Exotic Yang-Mills dilaton gauge theories

Stephen C. Anco

Department of Mathematics, Brock University,
St Catharines, ON Canada L2S 3A1

Abstract

An exotic class of nonlinear $p$-form nonabelian gauge theories is studied, arising from the most general allowed covariant deformation of linear abelian gauge theory for a set of massless 1-form fields and 2-form fields in four dimensions. These theories combine a Chapline-Manton type coupling of the 1-forms and 2-forms, along with a Yang-Mills coupling of the 1-forms, a Freedman-Townsend coupling of the 2-forms, and an extended Freedman-Townsend type coupling between the 1-forms and 2-forms. It is shown that the resulting theories have a geometrically interesting dual formulation that is equivalent to an exotic Yang-Mills dilaton theory involving a nonlinear sigma field. In particular, the nonlinear sigma field couples to the Yang-Mills 1-form field through a generalized Chern class 4-form term.
I. INTRODUCTION

There has been much recent interest in the systematic construction of nonlinear $p$-form gauge theories with couplings among $p$-form fields for $p \geq 2$ in $n$ dimensions. The construction begins with deforming the linear abelian gauge theory of $p$-form fields by addition of cubic terms in the free field Lagrangian and linear terms in the abelian gauge symmetries such that gauge invariance of the theory is maintained to first-order. All allowed first-order deformation terms can be obtained as solutions of determining equations derived from the condition of gauge invariance. The full nonlinear theory consists of completing the deformation of the gauge symmetries and Lagrangian to all orders [1].

A classification of first-order deformations in $n$ dimensions has been derived in Ref. [2, 3] by formulating and solving the relevant determining equations in the setting of BRST cohomology [4, 5]. In particular, for $p$-form fields with $p \geq 2$, the allowed cubic terms for deforming the $n$-form Lagrangian are classified in terms of the abelian $p$-form field strengths as follows: Freedman-Townsend type couplings (if $n \geq 3$), which are quadratic in the dual field strengths; Chapline-Manton type couplings (if $n \geq 4$), which are linear in the dual field strengths and at least linear in the field strengths; higher derivative generalizations (if $n \geq 5$), which are quadratic in the dual field strengths and at least linear in the field strengths. When 1-form fields are considered, there is also the Yang-Mills coupling (if $n \geq 2$), which is linear in the dual field strength of the 1-form. The full nonlinear gauge theories for each of these separate types of couplings are well-known. However, there has been less effort to-date in the study of theories that combine the different types of couplings together. Such theories are expected to have somewhat exotic features, as exhibited in the cases of combined Yang-Mills/Freedman-Townsend couplings in $n = 3$ and $n = 4$ dimensions studied in Ref. [8, 9].

In this Letter, full nonlinear gauge theories for the cases of a Chapline-Manton coupling combined with Freedman-Townsend and Yang-Mills couplings in $n = 4$ dimensions are explored. The main result is to show that these theories have an interesting geometrical formulation and duality which, in general, is equivalent to an exotic Yang-Mills dilaton theory involving a nonlinear sigma field. In particular, the nonlinear sigma field is found to couple to the Yang-Mills 1-form through a generalized Chern class term [10].
II. PRELIMINARIES

We start from the results of the analysis in Ref. [9] for geometrical first-order deformations [12] of linear abelian gauge theory of a set of massless 1-forms $A^a$ ($a = 1, \ldots, k$) and 2-forms $B^{a'}$ ($a' = 1, \ldots, k'$) with field strength 2-forms $F^a = dA^a$ and 3-forms $H^{a'} = dB^{a'}$ on a four-dimensional manifold $M$. (Throughout, $d$ is the exterior derivative operator, $\ast$ denotes the Hodge dual operator with $\ast^2 = \pm 1$, and products of $p$-forms are understood to be wedge products. All constructions will be local, i.e. in a single coordinate chart, with respect to $M$.) The undeformed Lagrangian and gauge symmetries are given by

$$(2.1) \quad \mathcal{L} = \ast F^a F^b g_{ab} + \ast H^{a'} H^{b'} g_{a'b'}$$

with coefficients $g_{ab}, g_{a'b'} = \text{diag}(\pm 1, \ldots, \pm 1)$, and

$$\delta^{(0)} \xi A^a = d\xi^a, \quad \delta^{(0)} \chi^{a'} = d\chi^{a'}, \quad \delta^{(0)} \xi A^a = \delta^{(0)} \chi^{a'} = 0,$$  

where $\xi^a$ is a set of arbitrary 0-forms (i.e. functions) and $\chi^{a'}$ is a set of arbitrary 1-forms on $M$.

**Proposition 1** The separate Yang-Mills, (extended) Freedman-Townsend, Chapline-Manton first-order deformations of linear abelian $p$-form gauge theory ($p = 1, 2$) on $M$ consist of the terms [2, 9]

$$(2.3) \quad \mathcal{L} = \left( \frac{1}{2} a_{abc} \ast F^a A^b A^c \right)_{\text{YM}} + \left( -\frac{1}{2} c_{a'b'c'} \ast H^{a'} H^{b'} B^{c'} \right)_{\text{FT}}$$

$$+ \left( b_{ab'c'} \ast F^a \ast H^{b'} A^c \right)_{\text{extFT}} + \left( -c_{a'bc} \ast H^{a'} F^b A^c \right)_{\text{CM}}$$

and

$$(2.4) \quad \delta^{(1)} \xi A^a = (a^a_{bc} A^b A^c)_{\text{YM}} + (b^a_{b'c} \ast H^{b'} \xi^c)_{\text{extFT}},$$

$$(2.5) \quad \delta^{(1)} \chi^{a'} = (-b_{a'b'c} \ast F^b \xi^c)_{\text{extFT}} + (c_{a'bc} \ast H^{a'} F^b A^c)_{\text{CM}},$$

$$(2.6) \quad \delta^{(1)} \xi A^a = 0, \quad \delta^{(1)} \chi^{a'} = (c^{a'}_{b'c'} \ast H^{b'} \chi^{c'})_{\text{FT}},$$

where the coupling constants satisfy the algebraic relations

$$a_{a(bc)} = 0, \quad a_{ab'[c']de} = 0, \quad c_{(a'b)c'} = 0, \quad c_{[d'e'c']b'c'd'} = 0,$$  

$$2b_{a[b'[c']d']e} = b_{a[e'c']b'd'}, \quad c_{a'[bc]} = 0.$$
Additional algebraic relations on the coupling constants arise as integrability conditions for the existence of second-order deformation terms when one considers deformations that combine pure Yang-Mills, (extended) Freedman-Townsend, and Chapline-Manton couplings.

**Theorem 2** The necessary and sufficient algebraic obstructions for existence of mixed Yang-Mills, (extended) Freedman-Townsend, Chapline-Manton second-order deformations are given by

\[ 2a_{ab'[e]d'}b^{b'} = b_{d'b}a^{b}[e]c, \quad (2.9) \]

\[ e_{a'b'[d]}(b^{c})_{d'e} = 0, \quad (2.10) \]

\[ c_{d'e'}e_{c'}a_{b} = 2e_{d'[ca]b'}^{c'} + 2e_{d'[cb]b'}^{c'}a'. \quad (2.11) \]

The relations (2.7) to (2.11) impose a certain natural algebraic structure on the internal vector spaces, \( \mathcal{A} = \mathbb{R}^{k}, \mathcal{A}' = \mathbb{R}^{k'} \), associated with the set of fields \( A^a, B^{a'} \). Specifically, this structure is given by:

(i) \( a_{bc}^a \) defines structure constants of a Lie bracket \([\cdot, \cdot]_A\) from \( \mathcal{A} \times \mathcal{A} \) into \( \mathcal{A} \) such that \( g_{ab} \) is an invariant metric on \( \mathcal{A} \), i.e.

\[ [u, v]_A = -[v, u]_A, \quad [u, [v, w]]_A + \text{cyclic terms} = 0, \quad (2.12) \]

\[ g(u, [v, w]_A) = g([u, v]_A, w); \quad (2.13) \]

(ii) \( c_{d'e'}^c \) defines structure constants of a Lie bracket \([\cdot, \cdot]_{A'}\) from \( \mathcal{A}' \times \mathcal{A}' \) into \( \mathcal{A}' \), i.e.

\[ [u', v']_{A'} = -[v', u']_{A'}, \quad [u', [v', w']_{A'}]_{A'} + \text{cyclic terms} = 0, \quad (2.14) \]

while \( g_{a'bc} \) need not be an invariant metric;

(iii) \( b^a_{d'c} \) defines a linear map \( b(\cdot) \) from \( \mathcal{A}' \times \mathcal{A} \) into \( \mathcal{A} \) given by, jointly, a representation of the Lie algebra \( \mathcal{A}' \) on the vector space \( \mathcal{A} \) and a derivation of the Lie algebra \( \mathcal{A} \), i.e.

\[ [b(u'), b(v')] = b([u', v']_{A'}), \quad b(w')[u, v]_A = [b(w')u, v]_A + [u, b(w')v]_A; \quad (2.15) \]

(iv) \( e^{a'}_{bc} \) defines a symmetric product \( e(\cdot, \cdot) \) from \( \mathcal{A} \times \mathcal{A} \) into \( \mathcal{A}' \) that is invariant with respect to the Lie bracket on \( \mathcal{A} \)

\[ e(u, v) = e(v, u), \quad e(u, [v, w]_A) = e([u, v]_A, w) \quad (2.16) \]
and such that the related symmetric linear map \( e^S(\cdot) \) from \( \mathcal{A}' \times \mathcal{A} \) into \( \mathcal{A} \) defined by \( e_{ab}^c \) intertwines with the representation of \( \mathcal{A}' \) via

\[
e^{S([u', v']_{\mathcal{A}'}, [v', u']_{\mathcal{A}}) - [e^S(u'), b^A(v')] - [b^A(u'), e^S(v')] = \{e^S(u'), b^S(v')\} - \{b^S(u'), e^S(v')\} \quad (2.17)
\]

where \( b^S(\cdot) = \frac{1}{2}(b(\cdot) + b^T(\cdot)) \) and \( b^A(\cdot) = \frac{1}{2}(b(\cdot) - b^T(\cdot)) \) are the symmetric and skew parts of the representation \( b(\cdot) \).

By a generalization of the algebraic analysis of Ref. [9] applied to (i)-(iv), one obtains the following result.

**Proposition 3** Suppose the metrics given by \( g_{ab}, g'_{a'b'} \) on \( \mathcal{A}, \mathcal{A}' \) are positive definite. Then the Lie algebra \( \mathcal{A} \) is compact semisimple or abelian. In the semisimple case for \( \mathcal{A} \), Eq. (2.13) is satisfied by the Cartan-Killing metric \( g(u, v) = -\text{tr}(ad_A(u)ad_A(v)) \equiv \text{tr}_A(uv) \) where \( ad_A(\cdot) \) is the adjoint representation of \( \mathcal{A} \), i.e.

\[
g_{ab} = -a^d_{ca}a^c_{db}, \quad (2.18)
\]

and Eq. (2.13) is satisfied in terms of \( ad_A(\cdot) \) by \( b(w') = ad_A(h(w')) = -b^T(w') \) with \( h(\cdot) \) being a Lie-algebra homomorphism of \( \mathcal{A}' \) into \( \mathcal{A} \), \( [h(u'), h(v')]_{\mathcal{A}} = h([u', v']_{\mathcal{A}'}, i.e.

\[
b^a_{b'c} = a^a_{bc}h^b_{b'}, \quad h^b_{b'}h^c_{c'}a^a_{bc} = h^a_{a'}c_{b'c'}a'. \quad (2.19)
\]

Moreover, Eqs. (2.16) and (2.17) are then satisfied by \( e(u, v) = e' \otimes g(u, v) \) for any vector \( e' \) in \( \mathcal{A}' \) orthogonal to the commutator ideal \( DA' = [A', A'] \), i.e.

\[
e^{a'}_{bc} = e^{a'}_{bc}g_{bc}, \quad e'_{a'c_{b'c}}a' = 0 \quad (2.20)
\]

(and hence \( e' = 0 \) whenever \( \mathcal{A}' = DA' \)). However, in the abelian case for \( \mathcal{A} \), Eqs. (2.16) and (2.17) are instead satisfied by \( e^S(w') = eb^S(w') \) for any constant \( e \) if \( b^S(\cdot) \neq 0 \) or by \( e^S(w') = g(e', w')id_A \) for any vector \( e' \) as above if \( b^S(\cdot) = 0 \), i.e.

\[
e^a_{b'c} = \begin{cases} 
\frac{1}{2}(b^e_{a'b} + b_{b'a}) & \text{if } b_{(c|a')(b)} \neq 0 \\
\frac{1}{2}(b^c_{a'c} - b_{b'a'}) & \text{if } b_{(c|a')(b)} = 0 
\end{cases} \quad (2.21)
\]

which is seen from a comparison of Eq. (2.17) with the symmetric part of Eq. (2.8).

The aim now is to write down a class of nonlinear gauge theories giving a complete deformation to all orders (in particular, there are no further algebraic obstructions). To
illustrate the nature and essential pattern for this class of theories, we first examine the Chapline-Manton coupling of an abelian 1-form $A$ and abelian 2-form $B$, with field strengths $F = dA, H = dB$.

The Lagrangian is given by the 4-form $L = \ast FF + \ast (H - eFA)(H - eFA)$ where $e$ is a coupling constant. This Lagrangian is invariant under the separate abelian gauge symmetries $\delta_\xi A = d\xi, \delta_\xi B = eF \xi$, for arbitrary 0-forms $\xi$, and $\delta_\chi A = 0, \delta_\chi B = d\chi$, for arbitrary 1-forms $\chi$. The field equations obtained from $L$ are given by

$$0 = d\ast F + e(2F(\ast H - e\ast(FA)) - Ad(\ast H - e\ast(FA))), \quad 0 = d(\ast H - e\ast(FA)). \quad (2.22)$$

This theory describes a nonlinear deformation of abelian Maxwell/Freedman-Townsend gauge theory for $A, B$. It has a dual formulation in terms of a scalar field introduced through the $B$ field equation,

$$\ast (H - eFA) = d\phi. \quad (2.23)$$

Then the $A$ field equation becomes

$$d\ast F = -2eFd\phi. \quad (2.24)$$

The $\phi$ field equation is obtained by elimination of $H$ in equation $(2.23)$, which yields

$$d\ast d\phi = \mp eFF. \quad (2.25)$$

These field equations for $\phi$ and $A$ arise equivalently from the Lagrangian 4-form

$$L^{\text{dual}} = \ast FF + (2eG \pm \ast d\phi)d\phi \quad (2.26)$$

where $G = AF = AdA$ is the abelian Chern-Simons 3-form, satisfying $dG = FF$. Thus, $\phi$ couples to $A$ through a Chern-Simons term $Gd\phi = \phi FF$ (to within a trivial exact term) where $FF$ is the abelian Chern class 4-form on $M$.

The Lagrangian $L^{\text{dual}}$ has a more direct derivation from the nonlinear gauge theory for $A$ and $B$ by passing to a 1st order formalism $L^{1st} = \ast FF + (2eG \pm \ast K)K + 2BdK$ with the 1-form $K$ being an auxiliary field variable. Elimination of $K$ via its field equation $K + e\ast G = \ast dB$ leads to the original Lagrangian $L$ in terms of $A, B$. Alternatively, through the $B$ field equation $dK = 0$, the introduction of $\phi$ via $K = d\phi$ yields the dual Lagrangian $L^{\text{dual}}$ for $\phi, A$. 

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This 1st order formulation will now be exploited for generalizing the Chapline-Manton coupling to a set of 1-forms \( A^a \) \((a = 1, \ldots, k)\) and 2-forms \( B^{a'} \) \((a' = 1, \ldots, k')\) with a Yang-Mills coupling on \( A^a \), a Freedman-Townsend coupling on \( B^{a'} \), and an extended Freedman-Townsend coupling between \( A^a \) and \( B^{a'} \).

III. CONSTRUCTION OF THE CLASS OF NONLINEAR THEORIES

We begin by writing down the nonabelian Yang-Mills generalization of the 1st order Lagrangian for the extended Freedman-Townsend gauge theory of 1-forms \( A^a \) and 2-forms \( B^{a'} \), similarly to the abelian theory discussed in Ref. [2]. The 1st order formulation uses auxiliary 1-forms \( K^{a'} \) whose role is a Freedman-Townsend nonlinear field strength. Let the 2-forms \( F_A^a = dA^a + \frac{1}{2} a^a_{bc} A^b A^c \) denote the nonabelian Yang-Mills field strength. Then the Lagrangian is given by the 4-form

\[
L_{\text{YM/exFT}}^{1\text{st}} = \ast J^a J^b g_{ab} + (2 B^{a'} R_{K}^{b'} \pm \ast K^{a'} K^{b'}) g_{a'b'},
\]

(3.1)

where

\[
R_{K}^{a'} = dK^{a'} + \frac{1}{2} c_{cc'} a^{a'} K^{b'} K^{c'}
\]

(3.2)

and

\[
J^a = F_A^a + b_{Dc} K^{b'} A^c \equiv D_K A^a + \frac{1}{2} a^a_{bc} A^b A^c.
\]

(3.3)

Geometrically, \( R_{K}^{a'} \) is the curvature of the Freedman-Townsend connection \( K^{a'} \), while \( J^a \) is a generalized curvature of the Yang-Mills connection \( A^a \) involving a Freedman-Townsend coupling through the covariant derivative operator \( D_K \) associated with the connection \( K^{a'} \).

(The underlying gauge groups for these connections are the unique local Lie groups whose Lie algebras are \( \mathcal{A}', \mathcal{A} \).) This 1st order Lagrangian is invariant under combined Yang-Mills and extended Freedman-Townsend gauge symmetries. A Chapline-Manton coupling is now introduced in this theory as follows. First, let \( G_A^{a'} = e_{bc}^{a'} (A^b dA^c + \frac{1}{3} a^c_{de} A^d A^e) \) denote a nonabelian Chern-Simons 3-form, satisfying \( dG_A^{a'} = e_{bc}^{a'} F_A^b F_A^c \) due to property (2.16), and next define the related 3-form

\[
G^{a'} = G_A^{a'} - e_{bc}^{a'} b_{Dc} A^b A^d K^{b'} = e_{bc}^{a'} A^b (D_K A^c + \frac{1}{3} a^c_{de} A^d A^e).
\]

(3.4)

Then add the coupling term

\[
L_{\text{CM}}^{1\text{st}} = 2 G^{a'} K^{b'} g_{a'b'}
\]

(3.5)
to the 1st order Lagrangian, yielding the complete Lagrangian $L_{YM/exFT}^{1st} + L_{CM}^{1st}$ for the theory. The gauge symmetries under which this Lagrangian is invariant (to within an exact 4-form) are given by

$$\delta_\chi B^{a'} = d\chi^{a'} + e^{a'}_{\nu e} K^{b'} {\chi^{c'}} = D_K \chi^{a'},$$

$$\delta_\xi B^{a'} = e^{a'}_{bc} D_K \xi^c - b^{a'}_b (\ast J^b + 2 e^{a'}_{\nu d} A^{d'} K^{a''}) \xi^c,$$

$$\delta_\chi A^a = D_K \xi^a + a^{a'}_{bc} A^b \xi^c = D_{K+} A^a,$$

$$\delta_\chi K^{a'} = \delta_\xi K^{a'} = 0.$$  

Here $\delta_\chi$ is a Friedman-Townsend gauge symmetry, and $\delta_\xi$ is a Yang-Mills generalization of an extended Friedman-Townsend gauge symmetry combined with a Chapline-Manton gauge symmetry. (The commutators are given by $[\delta_\chi, \delta_\xi] = \delta_\xi$, with $\xi^a = a^a_{bc} \xi^b_1 \xi^c_2$, and $[\delta_\chi, \delta_\chi] = [\delta_\xi, \delta_\chi] = 0$, to within trivial symmetries proportional to the field equations. Hence the gauge symmetry algebra is $\mathcal{A} \times U(1)^k$.)

By elimination of $K^{a'}$ via its field equation

$$K^{a'} + \ast G_{a'} - b^{a'}_b \ast (A^c (\ast J^b + e^{a'}_{\nu e} A^e K^{d'})) = \ast D_K B^{a'},$$

we obtain a nonlinear gauge theory for $A^a, B^{a'}$, with the Lagrangian

$$L_{YM/exFT/CM} = K^{a'} (H^{b'} - G^{b'}) g_{a'b'} + \ast J^a F^{a} g_{a'b}$$

which is invariant under the previous gauge symmetries on $A^a, B^{a'}$. As one sees by Eq. (3.11), the 1-form $K^{a'} = Y_{A, B'} ^{-1} a'_{b'}(\ast H^{b'} - \ast G^{b'})$ in this theory is a nonpolynomial field strength for $B^{a'}$ given in terms of the inverse of a symmetric linear map defined on $\mathcal{A}'$-valued 1-forms by

$$Y_{A, B'} ^{a'} = g_{b'}^{a'} \ast (B^{a'} \cdot) + \ast b^{a'}_d b^{d'}_c \ast (A^b (A^c \cdot)) + (e^{a'}_{\nu d} b^{d'}_{\nu e} - e^{a'}_{\nu d} b^{d'}_{\nu e}) \ast (A^b A^c \cdot).$$

**Theorem 4** The class of nonlinear gauge theories (3.6) to (3.12) comprise the most general deformation (to all orders) of linear abelian gauge theory (2.1) and (2.2) determined by combining Yang-Mills, extended Friedman-Townsend, and Chapline-Manton first-order deformations (2.5) for a set of 1-forms $A^a$ and 2-forms $B^{a'}$ on $M$.

This class of theories is equivalent to an exotic Yang-Mills nonabelian dilaton theory obtained from the 1st order Lagrangian by eliminating $B^{a'}$ via its field equation $R^{a'}_K = 0$, as
follows. Since $R^a_K$ is the curvature of $K^a$, we see that $K^a$ is a flat connection and hence is of form

$$K^a = e^a_\mu(\varphi) d\varphi^\mu$$  \hspace{1cm} (3.13)

in terms of a nonlinear sigma field $\varphi^\mu$ given by a map from $M$ into the local Lie group $G_A$, determined by the Lie algebra $A$, where $e^a_\mu(\varphi)$ is a left-invariant basis of 1-forms (i.e. a frame) on $G_A$, with $2\partial_\nu e^a_\mu = c^b_\nu c^c_\mu e^b_\nu$. This is an extension of the well-known geometrical equivalence [6] of nonlinear sigma theory for $\varphi^\mu$ based on a Lie group target $G_A$ and pure Freedman-Townsend gauge theory for $B^a$ involving the Lie algebra $A$ of $G_A$.) Consequently, once $B^a$ is eliminated in the 1st order Lagrangian, we obtain the dual Lagrangian

$$L_{\text{dual}} = * J^a J^b g_{ab} + (2G^d e^b_\mu(\varphi) \pm * d\varphi^\mu) d\varphi^\nu g_{\mu\nu}(\varphi)$$  \hspace{1cm} (3.14)

where

$$g_{\mu\nu}(\varphi) = e^a_\mu(\varphi) e^b_\nu(\varphi) g_{ab}$$  \hspace{1cm} (3.15)

is the left-invariant metric on $G_A$ associated with the Lie algebra metric $g_{ab}$, and here $J^a$ and $G^a$ are now defined in terms of the covariant derivative $D_K = d + b^a e^b_\mu d\varphi^\mu$. Note $e^a_\mu$ is the coframe of $e^a_\mu$ on $G_A$. The field equations for $A^a$ and $\varphi^\mu$, after some simplifications using Eqs. (2.10) to (2.14), are given by

$$D_{a'} J^a = e^b_\mu(\varphi) d\varphi^\mu (b^a_{\mu} a' \ast J^c - 2e^b_{\nu} e^a_\mu J^c),$$  \hspace{1cm} (3.16)

$$\pm (d*d\varphi^\nu + \Gamma_{\nu\sigma}^\mu(\varphi) d\varphi^\mu d\varphi^\sigma = e^b_\mu(\varphi)(b^a_{\mu} e^c_\nu J^b J^c - e^{a'}_{\mu b c} J^b J^c),$$  \hspace{1cm} (3.17)

where $\Gamma_{\nu\sigma}^\mu(\varphi) = g^{\mu\nu}(\varphi)(\partial_{\nu} g_{\sigma\tau}(\varphi) - \frac{1}{2} \partial_{\tau} g_{\nu\sigma}(\varphi))$ is the Christoffel symbol of the Lie-group metric $g_{\mu\nu}(\varphi)$. (Here, geometrically, $d + \Gamma_{\nu\sigma}^\mu(\varphi) d\varphi^\nu = \nabla_g$ is the pullback to $M$ of the unique torsion-free derivative operator on $G_A$ determined by the metric.) Thus, $\varphi^\mu$ couples to $A^a$ through the 4-form terms $e^d_{bc} J^b J^c$ and $b^a_{\mu} e^c_\nu J^b J^c$.

From the manifest similarity between $b^a_{\mu} e^c_\nu J^b J^c$ and the generalized Yang-Mills term $\ast J^b J^c g_{bc} \equiv L_{\text{YM}}$ in the Lagrangian $L_{\text{dual}}$, the 4-form $b^a_{\mu} e^c_\nu J^b J^c$ is seen to describe a dilaton type coupling between $\varphi^\mu$ and $A^a$ in the field equations. Further discussion of the nature of this coupling is given at the end of the next section.

In comparison, the 4-form $e^a_{\mu bc} J^b J^c$ in the field equations describes an exotic type of dilaton coupling, related to a generalized Chern class term as follows. Let

$$G^a_K = G^a_A - (e^a_{\mu bc} b^c_{\nu d} + b^a_{\mu b c} b^c_{\nu d}) A^b A^d K^{b'd}$$  \hspace{1cm} (3.18)
as given by the variational derivative of the Chapline-Manton term $L_{CM}^{1st}$ with respect to $K^{a'}$, i.e. $\delta_K L_{CM}^{1st} = 2 G_{K}^{a'} \delta K^{ab'} g_{a'b'}$. Likewise, let

$$
\tilde{G}_{K}^{a'} = b_{b'}^{a'} c^{A'} \ast J^{b} = b_{b'}^{a'} c^{A'} (D_{K} A^{b} + \frac{1}{2} g_{A'}^{d} A^{d} A^{c})
$$

(3.19)

obtained from the Yang-Mills term $L_{YM}$.

**Proposition 5** The 3-forms $G_{K}^{a'}$ and $\tilde{G}_{K}^{a'}$ are, jointly, potentials for the gauge invariant 4-forms $e^{a'}_{bc} J^{b} J^{c}$ and $b_{b'}^{a'} c^{A'} \ast J^{b} J^{c}$, satisfying the relation

$$
D_{K} (G_{K}^{a'} - \tilde{G}_{K}^{a'}) = e^{a'}_{bc} J^{b} J^{c} - b_{b'}^{a'} c^{A'} J^{b} J^{c}
$$

(3.20)

on all solutions $A^{a}$ of the field equations. This is a generalization of the relation [11] between the Chern-Simons 3-form $G_{A}^{a'}$ and the Yang-Mills Chern class 4-form $e^{a'}_{bc} F_{A}^{b} F_{A}^{c} = d G_{A}^{a'}$ (recall, here, that $e^{a'}_{bc}$ acts as an invariant metric on the Yang-Mills Lie algebra $A$). Therefore, it follows that $G_{K}^{a'}$ has the geometrical role of a generalized nonabelian Chern-Simons 3-form, determining a generalized Chern class 4-form $e^{a'}_{bc} J^{b} J^{c}$ through the relation (3.20).

**IV. GEOMETRICAL FORMULATION**

In Theorem 4, the special cases of a pure Yang-Mills/Chapline-Manton deformation, a pure Freedman-Townsend/Chapline-Manton deformation, and pure extended Freedman-Townsend/Chapline-Manton deformations, each lead to interesting nonlinear theories with noteworthy geometrical features, as will now be described. For this purpose it is convenient to employ an index-free notation and work entirely with $A$, $A'$-valued 1-forms and 2-forms (denoted by bold face field variables).

**A. Yang-Mills/Chapline-Manton theory**

From Theorem 2 and Proposition 3, the nontrivial algebraic structure for this type of deformation is given by a semisimple Lie bracket $[\cdot, \cdot]_{A}$ and a symmetric bilinear map $e(\cdot, \cdot) = e' \otimes g(\cdot, \cdot)$ where $g(\cdot, \cdot) = tr_{A} (\cdot \otimes \cdot)$ is the Cartan-Killing metric of $A$ and $e'$ is any vector in $A'$, with $b(\cdot) = 0$ and $[\cdot, \cdot]_{A'} = 0$. Accordingly, without loss of essential generality, we let $A'$ be a normed one-dimensional abelian Lie algebra $U(1)$ spanned by $e'$, so the field variables then consist of a $A$-valued Yang-Mills 1-form $A$ and a $U(1)$-valued 2-form $B$. 
The Lagrangian is given by the 4-form

\[ L_{YM/CM} = \text{tr}_A (\ast F_A F_A) + \ast (H - eG_A) (H - eG_A) \]  

(4.1)

where \( F_A = dA + \frac{1}{2} [A, A]_A \), \( H = dB \) are the Yang-Mills curvature 2-form and abelian field strength 3-form, respectively, and \( G_A = \text{tr}_A (AdA + \frac{1}{3} A[A, A]_A) \) is the nonabelian Chern-Simons 3-form \([11]\), while \( e \) is a coupling constant equal to the norm of \( e' \). This Lagrangian is separately invariant under the abelian gauge symmetry

\[ \delta \chi A = 0, \quad \delta \chi B = d\chi, \]  

(4.2)

for arbitrary \( U(1) \)-valued 1-forms \( \chi \), and under the combined Yang-Mills/Chapline-Manton gauge symmetry

\[ \delta \xi A = d\xi + [A, \xi]_A \equiv D_A \xi, \quad \delta \xi B = e \text{tr}_A (Ad\xi), \]  

(4.3)

for arbitrary \( A \)-valued 0-forms \( \xi \). This theory describes a nonlinear deformation of non-abelian Yang-Mills gauge theory and abelian Freedman-Townsend gauge theory for \( A, B \). Its dual formulation involves a linear \( U(1) \) sigma (i.e. scalar) field \( \phi \) introduced through the \( B \) field equation by

\[ \ast H - e\ast G_A = d\phi. \]  

(4.4)

The dual Lagrangian for \( \phi \) and \( A \) is given by the 4-form

\[ L_{YM/CM}^{\text{dual}} = \text{tr}_A (\ast F_A F_A) + (2eG_A \pm \ast d\phi) d\phi. \]  

(4.5)

This yields the field equations

\[ D_A \ast F_A = -2eF_A d\phi, \quad d\ast d\phi = \mp e \text{tr}_A (F_A F_A). \]  

(4.6)

Hence, \( \phi \) couples to \( A \) through the nonabelian Chern class 4-form \( \text{tr}_A (F_A F_A) \) \([11]\).

Thus a pure Yang-Mills/Chapline-Manton deformation describes an exotic nonabelian Yang-Mills scalar dilaton theory.

**B. Freedman-Townsend/Chapline-Manton theory**

For this type of deformation, the nontrivial algebraic structure given by Theorem 2 and Proposition 3 consists of a nonabelian Lie bracket \([\cdot, \cdot]_A'\) and a symmetric linear map \( e^S(\cdot) \)
that annihilates the commutator ideal $D\mathcal{A}' = [\mathcal{A}', \mathcal{A}]$, with $b(\cdot) = 0$ and $[\cdot, \cdot]_\mathcal{A} = 0$. Consequently, for there to exist a nontrivial such map $e^S(\cdot)$, it is necessary that $D\mathcal{A}'$ be a proper Lie subalgebra of $\mathcal{A}'$ and that the cokernel of $e^S(\cdot)$ belong to the orthogonal complement of $D\mathcal{A}'$ in $\mathcal{A}$, which we denote by $D^\perp \mathcal{A}'$. Note it then follows that $[D^\perp \mathcal{A}', D\mathcal{A}'] \subseteq D\mathcal{A}'$ and $e^S(D\mathcal{A}') = 0$ while the linear maps $e^S(u')$ for each $u'$ in $D^\perp \mathcal{A}'$ are, by diagonalization, a sum of one-dimensional projection operators in $\mathcal{A}$. These algebraic relations are satisfied if, with little loss of generality, we let $D^\perp \mathcal{A}' = U(1)$ and take $\mathcal{A}'$ to be the semidirect product $U(1) \rtimes \mathcal{S}'$ of an abelian Lie algebra $U(1)$ and a semisimple Lie algebra $\mathcal{S}' = D\mathcal{A}'$ such that $U(1)$ and $\mathcal{S}'$ are orthogonal subspaces in $\mathcal{A}'$ with respect to a fixed metric $g'(\cdot, \cdot)$ whose restriction to $\mathcal{S}'$ is the Cartan-Killing metric, where the $U(1)$ action on $\mathcal{S}'$ is given by an adjoint representation $ad_{\mathcal{S}'}(e')$ using a vector $e'$ in $\mathcal{S}'$. We also then let $\mathcal{A} = U(1)$ and $e^S(\cdot) = g'(e', \cdot)\text{id}_\mathcal{A}$ where $e'$ belongs to $U(1)$ in $\mathcal{A}'$ and $\text{id}_\mathcal{A}$ is the identity map on $\mathcal{A}$. Correspondingly, the field variables are a $U(1)$-valued 1-form $A$, and a $U(1)$-valued 2-form $B$ along with a $\mathcal{S}'$-valued 2-form $B$ as associated with $\mathcal{A}' = U(1) \rtimes \mathcal{S}'$.

To proceed, let $F = dA$ and $G = AdA$ denote the $U(1)$ Yang-Mills (i.e. Maxwell) field strength 2-form and $U(1)$ Chern-Simons 3-form, respectively. Introduce as nonlinear Freedman-Townsend field strengths a $U(1)$-valued 1-form $K$ and $\mathcal{S}'$-valued 1-form $K$, which include a Chapline-Manton coupling, given by

$$K = *(D_KB + \frac{1}{e}K[e', B]_{\mathcal{S}'})$$

$$K = *(dB + \frac{1}{e}g'(e', [K, B]_{\mathcal{S}'}) - eG),$$

(4.7)

together with the covariant derivative

$$D_K = d + [K, \cdot]_{\mathcal{S}'}$$

(4.8)

where $e = g'(e', e')^{1/2}$ is a coupling constant. Now the Lagrangian is given by the 4-form

$$L_{\text{FT/CM}} = *FdA + tr_{\mathcal{S}'}(KdB) + K(dB - eG)$$

(4.9)

with $tr_{\mathcal{S}'}(\cdot \otimes \cdot) = g'(\cdot, \cdot)|_{\mathcal{S}'}$ denoting the Cartan-Killing metric on $\mathcal{S}'$. This Lagrangian is separately invariant (to within an exact 4-form) under the Freedman-Townsend gauge symmetries

$$\delta_\chi B = d\chi, \quad \delta_\chi B = 0, \quad \delta_\chi A = \delta_\chi A = 0,$$

(4.10)

$$\delta_\chi B = \frac{1}{e}g'(e', [K, \chi]_{\mathcal{S}'})$$

$$\delta_\chi B = D_K\chi + \frac{1}{e}K[e', \chi]_{\mathcal{S}'},$$

(4.11)
for arbitrary $U(1)$-valued 1-forms $\chi, \chi'$, and under the combined Maxwell/Chapline-Manton gauge symmetry
\[
\delta_\xi A = d\xi, \quad \delta_\xi B = eF\xi, \quad \delta_\xi B = 0, \quad (4.12)
\]
for arbitrary $U(1)$-valued 0-forms $\xi$. This theory describes a nonlinear deformation of non-abelian Freedman-Townsend gauge theory for $B, \mathbf{B}$ and Maxwell (i.e. $U(1)$ Yang-Mills) gauge theory for $A$.

The dual formulation of this theory, obtained through the $B, \mathbf{B}$ field equations, involves both a scalar field $\phi$ and a chiral (nonlinear sigma) field $U$ given by
\[
K = d\phi, \quad ad_{\mathcal{S}'}(K) = U^{-1}dU - e'_{\mathcal{S}'}d\phi \quad (4.13)
\]
where $U$ is a matrix belonging to the adjoint representation of the local semisimple Lie group determined by the Lie algebra $\mathcal{S}'$, and $e'_{\mathcal{S}'}$ denotes the matrix $\frac{1}{2}ad_{\mathcal{S}'}(e')$. The dual Lagrangian is given by the 4-form
\[
L_{\text{exFT}}^{\text{dual}} = \ast FF + 2(eG \pm \ast d\phi)d\phi \pm tr_{\mathcal{A}'}(U^{-1}\ast dU(U^{-1}dU - 2e'_{\mathcal{S}'}d\phi)), \quad (4.14)
\]
which yields the field equations
\[
d\ast F = -2eFd\phi, \quad d\ast d\phi = \mp eFF, \quad (4.15)
\]
\[
d\ast dU = dUU^{-1}\ast dU + U[U^{-1}dU, e'_{\mathcal{S}'}]\ast d\phi \mp eFFUe'_{\mathcal{S}'}.
\]
Hence, $\phi$ and $U$ couple to $A$ through the $U(1)$ Chern class 4-form $FF$.

Therefore a pure Freedman-Townsend/Chapline-Manton deformation describes an exotic Maxwell scalar/chiral dilaton theory.

C. Extended Freedman-Townsend theory

For an extended Freedman-Townsend deformation by itself, the most natural algebraic structure is given by a semisimple Lie algebra $\mathcal{A}'$ together with its adjoint representation $b(\cdot) = ad_{\mathcal{A}'}(\cdot)$ on the vector space $\mathcal{A} \simeq \mathcal{A}'$ (i.e. $\mathbf{h} = \text{id}$ is a vector space isomorphism), and an invariant metric $g(\cdot, \cdot)$ on both $\mathcal{A}', \mathcal{A}$. It is worth describing this type of deformation in more detail, since it has some novel features of its own. The Lagrangian is given in terms of an $\mathcal{A}$-valued 1-form $A$ and an $\mathcal{A}'$-valued 2-form $B$ by
\[
L_{\text{exFT}} = tr_{\mathcal{A}}(\ast JF) + tr_{\mathcal{A}'}(KH) \quad (4.17)
\]
where $F = dA$, $H = dB$ are abelian field strengths, and the nonlinear Freedman-Townsend field strength 2-form $*J$ and 1-form $K$ are defined by

$$J = D_K A, \quad K = *(D_K B + ad_{\mathcal{A}'}(*J)A),$$

(4.18)

using the covariant derivative

$$D_K = d + ad_{\mathcal{A}'}(K).$$

(4.19)

This Lagrangian is invariant (to within an exact 4-form) under the extended Freedman-Townsend gauge symmetries

$$\delta_{\chi} A = 0, \quad \delta_{\chi} B = D_K \chi,$$

(4.20)

$$\delta_{\xi} A = D_K \xi, \quad \delta_{\xi} B = ad_{\mathcal{A}'}(*J)\xi,$$

(4.21)

for arbitrary $\mathcal{A}$-valued 0-forms $\xi$, $\mathcal{A}'$-valued 1-forms $\chi$. The field equations for $A, B$

$$D_K *J = 0, \quad dK + \frac{1}{2}[K, K]_{\mathcal{A}'} = 0$$

(4.22)

yield a dual formulation

$$d* dU = dUU^{-1} * dU, \quad d* d(UA) = 0$$

(4.23)

in terms of the chiral field $U$ given by a matrix belonging to the adjoint representation of the local semisimple Lie group associated with the Lie algebra $\mathcal{A}'$, where

$$K = U^{-1} dU, \quad J = U^{-1} d(UA).$$

(4.24)

Hence, extended Freedman-Townsend gauge theory is equivalent to decoupled theories of a chiral field $U$ and an abelian Yang-Mills field $\tilde{A} = UA$ where $\tilde{F} = d\tilde{A}$ is the abelian Yang-Mills field strength. The structure of this theory, however, is incompatible with a Chapline-Manton coupling, as will now be demonstrated. From Proposition 3, the inclusion of a Chapline-Manton deformation involves a symmetric linear map $e^S(\cdot)$ given by either of the two algebraic relations in Eq. (2.21). But, here, notice the first relation becomes trivial, $e^S(\cdot) = eb^S(\cdot) = 0$, since $b(\cdot) = ad_{\mathcal{A}'}(\cdot) = -ad_{\mathcal{A}'}^T(\cdot) = -b^T(\cdot)$, while the second relation $e^S(\cdot) = g(e', \cdot)id_{\mathcal{A}}$ cannot be satisfied because there is no nonzero vector $e'$ orthogonal to $D\mathcal{A}' = [\mathcal{A}', \mathcal{A}']_{\mathcal{A}'} = \mathcal{A}'$ in a semisimple Lie algebra.

This proves a no-go result for combining a Chapline-Manton deformation with an extended Freedman-Townsend deformation in the natural case where the algebraic structure
is based on a semisimple Lie algebra \( \mathcal{A}' \). However, if a non-semisimple Lie algebra \( \mathcal{A}' \) is considered, then there exists a consistent nontrivial extended Freedman-Townsend/Chapline-Manton deformation. In particular, the algebraic structure required by Proposition 3 is satisfied by the non-semisimple example for \( \mathcal{A}' \) used above in the pure Freedman-Townsend/Chapline-Manton deformation if we take \( b(\cdot)e = e^S(\cdot) = g(e'_\perp, \cdot)id_A \) for any vector \( e'_\perp \) in \( U(1) \), where \( \mathcal{A}' \) is the semidirect product \( U(1) \times S' \) and \( A = U(1) \). Here, as above, \( U(1) \) and \( S' \) are orthogonal subspaces in \( \mathcal{A}' \), with the \( U(1) \) action on \( S' \) given by an adjoint representation \( ad_{S'}(e') \equiv e e'_S \) involving a fixed vector \( e' \) in \( S' \), and a coupling constant \( e \) equal to the norm of \( e' \).

This non-semisimple algebraic structure provides a generalization of the pure Freedman-Townsend/Chapline-Manton theory for \( A, B, B' \) to a theory with an extended Freedman-Townsend coupling. The dual formulation for this deformation is given by just replacing \( F = dA \) with \( J = D_K A = dA + d\phi A \) in the Lagrangian \( L_{\text{dualFT/CM}} \). Under this replacement, since \( A = U(1) \) is abelian, the \( U(1) \)-valued 3-form

\[
\frac{1}{e}G_K = AD_K A = A(dA + d\phi A) = AdA = G_A
\]  

(4.25)

is equal to the \( U(1) \) Chern-Simons 3-form \( G_A \). The resulting field equations are given by

\[
d\ast J = (\ast J - 2eJ)d\phi, \quad \pm d\ast d\phi = \ast J J - e J J,
\]

(4.26)

\[
d\ast dU = dUU^{-1}\ast dU \pm (\ast J J - eJ J)U e'_{S'}.
\]

(4.27)

Hence, in this theory, the abelian 1-form \( A \) couples to a scalar dilaton \( \phi \) and a chiral dilaton \( U \) through a generalized \( U(1) \) Chern class 4-form \( JJ = FF - 2G_A d\phi \) as well as a standard Maxwell type 4-form \( \ast JJ \), which are related by \( d(eG_A - A \ast D_K A) = eJ J - \ast J J \) on solutions \( A \). In particular, since \( D_K = d + d\phi \), it follows that \( J = e^{-\phi} \tilde{J} \) and \( A = e^{-\phi} \tilde{A} \) where \( \tilde{J} = d\tilde{A} \) is the Maxwell field strength of \( \tilde{A} \). Thus, the effect of adding an extended Freedman-Townsend coupling produces a standard dilaton coupling of \( \phi \) to \( A \) in the pure Freedman-Townsend/Chapline-Manton deformation.

Finally, the geometrical structure of this theory illustrates a general feature that the Chapline-Manton and (extended) Freedman-Townsend deformations possess opposite parity. Specifically, under a parity operator defined by \( \mathcal{P} = \mathcal{P}^2 \) such that \( \mathcal{P}d = d\mathcal{P} \) and \( \mathcal{P} \ast = -\ast \mathcal{P} \), it follows from Proposition 1 that if \( A^a \to \mathcal{P} A^a, B^{a'} \to \mathcal{P} B^{a'} \), then \( L_{\text{YM}} \to -\mathcal{P} L_{\text{YM}}, L_{\text{FT}} + L_{\text{exFT}} \to \mathcal{P}(L_{\text{FT}} + L_{\text{exFT}}), \) and \( L_{\text{CM}} \to -\mathcal{P} L_{\text{CM}}. \)
V. CONCLUDING REMARKS

The most general class of covariant nonabelian gauge theories of a set of coupled massless 1-form and 2-form fields in four dimensions has been constructed in this Letter, arising from the recent classification of geometrical nonlinear deformations of the linear abelian gauge theory of such fields \[2,3,9,12\]. These deformations comprise a Yang-Mills coupling of the 1-forms, a Freedman-Townsend coupling of the 2-forms, an extended Freedman-Townsend type coupling between the 1-forms and 2-forms, in addition to a Chapline-Manton type coupling of the 1-forms with the 2-forms. The well known duality \[6\] of nonabelian Freedman-Townsend gauge field theory of 2-forms and nonlinear sigma field theory for Lie group targets carries over to the more general gauge field theories of 1-forms and 2-forms presented here. In particular, due to the presence of the Chapline-Manton coupling, the dual formulation of these theories is found to describe a class of exotic Yang-Mills dilaton theories in which the nonlinear sigma field has a novel type of dilaton coupling to the Yang-Mills 1-form field through a generalized Chern class term. This Yang-Mills dilaton coupling gives an interesting generalization of nonlinear sigma field theory into Lie group targets on four dimensional manifolds.

In the case of a Euclidean manifold (namely, \(\ast^2 = 1\)), the Yang-Mills dilaton field equations represent harmonic maps coupled to elliptic equations for a Yang-Mills connection (modulo gauge conditions), while in the case of a Lorentzian manifold (namely, \(\ast^2 = -1\)), the field equations instead represent wave maps coupled to hyperbolic equations for a Yang-Mills connection (modulo gauge conditions). Since the field equations in both cases involve a generalized Chern class term, the solutions of the resulting elliptic and hyperbolic nonlinear systems may be expected to possess some analytic features that depend on global properties of the bundle of Yang-Mills connections and the underlying four-manifold. As such, these systems should prove to be of interest of study in areas of mathematical physics related to the Yang-Mills equations, harmonic maps and wave maps.

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[10] In pure Yang-Mills theory on a $n$-dimensional manifold $M$ with a gauge group $G$, the Chern class is the gauge invariant $n$-form given by the exterior product of the Yang-Mills curvature 2-form and its $n-2$-form Hodge dual, contracted with the Cartan-Killing metric of the Lie algebra $A$ of $G$. The integral of the Chern class over $M$ defines a global differential invariant, associated with the principle bundle on $M$ whose structure group is $G$. See, e.g., Ref. [11].

[11] Y. Choquet-Bruhat and C. DeWitt-Morette, *Analysis, Manifolds and Physics Part II: 92 Applications* (North-Holland, 1989).

[12] Here, geometrical deformations mean that all deformation terms are locally constructed in a covariant manner entirely from the $p$-form fields and their associated gauge symmetry parameters (arbitrary $p-1$-forms on $M$), using only the Hodge dual $\ast$ and exterior derivative $d$ operators.