectation.

**Theorem 7.** For every \( d \) the following inequality holds:

\[
P \left( \left| \#(d) - E\#(d) \right| > \sqrt{N n \log n} \right) = O \left( \frac{1}{\log n} \right).
\]

As before, for \( d = o \left( \left( \frac{n}{N \log n} \right)^{\frac{1}{\gamma+1}} \right) \) we have \( \sqrt{N n \log n} = o \left( E\#(d) \right) \) and Theorem 7 gives the concentration.

We prove Theorem 6 in Sections 4.1.2 and 4.1.3. Theorem 7 is proved in Section 4.1.4.

4.1.2 Concentration of the overall attractiveness

We fix some \( n \) and \( N = N(n) \).

By \( Q(t) \) we denote the total attractiveness of all vertices at \( t \)-th step, i.e.,

\[
Q(t) = \sum_{k=1}^{t-1} q(k) e^{-\frac{t-k-1}{N}}.
\]

The average value of \( Q(t) \) is

\[
E Q(t) = E \zeta \sum_{k=0}^{t-2} e^{-\frac{k}{N}} = E \zeta \frac{1 - e^{-\frac{t-1}{N}}}{1 - e^{-\frac{1}{N}}} = NE\zeta \left( 1 + O \left( e^{-t/N} \right) + O \left( 1/N \right) \right).
\]
If \( t > N \log N \), then
\[
E_Q(t) = NE\zeta \left( 1 + O(1/N) \right).
\]

Again, by \( W^q_p(i) \) we denote the total attractiveness of all vertices when the age of \( p \) equals \( i \) given the quality \( q \) of the vertex \( p \).

**Theorem 8.** Fix some positive constant \( c \). Let \( \varphi(N) \) be any function such that \( \varphi(N) > \log(CN) \) for some \( C > 0 \). Then for any \( p > N\varphi(N) \) with \( |q(p) - E\zeta| \leq N^c/3 \) we have:

1) if \( \gamma > 2 \), then
\[
P\left( \max_{1 \leq i \leq N\varphi(N)} |W^q_p(i) - NE\zeta| \geq N^c \right) = O\left( e^{2\varphi(N)N^{1-2c}} \right);
\]
2) if \( 1 < \gamma \leq 2 \) and \( 1 < \alpha < \gamma \), then
\[
P\left( \max_{1 \leq i \leq N\varphi(N)} |W^q_p(i) - NE\zeta| \geq N^c \right) = O\left( e^{\alpha\varphi(N)N^{1-\alpha c}} \right).
\]

**Proof.**

Note that \( E \left( Q(p+i+1) \mid Q(p+i) \right) = Q(p+i)e^{-\frac{1}{N}} + E\zeta \). Therefore \( X_i = e^\frac{i}{N} \left( Q(p+i) - \frac{E\zeta}{1-e^{-\frac{1}{N}}} \right) \) is a martingale. Indeed,
\[
E(X_{i+1} \mid X_i) = \left( Q(p+i)e^{-\frac{1}{N}} + E\zeta \right) e^{\frac{i+1}{N}} - \frac{e^{\frac{i+1}{N}}E\zeta}{1-e^{-\frac{1}{N}}} = X_i.
\]

So, we can apply Doob’s inequality for a submartingale \( |X_i| \):
\[
P\left( \max_{1 \leq i \leq N\varphi(N)} \left| e^\frac{i}{N} \left( Q(p+i) - \frac{E\zeta}{1-e^{-1/N}} \right) \right| \geq x \right) \leq \frac{E \left( e^{N\varphi(N)} \left( Q(p+N\varphi(N)) - \frac{E\zeta}{1-e^{-1/N}} \right) \right)^\beta}{x^\beta},
\]
where \( \beta > 1 \).

If \( \gamma > 2 \), then we take \( \beta = 2 \) and have
\[
P\left( \max_{1 \leq i \leq N\varphi(N)} \left| Q(p+i) - \frac{E\zeta}{1-e^{-1/N}} \right| \geq N^c/3 \right) \leq \frac{E \left( e^{N\varphi(N)} \left( Q(p+N\varphi(N)) - \frac{E\zeta}{1-e^{-1/N}} \right) \right)^2}{x^2},
\]

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\[ \leq \frac{9e^{2\varphi(N)}E \left( Q(p + N\varphi(N)) - \frac{E\zeta}{1 - e^{-1/N}} \right)^2}{N^{2c}}. \]

Using
\[ \frac{E\zeta}{1 - e^{-1/N}} = EQ(p + N\varphi(N)) + E\zeta \sum_{k=p+N\varphi(N)-1}^{\infty} e^{-\frac{k}{N}}, \]
we get
\[ E \left( Q(p + N\varphi(N)) - \frac{E\zeta}{1 - e^{-1/N}} \right)^2 = \]
\[ = E\left( Q(p + N\varphi(N)) - EQ(p + N\varphi(N)) \right)^2 + (E\zeta)^2 \left( \frac{e^{-p+N\varphi(N)-1}}{1 - e^{-1/N}} \right)^2 = \]
\[ = Var(\zeta) \sum_{k=0}^{p+N\varphi(N)-2} e^{-\frac{2k}{N}} + O \left( \frac{e^{-2(N\log(CN)+N\varphi(N))}}{(1 - e^{-1/N})^2} \right) = \]
\[ = O \left( \frac{1}{1 - e^{-2/N}} + e^{-2\varphi(N)} \right) = O(N). \]

So,
\[ P \left( \max_{1 \leq i \leq N\varphi(N)} \left| Q(p + i) - \frac{E\zeta}{1 - e^{-1/N}} \right| \geq N^c/3 \right) = O \left( e^{2\varphi(N)} N^{1-2c} \right). \]

Now we can estimate \( W^q_p(i) \) which is \( Q(p + i) \) given the quality \( q(p) \) of the vertex \( p \). We have \( |q(p) - E\zeta| \leq N^c/3 \) and \( \left| \frac{E\zeta}{1 - e^{-1/N}} - NE\zeta \right| \leq N^c/3 \) for large \( N \), therefore
\[ P \left( \max_{1 \leq i \leq N\varphi(N)} |W^q_p(i) - NE\zeta| \geq N^c \right) = O \left( e^{2\varphi(N)} N^{1-2c} \right). \]

Similarly, for \( \gamma \leq 2 \) we take \( \beta = \alpha \) and using Lemma 1 we get
\[ P \left( \max_{1 \leq i \leq N\varphi(N)} |W^q_p(i) - NE\zeta| \geq N^c \right) = O \left( e^{\alpha\varphi(N)} N^{1-\alpha} \right). \]
4.1.3 Expectation

Let \( \varphi(N) \) be any function such that \( \varphi(N) > \log(CN) \) for some \( C > 0 \). Let \( \rho(d, q) \) be the conditional probability that a vertex \( p \) such that \( N\varphi(N) \leq p \leq n - N\varphi(N) + 1 \) has an in-degree \( d \) given a quality \( q \) of this vertex, i.e.,

\[
\rho(d, q) = P(d_{in}(p) = d | q(p) = q).
\]

We omit \( n \) and \( p \) in the notation \( \rho(d, q) \) because, as we will see, we get similar bounds for \( \rho(d, q) \) for all \( p \) such that \( N\varphi(N) \leq p \leq n - N\varphi(N) + 1 \). Using this notation, we get the following equality:

\[
E \#_{in}(d) = (n - 2N\varphi(N)) \int_a^\infty f(q) \rho(d, q) dq + r(N),
\]

where \( f(q) \) is the density function of Pareto distribution and \( r(N) \), \( 0 \leq r(N) \leq 2N\varphi(N) \) is the error term.

Let \( r \) and \( c \) be some constants such that \( 0 < r < 1/2 \) and \( 1/2 < c < 1 \). As in Section 3.1.3, we split the integral

\[
I = \int_a^\infty f(q) \rho(d, q) dq = \int_a^{N_r} f(q) \rho(d, q) dq + \int_{N_r}^\infty f(q) \rho(d, q) dq = I_1 + I_2
\]

and

\[
I_2 \leq \int_{N_r}^\infty f(q) dq = \frac{a^r}{N^{r\gamma}}.
\]

The event \( A \) is defined as in Section 3.1.3:

\[
A = \left\{ \max_{1 \leq i \leq N\varphi(N)} |Q(p + i) - NE\zeta| \leq N^c \right\}.
\]

We again split \( I_1 \) into two integrals:

\[
I_1 = \int_a^{N_r} f(q) \rho_A(d, q) P(A_q) dq + \int_{N_r}^{N} f(q) \rho_A(d, q) P(\bar{A}_q) dq = I^1_1 + I^2_1,
\]

where

\[
A_q = [A | q(p) = q] = \left\{ \max_{1 \leq i \leq N\varphi(N)} |W^q_p(i) - NE\zeta| \leq N^c \right\},
\]

\[
\bar{A}_q = [\bar{A} | q(p) = q] = \left\{ \max_{1 \leq i \leq N\varphi(N)} |W^q_p(i) - NE\zeta| > N^c \right\}.
\]
We can estimate
\[ I_1^2 \leq \max_{q \leq N^r} P(\bar{A}_q) \]  
and for \( q \leq N^r \) Theorem 8 gives the upper bound for \( P(\bar{A}_q) \) (since \( N^r < N^c/3 \) and \( |q(p) - E\zeta| \leq N^r \) for large \( N \)):

\[ \max_{q \leq N^r} P(\bar{A}_q) = O\left(e^{2\phi(N)N^{1-2c}}\right) \quad \text{if} \quad \gamma > 2, \tag{11} \]

\[ \max_{q \leq N^r} P(\bar{A}_q) = O\left(e^{\alpha\phi(N)N^{1-\alpha c}}\right) \quad \text{if} \quad 1 < \gamma \leq 2, \tag{12} \]

where \( 1 < \alpha < \gamma \).

Let us now focus on \( I_1 \). Consider an event \( R_q^p(k) \) that there is an edge from at least one vertex \( p + i \) with \( i \geq k \) to a vertex \( p \) with a quality \( q \). Then for \( k > N \) conditional probability of \( R_q^p(k) \) given \( A_q \) can be estimated as follows

\[
P(\{R_q^p(k) \mid A_q\}) \leq \sum_{i=k}^{\infty} P(\text{edge } (p + i, p) \text{ belongs to } G_n \mid A_q) \leq \sum_{i=k}^{\infty} \frac{mq e^{-i/N}}{aN/2} = O\left(q e^{-k/N}\right).
\]

This estimate means that the most contribution to the final degree of a vertex is made during the first several steps after its appearance and we have the following bounds for \( \rho_A(d, q) \):

\[ \rho_A(d, q) = \rho_{\pm}(d, q) + O\left(q e^{-\phi(N)}\right), \]

where \( \rho_{\pm}(d, q) \) are lower and upper bounds for the probability that a vertex \( p \) with a quality \( q \) has an in-degree \( d \) in \( \bar{G}_n^{p+N\varphi(N)} \) given \( A_q \). We can estimate \( \rho_{\pm}(d, q) \) in the following way. A vertex \( p \) has an in-degree \( d \) in \( \bar{G}_n^{p+N\varphi(N)} \) if \( d \) edges out of \( m\varphi(N)N \) are connected to this vertex and others are not. For every set of indexes \( 0 \leq i_1 < \ldots < i_d \leq m\varphi(N)N \) we should multiply the probabilities that the corresponding edges go to the vertex \( p \). Given \( A_q \), these probabilities can be estimated by \( \frac{qe^{-\left(i_j/m\right)/N}}{Ne^{-\varphi(N)}} \). And we should also multiply the obtained product by the probabilities that other edges are not connected to \( p \), i.e., \( \left(1 - \frac{qe^{-\left(i_j/m\right)/N}}{Ne^{-\varphi(N)}}\right) \) for the corresponding indexes \( i \). Finally, we get:
\[
\rho_\pm(d, q) = \prod_{i=0}^{m\varphi(N)N} \left(1 - \frac{qe^{-[i/m]}}{N\zeta \pm N^c} \right) \sum_{0 \leq i_1 < \cdots < i_d \leq m\varphi(N)N} \prod_{j=1}^{d} \frac{qe^{-(i_j/m)}}{N\zeta \pm N^c}.
\]

Now we put
\[
S_\pm(d, q) := \int_a^{N^r} f(q)\rho_\pm(d, q) dq.
\]

Using this notation, we can estimate \(I_1\) in the following way:
\[
I_1 \leq \int_a^{N^r} f(q)\rho_A(d, q) dq \leq S_+ + O\left(\int_a^{\infty} f(q)qe^{-\varphi(N)} dq\right),
\]
(13)
\[
I_1 \geq \left(1 - \max_{q \leq N^r} \mathcal{P}(\bar{A}_q)\right)S_- + O\left(\int_a^{\infty} f(q)qe^{-\varphi(N)} dq\right).\]
(14)

We estimate \(S_\pm\) in the following way.

**Lemma 3.** Assume that both \(d\) and \(N\) grow, \(d = o(N^{1-c})\), \(d = o(e^{\varphi(N)})\), and \(q \leq N^r\), then
\[
S_\pm(d, q) = \frac{\gamma}{d^\gamma} \left(\frac{(\gamma - 1)m}{\gamma}\right)^\gamma (1 + o(1)).
\]

We placed the proof of this technical lemma to the appendix.

Finally, using Equations (8), (9), (10), (13), and (14), we get
\[
\left(1 - \frac{2N\varphi(N)}{n}\right) \left(1 - \max_{q \leq N^r} \mathcal{P}(\bar{A}_q)\right)S_- + O\left(\int_a^{\infty} f(q)qe^{-\varphi(N)} dq\right) \leq \frac{\mathbb{E}(d + m)}{n} \leq \frac{S_+ + O\left(\int_a^{\infty} f(q)qe^{-\varphi(N)} dq\right) + \max_{q \leq N^r} \mathcal{P}(\bar{A}_q) + \frac{a^\gamma}{N^{\gamma\gamma}} + \frac{2N\varphi(N)}{n}}{n}.
\]

We want all remainder terms to be \(o(d^{-\gamma-1})\). In order to do this, we need to find the proper values of \(c\) and \(\varphi(n)\). Note that we have already assumed that \(d = o(N^{1-c})\) and \(d = o(e^{\varphi(N)})\).
1. \[ \int_0^\infty f(q)O\left(qe^{-\varphi(N)}\right) dq = O\left(e^{-\varphi(N)}\right) = o\left(d^{-\gamma-1}\right) \text{ if } d^{\gamma+1} = o\left(e^\varphi(N)\right). \]

2. \[ \frac{\alpha^2}{N^{2\gamma}} = o\left(d^{-\gamma-1}\right) \text{ if } d = o\left(N^{\gamma/(\gamma+1)}\right). \] Put \( r = \frac{\alpha}{22} \), then we have \( d = o\left(N^{\gamma/(\gamma+1)}\right) \) under the conditions of the theorem. Since \( N^{9\gamma}/22(\gamma+1) \geq N^{1/(3\gamma+5)} \) for \( \gamma > 2 \) and \( N^{9\gamma}/22(\gamma+1) \geq N^{\frac{\alpha+1}{\alpha+(\gamma+1)(\alpha+1)}} \) for \( 1 \leq \gamma \leq 2 \).

3. For \( \gamma > 2 \), \( \max_{q \leq N^r} P(A_q) = O\left(e^{2\varphi(N)}N^{1-2c}\right) = o\left(d^{-\gamma-1}\right) \) if \( e^{2\varphi(N)} = o\left(N^{2c-1}d^{-\gamma-1}\right) \). Here we used Equation (11).

   For \( \gamma \leq 2 \), \( \max_{q \leq N^r} P(A_q) = O\left(e^{\alpha\varphi(N)}N^{1-\alpha c}\right) = o\left(d^{-\gamma-1}\right) \) if \( e^{\alpha\varphi(N)} = o\left(N^{\alpha c-1}d^{-\gamma-1}\right) \). Here we used Equation (12).

4. \( N^{\varphi(N)}/n = o\left(d^{-\gamma-1}\right) \) if \( d = o\left(\left(\frac{n}{N^{\varphi(N)}}\right)^{\frac{1}{\gamma+1}}\right) \). This holds under the conditions of the theorem.

Consider the case \( \gamma > 2 \) and take \( c = \frac{3\gamma+4}{3\gamma+5} \), \( \varphi(N) = \log \frac{N(\gamma+1)}{3\gamma+5} \). Then for \( d = o\left(N^{1/(3\gamma+5)}\right) \) all the conditions hold.

Consider the case \( \gamma \leq 2 \) and take \( c = \frac{1+(\gamma+1)(\alpha+1)}{\alpha+(\gamma+1)(\alpha+1)} \), \( \varphi(N) = \log \frac{N(\alpha-1)(\gamma+1)}{\alpha+(\gamma+1)(\alpha+1)} \). Then for and \( d = o\left(N^{\frac{\alpha+1}{\alpha+(\gamma+1)(\alpha+1)}}\right) \) all the conditions hold.

### 4.1.4 Concentration

We prove Theorem 7 using Chebyshev’s inequality. In order to apply this inequality we first estimate \( \text{Var}(\#(d)) \):

\[
\text{Var}(\#(d)) = \sum_{i,j=1}^n \left( P(d_n(i) = d, d_n(j) = d) - P(d_n(i) = d) P(d_n(j) = d) \right).
\]

Let us estimate the difference \( P(d_n(i) = d, d_n(j) = d) - P(d_n(i) = d) P(d_n(j) = d) \) for \( i < j \).

Note that

\[
P(d_j(i) = d, d_n(j) = d) = P(d_j(i) = d) P(d_n(j) = d).
\]

In order to prove this we first show that (15) holds given all the qualities \( q_1, \ldots, q_n \) and then integrate over all qualities. Given the qualities, \( P(d_j(i) = d, d_n(j) = d) \) is the sum over all \( mi < i_1 < \ldots < i_d \leq mj, mj < j_1 < \ldots < j_d \leq mn \) of the probabilities that the corresponding edges \([i_k/m], i\)
and \((\lfloor kj/m \rfloor, j)\) are drawn and all other edges \((i', i)\) with \(i < i' \leq j\) and \((j', j)\) with \(j < j' \leq n\) are absent. Since qualities are fixed, these events are independent and \(P(d_j(i) = d, d_n(j) = d) = P(d_j(i) = d) P(d_n(j) = d)\).

Let \(R_p(k)\) be the event that there is an edge from at least one vertex \(p + i\) with \(i \geq k\) to a vertex \(j\). Then

\[
P(d_n(i) = d, d_n(j) = d) - P(d_n(i) = d) P(d_n(j) = d) \leq
\]

\[
= P(d_j(i) = d) P(d_n(j) = d) + P(R_i(j - i)) - P(d_n(i) = d) P(d_n(j) = d) \leq
\]

\[
= 2 P(R_i(j - i)) = 2 \int_a^\infty R_i^q(j - i) f(q) dq =
\]

\[
= O \left( \int_a^\infty q^{-\gamma - 1} e^{-\frac{j-i}{N}} dq \right) = O \left( e^{-\frac{j-i}{N}} \right).
\]

Finally,

\[
Var(\#(d)) = O \left( \sum_{1 \leq i \leq j \leq n} e^{-\frac{j-i}{N}} \right) = O \left( Nn \right).
\]

Applying Chebyshev’s inequality we get

\[
P(\left| \#(d) - E\#(d) \right| > \sqrt{Nn \log n}) = O \left( \frac{Var(\#(d))}{Nn \log n} \right) = O \left( \frac{1}{\log n} \right).
\]

4.2 Recency property

In this section, we show that the behavior of \(e(T)\) for the model with exponential popularity decay is realistic. It was shown in [11] that \(e(T)\) decreases exponentially with \(T\) in real data.

First, we compute the expectation of \(e(T)\). The following theorem holds.

**Theorem 9.** For any integer \(T\)

\[
Ee(T) = e^{-\frac{T}{N}} + O \left( \frac{N}{n} \right).
\]
Indeed, the probability that an edge from a vertex $k$ goes to a vertex $i$ with $k - i > T$ equals $e^{-\frac{T}{N}} + O\left(e^{-\frac{T}{N}}\right)$. From this Theorem 9 follows.

Exactly as in the Section 3.2, we can use Chebyshev’s inequality to prove the concentration.

**Theorem 10.** For any integer $T$

\[
P\left(|e(T) - \mathbb{E}e(T)| \geq \sqrt{\frac{\log n}{n}}\right) \leq \frac{1}{m \log n}.
\]

These theorems mean that $e(T)$ decays exponentially, as it was observed in real data.

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Appendix

Proof of Lemma 2

First, we prove the following lemma.

Lemma 4. Denote $\frac{N^c}{NE\zeta}$ by $\varepsilon$, then

$$S_\pm = \frac{\gamma a^\gamma (mN\zeta^d)}{(NE\zeta^d)} B (d - \gamma, mN - d + 1) \frac{(1 \mp \varepsilon)^{d-\gamma}}{(1 \pm \varepsilon)^d}.$$  

\begin{equation}
\cdot \left(1 + O\left(\frac{(1 - \frac{\varepsilon}{2+\varepsilon})^{mN-d+1}(mN)^{d-\gamma}}{\Gamma(d - \gamma)(mN - d + 1)}\right) + O\left(\frac{am}{\Gamma(d - \gamma + 1)}\right)\right). \quad (16)
\end{equation}

Proof. Let us rewrite $S_\pm$ using the incomplete beta-function $B(x; a, b)$

$$S_\pm = \int_a^{N^c/2} \frac{\gamma a^\gamma (mN\zeta^d)}{q^{\gamma+1}} \left(\frac{q}{NE\zeta \pm N^c}\right)^d \left(1 - \frac{q}{NE\zeta \mp N^c}\right)^{mN-d} dq =$$

$$= \frac{\gamma a^\gamma (mN\zeta^d)}{(NE\zeta \pm N^c)^d} \cdot \int_a^{N^c/2} \left(\frac{q}{NE\zeta \mp N^c}\right)^{d-\gamma-1} \left(1 - \frac{q}{NE\zeta \pm N^c}\right)^{mN-d} dq =$$
= \frac{\gamma a^\gamma \left( \frac{mN}{d} \right) (NE\zeta \mp Nc)^{d-\gamma}}{(NE\zeta \pm Nc)^d} \cdot \int_{\frac{Nc}{NE\zeta \mp Nc}}^{\frac{Nc}{NE\zeta \mp Nc}} x^{d-\gamma-1}(1-x)^{mN-d}dx =

= \frac{\gamma a^\gamma \left( \frac{mN}{d} \right) (NE\zeta \mp Nc)^{d-\gamma}}{(NE\zeta \pm Nc)^d} \left( B\left( \frac{Nc/2}{NE\zeta \mp Nc}; d - \gamma, mN - d + 1 \right) - B\left( \frac{a}{NE\zeta \mp Nc}; d - \gamma, mN - d + 1 \right) \right).

Now we substitute \( \frac{Nc}{NE\zeta} \) by \( \varepsilon \) and get

\[ S_\mp = \frac{\gamma a^\gamma \left( \frac{mN}{d} \right) (1 \mp \varepsilon)^{d-\gamma}}{(NE\zeta)^{\gamma}} \left( \frac{\varepsilon}{2 \mp 2\varepsilon} \right)^d \left( B\left( \frac{\varepsilon}{2 \mp 2\varepsilon}; d - \gamma, mN - d + 1 \right) - B\left( \frac{a}{NE\zeta(1 \mp \varepsilon)}; d - \gamma, mN - d + 1 \right) \right). \]

We will use the following estimates for the incomplete beta-function:

\[ B(x; a, b) = \int_0^x t^{a-1}(1-t)^{b-1}dt = O \left( \int_0^x t^{a-1}dt \right) = O \left( \frac{x^a}{a} \right), \]

\[ B(x; a, b) = B(a, b) - \int_x^1 (1-t)^{b-1}dt = B(a, b) + O \left( \frac{(1-x)^b}{b} \right). \]

These estimates give us

\[ S_\mp = \frac{\gamma a^\gamma \left( \frac{mN}{d} \right) (1 \mp \varepsilon)^{d-\gamma}}{(NE\zeta)^{\gamma}} \left( 1 \mp \varepsilon \right)^d \left( B\left( d - \gamma, mN - d + 1 \right) + O \left( \frac{(1 - \frac{\varepsilon}{2 \mp 2\varepsilon})^{mN-d+1}}{mN - d + 1} \right) \right) - \]

\[ + O \left( \frac{\left( \frac{a}{NE\zeta(1 \mp \varepsilon)} \right)^{d-\gamma}}{d - \gamma} \right) \]
We need to factor out beta-function to finish the proof. We will use the fact that \( \frac{1}{B(d-\gamma,mN-d+1)} = \frac{\Gamma(mN+1-\gamma)}{\Gamma(d-\gamma)\Gamma(mN-d+1)} = O \left( \frac{(mN)^{d-\gamma}}{\Gamma(d-\gamma)} \right). \)

\[
S_+ = \gamma a^\gamma \frac{(mN)}{(N\varepsilon\zeta)}^\gamma B \left(d - \gamma, mN - d + 1 \right) \frac{(1 \mp \varepsilon)^{d-\gamma}}{(1 \pm \varepsilon)^d} \cdot \\
\cdot \left(1 + O \left( \frac{(1 - \frac{\varepsilon}{2mN})^{mN-d+1} (mN)^{d-\gamma}}{\Gamma(d-\gamma)(mN - d + 1)} \right) + O \left( \frac{(am \varepsilon^{1+c})^{d-\gamma}}{\Gamma(d-\gamma+1)} \right) \right).
\]

As it follows from the proof, constants hidden in both \( O(\ldots) \) in Equation (16) are equal 1.

Recall that \( \varepsilon = N^{c-1}/E\zeta \). Let us simplify Equation (16):

1. \( \frac{(1 \mp \varepsilon)^{d-\gamma}}{(1 \pm \varepsilon)^d} = \frac{(1 \mp N^{c-1}/E\zeta)^{d-\gamma}}{(1 \pm N^{c-1}/E\zeta)^d} = 1 + o(1) \) if \( d = o(N^{1-c}) \).

2. \( O \left( \frac{(1 - \frac{\varepsilon}{2mN})^{mN-d+1} (mN)^{d-\gamma}}{\Gamma(d-\gamma)(mN - d + 1)} \right) = O \left( \frac{\varepsilon^{(mN-d+1)\log(1 - \frac{N^{c-1}}{2E\zeta(2N^{c-1})}) + (d-\gamma-1)\log(mN)}}{\Gamma(d-\gamma)(1 - \frac{aE\zeta}{mN})} \right) \)

\[= O \left( \frac{-mN\varepsilon^{(1-d/mN)} + (d-\gamma-1)\log(mN)}}{\Gamma(d-\gamma)(1 - \frac{aE\zeta}{mN})} \right) = O \left( \frac{-mN\varepsilon^{(1-o(1))} + d\log(mN)}}{\Gamma(d-\gamma)(1 - o(1))} \right) = o(1) \]

if \( d < \frac{mN\varepsilon}{2E\zeta \log(mN)} \) which is true for sufficiently large \( N \) as soon as \( d = o(N^{1-c}) \) and \( 1/2 < c < 1 \).

3. \( O \left( \frac{am \varepsilon^{1+c}}{\Gamma(d-\gamma+1)} \right) = o(1) \) if \( d \) and \( N \) grow.

4. \( \frac{\gamma a^\gamma \frac{(mN)}{(N\varepsilon\zeta)}^\gamma B \left(d - \gamma, mN - d + 1 \right)}{\Gamma(mN+1)\Gamma(mN-d+1)} \frac{\Gamma(mN+1-\gamma)}{\Gamma(d-\gamma)\Gamma(mN-d+1)} = \)

\[= \frac{\gamma a^\gamma}{d^{d+1}} \frac{am \varepsilon^\gamma}{\Gamma(d+1)} \left(1 + o(1)\right) \] if \( d \) and \( N \) grow.

It remains to note that \( E\zeta = \frac{\gamma a}{\gamma - 1} \), therefore \( \frac{am \varepsilon^\gamma}{\Gamma(d+1)} \frac{am \varepsilon^\gamma}{\Gamma(d+1)} \frac{\Gamma(mN+1)\Gamma(mN-d+1)}{\Gamma(d-\gamma)\Gamma(mN-d+1)} = \)

\[= \frac{\gamma a^\gamma}{d^{d+1}} \left( \frac{(\gamma - 1)m}{\gamma} \right)^\gamma. \]

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Proof of Lemma 3

First, we prove the following lemma.

**Lemma 5.** Under the condition of Lemma 3 (i.e., \(d = o(N^{1-c})\), \(d = o(e^{\varphi(N)})\), and \(q \leq N^r\)) we have

\[
\rho_{\mp}(d, q) = \left(1 + O\left(\frac{q^2}{N}\right) + o(1)\right) \left(\frac{qm}{\mathbb{E}\zeta}\right)^{d} e^{-\frac{qm}{E\zeta}} \cdot d!
\]

**Proof.** Recall that

\[
\rho_{\mp}(d, q) = \prod_{i=0}^{m\varphi(N)N} \left(1 - \frac{qe^{-\frac{i/m}{N}}}{NE\zeta \pm N^c}\right) \sum_{0 \leq i_1 < \ldots < i_d \leq m\varphi(N)N} \prod_{j=1}^{d} \frac{qe^{-\frac{i_j/m}{N}}}{NE\zeta \pm N^c}.
\]

Note that

\[
\prod_{i=0}^{m\varphi(N)N} \left(1 - \frac{qe^{-\frac{i/m}{N}}}{NE\zeta \pm N^c}\right) = \prod_{i=0}^{m\varphi(N)N} \left(1 - \frac{qe^{-\frac{i/m}{N}} e^{O\left(\frac{1}{N}\right)}}{NE\zeta \pm N^c}\right) =
\]

\[
= \exp \left(\sum_{i=0}^{m\varphi(N)N} \log \left(1 - \frac{qe^{-\frac{i/m}{N}} (1 + O\left(\frac{1}{N}\right))}{NE\zeta \pm N^c}\right)\right) =
\]

\[
= \exp \left(- \sum_{i=0}^{m\varphi(N)N} \left(\frac{qe^{-\frac{i/m}{N}}}{NE\zeta \pm N^c} + O\left(\frac{q^2 e^{-2i/m}{N}}{N^2}\right)\right)\right) =
\]

\[
= \exp \left(- \frac{q(1 - e^{-\varphi(N)})}{(1 - e^{-mN})(NE\zeta \pm N^c)} + O\left(\frac{q(1 - e^{-\varphi(N)})}{N^2(1 - e^{-mN})}\right) + O\left(\frac{q^2 (1 - e^{-2\varphi(N)})}{N^2(1 - e^{-mN})}\right)\right) =
\]

\[
= \left(1 + O\left(\frac{q^2}{N}\right) + O\left(N^{c-1}\right) + O\left(e^{-\varphi(N)}\right)\right) \exp \left(-\frac{qm}{E\zeta}\right) =
\]

\[
= (1 + o(1)) \exp \left(-\frac{qm}{E\zeta}\right).
\]
Here we used the fact that $q \leq N^{r} < N^{1/2}$ since it allows us to estimate $\exp \left( O \left( \frac{q^{2}}{N} \right) \right)$ as $1 + O \left( \frac{q^{2}}{N} \right)$.

Let us continue

$$
\prod_{j=1}^{d} \frac{q}{1 - \frac{q \cdot \lfloor \frac{m}{N} \rfloor}{N E \zeta + N}} = \left( \frac{q}{N E \zeta} \right)^{d} \left( 1 + O \left( \frac{d q}{N} \right) + O \left( d N^{c-1} \right) \right) = \left( \frac{q}{N E \zeta} \right)^{d} \left( 1 + o(1) \right), \quad (19)
$$

$$
\sum_{0 \leq i_{1} < \ldots < i_{d} \leq m \phi(N)N} \prod_{j=1}^{d} e^{-\frac{|i_{j}/m|}{N}} = \sum_{0 \leq i_{1} < \ldots < i_{d} \leq m \phi(N)N} e^{-\frac{|i_{1}/m|}{N}} \left( 1 + O \left( \frac{d}{N} \right) \right) = \sum_{0 \leq i_{1} < \ldots < i_{d} \leq m \phi(N)N} e^{-\frac{|i_{1}/m|}{N}} (1 + o(1)). \quad (20)
$$

It remains to estimate $\sum_{0 \leq i_{1} < \ldots < i_{d} \leq m \phi(N)N} e^{-\frac{|i_{1}/m|}{N}}$. We use the following notation:

$$
F(k, d) = \sum_{0 \leq i_{1} < \ldots < i_{d} \leq m \phi(N)N} e^{-\frac{|i_{1}/m|}{N}}.
$$

**Lemma 6.** If $d(k + d) = o(N)$ and $k + d = o \left( e^{\phi(N)} \right)$, then

$$
F(k, d) = \frac{(m N)^{d}(k - 1)!}{(k + d - 1)!} (1 + o(1)).
$$

**Proof.** Note that

$$
F(k, 1) = \sum_{0 \leq i_{1} \leq m \phi(N)N} e^{-\frac{k i_{1}}{m N}} = \frac{1 - e^{-k \phi(N) - \frac{k}{m N}}}{1 - e^{-1}}.
$$

Let us get a recurrent formula for $F(k, d)$:

$$
F(k, d) = \sum_{0 \leq i_{1} < \ldots < i_{d} \leq m \phi(N)N} e^{-\frac{|i_{1}/m|}{N} - \frac{k i_{d}}{m N}} =
$$

$$
= \sum_{0 \leq i_{1} < \ldots < i_{d-1} \leq m \phi(N)N} m \phi(N)N \sum_{i_{d} = i_{d-1} + 1}^{m \phi(N)N} e^{-\frac{k i_{d}}{m N}} =
$$
It is easy to get an upper bound for \( F(k, d) \)

\[
F(k, d) \leq \frac{e^{-k/mN}}{1 - e^{-k/mN}} F(k + 1, d - 1) \leq \ldots \leq
\]

\[
\leq \frac{e^{-(2k+d-2)(d-1)}}{2mN} F(k + d - 1, 1) \leq \frac{e^{-(2k+d-2)(d-1)}}{2mN} =
\]

\[
\frac{e^{-[2k+d-2](d-1)}}{2mN} \frac{(mN)^d(k-1)!}{(k+d-1)!} \left( 1 + O \left( \frac{(k+d)d}{N} \right) \right).
\]

Using this upper bound and the recurrent formula above we can find a lower bound. Assume that

\[
F(k, d) = \frac{e^{-k/mN}}{1 - e^{-k/mN}} \left( F(k + 1, d - 1) - e^{-k\varphi(N)} F(1, d - 1) \right) = \ldots =
\]

\[
= \frac{e^{-(2k+d-2)(d-1)}}{2mN} F(k + d - 1, 1) -
\]

\[
- \sum_{i=1}^{d-1} \frac{e^{-(2k+i-1)i}}{2mN} e^{-i\varphi(N)} F(1, d - i) =
\]

\[
= \frac{(mN)^d(k-1)!}{(k+d-1)!} \left( 1 + O \left( \frac{(k+d)d}{N} \right) \right) -
\]

\[
- \sum_{i=1}^{d-1} e^{-i\varphi(N)} \frac{(mN)^d-i}{(d-i)!} \frac{(mN)^i(k-1)!}{(k+i-1)!} \left( 1 + O \left( \frac{(d-i)^2}{N} \right) + O \left( \frac{(k+i)i}{N} \right) \right) =
\]

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\[
\frac{(mN)^d(k-1)!}{(k+d-1)!} \left( 1 + o(1) - \sum_{i=1}^{d-1} \frac{e^{-(k+i-1)\varphi(N)} (k+d-1)!}{(d-i)!(k+i-1)!} (1 + o(1)) \right) \geq \frac{(mN)^d(k-1)!}{(k+d-1)!} \left( 1 + o(1) - (1 + o(1)) \sum_{i=1}^{d-1} \frac{(k+d-1) e^{-\varphi(N)} k^{i-1}}{(k+i-1)!} \right) = \\
= \frac{(mN)^d(k-1)!}{(k+d-1)!} \left( 1 + o(1) + O \left( \frac{(k+d-1) e^{-\varphi(N)} k^{i-1}}{k!} \right) \right) = \\
= \frac{(mN)^d(k-1)!}{(k+d-1)!} (1 + o(1)) .
\]

Finally, taking into account Equations (17)-(20) and Lemma 6, we get

\[
\rho_\pm(d, q) = e^{-\frac{qm}{E\zeta}} \left( \frac{q}{NE\zeta} \right)^d F(1, d) (1 + o(1)) = \\
= e^{-\frac{qm}{d!}} \left( \frac{q}{NE\zeta} \right)^d (1 + o(1)) = (1 + o(1)) \left( \frac{qm}{E\zeta} \right)^d \frac{e^{-\frac{qm}{d!}}}{d!} .
\]

Now we can estimate

\[
S_\pm(d, q) = \int_a^{Nr} \int_a^{N\gamma} f(q) \rho_\pm(d, q) dq = \\
= \int_a^{Nr} \left( 1 + o(1) \right) \gamma a^{\gamma} \frac{q^{\gamma}}{q^{\gamma+1}} \left( \frac{qm}{E\zeta} \right)^d e^{-\frac{qm}{d!}} dq .
\]

We get an incomplete gamma function:

\[
S_\pm(d, q) = \left( 1 + o(1) \right) \gamma a^{\gamma} m^{\gamma} d! (E\zeta)^\gamma \int_a^{N\gamma} \int_a^{N\gamma/m} x^{d-\gamma-1} e^{-x} dx = \\
= \frac{\gamma(1 + o(1))}{\Gamma(d+1)} \left( \frac{am}{E\zeta} \right)^\gamma \left( \Gamma \left( d - \gamma; \frac{am}{E\zeta} \right) - \Gamma \left( d - \gamma; \frac{N\gamma m}{E\zeta} \right) \right) =
\]

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\[(1 + o(1)) \frac{\gamma \Gamma(d - \gamma)}{\Gamma(d + 1)} \left( \frac{am}{E \zeta} \right)^\gamma = \frac{\gamma}{d^{\gamma+1}} \left( \frac{am}{E \zeta} \right)^\gamma (1 + o(1)) = \frac{\gamma}{d^{\gamma+1}} \left( \frac{(\gamma - 1)m}{\gamma} \right)^\gamma (1 + o(1)).\]