HOW SMOOTH CAN CONVEX CHAOTIC BILLIARD TABLES BE?

LEONID A. BUNIMOVICH AND ALEXANDER GRIGO

ABSTRACT. We solve the longstanding problem of smoothing a stadium billiard. Besides our results demonstrate why there were no clear conjectures how much the stadium’s boundary must be smoothened to destroy chaotic dynamics. To do that we needed to extend standard KAM theory to analyze stability of periodic orbits, because of the low smoothness of the system. In fact, the stadium has a $C^1$ boundary, and we show that $C^2$ smoothing results in appearance of elliptic periodic orbits.

CONTENTS

1. Introduction 1
2. Setup and Statement of the Main Results 3
3. Elliptic Periodic Orbits for Small Separations 5
   3.1. Existence of certain periodic orbits 5
   3.2. Definition of the smoothening and quantitative analysis of certain periodic orbits 6
   3.3. Stability of the periodic orbits 12
   3.4. Existence and stability of certain periodic orbits for small smoothening regions 14
   3.5. Sufficient conditions for the existence of elliptic orbits. 17
4. Elliptic Periodic Orbits for Large Separations 19
   4.1. Construction of symmetric periodic orbits and corresponding return maps 20
   4.2. Linear stability analysis 25
   4.3. Local analysis of the return map along periodic orbits 26
   4.4. Nonlinear stability analysis 29
   4.5. Verification of Assumption 4.1 and an explicit stability criterion 33
5. Proofs of Theorem 2.2 and Theorem 2.3 37
6. Conclusions 39
References 40
Appendix A. Notation and some facts about planar billiards 41

1. INTRODUCTION

In 1973 Lazutkin [14] showed that for strictly convex billiard tables $Q$ with a boundary $\partial Q$ of class $C^{555}$ there exists an uncountable family of caustics near the

L.B. was partially supported by the NSF grant DMS-1600568.
A.G. was partially supported by the NSF grant DMS-1413428.
boundary. The presence of these caustics prevents the billiard dynamics from being ergodic and gives rise to nearly integrable motion close to the boundary.

Shortly after, in 1974 it was shown in [1, 2] (see also [3]) that there are convex billiard tables on which the billiard dynamics is hyperbolic and ergodic. In billiards with focusing boundary components the mechanism that creates the hyperbolicity is the mechanism of defocusing. The most famous and best studied convex billiard in this class is the stadium [2, 3] whose boundary consists of two semi-circles connected by two straight line segments. For any (nonzero) length of the line segments the resulting billiard is hyperbolic and ergodic. Note that the boundary of the stadium is (globally) $C^1$.

These two results are in sharp contrast to each other. The difference is due to the smoothness of the boundary of the billiard table. If the boundary of a convex table is smooth enough then the presence of caustics prevents ergodicity. This was first proved by Lazutkin for $C^{5,3}$ smooth boundaries. Later R. Douady [11] lowered the smoothness requirement to $C^6$ boundaries. On the other hand, if the boundary is assumed to be only $C^1$ smooth, then there are (continuous families of) convex billiard tables with hyperbolic and ergodic billiard dynamics. Therefore, the natural question is: which class of smoothness of the boundary of the billiard table separates convex billiards with completely chaotic dynamics from non-ergodic dynamics with elliptic islands? This question was raised immediately after the appearance of the stadium billiard in 1974 by Alekseev, Anosov, Arnold, Katok, Moser and others. Nevertheless, this question remained open without clear conjectures. The results of our paper answer this long standing problem. Moreover, we clarify why there were no clearly stated conjectures. Namely, we show that there are essentially two different mechanisms, which create elliptic periodic orbits for short and long separations of the two curved parts of the boundary, respectively.

Various classes of hyperbolic (chaotic) billiards with boundary containing focusing components were constructed through the years (see e.g. [8] and references therein). The main property/restriction though is that all focusing components in such billiards must be absolutely focusing. Recall that a focusing components $\Gamma$ is absolutely focusing if any infinitesimal beam of parallel rays falling onto $\Gamma$ will leave $\Gamma$, after a series of consecutive reflections, as a focusing beam [4, 5, 10].

In [6] we showed that as soon as non-absolutely focusing boundary components are present the mechanism of defocusing fails, and hence stable periodic orbits can appear. Observe that focusing boundary components with vanishing curvature, which appear when making the boundary of the stadium $C^2$ smooth, are never absolutely focusing. Therefore it seems natural to attack the above question from this point of view, especially because the hyperbolicity in the stadium billiard is generated solely by the defocusing mechanism. Note that since the curvature of each boundary component is assumed to be continuous it follows that the contact between the straight line segments and the curved boundary component is either $C^2$ or $C^1$, depending on whether the curvature of the curved boundary component vanishes or not. An intermediate regularity of type $C^{1+\alpha}$ is not possible.

However, the problem addressed in this paper is much more challenging than the one in [6]. This is because there one was allowed to design the boundary of the billiard table. Here the problem is that we only allow for small perturbation of a given boundary. Moreover, due to the (insufficient) $C^2$ smoothness standard methods of perturbation theory cannot be applied to establish stability of periodic orbits.
in case of small separations and new (low smoothness) results must be obtained. Among new technical tools and results we used to address the low smoothness is a convenient system of coordinates which could be used for analytic and numerical studies of more general not sufficiently smooth Hamiltonian systems.

The structure of the paper is as follows. In Section 2 we formulate the problem more precisely and state the main results. In Section 3 we study the dynamics of the billiard in $C^2$-stadia for small separations of the two curved boundary segments. The corresponding analysis of the dynamical properties for large separations is carried out in Section 4. The last Section 6 contains some concluding remarks.

2. Setup and Statement of the Main Results

The billiard tables we consider are obtained by the following construction. Take a semi-circle and replace some segment near its endpoints by a convex curve whose curvature continuously tends to zero near its endpoint, and is such that the resulting focusing curve $\Gamma$ remains symmetric about the central axis just as the semi-circle was. Consider two identical copies of the focusing boundary component $\Gamma$. By connecting their endpoints using straight line segments of length $L$ we obtain a continuous family (parametrized by $L$) of $C^2$-smooth stadium-like billiard tables, or simply $C^2$-stadia. See Fig. 1 for the illustration of this construction.

![Figure 1. Smooth stadium like billiard tables.](image_url)

Our strategy of proof is the following. First, we need to carefully construct a family of periodic orbits, which have reflections off the smoothened part of the boundary. It is worthwhile to also mention that not all such periodic orbits are stable. Then we establish linear stability of the constructed periodic orbits, and then their nonlinear stability using standard methods from normal form theory.

The length $L$ of the straight line segments will be referred to as the separation distance between the curved boundary components. The part of the resulting billiard table, which is obtained by completing the two parallel line segments to a rectangle will be referred to as the rectangular channel.

In this paper we prove two main results about the dynamics of the billiard in $C^2$-stadia. The first result concerns small separation distances $L$. Due to the assumed symmetry of $\Gamma$ the billiard table with $L = 0$ has a two-periodic orbit reflecting off the endpoints of $\Gamma$. Since the curvature at the endpoints of $\Gamma$ vanishes this two-periodic orbit is parabolic. If this orbit was non-linearly stable one would hope that for small enough $L$ there will be a stable periodic orbit on the resulting $C^2$-stadium. However, the billiard map of the $C^2$-stadium is globally only of class $C^1$, which is not regular enough for KAM-type of arguments to apply directly. Since we are interested in boundary components $\Gamma$ that differ from semi-circles only on short
segments near the endpoints we introduce in Lemma 3.2 and Corollary 3.3 a scaling parameter $\alpha$ and a “shape function” $h$ such that the arc length $s_\alpha$ of the smoothened segment of $\Gamma$ is proportional to $\alpha$, and the curvature $K$ on the smoothened segment is given by scaling $h$, i.e. $K(s)$ is obtained by evaluating $-h(\frac{s}{s_\alpha})$. In Section 3 we prove that for fixed $h$ the billiard in the $C^2$-stadium corresponding to small enough $\alpha$ and small enough separations $L$ will have elliptic periodic orbits. More precisely, we prove the following

**Theorem 2.1** (Short Separations). Suppose $h$ is such that the curvature of the smoothened boundary component decreases monotonically to zero so that its arc length is less than twice the arc length of the circular segment it replaces. Suppose further that $h$ satisfies the non-degeneracy condition (C) on page 18. Then for all sufficiently small values of $\alpha$ there exists $L_{h,\alpha} > 0$ such that for any separation distance $L$ with $0 \leq L < L_{h,\alpha}$ the resulting $C^2$-stadium has an elliptic periodic orbit.

Since the boundary of the smoothened stadium is $C^2$, changing $L$ from zero to a nonzero value represents a $C^1$ perturbation of the billiard map. Recall that for the usual stadium the corresponding perturbation is only $C^0$. In view of perturbation theory the existence of elliptic periodic orbits due to the increased smoothness of perturbation not too surprising. However, as we already pointed out, it seems not possible to obtain a proof of Theorem 2.1 by a direct application of standard results of perturbation theory.

For the usual stadium billiard the billiard dynamics is hyperbolic and ergodic not only for arbitrarily short, but also for arbitrarily large separations of the two semi-circles. This is due to the mechanism of defocusing [2, 1, 3], which actually becomes more efficient the larger the separation distance is. Therefore, our second main result Theorem 2.2 is less expected than Theorem 2.1.

**Theorem 2.2** (Large Separations).

(i) Let $\Gamma$ be obtained from a semi-circle by smoothing a sufficiently small neighborhood of its endpoints. Then there exist constants $\delta, a, b > 0$ and $N \in \mathbb{N}$ such that for any separation distance $L$ with $L \in \bigcup_{n \geq N} [a + nb, a + nb + \delta]$ the resulting $C^2$-smooth stadium-like billiard table has elliptic periodic orbits.

(ii) There exists an open (in the $C^5$ topology) set $G$ of smoothenings of a semi-circle, that includes arbitrarily short smoothenings of that semi-circle, such that for any $\Gamma \in G$ the corresponding $C^2$-smooth stadium-like billiard table has a nonlinearly stable periodic orbit for all separation distances $L \geq L_\Gamma$.

Unlike the elliptic periodic orbits for small separations, the elliptic periodic orbits of Theorem 2.2 are in no sense obtained by a perturbation for the billiard table corresponding to $L = 0$. In fact, even the existence of elliptic periodic orbits for some (arbitrary) large separation distance is not obvious. So it is quite surprising that elliptic periodic orbits actually exist for a large and very regular set of separation distances, which for an open set of smoothenings is even the set of all sufficiently large separation distances!

The key observation behind the construction of these orbits is the fact that the curved boundary component $\Gamma$ is non-absolutely focusing. The elliptic periodic
orbits of Theorem 2.2 naturally correspond to those constructed in [6]. Recall that the general design principle for hyperbolic billiards [1, 2, 19, 16, 5, 10] requires the focusing boundary components to be absolutely focusing. This is because the only general principle known to ensure hyperbolicity with focusing boundary components is the defocusing mechanism. Since Theorem 2.2 shows that violating the absolutely focusing property of $\Gamma$ cannot be compensated for by making the separation distance arbitrarily large we see that the absolute focusing property is not only necessary for the general design principle of hyperbolic billiards with focusing components [6], but also in such rigid settings as in the stadium billiard family, which has only the separation distance as a free parameter.

Finally, combining our results for large separation with a special construction of stable periodic orbits for small separation distances we show the following:

**Theorem 2.3 (All Separations).** There exists an open (in the $C^5$ topology) set $G$ of smoothenings of a semi-circle, that includes arbitrarily short smoothenings of that semi-circle, such that for any $\Gamma \in G$ the corresponding $C^2$-smooth stadium-like billiard table has a nonlinearly stable periodic orbit for all separation distances $L \geq 0$.

### 3. Elliptic Periodic Orbits for Small Separations

In this section we will prove Theorem 2.1. To explain the underlying idea, consider the billiard table where the two curved boundary components are not separated. In the case of the usual stadium this would be a circle. Suppose that there exists an elliptic periodic orbit, which has no reflection off either of the two points where the two curved boundary components are joined. Then separating the two curved boundary components by adding straight line segments to the boundary of the billiard table represents a smooth perturbation of the billiard map in a neighborhood of the periodic orbit. Therefore, the ellipticity of the periodic orbit implies that it persists as an elliptic periodic orbit for all sufficiently small separations of the two curved boundary components, and thus proves Theorem 2.1.

Therefore, the strategy of proof we adopt is the following. In all of this section we assume that the two curved boundary components are joined at their endpoints, where the curvature vanishes. This results in a billiard table with a globally $C^2$ smooth boundary, which is symmetric about the vertical and horizontal axis, recall Fig. 1 and see also Fig. 2 below. In a first step we establish the existence of a certain class of periodic orbits. In a second step we derive a criterion on the smoothening that guarantees that these periodic orbits are elliptic. From this we then obtain, as described above, the existence of elliptic periodic orbits for all sufficiently small separation distances.

#### 3.1. Existence of certain periodic orbits.

First we explain the special periodic orbits we consider for the analysis of the dynamics of the billiard for small separations. As already mentioned before, we assume here as we will continue throughout the entire section, that the two smoothened segments are not separated. Let $n \geq 0$ be some integer, and choose $n + 2$ points on $\Gamma$ in counterclockwise orientation. Denote the corresponding arc length parameters by $s_0, \ldots, s_{n+1}$. Let the first and the last points in this sequence coincide with the midpoints and the endpoint of $\Gamma$, respectively. By changing $s_1, \ldots, s_n$, while preserving their order, we can maximize the length of the broken line connecting these $n + 2$ points. And because the length
of such a broken line is a generating function for the billiard dynamics \cite{13, 18, 17}.

any maximizing configuration represents a segment of a billiard trajectory \( \gamma_s \). Since the billiard table is symmetric about the horizontal and the vertical axis, and since \( \gamma_s \) has the midpoint of \( \Gamma \) as its first point and the endpoint of \( \Gamma \) as its last point it then follows by symmetry that the \( \gamma \) is part of a periodic billiard trajectory of period \( 4(n+1) \). And this periodic orbit is also symmetric about the vertical and horizontal symmetry axis of the billiard table.

**Lemma 3.1** (Existence of certain periodic orbits – qualitative version). For every fixed smoothening of the circular segments there exists an integer \( n_{\Gamma} \geq 0 \) such that for every \( 0 \leq n \leq n_{\Gamma} \) there exists a \( 4(n+1) \)-periodic orbit that is symmetric about the horizontal symmetry axis of the billiard table, but not the vertical symmetric axis of the billiard table. Furthermore, these orbits have reflections off the midpoints of the curved boundary components, but not off their endpoints. An illustration is such a periodic orbit is given in Fig. 2 below.

**Proof.** In the preceding discussion we noted that for any \( n \geq 0 \) a periodic orbit (corresponding to \( \gamma_s \)) exists, which is symmetric about the horizontal and vertical symmetry axis, and with reflections off the midpoint and the endpoints of \( \Gamma \). Clearly, for any smoothening of a semi-circle near its endpoints there exists an integer \( n_{\Gamma} \geq 0 \), such that for every \( 0 \leq n \leq n_{\Gamma} \) only the point corresponding to \( s_{n+1} \) in \( \gamma_s \) is off the smoothened part of \( \Gamma \). More precisely, \( s_0, \ldots, s_n \) correspond to points off the interior of the circular part of \( \Gamma \).

Let \( \psi_n \) denote the angle between the horizontal axis and the line segment connecting the point \( \Gamma(s_n) \) with the center point of the circle, see Fig. 2. And since \( \Gamma(s_{n+1}) \) corresponds to the endpoint of \( \Gamma \), at which the curvature vanishes by assumption on the smoothening, it follows that increasing the angle \( \psi_n \) while keeping \( \Gamma(s_0) \) fixed, will move the point \( \Gamma(s_{n+2}) \) towards the vertical symmetry axis, and gives rise to a perturbed billiard trajectory piece \( \gamma \). Of course, \( \gamma \) is no longer part of a \( 4(n+1) \)-periodic orbit anymore. And since the new point of reflection \( \Gamma(s_{n+2}) \) is (for small increments of \( \psi_n \)) still off the circular part of the boundary we obtain that \( \Gamma(s_{2n+2}) \) moves away from the midpoint of the curved segment in clockwise direction (i.e. the points “moves upwards”; cf Fig. 2).

However, increasing \( \psi_n \) even further will eventually force \( \Gamma(s_{2n+2}) \) to move in the opposite direction. By continuity it eventually must cross again the midpoint of the curved boundary component. And due to the symmetry of the billiard table about the horizontal axis this will result in a \( 4(n+1) \)-periodic orbit, which is symmetric about the horizontal symmetry axis of the billiard table, but which is not symmetric about the vertical. Such a periodic orbit is illustrated in Fig. 2. \( \square \)

Note, however, that Lemma 3.1 does not make a statement about the location of the \( n+1 \)-st reflection (see Fig. 2). In particular, there is no guarantee that a periodic orbit given by Lemma 3.1 has a reflection off the smoothened part of the boundary. In particular, the stability type of these periodic orbits is not clear at all. In fact, it is not too difficult to see that these periodic orbits could be hyperbolic, if the part of the boundary corresponding to the smoothened region is sufficiently long compared to the part of the circle it replaces. In particular, if all reflections are off the circular parts any such periodic orbit must be hyperbolic.
Therefore, in order to prove the existence of elliptic periodic orbits for small separations as stated in Theorem 2.1 we need to quantify more precisely the smoothening of the boundary. This is the aim of the next part.

3.2. Definition of the smoothening and quantitative analysis of certain periodic orbits. Let \( \Gamma \) be obtained by smoothening a semi-circle of radius \( \rho \) near its two endpoints. Let \( \alpha \) denote the angle corresponding to the part near the endpoint of the semi-circle that got replaced by the smoothed out part, whose arc length will be denoted by \( s_\alpha \). Recall Fig. 2. The arc length parameter and the curvature along \( \Gamma \) will be denoted by \( s \) and \( K(s) \), respectively. We count \( s \) starting at zero at the point where the circular part of \( \Gamma \) ends and the smoothed part starts, and \( s \) increases in value towards the endpoint of \( \Gamma \). Then the following Lemma 3.2 provides a characterization of the smoothened part of \( \Gamma \).

**Lemma 3.2** (Normal form of the smoothened segment). There exists a continuous function

\[
h : [0, 1] \rightarrow [0, \infty), \quad h(0) = 1, \quad h(1) = 0
\]

such that

\[
K(s) = -\frac{1}{\rho} h\left(\frac{s}{\rho \alpha} \int_0^1 h(\xi) \, d\xi\right)
\]

for all \( 0 \leq s \leq \frac{\rho \alpha}{\int_0^1 h(\xi) \, d\xi} = s_\alpha \)

holds for the curvature \( K \) and the arc length \( s_\alpha \) of the smoothened part of \( \Gamma \), respectively.

**Proof.** Since every plane curve is, up to rigid motion, uniquely specified by its curvature, it is \( K \) that needs to be characterized. Since the smoothened part of \( \Gamma \) of arc length \( s_\alpha \) must (by the very definition of \( \alpha \)) enclose the angle \( \alpha \) it follows that the curvature must satisfy

\[
K(0) = -\frac{1}{\rho}, \quad K(s_\alpha) = 0, \quad \alpha = \int_0^{s_\alpha} -K(s) \, ds, \quad K(s) \leq 0 \quad \forall \ 0 \leq s \leq s_\alpha.
\]
These conditions on \( K \) are equivalent to the existence of a continuous function \( h : [0, 1] \to [0, \infty) \) such that
\[
h(0) = 1, \quad h(1) = 0, \quad K(s) = -\frac{1}{\rho} h\left(\frac{s}{s_\alpha}\right) \quad 0 \leq s \leq s_\alpha
\]
in which case it follows that
\[
s_\alpha = \frac{\rho \alpha}{\int_0^1 h(\xi) \, d\xi}.
\]

\[\square\]

**Corollary 3.3** (Explicit form of the smoothened segment). Let \( h \) be as in Lemma 3.2. Then smoothened part of the boundary component \( \Gamma \) can be written as \( \Gamma(s_\alpha \zeta) = \Gamma_\alpha(\zeta) \) for \( 0 \leq \zeta \leq 1 \), where
\[
\frac{1}{\rho} \Gamma_\alpha^x(\zeta) = \frac{\int_0^1 \cos[\alpha \Theta(\xi)] \, d\xi}{\int_0^1 h(\xi) \, d\xi}, \quad \frac{1}{\rho} \Gamma_\alpha^y(\zeta) = \cos \alpha + \frac{\int_0^\zeta \sin[\alpha \Theta(\xi)] \, d\xi}{\int_0^1 h(\xi) \, d\xi}
\]
where \( \Theta(\zeta) = \int_0^\zeta h(\xi) \, d\xi \int_0^1 h(\xi) \, d\xi \)
for \( 0 \leq \zeta \leq 1 \).

**Proof.** By Lemma 3.2 we obtain the expression for the curvature \( K \) of \( \Gamma \) in terms of the function \( h \). And since the smoothened part of the boundary component \( \Gamma \) can be written in terms of its curvature as \( \Gamma(s) = \delta_\alpha + \int_0^s (-\cos \theta(r)) \sin \theta(r) \, dr \), \( \theta(s) = \alpha - \int_0^s K(r) \, dr \) for all \( 0 \leq s \leq s_\alpha \), it follows that
\[
\Gamma_\alpha(\zeta) = \left(\frac{\delta_\alpha}{\rho \cos \alpha}\right) + \frac{\rho \alpha}{\int_0^1 h(\xi) \, d\xi} \int_0^\zeta \left(-\cos[\alpha \Theta(\xi)]\right) \sin[\alpha \Theta(\xi)] \, d\xi
\]
for all \( \Theta(\zeta) = 1 - \frac{\int_0^\zeta h(\xi) \, d\xi}{\int_0^1 h(\xi) \, d\xi} \equiv \frac{\int_0^\zeta h(\xi) \, d\xi}{\int_0^1 h(\xi) \, d\xi} \)

The expression for \( \delta_\alpha \) is given by the condition \( \Gamma_x(s_\alpha) = 0 \), see Fig. 2, i.e.
\[
\delta_\alpha = \frac{\int_0^1 \cos[\alpha \Theta(\xi)] \, d\xi}{\int_0^1 h(\xi) \, d\xi}, \quad \Theta(\zeta) = \frac{\int_0^\zeta h(\xi) \, d\xi}{\int_0^1 h(\xi) \, d\xi}
\]
and hence shows the claimed expression for \( \Gamma \) after substituting this expression for \( \delta_\alpha \) in the above form of \( \Gamma_\alpha \). \[\square\]

Furthermore, upon inspection of Fig. 2 we see that the location \( c_\alpha \) of the center point of the circular part of \( \Gamma \) satisfies \( c_\alpha = \frac{1}{\rho} \Gamma_x(0) - \sin \alpha \), so that the explicit parametrization of \( \Gamma \) as given in Corollary 3.3 yields
\[
\frac{c_\alpha}{\rho} = \alpha \frac{\int_0^1 \cos[\alpha \Theta(\xi)] \, d\xi}{\int_0^1 h(\xi) \, d\xi} - \sin \alpha \quad \text{for} \quad \alpha > 0.
\]

With this more quantitative formulation of the smoothening of the circular boundary components we arrive at a quantitative version of Lemma 3.1. In light of the discussion after Lemma 3.1 of the stability of such periodic orbits we are...
mainly interested in the case where out of the point of reflections corresponding to
$s_0, \ldots, s_{n+2}$ only the $n + 1$-st reflection is off the smoothened part of $\Gamma$, whereas
the remaining reflections are off the interior of the circular part of $\Gamma$.

**Proposition 3.4** (Existence of certain periodic orbits – quantitative version). Let
$n \geq 0$ be some fixed integer. Then a $4(n + 1)$-periodic orbit as in Lemma 3.2
(and illustrated in Fig. 2) such that only the $n + 1$-st reflection in the sequence of
reflections $s_0, \ldots, s_{n+2}$ is off the smoothened boundary component $\Gamma$ exists if and
only if

\[
\frac{1}{\rho} \Gamma^x_\alpha (\zeta) = \cos \left( \frac{\varphi_{n+1}}{2n + 1} \right) \sin \Theta(\zeta) \cos \left( \frac{\alpha \Theta(\zeta)}{2n + 1} \right)
\]

\[
\quad + \frac{\alpha \Theta(\zeta)}{2n + 1} \sin \left( \frac{\alpha \Theta(\zeta)}{2n + 1} \right) \cos \Theta(\zeta)
\]

\[
\quad + \frac{c_\alpha}{\rho} \sin \alpha \Theta(\zeta) \cos \alpha \Theta(\zeta) \left( \tan \varphi_{n+1} + \frac{1}{\tan \varphi_{n+1}} \right)
\]

\[
\frac{1}{\rho} \Gamma^y_\alpha (\zeta) = \cos \left( \frac{\varphi_{n+1}}{2n + 1} \right) \cos \left( \frac{\alpha \Theta(\zeta)}{2n + 1} \right) \cos \alpha \Theta(\zeta)
\]

\[
\quad - \frac{\alpha \Theta(\zeta)}{2n + 1} \sin \left( \frac{\alpha \Theta(\zeta)}{2n + 1} \right) \sin \alpha \Theta(\zeta)
\]

\[
\quad + \frac{c_\alpha}{\rho} \cos \left( \frac{2\alpha \Theta(\zeta)}{2n + 1} - \sin \alpha \Theta(\zeta) \tan \varphi_{n+1} \right)
\]

have a solution for $0 \leq \zeta \leq 1$ and $0 < \varphi_{n+1} < \frac{\pi}{2}$ subject to the constraints

\[0 < \psi_n, \psi_{n+2} < \frac{\pi}{2} - \alpha, \quad 0 < \varphi - \psi_n, \varphi_{n+2} - \psi_{n+2}\]

where

\[\varphi_n = \frac{\pi}{2} - \frac{\varphi_{n+1} - \alpha \Theta(\zeta)}{2n + 1}, \quad \varphi_{n+2} = \frac{\pi}{2} - \frac{\varphi_{n+1} + \alpha \Theta(\zeta)}{2n + 1}\]

\[
\psi_n = n \left( \pi - 2 \varphi_n \right), \quad \psi_{n+2} = n \left( \pi - 2 \varphi_{n+2} \right)
\]

\[
\tau_{n+1,n+2} = \frac{1}{\rho} \left( \Gamma^x_\alpha (\zeta) + c_\alpha \right) \left( \Gamma^y_\alpha (\zeta) \right) + \cos \varphi_{n+2}
\]

\[
\tau_{n,n+1} = \frac{1}{\rho} \left( \Gamma^y_\alpha (\zeta) - c_\alpha \right) \left( \Gamma^x_\alpha (\zeta) \right) + \cos \varphi_n
\]

and $\alpha > 0$.

**Proof.** For such a periodic orbit the number of reflections off each of the two circular
parts of the boundary of the billiard table arcs is $2n + 1$, recall Fig. 2. The notation
we will use below is illustrated in Fig. 3 which is Fig. 2 adapted to our setting where
only $s_{n+1}$ is off the smoothened part.
By inspecting Fig. 3 we see that at the \( n+1 \)-st reflection a necessary and sufficient condition for a specular reflection is that

\[
\pi - 2 \varphi_{n+1} = \varphi_{n+2} - \psi_{n+2} + \varphi_n - \psi_n
\]

\[0 < \varphi_{n+1} < \frac{\pi}{2}, \quad 0 < \psi_n, \psi_{n+2} < \frac{\pi}{2} - \alpha, \quad 0 < \varphi_n - \psi_n, \varphi_{n+2} - \psi_{n+2}\]

\[
\Gamma^\alpha(\zeta) = \begin{pmatrix} -c_\alpha - \rho \cos \psi_{n+2} \\ \rho \sin \psi_{n+2} \end{pmatrix} + \tau_{n+1,n+2} \begin{pmatrix} \cos(\varphi_{n+2} - \psi_{n+2}) \\ \sin(\varphi_{n+2} - \psi_{n+2}) \end{pmatrix}
\]

\[
\Gamma^{\alpha}(\zeta) = \begin{pmatrix} c_\alpha + \rho \cos \psi_n \\ \rho \sin \psi_n \end{pmatrix} + \tau_{n,n+1} \begin{pmatrix} -\cos(\varphi_n - \psi_n) \\ \sin(\varphi_n - \psi_n) \end{pmatrix}
\]

hold. The conditions \( 0 < \psi_n, \psi_{n+2} < \frac{\pi}{2} - \alpha \) and \( 0 < \varphi_n - \psi_n, \varphi_{n+2} - \psi_{n+2} \) are imposed to guarantee that there is only one reflection off the smoothened boundary component \( \Gamma \).

Once the trajectory forms a valid billiard trajectory a necessary and sufficient condition for it to form a periodic orbit as shown in Fig. 2 is that the angles \( \psi_n \) and \( \psi_{n+2} \) must be chosen so that

\[
\psi_n = n \left( \pi - 2 \varphi_n \right), \quad \psi_{n+2} = n \left( \pi - 2 \varphi_{n+2} \right)
\]

hold. This is due to the explicit solution of the billiard map along a circular arc.

Using the parametrization of \( \Gamma \) as given in Corollary 3.3 shows that

\[
\varphi_{n+1} = \frac{\pi}{2} - \alpha \Theta(\zeta) - (\varphi_{n+2} - \psi_{n+2}) = \frac{\pi}{2} + \alpha \Theta(\zeta) - (\varphi_n - \psi_n)
\]

for the expression for \( \varphi_{n+1} \) in terms of \( \zeta \), where the equality of the two expressions on the right is due to the above conditions on the angles in order to form a specular reflection.

Finally, note that by taking projections of the four equations

\[
\frac{1}{\rho} \Gamma^\alpha(\zeta) = \begin{pmatrix} -c_\alpha - \rho \cos \psi_{n+2} \\ \rho \sin \psi_{n+2} \end{pmatrix} + \tau_{n+1,n+2} \begin{pmatrix} \cos(\varphi_{n+2} - \psi_{n+2}) \\ \sin(\varphi_{n+2} - \psi_{n+2}) \end{pmatrix}
\]

\[
\frac{1}{\rho} \Gamma^{\alpha}(\zeta) = \begin{pmatrix} c_\alpha + \rho \cos \psi_n \\ \rho \sin \psi_n \end{pmatrix} + \tau_{n,n+1} \begin{pmatrix} -\cos(\varphi_n - \psi_n) \\ \sin(\varphi_n - \psi_n) \end{pmatrix}
\]
we can separate the parts depending on $\tau_{n+1,n+2}$ and $\tau_{n+1,n+2}$ from the parts depending only on $\zeta$ and the various angles. This proves the two equations for the free paths

\[
\frac{\tau_{n+1,n+2}}{\rho} = \frac{1}{\rho} \begin{pmatrix} \Gamma_x^\alpha(\zeta) + c_\alpha \\ \Gamma_y^\alpha(\zeta) \end{pmatrix} \begin{pmatrix} \cos(\varphi_{n+2} - \psi_{n+2}) \\ \sin(\varphi_{n+2} - \psi_{n+2}) \end{pmatrix} + \cos \varphi_{n+2}
\]

\[
\frac{\tau_{n,n+1}}{\rho} = \frac{1}{\rho} \begin{pmatrix} \Gamma_x^\alpha(\zeta) - c_\alpha \\ \Gamma_y^\alpha(\zeta) \end{pmatrix} \begin{pmatrix} -\cos(\varphi_n - \psi_n) \\ \sin(\varphi_n - \psi_n) \end{pmatrix} + \cos \varphi_n
\]

and the two equations

\[
\frac{1}{\rho} \begin{pmatrix} \Gamma_x^\alpha(\zeta) + c_\alpha \\ \Gamma_y^\alpha(\zeta) \end{pmatrix} \begin{pmatrix} -\sin(\varphi_{n+2} - \psi_{n+2}) \\ \cos(\varphi_{n+2} - \psi_{n+2}) \end{pmatrix} = \sin \varphi_{n+2}
\]

\[
\frac{1}{\rho} \begin{pmatrix} \Gamma_x^\alpha(\zeta) - c_\alpha \\ \Gamma_y^\alpha(\zeta) \end{pmatrix} \begin{pmatrix} \sin(\varphi_n - \psi_n) \\ \cos(\varphi_n - \psi_n) \end{pmatrix} = \sin \varphi_n
\]

for the angles and $\zeta$. Since we have already shown that

\[
\varphi_{n+2} - \psi_{n+2} = \frac{\pi}{2} - \alpha \Theta(\zeta) - \varphi_{n+1} \quad \text{and} \quad \varphi_{n} - \psi_{n} = \frac{\pi}{2} + \alpha \Theta(\zeta) - \varphi_{n+1}
\]

we can eliminate $\psi_n$ and $\psi_{n+2}$ and obtain the two claimed equations for the free paths and the following two equations for the angles and $\zeta$

\[
\frac{1}{\rho} \begin{pmatrix} \Gamma_x^\alpha(\zeta) + c_\alpha \\ \Gamma_y^\alpha(\zeta) \end{pmatrix} \begin{pmatrix} -\cos[\varphi_{n+1} + \alpha \Theta(\zeta)] \\ \sin[\varphi_{n+1} + \alpha \Theta(\zeta)] \end{pmatrix} = \sin \varphi_{n+2}
\]

\[
\frac{1}{\rho} \begin{pmatrix} \Gamma_x^\alpha(\zeta) - c_\alpha \\ \Gamma_y^\alpha(\zeta) \end{pmatrix} \begin{pmatrix} \cos[\varphi_{n+1} - \alpha \Theta(\zeta)] \\ \sin[\varphi_{n+1} - \alpha \Theta(\zeta)] \end{pmatrix} = \sin \varphi_n .
\]

Using the already obtained relations $\psi_n = n(\pi - 2\varphi_n)$ and $\psi_{n+2} = n(\pi - 2\varphi_{n+2})$ one gets

\[
(2n+1) \varphi_{n+2} = n\pi + \frac{\pi}{2} - \alpha \Theta(\zeta) - \varphi_{n+1}
\]

\[
(2n+1) \varphi_{n} = n\pi + \frac{\pi}{2} + \alpha \Theta(\zeta) - \varphi_{n+1}
\]

and hence

\[
\varphi_{n+2} = \frac{\pi}{2} - \frac{\varphi_{n+1} + \alpha \Theta(\zeta)}{2n+1} \quad \text{and} \quad \varphi_{n} = \frac{\pi}{2} - \frac{\varphi_{n+1} - \alpha \Theta(\zeta)}{2n+1}
\]

which we substitute in the two equations for $\zeta$ and the angles and obtain

\[
\frac{1}{\rho} \begin{pmatrix} \Gamma_x^\alpha(\zeta) + c_\alpha \\ \Gamma_y^\alpha(\zeta) \end{pmatrix} \begin{pmatrix} -\cos[\varphi_{n+1} + \alpha \Theta(\zeta)] \\ \sin[\varphi_{n+1} + \alpha \Theta(\zeta)] \end{pmatrix} = \cos \left[\frac{\varphi_{n+1} + \alpha \Theta(\zeta)}{2n+1}\right]
\]

\[
\frac{1}{\rho} \begin{pmatrix} \Gamma_x^\alpha(\zeta) - c_\alpha \\ \Gamma_y^\alpha(\zeta) \end{pmatrix} \begin{pmatrix} \cos[\varphi_{n+1} - \alpha \Theta(\zeta)] \\ \sin[\varphi_{n+1} - \alpha \Theta(\zeta)] \end{pmatrix} = \cos \left[\frac{\varphi_{n+1} - \alpha \Theta(\zeta)}{2n+1}\right]
\]

These two equations determine $\zeta$ and $\varphi_{n+1}$ as functions of $\alpha$. All other angles and the free paths are then determined through $\zeta$ and $\varphi_{n+1}$.
Adding and subtracting these two equations yields
\[\frac{1}{\rho} \left( \Gamma_1^{\alpha}(\zeta) \right) \cdot \left( \sin \varphi_{n+1} \sin[\alpha \Theta(\zeta)] \right) = \cos \left[ \frac{\varphi_{n+1}}{2n+1} \right] \cos \left[ \frac{\alpha \Theta(\zeta)}{2n+1} \right] + \frac{c_\alpha}{\rho} \cos \varphi_{n+1} \cos[\alpha \Theta(\zeta)]\]
\[\frac{1}{\rho} \left( \Gamma_1^{\alpha}(\zeta) \right) \cdot \left( -\cos \varphi_{n+1} \cos[\alpha \Theta(\zeta)] \right) = -\sin \left[ \frac{\varphi_{n+1}}{2n+1} \right] \sin \left[ \frac{\alpha \Theta(\zeta)}{2n+1} \right] - \frac{c_\alpha}{\rho} \sin \varphi_{n+1} \sin[\alpha \Theta(\zeta)]\]
which we now can easily solve for \(\Gamma_1^{\alpha}\) and \(\Gamma_2^{\alpha}\) and obtain the claimed equations for \(\zeta\) and \(\varphi_{n+1}\). \(\square\)

3.3. Stability of the periodic orbits. Proposition 3.4 provides a quantitative description of the periodic orbits like the one depicted in Fig. 2. And since for these periodic orbits all but two reflections are off circular boundary components the analysis of their stability becomes manageable. The first step is to compute the monodromy matrix corresponding to such a periodic orbit.

**Proposition 3.5** (Monodromy matrix). For any \(n \geq 0\) the monodromy matrix \(M_\alpha\) of the \(4(n+1)\)-periodic orbit of Proposition 3.4 is given by
\[
M_\alpha = \begin{pmatrix}
1 & \tau_{n,n+1} & 0 & 0 \\
0 & 1 & \tau_{n+1,n+2} & 0 \\
\frac{2h(\zeta)}{\rho \cos \varphi_{n+1}} & -1 & 1 & \frac{2h(\zeta)}{\rho \cos \varphi_{n+1}} \\
\frac{2h(\zeta)}{\rho \cos \varphi_{n+1}} & -1 & 1 & \frac{2h(\zeta)}{\rho \cos \varphi_{n+1}}
\end{pmatrix} J_c(\varphi_{n+2}) \cdot J_c(\varphi_n)
\]

where \(J_c(\varphi)\) is given by
\[
J_c(\varphi) = \begin{pmatrix}
-1 & -4n & \frac{4n \rho \cos \varphi}{\rho} \\
\frac{4n \rho \cos \varphi}{\rho} & 1 & -1 - 4n
\end{pmatrix}
\]

and \(\zeta, \varphi_n, \varphi_{n+1}, \varphi_{n+2}, \tau_{n,n+1}, \tau_{n+1,n+2}\) are as in Proposition 3.4.

**Proof.** Along each of the two circular boundary components there are \(2n+1\) reflections.

If \(\varphi\) denotes the angle of reflection on one of the two circular boundary components, then
\[
J_c(\varphi) = \begin{pmatrix}
\frac{-1}{\rho \cos \varphi} & 0 & 0 \\
\frac{2}{\rho \cos \varphi} & 1 & -1
\end{pmatrix}^2
\]

is the corresponding linearization of the billiard flow starting right before the first reflection and ending right after the last reflection.
By inspection of Fig. 2 and Fig. 3 we see that the monodromy matrix of the periodic orbit is given by
\[ M_\alpha = \begin{pmatrix} 1 & \tau_{n,n+1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -\frac{\mathcal{K}(s_{n+1})}{\cos \varphi_{n+1}} & -1 \end{pmatrix} \begin{pmatrix} 1 & \tau_{n+1,n+2} \\ 0 & 1 \end{pmatrix} J_c(\varphi_{n+2}) \cdot \begin{pmatrix} 1 & \tau_{n+1,n+2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -\frac{\mathcal{K}(s_{n+1})}{\cos \varphi_{n+1}} & -1 \end{pmatrix} \begin{pmatrix} 1 & \tau_{n,n+1} \\ 0 & 1 \end{pmatrix} J_c(\varphi_{n}) \]
and since \( \mathcal{K}(s_{n+1}) = -\frac{1}{p} h(\zeta) \) we obtain the claimed expression for \( M_\alpha \).
\[ \square \]

Since the billiard flow is area preserving the eigenvalues of the monodromy matrix are related to its trace. Using the result of Proposition 3.5 for the explicit form of the monodromy matrix of the periodic orbit we arrive at the following stability criterion.

**Corollary 3.6** (Criterion for stability). *In the special case*

\[
\varphi_n = \varphi_{n+1} = \varphi_{n+2} = \frac{\pi}{4} \frac{2n + 1}{n + 1}, \quad \tau_{n,n+1} = \tau_{n+1,n+2} = 2 \rho \cos \varphi_{n+1}
\]

the result of Proposition 3.5 for the monodromy matrix \( M_\alpha \) implies that

\[
\left| \frac{1}{2} \operatorname{tr} M_\alpha \right| < 1 \iff \frac{n + \frac{3}{2}}{n + 1} < h(\zeta) < 1 \quad \text{and} \quad h(\zeta) \neq \frac{n + \frac{3}{2}}{n + 1}
\]
hold for all \( n \geq 0 \).

**Proof.** Denote \( 2 \rho \cos \varphi_{n+1} \) by \( \tau \). Then, under the stated assumptions on the angles and the free paths, the result of Proposition 3.5 for the monodromy matrix becomes

\[
M_\alpha = \left[ \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -\frac{h(\zeta)}{\rho \cos \varphi} & -1 \end{pmatrix} \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} J_c(\varphi) \right]^2
\]

\[
= \left[ \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -\frac{h(\zeta)}{\rho \cos \varphi} & -1 \end{pmatrix} \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -4n \rho \cos \varphi \\
2n + 1 \rho \cos \varphi & -1 - 4n \end{pmatrix} \right]^2
\]

\[
= \left[ \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -\frac{h(\zeta)}{\rho \cos \varphi} & -1 \end{pmatrix} \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -4n \rho \cos \varphi \\
4\frac{2n + 1}{\rho \cos \varphi} & -1 - 4n \end{pmatrix} \right]^2
\]

\[
= \left( -7 - 12n + 4(3 + 4n) h \quad 2\tau \left[ 1 + 3n - 2 (1 + 2n) h \right] \right)^2
\]

where \( h \) is short hand notation for \( h(\zeta) \). In particular,

\[
\operatorname{tr} M_\alpha = \left[ -7 - 12n + 4(3 + 4n) h + 1 + 4n - 4 (1 + 2n) h \right]^2 - 2
\]

\[
= 4 \left[ 3 + 4n - 4 (1 + n) h \right]^2 - 2
\]
follows for the trace of $M_\alpha$. Hence

$$
\left| \frac{1}{2} \text{tr } M_\alpha \right| < 1 \iff -1 < 2 \left[ -3 - 4n + 4(1 + n) h \right]^2 - 1 < 1
\iff 0 < \left| -3 - 4n + 4(1 + n) h \right| < 1
\iff -1 < -3 - 4n + 4(1 + n) h < 1, \quad h \neq \frac{n + \frac{3}{4}}{n + 1}
\iff \frac{n + \frac{3}{4}}{n + 1} < h < 1, \quad h \neq \frac{n + \frac{3}{4}}{n + 1},
$$

which finishes the proof. \hfill \square

3.4. **Existence and stability of certain periodic orbits for small smoothening regions.** The result of Lemma 3.2 showed that the arc length $s_\alpha$ of the smoothened boundary component is proportional to $\alpha$, provided that $h$ is kept fixed. Therefore, the asymptotics as $\alpha$ tends to zero correspond to $C^2$-stadium, which are $C^1$-close to the usual stadium.

In the usual stadium all periodic orbits are hyperbolic. The aim of this part of the paper is to show that for arbitrarily small smoothening of the stadium elliptic periodic orbits exist for sufficiently small separations $L$ of the two curved boundary components. And as outlined at the beginning of this section we will actually show the existence of elliptic periodic orbits for the smoothed stadium where $L = 0$, i.e. where the two curved boundary component are joined at their endpoints without additional straight line segments. A criterion for the existence of certain periodic orbits was established in Proposition 3.4.

Since we are interested in a small smoothening, i.e. the smoothened boundary component $\Gamma$ should differ from a semi-circle only on small segments near the endpoints we need to consider small values of $\alpha$. For sufficiently small $\alpha$ the following Proposition 3.7 provides an asymptotic description of the periodic orbits considered in Proposition 3.4.

**Proposition 3.7** (Existence of the orbit for small $\alpha$). Let $n \geq 0$ and a continuous function $h: [0, 1] \to [0, \infty)$ with $h(0) = 1$ and $h(1) = 0$ be fixed. Then a $4(n + 1)$-periodic orbit as in Proposition 3.4 exists for sufficiently small values of $\alpha$ if and only if

$$
\frac{2n + 1}{2n + 2} (1 - \zeta) = \int_\zeta^1 h(\xi) d\xi
$$

has a solution $0 < \zeta < 1$. In that case

$$
\zeta(\alpha) = \zeta_* + O(\alpha) \quad \text{and} \quad \varphi_{n+1}(\alpha) = \pi \frac{2n + 1}{4 \frac{n + 1}{n + 1}} + O(\alpha)
$$

holds uniformly for any solution and all sufficiently small values of $\alpha$.

**Proof.** Combining the explicit form of $\Gamma_\alpha$ as given in Corollary 3.3 with the result of Proposition 3.4 for the existence of the periodic orbit we see that a periodic orbit as shown in Fig. 2 exists with the $n + 1$-st reflection off the interior of the
smoothed part of $\Gamma$ if and only if the system of two equations

\[
\alpha \int_0^1 \frac{\cos[\alpha \Theta(\xi)]}{h(\xi)} d\xi = \frac{\cos \left[ \frac{\varphi_{n+1}}{2n+1} \right]}{\sin \varphi_{n+1}} \sin[\alpha \Theta(\zeta)] \cos \left[ \frac{\alpha \Theta(\zeta)}{2n+1} \right] \\
+ \left[ \alpha \int_0^1 \frac{\cos[\alpha \Theta(\xi)]}{h(\xi)} d\xi - \sin \alpha \right] \sin[\alpha \Theta(\zeta)] \cos[\alpha \Theta(\zeta)] \left[ \tan \varphi_{n+1} + \frac{1}{\tan \varphi_{n+1}} \right] \\
+ \frac{\sin \left[ \frac{\varphi_{n+1}}{2n+1} \right]}{\cos \varphi_{n+1}} \sin \left[ \frac{\alpha \Theta(\zeta)}{2n+1} \right] \cos[\alpha \Theta(\zeta)]
\]

and

\[
\cos \alpha + \alpha \int_0^1 \frac{\sin[\alpha \Theta(\xi)]}{h(\xi)} d\xi = \frac{\cos \left[ \frac{\varphi_{n+1}}{2n+1} \right]}{\cos \varphi_{n+1}} \sin[\alpha \Theta(\zeta)] \cos \left[ \frac{\alpha \Theta(\zeta)}{2n+1} \right] \cos[\alpha \Theta(\zeta)] \\
- \frac{\sin \left[ \frac{\varphi_{n+1}}{2n+1} \right]}{\cos \varphi_{n+1}} \sin \left[ \frac{\alpha \Theta(\zeta)}{2n+1} \right] \sin[\alpha \Theta(\zeta)] \\
+ \left[ \alpha \int_0^1 \frac{\cos[\alpha \Theta(\xi)]}{h(\xi)} d\xi - \sin \alpha \right] \left[ \cos \left[ \frac{\alpha \Theta(\zeta)}{2n+1} \right] \cos \left[ \frac{\alpha \Theta(\zeta)}{2n+1} \right] \tan \varphi_{n+1} \right] \\
\]

has a solution for $0 < \zeta < 1$ and $0 < \varphi_{n+1} < \frac{\pi}{2}$, where we used (1) to eliminate $c_\alpha$ in terms of $h$.

Since $h$ and $n$ are fixed these two equations become

\[
\frac{1 - \zeta}{\int_0^1 h(\xi) d\xi} = \frac{\cos \left[ \frac{\varphi_{n+1}}{2n+1} \right]}{\sin \varphi_{n+1}} \Theta(\zeta) + \frac{\sin \left[ \frac{\varphi_{n+1}}{2n+1} \right]}{\cos \varphi_{n+1}} \Theta(\zeta) \left[ \frac{\cos \varphi_{n+1}}{2n+1} \right] \Theta(\zeta) \\
+ \alpha \Theta(\zeta) \int_0^1 \frac{[1 - h(\xi)]}{h(\xi)} d\xi \left[ \tan \varphi_{n+1} + \frac{1}{\tan \varphi_{n+1}} \right] + \mathcal{O}(\alpha^2) \\
= \frac{\cos \left[ \frac{\varphi_{n+1}}{2n+1} \right]}{\sin \varphi_{n+1}} \Theta(\zeta) + \alpha \int_0^1 \frac{[1 - h(\xi)]}{h(\xi)} d\xi \frac{1}{\tan \varphi_{n+1}} + \mathcal{O}(\alpha^2)
\]

(2)

where the terms $\mathcal{O}(\alpha^2)$ are uniform in $\zeta$ as $\alpha$ tends to zero.

In the limit $\alpha \to 0$ these take on the form

\[
\frac{1 - \zeta}{\int_0^1 h(\xi) d\xi} = \frac{\cos \left[ \frac{\varphi_{n+1}}{2n+1} \right]}{\sin \varphi_{n+1}} \Theta(\zeta) + \frac{\sin \left[ \frac{\varphi_{n+1}}{2n+1} \right]}{\cos \varphi_{n+1}} \Theta(\zeta) \left[ \frac{\cos \varphi_{n+1}}{2n+1} \right] \Theta(\zeta), \quad 1 = \frac{\cos \left[ \frac{\varphi_{n+1}}{2n+1} \right]}{\sin \varphi_{n+1}}
\]

so that the second equation for $\varphi_{n+1}$ yields

\[
\varphi_{n+1}|_{\alpha=0} = \frac{\pi}{4} \frac{2n+1}{n+1}
\]

as the only solution. This implies that

\[
\sin \left[ \frac{\varphi_{n+1}}{2n+1} \right] = \sin \left[ \frac{\pi}{4} \frac{2n+1}{n+1} \right] = \cos \left[ \frac{\pi}{4} \frac{2n+1}{n+1} \right] = \cos \left[ \frac{\pi}{4} \frac{2n+1}{n+1} \right] = \cos \varphi_{n+1}
\]

holds at $\alpha = 0$. Hence the first of the above two equations becomes

\[
\frac{1 - \zeta}{\int_0^1 h(\xi) d\xi} = \frac{2n+2}{2n+1} \Theta(\zeta) \quad \text{or simply} \quad \frac{2n+1}{2n+2} (1 - \zeta) = \int_0^1 h(\xi) d\xi
\]

where we used the definition of $\Theta$ in the last step.
Because the error terms are uniform in $\zeta$, and because the implicit function theorem applies to (2) we see that (2) has a solution for sufficiently small values of $\alpha$ if and only if (3) has a solution for $0 < \zeta < 1$. □

The asymptotic description of the periodic orbits established in Proposition 3.7 corresponds precisely to the special case of the stability analysis presented in Corollary 3.6. Therefore we obtain the following existence and stability result for periodic orbits as in Fig. 2 for all sufficiently small smoothing regions.

**Theorem 3.8** (Asymptotic existence and stability). Let $n \geq 0$ and $h: [0, 1] \rightarrow [0, \infty)$ with $h(0) = 1$ and $h(1) = 0$ be fixed. For all sufficiently small values of $\alpha$ there exists a $4(n + 1)$ periodic orbit as in Proposition 3.4 if

$$\frac{2n + 1}{2n + 2} (1 - \zeta) = \int_{\zeta}^{1} h(\xi) \, d\xi$$

has a solution $0 < \zeta_* < 1$. Furthermore, this orbit is linearly stable if

$$\frac{n + \frac{1}{2}}{n + 1} < h(\zeta_*) < 1 \quad \text{and} \quad h(\zeta_*) \neq \frac{n + \frac{3}{2}}{n + 1}$$

holds.

Proof. Proposition 3.7 shows that a periodic orbit corresponding to some $\zeta(\alpha)$ and $\varphi_{n+1}(\alpha)$ is equivalent to the existence of a solution $0 < \zeta_* < 1$ to

$$\frac{2n + 1}{2n + 2} (1 - \zeta) = \int_{\zeta}^{1} h(\xi) \, d\xi$$

and in this case such a solution must satisfy

$$\zeta(\alpha) = \zeta_* + O(\alpha), \quad \varphi_{n+1}(\alpha) = \frac{\pi}{4} \frac{2n + 1}{n + 1} + O(\alpha)$$

uniformly in the choice of the solution $\zeta(\alpha)$ and $\varphi_{n+1}(\alpha)$.

This together with (the second part of) Proposition 3.4 implies that

$$K(s_{n+1}(\alpha)) = -\frac{1}{\rho} h(\zeta(\alpha)) = -\frac{1}{\rho} h(\zeta_*) + O(\alpha)$$

$$\varphi_n = \frac{\pi}{2} \frac{\varphi_{n+1}}{2n + 1} + O(\alpha) = \frac{\pi}{4} \frac{2n + 1}{n + 1} + O(\alpha)$$

$$\varphi_{n+2} = \frac{\pi}{2} \frac{\varphi_{n+1}}{2n + 1} + O(\alpha) = \frac{\pi}{4} \frac{2n + 1}{n + 1} + O(\alpha)$$

$$\frac{\tau_{n+1,n+2}}{\rho} = \cos \varphi_{n+1} + \cos \varphi_{n+2} + O(\alpha) = 2 \cos \left[ \frac{\pi}{4} \frac{2n + 1}{n + 1} \right] + O(\alpha)$$

$$\frac{\tau_{n,n+1}}{\rho} = \cos \varphi_{n+1} + \cos \varphi_{n} + O(\alpha) = 2 \cos \left[ \frac{\pi}{4} \frac{2n + 1}{n + 1} \right] + O(\alpha)$$

hold uniformly.

In particular, the periodic orbit is linearly stable if and only if $|\frac{1}{2} \text{tr} M_\alpha| < 1$. Due to the uniformity of the error terms we can apply Corollary 3.6 and obtain

$$\left| \frac{1}{2} \text{tr} M_\alpha \right| < 1 \iff \frac{n + \frac{1}{2}}{n + 1} < h(\zeta_*) < 1 \quad \text{and} \quad h(\zeta_*) \neq \frac{n + \frac{3}{2}}{n + 1}$$

for the trace of the monodromy matrix $M_\alpha$ for all sufficiently small values of $\alpha$. □
3.5. **Sufficient conditions for the existence of elliptic orbits.** While the result of Theorem 3.8 characterizes the existence of certain periodic orbits and establishes their stability, it does not say anything about how to choose the smoothening, i.e. the function \( h \), in order to obtain elliptic periodic orbits. Therefore, to prove our main result Theorem 2.1 we still need to describe the smoothenings, that result in billiard tables which have elliptic orbits. From the result of Theorem 3.8 we already know that we need to study equations of the form \( \frac{2n+1}{2} (1 - \zeta) = \int_{\zeta}^{1} h(\xi) \, d\xi \). The following Lemma 3.9, Corollary 3.10, and Corollary 3.11 describe solutions to this equation. To simplify notation, for a given continuous function \( h : [0,1] \to \mathbb{R} \) we denote by \( H(\zeta) \) the function

\[
H(\zeta) = \int_{0}^{1} h(t\zeta + 1 - t) \, dt .
\]

**Lemma 3.9** (Auxiliary monotonicity result). If \( h \) is non-increasing with \( h(0) = 1 \) and \( h(1) = 0 \), then

1. The function \( H \) is continuous and non-increasing with \( H(0) > 0 \) and \( H(1) = 0 \).
2. If \( \zeta \in [0,1] \) is such that \( H(\zeta) = H(0) \), then \( h(\xi) = 1 \) for all \( \xi \in [0,\zeta] \).
3. If \( \zeta \in [0,1] \) is such that \( H(\zeta) = H(1) \), then \( h(\xi) = 0 \) for all \( \xi \in [\zeta,1] \).
4. For all \( \zeta \in [0,1] \) the inequality \( H(\zeta) \leq h(\zeta) \) holds, where equality holds if and only if \( H(\zeta) = 0 \).

**Proof.** Let \( \zeta_1 \leq \zeta_2 \) be arbitrary. Then for any \( t \in [0,1] \) we have \( t\zeta_1 + 1 - t \leq t\zeta_2 + 1 - t \), so that the monotonicity of \( h \) implies

\[
H(\zeta_1) = \int_{0}^{1} h(t\zeta_1 + 1 - t) \, dt \geq \int_{0}^{1} h(t\zeta_2 + 1 - t) \, dt = H(\zeta_2) ,
\]

i.e. \( H \) is non-increasing. And \( H(0) > 0 \) and \( H(1) = \int_{0}^{1} h(1) \, dt = h(1) = 0 \) follows from the assumption on \( h \).

Furthermore, suppose that \( H(\zeta) = H(0) \) for some \( \zeta \in [0,1] \). Then

\[
0 = H(0) - H(\zeta) = \int_{0}^{1} [h(1-t) - h(t\zeta + 1 - t)] \, dt
\]

and the monotonicity of \( h \) imply that \( h(1-t) = h(t\zeta + 1 - t) \) must hold for all \( 0 \leq t \leq 1 \). In particular, for \( t = 1 \) this becomes \( h(0) = h(\zeta) \), and therefore \( h(\xi) = h(0) = 1 \) for all \( \xi \in [0,\zeta] \) follows from the monotonicity of \( h \).

Similarly, suppose that \( H(\zeta) = H(1) \) for some \( \zeta \in [0,1] \). Then

\[
0 = H(\zeta) - H(1) = \int_{0}^{1} [h(t\zeta + 1 - t) - h(1)] \, dt
\]

and the monotonicity of \( h \) imply that \( h(t\zeta + 1 - t) = h(1) = 0 \) must hold for all \( 0 \leq t \leq 1 \), i.e. \( h(\xi) = 0 \) for all \( \xi \in [\zeta,1] \).

Finally, the inequality

\[
H(\zeta) = \int_{0}^{1} h(t\zeta + 1 - t) \, dt = h(\zeta) - \int_{0}^{1} [h(\zeta) - h(t\zeta + 1 - t)] \, dt \leq h(\zeta)
\]

holds for all \( \zeta \), because \( h \) is non-increasing. Hence the equality \( H(\zeta) = h(\zeta) \) holds if and only if \( h(\xi) = h(1) = 0 \) for all \( \zeta \leq \xi \leq 1 \), i.e. if and only if \( H(\zeta) = 0 \). \( \square \)

**Corollary 3.10.** Let \( h \) as in Lemma 3.9 and let \( \beta \) be a real number.
The equation $\beta (1 - \zeta) = \int_{\zeta}^{1} h(\xi) d\xi$ always has the solution $\zeta = 1$, regardless of the value of $\beta$.

(2) The equation $\beta (1 - \zeta) = \int_{\zeta}^{1} h(\xi) d\xi$ has a solution $\zeta \in [0, 1)$ if and only if $\zeta$ is a solution to $\beta = H(\zeta)$.

(3) The equation $\beta = H(\zeta)$ has a solution $\zeta \in [0, 1]$ if and only if $0 \leq \beta \leq \int_{0}^{1} h(\xi) d\xi$.

Proof. Clearly, the special value $\zeta = 1$ is always a solution to $\beta (1 - \zeta) = \int_{\zeta}^{1} h(\xi) d\xi$. And for $\zeta \neq 1$, the change of variables $t \mapsto \xi = t \zeta + 1 - t$ shows that the equation $\beta (1 - \zeta) = \int_{\zeta}^{1} h(\xi) d\xi$ is equivalent to $\beta = \int_{0}^{1} h(t \zeta + 1 - t) dt$, which is nothing else but $\beta = H(\zeta)$.

By Lemma 3.9, we know that $H$ is continuous and non-increasing. Therefore $\beta = H(\zeta)$ has a solution $\zeta \in [0, 1]$ if and only if $H(1) = 0 \leq \beta \leq H(0) = \int_{0}^{1} h(\xi) d\xi$ holds.

Corollary 3.11. Let $h$ be as in Lemma 3.9 and suppose that $h$ satisfies the additional assumption $h(\zeta) < h(0)$ for all $\zeta > 0$. If $0 < \beta < \int_{0}^{1} h(\xi) d\xi$, then the solution $\zeta_\beta$ to the equation $\beta = H(\zeta)$ satisfies $0 < \zeta_\beta < 1$ and $\beta < h(\zeta_\beta) < 1$.

Proof. The existence of a solution $\zeta_\beta$ to $\beta = H(\zeta)$ was already established in Corollary 3.10. Since $H(\zeta_\beta) = \beta \neq 0 = H(1)$, the result of Lemma 3.9 shows that $\beta = H(\zeta_\beta) < h(\zeta_\beta)$ holds. In particular it follows that the monotonicity of $h$ implies $\zeta_\beta < 1$, because $h(1) = 0$ and $0 < \beta < h(\zeta_\beta)$.

On the other hand, since $H(\zeta_\beta) = \beta < H(0)$ the monotonicity of $H$, recall Lemma 3.9 implies that $\zeta_\beta > 0$. The assumed property on $h$ that $h$ is strictly decreasing at $\zeta = 0$ then implies that $h(\zeta_\beta) < h(0) = 1$, which finishes the proof.

In order to proceed we need to introduce a certain non-degeneracy condition for $h$. In the following Theorem 3.12 we will consider continuous non-increasing functions $h : [0, 1] \to \mathbb{R}$ that satisfy the non-degeneracy condition

there exists an integer $n_\ast \geq 0$ such that

$$\frac{2n_\ast + 1}{2n_\ast + 2} < \int_{0}^{1} h(\xi) d\xi$$

(C)

and for all $\zeta$ with

$$\frac{2n_\ast + 1}{2n_\ast + 2} = \int_{\zeta}^{1} h(\xi) d\xi$$

we have $h(\zeta) \neq \frac{2n_\ast + \frac{1}{2}}{2n_\ast + 2}$.

Using the results of Lemma 3.9, Corollary 3.10 and Corollary 3.11 we obtain the following sufficient condition for the existence of elliptic periodic orbits.

Theorem 3.12 (Sufficient condition for the existence of elliptic periodic orbits). Suppose that $h : [0, 1] \to \mathbb{R}$ is continuous non-increasing with

$$h(0) = 1, \quad h(1) = 0, \quad \frac{1}{2} < \int_{0}^{1} h(\xi) d\xi, \quad h(\zeta) < h(0) \quad \text{for all } \zeta > 0.$$  

Suppose further that $h$ satisfies the non-degeneracy condition (C). Then there exists a linearly stable 4 ($n_\ast + 1$)-periodic orbit for all sufficiently small values of $\alpha$.

Proof. By Corollary 3.11 we know that under the stated assumptions on $h$ there exists a solution $\zeta_\ast$ to $H(\zeta_\ast) = \frac{2n_\ast + 1}{2n_\ast + 2}$, which satisfies

$$0 < \zeta_\ast < 1 \quad \text{and} \quad \frac{n_\ast + \frac{1}{2}}{n_\ast + 1} < h(\zeta_\ast) < 1.$$
Therefore, Theorem 3.8 proves that there exists a linearly stable \( (n + 1) \)-periodic orbit (as shown in Fig. 2 with one reflection off the smooth boundary component \( \Gamma \)) for all sufficiently small values of \( \alpha \).

**Proof of Theorem 2.1.** At this point we would like to make a few comments on Theorem 3.12 and at the same time prove the announced Theorem 2.1.

First of all, Theorem 3.8 provides a characterization of the existence and stability of certain periodic orbits for small smoothening regions for fixed \( h \).

Under the assumption that \( h \) is non-increasing, that is to say that the curvature of the smoothened part of \( \Gamma \) tends to zero monotonically, the result of Corollary 3.11 shows the following. If there exists a solution to the equation for \( \zeta \) in Theorem 3.8, then it automatically satisfies the stability criterion of Theorem 3.8. In other words, as long as the curvature of the smoothened part of \( \Gamma \) is monotone, the existence of a periodic orbit as shown in Fig. 2 automatically implies its ellipticity. The condition \( \frac{1}{2} < \int_0^1 h(\xi) \, d\xi \) on \( h \) in Theorem 3.12 then guarantees that there exists such a periodic orbit.

Secondly, recall that by Lemma 3.2 the arc length of the smoothened part of the boundary, denoted by \( s_\alpha \), satisfies

\[
\frac{\alpha \rho}{s_\alpha} = \int_0^1 h(\xi) \, d\xi
\]

for all \( \alpha > 0 \). Note further that \( s_\alpha^{\text{circle}} = \alpha \rho \) is precisely the arc length of the piece of the circular boundary component that got replaced by the smoothened part.

Therefore the condition \( \frac{1}{2} < \int_0^1 h(\xi) \, d\xi \) in Theorem 3.12 can be recast as

\[
\int_0^1 h(\xi) \, d\xi = \frac{s_\alpha^{\text{circle}}}{s_\alpha} > \frac{1}{2} \quad \text{i.e.} \quad s_\alpha < 2 s_\alpha^{\text{circle}}.
\]

Therefore, the meaning of the result of Theorem 3.12 is that if the curvature monotonically decreases to zero so that the smoothened part of \( \Gamma \) is less than twice the length of the original circular segment it replaces, then there exists an elliptic periodic orbit as is shown in Fig. 2.

Furthermore, if the smoothened segment is larger than twice the corresponding circular part (the curvature is still non-increasing), then the corresponding periodic orbits still exists (see Lemma 3.1), but have no reflection off the smoothened part. Hence they are unstable, as they are also present in the usual stadium.

Finally, as was already mentioned at the very beginning of this section, since elliptic periodic orbits persist as elliptic periodic orbits for small enough separations of the two curved boundary components we obtain Theorem 2.1.

**4. Elliptic Periodic Orbits for Large Separations**

In order to find stable periodic orbits for large separations of the two curved boundary components we exploit the fact that the curved components are not absolutely focusing. In particular, the first step will be to find parabolic periodic orbits. This will be accomplished by constructing parts of billiard trajectories that correspond to a complete sequence of reflections off the curved boundary segment such that an infinitesimal parallel beam falling onto a focusing component will component being again a parallel beam.
It will be shown that every such part of a billiard trajectory gives rise to a parabolic periodic orbit on a sequence of stadium-like billiard tables. Then a perturbation argument is applied to establish existence of elliptic periodic orbits for large set of separation distances of the two curved boundary segments.

Finally, the nonlinear stability of these orbits will be established by computing the corresponding Birkhoff. To guarantee the $C^4$-smoothness of the billiard map, which will be used in the nonlinear stability analysis, we make throughout this section the standing assumption that the curved boundary component $\Gamma$ is of class $C^5$. However, globally the boundary of the billiard table is, generally, only of class $C^2$, because we do not assume that also the derivative of the curvature vanishes at the endpoints of $\Gamma$.

4.1. Construction of symmetric periodic orbits and corresponding return maps. The first step in the construction of stable periodic orbits is to use the symmetry of the smoothed out circular boundary component $\Gamma$ about the horizontal axis to construct a special part of a billiard trajectory. This serves as a building block in the construction of parabolic periodic orbits.

The actual construction of these special trajectory pieces is not particularly complicated. However, we prefer to postpone the description of this construction to Section 4.5 for two reasons. Firstly, the particular details are not relevant for our results on the existence of nonlinearly stable periodic orbits, and so we prefer to first provide those general results. In particular, only after these general results on nonlinear stability we know what we need to verify in specific examples. And this is the second reason for postponing the details of the construction of these special trajectory pieces, so that we can verify at once all conditions we need for the existence and nonlinear stability results.

Therefore, instead of showing the existence of certain trajectory segments we make the following assumption on the smoothed-out curved boundary component $\Gamma$:

**Assumption 4.1.** There exists a complete sequence of reflections off of $\Gamma$ that is symmetric about the horizontal axis. Furthermore, denote the initial pre-collisional coordinates by $\hat{s}_0, \hat{\omega}_0$ and final post-collisional coordinates by $\hat{s}_1, \hat{\omega}_1$, then $\frac{3}{2} \pi < \hat{\omega}_0 < 2\pi$, $\hat{\omega}_1 = 3\pi - \hat{\omega}_0$, and the linearization of the corresponding billiard flow (in terms of the usual Jacobi coordinates) is given by $\begin{pmatrix} -1 & \beta \\ 0 & -1 \end{pmatrix}$ with $\beta \neq 0$.

In Section 4.5 we show that as long as the smoothed-out portion of $\Gamma$ is short enough, i.e. as long as $\Gamma$ is a circle segment except for short segments near its endpoints, $\Gamma$ satisfies Assumption 4.1.

In the following we adopt the following convention: Given a complete sequence of reflections off of $\Gamma$, let $s_0$ denote the arc length parameter of the first reflection and let $\omega_0$ denote the angle corresponding to the pre-collisional flow direction $(\cos \omega_0, \sin \omega_0)$. Similarly, let $s_1$ denote the arc length parameter corresponding to last point of reflection and let $\omega_1$ denote the angle corresponding to the post-collisional flow direction.
The next step in the construction of stable periodic orbits is to use specialized coordinates \((x_0, y_0)\) and \((x_1, y_1)\) in a neighborhood of \((\hat{s}_0, \hat{\omega}_0)\) and \((\hat{s}_1, \hat{\omega}_1)\), respectively,

\[
\begin{align*}
x_0 &= [\Gamma(s_0) - \Gamma(\hat{s}_0)] \cdot \begin{pmatrix} \cos \omega_0 \\ \sin \omega_0 \end{pmatrix}^\perp, \quad y_0 = \omega_0 - \hat{\omega}_0, \\
x_1 &= [\Gamma(s_1) - \Gamma(\hat{s}_1)] \cdot \begin{pmatrix} \cos \omega_1 \\ \sin \omega_1 \end{pmatrix}^\perp, \quad y_1 = \omega_1 - \hat{\omega}_1,
\end{align*}
\]

where we make use of the notation \(\perp\)

\[
\begin{pmatrix} a \\ b \end{pmatrix}^\perp = \begin{pmatrix} -b \\ a \end{pmatrix}
\]

for any two \(a, b \in \mathbb{R}\),

which will be used throughout the rest of the paper. These coordinates are adapted to the billiard flow, and more details are given Section A. In terms of these coordinates the billiard map \((x_0, y_0) \mapsto (x_1, y_1)\) near the reference orbit \((\hat{s}_0, \hat{\omega}_0) \mapsto (\hat{s}_1, \hat{\omega}_1)\) can be expressed as

\[
(x_1, y_1) = F(x_0, y_0),
\]

which is a well-defined \(C^4\) map in a neighborhood of \((0, 0)\) and satisfies

\[
F(0, 0) = (0, 0), \quad DF(0, 0) = \begin{pmatrix} -1 & a_{01} \\ 0 & -1 \end{pmatrix} \quad \text{with} \quad a_{01} \neq 0.
\]

In fact, by construction \(F\) is area and orientation preserving, i.e.

\[
\det DF(x_0, y_0) = 1
\]

for all \((x_0, y_0)\). Furthermore, the symmetry of the boundary component \(\Gamma\) and the symmetry of the reference orbit \((\hat{s}_0, \hat{\omega}_0) \mapsto (\hat{s}_1, \hat{\omega}_1)\) ensure that \(F\) has the following symmetry property

\[
J \circ F \circ J \circ F(x_0, y_0) = (x_0, y_0) \quad \text{where} \quad J(x_0, y_0) = (x_0, -y_0)
\]
for all \((x_0, y_0)\).

**Lemma 4.2.** In some neighborhood of 0 there exists a unique function \(x = f(y)\), which satisfies

\[
F(f(y), y) = (f(y), -y), \quad f(0) = 0, \quad f'(0) = \frac{1}{2} a_{01} \neq 0
\]

for all \(y\) in the domain of \(f\).

**Proof.** By the implicit function theorem there exists a neighborhood \(U\) of 0 and a uniquely determined invertible function \(f\) defined on \(U\) such that

\[
f(y) = F_x(f(y), y) \quad \text{with} \quad f(0) = 0, \quad f'(0) = \frac{1}{2} a_{01} \neq 0
\]

for all \(y\) in \(U\). The symmetry condition (8) of \(F\) then implies

\[
(f(y), y) = J \circ F \circ J \circ F(f(y), y).
\]

By definition of \(J\) we have

\[
J \circ F \circ J \circ F(f(y), y) = J \circ F(f(y), -F_y(f(y), y))
\]

Hence the symmetry property of \(F\) takes on the form

\[
f(y) = F_x(f(y), -F_y(f(y), y)), \quad y = -F_y(f(y), -F_y(f(y), y)).
\]

The local uniqueness of \(y \rightarrow f(y)\), i.e.

\[
x = F_x(x, y) \iff x = f(y)
\]

implies

\[
f(y) = f(-F_y(f(y), y))
\]

Since \(f\) is invertible we get

\[
y = -F_y(f(y), y),
\]

which means

\[
F(f(y), y) = (f(y), -y).
\]

\(\square\)

Now we are in the position to complete the construction of families of certain symmetric periodic orbits. Given \(\bar{y}_0\) sufficiently close to 0 we choose \(\bar{x}_0 = f(\bar{y}_0)\).

By Lemma 4.2 the image \((\bar{x}_1, \bar{y}_1) = F(\bar{x}_0, \bar{y}_0)\) satisfies

\[
(\bar{x}_1, \bar{y}_1) = (\bar{x}_0, -\bar{y}_0).
\]

Let \((\bar{s}_0, \bar{\omega}_0)\) and \((\bar{s}_1, \bar{\omega}_1)\) denote the arc length and direction angle corresponding to \((\bar{x}_0, \bar{y}_0)\) and \((\bar{x}_1, \bar{y}_1)\), respectively. In particular, for \((\bar{x}_0, \bar{y}_0) = (0, 0)\) we have \((\bar{s}_0, \bar{\omega}_0) = (\bar{s}_0, \bar{\omega}_0)\) and \((\bar{s}_1, \bar{\omega}_1) = (\bar{s}_1, \bar{\omega}_1)\).

Upon unfolding the billiard table in the vertical direction (i.e. about the parallel walls) it is easily seen that the additional reflections inside the parallel channel correspond to a free flight on the unfolded table. This construction is illustrated by Fig. 5. In order for \((\bar{x}_0, \bar{y}_0)\) and \((\bar{x}_1, \bar{y}_1)\) to belong to a periodic orbit for any integer \(n = 1, 2, \ldots\) the separation distance \(\bar{L}_n\) of the two curved boundary segments must be a specially chosen value. Before getting to the expression for \(\bar{L}_n\) note that due to the symmetry of the billiard table about the vertical center axis the free flight \((s_1, \omega_1) \rightarrow (s_2, \omega_2)\) across the channel of length \(L\) connecting the two curved boundary components can be expressed as a map from the post-collisional state \((s_1, \omega_1)\) (after the last in a sequence of reflection off of \(\Gamma\)) to the pre-collisional state
Figure 5. The solid black lines in Fig. 5a indicate the boundary of the “folded” billiard table. The construction of a periodic orbit by closing up a complete symmetric sequence of reflections off the curved boundary component $\Gamma$ is then obtained by unfolding the billiard in the horizontal and vertical direction, which is indicated by the dashed lines.

$(s_2, \omega_2)$ of the first in a sequence of reflection off of the same boundary component $\Gamma$

$$\Gamma(s_1) + \tau \begin{pmatrix} \cos \omega_1 \\ \sin \omega_1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \Gamma(s_2) - \begin{pmatrix} L \\ nW \end{pmatrix}, \quad \omega_2 = 3\pi - \omega_1,$$

where $\tau$ is the length of the free flight. In terms of the local flow coordinates $\Gamma$ corresponding to $(s_1, \omega_1)$, $(s_2, \omega_2)$ the expression (11) for the free flight becomes

$$-x_2 = x_1 + \left[ 2 \Gamma(\hat{s}_1) + \left( \frac{L}{nW} \right) \right] \cdot \begin{pmatrix} \cos(\hat{\omega}_1 + \hat{y}_1) \\ \sin(\hat{\omega}_1 + \hat{y}_1) \end{pmatrix}, \quad -y_2 = y_1.$$

Lemma 4.3. The part of a billiard trajectory corresponding to $(\bar{x}_0, \bar{y}_0) \mapsto (\bar{x}_1, \bar{y}_1)$ is part of a periodic orbit if and only if the separation distance $L$ of the two curved boundary segments has the form

$$L \equiv L_n(\bar{y}_0) = \frac{2 f(\bar{y}_0) + n W \cos(\bar{\omega}_1 - \bar{y}_0) + 2 \Gamma(\bar{s}_1) \cdot \begin{pmatrix} \cos(\bar{\omega}_1 - \bar{y}_0) \\ \sin(\bar{\omega}_1 - \bar{y}_0) \end{pmatrix}}{\sin(\bar{\omega}_1 - \bar{y}_0)}.$$

for any $n = 0, 1, \ldots$.

Proof. By (10) we have $(\bar{x}_1, \bar{y}_1) = (\bar{x}_0, -\bar{y}_0)$. Therefore, it follows from (12) that $(\bar{x}_0, \bar{y}_0) = (\bar{x}_2, \bar{y}_2)$ holds if and only if

$$-\bar{x}_0 = \bar{x}_0 + \left[ 2 \Gamma(\hat{s}_1) + \left( \frac{L}{nW} \right) \right] \cdot \begin{pmatrix} \cos(\hat{\omega}_1 - \hat{y}_0) \\ \sin(\hat{\omega}_1 - \hat{y}_0) \end{pmatrix}.$$
Solving for \( L \) gives

\[
L = \frac{2\vec{x}_0 + nW \cos(\hat{\omega}_1 - \vec{y}_0) + 2\Gamma(\hat{s}_1) \cdot \left( \frac{\cos(\hat{\omega}_1 - \vec{y}_0)}{\sin(\hat{\omega}_1 - \vec{y}_0)} \right)}{\sin(\hat{\omega}_1 - \vec{y}_0)}.
\]

Finally, recall that \( \vec{x}_0 = f(\vec{y}_0) \) which proves the claim. \( \square \)

For tables with separation distance equal to \( L_n(\vec{y}_0) \) we thus have a periodic orbit corresponding to \( \vec{y}_0 \), whose stability we will investigate in the remaining part of this section. For this purpose we introduce local coordinates in a neighborhood of \( \vec{x}_0, \vec{y}_0 \) and of \( \vec{x}_1, \vec{y}_1 \) by

\[
\delta x_0 = x_0 - \vec{x}_0, \quad \delta y_0 = y_0 - \vec{y}_0, \quad \delta x_1 = x_1 - \vec{x}_1, \quad \delta y_1 = y_1 - \vec{y}_1.
\]

**Proposition 4.4.** For any \( n = 0, 1, \ldots \), on a table with separation distance equal to \( L_n(\vec{y}_0) \) the free flight map has the form

\[
-\delta x_2 = \delta x_1 + \left[ 2 f(\vec{y}_0) \left( \sin \delta y_1 \frac{\sin \delta y_1}{1 + \cos \delta y_1} - \cot(\hat{\omega}_1 - \vec{y}_0) \right) - \frac{2 \Gamma_y(\hat{s}_1) + nW}{\sin(\hat{\omega}_1 - \vec{y}_0)} \right] \sin \delta y_1
\]

\[
-\delta y_2 = \delta y_1
\]

in a neighborhood of \( \vec{y}_0 \).

**Proof.** For \( L = L_n(\vec{y}_0) \) the expression (12) for the free flight map becomes

\[
-x_2 = x_1 + \left[ 2 \Gamma(\hat{s}_1) + \left( \frac{L_n(\vec{y}_0)}{nW} \right) \right] \cdot \left( \frac{\cos(\hat{\omega}_1 + y_1)}{\sin(\hat{\omega}_1 + y_1)} \right), \quad -y_2 = y_1.
\]

Using the definition of the local coordinates near the periodic orbit it then follows that

\[
-\delta y_2 = -[y_2 - \vec{y}_0] = [y_2 - \vec{y}_0] = y_1 + \vec{y}_0 = y_1 - \vec{y}_1 = \delta y_1
\]

and

\[
-\delta x_2 = -[x_2 - \vec{x}_2]
\]

\[
= x_1 + \left[ 2 \Gamma(\hat{s}_1) + \left( \frac{L_n(\vec{y}_0)}{nW} \right) \right] \cdot \left( \frac{\cos(\hat{\omega}_1 + \vec{y}_1 + \delta y_1)}{\sin(\hat{\omega}_1 + \vec{y}_1 + \delta y_1)} \right)
\]

\[
- \vec{x}_1 - \left[ 2 \Gamma(\hat{s}_1) + \left( \frac{L_n(\vec{y}_0)}{nW} \right) \right] \cdot \left( \frac{\cos(\hat{\omega}_1 + \vec{y}_1)}{\sin(\hat{\omega}_1 + \vec{y}_1)} \right)
\]

\[
= \delta x_1 + \left[ 2 \Gamma(\hat{s}_1) + \left( \frac{L_n(\vec{y}_0)}{nW} \right) \right] \cdot \left( \frac{\cos(\hat{\omega}_1 + \vec{y}_1 + \delta y_1) - \cos(\hat{\omega}_1 + \vec{y}_1)}{\sin(\hat{\omega}_1 + \vec{y}_1 + \delta y_1) - \sin(\hat{\omega}_1 + \vec{y}_1)} \right) \cdot \sin \delta y_1.
\]

By making use of

\[
\left( \frac{\cos(\hat{\omega}_1 + \vec{y}_1 + \delta y_1) - \cos(\hat{\omega}_1 + \vec{y}_1)}{\sin(\hat{\omega}_1 + \vec{y}_1 + \delta y_1) - \sin(\hat{\omega}_1 + \vec{y}_1)} \right) \cdot \sin \delta y_1 = - \left( \frac{\cos(\hat{\omega}_1 + \vec{y}_1)}{\sin(\hat{\omega}_1 + \vec{y}_1)} \right) \left[ 1 - \cos \delta y_1 \right]
\]

\[
- \left( \frac{\cos(\hat{\omega}_1 + \vec{y}_1)}{\sin(\hat{\omega}_1 + \vec{y}_1)} \right) \sin \delta y_1
\]
we get

\[-\delta x_2 = \delta x_1 - [1 - \cos \delta y_1] \left[ 2 \Gamma(\hat{s}_1) + \left( \frac{L_n(\bar{y}_0)}{nW} \right) \right] \cdot \left( \frac{\cos(\hat{\omega}_1 + \bar{y}_1)}{\sin(\hat{\omega}_1 + \bar{y}_1)} \right) \]

\[-\sin \delta y_1 \left[ 2 \Gamma(\hat{s}_1) + \left( \frac{L_n(\bar{y}_0)}{nW} \right) \right] \cdot \left( \frac{\cos(\hat{\omega}_1 + \bar{y}_1)}{\sin(\hat{\omega}_1 + \bar{y}_1)} \right) \]

\[= \delta x_1 - [1 - \cos \delta y_1] [-\bar{x}_2 - \bar{x}_1] - \sin \delta y_1 \left[ 2 \Gamma(\hat{s}_1) + \left( \frac{L_n(\bar{y}_0)}{nW} \right) \right] \cdot \left( \frac{\cos(\hat{\omega}_1 + \bar{y}_1)}{\sin(\hat{\omega}_1 + \bar{y}_1)} \right) \]

\[= \delta x_1 + 2 [1 - \cos \delta y_1] \bar{x}_0 - \sin \delta y_1 \left[ 2 \Gamma_x(\hat{s}_1) + L_n(\bar{y}_0) \right] \cos(\hat{\omega}_1 - \bar{y}_0) + \left[ 2 \Gamma_y(\hat{s}_1) + nW \right] \sin(\hat{\omega}_1 - \bar{y}_0) \]

Using the expression for $L_n(\bar{y}_0)$ given in Lemma 4.3 we conclude that

\[-\delta x_2 = \delta x_1 + 2 f(\bar{y}_0) \left[ \frac{1 - \cos \delta y_1}{\sin \delta y_1} - \frac{\cos(\hat{\omega}_1 - \bar{y}_0)}{\sin(\hat{\omega}_1 - \bar{y}_0)} \right] \sin \delta y_1 - \frac{2 \Gamma_y(\hat{s}_1) + nW}{\sin(\hat{\omega}_1 - \bar{y}_0)} \sin \delta y_1 , \]

which finishes the proof. \(\square\)

The result of Proposition 4.4 is an expression for the free flight across the table with separation distance equal to $L_n(\bar{y}_0)$ in a neighborhood of the periodic orbit corresponding to $\bar{y}_0$. This provides the mapping $(\delta x_1, \delta y_1) \mapsto (\delta x_2, \delta y_2)$, which maps a neighborhood of $(\bar{x}_1, \bar{y}_1)$ to some neighborhood of $(\bar{x}_0, \bar{y}_0)$. Introduce the map $(\delta x_1, \delta y_1) = F_{\bar{y}_0}(\delta x_0, \delta y_0)$, defined in some neighborhood of $(0,0)$, as

\[(14) \quad F_{\bar{y}_0}(\delta x_0, \delta y_0) = F(f(\bar{y}_0) + \delta x_0, \bar{y}_0 + \delta y_0) - F(f(\bar{y}_0), \bar{y}_0). \]

Therefore, composing the map $F_{\bar{y}_0}$, defined in (14), with the local expression of the free flight provided in Proposition 4.4 we obtain the first return map $T_{\bar{y}_0}$ on the table with separation distance equal to $L_n(\bar{y}_0)$ along the periodic orbit corresponding to $\bar{y}_0$. Indeed, the explicit expression for $(\delta x_2, \delta y_2) = T_{\bar{y}_0}(\delta x_0, \delta y_0)$ readily follows from the above as

\[(15) \quad (\delta x_1, \delta y_1) = F_{\bar{y}_0}(\delta x_0, \delta y_0) \]

\[-\delta x_2 = \delta x_1 + 2 f(\bar{y}_0) \left[ \frac{\sin \delta y_1}{1 + \cos \delta y_1} - \cot(\hat{\omega}_1 - \bar{y}_0) \right] - \frac{2 \Gamma_y(\hat{s}_1) + nW}{\sin(\hat{\omega}_1 - \bar{y}_0)} \sin \delta y_1 \]

\[-\delta y_2 = \delta y_1 , \]

which is well defined in a neighborhood of $(0,0)$ (which corresponds to a neighborhood of $(x_0, y_0)$ in terms of $(x, y)$-coordinates). We finish this section by recording that the above construction proves:

**Lemma 4.5.** The first return map $T_{\bar{y}_0}$ is an orientation and area preserving map satisfying $T_{\bar{y}_0}(0,0) = (0,0)$.

### 4.2. Linear stability analysis.

In view of Lemma 4.5 the area preservation property of $T_{\bar{y}_0}$ implies that the periodic orbit corresponding to $\bar{y}_0$ on the table with separation distance equal to $L_n(\bar{y}_0)$ is linearly stable if

\[(16) \quad |\text{tr} \, DT_{\bar{y}_0}(0,0)| < 2. \]
It follows immediately from the definition of $T_{\tilde{y}_0}$ that
\[\text{tr } DT_{\tilde{y}_0}(0,0) = -\text{tr } DF(f(\tilde{y}_0),\tilde{y}_0) + \frac{2f(\tilde{y}_0) \cos(\omega_1 - \tilde{y}_0) + 2\Gamma_y(s_1) + nW\frac{\partial F_y}{\partial x}(f(\tilde{y}_0),\tilde{y}_0)}{\sin(\omega_1 - \tilde{y}_0)} .\]
\[(17)\]
In particular, for the periodic orbit to be stable we need the value of the expressions
\[nW\frac{\partial F_y}{\partial x}(f(\tilde{y}_0),\tilde{y}_0)\]
to be not too large in absolute value. Hence, for large values of $n$ it follows that
\[|\tilde{y}_0| \ll 1\]
must hold as $n \to \infty$ provided that
\[nW\frac{\partial F_y}{\partial x}(f(\tilde{y}_0),\tilde{y}_0) = O(1) .\]
In other words, the periodic orbit corresponding to $\tilde{y}_0$ on a table with parameter $n$ can only be stable if $\tilde{y}_0 \to 0$ as $n \to \infty$. Therefore, we will focus our attention to a local description of $T_{\tilde{y}_0}$ for small values of $\tilde{y}_0$.

By combining Lemma 4.3 and (17) the following fact relating $\text{tr } DT_{\tilde{y}_0}(0,0)$ and $L_n(\tilde{y}_0)$ follows immediately.

**Lemma 4.6.** For all $\tilde{y}_0$
\[L_n(\tilde{y}_0) = L_n(0) + \frac{2\Gamma_y(s_1) + nW\sin\tilde{y}_0}{\sin(\omega_1 - \tilde{y}_0)} \sin\tilde{\omega}_1 + \frac{2f(\tilde{y}_0)}{\sin(\omega_1 - \tilde{y}_0)}\]
\[= L_n(0) \frac{\tan\tilde{\omega}_1}{\tan(\omega_1 - \tilde{y}_0)} + 2\frac{\tan(\omega_1 - \tilde{y}_0)^{\perp}}{\sin(\omega_1 - \tilde{y}_0)} f(\tilde{y}_0) + [1 - \frac{\cos(\omega_1 - \tilde{y}_0)}{\cos(\omega_1)}] \Gamma(s_1) \cdot \frac{\cos\tilde{\omega}_1}{\sin\tilde{\omega}_1}.\]

Furthermore, the relation
\[L_n(\tilde{y}_0) - L_n(0) = \frac{\text{tr } DT_{\tilde{y}_0}(0,0) + \text{tr } DF(f(\tilde{y}_0),\tilde{y}_0)}{\sin\tilde{\omega}_1} \frac{\sin\tilde{y}_0}{\frac{\partial F_y}{\partial x}(f(\tilde{y}_0),\tilde{y}_0)} - \frac{2f(\tilde{y}_0) \cos(\omega_1 - \tilde{y}_0)}{\sin\omega_1 \sin(\omega_1 - \tilde{y}_0)} \frac{\sin\tilde{y}_0 + 2f(\tilde{y}_0)}{\sin(\omega_1 - \tilde{y}_0)}\]
holds for all $\tilde{y}_0$ for which $\frac{\partial F_y}{\partial x}(f(\tilde{y}_0),\tilde{y}_0) \neq 0$.

### 4.3. Local analysis of the return map along periodic orbits.

In order to proceed with the stability analysis of the periodic orbit corresponding to $\tilde{y}_0$ on the table with separation distance equal to $L_n(\tilde{y}_0)$ we only need a local description of the corresponding first return map $T_{\tilde{y}_0}$. From its explicit expression given in (15) we see that the local representation of the return map is determined by the local form of $F_{\tilde{y}_0}$, which is in turn determined by a local description of $F$.

Since the boundary component $\Gamma$ is of class $C^5$ the map $F$ is of class $C^4$, and hence in some neighborhood of $(0,0)$ it has the form
\[F(x_0, y_0) = \left(\begin{array}{c} -x_0 + a_{01} y_0 + a_{20} x_0^2 + a_{11} x_0 y_0 + \ldots + a_{04} y_0^4 \\ -y_0 + b_{20} x_0^2 + b_{11} x_0 y_0 + \ldots + b_{04} y_0^4 \end{array}\right) + \omega_4(x_0, y_0),\]
\[(19)\]
where $a_{i0} = -1$, $a_{01} \neq 0$, $b_{i0} = 0$, $b_{01} = -1$. Clearly, this local representation of $F$ only incorporates (6). However, (7) and (8) give additional restrictions on the possible values of the various coefficients $a_{ij}, b_{ij}$ in (19). Indeed, substituting (19)
in (7) and (8) yields after some straightforward but rather tedious computation the following:

**Proposition 4.7.** The conditions (6), (7), (8) hold to order 3 if and only if

\[ \begin{align*} &a_{11} = 0, \quad b_{20} = 0, \quad b_{11} = -2a_{20}, \quad b_{22} = a_0 a_{20} \\
&b_{30} = -2 \frac{a_{20} + a_{30}}{a_{01}}, \quad b_{21} = 3 a_{30} + 2 a_{20}^2, \quad b_{12} = \frac{2a_{12} - 2a_{02}a_{20} - 3a_{01}^2 a_{30}}{a_{01}} \end{align*} \]

for the coefficients in (19), where it is assumed that \( a_{01} \neq 0 \).

As an immediate consequence of Lemma 4.2, (19), and Proposition 4.7 we obtain the local representation of \( f \) as

\[ f(y) = \frac{a_{01}}{2} y + \left( \frac{a_{02}}{2} + \frac{a_{01}^2 a_{20}}{8} \right) y^2 + \left( \frac{a_{02}}{2} + \frac{a_{01} a_{20}}{2} + \frac{a_{01}^3 a_{20}}{16} + \frac{a_{01}^3 a_{30}}{16} \right) y^3 + O(y^4), \]

which holds for all \( y \) in a neighborhood of 0.

**Lemma 4.8.** Suppose that \( a_{20} \neq 0 \). Then there exists an integer \( n_0 \in \mathbb{N} \) and \( \check{Y}_n \in C^4[-1,5] \) such that for all \( n \geq n_0 \) the equality \( \text{tr} \, D^2 \check{Y}_n(0,0) = 2 - t \) holds if and only if \( \check{y}_0 = \check{Y}_n(t) \), and all all derivatives of \( \check{Y}_n(t) \) are uniformly bounded in \( n \geq n_0 \).

**Proof.** Let \( 2 - t = \text{tr} \, D^2 \check{Y}_n(0,0) \). Then we can re-write (17) as

\[ t = H_n(\check{y}_0) \]

where

\[ H_n(\check{y}_0) = -2 - \text{tr} \, D^2 f(\check{y}_0, \check{y}_0) + \frac{2 \cos(\check{\omega}_1 - \check{y}_0) + 2 \Gamma_y(\check{s}_1) + n W}{\sin(\check{\omega}_1 - \check{y}_0)} \frac{\partial F_y}{\partial x}(f(\check{y}_0), \check{y}_0). \]

Since \( \frac{\partial F_y}{\partial x}(f(\check{y}_0), \check{y}_0)|_{\check{y}_0=0} = 0 \) and \( \text{tr} \, D^2 f(\check{y}_0, \check{y}_0)|_{\check{y}_0=0} = -2 \) it follows that \( t = 0 \), \( \check{y}_0 = 0 \) satisfy the above equation \( -t = H_n(\check{y}_0) \) for every \( n \).

Notice that

\[ H_n'(\check{y}_0)|_{\check{y}_0=0} = - \frac{d}{d\check{y}_0} \text{tr} \, D^2 f(\check{y}_0, \check{y}_0)|_{\check{y}_0=0} + 2 \frac{\Gamma_y(\check{s}_1) + n W}{\sin \check{\omega}_1} \frac{d}{d\check{y}_0} \frac{\partial F_y}{\partial x}(f(\check{y}_0), \check{y}_0)|_{\check{y}_0=0} \]

\[ = -2 a_{20} \left[ a_{01} + 2 \frac{\Gamma_y(\check{s}_1) + n W}{\sin \check{\omega}_1} \right] \]

where we used

\[ \text{tr} \, D^2 f(x,y) = -2 + 2 a_{01} a_{20} y + O_2(x,y) \]

and

\[ \frac{\partial F_y}{\partial x}(x,y) = -2 a_{20} y + O_2(x,y), \]

which follow immediately from (19), Proposition 4.7.

Since we assume that \( a_{20} \neq 0 \) we conclude that \( H_n'(\check{y}_0)|_{\check{y}_0=0} \neq 0 \) for but possibly one \( n \). Hence, for all but possibly one value of \( n \), the implicit function theorem guarantees the existence of \( \check{Y}_n(t) \), defined in some neighborhood of \( t = 0 \), such that

\[ t = H_n(\check{Y}_n(t)), \quad \check{Y}_n(0) = 0, \]
which provides the relation \( \bar{y}_0 = \bar{Y}_n(2 - \Theta) \) between the trace of the monodromy matrix and the choice of \( \bar{y}_0 \) that realizes the corresponding periodic orbit. Furthermore, \( \bar{Y}_n(t) \) is defined (at least) on the largest interval containing zero on which the initial value problem

\[
\frac{d}{dt} \bar{Y}_n(t) = -\frac{1}{H'_n(\bar{Y}_n(t))}, \quad \bar{Y}_n(0) = 0
\]

has a well-defined solution. Since we are interested in guaranteeing that \( \bar{Y}_n(t) \) is well-defined for at least all \( 0 \leq t \leq 2 \) we seek to find estimates on the domain on which the above initial value problem as a well-defined solution. For this purpose we simplify notation and introduce

\[
h_1(\bar{y}_0) = \frac{1}{\sin(\hat{\omega}_1 - \bar{y}_0)} \frac{\partial F_{\bar{y}}}{\partial x}(f(\bar{y}_0), \bar{y}_0)
\]

\[
h_2(\bar{y}_0) = -2 - \text{tr } Df(\bar{y}_0, \bar{y}_0) + 2 f(\bar{y}_0) \cos(\hat{\omega}_1 - \bar{y}_0) h_1(\bar{y}_0)
\]

so that the initial value problem for \( \bar{Y}_n(t) \) takes on the form

\[
\frac{d}{dt} \bar{Y}_n(t) = -\frac{1}{h_2'(\bar{Y}_n(t)) + [2 \Gamma(\hat{\omega}_1) + n W] h_1'(\bar{Y}_n(t))}, \quad \bar{Y}_n(0) = 0.
\]

As we had noted above already, using the assumption \( a_{20} \neq 0 \),

\[
h_1(\bar{y}_0) = -2 \frac{a_{20}}{\sin \hat{\omega}_1} [\bar{y}_0 + R_1(\bar{y}_0)], \quad R_1(\bar{y}_0) = \mathcal{O}(\bar{y}_0^2),
\]

\[
h_2(\bar{y}_0) = -2 a_{20} [a_{01} \bar{y}_0 + R_2(\bar{y}_0)], \quad R_2(\bar{y}_0) = \mathcal{O}(\bar{y}_0^2),
\]

where \( R_1, R_2 \) are \( C^4 \) functions independent of the value of \( n \). Therefore, the initial value problem for \( \bar{Y}_n(t) \) takes on the form

\[
\frac{d}{dt} \bar{Y}_n(t) = \frac{1}{2 a_{20}} \frac{1}{a_{01} + R'_1(\bar{Y}_n(t))} + \frac{1}{\sin \hat{\omega}_1} \frac{1}{[1 + \mathcal{O}(n^{-1})]}, \quad \bar{Y}_n(0) = 0,
\]

where \( R'_1(\bar{y}_0) = \mathcal{O}(\bar{y}_0), R'_2(\bar{y}_0) = \mathcal{O}(\bar{y}_0) \). From this it is now clear that there exists an integer \( n_0 \in \mathbb{N} \) such that \( \bar{Y}_n(t) \) is well-defined for all \(-1 \leq t \leq 5\), and is of class \( C^4 \) with all derivatives uniformly bounded in \( n \geq n_0 \). This completes the proof. \( \square \)

For large values of \( n \geq n_0 \) the initial value problem for \( \bar{Y}_n(t) \) that was considered in the above proof of Lemma \( \text{[L]} \) shows that \( \bar{Y}_n(t) = \mathcal{O}(\frac{1}{n^5}) \) uniformly in \(-1 \leq t \leq 5\). This allows for the asymptotic expansion in \( \frac{1}{n} \)

\[
\frac{d}{dt} \bar{Y}_n(t) = \frac{1}{2 a_{20}} \frac{\sin \hat{\omega}_1}{n W} [1 + \mathcal{O}(n^{-1})], \quad \bar{Y}_n(0) = 0,
\]

because \( R'_1(\bar{Y}_n(t)) = \mathcal{O}(n^{-1}) \) uniformly for \(-1 \leq t \leq 5\). Integrating this equation yields

\[
\bar{Y}_n(t) = \frac{t}{2 a_{20}} \frac{\sin \hat{\omega}_1}{n W} + \mathcal{O}(n^{-2}), \quad \bar{Y}_n(0) = 0
\]

uniformly in \(-1 \leq t \leq 5\).

**Proposition 4.9.** For \( n \geq n_0 \) and \(-3 \leq \Theta \leq 3\) the periodic orbit corresponding to \( \bar{y}_0 = \bar{Y}_n(2 - \Theta) \) has a monodromy matrix with trace \( \Theta \), and the separation distance \( L_n \) of the two curved boundary segments has the asymptotic expression

\[
L_n(\bar{y}_0) - L_n(0) = \frac{2 - \Theta}{2 a_{20} \sin \hat{\omega}_1} + \mathcal{O}(n^{-1})
\]
as \( n \to \infty \), which is uniform in \(-3 \leq \Theta \leq 3\).

**Proof.** Fix \( n \geq n_0 \). Then for any \(-3 \leq \Theta \leq 3\) it follows from Lemma 4.8 that the value \( \bar{y}_0 = \bar{Y}_n(2 - \Theta) \) corresponds to a periodic orbit whose monodromy matrix has trace \( \Theta \). By Lemma 4.6

\[
L_n(\bar{Y}_n(2 - \Theta)) - L_n(0) = \frac{\Theta + \text{tr} Df(\bar{y}_0), \bar{y}_0}{\sin \omega_1} \sin \bar{y}_0 \bigg|_{\bar{y}_0 = \bar{Y}_n(2 - \Theta)} \\
- 2 f(\bar{y}_0) \cos(\hat{\omega}_1 - \bar{y}_0) \sin \bar{y}_0 \bigg|_{\bar{y}_0 = \bar{Y}_n(2 - \Theta)} \\
+ 2 \frac{f(\bar{y}_0)}{\sin(\hat{\omega}_1 - \bar{y}_0)} \bigg|_{\bar{y}_0 = \bar{Y}_n(2 - \Theta)}
\]

which yields with (21), (22)

\[
L_n(\bar{Y}_n(2 - \Theta)) - L_n(0) = \frac{2 - \Theta}{2 \sigma_{20} \sin \hat{\omega}_1} + O(n^{-1})
\]

uniformly in \(-3 \leq \Theta \leq 3\) as \( n \to \infty \).

The result of Proposition 4.9 shows in particular, that for all \( n \geq n_0 \) there exists elliptic periodic orbits on the table with separation distance \( L_n \) as long as

\[
L_n = L_n(0) + \frac{2}{\sigma_{20} \sin \hat{\omega}_1} + O(n^{-1})
\]

as \( n \to \infty \). And since the values \( (L_n(0))_n \) are equidistant it follows that there are elliptic periodic orbits on the smoothed out stadium for all separation distances from an infinite sequence of equidistantly spaced intervals of fixed length in \((0, \infty)\).

### 4.4. Nonlinear stability analysis

The results in this section will address the non-linear stability of the elliptic periodic orbits that were considered in Proposition 4.9. The main tool we use is the Birkhoff normal form. We record the following abstract result:

**Lemma 4.10** (Normal form expansion and nonlinear stability – [15, 9, 12]). Let \( T(x, y) \) be an area-preserving \( C^4 \) mapping with an elliptic fixed point at the origin

\[
T(x, y) = \begin{pmatrix}
A_{10} x + A_{01} y + A_{20} x^2 + A_{11} x y + \ldots + A_{03} y^3 \\
B_{10} x + B_{01} y + B_{20} x^2 + B_{11} x y + \ldots + B_{03} y^3
\end{pmatrix} + O_4(x, y)
\]

and let \( \lambda = e^{\pm i \phi} \) denote the complex eigenvalues of \( dT(0, 0) \), where the sign of \( \phi \) is chosen such that \( A_{01} \sin \phi > 0 \). If \( \lambda^2, \lambda^3, \lambda^4 \neq 1 \), then there exists a real-analytic canonical change of coordinates \( (x, y) \mapsto z \in \mathbb{C} \) taking \( T \) into its Birkhoff normal form \( z \mapsto \lambda z e^{A |z|^2} + O(|z|^4) \). The expression for the first Birkhoff coefficient \( A \) reads

\[
A = \text{Im} c_{21} + \frac{\sin \phi}{\cos \phi - 1} \left( 3 |c_{20}|^2 + \frac{2}{\cos \phi + 1} |c_{02}|^2 \right)
\]
where

\[
8 \text{ Im } c_{21} = A_{10} \left[ -A_{21} + 3 \frac{B_{10} A_{03}}{A_{01}} - 3 \frac{A_{01} B_{30}}{B_{10}} + B_{12} \right] - B_{10} \left[ A_{12} - 3 \frac{A_{01} A_{30}}{B_{10}} - \frac{A_{01} B_{21}}{B_{10}} + 3 B_{03} \right]
\]

\[
16 |c_{20}|^2 = \sqrt{-\frac{A_{01}}{B_{10}} \left[ B_{10} A_{02} + A_{20} + B_{11} \right]^2} + \sqrt{-\frac{B_{10}}{A_{01}} \left[ B_{10} A_{02} + A_{20} + B_{11} \right]^2}
\]

\[
16 |c_{02}|^2 = \sqrt{-\frac{A_{01}}{B_{10}} \left[ B_{10} A_{02} + A_{20} - B_{11} \right]^2} + \sqrt{-\frac{B_{10}}{A_{01}} \left[ B_{10} A_{02} + A_{20} - B_{11} \right]^2}
\]

are given in terms of the $A_{ij}$ and $B_{ij}$. Furthermore, if $A \neq 0$, then the elliptic fixed point at the origin is nonlinearly stable.

In order to proceed with the nonlinear stability analysis of the first return map $\bar{T}_{\bar{y}_0}$ we note that (15) and Proposition 4.9 imply

(23a)

\[
(\delta x_1, \delta y_1) = \bar{F}(\delta x_0, \delta y_0), \quad \bar{y}_0 = \bar{\Upsilon}_n(2 - \Theta)
\]

\[
-\delta x_2 = \delta x_1 + h(\Theta, n) \delta y_1^2 \left[ 1 + \mathcal{O}(\delta y_1^2) \right] - g(\Theta, n) \left[ \delta y_1 - \frac{1}{6} \delta y_1^3 + \mathcal{O}(\delta y_1^4) \right]
\]

\[
-\delta y_2 = \delta y_1
\]

where the $\mathcal{O}$-terms are uniform, and we introduced the short-hand notations

(23b)

\[
h(\Theta, n) = f(\bar{\Upsilon}_n(2 - \Theta))
\]

\[
g(\Theta, n) = 2 h(\Theta, n) \cot(\hat{\omega}_1 - \bar{\Upsilon}_n(2 - \Theta)) + \frac{2 \Gamma_y(\hat{s}_1) + n W}{\sin(\hat{\omega}_1 - \bar{\Upsilon}_n(2 - \Theta))}.
\]

The next observation is that we can write $\bar{F}_{\bar{y}_0}$ in terms of a power series as

(24a)

\[
\bar{F}_{\bar{y}_0}(\delta x_0, \delta y_0) = \left( \bar{a}_{10} \delta x_0 + \bar{a}_{01} \delta y_0 + \bar{a}_{20} \delta x_0^2 + \ldots + \bar{a}_{03} \delta y_0^3 \right) + \mathcal{O}(\delta x_0, \delta y_0),
\]

where

(24b)

\[
\bar{a}_{kl}(\Theta, n) = \frac{1}{k! l!} \frac{\partial^k}{\partial x^k} \frac{\partial^l}{\partial y^l} F_x(f(\bar{y}_0), \bar{y}_0)
\]

\[
\bar{b}_{kl}(\Theta, n) = \frac{1}{k! l!} \frac{\partial^k}{\partial x^k} \frac{\partial^l}{\partial y^l} F_y(f(\bar{y}_0), \bar{y}_0)
\]

and $\bar{y}_0 = \bar{\Upsilon}_n(2 - \Theta)$. 

Combining (23) and (24) yields

\begin{align*}
A_{10} &= -\bar{a}_{10} + g(\Theta, n) \bar{b}_{10} \\
A_{01} &= -\bar{a}_{01} + g(\Theta, n) \bar{b}_{01} \\
A_{20} &= -\bar{a}_{20} + g(\Theta, n) \bar{b}_{20} - h(\Theta, n) \bar{b}_{10}^2 \\
A_{11} &= -\bar{a}_{11} + g(\Theta, n) \bar{b}_{11} - 2 h(\Theta, n) \bar{b}_{10} \bar{b}_{01} \\
A_{02} &= -\bar{a}_{02} + g(\Theta, n) \bar{b}_{02} - h(\Theta, n) \bar{b}_{01}^2 \\
A_{30} &= -\bar{a}_{30} + g(\Theta, n) \bar{b}_{30} - 2 h(\Theta, n) \bar{b}_{10} \bar{b}_{20} - \frac{1}{6} g(\Theta, n) \bar{b}_{10}^3 \\
A_{21} &= -\bar{a}_{21} + g(\Theta, n) \bar{b}_{21} - 2 h(\Theta, n) (\bar{b}_{10} \bar{b}_{11} + \bar{b}_{01} \bar{b}_{20}) - \frac{1}{2} g(\Theta, n) \bar{b}_{10}^2 \bar{b}_{01} \\
A_{12} &= -\bar{a}_{12} + g(\Theta, n) \bar{b}_{12} - 2 h(\Theta, n) (\bar{b}_{10} \bar{b}_{02} + \bar{b}_{01} \bar{b}_{11}) - \frac{1}{2} g(\Theta, n) \bar{b}_{10} \bar{b}_{01}^2 \\
A_{03} &= -\bar{a}_{03} + g(\Theta, n) \bar{b}_{03} - 2 h(\Theta, n) \bar{b}_{01} \bar{b}_{02} - \frac{1}{6} g(\Theta, n) \bar{b}_{01}^3 \\
B_{kl} &= -\bar{b}_{kl}
\end{align*}

for the coefficients in the series expansion of the first return map \( T_{y_0} \) (using the same notation as in Lemma 4.10).

With (22) and (20) the expressions for \( h(\Theta, n) \) and \( g(\Theta, n) \) defined in (23) take on the form

\begin{align*}
\begin{aligned}
\frac{n}{\Theta} h(\Theta, n) &= \frac{a_{01}}{W a_{20}} \frac{2 - \Theta}{4} \sin \hat{\omega}_1 + \mathcal{O}(n^{-1}) \\
\frac{1}{n} g(\Theta, n) &= \frac{W}{\sin \hat{\omega}_1} + \mathcal{O}(n^{-1})
\end{aligned}
\end{align*}

as \( n \to \infty \), uniformly in \( -3 \leq \Theta \leq 3 \). Similarly, the asymptotic forms of \( \bar{a}_{kl}(\Theta, n) \) and \( \bar{b}_{kl}(\Theta, n) \), as defined in (24), are given by

\begin{align*}
\bar{a}_{k,l}(\Theta, n) &= a_{k,l} + \frac{(k + 1) a_{01} a_{k+1,l} + 2 (l + 1) a_{k,l+1} - \Theta \sin \hat{\omega}_1}{4 n W} + o(n^{-1}) \\
\bar{b}_{k,l}(\Theta, n) &= b_{k,l} + \frac{(k + 1) a_{01} b_{k+1,l} + 2 (l + 1) b_{k,l+1} - \Theta \sin \hat{\omega}_1}{4 n W} + o(n^{-1})
\end{align*}

as \( n \to \infty \), uniformly in \( -3 \leq \Theta \leq 3 \).

**Proposition 4.11.** Assuming that \( a_{20} \neq 0 \) and \( a_{30} \neq -a_{20}^2 \) (i.e. \( b_{30} \neq 0 \)). Then the first Birkhoff coefficient \( A \) of the periodic orbit corresponding to \( \Theta \) and \( n \) satisfies

\[
\frac{1}{n^2} A = -\frac{1}{2 - \Theta} \frac{3 b_{30} W^2}{8 \sin^2 \hat{\omega}_1} + o(1)
\]

as \( n \to \infty \), uniformly in \( \Theta \) in any compact subset of \( (-2, 2) \setminus \{-1, 0\} \).
Proof. Using the expressions derived in (25), (26), (27) as well as the symmetry relations given in Proposition 4.7 we obtain
\[ B_{k,l} = -b_{k,l} - (k + 1) a_{01} b_{k+1,l} + 2 (l + 1) b_{k,l+1} \frac{2 - \Theta \sin \hat{\omega}_1}{a_{20}} + o(n^{-1}) \]
\[ \frac{1}{n} A_{k,l} = \mathcal{O}(1), \quad A_{10} = \Theta - 1 + o(1), \quad A_{20} = -\frac{4 - \Theta}{2} a_{20} + o(1) \]
\[ \frac{1}{n^2} A_{01} A_{20} = -b_{30} \frac{W^2}{\sin^2 \hat{\omega}_1} + o(1) \]
uniformly in \(-3 \leq \Theta \leq 3\), and also
\[ -\frac{A_{01}}{B_{10}} = n^2 \left[ \frac{W^2}{(2 - \Theta) \sin^2 \hat{\omega}_1} + o(1) \right]. \]
Using the same notation as in Lemma 4.10 we therefore obtain
\[ \frac{1}{n^2} 8 \text{Im} c_{21} = -\frac{1}{2 - \Theta} \frac{3 b_{30} W^2}{\sin^2 \hat{\omega}_1} + o(1), \quad \frac{1}{n} 16 |c_{20}|^2 = \mathcal{O}(1), \quad \frac{1}{n} 16 |c_{02}|^2 = \mathcal{O}(1). \]
In particular, this readily implies the claimed expression for the Birkhoff coefficient, which finishes the proof.

With the results of Proposition 4.9 and Proposition 4.11 and the general stability result Lemma 4.10 we immediately obtain the existence of nonlinearly stable periodic orbits for large set of separation distances of the two curved boundary components. A summary of our results so far is given in the following Theorem 4.12.

**Theorem 4.12** (Range of existence of stable symmetric periodic orbits). Suppose that \( \Gamma \) satisfies Assumption 4.1 and suppose that the corresponding part of a billiard trajectory satisfies \( a_{20} \neq 0 \) and \( a_{30} \neq -a_{20}^2 \) (i.e. \( b_{30} \neq 0 \)). For every \( 0 < \epsilon < \frac{1}{2} \) there exists \( N_\epsilon \geq n_0 \) such that on any billiard table with separation distance \( L \) in the set
\[ \bigcup_{n \geq N_\epsilon} \left\{ L_0(0) + \frac{n W}{\tan \hat{\omega}_1} + \frac{1}{2 a_{20} \sin \hat{\omega}_1} I_\epsilon \right\}, \quad I_\epsilon = [\epsilon, 2 - \epsilon] \cup [2 + \epsilon, 3 - \epsilon] \cup [3 + \epsilon, 4 - \epsilon] \]
there exist nonlinearly stable periodic orbits in the sense that their first Birkhoff coefficient is nonzero.

**Proof.** Proposition 4.9 provides an asymptotic expression for the range of separation distances on which elliptic periodic orbits exit. In particular, for every \( 0 < \epsilon < \frac{1}{2} \) there exists \( N_\epsilon \geq n_0 \) such that there exist elliptic periodic orbits on the billiard table with separation distance \( L \) in the set
\[ \bigcup_{n \geq N_\epsilon} \left\{ L_0(0) + \frac{n W}{\tan \hat{\omega}_1} + \frac{2 - \Theta}{2 a_{20} \sin \hat{\omega}_1} : -2 + \epsilon \leq \Theta \leq 2 - \epsilon \right\}, \]
where we used Lemma 4.3 to express \( L_n(0) \) explicitly in terms of \( n \).

By Proposition 4.11 it then follows that these periodic orbits are also nonlinearly stable in the sense that their first Birkhoff coefficient is nonzero, provided that \( N_\epsilon \) is large enough, and that the range of \( \Theta \) excludes lower order resonances. The latter can be achieved by using
\[ \Theta \in [-2 + \epsilon, -1 - \epsilon] \cup [-1 + \epsilon, -\epsilon] \cup [\epsilon, 2 - \epsilon] \]
instead of \( -2 + \epsilon \leq \Theta \leq 2 - \epsilon \), again assuming that the value of \( N_\epsilon \) is large enough. This finishes the proof. \qed
Theorem 4.12 shows the billiard table has nonlinearly stable periodic orbits for all separation distances taken from a set that is formed by a sequence of equidistantly spaced copies of the three closed intervals forming the set \( \frac{1}{2 \alpha_{20} \sin \omega_1} I_0 \). The restriction on the separation distance imposed by using \( I_0 \) instead of \([\epsilon, 4 - \epsilon]\) in Theorem 4.12 is a technical condition to guarantee the nonlinear stability in the sense of the nonvanishing of the first Birkhoff coefficient \( A \). This condition is most likely not optimal, because it was shown in Proposition 4.11 that \( |A| \) actually grows in \( n \) for fixed values of \( \Theta \). Therefore, to guarantee that, say, \( |A| \geq 1 \), we can probably allow for the range of \( \Theta \) to approach the resonance values \(-1, 0\) as \( n \) increases. However, since any condition to guarantee nonlinear stability is of technical nature we have not tried to optimize this part of Theorem 4.12.

Finally we remind the reader that the assertion of Theorem 4.12 is conditional to \( \Gamma \) satisfying Assumption 4.1, in other words, it is conditional to the existence of particular billiard trajectory piece. The following Section 4.5 addresses the question of validity of Assumption 4.1.

### 4.5. Verification of Assumption 4.1 and an explicit stability criterion.

In this section we address the validity of Assumption 4.1. In view of Fig. 4 the simplest part of a billiard trajectory required for \( \Gamma \) to satisfy Assumption 4.1 has 3 reflections off of \( \Gamma \). Namely, one at its center, and two more near the end points. In other words, the existence of such a billiard trajectory is a sufficient criterion for \( \Gamma \) to satisfy Assumption 4.1.

For definiteness, let \( s_0, s_1, s_2 \) denote the arc length parameters of the three points of reflection, and let \( \varphi_0, \varphi_1, \varphi_2 \) denote the corresponding angles of reflection. By symmetry \( \varphi_0 = \varphi_1 \), and \( s_1 = \frac{1}{2} |\Gamma| \). Furthermore, by the two free paths \( \tau_{01} \) and \( \tau_{12} \) corresponding to the three points of reflection satisfy \( \tau_{01} = \tau_{12} \), again by the assumed symmetry of the trajectory piece.

In particular, the linearization of the billiard flow from the pre-collisional state at \( s_0 \) to the post-collisional state at \( s_2 \) is given by

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = - \begin{pmatrix} 1 & 0 \\ R_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \tau_{01} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ R_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \tau_{01} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ R_0 & 1 \end{pmatrix}.
\]

Evaluating yields

\[
a = d = -1 - \tau_{01} (R_1 + 2R_0 + \tau_{01} R_0 R_1) = 1 - (1 + \tau_{01} R_0) (2 + \tau_{01} R_1),
\]

\[
b = -\tau_{01} (2 + \tau_{01} R_1),
\]

\[
c = -(1 + \tau_{01} R_0) (R_1 + 2R_0 + \tau_{01} R_0 R_1).
\]

In particular we obtain that

\[
a = -1 \quad \text{and} \quad c = 0 \iff R_0 = -\frac{R_1}{2 + \tau_{01} R_1}.
\]

That is to say that for Assumption 4.1 to be satisfied for this particular type of trajectory piece the relation \( R_0 = -\frac{R_1}{2 + \tau_{01} R_1} \) must hold.

**Lemma 4.13.** For any \( \Gamma \) there exists \( s_0 \in (0, s_1) \) such that \( R_0 = -\frac{R_1}{2 + \tau_{01} R_1} \).

**Proof.** Varying \( s_0 \) on \( (0, s_1) \) makes \( q(s_0) = \tau_{01} R_0 + \frac{\tau_{01} R_1}{2 + \tau_{01} R_1} \) a well-defined continuous function. Clearly \( \lim_{s_0 \to 0} q(s_0) = \infty \). And since for a circle we have that...
$\tau R = -4$ it follows that $\lim_{s_0 \to s_1} q(s_0) = -2$. By continuity of $q$ it thus follows that there must be a $s_0 \in (0, s_1)$ for which $q(s_0) = 0$, which finishes the proof. □

While the existence of $s_0$ as in Lemma 4.13 is necessary for this special part of a billiard trajectory to be as required by Assumption 4.1, it is not sufficient. This is due to the fact that Lemma 4.13 does not necessarily guarantee that there are no other reflections off $\Gamma$ prior to the reflection at $s_0$.

In order to proceed we need to recall the notion of absolute focusing, which was introduced in [10][4]. A finite piece of a billiard trajectory is absolutely focused, if an initially parallel beam, when sent along this billiard trajectory, is focused right after each reflection, and has a conjugate point in between any two consecutive reflections. Furthermore, a focusing boundary component is called absolutely focusing, if every complete sequence of reflections off it is absolutely focused. In particular, arcs of a circle are absolutely focusing, and so are sufficiently short segments of an arbitrary focusing boundary component, as well as any subsegments of absolutely focusing components.

The key observation now is that if the three reflections $s_0, s_1, s_2$ as in Lemma 4.13 were all on an absolutely focusing subsegment of $\Gamma$, then an initially parallel beam entering right before the first reflection at $s_0$ would have to be focusing right after the reflection at $s_2$, [10][4]. However, by Lemma 4.13 it is parallel, and hence $s_0, s_1, s_2$ cannot be on an absolutely focusing subsegment of $\Gamma$. This observation now implies our main criterion, given in Lemma 4.14 below, to verify Assumption 4.1.

**Lemma 4.14.** Suppose that $\Gamma_{abs}$ is an absolutely focusing subsegment of $\Gamma$, which is symmetric about the horizontal axis. With the notation as shown in Fig. 6, suppose that the angle enclosed by $\Gamma_{abs}$ is at least as large as $\pi - \varphi_{end}$, and that $W_{abs} \geq \frac{\Gamma - \Gamma_{abs}}{2} \tan \varphi_{abs}$. Then $\Gamma$ satisfies Assumption 4.1.

**Proof.** There are two types of additional reflections possible. The first kind are reflections off of $\Gamma$ on the segment connecting $\Gamma(0)$ and $\Gamma(s_0)$. The second kind are reflections off of $\Gamma$ on the segment connecting $\Gamma(|\Gamma|)$ and $\Gamma(s_2)$.
A sufficient condition to avoid additional reflections of the first kind is $0 \leq 2\pi - \omega_0 \leq \frac{\pi}{2}$, where $\omega_0$ denotes the direction angle corresponding to the pre-collisional velocity at $s_0$. Due to the geometry

$$2\pi - \omega_0 = \varphi_1 - 2\theta(s_0), \quad \frac{d\varphi_1}{ds_0} > 0, \quad \frac{d\theta(s_0)}{ds_0} > 0, \quad 0 \leq \varphi_1, \theta(s_0) \leq \frac{\pi}{2},$$

and hence

$$\varphi_{\text{end}} - 2\theta(s_{\text{abs}}) \leq 2\pi - \omega_0 \leq \varphi_1 \leq \frac{\pi}{2}.$$ 

And since the angle enclosed by the $\Gamma_{\text{abs}}$ is $\pi - 2\theta(s_{\text{abs}})$ it thus follows from our assumption that $0 \leq 2\pi - \omega_0 \leq \frac{\pi}{2}$ holds.

It remains to rule out additional reflections of the second kind. Since $\Gamma$ is convex, and we already verified that $0 \leq 2\pi - \omega_0 \leq \frac{\pi}{2}$ it follows that

$$\Gamma(s_0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Gamma(s_{\text{abs}}) - \begin{pmatrix} q \\ 0 \end{pmatrix} + t \begin{pmatrix} \cos \omega_0 \\ \sin \omega_0 \end{pmatrix}$$

for some $q \geq s_{\text{abs}}$

is sufficient to rule out additional reflections of the second kind. Eliminating $t$ yields

$$q = [\Gamma(s_{\text{abs}}) - \Gamma(s_0)] \cdot \begin{pmatrix} 1 \\ 1 \tan \omega_0 \end{pmatrix} + \frac{W_{\text{abs}}}{1 - \tan \omega_0}$$

and $q \geq s_{\text{abs}}$.

where we used the fact that $W_{\text{abs}} = -2\Gamma_y(s_{\text{abs}})$. Since $\tan \omega_0 < 0$ the first term of the right-hand-side is clearly positive, hence $q \geq \frac{W_{\text{abs}}}{\tan \omega_0}$. Furthermore, using again the relation $2\pi - \omega_0 = \varphi_1 - 2\theta(s_0)$ yields $-\tan \omega_0 = \tan(\varphi_1 - 2\theta(s_0))$. The monotonicity property implies $\varphi_1 - 2\theta(s_0) \leq \varphi_1 \leq \varphi_{\text{abs}}$, so that $q \geq \frac{W_{\text{abs}}}{\tan \varphi_{\text{abs}}}$. Therefore, a sufficient condition for $q \geq s_{\text{abs}}$ is to require $W_{\text{abs}} \geq s_{\text{abs}} \tan \varphi_{\text{abs}}$.

Since $s_{\text{abs}} = \frac{\Gamma_{\text{end}} - \Gamma_{\text{abs}}}{2}$ this finishes the proof.

Because we are particularly interested in smoothening a semi-circle of some radius $R$ in a small neighborhood of its endpoints, we consider again the same construction as in Fig. 2. The following Corollary 4.15 shows that if $\Gamma$ is a semi-circle except for a sufficiently small segment near its endpoints, then it satisfies Assumption 4.14.

**Corollary 4.15.** Every local smoothing $\Gamma$ of a semi-circle $\Gamma_c$ as shown in Fig. 2 with

$$2\alpha \leq \varphi_{\text{end}} \quad \text{and} \quad \frac{|\Gamma|}{|\Gamma_c|} \leq 1 + \frac{4}{\pi} \frac{\cos \alpha}{\tan(\frac{\pi}{4} + \frac{\alpha}{2})} - \frac{2\alpha}{\pi}$$

satisfies Assumption 4.14.

**Proof.** For any $\alpha$ we can take $\Gamma_{\text{abs}}$ to be the remaining circular segment of radius $R$. In particular, $\Gamma_{\text{abs}}$ encloses an angle of $\pi - 2\alpha$, and $\varphi_{\text{abs}} = \frac{\pi}{4} + \frac{\alpha}{2}$, $|\Gamma_{\text{abs}}| = (\pi - 2\alpha) R$, $W_{\text{abs}} = 2R \cos \alpha$.

Thus our assumption $2\alpha \leq \varphi_{\text{end}}$ implies the assumption of Lemma 4.14 that the angle enclosed by $\Gamma_{\text{abs}}$ is at least as large as $\pi - \varphi_{\text{end}}$. Furthermore, the second condition of Lemma 4.14, i.e. $W_{\text{abs}} \geq \frac{|\Gamma| - |\Gamma_{\text{abs}}|}{2} \tan \varphi_{\text{abs}}$ takes on the explicit form

$$2R \cos \alpha \geq \frac{|\Gamma| - (\pi - 2\alpha) R}{2} \tan \left(\frac{\pi}{4} + \frac{\alpha}{2}\right).$$

\[\square\]
So far we have proved a sufficient condition for \( \Gamma \) to satisfy Assumption 4.1 by showing the existence of a particular type of billiard trajectory as required by Assumption 4.1. In the following Theorem 4.16 makes the stability result of Theorem 4.12 explicit for this particular setting, were we use the notation \( R^{(1)} \), \( R^{(2)} \) introduced in (A.9) in Section A.

**Theorem 4.16** (Range of existence of special stable symmetric periodic orbits). Suppose that \( \Gamma \) is as in Lemma 4.14, and suppose further that the two non-degeneracy conditions

\[
R^{(1)}_0 \neq -\frac{1}{\tau_{01} R_1} \left[ \frac{R_1}{2 + R_1 \tau_{01}} \right]^2 \left[ 3 \tau_{01} R_1 \tan \varphi_0 + 4 \tan \varphi_1 + 4 \tan \varphi_0 \right]
\]

and

\[
R^{(2)}_0 \neq \frac{1}{2 \left[ 2 + \tau_{01} R_1 \right]^2} \left[ -16 R^{(2)}_0 + 3 \tau_{01} \left[ R^{(1)}_0 \right]^2 \left[ 2 + \tau_{01} R_1 \right]^5 + 2 R^{(1)}_0 \left[ 2 + \tau_{01} R_1 \right]^3 \left[ (14 + 9 \tau_{01} R_1) \tan \varphi_0 + 12 \tan \varphi_1 \right] + 3 R^{(2)}_1 \left[ 3 \tan \varphi_0 (2 + \tau_{01} R_1) + 4 \tan \varphi_1 \right]^2 + 2 (2 + \tau_{01} R_1) \right] \]

are satisfied. Then, for every \( 0 < \epsilon < \frac{1}{2} \) there exists \( N_\epsilon \geq n_0 \) such that on any billiard table with separation distance \( L \) in the set

\[
\bigcup_{n \geq n_\epsilon} \left\{ L_0(0) + \frac{n W}{\tan(\varphi_1 - 2 \theta(s_0))} + \frac{1}{2 \kappa \sin(\varphi_1 - 2 \theta(s_0))} I_\epsilon \right\},
\]

where

\[
I_\epsilon = [\epsilon, 2 - \epsilon] \cup [2 + \epsilon, 3 - \epsilon] \cup [3 + \epsilon, 4 - \epsilon]
\]

\[
\kappa = \frac{1}{4} \tau_{01} R^{(1)}_0 \left[ 2 + \tau_{01} R_1 \right] + \frac{1}{4} \frac{R_1}{2 + \tau_{01} R_1} \left[ 3 \tau_{01} R_1 \tan \varphi_0 + 4 \tan \varphi_1 + 4 \tan \varphi_0 \right]
\]

there exist nonlinearly stable periodic orbits in the sense that their first Birkhoff coefficient is nonzero.

**Proof.** The billiard trajectory considered in Lemma 4.14 satisfies \( R_0 = -\frac{R_1}{2 + \tau_{01} R_1} \). Furthermore, the symmetry about the horizontal axis of both the billiard trajectory and of \( \Gamma \) imply

\[
\tau_{12} = \tau_{01} \ , \ \varphi_2 = \varphi_0 \ , \ R_2 = R_0 \ , \ R^{(1)}_2 = -R^{(1)}_0 \ , \ R^{(2)}_2 = R^{(2)}_0 \,
\]

and

\[
\kappa'(s_1) = 0 \quad \text{and hence} \quad R^{(1)}_1 = 0 .
\]

In order to determine the nonlinear stability of the corresponding periodic orbits using the result of Theorem 4.12 we need to compute \( a_{20} \) and \( b_{30} \). The expansion of the free flight map and of the reflection map are provided in Lemma (A.4). Thus, composing these expansion along the special billiard trajectory segment we consider
we obtain after a straightforward but lengthy computation that
\[ a_{01} = -\tau_0 \left[ 2 + \tau_0 R_1 \right], \]
\[ a_{20} = -\frac{1}{4} \tau_0 R_0^{(1)} \left[ 2 + R_1 \tau_0 \right] - \frac{1}{4} \frac{R_1}{2 + R_1 \tau_0} \left[ 3 \tau_0 R_1 \tan \varphi_0 + 4 \tan \varphi_1 + 4 \tan \varphi_0 \right], \]
\[ b_{30} = \frac{1}{24} \left( \frac{2 + \tau_0 R_1}{R_1} \right)^3 \left[ -16 R_1^{(2)} - 2 R_0^{(2)} \left[ 2 + \tau_0 R_1 \right]^4 + 3 \tau_0 \left[ R_0^{(1)} \right]^2 \left[ 2 + \tau_0 R_1 \right]^5 \right. \]
\[ + 2 R_0^{(1)} R_1 \left[ 2 + \tau_0 R_1 \right]^3 \left[ (14 + 9 \tau_0 R_1) \tan \varphi_0 + 12 \tan \varphi_1 \right] \]
\[ \left. + 3 R_1^3 \left[ 3 \tan \varphi_0 \left( 2 + \tau_0 R_1 \right) + 4 \tan \varphi_1 \right]^2 + 2 \left( 2 + \tau_0 R_1 \right) \right], \]
where we eliminated \( R_0 \) using the relation \( R_0 = -\frac{R_1}{2 + \tau_0 R_1} \). In particular, \( a_{20} \neq 0 \) and \( b_{30} \neq 0 \) if and only if the two stated assumptions on \( R_0^{(1)} \) and \( R_0^{(2)} \) hold, respectively. With Theorem 4.12 the claimed existence and stability result follows when noting that
\[ \varphi_0 = \frac{\pi}{2} - \varphi_1 + \theta(s_0), \quad \dot{\omega}_1 = 3\pi - \omega_0 = \pi + \varphi_1 - 2\theta(s_0) \]
due to the specific geometry of the billiard trajectory piece we consider, hence
\[ \sin \dot{\omega}_1 = -\sin(\varphi_1 - 2\theta(s_0)), \quad \text{and} \quad \tan \dot{\omega}_1 = \tan(\varphi_1 - 2\theta(s_0)). \]
\( \square \)

5. Proofs of Theorem 2.2 and Theorem 2.3

Now we are ready to prove our final main results Theorem 2.2 and Theorem 2.3.

Theorem 2.2. By Lemma 4.14 and specifically its Corollary 4.15 any sufficiently short smoothening of a semi-circle satisfies the assumptions of of Theorem 4.16. Therefore, claim (i) of Theorem 2.2 follows.

Clearly, if necessary the smoothening can be (locally) modified such that the point \( \Gamma(s_0) \), the corresponding tangent direction \( \theta(s_0) \), the curvature \( K(s_0) \), the length of the smoothed region, and the endpoint of \( \Gamma \) remain unaltered, while the first and second derivatives of the curvature \( K'(s_0) \), \( K''(s_0) \) can be adjusted to take on any prescribed value.

This allows to make sure that the conditions of Theorem 4.16 for the existence of nonlinearly stable periodic orbits are satisfied for arbitrarily small values of \( \kappa \). For \( \kappa \) sufficiently small the intervals for \( L \) for which existence of nonlinearly stable periodic orbits is guaranteed overlap. Therefore, there exist nonlinearly stable periodic orbits for all sufficiently large separations. Finally, note that the nonlinear stability persists for \( C^5 \) perturbation of such a boundary component, because of the nonvanishing of the first Birkhoff coefficient is an open condition. Thus we obtain a proof of item (ii) of Theorem 2.2. \( \square \)

Theorem 2.3. Move a horizontal trajectory segment from the joint point of the circular part and the smoothing towards the interior of the circle such that a periodic orbit with only reflections off the circular part of \( \Gamma \) forms, as shown in Fig. 7. This results in a billiard trajectory that falls onto the circular boundary component along the horizontal direction and is symmetric about the horizontal symmetry axis of the boundary component. Let the number of reflections off the boundary component be equal to \( n + 2 \), and denote points of reflection by \( s_0, s_1, \ldots, s_{n+1} \), see Fig. 7.

For reflections off the circle we have \( \tau R = -4 \). Now we perturb the boundary component in a symmetric manner in a small neighborhood of \( s_0 \) and \( s_{n+1} \) such
that the location and the tangent of the boundary at this point remains unchanged.

Denote the curvature at these two points by $K_0$, so that

$$\frac{\mathcal{R}_0}{\mathcal{K}} = \frac{K_0}{\mathcal{K}} = \frac{\rho}{\rho_0}, \quad \mathcal{R}_0 = -\frac{4}{\tau} \frac{\rho}{\rho_0}.$$ 

Since the beam falls horizontally onto the boundary component and is symmetric about the horizontal we obtain a periodic orbit for any separation distance $L$.

Denote the length of the free path between the two boundary components by $l$, see Fig. 7. Then the linearization along the periodic orbit is given by $M^2$, where

$$M = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -\mathcal{R}_0 & -1 \end{pmatrix} \left[ \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -\mathcal{R} & -1 \end{pmatrix} \right]^n \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -\mathcal{R}_0 & -1 \end{pmatrix}.$$ 

Simplifying the expression for $M$ yields

$$M = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \tau & -\tau \\ \tau & -1 \end{pmatrix} \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -\mathcal{R}_0 & -1 \end{pmatrix}.$$ 

and hence

$$\text{tr} M^2 = (\text{tr} M)^2 - 2, \quad \text{tr} M = 2 + [2n + (n + 1) \mathcal{R}_0 \tau] \frac{L}{\tau}.$$ 

Using $\mathcal{R}_0 = -\frac{4}{\tau} \frac{\rho}{\rho_0}$ it follows that

$$\frac{1}{2} \text{tr} M = 1 + 2 \frac{n - 2 (n + 1) \frac{\rho}{\rho_0}}{1 + (1 - 2 \frac{\rho}{\rho_0}) \frac{L}{\tau}}.$$ 

Therefore,

$$\frac{1}{2} \text{tr} M < 1 \iff -1 - 4(n + 1) \left[ \frac{\rho}{\rho_0} - \frac{1}{2} \right] + 8(n + 1) \left[ \frac{\rho}{\rho_0} - \frac{n}{2(n + 1)} \right] \left[ \frac{\rho}{\rho_0} - \frac{1}{2} \right] \frac{L}{\tau} < 1.$$ 

\[\frac{\rho}{\rho_0} - \frac{n}{2(n + 1)} \left[ \frac{\rho}{\rho_0} - \frac{1}{2} \right] \frac{L}{\tau} < \frac{1}{2} \left[ \frac{\rho}{\rho_0} - \frac{n}{2(n + 1)} \right]\]
and
\[
\frac{1}{2} \text{tr } M > -1 \iff -1 - 4(n + 1) \left[ \frac{\rho_0}{\rho_0 - \frac{1}{2}} \right] + 8(n + 1) \left[ \frac{\rho_0}{\rho_0 - \frac{n}{2(n + 1)}} \right] \left[ \frac{\rho_0 - \frac{1}{2}}{\rho_0 - \frac{1}{2}} \right] > -1
\]
\[
\iff \left[ \frac{\rho_0}{\rho_0 - \frac{n}{2(n + 1)}} \right] \left[ \frac{\rho_0 - \frac{1}{2}}{\rho_0 - \frac{1}{2}} \right] > \frac{1}{2} \left[ \frac{\rho_0 - \frac{1}{2}}{\rho_0 - \frac{1}{2}} \right]
\]

Choosing the radius of curvature \( \rho_0 \) at \( s_0 \) such that
\[
\frac{n}{2(n + 1)} < \frac{\rho}{\rho_0} < \frac{1}{2}
\]
yields
\[
\frac{1}{2} \text{tr } M < 1 \quad \text{for all } \quad l \geq 0
\]
and
\[
\frac{1}{2} \text{tr } M > -1 \iff \frac{l}{\tau} < \frac{1}{2} \rho_0 \frac{1}{n} \frac{n}{2(n + 1)}.
\]
Therefore, the periodic orbit remains elliptic for all \( l \geq 0 \) with \( l < \frac{\tau}{2} \frac{1}{\rho_0} \frac{n}{2(n + 1)} \). This upper limit can be made as large as desired by choosing \( \rho_0 \) such that \( \frac{\rho}{\rho_0} - \frac{1}{n} \frac{n}{2(n + 1)} \) is positive and near zero, i.e. by choosing \( \rho_0 \approx 2(1 + \frac{1}{n}) \rho \).

Provided that we change the small smoothening of the circular boundary component to a slightly larger smoothening region we can combine the above construction without changing the orbits constructed in item (i) of Theorem 2.2. By possibly adjusting the above construction we can again guarantee the nonlinear stability of these periodic orbits. Hence we can find arbitrarily short smoothenings of a circular boundary component such that the corresponding smooth stadium has a nonlinearly stable periodic orbit for all separation distances. And since the nonvanishing of the first Birkhoff coefficient is an open condition for \( C^5 \) perturbation of \( \Gamma \) we obtain an open set of smoothenings with the property that the corresponding smooth stadium has nonlinearly stable periodic orbits for all separation distances. This finishes the proof of Theorem 2.3.

\[\Box\]

6. Conclusions

Since the appearance of [14] and [2, 3] it is known that the dynamics of the billiards depends on the smoothness of the boundary in an essential way.

More precisely, as long as the boundary of a strictly convex two-dimensional billiard table is of class \( C^6 \) it was shown in [11] that a positive measure family of caustics is present near the boundary of the billiard table. In particular, such billiards are never ergodic.

The results of [2, 3] show that the if the boundary of convex billiards is allowed to be only \( C^1 \), then the resulting billiards may be hyperbolic and ergodic. In fact, the \( C^1 \) smoothness is only imposed at a few isolated points, and away from these points the boundary can be \( C^\infty \) or even analytic.

A standard example of billiards considered in [2, 3] is the stadium billiard. Our result of Theorem 2.2 shows that if at the endpoints of the circular boundary segments of the stadium the curvature is smoothed out, then elliptic periodic orbits are present for a large class of such \( C^2 \) stadium like billiards.
By our assumption the curvature is continuous on each boundary component. This implies that the global smoothness of the boundary is either $C^1$ or $C^2$; no intermediate fractional smoothness $C^{1+\alpha}$ is possible. Either there is a point on the boundary where the curvature has a jump, or not. For global $C^1$ or $C^0$ smoothness of the boundary of the billiard table [2, 3] provides large classes of completely hyperbolic and ergodic billiards. Therefore, our results indicate that for convex billiards with piecewise smooth ($C^3$ is enough) and globally $C^2$-smooth boundary elliptic periodic orbits are expected to be present. Hence global $C^2$ smoothness represents the critical smoothness for elliptic dynamics to be present in convex billiards.

References

[1] L. A. Bunimovich. Billiards that are close to scattering billiards. *Mat. Sb. (N.S.),* 94(136):49–73, 159, 1974.
[2] L. A. Bunimovich. The ergodic properties of certain billiards. *Funkcional. Anal. i Priložen.,* 8(3):73–74, 1974.
[3] L. A. Bunimovich. On the ergodic properties of nowhere dispersing billiards. *Comm. Math. Phys.,* 65(3):295–312, 1979.
[4] L. A. Bunimovich. On absolutely focusing mirrors. In *Ergodic theory and related topics, III (Güstrow, 1990),* volume 1514 of Lecture Notes in Math., pages 62–82, Springer, Berlin, 1992.
[5] L. A. Bunimovich. Absolute focusing and ergodicity of billiards. *Regul. Chaotic Dyn.,* 8(1):15–28, 2003.
[6] Leonid A. Bunimovich and Alexander Grigo. Focusing components in typical chaotic billiards should be absolutely focusing. *Comm. Math. Phys.,* 293(1):127–143, 2010.
[7] Nikolai Chernov and Roberto Markarian. *Chaotic billiards,* volume 127 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2006.
[8] Gianluigi Del Magno and Roberto Markarian. On the Bernoulli property of planar hyperbolic billiards. *Comm. Math. Phys.,* 350(3):917–955, 2017.
[9] Mário Jorge Dias Carneiro, Sylvie Oliffson Kamphorst, and Sônia Pinto de Carvalho. Elliptic islands in strictly convex billiards. *Ergodic Theory Dynam. Systems,* 23(3):799–812, 2003.
[10] Victor J. Donnay. Using integrability to produce chaos: billiards with positive entropy. *Comm. Math. Phys.,* 141(2):225–257, 1991.
[11] R. Douady. *Applications du théorème des tores invariants. Thèse de 3 e cycle.* PhD thesis, Université Paris VII, 1982.
[12] Sylvie Oliffson Kamphorst and Sônia Pinto-de Carvalho. The first Birkhoff coefficient and the stability of 2-periodic orbits on billiards. *Experiment. Math.,* 14(3):299–306, 2005.
[13] Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems,* volume 54 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza.
[14] V. F. Lazutkin. Existence of caustics for the billiard problem in a convex domain. *Izv. Akad. Nauk SSSR Ser. Mat.,* 37:186–216, 1973.
[15] Vladimir F. Lazutkin. *KAM theory and semiclassical approximations to eigenfunctions,* volume 24 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)].* Springer-Verlag, Berlin, 1993. With an addendum by A. I. Shnirelman.
[16] Roberto Markarian. Billiards with Pesin region of measure one. *Comm. Math. Phys.,* 118(1):87–97, 1988.
[17] Karl Friedrich Siburg. *The principle of least action in geometry and dynamics,* volume 1844 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2004.
[18] Serge Tabachnikov. *Geometry and billiards,* volume 30 of *Student Mathematical Library,* American Mathematical Society, Providence, RI, 2005.
[19] Maciej Wojtkowski. Principles for the design of billiards with nonvanishing Lyapunov exponents. *Comm. Math. Phys.,* 105(3):391–414, 1986.
Appendix A. Notation and Some Facts about Planar Billiards

Below we collect some well-known facts about billiards \( \mathcal{F} \), and derive the higher order expansions of the billiard map. The latter will be used extensively in the study of nonlinear stability of periodic orbits, by means of Birkhoff normal form expansion. In what follows we will use the notations

\[
\Gamma'(s) = \mathcal{T}(s), \quad \mathcal{T}(s) = \begin{pmatrix} \cos \theta(s) \\ \sin \theta(s) \end{pmatrix},
\]

(A.1)

\[
\mathcal{N}(s) = \begin{pmatrix} -\sin \theta(s) \\ \cos \theta(s) \end{pmatrix}, \quad \theta'(s) = -\mathcal{K}(s).
\]

Lemma A.1 (Billiard map). Let \((\bar{s}_{i+1}, \bar{\varphi}_{i+1}) = \mathcal{F}(\bar{s}_i, \bar{\varphi}_i)\) be a given segment of a billiard trajectory. Then the billiard map \((s_{i+1}, \varphi_{i+1}) = \mathcal{F}(s_i, \varphi_i)\) is (locally) determined by the relations

\[
\int_{s_i}^{s_{i+1}} \cos[\varphi_{i+1} + \theta(\bar{s}_{i+1}) - \theta(\sigma) - \varphi_i + \varphi_i + \theta(s_i) - \theta(\bar{s}_i)] d\sigma
\]

\[
= -\int_{\bar{s}_i}^{s_i} \cos[\varphi_i + \theta(\sigma) - \theta(s_i)] d\sigma - \bar{\tau}_{i, i+1} \sin[\varphi_i - \varphi_i - \theta(s_i) + \theta(\bar{s}_i)]
\]

and

\[
\varphi_{i+1} - \varphi_{i+1} = \varphi_i - \varphi_i + \theta(s_i) - \theta(s_i) + \theta(s_{i+1}) - \theta(s_{i+1}),
\]

where \(\theta'(s) = -\mathcal{K}(s)\) for any \(s\). The free path is then given by

\[
\tau_{i, i+1} = \int_{s_i}^{s_{i+1}} \sin[\varphi_{i+1} + \theta(\bar{s}_{i+1}) - \theta(\sigma) - \varphi_i + \varphi_i + \theta(s_i) - \theta(\bar{s}_i)] d\sigma
\]

\[
- \int_{\bar{s}_i}^{s_i} \sin[\varphi_i + \theta(\sigma) - \theta(s_i)] d\sigma + \bar{\tau}_{i, i+1} \cos[\varphi_i - \varphi_i - \theta(s_i) + \theta(\bar{s}_i)].
\]

Proof. Given \(s_i\) and \(\varphi_i\) then arc length parameter \(s_{i+1}\) of the next point of reflection is determined by

\[
\Gamma(s_{i+1}) = \Gamma(s_i) + \tau_{i, i+1} \left[ \mathcal{T}(s_i) \sin \varphi_i + \mathcal{N}(s_i) \cos \varphi_i \right]
\]

for some \(\tau_{i, i+1} > 0\). Using the parametrization of the boundary in terms of its curvature \(\mathcal{A.1}\) we have that

\[
\mathcal{T}(s_i) \sin \varphi_i + \mathcal{N}(s_i) \cos \varphi_i = \begin{pmatrix} \sin[\varphi_i - \theta(s_i)] \\ \cos[\varphi_i - \theta(s_i)] \end{pmatrix}
\]

and hence

\[
\Gamma(s_{i+1}) = \Gamma(s_i) + \tau_{i, i+1} \begin{pmatrix} \sin[\varphi_i - \theta(s_i)] \\ \cos[\varphi_i - \theta(s_i)] \end{pmatrix}.
\]

Therefore, the values of \(s_{i+1}\) and \(\tau_{i, i+1}\) are (locally) determined by

\[
\Gamma(s_{i+1}) - \Gamma(s_{i+1}) = \Gamma(s_i) - \Gamma(s_i) + \tau_{i, i+1} \begin{pmatrix} \sin[\varphi_i - \theta(s_i)] \\ \cos[\varphi_i - \theta(s_i)] \end{pmatrix}
\]

\[
- \bar{\tau}_{i, i+1} \begin{pmatrix} \sin[\varphi_i - \theta(s_i)] \\ \cos[\varphi_i - \theta(s_i)] \end{pmatrix}.
\]

(A.2)
Eliminating $\tau_{i,i+1}$ yields
\[
[\Gamma(s_{i+1}) - \Gamma(\bar{s}_{i+1})] \cdot \left( -\frac{\cos[\varphi_i - \theta(s_i)]}{\sin[\varphi_i - \theta(s_i)]} \right) = [\Gamma(s_i) - \Gamma(\bar{s}_i)] \cdot \left( -\frac{\cos[\varphi_i - \theta(s_i)]}{\sin[\varphi_i - \theta(s_i)]} \right) - \bar{\tau}_{i,i+1} \sin[\varphi_i - \bar{\varphi}_i - \theta(s_i) + \theta(\bar{s}_i)] ,
\]
which determines (locally) $s_{i+1}$. Writing the differences $\Gamma(s_i) - \Gamma(\bar{s}_i)$ and $\Gamma(s_{i+1}) - \Gamma(\bar{s}_{i+1})$ in terms of the integral over the corresponding tangent vector we obtain
\[
(A.3) \quad \int_{\bar{s}_{i+1}}^{s_{i+1}} \cos[\varphi_i + \theta(\sigma) - \theta(s_i)] \, d\sigma = \int_{\bar{s}_i}^{s_i} \cos[\varphi_i + \theta(\sigma) - \theta(s_i)] \, d\sigma + \bar{\tau}_{i,i+1} \sin[\varphi_i - \bar{\varphi}_i - \theta(s_i) + \theta(\bar{s}_i)] .
\]
Furthermore, the angle of reflection $\varphi_{i+1}$ is determined by
\[
(A.4) \quad \varphi_{i+1} + \theta(s_{i+1}) = \pi - \varphi_i + \theta(s_i) .
\]
Using (A.4) we obtain
\[
\varphi_i + \theta(\sigma) - \theta(s_i) = \varphi_i - \bar{\varphi}_i + \theta(\sigma) - \theta(\bar{s}_{i+1}) - \theta(s_i) + \bar{\varphi}_i + \theta(\bar{s}_{i+1}) = \pi - \bar{\varphi}_{i+1} + \varphi_i - \varphi_i + \theta(\sigma) - \theta(\bar{s}_{i+1}) - \theta(s_i) + \theta(\bar{s}_i) .
\]
Substituting this back into equation (A.3) for $s_{i+1}$ proves the second relation for the billiard map.

To finish the proof, note that (A.2) also shows that
\[
[\Gamma(s_{i+1}) - \Gamma(\bar{s}_{i+1})] \cdot \left( \frac{\sin[\varphi_i - \theta(s_i)]}{\cos[\varphi_i - \theta(s_i)]} \right) = [\Gamma(s_i) - \Gamma(\bar{s}_i)] \cdot \left( \frac{\sin[\varphi_i - \theta(s_i)]}{\cos[\varphi_i - \theta(s_i)]} \right) + \tau_{i,i+1} - \bar{\tau}_{i,i+1} \cos[\varphi_i - \bar{\varphi}_i - \theta(s_i) + \theta(\bar{s}_i)] ,
\]
which can be simplified to
\[
\int_{\bar{s}_{i+1}}^{s_{i+1}} \sin[\varphi_i - \theta(s_i) + \theta(\sigma)] \, d\sigma = \int_{\bar{s}_i}^{s_i} \sin[\varphi_i - \theta(s_i) + \theta(\sigma)] \, d\sigma + \tau_{i,i+1} - \bar{\tau}_{i,i+1} \cos[\varphi_i - \bar{\varphi}_i - \theta(s_i) + \theta(\bar{s}_i)] .
\]
From (A.4) we obtain $\bar{\varphi}_{i+1} + \theta(\bar{s}_{i+1}) = \pi - \bar{\varphi}_i + \theta(\bar{s}_i)$, and hence can rewrite the previous relation as
\[
\int_{\bar{s}_{i+1}}^{s_{i+1}} \sin[\pi - \bar{\varphi}_{i+1} + \varphi_i - \bar{\varphi}_i - \theta(s_i) + \theta(\bar{s}_i) + \theta(\sigma) - \theta(\bar{s}_{i+1})] \, d\sigma = \int_{\bar{s}_i}^{s_i} \sin[\varphi_i - \theta(s_i) + \theta(\sigma)] \, d\sigma + \tau_{i,i+1} - \bar{\tau}_{i,i+1} \cos[\varphi_i - \bar{\varphi}_i - \theta(s_i) + \theta(\bar{s}_i)] .
\]
Simplifying yields the claimed expression, and finishes the proof. \hfill \Box

The billiard flow in Cartesian coordinates $x$, $y$ on the billiard table $Q$ and velocity vector $v_x = \cos \omega$, $v_y = \sin \omega$ preserves the form $dx \wedge dy \wedge d\omega$. Near such a point $(\bar{x}, \bar{y}, \bar{\omega})$ one can then locally define the so-called Jacobi coordinates $\eta$, $\xi$, $\omega$ (e.g. [7])
\[
\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} + \eta v + \xi v^\perp , \quad v = \begin{pmatrix} \cos \omega \\ \sin \omega \end{pmatrix}
\]
or simply
\[
\eta = [x - \bar{x}] \cos \omega + [y - \bar{y}] \sin \omega \\
\xi = -[x - \bar{x}] \sin \omega + [y - \bar{y}] \cos \omega.
\]

(A.5)

It is straightforward to check that the billiard flow preserves the form \(d\eta \wedge d\xi \wedge d\omega\).

Restricting the points \((\bar{x}, \bar{y})\) and \((x, y)\) to some boundary component \(\Gamma\), i.e. \(\Gamma(\bar{s}) = (\bar{x}, \bar{y})\) and \(\Gamma(s) = (x, y)\), the (local) Jacobi coordinates induce the local coordinates \((x, y)\) on \(\Gamma\) in place of the arc length parameter \(s\) and the angle \(\omega\)
\[
x = x(s, \omega) = [\Gamma(s) - \Gamma(\bar{s})] \cdot \left(\begin{array}{c}
-\sin \omega \\
\cos \omega
\end{array}\right), \\
y = y(s, \omega) = \omega - \bar{\omega}
\]

and
\[
z = z(s, \omega) = [\Gamma(s) - \Gamma(\bar{s})] \cdot \left(\begin{array}{c}
\cos \omega \\
\sin \omega
\end{array}\right),
\]

which corresponds to \(\eta\). Clearly, these coordinates are well defined away from tangencies, and can be expressed as
\[
x = x(s, \omega) = \int_{\bar{s}}^{s} \sin[\theta(\sigma) - \omega] \, d\sigma, \\
y = y(s, \omega) = \omega - \bar{\omega}
\]

(A.6)

where we used the (local) parametrization \((A.1)\) of \(\Gamma\) in terms of the curvature \(K\).

Note also that \((A.6)\) implies
\[
dx = \sin[\theta(s) - \omega] \, ds - z \, d\omega, \\
dy = d\omega
\]

and hence
\[
dx \wedge dy = \sin[\theta(s) - \omega] \, ds \wedge d\omega = \cos \varphi \, ds \wedge d\varphi,
\]

where
\[
\varphi \equiv \varphi(s, \omega) = \theta(s) + \frac{\pi}{2} - \omega
\]

(A.8)

denotes the angle of reflection corresponding at the point \(\Gamma(s)\), i.e. the angle between the flow direction given by \(\omega\) and the normal line spanned by \(N(s)\), which is counted positively in direction of tangent vector \(\Gamma'(s) = T(s) = \left(\begin{array}{c}
\cos \theta(s) \\
\sin \theta(s)
\end{array}\right)\).

Lemma A.2 (Billiard dynamics in local Jacobi coordinates).

(i) (Free flight) Suppose that \((\bar{x}_1, \bar{y}_1) = (\bar{x}_0, \bar{y}_0) + \bar{\tau}_{0,1} (\cos \bar{\omega}_0, \sin \bar{\omega}_0), (\bar{x}_1, \bar{y}_1) = \Gamma_1(\bar{s}_1)\). Then the billiard flow at the moment right before the next reflection off of \(\Gamma_1\) is locally given by
\[
x_1 = x_0 + \bar{\tau}_{0,1} \sin y_0, \\
y_1 = y_0
\]

where \(\bar{\omega}_1 = \bar{\omega}_0\) for the flow direction at \((\bar{x}_1, \bar{y}_1)\). The corresponding expressions for \(\tau_{0,1}\) and \(z_1\) read
\[
\tau_{0,1} = z_1 - z_0 + \bar{\tau}_{0,1} \cos y_0, \\
z_1 = \int_{\bar{s}_1}^{s_1} \cos[\theta(\sigma) - \bar{\omega}_0 - y_0] \, d\sigma.
\]
If in addition \((\bar{x}_0, \bar{y}_0) = \Gamma_0(\bar{s}_0)\), and the initial points are restricted to \(\Gamma_0\), then locally

\[
x_1 = x_0 + \bar{\tau}_{0,1} \sin y_0, \quad y_1 = y_0,
\]

and

\[
\tau_{0,1} = z_1 - z_0 + \bar{\tau}_{0,1} \cos y_0, \quad z_i = \int_{\bar{s}_i}^{s_i} \cos[\theta(\sigma) - \bar{\omega}_0 - y_0] \, d\sigma \quad (i = 0, 1)
\]
is the expression for the corresponding free path.

(ii) (Reflection) Let \((\bar{x}, \bar{y}) = \Gamma(\bar{s})\) and \(\bar{\omega}\) be given. Then the reflection off of \(\Gamma\) is locally given by

\[
x^+ = \int_{\bar{s}}^{s} \sin[\theta(\sigma) - 2\theta(s) + y^- + \bar{\omega}^-] \, d\sigma
\]

\[
y^+ = 2 \big[\theta(s) - \theta(\bar{s})\big] - y^-,
\]

where \(s \equiv s(x^-, y^-)\) is determined by

\[
x^- = \int_{\bar{s}}^{s} \sin[\theta(\sigma) - y^- - \bar{\omega}^-] \, d\sigma
\]

and \(\bar{\omega}^+ = 2\theta(\bar{s}) - \bar{\omega}^-\).

(iii) (Invariant form) Both, a free flight and a reflection preserve the form \(dx \wedge dy\), and hence so does the billiard map.

Proof. Locally the billiard flow is determined by

\[
\Gamma_1(s_1) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \tau_{0,1} \begin{pmatrix} \cos \omega_0 \\ \sin \omega_0 \end{pmatrix}, \quad \omega_1 = \omega_0
\]

where \((x_0, y_0, \omega_0)\) is assumed to be close to \((\bar{x}_0, \bar{y}_0, \bar{\omega}_0)\). Therefore, this is equivalent to

\[
\Gamma_1(s_1) - \Gamma_1(\bar{s}_1) = \begin{pmatrix} x_0 - \bar{x}_0 \\ y_0 - \bar{y}_0 \end{pmatrix} + \tau_{0,1} \begin{pmatrix} \cos \omega_0 \\ \sin \omega_0 \end{pmatrix} - \bar{\tau}_{0,1} \begin{pmatrix} \cos \bar{\omega}_0 \\ \sin \bar{\omega}_0 \end{pmatrix}
\]

\[
\omega_1 = \omega_0, \quad \bar{\omega}_1 = \bar{\omega}_0.
\]

Hence

\[
[\Gamma_1(s_1) - \Gamma_1(\bar{s}_1)] \cdot \begin{pmatrix} -\sin \omega_0 \\ \cos \omega_0 \end{pmatrix} = \begin{pmatrix} x_0 - \bar{x}_0 \\ y_0 - \bar{y}_0 \end{pmatrix} \cdot \begin{pmatrix} -\sin \omega_0 \\ \cos \omega_0 \end{pmatrix} + \tau_{0,1} \sin[\omega_0 - \bar{\omega}_0]
\]

\[
\omega_1 - \bar{\omega}_1 = \omega_0 - \bar{\omega}_0
\]

and

\[
[\Gamma_1(s_1) - \Gamma_1(\bar{s}_1)] \cdot \begin{pmatrix} \cos \omega_0 \\ \sin \omega_0 \end{pmatrix} = \begin{pmatrix} x_0 - \bar{x}_0 \\ y_0 - \bar{y}_0 \end{pmatrix} \cdot \begin{pmatrix} \cos \omega_0 \\ \sin \omega_0 \end{pmatrix} + \tau_{0,1} \sin[\omega_0 - \bar{\omega}_0] .
\]

In terms of the (local) Jacobi coordinates (A.3) and (A.6) this takes on the form

\[
x_1 = \xi_0 + \bar{\tau}_{0,1} \sin y_0, \quad y_1 = y_0
\]

\[
z_1 = \eta_0 + \bar{\tau}_{0,1} \cos y_0, \quad z_1 = \int_{\bar{s}_1}^{s_1} \cos[\theta(\sigma) - \bar{\omega}_0 - y_0] \, d\sigma
\]

and since restricting the initial points to \(\Gamma_0\) makes \(\xi_0 = x_0\) and

\[
\eta_0 = \zeta_0 = \int_{\bar{s}_0}^{s_0} \cos[\theta(\sigma) - \bar{\omega}_0 - y_0] \, d\sigma
\]

the first claim follows.
Let \((x^-, y^-) = \Gamma(s^-)\) and \(\omega^-\) be any point (near the reference point). Then the reflection about \(\Gamma\) is given by \(s^+ = s^-, \omega^+ = 2\theta(s) - \omega^-\). Since the corresponding Jacobi coordinates \((A.6)\) are given by

\[
x^- = \int_{\bar{s}}^{s} \sin[\theta(\sigma) - \omega^-] d\sigma, \quad y^- = \omega^- - \bar{\omega}^-\]

and

\[
x^+ = \int_{\bar{s}}^{s} \sin[\theta(\sigma) - \omega^+] d\sigma = \int_{\bar{s}}^{s} \sin[\theta(\sigma) - 2\theta(s) + \omega^-] d\sigma
\]

\[
y^+ = \omega^+ - \bar{\omega}^+ = 2\theta(s) - 2\theta(\bar{s}) - \omega^- + \bar{\omega}^- = 2[\theta(s) - \theta(\bar{s})] - y^- .
\]

This proves the second claim.

The invariance of \(dx \wedge dy\) under the free flight is readily verified. To see the invariance of this form under the reflection map observe that it follows from the above that

\[
dx^- = \sin[\theta(s) - y^- - \bar{\omega}^-] ds
\]

\[
dy^+ = 2\theta'(s) ds - dy^-
\]

\[
dx^+ = \left[ \int_{\bar{s}}^{s} \cos[\theta(\sigma) - 2\theta(s) + y^- + \bar{\omega}^-] d\sigma \right] [dy^- - 2\theta'(s) ds]
\]

\[
+ \sin[-\theta(s) + y^- + \bar{\omega}^-] ds
\]

\[
= - \left[ \int_{\bar{s}}^{s} \cos[\theta(\sigma) - 2\theta(s) + y^- + \bar{\omega}^-] d\sigma \right] dy^+ - dx^- .
\]

Therefore,

\[
dx^+ \wedge dy^+ = -dx^- \wedge dy^+ = -dx^- \wedge [2\theta'(s) ds - dy^-] = dx^- \wedge dy^-
\]

follows, which finishes the proof of the third claim. \(\square\)

For any given \(\bar{s}\) and \(\bar{\varphi}\) introduce the following notation

\[
(A.9) \quad R = \frac{2K(\bar{s})}{\cos \bar{\varphi}}, \quad R^{(1)} = \frac{4K'(\bar{s})}{\cos^2 \bar{\varphi}}, \quad R^{(2)} = \frac{8K''(\bar{s})}{\cos^3 \bar{\varphi}},
\]

which will be used in expressing the local expansion of the billiard dynamics.

**Lemma A.3** (Asymptotic expansions of the change of coordinates). Fix \(\bar{s}\) and \(\bar{\varphi}\), where \(\theta(\bar{s}) = \bar{\varphi} - \frac{\pi}{2} + \bar{\omega}\).

(i) The change of coordinates \((A.6)\) has the asymptotic expansion

\[
x(s, \omega) = - \cos \bar{\varphi} (s - \bar{s}) - \tan \bar{\varphi} [\cos \bar{\varphi} (s - \bar{s})] (\omega - \bar{\omega}) - \frac{1}{4} \tan \bar{\varphi} R [\cos \bar{\varphi} (s - \bar{s})]^2
\]

\[
+ \frac{1}{2} [\cos \bar{\varphi} (s - \bar{s})] (\omega - \bar{\omega})^2 + \frac{1}{4} R [\cos \bar{\varphi} (s - \bar{s})]^2 (\omega - \bar{\omega})
\]

\[
+ \frac{1}{24} [R^2 - \tan \bar{\varphi} R^{(1)}] [\cos \bar{\varphi} (s - \bar{s})]^3
\]

\[+ O_4(s - \bar{s}, \omega - \bar{\omega})\]

\[
y(s, \omega) = \omega - \bar{\omega} .
\]
(ii) The inverse of the change of coordinates (A.6) has the asymptotic expansion

\[ s(x, y) - \bar{s} \cos \bar{\phi} = -x + \tan \bar{\phi} x y - \frac{1}{4} \tan \bar{\phi} R x^2 \]

\[ - \frac{1}{2} [1 + 2 \tan^2 \bar{\phi}] x y^2 + \frac{1}{4} [1 + 3 \tan^2 \bar{\phi}] R x^2 y \]

\[ - \frac{1}{24} \left[ [1 + 3 \tan^2 \bar{\phi}] R^2 - \tan \bar{\phi} R^{(1)} \right] x^3 + O_4(x, y) \]

\[ \omega(x, y) = \bar{\omega} + y. \]

(iii) The expression for \( z \) in terms of \( x \) and \( y \) has the asymptotic expansion

\[ \frac{z(x, y)}{1 + \tan^2 \bar{\phi}} = -\frac{\tan \bar{\phi}}{1 + \tan^2 \bar{\phi}} x + x y - \frac{1}{4} R x^2 \]

\[ - \tan \bar{\phi} x y^2 + \frac{3}{4} R \tan \bar{\phi} x^2 y - \frac{1}{24} [3 \tan \bar{\phi} R^2 - R^{(1)}] x^3 \]

\[ + O_4(x, y). \]

Proof. Recall the definition of the Jacobi coordinates (A.6). With (A.8) we have

\[ x(s, \omega) = \sin[\theta(s) - \omega] (s - \bar{s}) - \cos[\theta(s) - \omega] K(s) \frac{1}{2} (s - \bar{s})^2 \]

\[ - \left[ \sin[\theta(s) - \omega] K(s)^2 + \cos[\theta(s) - \omega] K'(s) \right] \frac{1}{6} (s - \bar{s})^3 + O(s - \bar{s})^4 \]

\[ = - \frac{\cos[\bar{\omega} + \omega - \bar{\phi}]}{\cos \bar{\phi}} \left[ \cos \bar{\phi} (s - \bar{s}) \right] - \frac{1}{4} R \frac{\sin[\bar{\omega} + \omega - \bar{\phi}]}{\cos \bar{\phi}} [\cos \bar{\phi} (s - \bar{s})]^2 \]

\[ + \frac{1}{24} \left[ \frac{\cos[\bar{\omega} + \omega - \bar{\phi}]}{\cos \bar{\phi}} R^2 - \frac{\sin[\bar{\omega} + \omega - \bar{\phi}]}{\cos \bar{\phi}} R^{(1)} \right] [\cos \bar{\phi} (s - \bar{s})]^3 \]

\[ + O(s - \bar{s})^4. \]

and

\[ z(s, \omega) = \cos[\theta(s) - \omega] (s - \bar{s}) + \sin[\theta(s) - \omega] K(s) \frac{1}{2} (s - \bar{s})^2 \]

\[ - \left[ \cos[\theta(s) - \omega] K(s)^2 - \sin[\theta(s) - \omega] K'(s) \right] \frac{1}{6} (s - \bar{s})^3 + O(s - \bar{s})^4 \]

\[ = \frac{\sin[\bar{\omega} + \omega - \bar{\phi}]}{\cos \bar{\phi}} \left[ \cos \bar{\phi} (s - \bar{s}) \right] - \frac{1}{4} R \frac{\cos[\bar{\omega} + \omega - \bar{\phi}]}{\cos \bar{\phi}} [\cos \bar{\phi} (s - \bar{s})]^2 \]

\[ - \frac{1}{24} \left[ \frac{\sin[\bar{\omega} + \omega - \bar{\phi}]}{\cos \bar{\phi}} R^2 + \frac{\cos[\bar{\omega} + \omega - \bar{\phi}]}{\cos \bar{\phi}} R^{(1)} \right] [\cos \bar{\phi} (s - \bar{s})]^3 \]

\[ + O(s - \bar{s})^4. \]

Hence it follows that \( s = s(x, y) \) is locally given by

\[ \cos \bar{\phi} (s - \bar{s}) = -\frac{\cos \bar{\phi}}{\cos[\bar{\phi} - y]} x - \frac{1}{4} R \tan[\bar{\phi} - y] \frac{\cos^2 \bar{\phi}}{\cos^2[\bar{\phi} - y]} x^2 \]

\[ - \frac{1}{24} \left[ [1 + 3 \tan^2 [\bar{\phi} - y]] R^2 - \tan[\bar{\phi} - y] R^{(1)} \right] \frac{\cos^3 \bar{\phi}}{\cos^3[\bar{\phi} - y]} x^3 \]

\[ + O(x^4). \]

Expanding also with respect to \( y \) yields the claimed expression for \( z_\omega(s - \bar{s}) \).

Substituting this expression for \( s \) back into the above expansion for \( z \) yields the claimed expression for \( z(x, y) \). \( \square \)
Lemma A.4 (Asymptotic expansions of the billiard dynamics). Using the notation as in Lemma A.2 we have the following asymptotic expansions:

(i) (Free flight) The billiard flow starting at $\Gamma$ at the moment right before hitting $\Gamma_1$ is locally given by

\[
x_1 = x_0 + \bar{\tau}_{0,1} y_0 - \frac{1}{6} \bar{\tau}_{0,1} y_0^3 + \mathcal{O}(y_0^5), \quad y_1 = y_0,
\]

where $\bar{\omega}_1 = \bar{\omega}_0$.

(ii) (Reflection) Let $(\bar{x}, \bar{y}) = \Gamma(s)$ and $\bar{\omega}$ be given. Then the reflection off of $\Gamma$ is locally given by

\[
-x_+ = x_- - \frac{1}{2} \tan \bar{\varphi} R x_- - \frac{1}{2} [1 + 2 \tan^2 \bar{\varphi}] R x_- y_- \\
- \frac{1}{12} [3 [1 + \tan^2 \bar{\varphi}] R^2 + 2 \tan \bar{\varphi} R^{(1)}] x_-^3 \\
+ \mathcal{O}_4(x_-, y_-)
\]

\[
-y_+ = R x_- + y_- + \tan \bar{\varphi} R x_- y_- + \frac{1}{4} [\tan \bar{\varphi} R^2 + R^{(1)}] x_-^2 \\
+ \frac{1}{2} [1 + 2 \tan^2 \bar{\varphi}] R x_- y_- \\
+ \frac{1}{4} [1 + 3 \tan^2 \bar{\varphi}] R^2 + 2 \tan \bar{\varphi} R^{(1)}] x_-^2 y_- \\
+ \frac{1}{24} [1 + 3 \tan^2 \bar{\varphi}] R^3 + 4 \tan \bar{\varphi} R R^{(1)} + R^{(2)}] x_-^3 \\
+ \mathcal{O}_4(x_-, y_-)
\]

where $R$, $R^{(1)}$, $R^{(2)}$, $\bar{\varphi}$ correspond to the state right after the reflection.

Proof. The expansion of the expression for the free flight is trivial.

To obtain the expansion of the reflection observe first that by Lemma A.2 we have $\bar{\omega}^+ = 2 \theta(\bar{s}) - \bar{\omega}^-$. Therefore (A.8) shows that

\[
\bar{\varphi}^+ = \theta(\bar{s}) + \frac{\pi}{2} - \bar{\omega}^+ = \frac{\pi}{2} - \theta(\bar{s}) + \bar{\omega}^- = \pi - \bar{\varphi}^- 
\]

and thus $\cos \bar{\varphi}^+ = -\cos \bar{\varphi}^-$. Consequently $R^+ = -R^-$, $R^{(1)}_+ = R^{(1)}_-$, $R^{(2)}_+ = -R^{(2)}_-$.

Using the result of Lemma A.3 we can solve

\[
x_- = \int_s^\bar{s} \sin[\theta(\sigma) - y_- - \bar{\omega}_-] d\sigma
\]

for $s$

\[
\cos \bar{\varphi}^- (s - \bar{s}) = -x_- + \tan \bar{\varphi}^- x_- y_- - \frac{1}{4} \tan \bar{\varphi}^- R_- x_-^3 \\
- \frac{1}{2} [1 + 2 \tan^2 \bar{\varphi}^-] x_- y_- + \frac{1}{4} [1 + 3 \tan^2 \bar{\varphi}^-] R_- x_-^2 y_- \\
- \frac{1}{24} [1 + 3 \tan^2 \bar{\varphi}^-] R_-^2 - \tan \bar{\varphi}^- R^{(1)}_+] x_-^3 + \mathcal{O}_4(x_-, y_-),
\]
which can be rewritten as
\[
\cos \varphi_+ (s - \bar{s}) = x_- + \tan \varphi_+ x_- y_- + \frac{1}{4} \tan \varphi_+ R_+ x_-^2 \\
+ \frac{1}{2} \left[ 1 + 2 \tan^2 \varphi_+ \right] x_- y_-^2 + \frac{1}{4} \left[ 1 + 3 \tan^2 \varphi_+ \right] R_+ x_-^2 y_- \\
+ \frac{1}{24} \left[ 1 + 3 \tan^2 \varphi_+ [R_+^2 + \tan \varphi_+ R_+^{(1)}] \right] x_-^3 + O_4(x_-, y_-)
\]

By Lemma A.2 we have
\[
y_+ = 2 [\theta(s) - \theta(\bar{s})] - y_-, \text{ whose expansion}
- y_+ = -2 [\theta'(\bar{s})(s - \bar{s}) + \frac{1}{2} \theta''(\bar{s})(s - \bar{s})^2 + \frac{1}{6} \theta'''(\bar{s})(s - \bar{s})^3] + y^- + O(s - \bar{s})^4
= R_+ [\cos \varphi_+ (s - \bar{s})] + \frac{1}{4} R_+^{(1)} [\cos \varphi_+ (s - \bar{s})]^2 + \frac{1}{24} R_+^{(2)} [\cos \varphi_+ (s - \bar{s})]^3 \\
+ y^- + O(s - \bar{s})^4
\]
therefore takes on the claimed form in terms of \(x_-, y_-\).

For the corresponding \(x_+\) it follows from Lemma A.2 (or directly from (A.6)) that
\[
x_+ = \int_{\bar{s}}^s \sin[\theta(\sigma) - y_+ - \bar{\omega}_+] d\sigma.
\]

Therefore, the above expansions of \(s\) and \(y^+\), when combined with the result of Lemma A.3
\[
-x_+ = \cos \varphi_+ (s - \bar{s}) + \tan \varphi_+ [\cos \varphi_+ (s - \bar{s})] y_+ + \frac{1}{4} \tan \varphi_+ R_+ [\cos \varphi_+ (s - \bar{s})]^2 \\
- \frac{1}{2} [\cos \varphi_+ (s - \bar{s})] y_+^2 - \frac{1}{4} R_+ [\cos \varphi_+ (s - \bar{s})]^2 y_+ \\
- \frac{1}{24} R_+^2 - \tan \varphi_+ R_+^{(1)}] [\cos \varphi_+ (s - \bar{s})]^3 \\
+ O_4(s - \bar{s}, y_+).
\]

Dropping the subscript + for \(R_+, R_+^{(1)}, R_+^{(2)}, \varphi_+\) yields the claimed expression for the local expansion of \(x_+\) and \(-y_+\). \(\square\)

School of Mathematics, Georgia Institute of Technology, Atlanta GA 30332, USA
E-mail address: bunimovh@math.gatech.edu

University of Oklahoma, Department of Mathematics, Norman OK 73019, USA
E-mail address: grigo@math.ou.edu