C-OPH: Improving the Accuracy of One Permutation Hashing (OPH) with Circulant Permutations

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Abstract

Minwise hashing (MinHash) is a classical method for efficiently estimating the Jaccard similarity in massive binary (0/1) data. To generate $K$ hash values for each data vector, the standard theory of MinHash requires $K$ independent permutations. Interestingly, the recent work on “circulant MinHash” (C-MinHash) (Li and Li, 2021b) has shown that merely two permutations are needed. The first permutation breaks the structure of the data and the second permutation is re-used $K$ time in a circulant manner. Surprisingly, the estimation accuracy of C-MinHash is proved to be strictly smaller than that of the original MinHash. The more recent work (Li and Li, 2021a) further demonstrates that practically only one permutation is needed. Note that those two papers (Li and Li, 2021a,b) are different from the well-known work on “One Permutation Hashing (OPH)” published in NIPS’12 (Li et al., 2012). OPH and its variants using different “densification” schemes are popular alternatives to the standard MinHash. The densification step is necessary in order to deal with empty bins which exist in One Permutation Hashing.

In this paper, we propose to incorporate the essential ideas of C-MinHash to improve the accuracy of One Permutation Hashing. Basically, we develop a new densification method for OPH, which achieves the smallest estimation variance compared to all existing densification schemes for OPH. Our proposed method is named C-OPH (Circulant OPH). After the initial permutation (which breaks the existing structure of the data), C-OPH only needs a “shorter” permutation of length $D/K$ (instead of $D$), where $D$ is the original data dimension and $K$ is the total number of bins in OPH. This short permutation is re-used in $K$ bins in a circulant shifting manner. It can be shown that the estimation variance of the Jaccard similarity is strictly smaller than that of the existing (densified) OPH methods.

Additionally, our work leads to an interesting and useful consequence. That is, if we neglect the cost of the initial permutation needed by the bin-dividing step, then we actually just need “$1/K$” permutation instead of one permutation. It turns out that the initial permutation can be safely replaced by an approximate permutation such as 2-universal (2U) hashing, as verified by an extensive empirical study.
1 Introduction

With the explosive growth in the scale of data, efficient methods for search, data storage, and large-scale machine learning have become increasingly important. The method of minwise hashing (MinHash) (Broder, 1997; Broder et al., 1998; Li and König, 2011) is a classical and popular hashing algorithm for binary 0/1 data. The hashed values of two data points $v, w \in \{0,1\}^D$, which are non-negative integers, have collision probability equal to the Jaccard similarity (resemblance) between the two data points defined as

$$J(v, w) = \frac{\sum_{i=1}^D \mathbb{1}\{v_i = w_i = 1\}}{\sum_{i=1}^D \mathbb{1}\{v_i + w_i \geq 1\}}.$$  

When using MinHash in practice, in order to ensure reliable accuracy, one typically needs to generate hundreds to thousands hash values per data point depending on the applications. We use $K$ to denote the target number of hashes (per data point). Operationally, to achieve a strict performance guarantee that matches the theory, the standard MinHash requires $K$ independent permutations, with each hash value being the minimal permuted index of the non-zero elements of the data vector.

Over the past decades, MinHash and variants have been widely used in numerous applications, for similarity estimation, approximate nearest neighbor search, duplicate detection, clustering, classification, image retrieval, database systems, etc. (Broder et al., 1997; Charikar, 2002; Fetterly et al., 2003; Das et al., 2007; Buehrer and Chellapilla, 2008; Bendersky and Croft, 2009; Lee et al., 2010; Li et al., 2011; Deng et al., 2012; Chum and Matas, 2012; Srivastava and Li, 2012; Tamsersoy et al., 2014; Zhu et al., 2017; Nargesian et al., 2018; Wang et al., 2019; Lemiesz, 2021; Tseng et al., 2021; Feng and Deng, 2021; Jia et al., 2021).

Here, we should also mention that, in the very early development of minwise hashing, only one permutation was used by storing the first $K$ non-zero locations after the permutation (Broder, 1997; Broder et al., 1997). Later Li and Church (2005) proposed maximum likelihood estimators to significantly improve the estimation accuracy, and Li and Church (2007) further extended the method to estimating three-way and multi-way associations. However, because the hashed values in Broder (1997); Broder et al. (1997); Li and Church (2005, 2007) did not form a metric space (i.e., satisfy the triangle inequality), they could not be used for numerous important applications that require metric space.

1.1 Circulant Minwise Hashing (C-MinHash)

Recently, Li and Li (2021b) proposes a convenient variant of MinHash that rigorously reduces the number of independent permutations required to merely 2. In the so-called Circulant MinHash method, noted as C-MinHash-(σ, π), the first permutation $σ$ is used for initially permuting the data (pre-processing), and the second (independent) permutation $π$ is used for generating the hash values, repeatedly for $K$ times by circulant shifts. For example, if $π = [3, 2, 4, 1]$, then $π_{-1} = [1, 3, 2, 4]$ is the permutation shifted rightwards by 1 element, and so on. C-MinHash uses $π_{-1}, π_{-2}, ..., $ to generate the hash values, instead of the independent permutations as in the standard MinHash. The concrete procedure is summarized in Algorithm 1.

| Algorithm 1: C-MinHash-(σ, π) |
|---|
| 1 **Input:** Binary data vector $v \in \{0,1\}^D$, Permutation vectors $π$ and $σ$: $[D] \rightarrow [D]$ |
| 2 **Output:** Hash values $h_1(v), ..., h_K(v)$ |
| 3 Initial permutation: $v' = σ(v)$ |
| 4 For $k = 1$ to $K$ |
| 5 Shift $π$ circulant rightwards by $k$ units: $π_k = π_{-k}$ |
| 6 $h_k(v) \leftarrow \min_{i: v'_i \neq 0} \pi_{-k}(i)$ |
| 7 End For |

Surprisingly, it was proved in Li and Li (2021b) that using merely 2 permutations, the Jaccard estimator of C-MinHash has uniformly smaller variance than that of the classical MinHash (with $K$ independent permutations). Moreover, in a followup work (Li and Li, 2021a), they further verified by extensive empirical results that actually one can more conveniently use merely one permutation, i.e., simply letting $σ = π$ in Algorithm 1.
That is, one can use $\pi$ for both initialization (pre-processing) and circulant hashing. While the dependence between initial permutation and those for hashing makes the theoretical analysis very complicated, the authors of (Li and Li, 2021a) provided the exact expression of the mean estimation of C-MinHash-$\pi$-$\pi$. Even though the Jaccard estimator of C-MinHash-$\pi$-$\pi$ is biased, the bias is too small to have any noticeable impact on the overall mean square error (MSE), where MSE = variance + bias$^2$.

Practically speaking, C-MinHash provides a convenient strategy for the design of hashing systems. For instance, when the data space is not ultra large (e.g., $\leq 2^{30}$, a billion features), we can simply save one “permutation vector” to generate all $K$ hash values. A vector of length $2^{30}$ can be easily stored, even in GPU memory. However, saving thousands of such permutations might be still unrealistic and wasteful. The benefit of using exact permutations, instead of approximation by hash functions, is that the empirical performance would always match the theory. We will elaborate more on this point later in the paper.

1.2 One Permutation Hashing (OPH)

While C-MinHash methods need to use only one or two permutations, it should be clear that they are different from the work on “one permutation hashing (OPH)” and its many variants of “densification” schemes. The idea of OPH (Li et al., 2012) is to randomly divide the data vector of size $D$ into $K$ equal-sized bins (by applying a random permutation $\sigma$). Then, minwise hashing is applied in each bin, producing $K$ hash values from $K$ bins. An illustrative example is presented in Figure 1. The theoretical correctness of this approach is built upon the proof that, the expected “bin-wise” Jaccard similarity in each simultaneously non-empty bin is the same as the true $J$ of the whole data vectors $v, w$ (Li et al., 2012).

![Figure 1: A toy example of One Permutation Hashing (OPH) provided by Li et al. (2019) on three binary vectors, $v_1, v_2, v_3$. The data vector $v$ is first randomly split into $K = 4$ bins, $B_1, ..., B_4$, by a random permutation $\sigma$. Then, the hash value is taken as the smallest non-zero index within each bin (effectively doing a minwise hashing within the bin). If a bin is empty, we record “E”. For instance, $h(v_1) = [E, 5, 11, 14]$.](image)

Note that, as shown in Figure 1, OPH really only needs one permutation (if the number of hash values is smaller than $K$). After the data vector is permuted by $\sigma$ and broken into $K$ bins, the “small permutation” within each bin is already completed. Nevertheless, for the convenience of explaining our proposed idea in this paper, we can still conceptually view OPH as a “two-permutation” scheme. That is, the initial permutation $\sigma$ is applied and the data vector is divided after the permutation. A second permutation $\pi$ is broken evenly into $K$ small permutations and each small permutation is used in one bin to conduct minwise hashing within the bin. As proved in (Li et al., 2012), for OPH, we can simply let $\pi = \sigma$. This interpretation would be helpful to understand the densification and re-randomization procedures which will soon be introduced.

There are possibly empty bins, especially for relatively sparse data vectors, as reflected in Figure 1. To tackle the problem of empty bins, Shrivastava and Li (2014a, b) developed two “densification” schemes, which were later improved by Shrivastava (2017); Li et al. (2019), among other works.
1.3 Summary of Contributions

In this work, we seek to improve the estimation accuracy of the existing OPH framework as well its densification methods, by incorporating the essential idea of circulant MinHash (Li and Li, 2021b,a). After the initial bin-dividing process, we use a “small” permutation of size $D/K$, where $D$ is the data dimension and $K$ is the number of bins. Without loss of generality, we assume $D/K$ is an integer (otherwise we can always pad zeros). This permutation is re-used in all bins in a circulant shifting manner. We derive the precise variance formula of the Jaccard estimator, and show that C-OPH reduces the prior known optimal Jaccard estimation variance of densified OPH (Li et al., 2019). Numerical results are provided to validate the theory.

In addition to achieving improved accuracy, there is also another beneficial consequence. That is, if we neglect the cost of the initial permutation for the bin-dividing procedure, we actually only need “1/$K$ permutation” to generate $K$ hash values, instead of using one permutation of size $D$. It turns out that the initial permutation can be safely replaced by an approximation such as the 2-universal (2U) hashing and other approximate hashing methods such as 4-universal (4U) hashing. Consider the original Circulant MinHash with two permutations C-MinHash-$(\sigma, \pi)$. If we replace the initial permutation $\sigma$ by 2U hashing, it becomes C-MinHash-$(2U, \pi)$. Likewise, if we replace the second permutation $\pi$ by 2U hashing, we obtain C-MinHash-$(\sigma, 2U)$. Our experimental study has shown that C-MinHash-$(2U, \pi)$ achieves essentially the same accuracy as C-MinHash-$(\sigma, \pi)$. The accuracy of C-MinHash-$(\sigma, 2U)$, however, can be much worse than C-MinHash-$(\sigma, \pi)$, in certain datasets.

The above observation provides the intuition for developing C-OPH-$(2U, \pi/K)$. That is, in Circulant OPH, we use the 2U hashing for the initial permutation and use one single (small) permutation of size $D/K \leq 10$ to generate hashes from the bins, in a circulant shifting fashion. This strategy would allow practitioners to be able to use C-OPH-$(2U, \pi/K)$ in datasets of much higher dimensions. For example, consider a dataset with $D = 2^{40}$ and we let $K = 2^{10}$. Then we just need a small permutation of size $D/K = 2^{30}$ for this very high dimensional dataset, and $2^{30}$ is small enough even for the GPU memory.

2 Background: Densified One Permutation Hashing (OPH)

We first provide the details about OPH and its densification schemes. The generic framework of OPH is provided in Algorithm 2, where $[K]$ denotes the set $\{1, \ldots, K\}$. For illustration, we present ReDen (Re-randomized Densified), the most recent and theoretically the most accurate densified OPH (Li et al., 2019). Here, we consider a flexible implementation of OPH to allow the number of bins $K$ and the number of hash values $M$ to be different. This is a convenient and practical setting where one may hope to continue generating more than $K$ hashes after each bin has contributed one hash.

The general procedure of ReDen is as follows:

1. **Bin split and permutation generation**: use a permutation $\sigma$ to randomly split the data vector $v$ into $K$ bins, where $B_k = \{ j : (i-1)\frac{D}{K} + 1 \leq \sigma(j) \leq i\frac{D}{K} \}$. Assign each hash $k = 1, \ldots, M$ an independent “bin-wise” permutation $\pi^{(k)} : [D/K] \mapsto [D/K]$.

2. **First scan**: for $k = 1, \ldots, M$, select a bin $i_k \in [K]$ by some strategy. If bin $i_k$ is non-empty, set $h_k(v) = \min_{j : j \in B_{i_k}, v_j \neq 0} \pi^{(k)}(\sigma(j) - (i_k-1)\frac{D}{K}) + (i_k-1)\frac{D}{K}$, where the addition is to recover the original index to prevent accidental collision; otherwise, set $h_k(v) = E$.

3. **Second scan and Re-randomized Densification**: We do another screening over hash samples $h_k$, $k = 1, \ldots, M$: if the $k$-th hash is empty ("$E"$), find a non-empty bin $B_{i_k'}$. Densify the $k$-th hash by $h_k(v) = \min_{j : j \in B_{i_k'}, v_j \neq 0} \pi^{(k)}(\sigma(j) - (i_k'-1)\frac{D}{K}) + (i_k'-1)\frac{D}{K}$.

Note that, in the densification process, ReDen applies independent permutation $\pi^{(k)} \| \pi^{(k')}$ to get the hash sample for an empty bin $k$. That said, the “local” permutations used within bins for producing the hash for each $h_k$ are all independent. Moreover, as mentioned in Section 1.2, the permutation $\sigma$ used for random bin split also implies the independent bin-wise permutations $\pi^{(1)} \ldots, \pi^{(M)}$ as required, when $M \leq K$. This is because, for the permutation $\sigma$ on $[D]$, its “partial” sub-permutations split by consecutive
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[Li et al., 2019]

The exact implementation is more complicated in practice, and we refer interested readers to the aforementioned papers for more details. It is known that the uniform sampling strategy would lead to reduced Jaccard estimation variance. Therefore, in our paper, we will focus on this densification bin selection routine. Note that, the main contribution of this work, which is the improvement brought by using circulation in OPH, is valid for all types or combinations of bin selection methods.

| Algorithm 2: Generic One Permutation Hashing (OPH) with Re-randomized Densification (Re-Den) |
|---|
| **Input:** Binary data vector \( v \in \{0, 1\}^D \); Number of bins \( K \); Number of hash values \( M \) |
| **Output:** OPH hashes \( h_1(v), ..., h_M(v) \) |

3. Use a permutation \( \sigma \) to randomly split \([D]\) into \( K \) equal-size bins \( B_1, ..., B_K \),

\[
B_i = \{ j : (i - 1) \frac{D}{K} + 1 \leq \sigma(j) \leq i \frac{D}{K} \}
\]

4. Assign each hash an independent permutation \( \pi^{(i)} : [D/K] \mapsto [D/K], i \in [M] \)

For \( k = 1 \) to \( M \)

6. Select a bin \( i_k \in [K] \) by some strategy (option (i) or (ii)) \hspace{1cm} //First scan bin selection

7. If \( B_{i_k} \) is not empty

8. \( h_k(v) \leftarrow \min_{j \in B_{i_k}}:v_j \neq 0 \pi^{(k)}(\sigma(j) - (i_k - 1) \frac{D}{K}) + (i_k - 1) \frac{D}{K} \) \hspace{1cm} //MinHash within non-empty bin

9. Else

10. \( h_k(v) = E \) \hspace{1cm} //Empty bin

11. End If

12. End For

13. For \( k = 1 \) to \( M \)

14. If \( h_k(v) = E \)

15. Select a non-empty bin \( i'_k \in [K] \) \hspace{1cm} //Densification bin selection

16. \( h_k(v) \leftarrow \min_{j \in B_{i'_k} : v_j \neq 0} \pi^{(k)}(\sigma(j) - (i'_k - 1) \frac{D}{K}) + (i'_k - 1) \frac{D}{K} \) \hspace{1cm} //Re-randomized Densification

17. End If

18. End For

Bins (local order) are also perfectly random and independent. Therefore, for the \( k \)-th hash, applying \( \sigma \) for bin split implicitly accomplishes the role of applying an additional bin-wise permutation \( \pi^{(k)} \). Consequently, in practical implementation, when \( M \leq K \), we only need one long permutation \( \sigma \), which achieves bin split and implies all the bin-wise short permutations. When \( M > K \), we will need additional \((M - K)\) “bin-wise” permutations for the purpose of re-randomization.

2.1 Bin Selection Strategies

In Algorithm 2, two important ingredients are the first scan bin selection and the densification bin selection, as depicted by line 6 and line 15, respectively. Regarding the first scan bin selection (line 6), there are two available strategies:

(i) We strictly pick the \( K \) bins one by one. For hash value \( h_k \) with \( k > K \), we apply a rotation strategy, i.e., to pick the \( \{\text{mod}(k - 1, K) + 1\} \)-th bin.

(ii) For every \( k \in [M] \), we uniformly randomly choose a bin out of \( K \) bins.

Strategy (i) can be regarded as fixing the number of times each bin is chosen, while strategy (ii) is more flexible. Strategy (i) typically has smaller Jaccard estimation variance than (ii) (Li et al., 2019). Therefore, in the remainder, we will stick to strategy (i) as the scanning bin selection for all the analysis and simulations.

For the densification bin selection strategy (line 15), the earlier work proposed to rotationally select a nearest non-empty bin in a clock-wise direction. Later, Shrivastava (2017) proposed to uniformly randomly select non-empty bins for densification through 2-universal hashing, which is also adopted in Li et al. (2019). The exact implementation is more complicated in practice, and we refer interested readers to the aforementioned papers for more details.
2.2 Densification Strategies

Line 16 of Algorithm 2 is another important component of OPH, which is the strategy to generate the (densified) hash value once the non-empty bin is selected. In ReDen, the re-randomization procedure proposes to re-do an independent bin-wise permutation. For example, if the hash \( h_k \) is empty (i.e., \( h_k(v) = E \)) and the \( j \)-th bin is chosen for densifying \( h_k \), then \( \pi(k) \) is applied to the data in bin \( B_j \) to get \( h_k(v) \). In earlier works, e.g., Shrivastava and Li (2014a); Shrivastava (2017), for densification, the hash value is directly copied from the non-empty bin to the empty bin, i.e., \( h_k(v) \leftarrow h_j(v) \). However, this strategy leads to larger estimation variance especially on highly sparse data, since the densification of many empty bins would output many identical hash values. Intuitively speaking, empty bins would not provide much useful additional information of the data. Overall, the ReDen method as presented in Algorithm 2 with option (i) for bin selection and re-randomized densification is the most recent framework that achieves the smallest Jaccard similarity estimation variance among all densified OPH schemes.

3 C-OPH: Circulant One Permutation Hashing

In this section, we present our proposed C-OPH method, inspired by the idea of circulant minwise hashing.

3.1 Approach: An Improved Densification Scheme

As shown in Algorithm 3, the proposed Circulant One Permutation Hashing (C-OPH) method admits the same “bin-wise hashing + densification” protocol as ReDen. The key difference is the bin-wise permutations used for hashing. Recall that in Algorithm 2, to generate each hash \( h_k(v) \), we use an independent length-\( D/K \) permutation \( \pi(k) : [D/K] \mapsto [D/K], k = 1, \ldots, M \). In the proposed C-OPH protocol, we indeed only need one length-\( D/K \) permutation, \( \pi \), which is used circulantly in the same spirit as in C-MinHash. That is, we use \( \pi \rightarrow k \) to generated the \( k \)-th hash, for both the first scan and the densification. We provide an illustrative example in Figure 2 of C-OPH. Here, to make sure that no circulant permutation is repeatedly used, we assume that \( KM \leq D \), which is also needed for establishing rigorous theoretical results.

### Algorithm 3: Circulant One Permutation Hashing (C-OPH-(\( \sigma, \pi \))

1. **Input:** Binary data vector \( v \in \{0, 1\}^D \); Number of bins \( K \); Number of hash values \( M \)
2. **Permutation** \( \sigma : [D] \mapsto [D], \pi : [D/K] \mapsto [D/K] \)
3. **Output:** C-OPH hashes \( h_1(v), \ldots, h_M(v) \)

4. Use \( \sigma \) to randomly split \([D]\) into \( K \) equal-size bins \( B_1, \ldots, B_K \), \( B_i = \{ j : (i-1)\frac{D}{K} + 1 \leq \sigma(j) \leq i\frac{D}{K} \} \)
5. For \( k = 1 \) to \( M \)
6. Select a bin \( i_k = mod(k-1, K) + 1 \) by option (i)
7. If \( B_{i_k} \) is not empty
8. \( h_k(v) \leftarrow \min_{j \in B_{i_k}, v_j \neq 0} \pi \rightarrow k (\sigma(j) - (i_k - 1)\frac{D}{K}) + (i_k - 1)\frac{D}{K} \) //MinHash within non-empty bin
9. Else
10. \( h_k(v) = E \) //Empty bin
11. End If
12. End For
13. For \( k = 1 \) to \( M \)
14. If \( h_k(v) = E \)
15. Select a non-empty bin \( i'_k \in [M] \) uniformly at random
16. \( h_k(v) \leftarrow \min_{j \in B_{i'_k}, v_j \neq 0} \pi \rightarrow k (\sigma(j) - (i'_k - 1)\frac{D}{K}) + (i'_k - 1)\frac{D}{K} \) //C-OPH Densification
17. End If
18. End For
Figure 2: An illustration of the proposed C-OPH strategy. The data vector $v$ is first randomly split into $K = 4$ bins. For ReDen (Li et al., 2019), we use independent permutation $\pi^{(1)}, \ldots, \pi^{(4)}$ to perform MinHash within each bin. For C-OPH, we use circulant permutations $\pi_{\rightarrow 1}, \ldots, \pi_{\rightarrow 4}$ instead. In this toy example, the 2nd bin $B_2$ is empty. In both approaches, we randomly select a non-empty bin ($B_4$ for ReDen, and $B_1$ for C-OPH), and min-hash the data in that bin with the permutation associated with $B_2$.

3.2 Theoretical Analysis

We prove that the prior-known minimal Jaccard estimation variance among OPH methods, which is achieved by ReDen, can be further reduced by the proposed C-OPH. For conciseness, we assume $M = K$ and $K^2 \leq D$.

For both ReDen (Algorithm 2) and C-OPH (Algorithm 3), the Jaccard estimator between two binary data vectors $v, w \in \{0, 1\}^D$ is in the form of average hash collisions,

$$\hat{J}(v, w) = \frac{1}{K} \sum_{k=1}^{K} \mathbb{1}\{h_k(v) = h_k(w)\}. \quad (2)$$

Denote $\hat{J}_{\text{ReD}}$ and $\hat{J}_{\text{COPH}}$ as the estimator resulting from Algorithm 2 and Algorithm 3 (both with same bin selection options as in Algorithm 3), respectively. Suppose the length of each bin $d = D/K$ is an integer.

Throughout the paper, we denote

$$a = \sum_{i=1}^{D} \mathbb{1}\{v_i = 1 \text{ and } w_i = 1\}, \quad f = \sum_{i=1}^{D} \mathbb{1}\{v_i = 1 \text{ or } w_i = 1\}.$$  

The following conditional probability regarding the bin split would be needed for our analysis.

**Lemma 3.1.** Let $d = D/K$. Define function $H$ with the following recursion: $\forall 1 \leq k \leq K$, integer $n > 0$,

$$H(k, n|d) = \sum_{j=1}^{d \wedge (n-k+1)} \binom{d}{j} H(k-1, n-j|d), \quad H(1, n|d) = \binom{d}{n}.$$  

Then, conditional on the event that Algorithm 3 has $m$ non-empty bins, denote $\bar{a} = \sum_{i \in B_k} \mathbb{1}\{v_i + w_i = 2\}$, $\bar{f} = \sum_{i \in B_k} \mathbb{1}\{v_i + w_i = 1\}$ in a non-empty bin $B_k$. For $0 \leq a' \leq \min\{a, d\}$, $1 \leq f' \leq \min\{f, d\}$, $a' \leq f'$, we have the conditional distribution as

$$P\left[\bar{a} = a', \bar{f} = f'|m\right] = \frac{\binom{d}{f'} H(m-1, f - f'|d)}{H(m, f|d)} \cdot \frac{\binom{a}{f' - a}}{\binom{f'}{f'}}. \quad (3)$$

**Proof.** By Lemma 1 of Li et al. (2019), we know that conditional on the event that there are $m$ non-empty bins (denoted as “$m$” in the formulas for simplicity), the marginal distribution of $\bar{f}$ follows

$$P\left[\bar{f} = f'|m\right] = \frac{\binom{d}{f'} H(m-1, f - f'|d)}{H(m, f|d)}.$$  

7
Thus, the joint distribution can be characterized by

\[
P \left[ \tilde{a} = a', \tilde{f} = f' \mid m \right] = P[\tilde{a} = a' \mid \tilde{f} = f', m] P[\tilde{f} = f' \mid m]
\]

\[
= \frac{\binom{f}{\tilde{a}} H(m - 1, f - f' \mid d)}{H(m, f \mid d)} \cdot \frac{\binom{f}{\tilde{f}} \binom{f - \tilde{a}}{\tilde{f} - \tilde{a}} \binom{f - \tilde{f}}{f - \tilde{f}}}{\left( \binom{f}{\tilde{f}} \right)}.
\]

due to the fact that given \( \tilde{f} \) within a non-empty bin, \( \tilde{a} \) follows a hyper-geometric distribution \( \text{Hyper}(f, a, f', a') \).

The variance of \( \hat{J}_{\text{CPH}} \) is provided as below.

**Theorem 3.2.** Denote \( d = D/K \). Assume \( K^2 \leq D \) and \( f \leq \frac{K - 1}{K} D \). Let \( N_{\text{emp}} = K - m \) be the number of empty bins after the bin split, \( J = \frac{a}{f} \) and \( J = \frac{a}{f} - 1 \). We have

\[
\text{Var}[\hat{J}_{\text{CPH}}] = \mathbb{E}\left[ KJ + N_{\text{emp}}(2K - N_{\text{emp}} - 1)E_1 + (K - N_{\text{emp}})(K - N_{\text{emp}} - 1)J\tilde{J} \right],
\]

with

\[
E_1 = \frac{1}{m} \sum_{(a', f', d) \in \Omega(m)} \tilde{E}(a', f', d)p(a', f' \mid m) + \frac{m - 1}{m} \tilde{J}J,
\]

where \( \Omega(m) \) is the set of all possible \((\tilde{a}, \tilde{f})\) defined as in Lemma 3.1 within a non-empty bin, and \( p(a', f' \mid m) \) is given by (3). Additionally, for any \( 1 \leq a' \leq f' \leq d \),

\[
\tilde{E}(a', f', d) = \sum_{\{l_0, l_2, g_0, g_1\}} \left\{ \frac{l_0}{f' + g_0 + g_1} + \frac{a(g_0 + l_2)}{(f' + g_0 + g_1)f'} \right. \\
\times \left. \sum_{s=1}^{d-f'-1} \binom{d-f'}{s} \binom{d-f'-s}{d-f'-1} \binom{s}{n_1} \binom{d-f'-s}{n_2} \binom{a'}{n_3} \binom{a'-1}{n_4} \binom{a'-l_1-l_2}{d-f'-1} \right\},
\]

where

\[
\begin{align*}
    n_1 & = g_0 - (d - f' - s - g_1), \quad n_2 = d - f' - s - g_1, \\
    n_3 & = l_1 - (d - f' - s - g_1), \quad n_4 = l_1 - (d - f' - s - g_1),
\end{align*}
\]

and the feasible set of \( \{l_0, l_2, g_0, g_1\} \) satisfies the intrinsic constraints that for non-negative \( l_0, g_0, g_1, h_0 \),

\[
\begin{align*}
l_0 + l_1 + l_2 & = l_0 + g_0 + h_0 = a', \\
g_0 + g_1 + g_2 & = l_1 + g_1 + h_1 = d - f', \\
h_0 + h_1 + h_2 & = l_2 + g_2 + h_2 = f' - a'.
\end{align*}
\]

In particular, it holds that \( \text{Var}[\hat{J}_{\text{CPH}}] < \text{Var}[\hat{J}_{\text{RD}}] \).

**Remark 3.1.** In Theorem 3.2, the expectation is with respect to the number of empty bins, \( N_{\text{emp}} \). The exact probability distribution of this random variable is

\[
\text{Pr}[N_{\text{emp}} = j] = \sum_{\ell=0}^{K-j} (-1)\ell \binom{K}{j} \binom{K-j}{\ell} \binom{D(1 - (j + \ell)/K)}{f} / \binom{D}{f}.
\]

We refer interested readers to Li et al. (2019) for the detailed derivation.

**Proof.** Firstly, we separate the event of matching hash values into two distinct events. Denote \( C_k^f \) the indicator of hash collision at bin \( k \) when \( k \) is empty, and \( C_k^N \) the indicator of collision when \( k \) is simultaneously
non-empty. Consequently, we have $C_k = C_k^N + C_k^E$. Denote $I_{emp,k}$ as the indicator function of the $k$-th bin being empty. According to Li et al. (2012), for non-empty bins we have

$$
E(C_k^N | I_{emp,k} = 0) = E(C_k^E | I_{emp,k} = 0) = J
$$

$$
E[(C_k^N)^2 | I_{emp,k} = 0] = E[(C_k^E)^2 | I_{emp,k} = 0] = J.
$$

Based on above notations, for both schemes we can write $\hat{J} = \frac{1}{K} \sum_{k=1}^{K} (C_k^E + C_k^N)$. For any unbiased estimator, we can expand

$$\text{Var}(\hat{J}) = E\left( \frac{1}{K^2} \left( \sum_{k=1}^{K} (C_k^E + C_k^N) \right)^2 \right) - J^2 = \frac{1}{K^2} A - J^2. \quad (4)$$

It suffices to analyze $A$. Conditional on the event that the number of non-empty bins $K - N_{emp} = m$, we have

$$A = E[E[(\sum_{k=1}^{K} (C_k^E + C_k^N))^2 | K - N_{emp} = m]].$$

For the ease of notation, we simply let $m$ denote the event $\{K - N_{emp} = m\}$. One important fact of C-Oph is that, the expectation of $C_k^N C_j^N$ for non-empty bins $B_i, B_j$ where $\pi$ is used circulantly equals that of ReDen where two independent permutations are applied to $B_i$ and $B_j$ respectively. This is because in the algorithms, the bin split is uniform at random. Keeping this in mind, for both ReDen and C-Oph we have

$$A = E[E[(\sum_{k=1}^{K} (C_k^E + C_k^N))^2 | m]]$$

$$= E(E[\sum_{k=1}^{K} ((C_k^E)^2 + (C_k^N)^2) + \sum_{i \neq j} C_i^E C_j^E$$

$$+ 2 \sum_{i \neq j} C_i^E C_j^N + \sum_{i \neq j} C_i^N C_j^N | m]]$$

$$= E[KJ + N_{emp}(N_{emp} - 1)E_1 + 2N_{emp}(K - N_{emp})E_1$$

$$+ (K - N_{emp})(K - N_{emp} - 1)\tilde{J}]$$

$$= E[KJ + N_{emp}(2K - N_{emp} - 1)E_1$$

$$+ (K - N_{emp})(K - N_{emp} - 1)\tilde{J}], \quad (5)$$

where $E_1 = E[C_i^E C_j^E | m]$, for $i \neq j$. Since $f \leq \frac{K}{2}D$, this expectation is always positive. Denote event $\Upsilon$ as bin $i$ and bin $j$ choosing the same non-empty bin in the densification bin selection procedure. Expanding the expectation yields

$$E_1 = P[C_i^E = C_j^E = 1 | m]$$

$$= P[C_i^E = C_j^E = 1 | \Upsilon, m]P[\Upsilon | m] + P[C_i^E = C_j^E = 1 | \Upsilon, m]P[\Upsilon | m]$$

$$= \frac{1}{m} P[C_i^E = C_j^E = 1 | \Upsilon, m] + \frac{m - 1}{m} \tilde{J}.$$

Suppose both bins select bin $k$ (non-empty). Let $I_k$ be the index set of bin $k$. Denote $a_k = |\{i \in I_k : v_i + w_i = 2\}|$, $f_k = |\{i \in I_k : v_i + w_i = 1\}|$ as the number of two types of simultaneously non-zero elements in bin $k$. Let $\Omega(m)$ be the collection of possible $(a_k, f_k)$ conditional on $m$ non-empty bins. We proceed with

$$E_1 = \frac{1}{m} \sum_{(a', f') \in \Omega(m)} \left\{ P[C_i^E = C_j^E = 1 | (a_k, f_k) = (a', f'), \Upsilon, m] \right\} \times P[(a_k, f_k) = (a', f') | \Upsilon, m] + \frac{m - 1}{m} \tilde{J}. \quad (6)$$
In (6), the second probability is precisely the conditional probability given by Lemma 3.1. The first probability can be computed as \( \mathcal{E}(a', f', d) \) by Lemma 3.3 in Li and Li (2021b), applied to the selected non-empty bin for densification. We refer interested readers to Li and Li (2021b) for details. Aggregating parts together gives the exact formula of the variance of C-OPH Jaccard estimator.

To show that \( \text{Var}[\tilde{J}_{\text{COPH}}] \) is smaller than \( \text{Var}[\tilde{J}_{\text{ReDen}}] \), notice that in (6), the distributions of \((a_k, f_k)\) and \(m\) only depend on the bin split procedure which is shared by both ReDen and C-OPH. Therefore, using circulant permutations in fact only affects the term

\[
B \triangleq P[C_i^E = C_j^E = 1|(a_k, f_k) = (a', f'), \mathcal{Y}, m],
\]

for each \((a', f')\), consider bin \(k\) as a length-\(D/K\) data vector with within-bin Jaccard index \(J' = a'/f'\). Since \(K^2 \leq D\), applying the arguments of Theorem 3.4 in Li and Li (2021b) within the bin, we know that C-OPH gives smaller term \(B\) than ReDen by the variance reduction of circulant permutations applied within the bin. Now combining (4) and (5) proves the desired result.

Theorem 3.2 says that, C-OPH can further improve the prior known smallest variance of the OPH-type Jaccard estimators, which is the main theoretical merit of the proposed C-OPH. It is easy to show that this strict variance reduction holds for arbitrary \(K\) and \(M\) with \(MK \leq D\), which is usually true in practice for high-dimensional data. Moreover, even if this condition is not present, the variance of C-OPH can still be considerably smaller. In line 8 and line 16 of Algorithm 3, we can change \(\pi_{\rightarrow k}\) into \(\pi_{\rightarrow (k+kd/D)}\)", where the term \([kd/D]\) is to deploy an additional periodic shifting. For example, suppose \(d = \frac{D}{K} = \frac{M}{f}\) (i.e., \(MK = 2D\)). In the first \(k = d = \frac{M}{f}\)-th hashes, the permutation used to generate hash from, e.g., \(B_1\), is \(\pi_{\rightarrow 1}\). With the shifting term, the permutation used in the last \(\frac{M}{f}\) hash values for \(B_1\) would be \(\pi_{\rightarrow (k/K)} = \pi_{\rightarrow (d+2)}\), which is different from the first permutation \(\pi_{\rightarrow 1}\) used for \(B_1\). Otherwise (without this shifting term), since \(\pi_{\rightarrow (d+1)} = \pi_{\rightarrow 1}\), the permutations (for \(B_1\)) will repeat, leading to exactly the same hash values and consequently larger estimation variance. However, if \(B_1\) is an empty bin, then we still have \(1/m\) chance of using the same permutation to generate hashes from a same non-empty bin (densification bin collision), where \(m\) is the number of non-empty bins in total. This would slightly increase the variance of C-OPH from the rigorous theoretical perspective. Nonetheless, in most practical cases where \(f = |v \cup w|\) is not very small (i.e., not extremely sparse data in the absolute scale), the impact of the densification bin collision would be negligible (e.g., see our empirical verification). To sum up, the variance reduction of C-OPH is valid rigorously when \(MK \leq D\), and is true empirically when \(MK > D\).

![Figure 3: Empirical Mean Squared Error (MSE) of simulated data pairs, \(D = 2^{12}, f = |v \cup w| = 2^{11},\) with various \(J\). 1st row: \(K = 2^5\), 2nd row: \(K = 2^7\). We confirm that the MSE of C-OPH is smaller than that of ReDen in all cases.](image-url)
In Figure 3, we plot the empirical MSE of C-OPH versus ReDen on several synthetic binary data pairs to demonstrate the variance reduction effect and justify the theory. Here we simulate data vectors with various dimensionality $D$ and Jaccard similarity values. The bin split is realized by a random permutation on all data elements. We validate that C-OPH can indeed incur smaller estimation variance than ReDen. More numerical examples on real-world data can be found in Section 4 (Figure 5 and 6).

4. Practical Implementation

So far, our discussion has assumed that we have access to a uniformly random bin split. One can certainly achieve this by doing a random permutation on the data vector, and then split non-zero elements into different bins based on their permuted location. In practice, however, applying exact permutations might not be desirable when the data dimensionality is extremely large, due to the high memory consumption to store the “permutation vector”. Instead, using hash functions to approximate random permutations is a common choice, which waves the need to store explicitly the permutation. In this section, we provide some discussion on the practical implementation of C-OPH, to show how hash functions can be effectively used in C-OPH. Recall that, C-MinHash (Li and Li, 2021b,a) can reduce the $K$ permutations used in standard MinHash to merely one permutation, allowing the efficient usage of one “permutation vector” such that the practical estimation always matches the theory. Interestingly, when combined with OPH, our proposed C-OPH can further reduce the number of permutations used to just “1/$K$”, practically speaking.

4.1 When shall we use hash functions?

In many applications involving very high ($D$) dimensional data, storing exact random permutations may be too costly. A simple alternative is to use simple hash functions to approximate the permutations. It is well-known that, in some cases, simply replacing the permutations with hash functions in MinHash may lead to undesirable and systematic estimation inconsistencies. However, interestingly as shown in Figure 4, for C-MinHash-$(\sigma, \pi)$, we can safely replace the first permutation $\sigma$ by simple 2-universal (2U) hashing.

![Figure 4: Mean Squared Error (MSE) of C-MinHash-$(\sigma, \pi)$, C-MinHash-$(2U, \pi)$ and C-MinHash-$(\sigma, 2U)$ on data pairs from the Words dataset (Li and Church, 2005), with various $J$. We see that replacing the initial (pre-processing) permutation $\sigma$ by a 2U hash function (i.e., C-MinHash-$(2U, \pi)$) gives virtually the same estimation MSE as C-MinHash-$(\sigma, \pi)$. However, replacing the second permutation $\pi$ by 2U (i.e., C-MinHash-$(\sigma, 2U)$) leads to larger estimation errors. In each plot, the two vectors from Li and Church (2005) denote whether the words appear in the documents of the repository.](image)

Here, the 2U hash function is defined by $H : [D] \mapsto [D]$ for a non-negative integer $x \in [D]$: $H(x) = ax + b \mod p$, where $p > D$ is a prime number, $a$ and $b$ are uniformly chosen from $\{0, 1, \ldots, p - 1\}$ and $a$ is odd. There are also known tricks to avoid the modulo operations. As shown in Figure 4, for C-MinHash-$(\sigma, \pi)$, we can safely replace by the first permutation $\sigma$ by 2U hash. However, we can observe obvious performance deviations once we place the second permutation $\pi$ by 2U hash (i.e., C-MinHash-$(\sigma, 2U)$). We have experimented with other simple hashing functions including 4-universal hashing and murmur hashing, but the observations are essentially very similar to using 2U hashing.
4.2 From One Permutation to “1/K” Permutation

Therefore, we naturally develop C-OPH-(2U, π). That is, we can replace the initial permutation σ in the original C-OPH-(σ, π). This means, we actually just need “1/K” permutation instead of one permutation. This might need to a substantial convenience in practice. For example, if $D = 2^{40}$ and $K = 2^{10}$, then we just need one short permutation of size $D/K = 2^{30}$, which is small enough even for GPU memory.

Figure 5 and Figure 6 provide another set of experiments for sanity check: (i) C-OPH-(σ, π) improves “ReDen” the prior best densification (Li et al., 2019); (2) The “1/K permutation version C-OPH-(2U, π) performs essentially the same as C-OPH-(σ, π), using the same “Words” dataset (Li and Church, 2005). From the plots with various $K$, we validate again that the proposed C-OPH provides smaller estimation MSEs than ReDen. Furthermore, using hash function for bin split (C-OPH-(2U, π)) essentially gives same empirical accuracy as the standard C-OPH-(σ, π).

Figure 5: Mean Squared Error (MSE) of ReDen, C-OPH-(σ, π) and C-OPH-(2U, π) on word pairs from the Words dataset, $K = 2^7$. We see that C-OPH improves the MSE of ReDen, and using hash function to perform bin split empirically gives same MSE as using perfectly random permutation.
Figure 6: Mean Squared Error (MSE) of ReDen, C-OPH-($\sigma, \pi$) and C-OPH-($2U, \pi$) on word pairs from the Words dataset, $K = 2^{10}$. We see that C-OPH improves the MSE of ReDen, and using hash function to perform bin split empirically gives same MSE as using perfectly random permutation.

5 Conclusion

The popular minwise hashing (MinHash) method has been improved by the so-called Circulant MinHash (C-MinHash) (Li and Li, 2021b,a). To generate $K$ hash values, C-MinHash only needs two permutations or even just one permutation, instead of using $K$ independent permutations as required by the standard MinHash. It is clear that C-MinHash is different from the previous known work on “One Permutation Hashing” (OPH) (Li et al., 2012) and its variants (due to different “densification” schemes). In this paper, by incorporating the central ideas of circulant permutations, we propose Circulant OPH (C-OPH) to improve the accuracy of the state-of-the-art densified OPH, i.e., the “ReDen” method developed in Li et al. (2019).

The basic idea of Circulant OPH (C-OPH) is to first randomly divide the data vectors into equal-sized $K$ bins, then use a smaller permutation (of size $D/K$, where $D$ is the data dimension) to generate the required number of hash values in a circulant manner. We show that the proposed C-OPH achieves a smaller estimation variance than “ReDen”, the previous best (most accurate) densified OPH algorithm. In addition to achieving improved estimation accuracy, another interesting benefit of C-OPH is that, practically speaking, C-OPH just needs “$1/K$” permutation instead of one permutation. This consequence would be useful in practice. For example, consider a dataset with $D = 2^{40}$ and $K = 2^{10}$. Then we just need a small permutation of size $D/K = 2^{30}$ for this high-dimensional dataset, and $2^{30}$ is small even for the GPU memory.
References

Michael Bendersky and W. Bruce Croft. Finding text reuse on the web. In *Proceedings of the Second International Conference on Web Search and Web Data Mining (WSDM)*, pages 262–271, Barcelona, Spain, 2009.

Andrei Z. Broder. On the resemblance and containment of documents. In *Proceedings of the Conference on Compression and Complexity of SEQUENCES*, pages 21–29, Positano, Amalfitan Coast, Salerno, Italy, 1997.

Andrei Z. Broder, Steven C. Glassman, Mark S. Manasse, and Geoffrey Zweig. Syntactic clustering of the web. *Comput. Networks*, 29(8-13):1157–1166, 1997.

Andrei Z. Broder, Moses Charikar, Alan M. Frieze, and Michael Mitzenmacher. Min-wise independent permutations. In *Proceedings of the Thirtieth Annual ACM Symposium on the Theory of Computing (STOC)*, pages 327–336, Dallas, TX, 1998.

Gregory Buehrer and Kumar Chellapilla. A scalable pattern mining approach to web graph compression with communities. In *Proceedings of the International Conference on Web Search and Web Data Mining (WSDM)*, pages 95–106, Stanford, CA, 2008.

Moses S. Charikar. Similarity estimation techniques from rounding algorithms. In *Proceedings on 34th Annual ACM Symposium on Theory of Computing (STOC)*, pages 380–388, Montreal, Canada, 2002.

Ondrej Chum and Jiri Matas. Fast computation of min-hash signatures for image collections. In *Proceedings of the 2012 IEEE Conference on Computer Vision and Pattern Recognition (CVPR)*, pages 3077–3084, Providence, RI, 2012.

Abhinandan Das, Mayur Datar, Ashutosh Garg, and Shyamsundar Rajaram. Google news personalization: scalable online collaborative filtering. In *Proceedings of the 16th International Conference on World Wide Web (WWW)*, pages 271–280, Banff, Canada, 2007.

Fan Deng, Stefan Siersdorfer, and Sergej Zerr. Efficient jaccard-based diversity analysis of large document collections. In *Proceedings of the 21st ACM International Conference on Information and Knowledge Management (CIKM)*, pages 1402–1411, Maui, HI, 2012.

Weiqi Feng and Dong Deng. Align: Aligning all-pair near-duplicate passages in long texts. In *Proceedings of the International Conference on Management of Data (SIGMOD)*, pages 541–553, Virtual Event, China, 2021.

Dennis Fetterly, Mark Manasse, Marc Najork, and Janet L. Wiener. A large-scale study of the evolution of web pages. In *Proceedings of the Twelfth International World Wide Web Conference (WWW)*, pages 669–678, Budapest, Hungary, 2003.

Peng Jia, Pinghui Wang, Junzhou Zhao, Shuo Zhang, Yiyan Qi, Min Hu, Chao Deng, and Xiaohong Guan. Bidirectionally densifying LSH sketches with empty bins. In *Proceedings of the International Conference on Management of Data (SIGMOD)*, pages 830–842, Virtual Event, China, 2021.

David C. Lee, Qifa Ke, and Michael Isard. Partition min-hash for partial duplicate image discovery. In *Proceedings of the 11th European Conference on Computer Vision (ECCV), Part I*, pages 648–662, Heraklion, Crete, Greece, 2010.

Jakub Lemiesz. On the algebra of data sketches. *Proc. VLDB Endow.*, 14(9):1655–1667, 2021.

Ping Li and Kenneth Ward Church. Using sketches to estimate associations. In *Proceedings of the Conference on Human Language Technology and the Conference on Empirical Methods in Natural Language Processing (HLT/EMNLP)*, pages 708–715, Vancouver, Canada, 2005.

Ping Li and Kenneth Ward Church. A sketch algorithm for estimating two-way and multi-way associations. *Comput. Linguistics*, 33(3):305–354, 2007.
Ping Li and Arnd Christian König. Theory and applications of \( b \)-bit minwise hashing. *Commun. ACM*, 54 (8):101–109, 2011.

Ping Li, Anshumali Shrivastava, Joshua Moore, and Arnd Christian König. Hashing algorithms for large-scale learning. In *Advances in Neural Information Processing Systems (NIPS)*, pages 2672–2680, Granada, Spain, 2011.

Ping Li, Art B Owen, and Cun-Hui Zhang. One permutation hashing. In *Advances in Neural Information Processing Systems (NIPS)*, pages 3122–3130, Lake Tahoe, NV, 2012.

Ping Li, Xiaoyun Li, and Cun-Hui Zhang. Re-randomized densification for one permutation hashing and bin-wise consistent weighted sampling. In *Advances in Neural Information Processing Systems 32: Annual Conference on Neural Information Processing Systems 2019, NeurIPS 2019, 8-14 December 2019, Vancouver, BC, Canada*, pages 15900–15910, 2019.

Xiaoyun Li and Ping Li. C-MinHash: Practically reducing two permutations to just one. arXiv preprint arXiv:2109.04595, 2021a.

Xiaoyun Li and Ping Li. C-MinHash: Rigorously reducing \( k \) permutations to two. arXiv preprint arXiv:2109.03337, 2021b.

Fatemeh Nargesian, Erkang Zhu, Ken Q. Pu, and Renée J. Miller. Table union search on open data. Proc. VLDB Endow., 11(7):813–825, 2018.

Anshumali Shrivastava. Optimal densification for fast and accurate minwise hashing. In *Proceedings of the 34th International Conference on Machine Learning (ICML)*, pages 3154–3163, Sydney, Australia, 2017.

Anshumali Shrivastava and Ping Li. Fast near neighbor search in high-dimensional binary data. In *Proceedings of European Conference on Machine Learning and Knowledge Discovery in Databases (ECML-PKDD)*, pages 474–489, Bristol, UK, 2012.

Anshumali Shrivastava and Ping Li. Densifying one permutation hashing via rotation for fast near neighbor search. In *Proceedings of the 31th International Conference on Machine Learning (ICML)*, Beijing, China, 2014a.

Anshumali Shrivastava and Ping Li. Improved densification of one permutation hashing. In *Proceedings of the Thirtieth Conference on Uncertainty in Artificial Intelligence (UAI)*, Quebec City, Canada, 2014b.

Acar Tumersoy, Kevin A. Roundy, and Duen Horng Chau. Guilt by association: large scale malware detection by mining file-relation graphs. In *Proceedings of the 20th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining (KDD)*, pages 1524–1533, New York, NY, 2014.

Tom Tseung, Laxman Dhulipala, and Julian Shun. Parallel index-based structural graph clustering and its approximation. In *Proceedings of the International Conference on Management of Data (SIGMOD)*, pages 1851–1864, Virtual Event, China, 2021.

Pinghui Wang, Yiyan Qi, Yuanming Zhang, Qiaozhu Zhai, Chenxu Wang, John C. S. Lui, and Xiaohong Guan. A memory-efficient sketch method for estimating high similarities in streaming sets. In *Proceedings of the 25th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining (KDD)*, pages 25–33, Anchorage, AK, 2019.

Erkang Zhu, Ken Q. Pu, Fatemeh Nargesian, and Renée J. Miller. Interactive navigation of open data linkages. Proc. VLDB Endow., 10(12):1837–1840, 2017.