An Iterative Approach to Twisting and Diverging, Type N, Vacuum Einstein Equations: A (Third-Order) Resolution of Stephani’s ‘Paradox’

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Abstract

In 1993, a proof was published, within this journal, that there are no regular solutions to the linearized version of the twisting, type-N, vacuum solutions of the Einstein field equations. While this proof is certainly correct, we show that the conclusions drawn from that fact were unwarranted, namely that this irregularity caused such solutions not to be able to truly describe pure gravitational waves. In this article, we resolve the paradox—since such first-order solutions must always have singular lines in space for all sufficiently large values of $r$—by showing that if we perturbatively iterate the solution up to the third order in small quantities, there are acceptable regular solutions. That these solutions become flat before they become non-twisting tells us something interesting concerning the general behavior of solutions describing gravitational radiation from a bounded source.

1. Introduction.

The generic behavior of gravitational radiation from a bounded source is clearly an important physical problem. Even reasonably far from that source, however, type-N solutions of the vacuum field equations must have both non-zero twist as well as reasonable asymptotic behavior in order to provide an exact description of that radiation. Such solutions would provide small laboratories to better understand the complete nature of the singularities of type-N solutions, and could also be used to check numerical solutions that include gravitational radiation. It is therefore reasonable that there is considerable interest in this problem. In addition to non-zero values for the twist parameter, interesting solutions must also have appropriate asymptotic behavior; the only currently-known solution to the twisting problem, due to Hauser [1], does not have this asymptotic behavior. As well the definitive relevance of homothetic vectors to the study of metrics of Petrov type N has been very well enunciated by McIntosh [2]. It is the existence of two homothetic
vectors—more exactly, an $H_2$ of symmetries, a non-Abelian group of (local) homothetic vectors for the manifold—that allows the defining equations to be reduced to an ordinary differential equation. By construction, this equation must contain a constant parameter, the homothetic parameter, which McIntosh called $n$. Hauser’s solution has the value of $5/2$ for the McIntosh parameter, $n$.

Although several distinct formulations of this problem already exist, none have yet been able to produce new solutions. We therefore believe that any exact solutions to this problem would be important in obtaining a better understanding of the general problem. Clearly Hans Stephani felt much the same way when he published his proof [3] that all, non-flat, first-order solutions of the twisting, type-N, vacuum Einstein field equations must always contain singular lines, i.e., places on the $\zeta, \bar{\zeta}$-sphere such that for sufficiently large values of $r$, and $u$, the analytic functions contained within the solution must a) be non-constant since otherwise the solution would be flat, and b) therefore must contain poles on that sphere which extend to infinite values of the affine parameter, $r$. Since it is true that such singular behavior is the rule for non-twisting Petrov type N solutions, Stephani then conjectured that all pure, Petrov type N solutions of the vacuum field equations would be insufficient to completely describe the propagation, in vacuum, of gravitational waves from a bounded source. We resolve this apparent paradox, in the discussion below, by defining an algorithm that allows us to extend Stephani’s discussion of the original Einstein field equations to an arbitrary order. We then find that at third-order we can indeed show the existence of a solution that is non-flat (of Petrov type N), twisting, and also regular for sufficiently large spheres.

2. The description of the vacuum field equations:

To most easily compare the equations to other forms, we give our presentation in the second variant of the usual null tetrad formalism as originated by Debney, Kerr, and Schild in Section 4 of their paper [4]; however, for ease of comparison to the work of Stephani, we (mostly) use the same symbols as he does, which come originally from the work of
Kramer, Stephani, MacCallum and Herlt [5]. Therefore we write the metric, \( g \), in terms of a complex null tetrad, \( e^\mu \) as follows:

\[
g = g_{\mu \nu} e^\mu \otimes e^\nu = 2e^1_s \otimes e^2_s + 2e^3_s \otimes e^4_s, \quad \overline{e^1} = e^2, \ e^3, \ e^4 \ \text{real},
\]

(2.1)

and use the overbar for complex conjugation. For any vacuum, type-N space-time with non-vanishing complex expansion, i.e., where \( Z \equiv -\Gamma_{421} \neq 0 \), Debney, Kerr, and Schild showed the existence of local coordinates \( \{ \zeta, \overline{\zeta} \} \), complex, and \( \{ r, u \} \), real, with \( r \) the affine parameter along the radiation trajectories. In terms of these coordinates, they showed that one can always write the null tetrad, \( e^\mu \), so that

\[
e^1 = \frac{1}{PZ} d\zeta, \quad e^2 = \frac{1}{PZ} d\overline{\zeta}, \quad e^3 = du + L d\zeta + \overline{T} d\overline{\zeta}, \quad e^4 = dr + W d\zeta + \overline{W} d\overline{\zeta} - H e^3,
\]

(2.2)

where the metric functions are given by

\[
Z^{-1} = (r - i\Sigma), \quad 2i \Sigma = P^2(\overline{DL} - DL),
\]

\[
W = -\frac{1}{Z} L, \quad D \equiv \partial_\zeta - L \partial_u
\]

(2.3)

\[
H = -r \partial_u (\ln P) + \frac{1}{2} K, \quad K = 2P^2 \text{Re}[D(\overline{DL} - DL)]
\]

where the subscripts indicate partial differentiation. Within this tetrad, setting \( P \equiv V_u \), the remaining Einstein vacuum field equations take the form

\[
\overline{D}\left\{ P^{-1} \partial_u \overline{DDV} \right\} = 0, \quad (2.4a)
\]

\[
\overline{DDDD} = DD\overline{DD} \quad (2.4b)
\]

Contrariwise, we insist that the solutions be non-flat, and have non-zero twist, which insists that both the following two quantities should not vanish:

\[
C^{(1)} \propto \partial_u \partial_u \{ P^{-1} \overline{DDV} \} \neq 0,
\]

\[
2i \Sigma = P^2 (\overline{DL} - DL) \neq 0.
\]

(2.5)
3. The description of the perturbation equations:

To define the perturbation scheme, we first notice [5] that one may always take a
gauge condition designed to maintain the function \( P \) at its simplest value, namely

\[
P = 1 + \frac{1}{2} \zeta \bar{\zeta} .
\]

(3.1)

As well, if we now append the requirement that \( L \equiv L(\zeta, \bar{\zeta}, u) = 0 \), we get exactly the
following form of the flat space-time metric:

\[
ds^2 = 2 \left\{ \frac{r}{1 + \frac{1}{2} \zeta \bar{\zeta}} \right\}^2 \left[ \zeta \otimes s \zeta + 2 \frac{du \otimes dr}{s} + \frac{du \otimes du}{s} \right],
\]

(3.2)

where \( \zeta \) and \( \bar{\zeta} \) simply define the (usual) stereographic coordinates on the complex sphere,
while \( r \) is the radial coordinate and \( u \) is the retarded time coordinate. Our perturbative
scheme can then be developed by thinking of \( L \) as being defined in terms of a repeti-
tive sequence of approximations, involving higher and higher order approximations, and
defining

\[
\Phi \equiv -P^{-1} DDV = L_{\zeta} + \frac{\bar{\zeta}}{1 + \frac{1}{2} \zeta \bar{\zeta}} L - L \partial_u L,
\]

(3.3)
in terms of them. Therefore, Eq. (2.4a) now take the form

\[
D \partial_u \Phi = 0 = \partial_{\zeta} \partial_u \Phi - L \partial_u^2 \Phi,
\]

(3.4)

It is now obvious that \( L^{(0)} = 0 \) implies that \( \Phi^{(0)} = 0 \), so that we may start with the
lowest-order approximation, denoted in this form. To find the next step—the first-order
approximation—we suppose that both \( L \) and \( \Phi \) may be expanded into series, and take
Eq. (3.3) as defining \( \Phi \) for us in terms of \( L \), and then take Eq. (3.4) as the constraint
which defines \( \Phi \) within the next order, so that we immediately have the constraint

\[
\partial_{\zeta} \partial_u \Phi^{(1)} = 0 \implies \Phi^{(1)} = \alpha^{(1)}(\zeta, u) + \beta^{(1)}(\zeta, \bar{\zeta}) .
\]

(3.5)
Inserting this expression back into Eq. (3.3), we obtain a defining relation for \( L^{(1)} \), namely
\[
\frac{\partial \zeta}{1 + \frac{1}{2} \zeta} L^{(1)} = \Phi^{(1)} + L^{(0)} \partial_u L^{(1)} + L^{(1)} \partial_u L^{(0)} = \alpha^{(1)}(\zeta, u) + \beta^{(1)}(\zeta, \overline{\zeta}) . \quad (3.6)
\]

The general solution of this equation, for \( L^{(1)} \), is given by
\[
L^{(1)} = (1 + \frac{1}{2} \zeta \overline{\zeta} )^{-2} \left\{ f^{(1)}(\zeta, u) + \int d\zeta (1 + \frac{1}{2} \zeta \overline{\zeta})^2 \left[ \alpha^{(1)}(\zeta, u) + \beta^{(1)}(\zeta, \overline{\zeta}) \right] \right\} . \quad (3.7)
\]

This allows us to write, now, the second-order steps in the form
\[
\frac{\partial \zeta}{1 + \frac{1}{2} \zeta} L^{(2)} = \Phi^{(2)} + L^{(1)} \partial_u L^{(1)} . \quad (3.8)
\]

The general solution of these equations is then easily given in the form
\[
\Phi^{(2)} = \int du \int d\zeta \frac{L^{(1)}}{\partial_u} \partial^2 \Phi^{(1)} ,
\]
\[
L^{(2)} = (1 + \frac{1}{2} \zeta \overline{\zeta} )^{-2} \left\{ f^{(2)}(\zeta, u) + \int d\zeta (1 + \frac{1}{2} \zeta \overline{\zeta})^2 \left[ \Phi^{(2)} + L^{(1)} \partial_u L^{(1)} \right] \right\} . \quad (3.9)
\]

At the \( n \)-th level, the equations to be solved are then just
\[
\frac{\partial \zeta}{1 + \frac{1}{2} \zeta} \Phi^{(n)} = \sum_{j=1}^{n-1} b^{(n-j)} \partial^2 \Phi^{(j)} ,
\]
\[
\frac{\partial \zeta}{1 + \frac{1}{2} \zeta} L^{(n)} = \Phi^{(n)} + \sum_{j=1}^{n-1} L^{(n-j)} \partial_u L^{(j)} . \quad (3.10)
\]

The general (\( n \)-th order) solution of the equations for \( \Phi \) and \( L \) are then given by the series, up to the \( n \)-th order, of the \( \Phi^{(j)} \) and the \( L^{(j)} \), determined as
\[
\Phi^{(n)} = \sum_{j=1}^{n-1} \int du \int d\zeta \frac{L^{(n-j)}}{\partial_u} \partial^2 \Phi^{(j)} ,
\]
\[
L^{(n)} = (1 + \frac{1}{2} \zeta \overline{\zeta} )^{-2} \left\{ f^{(n)}(\zeta, u) + \int d\zeta (1 + \frac{1}{2} \zeta \overline{\zeta})^2 \left[ \Phi^{(n)} + \sum_{j=1}^{n-1} L^{n-j} \partial_u L^{(j)} \right] \right\} . \quad (3.11)
\]

It now remains to consider Eq. (2.4b) which, in the current notation, can be written
\[
\text{Im} (DDP \Phi) = 0 . \quad (3.12)
\]
Expanding out the terms in $\mathcal{D}$, we obtain

$$P \text{Im} \left\{ \partial_\zeta \partial_\zeta \Phi + \frac{\zeta}{1 + \frac{1}{2} \zeta \zeta} \partial_\zeta \Phi - \mathcal{L} \partial_u \partial_\zeta \Phi - \mathcal{L} (\partial_u \partial_\zeta \Phi - \mathcal{L} \partial_u^2 \Phi) ight\} = 0 .$$  \hspace{1cm} (3.13)

However, we can use Eq. (3.3) and Eq. (3.4) to re-write substantially the terms in the above equation, giving us

$$\text{Im} \left\{ \partial_\zeta \partial_\zeta \Phi + \frac{\zeta}{1 + \frac{1}{2} \zeta \zeta} \partial_\zeta \Phi - \mathcal{L} \partial_u \Phi - \mathcal{L} \partial_u \partial_\zeta \Phi \right\} = 0 . \hspace{1cm} (3.14)$$

Therefore, applying the same procedures as we have been using so far, for a constraint on the $n$-th iterative step, we find the linear, inhomogeneous, pde for $\Phi^{(n)}$, namely

$$\text{Im} \left\{ \partial_\zeta \partial_\zeta \Phi^{(n)} + \frac{\zeta}{1 + \frac{1}{2} \zeta \zeta} \partial_\zeta \Phi^{(n)} + \mathcal{L} \partial_u \Phi^{(n)} + \mathcal{L} \partial_u \partial_\zeta \Phi^{(n)} \right\} = 0 . \hspace{1cm} (3.15)$$

This equation surely does have solutions, and we may conclude that our iterative procedure is indeed complete. Within the $n$-th order approximation, one has that

$$\Phi \approx \sum_{j=1}^{n} \Phi^{(n)} , \hspace{0.5cm} L \approx \sum_{j=1}^{n} L^{(n)} . \hspace{1cm} (3.16)$$

4. The first few approximate solutions

Returning, first, to the linearized, or first-order approximation, we of course have that the solution is given by Eq. (3.5) and Eq. (3.7), where $\alpha^{(1)} = \alpha^{(1)}(\zeta, u)$ and $f^{(1)} = f^{(1)}(\zeta, u)$, while the function $\beta^{(1)} = \beta^{(1)}(\zeta, \zeta)$ is allowed to be an solution of the equation

$$\text{Im} \left\{ \partial_\zeta \partial_\zeta \beta^{(1)} + \frac{\zeta}{1 + \frac{1}{2} \zeta \zeta} \partial_\zeta \beta^{(1)} \right\} = 0 . \hspace{1cm} (4.1)$$

Defining, now, the function $E = E(\zeta, u)$ such that

$$\partial_\zeta^3 E(\zeta, u) = \alpha^{(1)}(\zeta, u) , \hspace{1cm} (4.2)$$
we can quickly re-write $L^{(1)}$ in the form

$$L^{(1)} = B(\zeta, \bar{\zeta}) + \frac{C(\zeta, u)}{(1 + \frac{1}{2} \zeta \bar{\zeta})^2} + \frac{\zeta^2 E(\zeta, u)}{2(1 + \frac{1}{2} \zeta \bar{\zeta})^2} - \frac{\zeta \partial_\zeta E(\zeta, u)}{1 + \frac{1}{2} \zeta \bar{\zeta}} + \partial_\zeta^2 E(\zeta, u),$$

(4.3)

where $B(\zeta, \bar{\zeta}) \equiv \left(\frac{1}{(1 + \frac{1}{2} \zeta \bar{\zeta})^2}\right)^2 \int d\zeta \ (1 + \frac{1}{2} \zeta \bar{\zeta})^2 \beta^{(1)}(\zeta, \bar{\zeta}),$

and we have denoted our $f^{(1)}(\bar{\zeta}, u)$ by Stephani’s symbol $C(\bar{\zeta}, u)$, while we have used $E(\zeta, u)$ to denote his $D(\zeta, u)$. With these identifications, this is exactly the solution given by Stephani in [3]. Insisting that the solution be non-flat, we infer that it is necessary for the existence of a non-trivial, linearized Petrov type N solution that we must have

$$\partial_u^2 \partial_\zeta^3 \bar{E} \neq 0.$$  

(4.4)

However, as has already been pointed out by Stephani [3], this can happen if and only if the field is singular at some point on the $\zeta, \bar{\zeta}$-sphere. Every non-singular field defines a flat spacetime, only, at this level of approximation.

To see this in some more detail, we note, still following Stephani, that the invariants available to us are

$$J_1 \approx J_1^{(1)} = \frac{P^2}{2ir} \left\{ \partial_\zeta L^{(1)} - \partial_i \bar{L}^{(1)} \right\},$$

$$J_2 \approx J_2^{(0)} + J_2^{(1)} = \frac{1}{r^2} \left\{ 1 - P^2(\partial_\zeta \partial_u L^{(1)} + \partial_i \partial_u L^{(1)}) \right\}.$$  

(4.5)

Differentiating the first of these with respect to $u$, and comparing the result with the second, one easily concludes that if both of these are regular on the $\zeta, \bar{\zeta}$-sphere, then it must also be true that $P^2 \partial_\zeta \partial_u L^{(1)}$ must be regular there. Employing our equation, Eq. (4.3), for $L^{(1)}$, we easily calculate that this quantity is given by

$$P^2 \partial_\zeta \partial_u L^{(1)} = -\frac{\zeta}{1 + \frac{1}{2} \zeta \bar{\zeta}} \partial_u C(\bar{\zeta}, u) + \partial_\zeta \partial_u C(\bar{\zeta}, u) + \frac{\bar{\zeta}}{1 + \frac{1}{2} \zeta \bar{\zeta}} \partial_u E(\zeta, u) - \partial_\zeta \partial_u E(\zeta, u).$$  

(4.6)

We easily infer that this quantity is, then, regular on the sphere, if and only if the functions $C$ and $D$ are of the form

$$C(\bar{\zeta}, u) = \bar{\zeta} r(u) + \sigma(u) + \rho(\bar{\zeta}),$$

$$E(\zeta, u) = \zeta \delta(u) + \omega(u) + h(\zeta).$$  

(4.7)
However, the form given in Eq. (4.7) for $E(\zeta, u)$ would obviously lead to the space-time being flat, rather than, non-trivially, of Petrov type N. This result, given by Stephani [3], seems to lead to the following conclusion, which we refer to as Stephani’s ‘paradox,’ namely

So either twisting type N fields do not describe a radiation field outside a bounded source, or due to a mechanism not yet recognized, they cannot be linearized, or the naive interpretation of the coordinates (starting from their Newtonian limit) is wrong.

The most catastrophic of these options is the first possibility, i.e., that “twisting, type N fields do not describe a radiation field outside a bounded source.” In the next section, we find a solution for the twisting, type N field, at the third step of iteration. Our solution is in fact regular on the $\zeta, \bar{\zeta}$-sphere. This allows us to resolve the catastrophic argument cited above, by saying that it is indeed not true.

5. Example of a regular solution at the third level of approximation.

Referring back to Eq. (4.3) for $L^{(1)}$, we choose it to have the very simple form

$$L^{(1)} = \frac{a^{(1)}}{\zeta}, \quad \Rightarrow \quad \Phi^{(1)} = \frac{a^{(1)}}{\zeta} \left\{ \frac{\bar{\zeta}}{1 + \frac{1}{2} \zeta \bar{\zeta}} - \frac{1}{\zeta} \right\}, \quad a^{(1)} \text{ a complex constant.} \quad (5.1)$$

It is straightforward to check that this solution does indeed satisfy Eq. (3.15) for $n = 1$. Now, we may choose—see Eq. (3.9)—the second-order quantities to be given by

$$L^{(2)} = \frac{a^{(2)}}{\zeta} + \frac{\bar{\zeta} f^{(2)}(u)}{(1 + \frac{1}{2} \zeta \bar{\zeta})^2}, \quad \Rightarrow \quad \Phi^{(2)} = \frac{a^{(2)}}{\zeta} \left\{ \frac{\bar{\zeta}}{1 + \frac{1}{2} \zeta \bar{\zeta}} - \frac{1}{\zeta} \right\}. \quad (5.2)$$

Once again, it is evident that the constraint Eq. (3.15) is satisfied for $n = 2$. Now, using the $n$-th order equation for $n = 3$, we find that

$$L^{(3)} = \frac{1}{(1 + \frac{1}{2} \zeta \bar{\zeta})^2} \left\{ \bar{\zeta} f^{(3)}(u) + \int d\zeta \left[ (1 + \frac{1}{2} \zeta \bar{\zeta})^2 \Phi^{(3)} + \frac{\bar{\zeta} a^{(1)} d^2 f^{(2)}(u)}{du^2} \right] \right\}. \quad (5.3)$$
We may then set $a^{(3)}$ as yet another complex constant, and put

$$\phi^{(3)} = -\frac{a^{(1)}}{\zeta^2} \frac{d^2 f^{(2)}(u)}{du^2} + \frac{a^{(3)}}{\zeta} \left\{ \frac{\bar{\zeta}}{1 + \frac{1}{2} \zeta \bar{\zeta}} - 1 \right\}.$$  \hfill (5.4)

Once again, we may substitute this value for $\phi^{(3)}$ into Eq. (5.3) for $L^{(3)}$, which gives us its form:

$$L^{(3)} = \frac{\bar{\zeta} f^{(3)}(u)}{(1 + \frac{1}{2} \zeta \bar{\zeta})^2} + \frac{a^{(1)}}{\zeta} \frac{d^2 f^{(2)}(u)}{du^2} \frac{1 - \frac{1}{2} \zeta \bar{\zeta}}{1 + \frac{1}{2} \zeta \bar{\zeta}} + \frac{a^{(3)}}{\zeta}.$$  \hfill (5.5)

Inserting, as before, these values into Eq. (3.15), we find that they satisfy that equation for $n = 3$.

We must now determine the values for the twist and the curvature that go along with this, third-order approximation for the complete solution. We have, immediately, that the twist, $\Sigma$, is given by

$$\Sigma^{(0)} = 0,$$

$$\Sigma^{(1)} = \frac{1}{2i} P^2 \left( \partial_u L^{(1)} - \partial_{\bar{u}} L^{(1)} \right) = 0,$$

$$\Sigma^{(2)} = \frac{1}{2i} P^2 \left( \partial_u L^{(2)} - \partial_{\bar{u}} L^{(2)} \right) = \frac{1 - \frac{1}{2} \zeta \bar{\zeta}}{1 + \frac{1}{2} \zeta \bar{\zeta}} \text{Im}(f^{(2)}(u)), \hfill (5.6)$$

$$\Sigma^{(3)} = \frac{1}{2i} P^2 \left( \partial_u L^{(3)} - L^{(1)} \partial_u L^{(2)} - \partial_{\bar{u}} L^{(3)} + L^{(1)} \partial_{\bar{u}} L^{(2)} \right)$$

$$= \frac{1 - \frac{1}{2} \zeta \bar{\zeta}}{1 + \frac{1}{2} \zeta \bar{\zeta}} \text{Im}(f^{(3)}(u)) - 2[\text{Re}(a^{(1)})] \text{Im} \left( \frac{df^{(2)}(u)}{du} \right),$$

along with the iterated values for the contributions to the (2-dimensional) curvature, $K$:

$$K^{(0)} = 2P^2 \text{Re} (\overline{DD \ln P}) = 1,$$

$$K^{(1)} = -2P^2 \text{Re}(\partial_u \overline{L^{(1)}}) = 0,$$

$$K^{(2)} = -2P^2 \text{Re}(\partial_u \overline{L^{(2)}}) = -2 \frac{1 - \frac{1}{2} \zeta \bar{\zeta}}{1 + \frac{1}{2} \zeta \bar{\zeta}} \text{Re} \left( \frac{df^{(2)}(u)}{du} \right), \hfill (5.7)$$

$$K^{(3)} = -2P^2 \text{Re}(\partial_u \overline{L^{(3)}} - L^{(1)} \partial_u^2 L^{(2)}) = -2 \frac{1 - \frac{1}{2} \zeta \bar{\zeta}}{1 + \frac{1}{2} \zeta \bar{\zeta}} \text{Re} \left( \frac{df^{(3)}(u)}{du} \right)$$

$$+ 4 [\text{Re}(a^{(1)})] \text{Re} \left( \frac{d^2 f^{(2)}(u)}{du^2} \right),$$
so that the total, 4-dimensional curvature, \( \bar{C}^{(1)} \equiv 2\Psi_4 \), is given iteratively by

\[
\Psi_4^{(0)} = 0, \quad \Psi_4^{(1)} = 0, \quad \Psi_4^{(2)} = 0, \\
\Psi_4^{(3)} = \frac{a^{(1)}}{r} \left( 1 + \frac{1}{2} \xi \zeta \zeta \right)^2 \frac{d^3 f^{(2)}(u)}{du^3}.
\]  

(5.8)

Bringing together the sum of all these quantities, to write the complete solution valid up through the third iteration step, we have

\[
\Phi \approx \frac{a_1}{\zeta} \left( \frac{\zeta}{1 + \frac{1}{2} \zeta \zeta} - 1 \right) - \frac{a_1}{\zeta^2} \frac{df_2(u)}{du}, \\
L \approx \frac{a_1}{\zeta} + \frac{\zeta}{(1 + \frac{1}{2} \zeta \zeta)^2} f_2(u) + \frac{1 - \frac{1}{2} \zeta \zeta a_1}{1 + \frac{1}{2} \zeta \zeta} \frac{df_2(u)}{du}, \\
\Sigma \approx \frac{1 - \frac{1}{2} \zeta \zeta}{1 + \frac{1}{2} \zeta \zeta} \text{Im}[f_2(u)] - 2\text{Re}(a_1)\text{Im} \left( \frac{df_2(u)}{du} \right); \\
K \approx 1 - 2 \frac{1 - \frac{1}{2} \zeta \zeta}{1 + \frac{1}{2} \zeta \zeta} \text{Re} \left( \frac{df_2(u)}{du} \right) + 4[\text{Re}(a_1)]\text{Re} \left( \frac{d^2 f_2(u)}{du^2} \right),
\]  

(5.9)

along with the curvature itself,

\[
\Psi_4 \approx \frac{a_1}{r} \left( 1 + \frac{1}{2} \xi \zeta \zeta \right)^2 \frac{d^3 f_2(u)}{du^3},
\]  

(5.10)

where we have defined

\[
a_1 \equiv a^{(1)} + a^{(2)} + a^{(3)}, \quad f_2(u) \equiv f^{(2)}(u) + f^{(3)}(u).
\]  

(5.11)

In order to actually determine the metric itself, we must, lastly, determine the function, \( W \), which, to this level of iteration, is given by

\[
W \approx -\frac{\zeta}{(1 + \frac{1}{2} \zeta \zeta)^2} \left( r \frac{df_2(u)}{du} + i\text{Im} f_2(u) \right) - \frac{a_1}{\zeta} \frac{1 - \frac{1}{2} \zeta \zeta}{1 + \frac{1}{2} \zeta \zeta} \left( r \frac{d^2 f_2(u)}{du^2} + i\text{Im} \frac{df_2(u)}{du} \right).
\]  

(5.12)

Inserting all this into the equation for the metric, \( g \), itself, we see that the metric does indeed appear to be everywhere regular on the \( \zeta, \bar{\zeta} \)-sphere. To completely show this, one needs only to change the coordinates in a neighborhood of the north pole, according to the usual rule, \( \zeta' \equiv 1/\zeta \), and \( \bar{\zeta}' \equiv 1/\bar{\zeta} \), which causes no trouble at all.
Gathering all our results together, we conclude that indeed one can find a **regular**, **twisting**, **non-flat**, **Petrov type N**, **vacuum** metric that is regular on the $\zeta, \bar{\zeta}$-sphere in the third order of approximation. Consequently it seems that the twisting, type N fields **can** describe a radiation field outside a bounded source. Of course it is quite interesting to determine what happens in the next iteration steps. We intend to consider this question soon.

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