The Chern-Gauss-Bonnet integral and asymptotic behavior

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Abstract: For $n \geq 4$, let $(\mathbb{R}^n, e^{2u}|dx|^2)$ be a complete even dimensional manifold with nonnegative $Q$ curvature and $Q e^{nu} \in L^1(\mathbb{R}^n)$. Suppose $C_0 e^{-|x|^{\beta}} \leq Q_g(x)$ at infinity for some $\beta \in (0, 2)$, then

$$\alpha := \frac{1}{c_n} \int_{\mathbb{R}^n} Q_g(y) e^{nu(y)} dy \leq 1$$

and

$$\chi(\mathbb{R}^n) - \frac{1}{c_n} \int_{\mathbb{R}^n} Q_g(y) e^{nu(y)} dy = \lim_{r \to \infty} \frac{\partial B_r(0)|_{g}}{n \omega_{n-1} |B_r(0)|_g},$$

where $c_n = 2^{n-2} \frac{(n-2)!}{\pi^{n/2}}$, $\omega_{n-1} = |S^{n-1}|$. In fact, the lower bound of $Q_g$ is sharp. Moreover, if $Q_g$ also satisfies $Q_g(x) \leq C_0 |x|^\gamma$ at infinity for some $\gamma > 0$, then $u(x) = -\alpha \log |x| + o(\log |x|)$.

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1 Introduction

The Gauss-Bonnet formula plays an important role in differential geometry and is related to many geometric concepts. Its classical version can be expressed as follows: suppose $(M, g)$ is a compact manifold without boundary, then

$$\int_M K_g dV_g = 2\pi \chi(M),$$

where $K_g$ is the Gauss curvature. For complete surfaces, Huber [13] and Finn [11] proved that if $K_g$ is absolutely integral, then

$$\int_M K_g dV_g \leq 2\pi \chi(M) \quad (1.1)$$

and

$$\chi(M) = \frac{1}{2\pi} \int_M K_g dV_g = \sum_{i=1}^{k} \nu_i, \quad (1.2)$$

where $\nu_i$ is the isoperimetric ratio at the $i$-th end of $(M, g)$, i.e,

$$\nu_i = \lim_{r \to \infty} \frac{\partial B_r|_{g}}{4\pi |B_r|_g}.$$
where \( B_r(0) \) contains a compact set \( K_i \) and \( M \setminus K_i \) is diffeomorphic to \( \mathbb{R}^2 \setminus B_1(0) \).

Subsequently, Chang, Qing and Yang \([3, 4]\) successfully generalized (1.1) and (1.2) to four dimensions. They showed that if \((\mathbb{R}^4, e^{2u}|dx|^2)\) is complete with absolutely integrable \( Q_g e^{4u} \), and supposed that \( R_g \) is nonnegative at infinity, then

\[
\int_{\mathbb{R}^4} Q_g(x) e^{4u(x)} dx \leq 4\pi^2 \tag{1.3}
\]

and

\[
1 - \frac{1}{4\pi^2} \int_{\mathbb{R}^4} Q_g(x) e^{4u(x)} dx = \lim_{r \to \infty} \frac{|\partial B_r(0)|_{g}^2}{4(2\pi)^{1/3}|B_r(0)|_{g}}. \tag{1.4}
\]

For \( Q \) curvature, it is a very interesting conception. There are numerous papers concerned with it, but we only list some famous references \([1, 5, 6, 9]\).

Recently, Wang \([21]\) has made significant progress on the isoperimetric inequalities in higher dimensions. Using a Sobolev inequality with weights, she has shown that assuming \((\mathbb{R}^n, e^{2u}|dx|^2)\) is complete and normal, i.e.,

\[
\alpha_1 := \int_{\mathbb{R}^n} Q_g^+(y) e^{nu(y)} dy < c_n
\]

and

\[
\alpha_2 := \int_{\mathbb{R}^n} Q_g^-(y) e^{nu(y)} dy < +\infty,
\]

then there exist \( C = C(n, \alpha_1, \alpha_2) \) such that for any bounded smooth domain \( \Omega \) in \( \mathbb{R}^n \),

\[
|\Omega|_g \leq C(n, \alpha_1, \alpha_2) |\partial \Omega|_{g^{1/n}}^{1/n}. \]

By studying the \( n \)-Laplace equation, a general Huber type theorem have been established by S. Ma and Qing \([19]\).

In this paper, we consider the manifold is \((\mathbb{R}^n, g = e^{2u}|dx|^2)\) with nonnegative \( Q \) curvature, \( n = 2m \geq 4 \) and denote

\[
\alpha = \frac{1}{c_n} \int_{\mathbb{R}^n} Q_g dV_g = \frac{1}{c_n} \int_{\mathbb{R}^n} Q_g(y) e^{nu(y)} dy \tag{1.5}
\]

and

\[
f(y) = Q_g(y) e^{nu(y)}, \tag{1.6}
\]

then we know

\[
(-\Delta)^m u = 2Q_g e^{nu} \quad \text{in} \quad \mathbb{R}^n. \tag{1.7}
\]

From the viewpoint of PDE, the above theorem can be expressed as follows: under some curvature decay assumption, the solution of the PDE (1.7) has some asymptotic behavior. In two dimension, Cheng and Lin \([2]\) demonstrated that if \( u \) solves

\[
-\Delta u = K(x) e^{2u(x)}
\]

with absolutely integrable \( K(x) e^{2u(x)} \), and

\[
C_0 e^{-|x|^\beta} \leq K(x) \leq C_0 |x|^{-\gamma}
\]

for \( \beta, \gamma > 0 \).
for some $\beta \in (0,1)$ or
\[ C_0|x|^{-\gamma} \leq -K(x) \leq C_0|x|^{\gamma} \]
for $|x| \gg 1$, then
\[ u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^n} \log\left( \frac{|y|}{|x-y|} K(y) e^{2u(y)} \right) dy + C \] (1.8)

and
\[ u(x) = -\alpha \log |x| + o(\log |x|). \] (1.9)

In four dimension, if $Q_g$ is a constant in (1.7), Lin [17] creatively gave a systematic approach to study the higher order critical exponent elliptic equation. In high dimension, the solutions are classified by Chang and Yang [7] and Wei and Xu [22] under $u(x) = o(|x|^2)$ constraint. Afterwards, L. Martinazzi [18] gave another nice proof and proposed some new ideas. For more general $Q$, you can also refer [8, 12].

In this paper, our motivations are twofold. First, we hope to generalize the results of Chang, Qing and Yang [3] to high dimension without the scalar curvature assumption. Second, we want to get the asymptotic behavior (1.9) at infinity in high dimension. So, what are the appropriate assumptions about $Q_g$? From many excellent papers mentioned above, we conclude the following assumptions:

(1) $e^{-|x|^\beta} \leq Q_g(x)$ for some $\beta \in (0,2)$ if $|x| \gg 1$. The necessity of this assumption can be seen in the following example, which was proposed by Chang, Qing and Yang [3]. Let $u(x) = \log \frac{1}{1+|x|^2} + |x|^2$, you can check the completeness of the metric, then
\[ \frac{1}{cn} \int_{\mathbb{R}^n} Q_g dV_g = \frac{1}{cn} \int_{\mathbb{R}^n} (-\Delta)^{n} u = 2 > 1. \]

In the above example $Q_g = e^{-|x|^\beta} Q_{g,n}$, which means that the decay of $Q_g$ does not seem to be too fast. In this paper, we prove that the exponent 2 is sharp. Especially, in two dimension, Cheng and Lin [2] have pointed out that $\beta < 1$ is necessary for (1.8) and (1.9) without completeness of metric, you can also refer Lemma 3.2. Let us explain that why $\beta \in (0,2)$ will imply the normal metric. Roughly speaking, $e^{-|x|^\beta} \leq Q_g(x)$ will give that
\[ u(x) = \frac{1}{4\pi^2} \int_{\mathbb{R}^n} \log \left( \frac{|y|}{|x-y|} Q_g(y) e^{4u(y)} \right) dy + C + \sum_{i=1}^{n} a_i x_i \]
\[ \sim -\alpha \log |x| + C + \sum_{i=1}^{n} a_i x_i. \]

But we can find a angle $\theta_0$ such that $u(r, \theta_0) \sim -c_0 r$, hence
\[ \int_{R_0}^{+\infty} e^{u(r, \theta_0)} dr < +\infty. \]

This is a contradiction with the completeness of $(\mathbb{R}^n, e^{2u}|dx|^2)$.

(2) $Q_g e^{\alpha u} \in L^1(\mathbb{R}^n)$. This is a natural assumption for nonnegative $Q_g$ curvature. In fact, this condition will imply that $Q_g$ has some decay at infinity. Formally speaking,
\[ u(x) \sim -\alpha \log |x| \quad \text{and} \quad \alpha \leq 1, \]

then
\[ Q_g(x) e^{\alpha u(x)} \sim \frac{Q_g(x)}{|x|^{\alpha}} \in L^1(\mathbb{R}^n). \]
Now we state our first theorem, the basic idea is somewhat inspired by [17] and [22].

**Theorem 1.1** Suppose \((\mathbb{R}^4, e^{2u}|dx|^2)\) satisfies
\[
\begin{cases}
Q_g \geq 0 \\
0 < C_0 e^{-|x|^\beta} \leq Q_g(x) \text{ for } |x| \gg 1, \text{ where } 0 < \beta < 1.
\end{cases}
\]
\[Q_g(x) e^{4u(x)} \in L^1(\mathbb{R}^n),
\]
\[u(x) = O(|x|^2).
\]
Then the metric is normal, i.e.,
\[u(x) = \frac{1}{4\pi^2} \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} Q_g(y) e^{4u(y)} dy + C.
\]

If we assume that the metric \((\mathbb{R}^n, e^{2u}|dx|^2)\) is complete, then we can establish the similarly results for all even dimension with \(\beta \in (0, 2).

**Theorem 1.2** Suppose \((\mathbb{R}^n, g = e^{2u}|dx|^2)\) is complete and satisfies
\[
\begin{cases}
C_0 e^{-|x|^\beta} \leq Q_g(x) \text{ for } |x| \gg 1, \text{ where } \beta \in (0, 2).
\end{cases}
\]
\[Q_g \geq 0 \text{ and } Q_g(x) e^{nu(x)} \in L^1(\mathbb{R}^n)
\]
Then,

(1) the metric is normal, i.e.,
\[u(x) = \frac{1}{c_n} \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} Q_g(y) e^{nu(y)} dy + C. \quad (1.10)
\]

(2)
\[
\int_{\mathbb{R}^n} Q_g(y) e^{nu(y)} dy \leq c_n. \quad (1.11)
\]

**Remark 1.1** We note that if \(R_g\) is nonnegative at infinity, then the inequality \((1.11)\) has been proved by H. Fang [10].

We consider some typical examples which satisfies all our assumptions. Let
\[u_\theta(x) = \theta \log \frac{2}{1 + |x|^2} \quad \text{for} \quad \theta \in (0, \frac{1}{2}],
\]
then \(g_\theta = e^{2u_\theta}|dx|^2\) is complete. Clearly,
\[Q_{g_\theta} = \frac{1}{2} e^{-nu_\theta} (-\Delta) u_\theta = \theta Q_{g_\theta} \left( \frac{2}{1 + |x|^2} \right)^{n(1-\theta)}
\]
and
\[
\int_{\mathbb{R}^n} Q_{g_\theta} e^{nu_\theta} = \theta \omega_n Q_{g_\theta} \leq c_n.
\]

Following the strategy of [3], we first suppose that \(u \) is radial function and try to establish \((1.12)\) and \((1.13)\). Next, we reduce the general case into the radial case. Finally, we prove that:

**Theorem 1.3** Let \((\mathbb{R}^n, e^{2u}|dx|^2)\) be a complete even dimensional manifold with nonnegative \(Q\) curvature and \(Q_g e^{nu} \in L^1(\mathbb{R}^n)\). Suppose \(C_0 e^{-|x|^\beta} \leq Q_g(x)\) at infinity for some \(\beta \in (0, 2),\) then
\[
\int_{\mathbb{R}^n} Q_g(y) e^{nu(y)} dy \leq c_n \quad (1.12)
\]
and
\[
1 - \frac{1}{c_n} \int_{\mathbb{R}^n} Q_g(y) e^{nu(y)} dy = \lim_{r \to \infty} \frac{\left| \partial B_r(0) \right|_{g_\nu}^{\frac{n}{n-1}}}{\nu_{n-1} |B_r(0)|_g}, \quad (1.13)
\]
Remark 1.2 With the same assumption as Theorem 1.3 and suppose \( \int_{\mathbb{R}^n} Q_g(y)e^{nu(y)}dy < c_n \), the isoperimetric inequality holds, i.e., there exists \( C(n, \alpha) \) such that for any bounded smooth domain \( \Omega \) in \( \mathbb{R}^n \)

\[
|\Omega| \leq C(n, \alpha)|\partial\Omega|^\frac{n}{n-1}.
\]

Suppose \( Q_g(x) \leq C_0|x|^{\gamma} \) at infinity, then we could obtain the high-dimensional asymptotic formula with respect to 1.9.

Theorem 1.4 Suppose \((\mathbb{R}^n, e^{2u}|dx|^2)\) is complete and \( u \) is a solution of

\[
(-\Delta)^m u(x) = 2Q_g(x)e^{nu(x)} \quad \text{in} \ \mathbb{R}^n.
\]

and satisfies

\[
\begin{cases}
C_0e^{-|x|^{\beta}} \leq Q_g(x) \leq C_0|x|^{\gamma} \quad \text{for} \ |x| \gg 1, \ \text{where} \ \gamma > 0 \ \text{and} \ \beta \in (0, 2).
\end{cases}
\]

\( Q_g \geq 0 \) and \( Q_g(x)e^{nu(x)} \in L^1(\mathbb{R}^n) \).

Then, for any \( \epsilon > 0 \), there exist \( R_\epsilon \gg 1 \) such that

\[
-(\alpha - \epsilon) \log |x| + C \geq u(x) \geq -\alpha \log |x| - C
\]  

(1.14) for \( |x| > R_\epsilon \).

The organization of this paper is as follows. In Section 2, we establish some crucial estimates for the integral equation. In Section 3, we first give a simple proof of normal metric in four dimension, then we focus on how to prove high-dimensional normal metric. In Section 4, following the method of [3], we first establish the radial version of Theorem 1.3, then we reduce the general case into the radial case. In section 5, we obtain important asymptotic formula (1.14).

2 Preliminaries

2.1 Weak harnack inequality of singular integral

In this section, we consider the integral equation

\[
v(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^\sigma}dy.
\]

Firstly, we introduce the conception of \( A_1 \) weight. A nonnegative function \( w \in A_1 \), for any ball \( B \subset \mathbb{R}^n \)

\[
\int_B w \leq C \inf_B w
\]

or equivalent to

\[
M(w)(x) = \sup_{r>0} \int_{B_r(x)} w \leq Cw(x) \quad \text{a.e.} \quad x \in \mathbb{R}^n.
\]

For the basic knowledge of \( A_p \) weight, you can refer the Chapter 5 of [20].

Lemma 2.1 If \( 0 < \sigma < n \), then \( \frac{1}{|x|^\sigma} \in A_1 \) weight.

Proof. For any \( B_r(x), \) if \( r < \frac{|x|}{4} \), then for any \( y \in B_r(x) \) we have \( \frac{1}{4} < |y| < \frac{|x|}{2} \). Thus,

\[
\int_{B_r(x)} \frac{1}{|y|^\sigma}dy \leq C \frac{1}{|x|^\sigma}.
\]
If \( r > \frac{|x|}{2} \), then
\[
\int_{B_r(x)} \frac{1}{|y|^\sigma} dy \leq \frac{1}{|B_r(x)|} \int_{B_{3r}(0)} \frac{1}{|y|^\sigma} dy \leq \frac{C}{r^\sigma} \leq \frac{C}{|x|^\sigma}.
\]
Then we know maximal function
\[
M \left( \frac{1}{|y|^\sigma} \right)(x) = \sup_{r>0} \int_{B_r(x)} \frac{1}{|y|^\sigma} dy \leq C \frac{1}{|x|^\sigma}.
\]
i.e., \( \frac{1}{|x|^\sigma} \in A_1 \).

\[\square\]

Lemma 2.2 Suppose \( f \) is a nonnegative function and \( 0 < k \leq n-2 \), then \( v_k(x) \in A_1 \) weight.
\[
C \int_{B_r} v_k \leq \inf_{\partial B_r} v_k \leq \int_{\partial B_r} v_k.
\] (2.1)

Proof. For any \( r > 0 \), we have
\[
\int_{B_r(x)} v_k(y) dy = \int_{B_r(x)} \int_{\mathbb{R}^n} \frac{f(z)}{|y-z|^k} dz dy = \int_{B_{r}(x)} \frac{1}{|y-z|^k} dy \int_{\mathbb{R}^n} f(z) dz 
\]
\[
\leq C \int_{\mathbb{R}^n} \frac{f(z)}{|x-z|^k} dz = C v_k(x).
\]
The above inequality follows by Lemma 2.1, i.e.,
\[
\int_{B_r(x)} \frac{1}{|y-z|^k} dy = \int_{B_r(0)} \frac{1}{|y+x-z|^k} dy \leq \frac{C}{|x-z|^k}.
\]
Then, \( M(v_k)(x) \leq C v_k(x) \), i.e., \( v_k \in A_1 \). From easy calculation, we know
\[
-\Delta v_k(x) = k(n+k-2) \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{k+2}} dy > 0 \quad \text{for} \quad 0 < k < n-2.
\]
And,
\[
-\Delta v_k = (n-2)\omega_{n-1} f \geq 0 \quad \text{for} \quad k = n-2.
\]
Hence, (2.1) follows by maximum principle and the definition of \( A_1 \) weight. \[\square\]

2.2 Estimate of singular integral

For even number \( n = 2m \), we consider the integral equation
\[
v(x) = \frac{1}{c_n} \int_{\mathbb{R}^n} \log \frac{|x-y|}{|y|} f(y) dy
\] (2.2)
where \( c_n = 2^{m-2} (m-1)! \pi^m \) and \( f \in L^1(\mathbb{R}^n) \) is a nonnegative continuous function. Denote
\[
\alpha = \frac{1}{c_n} \int_{\mathbb{R}^n} f(y) dy,
\]
obviously we have
\[
(-\Delta v)^m(x) = -2f(x) \quad \text{in} \quad \mathbb{R}^n.
\]
**Lemma 2.3** Suppose \( v \) satisfies (2.2), then for \( |x| \gg 1 \)
\[
v(x) \leq \alpha \log |x| + C,
\]
where \( C \) depends on \( \max B_1 f \).

**Proof.** We decompose \( \mathbb{R}^n \) to \( \mathbb{R}^n = A_1 \cup A_2 \cup A_3 \), where

\[
A_1 = \{ y \|y\| < |x|/2 \}, \quad A_2 = \{ y \|y - x\| < |x|/2 \}, \\
A_3 = \{ y \|y\| > |x|/2, \|y - x\| > |x|/2 \}.
\]

In \( A_2 \), we have \( |x - y| < |x|/2 < |y| \), then \( \frac{|x - y|}{|y|} \leq 0 \). In \( A_1 \), we know \( \frac{|x|}{2} \leq |x - y| \leq \frac{|x|}{2} \)
\[
\frac{1}{c_n} \int_{A_1} \log \frac{|x - y|}{|y|} f(y)dy \leq \log |x| \frac{1}{c_n} \int_{A_1} f(y)dy + C \int_{A_2} f(y)dy - \frac{1}{c_n} \int_{A_3} \log |y| f(y)dy \\
\leq \log |x| \frac{1}{c_n} \int_{A_1} f(y)dy + C \left( \frac{\max B_1 f}{c_n} \int_{|y| < 1} \log |y|dy \right) \\
\leq \log |x| \frac{1}{c_n} \int_{A_1} f(y)dy + C.
\]

In \( A_3 \), we get \( |x - y| \leq |y| + |x| \leq 3|y| \) and \( |y| \leq |x - y| + |x| \leq 3|x - y| \). Then,
\[
\frac{1}{c_n} \int_{A_3} \log \frac{|x - y|}{|y|} f(y)dy \leq C \int_{A_3} f(y)dy.
\]

So, we conclude that
\[
v(x) \leq \alpha \log |x| + C.
\]

\[\square\]

**Lemma 2.4** Suppose \( v \) satisfies (2.2) and \( f \) is a nonnegative continuous function, then for \( |x| \gg 1 \)
\[
v(x) \geq (\alpha - \epsilon) \log |x| - \frac{1}{c_n} \int_{|y - x| < 1} \log \frac{1}{|x - y|} f(y)dy.
\]

**Proof.** Similarly we decompose \( \mathbb{R}^n \) into \( \mathbb{R}^n = A_1 \cup A_2 \cup A_3 \cup A_4 \), where

\[
A_1 = \{ y \|y\| < R_0 \}, \quad A_2 = \{ y \|y - x\| < |x|/2 \} \\
A_3 = \{ y \|y\| > R_0, \|y - x\| > |x|/2, \|y\| < 2|x| \}, \quad A_4 = \{ y \|y\| > 2|x| \}.
\]

then
\[
\int_{\mathbb{R}^n} \log \frac{|x - y|}{|y|} f(y)dy = \sum_{i=1}^{4} \int_{A_i} \log \frac{|x - y|}{|y|} f(y)dy.
\]

\[
= I + II + III + IV. \tag{2.3}
\]

For \( A_1 \), we suppose \( 2R_0 \ll |x| \) then
\[
I \geq \int_{A_1} \log \frac{|x - y|}{|y|} f(y)dy \\
\geq \left( \int_{|y| < R_0} f(y)dy \right) \log |x| - (\log(2R_0) + C) \int_{|y| < R_0} f(y)dy. \tag{2.4}
\]

In \( A_2 \), \( |y| \gg R_0 \) and \( \frac{1}{2} |x| < |y| < \frac{3}{2} |x| \) then
\[
II = \int_{A_2} \log |x - y| f(y)dy - \int_{A_2} \log |y| f(y)dy
\]
\[ \geq \int_{|y-x|<1} \log |x-y| f(y) \, dy - \left( \int_{A_2} f(y) \, dy \right) \log |x| - C \]
\[ \geq \int_{|y-x|<1} \log |x-y| f(y) \, dy - \left( \int_{|y|>R_0} f(y) \, dy \right) \log |x| - C. \quad (2.5) \]

For \( A_3 \), then
\[ III = \int_{A_3} \log |x-y| f(y) \, dy - \int_{A_3} \log |y| f(y) \, dy \]
\[ \geq \left( \int_{A_3} f(y) \, dy \right) \log \left( \frac{|x|}{2} \right) - \left( \int_{A_3} f(y) \, dy \right) \log(2|x|) \]
\[ \geq -C \int_{A_3} f(y) \, dy. \quad (2.6) \]

For \( A_4 \), we know \( \frac{1}{2}|y| \leq |x-y| \leq \frac{3}{2}|y| \), then
\[ IV \geq -C \int_{A_4} f(y) \, dy. \quad (2.7) \]

Combated with (2.3), (2.4), (2.5), (2.6) and (2.7), we choose \( R_0 \) sufficiently big, then we complete the proof.

**Corollary 2.1** If \( f \in L^1(\mathbb{R}^n) \), then for \( R_0 \gg 1 \)
\[ \int_{\mathbb{R}^n \setminus B_{R_0}(0)} v^- \leq C. \]

**Proof.** By Lemma 2.4, we know
\[ \int_{\mathbb{R}^n \setminus B_{R_0}(0)} v^-(x) \, dx \leq C \int_{\mathbb{R}^n \setminus B_{R_0}(0)} \int_{|y-x|<1} \log \frac{1}{|x-y|} f(y) \, dy \, dx \]
\[ \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{|x-y|<1} \log \frac{1}{|x-y|} f(y) \, dy \, dx \]
\[ \leq C \int_{\mathbb{B}_1(y)} \log \frac{1}{|x-y|} \, dx \int_{\mathbb{R}^n} f(y) \, dy \leq C. \quad \square \]

**Lemma 2.5** Suppose \( v \) is given by (2.2), then
\[ (-\Delta)^k v(x) = \frac{d_k}{c_m} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{2k}} \, dy \quad \text{for } k = 1, \ldots, m-1, \]
where \( d_{k+1} = 2k(n-2k-2)d_k \) and \( d_1 = -(n-2) \).

**Proof.** For \( h \ll 1 \), we know
\[ \left| \frac{v(x + he_i) - v(x)}{h} \right| \leq \frac{C}{h} \int_{\mathbb{R}^n} \left( \frac{1 + |x + he_i - y| - |x-y|}{|x-y|} \right) f(y) \, dy \]
\[ \leq \frac{C}{h} \int_{\mathbb{R}^n} \left( \frac{|h + 2(x - y)|}{|x-y|} \right) f(y) \, dy \]
\[ \leq C \int_{\mathbb{R}^n} \left( \frac{1 + 2|x-y|}{|x-y|^2} \right) f(y) \, dy < +\infty. \]
So the Lebesgue dominated convergence theorem gives
\[ \frac{\partial v}{\partial x_i} = \lim_{h \to 0} \frac{v(x + h e_i) - v(x)}{h} = \frac{1}{c_n} \int_{\mathbb{R}^n} \frac{x_i - y_i}{|x - y|^2} f(y) dy. \]

Through a similar argument, we further prove
\[ \frac{\partial^2 v}{\partial x_i \partial x_j} = \frac{1}{c_n} \int_{\mathbb{R}^n} \left[ \frac{\delta_{ij}}{|x - y|^2} - \frac{2(x_i - y_i)(x_j - y_j)}{|x - y|^4} \right] f(y) dy. \]

So, we arrive
\[ -\Delta v(x) = -\frac{(n - 2)}{c_n} \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-2}} dy. \]

Since \((-\Delta)^k \log |x - y| = d_k|x - y|^{-2k}\), we obtain
\[ (-\Delta)^k v(x) = \frac{d_k}{c_n} \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{2k}} dy. \]

In the following, we focus on estimate the term \(\int_{\partial B_1(x)} (-\Delta)^k v(y) d\sigma(y)\), which will work in Lemma 5.1.

**Lemma 2.6** For \(n > k > 2\) and \(x \in \mathbb{R}^n\), there holds
\[ \int_{\mathbb{R}^n \setminus B_2(x)} \frac{1}{|x - y|^{n-2}|y - z|^k} dy \leq C_1 \chi_{B_2(x)}(z) + \frac{C_2 \chi_{B_3(z)}}{|x - z|^{k-2}}. \]

**Proof.** For simplicity, we let \(I = \int_{\mathbb{R}^n \setminus B_2(x)} \frac{1}{|x - y|^{n-2}|y - z|^k} dy\). First, if \(|x - z| \leq 2\), then \(|y - z| > |x - y| - |x - z| \geq 2\). Thus, we know
\[ |x - y| \leq |x - z| + |y - z| \leq 2|y - z| \]
and
\[ |y - z| \leq |x - y| + |x - z| \leq \frac{3}{2}|x - y|. \]
So,
\[ I \leq C \int_{\mathbb{R}^n \setminus B_2(x)} \frac{1}{|x - y|^{n+k-2}} \leq C_1. \]

Second, if \(|x - z| \geq 2\), splitting \(\mathbb{R}^n \setminus B_2(x)\) into \(\mathbb{R}^n \setminus B_2(x) = A_1 \cup A_2 \cup A_3\), where
\[ A_1 = \{ y \mid |x - y| \geq 4, |y - z| \leq |x - z|/2, \} , \]
\[ A_2 = \{ y \mid 4 \leq |x - y| \leq |x - z|/2, \} , \]
\[ A_3 = \{ y \mid |x - z|/2 < |x - y|, |x - z|/2 < |y - z| \}. \]
And we note that \(A_1, A_2\) may be null set. There two cases will happen.
Case 1: \(|x - z| \geq 8\), then \(A_1, A_2 \neq \emptyset\). We have
\[ I = \int_{A_1} + \int_{A_2} + \int_{A_3} \frac{1}{|x - y|^{n-2}|y - z|^k} dy \]
\[ := I_1 + I_2 + I_3. \]
In \(A_1\), we know
\[ |x - y| \leq |x - z| + |y - z| \leq \frac{3}{2}|x - z| \]
and
\[ |x - z| \leq |x - y| + |y - z| \leq |x - y| + \frac{|x - z|}{2}. \]

So,
\[ I_1 \leq \frac{C}{|x - z|^{n-2}} \int_{|y - z| < |x - z|/2} \frac{1}{|y - z|^k} \leq \frac{C}{|x - z|^{k-2}}. \]

Similarly, you can get
\[ I_2 \leq \frac{C}{|x - z|^{k-2}}. \]

In \( A_3 \),
\[ |x - y| \leq |x - z| + |y - z| \leq 3|y - z| \]

and
\[ |y - z| \leq |x - z| + |x - y| \leq 3|x - y|. \]

Thus, we obtain
\[ I_3 \leq C \int_{|y - z| > |x - z|/2} \frac{1}{|y - z|^{n+k-2}} dy \leq \frac{C}{|x - z|^{k-2}}. \]

Case 2: \( 2 \leq |x - z| \leq 8 \). Then we get
\[ I \leq \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-2}} dy = \int_{B_1} + \int_{B_2} + \int_{B_3} \frac{1}{|x - y|^{n-2}} dy, \]

where
\[ B_1 = \{ y | x - y | \leq 1 \}, \]
\[ B_2 = \{ y | y - z | \leq 1 \}, \]
\[ B_3 = \{ y | |x - z| > 1, |y - z| > 1 \}. \]

Similarly, you can get
\[ I \leq C, \] so we complete the proof. \( \square \)

**Lemma 2.7** For any \( u \in L^1_{loc}(\mathbb{R}^n) \), then for any \( R > 0 \) we have
\[ \int_0^R \frac{1}{|\partial B_r|} \int_{B_r} u(x) dx dr = \frac{1}{(n - 2)\omega_{n-1}} \int_{B_R} \left( \frac{1}{|x|^{n-2}} - \frac{1}{R^{n-2}} \right) u(x) dx. \]

**Proof.** By element calculation, we know
\[ \int_0^R \frac{1}{|\partial B_r|} \int_{B_r} u(x) dx dr = \int_0^R \frac{1}{|\partial B_r|} \int_0^r \int_{\partial B_s} u(x) d\sigma ds dr \]
\[ = \int_0^R \int_{\partial B_r} u(x) d\sigma dr \int_0^r \frac{1}{|\partial B_s|} ds dr \]
\[ = \frac{1}{(n - 2)\omega_{n-1}} \int_{B_R} \left( \frac{1}{|x|^{n-2}} - \frac{1}{R^{n-2}} \right) u(x) dx. \] \( \square \)
Lemma 2.8 If $v$ is defined by (2.2) and $f := -rac{1}{2}(-\Delta)^m v \in L^1(\mathbb{R}^n)$, then there exist $C$ which is independent of $x$ such that
\[ 0 \leq \int_{\partial B_4(x)} -(-\Delta)^i v(y)d\sigma(y) \leq C, \quad \text{for } i = 1, \ldots, m-1, \]
where $x \in \mathbb{R}^n$.

Proof. We argue it by induction. For $k = m-1$, we know
\[ -\int_{\partial B_4(x)} \frac{\partial}{\partial r}(-\Delta)^{m-1} v = \frac{1}{|\partial B_4(x)|} \int_{B_4(x)} (-\Delta)^m v. \]

Integral two sides from 0 to 4, then
\[ RHS = (-\Delta)^{m-1} v(x) - \int_{\partial B_4(x)} (-\Delta)^{m-1} v \\
= \frac{1}{(n-2)\omega_{n-1}} \int_{\mathbb{R}^n} \frac{(-\Delta)^m v(y)}{|x-y|^{n-2}} dy - \int_{\partial B_4(x)} (-\Delta)^{m-1} v \]
and by Lemma 2.7 we arrive
\[ LHS = \frac{1}{(n-2)\omega_{n-1}} \int_{B_4(x)} \left( \frac{1}{|x-y|^{n-2}} - \frac{1}{4^{n-2}} \right) (-\Delta)^m v(y)dy. \]

So, we get
\[ \int_{\partial B_4(x)} -(-\Delta)^{k-1} v = \frac{1}{(n-2)\omega_{n-1}} \left( \int_{\mathbb{R}^n \setminus B_4(x)} -(-\Delta)^k v(y)dy + \frac{1}{4^{n-2}} \int_{B_4(x)} (-\Delta)^k v(y)dy \right), \]
we get $k = m-1$. If we have know $k$ is right, we hope to prove $k-1$ is also true where $m-1 \geq k \geq 2$. Similarly, we know
\[ \int_{\partial B_4(x)} -(-\Delta)^{k-1} v = \frac{1}{(n-2)\omega_{n-1}} \left( \int_{\mathbb{R}^n \setminus B_4(x)} -(-\Delta)^k v(y)dy + \frac{1}{4^{n-2}} \int_{B_4(x)} (-\Delta)^k v(y)dy \right) \]
(2.8)

By Lemma 2.5 and Lemma 2.6, we conclude that
\[ \int_{\mathbb{R}^n \setminus B_4(x)} \frac{(-\Delta)^k v(y)}{|x-y|^{n-2}} dy = C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus B_4(z)} \frac{f(z)}{|x-y|^{n-2}|y-z|^{2k}} dydz \leq C \int_{B_2(z)} f(z)dz + C \int_{\mathbb{R}^n \setminus B_2(z)} \frac{f(z)}{|x-z|^{n+2k-2}} dz \leq C. \]

(2.9)

For the second term, we know
\[ -(-\Delta)^k v(y) = C \int_{\mathbb{R}^n} \frac{f(z)}{|y-z|^{2k}} dy = C v_{2k}(y) \in A_1. \]

By Lemma 2.2, we obtain
\[ \int_{\partial B_4(x)} -(-\Delta)^k v = C \int_{B_4(x)} v_{2k} \leq C \inf_{B_4(x)} v_{2k} = C \inf_{\partial B_4(x)} v_{2k} \]
\[ \leq C \int_{\partial B_4(x)} v_{2k} = C \int_{\partial B_4(x)} -(-\Delta)^k v \leq C, \]
(2.10)

where we use the induction for $k$. From (2.8) , (2.9) and (2.10), we complete the argument. □
Lastly, we list two Lemmas which are used to proof Lemma 5.1.

**Lemma 2.9 (Hardy-Littlewood-Sobolev)** If \( p, q \geq 1 \), \( 0 < \lambda < n \) and \( \frac{1}{p} + \frac{1}{q} + \frac{\lambda}{n} = 2 \), we suppose \( f \in L^p(\mathbb{R}^n) \) and \( g \in L^q(\mathbb{R}^n) \) then

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^\lambda} \, dx \, dy \leq C \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}
\]

**Lemma 2.10** If \( \Omega \) is a bounded domain and \( h \) solves

\[
\left\{
\begin{array}{ll}
(-\Delta)^m h = 2f & \text{in } \Omega \\
\end{array}
\right.
\]

where \( f \in L^1(\Omega) \). Then for any \( \delta \in (0, nc_n) \), there exist \( C_\delta \) such that

\[
\int_{\Omega} \exp \left( \frac{\delta |h(x)|}{\|f\|_{L^1(\Omega)}} \right) \, dx \leq C_\delta (\text{diam } \Omega)^n
\]

**Proof.** Let \( R = \text{diam } \Omega \) and \( \tilde{f} \) is a zero extension of \( f \) and we assume \( 0 \in \Omega \). Set

\[
w(x) = \frac{1}{c_n} \int_{\mathbb{R}^n} \log \frac{2R}{|x-y|} \tilde{f}(y) \, dy.
\]

By the Lemma 2.5, we know

\[(-\Delta)^k w(x) \geq 0 \quad \text{in } \Omega\]

for \( k = 0, 1, \cdots, m - 1 \) and

\[(-\Delta)^m w(x) \geq 2|f(x)| \quad \text{in } \Omega.
\]

By the maximum principle, we have \( |h(x)| \leq w(x) \). Applying the Jensen inequality, we obtain

\[
\int_{\Omega} \exp \left( \frac{\delta |h(x)|}{\|f\|_{L^1(\Omega)}} \right) \, dx \leq \int_{\Omega} \exp \left( \frac{\delta}{c_n} \int_{B_R} \log \frac{2R}{|x-y|} \tilde{f}(y) \, dy \right) \, dx \\
\leq C_\delta \left( \frac{2R}{\text{diam } \Omega} \right)^n \int_{\Omega} \int_{B_R} \frac{\tilde{f}(y)}{|x-y|^{\frac{n}{\lambda}}} \, dy \, dx \\
\leq C_\delta R^n.
\]

The last inequality follows by the Lemma 2.9.

### 3 Proof of normal metrics

#### 3.1 A simple proof of normal metrics in four dimension

For \((\mathbb{R}^4, g = e^{2u}|dx|^2)\), we consider the equation

\[(-\Delta)^2 u(x) = 2Q_g(x)e^{4u(x)} \quad \text{in } \mathbb{R}^n. \tag{3.1}\]

The main assumption:

\[
\left\{
\begin{array}{l}
Q_g \geq 0 \text{ and } 0 < C_0 e^{-|x|^\beta} \leq Q_g(x) \text{ for } |x| \gg 1, \text{ where } 0 < \beta < 1.
\end{array}
\right.
\]

\[Q_g(x)e^{nu(x)} \in L^1(\mathbb{R}^n), \quad u(x) = o(|x|^2). \tag{3.2}\]
Lemma 3.1 There holds
\[-\Delta u(x) = \frac{1}{2\pi^2} \int_{\mathbb{R}^4} \frac{Q_g(y)e^{4u(y)}}{|x-y|^2} \, dy, \quad \text{for} \ x \in \mathbb{R}^4.\]

Proof. Let \( w = u + v \), we know \( \Delta^2 w = 0 \). Then for any \( x_0 \in \mathbb{R}^4 \)
\[
\Delta w(x_0) = \int_{B_r(x_0)} \Delta w = \frac{1}{|B_r(x_0)|} \int_{\partial B_r(x)} \frac{\partial w}{\partial v}.
\]
So, integrating two sides, we obtain
\[
\frac{r^2}{8} \Delta w(x_0) = \int_{\partial B_r(x_0)} w - w(x_0).
\] (3.3)
By Jensen inequality and (3.3), we know
\[
\exp \left( \frac{r^2}{2} \Delta w(x_0) \right) = e^{-4w(x_0)} \exp \left( \int_{\partial B_r(x_0)} 4w \right)
\leq e^{-4w(x_0)} \int_{B_r(x_0)} e^{4w} \leq e^{-4w(x_0)} e^{\alpha^2} C_0 \int_{\partial B_r(x_0)} Q_g e^{4w}
\leq Ce^{-4w(x_0)} e^{4\alpha^2} e^{\alpha^2} \int_{\partial B_r(x_0)} Q_g e^{4w}.
\]
Then we get for \( R_0 \gg 1 \),
\[
\int_{R_0}^{+\infty} r^{3-4\alpha} \exp \left( \frac{r^2}{2} \Delta w(x_0) - r^2 \right) \, dr < \infty. \tag{3.4}
\]
Thus, using (3.4) we obtain \( \Delta w(x_0) \leq 0 \). So, the harmonic function theory implies that \( \Delta w(x) \equiv -C_2 \leq 0 \) i.e,
\[
-\Delta u(x) = \frac{1}{2\pi^2} \int_{\mathbb{R}^4} \frac{Q_g(y)e^{4u(y)}}{|x-y|^2} \, dy + C_2.
\]
where \( C_2 \geq 0 \). If \( C_2 > 0 \), then \( \Delta u(x) \leq -C_2 \). We consider the sphere average of \( u \),
\[
\bar{u}(r) = \frac{1}{\partial B_r(0)} u.
\]
We note that \( \frac{1}{r^3} (r^3 \bar{u}'(r))' \leq -C_2 \), therefore \( \bar{u}(r) \leq u(0) - \frac{C_2 r^2}{8} \). It is a contradiction with \( u(x) = o(|x|^2) \).

Lemma 3.2 Suppose \( u \) satisfies (3.2) and (3.1), then
\[
u(x) = -\frac{1}{4\pi^2} \int_{\mathbb{R}^4} \log \frac{|x-y|}{|y|} Q_g(y) e^{4u(y)} \, dy + C.
\]

Proof. Applying Lemma 2.3 and Lemma 3.1, we know \( \Delta(u + v) = 0 \) and \( u + v \leq o(|x|^2) \).
Then, we claim that: \( |u + v| = o(|x|^2) \).
Denote \( w = u + v \), without loss of generality we assume \( w(0) = 0 \). Then for any \( \epsilon > 0 \), there exist \( R_\epsilon \) such that
\[
w(x) \leq \epsilon |x|^2 \quad \text{for} \ |x| > R_\epsilon. \tag{3.5}
\]
For \( B_R(0) \) and \( R > 2R_\epsilon \), using Harnack inequality we know
\[
\sup_{B_{R/2}(0)} (\epsilon R^2 - w) \leq C(n) \inf_{B_{R/2}(0)} (\epsilon R^2 - w) \leq C(n) \epsilon R^2,
\]
then we obtain
\[
\sup_{B_{R/2}(0)} (-w) \leq C(n)\varepsilon R^2. \tag{3.6}
\]
From (3.5) and (3.6), we conclude that \(|w(x)| \leq C(n)\varepsilon |x|^2\) for \(|x| > R\). Standard elliptic estimate implies that for any fixed \(x_0 \in \mathbb{R}^n\)
\[
|\nabla^3 w|(x_0) \leq \frac{C(n)}{R^7} \int_{B_R(x_0)} |u| \leq \frac{C(n)\varepsilon}{R} \to 0
\]
as \(R \to \infty\), we get \(w(x) = p_0 + p_1(x) + p_2(x)\), where \(p_i(x)\) is a homogeneous polynomial of degree \(i\). In fact, \(p_2(x) \equiv 0\). We suppose \(p_2(x) = |x|^2 p_2(\theta)\), where \(\theta = \frac{x}{|x|} \in \mathbb{S}^3\). If \(p_2(\theta_0) > 0\) for some \(\theta_0 \in \mathbb{S}^3\), then
\[
C(n)\varepsilon |x|^2 \geq w(|x|\theta_0) \geq \frac{p_2(\theta_0)}{2} |x|^2 \quad \text{for} \quad |x| \gg 1.
\]
This is impossible, so \(p_2 \leq 0\). Similarly you can get \(p_2 \geq 0\), then we know \(p_2 \equiv 0\). Hence, \(w = u(x) + v(x) = \sum_{i=1}^4 a_i x_i + C\) and
\[
\frac{C_0}{|x|^{4\alpha}} e^{\sum_{i=1}^4 a_i x_i + 4|\cdot| |x|^2} = C_0 e^{-|\cdot| |x|^2} e^{4u(x)} \leq Q_\beta(x) e^{4u(x)} \quad \text{for} \quad |x| \gg 1.
\]
Since \(Q_\beta e^{4u}\) is absolutely integrable, then \(a_i = 0\) for \(i = 1, \cdots, 4\), we complete the proof. \(\Box\)

### 3.2 normal metrics in even dimension

For complete manifold \((\mathbb{R}^n, g = e^{2u}|dx|^2)\), we consider the equation
\[
(-\Delta)^m u(x) = 2Q_\beta(x) e^{nu(x)} \quad \text{in} \quad \mathbb{R}^n \tag{3.7}
\]
The main assumption:
\[
\begin{cases}
C_0 e^{-|\cdot| |x|^2} \leq Q_\beta(x) \quad \text{for} \quad |x| \gg 1, \quad \beta \in (0, 2).
\end{cases}
\]
\[
Q_\beta \geq 0 \quad \text{and} \quad Q_\beta e^{nu(x)} \in L^1(\mathbb{R}^n). \tag{3.8}
\]
In the following, inspired by L. Martinazzi [18] and Cheng, Lin [2], we give another type proof of the normal metric. Some similar ideas have appeared in [10, 14, 15].

We begin with mean value theorem and elliptic estimates for polyharmonic functions.

**Lemma 3.3** (Pizzetti formula, [18, Lemma 3]) Suppose \(\Delta^m h = 0\) in \(B_{2R}(x_0)\), then
\[
\int_{B_R(x_0)} h(y) dy = \sum_{i=0}^{m-1} c_i R^{2i} \Delta^i h(x_0),
\]
where \(c_i = \frac{n}{n+2} \frac{(n-2)!!}{(2i+2)(n-2i)!!} \) for \(i \geq 1\).

**Proof.** You can refer the Page of 310 of [18]. \(\Box\)

**Lemma 3.4** (L. Martinazzi, [18, Proposition 4]) If \(\Delta^m h = 0\) in \(B_4(0)\), then for any \(\beta \in [0, 1)\), \(p \geq 1\) and \(k \geq 1\), we have
\[
\begin{align*}
||h||_{W^{k,p}(B_1(0))} &\leq C(k, p)||h||_{L^1(B_4(0))}, \\
||h||_{C^k,B_1(0)} &\leq C(k, p)||h||_{L^1(B_4(0))}.
\end{align*}
\]

**Proof.** See the Page of 311 of [18]. \(\Box\)
Lemma 3.5 If $u$ solves (3.7) and satisfies (3.8), then
\[ u(x) + v(x) = p(x), \tag{3.9} \]
where $p(x)$ is a polynomial of degree at most $n - 2$.

Proof. Let $h = u + v$, for any fixed $x_0 \in \mathbb{R}^n$ by the elliptic estimate 3.4 we have
\[ |\nabla^{n-1} h(x_0)| \leq \frac{C}{R^{n-1}} \int_{B_R(x_0)} |h| = -\frac{C}{R^{n-1}} \int_{B_R(x_0)} h + \frac{2C}{R^{n-1}} \int_{B_R(x_0)} h^+ \]

Thanks to Lemma 3.3, we know
\[ \frac{C}{R^{n-1}} \int_{B_R(x_0)} h = O(R^{-1}) \]

Now we focus on the second part, applying Lemma 2.3 and Jensen inequality we obtain
\[ \frac{1}{R^{n-1}} \int_{B_R(x_0)} h^+ \leq \frac{1}{R^{n-1}} \int_{B_R(x_0)} u^+ + C \frac{\log R}{R^{n-1}} \leq \frac{C}{R^{n-1}} \log \int_{B_R(x_0)} e^{nu^+} + C \frac{\log R}{R^{n-1}}. \tag{3.10} \]

Since
\[ \int_{B_R(x_0)} e^{nu^+} \leq \int_{B_R(x_0)} e^{nu} + 1 \leq C(R_0, x_0) + CR^n + C \int_{B_R \setminus B_{R_0}(x_0)} e^{\beta|z|^2} e^{nu} \leq C(R_0, x_0) + CR^n + Ce^{R^\beta} \tag{3.11} \]

for $R \gg 1$, from (3.10) and (3.11) we arrive
\[ \frac{1}{R^{n-1}} \int_{B_R(x_0)} h^+ \leq \frac{C}{R^{n-1-\beta}} + \frac{C \log R}{R^{n-1}} \to 0 \tag{3.12} \]
as $R \to \infty$. \hfill \Box

Remark 3.1 For (3.12), we have use the fact $n \geq 4$.

Lemma 3.6 Suppose $(\mathbb{R}^n, e^{2u}|dx|^2)$ is complete and satisfies (3.8), then the following two statements holds.

1) For $p(x) = p(r, \theta)$ in Lemma 3.5, there is no $\theta \in S^{n-1}$ such that
\[ \lim_{r \to \infty} \frac{p(r, \theta)}{r^k} = C(k, \theta) > 0 \quad \text{for} \quad k \geq 2. \tag{3.13} \]

2) For $p(x) = p(r, \theta)$ in Lemma 3.5, there is no $\theta \in S^{n-1}$ such that
\[ \lim_{r \to \infty} \frac{p(r, \theta)}{r^s} \leq -C(s, \theta) < 0 \quad \text{for} \quad s > 0. \tag{3.14} \]
Proof. For (1), if not. Thus, for any \( r > R_0 \) there exist \( x_r \) such that \( |x_r| = r, \frac{r}{|x_r|} = \theta_0 \in S^{n-1} \), where \( \theta_0 \) is independent of \( r \). And

\[
p(x) \geq c_0 r^k \text{ for } x \in B_{1/r^{n-3}}(x_r),
\]

this follows by \( |\nabla p(x)| \leq C|x|^{k-1} \leq C|x|^{n-3} \). Then, thanks to Lemma 2.3, we know

\[
u(x) = -v(x) + p(x) \geq c_0 r^k - \alpha \log r - C \geq \frac{c_0}{2} r^k \text{ for } x \in B_{1/r^{n-3}}(x_r).
\]

But we get

\[
\int_{\mathbb{R}^n \setminus B_{R_0}} Q_\sigma e^{nu} \geq \int_{R_0}^{+\infty} \int_{\partial B_r \cap B_{3-n}(x_r)} Q_\sigma e^{nu} d\sigma dr \\
\geq C \int_{R_0}^{+\infty} \frac{e^{nu} r^{k-r^\beta}}{r^{n-3}\beta} = +\infty.
\]

where we use \( \beta < 2 \). This is a contradiction with \( \int_{\mathbb{R}^n} Q_\sigma e^{nu} < +\infty \).

For (2), if it’s wrong. There exist \( \theta_0 \in S^{n-1} \) such that

\[
\lim_{r \to \infty} \frac{p(r, \theta_0)}{r^k} \leq -C(s, \theta_0) < 0 \text{ for } s > 0.
\]

we consider the length of curve near infinity, denote

\[
III = \int_{R_0+1}^{+\infty} e^{u(r, \theta_0)} dr,
\]

where \( R_0 \gg 1 \). From Lemma 2.4, we know

\[
III \leq C \int_{R_0+1}^{+\infty} \frac{e^{p(r, \theta_0)}}{r^{(n-\alpha)}} II(r) dr,
\]

where

\[
II(r) = \exp \left( \frac{1}{c_n} \int_{|r\theta_0-y| < 1} \log \frac{1}{|r\theta_0 - y|} f(y) dy \right).
\]

Since for any \( r > R_0 + 1 \), then \( y \in \mathbb{R}^n \setminus B_{R_0} \). We rewrite \( II(r) \) term as

\[
II(r) = \exp \left( \int_{\mathbb{R}^n \setminus B_{R_0}} \sigma(R_0) \chi_{|r\theta_0-y| < 1} \log \frac{1}{|r\theta_0 - y|} \frac{f(y)}{c_n \sigma(R_0)} dy \right),
\]

where \( \sigma(R_0) = \frac{1}{c_n} ||f||_{L^1(\mathbb{R}^n \setminus B_{R_0})} < \frac{1}{2} \). For \( dv(y) = \frac{f(y)}{||f||_{L^1(\mathbb{R}^n \setminus B_{R_0})}} dy \), we apply the Jensen inequality to get

\[
II(r) \leq \int_{\mathbb{R}^n \setminus B_{R_0}} \exp \left( \sigma(R_0) \chi_{|r\theta_0-y| < 1} \log \frac{1}{|r\theta_0 - y|} \right) \frac{f(y)}{c_n \sigma(R_0)} dy.
\]

Plug (3.17) into (3.16) and by the Fubini’s Theorem, we conclude

\[
III \leq C \int_{\mathbb{R}^n \setminus B_{R_0}} (III_1(y) + III_2(y)) \frac{f(y)}{||f||_{L^1(\mathbb{R}^n \setminus B_{R_0})}} dy,
\]

where

\[
III_1(y) = \int_{I_s \cap (R_0+1, +\infty)} \frac{e^{p(r, \theta_0)}}{r^{(n-\alpha)}} |r\theta_0 - y|^{\sigma(R_0)} dr,
\]

\[
III_2(y) = \int_{\mathbb{R}^n \setminus B_{R_0}} \frac{e^{p(r, \theta_0)}}{r^{(n-\alpha)}} |r\theta_0 - y|^{\sigma(R_0)} dr.
\]
III_2(y) = \int_{(R_0 + 1, +\infty) \setminus I_y} \frac{e^{p(r, \theta_0)}}{r^{(\alpha - \epsilon_0)}} dr

and

I_y = \{r \theta_0 | R_0 + 1 < r < +\infty \} \cap B_1(y).

For any fixed \( y \in \mathbb{R}^n \setminus B_{R_0} \), denoted the center of \( I_y \) by \( y^* \), i.e., \( y^* \in \{r \theta_0 | R_0 + 1 < r < +\infty \} \) and \((y^* - y) \cdot \theta_0 = 0\). Note that for any fixed \( y \in \mathbb{R}^n \), \(|r \theta_0 - y^*| \leq |r \theta_0 - y| \) and \(|I_y| \leq 2\), so we know

\[
III_1 \leq \int_{(|y^* - 1, |y^*| + 1)} \frac{e^{p(r, \theta_0)}}{r^{(\alpha - \epsilon_0)} |r \theta_0 - y^*|^{\sigma(R_0)}} \leq C(\theta_0) \int_{(|y^* - 1, |y^*| + 1)} \frac{1}{|r \theta_0 - y^*|^{1/2}} dr \leq C(\theta_0). \tag{3.19}
\]

Clearly,

\[
III_2 = \int_{(R_0 + 1, +\infty) \setminus I_y} \frac{e^{p(r, \theta_0)}}{r^{(\alpha - \epsilon_0)}} dr \leq C(\theta_0), \tag{3.20}
\]

then (3.15), (3.18), (3.19) and (3.20) imply \( III < +\infty \). This is a contradiction with the completeness of \((\mathbb{R}^n, e^{2u}|dx|^2)\).

**Theorem 3.1** Suppose \((\mathbb{R}^n, g = e^{2u}|dx|^2)\) is complete and satisfies

\[
\begin{align*}
C_0 e^{-|x|^\beta} &\leq Q_g(x) \text{ for } |x| \gg 1, \text{ where } \beta \in (0, 2). \\
Q_g &\geq 0 \text{ and } Q_g(x)e^{au(x)} \in L^1(\mathbb{R}^n)
\end{align*}
\]

Then,

(1) the metric is normal, i.e,

\[
u(x) = \frac{1}{c_n} \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} Q_g(y)e^{au(y)} dy + C. \tag{3.21}
\]

(2)

\[
\int_{\mathbb{R}^n} Q_g(y)e^{au(y)} dy \leq c_n. \tag{3.22}
\]

**Proof.** For (1), let \( p(x) = p_0 + |x| p_1(\theta) + \cdots + |x|^k p_k(\theta) \), where \( \theta = \frac{x}{|x|} \in S^{n-1} \) and \( k = \deg p \leq n - 2 \), we always assume \( k \geq 1 \).

Firstly, we claim that \( k \) is an even number. If \( k \) is an odd number, there exists \( \theta_0 \in S^{n-1} \) such that \( p_k(\theta_0) < 0 \), thus

\[
\lim_{r \to \infty} \frac{p(r, \theta_0)}{r^k} = p_k(\theta_0) < 0.
\]

This cause a contradiction with (2) in Lemma 3.6.

Secondly, \( \sup_{\theta \in S^{n-1}} p_k(\theta) \leq 0 \). Otherwise, there exists \( \theta_0 \in S^{n-1} \) such that \( p_k(\theta_0) > 0 \), hence

\[
\lim_{r \to \infty} \frac{p(r, \theta_0)}{r^k} = p_k(\theta_0) > 0.
\]

where \( k \geq 2 \). It’s a contradiction with (1) in Lemma 3.6. Clearly, \( p_k(\theta) \not\equiv 0 \), then there exists \( \theta_0 \in S^{n-1} \) such that \( p_k(\theta_0) < 0 \), this is also impossible!
For (2), if \( \alpha > 1 \), by the proof of Lemma 3.6, we obtain
\[
III \leq C \int_{\mathbb{R}^n \setminus B_{R_0}} (III_1(y) + III_2(y)) \frac{f(y)}{\|f\|_{L^1(\mathbb{R}^n \setminus B_{R_0})}} dy,
\]
where
\[
III_1(y) = \int_{I_y \cap (R_0 + 1, +\infty)} \frac{1}{r^{(\alpha - \epsilon_0)}|\theta - y|^{\sigma(R_0)}} dr,
III_2(y) = \int_{(R_0 + 1, +\infty) \setminus I_y} \frac{1}{r^{(\alpha - \epsilon_0)}} dr.
\]
We choose \( R_0 \gg 1 \), then \( \epsilon_0 \ll 1 \), \( \sigma(R_0) \ll 1 \) and \( \alpha - \epsilon_0 > 1 \). Note that,
\[
\int_{-1}^{1} \frac{1}{|r|^{1/2}} dr < +\infty \quad \text{and} \quad \int_{R_0 + 1}^{+\infty} \frac{1}{r^{\alpha - \epsilon_0}} dr < +\infty.
\]
We know \( III < +\infty \), this is a contradiction with the completeness of \((\mathbb{R}^n, e^{2u}|dx|^2)\). \( \Box \)

We point out that (3.21) and (3.22) will automatically imply the positivity of the scalar curvature.

**Lemma 3.7** Suppose the same condition as Theorem 3.1, then
\[
R_g(x) \geq -(n - 1) \Delta u(x) e^{-2u(x)} > 0.
\]
**Proof.** By the conformal change of scalar curvature, we obtain
\[
R_g(x) = e^{-2u} (n - 1) \left( -2 \Delta u(x) - (n - 2)|\nabla u(x)|^2 \right)
= e^{-2u + 2C} (n - 1) \left( 2 \Delta v(x) - (n - 2)|\nabla v(x)|^2 \right).
\]
Clear,
\[
2 \Delta v(x) - (n - 2)|\nabla v(x)|^2
= \frac{2(n - 2)}{c_n} \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^2} - \frac{(n - 2)}{c_n^2} \sum_{i=1}^{n} \left( \int_{\mathbb{R}^n} \frac{(x_i - y_i) f(y)}{|x - y|^2} \right)^2.
\]
Notice that Theorem 5.1 (3) implies that \( \int_{\mathbb{R}^n} f(y) dy \leq c_n \), then
\[
\frac{(n - 2)}{c_n} \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^2} - \frac{(n - 2)}{c_n^2} \sum_{i=1}^{n} \left( \int_{\mathbb{R}^n} \frac{(x_i - y_i) f(y)}{|x - y|^2} \right)^2
\geq \frac{(n - 2)}{c_n} \int_{\mathbb{R}^n} f(y) \frac{f(y)}{|x - y|^2} - \frac{(n - 2)}{c_n^2} \sum_{i=1}^{n} \left( \int_{\mathbb{R}^n} \frac{(x_i - y_i) f(y)}{|x - y|^2} \right)^2
\geq 0.
\]
The final step follows by Cauchy-Schwarz inequality. From (2) in Theorem 5.1, we arrive
\[
-\Delta u = \frac{(n - 2)}{c_n} \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^2}.
\]
Combated with (3.23), (3.24), (3.25) and (3.26), we finishing the proof. \( \Box \)
4 Chern-Gauss-Bonnet formula in even dimension

In this section, we prove the Chern-Gauss-Bonnet formula by two steps. Firstly, we assume that \( u(x) = u(|x|) \), since we have normal condition (3.21) and inequality (3.22), we will take a very direct approach to prove it. Secondly, using Lemma 4.3, we can reduce the general case into radial case. In this section, we assume:

\[
\begin{align*}
    u(x) &= \frac{1}{c_n} \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} Q_g(y) e^{nu(y)} \, dy + C \quad (4.1) \\
    \alpha &= \frac{1}{c_n} \int_{\mathbb{R}^n} Q_g(y) e^{nu(y)} \, dy \leq 1.
\end{align*}
\]

4.1 Radial symmetric case

Lemma 4.1 If \( u(x) = u(|x|) \), then

\[
\lim_{r \to \infty} ru'(r) = -\frac{1}{c_n} \int_{\mathbb{R}^n} f(y)dy.
\]

Proof. Since

\[
ru'(r) + \frac{1}{c_n} \int_{\mathbb{R}^n} f(y)dy = \frac{1}{c_n} \int_{\mathbb{R}^n} \frac{y \cdot (y-x)}{|x-y|^2} f(y)dy,
\]

we only need to proof \( \int_{\mathbb{R}^n} \frac{y \cdot (y-x)}{|x-y|^2} f(y)dy \to 0 \) as \( |x| \to \infty \). By easy calculation, we know

\[
\begin{align*}
    I := \int_{\mathbb{R}^n} \frac{y \cdot (y-x)}{|x-y|^2} f(y)dy &= \int_{0}^{+\infty} \int_{\partial B_r(0)} \frac{y \cdot (y-x)}{|x-y|^2} f(y)d\sigma(y)dr \\
    &= \int_{0}^{+\infty} f(s) \int_{\partial B_r(0)} \frac{y \cdot (y-x)}{|y||x-y|^2} d\sigma(y)ds \\
    &= \int_{0}^{+\infty} f(s) \int_{\partial B_r(0)} \frac{\partial \log |x-y|}{\partial \nu} d\sigma(y)ds. \quad (4.2)
\end{align*}
\]

We claim that for \( x \notin \partial B_r(0) \), we have

\[
\int_{\partial B_r(0)} \frac{\partial \log |x-y|}{\partial \nu} = \int_{B_r(0)} \Delta \log |x-y|dy = (n-2) \int_{B_r(0)} \frac{1}{|x-y|^2}dy. \quad (4.3)
\]

First, if \( x \notin B_r(0) \) the above identity is trivial. Suppose \( x \in B_r(0) \), then for any \( s - |x| > \rho > 0 \), we obtain

\[
\int_{\partial B_r(0)} \frac{\partial \log |x-y|}{\partial \nu} - \int_{\partial B_r(x)} \frac{\partial \log |x-y|}{\partial \nu} = \int_{B_r(0) \setminus B_r(x)} \Delta \log |x-y|dy.
\]

Since

\[
\int_{\partial B_r(x)} \frac{\partial \log |x-y|}{\partial \nu} = \omega_{n-1} \rho^{-n-2} \to 0
\]

and

\[
\int_{B_r(x)} \Delta \log |x-y|dy = (n-2) \int_{B_r(x)} \frac{1}{|x-y|^2}dy = \omega_{n-1} \rho^{-n-2} \to 0.
\]
Thus, combated (4.2) with (4.3), we know
\[ I = (n - 2) \int_0^{+\infty} f(s) s \int_{B_x(0)} \frac{1}{|x - y|^2} dy ds. \]

For any \( \epsilon > 0 \), let \( |x| > \frac{1}{2\epsilon} \) we get
\[
I \leq C \int_0^{\epsilon|x|} + \int_{\epsilon|x|}^{+\infty} f(s) s \int_{B_x(0)} \frac{1}{|x - y|^2} dy ds \\
\leq \frac{C}{(1 - \epsilon)^2 |x|^2} \int_0^{\epsilon|x|} f(s) s^{n+1} dr + C \int_{\epsilon|x|}^{+\infty} f(s) s \int_{B_x(0)} \frac{1}{|y|^2} dy dr \\
\leq \frac{C\epsilon^2}{(1 - \epsilon)^2} \int_0^{+\infty} f(s) s^{n-1} dr + C \int_{\epsilon|x|}^{+\infty} f(s) s^{n-1} ds \\
\leq C\epsilon^2 ||f||_{L^1(\mathbb{R}^n)} + C \int_{\mathbb{R}^n \setminus B_{\epsilon|x|}(0)} f(y) dy,
\]
where the second inequality follows by Rearrangement inequality in Chapter 3 of [16],
\[
\int_{B_x(0)} \frac{1}{|x - y|^2} dy \leq \int_{B_x(0)} \frac{1}{|y|^2} dy \leq C s^{n-2}. \tag{4.4}
\]

Let \( \epsilon \to 0 \), we know \( I \to 0 \).

**Lemma 4.2** If \((\mathbb{R}^n, e^{2u}|dx|^2)\) is complete, \( u(x) = u(|x|) \) then
\[
1 - \frac{1}{cn} \int_{\mathbb{R}^n} Q_g(y)e^{nu(y)} dy = \lim_{r \to \infty} \left( \frac{\int_{\partial B_r} e^{(n-1)u}}{n \omega_{n-1}^\frac{1}{n-1} \int_{B_r} e^{nu}} \right) = \lim_{r \to \infty} \left( \frac{\int_{\partial B_r} e^{(n-1)u}}{\omega_{n-1}^\frac{1}{n-1} \int_{B_r} e^{nu}} \right) = 1 - \frac{1}{cn} \int_{\mathbb{R}^n} Q_g(y)e^{nu(y)} dy \tag{4.4}
\]

**Proof.** Applying Lemma 2.3 and (4.1), then for \( |x| \gg 1 \) we have
\[
u(x) \geq -\alpha \log |x| - C.
\]

Since \( \alpha \leq 1 \), then
\[
\int_{B_r} e^{nu} \to \infty \quad \text{and} \quad \frac{d}{dr} \int_{B_r} e^{nu} = \int_{\partial B_r} e^{nu} > 0. \tag{4.5}
\]

Clearly,
\[
\frac{d}{dr} \left( \frac{\int_{\partial B_r} e^{(n-1)u}}{n \omega_{n-1}^\frac{1}{n-1} \int_{B_r} e^{nu}} \right) = \frac{n-1}{n} \left( \frac{\int_{\partial B_r} e^{(n-1)u}}{n \omega_{n-1}^\frac{1}{n-1} \int_{B_r} e^{nu}} \right) \left( \int_{\partial B_r} \frac{\partial u}{\partial r} e^{(n-1)u} + \frac{n-1}{n-1} \int_{\partial B_r} e^{(n-1)u} \right) \\
= \left( \int_{\partial B_r} e^{(n-1)u} \right) \frac{n-1}{n} \left( \int_{\partial B_r} \frac{\partial u}{\partial r} e^{(n-1)u} + \int_{\partial B_r} e^{(n-1)u} \right) \\
= \frac{\int_{\partial B_r} e^{(n-1)u}}{\omega_{n-1}^\frac{1}{n-1} \int_{B_r} e^{nu}} \left( \int_{\partial B_r} \frac{\partial u}{\partial r} e^{(n-1)u} + \int_{\partial B_r} e^{(n-1)u} \right) \\
= e^{u(r)} \frac{\omega_{n-1}^\frac{1}{n-1} \frac{\partial u}{\partial r} e^{(n-1)u} + e^{(n-1)u(r)}}{\omega_{n-1}^\frac{1}{n-1} \int_{\partial B_r} e^{nu(r)}} = 1 + ru'(r).
\]

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Now, from Lemma 4.1, we get
\[
\lim_{r \to \infty} \frac{d}{dr} \left( \frac{\int_{\partial B_r} e^{(n-1)u}}{\omega^{n-1}_n} \right)^{\frac{n}{n-1}} = 1 - \frac{1}{c_n} \int_{\mathbb{R}^n} Q_g(y) e^{nu(y)} dy. \tag{4.6}
\]

Combating (4.5), (4.6) and L’Hôpital’s rule imply that
\[
1 - \frac{1}{c_n} \int_{\mathbb{R}^n} Q_g(y) e^{nu(y)} dy = \lim_{r \to \infty} \frac{\int_{\partial B_r} e^{(n-1)u}}{\omega^{n-1}_n} \left( \int_{\partial B_r} e^{nu} \right) = 1 - \frac{1}{c_n} \hat{R} Q_g(y) e^{nu(y)} dy. \tag{4.7}
\]

\[\square\]

**Theorem 4.1** Suppose that \( u \) is radial function, \( (\mathbb{R}^n, e^{2u} |dx|^2) \) is complete with
\[
\begin{align*}
C_0 e^{-|x|^\beta} &\leq Q_g(x) \text{ for } |x| \gg 1, \text{ where } \beta \in (0, 2), \\
Q_g &\geq 0 \text{ and } Q_g(x) e^{nu(x)} \in L^1(\mathbb{R}^n)
\end{align*}
\]

Then
\[
\int_{\mathbb{R}^n} Q_g(y) e^{nu(y)} dy \leq c_n
\]

and
\[
1 - \frac{1}{c_n} \int_{\mathbb{R}^n} Q_g(y) e^{nu(y)} dy = \lim_{r \to \infty} \frac{|\partial B_r(0)|_g^{\frac{n}{n-1}}}{\omega^{n-1}_n |B_r(0)|_g}.
\]

**Proof.** See the Theorem 5.1 and Lemma 4.2. \(\square\)

### 4.2 General case

For \( u \in C^\infty(\mathbb{R}^n) \), we define the spherical average
\[
\bar{u}(r) = \frac{\int_{\partial B_r(0)} u(y) d\sigma(y)}{\omega^{n-1}_n r^{n-1}}
\]

The following Lemma 4.3 is proved by Chang, Qing and Yang [3] in four dimension, but there is no difference for high dimension, so we omit the details.

**Lemma 4.3 (Chang Qing and Yang, [3, Lemma 3.2])** Suppose \( e^{2u} |dx|^2 \) is normal metric. Then for any \( k > 0 \),
\[
1 - \frac{1}{c_n} \int_{\mathbb{R}^n} Q_g(y) e^{nu(y)} dy = \lim_{r \to \infty} \frac{|\partial B_r(0)|_g^{\frac{n}{n-1}}}{\omega^{n-1}_n |B_r(0)|_g}.
\]

**Proof.** See the Page 523 of [3]. \(\square\)

**Lemma 4.4** If \( (\mathbb{R}^n, g = e^{2u} |dx|^2) \) is complete and satisfies
\[
\begin{align*}
C_0 e^{-|x|^\beta} &\leq Q_g(x) \text{ for } |x| \gg 1, \text{ where } \beta \in (0, 2), \\
Q_g &\geq 0 \text{ and } Q_g(x) e^{nu(x)} \in L^1(\mathbb{R}^n)
\end{align*}
\]

then \( (\mathbb{R}^n, \tilde{g} = e^{\alpha u(r)} |dx|^2) \) also satisfies the above properties. Moreover \( \alpha_g = \alpha_g \), namely,
\[
1 - \frac{1}{c_n} \int_{\mathbb{R}^n} Q_g(x) e^{nu(x)} dx = \frac{1}{c_n} \int_{\mathbb{R}^n} Q_g(x) e^{nu(x)} dx. \tag{4.8}
\]
**Proof.** Since \((\mathbb{R}^n, g = e^{2u}|dx|^2)\) is complete, then for any \(\theta \in S^{n-1}\),
\[
\int_0^{+\infty} e^{u(r, \theta)} dr = +\infty.
\]
And we know \(\int_{\partial B_r} e^{u(x)} d\sigma(x) = \int_{\partial B_1(0)} e^{u(r, \theta)} d\sigma(\theta)\), by Fubini’s theorem, then we know
\[
+\infty = \int_{\partial B_1(0)} \int_0^{+\infty} e^{u(r, \theta)} dr d\sigma(\theta) = \int_0^{+\infty} \int_{\partial B_r(0)} e^{u(x)} d\sigma(x) dr.
\]
Combated with Lemma 4.3 we get \(\int_{\partial B_r} e^{u(x)} d\sigma(x) = e^{\bar{u}(r)} e^{\alpha(1)}\), then
\[
\int_0^{+\infty} e^{\bar{u}(r)} dr = +\infty.
\]
(4.9)
i.e, \((\mathbb{R}^n, g = e^{n\bar{u}(r)}|dx|^2)\) is complete. Here it is apparent that
\[
2Q_g(r) = e^{-n\bar{u}(r)} (-\Delta)^m \bar{u} = e^{-n\bar{u}(r)} \int_{\partial B_r(0)} (-\Delta)^m u
\]
\[
= e^{-n\bar{u}(r)} \int_{\partial B_r(0)} 2Q_g e^{nu} \geq 0.
\]
(4.10)
By Lemma 4.3, we know
\[
\frac{1}{|\partial B_r|} \int_{\partial B_r} e^{nu} = e^{n\bar{u}(r)} e^{\alpha(1)}.
\]
So,
\[
Q_g(r) \geq C_0 e^{-r^\beta} e^{-n\bar{u}(r)} \int_{\partial B_r(0)} e^{nu} \geq C'_0 e^{-r^\delta} \quad \text{for } r \gg 1.
\]
(4.11)
For the last property,
\[
\int_{\mathbb{R}^n} Q_g(x)e^{n\bar{u}(x)} dx = \int_{\mathbb{R}^n} (-\Delta)^m \bar{u}(x) dx
\]
\[
= \int_0^{+\infty} \int_{\partial B_r} (-\Delta)^m \bar{u} d\sigma dr = \int_0^{+\infty} |\partial B_r| (-\Delta)^m \bar{u} dr
\]
\[
= \int_0^{+\infty} |\partial B_r| \int_{\partial B_r} (-\Delta)^m u d\sigma dr = \int_{\mathbb{R}^n} (-\Delta)^m u = \int_{\mathbb{R}^n} Q_g e^{nu}
\]
(4.12)
With (4.9), (4.10), (4.11) and (4.12), we complete the argument. 
\(\square\)

**Remark 4.1** We note that the formula (4.8) is very important, it determines whether we can translate the general case into the radial symmetric case. Namely, the \(\alpha\) in Lemma 4.2 and the \(\alpha\) in Theorem 4.2 are the same.

**Theorem 4.2** Suppose \((\mathbb{R}^n, g = e^{2u}|dx|^2)\) is complete and satisfies
\[
\begin{cases}
C_0 e^{-|x|^\beta} \leq Q_g(x) & \text{for } |x| \gg 1, \text{ where } \beta \in (0, 2), \\
Q_g \geq 0 \text{ and } Q_g(x) e^{nu(x)} \in L^1(\mathbb{R}^n)
\end{cases}
\]
then
\[
1 - \frac{1}{e_n} \int_{\mathbb{R}^n} Q_g(y) e^{nu(y)} dy = \lim_{r \to \infty} \frac{|\partial B_r(0)|^\frac{\gamma - 1}{\gamma}}{n \omega_{n-1} |B_r(0)|^\gamma}
\]
Proof. Similarly, by $\alpha \leq 1$ we know

$$\int_{B_r} e^{nu} \to \infty \quad \text{and} \quad \frac{d}{dr} \int_{B_r} e^{nu} = \int_{\partial B_r} e^{nu} > 0. \quad (4.13)$$

Clearly,

$$\frac{d}{dr} \left( \int_{\partial B_r} e^{(n-1)u(r)} \right) = \frac{n\omega_{n-1}}{n} \frac{d}{dr} \int_{B_r} e^{nu}$$

$$= \left( \int_{\partial B_r} e^{(n-1)u(r)} \right) \left( \int_{\partial B_r} \frac{\partial u(r)}{\partial r} e^{(n-1)u(r)} + \frac{1}{r} \int_{\partial B_r} e^{(n-1)u(r)} \right). \quad (4.14)$$

By the Lemma 4.2, we obtain

$$\lim_{r \to \infty} \left( \int_{\partial B_r} e^{(n-1)u(r)} \right) \left( \int_{\partial B_r} \frac{\partial u(r)}{\partial r} e^{(n-1)u(r)} + \frac{1}{r} \int_{\partial B_r} e^{(n-1)u(r)} \right) = 1 - \frac{1}{c_n} \int_{\mathbb{R}^n} Q_g(y)e^{nu(y)}dy. \quad (4.15)$$

Lemma 4.3 implies that

$$\lim_{r \to \infty} \int_{\partial B_r} e^{ku(r)} = 1 \quad \text{for} \quad k > 0. \quad (4.16)$$

Combating with (4.14), (4.15) and (4.16), we have

$$\lim_{r \to \infty} \frac{d}{dr} \left( \int_{\partial B_r} e^{(n-1)u(r)} \right) = 1 - \frac{1}{c_n} \int_{\mathbb{R}^n} Q_g(y)e^{nu(y)}dy. \quad (4.17)$$

The final result follows by (4.15) and (4.17).

5 asymptotic behavior at infinity

In this section, we suppose $(\mathbb{R}^n, e^{2u}|dx|^2)$ is complete and $u$ is a solution of

$$(-\Delta)^n u(x) = 2Q_g(x)e^{nu(x)} \quad \text{in} \quad \mathbb{R}^n.$$ 

and satisfies

$$u(x) = \frac{1}{c_n} \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} Q_g(y)e^{nu(y)}dy + C \quad (5.1)$$

and

$$0 \leq Q_g(x) \leq C_0|\alpha|^{\gamma} \quad \text{for} \quad \gamma > 0.$$ 

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For any $k \in \mathbb{N}^+$, we consider the Green’s function of $(-\Delta)^k$ on $B_4(0)$. Let

$$
\begin{cases}
(-\Delta)^k G = \delta_0 & \text{in } B_4(0) \\
G = 0, -\Delta G = 0, \cdots, (-\Delta)^{k-1} G = 0 & \text{on } \partial B_4(0),
\end{cases}
$$

in fact $G$ is a radial function. For any $\Delta^k h(x) = 0$ in $B_8(0)$, then

$$
\begin{align*}
    h(0) &= \int_{B_4} (-\Delta)^k G h = \int_{B_4} (-\Delta)^k G(-\Delta) h - \int_{\partial B_4} h \frac{\partial}{\partial r} (-\Delta)^{k-1} G \\
    &= -\int_{\partial B_4} h \frac{\partial}{\partial r} (-\Delta)^{k-1} G - \int_{B_4} (-\Delta) h \frac{\partial}{\partial r} (-\Delta)^{k-2} G = \cdots \\
    &= \sum_{i=0}^{k-1} \int_{\partial B_4} (-\Delta)^i h \left( -\frac{\partial}{\partial r} \right) (-\Delta)^{k-i-1} G
\end{align*}
$$

Then we claim that $-\frac{\partial}{\partial r} (-\Delta)^{k-1-i} G = c_i > 0$ on $\partial B_4(0)$ for $i = 0, 1, \cdots, k-1$. Let $h_i$ solves

$$
\begin{cases}
(-\Delta)^k h_i = 0 & \text{in } B_4(0) \\
h_i = 0, \cdots, (-\Delta)^{k-1} h_i = 1, \cdots, (-\Delta)^{k-1} h_i = 0 & \text{on } \partial B_4(0),
\end{cases}
$$

we know $h_i(x) > 0$ in $B_4(0)$, then

$$
h_i(0) = \int_{\partial B_4} -\frac{\partial}{\partial r} (-\Delta)^{k-1-i} G = |\partial B_4| c_i > 0.
$$

**Lemma 5.1** For any $\epsilon > 0$, there exist $R_\epsilon \gg 1$ if $|x| > R_\epsilon$, 

$$
\int_{B_4(x)} \log \frac{1}{|x-y|} Q_n(y) e^{nu(y)} \leq \epsilon \log |x|
$$

and

$$
\alpha \log |x| + C \geq v(x) \geq (\alpha - \epsilon) \log |x| - C.
$$

**Proof.** For $|x| > R_0 \gg 1$, then we solve the PDE

$$
\begin{cases}
(-\Delta)^m l = 2f & \text{in } B_4(x) \\
l = (-\Delta) l = \cdots = (-\Delta)^{m-1} l = 0 & \text{on } \partial B_4(x)
\end{cases}
$$

By Lemma 2.10, we know there exist $q$ sufficiently big such that

$$
\int_{B_4(x)} e^{vq(y)} dy \leq C
$$

Let $h(y) = -(v(y) + l(y))$, then it solves

$$
\begin{cases}
(-\Delta)^m h = 0 & \text{in } B_4(x) \\
h = -v, (-\Delta) h = \Delta v, \cdots, (-\Delta)^{m-1} h = -(-\Delta)^{m-1} v & \text{on } \partial B_4(x)
\end{cases}
\quad \square
$$

Then, let $G$ be the Green’s function of $(-\Delta)^{m-1}$ at $x$, i.e,

$$
\begin{cases}
(-\Delta)^{m-1} G = \delta_x & \text{in } B_4(x) \\
G = 0, -\Delta G = 0, \cdots, (-\Delta)^{m-2} G = 0 & \text{on } \partial B_4(x).
\end{cases}
$$

By (5.2) and Lemma 2.8 we know

$$
-\Delta h(x) = \sum_{i=0}^{m-2} \int_{\partial B_4(x)} (-\Delta)^{i+1} h \left( -\frac{\partial}{\partial r} \right) (-\Delta)^{m-2-i} G
$$
\[
= \sum_{i=1}^{m-1} c_{i} \int_{\partial B_{r_{i}}(x)} (-\Delta)^{i} h = \sum_{i=1}^{m-1} c_{i} \int_{\partial B_{r_{i}}(x)} (-\Delta)^{i} v \leq C
\]

For small \( r_{0}, z \in B_{r_{0}}(x) \), let \( G_{z} \) be the Green’s function at \( z \) i.e,

\[
\begin{cases}
(-\Delta)^{m-1} G_{z} = \delta_{z} & \text{in } B_{r_{i}}(x) \\
G_{z} = 0, -\Delta G_{z} = 0, \cdots, (-\Delta)^{m-2} G_{z} = 0 & \text{on } \partial B_{r_{i}}(x)
\end{cases}
\]

Then if \( r_{0} \) is sufficiently small, we know

\[
0 < \frac{c_{i}}{2} < -\frac{\partial}{\partial \nu} (-\Delta)^{m-2-i} G_{z} < 2c_{i} \quad \text{on } \partial B_{r_{i}}(x)
\]

for \( i = 0, 1, \cdots, m - 2 \). The Green’s function at \( x \) can be obtained by translating the Green’s function at the zero point. Hence, we note that \( r_{0} \) is independent of \( x \) ! Similarly, you can get

\[
-\Delta h(z) \leq C \quad \text{in } B_{r_{0}}(x).
\]

Using the Corollary 2.1, we have

\[
\int_{B_{r_{i}}(x)} h^{+} \leq \int_{B_{r_{i}}(x)} v^{-} \leq C
\]

Standard elliptic estimate implies that

\[
\sup_{x \in B_{r_{0}/2}(x)} h \leq \int_{B_{r_{i}}(x)} h^{+} + C \sup_{x \in B_{r_{0}}(x)} -\Delta h \leq C.
\]

By (5.1), we know

\[
u = C - v = C + h + l \leq C + l \quad \text{in } B_{r_{0}/2}(x).
\]

Now, for any \( q > s > 1 \) and \( \beta > 0 \) we conclude

\[
\int_{B_{|x|}^{\beta}} (Q_{g} e^{nu})^{s} \leq C \int_{B_{|x|}^{\beta}} Q_{g}^{s} e^{\mu t} \leq C|x|^{\gamma s} \int_{B_{|x|}^{\beta}} e^{\mu t} \leq C|x|^{\gamma s} \left( \int_{B_{|x|}^{\beta}} e^{\mu t} \right)^{\frac{\gamma s}{\gamma}} \leq C|x|^{\gamma s - \frac{\beta(n - \gamma)}{4}}
\]

where we can choose \( \beta = \frac{\gamma s}{n(q - s)} \). For (5.5) and \( s = 2 \), we obtain

\[
\int_{B_{r_{i}}(x)} \log \frac{1}{|x - y|} Q_{g}(y) e^{nu(y)}dy \leq \left( \int_{B_{|x|}^{\beta}} \left( \log \frac{1}{|x - y|} \right)^{2} \right)^{\frac{1}{2}} \left( \int_{B_{|x|}^{\beta}} (Q_{g} e^{nu})^{2} \right)^{\frac{1}{2}} + \int_{B_{r_{i}}(x) \setminus B_{|x|}^{\beta}} \log \frac{1}{|x - y|} Q_{g}(y) e^{nu(y)}dy \leq C|x|^{-\frac{\beta(n - \gamma)}{2}} + \beta \log |x| \int_{B_{r_{i}}(x) \setminus B_{|x|}^{\beta}} Q_{g}(y) e^{nu(y)}dy \leq \log |x| + C
\]

where the last inequality we use the \( \int_{\mathbb{R}^{n}} Q_{g} e^{nu} < +\infty \) and \( |x| \gg 1 \). Combated with Lemma 2.3, Lemma 2.4 and (5.6), we get the estimate (5.3).
Theorem 5.1 Suppose \((\mathbb{R}^n, e^{2u}|dx|^2)\) is normal with absolutely integrable \(Q_g(x)e^{nu(x)}\), if
\[
0 \leq Q_g(x) \leq C_0|x|^{\gamma} \quad \text{for} \quad |x| \gg 1, \quad \text{where} \quad \gamma > 0.
\] (5.7)
Then, for any \(\epsilon > 0\), there exist \(R_\epsilon \gg 1\) such that
\[
-(\alpha - \epsilon) \log |x| + C \geq u(x) \geq -\alpha \log |x| - C
\]
for \(|x| > R_\epsilon\).

Proof. The asymptotic formula follows by Lemma 2.3, Lemma 2.4 and Lemma 5.1. □

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References

[1] T. Branson Sharp inequalities, the functional determinant, and the complementary series. Trans. Amer. Math. Soc. 347 (1995), no. 10, 3671–3742.

[2] K. Cheng and C. Lin, On the asymptotic behavior of solutions of the conformal Gaussian curvature equations in \(\mathbb{R}^2\) Math. Ann. 308 (1997), no. 1, 119–139.

[3] S.-Y. A. Chang, J. Qing and P. C. Yang, On the Chern-Gauss-Bonnet integral for conformal metrics on \(\mathbb{R}^4\) Duke Math. J. 103 (2000), no. 3, 523–544.

[4] S.-Y. A. Chang, J. Qing and P. C. Yang, Compactification of a class of conformally flat 4-manifold. Invent. Math. 142 (2000), no. 1, 65–93.

[5] S.-Y. A. Chang, M. Gursky and P. C. Yang, Regularity of a fourth order nonlinear PDE with critical exponent. Amer. J. Math. 121 (1999), no. 2, 215–257.

[6] S.-Y. A. Chang and P. C. Yang, Extremal metrics of zeta function determinants on 4-manifolds. Ann. of Math. (2) 142 (1995), no. 1, 171–212.

[7] S.-Y. A. Chang and P. C. Yang, On uniqueness of solutions of nth order differential equations in conformal geometry. Math. Res. Lett. 4 (1997), no. 1, 91–102.

[8] X. Chen and X. Xu, Q-curvature flow on the standard sphere of even dimension. J. Funct. Anal. 261 (2011), no. 4, 934–980.

[9] C. Fefferman and C. Graham, The ambient metric. Annals of Mathematics Studies 178 Princeton University Press, Princeton, NJ, 2012. x+113 pp.

[10] H. Fung On a conformal Gauss-Bonnet-Chern inequality for LCF manifolds and related topics. Calc. Var. Partial Differential Equations 23 (2005), no. 4, 469–496.

[11] R. Finn, On a class of conformal metrics, with application to differential geometry in the large. Comment. Math. Helv. 40 (1965), 1–30.

[12] Y. Ge and X. Xu, Prescribed Q-curvature problem on closed 4-Riemannian manifolds in the null case. Calc. Var. Partial Differential Equations 31 (2008), no. 4, 549–555.

[13] A. Huber, On subharmonic functions and differential geometry in the large. Comment. Math. Helv. 52 (1957), 13–72.

[14] A. Hyder, G. Mancini and L. Martinazzi, Local and nonlocal singular Liouville equations in Euclidean spaces. Int. Math. Res. Not. IMRN 2021, no. 15, 11393–11425.

[15] T. Jin, A. Maalaoui, L. Martinazzi and J. Xiong, Existence and asymptotics for solutions of a non-local Q-curvature equation in dimension three. Calc. Var. Partial Differential Equations 52 (2015), no. 3-4, 469–488.

[16] E. Lieb, M. Loss, Analysis. Second edition. Graduate Studies in Mathematics 14. American Mathematical Society, Providence, RI, 2001. xxii+346 pp. ISBN: 0-8218-2783-9

[17] C. Lin, A classification of solutions of a conformally invariant fourth order equation in \(\mathbb{R}^n\). Comment. Math. Helv. 73 (1998), no. 2, 206–231.

[18] L. Martinazzi, Classification of solutions to the higher order Liouville’s equation on \(\mathbb{R}^{2m}\) Math. Z. 263 (2009), no. 2, 307–329.
[19] S. Ma and J. Qing On Huber-type theorems in general dimensions. Adv. Math. 395 (2022), Paper No. 108145, 37 pp.

[20] E. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. With the assistance of Timothy S. Murphy. Princeton Mathematical Series, Princeton Mathematical Series 43 Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993. xiv+695 pp.

[21] Y. Wang, The isoperimetric inequality and Q-curvature. Adv. Math. 281 (2015), 823–844.

[22] J. Wei and X. Xu Classification of solutions of higher order conformally invariant equations. Math. Ann. 313 (1999), no. 2, 207–228.

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