Parity breaking in 2+1 dimensions and finite temperature

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Abstract

An expansion in the number of spatial covariant derivatives is carried out to compute the $\zeta$-function regularized effective action of 2+1-dimensional fermions at finite temperature in an arbitrary non-Abelian background. The real and imaginary parts of the Euclidean effective action are computed up to terms which are ultraviolet finite. The expansion used preserves gauge and parity symmetries and the correct multivaluation under large gauge transformations as well as the correct parity anomaly are reproduced. The result is shown to correctly reproduce known limiting cases, such as massless fermions, zero temperature, and weak fields as well as exact results for some Abelian configurations. Its connection with chiral symmetry is discussed.

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I. INTRODUCTION

Three-dimensional field theories find a direct physical application in the context of condensed matter [1] and also appear as the high temperature limit of 3+1-dimensional theories [2]. From the theoretical point of view, odd-dimensional field theories have some remarkable properties. They are free from scale anomalies [3] and so the corresponding coupling constants do not run at one-loop [4]. Gauge theories in odd-dimensional spaces naturally admit a local term which is of topological nature, namely, the Chern-Simons secondary characteristic class (CS). Such terms are invariant under topologically small gauge transformations, induce a parity breaking in the effective action and give a mass and spin to the gauge boson [5]. Moreover, the so called topological mass of the photon in QED$_3$ is not renormalized beyond one loop order [6]. In the non-Abelian case, the CS term may change by integer multiples of $2\pi i$ under large gauge transformations, implying a quantization condition on the mass to coupling constant ratio [5]. The quantization of the CS coefficient is in complete analogy with the quantization of the topological Wess-Zumino-Witten action (WZW) in even-dimensional space-times [7,8].

When fermions are coupled to the gauge fields, the CS term is induced by quantum fluctuations [9–11]. Preservation of gauge invariance requires then the presence of an appropriate CS counter term which introduces a parity anomaly in the effective action [4]. The parity anomaly has the same technical meaning as for axial anomalies in even dimensional theories, that is, the anomalous term breaks a symmetry that was present at the classical level. In the chiral case the fermions induce a WZW term and enforcement of vector gauge invariance introduces an axial anomaly [12]. The dimensional ladder relating chiral, parity and gauge anomalies for massless fermions in space-times of dimensions $D+2$, $D+1$ and $D$ has been elucidated in [13–15]. The relation between axial anomalies in $D+1$ dimensions and effective actions in $D$ dimensions using trace identities, both for zero and finite temperature and density, has been studied in [10,16–18]. It is also noteworthy that induced CS terms appear in four dimensional theories at finite temperature and density [19].

The exact parity-violating piece of the effective action for massless fermions in odd-dimensional space-times was obtained in [14] in terms of the $\eta$-invariant. This result follows from the use of the $\zeta$-function regularization [20] which automatically preserves strict gauge invariance under large and small gauge transformations. The $\eta$-invariant measures the spectral asymmetry of the corresponding Dirac operator, and through an application of the Atiyah-Patodi-Singer index theorem [21,14], it can be related to the CS term corrected by the index of the Dirac operator extended to one more dimension. Actually, the massless exact result holds for zero as well as for finite temperature. Indeed, it applies to fairly general space-time manifolds and, within the imaginary time formulation, the finite temperature amounts to a $S^1$ compactification in the time direction.

For massive fermions there is no exact closed solution at zero temperature but some exact results are known, including the presence of a correctly quantized CS term (see e.g. [22,23]). At finite temperature, however, inconsistencies did appear in the first calculations of the induced CS term in 2+1 dimensions both in the Abelian and non-Abelian cases [16,24]. Since the CS term is a local polynomial, the calculations to extract this term were usually done using a combination of perturbative and derivative expansions (see also [18] for a recent
calculation of this type at finite temperature and density, and \cite{25} for perturbative calculations without the low momentum approximation). The result of these calculations yields a CS term with a coefficient which is a smooth function of the temperature and this breaks invariance under large gauge transformations. The necessity of gauge invariance was emphasized in \cite{26}. In \cite{27} it was argued that perhaps the exact (i.e., non-perturbative) coefficient is actually a stepwise function of the temperature so that only properly quantized values appear and in any case only those quantized values would contribute after the functional integration on the gauge fields (this observation, however, does not solve the problem when the gauge fields are external). Replacing the CS by its gauge invariant version, namely, the $\eta$-invariant, does not solve the problem either. This is because this quantity is gauge invariant at the price of displaying jumps of $\pm 2$ coming from its index contribution. However, if the spectrum of the Dirac operator is discrete and sufficiently well behaved, the renormalized fermionic determinant will be free from discontinuities under continuous deformations of the gauge configuration, therefore the exact effective action can only have jump discontinuities which are multiples of $2\pi i$ (or multiples of $i\pi$ if some eigenvalue vanishes during the deformation; this is the generic case for massless fermions). This leads again to a quantization of the coefficient in the $\eta$-invariant.

This apparent paradox has recently been solved: in Ref. \cite{28} it is shown, in the exactly solvable 0+1-dimensional Abelian case, that the exact result is free of pathologies and consistent with gauge invariance. Large gauge transformations (which exists in this Abelian model due to the compactification introduced by the temperature) mix different orders of perturbation theory. In Refs. \cite{29} the 2+1 dimensional case is considered. There it is shown that, using the $\zeta$-function prescription, the effective action at finite temperature can be regularized in a completely gauge invariant manner. Furthermore, the imaginary part of Euclidean the effective action is computed explicitly for some special configurations where separation of the time and space variables is possible and the gauge fields are effectively Abelian. In Ref. \cite{30} another calculation is carried out for such configurations, this time by integration of the chiral anomaly of the associated two dimensional problem (the calculation is extended to arbitrary odd dimensions in \cite{31}). These results check invariance under both large and small gauge transformations and display the expected parity anomaly. Related work, both on odd and even dimensional space-times, prompted by the insights in those papers can be found in \cite{32--34}.

In the present work we study the Euclidean $\zeta$-function regularized effective action of fermions in 2+1 dimensions in the presence of arbitrary external non-Abelian gauge field configurations and at finite temperature. The calculation is done not only for the imaginary part, which contains the topological terms, but also for the real part.

Besides the $\zeta$-function regularization, the main ingredients of the calculation are, first, the use of a Wigner transformation to represent the operators and second, an expansion in the number of spatial covariant derivatives. The Wigner representation method was introduced in \cite{22} and is closely related to the method of symbols for pseudo-differential operators. (An improved symbols method and many references can be found in \cite{35}.) The method is adapted here to treat the finite temperature case, and has been used also for 1+1- and 3+1-dimensional fermions at finite temperature in \cite{34} and for non-local Dirac operators
The combination of $\zeta$-function and Wigner transformation yields a computational setup which is suitable to obtain the effective action using different expansions. The other ingredient is the use of a derivative expansion. In order to preserve gauge invariance, it is, of course, necessary to expand in terms of the covariant derivatives. This would be sufficient at zero temperature, however, at finite temperature because the frequencies take on discrete values only, an expansion in the number of time-like covariant derivatives breaks invariance under large gauge transformations and so such derivatives have to be treated non-perturbatively. Therefore, we will consider an expansion in the number of spatial covariant derivatives only. The advantages of such an expansion are, first, that it preserves gauge and parity invariances, i.e., terms of different order do not mix under such transformations, and second, each order increases the degree of ultraviolet convergence. Therefore, computing the terms which are ultraviolet divergent, as we will do, yields all contributions which may contain anomalies and multivaluation under large gauge transformations. In particular, topological terms are included there. The remaining terms are strictly gauge and parity invariant.

For technical reasons, the calculation is carried out in the gauge $\partial_0 A_0 = 0$ which is shown to exist for any given configuration, although it is not unique. We check that different $\partial_0 A_0 = 0$ gauges yield compatible result. The calculation is not based on integrating the axial anomaly in two dimensions or similar ideas. Except for the use a particular gauge, it is a direct and systematic calculation which uses only the definitions of $\zeta$-function, derivative expansion, etc. This allows to obtain both the topological and non-topological terms. The expansion can be carried out to any order in principle, although, of course, higher orders require an increasing amount of work which quickly becomes prohibitive. As mentioned, in the present work we compute the terms with ultraviolet divergences. This amounts to zero and two derivatives for the real part and two derivatives for the imaginary part. We check that our result is consistent with gauge invariance, more precisely, it transforms as the CS term, and yields the correct parity anomaly, which is shown to be temperature independent. Further, it reduces to the result in [30] for the same configurations, has the correct zero temperature limit and reproduces the exact result for massless fermions.

Another related expansion is also considered which consists in carrying out a further expansion in powers of the time-like covariant derivatives which are inside commutators (as will clear below, the space-like covariant derivatives were already inside commutators due to gauge invariance since the space is not compactified). This new expansion still preserves gauge and parity invariances and higher order terms are increasingly convergent. At leading order, the result is a simpler expression which still contains all topological terms, saturates the parity anomaly and possesses the same multivaluation of the full effective action. This

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1It should be noted, however, that the $\zeta$-function is not particularly convenient to compute the effective action in odd-dimensional space-times at zero temperature. This is because there is no pathology free way to choose the branch cut in $z^s$ so as to avoid both the spectrum of the Dirac operator, which is continuous under most expansions, and the spectrum of the parity transformed Dirac operator. This regularization is suitable at finite temperature since then the spectrum becomes discrete for practical purposes.
result also reproduces that of [30], as well as the zero temperature and zero mass limits. The same expansion has been used in [34] for even-dimensional fermions at finite temperature.

The paper is organized as follows. In section II some general considerations are made regarding the effective action, its relevant symmetries and the Wigner transformation method. In section III the 0+1-dimensional case is revised. This allows to illustrate some exact results as well as the method before going to the three-dimensional case. In section IV the main results of the three-dimensional problem are presented. For the leading term of the imaginary part, which is the finite temperature version of the induced CS term, we check that it possesses all expected correct properties under symmetry transformations and limiting cases. In section V we present the explicit computation of the effective action at lowest order, that is, without derivatives. This term is purely real. Section VI contains the computation of the terms which are real and with two derivatives. Section VII contains the similar calculation for the imaginary part. Appendix A proves the existence of the $A_0$-stationary gauge and finds the allowed gauge transformations within that gauge. Finally, Appendix B introduces some mathematical results used in the main text.

II. GENERAL CONSIDERATIONS

A. The Dirac operator

The most general Dirac operator in a flat three-dimensional Euclidean space has the form

$$D = \gamma_\mu D_\mu + M$$  \hspace{1cm} (1)

where $D_\mu = \partial_\mu + A_\mu$ is the covariant derivative and $A_\mu(x)$ and $M(x)$ are matrices in some internal space to be referred to as flavor. Our convention for the Dirac matrices will be

$$\gamma^\dagger_\mu = \gamma_\mu, \quad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}.$$ \hspace{1cm} (2)

In 2+1 dimensions there are two inequivalent irreducible representations of the Dirac algebra labeled by $\eta = \pm 1$, defined as $\gamma_0\gamma_1\gamma_2 = i\eta$. For instance, $\gamma_0 = \eta\tau_3$, $\gamma_{1,2} = \tau_{1,2}$ where $\tau_i$ stand for the Pauli matrices. To ensure unitarity in Minkowski space, $A_\mu$ is required to be anti-Hermitian and $M$ Hermitian. At finite temperature $T$, the Dirac operator $D$ acts on the space of fermionic wave functions $\psi(x)$, which are antiperiodic in the Euclidean time direction with period $\beta = 1/T$. Correspondingly, the external bosonic fields $A_\mu$ and $M$ are periodic. Further restrictions on $D$ comes by imposing continuity in all involved functions (fermionic wave functions, field configurations, gauge transformations, etc). In addition, we will occasionally assume that the space is compactified so that it is topologically a two dimensional sphere. This implies a topology $M_3 = S^2 \times S^1$ for the space-time. The class of Dirac operators to be considered in each case must be a subset of the above and should be sufficiently large as to be invariant under the relevant symmetries of the problem.

B. The effective action

The unrenormalized partition functional is
\[ Z(D) = \int \mathcal{D}\psi\mathcal{D}\bar{\psi} e^{-\int d^3x \bar{\psi} D\psi}. \]  

(3)

Formal integration of the fermions gives \( Z = \text{Det}(D) \) and thus the unrenormalized effective action is

\[ W_{\text{bare}}(D) = -\log Z(D) = -\text{Tr} \log(D). \]  

(4)

Upon regularization and renormalization of the ultraviolet divergences, a well-defined and finite effective action \( W \) is obtained. However, the renormalization procedure is not unique and thus there will be several functions \( W(D) \) all of them qualifying as valid effective actions for the same original action \( \int d^3x \bar{\psi} D\psi \). They are constrained to reproduce the same ultraviolet finite terms since such terms can be computed without regularization and are therefore unambiguous. This implies that the fourth order variation should be common to all \( W(D) \). This is because for arbitrary commuting variations of \( D \)

\[ \delta_1\delta_2\delta_3\delta_4 W(D) = \text{Tr}(\delta_1DD^{-1}\delta_2DD^{-1}\delta_3DD^{-1}\delta_4DD^{-1}) + 5 \text{ permutations} \]  

(5)

is ultraviolet finite in 2+1 dimensions. As a consequence, the (renormalized) ultraviolet divergent terms, responsible of the ambiguity in the renormalization, must vanish after a fourth order variation. Thus the allowed ambiguity introduced by the renormalization is a local polynomial action of canonical dimension at most three constructed with the fields \( M, A_\mu \) and their derivatives. This is also clear in perturbation theory since all Feynman diagrams become convergent after four derivatives in the fields or the external momenta. Of course, in order to compare the effective action for different Dirac operators, it is essential to use the same renormalization for the whole class of Dirac operators \( D \) considered. Only in this case can be meaningful the statement that \( W(D) \), as a function of \( D \), is unique modulo a local polynomial. All these remarks are well-known but they will be needed later and non trivial consequences will be extracted from them.

A set of renormalizations are based on the spectrum of the Dirac operator. Let \( D\psi_n = \lambda_n\psi_n \) be the eigenvalue equation for \( D \). The ultraviolet divergent part of the spectrum comes from the term \( \gamma_\mu \partial_\mu \) of the Dirac operator and thus it lies on \( \pm i\infty \). In practice we will take \( M(x) = m \), a constant c-number, and in this case the spectrum will lie on the line \( m + i\mathbb{R} \). The bare effective action can be expressed as

\[ W_{\text{bare}} = -\sum_n \log(\lambda_n). \]  

(6)

There is a subtlety at finite temperature. In this case the frequency takes discrete values only and this prevents to take a derivative with respect to it. Nevertheless, since the ultraviolet convergence also increases by taking finite differences, the argument applies also for finite temperature and the allowed ambiguity is still a local polynomial.

As an extreme case of ignoring this restriction, consider adding to the effective action a different “constant” for each \( D \). This would result in a completely arbitrary function \( W(D) \).
In this form, the sum over eigenvalues is ultraviolet divergent and must be renormalized. However, the renormalization only affects the ultraviolet part of the spectrum so any particular eigenvalue still contributes with $-\log \lambda_n$ to the renormalized effective action. This shows that qualitatively $W(D)$ is to be understood as a many-valuated function defined modulo $2\pi i$ on the manifold of (sufficiently regular) Dirac operators to be denoted by $\mathcal{M}$. The branching points correspond to the Dirac operators which are singular, that is, with a zero eigenvalue.

A mathematically well founded choice for the renormalization prescription, which we will adopt here, is the $\zeta$-function version of $W$, namely,

$$W(D) = -\sum_n \lambda_n^s \log \lambda_n \bigg|_{s=0}.$$  \hspace{1cm} (7)

(The same branch is to be taken for the two functions $z^s$ and $\log(z)$ and all the eigenvalues.) This expression is to be understood as an analytical extension on the variable $s$ from $\text{Re}(s) < -3$ and a sufficient condition to be well defined is that the eigenvalues are all on the same Riemann sheet of the logarithm function and off the branch cut \cite{20}. Once the branch cut, $\Gamma$, for the function $-z^s \log(z)$ on the complex plane is chosen, $W(D)$ becomes completely well defined but it will, of course, have a spurious discontinuity along the cut. More precisely, the cut on the complex plane will produce on $\mathcal{M}$ a branch cut stemming from each of the singular Dirac operators. The corresponding cut manifold will be denoted by $\mathcal{M}_\Gamma$. On the complex plane and upon analytical extension, two different choices of the cut, $\Gamma_1$ and $\Gamma_2$, will yield the same function $-z^s \log(z)$. This is no necessarily the case on $\mathcal{M}$. The two choices of the cut will yield the same function $W(D)$ upon analytical extension if and only if $\Gamma_1$ and $\Gamma_2$ can be deformed into each other crossing at most a finite number of eigenvalues. Because the ultraviolet sector of the spectrum lies on $\pm i\infty$ there are essentially only two inequivalent prescriptions, namely, choosing the branch cut of the logarithm along the negative real axis or along the positive real axis. The two choices will be labeled by $\sigma = +1$ and $\sigma = -1$ and the two cut manifolds by $\mathcal{M}_{\sigma=1}$ and $\mathcal{M}_{\sigma=-1}$, respectively.

Analogously to the theory of many-valued functions on the complex plane, an alternative way to achieve both one-valuedness and analyticity of $W(D)$ is to work on the simply connected Riemann surface manifold associated to $\mathcal{M}$ (i.e., its the universal covering space) which will be denoted by $\tilde{\mathcal{M}}$. From the previous discussion it follows that the two functions $W_{\sigma=\pm 1}(D)$ defined on their common Riemann surface $\tilde{\mathcal{M}}$ by analytical extension represent two different ($\zeta$-function) renormalizations of the effective action and thus must differ by a local polynomial function on $\tilde{\mathcal{M}}$.

Another comment refers to symmetries. As usual, a symmetry is defined as any transformation of the Dirac operator which leaves invariant the class of allowed Dirac operators and which can be compensated by a corresponding transformation in the fermionic wave functions so that the action $\int d^3x \bar{\psi} D\psi$ remains unchanged. According to this definition, symmetries leave invariant the classical effective action (defined as the action evaluated at its minimum with respect to the fermionic fields). A non-vanishing variation of the quantum effective action implies a quantum anomaly for the symmetry. The anomaly must be a local polynomial. Indeed, if $W(D)$ is any definition of the effective action and $D^\Omega$ is the transformed Dirac operator with $\Omega$ independent of $D$, the function $W(D^\Omega)$ also qualifies as an admissible definition of the effective action of $D$. Therefore, the anomaly $W(D^\Omega) - W(D)$ must be a local polynomial. Instances of this are the chiral anomaly in even dimensional
space-times or the parity anomaly in odd dimensions. It also applies to the $2\pi in$ multivaluation under large gauge transformations.

In practice, the $\zeta$-function renormalization is carried out by introducing the quantity

$$\Omega_s(D) = \text{Tr}(D^s) = \sum_n \lambda_n^s,$$  \hspace{1cm} (8)

which is ultraviolet finite if $\text{Re}(s) < -3$ and a meromorphic function of $s$ with simple poles in $s = -3, -2, -1$ \cite{20}. Then, the $\zeta$-function prescription is

$$W(D) = -\frac{d}{ds} \Omega_s(D) \bigg|_{s=0}. \hspace{1cm} (9)$$

The precise definition of the function $z^s$ requires cutting the complex plane along a ray characterized by an angle $\theta$. Applying Cauchy’s theorem \cite{20}

$$\Omega_{s,\Gamma}(D) = -\text{Tr} \int_{\Gamma} \frac{dz}{2\pi i} z^s D - z.$$  \hspace{1cm} (10)

The trace Tr is to be taken in the fermionic Hilbert space which is the tensor product of space, time, flavor and Dirac. The integration path $\Gamma$ follows the ray $\theta$ starting from infinity, encircles zero clockwise and goes back to infinity along the ray $\theta - 2\pi$, so that if the spectrum of the operator $D$ were bounded $\Gamma$ could be deformed to enclose it anticlockwise. The choices $\theta = \pi$ and $2\pi$ correspond to $\sigma = +1$ and $-1$ respectively. As noted, any value $\pi/2 < \theta < 3\pi/2$ gives the same function $\Omega_{s,\sigma=+1}$ on the Riemann, surface, and similarly any $3\pi/2 < \theta < 5\pi/2$ yields the same function $\Omega_{s,\sigma=-1}$. Besides, The same effective action follows from taking $\theta$ or $\theta + 2\pi n$ for integer $n$. (See Appendix B.)

C. Gauge transformations

Gauge transformations are defined as $D^U = U^{-1}DU$, where $U(x)$ is a matrix valued function acting as a multiplicative operator on the fermionic wave functions. More explicitly, $D^U = \gamma_\mu (\partial_\mu + A^U_\mu) + M^U$, with

$$A^U_\mu(x) = U^{-1}(x)(\partial_\mu + A_\mu(x))U(x), \quad M^U(x) = U^{-1}(x)M(x)U(x). \hspace{1cm} (11)$$

$U(x)$ belongs to some gauge group $G$ which is a subgroup of $U(N_f)$, $N_f$ being the number of flavors. Correspondingly $A_\mu$ must be in the Lie algebra of $G$ and likewise the class of matrices $M(x)$ must also be closed under gauge transformations. (This is trivially satisfied if $M$ is just a c-number.) Besides, $U(x)$ must be continuous on the space-time manifold and in particular periodic as a function of $x_0$.

\footnote{It follows that it is always possible to define effective actions free from anomalies associated to compact groups, namely, by taking an average over all symmetry transformed configurations. This applies in particular to parity transformations, however, the average of the two effective actions defined modulo $2\pi i$ will have an ambiguity $i\pi$ in general, so gauge invariance is not ensured.}
Being similarity transformations, gauge transformations leave the spectrum of $D$ invariant. As a consequence, the functions $W_{\sigma=\pm1}(D)$ defined on the cut manifolds $\mathcal{M}_{\sigma=\pm1}$ are strictly gauge invariant by construction since these functions depend solely on the spectrum of $D$. In this sense, the $\zeta$-function renormalization prescription trivially preserves gauge invariance. On the Riemann surface $\tilde{\mathcal{M}}$ the statement is less trivial. Let $D$ be continuously deformed along a path connecting two field configurations which are related by a gauge transformation. This will induce a corresponding bounded path on the complex plane for each eigenvalue and, since the spectrum is unchanged at the end, the net effect on the eigenvalues can be at most a permutation. During its walk, a finite number of eigenvalues will cross the branch cut, each time adding a net $\pm 2\pi i$, to the effective action. In summary, the functions $W_{\sigma=\pm1}(D)$ extended to $\tilde{\mathcal{M}}$ are gauge invariant modulo $2\pi i$. It should be noted that, by continuity, the effect of a gauge transformation may only depend on the homotopy class of the gauge transformation and on the homotopy class of the initial field configuration. In particular, only topologically large gauge transformations may change the effective action by a non-vanishing integer multiple of $2\pi i$. We are assuming that no branching point lies on the path followed by the eigenvalues. In the massless case this cannot be avoided since the spectrum is purely imaginary. In this particular case the variation in $W_{\sigma=\pm1}(D)$ is a multiple of $i\pi$ instead.

For convenience, the calculation of the effective action will be carried out in an $A_0$-stationary gauge, i.e., a gauge such that $\partial_0 A_0(x) = 0$. Such a gauge always exits for any given gauge configuration, therefore there are no loss of generality by making this choice. This result is proven in Appendix A. In the same appendix, it is shown that the gauge transformation needed to bring a gauge field configuration to an $A_0$-stationary gauge can always be chosen to be topologically small in the Abelian case. On the other hand, for simply connected gauge groups large transformations may be necessary, depending on the given gauge configuration.

To check gauge invariance of our calculation, it will also be necessary to find the most general gauge transformations which preserve the gauge fixing condition. As shown in Appendix A, these transformations have the form

$$U(x) = \exp(-x_0 A_0(x)) \exp(x_0(A_0(x) + \Lambda(x))) U_0(x).$$

Here $U_0(x)$ is an arbitrary time-independent gauge transformation. $\Lambda$ is a time-independent function taking values in the Lie algebra of the gauge group. It is only restricted by continuity of $U(x)$, which requires

$$\exp(\beta A_0(x)) = \exp(\beta(A_0(x) + \Lambda(x))).$$

Therefore, the most general gauge transformation preserving the $A_0$-stationarity condition is composed of a particular time-dependent transformation subject to the condition in eq. (13) followed by a stationary gauge transformation. The corresponding zeroth component of the transformed field takes the form

\[5\text{Recall that each particular eigenvalue adds } -\log \lambda_n \text{ to the renormalized effective action; the renormalization, } \lim_{s \to 0} \sum_n \lambda_n^s \log \lambda_n, \text{ leaves no trace on any given bounded region of the complex plane.}\]
\[ A_0^U(x) = U_0^{-1}(x)(A_0(x) + \Lambda(x))U_0(x). \] (14)

The time-dependent transformations \( e^{-x_0A_0}e^{x_0(A_0+\Lambda)} \) will be called discrete gauge transformations for the stationary field \( A_0 \) (since generically they form a discrete set due to eq. (13)).

In general the condition on \( \Lambda \) can be simplified. From eq. (13), it follows that both \( A_0(x) \) and \( \Lambda(x) \) must commute with \( \exp(\beta A_0(x)) \). If the spectrum of \( \exp(\beta A_0(x)) \) is nowhere degenerated, this unitary matrix takes a diagonal form in an \( x \)-dependent basis in flavor space that is essentially unique, thus \( A_0(x) \) and \( \Lambda(x) \) must also be diagonal in the same basis and therefore they commute with each other. In this case the condition on \( \Lambda \) becomes

\[ [A_0(x), \Lambda(x)] = 0, \quad \exp(\beta \Lambda(x)) = 1. \] (15)

The second condition means that the eigenvalues of \( \Lambda(x) \) are of the form \( \lambda_j = \frac{2\pi in_j}{\beta} \), for integer \( n_j \). Such integers are \( x \)-independent by continuity. Correspondingly, in this case, the allowed discrete transformations take the simpler form

\[ U(x) = \exp(x_0\Lambda(x)). \] (16)

This form is the generic one. It applies if the spectrum of \( \exp(\beta A_0(x)) \) is nowhere degenerated, or if the points of degeneracy can be resolved by continuity.

The stationary gauge transformations are always topologically small since the second homotopy group is trivial for any Lie group. Whether the discrete transformations are topologically large or not depends on several factors. If the gauge group is not simply connected, for instance the Abelian case, \( U(1) \), and \( \Lambda \) is not zero, such transformations are necessarily large, since the loop around \( S^1 \) for each \( x \) cannot be contracted to the identity. If the gauge group is simply connected, for instance \( SU(N_f) \), the discrete transformation may be large or small depending on \( \Lambda(x) \). For instance, if \( \Lambda \) is \( x \)-independent, or becomes so after a continuous deformation, the discrete transformation will be topologically small. In particular, this holds if \( \Lambda(x) \) can be diagonalized using a similarity transformation which is continuous on \( S^2 \), since a diagonal \( \Lambda \) must be \( x \)-independent due to eq. (13). On the other hand, to see that there are large discrete transformation in general, consider the discrete transformations from \( S^2 \times S^1 \) into \( SU(2) \),

\[ U(x, x_0) = \exp\left(\frac{2\pi n x_0}{\beta} i \tau x \right), \quad n \in \mathbb{Z} \] (17)

where \( \tau \) are the Pauli matrices and \( x \) belongs to the unit sphere \( S^2 \) in \( \mathbb{R}^3 \). \( U(x) \) covers \( SU(2) \) \( 2n \) times (it is \( 2n \) to one and it maps a positively oriented volume element on \( S^2 \times S^1 \) into a positively oriented volume element on \( SU(2) \)). Equivalently, one can compute the normalized Wess-Zumino-Witten 3-form (relevant in chiral theories in 1+1 dimensions) which is closed and thus invariant under deformations,

\[ \Gamma_{\text{wzw}}(U) = -\frac{i}{12\pi} \int_{M_3} \text{tr}(R^3) = -4i\pi n, \quad R = U^{-1}dU. \] (18)

Therefore, such discrete transformations are all homotopically inequivalent and large for non vanishing \( n \). The Wess-Zumino-Witten action appears naturally in this context since the Chern-Simons action
\( W_{\text{CS}}(A) = \frac{i}{4\pi} \int_{M_3} \text{tr}(AF - \frac{1}{3}A^3), \)  

(19)

(where \( F = dA + A^2 \)) evaluated for a pure gauge field \( A = U^{-1}dU = R \) is just \( \Gamma_{\text{WZW}}(U). \) As noted in the introduction, the Chern-Simons action can only change under gauge transformations which are topologically large.

D. Parity transformation

The parity transformed of \( D \) will be defined as \( D^P = \gamma_\mu(\partial_\mu + A^P_\mu) + M^P \) with

\[
A^P_0(x) = -A_0(x^P), \quad A(x) = +A(x^P), \quad M^P(x) = -M(x^P), \quad x^P = (-x_0, x).
\]

(20)

This is the definition taken in [14]. All definitions of parity consist in reversing an odd number of coordinates and are related by a similarity transformation thereby being equivalent.

The fermionic field transforms as \( \psi^P(x) = \gamma_0\psi(x^P) \) and the eigenvalue equation becomes \( D^P\psi^P = -\lambda_n\psi^P. \) Since the eigenvalues are changed, \( W(D) \) needs not be parity invariant within the \( \zeta \)-function regularization. This allows for a parity anomaly which is defined as

\[
\mathcal{A}_P(D) = W(D) - W(D^P).
\]

(21)

As noted at the end of subsection II B, the parity anomaly is due to the ultraviolet divergences present at the quantum level and hence is a local polynomial in the fields. The same result follows from noting that the two choices for the cut, \( \sigma = \pm 1 \), are related by a parity transformation and thus

\[
W_\sigma(D^P) = W_{-\sigma}(D).
\]

(22)

As a consequence, the anomaly can also be written as \( \mathcal{A}_P(D) = W_\sigma(D) - W_{-\sigma}(D) \), which, as noted above, is a local polynomial action on \( \tilde{\mathcal{M}}. \)

Under parity, the action can be split into a parity preserving part plus a parity violating part, or equivalently, a \( \sigma \)-even plus a \( \sigma \)-odd component

\[
\begin{align*}
W_{\text{even}}(D) &= \frac{1}{2}(W_{\sigma}(D) + W_{-\sigma}(D)), \\
W_{\text{odd}}(D) &= \frac{1}{2}\mathcal{A}_P(D).
\end{align*}
\]

(23)

Since \( W_{\text{odd}} \) is just a local polynomial, it can be removed by counterterms to end up with a parity preserving effective action. However, this can be incompatible with gauge invariance. By construction \( W_\sigma \) and \( \mathcal{A}_P \) are gauge invariant modulo \( 2\pi i \), but \( W_{\text{even}} \) and \( W_{\text{odd}} \) can separately change by a multiple of \( i\pi \) in general. When this happens, adding counterterms to remove \( W_{\text{odd}} \) spoils gauge invariance of the effective action. Parity and gauge invariances cannot be both enforced in general [9].

Because the parity anomaly is due to divergences in the ultraviolet sector of the theory, it is to be expected that it is essentially unchanged by introducing a finite temperature, which only affects the infrared sector through the periodic boundary conditions. The same argument applies for other quantum anomalies. In particular, it is well established in the literature (e.g. [37]) that the axial anomaly takes the Bardeen form independently of the temperature. Remarkably, the analogous result can be proven for the parity anomaly, even
without a detailed calculation, namely, the form of the essential parity anomaly at finite temperature is given by the Chern-Simons action. By essential it is meant the part of the anomaly which is not removable by gauge invariant counter terms (i.e., the analogous of the Bardeen anomaly in the chiral case). To prove this result, we start with the parity anomaly at zero temperature. It is given by the Chern-Simons action which, of course, is a local polynomial, parity odd and gauge invariant modulo \(2\pi i\). Therefore, at zero temperature, \(W_{\text{odd}} = \pm \frac{1}{2} W_{\text{CS}}\). At finite temperature, \(W_{\text{odd}}\) will be given by \(\pm \frac{1}{2} W_{\text{CS}}\) plus a local polynomial with a smooth temperature dependence. By continuity this local polynomial must be strictly gauge invariant (not just modulo \(i\pi\)) and thus it can be removed by a counterterm without destroying gauge invariance, leaving \(A_P = \pm W_{\text{CS}}\) at any temperature. This conclusion is confirmed by our detailed calculation below and also follows from eq. (5) of [29] (noting the different definitions of parity violating terms in both works). The same argument applies to any other smooth dependence such as that introduced by a finite density or a change in the mass of the fermion.

E. Pseudo-parity transformation

It is also of interest to introduce a pseudo-parity transformation [8], defined as

\[
A_0(x) \rightarrow -A_0(x_P), \quad A(x) \rightarrow +A(x_P), \quad M(x) \rightarrow +M(x_P).
\]  

(24)

It is clear that the pseudo-parity odd component of the effective action is that with an odd number of zeroth Lorentz indices and thus is the component containing the Levi-Civita pseudo-tensor. It also corresponds to the imaginary part of the Euclidean effective action. The CS action is pseudo-parity odd. The \(\zeta\)-function prescription yields a parity anomaly both in the pseudo-parity even and odd components, but only the latter is essential in the sense that it cannot be removed without breaking gauge invariance. This is because the even component can be renormalized preserving gauge and parity invariances simultaneously [13,14]. Note that in the massless case, parity and pseudo-parity transformations coincide.

F. Wigner transformation method

The (asymmetric) Wigner transformation method is described in great detail in [23]. The corresponding formula for the zero-temperature (i.e., uncompactified) case is given in that reference and only a variation is needed to cope with the compactification in the zeroth direction. We concentrate on the Hilbert spaces of periodic and antiperiodic wave functions of time defined on \(S^1\). Let us introduce the fermionic and bosonic Matsubara frequencies

\[
\omega_n = \frac{2\pi i}{\beta} (n + \frac{1}{2}), \quad \kappa_n = \frac{2\pi in}{\beta}.
\]  

(25)

The states \(|\omega\rangle = e^{\omega x_0}\) are periodic if \(\omega = \kappa_n\) and antiperiodic if \(\omega = \omega_n\). In both cases they are normalized to \(\sqrt{\beta}\). A general operator acting on the antiperiodic space will be of the form \(f(\hat{x}_0, \hat{\partial}_0)\) where \(\hat{x}_0\) and \(\hat{\partial}_0\) denote the time and frequency operators respectively, and \(f\) is a periodic function of \(\hat{x}_0\). Then the trace of \(f\) is given by
\[
\text{Tr}(f(\hat{x}_0, \hat{\partial}_0)) = \frac{1}{\beta} \sum_{n \in \mathbb{Z}} \langle \omega_n | f(\hat{x}_0, \hat{\partial}_0) | \omega_n \rangle \\
= \frac{1}{\beta} \sum_{n \in \mathbb{Z}} \langle 0 | e^{-\omega_n \hat{x}_0} f(\hat{x}_0, \hat{\partial}_0) e^{\omega_n \hat{x}_0} | 0 \rangle \\
= \frac{1}{\beta} \sum_{n \in \mathbb{Z}} \langle 0 | f(\hat{x}_0, \hat{\partial}_0 + \omega_n) | 0 \rangle.
\] (26)

Here \(|0\rangle\) is the zero frequency state. Note that \(|0\rangle\) is periodic rather than antiperiodic. For the non compact coordinates the treatment is entirely similar, \(D_i\) is replaced by \(D_i + p_i\), there is an integral over momenta and \(|0\rangle\) is the zero momentum state \([23]\). Therefore, using eq. (10), the \(\zeta\)-function can then be written as

\[
\Omega_{s,r}(D) = - \int \frac{d^2 k}{(2\pi)^2} \frac{1}{\beta} \sum_n \int_\mathbb{R} \frac{dz}{2\pi i} z^s \text{tr}(0) \frac{1}{\hat{p} + D - z | 0 \rangle}.
\] (27)

Here \(p = ik\) (where \(k \in \mathbb{R}^2\)), \(p_0 = \omega_n\) and \(|0\rangle\) represents the spinless and flavorless zero energy-momentum state normalized as \(\langle x | 0 \rangle = 1\). In particular \(\partial_\mu | 0 \rangle = \langle 0 | \partial_\mu = 0\), and also \(\langle 0 | 0 \rangle = \int d^3 x\), and moreover, if \(f(\hat{x})\) is a multiplicative operator, i.e., it does not contain derivatives,

\[
\langle 0 | f(\hat{x}) | 0 \rangle = \int d^3 x f(x).
\] (28)

The formula just given for \(\Omega_{s,r}(D)\) is completely finite in the ultraviolet sector and exact if a space-time topology \(\mathbb{R}^2 \times S^1\) is assumed.

G. Covariant derivative expansion

We will find convenient to use an \(A_0\)-stationary gauge to compute the effective action. This is allowed due to gauge invariance, however, since the \(A_0\)-stationary gauge is not unique, we will have to check that the result is the same for any of the gauge copies, i.e., invariant under time-independent and discrete gauge transformations. It should be noted that the integrand in eq. (27) is not directly gauge invariant due to the presence of \(|0\rangle\); gauge invariance requires integration over momentum and sum over frequencies \([23]\). Roughly speaking, invariance under time independent transformations means that \(D_i\) appears only inside commutators and this is equivalent to say that the result is unchanged if \(D_i\) is replaced by \(D_i + a_i\), \(a_i\) being an arbitrary \(x\)-independent \(c\)-number, but precisely this invariance is ensured by the integration over momentum since \(a\) can always be absorbed in \(p\). Similarly, invariance under the discrete gauge transformations, \(A_0 \rightarrow A_0 + \lambda\), is ensured by the sum over discrete frequencies, since \(A_0\) can be absorbed in \(\omega_n\) (recall that the spectrum of \(\lambda\) is quantized). We note that a recently proposed technique, due to Pletnev and Banin \([35]\), allows to bring the expression to one where \(D_i\) appears inside commutators from the beginning, i.e., prior to momentum integration. Presently this technique applies to covariant derivatives in non compact directions.

In order to carry out the traces, sums and integrations implied in eq. (27) we will perform a formal expansion in powers of the spatial covariant derivatives, \(D_i, i = 1, 2\). However, it
is important to remark that such an expansion is not only formal; it is meaningful for the effective action functional itself beyond the particular computational procedure followed. It actually means to change the bosonic field configuration to \( A_0(\lambda x, x_0) \), \( \lambda A(\lambda x, x_0) \) and \( M(\lambda x, x_0) \), and then count powers of \( \lambda \). Since this scaling preserves full gauge invariance (namely, by scaling \( U \) too) the gradient expansion is gauge invariant order by order. Formally, the scaling corresponds to the replacement \( D_i \rightarrow \lambda D_i \) and gauge invariance is preserved since \( \lambda a_i \) can still be compensated by a change of variables in \( k_i \). The situation is completely different for a formal expansion in powers of \( D_0 \). First of all, it would only be formal, since scaling \( x_0 \rightarrow \lambda x_0 \) violates the boundary conditions for most field configurations (because \( \beta \) is not scaled). At a formal level, scaling \( D_0 \) would imply a similar scaling in \( \Lambda \) which will no longer be properly quantized and it could not be compensated by a shift in \( \omega_n \).

Due to rotational invariance, the proposed gradient expansion contains even orders only. In this work we explicitly work out the zeroth and second order terms, \( W_0(D) \) and \( W_2(D) \) respectively. Since all higher order terms are ultraviolet finite and the different orders do not mix under gauge or parity transformations, all anomalies are contained in these two terms. The pseudo-parity odd component starts at second order since the Levi-Civita pseudo-tensor requires the presence of at least two gradients. Thus the leading pseudo-parity odd term, \( W_2^-(D) \), is responsible for all essential anomalies in the effective action, whereas \( W_0 \) and \( W_2^+ \) may contain at most removable anomalies.

There is another expansion which is also of great interest in the present context. As will be clear from the calculation, the final result at each order in the spatial gradient expansion comes as a function of \( A_0 \) and \( D_0 \) which is defined as \([D_0,\ ]\). (Note that \( A_0 \) and \( D_0 \) commute in a \( A_0 \)-stationary gauge.) It is the expansion in powers of \( A_0 \) the one that would break gauge invariance under discrete gauge transformations, and so \( A_0 \) should be treated nonperturbatively. On the other hand, expanding in powers of \( D_0 \) does not break any symmetry of the problem (excepting, of course, Lorentz invariance in the limit of zero temperature). Each extra power of \( D_0 \) or \( D_i \) increases the degree of convergence in the ultraviolet sector. Therefore, at finite temperature, it is natural to consider a double expansion in powers of \( D_i \) and \( D_0 \) which play a similar role as a standard space-time gradient expansion which is often used at zero temperature. (In fact, \( D_i \) always appears inside of commutators and so the expansion in powers of \( D_i \) can be seen also as an expansion in powers of \( D_i = [D_i, \ ] \)). This same idea has been applied in the case of 3+1-dimensional fermions at finite temperature [34].

### III. THE 0+1-DIMENSIONAL CASE

Before going to 2+1 dimensions, we will study the simpler 0+1-dimensional case. This will allow us to illustrate some of the previous remarks as well as the method.

In 0+1 dimensions, the Dirac operator is \( \gamma_0 D_0 + m \) with \( \gamma_0 = \gamma = \pm 1 \), a \( 1 \times 1 \) dimensional matrix in Dirac space. We have already taken \( M(x) = m \), a constant c-number. We start by fixing the gauge to be \( A_0 \)-stationary. For convenience we will use units \( \beta = 2 \). A direct application of the Wigner transformation formula (i.e., eq. (27) adapted to 0+1 dimensions) gives:
\[
\Omega_{s,\Gamma}(D) = -\frac{1}{2} \sum_n \int_{\Gamma} \frac{dz}{2\pi i} z^s \text{tr} \langle 0 | \frac{1}{\gamma_0 (\omega_n + D_0) + m - z} | 0 \rangle \\
= -\sum_n \int_{\Gamma} \frac{dz}{2\pi i} z^s \text{tr} \left( \frac{1}{\eta (\omega_n + A_0) + m - z} \right). 
\] (29)

We have replaced \( D_0 \) by \( A_0 \) since everything is time independent and \( \partial_0 \) vanishes on \( |0\rangle \), and \( \langle 0|0\rangle = \beta = 2 \) has been used.

In order to sum over frequencies, we first transform the series in \( \Omega_{s,\Gamma}(D) \) into a convergent one by performing a subtraction

\[
\Omega_{s,\Gamma}(D) = -\int_{\Gamma} \frac{dz}{2\pi i} z^s \text{tr} \left( \sum_n \left( \frac{1}{\eta (\omega_n + A_0) + m - z} - \frac{1}{\eta (\omega_n + A_0) + m} \right) \right). 
\] (30)

This is justified due to the identity

\[
\int_{\Gamma} \frac{dz}{2\pi i} z^s q(z) = 0, 
\] (31)

for any polynomial \( q(z) \), which comes from closing \( \Gamma \) with the circumference at infinity. This and similar formulas are to be understood upon analytical extension from sufficiently negative \( \text{Re}(s) \). Then, the following formula can be used

\[
\sum_n \left( \frac{1}{x_1 + \omega_n} - \frac{1}{x_2 + \omega_n} \right) = \tanh(x_1) - \tanh(x_2) 
\] (32)

and the sum over frequencies yields

\[
\Omega_{s,\Gamma}(D) = -\int_{\Gamma} \frac{dz}{2\pi i} z^s \text{tr} \tanh(m + \eta A_0 - z). 
\] (33)

The final step is to carry out the integration over \( z \) and apply \(-d/ds|_{s=0}\) to obtain the effective action. This can be done using the functions

\[
\Omega_{\Gamma}(\omega, s) = -\int_{\Gamma} \frac{dz}{2\pi i} z^s \tanh(\omega - z). 
\] (34)

The properties of these and related functions are summarized in Appendix B. As shown there

\[
\Omega_\sigma(\omega, 0) = 0 \\
\Omega'_\sigma(\omega, 0) = \log(e^{-2\sigma \omega} + 1). 
\] (35)

Here the prime refers to derivative with respect to \( s \) and \( \sigma = \pm 1 \) refers to the two possible inequivalent choices of \( \Gamma \). Two remarks can be made regarding the first of these equations. First, the vanishing of \( \Omega_\sigma(\omega, 0) \) guarantees that taking \( \Gamma \) along the ray \( \theta \) or \( \theta + 2\pi n \) yields the same final result for the effective action. Second, it also implies that there is no scale anomaly in 0+1 dimensions. Indeed, the scale anomaly can be associated to a non trivial dependence of the effective action on some mass parameter \( M_0 \) introduced by using \((z/M_0)^s\) instead of \( z^s \) in the definition of the \( \zeta \)-function. Such dependence cancels in odd dimensions.
The final exact result for the effective action in 0+1 dimensions, restoring arbitrary units, is thus

\[ W_\sigma(D) = -\text{tr} \log(e^{-\sigma \beta(m + \eta A_0)} + 1). \]  (36)

Recalling that under gauge transformations the eigenvalues of \( A_0 \) may change at most by integer multiples of \( 2\pi i/\beta \) it follows that \( W_\sigma(D) \) is manifestly gauge invariant modulo \( 2\pi i \) on the Riemann surface \( \tilde{M} \) and strictly gauge invariant on each Riemann sheet \( M_{\sigma} \). This latter property also follows from eq. (33) since the integrand there is strictly periodic; this is a direct consequence of the sum over \( \omega_n \). As noted in [28], a perturbative expansion in powers of \( A_0 \) would display a spurious breaking of gauge invariance at finite temperature. This is because the perturbative condition, \( A_0 \) small, is not preserved by large gauge transformations. This simply means that a truncated series expansion of a periodic function is not itself a periodic function in general.

The behavior under parity is better exposed by introducing the functions \( \phi_n \) of Appendix B. In the present case

\[ \Omega'(\omega, 0) = \phi_0(\omega) - \sigma \omega, \]
\[ \phi_0(\omega) = \log(2 \cosh(\omega))). \]  (37)

The parity transformation corresponds here to \( \omega \to -\omega \). In agreement with our previous considerations, the term that breaks parity, \( -\sigma \omega \), is a polynomial. Also, a parity transformation is compensated by a change \( \sigma \to -\sigma \), in agreement with eq. (22). The effective action can then be rewritten as

\[ W_\sigma(D) = -\text{tr} \log(2 \cosh(\frac{\beta}{2}(m + \eta A_0))) + \eta \sigma \frac{\beta}{2} \text{tr}(A_0) + \sigma N_f \frac{\beta}{2} m. \]  (38)

The first term coincides with the result in [28] after subtraction of a \( A_0 \)-independent term (namely, the same expression with \( A_0 = 0 \)). It preserves parity but may break gauge invariance since it changes by integer multiples of \( i\pi \). This is compensated by the second term which is odd under pseudo-parity and breaks both gauge and parity symmetries. It is a local polynomial that introduces a parity anomaly. It cannot be removed without breaking gauge invariance. The last term is also a local polynomial that breaks parity but it can be removed preserving gauge invariance. This term illustrates that the \( \zeta \)-function prescription may yield unessential pseudo-parity even contributions to the anomaly; the same holds for chiral symmetry in even dimensions. The (essential) parity anomaly is thus

\[ A_P = \eta \sigma \int dx_0 \text{tr}(A_0) = \eta \sigma W_{CS}, \]  (39)

where \( W_{CS} \) is the 0+1-dimensional version of the Chern-Simons action. This checks that the essential parity anomaly is a local polynomial and temperature independent. When the gauge field configurations are traceless, \( W_{CS} \) vanishes identically. This is consistent with the fact that in this case the gauge group must be a subgroup of SU(\( N_f \)) which is simply connected and does not have large gauge transformations in 0+1 dimensions. (The gauge group itself may have large gauge transformations but the effective action must be strictly gauge invariant since one could always deform the configuration within SU(\( N_f \)).)
The effective action can be written as the sum of components with well defined pseudo-parity, namely, \( W_{\sigma}^{\pm}(-A_0) = \pm W_{\sigma}^{\pm}(A_0) \). After removing the term \( \sigma N_f \beta m/2 \), the pseudo-parity even component is gauge and parity invariant. The odd component can be identified as the anomalous part of the effective action at finite temperature in 0+1 dimensions since it contains all the anomalies

\[
W_{\text{anom}}(D) = W_{\sigma}^{-}(D) = \frac{\eta}{4} \int dx_0 \text{tr}(\Phi_{\sigma}),
\]

where

\[
\Phi_{\sigma} = \frac{2}{\beta} \Omega_{\sigma} (\frac{\beta}{2} (m - A_0), 0) - \text{p.p.c.}
\]

\[
= \frac{2}{\beta} \left[ \log \left( e^{-\sigma \beta(m-A_0)} + 1 \right) - \log \left( e^{-\sigma \beta(m+A_0)} + 1 \right) \right].
\]

In this expression, p.p.c. stands for pseudo-parity conjugate, \( A_0 \rightarrow -A_0 \). Besides, the branch cut of the logarithm is to be taken along the negative real axis to obtain the manifolds \( \mathcal{M}_{\sigma} \) for each \( \sigma \). Alternatively \( W_{\text{anom}}(D) \) can also be written as

\[
W_{\text{anom}}(D) = \eta \left( \frac{\sigma}{2} W_{\text{CS}} + \Gamma_{\text{odd}} \right)
\]

with

\[
\Gamma_{\text{odd}} = -\text{tr} \left[ \tanh^{-1} \left( \tanh \left( \frac{\beta m}{2} \right) \tanh \left( \frac{\beta A_0}{2} \right) \right) \right].
\]

This same function appears also in exact 2+1-dimensional results \[30\] (cf. subsection \[IV C\]).

The anomalous action \( W_{\text{anom}}(D) \) resulting from the calculation is to be compared with the naive temperature independent action

\[
\frac{1}{2} \eta \left( \sigma - \varepsilon(m) \right) W_{\text{CS}},
\]

which also saturates the parity and gauge variations of the full effective action but displays a singularity at \( m = 0 \) which is spurious at finite temperature. In fact this is the zero temperature limit of \( W_{\text{anom}} \). On the other hand, the perturbative result follows from using

\[
\Phi_{\sigma} = \frac{4\sigma}{e^{\sigma \beta m} + 1} A_0 + O(A_0^3).
\]

Substituting in \( W_{\text{anom}} \) yields

\[
W_{\text{anom}}(D) = \frac{1}{2} \eta \left( \sigma - \tanh \left( \frac{\beta m}{2} \right) \right) W_{\text{CS}} + O(A_0^3).
\]

This is the usual perturbative result which violates gauge invariance under homotopically non trivial gauge transformations.

It is noteworthy that in 0+1 dimensions the exact result can also be written in closed form without gauge fixing

\[
W_{\sigma}(D) = -\text{tr} \log(e^{-\sigma \beta m} \Omega^{\sigma} + 1), \quad \Omega = T e^{-\int_0^\beta A_0(x_0) dx_0}.
\]
IV. SUMMARY OF THE MAIN RESULTS AND DISCUSSION

In this section we will summarize the results for the 2+1-dimensional case. The details of the calculation are given in next sections. The calculation proceeds by: (i) Expanding the expression in eq. (27) to select terms with at most two spatial covariant derivatives. (ii) Taking the Dirac trace. To make this step simple, we assume the scalar field $M(x)$ to be a constant c-number mass $m$. (iii) Moving the operators $D_i$ to the right. This generates gauge covariant objects of the form $[D_i, X]$. (iv) Carrying out the $k$ integration. This kills all remaining gauge non invariant objects $\text{tr}(0| XD_i|0)$. This step requires to diagonalize $D_0$ in the Hilbert space of flavor and time (with $x$ fixed) and can be done in an $A_0$-stationary gauge. (v) Summing over fermionic frequencies $\omega_n$. And (vi) integrating over $z$ in closed form whenever possible.

$W_0(D)$ denotes the zeroth order term in the expansion in powers of $D_i$, $W_2^+(D)$ denote the pseudo-parity even and odd second order terms, respectively. The dependence on $\sigma$ and $\beta$ will not be displayed explicitly.

$$W_0(D) = \frac{1}{4\pi} \int d^3x \text{tr} \left[ \left( \frac{2}{\beta} \right)^2 m\phi_1(\frac{\beta}{2}(m - A_0)) - \left( \frac{2}{\beta} \right)^3 m\phi_2(\frac{\beta}{2}(m - A_0)) - \frac{1}{3} \sigma m^3 \right] + \text{p.p.c.}$$  \hspace{1cm} (48)$$

The functions $\phi_n(\omega)$ are of the form $\partial_{\omega}^{-(n+1)} \tanh(\omega)$, that is, they are primitives of $\tanh(\omega)$, with appropriate integration constants so that they are even or odd functions of $\omega$. Their proper definition is given in Appendix B. The function $\phi_0(\omega)$ has already appeared in the 0+1-dimensional case. The notation p.p.c. stands for pseudo-parity conjugate, $A_0 \rightarrow -A_0$, and after taking $\int d^3x \text{tr}$ it coincides with the complex conjugate.

$$W_2^+(D) = \frac{1}{4\pi} \int dx^3 \text{tr} \left[ \left( \frac{2}{\beta} \right)^2 \phi_1(\frac{\beta}{2}(m - A_0)) A_i \frac{1}{D_0} A_i - \frac{2}{\beta} \phi_0(\frac{\beta}{2}(m - A_0)) A_i \left( \frac{1}{4} + \frac{1}{2mD_0} \right) A_i \right]$$

$$+ \int_0^{-\sigma\infty} dt \tanh(\frac{\beta}{2}(m - t - A_0)) A_i \left( \frac{1}{4D_0 + \frac{m^2}{D_0}} \frac{1}{2(m - t) + D_0} A_i \right) + \text{p.p.c.}$$  \hspace{1cm} (49)$$

The symbol $D_0$ denotes the covariant derivative $D_0(X) = [D_0, X]$ and p.p.c. now includes $D_0 \rightarrow -D_0$. In addition, we have introduced the new field $A_i(x)$ which is defined as any solution (in the space of matrix valued functions, i.e., multiplicative operators in $x$ and $x_0$ spaces) of the equation

$$D_0 E_i = D_0^2 A_i.$$  \hspace{1cm} (50)$$

Where $E_i(x) = F_{0i} = [D_0, D_i]$ is the electric field. Essentially, $A_i = D_0^{-1} E_i$. However, since the operator $D_0$ is singular, $D_0^{-1} E_i$ either does not exist or is not unique. Because $A_i$ is not unique, we will have to check that the final result is independent of the particular solution taken. This check is done below. Clearly, the space of solutions of this equation is gauge and parity invariant.

$$W_2^-(D) = \frac{ie}{8\pi} \int d^3xe_{ij} \text{tr} \left[ \left( \frac{1}{2} F_{ij} - A_i A_j \right) \Phi_0 - \sigma A_i D_0 A_j \right]$$

$$+ \int_0^{-\sigma\infty} dt A_i \left( \frac{\tanh(\frac{\beta}{2}(m - t + A_0))}{2(m - t) + D_0} - \frac{\tanh(\frac{\beta}{2}(m - t - A_0))}{2(m - t) - D_0} \right) A_j.$$  \hspace{1cm} (51)$$
The quantity \( \Phi_\sigma \) was introduced in eq. (41) and \( F_{ij} = [D_i, D_j] \).

As already noted at the end of subsection II G, a further expansion can be done in powers of \( D_0 \). This expansion can be regarded as one in powers of \( D_\mu = [D_\mu, \cdot] \), since \( D_i \) appears only in commutators. Such an expansion preserves two essential properties of the original expansion in powers of \( D_i \): first, it does not break gauge or parity symmetries and second, higher order terms are increasingly ultraviolet convergent.

Clearly, \( W_0 \) is already independent of \( D_0 \). The expansion of \( W^\pm_2 \) is less obvious and is done in their respective sections. For the pseudo-parity even component we find (cf. subsection VI F)

\[
W^+_2, \text{leading}(D) = -\frac{1}{24\pi} \int d^3x \, \text{tr} \left[ \frac{\tanh(\frac{\beta}{2}(m - A_0))}{2m} - \frac{\beta}{2} \frac{1}{4 \cosh^2(\frac{\beta}{2}(m - A_0))} \right] E_i^2 + \text{p.p.c. (52)}
\]

On the other hand, for the pseudo-parity odd sector, the leading order in \( D_0 \) is (cf. section VII):

\[
W_{\text{anom}}(D) = \frac{i\eta}{8\pi} \int d^3x \epsilon_{ij} \text{tr} \left[ \left( \frac{1}{2} F_{ij} - A_i A_j \right) \Phi_\sigma \right. \\
\left. + \frac{1}{2} \left( \tanh(\frac{\beta}{2}(m - A_0)) + \tanh(\frac{\beta}{2}(m + A_0)) - 2\sigma \right) A_i D_0 A_j \right]. \tag{53}
\]

This component of the action differs from \( W^-_2 \) by terms which are ultraviolet convergent, gauge and parity invariant and free of multivaluation, thus it is natural to identify it with the purely anomalous component of the effective action: it is the ultraviolet logarithmically divergent part of the effective action that contains all the essential anomaly.

In the remainder of this section we will discuss the properties of \( W_{\text{anom}}(D) \) since this term has attracted more attention in the literature. The discussion of \( W_0 \) and \( W^\pm_2 \) will be made in their respective sections. Nevertheless, it can be pointed out that the pseudo-parity even components are strictly gauge invariant, not just modulo \( 2\pi i \). This was to be expected since the only allowed ambiguity in the \( \zeta \)-function regularized action is \( 2\pi i n \) which is purely imaginary and \( W^+ \) is real. Also they are parity invariant, except for the term \(-\frac{1}{3}\sigma m^3 \) in \( W_0 \) which is is removable by counterterms. On the other hand, \( W^-_2 \) has the same parity and gauge transformations as \( W_{\text{anom}}(D) \). No scale anomaly is present in the effective action.

First of all, let us check that \( W_{\text{anom}}(D) \) does not depend on the particular solution taken for \( A_i(x) \). In order to show this, note first that if \( A_0(x) \) is not only stationary but also diagonal \([A_0(x), \partial_i A_0(x)] = 0\), and a solution is simply \( A_i = A_i \). In an arbitrary \( A_0 \)-stationary gauge, an explicit solution is \( A_i = A_i - U \partial_i U^{-1} \), where \( U(x) \) is any of the stationary gauge transformations bringing \( A_0(x) \) to diagonal form. Adding an arbitrary solution of \( D_0(X) = 0 \), the most general form of \( A_i \) is

\[
A_i(x) = A_i(x) - U(x) \partial_i U^{-1}(x) + B_i(x), \quad [A_0(x), B_i(x)] = 0. \tag{54}
\]
$B_i(x)$ is time-independent and commutes with $A_0(x)$ at each point and is otherwise arbitrary. An explicit calculation shows that, as expected, the field $A_i$ transforms covariantly under stationary and discrete gauge transformations, modulo redefinitions of $B_i$. It is invariant under parity transformations. Using antisymmetry of $\epsilon_{ij}$ and the cyclic property of the trace, it easy to see that $W_{\text{anom}}(D)$ is uniquely defined, that is, the dependence on $B_i(x)$ cancels in eq. (53). We remark that $A_i$ depends both on $x$ and $x_0$, in general.

A. Parity variation

Let us study the properties of $W_{\text{anom}}(D)$ under parity transformations. As expected, replacing $D$ by $D^P$ and $\sigma$ by $-\sigma$ leaves the formula unchanged thus eq. (22) is verified. Therefore, the anomaly can be obtained from the relation $A_P(D) = W_{\sigma}(D) - W_{-\sigma}(D)$. Using the property

$$\Phi_{\sigma} - \Phi_{-\sigma} = 4\sigma A_0,$$

and substituting in $W_{\text{anom}}(D)$, the parity anomaly is

$$A_P(D) = \eta \sigma \frac{i}{4\pi} \int d^3x \epsilon_{ij} \text{tr} \left[ (F_{ij} - 2A_iA_j)A_0 - A_iD_0A_j \right].\tag{56}$$

This expression can be simplified by using the following replacement (valid inside $\int d^3x \text{tr}$)

$$\epsilon_{ij}A_iD_0A_j = \epsilon_{ij}(A_i\partial_0A_j - 2A_iA_jA_0).\tag{57}$$

This follows from noting that $A_i\partial_0A_j$ can be replaced by $A_i\partial_0A_j$ since $A_i$ and $A_i$ differ only by a stationary field (cf. eq. (54)) and integration by parts. Then, the terms containing $A_i$ cancel and the final expression for the parity anomaly becomes

$$A_P(D) = \eta \sigma \frac{i}{4\pi} \int d^3x \epsilon_{ij} \text{tr}(A_0F_{ij} - A_i\partial_0A_j) = \eta \sigma W_{\text{CS}}(D).\tag{58}$$

$W_{\text{CS}}$ being the Chern-Simons action. This checks that the parity anomaly has a temperature and mass independent form and is a local polynomial, as expected from general arguments given in subsection II D.

B. Gauge variation

Any correct calculation must be consistent with gauge invariance, however, since we have partially fixed the gauge, we can only check invariance under the restricted set of transformations considered in subsection II C, i.e., time-independent and discrete gauge

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6This is the generic case. If two different eigenvalues of $A_0(x)$ differ by $2\pi in/\beta$ in a sufficiently large region of the space-time, $D_0(X) = 0$ will admit time-dependent and not $A_0$-commuting solutions as well.
transformations. In other words, it must be verified that bringing the gauge configuration to two different $A_0$-stationary gauges gives the same result modulo $2\pi i$. As has been noted in subsection II C, for some gauge configurations all allowed gauge transformations (i.e., consistent with the condition $\partial_0 A_0 = 0$) are topologically small and on the other hand, going to a $A_0$-stationary gauge may require large transformations. Thus our calculation does not constitute a proof that the effective action is invariant under all gauge transformations. This fact has already been established as a consequence of using the $\zeta$-function renormalization prescription and the fact that gauge transformations do not change the spectrum of the Dirac operator.

In order to study the gauge variation of $W_{\text{anom}}(D)$, note that all the quantities there transform covariantly under time-independent gauge transformations. In particular, $A_i$ is defined by a covariant equation. Regarding discrete gauge transformations, $A_0$ is the only quantity which transforms inhomogenously, namely $A_0^U(x) = A_0(x) + \Lambda(x)$. The terms with tanh in $W_{\text{anom}}(D)$ depend on $\exp(\beta A_0)$ and this quantity is unchanged by discrete gauge transformations, cf. eq (13), hence the last term in $W_{\text{anom}}(D)$ is invariant. The only variation may come through $\Phi_{\sigma}$. This quantity also depends on $\exp(\beta A_0)$, thus if the argument of the logarithm is chosen always on the same Riemann sheet, i.e., on the cut manifold $\mathcal{M}_\sigma$, this quantity is trivially invariant. On the other hand, on the Riemann surface $\tilde{\mathcal{M}}$ one can consider a path connecting D and $D^U$, for instance the path $A_t^U = A_0 + t\Lambda$ with $0 \leq t \leq 1$ and keeping $m$ fixed. Let us consider first the generic non-degenerated case discussed in subsection II C for which eqs. (15) apply, then a simple analysis shows that

$$\Phi_{\sigma}^U - \Phi_{\sigma} = 4\sigma \Theta(-\sigma m)\Lambda, \quad \text{on} \quad \tilde{\mathcal{M}}. \quad (59)$$

This implies that the corresponding gauge variation of $W_{\text{anom}}(D)$ is of the form $(\sigma - \varepsilon(m))$ times something independent of $\sigma$. (Here $\Theta$ and $\varepsilon$ stand for the step and sign functions, respectively.) Because it has already been established in the previous subsection that the $\sigma$-dependent part of the action is $\frac{1}{2} \eta\sigma W_{\text{CS}}$, it follows that $W_{\text{anom}}(D)$ must have precisely the same gauge variation as $\frac{1}{2} \eta(\sigma - \varepsilon(m))W_{\text{CS}}$, that is

$$W_{\text{anom}}(D^U) - W_{\text{anom}}(D) = \eta \sigma \Theta(-\sigma m)(W_{\text{CS}}(D^U) - W_{\text{CS}}(D)), \quad \text{on} \quad \tilde{\mathcal{M}}. \quad (60)$$

Observe that when $\sigma m > 0$ the formula predicts strict gauge invariance. This is correct since in this case the branch cut $\Gamma$ and the spectrum do not intersect and so there is no flux of eigenvalues through the branch cut when going from D to $D^U$. As noted previously, the variation of the Chern-Simons action is $2\pi i$ times an integer which depends on the topological numbers of the gauge transformation and the gauge field configuration. Then, as expected, both $W_{\text{anom}}$ and $A_P$ are gauge invariant modulo $2\pi i$.

For degenerated configurations (defined in subsection II C), $A_0$ no longer needs to commute with $A_0$, however, by inspection of $\Phi_{\sigma}$ and unitarity of $\exp(\beta A_0)$, it follows that there will be no jump in the logarithm unless $\sigma m < 0$ and in this case the jump is odd in $\sigma$, therefore the same factor $\sigma \Theta(-\sigma m)$ is obtained and the same argument as before applies.

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It is interesting to note that the two tanh terms in $W_{\text{anom}}$ can be combined into $$\frac{\sinh(\beta m)}{\cosh(\beta m) + \cosh(\beta A_0)}$$ which is the same combination appearing in [18] replacing $A_0$ by the chemical potential.
C. Abelian-like reductions

As a check of our formula, let us assume that \( A \) is also time-independent and commutes everywhere with \( A_0 \). In this case, the field \( A_i \) has (or can be chosen to have) the same properties. As a consequence all terms with \( A_i \) vanish due to antisymmetry and our result for \( W_{\text{anom}}(D) \) reduces to

\[
W_{\text{anom}}(D) = \frac{i\eta}{16\pi} \int d^3 x \epsilon_{ij} \text{tr} \left( F_{ij} \Phi_\sigma \right).
\]

(61)

This expression holds also for \( W_2^- \). Separating the even and odd components under \( m \to -m \), this can be rewritten as

\[
W_{\text{anom}}(D) = \eta \left( \frac{\sigma}{2} W_{CS} + \Gamma_{\text{odd}} \right),
\]

(62)

where

\[
\Gamma_{\text{odd}} = -\frac{i}{4\pi} \int d^2 x \epsilon_{ij} \text{tr} \left[ F_{ij} \tanh^{-1} \left( \tanh \left( \frac{\beta m}{2} \right) \tanh \left( \frac{\beta A_0}{2} \right) \right) \right]
\]

(63)

is the expression introduced in Ref. [30]. As proven there (see also [29,31,33]), when \( A_0 \) is in addition \( x \)-independent, there are no higher order corrections and the right-hand side of eq. (61) gives the full pseudo-parity odd component of the effective action. (Recall that according to our definitions, \( \Gamma_{\text{odd}} \) is even under parity since \( m \) and \( A_0 \) are both odd. \( \Gamma_{\text{odd}} \) is an odd function of \( m \).)

D. Zero temperature limit

Using the zero temperature limit of \( \Phi_\sigma \),

\[
\Phi_\sigma = 4\sigma \Theta(-\sigma m) A_0, \quad (T = 0),
\]

(64)

and the replacement in eq. (57), it is straightforward to derive the zero temperature limit of our formula, namely

\[
W_{\text{anom}}(D) = \frac{1}{2} \eta (\sigma - \varepsilon(m)) W_{CS}, \quad (T = 0).
\]

(65)

This is the standard result at zero temperature (see e.g. [22,23]).

E. Massless fermions

We may also consider massless fermions. In this case the terms with \( \tanh \) cancel in \( W_{\text{anom}}(D) \). Also, \( \Phi_\sigma \) becomes an odd function of \( \sigma \) on \( \tilde{M} \). Since all terms in the action are odd under \( \sigma \to -\sigma \), it follows from subsection IV A that

\[
W_{\text{anom}}(D) = \frac{1}{2} \eta \sigma W_{CS}, \quad \text{for } m = 0 \text{ and on } \tilde{M}.
\]

(66)
This formula holds for $W^{-2}$ too. This refers to the formula after analytical extension, i.e., on $\tilde{M}$. Keeping track of the Riemann sheet in the logarithm in $\Phi_{\sigma}$ adds a term $i\pi n$, so that the trivial gauge invariance of the $\zeta$-function prescription is preserved on $M_{\sigma}$. The full result is an application of the Atiyah-Singer index theorem [14,29]. Because in the massless case parity and pseudo-parity transformations coincide, $W^{-}(D)$ equals $\frac{1}{2}A_{P}(D)$. Thus, eq. (66) is in fact exact to all orders [14].

**F. Perturbative result**

Next, let us show that our result for the pseudo-parity odd part reproduces the results obtained using perturbation theory at lowest orders. As noted in the Introduction, the latter approach yields a renormalization factor in front of the Chern-Simons action which is not quantized, and so breaks gauge invariance under large gauge transformations. To obtain the perturbative result we should retain terms of zeroth or first order in $A_{0}$ in $W_{\text{anom}}(D)$. Use of the replacement in eq. (57) and the expansion of $\Phi_{\sigma}$ in eq. (45), gives

$$W_{\text{anom}}(D) = i\eta \int d^{3}x \epsilon_{ij} \text{tr} \left[ \frac{4\sigma}{e^{\sigma\beta m} + 1} \left( \frac{1}{2} F_{ij} - A_{i}A_{j} \right) A_{0} ight. + \left( \tanh(\frac{\beta m}{2}) - \sigma \right) \left( A_{i} \partial_{0} A_{i} - 2 A_{i} A_{j} A_{0} \right) + O(A_{0}^{3}).$$

(67)

It trivial to check that the terms with $A_{i}$ cancel, and the final result can be expressed as

$$W_{\text{anom}}(D) = \frac{1}{2} \eta \left( \sigma - \tanh(\frac{\beta m}{2}) \right) W_{\text{CS}} + O(A_{0}^{3}),$$

(68)

which is the standard perturbative result [24]. It has the same form as in the 0+1-dimensional case, eq. (46).

**G. Relation to the chiral case**

There are strong similarities with the situation of chiral anomalies in even dimensions. There the (consistent) chiral anomaly is defined as the variation of the effective action under chiral transformations. The chiral anomaly contains an essential part which can only be derived from a non polynomial action. As it is well-known, at zero temperature this is the gauged WZW action which again is pseudo-parity odd [8]. As already noted, the chiral anomaly has also been shown to be temperature independent (see e.g. [37]). This observation has lead to propose that the same gauged WZW action is the full anomalous action at finite temperature too [38]. The findings in 0+1 and 2+1 dimensions suggest that this is not the case. Indeed, from the point of view of parity anomaly saturation and gauge invariance, the naive result given by eq. (55) is entirely sufficient, however the correct result, eq. (53), has an explicit temperature dependence. It is worth noting that the naive formula has a singularity along the line $m = 0$ which is spurious at finite temperature, since there are no infrared singularities, and is not present in the full result. If a similar situation takes place in the chiral case, the amplitude corresponding to anomalous processes (but not the anomaly
itself) will show a smooth temperature dependence. (By anomalous processes it is meant those processes driven by a logarithmically divergent pseudo-parity odd amplitude.) Such dependence has actually been found in Ref. [39]. A study of the chiral case along the lines followed here has been carried out in [34], with the result that the anomalous component of the effective action has indeed a non trivial temperature dependence.

V. THE EFFECTIVE ACTION AT ZEROTH ORDER

In this section we will compute the 2+1-dimensional effective action at lowest order in the derivative expansion. For simplicity we will use units $\beta = 2$. After setting $D_i$ to zero, eq. (27) yields

$$\Omega_{s,0}(D) = -\int \frac{d^2 k}{(2\pi)^2} \frac{1}{2} \sum_n \int_{\Gamma} \frac{dz}{2\pi i} z^s \text{tr} \langle 0 | \frac{1}{\gamma_0 Q + \gamma p + \mu} | 0 \rangle. \quad (69)$$

Here $Q = \omega_n + D_0$ and $\mu = M(x) - z$. As mentioned, we will take the scalar field $M(x) = m$. Then, $Q$, $p$ and $\mu$ commute with each other and $\gamma_0 Q + \gamma p + \mu$ can be brought to the numerator

$$\frac{1}{\gamma_0 Q + \gamma p + \mu} = \frac{\mu - \gamma_0 Q - \gamma p}{\Delta}, \quad \mu = m - z, \quad \Delta = \mu^2 - Q^2 + k^2. \quad (70)$$

The Dirac trace can be computed immediately yielding

$$\Omega_{s,0}(D) = -\int \frac{d^2 k}{(2\pi)^2} \sum_n \int_{\Gamma} \frac{dz}{2\pi i} z^s \text{tr} \langle 0 | \mu \frac{1}{\Delta} | 0 \rangle. \quad (71)$$

Here the trace refers to flavor only. In order to perform the momentum integrals it is convenient to use integration by parts:

$$-\int_{\Gamma} \frac{dz}{2\pi i} z^s f(z) = \frac{1}{s + 1} \int_{\Gamma} \frac{dz}{2\pi i} z^{s+1} f'(z), \quad (72)$$

which holds provided that $z^s f(z)$ vanishes at infinite on $\Gamma$ for sufficiently negative $\text{Re}(s)$. Then

$$\Omega_{s,0}(D) = -\int \frac{d^2 k}{(2\pi)^2} \sum_n \int_{\Gamma} \frac{dz}{2\pi i} z^s \text{tr} \langle 0 | \left( -\frac{z^2}{(s + 1)(s + 2)} \frac{2\mu}{\Delta^2} - \frac{z}{(s + 1)} \frac{2\mu^2}{\Delta^2} \right) | 0 \rangle$$

$$= -\int \frac{d^2 k}{(2\pi)^2} \sum_n \int_{\Gamma} \frac{dz}{2\pi i} z^{s+1} \text{tr} \langle 0 | \left( \frac{z}{s + 2} - \frac{m}{s + 1} \right) \frac{2\mu}{\Delta^2} | 0 \rangle. \quad (73)$$

The momentum integral can be done using

$$\int \frac{d^2 k}{(2\pi)^2} \frac{k^{2n}}{(k^2 + M^2)^N} = \frac{\Gamma(n + 1)\Gamma(N - n - 1)}{4\pi\Gamma(N)} (M^2)^{n-N+1} \quad (74)$$

\footnote{The subindex 0 in $\Omega_{s,0}(D)$ stands for zeroth order. We will no longer make explicit the dependence on $\Gamma$.}
and it yields
\[
\Omega_{s,0}(D) = -\frac{1}{4\pi} \sum_n \int \frac{dz}{2\pi i} z^{s+1} \text{tr}(0) \left( \frac{z}{s+2} - \frac{m}{s+1} \right) \frac{2\mu}{\mu^2 - Q^2} [0].
\] (75)

The sum over frequencies can be done straightforwardly by using eq. (32)
\[
\Omega_{s,0}(D) = -\frac{1}{4\pi} \int d^3 x \int \frac{dz}{2\pi i} z^{s+1} \text{tr} \left( \frac{z}{s+2} - \frac{m}{s+1} \right) \tanh(\mu - A_0) + \text{p.p.c.}
\] (76)

Again, p.p.c. stands for pseudo-parity conjugate. In addition, we have used that 
\( A_0 \) is time independent and so \( D_0 \) can be replaced by \( A_0 \).

Finally, the integration over \( z \) can be done using the functions \( \Omega_\Gamma(\omega,s) \) introduced in 
eq. (34) and Appendix B. Thus
\[
\Omega_{s,0}(D) = \frac{1}{4\pi} \int d^3 x \text{tr} \left( \frac{1}{s+2} \Omega_\Gamma(m - A_0, s + 2) - \frac{m}{s+1} \Omega_\Gamma(m - A_0, s + 1) \right) + \text{p.p.c.}
\] (77)

To obtain the effective action it remains to apply \(-\frac{d}{ds}|_{s=0}\). As shown in Appendix B
\[
\Omega_\sigma(\omega, n) = 0, \quad n = 0, 1, 2, \ldots,
\]
\[
\Omega'_\sigma(\omega, 1) = \phi_1(\omega) - \sigma \left( \frac{1}{2} \omega^2 - \frac{1}{6} \left( \frac{i\pi}{2} \right)^2 \right),
\]
\[
\frac{1}{2!} \Omega'_{\sigma}(\omega, 2) = \phi_2(\omega) - \sigma \left( \frac{1}{6} \omega^3 - \frac{1}{6} \left( \frac{i\pi}{2} \right)^2 \omega \right).
\] (78)

Here the prime refers to derivative with respect to \( s \) and \( \sigma = \pm 1 \) refers to the two possible 
equivalent choices of \( \Gamma \). The functions \( \phi_n(\omega) \) are defined for every integer \( n \) and satisfy
\[
\phi_{-1}(\omega) = \tanh(\omega),
\]
\[
\phi'_n(\omega) = \phi_{n-1}(\omega),
\]
\[
\phi_n(-\omega) = (-1)^n \phi_n(\omega).
\] (79) (80) (81)

(See Appendix B for the fixing of the integration constants for non negative \( n \).) The function
\( \phi_0(\omega) \) has already appeared in the 0+1-dimensional case and similar comments can be made here: for non negative integer \( n \), the terms \( \phi_n(\omega) \) in \( \Omega'_\Gamma(\omega, n) \) are those preserving parity 
and the breaking comes from the polynomial term which is odd under \( \sigma \rightarrow -\sigma \).

After taking the \(-\frac{d}{ds}|_{s=0}\) in eq. (77) and using eqs. (78), the effective action at zeroth 
order takes the form
\[
W_0(D) = \frac{1}{4\pi} \int d^3 x \text{tr} \left( m\phi_1(m - A_0) - \phi_2(m - A_0) - \frac{1}{3} \sigma m^3 \right) + \text{p.p.c.}
\] (82)

Recalling that \( m \) and \( A_0 \) change sign under parity, it follows that this action preserves parity 
except for the last term \( \sigma m^3 \), which is similar to the term \( \sigma m \) in 0+1 dimensions. Since this 
term is polynomial and strictly gauge invariant it can be removed from the action.

Next we should check the gauge invariance of \( W_0(D) \). Under time-independent gauge 
transformations \( A_0 \) changes by a similarity transformation and this leaves \( W_0(D) \) invariant 
due to the cyclic property of the trace. Under discrete gauge transformations the spectrum
of $A_0$ changes by an integer multiple of $i\pi$. The strict gauge invariance of $\Omega_{s,0}(D)$, and hence of $W_0(D)$, on the manifold $\mathcal{M}_\sigma$ (i.e., the complex plane cut along $\Gamma$) follows immediately from eqs. (34) and (77) since tanh is periodic and so the functions $\Omega_\Gamma(\omega, s)$ are also periodic. The strict gauge invariance of $W_0(D)$ also on the Riemann surface $\tilde{M}$ can be checked directly using the formula (cf. Appendix B)

$$\phi_n(\omega + \frac{i\pi}{2}) - \phi_n(\omega - \frac{i\pi}{2}) = i\pi \varepsilon(m) \frac{\omega^n}{n!}, \quad n = 0, 1, 2, \ldots$$

(83)

where we have used the notation $m = \text{Re}(\omega)$ (since in practice $\omega = m \mp A_0$ and $A_0$ is anti-Hermitian), $\varepsilon(x)$ is the sign of $x$, and the difference refers to the straight path from $\omega - \frac{i\pi}{2}$ to $\omega + \frac{i\pi}{2}$ on the complex plane. This identity can be used to show that the combination appearing in $W_0(D)$, $m\phi_1(\omega) - \phi_2(\omega) + \text{c.c.}$ (where c.c. stands for complex conjugate) is a periodic function.

VI. THE PSEUDO-PARITY EVEN EFFECTIVE ACTION AT SECOND ORDER

A. Dirac degrees of freedom

Starting from eq. (27), the second order takes the form (we use units $\beta = 2$)

$$\Omega_{s,2}(D) = - \int \frac{d^2k}{(2\pi)^2} \frac{1}{2} \sum_n \int \frac{dz}{2\pi i} z^n \text{tr}\langle 0 | \left( \frac{1}{\gamma_0 Q + \gamma P + \mu} \gamma D \right)^2 \frac{1}{\gamma_0 Q + \gamma P + \mu} | 0 \rangle. \quad (84)$$

Where $Q$ and $\mu$ were introduced in the previous section. Using again eq. (70), the Dirac trace can be computed immediately. Since we are interested in the pseudo-parity even sector, $W_2^+(D)$, we keep only terms without the Levi-Civita pseudo-tensor $\epsilon_{ij}$. The resulting expression is simplified using that $k_i k_j$ is equivalent to $\frac{1}{2} k^2 \delta_{ij}$ within the two-dimensional momentum integral. This yields

$$\Omega_{s,2}^+(D) = - \int \frac{d^2k}{(2\pi)^2} \sum_n \int \frac{dz}{2\pi i} z^n \mu \text{tr}\langle 0 | \left( (\mu^2 - k^2) \frac{1}{\Delta} D_i \frac{1}{\Delta} D_i \frac{1}{\Delta} ight. \\
+ \frac{Q}{\Delta} D_i \frac{1}{\Delta} D_i \frac{Q}{\Delta} - \frac{1}{\Delta} D_i \frac{Q}{\Delta} Q_i \frac{Q}{\Delta} - \frac{Q}{\Delta} D_i \frac{Q}{\Delta} D_i \frac{1}{\Delta} \right) | 0 \rangle. \quad (85)$$

Here the trace refers to flavor only.

B. Space-time and flavor degrees of freedom

Next, the spatial covariant derivatives are brought to the right producing the quantity $E_i = [Q, D_i] = F_0$. We have integrated by parts terms of the form $[D_i, E_i]$ in order to produce a simpler expression. In this form the following expression is derived:
\[ \Omega^+(D) = - \int \frac{d^2 k}{(2\pi)^2} \sum_n \int \frac{dz}{2\pi i z^\mu} \mu \text{ tr}(0) \left[ (\mu^2 - k^2 - Q^2) \frac{1}{\Delta^2} D_i^2 \right. \\
+ \left. \left( \frac{1}{\Delta^2} E_i Q - \frac{Q}{\Delta^2} E_i \frac{1}{\Delta^2} + \frac{Q^2}{\Delta^2} E_i \frac{1}{\Delta^2} + \frac{Q^2}{\Delta^2} E_i \frac{1}{\Delta^2} + 2 \frac{Q^2}{\Delta^2} \right) i D_i \right. \\
- \left. (\mu^2 - k^2) \left( \frac{1}{\Delta^2} E_i \frac{Q}{\Delta^2} + \frac{Q}{\Delta^2} E_i \frac{1}{\Delta^2} + \frac{Q^2}{\Delta^2} E_i \frac{1}{\Delta^2} + 2 \frac{Q^2}{\Delta^2} \frac{Q}{\Delta^2} \right) \right] |0\rangle. \quad (86) \]

Several remarks are in order here. First, the terms with explicit \( D_i \) break gauge invariance and will be shown to cancel after momentum integration. That these terms break gauge invariance can be seen from \( D_i |0\rangle = A_i |0\rangle \). The breaking is due to the state \( |0\rangle \). The terms of the form \( E_i E_i \) are explicitly invariant under time-independent gauge transformations, since both \( E_i \) and \( Q = D_0 + \omega_n \) transform covariantly under such transformations. For these terms, the operators appearing in \( \langle 0|0 \rangle \) are purely multiplicative in \( x \)-space since all \( \partial_i \) operators appear inside commutators. Finally, to obtain a more compact expression, the cyclic property has been used in the gauge invariant terms. This is an important point of the present formalism and deserves further clarification. The gauge invariant terms contain the construction

\[ \langle X \rangle = \text{ tr}(0|X|0), \quad (87) \]

where \( X \) is purely multiplicative in \( x \)-space. In this case, we can consider \( \langle X \rangle \) for each \( x \) separately and use \( |0\rangle \) to refer to the zero-frequency state in the Hilbert space of functions of \( x_0 \). The point is that \( \langle \rangle \) is not a trace in \( x_0 \)-space and thus the cyclic property holds only in a restricted form. Nevertheless two cyclic properties can be used:

**Rule 1.** If \( X(D_0) \) is a function of \( D_0 \) only and \( Y \) is a multiplicative operator in \( x \)-space,

\[ \langle XY \rangle = \langle YX \rangle. \quad (88) \]

**Rule 2.** If \( X_1(D_0) \) and \( X_2(D_0) \) are functions of \( D_0 \) only, and \( Y_1 \) and \( Y_2 \) are multiplicative operators both in \( x \) and \( x_0 \) the following substitution applies

\[ \sum_n \langle X_1(Q)Y_1(Q)X_2(Q)Y_2 \rangle = \sum_n \langle X_2(Q)Y_2X_1(Q)Y_1 \rangle, \quad (89) \]

provided that the sum over \( n \) is sufficiently convergent. (The sum refers to the \( n \) dependence in \( Q = D_0 + \omega_n \).)

To see this, we exploit that \( A_0 \) is stationary to introduce the basis of the flavor-time space (for given \( x \)) formed with eigenstates of \( Q, |\alpha, \ell\rangle \), namely

\[ A_0 |\alpha \rangle = a_\alpha |\alpha \rangle, \quad \langle x_0 | \ell \rangle = e^{\kappa_\ell x_0} \quad (\kappa_\ell = i \pi \ell), \]

\[ |\alpha, \ell \rangle = |\alpha \rangle \otimes |\ell \rangle, \]

\[ Q |\alpha, \ell \rangle = (\omega_n + a_\alpha + \kappa_\ell) |\alpha, \ell \rangle. \quad (90) \]
Then, the first rule follows from
\[ \langle X(D_0)Y \rangle = \sum_{\alpha} X(a_{\alpha}) \langle \alpha, 0 | Y | \alpha, 0 \rangle = \langle Y X(D_0) \rangle . \] (91)

And the second rule comes from
\[
\sum_n \langle X_1(Q)Y_1X_2(Q)Y_2 \rangle = \sum_n \sum_{\alpha,\beta} \sum_{\ell} X_1(\omega_n + a_{\alpha}) \langle \alpha, 0 | Y_1(\beta, \ell) X_2(\omega_n + a_{\beta} + \kappa_\ell) \langle \beta, \ell | Y_2 \rangle \rangle \\
= \sum_n \sum_{\alpha,\beta} \sum_{\ell} X_2(\omega_n + a_{\alpha} - \kappa_\ell) \langle \alpha, -\ell | Y_2 | \beta, 0 \rangle X_1(\omega_n + a_{\beta}) \langle \beta, 0 | Y_1 \rangle \rangle \\
= \sum_n \sum_{\alpha,\beta} \sum_{\ell} X_2(\omega_n + a_{\alpha}) \langle \alpha, 0 | Y_2 | \beta, \ell \rangle X_1(\omega_n + a_{\beta} + \kappa_\ell) \langle \beta, \ell | Y_1 \rangle \rangle \\
= \sum_n \langle X_2(Q)Y_2X_1(Q)Y_1 \rangle . \] (92)

The first equality uses that \( Y_{1,2} \) are multiplicative in \( \mathbf{x} \)-space. The second equality comes from exchanging \( \alpha \) with \( \beta \) and \( \ell \) by \(-\ell\). The third equality uses that \( Y_{1,2} \) are multiplicative operators in \( x_0 \)-space and also \( n \) has been shifted to \( n + \ell \). This last step means that the formula only needs to hold if \( \sum_n \langle X_1(Q)Y_1X_2(Q)Y_2 \rangle \) remains unchanged under the replacement \( Q \to Q + \omega_r \) (the same shift for all \( Q \) in the expression) which will be true if the sum over \( n \) is convergent.

We remark that the present formalism actually only requires \( A_0(\mathbf{x}) \) to be everywhere diagonalizable but not necessarily anti-Hermitian or even normal. This is relevant if one wants to study the finite density case which requires introducing a chemical potential. Effectively this amounts to add a constant Hermitian term to \( A_0 \) (see e.g. [37,18]).

C. Momentum degrees of freedom

The momentum integration can be carried out immediately for the term with \( D_1^2 \) using eq. (74) and this term vanish identically. For the other non-gauge invariant terms of the form \( E_iD_i \), we can replace \( D_i \) by \( A_i \) since \( \partial_i | 0 \rangle = 0 \). Thus all remaining operators in \( \langle 0 | 0 \rangle \) are multiplicative in \( \mathbf{x} \)-space.

In order to integrate over \( \mathbf{k} \), we can again make use of the flavor-time basis \( | \alpha, \ell \rangle \) so that only \( \alpha \)-numbers are involved. However, the integrand is a sum of terms of the form \( \langle X E_i X' E_i \rangle \) or \( \langle X E_i X' A_i \rangle \), where \( X, X' \) are functions of \( Q \) and this allows to use an equivalent and preferable method, namely, to use the label 1 to denote operators in the position \( X \) and the label 2 for operators in the position \( X' \). More generally, operators \( X'' \) in the position \( \langle X E_i X' E_i X'' \rangle \) would also carry a label 1 due to the first cyclic property, eq. (88).

That is
\[ \langle X E_i X' E_j X'' \rangle = \langle X_1 X'_2 X''_1 E_i E_j \rangle . \] (93)

The second cyclic property, eq. (89), then implies
\[ \langle Z(Q_1, Q_2) E_i E_j \rangle = \langle Z(Q_2, Q_1) E_j E_i \rangle \] (94)
provided that \( Z(x_1, x_2) = Z(x_1 + \omega_r, x_2 + \omega_r) \). With this ordering convention \( Q_1 \) and \( Q_2 \) are commuting objects and the momentum integrals in eq. (86) can be performed straightforwardly. All required integrals can be derived from eq. (74) and

\[
\int \frac{d^2k}{(2\pi)^2 (k^2 + M_1^2)(k^2 + M_2^2)} = \frac{1}{4\pi} \log\left(\frac{M_1^2}{M_2^2}\right)
\]

(95)

A non trivial check of the calculation is that all non-gauge invariant terms of the form \( \langle X_E_i X'A_i \rangle \) cancel at this step. The result can be written in the following form

\[
\Omega_{s,2}^+(D) = \frac{1}{4\pi} \sum_n \int_{\Gamma} \frac{dz}{2\pi i} z^n \mu \text{tr} \langle 0 | \frac{H(Q_1, Q_2)}{(Q_1 - Q_2)^2} E_i^2 | 0 \rangle,
\]

(96)

where

\[
H(x_1, x_2) = \frac{1}{x_1^2 - x_2^2} \log \left( \frac{\mu^2 - x_1^2}{\mu^2 - x_2^2} \right) + \frac{\mu^2 - x_1 x_2}{(\mu^2 - x_1^2)(\mu^2 - x_2^2)}.
\]

(97)

The symmetry of \( H(x_1, x_2) \) under exchange of its arguments is a direct consequence of the cyclic property, eq. (94).

D. The frequency degree of freedom

In order to sum over frequencies, we first apply integration by parts, eq. (72), to transform the logarithmic term in \( H(x_1, x_2) \) into a rational function. The resulting rational functions can then be reduced to a sum of simple poles and summed over \( n \) using the identity in eq. (32). The relevant formula is

\[
\mu \sum_n H(x_1 + \omega_n, x_2 + \omega_n) \simeq \left( \frac{\mu}{2} + \frac{z}{s + 1} \frac{\mu}{x_1 - x_2} + \frac{z^2}{(s + 1)(s + 2)} \frac{1}{x_1 - x_2} \right) \frac{\tanh(\mu - x_1)}{2\mu - x_1 + x_2} + X_{1,2} + \text{p.p.c.}
\]

(98)

The symbol \( \simeq \) is used since this substitution only holds within \( \int dz \, z^s \) due to the integration by parts. \( X_{1,2} \) means the same expression but exchanging the labels 1 and 2. Finally, p.p.c. stands for \( x_1, x_2 \to -x_1, -x_2 \). When this formula is used in eq. (93), \( x_1 = D_{01} \) and \( x_2 = D_{02} \) (where \( D_{01} \) and \( D_{02} \) stand for \( D_0 \) at positions 1 and 2 respectively) and so p.p.c. yields the pseudo-parity conjugate.

In order to integrate over \( z \) it is convenient to extract the terms which diverge as a polynomial for large \( z \):

\[
\mu \sum_n H(x_1 + \omega_n, x_2 + \omega_n) \simeq \frac{1}{2} \left[ \frac{z}{s + 2} \frac{1}{x_1 - x_2} + \frac{1 + s + 1}{2s + 2} \frac{1}{(s + 1)(s + 2)} \frac{1}{x_1 - x_2} \right. \\
+ \left. \frac{1}{2s + 2} (x_1 - x_2) + \frac{s}{(s + 1)(s + 2)} m + \frac{2}{(s + 1)(s + 2)} \frac{m^2}{x_1 - x_2} \right] \frac{1}{2\mu - x_1 + x_2}
\]

\times \tanh(\mu - x_1) + X_{1,2} + \text{p.p.c.}
\]

(99)
E. The proper time degree of freedom

Next we have to integrate over $z$ and apply $-d/ds|_{s=0}$. The terms in eq. (99) which are of the form $(az + b) \tanh(\omega - z)$ are immediately worked out using the function $\Omega_{\sigma}(\omega, s)$. For the terms of the form $\tanh(\omega - z)/(\omega' - z)$ we will need some identities. Let $f(z)$ be a meromorphic function such that $zf(z) \to 0$ as $z \to \sigma\infty$, and $g(s)$ analytic at $s = 0$, then

$$
\left( g(s) \int_{\Gamma} \frac{dz}{2\pi i} z^s f(z) \right)_{s=0} = 0 ,
$$

(100)

and

$$
-\frac{d}{ds} \left( g(s) \int_{\Gamma} \frac{dz}{2\pi i} z^s f(z) \right)_{s=0} = g(0) \int_{0}^{-\sigma\infty} dt f(t) .
$$

(101)

The first equality follows from taking $s = 0$ directly in the integrand, since everything is convergent. The second equality follows from previous one. In our case, the two terms in eq. (99) which are of the form $\tanh(\omega - z)/(\omega' - z)$ which are even on the explicit $m$ are convergent for large $z$ after adding their pseudo-parity conjugate, so eq. (101) applies directly to them. The remaining term is not directly convergent. It is of the form

$$
I_2(\omega, \omega') = -\frac{d}{ds} \left( sg(s) \int_{\Gamma} \frac{dz}{2\pi i} z^s \frac{\tanh(\omega - z)}{\omega' - 2z} \right)_{s=0} ,
$$

(102)

which can be rewritten as

$$
I_2 = -\frac{d}{ds} \left( sg(s) \int_{\Gamma} \frac{dz}{2\pi i} z^s \left( \frac{\tanh(\omega - z) - \sigma}{\omega' - 2z} + \frac{\sigma}{\omega' - 2z} \right) \right)_{s=0} .
$$

(103)

The subtracted term is convergent and thus it vanishes due to eq. (101). The remainder can be computed explicitly

$$
I_2 = g(0) \frac{\sigma}{2} .
$$

(104)

The effective action takes the form

$$
W_2^+(D) = \frac{1}{8\pi} \text{tr}(0) \left[ \frac{1}{2} \Omega'_{\sigma}(m - D_{01}, 1) + \left( \frac{1}{4} - \frac{1}{2} \frac{m}{D_{01} - D_{02}} \right) \Omega'_{\sigma}(m - D_{01}, 0) \right.
$$

$$
+ \frac{\sigma m}{4} + \int_{0}^{-\sigma\infty} dt \left( \frac{1}{4} (D_{01} - D_{02}) + \frac{m^2}{D_{01} - D_{02}} \right) \frac{\tanh(m - t - D_{01})}{2(m - t) - D_{01} + D_{02}} \]
$$

$$
\times \frac{E_i^2}{(D_{01} - D_{02})^2} |0 \rangle + X_{1,2} + \text{p.p.c.}
$$

(105)

This expression can be transformed by making some observations. First, due to the cyclic property, the terms $X_{1,2}$ just give a factor 2 to the action. Second, recalling that $D_0(X) = [D_0, X]$, the cyclic properties imply

$$
D_{01} - D_{02} = D_{01} = -D_{02} .
$$

(106)

For instance
\[\langle (D_{01} - D_{02})^2 XY \rangle = \langle D_{0}^2 XY - 2D_0 XD_0 Y + XD_0^2 Y \rangle \]
\[= \langle X[D_0, [D_0, Y]] \rangle \]
\[= \langle XD_0^2 Y \rangle \]
\[= \langle D_{02}^2 XY \rangle \]. \tag{107}\]

Therefore, \(D_{02}\) can be replaced everywhere by \(D_{01} + D_{02}\). Third, because \(A_0\) commutes with \(\partial_0\) and \(\partial_0|0\rangle = \langle 0|\partial_0 = 0\), the symbol \(D_{01}\) can be replaced by \(A_{01}\) everywhere. Fourth, we make use of the fields \(A_i(x)\) which were defined in eq. (50). The relation \(D_0 E_i = D_{0i}^2 A_i\) allows to make the replacements
\[E_i \frac{1}{D_0^2} E_j = -A_i A_j, \quad E_i \frac{1}{D_0} E_j = -A_i D_0 A_j. \tag{108}\]
(The minus signs come from by part integration.) Finally, we can make use of the functions \(\phi_n(\omega)\) instead of \(\Omega_i'(\omega, n)\) (see Appendix B). After these manipulations, the pseudo-parity even effective action at second order can be written as
\[W_2^+(D) = \frac{1}{4\pi} \int dx^3 \text{tr} \left[ \frac{1}{2} \phi_1 (m - A_0) A_i \frac{1}{D_0} A_i - \phi_0 (m - A_0) A_i \left( \frac{1}{4} + \frac{m}{2D_0} \right) \right] A_i \]
\[+ \int_0^{-\sigma_\infty} dt \text{tanh}(m - t - A_0) A_i \left( \frac{1}{4D_0 + \frac{m^2}{D_0}} \right) \frac{1}{2(m - t) + D_0 A_i} \] + p.p.c. \tag{109}\]
(Here we are no longer using an ordering prescription with labels 1 and 2; the position of the operators is that given literally by the formula.)

**F. Transformation properties of the result and limit cases**

First of all, let us show that \(W_2^+(D)\) does not contribute to the gauge or parity anomalies. Because the equation defining \(A_i\) (eq. (50)) is gauge and parity covariant, we can choose \(A_i\) to transform covariantly too. The gauge invariance under time-independent transformations is immediate since \(A_0, A_i\) and \(D_0\) transform covariantly. Under discrete transformations, \(A_i\) and \(D_0\) remain invariant, whereas \(A_0\) changes by integer multiples of \(i\pi\) thus the integral term in \(W_2^+(D)\) is invariant due to periodicity of the hyperbolic tangent. For the terms containing \(\phi_n\), the invariance under discrete gauge transformations follows from eqs. (83) through a cancellation among the terms with \(\phi_0, \phi_1\) and their p.p.c. Since the cancellation is non trivial it provides a check of the calculation.

The parity invariance of the terms containing \(\phi_n\) in \(W_2^+(D)\) follows immediately from the parity properties of these functions, eq. (81). On the other hand, in the term containing the integral over \(t\), a parity transformation can be seen to be equivalent to change \(\sigma \rightarrow -\sigma\), in agreement with eq. (22). Thus the parity violating contribution would come from extending the integral on \(t\) to the range \(-\infty < t < \infty\). Taking into account the p.p.c. term, such integral vanishes because it is convergent and the integrand is an odd function of \(m - t\).

Next, let us check that \(W_2^+(D)\) only depends on \(E_i(x)\) and not on the particular solution taken for the field \(A_i(x)\). Recalling the definition of \(A_i\) (cf. eq. (50)) this requires that \(A_i A_i\) should appear with at least two powers of the operator \(D_0\). The identity \(D_{02} = Q_2 - Q_1\)
(eq. (106)) then requires in eq. (96) that the function $H(x_1, x_2)$ should be of order $(x_1 - x_2)^2$ as $x_1 - x_2 \to 0$. In fact this is the case,

$$\lim_{x_1, x_2 \to x} \frac{H(x_1, x_2)}{(x_1 - x_2)^2} = \frac{3\mu^2 + x^2}{6(\mu^2 - x^2)^3}.$$  

In eq. (109) the dependence through $E_i$ is not manifest because we have used the symmetry between the labels 1 and 2 to simplify the expression. To check this property directly from the final expression for $W^{2+}_{(D)}$ it is necessary to restore $D_{01}, D_{02}$, add $X_{1,2}$ (dividing by two), eliminate $D_{02}$ in favor of $D_0$ and then expand in powers of $D_0$. (This procedure is carried out in detail for $W_{2-}$ in section VII.) After doing this, it is indeed found that the necessary cancellations take place and only terms of order $A_i D_0^2 A_i$ survive. These cancellations among different terms are again a non-trivial check of the result. Retaining the lowest order in the expansion on $D_0$ just described yields

$$W^{2+}_{\text{leading}}(D) = -\frac{1}{24\pi} \int d^3 x \text{tr} \left[ \frac{\tanh(m - A_0)}{2m} - \frac{1}{4 \cosh^2(m - A_0)} \right] E_i^2 + \text{p.p.c.}$$  

It is noteworthy that the terms in the expansion combine in such a way that the integral over $t$ present in the original expression can be done in closed form by integration by parts. It has been checked that the same final expression is obtained starting directly from eq. (110).

In fact, the interest of $W^{2+}_{\text{leading}}$ goes beyond checking explicitly the independence of $W^{2+}_2(D)$ on the spurious degrees of freedom contained in the variables $A_i$. $W^{2+}_{\text{leading}}$ has been obtained by making first an expansion in powers of $D_i$ to second order and then keeping the leading order in an expansion in powers of $D_0$. As noted at the end of subsection III.C, this double expansion in $D_i$ and $D_0$ is the natural finite temperature version of the standard space-time gradient expansion at zero temperature (which in the absence of other non-gauge fields, is also equivalent to a $1/m$ expansion). No symmetry has been broken by the further expansion in powers of $D_0$ and $W^{2+}_{\text{leading}}$ is manifestly gauge and parity invariant. On the other hand, expanding in powers of $A_0$, as implied by standard perturbation theory, would break invariance under discrete gauge transformations.

In the particular case of Abelian and stationary fields (i.e., time-independent $A$) $D_0$ vanishes so $W^{2+}_{\text{leading}}$ does in fact coincide with $W^{2+}_2$.

It is also of interest to examine the zero temperature limit of $W^{2+}_2(D)$. As shown in Appendix B,

$$\lim_{\beta \to \infty} \phi_{n-1}(\omega) = \frac{1}{n!} \omega^n \varepsilon(m), \quad n = 0, 1, 2, \ldots.$$  

(Recall that we are using units $\beta = 2$ so the zero temperature limit corresponds to large $\omega = m - A_0$). After some non trivial cancellations, one finds

$$W^{2+}_2(D) = -\frac{1}{4\pi} \int d^3x \text{tr} E_i \left[ \frac{1}{8|m|} + \frac{1}{2}(4m^2 + D_0^2) \int_{|m|}^{+\infty} dt \frac{1}{4t^2(4t^2 - D_0^2)} \right] E_i, \quad (T = 0).$$  

This result depends only on $E_i$ (and not $A_i$) and is free of parity or gauge anomalies. Full Lorentz invariance, which is recovered at zero temperature, is no supported since it has been
broken by making an expansion on the spatial gradient only. In order to compare with direct zero temperature calculations, we will only retain terms of dimension 4, or equivalently, the leading term in a $1/m$ expansion. This gives

$$W_2^+(D) = -\frac{1}{24\pi} \frac{\varepsilon(m)}{m} \int d^3x \, tr \, E_i^2 + O\left(\frac{1}{m^3}\right), \quad (T = 0).$$

(114)

(Of course, the same result is obtained taking the zero temperature limit of $W_{2,\text{leading}}^+(D)$ in eq. (111)). Then Lorentz invariance implies

$$W^+(D) = -\frac{1}{48\pi} \frac{\varepsilon(m)}{m} \int d^3x \, tr \, F_{\mu\nu}^2 + O\left(\frac{1}{m^3}\right), \quad (T = 0).$$

(115)

(Up to constant terms coming from $W_0(D)$.) It is an interesting check of the calculation that precisely the same result is obtained from a direct zero temperature calculation (using for instance the $1/m$ expansion coefficients given in [23]).

VII. THE PSEUDO-PARITY ODD EFFECTIVE ACTION AT SECOND ORDER

In this section we will compute the leading contribution to the imaginary part of the effective action, which comes from the second order in the gradient expansion. This leading term is particularly interesting since it contains the essential parity anomaly as well as the multivaluation under large gauge transformations.

Many of the ideas needed to carry out the calculation have already appeared in the previous section, so they do not need to be repeated here. We start from eq. (84) (recall that we use units $\beta = 2$). Use of eq. (70) allows to compute the Dirac trace. Keeping just the terms with $\epsilon_{ij}$ yields

$$\Omega_{s,2}(D) = -i \eta \int \frac{d^2k}{(2\pi)^2} \sum_n \int \frac{dz}{2\pi i} z^n \epsilon_{ij} \text{tr}(0) \left( \frac{Q}{\Delta} D_i \frac{Q}{\Delta} D_j \frac{Q}{\Delta} \right)$$

$$+ (\mu^2 - k^2) \frac{1}{\Delta} D_i \frac{Q}{\Delta} D_j \frac{1}{\Delta} - \mu^2 \frac{1}{\Delta} D_i \frac{1}{\Delta} D_j \frac{Q}{\Delta} - \mu^2 \frac{Q}{\Delta} D_i \frac{1}{\Delta} D_j \frac{1}{\Delta} ) |0\rangle,$n

(116)

where the trace refers to flavor only. (This expression is analogous to that in eq. (54) for the pseudo-parity even part.)

Next, the spatial covariant derivatives are brought to the right. It has been found to be best to move $D_i$ first and then $D_j$ to generate a smaller number of terms. This involves the quantities $E_i$ and $F_{ij} = [D_i, D_j]$, and also $A_i$ appears due to $D_i |0\rangle = A_i |0\rangle$. In this form the following expression is derived:
\[ \Omega_{s,2}^{-}(D) = -i \eta \int \frac{d^2 k}{(2\pi)^2} \sum_n \int_\Gamma \frac{dz}{2\pi i} \Delta z^4 \text{tr}(0) \left( \left( \frac{Q}{\Delta^3} F + \frac{Q^3}{\Delta^3} F \right) \left( -\frac{1}{\Delta} E \frac{Q}{\Delta^2} E + \mu^2 \frac{1}{\Delta} E \frac{Q}{\Delta^3} E - \frac{1}{\Delta} E \frac{Q^3}{\Delta^3} E \right) \right) |0\rangle. \]

This expression is analogous to that in eq. (86) for \( \Omega_{s,2}^{+}(D) \) and many of the remarks made there apply here too. A standard differential form notation has been adopted for the spatial indices, i.e., \( dx_1 \) and \( dx_2 \) are anticommuting, \( d^2 x_{ij} = dx_i dx_j \), \( D = D_i dx_i \), \( A = A_i dx_i \), \( E = E_i dx_i \) and \( F = D^2 \). The operator inside \( \langle 0|0 \rangle \) is multiplicative in \( x \)-space, thus the part of \( \langle 0|0 \rangle \) that has been replaced by \( \int d^2 x \) and \( d^2 x \) has been included in the differential forms, so \( |0\rangle \) now refers to the \( x_0 \)-space only. The terms with explicit \( A \) break gauge invariance and will cancel later. Finally, to obtain a more compact expression, the two cyclic properties have been used.

To carry out the momentum integration, we again make use of the method of labelling the \( Q \) operators with labels 1 or 2 according to their position. Note, however, that now

\[ \langle Z(Q_1, Q_2) E^2 \rangle = -\langle Z(Q_2, Q_1) E^2 \rangle, \]

the extra minus sign coming from \( E \) being a 1-form. Since all quantities are now commuting (or anti-commuting) momentum integration is straightforward. A non trivial check of the calculation is that all non-gauge invariant terms \( \langle X_1 E X_2 A \rangle \) cancel at this step. The result can be written in the following form

\[ \Omega_{s,2}^{-}(D) = \frac{i \eta}{4\pi} \sum_n \int_\Gamma \frac{dz}{2\pi i} \Delta z^4 \text{tr}(0) \left( \frac{Q}{\mu^2 - Q^2} F - \frac{1}{2} \frac{H(Q_1, Q_2)}{Q_1 - Q_2} E^2 \right) |0\rangle, \]

where \( H(x_1, x_2) \) is the same function as in the pseudo-parity even case\( ^4 \), eq. (74). Note that in this case \( E^2 \) is odd under exchange of the labels 1 and 2 (since \( E \) is a 1-form), so again the total expression is even under this exchange.

In order to sum over frequencies we need a new identity for the term with \( F \), namely\( ^9 \)

---

\(^9\) This suggests that the second order pseudo-parity even and odd components are closely related and perhaps connected through some kind of analytical extrapolation. Such relations exist in 1+1 dimensions and allow to exactly compute the corresponding Weyl determinant \( ^7 \).

\(^10\) The series in the left-hand side is not absolutely convergent, nevertheless it will be so upon subtraction of a suitable \( z \)-independent term, as in eq. (30). Thus this equality holds inside \( \int_\Gamma dz^4 z^n \).
\[
\sum_n \frac{x + \omega_n}{\mu^2 - (x + \omega_n)^2} \simeq \frac{1}{2} \tanh(\mu - x) - \text{p.p.c.}.
\] (120)

In the term with \(H(Q_1, Q_2)\) we again use integration by parts to eliminate the logarithm. This yields
\[
\sum_n H(x_1 + \omega_n, x_2 + \omega_n) \simeq \left( \frac{z}{s + 1} \frac{1}{x_1 - x_2} + \frac{1}{2} \right) \frac{\tanh(\mu - x_1)}{2\mu - x_1 + x_2} + X_{1,2} + \text{p.p.c.}.
\] (121)

Note that this result multiplied by \(\mu = m - z\) needs to coincide with that in eq. (98) only inside \(\int dz \, z\), since integration by parts has been used in both cases. In analogy to eq. (99), it is convenient to rewrite this expression as
\[
\sum_n H(x_1 + \omega_n, x_2 + \omega_n) \simeq \left[ -\frac{1}{2} \frac{1}{s + 1} \frac{1}{x_1 - x_2} + \left( \frac{1}{2} \frac{s}{s + 1} + \frac{1}{s + 1} \frac{m}{x_1 - x_2} \right) \frac{1}{2\mu - x_1 + x_2} \right] \times \tanh(\mu - x_1) + X_{1,2} + \text{p.p.c.}
\] (122)

After carrying out the \(z\)-integration, we arrive at a formula analogous to eq. (105), namely
\[
W_2^-(D) = \frac{in}{4\pi} \text{tr}(0) \left[ \frac{1}{2} \Omega'_\sigma(m - D_0, 0) F + \frac{1}{2} \left( \frac{1}{2 D_{02}} \Omega'_\sigma(m - D_{01}, 0) \right) + \frac{\sigma}{4} - \frac{m}{D_{02}} \int_0^{-\sigma\infty} dt \frac{\tanh(m - t - D_{01})}{2(m - t) + D_{02}} + X_{1,2} \right] \left( \frac{E^2}{D_{02}} \right) \langle 0 \rangle - \text{p.p.c.}.
\] (123)

The contribution \(X_{1,2}\) refers to everything inside the parenthesis and it gives a factor 2 due to the cyclic property. The final form is obtained by further replacing \(D_{01}\) by \(A_{01}\), \(E^2\) by \(-D_{02}^2 A^2\) and \(\langle 0 | 0 \rangle\) by \(\int dx_0\). This gives
\[
W_2^-(D) = \frac{in}{8\pi} \int dx_0 \text{tr} \left[ (F - A^2) \Phi_\sigma - \sigma A D_{01} A \right. \\
\left. + 2m \int_0^{-\sigma\infty} dt A \left( \frac{\tanh(m - t + A_0)}{2(m - t) + D_0} - \frac{\tanh(m - t - A_0)}{2(m - t) - D_0} \right) \right] A \right] \text{tr}(0) - \text{p.p.c.},
\] (124)

where \(\Phi_\sigma = \Omega'_\sigma(m - A_0, 0)\) - p.p.c. was introduced in eq. (111). Note that \(d^2 x\) is included in the differential forms.

In order to obtain the leading term in a further expansion in powers of \(D_0\) (the analogous of \(W_{2, \text{leading}}^+(D)\)) we go back to the symmetrized formula, eq. (123), then express the operator \(D_{02}\) in \(X_{1,2}\) as \(D_{01} + D_{02}\) and carry out an expansion in powers of \(D_{02}\) keeping the leading order. At the end \(D_{01}\) can be replaced by \(A_{01}\). This needs only be done in the term containing the integral over \(t\):
\[ m \int_{0}^{-\sigma} dt \left[ \frac{\tanh(m - t - D_{01})}{2(m - t) + D_{02}} - X_{1,2} \right] \]
\[ m \int_{0}^{-\sigma} dt \left[ \frac{\tanh(m - t - D_{01})}{2(m - t) + D_{02}} - \frac{\tanh(m - t - D_{01} - D_{02})}{2(m - t) - D_{02}} \right] \]
\[ = m \int_{0}^{-\sigma} dt \left[ \frac{\tanh(m - t - D_{01})}{2(m - t)^2} + \frac{1}{\cosh^2(m - t - D_{01})} \frac{1}{2(m - t)} \right] D_{02} + O(D_{02}^2) \]
\[ = \frac{1}{2} \tanh(m - D_{01}) D_{02} + O(D_{02}^2). \] (125)

So finally the result is
\[ W_{\text{anom}}(D) = \frac{i \eta}{8\pi} \int dx_0 \text{tr} \left[ (F - A^2) \Phi \sigma \right] + \frac{1}{2} \left( \tanh(m - A_0) + \tanh(m + A_0) - 2\sigma \right) A D_{0} A \right]. \] (126)

Let us finish by discussing the properties of \( W_{-2} \) paralleling the discussion of \( W_{\text{anom}} \) in section [4]. Using the same arguments as for \( W_{\text{anom}} \) it follows that \( W_{-2} \) is well defined, i.e., it does not depend on the choice of \( A_i(x) \). This implies that it is possible to express all terms not contained in \( W_{\text{anom}} \) in terms of the electric field.

The term of \( W_{-2} \) with the integral over \( t \) is actually independent of \( \sigma \) since using \( +\infty \) or \( -\infty \) as the upper limit of the integral gives the same result. This follows because the integrand is an odd function of \( m - t \). Therefore it is parity invariant and \( W_{-2} \) has the same parity anomaly as \( W_{\text{anom}} \). Likewise, the difference between \( W_{-2} \) and \( W_{\text{anom}} \) is strictly gauge invariant.

Eq. (121) for Abelian-like configurations also holds for \( W_{-2} \), since the terms without \( F \) cancel. In the massless case, the term with an integral over \( t \) cancels and so eq. (126) still holds for \( W_{-2} \).

The zero temperature limit of \( W_{-2} \) is given by
\[ W_{-2}(D) = \eta \sigma \Theta(-\sigma m) W_{CS} - \frac{i \eta}{8\pi} \int dx_0 \text{tr} \int_{|m|}^{\infty} dt E \frac{m D_0}{t^2(4t^2 - D_0^2)} E, \quad (T = 0). \] (127)
The first term was already present in \( W_{\text{anom}} \). Since the second term is of order \( 1/m^2 \) in an inverse mass expansion, it is ultraviolet finite.

The zero temperature limit of \( W_{-2}(D) \) is not directly Lorentz invariant. Restoring this invariance requires including higher order terms in the gradient expansion. Nevertheless, keeping the first non trivial \( 1/m \) contribution in eq. (127), the only Lorentz invariant completion is
\[ W^{-}(D) = \eta \sigma \Theta(-\sigma m) W_{CS} - \frac{i \eta \varepsilon(m)}{8\pi} \int d^3x \varepsilon_{\mu \nu \alpha} \text{tr}(F_{\beta \mu} D_{\alpha} F_{\beta \nu}) + O \left( \frac{1}{m^3} \right), \quad (T = 0). \] (128)

Precisely the same result is obtained through a direct zero temperature calculation using the formalism of Ref. [23]. This is another non trivial test of the calculation.
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APPENDIX A: THE \( A_0 \)-STATIONARY GAUGES

Let the space-time have the topology \( M_{d+1} = M_d \times S^1 \), \( d \) being the space dimension. All functions involved are assumed to be continuous on \( M_{d+1} \) (and thus, in particular, periodic in the time coordinate) unless otherwise stated. Let \( A_\mu(x) \) be a gauge configuration on \( M_{d+1} \). We want to show that there exists another configuration \( B_\mu(x) \) in the \( A_0 \)-stationary gauge (i.e., \( \partial_0 B_0 = 0 \)) related to \( A_\mu \) by a gauge transformation. To do this, let us construct the following time-independent field

\[
\Omega(x) = T \exp \left( - \int_0^\beta A_0(x, x_0) dx_0 \right). \tag{A1}
\]

(The symbol \( T \) stands for time ordered product.) As is well-known, \( \Omega \) transforms covariantly (in fact \( \text{tr}(\Omega) \) is just the Polyakov loop)

\[
\Omega^U(x) = U^{-1}(x, 0) \Omega(x) U(x, 0). \tag{A2}
\]

Thus, \( \Omega \) is gauge invariant modulo stationary gauge transformations. In view of this, we demand that \( B_\mu \) should yield the same field \( \Omega \) as \( A_\mu \), that is,

\[
\Omega(x) = \exp (-\beta B_0(x)). \tag{A3}
\]

That the function \( B_0(x) \) above exists (with continuity on \( M_d \)) follows from \( \Omega \) being an homotopically trivial application from \( M_d \) into \( G \). In fact,

\[
U_1(x, t) = T \exp \left( - \int_0^t A_0(x, t') dt' \right), \quad 0 \leq t \leq \beta \tag{A4}
\]

defines one such deformation of \( \Omega \) to the identity element of \( G \) with \( t \) as interpolating parameter. It only remains to show that there actually exists a gauge transformation bringing \( A_\mu \) to \( B_\mu \). This is achieved with

\[
U(x, x_0) = U_1(x, x_0) \exp (x_0 B_0(x)). \tag{A5}
\]

The first factor, \( U_1 \), brings \( A_0 \) to zero but breaks periodicity in Euclidean time. The second factor \( e^{x_0 B_0} \) yields \( A_0^U = B_0 \) and reestablishes the continuity of \( U \) on \( M_{d+1} \). Therefore, an \( A_0 \)-stationary gauge always exists.

Next, let us determine the remaining gauge freedom within an \( A_0 \)-stationary gauge. Let \( A_\mu \) and \( B_\mu \) be two \( A_0 \)-stationary configurations and let \( U \) be a gauge transformation bringing \( A_\mu \) to \( B_\mu \). Then

\[
B_0(x) = U^{-1}(x) \partial_0 U(x) + U^{-1}(x) A_0(x) U(x). \tag{A6}
\]
In terms of the auxiliary variable \( V(x) = \exp(x_0 A_0(x)) U(x) \), the previous equation becomes
\[
B_0(x) = V^{-1}(x) \partial_0 V(x) .
\]
Its general solution is
\[
V(x) = U_0(x) \exp(x_0 B_0(x)) ,
\]
where \( U_0 \) is an arbitrary time-independent gauge transformation. In terms of \( U(x) \)
\[
U(x) = \exp(-x_0 A_0(x)) U_0(x) \exp(x_0 B_0(x)) .
\]
This is the most general “gauge transformation” bringing \( A_\mu \) into a \( A_0 \)-stationary gauge, and depends on the two arbitrary time-independent fields \( B_0 \) and \( U_0 \). However, \( U \) will not be a true gauge transformation unless it is also continuous on \( M_{d+1} \). To impose this restriction, let us change variables from \( B_0(x) \) to \( \Lambda(x) \) by means of
\[
B_0(x) = U_0^{-1}(x) (A_0(x) + \Lambda(x)) U_0(x) .
\]
This allows to write \( U(x) \) more conveniently as
\[
U(x) = \exp(-x_0 A_0(x)) \exp(x_0 (A_0(x) + \Lambda(x))) U_0(x) .
\]
The continuity of \( U(x) \) follows from requiring \( U(x, 0) = U(x, \beta) \), which is equivalent to
\[
\exp(\beta A_0(x)) = \exp(\beta (A_0(x) + \Lambda(x))) .
\]
Finally, let us show that for simply connected gauge groups, it is not always possible to bring any given gauge field configuration to the \( A_0 \)-stationary gauge using only small transformations. To show this, let us assume that the initial configuration \( A_\mu \) is already \( A_0 \)-stationary. Let us perform a large gauge transformation on it, yielding the configuration \( A''_\mu \), and then bring this configuration again to an \( A_0 \)-stationary gauge to finally obtain a configuration \( A'_\mu \). Without loss of generality it can be assumed that \( A_\mu \) and \( A'_\mu \) are related by a discrete transformation of \( A_0(x) \) (i.e., with no time-independent part). We want to show that the gauge transformation bringing \( A''_\mu \) to \( A'_\mu \) cannot always be chosen to be small. Indeed, if it were small, \( A_\mu \) and \( A'_\mu \) would be related by a large gauge transformation which is also a discrete transformation for \( A_0(x) \). However, this not always possible if the gauge group is simply connected. This can be seen by taking \( A_0(x) \) to be a constant diagonal matrix such that the spectrum of \( \exp(\beta A_0(x)) \) is not degenerated. Then, the allowed \( \Lambda(x) \) for this configuration will be also diagonal and constant and thus \( e^{x_0 \Lambda} \) will necessarily be small, that is, no large discrete transformation exists for such \( A_0(x) \). On the other hand, in the Abelian \( U(1) \) case, it is always possible to deform \( A_0(x) \) to a time-independent one preserving the Polyakov loop and no large transformations are required.

**APPENDIX B: AUXILIARY FUNCTIONS**

The functions \( \Omega_\Gamma(\omega, s) \) are defined through the formula

\[38\]
\[ \Omega_\Gamma(\omega, s) = -\int_\Gamma \frac{dz}{2\pi i} \omega^n \tanh(\omega - z). \] (B1)

The integral is defined for \( \text{Re}(s) < -1 \) and extends to a meromorphic function of \( s \) with a simple pole at \( s = -1 \). The integration path \( \Gamma \) follows the ray \( \theta \) starting from infinity, encircles zero clockwise and goes back to infinity along the ray \( \theta - 2\pi \) on the \( z \)-complex plane. Because the function \( \tanh(z) \) has simple poles at \( z = \omega_n = i\pi (n + \frac{1}{2}) \), the integral is not defined when \( \omega + \omega_n \) lies on \( \Gamma \) for some \( n \). As a consequence, the complex plane corresponding to the variable \( \omega \) is cut along rays with angle \( \theta \) stemming from the points \( \omega = \omega_n \) which are branching points of the function \( \Omega_\Gamma(\omega, s) \) (unless \( s \) is an integer). Let us denote the \( \omega \)-complex plane so cut by \( C_\sigma \), and the corresponding Riemann surface by \( \tilde{C} \). This latter manifold is common to all values of \( \theta \).

The angle \( \theta \) can take any real value excepting \( \theta = \pm \pi/2 \) (mod. \( 2\pi \)) (i.e., \( \Gamma \) on the imaginary axis). The function \( \omega^n \) is defined taking \( \text{arg}(\omega) \) in the range \( ]\theta - 2\pi, \theta[ \). Correspondingly, we will use the notation \( \omega^n \), \( \text{arg}_\theta(\omega) \) and \( \Omega_\theta(\omega, s) \). Applying Cauchy’s theorem, the function can then be written as

\[ \Omega_\theta(\omega, s) = \sum_{n \in \mathbb{Z}} (\omega - \omega_n)^n. \] (B2)

If \( \theta \) is shifted to \( \theta' \) but without crossing the limits \( \pm \pi/2 \) (mod. \( 2\pi \)), only a finite number of terms in the series get modified and the two functions \( \Omega_\theta, \Omega_{\theta'} \) are related by analytical continuation, that is, they coincide on the \( \tilde{C} \). On the other hand, from the definition it follows immediately that

\[ \Omega_{\theta + 2\pi n}(\omega, s) = e^{2\pi ins} \Omega_\theta(\omega, s), \quad (n \in \mathbb{Z}) \] (B3)

As will be shown below, \( \Omega_\Gamma(\omega, k) = 0 \) for \( k = 0, 1, 2, \ldots \), so \( \Omega'_{\theta + 2\pi n}(\omega, k) = \Omega_\theta(\omega, k) \) (where the prime refers to the derivation on \( s \)). Thus, all angles differing by an integer multiple of \( 2\pi \) give the same effective action and are equivalent. It follows that all practical cases are covered by taking \( \theta = \pi \) and \( \theta = 2\pi \), which correspond to \( \sigma = 1 \) and \( \sigma = -1 \) respectively.

The \( \omega \)-complex planes \( C_\sigma \) are cut along rays which are parallel to the real axis, start at \( \omega = \omega_n \) and go to the right for \( \sigma = +1 \) and to the left for \( \sigma = -1 \). To describe the functions \( \Omega_\sigma(\omega, s) \) on the Riemann surface \( \tilde{C} \), it will prove convenient to introduce the \( \omega \)-complex plane cut in another way, which will be denoted by \( C_p \) and is defined as follows. Starting from any of \( C_\sigma \) planes, rotate upwards the rays which are above the real axis so that these rays lie now on the positive imaginary axis, and likewise rotate downwards the rays below the real axis to put them on the negative imaginary axis. Since the branching points which are nearest to the origin are \( \omega = \pm \omega_0 = \pm i\pi/2 \), the functions \( \Omega_\sigma(\omega, s) \) extended to \( C_p \) are analytical for all \( \omega \) except when \( \omega \) is purely imaginary and above \( \omega_0 \) or below \( -\omega_0 \). The plane \( C_p \) has the property of supporting parity transformations, which correspond to \( \omega \to -\omega \) (since in practice \( \omega = m \pm A_0 \)) and also discrete gauge transformations, \( \omega \to \omega + i\pi \), where the two points are connected by a straight line. Besides, for each value of \( \sigma \), the function \( \Omega_\sigma(\omega, s) \) takes the same value on \( C_\sigma \) and \( C_p \) when \( \omega \) is in the half-plane \( \sigma \text{Re}(\omega) > 0 \).

The functions \( \Omega_\sigma(\omega, s) \) can be directly related to the Hurwitz function \( \zeta(z, q) \) (p. 1073 and ff.):

\[ \zeta(z, q) = \sum_{n=0}^{\infty} (q + n)^{-z}, \] (B4)
where the subindex $\pi$ means $|\arg(q + n)| < \pi$. This function is analytical (as a function of $q$) except on the negative real axis. The series representation of $\Omega_\sigma$, eq. (B2), allows to write

$$
\Omega_+(\omega, s) = (-i\pi)^s \zeta\left(-s, \frac{1}{2} - \frac{\omega}{i\pi}\right) + (i\pi)^s \zeta\left(-s, \frac{1}{2} + \frac{\omega}{i\pi}\right)
$$

$$
\Omega_-(\omega, s) = e^{2\pi i s} (-i\pi)^s \zeta\left(-s, \frac{1}{2} - \frac{\omega}{i\pi}\right) + (i\pi)^s \zeta\left(-s, \frac{1}{2} + \frac{\omega}{i\pi}\right),
$$

(B5)

where all arguments are to be taken on $[-\pi, \pi]$ and the equalities hold on $C_\rho$. An immediate consequence is that the functions $\Omega_+(\omega, s)$ and $\Omega_-(\omega, s)$ coincide everywhere when $s$ is an integer. Using the identities

$$
\zeta(-n, q) = -\frac{B_{n+1}(q)}{n + 1}, \quad B_n(1 - x) = (-1)^n B_n(x), \quad n = 0, 1, 2, \ldots
$$

(B6)

where $B_n(x)$ are the Bernoulli polynomials, it follows immediately from eq. (B5) that

$$
\Omega_\pm(\omega, n) = 0, \quad n = 0, 1, 2, \ldots
$$

(B7)

and furthermore,

$$
\Omega'_+(0, n) = \Omega'_-(0, n), \quad n = 0, 2, 4, \ldots
$$

(B8)

$$
\Omega'_\sigma(0, n) = -\sigma(i\pi)^n \frac{B_{n+1}(\frac{1}{2})}{n + 1}, \quad n = 1, 3, 5, \ldots
$$

(B9)

where the prime refers to derivative with respect to $s$. Another useful property follows from the identity

$$
\partial_\omega \Omega'_\sigma(\omega, s) = \Omega_\sigma(\omega, s - 1) + s \Omega'_\sigma(\omega, s - 1),
$$

(B10)

which implies

$$
\partial_\omega \Omega'_\sigma(\omega, n + 1) = (n + 1) \Omega'_\sigma(\omega, n), \quad n = 0, 1, 2, \ldots
$$

(B11)

The functions $\Omega'_\sigma(\omega, n)$ for non negative integer $n$ can be expressed in terms of simple integrals as follows

$$
\Omega'_\sigma(\omega, n) = -\frac{d}{ds} \left( \int_{\Gamma} \frac{dz}{2\pi i} z^s \tanh(\omega - z) \right)_{s=n}
$$

$$
= -\frac{d}{ds} \left( \int_{\Gamma} \frac{dz}{2\pi i} z^s [z^n (\tanh(\omega - z) - \sigma)] \right)_{s=0}
$$

$$
= \int_{\sigma\infty}^{-\sigma\infty} dt t^n (\tanh(\omega - t) - \sigma)
$$

$$
= \int_{\sigma\infty}^{\omega} dt (\omega - t)^n (\tanh(t) - \sigma), \quad n = 0, 1, 2, \ldots
$$

(B12)

The defining integral has been converted into a convergent one by using eq. (31) and then eq. (101) has been applied. This result refers to $\omega$ in $C_\sigma$ and in the integrals over $t$, the integration path is to be taken parallel to the real axis. (A similar method would also yield
the relations in eqs. (B7, B8, B9). In particular, for \( n = 0 \) the integral can be done in closed form

\[
\Omega'_\sigma(\omega, 0) = \log(e^{-2\sigma \omega} + 1),
\]

which can be rewritten as

\[
\Omega'_\sigma(\omega, 0) = \phi_0(\omega) - \sigma \omega,
\]

\[
\phi_0(\omega) = \log(2 \cosh(\omega)).
\]

Next, let us introduce the functions \( \phi_n(\omega) \) for arbitrary integer \( n \). They are defined on \( C_p \) by the relations

\[
\phi_{-1}(\omega) = \tanh(\omega),
\]

\[
\phi'_n(\omega) = \phi_{n-1}(\omega),
\]

\[
\phi_n(-\omega) = (-1)^n \phi_n(\omega),
\]

\[
\phi_n(0) = \frac{1}{n!} \Omega'_{\pm}(0, n), \quad n = 0, 2, 4, \ldots
\]

The lowest order relations are

\[
\Omega'_\sigma(\omega, 0) = \phi_0(\omega) - \sigma \omega,
\]

\[
\Omega'_\sigma(\omega, 1) = \phi_1(\omega) - \sigma \left( \frac{1}{2} \omega^2 - \frac{1}{6} \left( \frac{i\pi}{2} \right)^2 \right),
\]

\[
\frac{1}{2!} \Omega'_\sigma(\omega, 2) = \phi_2(\omega) - \sigma \left( \frac{1}{6} \omega^3 - \frac{1}{6} \left( \frac{i\pi}{2} \right)^2 \omega \right).
\]

These relations make explicit the transformation properties of the functions \( \Omega'_\sigma(\omega, n) \) under parity, \( \omega \to -\omega \). The polynomial part is responsible for an anomalous violation of parity. Also, it follows
\[ \Omega'_\sigma(\omega, n) = (-1)^n \Omega'_{-\sigma}(-\omega, n) \] (B28)

that reflects eq. (B2).

Next, let us discuss the periodicity properties of these functions, which are related to the discrete gauge transformations. Since \( \tanh(\omega) \) is a periodic function with period \( i\pi \), so are the functions \( \Omega_\sigma(\omega, s) \) on \( C_\sigma \). This reflects the trivial gauge invariance of the \( \zeta \)-function regularization due to the invariance of the spectrum of the Dirac operator. Let us consider now the periodicity on \( C_p \). Since \( \Omega_\sigma(\omega, s) \) takes the same values on \( C_\sigma \) and \( C_p \) on the half-plane \( \sigma \text{Re}(\omega) > 0 \), it follows that this function is periodic on that half-plane of \( C_p \). For the other half-plane, \( \sigma \text{Re}(\omega) < 0 \), take \( \omega \) in the strip \(-\pi < \text{Im}(\omega) < \pi\) (which is common to \( C_\sigma \) and \( C_p \)). Then, the value of \( \Omega_\sigma(\omega + i\pi, s) \) computed by taking \( \omega + i\pi \) in \( C_p \) will be different from the value arrived at on \( C_\sigma \) and this difference comes solely from the term \( n = 0 \) in eq. (B2). For instance, for \( \sigma = +1 \), the contribution of this term on \( C_\sigma \) would be \( (\omega + i\pi - \omega_0)^s \pi \), whereas on \( C_p \) the contribution is instead \( (\omega + i\pi - \omega_0)^s \pi \). Noting that \( \Omega_\sigma(\omega + i\pi, s) = \Omega_\sigma(\omega, s) \) on \( C_\sigma \) and also that \( \Omega_\sigma(\omega, s) \) takes the same value on \( C_\sigma \) and \( C_p \) because of \(-\pi < \text{Im}(\omega) < \pi\), it is easily established that, for \( \omega + i\pi \in C_p \),

\[ \Omega_\sigma(\omega + i\pi, s) - \Omega_\sigma(\omega, s) = (e^{-\sigma 2\pi i s} - 1)(\omega + \omega_0)^s \Theta(-\sigma m). \] (B29)

Here \( \omega_0 = i\pi/2 \) and \( m = \text{Re}(\omega) \). After analytical extension beyond the strip \(-\pi < \text{Im}(\omega) < \pi\), this relation holds on the whole plane \( C_p \). From here it follows the useful relation

\[ \Omega'_\sigma(\omega + \omega_0, n) - \Omega'_\sigma(\omega - \omega_0, n) = -\sigma 2\pi i \omega^n \Theta(-\sigma m), \quad \omega \in C_p, \quad n \in \mathbb{Z}. \] (B30)

Where \( \Theta(x) \) is the step function. Using that, on \( C_p, \phi_n(\omega) \) is the component of \( \Omega'_\sigma(\omega, n)/n! \) which is even in \( \sigma \), we also find

\[ \phi_n(\omega + \omega_0) - \phi_n(\omega - \omega_0) = i\pi \frac{\omega^n}{n!} \varepsilon(m), \quad \omega \in C_p, \quad n = 0, 1, 2, \ldots, \] (B31)

where \( \varepsilon(x) \) stands for the sign of \( x \).

Finally, let us study the zero temperature limit properties of these functions. This corresponds to the large \( \omega \) limit. Again we start with \( \omega \) in the strip \(-\pi < \text{Im}(\omega) < \pi\), then from eq. (B32) it follows that

\[ \Omega'_\sigma(\omega, n) \sim_{\omega \to \infty} -2\sigma \frac{\omega^{n+1}}{n+1} \Theta(-\sigma m), \quad \omega \in C_p, \quad n = 0, 1, 2, \ldots \] (B32)

By analytical extension, this relation holds on \( C_p \). Moreover, for positive \( \sigma m \), the zero limit has only exponentially decreasing corrections. Then, from eq. (B20) it follows that

\[ \phi_{n-1}(\omega) \sim_{\omega \to \infty} \varepsilon(m) P_n(\omega), \quad \omega \in C_p, \quad n = 0, 1, 2, \ldots \] (B33)

and in particular

\[ \phi_{n-1}(\omega) = \frac{\omega^n}{n!} \varepsilon(m) \left(1 + O \left(\frac{1}{\omega}\right)\right), \quad \omega \in C_p, \quad n = 0, 1, 2, \ldots \] (B34)
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