Hilbert schemes and multiple $q$-zeta values

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Abstract

We present several conjectures on multiple $q$-zeta values and on
the role they play in certain problems of enumerative geometry.

1 Multiple $q$-zeta values

1.1

Multiple zeta values, that is, series of the form
\[
\zeta(s) = \zeta(s_1, \ldots, s_k) = \sum_{n_1 > n_2 > \cdots > n_k} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}},
\]
where $s_i$ are nonnegative integers and $s_1 > 1$, drew the attention of some of
the greatest mind in mathematics, starting with Euler, see e.g. [20]. Many re-
markable properties, striking applications, and unexpected connections were
found for them, some conjectural, some proven. For an introduction to their
motivic and representation-theoretic aspects, the reader may turn, for exam-
ple, to chapter 25 in [1] and the Appendix in [5], respectively.

1.2

In this paper, we work with $q$-deformations of the series (1). There already
exists a large body of work on various flavors of such $q$-deformations, see e.g.
[2, 17, 21]. The one we use here has the form
\[
Z(s) = \sum_{n_1 > n_2 > \cdots > n_k} \prod (n_i)^{-s_i}, \quad (n)^{-s} = \frac{p_s(q^n)}{(1 - q^n)^s},
\]
where \( p_s \) is a nonzero polynomial of degree \( s \) without constant term.

The linear span of the series (2) is independent of the choice of the
numerators \( p_s \). However, to discuss a conjectural \( \mathbb{Z}/2 \)-grading, it is convenient
to choose the numerators palindromic, that is, satisfying \( t^s p_s(1/t) = p_s(t) \),
which is only possible if \( s > 1 \). In what follows, we assume that \( s_i \geq 2 \) and define

\[
p_s(t) = \begin{cases} 
  t^{s/2}, & s = 2, 4, 6, \ldots, \\
  t^{(s-1)/2}(1 + t), & s = 3, 5, 7, \ldots.
\end{cases}
\]

1.3

We define \( q\text{MZV} \) as the \( \mathbb{Q} \)-subalgebra of \( \mathbb{Q}[\![q]\!] \) spanned by the series \( Z(s) \)
with \( s_i \geq 2 \). This algebra is filtered by weight, where

\[
\text{weight } Z(s) = \sum s_i.
\]

Note that

\[
\text{QM} = \mathbb{Q}[Z(2), Z(4), Z(6)] \subset q\text{MZV}
\]

is the classical ring of quasimodular forms \([8]\), the analog of \( \mathbb{Q}[\pi^2] \subset \text{MZV} \).
It contains all \( Z(2k) \) and with proper normalization (which includes correct
constant terms), is \( \mathbb{Z} \)-graded by weight. I don’t know whether there is a way
to extend this grading to all of \( q\text{MZV} \).

Clearly,

\[
(1 - q)^\text{weight } Z(s) \to 2^{\#\text{odd}(s)} \zeta(s), \quad q \to 1,
\]

which defines a homomorphism \( \text{gr}_{q\text{MZV}} \to \text{MZV} \), where \( \text{gr}_{q\text{MZV}} \) is the
associated graded algebra for the weight filtration of \( q\text{MZV} \) and \( \#\text{odd}(s) \) is the
number of odd terms in \( s \). This homomorphism has a large kernel, already
for \( \text{QM} \to \mathbb{Q}[\pi^2] \). Since \( \zeta(s) > 0 \), we see that \( Z(s) \) is not contained in any
smaller weight filtration subspace.

1.4

We propose the following

**Conjecture 1.** The algebra \( q\text{MZV} \) is spanned by \( Z(s) \) with \( 2 \leq s_i \leq 5 \), \( \mathbb{Z}/2 \)-
graded by weight, and stable under the operator \( q \frac{d}{dq} \) that increases the weight.
by 2. The Hilbert series of the graded algebra \( \text{grqMZV} \) equals
\[
\sum_k t^k \dim_{\mathbb{Q}} \text{grqMZV}_k = \frac{1}{1 - t^2 - t^3 - t^4 - t^5 + t^8 + t^9 + t^{10} + t^{11} + t^{12}}. \tag{3}
\]

The first statement here may be compared to a conjecture of Hoffman [7], proven by F. Brown [3], that says \( \zeta(s) \) with \( s_i \in \{2, 3\} \) span MZV. However, while such \( \zeta(s) \) are conjectured to be a basis of MZV, the Hilbert series (3) predicts relations among \( Z(s) \) with \( 2 \leq s_i \leq 5 \).

## 2 Hilbert schemes of surfaces

### 2.1

Let \( S \) be a nonsingular quasi-projective surface. The Hilbert scheme \( \text{Hilb}(S, n) \) parametrizes 0-dimensional subschemes \( \mathcal{G} \subset S \) of length \( n \), see e.g. [6, 12, 13] for an introduction. It is an irreducible nonsingular quasi-projective variety of dimension \( 2n \).

The geometry of \( \text{Hilb}(S, n) \), and in particular, the characteristic numbers of natural vector bundles on it are of great interest to algebraic geometers and mathematical physicists. These characteristic numbers may be defined if \( S \) is proper, or if there is a torus action on \( S \) with proper fixed locus. In the latter case, they take values in localized \( G \)-equivariant cohomology of a point. Here we fix a connected reductive algebraic group \( G \) that acts on \( S \) so that the fixed-point set of its maximal torus is proper. It is convenient not to assume this action faithful.

### 2.2

A line bundle \( \mathcal{L} \) on \( S \) defines a rank \( n \) vector bundle on \( \text{Hilb}(S, n) \) with fiber
\[
\mathcal{L}^{[n]} \mid_{\mathcal{G}} = H^0(\mathcal{O}_S \otimes \mathcal{L}).
\]

These vector bundles are called tautological.

The tangent bundle \( \mathcal{T} \) to the Hilbert scheme is described by
\[
\mathcal{T} \mid_{\mathcal{G}} = \chi(\mathcal{O}_S) - \chi(\mathcal{I}_S, \mathcal{I}_G),
\]
where \( \mathcal{I}_S \) is the ideal sheaf of \( S \) and
\[
\chi(A, B) = \sum (-1)^i \text{Ext}^i(A, B).
\]
From this, and the Grothendieck-Riemann-Roch theorem, it is possible to express the characteristic classes of \( T \) in terms of those of \( \mathcal{O}^n \). For our purposes, however, such reduction appears quite impractical and it is perhaps best to keep the characteristic classes of \( T \) separate from those of tautological bundles.

Further, the tangent bundle \( T \) may be twisted by a line bundle \( \mathcal{M} \in \text{Pic}(S) \) as follows
\[
T(\mathcal{M})|_S = \chi(\mathcal{M}) - \chi(\mathcal{I}_S, \mathcal{I}_S \otimes \mathcal{M}).
\]
In particular, if \( \mathcal{M} \) is a pure \( G \)-character then \( T(\mathcal{M}) = T \otimes \mathcal{M} \) and then \( \mathcal{M} \) would be called the mass of the adjoint matter in Nekrasov theory [14].

### 2.3

Fix \( \mathcal{L}, \mathcal{M} \), a characteristic class \( f \) and form the following generating function
\[
\langle f \rangle = \sum_n q^n \int_{\text{Hilb}(S, n)} f(\mathcal{L}^n) \text{Euler}(T(\mathcal{M})).
\]
By construction
\[
\langle f \rangle \in H^*_G(\text{pt})_{\text{loc}}[[q]].
\]
Since \( \mathcal{M} \) may be always additionally twisted by a character, this is a generating function for the integrals of arbitrary characteristic class of \( \mathcal{L}^n \) against a Chern class of \( T(\mathcal{M}) \). Precisely these combinations often come up in practice, see e.g. [9], but not always packaged in this particular form.

Note that for a nontrivial group \( G \), the ring \( H^*_G(\text{pt})_{\text{loc}} \) is nonzero in all degrees, so the degrees of characteristic classes in (4) don’t need to sum up to the dimension of the Hilbert scheme.

### 2.4

It was shown in [14] that
\[
\langle 1 \rangle = \prod_{n>0} (1 - q^n)^{\delta}, \quad \delta = - \int_S c_2(TS \otimes \mathcal{M}),
\]
where $TS$ is the tangent bundle of $S$. We define

$$\langle f \rangle' = \langle f \rangle / (1)$$

and define the connected generating functions by

$$\langle ch_a \cdot ch_b \rangle^0 = \langle ch_a \cdot ch_b \rangle' - \langle ch_a \rangle' \langle ch_b \rangle'$$

$$\langle ch_a \cdot ch_b \cdot ch_c \rangle^0 = \langle ch_a \cdot ch_b \cdot ch_c \rangle' - \langle ch_a \cdot ch_b \rangle' \langle ch_c \rangle'$$

$$- \langle ch_a \cdot ch_c \rangle' \langle ch_b \rangle' - \langle ch_b \cdot ch_c \rangle' \langle ch_a \rangle' + 2 \langle ch_a \rangle' \langle ch_b \rangle' \langle ch_c \rangle' ,$$

et cetera. Here $ch_a$ are the components of the Chern character.

The connected generating functions satisfy better integrality:

$$\langle f \rangle^0 \in k[[q]],$$

where $k$ is the image of the map

$$H_G^*(S) \ni \gamma \mapsto \int_S \gamma \cup c_2(TS \otimes \mathcal{M}) \in H_G^*(pt_{loc}).$$

### 2.5

These $q$-series are the subject of the following

**Conjecture 2.** The series $\langle f \rangle'$ is a multiple $q$-zeta value of the same weight as $f$, where

$$\text{weight } ch_k = k + 2.$$

*In the case $\mathcal{L} = \mathcal{M}_S^{-1/2}$, it is also of the same parity as the weight of $f$.*

Note that

$$\langle f \cdot ch_b \rangle' = q \frac{d}{dq} [\langle f \rangle' + \ln(1)],$$

which is why we were interested in the action of the operator $q \frac{d}{dq}$ on $qMZV$.

### 2.6

Among the conjectures presented in this paper, Conjecture 2 appears the most accessible, perhaps using the techniques developed in [4]. In fact, prompted by this conjecture, it was already shown in [4] that

$$\langle c_1(\mathcal{O}^{[n]}) \rangle' = \frac{1}{2} (Z(2) - Z(3)) \int_S (c_1 c_2 - c_3)(TS \oplus \mathcal{M}).$$
The main result of [4] computes (4) as the trace over the Fock space of a product of a certain vertex operator and a certain integral of motion of the second quantized trigonometric Calogero system. Formulas for the latter in terms of bosonic operators may be derived systematically using, for example, the formulas of [18] or, alternatively, using many other approaches developed in the literature. The trace can then be explicitly computed as a multiple \( q \)-series, similar to the series
\[
\sum_{k,l>0} \frac{q^{k+l}}{(1-q^k)(1-q^l)(1-q^{k+l})},
\]
which was studied in [4]. The problem is thus reduced to showing that certain rather concrete series lie in \( q \text{MZV} \). It is hard to know without trying how difficult this task would be.

3 Hilbert schemes of threefolds

3.1

Now let \( X \) be a nonsingular quasiprojective threefold and consider its Hilbert scheme of points \( \text{Hilb}(X,n) \). This is a quite singular reducible scheme, however, as one of the simplest moduli space in Donaldson-Thomas theory [19] it has a perfect obstruction theory (here, of virtual dimension 0) and the corresponding 0-dimensional virtual fundamental class.

In parallel to (4) we define
\[
\langle f \rangle_{3D} = \sum_n (-q)^n \int_{[\text{Hilb}(X,n)]_{\text{vir}}} f(\mathcal{L}^{[n]}),
\]
where \( \mathcal{L}^{[n]} \) is a rank \( n \) bundle on \( \text{Hilb}(X,n) \) defined in the same way as before.

For example, \( X \) could be the total space of a line bundle \( \mathcal{M} \) over a surface \( S \) and then \( \text{Hilb}(S,n) \) is one of the component of \( \text{Hilb}(X,n)^{\mathbb{C}^\times} \), where \( \mathbb{C}^\times \) acts by scaling \( \mathcal{M} \). The contribution of this component to (5) is closely related to the integrals considered in Section 2. However, the precise packaging of those integrals in the generating function is different, as will be the functional nature of the series (5).
3.2

The evaluation
\[
\langle 1 \rangle_{3D} = \prod_{n>0} (1 - q^n)^n \delta_{3D}, \quad \delta_{3D} = \int_X (c_1 c_2 - c_3),
\]
was checked for \( X = \mathbb{C}^3 \) in [11] and conjectured in general. It was proven for all \( X \) in [10], another proof was announced by Jun Li. As before, we set
\[
\langle f \rangle'_{3D} = \langle f \rangle_{3D} / \langle 1 \rangle_{3D}
\]
and define connected generating functions as in Section 2. We also define a sequence of \( H^*_{G}(pt) \)-submodules
\[
H^*_{G}(pt)_{loc} \supset k_1 \supset k_2 \supset \ldots
\]
as the images of the maps
\[
H^*_{G}(X) \ni \gamma \mapsto \int_X \gamma \cup (c_1 c_2 - c_3)^d, \quad q = 1, 2, \ldots
\]

3.3

Conjecturally, the generating functions (7) belong to the algebra
\[
oqZ = \mathbb{Q} \left[ \left( q \frac{d}{dq} \right)^l Z(2k + 1) \right]_{k \geq 1, l \geq 0} \subset qMZV
\]
generated by the odd \( q \)-zeta values and their derivatives. This is the \( q \)-analog of the subalgebra
\[
oZ = \mathbb{Q}[\zeta(3), \zeta(5), \zeta(7), \ldots] \subset MZV,
\]
which, by standard transcendence conjectures, is a free commutative \( \mathbb{Q} \)-algebra on its generators. A parallel conjecture for (8) would be that it is also a free commutative algebra on its generators. This allows us to define a grading by 3D weight and depth by setting
\[
\text{weight}_{3D} \left( q \frac{d}{dq} \right)^l Z(2k + 1) = 2k + 1 + l, \quad \text{(9)}
\]
\[
\text{depth} \left( q \frac{d}{dq} \right)^l Z(2k + 1) = 1
\]
on generators.
3.4

Conjecture 3. Connected generating functions satisfy
\[
\left\langle \prod \text{ch}_{k_i} \right\rangle^o \in \sum_d \mathbb{H}_d \otimes \mathbb{Q} \mathfrak{oqZ}_{\leq 2 + \sum (k_i+1), \geq d},
\]
where first subscript denotes an upper bound on the 3D weight \( \langle \rangle \) while the second denotes the depth \( d \).

Presumably, the techniques used to prove (6) will similarly reduce the general case of this conjecture to the case \( X = \mathbb{C}^3 \).

3.5

For \( X = \mathbb{C}^3 \), any given instance of this conjecture may be attacked, at least in principle, by expanding in a series in
\[ t_{12} = t_1 + t_2 \in H^2_{\text{GL}(3)}(\text{pt}), \]
where \( t_i \) are the weights of the coordinates of \( \mathbb{C}^3 \). For connected functions, we have
\[
\left\langle \prod \text{ch}_{k_i} \right\rangle^o \in \delta_{3\text{D}} \cdot H^2_{\text{GL}(3)}(\text{pt})[[q]], \quad \delta_{3\text{D}} = \frac{(t_1+t_2)(t_1+t_3)(t_2+t_3)}{t_1t_2t_3},
\]
and so the expansion in \( t_{12} \) is a finite expansion.

Let \( \pi \) is a 3-dimensional partition, presented as piles of boxes \( \boxtimes = (i, j, k) \) of height \( \pi_{ij} \), that is
\[ 1 \leq k \leq \pi_{ij}, \]
placed over the squares \( \square = (i, j) \) of a 2-dimensional partition \( \lambda \) lying in the \((x_1, x_2)\)-plane. The level sets of the function \( \pi \) define a a decomposition of \( \lambda \) into skew diagrams
\[ \emptyset \subset \cdots \subset \lambda'' \subset \lambda' \subset \lambda. \]

For a skew diagram, we define its rank as the minimal number of rim hooks needed to decompose it, see Section 4.5 of [16]. We define the total rank of \( \pi \) by
\[ \rho(\pi) = \sum \text{rank}(\lambda^{(i)}/\lambda^{(i+1)}) \]
It can then be shown [16] that
\[ \text{ord}_{t_{12}} \text{contribution}(\pi) = \rho(\pi) \]
where order is the order of vanishing along \( t_{12} = 0 \) and the contribution of \( \pi \) is the weight at \( \pi \) of the virtual fundamental cycle in the equivariant localization formula.

Terms of small degree in \( t_{12} \) thus come from 3-dimensional partitions of small total rank, and it may be possible to analyze them directly.

### 3.6

For example, consider the computation of

\[
\langle \text{ch}_0 \rangle' \in \delta_{3D} \cdot \mathbb{Q}[[q]],
\]

for which it is enough to compute the linear term in \( t_{12} \). A 3-dimensional partition \( \pi \) has total rank 1 if and only if

1. partition \( \lambda \) is a hook, and
2. the function \( \pi_{ij} \) is a constant, say, \( c \).

Denoting \( n = |\pi| \) the size of \( n \), we see that \( c \) is a divisor of \( n \) and, for given \( c \), there are exactly \( n/c \) choices of \( \lambda \). One further checks that

\[
\left( \frac{t_3}{t_{12}} \text{contribution}(\pi) \right)_{t_{12}=0,t_3=0} = \frac{(-1)^{n+1}}{c}.
\]

Since \( \text{ch}_0 = n \), we obtain

\[
\langle \text{ch}_0 \rangle' = -\delta_{3D} \sum_n q^n \sum_{c|n} (n/c)^2 = -\delta_{3D} Z(3),
\]

recovering (6).

### 3.7

It is interesting to find out to what extent the Gromov-Witten/Donaldson-Thomas correspondence of [11] holds for the functions (7).

Object parallel to the connected functions \( \langle \prod \text{ch}_k \rangle^0 \) on the Gromov-Witten side are the following Hodge integrals

\[
\left\langle \prod_{i=1}^{n} \tau_{k_i+1} \right\rangle = \sum_{g \geq 0} u^{2g-2} \int_{\mathcal{M}_{g,n}} \Lambda(t_1) \Lambda(t_2) \Lambda(t_3) \prod_{i=1}^{n} \psi_i^{k_i+1} \quad (11)
\]
where $\overline{M}_{g,n}$ is the Deligne-Mumford moduli space of stable $n$-pointed genus $g$ curves $C$, $$\Lambda(t) = t^g - t^{g-1}\lambda_1 + \cdots \pm \lambda_g, \quad \lambda_i = c_i(H^0(\omega_C)),$$
is, up to normalization, the Chern polynomial of the Hodge bundle, and $\psi_i \in H^2(\overline{M}_{g,n})$ is 1st Chern class of the line bundle formed by the cotangent line at the $i$th marked point.

3.8

A correspondence would involve knowing the top and the bottom arrow in the following diagram

$$ Q[\text{ch}_k] \xrightarrow{\Phi} Q[i][\tau_k] \xrightarrow{\psi} \mathbb{Q}$$

$$ k \otimes \mathbb{Q}[Z] \xrightarrow{\phi} k \otimes \mathbb{Q}[u^{-1}, u]$$
in which $\phi$, but not $\Phi$, should be an algebra homomorphism. For the top map $\Phi$ there are certain general principles stated in [11] and a also concrete explicit proposal [15]. In the usual GW/DT story, the bottom map is given by the expansion of a rational function of $q$ in a Laurent series in $u$ via the substitution $q = e^{iu}$. This cannot literally work in the present case for at least two reasons.

First, certain factors of $\frac{1}{2}$ appear consistently in the treatment of the GW/DT correspondence for Hilbert scheme of points (as opposed to curves of nonzero degree). This concerns even the series (6). Second, the asymptotic expansion of the odd zeta values contains transcendental terms, namely

$$ Z(2k+1) \sim \frac{(2k)! \zeta(2k+1)}{(-\log q)^{2k+1}} - \sum_{n=0}^{\infty} \frac{B_{2n+2k} B_{2n}}{(2n)! (2n+2k)} (-\log q)^{2n-1},$$
as $q \uparrow 1$. If one sets

$$ \phi(Z(2k+1)) = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{B_{2n+2k} B_{2n}}{(2n)! (2n+2k)} (-iu)^{2n-1},$$
then for small examples one can find an agreement between the Donaldson-Thomas and Gromov-Witten computations. However, I am not convinced that it would work in general.
Instead, I would like to pose the determination of the functional nature of the series (11) as an open problem, the solution to which may very well require a generalization of the universe of asymptotic expansions of $q$-zeta values.

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