LOCALLY DEFINABLE AND APPROXIMATE SUBGROUPS OF SEMIALGEBRAIC GROUPS

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Abstract. We prove the following instance of a conjecture stated in [10]. Let $G$ be an abelian semialgebraic group over a real closed field $R$ and let $X$ be a semialgebraic subset of $G$. Then the group generated by $X$ contains a generic set and in particular is divisible.

More generally, the same result holds when $X$ is definable in any o-minimal expansion of $R$ which is elementarily equivalent to $\mathbb{R}_{an,exp}$.

We observe that the above statement is equivalent to saying: there exists an $m$ such that $\sum_{i=1}^{m} (X - X)$ is an approximate subgroup of $G$.

1. Introduction

Locally definable groups arise naturally in the study of definable groups in o-minimal structures. In this paper we are mostly interested in definably generated groups, namely locally definable groups which are generated by definable sets (see Section 2 for basic definitions). An important example of such groups is the universal cover of a definable group. Indeed, a definable group in an o-minimal structure can be endowed with a definable manifold structure making the group into a topological group and then, similarly to the Lie context, one can construct its universal covering group, in the category of locally definable groups, see [9]. This universal covering is generated by a definable set.

The universal covering is an example of a locally definable group $U$ with a definable (left) generic set $X$; that is, a definable set such that $AX = U$ for some countable subset $A \subseteq U$ (see [11, Lemma 1.7]). In [10], it was conjectured that every definably generated abelian group is of this form:

Conjecture 1.1. Let $U$ be an abelian, connected, definably generated group. Then $U$ contains a definable generic set.

Note that by [10, Claim 3.11], we may assume in the above conjecture that $U$ is generated by a definably compact set.

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It has been shown in recent papers that the above conjecture can be re-stated in several ways (see for example [3]). We will be using the equivalences below, for which we first need a definition.

**Definition 1.2.** Given \( U \) an abelian, connected, definably generated group we say that a locally definable normal subgroup \( \Gamma < U \) is a lattice if \( \dim(\Gamma) = 0 \) and \( U/\Gamma \) is definable; that is, there exist a definable group \( G \) and a locally definable surjective homomorphism from \( U \) onto \( G \), whose kernel is \( \Gamma \).

**Fact 1.3** ([10, Proposition 3.5] and [11, Theorem 2.1]). Let \( U \) be an abelian, connected, definably generated group. Then the following are equivalent:

1. \( U \) contains a definable generic set.
2. \( U \) admits a lattice.
3. \( U \) admits a lattice isomorphic to \( \mathbb{Z}^k \), for some \( k \).

Moreover, each of the above clauses implies that \( U \) is divisible.

In this note, we study Conjecture [11] for definably generated subgroups of definable groups. To that aim, we introduce the following notion.

**Definition 1.4.** Let \( M \) be an o-minimal structure. We say that an abelian locally definable group \( G \) has the generic property with respect to \( M \) if every definably generated subgroup of \( G \) contains a definable generic set. We omit the reference to \( M \) if it is clear from the context (see Remark 2.3).

The main result of [11] can be stated as follows:

**Fact 1.5.** Let \( R \) be an o-minimal expansion of a real closed field \( R \). Then \( \langle R^n, + \rangle \) has the generic property with respect to \( R \).

Our first result, in Section 3, is that the generic property can be lifted under the presence of an exact sequence (Theorem 3.4).

**Theorem.** Let \( M \) be an o-minimal structure. Assume that we are given an exact sequence of abelian locally definable groups and maps.

\[
0 \rightarrow \mathcal{H} \xrightarrow{i} \mathcal{G} \xrightarrow{\pi} \mathcal{V} \rightarrow 0
\]

If \( \mathcal{V} \) and \( \mathcal{H} \) have the generic property, then so does \( \mathcal{G} \).

This is a useful criterion that can be applied inductively in certain situations. As a corollary, we prove (Subsection 3.1):

**Theorem.** Let \( M \) be an o-minimal structure.

1. If \( G \) is a definable abelian torsion-free group, then \( G \) has the generic property.
2. If \( M \) expands a real closed field \( R \) and \( G \subseteq \text{Gl}(n, R) \) is a definable abelian linear group, then \( G \) has the generic property.

In Section 4 we apply the above lifting result to study definably generated subgroups of semialgebraic groups. In order to formulate the next result, recall that \( \mathbb{R}_{an,exp} \) is the expansion of the real field by the real exponential
map and all restrictions of real analytic functions to the closed unit box in $\mathbb{R}^n$, for all $n \in \mathbb{N}$. By [7], it is o-minimal. The following is the main result (Theorem 4.5) of the paper, which generalizes Fact 1.5 above.

**Theorem.** Let $\mathcal{R}$ be an o-minimal expansion of a real closed field $R$ such that the theory $\text{Th}(\mathcal{R}) \cup \text{Th}(\mathbb{R}_{\text{an,exp}})$ is consistent and has an o-minimal completion. Any abelian semialgebraic group $G$ has the generic property with respect to $\mathcal{R}$.

In particular, any semialgebraically generated subgroup of $G$ contains a semialgebraic generic set.

A special case of the above result is when $\mathcal{R}$ is elementarily equivalent to $\mathbb{R}_{\text{an,exp}}$.

A crucial key case of the above theorem is when $G$ is an abelian variety. In [18] the authors prove the definability in $\mathbb{R}_{\text{an,exp}}$, on appropriate domains, of embeddings of families of abelian varieties into projective spaces. From those results it is possible to extract the following non-standard property of abelian varieties.

**Fact 1.6.** [18] Let $\mathcal{R} = \langle R, \ldots \rangle$ be a model of $\text{Th}(\mathbb{R}_{\text{an,exp}})$ and let $A \subseteq \mathbb{P}^N(K)$, $K = R(i)$, be an embedded abelian variety of dimension $g$. Then there exist a locally definable subgroup $G$ of $\mathbb{R}^g$ and a locally definable covering homomorphism $p : G \to A$.

For the sake of completeness, we provide a proof of the above fact in Appendix 5. Another important ingredient is the work of E. Barriga on semialgebraic groups, [2], which we recall in Fact 4.4.

1.1. **The connection to approximate subgroups.** Approximate subgroups have been studied extensively in various fields including model theory, see for example [5] and [12].

**Definition 1.7.** Given a group $G$, and $k \in \mathbb{N}$, a set $X \subseteq G$ is called a $k$-approximate group if $X$ is $X = X^{-1}$ and there is a finite set $A \subseteq G$ of cardinality $k$ such that $X \cdot X \subseteq A \cdot X$. We say that $X$ is an approximate group if it is $k$-approximate for some $k \in \mathbb{N}$.

As we observe in Remark 2.3 below, the existence of a generic set inside a definably generated group $(X) \subseteq G$ is equivalent to saying that there exists an $m$ such that the set $X(m)$ (the addition of $X - X$ to itself $m$ times) is an approximate group. Thus our various results and conjectures can be re-formulated in the language of approximate subgroups. For example, Conjecture 1.1 can be re-formulated as follows.

**Conjecture 1.8.** Let $\mathcal{U}$ be a locally definable abelian group in an o-minimal structure and $X \subseteq \mathcal{U}$ a definable set. Then there exists $m \in \mathbb{N}$ such that $X(m)$ is an approximate group.

Our main result above (Theorem 4.5) easily implies the following uniformity statement:
Theorem 1.9. Let $\mathcal{R} = (\mathbb{R}; <, +, \ldots)$ be an o-minimal expansion of $\mathbb{R}_{\text{an,exp}}$. Let $\{G_t : t \in T\}$ an $\mathbb{R}_{\text{an,exp}}$-definable family of semialgebraic abelian groups, and $\{X_t : t \in T\}$ an $\mathcal{R}$-definable family, with each $X_t \subseteq G_t$. Then there is $k \in \mathbb{N}$, such that for every $t \in T$, the set $X_t(k)$ is a $k$-approximate subgroup of $G$.

In Conjecture 1.8 we restricted our discussion to definable sets in o-minimal structures, but the same problem could be formulated for arbitrary smooth curves in $\mathbb{R}^n$.

Question 1.10. Let $X \subseteq \mathbb{R}^n$ be a connected smooth curve. Is there $m \in \mathbb{N}$ such that $X(m)$ is an approximate subgroup of $\langle \mathbb{R}^n, + \rangle$?

Let us see that when $X$ is compact the answer to the above question is positive: Indeed, without loss of generality, $0 \in X$ and $X$ is given by $\gamma : [0, 1] \to \mathbb{R}^n$. Moreover, we can assume that $\mathbb{R}^n$ is the minimal linear space containing $X$. Thus, there are $t_1, \ldots, t_n$ such that $\gamma(t_1), \ldots, \gamma(t_n)$ form a basis for $\mathbb{R}^n$ (otherwise, $\mathbb{R}^n$ would not be minimal).

It follows that the map $(x_1, \ldots, x_n) \mapsto x_1 + \cdots + x_n \in \mathbb{R}^n$ is a submersion at the point $(\gamma(t_1), \ldots, \gamma(t_n))$ and hence the point $\gamma(t_1) + \cdots + \gamma(t_n)$ is an internal point of $X(n)$ inside $\mathbb{R}^n$. Since $X(n) + X(n)$ is compact it can be covered by finitely many translates of $X(n)$, so that $X(n)$ is an approximate subgroup.

Note that even if the answer to Question 1.10 is positive, one does not expect any uniformity statement such as that of Theorem 1.9 to hold on this level of generality.

We finish this part of the introduction by pointing out that one cannot expect a positive answer to the above question without the model theoretic (o-minimality) or the topological (smoothness) assumptions. The example was suggested to us by P. Simon. A similar example was also proposed by E. Breuillard.

Example 1.11. Let $G = \mathbb{R}^\mathbb{N}$ with coordinate-wise addition and let $X \subseteq \mathbb{R}^\mathbb{N}$ be the set of all elements with at most one nonzero coordinate. We claim that for no $n$ is the set $X(n)$ an approximate subgroup. Indeed, assume that the set $X(n+1)$ is covered by finitely many translates of $X(n)$. Let $p : \mathbb{R}^\mathbb{N} \to \mathbb{R}^{2(n+1)}$ be the projection onto the first $2(n+1)$ coordinates. The set $p(X(n))$ consists of the tuples with at most $2n$ coordinates different than 0, so for any finite subset $A$ of $\mathbb{R}^\mathbb{N}$ we have that $p(A + X(n)) = p(A) + p(X(n))$ has dimension $2n$. On the other hand, $p(X(n+1)) = \mathbb{R}^{2(n+1)}$, a contradiction.

Because $\langle \mathbb{R}^\mathbb{N}, + \rangle$ is isomorphic as a group to $\langle \mathbb{R}, + \rangle$, we can also find a set $X \subseteq \mathbb{R}$ such that for no $n$ is the set $X(n)$ an approximate subgroup.

1.2. The non-abelian case. It has been shown in [3] that Fact 1.3 fails for non-abelian groups. More precisely, it was shown that every definable centerless subgroup, in a sufficiently saturated o-minimal structure, contains a definably generated subgroup with a definable generic set, which is not
the cover of any definable group. However, as far as we know the following question is still open.

**Question 1.12.** Let $U$ be a definably generated group in an o-minimal structure. Does $U$ contain a generic set?

In [13, Section 7] there is a discussion of locally definable (called Ind-definable) groups and it is shown (see Proposition 7.8 there) that every definable group contains a definably generated subgroup $U$ of the same dimension which contains a definable generic set.

2. Preliminaries

Let $M$ be an arbitrary $\kappa$-saturated o-minimal structure for $\kappa$ sufficiently large. By *bounded* cardinality, we mean cardinality smaller than $\kappa$. We refer the reader to [1] and [8] for the basics concerning locally definable groups. A **locally definable group** is a group $(U, \cdot)$ whose universe is a directed union $U = \bigcup_{k \in \mathbb{N}} X_k$ of definable subsets of $M^n$ for some fixed $n$, and for every $i, j \in \mathbb{N}$, the restriction of group multiplication to $X_i \times X_j$ is a definable function (by saturation, its image is contained in some $X_k$). The dimension of $U$ is by definition $\dim(U) = \max\{\dim(X_k) : k \in \mathbb{N}\}$.

A map $\phi : U \to H$ between locally definable groups is called **locally definable** if for every definable $X \subseteq U$ and $Y \subseteq H$, the set $\text{graph}(\phi) \cap (X \times Y)$ is definable. Equivalently, the restriction of $\phi$ to any definable set is a definable map. If $\phi$ is surjective, then there exists a locally definable section $s : H \to U$ of $\phi$.

For a locally definable group $U$, we say that $V \subseteq U$ is a **compatible subset of** $U$ if for every definable $X \subseteq U$, the intersection $X \cap V$ is a definable set (note that in this case $V$ itself is a bounded union of definable sets). We say that $U$ is **connected** if there is no proper compatible subgroup of bounded index. Every locally definable group $U$ has a connected component $U^0$, that is, a connected compatible subgroup of $U$ of the same dimension ([8]). Note that we still use the term “definably connected” when referring to definable sets. Note also that if $\phi : U \to V$ is a locally definable homomorphism between locally definable groups, then $\ker(\phi)$ is a compatible locally definable normal subgroup of $U$. In fact, the following holds.

**Fact 2.1.** ([8, Theorem 4.2]) If $U$ is a locally definable group and $H \subseteq U$ is a locally definable normal subgroup then $H$ is a compatible subgroup of $U$ if and only if there exists a locally definable surjective homomorphism of locally definable groups $\phi : U \to V$ whose kernel is $H$.

In Definition [13] we introduced the notion of an abelian locally definable group having the generic property. Now, we stress some easy properties regarding that notion. For that, we need the following notation that will be used throughout the paper.
Notation 2.2. Let $G$ be an abelian group and $X$ a subset. The set $X(m)$ denotes the addition of $X - X$ to itself $m$ times. We say that $X$ is symmetric if $X = -X$.

Remark 2.3. 1) An abelian locally definable group $G$ has the generic property if and only if for every definable subset $Y \subseteq G$, there are $m, k \in \mathbb{N}$ and $0 \in A \subseteq Y(2m)$, $|A| \leq k$, such that $Y(m) + Y(m) \subseteq A + Y(m)$. In particular, $Y(m)$ is a $k$-approximate group.

2) If $G$ has the generic property and $H$ is a locally definable subgroup of $G$, then $H$ has also the generic property.

3) Let $G$ and $V$ be abelian locally definable groups, and let $\pi : G \to V$ be a surjective locally definable homomorphism. If $G$ has the generic property, then $V$ has the generic property. Indeed, for $X \subseteq V$ definable, let $Y \subseteq G$ be any definable set with $\pi(Y) = X$ (such $Y$ exists by saturation). Since $G$ has the generic property, there are $m, k \in \mathbb{N}$ and a set $0 \in A \subseteq Y(2m)$, $|A| \leq k$ such that $Y(m) + Y(m) \subseteq A + Y(m)$. In particular, we get that $0 \in \pi(A) \subseteq X(m)$ and

$$X(m) + X(m) = \pi(Y(m) + Y(m)) \subseteq \pi(A + Y(m)) = \pi(A) + X(m),$$

as required.

4) Let $\mathcal{M}'$ be an o-minimal expansion of $\mathcal{M}$. By 1) above, if $G$ is a locally definable group in $\mathcal{M}$ with the generic property with respect to $\mathcal{M}'$, then $G$ has the generic property with respect to $\mathcal{M}$.

We can now formulate:

Proposition 2.4. Let $G$ be a locally definable group in $\mathcal{M}$ (which is still sufficiently saturated). Then the following are equivalent:

1) $G$ has the generic property.

2) For every definable family $\{X_t : t \in T\}$ of subsets of $G$ there exist $N, k \in \mathbb{N}$ such that for every $t \in T$, there exists a subset $A \subseteq X_t(2m)$, of size at most $k$ such that $X_t(m) - X_t(m) \subseteq A + X_t(m)$.

In particular, $G$ has the generic property in $\mathcal{M}$ if and only if it has the generic property in any/some elementary extension of $\mathcal{M}$.

Proof. This follows immediately from Remark 2.3 (1) and saturation. □

Remark 2.5. While we focus here on o-minimal structures, the notions we defined make sense in any sufficiently saturated structure, in which case Remark 2.3 as well as Proposition 2.4 are also true.

3. Group extensions

In this section we study the existence of definable generic sets when dealing with abelian group extensions, in an arbitrary o-minimal structure $\mathcal{M}$. As a corollary, we prove that definably generated subgroups of abelian
torsion-free definable groups contain definable generic sets. When \( \mathcal{M} \) expands a real closed field \( R \), we deduce a similar result for definable linear groups over \( R \).

**Proposition 3.1.** Assume that we are given an exact sequence of abelian locally definable groups and maps,

\[
0 \rightarrow \mathcal{H} \xrightarrow{i} \mathcal{G} \xrightarrow{\pi} \mathcal{V} \rightarrow 0,
\]

where \( \mathcal{V} \) is connected and admits a lattice. Let \( Y \subseteq \mathcal{V} \) be a definable generic set and \( s : Y \rightarrow \mathcal{G} \) a definable section. Then the intersection \( \langle s(Y) \rangle \cap \mathcal{H} \) is definably generated.

**Proof.** By [11, Lemma 1.7], \( \mathcal{V} = \langle Y \rangle \). In particular, \( \pi \) sends the group \( \langle s(Y) \rangle \) onto \( \mathcal{V} \).

Henceforth we will use that given a definable set \( Z \subseteq \mathcal{V} \), we can assume that \( Z \subseteq Y \). Indeed, by saturation there is \( n \) such that \( Z \subseteq Y(n) \) and by definable choice there is a section \( r : Z \rightarrow s(Y)(n) \subseteq \langle s(Y) \rangle \). Thus we can extend the section \( s : Y \rightarrow \mathcal{G} \) to a section \( \bar{s} : Y \cup (Z \setminus Y) \rightarrow \mathcal{G} \) via \( r \) in such a way that \( \langle s(Y) \rangle = \langle \bar{s}(Y \cup Z) \rangle \). Therefore we can work with the generic set \( Y \cup Z \) instead of \( Y \), as required. For example, we can assume that \( Y \cap 0 \) is symmetric (extend the section \( s \) to the set \( -Y \cup \{0\} \)). Moreover, we can set \( s(0) = 0 \). For, let \( y_0 := s(0) \in \mathcal{H} \) and consider the definable section \( \bar{s} : Y \rightarrow \mathcal{G} \) such that \( \bar{s} = s \) in \( Y \setminus \{0\} \) and \( \bar{s}(0) = 0 \). If \( \bar{D} \) is a definable set which generates \( \langle \bar{s}(Y) \rangle \cap \mathcal{H} \), then \( D := \bar{D} \cup \{y_0\} \) generates \( \langle s(Y) \rangle \cap \mathcal{H} \), as required.

By Fact [11,3] and since \( Y \) is generic, the locally definable group \( \mathcal{V} \) admits a lattice \( \Gamma \cong \mathbb{Z}^k \). Since \( \mathcal{V}/\Gamma \) is definable and \( Y \) generic in \( \mathcal{V} \), there is a finite set \( A \subseteq \mathcal{V} \) such that \( Y + A + \Gamma = \mathcal{V} \). Indeed, to see that note that the image of \( Y \) in \( \mathcal{V}/\Gamma \) is a generic so finitely many translates of it cover the group. Now, without loss, we can assume that \( A \) contains a fixed set of generators \( \gamma_1, \ldots, \gamma_k \) of \( \Gamma \). Therefore we can assume that \( Y + \Gamma = \mathcal{V} \) and \( \gamma_1, \ldots, \gamma_k \in Y \) (extending the section \( s \) to \( Y + A \)).

Let \( \Delta = \langle s(\gamma_1), \ldots, s(\gamma_k) \rangle \) and note that \( \pi|_{\Delta} : \Delta \rightarrow \Gamma \) is an isomorphism. Consider the symmetric finite set (notice that \( Y(2) \cap \Gamma \) is finite)

\[
\Delta_0 := \{ \delta \in \Delta : \pi(\delta) \in Y(2) \}
\]

and the definable set

\[
D := (\Delta_0 + s(Y)(2)) \cap \mathcal{H} \subseteq \langle \Delta + s(Y) \rangle \cap \mathcal{H} = \langle s(Y) \rangle \cap \mathcal{H}.
\]

Note that \( 0 \in D \), and we now claim that \( D \) generates \( \langle s(Y) \rangle \cap \mathcal{H} \). To prove that, it is sufficient to show the following:

**Claim.** For all \( n \) and for every \( \delta_1, \ldots, \delta_2^n, \gamma_1, \ldots, y_2^n \in Y \), if \( \Sigma_{i=1}^{2^n} \delta_i + s(y_i) \in \mathcal{H} \) then \( \Sigma_{i=1}^{2^n} \delta_i + s(y_i) \in \langle D \rangle \).

Indeed, granted the claim, pick \( \Sigma_{i=1}^{m} s(y_i) \in \langle s(Y) \rangle \cap \mathcal{H} \). Define \( \delta_1 = \cdots = \delta_{2^n} = 0 \) and \( y_{2^n+1} = \cdots = y_{2m} = 0 \). Since \( \Sigma_{i=1}^{2^n} \delta_i + s(y_i) = \Sigma_{i=1}^{m} s(y_i) \in \mathcal{H} \) we deduce \( \Sigma_{i=1}^{m} s(y_i) \in \langle D \rangle \), as required.
Proof of the claim. By induction on \( n \). The case \( n = 0 \) gives \( \pi(\delta_1) + y_1 = 0 \), hence \( \pi(\delta_1) \in Y \subseteq Y(2) \), so \( \delta_1 \in \Delta_0 \). Therefore \( \delta_1 + s(y_1) \in D \).

Assume now that \( \Sigma_{i=1}^{2^n} \delta_1 + s(y_i) \in \mathcal{H} \). We want to show that \( \Sigma_{i=1}^{2^n} \delta_1 + s(y_i) \) is in \( \langle D \rangle \). We write the sum in pairs:

\[
\Sigma_{i=1}^{2^n} (\delta_1 + s(y_i)) = \Sigma_{k=1}^{2^n-1} (s(y_{2k-1}) + s(y_{2k}) + \delta_{2k-1} + \delta_{2k}).
\]

Now, because \( Y + \Gamma = \mathcal{V} \), for each \( k = 1, \ldots, 2^n-1 \) there is \( w_k \in Y \) and \( \beta_k \in \Gamma \) such that \( y_{2k-1} + y_{2k} = \beta_k + w_k \).\footnote{Note that this implies \( \beta_k \in \Delta \) be such that \( \pi(\beta_k) = \beta_k \). Note that \( \beta_k \in Y(2) \), so that \( \alpha_k \in \Delta_0 \). Hence,}

\[
(s(y_{2k-1}) + s(y_{2k}) - \alpha_k - s(w_k)) \in D.
\]

Also because the image under \( \pi \) of \( s(y_{2k-1}) + s(y_{2k}) - \alpha_k - s(w_k) \) is 0, it belongs to \( \mathcal{H} \).

Thus the above sum also equals

\[
\Sigma_{i=1}^{2^n} (\delta_1 + s(y_i)) = \Sigma_{k=1}^{2^n-1} (s(y_{2k-1}) + s(y_{2k}) - \alpha_k - s(w_k)) + \Sigma_{k=1}^{2^n-1} (\delta_{2k-1} + \delta_{2k} + \alpha_k + s(w_k)).
\]

We already showed that \( \Sigma_{k=1}^{2^n-1} (s(y_{2k-1}) + s(y_{2k}) - \alpha_k - s(w_k)) \in \langle D \cap \mathcal{H} \rangle \), so if we denote \( \tilde{\delta}_k := \delta_{2k-1} + \delta_{2k} + \alpha_k \in \Delta \) then

\[
\Sigma_{k=1}^{2^n-1} (\tilde{\delta}_k + s(w_k)) \in \mathcal{H},
\]

and it remains to see that it belongs to \( \langle D \rangle \). This follows by induction, so the claim is proved and with it Proposition 3.1. \( \square \)

**Proposition 3.2.** With \( \mathcal{H}, \mathcal{G} \) and \( \mathcal{V} \) as in Proposition 3.1, assume that \( X \subseteq \mathcal{G} \) is a definable set with \( \langle \pi(X) \rangle = \mathcal{V} \). Then \( \langle X \rangle \cap \mathcal{H} \) is definably generated.

**Proof.** Since \( \mathcal{V} \) admits a lattice it contains a definable generic set \( Y \). Without loss we can assume that \( \pi(X)(1) \subseteq Y \). By saturation \( Y \subseteq \pi(X)(\ell) \) for some \( \ell \in \mathbb{N} \) and therefore by definable choice we can pick a section \( s : Y \to \langle X \rangle \).

Moreover, we can assume that \( s(\pi(X)) \subseteq X \). Let \( E := X(1) \cap \mathcal{H} \). By Proposition 3.1 we have that \( \mathcal{H}_0 := \langle s(Y) \rangle \cap \mathcal{H} \) is definably generated. Thus, to prove that \( \langle X \rangle \cap \mathcal{H} \) is definably generated it suffices to show that \( \langle X \rangle \cap \mathcal{H} = \langle E \rangle + \mathcal{H}_0 \).

To that aim, pick \( x_1, \ldots, x_n \in X \) such that \( \Sigma_{i=1}^n x_i \in \mathcal{H} \). We can write

\[
\Sigma_{i=1}^n x_i = \Sigma_{i=1}^n (x_i - s(\pi(x_i))) + \Sigma_{i=1}^n s(\pi(x_i)).
\]

Note that \( x_i - s(\pi(x_i)) \in E \) for each \( i = 1, \ldots, n \) and therefore \( \Sigma_{i=1}^n (x_i - s(\pi(x_i))) \) is \( \in \langle E \rangle \). Moreover, \( \Sigma_{i=1}^n s(\pi(x_i)) = \Sigma_{i=1}^n x_i - \Sigma_{i=1}^n (x_i - s(\pi(x_i))) \in \mathcal{H} \). Since also \( s(\pi(x_i)) \in s(\pi(X)) \subseteq s(Y) \) for each \( i = 1, \ldots, n \), we get \( \Sigma_{i=1}^n s(\pi(x_i)) \in \mathcal{H}_0 \) and so \( \Sigma_{i=1}^n x_i \in \langle E \rangle + \mathcal{H}_0 \), as required. \( \square \)

Before the main corollary we need also the following lemma.
Lemma 3.3. If $G$ is a definably generated abelian group, then its connected component is definably generated by a definably connected set. In particular, if every connected definably generated subgroup of $G$ contains a definable generic set then $G$ has the generic property.

Proof. Let $G$ be a locally definable group and $X \subseteq G$ be a definable set which generates $H$. Let $X_1, \ldots, X_k$ be its connected components. Fix an element $a_i$ in each $X_i$, and let $\Gamma = \langle a_1, \ldots, a_k \rangle$. Consider the connected set $\tilde{X} = \bigcup X_i - a_i$, and notice that $\langle X \rangle = \langle \tilde{X} \rangle + \Gamma$. Since $\langle \tilde{X} \rangle$ is a locally definable subgroup of $G$ of bounded index, it must be its connected component.

For the second part of the statement, let $H$ be a definably generated subgroup of $G$. Then, by what we just showed, its connected component $H^0$ is definably generated and therefore by hypothesis it contains a definable generic set $Y$, that is, there is a bounded $A \subseteq H^0$ such that $A + X = H^0$. Since $H^0$ has bounded index in $H$, there is a bounded $B \subseteq H$ such that $B + H^0 = H$. In particular $A + B + X = H$, as required. $\square$

Theorem 3.4. Assume that we are given an exact sequence of abelian locally definable groups and maps.

$$0 \rightarrow H \xrightarrow{i} G \xrightarrow{\pi} V \rightarrow 0$$

If $V$ and $H$ have the generic property then $G$ has the generic property.

Proof. By Lemma 3.3 it is sufficient to consider subgroups of $G$ which are generated by definably connected sets. Let $X$ be a definably connected set. Since $\pi(\langle X \rangle) = \langle \pi(X) \rangle$ is a definably generated connected group, we have the exact sequence of locally definable groups

$$0 \rightarrow \langle X \rangle \cap H \rightarrow \langle X \rangle \rightarrow \langle \pi(X) \rangle \rightarrow 0$$

By hypothesis the connected group $\langle \pi(X) \rangle$ contains a definable generic set, that is, there exists a definable set $Z_1 \subseteq \langle X \rangle$ such that $\pi(Z_1)$ is generic in $\langle \pi(X) \rangle$. In particular the group $\langle \pi(X) \rangle$ admits a lattice (see Fact 1.3) and therefore by Proposition 3.2 the group $\langle X \rangle \cap H$ is definably generated. Again by hypothesis and by Lemma 3.3 we have that $\langle X \rangle \cap H$ contains a definable generic set $Z_2$. Finally, it is not hard to see that $Z_1 + Z_2$ is generic in $\langle X \rangle$. $\square$

3.1. Some applications of Theorem 3.4. First, we can study definably generated subgroups of abelian torsion-free definable groups (see basic facts on torsion-free groups definable in o-minimal structures in Section 2.1 in [17]).

Corollary 3.5. Any abelian torsion-free definable group in an o-minimal structure $M$ has the generic property with respect to $M$.

Proof. We prove it by induction on $\dim(G)$. Assume first that $\dim(G) = 1$ and prove first a more general result:
Lemma 3.6. If $U$ is a 1-dimensional torsion-free locally definable group then it has the generic property.

Proof. By [8, Corollary 8.3], the group $U$ can be linearly ordered. By Lemma 3.3 it suffices to study a subgroup generated by a set of the form $(-b, b) := \{ x \in G : -b < x < b \}$, that is,

$$\langle (-b, b) \rangle = \bigcup_{n \in \mathbb{N}} (-nb, nb).$$

It is easy to verify that the group $\Gamma = \mathbb{Z}b$ is a lattice in $\langle (-b, b) \rangle$ because $\langle (-b, b) \rangle/\Gamma$ is isomorphic to the definable group $([0, b), \text{mod } b)$.

Now, assume that $\dim(G) > 1$. Then, by [19], there exists a subgroup $H$ of $G$ of dimension 1. In particular, we have the exact sequence

$$0 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 0.$$

Since $H$ and $G/H$ are abelian torsion-free definable groups smaller dimension it follows by induction that they have the generic property. By Theorem 3.4, so does $G$. □

Next we prove:

Corollary 3.7. Let $\mathcal{M}$ be an o-minimal expansion of a real closed field and $G \subseteq \text{Gl}(n, R)$ a definable abelian linear group. Then $G$ has the generic property with respect to $\mathcal{M}$.

Proof. We may assume that $G$ is definably connected. By [15] Proposition 3.10], $G$ is definably isomorphic to a semialgebraic linear group, and hence it is the connected component of $H(R)$ for some abelian linear subgroup of $GL(n, K)$, defined over $R$ (here $K$ is the algebraic closure of $R$). By [15] Fact 3.1], $G$ is semialgebraically isomorphic to a group of the form $\mathcal{T}_m \times (\mathbb{R}^*_0)^k \times (\mathbb{R}^+)^n$, where $T = SO(2, R)$.

By Corollary 3.5 the group $(\mathbb{R}^*_0)^k \times (\mathbb{R}^+)^n$ has the generic property, so by Theorem 3.2 it is enough to show that $T = SO(2, R)$ has the generic property. The universal covering of $T$ is a torsion-free1-dimensional locally definable group, so by Lemma 3.6 it has the generic property. Thus $G$ has the generic property. □

4. SEMIALGEBRAIC GROUPS

The main purpose of this section is to show, see Theorem 4.5 below, that every semialgebraic abelian group over a real closed field $R$ has the generic property with respect to certain o-minimal expansions of $R$, which we now fix.

In the rest of the section, we fix $\mathcal{R}$ to be an o-minimal $L$-structure expanding a real closed field $R$ such that the $(L \cup L_{\text{an}, \text{exp}})$-theory $\text{Th}(\mathcal{R}) \cup T_{\text{an,exp}}$ is consistent and has an o-minimal completion, call it $T_0$. We denote by $K := R(i)$ its algebraic closure.
For example, any real closed field, or more generally a structure elementarily equivalent to $\mathbb{R}_{\text{an,exp}}$ clearly satisfies the above.

We start by analysing the case of abelian varieties.

**Proposition 4.1.** Every embedded abelian variety $A \subseteq \mathbb{P}^N(K)$ over $K$ has the generic property with respect to $\mathcal{R}$.

**Proof.** First, note that by our assumptions there exists an elementary extension $\mathcal{R} \prec \mathcal{R}'$ such that $\mathcal{R}'$ can be expanded to a model of $T_0$. Furthermore, we may assume that this structure is $\aleph_1$-saturated. By Proposition 2.4 and the fact that the generic property is preserved by reducts, it is sufficient to prove the result in $\mathcal{R}'$. Thus, all in all, we can assume that $\mathcal{R}$ is an $\aleph_1$-saturated model of $T_0$.

By Fact 1.6, there exist a locally definable subgroup $G$ of $R^g$ and a locally definable covering homomorphism $p : G \to A$. Thus, by Fact 1.5 and Remark 2.3, the group $A$ has the generic property. □

**Proposition 4.2.** Let $H$ be an irreducible abelian $K$-algebraic group. Then $H$ has the generic property with respect to $\mathcal{R}$.

**Proof.** As in the proof of Proposition 4.1 we can assume that $\mathcal{R}$ is an $\aleph_1$-saturated model of $T_0 := \text{Th}(\mathcal{R}) \cup T_{\text{an,exp}}$.

By Corollary 3.7, the result is true when $H$ is linear (notice that every linear subgroup of $GL(n, K)$ can be viewed as a linear subgroup of $GL(m, R)$ for some $R$).

For the general case, by Chevalley theorem’s, there are a linear group $L$ and an abelian variety $A$ such that

$$0 \to L \to H \to A \to 0.$$ 

Thus, by Corollary 3.4 and Propositions 4.1 the group $H$ also has the generic property. □

**Remark 4.3.** If $H$ is an irreducible $K$-algebraic defined over $R$, then by Remark 2.3 and the result above, the group of $R$-rational points $H(R)^0$ has the generic property.

Before reaching our main theorem we recall the following result of Barriga, [2, Theorem 7.2], which describes every semialgebraic group in terms of the $R$-points of an associated algebraic group over $R$.

**Fact 4.4.** Let $G$ be a definably compact semialgebraic abelian group over $R$. Then there exists a $K$-algebraic group $H$ defined over $R$, an open connected locally semialgebraic subgroup $W$ of the o-minimal universal covering group $H(R)^0$ of the connected component of $H(R)$, and a locally semialgebraic surjective covering homomorphism $\theta : W \to G$, with 0-dimensional kernel.

**Theorem 4.5.** For $\mathcal{R}$ an o-minimal structure expanding a real closed field $R$, as before, let $G$ be an abelian semialgebraic group over $R$. Then $G$ has the generic property with respect to $\mathcal{R}$. In particular, any semialgebraically generated subgroup of $G$ contains a generic semialgebraic subset.
Proof. By Lemma 3.3 it is enough to show that every locally definable subgroup of $G$ generated by a definably connected set contains a definable generic subset. Thus, we can assume that $G$ is connected.

By [6, Theorem 1.2] the quotient of $G$ by its maximal semialgebraic torsion-free subgroup $\mathcal{N}(G)$ is definably compact. Then by Theorem 3.4 and Corollary 3.5 we can assume that $G$ is definably compact.

Using the notation of Fact refbarriga above, we have a covering homomorphism $\theta : \mathcal{W} \to \mathcal{G}$, with $\mathcal{W}$ a definably generated subgroup of the locally definable group $\mathcal{H}(\mathbb{R})^o$.

Denote by $p : \mathcal{H}(\mathbb{R})^o \to H(\mathbb{R})^o$ the universal covering map. We have the exact sequence

$$0 \to \ker(p) \to \mathcal{H}(\mathbb{R})^o \to H(\mathbb{R})^o \to 0.$$ 

Note that $\ker(p)$ is discrete and therefore its only semialgebraically generated connected subgroup is the trivial one, so by Lemma 3.3 the group $\ker(p)$ has the generic property. Thus, by Theorem 3.4 and Remark 4.3, we deduce that $\mathcal{H}(\mathbb{R})^o$ has the generic property. In particular, by Remark 2.3 the same is true for $\mathcal{W}$, and so for $G$, as required. □

5. Appendix: Abelian Varieties

Fact 1.6 is a consequence of the results in [19]. Maybe not in this form, we believe it is well-known by the experts. For example, a similar statement is used in [20, §5.2.2 and §5.3] in order to construct differential analytic operators related with the inverse of certain complex algebraic covering maps, the latter appearing naturally when studying the action of an algebraic group on an algebraic variety. For the sake of completeness, we provide a proof in this appendix. As in [19], we quote several facts concerning abelian varieties, see [4] for details.

For a positive $g \in \mathbb{N}$, by a complex $g$-torus we mean the quotient group $\mathbb{C}^g/\Lambda$ where $\Lambda$ is a lattice, i.e., a subgroup of $(\mathbb{C}^g, +)$ generated by $2g$ vectors which are $\mathbb{R}$-linearly independent. It is a compact complex Lie group of dimension $g$. A torus $\mathbb{C}^g/\Lambda$ is called an abelian variety if it is biholomorphic with a projective variety in $\mathbb{P}^k(\mathbb{C})$ for some $k$.

Let us denote by $\mathbb{H}_g$ the set of $g \times g$ symmetric matrices with a positive definite imaginary part. Then the well-known Riemann criterion establishes that a complex $g$-torus $E$ is an abelian variety if and only if it is biholomorphic with a torus $\mathbb{C}^g/(\tau\mathbb{Z}^g + D\mathbb{Z}^g)$ where $\tau \in \mathbb{H}_g$ and $D$ is a diagonal matrix $D = \text{diag}(d_1, \ldots, d_g)$ with positive integers $d_1|d_2| \cdots |d_g$. We call $D$ a polarization type of $E$.

Note that if $E$ has polarization $D$, then it also has polarization $kD$ for any positive integer $k$. Indeed, the map $\mathbb{C}^g/(\tau\mathbb{Z}^g + D\mathbb{Z}^g) \to \mathbb{C}^g/(k\tau\mathbb{Z}^g + kD\mathbb{Z}^g) : z \mapsto kz$ is a biholomorphism and therefore being biholomorphic with $\mathbb{C}^g/(\tau\mathbb{Z}^g + D\mathbb{Z}^g)$ is equivalent to being biholomorphic with $\mathbb{C}^g/(k\tau\mathbb{Z}^g + kD\mathbb{Z}^g)$. In particular, by taking $k = 6$ above, any abelian variety has a
polarization type $D = \text{diag}(d_1, \ldots, d_g)$ satisfying $d_1 \geq 4$, $2|d_1$ and $3|d_1$. If a polarization type satisfies the latter we call it a Baily-polarization.

Let us show that the family of abelian varieties with a fixed Baily-polarization is constructible.

First, let us recall the definition of theta functions. In some occasions, we will identify the symmetric $g \times g$ matrices with $\mathbb{C}^{n}$ for $n := \frac{2(g+1)}{2}$, and therefore we will view $H_g$ as a subset of $\mathbb{C}^n$. The well-defined function

$$\vartheta : \mathbb{C}^g \times H_g \to \mathbb{C}, \quad (z, \tau) \mapsto \sum_{n \in \mathbb{Z}^g} \exp \left( \pi i (n \tau n) + 2t'nz \right)$$

is homomorphic and $\mathbb{Z}^g$-periodic in $z$ and $(2\mathbb{Z})^n$-periodic in $\tau$. For any $a \in \mathbb{R}^g$ the associated Riemann Theta function is

$$\vartheta_a : \mathbb{C}^g \times H_g \to \mathbb{C}, \quad (z, \tau) \mapsto \vartheta_a(z, \tau) = \exp \left( \pi i (a \tau a) + 2t'az \right) \vartheta(z + \tau a, \tau).$$

A consequence of the classical Lefschetz Theorem is the following. Fix a polarization $D = \text{diag}(d_1, \ldots, d_g)$ with $d_1 \geq 3$ and fix a set of representatives $\{c_0, \ldots, c_N\}$ of the cosets of $\mathbb{Z}^g$ in the group $D^{-1} \mathbb{Z}^g$. Then

$$\varphi^D : \mathbb{C}^g \times H_g \to \mathbb{P}^N(\mathbb{C})$$

is a well-defined holomorphic map, and given $\tau \in H_g$ we have that

$$\varphi^D_\tau : \mathbb{C}^g \to \mathbb{P}^N(\mathbb{C}), \quad z \mapsto \varphi^D_\tau(z) := \varphi^D(z, \tau)$$

is an $(\tau \mathbb{Z}^g + D\mathbb{Z}^g)$-periodic immersion and induces an analytic embedding of the abelian variety $\mathcal{E}^D_\tau = \mathbb{C}^g / (\tau \mathbb{Z}^g + D\mathbb{Z}^g)$ into $\mathbb{P}^N(\mathbb{C})$.

Moreover, if we denote

$$\Psi^D : H_g \to \mathbb{P}^N(\mathbb{C}), \quad \tau \mapsto \varphi^D_\tau(0, \tau),$$

and

$$\Phi^D : \mathbb{C}^g \times H_g \to \mathbb{P}^N(\mathbb{C}) \times \mathbb{P}^N(\mathbb{C}), \quad (z, \tau) \mapsto (\varphi^D(z, \tau), \Psi^D(\tau))$$

then we have the following properties for Baily-polarizations, see [3, Theorem 8.10.1. and Remark 8.10.4].

**Fact 5.1.** Let $D$ be a Baily-polarization.

1) The map $\Psi^D$ is an immersion, the set $\Psi^D(H_g)$ is a Zariski open subset of an algebraic subvariety of $\mathbb{P}^N(\mathbb{C})$. Moreover, if $\Psi^D(\tau) = \Psi^D(\tau')$ then

$$\varphi^D_\tau(\mathcal{E}^D_\tau) = \varphi^D_\tau'(\mathcal{E}^D_\tau').$$

2) The image of $\Phi^D$ is a Zariski open subset of an algebraic subvariety of $\mathbb{P}^N(\mathbb{C}) \times \mathbb{P}^N(\mathbb{C})$. 
Remark 5.2. In other words, if we denote by $P^D := \Psi^D(\mathbb{H}_q)$ and for each $p := \Psi^D(\tau) \in P^D$ we denote $A_p^D := \varphi^D(\mathcal{E}_D^p)$, then the above result ensures that the family

$$A^D := \{A_p^D : p \in P^D\}$$

is constructible.

Now, let us introduce abelian varieties in the broader context of algebraically closed fields. Let $K$ be an algebraic closed field. An embedded $g$-dimensional abelian variety $A$ of $\mathbb{P}^N(K)$ is an irreducible projective subvariety of $\mathbb{P}^N(K)$ of dimension $g$ together with two regular maps $m : A \times A \rightarrow A$ and $i : A \rightarrow A$ such that $m$ endows $A$ with a group structure (which must be abelian) and $i$ is its inverse.

Henceforth, when we write that $A = \{A_t : t \in T\}$ is a constructible family of embedded $g$-dimensional abelian varieties of $\mathbb{P}^N(K)$, we mean that each $A_t$ is an irreducible projective subvariety of $\mathbb{P}^N(K)$ of dimension $g$ and that there are a constructible families $\{m_t : t \in T\}$ and $\{i_t : t \in T\}$ of regular maps $m_t : A_t \times A_t \rightarrow A_t$ and $i_t : A_t \rightarrow A_t$ such that each $A_t$ with $m_t$ and $i_t$ is an abelian variety.

**Proposition 5.3.** Let $A = \{A_t : t \in T\}$ be a constructible family without parameters of embedded $g$-dimensional abelian varieties of $\mathbb{P}^M(\mathbb{C})$. Then there exist finitely many polarizations $D_1, \ldots, D_k$ and $d \in \mathbb{N}$ such that for any $t \in T$ there exists an isomorphism of degree less than $d$ between $A_t$ and an abelian variety in $A^{D_j}$ for some $j = 1, \ldots, k$.

**Proof.** For each $t \in T$, the abelian variety $A_t$ is biholomorphic, as a complex Lie group, to a $g$-torus satisfying the Riemann criterion, so we can associate a Baily-polarization $D$ to $A_t$. In particular, $A_t$ is bi-holomorphic isomorphic to an abelian variety in $A^D$. By Chow’s theorem, they are actually bi-regularly isomorphic. We claim that only finitely many Baily-polarizations $D_1, \ldots, D_k$ are needed to cover all the abelian varieties in $A$.

Indeed, assume that it is not true. For any fixed Baily-polarization $D$ and for any $d \in \mathbb{N}$ consider the formula $F^D_d(t)$ in the language of rings that says that $t \in T$ and for every $p \in P^D$ there is no bi-regular isomorphism from $A_t$ to $A_p^D$ of degree $d$. Now, consider

$$q(t) = \{F^D_d(t) : D \text{ a Baily-polarization } \& d \in \mathbb{N}\}.$$ 

By our assumption, we get that $q$ is a partial type over a countable set of parameters. On the other hand, the complex field $\mathbb{C}$ in the language of rings is $\aleph_1$-saturated. Thus, there is $t_0 \in T$ that realises the type $q$.

In other words, the abelian variety $A_{t_0}$ is not isomorphic to $A_p^D$ for any Baily-polarization $D$ and $p \in P^D$, which is a contradiction. It follows that there exist $D_1, \ldots, D_k$ Baily-polarizations such that for any $t \in T$ we have that $A_t$ is bi-regularly isomorphic to an abelian variety in $A^{D_j}$ for some $j \in \{1, \ldots, k\}$. Moreover, we have that the degree of the isomorphism
is bounded. Otherwise, we would get a contradiction by considering the partial type
\[ q(t) := \{ F_{d_j}(t) : j = 1, \ldots, k & d \in \mathbb{N} \}, \]
as required. \(\square\)

**Remark 5.4.** Note that with the notation used in Proposition 5.3, for each \(j \in \{1, \ldots, k\}\) the set \(T_j\) of \(t \in T\) such that there exists a bi-regular isomorphism of degree less than \(d\) between \(A_t\) and an abelian variety in \(A^{D_j}\) is constructible without parameters.

In the rest of this section, we fix a real closed field \(R\), and let \(K = R(i)\) be its algebraic closure.

We will use the obvious identification of \(K^n\) with \(R^{2n}\). We say that a subset of \(K^n\) is semialgebraic over \(C \subseteq R\) if it is semialgebraic over \(C\) as a subset of \(R^{2n}\). Note that if \(X\) is a constructible subset of \(K^n\) over \(A \subseteq K\) then it is clearly semialgebraic over the real and imaginary parts of the elements in \(A\). Finally, note that \(R^n\) can be identified with the real part of \(K^n\), and that by elimination of quantifiers of the theory of real closed fields, a subset of \(R^n\) is semialgebraic in the usual sense if and only if it is semialgebraic as a subset of \(K^n\).

Henceforth, given a semialgebraic subset \(T\) of \(K^n\), when we write that \(A = \{ A_t : t \in T \}\) is a semialgebraic family of embedded \(g\)-dimensional abelian varieties of \(\mathbb{P}^N(K)\), we mean that each \(A_t\) is an irreducible projective subvariety of \(\mathbb{P}^N(K)\) of dimension \(g\), that the family \(A\) is semialgebraic in the obvious (complex) sense, that is, the set
\[ \{(x, t) : t \in T, x \in A_t\} \]
is a semialgebraic subset of \(\mathbb{P}^M(\mathbb{C}) \times T\), and that there are also semialgebraic families \(\{ m_t : t \in T \}\) and \(\{ i_t : t \in T \}\) of regular maps \(m_t : A_t \times A_t \to A_t\) and \(i_t : A_t \to A_t\) such that each \(A_t\) with \(m_t\) and \(i_t\) is an abelian variety. Note that it only makes sense to say that a semialgebraic family is defined over a real tuple.

**Lemma 5.5.** Let \(g, d \in \mathbb{N}\), let \(D\) be a polarization and \(A = \{ A_t : t \in T \}\) be a semialgebraic family of embedded \(g\)-dimensional abelian varieties of \(\mathbb{P}^N(K)\) such that for every \(t \in T\) there exists a bi-regular isomorphism of degree less than \(d\) between \(A_t\) and an abelian variety in \(A^D\). Then there exists a semialgebraic family \(\{ g_t : t \in T \}\) of bi-regular isomorphisms \(g_t\) from \(A_t\) to an abelian variety in \(A^D\).

**Proof.** As in Remark 5.4, given \(t \in T\) and \(p \in P^D\), denote by \(I_{t,p}\) the non-empty constructible set of bi-regular isomorphisms of degree less than \(d\) from \(A_t\) to \(A_{p}^{D}\), and note that \(\{ I_{t,p} : t \in T, p \in P^D \}\) is a semialgebraic family. Thus, by definable Skolem functions of the theory of real closed fields, there is a semialgebraic map
\[ p : T \to P^D \]
and for each $t \in T$ a bi-regular isomorphism

$$g_t : A_t \to \mathbb{A}^D_{p(t)}$$

such that the family $\{g_t : t \in T\}$ is semialgebraic, as required. □

Given a polarization $D$ and $\tau \in \mathbb{H}_g$, we denote by $E^D_\tau \subseteq \mathbb{C}^g$ the fundamental parallelogram of $\mathcal{E}^D_\tau$. The following fact follows from [18, Theorem 8.10] (see also the comments above it), and recall that $P^D$ denotes $\Psi^D(\mathbb{H}_g) \subseteq \mathbb{P}^N(\mathbb{C})$.

**Fact 5.6.** Let $D$ be a Baily-polarization. Then there is a set $S \subseteq \mathbb{H}_g$ such that

$$\Psi^D|_S : S \to P^D$$

is a surjective map definable in $\mathbb{R}_{an,exp}$ and such that there is a family $\{h^D_\tau : \tau \in S\}$ definable in $\mathbb{R}_{an,exp}$ with

$$h^D_\tau : E^D_\tau \to \mathbb{P}^N(\mathbb{C})$$

an embedding of the abelian variety $E^D_\tau$ into the projective space $\mathbb{P}^N(\mathbb{C})$.

**Theorem 5.7.** Let $A = \{A_t : t \in T\}$ be a semialgebraic family without parameters of embedded $g$-dimensional abelian varieties of $\mathbb{P}^M(\mathbb{C})$. Let $d \in \mathbb{N}$ and $D$ be a polarization such that for each $t \in T$ there exists a bi-regular isomorphism of degree less than $d$ between $A_t$ and an abelian variety in $A^D$.

Then there is definable in $\mathbb{R}_{an,exp}$ a family of analytic maps

$$\{h_t : t \in T\}$$

where $h_t : E_t \to A_t$ is such that $E_t$ is a fundamental parallelogram in $\mathbb{C}^g$ of an abelian variety $\mathcal{E}_t$ and $h_t$ induces an isomorphism between $\mathcal{E}_t$ and $A_t$.

**Proof.** By Lemma 5.5 there is a semialgebraic map

$$p : T \to P^D$$

and for each $t \in T$ a bi-regular isomorphism

$$g_t : A_t \to \mathbb{A}^D_{p(t)}$$

such that the family $\{g_t : t \in T\}$ is semialgebraic. On the other hand, pick $S$ a definable set in $\mathbb{R}_{an,exp}$ as in Fact 5.6 and consider a definable section

$$s : P^D \to S$$

of $\Psi^D|_S : S \to P^D$. Finally, define

$$h_t := h^D_{s(p(t))} \circ g_t^{-1}$$

where $h^D_\tau : E^D_\tau \to \mathbb{P}^N(\mathbb{C})$, $\tau \in S$, are the embeddings given by Fact 5.6 □
Proof of Fact 1.6. Let $A \subseteq \mathbb{P}^N(K)$ be an embedded abelian variety. Let $c \in R^t$ be a tuple of coefficients defining algebraically the variety $A$, the regular group operation $\mathfrak{m} : A \times A \to A$ and the inverse map $\mathfrak{i} : A \to A$. We can replace the parameter $c$ by a tuple $t$ of free variables and therefore we obtain (without parameters) a constructible family $\mathcal{A} = \{A_t : t \in T\}$ of irreducible projective subvarieties of $\mathbb{P}^N(K)$ of dimension $g := \dim(A)$, and constructible families $\mathcal{F} = \{\mathfrak{m}_t : t \in T\}$ and $\{\mathfrak{i}_t : t \in T\}$ of regular maps such that $\mathfrak{m}_t$ and $\mathfrak{i}_t$ endow $A_t$ with a group structure. Note that $c \in T$, for which we get $A_c = A$, $\mathfrak{m}_c = \mathfrak{m}$ and $\mathfrak{i}_c = \mathfrak{i}$. Consider the realization $\mathcal{A}(\mathbb{C})$ of $\mathcal{A}$ in $\mathbb{C}$. By Proposition 5.3 we can assume that there exist $d \in \mathbb{N}$ and a polarization $D$ such that for each $t \in T(\mathbb{C})$ there exists a bi-regular isomorphism of degree less than $d$ between $A_t(\mathbb{C})$ and an abelian variety in $\mathcal{A}^D$.

By Theorem 5.7 there is a definable family in $\mathbb{R}_{an,exp}$ of maps
\[ \{h_t : E_t \to A_t | t \in T(\mathbb{C})\} \]
where $E_t$ is the fundamental parallelogram in $\mathbb{C}^g$,
\[ E_t := \{ \sum_{j=1}^{2g} t_j v_j(t) : 0 \leq t_j < 1 \} \]
of the $R$-independent vectors $v_1(t), \ldots, v_{2g}(t)$ and $h_t$ induces an analytic isomorphism $f_t$ between $\mathcal{E}_t := \mathbb{C}^g / \langle v_1(t), \ldots, v_{2g}(t) \rangle$ and $A_t$. Let us regard $\mathcal{E}_t$ properly as a definable group. To that aim, first note that for each $t \in T$ the set $\Gamma_t := E_t(2) \cap \langle v_1(t), \ldots, v_{2g}(t) \rangle$ is finite, and therefore the family $\{\Gamma_t : t \in T\}$ is definable. Consider also the definable group on $E_t$ given by $x \oplus_t y = z$ if and only if $z = (x + y) \in \Gamma_t$. Clearly, the definable group operation $(E_t, \oplus_t)$ is isomorphic to $\mathcal{E}_t$, and the map $h_t$ is a definable isomorphism between $(E_t, \oplus_t)$ and $A$.

Back to $\mathcal{R}$, we obtain the $R$-linear independent vectors $v_1, \ldots, v_{2g}$ where $v_i := v_i(c)$ for $i = 1, \ldots, 2g$. If we denote $E := E_c$, $\Gamma := \Gamma_c$, $\oplus := \oplus_c$ and $h := h_c$, we have that $h : E \to A$ is a definable isomorphism between $(E, \oplus)$ and $A$. On the other hand, $(E, \oplus)$ is clearly isomorphic to $\mathcal{E} := \langle E \rangle / \langle \mathbb{Z} v_1 + \cdots + \mathbb{Z} v_{2g} \rangle$, and therefore for $\mathcal{G} := \langle E \rangle$ we have a definable covering homomorphism $p : \mathcal{G} \to A$, as desired.

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