Incoherent tunneling effects in a one-dimensional quantum walk

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Abstract
In this article we investigate the effects of shifting position decoherence, arising from the incoherent tunneling effect in the experimental realization of the quantum walk, on the one-dimensional discrete time quantum walk. We show that in the regime of this type of noise the quantum behavior of the walker does not vanish, in contrast to the coin decoherence for which the walker undergoes a quantum-to-classical transition even for weak noise. In particular, we show that the quadratic dependence of the variance on the time and also the coin–position entanglement, i.e. two important quantum aspects of the coherent quantum walk, are preserved in the presence of tunneling decoherence. Furthermore, we present an explicit expression for the probability distribution of the decoherent one-dimensional quantum walk in terms of the corresponding coherent probabilities, and show that this type of decoherence smooths the probability distribution.

Keywords: quantum walk, entanglement, decoherence

(Some figures may appear in colour only in the online journal)

1. Introduction

The quantum walk (QW) is the quantum version of the classical random walk (CRW). The superposition property of quantum systems allows one to define a quantum walker which can walk in all possible paths simultaneously, and the interference of these paths creates some important differences between the QW and the CRW. Notable differences between the QW and the CRW are the quadratic dependence of the variance on the number of steps and the
complex oscillatory probability distribution in the QW instead of the linear variance
dependence and the binomial probability distribution in the CRW. These differences have
been used to present several quantum algorithms in order to solve some specific problems [1–
6] with better performance than the best known classical versions.

Two types of QWs have been introduced: discrete time [7] and continuous time [8]. The
relation between these two types of QWs was an open problem for many years, but the
relation between them has been better understood since 2008 [9, 10]. Many works have been
devoted to studying QWs, both from the theoretical as well as the experimental view points.
Many aspects of QWs have been studied theoretically, for example, some research has
focused on the network over which the walks take place, in which context the QW on a line
has been well studied [11–16], and other topologies such as cycles [17, 18], two-dimensional
lattices [19–22], or n-dimensional hypercubes [23, 24] have also been investigated.

The study of quantum entanglement and its connection with the QW is another important
research area which has recently attracted much attention. The effect of entanglement of the
coin subspace on the QW [13, 21, 25], entanglement between the coin and the position
subspaces [20, 26, 27] and QWs as entanglement generators [29, 28] are examples of these
studies.

In addition to all of the theoretical studies, the experimental implementation and reali-
zation of the QW have also received much attention [30–35]. In practice, due to the effects
of the environment, the preparation of a pure quantum state is not possible and so the quantum
properties could be damped out. Therefore, it is very important to formulate and quantify the
influence of decoherence on the QW [32, 36–44]. Moreover, the study of decoherent QWs
can be considered as a link between theoretical investigation and the experimental
implementation.

It is known that the quantum properties of QWs, such as the quadratic dependence of
variance on the time, make the QW more powerful than the CRW, but these properties are
very fragile when they are exposed to decoherence or noise. Brun et al [39] have shown that
the quadratic term of variance vanishes, in the long time limit, even for weak decoherence of
the coin subspace. On the other hand, the linear term remains greater than the classical one for
any reasonable strength of noise. This means that if the noise in the system is less than some
certain value, the QW will spread faster than the CRW.

Since the unavoidable noise in quantum systems leads to the emergence of classical
behavior, some authors have tried to use such classical behavior. For example, Kendon et al
[37] have shown that a weak noise, both on the coin and the position factor-spaces, can be
useful in quantum algorithms. They have explained that a general weak noise can smooth and
flatten the probability distribution. Of course, we should note that this smoothness occurs at
the cost of losing the speed of spreading (variance). On the other hand, other routes to
classical behavior of the QW have been investigated and compared to classical behavior
emerging by decoherence. Brun et al [39] have shown that a multiple coin version of the QW
retains ‘quantum’ quadratic growth of the variance, except in the limit of a new coin for
every step.

In this paper we use our analytical expression for variance, presented in [44], and show
that the shifting position decoherence, arising from the incoherent tunneling effect, does not
change the quadratic behavior of the variance. We show also that although this quadratic term
depends on the initial state, it can not be removed even for initial mixed states. Furthermore,
we show that this type of decoherence smooths the probability distribution and we present a
closed formula for the probability distribution in terms of the probabilities of the corre-
spanding coherent QW.
This work is organized as follows. Section 2 gives a brief review of the one-dimensional (1D)QW and decoherence. Our calculations and results for incoherent tunneling effects are in section 3. In section 3.1 we obtain an expression for the variance of 1DQWs in the presence of incoherent tunneling noise. The entanglement properties between the coin and the position degrees of freedom are studied in section 3.2. A closed formula for the probability distribution is presented in section 3.3 and finally we summarize our results and present our conclusions in section 4.

2. Background

The QW is the quantum version of the CRW where instead of coin flipping, we use the coin operator to make a superposition on the coin space, and instead of the walking we use the translation operator to move the quantum particle according to the coin’s degrees of freedom. In a one-dimensional QW we have two degrees of freedom in the coin space $H_c$, spanned by $\{|L\rangle, |R\rangle\}$, and infinite degrees of freedom in the position space $H_p$, spanned by $\{|i\rangle \; i = -\infty, \ldots, \infty\}$. The walker Hilbert space $H_W$ is defined as the tensor product of the coin space $H_c$ and the position space $H_p$, i.e. $H_W = H_p \otimes H_c$. Each step of the QW is constructed by the unitary operator $U_c$, making a superposition on the coin space, followed by the translation operator $S$, moving the particle according to the coin state, i.e

$$U_w = S (I \otimes U_c),$$

where

$$S = \sum_x |x + 1\rangle \langle x| \otimes |R\rangle \langle R| + |x - 1\rangle \langle x| \otimes |L\rangle \langle L|.$$  

Therefore, the unitary evolution of the quantum walker is as follows

$$|\Psi(t + 1)\rangle = U_w |\Psi(t)\rangle \rightarrow |\Psi(t)\rangle = U_w |\Psi(0)\rangle.$$  

This is, in fact, a coherent evolution of the system but, in practice, it is not possible to isolate the system from the environment and, in general, the environment affects the coherent evolution of the system. Generally, decoherence is used to estimate deviation from the ideal case in which the effects of environment are neglectable. One important approach to investigating decoherence is the so-called Kraus representation [45]. Let us define $H_E$ as the Hilbert space of the environment, spanned by $\{|e_n\rangle, \; n = 0, \ldots, m\}$ where $m$ is the dimension of the environment’s Hilbert space. Now, in order to study the time evolution of the system we should consider the time evolution of the whole system, i.e. system + environment, defined on the Hilbert space $H = H_W \otimes H_E$, and obtain the state of the system by tracing out over the environment’s degrees of freedom, i.e.

$$\rho_{sys} = T_{H_{env}}(U \rho U^\dagger).$$

Here $U$ acts both on the system and the environment Hilbert spaces. Without loss of generality we assume that the initial state of the whole system is $\rho = \rho_0 \otimes |\chi_0\rangle \langle \chi_0|$, where $|\chi_0\rangle$ is the initial state of the environment. So we can write equation (4) as

$$\rho_{sys} = \sum_{n=0}^m \langle e_n | \ U | \chi_0\rangle \rho_0 \langle \chi_0 | U^\dagger | e_n\rangle = \sum_{n=0}^m E_n \rho_0 E_n^\dagger,$$

where $E_n = \langle e_n | \ U | \chi_0\rangle$, $n = 0, 1, \ldots, m$ are the so-called Kraus operators. These operators satisfy the following completeness relation
By definition of the Kraus operators, one step of the walking can be written as follows
\[ \rho(t + 1) = \sum_{n=0}^{m} E_n \rho(t) E_n^\dagger. \]

For \( t \) steps, we can write
\[ \rho(t) = \sum_{n_1=0}^{m} \cdots \sum_{n_t=0}^{m} E_{n_t} \cdots E_{n_2} \rho(0) E_{n_1}^\dagger \cdots E_{n_2}^\dagger. \]

It is worth noting that equation (8) is general and the Kraus operators \( E_n \) include all information about all types of memory-less evolution. It follows, therefore, that the coin operator, the translation operator and the environment effects all are embedded in \( E_n \) and we have not assumed any restriction yet.

To make any progress, we should therefore find the Kraus operators for our system defined in equation (5), and use equation (8) in order to obtain the final state \( \rho(t) \). Evidently, the \( E_n \) are operators that act on the system (coin + position) Hilbert space, and therefore we can write the general form of \( E_n \) as follows
\[ E_n = \sum_{x,x',j,l} a^{(n)}_{x,x',j,l} |x\rangle \langle x' | \otimes | i \rangle \langle j | \]
\[ = \sum_{x} \sum_{l} a^{(n)}_{x,j,l} |x + l \rangle \langle x | \otimes | i \rangle \langle j |, \]

where \( x, l = -\infty, \cdots, \infty \) and \( i, j = \{ L, R \} \).

We have shown in [44], that a reasonable suggestion on the coefficient \( a^{(n)} \) of equation (9) enables us to derive useful analytical expressions for the first and second moments of position. In the remainder of this section, we briefly introduce the method. To begin with, we use the Fourier transformation
\[ |x\rangle = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-ikx} |k\rangle, \]
and write equation (9) in the \( k \)-space as
\[ \tilde{E}_n = \sum_{x,j,l} a^{(n)}_{x,l,j} \int \frac{dk' dk}{4\pi^2} e^{-ik'x} e^{-i(k-k')x} |k\rangle \langle k' | \otimes | i \rangle \langle j | \]

If we assume that the coefficients \( a^{(n)}_{x,l,j} \) do not depend on the coordinate \( x \), then one can use the orthonormalization relation \( \sum_x e^{-i(k-k')x} = 2\pi \delta(k-k') \) and carry out the summation on \( x \). This is, of course, a reasonable assumption for a large family of QWs, including the QW on homogeneous position space. With this assumption, we have
\[ \tilde{E}_n = \int \frac{dk}{2\pi} |k\rangle \langle k | \otimes C_n(k), \]
where
\[ C_n(k) = \sum_{l,j} a^{(n)}_{l,j} e^{-ikl} | j \rangle \langle j |. \]
Now, if we write the general form of $\rho_0$ in the $k$-space as

$$\rho_0 = \int \frac{dk dk'}{4\pi^2} |k\rangle \langle k| \otimes |\psi_0\rangle \langle \psi_0|,$$

we obtain, from equation (7), the following form for the first step of walking

$$\rho' = \int \frac{dk dk'}{4\pi^2} |k\rangle \langle k'| \otimes \sum_n C_n(k) |\psi_0\rangle \langle C_n^+(k')|,$$

$$= \int \frac{dk dk'}{4\pi^2} |k\rangle \langle k'| \otimes \mathcal{L}_{k,k'} |\psi_0\rangle \langle \psi_0|,$$

where the superoperator $\mathcal{L}_{k,k'}$ is introduced by

$$\mathcal{L}_{k,k'} \tilde{O} = \sum_n C_n(k) \tilde{O} C_n^+(k').$$

Therefore, after $t$ steps walking, we find

$$\rho(t) = \int \frac{dk dk'}{4\pi^2} |k\rangle \langle k'| \otimes \mathcal{L}_{k,k'}^t |\psi_0\rangle \langle \psi_0|,$$

as the state of the walker and

$$P(x, t) = \int \frac{dk dk'}{4\pi^2} \langle x| \langle k'|k\rangle \rangle \text{Tr}(\mathcal{L}_{k,k'}^t \rho_0)$$

$$= \int \frac{dk dk'}{4\pi^2} e^{-i\langle k'|k\rangle} \text{Tr}(\mathcal{L}_{k,k'}^t \rho_0),$$

as the probability of finding the walker at the position $x$. Note that the completeness relation on the Kraus operators, given in equation (6), now implies that the coin operators $C_n(k)$ satisfy the same relation as

$$\sum_n C_n^+(k) C_n(k) = I.$$ 

Equation (19) can be used to prove another important property of $\mathcal{L}_{k,k'}$, i.e. the *trace preserving* condition

$$\text{Tr}(\mathcal{L}_{k,k'}^m \tilde{O}) = \text{Tr}(\tilde{O}),$$

for arbitrary operator $\tilde{O}$.

By definition, the $m$th moment of the probability distribution $p(x, t)$ is as follows

$$\langle x^m \rangle = \sum_x x^m p(x, t),$$

where equation (21) can be used, together with equation (18), to write the first and second moments as

$$\langle x \rangle = -\frac{i}{2\pi} \int \frac{dk dk'}{dk} \frac{d\delta}{dk} \text{Tr}(\mathcal{L}_{k,k'}^t \rho_0),$$

$$\langle x^2 \rangle = \frac{1}{2\pi} \int \frac{dk dk'}{dk} \frac{d^2\delta}{dk dk'} \text{Tr}(\mathcal{L}_{k,k'}^t \rho_0).$$
Now, by using the relations
\[ \frac{d}{dk} \text{Tr}(L_{kk} \rho) = \text{Tr}(\mathcal{G}_{kk} \rho) \]
\[ \frac{d}{dk'} \text{Tr}(L_{kk'} \rho) = \text{Tr}(\mathcal{G}^*_{kk'} \rho), \]
where
\[ \mathcal{G}_{k,k'} = \sum_n \frac{dC_n(k)}{dk} \bar{O} C_n^*(k'), \]
one can carry out the integrations of equation (22) and, after some calculations, obtain the following relations for the first and second moments
\[ \langle x \rangle_k = i \int_{-\pi}^{\pi} \frac{dk}{2\pi} \sum_{m=1}^{t} \text{Tr} \{ \mathcal{G}_k (L_k^{m-1} |\psi_0\rangle \langle \psi_0|) \} \]
\[ \langle x^2 \rangle_k = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \sum_{m=1}^{t} \text{Tr} \{ \mathcal{G}_k (L_k^{m-1} \mathcal{L}_k^{m-1} |\psi_0\rangle \langle \psi_0|) \}
+ \mathcal{G}_k L_k^{m-1} (\mathcal{G}_k L_k^{m-1} |\psi_0\rangle \langle \psi_0|) \}
+ \int_{-\pi}^{\pi} \frac{dk}{2\pi} \sum_{m=1}^{t} \text{Tr} \{ \mathcal{J}_k (L_k^{m-1} |\psi_0\rangle \langle \psi_0|) \}. \]  
\[ \mathcal{J}_k = \frac{d\mathcal{G}^*_k}{dk} |k'=-k = \sum_n \frac{dC_n(k)}{dk} \bar{O} \frac{dC_n^*(k')}{dk'} |k'=-k. \]

3. Calculations and results

In this section, by using the analytical expressions for the first and second moments given in equation (25), we investigate the variance of the 1DQW in the presence of incoherent tunneling noise, i.e. a type of decoherence which is applied on the position subspace.

3.1. Variance of decoherent 1DQW

Consider a 1DQW such that after each step of walking, the walker can move to the nearest neighbors with probability \( p \). This phenomenon occurs in the experimental set-up because of the incoherent tunneling effect. The effect of such noise on the probability distribution has investigated, numerically, by Dür et al [32]. In this section we calculate, analytically, the variance of a 1DQW in the presence of this kind of noise and investigate some important features of it.

Let us assume that, after each step in the 1DQW, the walker tunnels to the left or to the right site with probability \( p \). Therefore, it is clear that the walker does not move with probability \((1-p)\), unlike the coherent QW, all sites may be occupied. In the presence of such tunneling, one step of the walking can be written as
where \( x = \sum_\xi x \mid x \pm 1 \rangle \langle x \mid \otimes I_c \) and \( U_w \) is the coherent walking operator, given by equation (1). For \( p = 0 \), the system is exactly the same as the coherent QW and the evolution of the system is unitary, so the final state of the system remains pure. On the other hand, for \( p \neq 0 \) the evolution is nonunitary and decoherence occurs.

We should note that although \( S_x U_w \) in equation (27) changes the length of the steps from 1 to 0 or 2, it is not equivalent to the quantum walker with an imperfect translation operator in which imperfect responses change the length of steps to 0 or 2. For example assume an imperfect translation operator which instead of moving 1 step, randomly does not move the walker or moves it 2 steps. In this case we will have some Kraus operators in the form of

\[
\sum_x \langle x + 2 \mid \otimes |R\rangle \langle R| + |x - 1 \rangle \langle x \mid \otimes |L\rangle \langle L|.
\]

or

\[
\sum_x \langle x \mid \otimes |R\rangle \langle R| + |x - 1 \rangle \langle x \mid \otimes |L\rangle \langle L|.
\]

etc, which do not appear in equation (27).

Generally, decoherence is a measure of the reduction of purity of the system state and, therefore, one can use the purity (or coherence)

\[
\mathcal{C}(t) = \text{Tr}[\rho^2(t)],
\]

as a measure of coherence. \( \mathcal{C}(t) \) has the limiting values 1 and 1/N for pure and maximally mixed states, respectively, where \( N \) is the dimension of the space that the density matrix \( \rho(t) \) is supported on. Therefore, a purity of less than 1 is a signature of decoherent evolution. In figure (1), we have plotted the coherence of the system as a function of time in the presence of the incoherent tunneling noise of equation (27), for different values of the noise strength \( p \). As we expect, for \( p = 0 \) the evolution of QW takes place coherently, while by increasing \( p \) the system loses its coherence very fast.
Now by comparing equation (27) with equation (7), we can write the Kraus operators as

\[ E_1 = \sqrt{1 - p} U_w \]

\[ E_2 = \frac{p}{\sqrt{2}} S_+ U_w \]

\[ E_3 = \frac{p}{\sqrt{2}} S_- U_w. \]

All the above Kraus operators have the form given by equation (9), therefore, by using equation (13) we find the \( C_i \) matrices as follows

\[
C_1 = \sqrt{\frac{1 - p}{2}} \left( e^{-ik} \ e^{ik} \right) \\
C_2 = \sqrt{\frac{p}{2}} \left( e^{-2ik} \ e^{2ik} \right) \\
C_3 = \sqrt{\frac{p}{2}} \left( 1 \ e^{2ik} \ e^{-2ik} \right).
\]

where equation (19) is clearly satisfied.

By using Bloch representation [45], we can represent any arbitrary two-by-two density matrix by a four-dimensional column vector \( \vec{\rho} \) as

\[
\vec{\rho} = \sum_{i=0}^{3} r_i \sigma_i,
\]

where \( \sigma_i \) are Pauli matrices and \( r_i = \text{Tr}(\sigma_i \vec{\rho})/2 \). In this representation any trace preserve quantum operator \( \varepsilon \) is equivalent to a map of the form

\[
\varepsilon(\vec{\rho}) = \vec{\rho}' \Leftrightarrow \vec{r}' = M \vec{r} + \vec{c}
\]

where \( M \) is a \( 4 \times 4 \) matrix and \( \vec{c} \) is a constant vector. This is an affine map, mapping the Bloch sphere into itself [45].

We use the affine map and represent the action of \( L_k, G_k, G_k^\dagger \) and \( J_k \) on an arbitrary two-by-two matrix \( \vec{\rho} \) as follows

\[
L_k \vec{\rho} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin(2k) & \cos(2k) \\ 0 & -\cos(2k) & \sin(2k) \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{pmatrix}
\]

\[
G_k \vec{\rho} = \begin{pmatrix} 0 & -i & 0 & 0 \\ 0 & \cos(2k) & 0 & 0 \\ 0 & 0 & \sin(2k) & -\sin(2k) \\ -i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{pmatrix}
\]

\[
J_k \vec{\rho} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q \sin(2k) & q \cos(2k) \\ 0 & 0 & -q \cos(2k) & q \sin(2k) \\ 0 & 1 + p & 0 & 0 \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{pmatrix},
\]

where \( q = 1 - p \) and \( G_k^\dagger = G_k^\dagger \), which is obtained from the hermiticity of the Pauli matrices.
Now, according to equation (25), in order to calculate moments and variance, we need to find the mth power of $\mathcal{L}_k$. But, here, the method of non-trivial submatrix $M_k$, which is introduced in [38], is not applicable because $\det(I - M) = 0$, i.e. $I - M$ is not invertible. Fortunately, the eigenvalues and eigenvectors of $\mathcal{L}_k$ are sufficiently simple to calculate $\mathcal{L}_k^m$ directly. Straightforward calculations show that the eigenvalues and the corresponding eigenvectors of matrix $\mathcal{L}_k$, given in equation (37), are as follows, respectively,

$$
\lambda_1 = \lambda_2 = 1, \quad \lambda_3 = e^{i(\theta + \pi)}, \quad \lambda_4 = e^{-i(\theta + \pi)},
$$

and

$$
\begin{align*}
|e_1\rangle &= \frac{1}{N_1} \begin{pmatrix} \sin(k) \\ \cos(k) \end{pmatrix}, \\
|e_2\rangle &= \frac{1}{N_2} \begin{pmatrix} -2(\cos^2(k) + 1) \\ \sin(2k) \end{pmatrix}, \\
|e_3\rangle &= |e_4\rangle^* = \frac{1}{N_3} \begin{pmatrix} 0 \\ \sin(2k)e^{2i\theta} \end{pmatrix},
\end{align*}
$$

where $\cos(\theta) = \cos^2(k)$ and $N_i$ are the normalization factors. With the help of the eigenvectors and the spectral decomposition, we are able to find any power of matrix $\mathcal{L}_k$, as it appears in equation (25).

In order to calculate the first moment $\langle x \rangle$, we need

$$
\Gamma = \sum_{m=1}^{t} \mathcal{L}_k^{m-1} = \sum_{m=1}^{t} \{ |e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| \\
+ e^{i(\theta + \pi)(m-1)}|e_3\rangle\langle e_3| + e^{-i(\theta + \pi)(m-1)}|e_4\rangle\langle e_4| \}
= t\{ |e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| + (1 - e^{i(\theta + \pi)}/(1 + e^{i\theta}))(|e_3\rangle\langle e_3| \\
+ e^{-i(\theta + \pi)}/(1 + e^{-i\theta})(|e_4\rangle\langle e_4| \}
= t\{ |e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| + 2\Re\left(\frac{1}{1 + e^{i\theta}}|e_3\rangle\langle e_3| \right) \}.
$$

Here, in the last equality, we used the fact that the fourth eigenvalue and eigenvector are complex conjugates of the third one, respectively (see equations (40) and (41)). Also we neglect the oscillatory term $e^{i(\theta + \pi)}$, because according to the stationary phase theorem [11] the oscillatory term can be neglected in the long time limit. By putting equation (41) into equation (42) and after some simplification, we find the following matrix form for $\Gamma$.
\[
\Gamma = \frac{1}{\Delta} \begin{pmatrix}
\Delta f & \sin(2k) & 0 & 0 \\
0 & f & \cos^2(k) + t \sin^2(k) & g \\
0 & g - \sin(2k) & 0 & f - 2 \\
0 & f - 2 \cos^2(k) & g - \sin(2k) & f
\end{pmatrix}
\] (43)

where
\[
\Delta = 2(\cos^2(k) + 1) \\
f' = 2t \cos^2(k) + 1 \\
g = t \sin(2k)
\]

We can, therefore, write the first moment \( \langle x \rangle \), given in equation (25), as follows
\[
\langle x \rangle = \frac{i}{2\pi} \int_{-\pi}^{\pi} dk \langle 0 \mid -2i 0 0 \rangle \Gamma \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix} \\
= \left[ (2 - \sqrt{2})t + \frac{1}{\sqrt{2}} \right] r_1 + \left[ (2 - \sqrt{2})t - \frac{1}{\sqrt{2}} \right] r_3.
\] (44)

Note, however, that in calculating this moment from equation (25) only the first row of \( G_k \) gives the nonzero contribution. Clearly, the above expression for \( \langle x \rangle \) does not contain \( p \), which implies that this type of noise cannot change the first moment.

Now, in order to calculate \( \langle x^2 \rangle \), we first calculate the last term of equation (25) as follows
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} dk \sum_{m=1}^{t} \text{Tr} \left( \mathcal{J}_k (\mathcal{L}_k^{m-1}) \langle \psi_0 \rangle \langle \psi_0 \rangle \right) \\
= \frac{1}{\pi} \int_{-\pi}^{\pi} dk (1 + p 0 0 0) \Gamma \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix} \\
= t (1 + p).
\] (45)

On the other hand, the first term of equation (25) can be written as follows
\[
\int_{-\pi}^{\pi} \frac{dk}{2\pi} \sum_{m=1}^{t} \sum_{m'=1}^{m-1} \text{Tr} \left( G_k^+ \mathcal{L}_k^{m-m'-1} (G_k \mathcal{L}_k^{m'-1}) \langle \psi_0 \rangle \langle \psi_0 \rangle \right) + \mathcal{G}_k \mathcal{L}_k^{m-m'-1} (G_k^+ \mathcal{L}_k^{m'-1}) \langle \psi_0 \rangle \langle \psi_0 \rangle \\
= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \sum_{m=1}^{t} \sum_{m'=1}^{m-1} (0 i 0 0) \mathcal{L}_k^{m-m'-1} (G_k - G_k^+) \mathcal{L}_k^{m'-1} \begin{pmatrix}
1/2 \\
0 \\
0 \\
0
\end{pmatrix} \\
= t (1 + p).
\] (46)

where we have used the fact that \( G_k^+ = G_k^+ \). From equation (38) the exact form of \( G_k - G_k^+ \) is
\[
(G_k - G_k^+) \hat{O} = \begin{pmatrix}
0 & -2i 0 0 \\
0 & 0 0 0 \\
0 & 0 0 0 \\
-2i & 0 0 0
\end{pmatrix} \begin{pmatrix}
r_0 \\
r_1 \\
r_2 \\
r_3
\end{pmatrix}.
\] (47)
Also from the trace preserving property of $\mathcal{L}_k$, we have

$$
\mathcal{L}_k^{m-1} \begin{pmatrix} 1/2 \\ r_1 \\ r_2 \\ r_3 \\ r_1' \\ r_2' \\ r_3' \end{pmatrix} = \begin{pmatrix} 1/2 \\ r_1' \\ r_2' \\ r_3' \end{pmatrix}.
$$

(48)

So, easily, we obtain

$$
(\mathcal{G}_k - \mathcal{G}_k') \mathcal{L}_k^{m-1} |\psi_0\rangle \langle \psi_0| = \begin{pmatrix} -2ir_1' \\ 0 \\ 0 \\ -i \end{pmatrix}.
$$

(49)

Putting this into equation (46), we get

$$
\int_{-\pi}^{\pi} \frac{dk}{\pi} (0 \ 1 \ 0 \ 0) \Gamma' \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

(50)

where $\Gamma'$ is defined by

$$
\Gamma' = \sum_{m=1}^{t} \sum_{m'=1}^{m-1} \mathcal{L}_k^{m-m'-1} = \sum_{m=1}^{m-1} \sum_{m'=1}^{m-1} \{ |e_1\rangle \langle e_1| + |e_2\rangle \langle e_2| + e^{i(\theta+\pi)(m-m'-1)}|e_3\rangle \langle e_3| + e^{-i(\theta+\pi)(m-m'-1)}|e_4\rangle \langle e_4| \}

= \frac{t}{2} (t-1) \{|e_1\rangle \langle e_1| + |e_2\rangle \langle e_2| \}

+ 2\Re \left\{ \left( \frac{t}{1+e^{i\theta}} - \frac{1}{1+e^{i\theta}} \right) |e_3\rangle \langle e_3| \right\}.
$$

(51)

From equation (50) it is clear that we need only the term $\Gamma'_{2,4}$, which is obtained as follows

$$
\Gamma'_{2,4} = \frac{t^2 (\cos^2(k) + \cos^2(k)) + 1}{2(\cos^2(k) + 1)^2} - \frac{t}{2}.
$$

(52)

Substituting this result into equation (50), carrying out the integration and putting everything together, we find the following form for the second moment

$$
\langle x^2 \rangle = t (1 + p) + \left( 1 - \frac{1}{\sqrt{2}} \right) t^2 - t + \frac{3\sqrt{2}}{8}

= \left( 1 - \frac{1}{\sqrt{2}} \right) t^2 + tp + \frac{3\sqrt{2}}{8}.
$$

(53)

It is worth noting that the second moment does not depend on the initial state and, surprisingly, it involves the quadratic term $t^2$ which is independent of the noise strength $p$. This means that whenever the position space is subjected to incoherent tunneling noise the quantum feature of the variance, i.e. the quadratic dependence of the variance on the time, is preserved, in contrast to the case that the coin is subjected to noise in which for long time walking the system becomes classical and its variance becomes linear [38].
Let us look at the variance more precisely. From equations (44) and (53) we have

\[ V = \langle x^2 \rangle - \langle x \rangle^2 = At^2 + Bt + C, \]  

(54)

where

\[ A = \alpha - 4\alpha^2(r_3 + r_i)^2 \]
\[ B = 2\sqrt{2}\alpha(r_3^2 - r_i^2) + p \]
\[ C = -(r_3 - r_i)^2/2 + 3\sqrt{2}/8, \]

(55)

in which \( \alpha = 1 - 1/\sqrt{2} \). In the long time limit and for \( A \neq 0 \), we can neglect the \( B \) and \( C \) coefficients. The equation (54) shows that the variance contains the quadratic term \( t^2 \) which is independent of the noise strength \( p \). This means that in the presence of incoherent tunneling noise the variance is exactly the same as the one in the coherent QW. A question that may arise here: is it possible to find initial states for which \( A = 0 \) and \( B \neq 0 \)? or, in other words: is it possible to find initial states such that the variance is classical? It is not difficult to show that it is impossible to have \( A = 0 \) even for mixed initial states. Let us consider the following general form for the initial state \( \rho_0 \)

\[ \rho_0 = \sum_{i=0}^{3} \alpha_i |r_i \rangle \langle r_i | = \begin{pmatrix} r_0 + r_3 & r_1 - ir_2 \\ r_1 + ir_2 & r_0 - r_3 \end{pmatrix}, \]

(56)

where \( r_0 = 1/2 \), because of the normalization, and the positivity of \( \rho_0 \) implies that

\[ r_1^2 + r_2^2 + r_3^2 \leq 1/4 \implies r_1^2 + r_3^2 \leq 1/4. \]

(57)

This means that \( |r_1|, |r_3| \leq 1/2 \) or \( 2r_1r_3 \leq 1/2 \), so

\[ (r_1 + r_3)^2 = r_1^2 + r_3^2 + 2r_1r_3 \leq \frac{3}{4}. \]

(58)

On the other hand, the coefficient \( A \) in equation (55) vanishes only for

\[ (r_1 + r_3)^2 = \frac{1}{4\alpha} \approx 0.85, \]

(59)

which is, obviously, in conflict with the condition given by equation (58). Consequently, not only does the incoherent tunneling noise, which is noise on the position subspace of the QW, not change the quadratic behavior of the variance (\( p \) does not appear in the coefficient \( A \) in equation (55)), but it also increases the variance by a small amount (as a linear term in the coefficient \( B \)). Furthermore, although the quadratic term depends on the initial state, it can never vanish by tuning the initial state and it ranges between the minimum \( 3\alpha^2 \) and the maximum \( \alpha \). Now, in order to find the initial state for which the variance is maximum, let us consider the following generic form for the pure initial state

\[ |\psi_\theta \rangle = \cos \theta |R \rangle + e^{i\phi} \sin \theta |L \rangle, \]

(60)

where \( 0 \leq \theta \leq \pi \) and \( 0 \leq \phi < 2\pi \). For this initial state, the elements of \( r_i \) in the affine map representation defined in equation (35) are as follows

\[ r_0 = \frac{1}{2} \]
\[ r_1 = \frac{1}{2} \cos \phi \sin 2\theta \]
\[ r_2 = \frac{1}{2} \sin \phi \sin 2\theta \]
\[ r_3 = \frac{1}{2} \cos 2\theta. \]

(61)
Using these $n_1$, $n_3$ in the definition of $A$, given in equation (55), and maximizing it, we find that for all $\theta$, $\phi$, satisfying $\phi = -\cos^{-1}(\cot 2\theta)$, the variance is maximum. In figure 2, we have plotted variance versus time for various initial states and have compared the results with the numerical simulation. The figure shows that there are excellent matching between the theoretical result of equation (54) and the direct numerical calculations. An interesting point which we would like to emphasize here is that, although our expression have been derived for the long time limit, the very fast decay of the oscillatory term implies that we have very good matching even after a few steps (see figure 2).

3.2. Entanglement in decoherent QW

In the previous subsection we showed that the incoherent tunneling noise preserves the quantum property of variance. In this subsection we turn our attention to another quantum property of the system, namely the entanglement between the coin and the position degrees of freedom in the presence of incoherent tunneling noise. We also compare the result with the case in which only the coin is subjected to decoherence. The model that we choose for the coin decoherence is defined by

$$
|P_I R \rangle_R \langle I R| + |P_I L \rangle_R \langle I L|
$$

where $P_I = I_r \otimes |R \rangle \langle R|$ and $P_L = I_r \otimes |L \rangle \langle L|$ are projections on the coin subspace. Equation (62) describes the situation in which, after each step of walking, a measurement can be performed on the coin subspace of the walker with probability $p$. The probability distribution and the variance of this type of decoherence have been investigated in [38].

As we mentioned before, in the presence of noise, the state of the system becomes mixed and, therefore we use the negativity of the partial transpose of the density matrix as a suitable measure of entanglement which is, of course, a computable measure for mixed bipartite systems [46]. Negativity is based on the Peres–Horodecki criterion for separability [47, 48] and measures the degree to which the partial transpose of $\rho$ fails to be positive, i.e. the

![Figure 2. Variance versus $t$ for different initial states of equation (60). The continuous curves are the results of theoretical formula (54) and the corresponding discrete symbols are the direct numerical calculations.](image-url)
The absolute value of the sum of negative eigenvalues of the partial transpose of the density matrix

$$N = \frac{1}{2} \left( \sum_i |\lambda_i^r| - 1 \right) = \sum_i |\lambda_i^r|,$$

(63)

where $\lambda_i^r$ and $\lambda_i^l$ are eigenvalues and negative eigenvalues of the matrix $\rho'(t)$, respectively. Here $\rho'(t)$ is the partial transpose of $\rho(t)$, which is obtained from the density matrix $\rho(t)$ by taking the transpose with respect to one of the subsystems, say the first subsystem, i.e.

$$\rho'_{x,yb} = \rho_{xb,yc},$$

(64)

where $x, y$ and $b, c$ denote the position and the coin state indices, respectively.

In Figure 3, we have plotted the negativity of the system in terms of the noise strength $p$, both for the case that the position is subjected to the noise given in equation (27) and the case that the coin is subjected to the noise given in equation (62). Comparing the two negativities, it is clear from the figure that the system with coin-only decoherence is more influenced by the noise and loses its negativity very fast. Moreover, although the negativity of the system with coin-only decoherence decays to zero as $p$ increases, this situation is very different for the system with position-only decoherence and, indeed, for such a system the negativity converges to a significant value, never less than $\approx 0.7$, even for the maximum rate of noise. This means that the system with incoherent tunneling noise never loses all quantum behavior, but the system with coin decoherence will. In Figure 4, we have plotted the negativity of the system as a function of time, in the presence of tunneling noise for various $p$. The figure clearly shows the convergence behavior of the negativity for this type of noise.

Note that the flat behavior of the negativity in coherent case, i.e. $p = 0$, changes to descending (ascending) behavior for $p < 0.5$ ($p > 0.5$) (see figure 4).
3.3. Probability distribution of a decoherent QW

In this subsection we derive an analytical expression for the probability distribution of a decoherent QW in terms of the corresponding coherent probability distributions. To begin with, we first rewrite equation (18) in the coherent case \( p = 0 \) as

\[
P_0(x, t) = \int \frac{dk}{2\pi} \frac{dk'}{2\pi} e^{-ix(k' - k)} \operatorname{Tr}(\mathcal{W}_{kk'}^\dagger \rho_0).
\]  

(65)

Note that we use \( P_0(x, t) \) and \( \mathcal{W}_{kk'} \) instead of \( P(x, t) \) and \( \mathcal{L}_{kk'} \), respectively, in order to denote the corresponding quantities for the coherent QW. It is clear from equations (16) and (34) that

\[
\mathcal{W}_{kk'} \rho_0 = U(k) \rho_0 U(k')^\dagger,
\]

(66)

where \( U(k) \) is the Fourier form of the unitary transformation of the Hadamard walk in the \( k \)-space, i.e.

\[
U(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-ik} & e^{-ik} \\ e^{ik} & -e^{ik} \end{pmatrix}.
\]

(67)

Accordingly, all \( C_i \) given in equation (34) can be rewritten as follows

\[
C_1 = \sqrt{1 - p} U(k) \\
C_2 = \frac{p}{\sqrt{2}} e^{-ik} U(k) \\
C_3 = \frac{p}{\sqrt{2}} e^{ik} U(k).
\]

(68)
We can, therefore, write

$$\mathcal{L}_{k,k'}\rho_0 = \sum_{n} C_n(k)\rho_0 C^*_n(k')$$

$$= \left[ 1 - p + \frac{p}{2} (e^{i(k'-k)} + e^{-i(k'-k)}) \right] \mathcal{V}_{k,k'}\rho_0.$$  \hspace{1cm} (69)$$

Putting this into equation (18) and using equation (65), we find the following form for the probability distribution of the decoherent QW

$$P(x, t) = \sum_{n=0}^{t} \sum_{m=0}^{n} \binom{n}{m} (1 - p)^{t-n} \left( \frac{p}{2} \right)^n P_0(x + 2m - n, t),$$ \hspace{1cm} (70)$$

where $\binom{r}{s} = \frac{r!}{s!(r-s)!}$ is the binomial coefficient.

Equation (70) shows that, after $t$ steps of decoherent walking, the probability of finding the walker at site $x$ can be written as a linear combination of the probabilities of the corresponding coherent walking, but at some other sites ranging from $x - t$ to $x + t$. The equation can be also used in order to show the smooth property of the distribution and all site occupation properties. A comparison of this formula and the numerical simulation shows that there is an excellent agreement between them. Figure (5) shows the probability distribution in terms of $x$, in the presence of position decoherence with different noise strengths. It is worth mentioning that although $P_0(x, t) = 0$ for odd (even) sites after even (odd) steps, it is evident from equation (70) that the probability $P(x, t)$ of finding the walker in all sites $x$ and $0 < p < 1$ is nonzero (see figure 5). In the totally decoherent case, i.e. $p = 1$, only the terms with $n = t$ have a nonzero contribution in the summation of equation (70), leading therefore to
This equation shows that in the case $p = 1$, unlike the coherent case $p = 0$, only even sites are occupied for any $t$. The reason is that if $t = 2k$, then only for $x = 2l$ we have $P(x + 2m - t, t) = P(2s, 2k) = 0$ with $s = l + m - k$. On the other hand, if $t = 2k + 1$, again only for $x = 2l$, we have $P(x + 2m - t, t) = P(2s - 1, 2k + 1) = 0$. This means that the walker influenced by full tunneling noise cannot be found in an odd position after any number of steps. Numerical calculations show that for $p \gtrless 0.97$ the probability distribution loses its smoothness and converges to the case $p = 1$ in which only even sites are occupied. Figure 6 shows the probability distribution for some large values of $p$. It is evident from figures 5 and (6) that for very weak and very strong noise the probability distribution is rough, but for the other strengths of noise the incoherent tunneling effect can smooth the probability distribution.

At the end of this section, we would like to emphasize that the expression in equation (70) is independent of the type of walking. It means that for any type of 1DQW with a known coherent probability distribution, one can use this expression to find the effects of position noise on it. Several types of 1DQWs such as the biased Hadamard walk [49], the QW with an SU(2) coin operator [50], the 1DQW with entangled coin [13], the many coin QW [51] and so on have been introduced and have been well studied. Some aspects of specific types of 1DQWs can be combined with the smoothness property of the position decoherence and open new useful features in the study of QWs. For example, Kendon et al [37] have shown that the weak decoherence, both on the position and on the coin subspaces, can produce a flat probability distribution which is useful in quantum information and computation procedures. In their work, they used the weak noise on the coin subspace in order to generate a peak around the origin (classical probability distribution) in addition to two peaks of symmetric probability distribution (quantum probability distribution). They then applied a
noise on the position subspace in order to smooth these peaks and generated a smooth flat distribution. On the other hand, Brun et al have shown that although the QW with weak noise on the coin subspace spreads faster than the CRW, the variance is linear [38]. One of the promising aspects of the current work is that we could smooth the probability distribution without the loss of the quantum features. For example, we can use many coins [51] or the 1DQW with entangled coins [13], both having probability distributions with some peaks, and generate a smooth flat probability distribution by tuning the noise on the position subspace.

4. Summary and conclusions

By using the analytical expressions for the moments of the distribution introduced in [44], we have investigated the incoherent tunneling effect in the 1DQW. Our analytical calculation shows that the shifting noise on the position subspace of the QW does not eliminate the quantum behavior of the system. We have provided two pieces of evidence for our claim. First, we have shown that the quadratic dependence of the variance on time never vanished, even for a mixed initial state. Second, we have calculated the entanglement between the coin and the position and have shown that unlike coin-only decoherence, in which the entanglement goes to zero very fast by increasing the strength of the noise, the entanglement in position-only decoherence converges to a significant value, even for a maximum rate of noise. Furthermore, we have derived an exact expression for the probability distribution of the QW in the presence of the incoherent tunneling effect, in terms of the corresponding coherent probability distributions. Our results have shown that the effect of the incoherent tunneling noise on the QW is the smoothness of the probability distribution, such that this smoothness occurs for all noise strengths $p$, except for $p$ near 1 for which the smoothness is broken.

Accordingly, in all experimental realizations of QWs where the quantum behavior, such as the speed of spreading or the entanglement, plays an important role, one does not need worry about the noise on the position subspace, because this type of decoherence does not remove the quantumness of the system, but significant classicality will happen whenever the coin is subjected to the decoherence. This paper shows that the position-only noise not only does not make the transition from quantum to classical, but also that it can smooth the probability distribution, a task which is useful in quantum information and quantum computation processes.

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