THE ALGEBRA OF OBSERVABLES IN NONCOMMUTATIVE DEFORMATION THEORY

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Abstract. We consider the algebra $O(M)$ of observables and the (formally) versal morphism $\eta: A \to O(M)$ defined by the noncommutative deformation functor $\text{Def}_M$ of a family $M = \{M_1, \ldots, M_r\}$ of right modules over an associative $k$-algebra $A$. By the Generalized Burnside Theorem, due to Laudal, $\eta$ is an isomorphism when $A$ is finite dimensional, $M$ is the family of simple $A$-modules, and $k$ is an algebraically closed field. The purpose of this paper is twofold: First, we prove a form of the Generalized Burnside Theorem that is more general, where there is no assumption on the field $k$. Secondly, we prove that the $O$-construction is a closure operation when $A$ is any finitely generated $k$-algebra and $M$ is any family of finite dimensional $A$-modules, in the sense that $\eta_B: B \to O_B(M)$ is an isomorphism when $B = O(M)$ and $M$ is considered as a family of $B$-modules.

1. Introduction

Let $k$ be a field, let $A$ be a finite dimensional associative algebra over $k$, and let $M = \{M_1, \ldots, M_r\}$ be the family of simple right $A$-modules, up to isomorphism. We consider the algebra homomorphism

$$\rho: A \to \bigoplus_{i=1}^r \text{End}_k(M_i)$$

given by right multiplication of $A$ on the family $M$. By the extended version of the classical Burnside Theorem, $\rho$ is surjective when $k$ is algebraically closed, and if $A$ is semisimple, then it is an isomorphism. We remark that Artin-Wedderburn theory gives a version of the theorem that holds over any field:

Theorem (Classical Burnside Theorem). Let $A$ be a finite dimensional $k$-algebra, and let $\{M_1, \ldots, M_r\}$ be the family of simple right $A$-modules. If $\text{End}_A(M_i) = k$ for $1 \leq i \leq r$, then $\rho: A \to \bigoplus_i \text{End}_k(M_i)$ is surjective.

In Laudal [3], a generalization called the Generalized Burnside Theorem was obtained. This is a structural result for not necessarily semisimple algebras, and the essential idea of Laudal was to replace $\rho$ with the versal morphism $\eta$ defined by noncommutative deformations of modules. Let us recall the construction:

Let $A$ be an arbitrary associative $k$-algebra, let $M = \{M_1, \ldots, M_r\}$ be a family of right $A$-modules, and consider the noncommutative deformation functor $\text{Def}_M$. This functor has a pro-representing hull $H$ and a versal family $M_H$ if $M$ is a swarm. Following Laudal [3], we define the algebra of observables of a swarm $M$ to be $O(M) = \text{End}_H(M_H) \cong \langle H_{ij} \otimes_k \text{Hom}_k(M_i, M_j) \rangle$, and its versal morphism to be the
algebra homomorphism \( \eta : A \to O(M) \) given by right multiplication of \( A \) on the versal family \( M_H \). It fits into the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\eta} & (H_{ij} \otimes_k \text{Hom}_k(M_i, M_j)) \\
\downarrow^{\rho} & & \downarrow^{\oplus_{i=1}^r \text{End}_k(M_i)} \\
\oplus_{i=1}^r \text{End}_k(M_i) & & 
\end{array}
\]

where \( \rho : A \to \oplus_{i=1}^r \text{End}_k(M_i) \) is the algebra homomorphism given by right multiplication of \( A \) on the family \( M \). By Theorem 1.2 in Laudal [3], it follows that \( \eta \) is an isomorphism when \( A \) is finite dimensional, \( M \) is the family of simple \( A \)-modules, and \( k \) is algebraically closed. In this paper, we prove a more general version of this result:

**Theorem (Generalized Burnside Theorem).** Let \( A \) be a finite dimensional \( k \)-algebra, and let \( M \) be the family of simple right \( A \)-modules, up to isomorphism. The versal morphism \( \eta : A \to O(M) \) is injective. If \( \text{End}_A(M_i) = k \) for \( 1 \leq i \leq r \), then \( \eta \) is an isomorphism. In particular, \( \eta \) is an isomorphism if \( k \) is algebraically closed.

In case \( D_i = \text{End}_A(M_i) \) is a division algebra with \( \dim_k D_i > 1 \) for some simple module \( M_i \), it is often not difficult to describe the image of \( \eta \) as a subalgebra of \( O(M) \), and we shall give examples. As an application of the theorem, we introduce the standard form of any finite dimensional algebra \( A \), given as

\[
A \cong O(M) = (H_{ij} \otimes_k \text{Hom}_k(M_i, M_j))
\]

when \( \text{End}_A(M_i) = k \) for \( 1 \leq i \leq r \), or as a subalgebra of \( O(M) \) in general.

Let \( A \) be any finitely generated \( k \)-algebra and let \( M \) be any family of finite dimensional right \( A \)-modules. In this more general situation, the versal morphism \( \eta : A \to O(M) \) is not necessarily an isomorphism. However, we may consider the algebra \( B = O(M) \) of observables, and \( M \) as a family of right \( B \)-modules, and iterate the process. We prove that the operation \((A, M) \mapsto (B, M)\) has the following closure property:

**Theorem (Closure Property).** Let \( A \) be a finitely generated \( k \)-algebra, let \( M \) be a family of finite dimensional \( A \)-modules, and let \( B = O(M) \). Then the versal morphism \( \eta^B : B \to O^B(M) \) of \( M \), considered as a family of right \( B \)-modules, is an isomorphism.

One may consider a noncommutative algebraic geometry where the closed points are represented by simple modules; see for instance Laudal [3]. With this point of view, one may use versal morphisms \( \eta : A \to O(M) \) for families \( M \) of \( A \)-modules to construct noncommutative localization homomorphisms \( \eta_s : A \to A_s \) for any \( s \in A \). We explain this construction in Section 6. These localization maps are universal \( S \)-inverting localization maps, where \( S = \{1, s, s^2, \ldots \} \), and can be used as an essential building block for structure sheaves on noncommutative schemes.

2. **Noncommutative deformations of modules**

Let \( A \) be an associative algebra over a field \( k \). For any right \( A \)-module \( M \), there is a deformation functor \( \text{Def}_M : I \to \text{Sets} \) defined on the category \( I \) of commutative Artinian local \( k \)-algebras \( R \) with residue field \( k \). We recall that \( \text{Def}_M(R) \) is the set of equivalence classes of pairs \((M_R, \tau_R)\), where \( M_R \) is an \( R \)-flat \( R \times A \) bimodule.
on which \( k \) acts centrally, and \( \tau_R : k \otimes_R M_R \to M \) is an isomorphism of right \( A \)-modules. Deformations in \( \text{Def}_M(R) \) are called \textit{commutative deformations} since the base ring \( R \) is commutative.

\textit{Noncommutative deformations} were introduced in Laudal [3]. The deformations considered by Laudal are defined over certain noncommutative base rings instead of the commutative base rings in \( I \). In what follows, we shall give a brief account of noncommutative deformations of modules. We refer to Laudal [3], Eriksen [2] and Eriksen, Laudal, Siqveland [1] for further details.

For any positive integer \( r \) and any family \( M = \{M_1, \ldots, M_r\} \) of right \( A \)-modules, there is a \textit{noncommutative deformation functor} \( \text{Def}_M : \mathfrak{a}_r \to \text{Sets} \), defined on the category \( \mathfrak{a}_r \) of noncommutative Artinian \( r \)-pointed \( k \)-algebras with exactly \( r \) simple modules (up to isomorphism). We recall that an \( r \)-pointed \( k \)-algebra \( R \) is one fitting into a diagram of rings \( k^r \to R \to k^r \), where the composition is the identity. The condition that \( R \) has exactly \( r \) simple modules holds if and only if \( \overline{R} \cong k^r \), where \( \overline{R} = R/I(R) \) and \( I(R) \) denotes the Jacobson radical of \( R \).

The noncommutative deformations in \( \text{Def}_M(R) \) are equivalence classes of pairs \( (M_R, \tau_R) \), where \( M_R \) is an \( R \)-flat \( R \)-\( A \) bimodule on which \( k \) acts centrally, and \( \tau_R : k^r \otimes_R M_R \to M \) is an isomorphism of right \( A \)-modules with \( M = M_1 \oplus \cdots \oplus M_r \).

In concrete terms, an algebra homomorphism \( \eta : A \to \text{End}_R(M_R) \cong (R_{ij} \otimes_k \text{Hom}_k(M_i, M_j)) \) that lifts \( \rho : A \to \oplus_i \text{End}_k(M_i) \). Explicitly, we interpret \( \eta(a) \) as a right action of \( a \) on \( M_R \) via

\[
\eta_R(a) = \sum_i e_i \otimes \rho_i + \sum_{i,j,l} r_{ij}^l \otimes a_{ij}^l \iff (e_i \otimes m_i)a = e_i \otimes (m_i a) + \sum_{j,l} r_{ij}^l \otimes a_{ij}^l (m_i)
\]

where \( \rho_i : A \to \text{End}_k(M_i) \) is the algebra homomorphism given by the right action of \( A \) on \( M_i \) such that \( \rho = (\rho_1, \ldots, \rho_r) \), and where \( r_{ij}^l \in R_{ij} \) and \( a_{ij}^l \in \text{Hom}_k(M_i, M_j) \). Deformations in \( \text{Def}_M(R) \) can therefore be represented by commutative diagrams

\[
\begin{array}{ccc}
A & \xrightarrow{\eta_R} & (R_{ij} \otimes_k \text{Hom}_k(M_i, M_j)) \\
\rho \downarrow & & \downarrow \rho \\
\oplus_{i=1}^r \text{End}_k(M_i) & & \\
\end{array}
\]

These deformations are called \textit{noncommutative deformations} since the base ring \( R \) is noncommutative.

For any \( r \)-pointed algebra \( R \), with structural maps \( k^r \to R \to k^r \), we write \( I(R) = \ker(R \to k^r) \). Recall that the pro-category \( \widehat{\mathfrak{a}}_r \) is the full subcategory of the category of \( r \)-pointed algebras consisting of algebras \( R \) such that \( R/I(R)^n \) is Artinian for all \( n \) and such that \( R \) is complete in the \( I(R) \)-adic topology.

The family \( M = \{M_1, \ldots, M_r\} \) is called a \textit{swarm} if \( \dim_k \text{Ext}_A^1(M, M) \) is finite. In this case, the noncommutative deformation functor \( \text{Def}_M \) has a pro-representing hull \( H \) in the pro-category \( \widehat{\mathfrak{a}}_r \) and a versal family \( M_H \in \text{Def}_M(H) \); see Theorem 3.1 in Laudal [3]. The defining property of the miniversal pro-couple \( (H, M_H) \) is that
the induced natural transformation

$$\phi : \text{Mor}(H, -) \to \text{Def}_M$$

on $a_r$ is smooth (which implies that $\phi_R$ is surjective for any $R$ in $a_r$), and that $\phi_R$ is an isomorphism when $J(R)^2 = 0$. The miniversal pro-couple $(H, M_H)$ is unique up to (non-canonical) isomorphism.

Let $M$ be a swarm of right $A$-modules, and let $(H, M_H)$ be the miniversal pro-couple of the noncommutative deformation functor $\text{Def}_M : a_r \to \text{Sets}$. We define the algebra of observables of $M$ to be $O(M) = \text{End}_H(M_H) \cong (H_{ij} \hat{\otimes}_k \text{Hom}_k(M_i, M_j))$ where $\hat{\otimes}$ is the completed tensor product (the completion of the tensor product), and write $\eta : A \to O(M)$ for the induced versal morphism, giving the right $A$-module structure on $M_H$. By construction, it fits into the commutative diagram

\[ A \xrightarrow{\eta} (H_{ij} \hat{\otimes}_k \text{Hom}_k(M_i, M_j)) \xrightarrow{\oplus_{i=1}^r \text{End}_k(M_i)} O(M) \]

**Remark 1.** Notice that the diagram extends the right action of $A$ on the family $M$ to a right action of $O(M)$, such that $M$ is a family of right $O(M)$-modules.

**Remark 2.** For any $R$ in $a_r$ and any deformation $M_R \in \text{Def}_M(R)$, there is a morphism $u : H \to R$ in $\hat{a}_r$ such that $\text{Def}_M(u)(M_H) = M_R$ by the versal property, and the deformation $M_R$ is therefore given by the composition $\eta_R = u^* \circ \eta$ in the diagram

\[ A \xrightarrow{\eta} (R_{ij} \hat{\otimes}_k \text{Hom}_k(M_i, M_j)) \xrightarrow{u^* = u \otimes \text{id}} O(M) \]

In this sense, the versal morphism $\eta : A \to O(M)$ determines all noncommutative deformations of the family $M$.

### 3. Iterated extensions and injectivity of the versal morphism

Let $E$ be a right $A$-module and let $r \geq 1$ be a positive integer. If $E$ has a cofiltration of length $r$, given by a sequence

$$E = E_r \xrightarrow{f_r} E_{r-1} \to \cdots \to E_2 \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0 = 0$$

of surjective right $A$-module homomorphisms $f_i : E_i \to E_{i-1}$, then we call $E$ an *iterated extension* of the right $A$-modules $M_1, M_2, \ldots, M_r$, where $M_i = \ker(f_i)$. In fact, the cofiltration induces short exact sequences

$$0 \to M_i \to E_i \xrightarrow{f_i} E_{i-1} \to 0$$

for $1 \leq i \leq r$. Hence $E_1 \cong M_1$, $E_2$ is an extension of $E_1$ by $M_2$, and in general, $E_i$ is an extension of $E_{i-1}$ by $M_i$. 
Let $M = \{M_1, \ldots, M_r\}$ be a swarm of right $A$-modules, and let $\text{Def}_M : a_r \to \text{Sets}$ be its noncommutative deformation functor. Then $\text{Def}_M$ has a miniversal pro-couple $(H, M_H)$, and we consider the induced versal morphism $\eta : A \to \mathcal{O}(M)$ and its kernel $K = \ker(\eta)$.

We note that Theorem 3.2 in Laudal [3] holds without assumptions on the base field $k$, since the construction that precedes this theorem works over any field. From this observation, we obtain the following lemma:

**Lemma 3.** Let $M$ be a swarm of right $A$-modules. For any iterated extension $E$ of the family $M$, we have that $E \cdot K = 0$.

Let $A$ be a finite dimensional $k$-algebra and let $M$ be the family of all simple right $A$-modules, up to isomorphism. Then $M$ is a swarm, and we consider the versal morphism $\eta : A \to \mathcal{O}(M)$. If $k$ is algebraically closed, then the versal morphism $\eta$ is injective by Corollary 3.1 in Laudal [3]. Using Lemma 3, we generalize this result:

**Proposition 4.** If $A$, considered as a right $A$-module, is an iterated extension of a swarm $M$, then the versal morphism $\eta : A \to \mathcal{O}(M)$ is injective. In particular, $\eta$ is injective when $A$ is a finite dimensional algebra and $M$ is the family of simple right $A$-modules.

**Proof.** If $A$ is an iterated extension of $M$, then $1 \cdot K = 0$ by Lemma 3 and this implies that $K = 0$. If $A$ is finite dimensional, then the right $A$-module $A$ has finite length, and it is an iterated extension of the simple modules. \qed

We remark that our proof, based on Lemma 3, holds whenever there is an element $e \in E$ such that $a \mapsto e \cdot a$ defines an injective right $A$-module homomorphism $A \to E$. This means that $\eta : A \to \mathcal{O}(M)$ is injective if there is an iterated extension $E$ of $M$ such that $E$ contains a copy of $A_A$.

## 4. The Generalized Burnside Theorem

Let $A$ be a finite dimensional $k$-algebra, and let $M = \{M_1, \ldots, M_r\}$ be the family of simple right $A$-modules, up to isomorphism. Then $M$ is a swarm, and we consider the versal morphism $\eta : A \to \mathcal{O}(M)$ and the commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\eta} & (H_{ij} \otimes_k \text{Hom}_k(M_i, M_j)) \\
& \searrow \rho & \\
& & \oplus_{i=1}^r \text{End}_k(M_i)
\end{array}
$$

Clearly, $\rho$ factors through $A/ J(A)$, and if $\text{End}_A(M_i) = k$ for $1 \leq i \leq r$, then $A/ J(A) \to \oplus_i \text{End}_k(M_i)$ is an isomorphism by the Artin-Wedderburn theory for semisimple algebras. This proves the Classical Burnside Theorem mentioned in the introduction. By Theorem 3.4 in Laudal [3], the versal morphism $\eta : A \to \mathcal{O}(M)$ is an isomorphism when $k$ is algebraically closed. We generalize this result:

**Theorem 5.** Let $A$ be a finite dimensional $k$-algebra and let $M$ be the family of simple right $A$-modules, up to isomorphism. Then $\eta : A \to \mathcal{O}(M)$ is injective, and it is an isomorphism if $\text{End}_A(M_i) = k$ for $1 \leq i \leq r$. In particular, the versal morphism $\eta : A \to \mathcal{O}(M)$ is an isomorphism if $k$ is algebraically closed.
Proof: By Proposition 3, the versal morphism $\eta$ is injective, and it is enough to prove that $\eta$ is surjective when $\text{End}_k(M_i) = k$ for $1 \leq i \leq r$. Note that $\eta$ maps the Jacobson radical $J(A)$ of $A$ to the Jacobson radical $J = (J(H))_{\ij} \otimes_k \text{Hom}_k(M_i, M_j)$ of $O(M)$. Moreover, $A$ is $J(A)$-adic complete since it is finite dimensional, and $O(M)$ is clearly $J$-adic complete. By a standard result for filtered algebras, it is therefore sufficient to show that $\text{gr}_1(\eta) : J(A)/J(A)^2 \rightarrow J/J^2$ is surjective, since $\text{gr}_0(\eta) : A/J(A) \rightarrow \oplus \text{End}_k(M_i)$ is an isomorphism by the Classical Burnside Theorem. We notice that

$$J/J^2 \cong ((J(H)/J(H)^2)_{\ij} \otimes_k \text{Hom}_k(M_i, M_j)) \cong (\text{Ext}_A^1(M_i, M_j)^* \otimes_k \text{Hom}_k(M_i, M_j))$$

since $J(H)/J(H)^2$ is the dual of the tangent space $(\text{Ext}_A^1(M_i, M_j))$ of $\text{Def}_M$. We note that Lemma 3.7 in Laudal [3] holds over any field. Hence the map

$$J(A)/J(A)^2 \rightarrow (\text{Ext}_A^1(M_i, M_j)^* \otimes_k \text{Hom}_k(M_i, M_j))$$

induced by $\eta$ is an isomorphism, and this completes the proof. 

5. The Closure Property

Let $A$ be a finitely generated $k$-algebra of the form $A = k\langle x_1, \ldots, x_d \rangle/I$, and let $M = \{M_1, \ldots, M_r\}$ be a family of finite dimensional right $A$-modules. Then $M$ is a swarm, since

$$\dim_k \text{Ext}_A^1(M_i, M_j) \leq \dim_k \text{Der}_k(A, \text{Hom}_k(M_i, M_j)) \leq \dim_k \text{Hom}_k(M_i, M_j)^d$$

The last inequality follows from the fact that any derivation $D : A \rightarrow \text{Hom}_k(M_i, M_j)$ is determined by $D(x_i) \in \text{Hom}_k(M_i, M_j)$ for $1 \leq i \leq d$. We consider the algebra of observables $B = O(M)$ of the swarm $M$, and write $\eta : A \rightarrow B$ for its versal morphism. In general, $M = \{M_1, \ldots, M_r\}$ is a family of right $B$-modules via $\eta$.

Lemma 6. The family $M = \{M_1, \ldots, M_r\}$ of right $B$-modules is the simple right $B$-modules, and it is swarm of $B$-modules.

Proof. It follows from the Artin-Wedderburn theory that $M = \{M_1, \ldots, M_r\}$ is the family of simple modules over

$$\overline{B} = B/J(B) \cong (H/J(H) \otimes_k \text{Hom}_k(M_i, M_j)) \cong \oplus \text{End}_k(M_i).$$

Since $B$ and $\overline{B} = B/J(B)$ have the same simple modules, it follows that $M$ is the family of simple right $B$-modules. We have that $\text{Ext}_B^1(M_i, M_j)$ is a quotient of $\text{Der}_k(B, \text{Hom}_k(M_i, M_j))$, and any derivation $D : B \rightarrow \text{Hom}_k(M_i, M_j)$ satisfies $D(J^2) = JD(J) + D(J)J = 0$ when $J = J(B)$ since $M$ is the family of simple $B$-modules. From the fact that

$$B/J^2 \cong ((H/J(H)^2)_{\ij} \otimes_k \text{Hom}_k(M_i, M_j))$$

is finite dimensional, and in particular a finitely generated $k$-algebra, it follows from the argument preceding the lemma that $M$ is a swarm of $B$-modules. 

In this situation, we may iterate the process. Since $M$ is a swarm of right $B$-modules, the noncommutative deformation functor $\text{Def}_M^B$ of $M$, considered as a family of right $B$-modules, has a universal pro-couple $(H^B, M^B_B)$. We write $O^B(M) = \text{End}_B(M^B_B) \cong (H^B_{\ij} \otimes_k \text{Hom}_k(M_i, M_j))$ for its algebra of observables and $\eta^B : B \rightarrow O^B(M)$ for its versal morphism.
Theorem 7. Let $A$ be a finitely generated $k$-algebra, let $M = \{M_1, \ldots, M_r\}$ be a family of finite dimensional $A$-modules, and let $B = \mathcal{O}(M)$. Then the versal morphism $\eta^B : B \to \mathcal{O}^B(M)$ of $M$, considered as a family of right $B$-modules, is an isomorphism.

Proof. Since $M$ is a swarm of $A$-modules and of $B$-modules, we may consider the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\eta} & B = \mathcal{O}(M) & \xrightarrow{\eta^B} & C = \mathcal{O}^B(M) \\
& \downarrow{\rho} & \downarrow{} & \downarrow{} & \downarrow{\oplus \text{End}_k(M_i)} \\
& & \oplus \text{End}_k(M_i) & & \\
\end{array}
\]

The algebra homomorphism $\eta^B$ induces maps $B/J(B)^n \to C/J(C)^n$ for all $n \geq 1$, and it is enough to show that each of these induced maps is an isomorphism. For $n = 1$, we have

\[
B/J(B) \cong C/J(C) \cong \oplus \text{End}_k(M_i)
\]

so it is clearly an isomorphism for $n = 1$. For $n \geq 2$, we have that $B_n = B/J(B)^n$ is a finite dimensional algebra with the same simple modules as $B$ since $M_i J^n = 0$. We may therefore consider the versal morphism of the swarm $M$ of right $B_n$-modules, which is an isomorphism by the Generalized Burnside Theorem since $\text{End}_B(M_i) = k$ for $1 \leq i \leq r$. Finally, any derivation $D : B \to \text{Hom}_k(M_i, M_j)$ satisfies $D(J^n) = 0$ when $n \geq 2$. Therefore, we have that

\[
\text{Ext}^1_{B_n}(M_i, M_j) \cong \text{Ext}^1_B(M_i, M_j)
\]

and this implies that $B/J(B)^n \to C/J(C)^n$ coincides with the versal morphism of the swarm $M$ of right $B_n$-modules. It is therefore an isomorphism. \qed

Theorem 7 implies that the assignment $(A, M) \mapsto (B, M)$ is a closure operation when $A$ is a finitely generated $k$-algebra and $M = \{M_1, \ldots, M_r\}$ is a family of finite dimensional right $A$-modules. In other words, the algebra $B = \mathcal{O}(M)$ has the following properties:

1. The family $M$ is the family of simple right $B$-modules.
2. The family $M$ has exactly the same module-theoretic properties, in terms of extensions and matric Massey products, considered as a family of $B$-modules and as a family of $A$-modules.

Moreover, these properties characterize the algebra of observables $B = \mathcal{O}(M)$.

Remark 8. Assume that $k$ is a field that is not algebraically closed. When $A$ is a finite dimensional $k$-algebra and $M$ is the family of simple right $A$-modules, it could happen that the division algebra $D_i = \text{End}_A(M_i)$ has dimension $\dim_k D_i > 1$ for some simple $A$-modules $M_i$. In this case, $\eta : A \to \mathcal{O}(M)$ is not necessarily an isomorphism. However, if the subfamily $M' = \{M_i : \text{End}_A(M_i) = k\} \subseteq M$ is non-empty, we may consider the algebra $B = \mathcal{O}(M')$, and it follows from the closure property that $\eta : B \to \mathcal{O}^B(M')$ is an isomorphism. This means that the Generalized Burnside Theorem holds for the family $M'$ of right $B$-modules.
6. Noncommutative localizations via the algebra of observables

Let $A$ be a finitely generated $k$-algebra, and denote by $X = \text{Simp}(A)$ the set of (isomorphism classes of) simple finite dimensional right $A$-modules. For any $s \in A$, we write

$$D(s) = \{ M \in X : M \twoheadrightarrow M \text{ is invertible} \} \subseteq X.$$  

We note that $\{ D(s) \}_{s \in A}$ is a base for a topology on $X$, since $D(s) \cap D(t) = D(st)$, which we call the Jacobson topology on $X = \text{Simp}(A)$.

For any inclusion $M \subseteq M'$ of finite subsets of $D(s)$, there is a surjective algebra homomorphism $\mathcal{O}(M') \rightarrow \mathcal{O}(M)$. We may consider the algebra homomorphism

$$\eta_s : A \rightarrow \lim_{M \subseteq D(s)} \mathcal{O}(M)$$

where the projective limit is taken over all finite subsets $M \subseteq D(s)$. Notice that $\eta_s(s)$ is a unit, since it is a unit in $\mathcal{O}(M)$ for any finite subset $M \subseteq D(s)$. We define $A_s$ to be the subring of the projective limit

$$\lim_{M \subseteq D(s)} \mathcal{O}(M)$$

generated by $\eta_s(A)$ and $\eta_s(s)^{-1}$. By abuse of notation, we write $\eta_s$ for the algebra homomorphism $\eta_s : A \rightarrow A_s$ into the subring $A_s$.

Let $S$ be the multiplicative subset $S = \{ 1, s, s^2, \ldots \} \subseteq A$. Then $\eta_s : A \rightarrow A_s$ is an $S$-inverting algebra homomorphism, and it has the following universal property: If $\phi : A \rightarrow B$ is any $S$-inverting algebra homomorphism, then there is a unique algebra homomorphism $\phi_s : A_s \rightarrow B$ such that $\phi_s \circ \eta_s = \phi$. We remark that $A_s$ is a finitely generated $k$-algebra, generated by the images of the generators of $A$ and $\eta_s(s)^{-1}$. In general, it is not a (left or right) ring of fractions.

7. Applications

Let $A$ be a finite dimensional $k$-algebra. We consider the family $M = \{ M_1, \ldots, M_r \}$ of simple right $A$-modules. By the Generalized Burnside Theorem, $A$ can be written in standard form as

$$A \cong \text{im}(\eta) \subseteq (H_{ij} \otimes_k \text{Hom}_k(M_i, M_j)) = \mathcal{O}(M)$$

If $\text{End}_A(M_i) = k$ for $1 \leq i \leq r$, then the standard form of $A$ is $A \cong \mathcal{O}(M)$, and in general, it is a subalgebra of $\mathcal{O}(M)$.

The standard form can, for instance, be used to compare finite dimensional algebras and determine when they are isomorphic. Let us illustrate this with a simple example. Let $k$ be a field, and let $A = k[G]$ be the group algebra of $G = \mathbb{Z}_3$. In concrete terms, we have that $A \cong k[x]/(x^3 - 1)$, and over a fixed algebraic closure $\overline{k}$ of $k$, we have that

$$x^3 - 1 = (x - 1)(x^2 + x + 1) = (x - 1)(x - \omega)(x - \omega^2)$$

with $\omega \in \overline{k}$. If $\text{char}(k) \neq 3$ and $\omega \in k$, then the simple $A$-modules are given by $M = \{ M_0, M_1, M_2 \}$, where $M_i = A/(x - \omega^i)$. Furthermore, a calculation shows that $\text{Ext}_A^1(M_i, M_j) = 0$ for $0 \leq i, j \leq 2$. Hence, the noncommutative deformation functor $\text{Def}_M$ has a pro-representing hull $H = k^3$ (it is rigid), and the versal morphism $\eta : A \rightarrow \mathcal{O}(M)$ is an isomorphism. The standard form of $A$ is therefore given
by
\[ A = k[Z_3] \cong k^3 = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}. \]

If char\( (k) = 3 \), then \( M_0 \) is the only simple \( A \)-module since \( x^3 - 1 = (x - 1)^3 \), and we find that \( \text{Ext}^1_A(M_0, M_0) = k \). In this case, it turns out that \( H \cong k[[t]]/(t^3) \), and the standard form of \( A \) is given by \( A = k[Z_3] \cong k[t]/(t^3) \). In both cases, it follows from the Generalized Burnside Theorem that \( \eta \) is an isomorphism, since \( \text{End}_A(M) = k \) for all the simple \( A \)-modules \( M \).

If char\( (k) \neq 3 \) and \( \omega \notin k \), then the simple \( A \)-modules are given by \( M = \{ M, N \} \), where \( M = M_0 = A/(x - 1) \) is 1-dimensional, and \( N = A/(x^2 + x + 1) \cong k(\omega) = K \) is 2-dimensional. In this case, we have that \( \text{End}_A(M) = k \) and \( \text{End}_A(N) = K \), and we find that the standard form of \( A \) is given by
\[ H = \begin{pmatrix} k & 0 \\ 0 & 0 \\ k \end{pmatrix} \Rightarrow A \cong \text{im}(\eta) = \begin{pmatrix} k & 0 \\ 0 & K \end{pmatrix} \subseteq \text{O}(M) = \begin{pmatrix} k & 0 \\ 0 & \text{End}_k(K) \end{pmatrix}. \]

It follows from Proposition 4 that \( \eta : A \to \text{O}(M) \) is injective. However, it is not an isomorphism in this case.

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