About the unification type of simple symmetric modal logics

Philippe Balbiani and Çiğdem Gencer

1 Institut de recherche en informatique de Toulouse
CNRS — Toulouse University
Toulouse, France

2 Faculty of Arts and Sciences
Aydin University
Istanbul, Turkey

Abstract

The unification problem in a normal modal logic is to determine, given a formula $\varphi$, whether there exists a substitution $\sigma$ such that $\sigma(\varphi)$ is in that logic. In that case, $\sigma$ is a unifier of $\varphi$. We shall say that a set of unifiers of a unifiable formula $\varphi$ is complete if for all unifiers $\sigma$ of $\varphi$, there exists a unifier $\tau$ of $\varphi$ in that set such that $\tau$ is more general than $\sigma$. When a unifiable formula has no minimal complete set of unifiers, the formula is nullary. In this paper, we prove that $\mathbf{KB}, \mathbf{KDB}$ and $\mathbf{KTB}$ possess nullary formulas.

1 Introduction

The unification problem in a normal modal logic is to determine, given a formula $\varphi$, whether there exists a substitution $\sigma$ such that $\sigma(\varphi)$ is in that logic. In that case, $\sigma$ is a unifier of $\varphi$. We shall say that a set of unifiers of a formula $\varphi$ is complete if for all unifiers $\sigma$ of $\varphi$, there exists a unifier $\tau$ of $\varphi$ in that set such that $\tau$ is more general than $\sigma$. An important question is the following [1, 16]: when a formula is unifiable, has it a minimal complete set of unifiers? When the answer is “no”, the formula is nullary. When the answer is “yes”, the formula is unitary, or finitary, or infinitary depending on the cardinalities of its minimal complete sets of unifiers. A normal modal logic is called nullary if it possesses a nullary formula. Otherwise, it is called unitary, or finitary, or infinitary depending on the types of its unifiable formulas. We usually distinguish between elementary unification and unification with parameters. In elementary unification, all variables are likely to be replaced by formulas when one applies a substitution. In unification with parameters, some variables — called parameters — remain unchanged.

It is known that $\mathbf{S}_5$ is unitary [1], $\mathbf{KT}$ is nullary [6], $\mathbf{KD}$ is nullary [7], $\mathbf{Alt}_1$ is
nullary [9], S4.3 is unitary [18], transitive normal modal logics like K4 are finitary [22] and K is nullary [26], though the nullariness character of KT and KD has only been obtained within the context of unification with parameters. Taking a look at the literature about unification types in normal modal logics [1, 16], one will quickly notice that much remains to be done. For instance, the types of simple Church-Rosser normal modal logics like KG, KDG and KTG are unknown. Even, for all \( k \in \mathbb{N} \) such that \( k \geq 2 \), the type of the least normal modal logic containing \( \Box k \bot \) is unknown. In this paper, we adapt to KB, KDB and KTB the argument of Ježabek [26] showing K is nullary, though the nullariness character of KB, KDB and KTB will only be obtained within the context of unification with parameters. We assume the reader is at home with tools and techniques in modal logic. For more on this, see Blackburn et al. [11], or Chagrov and Zakharyaschev [12], or Chellas [13].

2 Syntax

In this section, we present the syntax of normal modal logics.

Formulas Let \( \text{VAR} \) be a nonempty countable set of propositional variables (with typical members denoted \( x, y, \) etc) and \( \text{PAR} \) be a nonempty countable set of propositional parameters (with typical members denoted \( p, q, \) etc). Atoms (denoted \( \alpha, \beta, \) etc) are variables or parameters. The set \( \text{FOR} \) of all formulas (with typical members denoted \( \varphi, \psi, \) etc) is inductively defined as follows:

\[
\varphi, \psi ::= x \mid p \mid \bot \mid \neg \varphi \mid (\varphi \lor \psi) \mid \Box \varphi.
\]

We adopt the standard rules for omission of the parentheses. The Boolean connectives \( \top, \land, \rightarrow \) and \( \leftrightarrow \) are defined by the usual abbreviations. For all parameters \( p \), we write “\( p^0 \)” to mean “\( \neg p \)” and we write “\( p^1 \)” to mean “\( p \)”.

Let \( p, q \) be fixed distinct parameters.

Let \( \Box \) and \( \square \) be the modal connectives defined as follows:

\[
\Box \varphi ::= (p^0 \land q^0 \rightarrow \square(p^1 \land q^0 \rightarrow \square(p^0 \land q^1 \rightarrow \square(p^0 \land q^0 \rightarrow \varphi)))],
\]

\[
\square \varphi ::= (p^0 \land q^0 \rightarrow \square(p^0 \land q^0 \rightarrow \square(p^0 \land q^0 \rightarrow \square(p^0 \land q^0 \rightarrow \varphi))))].
\]

For all \( k \in \mathbb{N} \), the modal connectives \( \Box^k \) and \( \square^k \) are inductively defined as follows:

\[
\Box^0 \varphi ::= \varphi,
\]

\[
\Box^{k+1} \varphi ::= \Box \Box^k \varphi,
\]

\[
\square^0 \varphi ::= \varphi,
\]

\[
\square^{k+1} \varphi ::= \square \square^k \varphi.
\]

\[1\text{In this paper, we follow the same conventions as in [11, 12, 13] for talking about normal modal logics: S5 is the least normal modal logic containing the formulas usually denoted (T), (4) and (B). KT is the least normal modal logic containing the formula usually denoted (T), etc.}
For all $k \in \mathbb{N}$, the modal connectives $\Box^k$ and $\Diamond^k$ are inductively defined as follows:

- $\Box^0 \varphi := \top$,
- $\Box^{k+1} \varphi := (\Box^k \varphi \land \Diamond^k \varphi)$,
- $\Diamond^0 \varphi := \top$,
- $\Diamond^{k+1} \varphi := (\Box^k \varphi \land \Diamond^k \varphi)$.

**Degrees** The degree of a formula $\varphi$ (in symbols $\text{deg}(\varphi)$) is the nonnegative integer inductively defined as follows:

- $\text{deg}(x) = 0$,
- $\text{deg}(p) = 0$,
- $\text{deg}(\bot) = 0$,
- $\text{deg}(\neg \varphi) = \text{deg}(\varphi)$,
- $\text{deg}(\varphi \lor \psi) = \max\{\text{deg}(\varphi), \text{deg}(\psi)\}$,
- $\text{deg}(\Box \varphi) = \text{deg}(\varphi) + 1$.

**Lemma 1** Let $\varphi$ be a formula.

1. $\text{deg}(\Box \varphi) = \text{deg}(\varphi) + 3$,
2. $\text{deg}(\Diamond \varphi) = \text{deg}(\varphi) + 3$,
3. for all $k \in \mathbb{N}$, $\text{deg}(\Box^k \varphi) = \text{deg}(\varphi) + 3k$.
4. for all $k \in \mathbb{N}$, $\text{deg}(\Diamond^k \varphi) = \text{deg}(\varphi) + 3k$.
5. for all $k \in \mathbb{N}$, if $k = 0$ then $\text{deg}(\Box^k \varphi) = 0$ else $\text{deg}(\Box^k \varphi) = \text{deg}(\varphi) + 3(k - 1)$.
6. for all $k \in \mathbb{N}$, if $k = 0$ then $\text{deg}(\Diamond^k \varphi) = 0$ else $\text{deg}(\Diamond^k \varphi) = \text{deg}(\varphi) + 3(k - 1)$.

**Proof:** (1) and (2): Left to the reader. (3)–(6): By induction on $k$. ⊡

**Substitutions** A substitution is a function $\sigma$ associating to each variable $x$ a formula $\sigma(x)$. Following the standard assumption considered in the literature about the unification problem in normal modal logics [1, 16], we will always assume that substitutions move at most finitely many variables. For all formulas $\varphi(x_1, \ldots, x_m, p_1, \ldots, p_n)$, let $\sigma(\varphi(x_1, \ldots, x_m, p_1, \ldots, p_n))$ be $\varphi(\sigma(x_1), \ldots, \sigma(x_m), p_1, \ldots, p_n)$. The composition $\sigma \circ \tau$ of the substitutions $\sigma$ and $\tau$ is the substitution associating to each variable $x$ the formula $\tau(\sigma(x))$. 

3
3 Semantics

In this section, we present the semantics of normal modal logics.

Frames and models  A frame is a couple $F = (W, R)$ where $W$ is a non-empty set of states and $R$ is a relation on $W$. We shall say that a frame $F = (W, R)$ is symmetric if for all $s, t \in W$, if $sRt$ then $tRs$. We shall say that a frame $F = (W, R)$ is serial if for all $s \in W$, there exists $t \in W$ such that $sRt$. We shall say that a frame $F = (W, R)$ is reflexive if for all $s \in W$, $sRs$. Remark that reflexive frames are serial. A model based on a frame $F = (W, R, V)$ is a triple $M = (W, R, V)$ where $V$ is a function assigning to each variable $x$ a subset $V(x)$ of $W$ and to each parameter $p$ a subset $V(p)$ of $W$. Given a model $M = (W, R, V)$, the satisfiability of a modal formula $\varphi$ at $s \in W$ (in symbols $M, s \models \varphi$) is inductively defined as follows:

- $M, s \models x$ iff $s \in V(x)$,
- $M, s \models p$ iff $s \in V(p)$,
- $M, s \not\models \perp$,
- $M, s \models \neg \varphi$ iff $M, s \not\models \varphi$,
- $M, s \models \varphi \lor \psi$ iff $M, s \models \varphi$, or $M, s \models \psi$,
- $M, s \models \Box \varphi$ iff for all $t \in W$, if $sRt$ then $M, t \models \varphi$.

Truth and validity  We shall say that a formula $\varphi$ is true in a model $M = (W, R, V)$ if $\varphi$ is satisfied at all $s \in W$. We shall say that a formula $\varphi$ is valid in a frame $F$ if $\varphi$ is true in all models based on $F$. We shall say that a formula $\varphi$ is valid in a class $C$ of frames if $\varphi$ is valid in all frames of $C$. Let $KB$ be the set of all formulas valid in the class of all symmetric frames. Let $KDB$ be the set of all formulas valid in the class of all serial symmetric frames. Let $KTB$ be the set of all formulas valid in the class of all reflexive symmetric frames. Obviously, $KB \subseteq KDB \subseteq KTB$. Moreover, $KB$ is the least normal modal logic containing all formulas of the form $\neg \varphi \rightarrow \Box \neg \varphi$, $KDB$ is the least normal modal logic containing all formulas of the form $\Box \neg \varphi \rightarrow \neg \Box \varphi$ and $\neg \varphi \rightarrow \Box \neg \varphi$ and $KTB$ is the least normal modal logic containing all formulas of the form $\Box \varphi \rightarrow \varphi$ and $\neg \varphi \rightarrow \Box \neg \varphi$. From now on, we write "frame" to mean "symmetric frame".

Lemma 2  For all $k \in \mathbb{N}$,

1. $\Box^k \top \in KB$.
2. $\Box^k \top \in KB$.
3. $\Box^{<k} \top \in KB$.
4. $\Box^{<k} \top \in KB$. 
Proof: By induction on $k$. $\neg$

Lemma 3 For all $k \in \mathbb{N}$,
1. $\Box^{k} \perp \notin \text{KB}$.
2. $\Box^{k} \perp \notin \text{KB}$.

Proof: Let $k \in \mathbb{N}$. Let $F = (W, R)$ where $W = \{0, \ldots, 3k\}$ and $R = \{(i, j) : |j - i| \leq 1\}$. Let $M = (W, R, V)$ where $V(p) = \{i : i = 1 \mod 3\}$, $V(q) = \{i : i = 2 \mod 3\}$ and for all atoms $\alpha$, if $\alpha \neq p$ and $\alpha \neq q$ then $V(\alpha) = 0$. The reader may easily verify that $M, 0 \notin \Box^{k} \perp$ and $M, 3k \notin \Box^{k} \perp$. Hence, $\Box^{k} \perp \notin \text{KB}$ and $\Box^{k} \perp \notin \text{KB}$. $\neg$

In the proof of Lemma 3, remark that the frame $F = (W, R)$ is reflexive.

Lemma 4 Let $\varphi$ be a formula. For all $k \in \mathbb{N}$,
1. $(\square^{k+1} \varphi \leftrightarrow \varphi \land \square \square^{k} \varphi) \in \text{KB}$.
2. $(\Box^{k+1} \varphi \leftrightarrow \varphi \land \Box \Box^{k} \varphi) \in \text{KB}$.

Proof: By induction on $k$. $\neg$

Lemma 5 For all $k, l \in \mathbb{N}$,

1. if $k > l$ then $(\Box^{k} \perp \rightarrow \Box^{l} \perp) \notin \text{KB}$.
2. if $k > l$ then $(\Box^{k} \perp \rightarrow \Box^{l} \perp) \notin \text{KB}$.

Proof: Let $k \in \mathbb{N}$. Suppose $k > l$. Let $F = (W, R)$ where $W = \{0, \ldots, 3l\}$ and $R = \{(i, j) : |j - i| \leq 1\}$. Let $M = (W, R, V)$ where $V(p) = \{i : i = 1 \mod 3\}$, $V(q) = \{i : i = 2 \mod 3\}$ and for all atoms $\alpha$, if $\alpha \neq p$ and $\alpha \neq q$ then $V(\alpha) = 0$. The reader may easily verify that $M, 0 \notin (\Box^{k} \perp \rightarrow \Box^{l} \perp)$ and $M, 3l \notin (\Box^{k} \perp \rightarrow \Box^{l} \perp)$. Hence, $(\Box^{k} \perp \rightarrow \Box^{l} \perp) \notin \text{KB}$ and $(\Box^{k} \perp \rightarrow \Box^{l} \perp) \notin \text{KB}$. $\neg$

In the proof of Lemma 5, remark that the frame $F = (W, R)$ is reflexive.

Lemma 6 For all formulas $\varphi, (\varphi \rightarrow \Box \varphi) \in \text{KB}$ iff $(\neg \varphi \rightarrow \Box \neg \varphi) \in \text{KB}$.

Proof: Let $\varphi$ be a formula such that $(\varphi \rightarrow \Box \varphi) \notin \text{KB}$ and $(\neg \varphi \rightarrow \Box \neg \varphi) \in \text{KB}$, or $(\varphi \rightarrow \Box \varphi) \in \text{KB}$ and $(\neg \varphi \rightarrow \Box \neg \varphi) \notin \text{KB}$.

— Case “$(\varphi \rightarrow \Box \varphi) \notin \text{KB}$ and $(\neg \varphi \rightarrow \Box \neg \varphi) \in \text{KB}”$: Let $F = (W, R)$ be a frame, $M = (W, R, V)$ be a model based on $F$ and $s \in W$ be such that $M, s \not|= (\varphi \rightarrow \Box \varphi)$. Hence, $M, s \not|= \varphi$ and $M, s \not|= \Box \varphi$. Let $t, u, v \in W$ be such that $sRt, tRu, uRv, M, s |= p^0 \land q^0, M, t |= p^1 \land q^0, M, u |= p^0 \land q^1, M, v |= p^0 \land q^0$ and $M, v \not|= \varphi$. Thus, $tRs, uRt$ and $vRu$. Moreover, $M, v \not|= \neg \varphi$. Since $(\neg \varphi \rightarrow \Box \neg \varphi) \in \text{KB}$, therefore $M, v \not|= (\neg \varphi \rightarrow \Box \neg \varphi) \in \text{KB}$. Since $M, v \not|= \neg \varphi$, therefore $M, v \not|= \Box \neg \varphi$. Since $tRs, uRt, vRu, M, s |= p^0 \land q^0, M, t |= p^1 \land q^0, M, u |= p^0 \land q^1$ and $M, v |= p^0 \land q^0$, therefore $M, s \not|= \neg \varphi$. Consequently, $M, s \not|= \varphi$: a contradiction.
4 Unification

In this section, we present unification in $\mathbf{KB}$.

**Unification problem**  We shall say that a substitution $\sigma$ is *equivalent* to a substitution $\tau$ (in symbols $\sigma \simeq \tau$) if for all variables $x$, $(\sigma(x) \leftrightarrow \tau(x)) \in \mathbf{KB}$. We shall say that a substitution $\sigma$ is more *general* than a substitution $\tau$ (in symbols $\sigma \preceq \tau$) if there exists a substitution $\upsilon$ such that $\sigma \circ \upsilon \simeq \tau$. Obviously, $\preceq$ contains $\simeq$. Moreover,

**Proposition 1** (Baader and Ghilardi [1], Dzik [16])

1. The binary relation $\simeq$ is reflexive, symmetric and transitive on the set of all substitutions.

2. the binary relation $\preceq$ is reflexive and transitive on the set of all substitutions.

We shall say that a set $\Sigma$ of substitutions is *minimal* if for all $\sigma, \tau \in \Sigma$, if $\sigma \preceq \tau$ then $\sigma \simeq \tau$. We shall say that a formula $\varphi$ is *unifiable* if there exists a substitution $\sigma$ such that $\sigma(\varphi) \in \mathbf{KB}$. In that case, $\sigma$ is a *unifier* of $\varphi$.

**Proposition 2** Let $\varphi$ be a formula. For all unifiers $\sigma$ of $\varphi$, there exists a unifier $\tau$ of $\varphi$ such that $\tau \preceq \sigma$ and for all variables $x$, if $x$ does not occur in $\varphi$ then $\tau(x) = x$.

**Proof:** Left to the reader. \(\blacksquare\)

We shall say that a set $\Sigma$ of unifiers of a unifiable formula $\varphi$ is *complete* if for all unifiers $\sigma$ of $\varphi$, there exists $\tau \in \Sigma$ such that $\tau \preceq \sigma$.

**Unification types** An important question is the following: when a formula is unifiable, has it a minimal complete set of unifiers? When the answer is “yes”, how large is this set? We shall say that a unifiable formula

- is *nullary* if there exists no minimal complete set of unifiers of $\varphi$,
- is *unitary* if there exists a minimal complete set of unifiers of $\varphi$ with cardinality $1$,
- is *finitary* if there exists a finite minimal complete set of unifiers of $\varphi$ but there exists no with cardinality $1$,
- is *infinitary* if there exists a minimal complete set of unifiers of $\varphi$ but there exists no finite one.
5 Playing with substitutions

For all \( k \in \mathbb{N} \), let \( \sigma_k \) and \( \tau_k \) be the substitutions inductively defined as follows:

\begin{itemize}
  \item \( \sigma_0(x) = \bot \),
  \item for all variables \( y \) distinct from \( x \), \( \sigma_0(y) = y \),
  \item \( \tau_0(x) = \top \),
  \item for all variables \( y \) distinct from \( x \), \( \tau_0(y) = y \),
  \item \( \sigma_{k+1}(x) = (x \land \Box \sigma_k(x)) \),
  \item for all variables \( y \) distinct from \( x \), \( \sigma_{k+1}(y) = y \),
  \item \( \tau_{k+1}(x) = \neg(\neg x \land \Box \neg \tau_k(x)) \),
  \item for all variables \( y \) distinct from \( x \), \( \tau_{k+1}(y) = y \).
\end{itemize}

These substitutions will be used in Section 6 to prove that \( \mathbf{KB} \) possesses nullary formulas.

Lemma 7 For all \( k \in \mathbb{N} \),

1. \( (\Box^k x \land \Box^k \bot \rightarrow \sigma_k(x)) \in \mathbf{KB} \).
2. \( (\Box^k \neg x \land \Box^k \bot \rightarrow \neg \tau_k(x)) \in \mathbf{KB} \).

Proof: By induction on \( k \). \( \dashv \)

Lemma 8 For all \( k \in \mathbb{N} \),

1. \( (\sigma_k(x) \rightarrow x) \in \mathbf{KB} \).
2. \( (\neg \tau_k(x) \rightarrow \neg x) \in \mathbf{KB} \).

Proof: By induction on \( k \). \( \dashv \)

Lemma 9 For all \( k \in \mathbb{N} \),

1. \( (\sigma_k(x) \rightarrow \Box \sigma_k(x)) \in \mathbf{KB} \).
2. \( (\neg \tau_k(x) \rightarrow \Box \neg \tau_k(x)) \in \mathbf{KB} \).

Proof: By induction on \( k \). \( \dashv \)

Lemma 10 For all \( k, l \in \mathbb{N} \),

1. if \( k \leq l \) then \( (\sigma_k(x) \rightarrow \Box^l \bot) \in \mathbf{KB} \).
2. if \( k \leq l \) then \( (\neg \tau_k(x) \rightarrow \Box^l \bot) \in \mathbf{KB} \).
Proof: By induction on $k$. \(\vdash\)

Lemma 11 For all $k, l \in \mathbb{N}$,

1. if $k > l$ then $(\sigma_k(x) \rightarrow \Box^l \bot) \notin KB$.
2. if $k > l$ then $(\neg\tau_k(x) \rightarrow \Box^l \bot) \notin KB$.

Proof: Let $k, l \in \mathbb{N}$.

(1): Suppose $k > l$ and $(\sigma_k(x) \rightarrow \Box^l \bot) \in KB$. Let $\nu$ be the substitution defined as follows:

- $\nu(x) = \top$,
- for all variables $y$ distinct from $x$, $\nu(y) = y$.

Since $(\sigma_k(x) \rightarrow \Box^l \bot) \in KB$, therefore $(\nu(\sigma_k(x)) \rightarrow \Box^l \bot) \in KB$. By Lemma 7, $(\Box^{<k} \land \Box^k \bot \rightarrow \sigma_k(x)) \in KB$. Hence, $(\Box^{<k} \nu(x) \land \Box^k \bot \rightarrow \nu(\sigma_k(x))) \in KB$. Since $\nu(x) = \top$, therefore by Lemma 2, $(\Box^k \bot \rightarrow \nu(\sigma_k(x))) \in KB$. Since $(\nu(\sigma_k(x)) \rightarrow \Box^l \bot) \in KB$, therefore $(\Box^k \bot \rightarrow \Box^l \bot) \in KB$. Thus, by Lemma 5, $k \neq l$: a contradiction.

(2): Suppose $k > l$ and $(\neg\tau_k(x) \rightarrow \Box^l \bot) \in KB$. Let $\nu$ be the substitution defined as follows:

- $\nu(x) = \bot$,
- for all variables $y$ distinct from $x$, $\nu(y) = y$.

Since $(\neg\tau_k(x) \rightarrow \Box^l \bot) \in KB$, therefore $(\nu(\neg\tau_k(x)) \rightarrow \Box^l \bot) \in KB$. By Lemma 7, $(\Box^{<k} \land \Box^k \bot \rightarrow \neg\tau_k(x)) \in KB$. Hence, $(\Box^{<k} \nu(x) \land \Box^k \bot \rightarrow \nu(\neg\tau_k(x))) \in KB$. Since $\nu(x) = \bot$, therefore by Lemma 2, $(\Box^k \bot \rightarrow \nu(\neg\tau_k(x))) \in KB$. Since $(\nu(\neg\tau_k(x)) \rightarrow \Box^l \bot) \in KB$, therefore $(\Box^k \bot \rightarrow \Box^l \bot) \in KB$. Thus, by Lemma 5, $k \neq l$: a contradiction. \(\vdash\)

Lemma 12 For all $k, l \in \mathbb{N}$,

1. $(\Box^k \bot \lor \neg\tau_l(x)) \notin KB$,
2. $(\Box^k \bot \lor \sigma_l(x)) \notin KB$.

Proof: Let $k, l \in \mathbb{N}$.

(1): Suppose $(\Box^k \bot \lor \neg\tau_l(x)) \in KB$. By Lemma 8, $(\neg\tau_l(x) \rightarrow \neg \bot) \in KB$. Since $(\Box^k \bot \lor \neg\tau_l(x)) \in KB$, therefore $(\Box^k \bot \lor \neg \bot) \in KB$. Let $\nu$ be the substitution defined as follows:

- $\nu(x) = \top$,
- for all variables $y$ distinct from $x$, $\nu(y) = y$.
Since \( \Box^k \bot \lor \neg x \in \text{KB} \), therefore \( \Box^k \bot \lor \neg v(x) \in \text{KB} \). Since \( v(x) = T \), therefore \( \Box^k \bot \in \text{KB} \): a contradiction with Lemma 3.

(2): Suppose \( \Box^k \bot \lor \sigma_l(x) \in \text{KB} \). By Lemma 8, \( (\sigma_l(x) \rightarrow x) \in \text{KB} \). Since \( (\Box^k \bot \lor \sigma_l(x)) \in \text{KB} \), therefore \( (\Box^k \bot \lor x) \in \text{KB} \). Let \( \nu \) be the substitution defined as follows:

\[
\begin{align*}
\nu(x) &= \bot, \\
\text{for all variables } y \text{ distinct from } x, \nu(y) &= y.
\end{align*}
\]

Since \( (\Box^k \bot \lor x) \in \text{KB} \), therefore \( (\Box^k \bot \lor \nu(x)) \in \text{KB} \). Since \( \nu(x) = \bot \), therefore \( \Box^k \bot \in \text{KB} \): a contradiction with Lemma 3. \( \dashv \)

Lemma 13 For all \( k, l \in \mathbb{N} \),

1. if \( k \leq l \) then \( (\Box^k \bot \land \sigma_l(x) \leftrightarrow \sigma_k(x)) \in \text{KB} \),
2. if \( k \leq l \) then \( (\Box^k \bot \land \neg \tau_l(x) \leftrightarrow \neg \tau_k(x)) \in \text{KB} \).

Proof: By induction on \( k \). \( \dashv \)

For all \( k \in \mathbb{N} \), let \( \lambda_k \) and \( \mu_k \) be the substitutions defined as follows:

\[
\begin{align*}
\lambda_k(x) &= (x \land \Box^k \bot), \\
\text{for all variables } y \text{ distinct from } x, \lambda_k(y) &= y, \\
\mu_k(x) &= \neg (\neg x \land \Box^k \bot), \\
\text{for all variables } y \text{ distinct from } x, \mu_k(y) &= y.
\end{align*}
\]

Lemma 14 For all \( k, l \in \mathbb{N} \),

1. if \( k \leq l \) then \( (\lambda_l(\sigma_k(x)) \leftrightarrow \sigma_k(x)) \in \text{KB} \),
2. if \( k \leq l \) then \( (\mu_l(\tau_k(x)) \leftrightarrow \tau_k(x)) \in \text{KB} \).

Proof: By induction on \( k \). \( \dashv \)

Lemma 15 For all \( k, l \in \mathbb{N} \),

1. if \( k \geq l \) then \( (\lambda_l(\sigma_k(x)) \leftrightarrow \sigma_l(x)) \in \text{KB} \),
2. if \( k \geq l \) then \( (\mu_l(\tau_k(x)) \leftrightarrow \tau_l(x)) \in \text{KB} \).

Proof: By induction on \( k \). \( \dashv \)

Lemma 16 For all \( k, l \in \mathbb{N} \),

1. if \( k \leq l \) then \( \sigma_l \circ \lambda_k \simeq \sigma_k \),
2. if \( k \leq l \) then \( \tau_l \circ \mu_k \simeq \tau_k \).
Proof: By Lemma 15. ⊥

Lemma 17 For all $k, l \in \mathbb{N}$,

1. if $k \leq l$ then $\sigma_k \preceq \sigma_l$.
2. if $k \leq l$ then $\tau_k \preceq \tau_l$.

Proof: By Lemma 16. ⊥

Lemma 18 For all $k, l \in \mathbb{N}$,

1. if $k < l$ then $\sigma_k \not\preceq \sigma_l$.
2. if $k < l$ then $\tau_k \not\preceq \tau_l$.

Proof: Let $k, l \in \mathbb{N}$.

1: Suppose $k < l$ and $\sigma_k \preceq \sigma_l$. Let $\lambda$ be a substitution such that $\sigma_k \circ \lambda \simeq \sigma_l$. Hence, $(\lambda(\sigma_k(x)) \leftrightarrow \sigma_l(x)) \in KB$. By Lemma 10, $(\sigma_k(x) \rightarrow \Box^k \bot) \in KB$. Thus, $(\lambda(\sigma_k(x)) \rightarrow \Box^k \bot) \in KB$. Since $(\lambda(\sigma_k(x)) \leftrightarrow \sigma_l(x)) \in KB$, therefore $(\sigma_l(x) \rightarrow \Box^k \bot) \in KB$. Consequently, by Lemma 11, $l \not\simeq k$: a contradiction.

(2): Suppose $k < l$ and $\tau_k \preceq \tau_l$. Let $\mu$ be a substitution such that $\tau_k \circ \mu \simeq \tau_l$. Hence, $(\mu(\tau_k(x)) \leftrightarrow \tau_l(x)) \in KB$. By Lemma 10, $(\neg \tau_k(x) \rightarrow \Box^k \bot) \in KB$. Thus, $(\mu(\neg \tau_k(x)) \rightarrow \Box^k \bot) \in KB$. Since $(\mu(\tau_k(x)) \leftrightarrow \tau_l(x)) \in KB$, therefore $(\neg \tau_l(x) \rightarrow \Box^k \bot) \in KB$. Consequently, by Lemma 11, $l \not\simeq k$: a contradiction. ⊥

Lemma 19 For all $k, l \in \mathbb{N}$,

1. $\sigma_k \not\preceq \tau_l$.
2. $\tau_k \not\preceq \sigma_l$.

Proof: Let $k, l \in \mathbb{N}$.

1: Suppose $\sigma_k \preceq \tau_l$. Let $\nu$ be a substitution such that $\sigma_k \circ \nu \simeq \tau_l$. Hence, $(\nu(\sigma_k(x)) \leftrightarrow \tau_l(x)) \in KB$. By Lemma 10, $(\sigma_k(x) \rightarrow \Box^k \bot) \in KB$. Thus, $(\nu(\sigma_k(x)) \rightarrow \Box^k \bot) \in KB$. Since $(\nu(\sigma_k(x)) \leftrightarrow \tau_l(x)) \in KB$, therefore $(\Box^k \bot \lor \neg \tau_l(x)) \in KB$: a contradiction with Lemma 12.

(2): Suppose $\tau_k \preceq \sigma_l$. Let $\nu$ be a substitution such that $\tau_k \circ \nu \simeq \sigma_l$. Hence, $(\nu(\tau_k(x)) \leftrightarrow \sigma_l(x)) \in KB$. By Lemma 10, $(\neg \tau_k(x) \rightarrow \Box^k \bot) \in KB$. Thus, $(\nu(\neg \tau_k(x)) \rightarrow \Box^k \bot) \in KB$. Since $(\nu(\tau_k(x)) \leftrightarrow \sigma_l(x)) \in KB$, therefore $(\Box^k \bot \lor \sigma_l(x)) \in KB$: a contradiction with Lemma 12. ⊥

6 About the nullariness of KB

In this section, we prove that the following formula is unifiable and nullary:
Lemma 20. Let $\sigma$ be a unifier of $\varphi$. For all $k \in \mathbb{N}$,

1. $(\sigma(x) \rightarrow \boxdot^k \sigma(x)) \in \text{KB}$.
2. $(\lnot \sigma(x) \rightarrow \square^k \lnot \sigma(x)) \in \text{KB}$.

Proof: By induction on $k$. ⊥

Lemma 21. For all $k \in \mathbb{N}$,

1. $\tau_k$ is a unifier of $\varphi$.
2. $\tau_k$ is a unifier of $\varphi$.

Proof: By Lemmas 6 and 9. ⊥

Lemma 22. Let $\nu$ be a substitution. If $\nu$ is a unifier of $\varphi$ then

1. for all $k \in \mathbb{N}$, the following conditions are equivalent: (a) $\sigma_k \circ \nu \simeq \nu$, (b) $\sigma_k \preceq \nu$, (c) $(\nu(x) \rightarrow \boxdot^k \bot) \in \text{KB}$.
2. for all $k \in \mathbb{N}$, the following conditions are equivalent: (d) $\tau_k \circ \nu \simeq \nu$, (e) $\tau_k \preceq \nu$, (f) $(\lnot \nu(x) \rightarrow \square^k \bot) \in \text{KB}$.

Proof: Suppose $\nu$ is a unifier of $\varphi$.

(1): Let $k \in \mathbb{N}$.

(a) ⇒ (b): Suppose $\sigma_k \circ \nu \simeq \nu$. Hence, $\sigma_k \preceq \nu$.
(b) ⇒ (c): Suppose $\sigma_k \preceq \nu$. Let $\nu'$ be a substitution such that $\sigma_k \circ \nu' \simeq \nu$.
Hence, $(\nu'(\sigma_k(x)) \leftrightarrow \nu(x)) \in \text{KB}$. By Lemma 10, $(\sigma_k(x) \rightarrow \boxdot^k \bot) \in \text{KB}$.
Thus, $(\nu'(\sigma_k(x)) \rightarrow \boxdot^k \bot) \in \text{KB}$. Since $(\nu'(\sigma_k(x)) \leftrightarrow \nu(x)) \in \text{KB}$, therefore $(\nu(x) \rightarrow \boxdot^k \bot) \in \text{KB}$.

(c) ⇒ (a): Suppose $(\nu(x) \rightarrow \boxdot^k \bot) \in \text{KB}$. Since $\nu$ is a unifier of $\varphi$, therefore by Lemma 20, $(\nu(x) \rightarrow \boxdot^k \nu(x)) \in \text{KB}$. Since $(\nu(x) \rightarrow \boxdot^k \bot) \in \text{KB}$, therefore $(\nu(x) \rightarrow \boxdot^k \nu(x) \land \boxdot^k \bot) \in \text{KB}$. By Lemma 7, $(\boxdot^k \nu(x) \land \boxdot^k \bot \rightarrow \sigma_k(x))$ \in \text{KB}.
Hence, $(\boxdot^k \nu(x) \land \boxdot^k \bot \rightarrow \nu(\sigma_k(x))) \in \text{KB}$. Since $(\nu(x) \rightarrow \boxdot^k \nu(x) \land \boxdot^k \bot) \in \text{KB}$, therefore $(\nu(\sigma_k(x)) \leftrightarrow \nu(x)) \in \text{KB}$. By Lemma 8, $(\sigma_k(x) \rightarrow x) \in \text{KB}$.
Thus, $(\nu(\sigma_k(x)) \rightarrow \nu(x)) \in \text{KB}$. Since $(\nu(x) \rightarrow \nu(\sigma_k(x))) \in \text{KB}$, therefore $(\nu(\sigma_k(x)) \leftrightarrow \nu(x)) \in \text{KB}$. Consequently, $\sigma_k \circ \nu \simeq \nu$.

(2): Let $k \in \mathbb{N}$.

(d) ⇒ (e): Suppose $\tau_k \circ \nu \simeq \nu$. Hence, $\tau_k \preceq \nu$.
(e) ⇒ (f): Suppose $\tau_k \preceq \nu$. Let $\nu'$ be a substitution such that $\tau_k \circ \nu' \simeq \nu$.
Hence, $(\nu'(\tau_k(x)) \leftrightarrow \nu(x)) \in \text{KB}$. By Lemma 10, $(\lnot \tau_k(x) \rightarrow \square^k \bot) \in \text{KB}$.
Thus, $(\nu'(\tau_k(x)) \rightarrow \square^k \bot) \in \text{KB}$. Since $(\nu'(\tau_k(x)) \leftrightarrow \nu(x)) \in \text{KB}$, therefore $(\lnot \nu(x) \rightarrow \square^k \bot) \in \text{KB}$.

(f) ⇒ (d): Suppose $(\lnot \nu(x) \rightarrow \square^k \bot) \in \text{KB}$. Since $\nu$ is a unifier of $\varphi$, therefore by Lemma 20, $(\lnot \nu(x) \rightarrow \square^k \lnot \nu(x)) \in \text{KB}$. Since $(\lnot \nu(x) \rightarrow \square^k \bot) \in \text{KB}$.
therefore $(-v(x) \rightarrow \Box^k v(x) \land \Box^k \bot) \in \mathbf{KB}$. By Lemma 7, $((\Box^k v(x) \land \Box^k \bot \rightarrow \neg \tau_k(x)) \in \mathbf{KB}$. Hence, $((\Box^k \neg v(x) \land \Box^k \bot \rightarrow v(\neg \tau_k(x))) \in \mathbf{KB}$. Since $(-v(x) \rightarrow \Box^k \neg v(x) \land \Box^k \bot) \in \mathbf{KB}$, therefore $(-v(x) \rightarrow v(\neg \tau_k(x))) \in \mathbf{KB}$. By Lemma 8, $(-\tau_k(x) \rightarrow \neg x) \in \mathbf{KB}$. Thus, $(v(\neg \tau_k(x)) \rightarrow \neg v(x)) \in \mathbf{KB}$. Since $(-v(x) \rightarrow v(\neg \tau_k(x))) \in \mathbf{KB}$, therefore $(v(\tau_k(x)) \leftrightarrow v(x)) \in \mathbf{KB}$. Consequently, $\tau_k \circ v \simeq v$.

**Lemma 23** Let $\sigma$ be a substitution. If $\sigma$ is a unifier of $\varphi$ then there exists $k \in \mathbb{N}$ such that $\sigma_k \preceq \sigma$, or $\tau_k \preceq \sigma$.

**Proof:** Suppose $\sigma$ is a unifier of $\varphi$. By Propositions 1 and 2, we can assume that for all variables $y$ distinct from $x$, $\sigma(y) = y$. Let $k \in \mathbb{N}$ be such that $\deg(\sigma(x)) \leq 3k$.

Suppose $\sigma_k \not\preceq \sigma$ and $\tau_k \not\preceq \sigma$. Since $\sigma$ is a unifier of $\varphi$, therefore by Lemma 22, $(\sigma(x) \rightarrow \Box^k \bot) \not\in \mathbf{KB}$ and $(\neg \sigma(x) \rightarrow \Box^k \bot) \not\in \mathbf{KB}$. Let $F = (W, R)$ be a frame, $M = (W, R, V)$ be a model based on $F$, $s \in W$, $F' = (W', R')$ be a frame, $M' = (W', R', V')$ be a model based on $F'$ and $s' \in W'$ be such that $M, s \not\models (\sigma(x) \rightarrow \Box^k \bot)$ and $M', s' \not\models (\neg \sigma(x) \rightarrow \Box^k \bot)$. Hence, $M, s \models \sigma(x)$, $M, s \not\models \Box^k \bot$, $M', s' \models \neg \sigma(x)$ and $M', s' \not\models \Box^k \bot$. Let $v_0, t_1, v_1, \ldots, t_k, u_k, v_k \in W$ and $t'_0, t'_1, u'_1, \ldots, t'_k, u'_k, v'_k \in W'$ be such that $s = v_0$, $s' = t'_0$ and for all $i \in \mathbb{N}$, if $i < k$ then

- $v_i R t_{i+1}$,
- $t_{i+1} R u_{i+1}$,
- $u_{i+1} R v_{i+1}$,
- $v'_i R' t'_{i+1}$,
- $t'_{i+1} R' u'_{i+1}$,
- $u'_{i+1} R' v'_{i+1}$,
- $M, v_i \models p_0 \land q_0$,
- $M, t_{i+1} \models p_1 \land q_0$,
- $M, u_{i+1} \models p_0 \land q_1$,
- $M, v_{i+1} \models p_0 \land q_0$,
- $M', v'_i \models p_0 \land q_0$,
- $M', t'_{i+1} \models p_0 \land q_0$,
- $M', u'_{i+1} \models p_1 \land q_0$,
- $M', v'_{i+1} \models p_0 \land q_0$. 

12
Let $M_s = (W_s, R_s, V_s)$ be the symmetric unravelling of $M$ around $s$ and $M_{s'} = (W_{s'}, R_{s'}, V_{s'})$ be the symmetric unravelling of $M'$ around $s'$. For more on this, see [11, Definition 4.51]. Since $M, s \models \sigma(x)$ and $M', s' \models \neg \sigma(x)$, therefore by [11, Proposition 2.14 and Lemma 4.52], $M_s, (v_0) \models \sigma(x)$ and $M_{s'}, (v_0) \models \neg \sigma(x)$. Let $F'' = (W'', R'')$ be the least frame containing the disjoint union of $(W_s, R_s)$ and $(W_{s'}, R_{s'})$ and such that for some new states $t$ and $u$,

- $(v_0, t_1, u_1, \ldots, t_k, u_k) \in R''t$,
- $tR''(v_0, t_1, u_1, \ldots, t_k, u_k, v_k)$,
- $tR''t$,
- $tR''u$,
- $uR''t$,
- $uR''u$,
- $uR''(v_0', t_1', u_1', \ldots, t_k', u_k', v_k')$,
- $(v_0', t_1', u_1', \ldots, t_k', u_k', v_k') \in R''u$.

Let $M'' = (W'', R'', V'')$ where

- $V''(p) = V_s(p) \cup V_{s'}(p) \cup \{t\}$,
- $V''(q) = V_s(q) \cup V_{s'}(q) \cup \{u\}$,
- for all atoms $\alpha$, if $\alpha \neq p$ and $\alpha \neq q$ then $V''(\alpha) = V_s(\alpha) \cup V_{s'}(\alpha)$.

Since $\deg(\sigma(x)) \leq 3k$, $M_s, (v_0) \models \sigma(x)$ and $M_{s'}, (v_0') \models \neg \sigma(x)$, therefore $M'', (v_0) \models \sigma(x)$ and $M'', (v_0') \models \neg \sigma(x)$. Since $\sigma$ is a unifier of $\varphi$, therefore $((\sigma(x) \rightarrow \boxdot \sigma(x)) \land (\neg \sigma(x) \rightarrow \boxdot \neg \sigma(x))) \in \mathbf{KB}$. Since $M'', (v_0) \models \sigma(x)$ and $M'', (v_0') \models \neg \sigma(x)$, considering that for all $i \in \mathbb{N}$, if $i < k$ then $M, v_i \models p^0 \land q^0, M, t_{i+1} \models p^0 \land q^0$, $M, v_{i+1} \models p^0 \land q^0, M, t_{i+1} \models p^0 \land q^0, M', v'_i \models p^0 \land q^0, M', t'_{i+1} \models p^0 \land q^1$, $M', v'_{i+1} \models p^1 \land q^0$ and $M'', v''_i \models p^0 \land q^0$, therefore $M'', (v_0, t_1, u_1, \ldots, t_k, u_k, v_k) \models \sigma(x)$ and $M'', (v_0', t'_1, u'_1, \ldots, t'_k, u'_k, v'_k) \models \neg \sigma(x)$. Since $((\sigma(x) \rightarrow \boxdot \sigma(x)) \land (\neg \sigma(x) \rightarrow \boxdot \neg \sigma(x))) \in \mathbf{KB}$, considering that $M, v_k \models p^0 \land q^0, M'', t \models p^0 \land q^0, M'', u \models p^0 \land q^0$ and $M', v'_k \models p^0 \land q^0$, therefore $M'', (v_0, t_1, u_1, \ldots, t_k, u_k, v_k) \models \neg \sigma(x)$ and $M'', (v_0', t'_1, u'_1, \ldots, t'_k, u'_k, v'_k) \models \sigma(x)$. Thus, $M'', (v_0, t_1, u_1, \ldots, t_k, u_k, v_k) \not\models \sigma(x)$ and $M'', (v_0', t'_1, u'_1, \ldots, t'_k, u'_k, v'_k) \not\models \neg \sigma(x)$: a contradiction. \rule{0pt}{0pt}

In the proof of Lemma 23, remark that the symmetric unravellings $M_s$ and $M_{s'}$ are serial when the models $M$ and $M'$ are serial. Moreover, when the models $M$ and $M'$ are reflexive, $M_s$ and $M_{s'}$ can be defined as their reflexive symmetric unravellings.

**Proposition 3** $\varphi$ is nullary.
Proof: Suppose $\varphi$ is not nullary. Let $\Sigma$ be a minimal complete set of unifiers of $\varphi$. By Lemma 21, $\sigma_0$ is a unifier of $\varphi$. Since $\Sigma$ is a complete set of unifiers of $\varphi$, therefore let $\sigma \in \Sigma$ be such that $\sigma \preceq \sigma_0$. Hence, by Lemma 23, let $k \in \mathbb{N}$ be such that $\sigma_k \preceq \sigma$, or $\tau_k \preceq \sigma$.

— Case “$\sigma_k \preceq \sigma$”: By Lemma 21, $\sigma_{k+1}$ is a unifier of $\varphi$. Since $\Sigma$ is a complete set of unifiers of $\varphi$, therefore let $\sigma' \in \Sigma$ be such that $\sigma' \preceq \sigma_{k+1}$. Since $\sigma_k \preceq \sigma$, therefore by Lemma 17, $\sigma' \preceq \sigma$. Since $\Sigma$ is a minimal set of unifiers of $\varphi$, therefore $\sigma' \simeq \sigma$. Since $\sigma_k \preceq \sigma$ and $\sigma' \preceq \sigma_{k+1}$, therefore $\sigma_k \preceq \sigma_{k+1}$: a contradiction with Lemma 18.

— Case “$\tau_k \preceq \sigma$”: Since $\sigma \preceq \sigma_0$, therefore $\tau_k \preceq \sigma_0$: a contradiction with Lemma 19.

7 Conclusion

In modal logic, the problem of checking the unifiability of formulas has been introduced as a special case of the problem of checking the admissibility of inference rules [29]. Intuitively, for an axiomatically presented modal logic, the admissibility problem asks whether a given inference rule can be added to the axiomatization of the logic without changing the associated set of derivable formulas. Its computability has been studied — for a limited number of normal modal logics like $K_4$, $GL$ and $S_4$ — by Jeřábek [25] and Rybakov [27]. Aside from these transitive normal modal logics and for the normal extensions of $S_5$, it is still unknown for numerous normal modal logics — for example $K$, $KD$ and $KT$ — whether the problem of checking the admissibility of inference rules is solvable. The significance of the unification type in the research on the problem of checking the unifiability of formulas stems from the fact that if a normal modal logic is unitary, or finitary then the problem of checking the admissibility of inference rules can be reduced to the problem of checking the unifiability of formulas.

In this paper, we have adapted to $KB$ the argument of Jeřábek [26] showing that $K$ is nullary, though the nullariness character of $KB$ have only been obtained within the context of unification with parameters. Seeing that the frames constructed in the proofs of Lemmas 3 and 5 are reflexive and the symmetric unravellings of the models constructed in the proof of Lemma 23 are serial when the considered models are serial, or can be forced to be reflexive when the considered models are reflexive, therefore on checking the proofs of our results, the reader may easily verify that our adaptation also applies in the case of $KDB$ and $KTB$ — one has only to replace “$KB$” by “$KDB$”, or “$KTB$”, “frame” by “serial frame”, or “reflexive frame”, etc. The nullariness character of $KB$, $KDB$ and $KTB$ constitutes an answer to questions put forward by Dzik [16]. Nevertheless, much remains to be done, seeing that, for instance, the types of simple Church-Rosser normal modal logics like $KG$, $KDG$ and $K TG$ are unknown and for all $k \in \mathbb{N}$ such that $k \geq 2$, the type of the least normal modal logic containing $\square^k \bot$ is unknown.
Acknowledgements

This paper has been written on the occasion of a 3-months visit of Çiğdem Gencer during the Fall 2018 in Toulouse that was financially supported by Université Paul Sabatier (“Professeurs invités 2018”). We make a point of thanking the colleagues of the Institut de recherche en informatique de Toulouse who contributed to the development of the work we present today. Special acknowledgement is also heartily granted to Maryam Rostamigiv (Toulouse University, France) and Tinko Tinchev (Sofia University St. Kliment Ohridski, Bulgaria) for their valuable remarks.

References

[1] BAADER, F., and S. GHI L AR DI, ‘Unification in modal and description logics’, Logic Journal of the IGPL 19:705–730, 2011.

[2] BAADER, F., and B. MORAWSKA, ‘Unification in the description logic $\mathcal{EL}$’, In: Rewriting Techniques and Applications, Springer 350–364, 2009.

[3] BAADER, F., and P. NAREND RAN, ‘Unification of concept terms in description logics’, Journal of Symbolic Computation 31:277–305, 2001.

[4] BABENYSHEV, S., V. RYBAKOV, R. SCHMIDT, and D. TISHKOVSKY, ‘A tableau method for checking rule admissibility in $\mathcal{S}_4$’, Electronic Notes in Theoretical Computer Science 262:17–32, 2010.

[5] BABENYSHEV, S., and V. RYBAKOV, ‘Unification in linear temporal logic $\mathcal{LTL}$’, Annals of Pure and Applied Logic 162:991–1000, 2011.

[6] BALBIAN I, P., ‘Remarks about the unification type of some non-symmetric non-transitive modal logics’, Logic Journal of the IGPL (to appear).

[7] BALBIAN I, P., and Ç. GENCER, ‘$KD$ is nullary’, Journal of Applied Non-Classical Logics 27:196–205, 2017.

[8] BALBIAN I, P., and Ç. GENCER, ‘Unification in epistemic logics’, Journal of Applied Non-Classical Logics 27:91–105, 2017.

[9] BALBIAN I, P., and T. TINCHEV, ‘Unification in modal logic $Alt_1$’, In: Advances in Modal Logic, College Publications 117–134, 2016.

[10] BALBIAN I, P., and T. TINCHEV, ‘Elementary unification in modal logic $KD_{45}$’, Journal of Applied Logic — IFCoLog Journal of Logics and their Applications 5:301–317, 2018.

[11] BLACKBURN, P., M. DE RIJKE, and Y. VENEMA, Modal Logic, Cambridge University Press, 2001.

[12] CHAGROV, A., and M. Zakharyaschev, Modal Logic, Oxford University Press, 1997.
[13] Chellas, B., Modal Logic. An Introduction, Cambridge University Press, 1980.

[14] Cintula, P., and G. Metcalfe, ‘Admissible rules in the implication-negation fragment of intuitionistic logic’, Annals of Pure and Applied Logic 162:162–171, 2010.

[15] Dzik, W., ‘Unitary unification of S5 modal logics and its extensions’, Bulletin of the Section of Logic 32:19–26, 2003.

[16] Dzik, W., Unification Types in Logic, Wydawnicto Uniwersytetu Slaskiego, 2007.

[17] Dzik, W., ‘Remarks on projective unifiers’, Bulletin of the Section of Logic 40:37–46, 2011.

[18] Dzik, W., and P. Wojtylak, ‘Projective unification in modal logic’, Logic Journal of the IGPL 20:121–153, 2012.

[19] Fernández Gil, O., ‘Hybrid Unification in the Description Logic $\mathcal{EL}$’, Master Thesis of Technische Universität Dresden, 2012.

[20] Gençer, Ç., and D. De Jongh, ‘Unifiability in extensions of $K4$’, Logic Journal of the IGPL 17:159–172, 2009.

[21] Ghilardi, S., ‘Unification in intuitionistic logic’, Journal of Symbolic Logic 64:859–880, 1999.

[22] Ghilardi, S., ‘Best solving modal equations’, Annals of Pure and Applied Logic 102:183–198, 2000.

[23] Ghilardi, S., and L. Sacchetti, ‘Filtering unification and most general unifiers in modal logic’, Journal of Symbolic Logic 69:879–906, 2004.

[24] Iemhoff, R., ‘On the admissible rules of intuitionistic propositional logic’, Journal of Symbolic Computation 66:281–294, 2001.

[25] Jeřábek, E., ‘Complexity of admissible rules’, Archive for Mathematical Logic 46:73–92, 2007.

[26] Jeřábek, E., ‘Blending margins: the modal logic $K$ has nullary unification type’, Journal of Logic and Computation 25:1231–1240, 2015.

[27] Rybakov, V., ‘A criterion for admissibility of rules in the model system $S4$ and the intuitionistic logic. Algebra and Logic 23:369–384, 1984.

[28] Rybakov, V., ‘Bases of admissible rules of the logics $S4$ and $\text{Int}$’, Algebra and Logic 24:55–68, 1985.

[29] Rybakov, V., Admissibility of Logical Inference Rules, Elsevier Science, 1997.

[30] Rybakov, V., ‘Construction of an explicit basis for rules admissible in modal system $S4$', Mathematical Logic Quarterly 47:441–446, 2001.
[31] Wolter, F., and M. Zakharyaschev, ‘Undecidability of the unification and admissibility problems for modal and description logics’, ACM Transactions on Computational Logic 9:25:1–25:20, 2008.