NOTE ON THE CHEN-LIN RESULT WITH LI-ZHANG METHOD.

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Abstract: We give a new proof of Chen-Lin result with Li-Zhang method.

1. INTRODUCTION AND RESULTS.

We set \( \Delta = -\partial_{11} - \partial_{22} \) the geometric Laplacian on \( \mathbb{R}^2 \).

On an open set \( \Omega \) of \( \mathbb{R}^2 \), with a smooth boundary, we consider the following problem:

\[
(P) \quad \begin{cases}
\Delta u = V e^u & \text{in } \Omega, \\
0 < a \leq V \leq b < +\infty
\end{cases}
\]

The previous equation is called, the Prescribed Scalar Curvature, in relation with conformal change of metrics. The function \( V \) is the prescribed curvature.

Here, we try to find some a priori estimates for sequences of the previous problem. Equations of the previous type were studied by many authors. We can see in [B-M], different results for the solutions of those type of equations with or without boundaries conditions and, with minimal conditions on \( V \), for example we suppose \( V \geq 0 \) and \( V \in L^p(\Omega) \) or \( V e^u \in L^p(\Omega) \) with \( p \in [1, +\infty] \). We can see in [B-M] the following important Theorem,

**Theorem A (Brezis-Merle).** If \((u_i)_i \) and \((V_i)_i \) are two sequences of functions relatively to the problem \((P)\) with, \(0 < a \leq V_i \leq b < +\infty\), then, for all compact set \( K \) of \( \Omega \),

\[
sup_K u_i \leq c = c(a, b, m, K, \Omega) \text{ if } \inf_{\Omega} u_i \geq m.
\]

A simple consequence of this theorem is that, if we assume \( u_i = 0 \) on \( \partial \Omega \) then, the sequence \((u_i)_i \) is locally uniformly bounded.

If, we assume \( V \) with more regularity, we can have another type of estimates, \( \sup + \inf \). It was proved, by Shafrir, see [S], that, if \((u_i)_i \), \((V_i)_i \) are two sequences of functions solutions of the previous equation without assumption on the boundary and, \(0 < a \leq V_i \leq b < +\infty\), then we have the following interior estimate:

\[
C(a/b) \sup_K u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).
\]

We can see in [C-L], an explicit value of \( C(a/b) \) is \( \sqrt{a/b} \).

Now, if we suppose \((V_i)_i \) uniformly Lipschitzian with \( A \) the Lipschitz constant, then, \( C(a/b) = 1 \) and \( c = c(a, b, A, K, \Omega) \), see Brézis-Li-Shafrir [B-L-S]. This result was extended for Hölderian sequences \((V_i)_i \), by Chen-Lin, see [C-L]. Also, we can see in [L], an extension of the Brézis-Li-Shafrir to compact Riemann surface without boundary. We can see in [L-S] explicit form, \((8\pi m, m \in \mathbb{N}^* \text{ exactly})\), for the numbers in front of the Dirac masses, when the solutions blow-up.

On open set \( \Omega \) of \( \mathbb{R}^2 \) we consider the following equation:

\[
\Delta u_i = V_i e^{u_i} \text{ on } \Omega.
\]

\[
0 < a \leq V_i \leq b < +\infty, \quad |V_i(x) - V_i(y)| \leq A|x - y|^s, \quad 0 < s < 1, \ x, y \in \Omega.
\]
**Theorem B** (*Chen-Lin*). For all compact \( K \subset \Omega \) and all \( s \in ]0, 1[ \) there is a constant \( c = c(a, b, A, s, K, \Omega) \) such that,

\[
\sup_K u_i + \inf_{\Omega} u_i \leq c, \ \forall \ i.
\]

Here we try to prove the previous theorem by the moving-plane method and Li-Zhang method.

We argue by contradiction, and we want to prove that:

\[
\exists R > 0, \ \text{such that} \ 4 \log R + \sup_{B_R(0)} u + \inf_{B_R(0)} u \leq c = c(a, b, A),
\]

Thus, by contradiction we can assume:

\[
\exists (R_i)_i, \ (u_i)_i, \ R_i \to 0, \ 4 \log R_i + \sup_{B_R(i)} u_i + \inf_{B_2R(i)} u_i \to +\infty,
\]

**The blow-up analysis**

Let \( x_0 \in \Omega \), we want to prove the theorem locally around \( x_0 \), we use the previous assertion with \( x_0 = 0 \). The classical blow-up analysis gives the existence of the sequence \((x_i)_i\) and a sequence of functions \((v_i)_i\) such that:

We set,

\[
\sup_{B_{R_i}(0)} u_i = u_i(\tilde{x}_i),
\]

\[
s_i(x) = 2 \log(R_i - |x - \tilde{x}_i|) + u_i(x), \ \text{and} \ s_i(x_i) = \sup_{B_{R_i}(x_i)} s_i, \ l_i = \frac{1}{2}(R_i - |x_i - \tilde{x}_i|).
\]

Also, we set:

\[
v_i(x) = u_i[x_i + xe^{-u_i(x_i)/2}] - u_i(x_i), \ \bar{V}_i(x) = V_i[x_i + xe^{-u_i(x_i)/2}],
\]

Then,

\[
\Delta v_i = \bar{V}_i e^{u_i},
\]

\[
v_i \leq 2 \log 2, \ v_i(0) = 0,
\]

\[
v_i \to v = \log \frac{1}{(1 + [V(0)/8]|x|^2)^2} \ \text{converge uniformly on each compact set of} \ \mathbb{R}^2
\]

with \( V(0) = \lim_{i \to +\infty} V_i(x_i) \).

The classical elliptic estimates and the classical Harnack inequality, we can prove the previous uniform convergence on each compact of \( \mathbb{R}^2 \).

**The Kelvin transform and the moving-plane method:** *Li-Zhang method*

For \( 0 < \lambda < \lambda_1 \), we define:

\[
\Sigma_\lambda = B(0, l_i M_i) - B(0, \lambda).
\]

First, we set:

\[
\bar{v}_i^\lambda = v_i^\lambda - 4 \log |x| + 4 \log \lambda = v_i \left( \frac{\lambda^2 x}{|x|^2} \right) + 4 \log \frac{\lambda}{|x|},
\]

\[
\bar{V}_i^\lambda = \bar{V}_i \left( \frac{\lambda^2 x}{|x|^2} \right)
\]

\[
M_i = e^{u_i(x_i)/2},
\]
and,
\[ w_\lambda = \bar{v}_i - \hat{v}_i^\lambda. \]
Then,
\[ \Delta \hat{v}_i^\lambda = V^\lambda e^{v_i^\lambda}, \]
\[ \min_{|y|=R_i M_i} \hat{v}_i^\lambda = u_i(x_i + r\theta) - u_i(x_i) + 2u_i(x_i) \geq \inf_{\Omega} u_i + u_i(x_i) \to +\infty, \]
and,
\[ \Delta (v_i - v_i^\lambda) = V_i e^{v_i} - e^{v_i^\lambda} + (V_i - V_i^\lambda)e^{v_i^\lambda}, \]
We have the following estimate:
\[ |V_i - V_i^\lambda| \leq AM_i^{-s}|x|^s \left[ 1 - \frac{\lambda^2}{|x|^2} \right]^s, \]
We take an auxiliary function \( h_\lambda : \)
Because, \( v_i(x^\lambda) \leq C(\lambda_1) < +\infty, \) we have,
\[ h_\lambda = C_1 M_i^{-s}\lambda^2(\log(\lambda/|x|)) + C_2 M_i^{-s}\lambda^{2+s}[1 - (\lambda/|x|)^{2-s}], \]
with \( C_1, C_2 = C_1, C_2(s, \lambda_1) > 0, \)
\[ h_\lambda = M_i^{-s}\lambda^2(1 - \lambda/|x|)(C_1 \frac{\log(\lambda/|x|)}{1 - \lambda/|x|} + C_2), \]
with, \( C_2' = C_2'(s, \lambda_1) > 0. \) We can choose \( C_1 \) big enough to have \( h_\lambda < 0. \)

**Lemma 1:** There is an \( \lambda_{r,0} > 0 \) small enough, such that, for \( 0 < \lambda < \lambda_{r,0} \), we have:
\[ w_\lambda + h_\lambda > 0. \]

We have,
\[ f(r, \theta) = v_i(r\theta) + 2\log r, \]
then,
\[ \frac{\partial f}{\partial r}(r, \theta) = < \nabla v_i(r\theta) | \theta > + \frac{2}{r}. \]
According to the blow-up analysis,
\[ \exists r_0 > 0, C > 0, \ |
\nabla v_i(r\theta) | \theta > | \leq C, \text{ for } 0 \leq r < r_0, \]
Then,
\[ \exists r_0 > 0, C' > 0, \frac{\partial f}{\partial r}(r, \theta) > C', \text{ for } 0 < r < r_0, \]
if \( 0 < \lambda < |y| < r_0, \)
\[ w_\lambda(y) + h_\lambda(y) = v_i(y) - v_i^\lambda(y) + h_\lambda(y) > C(\log |y| - \log |y^\lambda|) + h_\lambda(y), \]
by the definition of \( h_\lambda, \) we have, for \( C, C_0 > 0 \) and \( 0 < \lambda \leq |y| < r_0, \)
\[ w_\lambda(y) + h_\lambda(y) > (|y| - \lambda)C \frac{\log |y| - \log |y^\lambda|}{|y| - \lambda} - \lambda^{1+s}C_0 M_i^s, \]
but,
\[ |y| - |y^\lambda| > |y| - \lambda > 0, \text{ and } |y^\lambda| = \frac{\lambda^2}{|y|}, \]
thus,
\[ |y| - |y^\lambda| > |y| - \lambda > 0, \text{ and } |y^\lambda| = \frac{\lambda^2}{|y|}. \]
In the first step of the lemma 2, we have,

\[ v_i \geq \min v_i = C_1 \lambda \leq C_1(\lambda, r_0), \text{ if } r_0 \leq |y| \leq R_i M_i, \]

Thus, in \( r_0 \leq |y| \leq R_i M_i \) and \( \lambda \leq \lambda_1 \), we have,

\[ w_\lambda + h_\lambda \geq C_i - 4 \log \lambda + 4 \log r_0 - C_i \lambda_i \]

then, if \( \lambda \to 0 \), \(- \log \lambda \to +\infty\), and,

\[ w_\lambda + h_\lambda > 0, \text{ if } \lambda < \lambda_0, \lambda_0 \text{ (small)}, \text{ and } r_0 < |y| \leq R_i M_i, \]

By the maximum principle and the Hopf boundary lemma, we have:

**Lemma 2:** Let \( \tilde{\lambda}_k \) be a positive number such that:

\[ \tilde{\lambda}_k = \sup\{ \lambda < \lambda_1, w_\lambda + h_\lambda > 0 \text{ in } \Sigma_\lambda \}. \]

Then,

\[ \tilde{\lambda}_k = \lambda_1. \]

The blow-up analysis gives the following inequality for the boundary condition,

For \( y = |y/\theta| = R_i M_i \theta \), we have,

\[ w_\lambda(|y| = R_i M_i) + h_\lambda(|y| = R_i M_i) = \]

\[ = u_i(x_i + R_i \theta) - u_i(x_i) - v_i(y \lambda^i, |y| = R_i M_i) - 4 \log + 4 \log(R_i M_i) + C(s, \lambda_1) M_i^{-s} \lambda_i \]

\[ \geq 4 \log R_i + u_i(x_i) + \inf_{\Omega} u_i \to +\infty, \]

which we can write,

\[ w_\lambda(|y| = R_i M_i) + h_\lambda(|y| = R_i M_i) \geq \min_{\Omega} u_i + u_i(x_i) + 4 \log R_i - C(s, \lambda_1) \to +\infty, \]

because, \( 0 < \lambda \leq \lambda_1 \).
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