On some relations between ideals of nowhere dense sets in topologies on positive integers

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Abstract
We examine the ideals of nowhere dense sets in three topologies on the set of positive integers, namely Furstenberg’s, Rizza’s and the common division topology. We mainly concentrate on inclusions between these ideals, we present a diagram showing these and we explore all possible inclusions between them. We present a formula for the closure of a set in the common division topology. We answer a question posed by Kwela and Nowik (Topol Appl. 248:149–163, 2018) by constructing a set in \( I_G \setminus (I_K \cup I_F) \). Therefore, the main diagram of comparison between the ideals of nowhere dense sets in various topologies from the article by M. Kwela and A. Nowik is completed.

Keywords Furstenberg’s topology · Golomb’s topology · Kirch’s topology · Rizza’s topology · Common division topology · Arithmetic progressions · Nowhere dense sets · Ideals

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1 Preliminaries
The symbol \( \mathbb{N} \) and Primes denote the set of positive integers and the set of primes, respectively. Let \( [\mathbb{N}]^\mathbb{N} \) denote the collection of all infinite subsets of the set of natural numbers. For a subset \( A \) of a topological space, we use the symbols \( \text{cl}(A) \) and \( \text{int}(A) \) for the closure and the interior of \( A \), respectively. The symbols \( \Theta(n) \) and \( D(n) \) denote the set of all prime factors of \( n \), (so \( \Theta(n) = \{ p \in \text{Primes}: p | n \} \)) and the set of all divisors of \( n \), (so \( D(n) = \{ k \in \mathbb{N}: k | n \} \)), respectively. Let \( \mathbb{S} \mathbb{F} \) denote the set of square-free numbers (i.e., numbers not divisible by any
square greater than 1). Following [10], for all \(a, b \in \mathbb{N}\), the symbol \(\{an + b\}\) stands for the infinite arithmetic progressions: \(\{an + b\} = \{a \cdot n + b : n \in \mathbb{N} \cup \{0\}\}\). Moreover, define an abbreviation: \(\{an\} = \{an + a\}\). Let us say that a family \(\mathcal{F} \subseteq [\mathbb{N}]^2\) has the splitting property if for any \(F \in \mathcal{F}\) one can find \(F_1, F_2 \in \mathcal{F}\) such that \(F_1 \cup F_2 \subseteq F\) and \(F_1 \cap F_2 = \emptyset\).

Let us define various topologies on \(\mathbb{N}\):

- **Golomb’s topology** \((\mathbb{N}, D)\) with the base \(B_G = \{\{an + b\} : (a, b) = 1, b < a\}\),
- **Kirch’s topology** \((\mathbb{N}, D')\) with the base \(B_K = \{\{an + b\} : (a, b) = 1, b < a, a \in \mathbb{S}\}\),
- **Furstenberg’s topology** \((\mathbb{N}, T_F)\) with the base \(B_F = \{\{an + b\} : b \leq a\}\),
- **the common division topology** \((\mathbb{N}, T)\) with the base \(B_T = \{\{an + b\} : \Theta(a) \subseteq \Theta(b)\}\),
- **the division topology** (Rizza’s topology) \((\mathbb{N}, T')\) with the base \(B_{T'} = \{\{an\}\}\).

Furstenberg’s topology was first defined in [1] to present a topological proof of the existence of infinitely many prime numbers. This topology is metrizable, zero-dimensional and totally disconnected. Furstenberg’s topology was originally defined on the set of integers but in this paper, in order to make our presentation more unified, we trim this topology to \(\mathbb{N}\) and this topology is again Hausdorff but not regular. Kirch’s topology was defined in [3] and this topology is again Hausdorff but not regular. Kirch’s topology is weaker than Golomb’s topology and it is locally connected, as opposed to Golomb’s topology. Moreover, Rizza in [6] introduced the division topology. In [9] the author defined the common division topology \(T\) on \(\mathbb{N}\), stronger than the division topology \(T'\). Both topologies \(T\) and \(T'\) are \(T_0\), they are not \(T_1\), and they are connected—however, the common division topology \(T\) is not locally connected, as opposed to the division topology \(T'\). Let us notice that all these topologies have recently been studied by P. Szyszkwowska née Szczuka, e.g., in [7,8,10].

An ideal on \(\mathbb{N}\) is a family of subsets of \(\mathbb{N}\) closed under taking finite unions and subsets of its elements. We assume that an ideal is proper and contains all finite sets. Obviously, in any \(T_I\) topology without isolated points, the nowhere dense sets form an ideal. For each topology defined above, consider the ideal of nowhere dense sets: \(I_G, I_K, I_F, I_S, I_R\) in Golomb’s, Kirch’s, Furstenberg’s, the common division, and Rizza’s topology, respectively.

### 2 Results

In [5] the authors examined properties of the ideals \(I_G, I_K,\) and \(I_F\) and they asked if it is true that \(I_G \setminus (I_K \cup I_F) \neq \emptyset\) ([5, Problem 2.10]). It turns out that it is true. In the proof of the proposition below we present a simple solution of this problem.

**Proposition 2.1** \(I_G \setminus (I_K \cup I_F) \neq \emptyset\).

**Proof** We construct the set \(Ex\) as in the proof of [5, Theorem 2.9]. Namely, define

\[
C = \{\{an + b\} : B_K : \{an + b\} \subset \{2n + 1\}\}.
\]

Let \(\{C_k : k \in \mathbb{N}\}\) be an enumeration of \(C\). The set \(Ex = \{x_k : k \in \mathbb{N}\}\) is constructed as follows: for every \(k \in \mathbb{N}\) we pick \(x_k\) such that \(x_k \in C_k\) and \(x_k \in \{2^k n + 1\}\). In the proof of
[5, Theorem 2.9] it was shown that \(Ex \in (I_G \cap I_F)\setminus I_K\). Now let

\[ X = \{2n\} \cup Ex. \]

By [5, Example 2.6], \(\{2n\} \in I_G \setminus I_F\). So, \(X \in I_G\) as the sum of two nowhere dense sets in Golomb’s topology, but \(X \notin I_K\) since \(Ex \notin I_K\) and \(X \notin I_F\) since \(\{2n\} \notin I_F\). \(\square\)

The diagram below is described in [5]. Our recent example (Proposition 2.1) can be seen in this diagram describing the relations between the ideals \(I_K, I_G\) and \(I_F\). Hence, we finally completed the diagram.

In this diagram, \(Ex\) is the set constructed in [5, Theorem 2.9].

In the next part of this paper we will mainly focus on the relationships between the ideals \(I_R, I_S\) and \(I_F\). However, some relations between all five ideals will also be examined. At first, we present a proposition showing that Furstenberg’s, Rizza’s and the common division ideal are not disjoint.

**Proposition 2.2** There is an infinite set in \(I_R \cap I_S \cap I_F\).

**Proof** Clearly

- Primes \(\in I_S\) ([13, p. 97]),
- Primes \(\in I_R\) since \(\text{int}_{T'}(\text{cl}_{T'}(\text{Primes})) = \text{int}_{T'}(\text{Primes} \cup \{1\}) = \emptyset\),
- Primes \(\in I_F\) ([5, Example 2.2]).

\(\square\)

Moreover,

\[(I_F \cap I_S \cap I_R) \setminus (I_G \cup I_K) \neq \emptyset\]

since Primes \(\notin I_G \cup I_K\) ([5, Example 2.2]).

The next propositions will show that the ideals \(I_R, I_S\) and \(I_F\) differ significantly.

**Proposition 2.3** \((I_R \cap I_S) \setminus I_F \neq \emptyset\).

**Proof** We will show \(\{2n + 1\} \in (I_R \cap I_S) \setminus I_F\). First observe that \(\{2n + 1\} \notin I_F\). Moreover,

- \(\text{int}_{T'}(\text{cl}_{T'}(\{2n + 1\})) = \text{int}_{T'}(\{2n + 1\}) = \emptyset\) (we clearly have \(\forall_{U \in T'} U \cap \{2n\} \neq \emptyset\), which implies \(\forall_{U \in T'} U \setminus \{2n + 1\} \neq \emptyset\)).
- \(\text{int}_{T}(\text{cl}_{T}(\{2n + 1\})) = \text{int}_{T}(\{2n + 1\}) = \emptyset\) since

**Claim 2.4** \(\forall_{U \in T} U \cap \{2n\} \neq \emptyset\).
Indeed, if \( U \subseteq T \setminus \{ \emptyset \} \), then we can find \( \{ an + b \} \subseteq U \) such that \( \Theta(a) \subseteq \Theta(b) \). Let us consider the following cases:

- **a, b even**: then \( \{ an + b \} \cap \{ 2n \} \neq \emptyset \),
- **a, b odd**: then \( 2((a + b)) \) so again \( \{ an + b \} \cap \{ 2n \} \neq \emptyset \),
- **a even; b odd**: impossible, since \( \Theta(a) \subseteq \Theta(b) \),
- **a odd; b even**: then \( 2(2a + b) \), hence \( \{ an + b \} \cap \{ 2n \} \neq \emptyset \).

So, \( \forall U \subseteq T \setminus \{ 2n + 1 \} \neq \emptyset \). Consequently, \( \{ 2n + 1 \} \subseteq I_R \cap I_S \).

Moreover,

\[
(I_R \cap I_S) \setminus (I_F \cup I_G \cup I_K) \neq \emptyset.
\]

This follows from \( \{ 2n + 1 \} \subseteq D \) and \( \{ 2n + 1 \} \subseteq D' \). Therefore \( \{ 2n + 1 \} \notin I_G \) and \( \{ 2n + 1 \} \notin I_K \).

**Remark 2.5** If \( A \notin I_R \), then \( A \) is dense in \( (\mathbb{N}, T') \).

**Proof** If \( A \notin I_R \), then \( \text{int}_T(\text{cl}_{T'}(A)) \neq \emptyset \). So \( \{ an \} \subseteq \text{cl}_{T'}(A) \). By [12, Corollary 6.5], \( \mathbb{N} = \text{cl}_{T'}(\{ an \}) \subseteq \text{cl}_{T'}(A) \).

**Corollary 2.6** (Independently proved in [4]) *In the division (Rizza’s) topology all sets are either dense or nowhere dense.*

**Proposition 2.7** \( I_R \setminus (I_S \cup I_F) \neq \emptyset \).

**Proof** We will show \( \{ 4n + 2 \} \in I_R \setminus (I_S \cup I_F) \). Indeed, since \( \{ 4n + 2 \} \) is open in \( T \) and in \( I_F, \{ 4n + 2 \} \notin I_S \cup I_F \). By Remark 2.5, either \( \{ 4n + 2 \} \in I_R \) or \( \{ 4n + 2 \} \) is dense in \( (\mathbb{N}, T') \) but it cannot be dense as \( \{ 4n + 2 \} \cap \{ 4n \} = \emptyset \) and \( \{ 4n \} \in T' \).

Moreover,

\[
(I_R \cap I_G \cap I_K) \setminus (I_S \cup I_F) \neq \emptyset.
\]

Since \( I_K \subseteq I_G \) ([5, Theorem 2.5]), it is sufficient to show \( \{ 4n + 2 \} \in I_K \). By [11, Theorem 4.3], \( \text{cl}_{T'}(\{ 4n + 2 \}) = \{ 2n \} \), hence

\[
\text{int}_{T'}(\text{cl}_{T'}(\{ 4n + 2 \})) = \text{int}_{T'}(\{ 2n \}) = \emptyset.
\]

Indeed, if \( \text{int}_{T'}(\{ 2n \}) \neq \emptyset \), then there would exist \( \{ an + b \} \in B_K \) such that \( \{ an + b \} \subseteq \{ 2n \} \).

Consider the following cases:

- **a even**: then \( \{ an + b \} \subseteq \{ 2n + 1 \} \) (notice that \( (a, b) = 1 \)), a contradiction,
- **a even; b odd**: then \( a + b \notin \{ 2n \} \), impossible,
- **a, b even**: then \( 2a + b \notin \{ 2n \} \), impossible.

To prove the next proposition we will need the characterization of closed sets in the common division topology.

**Remark 2.8** There is a characterization of the closure of a set in the division topology, namely:

\[
\text{cl}_{T'}(A) = \bigcup_{a \in A} D(a) \quad \text{(see [6]).}
\]

Let us formulate an analogous characterization for the common division topology:

**Theorem 2.9** \( \text{cl}_{T}(A) = \{ x \in \mathbb{N} : \exists k \in \mathbb{N} \exists a_k \in A \ a_k \equiv x \pmod{x^k} \} \).
Proof Write $B = \{ x \in \mathbb{N} : \forall k \in \mathbb{N} \exists a_k \in A \ a_k = x \pmod{x^k} \}$. We have to prove $B = \text{cl}_T(A)$. Observe that $x \in B$ is equivalent to $\forall k \in \mathbb{N} \exists a_k \in A \ a_k \in \{ x^k n + x \}$ which in turn is equivalent to $\forall k \in \mathbb{N} \ A \cap \{ x^k n + x \} \neq \emptyset$.

If $x \notin B$, then $\exists k \in \mathbb{N} \ A \cap \{ x^k n + x \} = \emptyset$. Since $x \in \{ x^k n + x \}$ and $\{ x^k n + x \} \in T$, we have $x \notin \text{cl}_T(A)$.

Now suppose $x \in B$. If $x = 1$, then by [9, Proposition 3.1] $x \in \text{cl}_T(A)$. So, we can suppose $x \neq 1$. Let us choose $U \in T$ such that $x \in U$. By [10, Lemma 3.1], there exists $\{ cn + x \} \in B_T$ such that $\{ cn + x \} \subseteq U$. Since $\{ cn + x \} \in B_T$, we have $\Theta(c) \subseteq \Theta(x)$. Let $x = p_1^{a_1} \cdots p_m^{a_m}$ be a prime factor decomposition of $x$. Since $\Theta(c) \subseteq \Theta(x)$, without loss of generality we may assume that $c = p_1^{\beta_1} \cdots p_m^{\beta_m}$ is a prime factor decomposition of $c$, where $1 \leq i \leq m$. Let $k = \Pi_{j=1}^{m} \beta_j$. Then $c | x^k$, hence $\{ x^k n + x \} \subseteq \{ cn + x \}$. But since $\{ x^k n + x \} \cap A \neq \emptyset$, we have $U \cap A \supseteq \{ cn + x \} \cap A \supseteq \{ x^k n + x \} \cap A \neq \emptyset$. Thus, $x \in \text{cl}_T(A)$.

Corollary 2.10 $x \in \text{cl}_T(A)$ if and only if $\forall k \in \mathbb{N} \ A \cap \{ x^k n + x \} \neq \emptyset$.

Corollary 2.11 If $A \subseteq \mathbb{N}$ is a finite set, then $\text{cl}_T(A) = A \cup \{ 1 \}$.

Proof Let $x \notin A \cup \{ 1 \}$. Since $A$ is a finite set, $\exists k \in \mathbb{N} \forall a \in A \ x^k > a$. Then obviously $\{ x^k n + x \} \cap A = \emptyset$. By Corollary 2.10, $x \notin \text{cl}_T(A)$.

On the other hand, $A \subseteq \text{cl}_T(A)$ and since $\forall k \in \mathbb{N} \ A \cap \{ 1^k n + 1 \} = A \cap \mathbb{N} = A \neq \emptyset$, again by Corollary 2.10, $1 \in \text{cl}_T(A)$.

Proposition 2.12 $(I_S \cap I_F) \setminus I_R \neq \emptyset$.

Proof Let $A = \{ n! : n \in \mathbb{N} \}$. By [5, Example 2.1], $A \in I_F$. Now we will show $A \notin I_R$. Observe that $\text{cl}_T(A) = \mathbb{N}$. Indeed, fix $x \in \mathbb{N}$ and define $a = x!$. Then, by Remark 2.8, $x \in D(a) \subseteq \bigcup_{a' \in A} D(a') = \text{cl}_T(A)$. Therefore $A \notin I_R$.

Finally, we have to show $A \in I_S$. At first we show that $A$ is a closed set in $(\mathbb{N}, T)$. Suppose $x \in \text{cl}_T(A) \setminus A$. By Corollary 2.10, $\forall k \in \mathbb{N} \ A \cap \{ x^k n + x \} \neq \emptyset$. With $k = 2$ we obtain $A \cap \{ x^2 n + x \} \neq \emptyset$. Define $A_1 = \{ n! : n = 1, \ldots, x^2 - 1 \}$ and $A_2 = A \setminus A_1$. Since $\text{cl}_T(A) = \text{cl}_T(A_1) \cup \text{cl}_T(A_2)$ and $x \notin A_1 = A_1 \cup \{ 1 \} = \text{cl}_T(A_1)$ (by virtue of Corollary 2.11), so $x \notin \text{cl}_T(A_2)$. Observe that $A_2 \subseteq \{ x^2 n \}$, and since $x \in \text{cl}_T(A_2)$, by Corollary 2.10, we deduce $A_2 \cap \{ x^2 n + x \} \neq \emptyset$, which is impossible, since $\{ x^2 n \} \cap \{ x^2 n + x \} = \emptyset$.

Next, $\text{int}_T(\text{cl}_T(A)) = \text{int}_T(A) = \emptyset$, which yields $A \in I_S$.

Moreover, we obtain

$$(I_S \cap I_F \cap I_G \cap I_K) \setminus I_R \neq \emptyset.$$}

This follows from [5, Example 2.1], where it was shown that $A = \{ n! : n \in \mathbb{N} \} \in I_G \cap I_K$.

Proposition 2.13 $I_S \setminus (I_R \cup I_F) \neq \emptyset$.

Proof Let $A = \{ n! : n \in \mathbb{N} \}$ and $X = A \cup \{ 2n + 1 \}$. Then $X \in I_S$, since $A \in I_S$ and $\{ 2n + 1 \} \in I_S$. Observe that $X \notin I_R$ since $A \notin I_R$ and $X \notin I_F$ since $\{ 2n + 1 \} \notin I_F$ (cf. the proofs of Proposition 2.12 and Proposition 2.3).

Moreover,

$I_S \setminus (I_F \cap I_G \cap I_K \cap I_R) \neq \emptyset$.

This follows from $\{ 2n + 1 \} \notin I_K \cup I_G$.

All relations among the ideals $I_S, I_R$ and $I_F$ proven in this article can be seen in the following diagram:
Problem 2.14 Is it true that $({I}_R \cap {I}_F) \setminus {I}_S \neq \emptyset$?

Problem 2.15 Is it true that ${I}_F \setminus ({I}_R \cup {I}_S) \neq \emptyset$?

Finally, note that if the answer to Problem 2.14 is positive, then the answer to Problem 2.15 will also be positive.

Let us end the article with a result about the splitting property of the common division topology.

Namely, by Proposition 1.1 from [5] we know that any base for any Hausdorff topology without isolated points has the splitting property. However, the division and the common division topology is not Hausdorff, therefore it is natural to ask whether these two topologies have the splitting property. We obtain

Proposition 2.16 The family $B_T$ has the splitting property.

Proof Suppose $\{an + b\} \in B_T$. Then $\Theta(a) \subseteq \Theta(b)$. Consider two cases:

\[
\begin{align*}
a > 1: & \quad \text{Define } F_1 = \{a^2n + b\} \text{ and } F_2 = \{a^2n + (a + b)\}. \\
a = 1: & \quad \text{Define } F_1 = \{(b + 1)^2n\} \text{ and } F_2 = \{(b + 1)^2n + (b + 1)\}.
\end{align*}
\]

Then in both cases $F_1, F_2 \in B_T$, $F_1 \cap F_2 = \emptyset$ and $F_1 \cup F_2 \subseteq \{an + b\}$. \quad \square

On the other hand, the family $B_T'$ does not have the splitting property, since $ab \in \{an\} \cap \{bn\}$.

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