Non-invasive stabilization of periodic orbits in $O_4$-symmetrically coupled Van der Pol oscillators

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Abstract

Pyragas time delayed feedback control has proven itself as an effective tool to non-invasively stabilize periodic solutions. In a number of publications, this method was adapted to equivariant settings and applied to stabilize branches of small periodic solutions in systems of symmetrically coupled Landau oscillators near a Hopf bifurcation point. The form of the control ensures the non-invasiveness property, hence reducing the problem to finding a set of the gain matrices, which would guarantee the stabilization. In this paper, we apply this method to a system of Van der Pol oscillators coupled in a cube-like configuration leading to $O_4$-equivariance. We discuss group theoretic restrictions which help to shape our choice of control. Furthermore, we explicitly describe the domains in the parameter space for which the periodic solutions are stable.

Keywords: Time-delayed feedback, Pyragas control, equivariant Hopf bifurcation, non-invasive control, spatio-temporal symmetries, coupled oscillators.

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1. Introduction

Stabilization of unstable periodic solutions is a classical control problem. A control is called non-invasive if the controlled system has the same periodic solution as the uncontrolled system. An elegant method of non-invasive control due to Pyragas [1] is based on using a delayed phase variable. This control strategy suggests to transform an uncontrolled ordinary differential system

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$$\dot{x} = F(x), \quad x \in \mathbb{R}^N,$$
into the delayed differential system

\[ \dot{x} = F(x) + \mathcal{K}(x(t) - x(t - \tau)), \]

where \( \mathcal{K} \) is the gain matrix. Obviously, if the delay \( \tau \) equals the period \( T_* \) of a periodic solution \( x_* = x_*(t) \) to equation (1), then \( x_* \) is simultaneously a solution of the controlled equation (2), since the control term \( \mathcal{K}(x(t) - x(t - \tau)) \) vanishes on such solution. At the same time, Floquet multipliers of \( x_* \) are different for the delayed and non-delayed equations, which may allow for stabilization with a proper choice of the gain matrix \( \mathcal{K} \). Note that typically the period \( T_* \) of \( x_* \) is not known \textit{a priori}. However, a stable periodic solution \( x \) to (2) can usually be obtained for a range of delays \( \tau \) sufficiently close to \( T_* \). Further tuning of the delay until the period \( T \) of \( x \) coincides with the delay \( \tau \) can be used to achieve the non-invasive control.

A modification of the Pyragas control method that adapts it to symmetric (equivariant) setting has been developed in [2, 3, 4, 5]. Periodic solutions of symmetric systems come in orbits (generated by the action of the symmetry group \( G \) of the system) and can be classified according to their symmetric properties. In particular, every periodic solution is fixed by a specific subgroup \( H \) of the full group \( G \times S^1 \) of spatio-temporal symmetries. The control strategy proposed in [2, 3, 4, 5] is \textit{selective} in the sense that it acts non-invasively on periodic solutions with a specified period and symmetry group including a given element while deforming or eliminating other periodic solutions. As a simple example, a control

\[ \mathcal{K}(x(t) + x(t - \tau/2)) \]

(3)
can be used for non-invasive stabilization of \( \mathbb{Z}_2 \)-symmetric anti-periodic solutions \( x(t) = -x(t - T/2) \) of period \( T = \tau \), but this control does not vanish on \( \tau \)-periodic solutions that are not anti-periodic.

Since, in general, stability analysis of periodic solutions to delay differential equations, based on the usage of Floquet theory, is not well explored, in [6] it was suggested that complete stability analysis can be performed in the case of periodic solutions born via Hopf bifurcation. Following [6] stability analysis for systems of symmetrically coupled Landau oscillators (the Landau oscillator is equivalent to the normal form of the Hopf bifurcation) was carried out in [2, 3, 4, 5], and essentially exploits the idea outlined below. To conclude stability of a bifurcating branch of periodic solutions from the well-known exchange of stability results, it is enough to check the following three conditions:

(i) At the bifurcation point, the equilibrium is neutrally stable with neutral dimension two (genericity);
(ii) The purely imaginary eigenvalues of the equilibrium cross the imaginary axis transversally;

(iii) The branch bifurcates in the direction in which the equilibrium becomes unstable. In the case of coupled Landau oscillators, this is simple to check since the periodic solutions are explicitly given.

For large dimensional delayed systems, stability of the equilibrium can be difficult to verify. However, any application of the above control strategy to a specific symmetric system relies on the choice of one or several gain matrices. Since there is no general recipe for constructing those matrices, a possible approach to simplify analysis is to select a class of matrices depending on a small number of parameters. In particular, one can attempt to use diagonal (or block-diagonal) gain matrices, which allows one to factorize the characteristic quasipolynomial.

In equivariant settings, it is usually the case that the genericity condition (i) is violated. In [2, 3] the control (3) is generalized to

\[ \mathcal{K}(T_g x(t - 2\pi \theta \tau) - x(t)), \]  

where \( T_g \) is the matrix associated with the single spatial group element \( g \) and \( 2\pi \theta \tau \) is a rational fraction of the period which “compensates” the action of \( g \) on the selected periodic solution. Control (11) breaks the symmetry (in particular, it is not non-invasive on the whole orbit of the targeted UPO) and makes (i) possible to achieve for the controlled system. At the same time, as was highlighted in [3], for certain groups, (i) can never be achieved by (11). On the other hand, [7] suggested a general class of selective non-invasive equivariant Pyragas controls by taking a linear combination of controls of form (4), where \((g, \theta)\) varies amongst several group elements.

In this paper, as a case study, we consider Hopf bifurcations in a system of 8 coupled Van der Pol oscillators arranged in a cubic connection with a relatively complex group of permutational symmetries, \( G = \mathbb{Z}_2 \times O_4 \). This system possesses one stable and 55 unstable branches of periodic solutions, which emanate from the zero equilibrium at 4 bifurcation points as a bifurcation parameter \( \alpha \) is varied. The branches can be classified into 12 types of spatio-temporal symmetries, which have been described in [8] using the equivariant topological degree method presented in [9]. We adapt one class of controls presented in [7] with the objective to stabilize small periodic solutions from each branch using a selective control with the corresponding symmetry. We consider linear combinations of controls (11) where we choose the values of \((g_k, \theta_k)\) from the symmetry group of the targeted unstable periodic solution in such a way that \( \theta_k \) is constant. Also, we choose each \( \mathcal{K}_k \) to be the same real scalar matrix with one scalar tuning parameter—the control strength \( b \); another parameter is the coupling strength \( a \) in the uncontrolled system. It turns out that these controls
are sufficient for stabilizing unstable branches of all symmetry types except for one. Moreover, we obtain explicit expressions for stability domains in the \((a,b)\)-plane for each stabilizable branch. In Remark 3.4 we discuss a group-theoretic obstruction to this method and how this affects the branches which the chosen control fails to stabilize (cf. \[3\]).

Unlike systems of Landau oscillators, the system of Van der Pol oscillators does not yield an explicit expression for periodic solutions in the form of relative equilibria. However, this does not create extra difficulties, since the proofs are based on asymptotic analysis. The proofs follow the general scheme from \[6\].

The paper is organized as follows. In the next section, we describe symmetries of branches of periodic solutions for the system of interest and establish that all the four Hopf bifurcations giving rise to these branches are supercritical. Main results on stabilization of unstable branches by selective equivariant delayed control are presented in Section 3. Sections 4 and 5 contain proofs and conclusions. The symbols representing spatio-temporal symmetry groups are explained in the Appendix.

2. Uncontrolled system

In this paper, we consider the system of coupled Van der Pol oscillators

\[
\ddot{x} = (\alpha - x^2)\dot{x} - x + \frac{a}{2}\mathcal{B}\dot{x},
\]

(5)

where \(x \in W := \mathbb{R}^8\); \(\alpha\) is the bifurcation parameter, and the interaction matrix has the form\footnote{The system from \[8\] describing an electrical circuit of coupled oscillators can be reduced to \(\mathcal{B}\) by standard rescaling.}

\[
\mathcal{B} = \begin{pmatrix}
-3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -3 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -3 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & -3 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & -3 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & -3 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & -3 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & -3 \\
\end{pmatrix}
\]

The parameter \(a\) measures the coupling strength. In what follows, \(V := W \oplus W\) stands for the phase space; also, the notation \(x^3 = x \cdot x \cdot x\) is used for componentwise multiplication, \((x \cdot y)_j = x_j y_j\).
Table 1: Branches of solutions at Hopf bifurcation points

| Bifurcation point | The group $H^\varphi$ of spatio-temporal symmetries of the branch | Total number of branches at the bifurcation point $\alpha$ |
|-------------------|-------------------------------------------------------------|-----------------------------------------------------|
| $\alpha = 0$      | $^+ (S_4)$                                                  | 1                                                   |
| $\alpha = a$      | $(-D_4^j), (-D_3^j), (-D_2^j), (-Z_4^j), (-Z_3^j)$         | 27                                                  |
| $\alpha = 2a$     | $^+ D_4^j, (+D_3^j), (+D_2^j), (+Z_4^j), (+Z_3^j)$         | 27                                                  |
| $\alpha = 3a$     | $(-S_4^-)$                                                  | 1                                                   |

$j = 1, \ldots, 8$. For future reference, we denote the right hand side of (5) by $f(\alpha, a, x, \dot{x})$, which will allow us to use the notation

$$\ddot{x} = f(\alpha, a, x, \dot{x}) \quad (6)$$

In [8], system (5) was treated as an $S_4$-equivariant system, where $S_4 < S_8$ is the group of permutational symmetries of the cube preserving the orientation. If we include orientation reversing symmetries of the cube, this increases to $O_4 = S_4 \times \mathbb{Z}_2$. Noticing also that the right hand side of (5) is an odd function (i.e. it is equivariant with respect to $\mathbb{Z}_2$ acting antipodally), in this paper we consider system (5) with the full symmetry group $\mathbb{Z}_2 \times O_4$. Each element $(r, g) \in \mathbb{Z}_2 \times O_4$ is composed of $r = \pm 1$ and a permutation $g$ of 8 symbols. We will denote by $\mathcal{T}_g : W \rightarrow W$ the permutation matrix of $g$.

The spatio-temporal symmetries of a periodic function $x(t)$ are described by a subgroup $H < \mathbb{Z}_2 \times O_4$ and a homomorphism $\varphi : H \rightarrow S^1 \simeq \mathbb{R}/\mathbb{Z}$. This information is encoded in the graph of the homomorphism $\varphi$ which we will denote by $H^\varphi$. Put plainly, if $x(t)$ is a periodic function with period $T$ and symmetry group $H^\varphi$, then for each $(r, h) \in H$,

$$r \mathcal{T}_h x(t - \varphi(r, h)T) = x(t) \quad (7)$$

As it was shown in [8], system (5) undergoes 4 equivariant Hopf bifurcations giving rise to at least 56 branches of periodic solutions exhibiting different symmetry properties. Combining this result with the additional symmetry mentioned above allows us to describe the full symmetries of each branch (see Table 1 and Appendix for an explicit description of the groups listed in the second column).
To illustrate the meaning of these symmetries, let us take as an example the group
\[ \text{Group } -\mathbb{Z}_3^t = \{(1,0), (1,245)(386), 1/3), (1,254)(368), 2/3),
(1,17)(28)(35)(46), 0), (1,256843), 1/3),
(1,17)(234856), 2/3), (1,0), 1/2), (1,245)(386), 5/6),
(1,254)(368), 1/6), (1,17)(28)(35)(46), 1/2),
(1,17)(265843), 5/6), (1,17)(234856), 1/6) \}.

Suppose that \( x(t) \) is a \( T \)-periodic function admitting the spatio-temporal symmetry \( -\mathbb{Z}_3^t \). Then, the components of \( x \) respect certain relations. For example, for the element \((r,h,\varphi(r,h)) = (-1,17)(265843), 1/3) \in -\mathbb{Z}_3^t\), we have
\[
R_h = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
\]

and, according to (7),
\[
x_1(t) = -x_7(t - T/3) = x_1(t - 2T/3) = -x_7(t) = x_1(t - T/3) = -x_7(t - 2T/3),
x_2(t) = -x_6(t - T/3) = x_5(t - 2T/3) = -x_8(t) = x_4(t - T/3) = -x_3(t - 2T/3).
\]

The following statement plays an important role for the control problem.

**Theorem 2.1.** All branches described in Table 1 are born via supercritical Hopf bifurcations.

The proof can be obtained by combining a standard asymptotic argument with \( H \)-fixed point reduction. We sketch the proof for convenience of the reader.

**Proof:** Notice that due to equivariance, the space of \( H \)-fixed points
\[ V^H := \{ x \in V : h x = x \ \forall h \in H \} \]
is a flow invariant subspace of the phase space for any \( H < \mathbb{Z}_2 \times O_4 \). If \( x^* \) is a periodic solution with symmetry group \( H^\varphi \), then \( x^*(t) \in V^\phi H^\varphi \) for all \( t \) (cf. (8)), where
\[ \phi H^\varphi = \text{Ker} \varphi. \]
For each \( H^\varphi \) appearing in Table 1, system \([5]\) restricted to \( V_{aH^\varphi} \) undergoes a (non-equivariant) Hopf bifurcation whose sub/supercriticality coincides with that of the original system. In what follows, we distinguish between the generic and non-generic (non-equivariant) Hopf bifurcations in the restricted systems. Our analysis splits into 3 cases when the Hopf bifurcation is generic and one case related to the non-generic setting.

**Case 1:** \( H^\varphi = +S_4, -D_4^d, -D_4^d, +D_4^d, +S_4^- \). In this case, \( V_{aH^\varphi} = \mathbb{R} \oplus \mathbb{R} \) and system \([5]\) restricted to \( V_{aH^\varphi} \) is equivalent to the equation of a single Van der Pol oscillator, where the parameter is shifted by an integer multiple of \( a \) depending on the branch.

**Case 2:** \( H^\varphi = -Z_3^t \) or \( +Z_3^t \). In this case, \( V_{aH^\varphi} = \mathbb{R}^2 \oplus \mathbb{R}^2 \) and system \([5]\) restricted to \( V_{aH^\varphi} \) is equivalent to the system of two uncoupled Van der Pol oscillators. This system has a continuum of periodic solutions depending on the phase between the two oscillators. Although they all correspond to solutions of the original system, only the solutions for which the first oscillator is one quarter of the period out of phase with the second correspond to the solutions with the prescribed symmetry.

**Case 3:** \( H^\varphi = +D_3 \) or \( -D_3^d \). In this case, \( V_{aH^\varphi} = \mathbb{R}^2 \oplus \mathbb{R}^2 \) and system \([5]\) restricted to \( V_{aH^\varphi} \) is equivalent to the system of two asymmetrically coupled Van der Pol oscillators given by

\[
-D_3: \begin{cases} 
\dot{x}_1 - \alpha \dot{x}_1 + \dot{x}_1 x_1^2 + x_1 = \frac{3}{2}(\dot{x}_2 - \dot{x}_1), \\
\dot{x}_2 - \alpha \dot{x}_2 + \dot{x}_2 x_2^2 + x_2 = \frac{3}{2}(\dot{x}_1 - 5\dot{x}_2);
\end{cases}
\]

\[
+D_3^d: \begin{cases} 
\dot{x}_1 - \alpha \dot{x}_1 + \dot{x}_1 x_1^2 + x_1 = \frac{3}{2}(\dot{x}_2 - \dot{x}_1), \\
\dot{x}_2 - \alpha \dot{x}_2 + \dot{x}_2 x_2^2 + x_2 = \frac{3}{2}(\dot{x}_1 - \dot{x}_2).
\end{cases}
\]

**Case 4:** \( H^\varphi = -Z_3^t \) or \( +Z_3^t \). In this case, \( V_{aH^\varphi} = \mathbb{R}^4 \oplus \mathbb{R}^4 \) and system \([5]\) restricted to \( V_{aH^\varphi} \) undergoes a non-generic Hopf bifurcation. We will consider the following families of non-symmetric delayed differential equations with an additional parameter \( T \):

\[
-Z_3^t: \begin{cases} 
\dot{x}_1 - \alpha \dot{x}_1 + \dot{x}_1 x_1^2 + x_1 = \frac{3}{2} \left( \dot{x}_2 + \dot{x}_2(t - \frac{T}{4}) + \dot{x}_2(t - \frac{2T}{4}) - 3\dot{x}_1 \right), \\
\dot{x}_2 - \alpha \dot{x}_2 + \dot{x}_2 x_2^2 + x_2 = \frac{3}{2} \left( \dot{x}_1 - \dot{x}_2(t - \frac{T}{4}) - \dot{x}_2(t - \frac{2T}{4}) - 3\dot{x}_2 \right);
\end{cases}
\]

\[
+Z_3^t: \begin{cases} 
\dot{x}_1 - \alpha \dot{x}_1 + \dot{x}_1 x_1^2 + x_1 = \frac{3}{2} \left( \dot{x}_2 + \dot{x}_2(t - \frac{T}{4}) + \dot{x}_2(t - \frac{2T}{4}) - 3\dot{x}_1 \right), \\
\dot{x}_2 - \alpha \dot{x}_2 + \dot{x}_2 x_2^2 + x_2 = \frac{3}{2} \left( \dot{x}_1 + \dot{x}_2(t - \frac{T}{4}) + \dot{x}_2(t - \frac{2T}{4}) - 3\dot{x}_2 \right).
\end{cases}
\]

Clearly, \( T \)-periodic solutions to the original system with the spatio-temporal symmetry \(-Z_3^t\) (resp. \(+Z_3^t\)) are in one-to-one correspondence with \( T \)-periodic solutions to \([9]\) (resp. \([10]\)).
To establish supercriticality, in Cases 1 and 2 we recall that the branch of periodic solutions of a Van der Pol equation is supercritical, while for Cases 3 and 4 one can apply the standard techniques of asymptotic analysis. We will just give a detailed explanation for (3) since the other cases are analogous.

**Step 1:** By rescaling time $y(\beta t) = x(t)$, where $\beta = T/2\pi$, one obtains:

$$
\beta^2 \ddot{y}_1 - \alpha \beta \dot{y}_1 + \beta \dot{y}_1 \dot{y}_1 + y_1 = \frac{a\beta}{2} \left( \dot{y}_2 + \dot{y}_2 \left( t - \frac{2\pi}{3} \right) + \dot{y}_2 \left( t - \frac{4\pi}{3} \right) - 3\dot{y}_1 \right),
$$

$$
\beta^2 \ddot{y}_2 - \alpha \beta \dot{y}_2 + \beta \dot{y}_2 \dot{y}_2 + y_2 = \frac{a\beta}{2} \left( \dot{y}_1 - \dot{y}_2 \left( t - \frac{2\pi}{3} \right) - \dot{y}_2 \left( t - \frac{4\pi}{3} \right) - 3\dot{y}_2 \right).
$$

**Step 2:** We will take $r$ to be a small parameter and expand the parameters $\alpha$ and $\beta$ near the values $\alpha = a$ and $\beta = 1$ as follows:

$$
\alpha = a + \hat{\alpha} r^2 + o(r^2), \quad \beta = 1 + \hat{\beta} r^2 + o(r^2).
$$

The standard results about asymptotics of branches born at a Hopf point legitimize the absence of linear terms. We will now expand

$$
y_2 = r \cos t + r^3 \psi_2(t) + o(r^3),
$$

where $\psi_2$ is orthogonal to $\sin t$ and $\cos t$ in $L^2[0, 2\pi]$. Plugging this expression into the first equation shows that $y_1$ has only harmonics of order divisible by 3 and its expansion starts with $r^3$. This allows us to expand

$$
y_1 = r^3 \psi_1(t) + o(r^3),
$$

where $\psi_1(t)$ is orthogonal to $\sin t$ and $\cos t$.

**Step 3:** Projecting terms of order $r^3$ in the second equation onto the first Fourier mode gives the equation

$$
-2\hat{\beta} \cos t + (a\hat{\beta} + \hat{\alpha}) \sin t - \frac{1}{4} \sin t = a\hat{\beta} \sin t.
$$

From this it can be seen that $\hat{\alpha} = 1/4 > 0$, so the branch must be supercritical. □

**Remark 2.2.** Theorem [2.1] allows us to reduce the analysis of stability of periodic solutions to studying characteristic equations related to the zero equilibrium, from which the periodic solutions bifurcate.

3. Main Results

For the symmetry group $H^\varphi$, recall that $0H^\varphi = \ker \varphi$ (cf. [3]). We will denote by $t_0(H^\varphi)$ the smallest $t \in (0, 1)$ such that $t = \varphi(r, h)$ for some $(r, h) \in H$. Finally, define a set of spatial
Table 2: Domains of stability

| Symmetry of the branch | Domain $D$ of parameters for which the branch is stable |
|------------------------|------------------------------------------------------|
| $(-Z_4^-)$             | $0 < a < b$                                          |
| $(-Z_3^-)$             | $0 < a < b$                                          |
| $(+Z_4^+)$             | $0 < 2a < b$                                         |
| $(+Z_3^+)$             | $0 < a < \psi(b)$, where $\psi$ is described in Remark [3.3] and illustrated in Figure [1] |

Symmetries by

$$1H^\varphi = \varphi^{-1}(t_0(H^\varphi))$$

and by $|H|$ the cardinality of $H$.

**Theorem 3.1.** Suppose $x_\alpha^*$ is a branch of periodic solutions to (6) with symmetry $K^\varphi = -D_4^i$, $-D_2^d$, $-D_3^i$, $+D_4^d$, $+D_2^d$ or $-S_4^-$ which bifurcates from the zero solution $x = 0$ at $\alpha_o = ka$ (where $k = 1, 2, 3$ is given in Table [7]). Then, for every $b > ka$ there exists an $\alpha^* = \alpha^*(a, b) > \alpha_o$ such that $x_\alpha^*$ is an asymptotically stable solution of

$$\ddot{x} = f(\alpha, a, x, \dot{x}) + b \left(-\dot{x}(t) + \frac{1}{|1H^\varphi|} \sum (r,h) \in 1H^\varphi r\mathcal{T}_h \dot{x}(t)\right)$$

(11)

for every $\alpha \in (\alpha_o, \alpha^*)$.

**Theorem 3.2.** Suppose $x_\alpha^*$ is a branch of $T_\alpha$-periodic solutions to (6) with symmetry $H^\varphi = -Z_4^i$, $-Z_3^i$, $+Z_4^i$ or $+Z_3^i$ which bifurcates from $x = 0$ at $\alpha_o = ka$ (where $k = 1, 2$ is given in Table [7]). Then, there exists a domain $D \in \mathbb{R}^2_+$ such that for every point $(a, b) \in D$ there exists an $\alpha^* = \alpha^*(a, b) > \alpha_o$ such that $x_\alpha^*$ is an asymptotically stable solution of

$$\ddot{x} = f(\alpha, a, x, \dot{x}) + b \left(-\dot{x}(t) + \frac{1}{|1H^\varphi|} \sum (r,h) \in 1H^\varphi r\mathcal{T}_h \dot{x}(t - \tau_\alpha)\right)$$

(12)

for every $\alpha \in (\alpha_o, \alpha^*)$ with $\tau_\alpha = t_0(H^\varphi)T_\alpha$. Furthermore, for each $H^\varphi$, the domain $D$ is explicitly described in Table [8]

**Remark 3.3.** Consider the curve

$$(a, b) = (\gamma_1(s), \gamma_2(s)) = \left(\frac{(s^2 - 1)(1 + \cos(\frac{\pi}{3} s))}{2s \sin(\frac{\pi}{3} s)}, \frac{s^2 - 1}{s \sin(\frac{\pi}{3} s)}\right), \; s \in [1, 3),$$

(13)
which bounds the shaded domain $\mathcal{D}$ in Figure 1. By direct computation, it is easy to see that $\gamma_2(s)$ is monotonic on the interval $[1, 3)$, and therefore invertible. The function $\psi$ appearing in Table 2 is defined by $\psi := \gamma_1 \circ \gamma_2^{-1}$.

**Remark 3.4.** Since our analysis of stability of the bifurcating branch (with symmetry $H^\varphi$) in the controlled system relies on the standard exchange of stability results, we require that $\pm i$ has multiplicity one at the bifurcation point. For an element $(r, h, \theta) \in H^\varphi$ we will denote by $V^c_{(r,h,\theta)}$ the set of points in the complexification of center space which is fixed by $(r, h, \theta)$, where $\theta$ acts on the complexification by multiplication by $e^{2\pi i \theta}$. It was observed in [3] that for a control of the form $K(rT_h x(t - 2\pi \theta) - x(t))$, the above condition can be satisfied only if $\dim_{\mathbb{C}} V^c_{(r,h,\theta)} = 2$. For any subset $S := \{(r_k, h_k, \theta_k)\}$ of $H^\varphi$, the equivalent requirement for a linear combination of these controls is that

$$\dim_{\mathbb{C}} \bigcap_{(r_k, h_k, \theta_k) \in S} V^c_{(r_k, h_k, \theta_k)} = 2.$$ 

For the majority of the branches considered in this paper, although for a single group element $(r, h, \theta)$ this condition is not satisfied, it is satisfied if we consider a set $S$ of several group elements where $\theta_k \equiv \theta$ is the same for all $k$ (cf. [12]). However, in the case of $+D_3$ and $a = 0$, we have

$$\dim_{\mathbb{C}} \bigcap_{(r,h,\theta) \in +D_3} V^c_{(r_h,\theta)} = 4.$$
For this reason, our control fails to stabilize the branch with symmetry $^+D_3$. It is our conjecture (confirmed by numerical simulations) that this obstruction still exists for weak coupling.

4. Proofs

The $\mathbb{Z}_2 \times O_4$-isotypical decomposition of $V = W \oplus W$ is given by

$$W = W_1 \oplus W_2 \oplus W_3 \oplus W_4,$$

(14)

where $W_1$ (resp. $W_2$, $W_3$ and $W_4$) are mutually non-equivalent absolutely irreducible representations with $\dim W_1 = 1$ (resp. $\dim W_2 = 3$, $\dim W_3 = 3$ and $\dim W_4 = 1$). Take a basis $e_1 \in W_1$ (resp. $e_2$, $e_3$, $e_4 \in W_2$, $e_5$, $e_6$, $e_7 \in W_3$, $e_8 \in W_4$) and call the basis $e_1, \ldots, e_8 \subset W$ an isotypical basis for $W$. Observe that in any isotypical basis the linearization of system (5) at the origin is given by

$$\ddot{x} = A_0 \dot{x} - x$$

(15)

with

$$A_0 = \begin{pmatrix}
\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha - a & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha - a & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha - a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha - 2a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha - 2a & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \alpha - 2a & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha - 3a
\end{pmatrix}.$$ 

(16)

Hereafter, we will assume that the linearized system is of the form (15).

4.1. Proof of Theorem 3.1

Since the treatment of each branch relevant to this theorem follows the same lines, we restrict ourselves to the case when $H^\varphi = -D^d_2$ for which we have

$$\hat{D}^d_2 = \{ (1, ()), (-1, (13)(24)(57)(68)), (1, (15)(28)(37)(46)), (-1, (17)(26)(35)(48)), (-1, (17)(28)(35)(46)), (1, (15)(26)(37)(48)), (-1, (13)(57),), (1, (24)(68)) \}.$$
Then, it follows that the control term (written in the original basis) is represented by

\[ b \left( -\dot{x}(t) + \frac{1}{|0^{H^e}|} \sum_{h \in 0^{H^e}} \mathcal{H}_h \dot{x}(t) \right) = \frac{b}{4} \begin{pmatrix} -3 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\
0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -3 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & -3 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 \\
-1 & 0 & 1 & 0 & -1 & 0 & -3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{pmatrix} \dot{x}. \] (17)

Notice that \( x_\alpha \) bifurcates at the value \( \alpha_0 = a \) (see Table 1). Due to Theorem 3.2, to complete the proof, it is enough to show that if \( b > a \), then the unstable dimension of the trivial equilibrium of system (11) changes from zero to two as \( \alpha \) increases and passes \( \alpha_0 \). Combining (15) with (17) (written in the isotypical basis) allows us to write the linearization of (11) as

\[ \ddot{x} = (A_0 - bB_0)\dot{x} - x \]

with the matrix \( A_0 \) defined by (16) and

\[ B_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \]

Since \( b > a > 0 \), it is easy to see that for \( \alpha \) close to \( a \), all but one pair of eigenvalues have negative real part, while the real part of that pair increases as \( \alpha \) increases and passes \( \alpha_0 \). This completes the proof.

4.2. Proof of Theorem 3.2

The proof of Theorem 3.2 requires that for each \( H^e \) one computes the characteristic equation of the linearization of system (12) at the origin. The results of these computations done in an isotypical basis are presented in Table 3. Since the treatment of each branch appearing in Table
follows the same lines, we restrict ourselves to the case when $H^\varphi = +\mathbb{Z}_3^t$. Similarly to the proof of Theorem 3.1 our goal is to show that if $(a, b) \in D$, then the unstable dimension of the trivial equilibrium of system (12) changes from zero to two as $\alpha$ increases and passes $\alpha_0$. This goal is achieved in two steps.

**Step 1.** At this stage we show that for $\alpha = \alpha_0$ and any $(a, b) \in D$, the trivial equilibrium of system (12) has a two-dimensional center manifold and no unstable manifold. To this end, taking characteristic equations from Table 3 related to $H^\varphi = +\mathbb{Z}_3^t$, and putting $\alpha = \alpha_0 = 2a$ and $T_\alpha = 2\pi$ yields the following equations (here $\eta = e^{\frac{2\pi}{6}i}$):

\begin{align*}
\lambda^2 + (b - 2a)\lambda + 1 &= -b\lambda e^{-\frac{2\lambda\pi}{6}}, \\
\lambda^2 + (b - a)\lambda + 1 &= 0, \\
\lambda^2 + (b - a)\lambda + 1 &= 0, \\
\lambda^2 + (b - a)\lambda + 1 &= 0, \\
\lambda^2 + b\lambda + 1 &= -b\lambda e^{-\frac{2\lambda\pi}{6}}, \\
\lambda^2 + b\lambda + 1 &= b\eta\lambda e^{-\frac{2\lambda\pi}{6}}, \\
\lambda^2 + b\lambda + 1 &= b\eta\lambda e^{-\frac{2\lambda\pi}{6}}, \\
\lambda^2 + (a + b)\lambda + 1 &= 0.
\end{align*}

The spectrum of the zero equilibrium is the union of all the solutions $\lambda$ to these 8 equations. By inspection, if $b > a > 0$, then, except for (18)—(21), the above equations do not admit roots with non-negative real parts meaning that the corresponding polynomials are stable. Next, notice that for $b = 0$, equations (18)—(21) admit $\pm i$ as a root. Furthermore, for any $b$, $i$ remains a root of (20), while $-i$ remains a root of (21). Finally observe that, by implicit differentiation of equations (18)—(21) with respect to $b$ at $a = b = 0$ and $\lambda = \pm i$, it is easy to see that for any sufficiently small $b > 0$, all other roots of (18)—(21) lie in the left half plane.

To show that for any $(a, b) \in D$, the roots of equations (18)—(21) lie in the left half plane, we use a variant of Zero Exclusion Principle. Since $D$ contains the points $(0, b)$ for small $b > 0$, it is enough to show that as $(a, b)$ varies in $D$, no roots of equations (18)—(21) can ever pass through the purely imaginary axis. To this end, we plug $\lambda = is$ into each equation in turn. The points $(a, b)$ for which (18) admits a purely imaginary root form the set of curves given by

$$
\gamma(s) = (a(s), b(s)) = \left(\frac{(s^2 - 1)(1 + \cos(\frac{s\pi}{2}))}{2s \sin(\frac{s\pi}{2})}, \frac{s^2 - 1}{s \sin(\frac{s\pi}{2})}\right)
$$

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Table 3: Characteristic equations written in isotypical coordinates.

|   |         |         | +Z_{c4} | +Z_{t3} |
|---|---------|---------|---------|---------|
| 1 | \(\lambda^2 + (b - a)\lambda + 1 = 0\) | 0       | 0       | \(-b\lambda e^{-\frac{T\lambda}{b}}\) |
| 2 | \(\lambda^2 + (b + a - a)\lambda + 1 = ib\lambda e^{-\frac{T\lambda}{b}}\) | \(-b\lambda e^{-\frac{T\lambda}{b}}\) | 0       | 0       |
| 3 | \(\lambda^2 + (b + a - a)\lambda + 1 = -ib\lambda e^{-\frac{T\lambda}{b}}\) | \(b\lambda e^{\frac{2\pi i - T\lambda}{b}}\) | 0       | 0       |
| 4 | \(\lambda^2 + (b + a - a)\lambda + 1 = 0\) | \(b\lambda e^{-\frac{2\pi i - T\lambda}{b}}\) | 0       | 0       |
| 5 | \(\lambda^2 + (b + 2a - a)\lambda + 1 = 0\) | 0       | \(ib\lambda e^{-\frac{T\lambda}{b}}\) | \(-b\lambda e^{-\frac{T\lambda}{b}}\) |
| 6 | \(\lambda^2 + (b + 2a - a)\lambda + 1 = 0\) | 0       | \(-ib\lambda e^{-\frac{T\lambda}{b}}\) | \(b\lambda e^{\frac{2\pi i - T\lambda}{b}}\) |
| 7 | \(\lambda^2 + (b + 2a - a)\lambda + 1 = 0\) | 0       | 0       | \(b\lambda e^{-\frac{2\pi i - T\lambda}{b}}\) |
| 8 | \(\lambda^2 + (b + 3a - a)\lambda + 1 = 0\) | \(-b\lambda e^{-\frac{T\lambda}{b}}\) | 0       | 0       |

(cf. (13)). Since for each \(s\),

\[
\frac{a(s)}{b(s)} = \frac{1 + \cos(\frac{2\pi s}{3})}{2} \leq 1
\]

with equality iff \(s = 6k + 1\), for some integer \(k\) the segment of the curve corresponding to \(1 < s < 3\) lies above the straight line \(a = b\). Notice that if \(\gamma(s) = \gamma(t)\) for some \(t \neq s\), then either \(s = t + 6k\) or \(s = 6k - t\) for some integer \(k\). Combining this observation with monotonicity of \(s^2 - 1\) for \(s > 1\) and periodicity of the sinus function proves that \(\gamma(s)\) does not have self-intersection points. For this reason, we see that \(\mathcal{D}\) is bounded by \(\{\gamma(s) : s \in [0, 3)\} \cup \{a = 0\}\) (cf. Remark 3.3 and Figure 1).

By taking the absolute value of both sides of (19), it follows that (19) never admits a purely imaginary root. On the other hand, while (20) admits \(i\) as a root for all \((a,b)\), the same argument as for (19) shows that \(\lambda = i\) is the only purely imaginary root of (20). By differentiating (20) with respect to \(\lambda\), one concludes that for \(b > 0\), \(\lambda = i\) is a simple root. Replacing \(i\) by \(-i\), one can apply the same argument to (21).

To summarize Step 1: We showed that, in the case of \(-Z_{t3}^d\), at the bifurcation point \(\alpha = 2a\), if \(a,b > 0\), only one of the quasi-polynomials from Table 3 can admit roots with positive real part (namely equation 1). On the other hand, the boundary of \(\mathcal{D}\) is defined by the values of \((a,b)\) for which equation 1 admits purely imaginary roots. In the cases of \(-Z_{c4}^d\), \(-Z_{t3}^d\) and \(+Z_{c4}^d\), at the corresponding bifurcation points, all the quasi-polynomials from Table 3 do not admit roots with positive real parts. This explains why the case of \(-Z_{t3}^d\) was taken as the demonstrative example and why in Table 2 it has a seemingly peculiar entry.
**Step 2.** It is now left to show that as $\alpha$ increases and $T(\alpha)$ varies, the purely imaginary root $i$ (resp. $-i$) of equation 6 (resp. 7) in Table 3 moves into the right half-plane. To this end, following the idea suggested in [6], p. 326 (see also references therein), we will fix $a$ and $b$, and treat $\alpha$ and $T$ as independent bifurcation parameters. Let us show that in the $(\alpha, T)$-plane a Hopf curve passes through the point $(2a, 2\pi)$ with a vertical tangent line. In fact, substituting $i\omega$ into Table 3, equation 6, one obtains

$$1 - \omega^2 + (b + 2a - \alpha)i\omega = b\eta i\omega e^{-i\omega T}.$$  

The above equation implicitly defines $\alpha$ and $T$ as functions of $\omega$. Differentiating this equation with respect to $\omega$ and separating real and imaginary parts yields

$$-2 = \frac{b}{6}(2\pi + T'(\omega)), \quad \alpha'(\omega) = 0$$  

as desired. On the other hand, fixing $T = 2\pi$ and differentiating equation 6 from Table 3 with respect to $\alpha$ at $\alpha = 2a$ and $\lambda = i$ yields:

$$\lambda' = \frac{3}{6 + \pi b} > 0.$$  

Combining (22) and (23) implies that for any function $T = T(\alpha)$ with $T(2a) = 2\pi$, one has that $\lambda'$ evaluated at $\alpha = 2a$ and $\lambda = i$, is positive. The same argument can be used in the case of $-i$ as a root of Table 3, equation 7. Combining this with Theorem 2.1 and the standard exchange of stability results completes the proof.

5. **Conclusions**

We have considered a system of symmetrically coupled Van der Pol oscillators with $O_4$-permutational symmetry. This system possesses multiple branches of unstable periodic solutions with different symmetry properties. Using an equivariant Pyragas type delayed control introduced in [2, 3, 4] we proposed a specific form of the gain matrices, which ensures the non-invasive stabilization of periodic solutions near a Hopf bifurcation point for the branches of each symmetry type with one exception. We found explicitly stability domains of the controlled system in the parameter space. The failure of the control for branches with one specific type of symmetry can be associated with group theoretic restrictions considered in [3].

6. **Appendix**

In this Appendix, we will explain the symbols used in the main body of the text to denote spatio-temporal symmetry groups. For any $H < S_4 \times S^1$, we will define $-H < \mathbb{Z}_2 \times O_4 \times S^1$ and
$+H < \mathbb{Z}_2 \times O_4 \times S^1$ by

$$+H := H \times (\mathbb{Z}_2 \times O_1)^\alpha, \quad -H := H \times (\mathbb{Z}_2 \times O_1)^{\alpha z},$$

where

$$(\mathbb{Z}_2 \times O_1)^\alpha := \{(1, 1, 0), (1, (17)(28)(35)(46), 0), (-1, 0, 1/2), (-1, (17)(28)(35)(46), 1/2),\}$$

$$(\mathbb{Z}_2 \times O_1)^{\alpha z} := \{(1, 1, 0), (-1, (17)(28)(35)(46), 0), (-1, 0, 1/2), (1, (17)(28)(35)(46), 1/2)\}.$$  

All spatio-temporal symmetry groups which we deal with in this paper appear as either $+H$ or $-H$, where $H$ is among the following groups:

$S_4 := \{(1, 0), ((15)(28)(37)(46), 0), ((17)(26)(35)(48), 0), ((12)(35)(46)(78), 0), ((17)(28)(34)(56), 0),
\quad ((14)(28)(35)(67), 0), ((17)(23)(46)(58), 0), ((13)(24)(57)(68), 0), ((18)(27)(36)(45), 0),
\quad ((16)(25)(38)(47), 0), ((254)(368), 0), ((245)(386), 0), ((163)(457), 0), ((136)(475), 0), ((168)(274), 0),
\quad ((186)(247), 0), ((138)(275), 0), ((183)(257), 0), ((1234)(5678), 0), ((1432)(5876), 0),
\quad ((1265)(3874), 0), (1562)(3478), 0), (1485)(2376), 0), (1584)(2678), 0)\}$$

$D_5^1 := \{(1, 0), ((1234)(5678), 0), ((13)(24)(57)(68), 0), ((1432)(5876), 0),
\quad ((17)(26)(35)(48), 0), ((18)(27)(36)(45), 0), ((15)(28)(37)(46), 0), ((16)(25)(38)(47), 0)\}$$

$D_5^2 := \{(1, 0), (254)(368), 0), (245)(386), 0), ((17)(26)(35)(48), 0),
\quad ((17)(28)(34)(56), 0), (17)(23)(46)(58), 0)\}$$

$D_4^1 := \{(1, 0), (17)(26)(35)(48), 1), ((13)(24)(57)(68), 0), ((15)(28)(37)(46), 0)\}$$

$\mathbb{Z}_4^1 := \{(1, 0), ((1234)(5678), 0), ((13)(24)(57)(68), 0), ((1432)(5876), 0),
\quad ((1265)(3874), 0), (1562)(3478), 0), (1485)(2376), 0), (1584)(2678), 0)\}$$

$S_4^- := \{(1, 0), ((15)(28)(37)(46), 0), ((17)(26)(35)(48), 0), ((12)(35)(46)(78), 0), ((17)(28)(34)(56), 0),
\quad ((14)(28)(35)(67), 0), ((17)(23)(46)(58), 0), ((13)(24)(57)(68), 0), ((18)(27)(36)(45), 0),
\quad ((16)(25)(38)(47), 0), ((254)(368), 0), ((245)(386), 0), ((163)(457), 0), ((136)(475), 0), ((168)(274), 0),
\quad ((186)(247), 0), ((138)(275), 0), ((183)(257), 0), ((1234)(5678), 0), ((1432)(5876), 0),
\quad ((1265)(3874), 1/2), (1562)(3478), 0), (1485)(2376), 0), (1584)(2678), 0)\}.$
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