Non-relativistic Matrix Inflation

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Abstract

We reconsider a string theoretic inflationary model, where inflation is driven by $n$ multiple coincident $D3$-branes in the finite $n$ limit. We show that the finite $n$ action can be continued to the limit of large $n$, where it converges to the action for a wrapped $D5$-brane with $n$ units of $U(1)$ flux. This provides an important consistency check of the scenario and allows for more control over certain back-reaction effects. We determine the most general form of the action for a specific sub-class of models and examine the non-relativistic limits of the theory where the branes move at speeds much less than the speed of light. The non-Abelian nature of the world-volume theory implies that the inflaton field is matrix valued and this results in modifications to the slow-roll parameters and Hubble-flow equations. A specific small field model of inflation is investigated where the branes move out of an AdS throat, and observational constraints are employed to place bounds on the background fluxes.
1 Introduction

Modulo several signatures from the strong-coupling regime of QCD [1, 2, 3], as well as indirect evidence possibly emerging from terrestrial accelerators [57], string theory remains best-tested in the realm of early universe cosmology. Observations of the cosmic microwave background (CMB) provide strong evidence for the fiducial ΛCDM model of cosmology, together with a period of primordial inflation which set the initial conditions for the density fluctuations. Ideally primordial inflation occurred in an energy regime where degrees of freedom unique to string theory are excited; or, at the very least, string theory should provide the mechanism for an epoch of inflation that is consistent with current observations. An understanding of early-universe dynamics therefore provides a direct means of exploring aspects of string theory.

Given this motivation, inflationary model building within string theory has become an established research direction, yielding many different models. Despite the large number of proposals, this pursuit serves—at the very least—to constrain the parameter space and overall viability of string theory [58]. A simple dichotomy of these scenarios emerges from the origin of the inflaton, i.e., whether this field originates from the open or closed sector of the theory. In both cases, numerous novel phenomena can arise and, in this sense, string theory provides a natural explanation for possible effects which can not be understood within the context of vanilla, slow-roll inflation. Since such results reside mostly within the framework of effective field theory, one cannot be certain if string theory is the unique UV completion. Nonetheless, it is remarkable that the classes of models selected by string theory have such a wealth of unique and potentially detectable phenomena.

In this paper we focus on a popular form of brane inflation embedded in the open-string sector of string theory. This model of inflation associates the inflaton with a modulus parametrizing the separation of a probe brane from a stack of branes residing at the bottom of a warped throat. There has been immense excitement that such a model could lead to distinct observational signatures, particularly regarding a non-vanishing bi-spectrum in the primordial curvature perturbation (see [47, 48, 49, 50] for examples). However, there remain many open questions regarding the regime of validity of this scenario, at both the theoretical and observational levels [15, 27, 28].

In view of this, we go beyond the simplest approximation and consider a more complicated model comprised of $n$ multiple coincident branes that follow a single trajectory. Due to the nature of the world-volume theory living on these branes, the resulting field-theoretic analysis is automatically a non-Abelian theory, where the inflaton is now a scalar field transforming in an adjoint representation. One strong objection to using a theory of this form is that we wish the standard model to exist in another (warped) throat, and therefore a world-volume description of inflation is not desirable. However, we will argue that such a theory has a dual description in terms of a wrapped higher-dimensional brane, and therefore represents an important theory within the class of DBI-inflation models. Importantly, we will argue that the wrapped brane description is valid for large values of $U(1)$ flux by showing how the multi-brane action converges as we take the large $n$ limit of the finite $n$ theory.
Since the fully relativistic theory has been (relatively) well explored \([39, 40, 41, 14]\), we focus in the present work on the non-relativistic limit, where the velocity of the inflaton field is significantly smaller than the speed of light. It is in this sense that we refer to such a regime as ‘non-relativistic’ and this is distinct from the limit where the effective sound speed of inflaton field fluctuations is much less than that of light. In the standard DBI-inflation model, the non-relativistic limit of the theory is simply a canonical scalar field in the appropriate FRW-background. However, due to the non-commutative structure inherited from the non-Abelian field theory, the non-relativistic limit in our scenario is a non-canonical scalar field theory. Consequently, one should anticipate different physical effects when compared to the standard ‘slow roll’ inflationary scenario.

The paper is structured as follows. In section 2, we remind the reader of the string theoretic construction for this class of models. Firstly we argue that the large \(n\) limit of the multi-brane theory is precisely dual to a single wrapped \(D5\)-brane with \(n\)-units of \(U(1)\) flux. We then consider the finite \(n\) limit of the multi-brane theory, since this is analytically tractable, and proceed to argue that the finite \(n\) limit does indeed analytically converge with the large \(n\) theory in the correct scaling limit. This is the first time that such a result has been established in the literature. In section 3, we develop the framework for analyzing inflationary dynamics in the finite \(n\) theory, and construct a simple example of small-field inflation in an AdS background driven by an inverted harmonic oscillator potential. In section 4, we specialize to the \(n = 2\) theory and consider in some detail how the inflationary observables differ from those of the single brane model. We then discuss how the flow equations for such a model can be constructed. Finally, we conclude in section 5 with some general remarks and a discussion of future directions.

2 The Action for matrix cosmology

2.1 Matrix cosmology at large \(n\)

Numerous extensions of the vanilla DBI-inflation model have been proposed, yielding novel results and phenomena. These include, but are not limited to, multiple-field scenarios \([19, 20, 21, 26]\), multiple branes \([22, 23, 24]\), monodromies \([25]\), Wilson lines \([45, 46]\) and the inclusion of bulk form fields \([18]\). In this section, we are interested in the more conservative generalization of the model, where the solitary \(D3\)-brane is replaced by a solitary \(D5\)-brane wrapping a non-trivial two-cycle of the internal geometry (usually taken to be an \(S^2\)—note that this is a simply-connected space and therefore we must turn on some world-volume flux to stabilize the brane) \([27, 28]\). Whilst there has been significant progress in the field of flux compactification in the type II theory onto (conformal) Calabi-Yau manifolds, we will consider a simpler class of warped background geometries based upon the well understood conifold geometry

\[
ds^2 = h^2(\rho)ds_4^2 + h^{-2}(\rho)(d\rho^2 + \rho^2d\Omega_5^2).
\]

The basic idea is that fluxes in the compactification back-react to form a throat (parametrized by a radial coordinate \(\rho\)) over the base manifold, \(X_5\). We allow the brane to be localized (and flat) in the large dimensions, wrapping a two-cycle within the internal manifold. We further
assume that the brane is dynamical along the throat direction and freeze out any angular degrees of freedom.

Without specializing to a particular supergravity background solution for the warp factor, \( h(\rho) \), we know that generically one must turn on magnetic flux along the two-cycle directions in order to stabilize the configuration. The \( U(1) \) field strength must be proportional to the volume of the wrapped cycle, where the constant of proportionality is determined by the \( n \) units of flux threaded through the \( S^2 \):

\[
P^{(2)} = \frac{n}{2} \omega_2,
\]

\[
(2.2)
\]

where \( \omega_2 \) is the associated two-form of the sphere.

The corresponding action for a \( D5 \)-brane is then given by the usual DBI expression (see [29] for further discussion):

\[
S = -T_5 \int d^6x e^{-\Phi} \sqrt{-\det(\hat{G}_{ab} + \hat{B}_{ab} + \lambda F_{ab})} - \mu_5 \int \left( \sum_n \hat{C}^{(n)} e^\hat{B} \right) e^{\lambda F},
\]

\[
(2.3)
\]

where \( ^\hat{\cdot} \) denotes the pullback of that particular tensor to the world-volume, and \( \lambda \) is the inverse of the fundamental string tension which couples to the \( U(1) \) field strength. The field \( \Phi \) is the dilaton defining the string coupling constant in the low energy theory. However, since we are assuming that the throat is sourced by \( D3 \)-brane charge, we can set this term to unity without loss of generality. For more involved backgrounds, the dilaton will indeed be non-trivial and can lead to more complicated dynamics.

The second term in Eq. (2.3) corresponds to the coupling of the brane to the bulk form fields in the \( RR \)-sector once they are pulled-back to the world-volume. The presence of the summation indicates that there can be coupling to form fields of lower degree, provided that there is a non-zero \( B \) (or \( F \)) term. Note that \( T_5 \) and \( \mu_5 \) denote, respectively, the tension and charge carried by the brane, and that these are related via supersymmetry. By utilizing the above metric and gauge field ansatz we can compute the action for the wrapped \( D5 \)-brane. It is given by

\[
S = -4\pi T_5 \int d^4\xi \left( h^2 \sqrt{1 - h^{-4} \rho^2} \sqrt{\frac{1}{4} h^4 \lambda^2 n^2 + \rho^4 - h^4 n_2 \lambda} \right),
\]

\[
(2.4)
\]

where we have worked in physical coordinates and integrated out the directions along the \( S^2 \). For cosmologically relevant solutions one must minimally couple this term to the usual Einstein-Hilbert action. For a concrete embedding of this action into a particular string theory background we refer the reader to [28].

One of the many interesting features of string theory is the existence of duality symmetries, whereby a brane configuration can be related to another (equivalent) one. In this instance, we can employ our knowledge of the Myers dielectric effect [29] to understand how the above action is captured in the dual picture by \( n \) coincident \( D3 \)-branes expanding along a fuzzy two-sphere\(^4\). The full discussion is intricate and chronicled at length elsewhere (see [30] [34] for examples in non-trivial backgrounds). We therefore quote the results and refer the reader to [40] [41] for

\[^4\text{See the alternate proposal by [31].}\]
more details on the relation to cosmological model building. The validity of the Myers action using boundary fermions has been considered at length in [32, 33], and can certainly be trusted at leading order. An important related usage of the action is discussed in [35].

The action for $n$ coincident branes in the limit where $n \gg 1$ can be written as

$$S = -nT_3 \int d^4\xi h^4 \left( \sqrt{1 - h^{-4} \dot{\rho}^2} \sqrt{1 + \frac{4\rho^4 h^{-4}}{\lambda^2 n^2}} - 1 \right),$$

and it can be shown that this is exactly the same as the action (2.4) above by employing the known relationship between the brane tensions, $T_3 = 4\pi^2 \alpha'^5T_5$, and by identifying the large $n$ limit in both cases. This implies that in the (macroscopic) $D5$-brane theory, we must take the flux to be large. Since our background is relatively simple, one could also consider the S-dual theory, where the $D3$-branes expand into a wrapped $NS5$-brane. In this case, the question of where the standard model degrees of freedom reside becomes crucial since fundamental strings cannot end on the world-volume of the fivebrane. Within this context, a fivebrane model was recently proposed as a concrete realization of axionic monodromy inflation [59].

An important result is that the back-reaction of higher-dimensional branes becomes important in the relativistic regime [27]. Given our dual interpretation of the $D5$-brane action, this is intuitively obvious, since we are essentially taking $n \gg 1$, and this will typically back-react on the geometry and invalidate the probe-brane approximation. Thus, it is clearly desirable to construct a theory where $n$ is not taken to be extremely large. This should reduce the back-reaction and allow for more control over the theory. However, constructing the action for multiple $D3$-branes in the finite $n$ limit is a highly technical issue and has only been resolved in a few simple cases. Nonetheless, progress has been made and in the following subsection, the key results that are relevant to the present discussion are summarized. Full details of the construction can be found in [39].

2.2 Matrix cosmology at finite $n$

The tractability of the problem for finite $n$ is intimately tied to the symmetrized trace prescription (STr) associated with open string scattering amplitudes. This suggests that one must first average over all symmetric permutations before taking the gauge trace. Fortunately, one can make headway by identifying the scalar fields with a finite dimensional irrep of $SO(3) \sim SU(2)$, which leads to the following result [36, 37]:

$$\text{STr}(\alpha^i \alpha^j)^q = 2(2q + 1) \sum_{i=1}^{n/2} (2i - 1)^{2q} \quad n \text{ even}$$

$$= 2(2q + 1) \sum_{i=1}^{(n-1)/2} (2i)^{2q} \quad n \text{ odd}.$$  

At least to leading order. Examining the duality to higher orders is certainly interesting, but highly non-trivial.
As an aside we remark that for the flat D3-branes, the scalar fields parametrize the direction of a transverse $S^2$—assuming that the full, non-compact, background geometry admits a submanifold with an $SO(3)$ isometry. The corresponding action for $n$ multiple D3-branes is then given by the usual expansion of the Myers DBI action:

$$P_n = -T_3 \text{Str} \left( h^4(\rho) \sum_{k,p=0}^{\infty} (-Z \hat{R}^2)^k Y^p (\alpha^i \alpha^j)^{k+p} \left( \frac{1/2}{k} \right) \left( \frac{1/2}{p} \right) + V(\rho) \mathbf{1}_n - h^4(\rho) \mathbf{1}_n \right)$$

where we have defined the following terms

$$Z = \lambda^2 h^{-4}(\rho), \quad Y = 4 \lambda^2 R^4 h^{-4}(\rho), \quad \left( \frac{1/2}{q} \right) = \frac{\Gamma(3/2)}{\Gamma(3/2 - q) \Gamma(1 + q)}$$

and the fuzzy sphere radius $R$ is related to the physical radius $\rho$ and the number of branes via

$$\rho^2 = \lambda^2 R^2 (n-1)^2.$$  

Those familiar with such a construction should note that one usually relates the physical coordinate to the quadratic Casimir $(n^2 - 1)$. However, for finite $n$ there is a more precise definition of the physical radius in terms of ratios of operators [37]. This is a necessary requirement for the convergence of the theory, and agrees with the definition in the limit of large $n$. It should be noted that the radial coordinate $\rho$, the potential $V(\rho)$, and the warp factor $h(\rho)$ are all singlets under the symmetrized trace prescription. Moreover, the potential as written here is dimensionless, since we have absorbed a factor of the brane tension into its definition. It is also important to note that the physical radius is a function of $n$ in this model and this will play a role when the recursion relations are employed. The key point is that the scalar field, which we wish to identify with the inflaton, is now matrix valued and is in a finite dimensional representation of the corresponding gauge group.

The most important representation of $SU(2)$ is the fundamental (or two dimensional) one, since the representation is simply that of the Pauli spin matrices. It also turns out that the form of the symmetrized trace ansatz leads to a recursive structure for the theory, where we can construct the action for any $n$ once the action for $n = 2$ has been computed. Thus, the correct way to write the action is in terms of spin-$\frac{1}{2}$ variables as first elucidated in [39]. It is found that the action takes the form

$$P_n(Z,Y) = \sum_{k=1}^{(n-1+\delta_n)/2} P_2((2k - \delta_n)^2 Z, (2k - \delta_n)^2 Y) - nT_3 (V(\rho) - h^4(\rho))$$

where $\delta_n = 1$ when $n$ is even and $\delta_n = 0$ when $n$ is odd. A similar recursive structure exists for the energy density, $E_n(Z,Y)$ (more specifically the time-time component of the kinetic part of the energy-momentum tensor).

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6For recent additional work using the same idea we refer the interested reader to [56].
For completeness, we write down the recursion functions $P_2$ and $E_2$:

\[
P_2(Z,Y) = -\frac{2T_3h^4}{\sqrt{1+Y}} \left( \frac{1+2Y-(2+3Y)Z\dot{R}^2}{\sqrt{1-ZR^2}} \right),
\]

\[
E_2(Z,Y) = \frac{2T_3h^4}{\sqrt{1+Y}} \left( \frac{1+2Y-YZ\dot{R}^2}{(1-ZR^2)^{3/2}} \right),
\]

(2.11)

where $Z, Y$ are the functions defined in Eq. (2.8). Note that the algebraic forms of these functions are significantly different from those of the $n=1$ action and, consequently, one should anticipate different physics to emerge.

### 2.3 Convergence with the large $n$ limit

A crucial question that now arises is whether the finite $n$ action defined in Eq. (2.10) really does reproduce the action (2.5) in the large $n$ limit. We present here, for the first time, an argument which suggests that this is indeed the case. Our approach is to expand both actions as Taylor series and compare coefficients term by term. We focus on the NS-NS sector, since the RR-sector converges in a trivial manner. To proceed, we rewrite action (2.5) at large $n$ in the more compact form

\[
P = -nT_3h^4 \left( 1 + Y \right)^{1/2} \sqrt{1 - (n-1)^2 \dot{\psi}^2}
\]

(2.12)

where we have introduced the following variables:

\[
Y_0 = \frac{m_4^2}{\pi^2 T_3} \left( \frac{\phi}{h} \right)^4, \quad \dot{\psi}^2 = \frac{1}{(n-1)^2} \frac{\dot{\phi}^2}{T_3 h^4}, \quad \hat{C} = n^2 - 1, \quad Y = \frac{Y_0}{(n-1)^4}
\]

(2.13)

The canonical inflaton field is defined in terms of the throat geometry via the standard relation $\phi = \sqrt{\hat{T}_3 \rho}$.

Expanding the DBI action requires the introduction of the binomial coefficient, which we write as

\[
(1+x)^\alpha = \sum_{j=0}^\infty \binom{\alpha}{j} x^j, \quad \binom{\alpha}{j} = \frac{\alpha(\alpha-1)\ldots(\alpha-j+1)}{j!}
\]

(2.14)

For the remainder of this subsection we explicitly drop the summation sign, although the summation over Latin indices is always implied. The key point is to observe that the ratio $Y_0/\hat{C}$ will generally be small in the large $n$ limit. This implies that we can expand the action (2.12) up to the leading order terms in $(\dot{\psi}^2)^j Y^j$. It follows, upon expansion, that

\[
P \simeq -\frac{n(n-1)^2}{4(n^2-1)\hat{C} \dot{\psi}^2} \left[ \left( -\frac{1}{2} \right)^j \ldots \left( \frac{3-2j}{2} \right) Y_0^j \right] \left[ \left( -\frac{1}{2} \right)^j \ldots \left( \frac{3-2j}{2} \right) (-\dot{\psi}^2)^j \right].
\]

(2.15)
The action (2.15) is what we aim to reconstruct using the finite \( n \) formalism. We demonstrate this explicitly for odd \( n \) (the case of even \( n \) is analogous). The relevant \( n = 2 \) term, written in terms of the new variables (2.13), is given by

\[
P_2 = -\frac{2T_3h^4}{\sqrt{1 + Y}} \left( 1 + 2Y - (2 + 3Y)\dot{\psi}^2 \right) \left( 1 - \dot{\psi}^2 \right)^{-1/2}.
\] (2.16)

We now proceed to expand the velocity factor \( (1 - \dot{\psi}^2)^{-1/2} \) up to terms of order \( (\dot{\psi}^2)^j \). After some algebra, we find that the relevant term in the expansion is given by

\[
P_2 \simeq -\frac{2T_3h^4}{\sqrt{1 + Y}} \left( 1 + 2Y - (2 + 3Y)\dot{\psi}^2 \right) \left[ \left( \frac{1}{2} \right) \cdots \left( \frac{1-2i}{j!} \right) \right] \left( \frac{3-2i}{2} \right) \left( -\dot{\psi}^2 \right)^{j-1} \left( \frac{1}{2} \right) \cdots \left( \frac{(j-1)!}{2} \right) Y^j + \ldots
\] (2.17)

Performing the analogous expansion on the \( (1 + Y)^{-1/2} \) factor up to terms of order \( Y^i \) then yields the result:

\[
P_2 \simeq -\frac{2T_3h^4}{\sqrt{1 + Y}} \left[ \left( \frac{1}{2} \right) \cdots \left( \frac{3-2i}{2} \right) \right] \left( \frac{1}{2} \right) \cdots \left( \frac{(j+1)!}{2j!} \right) Q_\psi \ldots
\] (2.18)

However, we should recall that this is simply the expansion of the \( n = 2 \) action and we must employ the recursion relations (2.10) in order to generate the full structure for any value of \( n \). For odd \( n \), this is achieved through the rescaling

\[
\dot{\psi}^2 \rightarrow (2k)^2 \dot{\psi}^2, \quad Y \rightarrow (2k)^2 Y
\] (2.19)

Furthermore, we must also introduce a new summation over the \( k \) variable to obtain the complete expression for the leading order terms in \( (\dot{\psi}^2)^j Y^i \) in the action \( P_n \). Taking these considerations into account, we arrive at the expression

\[
P_n \simeq -2T_3h^4Q_\psi \left( \frac{1 + 2j + 2i}{4} \right)^{(n-1)/2} \sum_{k=1}^{(n-1)/2} 2^{(i+j)}k^{2(i+j)}.
\] (2.20)

To establish the correspondence with the large \( n \) action (2.15), we need to trade the sum over \( k \) for a function of \( n \). This is achieved by considering the well-known algebraic condition:

\[
\frac{n^{p+1}}{p + 1} < \sum_{k=1}^{n} k^p < \frac{(n + 1)^{p+1}}{p + 1}
\] (2.21)

It can be seen that in the limit of large \( n \) the lower and upper bounds converge. Thus, we can make the following algebraic identification

\[
\sum_{k=1}^{(n-1)/2} 2^{(i+j)}k^{2(i+j)} \sim \frac{(n - 1)^{(2j+2i+1)}}{2(1 + 2j + 2i)}.
\] (2.22)
Finally, inserting this relation into the action (2.20), collecting together all the terms and substituting for \( Y \) in terms of \( Y_0 \), leads us to the result

\[
P_n \simeq -\frac{T_3 h^4 (n-1)^{2j-2i+1}}{4!j!} \left[ \left( -\frac{1}{2} \right) \cdots \left( \frac{3-2j}{2} \right) Y_0 \right] \left[ \left( -\frac{1}{2} \right) \cdots \left( \frac{3-2j}{2} \right) (-\dot{\psi}^2)^j \right]
\]

(2.23)

Eq. (2.23) may be compared directly to the action (2.15) in the large \( n \) limit.

In conclusion, therefore, we have argued that the action for finite \( n \) does indeed converge to the full action at large \( n \), which is a non-trivial result. This indicates that the finite \( n \) action can be trusted (at least for the simple case we consider). Whilst the full structure of the non-Abelian DBI action remains unknown, the Myers action [29]—and therefore our finite \( n \) theory given by Eq. (2.10)—will be a good approximation to leading order in an \( \alpha' \) expansion. In the following section, we consider some of the consequences of this action for the inflationary scenario of the early universe.

3 Matrix inflation at finite \( n \)

3.1 General remarks

One of the main characteristics of DBI-driven inflation is that the inflaton can move relativistically and still drive a sustained period of accelerated expansion. This is possible because the warped metric redshifts all physical scales associated with the brane dynamics. This feature is frequently exploited to simplify the functional form of the solution by considering the so-called ‘ultra-relativistic limit’, where the kinetic contribution to the action is rewritten in terms of a function, \( \gamma \), corresponding to a generalization of the relativistic factor of special relativity.

In our scenario, the algebraic structure of the multi-brane action forces us to consider a more general version of this function, which we parametrize in terms of the inflaton \( \phi \) such that

\[
\gamma_k = \left( 1 - \frac{\dot{\phi}^2}{h^4 T_3} \left( \frac{2k - \delta_n}{n-1} \right)^2 \right)^{-1/2}
\]

(3.1)

Note that there is an implicit summation over \( k \) in this expression. Let us study this function on its own for the moment. It is trivial to see that increasing the number of branes, \( n \), increases the number of poles of the function. For example, we see that in the case of \( n = 7 \)

\[
\gamma_k = \left( 1 - \frac{\dot{\phi}^2}{h^4 T_3} \right)^{-1/2} + \left( 1 - \frac{4\dot{\phi}^2}{9h^4 T_3} \right)^{-1/2} + \left( 1 - \frac{\dot{\phi}^2}{9h^4 T_3} \right)^{-1/2}
\]

(3.2)

and it follows that \( \gamma \) diverges in the same limit as that of the usual DBI-inflation models, namely when \( \dot{\phi}^2 \sim h^4 T_3 \). The divergences arising from the higher order expansion in \( k \) are not physical. Consequently, it is the lowest term in \( k \) that determines the sound speed for inflaton field
fluctuations, at least if the analysis is limited to the behavior of the \( \gamma \)-function. The implication is that \( \gamma_k \sim \gamma \) in the ultra-relativistic regime.

If we include the coupling to Einstein-Hilbert gravity, we find that the energy density of the DBI-scalar is not conserved, due to the presence of the FRW scale factor. It is a standard result that
\[
\dot{E}_n = -3H(P_n + E_n),
\]
where \( H \) is the usual Hubble factor, and from this expression we can then write down the dynamics of the Hubble parameter as follows:
\[
\dot{H}_n = -3H(P + E)
\]
\[
\simeq 2T_3h^4 \sum_{k=1}^{(n-1+\delta_n)/2} \frac{\gamma_k(\gamma_k^2-1)}{\sqrt{1+Y(2k-\delta_n)^2}} \left( 3 + 4Y(2k-\delta_n)^2 - \frac{(\gamma_k^2-1)(2 + 3Y(2k-\delta_n)^2)}{\gamma_k^2} \right)
\]
This is a highly complicated sum over all \( k \) states. However, the analysis can be simplified somewhat if it is assumed that the generalized \( \gamma \)-function \((3.1)\) is large, i.e. \( \gamma_k \gg 1 \forall k \). If this condition is satisfied, we find that
\[
\dot{H}_n \sim -\frac{T_3h^4\gamma^3}{M_p^2} \sum_k \sqrt{1+(2k-\delta_n)^2Y}
\]
where we have explicitly included the dependence on the number of branes in the \( \dot{H} \) term and used the fact that \( \gamma_k \sim \gamma \) in this regime. This expression differs significantly from the corresponding limit of the single brane \((n = 1)\) solution.

### 3.2 Non-relativistic expansion

What happens to the expansion of the action in the non-relativistic limit? Related work for the single brane case is discussed in [54, 55]. Fortunately, one can see that the symmetrized trace operation commutes with the non-relativistic limit and therefore we can immediately write down the following equation of motion for the inflaton:
\[
\dot{\phi} \simeq -H'M_p^2 \left( \sum_{k=1}^{(n-1+\delta_n)/2} \left( \frac{2k-\delta_n}{n-1} \right)^2 \frac{3 + 4Y(2k-\delta_n)^2}{\sqrt{1+Y(2k-\delta_n)^2}} \right)^{-1}.
\]
Here a prime denotes differentiation with respect to \( \phi \) and we have assumed the validity of the Hamilton-Jacobi expansion when higher order terms in the velocity expansion are neglected. Given an expression for the time dependence of the inflaton, the corresponding slow-roll parameter \( \epsilon \equiv \dot{H}/H^2 \) can also be deduced:
\[
\epsilon \simeq M_p^2 \left( \frac{H'}{H} \right)^2 \left( \frac{3 + 4Y(2k-\delta_n)^2}{\sqrt{1+Y(2k-\delta_n)^2}} \right)^{-1}.
\]
It follows, therefore, that the slow-roll parameter is suppressed relative to that of canonical inflation by the additional sum in the denominator, which we will denote by \( \sum_k f_k \) for simplicity, i.e.,
\[
\epsilon \simeq \frac{M_p^2}{\sum_k f_k} \left( \frac{H'}{H} \right)^2.
\]
The maximal suppression occurs when $\sum f_k$ takes its largest value, although one must also be aware that $Y$ is itself a function of $n$, which complicates the analysis. Indeed, it is trivial to see that

$$Y = \frac{4\phi^4}{h^4\lambda^2 T_3^2(n-1)^4}$$

and so the dependence on the inflaton is essential in determining the dynamics.

In most cases of interest, the warped throats fall into one of two classes. The first is an AdS-type throat, arising for example from the near horizon limit of $D3$-branes. In this instance the warp factor takes the form $h \sim \phi$ and $Y$ is fixed at some non-negative value. The second class corresponds to regularized throats, where the back-reaction of the form fluxes are used to 'cap' the throat at the tip. A canonical example of such a background is given by the warped deformed conifold [6, 7]. The warping in this case tends to a constant at some finite value of the throat, and therefore we find that $Y$ is an increasing (decreasing) function of the inflaton depending upon whether we are considering small (large) field inflation. For the IR scenario (an example of small field inflation), one sees that $\sum f_k$ is rapidly dominated by smaller values of $n$ and therefore inflationary trajectories for $n = 2$ are preferred over those with larger $n$. In the UV (large field) case, on the other hand, we find that this function is sensitive to the precise value of the constants. Consequently, we cannot identify a priori whether more suppression occurs for smaller values of $n$, although one should remember that UV inflation is more severely constrained than the IR scenario.

In general, we can write the Hubble parameter in the following form:

$$3H^2 M_p^2 \simeq 2T_3 \sum_k W_k(1 + \varepsilon_k)$$

where the generalized effective potential is given by

$$W_k \simeq h^4(1 + 2Y(2k - \delta_n)^2) + \left(\frac{n}{n - 1 + \delta_n}\right) (V - h^4) \sqrt{1 + Y(2k - \delta_n)^2}$$

and the parameter

$$\varepsilon_k \simeq \frac{\phi^2}{2T_3(n - 1)^2 W_k} \frac{(3 + 4Y(2k - \delta_n)^2)}{\sqrt{1 + Y(2k - \delta_n)^2}}$$

quantifies the ratio of the kinetic and potential energies. It follows, therefore, that the generalized master (Friedmann) equation for the Hubble parameter can be expressed in the form

$$3H^2 M_p^2 - 2T_3 \sum_k W_k - \sum_k \frac{(3 + 4Y(2k - \delta_n)^2)}{\sqrt{1 + Y(2k - \delta_n)^2}} \frac{H^2 M_p^4}{(n - 1)^2} \left(\frac{1}{\sum_p f_p}\right)^2 \simeq 0$$

and this equation can be employed to determine the inflationary trajectories.

### 3.3 A matrix inflation model in AdS

In this subsection we investigate a concrete example of non-relativistic matrix inflation with the aim of identifying regions of parameter space that are consistent with cosmological observations.
Specifically, we consider IR (small field) inflation driven by an inverted harmonic oscillator potential, where the branes are propagating out of an AdS throat.

It is simple to show by expanding Eq. (2.10) that the leading order expression for the pressure can be written as

$$ P_n = Q \dot{\phi}^2 - n T_3 V(\phi) + T_3 h^4 \left( n - \sum_k \frac{2 + 4(2k - \delta_n)^2 Y}{\sqrt{1 + (2k - \delta_n)^2 Y}} \right) $$

where we have defined

$$ Q \equiv \frac{\sum_k \left(3 + 4(2k - \delta_n)^2 Y\right)(2k - \delta_n)^2}{(n-1)^2 \sqrt{1 + (2k - \delta_n)^2 Y}}. $$

For a class of AdS backgrounds, the warp factor $h(\phi) = \phi / (L \sqrt{T_3})$ is determined in terms of the AdS radius of curvature $L = \left[4\pi^4 g_s M K / (\text{Vol}(X_5) m_s^4)\right]^{1/4}$, where $g_s$ denotes the string coupling, $\text{Vol}(X_5)$ is the volume of the base and the background fluxes $M, K$ thread through the three-cycles in the compactification. Eq. (3.13) then implies that $Q$ is independent of $\phi$. We may therefore introduce a new field, $\psi = \sqrt{2Q} \phi$, in order to write the pressure in a canonical form. The effective potential energy of this field is then given by

$$ W = n T_3 V(\psi) - \frac{\text{Vol}(X_5) \psi^4}{2\pi M K Q^2} \left( n - \sum_k \frac{2 + 4(2k - \delta_n)^2 Y}{\sqrt{1 + (2k - \delta_n)^2 Y}} \right) $$

where we have written the explicit parameters

$$ Y = \frac{Y_0}{(n-1)^4}, \quad Y_0 = \frac{4\pi^2 g_s M K}{\text{Vol}(X_5)} $$

Since typically the product of fluxes must satisfy $MK \gg 1$, it follows that the second term in the effective potential is suppressed and therefore only the term proportional to $V(\psi)$ is expected to dominate the dynamics.

Indeed, if one assumes that the inflaton potential takes the form of a simple inverted harmonic oscillator, $V(\phi) = n T_3 V_0 (1 - \beta \phi^2 / M_p^2)$, where $\beta$ is an order one parameter of the theory, we may write

$$ W(\psi) = W_0 - \frac{\omega^2 \psi^2}{2} $$

where

$$ W_0 = n T_3 V_0, \quad \omega = \frac{\beta n T_3 V_0}{Q M_p^2}. $$

Coincident branes located near the tip of a warped throat are expected to be attracted towards branes residing in other throats through the generation of such a tachyonic potential. Since our theory is now in canonical form, we can adapt the usual inflationary tools to our current objective. During inflation the Gaussian curvature perturbation has an amplitude given by

$$ \mathcal{P}_R = \frac{W}{24\pi^2 M_p^6 \epsilon} $$

and spectral index

$$ 1 - n_s = 6\epsilon - 2\eta $$
where $\epsilon \equiv \frac{1}{2}M_p^2(W'/W)^2$ and $\eta \equiv M_p^2W''/W$ are the usual slow roll parameters obtained from derivatives of the effective potential. Near the maximum of the potential we see that

$$
\eta \approx -\frac{M_p^2\omega^2}{W_0}, \quad \epsilon \approx \frac{M_p^2\omega^4\psi^2}{2W_0^2} \sim \frac{\eta^2\psi^2}{2M_p^2}
$$

which implies that $\epsilon \ll |\eta| \ll 1$. The spectral index of the density perturbations can therefore be computed as

$$
1 - n_s = \frac{\beta}{Q}
$$

This has an interesting dependence on the number of coincident branes, since $Q$ has a local maximum when $n$ takes the lowest possible value ($n = 2$ for the even case), which suggests that $1 - n_s$ is initially small. As we increase the number of branes, we see that $1 - n_s$ also increases until it attains a local maximum around $n \sim 6$ (for the even case), before asymptotically tending to zero as $n$ increases further.

The above expressions assume implicitly that the slow-roll conditions hold until the very end of inflation, i.e., it is assumed that the branes continue to move non-relativistically. The self-consistency of this assumption can be verified by employing the effective field equation $3H\dot{\psi} \simeq -dW/d\psi$ and the Friedmann equation $3H^2 \simeq W_0/M_p^2$, together with Eq. (3.22). The non-relativistic limit, $\dot{\phi}^2 \ll T_3 h^4$, is then equivalent to

$$
\psi^2 \gg \beta L^4 T_3(1 - n_s)\frac{nT_3V_0}{3M_p^2}
$$

Since the inflaton is a monotonically increasing function in this scenario, it suffices to show that the bound (3.23) was satisfied when observable scales crossed the Hubble radius. The value of the field at that time is related to the normalization of the CMB power spectrum such that

$$
\frac{\psi_{\text{cmb}}^2}{M_p^2} = \frac{1}{3\pi^2(1 - n_s)^2\mathcal{P}_R} \frac{nT_3V_0}{M_p^4}
$$

and substituting Eq. (3.24) into the constraint (3.23) then yields the consistency condition

$$
\beta T_3 L^4(1 - n_s)^3\pi^2\mathcal{P}_R \ll 1
$$

This constraint can typically be satisfied for a large region of the physical parameter space.

The value of $\psi_{\text{cmb}}$ can also be determined from the scalar field equation of motion to be

$$
\psi_{\text{cmb}} = \psi_{\text{end}} e^{-(1 - n_s)N/2}
$$

This allows us to write the tensor-scalar ratio in the following form

$$
r = 16\epsilon = 2(1 - n_s)^2\frac{\psi_{\text{cmb}}^2}{M_p^2}
$$

suggesting that gravitational waves are negligible in this model.

We may gain further insight by specializing to the case of even or odd $n$. Let us choose the latter case for clarity and reconsider the constraint on the spectral index (3.22). We can clearly
see that since $Y$ is positive-definite, the definition of $Q$ given in Eq. (3.14) implies that this parameter is bounded from above such that

$$Q < \frac{(n-1)^{1/2}}{\pi} \left( \frac{12k^2}{(n-1)^2} + \frac{64k^4Y_0}{(n-1)^6} \right).$$  (3.28)

Utilizing the bound (2.21) enables us to write down a further constraint that is slightly weaker, but expressible simply as a function of $n$:

$$Q < \frac{12}{3(n-1)^2} \left( \frac{n+1}{2} \right)^3 + \frac{64Y_0}{5(n-1)^6} \left( \frac{n+1}{2} \right)^5 + \ldots$$  (3.29)

where we neglect the terms coming from higher orders in the expansion.

The WMAP five-year data implies that $n_s \geq 0.93$ $(2\sigma)$ when the gravitational wave background is negligible. This in turn imposes the condition $Q \geq 30\beta \gg n$ if we take $\beta \simeq O(1)$. Now, since the first term in the expansion of (3.29) is essentially $O(n)$, and therefore negligible compared to $Q$, the leading order bound is dominated by

$$Q \leq \frac{64Y_0}{5(n-1)^6} \left( \frac{n+1}{2} \right)^5$$  (3.30)

and this results in $Y_0$ being bounded from below:

$$Y_0 \geq \frac{5\beta}{(1-n_s)} \frac{(n-1)^6}{(n+1)^5}$$  (3.31)

In principle, therefore, condition (3.31) may be interpreted as an observational lower bound on the product of the fluxes $MK$ after ‘canonical’ values for the string coupling, $g_s \simeq 10^{-2}$, and base manifold volume, $\text{Vol}(X_5) \simeq \pi^3$, have been specified. It follows from Eq. (3.16) that

$$MK \geq \frac{400\beta}{(1-n_s)} \frac{(n-1)^6}{(n+1)^5}.$$  (3.32)

Moreover, the ratio $(n-1)^6/(n+1)^5$ is a monotonically increasing function of $n$ and this implies that consistency with observations would be difficult to achieve if condition (3.32) was not satisfied for $n = 3$. For example, invoking the central value for the spectral index inferred from WMAP, $n_s = 0.96$, and setting $\beta \simeq O(1)$ suggests that $MK > 630$. One should note that the fluxes depend on the inverses of both the string coupling and the spectral index, and a more weakly coupled theory therefore leads to a tighter bound on the fluxes. Likewise, it becomes more difficult to satisfy this condition as the spectrum becomes closer to scale invariance.

To summarize, in this model the spectral index of the density perturbations is determined by the parameter $Q$ that quantifies the non-Abelian nature of the multi-brane configuration. Consistency with observations is possible if the background fluxes are sufficiently large.

In the following section, we proceed to investigate the dynamics of the $n = 2$ configuration in more detail.
4 Cosmology of the two-dimensional representation

Given the recursive structure of the action in Eq. (2.10), we see that the two dimensional representation of SU(2) is the most important, corresponding in our matrix language to a configuration of two coincident D3-branes. Note that this is not the same model as that considered in [23], which considered two D3-branes separated by a distance larger than the string length. The world-volume gauge theory in that case is therefore U(1) × U(1), whereas our model possesses a U(2) symmetry. Note that this is the gauge group of the open string states on the world-volume, and should not be confused with the SO(3) ~ SU(2) symmetry group that parametrizes the transverse space.

4.1 The relativistic limit

To begin this section let us focus on the relativistic limit to highlight how the theory differs from the n = 1 solution. Unlike the case of a single D3-brane, the pressure and energy of the model have very different dependence upon the brane velocity. Given our generalized definition of γ_k, it is convenient to now introduce the following notation

\[ \gamma_2 = \frac{1}{\sqrt{1 - Z R^2}}. \]  

(4.1)

As in the case of n = 1 the sound speed of the Fourier modes, c_s, is reduced from unity—the relevant calculation proceeds in the usual manner and we find that

\[ c_s^2 = \frac{1}{2 \gamma_2^2} \left( \frac{2 \gamma_2^2 (3 + 4Y) - (\gamma_2^2 - 1)(4 + Y)}{\gamma_2^2 (3 + 4Y) - (\gamma_2^2 - 1)Y} \right) \]  

(4.2)

which is very different from that of the standard DBI inflation models (see [44] for example). In the ultra-relativistic limit the above expression simplifies to become

\[ c_s^2 \approx \frac{1}{6 \gamma_2^2} \left( \frac{2 + 7Y}{1 + Y} \right) \]  

(4.3)

and this should be compared to the \( c_s^2 \sim 1/\gamma^2 \) dependence of the n = 1 model. One sees that for \( Y \gg 1 \) the sound speed varies as \((7/6)\gamma_2^{-2}\), whilst in the converse limit it behaves as \((1/3)\gamma_2^{-2}\).

More importantly for cosmology, the sound speed modifies the tensor-scalar ratio as follows:

\[ r \approx \frac{16 \epsilon_H}{\gamma_2} \sqrt{\frac{1}{6} \left( \frac{2 + 7Y}{1 + Y} \right)}, \]  

(4.4)

with the presence of an additional enhancement factor due to the contribution of Y. Indeed, this accounts for the enhancement of \( r \), relative to single brane models, discussed in [39].

Given the modified expression for the sound speed, and the non-linear form of the action, we can compute the bi-spectrum arising from higher order interactions of the different Fourier modes

\[ \text{Note that this is the adjoint representation of the group.} \]
in the density perturbations. For simplicity, we focus on the equilateral triangle configuration to highlight the relative contribution to the non-Gaussian parameter $f_{nl}$. A standard calculation reveals that

$$f_{nl} \sim -\frac{85\gamma^2}{162} \left( \frac{\gamma^2(3 + 4Y) - Y(\gamma^2 - 1)}{2\gamma^2(3 + 4Y) - (\gamma^2 - 1)(4 + Y)} \right) + \frac{85}{324} - \frac{5(\gamma^2 - 1)}{81} \left( \frac{\gamma^2(5 + 6Y) - Y(\gamma^2 - 1)}{\gamma^2(3 + 4Y) - Y(\gamma^2 - 2)} \right).$$

(4.5)

This function is negative definite for all values of $\gamma^2$ and $Y$ in the relativistic limit. Moreover, one sees that the detectable range of $f_{nl}$ falls into the regime where $Y \ll 1$. For example, in $AdS$-type backgrounds, where $h \sim \phi$, the parameter $Y$ reduces to a constant determined by Eq. (3.16). For sufficiently large $MK/g_s$, we find that $Y$ is suppressed, and detectable levels of (negative) $f_{nl}$ are therefore more likely. Note that in the limit where $Y \to 0$, the functional form of the action becomes very similar to that of the single brane models. Thus, a small (but manifestly non-zero) value of $Y$ is crucial for generating a detectable signature of non-Gaussianity in the primordial curvature perturbation.

Although the highly non-linear nature of the multi-brane action leads to interesting observables, it is also of interest to investigate the nature of the inflationary trajectories. This can be achieved by calculating the variation of the Hubble parameter from Eq. (3.3), which takes the form

$$\dot{H} \sim -\frac{T_3 h^4 \gamma_2 (\gamma_2^2 - 1)}{M_p^2} \sqrt{1 + Y} \left( 3 + 4Y + \frac{(2 + 3Y)(\gamma_2^2 - 1)}{\gamma_2^2 - 1}\right).$$

(4.6)

After taking the ultra-relativistic limit of Eq. (4.6), as well as the corresponding limit in the definition of the energy density

$$E_2 \sim 2T_3 \left( h^4 \gamma_2^3 \sqrt{1 + Y} + V - h^4 \right)$$

(4.7)

we see that the ‘slow roll’ parameter, $\epsilon = \dot{H}/H^2$, becomes

$$\epsilon \sim \frac{3}{2} \left( 1 + \frac{(V - h^4)}{h^4 \gamma_2^3 \sqrt{1 + Y}} \right)^{-1} \left( 1 + \frac{1}{\gamma_2^2} \left( 1 + \frac{2}{1 + Y}\right) \right).$$

(4.8)

This leads to a novel phenomenon in multi-brane DBI models. The final term in parenthesis is essentially a term of order one suppressed by the relativistic factor and it effectively decouples in the relativistic limit. If the effective potential $(V - h^4)$ is much smaller than the kinetic term, inflation will never begin. On the other hand, if the potential term dominates—even for the fast rolling solution—we see that this is a concrete realization of eternal inflation.

### 4.2 The non-relativistic limit

We now proceed to consider the non-relativistic limit of the theory. In this limit, the energy density takes the following, almost canonical, form:

$$E_2 = \frac{2T_3}{\sqrt{1 + Y}} \left( W(\phi) + \frac{\dot{\phi}^2}{2T_3}(3 + 4Y) \right)$$

(4.9)
where the effective potential is given by
\[ W(\phi) = h^4(1 + 2Y - \sqrt{1 + Y}) + V(\phi)\sqrt{1 + Y}. \] (4.10)

We also note that the pressure, \( P_2 \), in the non-relativistic limit is given by
\[ P_2(Z,Y) = -\frac{2T_3}{\sqrt{1 + Y}} \left( W(\phi) - \frac{\dot{\phi}^2}{2T_3}(3 + 4Y) \right). \] (4.11)

As an aside, we briefly comment on the non-Gaussianities in this limit. In the \( n = 1 \) case, the non-relativistic limit reduces to a canonical scalar field theory, and therefore one expects that \( f_{nl} \) will be of the same order as the slow roll parameters. For \( n = 2 \), it is convenient to parametrize the non-relativistic expansion as \( \gamma_2 \sim 1 + \xi \) and focus on the correction term
\[ f_{nl} \sim -\frac{5\xi}{234} \left( \frac{210 + 167Y}{3 + 4Y} \right). \] (4.12)

One can see that this term is very small even in the limit where \( Y \gg 1 \).

Assuming the validity of the Hamilton-Jacobi approximation, we can use the continuity equation to obtain the following expression for the time-dependence of the scalar field
\[ \dot{\phi} = -M_p^2 H^2 \frac{\sqrt{1 + Y}}{1 + 4Y}, \] (4.13)
which is modified from that of the canonical theory by the \( Y \)-dependent terms. Interestingly, we can write the slow roll parameter \( \epsilon \) in in exactly the same way as the canonical model
\[ \epsilon = \frac{3\epsilon_2}{1 + \epsilon_2}, \] (4.14)
so that \( \epsilon_2 \leq 1/2 \) is required in order for inflation to occur. Recall that \( \epsilon_2 \) is the ratio of the kinetic energy to the potential energy in the theory and follows from the generalized definition in (3.11).

The final step for our analysis is to calculate the master (Friedmann) equation for the Hubble parameter. This is given by
\[ 3M_p^2 H^2 \frac{3 + 4Y}{\sqrt{1 + Y}} = 2T_3 W \left( \frac{3 + 4Y}{1 + Y} \right) = M_p^4 H'^2 \] (4.15)

which follows from the generalized expressions derived in section 3.2. Since an inflating solution requires that the terms on the right hand side are smaller than those on the left hand side, we can iteratively expand the Hubble parameter in a derivative series. The decoupled leading order solutions are given by
\[ 0 \sim 3M_p^2 H_0^2 - 2WT_3 \] (4.16)
\[ M_p^2 H_0'^2 \sim 6H_0 H_1 \frac{(3 + 4Y)}{\sqrt{1 + Y}} \]
\[ 2M_p^2 H_0'' H_1' \sim (2H_0 H_2 + H_1^2) \frac{(3 + 4Y)}{\sqrt{1 + Y}} \]
\[ M_p^2 (H_1'^2 + 2H_0'' H_2') \sim 6H_1 H_2 \frac{(3 + 4Y)}{\sqrt{1 + Y}} \]
for the first few terms in the expansion and this enables us to reconstruct the Hubble parameter at leading order as follows:

\[ H \sim \sqrt{\frac{2T_3W}{3M_p^2}} \left( 1 + \frac{1}{12} \frac{3(1 + Y)}{2T_3W} \frac{M_p^3W'}{W(3 + 4Y)} + \ldots \right) . \]  (4.17)

The validity of this expression requires that the second order terms are negligible in comparison, which is equivalent to the condition

\[ 1 \gg \frac{3M_p^2}{2} \frac{\sqrt{1 + Y}}{(3 + 4Y)} \left( \frac{W''}{W} - \frac{W'^2}{W^2} - \frac{W'Y'}{2W} \frac{(5 + 4Y)}{(1 + Y)(3 + 4Y)} - \frac{M_pW'}{12W} \sqrt{\frac{2}{3T_3W}} \right) . \]  (4.18)

One can see that in the asymptotic regimes for \( Y \), this constraint is essentially a bound on the values of \( W \) and its derivatives, exactly as in canonical slow roll inflation. This is particularly clear for the case where \( Y \) is constant, since this implies that the effective potential depends solely upon the derivatives of the warp factor \( h(\phi) \) and the scalar potential \( V(\phi) \). The limit where \( Y \rightarrow 0 \) is essentially that of the usual DBI scenario (modulo a field redefinition) and we can therefore be sure that inflation will occur. In the limit of large \( Y \), we see that the higher-order corrections to the Hubble parameter become arranged into a \( 1/Y \) expansion and are therefore suppressed relative to the leading order terms.

### 4.3 Dynamical trajectories

In this section, we are interested in studying the flow equations for the \( n = 2 \) solution along the research-lines of [42, 43, 44] in both the relativistic and non-relativistic limits. Recall that in the case of a single brane, the inflaton equation of motion (in the Hamilton-Jacobi formalism) is given by

\[ \dot{\phi} \simeq -\frac{2M_p^2H'}{\gamma} \]  (4.19)

and the first slow roll parameter \( \epsilon_H \) by

\[ \epsilon_H \simeq \frac{2M_p^2}{\gamma} \left( \frac{H'}{H} \right)^2 . \]  (4.20)

In the case of finite \( n \), the non-linear (and occasionally higher derivative) nature of the action prevents us from analyzing the flow equations in the usual manner. However, there should exist regions of parameter space where one can work with equivalent algebraic structures. As such, we may extend the relations first established in [44] by introducing the following generalized
flow parameters:

\[\epsilon(\phi) \sim \frac{2M_p^2}{f(\phi)} \left(\frac{H'}{H}\right)^2\]
\[\eta(\phi) \sim \frac{2M_p^2}{f(\phi)} H''\]
\[\kappa(\phi) \sim \frac{2M_p^2}{f(\phi)} H' f(\phi)'\]
\[l_{\lambda} \sim \left(\frac{2M_p^2}{f(\phi)}\right)^l \left(\frac{H'}{H}\right)^{l-1} \frac{1}{H} \frac{d^{l+1}H}{d\phi^{l+1}}\]
\[l_{\alpha} \sim \left(\frac{2M_p^2}{f(\phi)}\right)^l \left(\frac{H'}{H}\right)^{l-1} \frac{1}{f(\phi)} \frac{d^{l+1}f(\phi)}{d\phi^{l+1}}\]

(4.21)

where \(f(\phi)\) represents a generalized function that needs to be determined in a limit of the theory where higher derivative terms can indeed be neglected.\(^8\)

It is convenient for our discussion to note the following relations between the parameters:

\[\eta = 1_{\lambda}, \quad \xi = 2_{\lambda}, \quad \rho = 1_{\alpha}, \quad \sigma = 2_{\alpha}.\]

(4.22)

It then follows that the dynamics of these parameters can be established by solving the set of differential equations

\[
\begin{align*}
\frac{d\epsilon}{dN_e} &\sim -\epsilon(2\epsilon - 2\eta + \kappa) \\
\frac{d\eta}{dN_e} &\sim -\eta(\epsilon + \kappa) + \xi \\
\frac{d\kappa}{dN_e} &\sim -\kappa(2\kappa + \epsilon - \eta) + \epsilon\rho \\
\frac{d\rho}{dN_e} &\sim -2\rho\kappa + \sigma \\
\frac{d^l\lambda}{dN_e} &\sim -^l\lambda(l\kappa + l\epsilon - \eta(l - 1)) + ^{l+1}\lambda \\
\frac{d^l\alpha}{dN_e} &\sim -^l\alpha((l + 1)\kappa + (l - 1)\epsilon - (l - 1)\eta) + ^{l+1}\alpha,
\end{align*}
\]

where \(N_e\) represents the number of e-foldings before the end of inflation and is defined by the condition \(dN_e = -Hdt\). The critical (stable) points of the phase space trajectories then correspond to the vanishing of each term on the right hand sides.

A simple calculation shows that at leading order in the derivative expansion, the function \(f\) for the \(n = 2\) case becomes

\[f \sim \frac{2\gamma^3}{\sqrt{1 + \lambda}}.\]

(4.24)

\(^8\)The flow parameters used in this context are only applicable to the leading order dimension four terms. One can include the higher derivative operators, but the flow equations cannot then be written as simple functions of \((H'/H)\). We hope to return to this more general problem in a future publication.
This is significantly different to the analogous term for the single brane model, where $f \sim \gamma$. Moreover, since $Y$ can also depend on the field $\phi$ (provided $h$ is constant), we find that there is an interesting scenario where the branes are moving relativistically, but $f \sim$ constant. This feature will be important in understanding the full dynamical phase space flow. In the non-relativistic limit we can truncate the theory such that $\gamma_2 \sim 1$, and then we find that $f$ is only a function of $Y$. In this case, $f$ can be constant if the warping is of the AdS form $h \sim \phi$. Thus there are two solution branches that are of interest if we wish $f$ to be constant:

- Relativistic limit and taking $h \sim$ constant.
- Non-relativistic limit and taking $h \sim \phi$.

Note that the latter branch leads to the strongest constraint, since $f$ is always constant if the warping is of the AdS-type. Indeed, in this case we immediately see that $\kappa = 0 = t_0$ which reduces the dimension of the overall phase space. Since the flow parameter $\epsilon$ is first order, we take this to be the fundamental parameter for our analysis and outline the results below:

- $\epsilon = 0$
  
  Setting $\epsilon$ to be zero fixes $H'$ to be zero and therefore the scale factor can be trivially integrated to yield
  
  $$a(t) \sim e^{Ht}$$
  
  (4.25)

- $\epsilon \neq 0$
  
  Here we are forced to choose solution slices where $\eta = \epsilon$ and also $t \lambda = e^t$, as can be seen from the flow equations. Again from this we can infer that $H$ is fixed to be exponential in form
  
  $$H \sim H_0 \exp \left( \pm \sqrt{\frac{\epsilon f}{2M_0^2}} \phi \right)$$
  
  (4.26)

  where we must take the positive sign above and also assume that $\phi$ is a decreasing function if this is to lead to an inflationary trajectory. From this we can then reconstruct the solution for the inflaton field velocity with the result that $Ht \sim 1/\epsilon$. Therefore we obtain the usual power law trajectory with $a \sim t^{1/\epsilon}$ as in the literature [43, 52, 53].

These are the only possible behaviors in this regime. The final case to consider is when $f$ is no longer constant but scales in such a way such that $\epsilon, \kappa$ remain constant. In this instance we can solve most of the dynamical equations exactly, obtaining relations such as

$$\eta = \frac{1}{2}(2\epsilon + \kappa).$$

(4.27)

The key point is that one can reconstruct $f$ in terms of the flow parameters. It will be convenient to perform the analysis on a case by case basis. Let us first consider the non-relativistic regime of the theory, where $f \sim 2/\sqrt{1+Y}$. We note immediately that $h \sim \phi$ yields constant $Y$ and therefore constant $f$, forcing us to consider the situation where $h \sim $ constant. We now have two
choices depending upon whether we wish to consider small or large field inflation. In the former case it can be seen that $f \to 0$ as the branes move and therefore we obtain

$$f_{IR} \sim 8M_p^2 \epsilon f_s \frac{(\sqrt{f_s} \kappa \phi \pm \sqrt{8M_p \epsilon})^2}{(f_s \kappa^2 \phi^2 - 8M_p^2 \epsilon)^2}$$

(4.28)

where $f_s$ is a constant of integration. It can be verified that $f \to 0$ as $\phi$ diverges. For large field inflation, we can write the function as follows:

$$f_{UV} \sim 16M_p^2 \epsilon \frac{(\sqrt{2} \kappa \phi \pm \sqrt{8M_p \epsilon})^2}{(2\kappa^2 \phi^2 - 8M_p^2 \epsilon)^2}$$

(4.29)

which satisfies the required boundary conditions. The Hubble parameter can then be reconstructed trivially (in both cases) to yield

$$H \sim H_0 \exp \left( \mp \sqrt{\frac{\epsilon}{2M_p^2}} \int \sqrt{f} d\phi \right)$$

(4.30)

Since the resulting expressions are rather complicated, we do not write them explicitly. We note, however, that there are many different choices for inflating trajectories depending on the particular choice of constants (and their signs).

The relativistic case is more complicated, since $f$ can still remain constant if the various parameters scale in the right way. For simplicity, let us consider the case where $Y$ is constant and we are also in the large field branch of solution space. This requires us to consider a limit where $\gamma_2$ diverges when $\phi \to 0$ in such a way that we can re-arrange the equation to solve for the relativistic factor as a function of the inflaton

$$\gamma^3 \sim \frac{2M_p^2 \epsilon (1 + Y)}{9\kappa^2 \phi^2}.$$  

(4.31)

This yields the result that $\phi^2 \propto \exp(-3\kappa N)$ and implies that the field falls off significantly faster than in standard single brane inflation. After substituting this dependence into the definition of the flow equation for $\epsilon$, we find that

$$H \propto \phi^{-\xi}, \quad \xi = \frac{\epsilon}{\kappa} \sqrt{\frac{2(1 + Y)}{9}}$$

(4.32)

In this case, therefore, the inflationary dynamics are sensitive to the ratio $\epsilon/\kappa$, as in the single brane case, but also to the precise value of $Y$ which is, in turn, set by the value of the background throat charge.

5 Discussion

In this work we have considered the cosmological consequences of using the action for $n$ coincident $D3$-branes in the finite $n$ limit. We have argued that the action for $n$ coincident $D3$-branes, with $n \gg 1$ and with scalars transforming under the $n$ dimensional representation of $SO(3)$, is the
same as that arising from a single $D5$-brane wrapping an $S^2$ and carrying $n$ units of $U(1)$ flux. Through the usual notion of string duality, this implies that we can either consider cosmology on $n$ coincident $D3$-branes, or a single wrapped $D5$-brane with commutative flux.

One immediate consequence is that the probe limit may no longer be a valid description of the physics. On the macroscopic side, this is due to the addition of $U(1)$ flux through the $S^2$, which effectively contributes a large mass correction to the theory at large $n$. Therefore, it is preferable to consider theories with finite $n$ in order to minimize the effect of back-reaction. This is a non-trivial problem in general, with the only conjectured solution being that of the $SO(3)$ scalar representation, which has been shown to lead to interesting physics in its own right.

Given that the finite $n$ prescription and the large $n$ prescription are essentially decoupled from one another, we have endeavored in section 2 to show that the finite $n$ action does indeed converge to that at large $n$ in a certain limit. This is the first time that such an argument has been presented in the literature. How does one then interpret the finite $n$ theory from the macroscopic side? Although we have not attempted to answer such a question, we believe that the correct interpretation is that of a wrapped $D5$-brane which now carries non-commutative $U(1)$ flux on the world-volume through the introduction of a star product. Thus our $D3$-brane model should be dual to a class of non-commutative field theories which have been explored in the literature [60,61]. It would be interesting to explore this in more detail since this is a different theory to the $\kappa$-deformed models and therefore evades the torsional interpretation [64] of the Snyder algebra proposed in [63], which seems to rule out many non-commutative field theory models.

The above discussion applies for the relativistic theory, and one may be interested in the preservation (or not) of non-Abelian physics in the non-relativistic expansion. This was explored in sections 3 and 4 of the paper. In section 3, we considered various consequences for the inflationary scenario when an arbitrary number of coincident branes are present and demonstrated explicitly how the slow roll parameters are modified by the non-Abelian structure. We also considered a particular small-field inflationary model where the branes are moving out of an AdS-type throat. We found that observational bounds on the scalar spectral index can in turn constrain the fluxes in the supergravity theory.

In section 4, we discussed the case of $n = 2$ in an arbitrary background in both the relativistic and non-relativistic limits. In the latter case, we found that the level of non-Gaussianity in the primordial curvature perturbation is indeed suppressed, although corrected from the canonical slow roll models. We concluded by considering the flow trajectories for such a configuration and compared these to those of the standard, single brane DBI scenario.

We have not addressed the question of reheating in this scenario and this is an important topic to consider. Neither have we attempted to identify the location of the standard model. On the other hand, we have argued from the field theory perspective that a $U(n)$ field-theory with scalars transforming under $SO(3)_n$ is dual to a $U(1)$ field-theory with scalars transforming under another $U(1)$. This duality appears to be true for all $n$. It is tempting to suggest that one could then identify standard model-type states within a $U(1)$ theory (with additional gauge fields), after symmetry breaking. This could be a particularly useful description for transferring
inflationary energy into the particle sector.

Acknowledgements

We wish to thank Andrei Frolov and Steve Thomas for their comments. AB and JW are supported in part by NSERC of Canada.

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