MINIMIZATION OF THE LOWEST EIGENVALUE FOR A VIBRATING BEAM

QUANYI LIANG AND KAIRONG LIU
LMIB and School of Mathematics and Systems Science
Beihang University, Beijing, China

GANG MENG
School of Mathematical Sciences
University of Chinese Academy of Sciences, Beijing, China

ZHIKUN SHE
LMIB and School of Mathematics and Systems Science
Beihang University, Beijing, China

(Communicated by Jiangong You)

ABSTRACT. In this paper we solve the minimization problem of the lowest eigenvalue for a vibrating beam. Firstly, based on the variational method, we establish the basic theory of the lowest eigenvalue for the fourth order measure differential equation (MDE). Secondly, we build the relationship between the minimization problem of the lowest eigenvalue for the ODE and the one for the MDE. Finally, with the help of this built relationship, we find the explicit optimal bound of the lowest eigenvalue for a vibrating beam.

1. Introduction. The vibrating beam investigated in this paper is subject to an axial compressive load $\lambda$ which causes it to buckle. The beam is supported on an elastic foundation which provides, at each point $x$, an elastic destructive force $F(x)y$, which opposes restoration toward the line of no deflection and is directly proportional to the displacement $y$. From elementary beam theory, the natural modes of buckling of our problem are the eigenfunctions of a vibrating beam described by

$$y''''(t) + \lambda y''(t) + F(t)y(t) = 0, \quad t \in [0, 1],$$

subject to elastically constrained boundary conditions. Here, we will assume hinged-hinged boundary conditions

$$y(0) = y''(0) = 0 = y(1) = y''(1).$$

It is well-known that for an elastic destructive force (per unit length) $F \in \mathcal{L}^1 := \mathcal{L}^1([0, 1], \mathbb{R})$, problem (1)-(2) has a sequence of (real) eigenvalues

$$\lambda_1(F) < \lambda_2(F) < \cdots < \lambda_m(F) < \cdots .$$
The lowest eigenvalue \( \lambda_1 \) represents the smallest axial compressive force necessary to cause the beam to buckle. In this paper we are concerned with the lowest eigenvalue \( \lambda_1(F) \) and will give its optimal lower bound when the \( L^1 \) norm \( \|F\|_1 = \|F\|_{L^1([0,1])} \) is bounded. To this end, we will solve the following minimization problem

\[
L(r) := \inf \{ \lambda_1(F) : F \in B_1[r] \}. \tag{3}
\]

Here, for \( r \in (0, +\infty) \),

\[
B_1[r] := \{ F \in L^1 : \|F\|_1 \leq r \}
\]
is the ball of \( (L^1, \|\cdot\|_1) \). Once minimization problem (3) is solved, one has the following lower bound for \( \lambda_1(F) \):

\[
\lambda_1(F) \geq L(\|F\|_1) \quad \forall F \in L^1, \tag{4}
\]

which will be shown to be optimal in a certain sense.

Since the \( L^1 \) balls \( B_1[r] \) lack compactness even in the weak topology of \( L^1 \), we usually do not know whether minimization problem (3) can be attained by some potentials from \( B_1[r] \). To overcome this, different from the approach in [7, 14, 15, 16, 17], we here will extend the problem to the measure case.

Firstly, based on the variational method, we will establish the minimization characterization for the lowest eigenvalue \( \lambda_1(\mu) \) of the fourth order measure differential equation (MDE) described by

\[
dy^{(3)}(t) + \lambda y''(t) \, dt + y(t) \, d\mu(t) = 0, \quad t \in [0,1], \tag{5}
\]

with the corresponding boundary condition (2), arriving at the following first contribution of this paper.

**Theorem 1.1.** Let \( \mu \in \mathcal{M}_0 \) and the Rayleigh form

\[
R(u) = R_\mu(u) := \frac{\int_{[0,1]} (u'')^2 \, dt + \int_{[0,1]} u^2 \, d\mu(t)}{\int_{[0,1]} (u')^2 \, dt}\quad \text{for } u \in W^{2,2}_0 \setminus \{0\}. \tag{6}
\]

Problem (5)-(2) admits a lowest eigenvalue \( \lambda_1(\mu) \), which has the following minimization characterizations

\[
\lambda_1(\mu) = \min_{u \in W^{2,2}_0 \setminus \{0\}} R(u) = \min_{u \in \mathcal{H}^3_{00} \setminus \{0\}} R(u). \tag{7}
\]

Here, \( \mathcal{M}_0, W^{2,2}_0 \) and \( \mathcal{H}^3_{00} \) are as in (9),(25) and (26), respectively.

Secondly, we build a relation between the minimization problem of the lowest eigenvalue for the ODE (1) and the one for the MDE (5) and use this relationship to find the explicit optimal lower bound of the lowest eigenvalue for the ODE (1), arriving at the following main contribution of this paper.

**Theorem 1.2.** Given \( r > 0 \), one has

\[
L(r) = Q^{-1}(r).
\]

Here, the invertible elementary function \( Q : (-\infty, \pi^2) \to (0, +\infty) \) is defined as

\[
Q(x) := \frac{2x^{\frac{3}{2}} \sin x^{\frac{1}{2}}}{1 - \cos x^{\frac{1}{2}} - \frac{1}{2} x^{\frac{1}{2}} \sin x^{\frac{1}{2}}}. \tag{8}
\]
This paper is organized as follows. In Section 2, we will recall basic facts on measures, the Lebesgue-Stieltjes integral and the Riemann-Stieltjes integral. In Section 3, we will use the variational method to prove Theorem 1.1. In section 4, after obtaining the relationship between the minimization problem for ODE and the one for MDE, described by Theorem 4.1, we will find the explicit optimal bound of the lowest eigenvalue for the vibrating beam and thus prove Theorem 1.2.

2. Basic facts on measures. Let $I = [0, 1]$. For a function $\mu : I \to \mathbb{R}$, the total variation of $\mu$ (over $I$) is defined as

$$V(\mu, I) := \sup \left\{ \sum_{i=0}^{n-1} |\mu(t_{i+1}) - \mu(t_i)| : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1, \ n \in \mathbb{N} \right\}.$$ 

Let

$$\mathcal{M}(I, \mathbb{R}) := \{ \mu : I \to \mathbb{R} : \mu(0+) \text{ exists}, \ \mu(t+) = \mu(t) \ \forall t \in (0, 1), \ \int_I \mu < \infty \}$$

be the space of non-normalized $\mathbb{R}$-valued measures of $I$. Here, for any $t \in [0, 1)$, $\mu(t+) := \lim_{s \searrow t} \mu(s)$ is the right-limit. The space of (normalized) $\mathbb{R}$-valued measures is

$$\mathcal{M}_0(I, \mathbb{R}) := \{ \mu \in \mathcal{M}(I, \mathbb{R}) : \mu(0+) = 0 \}.$$ 

For simplicity, we write $V(\mu, I)$ as $\|\mu\|_V$. By the Riesz representation theorem [5], $(\mathcal{M}_0(I, \mathbb{R}), \|\cdot\|_V)$ is the same as the dual space of the Banach space $(C(I, \mathbb{R}), \|\cdot\|_\infty)$ of continuous $\mathbb{R}$-valued functions of $I$ with the supremum norm $\|\cdot\|_\infty$. In fact, $\mu \in (\mathcal{M}_0(I, \mathbb{R}), \|\cdot\|_V)$ defines $\mu^* \in (C(I, \mathbb{R}), \|\cdot\|_\infty)^*$ by

$$\mu^*(f) = \int_I f(t) d\mu(t), \quad f \in C(I, \mathbb{R}),$$

which refers to the Riemann-Stieltjes integral [1]. Moreover, one has

$$\|\mu\|_V = V(\mu, I) = \sup \left\{ \int_I f \ d\mu : f \in C(I, \mathbb{R}), \ \|f\|_\infty = 1 \right\}.$$ 

Lemma 2.1. ([11]) Let $\nu \in \mathcal{M}_0(I, \mathbb{R})$. Define

$$\hat{\nu}(t) := \begin{cases} -|\nu(0)| & \text{for } t = 0, \\ \nu(\nu, (0, t]) & \text{for } t \in (0, 1], \end{cases}$$

(11) Then $\hat{\nu} \in \mathcal{M}_0(I, \mathbb{R})$ satisfies $\|\nu\|_V = \hat{\nu}(1) - \hat{\nu}(0)$ and for any $f \in C(I, \mathbb{R})$ and $[a, b] \subset I$,

$$\left| \int_{[a,b]} f(s) \ d\nu(s) \right| \leq \int_{[a,b]} |f(s)| \ d\hat{\nu}(s).$$

(12)

Typical examples of measures are as follows.

- Let $\ell : I \to \mathbb{R}$ be $\ell(t) \equiv t$. Then $\ell$ yields the Lebesgue measure of $I$ and the Lebesgue integral. More generally, any $q \in L^1(I, \mathbb{R})$ induces an absolutely continuous measure defined by

$$\mu_q(t) := \int_{[0,t]} q(s) \ ds, \quad t \in I.$$ 

(13)

In this case, one has

$$\|\mu_q\|_V = \|q\|_1 = \|q\|_{L^1(I, \mathbb{R})},$$

(14)

and

$$\int_{I_0} f(t) \ d\mu_q(t) = \int_{I_0} f(t) q(t) \ dt = \int_{I_0} f(t) \ d\mu_q(t).$$
for any $f \in C(I, \mathbb{R})$ and subinterval $I_0 \subset I$.

- For $a = 0$, the unit Dirac measure at $t = 0$ is
  \[
  \delta_0(t) = \begin{cases} 
  -1 & \text{for } t = 0, \\
  0 & \text{for } t \in (0, 1].
  \end{cases}
  \] (15)

- For $a \in (0, 1]$, the unit Dirac measure at $t = a$ is
  \[
  \delta_a(t) = \begin{cases} 
  0 & \text{for } t \in [0, a), \\
  1 & \text{for } t \in [a, 1].
  \end{cases}
  \] (16)

In the space $\mathcal{M}_0(I, \mathbb{R})$ of measures, besides the usual topology induced by the norm $\|\cdot\|_V$, one has the following weak* topology $w^*$.

**Definition 2.2.** Let $\mu_0, \mu_n \in \mathcal{M}_0(I, \mathbb{R})$, $n \in \mathbb{N}$. We say that $\mu_n$ is weakly* convergent to $\mu_0$ if and only if one has
  \[
  \lim_{n \to \infty} \int_I f \, d\mu_n = \int_I f \, d\mu_0, \quad \forall f \in C(I, \mathbb{R}).
  \]

In general, a measure cannot be a limit of smooth measures in the norm $\|\cdot\|_V$. However, in the $w^*$ topology, the following conclusion holds.

**Lemma 2.3.** (6) Given $\mu_0 \in \mathcal{M}_0(I, \mathbb{R})$, there exists a sequence of measures $\{\mu_n\} \subset C^\infty(I, \mathbb{R}) \cap \mathcal{M}_0(I, \mathbb{R})$ such that $\mu_n \to \mu_0$ in $(\mathcal{M}_0(I, \mathbb{R}), w^*)$.

Moreover, if $\mu_0$ is increasing (decreasing) on $I$, then the sequence $\{\mu_n\}$ above can be chosen such that for each $n \in \mathbb{N}$, $\mu_n$ is increasing (decreasing) on $I$ and $\|\mu_n\|_V = \|\mu_0\|_V$.

Considering $q \in L^1(I, \mathbb{R})$ as a density, one has the measure or distribution given by (13). Since $\|\mu_q\|_V = \|q\|_1$,
  \[
  (L^1(I, \mathbb{R}), \|\cdot\|_1) \hookrightarrow (\mathcal{M}_0(I, \mathbb{R}), \|\cdot\|_V) \text{ is an isometric embedding.} \] (17)

Define
  \[
  B_0[r] := \{\mu \in \mathcal{M}_0 : \|\mu\|_V \leq r\}. \] (18)

Via (13), by the Hölder inequality and the isometrical embedding (17), one has the following result on the $L^1$ balls $B_1[r]$ and the $\mathcal{M}_0$ balls $B_0[r]$ [8].

**Lemma 2.4.** (8) Let $r > 0$. The following inclusion is proper
  \[
  B_1[r] \subset B_0[r].
  \]

As for the compactness of these balls in weak* topology, we have the following result.

**Lemma 2.5.** (5) Let $r > 0$. Then $B_0[r] \subset (\mathcal{M}_0(I, \mathbb{R}), w^*)$ is sequentially compact.

3. **Characterization for the lowest eigenvalue of MDE.** Firstly, given a measure $\mu \in \mathcal{M}_0 := \mathcal{M}_0(I, \mathbb{R})$, let us consider the fourth order linear MDE with the measure $\mu$, formulated as:
  \[
  d^3y(t) + \lambda d^2y(t) \, dt + y(t) \, d\mu(t) = 0, \quad t \in [0, 1].
  \] (19)
Definition 3.1. A function y : I → ℝ is called a solution to the equation (19) on
the interval I if
• y ∈ C := C(I, ℝ), and
• there exist (y₀, y₁, y₂, y₃) ∈ ℝ⁴ and functions y⁽¹⁾, y⁽²⁾, y⁽³⁾ : [0, 1] → ℝ such that
  the following are satisfied

\[
y(t) = y₀ + \int_{[0,t]} y₁(s) \, ds, \quad t ∈ [0,1],
\]
\[
y⁽¹⁾(t) = y₁ + \int_{[0,t]} y₂(s) \, ds, \quad t ∈ [0,1],
\]
\[
y⁽²⁾(t) = y₂ + \int_{[0,t]} y₃(s) \, ds, \quad t ∈ [0,1],
\]
\[
y⁽³⁾(t) = \begin{cases} y₃, & t = 0, \\ y₃ - \int_{[0,t]} λy⁽²⁾(s) \, ds - \int_{[0,t]} y(s) \, dµ(s), & t ∈ (0,1]. \end{cases}
\]

The initial condition of the MDE formulated by (19) can be written as

\[
(y(0), y⁽¹⁾(0), y⁽²⁾(0), y⁽³⁾(0)) = (y₀, y₁, y₂, y₃).
\]

Since we have assumed that y ∈ C, the right-hand sides of (20), (21), (22) are
the Lebesgue integral and right-hand side of (23) is the Lebesgue-Stieltjes integral,
respectively.

One can find a proof from [3, 13] based on the Kurzweil-Stieltjes integral to show
the existence and uniqueness of solution for (19)-(24).

Lemma 3.2. For each (y₀, y₁, y₂, y₃) ∈ ℝ⁴, problem (19)-(24) has the unique solution
y(t) defined on [0,1].

For p ∈ [1, ∞], let Lᵖ := Lᵖ([0,1], ℝ) be the Lebesgue space of real-valued functions
with the Lᵖ norm ∥·∥ₚ. For n ∈ N, let Wⁿ⁺ := Wⁿ⁺([0,1], ℝ) and

\[
Wⁿ⁺ := Wⁿ⁺([0,1], ℝ) = \{ y ∈ Wⁿ⁺ : y(0) = y(1) = 0 \}
\]

be the Sobolev spaces with the norm ∥·∥ₓ. For p = 2, Wⁿ⁺ and Wⁿ⁺ are denoted
simply by ℋⁿ and ℋⁿ, respectively, with the norm ∥·∥ₓ. Besides the
Sobolev spaces ℋⁿ and ℋⁿ, let us introduce

\[
ℋⁿ := \{ y ∈ ℋⁿ : y satisfies (2) \} = \{ y ∈ ℋⁿ : y(0) = y(1) = y''(0) = y''(1) = 0 \}.
\]

One has the proper inclusions ℋⁿ ⊂ ℋⁿ ⊂ ℋⁿ.

By the properties of Lebesgue integral and Lebesgue-Stieltjes integral, some
regularity results for solutions y(t) are as follows.

Corollary 1. Let y(t) be the solution of (19). Then y ∈ ℋⁿ and y⁽³⁾ ∈ M :=
M([0,1], ℝ). Hence,

y⁽¹⁾(t) = y⁽¹⁾(t) ∈ C¹ := C¹([0,1], ℝ), \quad y⁽²⁾(t) = y⁽²⁾(t) ∈ AC := AC([0,1], ℝ),

and y⁽³⁾(t) = y⁽³⁾(t) a.e. t ∈ [0,1]. Here' denotes the derivative with respect to t,
C¹([0,1], ℝ) is the space of continuously differentiable functions and AC([0,1], ℝ) is
the space of absolutely continuous functions.

We use y(t, y₀, y₁, y₂, y₃) to denote the unique solution of (19)-(24). Let

\[
ϕ₁(t) := y(t, 1, 0, 0, 0), \quad ϕ₂(t) := y(t, 0, 1, 0, 0),
\]
\[
ϕ₃(t) := y(t, 0, 0, 1, 0), \quad ϕ₄(t) := y(t, 0, 0, 0, 1),
\]
be the fundamental solutions of (19). By the linearity of (19) and the uniqueness of solution, one has that, for $t \in [0, 1]$,

\[
\begin{pmatrix}
y(t, y_0, y_1, y_2, y_3) \\
y^{(1)}(t, y_0, y_1, y_2, y_3) \\
y^{(2)}(t, y_0, y_1, y_2, y_3) \\
y^{(3)}(t, y_0, y_1, y_2, y_3)
\end{pmatrix}
= \begin{pmatrix}
\varphi_1(t) & \varphi_2(t) & \varphi_3(t) & \varphi_4(t) \\
\varphi_1^{(1)}(t) & \varphi_2^{(1)}(t) & \varphi_3^{(1)}(t) & \varphi_4^{(1)}(t) \\
\varphi_1^{(2)}(t) & \varphi_2^{(2)}(t) & \varphi_3^{(2)}(t) & \varphi_4^{(2)}(t) \\
\varphi_1^{(3)}(t) & \varphi_2^{(3)}(t) & \varphi_3^{(3)}(t) & \varphi_4^{(3)}(t)
\end{pmatrix}
\begin{pmatrix}
y_0 \\
y_1 \\
y_2 \\
y_3
\end{pmatrix}
= : N_\mu(t) \begin{pmatrix}
y_0 \\
y_1 \\
y_2 \\
y_3
\end{pmatrix}.
\]

For the second order linear MDE, the continuity of solutions in measures has been obtained in [8]. In a similar way, for the fourth order linear MDE, we can also easily prove the following conclusion.

**Theorem 3.3.** Let $y(t, \mu)$ be the solution of (19)-(24). Then the following solution mappings are continuous

\[
\begin{align*}
(M_0, w^*) &\to (C, \| \cdot \|_\infty), \quad \mu \to y(\cdot, \mu), \quad (27) \\
(M_0, w^*) &\to (C, \| \cdot \|_\infty), \quad \mu \to y^{(1)}(\cdot, \mu), \quad (28) \\
(M_0, w^*) &\to (C, \| \cdot \|_\infty), \quad \mu \to y^{(2)}(\cdot, \mu), \quad (29) \\
(M_0, w^*) &\to (M, w^*), \quad \mu \to y^{(3)}(\cdot, \mu). \quad (30)
\end{align*}
\]

Further, by Corollary 1, we have the following corollary.

**Corollary 2.** The following solution mapping

\[
(M_0, w^*) \to (C^2, \| \cdot \|_{H^2}), \quad \mu \to y(\cdot, \mu), \quad (31)
\]

is continuous, where $C^2 := C^2([0, 1], \mathbb{R})$.

Secondly, we consider eigenvalue problem of the fourth order equation (5) with the boundary condition (2). It is a standard result that any (possible) eigenvalue $\lambda \in \mathbb{R}$ of problem (5)-(2) with eigenfunction $u \in H^4_0$ must satisfy

\[
R(u) = \lambda, \quad (32)
\]

where $R(\cdot)$ is as in (6). We will show that problem (5)-(2) does admit the lowest eigenvalue. To this end, we need the following basic estimate.

**Lemma 3.4.** Assuming $u \in H^2_0 \setminus \{0\}$, we have

\[
R(u) \geq -\frac{\|\mu\|_V^2}{4}. \quad (33)
\]

**Proof.** Assuming $u \in H^2_0 \setminus \{0\}$, we have

\[
\|u\|^2_\infty \leq \left( \int_{[0,1]} |u'| \, dt \right)^2 \leq \int_{[0,1]} u'u' \, dt = u'u'_1 - \int_{[0,1]} uu'' \, dt \leq \int_{[0,1]} |uu''| \, dt \leq \|u\|_2 \|u''\|_2, \quad (34)
\]
and
\[ \int_{[0,1]} u^2 \, d\mu(t) \geq -\|\mu\|_V \|u\|_2^2 = -\frac{\|\mu\|_V \|u\|_2}{\sqrt{2}} \cdot \frac{\sqrt{2} \|u\|_\infty^2}{\|u\|_2} \]
\[ \geq -\frac{1}{2} \left( \frac{\|\mu\|_V \|u\|_2^2}{2} + \frac{2 \|u\|_\infty^4}{\|u\|_2^2} \right). \tag{35} \]

By (34) and (35), we have that
\[ \int_{[0,1]} (u'')^2 \, dt + \int_{[0,1]} u^2 \, d\mu(t) \geq \|u\|_\infty^4 / \|u\|_2^2 - \frac{1}{2} \left( \frac{\|\mu\|_V \|u\|_2^2}{2} + \frac{2 \|u\|_\infty^4}{\|u\|_2^2} \right) \]
\[ = -\frac{\|\mu\|_V^2 \|u\|_2^2}{4} \geq -\frac{\|\mu\|_V \|u'\|_2^2}{4}, \]
which implies
\[ R(u) \geq -\frac{\|\mu\|_V^2}{4} \quad \forall 0 \neq u \in \mathcal{H}_0^2. \]

\[ \square \]

Based on Lemma 3.4 and the variational method, we can now use \( R(u) \) to prove Theorem 1.1 for the minimization characterizations of the lowest eigenvalue as follows.

The proof of Theorem 1.1. By (33), one has
\[ \lambda_1 := \inf_{u \in \mathcal{H}_0^2 \setminus \{0\}} R(u) > -\infty. \tag{36} \]

Take a sequence \( \{u_n\} \subset \mathcal{H}_0^2 \) such that
\[ \|u'_n\|_\infty = 1 \text{ and } \lim_{n \to +\infty} R(u_n) = \lambda_1. \tag{37} \]

Then, by (37),
\[ \int_{[0,1]} (u''_n)^2 \, dt = R(u_n) \int_{[0,1]} u_n^2 \, dt - \int_{[0,1]} u_n^2 \, d\mu(t) \leq |R(u_n)| + \|\mu\|_V \]
is bounded. Combining with the assumption that \( \|u'_n\|_\infty = 1 \), it is easy to see that \( \{u_n\} \subset \mathcal{H}_0^2 \) is bounded. Since \( \mathcal{H}_0^2 \) is a Hilbert space and thus is compactly embedded into \( C^1 \), there exists a non-zero \( u_0 \in \mathcal{H}_0^2 \) such that
\[ u_n \to u_0 \text{ in } (\mathcal{H}_0^2, \mathcal{W}) \text{ and } u_n \to u_0 \text{ in } (C^1, \|\cdot\|_{C^1}), \]
going to a subsequence if necessary. Thus
\[ \int_{[0,1]} (u''_0)^2 \, dt = \lim_{n \to +\infty} \int_{[0,1]} u''_n u''_0 \, dt \]
\[ \leq \liminf_{n \to +\infty} \left( \int_{[0,1]} (u''_n)^2 \, dt \right)^{1/2} \left( \int_{[0,1]} (u''_n)^2 \, dt \right)^{1/2}. \]

This implies that
\[ \int_{[0,1]} (u''_0)^2 \, dt \leq \liminf_{n \to +\infty} \int_{[0,1]} (u''_n)^2 \, dt. \]
Hence

\[ R(u_0) \leq \liminf_{n \to +\infty} \frac{\int_{[0,1]} (u_0^\prime)^2 \, dt + \int_{[0,1]} u_0^2 \, d\mu(t)}{\int_{[0,1]} u_0^2 \, dt} \]

\[ = \liminf_{n \to +\infty} \frac{\int_{[0,1]} (u_n^\prime)^2 \, dt + \int_{[0,1]} u_n^2 \, d\mu(t)}{\int_{[0,1]} u_n^2 \, dt} \]

\[ = \liminf_{n \to +\infty} R(u_n) = \lambda_1. \]

Combining with (36), one has

\[ R(u_0) = \lambda_1 = \min_{u \in \mathcal{H}_2^0 \setminus \{0\}} R(u). \tag{38} \]

Take any \( \phi \in C_c^\infty := \{ \phi \in C^\infty([0,1]) : \text{supp} \phi \subset (0,1) \} \).

Then \( u_0 + s\phi \in \mathcal{H}_2^0 \setminus \{0\} \) for all \( s \in \mathbb{R} \) with \( |s| \) small enough.

As a function of \( s \), it follows from (38) that \( R(u_0 + s\phi) \) takes a minimum at \( s = 0 \).

Thus

\[ 0 = \left. \frac{dR(u_0 + s\phi)}{ds} \right|_{s=0} \]

\[ = \left. \frac{2}{\int_{[0,1]} u_0^2 \, dt} \left( \int_{[0,1]} u_0^2 \, dt \left( \int_{[0,1]} u_0^\prime \phi'' \, dt + \int_{[0,1]} u_0 \phi \, d\mu(t) \right) \right) \right|_{s=0} \]

\[ - \left. \left( \int_{[0,1]} (u_0^\prime)^2 \, dt + \int_{[0,1]} u_0^2 \, d\mu(t) \right) \int_{[0,1]} u_0^\prime \phi' \, dt \right|_{s=0} \]

\[ = \frac{2}{\int_{[0,1]} u_0^2 \, dt} \left( \int_{[0,1]} u_0^\prime \phi'' \, dt + \int_{[0,1]} u_0 \phi \, d\mu(t) - \lambda_1 \int_{[0,1]} u_0^\prime \phi' \, dt \right). \tag{39} \]

Here (38) is used and the derivative is found using definition (6) for \( R(u) \).

Since \( \phi \in C_c^\infty \), one has

\[ \phi'(t) = \int_{[0,t]} \phi''(s) \, ds \]

and

\[ \phi(t) = \int_{[0,t]} \left( \int_{[0,s]} \phi''(\tau) \, d\tau \right) \, ds = \int_{[0,t]} (t-s)\phi''(s) \, ds. \]

Then

\[ \int_{[0,1]} u_0^\prime(t)\phi'(t) \, dt = \int_{[0,1]} \left( \int_{[0,t]} u_0^\prime(t)\phi''(s) \, ds \right) \, dt \]

\[ = \int_{[0,1]} \left( \int_{[s,1]} u_0^\prime(t) \, dt \right) \phi''(s) \, ds \]

\[ = \int_{[0,1]} \left( \int_{[t,1]} u_0^\prime(s) \, ds \right) \phi''(t) \, dt. \]
and
\[
\int_{[0,1]} u_0(t) \phi(t) \, d\mu(t) = \int_{[0,1]} \left( \int_{[0,t]} (t-s)u_0(t) \phi''(s) \, ds \right) \, d\mu(t)
\]
\[
= \int_{[0,1]} \left( \int_{(s,1]} (t-s)u_0(t) \, d\mu(t) \right) \phi''(s) \, ds
\]
\[
= \int_{[0,1]} \left( \int_{(t,1]} (s-t)u_0(s) \, d\mu(s) \right) \phi''(t) \, dt.
\]
Substituting them into (39), we obtain
\[
\int_{[0,1]} \left( u''_0(t) + \int_{(t,1]} (s-t)u_0(s) \, d\mu(s) - \lambda_1 \int_{(t,1]} u'_0(s) \, ds \right) \phi''(t) \, dt = 0
\]
for all \( \phi \in C^\infty_c \). Hence \( u_0(t) \) satisfies
\[
u''_0(t) + \int_{(t,1]} (s-t)u_0(s) \, d\mu(s) - \lambda_1 \int_{(t,1]} u'_0(s) \, ds = ct + \hat{c} \quad \text{a.e. } t \in [0,1],
\]
where \( c, \hat{c} \) are some constants. Note that
\[
\int_{[0,t]} \left( \int_{[0,\tau]} u_0(s) \, d\mu(s) \right) \, d\tau = \int_{[0,t]} (t-s)u_0(s) \, d\mu(s)
\]
\[
= \int_{(t,1]} (s-t)u_0(s) \, d\mu(s) + \int_{[0,1]} (t-s)u_0(s) \, d\mu(s)
\]
\[
= \int_{(t,1]} (s-t)u_0(s) \, d\mu(s) + c_1 t + \hat{c}_1,
\]
where \( c_1 \) and \( \hat{c}_1 \) are constants. Hence equation (40) can be rewritten as
\[
u''_0(t) + \int_{[0,t]} \left( \int_{[0,\tau]} u_0(s) \, d\mu(s) \right) \, d\tau - \lambda_1 \int_{(t,1]} u'_0(s) \, ds = c_2 t + \hat{c}_2 \quad \text{a.e. } t \in [0,1],
\]
where \( c_2 = c - c_1 \) and \( \hat{c}_2 = \hat{c} - \hat{c}_1 \). By the properties of Lebesgue integral and Lebesgue-Stieltjes integral, one knows from (41) that \( u''_0(t) \) is absolutely continuous and satisfies
\[
u'''_0(t) + \int_{[0,t]} u_0(s) \, d\mu(s) + \lambda_1 u'_0(t) = \hat{c}_2 \quad \text{a.e. } t \in [0,1].
\]
Equation (42) implies that \( u_0 \in H^3 \) and therefore \( u_0 \in H^3_0 \). With the explanation to solutions of MDE, (42) further shows that \( u_0(t) \) is a non-zero solution of the MDE (5) with the choice \( \lambda = \lambda_1 \). Moreover, it is standard to verify that \( u_0(t) \) also satisfies the boundary condition \( u''_0(0) = u''_0(1) = 0 \) (see [2, p. 208]). Thus \( u_0 \in H^3_0 \) and \( \lambda_1 \) is necessarily an eigenvalue of problem (5)-(2) with the eigenfunction \( u_0 \). Because of (38) and the fact that \( u_0 \in H^3_0 \), \( \lambda_1 = \lambda_1(\mu) \) which is characterized as in (7).

Finally, due to result (32) for general eigenvalues, we know that \( \lambda_1(\mu) \) must be the lowest eigenvalue of problem (5)-(2).

Now, let us introduce the following ordering for measures. We say that measures \( \mu_2 \geq \mu_1 \) if
\[
\int_{[0,1]} f(t) \, d\mu_2(t) \geq \int_{[0,1]} f(t) \, d\mu_1(t) \quad \text{for all } f \in C_+: = \{ f \in C : f(t) \geq 0, \ t \in [0,1] \}.
\]
Then, as a consequence of (7) in Theorem 1.1, we can obtain the following result.

**Corollary 3.** Let $\mu_1, \mu_2 \in \mathcal{M}_0$. Then

$$\mu_2 \geq \mu_1 \Rightarrow \lambda_1(\mu_2) \geq \lambda_1(\mu_1).$$

Additionally, the continuity of the lowest eigenvalue in measures with the weak* topology can be proved by the same arguments as those in [4].

**Theorem 3.5.** As a nonlinear functional, $\lambda_1(\mu)$ is continuous in $\mu \in (\mathcal{M}_0, w^*)$.

4. Minimization of the lowest eigenvalue for a vibrating beam. In this section, for minimizing the lowest eigenvalue for the vibrating beam, we will firstly build the relationship between minimization problem of the lowest eigenvalue for the ODE (1) and the one for the MDE (5) as follows.

**Theorem 4.1.** Given $r > 0$, consider the following minimization problem

$$\tilde{L}(r) := \inf\{\lambda_1(\mu) : \mu \in B_0[r]\} = \min\{\lambda_1(\mu) : \mu \in B_0[r]\}.$$  \hspace{1cm} (43)

One has

$$L(r) = \tilde{L}(r).$$ \hspace{1cm} (44)

**Proof.** Given $F \in B_1[r]$, the measure $\mu_F \in \mathcal{M}_0$ is defined as (13). By (17), we have that $\mu_F \in B_0[r]$ is absolutely continuous with respect to the Lebesgue measure. Thus, for any $F \in B_1[r]$,

$$\tilde{L}(r) \leq \lambda_1(\mu_F) = \lambda_1(F),$$

which implies that

$$\tilde{L}(r) \leq L(r).$$ \hspace{1cm} (45)

On the other hand, there is $\bar{\mu} \in B_0[r]$ such that $\lambda_1(\bar{\mu}) = \tilde{L}(r)$ because $B_0[r]$ is sequentially compact in $(\mathcal{M}_0, w^*)$ and $\lambda_1(\mu)$ is continuous in $\mu \in (\mathcal{M}_0, w^*)$ by Theorem 3.5. According to the property of measures in Lemma 2.1 and the monotonicity of $\lambda_1(\mu)$ in Corollary 3, without loss of generality, we can assume that $\bar{\mu} = -\hat{\mu}$ is decreasing. Then, by Lemma 2.3, there exists a sequence of measures $\{\bar{\mu}_n\} \subset C^\infty \cap \mathcal{M}_0$ such that

$$\bar{\mu}_n \to \bar{\mu} \text{ in } (\mathcal{M}_0, w^*), \quad \frac{d\bar{\mu}_n(t)}{dt} = \bar{F}_n(t), \quad \|\bar{\mu}_n\|_V = \|\bar{F}_n\|_1 = \|\bar{\mu}\|_V \leq r.$$  \hspace{1cm} (46)

Therefore, by Theorem 3.5, we have

$$\tilde{L}(r) = \lambda_1(\bar{\mu}) = \lim_{n \to \infty} \lambda_1(\bar{\mu}_n) = \lim_{n \to \infty} \lambda_1(\bar{F}_n) \geq \lim_{n \to \infty} L(r) = L(r).$$ \hspace{1cm} (47)

From (45) and (46), we have that $L(r) = \tilde{L}(r)$. \hfill $\square$

Now, based on the relationship (44) between minimization problem of the ODE and the MDE described by Theorem 4.1, we can then use $\tilde{L}(r)$ to find the explicit optimal lower bound of the lowest eigenvalue for the vibrating beam (1) and thus prove Theorem 1.2 as follows.

**The proof of Theorem 1.2.** We will solve this extremal value problem in three steps.

**Step 1.** We will prove that

$$\tilde{L}(r) = \inf_{a \in (0, 1)} \lambda_1(-r\delta_a).$$
Given $\mu \in B_0[r]$, we take an eigenfunction $y(t)$ associated with $\lambda_1(\mu)$ which satisfies the normalization condition $\|y\|_2 = 1$. Then, there exists $a \in (0, 1)$ such that

$$\|y\|_{\infty} = \max_{t \in [0, 1]} |y(t)| = |y(a)|.$$ 

Moreover, we have that

$$\lambda_1(\mu) = \int_{[0, 1]} (y''')^2 \, dt + \int_{[0, 1]} y''^2 \, d\mu(t) \geq \int_{[0, 1]} (y''')^2 \, dt - \|\mu\| \|y\|_{\infty}^2$$

$$\geq \int_{[0, 1]} (y''')^2 \, dt - ry^2(a)$$

$$= \int_{[0, 1]} (y''')^2 \, dt + \int_{[0, 1]} y''^2 \, d(-r\delta_a(t)) \geq \lambda_1(-r\delta_a). \tag{47}$$

Here the last inequality in (47) follows from characterization (7) for $\lambda_1(-r\delta_a)$ since $\|y''\|_2 = 1$. Hence

$$\tilde{L}(r) = \inf_{a \in (0, 1)} \lambda_1(-r\delta_a).$$

**Step 2.** For fixed $a \in (0, 1)$ and $r > 0$, we will explicitly find the lowest eigenvalue for Dirac measure $-r\delta_a$ as follows:

$$\lambda_1(-r\delta_a) = \min\{\lambda \in \mathbb{R} : G(\lambda, a) - r = 0\}, \tag{48}$$

where $G : (-\infty, \pi^2] \times (0, 1) \to [0, +\infty)$ is defined as in (56).

To this end, we need to solve the following equation

$$dy^{(3)}(t) + \lambda y^{(2)}(t) \, dt - ry(t) \, d\delta_a(t) = 0, \quad t \in [0, 1]. \tag{49}$$

From the explanation to solutions of MDE, one knows that solutions $y(t)$ of (49) satisfies the classical ODE

$$y'''(t) + \lambda y''(t) = 0 \tag{50}$$

for $t$ on the intervals $[0, a)$ and $(a, 1]$. At $t = a$, one has the following relations:

$$\left\{ \begin{array}{ll}
y(a+) = y(a-), & y'(a+) = y'(a-), \\
y''(a+) = y''(a-), & y'''(a+) = y'''(a-) + ry(a-). \end{array} \right. \tag{51}$$

From the first two conditions of (2), let us consider the initial value

$$(y(0), y'(0), y''(0), y'''(0)) = (0, c_0, \dot{c}, \ddot{c}) \neq 0.$$ 

From ODE (50) on $[0, a)$, we obtain

$$y(t) = c_0 \varphi_2(t) + \dot{c} \varphi_4(t) = ct + \dot{c} \left( \frac{1}{\omega^2} t + \frac{-1}{\omega^3} \sin \omega t \right) = c_1 t + c_2 \sin \omega t \tag{52}$$

for $t \in [0, a)$ and $(c_1, c_2) \neq 0$. Here,

$$\omega := \left\{ \begin{array}{ll}
\sqrt[3]{\lambda} \in \mathbb{R} & \text{for } \lambda \geq 0, \\
\sqrt[3]{|\lambda|} \in \mathbb{C} & \text{for } \lambda < 0. \end{array} \right. \tag{53}$$

By (51), we have

$$y(a+) = z_0 := c_1 a + c_2 \sin \omega a, \quad y'(a+) = z_1 := c_1 + c_2 \omega \cos \omega a,$$

$$y''(a+) = z_2 := -c_2 \omega^2 \sin \omega a, \quad y'''(a+) = z_3 := -c_2 \omega^3 \cos \omega a + r(c_1 a + c_2 \sin \omega a).$$
By using this as the initial value at \( t = a \), we obtain from ODE (50) that
\[
y(t) = y(a^+)\varphi_1(t-a) + y'(a^+)\varphi_2(t-a) + y''(a^+)\varphi_3(t-a) + y'''(a^+)\varphi_4(t-a)
\]
\[
= z_0\varphi_1(t-a) + z_1\varphi_2(t-a) + z_2\varphi_3(t-a) + z_3\varphi_4(t-a)
\]
\[
= (c_1a + c_2\sin \omega a) \cdot 1
\]
\[
+ (c_1 + c_2 \cos \omega a)(t-a) + (-c_2 \omega^2 \sin \omega a)\left(\frac{1}{\omega^2} + \frac{1}{\omega^2} \cos \omega(t-a)\right)
\]
\[
+ (-c_2 \omega^3 \cos \omega a + r(c_1a + c_2 \sin \omega a))\left(\frac{1}{\omega^2}(t-a) + \frac{1}{\omega^3} \sin \omega(t-a)\right)
\]
\[
= c_1 + c_2(\sin \omega a \cos \omega(t-a) + \cos \omega a \sin \omega(t-a))
\]
\[
+ (r(c_1a + c_2 \sin \omega a))\left(\frac{1}{\omega^2}(t-a) + \frac{1}{\omega^3} \sin \omega(t-a)\right)
\]
\[
= c_1 + c_2 \sin \omega t + (r(c_1a + c_2 \sin \omega a))\left(\frac{1}{\omega^2}(t-a) + \frac{1}{\omega^3} \sin \omega(t-a)\right)
\]
and
\[
y''(t) = -c_2 \omega^2 \sin \omega t + r(c_1a + c_2 \sin \omega a)\frac{\sin \omega(t-a)}{\omega}
\]
for \( t \in (a, 1) \).

Now the last two conditions \( y(1) = y''(1) = 0 \) of (2) imply the following linear system for \((c_1, c_2)\):
\[
\begin{cases}
  c_1 t + c_2 \sin \omega a + r(c_1a + c_2 \sin \omega a)(\frac{1}{\omega^2}(1-a) + \frac{1}{\omega^3} \sin \omega(1-a)) = 0, \\
  -c_2 \omega^2 \sin \omega a + r(c_1a + c_2 \sin \omega a)\frac{\sin \omega(1-a)}{\omega} = 0.
\end{cases}
\]  
(54)

In order to make system (54) have non-zero solutions \((c_1, c_2)\), the corresponding determinant of (54) is necessarily zero. This yields the following equation
\[
G(\lambda, a) = r,
\]  
(55)
where \( G : (-\infty, \pi^2] \times (0, 1) \to [0, +\infty) \) is defined as
\[
G(\lambda, a) := \begin{cases}
  \frac{2\omega^3 \sin \omega}{3(1-a^2)\omega^2 - 2\omega a(1-a) \sin \omega} & \text{for } \lambda \neq 0 \\
  \frac{1}{(1-a^2)\omega} & \text{for } \lambda = 0.
\end{cases}
\]  
(56)

Then, by the existence of the lowest eigenvalue, we conclude that
\[
\lambda_1(-r\delta_a) = \min\{\lambda \in \mathbb{R} : G(\lambda, a) - r = 0\}.
\]

**Step 3.** We will prove that
\[
\lambda_1(-r\delta_{1/2}) = \inf_{a \in (0, 1)} \lambda_1(-r\delta_a) = Q^{-1}(r).
\]  
(57)

Here \( Q(\cdot) \) is as in (8).

It is easy to check that \( G(\lambda, a) \) is a well-defined real function of \((\lambda, a) \in (-\infty, \pi^2] \times (0, 1)\) with \( G(\pi^2, a) = 0 \) and \( G(\lambda, a) = G(\lambda, 1-a) \). By the monotonicity of the lowest eigenvalue in Corollary 3, we have the following property:

(P1) when \( a \in (0, 1) \) is fixed, \( G(\lambda, a) \) is decreasing in \( \lambda \in (-\infty, \pi^2] \).

Notice that
\[
\frac{\partial}{\partial a} G(\lambda, a) = \frac{\sin \omega(1-2a) - (1-2a) \sin \omega}{\omega^2 \sin \omega}, \quad \forall \lambda \in (-\infty, \pi^2].
\]

Hence,
\[
\frac{\partial}{\partial a} G(\lambda, 0) = \frac{\partial}{\partial a} G(\lambda, 1/2) = \frac{\partial}{\partial a} G(\lambda, 1) = 0.
\]
When \( a \in (0, 1/2) \) and \( \lambda \in (-\infty, \pi^2] \), we claim that
\[
\frac{\partial}{\partial a} \frac{1}{G(\lambda, a)} > 0.
\]
In fact, if \( \lambda > 0 \), then \( \sin \omega(1 - 2a) - (1 - 2a) \sin \omega > 0 \), which implies that
\[
\frac{\partial}{\partial a} \frac{1}{G(\lambda, a)} > 0.
\]
If \( \lambda < 0 \), then \( \sinh \omega(1 - 2a) - (1 - 2a) \sinh \omega < 0 \), which also implies that
\[
\frac{\partial}{\partial a} \frac{1}{G(\lambda, a)} > 0.
\]
So
\[
\frac{\partial}{\partial a} \frac{1}{G(\lambda, a)} \geq 0 \quad \forall a \in (0, 1/2],
\]
which means that \( G(\lambda, a) \) is increasing. Hence, we obtain another property as follows:
(P2) when \( \lambda \in (-\infty, \pi^2] \) is fixed, \( G(\lambda, a) \) is decreasing in \( a \in (0, 1/2) \).

By (48) and the properties (P1) and (P2), we have that
\[
\lambda_1(-r\delta_{1/2}) = \inf_{a \in (0,1)} \lambda_1(-r\delta_a).
\]

Moreover, one has that for fixed \( a = 1/2 \), \( r > 0 \),
\[
Q(\lambda_1(-r\delta_{1/2})) = r,
\]
where \( Q(\cdot) \) as in (8).

Thus, by Theorem 4.1, the conclusion finally holds.

**Remark 1.** To compute \( L(r) \), it suffices to solve the following optimization problem
\[
\min f(\lambda, a) = \lambda
\]
subject to the constraints
\[
G(\lambda, a) - r = 0, \quad 0 < a \leq 1/2,
\]
where \( G(\lambda, a) \) as in (56). In Figure 1, we have plotted \( L(r) \) as function of \( r \).

---

**Figure 1.** Function \( L(r) \) of \( r \).
Acknowledgments. The authors would like to thank Xin Jiang at Beihang University for helpful discussions.

REFERENCES

[1] M. Carter and B. van Brunt, The Lebesgue-Stieltjes Integral: A Practical Introduction, Springer-Verlag, New York, 2000.
[2] R. Courant and D. Hilbert, Methods of Mathematical Physics, Wiley, New York, 1953.
[3] Z. Halas and M. Tvrďý, Continuous dependence of solutions of generalized linear differential equations on a parameter, Funct. Differ. Equ., 16 (2009), 299–313.
[4] X. Jiang, K. Liu, G. Meng and Z. She, Continuity of the eigenvalues for a vibrating beam, Appl. Math. Lett., 67 (2017), 60–66.
[5] R. E. Megginson, An Introduction to Banach Space Theory, Graduate Texts in Mathematics, 183, Springer-Verlag, New York, 1998.
[6] G. Meng, Extremal problems for eigenvalues of measure differential equations, Proc. Amer. Math. Soc., 143 (2015), 1991–2002.
[7] G. Meng, P. Yan and M. Zhang, Minimization of eigenvalues of one-dimensional p-Laplacian with integrable potentials, J. Optim. Theory Appl., 156 (2013), 294–319.
[8] G. Meng and M. Zhang, Dependence of solutions and eigenvalues of measure differential equations on measures, J. Differential Equations, 254 (2013), 2196–2232.
[9] A. B. Mingarelli,Volterra-Stieltjes Integral Equations and Generalized Ordinary Differential Expressions, Lecture Notes Math., Vol. 989, Springer-Verlag, New York, 1983.
[10] P. Savoye, Equimeasurable rearrangements of functions and fourth order boundary value problems, Rocky Mountain J. Math., 26 (1996), 281–293.
[11] Š. Schwabik, Generalized Ordinary Differential Equations, World Scientific, Singapore, 1992.
[12] M. Tvrďý, Linear distributional differential equations of the second order, Math. Bohem., 119 (1994), 415–436.
[13] M. Tvrďý, Differential and integral equations in the space of regulated functions, Mem. Differential Equations Math. Phys., 25 (2002), 1–104.
[14] Q. Wei, G. Meng and M. Zhang, Extremal values of eigenvalues of Sturm-Liouville operators with potentials in $L^1$ balls, J. Differential Equations, 247 (2009), 364–400.
[15] P. Yan and M. Zhang, Continuity in weak topology and extremal problems of eigenvalues of the p-Laplacian, Trans. Amer. Math. Soc., 363 (2011), 2003–2028.
[16] M. Zhang, Extremal values of smallest eigenvalues of Hill’s operators with potentials in $L^1$ balls, J. Differential Equations, 246 (2009), 4188–4220.
[17] M. Zhang, Minimization of the zeroth Neumann eigenvalues with integrable potentials, Ann. Inst. H. Poincaré Anal. Non Linéaire, 29 (2012), 501–523.

Received for publication May 2017.

E-mail address: lquanyiyi@163.com
E-mail address: krliu@buaa.edu.cn
E-mail address: menggang@ucas.ac.cn
E-mail address: zhikun.she@buaa.edu.cn