AN OBSERVATION ON INITIALLY $\kappa$-COMPACT SPACES

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Abstract. In [2], Chaber has proved that countably compact spaces with a quasi-$G_\delta$-diagonal are compact. We prove that initially $\kappa$-compact spaces with a quasi-$G_\kappa$-diagonal are compact, for any infinite cardinal $\kappa$.

1. Introduction and Terminology

Chaber, in [2] has proved that countably compact spaces with a (quasi-) $G_\delta$-diagonal are compact. We observe that this result may be generalised by using any infinite cardinal instead of the first infinite cardinal $\omega$. For that purpose, we regard countable compactness as initial $\omega$-compactness and extend it naturally to initial $\kappa$-compactness, and the quasi-$G_\delta$-diagonal property to quasi-$G_\kappa$-diagonal property which allows us to conclude that initially $\kappa$-compact spaces with a quasi-$G_\kappa$-diagonal are still compact, for any infinite cardinal $\kappa$.

Throughout this paper, $\kappa$ is an infinite cardinal, $\omega$ is the first infinite ordinal and cardinal, and $X$ is a topological space. Let us recall some basic definitions. If $A \subseteq X$ and $\mathcal{H}$ is a family of subsets of $X$, then the star of $\mathcal{H}$ about $A$ is denoted by $\text{st}(A, \mathcal{H}) = \bigcup \{ H \in \mathcal{H} : H \cap A \neq \emptyset \}$. For $x \in X$, we write $\text{st}(x, \mathcal{H})$ instead of $\text{st}(\{x\}, \mathcal{H})$. A transfinite sequence $\{ \mathcal{O}_\alpha : \alpha \in \kappa \}$ of collections of open subsets of $X$ is said to be a quasi-$G_\kappa$-diagonal sequence for $X$ if for each $x, y \in X$ with $x \neq y$, there exists an $\alpha \in \kappa$ with $x \in \bigcup \{ O : O \in \mathcal{O}_\alpha \}$ and $y \notin \text{st}(x, \mathcal{O}_\alpha)$. If each $\mathcal{O}_\alpha$ is a cover of $X$, then the quasi-$G_\kappa$-diagonal sequence $\{ \mathcal{O}_\alpha : \alpha \in \kappa \}$ is called a $G_\kappa$-diagonal sequence for $X$. For $\kappa = \omega$, a (quasi-) $G_\omega$-diagonal sequence is called a (quasi-) $G_\delta$-diagonal sequence.

It is said that $X$ has a $G_\kappa$-diagonal if the diagonal $\Delta_X = \{(x, x) : x \in X \}$ of $X$ is a $G_\kappa$-set in $X \times X$, (that is, the set $\Delta_X$ is the intersection of $\kappa$-many open sets of $X \times X$). In [1], it was shown that $X$ has a $G_\delta$-diagonal if and only if $X$ has a $G_\delta$-diagonal sequence. It can easily be seen that $X$ has a $G_\kappa$-diagonal if and only if $X$ has a $G_\kappa$-diagonal sequence. The diagonal number $\Delta(X)$ of a space $X$ is the smallest infinite cardinal $\kappa$ such that the

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diagonal $\Delta_X$ of $X$ is the intersection of $\kappa$-many open sets of $X \times X$. Thus, if $X$ has a $G_\kappa$-diagonal, then $\Delta(X) \leq \kappa$.

Recall that a topological space $X$ is said to be initially $\kappa$-compact if every open cover of $X$ of cardinality not exceeding $\kappa$ has a finite subcover. For $\kappa = \omega$, initial $\omega$-compactness is equivalent to countable compactness. For a survey of initial $\kappa$-compactness see [5].

The cardinality of a set $A$ is denoted by $|A|$. Recall that the the weight of $X$ is denoted by $w(X)$ and is defined as the smallest cardinal number of the form $|\mathcal{B}|$, where $\mathcal{B}$ is a base for $X$.

2. Main Result

The following statement generalizes Chaber’s theorem in [2].

**Theorem 2.1.** An initially $\kappa$-compact space with a quasi $G_\kappa$-diagonal is compact.

**Proof.** Let $X$ be an initially $\kappa$-compact space, and let $\{\mathcal{O}_\alpha : \alpha \in \kappa\}$ be a quasi-$G_\kappa$-diagonal sequence for $X$. Suppose that $X$ is not compact. Let $\mathcal{M}$ be a maximal open cover of $X$ without any finite subcover, (that is, $\mathcal{M}$ is an open cover of $X$ without any finite subcover, and the cover $\mathcal{M} \cup \{O\}$ has a finite subcover for any open subset $O$ of $X$ with $O \notin \mathcal{M}$). Since the space $X$ is initially $\kappa$-compact, $\mathcal{M}$ has no subcover of cardinality at most $\kappa$. We claim that for each $x$ in $X$ there exists an $\alpha(x) \in \kappa$ such that $x \in \bigcup \mathcal{O}_{\alpha(x)}$ and $st(x, \mathcal{O}_{\alpha(x)}) \in \mathcal{M}$. To prove this claim, suppose that $st(x, \mathcal{O}_\alpha) \notin \mathcal{M}$ for all $\alpha \in \kappa$ satisfying $x \in \bigcup \mathcal{O}_\alpha$. Let $J = \{\alpha \in \kappa : x \in \bigcup \mathcal{O}_\alpha\}$. Since $st(x, \mathcal{O}_\alpha)$ is an open subset of $X$, the maximality of $\mathcal{M}$ gives us a finite subcover $\mathcal{H}_\alpha$ of the open cover $\mathcal{V}_\alpha = \mathcal{M} \cup \{st(x, \mathcal{O}_\alpha)\}$, for all $\alpha \in J$. So, we have a finite subfamily $\mathcal{W}_\alpha$ of $\mathcal{M}$ such that $\mathcal{H}_\alpha = \mathcal{W}_\alpha \cup \{st(x, \mathcal{O}_\alpha)\}$, for each $\alpha \in J$. Since $\mathcal{H}_\alpha$ is a cover of $X$ and $\{\mathcal{O}_\alpha : \alpha \in \kappa\}$ is a quasi-$G_\kappa$-diagonal sequence for $X$, we have $X \setminus \{x\} \subseteq \bigcup_{\alpha \in J} (\bigcup \mathcal{W}_\alpha)$. Take an $M \in \mathcal{M}$ with $x \in M$. It is clear that the family $\{W : W \in \mathcal{W}_\alpha, \alpha \in J\} \cup \{M\}$ is a subcover of $\mathcal{M}$ and its cardinality is at most $\kappa$. But this contradicts the fact $\mathcal{M}$ has no subcover of cardinality at most $\kappa$. Hence, our claim is true. So, choose an $\alpha(x) \in \kappa$ such that $x \in \bigcup \mathcal{O}_{\alpha(x)}$ and $st(x, \mathcal{O}_{\alpha(x)}) \in \mathcal{M}$, for each $x \in X$. Let $Y_\alpha = \{x \in X : \alpha(x) = \alpha\}$. Obviously, $X = \bigcup_{\alpha \in \kappa} Y_\alpha$.

Now, we claim that $Y_\alpha$ is covered by a finite subfamily of $\mathcal{M}$, for each $\alpha \in \kappa$. Take an $\alpha \in \kappa$. We have two cases:

Case 1: Suppose $st(Y_\alpha, \mathcal{O}_\alpha) \in \mathcal{M}$. Then $Y_\alpha$ is covered by $\{st(Y_\alpha, \mathcal{O}_\alpha)\}$.

Case 2: Suppose $st(Y_\alpha, \mathcal{O}_\alpha) \notin \mathcal{M}$. In this case, by the maximality of $\mathcal{M}$, we have a finite subfamily $\mathcal{S}$ of $\mathcal{M}$ such that $X = st(Y_\alpha, \mathcal{O}_\alpha) \cup (\bigcup \mathcal{S})$. Now, we claim that if $Y_\alpha \setminus (\bigcup \mathcal{S}) \neq \emptyset$, then there exists a finite subset $\{x_0, x_1, ..., x_n\}$ of $Y_\alpha$ such that $Y_\alpha \setminus (\bigcup \mathcal{S}) \subseteq \bigcup_{i=0}^n st(x_i, \mathcal{O}_\alpha)$. Indeed, take a point $x_0 \in Y_\alpha \setminus (\bigcup \mathcal{S})$. If $Y_\alpha \setminus (\bigcup \mathcal{S}) \setminus st(x_0, \mathcal{O}_\alpha) \neq \emptyset$, we can choose a point $x_1 \in Y_\alpha \setminus (\bigcup \mathcal{S}) \setminus st(x_0, \mathcal{O}_\alpha)$. If we can choose inductively a point $x_n \in Y_\alpha \setminus (\bigcup \mathcal{S}) \setminus \bigcup_{m<n} st(x_m, \mathcal{O}_\alpha)$, for each $n \in \omega$, then we obtain a sequence...
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$\{x_n : n \in \omega\}$ in the closed subspace $X \setminus (\bigcup \mathcal{S})$ of $X$. Since initial $\kappa$-compactness is hereditary with respect to closed subsets and every initially $\kappa$-compact space is countably compact, the sequence $\{x_n : n \in \omega\}$ has an accumulation point $x$ in $X \setminus (\bigcup \mathcal{S})$. Since $X = \text{st} (Y_\alpha, \mathcal{O}_\alpha) \cup (\bigcup \mathcal{S})$, we have $x \in \text{st} (Y_\alpha, \mathcal{O}_\alpha)$. So, there exists $O \in \mathcal{O}_\alpha$ with $x \in O$. Note that $|O \cap \{x_n : n \in \omega\}| \leq 1$. But this contradicts the fact that $x$ is an accumulation point of the sequence $\{x_n : n \in \omega\}$.

Since $X = \text{st} (Y_\alpha, \mathcal{O}_\alpha) \cup (\bigcup \mathcal{S})$, we have $x \in \text{st} (Y_\alpha, \mathcal{O}_\alpha)$. So, there exists $O \in \mathcal{O}_\alpha$ with $x \in O$. Note that $|O \cap \{x_n : n \in \omega\}| \leq 1$.

Hence, each $Y_\alpha$ is covered by a finite subfamily of $M$ of cardinality at most $\kappa$. This contradiction enables us to claim that $X$ is compact. □

Since $\Delta (X) = w (X)$, for a compact Hausdorff space $X$ (for example, in [4, 7.6. Corollary]), we can assert the following.

**Corollary 2.2.** If $X$ is a Hausdorff initially $\kappa$-compact space with a $G_\kappa$-diagonal, we have $w (X) \leq \kappa$.

Recall that the pseudocharacter of $X$ at a subset $A$, denoted by $\psi (A, X)$, is defined as the smallest cardinal number of the form $|U|$, where $U$ is a family of open subsets of $X$ such that $\bigcap U = A$. If $A = \{x\}$ is a singleton, then we write $\psi (x, X)$ instead of $\psi (\{x\}, X)$. The pseudocharacter of a space $X$ is defined to be $\psi (X) = \sup \{\psi (x, X) : x \in X\}$. Evidently, the diagonal number $\Delta (X)$ of a space $X$ is equal to the pseudocharacter of its square $X \times X$ at its diagonal $\Delta_X = \{(x, x) : x \in X\}$.

It is well known that, if $G$ is a topological group, we have $\Delta (G) = \psi (G)$. So, Theorem 2.1 enables us to claim the following.

**Corollary 2.3.** If $G$ is an initially $\kappa$-compact topological group with $\psi (G) \leq \kappa$, then $G$ is compact.

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