An Accurate 3D Analytic Model for Exoplanetary Photometry, Radial Velocity, and Astrometry

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Received 2021 August 30; revised 2021 November 9; accepted 2021 November 12; published 2022 January 26

Abstract

We developed and provide AnalyticLC, a novel analytic method and code implementation for dynamical modeling of planetary systems, including non-coplanar interactions, based on a disturbing function expansion to fourth order in eccentricities and inclinations. AnalyticLC calculates the system dynamics in 3D and the resulting model light-curve, radial-velocity, and astrometry signatures, enabling simultaneous fitting of these data. We show that for a near-resonant chain of three planets, where the two super-periods are close to each other, the TTVs of the pair-wise interactions cannot be directly summed to give the full system TTVs because the super-periods themselves resonate. We derive the simultaneous three planets correction and include it in AnalyticLC. We compare the model computed by AnalyticLC to synthetic data generated by an N-body integrator, and evaluate its accuracy. Depending on the maximal order of expansion terms kept, AnalyticLC computation time can be up to an order of magnitude faster than the state-of-the-art published N-body integrator TTVFast, with a smaller enhancement seen at higher order. The advantage increases for long-term observations as our approach’s computation time does not depend on the time span of the data. Depending on the system parameters, the photometric accuracy is typically a few ppm, significantly smaller than Kepler’s and other observatories’ typical data uncertainty. Our highly efficient and accurate implementation allows full inversion of a large number of observed systems for planetary physical and orbital parameters, presented in a companion paper.

Unified Astronomy Thesaurus concepts: Celestial mechanics (211); Exoplanets (498); Transits (1711)

1. Introduction

Since the first discoveries of extrasolar planets (Latham et al. 1989; Wolszczan & Frail 1992; Mayor & Queloz 1995), our knowledge of planetary systems is continuously growing. To date, more than 4400 confirmed extrasolar planets are known. The accumulating knowledge of the physical and orbital characteristics of individual planets and systems helps us to better understand the collective nature of planetary systems—which were found to be a natural and common consequence of star formation.

While the first detections of exoplanets were done mainly using the radial-velocity (RV) method, the Kepler mission (Borucki et al. 2010) provided a great leap in the number of planet detections by utilizing the transit method, yielding about 2300 new confirmed planets. Due to the large potential of these data sets, which continue to rapidly grow nowadays with the Transiting Exoplanet Survey Satellite mission (Ricker et al. 2010) and in the future Planetary Transits and Oscillations of stars mission (Rauer et al. 2014), efforts have been made in order to develop efficient methods for interpretation of these data sets.

The most direct way to invert RV or light-curve data to physical parameters is by running multiple N-body integrations driven by some nonlinear fitting process. Such a method was used for RV data (e.g., Rivera et al. 2005) and for photometric data (e.g., Mills & Fabrycky 2017; Mills et al. 2019; Freudenthal et al. 2018; Grimm et al. 2018) to fit for the parameters of several planetary systems. A drawback of this method is that as the observational time span increases, the modeling time becomes longer, making this a cumbersome, and sometimes even prohibitive, process.

In order to make the process of interpreting long-baseline photometric data more efficient, analytic methods were developed for data interpretation. In addition to the efficiency of these methods, they are powerful in providing us with a better understanding of the physical processes that govern the dynamics of planetary systems.

Since the time of mid-transit is usually a well-constrained property of an individual sufficiently deep transit event, transit timing variations (TTVs, Agol et al. 2005) were at the focus of many studies that use methods to invert them for planetary parameters—and planetary mass in particular. Nesvorný & Morbidelli (2008) used perturbation theory to analytically derive the TTV. This method was extended to eccentric and inclined orbits (Nesvorný 2009) and then implemented and used in the code TTVIM (Nesvorný & Beaugé 2010). Meschiari et al. (2009) developed a TTV-RV joint fitting software and used it to study the potential of their joint analysis (Meschiari & Laughlin 2010), and Payne et al. (2010) studied the magnitude of the TTV effect for various inclination regimes. Lithwick et al. (2012) explained the observed long-period TTV (“super-period”) near-first-order mean motion resonance (MMR) as the period of circulation of the line of conjunction between two interacting planets. The interpretation of the system parameters from this TTV pattern is impacted by a mass—eccentricity degeneracy, which can sometimes be broken by detecting the high-frequency TTV components, also called “chopping TTV” (Nesvorný & Vokrouhlický 2014; Deck & Agol 2015), or by detecting TTV to second order in eccentricity (Hadden & Lithwick 2016).
The ability of the TTV modeling to invert for the planetary masses led to the development of open-source software tools dedicated to model TTVs either by a full N-body integration (Deck et al. 2014) or by analytic calculation to first order in eccentricities (Agol & Deck 2016). Linial et al. (2018) constructed a modal decomposition method using a geometric approach to invert TTV data—also accurate to first order in eccentricity. Because of the abundance of TTVs in the Kepler population (Holczer et al. 2016; Ofir et al. 2018), TTV data were used to extract masses and eccentricities (or combinations of them) for a large number of planets, e.g., Hadden & Lithwick (2016, 2017), Jontof-Hutter et al. (2021), Yoffe et al. (2021), and many more.

While in many cases fitting times of mid-transit can yield a good understanding of planetary properties, this method does have drawbacks. As highlighted by Ofir et al. (2018), fitting transit times rather than flux does not to fully exploit the information encoded in the light curve, because many degrees of freedom are required in order to translate the full light curve to individual transit times. Fitting transit times also creates a bias that favors large planets (of clear transits that enable a good estimate of the mid-transit time) with strong TTV signals—while often the desired signals are those of small planets with shallow transits.

A second drawback of fitting times rather than flux is that it reduces the light-curve data to mid-transit times alone, erasing valuable information encoded in other types of transit variations, such as depth and duration. These are of great interest, as they can probe non-coplanar interactions within the system.

Both the transit and RV methods are limited in the ability to assess mutual 3D inclination: RV is inherently blind to inclination, while transit is inherently biased to selecting flat multi-transiting systems (Ragozzine & Holman 2010) and is symmetric to rotation about the line of sight. Therefore, many studies used a statistical approach to assess the mutual inclinations. Such an approach aims to characterize the dispersion of mutual inclinations (and/or eccentricities) rather than infer the values of individual systems (Fabrycky et al. 2014). Xie et al. (2016) have shown that the excess of singly transiting planets can be explained by a bimodal distribution that is characterized by a different dispersion of mutual inclinations and eccentricities for “hot” and “cold” systems. The dichotomic explanation for the dispersion of eccentricities and inclinations was further studied by He et al. (2019). These authors developed a framework for simulating planetary populations, and used it to fit a bimodal distribution to the Kepler planet population, taking into account a large set of observed quantities. In a following work (He et al. 2020), this dichotomic explanation for the excess of singly transiting planets has been revised by suggesting a non-dichotomous model based on angular momentum deficit (Laskar & Petit 2017). The AMD-stable model provides a more natural and physically-based theory than the dichotomic model. Millholland et al. (2021) performed a statistical analysis of the number of observed transit duration variations (TDVs; Shahaf et al. 2021) to show that a non-dichotomous model based on angular momentum deficit (Laskar & Petit 2017) is statistically preferred over a dichotomous model. Another point of view of the mutual inclinations within planetary systems was addressed by Masuda et al. (2020), who studied the mutual inclinations between inner rocky planets and outer giants by the occurrence rate of transits of cold Jupiters and their general abundance based on RV data; they find that systems with multiply transiting inner rocky planets are expected to posses a lower mutual inclination with their outer giants than singly transiting inner rocky planets.

These examples show how statistical methods can infer the general dispersion of mutual inclinations; however, constraining the mutual inclination of specific systems has been done only for a handful of cases (e.g., Mills & Fabrycky 2017). It is of high interest to pin-point systems that are likely to possess mutual inclinations, as this will enable the focusing of observational efforts. A sign of mutual inclinations is the existence of TDVs, or impact parameter variations (TbVs), since for low-eccentricity orbits $b$ and $D$ are more sensitive to nodal precession than to apsidal precession. In order to detect such signals in the data, which are typically weak and slow, a global light-curve model that contains both TTVs and TbVs should be utilized. A model that jointly fits TTVs and TbVs is a key to probing the existence of mutual inclinations in individual planetary systems.

Here, we present AnalyticLC, an analytic tool that accurately and efficiently calculates a full light-curve model, based on planetary physical and orbital parameters. The basic characteristics of the tool that guided its development are: (i) full light-curve modeling, exploiting the photometric data (as opposed to just transit times), (ii) computational speed to enable multidimensional inversions, (iii) an analytical treatment that sheds light on the dynamical processes manifested in the data, and (iv) applicability to multiple data types, including photometry, RV, and astrometry.

This paper is organized as follows. In Section 2 we describe the mathematical model of a single transit event, and link it to the instantaneous orbital elements. In addition, we present analytic formulae for TTVs that arise from the simultaneous interactions of three planets on a near-resonant chain with each other, a frequent scenario in the Kepler population. In Section 3 we test the accuracy limits of the model against full N-body integrations for two and three planet systems. We conclude in Section 4. In Appendix A we give the full mathematical derivation of the orbital elements, and in Appendix B we describe some code technicalities that enhance the calculation speed.

2. The Model

2.1. Overview

The model is based on constructing an approximate solution of Lagrange’s planetary equations of motion, and then translating the orbital element values to transit properties. Once the properties of each individual transit are known, the instantaneous flux is calculated using the Mandel–Agol formula (Mandel & Agol 2002). First, the disturbing function is expanded to the desired order in $e$ and $I$ (up to fourth order in AnalyticLC), and Lagrange’s planetary equations of motion are written (Murray & Dermott 1999). The secular terms (i.e., terms that do not depend on mean longitudes) are isolated, and the equations are solved for them analytically by matrix inversion (Murray & Dermott 1999, Chapter 7). This yields the so-called free eccentricity (Lithwick et al. 2012) and the corresponding “free inclination.” We note that the division into “free” and “forced” as used in this work follows the definition of Lithwick et al. (2012), not to be confused with the definitions of Murray & Dermott (1999) who used the terms “free” and “forced” to
describe two timescales in the motion of a test particle experiencing secular perturbations. Our treatment of the free elements differs from the one presented by Lithwick et al. (2012) in the sense that we allow secular motion, and hence the free values change in time, while in their method the free values are assumed to be constant in time. After solving for the secular motion of the free values, the equations are written for the non-secular terms only (these include both synodic terms and resonant terms), and they are solved by assuming that their right-hand sides are time dependent only through the mean longitudes, which vary on an orbital timescale. The other elements are assumed to be equal to their free values for the sake of integrating these equations. This technique is similar to the one applied by Hadden & Lithwick (2016), who used the variations in the orbital elements to construct analytic expressions for TTVs to second order in eccentricity under the assumption of coplanarity. Similar to their approach, here we expand on the TTV analysis by calculating the variations in all orbital elements and translating them to transit properties of each individual event, in addition to incorporating secular effects that give rise to slow, gradual variations in the transit shape.

Throughout the derivation we use astrocentric coordinates, as these are natural for describing a transit event; only for the computation of the RV/Astrometry model do we transform to barycentric coordinates. In Appendix A we give the detailed analytic derivation. The general steps of the AnalyticLC model are summarized in Table 1.

2.2. Orbital Geometry to Transit Properties

Let us begin by expressing the transit properties using the orbital elements. For a circular orbit with an orbital period $P$ an orbital radius $a$, a normalized impact parameter $b$, and mean motion $n = 2\pi/P$, the sky-plane coordinates around the transit are given by

$$y = \frac{a}{R_*} \sin \phi$$

and

$$z = b \cos \phi,$$

where the origin is at the center of the star, $x$ points to the observer, $y$ and $z$ form the plane of the sky, the phase is given by $\phi = n(t - t_{\text{mid}})$, $R_*$ is the stellar radius, $t$ is the time, and $t_{\text{mid}}$ is the time of mid-transit.

Inverting the relation between sky position and time yields the transit duration $T$ (time between the two points at which the distance between the centers of the planet and star is 1 stellar radius) and the ingress/egress time $\tau$, as follows:

$$T = 2 \frac{\pi}{n} \arcsin \sqrt{\frac{1 - b^2}{a^2 - b^2}}$$

and

$$\tau = \frac{1}{n} \left[ \arcsin \sqrt{\frac{(1 + r)^2 - b^2}{a^2 - b^2}} - \arcsin \sqrt{\frac{(1 - r)^2 - b^2}{a^2 - b^2}} \right].$$

Similar analytic expressions for the circular orbit case were given by Seager & Mallén-Ornelas (2003).

For an elliptic orbit the value of $a$ is replaced by the planet-star separation at mid-transit, and $n$ is replaced by the angular velocity at mid-transit, respectively:

$$d = \frac{a(1 - e^2)}{1 + e \cos f}$$

and

$$\dot{d} = n \frac{(1 + e \cos f)^2}{(1 - e^2)^{3/2}},$$

where $e$ is the eccentricity and $f$ is the true anomaly at mid-transit. The impact parameter for an elliptic orbit is given by

$$b = \frac{a}{R_*}(1 - e^2) \cos i,$$

where $i$ is the inclination with respect to the plane of the sky.

Assuming that along the transit one can treat the trajectory as a small arc of a circular path with a constant angular velocity (justified provided that $T \ll P$), one can calculate the star-planet sky-projected separation at any given time around mid-transit. The light curve is obtained from this distance using the Mandel–Agol formula (Mandel & Agol 2002).

In order to conveniently calculate the transit timing variations, we work in an axis system where the $x$-axis points to the observer, similar to previous works, e.g., Lithwick et al. (2012) and Hadden & Lithwick (2016). In this case, the mid-transit true longitude is 0, which allows for calculating the mid-transit times conveniently. The $y$-axis is defined such that the longitude of ascending node, $\Omega$, of the innermost planet, lies in the positive direction of $y$. The sky is the plane $yz$. The orbital inclination $I$ is the angle between the orbit normal and the $z$-axis. Under these conventions, the true anomaly at mid-transit is $f = -\varpi$, and hence $\cos f$ in the expressions above is replaced by $\cos \varpi$. The inclination of the orbit normal with respect to the line of sight $i$ is a projection of the angle $I$ between the $z$-axis and the orbit normal, specified in azimuth by $\Omega$ (the longitude of ascending node). Thus,

$$\cos i = \sin I \sin \Omega.$$  

For a transit to occur, $i$ should be close to $90^\circ$.

The transformations above enable translating the instantaneous orbital elements at mid-transit to a full description of the transit shape as a function of time. The orbital geometry is illustrated in Figure 1.

2.3. Orbital Elements Variations

The formulae above show how a set of the osculating elements $a$, $e$, $\varpi$, $I$, and $\Omega$ can fully describe the shape of any single transit under the assumption that they all vary slowly enough, relative to the transit duration, such that they are locally constant. The sixth orbital element, the mean longitude $\lambda = M + \varpi$ (where $M$ is the mean anomaly), affects the transit time, but not its shape. The problem of generating a full light-curve model is then reduced to calculating the orbital elements as a function of time, on timescales longer than $T$. We approximate them by using the disturbing function formalism. This has been done for the calculation of TTVs via the small variations in $\lambda$ and $z = e \exp(i\varpi)$ to second order in eccentricities by Hadden & Lithwick (2016), under the assumption of coplanarity. We extend this method to calculating the variations of all orbital elements to fourth power in
| #  | Step                                          | Technique                                      | Assumptions                                                                 | Outputs                                      |
|----|----------------------------------------------|-----------------------------------------------|----------------------------------------------------------------------------|----------------------------------------------|
| 1  | Secular orbital motion                       | matrix inversion                              | maintaining secular terms up to second order in $e, I$                      | motion of free eccentricities and inclinations |
| 2  | Near-MMR orbital motion                      | analytic approximate integration of Lagrange’s\ equations of motion | maintaining resonant terms up to fourth order in $e, I$; slow motion of all\ elements except mean longitudes | near-resonant motion of all elements          |
| 3  | Individual transit properties (Section 2.2)   | geometric calculation                         | along the transit, the local shape of the orbit is approximated by a circula\ r arc | individual transit parameters                |
| 4  | Light-curve generation                       | Mandel–Agol model; binning                    | Mandel–Agol model with two limb-darkening parameters                      | full light curve                              |

**Note.** The first two steps relate to the system dynamics; the third step translates the orbital motion to transit parameters; the fourth step computes a full light curve. RV and astrometry values are also calculated for any required time.
eccentricities and inclinations and in 3D. We summarize the method here; the full derivation is given in Appendix A.

The potential energy of the gravitational interactions among the planets can be described as a series sum over different frequencies. The secular terms, which do not depend on the mean longitudes, cause a drift in the eccentricity and inclination vectors—a slow motion with a frequency proportional to the orbital frequency of the perturbed planet and to the mass ratio of the perturber and the host star. Superimposed on this slow secular drift, the eccentricity and inclination vectors oscillate with various frequencies, including the strongest component at the inverse super-period, arising from the nearest first-order MMR (Lithwick et al. 2012). The full motion can be divided into near-resonant frequencies (as will be explained below, we do not treat the case of resonance locking) and synodic frequencies, which cause the so-called “chopping” effect (Deck & Agol 2015). The flow of AnalyticLC is composed of four steps, and incorporates both secular and near-MMR dynamics, described in Table 1.

An underlying important assumption in the approximate solution of Lagrange’s equations is that the conjunction longitude circulates rather than librates; in other words, the system is not locked in resonance. The out-of-resonance configuration is the overwhelmingly common case for the observed Kepler population. This observational finding has been explained by dissipation processes, which tend to push resonance-locked systems wide of resonance (e.g., Lithwick & Wu 2012; Millholland & Laughlin 2019).

2.4. Three Planet Interactions

Past works that analytically calculated the TTVs treated them as perturbations to first order in the perturber mass. In such a description, the perturbations are additive, and hence the total TTV that a planet experiences is the sum of TTVs applied by the different individual perturbers. In Appendix A we show that for a triplet of planets that constructs a chain of two near-first-order MMRs, new TTV patterns can arise that are not negligible in comparison to the first-order effect. In fact, if the two super-periods of the near-resonant-chain are similar, then this additional TTV amplitude can reach the amplitude of the individual near-first-order MMR TTVs, which were introduced by Lithwick et al. (2012). The additional TTV pattern is of second order in the planetary masses, but it also scales inversely with the sum or difference between the super-mean motions. In other words, if the super-periods match, they resonate themselves. We refer to this phenomenon as super-mean-motion resonance (SMMR), because it involves a resonance between the super-mean motions. Explicitly, if the inner planets are near a \( j:j−1 \) resonance, and the outer pair is

Figure 1. The orbital geometry and orbital parameters of a transit event, here illustrated for \( e = 0.1 \). In both panels the dashed line is the orbit, and the red solid lines are the circular arc approximation. (a) The orbit projected on the sky plane, with impact parameter \( b \) and planetary angular velocity \( \dot{\theta} \). (b) The xy plane. The ratio \( d/R_\star \) is the star-planet separation at mid-transit, in stellar radii—due to the orbital inclination in this case, the line connecting the planet and star is not in the xy plane. The x-axis points to the observer, and y is along the direction of motion at mid-transit.

Figure 2. Relative departure of the three planet TTV magnitude (\( \sigma_{TTV} \)) from the pair-wise TTV magnitude (\( \sigma_{PWTTV} \)). This figure shows that for a near-SMMR (Section 2.4) configuration, the pair-wise TTV approximation breaks, and that Equation (14) gives a good prediction of the regions at which the additional TTV pattern should be detectable. Blue, orange, and yellow circles represent the relative deviation of the innermost, intermediate, and outer planet TTV magnitudes from the pair-wise calculation as a function of the outer pair super-period. The inner pair super-period is indicated by a black dashed line. Within the gray region this deviation is not observable in the given time span (Equation (14)).
near a \( k: k - 1 \) resonance, then the additional TTV is given by these expressions:

\[
\delta t^{(2-3)} = \frac{P^3}{2\pi} \left( \frac{m_i m'_j}{m_* + m'_j} \right) n'' n' = \alpha_{12} \alpha_{23} \\
\times \frac{j - 1}{n''} f_{27}(\alpha_{23}) \left( f_{31}(\alpha_{12}) - 2\alpha_{12} \delta_{ij} \right) \\
\times \left( \frac{\sin(\lambda^j + \lambda^k - 2\omega^j)}{(n^j + n^k)^2} - \frac{\sin(\lambda^j - \lambda^k)}{(n^j - n^k)^2} \right),
\]

(9)

\[
\delta t^{(n-3)} = \frac{P^3}{2\pi} \left( \frac{m_i m'_j}{m_* + m'_j} \right) n'' n' = \alpha_{12} \alpha_{23} \\
\times \left( f_{31}(\alpha_{12}) - \frac{\delta_{ij}}{2\alpha_{12}} \right) \\
\times \left( \frac{j}{n^k} + \frac{1 - k}{n^j} \right) \sin(\lambda^j + \lambda^k - 2\omega^j) \\
+ \left( \frac{j}{n^k} - \frac{1 - k}{n^j} \right) \sin(\lambda^j - \lambda^k) \\
\times \frac{(n^j + n^k)^2}{(n^j - n^k)^2},
\]

(10)

and

\[
\delta t^{(n-1)} = \frac{P^3}{2\pi} \left( \frac{m_i m'_j}{m_* + m'_j} \right) n'' n' = \alpha_{12} \alpha_{23} \\
\times \left( \frac{k}{n''} f_{27}(\alpha_{23}) \left( f_{31}(\alpha_{12}) - \frac{\delta_{ij}}{2\alpha_{12}} \right) \right) \\
\times \left( \frac{\sin(\lambda^j + \lambda^k - 2\omega^j)}{(n^j + n^k)^2} - \frac{\sin(\lambda^j - \lambda^k)}{(n^j - n^k)^2} \right),
\]

(11)

where \( m \) is the planetary mass, \( m_* \) is the stellar mass, \( P \) is the orbital period, \( n \) is the mean motion, \( \alpha_{ij} \) is the ratio between the semimajor axes (of planets \( i \) and \( j \)), and \( f_{27} \) and \( f_{31} \) are functions of the Laplace coefficients, given by Murray & Dermott (1999, Appendix 2B). The unprimed quantities refer to the innermost planet (1), the primed quantities refer to the intermediate planet (2), and the double-primed quantities refer to the outermost planet (3). The upper scripts represent the planets that generate the cross interaction and on which the additional TTV depends; for example, \( \delta t^{(2-3)} \) is proportional to the masses of planets 2 and 3. The frequencies \( n^i \) and \( n^k \) are the super-mean motions, related to the super-periods of the near-resonant interactions (Lithwick et al. 2012). They are given as a function of the orbital mean motions by

\[
n^j = jn' + (1 - j)n
\]

(12)

and

\[
n^k = kn'' + (1 - k)n'.
\]

(13)

The full derivation for this TTV pattern is given in Appendix A.

The frequency of the three planet interaction TTV is equal to the sum of, or difference between, the super-mean motions of the two pairs. If the super-periods are too close and this frequency is too small relative to the duration of the observation, this TTV pattern will not be seen. In other words, if the observation period is less than \( \approx 0.25 \) of the three planets Table 2

| Parameter | \( P \) (days) | \( a \) (au) | \( e \) | \( I \) (°) | \( m \) (m⊕) | \( R \) (R⊕) |
|-----------|----------------|-------------|-------|--------|-------------|-------------|
| Planet 1  | 11.551         | 0.1         | 0.014 | 1.416  | 6            | 2.4495      |
| Planet 2  | 17.683         | 0.1328      | 0.014 | 3.04   | 9            | 3           |

Note. The two planets are near the 3:2 MMR, with a period ratio of about 1.531, equivalent to a normalized distance from resonance of \( \approx 0.02 \).
Figure 4. Individual transit properties of the inner planet of the simulated two planet system, showing the ability of AnalyticLC to reproduce a model based on an N-body integration. For each property, the blue circles indicate the results obtained from an N-body integration, while the orange ×'s are the values obtained from AnalyticLC, with the residuals shown in small black dots. (a) TTV, (b) planet-to-star separation at mid-transit, (c) impact parameter, (d) angular velocity, (e) transit duration, and (f) ingress-egress time.

Figure 5. Model accuracy in terms of light-curve flux, RV, and astrometry. As in former Figures, results from an N-body integration are shown in blue, while results from AnalyticLC are shown in orange. Residuals are shown in black, with text indicating the standard deviation σ of the residuals, which are much smaller than the typical precision of Kepler photometry and of current RV instruments. (a) Folded light curve about times of mid-transit (including TTV) of the inner planet. (b) Folded light curve about times of mid-transit (including TTV) of the outer planet. (c) RV values. (d, e) Astrometry values in absolute units; for actual data the stellar absolute displacement y*, z* would be translated to angular displacement. Note the different scales for y* and z*; for both cases the residuals are two orders of magnitude smaller than the magnitude of the variations.
TTV period, the curvature of the sine would not be caught. For a given inner super-period $P_j$ and an observation duration $D$, the nondetection limits for the three planet TTV as a function of the outer pair super-period $P_k$ are given by the conditions

$$\frac{4DP_i}{4D + p_i} \lesssim P_k \lesssim \frac{4DP_i}{4D - p_i}$$

(14)

where the factor 4 in both of these equations comes from the requirement that the observation time would be 4 times larger than the period of the three planets effect; this is a heuristic requirement, as the exact factor would depend on the exact phase in which the observation is performed.

Between these two limits, the phenomenon would be small. This is illustrated in Figure 2. In this figure we show the results of a computational experiment involving a system of three planets in a near-MMR chain. The orbital periods of the inner pair are kept constant such that their super-period is ≈418 days, while the orbital period of the outer planet varies such that the super-period of the outer pair varies between 160 and 1200 days. For each orbital configuration, the total TTVs are calculated by a single three planet $N$-body integration, and for each planet the standard deviation of the arising TTV, $\sigma_{\text{TTV}}$, is calculated. The pair-wise TTVs are calculated using three two planet integrations, and the standard deviation of this pair-wise calculated TTV is denoted $\sigma_{\text{PWTTV}}$. The quantities $\sigma_{\text{TTV}}$ and $\sigma_{\text{PWTTV}}$ reflect, in one number per orbital configuration, the magnitude of the TTV for the simultaneous three planet interaction and for the pair-wise interaction. Throughout all integrations, we kept the free eccentricities and inclinations constant by performing a fine adjustment to the initial conditions. When the super-period of the outer pair approaches that of the inner, the TTVs of all three planets deviate from the pair-wise calculation. When the super-periods are close enough to each other such that the SMMR effect is of too low a frequency for the observation time to capture, the effect is not seen in the TTV (it is absorbed by an adjusted mean period). The theoretical region at which the SMMR-TTV should not be detected (Equation (14)) matches nicely the results of the numerical experiment.

3. Testing the Model

In this section, we present a few of the tests we performed on the model to check its validity and its accuracy limits.

3.1. Comparison with N-body—Two Planets

The first test presented here shows the comparison of the analytic model with an $N$-body integration of a two planet system near the first-order 3:2 MMR with small eccentricities and inclinations. The $N$-body integration was done using MERCURY6 (Chambers 1999), which gives as an output the full set of orbital elements as a function of time. The mid-transit times were estimated by using weighed Keplerian arcs for the two points bracketing the transit, as done in the code TTVFaster (Deck et al. 2014). In Table 2, the main parameters of the modeled system are presented. The integration time span is 1500 days with a rather oversampled time step of 0.5 hr, similar to Kepler’s long cadence (Borucki et al. 2010).
In Figure 3 we show the accuracy of AnalyticLC calculation for the eccentricity and inclination vectors motion for the inner planet by comparing them to their values from an \(N\)-body integration performed by MERCURY6 (Chambers 1999). In order to perform the comparison, we translated the osculating orbital elements at which the \(N\)-body integration was initiated to the free elements that are used by AnalyticLC by calculating the forced elements and subtracting them from the osculating ones, as described in Appendix A. The eccentricity vector variations are dominated by the forced motion due to the near-resonant interaction, while the slow variation of the inclination is a result of secular interactions. The residuals are about two orders of magnitude smaller than the amplitudes of the orbital elements variations.

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As discussed in Section 2.2, the instantaneous orbital elements values are translated to transit parameters. In Figure 4 we show the accuracy of the transit properties of the inner planet in terms of TTV, planet-star separation, impact parameter, angular velocity, transit duration, and ingress-egress time. The TTV shape is composed of a fundamental sine-like oscillation at the super-period frequency, superimposed by a saw-tooth-like “chopping” pattern, due to the gravitational impulses at conjunction (Deck & Agol 2015). The standard deviation of transit time errors is about 3 s, much better than the typical uncertainty on mid-transit times of Kepler data, and two orders of magnitude smaller than the TTV semi-amplitude. The accuracy in the other transit properties (planet-to-star separation, impact parameter, angular velocity, transit duration, and ingress-egress time) is shown as well. All these properties together yield AnalyticLC’s accuracy in flux, which in this case is of order 1 ppm (shown in Figure 5)—about two orders of magnitude better than the typical Kepler precision. The sinusoidal nature of eccentricity components results from the truncated near-resonant terms of the series expansion. The long-term deviation in the inclination vector motion results from the truncated secular terms of the series expansion. These parameters are then translatable to direct observables; as shown in the Figure, AnalyticLC provides good accuracy.

3.2. Comparison with \(N\)-body—Three Planets

As described above, modeling planet pairs to a high accuracy cannot be naïvely extended to triplets by summing just the TTVs of three pairs. Below we show an example chosen to illustrate this effect. We model a system of three and calculate the TTVs with numerical (full \(N\)-body integration) and analytical methods (AnalyticLC and TTVFaster; Agol & Deck 2016) to evaluate this inaccuracy. Figure 6 shows the arising TTV pattern of each planet, calculated by an \(N\)-body integration (black), by TTVFaster (blue), and by AnalyticLC (orange). It is visually clear that the analytic methods capture the TTVs of all planets, which are composed of both a super-period fundamental TTV (Lithwick et al. 2012) and a saw-tooth like “chopping” TTV (Deck & Agol 2015). We next turn to inspecting the inaccuracy in times of mid-transit. Although the large-scale TTVs were well captured by all techniques, upon comparing with the full \(N\)-body simulation, structured residuals appear, shown in Figure 7. The inaccuracy pattern of TTVFaster (blue) is similar to the inaccuracy...
pattern of AnalyticLC when used without the three planets term (yellow). In order to prove that these remaining residuals are a direct result of the three planet simultaneous interaction, and not a result of a missing term in the pair-wise calculation, we performed a set of pair-wise N-body integrations and summed the TTVs resulting from them (purple); the arising residuals are similar to those for the analytic methods. Applying AnalyticLC including the three planet SMMR terms corrects (some of) this residual TTV pattern (orange, highlighted). We note that residuals with a similar pattern to the correction itself remain unresolved.

4. Summary and Future Prospects

In this work we presented AnalyticLC, an analytic method and code implementation for light-curve, RV, and astrometry modeling. The development of the method was motivated by the benefit of full light-curve modeling over fitting times of mid-transit, which can be used to detect small amplitude TTVs (Ofir et al. 2018) and also can be used to detect TTVs, which are a key observable for the detection of forces out of the plane. The method assumes that the system is out of resonance, which is the common case in the Kepler population (e.g., Fabrycky et al. 2014). The calculation in resonant terms is up to fourth order in eccentricities and inclinations, sufficiently accurate for systems of up to mild eccentricities, also common in the Kepler population (Fabrycky et al. 2014). For computational efficiency, the calculation can be truncated at lower orders. Beyond the question of calculation speed, such an analytic approach provides insight into the motion of the orbital elements, which is difficult to obtain from numerical integration.

We show that three planet interactions are important in some cases, in addition to the pair-wise interactions in the systems, and we have made progress toward correctly modeling this effect. We found that these terms correctly predict the morphology of the needed correction, but the predicted amplitude of the correction sometimes underestimates the magnitude seen in comparison to full N-body integrations. Because the basic pattern of the residuals is captured by our triple-interaction expression, we are optimistic that including additional terms will further improve the accuracy of the theory.

The computation time of AnalyticLC scales with the number of calculated points—but not with the temporal baseline of the simulation. This advantage becomes increasingly important as data sets begin to span multiple decades (already common in RV studies, and soon in photometry as well).

In a companion paper, we use this method to fit a model to a subset of the Kepler systems. The method of AnalyticLC can be further used to analyze combined data sets of photometry, RV, and astrometry.

Our code is available at https://github.com/yair111/AnalyticLC along with all necessary files and with a user manual.

This study was supported by the Helen Kimmel Center for Planetary Sciences and the Minerva Center for Life Under Extreme Planetary Conditions #13599 at the Weizmann Institute of Science.

Appendix A Orbital Elements and Transit Times Derivation

A.1. Equations of Motion

The mathematical formalism for deriving the orbital elements is based on expanding the disturbing function in powers of e and I, and then approximately solving Lagrange’s planetary equations. The task of finding the times of mid-transit has already been addressed in a similar manner by Hadden & Lithwick (2016) and Agol & Deck (2016). The derivation of times of mid-transit as presented below follows the approach of Hadden & Lithwick (2016), adding more terms to the summation and elucidating the role of the different terms in the expansion. Other transit properties are obtained from the instantaneous orbital elements; the expressions are given in Section 1.

The disturbing function is given as a series sum of cosines, with prefactors depending on the ratio $\alpha = a/a'$ multiplied by different powers of the eccentricities and inclinations. Each cosine argument includes a linear combination of the mean longitudes, longitudes of periapse, and longitudes of ascending nodes:

$$ R = Gm' \sum_{j=-\infty}^{j=\infty} S(a, a', e, e', I, I') \cos \varphi, \quad (A1) $$

where $\varphi$ is a linear combination of the angles $\lambda, \varpi, \Omega, \lambda', \varpi', \Omega'$. A similar expression gives $R'$ for the case of an internal perturber. The full derivation of the disturbing function is given in Murray & Dermott (1999).

Lagrange’s planetary equations are (Murray & Dermott 1999):

$$ \frac{da}{dt} = \frac{2}{na^3 e} \frac{\partial R}{\partial \lambda}, \quad (A2) $$

$$ \frac{de}{dt} = -\sqrt{1 - e^2} \left(1 - \sqrt{1 - e^2}\right) \frac{\partial R}{\partial e} - \sqrt{1 - e^2} \frac{\partial R}{\partial \varpi}, \quad (A3) $$

$$ \frac{d\lambda}{dt} = n \left(1 - \frac{3}{2} \frac{\partial R}{\partial a} + \frac{\sqrt{1 - e^2}}{na^2 e} \frac{\partial R}{\partial e} \right) $$

$$ \times \left(1 - \sqrt{1 - e^2}\right) \frac{\partial R}{\partial e} + \tan(1/2) \frac{\partial R}{\partial \varpi}, \quad (A4) $$

$$ \frac{d\Omega}{dt} = \frac{1}{na^3 \sqrt{1 - e^2} \sin I} \frac{\partial R}{\partial I}, \quad (A5) $$

$$ \frac{d\varpi}{dt} = \frac{\sqrt{1 - e^2}}{na^3 \sqrt{1 - e^2} \sin I} \frac{\partial R}{\partial I} + \frac{\tan(1/2)}{na^3 \sqrt{1 - e^2} \sin I} \frac{\partial R}{\partial \varpi}, \quad (A6) $$

$$ \frac{dl}{dt} = -\tan(1/2) \left(\frac{\partial R}{\partial \lambda} + \frac{\partial R}{\partial \varpi}\right) $$

$$ - \frac{1}{na^3 \sqrt{1 - e^2} \sin I} \frac{\partial R}{\partial \Omega}. \quad (A7) $$

We solve them in two steps:

(i) Solving the secular motion of the eccentricity and inclination vectors (Murray & Dermott 1999, Chapter 7; using a second-order expansion of the disturbing function; the next order correction is fourth power).

(ii) Solving the near-resonant interactions by approximating that the right-hand side is time dependent only through the mean longitudes.

2 We offer a reward for the capture of the wanted terms.
A subtlety is noted here regarding the calculation of the derivatives with respect to the semimajor axes. For these derivatives, we assume that the perturber’s semimajor axis is held constant, and then we represent the derivative with respect to the planetary semimajor axis by the derivative with respect to the mean motion of the longitude of conjunction.

$$\frac{\partial}{\partial \Delta} = \frac{1}{a^2} \frac{\partial}{\partial \Delta} \cos \left( \frac{1}{a^2} \Delta \right)$$

This enables the calculation of the derivatives of the Laplace coefficients, which are functions of \( \Delta \).

\section{A.2. General Approximate Solution}

In this section we describe the second step of the method: solution of Lagrange’s planetary equations (the first step, calculating the secular motion, is described in Murray & Dermott 1999, Chapter 7). We make use of the fact that the equations are linear in \( \mathcal{R} \), and hence sum the effects of individual terms in the general form. Given the orbital elements of the inner companion \( a, e, \varpi, I, \Omega, \) and \( \lambda \) and their counterparts denoted with a prime corresponding to the outer companion, we define the following quantities:

$$C_{jk} = e^A e^{\alpha' \Delta} s^B s' \cos(j' \lambda')$$

and

$$S_{jk} = e^A e^{\alpha' \Delta} s^B s' \sin(j' \lambda')$$

with \( s = \sin(I/2) \) and \( s' = \sin(I'/2) \); (Murray & Dermott 1999). The disturbing function of the inner planet, then, can be given by

$$\mathcal{R}_{jk} = \frac{Gm'}{a'} (f + f_E) C_{jk},$$

where \( f \) and \( f_E \) are functions that depend on \( \alpha = a/a' \) and on the specific values of \( j \) and \( k \), but not on the orbital elements. The separation into two parts comes from the distinction between the direct part and the indirect part of the disturbing function for an external perturber that exists for specific cosine arguments.

The disturbing function for the outer planet is given in a similar form:

$$\mathcal{R}'_{jk} = \frac{Gm}{a} (f + f_j) C_{jk} = \frac{Gm}{a} (\alpha f + \alpha f_j) C_{jk}.$$ (A11)

The factor \( f_j \) contains the additional terms that sometimes exist, which arise from the indirect part for an internal perturber.

If we assume that \( C_{jk} \) and \( S_{jk} \) depend on time only through the mean longitudes, we can establish the following relations:

$$\int C_{jk} \, dt = \frac{1}{n_{jk}} S_{jk} \quad \text{(A12)}$$

and

$$\int S_{jk} \, dt = -\frac{1}{n_{jk}} C_{jk}, \quad \text{(A13)}$$

where

$$n_{jk} = n' + (k - j)n$$

is the mean motion of the longitude of conjunction.

We note here again, as mentioned previously, that one must assume that the longitudes of conjunction circulate rather than librate for this calculation to hold.

Next, we express the derivatives of the single disturbing function term using \( C_{jk} \) and \( S_{jk} \):

$$\frac{\partial \mathcal{R}_{jk}}{\partial \lambda} = (j - k) S_{jk}, \quad \text{(A15)}$$

$$\frac{\partial \mathcal{R}_{jk}}{\partial e} = \frac{A}{e} C_{jk}, \quad \text{(A16)}$$

$$\frac{\partial \mathcal{R}_{jk}}{\partial \varpi} = C S_{jk}, \quad \text{(A17)}$$

$$\frac{\partial \mathcal{R}_{jk}}{\partial \omega} = \frac{B \cot(I/2)}{2} C_{jk}, \quad \text{(A18)}$$

and

$$\frac{\partial \mathcal{R}_{jk}}{\partial \Omega} = D S_{jk}, \quad \text{(A19)}$$

and

$$\frac{\partial \mathcal{R}_{jk}}{\partial a} = -\frac{1}{a'} \frac{\partial}{\partial a} (f + f_E) C_{jk}, \quad \text{(A20)}$$

where for the equations for \( \mathcal{R}_{jk} \) and \( \mathcal{R}'_{jk} \) we used the chain rule, as follows:

$$\frac{\partial \mathcal{R}_{jk}}{\partial I} = \frac{\partial \mathcal{R}_{jk}}{\partial s} \frac{\partial s}{\partial I} \quad \text{(A21)}$$

and

$$\frac{\partial \mathcal{R}_{jk}}{\partial a} = \frac{\partial \mathcal{R}_{jk}}{\partial \alpha} \frac{\partial \alpha}{\partial a} \quad \text{(A22)}$$

Now, we substitute these derivatives into Lagrange’s planetary equations and integrate them in time using Equations (A12) and (A13). This yields the variations in the orbital elements. In order to put the equations in a dimensionless form, we make use of Kepler’s law and define the relative masses

$$\mu = \frac{m}{m_*} \quad \text{(A23)}$$

and

$$\mu' = \frac{m'}{m_*} \quad \text{(A24)}$$

where \( m_* \) is the stellar mass.

The obtained expressions are:

$$\frac{\delta a}{a} = 2 \frac{\mu'}{1 + \mu} (f + f')(k - j) \frac{n}{n_{jk}} C_{jk}, \quad \text{(A25)}$$
\[ \delta \lambda = 3 \frac{\mu'}{1 + \mu'} (f + f_E)(j - k) \frac{n}{n_j} S_{jk} \]
\[- 2 \frac{\mu'}{1 + \mu'} \alpha (f + f_E) \frac{\partial f}{\partial \alpha} + \frac{\partial f_E}{\partial \alpha} \frac{n}{n_j} S_{jk} \]
\[+ \frac{\mu'}{1 + \mu'} (f + f_E) A \frac{\sqrt{1 - e^2} (1 - \sqrt{1 - e^2})}{e^2} n S_{jk} \]
\[+ \frac{\mu'}{1 + \mu'} (f + f_E) B \frac{1}{2} \frac{1}{\sqrt{1 - e^2}} n S_{jk} \]
\[= \delta e = \frac{\mu'}{1 + \mu'} (f + f_E)(j - k) \frac{n}{n_j} C_{jk} \]
\[A \frac{\sqrt{1 - e^2} (1 - \sqrt{1 - e^2})}{e^2} n S_{jk} \]
\[B \frac{1}{2} \frac{1}{\sqrt{1 - e^2}} n S_{jk} \]
\[= \frac{\mu'}{1 + \mu'} \alpha (f + f_E) A \frac{\sqrt{1 - e^2} (1 - \sqrt{1 - e^2})}{e^2} n S_{jk} \]
\[A \frac{\sqrt{1 - e^2} (1 - \sqrt{1 - e^2})}{e^2} n S_{jk} \]
\[B \frac{1}{2} \frac{1}{\sqrt{1 - e^2}} n S_{jk} \]

A similar derivation for the external companion yields:
\[ \delta e = \frac{\mu'}{1 + \mu'} (f + f_E)(j - k) \frac{n}{n_j} C_{jk} \]
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Taking the differential with respect to the variables $\lambda, z$ yields

$$\delta(e^k \sin(kM)) = kR\left(e^{ik\lambda}\left(\frac{z^k e^k \delta z + \frac{z^k}{i} \delta z^*}{i}\right)\right). \quad (A42)$$

There are two terms in the expression for $\theta$ for which the power of $e$ does not match the prefactor of $M$; for expressing their differential we make use of the identity $e^\theta = e^{\theta^*}$ and get:

$$\delta(e^3 \sin M) = (z^* \delta z + z \delta z^*)R\left(e^{i\lambda\theta}\left(\frac{e^{i\lambda\theta}}{i}\right)\right)$$

$$+ az \delta \delta z\left(e^{i\lambda\theta}\left(\frac{e^{i\lambda\theta}}{i}\right)\right). \quad (A43)$$

and

$$\delta(e^4 \sin 2M) = (z^* \delta z + z \delta z^*)R\left(e^{2i\lambda\theta}\left(\frac{(e^{2i\lambda\theta})^2}{i}\right)\right)$$

$$+ 2z \delta z^*\left(e^{2i\lambda\theta}\left(\frac{z^* \delta z + \frac{z^*}{i} \delta z^*}{i}\right)\right). \quad (A44)$$

Summing all of these differentials yields an expression for $\delta \theta$ as a function of the complex eccentricity and the mean longitudes:

$$\delta \theta = \delta \lambda + 2R\left(e^{i\lambda}\left(\frac{z^* \delta z + \frac{z^*}{i} \delta z^*}{i}\right)\right)$$

$$+ 5 \frac{2R}{4}\left(e^{2i\lambda}\left(\frac{z^* \delta z + \frac{z^*}{i} \delta z^*}{i}\right)\right)$$

$$+ 12 \frac{3R}{4}\left(e^{3i\lambda}\left(\frac{z^* \delta z + \frac{z^*}{i} \delta z^*}{i}\right)\right)$$

$$- \frac{1}{4}\left(z^* \delta z + z \delta z^*\right)R\left(e^{i\lambda\theta}\left(\frac{e^{i\lambda\theta}}{i}\right)\right)$$

$$- \left(z^* \delta z + z \delta z^*\right)\left(e^{i\lambda\theta}\left(\frac{e^{i\lambda\theta}}{i}\right)\right)$$

$$- 13\left(\frac{3R}{96}\left(e^{i\lambda\theta}\left(\frac{z^* \delta z + z \delta z^*}{i}\right)\right)\right)$$

$$- 11\left(\frac{24}{24} \left(e^{i\lambda\theta}\left(\frac{z^* \delta z + z \delta z^*}{i}\right)\right)\right)$$

$$+ 2z \delta z^*\left(e^{2i\lambda\theta}\left(\frac{z^* \delta z + \frac{z^*}{i} \delta z^*}{i}\right)\right). \quad (A45)$$

In fact, for the sake of calculating the TTV, we are interested only in this expression for $\theta = 0$ (and $\lambda \approx 0$).

Relating $\delta \theta$ to time is done from the expression for the angular velocity:

$$\dot{\theta} = n \frac{(1 + e \cos f)^2}{(1 - e^2)^{3/2}}. \quad (A46)$$

If we are interested only in the values for $\theta = 0$ (i.e., when transit occurs) we get $f = -\omega$ and hence

$$\dot{\theta}(\theta = 0) = n \frac{(1 + e \cos \omega)^2}{(1 - e^2)^{3/2}}. \quad (A47)$$

The TTV, denoted by $\delta t$, is the translation of the angular shift to temporal shift. A larger longitude implies an earlier transit and a negative $\delta t$. The explicit expression for the TTV is hence

$$\delta t = -\delta \theta \frac{P (1 - e^2)^{3/2}}{2\pi (1 + \Omega(z))^2}, \quad (A48)$$

with $\delta \theta$ defined above.

A.4. TTVs of Three Planets in a Near-resonant Chain

In this section we derive TTV patterns that are second order in mass that arise from the cross interaction between two pairs. These TTV patterns make the total TTV deviate from the sum of the pair-wise interactions. We analyze a system of three coplanar planets, assuming that the innermost planet and the intermediate planet are near the $j:j - 1$ MMR, and that the intermediate planet and the outermost planet are near the $k:k - 1$ MMR. We neglect the interaction between the innermost and outermost planet to first order in eccentricity, assuming that they are not near any first-order MMR—this is justified if the system is not too packed. We use the same variables as described above, but here we also add the double prime to account for the variables relating to the outermost planet. We refer here only to the terms in the disturbing function related to the $j:j - 1$ and $k:k - 1$ resonances (as in Lithwick et al. 2012). The averaged disturbing function in this case is:

$$\langle \mathcal{R} \rangle = \frac{Gm'}{a'}\left(f_{27}(\alpha_{12}) e \cos(j \lambda' + (1 - j) \lambda - \omega)\right)$$

$$+ \left(f_{31}(\alpha_{12}) - 2\alpha_{12} \delta_{12}\right) e^i \cos(j \lambda' + (1 - j) \lambda - \omega'), \quad (A49)$$

$$\langle \mathcal{R}' \rangle = \frac{Gm'}{a'}\left(f_{27}(\alpha_{12}) e \cos(j \lambda' + (1 - j) \lambda - \omega)\right)$$

$$+ \left(f_{31}(\alpha_{12}) - \frac{\delta_{12}}{2\alpha_{12}}\right) e^i \cos(j \lambda' + (1 - j) \lambda - \omega')$$

$$+ \frac{Gm''}{a''}\left(f_{27}(\alpha_{23}) e \cos(k \lambda'' + (1 - k) \lambda' - \omega)\right)$$

$$+ \left(f_{31}(\alpha_{23}) - 2\alpha_{23} \delta_{23}\right) e^i \cos(k \lambda'' + (1 - k) \lambda' - \omega''), \quad (A50)$$

and

$$\langle \mathcal{R}'' \rangle = \frac{Gm'}{a'}\left(f_{27}(\alpha_{23}) e \cos(k \lambda'' + (1 - k) \lambda' - \omega)\right)$$

$$+ \left(f_{31}(\alpha_{23}) - \frac{\delta_{23}}{2\alpha_{23}}\right) e^i \cos(k \lambda'' + (1 - k) \lambda' - \omega''). \quad (A51)$$

Solving these equations for the motion of the eccentricity vector gives the sum of the motions for each pair interaction, as described by Lithwick et al. (2012):

$$\delta z = \delta z^{(\text{free})} + \frac{Gm'}{na'a'n'f_{27}(\alpha_{12}) e^i \lambda'}, \quad (A52)$$
\[ z' = z^{(\text{free})} + \frac{Gm}{n'd^3n^3} \left( f_{s1}(\alpha_{12}) - \frac{\delta_{22}}{2\alpha_{22}} \right) e^{\lambda t} + \frac{Gm''}{n'd'^3d^3} f_{s27}(\alpha_{23}) e^{\lambda t}, \]  
(A53)

and

\[ z'' = z^{(\text{free})} + \frac{Gm'}{n'd'^3d'^3} \left( f_{s1}(\alpha_{23}) - \frac{\delta_{22}}{2\alpha_{23}} \right) e^{\lambda t}. \]  
(A54)

Heuristically, each first-order near-MMR interaction creates a forced eccentricity that draws a circle about the free eccentricity in the eccentricity plane, as previously described by Lithwick et al. (2012). We also derive the TTV in a similar manner: translating the variations in the semimajor axis to variations in the mean motion. The variations in the semimajor axis are given by Equation (A2), with analogous equations for the evolution of \( a', a''. \)

Deriving the disturbing functions with respect to the mean longitudes yields

\[ \frac{\partial R}{\partial \lambda} = \frac{Gm'}{a'} \left( j - 1 \right) f_{s27}(\alpha_{12}) e \sin(\lambda - \varpi) + \left( f_{s1}(\alpha_{12}) - 2\delta_{22} \right) e' \sin(\lambda - \varpi'), \]  
(A55)

\[ \frac{\partial R'}{\partial \lambda'} = -\frac{Gm'}{a'} f_{s27}(\alpha_{12}) e \sin(\lambda - \varpi) + \left( f_{s1}(\alpha_{12}) - \frac{\delta_{22}}{2\alpha_{22}} \right) e' \sin(\lambda - \varpi'), \]  
(A56)

where \( \lambda' = \lambda' + (1 - j)\lambda \) and \( \lambda'' = \lambda'' + (1 - k)\lambda', \) following the notations of Lithwick et al. (2012).

The variations in the mean longitudes are given approximately by

\[ \frac{d\lambda}{dt} \approx n \left( 1 - \frac{3}{2} \delta a \right), \]  
(A58)

where \( \delta a \) is the variation in \( a \) about its mean value; other terms also appear, but they would have a larger denominator as will be evident soon and hence we neglect them here. Hence, to obtain the variations in \( \lambda \) that cause the second-order-in-mass TTVs, we integrate to get

\[ \delta \lambda \approx -\frac{3}{2} \int \delta a \, dt, \]  
(A59)

where \( \delta a \) is given by Equation (A25).

This expression is written as an indefinite integral without the two constants of integration, because they are absorbed into the definition of the linear ephemeris parameters (orbital period and reference time).

The derivation for \( \delta \lambda', \delta \lambda'' \) is analogous.

Let us now inspect the expression for \( \partial R / \partial \lambda \). If we treat \( e, e' \) as constant in time, we simply obtain the sum of two near-MMR interactions of Lithwick et al. (2012). However, we are interested in the cross interaction between the two pairs; plugging in the solution of \( z' \) using \( e' = z' e^{-\varpi} \), we get

\[ \frac{\partial R}{\partial \lambda} = \frac{Gm'}{a'} (j - 1) f_{s27}(\alpha_{12}) e \sin(\lambda - \varpi) + \left( f_{s1}(\alpha_{12}) - 2\delta_{22} \right) e' \sin(\lambda - \varpi'), \]  
(A60)

We take the real part of both sides and use the angle-sum identity to integrate this expression in time, yielding a solution for \( \delta \lambda \). The terms discussed here are second order in mass, and hence will be significant only if they attain a small denominator due to the integration in time. Therefore, we treat only the second-order denominator arising from the double integration and neglect other terms in the equation for \( d\lambda / dt \).

Finally, the integration yields the cross term, for which we use the superscript (2-3) to emphasize that this is the cross term arising from the \( m'm'' \) term in mass (the interaction between planets 2 and 3):

\[ \delta \lambda^{(2-3)} = -\frac{3}{2} \frac{Gm'm''}{n'^2d'^3d''^3} b \left( \frac{\sin(\lambda' + \lambda - 2\varpi')}{(n' + n'')^2} - \frac{\sin(\lambda' - \lambda')}{(n' - n'')^2} \right), \]  
(A61)

Using \( \delta t \approx -P \delta \lambda / 2\pi \), and using Kepler’s law \( n^2a^3 = G(m + a), \) \( n'^2a'^3 = G(m + a') \), we get

\[ \delta t^{(2-3)} = \frac{P}{2\pi} \left( \frac{m'm''}{2n'^2a'^3n''^2d''^3} \left( \frac{1}{n'^2} - \frac{1}{n''^2} \right) f_{s27}(\alpha_{12}) - 2\delta_{22} \right) \times \left( \frac{\sin(\lambda' + \lambda - 2\varpi')}{(n' + n'')^2} - \frac{\sin(\lambda' - \lambda')}{(n' - n'')^2} \right). \]  
(A62)

In a similar manner, we obtain

\[ \delta t^{(1-3)} = \frac{P'}{2\pi} \left( \frac{mm''}{2n^2(m + m')^2} \right) n'^3a'^3 f_{s27}(\alpha_{23}) \times \left( f_{s1}(\alpha_{12}) - \frac{\delta_{22}}{2\alpha_{12}} \right) \times \left( \frac{j}{n'^k} + \frac{1 - k}{n'^l} \right) \sin(\lambda' + \lambda - 2\varpi') \times \left( \frac{j}{n''^k} + \frac{1 - k}{n''^l} \right) \sin(\lambda' - \lambda'), \]  
(A63)
and
\[
\delta_{f}^{m(1-2)} = \frac{P^n}{2\pi} \frac{3}{2} \frac{mn'}{(m_{*} + m)(m_{*} + m'^{*})} \\
\times n'^{-2} f_{n'} f_{77} (\alpha \theta_{23}) f_{31} (\alpha_{12}) - \frac{\delta_{f}^{2}}{2\alpha_{12}} \\
\times \left( \frac{\sin(\lambda' + \lambda - 2\omega')}{(n' + n)^2} - \frac{\sin(\lambda' - \lambda)}{(n' - n)^2} \right)
\]
(A64)

Appendix B
Implementation Considerations

The code was implemented in MATLAB, with modular structure and embedded documentation. It takes advantage of vectorized operations where practical.

For computational efficiency, two supporting mechanisms were implemented. The first is a precalculated table of the Laplace coefficients and their derivatives, which avoids the need for a direct calculation of the integral at each function evaluation. The table is constructed once and the values are used many times during the calculation.

The second mechanism is a look-up table for the Mandel – Agol function values (Mandel & Agol 2002). The Mandel – Agol model requires four inputs: planet-to-star radii ratio, sky-projected distance in stellar radii, and two limb-darkening coefficients. Since, in practice, the limb-darkening coefficients are given, we construct a look-up table for a grid of planet radius and separation values and perform linear interpolation within the grid. For a grid of 100 planetary radius values and 500 planet-to-star distance values, the typical accuracy cost in the relative flux is of order 10^{-7}, about three orders of magnitude smaller than Kepler’s typical long-cadence error.

We performed timing tests of the code by running a large number of simulations. The calculation time depends on various parameters, such as the number of transits, the number of planets, the order of the expansion, etc. To get an idea of the code’s performance, we compared the running time of AnalyticLC to the running time of TTVFast for a two planet system with orbital periods of 11.5 and 17.7 days, a typical Kepler system. Compared to TTVFast, the typical running time of AnalyticLC was roughly five times faster when using the first- or second-order expansions. The gain in speed for the third-order calculation was a factor of two, and the fourth-order calculation did not yield a speed-up. The reason for this trend is that the number of disturbing function terms is not uniform among all orders. Therefore, in terms of efficiency, a significant gain relative to the state-of-the-art N-body integrator TTVFast is obtained when using the first- or second-order calculation.

References

Agol, E., & Deck, K. 2016, ApJ, 818, 177
Agol, E., Steffen, J., Sari, R., & Clarkson, W. 2005, MNRAS, 359, 567
Borucki, W. J., Koch, D., Basri, G., et al. 2010, Sci, 327, 977
Chambers, J. E. 1999, MNRAS, 304, 793
Deck, K. M., & Agol, E. 2015, ApJ, 802, 116
Deck, K. M., & Agol, E. 2016, ApJ, 821, 96
Deck, K. M., Agol, E., Holman, M. J., & Nesvorný, D. 2014, ApJ, 787, 132
Fabrycky, D. C., Lissauer, J. J., Ragozzine, D., et al. 2014, ApJ, 790, 146
Freudenthal, J., von Essen, C., Dreizler, S., et al. 2018, A&A, 618, A41
Grimm, S. L., Demory, B.-O., Guillon, M., et al. 2018, A&A, 613, A68
Hadden, S., & Lithwick, Y. 2016, ApJ, 828, 44
Hadden, S., & Lithwick, Y. 2017, AJ, 154, 5
He, Matthias Y., Ford, Eric B., & Ragozzine, Darin 2019, MNRAS, 490, 4575
He, Matthias Y., Ford, Eric B., Ragozzine, Darin, & Carrera, Daniel 2020, AJ, 160, 276
Holczer, T., Mazeh, T., Nachmani, G., et al. 2016, ApJS, 225, 9
Jontof-Hutter, D., Wagner, A., Ford, E. B., et al. 2021, AJ, 161, 246
Laskar, J., & Petit, A. C. 2017, A&A, 605, A72
Latham, D. W., Mazeh, T., Stefanik, R. P., Mayor, M., & Burki, G. 1989, Natur, 339, 38
Linial, I., Gilbaum, S., & Sari, R. 2018, ApJ, 860, 16
Lithwick, Y., & Wu, Y. 2012, ApJ, 756, L11
Lithwick, Y., Xie, J., & Wu, Y. 2012, ApJ, 761, 122
Mandel, K., & Agol, E. 2002, ApJ, 580, L171
Masuda, K., Winn, J. N., & Kawahara, H. 2020, AJ, 159, 38
Mayor, M., & Queloz, D. 1995, Natur, 378, 355
Meschiari, S., & Laughlin, G. P. 2010, ApJ, 718, 543
Meschiari, S., Wolf, A. S., Rivera, E. R., et al. 2009, PASP, 121, 1016
Milholland, S., & Laughlin, G. 2019, NatAs, 3, 424
Milholland, S. C., He, M. Y., Ford, E. B., et al. 2021, AJ, 162, 166
Mills, S. M., & Fabrycky, D. C. 2017, AJ, 153, 45
Mills, S. M., Howard, A. W., Weiss, L. M., et al. 2019, AJ, 157, 145
Murray, C. D., & Dermott, S. F. 1999, Solar System Dynamics (Cambridge: Cambridge Univ. Press)
Nesvorný, D. 2009, ApJ, 701, 1116
Nesvorný, D., & Beaugé, C. 2010, ApJ, 709, L44
Nesvorný, D., & Morbidelli, A. 2008, ApJ, 688, 636
Nesvorný, D., & Vokrouhlický, D. 2014, ApJ, 790, 58
Oifir, A., Xie, J.-W., Jiang, C.-F., Sari, R., & Aharonson, O. 2018, ApJS, 234, 9
Payne, M. J., Ford, E. B., & Veras, D. 2010, ApJL, 712, L86
Ragozzine, D., & Holman, M. J. 2016, arXiv:1006.3727
Rauer, H., Catala, C., Aerts, C., et al. 2014, ESA, 38, 249
Ricker, G. R., Latham, D. W., Vanderspek, R. K., et al. 2010, BAAS, 42, 459
Rivera, E. R., Lissauer, J. J., Butler, R. P., et al. 2005, ApJ, 634, 625
Seager, S., & Mallén-Ornelas, G. 2003, ApJ, 585, 1038
Shahaf, S., Mazeh, T., Zucker, S., & Fabrycky, D. 2021, MNRAS, 505, 1293
Wolszczan, A., & Frail, D. A. 1992, Natur, 355, 145
Xie, J.-W., Dong, S., Zhu, Z., et al. 2016, PNAS, 113, 11431
Yoffe, G., Oifir, A., & Aharonson, O. 2021, ApJ, 908, 114