A Special Nonlinear Connection in Second Order Geometry

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1 Introduction

As shown in [21], nonlinear connections on bundles can be a powerful tool for integrating systems of differential equations. A way of obtaining them is that of deriving them out of the respective systems of DE’s, in particular, out of variational principles, [1], [10], [12]. In particular an ODE system of order 2 on a manifold $M$ induces a nonlinear connection on its tangent bundle. A remarkable example is here the Cartan nonlinear connection of a Finsler space, which has the property that its autoparallel curves correspond to geodesics of the base manifold:

$$\frac{\delta y^i}{dt} := \frac{dy^i}{dt} + N^i_j y^j = 0.$$ 

Further, an ODE system of order three determines a nonlinear connection on the second order tangent (jet) bundle $T^2M = J^2_0(\mathbb{R}, M)$. For instance, Craig-Synge equations (R. Miron, [10])

$$\frac{d^3 x^i}{dt^3} + 3G^i(x, \dot{x}, \ddot{x}) = 0,$$

lead to:

a) Miron’s connection:

$$M^i_j = \frac{\partial G^i}{\partial y^{(2)j}}, M^i_j = \frac{1}{2} \left( S M^i_j + M^i_m M^m_j \right),$$

where $S = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} - 3G^i \frac{\partial}{\partial y^{(2)i}}$. 

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b) Bucățaru’s connection

\[ M^i_j = \frac{\partial G^i}{\partial y^{(2)}_j}, M^i_j = \frac{\partial G^i}{\partial y^{(1)}_j}. \]

W.r.t. the last one, if \( G^i \) are the coefficients of a spray on \( T^2M \) (i.e., 3-homogeneous functions), then the Craig-Synge equations can be interpreted as:

\[ \frac{\delta y^{(2)}_i}{\delta t} = 0, \tag{2} \]

where \( \frac{\delta y^{(2)}_i}{\delta t} := \frac{dy^{(2)}_i}{dt} + M^i_j \frac{dy^{(1)}_j}{dt} + M^i_2 \frac{dx^j}{dt}. \)

In Miron’s and Bucățaru’s approaches, nonlinear connections on \( T^2M \), are obtained from a Lagrangian of order 2, \( L(x, \dot{x}, \ddot{x}) \), by computing the first variation of its integral of action.

Here, we propose a different approach, which, we consider, is at least as interesting as the above one from the point of view of Mechanics - just by having in view that the big majority of known Lagrangians are of order one.

Namely, we start with a first order Lagrangian \( L(x, \dot{x}) \) and compute its second variation; out of it, we obtain a nonlinear connection on \( T^2M \), such that the obtained distributions correspond to extremal curves and their deviations, respectively.

As a remark, our nonlinear connection is also suitable for modelling the solutions of a (globally defined) ODE system, not necessarily attached to a certain Lagrangian, together with their deviations.

More precisely, in the following our aims are:

1. to obtain the Jacobi equations for the trajectories

\[ \frac{\delta y^i}{\delta t} = F^i(x, y) \]

(for extremal curves of a 2-homogeneous Lagrangian \( L(x, \dot{x}) \) in presence of external forces).

2. (main result): to build a nonlinear connection such that:

\[ w \in \mathcal{X}(M) \text{ Jacobi field along } c \Leftrightarrow \frac{\delta w^{(2)}_i}{\delta t} = 0, \]

where \( \frac{d}{dt} \) denotes directional derivative w.r.t. \( \dot{c} \) and \( \frac{\delta w^{(2)}_i}{\delta t} = \frac{1}{2} \frac{d^2 w^i}{dt^2} + M^i_1 \frac{dw^i}{dt} + M^i_2 w^j. \)

Properties:

I. For Finsler spaces \( (M, F) \), \( c \) is a geodesic of \( M \) if and only if its extension \( T^2M \) is horizontal.

II. For a vector field \( w \) along a geodesic \( c \) on \( M \), we have:
1. \( \frac{\delta w^i}{dt} = 0 \) if and only if \( w \) is parallel along \( \dot{c} = y \).

2. \( \frac{\delta w^{(2)i}}{dt} = 0 \) if and only if \( w \) is a Jacobi field along \( c \).

2 Tangent Bundle and 2-Tangent Bundle

Let \( M \) be a real differentiable manifold of dimension \( n \) and class \( C^\infty \); the coordinates of a point \( x \in M \) in a local chart \( (U, \phi) \) will be denoted by \( \phi (x) = (x^i), \ i = 1, \ldots, n \). Let \( TM \) be its tangent bundle and \( (x^i, y^i) \) the coordinates of a point in a local chart.

The 2-tangent bundle (or 2-osculator bundle) \( (T^2M, \pi^2, M) \) is the space of jets of order two in a fixed point \( t_0 \in I \subset \mathbb{R} \), of functions \( f : I \to M, t \mapsto (f^i(t)) \), \((14)-(19)\).

In a local chart, a point \( p \) of \( T^2M \) will have the coordinates \( (x^i, y^i, y^{(2)i}) \).

This is, \( x^i = f^i(t_0), \ y^i = \dot{f}^i(t_0), \ y^{(2)i} = \frac{1}{2} \ddot{f}^i(t_0) \).

Obviously, \( (T^2M, \pi^2, M) \) is a differentiable manifold of class \( C^\infty \) and dimension \( 3n \), and \( TM = \text{Osc}^1M \) can be identified with a submanifold of \( T^2M \).

For a curve \( c : [0, 1] \to M, t \mapsto (x^i(t)) \), let us denote:

- by \( \hat{c} \) its extension to the tangent bundle \( TM : \)
  \( \hat{c} : [0, 1] \to M, t \mapsto (x^i(t), \dot{x}^i(t)) ; \)

  along \( \hat{c} \), there holds:
  \( y^i = \dot{x}^i(t); \)

- by \( \tilde{c} \) its extension to \( T^2M : \)
  \( \tilde{c} : [0, 1] \to T^2M, \ t \mapsto (x^i(t), \dot{x}^i(t), \frac{1}{2} \ddot{x}^i(t)) ; \)

  along such an extension curve, there holds
  \( y^i(t) = \dot{x}^i(t), \ y^{(2)i}(t) = \frac{1}{2} \ddot{x}^i(t). \)

3 Nonlinear connections on \( TM \)

A nonlinear (Ehresmann) connection on \( TM \) is a splitting of the tangent space \( T(TM) \) in every point \( p \in TM \) into a direct sum

\[ T_p(TM) = N(p) \oplus V(p), \]  \((3)\)

each one of dimension \( n \). This generates two distributions:
• the horizontal distribution \( N : p \mapsto N(p) \);
• the vertical distribution \( V : p \mapsto V(p) \).

A local adapted basis to the above decomposition is:

\[
B = \{ \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i} \},
\]

where:

\[
\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^i_j \frac{\partial}{\partial y^j}.
\] (4)

With respect to changes of local coordinates on \( M \), \( \frac{\delta}{\delta x^i} \) transform by the same rule as vector fields on \( M \):

\[
\frac{\delta}{\delta x^i} = \frac{\partial}{\partial \tilde{x}^i} \frac{\delta}{\delta x^i}.
\]

The dual basis of \( B \) is \( B^* = \{ dx^i, \delta y^i \} \), given by

\[
\delta y^i = dy^i + N^i_j dx^j.
\] (5)

The quantities \( N^i_j \) are called the coefficients of the nonlinear connection \( N \).

Any vector field \( X \in \mathcal{X}(TM) \) is represented in the local adapted basis as

\[
X = X^{(0)i} \frac{\delta}{\delta x^i} + X^{(1)i} \frac{\partial}{\partial y^i}.
\] (6)

Similarly, a 1-form \( \omega \in \mathcal{X}^*(TM) \) will be decomposed as

\[
\omega = \omega^{(0)i} dx^i + \omega^{(1)i} \delta y^i.
\] (7)

In particular, if \( \hat{c} : t \rightarrow (x^i(t), y^i(t)) \) is an extension curve to \( TM \), then its tangent vector field is

\[
T = \frac{dx^i}{dt} \delta^{(0)i} + \frac{\delta y^i}{dt} \delta^{(1)i}.
\] (8)

We should mention an important result, (R. Miron, [12]):

**Proposition 1** Let \( L = L(x, \dot{x}) \) be a nondegenerate Lagrangian: \( \det(\frac{\partial^2 L}{\partial y^i \partial y^j}) \neq 0 \), and \( g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j} \), the induced (Lagrange) metric tensor. Then, the equations of evolution of a mechanical system with the Lagrangian \( L \) and the external force field \( F = F_i(x, \dot{x}) dx^i \) are

\[
\frac{d^2 x^i}{dt^2} + 2G^i(x, \dot{x}) = \frac{1}{2} F^i(x, \dot{x}),
\] (9)

where

\[
2G^i = \frac{1}{2} g^{ik} (\frac{\partial^2 L}{\partial y^k \partial x^j} y^j - \frac{\partial L}{\partial x^j}),
\]

is the canonical semispray of the Lagrange space \((M, L)\) and \( F^i = g^{ij} F_j \).
In the following, we shall use the above result in the case when $G$ is a spray, i.e., its components $G^i$ are 2-homogeneous w.r.t. $y^j$:

$$2G^i = \frac{\partial G^i}{\partial y^j} y^j.$$ 

Then, [1], [5], [13], the quantities $N^i_j = \frac{\partial G^i}{\partial y^j}$ are the coefficients of a nonlinear connection on $TM$, and, moreover, the equations (9) take the form:

$$\frac{\delta y^i}{dt} = \frac{1}{2} F^i.$$  \hspace{1cm} (10)

In particular, if there are no external forces, $F^i = 0$, then the extremal curves $t \mapsto x^i(t)$ of the Lagrangian $L$ have horizontal extensions and vice-versa: horizontal extension curves $\hat{c}$ project onto solution curves of the Euler-Lagrange equations of $L$.

4 Nonlinear connections on $T^2M$

A nonlinear connection on $T^2M$ is a splitting of the tangent space in every point $p$ into three subspaces:

$$T_p(T^2M) = N_0(p) \oplus N_1(p) \oplus V_2(p),$$  \hspace{1cm} (11)

each one of dimension $n$. This generates three distributions:

- the horizontal distribution $N_0 : p \mapsto N(p)$;
- the $v_1$-distribution $N_1 : p \mapsto N_1(p)$;
- the $v_2$-distribution $V_2 : p \mapsto V_2(p)$.

We denote by $h = v_0$, $v_1$ and $v_2$ the projectors corresponding to the above distributions.

Let $B$ denote a local adapted basis to the decomposition (11):

$$B = \{ \delta_{(0)i} := \frac{\delta}{\delta x^i}, \delta_{(1)i} := \frac{\delta}{\delta y^{(0)i}}, \delta_{(2)i} := \frac{\delta}{\delta y^{(2)i}} \},$$

this is, $N_0 = Span(\delta_{(0)i})$, $N_1 = Span(\delta_{(1)i})$, $V_2 = Span(\delta_{(2)i})$. The elements of the adapted basis are locally expressed as

$$\begin{align*}
\delta_{(0)i} &= \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_{(1)i} \frac{\partial}{\partial y^j} - N^j_{(2)i} \frac{\partial}{\partial y^{(2)}}; \\
\delta_{(1)i} &= \frac{\delta}{\delta y^{(0)i}} = \frac{\partial}{\partial y^i} - N^j_{(1)i} \frac{\partial}{\partial y^{(2)}}; \\
\delta_{(2)i} &= \frac{\delta}{\delta y^{(2)i}} = \frac{\partial}{\partial y^{(2)i}}.
\end{align*}$$  \hspace{1cm} (12)
With respect to changes of local coordinates on $M$, $\delta_{(\alpha)i}$, $\alpha = 0, 1, 2$, transform by the same rule as vector fields on $M$:

$$\delta_{(\alpha)i} = \frac{\partial \tilde{x}_j}{\partial x^i} \delta_{(\alpha)j}.$$ (13)

The dual basis of $\mathcal{B}$ is $\mathcal{B}^* = \{dx^i, \delta y^{(1)i}, \delta y^{(2)i}\}$, given by

$$\delta y^{(0)i} = dx^i,$$

$$\delta y^{(1)i} = \delta y^i = dy^i + M^i_{(1)j} dx^j,$$

$$\delta y^{(2)i} = dy^{(2)i} + M^i_{(1)j} dy^{(1)j} + M^i_{(2)j} dx^j.$$ (14)

Also, w.r.t. local chart changes, $\delta y^{(\alpha)i}$ behave exactly as 1-forms on $M$:

$$\delta y^{(\alpha)i} = \frac{\partial x^i}{\partial \tilde{x}^j} \delta y^{(\alpha)j}.$$ (15)

The quantities $N^j_{(1)i}$, $N^j_{(2)i}$ are called the coefficients of the nonlinear connection $N$, while $M^i_{(1)j}$ and $M^i_{(2)j}$ are called its dual coefficients.

Then, a vector field $X \in \mathcal{X}(T^2M)$ is represented in the local adapted basis as

$$X = X^{(0)i}\delta_{(0)i} + X^{(1)i}\delta_{(1)i} + X^{(2)i}\delta_{(2)i},$$ (16)

with the three right terms (called $d$-vector fields) belonging to the distributions $N$, $N_1$ and $V_2$ respectively.

A 1-form $\omega \in \mathcal{X}^*(T^2M)$ will be decomposed as

$$\omega = \omega^{(0)}_i dx^i + \omega^{(1)}_i \delta y^i + \omega^{(2)}_i \delta y^{(2)i}.$$ (17)

Similarly, a tensor field $T \in \mathcal{T}^r_s(T^2M)$ can be split into components, named $d$-tensor fields, which transform w.r.t. local coordinate changes in the same way as tensors on $M$.

In particular, if $\tilde{c} : t \rightarrow (x^i(t), y^i(t), y^{(2)i}(t))$ is an extension curve, then its tangent vector field is

$$T = \frac{dx^i}{dt} \delta_{(0)i} + \frac{\delta y^{(1)i}}{dt} \delta_{(1)i} + \frac{\delta y^{(2)i}}{dt} \delta_{(2)i}. $$ (18)

Our aim is to give a precise meaning to the equality $v_2(\tilde{c}) = 0$.

5 Berwald linear connection

Let $G^i$ be the coefficients of a spray on $TM$, and

$$N^i_j = \frac{\partial G^i}{\partial y^j},$$

the coefficients of the induced nonlinear connection (on $TM$).  


Let also
\[ L^i_{jk}(x, y) = \frac{\partial N^i_j}{\partial y^k} = \frac{\partial G^i}{\partial y^j \partial y^k}, \]
the local coefficients of the induced Berwald linear connection on \( TM \).

Now, let on \( T^2M \), \( N^i_{(1)j} = N^i_j(x, y^{(1)}) \), as above. The Berwald connection on \( T^2M \), \([7]\), is the linear connection defined by
\[
\begin{align*}
D_{\delta (0)k} \delta (0)j & = L^i_{jk} \delta (0)i, \\
D_{\delta (0)j} \delta (0)k & = 0.
\end{align*}
\]
This is, with the notations in \([10]\), the coefficients of the Berwald linear connection are \( B_\Gamma(N) = (L^i_{jk}, 0, 0) \).

For extension curves \( \tilde{c} \), we can express the \( v_1 \) component \( \dot{y}^i \) (the geometrical acceleration, \([8]\)) by means of the Berwald covariant derivative:
\[
\frac{D\dot{y}^i}{dt} := D_{\dot{c}}y^i = \frac{\delta y^i}{dt}.
\]

Let \( T \) denote its torsion tensor, and:
\[
R^i_{jk} = v_1 T(\delta (0)k, \delta (0)j) = \delta (0)k N^i_j - \delta (0)j N^i_k;
\]
also, let \( \mathbb{R} \) be the curvature tensor, and
\[
\begin{align*}
R^i_{jk} \delta (0)i & = h \mathbb{R}(\delta (0)i, \delta (0)k) = L^i_{jk} - \delta_k L^i_{jl} + L^m_{jk} L^l_{ml} - L^m_{jl} L^i_{mk}, \\
P^i_{jk} \delta (0)i & = h \mathbb{R}(\delta (1)i, \delta (0)k) = \delta (1)i L^i_{jk}.
\end{align*}
\]
its local components, which will be used in the following.

6 Jacobi equations for systems with external forces

Let us consider the following PDE system
\[
A^i = \frac{\partial^2 X^i}{\partial x^k \partial x^j} y^j y^k + \left( 2 \delta^j_i y^{(2)k} + a^i_j(x, y, y^{(2)}) y^k \right) \frac{\partial X^j}{\partial x^k} + b^j_j(x, y, y^{(2)}) X^j = 0,
\]
where:
- the unknown functions \( X^i = X^i(x) \) are the local coordinates of a vector field on \( M \);
- w.r.t. local coordinate changes, \( A^i \) transform as: \( \tilde{A}^i = \frac{\partial \tilde{x}^j}{\partial x^i} A^j \) (i.e., \( A^i \) are the components of a d-vector field on \( T^2M \)).
Then, it is easy to check

**Proposition 2** The quantities

\[
M_{(1)}^{i,j}(x, y, y^{(2)}) = \frac{1}{2} a^{i}_{j}(x, y, y^{(2)}),
\]

\[
M_{(2)}^{i,j}(x, y, y^{(2)}) = \frac{1}{2} b^{i}_{j}(x, y, y^{(2)}),
\]

are the dual coefficients of a nonlinear connection on \( T^2 M \).

Now, let us interpret (20). If \( c : t \in [0, 1] \rightarrow M \) is a curve, \( \tilde{c} \) is its extension to \( T^2 M \), and \( X \) is a vector field along \( c \), then (20) can be reexpressed in the following way:

\[
\frac{d^2 X^i}{dt^2} + 2 M_{(1)}^{i,j} \frac{dX^j}{dt} + 2 M_{(2)}^{i,j} X^j = 0.
\]

It is now convenient to denote

\[
X^{(2)j} = \frac{1}{2} \frac{dX^j}{dt},
\]

with these notation, (20) can be brought to the much more familiar form

\[
\frac{\delta X^{(2)i}}{dt} = 0,
\]

which leads to the following interpretation: the integral curves of \( X \) have extensions of second order with vanishing \( v_2 \)-components.

Let us suppose that we know a priori a nonlinear connection on \( TM \), with 1-homogeneous coefficients \( N^{i}_j = \frac{\partial G^i}{\partial y^j} \). Let \( c : [0, 1] \rightarrow M, t \mapsto x^i(t) \) be a curve and \( \tilde{c} : [0, 1] \rightarrow TM, t \mapsto (x^i(t), y^i(t) = \dot{x}^i(t)) \) its extension to \( TM \). Let us suppose that \( x^i \) are solutions for the system of ODE’s

\[
\frac{\delta y^i}{dt} = F^i(x, y),
\]

where \( F^i \) are the components of a \( d \)-vector field on \( M \) (for commodity of notations, we omitted the \( \frac{1}{2} \) in front of \( F \))

Let \( x^i(t, u) \) be a variation of \( c \) (not necessarily with fixed endpoints) \( y^i = y^i = \frac{\partial x^i}{\partial t} \) and

\[
\partial x^i \bigg|_{u=0} = w^i(t)
\]

its associated deviation vector field.
Then, by commuting partial derivatives, we have as immediate consequences
\[
\frac{\partial y^i}{\partial u} = \frac{dw^i}{dt} = 2w^{(2)i}, \quad \frac{\partial^2 y^i}{\partial t^2} = 2\frac{d^2 w^i}{dt^2} = 2\frac{dw^{(2)i}}{dt}, \quad \text{etc.}
\]

Let us denote
\[
\frac{\delta y^i}{\partial t} = \frac{\partial y^i}{\partial t} + M^i_{\ j}(x, y) y^j, \\
\frac{\delta y^i}{\partial u} = \frac{\partial y^i}{\partial u} + M^i_{\ j}(x, y) w^j, \\
\frac{\delta w^i}{\partial t} = \frac{\partial w^i}{\partial t} + M^i_{\ j}(x, y) w^j;
\]
then, \(\frac{\delta y^i}{\partial t}, \frac{\delta y^i}{\partial u}\) and \(\frac{\delta w^i}{\partial t}\) define d-vector fields on \(T^2M\) (the covariant derivatives "with reference vector \(\dot{c}\)" of \(\dot{c}\) and \(w\)).

It is also immediate that the last two covariant derivatives coincide:
\[
\frac{\delta y^i}{\partial u} = \frac{\delta w^i}{\partial t},
\]
which can be interpreted in terms of Berwald connection on \(TM\) as
\[
D_{\frac{\partial c}{\partial u}}(h(\frac{\partial c}{\partial t})) = D_{\frac{\partial c}{\partial t}}(h(\frac{\partial c}{\partial u})).
\]

By applying \(D_{\frac{\partial c}{\partial t}}\) again to the above quality and by permuting covariant derivatives, we get

**Theorem 3** The components of the deviation vector fields \(w^i\) of the trajectories
\[
\frac{\delta y^i}{\partial t} = F^i(x, y),
\]
satisfy the Jacobi-type equation
\[
\frac{D^2 w^i}{dt^2} = DF^i_{\partial y} |_{u=0} + y^h y^j R^i_{\ h jk} w^k.
\]

By a direct computation, we obtain

**Proposition 4** The deviation vector fields of the trajectories \(\frac{\delta y^i}{\partial t} = F^i(x, y)\) are solutions of ODE system:
\[
\frac{d^2 w^i}{dt^2} + (2M^i_{\ j} - \frac{\partial F^i}{\partial y^j}) \frac{dw^j}{dt} + \left(\mathcal{C}(M^i_{\ j}) + M^i_{\ k} M^k_{\ j} - y^h y^k R^i_{\ h jk} + L^i_{\ kj} F^j - \frac{\partial F^i}{\partial x^j}\right) w^j = 0,
\]

where \( C = y^k \frac{\partial}{\partial x^k} + 2y^{(2)k} \frac{\partial}{\partial y^k} \)

By re-expressing the above in terms of the nonlinear connection, \( R^i_{hjk} y^h = R^i_{hjk} L^i_{kj} = \frac{\partial M^i_k}{\partial y^i} \), we get

**Theorem 5** 1. The quantities

\[
M^{(1)i}_{(1)j} = \frac{1}{2} (2M^i_j - \frac{\partial F^i}{\partial y^j}),
\]

\[
M^{(2)i}_{(2)j} = \frac{1}{2} (C(M^i_j) + M^i_k M^k_j - y^j R^i_{jk} + \frac{\partial M^i_k}{\partial y^j} F_j - \frac{\partial F^i}{\partial x^j})
\]

are the dual coefficients of a nonlinear connection on \( T^2 M \).

**Theorem 6** With respect to this nonlinear connection, the extensions of Jacobi fields attached to \( \gamma^i \) have vanishing \( v_2 \) components:

\[
\frac{1}{2} \frac{d^2 u^i}{dt^2} + M^{(1)i}_{(1)j} \frac{du^j}{dt} + M^{(2)i}_{(2)j} u^j = 0.
\]

By denoting

\[
u^{(1)i} = \frac{du^i}{dt}, \quad w^{(2)i} = \frac{1}{2} \frac{d^2 w^i}{dt^2},
\]

the above relations can be reexpressed as:

\[
\frac{\delta w^{(2)i}}{dt} = 0.
\]

**7 Deviations of geodesics**

Let \( N^i_j = \frac{\partial G^i}{\partial y^j} \) be a nonlinear connection on \( TM \), coming from a spray.

If \( F = 0 \), then we deal with deviations of autoparallel curves,

\[
\frac{\delta y^i}{dt} = 0.
\]

In this case,

\[
M^{(1)i}_{(1)j} = M^i_j,
\]

\[
M^{(2)i}_{(2)j} = \frac{1}{2} (C(M^i_j) + M^i_k M^k_j - y^j R^i_{jk}),
\]

and we notice that our nonlinear connection differs only by the term \(-y^j R^i_{jk}\) from Miron’s one. [10].
**Remark 7** Along an extension curve $\tilde{c}$, there hold the equalities

$$\frac{\delta y^i}{dt} = \frac{Dy^i}{dt}, \quad \frac{\delta y^{(2)i}}{dt} = \frac{D^2y^i}{dt^2},$$

where $\frac{D}{dt}$ denotes the covariant derivative associated to the Berwald connection.

**Remark 8** Also, for a vector field $w$ along the projection $c$ on $M$, we have

$$\frac{\delta w^i}{dt} = \frac{Dw^i}{dt}.$$

**Conclusions:**

1. $c$ is a geodesic if and only if its extension $T^2M$ is horizontal.

For a vector field $w$ along a geodesic $c$ on $M$, we have:

1. $\frac{\delta w^i}{dt} = 0$, if and only if $w$ is parallel along $\dot{c} = y$.

2. $\frac{\delta w^{(2)i}}{dt} = 0$ if and only if $w$ is a Jacobi field along $c$.

This is, the extension to $T^2M$ of a Jacobi field $w$ on $M$ will have the form

$$W = w^i\delta_{(0)i} + \frac{\delta w^i}{dt}\delta_{(1)i}.$$  

**8 External forces in locally Minkovsian spaces**

Let $(M, L(y))$ be a locally Minkovski space.

Then, $M^i_j = 0$, $L^i_{jk} = 0$ (for the Berwald connection), [1], [5]. In presence of an external force field, the evolution equations of a mechanical system will take the form

$$\frac{dy^i}{dt} = F^i(x, y).$$  \hspace{1cm} (25)

In this case, with the above notations, we have

$$M^i_{(1)j} = -\frac{1}{2} \frac{\partial F^i}{\partial y^j},$$

$$M^i_{(2)j} = -\frac{1}{2} \frac{\partial F^i}{\partial x^j},$$

and the deviations of trajectories (25) are, simply:

$$\frac{d^2w^i}{dt^2} - \frac{\partial F^i}{\partial y^j} \frac{dw^j}{dt} - \frac{\partial F^i}{\partial x^j} w^j = 0.$$
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