OPTIMAL CONTROL IN BOMBIERI’S AND TAMMI’S CONJECTURES

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Abstract. Let \( S \) stand for the usual class of univalent regular functions in the unit disk \( U = \{ z : |z| < 1 \} \) normalized by \( f(z) = z + a_2 z^2 + \ldots \) in \( U \), and let \( S^M \) be its subclass defined by restricting \( |f(z)| < M \) in \( U, \ M \geq 1 \). We consider two classical problems: Bombieri’s coefficient problem for the class \( S \) and the sharp estimate of the fourth coefficient of a function from \( S^M \). Using Löwner’s parametric representation and the optimal control method we give exact initial Bombieri’s numbers and derive a sharp constant \( M_0 \), such that for all \( M \geq M_0 \) the Pick function gives the local maximum to \( |a_4| \). Numerical approximation is given.

§1. Introduction

Let \( S \) stand for the class of all holomorphic and univalent functions \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) in the unit disk \( U = \{ z : |z| < 1 \} \). Its subclass of bounded maps \( |f(z)| < M, \ M \geq 1 \), we denote by \( S^M, S^\infty \equiv S \). During the long history of univalent functions the famous Koebe function

\[
K(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n \in S
\]

has been known to be extremal in many problems. A relevant sample is the most celebrated Bieberbach Conjecture [2] \( |a_n| \leq n \) proved by L. de Branges in 1984 [5,6]. In spite of many works about coefficient estimates in the class \( S \), there are some difficult problems that are still unsolved, in particular, the Bombieri problem and the sharp upper bound for \( |a_n|, n \geq 4 \), for the subclass \( S^M \) that we will deal with.

E. Bombieri [4] in 1967 posed the problem to find

\[
\sigma_{mn} := \lim_{f \to K} \inf_{f \in S} \frac{n - \text{Re} \ a_n}{m - \text{Re} \ a_m}, \ m, n \geq 2,
\]

1991 Mathematics Subject Classification. Primary 30C50. Secondary 30C70, 49K15.

Key words and phrases. extremal problem, bounded univalent function, Bombieri conjecture, optimal control.

The first author is supported by the Russian Foundation for Basic Research, grant # 01-01-00123 and the grant of the Ministry of Higher Education (Russia) # E02-1.0-178; the second author is supported by FONDECYT (Chile), projects # 1030373, #1040333, and UTFSM #12.03.23

Typeset by \textsc{AMS-}TEX

1
\( f \to K \) locally uniformly in \( U \). We call \( \sigma_{mn} \) the Bombieri numbers. He conjectured that \( \sigma_{mn} = B_{mn} \), where
\[
B_{mn} = \min_{\theta \in [0, 2\pi]} \frac{n \sin \theta - \sin(n\theta)}{m \sin \theta - \sin(m\theta)}.
\]
and proved that \( \sigma_{mn} \leq B_{mn} \) for \( m = 3 \) and \( n \) odd. It is noteworthy that D. Bshouty and W. Hengartner [7] proved Bombieri’s conjecture for functions from \( S \) having real coefficients in their Taylor’s expansion. Continuing this contribution by D. Bshouty and W. Hengartner, the conjecture for the whole class \( S \) has been recently disproved by R. Greiner and O. Roth [9] for \( n = 2, m = 3, f \in S \). Actually, they have got the sharp Bombieri number \( \sigma_{32} = (e - 1)/4e < 1/4 = B_{32} \).

It is easily seen that \( \sigma_{43} = \sigma_{23} = B_{23} = 0 \). Applying Löwner’s parametric representation for univalent functions and the optimal control method we will find the exact Bombieri numbers \( \sigma_{43}, \sigma_{24}, \sigma_{34} \) and their numerical approximations \( \sigma_{43} \approx 0.050057 \ldots, \sigma_{24} \approx 0.969556 \ldots, \) and \( \sigma_{34} \approx 0.791557 \ldots\) (the Bombieri conjecture for these permutations of \( m, n \) suggests \( B_{42} = 0.1, B_{24} = 1, B_{34} = 0.828427 \ldots \)). Of course, our method permits us to reprove the result of [9] about \( \sigma_{32} \).

Our next target is the fourth coefficient \( a_4 \) of a function from \( S^M \). An analogue of the Koebe function for this class is the Pick function
\[
P_M(z) = MK^{-1}(K(z)/M) = z + \sum_{n=1}^{\infty} p_n(M)z^n.
\]

The sharp estimate \( |a_2| \leq 2(1 - 1/M) = p_2(M) \) in the class \( S^M \) is rather trivial and has been obtained by G. Pick [12] in 1917. The next coefficient \( a_3 \) was estimated independently by A. C. Schaeffer and D. C. Spencer [14] in 1945 and O. Tammi [16] in 1953. The Pick function does not give the maximum to \( |a_3| \) and the estimate is much more difficult. M. Schiffer and O. Tammi [15] in 1965 found that \( |a_4| \leq p_4(M) \) for any \( f \in S^M \) with \( M > 300 \). This result was repeated by O. Tammi [18, page 210] in a weaker form \( (M > 700) \) and there it was conjectured that this constant could be decreased until 11. The case of function with real coefficients is simpler: the Pick function gives the maximum to \( |a_4| \) for \( M \geq 11 \) and this constant is sharp (see [17], [19, p.163]). By our suggested method we will show that the Pick function locally maximizes \( |a_4| \) on \( S^M \) if \( M > M_0 = 22.9569 \ldots \) and does not for \( 1 < M < M_0 \). This disproves Tammi’s conjecture.

\section*{§2. Preliminary statements}

The parametric representation of univalent functions is based on the Löwner differential equation and goes back to the famous Löwner’s paper [11] where the author using an idea of semigroups of conformal maps derived the equation
\[
\frac{dw}{dt} = -w\frac{e^{iu} + w}{e^{iu} - w}, \quad w|_{t=0} = z, \quad t \geq 0,
\]
where the control function \( u = u(t) \) is piecewise continuous in \( t \geq 0 \). One finds the foundations of the parametric method, e.g., in [1,8,13]. It is convenient to make the change of variables \( t \rightarrow 1 - e^{-t} \) and rewrite Löwner’s equation using the preceding notation for the independent variable as follows

\[
\frac{dw}{dt} = \frac{-w e^{i\mu} + w}{(1-t) e^{i\mu} - w}, \quad w|_{t=0} = z, \quad 0 \leq t \leq 1.
\]

A. C. Schaeffer and D. C. Spencer [14] were the first who used Löwner’s equation for the class \( S^M \) and proved that the integrals

\[
w = w(z, t) = (1-t)(z + a_2(t)z^2 + \ldots)
\]

of the equation (1) represent a dense subclass of functions \( f \in S^M \) by

\[
f(z) = Mw(z, 1-1/M).
\]

Representation (3) is valid for all \( M \geq 1 \) including \( M = \infty \) if the product in (3) is regarded as the limit as \( M \to \infty \). From now on, we will use the notation (2).

We remark that the case \( u = \pi \) in (1) corresponds to the Koebe function in the class \( S \) or to the Pick function in the class \( S^M \) by (3). Besides, the dense subclass of \( S^M \) represented by (3) contains all functions that give the boundary points of the coefficient region

\[
V_4^M = \{(a_2, a_3, \text{Re } a_4) : f \in S^M\}, \quad 1 \leq M \leq \infty.
\]

For given real numbers \( \mu \) and \( \nu \), we will consider the linear functional

\[
L(\mu, \nu; f) = a_2 + \mu a_3 + \nu a_4
\]

in \( S^M \). The Koebe function \( K(z) \) maximizes \( \text{Re } L(0,0; f) \) in \( S \), and similarly, the Pick function \( P_M(z) \) maximizes \( \text{Re } L(0,0; f) \) in \( S^M \). We will describe the set of \( \mu \) and \( \nu \) for which the local maximum of \( \text{Re } L(\mu, \nu; f) \) in \( S^M \) is attained by the Pick function and will apply this result to the extremal problems for the class \( S \) or \( S^M \) stated in the introduction.

We write \( a_k(t) \equiv x_{2k-3}(t) + ix_{2k-2}(t) \), \( k = 2, 3, 4 \). Substituting (2) into (1) we obtain the following differential equations

\[
\begin{align*}
\dot{x}_1(t) &= -2 \cos u, \quad x_1(0) = 0, \\
\dot{x}_2(t) &= 2 \sin u, \quad x_2(0) = 0, \\
\dot{x}_3(t) &= -4(x_1 \cos u + x_2 \sin u) + 2(t-1) \cos 2u, \quad x_3(0) = 0, \\
\dot{x}_4(t) &= 4(x_1 \sin u - x_2 \cos u) - 2(t-1) \sin 2u, \quad x_4(0) = 0, \\
\dot{x}_5(t) &= -2((2x_3 + x_1^2 - x_2^2) \cos u + 2(x_4 + x_1 x_2) \sin u) \\
&\quad + 6(t-1)(x_1 \cos 2u + x_2 \sin 2u) - 2(t-1)^2 \cos 3u, \quad x_5(0) = 0.
\end{align*}
\]
The extremal problem

\[ \text{Re } L(\mu, \nu; f) \rightarrow \max \]

in the class \( S^M \) is equivalent to the extremal problem

\[ x_1(1 - 1/M) + \mu x_3(1 - 1/M) + \nu x_5(1 - 1/M) \rightarrow \max \]

for solutions to the system (4). The parametric representation (4) for the coefficients generated by Löwner’s equation allows us to apply the classical variational methods [3] or Pontryagin’s maximum principle [10]. We introduce the Hamiltonian function in order to formulate the necessary extremum conditions for the problem (5)

\[
H(t, x, \Psi, u) = -2 \cos u \Psi_1 + 2 \sin u \Psi_2 \\
- (4(x_1 \cos u + x_2 \sin u) - 2(t - 1) \cos 2u) \Psi_3 \\
+ (4(x_1 \sin u - x_2 \cos u) - 2(t - 1) \sin 2u) \Psi_4 \\
- (2(2x_3 + x_1^2 - x_2^2) \cos u + 2(x_4 + x_1 x_2) \sin u) \\
- 6(t - 1)(x_1 \cos 2u + x_2 \sin 2u) + 2(t - 1)^2 \cos 3u) \Psi_5,
\]

where \( x = (x_1, \ldots, x_5)^T \) satisfies (4) and \( \Psi = (\Psi_1, \ldots, \Psi_5)^T \) satisfies the conjugate system

\[
\dot{\Psi}_1 = 4 \cos u \Psi_3 - 4 \sin u \Psi_4 + (4x_1 \cos u + 4x_2 \sin u - 6(t - 1) \cos 2u) \Psi_5, \\
\dot{\Psi}_2 = 4 \sin u \Psi_3 + 4 \cos u \Psi_4 - (4x_2 \cos u - 4x_1 \sin u + 6(t - 1) \sin 2u) \Psi_5, \\
\dot{\Psi}_3 = 4 \cos u \Psi_5, \\
\dot{\Psi}_4 = 4 \sin u \Psi_5, \\
\dot{\Psi}_5 = 0,
\]

and the transversality conditions

\[
\Psi_1(1 - 1/M) = 1, \quad \Psi_3(1 - 1/M) = \mu, \quad \Psi_5(1 - 1/M) = \nu, \\
\Psi_2(1 - 1/M) = \Psi_4(1 - 1/M) = 0.
\]

The optimal control function \( u^* \) corresponding to the extremal function \( f^* \) in (5) satisfies Pontryagin’s maximum principle

\[
\max_u H(t, x^*, \Psi^*, u) = H(t, x^*, \Psi^*, u^*), \quad 0 \leq t \leq 1 - 1/M,
\]

where \( (x^*, \Psi^*) \) is the solution to (4) and (7) with \( u = u^* \) in their right-hand sides. Hence, \( u^* \) is a root of the equation

\[
H_u(t, x, \Psi, u) = 0
\]

for \( x = x^* \) and \( \Psi = \Psi^* \).
Lemma 1. Let us suppose that a control function $u$ in (4) and (7-8) generates the solutions $x(t)$ and $\Psi(t)$ for which $u$ satisfies (9), is unique up to the $2\pi$-translation, and

(11) \quad H_{uu}(t, x, \Psi, \pi) \neq 0, \quad 0 \leq t \leq 1 - 1/M.

Let us denote by $(x(t, \xi), \Psi(t, \xi))$ solutions to (4) and (7) with the initial conditions $\Psi(0, \xi) = \Psi(0) + \xi$ and $u = u(t, \xi)$ in their right-hand sides satisfying the maximum principle (9). Then for $\xi \to 0$, we have the following asymptotic behaviour

$$\|(x(1 - 1/M, \xi), \Psi(1 - 1/M, \xi)) - (x(1 - 1/M), \Psi(1 - 1/M))\| = o(1),$$

where $\| \cdot \|$ is the Euclidean vector norm.

Proof. Since there exists a unique solution $u$ satisfying (9) and (11), the same is true for a slightly changed parameters of the function $H$. Therefore, equations (9) and (10) locally determine a unique continuous implicit function $u = u(t, x, \Psi)$ satisfying the maximum principle. Writing $u(t, \xi) = u(t, x(t, \xi), \Psi(t, \xi))$ we substitute it into (4) and (7). Now we apply the theorem on the continuous dependence of solutions of differential equations on the initial conditions and complete the proof of Lemma 1. □

If the Pick function $P_M$ is extremal for (5), then $u = \pi$ is the optimal control function, (4) and (7) give $(x(t), \Psi(t)) = (x^0(t), \Psi^0(t))$, where

(12) \quad x^0_1(t) = 2t, \quad x^0_3(t) = 5t^2 - 2t, \quad x^0_2(t) = x^0_4(t) = 0,

and

(13) \quad \Psi^0_1(t) = \nu \left( t - 1 + \frac{1}{M} \right)^2 + \left( \frac{14\nu}{M} - 8\nu - 4\mu \right) \left( t - 1 + \frac{1}{M} \right) + 1,

\quad \Psi^0_3(t) = -4\nu \left( t - 1 + \frac{1}{M} \right) + \mu, \quad \Psi^0_2(t) = \nu, \quad \Psi^0_4(t) = 0.

The conditions of Lemma 1 play the key role as a necessary local extremum condition for the Pick function $P_M$. To verify these we substitute (12) and (13) into the Hamiltonian function $H(t, x, \Psi, u)$ given by (6) and study the extremum properties of $H(t, u) = H(t, x^0(t), \Psi^0(t), u)$ which is just a cubic polynomial of $\cos u$. Let us describe a set of suitable real parameters $(\mu, \nu)$ satisfying Lemma 1. Let $D(M)$ denote the maximal domain in the $(\mu, \nu)$-plane which is starlike with the respect to the origin and satisfies the following conditions:

[i] $H(t, u)$ as a function of $y = \cos u$ attains its maximum on $[-1, 1]$ only at $y = -1$ for all $t \in [0, 1 - 1/M]$;

[ii] $H_{uu}(t, \pi) \neq 0, 0 \leq t \leq 1 - 1/M$.

We will consider $(\mu, \nu) \in D(M)$. The point $x^0(1 - 1/M)$ belongs to the boundary $\partial V^M_4$ of $V^M_4$ and is given by the Pick function. Each $x \in \partial V^M_4$ can be obtained as a solution to the system (4) with a certain optimal control function $u$. Lemma 1 admits a reverse formulation.
Lemma 2. Let \((\mu, \nu) \in D(M)\) and denote by \((x^-(t, \xi), \Psi^-(t, \xi))\) solutions to (4) and (7) with the boundary conditions \(\Psi^-(1-1/M, \xi) = \Psi^0(1-1/M) + \xi\). Let \(u = u^-(t, \xi)\) in their right-hand sides satisfy the maximum principle (9). If

\[
\|x^-(1-1/M, \xi) - x^0(1-1/M)\| = o(1), \quad \text{as } \xi \to 0,
\]

then

\[
\|\Psi^-(0, \xi) - \Psi^0(0)\| = o(1).
\]

The proof of Lemma 2 is similar to that of Lemma 1 reversing the direction of variation of \(t\) from \(1 - 1/M\) to 0 and noting that \(x^-(0) = x^0(0) = 0\).

Lemmas 1 and 2 imply that if \((x(t), \Psi(t))\) is given by (4) and (7), \((x(1-1/M), \Psi(1-1/M))\) is close to \((x^0(1-1/M), \Psi^0(1-1/M))\), and \(x(1-1/M) \in \partial V_4^M\), then \((x(t), \Psi(t))\) is equal to \((x(t, \xi), \Psi(t, \xi))\) for a certain \(\xi\) close to 0.

The principles of calculus of variations interpret geometrically the transversality conditions as an orthogonality property of \(\Psi(1-1/M)\) to all possible variations of \(x(1-1/M)\) in \(\partial V_4^M\). Nevertheless, we rigorously prove this fact for completeness.

Lemma 3. Let us suppose that \((\mu, \nu) \in D(M)\) and the initial conditions in (7) are

\[
\Psi(0, \xi) = \Psi^0(0) + \xi, \quad \xi = \varepsilon e, \quad \varepsilon > 0, \quad e = (e_1, \ldots, e_5)^T, \quad \|e\| = 1,
\]

and

\[
x(t, \xi) = x^0(t) + \varepsilon \delta x(t) + o(\varepsilon), \quad \varepsilon \to 0.
\]

Then \(\Psi^0(1-1/M)\) is orthogonal to \(\delta x(1-1/M)\), if \(\delta x(1-1/M) \neq 0\).

Proof. First we note that the conditions of Lemma 1 guarantee the differentiability of \(x(t, \varepsilon e)\) with respect to \(\varepsilon\) at \(\varepsilon = 0\). Therefore, the representation (15) is valid and according to Lemmas 1 and 2 the expansion (15) produces all possible variations \(\delta x(1-1/M)\) at \(x^0(1-1/M)\) associated with \(\Psi^0(1-1/M)\).

We denote the column of the functions in the right-hand side of (4) by \(g(t, x, u)\) and rewrite (4) in the vector form as

\[
\dot{x} = g(t, x, u), \quad x(0) = 0.
\]

The system (7) is equivalent to

\[
\dot{\Psi} = -\frac{\partial H}{\partial x}(t, x, \Psi, u), \quad \Psi(0) = \Psi^0(0) + \varepsilon e^T.
\]

Substituting (15) into (16) we obtain

\[
\frac{d\delta x}{dt} = g_x \delta x + g_u u_\varepsilon,
\]
by differentiating with respect to $\varepsilon$ at $\varepsilon = 0$. This, together with (4), (6), (10), and (17), imply that

$$\frac{d}{dt}((\delta x)^T \Psi) = (\delta x)^T g_x^T \Psi + u_{\varepsilon} g_u^T \Psi + (\delta x)^T \dot{\Psi} = 0,$$

because the second term is equal to $u_{\varepsilon} H_u(t, x, \Psi, u) = 0$, and the remaining terms give the zero sum since $\dot{\Psi} = -g_x^T \Psi$.

Thus, we see that $(\delta x)^T \Psi$ does not depend on $t$ and vanishes at $t = 0$ because $\delta x(0) = 0$.

This completes the proof of Lemma 3. □

§3. Local extremum conditions

Lemma 4. Under the conditions of Lemma 3 we suppose that in the vector $e = (e_1, \ldots, e_5)^T$, which corresponds to the variation of $\Psi^0(0)$ in (14), the coordinates $e_2$ and $e_4$ vanish. Then $\delta x(1 - 1/M) = 0$.

Proof. The condition $e_2 = e_4 = 0$ implies that the systems (4) and (7) have vanishing coordinate solutions $x_2(t) = x_4(t) = 0$ and $\Psi_2(t) = \Psi_4(t) = 0$. In this case the Hamilton function $H$ is a polynomial of $y = \cos u$, which has a unique maximum on $[-1, 1]$ at $y = -1$. Its derivative with respect to $y$ does not vanish at $y = -1$ for $\varepsilon > 0$ sufficiently small. Hence, $u \equiv \pi$ is a unique optimal control function for such $\varepsilon$ and $x(1 - 1/M, \xi) = x^0(1 - 1/M)$, that ends the proof of Lemma 4. □

By analogy with the expansion (15) in Lemma 3 we have the expansion

$$\Psi(t, \xi) = \Psi^0(t) + \varepsilon \delta \Psi(t) + o(\varepsilon), \quad \varepsilon \to 0.$$

Lemma 4 shows that the condition $e_2 = e_4 = 0$ implies $\delta x = 0$ and $\delta \Psi_2 = \delta \Psi_4 = 0$. Only $\Psi_1$, $\Psi_3$, and $\Psi_5$ can vary in this case. It follows from Lemmas 1-4 that we should consider variations $\Psi(0, \xi)$ by (14) with $e_1 = e_3 = e_5 = 0$ in order to study the character of the point $x^0(1 - 1/M) \in \partial V_M^4$.

Let us set $\xi = (0, p, 0, q, 0)^T$ with arbitrary real $p$ and $q$, and study $x(1 - 1/M, \xi)$ in a neighborhood of $x^0(1 - 1/M)$. In other words, we will solve the systems (4) and (7) with the initial conditions $x(0) = 0$ and (14), which we rewrite with coordinates

$$\begin{align*}
\Psi_1(0) &= 3\nu \left(1 - \frac{1}{M}\right) \left(3 - \frac{5}{M}\right) + 4\mu \left(1 - \frac{1}{M}\right) + 1, \\
\Psi_2(0) &= p, \\
\Psi_3(0) &= 4\nu \left(1 - \frac{1}{M}\right) + \mu, \\
\Psi_4(0) &= q, \\
\Psi_5(0) &= \nu.
\end{align*}$$

(18)

Let

$$F : (p, q) \to x_1(1 - 1/M) + \mu x_3(1 - 1/M) + \nu x_5(1 - 1/M)$$
be a real valued mapping from the \((p, q)\)-plane onto the linear combination of the components of the solution to the Cauchy problem for the systems (4) and (7) with the initial conditions (18). The control function \(u\) in the right-hand side of (4) and (7) satisfies the maximum principle. The mapping \(F\) is well defined in a neighborhood of \((0, 0)\) if \((\mu, \nu) \in D(M)\). In this case \(u = u(t, x, \Psi)\) is an implicit function defined by (10). Since \((x, \Psi)\) depends only on \((p, q)\), we denote by \(u(t, p, q) = u(t, x(p, q), \Psi(p, q))\). We note that \(F(0, 0) = \text{Re} L(\mu, \nu; P_M)\) and the values of \(F(p, q)\) correspond to those of \(\text{Re} L(\mu, \nu; f)\) with respect to the variations of \(P_M\) generated by the initial conditions \(\Psi_2(0) = p\) and \(\Psi_4(0) = q\).

**Theorem 5.** Let us suppose that \((\mu, \nu) \in D(M)\). If \(P_M\) locally maximizes \(\text{Re} L(\mu, \nu; f)\) in \(S(M)\), then

\[
F_{pp}(0, 0) \leq 0, \quad F_{pp}(0, 0) F_{qq}(0, 0) - F_{pq}(0, 0)^2 \geq 0.
\]

Conversely, if

\[
F_{pp}(0, 0) < 0, \quad F_{pp}(0, 0) F_{qq}(0, 0) - F_{pq}(0, 0)^2 > 0,
\]

then \(P_M\) locally maximizes \(\text{Re} L(\mu, \nu; f)\) in \(S(M)\).

**Proof.** We first claim that \((0, 0)\) is a critical point of \(F(p, q)\). Indeed, substituting \(u = u(t, p, q)\) in the three equations in (4) (for \(x_1, x_3, x_5\)) and differentiating them with respect to \(p\) and \(q\), we obtain differential equations for \((x_k)_p\) and \((x_k)_q\) for \(k = 1, 3, 5\), with vanishing initial conditions. Substituting there \(p = q = 0\) and \(u = \pi, x_2(t) = x_4(t) = 0\) we find that all derivatives \((\dot{x}_k)_p\) and \((\dot{x}_k)_q\) for \(k = 1, 3, 5\) are identically zeros and, hence, \(F_p(0, 0) = F_q(0, 0) = 0\).

The first statement of Theorem 5 means that the quadratic form of the second differential of \(F\) at \((0, 0)\) is negatively semi-definite which is the necessary condition of local extremum. Similarly, the second statement signifies that the above quadratic form is negative definite which is the sufficient local extremum condition. This completes the proof of Theorem 5. \(\square\)

The same reasoning can be made for the class \(S^M_R\) of functions \(f \in S^M\) with real Taylor coefficients \(a_n, n \geq 2\). The coefficients of an arbitrary boundary function \(f_R \in S^M_R\) for the set \(V^M_R = \{(a_2, a_3, a_4) : f \in S^M_R\}\) can be obtained by integrating the systems (4) and (7) with the control function \(u\) satisfying the maximum principle (9) and with vanishing initial values of \(\Psi_2(0)\) and \(\Psi_4(0)\). Therefore, variations of \(\Psi_2(0)\) and \(\Psi_4(0)\) are forbidden and the Pick function \(P_M \in S^M_R\) locally maximizes \(L(\mu, \nu; f)\) in \(S^M_R\) if \((\mu, \nu) \in D(M)\).

Now we will apply Theorem 5 to construct an analytic and numerical solution process. We need to calculate the partial derivatives \(u_p\) and \(u_q\) at \((0, 0)\) to evaluate the partial derivatives of \(F\) at \((0, 0)\). Differentiating (10) with respect to \(p\) and \(q\) we obtain

\[
H_{ux} x_p + H_{up} \Psi_p + H_{uu} u_p = 0,
\]
\[
H_{ux} x_q + H_{up} \Psi_q + H_{uu} u_q = 0,
\]

which leads us to the formulae

\[
u_p = -\frac{H_{ux} x_p + H_{up} \Psi_p}{H_{uu}}, \quad \nu_q = -\frac{H_{ux} x_q + H_{up} \Psi_q}{H_{uu}}.
\]
and

\[ u_q = - \frac{H_{ux}x_q + H_u \Psi_{q}}{H_{uu}}. \]

Direct calculation gives

\[ H_{uu}(t, x^0, \Psi^0, \pi) = -2 \left[ 16\nu^2 - 4 \left( 2\nu + \frac{4\nu}{M} - \mu \right) t + 2\nu + 1 - \frac{4(2\nu + \mu)}{M} + \frac{15\nu}{M^2} \right]. \]

Differentiating (6) with respect to corresponding variable \( u = \pi \) we find

\[ H_{ux_2}(t, x^0, \Psi^0, \pi) = 4 \left( \nu \left( t + 1 - \frac{4}{M} \right) + \mu \right), \]

\[ H_{ux_4}(t, x^0, \Psi^0, \pi) = 4\nu, \]

\[ H_{u\psi_2}(t, x^0, \Psi^0, \pi) = -2, \]

\[ H_{u\psi_4}(t, x^0, \Psi^0, \pi) = 4(1 - 3t). \]

Differentiating (7) with respect to \( p \) and \( q \) at \((0, 0)\) with \( u = \pi, (x_1)_p(t) = (x_1)_q(t) = x_2(t) = 0 \) we see that

\[ (\Psi_1)_p(t) = (\Psi_1)_q(t) = (\Psi_3)_p(t) = (\Psi_3)_q(t) = (\Psi_5)_p(t) = (\Psi_5)_q(t) = 0. \]

The formulae (22–26) allow us to calculate the numerators in (19) and (20) as

\[ 4 \left( \nu \left( t + 1 - \frac{4}{M} \right) + \mu \right) (x_2)_p + 4\nu(x_4)_p - 2(\Psi_2)_p + 4(1 - 3t)(\Psi_4)_p \]

and

\[ 4 \left( \nu \left( t + 1 - \frac{4}{M} \right) + \mu \right) (x_2)_q + 4\nu(x_4)_q - 2(\Psi_2)_q + 4(1 - 3t)(\Psi_4)_q \]

respectively.

From (4) and (7) we conclude that \( \dot{\Psi}_4 = 2\nu \dot{x}_2 \) yields the equalities \((\dot{\Psi}_4)_p = 2\nu(\dot{x}_2)_p\) and \((\dot{\Psi}_4)_q = 2\nu(\dot{x}_2)_q\). The initial conditions \((\Psi_4)_p(0) = 0\) and \((\Psi_4)_q(0) = 1\) imply that \((\Psi_4)_p = 2\nu(x_2)_p\) and \((\Psi_4)_q = 2\nu(x_2)_q + 1\).

We substitute the last relations and (21) into (19–20), and finally, get

\[ u_p = \frac{((3 - 5t - 4/M)\nu + \mu)2y_4 + 2\nu y_5 - y_6}{16\nu^2 - 4(2\nu + 4\nu/M - \mu)t + 2\nu + 1 - 4(2\nu + \mu)/M + 15\nu/M^2}, \]

where \( y_4 := (x_2)_p, y_5 := (x_4)_p, y_6 := (\Psi_2)_p, \) and

\[ u_q = \frac{((3 - 5t - 4/M)\nu + \mu)2y_{10} + 2\nu y_{11} - y_{12} + 2(1 - 3t)}{16\nu^2 - 4(2\nu + 4\nu/M - \mu)t + 2\nu + 1 - 4(2\nu + \mu)/M + 15\nu/M^2}. \]
where \( y_{10} := (x_2)_q, y_{11} := (x_4)_q, y_{12} := (Ψ_2)_q \).

Set \( y_1 := (x_1)_{pp}, y_2 := (x_3)_{pp}, \) and \( y_3 := (x_5)_{pp} \). Differentiating (4) twice with respect to \( p \) at \( u = π, x_2 = x_4 = Ψ_2 = Ψ_4 = (x_1)_p = (x_3)_p = 0 \) we obtain

\[
\dot{y}_1 = -2u_p^2, \quad y_1(0) = 0, \\
\dot{y}_2 = 4(y_1 + 2y_4 u_p - 2(2t - 1)u_p^2), \quad y_2(0) = 0, \\
\dot{y}_3 = 2(7t - 3)y_1 + 4y_2 - 4y_4^2 + 8(5t - 3)y_4 u_p + 8y_5 u_p \\
- 2(47t^2 - 46t + 9)u_p^2, \quad y_3(0) = 0.
\]

Similarly, we differentiate the remaining equations in (4) with respect to \( p \) and obtain

\[
\dot{y}_4 = -2u_p, \quad y_4(0) = 0, \\
\dot{y}_5 = 4(y_4 + (1 - 3t)u_p), \quad y_5(0) = 0.
\]

Finally, we differentiate the second equation in (7) with respect to \( p \) and get

\[
\dot{y}_6 = -4ν \left( \left( t + 1 - \frac{4}{M} + \frac{μ}{ν} \right) u_p + y_4 \right), \quad y_6(0) = 1.
\]

Summarizing, we have deduced an evaluation algorithm for \( F_{pp}(0, 0) \) expressed by the following theorem.

**Theorem 6.** Suppose \((μ, ν) ∈ D(M)\). Let \( y_1(t), \ldots, y_6(t), 0 ≤ t ≤ 1 - 1/M, \) be solutions to the Cauchy problem for the differential equations (29–34). Then the relation

\[
F_{pp}(0, 0) = y_1(1 - 1/M) + μy_2(1 - 1/M) + νy_3(1 - 1/M)
\]

is valid.

**Remark.** The subsystem (32–34) can be solved independently because these equations do not contain \( y_1, y_2, \) and \( y_3 \).

Calculation of \( F_{qq}(0, 0) \) and \( F_{pq}(0, 0) \) may be handled in much the same way. Let \( y_7 := (x_1)_{qq}, y_8 := (x_3)_{qq}, \) and \( y_9 := (x_5)_{qq} \). From (4) we have

\[
\dot{y}_7 = -2u_q^2, \quad y_7(0) = 0, \\
\dot{y}_8 = 4(y_7 + 2y_{10} u_q - 2(2t - 1)u_q^2), \quad y_8(0) = 0, \\
\dot{y}_9 = 2(7t - 3)y_7 + 4y_8 - 4y_{10}^2 \\
+ 8(5t - 3)y_{10} u_q + 8y_{11} u_q - 2(47t^2 - 46t + 9)u_q^2, \quad y_9(0) = 0.
\]

Differentiating the two even equations in (4) and the second equation in (7) with respect to \( q \) we get

\[
\dot{y}_{10} = -2u_q, \quad y_{10}(0) = 0, \\
\dot{y}_{11} = 4(y_{10} + (1 - 3t)u_q), \quad y_{11}(0) = 0, \\
\dot{y}_{12} = -4ν \left( \left( t + 1 - \frac{4}{M} + \frac{μ}{ν} \right) u_q + y_{10} \right) - 4, \quad y_{12}(0) = 0.
\]
Let \( y_{13} := (x_1)_{pq} \), \( y_{14} := (x_3)_{pq} \), and \( y_{15} := (x_5)_{pq} \). We continue in this fashion differentiating (4) with respect to \( p \), and subsequently, with respect to \( q \) and obtain

\[
\begin{align*}
\dot{y}_{13} &= -2u_pu_q, \quad y_{13}(0) = 0, \\
\dot{y}_{14} &= 4(y_{13} + y_4u_q + y_{10}u_p) - 8(2t - 1)u_pu_q, \quad y_{14}(0) = 0, \\
\dot{y}_{15} &= 2(7t - 3)y_{13} + 4y_{14} - 4y_4y_{10} + 4(5t - 3)(y_4u_q + y_{10}u_p) \\
&\quad + 4y_5u_q + 4y_{11}u_p - 2(47t^2 - 46t + 9)u_pu_q, \quad y_{15}(0) = 0.
\end{align*}
\]

Summing up the calculation process we derive the following theorem.

**Theorem 7.** Suppose \((\mu, \nu) \in D(M)\). Let \( y_7(t), \ldots, y_{12}(t), 0 \leq t \leq 1 - 1/M \), be solutions to the Cauchy problem for the differential equations (35–40). Then, the relation

\[
F_{qq}(0, 0) = y_7(1 - 1/M) + \mu y_8(1 - 1/M) + \nu y_9(1 - 1/M)
\]

holds. Let \( y_4(t), y_5(t), y_6(t), \) and \( y_{13}(t), y_{14}(t), y_{15}(t), 0 \leq t \leq 1 - 1/M \), be solutions to the Cauchy problem for the differential equations (32–34) and (41–43) respectively. Then, the relation

\[
F_{pq}(0, 0) = y_{13}(1 - 1/M) + \mu y_{14}(1 - 1/M) + \nu y_{15}(1 - 1/M)
\]

holds.

**Remark.** As in the remark after Theorem 6 we note that the subsystem (38–40) can be solved independently because these equations do not contain \( y_7, y_8, \) and \( y_9 \).

### §4. Explicit integration

An explicit integration of the systems in Theorems 6 and 7 is possible only in the case \( \nu = 0 \), i.e., when the linear functional \( L \) does not depend on \( a_4 \). In this case the two last equations in (4) and (7) disappear and the mapping \( F \) becomes a function of a variable \( p \) as

\[
F : p \rightarrow x_1(1 - 1/M) + \mu x_3(1 - 1/M).
\]

The criterion in Theorem 5 is reduced to the inequality \( F''(0) < 0 \), where \( F''(0) = y_1(1 - 1/M) + \mu y_2(1 - 1/M) \). The systems (29–34), (35–40), and (41–43) are reduced to the four equations (29), (30), (32), and (34). We substitute \( \nu = 0 \) into (34, 27) and obtain that

\[
\dot{y}_6 = -4\mu u_p, \quad y_6(0) = 1,
\]

and

\[
u_p = \frac{2\mu y_4 - y_6}{4\mu t + 1 - 4\mu/M}.
\]
Compare (32, 44) and observe that \( \dot{y}_6 = 2\mu \dot{y}_4 \), which implies
\[
y_6(t) = 2\mu y_4(t) + 1.
\]

This equation allows us to exclude \( y_6 \) from (45) and integrate the system of equations (29), (30), and (32) with
\[
(46) \quad u_p = \frac{-1}{4\mu t + 1 - 4\mu/M}.
\]

The equations (29) and (32) give
\[
(47) \quad y_1(t) = \frac{1}{2\mu} \left( \frac{1}{1 - 4\mu/M + 4\mu t} - \frac{1}{1 - 4\mu/M} \right)
\]
and
\[
(48) \quad y_4(t) = \frac{1}{2\mu} \log \frac{1 - 4\mu/M + 4\mu t}{1 - 4\mu/M}.
\]

Substituting (46) and (48) into (30) we integrate it, then, taking into account (47), we finally obtain that
\[
(49) \quad y_1(t) + \mu y_2(t) = \frac{-1}{2\mu} \left[ \log^2((1 + 4\mu - 8\mu/M)(1 - 4\mu/M)) + \log \frac{1 + 4\mu - 8\mu/M}{1 - 4\mu/M} \right] .
\]

Making use of (49) we formulate the following theorem as a corollary of Theorem 6.

**Theorem 8.** Suppose \((\mu, 0) \in D(M)\) and \(\mu\) satisfies the inequality
\[
(50) \quad \frac{1}{2\mu} \left[ \log^2((1 + 4\mu - 8\mu/M)(1 - 4\mu/M)) + \log \frac{1 + 4\mu - 8\mu/M}{1 - 4\mu/M} \right] > 0.
\]

Then, the Pick function \(P_M\) locally maximizes \(\text{Re} L(\mu, 0; f) = \text{Re} (a_2 + \mu a_3)\) in \(S^M\). If the left-hand side of (50) is negative, then \(P_M\) does not give a local maximum to \(\text{Re} L(\mu, 0; f)\) in \(S^M\).

We note that \(\{ \mu : (\mu, 0) \in D = D(\infty) \} = \{ \mu : \mu > -0.25 \}\). If \(M = \infty\), then the inequality (50) is of the form
\[
-\frac{1}{2\mu} (1 + \log(1 + 4\mu)) \log(1 + 4\mu) < 0,
\]
that is equivalent to the best possible inequality \(\mu > -\lambda_0 := -(e - 1)/4e\). This means that the Bombieri number \(\sigma_{32}\) is equal to \(\lambda_0\). This result has been obtained recently by R. Greiner and O. Roth [9].
§5. Bombieri’s number $\sigma_{42}$

Now we apply Theorems 5-7 to evaluate Bombieri’s number $\sigma_{42}$.

**Proposition.** Let $m, n \geq 2$ be fixed integers. Then

$$\sigma_{mn} = \sup \{ \lambda \in \mathbb{R} : \text{Re} (a_n - \lambda a_m), \text{is locally maximized on } S \text{ by } K(z) \}.$$

The proof of this statement is quite obvious and can be found, e.g., in [9].

According to Theorem 5, Bombieri’s number $\sigma_{42}$ is calculated as

$$- \inf \{ \nu' : F_{pp}(0, 0) < 0 \text{ and } F_{pp}(0, 0) F_{qq}(0, 0) - F_{pq}^2(0, 0) > 0 \text{ for } \mu = 0, \ M = \infty, \ \nu \in [\nu', 0] \}.$$

Theorems 6 and 7 reduce the problem to the solution of the equations

(51) \quad $y_1(1) + \nu y_3(1) = 0$,

or

(52) \quad $(y_1(1) + \nu y_3(1))(y_7(1) + \nu y_9(1)) - (y_{13}(1) + \nu y_{15}(1))^2 = 0$,

where $y_1(t), \ldots, y_{15}(t)$ are solutions to the Cauchy problem for the differential equations (29–43) with $\mu = 0$ and $M = \infty$. Thus we are able to formulate the following result.

**Theorem 9.** Bombieri’s number $\sigma_{42}$ is equal to the maximum of the negative roots of the equations (51) and (52) multiplied by $(-1)$, where $y_1(t), \ldots, y_{15}(t)$ are solutions to (29–43) for $\mu = 0$ and $M = \infty$.

To illustrate this result we give the numerical approximation $\sigma_{42} \approx 0.050057\ldots$

**Remark.** We used Wolfram’s *Mathematica* to evaluate numerically $\sigma_{42}$ as well as other Bombieri’s numbers in the next sections. More precisely, the combination of 4-th order Runge-Kutta and Adams methods is used. It allows to reach a higher precision at short computational time. Actually, the level of precision imply the number of iterations and can be prescribed as the machine precision. For solving the above systems the level $10^{-12}$ is reached in time of order of few minutes on a machine with a Pentium 4 processor (1800 Mhz) and having 512 Mb of RAM.

**Remark.** Note that $\{ \nu : (0, \nu) \in D = D(\infty) \} = \{ \nu : \nu > -0.1 \}$. O. Roth showed (private communication) that $\sigma_{32} \leq 0.050284\ldots$, the latter number is equal to $(2 - \text{Re } a_2)/(4 - \text{Re } a_4)$ for a function sequence which is critical in the problem of finding $\sigma_{32}$. 
§6. Bombieri’s numbers $\sigma_{24}$ and $\sigma_{34}$

The problem of finding Bombieri’s number $\sigma_{24}$ is reduced to description of $\nu$ for which the Koebe function $K$ minimizes $\text{Re} \ L(0, \nu; f)$ in $S$. To follow the preceding scheme we will write $L^1(\nu; f) = a_4 + \nu a_2$ in place of $L$. Now $\sigma_{24}$ is equal to the supremum over all real values $\lambda_{24}$, such that $\text{Re} \ L^1(-\lambda_{24}; f)$ is locally maximized by the Koebe function $K$ in $S$. Again we consider the system of equations (4), the Hamilton function $H(t, x, \Psi, u)$ given by (6), and the system of equations (7), where the transversality conditions (8) are replaced by

\begin{equation}
(53) \quad \Psi_1(1) = \nu, \quad \Psi_5(1) = 1, \quad \Psi_2(1) = \Psi_3(1) = \Psi_4(1) = 0.
\end{equation}

For $u = \pi$ let us denote the integrals for (4) and (7) by $(x^0(t), \Psi^1(t))$, where $x^0_1(t), \ldots, x^0_4(t)$ are given by (12) and

\begin{equation}
(54) \quad \Psi^1_1(t) = t^2 - 10t + 9 + \nu, \quad \Psi^1_3(t) = -4(t - 1), \quad \Psi^1_1(t) = 1, \quad \Psi^1_2(t) = \Psi^1_4(t) = 0.
\end{equation}

As before, the control function $u = u(t)$ satisfies the maximum principle (9) and, hence, the equation (10). We consider only $\nu \in D^1$, where $D^1$ denotes a maximal interval on the $\nu$-axis that satisfies the following conditions:

1. $H^1(t, u) = H(t, x^0(t), \Psi^1(t), u)$ as a function of $y = \cos u$ attains its maximum on $[-1, 1]$ only at $y = -1$ for all $t \in [0, 1]$;
2. $H^1_{uu}(t, \pi) \neq 0$, $0 \leq t \leq 1$.

Note that $D^1 = \{\nu : \nu > -1\}$.

Similarly to Section 3 we solve the systems (4) and (7) with the initial conditions $x(0) = 0$ and

\begin{equation}
(55) \quad \Psi_1(0) = 9 + \nu, \quad \Psi_2(0) = p, \quad \Psi_3(0) = 4, \quad \Psi_4(0) = q, \quad \Psi_5(0) = 1,
\end{equation}

where $p$ and $q$ are arbitrary real numbers. The initial conditions (55) are thought of as variations of (54) at $t = 0$.

Let

\begin{equation}
F^1 : (p, q) \rightarrow \nu x_1(1) + x_3(1)
\end{equation}

be a real valued mapping from the $(p, q)$-plane to the linear combination of the components of a solution to the Cauchy problem for (4) and (7) with the initial conditions (55) and the control $u$ satisfying the maximum principle. The mapping $F^1$ is well defined in a neighborhood of $(0, 0)$ if $\nu \in D^1$, and $u = u(t, x, \Psi)$ is an implicit function defined by (10). We denote by $u(t, p, q) = u(t, x(p, q), \Psi(p, q))$. Lemmas 1-4 and Theorem 5 admit an analogous formulation for the functional $L^1(\nu; f)$. So, if $K$ locally maximizes $\text{Re} \ L^1(\nu; f)$ in $S$ for $\nu \in D^1$, then

\begin{equation}
F^1_{pp}(0, 0) \leq 0, \quad F^1_{pp}(0, 0)F^1_{qq}(0, 0) - (F^1_{pq})^2(0, 0) \geq 0.
\end{equation}

Conversely, if

\begin{equation}
F^1_{pp}(0, 0) < 0, \quad F^1_{pp}(0, 0)F^1_{qq}(0, 0) - (F^1_{pq})^2(0, 0) > 0,
\end{equation}

then, $\nu \in D^1$ and $u(t, p, q)$ is the linear combination of the components of a solution to the Cauchy problem for (4) and (7) with the initial conditions (55) and the control $u$ satisfying the maximum principle.
then $K$ locally maximizes $\Re L^1(\nu; f)$ in $S$. Direct calculation gives

$$H_{uu}(t, x^0, \Psi^1, \pi) = -2(16t^2 - 8t + \nu + 2),$$
$$H_{ux_2}(t, x^0, \Psi^1, \pi) = 4(t + 1), \quad H_{ux_4}(t, x^0, \Psi^1, \pi) = 4,$$
$$H_{u\Psi_2}(t, x^0, \Psi^1, \pi) = -2, \quad H_{u\Psi_4}(t, x^0, \Psi^1, \pi) = 4(1 - 3t).$$

The formula (26) remains true as well as $(\Psi_4)_p = 2(x_2)_p$ and $(\Psi_4)_q = 2(x_2)_q + 1$. Substituting the last relations and (56–58) into (19) and (20) we obtain

$$u_p = \frac{(3 - 5t)2y_4 + 2y_5 - y_6}{16t^2 - 8t + \nu + 2},$$
$$u_q = \frac{(3 - 5t)2y_{10} + 2y_{11} - y_{12} + 2(1 - 3t)}{16t^2 - 8t + \nu + 2}.$$

Evidently, the formulae (29–33), (35–39), and (41–43) are valid for our case, whereas the equations (34) and (40) are transformed into

$$\dot{y}_6 = -4(t + 1)u_p - 4y_4, \quad y_6(0) = 1,$$
$$\dot{y}_{12} = -4(t + 1)u_q - 4y_{10} - 4, \quad y_{12}(0) = 0,$$

respectively.

Summing up above calculation for the evaluation algorithm we state that by analogy with Theorems 6 and 7 the problem of finding $\sigma_{24}$ is reduced to the solution of the equations

$$\nu y_1(1) + y_3(1) = 0$$

or

$$(\nu y_1(1) + y_3(1))(\nu y_7(1) + y_9(1)) - (\nu y_{13}(1) + y_{15}(1))^2 = 0$$

where $y_1(t), \ldots, y_{15}(t)$ are solutions to the Cauchy problem for the differential equations (29–33), (61), (35–39), (62), and (41–43) with $u_p$ and $u_q$ given by (59) and (60). We formulate the following result.

**Theorem 10.** Bombieri’s number $\sigma_{24}$ is equal to the maximum of negative roots of the equations (63) and (64) multiplied by $(-1)$, where $y_1(t), \ldots, y_{15}(t)$ are solutions to (29–33), (61), (35–39), (62), (41–43), and (59–60).

Similarly to Bombieri’s number $\sigma_{42}$, the numerical approximation for $\sigma_{24}$ is $0.969556\ldots$, which is the maximal negative root of (64).

In order to evaluate $\sigma_{34}$ we consider the functional $N(\mu; f) = a_4 + \mu a_3$ as soon as $\sigma_{34}$ is equal to the supremum over all real values $\lambda_{34}$ for which $\Re N(-\lambda_{34}; f)$ is locally maximized by
the Koebe function $K$ in $S$. Now the systems (4) and (7) are supplied with the transversality conditions

$$
\Psi_3(1) = \mu, \quad \Psi_5(1) = 1, \quad \Psi_1(1) = \Psi_2(1) = \Psi_4(1) = 0.
$$

For $u = \pi$ we denote the integrals of (4) and (7) by $(x_0(t), \Psi_2(t))$, where $x_1^0(t), \ldots, x_4^0(t)$ are given by (12) and

$$
\Psi_1^2(t) = t^2 - (10 + 4\mu)t + 9 + 4\mu, \quad \Psi_3^2(t) = -4(t - 1) + \mu,
$$

$$
\Psi_5^2(t) = 1, \quad \Psi_2^2(t) = \Psi_4^2(t) = 0.
$$

We consider only $\mu \in D^2$, where $D^2$ is a maximal interval on the $\mu$-axis which satisfies the conditions:

[i] $H^2(t, u) = H(t, x^0(t), \Psi^2(t), u)$, as a function of $y = \cos u$, attains its maximum in $[-1, 1]$ only at $y = -1$ for all $t \in [0, 1]$;

[ii] $H_{uu}^2(t, \pi) \neq 0, \quad 0 \leq t \leq 1$.

Note that $D^2 = \{\mu : \mu > -2(\sqrt{2} - 1)\}$.

Again we solve the systems (4) and (7) with $x(0) = 0$ and

$$
\Psi_1(0) = 9 + 4\mu, \quad \Psi_2(0) = p, \quad \Psi_3(0) = 4 + \mu, \quad \Psi_4(0) = q, \quad \Psi_5(0) = 1,
$$

where $p$ and $q$ are arbitrary real numbers. The conditions (67) are thought of as variations of (66) at $t = 0$.

Let

$$
F^2 : (p, q) \rightarrow \mu x_2(1) + x_3(1)
$$

be a real valued mapping from $(p, q)$-plane into the linear combination of components of the solution to the Cauchy problem for (4) and (7) with the initial conditions (67) and $u$ satisfying the maximum principle. We preserve preceding denotations for $u(t, p, q)$, and similarly to Theorem 5, for $\mu \in D^2$, we assert, that if $K$ locally maximizes $\Re N(\mu; f)$ in the class $S$, then

$$
F_{pp}^2(0, 0) \leq 0, \quad F_{pp}^2(0, 0)F_{qq}^2(0, 0) - (F_{pq}^2)^2(0, 0) \geq 0.
$$

Conversely, if

$$
F_{pp}^2(0, 0) < 0, \quad F_{pp}^2(0, 0)F_{qq}^2(0, 0) - (F_{pq}^2)^2(0, 0) > 0,
$$

then $K$ locally maximizes $\Re N(\mu; f)$ in $S$.

Direct calculation gives

$$
H_{uu}(t, x^0, \Psi^2, \pi) = -4(8t^2 - (4 - 2\mu)t + 1),
$$

$$
H_{ux_2}(t, x^0, \Psi^2, \pi) = 4(t + 1 + \mu), \quad H_{ux_4}(t, x^0, \Psi^2, \pi) = 4,
$$

$$
H_u\Psi_2(t, x^0, \Psi^2, \pi) = -2, \quad H_u\Psi_4(t, x^0, \Psi^2, \pi) = 4(1 - 3t).$$
The formulae (19), (20), and (68–70) lead to

\[ u_p = \frac{(3 - 5t + \mu)2y_4 + 2y_5 - y_6}{16t^2 - (8 - 4\mu)t + 2} \]

\[ u_q = \frac{(3 - 5t + \mu)2y_{10} + 2y_{11} - y_{12} + 2(1 - 3t)}{16t^2 - (8 - 4\mu)t + 2} \]

The formulae (34) and (40) are changed to

\[ \dot{y}_6 = -4(t + 1 + \mu)u_p - 4y_4, \quad y_6(0) = 1, \]

\[ \dot{y}_{12} = -4(t + 1 + \mu)u_q - 4y_{10} - 4, \quad y_{12}(0) = 0, \]

respectively.

Let us sum up the results and state, that similarly to Theorems 6 and 7, the problem of finding \( \sigma_{34} \) is reduced to the solution of the equations

\[ \mu y_2(1) + y_3(1) = 0, \]

or

\[ (\mu y_2(1) + y_3(1))(\mu y_8(1) + y_9(1)) - (\mu y_{14}(1) + y_{15}(1))^2 = 0, \]

where \( y_1(t), \ldots, y_{15}(t) \) are solutions to the Cauchy problem for the differential equations (29–33), (73), (35–39), (74), and (41–43) with \( u_p \) and \( u_q \) given by (71) and (72).

**Theorem 11.** Bombieri’s number \( \sigma_{34} \) is equal to the maximum of the negative roots of equations (75) and (76) multiplied by \( -1 \), where \( y_1(t), \ldots, y_{15}(t) \) are solutions of (29–33), (73), (35–39), (74), (41–43), and (71–72).

Numerical methods applied to (75) and (76) show that \( \sigma_{34} \approx 0.791557 \ldots \)

§7. *Fourth coefficient of bounded univalent functions*

In this section we will find a sharp constant \( M_0 \), such that for all \( M \geq M_0 \) the Pick function \( P_M \) gives the local maximum to \( \text{Re} \ a_4 \) (and to \( |a_4| \)) over all univalent functions \( f \in S^M \). In the class \( S^M_R \subset S^M \) of functions with real Taylor coefficients the Pick function gives the maximum to \( a_4 \) for \( M \geq 11 \) and this constant is sharp (see [17], [19, p.163]). That is why we can consider only \( M \geq 11 \). For this problem an analog of Theorem 5 holds and allows us to construct an evaluation algorithm. Therefore, we again consider the systems (4) and (7) supplied now with the transversality conditions

\[ \Psi_5(1 - 1/M) = 1, \quad \Psi_1(1 - 1/M) = \cdots = \Psi_4(1 - 1/M) = 0. \]
For $u = \pi$, let us denote the integrals of (4) and (7) by $(x^0(t), \Psi^3(t))$, where

$$
\Psi^3_1(t) = t^2 - \left(10 - \frac{16}{M}\right)t + 9 - \frac{24}{M} + \frac{15}{M^2}, \quad \Psi^3_3(t) = -4 \left(t - 1 + \frac{1}{M}\right),
$$

$$
\Psi^3_2(t) = 1, \quad \Psi^3_4(t) = \Psi^3_5(t) = 0.
$$

Varying conditions (78) at $t = 0$ we put

$$
\Psi_1(0) = 9 - \frac{24}{M} + \frac{15}{M^2}, \quad \Psi_2(0) = p, \quad \Psi_3(0) = 4 \left(1 - \frac{1}{M}\right), \quad \Psi_4(0) = q, \quad \Psi_5(0) = 1,
$$

where $(p, q) \in \mathbb{R}^2$.

Let $F^3 : (p, q) \to x_5(1 - 1/M)$ be a mapping from the $(p, q)$-plane into the axis of the third component of the solution to the Cauchy problem for (4) and (7) with the initial conditions (79) and $u$ satisfying the maximum principle. Preserving notations of Section 6 for $u(t, p, q)$, similarly to Theorem 5, we observe that for $M > 11$, if $P_M$ locally maximizes $\text{Re } a_4$ in $S(M)$, then

$$
F^3_{pp}(0, 0) \leq 0, \quad F^3_{pp}(0, 0)F^3_{qq}(0, 0) - (F^3_{pq})^2(0, 0) \geq 0,
$$

and conversely, if

$$
F^3_{pp}(0, 0) < 0, \quad F^3_{pp}(0, 0)F^3_{qq}(0, 0) - (F^3_{pq})^2(0, 0) > 0,
$$

then $P_M$ locally maximizes $\text{Re } a_4$ in $S(M)$.

Direct calculation gives

$$
H_{wu}(t, x^0, \Psi^3, \pi) = -2 \left(16t^2 - \left(8 + \frac{16}{M}\right)t + 2 - \frac{8}{M} + \frac{15}{M^2}\right),
$$

$$
H_{ux_x}(t, x^0, \Psi^3, \pi) = 4 \left(t + 1 - \frac{4}{M}\right), \quad H_{ux_4}(t, x^0, \Psi^3, \pi) = 4,
$$

$$
H_{w\Psi_2}(t, x^0, \Psi^3, \pi) = -2, \quad H_{w\Psi_4}(t, x^0, \Psi^3, \pi) = 4(1 - 3t).
$$

The formulae (19), (20), and (80–82) lead to

$$
u_p = \frac{(3 - 4/M - 5t)2y_4 + 2y_5 - y_6}{16t^2 - (8 + 16/M)t + 2 - 8/M + 15/M^2},
$$

$$
u_q = \frac{(3 - 4/M - 5t)2y_10 + 2y_11 - y_12 + 2(1 - 3t)}{16t^2 - (8 + 16/M)t + 2 - 8/M + 15/M^2}.$$
The formulae (34) and (40) are changed to

\[
\dot{y}_6 = -4 \left( t + 1 - \frac{4}{M} \right) u_p - 4y_4, \quad y_6(0) = 1,
\]

and

\[
\dot{y}_{12} = -4 \left( t + 1 - \frac{4}{M} \right) u_q - 4y_{10} - 4, \quad y_{12}(0) = 0,
\]

respectively.

Summing up the results we state, that similarly to Theorems 6 and 7, the problem of finding the best possible \(M\) for which the Pick function \(P_M\) locally maximizes \(\text{Re} \ a_4\) in \(S(M)\) is reduced to the solution of the equations

\[
y_3(1 - 1/M) = 0,
\]

or

\[
y_3(1 - 1/M)y_9(1 - 1/M) - y_{15}^2(1 - 1/M) = 0,
\]

where \(y_1(t), \ldots, y_{15}(t)\) are solutions to the Cauchy problem for the differential equations (29–33), (85), (35–39), (86), and (41–43) with \(u_p\) and \(u_q\) given by (83) and (84).

**Theorem 12.** The Pick function \(P_M\) locally maximizes \(\text{Re} \ a_4\) in \(S(M)\) if \(M > M_0\) and does not give a local maximum to \(\text{Re} \ a_4\) if \(M < M_0\), where \(M_0\) is the maximum of the roots of the equations (87) and (88), where \(y_1(t), \ldots, y_{15}(t)\) are solutions of (29–33), (85), (35–39), (86), (41–43), and (83–84).

Numerical methods applied to (87) and (88) show that \(M_0 \approx 22.9569\...\)

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