Universal post-quench dynamics at a quantum critical point

Pia Gagel, Peter P. Orth, and Jörg Schmalian

1 Institute for Theory of Condensed Matter, Karlsruhe Institute of Technology (KIT), 76131 Karlsruhe, Germany
2 Institute for Solid State Physics, Karlsruhe Institute of Technology (KIT), 76021 Karlsruhe, Germany

Dated: June 26, 2014

We consider an open system near a quantum critical point that is suddenly moved towards the critical point and investigate its non-equilibrium dynamics after the quench. In an intermediate time regime we identify and determine a critical exponent, not related to equilibrium exponents, that describes universal prethermalized dynamics. We discuss the implications of this quantum critical prethermalization for the dynamics of the order parameter and response functions. Our results demonstrate that quantum quenches are efficient tools to manipulate and study the universal non-equilibrium dynamics of quantum many-body systems.

Predicting the out-of-equilibrium dynamics of quantum many-body systems is a challenge of fundamental and practical importance. This research area has been boosted by a number of recent experiments in cold-atom gases [1] and scaled-up quantum-circuits [2], by ultra-fast pump-probe measurements in correlated materials [3, 4], and even by performing heavy-ion collisions that explore the quark-gluon plasma [5]. In this context, the universality near a quantum critical point (QCP), well established in and near equilibrium, comes with the potential to make quantitative predictions for strongly interacting systems far from equilibrium. For example, the quantum version [6–10] of the Kibble-Zurek mechanism of defect formation [11, 12] was developed for systems driven through a symmetry breaking QCP at a small, but finite rate. Similarly, near a QCP the long-time dynamics after a sudden change of Hamiltonian parameters, is governed by equilibrium exponents [13]. These phenomena occur in the regime of longest time scales.

Recently, however, many physical systems were identified which display novel dynamical behavior on intermediate time scales, a behavior often referred to as prethermalization [14–23]. The question arises whether one can expect universality with new critical exponents during prethermalization near a QCP. Such exponents would govern the non-equilibrium dynamics soon after a quantum-quench, where initial correlations are still important, and should therefore not be related to equilibrium exponents. Since universality is usually reserved for large time and length scales, it seems counterintuitive to expect it at shorter times. On the other hand, such behavior is known to emerge near classical phase transitions [24, 25] and is closely related to the problem of boundary layer universality as it occurs near surfaces and interfaces [26]. Here, the boundary layer would correspond to a “surface in time” [27, 28].

In this Letter, we show that the time evolution of observables in an open system that is suddenly moved to a QCP displays universal behavior. Their dynamics is governed by a new critical exponent. Such quantum critical prethermalization is a rare example of universal dynamics in a quantum system far from equilibrium. It opens the opportunity to probe the universality class of a quantum system by quantum-quench experiments and, as we will show, to manipulate the crossover from prethermalization to thermalization. While there are important differences between classical and quantum quenches, the analysis of this paper was motivated by the pioneering experiments of this paper was motivated by the pioneering

FIG. 1. (Color online) (a) Schematic description of the setup and quench protocol. The system refers to an interacting scalar ϕ4-field theory, coupled to a bath of harmonic oscillators held at zero temperature. The quantum quench corresponds to a sudden change r0,i → r0,c right to the QCP. (b) Schematic phase diagram of the system as a function of temperature T and mass. It contains a boundary layer universe (⟨ϕ⟩ 0) and unbroken (⟨ϕ⟩ 0) phase, which are separated at T = 0 by a QCP at δr = 0 denoted by C. Red arrows describe the quench protocol. Inset distinguishes three time regimes: very short times t < tγ, with non-universal dynamics, the universal prethermalized regime γ−r−(z−1) < t < δr−νz/z/ν where we study, and a long-time regime t > δr−νz/z/ν where dynamics is described by equilibrium critical exponents. (c) Closed time contour including branch along imaginary time τ leading to initial correlations at t = 0 between system and bath in equilibrium.
theory of classical dynamics in Ref. 24.

The quench protocol that underlies our analysis is indicated in Fig. 1(a-b). We consider a quantum many-body system that is coupled to an external bath of harmonic oscillators. Prior to the quench, the complete system is prepared in the $T = 0$ ground state $|\Psi_i\rangle$ of the initial Hamiltonian $H_i = H_{s,i} + H_b + H_{sb}$. Here $H_{s,i}$ is the initial Hamiltonian of the system, $H_b$ describes the external bath and $H_{sb}$ the coupling between system and bath. Next, we suddenly switch a parameter in the system Hamiltonian $H_{s,i} \rightarrow H_s$ in such a way that the ground state of the full Hamiltonian $H = H_s + H_b + H_{sb}$ is located right at a QCP (see Fig. 1(b)). The time evolution after the quench is governed by the new Hamiltonian $\{\Psi(t)\} = e^{-iHt} |\Psi_i\rangle$. The bath ensures that the system equilibrates at $T = 0$, which allows reaching the QCP for $t \rightarrow \infty$.

We consider a N-component scalar quantum field $\varphi(x,t)$ with components $\varphi_a$ ($a = 1 \cdots N$), that is coupled to a collection of harmonic oscillators representing the bath. The initial Hamiltonian of the system is

$$H_{s,i} = \frac{1}{2} \int d^d x \left( \pi^2 + r_{0,i} \varphi^2 + (\nabla \varphi)^2 + \frac{u}{2} \varphi^4 \right). \quad (1)$$

The quench corresponds to suddenly changing $r_{0,i} \rightarrow r_{0,c}$ to its value at the QCP. The distance to the critical point before the quench is $\delta r_i = r_{0,i} - r_{0,c}$. In Eq. (1) $\pi$ is the canonically conjugated momentum to $\varphi$. The Hamiltonian $H_{s,i}$ describes a transverse-field Ising model for $N = 1$, systems near a superconducting-insulator quantum phase transition, Josephson junction arrays and quantum antiferromagnets in an external magnetic field for $N = 2$, or quantum dimer systems for $N = 3$ [29]. The oscillator bath is described by $H_b = \frac{1}{2} \int d^d x \sum_l \left( \omega_l X_l^2 + P_l^2 \right)$, while the system-bath coupling is $H_{sb} = \sum_l c_l \int d^d x \varphi \cdot \varphi$. The effects of the bath can be described in terms of the spectral function $\eta(\omega) = -\int d^d k \frac{\xi_k^2}{(\omega + i0)^2 - \Omega_k^2}$. Here we consider

$$\text{Im} \, \eta(\omega) = \gamma \omega |\omega|^{\alpha-1} e^{-|\omega|/\omega_c} \quad (2)$$

with damping coefficient $\gamma$ and cut-off energy $\omega_c$. The exponent $\alpha$ determines the low-energy spectrum of the bath, where we consider ohmic ($\alpha = 1$) and sub-ohmic ($\frac{1}{2} < \alpha < 1$) damping. Possible applications are dissipative superconducting nanowires [30], the superfluid-insulator transition in cold-atom gases coupled to other bath-atoms [31], or low-dimensional Heisenberg spin dimers or transverse field Ising spins with strong quantum fluctuations and coupling to phonons.

To analyze the non-equilibrium dynamics we use the Keldysh formalism of many-body theory [32]. To prepare the full system in the ground state of $H_i$, the Keldysh contour is deformed as indicated in Fig. 1(c) [33]. It consists of a branch on the imaginary axis, governed by the Hamiltonian $H_i$, and the usual round trip contour on the real axis with Hamiltonian $H$. The quench corresponds to the passage from the equilibrium dynamics along the imaginary time axis to the horizontal non-equilibrium time evolution.

We start with general scaling arguments for the non-equilibrium dynamics after a quench towards the critical point. The scaling behavior will be confirmed using a perturbative renormalization group (RG) analysis later in the paper and a large-$N$ analysis summarized in the supplementary material. For the full retarded response function $G^R$ and the Keldysh correlation function $G^K$ (we use the notation of Ref. 32) one expects from dimensional arguments

$$G^R(K) (k,t,t') = \left( \frac{t}{t'} \right)^{\theta(\theta')} \frac{f^{R(K)}(k^2 \gamma^2/2,t/t')}{k^{2-\eta-\gamma^2/2}} \quad (3)$$

where $k$ denotes momentum, $z$ the dynamic critical exponent. Within the one-loop RG analysis of this paper, it follows immediately $z = 2/\alpha$ with bath-exponent $\alpha$. In an out-of-equilibrium state the correlation and response functions depend on both time variables. This gives rise to an additional dimensionless ratio $t/t'$ compared to scaling in equilibrium. The singular dependence on this ratio in $G^R$ and $G^K$ is characterized by exponents $\theta$ and $\theta'$, respectively. The scaling functions $f^R$ and $f^K$ depend only weakly on $t'/t$ if $t \gg t'$.

Let us motivate the emergence of the exponents further: at time $t$ after the quench from a non-critical initial state one expects correlations are limited by the length $\xi(t) \propto t^{1/\gamma}$. This gives rise to a mass $r(t) = \gamma a/t^{2/\gamma}$ in the propagator with dimensionless coefficient $a$. A straightforward perturbation theory in $a$ that includes scattering due to a time-dependent mass yields for $t \gg t'$:

$$G^R(t,t') = G^R_0(t - t') \left[ 1 + \theta \log(t/t') \right] \quad (4)$$

with

$$\theta = \frac{a \sin(\pi/\gamma)}{\Gamma(2/\gamma)} \quad (5)$$

and the bare Green’s function $G^R_0$ given in Eq. (6) below. Exponentiation of the logarithm leads to Eq. (3). A similar analysis for $G^K(t,t')$ yields $\theta' = \theta - \frac{2-\gamma}{2} z$.

In what follows we use a momentum-shell RG approach to sum up these logarithms in a controlled fashion and determine the exponents $\theta$ and $\theta'$. Our analysis proceeds in two steps. First, we determine the retarded and Keldysh Green’s functions $G^R_0(k,t,t')$ and $G^K_0(k,t,t')$, subject to a quench in the non-interacting limit ($u = 0$). These bare Green’s functions will, in the second step, be used to develop the RG analysis.

Bare response and correlation functions. In distinction to the equilibrium case, the non-interacting problem far from equilibrium is nontrivial in its own right. Integrating out the bath degrees of freedom leads to long-time
correlations causing memory effects that couple events long before and long after the quench (see Fig. 1(c)). Despite the quench, the bare retarded Green’s functions $G^R(t, t')$ on the round trip contour only depends on the time difference $t - t'$ between two events. For the Fourier transform follows the same result as in equilibrium

$$G^R_0(k, \omega) = \frac{1}{t_{0,c} + k^2 - \omega^2 + \eta(\omega)}.$$  \hfill (6)

Here, we consider the hierarchy of scales $\omega_c \gg \Lambda \gg t_\gamma^{-1} = \gamma z/2(z-1)$ such that the dynamics is dominated by the bath for $t > t_\gamma$. The information about the quench is contained in the Keldysh function $G^K_0$ that we express in terms of a memory function $M(k, t, t')$:

$$G^K_0(k, t, t') = \int_0^\infty ds' G_0(k, s, s') \times G^R_0(k, t - s) G^R_0(k, t' - s').$$ \hfill (7)

This is the out-of-equilibrium version of the fluctuation-dissipation theorem for our quench protocol. We determine $M(k, s, s')$ by solving the Heisenberg equations of motion of the field operators $\varphi(k, t)$ and computing $G^K_0(k, t, t') = -i \langle [\varphi(k, t), \varphi(k, t')] \rangle$. This is achieved via Laplace transformation and we find $\varphi(k, t) = \int_0^\infty dt \varphi(k, t)e^{i(\omega + i\delta_r)t} = F(k, t, \omega)G^K_0(k, \omega)$ with force operator

$$F(k, \omega) = \pi_0(k) - i\omega \varphi_0(k) + \xi(\omega).$$ \hfill (8)

The $t = 0$ initial values $\pi_0(k)$ and $\varphi_0(k)$ are determined by the pre-quench dynamics and

$$\xi(t) = -\sum_I \left[ e_I X_{0I} \cos(\Omega_I t) + \frac{e_I P_{0I}}{\Omega_I} \sin(\Omega_I t) \right]$$ \hfill (9)

is the operator of the bath-induced force. It contains the bath coordinate and momentum operators at the initial time $t = 0$: $X_{0I}$ and $P_{0I}$. It follows that the two-time Laplace transform of the memory function is given by the force-force correlation function

$$M(k, \omega, \omega') = -i \sum_{n=1}^N \left< [F_n(k, \omega), F_n(-k, \omega') ]_+ \right>/N.$$

For a classical critical system, the dynamics is determined entirely by the bath, which exerts random Langevin forces, the classical limit of $\xi(t)$, on the system. In contrast, the quantum dynamics must respect the Heisenberg uncertainty relation, reflected in the initial value of the operators $\varphi_0, X_{0I}$ and their canonically conjugated momenta $\pi_0, P_{0I}$ in the force operator $F(k, \omega)$. The mentioned long-time correlations between pre- and post-quench dynamics, indicated in the Keldysh contour in Fig. 1(c), correspond to mixed terms like $-i \left< [\pi_0(k), \varphi_0(-k, \omega')]_+ \right>$ in the memory function $M$. Of the numerous terms that emerge, many have contributions that diverge in the limit $\omega_c \to \infty$.

A straightforward but rather tedious analysis yields that all divergencies cancel and the memory function reads

$$M(k, \omega, \omega') = -\frac{G^K_0(k, \omega) + G^K_0(k, \omega')}{\omega + \omega' + i0^+} \times G^K_{0r}(k, \omega)^{-1}G^K_{0i}(k, \omega')^{-1}.$$ \hfill (10)

Here, the index $i$ refers to the retarded and the Laplace transform of the Keldysh Green’s function $G^K_{0i}(k, \omega)$ of the system in equilibrium before the quench. Without quench one recovers after a few steps that $G^K_0(k, t, t')$ in Eq. (7) indeed takes its equilibrium value, obeying the fluctuation-dissipation theorem. In the classical limit we obtain known results for an ohmic [24] and an sub-ohmic bath [25].

Renormalization group approach.— We now perform a momentum-shell RG approach. In full analogy to the equilibrium case we integrate out states in a shell with momenta $\Lambda/b < k < \Lambda$ with $b > 1$ and rescale fields, momenta and time variables according to $\varphi'(k', t') = b^{-\frac{z}{2} - 1} \varphi_0(k, t), k' = bk$, and $t' = b^{-z} t$ to achieve self-similarity. The small parameter controlling the calculation is the deviation from the upper critical dimension $\epsilon = 4 - d - z$. The mass $\delta_{r1}$ in the initial Hamiltonian is a strongly relevant perturbation and rapidly flows to large values. For the mass renormalization after the quench follows at one-loop

$$r'(t) = b^2 r(b^2 t) + u \frac{N + 2}{2} \int_0^\infty \frac{d^dk}{(2\pi)^d} iG^K(k, t, t) , \hfill (11)$$

where $r$ refers to momenta inside the shell. The time dependence of the mass term is a consequence of the broken time translation invariance. For a similar analysis of classical surface criticality, see Ref. [34]. We replace $\delta_{r1}$ and the dimensionless coupling constant $\tilde{u} = uK_d\Lambda^{-\epsilon}/\gamma^{z/2}$ with $K_d = \frac{1(d/2)}{2\pi^{d/2}(2\pi)^{d/2}}$ by their fixed-point values $\delta_{r1} = 0$ and $\tilde{u} = \frac{\epsilon}{(4 - d - z)}$, with coefficient $c_z = \frac{4 \sin(\pi/2)/2}{\sin^2(\pi/2)}(\pi/2)$. From Eq. (11) we obtain a differential equation for the time-dependent fixed-point mass $r^*(t)$:

$$2r^* + z\frac{dr^*}{dt} + \frac{(N + 2)\tilde{u}\Lambda^2}{2} f^K_1(A^z t/\gamma^{z/2}, 1) = 0 . \hfill (12)$$

The scaling function $f^K_1$ characterizes $G^K_0$ according to Eq. (3). The solution of Eq. (12) is

$$r^*(t) = \frac{\gamma a}{t^{2z/2}} - \frac{(N + 2)\tilde{u}\Lambda^2}{2z t^{2z/2}} \int_0^t dt' f^K_1(A^z t'/\gamma^{z/2}, 1) t'^{2z/2} .$$ \hfill (13)

with $t_0 = \gamma^{z/2}/\Lambda^{z} \ll \gamma$. Here, $a$ denotes the integration constant of the fixed-point equation. Since the system equilibrates, we obtain $r^K_0(A^z t/\gamma^{z/2} \to \infty) \to r^K_{eq}$, where $r^K_{eq}$ describes the equal-time Keldysh function in equilibrium after the quench. For a perturbative RG analysis a long range decay of the mass parameter cannot emerge. We can therefore use that $r^*(t)$ approaches the
exponents. Inset shows growth of correlation function to zero in a universal way described by equilibrium critical exponents. At longer times, \( \langle \varphi(t) \rangle \) decays to zero in a universal way described by equilibrium critical exponents.

In order to determine the coefficient \( a \) and evaluate the integral in Eq. (14), for an ohmic bath with \( \omega > \tau^{-d/2} \) obtained above and \( \eta = \mathcal{O} \left( \tau^{d/2} \right) \). The exponent \( \theta \) thus determines the universal out-of-equilibrium dynamics of the order parameter. If the correlation volume grows faster than the local correlations decay, \( \theta > 0 \), and the magnetization rises after a quench to the critical point (see Fig. 2(a)).

The time dependence of the order parameter can also be rationalized using crossover arguments \([26]\). With initial value \( \delta r_i \) and associated scaling dimension \( \kappa \) follows for the non-equilibrium magnetization

\[
\langle \varphi \rangle_i = \mathcal{O} \left( \tau^{1/\kappa} \delta r_i \right),
\]

with universal function \( \Phi \left( y \right) \). As indicated in Fig. 2(a), the magnetization decays to zero in the long time limit \( t \gg t_c = \delta r_i^{-z} \) with its equilibrium scaling dimension \( \langle \varphi(t) \rangle \propto t^{-\frac{\kappa}{z}} \), since \( \Phi \left( y \right) \propto y^\frac{\kappa}{z} \).

In conclusion, we determined universal behavior that governs quantum critical prethermalization. The intermediate time dynamics of a system that is suddenly brought out of equilibrium and moved to a nearby QCP is characterized by a new exponent \( \theta \) and post-quench distance to the QCP \( \delta r_i \) to the critical point prior to the quench. If we further compare the correlation length \( \xi_i \propto \delta r_i^{\frac{1}{\nu}} \) prior to the quench with \( \xi \left( t^* \right) \propto \delta r_i^{-\nu/\kappa} \) at the crossover between the prethermalized regime and equilibration, our result \( \theta > 0 \) implies \( \kappa > 1 \). Thus, for small \( \delta r_i \) follows \( \xi \left( t^* \right) \ll \xi_i \). The correlation length collapses after the quench and does not reach its pre-quench value during prethermalization. The growth of the order parameter \( \propto t^\theta \) is thus caused by the recovery of locally ordered regions after this collapse. Since the crossover time \( t^* \) diverges for weak quenches, an almost critical system, subject to a sudden change of its parameters, undergoes a universal out-of-equilibrium dynamics for arbitrarily long periods of time. Our analysis was performed for a quench at \( T = 0 \) and right to the QCP. The universal behavior is unchanged, however, for finite but small temperatures \( T \ll \delta r_i^{\nu z}/\gamma z^{1/2} \) and post-quench distance to the QCP \( \delta r_f \ll \delta r_i \).

In conclusion, we determined universal behavior that governs quantum critical prethermalization. The intermediate time dynamics of a system that is suddenly brought out of equilibrium and moved to a nearby QCP is characterized by a new exponent \( \theta \). This quantum critical prethermalization results from a collapse of correlations after the quench and extends over long times, depending on the initial distance from the critical point. Our results show that the universal dynamics far from equilibrium can be analyzed quantitatively and that a quantum-quench can be an efficient tool to manipulate and study quantum many-body systems near quantum critical points.
The Young Investigator Group of P.P.O. received financial support from the “Concept for the Future” of the KIT within the framework of the German Excellence Initiative.

[1] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. 80, 885 (2008).
[2] A. A. Houck, H. E. Türeci, and J. Koch, Nat. Phys. 8, 292 (2012).
[3] D. Fausti, R. I. Tobey, N. Dean, S. Kaiser, A. Dienst, M. C. Hoffmann, S. Pyon, T. Takayama, H. Takagi, and A. Cavalleri, Science 331, 189 (2011).
[4] C. L. Smallwood, J. P. Hinton, C. Jozwiak, W. Zhang, J. D. Koralek, H. Eisaki, D.-H. Lee, J. Orenstein, and A. Lanzara, Science 331, 189 (2011).
[5] I. Arsene and et al., Nuclear Physics A 757, 1 (2005).
[6] B. Damski, Phys. Rev. Lett. 95, 035701 (2005).
[7] S. Deng, G. Ortiz, and L. Viola, EPL (Europhysics Letters) 84, 67008 (2008).
[8] C. De Grandi, A. Polkovnikov, and A. W. Sandvik, Phys. Rev. B 84, 224303 (2011).
[9] A. Chandran, A. Erez, S. S. Gubser, and S. L. Sondhi, Phys. Rev. B 86, 1137 (2012).
[10] M. Kolodrubetz, B. K. Clark, and D. A. Huse, Phys. Rev. Lett. 109, 015701 (2012).
[11] T. Kibble, J. Phys. A 9, 1387 (1976).
[12] W. H. Zurek, Nature (London) 317, 505 (1985).
[13] C. De Grandi, V. Gritsev, and A. Polkovnikov, Phys. Rev. B 81, 012303 (2010).
[14] J. Berges, S. Borsányi, and C. Wetterich, Phys. Rev. Lett. 93, 142002 (2004).
[15] M. Eckstein, M. Kollar, and P. Werner, Phys. Rev. Lett. 103, 056403 (2009).
[16] M. Moeckel and S. Kehrein, Phys. Rev. Lett. 100, 175702 (2008).
[17] J. Sabio and S. Kehrein, New J. Phys. 12, 055008 (2010).
[18] A. Mitra, Phys. Rev. B 87, 205109 (2013).
[19] M. C. Bañuls, J. I. Cirac, and M. B. Hastings, Phys. Rev. Lett. 106, 050405 (2011).
[20] C. Kollath, A. M. Läuchli, and E. Altman, Phys. Rev. Lett. 98, 180601 (2007).
[21] S. R. Manmana, S. Wessel, R. M. Noack, and A. Muramatsu, Phys. Rev. Lett. 98, 210405 (2007).
[22] M. Marcuzzi, J. Marino, A. Gambassi, and A. Silva, Phys. Rev. Lett. 111, 197203 (2013).
[23] N. Tsuji, M. Eckstein, and P. Werner, Phys. Rev. Lett. 110, 136404 (2013).
[24] H. Janssen, B. Schaub, and B. Schmittmann, Z. Phys. B 73, 539 (1989).
[25] J. Bonart, L. F. Cugliandolo, and A. Gambassi, J. Stat. Mech. 2012, P01014 (2012).
[26] H. W. Diehl, Int. J. Mod. Phys. B 11, 3503 (1997).
[27] P. Calabrese and J. Cardy, J. Stat. Mech. 0706, 06008 (2007).
[28] A. Gambassi and P. Calabrese, EPL (Europhysics Letters) 95, 66007 (2011).
[29] S. Sachdev, Quantum Phase Transitions (Cambridge University Press, Cambridge, U.K., 1999).
[30] S. Sachdev, P. Werner, and M. Troyer, Phys. Rev. Lett. 92, 237003 (2004).
Supplemental Material

Self-consistent large-\(N\) theory for \(\theta\)

By analyzing the limit of large-\(N\), where \(N\) is the number of components of the quantum-field \(\varphi\), we summarize an alternative approach to determine the non-equilibrium exponent \(\theta\). The approach is complementary to the renormalization group (RG) calculation presented in the main paper. Taking the limit \(N \to \infty\) in the RG calculation, valid for arbitrary \(N\), we find that both approaches agree.

In the large-\(N\) limit, the set of coupled equations after the quench are:

\[
G^R(k,t,t') = G^R_0(k,t-t') + \int_t^{t'} ds G^R_0(k,t-s) \Sigma^R(k,s) G^R(k,s,t') \\
G^K(k,t,t') = G^K_0(k,t-t') + \int_t^{t'} ds G^K_0(k,t-s) \Sigma^R(k,s) G^K(k,s,t') \\
+ \int_0^t ds G^R_0(k,t-s) \Sigma^R(k,s) G^K(k,s,t')
\]

\[
\Sigma^R(t) = \frac{uN}{2} \int \frac{d^d k}{(2\pi)^d} i G^K(k,t,t). \tag{19}
\]

Here, the integration over momenta goes up to an upper cut-off \(\Lambda\). The time-dependent retarded self-energy \(\Sigma^R(t)\) can be understood as a time-dependent “mass” in the retarded propagator:

\[
r(t) \equiv r_0 + \Sigma^R(t) = r_0 + \frac{uN}{2} \int \frac{d^d k}{(2\pi)^d} i G^K(k,t,t). \tag{20}
\]

If in equilibrium, the system is right at the critical point with \(r(t) = r_{eq} = 0\), i.e. \(r_0\) takes the value \(r_0 = r_{0,c}\) that obeys:

\[
0 = r_{0,c} + \frac{uN}{2} \int \frac{d^d k}{(2\pi)^d} i G^K(k), \tag{21}
\]

with \(i G^K(k) = f^K_{eq} / (q^{2-z} t^{z/2})\) independent on the absolute time and determined by the fluctuation dissipation relation. Combining both expressions yields

\[
r(t) = \frac{uN}{2} \int \frac{d^d k}{(2\pi)^d} \left( i G^K(k,t,t) - i G^K_{eq}(k) \right). \tag{22}
\]

Based on dimensional arguments, we make the following ansatz for \(r(t)\):

\[
r(t) = \frac{\gamma a}{t^{z/2}}, \tag{23}
\]

valid for \(t\) larger than the microscopic time \(t_0 = (\sqrt{\gamma}/\Lambda)^z\). The pre-factor \(\gamma\), that determines the strength of the coupling to the bath, was chosen to ensure that the coefficient \(a\) is dimensionless. Once \(a\) is known, it follows from Eq.(4) and (5) of the main paper that

\[
\theta = -\frac{a \sin \left( \frac{\pi}{\Gamma \left( \frac{z}{2} \right)} \right)}{\Gamma \left( \frac{z}{2} \right)}. \tag{24}
\]

Next we demonstrate that this ansatz for \(r(t)\) is a self consistent solution of the coupled equations and show how to determine \(a\). To proceed, we use the scaling form Eq.(3) of the main paper:

\[
G^K(k,t,t) = \frac{f^K(k^z t^{z/2}, 1)}{k^{2-z} \gamma^{z/2}}. \tag{25}
\]

Inserting our ansatz for \(r(t)\) and this scaling form into Eq.(22) we obtain:

\[
\frac{\gamma a}{t^{z/2}} = \frac{1}{t^{z/2}} \frac{uN}{2} \frac{\gamma^{z/2}}{2z} K_d t^{z/2} \int_0^{N^z t^{z/2}} x^{\frac{z-2}{z}} dx \left( f^K(x, 1) - f^K_{eq} \right). \tag{26}
\]
The coefficient $K_d = \frac{1}{2\pi^{d/2}} \Gamma(d/2)$ results from the angular integration, $\epsilon = 4 - d - z$ determines the deviation from the upper critical dimension, and $x = k^2 t / \gamma^{z/2}$ is a dimensionless integration variable. For Eq. (26) to be correct, the coefficient $t^{z/2}$ in front of the integral must be cancelled by the time dependence that results from the upper cut-off $\lambda = \Lambda^2 t / \gamma^{z/2}$ of the integration. For $\epsilon > 0$, the integral converges for $\lambda \to \infty$ and we can split the integration according to: $\int^\lambda_0 dx \cdots = \int^\infty_0 dx \cdots - \int^\lambda_0 dx \cdots$. In order for Eq. (26) to be valid, it must hold that

$$\int^\infty_0 x^\frac{\epsilon - 1}{z} dx \left( f^K(x, 1) - f^K_{eq} \right) = 0$$

(27)

and

$$f^K(x \gg 1, 1) = f^K_{eq} - bx^{-2/z},$$

(28)

with some dimensionless coefficient $b$. If these two conditions are fulfilled, it follows

$$\int^\lambda_0 x^\frac{\epsilon - 1}{z} dx \left( f^K(x, 1) - f^K_{eq} \right) = - \int^\infty_\lambda x^\frac{\epsilon - 1}{z} dx \left( f^K(x, 1) - f^K_{eq} \right) \sim b \int^\infty_\lambda x^{-\frac{\epsilon}{z}} dx \sim \frac{b}{c} \gamma^{z/2} \Lambda^{-\epsilon} t^{-\epsilon/z},$$

(29)

where the second step is valid for large $\lambda$. Inserting this result into Eq. (26) the $t$-dependence on both sides of the equation is indeed the same. Comparing the coefficients, we obtain the self-consistency condition:

$$a = \frac{u N \Lambda^{-\epsilon}}{2 \gamma^{z/2}} K_d \frac{b}{\epsilon}.$$  

(30)

The remaining task is to determine the coefficient $b$ of the long-time behavior of the correlation function, introduced in Eq. (28). We determine this coefficient from the condition Eq. (27) in the limit of small $\epsilon$. To this end we expand $f^K = f^K_0 + f^K_1 + O(\epsilon^2)$. From an analysis of the non-interacting Keldysh function follows that the slow decay in Eq. (28) must come from interaction corrections, i.e. $b$ is at least of order $\epsilon$.

To leading order in $\epsilon$ holds

$$\int^\infty_0 x^\frac{\epsilon - 1}{z} dx (f^K_0(x, 1) - f^K_{0eq}) = C_0,$$

(31)

with some constant $C_0$ that is of order unity. Since $C_0$ is determined for $\epsilon = 0$ it can be obtained from the bare Green’s functions. To fulfill the condition Eq. (27), corrections of order $\epsilon$ in the correlation functions must accumulate to a correction of order unity in the integral over $x$ (since $x = k^2 t / \gamma^{z/2}$ the $x$-integration can be interpreted as a time-integration). This is exactly what follows from the slow long time behavior of Eq. (28), valid for $x$ larger than some finite scale $x_0 > 1$:

$$\int^\infty_0 x^\frac{\epsilon - 1}{z} dy (f^K_1(x) - f^K_{1eq}) = O(\epsilon) - b \int^\infty_{x_0} x^{-\frac{\epsilon}{z}} dx = -\frac{b}{\epsilon} z + O(\epsilon).$$

(32)

The condition Eq. (27) now determines the coefficient

$$b = \frac{\epsilon}{z} C_0,$$

(33)

and we obtain, using Eq. (30):

$$a = \frac{u N \Lambda^{-\epsilon} K_d}{2 \gamma^{z/2}} \int^\infty_0 x^\frac{\epsilon - 1}{z} dx \left( f^K_0(x, 1) - f^K_{0eq} \right).$$

(34)

Inserting at the critical point the fixed point value

$$u^* = \frac{4 \sin \left( \frac{\pi z}{2} \right) \gamma^{z/2}}{N z \left( 2 - z \right) K_d \sin^{z/2} \frac{\pi \Lambda^*}{z}} \epsilon \Lambda^*,$$
for the interaction and using Eq. (24), we obtain an expression the exponent $\theta$ expressed in terms of $f^K_0(x, 1)$, i.e. in terms of the bare Keldysh function $G^K_0(k, t, t)$. This function must be determined using the non-equilibrium quench protocol described in the main paper. The key observation of this section is that the self consistent large-$N$ theory is an alternative approach to determine $\theta$ and that Eq. (34) agrees with the RG result Eq.14 of the main paper in the limit $N \gg 1$. Both calculations yield the same value for the critical exponent $\theta$ in the limit for large $N$.

**Scaling limit and evaluation of the exponent**

From the RG-analysis in the main paper or the large-$N$ analysis of the supplementary section 1 follows that

$$
\theta = - \frac{\sin \left( \frac{\pi}{2} \right) u^* (N + 2) \Lambda^{-z} K_d I_0}{2 \pi \gamma^{z/2}}
$$

with

$$
I_0 = \int_0^\infty x^{2-z} dx \left( f^K_0(x, 1) - f^K_{0eq} \right)
$$

and

$$
u^* = \frac{4 \sin \left( \frac{\pi z}{2} \right) \gamma^{z/2}}{(N + 8) z (2 - z) K_d \sin^{z/2} \pi^z} e^\Lambda^z
$$

If one uses dimensionful quantities, it holds $I_0 = \frac{x^{2-z}}{\gamma^{z/2}} I$, with

$$
I = \int_0^\infty dt \left( iG^K_0(k, t, t) - iG^K_{0eq}(k) \right) t^{2-z}.
$$

For the Keldysh function we obtained the result:

$$
iG^K_0(k, t, t) = \int_0^\infty ds \int_0^\infty ds' M(s, s', \omega_0^2) G^R_0(k, t - s) G^R_0(k, t - s'),
$$

where the argument $\omega_0^2 = \delta r_i + k^2$ indicates that $M$ is obtained from equilibrium Green’s functions prior to the quench. $\delta r_i$ is a measure for the distance from the critical point for $t < 0$. Similarly, it holds that the equilibrium Keldysh function after the quench is given by

$$
iG^K_{0eq}(k) = \int_0^\infty ds \int_0^\infty ds' M(s, s', k^2) G^R_0(k, t - s) G^R_0(k, t - s'),
$$

i.e. the expression that follows if the system before and after the quench are in fact the same.

If we introduce

$$
\delta M(s, s') = M(s, s', \omega_0^2) - M(s, s', q^2),
$$

it follows

$$
I = \int_0^\infty ds \int_0^\infty ds' \delta M(s, s') \int_0^\infty dt t^{2-z} G^R_0(k, t - s) G^R_0(k, t - s').
$$

We express the time-dependent retarded Green’s functions via their expressions as function of frequency:

$$
G^R_0(k, t) = i \int_{-\infty}^{\infty} d\omega \text{Im} G^R_0(k, \omega + i0^+) e^{-i\omega t},
$$

and obtain:

$$
I = - \int_{-\infty}^{\infty} \frac{d\omega d\omega'}{\pi^2} \text{Im} G^R_0(k, \omega) \text{Im} G^R_0(k, \omega').
$$
\[
\times \int_0^\infty ds \int_0^\infty ds' \delta M (s, s') e^{i\omega s + i\omega' s'} \int_0^\infty dt t^{2-z} e^{-i\omega (t+t')}
\]

The integration over \( s \) and \( s' \) yield the two-time Laplace transform \( \delta M (\omega, \omega') \) of the memory function \( \delta M (t, t') \), while the integral over \( t \) can be performed analytically:

\[
\int_0^\infty dt t^{2-z} e^{-i\omega (t+t')} = \frac{\Gamma \left( \frac{2-z}{2} \right)}{(i (\omega + \omega' - i0^+))^z}.
\]

It follows

\[
I = -\frac{\Gamma \left( \frac{2-z}{2} \right)}{i z^2} \int_0^\infty d\omega d\omega' \frac{\text{Im} G^R_{0eq} (k, \omega) \text{Im} G^R_{0eq} (k, \omega') \delta M (\omega, \omega')}{(\omega + \omega' - i0^+)^z}.
\]

Our explicit result for the memory function \( M (\omega, \omega', k^2) \) is:

\[
M (\omega, \omega', q^2) = -\frac{G^R_{0eq} (k, \omega) + G^R_{0eq} (k, \omega')}{\omega + \omega' + i0^+} G^R (k, \omega)^{-1} G^R (k, \omega')^{-1}.
\]

If we consider \( M (\omega, \omega', \omega_0^2) \) prior to the quench we simply replace \( k \to \sqrt{\delta r_i} + k^2 \) everywhere in the last expression. \( G^R_{0eq} (k, \omega) \) is the Laplace transform of the bare Keldysh function in equilibrium:

\[
G^K_{0eq} (k, \omega) = \int_0^\infty dt e^{i(\omega + i0^+) t} G^K (k, t).
\]

Using the \( T = 0 \) fluctuation-dissipation relation

\[
G^K_{0eq} (t) = i \int_0^\infty \frac{d\epsilon}{\pi} e^{-i\epsilon t} \text{sign} (\epsilon) \text{Im} G^R (k, \epsilon),
\]

valid in equilibrium, it follows after a few steps of algebra that

\[
G^K_{0eq} (k, \omega) = \text{sign} (\omega) (G^R (k, \omega) - g (k, \omega)).
\]

with

\[
g (k, \omega) = 2 \int_0^\infty \frac{d\epsilon}{\pi} \frac{\text{Im} G^R (k, \epsilon)}{|\omega| + \epsilon}.
\]

This allows us to write:

\[
M (\omega, \omega', q^2) = -\frac{\text{sign} (\omega) n (\omega, \omega', q^2) + \text{sign} (\omega') n (\omega', \omega, q^2)}{\omega + \omega' + i0^+},
\]

where we introduced:

\[
n (\omega, \omega', q^2) = G^R (k, \omega')^{-1} - g (k, \omega) G^R (k, \omega)^{-1} G^R (k, \omega')^{-1}.
\]

These results yield

\[
\delta M (\omega, \omega') = -\frac{\text{sign} (\omega) \delta n (\omega, \omega') + \text{sign} (\omega') \delta n (\omega', \omega)}{\omega + \omega' + i0^+}
\]

where

\[
\delta n (\omega, \omega') = G^R (\omega_0, \omega')^{-1} - g (\omega_0, \omega) G^R (\omega_0, \omega)^{-1} G^R (\omega_0, \omega')^{-1}
- G^R (k, \omega')^{-1} + g (k, \omega) G^R (k, \omega)^{-1} G^R (k, \omega')^{-1}.
\]

Again, we use \( \omega_0 \) when we consider the bare Green’s function prior to the quench, i.e.

\[
G^R_{0} (k, \omega) = \frac{1}{-k^2 + \eta (\omega) - \eta (0)}
\]
\[ G^R_0(\omega_0, \omega) = \frac{1}{-\delta r_i - k^2 + \eta(\omega) - \eta(0)}. \]  

(54)

Here, we assumed that the \( \omega^2 \) term in Eq.(6) of the main paper can be neglected for \( t \gg t_\gamma \). The dynamics is then governed by coupling to the bath.

The retarded bath-self energy is given as

\[ \eta(\omega) = \eta(0) + \gamma \left( -\cot \frac{\pi}{2z} + \text{isign}(\omega) \right) |\omega|^{2/z}. \]  

(55)

Inserting the expressions for \( G^R_0(q, \omega) \) and \( G^R_0(\omega_0, \omega) \) in \( \delta n(\omega, \omega') \) we find:

\[ \delta n(\omega, \omega') = -\delta r_i - g(\omega_0, \omega) \left( -\delta r_i - k^2 + \eta(\omega) \right) \left( -\delta r_i - k^2 + \eta(\omega') \right) \\
+ g(k, \omega) \left( -k^2 + \eta(\omega) \right) \left( -k^2 + \eta(\omega') \right) \]  

(56)

To proceed we need to perform the scaling limit \( \delta r_i \to \infty \). It is useful to formulate these results in dimensionless variables. For the retarded function we have

\[ G^R_0(k, \omega) = \frac{1}{k^2} \phi^R \left( \gamma^{z/2} \omega / k^z \right) \]  

(57)

with \( \phi(x) = \frac{1}{|x|^{2/z}(\cot \frac{\pi}{2} + \text{isign}(x))^{-1}} \), which leads to

\[ g(k, \omega) = \frac{1}{k^2} \varphi \left( \gamma^{z/2} |\omega| / k^z \right) \]  

(58)

with

\[ \varphi(y) = -\frac{2}{\pi} \int_0^\infty \frac{dx}{y+x} \frac{x^{2/z}}{(1 + \cot \frac{\pi}{2} x^{2/z})^2 + x^{4/z}}. \]  

(59)

Note, one finds \( \varphi(0) = -1 \) for all \( z \). This result allows to perform the scaling limit \( \delta r_i \to \infty \): \( g(\omega_0, \omega) \to \frac{1}{\delta r_i + k^2} \varphi(0) = -\frac{1}{\delta r_i + k^2} \). Inserting this result we find that the scaling limit \( \delta r_i \to \infty \) can be performed and yields the nontrivial result:

\[ \delta n(\omega, \omega') = q^2 - \eta(\omega) - \eta(\omega') \\
+ g(q, \omega) \left( q^2 - \eta(\omega) \right) \left( q^2 - \eta(\omega') \right). \]  

(60)

Using our earlier dimensionless expressions, we write this as

\[ \delta n(\omega, \omega') = k^2 \delta \mu \left( \frac{\gamma^{z/2} \omega}{k^z}, \frac{\gamma^{z/2} \omega'}{k^z} \right), \]  

(61)

where

\[ \delta \mu(y, y') = \delta \mu_a(y, y') + \delta \mu_b(y, y') \]  

(62)

has been split according to

\[ \delta \mu_b(y, y') = 1 - h(y) - h(y'), \]
\[ \delta \mu_a(y, y') = \varphi(y) \left( 1 - h(y) \right) \left( 1 - h(y') \right), \]  

(63)

with \( h(y) = \left( -\cot \frac{\pi}{2z} + \text{isign}(y) \right) |y|^{2/z} \). We obtain for the dimensionless integral \( I_0 \)

\[ I_0 = \frac{\Gamma \left( \frac{z}{2} \right)}{\pi^{z/2}} \int_0^\infty \frac{dy'dy}{y + y' - i0^+} \frac{\Im \phi(y) \Im \phi(y')}{(y + y' - i0^+)^{2/z}} \]
\[ \times \left( \text{sign}(y) + \text{sign}(y') \right) \delta \mu_b(y, y') \\
+ \text{sign}(y) \delta \mu_a(y, y') + \text{sign}(y') \delta \mu_a(y', y) \right). \]  

(64)

Due to \( h(-y) = h^*(y) \) follows that \( I_0 = I_{0,a} + I_{0,b} \) is real, as expected. Here, we refer to the contributions to \( I_0 \) from \( \delta \mu_a \) as \( I_{0,a} \) and to the contributions due to \( \delta \mu_b \) as \( I_{0,b} \), respectively. For \( z \neq 2 \), we perform the above integration numerically. As shown next, for \( z = 2 \) \( I_0 \) can easily be determined analytically.
Analysis for $z = 2$

Next we analyze $I_0$ for $z = 2$. In this limit follows

$$\varphi(y) = -\frac{1 + \frac{2}{\pi} |y| \ln |y|}{1 + y^2}$$

and $\text{Im} \phi (y) = -\frac{y}{1+y^2}$, as well as $h (y) = iy$, which yields $\delta \mu_b (y, y') = 1 - i (y + y')$. This leads to:

$$I_{0b} = -4 \int_0^{\infty} \frac{dy dy'}{\pi^2} \frac{y}{1+y^2} \frac{y'}{1+y'^2} \frac{1}{y+y'} = -\frac{2}{\pi}.
$$

In addition, we have to analyze

$$I_{0a} = -4 \int_0^{\infty} \frac{dy dy'}{\pi^2} \frac{\text{Im} \phi (y) \text{Im} \phi (y') \varphi (y)}{y+y'}$$

$$-\quad 4 \mathcal{P} \int_0^{\infty} \frac{dy dy'}{\pi^2} \frac{\text{Im} \phi (y) \text{Im} \phi (y') (y-y') \varphi (y')}{(y-y')^2}$$

The first term gives $\frac{2}{3\pi}$. In the second term we must perform one integral via principle-value integration and obtain $\frac{1}{3\pi}$, such that $I_{0a} = \frac{1}{3\pi}$. This finally determines the integral $I_0 = -\frac{1}{2}$ of Eq. (36). Performing the limit $z \rightarrow 2$ in the coefficient in Eq. (35) in front of $I_0$, we finally obtain the result for the exponent given in the main paper:

$$\theta (z = 2) = \frac{1}{4} \frac{N + 2}{N + 8} \epsilon.$$

(68)