On Nonlocality, Lattices and Internal Symmetries

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Abstract

We study functional analytic aspects of two types of correction terms to the Heisenberg algebra. One type is known to induce a finite lower bound $\Delta x_0$ to the resolution of distances, a short distance cutoff which is motivated from string theory and quantum gravity. It implies the existence of families of self-adjoint extensions of the position operators with lattices of eigenvalues. These lattices, which form representations of certain unitary groups cannot be resolved on the given geometry. This leads us to conjecture that, within this framework, degrees of freedom that correspond to structure smaller than the resolvable (Planck) scale turn into internal degrees of freedom with these unitary groups as symmetries. The second type of correction terms is related to the previous essentially by "Wick rotation", and its basics are here considered for the first time. In particular, we investigate unitarily inequivalent representations.

In the context of string theory and quantum gravity the possible existence of a natural ultraviolet cutoff, e.g. at the Planck scale, has been widely discussed with various ansatze, see e.g. [1]-[16]. In particular, these studies include uncertainty relations of the form

$$\Delta x \Delta p \geq \frac{\hbar}{2}(1 + \beta(\Delta p)^2 + ...)$$

with corresponding corrections to the Heisenberg commutation relations of the form

$$[x,p] = i\hbar(1 + \beta p^2 + ...)$$

Interest has so far mainly rested on the case $\beta > 0$, as it is in this case that an ultraviolet regularising short distance behavior appears. In the generic case of corrections to $n$-dimensional commutation relations with Minkowski signature also correction
terms of the type of $\beta < 0$ are to be expected, at least for the temporal components. Let us therefore begin with a brief analysis of the case $\beta < 0$. The uncertainty relation yields $\Delta x_0 = 0$, as usual. However, taking the trace on both sides of Eq.2 shows that finite dimensional representations are no longer excluded. Indeed, there now exist even one-dimensional representations with $x$ represented as some arbitrary number and $p$ represented as $\pm |\beta|^{-1/2}$. Indeed, all finite dimensional representations reduce to sums of these cases: In the $p$ eigenbasis $p_{ij} = p_i \delta_{ij}$ and the commutation relations, $x_{rs}(p_r - p_s) = i\hbar \delta_{rs}(1 + |\beta|^2)$ yield $p_r = \pm |\beta|^{-1/2}$, thus $(x, p)_{rs} = 0$, so that $x$ is diagonalisable simultaneously with $p$, and we obtain $p_{rs} = \text{diag}(p_1, p_2, ..., p_n)$ and $x_{rs} = \text{diag}(x_1, x_2, ..., x_n)$ with $p_i \in \{-|\beta|^{-1/2}, |\beta|^{-1/2}\}$ and $x_i \in \mathbb{R}$. The infinite dimensional representations are harder to classify. Let us begin with the spectral representation of $p$:

$$p \psi(\lambda) = \lambda \psi(\lambda)$$  \hspace{1cm} (3)

$$x \psi(\lambda) = i\hbar \left(\frac{d}{d\lambda} + \beta \lambda \frac{d}{d\lambda}\right) \psi(\lambda)$$  \hspace{1cm} (4)

$$\langle \psi_1 | \psi_2 \rangle = \int_I d\lambda \psi_1^* (\lambda) \psi_2 (\lambda)$$  \hspace{1cm} (5)

We note that, as is easy to verify, exactly the family of operators $G$ defined through the integral kernel $(a, b \in C)$

$$G(\lambda, \lambda') = \left(a \Theta(\lambda - |\beta|^{-1/2}) + b \Theta(\lambda + |\beta|^{-1/2})\right) \delta(\lambda - \lambda')$$  \hspace{1cm} (6)

commute with both $x$ and $p$. Each $G$ is diagonal and constant apart from two steps where it cuts momentum space, and with it the representation, into three unitarily inequivalent parts. The representation which has the proper limit as $\beta \to 0$ is given by Eqs.3,4 with the integration interval $I := I_c = [-|\beta|^{-1/2}, |\beta|^{-1/2}]$. Thus, $p$ becomes a bounded self-adjoint operator. Let us calculate the defect indices of $x$ in this representation, i.e. the dimensions of the kernels of $(x^* \pm i)$, i.e. we check for square integrable solutions to (from now on we set $\hbar = 1$)

$$i \left(\partial_\lambda - |\beta| \lambda^2 \partial_\lambda + \lambda\right) \psi_\xi (\lambda) = \xi \psi_\xi (\lambda)$$  \hspace{1cm} (7)

with $\xi = \pm i$. The equation is solved by

$$\langle \lambda | \xi \rangle = \psi_\xi (\lambda) = N \left(1 - |\beta| \lambda^2 \right)^{-1/2} \left(1 - \sqrt{|\beta| \lambda}\right)^{i\xi/2} \left(1 + \sqrt{|\beta| \lambda}\right)^{-i\xi/2}$$  \hspace{1cm} (8)

which are non-square integrable on $I_c$ for all $\xi \in \mathbb{C}'$, in particular also for $\xi = \pm i$. Thus, the defect indices are $(0, 0)$, i.e. $x$ is still essentially self-adjoint with a unique spectral representation (recall that the operator $i \partial_\lambda$ which normally represents $x$ on
momentum space has defect indices (1,1) on the interval. The position eigenfunctions are given by Eq.8 for real $\xi$. With the continuum normalisation $N = (2\pi)^{-1/2}$ it is not difficult to verify orthonormalisation and completeness:

$$\int_{-|\beta|^{-1/2}}^{+|\beta|^{-1/2}} d\lambda \langle \xi | \lambda \rangle \langle \lambda | \xi' \rangle = \delta(\xi - \xi')$$ (9)

$$\int_{-\infty}^{+\infty} d\xi \langle \xi | \xi' \rangle = \delta(\lambda - \lambda')$$ (10)

The generalised Fourier factor given in Eq.8 yields the transformation $\psi(\xi) = \int_I d\lambda \langle \xi | \lambda \rangle \psi(\lambda)$ that maps momentum space wave functions $\psi(\lambda) = \langle \lambda | \psi \rangle$ to position space wave functions $\psi(\xi) = \langle \xi | \psi \rangle$. To summarise, we have found no short distance cutoff in positions, while we have found that momentum space becomes bounded.

Let us now turn to the case $\beta > 0$. As is well known, and as is easily derived from Eq.4, the position resolution $\Delta x$ now becomes finitely bounded from below: $\Delta x_0 = \hbar \sqrt{\beta}$. To be precise, for all normalised vectors $|\psi\rangle$ in a domain $D$ on which the commutation relations are represented the position uncertainty obeys $\Delta x_{|\psi\rangle} = \langle \psi | (x - \langle \psi | x | \psi \rangle)^2 | \psi \rangle^{1/2} \geq \Delta x_0$. A convenient representation is given by Eqs.3-5 with $I = \mathbb{R}$. Technically, on any dense domain $D$ in a Hilbert space $H$ on which the commutation relations hold the position operator can only be symmetric but not self-adjoint, as diagonalisability is excluded by the uncertainty relation (eigenvectors to an observable automatically have vanishing uncertainty in this observable). This also excludes the possibility of finite dimensional representations of the commutation relations (since in these symmetry and self-adjointness coincide), as could of course also be seen by taking the trace of both sides of Eq.2. Generally, in order to insure that expectation values of positions and momenta are real, we only consider corrections to the commutation relations which are consistent with an involution which acts on the generators as $x_i^* = x_i, p_i^* = x_i$. The involution then also insures that the deficiency indices of the $x_i$ (and $p_i$) on any dense domain $D$ on which the commutation relations hold are equal, implying that the $x_i$ do have self-adjoint extensions in $H$, though not in $D$. This functional analytic structure was first found in [6, 8]. The self-adjoint extensions now have been calculated explicitly for a number of cases, for our one-dimensional case here, in [12].

We remark that a finite lower bound on the standard deviation in positions is, interpretationally, an ensemble-based short distance regularisation (which could only appear in quantum theory). There exists a straightforward way of introducing these generalised commutation relations into the quantum field theoretical path integral [17] with the functional analysis of representations on wave functions extending to representations on fields (though the interpretation does of course not extend straightforwardly). We will in the following use the quantum mechanical rather than the field theoretical terminology, the analysis is the same. It has been shown that this ensemble cutoff does indeed regularise the ultraviolet in euclidean field theory [17]-[21]. Let
us now discuss further physical implications, related to internal symmetries. As we will see, the unobservability of localisation beyond the minimal uncertainty $\Delta x_0$ can be seen to represent a local symmetry, where degrees of freedom which correspond to small scale structure beyond the Planck scale turn into internal degrees of freedom.

Consider a $*$-representation (such as given by Eqs.3-5) of the commutation relation Eq.2 on a maximal dense domain $D$ in a Hilbert space $H$. Then $\mathbf{x}$ is merely symmetric, i.e. $D$ is smaller than the domain $D_{\mathbf{x}^*}$ of the adjoint operator $\mathbf{x}^*$ (which is not symmetric). The deficiency spaces $L_+, L_-$, i.e. the spaces spanned by eigenvectors of $\mathbf{x}^*$ with eigenvalues $+i$ and $-i$ are one-dimensional, i.e. the deficiency indices are $(1,1)$ (also in $n$ dimensions they are equal, due to the involution).

Thus, there exists a set of self-adjoint extensions of $\mathbf{x}$ which is in one-to-one correspondence with the set of unitary transformations $\tilde{U} : L_+ \to L_-$. (Recall that, by the usual procedure, each $\tilde{U}$ defines a unitary extension of the Cayley transform of $\mathbf{x}$, with the inverse Cayley transform then defining a self-adjoint extension of $\mathbf{x}$. On the eigenvalues, Cayley transforms are Moebius transforms.)

The $\tilde{U}$ differ exactly by the set $G$ of unitary transformations $U : L^+ \to L^+$. In general, for defect indices $(n,n)$, this is the unitary group $U(n)$, which we may here call the local group $G$. Thus, the set of self-adjoint extensions $\{\mathbf{x}_\alpha\}$ forms a representation of the local group. $\alpha$, which labels the self adjoint extensions, is a vector in the fundamental representation of $G$. The local group also acts on the set of spectra $\{\sigma_\alpha\}$ of the $\mathbf{x}_\alpha$. Let us denote the eigenvalues of the self-adjoint extension $\mathbf{x}_\alpha$ by $v_{\alpha}(r)$. Then, for any fixed $r$, we obtain an orbit $O(r) := \{v_{U,\alpha} | U \in G\}$ of eigenvalues under the action of $G$.

Let us consider the example of the one-dimensional case above. The scalar product of eigenvectors of $\mathbf{x}^*$ has been calculated in [12]:

$$\langle \xi | \xi' \rangle = \frac{2\sqrt{\beta}}{\pi(\xi - \xi') \sin \left(\xi' \sqrt{\beta} \frac{2}{\pi}\right)}$$

From its zeros we can read off the family of discrete spectra of the self-adjoint extensions:

$$\sigma_\alpha = \left\{v_{\alpha}(r) = (2r + s/\pi)\sqrt{\beta} | r \in \mathbb{N}\right\} \text{ where } \alpha = e^{is} \text{ with } s \in [0, 2\pi]$$

The spectra are equidistant and two self-adjoint extensions only differ by a shift of their lattice of eigenvalues. The local group is here the group of translations of the lattices of eigenvalues. Due to the periodicity of the lattice this group is topologically $S^1$, or $U(1)$. This reflects that in this case $L_+$ is one-dimensional and the self-adjoint extensions therefore form a representation of the local group $U(1)$.

Each choice of self-adjoint extension of the position operators therefore corresponds to a choice of lattice on which the physics takes place. However, the commutation relations also imply that the smallest uncertainty in positions becomes finite...
and large enough so that the actual choice of lattice cannot be resolved. Technically, all self-adjoint extensions of $x$ coincide when restricted to a domain $D$ on which the commutation relations hold.

If, therefore, with a physical state $|\psi\rangle \in D$ also some vector $\alpha$ is specified, as a choice of self-adjoint extension, the action should be invariant, i.e., we arrive at a global symmetry principle. The additional information given by $\alpha$ can be interpreted. Assume that the state of a particle is projected onto a state of maximal localisation ($\Delta x = \Delta x_0$) with position expectation $\xi$. Specifying $\alpha$ is to specify one point in the orbit of the eigenvalue $\xi$ under the action of the local group. As a convention one can specify that this is where the maximally localised particle is said to "actually" sit. This is consistent because the radius of the orbits of the eigenvalues is $\sqrt{\beta} = \Delta x_0$, i.e., of the size of the finite minimal uncertainty $\Delta x_0$, so that all these conventions, differing only by the action of the local group, cannot be distinguished observationally. For example, the pointwise multiplication of fields as discussed e.g. in [12] can be reformulated in terms of a choice of position eigenbasis, rather than the set of maximally localised fields. The gauge principle is that the action is invariant under the local group. We remark that the proof of ultraviolet regularity will still go through, since not only the fields of maximal localisation, but also the position eigenfields are normalisable.

On the other hand, the 'local' group may also be taken to act locally, i.e., we consider $|\psi\rangle \in D \otimes L_+$. It is unobservable whether one specifies one self-adjoint extension’s lattice here and another’s there, as long as the parallel transport of $\alpha$ is consistently defined. At large scales this should lead to the ordinary local gauge principle. We note that the local gauge group will be determined through the functional analysis of the position operators. This in turn depends on the choice of short distance structure as specified through the corrections to uncertainty- and commutation relations. In the physical case of the Minkowski signature further nontrivial structures can be expected to arise from the behavior of the coordinates which behave according to the $\beta < 0$ case, as discussed above.

To summarise, there is a possibility that internal symmetry spaces arise as deficiency spaces of position operators. Introducing $\Delta x_0 > 0$ the infinite dimensional Hilbert space of fields develops special dimensions that correspond to degrees of freedom that describe localisation beyond what can be resolved, and which can therefore be viewed as internal degrees of freedom. Basically, the idea is, that certain corrections to the uncertainty relations lead to physics on a whole set of possible lattices, while the choice of any particular lattice from the set cannot be resolved and does therefore correspond to an internal degree of freedom. This may be a new mechanism, or it may be a reformulation of the Kaluza Klein idea, in which case one may expect a deeper relation to string theory.
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