Quantum Kramers–Wannier Duality And Its Topology

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Abstract

We show for any oriented surface, possibly with a boundary, how to generalize Kramers–Wannier duality to the world of quantum groups. The generalization is motivated by quantization of Poisson–Lie T-duality from the string theory. Cohomologies with quantum coefficients are defined for surfaces and their meaning is revealed. They are functorial with respect to some gluing operations and connected with q-invariants of 3-folds.

1 Introduction and Summary

The original motivation for this work was to gain some understanding of Poisson–Lie (PL) T-duality \[1\], including zero-modes, at quantum level. Let us recall the relevant facts: Usual (i.e. abelian) T-duality emerges in the presence of an abelian group of symmetry and it is a continuous counterpart of Kramers–Wannier (KW) duality. Its generalization, PL T-duality, comes from an action of a PL group \(G\), which is not necessarily a symmetry group, but rather gives rise to a conserved charge (‘nonabelian momentum’) with values in the dual PL group \(\tilde{G}\). In the dual model, the roles of \(G\) and \(\tilde{G}\) are interchanged.

There are path integral \[2, 3, 4\] and renormalization group \[10\] arguments for PL T-duality at quantum level. However, they are far from being complete and suffer from problems with boundary conditions. In this respect, it would be nice to have a lattice counterpart of PL T-duality – quantum KW duality. We should expect emergence of quantum groups in quantization of PL T-duality that are deformations of the corresponding PL groups. It is necessary at least in the case of open strings \[7\].

The usual (abelian) KW duality is an immediate consequence of Poincaré duality between cohomology groups \(H^1(\Sigma; G)\) and \(H^1(\Sigma; \tilde{G})\). Here \(\Sigma\) is a closed oriented surface and \(G, \tilde{G}\) are mutually dual finite abelian groups. Surfaces with boundaries are also admissible, provided one uses appropriate relative cohomologies (to be specified below).

The only problem with quantum KW duality consists in defining a suitable generalization of \(H^1(\Sigma; G)\), if we replace \(G\) with a general (i.e. non-commutative and non-cocommutative) Hopf algebra, preserving Poincaré duality. In this paper, this is accomplished for finite quantum groups (finite–dimensional Hopf \(C^*\)-algebras). It is easy to define these “cohomologies” using a graph on \(\Sigma\) (via cocycles): ordering problems (coming from non-commutativity and non-cocommutativity) are absent here, because there is a natural (cyclic) ordering on the edges around a face (or running from a vertex). The real problem is to prove topological invariance of these “cohomologies”. We should understand what they really are.

Now I will try to present briefly quantum KW duality (pointing out the special abelian case) in the form that seems to be the most convenient.
Our basic objects are called neckfaces. These are compact oriented surfaces, possibly with boundary; on each boundary circle there are some black and some white points, called beads. The beads are located as on the figure, i.e. between any two black there is a white one and vice versa; the beads dissect the boundary into pieces called strings.

Now suppose we are given a finite quantum group (a finite-dimensional Hopf C*-algebra) $\mathcal{H}$. Then we will give a construction that provides a vector space $\eta(\Sigma)$ for any neckface $\Sigma$ ($\eta(\Sigma)$ is a non-zero finite-dimensional Hilbert space). It is the promised analogy of the relative cohomology group $H^1(\Sigma, B; G)$, where $B$ is the set of black beads (more precisely, it is an analogy of the vector space $C^*H^1(\Sigma, B; G)$). The correspondence $\Sigma \mapsto \eta(\Sigma)$ enjoys certain functoriality properties. We organize neckfaces into a category NEFA: morphisms (glueings) are maps $f : \Sigma_1 \to \Sigma_2$ that may glue pairs of strings of $\Sigma_1$ (white endpoints with white and black with black), but up to this they are orientation-preserving homeomorphisms. The beads of $\Sigma_2$ should be the images of the beads of $\Sigma_1$ that lie on $\partial \Sigma_2$. Now we can see $\eta$ as a functor from NEFA to the category of vector spaces. The properties of $\eta$ are described in the section 2; for now it is enough to know that $\eta(\Sigma_1 \cup \Sigma_2) = \eta(\Sigma_1) \otimes \eta(\Sigma_2)$.

To have an example in mind, the reader should read the section 3 (and without inconvenience, she or he may read the exact statements of definitions given in the section 2). The case $E = 1, F = G$ will correspond to $G$-spin models (when $\mathcal{H}$ is the group algebra $\mathbb{C}G$); if moreover $G$ is abelian, this will correspond to the usual KW duality.

The correspondence between $\mathcal{H}$ and $\eta$ is the following: let $D$ be a disk with 2 black and 2 white beads. Then $\mathcal{H} = \eta(D)$. The multiplication and comultiplication on $\mathcal{H}$ are given by two glueings $D \cup D \to D$ that give us (by functoriality) maps $\mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$; these are the multiplication and the dual multiplication (the adjoint of the comultiplication map $\mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, up to a positive factor – cf. the appendix):
the latter in the dual algebra $\tilde{\mathcal{H}}$ of functions on $G$. In the general case, one obtains the famous Hopf algebra coming from the triple $(G, E, F)$.

The other operations (unit, counit, $S$ and $*$) are also given by glueings; we just mention the antipode $S : \mathcal{H} \to \mathcal{H}$ that comes from rotation of $D$ for 180 degrees (notice that $S^2 = 1$ as it should be for finite quantum groups).

There is an important point in the way we recovered the Hopf algebra structure on $\eta(D)$: we had two maps $\mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ and one of them was claimed to be the multiplication while the other one to be the dual multiplication. The distinction was made only by the colour of the beads that were mapped inside $D$. If we made it in the other way, we would obtain another Hopf algebra structure (clearly the dual one) on the same space. This identification of $\mathcal{H}$ with $\tilde{\mathcal{H}}$ is given by the Fourier transform and this is the origin of Fourier transforms in KW duality. In our picture, KW duality will simply state that recolouring the beads corresponds to replacing $\mathcal{H}$ with $\tilde{\mathcal{H}}$. In the case of a commutative group $G$ we have $\eta(\Sigma) = \mathbb{C} H^1(\Sigma, B; G)$ ($B$ is the set of black beads); what we have said implies that this space is naturally identified with $\mathbb{C} H^1(\Sigma, W; \tilde{G})$ ($W$ is the set of white beads (that becomes $B$ after recolouring) and $\tilde{G}$ the dual group). The identification is given by Fourier transform (the groups $H^1(\Sigma, B; G)$ and $H^1(\Sigma, W; \tilde{G})$ are mutually dual via Poincaré duality).

After this preparation we can describe the ‘statistical models’ and their KW duality. Suppose we are given the surface $nD$ that is simply the disjoint union of $n$ copies of $D$ and a glueing $f : nD \to \Sigma$; for example $\Sigma$ may be a square (or a torus) with a tiling by squares coming from $f$. The model is given by specifying a Boltzmann weight $w \in \eta(nD) = \eta(D)^{\otimes n} = \mathcal{H}^{\otimes n}$. The ‘partition sum’ is an element of $\eta(\Sigma)$: it is simply $\eta(f)(w)$. If it is necessary, we may take its inner product with some $c \in \eta(\Sigma)$ to obtain a number; one may say that $c$ specifies the boundary and periodicity conditions.

The correspondence with the usual lattice models is the following: we have the surface $\Sigma$ tiled by quadrilaterals (the tiling comes from $f$); every vertex of the tiling has a colour (given by its preimage in $nD$ which is a set of beads of the same colour). We can construct two graphs on $\Sigma$: draw the diameter connecting the black beads on each $D$ in $nD$; their images form a graph $\Gamma \subset \Sigma$. If we make the same thing with the white beads, we obtain a graph $\tilde{\Gamma}$, dual to $\Gamma$. One can go also in the opposite direction, starting with $\Gamma$, making $\tilde{\Gamma}$ and finally arriving at the tiling $f$.

In the usual formulation of KW duality, there are $G$-valued spins at vertices of $\Gamma$ interacting through the edges; the dual model lives on $\tilde{\Gamma}$ and has $\tilde{G}$-valued spins (where $\tilde{G}$ is the dual of the finite abelian group $G$). Only internal spins are summed over (external spins form the boundary condition) and the spins have to be allowed to take values in an arbitrary principal $G$-bundle over $\Sigma$. The partition sum depends on these boundary and periodicity conditions. In the picture described above, we organized these sums to a single element of a vector space.

Recall that in our picture KW duality simply states that recolouring corresponds to transition to the dual Hopf algebra and that on $D$ the recolouring gives Fourier transform (recoloured $D$ is after a rotation again $D$). On $nD$ one has to make Fourier for each $D$.

There remain several things that should be added:
We presented the functor $\eta$ as coming from a finite quantum group $\mathcal{H}$ and showed how to reconstruct $\mathcal{H}$ from $\eta$. The functor $\eta$ was motivated as a generalization of cohomology to quantum coefficients. But our final formulation, given in the section 2, characterizes $\eta$ by a set of axioms (not involving Hopf algebras) and then the theorem, encompassing all our results, claims that there is a 1-to-1 (up to isomorphisms) correspondence between such functors $\eta$ and finite quantum groups (in fact it claims a little bit more -- that the two categories are equivalent).

In the PL T-duality, an important role is played by the Drinfeld double of the PL groups. It should be expected in the KW duality as well; in fact, in the case of $G$-spin models (with possibly nonabelian $G$) it was found that the order and disorder operators are organized into the Drinfeld double and also that the canonical $R$-matrix of the double gives the braid group statistics of such models [5]. The same is true generally and becomes self-evident when the double is realized as $\eta(\Sigma)$ for certain $\Sigma$ (an annulus with one black and one white bead on each boundary circle). This will be done in the section 5.

Finally, we can naturally generalize ‘statistical’ models. They were described by a glueing $f : nD \to \Sigma$ and a weight in $\eta(nD)$. Of course, it is possible to take an arbitrary glueing $f : \Sigma' \to \Sigma$ and $w \in \eta(\Sigma')$. If $\Sigma'$ is a disjoint union of disks (but this time with an arbitrary number of beads), we obtain duality for many-particle interactions (this interpretation is for $\mathcal{H} = \mathbb{C}G$). Just imagine (as we did in the case of $nD$) $\Sigma$ tiled by polygons with coloured vertices. At each black vertex there is a $G$-valued spin and at each polygon an interaction. This may be nice in renormalization, which consists just in writing the map $f : nD \to \Sigma$ as a composition $nD \to \Sigma' \to \Sigma$.

### 2 The Theorem

A neckface is a compact oriented surface $\Sigma$, possibly with a boundary, together with two finite subsets $B$ (black beads) and $W$ (white beads) of $\partial \Sigma$, such that

- $B \cap W = \emptyset$
- for each boundary circle $c$, $B \cap c \neq \emptyset$, $W \cap c \neq \emptyset$
- between any two black beads there is a white one and vice versa.

The part of $\partial \Sigma$ between two neighbouring beads is called a string.

The category NEFA has neckfaces as its objects; the morphisms (glueings) are continuous maps $f : \Sigma_1 \to \Sigma_2$ such that

- $f$ is onto
- $f|_{\text{int}\Sigma_1}$ is an orientation-preserving homeomorphism onto its image
- for any string $s \subset \Sigma_1$, either $f$ maps $s$ bijectively onto a string $s_2 \subset \Sigma_2$, preserving the colour of the endpoints, or there is another string $s' \subset \Sigma_1$ that gets glued with $s$ (white end with white and black with black)

Clearly, $f^{-1}(B_2) \subset B_1$, $f^{-1}(W_2) \subset W_1$; if $f^{-1}(B_2) = B_1$ and $f^{-1}(W_2) = W_1$, i.e. if no bead gets lost inside $\Sigma_2$, the glueing $f$ is called nice.

The object of category HIL are non-zero finite-dimensional Hilbert spaces; morphisms are linear maps. A morphism $f : H_1 \to H_2$ is projecting, if the restriction $(\ker f)^\perp \to H_2$ is unitary.

A functor $\eta : \text{NEFA} \to \text{HIL}$ is an $\eta$-functor if
1. $\eta(\Sigma) = \overline{\eta(\Sigma)}$ (\(\Sigma\) is \(\Sigma\) with the opposite orientation)

2. $\eta(\Sigma_1 \cup \Sigma_2) = \eta(\Sigma_1) \otimes \eta(\Sigma_2)$

3. for any glueing \(f\), $\eta(f)$ is projecting

4. if \(f\) is a nice glueing, $\eta(f)$ is invertible (and thus unitary)

5. (independent projection law) Let

\[
\begin{array}{c}
\Sigma_1 \\
\downarrow f_2 \\
\Sigma_2
\end{array}
\begin{array}{c}
f_3 \\
\downarrow g_2 \\
\Sigma_4
\end{array}
\begin{array}{c}
\Sigma_3 \\
\downarrow g_3 \\
\Sigma_2
\end{array}
\]

be a commutative square and let $f = g_2 \circ f_2 = g_3 \circ f_3$. Then in $\eta(\Sigma_1)$ the orthogonal projections onto $(\ker \eta(f_2))^\perp$, $(\ker \eta(f_3))^\perp$ commute. If moreover the only beads of $\Sigma_1$ that get lost inside $\Sigma_4$ are those that get lost either inside $\Sigma_2$ or in $\Sigma_3$ then

$$(\ker \eta(f_2))^\perp \cap (\ker \eta(f_3))^\perp = (\ker \eta(f))^\perp$$

(such squares will be called minimal).

**Remark:** The condition 4 could be avoided by adding to NEFA new morphisms generated by the old ones and inverses to nice glueings. These are no longer maps (they involve cutting). We will not do this because of Hauptvermutung-like problems, but heuristically it is important. At any rate, 4 is a special case of 5. Because $\eta(f)$ is projecting for any $f : \Sigma_1 \to \Sigma_2$, we may see $\eta(\Sigma_2)$ inside $\eta(\Sigma_1)$ as $(\ker \eta(f))^\perp$ and $f$ as the orthogonal projection. The condition 5 states that this is consistent inside commutative squares; if moreover the square is minimal, the total projection (corresponding to $f$ in condition 5) is simply the composition of the projections corresponding to $f_2$ and $f_3$.

Let $D$ be a disk with two black and two white beads; for any $\eta$-functor one can find a finite quantum group structure on the space $\eta(D)$ (cf. the appendix for the definition of finite quantum groups and their category FQG); this structure comes from glueing and will be described below. A part of our theorem claims that this gives us a bijective correspondence between $\eta$-functors and finite quantum groups, up to isomorphisms.

For the full statement of the theorem, we have to organize $\eta$-functors into category ETA. A morphism $\eta_1 \to \eta_2$ is a collection of linear maps $\eta_1(\Sigma) \to \eta_2(\Sigma)$ preserving the inner products (i.e. isometric injections) such that the diagram

\[
\begin{array}{c}
\eta_1(\Sigma) \\
\downarrow \eta_1(g) \\
\eta_1(\Sigma')
\end{array}
\begin{array}{c}
\eta_2(g) \\
\downarrow \eta_2(\Sigma') \\
\eta_2(\Sigma')
\end{array}
\]

commutes up to a positive factor for each glueing $g : \Sigma \to \Sigma'$. Now the map $\text{ETA} \to \text{FQG}, \eta \mapsto \eta(D)$ becomes a functor.
Theorem: The functor $\eta \mapsto \eta(D)$ is an equivalence of categories.

We have to add one thing to the statement of the theorem. For any neckface $\Sigma$, let $\Sigma^{ex}$ denote $\Sigma$ with exchanged colours of the beads. For any $\eta$-functor $\eta$ there is another $\eta$-functor $\tilde{\eta}$ given by $\tilde{\eta}(\Sigma) = \eta(\Sigma^{ex})$. Clearly, the FQG corresponding to $\tilde{\eta}$ is $\tilde{\mathcal{H}}$. This fact (that the automorphism $\eta \mapsto \tilde{\eta}$ is translated to the duality automorphism) is Kramers–Wannier duality.

Let us look at the role of the Fourier transform in KW duality. Clearly, $D^{ex}$ may be identified with $D$ after a rotation, i.e. we have a map $\mathcal{H} \rightarrow \tilde{\mathcal{H}}$. It is the Fourier transform: Consider the reflection of $D$ with respect to the north-west–south-east diameter. It is a map $D \rightarrow D^{ex}$. Using $\eta$, it gives us $\eta(D) \rightarrow \eta(D^{ex})$, i.e. $\mathcal{H} \rightarrow \tilde{\mathcal{H}}$. It exchanges multiplication with the dual one, $S$ with $S \circ \ast$, etc. So it is nothing else but the trivial map coming from the fact that $\tilde{\mathcal{H}} = \overline{\mathcal{H}}$ (cf. Appendix). Now compose this reflection with the reflection with respect to the vertical diameter (the one giving $\ast$ when mapped by $\eta$). The result is the mentioned rotation, and when mapped by $\eta$, it gives us the Fourier transform $\mathcal{H} \rightarrow \tilde{\mathcal{H}}$.

3 An Example

Let $G$ be a finite group and $E$, $F$ its subgroups such that every element $g \in G$ can be written uniquely as a product $g = ef$, $e \in E$, $f \in F$. By a $(G, E, F)$-bundle over a neckface $\Sigma$ we mean a principal $G$-bundle over $\Sigma$ with a specified lift of each string to the bundle; discontinuities at the beads are allowed but at the black beads they have to be from $E$ and at the white beads from $F$.

An automorphism of a $(G, E, F)$-bundle is an automorphism of the $G$-bundle preserving the liftings of the strings. Clearly, nontrivial automorphisms can exist only on the closed components of $\Sigma$.

Let us fix a triple $(G, E, F)$. Let $X(\Sigma)$ be the set of all $(G, E, F)$-bundle types over $\Sigma$. We put

$$\eta(\Sigma) = \mathbb{C}X(\Sigma)$$

(the free vector space over $X(\Sigma)$). The inner product on $\eta(\Sigma)$ is defined as follows: the basis $X(\Sigma)$ is orthogonal and for $x \in X(\Sigma)$,

$$(x, x) = \text{the number of automorphisms of } x.$$  

If $f : \Sigma_1 \rightarrow \Sigma_2$ is a glueing, we define $\eta(f) : \eta(\Sigma_1) \rightarrow \eta(\Sigma_2)$ as follows: for $x \in X(\Sigma_1)$ we try to extend the glueing $f$ to the bundle in such a way that if two strings are glued, we glue their lifts. If it is possible, the result is $\eta(f)(x) \in X(\Sigma_2)$; otherwise $\eta(f)(x) = 0$.

This functor is an $\eta$-functor, up to one problem: the maps $\eta(f)$ are projecting only after rescaling the inner products. But if we rescale each $\eta(f)$ by a positive number so as to obtain a projecting map, we obtain a genuine $\eta$-functor.

4 The proof

The strategy of the proof is as follows:

First we define the finite quantum group structure on $\mathcal{H} = \eta(D)$ for any $\eta$ (this definition should be considered as a part of the statement of the theorem) and verify that it obeys FQG axioms.

Then we prove that $\eta$ is specified by the FQG $\mathcal{H} = \eta(D)$ up to isomorphisms. The point is that any neckface can be glued from several copies of $D$. Representations of some basic glueings by $\eta$ are already present in $\mathcal{H}$ (by its definition); this is sufficient to restore $\eta$. 
Finally, we have to show that there is an $\eta$ for any $\mathcal{H}$. Our original motivation is used here – $\eta$ is constructed as a generalization of cohomologies to quantum coefficients.

In this proof we show explicitly only that $\eta \mapsto \eta(D)$ is a bijection from ETA to FQG, up to isomorphisms. In Section 4.3 we construct a functor $\text{FQG} \to \text{ETA}$. The fact that the composition $\text{ETA} \to \text{FQG} \to \text{ETA}$ is isomorphic to the identity follows from the proof immediately.

4.1 $\mathcal{H} = \eta(D)$ is a finite quantum group

First notice the following figure:

It represents a nice glueing $C \cup C \to C$, so we have an isomorphism $\eta(C) \otimes \eta(C) \to \eta(C)$. As a consequence, $\eta(C) = \mathbb{C}$. Also notice that $\eta(S^2)$ is 1-dimensional, because there is an obvious glueing $C \to S^2$.

Now we can define the FQG structure on $\eta(D)$. The multiplication and the dual multiplication were already given on the figure at page 2. The counit is given by the following picture:

It represents two maps $D \to C$; the bottom object is a sphere and I indicated on $S^2$ the image of $\partial D$ (rather than $\partial C$) to make clear that the diagram is commutative. According to the independent projection law, the two glueings are represented by the same $\epsilon : \eta(D) \to \eta(C) = \mathbb{C}$. The definition of $\tilde{\epsilon}$ is similar (just change the colour of the beads).

Finally, the involution $* : \eta(D) \to \eta(D)$ is given by the reflection with respect to the diameter connecting the white beads of $D$ and the antipode $S$ by rotating $D$ for 180 degrees.

Now we have to prove that we really defined a finite quantum group. The only problem is to show that the comultiplication acts as a homomorphism with respect to the multiplication. The next picture presents another definition of comultiplication, where this condition becomes evident:

It has the form $D \cup \Sigma \to \Sigma$ and a nice glueing $D \cup D \to \Sigma$ is indicated to show that $\eta(\Sigma) = \mathcal{H} \otimes \mathcal{H}$. We claim that the corresponding map $\mathcal{H} \otimes (\mathcal{H} \otimes \mathcal{H}) \to \mathcal{H}$ is
given by $a \otimes (b \otimes c) \mapsto a_{(1)}^* b \otimes a_{(2)}^* c$. In other words, we have to prove something about the following diagram (the dashed arrows indicate nice morphisms):

Consider the image of the diagram by the functor $\eta$. There are two paths from the leftmost object to the rightmost one; if we have to go against an arrow, we take the adjoint of the map. We have to prove that the lower path is just $\langle \varepsilon, \tilde{\varepsilon} \rangle$ times the upper path. So, consider the following diagram:

It was obtained from the previous one by nice glueings (the dashed arrows are now contracted to identities). It has the same representation by $\eta$, so that we have to prove the same thing about the new diagram. And it follows from the following general fact:

On this diagram, two glueings of the form $\Sigma' \rightarrow \Sigma$ are drawn (it represents only the relevant parts of general neckfaces); also a nice glueing $\Sigma \cup D \rightarrow \Sigma'$ is indicated. We claim that having mapped the diagram to HIL by $\eta$, going from the left to the right is just $\langle \varepsilon, \tilde{\varepsilon} \rangle$ times the identity. And the nice glueing $\Sigma \cup D \rightarrow \Sigma'$ makes it clear.
4.2 $\eta$ is determined uniquely by $H = \eta(D)$

If $\Sigma$ is a neckface then the space $\eta(\Sigma)$ becomes an $H$-module for every black bead of $\Sigma$ and an $\tilde{H}$-module for every white bead. The picture represents the black bead case:

Now suppose we are given a glueing $f : \Sigma' \to \Sigma$. These $H$ and $\tilde{H}$ module structures on $\eta(\Sigma')$ will enable us to write down the projector $\eta(\Sigma') \to (\ker \eta(f))$. The simplest case is the following:

Here the projector is given obviously by the action of $\epsilon \in H$ at the black bead (due to the definition of $\epsilon$). The recoloured version is similar.

Consider the following neckface (with a nice glueing $\Sigma' \to \Sigma$ indicated):

In $\Sigma'$ the black bead splits into 3 beads (3 is just an example); they are numbered against the orientation of $\Sigma$. By the nice glueing we have $\eta(\Sigma') \simeq \eta(\Sigma)$. Upon this identification, the action of $a \in H$ at the black bead of $\Sigma$ is the same as the action of $a(1) \otimes a(2) \otimes a(3)$ at the 3 black beads of $\Sigma'$. This follows immediately from the bottom figure on page 7. A similar thing is valid for any number and in the recoloured version.

Putting these two facts together, we can say something about the following type of glueing $\Sigma' \to \Sigma$:

The projector $\eta(\Sigma') \to (\ker \eta(f))$ is simply the action of $\epsilon(1) \otimes \epsilon(2) \otimes \epsilon(3)$ at the 3 black beads of $\Sigma'$.

Now we can state the general form of the projector $\eta(\Sigma') \to (\ker \eta(f))$. If $n$ black beads of $\Sigma'$ are mapped to a single point inside $\Sigma$, we have to act by $\epsilon(1) \otimes \epsilon(2) \ldots \otimes \epsilon(n)$ (as in the previous figure). We call this operator a black projector. Similar white projection has to be made at each image of white beads inside $\Sigma$. These projections are mutually commuting and their composition is the projector.
η(Σ') → (ker η(f))^⊥. This can be proved from the fact concerning the previous picture and from the independent projection law.

Now we can prove that the finite quantum group structure on \( \mathcal{H} = η(D) \) determines η uniquely. The point is that any neckface Σ can be glued from several copies of \( D \) and the \( \mathcal{H} \) and \( \mathcal{H} \) module structure on \( η(D) \) is given simply by the multiplication and the dual multiplication. Thus we know η(Σ) as a subspace of \( \mathcal{H}^\otimes n \) and we also know η(f) for any \( f : Σ_1 \to Σ_2 \).

### 4.3 Construction of η for a given \( \mathcal{H} = η(D) \):
Cohomologies with quantum coefficients

Now we know much about reconstructing η from its \( \mathcal{H} \). For any neckface \( Σ_1 \) we take a glueing \( f : nD \to Σ_1 \) and we already know the subspace \( η(Σ_1) \simeq (\ker η(f))^⊥ \subset \mathcal{H}^\otimes n \). Also, any glueing \( g : Σ_1 \to Σ_2 \) gives us a glueing \( nD \to Σ_2 \) (by composition), so we know \( η(Σ_2) \) and also \( η(g) \) (in this picture, \( η(Σ_2) \subset η(Σ_1) \) and \( η(g) \) is the orthogonal projection). The only thing we miss is the identification of \( η(Σ) \) for any two glueings \( nD \to Σ \) and \( mD \to Σ \): we see \( η(Σ) \) as a subspace of \( η(nD) \) and also of \( η(mD) \); we have to understand how to identify these subspaces, using the finite quantum group \( \mathcal{H} \) only. Clearly, this has to be possible (in the view of the previous section).

Now we simply reverse the line of thoughts. Suppose we are given \( \mathcal{H} \) and we want to define a corresponding η (we already know that η is unique up to an isomorphism). The method is rather straightforward: for any glueing \( f : nD \to Σ \) we define \( η(f)(Σ) \) as a subspace of \( \mathcal{H}^\otimes n \) in the way described above – as the common range of the projectors like \( e_{(1)} \otimes e_{(2)} \ldots \otimes e_{(n)} \). One easily verifies that these projectors are orthogonal and mutually commuting. For any glueing \( g : Σ \to Σ' \) we have \( η(gf)(Σ') \subset η(f)(Σ) \). We define \( η(f)(Σ) \to η(gf)(Σ') \) as the orthogonal projection. To define η itself we have to show unitary isomorphisms \( Ξ_{f_2,f_1} : η(f_1(Σ)) \to η(f_2(Σ)) \) such that

\[
Ξ_{f_3,f_2} \circ Ξ_{f_2,f_1} = Ξ_{f_3,f_1}.
\]

Then we may identify \( η(f_1(Σ))'s \) for any two \( f \)'s and thus define \( η(Σ) \). If \( g : Σ \to Σ' \) is a glueing, we define \( η(g) \) as \( η(f) \) for any \( f \). Of course we have to require

\[
η(gof_1)(Σ') \subset η(f_1(Σ))
\]

\[
Ξ_{gof_2,gof_1} \circ Ξ_{f_2,f_1} = Ξ_{gof_2,f_1}
\]

\[
η(gof_2)(Σ') \subset η(f_2(Σ))
\]

to be commutative for any \( f_{1,2} : n_{1,2}D \to Σ \). If (1) and (2) hold, we have a well defined functor η. Notice that it automatically satisfies all the axioms of an η-functor, except possibly for \( η(Σ) = η(Σ') \). If we can check this and the fact that the FQG derived from η is \( \mathcal{H} \), we are done.

All the rest of this section is devoted to a description of an appropriate Ξ.

We use the graphs corresponding to a glueing \( f : nD \to Σ \), such as on the figure on page 3. The graphs will be called \( Γ_f \) (black vertices) and \( ˜Γ_f \) (white vertices), or the black graph and the white graph. Let \( E_f \) denote the set of edges of \( Γ_f \) and \( ˜E_f \) the set of edges of \( ˜Γ_f \). We choose an orientation of each edge of \( Γ_f \) and \( ˜Γ_f \) in the following way: on each \( D \) we orientate the horizontal diameter (connecting the black beads) from the left to the right and the vertical diameter from the top to the bottom; the edges of \( Γ_f \) and \( ˜Γ_f \) are their images. There is a 1-to-1 correspondence between the edges of \( Γ_f \) (and also \( ˜Γ_f \)) and the disks among \( nD \).
We will consider intersection of graphs (always a black graph with a white graph, but not necessarily coming from the same glueing \( nD \to \Sigma \)). We will always suppose that the two graphs have only finitely many common points, none of them being a vertex, and that the edges intersect transversally. This can be achieved by a deformation. An intersection point is positive or negative according to the relative orientation of the intersecting edges; by definition, all the intersections of \( \Gamma_f \) with \( \tilde{\Gamma}_f \) are positive.

We will consider intersections of a \( \Gamma \) with graphs in a 1-parametric continuous family \( \tilde{\Gamma}_\lambda \), \( \lambda \in (0, 1) \), too. In this case we admit only finitely many \( \lambda \)'s for which the intersection of \( \Gamma \) and \( \tilde{\Gamma}_\lambda \) violate the previous condition, always in this way: either an edge of \( \tilde{\Gamma} \) is crossing a vertex of \( \Gamma \), or a vertex is crossing an edge, or two opposite crossing of edges disappear or appear; only one such exception is allowed for a fixed \( \lambda \):

This situation can be achieved by a small deformation.

Before describing the system \( \Xi \) itself, we describe a similar procedure for a combinatorial definition of \( CH^1(\Sigma, B; G) \). Recall (Section 3) that \( \eta(\Sigma) = CG \) for a finite abelian group \( G \) (\( B \) is the set of the black beads of \( \Sigma \)). Changing the procedure slightly we arrive at a combinatorial definition of \( CH^1(\Sigma, B; G) \) using the Hopf algebra structure of \( CG \) only. This is easily generalized to an arbitrary FQG (just by taking care of ordering problems). This generalization will be our definition of \( \Xi \).

One can define \( H^1(\Sigma, B; G) \) combinatorially as cocycles on the graph \( \Gamma_f \) modulo coboundaries, i.e. modulo action of \( G \) at the internal vertices of \( \Gamma_f \). Here the transfer of cocycles on \( \Gamma_{f_1} \) to \( \Gamma_{f_2} \) can be done in the following way: take the graph \( \Gamma_{f_1} \); if \( c_1 \) is a cocycle on \( \Gamma_{f_1} \), the corresponding cocycle \( c_2 \) on \( \Gamma_{f_2} \) assigns to an edge \( e_2 \) the product \( \prod c_1(e_1)^{\pm 1} \). Here the product runs over all the intersections of \( e_2 \) with the edges of \( \Gamma_{f_1} \). If it is the intersection with an edge \( e_1 \), we put to the product \( c_1(e_1)^{\pm 1} \) according the sign of the intersection (here \( e_1 \in E_{f_1} \) is the edge dual to \( e_1 \), i.e. corresponding to the same \( D \) in \( n_1D \)).

Now we come to a combinatorial definition of \( CH^1(\Sigma, B; G) \). Let \( f : nD \to \Sigma \) be a glueing. Any element of \( H^1(\Sigma, B; G) \) is represented by a cocycle on the graph \( \Gamma_f \), i.e. by an element of \( G^n \). Of course, this cocycle is not unique: we can act by an element of \( G \) at any internal vertex of \( \Gamma_f \). But suppose we take \( (CG)^{\otimes n} = CG^n \) instead of \( G^n \). In this space we can average over the action of \( G \); the averaging is nothing but the black projector. So we get an element of \( (CG)^{\otimes n} \) which is now unique. It is closed (i.e. it is a linear combination of cocycles) and coclosed (i.e. \( G \)-invariant at the internal vertices of \( \Gamma_f \)); therefore we may call it harmonic. By linearity, each element of \( CH^1(\Sigma, B; G) \) is represented by a unique harmonic element of \( (CG)^{\otimes n} \); in fact, \( CH^1(\Sigma, B; G) \) may by identified with the space of harmonic elements of \( (CG)^{\otimes n} \). Evidently, closedness is equivalent to invariance with respect to the white projectors, i.e. the space of harmonic elements of \( (CG)^{\otimes n} \).
is just $\eta_f(\Sigma)$. Generally, elements of the range of the black (white) projectors will be called coclosed (closed). The procedure given for $H^1(\Sigma, B; G)$ works for $CH^1(\Sigma, B; G)$ almost without any change: we just have to use comultiplication to make transformations like $g \mapsto g \otimes \ldots \otimes g$ linear. The procedure has to be concluded by the black projectors (i.e. averaging) to assure coclosedness. The details will be given below for a general $\mathcal{H}$.

Recall the definition of the black projectors. Let $P$ be a vertex of $\Gamma_f$ and let $e_1, e_2, \ldots, e_k$ be the edges incident with $P$, in the cyclical order, given by the orientation of $\Sigma$. Some edges may occur twice in this list. Let

$$ a = \bigotimes_{e \in E_f} a_e \in \mathcal{H}^\otimes E_f = \mathcal{H}^\otimes n. $$

If all the edges point to $P$, we form the coproduct $\epsilon(1) \otimes \ldots \otimes \epsilon(k)$ and multiply each $a_e$ by $\epsilon(i)$ from the right. If we allow arbitrary orientations of $e_i$'s, we multiply by $S(\epsilon(i))$ from the left instead, if $e_i$ is pointing from $P$.

Let $b \in \mathcal{H}$. Let us replace $\epsilon$ with $b$ in the definition of the black projector. Clearly, $a$ is coclosed at $P$ iff the result is $\langle \epsilon, b \rangle a$ for any $b$. A similar fact is of course valid for closedness (just using $\mathcal{H}$ instead of $\mathcal{H}$). It can be translated to the following formulation, very similar to closedness of a group-valued cochain.

Let $F$ be an internal face of $\Gamma_f$ (i.e. containing an internal vertex of $\Gamma_f$). Let $e_1, e_2, \ldots, e_k$ be the edges of $\Gamma_f$ forming the boundary $\partial F$ of the face $F$, in the order given by the orientation of the face. We make comultiplication on $a = \otimes a_e$ at the edges among $\partial F$ (it may happen that an edge is twice in $\partial F$; in that case we make the comultiplication twice). For the sake of clarity, first suppose that all the edges $e_1, e_2, \ldots, e_k$ have the orientation compatible with $F$. In that case, we take the first component of each coproduct and multiply them in the order $1, \ldots, k$; we have something like

$$ a_{e_1(1)} a_{e_2(1)} \ldots a_{e_k(1)} \otimes a_{e_1(2)} \otimes a_{e_2(2)} \otimes \ldots \otimes a_{e_k(2)} \bigotimes_{e \in \{e_i\}} a_e \in \mathcal{H} \otimes \mathcal{H}^\otimes E_f. $$

The condition states that this has to be $1 \otimes a$. If the edges $e_1, e_2, \ldots, e_k$ have arbitrary orientation, we also make the comultiplication on each $a_{e_i}$, but if the orientation of $e_i$ is opposite to $F$, we take from the coproduct $a_{e_i(2)}$ and put to the product $S(a_{e_i(2)})$; again we have to obtain $1 \otimes a$. If an edge $e_i$ is twice in $\partial F$, we take to the product $a_{e_i(1)}$ and $S(a_{e_i(3)})$ (the orientation of $e_i$ once agrees and once disagrees with $F$). The condition $a \mapsto 1 \otimes a$ is clearly independent of the choice of $e_i$.

We use this definition of closedness in what follows; it simplifies the proof.

Now we define the system of unitary isomorphisms $\Xi_{f_2, f_1} : \eta_{f_1}(\Sigma) \to \eta_{f_2}(\Sigma)$. Let $I_{f_2, f_1} = \Gamma_{f_2} \cap \Gamma_{f_1}$. Recall our requirements on the intersection of the graphs $\Gamma_{f_2}$ and $\Gamma_{f_1}$; if necessary, we deform $\Gamma_{f_1}$ so that they held. Later we shall prove that $\Xi_{f_2, f_1}$ is independent of deformations of the graphs.

We define $\Xi_{f_2, f_1}$ as the composition of several maps. First, let us define a map $\mathcal{H}^\otimes F_{f_1} \to \mathcal{H}^\otimes I_{f_2, f_1}$ (recall $\eta_{f_1}(\Sigma) \subset \mathcal{H}^\otimes F_{f_1} = \mathcal{H}^\otimes E_{f_1}$). For any edge $e_i \in \tilde{E}_{f_1}$, we take the points $p^e_{k_i} \in I_{f_2, f_1}$ lying on $e_i$, in the natural order given by the orientation of $e_i$. Given an element

$$ \bigotimes_{\tilde{e} \in \tilde{E}_{f_1}} a_{\tilde{e}} \in \mathcal{H}^\otimes \tilde{E}_{f_1}, $$

we form the coproduct

$$ \bigotimes_{\tilde{e} \in \tilde{E}_{f_1}} (a_{\tilde{e}(1)} \otimes \ldots \otimes a_{\tilde{e}(k_i)}) \in \mathcal{H}^\otimes I_{f_2, f_1} $$
(if $k_\ell = 0$, we act by the counit on $a_\ell$); $a_{\ell(m)}$ corresponds to $p^\ell_m$.

Now we define a map $\mathcal{H}^{\otimes I_{f_2,f_1}} \to \mathcal{H}^{\otimes I_{f_2,f_1}}$. We simply act by the antipode at the negative points of $I_{f_2,f_1}$.

The next map is $\mathcal{H}^{\otimes I_{f_2,f_1}} \to \mathcal{H}^{\otimes E_{f_2}}$. For any $e \in E_{f_2}$ take its points $p_1^e, \ldots, p_{13}^e \in I_{f_2,f_1}$ in the natural order. For

$$\bigotimes_{p \in I_{f_2,f_1}} a_p \in \mathcal{H}^{\otimes I_{f_2,f_1}}$$

form

$$\bigotimes_{e \in E_{f_2}} a_{p_1^e} \ldots a_{p_{13}^e}.$$

The final map $\mathcal{H}^{\otimes E_{f_2}} \to \mathcal{H}^{\otimes E_{f_2}}$ ensures coclosedness: it is just the composition of all the black projectors of $f_2$.

$\Xi_{f_2,f_1}$ is equal to the composition

$$\eta_{f_1}(\Sigma) \subset \mathcal{H}^{\otimes E_{f_1}} \to \mathcal{H}^{\otimes I_{f_2,f_1}} \to \mathcal{H}^{\otimes I_{f_2,f_1}} \to \mathcal{H}^{\otimes E_{f_2}} \to \mathcal{H}^{\otimes E_{f_2}}$$

times a positive factor $c_{f_2,f_1}$ ensuring unitarity. One easily sees that $\text{rng } \Xi_{f_2,f_1} \subset \eta_{f_3}(\Sigma)$: we only have to check the evident fact that the composition

$$\mathcal{H}^{\otimes E_{f_1}} = \mathcal{H}^{\otimes E_{f_1}} \to \mathcal{H}^{\otimes E_{f_2}} \to \mathcal{H}^{\otimes I_{f_2,f_1}} \to \mathcal{H}^{\otimes I_{f_2,f_1}} \to \mathcal{H}^{\otimes E_{f_2}}$$

preserves closedness. The factor $c_{f_2,f_1}$ is

$$c_{f_2,f_1} = \langle \epsilon, \bar{\epsilon} \rangle^{n_\Gamma_{f_2}}$$

where $n_\Gamma$ is the number of vertices of $\Gamma_{f_2}$.

First we have to prove that $\Xi_{f_2,f_1}$ is independent of deformations of the graphs; this is a straightforward consequence of the useful identity $\epsilon_1(a) \otimes \epsilon_2(a) = \epsilon_1(a) \otimes \epsilon_2(S(a))$. Next we have to check (1). We do it in the case when $f_2$ and $f_3$ are close to each other (in the sense specified below) and then use the fact that any $f_2$, $f_3$ can be connected by a sequence $f_2 = f^1, \ldots, f^n = f_3$ with close $f^i$ and $f^{i+1}$.

We say that gluings $f_{2,3} : n_{2,3}D \to \Sigma$ are close to each other if the graphs $\Gamma_{f_{2,3}}$ differ just by removing an edge separating two different faces or by contracting an edge connecting two different vertices. For close $f_{2,3}$ the equation (1) is straightforward; we just have to draw the graphs (in the case of contraction) so that the contracted graph follows closely the original one, as indicated on the figure:

$$\begin{array}{c}
\bullet \\
\bullet
\end{array}$$

Similarly, to prove that $\Xi_{f_2,f_1}$ is unitary, we may suppose $f_{1,2}$ are close to each other. We already know

$$\Xi_{f_2,f_1} \circ \Xi_{f_1,f_2} = \Xi_{f_2,f_2} = 1;$$

checking that $\Xi_{f_2,f_1}$ is the adjoint of $\Xi_{f_1,f_2}$ is straightforward.

Only few things remain now. Condition (2) holds evidently. We have to prove $\eta(\Sigma) = \bar{\eta}(\Sigma)$. Let $f : nD \to \Sigma$ be a glueing and let $r : D \to \overline{D}$ be the reflection with respect to the vertical diameter. Let $f' : nD \to \overline{\Sigma}$ be the composition $nD \xrightarrow{f} \overline{D} \xrightarrow{r} \overline{\Sigma}$. The map $\mathcal{H}^{\otimes n} : \mathcal{H}^{\otimes \Sigma} \to \mathcal{H}^{\otimes \overline{\Sigma}}$ sends $\eta_f(\Sigma)$ onto $\eta_{f'}(\overline{\Sigma})$. It is easy to check that it gives us a well-defined and functorial identification of $\eta(\Sigma)$ with $\bar{\eta}(\Sigma)$.

Finally, we have to check that the quantum group coming from $\eta$ is indeed $\mathcal{H}$. It is an easy exercise.
5 The role of Drinfeld double

Recall (the figure at the top of page 9) that $\eta(\Sigma)$ is a $H$-module for each black bead and a $\tilde{H}$-module for each white bead of $\Sigma$. These $H$ and $\tilde{H}$ actions obviously commute with each other with the exception of neighbouring beads, when the actions form Weyl algebra (cf. [6]). If we take a boundary circle of $\Sigma$, all actions combine together to an action of an associative algebra, which is $\eta$ of the following figure (multiplication is given by glueing the internal circle with the external circle of another copy of the neckface on the figure; the action on $\eta(\Sigma)$ is given by gluing the internal circle with the circle on $\Sigma$):

When the boundary circle contains just one black and one white bead, we will call it a dyonic hole. In this case the $H$ action and the $\tilde{H}$ action combine to an action of the Drinfeld double $Dr$, i.e. to $\eta(A)$, where $A$ (together with nice glueing $D \cup D \to A$ giving rise to $H \otimes \tilde{H} \to Dr$) is

Thus $\eta(\Sigma)$ becomes a $Dr$-module for every dyonic hole. $H$ and $\tilde{H}$ actions on $\eta(\Sigma)$ correspond to order and disorder operators and at dyonic holes they form Drinfeld doubles.

Of course, the Drinfeld double is not just an associative algebra, but a quasitriangular FQG; this will lead us to a braid group statistics of the dyonic holes, given by the $R$-matrix of $Dr$. The comultiplication in $Dr$ is drawn in the left part of the following picture:

It presents a neckface $\Sigma$ together with a nice glueing $A \cup A \to \Sigma$, giving $Dr \otimes Dr \to \eta(\Sigma)$. If $Dr$ acts on the external circle of $\Sigma$, we have a map $Dr \otimes \eta(\Sigma) \to \eta(\Sigma)$, i.e. $Dr \otimes (Dr \otimes Dr) \to Dr \otimes Dr$. As in the case of the comultiplication on $H$, it is given by $a \otimes (b \otimes c) \mapsto a_{(1)}b \otimes a_{(2)}c$.

The right half of the picture presents another glueing $A \cup A \to \Sigma$; it can be obtained by a map $\Sigma \to \Sigma$ exchanging the internal circles. This exchange corresponds to the opposite comultiplication on $Dr$; if we have the element of $\eta(\Sigma)$ which is $1 \otimes 1 \in Dr \otimes Dr$ in the left decomposition of $\Sigma$, it becomes the $R$-matrix $R \in Dr \otimes Dr$ in the right.
This exchange of two dyonic holes is the basic element from which the braid group statistics of such holes is built up. It even proves that $D_r$ is quasitriangular.

6 Conclusion

Recall that our original aim was to find a lattice quantization of PL T-duality. Clearly, it is still an open problem. The point is that we have to admit Hopf algebras that come from quantization of PL groups. Finite dimension is (at least technically) good, so $\mathcal{H}$ should be a (restricted) quantum group at a root of 1. The most important difference is that now $S^2 \neq 1$, as if turning $D$ for 360 degrees did not give the identity. The way out should be achieved by extending the category NEFA.

The functor $\eta$ is closely related to q-invariants of 3-folds computed from the Drinfeld double. In fact, $\eta(\Sigma)'s$ (for closed $\Sigma$'s) are the state spaces of a TFT giving rise to these invariants. It would be nice if it had topological applications for open $\Sigma$'s, too. It seems reasonable that for $\Sigma$'s with dyonic holes only, it is connected with ribbon tangles invariants.

There are good reasons to expect that the hypothetical vector spaces $\eta(\Sigma)$ for the case of a root of 1 are closely related to the state spaces of Chern-Simons theory, because 1. in PL T-duality, WZNW action in the double gives the symplectic structure on the phase space, and 2. in the finite quantum group case, the functor $\eta$ is closely related to invariants of 3-folds computed from the double $D_r(\mathcal{H})$. Perhaps a part of the extension of NEFA should consist in requiring neckfaces to carry a complex structure.

Appendix: Finite quantum groups and the category FQG

The aim of this appendix is to fix the notation and to describe non-standard definitions used in the text. We do not give standard definitions here (see e.g. [8]).

A finite quantum group (FQG) $\mathcal{H}$ is a finite-dimensional Hopf $C^*$-algebra. We always rescale the inner product so that the multiplication map $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ becomes projecting, i.e. $m|_{(\ker m)^+}$ is unitary.

The counit $\epsilon : \mathcal{H} \to \mathbb{C}$ is clearly projecting, i.e. it is given by the inner product with an $\epsilon \in \mathcal{H}$, $\|\epsilon\| = 1$ (usually, there is no confusion between these two $\epsilon$'s, except for the definition of morphisms in the category FQG; we make a warning there). The dual counit $\tilde{\epsilon} : \mathcal{H} \to \mathbb{C}$ is given by the inner product with a positive multiple $\tilde{\epsilon}$ of $1 \in \mathcal{H}$ such that $\tilde{\epsilon} : \mathcal{H} \to \mathbb{C}$ is projecting (or $\|\tilde{\epsilon}\| = 1$), i.e. $\tilde{\epsilon} = 1/\|1\|$. Because $\langle\epsilon, 1\rangle = 1$, we have $\|1\|\langle\epsilon, \tilde{\epsilon}\rangle = 1$.

The element $\epsilon \in \mathcal{H}$ satisfies the identity

\[ a \epsilon = \epsilon a = \langle\epsilon, a\rangle \epsilon \]

for any $a \in \mathcal{H}$; as a consequence,

\[ \epsilon(1) a \otimes \epsilon(2) = \epsilon(1) \otimes \epsilon(2) S(a) \]

(both sides are equal to $\epsilon(1) a(1) \otimes \epsilon(2) a(2) S(a(3))$). We use the standard notation $a(1) \otimes a(2)$ for the coproduct of $a$ and $a(1) \otimes \ldots \otimes a(n)$ for the $n-1$-fold coproduct of $a$.

The adjoint of the coproduct $\mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ gives another associative algebra structure $\mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$. We rescale it by $\langle\epsilon, \tilde{\epsilon}\rangle$ so that it becomes projecting. This map $\tilde{m}$ is called the dual multiplication.
The dual FQG $\tilde{\mathcal{H}}$ is $\overline{\mathcal{H}} = \mathcal{H}^*$, if we exchange product with the dual product, counit with the dual counit and $*$ with $S \circ *$ and take the complex conjugate of the maps. This definition is clearly equivalent to the standard one, up to rescaling.

*Fourier transform* $\mathcal{H} \to \tilde{\mathcal{H}}$ is simply $*$ when $\tilde{\mathcal{H}}$ is understood as the FQG $\mathcal{H}$.

We organize FQG’s into the category $\text{FQG}$. A morphism $\mathcal{H}_1 \to \mathcal{H}_2$ is a linear map preserving the inner product (an isometric injection). It has to preserve $S$ and $*$ and to preserve $m, \tilde{m}, \epsilon$ and $\tilde{\epsilon}$ up to a positive factor. Warning: in this definition, $\epsilon$ and $\tilde{\epsilon}$ are understood as maps $\mathcal{H} \to \mathbb{C}$ and not as elements of $\mathcal{H}$. For example, if $N \subset G \subset H$ are finite groups and $N$ is normal in $G$, there is a morphism between the group algebras $\mathbb{C}G/N \to \mathbb{C}H$. It is given by

$$gN \mapsto \frac{|G|^{\frac{1}{2}}}{|N||H|^{\frac{1}{2}}} \sum_{h \in gN} h.$$  

Notice that for any $G$, $\|g\|^2 = |G|$.

In FQG, the map $\mathcal{H} \to \tilde{\mathcal{H}}$ is a *covariant* automorphism. We call it *duality*.

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