The lifespan of solutions of semilinear wave equations with the scale-invariant damping in two space dimensions

Takuto Imai *, Masakazu Kato †, Hiroyuki Takamura ‡ and Kyouhei Wakasa §

Keywords: semilinear wave equation, scale-invariant damping, lifespan
MSC2010: primary 35L71, secondary 35B44

Abstract

In this paper, we study the initial value problem for semilinear wave equations with the time-dependent and scale-invariant damping in two dimensions. Similarly to the one dimensional case by Kato, Takamura and Wakasa in 2019, we obtain the lifespan estimates of the solution for a special constant in the damping term, which are classified by total integral of the sum of the initial position and speed.

The key fact is that, only in two space dimensions, such a special constant in the damping term is a threshold between “wave-like” domain and “heat-like” domain. As a result, we obtain a new type of estimate especially for the critical exponent.

*Accenture Japan Ltd, Harumi Triton Square Office Tower Z, 1-8-12 Harumi, Chuo-ku, Tokyo, 104-0053, Japan. e-mail: takuto.imai@accenture.com.
†College of Liberal Arts, Mathematical Science Research Unit, Muroran Institute of Technology, 27-1 Mizumoto-cho, Muroran, Hokkaido 050-8585, Japan. email: mkato@mmm.muroran-it.ac.jp.
‡Mathematical Institute, Tohoku University, Aoba, Sendai 980-8578, Japan. e-mail: hiroyuki.takamura.a1@tohoku.ac.jp.
§Department of Creative Engineering, National Institute of Technology, Kushiro College, 2-32-1 Otanoshiki-Nishi, Kushiro-Shi, Hokkaido 084-0916, Japan. e-mail: wakasa@kushiro-ct.ac.jp.
1 Introduction

We are concerned with the following initial value problem for semilinear wave
equations with the scale-invariant damping:

\[
\begin{align*}
\begin{cases}
v_{tt} - \Delta v + \frac{\mu}{1+t} v_t = |v|^p & \text{in } \mathbb{R}^n \times [0, \infty), \\
v(x,0) = \varepsilon f(x), \quad v_t(x,0) = \varepsilon g(x), & x \in \mathbb{R}^n,
\end{cases}
\end{align*}
\]

where \( v = v(x,t) \) is a real valued unknown function, \( \mu > 0, \ p > 1, \ n \in \mathbb{N} \),
the initial data \((f, g) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\) has compact support, and \( \varepsilon > 0 \)
is a “small” parameter.

It is interesting to look for the critical exponent \( p_c(n) \) such that

\[
\begin{align*}
\begin{cases}
p > p_c(n) \text{ (and may have an upper bound)} & \implies T(\varepsilon) = \infty, \\
1 < p \leq p_c(n) & \implies T(\varepsilon) < \infty,
\end{cases}
\end{align*}
\]

where \( T(\varepsilon) \) is the lifespan, the maximal existence time, of the energy solution
of (1.1) with an arbitrary fixed non-zero data. Then, we have the following
conjecture:

\[
\begin{align*}
\begin{cases}
\mu > \mu_0(n) & \implies p_c(n) = p_F(n) \quad \text{(heat-like)}, \\
\mu = \mu_0(n) & \implies p_c(n) = p_F(n) = p_S(n + \mu) \quad \text{(intermediate)}, \\
0 < \mu < \mu_0(n) & \implies p_c(n) = p_S(n + \mu) \quad \text{(wave-like)},
\end{cases}
\end{align*}
\]

where

\[
\mu_0(n) := \frac{n^2 + n + 2}{n + 2}.
\]

Here

\[
p_F(n) := 1 + \frac{2}{n}
\]

is the so-called Fujita exponent which is the critical exponent of the associated
semilinear heat equations \( v_t - \Delta v = v^p \), and

\[
p_S(n) := \begin{cases}
\infty & (n = 1), \\
\frac{n + 1 + \sqrt{n^2 + 10n - 7}}{2(n - 1)} & (n \geq 2)
\end{cases}
\]

is the so-called Strauss exponent which is the critical exponent of the associated
semilinear wave equations \( v_{tt} - \Delta v = |v|^p \). We note that \( p_S(n) \ (n \geq 2) \)
is the positive root of

\[
\gamma(p, n) := 2 + (n + 1)p - (n - 1)p^2 = 0
\]

where

\[
\mu_0(n) := \frac{n^2 + n + 2}{n + 2}.
\]
and $0 < \mu < \mu_0(n)$ is equivalent to $p_F(n) < p_S(n + \mu)$.

The conjecture (1.2) shows the critical situation of our problem in the following sense. If one replaces the damping term $\mu v_t/(1 + t)$ in (1.1) by $\mu v_t/(1 + t)^\beta$, then one can see that there is no such a $p_c(n)$, namely $T(\varepsilon) = \infty$ for any $p > 1$ when $\beta < -1$, the so-called over damping case. Moreover one has $p_c(n) = p_F(n)$ for any $\mu > 0$ when $-1 \leq \beta < 1$, the so-called effective damping case, and $p_c(n) = p_S(n)$ for any $\mu > 0$ (it can be any $\mu \in \mathbb{R}$) when $\beta > 1$, the so-called scattering damping case. Therefore one may say that the so-called scale-invariant case, $\beta = 1$, is an intermediate situation between wave-like, in which the critical exponent is related to $p_S(n)$, and heat-like, in which the critical exponent is $p_F(n)$. To see all the references above results, for example, see introductions of related papers to the scattering damping case, Lai and Takamura [15] (the sub-critical case), Wakasa and Yordanov [22] (the critical case), Liu and Wang [17] (partial result of the super-critical case).

For the conjecture (1.2), D’Abbicco [3] has obtained the heat-like existence partially with

$$
\mu \geq \begin{cases} 
5/3 & \text{for } n = 1 \text{ (cf. } \mu_0(1) = 4/3), \\
3 & \text{for } n = 2 \text{ (cf. } \mu_0(2) = 2), \\
n + 2 & \text{for } n \geq 3,
\end{cases}
$$

while Wakasugi [25] has obtained the blow-up parts in $1 < p < p_F(n)$ for $\mu \geq 1$ and $1 < p < p_F(n + \mu - 1)$ for $0 < \mu < 1$. In the damped case $\mu > 0$, his second result is the first blow-up result for super-Fujita exponents which are larger than $p_F(n)$.

In this paper, we consider a special case of $\mu = 2$. The speciality of this value is clarified by setting

$$
u(x, t) := (1 + t)^{\mu/2} v(x, t),$$

where $v$ is the solution to (1.1). Then, $u$ satisfies

$$
\begin{aligned}
nu - \Delta u + \frac{\mu(2 - \mu)}{4(1 + t)^2} u &= \frac{|u|^p}{(1 + t)^{\mu(p-1)/2}} \quad \text{in } \mathbb{R}^n \times [0, \infty), \\
u(x, 0) &= \varepsilon f(x), \quad \nu_t(x, 0) = \varepsilon \{\mu f(x)/2 + g(x)\}, \quad x \in \mathbb{R}^n,
\end{aligned}
$$

(1.7)

so that all the technics in the analysis on semilinear wave equations can be employed and we may discussed about not only the energy solution but also the classical solution. In fact, via this reduced problem (1.7), D’Abbicco, Lucente and Reissig [5] have proved the intermediate part of the conjecture (1.2) for $n = 2$ and the wave-like part for $n = 3$ when $\mu = 2$. We note that the
assumption of the radial symmetry is considered in [5] for the existence part in \( n = 3 \). Moreover, D’Abbicco and Lucente [4] have obtained the wave-like existence part of (1.2) for odd \( n \geq 5 \) when \( \mu = 2 \) also with radial symmetry.

In the case of \( \mu \neq 2 \), Lai, Takamura and Wakasa [16] have first studied
the wave-like blow-up of the conjecture (1.2) with a loss replacing \( \mu \) by \( \mu/2 \) in the sub-critical case. Initiating this result, Ikeda and Sobajima [8] have obtained the blow-up part of (1.2).

For the lifespan estimate, one may expect that

\[
T(\varepsilon) \sim \begin{cases} 
  C\varepsilon^{-(p-1)/(2-n(p-1))} & \text{for } 1 < p < p_F(n), \\
  \exp \left( C\varepsilon^{-(p-1)} \right) & \text{for } p = p_F(n)
\end{cases}
\]  

for the heat-like domain \( \mu > \mu_0(n) \) and

\[
T(\varepsilon) \sim \begin{cases} 
  C\varepsilon^{-(p-1)/(3-p)} & \text{for } 1 < p < p_S(n+n) \\
  \exp \left( C\varepsilon^{-(p-1)} \right) & \text{for } p = p_S(n+n)
\end{cases}
\]

for the wave-like domain \( 0 < \mu < \mu_0(n) \). Recall the definitions of \( \mu_0(n) \), \( p_F(n) \), \( p_S(n) \) and \( \gamma(p,n) \) in (1.3), (1.4), (1.5) and (1.6). Here \( T(\varepsilon) \sim A(\varepsilon, C) \) stands for the fact that there are positive constants, \( C_1 \) and \( C_2 \), independent of \( \varepsilon \) satisfying \( A(\varepsilon, C_1) \leq T(\varepsilon) \leq A(\varepsilon, C_2) \). Actually, (1.8) for \( n = 1 \) and \( \mu = 2 > \mu_0(1) = 4/3 \) is obtained by Wakasa [24], and (1.9) is obtained by Kato and Sakuraba [12] for \( n = 3 \) and \( \mu = 2 < \mu_0(3) = 14/5 \). One may refer Lai [14] for the existence part of weaker solution. Moreover, the upper bound of (1.8) in the sub-critical case is obtained by Wakasugi [25]. Also the upper bound of (1.9) is obtained by Ikeda and Sobajima [8] in the critical case, later it is reproved by Tu and Li [21], and Tu and Li [20] in the sub-critical case.

In the non-damped case of \( \mu = 0 \), it is known that (1.9) is true for \( n \geq 3 \), or \( p > 2 \) and \( n = 2 \). The open part around this fact is \( p = p_S(n) \) for \( n \geq 9 \). In other cases, (1.9) is still true if \( \int_{\mathbb{R}^n} g(x)dx = 0 \). On the other hand, we have

\[
T(\varepsilon) \sim \begin{cases} 
  C\varepsilon^{-(p-1)/2} & \text{for } n = 1, \\
  C\varepsilon^{-(p-1)/(3-p)} & \text{for } n = 2 \text{ and } 1 < p < 2, \\
  Ca(\varepsilon) & \text{for } n = 2 \text{ and } p = 2
\end{cases}
\]

if \( \int_{\mathbb{R}^n} g(x)dx \neq 0 \), where \( a = a(\varepsilon) \) is a positive number satisfying \( \varepsilon^2 a^2 \log(1+a) = 1 \). We note that the bounds in (1.10) are smaller than the one of the first line in (1.9) with \( \mu = 0 \) in each case. For all the references in the case of \( \mu = 0 \), see Introduction of Imai, Kato, Takamura and Wakasa [10].

The remarkable fact is that even if \( \mu \) is in the heat-like domain, the lifespan estimate for (1.1) is similar to the one for non-damped case. Indeed,
for $n = 1$ and $\mu = 2 > \mu_0(1) = 4/3$, Kato, Takamura and Wakasa [13] show that the result on (1.8) by Wakasa [24] mentioned above is true only if

$$\int_\mathbb{R} \{f(x) + g(x)\} \, dx \neq 0.$$ 

More precisely, they have obtained that

$$T(\varepsilon) \sim \begin{cases} 
C\varepsilon^{-2p(p-1)/\gamma(p,3)} & \text{for } 1 < p < 2, \\
Cb(\varepsilon) & \text{for } p = 2, \\
C\varepsilon^{-p(p-1)/(3-p)} & \text{for } 2 < p < 3, \\
\exp(C\varepsilon^{-p(p-1)}) & \text{for } p = p_F(1) = 3,
\end{cases} \quad (1.11)$$

if $\int_\mathbb{R} \{f(x) + g(x)\} \, dx = 0$, where $b = b(\varepsilon)$ is a positive number satisfying $\varepsilon^2b\log(1 + b) = 1$. We note that the bounds in (1.11) are larger than those in (1.8) with $n = 1$ and $\mu = 2$ in each case.

Our aim in this paper is to show the lifespan estimates for (1.1) in two dimensional case, $n = 2$, with $\mu = 2$ which is similar to one dimensional case as above. We note $p_c(2) = p_F(2) = p_S(2 + 2) = 2$ and $\mu_0(2) = 2$. More precisely, we shall show that

$$T(\varepsilon) \sim \begin{cases} 
\varepsilon^{-(p-1)/(4-2p)} & \text{for } 1 < p < 2, \\
\exp(\varepsilon^{-1/2}) & \text{for } p = 2
\end{cases} \quad (1.12)$$

if $\int_{\mathbb{R}^2} \{f(x) + g(x)\} \, dx \neq 0$, and

$$T(\varepsilon) \sim \begin{cases} 
\varepsilon^{-2p(p-1)/\gamma(p,4)} & \text{for } 1 < p < 2, \\
\exp(\varepsilon^{-2/3}) & \text{for } p = 2
\end{cases} \quad (1.13)$$

if $\int_{\mathbb{R}^2} \{f(x) + g(x)\} \, dx = 0$. We note that the critical cases in (1.12) and (1.13) are new in the sense that they are different from (1.8) and (1.9). (1.12) and (1.13) are announced in Introduction of Lai and Takamura [15], but there are typos in the exponents of $\varepsilon$ in the critical case.

The strategy of proofs in this paper is based on point-wise estimates of the solution. In the existence part, we employ the classical iteration argument for semilinear wave equations without damping term, which is first introduced by John [11] in three space dimensions, and its variant, which is developed by Imai, Kato, Takamura and Wakasa [10] in two space dimensions. In the blow-up part, we also employ an improved version of Kato’s lemma on ordinary differential inequality by Takamura [18] for the sub-critical cases. We note that, till now, the so-called test function method such as in Ikeda, Sobajima and Wakasa [9] cannot be applicable to delicate analysis to catch the logarithmic growth of the solution in the case of $p = 2$ in (1.10), (1.11), (1.12) and (1.13). Therefore we employ the so-called slicing method of the blow-up domain for the critical case, which is introduced by Agemi, Kurokawa and Takamura [1] to handle weakly coupled systems of non-damped semilinear wave equations with critical exponents.
This paper is organized as follows. In the next section, our goals, (1.12) and (1.13), are described in four theorems, and we introduce the linear decay estimate and basic lemmas for a-priori estimates. Section 3, or Section 4, is devoted to the proof of the lower bound, or upper bound, of the lifespan respectively.

2 Theorems and preliminaries

In this section, we state our results (1.12) and (1.13) in four theorems. After them, we list useful point-wise estimates of linear wave equations. For the sake of the simplicity, we assume that

\[
\text{supp}(f, g) \subset \{ x \in \mathbb{R}^2 : |x| \leq k \}, \; k \geq 1 \quad (2.1)
\]

throughout this paper.

The existence parts of our goals in (1.12) and (1.13) are guaranteed by the following two theorems. Recall the definitions of \( \mu_0(n) \), \( p_F(n) \), \( p_S(n) \) and \( \gamma(p, n) \) respectively in (1.3), (1.4), (1.5) and (1.6).

**Theorem 2.1** Let \( n = 2 \), \( \mu = \mu_0(2) = 2 \) and \( 1 < p \leq p_F(2) = p_S(4) = 2 \).

Assume that \( (f, g) \in C^3_0(\mathbb{R}^2) \times C^2_0(\mathbb{R}^2) \) satisfies (2.1). Then, there exists a positive constant \( \varepsilon_0 = \varepsilon_0(f, g, p, k) \) such that (1.7) admits a unique solution \( u \in C^2(\mathbb{R}^2 \times [0, T]) \) if \( p = 2 \), or the integral equation associated with (1.7) admits a unique solution \( u \in C^1(\mathbb{R}^2 \times [0, T]) \) otherwise, as far as \( T \) satisfies

\[
T \leq \begin{cases} 
\frac{c \varepsilon^{(p-1)/(4-2p)}}{2} & \text{if } 1 < p < 2, \\
\exp(c \varepsilon^{-1/2}) & \text{if } p = 2
\end{cases}
\]

for \( 0 < \varepsilon \leq \varepsilon_0 \), where \( c \) is a positive constant independent of \( \varepsilon \).

**Theorem 2.2** Suppose that the assumptions in Theorem 2.1 are fulfilled. Assume additionally that

\[
\int_{\mathbb{R}^2} \{ f(x) + g(x) \} \, dx = 0.
\]

Then, there exists a positive constant \( \varepsilon_0 = \varepsilon_0(f, g, p, k) \) such that (1.7) admits a unique solution \( u \in C^2(\mathbb{R}^2 \times [0, T]) \) if \( p = 2 \), or the integral equation associated with (1.7) admits a unique solution \( u \in C^1(\mathbb{R}^2 \times [0, T]) \) otherwise, as far as \( T \) satisfies

\[
T \leq \begin{cases} 
\frac{c \varepsilon^{-2p(p-1)/\gamma(p, 4)}}{2} & \text{if } 1 < p < 2, \\
\exp(c \varepsilon^{-2/3}) & \text{if } p = 2
\end{cases}
\]

for \( 0 < \varepsilon \leq \varepsilon_0 \), where \( c \) is a positive constant independent of \( \varepsilon \).
On the other hand, the blow-up parts of our goals in (1.12) and (1.13) are guaranteed by the following two theorems.

**Theorem 2.3** Let \( n = 2, \mu = \mu_0(2) = 2, 1 < p \leq 2 = p_F(2) = p_S(4). \) Assume that \((f, g) \in C_0^0(R^2) \times C_0^0(R^2)\) satisfies (2.1). Suppose that the integral equation associated with (1.7) has a solution \( u \in C^1(R^2 \times [0, T]) \) with \( \text{supp } u \subset \{(x, t) \in R^2 \times [0, \infty) : |x| \leq t + k\} \) Then, there exists a positive constant \( \varepsilon_1 = \varepsilon_1(f, g, p, k) \) such that the solution cannot exist whenever \( T \) satisfies

\[
T \geq \begin{cases} 
\varepsilon^{-1-1/2} & \text{if } 1 < p < 2, f(x) \equiv 0 \text{ and } g(x) \geq 0 (\neq 0), \\
\exp\varepsilon^{-1/2} & \text{if } p = 2 \text{ and } \int_{R^2} \{f(x) + g(x)\} dx > 0
\end{cases}
\]

for \( 0 < \varepsilon \leq \varepsilon_1 \), where \( c \) is a positive constant independent of \( \varepsilon \).

**Theorem 2.4** Suppose that the assumptions in Theorem 2.3 are fulfilled. Assume additionally that

\[
f(x) + g(x) \equiv 0.
\]

Then, there exists a positive constant \( \varepsilon_1 = \varepsilon_1(f, g, p, k) \) such that the solution cannot exist whenever \( T \) satisfies

\[
T \geq \begin{cases} 
\varepsilon^{-2p(p-1)/\gamma(p,4)} & \text{if } 1 < p < 2 \text{ and } f(x) \geq 0 (\neq 0), \\
\exp(\varepsilon^{-2/3}) & \text{if } p = 2 \text{ and } \int_{R^2} f(x) dx < 0
\end{cases}
\]

for \( 0 < \varepsilon \leq \varepsilon_1 \), where \( c \) is a positive constant independent of \( \varepsilon \).

From now on, we introduce some definitions and useful lemmas. For \((x, t) \in R^2 \times [0, \infty)\), we set

\[
u_L(x, t) := \frac{\partial}{\partial t} R(f|x, t) + R(f + g|x, t),
\]

\[
R(\phi|x, t) := \frac{1}{2\pi} \int_{|x-y| \leq t} \frac{\phi(y)}{\sqrt{t^2 - |x-y|^2}} dy = \frac{t}{2\pi} \int_{|\xi| \leq 1} \frac{\phi(x + t\xi)}{\sqrt{1 - |\xi|^2}} d\xi. \tag{2.2}
\]

When \((f, g) \in C_0^0(R^2) \times C_0^0(R^2)\), we note that \( u_L \) satisfies that

\[
\begin{cases} 
(u_L)_{tt} - \Delta u_L = 0 & \text{in } R^2 \times [0, \infty), \\
u_L(x, 0) = f(x), (u_L)_t(x, 0) = f(x) + g(x), & x \in R^2
\end{cases}
\]

in the classical sense, and it holds

\[
\text{supp } u_L \subset \{(x, t) \in R^2 \times [0, \infty) : |x| \leq t + k\}.
\]

We introduce the decay estimates for the solutions of (2.2) which will be used in the proof of Theorem 2.1 and Theorem 2.2. For the proof, see Lemma 2.1 in [10].
Lemma 2.1 (Imai, Kato, Takamura and Wakasa [10]). Let $u_L$ be the one in (2.2). Then, there exist positive constants $\tilde{C}_0 = \tilde{C}_0(k)$ and $C_0 = C_0(\|f\|_{W^{3,1}(\mathbb{R}^2)}, \|g\|_{W^{2,1}(\mathbb{R}^2)}, k)$ such that $u_L$ satisfies

$$\sum_{|\alpha| \leq 1} |\nabla^\alpha_x u_L(x, t)| \leq \int_{\mathbb{R}^2} \left\{ f(x) + g(x) \right\} dx \cdot \frac{\tilde{C}_0}{(t + |x| + 2k)^{1/2}(t - |x| + 2k)^{1/2}}$$

in $\mathbb{R}^2 \times [0, \infty)$.

Next, we prepare the following decay estimate which will be employed in the proof of Theorem 2.3 and Theorem 2.4.

Lemma 2.2 Let $u_L$ be the one in (2.2). For $t - |x| \geq 2k$ and $t \geq 4k$, there exists a positive constant $C = C(f, g)$ such that

$$|u_L(x, t) - \frac{1}{2\pi} \int_{\mathbb{R}^2} \left\{ f(x) + g(x) \right\} dx| \leq \frac{Ck}{(t + |x|)^{1/2}(t - |x|)^{3/2}}, \quad (2.3)$$

moreover that

$$\left| u_L(x, t) + \frac{t}{2\pi} \int_{\mathbb{R}^2} f(x) dx \right| \leq \frac{Ck}{(t + |x|)^{1/2}(t - |x|)^{3/2}} \quad (2.4)$$

provided $f(x) + g(x) \equiv 0$.

Proof. First we shall prove (2.4). Denote $r := |x|$. For $t - r \geq 2k$ and $t \geq 4k$, we shall split the domain into the interior domain $t \geq 2r$ and the exterior domain $t \leq 2r$. We set

$$D_{\text{int}} := \{(x, t) \in \mathbb{R}^2 \times [0, \infty) : t \geq 2r, \ t \geq 4k\},$$

$$D_{\text{ext}} := \{(x, t) \in \mathbb{R}^2 \times [0, \infty) : r + 2k \leq t \leq 2r\}.$$

First, we prove (2.4) in $D_{\text{int}}$. Since $f(x) + g(x) \equiv 0$, (2.1) and

$$|x - y| \leq r + |y| \leq \frac{r}{2} + k \leq t \quad \text{for} \ (x, t) \in D_{\text{int}} \ \text{and} \ |y| \leq k,$$
we can rewrite $u_L(x, t)$ in (2.2) as

$$u_L(x, t) = \frac{\partial}{\partial t} \left\{ \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{f(y)}{\sqrt{t^2 - |x - y|^2}} dy \right\} = -\frac{t}{2\pi} \int_{\mathbb{R}^2} \frac{f(y)}{(t^2 - |x - y|^2)^{3/2}} dy.$$  

This expression gives us

$$\left| \frac{2\pi}{t} u_L(x, t) + \frac{1}{(t^2 - r^2)^{3/2}} \int_{\mathbb{R}^2} f(y) dy \right| \leq \frac{1}{(t^2 - r^2)^{3/2}} \int_{\mathbb{R}^2} \left| h_1(x, y, t) \right| f(y) dy,$$

where

$$h_1(x, y, t) := (t^2 - r^2)^{3/2} - (t^2 - |x - y|^2)^{3/2}.$$  

Using the Taylor expansion with respect to $y$ at the origin, we get

$$h_1(x, y, t) = 3\left( t^2 - |x - \theta y|^2 \right)^{1/2} \left\{ -< x, y > + \theta |y|^2 \right\}$$  

with $0 < \theta < 1$. For $(x, t) \in D_{int}$ and $|y| \leq k$, we obtain

$$\left( t^2 - |x - \theta y|^2 \right)^{1/2} \leq t + r + |y| \leq \frac{3}{2} t + k$$  

and

$$\left| -< x, y > + \theta |y|^2 \right| \leq \left( \frac{t}{2} + k \right) k.$$  

From (2.6), (2.7) and (2.8), it follows that

$$|h_1(x, y, t)| \leq 3 \left( \frac{3}{2} t + k \right) \left( \frac{t}{2} + k \right) k \leq Ckt^2.$$  

Therefore, combining (2.5), (2.9) and

$$t + |x - y| \geq t - |x - y| \geq t - r - \frac{t}{4} \geq \frac{t - r}{2},$$

we have

$$\left| \frac{2\pi}{t} u_L + \frac{1}{(t^2 - r^2)^{3/2}} \int_{\mathbb{R}^2} f(y) dy \right| \leq \frac{Ckt^2}{(t + r)^{3/2}(t - r)^{9/2}}.$$  

Since $3(t - r) \geq t + r$ holds in $D_{int}$, we obtain (2.4) in $D_{int}$. 

9
Next, we prove (2.4) in $D_{ext}$. Here, we employ the following different representation formula from (2.2) which is established by (6.24) in Hörmander [7]:

$$u_L(x, t) = \frac{\partial}{\partial t} \left( \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{I(f)(s, \omega, z)}{\sqrt{s - \rho + \rho^2 z/2}} ds \right), \quad (2.10)$$

where $\omega = x/r \in S^1$, $\rho = r - t$, $z = 1/r$ and

$$I(f)(s, \omega, z) := \int_{s = <\omega, y> - |y|^2 z/2} f(y) dS_y.$$  

For $(x, t) \in D_{ext}$ and $|y| \leq k$, we have

$$|<\omega, y> - |y|^2 z/2| \leq |y| + |y|^2 z \leq \frac{5k}{4}. \quad (2.11)$$

Since

$$\rho - \frac{\rho^2 z}{2} = -\frac{(t + r)(t - r)}{2r} \leq -2k, \quad 1 + \rho z = -\frac{t}{r} \quad \text{and} \quad \frac{\partial}{\partial t} = -\frac{\partial}{\partial \rho},$$

it follows from (2.10) and (2.11) that

$$u_L = \frac{\partial}{\partial t} \left( \frac{1}{2\sqrt{2\pi}} \int_{-5k/4}^{5k/4} \frac{I(f)(s, \omega, z)}{\sqrt{s - \rho + \rho^2 z/2}} ds \right)$$

$$= -\frac{t}{4\sqrt{2\pi r^{3/2}}} \int_{-5k/4}^{5k/4} \frac{I(f)(s, \omega, z)}{(s - \rho + \rho^2 z/2)^{3/2}} ds.$$  

Hence, we obtain

$$\left| \frac{4\sqrt{2\pi r^{3/2}}}{t} u_L + \left\{ \frac{2r}{(t + r)(t - r)} \right\}^{3/2} \int_{\mathbb{R}^2} f(y) dy \right|$$

$$\leq \left\{ \frac{2r}{(t + r)(t - r)} \right\}^{3/2} \int_{-5k/4}^{5k/4} \frac{|h_2(\rho, s, z)| I(\|f\|)(s, \omega, z)}{(s - \rho + \rho^2 z/2)^{3/2}} ds, \quad (2.12)$$

where

$$h_2(\rho, s, z) := \left( -\rho + \frac{\rho^2 z}{2} \right)^{3/2} - \left( s - \rho + \frac{\rho^2 z}{2} \right)^{3/2}. \quad (2.13)$$
Making use of the Taylor expansion with respect to $s$ at the origin, we have from (2.13)

$$h_2(\rho, s, z) = -\frac{3}{2} \left( \theta s - \rho + \frac{\rho^2 z}{2} \right)^{1/2} s \quad \text{with } 0 < \theta < 1. \quad (2.14)$$

Since $\rho = r - t$ and $z = 1/r$, for $|s| \leq 5k/4$, we obtain

$$\left| \theta s - \rho + \frac{\rho^2 z}{2} \right|^{1/2} |s| \leq \left\{ \frac{5k}{4} + \frac{(t + r)(t - r)}{2r} \right\}^{1/2} \cdot \frac{5k}{4} \leq Ck(t + r)^{1/2}(t - r)^{1/2} r^{1/2}$$

and

$$s - \rho + \frac{\rho^2 z}{2} \geq \frac{5k}{4} + t - r \geq \frac{3}{8}(t - r). \quad (2.15)$$

Hence, for $(x, t) \in D_{\text{ext}}$, it follows from (2.12), (2.14), (2.15) and (2.16) that

$$\left| \frac{4\sqrt{2}\pi r^{3/2}}{t} u_L + \left\{ \frac{2r}{(t + r)(t - r)} \right\}^{3/2} \int_{\mathbb{R}^2} f(y)dy \right| \leq \frac{Ck}{(t + r)(t - r)^{5/2}} \int_{\mathbb{R}^2} |f(y)|dy.$$

Therefore, we obtain (2.4) in $D_{\text{ext}}$.

Finally, we show (2.3). It follows from the proof of Lemma 2.1 in [10] that

$$R(f + g|x, t) - \int_{\mathbb{R}^2} \left\{ f(x) + g(x) \right\} dx \leq \frac{Ck}{(t + r)^{1/2}(t - r)^{3/2}} \quad (2.17)$$

for $t - r \geq 2k$ and $t \geq 4k$, where $R(f + g|x, t)$ is defined in (2.2). From (2.2), (2.4) and $t - r \geq 2k$, we have

$$\left| \frac{\partial}{\partial t} R(f|x, t) \right| \leq \frac{Ck}{(t + r)^{1/2}(t - r)^{3/2}}. \quad (2.18)$$

From (2.2), (2.17) and (2.18), we obtain (2.3). This completes the proof. □

In what follows, we consider the following integral equation:

$$u(x, t) = u^0(x, t) + L(F)(x, t) \quad \text{for } (x, t) \in \mathbb{R}^2 \times [0, \infty), \quad (2.19)$$
where we set
\[ u^0 := \varepsilon u_L \]  
(2.20)

and
\[ L(F)(x, t) := \frac{1}{2\pi} \int_0^t \frac{t - \tau}{(1 + \tau)^{p-1}} d\tau \int_{|\xi| \leq 1} \frac{F(x + (t - \tau)\xi, \tau)}{\sqrt{1 - |\xi|^2}} d\xi \]  
(2.21)

for \( F \in C(\mathbb{R}^2 \times [0, \infty)) \). We note that (2.21) solves
\[
\begin{cases}
  u_{tt} - \Delta u = (1 + t)^{-\frac{p-1}{2}} F \
  u(x, 0) = 0, \quad u_t(x, 0) = 0
\end{cases}
\]
for \( F \in C^2(\mathbb{R}^2 \times [0, \infty)) \). Then, the following lemma is one of the basic tools.

**Lemma 2.3 (Agemi and Takamura [2])** Let \( L \) be the linear integral operator defined by (2.21) and \( \Psi = \Psi(|x|, t) \in C(\mathbb{R}^2 \times [0, \infty)) \). Then we have
\[ L(\Psi)(x, t) = L_1(\Psi)(r, t) + L_2(\Psi)(r, t), \quad r = |x|, \]
where \( L_j(\Psi) \) \( (j = 1, 2) \) are defined by
\[
L_1(\Psi)(r, t) := \frac{2}{\pi} \int_0^t (1 + \tau)^{-\frac{p-1}{2}} d\tau \int_{|r-r-\tau|}^{t-r-r} \lambda \Psi(\lambda, \tau) d\lambda \int_{|r-\tau|}^{t-\tau} \frac{\rho h(\lambda, \rho; r)}{(t - \tau)^2 - \rho^2} d\rho,
\]
\[
L_2(\Psi)(r, t) := \frac{2}{\pi} \int_0^{(t-r)^+} (1 + \tau)^{-\frac{p-1}{2}} d\tau \int_0^{t-r-r} \lambda \Psi(\lambda, \tau) d\lambda \int_{|r-\tau|}^{\lambda+r} \frac{\rho h(\lambda, \rho; r)}{(t - \tau)^2 - \rho^2} d\rho,
\]
where \( a_+ := \max\{a, 0\} \) and
\[
h(\lambda, \rho; r) := \left\{ (\rho^2 - (\lambda - r)^2)((\lambda + r)^2 - \rho^2) \right\}^{-\frac{1}{2}}.
\]
Moreover, the following estimates hold in \([0, \infty)^2\):
\[
|L_1(\Psi)(r, t)| \leq \frac{1}{\sqrt{2}} \int_0^t (1 + \tau)^{-\frac{p-1}{2}} d\tau \int_{|r-t+\tau|}^{t+r-t} \frac{\lambda |\Psi(\lambda, \tau)| d\lambda}{(\sqrt{r} \text{ or } \sqrt{\lambda}) \sqrt{\lambda} + \lambda - t + r},
\]
\[
|L_2(\Psi)(r, t)| \leq \int_0^{(t-r)^+} (1 + \tau)^{-\frac{p-1}{2}} d\tau \\
\times \int_0^{t-r-r} \frac{\lambda |\Psi(\lambda, \tau)| d\lambda}{(\sqrt{2r} \text{ or } \sqrt{t-r+\lambda-\tau}) \sqrt{t-r-\tau-\lambda}}.
\]

In order to construct our solution in the weighted \( L^\infty \) space, we define the following weighted functions:

\[
w_1(r, t) := \tau_+ (r, t)^{1/2} \tau_- (r, t)^{1/2}, \tag{2.22}
\]

\[
w_2(r, t) := \tau_+ (r, t)^{p_1} \tau_- (r, t)^{p_2} \left( \log 2 \frac{\tau_+ (r, t)}{\tau_- (r, t)} \right)^{-p_3} (\log \tau_- (r, t))^{-p_4}, \tag{2.23}
\]

\[
w_3(r, t) := \tau_+ (r, t)^{1/2} \tau_- (r, t)^{3/2}, \tag{2.24}
\]

where we set

\[
\tau_+(r, t) := \frac{t + r + 2k}{k}, \quad \tau_-(r, t) := \frac{t - r + 2k}{k} \tag{2.25}
\]

and

\[
p_1 := \min \left\{ \frac{3p - 4}{2}, \frac{1}{2} \right\}, \quad p_2 := \max \left\{ 0, \frac{3p - 5}{2} \right\}, \quad p_3 := 0 \quad (p \neq \frac{5}{3}), \quad p_4 := 1 \quad (p = \frac{5}{3}), \quad p_4 := \begin{cases} 0 & (1 < p < 2), \\ 1 & (p = 2). \end{cases} \tag{2.26}
\]

We remark that \( w_2 \) can be described as

\[
w_2(r, t)^{-1} = \begin{cases} \tau_+ (r, t)^{4-3p}/2 & (1 < p < 5/3), \\ \tau_+ (r, t)^{-1/2} \log \left( 2 \frac{\tau_+ (r, t)}{\tau_- (r, t)} \right) & (p = 5/3), \\ \tau_+ (r, t)^{-1/2} \tau_- (r, t)^{(5-3p)/2} & (5/3 < p < 2), \\ \tau_+ (r, t)^{-1/2} \tau_- (r, t)^{-1/2} \log \tau_- (r, t) & (p = 2). \end{cases} \tag{2.27}
\]

For these weighted functions, we denote the weighted \( L^\infty \) norms of \( V \) by

\[
\|V\|_i := \sup_{(x,t) \in \mathbb{R}^2 \times [0,T)} \{w_i(|x|, t)|V(x, t)|\}
\]

where \( i = 1, 2, 3 \).

Finally, we shall introduce some useful representations for \( L \). It is trivial that \( 1 + \tau \geq (2k + \tau)/(2k) \) is valid for \( \tau \geq 0 \) and \( k \geq 1 \). Setting \( \tau = (\alpha + \beta)/2 \geq 0 \) with \( \beta \geq -k \), we have

\[
1 + \tau \geq \frac{\alpha + 2k}{4k}. \tag{2.28}
\]

Changing the variables by

\[
\alpha = \tau + \lambda, \quad \beta = \tau - \lambda \tag{2.29}
\]
and extending the domain of \((\alpha, \beta)\)-integration, we obtain from Lemma 2.3 and (2.28)

\[
L_1(\Psi)(r, t) \leq \frac{Ck}{\sqrt{r}} \int_{-k}^{t-r} d\beta \int_{t-r}^{t+r} \frac{\{(\alpha + 2k)/k\}^{2-2p}|\Psi^*(\alpha, \beta)|}{\sqrt{\alpha - (t - r)}} d\alpha
\]  

and

\[
L_1(\Psi)(r, t) \leq C\sqrt{k} \int_{-k}^{t-r} d\beta \int_{t-r}^{t+r} \frac{\{(\alpha + 2k)/k\}^{(3-2p)/2}|\Psi^*(\alpha, \beta)|}{\sqrt{\alpha - (t - r)}} d\alpha,
\]  

where \(\Psi^*(\alpha, \beta) := \Psi((\alpha - \beta)/2, (\alpha + \beta)/2)\).

Similarly, we get

\[
L_2(\Psi)(r, t) \leq \frac{Ck}{\sqrt{r}} \int_{-k}^{t-r} d\beta \int_{t-r}^{t+r} \frac{\{(\alpha + 2k)/k\}^{2-2p}|\Psi^*(\alpha, \beta)|}{t - r - \alpha} d\alpha
\]

and

\[
L_2(\Psi)(r, t) \leq Ck \int_{-k}^{t-r} d\beta \int_{t-r}^{t+r} \frac{\{(\alpha + 2k)/k\}^{2-2p}|\Psi^*(\alpha, \beta)|}{\sqrt{t - r - \alpha}\sqrt{t - r - \beta}} d\alpha.
\]

### 3 Proof of Theorem 2.1 and Theorem 2.2

In this section, we prove Theorem 2.1 and Theorem 2.2. The proof is based on the classical iteration method in John [11]. Lemma 3.3 will be used to prove Theorem 2.1, whereas we prove Theorem 2.2 by using Lemma 3.4. First, we prepare the elementary inequalities in Lemma 3.1 and Lemma 3.2.

**Lemma 3.1** Let \(a_1 \in \mathbb{R}\) and \(k \geq 1\). For \(0 \leq r \leq t + k\), it holds

\[
\int_{t-r}^{t+r} \frac{\{(\alpha + 2k)/k\}^{a_1}}{\sqrt{\alpha - (t - r)}} d\alpha \leq C\sqrt{k} \times \begin{cases} 
\tau_+(r, t)^{a_1 + 1/2} & (a_1 > -1/2), \\
\log \left(\frac{2\tau_+(r, t)}{\tau_-(r, t)}\right) & (a_1 = -1/2), \\
\tau_-(r, t)^{a_1 + 1/2} & (a_1 < -1/2),
\end{cases}
\]

where \(\tau_+(r, t)\) and \(\tau_-(r, t)\) are defined in (2.25).
Proof. For $0 \leq r \leq t + k$, the integration by parts yields

\[
\int_{t-r}^{t+r} \frac{\{(\alpha + 2k)/k\}^{a_1}}{\sqrt{\alpha - (t - r)}} \, d\alpha \\
\leq 2\sqrt{2r} \left( \frac{t + r + 2k}{k} \right)^{a_1} + \frac{2|a_1|}{\sqrt{k}} \int_{t-r}^{t+r} \left( \frac{\alpha + 2k}{k} \right)^{a_1-1/2} \, d\alpha \\
\leq 2\sqrt{2k\tau_+(r, t)^{a_1+1/2}} + 2|a_1| \sqrt{k} \times \left\{ \begin{array}{ll}
\frac{1}{a_1 + 1/2} \left( \frac{t + r + 2k}{k} \right)^{a_1+1/2} & (a_1 > -1/2), \\
\log \left( \frac{t + r + 2k}{t - r + 2k} \right) & (a_1 = -1/2), \\
\frac{1}{a_1 + 1/2} \left( \frac{t - r + 2k}{k} \right)^{a_1+1/2} & (a_1 < -1/2).
\end{array} \right.
\]

This completes the proof.

Lemma 3.2 Let $a_1 \in \mathbb{R}$ and $k \geq 1$. For $0 \leq r \leq t + k$, it holds

\[
\int_{-k}^{t-r} \frac{\{(\alpha + 2k)/k\}^{a_1}}{\sqrt{t - r - \alpha}} \, d\alpha \leq C\sqrt{k} \times \left\{ \begin{array}{ll}
\tau_-(r, t)^{a_1+1/2} & (a_1 > -1), \\
\tau_-(r, t)^{-1/2} \log \tau_-(r, t) & (a_1 = -1), \\
\tau_-(r, t)^{-1/2} & (a_1 < -1),
\end{array} \right.
\]

(3.1)

where $\tau_+(r, t)$ and $\tau_-(r, t)$ are defined in (2.25).

Proof. For $a_1 \geq 0$, we obtain

\[
\int_{-k}^{t-r} \frac{\{(\alpha + 2k)/k\}^{a_1}}{\sqrt{t - r - \alpha}} \, d\alpha \leq \left( \frac{t + r + 2k}{k} \right)^{a_1} \int_{-k}^{t-r} \frac{1}{\sqrt{t - r - \alpha}} \, d\alpha \\
\leq 2\sqrt{k\tau_-(r, t)^{a_1+1/2}}.
\]

Hence, we obtain (3.1) for $a_1 \geq 0$.

For $a_1 < 0$, we show (3.1). Let $-k \leq t - r \leq k$, i.e., $k \leq t - r + 2k \leq 3k$. It follows that

\[
\int_{-k}^{t-r} \frac{\{(\alpha + 2k)/k\}^{a_1}}{\sqrt{t - r - \alpha}} \, d\alpha \leq \int_{-k}^{t-r} \frac{1}{\sqrt{t - r - \alpha}} \, d\alpha \\
\leq 3^{-a_1} \sqrt{k\tau_-(r, t)^{a_1+1/2}}.
\]

We get (3.1) for $a_1 < 0$ and $-k \leq t - r \leq k$. 

15
Let $t - r \geq k$ which implies $t - r \geq (t - r + 2k)/4$. Then, breaking the integral up into two pieces, we get
\[
\int_{t-r}^{t-r} \frac{(\alpha + 2k)/k}{\sqrt{t-r-\alpha}} d\alpha = \int_{(t-r)/2}^{(t-r)/2} \frac{(\alpha + 2k)/k}{\sqrt{t-r-\alpha}} d\alpha + \int_{t-r}^{t-r} \frac{(\alpha + 2k)/k}{\sqrt{t-r-\alpha}} d\alpha
\]
\[=: J_1 + J_2. \tag{3.2} \]
It is easy to see that
\[
J_1 \leq \sqrt{2}(t-r)^{-1/2} \int_{-k}^{(t-r)/2} \left( \frac{\alpha + 2k}{k} \right)^{a_1} d\alpha
\]
\[\leq 2\sqrt{2}k \times \begin{cases} 1/(a_1 + 1) \tau_-(r,t)^{a_1+1/2} & (a_1 > -1), \\ \tau_-(r,t)^{1/2} \log \tau_-(r,t) & (a_1 = -1), \\ 1/(a_1 + 1) \tau_-(r,t)^{1/2} & (a_1 < -1). \end{cases} \tag{3.3} \]
We obtain
\[
J_2 \leq \left( \frac{t-r}{2} + 2k \right)^{a_1} \int_{(t-r)/2}^{t-r} \frac{1}{\sqrt{t-r-\alpha}} d\alpha
\]
\[\leq 2^{-a_1} \left( \frac{t-r}{2} + 2k \right)^{a_1} \sqrt{2}(t-r)^{1/2}
\]
\[\leq 2^{-a_1+1/2} \sqrt{k} \tau_-(r,t)^{a_1+1/2}. \tag{3.4} \]
By (3.2), (3.3) and (3.4), we obtain the desired inequality in (3.1) for $a_1 < 0$ and $t - r \geq k$. This completes the proof. \qed

The following lemma contains one of the most essential estimates.

**Lemma 3.3** Let $1 < p \leq 2$ and $L$ be the linear integral operator defined by (2.21). Assume that $V \in C(R^2 \times [0, T])$ with $\text{supp } V \subset \{(x, t) \in R^2 \times [0, \infty) : |x| \leq t + k\}$. Then, there exists a positive constant $C_1$ independent of $k$ and $T$ such that
\[
\|L(|V|^p)\|_1 \leq C_1 k^2 \|V\|_1^p D_1(T), \tag{3.5}
\]
where $D_1(T)$ is defined by
\[
D_1(T) := \begin{cases} T_k^{-4p} & \text{if } 1 < p < 2, \\ (\log T_k)^2 & \text{if } p = 2 \end{cases} \tag{3.6}
\]
with $T_k := (T + 3k)/k$.  

16
Proof. In order to show the a-priori estimate (3.5), it is enough to prove

$$L_j(w_1^{-p}) \leq Ck^2 w_1^{-1}D_1(T) \quad \text{for } j = 1, 2,$$

(3.7)

where $L_j$ are defined in Lemma 2.3. By (2.22), (2.25) and (2.29), we have

$$w_1(\lambda, \tau) = \left(\frac{\alpha + 2k}{k}\right)^{1/2} \left(\frac{\beta + 2k}{k}\right)^{1/2}.$$

(3.8)

We shall prove (3.7) in the following two cases.

Case 1: $4r \geq t + r + 2k$.

First, we evaluate $L_1$. We get from (2.30) and (3.8)

$$\int_{t-r}^{t+r} \frac{((\alpha + 2k)/k)^{(4-3p)/2}}{\sqrt{\alpha - (t-r)}} d\alpha \int_{-k}^{r} \left(\frac{\beta + 2k}{k}\right)^{-p/2} d\beta.$$

(3.9)

From Lemma 3.1 and (3.11), we obtain

$$\int_{t-r}^{t+r} \frac{((\alpha + 2k)/k)^{(4-3p)/2}}{\sqrt{\alpha - (t-r)}} d\alpha \leq C \sqrt{k} \times \begin{cases} \tau_+(r, t)^{(5-3p)/2} & (1 < p < 5/3), \\ \tau_+(r, t)^{1/2} & (p = 5/3), \\ \tau_-(r, t)^{(5-3p)/2} & (5/3 < p \leq 2). \end{cases}$$

(3.10)

Here, for the inequality with $p = 5/3$, we used the following fact:

$$\log s \leq \frac{s\delta}{\delta} \quad \text{for } s \geq 1 \text{ and } \delta > 0.$$

(3.11)

The $\beta$-integral is estimated by

$$\int_{-k}^{r} \left(\frac{\beta + 2k}{k}\right)^{-p/2} d\beta \leq Ck \times \begin{cases} \tau_-(r, t)^{(2-p)/2} & (1 < p < 2), \\ \log \tau_-(r, t) & (p = 2). \end{cases}$$

(3.12)

Therefore, it follows from (3.9), (3.10), (3.12), (2.22) and (3.6) that

$$L_1(w_1^{-p}) \leq Ck^2 \tau_+(r, t)^{-1/2} \tau_-(r, t)^{-1/2} \times \begin{cases} \tau_+(r, t)^{4-2p} & (1 < p < 2), \\ \log \tau_-(r, t) & (p = 2) \end{cases} \leq Ck^2 w_1(r, t)^{-1}D_1(T).$$

Here, we have used that

$$\tau_+(r, t) \leq \frac{2t + 3k}{k} \leq 2T_k \quad \text{and} \quad T_k \geq 3.$$
Thus, we have proved (3.7) with \( j = 1 \) in Case 1.

Next, if \( t > r \), we investigate the integral \( L_2 \). From (2.32) and (3.8), we get

\[
L_2(w_1^{-p})(r, t) 
\leq \frac{Ck}{\sqrt{r}} \int_{-k}^{t-r} \frac{((\alpha + 2k)/k)^{(4-3p)/2}}{\sqrt{t-r-\alpha}} \, d\alpha \int_{-k}^{t-r} \left( \frac{\beta + 2k}{k} \right)^{-p/2} \, d\beta. \tag{3.13}
\]

From Lemma 3.2, we obtain

\[
\int_{-k}^{t-r} \frac{((\alpha + 2k)/k)^{(4-3p)/2}}{\sqrt{t-r-\alpha}} \, d\alpha 
\leq C\sqrt{k} \times \begin{cases} 
\tau_-(r, t)^{(5-3p)/2} & (1 < p < 2), \\
\tau_-(r, t)^{-1/2} \log \tau_-(r, t) & (p = 2). 
\end{cases} \tag{3.14}
\]

From (3.13), (3.14), (3.12), (2.22) and (3.6), it follows that

\[
L_2(w_1^{-p}) \leq Ck^2 \tau_+(r, t)^{-1/2} \times \begin{cases} 
\tau_-(r, t)^{(7-4p)/2} & (1 < p < 2), \\
\tau_-(r, t)^{-1/2} \{\log \tau_-(r, t)\}^2 & (p = 2) 
\end{cases} 
\leq Ck^2 w_1(r, t)^{-1} D_1(T).
\]

Hence, we obtain (3.7) with \( j = 2 \) in Case 1.

Case 2: \( 4r \leq t + r + 2k \), i.e., \( t + r + 2k \leq 2(t - r + 2k) \).

First, we estimate \( L_1 \). We have from (2.31) and (3.8)

\[
L_1(w_1^{-p}) 
\leq C\sqrt{k} \int_{t-r}^{t+r} \frac{((\alpha + 2k)/k)^{(3-1-p)/2}}{\sqrt{\alpha - (t-r)}} \, d\alpha \int_{-k}^{t-r} \left( \frac{\beta + 2k}{k} \right)^{-p/2} \, d\beta. \tag{3.15}
\]

We obtain

\[
\int_{t-r}^{t+r} \frac{((\alpha + 2k)/k)^{(3-1-p)/2}}{\sqrt{\alpha - (t-r)}} \, d\alpha 
\leq \left( \frac{t-r+2k}{k} \right)^{(3-1-p)/2} \int_{t-r}^{t+r} \frac{1}{\sqrt{\alpha - (t-r)}} \, d\alpha 
\leq 2\sqrt{2r\tau_-(r, t)^{(3-1-p)/2}} 
\leq C\sqrt{k}\tau_+(r, t)^{(4-3p)/2}. \tag{3.16}
\]
Therefore, it follows from (3.15), (3.16), (3.12), (2.22) and (3.6)

\[ L_1(w_1^{-p}) \leq C k^2 \tau_+(r, t)^{-1/2} \tau_-(r, t)^{-1/2} \times \begin{cases} \tau_+(r, t)^{4-2p} & (1 < p < 2), \\ \log \tau_-(r, t) & (p = 2) \end{cases} \]

Thus, we obtain (3.7) with \( j = 1 \) in Case 2.

Next, if \( t > r \), we evaluate \( L_2 \). From (2.33) and (3.8), we obtain

\[ L_2(w_1^{-p})(r, t) \leq C k \int_{-k}^{t-r} \frac{\{\alpha + 2k\}/k^{(4-3p)/2}}{\sqrt{t-r-\alpha}} d\alpha \int_{-k}^{t-r} \frac{\{\beta + 2k\}/k^{-p/2}}{\sqrt{t-r-\beta}} d\beta. \tag{3.17} \]

From Lemma 3.2, we have

\[ \int_{-k}^{t-r} \frac{\{\beta + 2k\}/k^{-p/2}}{\sqrt{t-r-\beta}} d\beta \leq C \sqrt{k} \times \begin{cases} \tau_-(r, t)^{-(p-1)/2} & (1 < p < 2), \\ \tau_-(r, t)^{-1/2} \log \tau_-(r, t) & (p = 2). \end{cases} \tag{3.18} \]

From (3.17), (3.14), (3.18), (2.22) and (3.6), we have (3.7) with \( j = 2 \) in Case 2. Therefore, the proof of Lemma 3.3 is completed. □

Finally, we state an a-priori estimate of mixed type.

Lemma 3.4 Let \( 1 < p \leq 2 \) and \( L \) be the linear integral operator defined by (2.21). Assume that \( V, V_0 \in C(R^2 \times [0, T)) \) with \( \text{supp} (V, V_0) \subset \{ (x, t) \in R^2 \times [0, \infty) : |x| \leq t + k \} \). Then, there exists a positive constant \( C_2 \) independent of \( k \) and \( T \) such that

\[ \| L(|V_0|^{\nu-\nu}|V|^\nu) \|_2 \leq C_2 k^2 \| V_0 \|_3^\nu \| V \|_2^\nu D_{2,\nu}(T), \tag{3.19} \]

where \( \nu = 0, p - 1, 1, p \) and

\[ D_{2,\nu}(T) := \begin{cases} T_k^{(5-3p)/2 + \delta p_3} & (\nu = 0, p - 1 \text{ and } 1 < p \leq 5/3), \\ T_k^{-3p + \delta p_3} & (\nu = 1 \text{ and } 1 < p \leq 5/3), \\ 1 & (\nu \leq 1 \text{ and } 5/3 < p \leq 2), \\ T_k^{(p,4)/2} & (\nu = p \text{ and } 1 < p < 2), \\ (\log T_k)^3 & (\nu = p \text{ and } p = 2), \end{cases} \tag{3.20} \]

where \( \delta \) stands for any positive constant and \( p_3 \) is defined in (2.26).
**Proof.** In order to show the a-priori estimate (3.19), it is enough to prove

\[ L_j(w_3^{(p-\nu)}w_2^{-\nu}) \leq C k^2 w_2^{-1} D_{2,\nu}(T) \quad \text{for } j = 1, 2, \quad (3.21) \]

where \( L_j \) are defined in Lemma 2.3. For \( \delta > 0 \), from (2.23), (2.24), (2.25), (2.29) and (3.11), we have

\[
\begin{align*}
    w_3(\lambda, \tau)^{-(p-\nu)}w_2(\lambda, \tau)^{-\nu} &\leq C \left( \frac{\alpha + 2k}{k} \right)^{-(p-\nu)/2-\nu p_1+\delta \nu p_3} \left( \frac{\beta + 2k}{k} \right)^{-3(p-\nu)/2-\nu p_2-\delta \nu p_3} \\
    &\times \left( \log \frac{\beta + 2k}{k} \right)^{\nu p_4}.
\end{align*}
\]

(3.22)

We shall prove (3.21) in the following two cases.

Case 1: \( 4r \geq t + r + 2k \).

First, we evaluate \( L_1 \). From (2.30) and (3.22), we get

\[
\begin{align*}
    L_1(w_3^{(p-\nu)}w_2^{-\nu}) &\leq C k \int_{t-r}^{t+r} \left( \frac{\alpha + 2k}{k} \right)^{p_5} \int_{-k}^{t-r} \left( \frac{\beta + 2k}{k} \right)^{p_6} \left( \log \frac{\beta + 2k}{k} \right)^{\nu p_4} d\beta, \quad (3.23)
\end{align*}
\]

where

\[
\begin{align*}
p_5 &:= \frac{4 - 3p}{2} + \nu \left( \frac{1}{2} - p_1 \right) + \delta \nu p_3, \\
p_6 &:= -\frac{3(p-\nu)}{2} - \nu p_2 - \delta \nu p_3.
\end{align*}
\]

(3.24)

(3.25)

We have from (3.24) and (2.26)

\[
p_5 = \begin{cases} 
    \frac{4 - 3p}{2} + \nu \left( \frac{5 - 3p}{2} \right) & (1 < p < 5/3), \\
    -\frac{1}{2} + \delta \nu & (p = 5/3), \\
    \frac{4 - 3p}{2} & (5/3 < p \leq 2).
\end{cases}
\]

(3.26)
From Lemma 3.1 and (3.26), we obtain

\[
\int_{t-r}^{t+r} \frac{((\alpha + 2k)/k)^{p_5}}{\sqrt{\alpha - (t - r)}} d\alpha \\
\leq C\sqrt{k} \times \begin{cases}
\tau_+(r, t)^{(1+\nu)(5-3p)/2} & (1 < p < 5/3), \\
\log \left(\frac{\tau_+(r, t)}{\tau_-(r, t)}\right) & (\nu = 0 \text{ and } p = 5/3), \\
\tau_+(r, t)^{\delta \nu} & (\nu > 0 \text{ and } p = 5/3), \\
\tau_-(r, t)^{(5-3p)/2} & (5/3 < p \leq 2).
\end{cases}
\tag{3.27}
\]

We get from (2.26) and (3.25)

\[
p_6 = \begin{cases}
-3(p - \nu)/2 - \delta \nu p_3 & (1 < p \leq 5/3), \\
-3(p - \nu)/2 - \nu(3p - 5)/2 & (5/3 < p \leq 2).
\end{cases}
\tag{3.28}
\]

From (3.28), (2.26) and (1.6), the \(\beta\)-integral is estimated by

\[
\int_{-\delta}^{\delta} \left(\frac{\beta + 2k}{k}\right)^{p_6} \left(\log \frac{\beta + 2k}{k}\right)^{\nu p_4} d\beta \\
\leq Ck \times \begin{cases}
1 & (\nu = 0 \text{ or } \nu = p - 1), \\
\tau_-(r, t)^{(5-3p)/2+p_2} & (\nu = 1), \\
\tau_-(r, t)^{1-\delta \nu p_3} & (\nu = p \text{ and } 1 < p \leq 5/3), \\
(\log \tau_-(r, t))^\delta & (\nu = p \text{ and } p = 2).
\end{cases}
\tag{3.29}
\]

Here, for the inequality with \(\nu = p\) and \(p = 5/3\), we took \(0 < \delta \nu < 1\). It follows from (3.23), (3.27), (3.29), (2.26), (2.27) and (3.20) that

\[
L_1(w_3^{(p-\nu)}w_2^{-\nu})(r, t)^{-1} \leq Ck^2w_2(r, t)^{-1} \\
\times \begin{cases}
T_k^{\nu(5-3p)/2+\delta \nu p_3} & (\nu = 0, p - 1 \text{ and } 1 < p \leq 5/3), \\
T_k^{5-3p+\delta \nu p_3} & (\nu = 1 \text{ and } 1 < p \leq 5/3), \\
1 & (\nu < 1 \text{ and } 5/3 < p \leq 2), \\
T_k^{\gamma(p, 4)/2} & (\nu = p \text{ and } 1 < p < 2), \\
(\log T_k)^2 & (\nu = p \text{ and } p = 2)
\end{cases}
\leq Ck^2w_2(r, t)^{-1} D_{2, \nu}(T).
\]

We obtain (3.21) with \(j = 1\) in Case 1.

Next, if \(t > r\), we investigate \(L_2\). From (3.32) and (3.32), we get

\[
L_2(w_3^{(p-\nu)}w_2^{-\nu})(r, t) \\
\leq Ck \int_{t-r}^{t+r} \frac{((\alpha + 2k)/k)^{p_5}}{\sqrt{\alpha - (t - r)}} d\alpha \\
\times \int_{-\delta}^{\delta} \left(\frac{\beta + 2k}{k}\right)^{p_6} \left(\log \frac{\beta + 2k}{k}\right)^{\nu p_4} d\beta.
\tag{3.30}
\]
From Lemma 3.2 and (3.26), we have

$$
\int_{-k}^{t-r} \frac{(\alpha + 2k)/k}{\sqrt{t - r - \alpha}} d\alpha \\
\leq C\sqrt{k} \times \begin{cases} \\
\tau_-(r, t)^{(\nu+1)(5-3p)/2+\delta\nu p_3} & (1 < p \leq 5/3), \\
\tau_-(r, t)^{(5-3p)/2} & (5/3 < p < 2), \\
\tau_-(r, t)^{-1/2} \log \tau_-(r, t) & (p = 2).
\end{cases}
$$

(3.31)

Making use of (3.30), (3.31), (3.29), (1.6), (2.26), (2.27) and (3.20), we get

$$
L_2(w_3^{(p-\nu)}w_2^{-\nu})(r, t) \\
\leq Ck^2\tau_+(r, t)^{-1/2} \\
\times \begin{cases} \\
\tau_-(r, t)^{(\nu+1)(5-3p)/2+\delta\nu p_3} & (\nu = 0, p - 1 \text{ and } 1 \leq p \leq 5/3), \\
\tau_-(r, t)^{3(5-3p)/2+\delta\nu p_3} & (\nu = 1 \text{ and } 1 \leq p \leq 5/3), \\
\tau_-(r, t)^{(5-3p)/2} & (\nu \leq 1 \text{ and } 5/3 < p < 2), \\
\tau_-(r, t)^{-1/2} \log \tau_-(r, t) & (\nu \leq 1 \text{ and } p = 2), \\
\tau_-(r, t)^{(5-3p)/2+\gamma(p,4)/2} & (\nu = p \text{ and } 1 < p < 2), \\
\tau_-(r, t)^{-1/2} (\log \tau_-(r, t))^{3/2} & (\nu = p \text{ and } p = 2)
\end{cases}
$$

(3.32)

Hence, we obtain (3.21) with $j = 2$ in Case 1.

Case 2: $4r \leq t + r + 2k$, i.e., $t + r + 2k < 2(t - r + 2k)$.

First, we evaluate $L_1$. From (3.31) and (3.32), we have

$$
L_1(w_3^{(p-\nu)}w_2^{-\nu}) \\
\leq C\sqrt{k} \int_{t-r}^{t+r} \frac{(\alpha + 2k)/k}{\sqrt{\alpha - (t - r)}} d\alpha \\
\times \int_{-k}^{t-r} \left( \frac{\beta + 2k}{k} \right)^{p_0} \left( \log \frac{\beta + 2k}{k} \right)^{\nu p_4} d\beta.
$$

(3.33)

From (3.24), we obtain

$$
\int_{t-r}^{t+r} \frac{(\alpha + 2k)/k}{\sqrt{\alpha - (t - r)}} d\alpha \\
\leq \tau_-(r, t)^{3(1-p)/2} \tau_+(r, t)^{\nu(1/2-p_1)+\delta\nu p_3} \int_{t-r}^{t+r} \frac{1}{\sqrt{\alpha - (t - r)}} d\alpha \\
\leq C\sqrt{k}\tau_+(r, t)^{p_0}.
$$
Therefore, it follows from (3.32), (3.33), (3.26), (3.29), (2.27) and (3.20) that
\[
L_1(w_3^{-(p-\nu)} w_2^{-\nu}) \leq CK^2 r_+^2 (r, t)^{(4-3p)/2}
\]
\[
\times \begin{cases}  
\tau_+(r, t)^{5-3p+\delta p_3} & (\nu = 0, p - 1 \text{ and } 1 < p \leq 5/3), \\
\tau_+(r, t)^{5-3p+\delta p_3} & (\nu = 1 \text{ and } 1 < p \leq 5/3), \\
1 & (\nu \leq 1 \text{ and } 5/3 < p \leq 2), \\
\tau_-(r, t)^{(p,4)/2} & (\nu = p \text{ and } 1 < p < 2), \\
(\log \tau_-(r, t))^3 & (\nu = p \text{ and } p = 2),
\end{cases}
\leq CK^2 w_2(r, t)^{-1} D_{2,\nu}(T).
\]

Thus, the proof of (3.21) with \( j = 1 \) in Case 2 is finished.

Next, if \( t > r \), we investigate \( L_2 \). From (2.33), (3.22), (3.24) and (3.25), we get
\[
L_2(w_3^{-(p-\nu)} w_2^{-\nu})(r, t) \leq C k \int_{-k}^{l-r} \frac{\{(\alpha + 2k)/k\}^{p_3}}{\sqrt{t-r-\alpha}} d\alpha \\
\times \int_{-k}^{l-r} \frac{\{(\beta + 2k)/k\}^{p_4}}{\sqrt{t-r-\beta}} \left( \log \frac{\beta + 2k}{k} \right)^{\nu p_4} d\beta.
\]

From Lemma 3.2 and (3.28), we have
\[
\int_{-k}^{l-r} \frac{\{(\beta + 2k)/k\}^{p_4}}{\sqrt{t-r-\beta}} \left( \log \frac{\beta + 2k}{k} \right)^{\nu p_4} d\beta \\
\leq C \sqrt{k} \times \begin{cases}  
\tau_+(r, t)^{-1/2} & (\nu = 0 \text{ or } \nu = p - 1), \\
\tau_+(r, t)^{p_1} & (\nu = 1), \\
\tau_-(r, t)^{1/2} & (\nu = p \text{ and } 1 < p < 5/3), \\
\tau_-(r, t)^{-\delta \nu + 1/2} & (\nu = p \text{ and } p = 5/3), \\
\tau_-(r, t)^{-1/2 + (p,4)} & (\nu = p \text{ and } 5/3 < p < 2), \\
\tau_-(r, t)^{-1/2 + (p,4)} \log \tau_-(r, t) & (\nu = p \text{ and } p = 2).
\end{cases}
\]

Here, for the inequality with \( \nu = p = 5/3 \), we took \( 0 < \delta \nu < 1 \). Thus, we obtain (3.21) with \( j = 2 \) in Case 2 by (3.34), (3.31), (3.35), (2.27) and (3.20). Therefore, the proof of Lemma 3.4 is completed. \hfill \Box

In the following, we prove Theorem 2.1 and Theorem 2.2. We remark that it is possible to construct a classical solution if \( p = p_S(4) = 2 \). However, its construction is almost the same as for \( C^1 \) solution. Therefore, we shall omit the proof.

**Proof of Theorem 2.1.** We define
\[
X := \left\{ u(x, t) \Bigg| \begin{array}{cc}
D_u^a u(x, t) \in C(\mathbb{R}^2 \times [0, T]), \\
\|D_u^a u\|_{1} < \infty \quad (|\alpha| \leq 1), \\
u(x, t) = 0 \quad (|x| \geq t + k)
\end{array} \right\},
\]

23
where $D_\alpha^\alpha = D_1^\alpha D_2^\alpha \ (\alpha = (\alpha_1, \alpha_2))$ and $D_k = \partial/\partial x_k \ (k = 1, 2)$. We can verify easily that $X$ is complete with respect to the norm

$$
\|u\|_X = \sum_{|\alpha| \leq 1} \|D_\alpha^\alpha u\|_1.
$$

Using the iteration method, we shall construct a solution of (1.7). We define the sequence of functions \{u_j\} by

$$
u_0 = u^0, \quad u_{j+1} = u^0 + L(|u_j|^p) \text{ for } j \geq 0,
$$
where $u^0$ is defined in (2.20). It follows from Lemma 1 in [6], p.236 that $u^0$ satisfies

$$
|D_\alpha^\alpha u^0(x, t)| \leq C(f, g)\varepsilon(t + r + 2k)^{-1/2}(t - r + 2k)^{-1/2}
$$
for $|\alpha| \leq 1$, where the positive constant $C(f, g)$ depends on $D_\alpha^\alpha g$ and $D_\beta^\beta f \ (|\beta| \leq 2)$. Hence, we find

$$
\|D_\alpha^\alpha u^0\|_1 \leq C(f, g)k^{-1}\varepsilon. \tag{3.36}
$$

As in [11], p.258, we see from Lemma 3.3 and (3.36) that if $\varepsilon$ satisfies

$$
C_1 k^2 D_1(T)\varepsilon^{p-1}\|u_L\|_1^{p-1} \leq \frac{1}{p2^p}, \tag{3.37}
$$
then \{u_j\} is a Cauchy sequence in $X$. Since $X$ is complete, there exists a function $u \in X$ such that \{${D^\alpha_\alpha u_j}$\} converges uniformly to $D^\alpha_\alpha u$ as $j \to \infty$. Clearly $u$ satisfies (2.19) with $F(x, t) = |u(x, t)|^p$. In view of (2.19) and (2.21), we note that $\partial u/\partial t$ can be expressed in terms of $D^\alpha_\alpha u \ (|\alpha| \leq 1)$. From the continuity of $D^\alpha_\alpha u$, the continuity of $\partial u/\partial t$ is also valid. Thus, from (3.36) and (3.37), Theorem 2.1 is proved by taking $\varepsilon$ is small. \hfill \Box

**Proof of Theorem 2.2.** We consider the following integral equation:

$$
U = L(|u^0 + U|^{p}), \tag{3.38}
$$

where $u^0$ is defined in (2.20). Suppose that we obtain a solution $U = U(x, t)$ of (3.38). Then, putting $u = U + u^0$, we get the solution of (2.19) with $F(x, t) = |u(x, t)|^p$, and its maximal existence time is the same as that of $U$. Thus, we have reduced the problem to the analysis of (3.38). Let $Y$ be the norm space defined by

$$
Y := \left\{ U(x, t) \left| \begin{array}{l}
D^\alpha_\alpha U(x, t) \in C(\mathbb{R}^2 \times [0, T]), \\
\|D^\alpha_\alpha U\|_2 < \infty \ (|\alpha| \leq 1), \\
U(x, t) = 0 \ (|x| \geq t + k)
\end{array} \right. \right\},
$$

24
which is equipped with the norm

$$\|U\|_Y = \sum_{|\alpha| \leq 1} \|D_\alpha^x U\|_2.$$ 

We shall construct a solution of the integral equation (3.38) in $Y$. We define the sequence of functions $\{U_j\}$ by

$$U_0 = 0, \quad U_{j+1} = L(\|u^0 + U_j\|^p) \quad (j = 0, 1, 2, \ldots). \quad (3.39)$$

From Lemma 2.1 and the condition $\int_{\mathbb{R}^2} (f + g)(x)dx = 0$, we see that there exists a positive constant $C_0$ such that

$$\|D_\alpha^x u^0\|_3 \leq C_0 \varepsilon \quad (|\alpha| \leq 1). \quad (3.40)$$

We put

$$C_3 := (2^p p^2 p)^{(p-1)} \max \{ C_2 k^2 M_0^{p-1}, (C_2 k^2 C_0^{p-1})^p, (C_2 k^2 M_0^{p-2} C_0)^{p/(p-1)} \} \quad (3.41)$$

and

$$M_0 := 2^p p^2 k^2 C_0^p C_2, \quad (3.42)$$

where $C_2$ is the constant given in Lemma 3.4. We take $\varepsilon$ and $T$ such that

$$C_3 \varepsilon^{p(p-1)} D_{2,p}(T) \leq 1. \quad (3.43)$$

Similarly to the proof of Theorem 1 in [10], we shall obtain

$$\|U_j\|_2 \leq 2M_0 \varepsilon^p \quad (3.44)$$

by induction. For $j = 0$, (3.44) holds. Assume that $\|U_j\|_2 \leq 2M_0 \varepsilon^p$ for some $j$. From (3.39), (3.19) with $\nu = 0$ and $\nu = p$, (3.40), (3.41), (3.42) and (3.44), we have

$$\|U_{j+1}\|_2 \leq 2^{p-1} \{ \|L(\|u^0|^{p})\|_2 + \|L(\|U_j\|^p)\|_2 \} \leq 2^{p-1} C_2 k^2 \{ \|u^0\|_3^p D_{2,0}(T) + \|U_j\|^p_2 D_{2,p}(T) \} \leq 2^{p-1} C_2 k^2 \{ (C_0 \varepsilon)^p + (2M_0 \varepsilon^p)^p D_{2,p}(T) \} \leq M_0 \varepsilon^p + M_0 C_3 \varepsilon^{p^2} D_{2,p}(T). \quad (3.45)$$

Thus, from (3.45), we obtain (3.44) under the conditions (3.43).

Next, we shall estimate the differences of $\{U_j\}$. From (3.39), we obtain

$$\|U_{j+1} - U_j\|_2 \leq 2^{p-1} p \|L(\|u^0|^{p-1}|U_j - U_{j-1}|)\|_2 + \|L(\|U_j|^{p-1} + |U_{j-1}|^{p-1})|U_j - U_{j-1}|\|_2 \}. \quad (3.46)$$
Then, in view of the definitions of $D_{2,1}(T)$ and $D_{2,p}(T)$ in (3.20), the estimate 
$D_{2,1}(T) \leq D_{2,p}(T)^{1/p}$ holds because of $5 - 3p + \nu p_3 \leq \gamma(p, 4)/2p$ and $T_k \geq 3$. Here, when $p = 5/3$, we take $\delta > 0$ such that $0 < \delta < 1/p$ in Lemma 3.4. Thus, from (3.19) with $\nu = 1$ and (3.40), we obtain

$$
\| L(\|u\|^{p-1}|U_j - U_{j-1}|) \|_2 \leq C_2k^2\|u\|_3^{p-1}\|U_j - U_{j-1}\|_2D_{2,1}(T)
\leq C_2k^2(\nu_0\varepsilon)^{p-1}D_{2,p}(T)^{1/p}\|U_j - U_{j-1}\|_2.
$$

(3.47)

We also get from (3.19) with $\nu = p$ and (3.44) that

$$
\| L(\|U_j\|^{p-1} + \|U_{j-1}\|^{p-1})|U_j - U_{j-1}|) \|_2 
\leq C_2k^2(\|U_j\|_2^{p-1} + \|U_{j-1}\|_2^{p-1})\|U_j - U_{j-1}\|_2D_{2,p}(T)
\leq 2C_2k^2(2M_0\varepsilon^p)\|U_j - U_{j-1}\|_2.
$$

(3.48)

Hence, we obtain from (3.46), (3.47), (3.48) and (3.41) that

$$
\|U_{j+1} - U_j\|_2 \leq \frac{1}{4}\{(C_3\varepsilon^{p-1})D_{2,p}(T))^{1/p} + C_3\varepsilon^{p-1})D_{2,p}(T))\}\|U_j - U_{j-1}\|_2
\leq \frac{1}{2}\|U_j - U_{j-1}\|_2
$$

(3.49)

provided (3.43) holds.

Similarly to the proof of (3.44) and (3.49), if we assume that (3.43) holds, then we obtain the following estimates:

$$
\|D_1U_j\|_2 \leq 2M_0\varepsilon^p,
$$

(3.50)

$$
\|D_1(U_{j+1} - U_j)\|_2 \leq C_4(j + 1)2^{-j(p-1)},
$$

(3.51)

where $C_4$ is a positive constant independent of $j$. We remark that in order to show (3.50) and (3.51), we also use the estimates (3.19) with $\nu = p - 1$ and $D_{2,p-1}(T) \leq D_{2,p}(T)^{(p-1)/(p+1)}$. For the actual proof, see the inequalities (4.15) and (4.25) in [10] which correspond the estimates (3.50) and (3.51) respectively. Then, from (3.49) and (3.51), we see that $\{U_j\}$ is a Cauchy sequence in $Y$ provided that (3.43) holds. We can verify easily that $Y$ is complete. Hence, there exists a function $U$ such that $\{U_j\}$ converges to $U$ in $Y$. Therefore, $U$ satisfies the integral equation (3.38).

Let us fix $\varepsilon_0$ as

$$
C_3\varepsilon_0^{p-1} \leq \begin{cases} 6^{-\gamma(p,4)/2} & (1 < p < 2), \\ (\log 6)^{-3} & (p = 2). \end{cases}
$$

(3.52)
For $0 < \varepsilon \leq \varepsilon_0$, if we assume that

$$C_3 \varepsilon^{p(p-1)} \leq \begin{cases} \left( \frac{2T}{k} \right)^{-\gamma(p,4)/2} & (1 < p < 2), \\ \log \left( \frac{2T}{k} \right)^{-3} & (p = 2) \end{cases}$$

(3.53)

then (3.43) holds. In fact, when $T \leq 3k$, (3.43) follows from (3.52). When $T > 3k$, (3.43) follows from (3.53). Hence, Theorem 2.2 follows immediately from (3.53). This completes the proof. \qed

4 Proof of Theorem 2.3 and Theorem 2.4

In this section, we prove Theorem 2.3 and Theorem 2.4. For the sub-critical case, we use an improved version of Kato’s lemma on ordinary differential inequality which was introduced by Takamura [18]. For the critical case, we apply the slicing iteration method which was introduced by Agemi, Kurokawa and Takamura [1]. From now on, let $u \in C^1(R^2 \times [0, T))$ be the solution of the integral equation associated with (1.7).

4.1 Proof of Theorem 2.3

We divide the proof of Theorem 2.3 into two cases, $1 < p < 2$ and $p = 2$. First, we shall handle the sub-critical case.

Proof of Theorem 2.3 with $1 < p < 2$. We shall follow the arguments in Section 4 of Takamura [18]. In order to obtain the estimates in Theorem 2.3, we shall take a look on the ordinary differential inequality for

$$F(t) := \int_{R^2} u(x,t) dx.$$

(1.7) with $\mu = 2$ and (2.1) imply that

$$F''(t) = \frac{1}{(1+t)^{p-1}} \int_{R^2} |u(x,t)|^p dx \quad \text{for} \quad t \geq 0. \quad (4.1)$$

Hence, the Hölder’s inequality and (2.1) yield that

$$F''(t) \geq \pi^{-(p-1)}(t+k)^{-3(p-1)} |F(t)|^p \quad \text{for} \quad t \geq 0. \quad (4.2)$$
Due to the assumption on the initial data in Theorem 2.3, \( f(x) \equiv 0, g(x) \geq 0 \) \((\neq 0)\), we have
\[
F(0) = 0, \quad F'(0) > 0. \tag{4.3}
\]
It follows from (4.3) in [18] that
\[
u(x, t) \geq \frac{\|g\|_{L^1(\mathbb{R}^2)}}{2\sqrt{2\pi}\sqrt{t + k\sqrt{t - |x| + k}}} \varepsilon \quad \text{for } k \leq |x| \leq t - k. \tag{4.4}
\]
From (4.1), it follows that
\[
F''(t) \geq \frac{1}{(1 + t)^{p-1}} \int_{k \leq |x| \leq t-k} |u(x, t)|^p dx \quad \text{for } t \geq 2k.
\]
Plugging (4.4) into the right-hand side of this inequality, we have that
\[
F''(t) \geq \left( \frac{\|g\|_{L^1(\mathbb{R}^2)}}{2\sqrt{2\pi}(t + k)^{3/2-1/p}} \varepsilon \right)^p \int_{k \leq |x| \leq t-k} \frac{1}{(t - |x| + k)^{p/2}} dx
\[
= \frac{2\pi \|g\|_{L^p(\mathbb{R}^2)}}{(2\sqrt{2\pi})^p(t + k)^{3p/2-1}} \varepsilon^p \int_{k}^{t-k} \frac{r}{(t - r + k)^{p/2}} dr. \tag{4.5}
\]
We evaluate the integral of the last term in (4.5). For \( t \geq 3k \), we obtain
\[
\int_{k}^{t-k} \frac{r}{(t - r + k)^{p/2}} dr \geq \frac{1}{2t^{p/2}} \{(t - k)^2 - k^2\}
\[
\geq \frac{1}{6}t^{2-p/2}. \tag{4.6}
\]
From (4.5) and (4.6), we obtain
\[
F''(t) \geq \frac{\|g\|_{L^1(\mathbb{R}^2)}^p}{3 \cdot 2^{3p-1} \pi^{p-1}} \varepsilon^p t^{3-2p} \quad \text{for } t \geq 3k.
\]
Integrating this inequality in \([3k, t]\), we get from (4.3)
\[
F'(t) > \frac{\|g\|_{L^p(\mathbb{R}^2)}^p}{3(4 - 2p)2^{3p-1} \pi^{p-1}} \varepsilon^p t^{4-2p} \quad \text{for } t \geq 4k.
\]
Hence, we obtain from (4.3)
\[
F(t) > D_1 \varepsilon^p t^{5-2p} \quad \text{for } t \geq 5k, \tag{4.7}
\]
where
\[
D_1 := \frac{\|g\|_{L^p(\mathbb{R}^2)}^p(1 - (3/4)^{4-2p})(1 - (4/5)^{5-2p})}{3(4 - 2p)(5 - 2p)2^{3p-1} \pi^{p-1}} > 0.
\]
In the sub-critical case, the following basic lemma is useful.
Lemma 4.1 (Takamura [18]) Let \( p > 1, a > 0 \) and \( q > 0 \) satisfy
\[
M := \frac{p - 1}{2}a - \frac{q}{2} + 1 > 0. \tag{4.8}
\]
Assume that \( F \in C^2([0, T]) \) satisfies
\[
\begin{align*}
F(t) &\geq At^a \quad \text{for} \quad t \geq T_0, \\
F''(t) &\geq B(t + k)^{-q}|F(t)|^p \quad \text{for} \quad t \geq 0, \\
F(0) &\geq 0, \quad F'(0) > 0,
\end{align*}
\tag{4.9}
\]
where \( A, B, k, T_0 \) are positive constants. Then, there exists a positive constant \( D_0 = D_0(p, a, q, B) \) such that
\[
T < 2^{2/M}T_1
\]
holds provided
\[
T_1 := \max \left\{ T_0, \frac{F(0)}{F'(0)}, k \right\} \geq D_0 A^{-(p-1)/(2M)}. \tag{4.10}
\]
This is exactly Lemma 2.1 in [18], so that we shall omit the proof here.

According to (4.2), (4.3) and (4.7), we are in a position to apply our situation to Lemma 4.1 with
\[
A = D_1 \varepsilon^p, \quad B = \pi^{1-p}, \quad a = 5 - 2p, \quad q = 3(p - 1)
\]
which imply that (4.8) yields
\[
M = \frac{p - 1}{2}(5 - 2p) - \frac{3(p - 1)}{2} + 1 = p(2 - p) > 0.
\]
If we set
\[
T_0 := D_0 A^{-(p-1)/(2M)} = D_0 D_1^{-(p-1)/(p(4-2p))} \varepsilon^{-(p-1)/(4-2p)}, \tag{4.11}
\]
we find that there is an \( \varepsilon_0 = \varepsilon_0(g, p, k) \) such that
\[
T_0 \geq \max \left\{ \frac{F(0)}{F'(0)}, 5k \right\} = 5k \quad \text{for} \quad 0 < \varepsilon \leq \varepsilon_0.
\]
This means that \( T_1 = T_0 \) in (4.10). Therefore, from (4.11), the conclusion of Lemma 4.1 implies
\[
T < 2^{2/M}T_1 = D_2 \varepsilon^{-(p-1)/(4-2p)},
\]
29
where
\[ D_2 := 2^{2/M} D_0 D_1^{-\frac{(p-1)/(p(4-2p))}{2}} > 0. \]

The proof of Theorem 2.3 with \(1 < p < 2\) is now completed. \(\square\)

**Proof of Theorem 2.3 with \(p = 2\).**

Let \(\bar{v}\) be the spherical mean of \(v \in C^0(\mathbb{R}^2 \times [0, \infty))\) with radius \(r\);
\[
\bar{v}(r, t) := \frac{1}{2\pi} \int_{|\omega|=1} v(r\omega, t) dS_\omega.
\]
We get the following inequality (for the proof, see [2], p.529):
\[
\bar{u}(r, t) \geq \bar{u}^0(r, t) + \frac{2}{\pi} \int_0^{t-r} d\tau (1 + \tau)^{-1} \int_0^{t-r-\tau} \lambda |\bar{u}|^2(\lambda, \tau) d\lambda \\
\times \int_{|\lambda|}^{\lambda+\rho} \sqrt{h(\lambda, \rho; r)((t-\tau)^2 - \rho^2)} d\rho,
\]
where
\[
h(\lambda, \rho; r) := (\rho^2 - (\lambda - r)^2)((\lambda + r)^2 - \rho^2).
\]
Since \(\int_{\mathbb{R}^2} \{f(x) + g(x)\} dx > 0\), from Lemma 2.2, there exist positive constants \(E_0\) and \(K\) such that
\[
u_L(x, t) \geq \frac{E_0}{\sqrt{(t+r)(t-r)}}
\]
for \(t-r \geq K \geq 1\). Making use of the positivity of the second term of right-hand side in (4.12), we get
\[
\bar{u}(r, t) \geq \bar{u}^0(r, t) \geq \frac{E_0\varepsilon}{\sqrt{(t+r)(t-r)}} \text{ for } t-r \geq K. \tag{4.13}
\]
We define
\[
\Sigma_j := \{(r, t) \mid t-r \geq Kl_j\}, \quad \Sigma_\infty := \{(r, t) \mid t-r \geq 2K\},
\]
where
\[
l_j := \sum_{k=0}^{j} 2^{-k} \quad (j = 0, 1, 2, \ldots).
\]
For \((r, t) \in \Sigma_0\), it follows from (4.12) and (4.13) that

\[
\bar{u}(r, t) \geq \frac{2}{\pi} \int_0^{t-r} d \tau (1 + \tau)^{-1} \int_0^{t-r-\tau} d \lambda \lambda |\bar{u}|^2(\lambda, \tau)
\]

\[
\times \frac{1}{\sqrt{(t - r - \tau + \lambda)(t + r - \tau - \lambda)}} \times \int_{|\lambda - r|}^{\lambda+r} \frac{\rho}{\sqrt{h(\lambda, \rho; r)}} d \rho.
\]

(4.14)

For \((r, t) \in \Sigma_0\), we introduce

\[
Q_j(r, t) := \{(\lambda, \tau) \in [0, \infty)^2 \mid Kl_j \leq \tau - \lambda, \ 0 \leq \lambda \leq t - r - \tau\}.
\]

Since

\[
\int_{|\lambda-r|}^{\lambda+r} \frac{\rho}{\sqrt{h(\lambda, \rho; r)}} d \rho = B\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\pi}{2},
\]

we get from (4.14)

\[
\bar{u}(r, t) \geq \frac{1}{\sqrt{(t + r)(t - r)}} \int \int_{Q_0(r, t)} (1 + \tau)^{-1} \lambda |\bar{u}|^2 d \lambda d \tau \quad \text{in } \Sigma_0.
\]

(4.15)

For \((r, t) \in \Sigma_j\), we have

\[
Q_j(r, t) \subset Q_0(r, t) \quad \text{and} \quad Q_j(r, t) \subset \Sigma_j.
\]

(4.16)

Since \(\Sigma_j \subset \Sigma_0\), it follows from (4.15) and (4.16) that

\[
\bar{u}(r, t) \geq \frac{1}{\sqrt{(t + r)(t - r)}} \int \int_{Q_j(r, t)} (1 + \tau)^{-1} \lambda |\bar{u}|^2(\lambda, \tau) d \lambda d \tau \quad \text{in } \Sigma_j.
\]

(4.17)

By using the induction argument, we will show

\[
\bar{u}(r, t) \geq \frac{d_j}{\sqrt{(t + r)(t - r)}} \log^{a_j} \left(\frac{t - r}{Kl_j}\right) \quad \text{in } \Sigma_j,
\]

(4.18)

where

\[
a_0 = 0, \quad a_{j+1} = 2a_j + 2, \quad d_0 = E_0 q, \quad d_{j+1} = \frac{d_j^2}{2^{3j+9}}.
\]

(4.19) (4.20)
From (4.13), it holds (4.18) with \( j = 0 \). We assume that (4.18) holds for one natural number \( j \) and \((r, t) \in \Sigma_{j+1}\). Substituting (4.18) into (4.17) and changing the variables by (2.29), we get

\[
\sqrt{(t+r)(t-r)}\bar{u}(r, t) \geq \frac{d_j^2}{8(2a_j+1)} \int_{K_{l_j+1}}^{t-r} d\alpha \int_{K_{l_j}}^{t-r} \log^2 a_{j+1} \left( \frac{\beta}{K_{l_j+1}} \right) d\beta
\]

and, then,

\[
\sqrt{(t+r)(t-r)}\bar{u}(r, t) \geq \frac{d_j^2}{8(2a_j+1)} \int_{K_{l_j+1}}^{t-r} d\alpha \alpha^{-2} \int_{K_{l_j}}^{t-r} \log^2 a_{j+1} \left( \frac{\beta}{K_{l_j+1}} \right) d\beta
\]

Solving (4.19) yields

\[
a_j = 2^{j+1} - 2. \quad (4.22)
\]

Since \((1 - l_j/l_{j+1}) = 2^{-(j+1)}/l_{j+1} \geq 2^{-(j+2)}\), we have

\[
\frac{1 - l_j/l_{j+1}}{8(a_{j+1})^2} \geq \frac{2^{-(j+2)}}{2^3 \cdot 2^{2j+4}} = \frac{1}{2^{3j+9}}. \quad (4.23)
\]

Therefore, from (4.21), (4.22) and (4.23), (4.18) holds for all natural numbers.

We get from (4.20)

\[
\log d_{j+1} = 2^{j+1} \log d_0 - (\log 2) \sum_{k=0}^{j} \left\{ (3(j - k) + 9)2^k \right\}.
\]

We obtain

\[
d_j = \exp \left\{ 2^j \left( \log d_0 - (\log 2) \sum_{k=0}^{j-1} \frac{(3(j - k) + 9)2^k}{2^j} \right) \right\}. \quad (4.24)
\]
The sum part in (4.24) converges as $j \to \infty$ by the d’Alembert’s criterion. Hence, there exists a constant $q$ such that it holds

$$d_j \geq \exp \left\{ 2^j \log (E_0 e^q \varepsilon) \right\}.$$  \hspace{1cm} (4.25)

Since $l_j \leq 2$, we get from (4.18), (4.22) and (4.25)

$$\sqrt{(t+r)(t-r)} \bar{u}(r,t) \geq \exp \left\{ 2^j J(r,t) \right\} \log^{-2} \left( \frac{t-r}{2K} \right) \text{ in } \Sigma_\infty, \hspace{1cm} (4.26)$$

where

$$J(r,t) := \log \left( \varepsilon \left\{ B^{-1} \log \left( \frac{t-r}{2K} \right) \right\}^2 \right) \quad \text{and} \quad B := E_0^{-1/2} e^{-q/2}. \hspace{1cm} (4.27)$$

We take $\varepsilon_0 > 0$ so small that

$$B\varepsilon_0^{-1/2} \geq \log(2K). \hspace{1cm} (4.28)$$

For a fixed $\varepsilon \in [0, \varepsilon_0)$, we suppose that $T$ satisfies

$$T \geq \exp \left( 4B\varepsilon^{-1/2} \right). \hspace{1cm} (4.29)$$

Next, we take $\tau > 0$ so that

$$T > \tau > \exp \left( 2B\varepsilon^{-1/2} \right) \quad \left( > 2K \right).$$

From (4.28) and (4.29), it follows that

$$\tau > 2K \exp(B\varepsilon^{-1/2}). \hspace{1cm} (4.30)$$

We get from (4.27) and (4.30)

$$J(0,\tau) = \log \left( \varepsilon \left\{ B^{-1} \log \left( \frac{\tau}{2K} \right) \right\}^2 \right) > 0. \hspace{1cm} (4.31)$$

Since $(0,\tau) \in \Sigma_\infty$, from (4.26) and (4.31), we get $u(0,\tau) \to \infty \ (j \to \infty)$.\hspace{1cm} \Box

4.2 Proof of Theorem 2.4

We divide the proof of Theorem 2.4 into two cases, $1 < p < 2$ and $p = 2$. First, we shall handle the sub-critical case.

Proof of Theorem 2.4 with $1 < p < 2$. Due to the assumption on the initial data in Theorem 2.4, $f(x) \geq 0 \ (\neq 0)$ and $f(x) + g(x) \equiv 0$, we have

$$F(0) > 0, \quad F'(0) = 0.$$  

For the key inequality, we employ the following proposition.
Proposition 4.1 Let $1 < p < 2$. Suppose that the assumptions in Theorem 2.4 are fulfilled. Then, there exists a positive constant $C_* = C_*(f, g, p, k)$ such that $F(t) = \int_{\mathbb{R}^2} u(x, t) dx$ satisfies
\[
F''(t) \geq C_* \varepsilon^p t^{2-3p/2} \quad \text{for } t \geq k.
\] (4.32)

Proof. It follows from Lemma 2.2 in [23] that
\[
F_1(t) \geq \frac{1}{2} \left(1 - e^{-2k}\right) \int_{\mathbb{R}^2} \varepsilon f(x) \phi_1(x) dx \quad \text{for } t \geq k,
\] (4.33)
where
\[
\phi_1(x) = \int_{S^1} e^{x \cdot \omega} d\omega \quad \text{and} \quad F_1(t) = e^{-t} \int_{\mathbb{R}^2} u(x, t) \phi_1(x) dx.
\]
From (2.4) and (2.5) in [23], we obtain
\[
F''(t) \geq C(t + k)^{2-3p/2} \left| F_1(t) \right|^p \quad \text{for } t \geq 0.
\] (4.34)
From (4.33) and (4.34), we obtain (4.32). This completes the proof. \(\square\)

In the sub-critical case, the following basic lemma is useful.

Lemma 4.2 (Takamura [18]) Assume that (4.9) is replace by
\[
F(0) > 0, \quad F'(0) = 0,
\]
and additionally that there is a $t_0 > 0$ such that
\[
F(t_0) \geq 2F(0).
\] (4.35)
Then, the conclusion of Lemma 4.1 is changed to that there exists a positive constant $\tilde{D}_0 = \tilde{D}_0(p, a, q, B)$ such that
\[
T < 2^{2/M} T_2
\]
holds provided
\[
T_2 := \max \{ T_0, t_0, k \} \geq \tilde{D}_0 A^{-(p-1)/(2M)}.
\] (4.36)
This is exactly Lemma 2.2 in [18], so that we shall omit the proof here.

Integrating (4.32) in $[k, t]$, we have
\[
F'(t) \geq \frac{C_*}{3 - 3p/2} \varepsilon^p \left( t^{3-3p/2} - k^{3-3p/2} \right) + F'(k)
\]
for \( t \geq k \) because of \( 1 < p < 2 \). Note that \( F'(k) \geq 0 \) follows from \( F''(t) \geq 0 \) for \( t \geq 0 \) and \( F'(0) = 0 \). Hence we obtain that

\[
F'(t) \geq C_* \left( 1 - \frac{2^{-3+3p/2}}{3-3p/2} \right) \varepsilon^p t^{3-3p/2} \quad \text{for } t \geq 2k.
\]

Integrating this inequality in \([2k, t]\) together with \( F(0) > 0 \), we get

\[
F(t) \geq D_3 \varepsilon^p t^{a-3p/2} \quad \text{for } t \geq 4k,
\]

where

\[
D_3 := \frac{C_* (1 - 2^{-3+3p/2})(1 - 2^{-4+3p/2})}{(3-3p/2)(4-3p/2)} > 0.
\]

From \( 2F(0) = 2\|f\|_{L^1(\mathbb{R}^2)} \varepsilon \) and (4.37), (4.35) in Lemma 4.2 is fulfilled with

\[
t_0 := D_4 \varepsilon^{-(p-1)/(4-3p/2)},
\]

if \( t_0 \geq 4k \), where

\[
D_4 := \left\{ 2\|f\|_{L^1(\mathbb{R}^2)} D_3^{-1} \right\}^{(4-3p/2)-1}.
\]

We are now in a position to apply our result here to Lemma 4.2 with special choices on all positive constants except for \( T_0 \) as

\[
A = D_3 \varepsilon^p, \quad B = \pi^{1-p}, \quad a = 4 - \frac{3}{2} p, \quad q = 3(p - 1)
\]

which imply that (4.8) yields

\[
M = \frac{p-1}{2} a - \frac{q}{2} + 1 = \frac{\gamma(p,4)}{4} > 0.
\]

If we set

\[
T_0 := D_5 A^{-(p-1)/(2M)} = D_5 D_3^{-2(p-1)/\gamma(p,4)} \varepsilon^{-2p(p-1)/\gamma(p,4)},
\]

then we find that there is an \( \varepsilon_0 = \varepsilon_0(f, g, n, p, k) > 0 \) such that

\[
T_0 \geq \max\{t_0, 4k\} \quad \text{for } 0 < \varepsilon \leq \varepsilon_0
\]

because of \( 2p/\gamma(p,4) < 1/(4 - 3p/2) \). This means that \( T_2 = T_0 \) in (4.36). Therefore, the conclusion of Lemma 4.2 implies

\[
T < 2^{2/M} T_2 = D_5 \varepsilon^{-2p(p-1)/\gamma(p,4)} \quad \text{for } 0 < \varepsilon \leq \varepsilon_0,
\]

35
where
\[ D_5 := 2^{8/\gamma(p,4)}D_0D_3^{-2(p-1)/\gamma(p,4)} > 0. \]

This completes the proof. \( \square \)

**Proof of Theorem 2.4 with \( p = 2 \).** Since
\[ f(x) + g(x) \equiv 0 \quad \text{and} \quad \int_{\mathbb{R}^2} f(x)dx < 0, \]
by Lemma 2.2, there exist positive constants \( \widetilde{E}_0 \) and \( \tilde{K} \geq 1 \) such that
\[ u_L(x,t) \geq \frac{\widetilde{E}_0}{(t+r)^{1/2}(t-r)^{3/2}} \quad \text{for } t-r \geq \tilde{K}. \]
For \( t-r \geq \tilde{K} \), we get from (4.12) that
\[ \bar{u}(r,t) \geq \frac{E_0^\varepsilon}{(t+r)^{1/2}(t-r)^{3/2}}. \quad (4.38) \]
We define the following domains:
\[ \tilde{\Sigma}_j := \left\{ (r,t) \in [0, \infty)^2 \mid t-r \geq 3\tilde{K}l_j \right\} \quad (j=0,1,2,\ldots), \]
\[ \tilde{\Sigma}_\infty := \left\{ (r,t) \in [0, \infty)^2 \mid t-r \geq 6\tilde{K} \right\}. \]
In the same way as to obtain (4.15), for \( t-r \geq \tilde{K} \), we get
\[ \bar{u}(r,t) \geq \frac{1}{\sqrt{(t+r)(t-r)}} \int_{\tilde{Q}_0(r,t)} (1 + \tau)^{-1} \lambda |\bar{u}|^2 d\lambda d\tau, \quad (4.39) \]
where
\[ \tilde{Q}_0(r,t) := \left\{ (\lambda, \tau) \in [0, \infty)^2 \mid \tilde{K} \leq \tau - \lambda, \ 0 \leq \lambda \leq t-r-\tau \right\}. \]
For \( (r,t) \in \tilde{\Sigma}_0 \), we set
\[ S(r,t) := \left\{ (\lambda, \tau) \in [0, \infty)^2 \mid \frac{5}{2}\tilde{K} \leq \tau + \lambda \leq t-r, \ \tilde{K} \leq \tau - \lambda \leq \frac{5}{4}\tilde{K} \right\}. \]
For \((r, t) \in \tilde{\Sigma}_0\), we have \(S(r, t) \subset \tilde{Q}_0(r, t)\). Substituting (4.38) into (4.39) and changing the variables by (2.29), we get

\[
\sqrt{(t + r)(t - r)} \tilde{u}(r, t) \geq \int \int_{S(r, t)} (1 + \tau)^{-1} \lambda |\tilde{u}|^2 d\lambda d\tau \\
\geq \frac{1}{2} \int_{5\tilde{K}/2}^{t-r} \int_{\tilde{K}}^{5\tilde{K}/4} \left( 1 + \frac{\alpha + \beta}{2} \right)^{-1} \left( \frac{\alpha - \beta}{2} \right) \left( \frac{\tilde{E}_0 \varepsilon}{\alpha^{1/2} \beta^{3/2}} \right)^2 d\beta.
\]

(4.40)

Since \(\alpha + \beta \leq 2\alpha, \alpha - \beta \geq \alpha - 5k/4 \geq \alpha/2\) and \(\tilde{K} \geq 1\), we get from (4.40)

\[
\sqrt{(t + r)(t - r)} \tilde{u}(r, t) \geq \frac{(\tilde{E}_0 \varepsilon)^2}{24} \int_{5\tilde{K}/2}^{t-r} \alpha^{-1} d\alpha \int_{\tilde{K}}^{5\tilde{K}/4} \beta^{-3/2} d\beta \\
\geq E_0^* \varepsilon^2 \log \left( \frac{t - r}{3\tilde{K}} \right) \text{ in } \tilde{\Sigma}_0,
\]

(4.41)

where \(E_0^* := \tilde{E}_0^2 / \left( 2^9 \tilde{K}^{1/2} \right)\). Analogously to the proof of Theorem 2.3 with \(p = 2\), we obtain from (4.41)

\[
\tilde{u}(r, t) \geq \frac{\tilde{d}_j}{\sqrt{(t + r)(t - r)}} \log \tilde{a}_j \left( \frac{t - r}{3\tilde{K}l_j} \right) \text{ in } \tilde{\Sigma}_j,
\]

where

\[
\tilde{a}_0 = 1, \quad \tilde{a}_{j+1} = 2\tilde{a}_j + 2, \\
\tilde{d}_0 = E_0^* \varepsilon^2, \quad \tilde{d}_{j+1} = \frac{\tilde{d}_j^2}{3 \cdot 2^{3j+9}}.
\]

This is the same form as (4.18), (4.19) and (4.20). Hence, we see that there exists a constant \(\tilde{q}\) such that

\[
\tilde{d}_j \geq \exp \left\{ 2^j \log \left( E_0^* \varepsilon^2 \right) \right\}.
\]

Since \(l_j \leq 2\), we get

\[
\sqrt{(t + r)(t - r)} \tilde{u}(r, t) \geq \exp \left\{ 2^j J(r, t) \right\} \log^{-2} \left( \frac{t - r}{6\tilde{K}} \right) \text{ in } \tilde{\Sigma}_\infty,
\]

where

\[
J(r, t) := \log \left\{ \varepsilon^2 \left\{ \tilde{B}^{-1} \log \left( \frac{t - r}{6\tilde{K}} \right) \right\}^3 \right\} \quad \text{and} \quad \tilde{B} := (E_0^*)^{-1/3} e^{-\tilde{q}/3}.
\]

In the same way as in the proof of Theorem 2.3 with \(p = 2\), we get the desired estimates. The proof is now completed. \(\square\)
Acknowledgement

This work started when the first author was a master course student in Future University Hakodate, the third author was working in Future University Hakodate, and the fourth author was working in Muroran Institute of Technology. The third author has been partially supported by Special Research Expenses in FY2017, General Topics (No.B21), Future University Hakodate, also by the Grant-in-Aid for Scientific Research (B) (No.18H01132) and (C) (No.15K04964), Japan Society for the Promotion of Science. Finally all the authors are grateful to the referee for precise reading and useful comments.

References

[1] R.Agemi, Y.Kurokawa and H.Takamura, Critical curve for p-q systems of nonlinear wave equations in three space dimensions, J. Differential Equations 167 (2000), no. 1, 87-133.

[2] R.Agemi and H.Takamura, The lifespan of classical solutions to nonlinear wave equations in two space dimensions, Hokkaido Math. J. 21 (1992), no. 3, 517-162.

[3] M.D’Abbicco, The threshold of effective damping for semilinear wave equations, Math. Methods Appl. Sci. 38 (2015), no. 6, 1032-1045.

[4] M.D’Abbicco and S.Lucente, NLWE with a special scale invariant damping in odd space dimension, Discrete Contin. Dyn. Syst. 2015, Dynamical systems, differential equations and applications. 10th AIMS Conference. Suppl., 312-319.

[5] M.D’Abbicco, S.Lucente and M.Reissig, A shift in the Strauss exponent for semilinear wave equations with a not effective damping, J. Differential Equations 259 (2015), no. 10, 5040-5073.

[6] R.T.Glassey, Existence in the large for $\square u = F(u)$ in two space dimensions, Math. Z. 178 (1981), no. 2, 233-261.

[7] L.Hörmander, Lectures on Nonlinear Hyperbolic Differential Equations, Mathématiques & Applications (Berlin) [Mathematics & Applications], 26. Springer-Verlag, Berlin, 1997.

[8] M.Ikeda and M.Sobajima, Life-span of solutions to semilinear wave equation with time-dependent critical damping for specially localized initial data, Math. Ann. 372 (2018), no. 3-4, 1017-1040.
[9] M.Ikeda, M.Sobajima and K.Wakasa, Test function method for blow-up phenomena of semilinear wave equations and their weakly coupled systems, J. Differential Equations, 267 (2019), no. 9, 5165–5201.

[10] T.Imai, M.Kato, H.Takamura and K.Wakasa, The sharp lower bound of the lifespan of solutions to semilinear wave equations with low powers in two space dimensions, K.Kato, T.Ogawa and T.Ozawa ed., “Asymptotic Analysis for Nonlinear Dispersive and Wave Equations”, Advanced Studies in Pure Mathematics 81 (2019), 31-53.

[11] F.John, Blow-up of solutions of nonlinear wave equations in three space dimensions, Manuscripta Math. 28 (1979), no. 1-3, 235-268.

[12] M.Kato and M.Sakuraba, Global existence and blow-up for semilinear damped wave equations in three space dimensions, Nonlinear Anal. 182 (2019), 209-225.

[13] M.Kato, H.Takamura and K.Wakasa, The lifespan of solutions of semilinear wave equations with the scale-invariant damping in one space dimension, Differential Integral Equations 32 (2019), no. 11-12, 659-678.

[14] N.-A.Lai, Weighted $L^2 - L^2$ estimate for wave equation and its applications, arXiv:1807.05109.

[15] N.-A.Lai and H.Takamura, Blow-up for semilinear damped wave equations with subcritical exponent in the scattering case, Nonlinear Anal. 168 (2018), 222-237.

[16] N.-A.Lai, H.Takamura and K.Wakasa, Blow-up for semilinear wave equations with the scale invariant damping and super-Fujita exponent, J. Differential Equations 263 (2017), no. 9, 5377-5394.

[17] M.Liu and C.Wang, Global existence for semiliner damped wave equations in relation with the Strauss conjecture, Discrete Contin. Dyn. Syst. 40 (2020), no. 2, 709-724.

[18] H.Takamura, Improved Kato’s lemma on ordinary differential inequality and its application to semilinear wave equations, Nonlinear Anal. 125 (2015), 227-240.

[19] H.Takamura and K.Wakasa, The sharp upper bound of the lifespan of solutions to critical semilinear wave equations in high dimensions, J. Differential Equations 251 (2011), no. 4-5, 1157-1171.
[20] Z.Tu and J.Lin, *A note on the blowup of scale invariant damping wave equation with sub-Strauss exponent*, arXiv:1709.00866.

[21] Z.Tu and J.Lin, *Life-span of semilinear wave equations with scale-invariant damping: critical Strauss exponent case*, Differential Integral Equations **32** (2019), no. 5-6, 249-264.

[22] K.Wakasa and B.Yordanov, *On the nonexistence of global solutions for critical semilinear wave equations with damping in the scattering case*, Nonlinear Anal. **180** (2019), 67-74.

[23] B.Yordanov and Q.S.Zhang, *Finite time blow up for critical wave equations in high dimensions*, J. Funct. Anal. **231** (2006), no. 2, 361-374.

[24] K.Wakasa, *The lifespan of solutions to semilinear damped wave equations in one space dimension*, Comm. Pure Appl. Anal. **15** (2016), no. 4, 1265-1283.

[25] Y.Wakasugi, *Critical exponent for the semilinear wave equation with scale invariant damping*, Fourier analysis, 375-390, Trends Math., Birkhäuser/Springer, Cham, 2014.