Motion of a thin elliptic plate under symmetric and asymmetric orthotropic friction forces

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Abstract
The anisotropy of a friction force is proved to be an important factor in various contact problems. We study the dynamical behavior of thin plates with respect to symmetric and asymmetric orthotropic friction. The terminal motion of plates with circular and elliptic contact areas is analyzed. The evaluation of friction forces for both symmetric and asymmetric orthotropic cases is shown using an analytic approach. Regular pressure distribution is considered. Differential equations are formulated and solved numerically for a number of initial conditions. The method used gives more accurate results compared to the previous study. Examples show the significant influence of friction force asymmetry on motion.

1. Introduction
The anisotropy of a friction force is an important factor in contact problems. The surfaces of a wide number of materials (crystals, composites, polymers, wood) are anisotropic due to their internal properties. Engineering materials are usually manufactured in such a way that oriented surface roughness textures appear. Smart materials with directional asymmetry have been developed [1]. Using an atomic force microscope (AFM) [2] as a way of obtaining frictional hodographs is shown. This information is related to the intensity and symmetry of the friction phenomenon. The contact process with wear, deformation, surface evolution, thermal and chemical variation, impact the frictional behavior of materials. A review [3] discusses the factors influencing friction forces and approaches to friction force modeling. The paper [4] presents examples of friction forces with respect to contact stress calculations in various computational tasks.

The influence of anisotropy at the contact interface has been widely investigated experimentally and theoretically during the last few decades by [5–8] and others. Examples of centrosymmetric and noncentrosymmetric friction are presented in [9]. In [10], the authors proposed a generalized Coulomb-like friction law and set up a series of experiments on parallelepiped test specimens with an asymmetric surface texture.

Various phenomenological friction models exist in the literature. The general mathematical model based on the Amounton–Coulomb law was proposed by [11]. Based on the symmetry properties of the friction tensors, the author described a different kind of anisotropic pattern on the surface, see [3, 12], using limited surfaces and Zhukovskii’s moment function to describe the planar sliding of a rigid body. The method based on the Zhukovskii function is suitable for homogeneous friction. The limited surface concept provides an opportunity to deduce the final slip motion direction, and could possibly be used for anisotropic friction. Based on the theory of the tensor function, [13] obtained the structure of the friction tensor for an arbitrary nonlinear case according to the relative sliding velocity. With the foundation of a statistical model of interaction of springs located in a plane with periodically inclined asperities, [14] developed the structure of the friction tensor. Anisotropic friction models have recently been used for the finite element analysis of contact problems in commercial software, for example [15–17].

This study deals with the symmetric and asymmetric orthotropic friction. We investigate the dynamical behavior of sliding and spinning disks on anisotropic surfaces. In [8], some experimental and theoretical results regarding the terminal motion of sliding spinning disks are presented. Sliding and spinning motions are also considered in [6]. However, both papers assume an isotropic friction force. We attempt to finalize the work we have done during the last few years. We investigated a circular area with symmetric
orthotropic friction and uniform pressure distribution in [18], an elliptic area with symmetric orthotropic friction and uniform pressure distribution in [19], linear pressure distribution in [20], a mass point with asymmetric friction in [21], a mass point and elliptic plate with asymmetric friction in [22] and a ring with asymmetric friction in [23]. In these works, we considered different analytic approaches to find out the friction force and friction moment and finally came to the idea that the method presented in the current paper was most suitable for studying the asymmetry of friction.

In the paper, we present descriptions regarding the developed theory for circular and elliptic thin plates under uniform pressure distribution. The effect of the asymmetry of friction force is taken into account and compared with the symmetric case. The asymmetry of friction is used in omni-directional vehicles [24] and robotics [25]. The elliptical contact area, which appears in railway problems (see [26]), in multi-body dynamics during the analysis of foot motion (see [27]) and other situations, is considered. All equations are evolved for this contact domain as a generalized form, which includes the circular area as a test base. The main results are presented for the elliptic contact area. The uniform pressure distribution case is the basis for understanding the proposed approach, which is currently being applied to more complicated pressure laws [28].

2. Formulation of the problem

2.1. Friction law

Let us consider the terminal motion of a thin plate on a horizontal plane with an anisotropic friction force. The term terminal motion originates from the work [8], meaning the problem of identifying the parameters of motion exactly before it finishes. The anisotropic friction force \( \mathbf{T} \) at a point \( M \) of a moving body according to [11] can be written in the following form:

\[
\mathbf{T} = -p_M \mathbf{F}(M) \mathbf{v}, \quad \mathbf{F}(M) = \begin{pmatrix} f_x & f_y \\ f_y & f_x \end{pmatrix},
\]

(1)

where \( p_M \) is the normal pressure at point \( M \), \( \mathbf{F}(M) \) is the friction matrix written with respect to a stationary coordinate system \( \text{Oxy} \) (see [5]), and \( \mathbf{v} \) is the velocity vector of point \( M \).

Friction is symmetric orthotropic in the case of the friction matrix, \( \mathbf{F}(M) \) is a tensor whose components are constant and do not depend on the orientation of contacting areas. This approximation is possible for the case when the hardness of one plane is greater than the hardness of another one, or one of the contact bodies has isotropic frictional properties. If the hardness of each material of the contacting pair is similar we should use a more complicated law for the friction force (see [5]). In this case, the stationary coordinate system \( \text{Oxy} \) obviously coincides with the principal directions of the orthotropic tensor \( \mathbf{F}(M) \) (1).

Friction is asymmetric orthotropic if, in the friction matrix, components differ in negative and positive directions of sliding with \( f_{x^+} \geq f_{x^-}, f_{y^+} \geq f_{y^-} \). Thus, in (1) we have:

\[
f_x = \begin{cases} f_{x^+}, & v_x \geq 0 \\ f_{x^-}, & v_x < 0 \end{cases} \quad \text{and} \quad f_y = \begin{cases} f_{y^+}, & v_y \geq 0 \\ f_{y^-}, & v_y < 0 \end{cases},
\]

where \( v_x, v_y \) are projections of velocity vector in \( \text{Oxy} \).

For both cases the term orthotropic means an assumption that \( f = 0 \).

2.2. Equations of motion

Let us introduce a moving coordinate system \( C \xi \eta \zeta \), associated with the plate. For the elliptic plate this coordinate system is associated with the principal axes of it. Axis \( C \xi \) is perpendicular to the sliding plane. The stationary coordinate system \( \text{Oxy} \) is selected thus, so that the friction matrix has the form (1) and the axes \( Ox \) and \( Oy \) are in the sliding plane. Let \( \varphi \) be an angle between \( Ox \) and \( C \xi \), \( \vartheta \) is an angle between the axis \( Ox \) and \( \mathbf{v} \), which is the velocity of the center of the mass of the plate, so it is at an angle to the trajectory of the center of mass for the elliptic plate

\[
\mathbf{v}_C = \mathbf{v}_C (\cos \varphi \mathbf{i} + \sin \varphi \mathbf{j}),
\]

(2)

where \( \mathbf{v}_C \) is the velocity value and \( \mathbf{i}, \mathbf{j} \) are the unit vectors of axes \( Ox \), \( Oy \). The vector of an angular velocity is \( \mathbf{\omega} = \omega \mathbf{k} \), where \( \omega = \dot{\varphi} \), \( \mathbf{k} \) is a unit vector of axis \( Oz \).

The Euler equation \( \mathbf{v}_M = \mathbf{v}_C + \mathbf{\omega} \times \mathbf{CM} \) allows us to write the following statements:

\[
x' = \xi \cos \varphi - \eta \sin \varphi, \quad v_x = v_C \cos \varphi - \omega y', \\
y' = \xi \sin \varphi + \eta \cos \varphi, \quad v_y = v_C \sin \varphi + \omega x', \\
h' = \eta \cos \vartheta - \xi \sin \vartheta, \quad v_\vartheta = \sqrt{v_x^2 + v_y^2} = 2v_C \omega h.
\]

(3)

With the anisotropic friction law (1) and equation (3) we can write projections of the total friction force vector \( \mathbf{T} \) and the total friction moment \( \mathbf{M} \) in the form:

\[
T_x = \int \tau_x d\xi d\eta, \quad T_y = \int \tau_y d\xi d\eta, \\
M_{C\xi} = \int (\tau_x' x' - \tau_y' y') d\xi d\eta, \\
\tau_x = -f_x p(\xi, \eta) v_x(\xi, \eta) v_y(\xi, \eta) / v_M(\xi, \eta), \\
\tau_y = -f_y p(\xi, \eta) v_x(\xi, \eta) v_y(\xi, \eta) / v_M(\xi, \eta),
\]

(4)

where \( \Omega \) is an integration area.

The equations of motion in the Frenet–Serret frame (the axes are defined by the trajectory of the mass center: the unit vector tangential to the curved points in the direction of motion, the normal unit vector is the derivative of the tangent with respect to the arclength parameter of the curve, divided by its length, the binormal is the cross product of the tangent and normal, and the direction is perpendicular to the sliding surface) with respect to (4) are as follows:

\[
mC = T_x = T_x \cos \varphi + T_y \sin \varphi, \\
mC v = T_y = -T_x \sin \varphi + T_y \cos \varphi, \\
I \dot{\mathbf{\omega}} = M_{C\xi},
\]

(5)
where m is the mass of the plate, I is the plate’s inertia moment about \(C_\xi\), \(T_x\) and \(T_y\) are the projections of the friction force vector \(\mathbf{T}\) on the tangential and normal axes respectively, and \(M_{C_\xi}\) is the friction moment about axis \(C_\xi\).

Although one can argue that equation (5) describes the motion without external forces, which sufficiently reduces the applicability of the proposed study, the particular interest of this research is the impact of friction force anisotropy and asymmetry, which might be hidden by other effects. In addition, the simplified problem helps to lay a basic understanding for the suggested theoretical approach, and is under further development now.

Let us rewrite the system (5) in the dimensionless form using the following relations:

\[
I = ma^dT^*, \quad \xi = a\xi^*, \quad \eta = a\eta^*, \quad v_C = v_C^*\sqrt{ag},
\]

\[
\omega = a\omega\sqrt{ag}, \quad t = t^*\sqrt{ag}, \quad \dot{\theta} = \frac{d\theta}{dt^*}\sqrt{g}, \quad p = p^*\frac{mg}{S}
\]

and let us introduce a variable \(\beta = \frac{\xi}{\eta}\) and a parameter \(\mu = f_\eta - f_\xi\). In these equations, parameter \(a\) is measurable: it is the length of the largest line from point \(C\) to the area boundary and \(S\) is a contact area.

Equation (5) in dimensionless form (asterisks are omitted):

\[
\frac{dv_C}{dt} = -\int\int p(\xi, \eta) \left[ \frac{\beta(f_\xi + \mu \sin \theta) + f_\eta + \mu \eta \xi}{s} \right] d\xi d\eta,
\]

\[
\frac{v_C}{dt} = -\int\int p(\xi, \eta) \left[ \frac{\beta(f_\xi + \mu \sin \theta - \eta \xi)}{s} \right] d\xi d\eta,
\]

\[
\frac{dw_\xi}{dt} = -\int\int p(\xi, \eta) \left[ \frac{\beta(f_\eta \xi + \mu \eta - \eta \xi)}{s} \right] d\xi d\eta,
\]

\[
\frac{dw_\eta}{dt} = -\int\int p(\xi, \eta) \left[ \frac{\beta(f_\eta + \mu \xi + f_\eta)}{s} \right] d\xi d\eta
\]

where

\[
s = \sqrt{\beta^2 + \xi^2 + \eta^2 + 2\eta \xi}, \quad s_0 = \xi \cos \varphi - \eta \sin \varphi, \quad s_1 = \xi \sin \varphi - \eta \cos \varphi, \quad s_2 = \xi \cos \varphi + \eta \sin \varphi, \quad s_3 = \xi \sin \varphi \sin \vartheta - \eta \cos \varphi \sin \vartheta, \quad s_4 = \xi \cos \varphi \cos \vartheta - \eta \sin \varphi \cos \vartheta.
\]

The system of equation (6) is general. It is possible to numerically evaluate this system directly. However, in most cases it is better to integrate the forces in the right part of the system, at least once. This accelerates the calculations and simplifies the analysis. We will study the uniform pressure distribution later, and only in the orthotropic case.

### 3. Friction force evaluation

#### 3.1. Symmetric orthotropic friction

We will evaluate the friction forces using the method proposed by Lurye in [29] and further developed in [30, 31]. This method allows us to easily define the areas containing points directed into the same quadrant, so the friction matrix is clarified for each region. This leads to simplification of the friction force and the moment integration.

Let us introduce a polar coordinate system, whose origin is in the instantaneous velocity center \(G\), the polar axis is a ray from the simultaneous velocity center through the plate center \(C\) and \(\gamma\) is a polar angle. We will distinguish two cases of center position with simultaneous velocity: an inside and an outside area, covered by the plate, see figure 1.

The velocity vector is the following:

\[
v = v (\cos(\vartheta + \gamma) \mathbf{i} + \sin(\vartheta + \gamma) \mathbf{j}),
\]

where \(v\) is a velocity value and \(\mathbf{i}, \mathbf{j}\) are unit vectors of the coordinate system axes.

The vector of the elementary friction force is:

\[
\tau = -p (f_\eta \cos(\vartheta + \gamma) \mathbf{i} + f_\xi \sin(\vartheta + \gamma) \mathbf{j})
\]

The friction force and moment taking into account (7) in a stationary coordinate system are the following:

\[
T_x = -p \int_{\gamma_1}^{\gamma_2} \int_{\eta_1}^{\eta_2} \cos(\vartheta + \gamma) r dr d\gamma,
\]

\[
T_y = -p \int_{\gamma_1}^{\gamma_2} \int_{\eta_1}^{\eta_2} \sin(\vartheta + \gamma) r dr d\gamma,
\]

\[
M_\xi = M_G - CG \cdot p \int_{\gamma_1}^{\gamma_2} \int_{\eta_1}^{\eta_2} \frac{\mu}{2} \cos(\gamma) r dr d\gamma,
\]

\[
M_\eta = M_G - CG \cdot p \int_{\gamma_1}^{\gamma_2} \int_{\eta_1}^{\eta_2} \frac{\mu}{2} \cos(\gamma) r dr d\gamma
\]

We assume that the plate has an elliptical shape with the semi-axes \(a\) and \(b\), where \(a\) is a major semi-axis, then

\[
m = \rho \pi ab, \quad \kappa = \sqrt{1 - e^2}, \quad I = \frac{\rho \pi ab(a + b)}{4}, \quad p = \frac{mg}{\pi ab},
\]

where \(m\) is the mass of the plate, \(\rho\) is the mass density of the plate, \(e\) is the ellipse eccentricity, \(p\) is a value of uniformly distributed pressure and \(g\) is the free fall acceleration.

Let us find the integration ranges in (8). We introduce the angle \(\Psi = \frac{\pi}{2} - \vartheta - \varphi\) (see figure 1). If the point \(G\) is inside the elliptical plate, the distance \(r\) from \(G\) to the border of the contacting area \(B\) can be found from:

\[
\frac{(GB \cos(\pi - (\Psi + \gamma)) - \xi_G)^2}{a^2} + \frac{(GB \sin(\pi - (\Psi + \gamma)) - \eta_G)^2}{b^2} = 1.
\]

Here the coordinates of the instantaneous velocity center are:

\[
\xi_G = x_G \cos \varphi + y_G \sin \varphi, \quad \eta_G = -x_G \sin \varphi + y_G \cos \varphi,
\]

\[
x_G = -\frac{v}{\omega} \sin \vartheta, \quad y_G = \frac{v}{\omega} \cos \vartheta.
\]

Thus, we receive:

\[
r = GB = \frac{\lambda_1 + abD_1}{\lambda_2},
\]

where

\[
\lambda_1 = b^2 \xi_G \cos(\Psi + \gamma) + a^2 \eta_G \sin(\Psi + \gamma),
\]
\[
\lambda_2 = b^2 \cos^2(\Psi + \gamma) + a^2 \sin^2(\Psi + \gamma),
\]
\[
D_1 = \sqrt{(b^2 - \eta_G^2) \cos^2(\Psi + \gamma) + (a^2 - \xi_G^2) \sin^2(\Psi + \gamma) + \xi_G \eta_G \sin(2(\Psi + \gamma))}.
\]

Angle \(\gamma\) in equation (9) takes values from 0 to \(2\pi\) (see [32]).

The same calculations for point \(G\) outside the area lead to the following:

\[
\begin{align*}
    r_1 &= GA = \frac{\lambda_1 - abD_1}{\lambda_2}, \\
    r_2 &= GB = \frac{\lambda_1 + abD_1}{\lambda_2},
\end{align*}
\]

(10)

Let us find angles \(\gamma_1\) and \(\gamma_2\) for that case. The points where line \(GA\) intersects the ellipse in the coordinate system \(C\xi\eta\) can be found from the system of equations:

\[
\begin{align*}
    \frac{\xi_G^2}{a^2} + \frac{\eta_G^2}{b^2} &= 1, \\
    \eta_A &= \xi_A k + \sigma,
\end{align*}
\]

where

\[
\sigma = \eta_G - \xi_G k, \quad k = \tan(\frac{\pi}{2} + \vartheta - \varphi + \gamma).
\]

The line will be tangential to the ellipse (points \(T_1\) and \(T_2\) on the picture) if the following equation is satisfied:

\[
\sigma^2 = a^2 k^2 + b^2.
\]

We receive the equation for parameter \(k\):

\[
(\xi_G^2 - a^2)k^2 - 2\eta_G \xi_G k + \eta_G^2 - b^2 = 0.
\]

Thus:

\[
k_{1,2} = \frac{\eta_G \xi_G \pm \sqrt{\xi_G^2 b^2 + a^2 \eta_G^2 - a^2 b^2}}{\xi_G^2 - a^2},
\]

and, finally,

\[
\gamma_{1,2} = \arctan(k_{1,2}) - \frac{\pi}{2} - \vartheta + \varphi. \quad (11)
\]

We can rewrite the forces in dimensionless form taking into account the following relations:

\[
I = \rho \pi a^4 I^*, \quad \xi = a \xi^*, \quad \eta = a k \eta^*, \quad \beta = \beta^* a, \quad p = p^* \frac{mg}{\pi a b}.
\]
where the asterisks are dedicated to dimensionless variables: $I^*$ is a dimensionless inertia moment, and $\xi^*, \eta^*$ are dimensionless coordinates (asterisks for these variables are omitted later).

Thus, we can write the components of the total friction force and the total moment in the case when the instantaneous velocity center lies inside the contact area:

\[
\begin{align*}
T_x^* &= -f_x \int_0^{2\pi} \cos(\theta + \gamma) \left( \frac{\kappa (\lambda^*_1 + D^*_1)^2}{2\lambda^*_2} \right) \, d\gamma, \\
T_y^* &= -f_y \int_0^{2\pi} \sin(\theta + \gamma) \left( \frac{\kappa (\lambda^*_1 + D^*_1)^2}{2\lambda^*_2} \right) \, d\gamma, \\
M^*_G &= -\int_0^{2\pi} \left( f_x + \mu \frac{\mu}{2} \cos(2\theta + 2\gamma) \right) \left( \frac{\kappa^2 (\lambda^*_1 + D^*_1)^3}{3\lambda^*_2^3} \right) \, d\gamma, \\
M^*_C &= M^*_G - \int_0^{2\pi} \beta \left( \frac{\mu}{2} \cos(\gamma) - \mu \frac{\mu}{2} \cos(2\theta + \gamma) + f_x \cos(\gamma) \right) \left( \frac{\kappa (\lambda^*_1 + D^*_1)^2}{2\lambda^*_2} \right) \, d\gamma.
\end{align*}
\]  
(12)

And in the case when the instantaneous velocity center is outside the contact area:

\[
\begin{align*}
T_x^* &= -f_x \int_{\gamma_1}^{\gamma_2} \cos(\theta + \gamma) \left( \frac{2\kappa D^*_1 \lambda^*_1}{\lambda^*_2^2} \right) \, d\gamma, \\
T_y^* &= -f_y \int_{\gamma_1}^{\gamma_2} \sin(\theta + \gamma) \left( \frac{2\kappa D^*_1 \lambda^*_1}{\lambda^*_2^2} \right) \, d\gamma, \\
M^*_G &= -\int_{\gamma_1}^{\gamma_2} \left( f_x + \mu \frac{\mu}{2} \cos(2\theta + 2\gamma) \right) \left( \frac{\kappa^2 (2D^*_1 + 6D^*_1 \lambda^*_2)}{3\lambda^*_2^3} \right) \, d\gamma, \\
M^*_C &= M^*_G - \int_{\gamma_1}^{\gamma_2} \beta \left( \frac{\mu}{2} \cos(\gamma) - \mu \frac{\mu}{2} \cos(2\theta + \gamma) + f_x \cos(\gamma) \right) \left( \frac{2\kappa D^*_1 \lambda^*_1}{\lambda^*_2^2} \right) \, d\gamma,
\end{align*}
\]  
(13)

where

\[
\begin{align*}
\lambda^*_1 &= \kappa \xi_G \cos(\Psi + \gamma) + \eta_G \sin(\Psi + \gamma), \\
\lambda^*_2 &= \kappa^2 \cos^2(\Psi + \gamma) + \sin^2(\Psi + \gamma), \\
D^*_1 &= \sqrt{\kappa^2 (1 - \xi_G^2) \cos^2(\Psi + \gamma) + (1 - \xi_G^2) \sin^2(\Psi + \gamma) + \kappa \xi_G \eta_D \sin(2(\Psi + \gamma))}.
\end{align*}
\]

### 3.2. Asymmetric orthotropic friction

In order to use equations (12) and (13), achieved in the previous section for the asymmetric case, we should evaluate other integration ranges. In this case it is important to know the directions of the sliding velocities. Each area on figure 2 corresponds to different cases of velocity orientation, and thus different friction coefficients.

We have to find rules for positioning the instantaneous velocity center $G$ in each area. Let us introduce points $p1, p2, p3, p4$, which are the coordinates of tangents to the ellipse parallel to axes $C_1$ and $C_2$.

In system $C \xi \eta$: the equation of line $p_3$ is $\eta = k_3 \xi + l_3$, with $k_3 = -\tan \varphi$. The coordinates of the contact point in system $C \xi \eta$ can be found from the following relations:

\[
\begin{align*}
\eta &= -\tan \varphi \xi + l_3, \\
\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} &= 1.
\end{align*}
\]

Taking $l_3 = \sqrt{a^2 k_3^2 + b^2}$ we obtain

\[
\begin{align*}
\xi_{p3} &= \frac{a^2 k_3}{\sqrt{a^2 k_3^2 + b^2}}, \\
\eta_{p3} &= k_3 \xi_{p3} + l_3,
\end{align*}
\]

and in system $C \xi y$ we get:

\[
\begin{align*}
x_{p3} &= \xi_{p3} \cos \varphi - \eta_{p3} \sin \varphi, \\
y_{p3} &= \xi_{p3} \sin \varphi + \eta_{p3} \cos \varphi.
\end{align*}
\]

The same idea is useful for three other points with:

\[
\begin{align*}
k_1 &= \cot \varphi, \\
l_1 &= \sqrt{a^2 k_1^2 + b^2}, \\
k_2 &= \cot \varphi, \\
l_2 &= -l_1, \\
k_4 &= -\tan \varphi, \\
l_4 &= -l_1.
\end{align*}
\]

Now take a look at area seven (see table 2). It is divided into four parts. Each part contains points with velocities directed to the same quadrant. This means that the coefficients of friction remain constant for each part. So, the projections of the friction force and the moment are:
The dissipative friction force has a negative power, thus, the motion with nonzero initial conditions terminates (the plate move a finite period of time). Thus, the second equation in the system (16) allows us to write the relation:

\[ T^*_n(\beta, \vartheta) \xrightarrow{t \to \tau^*_n} 0, \]

where \( \tau^*_n(\beta, \vartheta) = \frac{T^*_n(\beta, \vartheta)}{M^*_c(\beta, \vartheta)} \).

4. Selected results

4.1. General considerations

Let us divide the first equation of the system (6) by a third and let us introduce the derived right part \( \Phi_1(\beta, \vartheta) \):

\[
\frac{d\psi_C}{d\vartheta} = \Phi_1(\beta, \vartheta),
\]

\[
\psi_C \frac{d\vartheta}{dr} = T_n(\beta, \vartheta),
\]

where \( \Phi_1(\beta, \vartheta) = \frac{T^*_n(\beta, \vartheta)}{M^*_c(\beta, \vartheta)} \).

In (14) the area integral from (12) is partitioned into five integrals with constant friction coefficient values of \( \psi_0 = 0, \psi_1 = \alpha_1, \psi_2 = \alpha_2, \psi_3 = \alpha_3, \psi_4 = \alpha_4, \psi_5 = \alpha_5 \) and \( \sum \psi_i = 2\pi \) (see table 2).

Tables 1 and 2 show how equation (13) is divided into several parts. Thus, finally, we achieve equations for the friction force projections in the case of an instantaneous velocity center lying outside the area covered by the plate in the following form:

\[
T^*_x = - \sum_{i=0, j=1}^{i=5, j=5} \int_{\psi_i}^{\psi_j} \dot{f}^x \cos(\theta + \gamma) \left( \frac{\kappa (\lambda^*_1 + D^*_1 \gamma^2)}{2 \lambda^*_1} \right) d\gamma,
\]

\[
T^*_y = - \sum_{i=0, j=1}^{i=5, j=5} \int_{\psi_i}^{\psi_j} f^y \sin(\theta + \gamma) \left( \frac{\kappa (\lambda^*_1 + D^*_1 \gamma^2)}{2 \lambda^*_1} \right) d\gamma,
\]

\[
M^*_G = - \sum_{i=0, j=1}^{i=5, j=5} \int_{\psi_i}^{\psi_j} \left( f^x \frac{\mu^G}{2} - \frac{\mu^G}{2} \cos(2\theta + \gamma) + f^y \dot{\psi} \cos(\gamma) \right) \left( \frac{\kappa (\lambda^*_1 + D^*_1 \gamma^2)}{2 \lambda^*_1} \right) d\gamma,
\]

\[
M^*_C = M^*_G - \sum_{i=0, j=1}^{i=5, j=5} \int_{\psi_i}^{\psi_j} \beta \left( \frac{\mu^C}{2} \cos(2\theta + \gamma) + f^y \dot{\psi} \cos(\gamma) \right) \left( \frac{\kappa (\lambda^*_1 + D^*_1 \gamma^2)}{2 \lambda^*_1} \right) d\gamma.
\]

\[
(14)
\]

\[
(15)
\]
\[
\frac{dv}{d\omega} = \omega \frac{d\beta}{d\omega} + \beta,
\]

\[
\Phi_1(\beta, \vartheta) - \beta = \omega \frac{d\beta}{d\omega},
\]

\[
\frac{d\omega}{\omega} = -\frac{d\beta}{\beta - \Phi_1(\beta, \vartheta)}.
\]

Finally, we achieve:

\[
\omega = \omega_0 \exp \left[ -\int_{\beta_0}^{\beta_1} \frac{d\beta}{\beta - \Phi_1(\beta, \vartheta)} \right]. \tag{18}
\]

It is important to mention that function \(\Phi_1(\beta, \vartheta)\) depends not only on \(\beta\) and \(\vartheta\), but also on the shape of the contact area, the pressure distribution law \(p(\xi, \eta)\), the components of the friction matrix \(f_{\alpha \varphi} f_{\varphi \vartheta}\) and the angle \(\varphi\) (orientation of the body on the surface). Thus, the value of \(\beta_1\), when the integral in (18) becomes improper and seeks \(-\infty\), depends on the parameters of the mechanical system:

\[
\beta_1 \xrightarrow{t_\ast} \beta_\ast(\vartheta_\ast, \varphi_\ast, \Omega, f_{\alpha \varphi} f_{\varphi \vartheta}, p(\xi, \eta)). \tag{19}
\]

Summarizing, note that by the time \(t_\ast\), relation (17) and

\[
\beta - \Phi_1(\beta, \vartheta) \xrightarrow{t_\ast} 0 \quad \beta \rightarrow \beta_\ast \tag{20}
\]

should be achieved.

Furthermore, with fixed values of \(\beta = \tilde{\beta}\), equations \(T_n(\tilde{\beta}, \vartheta) = 0\) and (20) may have several solutions. However, both conditions (17) and (20) are achieved with singular \(\vartheta_\ast\), \(\beta_\ast\) [5], which depend on the initial conditions.

It is important to mention that from system (6), and introducing parameter \(\delta = \tilde{\beta}^{-1}\), we may derive the following system of equations:

\[
\frac{d\omega}{d\tau} = \Phi_2(\delta, \vartheta),
\]

\[
v_C \frac{d\vartheta}{d\tau} = T_n(\delta, \vartheta), \tag{21}
\]

where \(\Phi_2(\delta, \vartheta) = \frac{M_C(\delta, \vartheta)}{T_n(\delta, \vartheta)}\).

From system (21) we achieve:

\[
T_n(\delta, \vartheta) \xrightarrow{t_\ast} 0, \quad \delta \rightarrow \delta_\ast \tag{22}
\]

and

\[
v_C = v_{C0} \exp \left[ \int_{\delta_\ast}^{\delta_\ast} \frac{d\delta}{\Phi_2(\delta, \vartheta) - \delta} \right]. \tag{23}
\]

Finally, with the same reasoning as in (20) we obtain:

\[
\Phi_2(\delta, \vartheta) - \delta \xrightarrow{t_\ast} 0, \quad \delta \rightarrow \delta_\ast \tag{24}
\]
During the search limit values of $\vartheta_*$, $\beta_*$ with equations (17) and (20), it may occur that there are no roots. Thus, we should solve equations (22) and (24) to find $\vartheta_*$, $\beta_*$. Furthermore, because there is a strict dependence of plate motion on interrelations between the inertia moment and friction coefficients (see, for example, [18]) we should check the solution in both regions all the time.

### 4.2. Symmetric orthotropic friction. Specific initial conditions of motion

#### 4.2.1. Initial conditions of motion $\omega = 0$, $v \neq 0$

Let us get back to the system (5). With (3) and (4) it is possible to show that in the case $\omega = 0$, $v \neq 0$ we have:

$$\tau_x = - p \left( f_3 \cos \vartheta + f_1 \sin \vartheta \right),$$
$$\tau_y = - p \left( - f_3 \cos \vartheta + f_1 \sin \vartheta \right),$$

and, thus, we achieve the system:

$$m v_c = T_x \cos \vartheta + T_y \sin \vartheta = - p \int_\Omega \int_\Omega (\mu \sin^2 \vartheta + f_3) \cos \vartheta \cos \theta' \sin \theta' \, d\varsigma d\zeta,$$

$$m v_c = - T_x \sin \vartheta + T_y \cos \vartheta = - p \int_\Omega \int_\Omega (\mu \sin \vartheta \cos \vartheta - f) \cos \vartheta \cos \theta' \sin \theta' \, d\varsigma d\zeta,$$

$$I \omega = \int_\Omega \int_\Omega (\tau_x x - \tau_y y) \, d\varsigma d\zeta = - p \int_\Omega \int_\Omega (\xi K_1 + \eta K_2) \, d\varsigma d\zeta,$$

where

$$K_1 = (\mu \sin \vartheta \cos \varphi + f_3 \sin (\vartheta - \varphi) - f \cos (\vartheta - \varphi)),$$

$$K_2 = (- \mu \sin \vartheta \sin \varphi - f_3 \cos (\vartheta - \varphi) - f \cos (\vartheta - \varphi)).$$

\[\text{(25)}\]
\[ F_{\text{Surf. Topogr.: Metrol. Prop.}} \]

4.3. Symmetric and asymmetric orthotropic friction: numerical results

If the initial values of the angular and linear (sliding) velocities are nonzero, it is difficult to reach any analytic simplifications. In our paper [20], the system of equations (6) was solved in the \( (\xi, \eta) \) coordinate system, but with this approach the accuracy of results was not satisfied enough. Furthermore, at most final points the method tends to oscillate significantly, because near \( \beta = \beta_s \), a situation with a singularity may appear. So we switched to the Lurye method described here. After manipulations with force integrals according to the described method, we obtained a very stable and accurate numerical procedure. With this new approach we achieved a numerical solution for uniform pressure distribution and symmetric and asymmetric friction forces.

Table 3 shows the resulting values of the parameters of interest for circular and elliptic plates. The
Figure 3. Evolution of parameters $\beta$ and $\vartheta$ for circular (solid line) and elliptic (dashed line) plate for orthotropic friction: (1) $\mu = \mu_+ = 0.03$, (2) $\mu = \mu_+ = 0.18$.

Figure 4. The evolution of $\vartheta(\beta)$ for the elliptic ($e = 0.6$) plate (solid line—symmetric case, dashed line—asymmetric case): (1) $\mu = \mu_+ = 0.03$, (2) $\mu = \mu_+ = 0.18$.

The instantaneous velocity center for both types of plates is located in the same area. However, the velocity vector rotational angle $\vartheta_*$ is noticeably higher for the elliptic plate. The instantaneous velocity center position described by $\beta_*$ is lower for the elliptic plate compared with the circular one.

In our paper [19], the results for the symmetric orthotropic case were presented. For the symmetric case, $\beta_*$ is significantly lower for both circular and elliptic plates compared to the asymmetric example. $\beta$ shows about a 10 percent increase in value for the elliptic plate in the asymmetric case. Furthermore, for the circular plate in the symmetric orthotropic friction we have $\vartheta_*=0$. However, in the asymmetric case for both shapes, the $\vartheta_*$ values show that the velocity vector orients to the third quadrant, which is actually the one with the lowest coefficient of friction.

Figure 3 demonstrates the evolution of $\beta(t)$, $\vartheta(t)$ for symmetric and asymmetric cases. It was assumed that $a=1$, representing a major semi-axis of the elliptic plate, whose eccentricity is $e=0.6$, and for the circular plate $e=0$. The initial conditions taken in the example are: $v_0=1$, $\omega_0=1$, $\vartheta_0=\pi/4$, and the initial ellipse orientation angle is $\varphi_0=\pi/3$. The friction coefficient $\mu_+=f_y-\gamma f_x+\gamma f_y$ with $f_x=f_+=0.42$, $f_-=-0.5f_+\gamma$, $f_f=f_+=0.5f_y$. For both the symmetric and asymmetric cases, the sliding and spinning end simultaneously. This important outcome was also achieved for nonuniform pressure distributions: for a circular plate with respect to the isotropic friction force and axisymmetric normal pressure in [33], under linear pressure distribution in [34], and for an elliptic plate under linear pressure distribution and symmetric orthotropic friction in [20]. However, figure 3 shows a significant difference in the behavior of $\beta(t)$, $\vartheta(t)$ curves with respect to the asymmetry of the friction force. The shape factor is noticeably important for both symmetric and asymmetric cases: the elliptic plate moves for a shorter period of time with more velocity vector rotation (higher changes of
We assume that for low levels of anisotropy, the inertia moment of the body and the shape of the contact area have a big impact, so we see sharp changes in the $\beta$ curve. However, this should be investigated in more detail, as done in [5] for a circular plate and symmetric orthotropic friction. Figure 4 represents the behavior of $\vartheta(\beta)$ for the elliptic plate. It can be seen that even low values of anisotropy asymmetry in the friction have an impact on the motion: the plate rotates much more for the asymmetric case (dashed line) than for the symmetric one (solid line).

As discussed in section 4.1, the normal component of the friction force $T_n$ and $\beta - \Phi_1$ should reach zero at the same moment (equations (17) and (20)). However, for the asymmetric case equation (20) has more than one root. Thus, the $\beta$ curve in figure 3 shows a sudden variation in the very final period of motion, since the normal component of the friction force is not zero yet, but equation (20) crosses zero. Nonetheless, finally, rules (17) and (20) are achieved at the same moment, and the phase trajectory enters the coordinate center tangential to the linear velocity axes. Figure 5 illustrates this. The linear velocity is tangential to all the phase trajectories both for the symmetric and asymmetric cases at point $v = 0, \omega = 0$. It can be seen from the figures that the more anisotropy there is on the surface (higher $\mu$ value) the shorter the period is during which the plates move. However, compared to the results in [19] we see that for the asymmetric case the motion is longer and the change in $\beta$ is sharper. This is a result of the fact that the directions forward and down are directions with a lower coefficient of friction, so motion in that direction is easier, thus allowing the body to slide and spin longer.

5. Conclusion

- The problem of the terminal motion of a thin elliptic plate is formulated, taking into account the anisotropy of the friction force. The general analytic results show that it is possible to analyze the system of motion equations without specifying the pressure distribution. It is stated that until the terminal point two conditions (20) and (17) should be achieved simultaneously.
- The total friction force and the total moment evaluations are shown in the case of the thin elliptic plate under a uniform pressure distribution. Two cases are discussed: symmetric orthotropic friction and asymmetric orthotropic friction. Based on the Lurye method, the friction force is obtained for both cases. This method allows the separation of areas containing points whose velocities are directed at the same quadrant, so the components of the friction matrix are easily defined. This leads to significant simplifications of the numerical procedure.
- Some specific cases of initial conditions are analyzed separately with the aid of numerical study. It is shown that for symmetric orthotropic friction and the elliptic contact area under a uniform pressure distribution, if the initial motion is linear it stays linear till the end. In cases where the initial motion is rotational, it is rotational during the whole period of motion.
- Numerical results are presented for the symmetric and asymmetric friction forces and elliptic and circular contact areas under uniform pressure distribution. It is stated that sliding and spinning end simultaneously both in the symmetric and asymmetric cases. The figures show the significant influence of the asymmetry of friction forces on motion. The impact of the contact area is also pointed out.

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