Monte Carlo method for parabolic equations involving fractional Laplacian*

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Abstract

We apply the Monte Carlo method to solving the Dirichlet problem of linear parabolic equations with fractional Laplacian. This method exploits the idea of weak approximation of related stochastic differential equations driven by the symmetric stable Lévy process with jumps. We utilize the jump-adapted scheme to approximate Lévy process which gives exact exit time to the boundary. When the solution has low regularity, we establish a numerical scheme by removing the small jumps of the Lévy process and then show the convergence order. When the solution has higher regularity, we build up a higher-order numerical scheme by replacing small jumps with a simple process and then display the higher convergence order. Finally, numerical experiments including ten- and one hundred-dimensional cases are presented, which confirm the theoretical estimates and show the numerical efficiency of the proposed schemes for high dimensional parabolic equations.

Key words: Monte Carlo method; fractional Laplacian; linear parabolic equation; Lévy process; Jump-adapted scheme

1 Introduction

The fractional Laplacian, \((-\Delta)^s\), is a prototypical operator for modeling nonlocal and anomalous phenomenon which incorporates long range interactions \([3, 10, 13, 19, 21, 35, 36]\). It arises in many areas of applications, including models for turbulent flows, porous media flows, pollutant transport, quantum mechanics, stochastic dynamics, finance and so on \([8, 11, 12, 16, 26]\). Almost known deterministic numerical methods have been proposed for approximating solutions of parabolic problems with fractional Laplacian in low dimension (less than 3 dimension)\([11, 2]\), while seldom probabilistic approach is taken into account to numerically solve such parabolic and steady state problems.

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The aim of this article is to develop a Monte Carlo method for solving the terminal-boundary value problem in high dimensional cases in the following form

\[
\begin{aligned}
\left\{ \begin{array}{ll}
\frac{\partial u}{\partial t} - (-\Delta)^s u + b(t, x) \cdot \nabla u + c(t, x)u + f(t, x) = 0, & (t, x) \in [0, T) \times D, \\
u(T, x) = g(x), & x \in D, \\
u(t, x) = \chi(t, x), & (t, x) \in [0, T] \times (\mathbb{R}^n \setminus D),
\end{array} \right. \\
(1.1)
\end{aligned}
\]

where \( s \in (0, 1), T > 0, b(t, x) = (b_1(t, x), b_2(t, x), \ldots, b_n(t, x)) \) is an \( n \)-dimensional vector, \( D \) is a bounded region in \( \mathbb{R}^n (n \geq 3) \) and the fractional Laplacian is defined by a singular integral which coincides with Riesz derivative on the whole space \([6, 7, 14, 20, 28, 33]\).

\[
(-\Delta)^s u(t, x) = C(n, s) \text{P.V.} \int_{\mathbb{R}^n} \frac{u(t, x) - u(t, y)}{|x - y|^{n+2s}} \, dy. \\
(1.2)
\]

Here P.V. stands for the principle value and the constant \( C(n, s) \) is given by \([6]\),

\[
C(n, s) = \left( \int_{\mathbb{R}^n} \frac{1 - \cos \zeta_1}{|\zeta|^{n+2s}} \, d\zeta \right)^{-1} = \frac{s2^s \Gamma(\frac{n}{2} + s)}{\pi^{\frac{n}{2}} \Gamma(1 - s)} \\
(1.3)
\]

with \( \zeta_1 \) being the first component of \( \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathbb{R}^n \) and \( \Gamma \) representing the Gamma function.

If we let \( v(t, x) = u(T - t, x) \), then equation (1.1) is changed to the initial-boundary value problem

\[
\begin{aligned}
\left\{ \begin{array}{ll}
\frac{\partial v}{\partial t} + (-\Delta)^s v + \overline{b}(t, x) \cdot \nabla v + \overline{c}(t, x)v + \overline{f}(t, x) = 0, & (t, x) \in (0, T] \times D, \\
v(0, x) = g(x), & x \in D, \\
v(t, x) = \overline{\chi}(t, x), & (t, x) \in [0, T] \times (\mathbb{R}^n \setminus D),
\end{array} \right. \\
(1.4)
\end{aligned}
\]

where \( \bar{b}(t, x) = -b(T - t, x), \overline{c}(t, x) = -c(T - t, x), \overline{f}(t, x) = -f(T - t, x) \), and \( \overline{\chi}(t, x) = \chi(T - t, x) \).

Probabilistic numerical methods (usually implemented with Monte Carlo method) provide effective approaches for numerically solving partial differential equation in high dimension with/without fractional Laplacian as probabilistic methods do not require any discretization in space \([15, 27, 28, 29, 30, 31, 32]\).

Classical Laplace operator and fractional Laplacian are the infinitesimal generators of Brownian motion and symmetric 2s-stable process, respectively, which connect partial differential equations with stochastic processes. Muller \([25]\) first proposed walk-on-spheres method by simulating the paths of the Brownian motion in spheres to numerically solve the steady-state equation with classical Laplace operator. Kyprianou et al. \([18]\) utilized walk-on-spheres method to approximate solution of the steady-state equation with fractional Laplacian based on the distribution of symmetric 2s-stable process issued from the origin, when it first exits a unit sphere. However, the walk-on-spheres method generally can not generate the first exit time
from the domain $\mathbb{D}$. Therefore, it is difficult for walk-on-spheres method to solve parabolic problems numerically. Milstein and Tretyakov \cite{23, 24} approximated the trajectory of diffusion process with Brownian motion by Euler scheme which is a uniformly-time discretization scheme to solve classical parabolic problems with integer order derivative. \cite{9} gave a random walk algorithm for the Dirichlet problem for parabolic integro-differential equation where the kernel of the integro-differential operator has better regularities than the kernel of fractional Laplacian.

In this article, we can approximate the trajectory of the system of stochastic differential equations with symmetric $2\xi$-stable process to numerically solve fractional parabolic problems \cite{1.1}. However, simulating this system via Euler scheme suffers from two difficulties: First, the jump intensity of the symmetric $2\xi$-stable process is infinite, which means there is an infinite number of jumps in every interval of nonzero length; Second, the exit time of the process leaving the domain $\mathbb{D}$ is hard to obtain. To overcome the first difficulty, we utilize the idea of Asmussen and Rosiński \cite{5} and remove small jumps or replace small jumps with corresponding simple processes. Jump-adapted scheme \cite{9, 17, 22} is adapted to go through the second difficulty. Compared with classical Euler scheme, jump-adapted scheme uses adaptive non-uniform discretization based on the times of jumps of the driving process.

In this paper, we first give the probabilistic representation of the solution to equation \cite{1.1}, which is deeply connected with the system of stochastic differential equations with symmetric $2\xi$-stable process. We then consider a simple jump-adapted Euler scheme to approximate the trajectory of the stochastic process and obtain the numerical solution to equation \cite{1.1}. Furthermore, we propose a high-order jump-adapted scheme by replacing small jumps with simple process if the regularity of the solution $u(t,x)$ is good enough. In addition, we give the weak convergence of the simulated Lévy process and the error estimate, which is related to the jumping intensity and statistical error. In comparison with \cite{9}, we study the Dirichlet problem for the parabolic problem with fractional Laplacian where the kernel is more singular and the related symmetric $2\xi$-stable process do not exist any moments. Based on the different regularities of the solution $u(t,x)$, we give the corresponding numerical schemes. For the optimal error estimate of the numerical scheme, we require $u \in C^{1,3}([0,T] \times \mathbb{R}^n)$, while \cite{9} requires a solution of the auxiliary Dirichlet problem belonging to $C^{2,4}([0,T] \times \mathbb{R}^n)$.

The rest of the paper is organized as follows. In Section 2, the probabilistic representation of the solution to equation \cite{1.1} is given, which is related to the system of stochastic differential equations with $2\xi$-stable process. In Section 3, A simple jump-adapted Euler scheme is derived. A high-order jump-adapted scheme is proposed in Section 4. The weak convergence of the simulated process and error estimates are presented in the corresponding sections. In Section 5, numerical experiments are performed to verify the theoretical analysis. Finally, we summarize the main work in the last section.

In the following sections, we denote positive constants by $C_1$ and $C_2$ which may be dependent of the index $s$, but not necessarily the same at different situations.
2 Probabilistic representation

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space with a filtration \(F = \{\mathcal{F}_t\}_{t \in [0, T]}\) of \(\sigma\)-algebra satisfying the usual conditions. Consider the symmetric 2\(s\)-stable process \([4]\),

\[
dL_\eta = \int_{|y|<1} y\tilde{N}(d\eta, dy) + \int_{|y|\geq 1} yN(d\eta, dy),
\]

where \(N(d\eta, dy)\) is a Poisson random measure on \([0, T] \times \mathbb{R}_0^n\), \((\mathbb{R}_0^n = \mathbb{R}^n \setminus \{0\})\) with

\[
E[N(d\eta, dy)] = \nu(dy)d\eta = C(n, s)|y|^{n+2s}d\eta
\]

and

\[
\tilde{N}(d\eta, dy) = N(d\eta, dy) - \nu(dy)d\eta
\]

is the compensated Poisson random measure. The characteristic function is given by \([34]\),

\[
Ee^{i\langle \xi, L_t \rangle} = \exp \left[ t \int_{\mathbb{R}^n} (e^{i\langle \xi, y \rangle} - 1 - i\langle \xi, y \rangle) \nu(dy) \right]
= \exp \left[ tC(n, s)\int_{\mathbb{R}^n} \frac{\cos(\langle \xi, y \rangle) - 1}{|y|^{n+2s}} dy \right]
= e^{-t|\xi|^{2s}}. \tag{2.4}
\]

It can be easily got that the infinitesimal generator of \(L_t\) is \(-(-\Delta)^s\) \([4]\). Thus, fractional Laplacian is closely related to the symmetric 2\(s\)-stable process.

Consider the following system of stochastic differential equations,

\[
\begin{cases}
    dX_\eta = b(\eta, X_{\eta-})d\eta + dL_\eta, & X_t = x, \\
    dY_\eta = c(\eta, X_{\eta-})Y_\eta d\eta, & Y_t = 1, \\
    dZ_\eta = f(\eta, X_{\eta-})Y_\eta d\eta, & Z_t = 0.
\end{cases} \tag{2.5}
\]

Based on the above system, we give the probabilistic representation of the solution to equation \([1.1]\) in the following theorem.

**Theorem 2.1.** Let \(u(t, x)\) be the solution of equation \([1.1]\). Then \(u(t, x)\) can be given by

\[
u(t, x) = E\left[ u(T \wedge \tau_{t,x}, Y_{T \wedge \tau_{t,x}}) + Z_{T \wedge \tau_{t,x}} \right]
= E\left\{ I_{\{\tau_{t,x} < T\}} \left[ \chi(\tau_{t,x}, X_{\tau_{t,x}})Y_{\tau_{t,x}} + Z_{\tau_{t,x}} \right] \right\}
\]

where \(X^t_{\eta}, Y^t_{\eta} \) and \(Z^t_{\eta} \) are the solution of Cauchy problem \([2.5]\), \(\tau_{t,x} = \{ \eta \geq t, X^t_{\eta} \notin \mathbb{D} \} \) is the first exit time of \(X^t_{\eta} \) starting from \(x \in \mathbb{R}^n\) to the boundary \(\mathbb{D}\).
Proof. From Itô formula, we have
\[
\begin{aligned}
du(\eta, X_\eta) &= \frac{\partial u}{\partial \eta}(\eta, X_{\eta-})d\eta + \sum_{i=1}^{n} b_i(\eta, X_{\eta-}) \frac{\partial u}{\partial x_i}(\eta, X_{\eta-})d\eta \\
&+ \int_{|y| \geq 1} [u(\eta, X_{\eta-} + y) - u(\eta, X_{\eta-})] N(d\eta, dy) \\
&+ \int_{|y| < 1} [u(\eta, X_{\eta-} + y) - u(\eta, X_{\eta-})] \tilde{N}(d\eta, dy) \\
&+ \int_{|y| < 1} [u(\eta, X_{\eta-} + y) - u(\eta, X_{\eta-}) - (\nabla u(\eta, X_{\eta-}), y)] \nu(dy) d\eta.
\end{aligned}
\] (2.7)

Since
\[
dY_\eta = c(\eta, X_{\eta-}) Y_\eta d\eta \quad \text{and} \quad dY_\eta \cdot du(\eta, X_\eta) = 0,
\] (2.8)

it follows that
\[
\begin{aligned}
du(\eta, X_\eta)Y_\eta &= u(\eta, X_\eta) dY_\eta + Y_\eta du(\eta, X_\eta).
\end{aligned}
\] (2.9)

Thus, we obtain
\[
\begin{aligned}
\mathbb{E}\left\{ u(T \wedge \tau_\varepsilon, X_{T \wedge \tau_\varepsilon}) Y_{T \wedge \tau_\varepsilon} + Z_{T \wedge \tau_\varepsilon} - u(t, x) \right\} \\
= \mathbb{E}\left\{ \int_{t}^{T \wedge \tau_\varepsilon} \left[ \frac{\partial u}{\partial \eta}(\eta, X_{\eta-}) + \sum_{i=1}^{n} b_i(\eta, X_{\eta-}) \frac{\partial u}{\partial x_i}(\eta, X_{\eta-}) \\
+ \int_{y \in \mathbb{R}} \left[ u(\eta, X_{\eta-} + y) - u(\eta, X_{\eta-}) - I_{\{y < 1\}}(\nabla u(\eta, X_{\eta-}), y) \right] \nu(dy) \\
+ f(\eta, X_{\eta-}) + c(\eta, X_{\eta-}) u(\eta, X_{\eta-}) \right] \exp\left( \int_{t}^{\eta} c(v, X_{v-}) dv \right) d\eta \right\}
= 0,
\end{aligned}
\] (2.10)

where we have used
\[
\int_{|y| < 1} (\nabla u(\eta, X_{\eta-}), y) \nu(dy) = 0, \quad Y_\eta = \exp\left( \int_{t}^{\eta} c(v, X_{v-}) dv \right)
\] (2.11)

and equation (1.1). Thus we complete the proof. \qed

At last, we introduce some notations which will be used later.

For $\beta = [\beta] + \{\beta\}^+ > 0$, $[\beta] \in \mathbb{Z}^+ \cup \{0\}$, $\{\beta\}^+ \in [0, 1)$, let $C^\beta([0, T] \times \mathbb{R}^n)$ denote the space of measurable functions $u(t, x)$ on $\mathbb{R}^n$ for any $t \in [0, T]$ such that the norm
\[
|u(t, x)|_\beta = \sum_{|\gamma| \leq [\beta]} \sup_{(t, x) \in [0, T] \times \mathbb{R}^n} |\partial_x^\gamma u(t, x)| + \sup_{\gamma = [\beta]} \sup_{t, x, h \neq 0} \frac{|\partial_x^\gamma u(t, x + h) - \partial_x^\gamma u(t, x)|}{|h|^{\{\beta\}^+}}
\] (2.12)
is finite. $C^\beta(\mathbb{R}^n)$ is just the general Hölder space whose functions are defined on $\mathbb{R}^n$. 
Define $C^{1,\beta}([0, T] \times \mathbb{R}^n)$ as the space of measurable functions $u(t, x) \in C^\beta([0, T] \times \mathbb{R}^n)$ whose first-order derivative with respect to $t$ are bounded continuous on $[0, T]$ for any $x \in \mathbb{R}^n$ with natural norm. Just like [9], we give the following assumptions.

Assumption I. There exists a unique solution $u(t, x)$ to parabolic problem (1.1) such that

$$u(t, x) \in C^{1,\beta}([0, T] \times \mathbb{R}^n).$$

(2.13)

Assumption II. The functions $b_i(t, x)$ $(i = 1, \ldots, n)$, $c(t, x)$, $f(t, x)$, and their first derivatives with respect to $t$ and $x$ are uniformly bounded.

Assumption III. The functions $b_i(t, x)$ $(i = 1, \ldots, n)$, $c(t, x)$, $f(t, x)$, their first derivatives with respect to $t$ and $x$ and their second derivatives with respect to $x$ are uniformly bounded.

3 Jump-adapted Euler scheme

For $\varepsilon \in (0, 1)$, we approximate the symmetric $2s$-stable process

$$L_t = \int_0^t \int_{|y|<1} y \tilde{N}(d\eta, dy) + \int_0^t \int_{|y| \geq 1} y N(d\eta, dy)$$

(3.1)

by

$$L_\varepsilon^\eta = \int_0^t \int_{\varepsilon<|y|<1} y \tilde{N}(d\eta, dy) + \int_0^t \int_{|y| \geq 1} y N(d\eta, dy).$$

(3.2)

Unlike replacing small jumps by Brownian motion in [17, 22], we directly remove small jumps. In the following lemma, we give the weak convergent order.

Lemma 3.1. Let $s \in (0, 1)$, $h \in C^\beta(\mathbb{R}^n)$, $\beta \in (2s \vee 1, 2)$. Then there is a constant $C_1$, such that for $0 \leq t' \leq t \leq T$, we have

$$|E[h(L_{t'-\varepsilon}) - h(L_{t'-\varepsilon}^\varepsilon)]| \leq C_1 \varepsilon^{\beta-2s} |h|_\beta (t-t').$$

(3.3)

Proof. By Itô formula,

$$v(t', x) = E[h(x + L_{t'-\varepsilon}^\varepsilon)]$$

(3.4)

is the solution of the backward Kolmogorov equation with $0 \leq \eta \leq t$,

$$\begin{cases}
\frac{\partial v}{\partial \eta}(\eta, x) + \int_{|y|>\varepsilon} v(\eta, x+y) - v(\eta, x) - I_{\{|y|<1\}}(\nabla v(\eta, x), y) \nu(dy) = 0, \\
v(t, x) = h(x).
\end{cases}$$

(3.5)

We can get $v \in C^\beta([0, t] \times \mathbb{R}^n)$ for any $\eta \in [0, t]$ since $|v|_\beta \leq |h|_\beta$. By Itô formula
and (3.5), we have
\[
\begin{align*}
& \left| \mathbb{E} \left[ h(L_t - L_{t'}) - h(L_t^\varepsilon - L_{t'}^\varepsilon) \right] \right| \\
= & \left| \mathbb{E} \left[ v(t, L_t - L_{t'}) - v(t', 0) \right] \right| \\
= & \left| \mathbb{E} \left[ \int_t^{t'} \frac{\partial v}{\partial \eta}(\eta, L_{\eta-} - L_{t'}) d\eta + \int_t^{t'} \int_{y \in \mathbb{R}^n} \left[ v(\eta, L_{\eta-} - L_{t'} + y) \\
- v(\eta, L_{\eta-} - L_{t'}) - I_{\{\|y\| < 1\}}(\nabla v(\eta, L_{\eta-} - L_{t'}), y) \right] \nu(dy) d\eta \right] \right| \\
= & \left| \mathbb{E} \left[ \int_t^{t'} \int_{\|y\| < \varepsilon} \left[ v(\eta, L_{\eta-} - L_{t'} + y) \\
- v(\eta, L_{\eta-} - L_{t'}) - I_{\{\|y\| < 1\}}(\nabla v(\eta, L_{\eta-} - L_{t'}), y) \right] \nu(dy) d\eta \right] \right| \\
\leq & \int_t^{t'} \int_{\|y\| < \varepsilon} \int_0^1 \mathbb{E} \left| (\nabla v(\eta, L_{\eta-} - L_{t'} + \alpha y) - \nabla v(\eta, L_{\eta-} - L_{t'}), y) \right| d\alpha \nu(dy) d\eta \\
\leq & C_1(t - t')|v|_\beta \int_{\|y\| < \varepsilon} |y|^{-n-2s+\beta} dy \\
\leq & C_1(t - t')|h|_\beta \varepsilon^{\beta - 2s}. 
\end{align*}
\]

\(\square\)

Consider the following \(L_t^\varepsilon\)-jump adapted time discretization:
\[
\begin{cases}
\tau_0 = t, \\
\tau_{i+1} = \inf\{t > \tau_i : \Delta L_t^\varepsilon \neq 0\} \wedge \tau_{i,x} \wedge T, 
\end{cases}
\]
where \(\tau_{i,x} = \left\{ r \geq t, \tilde{X}_r \notin \mathbb{D} \right\} \) and \(\tilde{X}_r\) is given by the following system. For \(r \in (\tau_i, \tau_{i+1})\), we define
\[
\begin{align*}
\tilde{X}_r &= \tilde{X}_{\tau_i} + b(\tau_i, \tilde{X}_{\tau_i})(r - \tau_i), \\
\tilde{Y}_r &= \tilde{Y}_{\tau_i} + c(\tau_i, \tilde{X}_{\tau_i})\tilde{Y}_{\tau_i}(r - \tau_i), \\
\tilde{Z}_r &= \tilde{Z}_{\tau_i} + f(\tau_i, \tilde{X}_{\tau_i})\tilde{Y}_{\tau_i}(r - \tau_i), 
\end{align*}
\]
where the starting point \(\tilde{X}_t = x, \tilde{Y}_t = 1, \text{ and } \tilde{Z}_t = 0\). For \(r = \tau_{i+1}\), we define
\[
\begin{align*}
\tilde{X}_{\tau_{i+1}} &= \tilde{X}_{\tau_{i+1}-} + \Delta L_{\tau_{i+1}}^\varepsilon, \quad \text{where } \Delta L_{\tau_{i+1}}^\varepsilon = L_{\tau_{i+1}}^\varepsilon - L_{\tau_{i+1}-}^\varepsilon, \\
\tilde{Y}_{\tau_{i+1}} &= \tilde{Y}_{\tau_{i+1}-} + c(\tau_{i+1}, \tilde{X}_{\tau_{i+1}})\tilde{Y}_{\tau_{i+1}}(\tau_{i+1} - \tau_i), \\
\tilde{Z}_{\tau_{i+1}} &= \tilde{Z}_{\tau_{i+1}} + f(\tau_{i+1}, \tilde{X}_{\tau_{i+1}})\tilde{Y}_{\tau_{i+1}}(\tau_{i+1} - \tau_i). 
\end{align*}
\]
Algorithm 1 Algorithm for (3.10)

1: initialize: \( \tau_0 = t, \bar{X}_{\tau_0} = x, \bar{Y}_{\tau_0} = 1, \) and \( \bar{Z}_{\tau_0} = 0, i = 0. \)
2: while \( \tau_i < T \) and \( \bar{X}_{\tau_i} \in D \) do
3: \hspace{1em} Sample: jump time \( \tau \) (\( \tau \) is exponentially distributed with parameter \( \lambda_\varepsilon \) in (3.11)).
4: \hspace{1em} Set: \( \bar{X}_{\tau_{i+1}} = \bar{X}_{\tau_i} + b(\tau_i, \bar{X}_{\tau_i})\tau. \)
5: \hspace{1em} if \( \tau + \tau_i > T \) or \( \bar{X}_{\tau_{i+1}} \notin D \) then,
6: \hspace{2em} Set: \( \tau_{i+1} = \sup \{ \tau_i < t < T : \bar{X}_{\tau_i} + b(\tau_i, \bar{X}_{\tau_i})(t - \tau_i) \in D \}. \)
7: \hspace{1em} Evaluate: \( \bar{X}_{\tau_{i+1}} \) without jump, \( \bar{Y}_{\tau_{i+1}}, \bar{Z}_{\tau_{i+1}} \) according to (3.9).
8: \hspace{1em} Set: \( i = i + 1. \)
9: else
10: \hspace{1em} Sample: jump size \( J_\varepsilon \) according to density (2.2).
11: \hspace{1em} Evaluate: \( \bar{X}_{\tau_{i+1}}, \bar{Y}_{\tau_{i+1}}, \bar{Z}_{\tau_{i+1}} \) according to (3.9).
12: \hspace{1em} Set: \( \tau_{i+1} = \tau_i + \tau \) and \( i = i + 1. \)
13: end if
14: end while
15: if \( \tau_{i+1} < T \) then Set: \( \bar{\tau}_{t,x} = \tau_{i+1}, \) Evaluate: \( \chi(\bar{\tau}_{t,x}, \bar{X}_{\bar{\tau}_{t,x}})\bar{Y}_{\bar{\tau}_{t,x}} + \bar{Z}_{\bar{\tau}_{t,x}}. \)
16: else Set: \( \bar{\tau}_{t,x} = T, \) Evaluate: \( g(T, \bar{X}_T)\bar{Y}_T + \bar{Z}_T. \)
17: end if
18: Loop above algorithm \( N \) times.
19: Evaluate: \( u(t, x) \approx \frac{1}{N} \sum_{j=1}^{N} \left[ u(T \wedge \bar{\tau}_{t,x}, \bar{X}_{T \wedge \bar{\tau}_{t,x}})\bar{Y}_{T \wedge \bar{\tau}_{t,x}} + \bar{Z}_{T \wedge \bar{\tau}_{t,x}} \right]. \)

Thus, we approximate the solution \( u(t, x) \) of equation (1.1) by

\[
\mathbb{E} \left[ u(T \wedge \bar{\tau}_{t,x}, \bar{X}_{T \wedge \bar{\tau}_{t,x}})\bar{Y}_{T \wedge \bar{\tau}_{t,x}} + \bar{Z}_{T \wedge \bar{\tau}_{t,x}} \right].
\]  (3.10)

We summarize the method in Algorithm 1.

To give the error estimate, we need the following lemmas.

Lemma 3.2. Let \( \delta_i = (\bar{\tau}_{t,x} \wedge T) - \tau_i, \) and

\[
\lambda_\varepsilon = \nu(|y| > \varepsilon) = \frac{2^s \Gamma \left( \frac{n}{2} + s \right) \varepsilon^{-2s}}{\Gamma(1 - s) \Gamma \left( \frac{n}{2} \right)}
\]  (3.11)

be the jump intensity. Then

(i) there is a constant \( C_1, \) such that for any \( i \geq 0, \)

\[
C_1(\delta_i \wedge \lambda_\varepsilon^{-1}) \leq \mathbb{E} [\tau_{i+1} - \tau_i | \mathcal{F}_{\tau_i}] \leq \delta_i \wedge \lambda_\varepsilon^{-1},
\]  (3.12)

(ii) there is a constant \( C_2, \) such that for any \( i \geq 0, \)

\[
\mathbb{E} [(\tau_{i+1} - \tau_i)^{\frac{3}{2}} | \mathcal{F}_{\tau_i}] \leq C_2 \left( \delta_i^{\frac{1}{2}} \wedge \lambda_\varepsilon^{-\frac{1}{2}} \right) \mathbb{E} [\tau_{i+1} - \tau_i | \mathcal{F}_{\tau_i}],
\]  (3.13)
(iii) there is a constant $C_3$, such that for any $i \geq 0$,

$$
E\left[ (\tau_{i+1} - \tau_i)^2 | F_{\tau_i} \right] \leq C_3 E\left[ \delta_i^2 \wedge \lambda^{-2} \epsilon | F_{\tau_i} \right] \\
\leq C_3 (\delta_i^2 \wedge \lambda^{-2} \epsilon) E\left[ \tau_{i+1} - \tau_i | F_{\tau_i} \right],
$$

(3.14)

Proof. The proofs of (i) and (iii) are similar to those of Lemma 8 in [22]. We omit the proofs of (i) and (iii). For (ii), we have

$$
E\left[ (\tau_{i+1} - \tau_i)^{3/2} | F_{\tau_i} \right] = \lambda \int_0^{\delta_i} e^{-\lambda t} t^{3/2} dt + e^{-\lambda \epsilon \delta_i} \frac{3}{2}
$$

$$
\leq C_3 (\delta_i^2 \wedge \lambda^{-2} \epsilon) E\left[ \tau_{i+1} - \tau_i | F_{\tau_i} \right].
$$

(3.15)

□

We also need the following auxiliary lemma.

**Lemma 3.3.** Under the Assumption II, $\tilde{Y}_{\tau_i}$ defined in (3.9) is uniformly bounded by a deterministic formula

$$
|\tilde{Y}_{\tau_i}| < \bar{c}(T-t), \quad i = 0, 1, \ldots, n_{T\wedge \tilde{\tau}_{i,x}},
$$

(3.16)

where $\bar{c} = \max_{(t,x) \in [0,T] \times \Omega} c(t,x)$.

Proof. From (3.9) and $\tilde{Y}_{\tau_0} = \tilde{Y}_t = 1$, we get the required estimate as follows,

$$
|\tilde{Y}_{\tau_i}| = |\tilde{Y}_{\tau_{i-1}} + c(\tau_{i-1}, \tilde{X}_{\tau_{i-1}}) \tilde{Y}_{\tau_{i-1}} (\tau_i - \tau_{i-1})| \\
\leq |\tilde{Y}_{\tau_{i-1}}| |1 + \bar{c}(\tau_i - \tau_{i-1})| \leq |\tilde{Y}_{\tau_0}| e^{\bar{c}(T-t)} \leq \bar{c}(T-t).
$$

(3.17)

□

Now we prove the convergence theorem for Algorithm 1.

**Theorem 3.1.** Let $\varepsilon \in (0,1)$. Under Assumption I with $\beta \in (2s \lor 1, 2)$ and Assumption II, the error estimate

$$
E\left[ u(T \wedge \tilde{\tau}_{i,x}, \tilde{X}_{T \wedge \tilde{\tau}_{i,x}}) \tilde{Y}_{T \wedge \tilde{\tau}_{i,x}} + \tilde{Z}_{T \wedge \tilde{\tau}_{i,x}} - u(t, x) \right] \leq C_1 \varepsilon^{2s} + C_2 \varepsilon^{\beta - 2s}
$$

(3.18)

holds with $C_1$ and $C_2$ being constants independent of $t$ and $x$.

Proof.

$$
E\left[ u(T \wedge \tilde{\tau}_{i,x}, \tilde{X}_{T \wedge \tilde{\tau}_{i,x}}) \tilde{Y}_{T \wedge \tilde{\tau}_{i,x}} + \tilde{Z}_{T \wedge \tilde{\tau}_{i,x}} - u(t, x) \right]
$$

$$
= E\left\{ \sum_{i=0}^{n_{T \wedge \tilde{\tau}_{i,x}} - 1} u(\tau_{i+1}, \tilde{X}_{\tau_{i+1}}) \tilde{Y}_{\tau_{i+1}} + \tilde{Z}_{\tau_{i+1}} - u(\tau_i, \tilde{X}_i) \tilde{Y}_{\tau_i} - \tilde{Z}_{\tau_i} \right\}.
$$

(3.19)
By Itô formula, the martingale property, systems (3.8) and (3.9), we have

\[
\begin{align*}
\mathbb{E}\left[ u(T \wedge \tilde{\tau}_i, \tilde{X}_{T \wedge \tilde{\tau}_i}) \mathbb{Y}_{T \wedge \tilde{\tau}_i} + \tilde{Z}_{T \wedge \tilde{\tau}_i} \right] - u(t, x) \\
= \mathbb{E}\left\{ \sum_{i=0}^{n_{T \wedge \tilde{\tau}_i} - 1} u(\tau_{i+1}, \tilde{X}_{\tau_{i+1}^-}) \mathbb{Y}_{\tau_{i+1}} + \tilde{Z}_{\tau_{i+1}} - u(\tau_i, \tilde{X}_{\tau_i}) \mathbb{Y}_{\tau_i} - \tilde{Z}_{\tau_i} \\
+ \mathbb{Y}_{\tau_{i+1}} \left[ u(\tau_{i+1}, \tilde{X}_{\tau_{i+1}^-}) - u(\tau_{i+1}, \tilde{X}_{\tau_{i+1}^-}) \right] \right\} \\
= \mathbb{E}\left\{ \sum_{i=0}^{n_{T \wedge \tilde{\tau}_i} - 1} \mathbb{Y}_{\tau_i} \int_{\tau_i}^{\tau_{i+1}} \frac{\partial}{\partial \eta} u(\eta, \tilde{X}_{\eta^-}) + \sum_{j=1}^n b_j(\tau_i, \tilde{X}_{\tau_i}) \frac{\partial}{\partial x_j} u(\eta, \tilde{X}_{\eta^-}) d\eta \\
+ \mathbb{Y}_{\tau_i} c(\tau_i, \tilde{X}_{\tau_i}) u(\tau_{i+1}, \tilde{X}_{\tau_{i+1}^-}) \Delta \tau_i + \mathbb{Y}_{\tau_i} f(\tau_i, \tilde{X}_{\tau_i}) \Delta \tau_i \\
+ \mathbb{E}\left[ \int_{t}^{T \wedge \tilde{\tau}_i} \mathbb{Y}_{\eta} \int_{|y| > \varepsilon} \left[ u(\eta, \tilde{X}_{\eta^-} + y) - u(\eta, \tilde{X}_{\eta^-}) \right] N(d\eta, dy) \right] \right\} \\
= \mathbb{E}\left\{ \sum_{i=0}^{n_{T \wedge \tilde{\tau}_i} - 1} \mathbb{Y}_{\tau_i} \mathbb{E}\left[ \int_{\tau_i}^{\tau_{i+1}} \frac{\partial}{\partial \eta} u(\eta, \tilde{X}_{\eta^-}) + \sum_{j=1}^n b_j(\tau_i, \tilde{X}_{\tau_i}) \frac{\partial}{\partial x_j} u(\eta, \tilde{X}_{\eta^-}) d\eta \\
+ \int_{\tau_i}^{\tau_{i+1}} \int_{|y| > \varepsilon} \left[ u(\eta, \tilde{X}_{\eta^-} + y) - u(\eta, \tilde{X}_{\eta^-}) \right] \nu(dy) d\eta \\
+ c(\tau_i, \tilde{X}_{\tau_i}) \Delta \tau_i \int_{\tau_i}^{\tau_{i+1}} \int_{|y| > \varepsilon} \left[ u(\eta, \tilde{X}_{\eta^-} + y) - u(\eta, \tilde{X}_{\eta^-}) \right] \nu(dy) d\eta \\
+ c(\tau_i, \tilde{X}_{\tau_i}) u(\tau_{i+1}, \tilde{X}_{\tau_{i+1}^-}) \Delta \tau_i + f(\tau_i, \tilde{X}_{\tau_i}) \Delta \tau_i | \mathcal{F}_{\tau_i} \right] \right\} \\
= \mathbb{E}\left\{ \sum_i \mathbb{Y}_{\tau_i} \mathbb{E}\left[ B_i \mid \mathcal{F}_{\tau_i} \right] \right\}, \tag{3.20}
\end{align*}
\]
Since $u(t, x)$ satisfies (1.1), we obtain

$$
\mathbb{E} \left[ B_1 | \mathcal{F}_{\tau_i} \right]
= \mathbb{E} \left\{ \int_{\tau_i}^{\tau_{i+1}} \sum_{j=1}^{n} \left[ b_j(\tau_i, \tilde{X}_{\tau_i}) - b_j(\eta, \tilde{X}_{\eta-}) \right] \frac{\partial u}{\partial x_j} (\eta, \tilde{X}_{\eta-}) d\eta 
- \int_{\tau_i}^{\tau_{i+1}} \int_{|y|<\varepsilon} \left[ u(\eta, \tilde{X}_{\eta-} + y) - u(\eta, \tilde{X}_{\eta-}) - I_{|y|<1}(\nabla u(\eta, \tilde{X}_{\eta-}, y)) \right] \nu(dy) d\eta 
+ c(\tau_i, \tilde{X}_{\tau_i}) \Delta \tau_i \int_{\tau_i}^{\tau_{i+1}} \int_{|y|>\varepsilon} \left[ u(\eta, \tilde{X}_{\eta-} + y) - u(\eta, \tilde{X}_{\eta-}) \right] \nu(dy) d\eta 
+ \int_{\tau_i}^{\tau_{i+1}} c(\tau_i, \tilde{X}_{\tau_i}) u(\eta, \tilde{X}_{\eta-}) d\eta 
+ \int_{\tau_i}^{\tau_{i+1}} f(\tau_i, \tilde{X}_{\tau_i}) - f(\eta, \tilde{X}_{\eta-}) d\eta \right\} 
= \tilde{Y}_{\tau_i} \mathbb{E} \left[ B_{i_1} + B_{i_2} + B_{i_3} + B_{i_4} + B_{i_5} | \mathcal{F}_{\tau_i} \right].
$$

For the first term, one has

$$
| \mathbb{E} \left[ B_{i_1} | \mathcal{F}_{\tau_i} \right] |
\leq \mathbb{E} \left[ \sum_{j=1}^{n} \int_{\tau_i}^{\tau_{i+1}} \left| b_j(\tau_i, \tilde{X}_{\tau_i}) - b_j(\eta, \tilde{X}_{\eta-}) \left| \frac{\partial u}{\partial x_j} (\eta, \tilde{X}_{\eta-}) \right| d\eta \bigg| \mathcal{F}_{\tau_i} \right] 
\leq |u|_{\beta} \mathbb{E} \left[ \sum_{j=1}^{n} \int_{\tau_i}^{\tau_{i+1}} \left| b_j(\tau_i, \tilde{X}_{\tau_i}) - b_j(\eta, \tilde{X}_{\tau_i}) \right| + \left| b_j(\eta, \tilde{X}_{\tau_i}) - b_j(\eta, \tilde{X}_{\eta-}) \right| d\eta \bigg| \mathcal{F}_{\tau_i} \right] 
\leq |u|_{\beta} \mathbb{E} \left[ \sum_{j=1}^{n} C_1 \int_{\tau_i}^{\tau_{i+1}} (\eta - \tau_i) d\eta + C_2 \int_{\tau_i}^{\tau_{i+1}} |\tilde{X}_{\tau_i} - \tilde{X}_{\eta-}| d\eta \bigg| \mathcal{F}_{\tau_i} \right] 
\leq C_1 |u|_{\beta} \mathbb{E} \left[ \int_{\tau_i}^{\tau_{i+1}} (\eta - \tau_i) d\eta \bigg| \mathcal{F}_{\tau_i} \right] 
\leq C_1 \mathbb{E} \left[ (\tau_{i+1} - \tau_i)^2 \bigg| \mathcal{F}_{\tau_i} \right].
$$

(3.22)

Clearly, for the second term, it holds that

$$
| \mathbb{E} \left[ B_{i_2} | \mathcal{F}_{\tau_i} \right] |
\leq \mathbb{E} \left[ \int_{\tau_i}^{\tau_{i+1}} \int_{|y|<\varepsilon} \int_{0}^{1} \left( \nabla u(\eta, \tilde{X}_{\eta} + \alpha y) - \nabla u(\eta, \tilde{X}_{\eta}), y \right) d\alpha \nu(dy) d\eta \bigg| \mathcal{F}_{\tau_i} \right] 
\leq C_2 \mathbb{E} \left[ (\tau_{i+1} - \tau_i) \bigg| \mathcal{F}_{\tau_i} \right] |u|_{\beta} \int_{|y|<\varepsilon} |y|^{-n-2s+\beta} dy 
\leq C_2 \mathbb{E} \left[ (\tau_{i+1} - \tau_i) \bigg| \mathcal{F}_{\tau_i} \right] \varepsilon^{\beta-2s}.
$$

(3.23)
Next, we estimate the forth term.

\[
\begin{align*}
|E[B_{i3} | \mathcal{F}_{\tau_i}]| & \\
& \leq E \left[ \int_{\tau_i}^{\tau_{i+1}} c(\tau_i, \tilde{X}_{\tau_i}) u(\tau_{i+1}, \tilde{X}_{\tau_{i+1}-}) - c(\eta, \tilde{X}_{\eta}) u(\eta, \tilde{X}_{\eta-}) d\eta \big| \mathcal{F}_{\tau_i} \right] \\
& \leq E \left[ \int_{\tau_i}^{\tau_{i+1}} \left[ c(\tau_i, \tilde{X}_{\tau_i}) - c(\eta, \tilde{X}_{\eta}) \right] u(\eta, \tilde{X}_\eta) \right] \\
& \quad + \left[ u(\tau_{i+1}, \tilde{X}_{\tau_{i+1}-}) - u(\eta, \tilde{X}_{\eta-}) \right] c(\tau_i, \tilde{X}_{\tau_i}) d\eta \big| \mathcal{F}_{\tau_i} \right] \\
& \leq C_1 E \left[ (\tau_{i+1} - \tau_i)^2 \big| \mathcal{F}_{\tau_i} \right].
\end{align*}
\]

For the remaining part, we can easily get

\[
|E \left[ B_{i3} + B_{i5} \big| \mathcal{F}_{\tau_i} \right] | \leq C_1 E \left[ (\tau_{i+1} - \tau_i)^2 \big| \mathcal{F}_{\tau_i} \right].
\]

Combining inequalities (3.19)-(3.25) yields

\[
\begin{align*}
& \leq E \left\{ \sum |\tilde{Y}_{\tau_i}| E \left[ |B_{i1}| + |B_{i2}| + |B_{i3}| + |B_{i4}| \big| \mathcal{F}_{\tau_i} \right] \right\} . \\
& \leq E \left\{ \sup |\tilde{Y}_{\tau_i}| \left( \sum E \left[ C_1 (\tau_{i+1} - \tau_i)^2 + C_2 \varepsilon^{-2s} (\tau_{i+1} - \tau_i) \big| \mathcal{F}_{\tau_i} \right] \right) \right\} \\
& \leq C_1 \varepsilon^{2s} + C_2 \varepsilon^{\beta-2s}.
\end{align*}
\]

Thus, the proof is completed. \( \square \)

**Remark 3.1.** The probabilistic approach can be also applied to the Dirichlet problem for the steady-state equation

\[
\begin{cases}
-(-\Delta)^s u + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x) u + f(x) = 0, & x \in \mathbb{D}, \\
u(x) = g(x), & x \in \mathbb{R}^n \setminus \mathbb{D}.
\end{cases}
\]

The solution has the probabilistic representation below

\[
u(x) = E \left[ g(X_{\tau_x}^x) Y_{\tau_x}^{x,1} + Z_{\tau_x}^{x,1,0} \right],
\]

where \( X_{\tau_x}^x, Y_{\tau_x}^{x,1}, \) and \( Z_{\tau_x}^{x,1,0} (r \geq 0) \) are the solution of the Cauchy problem for the systems of SDEs,

\[
\begin{align*}
\frac{dX_r}{dt} &= b(X_r) dr + dL_r, \quad X_0 = x, \\
\frac{dY_r}{dt} &= c(X_r) Y_r dr, \quad Y_0 = 1, \\
\frac{dZ_r}{dt} &= f(X_r) Z_r dr, \quad Z_0 = 0.
\end{align*}
\]
Here $x \in \mathbb{D}$ and $\tau_x$ is the first exit time of the trajectory $X^x_t$ to the boundary of $\mathbb{D}$. We apply the jump-adapted Euler scheme to system (3.29) and get

$$
\begin{align*}
\tilde{X}_{\tau_i+1} &= \tilde{X}_{\tau_i} + b(\tilde{X}_{\tau_i})(\tau_{i+1} - \tau_i) + L^\varepsilon_{\tau_{i+1} - \tau_i}, \\
\tilde{Y}_{\tau_i+1} &= \tilde{Y}_{\tau_i} + c(\tilde{X}_{\tau_i})\tilde{Y}_{\tau_i}(\tau_{i+1} - \tau_i), \\
\tilde{Z}_{\tau_i+1} &= \tilde{Z}_{\tau_i} + f(\tilde{X}_{\tau_i})\tilde{Y}_{\tau_i}(\tau_{i+1} - \tau_i),
\end{align*}
$$

(3.30)

where $\tau_i, (i = 0, 1, \ldots, n_{\tau_x})$ are jump times of $L^\varepsilon_t$ in (3.2). Thus we can approximate (3.28) by

$$
\mathbb{E}\left[ g(\tilde{X}_{\tau_x})\tilde{Y}_{\tau_x} + \tilde{Z}_{\tau_x} \right],
$$

(3.31)

where $\tilde{\tau}_x$ is the exit time of the trajectory $\tilde{X}^x_t$ to the boundary of $\mathbb{D}$.

If $b_i(x), (i = 1, 2, \ldots, n), f(x),$ and $c(x)$, together with their first derivatives with respect to $x$ being uniformly bounded, then we have

$$
\left| \mathbb{E}\left[ g(\tilde{X}_{\tau_x})\tilde{Y}_{\tau_x} + \tilde{Z}_{\tau_x} \right] - u(x) \right| \leq C_1(\varepsilon^{2s} + \varepsilon^{\beta - 2s}),
$$

(3.32)

where $C_1$ is a constant.

### 4 High-order scheme and error estimate

In this section, we will give a higher-order jump-adapted scheme under Assumption I with $\beta \in [2, 3]$ and Assumption II. By the idea of [5], we approximate the symmetric $2s$-stable process $L_t$ in (2.1) by

$$
\tilde{L}^\varepsilon_t = \sigma_\varepsilon W_t + \int_0^t \int_{\varepsilon < |y| < 1} y\tilde{N}(d\eta, dy) + \int_0^t \int_{|y| \geq 1} yN(d\eta, dy) = \sigma_\varepsilon W_t + L^\varepsilon_t,
$$

(4.1)

where $W_t = (W^1_t, W^2_t, \ldots, W^n_t)$ is an $n$-dimensional Brownian motion,

$$
\sigma_\varepsilon\sigma_\varepsilon^T = C(n, s) \left( \int_{|y| < \varepsilon} y_{j_1} y_{j_2} \nu(dy) \right)_{1 \leq j_1, j_2 \leq n},
$$

(4.2)

and

$$
\sigma_\varepsilon = \left[ C(n, s) \varepsilon^{2-2s} \frac{n^{n/2}}{\Gamma(n/2 + 1)} \right]^{1/2} I := \bar{\sigma}_\varepsilon I.
$$

(4.3)

Here $I$ is an identity matrix of dimension $N$.

We have the weak convergent order in the following lemma.

**Lemma 4.1.** Let $s \in (0, 1), h \in C^3(\mathbb{R}^n)$, $\beta \in [2, 3]$. Then there is a constant $C_1$, such that for $0 \leq t' \leq t \leq T$, we have

$$
\left| \mathbb{E}\left[ h(L_{t-t'}) - h(\tilde{L}^\varepsilon_{t-t'}) \right] \right| \leq C_1 \varepsilon^{[\beta]-2s}|h|_{\beta}(t - t').
$$

(4.4)
The proof is similar to that of Lemma 6 in [22], so we omit it here.

Since the exact exit time of stochastic process with Brownian motion is hard to approximate, we replace $\sigma_\varepsilon W_t$ by $\sigma_\varepsilon \sqrt{t} \xi$, where $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ and $P(\xi_1 = \pm 1) = P(\xi_2 = \pm 1) = \cdots = P(\xi_n = \pm 1) = \frac{1}{2}$. Consider the following high-order jump-adapted time discretization:

$$
\begin{aligned}
\tau_0 &= t, \\
\tau_{i+1} &= \inf\{t > \tau_i : \Delta L^\varepsilon_t := L^\varepsilon_t - L^\varepsilon_{t-} \neq 0 \} \wedge \tau_{t,x} \wedge T,
\end{aligned}
$$

where $\tau_{t,x} = \{r \geq t, \bar{X}_r \notin \mathbb{D}\}$ and $\bar{X}_r$ is given by the following system. For $r \in [\tau_i, \tau_{i+1})$, define

$$
\begin{align*}
\bar{X}_r &= \bar{X}_{\tau_i} + b(\tau_i, \bar{X}_{\tau_i})(r - \tau_i) + \sigma_\varepsilon \sqrt{r - \tau_i} \xi, \\
\bar{Y}_r &= \bar{Y}_{\tau_i} + c(\tau_i, \bar{X}_{\tau_i}) \bar{Y}_{\tau_i}(r - \tau_i), \\
\bar{Z}_r &= \bar{Z}_{\tau_i} + f(\tau_i, \bar{X}_{\tau_i}) \bar{Y}_{\tau_i}(r - \tau_i),
\end{align*}
$$

where the starting point $\bar{X}_t = x$, $\bar{Y}_t = 1$, and $\bar{Z}_t = 0$. For $r = \tau_{i+1}$, we define

$$
\begin{aligned}
\bar{X}_{\tau_{i+1}} &= \bar{X}_{\tau_{i+1}} - \Delta L^\varepsilon_{\tau_{i+1}}, \\
\bar{Y}_{\tau_{i+1}} &= \bar{Y}_{\tau_i} + c(\tau_i, \bar{X}_{\tau_i}) \bar{Y}_{\tau_i}(\tau_{i+1} - \tau_i), \\
\bar{Z}_{\tau_{i+1}} &= \bar{Z}_{\tau_i} + f(\tau_i, \bar{X}_{\tau_i}) \bar{Y}_{\tau_i}(\tau_{i+1} - \tau_i).
\end{aligned}
$$

Thus, we approximate the solution $u(t, x)$ of equation (1.1) by

$$
\mathbb{E}\left[u(T \wedge \tau_{t,x}, \bar{X}_{T \wedge \tau_{t,x}}) \bar{Y}_{T \wedge \tau_{t,x}} + \bar{Z}_{T \wedge \tau_{t,x}}\right].
$$

We summarize the method in Algorithm 2.

Before proving the convergence theorem for Algorithm 2, we give the following lemma which is similar to Lemma 3.3.

**Lemma 4.2.** Under Assumption III, $\bar{Y}_{\tau_i}$ defined in (4.7) is uniformly bounded by a deterministic formula

$$
|\bar{Y}_{\tau_i}| < e^{c(T-t)}, \quad i = 0, 1, \ldots, n_{T \wedge \tau_{t,x}},
$$

where $c = \max_{(t, x) \in \mathbb{D}} c(t, x)$.

**Theorem 4.1.** Let $\varepsilon \in (0, 1)$. Under Assumption I with $\beta \in [2, 3]$ and III, the error estimate

$$
|\mathbb{E}\left[u(T \wedge \tau_{t,x}, \bar{X}_{T \wedge \tau_{t,x}}) \bar{Y}_{T \wedge \tau_{t,x}} + \bar{Z}_{T \wedge \tau_{t,x}}\right] - u(t, x)| \leq C_1 \varepsilon^{2s} + C_2 \varepsilon^{l(\beta) - 2s}
$$

holds with $C_1$ and $C_2$ being constants independent of $t$ and $x$. 
Algorithm 2 Algorithm for (4.8)

1: initialize: \( \tau_0 = t, \bar{X}_{\tau_0} = x, \bar{Y}_{\tau_0} = 1, \) and \( \bar{Z}_{\tau_0} = 0, i = 0. \)
2: while \( \tau_i < T \) and \( \bar{X}_{\tau_i} \in D \) do
3: Sample: jump time \( \tau \) (\( \tau \) is exponentially distributed with parameter \( \lambda_x \) in (3.11)) and \( \xi = (\xi_1, \xi_2, \ldots, \xi_n) \) with \( P(\xi_1 = \pm 1) = P(\xi_2 = \pm 1) = \ldots = P(\xi_n = \pm 1) = \frac{1}{2} \).
4: Set: \( \bar{X}_{\tau_i+1} = \bar{X}_{\tau_i} + b(\tau_i, \bar{X}_{\tau_i})\tau + \sigma_x \tau^{\frac{3}{2}} \xi. \)
5: if \( \tau + \tau_i > T \) or \( \bar{X}_{\tau_i+1} \notin D \) then,
6: Set: \( \tau_{i+1} = \sup \{ \tau_i < t < T : \bar{X}_t + b(\tau_i, \bar{X}_{\tau_i})(t - \tau_i) + \sigma_x \sqrt{t - \tau_i} \in D \}. \)
7: Evaluate: \( \bar{X}_{\tau_{i+1}}, \bar{Y}_{\tau_{i+1}}, \bar{Z}_{\tau_{i+1}} \) according to (4.6).
8: Set: \( i = i + 1. \)
9: else
10: Sample: jump size \( J_x \) according to density (2.2).
11: Evaluate: \( \bar{X}_{\tau_{i+1}}, \bar{Y}_{\tau_{i+1}}, \bar{Z}_{\tau_{i+1}} \) according to (4.6).
12: Set: \( \tau_{i+1} = \tau_i + \tau \) and \( i = i + 1. \)
13: end if
14: end while
15: if \( \tau_{i+1} < T \) then Set: \( \tau_{t,x} = \tau_{i+1}, \) Evaluate: \( \chi(\tau_{t,x}, \bar{X}_{\tau_{t,x}})Y_{\tau_{t,x}} + Z_{\tau_{t,x}}. \)
16: else Set: \( \tau_{t,x} = T, \) Evaluate: \( g(T, \bar{X}_T)Y_T + Z_T. \)
17: end if
18: Loop above algorithm \( N \) times.
19: Evaluate: \( u(t, x) \approx \frac{1}{N} \sum_{j=1}^{N} \left[ u(T \land \tau_{t,x}, \bar{X}_{T \land \tau_{t,x}}^j)\bar{Y}_{T \land \tau_{t,x}}^j + \bar{Z}_{T \land \tau_{t,x}}^j \right]. \)

Proof. We only prove the case of \( \beta = 3. \) Other cases can be proved similarly.

\[
\begin{align*}
\mathbb{E} \left[ u(T \land \tau_{t,x}, \bar{X}_{T \land \tau_{t,x}})\bar{Y}_{T \land \tau_{t,x}} + \bar{Z}_{T \land \tau_{t,x}} - u(t, x) \right] \\
= \mathbb{E} \left\{ \sum_{i=0}^{n_T \land \tau_{t,x}-1} \left[ u(\tau_{i+1}, \bar{X}_{\tau_{i+1}})\bar{Y}_{\tau_{i+1}} + \bar{Z}_{\tau_{i+1}} - u(\tau_i, \bar{X}_{\tau_i})\bar{Y}_{\tau_i} - \bar{Z}_{\tau_i} \right] \right\} \\
= \mathbb{E} \left\{ \sum_{i=0}^{n_T \land \tau_{t,x}-1} \bar{Y}_{\tau_{i+1}} \left[ u(\tau_{i+1}, \bar{X}_{\tau_{i+1}}) - u(\tau_{i+1}, \bar{X}_{\tau_{i+1}}) \right] \\
+ \bar{Y}_{\tau_i} \left[ u(\tau_{i+1}, \bar{X}_{\tau_{i+1}}) - u(\tau_i, \bar{X}_{\tau_i}) \right] + \bar{Y}_{\tau_i} c(\tau_i, \bar{X}_{\tau_i})u(\tau_{i+1}, \bar{X}_{\tau_{i+1}}) \Delta \tau_i \\
+ \bar{Y}_{\tau_{i+1}} \left[ u(\tau_{i+1}, \bar{X}_{\tau_{i+1}}) - u(\tau_{i+1}, \bar{X}_{\tau_{i+1}}) \right] + \bar{Y}_{\tau_i} f(\tau_i, \bar{X}_{\tau_i}) \Delta \tau_i \right\},
\end{align*}
\]

where

\[
\begin{align*}
\bar{X}_{\eta} &= \bar{X}_{\tau_i} + b(\tau_i, \bar{X}_{\tau_i})(\eta - \tau_i) + \sigma_x W_{\eta-\tau_i}, \quad \eta \in [\tau_i, \tau_{i+1}), \\
\bar{X}_{\tau_{i+1}} &= \bar{X}_{\tau_{i+1}} + \Delta L^x_{\tau_{i+1}},
\end{align*}
\]
and $\Delta \tau_i = \tau_{i+1} - \tau_i$.

By Itô formula, the martingale property and system (4.12), one gets

$$
\begin{align*}
|E\left[ u(T \wedge \tau_{i,x}, X_{T \wedge \tau_{i,x}}) - u(t, x) \right] &= E\left\{ \sum_{i=0}^{n_{T \wedge \tau_{i,x}} - 1} \int_{\tau_i}^{\tau_{i+1}} \left[ \frac{\partial u}{\partial \eta}(\eta, \tilde{X}_{\eta-}) + \sum_{j=1}^{n} b_j(\tau_i, \tilde{X}_{\tau_i}) \frac{\partial u}{\partial x_j}(\eta, \tilde{X}_{\eta-}) \right. \\
&\quad + \frac{1}{2} \sum_{j_1, j_2=1}^{n} (\sigma_{\varepsilon}^T \sigma_{\varepsilon})_{j_1, j_2} \frac{\partial^2 u(\eta, \tilde{X}_{\eta-})}{\partial x_{j_1} \partial x_{j_2}} \right] d\eta + \int_{\tau_i}^{\tau_{i+1}} c(\tau_i, \tilde{X}_{\tau_i}) u(\tau_{i+1}, \tilde{X}_{\tau_{i+1}}) \Delta \tau_i \\
&\quad + \int_{\tau_i}^{\tau_{i+1}} \left[ u(\tau_{i+1}, \tilde{X}_{\tau_{i+1}}) - u(\tau_{i+1}, \tilde{X}_{\tau_{i+1}}) \right] + \int_{\tau_i}^{\tau_{i+1}} f(\tau_i, \tilde{X}_{\tau_i}) \Delta \tau_i \\
&\quad + \int_{|y| \geq \epsilon} u(\eta, \tilde{X}_{\eta-} + y) - u(\eta, \tilde{X}_{\eta-}^j) N(d\eta, dy) \right| \\
&= E\left\{ \sum_{i=0}^{n_{T \wedge \tau_{i,x}} - 1} \int_{\tau_i}^{\tau_{i+1}} \left[ \frac{\partial u}{\partial \eta}(\eta, \tilde{X}_{\eta-}) + \sum_{j=1}^{n} b_j(\tau_i, \tilde{X}_{\tau_i}) \frac{\partial u}{\partial x_j}(\eta, \tilde{X}_{\eta-}) \right. \\
&\quad + \frac{1}{2} \sum_{j_1, j_2=1}^{n} (\sigma_{\varepsilon}^T \sigma_{\varepsilon})_{j_1, j_2} \frac{\partial^2 u(\eta, \tilde{X}_{\eta-})}{\partial x_{j_1} \partial x_{j_2}} \right] d\eta + c(\tau_i, \tilde{X}_{\tau_i}) u(\tau_{i+1}, \tilde{X}_{\tau_{i+1}}) \Delta \tau_i \\
&\quad + [1 + c(\tau_i, \tilde{X}_{\tau_i}) \Delta \tau_i] \left[ u(\tau_{i+1}, \tilde{X}_{\tau_{i+1}}) - u(\tau_{i+1}, \tilde{X}_{\tau_{i+1}}) \right] + f(\tau_i, \tilde{X}_{\tau_i}) \Delta \tau_i \\
&\quad + \int_{|y| \geq \epsilon} \left[ u(\eta, \tilde{X}_{\eta-} + y) - u(\eta, \tilde{X}_{\eta-}) \right] \nu(dy) d\eta | \mathcal{F}_{\tau_i} \right| \\
&= E\left\{ \sum_{i=0}^{n_{T \wedge \tau_{i,x}} - 1} \int_{\tau_i}^{\tau_{i+1}} \left[ \frac{\partial^2 u(\eta, \tilde{X}_{\eta-})}{\partial x_{j_1} \partial x_{j_2}} \right] d\eta \right. \\
&\quad - \int_{|y| < \epsilon} \left[ u(\eta, \tilde{X}_{\eta-} + y) - u(\eta, \tilde{X}_{\eta-}) - I_{|y| < 1}(\nabla u(\eta, \tilde{X}_{\eta-}), y) \right] \nu(dy) \right] \\
&\quad + c(\tau_i, \tilde{X}_{\tau_i}) \Delta \tau_i \int_{\tau_i}^{\tau_{i+1}} \int_{|y| > \epsilon} \left[ u(\eta, \tilde{X}_{\eta-} + y) - u(\eta, \tilde{X}_{\eta-}) \right] \nu(dy) d\eta \\
&\quad + [1 + c(\tau_i, \tilde{X}_{\tau_i}) \Delta \tau_i] \left[ u(\tau_{i+1}, \tilde{X}_{\tau_{i+1}}) - u(\tau_{i+1}, \tilde{X}_{\tau_{i+1}}) \right] \\
&= E\left[ B_1 | \mathcal{F}_{\tau_i} \right]
\end{align*}
$$

Since $u(t, x)$ satisfies (1.1), it holds that

$$
E\left[ B_1 | \mathcal{F}_{\tau_i} \right] = E\left\{ \int_{\tau_i}^{\tau_{i+1}} \left[ \frac{\partial^2 u(\eta, \tilde{X}_{\eta-})}{\partial x_{j_1} \partial x_{j_2}} \right] d\eta \right. \\
&\quad - \int_{|y| < \epsilon} \left[ u(\eta, \tilde{X}_{\eta-} + y) - u(\eta, \tilde{X}_{\eta-}) - I_{|y| < 1}(\nabla u(\eta, \tilde{X}_{\eta-}), y) \right] \nu(dy) \right] \\
&\quad + c(\tau_i, \tilde{X}_{\tau_i}) \Delta \tau_i \int_{\tau_i}^{\tau_{i+1}} \int_{|y| > \epsilon} \left[ u(\eta, \tilde{X}_{\eta-} + y) - u(\eta, \tilde{X}_{\eta-}) \right] \nu(dy) d\eta \\
&\quad + [1 + c(\tau_i, \tilde{X}_{\tau_i}) \Delta \tau_i] \left[ u(\tau_{i+1}, \tilde{X}_{\tau_{i+1}}) - u(\tau_{i+1}, \tilde{X}_{\tau_{i+1}}) \right]
$$
\[ + \int_{\tau_i}^{\tau_{i+1}} \sum_{j=1}^{n} \left[ b_j(\tau_i, \tilde{X}_n) - b_j(\eta, \tilde{X}_{\eta-}) \right] \frac{\partial u}{\partial x_j}(\eta, \tilde{X}_{\eta-}) \, d\eta \]
\[ + \int_{\tau_i}^{\tau_{i+1}} c(\tau_i, \tilde{X}_{\tau_i})u(\tau_{i+1}, \tilde{X}_{\tau_{i+1}}) - c(\eta, \tilde{X}_{\eta-})u(\eta, \tilde{X}_{\eta-}) \, d\eta \]
\[ + \int_{\tau_i}^{\tau_{i+1}} f(\tau_i, \tilde{X}_{\tau_i}) - f(\eta, \tilde{X}_{\eta-}) \, d\eta \mid_{\mathcal{F}_{\tau_i}} \]
\[ = \mathbb{E} \left[ \overline{B}_{13} + \overline{B}_{12} + \overline{B}_{i3} + \overline{B}_{i4} + \overline{B}_{i5} + \overline{B}_{i6} \mid \mathcal{F}_{\tau_i} \right], \quad (4.14) \]

For the first term, we obtain
\[
\left| \mathbb{E} \left[ \overline{B}_{13} \mid \mathcal{F}_{\tau_i} \right] \right| \\
= C(n, s) \left| \mathbb{E} \left[ \int_{\tau_i}^{\tau_{i+1}} \frac{1}{2} \sum_{j_1,j_2=1}^{n} \int_{|y|<\varepsilon} y_{j_1} y_{j_2} \frac{\partial^2 u}{\partial x_{j_1} \partial x_{j_2}}(\eta, \tilde{X}_{\eta-}) \, dy \left[ \frac{d\alpha}{|y|^{n+2s}} \right] \right] \right| \\
- \int_{|y|<\varepsilon} \left[ \int_{0}^{1} (\nabla u(\eta, \tilde{X}_{\eta-} + \alpha y) - \nabla u(\eta, \tilde{X}_{\eta-}), y) \, d\alpha \left[ \frac{dy}{|y|^{n+2s}} \right] \right] \mid_{\mathcal{F}_{\tau_i}} \right| \\
= C(n, s) \left[ \int_{\tau_i}^{\tau_{i+1}} \sum_{j_1,j_2=1}^{n} \int_{|y|<\varepsilon} \int_{0}^{1} \int_{0}^{1} \alpha y_{j_1} y_{j_2} \right. \\
\left. \times \left[ \frac{\partial^2 u}{\partial x_{j_1} \partial x_{j_2}}(\eta, \tilde{X}_{\eta-} + \alpha' \alpha y) - \frac{\partial^2 u}{\partial x_{j_1} \partial x_{j_2}}(\eta, \tilde{X}_{\eta-}) \right] \, d\alpha' \, d\alpha \left[ \frac{dy}{|y|^{n+2s}} \right] \right] \mid_{\mathcal{F}_{\tau_i}} \right| \\
\leq C_2 \mathbb{E} \left[ \int_{\tau_i}^{\tau_{i+1}} \int_{|y|<\varepsilon} \int_{0}^{1} \int_{0}^{1} D^2 u(\eta, \tilde{X}_{\eta-} + \alpha' \alpha y) - D^2 u(\eta, \tilde{X}_{\eta-}) \, d\alpha' \, d\alpha \left[ \frac{dy}{|y|^{n+2s-2}} \right] \right] \mid_{\mathcal{F}_{\tau_i}} \right| \\
\leq C_2 \varepsilon^{3-2s} \mathbb{E} \left[ (\tau_{i+1} - \tau_i) \mid \mathcal{F}_{\tau_i} \right]. \quad (4.15) \]

By Taylor expansion, it follows for the third part that
\[
\left| \mathbb{E} \left[ \overline{B}_{13} \mid \mathcal{F}_{\tau_i} \right] \right| \\
= \mathbb{E} \left\{ \left[ 1 + c(\tau_i, \tilde{X}_{\tau_i}) \Delta \tau_i \right] \left[ u \left( \tau_{i+1}, \tilde{X}_{\tau_{i+1}} \right) - u \left( \tau_{i+1}, \tilde{X}_{\tau_{i+1}} \right) \right] \mid \mathcal{F}_{\tau_i} \right\} \\
= \mathbb{E} \left\{ \left[ 1 + c(\tau_i, \tilde{X}_{\tau_i}) \Delta \tau_i \right] \\
\times \left[ u \left( \tau_{i+1}, \tilde{X}_{\tau_i} + \tilde{X}_{\tau_i} + \tilde{\sigma}_{\tau_i} \Delta \tau_i \right) \right] - u \left( \tau_{i+1}, \tilde{X}_{\tau_i} + L_{\Delta \tau_i} + \tilde{\sigma}_{\tau_i} W_{\Delta \tau_i} \right) \mid \mathcal{F}_{\tau_i} \right\} \\
= \mathbb{E} \left\{ \left[ 1 + c(\tau_i, \tilde{X}_{\tau_i}) \Delta \tau_i \right] \left[ \sum_{j_1=1}^{n} \frac{\partial u}{\partial x_{j_1}} \left( \tau_{i+1}, \tilde{X}_{\tau_i} + L_{\Delta \tau_i} \right) \right] \mid \mathcal{F}_{\tau_i} \right\} \\
+ \frac{1}{2!} \sum_{j_1,j_2=1}^{n} \frac{\partial^2 u}{\partial x_{j_1} \partial x_{j_2}} \left( \tau_{i+1}, \tilde{X}_{\tau_i} + L_{\Delta \tau_i} \right) \left[ \Delta \tau_i \tilde{\sigma}_{\tau_i} \tilde{\sigma}_{\tau_i} - \tilde{\sigma}_{\tau_i} W_{\Delta \tau_i} \tilde{\sigma}_{\tau_i} W_{\Delta \tau_i} \right] \}.
\[
\begin{align*}
+ \frac{1}{3!} \sum_{j_1, j_2, j_3 = 1}^n \frac{\partial^3 u(\tau_{i+1}, \bar{X}_{\tau_i} + L_{\Delta \tau_i}^\varepsilon + \theta_1(\Delta \tau_i)\frac{\xi}{2})}{\partial x_{j_1} \partial x_{j_2} \partial x_{j_3}} \left( (\Delta \tau_i)^{3/2} \overline{\sigma}_\varepsilon \xi_{j_1} \overline{\sigma}_\varepsilon \xi_{j_2} \overline{\sigma}_\varepsilon \xi_{j_3} \right) \\
- \frac{\partial^3 u(\tau_{i+1}, \bar{X}_{\tau_i} + L_{\Delta \tau_i}^\varepsilon + \theta_2 \sigma \varepsilon W_{\Delta \tau_i})}{\partial x_{j_1} \partial x_{j_2} \partial x_{j_3}} \left( \overline{\sigma}_\varepsilon W_{\Delta \tau_i}^{j_1} \overline{\sigma}_\varepsilon W_{\Delta \tau_i}^{j_2} \overline{\sigma}_\varepsilon W_{\Delta \tau_i}^{j_3} \right) \right] \bigg| \mathcal{F}_{\tau_i} \bigg) \bigg| \\
\leq C_2 \sigma_\varepsilon^3 \mathbb{E} \left[ (\tau_{i+1} - \tau_i)^{3/2} \bigg| \mathcal{F}_{\tau_i} \right] \leq C_2 \varepsilon^{3-2 \delta} \mathbb{E} \left[ (\tau_{i+1} - \tau_i) \bigg| \mathcal{F}_{\tau_i} \right], \quad (4.16)
\end{align*}
\]
in which we utilize the fact that \( \xi, W_1 \) and \( \tau_i (i = 1, 2, \ldots, n) \) are independent,
\[
\begin{align*}
\mathbb{E} \left[ W_{\Delta \tau_i} \right] &= \mathbb{E} \left[ (\Delta \tau_i)^{1/2} W_1 \right], \quad \mathbb{E} \left[ \overline{\sigma}_\varepsilon \xi_{j_1} \right] = \mathbb{E} \left[ \overline{\sigma}_\varepsilon W_{\tau_1}^{j_1} \right] = 0, \quad j_1 = 1, 2, \ldots, n, \\
\mathbb{E} \left[ \overline{\sigma}_\varepsilon \xi_{j_1} \overline{\sigma}_\varepsilon \xi_{j_2} \right] &= \mathbb{E} \left[ \overline{\sigma}_\varepsilon W_{\tau_1}^{j_1} \overline{\sigma}_\varepsilon W_{\tau_1}^{j_2} \right] = \begin{cases} 0, & j_1 \neq j_2, \\
\sigma_\varepsilon^2, & j_1 = j_2, \end{cases} \quad (4.17)
\end{align*}
\]
and
\[
\mathbb{E} \left[ (\overline{\sigma}_\varepsilon \xi_{j_1} \overline{\sigma}_\varepsilon \xi_{j_2} \overline{\sigma}_\varepsilon \xi_{j_3})^2 \right] \leq \sigma_\varepsilon^3, \quad \mathbb{E} \left[ (\overline{\sigma}_\varepsilon W_{\tau_1}^{j_1} \overline{\sigma}_\varepsilon W_{\tau_1}^{j_2} \overline{\sigma}_\varepsilon W_{\tau_1}^{j_3})^2 \right] \leq \sqrt{15} \sigma_\varepsilon^3.
\]
Now we estimate the fourth term
\[
\begin{align*}
\mathbb{E} \left[ B_{\tau_{i+1}} \bigg| \mathcal{F}_{\tau_i} \right] &= \mathbb{E} \left[ \int_{\tau_i}^{\tau_{i+1}} \sum_{j=1}^n \left[ b_j(\tau_i, \bar{X}_{\tau_i}) - b_j(\eta, \bar{X}_{\eta-}) \right] \frac{\partial u}{\partial x_j}(\eta, \bar{X}_{\eta-}) d\eta \bigg| \mathcal{F}_{\tau_i} \right] \\
&= \mathbb{E} \left[ \int_{\tau_i}^{\tau_{i+1}} \sum_{j=1}^n \left[ b_j(\tau_i, \bar{X}_{\tau_i}) - b_j(\eta, \bar{X}_{\eta-}) \right] \frac{\partial u}{\partial x_j}(\eta, \bar{X}_{\eta-}) d\eta \bigg| \mathcal{F}_{\tau_i} \right] \\
&\quad + \mathbb{E} \left[ \int_{\tau_i}^{\tau_{i+1}} \sum_{j=1}^n \left[ b_j(\tau_i, \bar{X}_{\tau_i}) - b_j(\eta, \bar{X}_{\eta-}) \right] \left[ \frac{\partial u}{\partial x_j}(\eta, \bar{X}_{\eta-}) - \frac{\partial u}{\partial x_j}(\eta, \bar{X}_{\eta-}) \right] d\eta \bigg| \mathcal{F}_{\tau_i} \right] \\
&= \mathbb{E} \left[ B_{\tau_{i+1}} + B_{\tau_i} \bigg| \mathcal{F}_{\tau_i} \right]. \quad (4.18)
\end{align*}
\]
By Itô formula and the martingale property again, one has
\[
\begin{align*}
\mathbb{E} \left[ B_{\tau_{i+1}} \bigg| \mathcal{F}_{\tau_i} \right] &= \mathbb{E} \left\{ \sum_{j=1}^n \int_{\tau_i}^{\tau_{i+1}} \frac{\partial u}{\partial x_j}(\eta, \bar{X}_{\tau_i}) \int_\eta^n \left[ \frac{\partial b_j}{\partial \eta'}(\eta', \bar{X}_{\eta'-}) + \sum_{j=1}^n b_j(\tau_i, \bar{X}_{\tau_i}) \frac{\partial b_j}{\partial x_{j_1}}(\eta', \bar{X}_{\eta'-}) \right] d\eta' d\eta \bigg| \mathcal{F}_{\tau_i} \right\} \\
&\quad + \frac{1}{2} \sum_{j_1, j_2 = 1}^n (\sigma_\varepsilon^T \sigma_\varepsilon)_{j_1, j_2} \int_{\tau_i}^{\tau_{i+1}} \left[ \frac{\partial^2 u}{\partial x_{j_1} \partial x_{j_2}}(\eta, \bar{X}_{\tau_i}) \right] d\eta' d\eta \bigg| \mathcal{F}_{\tau_i} \right\} \\
&\quad + \mathbb{E} \left[ \sum_{j=1}^n \int_{\tau_i}^{\tau_{i+1}} \frac{\partial u}{\partial x_j}(\eta, \bar{X}_{\tau_i}) \int_\eta^n \sum_{j_1=1}^n \frac{\partial b_j}{\partial x_{j_1}}(\eta', \bar{X}_{\eta'-}) dW_{\eta'-} \bigg| \mathcal{F}_{\tau_i} \right] \\
&\leq C_1 \mathbb{E} \left[ (\tau_{i+1} - \tau_i)^2 \bigg| \mathcal{F}_{\tau_i} \right], \quad (4.19)
\end{align*}
\]
where $C_1$ is a constant. It is also easy to get
\[
E\left[|B_{ij2}| \mathcal{F}_{\tau_i}\right] \leq C_1 E\left[(\tau_{i+1} - \tau_i)^2 \mathcal{F}_{\tau_i}\right],
\]  
(4.20)
where the conditions of $u(t, x)$ and $b_j(t, x)$ are used.

For the remaining part, by Taylor expansion and Itô formula, it is evident that
\[
E\left[|B_{i2} + B_{i5} + B_{i6}| \mathcal{F}_{\tau_i}\right] \leq C_1 E\left[(\tau_{i+1} - \tau_i)^2 \mathcal{F}_{\tau_i}\right].
\]  
(4.21)
Here $C_1$ is a constant.

Since Lemma 3.2 is still true by replacing $\delta_i = (\tau_{t,x} \wedge T) - \tau_i$. Thus, combining Lemmas 3.2 and 4.2, and (4.11)-(4.21), we can obtain the desired result.

**Remark 4.1.** (I) The convergent order is lower when $s$ is small. However, consider the following jump-adapted time discretization: Let $\Delta t = (T-t)/M$ be a mesh size, $M \in \mathbb{Z}^+$,
\[
\tau_{i+1} = \inf\{t > \tau_i : \Delta L^\varepsilon_t \neq 0\} \wedge (\tau_{i+1} + \Delta t) \wedge \tau_{t,X} \wedge T.
\]  
(4.22)
Then we can get the following error estimate
\[
|E\left[u(T \wedge \tau_{t,X}, X_{T \wedge \tau_{t,X}})Y_{T \wedge \tau_{t,X}} + Z_{T \wedge \tau_{t,X}} - u(t, x)\right] - u(t, x)| \leq C_1 \varepsilon^{2s} \Delta t + C_2 \varepsilon^{[\beta]-2s}.
\]  
(4.23)
Thus, if we let $\Delta t = \varepsilon$, we can get the first order convergence for small $s$.

(II) If coefficients satisfy Assumption II instead of Assumption I, the higher-order scheme will lose accuracy. At this time, we have the following estimate
\[
|E\left[u(T \wedge \tau_{t,X}, X_{T \wedge \tau_{t,X}}Y_{T \wedge \tau_{t,X}} + Z_{T \wedge \tau_{t,X}} - u(t, x)\right] - u(t, x)| \leq C_1 \varepsilon^{s} + C_2 \varepsilon^{[\beta]-2s}.
\]  
(4.24)

## 5 Numerical experiments

In this section, numerical examples are carried out by using jump-adapted scheme (3.10) and higher-order scheme (4.8) on an i5-8250U CPU. We approximate the expectation by Monte Carlo method, so that it will have statistical error $\frac{1}{\sqrt{N}}$, where $N > 0$ is the number of samples.

Although the derived schemes for equation (1.1) in more than two space dimensions, they are still suitable for the two spacial-dimension case.

**Example 5.1.** Let $\mathbb{D}$ be a unit ball in $\mathbb{R}^n$ centered at the origin. Consider the following fractional heat equation,
\[
\begin{aligned}
\frac{\partial u}{\partial t} - (-\Delta)^s u + f(t, x) &= 0, & (t, x) &\in (0, T) \times \mathbb{D}, \\
u(T, x) &= T(1 - |x|^2)^{1+s}, & x &\in \mathbb{D}, \\
u(t, x) &= 0, & (t, x) &\in [0, T] \times \mathbb{R}^n \setminus \mathbb{D},
\end{aligned}
\]  
(5.1)
where $s \in (0, 1)$, $n \geq 2$, $T = 1$, and
\[
f(t, x) = 2^{2s} \Gamma(2+s) \Gamma(n/2 + s) \Gamma(n/2)^{-1} \left(1 - (1 + 2s/n) |x|^2\right) t - (1 - |x|^2)^{1+s}.
\]  
(5.2)
The exact solution to (5.1) is

\[ u(t, x) = t(1 - |x|^2)^{1+s}. \]  

(5.3)

We set \( s = 0.25, 0.5, 0.75 \), the number of samples \( N = 10^4, \varepsilon = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \frac{1}{80} \), \( \frac{1}{160} \) for 2, 3, 4, 10 dimensional cases, and \( \varepsilon = \frac{1}{5}, \frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \frac{1}{80} \) for 100 dimensional case. We evaluate \( u(t, x) \) with \( t = 0.5, x = \frac{1}{n} \ast \text{ones}(n), n = 2, 3, 4, 10, 100 \). Table 1 gives errors \( \left| u(t, x) - \frac{1}{N} \sum_{j=1}^{N} u(T \wedge \tilde{\tau}_{t,x}, \tilde{X}_{T \wedge \tilde{\tau}_{t,x}}^j, \tilde{Y}_{T \wedge \tilde{\tau}_{t,x}}^j, \tilde{Z}_{T \wedge \tilde{\tau}_{t,x}}^j) \right| \) by jump-adapted scheme (3.10). The statistical error produced by Monte Carlo method does not reflect the error of the method by the large number of samples \( N = 10^4 \). It is clear to observe that the computational time slowly increases with the growth of dimension. In addition, many small jumps occur when index \( s \) grows, which costs a lot of computational time. Figure 1 shows the convergent order of the scheme (3.10) for \( u(t, x) \in C^{1,1+s}([0, T] \times \mathbb{R}^n) \) is \( 2s \wedge (1 - s) \). The numerical results are in good agreement with the theoretical analysis.

![Graph](image)

**Figure 1**: Numerical errors of the scheme (3.10) for Example 5.1. It shows that the convergent order is \( 2s \wedge (1 - s) \).
Table 1: The error in simulation, the average number of steps, and the computational time (secs.) for Example 5.1 by scheme (3.10) are given.

| $n$ | $\epsilon$ | $s = 0.25$ step time | $s = 0.5$ step time | $s = 0.75$ step time |
|-----|-------------|-----------------------|----------------------|----------------------|
| 2   | $\frac{1}{10}$ | 3.525E-02 1.71 0.39 | 1.452E-02 6.52 0.28 | 2.978E-02 6.62 0.53 |
|     | $\frac{1}{20}$ | 2.914E-02 2.20 0.20 | 1.012E-02 6.52 0.62 | 2.262E-02 16.6 1.15 |
|     | $\frac{1}{40}$ | 2.151E-02 2.90 0.25 | 7.725E-03 12.5 0.85 | 1.867E-02 43.2 2.90 |
|     | $\frac{1}{80}$ | 1.560E-02 3.94 0.29 | 5.648E-03 24.7 1.67 | 1.431E-02 117 8.06 |
|     | $\frac{1}{160}$ | 1.131E-02 5.37 0.40 | 4.101E-03 48.5 3.23 | 1.165E-02 323 21.3 |

| $n$ | $\epsilon$ | $s = 0.25$ step time | $s = 0.5$ step time | $s = 0.75$ step time |
|-----|-------------|-----------------------|----------------------|----------------------|
| 3   | $\frac{1}{10}$ | 5.679E-02 1.86 0.32 | 6.624E-03 4.36 0.67 | 9.215E-02 8.86 1.46 |
|     | $\frac{1}{20}$ | 4.347E-02 2.42 0.45 | 3.252E-03 8.20 1.10 | 6.331E-02 22.3 2.82 |
|     | $\frac{1}{40}$ | 3.171E-02 3.24 0.53 | 2.408E-03 15.7 2.07 | 4.439E-02 59.5 7.64 |
|     | $\frac{1}{80}$ | 2.034E-02 4.35 0.62 | 1.403E-03 31.2 3.98 | 3.105E-02 160 20.2 |
|     | $\frac{1}{160}$ | 1.484E-03 5.96 0.93 | 9.072E-04 62.2 3.98 | 2.639E-02 435 54.3 |

| $n$ | $\epsilon$ | $s = 0.25$ step time | $s = 0.5$ step time | $s = 0.75$ step time |
|-----|-------------|-----------------------|----------------------|----------------------|
| 4   | $\frac{1}{10}$ | 7.138E-02 1.95 1.06 | 1.338E-02 4.80 2.62 | 1.299E-01 10.0 4.64 |
|     | $\frac{1}{20}$ | 5.090E-02 2.56 1.32 | 8.977E-03 9.22 4.48 | 8.291E-02 25.6 12.3 |
|     | $\frac{1}{40}$ | 3.841E-02 3.41 1.64 | 6.184E-03 17.9 8.51 | 5.776E-02 67.8 29.0 |
|     | $\frac{1}{80}$ | 2.710E-02 4.63 2.53 | 4.095E-03 35.4 16.3 | 3.494E-02 182 84.4 |
|     | $\frac{1}{160}$ | 1.936E-02 6.27 3.04 | 2.901E-03 70.0 33.3 | 2.790E-02 501 229 |

| $n$ | $\epsilon$ | $s = 0.25$ step time | $s = 0.5$ step time | $s = 0.75$ step time |
|-----|-------------|-----------------------|----------------------|----------------------|
| 10  | $\frac{1}{10}$ | 8.282E-02 2.14 3.65 | 2.260E-02 6.06 10.0 | 2.199E-01 12.2 20.3 |
|     | $\frac{1}{20}$ | 6.203E-02 2.89 5.03 | 1.850E-02 11.6 17.3 | 1.431E-01 31.2 47.5 |
|     | $\frac{1}{40}$ | 4.790E-02 3.91 6.54 | 1.787E-02 22.8 37.1 | 9.988E-02 82.0 125 |
|     | $\frac{1}{80}$ | 3.502E-02 5.41 8.40 | 1.224E-02 45.0 70.4 | 6.776E-02 222 340 |
|     | $\frac{1}{160}$ | 2.550E-02 7.48 12.3 | 8.396E-03 89.1 138 | 5.639E-02 602 885 |

| $n$ | $\epsilon$ | $s = 0.25$ step time | $s = 0.5$ step time | $s = 0.75$ step time |
|-----|-------------|-----------------------|----------------------|----------------------|
| 100 | $\frac{1}{10}$ | 9.041E-02 1.89 31.0 | 8.495E-02 3.59 59.1 | 4.192E-01 5.46 91.0 |
|     | $\frac{1}{20}$ | 7.590E-02 2.55 41.7 | 5.571E-02 6.81 111 | 2.597E-01 13.0 210 |
|     | $\frac{1}{40}$ | 5.708E-02 3.58 60.8 | 4.111E-02 13.3 221 | 1.761E-01 33.1 552 |
|     | $\frac{1}{80}$ | 4.014E-02 4.88 79.0 | 3.051E-02 26.0 419 | 1.299E-01 87.9 1423 |
|     | $\frac{1}{160}$ | 2.805E-02 6.88 113 | 2.204E-02 50.8 831 | 1.032E-01 235 3935 |
Example 5.2. Let $\mathbb{D}$ be a unit ball in $\mathbb{R}^n$ centered at the origin. Consider the following parabolic problem with fractional Laplacian,

$$
\begin{aligned}
\begin{cases}
\frac{\partial u}{\partial t} - (-\Delta)^s u + \sum_{j=0}^{n} b_j(t, x) \frac{\partial u}{\partial x_j} + c(t, x) u + f(t, x) = 0, \\
u(T, x) = T(1 - |x|^2)^{1+s}, \\
u(t, x) = t,
\end{cases}
\end{aligned}
$$

where $n \geq 2$, $T = 1$,

$$
b_j(t, x) = t \sin(x_j),
$$

$$
c(t, x) = \frac{e^t}{1 + e^{-|x|}},
$$

and

$$
f(t, x) = \frac{2^{2s} \Gamma(2 + s) \Gamma(n/2 + s)}{\Gamma(n/2)} (1 - (1 + 2s/n) |x|^2) t
$$

$$
+ 2t(1 + s)(1 - |x|^2)^s \sum_{i=1}^{n} t \sin(x_i) x_i
$$

$$
- \frac{e^t |t(1 - |x|^2)^{1+s} + t|}{1 + e^{-|x|}} - (1 - |x|^2)^{1+s} - 1.
$$

The exact solution to (5.4) is

$$
u(t, x) = t(1 - |x|^2)^{1+s} + t. \quad (5.7)
$$

We still set $s = 0.25$, 0.5, 0.75, the number of samples $N = 10^4$, $\varepsilon = \frac{1}{10}$, $\frac{1}{20}$, $\frac{1}{40}$, $\frac{1}{80}$, $\frac{1}{160}$ for 2, 3, 4, 10 dimensional cases, and $\varepsilon = \frac{1}{5}$, $\frac{1}{10}$, $\frac{1}{20}$, $\frac{1}{40}$, $\frac{1}{80}$ for 100 dimensional case. We evaluate $u(t, x)$ with $t = 0.5$, $x = \frac{1}{n} \text{ones}(n)$, $n = 2, 3, 4, 10, 100$. The numerical results given in Table 2 shows the efficiency of the method (3.10) and coincides with the theoretical analysis.

Example 5.3. Let $\mathbb{D}$ be a unit ball in $\mathbb{R}^n$ centered at the origin. Consider the following fractional heat equation,

$$
\begin{aligned}
\begin{cases}
\frac{\partial u_i}{\partial t} - (-\Delta)^s u_i + f_i(t, x) = 0, \\
u_i(T, x) = T(1 - |x|^2)^{1+i+s}, \\
u_i(t, x) = 0,
\end{cases}
\end{aligned}
$$

where $i = 1, 2, n \geq 2$, and

$$
f_i(t, x) = -(1 - |x|^2)^{2+s} - \frac{C(n, s)B(-s, i + s + 2)\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} {}_2F_1 \left( s + \frac{n}{2}, -i - 1; \frac{n}{2}; |x|^2 \right) t. \quad (5.9)
$$
Table 2: The error in simulation, the average number of steps, and the computational time (secs.) for Example 5.2 by scheme (3.10) are given.

| n    | ε     | s = 0.25 | step | time | s = 0.5 | step | time | s = 0.75 | step | time |
|------|-------|----------|------|------|----------|------|------|----------|------|------|
| 2    | 1/10  | 4.844E-02| 1.40 | 0.39 | 2.966E-02| 3.06 | 0.68 | 3.635E-02| 5.69 | 1.12 |
|      | 1/20  | 3.345E-02| 1.86 | 0.46 | 1.699E-02| 5.65 | 1.15 | 2.388E-02| 14.5 | 2.75 |
|      | 1/40  | 2.367E-02| 2.48 | 0.59 | 4.984E-03| 10.8 | 2.06 | 1.639E-02| 38.1 | 7.29 |
|      | 1/80  | 1.693E-02| 3.41 | 0.75 | 1.348E-02| 21.2 | 3.93 | 1.349E-02| 103  | 19.1 |
|      | 1/160 | 1.192E-02| 4.69 | 0.96 | 9.345E-03| 42.3 | 7.90 | 1.110E-02| 281  | 54.3 |
| 3    | 1/10  | 1.045E-02| 1.67 | 0.75 | 3.756E-03| 3.88 | 1.50 | 1.230E-01| 7.74 | 2.59 |
|      | 1/20  | 1.382E-02| 2.18 | 0.90 | 1.837E-02| 7.31 | 2.48 | 7.749E-02| 20.0 | 6.37 |
|      | 1/40  | 1.880E-02| 2.98 | 1.21 | 5.084E-03| 14.2 | 4.70 | 3.775E-02| 53.3 | 16.8 |
|      | 1/80  | 1.379E-02| 4.13 | 1.64 | 5.084E-03| 28.3 | 9.71 | 2.674E-02| 145  | 45.2 |
|      | 1/160 | 1.025E-02| 5.63 | 2.15 | 3.438E-03| 55.7 | 18.8 | 2.253E-02| 393  | 125  |
| 4    | 1/10  | 9.976E-03| 1.80 | 1.71 | 3.682E-02| 4.37 | 3.89 | 1.551E-01| 8.98 | 7.98 |
|      | 1/20  | 1.905E-02| 2.38 | 2.20 | 1.703E-02| 8.44 | 7.03 | 9.613E-02| 23.2 | 19.7 |
|      | 1/40  | 1.240E-02| 3.23 | 2.98 | 9.508E-03| 16.4 | 14.0 | 6.017E-02| 61.5 | 61.7 |
|      | 1/80  | 8.541E-03| 4.42 | 4.06 | 6.951E-03| 32.2 | 26.8 | 3.844E-02| 167  | 145  |
|      | 1/160 | 6.875E-03| 6.11 | 5.46 | 4.833E-03| 64.4 | 53.9 | 2.759E-02| 455  | 394  |
| 10   | 1/10  | 1.364E-02| 2.05 | 3.56 | 4.021E-02| 5.50 | 9.14 | 2.227E-01| 11.3 | 18.7 |
|      | 1/20  | 1.050E-02| 2.76 | 4.59 | 2.478E-02| 10.7 | 18.7 | 1.441E-01| 28.9 | 46.7 |
|      | 1/40  | 7.874E-03| 3.80 | 7.04 | 1.374E-02| 21.0 | 35.4 | 9.340E-02| 77.2 | 131  |
|      | 1/80  | 4.314E-03| 5.19 | 8.57 | 8.585E-03| 41.9 | 69.9 | 7.153E-02| 207  | 344  |
|      | 1/160 | 2.094E-03| 7.10 | 11.9 | 6.186E-03| 82.9 | 139 | 5.607E-02| 563  | 931  |
| 100  | 1/10  | 2.543E-02| 1.79 | 31.3 | 8.199E-02| 3.44 | 60.5 | 4.181E-01| 5.38 | 91.5 |
|      | 1/10  | 1.774E-02| 2.49 | 45.2 | 5.281E-02| 6.56 | 111 | 2.554E-01| 12.8 | 216  |
|      | 1/10  | 1.264E-02| 3.43 | 58.5 | 3.641E-02| 12.8 | 219 | 1.784E-01| 33.0 | 1491 |
|      | 1/10  | 7.963E-03| 4.82 | 82.3 | 2.768E-02| 24.8 | 433 | 1.268E-01| 86.7 | 1490 |
|      | 1/10  | 5.403E-03| 6.69 | 115  | 1.977E-02| 49.2 | 827 | 9.952E-02| 234  | 4039 |
Here \( C(n, s) \) is given in equation (1.3) and \( 2F_1(a, b; c; z) \) is the hypergeometric function. The exact solution to (5.8) is

\[
  u_{1}(t, x) = t(1 - |x|^2)^{2+s}, \quad i = 1,
\]

and

\[
  u_{2}(t, x) = t(1 - |x|^2)^{3+s}, \quad i = 2.
\]

We set \( s = 0.25, 0.5, 0.75 \), \( \varepsilon = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32} \) for 2, 3, 4, 10 dimensional cases and \( \varepsilon = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32} \) for 100 dimensional case. We evaluate \( u(t, x) \) with \( t = 0.5, x = \frac{1}{n} \) * ones \((n)\), \( n = 2, 3, 4, 10, 100 \). To avoid statistical error, we set \( N = 4 \times 10^4 \) for the case of \( s = 0.75, \varepsilon = \frac{1}{32} \) and \( N = 3 \times 10^5 \) for the case of \( s = 0.75, \varepsilon = \frac{1}{64} \). In other cases, \( N \) is still set by \( 10^4 \). Tables 3 and 4 give numerical errors \( |u(t, x) - \frac{1}{N} \sum_{j=1}^{N} u(T \wedge \tau_{t,x}, \bar{X}_{T \wedge \tau_{t,x}}) Y_{T \wedge \tau_{t,x}}^j + \bar{Z}_{T \wedge \tau_{t,x}}^j| \) with \( f_1(t, x) \) and \( f_2(t, x) \) in (5.9) by higher-order jump-adapted scheme (4.8), respectively. Compared with steps and computational time in Table 3, the corresponding results in Table 4 almost shows no evident differences. This is due to the fact that being given different \( f_i(t, x) \) in equation (5.8) with the same index \( s \) will not change the trajectory of Lévy processes. Figures (a), (b), (c) in Figure 2 and figures (d), (e), (f) in the same figure show the convergent order of the high-order scheme (4.8) with different functions \( f_i(t, x) \), \( i = 1, 2 \) in equation (5.8), respectively. When \( s = 0.25, 0.5, 0.75 \), figures (a), (b), (d), (e) demonstrate the convergent order for equation (5.8) with \( f_1(t, x) \) and \( f_2(t, x) \) are the same, which coincides with the theoretical analysis. Compared with figure (c) for \( s = 0.75 \), figure (f) shows higher convergent order due to the higher regularity of the \( u(t, x) \).

6 Conclusion

We propose Monte Carlo method to solve the Dirichlet problem for the parabolic equation with fractional Laplacian (1.1). First, we give the probabilistic representation of the solution to parabolic equation (1.1) which is related to stochastic differential equations driven by symmetric stable Lévy process with jump. Then we obtain two jump-adapted schemes in which time discretization is based on jump times of Lévy process to approximate Lévy driven stochastic differential equations. The first scheme removes small jumps of the \( 2s \)-symmetric stable process while the second scheme replaces small jumps by the simple process \( \sigma_{\varepsilon \sqrt{t}} \xi \). Based on these two schemes, we give two numerical algorithms to solve parabolic equations (1.1). Convergence theorems for both schemes are proved. The convergence order of the second scheme (4.8) is higher than the first one (3.10), while the second scheme requires better regularity of the solution. Numerical experiments verify the theoretical analysis and show the efficiency of the proposed algorithms.
Table 3: The error in simulation, the average number of steps, and the computational time (secs.) for $i = 1$ of Example 5.3 by scheme (4.8) are given.

| $\varepsilon$ | $s = 0.25$ | step time | $s = 0.5$ | step time | $s = 0.75$ | step time |
|---------------|------------|------------|------------|------------|------------|------------|
| $\frac{1}{2}$ | 1.620E-02  | 1.26       | 0.60       | 1.248E-02  | 1.55       | 0.90       |
| $\frac{1}{4}$ | 1.484E-02  | 1.57       | 0.76       | 1.122E-02  | 2.78       | 1.26       |
| $\frac{1}{8}$ | 1.060E-02  | 2.02       | 0.92       | 7.045E-03  | 5.18       | 2.42       |
| $\frac{1}{16}$ | 7.483E-03  | 2.66       | 1.20       | 2.885E-03  | 10.0       | 4.41       |
| $\frac{1}{32}$ | 5.493E-03  | 3.54       | 1.60       | 1.528E-03  | 19.7       | 8.23       |

| $\varepsilon$ | $s = 0.25$ | step time | $s = 0.5$ | step time | $s = 0.75$ | step time |
|---------------|------------|------------|------------|------------|------------|------------|
| $\frac{1}{2}$ | 4.228E-02  | 1.34       | 0.79       | 1.447E-02  | 1.86       | 1.04       |
| $\frac{1}{4}$ | 3.200E-02  | 1.72       | 1.14       | 7.239E-03  | 3.36       | 1.96       |
| $\frac{1}{8}$ | 2.166E-02  | 2.22       | 1.34       | 4.424E-03  | 6.48       | 3.25       |
| $\frac{1}{16}$ | 1.535E-02  | 2.93       | 1.54       | 2.255E-03  | 12.6       | 6.26       |
| $\frac{1}{32}$ | 1.085E-02  | 3.97       | 2.12       | 1.166E-03  | 24.6       | 12.1       |

| $\varepsilon$ | $s = 0.25$ | step time | $s = 0.5$ | step time | $s = 0.75$ | step time |
|---------------|------------|------------|------------|------------|------------|------------|
| $\frac{1}{2}$ | 5.372E-02  | 1.39       | 1.20       | 1.128E-02  | 2.02       | 2.04       |
| $\frac{1}{4}$ | 3.994E-02  | 1.78       | 1.50       | 6.274E-03  | 3.76       | 3.14       |
| $\frac{1}{8}$ | 2.420E-02  | 2.31       | 1.90       | 1.870E-03  | 7.24       | 5.92       |
| $\frac{1}{16}$ | 1.615E-02  | 3.11       | 2.54       | 1.165E-03  | 14.2       | 11.3       |
| $\frac{1}{32}$ | 1.070E-03  | 4.18       | 3.42       | 5.535E-04  | 27.8       | 22.6       |

| $\varepsilon$ | $s = 0.25$ | step time | $s = 0.5$ | step time | $s = 0.75$ | step time |
|---------------|------------|------------|------------|------------|------------|------------|
| $\frac{1}{2}$ | 7.687E-02  | 1.49       | 3.45       | 9.497E-02  | 2.39       | 5.14       |
| $\frac{1}{4}$ | 4.493E-02  | 1.97       | 3.78       | 3.861E-03  | 4.67       | 9.03       |
| $\frac{1}{8}$ | 1.723E-02  | 2.61       | 4.96       | 9.114E-03  | 9.20       | 17.5       |
| $\frac{1}{16}$ | 1.305E-03  | 3.54       | 6.81       | 4.943E-03  | 18.1       | 37.8       |
| $\frac{1}{32}$ | 9.910E-03  | 4.90       | 9.76       | 2.287E-03  | 35.6       | 66.3       |

| $\varepsilon$ | $s = 0.25$ | step time | $s = 0.5$ | step time | $s = 0.75$ | step time |
|---------------|------------|------------|------------|------------|------------|------------|
| $\frac{1}{2}$ | 1.110E-01  | 1.26       | 21.6       | 1.996E-01  | 1.28       | 21.6       |
| $\frac{1}{4}$ | 1.048E-01  | 1.70       | 28.5       | 4.635E-02  | 2.56       | 44.6       |
| $\frac{1}{8}$ | 7.995E-02  | 2.34       | 40.8       | 2.198E-02  | 5.18       | 87.1       |
| $\frac{1}{16}$ | 5.291E-02  | 3.21       | 56.2       | 1.815E-02  | 10.2       | 169        |
| $\frac{1}{32}$ | 3.729E-02  | 4.49       | 77.7       | 1.242E-02  | 20.1       | 350        |
Table 4: The error in simulation, the average number of steps, and the computational time (secs.) for \(i = 2\) of Example 5.3 by scheme (4.8) are given.

| \(n = 2\) | \(\varepsilon\) | \(s = 0.25\) | step time | \(s = 0.5\) | step time | \(s = 0.75\) | step time |
|-----------|---------------|-------------|-----------|-------------|-----------|-------------|-----------|
| \(\frac{1}{4}\) | 3.680E-03 | 1.26 | 0.67 | 7.855E-03 | 1.54 | 0.78 | 3.315E-02 | 1.55 | 0.90 |
| \(\frac{1}{8}\) | 3.143E-03 | 1.58 | 0.82 | 6.639E-03 | 2.79 | 1.78 | 1.182E-02 | 3.82 | 1.81 |
| \(\frac{1}{16}\) | 2.219E-03 | 2.03 | 0.92 | 3.624E-03 | 5.22 | 2.60 | 4.429E-03 | 10.2 | 4.65 |
| \(\frac{1}{32}\) | 1.639E-03 | 2.66 | 1.28 | 2.364E-03 | 10.0 | 4.68 | 1.483E-03 | 27.8 | 49.7 |
| \(\frac{1}{64}\) | 1.203E-03 | 3.54 | 1.56 | 1.159E-03 | 19.6 | 8.17 | 5.387E-04 | 77.2 | 984 |

| \(n = 3\) | \(\varepsilon\) | \(s = 0.25\) | step time | \(s = 0.5\) | step time | \(s = 0.75\) | step time |
|-----------|---------------|-------------|-----------|-------------|-----------|-------------|-----------|
| \(\frac{1}{4}\) | 2.620E-02 | 1.35 | 0.78 | 6.874E-03 | 1.86 | 1.09 | 3.593E-03 | 1.89 | 1.21 |
| \(\frac{1}{8}\) | 2.263E-02 | 1.69 | 1.03 | 1.400E-03 | 3.40 | 1.84 | 6.062E-03 | 5.04 | 2.70 |
| \(\frac{1}{16}\) | 1.532E-02 | 2.20 | 1.18 | 6.189E-04 | 6.50 | 3.51 | 2.121E-03 | 13.7 | 7.37 |
| \(\frac{1}{32}\) | 1.057E-02 | 2.92 | 1.56 | 2.450E-04 | 12.5 | 2.70 | 6.192E-04 | 37.7 | 74.0 |
| \(\frac{1}{64}\) | 7.335E-03 | 3.92 | 2.04 | 9.949E-05 | 24.8 | 12.4 | 2.255E-04 | 105 | 1549 |

| \(n = 4\) | \(\varepsilon\) | \(s = 0.25\) | step time | \(s = 0.5\) | step time | \(s = 0.75\) | step time |
|-----------|---------------|-------------|-----------|-------------|-----------|-------------|-----------|
| \(\frac{1}{4}\) | 4.448E-02 | 1.39 | 1.23 | 5.686E-03 | 1.99 | 1.75 | 3.241E-02 | 2.12 | 1.96 |
| \(\frac{1}{8}\) | 3.190E-02 | 1.78 | 1.60 | 2.909E-03 | 3.77 | 3.56 | 1.197E-02 | 5.69 | 4.68 |
| \(\frac{1}{16}\) | 1.800E-02 | 2.30 | 1.92 | 1.392E-03 | 7.22 | 6.28 | 3.720E-03 | 15.3 | 12.7 |
| \(\frac{1}{32}\) | 1.244E-02 | 3.08 | 2.56 | 7.193E-04 | 14.1 | 11.2 | 8.106E-04 | 43.0 | 136 |
| \(\frac{1}{64}\) | 8.068E-03 | 4.19 | 3.43 | 3.245E-04 | 28.0 | 22.6 | 2.650E-04 | 120 | 2913 |

| \(n = 10\) | \(\varepsilon\) | \(s = 0.25\) | step time | \(s = 0.5\) | step time | \(s = 0.75\) | step time |
|-----------|---------------|-------------|-----------|-------------|-----------|-------------|-----------|
| \(\frac{1}{10}\) | 1.087E-01 | 1.18 | 2.46 | 1.035E-01 | 1.25 | 2.57 | 1.239E-01 | 0.81 | 2.25 |
| \(\frac{1}{5}\) | 7.280E-02 | 1.50 | 3.09 | 1.622E-02 | 2.39 | 4.76 | 1.569E-01 | 2.33 | 4.76 |
| \(\frac{1}{8}\) | 4.345E-02 | 1.96 | 4.06 | 8.147E-03 | 4.62 | 8.89 | 5.641E-03 | 6.54 | 13.0 |
| \(\frac{1}{16}\) | 2.120E-02 | 2.63 | 5.25 | 5.924E-03 | 9.03 | 18.0 | 2.259E-02 | 18.3 | 38.5 |
| \(\frac{1}{32}\) | 6.903E-03 | 3.55 | 6.82 | 6.192E-03 | 18.1 | 34.4 | 1.600E-02 | 50.7 | 384 |
| \(\frac{1}{64}\) | 4.523E-03 | 4.80 | 9.32 | 2.287E-03 | 35.7 | 67.6 | 7.617E-02 | 142 | 8247 |

| \(n = 100\) | \(\varepsilon\) | \(s = 0.25\) | step time | \(s = 0.5\) | step time | \(s = 0.75\) | step time |
|-----------|---------------|-------------|-----------|-------------|-----------|-------------|-----------|
| \(\frac{1}{2}\) | 9.560E-01 | 1.25 | 20.7 | 2.831E-01 | 1.29 | 21.0 | 7.816E-01 | 0.84 | 15.5 |
| \(\frac{1}{4}\) | 9.787E-02 | 1.68 | 28.8 | 7.095E-02 | 2.57 | 43.5 | 7.816E-01 | 2.42 | 43.0 |
| \(\frac{1}{8}\) | 8.131E-02 | 2.32 | 40.7 | 3.068E-02 | 5.17 | 87.0 | 8.823E-02 | 6.91 | 119 |
| \(\frac{1}{16}\) | 5.694E-02 | 3.21 | 53.2 | 2.482E-02 | 10.2 | 181 | 4.192E-02 | 19.6 | 327 |
| \(\frac{1}{32}\) | 3.867E-02 | 4.41 | 73.6 | 1.373E-02 | 20.0 | 338 | 2.650E-02 | 54.6 | 3679 |
Figure 2: Numerical errors of the scheme (4.8) for Example 5.3. It shows that if $u(t,x) \in C^{1,\beta}([0,T] \times \mathbb{R}^n)$, $\beta \in [2,3]$, the convergent order is $2s \wedge (|\beta| - 2s)$.
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References

[1] L. Aceto, P. Novati, Rational approximation to the fractional Laplacian operator in reaction-diffusion problems, SIAM J. Sci. Comput. 39(1) (2017) A214–A228.

[2] S. S. Alzahrani, A. Q. M. Khaliq, Fourier spectral exponential time differencing methods for multi-dimensional space-fractional reaction-diffusion equations, J. Comput. Appl. Math. 361 (2019) 157–175.

[3] H. Antil, T. J. Harlim, Fractional diffusion maps, Appl. Comput. Harmon. Anal. 54 (2021) 145–175.

[4] D. Applebaum, Lévy Processes and Stochastic Calculus, Cambridge University Press, Cambridge, UK, 2009.

[5] S. Asmussen, J. Rosinski, Approximations of small jumps of Lévy processes with a view towards simulation, J. Appl. Probab. 38 (2001) 482–493.

[6] M. Cai, C. P. Li, On Riesz derivative, Fract. Calc. Appl. Anal. 22(2) (2019) 287–301.

[7] M. Cai, C. P. Li, Regularity of the solution to Riesz-type fractional differential equation, Integr. Transf. Spec. Funct. 30(9) (2019) 711–742.

[8] S. Das, Functional Fractional Calculus, Springer-Verlag, Berlin, 2011.

[9] G. Deligiannidis, S. Maurer, M. V. Tretyakov, Random walk algorithm for the Dirichlet problem for parabolic integro-differential equation, BIT Numer. Math. 61 (2021) 1223–1269.

[10] P. Guo, C. B. Zeng, C. P. Li, Y. Q. Chen, Numerics for the fractional Langevin equation driven by the fractional Brownian motion, Fract. Calc. Appl. Anal. 16 (2013) 123–141.

[11] R. Herrmann, Fractional Calculus: An Introduction for Physicists, World Scientific, Singapore, 2011.

[12] R. Hilfer ed., Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[13] Y. Hu, C. P. Li, Y. B. Yan, Weak convergence of the L1 scheme for a stochastic subdiffusion problem driven by fractionally integrated additive noise, Appl. Numer. Math., 178 (2022), 192–215.

[14] C. Y. Jiao, A. Khaliq, C. P. Li, H. X. Wang, Difference between Riesz derivative and fractional Laplacian on the proper subset of $\mathbb{R}$, Fract. Calc. Appl. Anal. 24(6) (2021) 1716–1734.

[15] C. Y. Jiao, C. P. Li, H. X. Wang, Z. Q. Zhang, A modified walk-on-sphere method for high dimensional fractional Poisson equation, Numer. Methods Partial Differential Equations, 2022, 1–35, https://doi.org/10.1002/num.22927
[16] B. T. Jin, Y. B. Yan, Z. Zhou, Numerical approximation of stochastic time-fractional diffusion, ESAIM Math. Model. Numer. Anal. 53(4) (2019) 1245–1268.

[17] A. Kohatsu-Higa, P. Tankov, Jump-adapted discretization schemes for Lévy-driven SDEs, Stochastic Process. Appl. 120 (2010) 2258–2285.

[18] A. E. Kyprianou, A. Osojnik, T. Shardlow, Unbiased “walk-on-spheres” Monte Carlo methods for the fractional Laplacian, IMA J. Numer. Anal. 38 (2018) 1550–1578.

[19] C. P. Li, M. Cai, Theory and Numerical Approximations of Fractional Integrals and Derivatives. SIAM, Philadelphia, 2019.

[20] C. P. Li, Z. Q. Li, The finite-time blow-up for semilinear fractional diffusion equations with time $\psi$-Caputo derivative, J. Nonlinear Sci. 32(6) (2022) article 82 (42 pp.).

[21] R. Metzler, J. Klafter, The random walks guide to anomalous diffusion: a fractional dynamics approach, Phys. Rep. 339(1) (2000) 1–77.

[22] R. Mikulevicius, On the rate of convergence of simple and jump-adapted weak Euler schemes for Lévy driven SDEs, Stochastic Process. Appl. 122 (2012) 2730–2757.

[23] G. N. Milstein, M. V. Tretyakov, Numerical solution of the Dirichlet problem for nonlinear parabolic equations by a probabilistic approach, IMA J. Numer. Anal. 21 (2001) 887–917.

[24] G. N. Milstein, M. V. Tretyakov, Discretization of forward-backward stochastic differential equations and related quasi-linear parabolic equations, IMA J. Numer. Anal. 27 (2007) 24–44.

[25] M. E. Muller, Some continuous Monte Carlo methods for the Dirichlet problem, Ann. Math. Stat. 27(3) (1956) 569–589.

[26] K. B. Oldham, J. Spanier, The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order, Academic Press, New York, London, 1974.

[27] K. K. Sabelfeld, Monte Carlo Methods in Boundary Value Problems, Springer Verlag, Berlin, 1991.

[28] K. K. Sabelfeld, Expansion of random boundary excitations for the fractional Laplacian, Journal of Cosmology and Astroparticle Physics 10 (2008) 004. https://doi.org/10.1088/1475-7516/2008/10/004.

[29] K. K. Sabelfeld, Random walk on spheres algorithm for solving transient drift-diffusion-reaction problems, Monte Carlo Methods Appl. 23(3) (2017) 189–212.

[30] K. K. Sabelfeld, First passage Monte Carlo algorithms for solving coupled systems of diffusion-reaction equations, Appl. Math. Lett. 88 (2019) 141–148.

[31] K. K. Sabelfeld, I. A. Shalimova, Spherical and Plane Integral Operators for PDEs: Construction, Analysis, and Applications, De Gruyter, Berlin, 2013.

[32] K. K. Sabelfeld, N. A. Simonov, Stochastic Methods for Boundary Value Problems: Numerics for High-dimensional PDEs and Applications, De Gruyter, Berlin, 2016.
[33] S. G. Samko, A. A. Kilbas, O. I. Marichev, Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach Science Publ., Amsterdam 1993.

[34] K. Sato, Lévy Processes and Infinitely Divisible Distributions, Cambridge Univ. Press, Cambridge, 1999.

[35] C. T. Sheng, S. N. Ma, H. Y. Li, L. L. Wang, L. L. Jia, Nontensorial generalised Hermite spectral methods for PDEs with fractional Laplacian and Schrödinger operators, ESAIM Math. Model. Numer. Anal. 55(5) (2021) 2141–2168.

[36] J. Zhai, B. W. Zheng, Strichartz estimate of the solutions for the free fractional Schrödinger equation with spatial variable coefficient, Appl. Comput. Harmon. Anal. 46 (2019) 207–225.