Measurements of the Hubble constant and cosmic curvature with quasars: ultra-compact radio structure and strong gravitational lensing

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Although the Hubble constant \(H_0\) and spatial curvature \(\Omega_K\) have been measured with very high precision, they still suffer from some tensions. In this paper, we propose an improved method to combine the observations of ultra-compact structure in radio quasars and strong gravitational lensing with quasars acting as background sources to determine \(H_0\) and \(\Omega_K\) simultaneously. By applying the distance sum rule to the time-delay measurements of 7 strong lensing systems and 120 intermediate-luminosity quasars calibrated as standard rulers, we obtain stringent constraints on the Hubble constant \(H_0 = 78.3 \pm 2.9 \text{ km s}^{-1} \text{ Mpc}^{-1}\) and the cosmic curvature \(\Omega_K = 0.49 \pm 0.24\). On the one hand, in the framework of a flat universe, the measured Hubble constant \(H_0 = 73.6^{+1.8}_{-1.0} \text{ km s}^{-1} \text{ Mpc}^{-1}\) is strongly consistent with that derived from the local distance ladder, with a precision of 2%. On the other hand, if we use the local \(H_0\) measurement as a prior, our results are marginally compatible with zero spatial curvature \(\Omega_K = 0.23^{+0.12}_{-0.15}\) and there is no significant deviation from a flat universe. Finally, we also evaluate whether strongly lensed quasars would produce robust constraints on \(H_0\) and \(\Omega_K\) in the non-flat and flat \(\Lambda\)CDM model, if the compact radio structure measurements are available from VLBI observations.

I. INTRODUCTION

The Hubble constant \(H_0\), which sets the present expansion rate of the Universe, has been one of the most important cosmological parameters one attempted to measure. Although \(H_0\) has been measured with very high precision, these measurements are currently in tension with each other. In the framework of the flat \(\Lambda\) cold dark matter (\(\Lambda\)CDM) cosmological model which is proven to be well consistent with various observations [1–4], the Planck satellite provides a stringent constraint on the Hubble constant as \(H_0 = 67.4 \pm 0.5 \text{ km s}^{-1} \text{ Mpc}^{-1}\), based on the anisotropies of the Cosmic Microwave Background Radiation (CMBR) [5]. The other independent estimates of \(H_0\) have been obtained by local type Ia supernovae (SNe Ia) calibrated via the distance ladder, with the value of \(H_0 = 74.03 \pm 1.42 \text{ km s}^{-1} \text{ Mpc}^{-1}\) recently released by SH0ES (SNe, H0, for the Equation of State of dark energy) collaboration [6]. However, such significant 4.4 \(\sigma\) tension can not be solely attributed to systematics errors [6–8]. Therefore, any other independent way to measure the Hubble constant is of utmost importance.

As early as in 1964, it has been proposed [9] that observations of time delays from the gravitationally lensed supernovae could be used to determine \(H_0\). Even though, the lensed supernovae have eventually been detected [10, 11], the sample is far too small to contribute in solving of \(H_0\) tension. In fact, more promising candidates in this context are quasars. Due to the brightness and variable nature of quasars, some recent works focused on the lensed quasars to measure the so-called "time-delay distance" \((D_{\Delta t})\), a combination of three angular diameter distances between the observer, lens, and source sensitive to \(H_0\). More recently, the HOLICOW (\(H_0\) Lenses in COSMOGRAIL’s Wellspring) collaboration showed that real observations of time delays for six lensed quasars can be used to obtain the Hubble constant \(H_0 = 73.3^{+1.7}_{-1.3} \text{ km s}^{-1} \text{ Mpc}^{-1}\) in the spatially flat \(\Lambda\)CDM model [12]. The result is in perfect agreement with the local measurements of \(H_0\) from the SH0ES collaboration. However, the drawback of this method is that the fits on \(H_0\) are strongly model dependent, i.e., the value of \(H_0\) would shift to \(81.6^{+7.9}_{-5.3} \text{ km s}^{-1} \text{ Mpc}^{-1}\) when the equation of state (EoS) parameter of dark energy is treated as a free parameter.

In fact, the \(H_0\) tension suggests the possibility that there could be an inconsistency between the early-universe and late-universe in modern cosmological theories. Moreover, the recent studies [13–15] of the spatial curvature parameter \(\Omega_K\) also highlighted inconsistency. A higher lensing amplitude, \(A_{\text{lens}}\) in the CMB power spectra has been confirmed by the recent Planck 2018 results [5]. Closed Universe can provide a physical explanation for this effect, resulting with \(\Omega_K = -0.044^{+0.015}_{-0.015}\) constrained from the combination of the Planck temperature and polarization power spectra [5]. However, combining the Planck lensing with low-redshift baryon acoustic oscillations (BAO), changes this constraint to \(\Omega_K = 0.0007 \pm 0.0019\) precisely fitting the flat Universe as it was expected by the inflation theory. In further analysis, Di Valentino et al. [13] showed that under a closed Universe preferred by Planck, higher than generally estimated discordances arise for most of the local cosmological observables. However, it should be stressed here that this method also makes a strong assumption based on some specific dark energy model, i.e., the non-flat \(\Lambda\)CDM model. To better understand the discrepancy between the Hubble constant and cosmic curvature
measured locally and the value inferred from the Planck survey, fully model-independent measurements of $H_0$ and $\Omega_K$ are still required. For the discussions about $H_0$ tension, we refer the reader the following works [16–23].

In particular, the distance sum rule [24] provides an effective approach to determine the spatial curvature and the Hubble constant but without adopting any particular model. In the framework of such a theoretical framework, great efforts have been made in the recent studies to set limits on the cosmic curvature [25–29], based on the precise measurements of the source-lens/observer distance ratio in strong gravitational lensing (SGL) systems [4, 30] and luminosity distances in other different distance indicators [31, 32]. However, such methodology is weakly dependent on the Hubble constant, which enters not through a distance measure directly, but rather through a distance ratio [33]. Based on the precise measurements of time delays between multiple images, the first attempt to determine the spatial curvature and Hubble constant with SN Ia luminosity distances was presented in [34]. This approach was then extended by using another luminosity distance probe, i.e., the nonlinear relation between the ultraviolet (UV) and X-ray monochromatic luminosities satisfied by high-redshift quasars [35].

However, one should remember that for the implementation of the distance sum rule, it would be beneficial to use distance probes covering higher redshifts thus taking advantage of a larger sample of SGL systems. More importantly, the cosmological application of time delay measurements requires good knowledge of three angular diameter distances of the lensed quasar system (from observer to lens, from observer to source, and from lens to source). Therefore, a new promising window of opportunity would open up if we could extend the $H_0$ and $\Omega_K$ measurements by using a new, deeper astronomical probes acting as standard rulers in a redshift range well consistent with the lensed quasars [31].

In this paper, we will further study $H_0$ and $\Omega_K$ using the angular size of compact structure in radio quasars as standard rulers and the time delay from lensed quasars. By virtue of the distance sum rule, a cosmological model-independent analysis becomes possible. In order to investigate the influence of the cosmological model on the constraints of $H_0$ and $\Omega_K$, we will also perform a comparative analysis of the non-flat and flat $\Lambda$CDM models.

II. METHODOLOGY AND OBSERVATIONS

In the homogeneous and isotropic universe, the FLRW metric is applied to describe its spacetime:

$$ds^2 = c^2 dt^2 - \frac{a(t)^2}{1 - K r^2} dr^2 - a(t)^2 r^2 d\Omega^2,$$  \hspace{1cm} (1)

where $a(t)$ represents the scale factor and $c$ is the speed of light. Note that the cosmic curvature parameter $\Omega_K$ is determined by the dimensionless curvature $K$ and the Hubble constant $H_0$ as $\Omega_K = -K c^2/a_0^2 H_0^2$. In this analysis we focus on a specific strong lensing system with the background quasar (at redshift $z_s$) as the source and the early-type galaxy (at redshift $z_l$) acting as a lens. By introducing dimensionless comoving distances $d_{ls} \equiv d(z_l, z_s)$, $d_l \equiv d(0, z_l)$ and $d_s \equiv d(0, z_s)$, these three types of distances are connected as

$$\frac{d_{ls}}{d_s} = \sqrt{1 + \Omega_K d_l^2} - \frac{d_l}{d_s} \sqrt{1 + \Omega_K d_l^2}. \hspace{1cm} (2)$$

by virtue of the well-known distance sum rule in non-flat FLRW models. The original idea of cosmological application of the distance sum rule in general, and with respect to gravitational lensing data in particular, can be traced back to the paper of Ref. [24], which has been extensively discussed in more recent papers focused on testing the spatial curvature of the Universe [31] and thus the validity of the FLRW metric [25, 30]. Furthermore, one is able to rewrite Eq. (2) as

$$\frac{d d_s}{d_{ls}} = \frac{1}{\sqrt{1/d_l^2 + \Omega_K} - \sqrt{1/d_s^2 + \Omega_K}}. \hspace{1cm} (3)$$

Let us note that the dimensionless comoving distances $d_l$ are related to the (dimensioned) comoving distances $D_l = H_0 D/c$.

Time-delay measurements from lensed quasars.— In the framework of a strong lensing system with the background quasar as the source and the early-type galaxy acting as a lens, one of the typical feature is that time delays between lensed images ($\theta_i$ and $\theta_j$) are dependent on the time-delay distance ($D_{\Delta t} \equiv (1 + z_l) \frac{D_l c}{D_s}$) and the gravitational potential of the lensing galaxy as

$$\Delta t_{i,j} = \frac{D_{\Delta t}}{c} \Delta \phi_{i,j}, \hspace{1cm} (4)$$

where $\Delta \phi_{i,j} = [(\theta_i - \beta)^2/2 - \psi(\theta_i) - (\theta_j - \beta)^2/2 + \psi(\theta_j)]$ is the Fermat potential difference determined by the two-dimensional lensing potential $\psi$ and the source position $\beta$. Therefore, the time-delay distance, i.e., the combination of three angular angular diameter distances, could be rewritten as

$$D_{\Delta t} = \frac{c \Delta t_{i,j}}{\Delta \phi_{i,j}} = \frac{c d d_s}{H_0 d_{ls}}. \hspace{1cm} (5)$$

As can be clearly seen from Eq. (3) and (5), if the other two dimensionless comoving distances $d_l$ and $d_s$ can be measured, then the value of $H_0$ and $\Omega_K$ could be determined directly from the time-delay measurements of $\Delta t$ and well-reconstructed lens potential of $\Delta \phi$, without involving any specific cosmological model.

For the source of available quasar-galaxy lensing systems, we use the latest sample of strong-lensing systems with time delay observations, recently released by the H0LiCOW collaboration and the STRIDES collaboration with precise time-delay distance measurement for each lensing system [12, 37]. The seven lenses with measured
time delays consist of the following systems: B1608+656 \cite{38, 39}, RXJ1131-1231 \cite{40-42}, HE 0435-1223 \cite{42, 43}, 1206+4332 \cite{44}, WFT2033-4723 \cite{45}, PG 1115+080 \cite{42}, and DES J0408-5354 \cite{37}, while the source redshift covers the range of 0.654 < z < 2.375. The relevant information necessary to perform statistical analysis, including the redshifts of both lens and source, as well as the posterior distributions of the time-delay distances in the form of Monte Carlo Markov chains (MCMC) are summarized in Ref. \cite{12}. We remark here that a kernel density estimator was used to compute the posterior distributions of five lenses (L_{\Delta t}), while the D_{\Delta t} likelihood function for the lens B1608+656 was given as a skewed log-normal distribution, due to the absence of blind analysis with respect to the cosmological quantities of interest.

**Distance calibration from unlensed radio quasars.**—From the observational point of view, to obtain model-independent measurements of the distance d(z), one can turn to the objects of known (or standardizable) comoving size acting as standard rulers. In this paper, with the aim of deriving the distances to the lens and the source corresponding to their redshifts, we focus on the angular size of the compact radio structure in radio quasars, based on the very-long-baseline interferometry (VLBI) observations \cite{46}. After refining their selection technique and redshift measurements, Cao et al. \cite{47, 48} collected a final sample of 120 intermediate-luminosity radio quasars with reliable measurements of the angular size of the compact structure. Such recently compiled milliarcsecond compact radio-quasar catalog covering the redshift range 0.46 < z < 2.76 will be used for the analysis performed in this paper. The angular size of the compact structure in radio quasars, θ(z), can be expressed in terms of the angular diameter distance and the linear size of the standard ruler as

\[ \theta(z) = \frac{l_m}{D_A(z)}. \]

The angular diameter distance D_A(z) can then be extracted from the angular size of the compact radio quasars θ(z), combined with the linear size of the standard ruler calibrated to \( l_m = 11.0 \pm 0.4 \) pc through a new cosmology-independent technique (the well-measured angular diameter distances from the BAO) \cite{49}. Now for each lensing system, one can obtain the dimensionless distances to the lenses \( d_l \) and to the sources \( d_s \) from angular diameter distances to the quasars as: \( d_l = H_0/c(1 + z_l)D_A(z_l) \) and \( d_s = H_0/c(1 + z_s)D_A(z_s) \), respectively. Considering uncertainties of the angular size measurements, instead of matching objects by redshift, we decided to use quasars for reconstructing the dimensionless co-moving distance function \( d(z) \) parameterized by a third-order polynomial

\[ d(z) = z + a_1z^2 + a_2z^3, \]

with the initial conditions of \( d(0) = 0 \) and \( d'(0) = 1 \). The radio quasar sample is sufficient to reconstruct the profile of \( d(z) \) up to the redshifts \( z \sim 3 \), without confining ourselves to any specific cosmology \cite{31}. Note that the two coefficients \( (a_1, a_2) \) in a third-order polynomial will be optimized along with the Hubble constant \( (H_0) \) and cosmic curvature \( (\Omega_K) \). For the radio quasar sample, the posterior likelihood \( \mathcal{L}_{QSO} \sim \exp(-X_{QSO}^2/2) \) is constructed through the following formula of

\[ X_{QSO}^2 = \sum_{i}^{120} \frac{(\theta(z_i; d(z)) - \theta_{oi})^2}{\sigma_i^2}, \]

where \( \theta_{oi} \) is the observed angular size for the \( ith \) quasar with uncertainty of \( \sigma_i \). Following the error strategy proposed in Ref. \cite{47}, an additional 10% systematical uncertainty in the observed angular sizes is also assumed in computing \( \mathcal{L}_{QSO} \), in order to account for the intrinsic spread in linear sizes \cite{46}.

**III. RESULTS AND DISCUSSION**

Implementing Python MCMC module EMCEE \cite{50}, \( \Omega_K \) and \( H_0 \) parameters are determined by maximizing

**FIG. 1**: Constraints on the parameters \( H_0, \Omega_K, a_1 \) and \( a_2 \) with strong lensing systems and radio quasars, in the framework of distance sum rule.

**TABLE I**: Results for \( H_0, \Omega_K \) and the parameters of polynomial \( a_1, a_2 \) in the framework of distance sum rule.

| \( H_0 \) [km s\(^{-1}\) Mpc\(^{-1}\)] | \( \Omega_K \) | \( a_1 \) | \( a_2 \) |
|---|---|---|---|
| 78.3 ± 2.9 | 0.49 ± 0.24 | -0.364 ± 0.037 | 0.068 ± 0.014 |
| 73.6^{+0.8}_{-1.0} | 0 (fixed) | -0.404 ± 0.030 | 0.077 ± 0.013 |
| 74.03 (fixed) | 0.23^{+0.17}_{-0.17} | -0.412 ± 0.018 | 0.084 ± 0.010 |
under assumption of zero spatial curvature — supported by other improved model-independent methods referring to a distant past [49], we get stringent constraints on the Hubble constant as $H_0 = 73.6^{+1.8}_{-1.6}$ km s$^{-1}$ Mpc$^{-1}$, with the corresponding marginalized probability distribution presented in Fig. 2. In the framework of a non-flat and flat Universe, the results for the lensed + unlensed quasar data are summarized in Table I. Therefore, our model-independent $H_0$ constraints are well consistent with the recent determinations of $H_0$ from the Supernovae $H_0$ for the SH0ES collaboration [6], which is the most unambiguous result of the current lensed + unlensed quasar dataset. Let us note that, at the current observational level, quasars may achieve model-independent $H_0$ measurements at much higher redshifts (which is especially important in cosmology), compared with other popular astrophysical probes (including SNe Ia) adopted as distance indicators for providing the distance $d(z)$ [34].

Cosmic curvature $\Omega_K$ is a fundamental parameter for cosmology. In this paper, we have focused on model-independent measurement of $\Omega_K$ by applying distance sum rule to the time-delay measurements of 7 strong lensing systems and 120 intermediate-luminosity quasars calibrated as standard rulers. With the prior of $H_0 = 74.03$ km s$^{-1}$ Mpc$^{-1}$ determined locally via the distance ladder, our final assessment of the cosmic curvature with corresponding $1\sigma$ uncertainty is $\Omega_K = 0.23^{+0.13}_{-0.17}$. The results are shown in Fig. 2, which suggests that there is no significant signal indicating the deviation of the cosmic curvature $\Omega_K$ from zero (spatially flat geometry). Such unambiguous result of the available quasar observations is also in agreement with the recent analysis focusing on the source-lens/lens distance ratio in strong lensing systems [25, 27, 29] and theoretical Hubble diagram reconstructed by the Hubble parameter measurements [51–53].
in the framework of other model-independent $\Omega_K$ test. The constraining power of our method is more obvious when the large size difference between the samples is taken into consideration.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
 & $H_0$ [km s$^{-1}$ Mpc$^{-1}$] & $\Omega_K$ & $\Omega_m$ \\
\hline
Non-flat $\Lambda$CDM & $72.5 \pm 1.6$ & $-0.09^{+0.13}_{-0.15}$ & $0.287 \pm 0.055$ \\
Flat $\Lambda$CDM & $73.1^{+1.3}_{-1.3}$ & $-0.254^{+0.027}_{-0.025}$ & \\
\hline
\end{tabular}
\caption{Fitting results (68.3\% confidence level) for the open $\Lambda$CDM and flat $\Lambda$CDM.}
\end{table}

Let us remark on two aspects. Firstly, from the observational point of view, one can see that the 120 intermediate-luminosity quasars have perfect coverage of source redshifts in 7 SGL systems ($z \sim 3$), acting as distance indicators for providing the distance $d(z)$ on the right side of Eq. (3). Therefore, we propose to investigate the constraints on $H_0$ and $\Omega_K$ from the SGL time delay data and radio quasars. Secondly, the constraint result of $H_0$ from H0LiCOW is strongly dependent on cosmological models. In order to investigate how sensitive our results on $H_0$ and $\Omega_K$ are on the choice of cosmological model, we also perform a comparative analysis of the current lensed + unlensed quasar dataset in the non-flat and flat $\Lambda$CDM model. The results of our parameter estimation computations are summarized in Table II, and with $1\sigma$ and $2\sigma$ confidence level contours are shown in Fig. 3-4. From the combined analyses with lensed + unlensed quasar dataset, we find that the Hubble constant and the spatial curvature density parameter are constrained to be $H_0 = 72.5 \pm 1.6$ km s$^{-1}$ Mpc$^{-1}$ and $\Omega_K = -0.09^{+0.13}_{-0.15}$. In order to check the constraining power of the lensed + unlensed quasar data on the Hubble constant, we chose to assume zero spatial curvature and obtain $H_0 = 73.1^{+1.3}_{-1.3}$ km s$^{-1}$ Mpc$^{-1}$ in a flat universe. Now it is worthwhile to make some comments on the results obtained above. Firstly, it is interesting – and might even be significant – that the $H_0$ and $\Omega_K$ constraints listed in Table II are quite consistent with estimates for these parameters from most other data. More specifically, in broad terms, the estimated values of $H_0$ are in agreement with the standard ones reported by the SH0ES collaboration [6]. Secondly, let us note that for the flat and non-flat $\Lambda$CDM models, the matter density parameter are fitted to $\Omega_m = 0.287 \pm 0.055$ and $\Omega_m = 0.254^{+0.027}_{-0.025}$, respectively. Compared with Planck fitting results [5], the best-fit values for the present density parameters will considerably be improved, with the help of the quasar observations.

In this paper, we focused on the idea of constraining $H_0$ and $\Omega_K$ by using the observations of quasars: ultra-compact structure in intermediate-luminosity radio quasars from the very-long-baseline interferometry (VLBI) observations [46-48, 54-56] and the time-delay measurements of strong lensing systems with quasars acting as background sources [12]. Providing a better redshift coverage of SGL systems, our method opens a new possibility to quantitatively analyze current tensions concerning the values of the Hubble constant and the curvature parameter with multiple measurements of lensed and unlensed high-redshift quasars. Note that there are many potential ways, in which our technique might be improved by the discoverable lens population in future surveys such as the Dark Energy Survey (DES), Vera C. Rubin Observatory (Legacy Survey of Space and Time – LSST), and Euclid. For instance, LSST would enable the discovery of 3000 lensed quasars in the most optimistic discovery scenario, with precise measurements of time delays between multiple images [57]. Our method could also be extended to the SNe Ia-galaxy strong-lensing systems with exceptionally well characterized spectral sequences and thus very accurate time delays measured in lensed SNe Ia [58]. Following the recent analysis of the likely yields of LSST [59], there are 650 multiply imaged SNe Ia that could provide precise time delays with supplementary data points on their light curves. On the other hand, benefit from more recent VLBI imaging observations based on better UV-coverage [60], both current and future VLBI surveys will discover a large amount of intermediate-luminosity radio quasars, with the angular sizes of the compact structure observed at different frequencies [61]. In summary, the approach introduced in this paper offers a new model-independent way of simultaneously constraining both $H_0$ and $\Omega_K$ at much higher accuracy, with future surveys of strongly lensed quasars and high-quality radio astronomical observations of quasars [62].
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