Using the momentum space representation, we study the (2+1)-dimensional Duffin-Kemmer-Petiau oscillator for spin 0 particle under a magnetic field in the presence of a minimal length in the noncommutative space. The explicit form of energy eigenvalues are found, the wave functions and the corresponding probability density are reported in terms of the Jacobi polynomials. Additionally, we also discuss the special cases and depict the corresponding numerical results.

1. Introduction

The interest in the study of the minimal length uncertainty relation combination with the noncommutative space commutation relations in nonrelativistic wave equation and relativistic wave equation has drawn much attention [1-2] in recent years. Motivated by string theory, loop quantum gravity and quantum geometry [3-15], the modification of the ordinary uncertainty relation has become an appealing case of research. In the so-called minimal length formulation, Kempf et al [16-18] have shown that the minimal length can be introduced as an additional uncertainty in position measurement, so that the usual canonical commutation relation between position and momentum operator is substituted by $[x, p] = \iota \hbar (1 + \beta p^2)$, where $\beta$ is a small positive parameter called the deformation parameter. Recently, various topics have been studied in connection with the minimal length uncertainty relation [19-28]. On the other hand, the issue of noncommutative (NC) quantum mechanics has also been extensively discussed. The noncommutativity of space-time coordinates was first introduced by Snyder [29] aiming to improve the problem of infinite self-energy in quantum field theory, and the noncommutative geometry has been put forward because of the discovery in string theory and matrix model of M-theory [30]. Recently, various aspects of both NC classical [31] and quantum [32] mechanics have been extensively studied devoted to exploring the role of NC parameter in the physical observables [33-40]. Recently, the effect of the quantum gravity on the quantum mechanics by modifying the basic commutators among the canonical variables has been an attractive topic, therefore the combination of the minimal length uncertainty relation and the noncommutative space commutation relations is a colorful problem. In the present work, we are interesting to study the two sectors in the
framework of the relativistic DKP equation, and analyse the effects of them on the energy spectrum and the corresponding wave functions.

The organization of this work is as follows. In Section 2, we consider the DKP oscillator for spin 0 particle in the presence of a minimal length in the NC space. In Section 3, we study the problem under a magnetic field. In Section 4, we discuss some special cases of the solutions to check the validity of our results. Finally, the work is summarized in last section.

2. The (2 +1)-DKP Oscillator for Spin 0 Particle in the presence of the minimal length in NC space

In NC space, the canonical variables satisfy the following commutation relations:

\[
\begin{align*}
\left[ x_i^{(NC)}, x_j^{(NC)} \right] &= i \theta_{ij}, & \left[ p_i^{(NC)}, p_j^{(NC)} \right] &= 0, & \left[ x_i^{(NC)}, p_j^{(NC)} \right] &= i \hbar \delta_{ij}, (i, j = 1, 2),
\end{align*}
\]

(1)

with \( \theta_{ij} = \epsilon_{ij} \theta \) is the antisymmetric NC parameter, representing the noncommutativity of the space, and \( x_i^{(NC)}, p_i^{(NC)} \) are the coordinate and momentum operators in the NC space. By replacing the normal product with star product, the DKP equation in commutation space will change into the DKP equation in NC space as

\[
H(P, x) \star \psi(x) = E \psi(x).
\]

(2)

Usually, one way to deal with the problem of NC space is via the star product or Moyal Weyl product on the commutative space functions:

\[
(f \star g)(x) = \exp \left[ \frac{1}{2} \theta_{ij} \frac{\partial^x_i}{\partial^y_j} \right] f(x)g(y) \big|_{y-x},
\]

(3)

where \( f(x) \) and \( g(y) \) are arbitrary infinitely differentiable functions.

Then the Moyal-Weyl product can be replaced by a Bopp shift of the form

\[
x_i^{(NC)} = \hat{x}_i - \frac{1}{2h} \theta_{ij} \hat{p}_j, \quad p_i^{(NC)} = \hat{p}_i,
\]

(4)

therefore in the two dimensional NC space, Eq. (4) can be expressed as

\[
x^{(NC)} = \hat{x} - \frac{\theta}{2h} \hat{p}_y, \quad y^{(NC)} = \hat{y} + \frac{\theta}{2h} \hat{p}_x, \quad p_x^{(NC)} = \hat{p}_x, \quad p_y^{(NC)} = \hat{p}_y,
\]

(5)

where \( \hat{x}, \hat{y}, \hat{p}_x \) and \( \hat{p}_y \) are the position and momentum operators in the usual quantum mechanics respectively, which satisfy the canonical Heisenberg commutation relations.

Thus, the relativistic DKP equation for a free boson of mass \( m \) is given by [41-42]
where $\tilde{\beta} = (\beta^1, \beta^2, \beta^3)$ and $\beta^0$ is the DKP matrices which meet the following algebra relation:

$$\beta^\mu \beta^\nu + \beta^\nu \beta^\mu = g^\mu \beta^i + g^\nu \beta^i,$$

with $g^\mu = \text{diag}(1, -1, -1, -1)$ is the metric tensor in Minkowski space. For spin 0 particle, $\beta^i$ are 5 × 5 matrices expressed as

$$\beta^0 = \begin{bmatrix} I_0 & \tilde{0} \\ 0^T & \delta \end{bmatrix}, \quad \beta^i = \begin{bmatrix} 0 & \hat{\delta} \\ -(k_i)^T & \delta \end{bmatrix}, \quad i = 1, 2, 3,$$

with $\delta, \tilde{0}$ and $\hat{\delta}$ being $3 \times 3, 2 \times 3$, and $2 \times 2$ zero matrices, respectively. Other matrices in equation (8) are given as follows:

$$I_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad k_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad k_2 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad k_3 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The DKP oscillator is introduced by using substitution of the momentum operator $\vec{p}$ with $\tilde{p}$, where the additional term is linear in $r$, $w$ is the oscillator frequency, and $\eta^0 = 2(\beta^0)^2 - 1$ with $(\eta^0)^2 = 1$.

It is easy to get the 2D DKP oscillator from the above equation:

$$[c\tilde{\beta} \cdot (\tilde{p} - im\eta^0\tilde{r}) + mc^2]\psi = i\hbar \beta^0 \frac{\partial}{\partial t} \psi.$$

Given this situation above, considering the NC formalism and via the Bopp shift (4-5), the (2+1)-dimensional DKP oscillator equation in NC space becomes

$$[c\beta^1 (P^x_{\text{NC}}) - im\eta^0 x^{(\text{NC})}) + c\beta^2 (P^y_{\text{NC}}) - im\eta^y_{\text{NC}}) + mc^2] \psi = \beta^0 \psi.$$

For a boson of spin 0, the spinor $\psi$ is a vector with five components [43, 44], which reads

$$\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5)^T.$$

Substituting $\psi$ into (11) we have

$$-mc^2\psi_1 + E\psi_2 + c(P^x_{\text{NC}}) + imx^{(\text{NC})})\psi_3 + c(P^y_{\text{NC}}) + imy^{(\text{NC})})\psi_4 = 0,$$

$$E\psi_1 - mc^2\psi_2 = 0,$$
Combination of (13) gives

\[ c \left( p_{x}^{(NC)} - imw^{(NC)} \right) \psi_{1} + mc^{2} \psi_{3} = 0. \]

\[ c \left( p_{y}^{(NC)} - imw^{(NC)} \right) \psi_{1} + mc^{2} \psi_{4} = 0, \]

\[ mc^{2} \psi_{5} = 0. \]  

Combination of (13) gives

\[
\begin{align*}
\left[ \left[ \hat{p}_{x} + imw \left( \frac{\theta}{2\hbar} \hat{p}_{y} \right) \right] \left[ \hat{p}_{x} - imw \left( \frac{\theta}{2\hbar} \hat{p}_{y} \right) \right] + \left[ \hat{p}_{y} + imw \left( \frac{\theta}{2\hbar} \hat{p}_{x} \right) \right] \hat{p}_{y} - \\
imw \left( \frac{\theta}{2\hbar} \hat{p}_{x} \right) \right] + m^{2}c^{2} - \frac{E^{2}}{c^{2}} \right] \psi_{1} = 0. 
\end{align*}
\]

In addition, in the minimal length formalism, the Heisenberg algebra is given by

\[ [\hat{x}, \hat{p}] = i\hbar \delta_{ij}(1 + \alpha P^{2}). \]  

where \( \alpha \) is the minimal length positive parameter. Moreover, in the momentum space, the position vector and momentum vector can be expressed as

\[ \hat{x}_{i} = i\hbar \left[ (1 + \alpha P^{2}) \frac{\partial}{\partial\hat{p}_{ij}} \right], \hat{p}_{i} = P_{i}, \]  

substituting (15) and (16) into (14) we have

\[
\begin{align*}
\left[ \left( 1 + \frac{m^{2}w^{2}\theta^{2}}{4\hbar^{2}} \right) (P_{x}^{2} + P_{y}^{2}) + m^{2}w^{2}(x^{2} + y^{2}) - 2mwh(1 + \alpha P^{2}) - \frac{m^{2}w^{2}\theta}{\hbar}(1 + \alpha P^{2})L_{z} + \\
m^{2}c^{2} - \frac{E^{2}}{c^{2}} \right] \psi_{1} = 0. 
\end{align*}
\]

Now, in order to solve Eq. (17), an auxiliary wave function is defined as \( \psi_{1}(P, \theta) = e^{i\theta}z\phi(P) \), and for the sake of simplicity, we bring the problem into the polar coordinates, recalling that

\[ P_{x} = P \cos \theta, P_{y} = P \sin \theta, \]

\[
\frac{\partial}{\partial P_{x}} = \cos \theta \frac{\partial}{\partial P} - \frac{\sin \theta}{P} \frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial P_{y}} = \sin \theta \frac{\partial}{\partial P} + \frac{\cos \theta}{P} \frac{\partial}{\partial \theta}, \]

then Eq. (17) becomes

\[
\begin{align*}
\left\{ \left( 1 + \alpha P^{2} \right)^{2} \frac{\partial^{2}}{\partial P^{2}} + \frac{1}{P} \left( 1 + \alpha P^{2} \right)^{2} + 2\alpha P(1 + \alpha P^{2}) \right\} \frac{\partial}{\partial P} - \frac{P^{2}}{p^{2}} (1 + \alpha P^{2})^{2} + \\
\frac{\left( 1 + \frac{m^{2}w^{2}\theta^{2}}{4\hbar^{2}} \right) + 2mwh + m^{2}w^{2}\theta^{2} + \frac{E^{2}}{c^{2}} + 2mwh + m^{2}w^{2} \theta^{2}}{m^{2}w^{2}h^{2}} \right\} \phi(P) = 0.
\end{align*}
\]
with the help of a variable transformation
\[ q = \frac{1}{\sqrt{\alpha}} \tan^{-1}(\sqrt{\alpha} P). \] (20)

which will map the variable \( P \in (0, \infty) \) to \( q \in \left(0, \frac{\pi}{2\sqrt{\alpha}}\right)\), we simplify Eq. (19) to
\[
\left[ \frac{\partial^2}{\partial q^2} + \left(\mu + \frac{\mu}{v}\right) \sqrt{\alpha} \frac{\partial}{\partial q} - \alpha \ell^2 \frac{v^2}{\mu^2} + \alpha k' \left(\frac{v^2}{\mu^2} + \alpha \epsilon\right) \right] \phi(q) = 0,
\] (21)
where
\[ \mu = \sin(\sqrt{\alpha} q), \quad v = \cos(\sqrt{\alpha} q), \]
\[ k' = -\ell^2 + \frac{-\left(1 + \frac{m^2}{4h^2} + \frac{2mw\omega h + m^2 w^2 \theta \alpha \ell}{a^2 m^2 w^2 h^2} \right)}{e^2 - \frac{2m w \omega h + m^2 w^2 \theta \ell}{a m^2 w^2 h^2}} - 2 \ell^2. \] (22)

Next, for convenience, another auxiliary function is introduced by \( \phi(q) = v^3 F(q) \), with \( \lambda \) being a constant to be determined. Thus we have
\[
\left[ \frac{\partial^2}{\partial q^2} + \left(\mu + (1 - 2\lambda) \frac{\mu}{v}\right) \sqrt{\alpha} \frac{\partial}{\partial q} - \alpha \ell^2 \frac{v^2}{\mu^2} + \alpha [k' + \lambda(\lambda - 1) - \lambda] \left(\frac{v^2}{\mu^2} + \alpha (\epsilon' - 2\lambda)\right) \right] F(q) = 0.
\] (23)

Here, in order to simplify above mathematical expression, we select \( k' + \lambda(\lambda - 1) - \lambda = 0 \), then it leads to the following expression of \( \lambda \)
\[ \lambda = 1 + \sqrt{1 - k'}, \quad \lambda' = 1 - \sqrt{1 - k'}. \] (24)

Since the second solution leads to a non physically acceptable wave function, then equation (23) turns into
\[
\left[ \frac{\partial^2}{\partial q^2} + \left(\mu + (1 - 2\lambda) \frac{\mu}{v}\right) \sqrt{\alpha} \frac{\partial}{\partial q} - \alpha \ell^2 \frac{v^2}{\mu^2} + \alpha (\epsilon' - 2\lambda) \right] F(q) = 0.
\] (25)

Then with the help of another auxiliary function \( F(q) = \mu \zeta(q) \), thus eq. (25) reads
\[
\left[ \frac{\partial^2}{\partial q^2} + \left[(2\ell + 1) \frac{\mu}{\mu} + (1 - 2\lambda) \frac{\mu}{v}\right] \sqrt{\alpha} \frac{\partial}{\partial q} + \alpha (\epsilon' - 2\lambda - 2\lambda \ell) \right] \zeta(q) = 0.
\] (26)

Now we make a variable transformation by demanding \( z = 2\mu^2 - 1 \), where the variable interval is \( z \in (-1, 1) \), then one can obtain
\[
\left\{ (1 - z^2) \frac{\partial^2}{\partial z^2} + \left[ (\ell + 1 - \lambda) - (\ell + 1 + \lambda)z \right] \frac{\partial}{\partial z} + \frac{1}{4} (\varepsilon' - 2\lambda - 2\lambda\ell) \right\} \zeta(z) = 0. \tag{27}
\]

It is important to point out that the wave function we used here will be regular at \( z = \pm 1 \) on the condition that \( \zeta(z) \) is a polynomial, which is obtained by imposing the following constraint \( \frac{1}{4} (\varepsilon' - 2\lambda - 2\lambda\ell) = n(n + \ell + \lambda) \), with \( n \) being a non-negative integer. Then eq. (27) turns into
\[
\left\{ (1 - z^2) \frac{\partial^2}{\partial z^2} + \left[ (\ell + 1 - \lambda) - (\ell + 1 + \lambda)z \right] \frac{\partial}{\partial z} + n(n + \ell + \lambda) \right\} \zeta(z) = 0, \tag{28}
\]
whose solution can be written in terms of Jacobi polynomials as \( \zeta(z) = P_n^{(a,b)}(z) \), where \( a = \lambda - 1, b = \ell \). In this case, the energy eigenvalue of the system can be expressed as
\[
\varepsilon'' = 2\ell^2 + 4n^2 + 4n\ell + \left[ 4n + 2(\ell + 1) \right] \left( 1 + \sqrt{1 - k} \right), \tag{29}
\]
with \( \varepsilon'' = \frac{E^2 + 2m\omega_h c^2 - m_2c^4}{am^2c^2 + m_2c^4} \).

Therefore the energy eigenvalues can be derived from eq. (29) as
\[
E_{n\ell} = \pm mc^2 \left\{ \left( 4n^2 + 4n\ell + 2\ell^2 \right) \frac{a_0^2 c^2 c}{c^2} - \frac{2b_0}{mc^2} - \frac{w^2_\ell}{c^2} + 1 + \frac{2a_0^2 c^2}{c^2} (2n + \ell + 1) \right\}^{1/2} \left( 1 + \ell^2 + \left( \frac{1 + m^2 w^2_\ell}{4b_0^2} \right)^{-2} \right)^{-1/2}, n, \ell = 0, 1, 2, \ldots, \tag{30}
\]
furthermore, the wave function may be expressed by
\[
\psi_1(P) = N e^{i\ell\theta} \left( \frac{1}{1 + aP^2} \right)^{\lambda/2} \left( \frac{\alpha P^2}{1 + \alpha P^2} \right)^{\xi/2} P_n^{(a,b)} \left( \frac{2aP^2}{1 + aP^2} - 1 \right), n, \ell = 0, 1, 2, \ldots, \tag{31}
\]
Then the wave function of the system is
\[
\psi = N \begin{pmatrix}
\frac{1}{E_{n\ell}/mc^2} \\
- \left( \cos \theta + \frac{\theta}{2m} \sin \theta \right) P + h m w (1 + aP^2) \left( \cos \theta \frac{\partial}{\partial P} - \frac{\sin \theta}{P} \frac{\partial}{\partial \theta} \right) / mc \\
- \left( \sin \theta - \frac{\theta}{2m} \cos \theta \right) P + m \omega h (1 + aP^2) \left( \sin \theta \frac{\partial}{\partial P} + \frac{\cos \theta}{P} \frac{\partial}{\partial \theta} \right) / mc \\
0
\end{pmatrix} \psi_1(P, \theta). \]
Now the Jacobi polynomial \([45]\) is employed to obtain the other components wave:

\[
\frac{d\psi^{(ab)}(t)}{dt} = \frac{1}{2} (n + a + b + 1) P^{(a+1,b+1)}_{n-1}(t), \tag{33}
\]

we finally have

\[
\psi_2(P, \theta) = \frac{E_n}{mc^2} \psi_1(P, \theta) = N \frac{E_n}{mc^2} \psi_1(P, \theta) \exp\left(\frac{\theta}{2\hbar} \right) \left(1 + \alpha P^2\right)^{\frac{\ell}{2}} \left(1 + \alpha P^2\right)^{-\frac{1}{2}} P_n(\alpha, \beta) \frac{P_{n-1}^{(a+1,b+1)}}{P_{n-1}(\alpha, \beta)}.
\]

\[
\psi_3(P, \theta) = -\frac{\left(\cos \theta + i \frac{\theta}{2\hbar} \sin \theta\right)P + \hbar m w(1 + \alpha P^2) \left(\cos \theta \frac{\partial}{\partial P} - \sin \theta \frac{\partial}{\partial \theta}\right)}{mc} \psi_1(P, \theta)
\]

\[
\psi_4(P, \theta) = -\frac{\left(\sin \theta - i \frac{\theta}{2\hbar} \cos \theta\right)P + m w h(1 + \alpha P^2) \left(\sin \theta \frac{\partial}{\partial P} + \cos \theta \frac{\partial}{\partial \theta}\right)}{mc} \psi_1(P, \theta)
\]

\[
\psi_5(P, \theta) = 0.
\]  

(34)

After ending this part, we determine the normalization constant \(N\) by demanding

\[
\int_{-\infty}^{\infty} \frac{d^3P}{(1 + \alpha P^2)} \psi(P) \beta^0 \psi(P) = 1, \tag{35}
\]

besides, according to the following property of the Jacobi polynomial

\[
\int_{-1}^{1} dt (1 - t)^a (1 + t)^b \left[P_n^{(a,b)}(t)\right]^2 = \frac{2^{a+b+1} \Gamma(a+n+1) \Gamma(b+n+1)}{n!(a+b+1+2n) \Gamma(a+b+n+1)} \tag{36}
\]

we obtain

\[
N = \frac{\left[mc^2n!(2n+\lambda+\ell)(n+\lambda+\ell)\right]}{E_n \Gamma(n+\lambda+\ell)(n+\ell+1)} \left[\frac{1}{\Gamma(n+\lambda+\ell)(n+\ell+1)}\right]^{\frac{1}{2}}. \tag{37}
\]
then one can obtain the corresponding probability density of every component given by

\[ P_i' = \int_0^\infty \tilde{\psi}_i(p, 9) p^0 \psi_i(p, 9) dp \, d9, \quad i=1,2,3,4,5. \]  

(38)

3. The Problem under a magnetic field

Now, in the presence of an external magnetic field, i.e. \( \vec{A} = \left( -\frac{B_y(NC)}{2}, \frac{B_x(NC)}{2}, 0 \right) \), the Eq. (11) is transformed into

\[
\left[ \beta^0 \mathbf{E} - c \beta^1 \left( p_x^{(NC)} + \frac{eB_y^{(NC)}}{2c} - \text{im} \eta^0 x^{(NC)} \right) - c \beta^2 \left( p_y^{(NC)} - \frac{eB_x^{(NC)}}{2c} - \text{im} \eta^0 y^{(NC)} \right) - mc^2 \right] \vec{\psi} = 0,
\]

(39)

here the spinor \( \vec{\psi} \) is also a vector with five components which reads

\[
\vec{\psi} = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5)^T.
\]

(40)

Substituting \( \vec{\psi} \) into (39) one can obtain

\[
-mc^2 \vec{\psi}_1 + E \vec{\psi}_2 + c \left( p_x^{(NC)} + \frac{eB_y^{(NC)}}{2c} + \text{im} \omega x^{(NC)} \right) \vec{\psi}_3 + c \left( p_y^{(NC)} - \frac{eB_x^{(NC)}}{2c} + \text{im} \omega y^{(NC)} \right) \vec{\psi}_4 = 0,
\]

\[
E \vec{\psi}_1 - mc^2 \vec{\psi}_2 = 0,
\]

\[
c \left( p_x^{(NC)} + \frac{eB_y^{(NC)}}{2c} - \text{im} \omega x^{(NC)} \right) \vec{\psi}_1 + mc^2 \vec{\psi}_3 = 0,
\]

\[
c \left( p_y^{(NC)} - \frac{eB_x^{(NC)}}{2c} - \text{im} \omega y^{(NC)} \right) \vec{\psi}_1 + mc^2 \vec{\psi}_4 = 0,
\]

\[
mc^2 \vec{\psi}_5 = 0.
\]

(41)

Simplifying Eq. (41) gives

\[
\left\{ \left[ 1 + \frac{eB_y^{(NC)}}{4c} \right] \vec{p}_x + \frac{eB_y^{(NC)}}{2c} + \text{im} \omega x^{(NC)} \right\} \left[ \left( 1 + \frac{eB_y^{(NC)}}{4c} \right) \vec{p}_x + \frac{eB_y^{(NC)}}{2c} - \text{im} \omega x^{(NC)} \right] + \left[ \left( 1 + \frac{eB_y^{(NC)}}{4c} \right) \vec{p}_y - \frac{eB_y^{(NC)}}{2c} + \text{im} \omega y^{(NC)} \right] \left[ \left( 1 + \frac{eB_y^{(NC)}}{4c} \right) \vec{p}_y - \frac{eB_y^{(NC)}}{2c} - \text{im} \omega y^{(NC)} \right] + \frac{m^2 c^4 - \vec{E}^2}{c^2} \right\} \vec{\psi}_1 = 0.
\]

(42)
and considering Eq. (16) and Eq. (42) one can obtain

\[
\left\{ \left( 1 + \frac{eB\phi}{4hc} \right)^2 + \frac{m^2w^2\theta^2}{4h^2} - 2\hbar\alpha \left( m\omega \left( 1 + \frac{eB\phi}{4hc} \right) + \frac{eB\phi\omega}{4hc} \right) - \alpha h\ell \left[ \frac{eB}{c} \left( 1 + \frac{eB\phi}{4hc} \right) + \frac{m^2w^2\theta}{h}\right]
\]

\[
2\hbar\alpha \frac{eB\phi\omega}{4hc} \right) \right] p^2 + \left[ m^2w^2 + \left( \frac{eB}{2c} \right)^2 \right] (x^2 + y^2) + \frac{m^2c^4-E^2}{c^2} - 2\hbar \left( m\omega \left( 1 + \frac{eB\phi}{4hc} \right) + \frac{eB\phi\omega}{4hc} \right) -
\]

\[
\hbar\ell \left[ \frac{eB}{c} \left( 1 + \frac{eB\phi}{4hc} \right) + \frac{m^2w^2\theta}{h} - 2\hbar\alpha \frac{eB\phi\omega}{4hc} \right] \right\} \tilde{\psi}_1(P, \theta) = 0. \tag{43}
\]

Now, by a series of analogical algebraic operations, and for the sake of simplification, we just give the results:

the energy eigenvalues of the system are

\[
\tilde{E}_{n\ell} = \pm mc^2 \left\{ \frac{\alpha h^2}{m^2c^2} \left( 4\tilde{n}^2 + 4\tilde{n}\ell + 2\ell^2 \right) \left[ m^2w^2 + \left( \frac{eB}{2c} \right)^2 \right] - \frac{2\hbar}{m^2c^2} \left[ m\omega \left( 1 + \frac{eB\phi}{4hc} \right) + \frac{eB\phi\omega}{4hc} \right] -
\]

\[
\frac{\hbar^2}{m^2c^2} \left[ \frac{eB}{c} \left( 1 + \frac{eB\phi}{4hc} \right) + \frac{m^2w^2\theta}{h} - 2\hbar\alpha \frac{eB\phi\omega}{4hc} \right] + 1 + \frac{2\hbar h}{m^2c^2} \left( \alpha + \sqrt{\alpha^2 - \alpha^2\tilde{K}} \right) \left( m^2w^2 + \left( \frac{eB}{2c} \right)^2 \right) \right\}^{1/2}, \quad \tilde{n}, \ell = 0,1,2,\ldots \tag{44}
\]

where

\[
\tilde{K} = -\ell^2 - \left[ \left( 1 + \frac{eB\phi}{4hc} \right)^2 + \frac{m^2w^2\theta^2}{4h^2} - 2\hbar\alpha \left( m\omega \left( 1 + \frac{eB\phi}{4hc} \right) + \frac{eB\phi\omega}{4hc} \right) - \alpha h\ell \left[ \frac{eB}{c} \left( 1 + \frac{eB\phi}{4hc} \right) + \frac{m^2w^2\theta}{h} - 2\hbar\alpha \frac{eB\phi\omega}{4hc} \right] \right] \left( m^2w^2 + \left( \frac{eB}{2c} \right)^2 \right)^{1/2}. \tag{45}
\]

Figure 1: The energy eigenvalues versus $B$ ($\theta = 0.00003$, $m = 1$).
The wave function can be expressed as

$$\psi_1(P, \theta) = \hat{N} e^{i\theta} \left( \frac{1}{1 + \alpha P^2} \right)_{\frac{\lambda}{2}} \left( \alpha P^2 \right)_{\frac{\ell}{2}} \left[ p_{\hat{n}}^{(\alpha, \beta)} \right] \left( 2\alpha P^2 \right)_{\frac{2\alpha P^2}{1 + \alpha P^2} - 1}, \quad \hat{n}, \ell = 0, 1, 2, 3, 4 \ldots \tag{46}$$

thus the wave function of the system is

$$\psi = \left( \begin{array}{c} 1 \vspace{1em} E_{\hat{n}}/mc^2 \\
\hat{N} - \left\{ (1 + \frac{eB\theta}{4hc}) \cos \theta + \frac{imw\theta}{2\hbar} \sin \theta \right\} P + (1 + \alpha P^2) \left( \frac{ihB}{2c} \frac{\partial}{\partial P_x} + hmw \frac{\partial}{\partial P_y} \right) \right\} \frac{mc}{mc} \psi_1(P, \theta) \tag{47}$$

then the other components wave function of the system are

$$\psi_2(P, \theta) = \frac{E_{\hat{n}}}{mc^2} \psi_1(P, \theta) = \hat{N} \frac{E_{\hat{n}}}{mc^2} e^{i\theta} \left( \frac{1}{1 + \alpha P^2} \right)_{\frac{\lambda}{2}} \left( \alpha P^2 \right)_{\frac{\ell}{2}} \left[ p_{\hat{n}}^{(\alpha, \beta)} \right] \left( 2\alpha P^2 \right)_{\frac{2\alpha P^2}{1 + \alpha P^2} - 1}.$$

$$\overline{\psi_3}(P, \theta) = \frac{1}{mc} \left\{ (1 + \frac{eB\theta}{4hc}) \cos \theta + \frac{imw\theta}{2\hbar} \sin \theta \right\} P + (1 + \alpha P^2) \left( \frac{ihB}{2c} \frac{\partial}{\partial P_x} + hmw \frac{\partial}{\partial P_y} \right) \psi_1(P, \theta)$$

$$= \frac{1}{mc} \left\{ (1 + \frac{eB\theta}{4hc}) \cos \theta + \frac{imw\theta}{2\hbar} \sin \theta \right\} P + \left( \frac{ihB}{2c} \sin \theta + hmw \cos \theta \right) \left( -\lambda' \alpha P + \ell' \right)^{- \frac{beB'}{2c} \left( 1 + \alpha P^2 \right) \frac{\cos \theta}{P} - i\ell' \frac{hmw(1 + \alpha P^2) \sin \theta}{P} \right) \frac{p_{\hat{n}}^{(\lambda', \ell')}}{p_{\hat{n} - 1}^{(\lambda', \ell')}}.$$

$$\psi_4(P, \theta) = \frac{1}{mc} \left\{ (1 + \frac{eB\theta}{4hc}) \sin \theta - \frac{imw\theta}{2\hbar} \cos \theta \right\} P + (1 + \alpha P^2) \left( \frac{-ihB}{2c} \frac{\partial}{\partial P_x} + hmw \frac{\partial}{\partial P_y} \right) \psi_1(P, \theta)$$

$$= \frac{1}{mc} \left\{ (1 + \frac{eB\theta}{4hc}) \sin \theta - \frac{imw\theta}{2\hbar} \cos \theta \right\} P + \left( -\frac{ihB}{2c} \cos \theta + hmw \sin \theta \right) \left( -\lambda' \alpha P + \ell' \right)^{\frac{beB \sin \theta}{2c} \frac{\cos \theta}{P} + hmw \frac{\cos \theta}{P} \ell' \left( 1 + \alpha P^2 \right) \frac{p_{\hat{n}}^{(\lambda', \ell')}}{p_{\hat{n} - 1}^{(\lambda', \ell')}}.$$

$$+ \left( -\frac{ihB}{2c} \cos \theta + hmw \sin \theta \right) \frac{2\alpha P^2}{1 + \alpha P^2} \left( n + \ell + \lambda' \right)^{p_{\hat{n} - 1}^{(\lambda', \ell')}} \right\}.$$
\( \tilde{\psi}_5(P, \theta) = 0. \) \( (48) \)

The normalization constant \( N \) is

\[
N = a^{\frac{1}{2}} \frac{\Gamma (n + \lambda + \ell + 1)}{\Gamma (n + \lambda + 1) \Gamma (n + \ell + 1)},
\]

\( (49) \)

finally, the corresponding probability density of every component can be expressed as

\[
P_i = \int_{0}^{\infty} \int_{0}^{\infty} \tilde{\psi}_i(P, \theta) \psi_i(P, \theta) dp d\theta, \quad i=1,2,3,4,5.
\]

\( (50) \)

4. Special cases and discussions

In this section, the natural unit (\( \hbar = c = m = \omega = 1 \)) is employed. Now, let us check the special cases. First, when the minimal length parameter \( \alpha \to 0 \), the energy relation (44) reduces to

\[
\tilde{\mathcal{E}}_{\tilde{n}, \ell} = \pm mc^2 \left\{ 1 - \frac{2\hbar}{m^2 c^2} \left[ mw \left( 1 + \frac{eB \theta}{4hc} \right) + \frac{eB \theta mw}{4hc} \right] - \frac{\hbar \ell}{m^2 c^2} \left[ \frac{eB}{c} \left( 1 + \frac{eB \theta}{4hc} \right) + \frac{m^2 \omega^2 \theta}{h} \right] + \tilde{\mathcal{K}} \right\}^{\frac{1}{2}},
\]

\( \tilde{n}, \ell = 0, 1, 2, \ldots \)

where

\[
\tilde{\mathcal{K}} = \frac{2\hbar^2}{m^2 c^2} (2\tilde{n} + \ell + 1) \left( m^2 w^2 + \left( \frac{eB}{2c} \right)^2 \right)^{\frac{1}{2}} \left[ \left( 1 + \frac{eB \theta}{4hc} \right)^2 + \frac{m^2 \omega^2 \theta^2}{4h^2} \right] \frac{1}{2}.
\]

\( (51) \)

The energy has plotted the energy eigenvalues versus \( B \) in Figure 1. We see that the energy \( E \) increases monotonically with the magnetic field parameter \( B \) and the tendency of the spectrum can be observed for large numbers. It also shows that for one principal quantum number, the energy \( E \) increases with the increase of the azimuthal quantum number. The energy relation in the special case of \( \theta = 0 \), i.e. for vanishing NC parameter gives
Figure 2: The energy eigenvalues versus \( B(\alpha = 0.05, \ m = 1) \)

Figure 3: The energy eigenvalues versus \( B \) (\( m = 1 \)).

\[
E_{n\ell} = \pm mc^2 \left\{ \frac{\alpha h^2}{m^2c^2} \left( 4\ell^2 + 4\ell \ell' + 2\ell'^2 \right) \left[ m^2w^2 + \left( \frac{eb}{2c} \right)^2 \right]^2 - \frac{2hw}{m^2c^2} - \frac{h\ell(eB-\hbar\omega Bmw)}{m^2c^3} \right\} + 1 + \frac{2\hbar^2}{m^2c^2} (2\ell + \ell' + 1) \left( m^2w^2 + \left( \frac{eb}{2c} \right)^2 \right) \left( \alpha + \alpha\sqrt{1 - \ell^2} \right) \right\}^{\frac{1}{2}},
\]

(53)

where
From the result shown in Figure 2, we see that the energy eigenvalues also increase monotonically with the magnetic field variable $B$, and the profile shows that the energy eigenvalues first has a slow-growth and then rapidly increases. Finally, when both the noncommutative and minimal length parameters are absent, i.e. $\alpha = \theta = 0$, the energy spectrum degrades into

$$E_{\ell\ell'} = \pm mc^2 \left[ 1 - \frac{h\ell eB}{mc^2} - \frac{2hw}{mc^2} + \frac{h(4\ell+2\ell'+2)(w^2+(eB/2mc)^2)^{1/2}}{mc^2} \right]^{1/2},$$

obviously, it is strictly consistent with [46]. In Figure 3, we have depicted the energy values versus $B$. It is also observed that the energy $E$ increases monotonically with the magnetic field parameter $B$, and for one principal quantum number, the energy $E$ increases with the increase of the azimuthal quantum number.

5. Conclusions

This paper is devoted to study of the $(2 +1)$-dimensional Duffin-Kemmer-Petiau oscillator for spin 0 particle under a magnetic field in the presence of a minimal length in the NC space. We first analyze the DKP oscillator in the presence of a minimal length in NC space, by employing the momentum space representation, the energy spectrum are obtained as well as the wave functions and the corresponding probability densities of the system are reported in terms of the Jacobi polynomials. Subsequently, we generalize this quantum model into the framework of a magnetic field and report the corresponding results respectively. Finally, this quantum model for special cases are discussed and the numerical results are depicted respectively. It shows that the energy eigenvalues increase monotonically with the magnetic field variable $B$ for the minimal length parameter and the NC parameter respectively, and for one principal quantum number, the energy $E$ increases with the increase of the azimuthal quantum number.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments
This work is supported by the National Natural Science Foundation of China (Grant nos. 11465006 and 11565009).

References

[1] H. Hassanabadi, Z. Molaee, and S. Zarrinkamar, “Noncommutative Phase Space Schrödinger Equation with Minimal Length,” Advances in High Energy Physics, vol. 2014, no. 6, Article ID 459345, 2014.

[2] A. Boumali and H. Hassanabadi, “Exact solutions of the (2+1)-dimensional Dirac oscillator under a magnetic field in the presence of a minimal length in the noncommutative phase-space,” Zeitschrift Für Naturforschung A, vol. 70, no. 9, pp. 619-627, 2015.

[3] P. Pedram, “Exact ultra cold neutrons’ energy spectrum in gravitational quantum mechanics,” Eur. Phys J.C, vol. 73, no. 5, pp. 1-5, 2013.

[4] H. Hassanabadi, S. Zarrinkamar, and E. Maghsoodi, “Scattering states of Woods–Saxon interaction in minimal length quantum mechanics,” Phys. Lett. B, vol. 718, no. 5, pp. 678-682, 2012.

[5] D. Amati, M. Cialfaloni, and G. Veneziano, “Superstring collisions at planckian energies,” Phys. Lett. B, vol. 197, no. 8, pp. 81-88, 1987.

[6] D. Amati, M. Cialfaloni and G. Veneziano, “Can spacetime be probed below the string size?” Phys. Lett. B, vol. 216, no.7, pp. 41-47, 1989.

[7] D. J. Gross and P. F. Mende, “The high-energy behavior of string scattering amplitudes,” Phys. Lett. B, vol. 197, no. 6, pp. 129-134, 1987.

[8] P. Pedram, Europhysics Letters, “On the modification of Hamiltonians’ spectrum in gravitational quantum mechanics,” vol. 89, no. 5, pp. 50008-50013, 2010.

[9] P. Pedram, K.Nozari, S.H. Taheri, “The effects of minimal length and maximal momentum on the transition rate of ultra cold neutrons in gravitational field,” JHEP, vol. 2011, no. 6, pp. 747-752, 2011.

[10] K. Konishi, G. Paffuti, and P. Provero, “Minimum physical length and the generalized uncertainty principle in string theory,” Phys. Lett. B, vol. 234, no. 10, pp. 276-284, 1990.

[11] S. Benczik, L. N. Chang, D. Minic and T. Takeuchi, “Hydrogen-atom spectrum under a minimal-length hypothesis,” Phys. Rev. A, vol. 72, no.1, pp. 573-573, 2005.
[12] M. Kato, “Particle theories with minimum observable length and open string theory,” Phys. Lett. B, vol. 245, no. 5, pp. 43-47, 1990.

[13] Saurya Das, Elias C Vagenas, and Ahmed Farag Ali, “Discreteness of space from GUP II: Relativistic wave equations,” Phys. Lett. B, vol. 690, no.6, pp. 407-412, 2010.

[14] A. Kempf, G. Mangano, and R. B. Mann, “Hilbert space representation of the minimal length uncertainty relation,” Phys. Rev. D, vol. 52, no. 11, pp. 1108-1118, 1995.

[15] K. Nozari and A. Etemadi, “Minimal length, maximal momentum and Hilbert space representation of quantum mechanics,” Phys. Rev. D, vol. 85, no. 11, pp. 502-512, 2012.

[16] A. Kempf, “Uncertainty relation in quantum mechanics with quantum group symmetry,” J. Math. Phys. vol. 35, no. 14, pp. 4483-4496, 1994.

[17] H. Hinrichsen, Kempf A, “Maximal localization in the presence of minimal uncertainties in positions and in momenta,” J. Math. Phys. vol. 37, no. 17, pp. 2121-2137, 1996.

[18] A. Kempf, “Nonpointlike Particles in Harmonic Oscillators,” J. Phys. A: Math. Gen. vol. 30, no. 6, pp. 2093-2099, 1996.

[19] K. Nouicer, “Path integral for the harmonic oscillator in one dimension with nonzero minimum position uncertainty,” Phys. Lett. A, vol. 354, no. 7, pp. 399-405, 2006.

[20] K. Nozari and T. Azizi, “Gravitational induced uncertainty and dynamics of harmonic oscillator,” Gen. Rel. Grav. vol. 38, no. 7, pp. 325-331, 2006.

[21] F. Brau and F. Buisseret, “Minimal length uncertainty relation and gravitational quantum well,” Phys. Rev. D, vol. 74, no. 1, pp. 307-307, 2006.

[22] M. M. Stesko and V. M. Tkachuk, “Scattering problem in deformed space with minimal length,” Phys. Rev. A, vol. 76, no. 3, pp. 693-695, 2007.

[23] M.V. Battisti and G. Montani, “Quantum dynamics of the Taub universe in a generalized uncertainty principle framework,” Phys. Rev. D, vol. 77, no. 6, pp. 179-184, 2008.
[24] M.V. Battisti, G. Montani, “The Big-Bang singularity in the framework of a Generalized Uncertainty Principle,” Phys. Lett. B, vol. 656, no. 6, pp. 96-101, 2007.

[25] B. Majumder, “Effects of the modified uncertainty principle on the inflation parameters,” Physics Letters B, vol. 709, no. 4, pp. 133-136, 2012.

[26] A. Tawfik and A. Diab, “Effect of the generalized uncertainty principle on compact stars,” International Journal of Modern Physics D, vol. 23, no. 10, pp. 161-170, 2014.

[27] WANG Lun-Zhou, LONG Chao-Yun, LONG Zheng-Wen, “Quantization of Space in the Presence of a Minimal Length,” Commun. Theor. Phys. vol. 63, no. 6, pp. 709-714, 2015.

[28] Bing-Qian Wang, Chao-Yun Longa, Zheng-Wen Long, and Ting Xu, “Solutions of the Schrödinger equation under topological defects space-times and generalized uncertainty principle,” Eur. Phys. J. Plus, vol. 131, no. 10, pp. 378-388, 2016.

[29] H. S. Snyder, “Quantized space-time,” Physical Review, vol. 71, no. 1, pp. 38–41, 1947.

[30] T.Banks, W. Fischler, S. H. Shenker, and L. Susskind, “M theory as a matrix model: a conjecture,” Physical Review D, vol. 55, no. 8, pp. 5112–5128, 1997.

[31] A. E. F. Djemai, “Noncommutative classical mechanics,” International Journal of Theoretical Physics, vol. 43, no. 2, pp. 299–314, 2004.

[32] J. Gamboa, M. Loewe, F. Mendez, and J. C. Rojas, “Noncommutative quantum mechanics,” Physical Review D, vol. 64, no. 6, Article ID 067901, 2001.

[33] J. M. Romero, J. A. Santiago, and J. D. Vergara, “Newton’s second law in a noncommutative space,” Physics Letters. A, vol. 310, no. 1, pp. 9–12, 2003.

[34] M. Daszkiewicz and C. J. Walczyk, “Newton equation for canonical, Lie-algebraic, and quadratic deformation of classical space,” Physical Review D, vol. 77, no. 10, Article ID 105008, 2008.

[35] J. Jing, F. H. Liu, and J. F. Chen, “Classical and quantum mechanics in the generalized noncommutative plane,” EPL, vol. 84, no. 6, Article ID 61001, 2008.

[36] B. Muthukumar and P. Mitra, “Noncommutative oscillators and the commutative limit,” Physical Review D, vol. 66, no. 2, Article ID 027701, 3 pages, 2002.
[37] A. Kijanka and P. Kosinski, “Noncommutative isotropic harmonic oscillator,” Physical Review D, vol. 70, no. 12, Article ID 127702, 2004.

[38] J. Jing, S. H. Zhao, J. F. Chen, and Z.W. Long, “On the spectra of noncommutative 2D harmonic oscillator,” The European Physical Journal C, vol. 54, no. 4, pp. 685–690, 2008.

[39] A. Das, H. Falomir, M. Nieto, J. Gamboa, and F. M´endez, “Aharonov-Bohm effect in a class of noncommutative theories,” Physical Review D, vol. 84, no. 4, Article ID 045002, 2011.

[40] Zhi Wang, Zheng-Wen Long, Chao-Yun Long, and Wei Zhang, “On the Thermodynamic Properties of the Spinless Duffin-Kemmer-Petiau Oscillator in Noncommutative Plane,” Advances in High Energy Physics, vol. 2015, no. 9, Article ID 901675, 2015.

[41] Y. Nedjadi and R. C. Barrett, “The Duffin-Kemmer-Petiau oscillator,” Journal of Physics A: Mathematical and General, vol. 27, no. 12, pp. 4301–4315, 1994.

[42] Y. Nedjadi and R. C. Barrett, “On the properties of the Duffin-Kemmer-Petiau equation,” Journal of Physics G: Nuclear and Particle Physics, vol. 19, no. 1, pp. 87–98, 1993.

[43] H. Hassanabadi, B. H. Yazarloo, S. Zarrinkamar, and A. A. Rajabi, “Duffin-Kemmer-Petiau equation under a scalar Coulomb interaction,” Physical Review C, vol. 84, no. 6, Article ID 064003, 2011.

[44] L. B. Castro and L. P. de Oliveira, “Remarks on the spin-one duffin-kemmer-petiau equation in the presence of nonminimal vector interactions in (3 + 1) dimensions,” Advances in High Energy Physics, vol. 2014, Article ID 784072, 8 pages, 2014.

[45] I. S. Gradshteyn and I. M. Ryzhik, Tables of Integrals, Series and Products Academic, New York, 1980.

[46] A. Boumali, Lyazid Chetouani, and Hassan Hassanabadi, “Two-dimensional Duffin–Kemmer–Petiau oscillator under an external magnetic field,” Can. J. Phys. vol. 91, no. 11, pp. 1-11, 2013.