Diffusive effects on hydrodynamic Casson nanofluid boundary layer flow over a stretching surface

M I Anwar1,2, N Tanveer1, M Z Salleh3 and S Shafie4

1Department of Mathematics, Faculty of Science, University of Sargodha, Sargodha, Pakistan,
2Higher Education Department (HED), Lahore, Punjab, Pakistan,
3Faculty of Industrial Science and Technology, Universiti Malaysia Pahang, Gambang, Kuantan, Pahang, Malaysia.
4Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, Skudai, Johor, Malaysia.

E-mail: jeotarar@yahoo.com

Abstract. In this paper, diffusive effects on hydrodynamic Casson nanofluid boundary layer flow over a stretching surface is studied. A non-uniform magnetic field of strength is imposed in transverse direction. The arising nonlinear problem is governed by Casson fluid, Nanofluid and magnetic field parameter. The governing boundary layer equations are reduced into ordinary differential equations by using suitable similarity transformations. The transformed equations are solved numerically by using an implicit finite difference scheme known as the Keller-box method. A comparison of the obtained results is performed with the published results. It is found that velocity profiles are suppressed by increasing values of magmatic and Casson fluid parameter.

1. Introduction

In non-Newtonian fluids shear stresses and the rate of strain/deformation are not linearly related. Fluids such as lubricants, greases, multi grade oils, gypsum pastes, printer inks, ceramics, polymers, liquid detergents, blood, paints, fruit juices etc. change their viscosity under the stress and deviate from the classical Newton’s law of viscosity, Such fluids are called Casson fluid. Casson fluid exhibit the yield stress like an elastic solid. In 1959, Casson presented Casson fluid model for the flow of viscoelastic fluids.

In present few years non-Newtonian fluids have become very important due to its applications. S. Nadeem et al. [1] studied MHD boundary layer flow of Casson fluid over an exponentially permeable shrinking sheet. Shehzad et al. [3] analyzed the effect of mass transfer in MHD flow of Casson fluid over porous stretching sheet in the presence of chemical reaction and obtained a series of solutions for resulting non-linear flow. Further Pramanik [4] studied the Casson fluid over exponentially stretching sheet in the presence of thermal radiation to obtained same variations in velocity field and temperature.

Today more than ever numerous industries facing thermal challenges. Nanofluid are the new type of heat transfer fluid by the uniform and stable suspension of nanometer sized particles such as
Al₂O₃, Cu, or CuO into liquids. Choi (1995) described the new class of nanotechnology based heat transfer fluids that exhibit thermal properties superior to those of their host fluids for conventional particle fluid suspensions [5]. Nanomaterials have unique mechanical, optical, electrical, magnetic and thermal properties [5]. Buongiorno [6] has been studied that how to increase in thermal conductivity of nanofluid in comparison with the base fluid and found that the Brownian motion and thermophoresis effects in the base fluid enhanced the thermal conductivity of the liquid. Khan and Pop [7] firstly investigate the flow of the nanofluid together with Brownian and thermophoresis motion on the stretching surface.

The flow over a stretching surface has important applications in many engineering processes such as extrusion, melt-spinning, the hot rolling, wire drawing, glass fiber production, manufacture of plastic and rubber sheets, cooling of a large metallic plate in a bath, which may be an electrolyte etc. Sakiadis [8] introduced his pioneering work with a study of boundary layer behavior on continuous solid surfaces. Further Fox et al. [9] extended the work of Sakiadis and investigated the laminar boundary layer on a moving continuous surface with suction and injection and presented numerical solutions of the boundary layer equations for momentum, heat and mass transfer for various values of the parameters. Gupta et al. [10] enhanced the study and investigated heat and mass transfer on a stretching sheet with suction and blowing. All of above investigations are restricted Newtonian fluids.

Non-Newtonian fluids have many important industrial applications. Magyari et al. [11] described boundary layers on an exponentially stretching continuous surface with an exponential temperature distribution and examined both analytically and numerically. Bidin et al. [12] investigated numerically the effect of thermal radiation on the steady laminar two-dimensional boundary layer flow and heat transfer over an exponentially stretching sheet.

Motivated by the above investigations, the present work concerned with diffusive effects on hydrodynamic Casson nanofluid boundary layer flow over a stretching surface. By applying the suitable similarity transformations the system of nonlinear partial differential equations reduced into the system of nonlinear ordinary differential equations. Nondimensional physical parameters appear after reduction along with the system of coupled ordinary differential equations which governs the behavior of fluid and then solved numerically by using implicit method known as Keller Box method. A comparison with the published results and the behavior of each physical parameter are shown through tables and different figures.

2. Problem formulation

Consider a steady, viscous, incompressible, two-dimensional boundary layer flow of Casson nanofluid over an exponentially stretching sheet. The stretching and free stream velocities are assumed to be of the forms \( u_\infty(x) = ae^x \) and \( u_\infty(x) = 0 \) respectively, where \( a \) is constant, \( x \) is the coordinate measured along the stretching surface and \( l \) is the length of the sheet. The temperature \( T \) and the nanoparticles fraction \( C \) take constant values \( T_\infty \) and \( C_\infty \), respectively at the wall, whereas the ambient values of temperature \( T_\infty \) and the nanoparticles fraction \( C_\infty \) are attained as \( y \) tends to infinity as shown in figure 1. A non-uniform magnetic field of strength \( B(x) = B_0 e^{-\gamma y} \) is imposed in transverse direction (normal to the flow direction), where \( B_0 \) is the uniform magnetic field strength. It is assumed here that the induced magnetic field due to the motion of an electrically conducting fluid is negligible. Further, external electrical field is zero and the electric field due to the polarization of charges is negligible. The governing boundary layer equations are as follows:
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \]  
\[ u \left( \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = v \left( \frac{1}{\beta} \right) \dot{u}^2 - \frac{\sigma B^2(x)}{\rho f}, \]  
\[ \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{1}{\alpha} \frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho c_f} \frac{\partial q}{\partial y} + \left[ D_r \frac{\partial C}{\partial y} \frac{\partial T}{\partial y} + D_r \left( \frac{\partial T}{\partial y} \right)^2 \right] \]  
\[ + \frac{\mu}{\rho c_p} \left( 1 + \frac{1}{\beta} \right) \left( \frac{\partial u}{\partial y} \right)^2, \]  
\[ u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D_h \frac{\partial^2 C}{\partial y^2} + D_r \frac{\partial^2 T}{\partial y^2} - K_1 (C - C_{\infty}), \]

where the Rosseland approximation (for radiation flux) is defined as:

\[ q_r = \frac{4\sigma T^4}{3k^2}. \]  

Expanding \( T^4 \) in Taylor series about \( T_{\infty} \) and neglecting higher order terms results for:

\[ T^4 \approx 4T_{\infty}^4T - 3T_{\infty}^4. \]

After substituting equation (5) and (6), equation (3) reduces to:

\[ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \left( \frac{16\sigma^4}{3k^2} \right) \frac{\partial^2 T}{\partial y^2} + \mu \left( 1 + \frac{1}{\beta} \right) \left( \frac{\partial u}{\partial y} \right)^2. \]  

The subjected boundary conditions are:

\[ u = u^\infty(y) = \alpha e^{x^2}, v = 0, T = T^\infty(x), C = C^\infty(x) \quad \text{at} \quad y = 0, \]  
\[ u \rightarrow 0, v \rightarrow 0, T \rightarrow T_{\infty}, C \rightarrow C_{\infty} \quad \text{as} \quad y \rightarrow \infty. \]

The prescribed temperature and concentration on the surface of stretching sheet are assumed to be of the form \( T^\infty(x) = T_{\infty} + T_0 e^{x^2} \) and \( C^\infty(x) = C_{\infty} + C_0 e^{x^2} \), where \( T_0 \) and \( C_0 \) are the reference temperature and concentration respectively. The nonlinear partial differential equation are reduced into nonlinear ordinary differential equations. For the sake of this purpose the stream function \( \psi = \psi(x, y) \) is defined as:

\[ u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \]  

where the continuity equation (1) is satisfied identically. Using the similarity transformations defined as:

\[ \psi = \sqrt{2f(\eta)}, \quad \theta(\eta) = \frac{T - T_{\infty}}{T_{\infty} - T_{\infty}}, \]
\[ \phi(\eta) = \frac{C - C_\infty}{C_u - C_\infty}, \quad \eta = y \sqrt{\frac{a}{2\nu \ell}} e^{x/\ell}. \] (10)

On substituting equations (9) and (10) in equations (2), (4) and (7) reduced to the nonlinear ordinary differential equations

\[ \left(1 + \frac{1}{\beta}\right)f'' + ff' - 2f^{''2} - Mf' = 0, \] (11)

\[ Pr\theta'' + f\theta' - f'\theta + Ec\left(1 + \frac{1}{\beta}\right)f^{''2} + Nb\theta'\phi' + Nt\theta'\phi = 0, \] (12)

\[ \phi'' + Le\phi' - Le\phi + Nt\theta' = LeR\phi = 0, \] (13)

where

\[ v = \frac{\mu}{\rho_f}, Pr = \frac{v}{\alpha}, Le = \frac{v}{D_u}, Ec = \frac{u^2}{C(T_w - T_\infty)}, Nt = \frac{Nt}{Nb}, Nb = \frac{\frac{\partial D_u(C_w - C_\infty)}{v}}{v}, \]

\[ Nt = \frac{\frac{\partial D_u(T_w - T_\infty)}{vT_\infty}}{v}, R = \frac{2lK_1}{u}, \]

\[ M = \frac{2l\sigma B_0^2}{\alpha \rho_f}, Pr_N = \frac{1}{Pr^{1/3}} \left(1 + \frac{4}{3}N^{1/3}\right), N = \frac{4\sigma T_{w}^{3}}{kk^3}. \] (14)

The corresponding boundary conditions (8) are transformed to:

\[ f(\eta) = 0, f'(\eta) = 1, \theta(\eta) = 1, \phi(\eta) = 1 \quad \text{at} \quad \eta = 0, \]

\[ f'(\eta) \to 0, \theta(\eta) \to 0, \phi(\eta) \to 0 \quad \text{as} \quad \eta \to \infty. \] (15)

The important quantities of physical interest are velocity, heat and mass transfer respectively, which are presented in terms of Skin-friction coefficient \(C_f\), Nusselt number \(Nu\) and Sherwood number \(Sh\). Using the transformed variables (10), the non-dimensional expressions for the Skin friction coefficient \(C_f(0) = \left(1 + \frac{1}{\beta}\right) f''(0)\), reduced Nusselt number \(-\theta'(0)\) and reduced Sherwood number \(-\phi'(0)\) respectively defined as:

\[ C_f(0) = \sqrt{\frac{2l}{x}} Re_s C_f, -\theta'(0) = \frac{Nu}{\sqrt{(x/2l) Re_s}}, -\phi'(0) = \frac{Sh}{\sqrt{(x/2l) Re_s}}. \] (16)

Where \( Re_s = \frac{U_w x}{\nu} \) is the local Reynolds number based on the stretching velocity.

3. Numerical procedure

The numerical procedure for Keller Box method [15] for MHD Casson nanofluid is explained for the finite difference method, Newton’s method, block-elimination method and starting conditions.

4. Results and discussions

The coupled nonlinear ordinary differential equations (11) to (13) subjected to the boundary conditions (15) are solved numerically by using the finite difference scheme name as the Keller Box method. The numerical results for physical parameters of interest such as Brownian motion
parameter \( Nb \), thermophoresis parameter \( Nt \), Casson fluid parameter \( \beta \), Eckert number \( Ec \), Chemical reaction parameter \( R \), radiation parameter \( N \), Prandtl number \( Pr \), Lewis number \( Le \) and Hartmann number \( M \) are given in tabular form (Table 1 and Table 2).

Table 1 describes a comparison of the reduced Nusselt number \(-\dot{\theta}(0)\) with the results given by Bidin and Nazar [12] and Ishak [14]. Table 2 shows the variations of the reduced Nusselt number \(-\dot{\theta}(0)\), the reduced Sherwood number \(-\phi(0)\) and Skin-friction coefficient \( C_n(0)\) for different values of \( Nb, Nt, Pr, Le, \beta, Ec, M, N \) and \( R \). It is noted that the reduced Nusselt number \(-\dot{\theta}(0)\) decreases for increase in \( Nb, Ec, M \) and \( R \) whereas increases for increase in \( Nt, \beta, Pr, Le \) and \( N \). Where the reduced Sherwood number \(-\phi(0)\) decreases for increase in \( Nt, \beta, M, N \) whereas increases for increase in \( Nb, Ec, Le, Pr, R \). Further the skin-friction coefficient \( C_n(0)\) decreases for increase in \( Pr, M \) while increases for increasing values of \( \beta \) and \( Le \).

Figures 2 to 8 characterize the flow phenomenon of Casson nanofluid through different governing parameters. Figure 2 depicts velocity profile for the different values of \( \beta \) by taking fixed values of \( Nb, Nt, Le, M, N, Pr, Ec \). This is because the decreasing yield stress suppressed the velocity field. Figure 3 shows the effects of \( M \) on velocity profile \( f'\eta(\eta)\) for the fixed values of \( Nb, Nt, N, R, Pr, Le, \beta \) and \( Ec \). This figure shows that velocity profile \( f'\eta(\eta)\) decreases for increasing values of \( M \). As \( M \) increases, the Lorentz force which opposes the flow, also increases and leads to enhance the deceleration of flow.

The temperature profile is shown to decreasing for the several values of radiation parameter for the fixed values of \( Nb, Nt, R, M, Pr, Le, \beta \) and \( Ec \) in figure 4. As the rate of heat transfer enhanced with the radiation parameter \( N \), this cause the reduction in thermal boundary layer thickness. From figures 6 and 7 it is observed that temperature profile increases for increasing values of \( Nb \) and \( Nt \) respectively. This is due to zigzag motion of particles. In which particles gain the kinetic energy results in increase the collisions of particles. That’s why \( \theta(\eta)\) increases for increasing values of \( Nt \). As thermophoresis causes small particles to bedriven away from hot surface and move towards the cold surface. So, temperature profile in figure 7 increases. Figure 5 shows that \( \theta(\eta)\) decreases for increasing values of \( Pr \). Since \( Pr \) is the ratio of viscous diffusion rate to the thermal diffusion rate. So, higher Prandtl number causes to reduce the thermal diffusivity and consequently \( \theta(\eta)\) is decreased.

The influence of the chemical reaction parameter on the concentration profile is displayed in figure 8. It can be observed from the figure that concentration profile decreases with increasing values of chemical reaction parameter. A decrease in concentration profile is observed due to the increase in interfacial mass transfer.
Table 1. Comparison of the reduced Nusselt number $-\theta (0)$ when $Nb = Nt = Le = R = Ec = 0$ and $\beta \to \infty$ [1]

| Pr | $M$ | $N$ | Bidin and Nazar[12] | Ishak[14] | Present Results |
|----|-----|-----|---------------------|-----------|-----------------|
|    |     |     | $-\theta (0)$      | $-\theta (0)$ | $-\theta (0)$ |
| 1  | 0   | 0   | 0.9548             | 0.9548     | 0.9548          |
| 2  | 0   | 0   | 1.4714             | 1.4714     | 1.4714          |
| 3  | 0   | 0   | 1.8691             | 1.8691     | 1.8691          |
| 1  | 1.0 | 1.0 | 0.5315             | 0.5312     | 0.5312          |
| 1  | 1.0 | 1.0 | -                 | 0.8611     | 0.8611          |

Table 2. Variations of the local Nusselt number $-\theta (0)$, the local Sherwood number $-\phi (0)$ and Skin-friction coefficient $C_x (0)$

| Nb  | $Nt$ | $\beta$ | $Pr$ | $Ec$ | $Le$ | $M$ | $N$ | $R$ | $-\theta (0)$ | $-\phi (0)$ | $C_x (0)$ |
|-----|-----|--------|-----|-----|-----|-----|-----|-----|----------------|-------------|-----------|
| 0.1 | 0.1 | 5.0    | 5.0 | 5.0 | 0.1 | 0.1 | 0.1 | 0.1 | 1.0944         | 2.3719      | 1.2059    |
| 0.5 | 0.1 | 5.0    | 6.5 | 0.5 | 5.0 | 0.1 | 0.1 | 0.1 | 0.3212         | 2.6761      | 1.2059    |
| 0.1 | 0.5 | 5.0    | 6.5 | 5.0 | 1.0 | 0.1 | 0.1 | 0.1 | 0.6421         | 2.6694      | 1.2059    |
| 0.1 | 0.1 | 7.0    | 10.0| 0.5 | 5.0 | 0.1 | 0.1 | 0.1 | 1.0994         | 2.3595      | 1.2357    |
| 0.1 | 0.1 | 5.0    | 10.0| 0.5 | 5.0 | 0.1 | 0.1 | 0.1 | 1.0996         | 2.4181      | 1.2059    |
| 0.1 | 0.1 | 5.0    | 10.0| 5.0 | 9.0 | 0.1 | 0.1 | 0.1 | 0.3507         | 2.9240      | 1.2059    |
| 0.1 | 0.1 | 5.0    | 10.0| 5.0 | 3.0 | 0.1 | 0.1 | 0.1 | 0.9797         | 4.7852      | 1.2059    |

Figure 1. Physical model.

Figure 2. Velocity profile against $\eta$ for different values of $\beta$. 
5. Conclusions
Present study numerically investigated the radiation and chemical reaction effects on Casson type MHD nanofluid flow over an exponentially stretching sheet. Non-Newtonian fluids with the suspended nanoparticles have great importance due to their superior properties and are beneficial in many industrial and technological fields. The reduced Nusselt number \( -\theta(0) \) decreases for increase in \( N_t, \beta, \) and \( N \) whereas increases for increase in \( N_t, \beta, M, N \) and decreases for increase in \( N_t, \beta, M, N \) and increases for increase in \( \beta, \) and \( Le. \) Velocity, temperature and concentration profiles are suppressed by the increasing values of Casson fluid, Magnetic field, radiation and chemical reaction parameters.

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