A BRIEF INTRODUCTION TO OPERATOR QUANTUM ERROR CORRECTION

DEDICATED TO JOHN HOLBROOK ON THE OCCASION OF HIS 65TH BIRTHDAY

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ABSTRACT. We give a short introduction to operator quantum error correction. This is a new protocol for error correction in quantum computing that has brought the fundamental methods under a single umbrella, and has opened up new possibilities for protecting quantum information against undesirable noise. We describe the various conditions that characterize correction in this scheme.

1. Introduction

In this paper we give an introduction to some of the mathematical aspects of quantum error correction, with an emphasis on the unified approach – called operator quantum error correction – recently introduced in [1, 2]. The field of quantum error correction took flight during the mid 1990’s [3, 4, 5, 6]. A central goal of this young field is to help construct quantum computers via the development of schemes that allow for the protection of quantum information against the noise associated with evolution of quantum systems. As it turns out, many of the problems in quantum error correction have an operator theoretic flavour. Here we shall briefly discuss the fundamental error correction protocols in quantum computing. We also describe various conditions that characterize correction in Operator QEC, and provide a new operator proof for the main testable condition in this scheme.

First let us briefly discuss the basic setting for quantum computation. See [7, 8] as examples of more extensive introductions. To each quantum system, the postulates of quantum mechanics associate a Hilbert space $\mathcal{H} = \mathbb{C}^n$. The finite dimensional case is the current focus in quantum computing for experimental reasons. A two-level quantum system is represented on $\mathcal{H} = \mathbb{C}^2$. This could describe, for instance, the ground

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and excited energy states of an electron in an atom. These are the two classical states that we observe, corresponding to an orthonormal basis \( \{ |0\rangle, |1\rangle \} \) for \( \mathbb{C}^2 \). However, quantum mechanics dictates that any linear combination \( |\psi\rangle \) of these classical states is an allowable state, even though we only observe either the \(|0\rangle\) or \(|1\rangle\) state. When it is a non-trivial linear combination, \( |\psi\rangle \) is said to be in a superposition of the classical states. A unit vector \( |\psi\rangle \) is the fundamental quantum bit of information, also called a “qubit”. Equivalently, we could consider the rank one projection \( |\psi\rangle \langle \psi| \). The corresponding \( n \)-qubit composite system is realized on \( \mathcal{H} = (\mathbb{C}^2)^\otimes n = \mathbb{C}^{2^n} \) with orthonormal basis \( \{ |i_1 \cdots i_n\rangle \equiv |i_1\rangle \otimes \cdots \otimes |i_n\rangle : i_j \in \{0,1\} \} \) determined by the underlying two-level system.

More generally, we will only know that our system is in one of several states with various possibilities. So the direct generalization of a classical probability distribution in quantum information theory is a positive matrix \( \rho \) with trace equal to one, a so-called density matrix.

A fundamental problem in quantum computation is to physically manipulate the superpositions inherent in quantum systems, without collapsing or “decohering” them. To accomplish this, methods must be developed to correct the errors that occur as quantum information is transferred from one physical location to the next inside, for instance, a quantum computer. To deal with this problem we must discuss evolution of quantum systems, a subject to which we now turn.

2. Evolution of Quantum Systems and Error Correction

The reversibility postulate of quantum mechanics implies that evolution in a closed quantum system occurs via unitary maps. From the discrete perspective, if we take a snapshot of this evolution, then a density matrix \( \rho \) will encode the possible states of the system with various probabilities at a given time. An evolution of the system corresponds to a map \( \rho \mapsto U\rho U^\dagger \) for some unitary operator \( U \).

In the context of quantum computing, the quantum systems of interest are “open” as they are exposed to external environments during computations. In such cases, the open system is regarded as part of a larger closed quantum system given by the composite of the system and the environment. If \( \mathcal{H}_S \) and \( \mathcal{H}_E \) are the system and environment Hilbert spaces, then the closed system is represented on \( \mathcal{H} = \mathcal{H}_E \otimes \mathcal{H}_S \).

The characterization of evolution in open quantum systems requires first that density operators are mapped to density operators; i.e. probability densities are mapped to probability densities. Thus, such a map must be positive and trace preserving. However, this property must be
preserved when the system is exposed to all possible environments. In terms of the map, if $\mathcal{E}$ describes an evolution of the system, then the map $id_E \otimes \mathcal{E} : \mathcal{B}(\mathcal{H}_E \otimes \mathcal{H}_S) \rightarrow \mathcal{B}(\mathcal{H}_E \otimes \mathcal{H}_S)$ must also be positive and trace preserving for all $E$. Hence, the widely accepted working definition of a quantum operation (or evolution, or channel) on a Hilbert space $\mathcal{H}$, is a completely positive, trace preserving map $\mathcal{E}$ on $\mathcal{B}(\mathcal{H})$ (CPTP for short).

Deriving from a theorem of Choi [9] and Kraus [10], every CPTP map $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ has an “operator-sum representation” of the form $\mathcal{E}(\rho) = \sum_a E_a \rho E_a^\dagger$ for some set of (non-unique) operators $\{E_a\} \subseteq \mathcal{B}(\mathcal{H})$ with $\sum_a E_a^\dagger E_a = \mathbb{I}$. The $E_a$ are called the noise operators or errors associated with $\mathcal{E}$. In the context of quantum error correction, it is precisely the effects of these errors that must be mitigated. As a short hand, we write $\mathcal{E} = \{E_a\}$ when an error model for $\mathcal{E}$ is known.

Error correction in quantum computing is a much more delicate problem in comparison to its classical counterpart. As a simple observation, consider that the only errors that occur classically are some version of bit flips; e.g., $|0\rangle$ goes to $|1\rangle$ or vice-versa. More generally, in quantum computing subtleties arise from the fact that a given qubit can be corrupted to an infinite number of possible superpositions. In terms of operators on single bits or qubits for instance, whereas the Pauli bit flip matrix $X = (0 1 \begin{array}{c} 1 0 \end{array})$ is the fundamental classical error matrix, any unitary matrix is a possible error in quantum computing. Of course, there are many other issues, such as the fabled “No Cloning Theorem”. The linearity of quantum mechanics implies that the analogue of the classically well-used repetition code does not extend to arbitrary qubits $|\psi\rangle \mapsto |\psi\rangle \otimes |\psi\rangle$. Fortunately, methods have been, and are being, developed to overcome these challenges.

3. Standard Model of Quantum Error Correction

The “Standard Model” of quantum error correction [3, 4, 5, 6] involves triples $(\mathcal{R}, \mathcal{E}, \mathcal{C})$ where $\mathcal{C}$ is a subspace, a quantum code, of a Hilbert space $\mathcal{H}$ associated with a given quantum system, and the error $\mathcal{E}$ and recovery $\mathcal{R}$ are quantum operations on $\mathcal{B}(\mathcal{H})$.

Recall from the discussions above that we are forced by quantum mechanics to consider subspaces $\mathcal{C}$ as sets of codes, as linear combinations of classical codewords are perfectly allowable codewords in this setting. In the trivial case, when $\mathcal{E} = \{U\}$ is implemented by a single unitary error operator, the recovery is just the reversal operation $\mathcal{R} = \{U^\dagger\}$; that is,

$$\rho \xrightarrow{\mathcal{E}} U\rho U^\dagger \xrightarrow{\mathcal{R}} U^\dagger(U\rho U^\dagger)U = \rho.$$
Of course, here there is no need to restrict the input operators $\rho$.

More generally, the set $(\mathcal{R}, \mathcal{E}, \mathcal{C})$ forms an “error triple” if $\mathcal{R}$ undoes the effects of $\mathcal{E}$ on $\mathcal{C}$ in the following sense:

\begin{equation}
(\mathcal{R} \circ \mathcal{E})(\sigma) = \sigma \quad \forall \sigma \in \mathcal{B}(\mathcal{C}),
\end{equation}

where $\mathcal{C}$ is naturally regarded as embedded inside $\mathcal{H}$.

When there exists such an $\mathcal{R}$ for a given pair $\mathcal{E}, \mathcal{C}$, the subspace $\mathcal{C}$ is said to be correctable for $\mathcal{E}$. The existence of a recovery operation $\mathcal{R}$ of $\mathcal{E} = \{E_a\}$ on $\mathcal{C}$ is characterized by the following condition [5, 6]:

\begin{equation}
P_C E_a^\dagger E_a P_C = \lambda_{ab} P_C \quad \forall a, b,
\end{equation}

where $P_C$ is the projection of $\mathcal{H}$ onto $\mathcal{C}$. It is not hard to see that this condition is independent of the operator-sum representation for $\mathcal{E}$. We note that Eq. (2) is a special case of Eq. (10) below.

The motivating case of an error model that satisfies Eq. (2) occurs when the restrictions $E_a|_{\mathcal{H}-\mathcal{C}} = E_a|_{\mathcal{C}}$ of the noise operators to $\mathcal{C}$ are scalar multiples of unitary operators $U_a$, such that the subspaces $U_a \mathcal{C}$ are mutually orthogonal. In this situation the positive scalar matrix $\Lambda$ is diagonal. A correction operation here may be constructed by an application of the measurement operation determined by the subspaces $U_a \mathcal{C}$, followed by the reversals of the corresponding restricted unitaries $U_a P_C$. Specifically, if $P_a$ is the projection of $\mathcal{H}$ onto $U_a \mathcal{C}$, then $\mathcal{R} = \{U_a^\dagger P_a\}$ satisfies Eq. (11) for $\mathcal{E}$ on $\mathcal{C}$.

Let us discuss a simple example. Let $\mathcal{C}$ be the subspace of $\mathcal{H} = \mathbb{C}^8$ given by $\mathcal{C} = \text{span}\{\langle 000 \rangle, \langle 111 \rangle\}$. Let $\mathcal{E} = \{\frac{1}{\sqrt{3}} X_k : k = 1, 2, 3\}$ with the Pauli matrix $X$ and $X_1 = X \otimes 1 \otimes 1$, and similarly for $X_2, X_3$. In this case, $\Lambda = \frac{1}{3} \mathbb{1}_3$. The correction operation $\mathcal{R}$ may be constructed as above.

4. Noiseless Subsystems

To describe the notion of noiseless subsystems from [1, 2], we begin with a decomposition of the system Hilbert space

$$\mathcal{H} = \bigoplus_j \mathcal{H}_J^A \otimes \mathcal{H}_J^B,$$

where the “noisy subsystems” $\mathcal{H}_J^A$ have dimension $m_J$ and the “noiseless subsystems” $\mathcal{H}_J^B$ have dimension $n_J$. We focus on the case where information is encoded in a single noiseless sector of $\mathcal{B}(\mathcal{H})$, so

$$\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K}$$
with \( \dim(\mathcal{H}^A) = m \), \( \dim(\mathcal{H}^B) = n \) and \( \dim \mathcal{K} = \dim \mathcal{H} - mn \). We shall write \( \sigma^A \) for operators in \( \mathcal{B}(\mathcal{H}^A) \) and \( \sigma^B \) for operators in \( \mathcal{B}(\mathcal{H}^B) \).

Let \( \{ |\alpha_k\rangle : 1 \leq k \leq m \} \) be an orthonormal basis for \( \mathcal{H}^A \) and let

\[
\{ P_{kl} = |\alpha_k\rangle \langle \alpha_l| \otimes \mathbb{1}_n : 1 \leq k, l \leq m \}
\]

be the corresponding family of matrix units in \( \mathcal{B}(\mathcal{H}^A) \otimes \mathbb{1}_B \). Recall that the partial trace over \( A \) on \( \mathcal{H}^A \otimes \mathcal{H}^B \) is the quantum operation defined on elementary tensors by \( \text{Tr}_A(\sigma^A \otimes \sigma^B) = \text{Tr}(\sigma^A)\sigma^B \).

Define for a fixed decomposition \( \mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K} \) the operator semigroup

\[
(3) \quad \mathfrak{A} = \{ \sigma \in \mathcal{B}(\mathcal{H}) : \sigma = \sigma^A \otimes \sigma^B, \text{ for some } \sigma^A \text{ and } \sigma^B \}.
\]

For notational purposes, we assume that bases have been chosen and define the matrix units \( P_{kl} \) as above, so that \( P_k = P_{kk}, \ P_\mathfrak{A} \equiv P_1 + \ldots + P_m \) and \( P_\mathfrak{A} \mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B \). We also define a map \( \mathcal{P}_\mathfrak{A} \) by the action \( \mathcal{P}_\mathfrak{A}(\cdot) = P_\mathfrak{A}(\cdot)P_\mathfrak{A} \). The following result motivates the (generalized) definition of NS’s from \([1, 2]\). (See \([2]\) for a proof.)

**Lemma 4.1.** Given a fixed decomposition \( \mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K} \) and a quantum operation \( \mathcal{E} \) on \( \mathcal{B}(\mathcal{H}) \), the following three conditions are equivalent:

1. \( \forall \sigma^A \forall \sigma^B, \exists \tau^A : \mathcal{E}(\sigma^A \otimes \sigma^B) = \tau^A \otimes \sigma^B \)
2. \( \forall \sigma^B, \exists \tau^A : \mathcal{E}(\mathbb{1}^A \otimes \sigma^B) = \tau^A \otimes \sigma^B \)
3. \( \forall \sigma \in \mathfrak{A} : (\text{Tr}_A \circ \mathcal{P}_\mathfrak{A} \circ \mathcal{E})(\sigma) = \text{Tr}_A(\sigma) \).

**Definition 4.2.** The \( \mathcal{H}^B \) sector of the semigroup \( \mathfrak{A} \) encodes a noiseless subsystem for \( \mathcal{E} \) when it satisfies the equivalent conditions of Lemma 4.1.

The NS framework discussed here is a generalization of both the “Standard NS” \([15, 16, 17]\) and “Decoherence-Free Subspace” \([11, 12, 13, 14]\) methods of passive error correction, both of which are used for unital quantum operations. The method described here applies to all CPTP maps. See \([1, 2]\) for more discussions on this point.

As a simple example of how such subsystems naturally arise, let \( \Phi : \mathcal{B}(\mathcal{H}^A) \to \mathcal{B}(\mathcal{H}^A) \) be an arbitrary CPTP map and let \( \Psi : \mathcal{B}(\mathcal{H}^B) \to \mathcal{B}(\mathcal{H}^B) \) be CPTP with a Standard NS \( \mathcal{H}_0^B \subseteq \mathcal{H}^B \); i.e., \( \Psi(\rho) = \rho \) for all \( \rho \in \mathcal{B}(\mathcal{H}_0^B) \). Then \( \mathcal{H}_0^B \) encodes a noiseless subsystem inside \( \mathcal{H}^A \otimes \mathcal{H}^B \) for the map \( \mathcal{E} = \Phi \otimes \Psi : \mathcal{B}(\mathcal{H}^A \otimes \mathcal{H}^B) \to \mathcal{B}(\mathcal{H}^A \otimes \mathcal{H}^B) \).

To be of use in practical applications, we need testable conditions for a map \( \mathcal{E} = \{ E_a \} \) to admit a NS described by a semigroup \( \mathfrak{A} \). Towards this end, we have proved the following theorem.
Theorem 4.3. Let $E = \{E_a\}$ be a quantum operation on $B(H)$ and let $A$ be a semigroup in $B(H)$ as above. Then the following three conditions are equivalent:

1. The $H^B$ sector of $A$ encodes a noiseless subsystem for $E$.
2. The subspace $P_A H = H^A \otimes H^B$ is invariant for the operators $E_a$ and the restrictions $E_a|_{P_A H}$ belong to the algebra $B(H^A) \otimes \mathbb{1}^B$.
3. The following two conditions hold:

   \[ P_k E_a P_l = \lambda_{akl} P_{kl} \quad \forall a, k, l \]
   \[ \text{for some set of scalars } (\lambda_{akl}) \text{ and} \]
   \[ E_a P_A = P_A E_a P_A \quad \forall a. \]

Proof. Since the matrix units \{\[P_{kl}\}\} generate $B(H^A) \otimes \mathbb{1}^B$, it follows that (3) is a restatement of (2). Here we sketch the proof of the equivalence of (1) and (3), see [1, 2] for details. To prove the necessity of Eqs. (1), (3) for (1), it follows from properties of the map $\Gamma = \{P_{kl}\}$ and Lemma 4.1 that there exist scalars $\mu_{kiajl,k'\ell'}$ such that

\[ P_{ki} E_a P_{jl} = \sum_{k'\ell'} \mu_{kiajl,k'\ell'} P_{k'\ell'}. \]

Multiplying both sides of this equality on the right by $P_l$ and on the left by $P_k$, we see that $\mu_{kiajl,k'\ell'} = 0$ when $k \neq k'$ or $l \neq l'$. This implies Eq. (4) with $\lambda_{akl} = \mu_{kkall,kl}$. Equation (5) follows from Lemma 4.1 and consideration of the operator-sum representation for $E$.

On the other hand, if Eqs. (1), (3) hold, then for all $\sigma = P_A \sigma \in A$ we have

\[ E(\sigma) = \sum_{a, k, k'} P_k E_a \sigma E_a^\dagger P_{k'}. \]

This implies that for all $\sigma = \sigma^A \otimes \sigma^B \in A$,

\[ E(\sigma^A \otimes \sigma^B) = \sum_{a, k, k', l, l'} P_k E_a P_l (\sigma^A \otimes \sigma^B) E_a^\dagger P_{k'}. \]

Condition (1) now follows from the fact that the matrix units $P_{kl}$ act trivially on the $B(H^B)$ sector.
Table 1: Special Cases of Operator QEC

| \( \mathfrak{A} \) = subspace | Standard QEC |
|-----------------------------|-------------|
| \( \mathcal{R} = id \)     | (Generalized) NS |
| \( \mathcal{R} = id + \mathfrak{A} = algebra \) | Standard NS |
| \( \mathcal{R} = id + \mathfrak{A} = subspace \) | DFS |

5. Operator Quantum Error Correction

The Operator QEC approach consists of triples \((\mathcal{R}, \mathcal{E}, \mathfrak{A})\) where \(\mathcal{R}\) and \(\mathcal{E}\) are quantum operations on some \(\mathcal{B}(\mathcal{H})\), and \(\mathfrak{A}\) is a semigroup in \(\mathcal{B}(\mathcal{H})\) defined as above with respect to a fixed decomposition \(\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K}\).

**Definition 5.1.** Given a triple \((\mathcal{R}, \mathcal{E}, \mathfrak{A})\) we say that the \(\mathcal{H}^B\) sector of \(\mathfrak{A}\) is correctable for \(\mathcal{E}\) if

\[
(\text{Tr}_A \circ \mathcal{P}_A \circ \mathcal{R} \circ \mathcal{E})(\sigma) = \text{Tr}_A(\sigma) \quad \text{for all} \quad \sigma \in \mathfrak{A}.
\]

(7)

Equivalently, \((\mathcal{R}, \mathcal{E}, \mathfrak{A})\) is a correctable triple if the \(\mathcal{H}^B\) sector of the semigroup \(\mathfrak{A}\) encodes a noiseless subsystem for the error map \(\mathcal{R} \circ \mathcal{E}\).

Table 1 indicates the special cases captured by Operator QEC. Our choice of terminology here is motivated by the fact that correctable codes in this scheme take the form of operator algebras and operator semigroups. We point the reader to [2, 18] for examples of error triples on subsystems that require non-trivial recovery operations, and [19, 20] for other recent related work.

An important feature of Operator QEC is that a semigroup \(\mathfrak{A}\) is correctable exactly when the C*-algebra \(\mathfrak{A}_0 = 1 \otimes \mathcal{B}(\mathcal{H}^B)\) can be corrected precisely.

**Theorem 5.2.** Let \(\mathcal{E} = \{E_a\}\) be a quantum operation on \(\mathcal{B}(\mathcal{H})\) and let \(\mathfrak{A}\) be a semigroup in \(\mathcal{B}(\mathcal{H})\) as above. Then the \(\mathcal{H}^B\) sector of \(\mathfrak{A}\) is correctable for \(\mathcal{E}\) if and only if there is a quantum operation \(\mathcal{R}\) on \(\mathcal{B}(\mathcal{H})\) such that

\[
(\mathcal{R} \circ \mathcal{E})(\sigma) = \sigma \quad \forall \sigma \in 1 \otimes \mathcal{B}(\mathcal{H}^B).
\]

(8)

**Proof.** If Eq. (8) holds, then condition (2) of Lemma 4.1 holds for \(\mathcal{R} \circ \mathcal{E}\) with \(\tau^A = 1 \otimes \mathcal{B}(\mathcal{H}^B)\) and hence \(\mathfrak{A}\) is correctable for \(\mathcal{E}\). On the other hand, suppose that \(\mathfrak{A}\) is correctable for \(\mathcal{E}\) and condition (2) of Lemma 4.1 holds for \(\mathcal{R} \circ \mathcal{E}\). Note that the map \(\Gamma' = \{\frac{1}{\sqrt{m}}P_{kl}\}\) is trace preserving on \(\mathcal{B}(\mathcal{H}^A \otimes \mathcal{H}^B)\). Thus, from basic properties of the map \(\Gamma = \{P_{kl}\}\), we have for all \(\sigma^B\),

\[
(\Gamma' \circ \mathcal{R} \circ \mathcal{E})(1 \otimes \sigma^B) = \Gamma'(\tau^A \otimes \sigma^B) \propto 1 \otimes \sigma^B.
\]

(9)
By trace preservation the proportionality factor must be one, and hence Eq. 8 is satisfied for \((\Gamma' \circ R) \circ \mathcal{E}\). The map \(\Gamma'\) may be extended to a quantum operation on \(\mathcal{B}(\mathcal{H})\) by including the projection \(P^\perp_{\mathcal{A}}\) onto \(\mathcal{K}\) as a noise operator. As this does not effect the calculation 9, the result follows.

We next give a testable condition, Eq. (10), that characterizes correction in the Operator QEC regime. Notice that this is a generalization of Eq. (2) for Standard QEC. This condition was introduced in [1] and necessity was established. Sufficiency was proved in [2] up to a set of technical conditions, and more recently in [19] with full generality. (The work of [19] also links this condition with an interesting information theoretic condition.) Here we include a sketch of the proof of necessity from [2], and a new operator theoretic version of the proof of sufficiency sketched in [19]. We assume that matrix units \(\{P_{kl} = |\alpha_k\rangle\langle\alpha_l| \otimes 1_l\}\) inside \(\mathcal{B}(\mathcal{H}_A) \otimes 1_l\) have been chosen as above.

**Theorem 5.3.** Let \(\mathcal{E} = \{E_a\}\) be a quantum operation on \(\mathcal{B}(\mathcal{H})\) and let \(\mathfrak{A}\) be a semigroup in \(\mathcal{B}(\mathcal{H})\) as above. For the \(\mathcal{H}_B\) sector of \(\mathfrak{A}\) to be correctable for \(\mathcal{E}\), it is necessary and sufficient that there are scalars \(\Lambda = (\lambda_{abkl})\) such that

\[
P_k E_a^\dagger P_l = \lambda_{abkl} P_{kl} \quad \forall a, b, k, l.
\]

**Proof.** For necessity, note first that Theorem 5.2 gives us a CPTP map \(R\) on \(\mathcal{B}(\mathcal{H})\) such that \(R \circ \mathcal{E}\) acts as the identity channel on \(\mathfrak{A}_0 = \mathbb{1}_A \otimes \mathcal{B}(\mathcal{H}_B) \subseteq \mathcal{B}(\mathcal{H})\).

Suppose that \(R = \{R_b\}\). The noise operators for the operation \(R \circ \mathcal{E}\) are \(\{R_b E_a\}\), and using arguments similar to those of Theorem 4.3 (see [2] for details) we may find scalars \(\mu_{abkl}\) such that

\[
P_k R_b E_a P_l = \mu_{abkl} P_{kl} \quad \forall a, b, k, l.
\]

Consider the products

\[
\left( P_k R_b E_a P_l \right) \left( P_{k'} R_b E_{a'} P_{l'} \right) = \left( \frac{\mu_{abkl}}{\mu_{a'b'k'l'}} \right) \left( \frac{\mu_{a'b'k'l'}}{\mu_{abkl}} \right) P_{kl'}\quad \text{if } k = k',
\]

\[
0 \quad \text{if } k \neq k'.
\]

Now, the subspace \(\mathcal{C}\) can be shown to be invariant for the noise operators \(R_b E_a\). Hence for fixed \(a, a'\) and \(l, l'\) we use \(\sum_b R^\dagger_b R_b = \mathbb{1}\) to
obtain
\[
\left( \sum_{b,k} \mu_{abk} \mu_{a'bk'} \right) P_{ll'} = \sum_{b,k} \left( P_l E_a^\dagger R_b^\dagger P_k \right) \left( P_k R_b E_{a'} P_{l'} \right)
\]
\[
= \sum_b P_l E_a^\dagger R_b^\dagger P_{kl'} R_b E_{a'} P_{l'}
\]
\[
= P_l E_a^\dagger \left( \sum_b R_b^\dagger R_b \right) E_{a'} P_{l'}
\]
\[
= P_l E_a^\dagger E_{a'} P_{l'}
\]
The proof of necessity is completed by setting \( \lambda_{aa'll'} = \sum_{b,k} \mu_{abk} \mu_{a'bk'} \) for all \( a, a' \) and \( l, l' \).

For sufficiency, let us assume that Eq. (10) holds. Let \( \sigma_k = |\alpha_k\rangle \langle \alpha_k| \in \mathcal{B}(\mathcal{H}^A) \), for \( 1 \leq k \leq m \), and define a CPTP map \( \mathcal{E}_k : \mathcal{B}(\mathcal{H}^B) \to \mathcal{B}(\mathcal{H}) \) by \( \mathcal{E}_k(\rho^B) \equiv \mathcal{E}(\sigma_k \otimes \rho^B) \). With \( P \equiv P_{kl} \) and \( E_{a,k} \equiv E_a P|\alpha_k\rangle \), it follows that \( \mathcal{E}_k = \{ E_{a,k} \} \). We shall find a CPTP map that globally corrects all of the errors \( E_{a,k} \).

To do this, first note that we may define a CPTP map \( \mathcal{E}_B : \mathcal{B}(\mathcal{H}^B) \to \mathcal{B}(\mathcal{H}) \) with error model
\[
\mathcal{E}_B = \left\{ \frac{1}{\sqrt{m}} E_{a,k} : \forall a, \forall 1 \leq k \leq m \right\}.
\]
Then Eq. (10) and \( P = \sum_k P_k \) give us
\[
\mathbb{I}^B E_{a,k}^\dagger E_{b,l}^\dagger \mathbb{I}^B = \mathbb{I}^B \langle \alpha_k| P E_{a,k}^\dagger E_b P|\alpha_l\rangle \mathbb{I}^B
\]
\[
= \sum_{k',l'} \mathbb{I}^B \langle \alpha_k| P_{k',l'} E_{a,k}^\dagger E_{b} P_{l'} |\alpha_l\rangle \mathbb{I}^B
\]
\[
= \sum_{k',l'} \lambda_{a'bl'} \mathbb{I}^B \langle \alpha_k| P_{k',l'} |\alpha_l\rangle \mathbb{I}^B
\]
\[
= \lambda_{abl} \mathbb{I}^B.
\]
In particular, Standard QEC implies the existence of a CPTP map \( \mathcal{R} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}^B) \) such that \( (\mathcal{R} \circ \mathcal{E})(\rho^B) = \rho^B \) for all \( \rho^B \).

This implies that
\[
(\mathcal{R} \circ \mathcal{E})(\mathbb{I}^A \otimes \rho^B) = \mathcal{R} \left( \sum_k \mathcal{E}_k(\rho^B) \right)
\]
\[
= m \mathcal{R} \left( \sum_{k,a} \frac{1}{m} E_{a,k} \rho^B E_{a,k}^\dagger \right)
\]
\[
= m \mathcal{R} \circ \mathcal{E}_B(\rho^B) = m \rho^B.
\]
Hence we may define a CPTP ampliation map $I_A : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H})$ via $I_A(\rho^B) = \frac{1}{m}(\mathbb{1}^A \otimes \rho^B)$. Thus on defining $\mathcal{R}' \equiv I_A \circ \mathcal{R}$, we obtain

$$(\mathcal{R}' \circ \mathcal{E})(\mathbb{1}^A \otimes \rho^B) = \mathbb{1}^A \otimes \rho^B \quad \forall \rho^B \in \mathcal{B}(\mathcal{H}_B).$$

The result now follows from an application of Theorem 5.2. ■

6. Concluding Remark

The focus of research in quantum error correction has mainly been on finite dimensional problems to this point. Primarily this reflects the current status of experimental efforts to build quantum computers, and the fact that many scientists working in the area are closely linked with experimentalists. Thus, in the author’s opinion, there is an opportunity here for operator theorists. In particular, mathematicians working in the field have, for the most part, not had the luxury of exploring infinite dimensional aspects and extensions of the quantum error correction framework. It is expected that problems of this nature will eventually be of experimental interest, and we expect they would be of current mathematical interest.

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