Nonlocality in quantum time via modular operators

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We formalize the concept of the modular energy operator within the Page and Wooters timeless framework. As a result, this operator is elevated to the same status as the more studied modular operators of position and momentum. In analogy with dynamical nonlocality in space associated with the modular momentum, we introduce and analyze the nonlocality in time associated with the modular energy operator. Some applications of our formalization are provided through illustrative examples.

Nonlocality is a remarkable concept which has been attracting an ever-increasing interest from the community since the early days of quantum theory [1, 2] till now. Whether it is through Bell’s nonlocality [3, 4], quantum steering [5, 6], quantum entanglement in general [7] or, even more broadly, quantum discord [8–11], the topic has been central in the studies of quantum foundations, and with a good reason: since multiple experiments verified the quantum violation of Bell’s inequalities [12–19], it is believed that quantum mechanics is fundamentally different from classical mechanics. These studies have led to theoretical and technological breakthroughs [20–28]. Moreover, it is even possible to discuss entanglement in time [29–33].

The above type of nonlocality is associated with the preparation (or preparation and measurement) of systems. It can, then, be referred to as kinematic nonlocality. A different type of nonlocality, which was introduced using the concept of modular variables [34], is associated with the equations of motion obeyed by quantum systems and, hence, called dynamical nonlocality. Although very promising, as already shown in first applications to quantum information with continuous systems [35–38], these variables have not fully received yet the corresponding attention from a significant part of the community [39].

The most common types of modular variables considered in the literature are the modular position and the modular momentum [35–48]. In fact, letting ℓ and p₀ be parameters with dimensions of length and momentum, respectively, the modular operators

\[ e^{i\ell p_0/\hbar} \]  (1)

and

\[ e^{ip\tau/\hbar} \]  (2)

associated with the modular position and momentum, are studied in various scenarios, specially in interference effects. However, other types of modular variables are also considered. In the Methods section, further background is provided regarding these operators.

For this work, modular energy, which is related to dynamical nonlocality in time, is of particular interest. Formally, it does not have the same status as the other modular variables. While, say, a modular position is “naturally” associated with a modular momentum, a modular energy does not find an associated modular time because time is a parameter, and not an observable, in quantum mechanics. Still, it is known from relativity theory that momentum and energy are parts of a single property (the stress-energy tensor) of physical objects in spacetime. This line of reasoning, then, leads to the concept of modular energy, given by

\[ e^{iH_S\tau/\hbar}, \]  (3)

where \( \tau \) is a parameter with units of time and \( H_S \) is the Hamiltonian of the system of interest, was also introduced. It is called a modular operator since the time parameter \( \tau \) defines different modular energies and is associated with the energy mod \( 2\pi\hbar/\tau \). Although the operator in Eq. (3) is familiar from the study of quantum dynamics, as a modular variable, it is assumed to be a property of the system, which makes it more delicate conceptually compared to the modular variables of position and momentum, for instance. This is the case, in part, since time, the canonically conjugate variable of the Hamiltonian \( H \), is not an observable in quantum mechanics, as already mentioned. Nevertheless, the time evolution of the modular momentum, which is displayed in Eq. (36) and involves a dynamical notion of nonlocality in space, seems to suggest the existence of nonlocality in time for the evolution of the modular energy, i.e., the time derivative of the modular energy might, in general, depend on temporally remote events. It is this notion of temporal nonlocality that we wish to investigate in the current work.

This is a subject that may play a relevant role in the study of phenomena with some periodicity in time, like time crystals [49–58] and many others, as it is further discussed later in this work. Moreover, it may lead to the new discoveries, including novel insights into conservation laws in quantum mechanics [59].
In our analysis, envisioning a better formalization of the nonlocality in time, we employ a previously suggested framework of timeless quantum mechanics. Particularly relevant here is the approach introduced by Page and Wooters [60], which has recently been subjected to much scrutiny [61–66]. According to this approach, a (quantum) clock system is used as a reference for the evolution of the system of interest. Together, the clock and the system of interest are often assumed to be a closed system. It can be shown that the evolution of the main system with respect to the time given by the clock is typically unitary — even in a scenario with multiple clocks [64].

Hereon, we shall study the modular energy operator using the framework of timeless quantum mechanics. First, we review some characteristics of modular variables. Following that, we introduce the Page and Wooters’ timeless framework for quantum mechanics. With that, we introduce our formalization of modular energy in the timeless framework and, then, present and discuss some examples of its application.

RESULTS

Nonlocality in time within the timeless framework. The timeless framework [60] considers a clock system, whose state is given by a vector in a Hilbert space $H_A$, and the system whose evolution is studied, represented by a state in a Hilbert space $H_R$, where $R$ stands for the “rest.” The joint system $|\Psi\rangle \in H_A \otimes H_R$ is assumed to be closed and, hence, it is subject to the Wheeler-DeWitt equation,

$$ H_T|\Psi\rangle = 0, \quad (4) $$

where $H_T$ is the total Hamiltonian acting on systems $A$ and $R$. The double ket notation is used to denote the full isolated system.

Now, let $T_A$ be the time operator associated with clock $A$ and $H_A$ its free Hamiltonian, with $[T_A, H_A] = i\hbar$, which implies that $H_A = -i\hbar\partial/\partial T_A$. Observe that $T_A$ and $H_A$ are the analogous in $H_A$ to the usual $X$ and $P$ in standard quantum mechanics. Also, let $H_R$ be the free Hamiltonian of the system of interest, and let $H_{int}(T_A)$ represent the interaction between $A$ and $R$, which is analogous to the time-dependent terms of the evolution of $R$ in standard formulations of quantum mechanics. Then,

$$ H_T = H_A + H_R + H_{int}(T_A). \quad (5) $$

Replacing it in Eq. (4) and applying a scalar product by an eigenstate $|t_A\rangle$ of $T_A$ on the left, i.e., $|\psi(t_A)\rangle = \langle t_A|\Psi\rangle$, which results in a reduced state of $R$ conditioned on the state $|t_A\rangle$ of the clock $A$, it holds that

$$ i\hbar\frac{\partial}{\partial T_A}|\psi(t_A)\rangle = (H_R + H_{int}(t_A))|\psi(t_A)\rangle, \quad (6) $$

which is the Schrödinger equation for the system $R$ with time measured by the external clock $A$. As a result, $|\Psi\rangle$ can be written as

$$ |\Psi\rangle = \int dt_A \langle t_A| \otimes |\psi(t_A)\rangle, \quad (7) $$

where $|\psi(t_A)\rangle$ is the usual (normalized at each instant of time) state vector considered in quantum mechanics. Because $|\Psi\rangle$ contains information about $|\psi(t_A)\rangle$ at every $t_A$, it is referred to as the history state.

Observe that, since the system composed by $A$ and $R$ is assumed to be isolated, clock $A$ mediates the interaction of any other system in the universe with $R$, as already discussed in Ref. [63]. In particular, any change in the energy distribution of $R$ generated by the Hamiltonian in Eq. (9) corresponds to a change in the energy distribution of clock $A$.

Here, we consider the case studied in Ref. [64], where system $R$ was assumed to be composed of the main system of interest $S$ and a clock $B$, i.e., an internal clock to $R$. We, then, write

$$ H_R = H_B + H_S + H_{BS}(T_B), \quad (8) $$

where $H_{BS}(T_B)$, assumed to be such that $[H_B, H_{BS}(t)] = 0$ for a parameter $t$, generates the inner unitary transformation of system $S$ controlled by the time in clock $B$, i.e., changes on system $R$ when it is completely isolated. In this case, Eq. (5) and, as a consequence, Eq. (6) hold.

For simplicity, the term $H_{BS}(T_B)$ is assumed to be null in this work. This implies that the effective Hamiltonian of system $R$ from the perspective of clock $A$ is

$$ H_{eff}^A = H_B + H_S + H_{int}(t_A). \quad (9) $$

We also restrict to cases where $[H_B, H_{int}(T_A)] = 0$, i.e., $H_{int}$ is not a function of $T_B$.

Our goal is to obtain the dynamics of the systems from the perspective of clock $B$. Then, writing $\langle t_B|\Psi\rangle = |\varphi(t_B)\rangle \in H_A \otimes H_S$, it holds that

$$ i\hbar\frac{\partial}{\partial t_B}|\varphi(t_B)\rangle = (H_A + H_S + H_{int}(T_A))|\varphi(t_B)\rangle, \quad (10) $$

i.e., from $B$’s perspective, the effective Hamiltonian is

$$ H_{eff}^B = H_A + H_S + H_{int}(T_A) \quad (11) $$

From the Heisenberg equation of motion, we obtain the $t_B$-evolution, i.e., the evolution from the perspective of the clock $B$ of the modular energy $e^{-iH_{eff}^B/\hbar}$ of clock $A$, where $\tau$ is a parameter with units of time. Explicitly,

$$ \frac{d}{dt_B} e^{iH_{eff}^B/\hbar} = -\frac{i}{\hbar} e^{iH_{eff}^B/\hbar} \left[ H_{eff}^B, \frac{\partial}{\partial t_B} \right] $$

$$ = -\frac{i}{\hbar}[H_{int}(T_A + \tau I) - H_{int}(T_A)] e^{iH_{eff}^B/\hbar}. \quad (12) $$
This equation is to be compared with Eq. (36). It becomes particularly interesting when two other results are pointed out. The first is similar to the classical evolution of the modular momentum represented in Eq. (37). In fact, it can be checked that the classical dynamical equation for a modular energy leads to

\[
\frac{d}{dt_B}e^{2\pi i E_A/E_0} = \left\{e^{2\pi i E_A/E_0}, H_{eff}^B \right\}
\]

\[
= -i\frac{2\pi}{E_0} \frac{dH_{int}}{dt_A} e^{2\pi i E_A/E_0},
\]

which, again, depends only on (local) derivatives. The second result concerns the quantum evolution of the observable \( T_A \) from the perspective of clock \( B \). Observe that the Heisenberg equation of motion leads to

\[
\frac{d}{dt_B} T_A = -\frac{i}{\hbar} [T_A, H_{eff}^B] = I.
\]

This means that, if the two clocks start synchronized, they remain synchronized from the perspective of one another. More generally, the “flow of time” is the same in both clocks. Nevertheless, as evidenced by Eq. (12), with respect to system’s \( R \) proper time, the dynamics of the modular energy of the external system \( A \) depends on a future instant of time.

Observe that, instead of studying the modular energy in Eq. (3) that is typically discussed in the literature and refers to a variable associated with system \( S \), we consider the modular energy of clock \( A \). As already said, this clock plays the effective role of external systems interacting with \( R \). Then, we infer the nonlocality in time of the modular energy of \( R \) from the nonlocality of the modular energy of \( A \). More than that, we infer the modular energy of \( S \). In fact, it can be checked that \( dh_{eff}^B/dt_B \) vanishes, which, in particular, implies that \( e^{iH_{eff}^A \tau/\hbar} \) is conserved. Then, a change in the modular energy of \( A \) implies a change in the modular energy of \( S \). Because of it, the parameter \( \tau \) that appears in the definition of the modular energy of \( A \) is taken with system \( S \) in mind. While the parameter \( \ell \) in the modular momentum was chosen to be the separation between the slits (the spatial periodicity) in the analysis of the double-slit experiment discussed in the Methods section, the parameter \( \tau \) in the modular energy of clock \( A \) should be associated with a time periodicity of system \( S \). This last conclusion follows, again, from the complete uncertainty relation, also discussed in that section.

Furthermore, the complete uncertainty relation also implies that if the system is observed with a projective measurement, then its modular energy becomes completely uncertain. While in the case of the double-slit experiment the analogous result prevented information about a present potential to be acquired faster than light, here it blocks access to information about a future potential acting over the particle. Notably, both scenarios may be linked via a Lorentz boost.

**Operational meaning of the modular energy in the timeless framework.** To give an operational meaning to the modular energy, we first obtain an analog of the expected value of the modular position given by Eq. (33). For that, we first observe that the expected value of any observable \( O \) acting on \( S \) at a given instant of time \( t \) in clock \( A \) can be computed in the timeless framework as the expected value of \( \langle t \rangle \langle t \otimes I_B \otimes O \rangle \) for the state \( |\Psi \rangle \), i.e., \( \langle O \rangle = \langle \Psi | \langle t \rangle \langle t \otimes I_B \otimes O \rangle |\Psi \rangle \). In case \( O \) is the modular momentum, it can be checked that Eq. (33) is recovered.

More generally, omitting the tensor products, one may argue that the expected value of an operator \( O \) that acts on the entire space can be computed as

\[
\langle O \rangle = \frac{1}{2} \langle \Psi | \langle t \rangle \langle t | O \rangle |\Psi \rangle,
\]

where \( \cdot | \cdot \rangle \) is the anticommutator, which is added to include cases where \( | t \rangle \langle t | \) and \( O \) do not commute. If we assume that the state \( \psi \) of system \( R \) in Eq. (7) is such that \( \langle \psi(t_\tau) \rangle = e^{i\varphi} \langle \psi(t_{A_\tau}) \rangle \) for a certain time parameter \( \tau \) and every \( t_\tau \), the expected value of the modular energy associated with clock \( A \) at a given instant of time \( t \) is computed as

\[
\langle e^{iH_{A_\tau \tau}} \rangle = \frac{1}{2} \langle \langle \Psi | \langle t \rangle \langle t | e^{iH_{A_\tau \tau}} \rangle \rangle |\Psi \rangle
\]

\[
= \frac{1}{2} \langle \langle \Psi | (| t \rangle \langle t + \tau | + | t - \tau \rangle \langle t |) \rangle |\Psi \rangle
\]

\[
= \frac{1}{2} \langle \langle \psi(t) \psi(t + \tau) + \psi(t - \tau) |\psi(t) \rangle \rangle
\]

\[
= e^{i\varphi}.
\]

This result is analogous to the one in Eq. (33) of the Methods section. Moreover,

\[
\langle e^{iH_{R_\tau \tau}} \rangle = \frac{1}{2} \langle \langle \Psi | \langle t \rangle \langle t | e^{iH_{R_\tau \tau}} \rangle \rangle |\Psi \rangle
\]

\[
= \langle \langle \Psi | (| t \rangle \langle t | e^{iH_{R_\tau \tau}} \rangle) \rangle |\Psi \rangle
\]

\[
= \langle \psi(t + \tau) |\psi(t) \rangle
\]

\[
= e^{-i\varphi}.
\]

Note the difference in the sign of the phases in Eqs. (16) and (17). This is due to the conservation of the total modular energy. Moreover, the above also provides an operational meaning to this modular variable. Its expectation value can be obtained by performing two tomographies of the system \( S \): one at \( t \) and the other at \( t + \tau \). However, in some special cases, one may optimize this process. For instance, omitting clock \( B \)’s system, observe that for an eigenstate of energy \( |E\rangle \), the expected value \( \langle e^{iH_{S_\tau \tau}} \rangle \) is just the dynamical phase \( e^{iE_n \tau/\hbar} \). Then, if a system has a discrete energy spectrum and the state of \( S \) is \( |\psi \rangle = \sum_n c_n |E_n \rangle \), \( \langle e^{iH_{S_\tau \tau}} \rangle = \sum_n |c_n|^2 e^{iE_n \tau/\hbar} \). In this case, the tomography of \( |\psi \rangle \), together with its spectral analysis, suffice for the measurement of \( e^{iH_{S_\tau \tau}} \). Also, in
scenarios where the system can be placed in an interferometer, one may use a delay line in one arm, which results in a direct interference between $|ψ(t)|$ and $|ψ(t+τ)|$.

Observe that the state of clock $B$ was omitted in this discussion and $|ψ⟩$ was taken to be the state of system $S$. This can be done by assuming that clock $B$ starts disentangled to $S$. Then, the analog of Eq. (14) from the perspective of $A$, i.e., $dT_B/τ_A = I$ implies that the two systems remain disentangled and, moreover, that the displacement of clock $B$'s state remains unchanged throughout the dynamics.

In what follows, we use the proposed formulation in order to analyze two scenarios where nonocality in time seems to play a role.

**Particle and the piston.** Now, we shall study a thought experiment previously suggested in Ref. [41]. We start by introducing the standard description of it. For that, consider a long rectangular box with a movable piston on its right-hand side. Inside it, a quantum particle well-localized in a region much smaller than the length $ℓ$ of the box is moving back-and-forth in periodic motion with period $τ$ and negligible spreading. Also, assume a second box with two open sides is attached to the piston. This scenario is represented in Fig. 1.

Then, suppose that the particle inside the closed box is located at its left end, when at time $τ_1$, an external ball hits the box attached to the piston from its inside, putting the piston in a motion to the left. Later on, at an instant $τ_2$, the ball hits the other inner side of the box, stopping the piston after it had moved a distance $δℓ$. It is assumed that $τ_2−τ_1 ≪ τ$ in order to assure that the particle will remain distant from the piston during its translation.

Classically, it is expected that there will be no interaction between the particle and the ball. In a quantum treatment, however, it can be argued that there is an exchange (nonlocal in time) of modular energy between the two systems [41], and the explanation for that used the fact that, if $|ϕ(0)⟩$ is the initial state of the particle, in cycles where the piston does not move $|ϕ(τ)⟩ = e^{iα}|ϕ(0)⟩$, where $α ∈ [0, 2π]$ and, hence, denoting the Hamiltonian of the particle by $H_S$, $⟨ϕ(0)|e^{iH_Sτ}|ϕ(0)⟩ = ⟨ϕ(τ)|ϕ(0)⟩ = e^{−iα}$. However, if the ball hits the piston, the final position of the particle of a period $τ$ is shifted by a distance $2δℓ$ with respect to its initial position. As a result, $|ϕ(τ)⟩ = e^{i(α+2πδℓ/ℏ)}|ϕ(0)⟩$ and $⟨ϕ(0)|e^{iH_Sτ}|ϕ(0)⟩ = e^{−i(α+2πδℓ/ℏ)}$. Hereon, we show how to analyze this example within the timeless framework.

The time-independent Hamiltonian of the particle inside the closed box, i.e., its Hamiltonian in case the external ball does not interact with the piston can be written as

$$H_S = \frac{1}{2m} P^2 + V_I(X) + V_r(X), \quad (18)$$

where $V_I$ and $V_r$ are the potentials associated respectively with the left and the right walls of the box. For simplicity, one could take $V_r(x) = V_I(x−ℓ)$.

Now, since instants $τ_1$ and $τ_2$ refer to events related to external systems, we assume they are observed in clock $A$. Then,

$$H_{int}(T_A) = V_r(X + f(T_A)) − V_r(X), \quad (19)$$

where

$$f(t) = δℓ \left[ \frac{t−τ_1}{τ_2−τ_1} \Theta(t−τ_1) + \frac{τ_2−t}{τ_2−τ_1} \Theta(t−τ_2) \right] \quad (20)$$

and $Θ$ is the Heaviside step function.

With that, the Hamiltonian $H_{eff}^B$ in Eq. (9) becomes

$$H_{eff}^B = H_A + \frac{1}{2m} P^2 + V_I(X) + V_r(X + f(T_A)) \quad (21)$$

and, hence, it follows from Eq. (12) that, over the first half of the cycle in which the piston had its position changed

$$\frac{d}{dt} e^{iH_Aτ/ℏ} = −\frac{i}{ℏ} [V_r(X + δℓI) − V_r(X)] e^{iH_Aτ/ℏ}. \quad (22)$$

This means that, even while the particle is as far away inside the box from the piston as possible, there exists an exchange of modular energy between the exterior (the ball, represented by its clock $A$) and the system $R$ that depends on the final position of the piston. However, this exchange does not lead to superluminal communication because, as the complete uncertainty relation implies, an observation of the particle or the ball would make their modular energy completely uncertain.

**Nonlocal interaction in time between magnetic fields and a spin.** The scenario that will be considered in this section was the first example of nonlocal exchange of modular energy discussed in the literature. It was introduced in the seminal article [34], and later revisited in Ref. [41]. Our aim here is to characterize the nonlocality in time associated with this exchange in the timeless framework.
chosen in such a way that the effective rotation about $B$ the same instant of time. Now, it becomes possible to $B$ pulses $z$

In this case, the dynamics of the spin is simply given with period $\tau$. (c) Then, periodic pulses of magnetic field $B_x$ in the $x$ direction with period $\tau' \neq n\tau$, where $n \in \mathbb{Z}$, are also applied upon the spin, generating partial rotations around its $x$ axis. Since $\tau'$ is not a multiple of $\tau$, the spin does not flip. (d) However, the introduction of extra pulses of magnetic field $B_z$ in the $z$ direction with period $\tau'$ can be designed in order to flip the spin, even if the pulses in the $x$ and $z$ directions never coincide in time.

To start, consider a spin-1/2 particle with magnetic moment $\mu$ under the influence of a constant magnetic field $B_0$ in the $z$ direction. Then, its Hamiltonian is

$$H_S = \frac{\hbar}{2} \mu B_0 \sigma_z. \quad (23)$$

In this case, the dynamics of the spin is simply given by a rotation around the $z$ axis with angular frequency $\omega = \mu B_0$. This scenario is presented in Fig. 2(a).

Now, suppose that an additional magnetic field $B_x(t)$ consisting of periodic pulses with period $\tau'$ is added to the dynamics of the spin, as shown in Fig. 2(b). In this case, it can be asked whether a spin that starts in an eigenstate of $\sigma_z$ direction flips or not. To answer this question, one needs to observe that, under $H_S$, the system completes each cycle in a period $\tau = 2\pi/\omega = 2\pi/\mu B_0$. Then, with the addition of $B_x(t)$, if $\tau'$ is a multiple of that, i.e., $\tau' = n\tau$ for a non-zero integer $n$, the spin will eventually flip. Otherwise, the effect of $B_x(t)$ vanishes on average and the spin remains unchanged.

Assume, then, that $\tau' \neq n\tau$ and apply an additional periodic pulse of magnetic field in the $z$ direction $B_z(t)$, also with a period $\tau'$, as illustrated in Fig. 2(c). The pulses $B_x(t)$ and $B_z(t)$ are such that their product is $B_x(t)B_z(t) = 0$ for every $t$, i.e., they are not applied at the same instant of time. Now, it becomes possible to flip the spin again. In fact, $B_z(t)$ can be conveniently chosen in such a way that the effective rotation about the $z$ axis caused by $B_0$ and $B_z(t)$ has a period $T$ such that $\tau = mT$ for a non-zero integer $m$.

As can be understood from this description, and pointed out in Ref. [41], the analysis of the precession of the spin has a classical analog. However, the change of energy associated with the flip of the spin seems to be puzzling from a classical perspective. In fact, in Ref. [41], after a careful consideration of the problem, the authors concluded that, while making no net contribution to the energy, the field $B_z(t)$ seems to modify the way the particle and $B_z(t)$ exchange energy, allowing exchanges that are not multiples of $2\pi \hbar / \tau'$. Because of that, it is commonly claimed that the exchange of energy that causes the flip of the spin is nonlocal in time.

Here, the aim is to formalize this idea in the timeless framework. To start, observe that the time-dependent part of the evolution is described by the Hamiltonian

$$H_{\text{int}}(T_A) = \frac{\hbar}{2} \mu [B_x(T_A)\sigma_x + B_z(T_A)\sigma_z]. \quad (24)$$

Then, the effective Hamiltonian $H_{\text{eff}}^B$ is

$$H_{\text{eff}}^B = H_A + H_S + H_{\text{int}}(T_A)$$

$$= H_A + \frac{\hbar}{2} \mu [B_x(T_A)\sigma_x + (B_0 + B_z(T_A))\sigma_z] \quad (25)$$

and

$$\frac{d}{dt_B} e^{iH_A\tau'/\hbar} = 0. \quad (26)$$

However, $\tau'$ is a frequency associated with the external system, and it is being assumed that there exists no integer $n$ such that $\tau' = n\tau$, i.e., $\tau'$ is not a multiple of the “natural” period $\tau$ associated with the spin. Hence, the dynamics of the modular energy of interest is

$$\frac{d}{dt_B} e^{iH_A\tau'/\hbar} = -i\mu \left[ B_x(T_A + \tau)\sigma_x + B_z(T_A + \tau I)\sigma_z - B_x(T_A)\sigma_x - B_z(T_A)\sigma_z \right] e^{iH_A\tau'/\hbar}. \quad (27)$$

Assuming that each pulse associated with $B_x(t)$ and $B_z(t)$ has a sufficiently short duration, Eq. (27) leads to

$$\frac{d}{dt_B} e^{iH_A\tau'/\hbar} = -i\mu \left[ B_x(T_A + \tau I)\sigma_x - B_z(T_A)\sigma_z \right] e^{iH_A\tau'/\hbar}$$

(28)

if $B_z(T_A) = 0$, and

$$\frac{d}{dt_B} e^{iH_A\tau'/\hbar} = -i\mu \left[ B_z(T_A + \tau I)\sigma_z - B_z(T_A)\sigma_z \right] e^{iH_A\tau'/\hbar}$$

(29)

if $B_z(T_A + \tau I) = 0$.

These equations show how the modular energy of the system has its evolution at different instants affected simultaneously by the presence of the field $B_z$ and $B_x$. This nonlocal interaction in time between the fields and the spin is what modifies how they exchange energy.
DISCUSSION

We have developed and applied a formalization of the idea of modular energy within the timeless framework of quantum mechanics. This puts modular energy and time on an equal footing with the modular position and momentum. While our approach helps to clarify how dynamical nonlocality in time is present in the study of modular energy, our results might be just the starting point for a complete comprehension thereof.

For instance, like the modular position and momentum divide the phase space into periodic cells, modular energy and its associated modular time also divide the energy-time phase space in this way. From the conceptual point of view, the meaning of such a division may deserve further clarification, which could have ramifications in other areas of quantum mechanics, like time crystals.

Moreover, if an ultimate Planck scale limit on time is assumed, the framework developed here can be particularly helpful since the modular energy promotes translations of length \( \tau \) in time. Therefore, with the appropriate parameter \( \tau \), this modular variable and the ensuing nonlocality in time could play a role in our understanding of spacetime. More precisely, modular energy as formalized here may be an important tool in the understanding of recent studies in the direction of modular spacetime [67, 68].

Finally, quantum events with indefinite causal order have been attracting some attention recently [69–74]. It seems that modular energy, and its nonlocality in time, may help in the understanding of such scenarios, although the connections of these studies with the present work still need to be further clarified.

METHODS

As mentioned in the Introduction, two of the most common modular variables considered in the literature are the modular position and modular momentum, given, respectively, by

\[
X_{\text{mod}} \equiv X \mod 2\pi\hbar I/p_0 \tag{30}
\]

and

\[
P_{\text{mod}} \equiv P \mod 2\pi\hbar I/\ell, \tag{31}
\]

where \( I \) is the identity operator, \( X \) and \( P \) are the usual position and momentum operators, and \( \ell \) and \( p_0 \) are parameters with dimensions of length and momentum, respectively. The adjective modular in modular variables comes from the periodicity inherent to them. As it will be better explained in this Introduction, although the parameters \( \ell \) and \( p_0 \) can be, in principle, arbitrary, there is typically a “natural” choice for them.

These concepts were introduced as a tool in the study of interference in the Heisenberg picture [41, 42, 47]. More specifically, some of these variables consist of (nonlocal) properties of a particle whose expectation values may be functions of the relative phase between two coherent wavepackets — even while they are spatially separated. In particular, if a system is in the state

\[
|\psi(t)\rangle = \frac{1}{\sqrt{2}} \left( |\xi_1(t)\rangle + e^{i\varphi} |\xi_2(t)\rangle \right), \tag{32}
\]

where \( \xi_1 \) and \( \xi_2 \) are two wavepackets, it is possible to show that the expectation value of any power of \( X \) and \( P \) or products thereof (i.e., any polynomial operator of the form \( f(X, P) = \sum_{mn} a_{mn} X^m P^n \), where \( a_{mn} \in \mathbb{C} \) for every \( m, n \)) does not depend on \( \varphi \) if \( \xi_1 \) and \( \xi_2 \) are orthogonal to each other [41]. However, the modular operators in Eqs. (1) and (2), which are equivalent to \( e^{iX_{\text{mod}}p_0/\hbar} \) and \( e^{iP_{\text{mod}}\ell/\hbar} \), respectively, may include \( \varphi \) in their expectation values. It should be noticed that, while the operators defined in Eqs. (1) and (2) are non-Hermitian, observables can be easily defined from them.

To illustrate that, let \( \langle x|\xi_1 \rangle \equiv \xi(x) \) and \( \langle x|\xi_2 \rangle \equiv \xi(x-\ell) \), for some \( \xi(x) \) with small enough support around one of the slits. Then, it can be verified by direct computation that

\[
\langle e^{iPt/\hbar} \rangle = \frac{1}{2} e^{i\varphi}. \tag{33}
\]

This example can be understood, for instance, as the analysis of the double-slit experiment, where \( \ell \) is the separation between the slits.

Like any property, modular variables are bound to a conservation law [41]. However, differently from the standard position and momentum, it can be checked that, for \( p_0 = 2\pi\hbar/\ell \), the modular position and momentum defined in Eqs. (30) and (31) commute with each other, a property that was recently experimentally investigated [48]. While that seems to violate the uncertainty principle, this is not the case. The uncertainty principle is actually manifested in a different manner [41]. Specifically, \( P_{\text{mod}} \) and \( X_{\text{mod}} \) divide the phase space into periodic cells. Their commutation means that, for a given state, a point can be assigned to each cell. However, \( P \) and \( X \) remain unknown, since their modular counterparts do not provide information on which cell is associated with the state. This characteristic is illustrated in Fig. 3. Also, it is possible to introduce a complete uncertainty relation for modular variables [41, 47], which states that, for any (dimensionless) modular variable \( \Phi_{\text{mod}} \equiv \Phi \mod 2\pi I \), the expected value \( \langle e^{i\Phi} \rangle \) vanishes for every integer \( n \) if and only if the modular variable \( \Phi_{\text{mod}} \) is completely uncertain. In fact, this follows from the Fourier series expansion of the probability \( P_{\Phi_{\text{mod}}} \) of \( \Phi_{\text{mod}} \) assuming an arbitrary value \( \varphi \in [0, 2\pi] \), i.e.,

\[
P_{\Phi_{\text{mod}}} (\varphi) = \sum_{n \in \mathbb{Z}} c_n e^{in\varphi}, \tag{34}
\]
with the classical equation of motion, which leads to
\[ H \text{ Hamiltonian of the system of interest, where } \langle e^{iP\ell/\hbar} \rangle \text{ vanishes for the states } |\xi_1\rangle \text{ and } |\xi_2\rangle. \] Then, when their superposition is destroyed by a measurement, the modular momentum of the particle becomes completely uncertain. This, in turn, preserves relativistic causality.

At this point, we are in a good position to justify the choice of the parameter \( \ell \) as the distance between the slits, something not explained in the literature prior to this work. If \( \ell \) is much bigger than the separation between the slits, than \( \langle e^{iP\ell/\hbar} \rangle \) also vanishes for the state \( |\psi\rangle \), i.e., this modular variable remains uncertain throughout the experiment and, then, is not relevant in the analysis of it. When \( \ell \) approaches the length of separation of the slits, \( \langle e^{iP\ell/\hbar} \rangle \) becomes non-zero. Although they are, then, relevant, they do not provide as much information as the choice of \( \ell \) coinciding with the separation between the slits. This is the sense in which we said that there is typically a natural choice for the parameters associated with a modular variable.

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