Disordered Dirac Fermions: Multifractality Termination and Logarithmic Conformal Field Theories

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We reexamine in detail the problem of fermions interacting with a non-Abelian random vector potential. Without resorting to the replica or supersymmetry approaches, we show that in the limit of infinite disorder strength the theory possesses an exact solution which takes the form of a logarithmic conformal field theory. We show that the proper treatment of the locality conditions in the SU(2) theory leads to the termination of the multifractal spectrum, or in other words to the termination of the infinite hierarchies of negative-dimensional operators that were thought to occur. Based on arguments of logarithmic degeneracies, we conjecture that such a termination mechanism should be present for general SU(N). Moreover, our results lead to the conclusion that the previous replica solution of this problem yields incorrect results.

I. INTRODUCTION

There now exists a great deal of evidence that the wavefunctions in disordered systems near a localization-delocalization transition exhibit multifractal behaviour. In other words, the moments of the density of states scale not according to one given fractal dimension like in a simple fractal, but rather according to an infinite set of scaling exponents. Since the most interesting features of multifractality are beyond the reach of perturbation theory, one needs to employ non-perturbative approaches. In this context the exactly solvable problem of Dirac fermions in a random vector potential in two dimensions has proved to be a very instructive training ground. This model was first considered in \cite{1} in connection with the integer quantum Hall transition. Recently exact results for the full multifractal spectrum for Dirac fermions interacting with a random magnetic field (i.e. an Abelian vector potential) have been obtained from a Gaussian field theory in an ultrametric space \cite{2}, and reproduced by mapping to a random energy model \cite{3}.

The non-Abelian random vector potential case, originally introduced in \cite{4} as an effective model for low-energy excitations around the nodes of a d-wave superconductor, has very recently appeared in physical problems of a very different nature, namely in non-Hermitian quantum mechanics \cite{5}. There, it appears as the effective theory of a system driven by a strong imaginary vector potential in the limit of weak disorder. Non-Hermitian quantum mechanics describes such problems as anomalous diffusion in a random media \cite{6} and statistical mechanics of flux lines in superconductors \cite{7}. The multifractality is observed in impurity-averaged correlators of local moments of wavefunctions, for which expressions for the non-Abelian part need to be computed. Although these correlators were investigated in \cite{8,9}, we wish to report some new and surprising aspects of the problem that were overlooked. An earlier presentation of some initial results can be found in \cite{10}.

The non-Abelian Random Vector Potential (RVP) has already been treated by both standard approaches for disordered systems, namely the replica and SUSY ones. The SUSY treatment can be found in \cite{8}, while the replica treatment was put forward in \cite{4} and carried through in \cite{9}. It was first recognized in this publication that the associated conformal theory should be of a special class, namely a logarithmic one. However, both of these lines of investigation suffered from various setbacks. First of all, the SUSY approach seemingly generated an infinite series of operators with negative conformal dimensions for SU(N) randomness. On the other hand, the replica approach gave a different physical content, coming from the fact that the conformal blocks of the theory were slightly different. There was consequently an urgent need to provide a better comprehension of this system by performing some more careful and extensive investigations.

As was pointed out in \cite{11}, the disorder averaging in the non-Abelian RVP problem can be performed without using the SUSY or replica approaches in the limit of vanishing frequency and infinite disorder strength. This limit corresponds to the conformal limit of the theory, which was studied in \cite{8}. The RVP model thus is a critical disordered system, which makes it theoretically very interesting in its own right. In addition, the logarithmic nature of the CFT involved calls for better understanding.

Usually, CFTs are synonymous with power-law dependence of physical correlators. Much work has been done about the unitary minimal models, for which there exists a unitary finite-dimensional representation of the Virasoro algebra. However, recently it has become clear that logarithmic dependence can appear for models outside of this...
class. The first instance in which logarithms were introduced in conformal correlators is to be found in [12]. It was first recognized in [3] that this type of logarithmic behaviour was associated to non-diagonalizability of the Virasoro operator. Subsequent applications of logarithmic operators were found in critical disordered models [15,17], critical polymers and percolation [10,18], gravitationally dressed CFTs [19], two-dimensional magnetohydrodynamic turbulence [20,21], D-brane recoil in string theory [22,24], and were recently shown to correspond to a novel bulk excitation in the Quantum Hall state [25]. Further new developments about logarithmic CFTs can be found for example in [26–31].

This paper aims to resolve some important remaining issues related to fermions interacting with a non-Abelian random vector potential in the conformal limit. Our main result is to show that the correct treatment of the full conformal field theory that is obtained from the SUSY or direct treatment (which coincide) leads to the termination of the parabolic multifractal spectrum before the scaling exponents \( \tau(q) \) reach their maximal value. They were previously thought to follow this parabolic law to arbitrarily negative values.

The paper is organized as follows. In section II we introduce the model and present the conformal limit. In section III we manipulate the theory at the critical point and recall how to average over disorder without SUSY or replicas. In section IV we explicitly construct the free-field formulation of the theory for \( N > 2 \). Section V discusses the consequences on the replica solution, and we finish with conclusions in section X.

II. THE MODEL

We consider \( N \) species of Dirac fermions living in a 2+1-dimensional space and interacting through a disordered vector potential \( A_\mu \) belonging to an \( su(N) \) algebra \( \mathcal{A} \), to which they are coupled minimally. The disorder allows for hopping between the different species. Since the vector potential is time-independent, different Matsubara frequencies do not couple, and can be treated independently by a Euclidean two-dimensional theory with explicit frequency dependence. In fact, for a given realization of the disorder, the partition function is given by the fermionic path integral

\[
Z[A_\mu] = \int \mathcal{D}[\bar{\Psi}, \Psi] e^{-S[\bar{\Psi}, \omega, A_\mu]}
\]

with the Dirac action

\[
S[\bar{\Psi}, \omega, A_\mu] = \int d^2x \ \bar{\Psi}(x) \left( \mathbf{1} \otimes \dot{\Psi} - i \omega + i \mathcal{A} \right) \Psi(x)
\]

(since we are in a two-dimensional Euclidean space, we take the Pauli matrices as Dirac \( \gamma \) matrices, i.e. \( \mathcal{A} = A_\mu \otimes \sigma^i, \mu = 1, 2; \bar{\Psi} = \Psi^\dagger \sigma^1 \)).

Disorder-dependent single-particle Green’s functions can then be represented by the path integral

\[
G(x, y; \omega; A_\mu) = \frac{-i}{Z[A_\mu]} \left. \int \mathcal{D}[\bar{\Psi}, \Psi] \right| \Psi(x) \bar{\Psi}(y)e^{-S[\bar{\Psi}, \omega, A_\mu]}
\]

Physical quantities are obtained by performing the disorder averaging procedure on products of Green’s functions. We use the distribution functional

\[
\ln P[A_\mu] = -\frac{1}{g} \int d^2x \ \text{Tr}A_\mu(x)A_\mu(x)
\]

representing the usual \( \delta \)-correlated Gaussian white noise for the random vector potential.

The impurity-averaged Green’s function then reads

\[
G(x, y; \omega) = \overline{G(x, y; \omega; A_\mu)} = \int \mathcal{D}A_\mu G(x, y; \omega; A_\mu)P[A_\mu] = -i \int \mathcal{D}A_\mu \frac{P[A_\mu]}{Z[A_\mu]} \left. \int \mathcal{D}[\bar{\Psi}, \Psi] \right| \Psi(x) \bar{\Psi}(y)e^{-S[\bar{\Psi}, \omega, A_\mu]}
\]

In the limits of infinite disorder strength \( g \to \infty \) and of vanishing frequency \( \omega \to 0 \), the theory becomes conformally invariant. We will formulate and solve for correlation functions at this conformal point, and later use renormalization group arguments to infer the scaling behaviour of physical quantities like the density of states and its local moments away from criticality. We will use the notation \( G(x_1, ..., x_n) \) to denote the impurity-averaged \( n \)-point correlator at zero frequency.
III. THE THEORY AT THE CRITICAL POINT

In the following sections, we will be concerned with the step-by-step formulation of the theory at the conformally invariant critical point, i.e., for infinite disorder strength and vanishing frequency. The principles of this approach have been formulated by Bernard [11] and Mudry et al. [3]. In this work we show that the path integral is perfectly well-defined and negative-dimensional operators do not appear in the physical operator OPEs.

We start by separating the fermionic action into chiral parts:

\[
S[\Psi, A_\mu] = S_+[\Psi, A_+] + S_-[\Psi, A_-]
\]

\[
S_\pm[\Psi, A_\pm] = \int d^2x \bar{\Psi}_\pm \partial_x \pm iA_\pm \Psi_\pm(x)
\]

where we have used the holomorphic and antiholomorphic derivatives and fields (2\(\bar{\theta}\) = \(\partial_-\), 2\(\bar{\bar{\theta}}\) = \(\partial_+\))

\[
\partial_\pm = \partial_1 \pm i\partial_2
\]

\[
A_\pm = A_1 \pm iA_2
\]

where now \(A_\pm \in su^C(N) = sl(N, \mathbb{C})\), the complex extension \(A^C\) of \(A\). This translates, at the level of the path integration over the vector fields, into transforming the double integral of \(A_i\) over \(su(N)\) into a single integral over \(su^C(N)\). The non-compact nature of the \(SU^C(N)\) group manifold, in contrast to \(SU(N)\), will be of crucial importance later on.

We parametrize the vector fields by fields \(g_\pm\) belonging to the complex extension \(G^C\) of the group \(G = SU(N)\):

\[
A_\pm(x) = i\partial_\pm g_\pm(x)g_\pm^{-1}(x)
\]

The reality condition \(A_\pm^\dagger(x) = A_-(-x)\) translates into \(g_\pm^\dagger(x) = g_-^{-1}(x)\). From now on, we will use the notation \(g_+(x) = g(x)\).

Let us study, for a little while, the problem at fixed disorder. We can observe that the transformations

\[
\Psi_\pm(x) \rightarrow g_\pm(x)\Psi_\pm'(x), \quad \Psi_\pm^\dagger(x) \rightarrow \Psi_\pm'^\dagger(x)g_\pm^{-1}(x)
\]

completely decouple the fermions from the random vector potential. It maps the minimally coupled Dirac action for \(\Psi\) fermions into the free Dirac action for \(\Psi'\) fermions, i.e.

\[
S[\Psi, A_\mu] \rightarrow S[\Psi'] = \int d^2x [\bar{\Psi}'_\dagger \partial_- \Psi'_+ + \bar{\Psi}'_\dagger \partial_+ \Psi'_-]
\]

generating the correlators

\[
\langle \Psi'_+(z_1)\Psi'^\dagger_+(z_2) \rangle = \frac{1}{2\pi z_{12}}
\]

\[
\langle \Psi'_-(z_1)\Psi'^\dagger_-(z_2) \rangle = \frac{1}{2\pi z_{12}}
\]

The Jacobian for this transformation,

\[
\frac{D[\Psi, \bar{\Psi}]}{D[\Psi', \bar{\Psi}']} = \frac{\text{Det}[\partial + iA]}{\text{Det}[\partial]} \propto Z[A]
\]

has the very important property of being proportional to the partition function at fixed disorder (up to the anomalous determinant \(\text{Det}[\partial]\), which is irrelevant for computation of correlation functions but contributes to the total central charge), thus cancelling it when computing the correlations for fixed disorder, i.e.,

\[
G(x_1, \ldots) = \int DA_\mu \frac{1}{Z[A]} \int \mathclap{D[\Psi, \bar{\Psi}] \Psi \ldots e^{-S[\Psi, A]}} = \int \mathclap{DA_\mu \int D[\Psi', \bar{\Psi}'] g\Psi' \ldots e^{-S[\Psi']}}
\]

This removes the need to invoke either the replica or supersymmetry methods to perform explicitly the disorder averaging. This also implies that we are not dealing with a free energy functional, and that disconnected diagrams do appear in a perturbation expansion.
Using this parametrization in the Jacobian (14) gives
as
being the Haar measure on $du$
explicitly performed with the help of the Iwasawa decomposition of $SU_N$
to proceed to the factorization of $SU_N$ then factorizes into the product of the Haar measures on $SU(N)$, $H \in SU(\phi(N))/SU(N)$ (the reality condition states then that $H^{-1} = H^\dagger$). This procedure induces a non trivial Jacobian in the path integral:

$$\mathcal{D}A_\mu = \mathcal{D}A_1 \mathcal{D}A_2 = \mathcal{D}A_{\text{gauge}} \mathcal{D}A_{\text{coset}} = \mathcal{D}A_{\text{gauge}} \mathcal{D}H e^{2\pi N W[H^\dagger H]}$$  \hspace{1cm} (14)

where $\mathcal{D}H$ is the integration measure over the coset $SU_N(SU(N))$, and $N$ corresponds to the dual Coxeter number of $SU(N)$. $W[h]$ is the WZNW functional ($\partial B = S$) for the field $h \equiv H^\dagger H$ on level $k = -2N$ (note that the topological term then has positive sign):

$$W[h] = \frac{1}{8\pi} \int_S d^2x \text{Tr}[\partial_\mu h \partial_\mu h^{-1}] + \frac{i}{12\pi} \int_B d^3x \epsilon_{\mu\nu\lambda} \text{Tr}[\partial_\mu hh^{-1} \partial_\nu hh^{-1} \partial_\lambda hh^{-1}]$$  \hspace{1cm} (15)

Thus, the partition function for $N$ species of Dirac fermions interacting with an $su(N)$ random vector potential has completely factorized in the product of two independent sectors: a free fermion (disorder independent) part, and an $SU_C(N)$ part in which all disorder dependence is concentrated.

IV. COSET FACTORIZATION AND FREE FIELD FORMULATION OF THE WZNW MODEL

The important point to notice from the previous formulas is that, since the WZNW model depends only on the combination $H^\dagger H$ which is invariant under left-multiplication by $u \in SU(N)$, i.e. $H \rightarrow uH$, it is advantageous to proceed to the factorization of $SU_N(SU(N))$ into the subgroup $SU(N)$ and the coset $SU_C(N)/SU(N)$. This can be explicitly performed with the help of the Iwasawa decomposition of $SU_N(N)$ \cite{34}. The Haar measure on $SU_C(N)$ then factorizes into the product of the Haar measures on $SU_C(N)/SU(N)$ and $SU(N)$.

For simplicity, we specialize now to $N = 2$. Our derivation follows essentially the approach to WZNW models as free field theories to be found in \cite{32}. For $g \in SU(2)$, the Iwasawa decomposition takes the form

$$g = u \begin{pmatrix} e^{\phi/2} & 0 & 1 \\ 0 & e^{-\phi/2} & \mu_+ \\ 1 & 0 & 1 \end{pmatrix}$$  \hspace{1cm} (16)

with $u \in SU(2)$ and $\phi \in \mathbb{R}$, $\mu_+ \in \mathbb{C}$. The integral measure for this parametrization is then given by (see derivation in the Appendix)

$$dg = du \ e^{2\phi} d\phi d\mu_+ d\mu_-$$  \hspace{1cm} (17)

du being the Haar measure on $SU(2)$.

The field appearing in the Jacobian (14), being $SU(2)$ independent, can be expressed in terms of the fields ($\phi, \mu_+, \mu_-$) as

$$h = H^\dagger H = \begin{pmatrix} e^\phi & \mu_+ e^{\phi} \\ \mu_- e^\phi & e^{-\phi} + \mu_+ \mu_- e^\phi \end{pmatrix}$$  \hspace{1cm} (18)

Using this parametrization in the Jacobian (14) gives

$$kW[h] = \frac{-k}{4\pi} \int d^2x [(\partial_\mu \phi)^2 + e^{2\phi} \partial_- \mu_+ \partial_+ \mu_-]$$  \hspace{1cm} (19)

as the action for the $SU_C(2)/SU(2)$ fields ($\phi, \mu_+, \mu_-$). For negative $k$, this action is positive definite. Once again: even though the Jacobian (14) induces a WZNW model with a negative level, which is not well-defined at the level of the path integral for a group manifold with a positive metric, the fact that the WZNW field $h$ belongs to the coset $SU_C(2)/SU(2)$ which has a negative-definite metric makes the coset action a positive-definite functional. This is reminiscent of the coset constructions on non-compact manifolds to be found in \cite{33}, where the coset formed of the non-compact group and its maximally compact subgroup forms a well-defined unitary conformal theory when one considers a negative level for the underlying Kac-Moody algebra.
The WZNW model has the property of being invariant with respect to Kac-Moody chiral current algebras. The Kac-Moody currents, given by (for negative level)

\[ J(z) = -k \partial h h^{-1}, \quad \bar{J}(\bar{z}) = -k h^{-1} \bar{\partial} h \]  

have, in terms of the new fields, the representation

\[ J(z) = J^+ \sigma^- + J^- \sigma^+ + H \sigma \]

\[ J^+ = -k[2\mu_\pm \partial \phi + \partial \mu_\pm - \mu_+^2 e^{2\phi} \partial \mu_+] \]

\[ J^- = -ke^{2\phi} \partial \mu_+ \]

\[ H = -k[\partial \phi - \mu_- e^{2\phi} \partial \mu_+] \]

\[ \bar{H} = H^\dagger \quad \bar{J}^\pm = J^{\mp \dagger} \]  

In the path integration over \((\phi, \mu_+, \mu_-)\), the presence of the factor \(e^{2\phi}\) in the measure as well as the presence of the combination \(e^{2\phi} \partial \mu_+\) in the expression for the current \(J(z)\) make the change of variables

\[ \omega_-(z) = -ke^{2\phi} \partial \mu_+ \]  

very convenient, since it brings us to a path integration over free fields.

This and the properly regularized anomalous determinant associated to this transformation [32]

\[ \text{Det}[e^{2\phi} \partial] \rightarrow (\text{Det}[e^{-2\phi} \bar{\partial} e^{2\phi} \partial])^{1/2} \]  

which shifts the action for \(\phi\) by

\[ \frac{1}{4\pi} \int d^2 x [2(\partial_\mu \phi)^2 + R \phi] \]  

modify the coset action to

\[ S[\phi, \omega_-, \mu_-] = \frac{1}{4\pi} \int d^2 x [(-k - 2)(\partial_\mu \phi)^2 + R \phi] + \frac{1}{2\pi} \int d^2 x \omega_- \partial_\mu \mu_- \]  

i.e. the anomaly shifts \(k \rightarrow k + 2 \equiv -q^2\) (recall that in the given case \(k = -4, q^2 = 2\)) in front of \(\phi\) and introduces the Riemann curvature \(R\) of the manifold in the action for \(\phi\), which will change its conformal properties. By rescaling \(\phi \rightarrow \frac{1}{\sqrt{q}} \phi'\), the action and Operator Product Expansions for our free fields read

\[ S[\phi', \omega_-, \mu_-] = \frac{1}{8\pi} \int d^2 x [(\partial_\mu \phi')^2 + \sqrt{\frac{2}{q}} R \phi'] + \frac{1}{2\pi} \int d^2 x \omega_- \partial_\mu \mu_- \]  

\[ \phi'(z_1) \phi'(z_2) = -\ln z_{12}, \quad \bar{\phi}'(\bar{z}_1) \bar{\phi}'(\bar{z}_2) = -\ln \bar{z}_{12}, \quad \mu_-(z_1) \omega_-(z_2) = \frac{1}{z_{12}} \]  

in which we have performed the separation of \(\phi'\) in terms of holomorphic and anti-holomorphic parts \(\phi'(z, \bar{z}) = \phi(z) + \bar{\phi}(\bar{z})\), valid at the level of correlation functions.

The proper expression for the currents in terms of our free fields is then given by [23], but only after performing the same transformations as above on \(\phi\), i.e. the shift \(k \phi \rightarrow (k + 2) \phi\) and the rescaling \(\phi \rightarrow \frac{1}{\sqrt{q}} \phi'\):

\[ J^-(z) = \omega_-(z) \]

\[ J^+(z) = \sqrt{2} q \mu_-(z) \partial \phi'(z) - k \partial_\mu_-(z) - \mu_+^2(z) \omega_-(z) \]

\[ H(z) = \frac{1}{\sqrt{2}} q \partial \phi'(z) - \mu_- (z) \omega_-(z) \]  

In all of these, the usual normal-ordering procedure is implied. The currents have the correct OPEs for generators in the Cartan-Weyl basis of \(SU(2)_k\), i.e.
Thus, what we have done is the following: by using the Iwasawa decomposition, we have explicitly shown that the level \( k = -2N = -4 \cos \text{cosec} SU^C(2)/SU(2) \) possesses a Wakimoto free field representation with Cartan-Weyl generators carrying an affine \( SU(2) \) current algebra with level analytically continued to negative value.

The stress-energy tensor

\[
T(z) = -\frac{1}{q^2} \frac{1}{2} \left( J^+ J^- + J^- J^+ \right) + H^2 \]

is, in terms of our free fields,

\[
T(z) = -\frac{1}{2} \partial \phi' \partial \phi' - \frac{1}{\sqrt{2q}} \partial^2 \phi' - \omega_- \partial \mu_- \]

The central charge is readily checked to be \( c = 3k/(k+2) \) as for the normal case of positive \( k \). The \( \phi' \) part of \( T(z) \) is that of a Dotsenko-Fateev free field with imaginary background charge \( \alpha_0 = i/(2q) \). Vertex operators \( V_\alpha(z) = e^{i\phi'(z)} \) then have holomorphic conformal weight \( h_\alpha = -(\alpha/2)(\alpha + \sqrt{2}/q) \).

Primary operators \( V_G \) can be found by requiring the correct OPEs with \( T(z) \) and the generators of the Kac-Moody algebra. A highest-weight finite-dimensional representation is available, composed of the \( 2j+1 \) operators \( V_{jm}(z) \) given by

\[
V_{jm}(z) = e^{\pm \phi'(z)} \{-\mu_-(z)\}^{j+m} \]

for which the OPEs with the currents have the correct \( SU(2) \) form

\[
J^\pm(z_1) V_{jm}(z_2) = \frac{j \mp m}{z_{12}} V_{jm \pm 1}(z_2) \]

\[
H(z_1) V_{jm}(z_2) = \frac{m}{z_{12}} V_{jm}(z_2) \]

These primary operators all possess identical conformal weights

\[
h_{jm} = -\frac{\alpha}{2}(\alpha + \frac{\sqrt{2}}{q}) = -\frac{j(j+1)}{q^2} = \frac{j(j+1)}{k+2} \]

(note that \( \mu_- \) is a weight zero operator, as can be seen from its OPE with \( T(z) \), and consequently does not contribute to the weight of \( V_{jm} \). Weights are thus negative in any spin-\( j \) representation for our value \( k = -4 \).

To calculate correlation functions, we follow the general procedure outlined in \( \text{[33]} \). This starts with the introduction of dual operators \( \tilde{V}_{jm} \), generalizing the usual dual vertex operators of the Coulomb-gas formalism. The highest-weight operator \( \tilde{V}_{j-1} \) is given by

\[
\tilde{V}_{j-1}(z) = e^{-\frac{\alpha}{2}(3+j)\phi'(z)} \omega_-^{3+2j}(z) \]

Acting on this with \( J^+ \) produces the other states in the representation, but there is a slight setback: they do not have such a simple form as \( \text{[37]} \). To obtain a correlator involving another dual state than \( \text{[37]} \), the trick is to use the conformal Ward identities and derive it from another correlator involving \( \text{[37]} \). For future reference, we here give the two dual operators in the fundamental spin-\( 1/2 \) representation:

\[
\tilde{V}_-(z) \equiv \tilde{V}_{1/2,-1/2} = e^{-\frac{\alpha}{2} \phi'(z)} \omega_-^4(z) \\
\tilde{V}_+(z) \equiv \tilde{V}_{1/2,1/2} = 4\sqrt{2q} \partial \phi'(z) \omega_-^3(z) - \mu_-^4(z) \omega_-^2(z) - 12 \partial_\omega_-(z) \omega_-^2(z) \]
\[
\sum \alpha_i = -\frac{6}{\sqrt{2}q} = -3, \quad N_\omega - N_{\mu_-} = 3
\] (39)

with \( \alpha \) understood as the numerical coefficient of the vertex operator, used as in \( e^{\sqrt{2}q} \phi' \), and where \( N_\omega \) and \( N_{\mu_-} \) count these respective operators in the correlator. Note that this last condition is equivalent to the requirement that the correlator be a singlet state, e.g. that it contains an equal number of + and − indices in the spin-1/2 case.

The last ingredient we will need is the screening operator. It can be determined in two ways: first, it must commute with all current generators up to a total differential; but also, it can be found from the change of variables (24). Inverting this, we can see that over every noncontractable cycle on the Riemann surface, we must have

\[
\oint d\zeta \omega_-(\zeta)e^{-\sqrt{2}q}\phi'(\zeta) = -k \oint d\mu_+ = 0
\] (40)

Enforcing this condition can be done by putting in the path integral the (infinite) product, over all closed cycles, of \( \delta \) functions

\[
\prod_e \delta(\oint \omega_- e^{-\sqrt{2}q}\phi') = \int d\lambda \exp(i\lambda \oint \omega_- e^{-\sqrt{2}q}\phi')
\] (41)

Expanding in powers of \( \lambda \) then produces the screening charge insertions, which we will denote by the operator \( Q(z) \). We have explicitly constructed the holomorphic operators of the theory. An identical procedure can be done for the antiholomorphic part, using instead of (24) the conjugate transformation

\[
\omega_+(\bar{z}) = -ke^{2\phi} \delta \mu_-
\] (42)

This way, primary highest-weight states of the antiholomorphic currents can be constructed, as well as their duals, and a conjugate screening charge can be found. We will denote all of these by \( V, \) etc.

The matrix field \( h(z, \bar{z}) \) can then be expressed, in correlation functions, in terms of the tensor product of left and right primary spin-1/2 fields as

\[
h(z, \bar{z}) = \begin{pmatrix}
V_-(z)V_-(\bar{z}) & V_-(z)V_+(\bar{z}) \\
V_+(z)V_-(\bar{z}) & V_+(z)V_+(\bar{z})
\end{pmatrix}
\] (43)

The two-point functions of primary fields

\[
\langle V_\pm(z_1)V_\mp(z_2) \rangle = z_{12}^{3/4}
\] (44)

give for the coset two-point correlator

\[
\langle h_{ab}(z_1, \bar{z}_1)h^{-1}_{a'b'}(z_2, \bar{z}_2) \rangle = \delta_{a_1a_2}\delta_{b_1b_2}|z_{12}|^{3/2}
\] (45)

We have thus seen how coset correlations can be simply obtained by calculating the relevant conformal blocks of our primary fields. This will be done for the four-point function in the following section.

V. MULTI POINT CORRELATION FUNCTIONS

Let us start by moving back to our original physical problem, and see what we are now capable of doing. By considering the limiting case \( \bar{q} \to \infty \), we have decomposed our original fermions into free fermions \( \psi' \) together with \( SU^C(N)/SU(N) \) coset operators \( h \). Thus, by considering correlators of the local operators

\[
\mathcal{O}(z, \bar{z}) = \Psi^\dagger_{-a}\Psi_{+a} = \Psi^\dagger_{-a}h_{ab}\Psi^\prime_{+b} \\
\mathcal{O}^{-1}(z, \bar{z}) = \Psi^\prime_{+a}\Psi^-_{-a} = \Psi^\dagger_{+a}h_{ab}^{-1}\Psi^-_{-b}
\] (46)

we explicitly sum over the \( SU(N) \) indices, leaving only the coset operators once the decoupling transformation (3) has been done. The \( SU(N) \) path integration then simply factorizes out of the effective generating functional.

The two-point function
shows that the operator \( O \) has conformal weights \((1/8, 1/8)\) (i.e. a spinless operator of dimension 1/4). This reproduces the previously known result [3].

Let us now turn to the four-point function. By performing the fermionic contractions, we get

\[
\langle O(1)\hat{O}^{-1}(2)\hat{O}(3)\hat{O}^{-1}(4) \rangle \sim \frac{1}{|z_{12}|^2} \langle \text{Tr}[h(1)h^{-1}(2)]\text{Tr}[h(3)h^{-1}(4)] \rangle + \frac{1}{z_{14}z_{23}z_{12}z_{34}} \langle \text{Tr}[h(1)h^{-1}(2)h(3)h^{-1}(4)] \rangle + (2 \leftrightarrow 4)
\]

By using (43), we can see that all correlators appearing in (48) can be expressed in terms of the three vertex correlators

\[
C_1 = \langle V_+(z_1)V_-(z_2)V_+(z_3)V_-(z_4) \rangle \quad C_2 = \langle V_+(z_1)V_+(z_2)V_-(z_3)V_-(z_4) \rangle \quad C_3 = \langle V_-(z_1)V_+(z_2)V_+(z_3)V_-(z_4) \rangle
\]

These can be calculated using the prescription of the last section: we replace the operator in position 4 by its dual (note that, in the definitions above, we have performed an overall spin flip, which does not change the correlators but allows to use the more simple expression \( \hat{V}_- \), and insert the necessary screening charges (here, only one). We use the projection invariance to perform the analytic mapping \((z_1, z_2, z_3, z_4) \rightarrow (0, z, 1, \infty), z = \frac{z_{14}z_{23}}{z_{12}z_{34}}\), for which we get

\[
C_i(z_1\ldots z_4) = \left[ z_{13}z_{24} \right]^{3/4} C_i(z), \quad C_i(z) = \lim_{z_{\infty} \rightarrow \infty} z_{\infty}^{3/4} C_i(0, z, 1, z_{\infty})
\]

Performing the integrals (with \( C_i(z) = [z(1-z)]^{3/4} \tilde{C}_i(z) \)) yields

\[
\tilde{C}_a^1(z) = F(3/2, 3/2; 2; z) \\
\tilde{C}_a^2(z) = F(1/2, 3/2; 2; 1-z) \\
\tilde{C}_b^1(z) = \frac{1}{2} F(1/2, 3/2; 2; 1-z) \\
\tilde{C}_b^2(z) = \frac{1}{2} F(1/2, 3/2; 2; z)
\]

in which the two independent solutions for each correlator are labeled by the integration contours \( a : (0, z), b : (z, 1) \) used in the screening. The \( b \) contour solutions have logarithmic behaviour near \( z = 0 \), whereas the \( a \) ones have logarithmic behaviour near \( z = 1 \). We are thus in the presence of a logarithmic conformal field theory, whose first examples appeared in [4,5].

The “full” correlator, in which the \( \bar{z} \) dependence has been included, has to be built by combining the conformal blocks in such a way as to satisfy the crossing symmetry constraints, as well as the requirement of single-valuedness. Crossing symmetry is easily satisfied in view of the extremely simple monodromy properties of the solutions [5]. The constraint of single valuedness (for example, around \( z = 0 \)) forces the cross-multiplication of \( a \) and \( b \) solutions respectively in the \( z \) and \( \bar{z} \) sectors, without which terms of the form \( \ln z \ln \bar{z} \) would appear, which obviously are to be excluded in a single-valued correlator.

In fact, by explicitly multiplying out (43) and using the above solutions, we find, using the standard identities

\[
F(3/2, 3/2; 2; z) + \frac{1}{2} F(1/2, 3/2; 2; 2-z) = \frac{3}{2} (1-z) F(3/2, 5/2; 2; z) \\
F(3/2, 3/2; 2; z) - F(1/2, 3/2; 1; z) = \frac{3}{8} z F(3/2, 5/2; 3; z)
\]

that the four-point correlator can be written as a function of the conformal blocks
\[ F^a_i(z) = 2\sqrt{2}(1 - z)F(3/2, 5/2; 2; z) \]
\[ F^b_j(z) = (1 - z)F(3/2, 5/2; 3; 1 - z) \]
\[ F^c_k(z) = zF(3/2, 5/2; 3; z) \]
\[ F^d_l(z) = 2\sqrt{2}zF(3/2, 5/2; 2; 1 - z) \]  

(52)

These can be verified to be the conformal blocks that one would obtain by solving straightforwardly the SU(2)\textsubscript{\textminus 4} Knizhnik-Zamolodchikov equations. We have obtained them using a somewhat different route, through the coset parametrization. Thus, even though naively the SU(2)\textsubscript{\textminus 4} WZNW model seems ill-defined at the level of the path integral (since the kinetic term, for a negative level, has the wrong sign!), the fact that we are in reality working on the coset SU\textsuperscript{C}(2)/SU(2) which has been shown above to carry the same current algebra makes the use of the KZ equations valid. Even though we have proved this in detail only for \( N = 2 \), we can expect that the same will be true for \( N > 2 \) as well. One very important remark to make at this point is that these conformal blocks do not reproduce the conformal blocks obtained within the replica approach \[ 9 \]. Although the operator dimensions are correctly reproduced, the correlators and OPEs are incorrect. We will return to this point later on.

The full coset correlator

\[ H_{a_1...b_4}(z_1, z_2, z_3, z_4) = \frac{1}{\sqrt{2}}H_{a_1...b_4}(z, \bar{z}) \]

\[ = \langle h_{a_1b_1}(z_1, \bar{z}_1) h_{a_2b_2}^{-1}(z_2, \bar{z}_2) h_{a_3b_3}(z_3, \bar{z}_3) h_{a_4b_4}^{-1}(z_4, \bar{z}_4) \rangle \]

(53)

can now be built as mentioned above, by solving the monodromy problem, and insuring single-valuedness in the complex plane. The correlator can be projected as usual onto the singlets

\[ I_1 = \delta_{a_1a_2}\delta_{a_3a_4} \quad I_2 = \delta_{a_1a_4}\delta_{a_2a_3} \]

\[ H_{a_1...b_4}(z, \bar{z}) = \sum_{i,j=1,2} I_i H_{ij}(z, \bar{z}) \]

(54)

with, by single-valuedness,

\[ H_{ij}(z, \bar{z}) = \alpha|z(1 - z)|^{3/2}[\tilde{F}^a_i(z)\tilde{F}^b_j(\bar{z}) + (a \leftrightarrow b)] \]

(55)

where \( \alpha \) is some constant chosen to satisfy consistency with the normalization of the two-point function.

The full four-point correlator for the local operators \[ \[ 4 \], with free fermion contribution, finally reads

\[ \langle O(1)...O^{-1}(4) \rangle \propto \frac{1}{\sqrt{2}} \left\{ \frac{P(I_1)}{z_{12} z_{23} z_{24}} + \frac{P(I_2)}{z_{13} z_{23} z_4} \right\} \left\{ \frac{\tilde{P}(\tilde{I}_1)}{\tilde{z}_{12} \tilde{z}_{23} \tilde{z}_{24}} + \frac{\tilde{P}(\tilde{I}_2)}{\tilde{z}_{13} \tilde{z}_{23} \tilde{z}_{4}} \right\} H_{a_1...b_4}(z, \bar{z}) \]

(56)

where \( P \) and \( \tilde{P} \) are projectors onto the singlets, i.e. \( P(I_i)\tilde{P}(\tilde{I}_j)H_{a_1...b_4} = H_{ij} \).

For further discussion we shall need the four-point correlator of the physical field \( M \) (related to the local density of states), which is composed of the sum of the two operators \[ \[ 4 \]:

\[ M(z, \bar{z}) = \text{Tr}\bar{\Psi}\Psi = O(z, \bar{z}) + O^{-1}(z, \bar{z}) \]

(57)

The correlator we are interested in is made up of the different permutations of \( \[ 3 \] that appear when we make use of \[ \[ 7 \]. In fact, it reads

\[ \langle M(1)M(2)M(3)M(4) \rangle = 2\langle O(1)O^{-1}(2)O(3)O^{-1}(4) \rangle + (2 \leftrightarrow 3) + (3 \leftrightarrow 4) \]

(58)

The permutation \( 2 \leftrightarrow 3 \) maps \( z \) to \( 1/z \), whereas \( 3 \leftrightarrow 4 \) maps it to \( z/(z - 1) \). The analytic continuation of the conformal blocks \[ \[ 3 \] to these regions can be performed directly by using hypergeometric function identities, or more easily by modifying the original screening charge integration contours appropriately. We find

\[ F^a_i(1/z) = -z^{3/2}F^a_i(z) \]
\[ F^b_j(1/z) = -z^{3/2}F^b_j(z) \]
\[ F^c_k(1/z) = -4z^{3/2}F(1/2, 5/2; 2; z) \]
\[ F^d_l(1/z) = 2\sqrt{2}z^{3/2}F(1/2, 5/2; 2; 1 - z) \]  

(59)

and
\( \tilde F_q^h(z, \bar z) = 2\sqrt 2(1 - z)^{3/2}F(1/2, 5/2; 2; z) \)
\( \tilde F_q^h(z, \bar z) = -4(1 - z)^{3/2}F(1/2, 5/2; 2; 1 - z) \)
\( \tilde F_q^h(z, \bar z) = -(1 - z)^{3/2}\tilde F_q^h(\bar z) \)
\( \tilde F_q^h(z, \bar z) = -(1 - z)^{3/2}\tilde F_q^h(z) \)

These, together with (60) and (64), allow us to write the exact expression for the full four-point \( \mathcal M \) correlator in our model:

\[
\langle \mathcal M(1)\mathcal M(2)\mathcal M(3)\mathcal M(4) \rangle \sim \frac{1}{|z_{12}z_{34}|^{3/2}}|1 - z|^{3/2} \left[ \mathcal H_{11} + \frac{z}{1 - z} \mathcal H_{12} + \frac{z}{1 - z} \mathcal H_{21} + \frac{|z|^2}{|1 - z|^2} \mathcal H_{22} \right] \tag{61}
\]

where

\[
\mathcal H_{11}(z, \bar z) = H_{11}'(z, \bar z) + |z|^{-1}H_{11}'(\frac{1}{z}, \frac{1}{\bar z}) + |1 - z|^{-3}H_{11}'(\frac{z}{z - 1}, \frac{\bar z}{\bar z - 1})
\]
\[
\mathcal H_{12}(z, \bar z) = H_{12}'(z, \bar z) + z|z|^{-3}H_{12}'(\frac{1}{z}, \frac{1}{\bar z}) + -(1 - z)|1 - z|^{-3}H_{12}'(\frac{z}{z - 1}, \frac{\bar z}{\bar z - 1})
\]
\[
\mathcal H_{21}(z, \bar z) = H_{21}'(z, \bar z) + z|z|^{-3}H_{21}'(\frac{1}{z}, \frac{1}{\bar z}) + -(1 - z)|1 - z|^{-3}H_{21}'(\frac{z}{z - 1}, \frac{\bar z}{\bar z - 1})
\]
\[
\mathcal H_{22}(z, \bar z) = H_{22}'(z, \bar z) + |z|^{-3}H_{22}'(\frac{1}{z}, \frac{1}{\bar z}) + |1 - z|^{-1}H_{22}'(\frac{z}{z - 1}, \frac{\bar z}{\bar z - 1})
\]

(62)

with the correlators \( H'_{ij} \) given by

\[
H'_{ij}(z, \bar z) = \left[ \tilde F_q^a(z)\tilde F_q^b(\bar z) + \tilde F_q^b(z)\tilde F_q^a(\bar z) \right]
\]

(63)

From Eq. (61) we extract the OPE of \( \mathcal M \) with itself, which in turn determines the scaling of products of local densities of states.

VI. FUSION RULES AND OPERATOR ALGEBRA

We will start here by studying the fusion rules of coset operators \( h(z, \bar z) \). There are two cases of interest: the fusion \( hh^{-1} \), and \( hh \). These two cases can be obtained from the coset correlator

\[
\langle h_{a_1b_1}(z_1, \bar z_1)h_{a_2b_2}^{-1}(z_2, \bar z_2)h_{a_3b_3}(z_3, \bar z_3)h_{a_4b_4}^{-1}(z_4, \bar z_4) \rangle \sim \sum_{i,j} \mathcal I_i \mathcal I_j \left[ \tilde F_q^a(z)\tilde F_q^b(\bar z) + a \leftrightarrow b \right] \tag{64}
\]

by respectively considering the limits \( z_1 \to z_2 \) in the above correlator, and in the one with 2 and 3 permuted. These correspond to taking \( z \to 0 \) in the appropriate solutions, which involves in the second case the analytic continuations \( z \to 1/z \).

Using the formulas contained in the appendix, we find that the dominant contributions come from \( H_{11} \), associated with the \( \mathcal I_1, \mathcal I_1 \) singlets (in the second case, this means contracting 1 with 3, 2 with 4). This yields (we here omit for simplicity the index structure)

\[
h(z_1, \bar z_1)h^{-1}(z_2, \bar z_2) \sim |z_{12}|^{3/2} \left[ \frac{4/3}{z_{12}} A(z_2) + \frac{4/3}{\bar z_{12}} \bar A(\bar z_2) + 4\mathcal I + 2\bar D(z_2, \bar z_2) + \ln |z_{12}| C(z_2, \bar z_2) \right]
\]

(65)

in which \( \mathcal I \) is the identity operator. The correlators of the operators appearing in (65) are

\[
\langle A(z_1)A(z_2) \rangle \sim z_{12}^2; \quad \langle \bar A(\bar z_1)\bar A(\bar z_2) \rangle \sim \bar z_{12}^2
\]

(66)

\[
\langle \bar D(z_1, \bar z_1)\bar D(z_2, \bar z_2) \rangle \sim -c_1 - \ln |z_{12}|
\]
\[
\langle \bar D(z_1, \bar z_1)\bar C(z_2, \bar z_2) \rangle \sim 1
\]
\[
\langle \bar C(z_1, \bar z_1)\bar C(z_2, \bar z_2) \rangle = 0
\]

(67)
where $c_1$ is some constant, unimportant for our purposes.

Some comments are in order here. We notice that the most relevant operators appearing in the OPE (65), $\mathcal{A}(z)$ and $\mathcal{A}(\bar{z})$, possess conformal weights $(-1,0)$ and $(0,-1)$ respectively. Usually, the fusion rules for WZNW models would imply that the adjoint operator, whose conformal weights are

$$h_{ad} = \bar{h}_{ad} = \frac{N}{N+k} = -1$$

should appear in the OPE (65). But the term in the four-point function pointing to the presence of such an operator, corresponding to a term $\sim 1/|z|^2$ inside the brackets of (65), does not appear since we are not allowed, by the requirement of single valuedness, to multiply the $b$ contour solutions in the holomorphic and antiholomorphic sectors together. This requirement cannot be seen from the chiral conformal algebra: it comes out of the solution for the full correlator.

Thus, single valuedness in the logarithmic conformal field theory that we are considering here changes the scaling behaviour expected from considerations not taking it into account, as in [8]. In fact, the operators (46) exactly correspond to the local fields considered there.

We can now very easily derive the OPE of $O^{−1}$. It has the exact same logarithmic form as above, with the exception that the proper fermionic contractions modify the weight in front:

$$O(z_1, \bar{z}_1)O^{−1}(z_2, \bar{z}_2) \sim \frac{1}{|z_{12}|^{1/2}} \left[ \frac{4}{3} \frac{1}{z_{12}} \mathcal{A}(z_2) + \frac{4}{3} \frac{1}{z_{12}} \bar{\mathcal{A}}(\bar{z}_2) + 4 \bar{D}(z_2, \bar{z}_2) + \ln |z_{12}| \bar{C}(\bar{z}_2, z_2) \right]$$

(69)

This gives the new scaling formula

$$\langle [O^{−1}](1) [O^{−1}](2) \rangle \propto z^2 + \bar{z}^2$$

(70)

to be contrasted with what was obtained in [8]. The other scalings mentioned there will also be modified for $N = 2$. The correct expressions are given below.

The OPE of the local DOS are extracted from Eq.(61) (we normalize the $M$ two-point function). First, we can write the behaviour of (61) as $z_1 - z_2 = \epsilon \to 0$:

$$\langle [M](1)[M](2)[M](3)[M](4) \rangle = \frac{1}{|\epsilon|^3} \left[ 1 + \gamma \ln |\epsilon| + \gamma \ln \left| \frac{z_{34}}{z_{23}z_{24}} \right| + \ldots \right]$$

(71)

in which $\gamma$ is a nonzero constant whose exact value is irrelevant for our purposes (it could be obtained by carefully expanding (65)). The crucial fact is that the operator in the symmetric representation, with weights $(-1,-1)$, does not appear here again, like the adjoint operator in (65). There is one quick way of seeing this last result. As $z \to 0$, $
abla z = -\nabla \bar{z}$. Thus, summing the conformal blocks for $z$ and $\nabla z$ amounts to adding the OPE (61) for $z$ and $-\bar{z}$, which makes the antisymmetric $1/z$ part disappear. The consequence of this, when one considers the point-splitting procedure leading to $\mathcal{M}^4$, is that negative-dimensional operators simply do not come about in the multifractality for $SU(2)$.

Postulating for $\mathcal{M}$ an OPE of the form (we remove the tildes from the logarithmic operators, since we use a new normalization)

$$\mathcal{M}(1)\mathcal{M}(2) \sim \frac{1}{|z_{12}|^{1/2}} \left[ \mathcal{I} + D(2) + \frac{1}{2} \ln |z_{12}| C(2) + \ldots \right]$$

(72)

then yields

$$\langle C(1)\mathcal{M}(2)\mathcal{M}(3) \rangle = 2\gamma \frac{1}{|z_{23}|^{1/2}}$$

$$\langle D(1)\mathcal{M}(2)\mathcal{M}(3) \rangle = \gamma \frac{1}{|z_{23}|^{1/2}} \ln \left| \frac{z_{23}}{z_{12}z_{13}} \right|$$

$$\langle D(1)D(2) \rangle = -2\gamma \ln |z_{12}|$$

$$\langle D(1)C(2) \rangle = 2\gamma$$

$$\langle C(1)C(2) \rangle = 0$$

(73)
in turn leading to the OPEs

\[ D(1)D(2) = -2\gamma \ln |z| + \ldots, \quad D(1)C(2) = 2\gamma + \ldots, \quad C(1)C(2) = 0 \]
\[ D(1)M(2) = -\gamma \ln |z| |M(2) + \ldots, \quad C(1)M(2) = 2\gamma M(2) + \ldots \]  \hspace{1cm} (74)

which will be used later to infer the behaviour of the multifractality.

VII. \( N > 2 \) CASE

Let us now outline what happens for the \( N > 2 \) case. We have seen in the previous section that, for \( SU(2) \), the spectrum of operator dimensions is cut right at the beginning by the logarithmic locality conditions. We can wonder if the same holds in the \( N > 2 \) case.

The conformal blocks, solutions to the \( SU(N)_{-2N} \) Knizhnik-Zamolodchikov equations, are \((F^p_i(z) = [z(1 - z)]^{-1-1/N^2} \tilde{F}^p_i(z))\)

\[
\tilde{F}^a_i(z) = \sqrt{2N} (1 - z) F(2 - 1/N, 2 + 1/N; 2; z) \\
\tilde{F}^b_i(z) = (1 - z) F(2 - 1/N, 2 + 1/N; 3; 1 - z) \\
\tilde{F}^a_2(z) = z F(2 - 1/N, 2 + 1/N; z) \\
\tilde{F}^b_2(z) = \sqrt{2N} z [1 + 1/N] F(1/N, 2 + 1/N; 2; 1 - z) 
\]  \hspace{1cm} (75)

For the case \( N = 2 \), we had that the analytic continuations to \( 1/z \) were of logarithmic type. Here, however, the correct single-valued construction is like the standard one \( ^{[34]} \) since the analytic continuations

\[
\tilde{F}^a_i(1/z) = -\sqrt{2N} z^{-1}(1 - z) F(2 - 1/N, 2 + 1/N; 2; 1/z) \\
\tilde{F}^b_i(1/z) = -z^{1+1/N} (1 - z) F(1 + 1/N, 2 + 1/N; 3; 1/z) \\
\tilde{F}^a_2(1/z) = z^{-1} F(2 - 1/N, 2 + 1/N; 3; 1/z) \\
\tilde{F}^b_2(1/z) = \sqrt{2N} z^{1+1/N} F(1/N, 2 + 1/N; 2; 1 - z) 
\]  \hspace{1cm} (76)

are free of logarithms near \( z = 0 \) for \( N \neq 2 \). On the other hand, we have for \( \tilde{F}^b_i \) the blocks

\[
\tilde{F}^b_i(1/z) = (1 - z)^{-1} F(2 - 1/N, 2 + 1/N; 3; 1/z) \\
\tilde{F}^b_i(z) = -(1 - z)^{1+1/N} z F(1 + 1/N, 2 + 1/N; 3; z) \\
\tilde{F}^b_2(z) = -\sqrt{2N} z^{-1} z F(2 - 1/N, 2 + 1/N; 2; 1/z) 
\]  \hspace{1cm} (77)

The full correlator reads in this case

\[
\langle M(1)M(2)M(3)M(4) \rangle \sim \frac{[1 - z^{-2-2/N^2}]}{[z_1 z_2 z_3 z_4]^{2/N^2}} \left[ H_{11} + \frac{z}{1 - z} H_{12} + \frac{z^2}{1 - z^2} H_{21} + \frac{|z|^2}{|1 - z|^2} H_{22} \right] 
\]  \hspace{1cm} (78)

where

\[
H_{11}(z, \bar{z}) = H'_{11}(z, \bar{z}) + |z|^{-2+4/N^2} H'_{11}(\frac{1}{z}, \frac{1}{\bar{z}}) + |1 - z|^{-4(1-1/N^2)} H'_{11}(\frac{z}{z - 1}, \frac{z}{\bar{z} - 1}) \\
H_{12}(z, \bar{z}) = H'_{12}(z, \bar{z}) - |z|^{-4(1-1/N^2)} H'_{12}(\frac{1}{z}, \frac{1}{\bar{z}}) - (1 - \bar{z}) |1 - z|^{-4(1-1/N^2)} H'_{12}(\frac{z}{z - 1}, \frac{z}{\bar{z} - 1}) \\
H_{21}(z, \bar{z}) = H'_{21}(z, \bar{z}) - |z|^{-4(1-1/N^2)} H'_{21}(\frac{1}{z}, \frac{1}{\bar{z}}) - (1 - z) |1 - z|^{-4(1-1/N^2)} H'_{21}(\frac{z}{z - 1}, \frac{z}{\bar{z} - 1}) \\
H_{22}(z, \bar{z}) = H'_{22}(z, \bar{z}) + |z|^{-4(1-1/N^2)} H'_{22}(\frac{1}{z}, \frac{1}{\bar{z}}) + |1 - z|^{-2+4/N^2} H'_{22}(\frac{z}{z - 1}, \frac{z}{\bar{z} - 1}) 
\]  \hspace{1cm} (79)

with the correlators \( H'_{ij} \) given by
\[ H_{ij}^\prime(z, \bar{z}) = \tilde{F}_i^\alpha(z) \tilde{F}_j^\beta(\bar{z}) + \tilde{F}_i^\beta(z) \tilde{F}_j^\alpha(\bar{z}). \]  

(80)

As for the SU(2)−4 case, the \( b \) contour solutions have logarithmic behaviour near \( z = 0 \) for \( z \) and \( \bar{z} \) (see appendix). Moreover, this logarithmic behaviour is exactly of the same form, involving the same powers of \( z \). Thus, again, the operator in the adjoint representation, having conformal weights \((-1, -1)\), does not appear in the OPE of the decoupling field with its inverse, because of single valuedness.

However, when one considers the OPE of \( h \) with \( h \), for which the analytic continuation to \( 1/z \) is necessary, the difference with the SU(2)−4 becomes clear. In fact, the analytic continuations of \( \bar{F} \) to \( 1/z \), in contrast to the \( N = 2 \) case, are completely regular for \( N > 2 \), as we have seen above. Expanding the full correlator (78) around \( z = 0 \), we find that the operators in the symmetric (S) and antisymmetric (A) representations do appear, as proposed in [8]. These have conformal weights

\[ h_S = \bar{h}_S = -\frac{(N+2)(N-1)}{N^2}, \]

\[ h_A = \bar{h}_A = -\frac{(N-2)(N+1)}{N^2} \]  

(81)

Thus, we now have a negative dimensional operator on the physical level, in contrast to the \( N = 2 \) case. This operator has conformal weights

\[ h = \bar{h} = -\frac{1}{N}(1-2/N) \]  

(82)

For \( N = 2 \), this vanishes, in accordance with the work of the previous sections.

Logarithms in conformal field theories originate from degeneracies in the spectrum of conformal dimensions. When two or more operators have weights which become degenerate, they become distinct again by incorporating powers of \( \ln z \), which is a weight zero object. For the \( N = 2 \) case, three operators in the theory were of weight zero: the unit operator, the first descendant of the adjoint operator, and the operator in the antisymmetric representation. This high level of degeneracy has produced the logarithmic degeneracies which, through the requirements of single-valuedness (locality) of the correlator, have led us to discover the termination of the spectrum.

In the case of \( N > 2 \), we have just seen that the termination does not appear immediately. We can however speculate about the moment at which the termination should occur. This relies on basic arguments of representation theory. For SU(N)−2N, the conformal weights of the operators in a given representation characterised by a Young tableau with \( f_i \) boxes in the \( i \)-th row is given by the associated Casimir

\[ c_{\{f_i\}} = \frac{1}{2} \sum_{i=1}^{N} [f_i^2 + (N + 1 - 2i)f_i] - \frac{f^2}{2N}, \quad f = \sum_{i=1}^{N} f_i \]  

(83)

as (considering holomorphic and anti-holomorphic)

\[ h_{\{f_i\}} = -\frac{c_{\{f_i\}}}{N}, \quad \bar{h}_{\{f_i\}} = -\frac{\bar{c}_{\{f_i\}}}{N} \]  

(84)

Now it is easy to check that for a completely antisymmetric representation of SU(N), i.e. \( f_i = 1, i = 1, ..., N \), the value of the Casimir vanishes. The associated operator thus has vanishing conformal weights, which makes it degenerate with the unit operator and the descendant of the adjoint operator, like in the \( N = 2 \) case extensively presented above. This operator in the completely antisymmetric representation is generated only by a product of \( N \) fundamental representations (our basic operator), so will appear only for \( q = N \). We thus speculate that, for \( N > 2 \), the additional conformal weight degeneracy at \( q = N \) leads to logarithmic degeneracies which, exactly in the same way as previously, lead to the termination of further scaling.

VIII. TERMINATION OF THE PARABOLIC MULTIFRACTAL SPECTRUM

The relationship between the operator dimensions in the CFT and the multifractal spectrum were derived with the help of real-space renormalization group arguments in [8], where all the definitions and notations used here are put forward. Three types of multifractal spectra are defined: \( \tau(q) \), which is the disorder averaged logarithm of the inverse participation ratio of normalized wavefunctions; \( \tau^*(q) \) (the one we calculate here), which is the logarithm of the ratio
of disorder averaged unnormalized wavefunctions and disorder averaged normalization to the appropriate power, and \( \tau^*(q) \), which is the logarithm of the disorder averaged participation ratio of normalized wavefunctions. There exist strong constraints on the first of these, \( \tau(q) \). It can only be a monotonously increasing function of \( q \), which forces the cut of the parabolic approximation at a given point. The other functions have not been shown to have to obey to such conditions.

The only multifractal spectrum that can be calculated with the help of the present theory is \( \tau^*(q) \). It is related to the conformal weights \( (h_q, \tilde{h}_q) \) of the most relevant operator contained in the point-splitting definition of \( \mathcal{M}^q \) according to

\[
\tau^*(q) = (q - 1)(2 - \Delta_1) + \Delta_q - \Delta_1
\]

where \( \Delta_q = h_q + \tilde{h}_q \). These weights are obtained by considering the decay of the correlator

\[
\langle \mathcal{M}^q(1)\mathcal{M}^q(2) \rangle \sim z_{12}^{-2h_q} z_{12}^{-2\tilde{h}_q}
\]

Let us first recall the arguments used by \([8]\) to obtain these weights. The operator \( \mathcal{O} \) is made from the product of free fermions and of a field in the fundamental representation of \( SU(N-2) \). The fusion of the fermions is straightforward, but the fusion of the WZNW primary fields is a bit less obvious. Building the Young tableaux associated to the tensor product of \( q \) fundamental representations and evaluating the quadratic Casimir eigenvalues \( c_m \) of all the resulting irreducible representations, one obtains a list of operators having conformal weights \( h_m = -c_m/N \). It is then easy to identify the most relevant operator as the one in the completely symmetric representation, which has \( h_S = -\frac{N - 1}{2N^2} q(q + N) \). The resulting combined fermionic and WZNW weights lead to the provisional result \([8]\)

\[
h_q = \tilde{h}_q = \frac{q}{2} - \frac{N - 1}{2N^2} (q^2 + Nq)
\]

and led to the corresponding scaling exponents \( \tau^*(q) \) given by

\[
\tau^*(q) = (q - 1) \left( 2 - \frac{N - 1}{N^2} q \right)
\]

One might question the fact that this \( \tau^*(q) \) becomes arbitrarily negative for large values of \( q \). It attains its maximum value at \( q_{\text{max}} = N(1 - 1/N)^{-1} + 1/2 > N \), after which the parabola bends back down. Even though it is in principle not impossible that \( \tau^*(q) \) could take on negative values (see the discussion in \([8]\)), a limit to this behaviour would be most welcome.

However, we can see from the previous sections that the straightforward WZNW fusion rules are not applicable in our case. Namely, we can perform for example the explicit point-splitting procedure leading to \( \mathcal{M}^3 \) for the case \( N = 2 \) from the fusion rules \([12]\):

\[
\mathcal{M}^3(z) = \lim_{a \to 0} \mathcal{M}(z + a)\mathcal{M}(z)\mathcal{M}(z - a) = \lim_{a \to 0} |a|^{-1/2} \left[ I + D(z) + \frac{1}{2} \ln |a| C(z) + \ldots \right] \mathcal{M}(z - a)
\]

\[
= (\lim_{a \to 0} |a|^{-1/2}) \mathcal{M}(x) + \ldots \sim \mathcal{M}(x)
\]

showing explicitly that \( \langle \mathcal{M}^3(1)\mathcal{M}^3(2) \rangle \sim |z_{12}|^{-1/2} \), i.e. that \( h_3 = \tilde{h}_3 = 1/8 \), in contrast to the results of \([8]\), for which \( h_3 = \tilde{h}_3 = -3/8 \). Generalizing to arbitrary \( q \) (still for the specific case of \( N = 2 \)), we obtain that

\[
\mathcal{M}^{2p} \propto I, \quad \mathcal{M}^{2p+1} \propto \mathcal{M}
\]

as far as the correlators \( \langle \mathcal{M}^q(1)\mathcal{M}^q(2) \rangle \) are concerned. This implies the explicit termination of the parabolic law for the multifractality in the case of \( SU(2) \) vector potential randomness, which can be understood as a vanishing of OPE coefficients associated to operators with more relevant dimensions. This type of termination seems special to the \( SU(N) \) non-Abelian disorder case, since the Abelian multifractality, when computed from CFT, only involves expectation values of vertex operators of Gaussian distribution which do not carry the same conformal block structure as the ones that we have here. Note also that in the case of the \( gl(1, 1) \) WZNW model \([12]\), fields with arbitrarily negative conformal dimensions can be obtained from the fusion of a sufficient number of fundamental representations. These interesting differences still need to be more properly understood.
Finally, from the results and discussion of the previous section, we can conjecture that the multifractal spectrum for general \( N \) should be complemented by a condition of the sort
\[
q < q_c
\]
where \( q_c \) is the order at which the proper treatment of the locality conditions in the presence of logarithmic degeneracies kill off further scaling. Thus, above this critical value of \( q \), the parabolic law will not hold anymore. From our explicit results for \( SU(2) \), and comparing with the results from \[8\], we can say that \( 2 < q_c(SU(2)) < 3 \). This is consistent with the exact solution for \( \tau(q) \) itself \[10\].

IX. WHAT IS WRONG WITH REPLICAS?

In general, the theoretical treatment of disordered systems is a notoriously difficult task in view of the fact that it is necessary to average the logarithm of the partition function over the statistical ensemble in order to obtain physical results. Two general approaches have been developed to perform such types of calculations. First and foremost, the supersymmetry (SUSY) approach \[37\] is a mathematically sound technique that makes use of commuting and anticommuting fields simultaneously, making the partition function \( Z \) equal to one by definition. Its only setback is its limited applicability: interacting disordered systems cannot be represented by a SUSY field theory.

The alternative technique is the well-known replica approach, introduced in \[38\], which allows to rewrite the average of \( \ln Z \) as a tractable object by making use of the identity
\[
\ln Z = \lim_{r \to 0} \frac{Z^r - 1}{r}
\]
(i.e. by considering a replicated theory with \( r \) species of the original field. One then hopes that the physical properties of the disordered system are faithfully represented by the analytical continuation to \( r \to 0 \). However, the main problem is that the replica approach is not mathematically well-defined. Indeed, even though it has been used successfully quite often, the pathologies persist (for example, the systems obtained by choosing a positive or negative number of replicas are markedly different, making the limit \( r \to 0 \) completely ill-defined). It has been argued in \[39\] that failure of the replica trick may occur whenever the theory for a general integer value of \( r \) possesses different symmetries than the one at \( r = 0 \).

We are thus now in a position to compare our present exact results with the previous ones obtained with the replica approach \[9\]. The conformal blocks obtained there were solutions to \( SU(0)_N \) equations, and different to the ones obtained here. The OPE of the physical field \( \mathcal{M} \) (\( Q \) in \[9\]) was given by
\[
\mathcal{M}(1)\mathcal{M}(2) \sim \frac{1}{|z|^{2/N^2}} \left[ I - z(D(2) + C(2) \ln |z|) - \bar{z}(-D(2) + \bar{C}(2) \ln |\bar{z}|) + \ldots \right]
\]
where the chiral-like logarithmic operators had correlators given by the expressions
\[
\langle D(1)D(2) \rangle \sim \frac{1}{|z|^{12}} \left[ \ln |z|^{12} + c \right]
\]
\[
\langle D(1)C(2) \rangle \sim \frac{1}{|z|^{12}}
\]
\[
\langle C(1)C(2) \rangle = 0
\]
with similar expressions for \( \bar{C}, \bar{D} \) given by \( z \) substituted by \( \bar{z} \). Although the basic two-point function generated from this OPE will be the same as the one obtained through above, the higher-point functions will show marked differences with the exact ones, which makes all further arguments unreliable.

Thus, although the replica approach successfully accounts for the basic dimensionality of the operators, it does not correctly describe their higher-level correlations. The failure of the replica solution may be understood in terms of the interpretation put forward in \[39\], since the essential ingredient in the replica solution was the \( SU(r) \) symmetry (associated to the \( r \) different flavours of replicated fermions), which obviously is a different symmetry at \( r = 0 \) than at integer \( r \).
X. CONCLUSION

Let us give a brief summary of our results.

(a) We have demonstrated explicitly that for the SU(2) group, the basic WZNW fusion rules are modified by the proper treatment of the locality conditions in the presence of so-called logarithmic operators, such that operators with negative conformal dimensions do not appear in OPE of the local density of states that we are interested in for multifractality purposes;

(b) We have provided arguments based on the nature of the conformal field theory and its operator content, to conjecture that the mechanism for the termination of the multifractal spectrum that we propose should have occurred as well before \( q = N \) at the latest for general SU(\( N \)). In these theories, there will thus be a finite string of physical operators with negative dimensions influencing the multifractality, but not an infinite set as previously believed. The termination for \( N > 2 \) is conjectured to occur by the above mechanism before the scaling exponents \( \tau^*(q) \) reach their maximum value, which provides evidence that the multifractal spectrum \( \tau^*(q) \) obeys the same type of inequalities as the spectrum \( \tau(q) \) defined with normalized wavefunctions;

(c) We have shown that the replica approach fails to reproduce the correct nonperturbative multipoint correlation functions in the conformal limit.

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XII. APPENDIX 1: THE HAAR MEASURE

For the sake of clarity, we here include the derivation of the Haar measure on \( SU^c(2)/SU(2) \) as can be found in [34].

For a function \( f(g^C) \in C_c(G^C) \), where \( C_c(G^C) \) is the space of continuous functions of compact support on \( G^C \), there exists a function \( D(k,a,n) \) such that the group integration, after the Iwasawa decomposition, can be written as

\[
\int_{G^C} f(g^C) dg^C = \int_{KAN} f(kan) D(k,a,n) dk dan \tag{95}
\]

Since \( G^C \) is unimodular (i.e. its left invariant measure is also right invariant), the LHS does not change when we replace \( f(g^C) \) by \( f(k_1a_1n_1) \), \( k_1 \in K, n_1 \in N \). Then, it follows that \( D(k_1^{-1}k, an_1^{-1}) \equiv D(k,a,n) \), so \( D(k,a,n) \) is a function \( \delta(a) \) of \( a \) only. Therefore, for \( a_1 \in A \), we have

\[
\int_{G^C} f(g^C) dg^C = \int_{G^C} f(g^C a_1) dg = \int_{KAN} f(kana_1) \delta(a) dk dan
\]

\[
= \int_{KAN} f(kaa_1(\eta^{-1}n_1)) \delta(a) dk dan = \int_{KAN} f(k(a_1^{-1}n_1)) \delta(aa_1^{-1}) dk dan
\]

\[
= \int_{KAN} f(kan) \delta(aa_1^{-1}) dk da I^{*}(a_1)(dn) \tag{96}
\]

where \( I(a_1) \) is the automorphism \( n \rightarrow a_1na_1^{-1} \) of \( N \), and \( I^{*} \omega \) denotes the transform (pullback) of \( \omega \) by \( I \). We have made use in the last step of the equality

\[
\int_N (f \circ \Phi^{-1}) \omega = \int_M f \Phi^{*} \omega \tag{97}
\]

in which \( M \) and \( N \) are two oriented manifolds, and \( \Phi \) is a diffeomorphism between them.

Finally, by looking at
we see that \( I(a_1)^*(dn) = e^{2\phi_1}d\mu_1d\mu_2 = e^{2\phi_1}d\mu^+d\mu^- \), and choosing \( a_1 = a \) produces the Haar measure (17).

### XIII. APPENDIX 2: ANALYTIC CONTINUATIONS

We here give for reference the analytic continuation formulas used in the text, as well as the expansions of the relevant functions around \( z = 0 \).

The second solutions for the conformal blocks, labeled by their contour \( b \), are hypergeometric functions of the argument \( 1 - z \). When \( z \to 0 \), these behave logarithmically. The proper expressions become

\[
\begin{align*}
\tilde{F}_1^b(z) &= \frac{4}{\pi} \left[ \frac{4/3}{z} + \tilde{F}_2^a(z) \ln z - 4/3 + (1-z)K_{11}(z) \right] \\
\tilde{F}_2^b(z) &= \frac{\sqrt{2}}{\pi} \left[ \frac{16/3}{z} + \tilde{F}_2^a(z) \ln z + 4/3 + zK_{12}(z) \right] \\
F(1/2, 5/2; 2; 1-z) &= -\frac{1}{\pi} \left[ \frac{-4/3}{z} + F(1/2, 5/2; 2; z) \ln z + K_{01}(z) \right]
\end{align*}
\]

where

\[
K_{ij}(z) = \sum_{n=0}^{\infty} z^n \frac{(1/2 + i)_n (5/2)_n}{(n+j)!n!} [\psi(n+i+1/2) + \psi(n+5/2) - \psi(n+j+1) - \psi(n+1)]
\]

Near \( z = 0 \), we can then perform the expansions

\[
\begin{align*}
\tilde{F}_1^b(z) &= 2\sqrt{2}[1 + 7/8z + O(z^2)] \\
\tilde{F}_1^b(z) &= \frac{4}{\pi} \left[ \frac{4/3}{z} + \ln z - 4/3 + K_{11}(0) \right] + O(z \ln z) \\
\tilde{F}_2^b(z) &= z + 5/4z^2 + O(z^3) \\
\tilde{F}_2^b(z) &= \frac{\sqrt{2}}{\pi} \left[ \frac{16/3}{z} - 4/3 \right] + O(z \ln z)
\end{align*}
\]

\[
\begin{align*}
F(1/2, 5/2; 2; z) &= 1 + 5/8z + O(z^2) \\
F(1/2, 5/2; 2; 1-z) &= -\frac{1}{\pi} \left[ \frac{-4/3}{z} + \ln z + K_{01}(0) \right] + O(z \ln z)
\end{align*}
\]

used in the determination of the OPEs in the text.

For the \( N > 2 \) case, we have for example

\[
\tilde{F}_1^b(z) = \frac{2}{\Gamma(1-1/N)\Gamma(1+1/N)} \left[ \frac{N^2}{(N^2-1)z} + \tilde{F}_1^a(z) \ln z - \frac{N^2}{N^2-1} + (1-z)J(z) \right]
\]

with \( J(z) \) some function regular as \( z \to 0 \).

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