A new infinite family of irregular algebraic surfaces with canonical map of degree 8

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Abstract. In this note, we construct an unlimited family of irregular algebraic surfaces of general type with canonical map of degree 8, irregularity 1, and arbitrarily large geometric genus such that the image of the canonical map is not a surface of minimal degree.

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1. Introduction. Let $X$ be a minimal smooth complex surface of general type and denote by $\varphi_{|K_X|}: X \longrightarrow \mathbb{P}^{p_g(X)-1}$ the canonical map of $X$, where $K_X$ is the canonical divisor of $X$ and $p_g(X) = \dim H^0(X, K_X)$ is the geometric genus. The existence of surfaces of general type with non-birational canonical map has been studied intensively by many authors in the last decades. This problem is motivated by the work of Beauville [1]. We refer to the recent preprint by Mendes Lopes and Pardini [6] on the subject. We restrict our attention to the existence of surfaces of general type with canonical map of degree 8. It is known that for surfaces of general type, the degree $d$ of the canonical map is at most 9 if the holomorphic Euler-Poincaré characteristic $\chi(O_X)$ is bigger than or equal to 31. This was proven by Beauville in [1]. In 1986, Xiao improved this result by showing that the degree of the canonical map is less than or equal to 8 if the geometric genus of the surface is bigger than 132 [11]. The first unlimited family of surfaces with canonical map of degree 8 was found by Beauville [8]. These examples were constructed as double covers of the product surface of a non-hyperelliptic curve of genus 3 and the rational

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curve \( \mathbb{P}^1 \). In [8], some unlimited families of surfaces with canonical map of degree 8 were found as \( \mathbb{Z}^3_2 \)-covers of the first Hirzebruch surface \( \mathbb{F}_1 \) or its blow-up in a point. In all these known examples, the image of the canonical map is a surface of minimal degree. In this note, we construct an unlimited family of algebraic surfaces with \( d = 8 \) and arbitrarily large \( p_g \) such that the image of the canonical map is not of minimal degree.

**Theorem 1.1.** Let \( n \) be an integer with \( n \geq 3 \). There exist minimal surfaces of general type \( X \) satisfying

| \( K_X^2 \) | \( p_g (X) \) | \( q (X) \) | \( \deg (\text{im} \varphi|_{K_X}) \) | \( |K_X| \) |
|-----|-----|-----|----------------|-----|
| 16n | 2n  | 1   | 2n             | is base point free |

such that the canonical map \( \varphi|_{K_X} \) has degree 8.

In the above theorem, \( q (X) = \dim H^1 (X, K_X) \) is the irregularity of \( X \). These surfaces are constructed as \( \mathbb{Z}^3_2 \)-covers of the product surface of a smooth elliptic curve \( C \) and the rational curve \( \mathbb{P}^1 \). The building data \( \{ L_x, D_{\sigma} \}_{x, \sigma} \) of \( \mathbb{Z}^3_2 \)-covers (see Section 2) are chosen such that there is a character \( \chi' \) of \( \mathbb{Z}^3_2 \) with arbitrarily large \( h^0 (L_{\chi'} + K_{\mathbb{P}^1 \times C}) \) and that \( h^0 (L_C + K_{\mathbb{P}^1 \times C}) \) vanishes for all other characters \( \chi \) of \( \mathbb{Z}^3_2 \). From the decomposition of the space of 2-forms of the surfaces (see Proposition 2.2),

\[
H^0 (X, K_X) = H^0 (\mathbb{P}^1 \times C, K_{\mathbb{P}^1 \times C}) \oplus \bigoplus_{\chi \neq \chi_{000}} H^0 (\mathbb{P}^1 \times C, K_{\mathbb{P}^1 \times C} + L_C),
\]

such a choice of the building data allows to conclude that the canonical map of \( X \) factors through the \( \mathbb{Z}^3_2 \)-cover. Furthermore, we choose the divisor \( L_{\chi'} \) in \( \mathbb{P}^1 \times C \) such that \( L_{\chi'} + K_{\mathbb{P}^1 \times C} \equiv E + \sum_{i=1}^n F_{ii} \), where \( E \) is a general elliptic fiber and \( F_{ii} \) are distinct rational fibres of \( \mathbb{P}^1 \times C \). This leads to the fact that the map \( \varphi|_{E + \sum_{i=1}^n F_{ii}} \) is of degree one for all \( n \geq 3 \). Thus, the canonical map of \( X \) is of degree 8. We notice that the covers of such a product guarantee the irregularity of the result by pulling back 1-forms.

In our construction, if \( n = 2 \), we obtain a smooth minimal surface of general type \( X \) with \( K_X^2 = 32, p_g (X) = 4, q (X) = 1, \) and \( d = 16 \) since the linear system \( |E + F_{11} + F_{22}| \) is a map of degree 2 onto \( \mathbb{P}^1 \times \mathbb{P}^1 \). This surface was constructed as \( \mathbb{Z}^3_2 \)-cover of \( \mathbb{P}^1 \times \mathbb{P}^1 \) (see [7]). It is worth pointing out that C. Gleissner, R. Pignatelli, and C. Rito constructed a family of surfaces with \( K_X^2 = 24, p_g (X) = 3, q (X) = 1, \) and \( d = 24 \) ([3]). Their example has a very similar construction as \( \mathbb{Z}^3_2 \)-cover of \( \mathbb{P}^1 \times C \) branched on “fibers” of the obvious trivial fibrations.

Throughout this note, all surfaces are projective algebraic over the complex numbers. The linear equivalence of divisors is denoted by \( \equiv \). A character \( \chi \) of the group \( G \) is a homomorphism from \( G \) to \( \mathbb{C}^* \), the multiplicative group of the non-zero complex numbers. The rest of the notation is standard in algebraic geometry.
2. $\mathbb{Z}_2^3$-coverings. The construction of Abelian covers was studied by Pardini in [9]. For details about the building data of Abelian covers and their notations, we refer the reader to Section 1 and Section 2 of Pardini’s work ([9]). For the sake of completeness, we recall some facts on $\mathbb{Z}_2^3$-covers, in a form which is convenient for our later constructions.

We denote by $\chi_{j_1,j_2,j_3}$ the character of $\mathbb{Z}_2^3$ defined by

$$\chi_{j_1,j_2,j_3}(a_1,a_2,a_3) := e^{(\pi a_1 j_1)\sqrt{-1}}e^{(\pi a_2 j_2)\sqrt{-1}}e^{(\pi a_3 j_3)\sqrt{-1}}$$

for all $j_1, j_2, j_3, a_1, a_2, a_3 \in \mathbb{Z}_2$. A $\mathbb{Z}_2^3$-cover $X \rightarrow Y$ can be determined by a collection of non-trivial divisors $L_\chi$ labelled by characters of $\mathbb{Z}_2^3$ and effective divisors $D_\sigma$ labelled by non-trivial elements of $\mathbb{Z}_2^3$ of the surface $Y$. More precisely, from [9, Theorem 2.1], we can define $\mathbb{Z}_2^3$-covers as follows:

**Proposition 2.1.** Given a smooth projective surface $Y$, let $L_\chi$ be divisors of $Y$ such that $L_\chi \not\equiv O_Y$ for all non-trivial characters $\chi$ of $\mathbb{Z}_2^3$ and let $D_\sigma$ be effective divisors of $Y$ for all $\sigma \in \mathbb{Z}_2^3 \setminus \{(0,0,0)\}$ such that the total branch divisor $B := \sum_{\sigma \neq 0} D_\sigma$ is reduced. Then $\{L_\chi, D_\sigma\}_{\chi, \sigma}$ is the building data of a $\mathbb{Z}_2^3$-cover $f : X \rightarrow Y$ if and only if

$$L_\chi + L_{\chi'} \equiv L_{\chi\chi'} + \sum_{\chi(\sigma)=\chi'(\sigma)=-1} D_\sigma$$

(2.1)

for all non-trivial characters $\chi, \chi'$ of $\mathbb{Z}_2^3$.

For the reader’s convenience, we leave here the relations (2.1) of the reduced building data of $\mathbb{Z}_2^3$-covers:

\[
\begin{align*}
L_{100} + L_{100} & \equiv D_{100} + D_{110} + D_{111}, \\
L_{100} + L_{010} & \equiv D_{110} + D_{111} + L_{110}, \\
L_{100} + L_{001} & \equiv D_{101} + D_{111} + L_{101}, \\
L_{010} + L_{010} & \equiv D_{010} + D_{011} + D_{110} + D_{111}, \\
L_{010} + L_{001} & \equiv D_{011} + D_{111} + L_{011}, \\
L_{001} + L_{001} & \equiv D_{001} + D_{011} + D_{101} + D_{111}.
\end{align*}
\]

By [9, Theorem 3.1], if each branch component $D_\sigma$ is smooth and the total branch locus $B$ is a simple normal crossings divisor, the surface $X$ is smooth. Also from [9, Lemma 4.2, Proposition 4.2], we have:

**Proposition 2.2.** If $Y$ is a smooth surface and $f : X \rightarrow Y$ is a smooth $\mathbb{Z}_2^3$-cover with the building data $\{L_\chi, D_\sigma\}_{\chi, \sigma}$, the surface $X$ satisfies the following:

$$2K_X \equiv f^*\left(2K_Y + \sum_{\sigma \neq 0} D_\sigma\right);$$

$$f_\ast O_X = O_Y \bigoplus_{\chi \neq \chi^{000}} L_\chi^{-1}.$$
This implies that
\[ H^0(X, K_X) = H^0(Y, K_Y) \oplus \bigoplus_{\chi \neq \chi_{000}} H^0(Y, K_Y + L_\chi); \]
\[ K_X^2 = 2 \left( 2K_Y + \sum_{\sigma \neq 0} D_\sigma \right)^2; \]
\[ p_g(X) = p_g(Y) + \sum_{\chi \neq \chi_{000}} h^0(L_\chi + K_Y); \]
\[ \chi(O_X) = 8\chi(O_Y) + \sum_{\chi \neq \chi_{000}} \frac{1}{2} L_\chi(L_\chi + K_Y). \]

Moreover, the canonical linear system \(|K_X|\) is generated by
\[ f^*((K_Y + L_\chi)) + \sum_{\chi(\sigma) = 1} R_\sigma, \forall \chi \in J, \]
where \( J := \{ \chi' : |K_Y + L_{\chi'}| \neq \emptyset \} \) and \( R_\sigma \) is the reduced divisor supported on \( f^*(D_\sigma) \).

For the proof of the last statement of Proposition 2.2, we refer the reader to [3, Page 3].

3. Construction. Throughout this section, we denote by \( Y := \mathbb{P}^1 \times C \) the product surface of the rational curve \( \mathbb{P}^1 \) and a smooth elliptic curve \( C \). Let \( p_1 : \mathbb{P}^1 \times C \longrightarrow \mathbb{P}^1 \) be the projection of the product surface \( \mathbb{P}^1 \times C \) on \( \mathbb{P}^1 \).

We denote by \( E \) a general fiber of \( p_1 \). The canonical class of \( Y \) is \( K_Y \equiv -2E \).

Let \( E_1, E_2, \ldots, E_6 \) be distinct elliptic fibres and let \( F_1, F_2, \ldots, F_n, F'_1, F'_2, \ldots, F'_n, F''_1, F''_2, F''_3 \) be distinct rational fibres (with \( n \geq 3 \)) such that \( 2F''_1 \equiv 2F''_2 \equiv 2F''_3 \). Because the sum of two points in an elliptic curve is divisible by \( 2 \) in the Picard group, there are fibres \( F_{ii} \) such that \( 2F_{ii} \equiv F_i + F'_i \), for all \( i \in \{1, 2, \ldots, n\} \). We consider the following divisors:

\[ D_{100} := E_1 + E_2, \quad D_{101} := E_3 + E_4, \quad D_{110} := E_5 + E_6, \]
\[ D_{111} := \sum_{i=1}^{n} \left( F_i + F'_i \right), \]
\[ L_{100} := 3E + \sum_{i=1}^{n} F_{ii}, \quad L_{010} := E + \sum_{i=1}^{n} F_{ii} + \eta_1, \]
\[ L_{001} := E + \sum_{i=1}^{n} F_{ii} + \eta_2, \quad L_{110} := 2E + \eta_1, \]
\[ L_{101} := 2E + \eta_2, \quad L_{011} := 2E + \eta_3, \quad L_{111} := E + \sum_{i=1}^{n} F_{ii} + \eta_3, \]
where $\eta_1 := F_1'' - F_2''$, $\eta_2 := F_2'' - F_3''$, $\eta_3 := F_1'' - F_3'' = \eta_1 + \eta_2$ are non-trivial 2-torsions. These divisors $D_\sigma, L_\chi$ satisfy the following relations:

\[
L_{100} + L_{100} \equiv D_{100} + D_{101} + D_{110} + D_{111} \equiv 6E + \sum_{i=1}^{n} 2F_{ii},
\]

\[
L_{100} + L_{010} \equiv D_{110} + D_{111} + L_{110} \equiv 4E + \sum_{i=1}^{n} 2F_{ii} + \eta_1,
\]

\[
L_{100} + L_{001} \equiv D_{101} + D_{111} + L_{101} \equiv 4E + \sum_{i=1}^{n} 2F_{ii} + \eta_2,
\]

\[
L_{010} + L_{010} \equiv D_{100} + D_{111} \equiv 2E + \sum_{i=1}^{n} 2F_{ii},
\]

\[
L_{010} + L_{001} \equiv D_{111} + L_{011} \equiv 2E + \sum_{i=1}^{n} 2F_{ii} + \eta_3,
\]

\[
L_{001} + L_{001} \equiv D_{101} + D_{111} \equiv 2E + \sum_{i=1}^{n} 2F_{ii}.
\]

Thus by Proposition 2.1, the divisors $D_\sigma, L_\chi$ define a $\mathbb{Z}_2^3$-cover $f : X \to Y$. Because each branch component $D_\sigma$ is smooth and the total branch locus $B$ is a normal crossings divisor, the surface $X$ is smooth. Moreover, by Proposition 2.2, the surface $X$ satisfies the following:

\[
2K_X \equiv f^* \left( 2E + \sum_{i=1}^{n} \left( F_i + F_i' \right) \right).
\]

We notice that a surface is of general type and minimal if the canonical divisor is big and nef (see e.g. [5, Section 2]). Since the divisor $2K_X$ is the pull-back of a nef and big divisor, the canonical divisor $K_X$ is nef and big. Thus, the surface $X$ is of general type and minimal. Moreover, by Proposition 2.2, the invariants of $X$ are as follows:

\[
K_X^2 = 16n, p_g (X) = 2n, \chi (\mathcal{O}_X) = 2n, q (X) = 1.
\]

We show that the canonical map is of degree 8. By Proposition 2.2, we have the following decomposition:

\[
H^0 (X, K_X) = H^0 (Y, K_Y) \oplus \bigoplus_{\chi \neq \chi_{000}} H^0 (Y, K_Y + L_\chi).
\]

Moreover, the choice of the $L_\chi$’s yields that

\[
h^0 (L_\chi + K_Y) = 0
\]

for all $\chi \neq \chi_{100}$. By Proposition 2.2, the linear system $|K_X|$ is generated by

\[
f^* (K_Y + L_{100}) = f^* \left( E + \sum_{i=1}^{n} F_{ii} \right).
\]
This implies that the canonical map of the surface $X$ factors through the $\mathbb{Z}_2^3$-cover $f : X \longrightarrow Y$. Thus the following diagram commutes:

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow f & & \downarrow \varphi|_{K_Z} \\
\varphi|_{K_X} & & \varphi|_{K + \sum_{i=1}^n F_{ii}}
\end{array}
$$

Since the map $\varphi|_{E + \sum_{i=1}^n F_{ii}}$ is of degree one for all $n \geq 3$, the canonical map of $X$ is of degree 8.

**Remark 3.1.** Let $Z := X/\Gamma$ be the quotient surface of $X$, where $\Gamma := \langle (0,0,1), (0,1,0) \rangle$ is the subgroup of $\mathbb{Z}_2^3$. The surface $Z$ is a surface of general type whose only singularities are $12n$ nodes. Moreover, the canonical map $\varphi|_{K_Z}$ is a map of degree 2 (see [4, Theorem 5.1] and [2, Theorem 1.1]). The canonical map $\varphi|_{K_X}$ is the composition of the degree 4 quotient map $X \longrightarrow Z := X/\Gamma$ with the canonical map $\varphi|_{K_Z}$ of $Z$ (see e.g. [10, Example 2.1]).

**Remark 3.2.** In the above construction, there are four different possible choices for each $F_{ii}$. A different choice produces a different surface $X$.

The previous observation leads to the following interesting question:

**Question 3.3.** Are the surfaces in Remark 3.2 deformation equivalent?

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