Connected holonomy is lower semicontinuous

Olaf Müller

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Abstract

In this article, we show that the map assigning to a metric its connected holonomy class (conjugacy class of the connected component of the holonomy representation) is lower semicontinuous on the space of $C^2$ metrics on a compact manifold. The same is true for the entire holonomy group at metrics of compact holonomy.

1 Introduction and statement of the main results

A vital feature of control theory and its innumerable technical applications (like parking a car or steering a satellite via 2 or 3 nozzles) is the effect of holonomy. Let $M$ be an oriented $n$-dimensional manifold. If we consider holonomy of the Levi-Civita connection as a map $\text{Hol}$ on $\text{Met}(M)$ to the power set of $SO(n)$, then a question of fundamental importance is whether $\text{Hol}$ is continuous. This article will show that the answer is halfway "yes". Let $M$ be a compact $n$-dimensional manifold and let $\text{Met}(M)$ be the space of Riemannian metrics on $M$ of regularity $1$, equipped with the usual $C^2$ (metrizable) topology. Let $C_n$ be the set of conjugacy classes of closed subgroups of $SO(n)$. It is well-known that the maps $\text{Hol}, \text{Hol}^0 : \text{Met}(M) \to C_n, \text{Hol}(g) = [\text{Hol}_p(g)], \text{Hol}^0(g) = [\text{Hol}_p^0(g)] \forall g \in \text{Met}(M)$ (where we use a $g$-orthogonal frame to identify $T_pM$ with $\mathbb{R}^n$) do not depend on the choice of $p \in M$. We equip $C_n$ with the quotient space topology under $q : C(SO(n)) \to C_n$ where, for a group $K$, the space $C(K)$ resp. $FC(K)$ is the set of closed subgroups of $K$ resp. of closed subgroups of $K$ with finitely many connected components, both equipped with the Hausdorff distance (defined from a

\footnote{Humboldt-Universität zu Berlin, Institut für Mathematik, Unter den Linden 6, 10099 Berlin. Email: mullerol@math.hu-berlin.de}

\footnote{About the choice of this regularity, see the last paragraph of the article.
bi-invariant metric on $SO(n)$) and its induced topology (and the identity component $\text{Hol}_p^0(g)$ of $\text{Hol}_p^0(g)$ is a closed subgroup of $SO(n)$, thus a compact Lie group).

On $X := q(FC(SO(n)))$ there is a (partial) order by inclusion of representatives. Indeed, $\leq \subset X \times X$ defined by

$$\forall a, b \in X : (a \leq b : \iff \exists x \in a \exists y \in b : x \subset y)$$

is a partial order: In contrast to the absence of a Cantor-Bernstein Theorem in the entire category of groups (cf. next section), there is even a strong Cantor-Bernstein Theorem for Lie groups with finitely many connected components.

**Theorem 1** Let $K, H$ be Lie groups with finitely many connected components, let $f : K \to H$ and $g : H \to K$ be injective Lie group homomorphisms. Then $f$ and $g$ are Lie group isomorphisms. Furthermore, let $K$ and $H$ be closed subgroups of a compact Lie group $G$ such that there are $g, h \in G$ with $gKg^{-1} \subset H$ and $hHh^{-1} \subset K$. Then $gKg^{-1} = H$.

The fact that $X$ is ordered allows to speak of semicontinuity of maps to $X$. In this article, we are going to show:

**Theorem 2** For $M$ compact, $\text{Hol}^0 : \text{Met}(M) \to X$ is lower semi-continuous. Moreover, let $g \in \text{Met}(M)$ with $\text{Hol}(g)$ compact. Then $\text{Hol}$ is lower semi-continuous at $g$.

The restrictions of $H$ and $H^0$ to the space $FM$ of Ricci-flat metrics on $M$ are even known to be locally constant, cf. [1].

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## 2 Proofs

**Conventions:** All manifolds in this article are supposed to be finite-dimensional. Fix a compact oriented $n$-dimensional manifold $M$ throughout the article.

The Lie algebra of a Lie group $G$ is denoted by $\text{LA}(G)$, and, for $0 \leq k \leq n = \dim(V)$, the oriented $k$-Grassmannian of a vector space $V$ with a scalar

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2“strong” means that if there are injective morphisms $f : X \to Y$ and $g : Y \to X$ then $f$ and $g$ themselves are isomorphisms.
product $\langle \cdot , \cdot \rangle$ by $\text{Gr}_k(V)$, topologized by the metric $d_P(V,W) := |P_V - P_W|_{op}$ where $V,W$ are two $k$-dimensional subspaces, $P_K$ is the orthonormal projection onto $K$ for any subspace $K$ of $V$, and $| \cdot |_{op}$ is the operator norm of a linear map.

Following Klingenberg [4], we call a subset $A$ of a metric space $Z$ strongly convex iff

- For every $p, q \in A$ there is exactly one shortest curve $c_{pq} : [0; 1] \rightarrow Z$ from $x$ to $y$, and the image of $c_{pq}$ is in $A$.
- Every ball in $A$ is convex, i.e., $\forall p,q,x \in A : c_{pq}([0; 1]) \subset B(x, \max\{d(x,p),d(x,q)\})$.

For each $p \in Z$ we define the **convexity radius** $r_Z(p) := \sup\{\rho > 0 | B(p, \rho) \text{ strongly convex} \}$ at $p$. Then by definition, $r$ is a 1-Lipschitz function on $Z$. If $Z$ is a Riemannian manifold, then Whitehead’s classical result ensures that $r_Z$ is a positive function on $Z$. We define the **convexity radius** $r(Z) := \inf\{r_Z(x)|x \in Z\}$ of $Z$. If $Z$ is a compact Riemannian manifold, a consequence of the facts above is $r(Z) > 0$.

Let us first revise an example due to Tsit Yuen Lam showing that even in $\text{Gl}(2, \mathbb{R})$ there is no Cantor-Bernstein theorem: Consider the usual linear representation $\Phi$ of the affine group in one dimension given by $\Phi : Y := (\mathbb{R} \setminus \{0\}) \times \mathbb{R} \rightarrow \text{Gl}(\mathbb{R}^2)$ defined by $(x,y) \mapsto \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$. Now $G := \Phi(Y)$ is a subgroup of $\text{Gl}(2, \mathbb{R})$, and $H := \Phi(\{1\} \times \mathbb{Z})$ is a subgroup of $G$ and of $\text{Gl}(2, \mathbb{R})$. For $g := \Phi((2,0))$ we get $gHg^{-1} = \Phi(\{1\} \times 2\mathbb{Z}) \subseteq H$.

**Proof of Theorem** □. $d_1f : T_1K \rightarrow T_1H$ and $d_1g : T_1H \rightarrow T_1K$ are linear. Moreover, the fact that for the local diffeomorphism $\exp$ we have $\exp \circ d_1f = f \circ \exp$ implies that $d_1f$ is injective: Assume $d_1f(v) = 0$ for some $v \in T_1K$, then choose $U \subset T_1G$ such that $\exp|_U$ is a diffeomorphism onto its image. There is $\lambda \in (0; \infty)$ with $w := \lambda v \in U$ and still $d_1f(w) = 0$. Now we have $\exp(d_1f(v)) = 1$ but $(f \circ \exp)(v) \neq 1$. Consequently, $v = 0$.

By the Cantor-Bernstein Theorem in the category of vector spaces, both tangent spaces have the same dimension and both linear maps are isomorphisms. Moreover, the connected component $H_0$ of 1 in $H$ is contained in $f(K)$ by surjectivity of $\exp$ onto $H_0$ and surjectivity of $d_1f$. With an analogous argument, $f$ is surjective onto every connected component it meets. Therefore $\pi_0f : \pi_0K \rightarrow \pi_0H$ is injective. As both $\pi_0K$ and $\pi_0H$ are finite, they contain the same number of elements, and $\pi_0f$ is bijective. □
To formulate some of the results, it will be convenient to define, for a metric space \((Z, d)\) with base point \(x_0\), the Busemann metric \(d_1\) on the space \(C(Z)\) of closed subsets of \(Z\) by

\[
d_1 = d_1^{x_0, d_0} : C(Z) \times C(Z) \to \mathbb{R} \cup \{\infty\}, \text{such that, for all } A, B \in C(Z) : \\
d_1(A, B) = \sup \{|d_0(\{x\}, A) - d_0(\{x\}, B)| \cdot \exp(-d_0(x_0, x)); x \in Z\}
\]

Convergence w.r.t. \(d_1\) is equivalent to Hausdorff convergence of intersections with every fixed compact subset of \(Z\), and if \(d\) is Heine-Borel, then \((C(X), d_1)\) is a compact metric space (for details see e.g. [6], Sec. 4). It is easy to see that \(d_{\text{Bigr}}(V) = d_P\) for a Euclidean vector space \(V\). With these preparations, we can show that \(\exp^{-1}_x\) is upper semi-continuous:

**Theorem 3** Let \(M\) be a Riemannian manifold and \(x \in M\). Let \(d_g\) be the metric induced by the Riemannian metric, and let \(C(T_x M)\) be the set of closed subsets of \(T_x M\), equipped with the Busemann metric \(d_B\) induced by \(g_x\). Then \(\exp^{-1}_x : (M, d_g) \to (C(T_x M), d_B)\) is upper semi-continuous, i.e.:

\[
\forall p \in M \forall \epsilon > 0 \exists \delta > 0 \forall q \in B^{d_g}(p, \delta) : \exp^{-1}_x(q) \subset B^{d_B}(\exp^{-1}_x(q), \epsilon).
\]

**Proof.** In view of the last paragraph preceding the theorem, it suffices to show that such an inclusion holds w.r.t. the Hausdorff metric after intersecting both sides with compact subsets of \(T_x M\), say \(B(0, R)\) and \(B(0, R + 1)\) (if we choose \(\delta < 1\)), i.e., we have to show \(\exp^{-1}_x(q) \cap B(0, R) \subset B(\exp^{-1}_x(p), \epsilon)\) for sufficiently small \(\delta\), which means a bound \(R + 1\) on the length of all involved geodesics, and this follows directly from classical Jacobi field estimates (note that sectional curvature is bounded in \(\exp(B(0, R + 1))\) as the Riemannian curvature tensor is continuous). \(\square\)

**Remark.** Full continuity is not true: The other possible inclusion fails e.g. in the example of a unit sphere in which \(x\) and \(p := -x\) are points opposite to each other: Let \(x, p \neq q_n \to p\) as \(n \to \infty\), then \(\exp^{-1}_x(q_n) \cap \text{cl}(B(0, \frac{3}{2}\pi))\) all consist of one or two points but \(\exp^{-1}_x(p) \cap B(0, \frac{3}{2}\pi) = \partial B(0, \pi)\).

Furthermore, we will need some converse of the classical group-theoretical statement of Montgomery-Zippin ([3]), accounted for below in Th. 7. A naive guess would be that its hypothesis implies also the converse of its conclusion, i.e. that for any Lie group \(G\) and any compact Lie subgroup \(K\) there is a neighborhood \(U\) of \(K\) s.t. for every Lie subgroup \(H\) of \(G\) contained
in $U$ there is $g_U \in G$ with $g_U^{-1}Kg_U$ being a subgroup of $H$ — but this is obviously wrong, see e.g. the example of $K$ being the compact Abelian group $\mathbb{R}^2 = G = K$, which contains the irrational slope dense subgroups isomorphic to $\mathbb{R}$. Second, one would like to convert the hypothesis, such that $K$ is supposed to be in some neighborhood of the groups $H$, so we need some uniformity. We use the metric and ask: Is it true that for a Lie group $G$ with a bi-invariant Riemannian metric and a compact subgroup $K$ there is $\epsilon > 0$ s.t. for every subgroup $H$ of $G$ with $K \subset B(G, \epsilon)$ there is a $g \in G$ with $g^{-1}Kg \subset H$? Again, the example of finite subgroups $H$ of $K := \mathbb{S}^1$ disprove this. The assumption of connectedness of $H$ alone does not help, cf. the connected rational subgroups $H_n$ of $\mathbb{S}^1 \times \mathbb{S}^1$ of slope $1/n$ approximating arbitrarily well the entire group. But in this example, every $G$-ball around $1$ contains elements of $H_n$ far from $1$ in the intrinsic distance on $H_n$. This motivates us to include the \textit{intrinsic} distance in the hypothesis:

\textbf{Theorem 4} Let $G$ be a compact Lie group and let $K$ be a Lie subgroup of $G$. For all $L \in (0; \infty)$, there is $\epsilon \in (0; \infty)$ such that: For a Lie subgroup $H$ of $G$ with intrinsic diameter $\text{diam}_H(H) < L$ and

$$K \subset B_G(H, \epsilon)$$

there is $g_\epsilon \in B_G(1, \epsilon)$ with $g_\epsilon^{-1}Kg_\epsilon \subset H$.

Preparing a proof of Theorem 4, the following two theorems (probably already obtained elsewhere) slightly generalize results of Montgomery-Zippin.

\textbf{Theorem 5} Let $(Z, g)$ be a Riemannian manifold, let $x \in Z$. Then for each two geodesic curves $\gamma, \delta : [0; 1] \to B(x, r(x)/4)$ and for all $a, b \in [0; 1]$,

$$d(\gamma(a), \delta(b)) < \max\{d(\gamma(0), \delta(0)), d(\gamma(0), \delta(1)), d(\gamma(1), \delta(0)), d(\gamma(1), \delta(1))\}.$$  

\textbf{Proof.} We first consider $y \in B(x, r/4)$ and prove $d(y, c(s)) < d(y, c(0)), d(y, c(1)))$ for all geodesics $c$ with endpoints in $B(x, r/4)$ due to convexity of the ball $B(y, \max\{d(y, c(0)), d(y, c(1))\})$. We apply this to $y$ being the inner points on the geodesics $\gamma$ and $\delta$ and $c$ being the other geodesic in each case. \hfill $\square$

\textbf{Theorem 6} Let $(Z, g)$ be a Riemannian manifold and let $H \subset \text{Isom}(M, g)$ be compact. Let $x \in Z$ and let $r$ be the convexity radius of $(Z, g)$ at $x$. If $H(x) \subset U := B(x, r/2)$ then $H$ has a fix point in $U$.  

5
Proof. Let \( A := \{ p \in Z | H(p) \subset \text{cl}(U) \} \). As \( x \in A \), the latter is nonempty. Moreover, it is a closed subset of \( \text{cl}(U) \) and thus compact. Let \( q \in \text{cl}(U) \) with

\[
\text{diam}(H(q)) = \min \{ \text{diam}(H(p)) | p \in A \},
\]

which exists as \( p \mapsto \text{diam}H(p) \) is continuous. Assume that \( H(q) \neq \{ q \} \).

Then let \( \gamma \) be the unique geodesic between two distinct elements \( x, z \) of \( H(q) \), i.e., \( \gamma : [0; 1] \to U \) with \( \gamma(0) = x \) and \( \gamma(1) = z \). Choose \( a \in (0; 1) \) and let \( y := \gamma(a) \). Convexity implies that \( y \in A \). We claim that

\[
\text{diam}(H(y)) < \text{diam}(H(q)),
\]

in contradiction with Eq. 1. And in fact, convexity of \( \text{cl}(U) \) implies that \( H(y) \subset \text{cl}(U) \) and compactness of \( H \) ensures that there is an \( h \in H \) with \( \text{diam}(H(y)) = d(y, h(y)) \). If we define \( \delta := h \circ \gamma \) (so \( \delta(a) = h(y) \)), then the claim follows by application of Theorem 5 with \( a = b \), as \( \gamma(0), \gamma(1), \delta(0), \delta(1) \in H \cdot \gamma(0) \), so all terms in the maximum are \( \leq \text{diam}(H \cdot \gamma(0)) \).

As a corollary, we recover a version of the Montgomery-Zippin Theorem (cf. [5]) with explicit size of the neighborhood in the hypothesis, something we will later need in the proof of the converse version of the same theorem:

**Theorem 7 (Quantified Montgomery-Zippin Theorem)** Let \( G \) be a Lie group and let \( K \) be a compact Lie subgroup of \( G \). Let \( r_K \) be the convexity radius of \( G/K \) with a left-invariant metric \( g \). Let \( 0 < r < r_K \). Then for each Lie subgroup \( H \) of \( G \) with \( H \subset W := B(K, r) \), there is \( g_{r,H} \in B_G(1, r) \) s.t. \( g_{r,H}^{-1} H g_{r,H} \subset K \).

Proof. Let \( q : G \to G/K \). First note that \( [1] = 1 \cdot K = K \). Let \( H \) be a subgroup of \( G \) contained in \( W \). Then for all \( h \in H \), we have \( hK \subset W \), so in the quotient \( Z := G/K \) we have \( hK \in U := \{ wK | w \in W \} \subset G/K \). Thus Theorem 5 applied to \( x = q(1) \) implies that \( H \) has a fix point \( gK \) in \( U \), i.e. \( HgK = gK \), so \( g^{-1}HgK = K \), which implies \( g^{-1}Hg \subset K \).

Now we want to switch roles of \( H \) and \( K \) in the last step. To this aim, we first restrict ourselves to the case of \( G \) carrying a bi-invariant metric, which is the case for \( G = \text{Hol}^0 \), as the latter is a closed subgroup of \( SO(n) \) and thus compact, so a bi-invariant metric on \( G \) is given as the negative of the Killing form. Then Theorem 4 is implied by the following theorem:
Theorem 8 Let \( g \) be a bi-invariant metric on a Lie group \( G \), \( L \in (0; \infty) \). Then there is \( \epsilon > 0 \) such that for every subgroup \( H \neq G \) of \( G \) of intrinsic diameter \( < L \), the convexity radius \( r_H = r(G/H) \) of \( G/H \) around [1] satisfies \( r_H \geq \epsilon \).

Remark. The diameter bound cannot be omitted, as rational subgroups of tori show.

Proof. This will follow e.g. from Dibble’s [3] improvement

\[
\lambda(Z) = \min\{r_f(Z), \lambda(Z)/4\}
\]

of Klingenberg’s Lemma, where \( Z \) is a complete Riemannian manifold, \( r_f(Z) \) is the focal radius, which is \( \geq \frac{\delta}{\sqrt{3}} \) if \( \sec Z \leq \delta \), and \( \lambda(Z) \) is the length of the shortest nonconstant periodic geodesic in \( Z \). The map \( q \) as above is a Riemannian submersion. To verify the hypotheses of Dibble’s theorem, we first examine \( \sec_{G/H} \), which for every, possibly nonconnected, \( H \) only depends on the connected component of the identity of \( H \). It is well-known (see e.g. [8], Th 3.1) that for a bi-invariant metric \( \langle \cdot, \cdot \rangle \), under the identification of horizontal vector fields (i.e. vector fields with values in the orthogonal complement \( P := (T_1H)^\perp \) of \( T_1H \)) with vector fields in \( G/H \) and with \( p \) being the orthogonal projection \( T_1G \rightarrow P \), we get for the Riemannian curvature of \( G/H \) and for \( X, Y \) horizontal:

\[
\langle R(X, Y)X, Y \rangle = -\frac{3}{4} p([X, Y])^2 - \frac{1}{2} \langle [X, [X, Y]], Y \rangle - \frac{1}{2} \langle [Y, [Y, X]], X \rangle + |U(X, Y)|^2 - \langle U(X, X), U(Y, Y) \rangle
\]

where \( U : P \times P \rightarrow P \) is defined by \( 2U(X, Y), Z = \langle [Z, X], Y \rangle + \langle X, [Z, Y] \rangle \) for every horizontal \( Z \). As \( p \) does not increase the length, compactness of the unit sphere implies that there is a global bound on \( \sec \).

Now for the estimate on \( \lambda(G/H) \): Let \( G \) be a fixed \( n \)-dimensional Lie group. First we show a generalization of well-known results accounted for e.g. in [2], Sec. 1.4.

Lemma 1 Let \( a : \mathbb{N} \rightarrow LA(G) \setminus \{0\} \) with \( a(n) \rightarrow_{n \to \infty} 0 \) and \( \exp(a(n)) \in H_n \rightarrow_{n \to \infty} H_{\infty} \), and assume that \( H_n \) for all \( n \) and \( H_{\infty} \) are closed subgroups. Let \( a(n)/|a(n)| \rightarrow_{n \to \infty} v \in LA(G) \). Then \( \exp(tv) \in \exp(V_{\infty}) \) for all \( t \in \mathbb{R} \).

Proof of the lemma. Let \( t \in \mathbb{R} \), let \( c(n) := [t/|a(n)|] \in \mathbb{N} \). Then \( t \in I_n := [c(n)|a(n)|; (c(n) + 1)|a(n)|] \). As \( a(n) \) converges to 0, the length of the interval \( I_n \) converges to zero, thus \( a(n)/|a(n)| \rightarrow_{n \to \infty} t \) and, for all \( n \in \mathbb{N} \), 

\[ \]
\( H_n \ni (\exp a(n))^{c(n)} = \exp(c(n)a(n)) = \exp(c(n)|a(n)| \frac{a(n)}{|a(n)|}) \to_{n \to \infty} \exp(tv). \)

The assertion of the theorem follows from the above together with the fact
\( H_\infty = \{ x \in G \| b : N \to G : b(n) \in H_n \forall n \in N \land b(n) \to_{n \to \infty} x \} \)

**Lemma 2** Let \( D \in [0; \infty], k \in [1; n] \) and let \( L_D \) be the space of all Lie subalgebras \( V \) of \( LA(G) \) such that \( \exp(V) \) is of intrinsic diameter \( \leq D \). Then \( L_D \) is closed in \( \Gr_k(LA(G)) \).

**Proof.** Closedness of the Lie algebra condition is obvious, for the diameter condition let w.l.o.g. \( D < \infty \), then let a sequence of Lie subalgebras \( n \mapsto V_n \in L_D \) be given that converges to a Lie subalgebra \( V_\infty \). Let \( \Delta : \Gr_k(LA(G)) \to [0; \infty) \) be defined by \( \forall V \in \Gr_k(LA(G)) : \Delta(V) := d_{\mathbb{H}d}(\exp(B(0,D + 1) \cap V), \exp(B(0,D) \cap V)). \) Then \( \Delta \) is continuous, and \( \Delta(V_n) = 0 \) for all \( n \in \mathbb{N} \), thus \( \Delta(V_\infty) = 0 \) and thus \( \text{diam}(V_\infty) \leq D. \)

**Lemma 3** For all \( L > 0 \), there is \( \epsilon > 0 \) such that for all connected Lie subgroups \( H \) with intrinsic diameter \( < L \) we get
\[ H \cap B(1, \epsilon) \subset \exp(B_{T_1 H}(0, \epsilon)). \]

**Proof of the lemma.** The task can be reduced to showing
\[ \exp(B_{T_1 H \perp}(0, \epsilon)) \cap \{0\} \setminus H = \emptyset. \]
Indeed, let \( p \) minimize \( d(1, \cdot) \) on \( H \cap \text{cl}(B(1, \epsilon)) \setminus \exp(B_{T_1 H}(0, \epsilon)) \) and assume \( p \in \exp(B(0, \epsilon)) \setminus (T_1 H)^\perp \), then the first variational formula for geodesics implies that there is \( q \in \exp(T_1 H \cap B(0, \epsilon)) \) with \( d(q,p) < d(1,p) < \epsilon \), but \( q^{-1}p \in H \) and \( d(1,q^{-1}p) = d(q,p) \), thus \( q^{-1}p \in B(1, \epsilon) \) contradicting the minimizing assumption.

Now, assume the opposite of the last displayed equation. Then there is a sequence \( a \) of connected Lie subgroups \( H \) with intrinsic diameter \( < L \) and \( w(n) \in T_1 a(n)^\perp \setminus \{0\} \) such that \( \exp(w(n)) \in a(n) \) and \( w(n) \to_{n \to \infty} 0. \) Compactness of \( \partial B(0,1) \) implies that the sequence of the \( x(n) := w(n)/|w(n)| \) has a subsequence converging to some \( v \in T_1 G \) with \( |v| = 1. \) Furthermore, by the pidgeon hole principle we can restrict to a subsequence of \( k \)-dimensional subgroups, for some \( k. \) Then by compactness of \( \Gr_k(LA(G)) \) there is a further subsequence converging to some limit subspace \( V_\infty \), which by closedness of the subset of Lie subalgebras is a Lie subalgebra. Furthermore, we have \( v \perp V_\infty \) as \( v \perp LA(H_n) \) for all \( n \in \mathbb{N}. \) And exp
maps $V_\infty$ to a closed subgroup $H_\infty$ due to the uniform intrinsic diameter bound on the $V_n$ by Lemma 2. Then the application of the Lemma 1 yields $\exp(tv) \in H_\infty \forall t \in \mathbb{R}$, thus $v \in LA(H_\infty)$. But $v \perp LA(H_\infty)$, so $v = 0$ which contradicts $|v| = 1$. □

By homogeneity we can assume that the geodesic $c$ appearing in $\lambda(G/H)$ starts at $[1]$. It lifts to a geodesic $\tilde{c}$ of the same length starting at $1$ with differential in $P := (T_1H)^\perp$ and with an endpoint in $H$. We can uniformly bound $\{|v| : v \perp T_1H, \exp(v) \in H\}$ from below by the $\epsilon$ from Lemma 3, which finally concludes the proof of Th. □

As now by Th. the convexity radius of $G/H$ can be bound uniformly from 0 for all subgroups $H$ of intrinsic diameter $\leq L$, Th. 4 follows by swapping $K$ and $H$ in the proof of Th. 7. □

To finish the proof of Theorem 2 let us try to show that the hypotheses of Theorem 4 are satisfied if we approach a limit metric. It turns out that it is essential to take into account the minimal length of curves realizing a given element of the holonomy. Indeed, let $\epsilon > 0$. For each $A \in \text{Hol}(g_1)$ there is a curve $c_A$ with $P_{c_A}g_1 = A$. Then there is a $C^1$ neighborhood $U$ of $g_1$ with

$$\forall g \in U : A \in B(P_{c_A}g_1, \epsilon) \subset B(\text{Hol}(g), \epsilon).$$

But the size of the neighborhood depends on the $g_1$-length $l(c_A)$ of $c_A$. Thus, we consider $\lambda : \text{Hol}^0(g_1) \to \mathbb{R}$, $\lambda(A) := \inf\{l(c)|P_{c_A}g_1 = A\}$. As $\text{Hol}^0(g)$ is compact, a positive lower bound on $\lambda$ could be established if we knew that $\lambda$ was upper semi-continuous. Unfortunately, this is wrong i.g., which can be seen by examples of manifolds that are flat in a neighborhood of $p$. (Conversely, we can prove that $\lambda$ is lower semi-continuous by an easy application of the general fact that for each $f : A \to \mathbb{R}$ continuous and $g : A \to C$ continuous and $f$ bounded on preimages of $g$, we have $\mu : c \mapsto \inf\{f(a)|g(a) = c\}$ is lower semi-continuous). What helps us in the end is Wilkins’ result [9] that, whenever $\text{Hol}(g)$ is compact, we have

$$\forall g \in \text{Met}(M) \exists L_0 \in \mathbb{R} \forall A \in \text{Hol}(g) \exists c \in \Omega(M) : l(c) \leq L_0 \land P^{\delta}(c) = A.$$ 

Thus we get via the usual ODE estimates

$$\forall g \in \text{Met}(M) \forall \epsilon > 0 \exists \delta > 0 \forall h \in B_{\text{Met}(M)}(g, \delta) : \text{Hol}(g) \subset B_{SO(n)}(B_{\text{Hol}(h)}(1, L_0 + \epsilon), \epsilon). \quad (4)$$

9
more exactly, Wilkins’ construction provides a subset $A_g$ of interval subsets of one-parameter subgroups of curves of length $\leq L$ with $\text{Hol}(g) = H(A_g) := \{P_c|c \in A_g\}$. Then our hope could be that inspection of Wilkins’ proof shows a bound on the intrinsic diameter of the subgroups, as for the holonomy one only has to pull together the slings of the very explicit lassos. Then application of Theorem 4 would conclude the proof of Theorem 2. But this hope fails: It seems difficult to bound the intrinsic diameter of subgroups uniformly in a neighborhood. So we show the following stronger variant of Th. 4:

**Theorem 9** Let $G$ be a compact Lie group, let $K$ be a Lie subgroup of $G$. For all $L \in (0; \infty)$, there is $\epsilon \in (0; \infty)$ s.t. for a Lie subgroup $H$ of $G$ with

$$K \subset B_G(B_H(1, L), \epsilon) (\subset B_G(1, L + \epsilon))$$

there is $g_\epsilon \in B_G(1, \epsilon)$ with $g_\epsilon^{-1}Kg_\epsilon \subset H$.

**Proof.** We just mimick the proof of Th 4 — where the crucial point was establishing a lower diameter bound of the convex subset $B_{G/H}([1], r)$ — except that now we do not quotient out the whole subgroup $H$ but work in $G$ instead of in $G/H$ and focus on $H_r := B_H(1, r)$: We define a tubular neighborhood $U$ of $H_r$ of radius $\epsilon$ with intrinsically convex yet noncomplete fibers $F_h$ at elements $h$, identified with each other via right multiplication by $H_r$. We prove existence of this tubular neighborhood analogously to the proof of Theorem 8 and show: Let $g$ be a bi-invariant metric on a Lie group $G$, and let $L > 0$, then there is $\epsilon > 0$ such that for each proper subgroup $H$ of $G$, there is a tubular neighborhood $U$ of $H_r$ each of whose fibers contains an isometric copy of $B(0, \epsilon)$. More precisely, we first show a refinement of Th. 8:

**Lemma 4** Let $G$ be a Lie group with a bi-invariant metric. Let $L > 0$. Then there is $\epsilon > 0$ such that for every proper subgroup $H$ of $G$ the piece $H_L := B_H(1, L)$ has a tubular neighborhood of thickness $\geq \epsilon$.

**Proof of the lemma.** To prove the lemma, we would like to apply Lemma 3 to get a $\delta > 0$ with Eq. 3 but now we do not have a uniform bound on intrinsic diameter any more. Instead, we proceed by contradiction and assume a sequence of (w.l.o.g. connected) subgroups $H_n$ such that $K \subset B_G(B_{H_n}(1, L), 1/n)$ with $\text{conv}(G/H_n) < 1/n$ and associated Lie algebras of codimension $\geq 1$ which contains a subsequence that $d_B$-converges...
to some $H_\infty$ and corresponding $w(n) \perp T_1 H_n$ as in the proof of Lemma 3. Then we get a contradiction to positivity of the thickness of a normal neighborhood of $B_{H_\infty}(1, L)$ exactly as in the proof of Lemma 3 and Th. 8. In other words, there is a uniform upper bound for the length of self-intersecting geodesics from the central points in the fibers. We cannot directly use Dibble’s theorem as it only applies to complete manifolds; but it can painlessly be replaced with the corresponding local statement from [7, Cor.3.5], which uses self-intersecting instead of periodic geodesics, explicitly:

Let $(M, g)$ be a Riemannian manifold, let $x \in M$, let $\delta, \ell, r \in \mathbb{R}_{>0}$. If

- the ball $B := B^g(x, r)$ is compact,
- $\text{sec}_y(\sigma) \leq \delta$ holds for every $y \in B$ and every 2-plane $\sigma \subseteq T_y M$,
- every self-intersecting geodesic in $(M, g)$ which is contained in $B$ has length $\geq \ell$,

then $\text{conv}_g(x) \geq \frac{1}{2} \min\{\frac{\pi}{\sqrt{\delta}}, \frac{\ell}{2}, \frac{r^2}{2}\}$.

And, fortunately, self-intersecting geodesics are exactly what we considered above, concluding the proof.

As left and right multiplication commute with each other, the left multiplication by $K$, if the latter is contained in $B_G(H_r, \epsilon/2)$, descends to a well-defined isometric action $\rho$ on the fiber $F_1$ at 1: For every $f \in F_1$ and every $k \in K$ there is exactly one $h \in H_r$ with $kfh \in F_1$. We then adapt the proof of Th. 4, replacing $H$ with $H_r$, taking minima in the fiber $F_1$, and considering only intrinsic geodesics in $F_1$, showing that $\rho$ has a fix point $g$, implying $KgH_r \subset g \cdot H_r$ as before, concluding $g^{-1}Kg \subset H_r \cdot H_r^{-1} \subset H$.

Now the main result Th. 2 follows from Eq. 4 and Th. 9 (for $r > L := \text{diam}(\text{Hol}(g)) + 1$ and $G := SO(n), K := \text{Hol}(g), H := \text{Hol}(h)$).

Regarding the question of the optimal regularity assumption for our purposes, the choice of $C^2$ regularity of the metric has been done in view of the curvature-driven Jacobi vector field estimates in our proof of Th. 3, which needed bounds of the curvature in compact subsets, and of Wilkin’s result which we use and which in turn uses the Ambrose-Singer Theorem. As the latter, to our best knowledge, is available if the connection is at least $C^1$ such that curvature exists as a continuous quantity. It is, however, conceivable that our main result is valid for metrics of lower regularity.
References

[1] Bernd Amman, Klaus Kröncke, Hartmut Weiß, Frederik Witt: Holonomy rigidity for Ricci-flat metrics. Math. Z. 291, no. 1-2, 303 — 311 (2019). arXiv: 1512.07390

[2] Helga Baum: Eichfeldtheorie. Springer-Verlag (2009)

[3] James Dibble: The convexity radius of a Riemannian manifold. Asian J. Math. 21, no. 1, 169 — 174 (2017). arXiv:1412.0341

[4] Wilhelm Klingenberg: Riemannian Geometry, Berlin, New York: De Gruyter (2011). https://doi.org/10.1515/9783110905120

[5] Deane Montgomery, Leo Zippin: A theorem on Lie groups. Bull. Amer. Math. Soc. 48, 448 — 452 (1942)

[6] Olaf Müller: Topologies on the future causal completion, arXiv: 1909.03797

[7] Olaf Müller, Marc Nardmann: Every conformal class contain a metric of bounded geometry. Math. Ann. 363, 143 — 174 (2015). arXiv:1303.5957

[8] Yu. G. Nikonorov, E. D. Rodionov, and V. V. Slavskii: Geometry of homogeneous Riemannian manifolds. Journal of Mathematical Sciences, Vol. 146, No. 6, pp. 6313 — 6390 (2007)

[9] David R. Wilkins: On the length of loops generating holonomy groups. Bull. London Math. Soc. 23, no. 4, 372 — 374 (1991)