Abstract. A systematic approach to the description of gauge invariant charges is presented and applied to the construction of both the static colour charge configuration in QCD and the monopole solution in pure SU(2). The gauge invariant non-abelian monopole offers a new style of order parameter for monopole condensation.

1. Introduction

It is a fact of life, and one that has been exploited again and again at this meeting, that even gauge invariant objects are more transparent in a specific gauge. For example, the static interquark potential, which has a well known gauge invariant definition, is often best treated in Coulomb gauge (see, for example, [1, 2] and more recently [3]) as this gauge is, in some way, closely adapted to that physical system. Usually such a choice of gauge is a simple pragmatic decision based on the simplicity of the resulting calculation. However, what we’d like to argue is that the connection between such an adapted gauge and the correct gauge invariant description, often goes much deeper [4]. We will see that such appropriate gauge fixings, once recognised, can lead to an understanding of the dominant contribution to the gauge invariant description of the relevant physical degrees of freedom. For monopoles and vortices, where gauge invariant formulation do not yet
exist, understanding this route from gauge fixing to gauge invariance is of central importance if we are ever to understand fully their roles in the non-perturbative structure of QCD.

2. What is a static quark?

In order to map out the connection between static quarks and the Coulomb gauge, we need to make precise just what we mean by saying that a field $\Psi$ describes a static quark. First, to capture the requirement that quarks carry colour, the field $\Psi$ cannot be a coloured singlet under the global gauge transformation although it must be gauge invariant under the local gauge transformations. This tells us straight away that it cannot be identified with the matter field $\psi$ that enters directly into the formulation of the theory since under a gauge transformation we have $\psi \rightarrow \psi U = U^{-1}\psi$. What we must have is that $\Psi = h^{-1}\psi$, for some field dependent configuration $h^{-1}$, where under a gauge transformation we have

$$h^{-1} \rightarrow (h^{-1})U = h^{-1}U.$$  \hfill (1)

We call $h^{-1}$ a dressing for the charge. It incorporates the cloud of fields around any charge.

To impose the static condition that $\partial_0 \Psi = 0$, we need to realise that static means infinite mass and hence we can exploit the dynamical simplification that comes from the heavy quark effective theory. That is, we can use the equations of motion that the matter field is covariantly constant:

$$(\partial_0 + gA_0)\psi = 0.$$  \hfill (2)

Equations (1) and (2) are the fundamental conditions the dressing must satisfy in order to construct a static charge. Explicit solutions to them can be found in QED \cite{6,7}, and perturbative solutions to them can be constructed in QCD \cite{4}. The point to note here is that the resulting dressing has structure. The charged field factorises into the product of two separately gauge invariant terms

$$\Psi = h^{-1}\psi = \left(\tilde{T}e^{-K}\right)e^{-\chi}\psi.$$  \hfill (3)

The bracketed term involves an anti-time ordering while the rest of the expression is local in time but non-local in space. We call $e^{-\chi}$ the minimal

\footnotetext[1]{Alternatively, rather than infinite mass, we can consider asymptotic fields that are static. See the discussion in Ref.\cite{5}.}
part of the dressing as it is essential for the overall gauge invariance of the charge. Additional terms, such as $e^{-K}$ in (3), are not expected from the overall requirement of gauge invariance. Rather, they are needed to ensure the correct dynamical properties of the charge.

The significance of this factorisation can be seen in either the infra-red properties of the fields or in the forces between two such charges. It emerges [8] that in QED the minimal part of the dressing is responsible for controlling the soft infra-red structure of the theory, while the additional part deals with the phase divergences. In terms of forces [8, 9, 11], the minimal part gives the anti-screening contribution to the inter-quark potential, while the additional term is needed for the lesser screening forces. Given the dominance of anti-screening over screening, we see that the minimal part of the dressing is capturing the dominant glue content of a static charge.

There are many important and interesting properties of these fields, but the key thing to note here is that the minimal part of the dressing becomes the identity in Coulomb gauge. This important fact is most easily seen in QED where

$$\chi(x) \propto \int d^3z \phi_i(x - z) A_i(z) = \frac{\partial_i A_i}{\nabla^2}(x),$$  

(4)

and $\phi_i$ is the classical Coulombic electric field of a static charge.

We see that Coulomb gauge is the unique gauge that trivialises the dominant part of the static dressing. This now makes precise the sense in which the Coulomb gauge is adapted to the description of static charges. Knowing this connection can, in turn, provide an efficient means for calculating the minimal part of the dressing which is, as we’ve seen, essential for a gauge invariant description of the static charge.

This connection between gauge invariance and gauge fixing can be generalised to moving charges [12, 6, 7] and it can also be given a simple geometric and hence global interpretation [4]. The conclusion from such an analysis is that in QCD there is a global obstruction to the construction of a static coloured charge, and that this is how confinement is seen in this approach.

3. Gauge invariant monopoles

We now want to consider a pure non-abelian gauge theory and show how a monopole creation operator can be constructed. The are several reasons for wanting to do this, the most immediate being to allow for the construction of new order parameters with which the dual superconductor account of confinement can be tested. It is also, as we’ll see, an interesting theoretical study of the relation between classical solutions and quantum configurations in gauge theories.
To motivate our approach, we note that in Dirac’s original account of the construction of electric charges [13], he arrived at the minimal, abelian, static dressing (4) by noting that its commutator with the electric field operator generated the Coulombic electric field expected from a static charge. As we will see, a similar argument can be applied to monopoles.

Let us start again in the abelian theory. Suppose that \( f_i(x) \) is the classical Dirac monopole potential [14]. Then it is straightforward to see that the operator

\[
M = \exp \left( i \int dz f_i(z) E_i(z) \right),
\]

is gauge invariant and its equal-time commutator with the potential is

\[
[A_i(x), M] = f_i(x) M.
\]

So we can interpret \( M \) as a monopole creation operator for the pure abelian theory. Although this operator allows us to rederive many of the important properties of monopoles, the singularity of the potential \( f_i \) makes this an artificial construction in QED. This reflects the fact that we do not expect monopoles in such an abelian theory. However, in non-abelian theories regular monopole solutions are known to exist when spontaneous symmetry breaking occurs, and they are conjectured to exist and to play an important role in pure gauge theories (see for example, Ref [15]). With this in mind, we now investigate how (5) can be generalised to the non-abelian theory.

The naive extension of this simple construction to a non-abelian theory, where we replace the abelian potential \( f_i \) by a non-abelian one \( f^a_i \tau^a \) and the electric field \( E_i \) by its chromo-electric generalisation \( E^a_i \tau^a \), runs into two immediate problems. The first is to decide on how to generalise the classical Dirac monopole configuration. The second is maintaining gauge invariance since, as is well known, the chromo-electric field is not gauge invariant.

If we now specialise to a pure SU(2) gauge theory, then there is a natural candidate for a monopole configuration first written down by Wu and Yang [16]:

\[
f^a_i = \frac{1}{2} \varepsilon_{aib} x^b / r^2.
\]

This is a solution to the classical Yang-Mills equations of motion which, through a singular gauge transformation, can be related to the abelian monopole configuration. It should be noted, though, that it is an unstable

\[\text{Footnote: There is an interesting subtlety here since different gauge related potentials } f_i \text{ will yield different, but weakly equivalent monopole operators. Thus the ability to move the Dirac string is seen in the weak equivalence of the construction.}\]
solution \[ \mathbf{[17, 18]} \]. How this is modified or reflected in the quantum theory is, however, unknown.

The gauge non-invariance of the chromo-electric field seems a much more serious obstacle to the construction of a non-abelian generalisation of \( (5) \). Extending the dressing technique used in the description of a static charge, we will solve this problem by dressing the chromo-electric field

\[
E^a_{\tau^a} \rightarrow \tilde{E}^a_{\tau^a} = h^{-1} E^a_{\tau^a} h,
\]

where the dressing transforms as in \( (1) \) so that \( \tilde{E}^a_{\tau^a} \) is now gauge invariant. The monopole creation operator generalising \( (5) \) is then

\[
M = \exp \left( i \int dz f^a_i \tilde{E}^a_i(z) \right). \tag{9}
\]

Given that we want to describe a static monopole, i.e., it should not generate a chromo-electric field, we require \( [E^a_\tau(x), M] = 0 \). This will follow if \( h = h[E] \), i.e., the dressing must solely depend on the chromo-electric field. As such, we cannot simply use the dressing constructed in \( (3) \) to describe a static quark. To proceed, we recall the close relation between monopoles and abelian gauge fixing \[ \mathbf{[15]} \]. Following our method for describing the minimally dressed static quark, we will exploit this adapted class of gauge fixings to construct a gauge invariant chromo-electric field and hence the dressing needed in \( (8) \).

Gauge fixing in the chromo-electric sector is, as far as we know, not well studied. The interesting point here is that it is not possible to fully fix the gauge. In terms of constraints, one can easily see that it is impossible to construct a complete second class set out of Gauss’ law and functions of just the chromo-electric field. However, second class subsets can be found that are valid on regions of the phase space. To see how this works, consider the simple chromo-electric gauge

\[
E^1_3 = E^2_3 = 0. \tag{10}
\]

These, along with the components \( D_i E^1_i \) and \( D_i E^2_i \) of Gauss’ law, form a second class set of constraints as long as \( E^3_3 \neq 0 \). To implement this reduction then, we should restrict ourselves to the regions in phase space where either \( E^3_3 > 0 \) or \( E^3_3 < 0 \). If \( E^3_3 = 0 \), then we can either take it and one of the components in \( (10) \) as our gauge, or we can choose another gauge by looking at, say, the \( E^a_1 \) components of the chromo-electric field. In this way, through a patching process, we can implement a chromo-electric gauge fixing that is only ill defined on configurations which have zero field strength. We do not yet fully understand the effect of such instanton configurations on our monopole construction, so for the moment we will neglect them and, for simplicity, just consider the gauge \( (10) \) in the region \( E^3_3 > 0 \).
Having settled on a gauge that we know is adapted, or at least sympathetic, to the non-abelian monopole configuration, we now have to find the dressing needed in (8) by rotating our fields into the gauge fixed configuration. For a configuration space gauge fixing, such as the Coulomb gauge, we were guaranteed that the resulting dressing would at least locally satisfy the fundamental relation (1). The incompleteness of the chromo-electric gauge fixing, though, means that a little more work is needed to get the correct transformation properties of the dressing.

In terms of the dressed fields (8), we need to solve the equations \[ \tilde{E}^1_i = \tilde{E}^2_i = 0. \] Now
\[ \tilde{E}^3_i = -2E^b_3 \text{tr}(\tau^a h^{-1} \tau^b h) = E^b_3 R_{ab} \] (11)
where \( R_{ab} \) is a rotation matrix. Hence we wish to solve \( E^b_3 R_{1b} = 0 \) and \( E^b_3 R_{2b} = 0 \).

These two equations are simple vector equations and can be immediately solved as follows. Take
\[ R_{1b} = \varepsilon_{bcd} \hat{E}^c_3 \hat{\lambda}^d \] (12)
where
\[ \hat{E}^c_3 = \frac{E^c_3}{\sqrt{E^b_3 E^b_3}} \] (13)
and \( \hat{\lambda}^d \) is, for the moment, an arbitrary unit vector. Then
\[ R_{2b} = \varepsilon_{bcd} \hat{E}^c_3 R_{1d} \] (14)
and
\[ R_{3b} = \varepsilon_{bcd} R_{1c} R_{1d} = \hat{E}^b_3. \] (15)
These allow us to construct the rotation matrix and check gauge invariance of the resulting dressed chromo-electric field.

For the third colour component gauge invariance is immediate since
\[ \tilde{E}^3_i = \frac{E^b_3 E^b_3}{\sqrt{E^b_3 E^b_3}}. \] (16)
We further note that \( \tilde{E}^3_3 = \sqrt{E^b_3 E^b_3} \), which (with our restriction that \( E^b_3 > 0 \)) is just \( E^b_3 \) in the gauge (10).

For the other components of the dressed chromo-electric field, though, gauge invariance can only be ensured through a good choice of \( \hat{\lambda} \). From the definition (11) we have
\[ \tilde{E}^1_i = E^b_3 R_{1b} = \varepsilon_{bac} E^b_3 \hat{E}^a_3 \hat{\lambda}^c, \] (17)
and
\[ \tilde{E}^2_i = E^b_3 R_{2b} = E^b_3 \hat{E}^b_3 \hat{E}^a_3 \hat{\lambda}^a - E^b_3 \hat{\lambda}^b_3 \hat{E}^a_3 \hat{E}^a_3. \] (18)
Gauge invariance will follow if $\hat{\lambda}^c$ is proportional to $E^c$. However, from (12), we also need $\hat{\lambda}$ orthogonal to $\hat{E}^a_3$. There are various ways to satisfy these conditions for gauge invariance. For example, we could take

$$\hat{\lambda}^c = \frac{x_i E^c_i}{\sqrt{x_j E^d_j x_k E^e_k}}$$  \hspace{1cm} (19)

In summary, we have seen in this section how to generalise the abelian monopole creation operator (5) to yield a gauge invariant monopole operator (6). This was done by construction a chromo-electric dressing adapted to the chromo-electric gauge fixing (10).

4. Conclusions

An important contribution to the dressing approach to gauge invariance is the recognition that there are special adapted gauges that have a particular significance for the description of both chromo-electric and magnetic charges in non-abelian gauge theories. For electric charges in both QED and QCD, these adapted gauges followed naturally from a more fundamental dressing equation. Solving that equation factorises the dressing into a dominant anti-screening term (that controlled the soft infra-red sector) and an additional screening term. For a specific dynamical configuration for the charges, the adapted gauge trivialises the dominant part of the dressing. However, it should be stressed that in a scattering situation where charges with differing momentum must be dressed differently, there is no gauge in which all the different dressings so simplify.

In pure SU(2) theory we have seen how to go from chromo-electric gauge fixing to a gauge invariant monopole creation operator. As yet there is no analogous dynamical approach to this dressing. It is hoped, though, that through our recognition of the adapted gauge to this system we have also captured the dominant monopole configuration. This will allow us to further probe the role of monopoles in confinement.

References

1. V.N. Gribov, SLAC-TRANS-0176 (In Leningrad 1977, Proceedings of the 12th Winter LNP1 School on Nuclear and Elementary Particle Physics, 147-162).
2. S.D. Drell, Trans. NY Acad. Sci. Series II 40, 76 (1980), SLAC-PUB-2694, available at SLAC document server.
3. J. Greensite and C.B. Thorn, JHEP 02, 014 (2002), hep-ph/0112326.
4. M. Lavelle and D. McMullan, Phys. Rept. 279, 1 (1997), hep-ph/9509344.
5. R. Horan, M. Lavelle, and D. McMullan, J. Math. Phys. 41, 4437 (2000), hep-th/9909044.
6. R. Horan, M. Lavelle, and D. McMullan, Pramana J. Phys. 51, 317 (1998), hep-th/9810089. Erratum-ibid, 51 (1998) 235.
7. E. Bagan, M. Lavelle, and D. McMullan, Annals Phys. 282, 471 (2000), hep-ph/9909257.
8. E. Bagan, M. Lavelle, and D. McMullan, Annals Phys. 282, 503 (2000), hep-ph/9909262.
9. M. Lavelle and D. McMullan, Phys. Lett. B436, 339 (1998), hep-th/9805013.
10. E. Bagan, M. Lavelle, and D. McMullan, Phys. Lett. B477, 355 (2000), hep-th/0002051.
11. E. Bagan, M. Lavelle, D. McMullan, and S. Tanimura, Phys. Rev. D65, 105004 (2002), hep-ph/0107308.
12. E. Bagan, M. Lavelle, and D. McMullan, Phys. Rev. D56, 3732 (1997), hep-th/9602083.
13. P.A.M. Dirac, Can. J. Phys. 33, 650 (1955).
14. P.A.M. Dirac, Phys. Rev. 74, 817 (1948).
15. G. ’t Hooft, Monopoles, instantons and confinement, (1999), hep-th/0010225.
16. T.T. Wu and C.N. Yang, in Properties of Matter under Unusual Conditions, Ed. H. Mark and S. Fernbach (Interscience, New York 1969) pp 349.
17. T. Yoneya, Phys. Rev. D16, 2567 (1977).
18. R.A. Brandt and F. Neri, Nucl. Phys. B161, 253 (1979).