COUNTING COLLISIONS IN AN \( N \)-BILLIARD SYSTEM USING ANGLES BETWEEN COLLISION SUBSPACES

SEAN GASIOREK

Abstract. The principal angles between binary collision subspaces in an \( N \)-billiard system in \( m \)-dimensional Euclidean space are computed. These angles are computed for equal masses and arbitrary masses. We then provide a bound on the number of collisions in the planar 3-billiard system problem. Comparison of this result with known billiard collision bounds in lower dimensions is discussed.

1. INTRODUCTION

In a series of papers, [BFK1] and [BFK2] computed a uniform bound for the number of collisions in semi-dispersing billiards in terms of the minimum and maximum masses and radii of the billiard balls. We provide an alternate approach to bounding the number of collisions in an \( N \)-billiard system through a different billiard model, taking lessons from classical billiard dynamics, linear algebra, and geometry.

1.1. Linear Point Billiards. Motivated by the high-energy limit of the \( N \)-body problem, [FKM] outlined a formulation of the standard billiard dynamical system, though with several caveats, outlined below. We consider an \( N \) massive point particle system in \( m \)-dimensional Euclidean space \( \mathbb{R}^m \) by its configuration space

\[ E = (\mathbb{R}^m)^N \approx \mathbb{R}^N \otimes \mathbb{R}^m. \]

Within \( E \) there are \( \binom{N}{2} \) binary collision subspaces

(1) \[ \Delta_{ij} = \{ q = (q_1, \ldots, q_N) \in (\mathbb{R}^m)^N : q_i = q_j, i \neq j \} \subset E. \]

where the \( i^{th} \) and \( j^{th} \) particles collide. Call \( C = \bigcup_{i \neq j} \Delta_{ij} \) the collision locus. A billiard trajectory will be a polygonal curve \( \ell : \mathbb{R} \to E \), all of whose vertices are collisions (i.e. vertices of \( \ell \) lie in \( C \)).

When \( \ell \) intersects a collision subspace \( \Delta_{ij} \), it instantaneously changes direction by the law “angle of incidence equals angle of reflection,” given by equations (2) and (3) below. We call a collision point a time \( t \) for which \( \ell(t) \in \Delta_{ij} \) for some distinct \( 1 \leq i, j \leq N \). We assume collision points are discrete and that no edge of \( \ell \) lies within a collision subspace. The velocities \( v_-, v_+ \) of \( \ell \) immediately before and after
collision with $\Delta_{ij} \in C$ are well-defined and locally constant. They undergo a jump $v_- \mapsto v_+$ at collision. Define

$$\pi_{\Delta_{ij}} : E \to \Delta_{ij}$$

to be the orthogonal projection onto $\Delta_{ij}$. We require that each velocity jump follow the rules

(2) \[ \|v_\| = \|v_+\| \]

(3) \[ \pi_{\Delta_{ij}}(v_-) = \pi_{\Delta_{ij}}(v_+) \]

which we consider as conservation of energy and conservation of linear momentum, respectively. Without loss of generality, we assume the billiard trajectory $\ell$ has unit speed.

An astute reader will notice that equation (3) is ambiguous if the collision point $t_*$ belongs to more than one binary collision subspace. This is analogous to trying to define standard billiard dynamics at a corner pocket of a polygonal billiard table. However, in this paper we only explore the case with $N = 3$ particles, and hence triple collision is the only way $t_*$ could be in more than one collision subspace. [FKM] addresses this by agreeing to choose one of the collision subspaces to which $t_*$ will belong.

This construction also leads to non-deterministic dynamics. If the binary collision subspaces are codimension $d$, $d \geq 1$, for a given $v_0 \in E \setminus 0$ to a $t_\in \Delta_{ij}$, there is a $(d - 1)$-dimensional sphere’s worth of choices for outgoing velocities $v_+$. Even if $d = 1$, the dynamics are non-deterministic, as the 0-sphere consists of two choices. It is standard to turn this case into a deterministic process by requiring transversality: $v_+ \neq v_-$ at each collision. This is exactly what is done for $N$ particles on a line.

1.2. Tensor construction of the $N$-billiard system. Recall that a useful tool in the $N$-body problem is the mass metric on $\mathbb{R}^N$:

$$\langle v, w \rangle_M = \sum_{i=1}^{N} m_i v_i w_i$$

for vectors $v, w \in \mathbb{R}^N$ and masses $m_i > 0$. It follows from this definition that the kinetic energy

$$K(q) = \frac{1}{2} \langle \dot{q}, \dot{q} \rangle_M = \frac{1}{2} \sum_{i=1}^{N} m_i |\dot{q}_i|^2.$$ 

In the configuration space $\mathbb{R}^N \otimes \mathbb{R}^m$, we will use the mass metric on $\mathbb{R}^N$ and the standard Euclidean inner product on $\mathbb{R}^m$. Let $\varepsilon_i \in \mathbb{R}^N$ denote the $i^{th}$ standard basis vector. Using the mass metric, we see that $\|\varepsilon_i\|^2_M = m_i \langle \varepsilon_i, \varepsilon_i \rangle = m_i$. Furthermore, define $E_i := \frac{\varepsilon_i}{\sqrt{m_i}}$ so that $E_i$ is a unit vector in $\mathbb{R}^N$ with respect to the mass metric.

We consider a vector $(\varepsilon_1, \ldots, \varepsilon_N) \in \mathbb{R}^N$ as a vector which carries with it an index of each billiard ball along with its mass, $m_i$, and a vector $q_i \in \mathbb{R}^m$ as a vector which
indicates the location of the point mass \( m_i \) in \( \mathbb{R}^m \).

It will be useful to translate the definition of the binary collision subspace into our tensor product construction as the following span of orthonormal elements of \( \mathbb{R}^N \otimes \mathbb{R}^m \):

\[
\Delta_{ij} = \left\{ E_1 \otimes q_1, \ldots, \frac{\varepsilon_i + \varepsilon_j}{\sqrt{m_i + m_j}} \otimes q_i, \ldots, E_N \otimes q_N : q_k \in \mathbb{R}^m, \|q_k\| = 1 \right\}.
\]

Remark. In our collision subspaces, we’ll adopt the convention that the “location” index will match that of the smaller of the two point mass indices (e.g. the element \( \frac{\varepsilon_i + \varepsilon_j}{\sqrt{2}} \otimes q_i \) will be used instead of \( \frac{\varepsilon_i + \varepsilon_j}{\sqrt{2}} \otimes q_j \) as a basis element in \( \Delta_{ij} \)). And we will continue to assume that each \( q_k \) is itself a unit vector in \( \mathbb{R}^m \). We omit the word “span” and write subspaces \( U = \{u_1, \ldots, u_N\} \) to mean the \( \mathbb{R} \)-linear span, \( U = \text{span}\{u_1, \ldots, u_N\} \). We shall also write these subspaces in terms of an orthonormal basis (even though an orthogonal basis is good enough).

### 1.3. The Unfolding of Angles

If a particle is shot into a wedge of angle \( \alpha \), can such a particle be trapped inside the wedge for infinite time? The answer is negative, and moreover through a simple geometric argument we can provide an upper bound on the total number of collisions of the particle with the boundary.

**Theorem 1.** Consider a billiard trajectory inside a wedge with angle measure \( \alpha \). By “unfolding the angle,” the maximum number of collisions within the angle is \( \lceil \pi/\alpha \rceil \).

This can clearly be seen as follows. Consider an incoming billiard trajectory into the wedge angle \( \alpha \). Instead of reflecting the billiard trajectory inside the angle, we reflect the angle itself across the side of impact and unfold the billiard trajectory into a straight line, which is exactly the rule “angle of incidence equals angle of reflection.” This is illustrated in figure 1.

Our aim is to use this technique in higher dimensions to bound the total number of collisions by using the collision subspaces as the “walls” of the wedge.

### 1.4. Linear algebra and Principal Angles

Linear algebra provides a framework for computing the angle between linear subspaces of a vector space. Definitions 1 and 2 are equivalent, though it is worth noting that definition 2 is computationally more efficient, as outlined in \( \text{BG} \).

**Definition 1.** Let \( F, G \) be subspaces of \( \mathbb{R}^n \) with \( \dim(F) = p \geq \dim(G) = q \). The **Jordan angles** or **principal angles** \( \angle(F, G) = [\theta_1, \theta_2, \ldots, \theta_q] \), are given by

\[
\cos(\theta_1) = \max_{u \in F} \max_{\|u\|=1} \langle u, v \rangle \quad \text{for} \quad \langle u, v \rangle = \langle u_1, v_1 \rangle
\]

where \( \langle \cdot, \cdot \rangle \) is the standard Euclidean inner product. For each \( 1 < k \leq q \),

\[
\cos(\theta_k) = \max_{u \in F} \max_{v \in G} \langle u, v \rangle = \langle u_k, v_k \rangle \quad \text{where} \quad (u, u_k) = 0 \quad \text{and} \quad (v, v_k) = 0
\]
Figure 1. (a) The billiard trajectory enters the wedge angle $\alpha$. Note that in each of these pictures, $\alpha \approx 30.96^\circ \approx 0.54$ radians. (b) When the trajectory hits the first wall, we reflect the angle across the wall of impact and “unfold” the angle once. (c) The complete unfolding of the angle $\alpha$ and the billiard trajectory within $\alpha$. We have at most $\lceil \frac{\pi}{0.54} \rceil = 6$ collisions

for each $1 \leq i \leq k - 1$. The vectors $u_k, v_k$ which realize the angle $\theta_k$ are called principal vectors.

By construction, we have that $0 \leq \theta_1 \leq \ldots \leq \theta_q \leq \pi/2$.

**Definition 2.** Let $\mathcal{F} \in \mathbb{R}^{n \times p}$ and $\mathcal{G} \in \mathbb{R}^{n \times q}$ with $p \geq q$ be matrices whose columns form orthonormal bases for subspaces $F$ and $G$, respectively, of $\mathbb{R}^n$. Further, let the singular value decomposition (SVD) of $\mathcal{G}^\top \mathcal{F}$ be $U \Sigma V^\top$ where $U$ and $V$ are orthogonal matrices and $\Sigma$ is a $p \times q$ matrix with real diagonal entries $s_1, s_2, \ldots, s_q$ in decreasing order. Then the cosine of the Jordan angles or principal angles are given by $\cos \angle^*(F,G) = S(\mathcal{F}^\top \mathcal{G}) = [s_1, \ldots, s_q]$, where $\angle^*(F,G)$ denotes the vector of principal angles between $F$ and $G$ arranged in increasing order and $S(A)$ denotes the vector of singular values of the matrix $A$. Furthermore, the principal vectors associated to this pair of subspaces are given by the first $q$ columns of $F U$ and $G V$, respectively.
The next lemma and example demonstrate that the angles between linear subspaces follow similar properties to what one would expect in the standard Euclidean geometry.

**Lemma 1.** Let \( F, G \) be subspaces of \( \mathbb{R}^n \) with \( \dim(F) = p \) and \( \dim(G) = q \). Furthermore let \( \angle(F, G) \) and \( \angle'(F, G) \) denote the vector of principal angles between \( F \) and \( G \) in increasing order and decreasing order, respectively.

1. \( \angle(F, G) = \angle(G, F) \)
2. \( [0, \ldots, 0, \angle(F, G)] = [0, \ldots, 0, \angle(F^\perp, G^\perp)] \), with \( \max\{n-p-q,0\} \) zeros on the left and \( \max\{p+q-n,0\} \) zeros on the right.
3. \( [0, \ldots, 0, \angle(F, G^\perp)] = [0, \ldots, 0, \angle(F^\perp, G)], \) with \( \max\{q-p,0\} \) zeros on the left and \( \max\{p-q,0\} \) zeros on the right.
4. \( [\angle(F, G), \frac{\pi}{7}, \ldots, \frac{\pi}{7}] = [0, \ldots, 0, \frac{\pi}{7} - \angle'(F, G^\perp)] \), with \( \max\{p-q,0\} \) \( \frac{\pi}{7} \)'s on the left and \( \max\{p+q-n,0\} \) zeros on the right.

The first property follows from definition [1] and we do not provide proofs for the rest of the properties. Proofs can be found as Theorem 2.7 in [AK] or as Theorems 2.6-7 in [AJK].

**Example 1.** Consider subspaces \( L \) and \( M \) in \( \mathbb{R}^6 \).

- Suppose \( \dim(L) = \dim(M) = 2 \) and \( \angle(L, M) = [\frac{\pi}{3}, \frac{\pi}{2}] \). Then \( \angle(L^\perp, M^\perp) = [0, 0, \frac{\pi}{3}, \frac{\pi}{2}] \), and \( \angle(L, M^\perp) = \angle(L^\perp, M) = [0, \frac{\pi}{3}] \).
- Suppose \( \dim(L) = \dim(M) = 4 \) and \( \angle(L, M) = [0, 0, \frac{\pi}{3}, \frac{\pi}{2}] \). We conclude \( \angle(L^\perp, M^\perp) = [\frac{\pi}{4}, \frac{\pi}{3}] \) and \( \angle(L, M^\perp) = \angle(L^\perp, M) = [\frac{\pi}{4}, \frac{\pi}{3}] \).

Though a subtlety that isn’t obvious in the two examples above is that, given \( \angle(L, M) \), the angles that appear in the vector \( \angle(L^\perp, M^\perp) \) are taken from the list \( \angle(L, M) \) from largest to smallest.

For example, if \( \dim(L) = \dim(M) = 8 \) in \( \mathbb{R}^{13} \) and

\[
\angle(L, M) = \left[0, \frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}\right]
\]

then

\[
\angle(L^\perp, M^\perp) = \left[\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}\right]
\]

which are the five largest angles in the vector \( \angle(L, M) \).

The following corollary will be useful to us in due time, and follows immediately from Lemma 1.

**Corollary 1.** If \( U, V \subset \mathbb{R}^n \) are codimension 1 subspaces, then the nonzero angle between \( U \) and \( V \) is \( \angle(U^\perp, V^\perp) \). That is, the nonzero angle between these subspaces is precisely the angle between their normal vectors.

Our first goal is to compute the angles between the collision subspaces \( \Delta_{ij} \) and \( \Delta_{kl} \) for some \( i, j, k, l \in \mathbb{N} \). By our definitions of principal angles, we can see that there will be \( d(N-1) \) principal angles between the codimension \( d \) collision subspaces.
2. The Main Angle Theorems between Collision Subspaces

2.1. Equal Masses. We aim to prove the following theorem when considering the particles to have equal masses. Through scaling we can assume all masses to be unit. When all masses are unit, note that the mass metric is identical to the standard Euclidean metric.

**Theorem 2.** Let \( i,j,k,l \) be distinct integers satisfying \( 1 \leq i,j,k,l \leq N \).

a) The first \( d(N-2) \) principal angles between \( \Delta_{ij} \) and \( \Delta_{kl} \) are 0 and the remaining \( d \) principal angles are all \( \frac{\pi}{2} \).

b) The first \( d(N-2) \) principal angles between \( \Delta_{ij} \) and \( \Delta_{jk} \) are 0 and the remaining \( d \) principal angles are \( \frac{\pi}{3} \).

The proof is computational and appears in full detail in Appendix A. However, this theorem can be viewed as a corollary of the next theorem when the masses are arbitrary.

2.2. Arbitrary Masses. We now prove the analogous result for arbitrary masses in each of the \( N \) bodies.

**Theorem 3.** Let \( i,j,k,l \) be distinct integers satisfying \( 1 \leq i,j,k,l \leq N \).

a) The first \( d(N-2) \) principal angles between \( \Delta_{ij} \) and \( \Delta_{kl} \) are 0 and the remaining \( d \) principal angles are \( \frac{\pi}{2} \).

b) The first \( d(N-2) \) principal angles between \( \Delta_{ij} \) and \( \Delta_{jk} \) are 0 and the last \( d \) principal angles are

\[
\theta = \arccos \left( \frac{2(m_i + m_k) - m_j}{\sqrt{(m_i + m_j + 4m_k)(4m_i + m_j + m_k)}} \right).
\]

The proof that follows is an abbreviated version of the proof of Theorem 2 in Appendix A.

**Proof.** Recall the collision subspaces in tensor product form:

\[
\Delta_{ij} = \left\{ E_1 \otimes q_1, \ldots, \frac{\varepsilon_i + \varepsilon_j}{\sqrt{m_i + m_j}} \otimes q_i, \ldots, E_N \otimes q_N \right\}
\]

and

\[
\Delta_{kl} = \left\{ E_1 \otimes q_1, \ldots, \frac{\varepsilon_k + \varepsilon_l}{\sqrt{m_k + m_l}} \otimes q_k, \ldots, E_N \otimes q_N \right\}.
\]

As before, we intend to compute the angle between the subspaces \((\Delta_{ij} \cap \Delta_{kl})^\perp \cap \Delta_{ij}\) and \((\Delta_{ij} \cap \Delta_{kl})^\perp \cap \Delta_{kl}\). First, we find that

\[
\Delta_{ij} \cap \Delta_{kl} = \left\{ E_1 \otimes q_1, \ldots, \frac{\varepsilon_i + \varepsilon_j}{\sqrt{m_i + m_j}} \otimes q_i, \ldots, \frac{\varepsilon_k + \varepsilon_l}{\sqrt{m_k + m_l}} \otimes q_k, \ldots, E_N \otimes q_N \right\}.
\]
and hence

\[(\Delta_{ij} \cap \Delta_{kl})^\perp = \left\{ \frac{m_j \varepsilon_i - m_i \varepsilon_j}{\sqrt{m_i m_j (m_i + m_j)}} \otimes \beta_1, \frac{m_i \varepsilon_k - m_k \varepsilon_l}{\sqrt{m_k m_l (m_k + m_l)}} \otimes \beta_2 \right\} \]

for some arbitrary unit vectors \( \beta_1, \beta_2 \in \mathbb{R}^m \). Repeating the same calculations as before, we find that

\[(\Delta_{ij} \cap \Delta_{kl})^\perp \cap \Delta_{ij} = \left\{ \frac{m_i \varepsilon_k - m_k \varepsilon_l}{\sqrt{m_i m_k (m_k + m_i)}} \otimes \beta_2 \right\} \]

and

\[(\Delta_{ij} \cap \Delta_{kl})^\perp \cap \Delta_{kl} = \left\{ \frac{m_j \varepsilon_i - m_i \varepsilon_j}{\sqrt{m_i m_j (m_i + m_j)}} \otimes \beta_1 \right\} . \]

Just as before, we have that

\[[(\Delta_{ij} \cap \Delta_{kl})^\perp \cap \Delta_{ij}] \perp [(\Delta_{ij} \cap \Delta_{kl})^\perp \cap \Delta_{kl}] . \]

Therefore regardless of the individual masses, \( \Delta_{ij} \) and \( \Delta_{kl} \) are orthogonal to one another when \( i, j, k, l \) are distinct.

We now turn our attention to the case where one of the indices is the same across the two collision subspaces. Repeating the same calculations as earlier but keeping track of the mass terms, we find that

\[\Delta_{ij} \cap \Delta_{jk} = \left\{ E_1 \otimes q_1, \ldots, \frac{\varepsilon_i + \varepsilon_j + \varepsilon_k}{\sqrt{m_i + m_j + m_k}} \otimes q_j, \ldots, E_N \otimes q_N \right\} \]

and

\[(\Delta_{ij} \cap \Delta_{jk})^\perp = \left\{ \frac{m_j \varepsilon_i - m_i \varepsilon_j}{\sqrt{m_i + m_j}} \otimes \beta_1, \frac{\varepsilon_i + \varepsilon_j - 2 \varepsilon_k}{\sqrt{m_i + m_j + 4m_k}} \otimes \beta_2 \right\} . \]

Repeating the same calculations results in

\[(\Delta_{ij} \cap \Delta_{jk})^\perp \cap \Delta_{ij} = \left\{ \frac{\varepsilon_i + \varepsilon_j - 2 \varepsilon_k}{\sqrt{m_i + m_j + 4m_k}} \otimes \beta_2 \right\} \]

and

\[(\Delta_{ij} \cap \Delta_{jk})^\perp \cap \Delta_{kl} = \left\{ \frac{2 \varepsilon_i - \varepsilon_j + \varepsilon_k}{\sqrt{4m_i + m_j + m_k}} \otimes \beta_2 \right\} . \]

Therefore the nonzero angle between these two subspaces is given by

\[\cos(\theta) = \frac{2(m_i + m_k) - m_j}{\sqrt{(m_i + m_j + 4m_k)(4m_i + m_j + m_k)}} . \]
3. Billiard Trajectories and Collision Bounds

3.1. A Primer on Jacobi Coordinates and the Mass Metric. We follow the approach of sections 3 and 7 of [RM1]. We consider the planar 3-billiard ball problem whose configuration space is $\mathbb{C}^3$. A vector $q = (q_1, q_2, q_3) \in \mathbb{C}^3$ represents a located triangle with each of its components representing the vertices of the triangle.

**Definition 3.** The mass metric on the configuration space $\mathbb{C}^3$ is the Hermitian inner product

$$\langle v, w \rangle = m_1 \bar{v}_1 \bar{w}_1 + m_2 \bar{v}_2 \bar{w}_2 + m_3 \bar{v}_3 \bar{w}_3.$$

A translation of this located triangle $q$ by $c \in \mathbb{C}$ is given by the located triangle $q + c$ where $1 = (1, 1, 1)$. Define

$$\mathbb{C}^3_0 := 1^\perp = \{ q \in \mathbb{C}^3 : m_1 q_1 + m_2 q_2 + m_3 q_3 = 0 \}$$

to be the set of planar three-body configurations whose center of mass $q_{cm}$ is at the origin. This two-dimensional complex space represents the quotient space of $\mathbb{C}^3$ by translations.

**Definition 4.** The Jacobi coordinates for $\mathbb{C}^3_0 := \{ q \in \mathbb{C}^3 : q_{cm} = 0 \}$ are given by

$$v = \mu_1 (q_1 - q_2), \quad w = \mu_2 \left( q_3 - \frac{m_1 q_1 + m_2 q_2}{m_1 + m_2} \right)$$

where $\frac{1}{\mu_1^2} = \frac{1}{m_1} + \frac{1}{m_2}$ and $\frac{1}{\mu_2^2} = \frac{1}{m_3} + \frac{1}{m_1 + m_2}$.

These are normalized coordinates diagonalize the restriction of the mass metric to $\mathbb{C}^3_0$. From this we can define the complex linear projection

$$\pi_{tr} : \mathbb{C}^3 \to \mathbb{C}^2, \quad (q_1, q_2, q_3) \mapsto (v, w),$$

which realizes the metric quotient of $\mathbb{C}^3$ by translations.

It is worthwhile to note that using Jacobi coordinates and our map $\pi_{tr}$, all of the triple collision triangles $(q, q, q) \in \mathbb{C}^3$ are mapped to the origin.

3.2. The Main Collision Bound Theorems. Consider equal masses $M$ in the plane. This changes our linear projection into

$$\pi_{tr} : \mathbb{C}^3 \to \mathbb{C}^2, \quad (q_1, q_2, q_3) \mapsto \sqrt{M} \left( \frac{1}{\sqrt{2}} (q_1 - q_2), \sqrt{\frac{2}{3}} \left( q_3 - \frac{1}{2} (q_1 + q_2) \right) \right).$$

The mass $M$ is clearly a dilation factor, so we assume the mass to be unit henceforth.

Recall our codimension 2 collision subspaces are defined as follows:

$$\Delta_{12} = \{(q_1, q_2, q_3) \in \mathbb{C}^3 : q_1 = q_2 \}$$

$$\Delta_{23} = \{(q_1, q_2, q_3) \in \mathbb{C}^3 : q_2 = q_3 \}$$

$$\Delta_{13} = \{(q_1, q_2, q_3) \in \mathbb{C}^3 : q_1 = q_3 \}.$$
Example 2. Using our previous results, we can see that \( \angle(\Delta_{12}, \Delta_{23}) = [0, 0, \frac{\pi}{3}, \frac{\pi}{3}] \). In this case the principal vectors are

\[
\begin{align*}
\theta_1 &= 0 : \frac{1}{\sqrt{3}}(1, 1, 1) \text{ and itself} \\
\theta_2 &= 0 : \frac{1}{\sqrt{6}}(i, i, i) \text{ and itself} \\
\theta_3 &= \frac{\pi}{3} : \frac{1}{\sqrt{6}}(-1, -1, 2) \text{ and } \frac{1}{\sqrt{6}}(-2, 1, 1) \\
\theta_4 &= \frac{\pi}{3} : \frac{1}{\sqrt{6}}(-i, -i, 2i) \text{ and } \frac{1}{\sqrt{6}}(-2i, i, i).
\end{align*}
\]

The image of these subspaces under \( \pi_{tr} \) are

\[
\begin{align*}
\Delta_{12}^0 &= \{(v, w) \in \mathbb{C}_0^2 : v = 0\} = \text{span}_\mathbb{C}\{(0, 1)\} \\
\Delta_{23}^0 &= \{(v, w) \in \mathbb{C}_0^2 : w = -\frac{1}{\sqrt{3}}v\} = \text{span}_\mathbb{C}\left\{\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)\right\} \\
\Delta_{13}^0 &= \{(v, w) \in \mathbb{C}_0^2 : w = \frac{1}{\sqrt{3}}v\} = \text{span}_\mathbb{C}\left\{\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)\right\}
\end{align*}
\]

Each of these codimension 2 (and dimension 2) subspaces are planes in \( \mathbb{C}_0^2 \).

Through this reduction via Jacobi coordinates, the angles between these subspaces are \( \angle(\Delta_{12}^0, \Delta_{23}^0) = \angle(\Delta_{12}^0, \Delta_{13}^0) = \angle(\Delta_{23}^0, \Delta_{13}^0) = [\frac{\pi}{3}, \frac{\pi}{3}] \).

And to further our previous example, we can observe that the principal vectors for \( \angle(\Delta_{12}^0, \Delta_{23}^0) \) are \((0, 1)\) and \((-\frac{\sqrt{3}}{2}, \frac{1}{2})\) for the first angle, and \((0, i)\) and \((-\frac{\sqrt{3}}{2}i, \frac{1}{2}i)\) for the second angle. But in fact one can check that indeed

\[
\pi_{tr}\left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right) = (0, 1)
\]
and
\[ \pi_{tr} \left( \left( -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right) \right) = \left( -\frac{\sqrt{3}}{2}, \frac{1}{2} \right). \]

Since \( \pi_{tr} \) is linear, we know that the image of the other pair of principal vectors between \( \Delta_{12} \) and \( \Delta_{23} \) will also be the principal vectors between \( \Delta_{012} \) and \( \Delta_{023} \). That is, the image of a nonzero principal vector under \( \pi_{tr} \) is still a principal vector!

**Theorem 4.** In the equal mass planar 3-billiard problem, there can be at most 3 collisions.

To prove the theorem, we need to following lemma.

**Lemma 2.** Let \( V_1 \) and \( V_2 \) be arbitrary vectors in two collision subspaces in \( \mathbb{C}^2 \) and let \( \hat{\theta} \) denote the angle between the vectors \( V_1 \) and \( V_2 \). Then \( \frac{\pi}{3} \leq \hat{\theta} \leq \frac{\pi}{2} \).

**Proof.** Without loss of generality we consider two of our three collision subspaces, namely \( \Delta_{012} \) and \( \Delta_{023} \) and consider two arbitrary vectors \( V_1 \) and \( V_2 \) in \( \Delta_{012} \) and \( \Delta_{023} \), respectively.

Recall the definition of principal angles:
\[ \cos(\theta_1) = \max_{u \in \Delta_{012}} \max_{v \in \Delta_{023}} \langle u, v \rangle := \langle u_1, v_1 \rangle, \]

where \( \|u\| = 1 \) and \( \|v\| = 1 \).

**Figure 3.** The two subspaces \( \Delta_{012} \) and \( \Delta_{023} \) along with their orthonormal basis vectors. In our proof we look to measure the angle between arbitrary vectors \( V_1 \in \Delta_{012} \) and \( V_2 \in \Delta_{023} \).
Let \( \ell \) be a collision sequence. From our earlier calculations, we know that \( \cos(\theta_1) = \frac{2}{3} \) and that our principal angles always satisfy \( 0 \leq \theta_i \leq \frac{\pi}{2} \). Because \( \cos(\theta) \) is a decreasing function on the interval \( 0 \leq \theta \leq \frac{\pi}{2} \), we see that

\[
\cos(\theta_1) = \langle u_1, v_1 \rangle \geq \langle V_1, V_2 \rangle = \cos(\hat{\theta}),
\]

because the inner product is maximized. Hence the possible angles between the vectors \( V_1 \) and \( V_2 \) must satisfy \( \frac{\pi}{2} \leq \hat{\theta} \leq \frac{\pi}{2} \). The argument and calculation is the same if we choose any pair of these collision subspaces. This proves the lemma.

In fact, the angle \( \hat{\theta} = \frac{\pi}{2} \) can be realized if we let \( V_1 = (0, \sqrt{3} - \frac{1}{2} i) \) and \( V_2 = (\frac{\sqrt{3} + i}{2}, \frac{1 + \sqrt{3} i}{2}) \).

Using the preceding lemma, the proof of the theorem is short and follows from the “unfolding the angle” argument. We also refer to a sequence of collisions by the order in which the binary collisions occur. For example, the collision sequence (12)(23) indicates that \( \ell(t) \) has a collision point in \( \Delta_{12} \) first and then \( \Delta_{23} \) second.

**Proof.** If we now consider an arbitrary piecewise linear trajectory \( \ell(t) \) in \( \mathbb{C}_0^2 \), we aim to maximize the number of collisions. Again without loss of generality, assume \( \ell(t) \) intersects \( \Delta_{12}^0 \) first.

Suppose \( \ell(t) \) intersects \( \Delta_{12}^0 \) at time \( t = t_1 \) and let \( V_1 \) be a vector in \( \Delta_{12}^0 \) whose endpoint is this point of intersection, \( \ell(t_1) \). This trajectory may change direction as it is piecewise linear. If the next subspace it intersects can be either \( \Delta_{23}^0 \) or \( \Delta_{13}^0 \) is \( \Delta_{23}^0 \) (it certainly cannot stay within \( \Delta_{23}^0 \) as that would mean that the billiard balls 1 and 2 have stuck together after their collision, and \( \ell(t) \) also cannot leave \( \Delta_{13}^0 \) and come back to hit \( \Delta_{12}^0 \) again as this “recollision” is forbidden by our billiard rules). So without loss of generality, assume \( \ell(t) \) next visits \( \Delta_{23}^0 \) at time \( t = t_2 \), and let the vector \( V_2 \in \Delta_{23}^0 \) be a vector whose endpoint is this second point of intersection, \( \ell(t_2) \). We know from the preceding lemma that the angle \( \hat{\theta}_1 \) between \( V_1 \) and \( V_2 \) satisfies \( \frac{\pi}{3} \leq \hat{\theta}_1 \leq \frac{\pi}{2} \).

We now repeat this process again. From \( \ell(t_2) \), the trajectory \( \ell(t) \) can now travel to either \( \Delta_{23}^0 \) or \( \Delta_{13}^0 \). Without loss of generality, assume \( \Delta_{13}^0 \) is the next subspace. Let \( \ell(t) \) intersect \( \Delta_{13}^0 \) at \( \ell(t_3) \) for some time \( t = t_3 \) and let \( V_3 \) be a vector in \( \Delta_{13}^0 \) whose endpoint is at the point of intersection \( \ell(t_3) \). Applying our lemma again, the angle \( \hat{\theta}_2 \) between \( V_2 \) and \( V_3 \) satisfies \( \frac{\pi}{3} \leq \hat{\theta}_2 \leq \frac{\pi}{2} \).

From there \( \ell(t) \) cannot hit any more collision subspaces. At best, \( \hat{\theta}_1 + \hat{\theta}_2 = \frac{2\pi}{3} \), and any third angle will add at least \( \frac{\pi}{3} \) by the previous lemma. So if we glue together the sectors spanned by \( V_1 \) and \( V_2 \), and \( V_2 \) and \( V_3 \) and flatten this angle, by the “unfold the angle” argument earlier, the trajectory \( \ell(t) \) can hit no more subspaces. This leaves us with a collision bound of \( \left\lceil \frac{\pi}{\pi/3} \right\rceil = 3 \) possible collisions. \( \square \)
3.3. An Arbitrary Mass Collision Bound Theorem. Considering arbitrary masses and reusing the proof of Theorem 4 we state the following:

Theorem 5. For three arbitrary point-masses $m_i, m_j, m_k$, the maximum number of collisions is

$$
\left\lfloor \frac{\pi}{\arccos \left( \frac{2(m_i + m_k) - m_j}{\sqrt{(m_i + m_j + 4m_k)(4m_i + m_j + m_k)}} \right)} \right\rfloor.
$$

This expression can be simplified slightly. Notice the expression above is symmetric in $m_i$ and $m_k$, so write $m_i = \alpha m_j$ and $m_k = \beta m_j$. The expression then is not directly dependent upon the masses but on the relative ratios of the masses, $\alpha$ and $\beta$:

$$
\left\lfloor \frac{\pi}{\arccos \left( \frac{2(\alpha + \beta) - 1}{\sqrt{(\alpha + 1 + 4\beta)(4\alpha + 1 + \beta)}} \right)} \right\rfloor.
$$

This expression also provides an interesting bound on the number of collisions. As seen in figure 5, the number of collisions only seems to change when $\beta$ is large with $\alpha \ll \beta$ or vice-versa.

3.4. A Detour: The Foch Sequence. A result by Murphy and Cohen [MC1], [MC2] states that the bound of the 3-billiard ball (seen as spheres) problem in $\mathbb{R}^n$ is four. The four-collision sequence $(12)(23)(12)(13)$ is called the Foch sequence, see
Figure 5. A contour plot for the maximum number of collisions when \((\alpha, \beta) \in (0, 10] \times (0, 10] \) and \(m_j = 1\).

Their proof is geometric and makes conditions on the locations of one of the balls in terms of the radii of the other billiard balls. However, it is interesting to note that despite how it may look at an initial glance, this Foch sequence does not contradict Theorem \(4\). In our problem, we are treating the balls as point masses, and so no such considerations are necessary. In fact, if the radii shrink to zero and follow the details of their proof, the Foch sequence is no longer possible, and the collision bound jumps from 4 to 3 when the radii reach zero.

4. Lines Intersection Collision Subspaces

4.1. Results in \(\mathbb{R}^N\). Motivated by the results in section \(3\), we aim to bound the number of times a billiard trajectory can intersect a collision subspace more generally. In this section, we treat the billiard trajectory as a line in \(E\), not a piecewise linear trajectory. To start, we bound the number of times a line can intersect arbitrary subspaces of codimension \(d\). We also make the assumption that the line does not lie within any of the codimension \(d\) subspaces.

Lemma 3. In \(\mathbb{R}^N\) there are \(N\) mutually orthogonal hyperplanes. The \(N - 1\) principal angles between any two such (distinct) hyperplanes are all 0 except the last angle which is \(\pi/2\).
Note: We use the word “orthogonal” throughout to mean that the only nonzero principal angles are $\frac{\pi}{2}$ (which is different than the orthogonal complement of the subspaces).

Proof. Let $\{v_1, \ldots, v_N\}$ be an orthonormal (or just orthogonal) basis for $\mathbb{R}^N$. Define the hyperplanes by $H_i = \text{span}\{v_i\}^\perp$ for $1 \leq i \leq N$. The statement about the principal angles follows from definition 1. $\square$

**Proposition 1.** A line can intersect at most $N$ hyperplanes in $\mathbb{R}^N$.

Proof. First, we note that we plan to avoid the possibility that the trajectory hits any intersection between hyperplanes (e.g. in $\mathbb{R}^3$, a trajectory which hits the $x$-axis, $y$-axis, $z$-axis, or the origin is not allowed). Let $\ell(t) = a + tu$ denote the linear trajectory and let $\{v_1, \ldots, v_N\}$ be an orthonormal basis for $\mathbb{R}^N$. Define the $N$ hyperplanes as

$$H_i = \text{span}\{v_i\}^\perp.$$ 

Without loss of generality, we may choose coordinates such that the point $a \in \mathbb{R}^N \setminus \left( \bigcup_{i=1}^N H_i \right)$ has all positive coordinates. Furthermore, choose $u = c_1v_1 + \cdots + c_N v_N$ where each coefficient $c_i \neq 0$ for $1 \leq i \leq N$. That is, we choose $u$ to be a vector which is not parallel to any of the hyperplanes $H_i$. By construction, we now see that as $\ell(t)$ crosses the hyperplane $H_i$, the $i^{th}$ component of $\ell(t)$ will change from positive to negative, and more importantly, will never become positive again. By the choice of $u$ and the point $a$, we know that eventually all components of $\ell(t)$ will become negative beyond some finite time $0 < T < \infty$. That means that...
for all $t > T$, $\ell(t)$ will not intersect any more hyperplanes. By counting the sign changes in each component, we see that this happens at most $N$ times. □

The previous propositions and lemmas are a special case of the following lemma. The proof of the theorem is constructive, and motivated by the fact that in $\mathbb{R}^3$, a line that does not go through the origin can only intersect at most two of the coordinate axes (which is the case $N = 3$ and $d = 2$ below).

**Proposition 2.** A line can intersect at most $N - d + 1$ mutually orthogonal codimension $d$ subspaces of $\mathbb{R}^N$.

**Proof.** Fix an integer $1 \leq d < N$ as the codimension of our subspaces. Let $\mathcal{B} = \{v_1, \ldots, v_N\}$ be an orthonormal basis for $\mathbb{R}^N$. For each $\mathcal{I} \subset \{1, \ldots, N\}$ with $|\mathcal{I}| = d$, define

$$S_\mathcal{I} = \text{span}\{v_i \in \mathcal{B} : i \notin \mathcal{I}\}.$$ 

That is, $S_\mathcal{I}$ is the codimension $d$ subspace created by removing the $d$ basis vectors from $\mathcal{B}$ whose indices are elements of $\mathcal{I}$.

If for some $t_0 > 0$, $\ell(t_0)$ has $d$ components equal to zero simultaneously, then we consider $\ell(t)$ to have intersected the subspace $S_\mathcal{I}$ where $\mathcal{I}$ is the set of indices of the components which are equal to zero at time $t = t_0$. If more than $d$ components of $\ell(t_0)$ are zero, then $\ell(t)$ has intersected an intersection of subspaces $S_\mathcal{I}$, which we do not allow for the moment. For example, if $N = 4$ and $d = 2$, we would not allow for $\ell(t) = (0, 0, 0, w)$ for all $t$ and for all $w$, as this would mean that the trajectory $\ell(t)$ would intersect $S_{12} \cap S_{13} \cap S_{23}$.

Construct the linear trajectory $\ell(t) = a - t\vec{u}$ in the following way: Choose $\vec{u} = (0, \ldots, 0, 1, \ldots, 1)$, where the first $d - 1$ entries of $\vec{u}$ are zero and the last $N - d + 1$ entries are 1. Without loss of generality, choose $a \in \mathbb{R}^N$ such that the first $d - 1$ entries of $a$ are zero and the rest are distinct, positive real numbers. That is, choose

$$a = (0, \ldots, 0, a_d, a_{d+1}, \ldots, a_N)$$

so that the first $d - 1$ entries are 0 and $a_i \neq a_j$ for distinct $d \leq i, j \leq N$. Then for each $a_i$, we know

$$\ell(a_i) = (0, \ldots, 0, a_d, a_{d+1}, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_N)$$

which means that $\ell(t)$ intersects the subspace $S_{\{1, \ldots, d-1, i\}}$. And because the $a_i$ are distinct, $\ell(t)$ will intersect $S_{\{1, \ldots, d-1, i\}}$ for each $d \leq i \leq N$ exactly once. Just as before, the $i^{th}$ component of $\ell(t)$ will change sign after each time $t = a_i$. This will produce $N - (d - 1) = N - d + 1$ such sign changes, and hence $\ell(t)$ will intersect the subspaces $N - d + 1$ times. By construction, for $t > \max\{a_i : d \leq i \leq N\}$, all nonzero components of $\ell(t)$ will be negative and will never become positive again. Because the signs will never change beyond this point, there cannot be any more subspace intersections. □
4.2. Results in $\mathbb{R}^3$. Knowing that the angles between our unit-mass collision subspaces are combinations of 0’s, $\frac{\pi}{3}$’s, and $\frac{\pi}{2}$’s, we consider which such configurations are geometrically, ignoring physical conditions that could constrain our system.

**Proposition 3.** Consider planes $H_i \subset \mathbb{R}^3$ which all contain the origin (so they are subspaces). Intersect $H_i$ with the unit sphere $S^2$ creates a great circle $C_i$. If $P_1$ and $P_2$ are the two points of intersection of $H_i$ and $H_j$, the nonzero angle between these planes can be measured by measuring the angle between tangent vectors to each $C_i$ at $P_1$ or $P_2$.

**Proof.** Let $H_1, H_2$ be two distinct planes which form a polyhedral cone (whose “point” is at the origin). By the definition of principal angles, the first of the two angles between the planes $H_1$ and $H_2$ is zero. The line of intersection of these two planes is spanned by a single vector, say $v_1$, and $v_1$ can be seen as lying in both planes simultaneously.

To find the second principal angle, we need to first consider all vectors orthogonal to $v_1$, which can be considered as the plane $(v_1)^\perp$. But we can translate this plane throughout $\mathbb{R}^3$ as we wish, and this plane can be the tangent plane to the unit sphere at the points $\{P_1, P_2\} = \text{span}\{v_1\} \cap S^2$. But then the second angle formed between $H_1$ and $H_2$ is exactly the angle between the tangent vectors to the great circles $C_1 = H_1 \cap S^2$ and $C_2 = H_2 \cap S^2$ at say, $P_1$, because these tangent vectors live in the plane $(v_1)^\perp$. Therefore we can measure the second principal angle between planes in $\mathbb{R}^3$ by computing the angles between the tangent vectors to the points of intersection of the two great circles. \hfill \Box

**Proposition 4.** In $\mathbb{R}^3$ it is impossible for a collection of three codimension 1 subspaces to all have mutual principal angles of 0 and $\frac{\pi}{3}$. That is, it is impossible for a polyhedral cone with 3 or more faces in $\mathbb{R}^3$ to have faces whose principal angles are 0 and $\frac{\pi}{3}$.

**Proof.** Consider a polyhedral cone in $\mathbb{R}^3$ with three faces. Cut the planes by the unit sphere $S^2$ to get a collection of three great circles. By the preceding proposition, if all of the angles between the tangent vectors to the great circles were indeed $\frac{\pi}{3}$, this would create a spherical triangle, all of whose interior angles are $\frac{\pi}{3}$. This contradicts the fact that the sum of the angles of a spherical triangle is always greater than $\pi$. Thus no such configuration of planes can exist in $\mathbb{R}^3$. \hfill \Box

It is worth noting that this proposition should be contrasted with the basic example of $\mathbb{R}^3$ seen as the configuration space of three point masses on a line. We can consider the three collision subspaces $\Delta_{12}$, $\Delta_{23}$, and $\Delta_{13}$ as the planes $x = y$, $x = z$, and $y = z$, respectively. These three planes intersect in a common line, $\text{span}\{(1,1,1)\}$ representing triple collision. However, this configuration does not create a polyhedral cone as the great circles created by intersection of the collision subspaces with the unit sphere $S^2$ have common intersection points $(1,1,1)$ and $(-1,-1,-1)$. Seen as a tilted north and south poles, these great circles divide the sphere into six spherical sectors along tilted meridians. However, in this case it
is easily seen that there can be at most 3 intersection points between these three planes and a line.

**Figure 7.** $S^2$ tiled by $\frac{\pi}{3} - \frac{\pi}{2} - \frac{\pi}{2}$ spherical triangles

**Proposition 5.** In $\mathbb{R}^3$, it is possible to create a polyhedral cone with three codimension 1 subspaces as faces to have pairwise nonzero principal angles $\frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$. In such a polyhedral cone, a billiard trajectory can have at most four collisions.

**Proof.** We again use spherical geometry. Consider planes $H_1, H_2, H_3$ given by $z = 0$, $y = 0$, and $y = \sqrt{3}x$, respectively, in $\mathbb{R}^3$. We clearly have

$$\angle(H_1, H_2) = \left[0, \frac{\pi}{2}\right], \angle(H_1, H_3) = \left[0, \frac{\pi}{2}\right], \text{ and } \angle(H_2, H_3) = \left[0, \frac{\pi}{3}\right].$$

Intersecting these planes with the unit sphere $S^2$ creates great circles $C_1, C_2, C_3$ which bound a spherical triangle (among others) in the first octant with interior angles $\frac{\pi}{3}$ at the “north pole” and angles of $\frac{\pi}{2}$ at the “equator.”

Luckily we can tile the sphere with 12 such $\frac{\pi}{3} - \frac{\pi}{2} - \frac{\pi}{2}$ spherical triangles. This only necessitates the addition of a fourth plane, $H_4$ given by $y = -\sqrt{3}x$. These four planes $H_1, H_2, H_3, H_4$ can each only be intersected once by a linear trajectory in $\mathbb{R}^3$, which correspond to a bound of four collisions of a billiard trajectory. \qed

One should also note that, even though specific equations of planes are given in the proof above, we can consider many other tilings of $S^2$ with $\frac{\pi}{3} - \frac{\pi}{2} - \frac{\pi}{2}$ spherical triangles by $SO(3)$ acting on this “base” configuration. Though it should be noted that $SO(3)$ acting on the base configuration does not result in all such tilings.
Proposition 6. In $\mathbb{R}^3$, it is possible to create a polyhedral cone with three codimension 1 subspaces as faces to have pairwise nonzero principal angles $\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}$. In such a polyhedral cone, a billiard trajectory can have at most six collisions.

Proof. The argument is nearly identical to the previous proof, though with appropriate adjustments made. Consider planes $H_1, H_2, H_3$ given by $x = 0$, $y = 0$, and $-\frac{1}{2}x - \frac{1}{2}y + \frac{1}{\sqrt{2}}z = 0$, respectively, in $\mathbb{R}^3$. We clearly have

$$\angle(H_1, H_2) = \left[0, \frac{\pi}{2}\right], \angle(H_1, H_3) = \left[0, \frac{\pi}{3}\right], \text{ and } \angle(H_2, H_3) = \left[0, \frac{\pi}{3}\right].$$

Intersecting these planes with the unit sphere $S^2$ creates great circles $C_1, C_2, C_3$ which bound a spherical triangle (among others) in the first octant with interior angles $\frac{\pi}{2}$ at the “north pole” and angles of $\frac{\pi}{3}$ above the “equator.”

Luckily we can tile the sphere with 24 such $\frac{\pi}{3} - \frac{\pi}{3} - \frac{\pi}{2}$ spherical triangles. This necessitates three additional planes, $H_4, H_5, H_6$ given by

$$H_4 : -\frac{1}{2}x + \frac{1}{2}y - \frac{1}{\sqrt{2}}z = 0$$

$$H_5 : \frac{1}{2}x - \frac{1}{2}y - \frac{1}{\sqrt{2}}z = 0$$

$$H_6 : \frac{1}{2}x + \frac{1}{2}y + \frac{1}{\sqrt{2}}z = 0.$$ 

These six planes $H_1, H_2, H_3, H_4, H_5, H_6$ can each only be intersected once by a linear trajectory in $\mathbb{R}^3$, which correspond to a bound of six collisions of a billiard trajectory. \qed
Just as before, we can consider many other tilings of $S^2$ with $\frac{\pi}{3} + \frac{\pi}{3} + \frac{\pi}{3}$ spherical triangles as arising from $SO(3)$ acting on this “base” configuration, though again this does not account for all such tilings of the sphere.

Remark. These propositions about lines intersecting subspaces should be contrasted with the earlier statements and the theorems in the next subsection, as $\mathbb{R}^3$ can only be the configuration space for three point masses on a line ($N = 3, m = 1$) or one point mass in $\mathbb{R}^3$, for which there are no collisions. If the particles are noninteracting (i.e. can pass through one another) then having three binary collision subspaces makes sense, otherwise the left-most and right-most particles could never interact except at triple collision. Interpreting this as a physical system requires the additional conditions that $x \leq y$ and $y \leq z$ if the particles are interacting. A similar unfold-the-angle argument leads to $3$ as the maximum number of collisions.

4.3. Billiard Bounds on a Line. To compare and contrast the results of the previous two subsections and address the remark above, we provide a summary below for known results about collision bounds for billiards on a line.

Definition 5. On a line, we call 1-dimensional spheres hard rods. Through a transformation, we may also treat these hard rods as point masses without affecting their masses.

Theorem 6 ([MC2] - 1993). For $N$ equal mass hard rods on a line, no more than $\binom{N}{2}$ collisions can occur.

Theorem 7 ([LC1] - 2007). Consider $N$ hard rods on a line with masses $m_i$, $1 \leq i \leq N$. If $m_i \geq \sqrt{m_{i-1}m_{i+1}}$, $2 \leq i \leq N - 1$, then the maximum number of collisions that can occur is $\binom{N}{2}$.

Corollary 2 ([LC1] - 2007). Consider $N$ hard rods on a line with masses $m_i$, $1 \leq i \leq N$. If $m_i \geq \frac{m_{i-1} + m_{i+1}}{2}$, $2 \leq i \leq N - 1$, then the maximum number of collisions that can occur is $\binom{N}{2}$.

In the equal-mass case, these three results agree that $\binom{N}{2}$ is the bound on the number of collisions, which matches physical intuition, and matches our commentary at the end of the previous section.

Theorem 8 ([Gal1] - 1981). Consider $N$ hard rods on a line with maximum and minimum masses $m_{\text{max}}$ and $m_{\text{min}}$, respectively. Then the number of collisions is at most

$$2 \left( 8N^2(N - 2)\frac{m_{\text{max}}}{m_{\text{min}}} \right)^{N-2}.$$ 

It is interesting to note that for $m_{\text{max}} = m_{\text{min}}$, this bound is significantly larger than the known maximum bound of $\binom{N}{2}$.

5. Future Work

5.1. Barriers to Entry and Next Steps. The work in this paper is the result of attempts to solve the original problem: bounding the number of collisions in
an $N$-billiard system using the computed angles between collision subspaces. The main collision theorems were possible due to the symplectic reduction using Jacobi coordinates, which reduced the angles between the reduced collision subspaces to all be nonzero. When $N > 3$ and $m > 2$, similar reduction techniques produces angles with measure 0 between subspaces. To the author, it seems that the presence of 0 as an angle between subspaces is a significant problem. Any suggestions or ideas are welcome.

This model is also inherently simplistic. Geometric and physical considerations are easily ignored (e.g. when considering hard rods on a line which cannot pass one another), but the bounds may still exist. Is there another interpretation of the system that more closely matches a physical system? A more rigorous study of this model to include such constraints is a logical next step.

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Appendix A. Proof of Theorem 2

To prove Theorem 2 we begin with a short lemma.

Lemma 4. $\dim(\Delta_{ij} \cap \Delta_{kl}) = d(N - 2) = \text{number of angles which are 0}.$

Proof. This is obvious by the definition of $\Delta_{ij}$. We know $\Delta_{ij} \cap \Delta_{kl}$ can have an orthonormal basis $\left\{ E_1 \otimes q_1, \ldots, \frac{\varepsilon_i + \varepsilon_j}{\sqrt{2}} \otimes q_i, \ldots, \frac{\varepsilon_k + \varepsilon_l}{\sqrt{2}} \otimes q_k, \ldots, E_N \otimes q_N \right\}$, which has dimension $Nd - 2d = d(N - 2)$. And based on our definition of principal angles, the number of angles which are zero will be exactly the dimension of the intersection of the two subspaces. $\Box$

From our proof it follows that $\dim(\Delta_{ij} \cap \Delta_{jk}) = d(N - 2) = \text{number of angles which are 0}.$ But to help prove our theorem, we make the following two observations:

- The cosine expression for nonzero angles in Theorem 2 part (a) is equivalent to showing that $\left[ (\Delta_{ij} \cap \Delta_{kl})^\perp \cap \Delta_{ij} \right] \perp \left[ (\Delta_{ij} \cap \Delta_{kl})^\perp \cap \Delta_{kl} \right].$
- The cosine expression for nonzero angles in Theorem 2 part (b) is equivalent to computing the angle between $\left[ (\Delta_{ij} \cap \Delta_{jk})^\perp \cap \Delta_{ij} \right]$ and $\left[ (\Delta_{ij} \cap \Delta_{jk})^\perp \cap \Delta_{jk} \right].$

Proof. We now aim to compute each of the terms above. Recall the binary collision subspaces in tensor form:

$$\Delta_{ij} = \left\{ E_1 \otimes q_1, \ldots, \frac{\varepsilon_i + \varepsilon_j}{\sqrt{2}} \otimes q_i, \ldots, E_N \otimes q_N : q_s \in \mathbb{R}^m, \|q_s\| = 1, 1 \leq s \leq N \right\}$$
\[ \Delta_{kl} = \left\{ E_1 \otimes q_1, \ldots, \frac{\varepsilon_k + \varepsilon_l}{\sqrt{2}} \otimes q_k, \ldots, E_N \otimes q_N : q_s \in \mathbb{R}^m, \|q_s\| = 1, 1 \leq s \leq N \right\}. \]

It follows that
\[ \Delta_{ij} \cap \Delta_{kl} = \left\{ E_1 \otimes q_1, \ldots, \frac{\varepsilon_i + \varepsilon_j}{\sqrt{2}} \otimes q_i, \ldots, \frac{\varepsilon_k + \varepsilon_l}{\sqrt{2}} \otimes q_k, \ldots, E_N \otimes q_N \right\} \]
and hence
\[ (\Delta_{ij} \cap \Delta_{kl})^\perp = \left\{ \frac{\varepsilon_i - \varepsilon_j}{\sqrt{2}} \otimes \beta_1, \frac{\varepsilon_k - \varepsilon_l}{\sqrt{2}} \otimes \beta_2 \right\} \]
where \( \beta_1 \) and \( \beta_2 \) are arbitrary unit vectors in \( \mathbb{R}^m \). The \( \beta_i \)'s can be chosen arbitrarily because the fact that \( (S \otimes \mathbb{R}^m)^\perp = S^\perp \otimes \mathbb{R}^m \) together with the way inner products of tensor products are computed (i.e. \( \langle a \otimes b, c \otimes d \rangle = \langle a, c \rangle \langle b, d \rangle \)), means that in our construction, we only concern ourselves with orthogonality in the first component of our tensor product.

We now explicitly compute \( (\Delta_{ij} \cap \Delta_{kl})^\perp \cap \Delta_{ij} \):

Let \( v \in (\Delta_{ij} \cap \Delta_{kl})^\perp \cap \Delta_{ij} \). Then for scalars \( a, b, c, d, e \in \mathbb{R} \),
\[ v = a \left( \frac{\varepsilon_i - \varepsilon_j}{\sqrt{2}} \otimes \beta_1 \right) + b \left( \frac{\varepsilon_k - \varepsilon_l}{\sqrt{2}} \otimes \beta_2 \right) \]
and
\[ v = c \left( \frac{\varepsilon_i + \varepsilon_j}{\sqrt{2}} \otimes q_i \right) + d(\varepsilon_k \otimes q_k) + e(\varepsilon_l \otimes q_l). \]
Then
\[ 0 = v - v \]
\[ = a \left( \frac{\varepsilon_i - \varepsilon_j}{\sqrt{2}} \otimes \beta_1 \right) + b \left( \frac{\varepsilon_k - \varepsilon_l}{\sqrt{2}} \otimes \beta_2 \right) - c \left( \frac{\varepsilon_i + \varepsilon_j}{\sqrt{2}} \otimes q_i \right) - d(\varepsilon_k \otimes q_k) - e(\varepsilon_l \otimes q_l) \]
\[ = (\varepsilon_i - \varepsilon_j) \otimes \frac{a}{\sqrt{2}} \beta_1 + (\varepsilon_k - \varepsilon_l) \otimes \frac{b}{\sqrt{2}} \beta_2 - (\varepsilon_i + \varepsilon_j) \otimes \frac{c}{\sqrt{2}} q_i - \varepsilon_k \otimes dq_k - \varepsilon_l \otimes eq_l \]
\[ = \varepsilon_i \otimes \left( \frac{a}{\sqrt{2}} \beta_1 - \frac{c}{\sqrt{2}} q_i \right) + \varepsilon_j \otimes \left( -\frac{a}{\sqrt{2}} \beta_1 - \frac{c}{\sqrt{2}} q_i \right) \]
\[ + \varepsilon_k \otimes \left( \frac{b}{\sqrt{2}} \beta_2 - dq_k \right) + \varepsilon_l \otimes \left( -\frac{b}{\sqrt{2}} \beta_2 - eq_l \right) \]

This implies each term on the right of each tensor product must be zero:
\[ 0 = \frac{a}{\sqrt{2}} \beta_1 - \frac{c}{\sqrt{2}} q_i \]
\[ 0 = -\frac{a}{\sqrt{2}} \beta_1 - \frac{c}{\sqrt{2}} q_i \]
\[ 0 = \frac{b}{\sqrt{2}} \beta_2 - dq_k \]
\[ 0 = -\frac{b}{\sqrt{2}} \beta_2 - eq_i. \]

The first two equations imply that \( a = c = 0 \), and the last two equations imply that \( \frac{b}{\sqrt{2}} \beta_2 = dq_k = -eq_i \). Thus

\[ (\Delta_{ij} \cap \Delta_{kl})^\perp \cap \Delta_{ij} = \left\{ \frac{\varepsilon_k - \varepsilon_l}{\sqrt{2}} \otimes \beta_2 \right\}. \]

A similar calculation yields

\[ (\Delta_{ij} \cap \Delta_{kl})^\perp \cap \Delta_{kl} = \left\{ \frac{\varepsilon_i - \varepsilon_j}{\sqrt{2}} \otimes \beta_1 \right\}. \]

From here it is easy to see the orthogonality argument. We clearly have that these two spaces are orthogonal to one another

\[ [(\Delta_{ij} \cap \Delta_{kl})^\perp \cap \Delta_{ij}] \perp [(\Delta_{ij} \cap \Delta_{kl})^\perp \cap \Delta_{kl}]. \]

In fact, since

\[ (\Delta_{ij} \cap \Delta_{kl})^\perp = \left\{ \frac{\varepsilon_i - \varepsilon_j}{\sqrt{2}} \otimes \beta_1, \frac{\varepsilon_k - \varepsilon_l}{\sqrt{2}} \otimes \beta_2 \right\} \]

we can see that this decomposes into a direct orthogonal sum of the two subspaces above. This along with Lemma 4 completes the proof of part (a) of Theorem 2.

We now focus on proving part (b) of Theorem 2. The calculations are similar. This means

\[ \Delta_{ij} \cap \Delta_{jk} = \left\{ E_1 \otimes q_i, \ldots, \frac{\varepsilon_i + \varepsilon_j + \varepsilon_k}{\sqrt{3}} \otimes q_i, \ldots, E_N \otimes q_N \right\} \]

and

\[ (\Delta_{ij} \cap \Delta_{jk})^\perp = \left\{ \frac{\varepsilon_i - \varepsilon_j}{\sqrt{2}} \otimes \beta_1, \frac{\varepsilon_i + \varepsilon_j - 2\varepsilon_k}{\sqrt{6}} \otimes \beta_2 \right\} \]

for some unit vectors \( \beta_1, \beta_2 \in \mathbb{R}^m \).

We now explicitly compute \( (\Delta_{ij} \cap \Delta_{jk})^\perp \cap \Delta_{ij} \).

Let \( w \in (\Delta_{ij} \cap \Delta_{jk})^\perp \cap \Delta_{ij} \). Then for scalars \( a, b, c, d \in \mathbb{R} \),

\[ w = a \left( \frac{\varepsilon_i - \varepsilon_j}{\sqrt{2}} \otimes \beta_1 \right) + b \left( \frac{\varepsilon_i + \varepsilon_j - 2\varepsilon_k}{\sqrt{6}} \otimes \beta_2 \right) \]

and

\[ w = c \left( \frac{\varepsilon_i + \varepsilon_j}{\sqrt{2}} \otimes q_i \right) + d(\varepsilon_k \otimes q_k). \]

Then

\[ 0 = w - w = a \left( \frac{\varepsilon_i - \varepsilon_j}{\sqrt{2}} \otimes \beta_1 \right) + b \left( \frac{\varepsilon_i + \varepsilon_j - 2\varepsilon_k}{\sqrt{6}} \otimes \beta_2 \right) - c \left( \frac{\varepsilon_i + \varepsilon_j}{\sqrt{2}} \otimes q_i \right) - d(\varepsilon_k \otimes q_k) \]
\[
= (\varepsilon_i - \varepsilon_j) \otimes \frac{a}{\sqrt{2}} \beta_1 + (\varepsilon_i + \varepsilon_j - 2\varepsilon_k) \otimes \frac{b}{\sqrt{6}} \beta_2 - (\varepsilon_i + \varepsilon_j) \otimes \frac{c}{\sqrt{2}} q_i - \varepsilon_k \otimes dq_k \\
= \varepsilon_i \otimes \left( \frac{a}{\sqrt{2}} \beta_1 + \frac{b}{\sqrt{6}} \beta_2 - \frac{c}{\sqrt{2}} q_i \right) + \varepsilon_j \otimes \left( -\frac{a}{\sqrt{2}} \beta_1 + \frac{b}{\sqrt{6}} \beta_2 - \frac{c}{\sqrt{2}} q_i \right) \\
+ \varepsilon_k \otimes \left( -\frac{2b}{\sqrt{6}} \beta_2 - dq_k \right).
\]

This implies each term on the right of each tensor product must be zero:

\[
0 = \frac{a}{\sqrt{2}} \beta_1 + \frac{b}{\sqrt{6}} \beta_2 - \frac{c}{\sqrt{2}} q_i \\
0 = -\frac{a}{\sqrt{2}} \beta_1 + \frac{b}{\sqrt{6}} \beta_2 - \frac{c}{\sqrt{2}} q_i \\
0 = -\frac{2b}{\sqrt{6}} \beta_2 - dq_k.
\]

The first two equations imply that \(a = 0\), and hence

\[
(\Delta_{ij} \cap \Delta_{jk})^\perp \cap \Delta_{ij} = \left\{ \frac{\varepsilon_i + \varepsilon_j - 2\varepsilon_k}{\sqrt{6}} \otimes \beta_2 \right\}.
\]

A similar calculation yields that

\[
(\Delta_{ij} \cap \Delta_{jk})^\perp \cap \Delta_{jk} = \left\{ 2\varepsilon_i - \varepsilon_j - \varepsilon_k \sqrt{2} \otimes \beta_2 \right\}.
\]

It’s now easily seen that the nonzero angle between these two spaces is indeed \(\frac{\pi}{3}\). This completes the proof of Theorem 2. □

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