The Distance Coloring of Graphs

Lian-Ying Miao\(^1\); Yi-Zheng Fan\(^2\)

\(^1\)Institute of Mathematics, China University of Mining and Technology, Xuzhou 221116, P. R. China
\(^2\)School of Mathematical Sciences, Anhui University, Hefei 230601, P. R. China

Abstract: Let \(G\) be a connected graph with maximum degree \(\Delta \geq 3\). We investigate the upper bound for the chromatic number \(\chi_\gamma(G)\) of the power graph \(G^\gamma\). It was proved that \(\chi_\gamma(G) \leq \Delta \left(\frac{(\Delta-1)^\gamma-1}{\Delta-2}\right) + 1 =: M + 1\) with equality if and only if \(G\) is a Moore graph. If \(G\) is not a Moore graph, and \(G\) holds one of the following conditions: (1) \(G\) is non-regular, (2) the girth \(g(G) \leq 2\gamma - 1\), (3) \(g(G) \geq 2\gamma + 2\), and the connectivity \(\kappa(G) \geq 3\) if \(\gamma \geq 3\), \(\kappa(G) \geq 4\) but \(g(G) > 6\) if \(\gamma = 2\), (4) \(\Delta\) is sufficiently large than a given number only depending on \(\gamma\), then \(\chi_\gamma(G) \leq M - 1\). By means of the spectral radius \(\lambda_1(G)\) of the adjacency matrix of \(G\), it was shown that \(\chi_2(G) \leq \lambda_1(G)^2 + 1\), with equality holds if and only if \(G\) is a star or a Moore graph with diameter 2 and girth 5, and \(\chi_\gamma(G) < \lambda_1(G)^\gamma + 1\) if \(\gamma \geq 3\).

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1 Introduction

Let \(G = (V(G), E(G))\) be a graph. A vertex \(k\)-coloring of \(G\) is a mapping from \(V(G)\) to the set \(\{1, 2, \ldots, k\}\) such that any two adjacent vertices are mapped to different integers. The smallest integer \(k\) for which a \(k\)-coloring exists is called the chromatic number of \(G\), denoted by \(\chi(G)\). The \(\gamma\)-th power of the graph \(G\), denoted by \(G^\gamma\), is a graph on the same vertex set as \(G\) such that two vertices are adjacent in \(G^\gamma\) if and only if their distance in \(G\) is at most \(\gamma\). The \(\gamma\)-distance \(k\)-coloring, also called distance \((\gamma, k)\)-coloring, is a \(k\)-coloring of the graph \(G^\gamma\) (that is, any two vertices within distance \(\gamma\) in \(G\) receive different colors). The \(\gamma\)-distance chromatic number of \(G\) is exactly the chromatic number of \(G^\gamma\), denoted by \(\chi_\gamma(G)\). Clearly \(\chi(G) = \chi_1(G) \leq \chi_\gamma(G) = \chi(G^\gamma)\).

The distance coloring was introduced by Florica Kramer and Horst Kramer [11, 12], and a recent survey on this topic was also given by them; see [13] for more details. The \(\gamma\)-distance coloring of graphs has a good application in the frequency assignment problem (or radio channel...
Conjecture 1.1 [20] Let $G$ be a planar graph with maximum degree $\Delta$. Then

$$
\chi_2(G) \leq \begin{cases} 
7, & \text{if } \Delta = 3, \\
\Delta + 5, & \text{if } 4 \leq \Delta \leq 7, \\
\lfloor \frac{3}{2} \Delta \rfloor + 1, & \text{if } \Delta \geq 8.
\end{cases}
$$

Some work has been done on the case $\Delta = 3$, as listed in [10, Problem 2.18]. The conjecture is still open, even for the case of $\Delta = 3$. Many upper bounds on $\chi_2(G)$ for planar graphs $G$ in terms of $\Delta$ have been obtained in the last about two decades. The best known upper bound so far has been found by Molloy and Salavatipour [15]: $\chi_2(G) \leq \lfloor \frac{9}{8} \Delta \rfloor + 78$. Havet et al. [6] proved that $\chi_2(G) \leq \frac{3}{2} \Delta (1 + o(1))$ when $\Delta \to \infty$.

We review some results below on the general $\gamma$-distance chromatic number. It was noted by Skupień [18] that the well-known Brooks’ theorem can provide the following upper bound:

$$
\chi_\gamma(G) \leq 1 + \Delta(G^\gamma) \leq 1 + \Delta \sum_{k=1}^{\gamma} (\Delta - 1)^{k-1} = 1 + \frac{\Delta(\Delta - 1)^\gamma - 1}{\Delta - 2}. \quad (1.1)
$$

If $G$ is planar, Jendrol and Skupień [9] improved it as $\chi_\gamma(G) \leq 6 + \frac{3\Gamma + 3}{1 - 2}(\Gamma - 1)^{\gamma - 1} - 1$, where $\Gamma = \max\{8, \Delta\}$. Agnarsson and Hållérsson [1] proved that $G^\gamma$ is $O(\Delta^{|\gamma/2|})$-colorable for any fixed $\gamma$. Some authors studied $\chi_\gamma(G)$ for special graphs arisen from applications, such as square lattice [5] and hexagonal lattice [14]. Other related work could be found in [2] and [8]. It was proved by Sharp [17] that for fixed $\gamma \geq 2$ the distance coloring problem is polynomial time for $k \leq \lfloor 3\gamma/2 \rfloor$ and NP-hard for $k > \lfloor 3\gamma/2 \rfloor$.

Though for some classes of graphs $G$, $\chi_\gamma(G)$ is much smaller relative to the order $\Delta^\gamma$, e.g. planar graphs. There are a lot of graphs whose distance chromatic numbers have the order $\Delta^\gamma$; e.g. Moore graphs (see Theorem 2.1 below). In this paper we discuss the upper bound of $\chi_\gamma(G)$ for a general $\gamma \geq 2$. These bounds are investigated in two aspects: one is to use the maximum degree subject to the conditions such as minimum degree, girth, connectivity, the other is to use the spectral radius of the adjacency matrix of $G$.

Some notations are introduced as follows. Let $G$ be a graph. Denote by $G[U]$ the subgraph of $G$ induced by the vertices of $U \subseteq V(G)$. For a vertex $v \in V(G)$, denote by $N^G_k(v)$ the set of vertices of $G$ with distance $k$ from $v$. Clearly, $N^G_k(v) = N_G(v)$ is exactly the neighborhood of $v$ in $G$. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$. The maximum (resp. minimum) degree of $G$ is denoted by $\Delta(G)$ (resp. $\delta(G)$). The distance between two vertices $u$ and $v$ in $G$ is denoted by $dist_G(u, v)$. The diameter, girth, connectivity and clique number of $G$ are denoted by $diam(G), g(G), \kappa(G), \omega(G)$ respectively. If $G$ contains no cycles, then we define $g(G) = +\infty$. The superscript or subscript $G$ may be omitted when it is non-ambiguous.
2 The upper bound of $\chi_\gamma(G)$ in terms of maximum degree

When $\Delta = 2$, there exist only two connected graphs of order $n$: the path $P_n$ and the cycle $C_n$. It is easy to get the result:

1. $\chi_\gamma(P_n) = \min\{n, \gamma + 1\}$;
2. $\chi_\gamma(C_n) = \gamma + 1$ if $n \mod (\gamma + 1) = 0$, and $\chi_\gamma(C_n) = \min\{i + 1 \geq \gamma + 2 | n \mod i \leq n/i\}$.

From now on it is assumed that $\gamma \geq 2$ when discussing $\chi_\gamma(G)$, and all graphs are connected with $\Delta \geq 3$. We denote by $M$ the maximum possible degree of the graph $G$ as follows:

$$M = \Delta \frac{(\Delta - 1)^\gamma - 1}{\Delta - 2}.$$ 

First we characterize the graph $G$ for which $\chi_\gamma(G)$ attains the maximum value $M + 1$.

**Theorem 2.1** Let $G$ be a graph. Then

$$\chi_\gamma(G) \leq M + 1,$$

with equality if and only if $G$ is $\Delta$-regular of order $M + 1$, $g(G) = 2\gamma + 1$, $diam(G) = \gamma$, i.e., $G$ is a Moore graph with degree $\Delta$ and diameter $\gamma$.

**Proof.** The upper bound follows from (1.1). We now discuss the equality case. For the necessity, assume $\chi_\gamma(G) = M + 1 \geq \Delta(G^\gamma) + 1$. Then, by Brooks’ Theorem, $G^\gamma$ is a complete graph of order $M + 1$, and every vertex is at distance at most $\gamma$ of the other $M$ vertices, which is maximum obtained when considering that $G$ is a complete $M$-ary tree. Thus for every vertex $v$ in $G$, the graph $G$ is a complete $M$-ary tree except for edges between vertices in $N^\gamma(v)$. It follows trivially that $G$ is $\Delta$-regular, $g(G) = 2\gamma + 1$, $diam(G) = \gamma$, i.e. $G$ is a Moore graph.

The sufficiency is easily obtained from the fact that $G^\gamma$ is a complete graphs of order $M + 1$ as $diam(G) = \gamma$. 

**Corollary 2.2** Let $G$ be a graph with $g(G) \neq 2\gamma + 1$. Then $\chi_\gamma(G) \leq M$.

**Proof.** By Theorem 2.1, $\chi_\gamma(G) \leq M + 1$, with equality only if $g(G) = 2\gamma + 1$. So, if $g(G) \neq 2\gamma + 1$, surely $\chi_\gamma(G) \leq M$. 

Next we give some sufficient conditions for a graph $G$ such that $\chi_\gamma(G) \leq M - 1$. We will use the idea of saving a color at a vertex $v$, motivated by Cranston and Kim’s work on list-coloring the square of subcubic graphs. A partial (proper) coloring is the same as a proper coloring except that some vertices may be uncolored. Given a graph $G$ and partial coloring of $G^\gamma$, we define excess($v$) to be $1 + ($the number of colors available at vertex $v$) – ($the number of uncolored neighbors of $v$ in $G^\gamma$). Note that for any graph $G$ and any such partial $(M - 1)$-coloring, every vertex $v$ has excess($v$) $\geq 0$. Similar to the proof of Lemmas 3 and 4 in [3], we easily get the following two lemmas.

**Lemma 2.3** For any edge $uv$ of $G$, $\chi(G^\gamma - \{u, v\}) \leq M - 1$. 


**Lemma 2.4** Let $G$ be a graph with a partial $(M-1)$-coloring. Suppose that $u$ and $v$ are uncolored, are adjacent in $G^v$, and that excess$(u) \geq 1$ and excess$(v) \geq 2$. If we can order the uncolored vertices so that each vertex except $u$ and $v$ is succeeded in the order by at least two adjacent vertices in $G^v$, then we can finish the partial coloring.

A simple but useful instance where Lemma 2.4 applies is when the uncolored vertices induce a connected subgraph and the vertices $u$ and $v$ are adjacent in the subgraph. In this case, we order the vertices by decreasing distance (within the subgraph) from the edge $uv$.

**Corollary 2.5** If $G$ is a non-regular graph, then $\chi_\gamma(G) \leq M - 1$.

**Proof.** Let $v$ be a vertex with minimum degree $\delta \leq \Delta - 1$, and let $u$ be a neighbor of $v$. Note that $d_{G^v}(u) \leq M - [1 + (\Delta - 1) + \cdots + (\Delta - 1)^{\gamma-2}] \leq M - 1$ and $d_{G^v}(v) \leq M - [1 + (\Delta - 1) + \cdots + (\Delta - 1)^{\gamma-1}] \leq M - \Delta \leq M - 3$. So, if giving a partial $(M-1)$-coloring of $G^v - \{u, v\}$, then excess$(u) \geq 1$ and excess$(v) \geq 2$. The result follows by Lemma 2.4.

**Corollary 2.6** If $G$ is a graph with $g(G) \leq 2\gamma - 1$, then $\chi_\gamma(G) \leq M - 1$.

**Proof.** Let $C$ be a cycle of $G$ with length $g(G)$. For any two adjacent vertices $u$ and $v$ on $C$, $d_{G^v}(u) \leq M - 2$ and $d_{G^v}(v) \leq M - 2$. So, if giving a partial $(M-1)$-coloring of $G^v - \{u, v\}$, then excess$(u) \geq 2$ and excess$(v) \geq 2$. The result follows by Lemma 2.4.

**Corollary 2.7** Let $G$ be a graph with $g(G) \geq 2\gamma + 2$. If $\gamma \geq 3$ and $\kappa(G) \geq 3$, or $\gamma = 2$ and $\kappa(G) \geq 4$ but $g(G) > 6$, then $\chi_\gamma(G) \leq M - 1$.

**Proof.** First suppose that $\gamma \geq 3$ and $\kappa(G) \geq 3$. Let $C$ be a cycle of $G$ with length $g(G)$. Arbitrarily choose a path $P$ of length $\gamma$ on $C$ connecting two vertices $v_1$ and $x_1$, and choose a vertex $x_2$ lying on $P$ that is adjacent to $v_1$. Let $y_1, y_2$ be two vertices of $C$ outside $P$, where $y_1$ is adjacent to $v_1$ and $y_2$ is adjacent to $y_1$. As $\kappa(G) \geq 3$ and $g(G) \geq 2\gamma + 2$, there exists a path $P'$ of length $\gamma - 1$ outside $C$ connecting $v_1$ and $x_2$. See the left graph in Fig. 2.1 for these labeled vertices.

We color $x_1, y_1$ with color 1, and $x_2, y_2$ with color 2. As $\kappa(G) \geq 3$, the subgraph $G - \{y_1, y_2\}$ is connected. Order the vertices of $G - \{y_1, y_2\}$ by decreasing distance (within the subgraph) from the edge $v_1v_2$. Then $x_1, x_2$ are in the distance class $\gamma - 1 \geq 2$. For any uncolored vertex $w$ other than $v_1$ and $v_2$, $w$ is always succeeded by at least two adjacent (uncolored) vertices in $G^v$. Since $v_2$ has two neighbors $x_2, y_2$ in $G^d$ colored by the same color, we have excess$(v_2) \geq 1$. Since $v_1$ has 4 neighbors $x_1, y_1, x_2, y_2$ in $G^d$ colored by 2 colors, we have excess$(v_1) \geq 2$. The result follows by Lemma 2.4.

Next suppose $\gamma = 2$ and $\kappa(G) \geq 4$ but $g(G) > 6$. By a similar discussion as the above, $G$ contains an induced subgraph $G_2$ as listed in the right side of Fig. 2.1. We color $x_1, x_2, x_3$ with color 1. As $\kappa(G) \geq 4$, the subgraph $G - \{x_1, x_2, x_3\}$ is connected. Order the vertices of $G - \{x_1, x_2, x_3\}$ by decreasing distance (within the subgraph) from the edge $v_1v_2$. Then every vertex except $v_1, v_2$ is succeeded by at least two adjacent (uncolored) vertices in $G^v$. Note excess$(v_2) \geq 1$ and excess$(v_1) \geq 2$. We can greedily finish the coloring by Lemma 2.4.
For the case (2), by Lemma 2.8, \( \Delta(G) \geq \Delta_0 \) and \( \omega(G) \leq \Delta(G) - 1 \) then \( \chi(G) \leq \Delta(G) - 1 \). In fact, \( \Delta_0 = 10^{14} \) will do.

**Lemma 2.9** If \( G \) is a connected graph with maximum degree \( \Delta \geq 3 \) and \( G \) is not a Moore graph, then \( G^\gamma \) cannot properly contain a clique of size \( M \).

**Proof.** If \( G \) is a counterexample, that is, \( G^\gamma \) properly contains a copy of \( K_M \), denoted by \( H \). Note that \( \chi(G) \geq M \), so by Corollary 2.5 \( G \) is a regular graph. Choose adjacent vertices \( u \) and \( v_1 \) such that \( v_1 \in V(H) \) and \( u \notin V(H) \). Note that \( |\bigcup_{j=1}^\gamma N^j(v_1)| \leq M \), so all vertices in \( \bigcup_{j=1}^\gamma N^j(v_1) \setminus \{u\} \subseteq V(H) \). Label the neighbors of \( u \) as \( v_1, v_2, \ldots, v_\Delta \), all belonging to \( H \). Note that \( v_i \) cannot lies on a cycle of length smaller than \( 2\gamma + 1 \); otherwise, \( d_{G^\gamma}(v_i) \leq M - 1 \) and hence \( d_H(v_i) \leq M - 2 \) as \( u \notin V(H) \), which yields a contradiction.

Now \( \bigcup_{j=1}^{\gamma-1} N^j(v_i) \setminus \{u\} \subseteq V(H) \) for all \( v_i \). Since no \( v_i \) lies on a cycle of length smaller than \( 2\gamma + 1 \), for any \( i \neq k \), \( \bigcup_{j=1}^{\gamma-1} N^j(v_i) \) and \( \bigcup_{j=1}^{\gamma-1} N^j(v_k) \) share no common vertices except \( u \). So, the vertices \( \left( \bigcup_{i=1}^\gamma \{v_i\} \cup \left( \bigcup_{j=1}^{\gamma-1} N^j(v_i) \right) \right) \setminus \{u\} \) are exactly \( M \) vertices of \( H \). But \( u \) is within distance \( \gamma \) of each of these \( M \) vertices of \( H \). Hence, adding \( u \) to \( H \) yields a clique of size \( M + 1 \), which implies \( G \) is a Moore graph. \( \blacksquare \)

**Theorem 2.10** There exists a \( \Delta_0 \) such that if \( \Delta \) is odd and \( \Delta \geq \Delta_0 \) then \( \chi(G) \leq M - 1 \) except \( G \) is a Moore graph. In fact, \( \Delta_0 = (10^{14} + 1)\frac{1}{\gamma} + 1 \) will do.

**Proof.** Assuming to the contrary, let \( G \) be a non-Moore graph with maximum degree \( \Delta \geq \Delta_0 = (10^{14} + 1)\frac{1}{\gamma} + 1 \) and \( \chi(G) \geq M \). Note \( M \) is an odd number as \( \Delta \) is odd, and \( M > (\Delta - 1)^\gamma - 1 \geq 10^{14} \). By Corollary 2.5 \( G \) is \( \Delta \)-regular.

By Brooks’ theorem, \( M \geq \Delta(G^\gamma) \geq \chi(G^\gamma) - 1 \geq M - 1 \). So we have three cases: (1) \( \Delta(G^\gamma) = M - 1 \) and \( \chi(G^\gamma) = M \), (2) \( \Delta(G^\gamma) = M \) and \( \chi(G^\gamma) = M \), and (3) \( \Delta(G^\gamma) = M \) and \( \chi(G^\gamma) = M + 1 \). For the case (1), also by Brooks’ theorem, \( G^\gamma \) is a complete graph of order \( M \). But the sum of the degrees of vertices of \( G \) is \( M \cdot \Delta \), which is an odd number; a contradiction. For the case (2), by Lemma 2.8 \( G^\gamma \) properly contains a clique of size \( M \), which contradicts to Lemma 2.9. The last case cannot occur; otherwise by Theorem 2.11 \( G \) is a Moore graph. \( \blacksquare \)
Remark 1: In this section we give some sufficient conditions for a graph $G$ such that $\chi_\gamma(G) \leq M - 1$. We suspect that there exit no graphs $G$ with $\chi_\gamma(G) = M$.

Conjecture 1 If $G$ is a connected graph with maximum degree $\Delta \geq 3$ and $G$ is not a Moore graph, then $\chi_\gamma(G) \leq M - 1$.

Cranston and Kim \cite{3} conjectured that $\chi_l(G^2) \leq \Delta^2 - 1$, where $\chi_l(G^2)$ is the list-chromatic number of $G^2$. If their conjure is true, then we will have $\chi_2(G) = \chi(G^2) \leq \chi_l(G^2) \leq \Delta^2 - 1$, which implies that Conjecture 1 will hold for $\gamma = 2$.

Conjecture 2 If $G$ is a connected graph with maximum degree $\Delta \geq 3$ and $g(G) = 2\gamma$, then $\chi_\gamma(G) \leq M - 1$.

By Corollary 2.5 it suffices to consider the regular graphs. Also, if $v$ lies on a $2\gamma$-cycle, then $v$ cannot lie on any other $2\gamma$-cycle because $d_{G^\gamma}(v) \leq M - 2$ otherwise. Surely Conjecture 1 implies Conjecture 2 as $g(G) = 2\gamma$ implies that $G$ is not a Moore graph.

Conjecture 3 If $G$ is a connected graph with maximum degree $\Delta \geq 3$, then $G^\gamma \neq K_M$.

Erdős et al. \cite{4} have proved Conjecture for the case of $\gamma = 2$. If Conjecture 3 is true, then Lemma 2.9 would be: $\omega(G^\gamma) \leq M - 1$, which will generalize the Lemma 22 of \cite{3}; also we can delete the limitation that $\Delta$ is odd in Theorem 2.10.

If $G$ is a counterexample of Conjecture 3, then it suffices to consider regular graph with girth $g(G) \geq 2\gamma$ by Corollaries 2.5 and 2.6. Also we find $diam(G) = \gamma$. So $g(G) = 2\gamma$, and each vertex lies on exactly one $2\gamma$-cycle. However, if we can prove Conjecture 2, then $\chi_\gamma(G) \leq M - 1$, and hence $G^\gamma \neq K_M$. So Conjecture 2 implies Conjecture 3.

If $\Delta$ is sufficiently large, then Conjecture 3 also implies Conjecture 1, and hence three conjectures are equivalent. Let $G$ be a counterexample of Conjecture 1. Then $\chi_\gamma(G) = M$, which implies $G$ is regular and $g(G) \geq 2\gamma$ by Corollaries 2.5 and 2.6. Hence, $\Delta(G^\gamma)$ equals $M - 1$ or $M$. If $\Delta(G^\gamma) = M - 1$, then by Brooks’ theorem $G^\gamma = K_M$; a contradiction to Conjecture 3. If $\Delta(G^\gamma) = M$ and $\Delta$ is sufficiently large, then by Lemma 2.8, $G^\gamma$ properly contains $K_M$; a contradiction of Lemma 2.9.

3 Upper bound of $\chi_\gamma(G)$ in terms of spectral radius

Let $A$ be an entrywise nonnegative matrix. By Perron-Frobenius theorem, there exists an entrywise nonnegative eigenvector of $A$ corresponding the largest eigenvalue or spectral radius; this vector is also called the Perron vector of $A$. Let $A = [a_{ij}], B = [b_{ij}]$ be two matrices of same size. Write $A \leq B$ if $a_{ij} \leq b_{ij}$ for any $i, j$.

Let $G$ be a graph on $n$ vertices. The adjacency matrix of $G$, denoted by $A(G)$, is defined as a symmetric $(0, 1)$-matrix of order $n$, where $A(G)_{uv} = 1$ if and only if $u$ is adjacent to $v$. Observe that $A(G)^k_{uv}$ is the number of walks with length $k$ in $G$ from $u$ to $v$. So we have

$$A(G^\gamma) \leq A(G) + A(G)^2 + \cdots + A(G)^\gamma.$$  \hspace{1cm} (3.1)
Denote by $\lambda_1(G)$ (resp. $\lambda_1(A)$) the spectral radius or the largest eigenvalue of $A(G)$ (resp. a square matrix $A$). The degree matrix $D(G)$ of $G$ is a diagonal matrix whose diagonal elements are the degrees of the vertices of $G$. The Laplacian matrix $L(G)$ of $G$ is defined as $L(G) = D(G) - A(G)$, which is singular and positive semi-definite. If $G$ is connected, then 0 is a simple eigenvalue of $L(G)$ with the all-one vector $1$ as the corresponding eigenvector.

**Lemma 3.1** [21] Let $G$ be a graph. Then $\chi(G) \leq \lambda_1(G) + 1$, with equality if and only if $G$ is a complete graph or an odd cycle.

By Lemma 3.1 and (3.1), we easily get

$$\chi_{\gamma}(G) = \chi(G^\gamma) \leq \lambda_1(G) + \lambda_1(G)^2 + \cdots + \lambda_1(G)^\gamma + 1 = \frac{\lambda_1(G)^{\gamma+1} - 1}{\lambda_1(G) - 1}. \quad (3.2)$$

However, the upper bound in (3.2) is too large. We improve it in the following, before we prove some basic facts.

**Lemma 3.2** Let $A, B$ be two nonzero symmetric nonnegative square matrix of same order. Then

$$\lambda_1(AB) \leq \lambda_1(A)\lambda_1(B),$$

with equality if and only if $A^2, B^2, AB$ share a common Perron vector.

**Proof.** There exists a unit Perron vector $x$ such that $\lambda_1(AB) = x^T A B x$. So, by Cauchy-Schwarz inequality

$$\lambda_1(AB) = x^T A B x \leq \|Ax\| \cdot \|Bx\| = (x^T A^2 x)^{1/2} \cdot (x^T B^2 x)^{1/2} \leq \lambda_1(A)\lambda_1(B),$$

with equality if and only if $A^2 x = \lambda_1(A)^2 x, B^2 x = \lambda_1(B)^2 x$, i.e. $A^2, B^2, AB$ share a common Perron vector.$\blacksquare$

**Lemma 3.3** Let $G$ be a connected graph on at least 3 vertices. Then

1. $A(G^2) \preceq A(G)^2 - L(G)$ with equality if and only if $g(G) \geq 5$.
2. If $\gamma \geq 3$, then

$$A(G^\gamma) \preceq A(G^{\gamma-1})A(G) - A(G)(D(G) - I) - L(G),$$

$$A(G^\gamma) \preceq A(G)A(G^{\gamma-1}) - (D(G) - I)A(G) - L(G),$$

both with equalities if and only if $g(G) \geq 2\gamma + 1$.

**Proof.** (1) We know that $A(G^2) \preceq A(G) + A(G)^2$. But $A(G)^2$ contains nonzero diagonal entries. In fact, $A(G)^2_{uv} = d(v)$. So $A(G^2) \preceq A(G) + A(G)^2 - D(G) = A(G)^2 - L(G)$. Surely both sides have zero trace. If the equality holds, then $A(G)_{uv} = 1$ implies $A(G)^2_{uv} = 0$, that is, $G$ contains no $C_3$. Furthermore, if $A(G)_{uv}^2 \neq 0$, then $A(G)^2_{uv} = 1$, i.e. there is exactly one path of length 2 from $u$ to $v$. Hence $G$ contains no $C_4$. 

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Conversely, if $G$ contains no $C_3$ or $C_4$, then for each edge $uv$ of $G^2$ (equivalently $A(G^2)_{uv} = 1$), $u$ and $v$ are joined by an edge or by exactly one path of length 2 (but cannot happen at the same time). So $[A(G)^2 - L(G)]_{uv} = 1$. If $uv$ ($u \neq v$) is not an edge of $G^2$ (equivalently $A(G^2)_{uv} = 0$), then $d(u, v) \geq 3$, and hence $[A(G)^2 - L(G)]_{uv} = 0$.

(2) Assume that $uv$ is an edge in $G^\gamma$. If $uv$ is not an edge in $G$, there exists a path $W$ of length $l$ connecting $u$ and $v$, where $2 \leq l \leq \gamma$. Writing $W = uw_1 \cdots w_{l-1}v$, then $uw_{l-1}$ is an edge of $G^\gamma$ and $w_{l-1}v$ is an edge of $G$, and hence $A(G^\gamma)_{uv} \leq [A(G^{\gamma - 1}) \cdot A(G)]_{uv}$. Surely, $uw_1$ is an edge of $G$ and $w_1v$ is an edge of $G^\gamma$, and therefore $A(G^\gamma)_{uv} \leq [A(G) \cdot A(G^{\gamma - 1})]_{uv}$. Observing that the $uv$-entries of $A(G)(D(G) - I), (D(G) - I)A(G), L(G)$ are all zeros, so two inequalities hold in this case. If $uv$ is also an edge of $G$, then

\[
[A(G^{\gamma - 1})A(G) - A(G)(D(G) - I)]_{uv} \geq \sum_{w \in N(v) \setminus \{u\}} A(G^{\gamma - 1})_{uw}A(G)_{uw} - [d(v) - 1] = 0;
\]

\[
[A(G)A(G^{\gamma - 1}) - (D(G) - I)A(G)]_{uv} \geq \sum_{w \in N(u) \setminus \{v\}} A(G)_{uw}A(G^{\gamma - 1})_{uw} - [d(u) - 1] = 0.
\]

So the two inequalities also hold in this case. The equality cases can be checked directly. ■

Let $G$ be a graph and let $v \in V(G)$. The 2-degree of $v$ in $G$, denoted by $d^{(2)}(v)$, is defined by $d^{(2)}(v) = \sum_{u \in N(v)} d(u)$. One can find $d^{(2)}(v)$ is exactly the sum of the row of $A(G)^2$ corresponding to the vertex $v$. The graph $G$ is called 2-degree regular if all vertices have the same 2-degrees, or equivalently $A(G)^2$ has the constant row sums. If the girth $g(G) \geq 5$, then $d^{(2)}(v) = d_{G^2}(v)$. So, in this case, $G$ is 2-degree regular if and only if $G^2$ is regular.

**Lemma 3.4** Let $G$ be a connected graph on at least 3 vertices. Then

1. $\lambda_1(G^2) \leq \lambda_1(G)^2$ with equality if and only if $G$ is 2-degree regular with $g(G) \geq 5$. In particular, if, in addition, $G$ is non-bipartite, then the equality holds if and only if $G$ is regular with $g(G) \geq 5$.

2. If $\gamma \geq 3$, $\lambda_1(G^\gamma) < \lambda_1(G^{\gamma - 1})\lambda_1(G)$ and $\lambda_1(G^\gamma) < \lambda_1(G)^\gamma$.

**Proof.** (1) By Lemma 3.3(1) and the theory of nonnegative matrices

$$
\lambda_1(A(G^2)) \leq \lambda_1(A(G)^2 - L(G)),
$$

with equality if and only if $A(G^2) = A(G)^2 - L(G)$ as $G^2$ is connected or $A(G^2)$ is irreducible. Hence, also by Lemma 3.3(1), the equality in (3.3) holds if and only if $g(G) \geq 5$.

By Wely’s inequality,

$$
\lambda_1(A(G)^2 - L(G)) \leq \lambda_1(A(G)^2) + \lambda_1(-L(G)) = \lambda_1(G)^2,
$$

with equality if and only if $A(G)^2 - L(G), A(G)^2$, and $-L(G)$ share a common eigenvector corresponding to their largest eigenvalues. But $L(G)$ has a simple least eigenvalue 0 with 1 as the corresponding eigenvector. So the equality in (3.4) holds if and only if $A(G)^2$ has 1 as an eigenvector, or equivalently $G$ is 2-degree regular. The first part of assertion (1) now follows by combining the above discussion.
If $G$ is non-bipartite, then $A(G)^2$ is irreducible, and has a unique Perron vector (up to multiples), which is necessarily the Perron vector of $A(G)$. So, in this case the equality in (3.4) holds if and only if $1$ is a Perron vector of $A(G)$, which implies that $G$ is regular.

(2) By Lemma 3.3(2),
\[ 2A(G^\gamma) \leq A(G^{\gamma-1})A(G) + A(G)A(G^{\gamma-1}) - A(G)(D(G) - I) - (D(G) - I)A(G) - 2L(G). \]

By Wely's inequality,
\[
2\lambda_1(G^\gamma) \leq \lambda_1\left(A(G^{\gamma-1})A(G) + A(G)A(G^{\gamma-1}) - A(G)(D(G) - I) + (D(G) - I)A(G)\right)
\leq \lambda_1\left(A(G^{\gamma-1})A(G)\right) + \lambda_1\left(A(G)A(G^{\gamma-1})\right)
\leq 2\lambda_1(G^{\gamma-1})\lambda_1(G),
\]
where the last inequality follows from Lemma 3.2. So, by the first result,
\[
\lambda_1(G^\gamma) < \lambda_1(G^{\gamma-1})\lambda_1(G) \leq \lambda_1(G^{\gamma-2})\lambda_1(G)^2 \leq \cdots \leq \lambda_1(G^2)\lambda_1(G)^{\gamma-2} \leq \lambda_1(G)^\gamma.
\]

**Theorem 3.5** Let $G$ be a connected graph on at least 3 vertices. Then $\chi_2(G) \leq \lambda_1(G)^2 + 1$, with equality holds if and only if $G$ is a star or a Moore graph with diameter 2 and girth 5. If $\gamma \geq 3$, then $\chi_\gamma(G) < \lambda_1(G)^\gamma + 1$.

**Proof.** By Lemmas 3.1 and 3.4
\[
\chi_d(G) = \chi(G^\gamma) \leq \lambda_1(G^\gamma) + 1 \leq \lambda_1(G)^\gamma + 1,
\]
where the last inequality is strict if $\gamma \geq 3$. In the case of $\gamma = 2$, we have
\[
\chi_2(G) \leq \lambda_1(G^2) + 1 \leq \lambda_1(G)^2 + 1. \tag{3.5}
\]

If (3.5) holds equalities, then from the first equality, $G^2$ is complete which implies that $diam(G) \leq 2$; and from the second equality, $G$ is 2-degree regular and $g(G) \geq 5$. If $G$ is bipartite, then $G$ must be a tree with diameter at most 2, which implies $G$ is a star; otherwise, $G$ contains a cycle and $g(G) \geq 6$, which implies $diam(G) \geq 3$, a contradiction. If $G$ is non-bipartite, then $G$ is regular by Lemma 3.4(1). Hence $G$ is a Moore graph with diameter 2 and girth 5. For the sufficiency, if $G$ is a star, the result holds obviously. If $G$ is a Moore graph with degree $\Delta$, by Theorem 2.1 $\chi_2(G) = \Delta^2 + 1$. Observing $\lambda_1(G) = \Delta$, we get the equality. ■

**Remark 2:** If $G$ is $\Delta$-regular, then $\chi_2(G) \leq \Delta^2 + 1$ by Theorem 2.1 which is consistent with the bound $\lambda_1(G)^2 + 1$ since $\lambda_1(G) = \Delta$ in this case. Otherwise, by Corollary 2.5 $\chi_2(G) \leq \Delta^2 - 1$. In this case $\lambda_1(G) < \Delta$. A special example is that $G$ is a star on $n$ vertices. Then Corollary 2.5 gives $\chi_2(G) \leq n^2 - 2n$, while Theorem 3.5 gives $\chi_2(G) \leq n$, that latter of which is an equality as the $G^2$ is complete.

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