Nonremovable Sets for Hölder Continuous Quasiregular Mappings in the Plane

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1. Introduction

Let \( \alpha \in (0,1) \). A function \( f : \mathbb{C} \to \mathbb{C} \) is said to be locally \( \alpha \)-Hölder continuous, that is, \( f \in \text{Lip}_\alpha(\mathbb{C}) \), if
\[
|f(z) - f(w)| \leq C|z - w|^{\alpha}
\]
whenever \( z, w \in \mathbb{C} \) and \( |z - w| < 1 \). A set \( E \subset \mathbb{C} \) is said to be removable for \( \alpha \)-Hölder continuous analytic functions if every function \( f \in \text{Lip}_\alpha(\mathbb{C}) \), holomorphic on \( \mathbb{C} \setminus E \), is actually an entire function. It turns out that there is a characterization of these sets \( E \) in terms of Hausdorff measures. For \( \alpha \in (0,1) \), Dolženko [7] proved that a set \( E \) is removable for \( \alpha \)-Hölder continuous analytic functions if and only if \( \mathcal{H}^{1+\alpha}(E) = 0 \). When \( \alpha = 1 \), we deal with the class of Lipschitz continuous analytic functions. Although the same characterization holds, a more involved argument, due to Uy [12], is needed to show that sets of positive area are not removable.

The same question may be asked in the more general setting of \( K \)-quasiregular mappings. Given a domain \( \Omega \subset \mathbb{C} \) and \( K \geq 1 \), one says that a mapping \( f : \Omega \to \mathbb{C} \) is \( K \)-quasiregular in \( \Omega \) if \( f \) is a \( W_{loc}^{1,2}(\Omega) \) solution of the Beltrami equation
\[
\bar{\partial}f(z) = \mu(z)\partial f(z)
\]
for almost every \( z \in \Omega \); here \( \mu \), the Beltrami coefficient, is a measurable function such that \( |\mu(z)| \leq \frac{K-1}{K+1} \) at almost every \( z \in \Omega \). If \( f \) is a homeomorphism, then \( f \) is said to be \( K \)-quasiconformal. When \( \mu = 0 \), we recover the classes of analytic functions and conformal mappings on \( \Omega \), respectively.

It was shown in [6] that if \( E \) is a compact set satisfying \( \mathcal{H}^d(E) = 0 \) for \( d = \frac{1+\alpha K}{1+\alpha} \), then \( E \) is removable for \( \alpha \)-Hölder continuous \( K \)-quasiregular mappings. This means that any function \( f \in \text{Lip}_\alpha(\mathbb{C}) \), \( K \)-quasiregular in \( \mathbb{C} \setminus E \), is actually \( K \)-quasiregular on the whole plane. To look for results in the converse direction, one observes that any compact set \( E \) with \( \mathcal{H}^{1+\alpha}(E) > 0 \) is nonremovable for holomorphic functions and hence also for \( K \)-quasiregular mappings in \( \text{Lip}_\alpha \). We are thus interested in dimensions between \( d \) and \( 1 + \alpha \). In this paper we show that the index \( d \) is sharp in the following sense: Given \( \alpha \in (0,1) \) and \( K \geq 1 \), for any \( t > d \) there exist (i) a compact set \( E \) of dimension \( t \) and (ii) a function \( f \in \text{Lip}_\alpha(\mathbb{C}) \)

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that is \( K \)-quasiregular in \( \mathbb{C} \setminus E \) yet has no \( K \)-quasiregular extension to \( \mathbb{C} \). In other words, we will construct nonremovable sets of any dimension exceeding \( d \).

We first have a look at the case \( K = 1 \). Given a compact set \( E \) with \( \mathcal{H}^{1+\alpha}(E) > 0 \), by Frostman’s lemma (see e.g. [10, p. 112]) there exists a positive Radon measure \( \nu \) supported on \( E \) such that \( \nu(B(z,r)) \leq C r^{1+\alpha} \) for any \( z \in E \). Thus, the function \( h = \frac{1}{\pi} \ast \nu \) is \( \alpha \)-Hölder continuous everywhere, is holomorphic outside the support of \( \nu \), and has no entire extension.

A similar situation is found in the limiting case \( \alpha = 0 \), where \( \text{Lip}_\alpha(\mathbb{C}) \) should be replaced by \( \text{BMO}(\mathbb{C}) \). In this case, a set \( E \) is called \textit{removable} for \( \text{BMO} K \)-quasiregular mappings if every \( \text{BMO}(\mathbb{C}) \) function \( f \) that is \( K \)-quasiregular on \( \mathbb{C} \setminus E \) is actually \( K \)-quasiregular on the whole plane. When \( K = 1 \), Král [9] characterized these sets as those with zero length. When \( K > 1 \), it is known [3; 5] that sets with \( \mathcal{H}^{2/(K+1)}(E) = 0 \) are removable for \( \text{BMO} K \)-quasiregular mappings. In fact, the appearance of this index \( \frac{2}{K+1} \) is not strange. Astala [2] has shown that, for any \( K \)-quasiconformal mapping \( \phi \) and any compact set \( E \),

\[
\frac{1}{K} \left( \frac{1}{\dim(E)} - \frac{1}{2} \right) \leq \frac{1}{\dim(\phi(E))} - \frac{1}{2} \leq K \left( \frac{1}{\dim(E)} - \frac{1}{2} \right). \tag{2}
\]

Furthermore, both equalities are always attainable. In particular, sets of dimension \( \frac{2}{K+1} \) are \( K \)-quasiconformally mapped to sets of dimension at most 1, which is the critical point for the analytic \( \text{BMO} \) situation. Hence, from equality at (2), for any \( t > \frac{2}{K+1} \) there exist a compact set \( E \) of dimension \( t \) and a \( K \)-quasiconformal mapping \( \phi \) that maps \( E \) to a compact set \( \phi(E) \) with dimension

\[
t' = \frac{2Kt}{2 + (K - 1)t} > 1.
\]

In particular, \( \mathcal{H}^{1}(\phi(E)) > 0 \). As before, we have a positive Radon measure \( \nu \) supported on \( \phi(E) \), with linear growth, whose Cauchy transform \( h = \frac{1}{\pi} \ast \nu \) is holomorphic on \( \mathbb{C} \setminus E \) and has a \( \text{BMO}(\mathbb{C}) \) extension that is not entire. Now, since \( \text{BMO}(\mathbb{C}) \) is invariant under quasiconformal changes of variables [11], the composition \( g = h \circ \phi \) is a \( \text{BMO}(\mathbb{C}) \) \( K \)-quasiregular mapping on \( \mathbb{C} \setminus E \) that has no \( K \)-quasiregular extension to \( \mathbb{C} \). In other words, the set \( E \) is not removable for \( \text{BMO} K \)-quasiregular mappings. This argument shows that the index \( \frac{2}{K+1} \) is somewhat critical for the \( \text{BMO} K \)-quasiregular problem.

Our plan is to repeat the foregoing argument after first replacing \( \text{BMO}(\mathbb{C}) \) with \( \text{Lip}_\alpha(\mathbb{C}) \). That is, given any dimension \( t > \frac{2+\alpha K}{1+\alpha} \), we will construct a compact set \( E \) of dimension \( t \) and a \( \text{Lip}_\alpha(\mathbb{C}) \) function that is \( K \)-quasiregular on \( \mathbb{C} \setminus E \) but not on \( \mathbb{C} \). We will start with a compact set \( E \) of dimension \( t \) and a \( K \)-quasiconformal mapping \( \phi \) such that \( \dim(\phi(E)) = t' = \frac{2Kt}{2 + (K - 1)t} \). Then, we will show that there are \( \text{Lip}_\beta(\mathbb{C}) \) functions for some \( \beta > 0 \), analytic outside of \( \phi(E) \), that in turn induce (by composition) \( K \)-quasiregular functions on \( \mathbb{C} \setminus E \) with some global Hölder continuity exponent. This construction will encounter two obstacles. First, the extremal dimension distortion of sets of dimension \( \frac{2+\alpha K}{1+\alpha} \) through \( K \)-quasiconformal mappings is not exactly \( 1 + \alpha \), the critical number in the analytic setting (this was so for \( \alpha = 0 \)). Second, the composition of \( \beta \)-Hölder continuous functions with
$K$-quasiconformal mappings is only in $\text{Lip}_{\beta/K}(C)$, so there is some loss of regularity that might be critical. To avoid these troubles, we will construct in an explicit way the mapping $\phi$. This concrete construction allows us to show that $\phi$ exhibits an exponent of Hölder continuity given by

$$\frac{t}{t'} = \frac{1}{K} + \frac{K - 1}{2K},$$

which is larger than the usual $\frac{1}{K}$ obtained from Mori’s theorem. This regularity will be sufficient for our purposes. On the other hand, if $\dim(E) = t$ and $\dim(\phi(E)) = t'$ then it is natural to expect $\phi$ to be $\text{Lip}_{t/t'}$.

2. Extremal Distortion

Throughout this section, $D(z, r)$ will denote the open disk of center $z$ and radius $r$. By $\text{diam}(D)$ we mean the diameter of the disk $D$, and $\lambda D$ will denote the disk concentric with $D$ having diameter $\text{diam}(\lambda D) = |\lambda| \text{diam}(D)$. By $\mathbb{D}$ we will mean the unit disk, and $Jf$ will denote the Jacobian determinant of the function $f$.

Recall that a Cantor-type set $E$ of $m$ components is the only compact set that is invariant under a fixed family of $m$ similitudes,

$$\psi_j : \mathbb{D} \to \mathbb{D},$$

$$z \mapsto \psi_j(z) = a_j + b_j z,$$

with $a_j, b_j \in \mathbb{C}$ for all $j = 1, \ldots, m$ and such that $D_i = \psi_i(\mathbb{D})$ are disjoint disks and $D_i \subset \mathbb{D}$. In other words, $E \subset \mathbb{D}$ is the only solution to the equation

$$E = \bigcup_{j=1}^m \psi_j(E).$$

Constructively, we have

$$E = \bigcap_{N=1}^\infty \left( \bigcup_{\ell(J)=N} \psi_J(\mathbb{D}) \right),$$

where $\psi_J = \psi_{j_1} \circ \cdots \circ \psi_{j_N}$ for any chain $J = (j_1, \ldots, j_N)$ of length $\ell(J) = N$ of members of $\{1, \ldots, m\}$. The Hausdorff dimension of $E$ is the only solution $d$ to the equation

$$\sum_{j=1}^m |\psi'_j|^d = 1.$$ 

Under the additional assumption $|\psi'_j| = r$ for all $i$, one easily obtains

$$\dim(E) = d = \frac{\log m}{\log(1/r)}.$$ 

Another typical situation appears when the image sets $D_i$ are uniformly distributed in $D$; then there is a constant $C$ such that, for any disk $D$,
where the sum runs over all disks $D_i$ that intersect $D$. In this case, $E$ is said to be a regular Cantor-type set. For these sets, $0 < \mathcal{H}^t(E) < \infty$.

One of the main results in [2] is the sharpness of the dimension distortion equation (2). To obtain the equality there, the author distorted holomorphically a fixed Cantor-type set $F$. This deformation defined actually a holomorphic motion on $F$. An interesting extension result, known as the $\lambda$-lemma, allows this motion to be extended quasiconformally from $F$ to the whole plane. This procedure avoided most of the technicalities and gave the desired result in a surprisingly direct way.

However, since we look both for extremal dimension distortion and higher Hölder continuity, $\phi$ must be constructed explicitly. Thus, let $t \in (0, 2)$ and $K \geq 1$ be fixed numbers, and denote $t' = \frac{2K}{2 + (K - 1)t}$. As in [2], we first give a $K$-quasiconformal mapping $\phi$ that maps a regular Cantor set $E$ of dimension $t$ to another regular Cantor set $\phi(E)$ for which $\dim(\phi(E))$ is as close as we want to $t'$.

**Proposition 1.** Given $t \in (0, 2)$, $K \geq 1$, and $\varepsilon > 0$, there exist a compact $E \subset \overline{\mathbb{D}}$ and a $K$-quasiconformal mapping $\phi : \mathbb{C} \to \mathbb{C}$ with the following properties:

1. $\phi$ is the identity mapping on $\mathbb{C} \setminus \overline{\mathbb{D}}$;
2. $E$ is a self-similar Cantor set, constructed with $m = m(\varepsilon)$ similarities;
3. $\dim(\phi(E)) \geq t' - \varepsilon$;
4. $J\phi \in L^p_{\text{loc}}(\mathbb{C})$ if and only if $p \leq \frac{K}{K - 1}$;
5. $|\phi(z) - \phi(w)| \leq Cm^{1/t - 1/t'}|z - w|^{1/t'}$ whenever $|z - w| < 1$.

**Proof.** Our construction follows the scheme in [4]. Thus, we will obtain $\phi$ as a limit of a sequence of $K$-quasiconformal mappings

$$\phi = \lim_{N \to \infty} \phi_N,$$

where every $\phi_N$ will act at the $N$th step of the construction of $E$. More precisely, both $E$ and $\phi(E)$ will be regular Cantor sets associated to two fixed families of similitudes $(\varphi_j)_{j=1,\ldots,m}$ and $(\psi_j)_{j=1,\ldots,m}$. At the $N$th step, $\phi_N$ will map each generating disk of $E$, $\varphi_{j_1}\ldots j_N(\overline{\mathbb{D}})$, to the corresponding generating disk of the image set, $\psi_{j_1}\ldots j_N(\overline{\mathbb{D}})$. Since $\phi$ is supposed to be $K$-quasiconformal and to give extremal distortion of dimension, we think about using a typical radial stretching,

$$f(z) = z|z|^{1/K - 1},$$

conveniently modified. It turns out that this radial stretching $f$ is extremal for some basic properties of $K$-quasiconformal mappings, such as Hölder continuity. In order to find $\phi$ in a better Hölder space (this fails for $f$), we will replace $f$ by a linear mapping in a small neighborhood of its singularity. This change will not affect the exponent of integrability but will enable some improvement on the Hölder exponent.

Take $m \geq 100$ and consider $m$ disjoint disks inside of $\overline{\mathbb{D}}$, $D(z_i, r)$, uniformly distributed and all with the same radius $r = r_m$. By taking $m$ large enough, we
may always assume that $c_m = mr^2 \geq \frac{1}{2}$. Given any $\sigma \in (0, 1)$ to be determined later, we can consider $m$ similitudes
$$\varphi_i(z) = z_i + \sigma rz, \; \; z \in \mathbb{D},$$
and denote, for every $i = 1, \ldots, m$,
$$D_i = \frac{1}{\sigma} \varphi_i(\mathbb{D}) = D(z_i, r_1),$$
$$D'_i = \varphi_i(\mathbb{D}) = D(z_i, \sigma r_1);$$
here we have written $r_1 = r$. We define
$$g_1(z) = \begin{cases} 
\sigma^{1/K-1}(z - z_i) + z_i & \text{if } z \in D'_i, \\
\frac{z - z_i}{r_1}^{1/K-1}(z - z_i) + z_i & \text{if } z \in D_i \setminus D'_i, \\
z & \text{otherwise.}
\end{cases}$$
It may be easily seen that $g_1$ defines a $K$-quasiconformal mapping that is conformal everywhere except on each ring $D_i \setminus D'_i$. Moreover, if we put
$$\psi_i(z) = z_i + \sigma^{1/K}rz, \; \; z \in \mathbb{D},$$
then $g_1$ maps every $D_i$ to itself while each $D'_i$ is mapped to $D''_i = \psi_i(\mathbb{D})$; see Figure 1. Now we denote $\phi = g_1$.

At the second step, we repeat this procedure inside of every $D''_i$ and leave the rest fixed. That is, we define $g_2$ on the target set of $\phi_1$ and then construct $\phi_2$ as
$$\phi_2 = g_2 \circ \phi_1.$$ 
To do this more explicitly, we denote
$$D_{ij} = \frac{1}{\sigma} \varphi_i(\varphi_j(\mathbb{D})) = D(z_{ij}, r_2),$$
$$D'_{ij} = \varphi_i(\varphi_j(\mathbb{D})) = D(z_{ij}, \sigma r_2);$$
a computation shows that \( r_2 = \sigma^{1/K} r \). Now we define

\[
g_2(z) = \begin{cases} 
\sigma^{1/K-1}(z - z_{ij}) + z_{ij} & \text{if } z \in D_{ij}', \\
\left| z - z_{ij} \right|^{1/K-1} (z - z_{ij}) + z_{ij} & \text{if } z \in D_{ij} \setminus D_{ij}', \\
\frac{z - z_{ij}}{r_2} & \text{otherwise.}
\end{cases}
\]

By construction, \( g_2 \) is \( K \)-quasiconformal on \( C \), is conformal outside a union of \( m^2 \) rings, and maps \( D_{ij}' \) to \( D_{ij}'' = \psi_{ij}(\mathbb{D}) \) while every point outside of the disks \( D_{ij} \) remains fixed under \( g_2 \); see Figure 2. Thus, the composition \( \phi_2 = g_2 \circ \phi_1 \) (see Figure 3) is still \( K \)-quasiconformal and agrees with the identity outside of \( D \); moreover,

\[
\phi_2(\psi_{ij}(\mathbb{D})) = \psi_{ij}(\mathbb{D})
\]

for any \( i, j = 1, \ldots, m \).
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After \(N - 1\) steps, we will define \(g_N\) on the target side of \(\phi_N^{-1}\). For each multi-index \(J = (j_1, \ldots, j_N)\) of length \(\ell(J) = N\), we denote

\[
D_J = \frac{1}{\sigma} \phi_N^{-1}(\psi_J(\mathbb{D})) = D(z_J, r_N),
\]

\[
D'_J = \phi_N^{-1}(\psi_J(\mathbb{D})) = D(z_J, \sigma r_N);
\]

now \(r_N = \sigma^{1/K} r_{N-1}\). Then the mapping

\[
g_N(z) = \begin{cases} 
\sigma^{1/K-1} (z - z_J) + z_J & \text{if } z \in D'_J, \\
\left|\frac{z - z_J}{r_N}\right|^{1/K-1} (z - z_J) + z_J & \text{if } z \in D_J \setminus D'_J, \\
z & \text{otherwise,}
\end{cases}
\]

is \(K\)-quasiconformal on the plane and conformal outside a union of \(m_N\) rings. Furthermore, \(g_N(D_J) = D_J\) and \(g_N(D'_J) = D''_J\), where \(D''_J = \psi_J(\mathbb{D})\).

As a consequence, the composition \(\phi_N = g_N \circ \phi_{N-1}\) is also \(K\)-quasiconformal and

\[
\phi_N(\psi_J(\mathbb{D})) = \psi_J(\mathbb{D}).
\]

With this procedure, it is clear that the sequence \(\phi_N\) is uniformly convergent to a homeomorphism \(\phi\). It is also clear that \(\phi\) has distortion bounded by \(K\) almost everywhere and, in fact, that \(\phi\) is a \(K\)-quasiconformal mapping. By construction, \(\phi\) maps the regular Cantor set

\[
E = \bigcap_{N=1}^{\infty} \left( \bigcup_{\ell(J) = N} \psi_J(\mathbb{D}) \right)
\]

to

\[
\phi(E) = \bigcap_{N=1}^{\infty} \left( \bigcup_{\ell(J) = N} \psi_J(\mathbb{D}) \right),
\]

which obviously is also a regular Cantor set. If now we choose \(\sigma\) so that

\[
m(\sigma r)^t = 1
\]

we directly obtain \(0 < H^t(E) < \infty\) as well as

\[
\dim(\phi(E)) = \frac{1}{t'} + \frac{K - 1}{2K} \log(1/mr^2) / \log m.
\]

Since \(c_m \geq \frac{1}{2}\) for all \(m\), we can always get

\[
\dim(\phi(E)) \geq t' - \varepsilon
\]

simply by increasing \(m\) if needed.

Now we must look at the regularity properties of our mapping \(\phi\). To do so, we introduce the following notation. Put \(G^0 = \mathbb{D}\), and denote by \(P^N_J\) and \(G^N_J\) (respectively) the peripheral and generating disks of generation \(N\). That is, for any chain \(J = (j_1, \ldots, j_N)\),
\[ P_J^N = \frac{1}{\sigma} \phi_J(D), \quad G_J^N = \psi_J(D). \]

With this notation, \( D_J = \phi_{N-1}(P_J^N), D'_J = \phi_{N-1}(G_J^N), \) and \( D''_J = \phi_N(G_J^N). \)

Now take any \( p \) such that \( J\phi \in L^p_{\text{loc}}(\mathbb{C}) \). Of course, we can assume \( p \geq 1 \).

Then, one may decompose the \( p \)-mass of \( J\phi \) over \( \mathbb{D} \) as follows:

\[
\int_{\mathbb{D}} J\phi(z)^p \, dA(z) = \int_{\mathbb{D} \setminus \bigcup \mathcal{P}_1} J\phi(z)^p \, dA(z) + \sum_{i=1}^m \int_{\mathcal{P}_1 \setminus \mathcal{G}_i} J\phi(z)^p \, dA(z) + \sum_{i=1}^m \int_{\mathcal{G}_i} J\phi(z)^p \, dA(z). 
\]

Since \( \phi = \phi_1 \) on \( \mathbb{C} \setminus \bigcup \mathcal{G}_i \), it follows that

\[
\int_{\mathbb{D}} J\phi(z)^p \, dA(z) = \int_{\mathbb{D} \setminus \bigcup \mathcal{P}_1} J\phi_1(z)^p \, dA(z) + m \int_{\mathcal{P}_1 \setminus \mathcal{G}_i} J\phi_1(z)^p \, dA(z) + m \int_{\mathcal{G}_i} J\phi(z)^p \, dA(z),
\]

where \( \mathcal{P}_1 \) and \( \mathcal{G}_i \) denote (respectively) any of the first-generation peripheral and generating disks. One may repeat this argument for the last integral, which by a recursive argument yields

\[
\int_{\mathbb{D}} J\phi(z)^p \, dA(z) = \sum_{N=0}^\infty m^N \int_{G_N \setminus \bigcup \mathcal{P}_j} J\phi_{N+1}(z)^p \, dA(z) + \sum_{N=1}^\infty m^N \int_{P_N \setminus G_N} J\phi_N(z)^p \, dA(z);
\]

here, as before, \( P_N \) and \( G_N \) denote (respectively) any \( N \)th-generation peripheral or generating disk.

Now we compute separately the integrals in both sums. On one hand, if \( J = (j_1, \ldots, j_N) \) then

\[
\int_{P_N \setminus G_N} J\phi_N(z)^p \, dA(z) = \int_{P_N \setminus G_N} Jg_N(r, \phi_{N-1}(z))^p J\phi_{N-1}(z)^p \, dA(z)
\]

\[
= \int_{D_J \setminus D'_J} Jg_N(w)^p J\phi_{N-1}(\phi_{N-1}(z))^{p-1} \, dA(w)
\]

\[
= (\sigma J)^{1-K} (N-1)(p-1) \int_{D_J \setminus D''_J} Jg_N(w)^p \, dA(w)
\]

\[
= r^{2N} \sigma^{(N-1)y} \frac{2\pi}{K^p} \left| \frac{1 - \sigma^y}{y} \right|
\]

under the additional assumption \( p \neq \frac{K}{K-1} \), here \( y = 2p(\frac{1}{K} - 1) + 2 \). If \( p = \frac{K}{K-1} \), then
\[
\int_{P^N \setminus G^N_j} J\phi_N(z)^{K/(K-1)} dA(z) = r^{2N} \frac{2\pi}{K^{K/(K-1)}} \log \frac{1}{\sigma}.
\]

On the other hand, for any value of \( p \),
\[
\int_{G^N_j \cup \bigcup_{i < j} P^{N+1}_{(i,j)}} J\phi_N(z)^p dA(z) = \int_{G^N_j \cup \bigcup_{i < j} P^{N+1}_{(i,j)}} Jg_N(\phi_N(z))^p J\phi_N(z)^p dA(z)
\]
\[
= \int_{D^N_j \cup D(j,i)} Jg_N(\phi_N^{-1}(w))^p J\phi_N^{-1}(w))^{p-1} dA(w)
\]
\[
= (\sigma^{1/K-1} 2^{N(p-1)} \int_{D^N_j \cup D(j,i)} 1 dA(w)
\]
\[
= (\sigma^{1/K-1} 2^{N(p-1)} |D^N_j \cup D(j,i)|
\]
\[
= r^{2N} \sigma^{N(p-1)} (1-c_m).
\]

Thus, for any \( p \neq \frac{K}{K-1} \),
\[
\int_D J\phi(z)^p dA(z) = \left( \pi (1-c_m) + c_m \frac{2\pi}{K^p} \left| 1 - \sigma^p \right| \right) \sum_{N=0}^\infty (c_m \sigma^p)^N.
\]

Since \( p \) is such that \( J\phi \in L^p_{\text{loc}}(C) \), we necessarily have \( \sigma^p < 1/c_m \). For \( m \) large enough, this is equivalent to \( \gamma > 0 \); that is, \( p < \frac{K}{K-1} \). At the critical point \( p = \frac{K}{K-1} \),
\[
\int_D J\phi(z)^{K/(K-1)} dA(z) = \left( \pi (1-c_m) + c_m \frac{2\pi}{K^{K/(K-1)}} \log \frac{1}{\sigma} \right) \sum_{N=0}^\infty (c_m)^N,
\]
which will always converge for any fixed value of \( m \). This shows that we can choose \( m \) large enough so that \( J\phi \in L^p_{\text{loc}}(D) \) if and only if \( p \leq \frac{K}{K-1} \).

Finally, it remains only to check that \( \phi \) is Hölder continuous with exponent \( \gamma = t/t' \). By means of Poincaré inequality together with the quasiconformality of \( \phi \), it is enough [8, p. 64] to show that, for any disk \( D \),
\[
\int_D J\phi(z) dA(z) \leq C \text{diam}(D)^{2t/t'}.
\]

Hence, for some fixed disk \( D \), take \( N \) such that \( (\sigma r)^N \leq \frac{1}{2} \text{diam}(D) < (\sigma r)^{N-1} \). Then
\[
\int_D J\phi(z) dA(z) \leq \int_D \bigcup G^N_j J\phi(z) dA(z) + \int_{\bigcup G^N_j} J\phi(z) dA(z),
\]
where the union \( \bigcup G^N_j \) runs over all disks \( G^N_j \) such that \( G^N_j \cap D \neq \emptyset \). On \( D \setminus \bigcup G^N_j \), we easily see that
\[
J\phi = J\phi_N \leq \frac{1}{K} (\sigma^{1/K-1})^{2N}.
\]
On the other hand, recall that $\phi(G_N^J) = \phi_N(G_N^J)$ are disks of radius $(\sigma r^{K-1})^N$. Hence,
\[
\int_{\bigcup J} J\phi(z) dA(z) = \sum_J \int_{G_N^J} J\phi(z) dA(z) = \sum_J |\phi_N(G_N^J)| = \sum_J \pi(\sigma K r)^{2N} \left( \frac{1}{2} \text{diam}(D) \right)^{2t/t'}.
\]
and it just remains to bound $\sum_J \left( \frac{(\sigma r)^N}{\frac{1}{2} \text{diam}(D)} \right)^{2t/t'}$. Actually, this is equivalent to finding some constant $C$ such that
\[
\sum_{G_N^J \cap D \neq \emptyset} \text{diam}(G_N^J)^{2t/t'} \leq C \text{diam}(D)^{2t/t'}.
\]
But the disks $G_N^J$ come from a self-similar construction that is said to give a regular Cantor set of dimension $t$. In particular, they may be chosen uniformly distributed so that the $t$-dimensional packing condition is satisfied:
\[
\sum_{G_N^J \cap D \neq \emptyset} \text{diam}(G_N^J)^t \leq C \text{diam}(D)^t.
\]
It is easy to show that this condition implies the $s$-dimensional one for all $s > t$ (in particular, for $s = 2t/t'$). Hence, the constant $C$ exists and is independent of $m$. Thus, what we finally obtain is that
\[
\int_D J\phi(z) dA(z) \leq \frac{c(K-1/2)}{K \sigma m} \left( \frac{1}{2} \text{diam}(D) \right)^{2t/t'},
\]
and the result follows. □

**Corollary 2.** Let $K \geq 1$ and $t \in (0, 2)$, and denote $t' = \frac{2Kt}{2t+(K-1)'}$. There exist a $t$-dimensional compact set $E$ and a $K$-quasiconformal mapping $\phi: \mathbb{C} \to \mathbb{C}$ such that
1. $\mathcal{H}^{t'}(E)$ is $\sigma$-finite,
2. $\text{dim}(\phi(E)) = t'$, and
3. $|\phi(z) - \phi(w)| \leq C |z - w|^{t'/t}$ whenever $|z - w| < 1$. 

Proof. Given $\varepsilon > 0$, $K \geq 1$, and $t \in (0, 2)$, let $\phi : \mathbb{C} \to \mathbb{C}$ and $E$ be as in Proposition 1. Then, for any fixed $r > 0$, the mapping

$$
\psi_r(z) = r\phi(z/r)
$$

and the set $E_r = rE$ exhibit the same properties as $\phi$ and $E$, since neither $K$-quasi-conformality nor Hausdorff dimension is modified through dilations. However, when computing the new $\text{Lip}_{t/t'}$ constant, if $|z - w| < r$ then

$$
|\psi_r(z) - \psi_r(w)| = r|\phi(z/r) - \phi(w/r)| \leq Cm^{1/1' - 1/t' - 1/1'}r^{1 - 1/t'}|z - w|^{1/1'}.
$$

Thus, as in [2], let $D_j = D(z_j, r_j)$ be a countable disjoint family of disks inside of $\mathbb{D}$, and let $\varepsilon_j$ be a sequence of positive numbers, $\varepsilon_j \to 0$ as $j \to \infty$. For each $j$, let $\phi_j$ and $E_j$ be as in Proposition 1, so that $\dim(\phi_j(E_j)) \geq t' - \varepsilon_j$. In particular, each $E_j$ is a regular Cantor set of $m_j$ components. Denote then $\psi_j(z) = r_j\phi_j(\frac{z - z_j}{r_j})$ and $F_j = z_j + r_jE_j$, and define

$$
\psi(z) = \begin{cases} 
\psi_j(z) & \text{if } z \in D_j, \\
\text{otherwise.} & \end{cases}
$$

By construction, $\psi$ is a $K$-quasiconformal mapping: it maps the set $F = \bigcup_j F_j$ to the set $\psi(F) = \bigcup_j \psi_j(F_j)$. Moreover, $\mathcal{H}^t(F)$ is $\sigma$-finite and

$$
\dim(\phi(F)) = \sup \dim(\psi_j(F_j)) = t'.
$$

Finally, assume that $z$ lies inside some fixed $D_k$ and that $w \in \mathbb{D} \setminus \bigcup_j D_j$. Then, consider the line segment $L$ between $z$ and $w$, and denote $\{z_k\} = L \cap \partial D_k$. Then both $z_k$ and $w$ are fixed points for $\psi$, so that

$$
|\psi(z) - \psi(w)| \leq |\psi(z) - \psi(z_k)| + |\psi(z_k) - \psi(w)| \\
\leq Cm^{1/1' - 1/t'}r_k^{1 - 1/t'}|z - z_k|^{1/1'} + |z_k - w|.
$$

Since we are still free to choose radii $r_j$, we may do so such that

$$
m_k^{1/1' - 1/t'}r_k^{1 - 1/t'} < 1
$$

or, equivalently, $m_k^{1/t'} < 1$. Under this assumption, we finally get

$$
|\psi(z) - \psi(w)| \leq (C + 1)|z - w|^{1/1'}
$$

whenever $|z - w| < 1$. This clearly shows that $\psi \in \text{Lip}_{t/t'}(\mathbb{C})$. \qed

Although the set in Corollary 2 is more critical than the one we constructed in Proposition 1 (in the sense that the first gives precisely the extremal dimension distortion), both do the same work when studying nonremovable sets for Hölder continuous quasiregular mappings.

Corollary 3. Let $K \geq 1$ and $\alpha \in (0, 1)$. Then, for any $t > 2^{\frac{1+\alpha}{1+1/K}}$, there exists a compact set $E$ with $0 < \mathcal{H}^t(E) < \infty$ that is nonremovable for $K$-quasiregular mappings in $\text{Lip}_\alpha$. 
Proof. Let $E$ and $\phi$ be such that $\dim \phi(E) \geq t' - \varepsilon > 1$ for some sufficiently small $\varepsilon$. Hence, by Frostman’s lemma, we can construct a positive Radon measure $\mu$ supported on $\phi(E)$ with growth $t' - 2\varepsilon$. Its Cauchy transform $g = C\mu$ defines a holomorphic function on $\mathbb{C} \setminus \phi(E)$, not entire and with a Hölder continuous extension to the whole plane, with exponent $t' - 2\varepsilon - 1$. Set

$$f = g \circ \phi.$$ 

Clearly, $f$ is $K$-quasiregular on $\mathbb{C} \setminus E$ and has no $K$-quasiregular extension to $\mathbb{C}$. Indeed, if $\tilde{f}$ extends $f$ $K$-quasiregularly to $\mathbb{C}$ then $\tilde{g} = \tilde{f} \circ \phi^{-1}$ would provide an entire extension of $g$, which is impossible. Furthermore, $f$ is Hölder continuous with exponent

$$(t' - 2\varepsilon - 1) \frac{t}{t'} = t - (2\varepsilon + 1) \frac{t}{t'}.$$ 

Thus, we just need $\varepsilon > 0$ small enough that

$$t - (2\varepsilon + 1) \frac{t}{t'} \geq \alpha;$$ 

but this inequality is equivalent to

$$\left( t - 2 \frac{1 + \alpha K}{1 + K} \right) \geq \varepsilon \frac{2}{K + 1} (2 + (K - 1)t)$$

and so the proof is complete.

Something similar may be said when dealing with finite distortion mappings. Recall that if $\Omega \subset \mathbb{C}$ is an open set then a mapping of finite distortion on $\Omega$ is a function $f : \Omega \rightarrow \mathbb{C}$ in the Sobolev class $W^{1,1}_{\text{loc}}(\mathbb{C})$ with locally integrable Jacobian, $Jf \in L^1_{\text{loc}}(\mathbb{C})$, and such that there exists a measurable function $K_f : \Omega \rightarrow [1, \infty]$, called the distortion function of $f$, that is finite almost everywhere and for which

$$|Df(z)|^2 \leq K_f(z) Jf(z)$$

at almost every $z \in \Omega$. When $K_f \in L^\infty$ and $\|K_f\|_\infty = K$, one recovers the class of $K$-quasiregular mappings. However, weaker assumptions on $K_f$ also give interesting results. The most typical situation appears when we ask the distortion function $K_f$ to be such that

$$\exp\{K_f\} \in L^p_{\text{loc}}(\mathbb{C})$$

for some $p$ large enough. Then we say that $K_f$ is exponentially integrable and that $f$ is a mapping of exponentially integrable distortion. In [6], it was shown that compact sets $E$ with $\sigma$-finite $H^{2\alpha}(E)$ are removable for $\alpha$-Hölder continuous mappings of exponentially integrable distortion.

Corollary 4. Let $\alpha \in (0, 1)$. For any $t > 2\alpha$ there exist a compact set $E$ of dimension $t$ and a function $f \in \text{Lip}_\alpha(\mathbb{C})$ that defines a mapping of exponentially integrable distortion $\mathbb{C} \setminus E$ and has no finite distortion extension to $\mathbb{C}$.
Proof. If \( t > 2\alpha \), then there exists \( K \geq 1 \) such that \( t > 2^{1+\alpha K} \). Thus, we have a compact set \( E \) of dimension \( t \) and a \( \text{Lip}_\alpha(\mathbb{C}) \) function \( f \) that is \( K \)-quasiregular on \( \mathbb{C} \setminus E \) but not on \( \mathbb{C} \). Of course, \( f \) is a mapping of exponentially integrable distortion on \( \mathbb{C} \setminus E \), with distortion function \( K_f \) essentially bounded by \( K \). If \( f \) extended to a mapping of finite distortion on \( \mathbb{C} \), then in particular we would have \( Jf \in L^1_{\text{loc}}(\mathbb{C}) \). But then, since \( K_f \leq K \) at almost every point, this would imply that actually \( f \) extends \( K \)-quasiregularly.

At this point, it should be said that above the critical index \( 2^{1+\alpha K} \) one might find also some removable set. For instance, an unpublished result of S. Smirnov shows that if \( E = \partial \mathbb{D} \) and \( \phi \) is a \( K \)-quasiconformal mapping then
\[
\dim(\phi(E)) \leq 1 + \left( \frac{K - 1}{K + 1} \right)^2,
\]
which is better than the usual dimension distortion equation (2). Hence, if we choose \( K \geq 1 \) small enough then there exists an \( \alpha \) that satisfies
\[
K \left( \frac{K - 1}{K + 1} \right)^2 < \alpha < \frac{K - 1}{2K}.
\]
For those values of \( \alpha \), the set \( E = \partial \mathbb{D} \) is removable for \( \alpha \)-Hölder continuous \( K \)-quasiregular mappings, although
\[
2^{1+\alpha K} \frac{1}{1+K} < \dim(E).
\]
This suggests that everything could happen between \( 2^{1+\alpha K} \frac{1}{1+K} \) and \( 1 + \alpha \).

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