TOPOLOGICAL OBSTRUCTIONS TO NONNEGATIVE SCALAR CURVATURE AND MEAN CONVEX BOUNDARY

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Abstract. We study topological obstructions to the existence of a Riemannian metric on manifolds with boundary such that the scalar curvature is nonnegative and the boundary is mean convex. In particular, we construct many examples of compact manifolds with boundary which admit no Riemannian metric with nonnegative scalar curvature and mean convex boundary.

1. Introduction

A central problem in modern differential geometry concerns the connection between curvatures and topology of a manifold. Consider, firstly, the case of surfaces. Let \((M^2, g)\) be a compact two-dimensional Riemannian manifold with nonempty boundary \(\partial M\). The Gauss-Bonnet theorem states that

\[ \int_M K\, da + \int_{\partial M} k_g\, ds = 2\pi \chi(M), \]

where \(K\) denotes the Gaussian curvature, \(k_g\) is the geodesic curvature of the boundary, \(\chi(M)\) is the Euler characteristic, \(da\) is the element of area and \(ds\) is the element of length. Note that the topological invariant \(\chi(M)\) gives a topological obstruction to the existence of a certain type of Riemannian metrics on the surface \(M^2\). For instance, a compact surface \(M^2\) with negative (nonpositive) Euler characteristic does not admit a Riemannian metric with nonnegative (positive) Gaussian curvature and nonnegative geodesic curvature.

In higher dimensions, the relationship between curvatures and the topology of a manifold is much more complicated. The scalar curvature of the manifold and the mean curvature of its boundary are commonly studied. A classical theorem due to Gromov [2], assuming no control on the boundary, states that every compact manifold with non-empty boundary admits a Riemannian metric of positive sectional curvature. However, there are topological obstructions if one further

\[ \text{Partially supported by CNPq.} \]
imposes convexity restrictions on the boundary. For instance, it is a result of Gromoll [1] that a compact Riemannian manifold of positive sectional curvature with non-empty convex boundary is diffeomorphic to the standard disc. However, positive sectional curvature is more restrictive than the scalar curvature. In dimension 3, a 3-dimensional manifold with positive Ricci curvature and convex boundary (positive definite second fundamental form) is diffeomorphic to a 3-ball. For the scalar curvature and mean curvature of the boundary, it is not clear which topological obstructions can appear. In particular, the existence of compact, orientable and essential surface properly embedded in $M$ which is not a disk or a cylinder is a topological obstruction for existence of a metric with nonnegative scalar curvature and mean convex boundary in $M$. For instance, define $M = S^1 \times \Sigma$, where $\Sigma$ is a compact, connected and orientable surface with boundary $\partial \Sigma \neq \emptyset$ which is not a disk or a cylinder. Note that $\Sigma$ is essential in $M$. Hence, there is no metric on $M$ with nonnegative scalar curvature and convex boundary. However, if $M$ has an essential cylinder, can exists such metric on $M$. Again, for instance, consider the manifold $[0,1] \times T^2$ with the product metric $g := dt^2 + \delta$, where $dt^2$ is the standard metric on $[0,1]$ and $\delta$ is the flat metric on $T^2$. Such Riemannian manifold is flat with totally geodesic boundary. In this notes, we show that the existence of a compact, orientable and essential surface properly embedded in $M$ which is not a disk, is a topological obstruction for the existence of a metric with nonnegative scalar curvature and strictly mean convex boundary or with positive scalar curvature and mean convex boundary in $M$. Furthermore, we show that the existence of an essential cylinder properly embedded in a compact and orientable 3-dimensional Riemannian manifold with nonnegative scalar curvature and mean convex boundary forces the metric to be flat with totally geodesic boundary. To be more precise, we prove the following result:

**Theorem 1.1.** Let $M^3$ be a connected, compact and orientable 3-dimensional manifold with non-empty boundary $\partial M$. Assume that the connected components of $\partial M$ are spheres or incompressible tori, but at least one of the components is a torus. Then there is no Riemannian metric on $M^3$ with nonnegative scalar curvature and strictly mean convex boundary or positive scalar curvature and mean convex boundary. In particular, if there exists a Riemannian metric $g$ on $M^3$ with nonnegative scalar curvature and mean convex boundary then $(M^3, g)$ should be flat with totally geodesic boundary.

As a direct consequence of the theorem above, we obtain that the 3-dimensional manifolds $(S^1 \times \tilde{T}^2) \# N$ and $(S^1 \times \tilde{T}^2) \# (I \times S^2)$ have no
metric with nonnegative scalar curvature and mean convex boundary, where \( T^2 \) is a torus minus an open disk, \( I = [a,b] \) and \( N \) is a closed, connected and orientable 3-dimensional manifold. Moreover, the 3-dimensional manifolds \((I \times T^2)\#(I \times T^2)\), \((S^1 \times T^2)\#(S^1 \times T^2)\), \((I \times T^2)\#(S^1 \times T^2)\), \((I \times T^2)\#(I \times S^2)\), \((S^1 \times T^2)\#(I \times S^2)\) have no metric with nonnegative scalar curvature and strictly mean convex boundary or with positive scalar curvature and mean convex boundary. Also, let \( N \) be a closed 3-dimensional manifold. Then the manifold \((I \times T^2)\#N\) has no metric with nonnegative scalar curvature and strictly mean convex boundary. If it has a metric with nonnegative scalar curvature and mean convex boundary, it should be flat with totally geodesic boundary. Thus, from this last claim, we can glue two copies of \((I \times T^2)\#N\) along the boundary and build a flat closed 3-dimensional manifold which is a connected sum of a 3-dimensional torus and a closed 3-dimensional manifold.

For high dimension \( 3 \leq n \leq 7 \), we want to prove that, if \( N \) is a closed \( n \)-dimensional manifold, then \((T^{n-2} \times T^2)\#N\) does not admit a metric of nonnegative scalar curvature and mean convex boundary and \((I \times T^{n-1})\#N\) does not admit a metric of positive scalar curvature and mean convex boundary.

Finally, we study free-boundary \( k \)-slicing in order to extend the results above for any dimension \( n \geq 3 \).

2. Preliminaries and Technical Results

Let \((M^n, g)\), \( n \geq 3 \), be a compact orientable Riemannian manifold with non-empty boundary \( \partial M \). Assume that \( R_g, H_g \geq 0 \) and \( Vol_g(M) = 1 \), where \( H_g \) denote the mean curvature of \( \partial M \) with respect to the outward unit normal vector. For each Riemannian metric \( \tilde{g} \) on \( M \) consider \( \lambda(\tilde{g}) \in \mathbb{R} \) and \( \Phi_{\tilde{g}} \in \mathcal{C}^\infty(M) \) satisfying:

\[
\begin{align*}
-\Delta_{\tilde{g}} \Phi_{\tilde{g}} + c_n R_{\tilde{g}} \Phi_{\tilde{g}} &= \lambda(\tilde{g}) \Phi_{\tilde{g}} \\
\frac{\partial \Phi_{\tilde{g}}}{\partial \nu_{\tilde{g}}} &= -2 c_n H_{\tilde{g}} \Phi_{\tilde{g}} \\
\int_M \Phi_{\tilde{g}} dv_{\tilde{g}} &= 1
\end{align*}
\]

where \( \nu_{\tilde{g}} \) denote the outward unit normal vector of the boundary \( \partial M \) in \((M, \tilde{g})\) and \( c_n := \frac{(n-2)}{4(n-1)} \). Note that, as we are considering, we can assume that \( \Phi_{\tilde{g}} > 0 \).
Moreover, note that
\[
\lambda(\tilde{g}) = -\int_M \Delta_g \Phi_g dv_{\tilde{g}} + c_n \int_M R_g \Phi_g dv_{\tilde{g}}
\]
\[
= -\int_{\partial M} \frac{\partial \Phi_g}{\partial n} d\sigma_{\tilde{g}} + c_n \int_M R_g \Phi_g dv_{\tilde{g}}.
\]

Therefore,
\[
\lambda(\tilde{g}) = 2c_n \int_{\partial M} \Phi_g H_g d\sigma_{\tilde{g}} + c_n \int_M R_g \Phi_g dv_{\tilde{g}}.
\]

**Lemma 2.1.** Let \((M^n, g)\), \(n \geq 3\), be a compact orientable Riemannian manifold with non-empty boundary \(\partial M\) such that \(R_g, H_g \geq 0\) and \(Vol_g(M) = 1\). If \(\lambda(g) = 0\) then
\[
D\lambda_g(h) = -c_n \int_{\partial M} \langle h, B_g \rangle d\sigma_g - c_n \int_M \langle h, Ric_g \rangle dv_g,
\]
for every symmetric tensor \(h \in \mathcal{T}^{(2,0)}(M)\).

**Proof.** Firstly, note that \(\lambda(g) = 0\) implies that \(R_g \equiv 0\), \(H_g \equiv 0\) and \(\Phi_g \equiv 1\). Let \(h \in \mathcal{T}^{(2,0)}(M)\) be a symmetric tensor. Consider \(g(t)\) for each \(t \in (-\epsilon, \epsilon)\) a smooth family of Riemannian metrics on \(M\) in a such way that \(g(0) = g e g'(0) = h\). Denote by
\[
\lambda(t) := \lambda(g(t)), \quad R(t) := R_{g(t)} \quad \text{and} \quad H(t) := H_{g(t)}.
\]
As \(R_g \equiv 0\), \(H_g \equiv 0\) and \(\Phi_g \equiv 1\), we obtain that
\[
D\lambda_g(h) = \lambda'(0) = 2c_n \int_{\partial M} H'_g(0) d\sigma_g + c_n \int_M R'(0) dv_g.
\]

We divide the proof in some steps.

**Step 1:** We have that
\[
R'(t) = -\langle h(t), Ric_{g(t)} \rangle + div_{g(t)}(div_{g(t)}(h(t))) - d(tr_{g(t)} h(t))), \quad h(t) := g'(t).
\]
Hence,
\[ D\lambda_g(h) = 2c_n \int_{\partial M} H'(0)d\sigma_g - c_n \int_M \langle h, Ric_g \rangle dv_g + \int_M \text{div}_g(\text{div}_g(h) - d(tr_g h))dv_g \]
\[ = 2c_n \int_{\partial M} H'(0)d\sigma_g - c_n \int_M \langle h, Ric_g \rangle dv_g + \int_M \langle ((\text{div}_g(h))\# - (d(tr_g h))\#, \nu \rangle d\sigma_g \]
\[ = c_n \int_{\partial M} (2H'(0) + X)d\sigma_g - c_n \int_M \langle h, Ric_g \rangle dv_g, \]

where \( \nu := \nu_g \) and \( X := \langle ((\text{div}_g(h))\# - (d(tr_g h))\#, \nu \rangle \).

Einstein convention and notation:

- Without a summation symbol, lower and upper index indicate a summation from 1 to \( n - 1 \).
- \( \nabla^t \) denote the Riemannian connection of \( (M, g(t)) \), \( \nabla := \nabla^0 \);
- \( B_t \) denote the second fundamental form of \( \partial M \) in \( (M, g(t)) \).

Consider \((x_1, \cdots, x_n)\) a local chart on \( M \) such that \((x_1, \cdots, x_{n-1})\) is a local chart on \( \partial M \) and \( \partial_n = \nu \).

**Step 2:** Computation of \( X \) in \( \partial M \).

We have that
\[
\text{d}(tr_g h) = \sum_{k=1}^{n} \partial_k \left( \sum_{i,j=1}^{n} g^{ij} h_{ij} \right) dx^k \quad \text{and} \quad \text{div}_g(h) = \sum_{k=1}^{n} (\text{div}_g(h))_k dx^k.
\]

It follows that
\[
\langle \text{d}(tr_g h)\# \rangle = \sum_{i,j,k,l=1}^{n} g^{lk} \partial_k (g^{ij} h_{ij}) \partial_l,
\]

and
\[
\langle \text{div}_g(h)\# \rangle = \sum_{k,l=1}^{n} g^{lk} (\text{div}_g(h))_k \partial_l = \sum_{i,j,k,l=1}^{n} g^{lk} g^{ij} (\nabla_i h)_{jk} \partial_l.
\]

Thus,
\[
\langle \text{div}_g(h)\# \rangle - \langle \text{d}(tr_g h)\# \rangle = \sum_{l=1}^{n} \left\{ \sum_{i,j,k,l=1}^{n} (g^{lk} g^{ij} (\nabla_i h)_{jk} - g^{lk} \partial_k (g^{ij} h_{ij})) \right\} \partial_l.
\]
In \( \partial M \), we get that \( g_{nn} = g^{nn} = 1 \) and \( g_{ln} = g^{ln} = 0 \), for every \( l = 1, \cdots n-1 \). Hence,

\[
X = \sum_{i,j=1}^{n} (g^{ij}(\nabla_i h)_{jn} - \nu(g^{ij} h_{ij}))
\]

\[
= g^{ij}(\nabla_i h)_{jn} + \nu(h_{nn}) - \nu(g^{ij} h_{ij}) - \nu(h_{nn})
\]

\[
= g^{ij}(\nabla_i h)_{jn} - \nu(g^{ij}) h_{ij} - g^{ij} \nu(h_{ij})
\]

\[
= g^{ij}(\nabla_i h)_{jn} + g^{ik} g^{jl} \nu(g_{kl}) h_{ij} - g^{ij} \nu(h_{ij})
\]

\[
= g^{ij}(\nabla_i h)_{jn} + 2 g^{ik} g^{jl} (B g)_{kl} (h)_{ij} - g^{ij} \nu(h_{ij})
\]

\[
= g^{ij}(\nabla_i h)_{jn} + 2 \langle h, B g \rangle - g^{ij} \nu(h_{ij})
\]

\[
= g^{ij}(\nabla_i h)_{jn} + 2 \langle h, B g \rangle - g^{ij} \nu(h_{ij})
\]

**Step 3:** Computation of \( H'(0) \):

We have \( H(t) = g^{ij} (B_t)_{ij} \). Hence,

\[
H'(t) = \frac{d}{dt} (g^{ij} (B_t)_{ij}) + g^{ij} \frac{d}{dt} ((B_t)_{ij})
\]

\[
= - g^{ik} g^{jl} \frac{d}{dt} (h_{kl})_{ij} + tr \left( \frac{d}{dt} B_t \right) - (h(t), B(t)) + tr \left( \frac{d}{dt} B_t \right).
\]

Let's focus our attention on \( tr \left( \frac{d}{dt} \bigg|_{t=0} B_t \right) \). We have that

\[
(B_t)_{ij} = - g_t (v_{g_t}, \nabla_i \partial_j).
\]

\[
\Rightarrow \frac{d}{dt} \bigg|_{t=0} (B_t)_{ij} = - h(v, \nabla_i \partial_j) - \left( \frac{d}{dt} \bigg|_{t=0} (v_{g_t}, \nabla_i \partial_j) - \nu, \frac{d}{dt} \bigg|_{t=0} (\nabla_i \partial_j) \right),
\]

where \( g = \langle ., . \rangle \).

**Claim 1.** For every \( X, Y, Z \in \mathcal{X}(M) \), we obtain that

\[
2 \left( \frac{d}{dt} \bigg|_{t=0} (\nabla^i_X Y), Z \right) = (\nabla^i_X h)(Y, Z) + (\nabla^i_Y h)(X, Z) - (\nabla^i_Z h)(X, Y).
\]

It follows from the claim 1 that

\[
\frac{d}{dt} \bigg|_{t=0} (B_t)_{ij} = - h(v, \nabla_i \partial_j) - \left( \frac{d}{dt} \bigg|_{t=0} (v_{g_t}, \nabla_i \partial_j) - \nu, \frac{d}{dt} \bigg|_{t=0} (\nabla_i \partial_j) \right) - \frac{1}{2} (\nabla^i_j h)_{jn}
\]

\[- \frac{1}{2} (\nabla^i_j h)_{in} + \frac{1}{2} (\nabla^i v h)_{ij}.
\]
Claim 2. In $\partial M$,

$$\frac{d}{dt}\bigg|_{t=0} (\nu_{g_t}) = -g^{kl} h_{nk} \partial_l - \frac{1}{2} h_{nn} \nu.$$

Proof. In $\partial M$, we have $(g_t)_{nk} = 0$ and $(g_t)_{nn} = 1$, for all $k = 1, \ldots, n - 1$ and $t \in (-\epsilon, \epsilon)$. Thus,

$$0 = \frac{d}{dt}\bigg|_{t=0} (g_t)_{nk} = h_{nk} + \left< \frac{d}{dt}\bigg|_{t=0} (\nu_{g_t}), \partial_k \right>,
$$

and

$$0 = \frac{d}{dt}\bigg|_{t=0} (g_t)_{nn} = h_{nn} + 2 \left< \frac{d}{dt}\bigg|_{t=0} (\nu_{g_t}), \nu \right>.$$ 

Moreover, for all $k = 1, \ldots, n - 1$, we have that

$$\left< \frac{d}{dt}\bigg|_{t=0} (\nu_{g_t}), \nu \right> = -\frac{1}{2} h_{nn},$$

and

$$\left< \frac{d}{dt}\bigg|_{t=0} (\nu_{g_t}), \partial_k \right> = -h_{nk}.$$ 

Denote by

$$\frac{d}{dt}\bigg|_{t=0} (\nu_{g_t}) = \sum_{l=1}^{n} a_l \partial_l.$$ 

Note that

$$a_n = \left< \frac{d}{dt}\bigg|_{t=0} (\nu_{g_t}), \nu \right> = -\frac{1}{2} h_{nn}.$$ 

Also,

$$-h_{nk} = \left< \frac{d}{dt}\bigg|_{t=0} (\nu_{g_t}), \partial_k \right> = \sum_{i=1}^{n-1} a_i g_{ki}, \ \forall k = 1, \ldots, n - 1.$$ 

$$\Rightarrow a_l = -g^{lk} h_{nk}, \ \forall l = 1, \ldots, n - 1.$$ 

Hence,

$$\frac{d}{dt}\bigg|_{t=0} (\nu_{g_t}) = \sum_{l=1}^{n-1} a_l \partial_l + a_n \nu = -g^{lk} h_{nk} \partial_l - \frac{1}{2} h_{nn} \nu.$$ 

$$\square$$
It follows from the claim 2 that
\[
\left\langle \frac{d}{dt}\bigg|_{t=0} \nu_{g_t}, \nabla_i \partial_j \right\rangle = -g^{lk} h_{nk} \langle \nabla_i \partial_j, \partial_l \rangle - \frac{1}{2} h_{nn} \langle \nabla_i \partial_j, \nu \rangle \\
= -g^{lk} h_{nk} \Gamma_{ij}^m g_{ml} + \frac{1}{2} h_{nn} (B_g)_{ij} \\
= -h_{nk} \Gamma_{ij}^k + \frac{1}{2} h_{nn} (B_g)_{ij}
\]

However,
\[
-h(\nabla_i \partial_j, \nu) = -h_{nk} \Gamma_{ij}^k - h_{nn} \Gamma_{ij}^n = (B_g)_{ij} h_{nn} - h_{nk} \Gamma_{ij}^k,
\]
since
\[
\Gamma_{ij}^n = \frac{1}{2} \sum_{k=1}^{n} g^{nk} \{ \partial_k g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} \} = -\frac{1}{2} \nu(g_{ij}) = -(B_g)_{ij}.
\]

It implies that
\[
\left\langle \frac{d}{dt}\bigg|_{t=0} \nu_{g_t}, \nabla_i \partial_j \right\rangle = -h(\nabla_i \partial_j, \nu) - (B_g)_{ij} h_{nn} + \frac{1}{2} h_{nn} (B_g)_{ij} \\
= -h(\nabla_i \partial_j, \nu) - \frac{1}{2} h_{nn} (B_g)_{ij}.
\]

Hence,
\[
\frac{d}{dt}\bigg|_{t=0} (B_t)_{ij} = -\frac{1}{2} (\nabla_i h)_{jn} - \frac{1}{2} (\nabla_j h)_{in} + \frac{1}{2} (\nabla_\nu h)_{ij} + \frac{1}{2} h_{nn} (B_g)_{ij}.
\]

Consequently,
\[
tr \left( \frac{d}{dt}\bigg|_{t=0} (B_t) \right) = -g^{ij} (\nabla_i h)_{jn} + \frac{1}{2} g^{ij} (\nabla_\nu h)_{ij} + \frac{1}{2} h_{nn} H_g.
\]

As $H_g = 0$, we obtain that
\[
2H'(0) = -2 \langle h, B_g \rangle + 2tr \left( \frac{d}{dt}\bigg|_{t=0} (B_t) \right) \\
= -2 \langle h, B_g \rangle - 2g^{ij} (\nabla_i h)_{jn} + g^{ij} (\nabla_\nu h)_{ij} \\
= -2 \langle h, B_g \rangle - 2g^{ij} (\nabla_i h)_{jn} + g^{ij} \nu(h_{ij}) - 2g^{ij} h(\nabla_\nu, \partial_j).
\]

Claim 3. In $\partial M$,
\[
g^{ij} h(\nabla_i \nu, \partial_j) = \langle h, B_g \rangle.
\]
Proof. Denote by
\[ \nabla_i \nu = \sum_{k=1}^{n} \Gamma_{ik}^k \partial_k. \]
Note that, in \( \partial M \), we have
\[ \Gamma_{in}^n = 0 \quad \text{and} \quad \Gamma_{in}^k = g^{mk}(B_g)_{im}, \quad \forall k = 1, \ldots, n-1. \]
Then, in \( \partial M \), we obtain that
\[ g^{ij} h(\nabla_i \nu, \partial_j) = g^{ij} g^{mk}(B_g)_{im} h_{kj} = \langle h, B_g \rangle. \]
\( \square \)
It follows from the claim 3 that
\[ 2H'(0) = -4 \langle h, B_g \rangle - 2g^{ij}(\nabla_i h)_{jn} + g^{ij} \nu(h_{ij}). \]
Therefore,
\[ 2H'(0) + X|_{\partial M} = -2 \langle h, B_g \rangle - g^{ij}(\nabla_i h)_{jn}. \]
Claim 4. In \( \partial M \), we have
\[ g^{ij}(\nabla_i h)_{jn} = -\langle h, B_g \rangle + \text{div}_g^{\partial M}(\omega), \]
for some \( \omega \in \Omega^1(\partial M) \).
Proof. It follows from the claim 3 that, in \( \partial M \),
\[ g^{ij}(\nabla_i h)_{jn} = g^{ij} \partial_i(h_{jn}) - g^{ij} h(\nabla_i \partial_j, \nu) - g^{ij} h(\nabla_i \nu, \partial_j) \]
\[ = g^{ij} \partial_i(h_{jn}) - g^{ij} h(\nabla_i \partial_j, \nu) - \langle h, B_g \rangle. \]
For \( 1 \leq i, j \leq n-1 \), in \( \partial M \), we can write
\[ \nabla_i \partial_j = (B_g)_{ij} \nu + \nabla_i \partial_j, \]
where \( \nabla \) is the Riemannian connection of \( (\partial M, g) \).
Hence, since \( H_g \equiv 0 \), we obtain that
\[ g^{ij} h(\nabla_i \partial_j, \nu) = h_{in} H_g + g^{ij} h(\nabla_i \partial_j, \nu) = g^{ij} h(\nabla_i \partial_j, \nu). \]
This implies that, in \( \partial M \),
\[ g^{ij}(\nabla_i h)_{jn} = g^{ij} \partial_i(h_{jn}) - g^{ij} h(\nabla_i \partial_j, \nu) - \langle h, B_g \rangle. \]
Define \( \omega \in \Omega^1(\partial M) \) as
\[ \omega := h(\cdot, \nu)|_{\partial M}. \]
Note that
\[ \text{div}_g^{\partial M}(\omega) = g^{ij}(\nabla_i \omega)_j = g^{ij} \partial_i(\omega_j) - g^{ij} \omega(\nabla_i \partial_j) \]
\[ = g^{ij} \partial_i(h_{jn}) - g^{ij} h(\nabla_i \partial_j, \nu). \]
Then, in $\partial M$,
\[ g^{ij}(\nabla_i h)_{jn} = -\langle h, B_g \rangle + \text{div}^g_{\partial M}(\omega). \]

It follows from (2.1) and 4 that
\[ 2H'(0) + X|_{\partial M} = -\langle h, B_g \rangle - \text{div}^g_{\partial M}(\omega). \]

Hence,
\[ D\lambda_g(h) = -c_n \int_{\partial M} \langle h, B_g \rangle d\sigma_g - c_n \int_M \langle h, \text{Ric}_g \rangle dv_g - c_n \int_{\partial M} \text{div}^g_{\partial M}(\omega) d\sigma_g. \]

We conclude, since $\partial M$ is a closed manifold, that
\[ D\lambda_g(h) = -c_n \int_{\partial M} \langle h, B_g \rangle d\sigma_g - c_n \int_M \langle h, \text{Ric}_g \rangle dv_g. \]

Proposition 2.2. Let $(M^n, g)$, $n \geq 3$, be a compact orientable Riemannian manifold with boundary such that $R_g, H_g \geq 0$, $\text{Vol}_g(M) = 1$ and $\lambda(g) = 0$. The metric $g$ is a critical point of the functional $\lambda$ if and only if $(M, g)$ is Ricci flat with totally geodesic boundary.

Remark 2.3. Let $(M^n, g)$, $n \geq 3$, be a compact orientable Riemannian manifold with boundary. Define the following pair of operators acting in $C^\infty(M)$:
\[
\begin{align*}
L_g &= -\Delta_g \varphi + c_n R_g \varphi \quad \text{in } M \\
T_g &= \frac{\partial \varphi}{\partial \nu} + 2c_n H_g^M \varphi \quad \text{on } \partial M
\end{align*}
\]

where $\nu$ denotes the outward unit normal vector of the boundary $\partial M$ in $(M,g)$ and $c_n := \frac{(n-2)}{4(n-1)}$. Consider the first eigenvalue $\lambda_1(M, g)$ of $L_g$ with boundary condition $T_g$:
\[
\begin{align*}
\lambda_1(M, g) &= \inf_{0 \neq \varphi \in H^1(M)} \frac{\int_M (|\nabla_g \varphi|^2 + c_n R_g \varphi^2) dv_g + 2c_n \int_{\partial M} H_g^M \varphi^2 d\sigma_g}{\int_M \varphi^2 dv_g}.
\end{align*}
\]
We can choose a positive function \( \varphi \in C^{\infty}(M) \) solution of (2.2). The conformal metric \( h = \varphi^{\frac{4}{n-2}}g \) is such that

\[
\begin{align*}
R_h &= \lambda_1(M,g)\varphi^{-\frac{4}{n-2}} \text{ in } M \\
H_{h \mid \partial M} &= 0 \text{ on } \partial M
\end{align*}
\]

In particular, this implies that if \( \lambda_1(M,g) > 0 \) then \( R_h > 0 \) and \( H_{h \mid \partial M} \equiv 0 \).

**Proposition 2.4.** Let \((M^n, g), n \geq 3,\) be a compact orientable Riemannian manifold with boundary such that \( R_g \geq 0 \) and \( H_g \geq 0 \). Then either \( M \) admits a metric with positive scalar curvature and minimal boundary or \((M,g)\) is Ricci flat with totally geodesic boundary.

**Proof.** We can assume that \( \text{Vol}_g(M) = 1 \). Hence, we obtain \( \lambda(g) \geq 0 \). If \( \lambda(g) > 0 \), then there exists a metric on \( M \) with positive scalar curvature and minimal boundary (See remark 2.3).

Then, assume that \( \lambda(g) = 0 \). If \( D\lambda_g \equiv 0 \), we have that \( g \) is a critical point of the functional \( \lambda \). Consequently, \( \text{Ric}_g \equiv 0 \) and \( B_g \equiv 0 \). If \( D\lambda_g \not\equiv 0 \), there exists a symmetric tensor \( h_0 \in \mathcal{T}^{(2,0)}(M) \) such that \( D\lambda_g(h_0) > 0 \). Consider a family of metrics on \( M \), \( g(t) = g + th_0, t \in (-\epsilon, \epsilon) \). Since \( \lambda(0) = D\lambda_g(h_0) > 0 \), we obtain that there exists \( \theta \in (0, \epsilon) \) such that the function \( t \in (-\theta, \theta) \mapsto \lambda(t) \in \mathbb{R} \) is an increase function. Since \( \lambda(g) = 0 \), we get that \( \lambda(t) > 0 \) for all \( t \in (0, \theta). \) Therefore, for each \( t \in (0, \theta) \) there is a metric \( \tilde{g}_t \) on \( M \) such that \( R_{\tilde{g}_t} > 0 \) and \( H_{\tilde{g}_t \mid \partial M} \equiv 0 \) (See remark 2.3). \( \square \)

**Proposition 2.5.** Let \((M^{n+1}, g), n \geq 2,\) be a compact orientable Riemannian manifold with non-empty boundary such that \( R_g > 0 \) and \( H_g \geq 0 \). Then every free-boundary stable minimal hypersurface in \( M \) has a metric with positive scalar curvature and minimal boundary.

**Proof.** Consider \( \Sigma \) a compact orientable free-boundary stable minimal in \( M \). It follows from the second variation formula for the volume that

\[
\int_{\Sigma} |\nabla \varphi|^2 dv_g \geq \int_{\Sigma} \varphi^2 (\text{Ric}_g(N,N) + |B^g_{\Sigma}|^2) dv_g + \int_{\partial \Sigma} \varphi^2 B^g_{\partial M}(N,N) dv_{\sigma_g}
\]

for every \( \varphi \in C^\infty(\Sigma) \), where \( N \) denote a unit vector field on \( \Sigma \) in \((M,g)\). As \( R_g > 0 \), it follows from the Gauss equation that

\[
\text{Ric}_g(N,N) + |B^g_{\Sigma}|^2 = \frac{1}{2}(R_g - R^g_{\Sigma} + |B^g_{\Sigma}|^2) > -\frac{1}{2} R^g_{\Sigma}.
\]

Hence

\[
\int_{\Sigma} |\nabla \varphi|^2 dv_g > -\frac{1}{2} \int_{\Sigma} \varphi^2 R^g_{\Sigma} dv_g + \int_{\partial \Sigma} \varphi^2 B^g_{\partial M}(N,N) dv_{\sigma_g}
\]
for every $\varphi \in C^\infty(\Sigma)$. Since $H_g^{\partial M} \geq 0$, and $\Sigma$ teis a free-boundary hypersurface in $(M, g)$, we obtain

$$B_g^{\partial M}(N, N) = H_g^{\partial M} - H_g^{\partial \Sigma} \geq -H_g^{\partial \Sigma}.$$ 

Thus,

$$\int_{\Sigma} |\nabla \varphi|^2 dv_g > -\frac{1}{2} \int_{\Sigma} \varphi^2 R_g^\Sigma dv_g - \int_{\partial \Sigma} \varphi^2 H_g^{\partial \Sigma} d\sigma_g$$

for every $\varphi \in C^\infty(\Sigma)$. Consequently,

$$\int_{\Sigma} |\nabla \varphi|^2 dv_g + c_n \int_{\Sigma} \varphi^2 R_g^\Sigma dv_g + 2c_n \int_{\partial \Sigma} \varphi^2 H_g^{\partial \Sigma} d\sigma_g > (1 - 2c_n) \int_{\Sigma} |\nabla \varphi|^2 dv_g$$

for every $0 \neq \varphi \in H^1(\Sigma)$, where $c_n = \frac{n-2}{4(n-1)}$. It follows that

$$\lambda = \inf_{0 \neq \varphi \in H^1(\Sigma)} \frac{\int_{\Sigma} |\nabla \varphi|^2 dv_g + c_n \int_{\Sigma} \varphi^2 R_g^\Sigma dv_g + 2c_n \int_{\partial \Sigma} \varphi^2 H_g^{\partial \Sigma} d\sigma_g}{\int_{\Sigma} \varphi^2 dv_g} > 0.$$ 

Therefore, there exists a metric on $\Sigma$ with positive scalar curvature and minimal boundary (See remark 2.3). \[\square\]

3. 3-DIMENSIONAL CASE

Let $M$ be a smooth 3-dimensional manifold. A sphere which is either properly embedded in $M$ or contained in $\partial M$ is said to be incompressible if it does not bound a ball in $M$. A surface $\Sigma$ which is either properly embedded in $M$ or contained in $\partial M$ and which is not a sphere, is said to be incompressible if the homomorphism $\pi_1(\Sigma) \hookrightarrow \pi_1(M)$ is injective. A surface $\Sigma$ with non-empty boundary embedded in $M$ such that $\partial \Sigma \subset \partial M$, is said to be $\partial$-incompressible if the homomorphism $\pi_1(\Sigma, \partial \Sigma) \hookrightarrow \pi_1(M, \partial M)$ is injective. A properly embedded surface $\Sigma$ with non-empty boundary in $M$ is said to be essential if it is incompressible and $\partial$-incompressible.

**Theorem 3.1** (Chen, Fraser e Pang, [3]). Let $(M^3, g)$ be a compact orientable 3-dimensional Riemannian manifold with non-empty boundary. If $\Sigma$ is a connected, compact and orientable surface, which is not a disk and $f : \Sigma \to M$ is a continuous map with $f(\partial \Sigma) \subset \partial M$ such that

$$f_* : \pi_1(\Sigma) \to \pi_1(M) \text{ and } f^\partial : \pi_1(\Sigma, \partial \Sigma) \to \pi_1(M, \partial M),$$

are injectives, then there exists a free-boundary minimal immersion $F : \Sigma \to M$ with $F(\partial \Sigma) \subset \partial M$ and it minimizes area among the maps $h : \Sigma \to M$ with $h(\partial \Sigma) \subset \partial M$ such that $h_*$ and $h^\partial$ are injectives.
**Theorem 3.2** (Chen, Fraser e Pang, [3]). Let \((M^3, g)\) be a compact orientable 3-dimensional Riemannian manifold with non-empty boundary such that \(H_g \geq 0\). If \(\Sigma\) is a compact orientable surface with non-empty boundary and \(f : (\Sigma, \partial \Sigma) \rightarrow (M, \partial M)\) is a free-boundary stable minimal immersion, then

1. If \(R_g > 0\), we obtain that \(\Sigma\) is a disk.
2. If \(R_g \geq 0\), we obtain that either \(\Sigma\) is a disk or \((\Sigma, g)\) is a flat cylinder with totally geodesic boundary.

Define \(\tilde{C}_3\) as the set of all smooth, compact and orientable 3-dimensional manifolds with non-empty boundary such that there is no continuous map \(f : (\Sigma, \partial \Sigma) \rightarrow (M, \partial M)\) with \(f_*\) and \(f^\partial_*\) are injectives, where \(\Sigma\) is a compact orientable surface with genus \(l\) and \(k\) boundary components satisfying: \(l = 0\) and \(k \geq 2\), or \(l \geq 1\) and \(k \geq 1\).

Let \(M\) be a compact orientable 3-dimensional manifold. As a consequence of the Theorems 3.2 e 3.1 we obtain that if \(M \notin \tilde{C}_3\) then there is no metric on \(M\) with nonnegative scalar curvature and mean convex boundary.

Define \(\overline{C}_3\) as the set of all smooth, compact and orientable three-dimensional manifolds with non-empty boundary such that there is no continuous map \(f : (\Sigma, \partial \Sigma) \rightarrow (M, \partial M)\) with \(f_*\) and \(f^\partial_*\) injective, where \(\Sigma\) is a compact orientable surface with genus \(l\) and \(k\) boundary components satisfying \(l = 0\) and \(k \geq 2\), or \(l \geq 1\) and \(k \geq 1\).

**Example 3.3.** Let \(M\) be a orientable 3-dimensional handlebody. We have that \(\pi_1(M, \partial M) = 0\). It follows that \(M \in \overline{C}_3\). In particular, the solid torus \(S^1 \times D^2\) is in the set \(\overline{C}_3\).

**Example 3.4.** Consider the 3-dimensional manifold \(M = S^1 \times \Sigma\), where \(\Sigma \neq D^2\) is a compact, connected and orientable surface such that \(\partial \Sigma \neq \emptyset\). Note that \(\Sigma\) is essential in \(M\). Therefore, \(M \notin \overline{C}_3\).

**Example 3.5.** Consider the 3-dimensional manifold \(M = I \times S^2\). Since \(M\) is simply connected, we have that \(M \in \overline{C}_3\).

**Example 3.6.** Consider the 3-dimensional manifold \(M = I \times S\), where \(S\) is a closed, connected and orientable surface with positive genus. Let \(\Sigma = I \times \gamma\), where \(\gamma\) is a closed curve which represents a non-trivial class in \(\pi_1(S)\) and bounds a "hole" in \(S\). Note that \(\Sigma\) is an essential cylinder which is properly embedded in \(M\). Therefore, \(M \notin \overline{C}_3\).

**Lemma 3.7.** Let \((M, g)\) be a compact, orientable and connected 3-dimensional Riemannian manifold such that \(g\) is flat with totally geodesic boundary. Then, \(M \notin \overline{C}_3\).
Proof. It follows from the theorem 5 in [9] that either \( M \) is diffeomorphic to a 3-dimensional handlebody or \( M \) is covered by \( I \times T^2 \). Since \( (M, g) \) is flat with totally geodesic boundary, we have that \( (\partial M, g) \) is a flat surface. Assume the \( M \) is a 3-dimensional handlebody. This case, we have that \( \partial M \) is connected. It follows from the Gauss-Bonnet theorem that \( \partial M \) is a 2-dimensional torus. This implies that \( M = S^1 \times D^2 \) and \( \partial M \) is a stable minimal flat torus in \( (M, g) \). But, this is a contradiction (see Theorem 8 in ([9])). Therefore, \( M \) is covered by \( I \times T^2 \). Consider then \( p : I \times T^2 \to M \) a covering map and \( C \) an essential cylinder which is properly embedded in \( I \times T^2 \). Define \( f = p \circ i : (C, \partial C) \to (M, \partial M) \), where \( i : C \to I \times T^2 \) is the inclusion map. Note that \( f_* \) and \( f^\partial \) are injective. Therefore, \( M \not\in \mathcal{C}_3 \). \( \square \)

**Theorem 3.8.** Let \( M^3 \) be a connected, compact and orientable 3-dimensional manifold with non-empty boundary \( \partial M \). Assume that the connected components of \( \partial M \) are spheres or incompressible tori, but at least one of the components is a torus. Then \( M \not\in \mathcal{C}_3 \).

**Proof.** First, \( M \) contains a compact, connected and orientable properly embedded incompressible surface \( \Sigma \) with boundary \( \partial \Sigma \neq \emptyset \) such that \( 0 \neq [\partial \Sigma] \in H_1(\partial M) \) (see lemma 6.8 in [8]).

1. \( \Sigma \) is not a disk.

In fact, assume that \( \Sigma \) is a disk. As \( 0 \neq [\partial \Sigma] \in H_1(\partial M) \), we have that \( \partial \Sigma \) represents a non-trivial class in \( \pi_1(\partial M) \). Hence, \( \partial \Sigma \) are in a connected component \( S \) of \( \partial M \) which is a torus (see the figure 1).

![Figure 1. \([\partial \Sigma] \in \pi_1(\partial M)\) is not trivial](image)

Note that the inclusion \( i : \Sigma \to M \) represents a non-trivial class of \( \pi_2(\Sigma, \partial \Sigma) \). As the connected components of \( \partial M \) are incompressible tori or spheres, we have that \( \pi_1(\partial M, x_0) \hookrightarrow \pi_1(M, x_0) \) is injective, for
every $x_0 \in \partial M$. Thus, we have the following exact sequence

$$
\cdots \to \pi_2(\partial M) \to \pi_2(M) \to \pi_2(M, \partial M) \to 0.
$$

This implies that map $\pi_2(M) \hookrightarrow \pi_2(M, \partial M)$ is onto. Hence, there exists $f : \Sigma \to M$ with $f(\partial \Sigma) = x_0 \in S$ and $0 \neq [f] \in \pi_2(M)$ such that $[i] = [f]$ in $\pi_2(M, \partial M)$. Consequently, $\partial \Sigma$ represents a non-trivial class in $\pi_1(\partial M)$. However, this is a contradiction. Then, $\Sigma$ is not a disk.

As $\Sigma$ is an incompressible surface, which is not a disk, we have that each connected component of $\partial \Sigma$ represents a non-trivial class in $\pi_1(\partial M)$. This implies that $\partial \Sigma$ is contained in the union of the connected components of $\partial M$ whose are torus. Hence, either $\Sigma$ is $\partial$-incompressible or it is a cylinder $\partial$-compressible (see lemma 1.10 in [7]).

2. $\Sigma$ is not a $\partial$-compressible cylinder.

In fact, suppose that $\Sigma$ is a $\partial$-compressible cylinder. In this case, we have that connected components $\alpha_1$ and $\alpha_2$ of $\partial \Sigma$ are contained in a same torus of $\partial M$. Consequently, we have only two possible situation for the circles $\alpha_1$ and $\alpha_2$, as we can see in the figures below.

Note that in both situation we have that $\alpha_1$ and $\alpha_2$ are homologous in $\partial M$. Hence, we have that $\partial \Sigma$ represents a non-trivial class in $H_1(\partial M)$. It follows that $\Sigma$ is not a $\partial$-compressible cylinder.

Hence, $\Sigma$ is an essential surface in $M$ which is not a disk. Therefore, $M \notin \mathcal{C}_3$. \qed

**Remark 3.9.**
A. Note that, it follows from the proposition 3.8, if the number of the incompressible tori in $\partial M$ is exactly one then the essential surface $\Sigma$ can not be a cylinder. Consequently, $M \notin \tilde{C}_3$.

B. The incompressibility condition of at least one torus of $\partial M$ in the proposition above is necessary. Actually, just consider the 3-dimensional manifold $M = S^1 \times D^2$. Note that the connected component of $\partial M$ is a compressible torus and $M \in C_3$.

**Corollary 3.10.** Let $M_1, \cdots, M_k$ be compact, connected and orientable 3-dimensional manifolds as in proposition 3.8, and $N_1, \cdots, N_s$ closed, connected and orientable 3-dimensional manifolds. For every integer $l \geq 0$, we have that

1. $M_1 \# \cdots \# M_k \# \natural \Bbb B^3 \notin \tilde{C}_3$,
2. $M_1 \# \cdots \# M_k \# N_1 \# \cdots \# N_s \# \natural \Bbb B^3 \notin \tilde{C}_3$.

Moreover, if the number of the incompressible tori in $\partial M_1$ is exactly one then

3. $M_1 \# \natural \Bbb B^3 \notin \tilde{C}_3$,
4. $M_1 \# N_1 \# \cdots \# N_s \# \natural \Bbb B^3 \notin \tilde{C}_3$.

**Example 3.11.** Define the 3-dimensional manifolds $M_1 = (S^1 \times \hat{T}^2) \# N$ and $M_2 = (S^1 \times \hat{T}^2) \# (I \times S^2)$, where $\hat{T}^2$ is a torus minus an open disk and $N$ is a closed, connected and orientable 3-dimensional manifold. It follows from the corollary 3.10 that $M_1, M_2 \notin \tilde{C}_3$. This implies that $M_1$ and $M_2$ have no metric with nonnegative scalar curvature and mean convex boundary.

**Proposition 3.12.** Let $(M^3, g)$ be a compact, connected and orientable 3-dimensional Riemannian manifold with boundary such that $R_g \geq 0$ and $H_g \geq 0$. Then either $M \in \tilde{C}_3$ or $(M, g)$ is flat with totally geodesic boundary. In particular, if $R_g > 0$ and $H_g \geq 0$ or $R_g \geq 0$ and $H_g > 0$ then $M \in \tilde{C}_3$.

**Proof.** Note that as $R_g \geq 0$ and $H_g \geq 0$, we have that $M \not\in \tilde{C}_3$. Assume that $M \not\in \tilde{C}_3$ and $g$ is not flat or $B_g^{\partial M} \neq 0$. Since $M \in \tilde{C}_3$, we obtain that $M$ possesses an embedded essential cylinder $C$. As $g$ is not flat or $B_g^{\partial M} \neq 0$, it follows from the proposition 2.4 there exists a Riemannian metric $h$ on $M$ such that $R_h > 0$ and $H_h^{\partial M} \equiv 0$. It follows from the Theorem 3.1 that there exists a stable free-boundary minimal immersion $f : (C, \partial C) \to (M, \partial M)$ with respect to the metric $h$. Hence, from Theorem 3.2, we have a contradiction. This implies that $M \in \tilde{C}_3$ or $(M, g)$ flat with totally geodesic boundary. It follows from the lemma 3.7 that either $M \in \tilde{C}_3$ or $(M, g)$ is flat with totally geodesic boundary.
Example 3.13. Consider the 3-dimensional manifold $I \times S$, where $S$ is a closed, connected and orientable surface with positive genus. It follows from the proposition 3.12 that there is no metric on $I \times S$ with positive scalar curvature and mean convex boundary or nonnegative scalar curvature and strictly mean convex boundary. In particular, there is no such metric on $I \times T^2$.

Example 3.14. It follows from the corollary 3.10 that the 3-dimensional manifolds below are not in the set $\mathcal{C}_3$.

1. $(I \times T^2) \# (I \times T^2)$
2. $(S^1 \times \tilde{T}^2) \# (S^1 \times T^2)$
3. $(I \times T^2) \# (S^1 \times \tilde{T}^2)$
4. $(I \times T^2) \# (I \times S^2)$
5. $(S^1 \times \tilde{T}^2) \# (I \times S^2)$
6. $(I \times T^2) \# N$, where $N$ is a closed, connected and orientable 3-dimensional manifold.

Therefore, it follows from the proposition 3.12 that these manifolds have no metric with nonnegative scalar curvature and strictly mean convex boundary or with positive scalar curvature and mean convex boundary.

4. n-DIMENSIONAL CASE, $3 \leq n \leq 7$

In this section, we want to study possible generalisation of some results on the existence of certain metrics on 3-dimensional manifolds to manifolds with dimension not greater than seven. The following theorem is a very important result from geometric measure theory which plays a fundamental role in our investigations.

Theorem 4.1 (See Chapter 8 in [10] and Theorem 5.4.15 in [5]). Let $(M^n, g)$, $3 \leq n \leq 7$, be a compact orientable Riemannian manifold with non-empty boundary. Assume that $\alpha \in H_{n-1}(M, \partial M)$ is a nontrivial element. Then there exists a free-boundary stable minimal hypersurface $\Sigma$ properly embedded in $M$ which represents the class $\alpha$.

For $n \geq 4$, we define inductively the set $\mathcal{C}_n$ as the set of all smooth, compact and orientable n-dimensional manifolds with non-empty boundary $M$ such that every nontrivial class $\alpha \in H_{n-1}(M, \partial M)$ can be represented by a compact orientable hypersurface with boundary $\Sigma$ such that $\Sigma \in \mathcal{C}_{n-1}$.

Theorem 4.2. Let $M^n$, $3 \leq n \leq 7$, be a compact orientable manifold with non-empty boundary such that $M \notin \mathcal{C}_n$. Then there is no metric on $M$ with nonnegative scalar curvature and mean convex boundary.
Proof. We note that it follows from a theorem above the it is true for \( n = 3 \). We proof by induction on \( n \). Assume the result is valid for \( n - 1 \). Assume there exists a metric \( g \) on \( M \) such that \( R_g \geq 0 \) and \( H_{g\mathcal{M}} \geq 0 \). It follows from proposition 2.4 that two cases can occurs.

(1) There exists a metric \( h \) on \( M \) such that \( R_h > 0 \) and \( H_{\partial M} \equiv 0 \). In this case, since \( M \not\in \tilde{\mathcal{C}}_n \), there exists a nontrivial class \( \alpha \in H_{n-1}(M, \partial M) \) which can be represented by a compact orientable hypersurface \( \Sigma \) with boundary such that \( \Sigma \not\in \tilde{\mathcal{C}}_{n-1} \). From theorem 4.1, we can assume that \( \Sigma \) is a free-boundary minimal stable hypersurface in \((M, h)\). From the proposition 2.5 there exists a metric on \( \Sigma \) with positive scalar curvature and minimal boundary. However, this is a contradiction since \( \Sigma \not\in \tilde{\mathcal{C}}_{n-1} \) and from the induction hypothesis does not exists such metric.

(2) Assume \( \text{Ric}_g \equiv 0 \) and \( B_{g\mathcal{M}} \equiv 0 \). Arguing as in the item (1), we obtain there exists a compact orientable free-boundary stable minimal hypersurface \( \Sigma \) in \((M, g)\) such that \( \Sigma \not\in \tilde{\mathcal{C}}_{n-1} \). Since \( \Sigma \) is free-boundary in \((M, g)\), we have

\[
H_{g\Sigma} = H_{g\mathcal{M}} - B_{g\mathcal{M}}(N, N) \equiv 0,
\]

where \( N \) is a unit vector field of \( \Sigma \) em \((M, g)\). Also, it follows from the Gauss equation and of the stability of \( \Sigma \) that \( R_{g\Sigma} \equiv 0 \). However, this is a contradiction, since \( \Sigma \not\in \tilde{\mathcal{C}}_{n-1} \), from the induction hypothesis, and does not exists a such metric on \( \Sigma \) with null scalar curvature and minimal boundary.

\( \Box \)

Example 4.3. Consider the \( n \)-dimensional manifold \( M^n = T^{n-2} \times \Sigma \), where \( \Sigma \) is a compact, connected and orientable surface with boundary which is not a disk or a cylinder. We have the following chain of submanifolds

\[
S^1 \times \Sigma \subset T^2 \times \Sigma \subset \cdots \subset T^{n-3} \times \Sigma \subset M.
\]

Since the surface \( \Sigma \) is essential in \( S^1 \times \Sigma \), we obtain that \( S^1 \times \Sigma \not\in \tilde{\mathcal{C}}_3 \).

Hence \( M \not\in \tilde{\mathcal{C}}_n \), for every \( n \geq 3 \). It follows from theorem 4.2 that there is no metric on \( M \) with nonnegative scalar curvature and mean convex boundary, if \( 3 \leq n \leq 7 \).

For \( n \geq 4 \), we define inductively \( \overline{\mathcal{C}}_n \) as the set of all smooth, compact and orientable \( n \)-dimensional manifolds with non-empty boundary \( M \) such that every nontrivial class \( \alpha \in H_{n-1}(M, \partial M) \) can be represented by a compact orientable hypersurface with boundary \( \Sigma \) such that \( \Sigma \in \overline{\mathcal{C}}_{n-1} \).
Proposition 4.4. Let $M^n$, $3 \leq n \leq 7$, be a compact orientable manifold with non-empty boundary such that $R_g \geq 0$ and $H_g \geq 0$. Then $M \in \overline{C}_n$ or $(M, g)$ is Ricci flat with totally geodesic boundary. In particular, if $R_g > 0$ and $H_g \geq 0$, or $R_g \geq 0$ and $H_g > 0$, then $M \in \overline{C}_n$.

Proof. It follows from the proposition 3.12 that the result is valid for $n = 3$. Let's do it by induction on $n$. Assume the result is valid for $n - 1$. Suppose that $\text{Ric}_g \not\equiv 0$ or $B_g \not\equiv 0$ and $M \not\in \overline{C}_n$. It follows from the proposition 2.4 that there exists a metric $h$ on $M$ such that $R_h > 0$ and $H_h \equiv 0$. Since $M \not\in \overline{C}_n$, from the theorem 4.1 we have there exists a compact orientable free-boundary minimal hypersurface $\Sigma$ in $(M, h)$ such that $\Sigma \not\in \overline{C}_{n-1}$. From the induction hypothesis we have that $\Sigma$ does not admit a metric with positive scalar curvature and minimal boundary. This is a contradiction with the proposition 2.5. Therefore, $M \in \overline{C}_n$ or $(M, g)$ is Ricci flat with totally geodesic boundary. □

Example 4.5. Consider the $n$-dimensional manifold $M^n = T^{n-2} \times \Sigma$, where $\Sigma$ is a cylinder. We have the following chain of submanifolds

$$S^1 \times \Sigma \subset T^2 \times \Sigma \subset \cdots \subset T^{n-3} \times \Sigma \subset M.$$ 

Note that, as the surface $\Sigma$ is an essential cylinder in $S^1 \times \Sigma$, we have that $S^1 \times \Sigma \not\in \overline{C}_3$. Thus, $M \not\in \overline{C}_n$, for every $n \geq 3$. It follows from the proposition 4.4 that there exists no metric on $T^{n-2} \times \Sigma$ with positive scalar curvature and mean convex boundary or nonnegative scalar curvature with strictly mean convex boundary, for $3 \leq n \leq 7$.

Example 4.6. Consider the $n$-dimensional manifold $I \times T^{n-1}$ and the chain of submanifolds

$$I \times T^2 \subset I \times T^3 \subset \cdots \subset I \times T^{n-1}.$$ 

Since $I \times T^2 \not\in \overline{C}_3$, we have that $I \times T^{n-1} \not\in \overline{C}_n$ for all $n \geq 3$. Hence, from the proposition 4.4, there exists no metric on $I \times T^{n-1}$ with positive scalar curvature and mean convex boundary or nonnegative scalar curvature with strictly mean convex boundary, for $3 \leq n \leq 7$.

Theorem 4.7 (Fraser e Li, [6]). Let $(M^n, g)$ be a compact Riemannian manifold with non-empty boundary $\partial M \neq \emptyset$. Assume that $(M, g)$ has nonnegative Ricci curvature and strictly convex boundary with respect to the inward unit normal vector. Then $\partial M$ is connected and $i_* : \pi_1(\partial M) \to \pi_1(M)$ is onto.

Remark 4.8. The theorem above is also valid if $(M, g)$ has positive Ricci curvature and convex boundary.

Corollary 4.9. Let $M^n$, $n \geq 3$, a smooth compact, connected and orientable manifold with non-empty boundary $\partial M \neq \emptyset$. We have that...
(1) If there is a map \( F : M \to \mathbb{T}^{n-2} \times \mathbb{T}^2 \) with \( \deg(F) = 1 \), then there is no metric on \( M \) with nonnegative Ricci curvature and strictly convex boundary.

(2) If there is a map \( F : M \to I \times \mathbb{T}^{n-1} \) with \( \deg(F) \neq 0 \), then there is no metric on \( M \) with nonnegative Ricci curvature and strictly convex boundary or positive Ricci curvature and convex boundary.

**Proof.**

(1) Denote by \( N \) the manifold \( \mathbb{T}^{n-2} \times \mathbb{T}^2 \). Suppose there exists such metric on \( M \). It follows from the theorem 4.7 that \( \partial M \) is connected and \( i_* : \pi_1(\partial M) \to \pi_1(M) \) is onto. Define the map \( f := F|_{\partial M} : \partial M \to \partial N \). Since \( \partial M \) is connected, we have \( \deg(f) = 1 \). Hence, the homomorphisms \( F_* : \pi_1(M) \to \pi_1(N) \) and \( f_* : \pi_1(\partial M) \to \pi_1(\partial N) \) are onto (see [8], Lemma 15.12). Therefore, since the diagram

\[
\pi_1(M) \xrightarrow{F_*} \pi_1(N) \\
\downarrow i_* \quad \quad \quad \downarrow j_* \\
\pi_1(\partial M) \xrightarrow{f_*} \pi_1(\partial N)
\]

commutes, we obtain that \( j_* \) is also onto. Note that \( \partial N \) is incompressible in \( N \). Then, \( j_* \) is an isomorphism. This is a contradiction since

\[
\pi_1(N) \cong \mathbb{Z}^{n-2} \times (\mathbb{Z} \ast \mathbb{Z}) \quad \text{and} \quad \pi_1(\partial N) \cong \pi_1(T^{n-1}) \cong \mathbb{Z}^{n-1}.
\]

(2) We have that \( \deg(F) \neq 0 \) implies that \( \partial M \) is not connected. Then the result follows from the theorem 4.7.

\[\square\]

5. **Free boundary minimal \( k \)-slicings**

All the manifolds considered here are compact, connected and orientable.

6. **Definition and Examples**

Let \( (M^n, g) \) be a Riemannian manifold with non-empty \( \partial M \neq \emptyset \). Assume that \( H_{n-1}(M, \partial M) \neq 0 \). Thus, we can choose a compact properly embedded free-boundary hypersurface \( \Sigma_{n-1} \subset M \) which minimizes volume in \( (M, g) \). Choose \( u_{n-1} > 0 \) a first eigenfunction for the second
variation of the volume \( S_{n-1}(.,.) \) of \( \Sigma_{n-1} \) in \((M, g)\). Define \( \rho_{n-1} = u_{n-1} \) and the weighted volume functional \( V_{\rho_{n-1}} \) for hypersurfaces of \( \Sigma_{n-1} \),

\[
V_{\rho_{n-1}}(\Sigma) = \int_{\Sigma} \rho_{n-1} d\mathcal{H}^{n-2},
\]

where \( d\mathcal{H}^{n-2} \) denote the Hausdorff \((n-2)\)-dimensional measure inducted from the space \((M, g)\). Assume there exists a properly embedded free-boundary hypersurface \( \Sigma_{n-2} \) in \( \Sigma_{n-1} \) which minimizes the weighted volume functional \( V_{\rho_{n-1}} \) in \( (\Sigma_{n-1}, g) \). Choose a first eigenfunction \( u_{n-2} > 0 \) for the second variation of the volume, \( S_{n-2}(.,.) \), \( V_{\rho_{n-1}} \). Define \( \rho_{n-2} = \rho_{n-1} u_{n-2} \). Assume that we can keep doing this, inductively. Hence, we obtain a family of free-boundary minimal submanifolds \( \Sigma_k \subset \cdots \subset \Sigma_{n-1} \subset \Sigma_n := (M, g) \), which was constructed by choosing, for each \( j \in \{k, \cdots, n-1\} \), a properly embedded free-boundary hypersurface \( \Sigma_j \) in \( \Sigma_{j+1} \) which minimizes the weighted volume functional \( V_{\rho_{j+1}} \) in \( (\Sigma_{j+1}, g) \), where \( \Sigma_n := M \) and \( \rho_{j+1} := \rho_{j+2} u_{j+1} = u_{j+1} u_{j+2} \cdots u_{n-1} \). We call such family of free-boundary minimal hypersurfaces a free-boundary minimal \( k \)-slicing in \( (M, g) \).

**Example 6.1.** Let \((N^k, g)\) be a \( k \)-dimensional Riemannian manifold with boundary \( \partial N \neq \emptyset \). Consider the following \( n \)-dimensional Riemannian manifold \( (N \times T^{n-k}, g + \delta) \), where \( \delta \) is the flat metric on the torus \( T^{n-k} \). The family of smooth hypersurfaces

\[
N \subset N \times S^1 \subset N \times T^2 \subset \cdots \subset N \times T^{n-k-1} \subset (N \times T^{n-k}, g + \delta),
\]

where \( \rho_j \equiv u_j \equiv 1 \), for every \( j = k, \cdots, n-1 \), is a free-boundary minimal \( k \)-slicing in \( (N \times T^{n-k}, g + \delta) \).

### 7. Geometric formulas for the regular points

Let \((M^n, g)\) be a Riemannian manifold with boundary \( \partial M \neq \emptyset \). Consider a free-boundary \( k \)-slicing in \( M \):

\[
\Sigma_k \subset \cdots \subset \Sigma_{n-1} \subset \Sigma_n = M.
\]

For simplicity, we assume that \( k \)-slicing is smooth.

**Notation:**

- \( R_j := \) scalar curvature of \((\Sigma_j, g)\).
- \( H_j := \) mean curvature of \( \Sigma_j \) em \( (\Sigma_{j+1}, g) \)
- \( \nu_j := \) smooth unit vector field of \( \Sigma_j \) in \((\Sigma_{j+1}, g)\).
- \( \eta_j := \) outer unit smooth vector field on the boundary \( \partial \Sigma_j \) in \((\Sigma_j, g)\).
- \( B_j := \) Second fundamental form of \( \Sigma_j \) in \((\Sigma_{j+1}, g)\).
\[ B^{\partial \Sigma_j} := \text{Second fundamental form of } \partial \Sigma_j \text{ in } (\Sigma_j, g). \]
\[ H^{\partial \Sigma_j} := \text{mean curvature of } \partial \Sigma_j \text{ in } (\Sigma_j, g). \]

**Lemma 7.1.** Let \((M^{n+1}, g)\) be a Riemannian manifold with boundary \(\partial M \neq \emptyset\) and \(\Sigma\) a free-boundary hypersurface in \(M\). Denote by \(\nu\) an unit vector field of \(\Sigma\) in \((M, g)\), and \(\eta\) and \(\bar{\eta}\) the outer unit vector field of \(\partial \Sigma\) in \((\Sigma, g)\) and \(\partial M\) in \((M, g)\), respectively. We have that

1. \(\eta = \bar{\eta}\) em \(\partial \Sigma\).
2. \(B^{\partial \Sigma}(X,Y) = B^{\partial M}(X,Y), \quad \forall X,Y \in T(\partial \Sigma)\)
3. \(H^{\partial \Sigma} = H^{\partial M} - B^{\partial M}(\nu, \nu)\).

**Proof.**

1. Since \(\Sigma\) is a free-boundary hypersurface in \(M\), we obtain that \(\partial \Sigma \subset \partial M\) and \(\nu \perp \eta\) in \(\partial \Sigma\). Hence, for every \(x \in \partial \Sigma\), we have that \(\bar{\eta}(x) \in T_x \Sigma\) and \(\bar{\eta}(x) \perp v\), for every \(v \in T_x \partial \Sigma\), since \(T_x \partial \Sigma \subset T_x \partial M\). Thus, \(\eta = \bar{\eta}\) in \(\partial \Sigma\).

2. Denote \(\nabla^M\) and \(\nabla^\Sigma\) the Riemannian connection of \((M, g)\) and \((\Sigma, g)\), respectively. Let \(x \in \partial \Sigma\) and \(X,Y \in T_x(\partial \Sigma)\). We have that

\[
B^{\partial M}(X,Y) = -\langle(\nabla^M_X \eta)(x), Y\rangle = -\langle(\nabla^M_X \bar{\eta})(x), Y\rangle,
\]

where \((\nabla^M_X \bar{\eta})(x)\) is the tangent component of \((\nabla^M_X \bar{\eta})(x)\) with respect to the decomposition

\[
T_x M = T_x \Sigma \oplus N_x(\Sigma, M).
\]

It follows from the item 1 that \(\eta = \bar{\eta}\) along any curve in \(\partial \Sigma \subset M\) passing at \(x\) with velocity \(X\). Hence,

\[
\nabla^M_X \bar{\eta}(x) = \nabla^M_X \eta(x).
\]

We obtain

\[
B^{\partial M}(X,Y) = -\langle(\nabla^M_X \eta)(x), Y\rangle = \langle(\nabla^\Sigma_X \eta)(x), Y\rangle = B^{\partial \Sigma}(X,Y).
\]

3. Let \(x \in \partial \Sigma\) and \(\{e_1, \ldots, e_{n-1}\}\) an orthonormal basis of \(T_x \partial \Sigma\). It follows from the item 2 that

\[
H^{\partial \Sigma}(x) = \sum_{i=1}^{n-1} B^{\partial \Sigma}(e_i, e_i)
\]
\[
= \sum_{i=1}^{n-1} B^{\partial M}(e_i, e_i) + B^{\partial M}(\nu, \nu)(x) - B^{\partial M}(\nu, \nu)(x)
\]
\[
= H^{\partial M}(x) - B^{\partial M}(\nu, \nu)(x)
\]
Remark 7.2. Since $\Sigma_j$ is a free-boundary hypersurface in $(\Sigma_{j+1}, g)$, for every $j = k, \cdots, n - 1$, it follows from the Lemma 7.1 that we have

(1) $\eta_j = \eta$ em $\partial \Sigma_j$, onde $\eta: = \eta_n$,

(2) Se $j \leq p \leq n - 1$, ento $\nu_p \perp \eta_j$ e $\eta_p = \eta_j$ in $\partial \Sigma_j$,

(3) $H^{\partial \Sigma_j} = H^{\partial \Sigma_{j+1}} - B^{\partial \Sigma_{j+1}}(\nu_j, \nu_j) = H^{\partial M} - \sum_{p=j}^{n-1} B^{\partial \Sigma_{p+1}}(\nu_p, \nu_p)$.

Lemma 7.3. Let $(M^{n+1}, g)$ be Riemannian manifold, $f \in C^\infty(M)$ and $S$ a compact hypersurface of $M$. The mean curvature of $\Sigma$ in $M$, with respect to the weighted volume functional

$$V_f(\Sigma) = \int_\Sigma e^f dv_g,$$

is given by

$$H_f = H - \langle \nabla_M f, \nu \rangle,$$

where $\nu$ and $H$ denote a unit vector field and the mean curvature of $\Sigma$ in $(M, g)$, respectively.

It follows from the Lemma 7.3 and the fact that $\Sigma_j$ is a minimal hypersurface of $\Sigma_{j+1}$ with respect to the weighted volume functional

$$V_{j+1}(\Sigma) = \int_\Sigma \rho_{j+1} d\mu^j = \int_\Sigma e^{(\log \rho_{j+1})} d\mu^j,$$

that

$$H_j = \langle \nabla_{j+1} \log \rho_{j+1}, \nu_j \rangle.$$

For each $j \in \{k, \cdots, n - 1\}$, define on $\Sigma_j \times T^{n-j}$ a metric

$$\hat{g}_j = g + \sum_{p=j}^{n-1} u_p^2 dt_p^2.$$

Observe that $\Sigma_j \times T^{n-j-1}$ is a free-boundary minimal hypersurface in $(\Sigma_{j+1} \times T^{n-j-1}, \hat{g}_{j+1})$.

Notation:

- $\hat{B}_j$: second fundamental form of $\Sigma_j \times T^{n-j-1}$ in $(\Sigma_{j+1} \times T^{n-j-1}, \hat{g}_{j+1})$.
- $\hat{H}_j$: mean curvature of $\Sigma_j \times T^{n-j-1}$ in $(\Sigma_{j+1} \times T^{n-j-1}, \hat{g}_{j+1})$.
- $\hat{R}_j$: scalar curvature of $(\Sigma_j \times T^{n-j-1}, \hat{g}_{j+1})$.
- $\hat{R}_j$: scalar curvature of $(\Sigma_{j+1} \times T^{n-j-1}, \hat{g}_{j+1})$.
- $\hat{B}_j^\partial$: Second fundamental form of $\partial \Sigma_j \times T^{n-j-1}$ in $(\Sigma_j \times T^{n-j-1}, \hat{g}_{j+1})$. 
Lemma 7.4. We have that
\[ \tilde{B}_j = B_j - \sum_{p=j+1}^{n-1} u_p \nu_j(u_p) dt_p^2 \quad \text{and} \quad \tilde{B}_j^\alpha = B_j^{\alpha \Sigma_j} - \sum_{p=j+1}^{n-1} u_p \eta_j(u_p) dt_p^2. \]

In particular,
\[ |\tilde{B}_j|^2 = |B_j|^2 + \sum_{p=j+1}^{n-1} (\nu_j(\log u_p))^2 \quad \text{and} \quad \tilde{B}_j^{\alpha \Sigma_j}(\nu_j, \nu_j) = B_j^{\alpha \Sigma_j}(\nu_j, \nu_j). \]

Lemma 7.5. We have that \( \Sigma_j \times T^{n-j-1} \) is a minimal hypersurface in \( (\Sigma_{j+1} \times T^{n-j-1}, \hat{g}_{j+1}) \), i.e., \( \tilde{H}_j = 0 \).

Proof. Consider \( \{e_1, \ldots, e_{n-1}\} \) an orthonormal basis of \( T(\Sigma_j \times T^{n-j-1}) \) such that \( \{e_1, \ldots, e_j\} \) is an orthonormal basis of \( T\Sigma_j \) and
\[ e_l := \frac{\partial t_l}{u_l}, \quad l = j + 1, \ldots, n - 1. \]

From the Lemma 7.4, we have that
\[ \tilde{H}_j = \sum_{l=1}^j \tilde{B}_j(e_l, e_l) + \sum_{l=j+1}^{n-1} \tilde{B}_j(e_l, e_l) \]
\[ = \sum_{l=1}^j B_j(e_l, e_l) - \sum_{p=j+1}^{n-1} \nu_j(u_p) u_p \]
\[ = H_j - \sum_{p=j+1}^{n-1} \langle \nabla_j \log u_p, \nu_j \rangle \]
\[ = H_j - \langle \nabla_j \log \rho_{j+1}, \nu_j \rangle. \]

The equality (7.1) implies that \( \tilde{H}_j = 0 \). \( \square \)

Note that, for every hypersurface \( \Sigma \subset \Sigma_{j+1} \), we obtain
\[ (7.2) \quad \text{Vol}(\Sigma \times T^{n-j-1}, \hat{g}_{j+1}) = \int_{\Sigma} \rho_{j+1} \mu^j = V_{\rho_{j+1}}(\Sigma). \]

Denote by \( S_j \) and \( \tilde{S}_j \) the quadratic forms associated to the second variation formula of \( \Sigma_j \) in \( \Sigma_{j+1} \) with respect to the weighted volume functional \( V_{\rho_{j+1}} \) and of \( \Sigma_j \times T^{n-j-1} \) in \( (\Sigma_{j+1} \times T^{n-j-1}, \hat{g}_{j+1}) \), respectively. From the Lemma 7.5, and equality (7.2), we have that
\[ (7.3) \quad S_j(\varphi, \varphi) = \tilde{S}_j(\varphi, \varphi) = \int_{\Sigma_j} (|\nabla_j \varphi|^2 - c_j \varphi^2) \rho_{j+1} \mu^j - \int_{\partial \Sigma_j} \varphi^2 B^{\alpha \Sigma_{j+1}}(\nu_j, \nu_j) \rho_{j+1} \mu^{j-1}, \]
for every $\varphi \in C^\infty(\Sigma_j)$, where
\[
c_{j} := \frac{1}{2}(\hat{R}_{j+1} - \tilde{R}_{j} + |\tilde{B}_{j}|^2).
\]
Define the quadratic form
\[
Q_{j}(\varphi, \varphi) = S_{j}(\varphi, \varphi) + \frac{3}{8} \int_{\Sigma_j} \left[ |\tilde{B}_{j}|^2 + \frac{1}{3n} \sum_{p=j+1}^{n-1} \left( |\nabla_{j} \log u_{p}|^2 + |\tilde{B}_{p}|^2 \right) \right] \varphi^2 \rho_{j+1} d\mu^j
\]
\[
= - \int_{\Sigma_j} \varphi \tilde{L}_{j}(\varphi) \rho_{j+1} d\mu^j + \int_{\partial \Sigma_{j}} \varphi \left( \frac{\partial \varphi}{\partial \eta_{j}} - \varphi B^{\partial \Sigma_{j+1}}(\nu_{j}, \nu_{j}) \right) \rho_{j+1} d\mu^{j-1},
\]
where $\tilde{L}_{j} : C^\infty(\Sigma_{j}) \rightarrow C^\infty(\Sigma_{j})$ is a differential operator given by
\[
\tilde{L}(\varphi) = \tilde{\Delta}_{j} \varphi + \frac{1}{2}(\hat{R}_{j+1} - \tilde{R}_{j}) \varphi + \frac{1}{8} \left( |\tilde{B}_{j}|^2 - \frac{1}{n} \sum_{p=j+1}^{n-1} \left( |\nabla_{j} \log u_{p}|^2 + |\tilde{B}_{p}|^2 \right) \right) \varphi,
\]
where $\tilde{\Delta}_{j}$ denote the Laplacian operator of $(\Sigma_{j} \times T^{n-j-1}, \hat{g}_{j+1})$.

Since $\Sigma_{j}$ minimizes the volume in $\Sigma_{j+1}$ with respect to the weighted volume functional $V_{\rho_{j+1}}$, we have that $S_{j}(\varphi, \varphi) \geq 0$, for every $\varphi \in C^\infty(\Sigma_{j})$. Consequently, $Q_{j}(\varphi, \varphi) \geq 0$, $\forall \varphi \in C^\infty(\Sigma_{j})$. It follows that the first eigenvalue $\lambda_{j}$ of the quadratic form $Q_{j}$ is nonnegative and the eigenfunctions are positive. From now on, we choose $u_{j}$ as a first eigenfunction of $Q_{j}$, i.e., $u_{j} \in C^\infty(\Sigma_{j})$ satisfying
\[
(7.4) \quad \begin{cases}
\tilde{L}_{j}(u_{j}) = -\lambda_{j} u_{j} \\
\frac{\partial u_{j}}{\partial \eta_{j}} = u_{j} B^{\partial \Sigma_{j+1}}(\nu_{j}, \nu_{j})
\end{cases}
\]

**Lemma 7.6** (Schoen and Yau, [12]). We have that
\[
\tilde{R}_{j} = R_{j} - 2 \sum_{p=j+1}^{n-1} u_{p}^{-1} \Delta_{j} u_{p} - 2 \sum_{j+1 \leq p \leq n-1} (\nabla_{j} \log u_{p}, \nabla_{j} \log u_{q}),
\]
or, equivalently,
\[
\tilde{R}_{j} = R_{j} - 4 \rho_{j+1}^{\frac{1}{2}} \Delta_{j} (\rho_{j+1}^{\frac{1}{2}}) - \sum_{p=j+1}^{n-1} |\nabla_{j} \log u_{p}|^2.
\]

**Lemma 7.7** (Schoen and Yau, [12]). We have that
\[
\tilde{R}_{j} = R_{n} + 2 \sum_{p=j}^{n-1} \lambda_{p} + \frac{1}{4} \sum_{p=j}^{n-1} \left[ |\tilde{B}_{p}|^2 - \frac{1}{n} \sum_{q=p+1}^{n-1} (|\nabla_{p} \log u_{q}|^2 + |\tilde{B}_{q}|^2) \right].
\]
Lemma 7.8 (Schoen and Yau, [12]). If $R^n \geq c$, for some constant $c > 0$, then

$$|\tilde{B}_j|^2 + \tilde{R}_{j+1} \geq c - \frac{1}{4} \sum_{p=j+1}^{n-1} |\nabla_j \log u_p|^2.$$ 

Proposition 7.9. If $R_n \geq c > 0$ and $H^{\partial M} \geq 0$ then

$$4 \int_{\Sigma_j} |\nabla_j \varphi|^2 d\mu_j \geq -2 \int_{\partial \Sigma_j} \phi^2 H^{\partial \Sigma_j} d\mu_{j-1} + \int_{\Sigma_j} \varphi^2 \left(c + \frac{1}{8} \sum_{p=j+1}^{n-1} |\nabla_j \log u_p|^2 - R_j\right) \mu_j,$$

for every $\varphi \in C^\infty(\Sigma_j)$ and $j = k, \ldots, n - 1$.

Proof. Since $\Sigma_j$ minimizes the volume in $\Sigma_{j+1}$ with respect to the weighted volume functional $V_{\rho_j+1}$, we have that $S_j(\varphi, \varphi) \geq 0$, $\forall \varphi \in C^\infty(\Sigma_j)$. Hence, from the equality (7.3) we have that

$$4 \int_{\Sigma_j} |\nabla_j \varphi|^2 \rho_{j+1} d\mu_j \geq 2 \int_{\Sigma_j} |\nabla_j \varphi|^2 \rho_{j+1} d\mu_j \geq 2 \int_{\Sigma_j} c \phi^2 \rho_{j+1} d\mu_j$$

$$+ 2 \int_{\partial \Sigma_j} \phi^2 B^{\partial \Sigma_{j+1}} (\nu_j, \nu_j) \rho_{j+1} d\mu_{j-1},$$

for every $\varphi \in C^\infty(\Sigma_j)$. It follows from the lemmas 7.6 and 7.8 that

$$2c_j = \tilde{R}_{j+1} - \tilde{R}_j + |\tilde{B}_j|^2$$

$$\geq c - \frac{1}{4} \sum_{p=j+1}^{n-1} |\nabla_j \log u_p|^2 - R_j + 4\rho_{j+1}^{-\frac{1}{2}} \Delta_j (\rho_{j+1}^{\frac{1}{2}}) + \sum_{p=j+1}^{n-1} |\nabla_j \log u_p|^2$$

$$= c + \frac{3}{8} \sum_{p=j+1}^{n-1} |\nabla_j \log u_p|^2 - R_j + 4\rho_{j+1}^{-\frac{1}{2}} \Delta_j (\rho_{j+1}^{\frac{1}{2}})$$

Thus,

$$4 \int_{\Sigma_j} |\nabla_j \varphi|^2 \rho_{j+1} d\mu_j \geq \int_{\Sigma_j} \left(c + \frac{3}{8} \sum_{p=j+1}^{n-1} |\nabla_j \log u_p|^2 - R_j\right) \phi^2 \rho_{j+1} d\mu_j$$

$$+ 4 \int_{\Sigma_j} \rho_{j+1}^{\frac{1}{2}} \Delta_j (\rho_{j+1}^{\frac{1}{2}}) \phi^2 d\mu_j$$

$$+ 2 \int_{\partial \Sigma_j} \phi^2 B^{\partial \Sigma_{j+1}} (\nu_j, \nu_j) \rho_{j+1} d\mu_{j-1},$$
for every $\varphi \in C^\infty(\Sigma_j)$. Replacing $\varphi$ by $\varphi \rho_j^{-\frac{1}{2}}$ at the last inequality, we obtain that
\[
4 \int_{\Sigma_j} \nabla_j (\varphi \rho_j^{-\frac{1}{2}})^2 \rho_{j+1} d\mu^j \geq \int_{\Sigma_j} \left( c + \frac{3}{8} \sum_{p=j+1}^{n-1} |\nabla_j \log u_p|^2 - R_j \right) \varphi^2 d\mu^j
+ 4 \int_{\Sigma_j} \rho_j^{-\frac{3}{2}} \Delta_j (\rho_j \varphi^2) d\mu^j
+ 2 \int_{\partial \Sigma_j} \varphi^2 B^{\partial \Sigma_j} (\nu_j, \nu_j) d\mu^{j-1}.
\]
(7.6)

Observe that
\[
\nabla_j (\varphi \rho_j^{-\frac{1}{2}}) = \varphi \nabla_j \rho_j^{-\frac{3}{2}} + \rho_j^{-\frac{3}{2}} \nabla_j \varphi
\]

This implies que,
\[
|\nabla_j (\varphi \rho_j^{-\frac{1}{2}})|^2 = \rho_j^{-1} |\nabla_j \varphi|^2 + \varphi^2 |\nabla_j \rho_j^{-\frac{1}{2}}|^2 + 2 \varphi \rho_j^{-\frac{3}{2}} \langle \nabla_j \rho_j^{-\frac{1}{2}}, \nabla_j \varphi \rangle
\]

Thus,
\[
\rho_{j+1} |\nabla_j (\varphi \rho_j^{-\frac{1}{2}})|^2 = |\nabla_j \varphi|^2 + \varphi^2 \rho_{j+1} |\nabla_j \rho_j^{-\frac{1}{2}}|^2 + \langle \nabla_j \log \rho_j^{-\frac{1}{2}}, \nabla_j (\varphi^2) \rangle
\]

Using integration by parts, we have that
\[
\int_{\Sigma_j} \langle \nabla_j \log \rho_j^{-\frac{1}{2}}, \nabla_j (\varphi^2) \rangle d\mu^j = -\int_{\Sigma_j} \varphi^2 \Delta_j \log \rho_j^{-\frac{1}{2}} d\mu^j
+ \int_{\partial \Sigma_j} \varphi^2 \partial (\log \rho_j^{-\frac{1}{2}}) \frac{\partial}{\partial \eta_j} d\mu^{j-1}
= + \int_{\Sigma_j} \varphi^2 \rho_j^{-\frac{3}{2}} \Delta_j \rho_j^{-\frac{1}{2}} d\mu^j
- \int_{\Sigma_j} \varphi^2 |\nabla_j \log \rho_j^{-\frac{1}{2}}|^2 d\mu^j
+ \int_{\partial \Sigma_j} \varphi^2 \partial (\log \rho_j^{-\frac{1}{2}}) \frac{\partial}{\partial \eta_j} d\mu^{j-1}
= -\int_{\Sigma_j} \varphi^2 |\nabla_j \log \rho_j^\frac{1}{2}|^2 d\mu^j
+ \int_{\Sigma_j} \varphi^2 \rho_j^{-\frac{3}{2}} \Delta_j \rho_j^\frac{1}{2} d\mu^j
+ \int_{\partial \Sigma_j} \varphi^2 \partial (\log \rho_j^{-\frac{1}{2}}) \frac{\partial}{\partial \eta_j} d\mu^{j-1}.
Then,

\[
4 \int_{\Sigma_j} \rho_{j+1} |\nabla_j (\varphi \rho_{j+1}^{-\frac{1}{2}})|^2 d\mu^j = 4 \int_{\Sigma_j} |\nabla_j \varphi|^2 d\mu^j \\
+ 4 \int_{\Sigma_j} \varphi^2 \rho_{j+1} |\nabla_j \rho_{j+1}^{-\frac{1}{2}}|^2 d\mu^j \\
- 4 \int_{\Sigma_j} \varphi^2 |\nabla_j \log \rho_{j+1}^\frac{1}{2}|^2 d\mu^j \\
+ 4 \int_{\Sigma_j} \varphi^2 \rho_{j+1}^{-\frac{1}{2}} \Delta_j \rho_{j+1}^\frac{1}{2} d\mu^j \\
+ 4 \int_{\partial \Sigma_j} \varphi^2 \frac{\partial (\log \rho_{j+1}^{-\frac{1}{2}})}{\partial \eta_j} d\mu^{j-1}
\]

Since,

\[
\nabla_j \rho_{j+1}^{-\frac{1}{2}} = -\rho_{j+1}^{-1} \nabla_j \rho_{j+1}^\frac{1}{2},
\]

we obtain that

\[
\rho_{j+1} |\nabla_j \rho_{j+1}^{-\frac{1}{2}}|^2 = |\nabla_j \log \rho_{j+1}^\frac{1}{2}|^2.
\]

This implies that

\[
4 \int_{\Sigma_j} \rho_{j+1} |\nabla_j (\varphi \rho_{j+1}^{-\frac{1}{2}})|^2 d\mu^j = 4 \int_{\Sigma_j} |\nabla_j \varphi|^2 d\mu^j \\
+ 4 \int_{\Sigma_j} \varphi^2 \rho_{j+1}^{-\frac{1}{2}} \Delta_j \rho_{j+1}^\frac{1}{2} d\mu^j \\
+ 4 \int_{\partial \Sigma_j} \varphi^2 \frac{\partial (\log \rho_{j+1}^{-\frac{1}{2}})}{\partial \eta_j} d\mu^{j-1}.
\]

Placing the equality above in the inequality (7.5), we obtain that

\[
4 \int_{\Sigma_j} |\nabla_j \varphi|^2 d\mu^j \geq \int_{\Sigma_j} \left( c + \frac{3}{8} \sum_{p=j+1}^{n-1} |\nabla_j \log u_p|^2 - R_j \right) \varphi^2 d\mu^j \\
+ 2 \int_{\partial \Sigma_j} \varphi^2 \left( B^{\partial \Sigma_{j+1}}(\nu_j, \nu_j) - 2 \frac{\partial (\log \rho_{j+1}^{-\frac{1}{2}})}{\partial \eta_j} \right) d\mu^{j-1} \\
= \int_{\Sigma_j} \left( c + \frac{3}{8} \sum_{p=j+1}^{n-1} |\nabla_j \log u_p|^2 - R_j \right) \varphi^2 d\mu^j \\
+ 2 \int_{\partial \Sigma_j} \varphi^2 \left( B^{\partial \Sigma_{j+1}}(\nu_j, \nu_j) + \langle \nabla_j \log \rho_{j+1}, \eta_j \rangle \right) d\mu^{j-1},
\]
for every \( \varphi \in C^\infty(\Sigma_j) \). From the remark 7.2, we obtain that

\[
-H^{\partial \Sigma_j} = -H^{\partial M} + \sum_{p=j}^{n-1} B^{\partial \Sigma_{p+1}}(\nu_p, \nu_p)
\]

\[
\leq \sum_{p=j}^{n-1} B^{\partial \Sigma_{p+1}}(\nu_p, \nu_p)
\]

\[
= B^{\partial \Sigma_{j+1}}(\nu_j, \nu_j) + \sum_{p=j+1}^{n-1} B^{\partial \Sigma_{p+1}}(\nu_p, \nu_p).
\]

**Proposition 7.10.** For every \( p \geq j + 1 \), we have that, in \( \partial \Sigma_j \),

\[
B^{\partial \Sigma_{p+1}}(\nu_p, \nu_p) = \langle \nabla_j \log u_p, \eta_j \rangle.
\]

In fact, from (7.4), we obtain that

\[
B^{\partial \Sigma_{p+1}}(\nu_p, \nu_p) = \frac{1}{u_p} \frac{\partial u_p}{\partial \eta_p} = \langle \nabla_p \log u_p, \eta_p \rangle,
\]

in \( \partial \Sigma_p \), for every \( p = k, \ldots, n - 1 \). Consider \( p \geq j + 1 \). As \( \partial \Sigma_j \subset \partial \Sigma_p \), we have

\[
B^{\partial \Sigma_{p+1}}(\nu_p, \nu_p) = \langle \nabla_p \log u_p, \eta_j \rangle,
\]

in \( \partial \Sigma_j \), since we have \( \eta_p = \eta_j \) in \( \partial \Sigma_j \) (see remark 7.2). In \( \Sigma_j \), we can write

\[
\nabla_p \log u_p = \nabla_j \log u_p + \sum_{l=j}^{p-1} \langle \nabla_p \log u_p, \nu_l \rangle \nu_l.
\]

Hence, in \( \partial \Sigma_j \), we have that

\[
B^{\partial \Sigma_{p+1}}(\nu_p, \nu_p) = \langle \nabla_j \log u_p, \eta_j \rangle + \sum_{l=j}^{p-1} \langle \nabla_p \log u_p, \nu_l \rangle \langle \nu_l, \eta_j \rangle.
\]

However, we have \( \eta_j \perp \nu_l \) in \( \partial \Sigma_j \), for every \( j \leq l \leq n - 1 \) (see remark 7.2). Thus, we have

\[
-H^{\partial \Sigma_j} \leq B^{\partial \Sigma_{j+1}}(\nu_j, \nu_j) + \langle \nabla_j \log \rho_{j+1}, \eta_j \rangle.
\]

Consequently,

\[
4 \int_{\Sigma_j} |\nabla_j \varphi|^2 \mu^j \geq -2 \int_{\partial \Sigma_j} \varphi^2 H^{\partial \Sigma_j} d\mu^{j-1}
\]

\[
+ \int_{\Sigma_j} \varphi^2 \left( c + \frac{3}{8} \sum_{p=j+1}^{n-1} |\nabla_j \log u_p|^2 - R_j \right) \mu^j.
\]

\[\square\]
Theorem 7.11. Let $(M^n, g)$ be Riemannian manifold with boundary $\partial M \neq \emptyset$ such that $R_g > 0$ and $H^\partial M \geq 0$. Consider a minimal free-boundary $k$-slinting in $(M, g)$,

$$\Sigma_k \subset \cdots \subset \Sigma_{n-1} \subset M.$$ 

Then:

1. For $3 \leq k \leq j \leq n - 1$, if $\Sigma_j$ is smooth, there exists on $\Sigma_j$ a metric with positive scalar curvature and minimal boundary.

2. If $k = 2$ and $\Sigma_2$ is smooth, its connected components are all disks.

Proof.

1. Let $j \in \{k, \cdots, n - 1\}$ be such that $\Sigma_j$ is smooth. It follows from the remark 2.3 that it’s enough to show that

$$\lambda(\Sigma_j, g) = \inf_{0 \neq \varphi \in H^1(\Sigma_j)} \frac{\int_{\Sigma_j} |\nabla_j \varphi|^2 dv_g + 2k_j \int_{\partial \Sigma_j} \varphi^2 H_i^\partial \sigma_g + k_j \int_{\Sigma_j} \varphi^2 R_j dv_g}{\int_{\Sigma_j} \varphi^2 dv_g},$$

is positive, where $k_j = \frac{j^2 - 2}{(j - 1)^2} > 0$. It follows from the Proposition 7.9 that

$$-4k_j \int_{\Sigma_j} |\nabla_j \varphi|^2 dv_g < 2k_j \int_{\partial \Sigma_j} \varphi^2 H_i^\partial \sigma_g + k_j \int_{\Sigma_j} \varphi^2 R_j dv_g,$$

for every $\varphi \in C^\infty(\Sigma_j)$ and $\varphi \neq 0$. This implies that

$$\frac{\int_{\Sigma_j} |\nabla_j \varphi|^2 dv_g + 2k_j \int_{\partial \Sigma_j} \varphi^2 H_i^\partial \sigma_g + k_j \int_{\Sigma_j} \varphi^2 R_j dv_g}{\int_{\Sigma_j} \varphi^2 dv_g} > (1 - 4k_j) \frac{\int_{\Sigma_j} |\nabla_j \varphi|^2 dv_g}{\int_{\Sigma_j} \varphi^2 dv_g},$$

for every $\varphi \in C^\infty(\Sigma_j)$ and $\varphi \neq 0$. Hence, $\lambda(\Sigma_j, g) > 0$.

2. It follows from the Proposition 7.9

$$4 \int_{\Sigma_2} |\nabla_2 \varphi|^2 dv_g > -2 \int_{\partial \Sigma_2} \varphi^2 H_i^\partial \sigma_g - 2 \int_{\Sigma_2} \varphi^2 K dv_g,$$

for every $\varphi \in C^\infty(\Sigma_2)$ and $\varphi \neq 0$, since $R_2 = 2K$, where $K$ denote the Gaussian curvature of $(\Sigma_2, g)$. In particular, for $\varphi \equiv 1$ we have that

$$(7.7) \quad \int_{\partial \Sigma_2} H_i^\partial \sigma_g + \int_{\Sigma_2} K dv_g > 0.$$
Let $S$ be a connected component of $\Sigma_2$. From the Proposition (7.7), and from the Gauss-Bonnet Theorem, we obtain $\chi(S) > 0$. This implies that $S$ is a disk.

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