Some Cosmological Solutions of a Nonlocal Modified Gravity

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Abstract. We consider nonlocal modification of the Einstein theory of gravity in framework of the pseudo-Riemannian geometry. The nonlocal term has the form $\mathcal{H}(R)F(\Box)G(R)$, where $\mathcal{H}$ and $G$ are differentiable functions of the scalar curvature $R$, and $F(\Box) = \sum_{n=0}^{\infty} f_n \Box^n$ is an analytic function of the d’Alambert operator $\Box$. Using calculus of variations of the action functional, we derived the corresponding equations of motion. Cosmological solutions are found for the case when the Ricci scalar $R$ is constant.

1. Introduction

Although very successful, Einstein theory of gravity is not a final theory. There are many its modifications, which are motivated by quantum gravity, string theory, astrophysics and cosmology (for a review, see [1]). One of very promising directions of research is nonlocal modified gravity and its applications to cosmology (as a review, see [2, 3] and [4]). To solve cosmological Big Bang singularity, nonlocal gravity with replacement $R \to R + CRF(\Box)R$ in the Einstein-Hilbert action was proposed in [5]. This nonlocal model is further elaborated in the series of papers [6–12].

In [13] we introduced a new approach to nonlocal gravity given by the action

$$S = \int d^4x \sqrt{-g} \left( \frac{R}{16\pi G} + R^{-1}F(\Box)R \right),$$

where the d’Alembert operator is $\Box = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu$. The nonlocal term $R^{-1}F(\Box)R = f_0 + R^{-1} \sum_{n=1}^{\infty} f_n \Box^n R$ contains $f_0$ which can be connected with the cosmological constant as $f_0 = -\frac{\Lambda}{16\pi G}$. This term is also invariant under transformation $R \to CR$, where $C$ is a constant, i.e. this nonlocality does not depend on magnitude of the scalar curvature $R \neq 0$.

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In this paper we consider $n$-dimensional pseudo-Riemannian manifold $M$ with metric $g_{\mu\nu}$ of signature $(n_-, n_+)$. Our nonlocal gravity model here is larger than (1) and given by the action

$$S = \int_M \left( \frac{R - 2\Lambda}{16\pi G} + \mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R) \right) \sqrt{|g|} d^n x,$$

which is a functional of metric (gravitational field) $g_{\mu\nu}$, where $\mathcal{H}$ and $\mathcal{G}$ are differentiable functions of the scalar curvature $R$, and $\Lambda$ is cosmological constant.

2. Variation of the action functional

Let us introduce the following auxiliary functionals

$$S_0 = \int_M (R - 2\Lambda) \sqrt{|g|} d^n x, \quad S_1 = \int_M \mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R) \sqrt{|g|} d^n x.$$  (3)

Then the variations of $S_0$ and $S_1$ can be considered separately and the variation of (2) can be expressed as

$$\delta S = \frac{1}{16\pi G} \delta S_0 + \delta S_1.$$  (4)

Note that variations of the metric tensor elements and their first derivatives are zero on the boundary of manifold $M$, i.e. $\delta g_{\mu\nu}|_{\partial M} = 0$, $\delta \partial g_{\mu\nu}|_{\partial M} = 0$.

**Lemma 2.1.** Let $M$ be a pseudo-Riemannian manifold. Then the following basic relations hold:

$$\frac{\partial g_{\mu\nu}}{\partial x^\alpha} = -g^{\mu\sigma}\Gamma^\nu_{\sigma\alpha} - g^{\nu\alpha}\Gamma^\mu_{\sigma\alpha}, \quad \delta g = g^{\mu\nu}\delta g_{\mu\nu} = -g_{\mu\nu}\delta g^{\mu\nu},$$

$$\Gamma^\mu_{\nu\sigma} = \frac{\partial}{\partial x^\nu} \ln \sqrt{|g|}, \quad \delta \sqrt{|g|} = -\frac{1}{2} g^{\mu\nu} \sqrt{|g|} \delta g^{\mu\nu},$$

$$\Box = \nabla^\mu \nabla_\mu = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} g^{\mu\nu} \partial_\nu), \quad \delta R = R_{\mu\nu} \delta g^{\mu\nu} + g_{\mu\nu} \Box \delta g^{\mu\nu} - \nabla_\mu \nabla_\nu \delta g^{\mu\nu}.$$  (6)

**Lemma 2.2.** On the manifold $M$ holds

$$\int_M g^{\mu\nu} \delta R_{\mu\nu} \sqrt{|g|} d^n x = 0.$$  (7)

**Proof.** Let $W^\nu = -g^{\mu\alpha} \delta \Gamma^\nu_{\mu\alpha} + g^{\mu\nu} \delta \Gamma^\alpha_{\mu\alpha}$. Then it follows

$$\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^\nu} (\sqrt{|g|} W^\nu) = \frac{\partial W^\nu}{\partial x^\nu} + W^\nu \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^\nu} \sqrt{|g|}. $$  (5)

Using Lemma 2.1 we get

$$\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^\nu} (\sqrt{|g|} W^\nu) = -\frac{\partial}{\partial x^\nu} (g^{\mu\alpha} \delta \Gamma^\nu_{\mu\alpha}) + \frac{\partial}{\partial x^\nu} (g^{\mu\nu} \delta \Gamma^\alpha_{\mu\alpha}) + (-g^{\mu\nu} \delta \Gamma^\alpha_{\mu\alpha} + g^{\mu\nu} \delta \Gamma^\alpha_{\mu\alpha}) \Gamma^\beta_{\gamma\nu\nu}$$

$$= -\frac{\partial g^{\mu\alpha}}{\partial x^\nu} \delta \Gamma^\nu_{\mu\alpha} - g^{\mu\nu} \frac{\partial \Gamma^\nu_{\mu\alpha}}{\partial x^\nu} + \frac{\partial g^{\mu\nu}}{\partial x^\nu} \delta \Gamma^\alpha_{\mu\alpha} + g^{\mu\nu} \frac{\partial \Gamma^\alpha_{\mu\alpha}}{\partial x^\nu} + (-g^{\mu\nu} \delta \Gamma^\alpha_{\mu\alpha} + g^{\mu\nu} \delta \Gamma^\alpha_{\mu\alpha}) \Gamma^\beta_{\gamma\nu\nu}.$$  (6)
Moreover, using again Lemma 2.1 we obtain
\[
\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^\nu} \sqrt{|g|} W^\nu = \frac{g^{ho
u} \Gamma^\mu_\rho_\nu + g^\mu_\nu \delta^\nu_\mu - g^\mu_\rho \Gamma^\nu_\mu_\rho - g^\mu_\rho \Gamma^\rho_\mu_\nu - g^\mu_\rho \delta^\nu_\mu - g^\mu_\rho \delta^\rho_\mu}{\sqrt{|g|}} + g^\mu_\rho \frac{\partial \Gamma^\nu_\mu_\rho}{\partial x^\nu} + g^\mu_\rho \frac{\partial \Gamma^\nu_\rho_\mu}{\partial x^\nu}
\]
\[
- g^\mu_\rho \frac{\partial \Gamma^\nu_\rho_\mu}{\partial x^\nu} + g^\mu_\rho \frac{\partial \Gamma^\nu_\rho_\mu}{\partial x^\nu}
\]
\[
g^\mu_\nu \left( - \frac{\partial \Gamma^\mu_\rho_\nu}{\partial x^\nu} + \frac{\partial \Gamma^\mu_\rho_\nu}{\partial x^\nu} + \Gamma^\rho_\mu \delta^\nu_\mu + \Gamma^\rho_\mu \delta^\nu_\mu + \Gamma^\rho_\mu \delta^\rho_\mu - \Gamma^\rho_\mu \delta^\rho_\mu \right)
\]
\[
g^\mu_\nu \delta \left( - \frac{\partial \Gamma^\mu_\rho_\nu}{\partial x^\nu} + \frac{\partial \Gamma^\mu_\rho_\nu}{\partial x^\nu} + \Gamma^\rho_\mu \delta^\nu_\mu - \Gamma^\rho_\mu \delta^\rho_\mu \right) = g^\mu_\nu \delta R_{\mu\nu}.
\] (7)

Finally, we have
\[
g^\mu_\nu \delta R_{\mu\nu} = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^\nu} \sqrt{|g|} W^\nu \textrm{ and } \int_M g^\mu_\nu \delta R_{\mu\nu} \sqrt{|g|} d^nx = \int_M \frac{\partial}{\partial x^\nu} \sqrt{|g|} W^\nu d^nx.
\] (8)

Using the Gauss-Stokes theorem one obtains
\[
\int_M \frac{\partial}{\partial x^\nu} \sqrt{|g|} W^\nu d^nx = \int_{\partial M} W^\nu d\sigma_v.
\] (9)

Since \( \delta g_{\mu\nu} = 0 \) and \( \delta \partial \chi g_{\mu\nu} = 0 \) at the boundary \( \partial M \) we have \( W^\nu|_{\partial M} = 0 \). Then we have \( \int_{\partial M} W^\nu d\sigma_v = 0 \), that completes the proof. \( \square \)

**Lemma 2.3.** The variation of \( S_0 \) is
\[
\delta S_0 = \int_M G_{\mu\nu} \sqrt{|g|} g^\mu_\nu d^nx + \Lambda \int_M g^\mu_\nu \sqrt{|g|} \delta g^\mu_\nu d^nx,
\] (10)

where \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \) is the Einstein tensor.

**Proof.** The variation of \( S_0 \) can be found as follows
\[
\delta S_0 = \int_M \delta (\sqrt{g}) (R - 2\Lambda) \sqrt{|g|} d^nx = \int_M \delta (\sqrt{g}) R \sqrt{|g|} d^nx - 2\Lambda \int_M \delta \sqrt{|g|} d^nx
\]
\[
= \int_M \left( \sqrt{|g|} \delta R + R \sqrt{|g|} d^nx + \Lambda \delta g_{\mu\nu} \sqrt{|g|} \delta g^\mu_\nu \right) d^nx
\]
\[
= \int_M \left( \sqrt{|g|} \delta (g^\mu_\nu R_{\mu\nu}) - \frac{1}{2} R \sqrt{|g|} \delta g_{\mu\nu} d^nx + \Lambda \delta g_{\mu\nu} \sqrt{|g|} \delta g^\mu_\nu \right) d^nx
\]
\[
= \int_M G_{\mu\nu} \sqrt{|g|} d^nx + \Lambda \int_M g^\mu_\nu \sqrt{|g|} \delta g^\mu_\nu d^nx + \int_M \delta R_{\mu\nu} \sqrt{|g|} d^nx.
\] (11)

Using Lemma 2.2 from the last equation we obtain the variation of \( S_0 \). \( \square \)

**Lemma 2.4.** For any scalar function \( h \) we have
\[
\int_M \sqrt{|g|} h R d^nx = \int_M \left( h R_{\mu\nu} + g_{\mu\nu} \Box h - \nabla_\nu \nabla_\mu h \right) \delta g^\mu_\nu \sqrt{|g|} d^nx.
\] (12)

**Proof.** Using Lemma 2.1 for any scalar function \( h \) we have
\[
\int_M \sqrt{|g|} d^nx = \int_M \left( h R_{\mu\nu} \delta g^\mu_\nu + h g_{\mu\nu} \Box \delta g^\mu_\nu - h \nabla_\mu \nabla_\nu \delta g^\mu_\nu \right) \sqrt{|g|} d^nx.
\] (13)
The second and third term in this formula can be transformed in the following way:

\[
\int_M h g_{\mu\nu}(\square g^{\mu\nu}) \sqrt{|g|} \, d^nx = \int_M g_{\mu\nu}(\square h) g^{\mu\nu} \sqrt{|g|} \, d^nx, \tag{14}
\]

\[
\int_M h \nabla_\mu \nabla_\nu g^{\mu\nu} \sqrt{|g|} \, d^nx = \int_M \nabla_\mu h \, g^{\mu\nu} \sqrt{|g|} \, d^nx. \tag{15}
\]

To prove the first of these equations we use the Stokes theorem and obtain

\[
\int_M h g_{\mu\nu}(\square g^{\mu\nu}) \sqrt{|g|} \, d^nx = \int_M h g_{\mu\nu} \nabla^\alpha g^{\mu\nu} \sqrt{|g|} \, d^nx - \int_M \nabla_\alpha (h g_{\mu\nu}) \nabla^\alpha g^{\mu\nu} \sqrt{|g|} \, d^nx
\]

\[
= \int_M g_{\mu\nu} \nabla_\alpha h \, g^{\mu\nu} \sqrt{|g|} \, d^nx = \int_M g_{\mu\nu} \square h \, g^{\mu\nu} \sqrt{|g|} \, d^nx. \tag{16}
\]

Here we have used \(\nabla_\mu g_{\mu\nu} = 0\) and \(\nabla^\alpha g_{\mu\nu} = \square g_{\mu\nu} = \square\) to obtain the last integral.

To obtain the second equation we first introduce vector

\[N^\mu = h \nabla_\nu g^{\mu\nu} - \nabla_\nu h g^{\mu\nu}.\tag{17}\]

From the above expression follows

\[
\nabla_\mu N^\mu = \nabla_\mu (h \nabla_\nu g^{\mu\nu} - \nabla_\nu h g^{\mu\nu}) = \nabla_\mu h \nabla_\nu g^{\mu\nu} + h \nabla_\mu \nabla_\nu g^{\mu\nu} - \nabla_\mu \nabla_\nu h \, g^{\mu\nu} - \nabla_\nu \nabla_\mu h \, g^{\mu\nu} - \nabla_\mu h \nabla_\nu g^{\mu\nu} - \nabla_\nu h \nabla_\mu g^{\mu\nu}.
\]

Integrating \(\nabla_\mu N^\mu\) yields \(\int_M \nabla_\mu N^\mu \sqrt{|g|} \, d^nx = \int_{\partial M} N^\mu n_\mu d\partial M\), where \(n_\mu\) is the unit normal vector. Since \(N^\mu|_{\partial M} = 0\) we have that the last integral is zero, which completes the proof. \(\square\)

**Lemma 2.5.** Let \(\theta\) and \(\psi\) be scalar functions such that \(\delta \psi|_{\partial M} = 0\). Then one has

\[
\int_M \theta \delta \psi \sqrt{|g|} \, d^nx = \frac{1}{2} \int_M g^\mu\nu \partial_\mu \theta \partial_\nu \psi g_{\mu\nu} \delta g^{\mu\nu} \sqrt{|g|} \, d^nx - \int_M \partial_\mu \theta \partial_\nu \psi \delta g^{\mu\nu} \sqrt{|g|} \, d^nx + \int_M \square \delta \psi \sqrt{|g|} \, d^nx + \frac{1}{2} \int_M g_{\mu\nu} \theta \square \psi \delta g^{\mu\nu} \sqrt{|g|} \, d^nx. \tag{19}\]

**Proof.** Since \(\theta\) and \(\psi\) are scalar functions such that \(\delta \psi|_{\partial M} = 0\) we have

\[
\int_M \theta \delta \psi \sqrt{|g|} \, d^nx = \int_M \partial_\alpha \delta (\sqrt{|g|} g^{\mu\nu} \partial_\nu \psi) \, d^nx + \int_M \theta \delta \left( \frac{1}{\sqrt{|g|}} \right) \partial_\alpha (\sqrt{|g|} g^{\mu\nu} \partial_\nu \psi) \sqrt{|g|} \, d^nx
\]

\[
= \int_M \partial_\alpha (\theta \delta (\sqrt{|g|} g^{\mu\nu} \partial_\nu \psi)) \, d^nx - \int_M \partial_\alpha \theta \, \delta (\sqrt{|g|} g^{\mu\nu} \partial_\nu \psi) \, d^nx + \frac{1}{2} \int_M g_{\mu\nu} \theta \delta \psi \delta g^{\mu\nu} \sqrt{|g|} \, d^nx. \tag{20}\]
It is easy to see that \( \int_M \partial_s \left( \nabla^s \left( \sqrt{g} \nabla^s \partial_s \psi \right) \right) \, d^n x = 0 \). From this result it follows

\[
\int_M \nabla^s \partial_s \sqrt{g} \, d^n x = - \int_M g^{s\tau} \partial_s \nabla^s \partial_s \sqrt{g} \, d^n x - \int_M \partial_s \nabla^s \partial_s g^{s\tau} \sqrt{g} \, d^n x \\
\quad - \int_M g^{s\tau} \partial_s \sqrt{g} \partial_s \psi \, d^n x + \frac{1}{2} \int_M g_{\mu\nu} \partial_\psi \sqrt{g} \, d^n x \\
= \frac{1}{2} \int_M g^{s\tau} \partial_s \psi \sqrt{g} \, d^n x + \int_M \partial_\psi \sqrt{g} \, d^n x \\
\quad - \int_M \partial_\psi (g^{s\tau} \sqrt{g}) \partial_s \theta \psi \, d^n x + \int_M \partial_\psi (g^{s\tau} \sqrt{g}) \Delta \psi \, d^n x \\
\quad + \frac{1}{2} \int_M g_{\mu\nu} \partial_\psi \sqrt{g} \, d^n x \\
= \frac{1}{2} \int_M g^{s\tau} \partial_s \psi \sqrt{g} \, d^n x + \int_M \partial_\psi \sqrt{g} \, d^n x \\
\quad - \int_M \partial_\psi \sqrt{g} \, d^n x \\
\quad + \frac{1}{2} \int_M g_{\mu\nu} \partial_\psi \sqrt{g} \, d^n x. \\
\tag{21}
\]

At the end we have that

\[
\int_M \partial_\psi \sqrt{g} \, d^n x = \frac{1}{2} \int_M g_{\mu\nu} \partial_\psi \sqrt{g} \, d^n x + \int_M \partial_\psi \sqrt{g} \, d^n x + \frac{1}{2} \int_M g_{\mu\nu} \partial_\psi \sqrt{g} \, d^n x. \\
\tag{22}
\]

Now, after this preliminary work we can get the variation of \( S_1 \).

**Lemma 2.6.** The variation of \( S_1 \) is

\[
\delta S_1 = \frac{1}{2} \int_M g_{\mu\nu} \nabla_\mu \Gamma (\partial_\tau \nabla_\nu \partial_\psi \sqrt{g}) \, d^n x + \int_M \left( R_{\mu\nu} \Phi - K_{\mu\nu} \phi \right) \delta g^{\mu\nu} \sqrt{g} \, d^n x \\
+ \frac{1}{2} \sum_{\alpha=1}^{n-1} \sum_{i=0}^{n-1} \int_M g_{\mu\tau} \left( \nabla_\mu \Gamma \left( \partial_\tau \nabla_\nu \partial_\psi \sqrt{g} \right) - \partial_\nu \Gamma \left( \partial_\tau \nabla_\mu \partial_\psi \sqrt{g} \right) \right) \delta g^{\mu\nu} \sqrt{g} \, d^n x,
\]

where \( K_{\mu\nu} = V_\mu V_\nu - g_{\mu\nu} \partial_\tau \phi \), \( \Phi = \nabla_\mu \Gamma (\partial_\tau \nabla_\nu \partial_\psi \sqrt{g}) + \nabla_\mu \Gamma (\partial_\nu \partial_\psi \sqrt{g}) \partial_\tau \phi \) and \( \Gamma \) denotes derivative with respect to \( R \).

**Proof.** The variation of \( S_1 \) can be expressed as

\[
\delta S_1 = \int_M \left( \nabla_\mu \Gamma (\partial_\tau \nabla_\nu \partial_\psi \sqrt{g}) \delta \sqrt{g} \right) \, d^n x + \left( \nabla_\mu \Gamma (\partial_\tau \nabla_\nu \partial_\psi \sqrt{g}) \delta \sqrt{g} \right) \, d^n x.
\tag{23}
\]

For the first two integrals in the last equation we have

\[
I_1 = \int_M \nabla_\mu \Gamma (\partial_\tau \nabla_\nu \partial_\psi \sqrt{g}) \, d^n x = - \frac{1}{2} \int_M g_{\mu\nu} \nabla_\mu \Gamma (\partial_\tau \nabla_\nu \partial_\psi \sqrt{g}) \, d^n x \tag{24}
\]

\[
I_2 = \int_M \nabla_\mu \Gamma (\partial_\tau \nabla_\nu \partial_\psi \sqrt{g}) \, d^n x = \int_M \nabla_\nu \Gamma (\partial_\tau \nabla_\mu \partial_\psi \sqrt{g}) \, d^n x.
\]
Using (19) together and obtain

$$I_2 = \int_M \left( R_{\mu\nu} \mathcal{H}'(R) F(\square) G(R) - K_{\mu\nu}(\mathcal{H}'(R) F(\square) G(R)) \right) \delta g^{\mu\nu} \sqrt{|g|} \, d^n x. \quad (25)$$

The third integral can be presented using linear combination of the following integrals

$$J_n = \int_M \mathcal{H}(R) \delta(\square^n G(R)) \sqrt{|g|} \, d^n x. \quad (26)$$

$$J_0$$ is the integral of the same form as $$I_2$$ so

$$J_0 = \int_M \left( R_{\mu\nu} \mathcal{G}'(R) \mathcal{H}(R) - K_{\mu\nu}(\mathcal{G}'(R) \mathcal{H}(R)) \right) \delta g^{\mu\nu} \sqrt{|g|} \, d^n x. \quad (27)$$

For $$n > 0$$, we can find $$J_n$$ using (19). In the first step we take $$\theta = \mathcal{H}(R)$$ and $$\psi = \square^{n-1} G(R)$$ and obtain

$$J_n = \frac{1}{4} \sum_{i=0}^{n-1} \int_M \left( g_{\mu\nu} \partial_\mu \mathcal{H}(R) \partial_\nu \square^{n-1} G(R) + \delta g^{\mu\nu} \sqrt{|g|} \, d^n x - \int_M \partial_\mu \mathcal{H}(R) \partial_\nu \square^{n-1} G(R) \delta g^{\mu\nu} \sqrt{|g|} \, d^n x \right. + \int_M \mathcal{H}(R) \delta \square^{n-1} G(R) \sqrt{|g|} \, d^n x + \frac{1}{2} \int_M g_{\mu\nu} R_{\mu\nu} \mathcal{H}(R) \square^n G(R) \delta g^{\mu\nu} \sqrt{|g|} \, d^n x. \quad (28)$$

In the second step we take $$\theta = \square \mathcal{H}(R)$$ and $$\psi = \square^{n-2} G(R)$$ and get the third integral in this formula, etc. Using (19) n times one obtains

$$J_n = \frac{1}{2} \sum_{i=0}^{n-1} \int_M \left( g_{\mu\nu} \partial^\mu \mathcal{H}(R) \partial_\nu \square^{n-1} G(R) + \delta g^{\mu\nu} \sqrt{|g|} \, d^n x \right. + \int_M \left( R_{\mu\nu} \mathcal{G}'(R) \square^n \mathcal{H}(R) - K_{\mu\nu}(\mathcal{G}'(R) \square^n \mathcal{H}(R)) \right) \delta g^{\mu\nu} \sqrt{|g|} \, d^n x. \quad (29)$$

Using the equation (12) we obtain the last integral in the above formula. Finally, one can put everything together and obtain

$$\delta S_1 = I_1 + I_2 + \sum_{n=0}^{\infty} f_n J_n = -\frac{1}{2} \int_M g_{\mu\nu} \mathcal{H}(R) F(\square) G(R) \delta g^{\mu\nu} \sqrt{|g|} \, d^n x + \int_M \left( R_{\mu\nu} \Phi - K_{\mu\nu} \Phi \right) \delta g^{\mu\nu} \sqrt{|g|} \, d^n x \right. + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \int_M \left( g_{\mu\nu} \partial^\mu \mathcal{H}(R) \partial_\nu \square^{n-1} G(R) + \delta^i \mathcal{H}(R) \square^{n-1} G(R) \right) \delta g^{\mu\nu} \sqrt{|g|} \, d^n x. \quad (30)$$

$$\square$$

**Theorem 2.1.** The variation of the functional (2) is equal to zero iff

$$\frac{G_{\mu\nu} + \Lambda g_{\mu\nu}}{16\pi G} - \frac{1}{2} g_{\mu\nu} \mathcal{H}(R) F(\square) G(R) + \left( R_{\mu\nu} \Phi - K_{\mu\nu} \Phi \right) \right. + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \left( g_{\mu\nu} \partial^\mu \mathcal{H}(R) \partial_\nu \square^{n-1} G(R) - 2 \partial_\mu \partial^i \mathcal{H}(R) \partial_\nu \square^{n-1} G(R) + g_{\mu\nu} \partial^i \mathcal{H}(R) \square^{n-1} G(R) \right) = 0. \quad (31)$$

**Proof.** Since we have $$\delta S = \frac{1}{16\pi G} \delta S_0 + \delta S_1$$ the theorem follows from Lemmas 2.3 and 2.6. $$\square$$
3. Signature (1, 3)

In the physics settings, where functional $S$ represents an action, theorem 2.1 gives the equations of motion. From this point we assume that manifold $M$ is the four-dimensional homogeneous and isotropic one with signature $(1, 3)$. Then the metric has the Friedmann-Lemaître-Robertson-Walker (FLRW) form:

$$\text{d}s^2 = -\text{d}t^2 + a(t)^2 \left(\frac{\text{d}r^2}{1 - kr^2} + r^2 \text{d}\theta^2 + r^2 \sin^2 \theta \text{d}\varphi^2\right).$$ (32)

**Theorem 3.1.** Suppose that manifold $M$ has the FLRW metric. Then the expression (31) has two linearly independent equations:

$$\frac{4\Lambda - R}{16\pi G} - 2\mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R) + (R\Phi + 3\Phi)$$

$$+ \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \left(\beta^l \Box^l \mathcal{H}(R)\partial_{\mu} \Box_{\nu}^n \mathcal{G}(R) + 2\Box^l \mathcal{H}(R) \Box_{\nu}^n \mathcal{G}(R)\right) = 0,$$ (33)

$$\frac{G_{00} + \Lambda g_{00}}{16\pi G} - \frac{1}{2} g_{00} \mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R) + (R_{00} \Phi - K_{00} \Phi)$$

$$+ \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \left( g_{00} \beta^l \Box^l \mathcal{H}(R)\partial_{\mu} \Box_{\nu}^n \mathcal{G}(R) - 2\beta^l \Box^l \mathcal{H}(R) \partial_{\nu} \Box_{\nu}^n \mathcal{G}(R) + g_{00} \Box^l \mathcal{H}(R) \Box_{\nu}^n \mathcal{G}(R)\right) = 0.$$ (34)

**Proof.** The FLRW metric satisfies $R_{\mu\nu} = \frac{\kappa}{2} g_{\mu\nu}$ and scalar curvature $R = 6 \left(\frac{\dot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right)$ depends only on $t$, hence equations (31) for $\mu \neq \nu$ are trivially satisfied. On the other hand, equations with indices 11, 22 and 33 can be rewritten as

$$g_{\mu\nu} \left(\frac{\dot{R}}{R} + \frac{\dot{\Lambda}}{8\pi G} - \mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R) + \frac{R}{2} \Phi + \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \left(\beta^l \Box^l \mathcal{H}(R)\partial_{\mu} \Box_{\nu}^n \mathcal{G}(R) + \Box^l \mathcal{H}(R) \Box_{\nu}^n \mathcal{G}(R)\right)\right) = 0.$$

Therefore these three equations are linearly dependent and there are only two linearly independent equations. The most convenient choice is the trace and 00-equation.

**Corollary 3.1.** For $\mathcal{H}(R) = R^p$ and $\mathcal{G}(R) = R^q$ the action (2) becomes

$$S = \int_M \left(\frac{R - 2\Lambda}{16\pi G} + R^p \mathcal{F}(\Box) R^q\right) \sqrt{|g|} \text{d}^4x,$$ (35)

and equations of motion are

$$\frac{1}{16\pi G} (G_{\mu\nu} + \Lambda g_{\mu\nu}) - \frac{1}{2} g_{\mu\nu} R^q \mathcal{F}(\Box) R^q + \left(R_{\mu\nu} \Phi - K_{\mu\nu} \Phi\right)$$

$$+ \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \left( g_{\mu\nu} \beta^l \Box^l R^p \partial_\alpha \Box_{\nu}^{n-1} R^q - 2 \beta^l \Box^l R^p \partial_\alpha \Box_{\nu}^{n-1} R^q + g_{\mu\nu} \Box^l \mathcal{H}(R) \Box_{\nu}^n \mathcal{G}(R)\right) = 0,$$ (36)

where $\Phi = p R^{p-1} \mathcal{F}(\Box) R^q + q R^{q-1} \mathcal{F}(\Box) R^q$.

**Corollary 3.2.** For $\mathcal{H}(R) = R^p$ and $\mathcal{G}(R) = R^q$ the equations of motion (36) are equivalent to the following two equations:

$$\frac{1}{16\pi G} (4\Lambda - R) - 2R^p \mathcal{F}(\Box) R^q + (R\Phi + 3\Box \Phi) + \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \left( \beta^l \Box^l R^p \partial_\alpha \Box_{\nu}^{n-1} R^q + 2 \Box^l \mathcal{H}(R) \Box_{\nu}^n \mathcal{G}(R)\right) = 0,$$ (37)
\[
\frac{1}{16\pi G}(G_{00} + \Lambda g_{00}) - \frac{1}{2} g_{00} R^0_i F_i^0 + (R_{00} \Phi - K_{00} \Phi) \\
+ \frac{1}{2} \sum_{n=1}^{\infty} f_n \left( g_{00} \partial^2 \partial \partial R_0^0 \partial \partial \partial^{-1} R_i^0 + g_{00} \partial \partial \partial^{-1} R_i^0 + 2 \partial \partial \partial \partial \partial \partial \partial \partial \partial^{-1} R_i^0 \right) = 0. 
\] 

(38)

4. Constant scalar curvature

**Theorem 4.1.** Let \( R = R_0 \) = constant. Then, solution of equations of motion (36) has the form

1. For \( R_0 > 0 \), \( a(t) = \sqrt{\frac{6k}{R_0} + \sigma e^{\frac{t}{\sqrt{R_0}}} + \tau e^{-\frac{t}{\sqrt{R_0}}}} \), where \( 9k^2 = R_0^2 \sigma \tau, \sigma, \tau \in \mathbb{R} \).
2. For \( R_0 = 0 \), \( a(t) = \sqrt{-kt^2 + \sigma t + \tau} \), where \( \sigma^2 + 4k\tau = 0 \), \( \sigma, \tau \in \mathbb{R} \).
3. For \( R_0 < 0 \), \( a(t) = \sqrt{\frac{6k}{R_0} + \sigma \cos \left( \frac{-R_0}{3} t + \tau \sin \left( \frac{-R_0}{3} t \right) \right)} \), where \( 36k^2 = R_0^2 (\sigma^2 + \tau^2) \), \( \sigma, \tau \in \mathbb{R} \), where \( k \) is curvature parameter.

**Proof.** Since \( R = R_0 \) one has

\[
6 \left( \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + k \frac{\dot{a}}{a} \right) = R_0. \] 

(39)

The change of variable \( b(t) = \dot{a}^2(t) \) yields second order linear differential equation with constant coefficients

\[
3 \ddot{b} - R_0 b = -6k. \] 

(40)

Depending on the sign of \( R_0 \) we have the following solutions for \( b(t) \)

\[
R_0 > 0, \quad b(t) = \frac{6k}{R_0} + \sigma e^{\frac{t}{\sqrt{R_0}}} + \tau e^{-\frac{t}{\sqrt{R_0}}}, \\
R_0 = 0, \quad b(t) = -kt^2 + \sigma t + \tau, \\
R_0 < 0, \quad b(t) = \frac{6k}{R_0} + \sigma \cos \left( \frac{-R_0}{3} t + \tau \sin \left( \frac{-R_0}{3} t \right) \right). 
\] 

(41)

Putting \( R = R_0 = \text{const} \) into (37) and (38) one obtains the following two equations:

\[
f_0 R_0^{p+q}(p + q - 2) = \frac{R_0 - 4\Lambda}{16\pi G}, \quad f_0 R_0^{p+q-1} \left( \frac{1}{2} R_0 + (p + q) R_{00} \right) = \frac{-G_{00} + \Lambda}{16\pi G}. 
\] 

(42)

Equations (42) will have a solution if and only if

\[
R_0^{p+q-1} (R_0 + 4R_{00}) (R_0 + (2\Lambda - R_0)(p + q)) = 0. 
\] 

(43)

In the first case we take \( R_0 + 4R_{00} = 0 \) that yields the following conditions on the parameters \( \sigma \) and \( \tau \):

\[
R_0 > 0, \quad 9k^2 = R_0^2 \sigma \tau, \\
R_0 = 0, \quad \sigma^2 + 4k\tau = 0, \\
R_0 < 0, \quad 36k^2 = R_0^2 (\sigma^2 + \tau^2). 
\] 

\( \square \)
Solutions given in \((41)\) together with conditions \((44)\) restrict the possibilities for the parameter \(k\).

**Theorem 4.2.**

1. If \(R_0 > 0\) then for \(k = 0\) there is solution with constant Hubble parameter, for \(k = +1\) the solution is \(a(t) = \sqrt{\frac{12}{R_0}} \cosh \left( \frac{1}{2} \sqrt{\frac{R_0}{3}} t + \varphi \right)\) and for \(k = -1\) it is \(a(t) = \sqrt{\frac{12}{R_0}} \sinh \left( \frac{1}{2} \sqrt{\frac{R_0}{3}} t + \varphi \right)\), where \(\sigma + \tau = \frac{6}{R_0} \cos \varphi\) and \(\sigma - \tau = \frac{6}{R_0} \sin \varphi\).
2. If \(R_0 = 0\) then for \(k = 0\) the solution is \(a(t) = \sqrt{t}\) const and for \(k = -1\) the solution is \(a(t) = |t + \frac{\varphi}{2}|\).
3. If \(R_0 < 0\) then for \(k = -1\) the solution is \(a(t) = \sqrt{-\frac{12}{R_0}} \cos \left( \frac{1}{2} \sqrt{-\frac{R_0}{3}} t - \varphi \right)\), where \(\sigma = \frac{6}{R_0} \cos \varphi\) and \(\tau = \frac{6}{R_0} \sin \varphi\).

**Proof.** Let \(R_0 > 0\). Set \(k = 0\) then we obtain solution with constant Hubble parameter. Alternatively, if we set \(k = +1\) then there is \(\varphi\) such that \(\sigma + \tau = \frac{6}{R_0} \cos \varphi\) and \(\sigma - \tau = \frac{6}{R_0} \sin \varphi\). Moreover, we obtain

\[
b(t) = \frac{12}{R_0} \cosh^2 \left( \frac{1}{2} \sqrt{\frac{R_0}{3}} t + \varphi \right), \quad a(t) = \frac{12}{R_0} \cosh \left( \frac{1}{2} \sqrt{\frac{R_0}{3}} t + \varphi \right).
\]

At the end, if we set \(k = -1\) one can transform \(b(t)\) to

\[
b(t) = \frac{12}{R_0} \sinh^2 \left( \frac{1}{2} \sqrt{\frac{R_0}{3}} t + \varphi \right), \quad a(t) = \frac{12}{R_0} \sinh \left( \frac{1}{2} \sqrt{\frac{R_0}{3}} t + \varphi \right).
\]

Let \(R_0 = 0\). If \(k = 0\) then function \(b(t)\) and consequently \(a(t)\) become constants which leads to a solution \(a(t) = \sqrt{t}\) const. On the other hand if \(k \neq 0\) then we can write

\[
b(t) = -k(t - \frac{\varphi}{2k}).
\]

If \(k = +1\) then there are no solutions for the scale factor \(a(t)\), because \(b(t) \leq 0\). On the other hand, when \(k = -1\) the scalar factor becomes

\[
a(t) = |t + \frac{\varphi}{2}|.
\]

In the last case, let \(R_0 < 0\). If \(k = -1\) we can find \(\varphi\) such that \(\sigma = \frac{6}{R_0} \cos \varphi\) and \(\tau = \frac{6}{R_0} \sin \varphi\) and rewrite \(a(t)\) and \(b(t)\) as

\[
b(t) = \frac{-12}{R_0} \cos^2 \left( \frac{1}{2} \sqrt{-\frac{R_0}{3}} t - \varphi \right), \quad a(t) = \frac{-12}{R_0} \cos \left( \frac{1}{2} \sqrt{-\frac{R_0}{3}} t - \varphi \right).
\]

In the case \(k = +1\) one can transform \(b(t)\) to \(b(t) = \frac{12}{R_0} \sinh^2 \left( \frac{1}{2} \sqrt{-\frac{R_0}{3}} t - \varphi \right)\), which is non positive and hence yields no solutions.

**Theorem 4.3.** If in \((43)\) we take

\[
R_0^{p^2 - 1}(R_0 + (p + q)(2\Lambda - R_0)) = 0
\]

then:

1. For \(p + q \geq 1\) there is obvious solution \(R_0 = 0\). In particular if \(p + q = 1\) then \((50)\) is satisfied for any \(R_0 \neq 0\) if \(\Lambda = 0\).
2. For \(p + q = 0\) there is no solution.
3. For \(p + q \neq 0, 1\) there is a unique value \(R_0 = \frac{2\Lambda(p + q)}{p + q - 1}\) that gives a solution. Since \(p\) and \(q\) are integers the value of \(R_0\) in the last equation is always positive, and for \(k = 0\) the solution \(b(t)\) is a linear combination of exponential functions.

Proof of this theorem is evident.
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