CATEGORICAL ACTIONS AND MULTICILITIES IN THE DELIGNE CATEGORY $\text{Rep}(GL_t)$

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ABSTRACT. We study the categorical type A action on the Deligne category $D_t = \text{Rep}(GL_t)$ ($t \in \mathbb{C}$) and its "abelian envelope" $\mathcal{V}_t$ constructed in [EHS].

For $t \in \mathbb{Z}$, this action categorifies an action of the Lie algebra $\mathfrak{sl}_2$ on the tensor product of the Fock space $\mathfrak{F}$ with $\mathfrak{F}^\vee$, its restricted dual "shifted" by $t$, as was suggested by I. Losev. In fact, this action makes the category $\mathcal{V}_t$ the tensor product (in the sense of Losev and Webster, [BLW]) of categorical $\mathfrak{sl}_2$-modules $\text{Pol}$ and $\text{Pol}^\vee$. The latter categorify $\mathfrak{F}$ and $\mathfrak{F}^\vee$ respectively, the underlying category in both cases being the category of stable polynomial representations (also known as the category of Schur functors), as described in [HY, LW], and further developed by Rouquier ([R]), Khovanov and Lauda ([KL]) for explicit definitions and details.

When $t \notin \mathbb{Z}$, the Deligne category $D_t$ is abelian semisimple, and the type A action induces a categorical action of $\mathfrak{sl}_2 \times \mathfrak{sl}_2$. This action categorifies the $\mathfrak{sl}_2 \times \mathfrak{sl}_2$-module $\mathfrak{F} \boxtimes \mathfrak{F}^\vee$, making $D_t$ the exterior tensor product of the categorical $\mathfrak{sl}_2$-modules $\text{Pol}$, $\text{Pol}^\vee$.

Along the way we establish a new relation between the Kazhdan-Lusztig coefficients and the multiplicities in the standard filtrations of tilting objects in $\mathcal{V}_t$.

1. Introduction

In this paper we study the categorical type A action on the Deligne categories $\text{Rep}(GL_t)$, and the induced categorical $\mathfrak{sl}_2$-action on their abelian versions $\mathcal{V}_t$ defined in [EHS] for $t \in \mathbb{Z}$.

1.1. Categorical actions. The theory of categorical type A actions was first introduced by Chuang and Rouquier in [CR], and further developed by Rouquier ([R]), Khovanov and Lauda (see for example [KL]), Brundan, Losev, Webster and others (see [BLW] [HY, LW], and the review [L]).

According to this theory, a categorical type A action on an abelian category $\mathcal{A}$ is an adjoint pair of (exact) endofunctors $E, F$ of $\mathcal{A}$, and natural transformations $\tau \in \text{End}(F^d)$, $x \in \text{End}(F)$, defining an action of the degenerate affine Hecke algebra $dAHA_d$ on $F^d$ for any $d$.

If the generalized eigenvalues of $x \in \text{End}(F)$ are integers, the type A action induces an $\mathfrak{sl}_2$-action. It is given by the decompositions $F = \sum_{a \in \mathbb{Z}} F_a$, $E = \sum_{a \in \mathbb{Z}} E_a$, corresponding to the generalized eigenvalues $a$ of $x \in \text{End}(F)$ and the induced natural transformation $x^\vee \in \text{End}(E)$. The obtained functors $E_a, F_a$ are once again an adjoint pair; furthermore, we require that $E_a$ be isomorphic to the right adjoint of $F_a$. On the level of the Grothendieck group of $\mathcal{A}$, the functors $E_a, F_a$ induce an action of $\mathfrak{sl}_2$ on $\text{Gr}(\mathcal{A})$ through its generators $e_a, f_a$ ($a \in \mathbb{Z}$). The natural transformation $\tau$ serves to categorify the relations $[e_a, f_b] = 0$ for $a \neq b$, and $[e_a, f_a] = h_a$. See [CR, R, BLW] for explicit definitions and details.

Several important examples of categorical $\mathfrak{sl}_2$-actions have been found; in particular, it was shown that the category $\text{Rep}(\mathfrak{gl}(m|n))$ of finite-dimensional representations of the Lie superalgebra $\mathfrak{gl}(m|n)$ carries a natural action of $\mathfrak{sl}_2$, where $F, E$ are given by tensoring with the standard representation $\mathbb{C}^m \otimes \mathbb{C}^n$ of $\mathfrak{sl}_2$.

Then the (complexified) Grothendieck group of $\text{Rep}(\mathfrak{gl}(m|n))$ corresponds to the representation $\Lambda^m \mathbb{C}^2 \otimes \Lambda^n (\mathbb{C}^2)^*$ of $\mathfrak{sl}_2$.

1.2. Deligne categories. In [DM], Deligne and Milne constructed a family of rigid symmetric monoidal categories $\text{Rep}(GL_t)$ (denoted by $D_t$ in this paper), parametrized by $t \in \mathbb{C}$. These categories satisfy several properties:

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• The category $D_t$ is the universal Karoubian additive symmetric monoidal category generated by a dualizable object of dimension $t$. We will denote this object by $V_t$.

• For $t = n \in \mathbb{Z}_+$, this universal property provides a symmetric monoidal functor $S_n : D_{t=n} \rightarrow Rep(GL(n))$, $V_t \mapsto \mathbb{C}^n \ (Rep(GL(n))$ is the category of finite-dimensional representations of $GL(n))$. This functor is full and essentially surjective; due to this fact, one can consider $D_t$ as a polynomial family “interpolating” the categories $Rep(GL(n))$.

• For $t$ non-integer, these are semisimple tensor categories.

The above universal property also provides symmetric monoidal functors $S_{m,n} : D_t \rightarrow Rep(GL(m|n))$, $V_t \mapsto \mathbb{C}^{m|n}$ whenever $t \in \mathbb{Z}$, $m,n \in \mathbb{Z}_+$, $m - n = t$ (here $Rep(GL(m|n))$ is the category of finite-dimensional representations of the algebraic supergroup $GL(m|n)$).

For $t \in \mathbb{Z}$ the category $D_t$ is Karoubian but not abelian. To remedy this, Hinich, Serganova and the author constructed in [LHS] a tensor category $V_t$ which satisfies a universal property: given a tensor category $C$, the exact tensor functors $V_t \rightarrow C$ classify the $t$-dimensional objects in $C$ not annihilated by any Schur functor (if $t \notin \mathbb{Z}$, this condition is vacuous).

The category $D_t$ embeds into $V_t$ for any $t$ as a full rigid symmetric monoidal subcategory. When $t \notin \mathbb{Z}$, the category $V_t$ is defined to be just $D_t$.

In addition to the properties above, the construction of the category $V_t$ demonstrates a certain stabilization phenomenon in the categories of representations of algebraic supergroups $GL(m|n)$, and $V_t$ can be seen as a “stable” inverse limit of $Rep(GL(m|n))$ for $m - n = t$ as $m, n \rightarrow \infty$.

In the category $V_t$ we distinguish three sets of isomorphism classes of objects, all three parametrized by the set of bipartitions of arbitrary size (denoted by $\lambda = (\lambda^*, \lambda^\vee)$). The three sets are: simple objects $L(\lambda)$, standard objects $V(\lambda)$ and tilting objects $T(\lambda)$.

When $t \notin \mathbb{Z}$, we have: $L(\lambda) = V(\lambda) = T(\lambda)$.

1.3. The categorical type A action on $D_t$ is given by functors $F, E \in \text{End}(D_t)$, $F := V_t \otimes (\cdot)$, $E := V_t^* \otimes (\cdot)$; the natural transformations $\tau, \chi$ are described in Construction [6.2]. One should beware that the category $D_t$ is not abelian, and the eigenvalues of the operator $\chi$ are not necessarily integral, making it a generalized type A action in the sense of [R, Section 5.1.1].

This action extends to a generalized type A action on the abelian category $V_t$, and the functors $S_{m,n} : D_{t=m-n} \rightarrow Rep(GL(m|n))$, $V_t \mapsto \mathbb{C}^{m|n}$ respect the type A actions on these categories.

1.4. Results. Our first main result is the description of the categorical actions on $D_t$ and $V_t$.

Consider the Fock space representation $\rho_\mathfrak{g} : \mathfrak{sl}_2 \rightarrow \text{End}(\mathfrak{g})$, with a basis $\{v_a\}$ of weight vectors parametrized by partitions of arbitrary size. We will also consider the “twisted” dual Fock representation $\rho^\vee_\mathfrak{g} : \mathfrak{sl}_2 \rightarrow \text{End}(\mathfrak{g}^\vee)$ (denoted $\mathfrak{g}^\vee$ for short), where $f_a, e_a$ act by $\rho_\mathfrak{g}(f_a)^T, \rho_\mathfrak{g}(e_a)^T$.

Next, for $t \in \mathbb{Z}$, consider the representation $\mathfrak{h}^\vee_t$ of $\mathfrak{sl}_2$: the underlying vector space is $\mathfrak{g}^\vee$, with $f_a, e_a$ acting by $\rho^\vee_\mathfrak{g}(f_{a+1}), \rho^\vee_\mathfrak{g}(e_{a+1})$.

The following statements were suggested by I. Losev:

Theorem 1.

1. Let $t \notin \mathbb{Z}$ (so $D_t$ is semisimple).

The functors $F = V_t \otimes (\cdot), E = V_t^* \otimes (\cdot)$ induce an action of the Lie algebra $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ on the complexified Grothendieck group $\mathbb{C} \otimes_{\mathbb{Z}} \text{Gr}(D_t)$.

As a $\mathfrak{sl}_2 \times \mathfrak{sl}_2$-module,

$\mathbb{C} \otimes_{\mathbb{Z}} \text{Gr}(D_t) \cong \mathfrak{h} \otimes \mathfrak{h}^\vee, \ [T(\lambda)] \mapsto v_{\lambda^\vee} \otimes v_{\lambda^\vee}$

2. Let $t \in \mathbb{Z}$. The functors $F = V_t \otimes (\cdot), E = V_t^* \otimes (\cdot)$ induce an action of the Lie algebra $\mathfrak{sl}_2$ on the complexified Grothendieck group $\mathbb{C} \otimes_{\mathbb{Z}} \text{Gr}(V_t)$.

We then have an isomorphism of $\mathfrak{sl}_2$-modules

$\mathbb{C} \otimes_{\mathbb{Z}} \text{Gr}(V_t) \cong \mathfrak{h} \otimes \mathfrak{h}^\vee, \ [V(\lambda)] \mapsto v_{\lambda^\vee} \otimes v_{\lambda^\vee}$

1The terminology comes from the theory of highest-weight categories. The category $V_t$ is not a highest-weight category (it does not have projectives nor injectives), but it is “lower highest-weight”, meaning that it has a filtration by full subcategories which are highest-weight. The objects $V(\lambda), T(\lambda)$ lying in these subcategories are respectively standard and tilting objects.
In fact, we prove a stronger statement. Let Pol be the category of polynomial representations of \(\mathfrak{gl}(\infty)\) (alternatively, this can be described as the category of all Schur functors, or the free Karoubian additive symmetric monoidal \(\mathbb{C}\)-linear category with one generator). This category has an \(\mathfrak{sl}_2\)-action categorifying the Fock space representation \(\tilde{\mathcal{F}}\) (see [HY, L]). One can similarly define a twisted \(\mathfrak{sl}_2\)-action on Pol categorifying the Fock space \(\mathcal{F}_t\). By abuse of notation, we denote the category Pol with these actions by Pol, Pol\(^t\) respectively.

With this notation, we can now state our second result, showing that \(\mathcal{V}_t\) is a tensor product categorification in the sense of Losev and Webster (see [LW, Remark 3.6], and details in Definition 10.2.3).

**Theorem 2.** Let \(t \in \mathbb{Z}\). The category \(\mathcal{V}_t\) with the action of \(\mathfrak{sl}_2\) is a tensor product categorification Pol \(\otimes\) Pol\(^t\).

Moreover, is unique in the following sense: consider a lower highest-weight category \(\mathcal{C}\) (see Definition 10.2.3) with an \(\mathfrak{sl}_2\)-action \((F', E', \tau')\) making it a tensor product categorification Pol \(\otimes\) Pol\(^t\) in the sense of Definition 10.2.3 and such that the natural transformation \(\tau'_{E'F'} : E'F' \to F'E'\) induced by \(\tau'\) is an isomorphism.

Then we have a strongly equivariant equivalence \(\mathcal{C} \cong \mathcal{V}_t\).

**Remark 1.4.1.** Of course, when \(t \notin \mathbb{Z}\), \(\mathcal{D}_t\) is an (exterior) tensor product of the categorical \(\mathfrak{sl}_2 \times \mathfrak{sl}_2\)-module Pol \(\boxtimes\) Pol\(^t\)\(^\vee\) in the sense of Losev and Webster. The uniqueness of such a tensor product categorification follows from the uniqueness of the categorification of the \(\mathfrak{sl}_2\)-representation \(\tilde{\mathcal{F}}\), which is straightforward (the weight spaces in \(\tilde{\mathcal{F}}\) being one-dimensional).

In a separate result, we establish a connection between multiplicities in standard filtrations of tilting objects and multiplicities of the Jordan-Holder components of the standard modules.

**Theorem 3.** Let \(t \in \mathbb{C}\), and let \(\lambda = (\lambda^\bullet, \lambda^\circ), \mu = (\mu^\bullet, \mu^\circ)\) be two bipartitions, regarded as pairs of Young diagrams.

Then
\[
(T(\lambda) : V(\mu))_{\mathcal{V}_t} = [V(\lambda) : L(\mu)]_{\mathcal{V}_t}
\]
where the LHS denotes the multiplicity of the standard object \(V(\mu)\) in the standard filtration of the standard object \(T(\lambda)\), and the RHS denotes the multiplicity of the simple object \(L(\lambda)\) among the Jordan-Holder components of \(V(\mu)\).

**Remark 1.4.2.** When \(t \notin \mathbb{Z}\), these multiplicities are \(\delta_{\lambda, \mu}\).

Moreover, we show that these multiplicities coincide with the multiplicity \(D^\lambda_{\mathfrak{sl}_2}(t)\) appearing in [CW, Section 6]. This statement is related to the existence of an exact faithful SM functor \(\text{Rep}_{\mathfrak{sl}_2}(GL_\infty) \to \mathcal{V}_t, \text{Rep}_{\mathfrak{sl}_2}(GL_\infty)\) being the category of algebraic representations of \(GL_\infty\) as described in [DPS, SS]. The standard objects \(V(\lambda)\) in \(\mathcal{V}_t\) are images of the simple objects in the category \(\text{Rep}_{\mathfrak{sl}_2}(GL_\infty)\), and thus are “flat” with respect to the parameter \(t\). This allows one to replace the “lifting” techniques from [CO, CW] (passing from a specific value of the parameter \(t\) to a generic value) by computations in the category \(\text{Rep}_{\mathfrak{sl}_2}(GL_\infty)\), aided by its embedding into the category \(\mathcal{D}_t\) for generic \(t\).

This theorem also allows us to obtain partial results on the action of the functors \(F_a, E_a\) on tilting objects \(T(\lambda)\) (see Section 9.3).

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2. **Notation**

The base field throughout the paper will be \(\mathbb{C}\). All the categories considered will be \(\mathbb{C}\)-linear.
2.1. Young diagrams and partitions. We will use the notions of partition and Young diagram interchangeably, and denote these by small-case greek letters. The set of all Young diagrams will be denoted by $\mathcal{P}$.

A bipartition is a pair of partitions, denoted by $\lambda = (\lambda^\bullet, \lambda^\circ) \in \mathcal{P} \times \mathcal{P}$.

Given a partition $\nu \vdash n$, we denote by $\nu_i, i \geq 0$ the parts of $\nu$ in non-increasing order ($\nu_i = 0$ for $i >> 0$). We also denote $|\nu| := n$, $\ell(\nu) := \max\{i : \nu_i \neq 0\}$, and by $\nu^T$ the transpose of $\nu$ (e.g. the transpose of $\begin{array}{c} \boxed{a} \\ \boxed{b} \end{array}$ is $\begin{array}{c} \boxed{b} \\ \boxed{a} \end{array}$).

For a bipartition $\lambda = (\lambda^\bullet, \lambda^\circ)$, we denote $|\lambda| := |\lambda^\bullet| + |\lambda^\circ|$.

Given a cell $(i,j)$ in the Young diagram of a bipartition $\nu$, we denote the content of this cell by $ct(\square(i,j)) := i - j$.

Given a Young diagram $\nu$, we denote by $\nu + \square$ (resp. $\nu - \square$) the set of all Young diagrams obtained by adding (resp. removing) a box from $\nu$. The sets $\nu \pm \square_a$ are disjoint unions of subsets $\nu \pm \square_a \ (a \in \mathbb{C})$; the subset $\nu \pm \square_a$ contains the Young diagrams obtained by adding/removing a box of content $a$ from $\nu$ (it is empty if $a \notin \mathbb{Z}$, otherwise it contains at most one element).

When considering a bipartition $\lambda = (\lambda^\bullet, \lambda^\circ)$, we will use the notation

$$\lambda \pm \square_a := \{\mu \in \mathcal{P} \times \mathcal{P} : \mu^\bullet \in \lambda^\bullet \pm \square_a, \mu^\circ = \lambda^\circ\}$$

and

$$\lambda \pm \square_a := \{\mu \in \mathcal{P} \times \mathcal{P} : \mu^\circ \in \lambda^\circ + \square_a, \mu^\bullet = \lambda^\bullet\}$$

Whenever a set of objects $\{X_\lambda\}$ in an additive category is parametrized by partitions (or bi-partitions), we denote by $X_{\nu \pm \square_a}$ the direct sum of objects corresponding to the partitions (or bi-partitions) in the set $\nu \pm \square_a$. In particular, if the set $\nu \pm \square_a$ is empty, then the object is zero.

2.2. Tensor categories. We will use the notions of a rigid symmetric monoidal $\mathbb{C}$-linear category (symmetric monoidal will be abbreviated as SM) and of symmetric monoidal functors (SM functor for short) as defined in [D1, ECGNO]. In such a category $\mathcal{C}$ we denote by $\mathbb{1}$ the unit object, by $\sigma_{C_1, C_2}$ the symmetry morphisms, and by $ev, coev$ the evaluation and coevaluation morphisms.

A (symmetric) tensor category (after [D1, ECGNO]) will stand for an abelian rigid SM category where the bifunctor $\otimes$ is bilinear on morphisms and where $\text{End}_\mathcal{C}(1) = \mathbb{C}$.

2.3. Representations of a the Lie algebra object $gl(V)$. Let $\mathcal{C}$ be a rigid SM category, and let $V \in \mathcal{C}$. The object $gl(V) := V \otimes V^*$ is a Lie algebra object, and one can consider the category $Rep_\mathcal{C}(gl(V))$ of representations of $gl(V)$ in $\mathcal{C}$, which is again a rigid SM category.

The objects of $Rep_\mathcal{C}(gl(V))$ are pairs $(M, \rho)$ where $\rho : gl(V) \otimes M \rightarrow M$. In particular, we have objects $V, V^*$, the standard and co-standard representations of $gl(V)$.

2.4. Representations of the general linear Lie superalgebra. Throughout the paper, we will consider the tensor category $Rep(gl(m|n))$ of finite-dimensional representations of the general linear Lie superalgebra $gl(m|n)$ (together with even morphisms) in the sense of [EHS, Section 3]. This category is “half” of the usual category of finite-dimensional representations of $gl(m|n)$ which are integrable over $GL(m|n)$; it contains those representations of $gl(m|n)$ on which the $\mathbb{Z}_2$-grading is given by the action of the element $diag(1, \ldots, 1, -1, \ldots, -1)$ from the supergroup $GL(m|n)$. Any object in the category $Rep(gl(m|n))$ is a subquotient of a finite direct sum of mixed tensor powers of $\mathbb{C}^{m|n}$.

The category $Rep_{\mathcal{C}}(gl(m|n))$ is a highest weight category.

To describe the weight poset of this category, let us introduce the following notation:

- Let $w_1, \ldots, w_m, w_{m+1}, \ldots w_{m+n}$ be the basis of the standard representation $W := \mathbb{C}^{m|n}$.
- Let $E_{i,j}, 1 \leq i, j \leq m + n$ be the corresponding matrix units basis of $gl(m|n)$.
- Let $\mathfrak{h}$ be the Cartan subalgebra consisting of diagonal matrices, and consider the Borel subalgebra of upper-triangular matrices in the above basis.
- Let $\delta_1, \ldots, \delta_{m+n}$ be the basis of $\mathfrak{h}^*$ dual to $E_{1,1}, \ldots, E_{m+n,m+n}$. The root system is $\{\delta_i - \delta_j | i \neq j\}$, with positive roots $\delta_i - \delta_j, i < j$. 


The simple modules $L(a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n)$ are then parametrized by pairs of weakly-decreasing integer sequences $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n$. The highest weight of such a module is $\sum_{i=1}^{m} a_i \delta_i + \sum_{j=1}^{n} b_j \delta_{m+j}$.

We denote the standard objects in the category $\text{Rep}(\mathfrak{gl}(m|n))$ by $K(a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n)$ (these are called Kac modules).

We now describe two (partial) correspondences between bipartitions and simple $\mathfrak{gl}(m|n)$-modules. For each bipartition $\lambda := (\lambda^\ast, \lambda^\circ)$ we define a $\mathfrak{gl}(m|n)$-module $L(\lambda)$, which is either simple or zero.

**Case** $m, n \neq 0$. Given a bipartition $\lambda$ such that $\ell(\lambda^\ast) \leq m, \ell(\lambda^\circ) \leq n$, let

$$L(\lambda) := L(a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n)$$

and

$$K(\lambda) := K(a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n)$$

where $a_i := \lambda_i^\ast$, $b_i := -\lambda_{n-i+1}^\circ$ for any $i$.

We will denote: $\bar{\lambda} := \sum_{i=1}^{m} a_i \delta_i + \sum_{j=1}^{n} b_j \delta_{m+j}$.

For other bipartitions we put $L(\lambda) := 0, K(\lambda) := 0$.

**Case** $n = 0$. In this case $\text{Rep}(\mathfrak{gl}(m|0))$ is the semisimple category of all rational representations of $GL(m)$. Given a bipartition $\lambda$ such that $\ell(\lambda^\ast) + \ell(\lambda^\circ) \leq m$, we define:

$$L(\lambda) := L(a_1, a_2, \ldots, a_m)$$

where $a_i := \lambda_i^\ast$ for $i \leq \ell(\lambda^\ast)$, and $a_{m-i+1} := -\lambda_i^\circ$ for $i \leq \ell(\lambda^\circ)$.

We will denote: $\bar{\lambda} := \sum_{i=1}^{m} a_i \delta_i$.

For other bipartitions we put $L(\lambda) := 0$.

**Example 2.4.1.** The bipartition $\lambda = \begin{bmatrix} \lambda_1 & \vdots & \lambda_r \\ \vdots & \ddots & \vdots \\ \lambda_r & \ldots & \lambda_m \end{bmatrix}$ corresponds to $L(1, 1, 0, \ldots, 0, -1, -1)$ if $m \geq 4$, $n = 0$. On the other hand, $\lambda$ corresponds to $L(1, 1, 0, \ldots, 0, -2)$ if $m, n \geq 2$.

This notation is convenient in view of the (straightforward) lemma below:

**Lemma 2.4.2.** Let $\lambda := (\lambda^\ast, \lambda^\circ)$ be a bipartition, and consider $m, n$ such that either $\ell(\lambda^\ast) \leq m, \ell(\lambda^\circ) \leq n$ or $\ell(\lambda^\ast) + \ell(\lambda^\circ) \leq m, n = 0$.

For any such $m, n$, let $W = \mathbb{C}^m|n$ be the standard representation of $\mathfrak{gl}(m|n)$. Then the maximal weight of $S^{\lambda_\ast}W \otimes S^{\lambda_\circ}W^\ast$ is

$$\bar{\lambda} := \sum_{i=1}^{m} a_i \delta_i + \sum_{j=1}^{n} b_j \delta_{m+j}$$

where $a_i, b_i$ are defined above. This weight occurs with multiplicity 1.

In other words, there exists a unique (up to scalar) non-zero homomorphism

$$K(a_1, \ldots, a_m, b_1, \ldots, b_n) \longrightarrow S^{\lambda_\ast}W \otimes S^{\lambda_\circ}W^\ast$$

3. Preliminaries

### 3.1. Deligne categories $D_t$

We briefly recall some facts about the Deligne categories $D_t$ (also known as $\text{Rep}(GL_t)$).

Let $t \in \mathbb{C}$.

**3.1.1. Construction.** Consider the free SM $\mathbb{C}$-linear category $OB$ generated by one dualizable object $V$; this is the oriented Brauer category, as described in [BCNR]. Next, consider the quotient $D_t^0$ of this category by the relation $\dim(V) = t$ (for any $t \in \mathbb{C}$). The category $D_t^0$ (denoted by Deligne as $\text{Rep}_D(\mathfrak{gl}_t)$, see [D2, Section 10]) is freely generated, as a $\mathbb{C}$-linear category, by one dualizable object of dimension $t$. We denote by $V_t$ and $V_t^\ast$ the $t$-dimensional generator and its dual. The objects in this category will be mixed tensor powers $V_t^\otimes p \otimes V_t^\ast \otimes q$, and the Hom-spaces will have a diagrammatic basis, generated by tensor products of the morphisms $\text{Id}_{V_t}, \sigma_{V_t, V_t^\ast}, \sigma_{V_t^\ast, V_t}, ev_{V_t}, coev_{V_t}$ and their duals, with the relation

$$\dim V_t := ev_{V_t} \circ \sigma_{V_t, V_t^\ast} \circ coev_{V_t} = t.$$
We will denote by $D_t$ the Karoubi additive envelope of the above “free” category, which is obtained by adding formal direct sums and images of idempotents. The category $D_t$ is a Karoubian additive rigid SM category, also called the Deligne’s category $Rep(GL_t)$. Its structure is studied in [2], and it is that the indecomposable objects of $D_t$ (up to isomorphism) are parametrized by the set of all bipartitions. We will denote the indecomposable object corresponding to the bipartition $\lambda$ by $T(\lambda)$.

**Example 3.1.1.** For any $t \in \mathbb{C}$, $T(\varnothing, \varnothing) = 1$, $T(\varnothing, \Box) = V_t$, $T(\Box, \Box) = V_t^*$. When $t \neq 0$, we have: $T(\varnothing, \varnothing) + T(\Box, \Box) = V_t \otimes V_t^*$. On the other hand, when $t = 0$, we have: $T(\Box) = V_t \otimes V_t^*$.

**3.1.2. Universal property.** The category $D_t$ possesses a universality property: given a Karoubian additive rigid SM category $C$, together with a fixed object $V \in C$ of dimension $t$, there exists an essentially unique additive SM functor $S_t : D_t \longrightarrow C$ such that $V_t \mapsto V$.

In particular, for $t \in \mathbb{Z}$ and for any $m, n \in \mathbb{Z}_+$ such that $m - n = t$, there exists a SM functor

$$S_{m,n} : D_t \longrightarrow Rep(gl(m|n)), \quad V_t \mapsto \mathbb{C}^{m|n}$$

In the special case $t = d \in \mathbb{Z}_+$, the functor $S_{d,0}$ takes the indecomposable object $T(\lambda)$ to the $GL(d)$-module $L(\lambda)$.

**3.1.3. Abelian version.** The category $D_t$ is a priori not necessarily abelian (thus not a tensor category). Yet it turns out that for $t \notin \mathbb{Z}$, this category is indeed abelian and even semisimple.

For $t \in \mathbb{Z}$, this is not the case. Fortunately, it turns out the one can construct a tensor category $V_t$ into which $D_t$ embeds as a full rigid SM subcategory. We will set $V_t := D_t$ when $t \notin \mathbb{Z}$.

The collection $\{V_t, Rep(gl(m|n))|m - n = t\}$ acts as a system of abelian envelopes of the Deligne category $D_t$ (see [EHS] for details).

When $t \in \mathbb{Z}$, the category $V_t$ is constructed in [EHS] as a certain limit of categories $Rep(gl(m|n))$ with $m - n = t$. For any $m, n$ such that $m - n = t$, there exists an additive SM functor (not exact!)

$$F_{m|n} : V_t \longrightarrow Rep(gl(m|n))$$

such that the composition of $F_{m|n}$ with the embedding $D_t \hookrightarrow V_t$.

The functors $F_{m|n}$ are “local” equivalences: the categories $V_t, Rep(gl(m|n))$ have natural $\mathbb{Z}_+$-filtrations on objects which are preserved by the functors $F_{m|n}$, and the latter induce an equivalence between each filtration $V_t^k$ and $Rep^K(gl(m|n))$ for $m, n >> k$.

In the category $V_t$ we distinguish three types of objects, all defined up to isomorphism: the simple objects, denoted by $L(\lambda)$, the “standard objects”, denoted by $V(\lambda)$ and the objects $T(\lambda)$. The three types are parametrized by arbitrary bipartitions. The name “standard” comes from the fact that the subcategories $V_t^k$ are highest-weight categories, with objects $L(\lambda), V(\lambda), T(\lambda)$ (for $|\lambda^*| + |\lambda| \leq k$) playing the roles of simples, standard and tilting objects respectively.

**Remark 3.1.2.** When $t \notin \mathbb{Z}$, the objects $L(\lambda), V(\lambda), T(\lambda)$ are defined to be isomorphic.

Finally, let us say a few words about the $gl(m|n)$-modules $V(\lambda) := F_{m|n}(V(\lambda))$ in $Rep^K(gl(m|n))$ (here $k \geq |\lambda|$ is fixed, $m, n >> k, m - n = t$). These modules appear as submodules of $S_{\lambda^*}W \otimes S_{\lambda}^*W$, where $W = \mathbb{C}^{m|n}$ is the standard representation of $gl(m|n)$.

In fact, we have the following description (see [EHS]):

**Lemma 3.1.3.** $V(\lambda)$ is a highest weight module, with highest weight $\check{\lambda} := \sum_{i=1}^{m} a_i \delta_i + \sum_{j=1}^{n} b_j \delta_{m+j}$ where $a_i := \lambda^*_i$, $b_i := -\lambda_{n-i+1}^*$ for any $i$.

In other words, there exists a unique (up to scalar) non-zero homomorphism

$$K(\lambda) \longrightarrow S_{\lambda^*}W \otimes S_{\lambda}^*W$$

with image $V(\lambda)$. In fact, $V(\lambda)$ is the maximal quotient of $K(a_1, \ldots, a_m, b_1, \ldots, b_n)$ lying in $Rep^K(gl(m|n))$.

See [EHS] Section 4.7] about the “local” highest weight structure of $V_t$.
3.1.4. Deligne categories with formal parameter. One can construct a similar Deligne category for a formal variable \( T \) instead of complex parameter \( t \); the obtained category \( D_T \) is a \( \mathbb{C}((T)) \)-linear semisimple tensor category. The simple objects (up to isomorphism) are again labeled by all bipartitions; by abuse of notation, we will denote them by \( \mathbf{T}(\mu) \) as well.

The connection between the Deligne categories \( D_t \) and \( D_T \) is manifested by the existence of a homomorphism of rings with scalar products

\[
\text{Lift}_t : K_0(D_t) \to K_0(D_T)
\]

\((K_0 \text{ stands for the split Grothendieck ring}) \) which respects duals and takes \( V_t \) to \( V_T \). This map was defined in [CW, Section 6].

The existence of this map is based on the fact that for any \( r', s' \in \mathbb{Z}_+ \), any idempotent \( e \) in the walled Brauer algebra

\[
\text{Br}_\mathbb{C}(r', s') = \text{End}_{D_t}(V_t^\otimes r' \otimes V_t^\ast \otimes s')
\]

over \( \mathbb{C} \) can be lifted to an idempotent \( \tilde{e} \) in the walled Brauer algebra \( \text{Br}_{\mathbb{C}((T))}(r', s') = \text{End}_{D_T}(V_T^\otimes r' \otimes V_T^\ast \otimes s') \) over \( \mathbb{C}((T)) \), so that \( \tilde{e}|_{T=1} = e \) (cf. [CO, Section 3.2]). We will denote by \( D_\mu^\lambda(t) \) the multiplicity of \( \mathbf{T}(\mu) \) inside \( \text{Lift}_t(\mathbf{T}(\lambda)) \). This multiplicity has been shown to be either 1 or 0, and [CW, Section 6] contains an explicit algorithm for computing \( D_\mu^\lambda(1) \) (see also Section 3).

For a fixed bipartition \( \mu \) and \( t \not\in \{0, \pm 1, \pm 2, \ldots, \pm |\mu^\ast| + |\mu^\circ|\} \), it was proved that \( \text{Lift}_t(\mathbf{T}(\mu)) = \mathbf{T}(\mu) \), and thus \( D_\mu^\lambda(t) = \delta_{\lambda, \mu} \) for almost all \( t \).

3.2. Lie algebra \( \mathfrak{sl}_2 \). Let \( \mathfrak{sl}_2 \) be the Lie algebra of \( \mathbb{Z} \times \mathbb{Z} \)-matrices with finitely-many non-zero entries and trace zero. This Lie algebra is generated by the elements \( f_a := E_{a+1,a} \) and \( e_a := E_{a,a+1} \), where \( a \in \mathbb{Z} \).

We will use the same definitions as in [BLW, Section 2.2]. The weight lattice of \( \mathfrak{sl}_2 \) is \( \mathbb{Z}\varpi_a \) where \( \varpi_a \) is the \( a \)-th fundamental weight. The \( a \)-th simple root is

\[
\alpha_a = 2\varpi_a - \varpi_{a-1} - \varpi_{a+1}
\]

These span the root lattice inside the weight lattice. We define a dominance order \( \geq \) on the lattice of weights by setting \( \beta \geq \gamma \) whenever \( \beta - \gamma \) is a finite combination of simple roots with non-negative integral coefficients.

We denote by \( \mathbb{C}^\mathbb{Z} := \text{span}_\mathbb{C}\{u_i|i \in \mathbb{Z}\} \) the tautological representation of \( \mathfrak{sl}_2 \). We have:

\[
f_a \cdot u_i = \delta_{a,i} u_{a+1}, \quad e_a \cdot u_i = \delta_{a,i} u_{a-1}
\]

The weight of the vector \( u_i \) is \( \varpi_i - \varpi_{i-1} \).

3.3. Fock space. The Fock space is a \( \mathbb{C} \)-vector space with a basis consisting of infinite wedges \( u_I := u_{i_0} \wedge u_{i_{-1}} \wedge u_{i_{-2}} \wedge \ldots (i_s \in \mathbb{Z} \text{ for any } s) \) where \( I := (i_0, i_{-1}, i_{-2}, \ldots) \) is a strictly decreasing infinite sequence satisfying: \( i_{-s} = -s \) for \( s >> 0 \).

This space has an obvious action of \( \mathfrak{sl}_2 \) on it.

We will also use another notation for this basis, indexing by partitions of arbitrary size:

\[
v_\nu := u_{\nu_1} \wedge u_{\nu_2-1} \wedge \ldots \wedge u_{\nu_{s+1}-s} \wedge \ldots
\]

In the new notation the action of \( \mathfrak{sl}_2 \) on \( \mathfrak{F} \) is given by

\[
f_a \cdot v_\nu := v_{\nu + a}, \quad e_a \cdot v_\nu := v_{\nu - a}
\]

for any partition \( \nu \) and any \( a \in \mathbb{Z} \).

Note that \( v_\nu \in \mathfrak{F} \) is a weight vector, i.e. \( h_a := [e_a, f_a] \in \mathfrak{sl}_2 \) acts on \( v_\nu \in \mathfrak{F} \) by a scalar \( n_a(\nu) \in \{0, \pm 1\} \), where

\[
n_a(\nu) = \begin{cases} 1 & \text{if } \nu + a \neq \emptyset, \\ -1 & \text{if } \nu - a \neq \emptyset, \\ 0 & \text{else} \end{cases}
\]

We denote the weight \((n_a(\nu))_{a \in \mathbb{Z}}\) of \( v_\nu \) by \( \omega_\nu \).
3.3.1. **Twisted and shifted duals.** We will also consider the twisted dual Fock space representation \( \tilde{\mathfrak{g}}^\vee \) of \( \mathfrak{sl}_2 \).

The space \( \tilde{\mathfrak{g}}^\vee \) is isomorphic to \( \tilde{\mathfrak{g}} \) as a vector space, with the action of \( \mathfrak{sl}_2 \) given by \( f_a, v_\lambda := v_{\lambda - \square - a}, e_a, v_\lambda := v_{\lambda + \square - a} \).

Finally, for \( t \in \mathbb{Z} \), we will consider the "shifted dual" representation \( \tilde{\mathfrak{g}}^\vee_t \) of \( \mathfrak{sl}_2 \): the underlying vector space is \( \tilde{\mathfrak{g}}^\vee \), with \( f_a, e_a (a \in \mathbb{Z}) \) acting by \( f_a, v_\lambda := v_{\lambda - \square - (a+t)}, e_a, v_\lambda := v_{\lambda + \square - (a+t)} \).

3.3.2. **Fock space as a "stable" inverse limit.** We will now show that the space \( \tilde{\mathfrak{g}} \) is a "stable" inverse limit of a system of subspaces of the exterior powers \( \wedge^n \mathbb{C}^Z \), \( n \geq 0 \). We will explain the precise meaning below.

The space \( \wedge^n \mathbb{C}^Z \) has a basis \( u_I := u_{i_0} \wedge u_{i_1} \wedge \ldots \wedge u_{i_n} \) indexed by decreasing sequences \( I \) of length \( n \). We have \( \mathfrak{sl}_2 \)-equivariant maps

\[
\phi_{n+1} : \wedge^{n+1} \mathbb{C}^Z \to \wedge^n \mathbb{C}^Z, \quad u_{i_0} \wedge u_{i_1} \wedge \ldots \wedge u_{i_{n+1}} \mapsto u_{i_0} \wedge u_{i_1} \wedge \ldots \wedge u_{i_n} 
\]

and

\[
\pi_n : \tilde{\mathfrak{g}} \to \wedge^n \mathbb{C}^Z, \quad u_I \mapsto u_{i_0} \wedge u_{i_1} \wedge \ldots \wedge u_{i_n} 
\]

Now, these maps define an inverse limit \( \lim_{n \to \infty} \wedge^n \mathbb{C}^Z \) and a map \( \tilde{\mathfrak{g}} \to \lim_{n \to \infty} \wedge^n \mathbb{C}^Z \), but this map will not be an isomorphism (e.g. \( u_1 \wedge u_0 \wedge u_{-1} \wedge \ldots \) will not lie in the image).

To fix this, consider a system of \( \mathbb{Z}_+ \)-parametrized subspaces in \( \wedge^n \mathbb{C}^Z \) and in \( \tilde{\mathfrak{g}} \) given by the so-called "energy" function on the basis:

Let the energy of the vector \( u_I \) be \( \sum s_i - s \). Then the subspaces \( \wedge^n \mathbb{C}^Z(\nu) \) and \( \tilde{\mathfrak{g}}(\nu) \) are defined as the span of vectors \( u_I \) of energy at most \( \nu \), for which \( i_s + s \geq 0 \) for all \( s \).

**Remark 3.3.1.** In the latter case we have: \( \tilde{\mathfrak{g}}(\nu) = \text{span}_I \{ v_\lambda \mid |\lambda| \leq \nu \} \).

**Remark 3.3.2.** Notice that the above subspaces are not preserved by the action of \( \mathfrak{sl}_2 \).

Then \( \tilde{\mathfrak{g}} = \lim_{k \to \infty} \tilde{\mathfrak{g}}(k) \), and the maps \( \phi_{n+1}, \pi_n \) preserve these systems of subspaces. Moreover, \( \pi_n|_{\tilde{\mathfrak{g}}(k)}, \phi_{n+1}|_{\wedge^{n+1} \mathbb{C}^Z(k)} \) become isomorphisms for \( n \geq k - 1 \), and thus

\[
\tilde{\mathfrak{g}} \cong \lim_{k \to \infty} \lim_{n \to \infty} \wedge^n \mathbb{C}^Z(\nu)
\]

Note that this is not an isomorphism of \( \mathfrak{sl}_2 \)-modules.

**Remark 3.3.3.** This isomorphism has a categorical version, describing the category \( \mathcal{P}ol \) of "stable" polynomial \( GL \)-modules as a special limit of the categories of polynomial \( GL_n \)-modules. See [HY] for details.

4. **Weight diagrams**

In this section we briefly recall the definitions of weight diagrams given in [EHS], [CW]. The weight diagrams were originally defined and used in the representation theory of superalgebras and Khovanov arc algebras (see [BS] [MS]), and provide a powerful combinatorial tool when studying the highest-weight structure of the categories \( \text{Rep}(\mathfrak{gl}(m|n)) \) and (locally) \( \mathcal{V}_I \).

4.1. **Two types of diagrams.** Let \( \lambda \) be a bipartition, and let \( t \in \mathbb{C} \). Below we define two diagrams corresponding to \( \lambda \), called \( d_\lambda \) and \( d'_\lambda \), which represent a labeling of integers by symbols \( \times, \bigcirc, <, > \).

For the weight diagram \( d_\lambda \), we will use the same notation as in [EHS]. This notation differs slightly from the notation in [BS] [MS]: in [BS] the symbols are permuted (see [MS] for a dictionary) and the diagrams are shifted.
Consider two infinite sequences 
\[ C := \{ \lambda^i + t - i \}_{i \geq 1}, \quad D := \{ \lambda^i - i \}_{i \geq 1}. \]

Consider the map 
\[ f_\lambda : \mathbb{Z} \to \{ \times, >, <, \circ \} \]
defined by
\[
(2) \quad f_\lambda(s) = \begin{cases} 
\circ, & \text{if } s \notin C \cup D \\
> , & \text{if } s \in C \setminus D \\
< , & \text{if } s \in D \setminus C \\
\times, & \text{if } s \in C \cap D.
\end{cases}
\]

To the map \( f_\lambda \) we associate a diagram \( d_\lambda \) where every integer \( s \) is labeled by \( f_\lambda(s) \).

**Example 4.1.1.** Let \( t = 0, \lambda = (\emptyset, \emptyset) \). The diagram \( d_\lambda \) is then

\[
\begin{array}{cccccccc}
\times & -5 & \times & -4 & \times & -3 & \times & -2 & \times & -1 & \circ & 0 & \circ & 1 & \circ & 2 & \circ & 3 & \circ & 4 & \circ & 5 \\
\end{array}
\]

**Example 4.1.2.** Let \( t = 1, \lambda = (\emptyset, \emptyset) \). The diagram \( d_\lambda \) is then

\[
\begin{array}{cccccccc}
\times & -5 & \times & -4 & \times & -3 & \times & -2 & > & -1 & \circ & 0 & \circ & < & 2 & \circ & 3 & \circ & 4 & \circ & 5 \\
\end{array}
\]

Note that \( f_\lambda(s) = \circ \) for all \( s > 0 \). If \( t \notin \mathbb{Z} \), then the diagram \( d_\lambda \) contains only the symbols \( \circ, < \). If \( t \in \mathbb{Z} \), then \( f_\lambda(s) = \times \) for all \( s < 0 \).

Next, consider the (infinite) sets 
\[ C' := \{ \lambda^i + t - i \}_{i \geq 1}, \quad D' := \mathbb{Z} \setminus \{ i - \lambda^i - 1 \}_{i \geq 1}. \]

Consider the map 
\[ f'_\lambda : \mathbb{Z} \to \{ \times, >, <, \circ \} \]
defined by
\[
(3) \quad f'_\lambda(s) = \begin{cases} 
\circ, & \text{if } s \notin C' \cup D' \\
> , & \text{if } s \in C' \setminus D' \\
< , & \text{if } s \in D' \setminus C' \\
\times, & \text{if } s \in C \cap D.
\end{cases}
\]

To the map \( f'_\lambda \) we associate a diagram \( d'_\lambda \) where every integer \( s \) is labeled by \( f'_\lambda(s) \).

**Example 4.1.3.** Let \( t = 0, \lambda = (\emptyset, \emptyset) \). The diagram \( d'_\lambda \) is then

\[
\begin{array}{cccccccc}
\times & -5 & \times & -4 & \times & -3 & \times & -2 & \times & -1 & \circ & 0 & \circ & 1 & \circ & 2 & \circ & 3 & \circ & 4 & \circ & 5 \\
\end{array}
\]

**Example 4.1.4.** Let \( t = 1, \lambda = (\emptyset, \emptyset) \). The diagram \( d'_\lambda \) is then

\[
\begin{array}{cccccccc}
\times & -5 & \times & -4 & \times & -3 & \times & -2 & > & -1 & \circ & 0 & \circ & < & 2 & \circ & 3 & \circ & 4 & \circ & 5 \\
\end{array}
\]

**Example 4.1.5.** Let \( t = 1, \lambda = (\emptyset, \emptyset) \). The diagram \( d'_\lambda \) is then

\[
\begin{array}{cccccccc}
\times & -5 & \times & -4 & \times & -3 & \times & -2 & > & -1 & \circ & 0 & \circ & < & 1 & \circ & 2 & \circ & 3 & \circ & 4 & \circ & 5 \\
\end{array}
\]

**Remark 4.1.6.** The definition of \( d'_\lambda \) corresponds to the diagrams (shifted by \( t \)) defined in [CW, Section 6], but with the symbols permuted. Here is a dictionary:

| Notation of [CW] | \( \lor \) | \( \land \) | \( \times \) | \( \circ \) |
|------------------|-------|-------|-------|-------|
| Our notation     | \( \circ \) | \( \times \) | \( > \) | \( < \) |
4.2. Relation between the two weight diagrams.

**Notation 4.2.1.** Let $\lambda, \mu$ be a bipartition. We denote by $\lambda^\vee := (\lambda^\bullet, \lambda^{\circ\vee})$, where $\lambda^{\circ\vee}$ is the transposed Young diagram $\lambda^\circ$.

**Lemma 4.2.2.** The diagrams $d_\lambda$ and $d^\vee_\lambda$ coincide.

**Proof.** We need to show that the set $D$ for $\lambda$ and the set $D'$ for $\lambda^\vee$ coincide. That is, that we want to show that for any Young diagram $\kappa := \lambda^\circ$, we have

$$\{\kappa_i - i\}_{i \geq 1} \cup \{i - \kappa_i^{\circ\vee} - 1\}_{i \geq 1} = \mathbb{Z}$$

First, we check that these two sets do not intersect. Indeed, assume that $\kappa_i - i = j - \kappa_j^{\circ\vee} - 1$ for some $i, j$. Then we would have $\kappa_i + \kappa_j^{\circ\vee} + 1 = i + j$. Yet this is impossible: $\kappa_i < j$ iff $\kappa_j^{\circ\vee} < i$, and thus $\kappa_i + \kappa_j^{\circ\vee} + 1$ is either at most $i + j - 1$, or at least $i + j + 1$.

Next, we show that the union of the two sets above is indeed $\mathbb{Z}$. Let $N = |\kappa|$. Then for any $j > N$, we have: $j = (j + 1) - \kappa_{j+1} - 1$, $-j = \kappa_j - j$ (the first statement is true even for $j = N$). Thus

$$Z \setminus \{-N, -N + 1, \ldots, N - 2, N - 1\} = \{j | j \geq N\} \cup \{-j | j > N\} \subset \{\kappa_i - i\}_{i \geq 1} \cup \{i - \kappa_i^{\circ\vee} - 1\}_{i \geq 1}$$

The set difference between the RHS and the LHS in the above inclusion is $\{\kappa_i - i\}_{1 \leq i \leq N} \cup \{i - \kappa_i^{\circ\vee} - 1\}_{1 \leq i \leq N}$. This is a set of $2N$ elements, and thus it covers precisely the set $\{-N, -N + 1, \ldots, N - 2, N - 1\}$.

$\square$

4.3. Diagrams and the Fock space. Finally, we prove a combinatorial auxiliary result relating weights in the Fock space representation of $\mathfrak{sl}_2$ and weight diagrams. This will be used to prove Theorem 10.2.6.

**Definition 4.3.1.** The core $\text{core}(d'_\lambda)$ of the diagram $d'_\lambda$ is the diagram obtained from $d'_\lambda$ by replacing all the symbols $\times$ with $\circ$. The core $\text{core}(d_\lambda)$ of $d_\lambda$ is defined in the same way.

**Lemma 4.3.2.** Let $t \in \mathbb{C}$. Let $\lambda, \mu$ be two bipartitions, such that $\text{core}(d'_\lambda) = \text{core}(d'_\mu)$. Then

$$n_a(\lambda^\bullet) - n_{-(a+t)}(\lambda^\circ) = n_a(\mu^\bullet) - n_{-(a+t)}(\mu^\circ)$$

for each $a \in \mathbb{Z}$ where $n_a(\nu)$ equals 1 if $\nu + \Box_a \neq \emptyset$, $-1$ if $\nu - \Box_a \neq \emptyset$, and zero otherwise, as defined in Section 3.3.

**Proof.** First of all, recall that by Lemma 4.2.2, $d'_\lambda = d_{\lambda^\vee}$, so $\text{core}(d_{\lambda^\vee}) = \text{core}(d_{\mu^\vee})$, and

$$n_a(\lambda^\bullet) - n_{-(a+t)}(\lambda^\circ) = n_a(\lambda^\bullet) - n_{-(a+t)}(\lambda^{\circ\vee})$$

$$n_a(\mu^\bullet) - n_{-(a+t)}(\mu^\circ) = n_a(\mu^\bullet) - n_{-(a+t)}(\mu^{\circ\vee})$$

Thus we wish to show that $\text{core}(d_{\lambda^\vee}) = \text{core}(d_{\mu^\vee})$ implies

$$n_a(\lambda^\bullet) - n_{a+t}(\lambda^{\circ\vee}) = n_a(\mu^\bullet) - n_{a+t}(\mu^{\circ\vee})$$

Consider the sets

$$C^\lambda := \{\lambda_i^\bullet - i\}, \quad D^\lambda := \{\lambda_i^{\circ\vee} - i - t\}, \quad C^\mu := \{\mu_i^\bullet - i\}, \quad D^\mu := \{\mu_i^{\circ\vee} - i - t\}$$

(notice the shift by $t$ with respect to the previous definitions).

Let $\phi^C_\lambda, \phi^D_\lambda : \mathbb{Z} \to \{0, 1\}$ be the characteristic functions of the sets $C^\lambda$ and $D^\lambda$ respectively, and similarly for $\phi^C_\mu, \phi^D_\mu$.

The condition $\text{core}(d_{\lambda^\vee}) = \text{core}(d_{\mu^\vee})$ implies that

$$\phi^C_\lambda - \phi^D_\lambda = \phi^C_\mu - \phi^D_\mu$$

Now,

$$n_a(\lambda^\bullet) = \begin{cases} 1 & \text{if } a - 1 \in C^\lambda, \ a \notin C^\lambda \\ -1 & \text{if } a \in C^\lambda, \ a - 1 \notin C^\lambda \\ 0 & \text{else} \end{cases}$$
n_{\alpha}(\lambda^\bullet) = \phi_{\lambda}^C(a - 1) - \phi_{\lambda}^D(a)

Similarly,

n_{\alpha+t}(\lambda^{\circ\vee}) = \phi_{\lambda}^D(a - 1) - \phi_{\lambda}^C(a)

Writing out similar identities for \mu, we obtain:

n_{\alpha}(\lambda^\bullet) - n_{\alpha+t}(\lambda^{\circ\vee}) = \phi_{\mu}^C(a - 1) - \phi_{\mu}^C(a) - (\phi_{\mu}^D(a - 1) - \phi_{\mu}^D(a)) =

= \phi_{\mu}^C(a - 1) - \phi_{\mu}^C(a) - (\phi_{\mu}^D(a - 1) - \phi_{\mu}^D(a)) = n_{\alpha}(\mu^\bullet) - n_{\alpha+t}(\mu^{\circ\vee})

as required.

5. Multiplicities in the category \mathcal{V}_t

In this section we multiplicities in standard filtrations and Jordan-Holder filtrations in category \mathcal{V}_t. Recall that the objects \mathbf{T}(\lambda) in \mathcal{V}_t have a standard filtration by objects \mathbf{V}(\mu), which is induced from the standardly-filtered objects \mathbf{V}_{t}^{\otimes |\lambda^\bullet|} \otimes \mathbf{V}_{t}^{\otimes |\lambda^{\circ\vee}|} (see [EHS] for details).

**Theorem 5.0.1.** Let \( t \in \mathbb{C} \), and let \( \lambda, \mu \) be two bipartitions. Then

\[
(\mathbf{T}(\lambda) : \mathbf{V}(\mu))_{\mathcal{V}_t} = [\mathbf{V}(\lambda) : \mathbf{L}(\mu)]_{\mathcal{V}_t}
\]

where the LHS denotes the multiplicity of the standard object \( \mathbf{V}(\mu) \) in the standard filtration of the standard object \( \mathbf{T}(\lambda) \), and the RHS denotes the multiplicity of the simple object \( \mathbf{L}(\lambda) \) among the Jordan-Holder components of \( \mathbf{V}(\mu) \).

Moreover, these multiplicities coincide with the multiplicity \( D^\lambda_{\mu}(t) \) (see Section 3.1.4 for definition, and [CW, Section 6]).

**Proof.** Denote:

\[
K^\lambda_{\mu}(t) := (\mathbf{T}(\lambda) : \mathbf{V}(\mu))_{\mathcal{V}_t}, \quad \bar{K}^\lambda_{\mu}(t) = [\mathbf{V}(\lambda) : \mathbf{L}(\mu)]_{\mathcal{V}_t}
\]

The multiplicities \( \bar{K}^\lambda_{\mu}(t) \) can be shown to be either 1 or 0, and can be computed using cap diagrams (cf. original definitions in [BS, MS]). We will use the cap diagrams as defined in [EHS, Section 3.5, Section 4.5]. The cap diagram corresponding to \( d^\lambda_{\alpha} \) is the diagram \( d^\lambda_{\alpha} \) together with additional arcs ("caps") which have a left end at \( \times \) and a right end at \( \circ \). These caps are required to satisfy the following conditions:

1. The caps should not intersect each other.
2. Any \( \circ \) which is inside a cap should be the right end of some other cap.
3. Any \( \times \) should be the left end of exactly one cap.

In [EHS, Lemma 4.5.2] it is shown that \( \bar{K}^\lambda_{\mu}(t) = 1 \) iff one can obtain \( d^\lambda_{\alpha} \) from \( d^\mu_{\alpha} \) by moving finitely many crosses in the cap diagram of \( d^\mu_{\alpha} \) from the left end of a cap to the right end of this cap.

Let us give a sketch of the proof: using Lemma 3.1.3, one reduces the problem to the computation of Kazhdan-Lusztig coefficients in the \( gl(m|n) \)-module \( K(\lambda) \). A simple module \( L(\mu) \) sits in \( K(\lambda) \) precisely if one can obtain the diagram \( d_{\lambda, \nu} \) from \( d_{\mu, \nu} \) by moving finitely many crosses from the left end of a cap to the right end of this cap (cf. [MS]). By Lemma 3.2.2, we have \( d_{\lambda, \nu} = d_{\lambda, \nu} \), which proves the required statement.

**Remark 5.0.2.** The transposition of the second Young diagram in the bipartitions appearing above comes from the different choice of simple roots in [EHS].

In [CW, Section 6], it is shown that lift_t(\mathbf{T}(\lambda)) = \bigoplus_{\mu} \mathbf{T}(\mu)^{\oplus D^\lambda_{\mu}(t)} (see Section 3.1.4 for definition of lift_t), where \( D^\lambda_{\mu}(t) \) is 1 or 0, and it can be computed explicitly using cap diagrams: namely, \( D^\lambda_{\mu}(t) = 1 \) iff \( d^\lambda_{\alpha} \) can be obtained from \( d^\mu_{\alpha} \) by moving finitely many \( \times \) in the cap diagram of \( d^\mu_{\alpha} \) from the left end of a cap to the right end of this cap.

\[\text{Note that the caps we describe are "complimentary" to the caps used in [CW]: the latter would have } \times \text{ on the right end and } \circ \text{ on the left end, which means that after switching the symbols they would become our caps.}\]
We will denote by \( b_t^\lambda(t) \) the multiplicities in the decomposition

\[
Y(\lambda) = \bigoplus_{\mu} T(\mu)^{\otimes b_t^\lambda(t)}
\]

We consider the objects \( T(\lambda), Y(\lambda) := S^{\lambda^*} V_t \otimes S^{\lambda^*} V_T^* \) in \( D_T \), where \( T \) is a formal parameter. We will denote by \( B^\lambda_{\mu} \) the corresponding multiplicity in this case.

One can immediately see from the definition of \( \text{lift}_t \) that \( \text{lift}_t(Y(\lambda)) = Y(\lambda) \) in \( D_T \), and therefore

\[
\bigoplus_{\mu} \text{lift}_t(T(\mu))^{\otimes b_t^\lambda(t)} = \bigoplus_{\mu'} T(\mu')^{\otimes \sum_{\mu} b_t^\lambda(t)D^\mu_{\mu'}(t)} = \text{lift}_t(Y(\lambda)) = Y(\lambda) = \bigoplus_{\mu'} T(\mu')^{\otimes B^\lambda_{\mu'}}
\]

Considering \( b, B, D \) as (infinite) matrices whose entries are numbered by pairs of bipartitions, we obtain: \( B = b(t)D(t) \).

Next, it was shown in [CW Corollary 7.1.2] that

\[
B^\lambda_{\mu} = \sum_{\kappa \in \mathcal{P}} LR^\lambda_{\mu, \kappa} \cdot LR^0_{\kappa},
\]

where \( LR \) denote the Littlewood-Richardson coefficients, and the sum is over all the partitions \( \kappa \).

On the other hand, recall that \( Y(\lambda) \subset V_t^{\otimes |\lambda^*|} \otimes V_T^{\otimes |\lambda^0|} \) is a standardly-filtered object in \( \mathcal{V}_t \), and the multiplicities of standards in this filtration are known, due to [EHS Section 3.1, Lemma 4.3.4], [PS]:

\[
(Y(\lambda) : V(\mu)) = \sum_{\kappa \in \mathcal{P}} LR^\lambda_{\mu, \kappa} \cdot LR^0_{\kappa}
\]

Thus \( (Y(\lambda) : V(\mu)) \) is the \( (\lambda, \mu) \) entry in \( b(t)D(t) \).

Now,

\[
(Y(\lambda) : V(\mu)) = \sum_{\mu'} [Y(\lambda) : T(\mu')] [T(\mu') : V(\mu)] = \sum_{\mu'} b^\lambda_{\mu'}(t)K^\mu_{\mu'}(t)
\]

Hence

\[
b(t)D(t) = b(t)K(t)
\]

We claim that the matrix \( b(t) \) is invertible. Indeed, we can order the set of bipartitions by total size \( (|\lambda| := |\lambda^*| + |\lambda^0|) \); bipartitions of the same size can be ordered arbitrarily. Then it is easy to see that \( B, D(t) \) are matrices which are lower-triangular (with finitely many non-zero entries in each column), and all entries on the diagonal are 1 (cf. [CW Section 6]). Thus \( B, D(t) \) are invertible, and so is \( b(t) \). We conclude that

\[
D(t) = K(t)
\]

which, together with the fact that \( \tilde{K}^\lambda_{\mu}(t) = D^\lambda_{\mu}(t) \), completes the proof.

\[\square\]

**Example 5.0.3.** Let \( t = 0, \lambda = (\square, \square), \mu = (\varnothing, \varnothing) \). Then \( T(\square, \square) = V_t \otimes V_T^* \), \( T(\varnothing, \varnothing) = V(\varnothing, \varnothing) = L(\varnothing, \varnothing) = 1 \). Then

\[
(T(\square, \square) : V(\varnothing, \varnothing))_{\bar{V}_t} = (V(\square, \square) : 1)_{\bar{V}_t} = 1
\]

while

\[
(T(\varnothing, \varnothing) : V(\bar{\square}, \square))_{\bar{V}_t} = (V(\varnothing, \varnothing) : L(\square, \square))_{\bar{V}_t} = 0
\]

As a corollary, we obtain the following formula for the dimension of the Hom-space for tilting modules:
Corollary 5.0.4. Let $\lambda, \mu$ be two bipartitions. Then

$$\dim \text{Hom}_{V}(T(\lambda), T(\mu)) = \sum_{\nu \in P \times P} (T(\lambda) : V(\nu))_{V_{i}} (T(\mu) : V(\nu))_{V_{i}}$$

Remark 5.0.5. In any highest weight category, we have

$$\dim \text{Hom}_{V}(T_{1}, T_{2}) = \sum_{\nu} (T_{1} : \Delta(\nu))_{V_{i}} (T_{2} : \nabla(\nu))_{V_{i}}$$

where $T_{1}, T_{2}$ are tilting objects, and $\Delta(\nu), \nabla(\nu)$ are the standard and costandard objects corresponding to weight $\nu$.

In our case, in the (local) highest-weight subcategories $V_{k}$ there is an exact duality functor taking standard objects to co-standard objects; this makes Corollary 5.0.4 an immediate consequence of the above formula.

Proof. It was proved in [CW, Section 6] that

$$\dim \text{Hom}_{V}(T(\lambda), T(\mu)) = \sum_{\nu} D_{\lambda}^{\mu}(t) D_{\nu}^{\mu}(t)$$

We have just proved that $D_{\lambda}^{\mu}(t) = (T(\lambda) : V(\nu))_{V_{i}}$ (this value is either 0 or 1), and the statement follows. $\square$

6. Categorical $\mathfrak{sl}_{\mathbb{Z}}$-action in tensor categories

The notions of categorical actions which we will use will be based on the definitions in [CR, R, BLW]. We will also define the notion of a categorical action of $\mathfrak{sl}_{\mathbb{Z}}$ corresponding to a rigid SM category, in the sense of [R, Section 5.1.1] (without the requirement that the category be abelian, nor that the complexified (split) Grothendieck group gives an integral representation of $\mathfrak{sl}_{\mathbb{Z}}$).

6.1. Categorical type A action. We start with the following definition, which is a slightly stronger form of the definition of a type A action given by Rouquier in [R, Section 5.1.1].

Definition 6.1.1. Let $\mathcal{A}$ be an additive category. A categorical type A action on $\mathcal{A}$ consists of the data $(F, E, x, \tau)$, where $(E, F)$ are an adjoint pair of (additive) endofunctors of $\mathcal{A}$, $x \in \text{End}(F)$, $\tau \in \text{End}(F^{2})$ and these satisfy:

- $F$ is isomorphic to the left adjoint of $E$,
- For any $d \geq 2$, the natural transformations $x, \tau$ define an action of the degenerate affine Hecke algebra on $F^{d}$ by

$$\text{dAHA}_{d} = \mathbb{C}[x_{1}, \ldots, x_{d}] \otimes \mathbb{C}[S_{d}] \to \text{End}(F^{d})$$

$$x_{i} \mapsto F^{d-i} x F^{i-1}$$

$$i, i + 1 \mapsto F^{d-i-1} \tau F^{i-1}$$

The definition of a functor between categorical $\mathfrak{sl}_{\mathbb{Z}}$-modules is the same as in [CR, 5.2.1].

In the spirit of [R, BLW], if $\mathcal{A}$ is Karoubian, we can consider the generalized eigenspaces of the operator $x$. This gives us a decomposition $F = \bigoplus_{a \in \mathbb{C}} F_{a}$, where $F_{a}$ is the generalized eigenspace of $x$ corresponding to eigenvalue $a$. Similarly, one can define a decomposition $E = \bigoplus_{a \in \mathbb{C}} E_{a}$ where $E_{a}$ is adjoint to $F_{a}$.

Notice that this does not necessarily give a categorical $\mathfrak{sl}_{\mathbb{Z}}$-action on $\mathcal{A}$ as described in [R, BLW], since the eigenvalues of $x$ are not necessarily integers.

\[\text{In [R], the first condition is replaced by a requirement that natural transformations } E_{a}F_{a} \to F_{a}E_{a} \text{ defined through } x, \tau \text{ would be invertible.}\]
6.2. Categorical type A action in $\text{Rep}_C(\mathfrak{gl}(V))$. Let $C$ be a rigid SM category, and fix $V \in C$. Consider the Lie algebra object $\mathfrak{gl}(V)$ and the category $\text{Rep}_C(\mathfrak{gl}(V))$ of representations of $\mathfrak{gl}(V)$ in $C$.

We construct a categorical type A action on $\text{Rep}_C(\mathfrak{gl}(V))$ as follows:

**Construction 6.2.1.**

- Define the functors $F := V \otimes (-)$, $E := V^* \otimes (-)$.
- Define the natural transformation $\tau := \sigma_{V,V} \otimes \text{Id} \in \text{End}(F^2)$.
- Define the natural transformation $\lambda \in \text{End}(F)$ by: $x_{(M,\rho)} \in \text{End}(V \otimes M)$ corresponds to $\rho$ under the isomorphism
  \[
  \text{End}(V \otimes M) \cong \text{Hom}_C(V \otimes V^* \otimes M, M)
  \]

By definition of rigidity, the functors $E, F$ are biadjoint. The action of the degenerate affine Hecke algebra acts on $F^d$ follows from the axioms of the symmetry morphism $\sigma_{-,-}$, together with the following lemma:

**Lemma 6.2.2.** The natural transformations $x, \tau$ satisfy: $\tau \circ x F - F x \circ \tau = \text{Id}$, $\tau \circ F x - x F \circ \tau = -\text{Id}$.

**Proof.** The proof is a straightforward computation, relying on the fact that for any $(M,\rho)$,
\[
x_{V \otimes(M,\rho)} = \sigma_{V,V} \otimes \text{Id}_M + (\sigma_{V,V} \otimes \text{Id}_M) \circ (\text{Id}_V \otimes x_{(M,\rho)}) \circ (\sigma_{V,V} \otimes \text{Id}_M)
\]
This implies that $xF = \tau + \tau \circ Fx$ and the statement follows.

**Remark 6.2.3.** As in [CR, Section 7.4], one can alternatively define $x$ using the Casimir natural transformation of the identity functor on $\text{Rep}_C(\mathfrak{gl}(V))$. The Casimir $C \in \text{End}(\text{Id}_{\text{Rep}_C(\mathfrak{gl}(V))})$ is defined as follows: for every $(M,\rho) \in \text{Rep}_C(\mathfrak{gl}(V))$, $C_{(M,\rho)}$ is the composition
\[
M \xrightarrow{\text{cor}_{\mathfrak{gl}(V)} \otimes \text{Id}_M} \mathfrak{gl}(V) \otimes \mathfrak{gl}(V) \otimes M \xrightarrow{\text{Id}_{\mathfrak{gl}(V)} \otimes \rho} \mathfrak{gl}(V) \otimes M \xrightarrow{\rho} M
\]
We then have
\[
x_M = \frac{1}{2}(C_{V \otimes M} - C_V \otimes \text{Id}_M - \text{Id}_V \otimes C_M)
\]

Here is an example which illustrates the effect of the natural transformation $x$ in $\text{Rep}_C(\mathfrak{gl}(V))$:

**Example 6.2.4.** Assume that $C$ is Karoubian (all idempotents split). Consider the decomposition
\[
F(V^\otimes n - 1) = V^\otimes n \cong \bigotimes_{\mu} S^\mu V \otimes \mu
\]
The action of $x_{V^\otimes n - 1}$ is given by a collection of endomorphisms $\phi_{\mu} \in \text{End}(\mu)$, such that $\phi_{\mu}$ commutes with the action of $G := S_{(2,3,...,n-1)} \subset S_n$. Thus $\phi_{\mu}$ is diagonalizable with eigenspaces given by irreducible $G$-summands of $\mu$. Identifying $G \cong S_{n-1}$, we get
\[
\text{Res}^n_G(\mu) = \bigoplus_{\lambda \in \mu} \lambda
\]
where the sum is over Young diagrams obtained from $\mu$ by erasing one box. The eigenvalue corresponding to eigenspace $\lambda$ is then $c(\mu - \lambda)$.

Now,
\[
F(S^\lambda V) = V \otimes S^\lambda V \cong \bigoplus_{\mu \subseteq \lambda + \square} S^\mu V
\]
In fact, we have:
\[
F(S^\lambda V \otimes \lambda) \cong \bigoplus_{\mu \subseteq \lambda + \square} S^\mu V \otimes \lambda \subset \bigoplus_{\mu \subseteq \lambda + \square} S^\mu V \otimes \mu
\]
Thus $x_{S^\lambda V}$ acts on each direct summand $S^\mu \subset F(S^\lambda V)$ ($\mu \in \lambda + \square$) by the scalar $c(\mu - \lambda)$.  

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From now on we let $\mathcal{C}$ be a tensor category (in particular, abelian), and fix $V \in \mathcal{C}$. The category $\text{Rep}_c(\mathfrak{gl}(V))$ is again a tensor category.

We will now consider the full tensor subcategory $\text{Rep}_c(\mathfrak{GL}(V), \varepsilon)$ of $\text{Rep}_c(\mathfrak{gl}(V))$ (see [D2, EHS Section 7] for definition). This subcategory is defined as the category of representations in $\mathcal{C}$ of a certain Hopf algebra object, which is the analogue of $\mathcal{O}(\mathfrak{GL}(W))$ for a vector space $W$. In particular, any subquotient of a direct sum of mixed tensor powers of $V$ has a natural structure making it an object of $\text{Rep}_c(\mathfrak{gl}(V))$, and it can be shown that the objects of $\text{Rep}_c(\mathfrak{GL}(V), \varepsilon)$ are all of this form (cf. [D2, Appendix]).

The subcategory $\text{Rep}_c(\mathfrak{GL}(V), \varepsilon)$ is preserved by the type A action defined above.

**Example 6.2.5.**

a) Let $(\mathcal{C}, V) := (S\text{Vec}, \mathbb{C}^{m\mid n})$. In this case $\text{Rep}_c(\mathfrak{GL}(V), \varepsilon) = \text{Rep}(\mathfrak{gl}(m\mid n))$ and the induced type A action on $\text{Rep}(\mathfrak{gl}(m\mid n))$ coincides with $\mathfrak{sl}_2$ action as in [BLW] (see Section 6.3).

b) Let $(\mathcal{C}, V) := (V_t, V_i) (t \in \mathbb{C})$. In this case $\text{Rep}_c(\mathfrak{GL}(V), \varepsilon) \cong V_t$ (see [EHS Section 7]), which defines an type A action on $V_t$, studied below.

We will be interested in the categories $\text{Rep}_c(\mathfrak{GL}(V), \varepsilon)$ since the above examples are universal among the pairs $(\text{Rep}_c(\mathfrak{GL}(V), \varepsilon), V)$ (see [EHS]).

The following lemma is straightforward (cf. [D1 Sect. 7], [EHS Section 7.2]).

**Lemma 6.2.6.** Let $(\mathcal{C}, V), (\mathcal{C}', V')$ be a pair of tensor categories with a fixed object. Consider their respective type A actions, and let $G : \mathcal{C} \rightarrow \mathcal{C}'$ be an SM functor such that $V \rightarrow V'$. Then $G$ induces an equivariant SM functor $G : \text{Rep}_c(\mathfrak{gl}(V)) \rightarrow \text{Rep}_c(\mathfrak{gl}(V'))$, and an equivariant SM functor $G : \text{Rep}_c(\mathfrak{GL}(V), \varepsilon) \rightarrow \text{Rep}_c(\mathfrak{GL}(V'), \varepsilon)$.

**Corollary 6.2.7.** Let $(\mathcal{C}, V)$ be a tensor category with a fixed object, and set $t := \dim V$. The additive SM functors

$$D_{\mathfrak{sl}_2} S_{\mathfrak{sl}_2} \rightarrow \text{Rep}_c(\mathfrak{GL}(V), \varepsilon)$$

are equivariant with respect to the type A action.

As it was mentioned before, if the eigenvalues of the operator $x$ on $\text{Rep}_c(\mathfrak{GL}(V), \varepsilon)$ are integers, then the functors $F_n, E_n$ define an $\mathfrak{sl}_2$-action on the Grothendieck group of the abelian category $\text{Rep}_c(\mathfrak{GL}(V), \varepsilon)$, in the sense of [EHS, BLW].

In the Example 6.2.5 (a), this is indeed the case. We will show that

**Proposition 6.2.8.** The eigenvalues of $x$ in $V_t$ are integers iff $t \in \mathbb{Z}$.

*Proof. * It will be shown in Sections 7 and 8 that the set of eigenvalues of $x$ is $\mathbb{Z} \cup \mathbb{Z} - t$. $\square$

Thus the type A action on $V_t$ defines an $\mathfrak{sl}_2$-action whenever $t \in \mathbb{Z}$.

We now formulate a corollary, which is a direct consequence of [EHS, Theorem 7.1.2]. This corollary essentially states that any categorical type A action on a category of the form $\text{Rep}_c(\mathfrak{GL}(V), \varepsilon)$ which originates in an type A action can be described using the categorical type A actions on the tensor categories $V_t$, $\text{Rep}(\mathfrak{gl}(m\mid n))$.

Recall that tensor categories which are finite-length abelian categories and have finite-dimensional Hom-spaces are called pre-Tannakian (see [D1, Section 8.1] for definition).

**Corollary 6.2.9.** Let $\mathcal{C}$ be a pre-Tannakian tensor category, and let $V \in \mathcal{C}$ whose dimension $t := \dim(V)$ is an integer. If $V$ is "torsion-free", i.e. $S^\lambda V \neq 0$ for any $\lambda$, then there exists an equivariant equivalence of $\mathfrak{sl}_2$-categorical modules

$$\text{Rep}_c(\mathfrak{GL}(V), \varepsilon) \cong V_t$$

If $V$ is "torsion", i.e. there exists $\lambda$ such that $S^\lambda V = 0$, then there exists an equivariant equivalence of $\mathfrak{sl}_2$-categorical modules

$$\text{Rep}_c(\mathfrak{GL}(V), \varepsilon) \cong \text{Rep}(\mathfrak{gl}(m\mid n))$$

This is “half” of the category of all finite-dimensional representations of $\mathfrak{gl}(m\mid n)$, as explained in Section 6.4.
for some \( m,n \in \mathbb{Z}_+ \), \( m - n = t \).

Remark 6.2.10. A similar result holds when \( t := \dim(V) \) is not an integer, but as it is seen in Theorem 8.0.2 this is an equivariant equivalence of \( \mathfrak{sl}_2 \times \mathfrak{sl}_2 \)-categorical modules.

6.3. Categorical action on \( \text{Rep}(\mathfrak{gl}(m|n)) \). We now describe two special cases of the above construction: a categorical \( \mathfrak{sl}_2 \)-action on \( \text{Rep}(\mathfrak{gl}(m|n)) \) and specifically on \( \text{Rep}(\mathfrak{gl}_m) \) (for explicit descriptions of these actions, see e.g. [CR, BLW]).

Example 6.3.1. Case of \( \text{Rep}(\mathfrak{gl}_m) \): Recall that the singular vectors in \( L(\lambda) \otimes \mathbb{C}^m \) have weights \( \tilde{\lambda} + \varepsilon_i \) for different \( i \); in terms of bipartitions, it means that

\[
F(L(\lambda)) \cong \bigoplus_{\mu \in \lambda + \mathfrak{B}} L(\mu)
\]

Moreover, recalling the definition of \( x \) through the Casimir operator, one can immediately see that the eigenvalue of \( x_{L(\lambda)} \) on \( L(\mu) \) is \( \tilde{\lambda}_i - i + 1 \), where \( i \) is such that \( \tilde{\mu} = \tilde{\lambda} + \varepsilon_i \). Again, in terms of bipartitions, this value is equal to \( ct(\mu^* - \lambda^*) \) if \( \mu \in \lambda + \mathfrak{B} \), and is equal to \( -ct(\lambda^0 - \mu^0) - m \) if \( \mu \in \lambda - \square \).

Thus

\[
F_a(L(\lambda)) = \bigoplus_{\mu \in \lambda + \mathfrak{B}} L(\mu) \oplus \bigoplus_{\mu \in \lambda - \square - m - a} L(\mu)
\]

(notation as in Section 2.1), and there is an isomorphism of \( \mathfrak{sl}_2 \)-modules

\[
\mathbb{C} \otimes_\mathbb{Z} \text{Gr}(\text{Rep}(\mathfrak{gl}_m)) \cong \wedge^m \mathbb{C}^\mathbb{Z}
\]

A similar situation appears in the general case of \( \text{Rep}(\mathfrak{gl}(m|n)) \) (cf. proof of Proposition 9.1.1 it was shown in [BLW] that in this case there is an isomorphism of \( \mathfrak{sl}_2 \)-modules

\[
\mathbb{C} \otimes_\mathbb{Z} \text{Gr}(\text{Rep}(\mathfrak{gl}(m|n))) \cong \wedge^m \mathbb{C}^\mathbb{Z} \otimes \wedge^n (\mathbb{C}^\mathbb{Z})^*
\]

sending Kac modules to pure wedges.

7. Operator x

7.1. Eigenvalues of the operator \( x \). We will now describe the generalized eigenspaces of the natural transformation \( x \) when \( t \notin \mathbb{Z} \).

Proposition 7.1.1. Let \( \lambda \) be a bipartition, and let \( t \notin \mathbb{Z} \). The generalized eigenspace of \( x_{T(\lambda)} \) corresponding to \( a \in \mathbb{C} \) is

\[
\bigoplus_{\mu \in \lambda + \mathfrak{B}} T(\mu) \oplus \bigoplus_{\mu \in \lambda - \square - (a + t)} T(\mu)
\]

(notation as in Section 2.7).

Remark 7.1.2. Note that since \( t \notin \mathbb{Z} \) and thus \( \text{Rep}(GL_t) \) is semisimple, the operator \( x_{T(\lambda)} \) is diagonalizable.

Proof. Fix \( \lambda \vdash (r,s) \), and let \( t \in \mathbb{C} \setminus \{0, \pm 1, \pm 2, \ldots, \pm (r + s + 1)\} \).

The statement of the proposition is equivalent to computing the generalized eigenvalues of \( x_{T(\lambda)} \). Each summand \( T(\mu) \) of \( F(T(\lambda)) \) is indecomposable, and hence corresponds to a generalized eigenvalue of \( x_{T(\lambda)} \).

We would like to say that the eigenvalue of \( x_{T(\lambda)} \) on \( T(\mu) \) depends polynomially on \( t \); we will compute separately the eigenvalue in the special case of \( t \in \mathbb{Z}_{>0} \), and the polynomiality would allow us to interpolate this result to all values of \( t \).

We will denote \( x_{\mu,t}^\lambda = p \circ x_{T(\lambda)} \circ i \) where \( i, p \) are the inclusion and the projection to of the direct summand \( T(\mu) \subset T(\lambda) \otimes V_t \) in \( D_t \).

Case \( t = d \in \mathbb{Z}_{>r+s} \):
In this case \( S_{t=d}(\mathbf{T}(\mu)) = L(\mu) \) (see Notation 2.4). Then
\[
S_{t=d}(F(\mathbf{T}(\lambda))) = F(L(\lambda)) \cong \bigoplus_{\mu \in \lambda + \Box - \Box} L(\mu)
\]

By Section 6.3, the generalized eigenvalue of \( x_{\mu}(\lambda) \) on \( L(\mu) \) is \( ct(\mu^* - \lambda^*) \) if \( \mu \in \lambda + \Box \), and is equal to \(-ct(\lambda^0 - \mu^0) - d \) if \( \mu \in \lambda - \Box \). By Corollary 6.2.7, the generalized eigenvalue of \( x_{\lambda}(\lambda) \) on \( \mathbf{T}(\mu) \) is the same.

We will now use the map \( \text{Lift}_t : K_0(D_t) \to K_0(D_T) \) defined in Section 3.1.4. As it was mentioned before, for any \( t \notin \{0, \pm 1, \pm 2, \ldots, \pm (|\mu^*| + |\mu^0|)\} \), we have: \( \text{Lift}_t(\mathbf{T}(\mu)) = \mathbf{T}(\mu) \).

Consider the idempotent \( \epsilon_{\mu}^\lambda \in Br_C(r + 1, s) \) which is the projector onto the multiplicity space of \( T^\mu \) in \( V \otimes \mathbf{T}(\lambda) \subset V^s_t \oplus V^s_t \) (by abuse of notation, we will denote by \( \epsilon_{\mu}^\lambda \) the corresponding idempotent in \( Br_C(T) \)) \((r + 1, s)\) as well.

In \( D_T \), we have (see also [CW, Section 7]):
\[
V_T \otimes \mathbf{T}(\lambda) \cong \bigoplus_{\mu \in \lambda + \Box} \mathbf{T}(\mu) \oplus \bigoplus_{\mu \in \lambda - \Box} \mathbf{T}(\mu)
\]
Together these two facts imply that for \( t \in \mathbb{C} \setminus \{0, \pm 1, \pm 2, \ldots, \pm (r + s + 1)\} \), a similar decomposition holds in \( D_t \):
\[
F(\mathbf{T}(\lambda)) = V_t \otimes \mathbf{T}(\lambda) \cong \bigoplus_{\mu \in \lambda + \Box} \mathbf{T}(\mu) \oplus \bigoplus_{\mu \in \lambda - \Box} \mathbf{T}(\mu)
\]
Thus the idempotents \( \epsilon_{\mu}^\lambda \) are primitive in \( D_T \) and for the above values of \( t \), and the lifting of \( \epsilon_{\mu}^\lambda \) is exactly \( \epsilon_{\mu}^\lambda \in Br_C(T) \) \((r + 1, s)\) whenever \( t \in \mathbb{C} \setminus \{0, \pm 1, \pm 2, \ldots, \pm (r + s + 1)\} \).

**Computation of \( x_{\mu,t}^\lambda \) for all \( t \in \mathbb{C} \setminus \{0, \pm 1, \pm 2, \ldots, \pm (r + s + 1)\} \):** We will now describe how to lift the endomorphism \( x_{\mu,t}^\lambda \) to an endomorphism in \( D_T \).

We have
\[
x_{\mu,t}^\lambda = e_{\mu}^\lambda x_{V^s_t \otimes V^s_t \otimes s} \epsilon_{\mu}^\lambda
\]
as an endomorphism in \( \text{End}_{D_t}(\mathbf{T}(\mu)) \subset \text{End}_{D_t}(V^s_t \otimes V^s_t \otimes s) \). First, consider the morphism \( x_{V^s_t \otimes V^s_t \otimes s} \) in \( D_T \), where \( T \) is a formal variable. We have
\[
x_{V^s_t \otimes V^s_t \otimes s}|_{T=t} = x_{V^s_t \otimes V^s_t \otimes s}
\]
(this is obvious, since the parameter \( t \) is not involved in the definition of \( x_{V^s_t \otimes V^s_t \otimes s} \)).

Secondly, we define an endomorphism
\[
x_{\mu,T}^\lambda := e_{\mu}^\lambda x_{V^s_t \otimes V^s_t \otimes s} \epsilon_{\mu}^\lambda \in \text{End}_{D_T}(\mathbf{T}(\mu)) \subset \text{End}_{D_T}(V^s_t \otimes V^s_t \otimes s)
\]
This endomorphism does not depend on \( t \). Since \( \mathbf{T}(\mu) \) is simple, this is in fact a scalar multiple of \( \epsilon_{\mu}^\lambda \); we will denote the corresponding scalar (a formal Laurent series in \( T \)) by \( \chi_{\mu}^\lambda \). The previous paragraph implies that the endomorphism \( x_{\mu,t}^\lambda \) is the lift of \( x_{\mu,T}^\lambda \) for \( t \notin \{0, \pm 1, \pm 2, \ldots, \pm (r + s + 1)\} \).

This immediately implies that \( x_{\mu,t}^\lambda = \chi_{\mu}^\lambda \) for any \( t \) as above. Applying this to the case \( t = d \in \mathbb{Z}_{d>0} \), we conclude that
\[
\chi_{\mu}^\lambda = \begin{cases} 
ct(\mu^* - \lambda^*) & \text{if } \mu \in \lambda + \Box \\
nct(\lambda^0 - \mu^0) - d & \text{if } \mu \in \lambda - \Box \\
0 & \text{else}
\end{cases}
\]
\[\Box\]

\[^{\text{We stress that these idempotents depend on } \lambda.}\]
8. Categorical sl₂-actions on $\mathcal{D}_t$

We will now define a categorical sl₂-action on the category $\mathcal{D}_t$.

**Definition 8.0.1.** Let $a \in \mathbb{C}$. We define the endofunctor $F_a$ of $\mathcal{D}_t$ so that $F_a(M)$ is the generalized $a$-eigenspace of $x_M$.

We have:

$$F = \bigoplus_{a \in \mathbb{C}} F_a$$

The endofunctor $E_a$ can be defined similarly as a direct summand of $E$; it is the left adjoint to $F_a$ and isomorphic to the right adjoint of $F_a$.

From now and until the end of this section, assume that $t \notin \mathbb{Z}$. The category $\mathcal{D}_t$ is then semisimple, and on simple objects, the actions of $E_a, F_a$ are given by Proposition 7.1.1.

$$F_a(T(\lambda)) = \bigoplus_{\mu \in \lambda + \mathbb{N}_a} T(\mu) + \bigoplus_{\mu \in \lambda - \mathbb{N}_{\lambda - (a + t)}} T(\mu)$$

$$E_a(T(\lambda)) = \bigoplus_{\mu \in \lambda + \mathbb{N}_a} T(\mu) + \bigoplus_{\mu \in \lambda - \mathbb{N}_{\lambda - (a + t)}} T(\mu)$$

One immediately observes that $F_a, E_a = 0$ when $a \notin \mathbb{Z} \cup \mathbb{Z} - t$.

In this case, the union $\mathbb{Z} \cup \mathbb{Z} - t$ is disjoint, which implies that there are two ”separate” copies of sl₂ acting on $\mathcal{D}_t$. Indeed, when $t \notin \mathbb{Z}$, one can define two commuting natural transformations $x', x'' \in \text{End}(F)$ such that $x = x' + x''$, and $x'$ has eigenvalues in $\mathbb{Z}$, while $x''$ has eigenvalues in $\mathbb{Z} - t$.

The construction of Rouquier (see [R], [CR, 7.4]) then gives an action of $\mathfrak{sl}_2$ on the Grothendieck group $\mathbb{C} \otimes \mathbb{Z} \text{Gr}(\mathcal{D}_t)$.

**Theorem 8.0.2.** Let $t \notin \mathbb{Z}$ (so $\mathcal{D}_t$ is semisimple). The functors $F'_a, F''_a, E'_a, E''_a$ for $a \in \mathbb{Z}$ define an action of the Lie algebra $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ on the complexified Grothendieck group $\mathbb{C} \otimes \mathbb{Z} \text{Gr}(\mathcal{D}_t)$.

As a $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$-module, $\mathbb{C} \otimes \mathbb{Z} \text{Gr}(\mathcal{D}_t)$ is isomorphic to the tensor product $\mathfrak{F} \otimes \mathfrak{F}^\vee$, with $[T(\lambda)]$ corresponding to $v_{\lambda^*} \otimes v_{\lambda'^*}$.

9. Categorical sl₂-actions on $\mathcal{V}_t$ for integer $t$

Let $t \in \mathbb{Z}$. Consider the sl₂-action on the category $\mathcal{V}_t$, induced by the sl₂-action described in Section 8.

Recall that the category $\mathcal{V}_t$, defined in Section 3.1.3 is essentially ”glued” from pieces of categories $\text{Rep}(\mathfrak{gl}(m|n))$ ($m - n = t$) using the functors $F_{m|n} : \mathcal{V}_t \to \text{Rep}(\mathfrak{gl}(m|n))$, $V_t \mapsto \mathbb{C}^{|m|n}$; these provide “local” equivalences, which allows us to reduce the study of the sl₂-action on $\mathcal{V}_t$ to “local” studies in $\text{Rep}(\mathfrak{gl}(m|n))$ for $m, n >> 0$. The functors $F_{m|n}$ are equivariant (a direct consequence of 6.2.6).

The term “local equivalences” means the following: the categories $\mathcal{V}_t$ and $\text{Rep}(\mathfrak{gl}(m|n))$ have $\mathbb{Z}_+$-filtrations, so that

$$\mathcal{V}_t^k \cong \text{Rep}^k(\mathfrak{gl}(m|n)) \cong \text{Rep}^k(\mathfrak{gl}(m - 1|n - 1))$$

for $m, n >> k$. The subcategories $\mathcal{V}_t^k$ (resp. $\text{Rep}^k(\mathfrak{gl}(m|n))$) are defined as full subcategories of $\mathcal{V}_t$ (resp. $\text{Rep}(\mathfrak{gl}(m|n))$) containing all the subquotients of finite direct sums of mixed tensor powers $V_{t}^{|m|n} \otimes V_{s}^{|m|n}$, $r + s \leq k$ (resp. $(\mathbb{C}^{|m|n})^{|m|n} \otimes (\mathbb{C}^{|m|n})^{|m|n}$).

**Remark 9.0.1.** These subcategories are not preserved by the functors $F, E$.  

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Using this fact we will now compute the action of functors $F_a \in \text{End}(V_t)$ on the standard objects $V(\lambda)$.

9.1. **Action on standard objects.** We now compute the action of $F_a$ on a standard object $V(\lambda) \in V_t$. The following proposition is valid for any $t \in \mathbb{C}$.

**Proposition 9.1.1.** The object $F_a(V(\lambda))$ in $V_t$ is standardly-filtered:
\[
0 \rightarrow V(\lambda + \square_a) \rightarrow F_a(V(\lambda)) \rightarrow V(\lambda - \square_{-(a+t)}) \rightarrow 0.
\]

**Proof.** If $t \notin \mathbb{Z}$, then $V(\lambda) = \mathbf{T}(\lambda)$, and the statement is true (see Section 3), so we will assume that $t \in \mathbb{Z}$.

First, recall from [EHS] Corollary 3.3.3 that
\[
(6) \quad 0 \rightarrow \bigoplus_{\mu \in \lambda - \square} V(\mu) \rightarrow F(V(\lambda)) \rightarrow \bigoplus_{\mu \in \lambda - \square} V(\mu) \rightarrow 0.
\]

Therefore $F_a(V(\lambda))$ is filtered by standard objects for every $a \in \mathbb{C}$.

To find out which $V(\mu)$ appear in $F_a(V(\lambda))$, let $m > > 0$ and let $n := m - t$. Denote by $W := \mathbb{C}^m[1/n]$ the tautological representation of $\mathfrak{gl}(m|n)$. Its even and odd parts will be denoted $W_0, W_1$.

Recall from Lemma 3.1.3 that $V(\lambda) := F_{m|n}(V(\lambda))$ is a highest-weight $\mathfrak{gl}(m|n)$-module, the image of the homomorphism from the Kac module $K(\lambda)$ to $S^\lambda W \otimes S^\lambda W^*$.

Now, the action of the endofunctor $F_a \in \text{End}(\text{Rep}(\mathfrak{gl}(m|n)))$ on the Kac module $K(\lambda)$ can be described very explicitly, as was done in [BLW].

Consider the decomposition of the Lie superalgebra
\[
\mathfrak{gl}(m|n) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1
\]
where $\mathfrak{g}_0$ is even part of the superspace $\mathfrak{gl}(m|n)$, and $\mathfrak{g}_1 \cong V_0 \otimes (V_1)^*$ (so $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ is the odd part of $\mathfrak{gl}(m|n)$). Then
\[
F(K(\lambda)) = U_{\mathfrak{g}(m|n)} (L_{\mathfrak{g}_0}(\lambda^*) \otimes L_{\mathfrak{g}_1}(\lambda^0)) \otimes W = U_{\mathfrak{g}(m|n)} (L_{\mathfrak{g}_0}(\lambda^*) \otimes L_{\mathfrak{g}_1}(\lambda^0) \otimes W)
\]

Since $W$ has a $\mathfrak{g}_0 \oplus \mathfrak{g}_1$-filtration with subquotients $W_0, W_1$, the $\mathfrak{gl}(m|n)$-module $F(K(\lambda))$ has a filtration by Kac modules with highest weights lying in the set $\{\lambda + \delta_j| j = 1, \ldots, m+n\}$.

In other words, $F(K(\lambda))$ has a filtration by Kac modules $K(\mu)$, where $\mu \in \lambda + \square \lambda - \square$.

Thus $F_a(K(\lambda))$ has a filtration by Kac modules. It was computed in [BLW] that the Kac modules $K(\mu)$ appearing in $F_a(K(\lambda))$ have $\bar{\mu} = \bar{\lambda} + \delta_j$ where $\bar{\lambda}_j - j + 1 = a$ if $1 \leq j \leq m$, and $-\lambda_j + j - 2m = a + 1$. Translating these into the language of bipartitions, we obtain: $\mu \in \lambda + \square_a \sqcup \lambda - \square_{-(a+t)}$, and hence $F_a(V(\lambda))$ has a filtration by $V(\mu)$ for the same bipartitions $\mu$. The filtration (6) now implies the required result.

**Remark 9.1.2.** One can similarly show that the object $E_a(V(\lambda)) \subset V_t^* \otimes V(\lambda)$ is standardly-filtered:
\[
0 \rightarrow V(\lambda + \square_{-(a+t)}) \rightarrow E_a(V(\lambda)) \rightarrow V(\lambda - \square_a) \rightarrow 0
\]

9.2. **Grothendieck group of $V_t$ in case of integer $t$.** Once again, let $t \in \mathbb{Z}$, and recall the definition of the shifted representation $\mathfrak{F}_t^\lambda$ of $\mathfrak{sl}_2$ (see Section 3.3.1).

The tensor product $\mathfrak{F} \otimes \mathfrak{F}_t^\lambda$ is again an $\mathfrak{sl}_2$-module, and Proposition 9.1.1 implies:

**Theorem 9.2.1.** Let $t \in \mathbb{Z}$. The functors $F_a, E_a$ for $a \in \mathbb{Z}$ define an action of the Lie algebra $\mathfrak{sl}_2$ on the complexified Grothendieck group $\mathbb{C} \otimes_{\mathbb{Z}} \text{Gr}(V_t)$.

We then have an isomorphism of $\mathfrak{sl}_2$-modules
\[
\mathbb{C} \otimes_{\mathbb{Z}} \text{Gr}(V_t) \cong \mathfrak{F} \otimes \mathfrak{F}_t^\lambda, \quad [V(\lambda)] \mapsto v_{\lambda^*} \otimes v_{\lambda^0}
\]
Remark 9.2.2. The Grothendieck group of the full subcategory \( \mathcal{V}_t^k \subset \mathcal{V}_t \) corresponds to the subspace  
\[
\lim_{r, s: r+s \leq k} \mathfrak{F}(r) \otimes \mathfrak{F}^\vee(s) \subset \mathfrak{F} \otimes \mathfrak{F}^\vee
\]
(see Section 3.3.2 for definition of subspace \( \mathfrak{F}(r) \subset \mathfrak{F} \); the subspace \( \mathfrak{F}(s) \subset \mathfrak{F}^\vee \) is defined analogously).

The functor \( F_m[n]: \mathcal{V}_t \to \text{Rep}(\mathfrak{gl}(m|n)) \) is not exact, so it does not correspond to a map between \( \mathfrak{F} \otimes \mathfrak{F}^\vee \) and \( \wedge^m \mathbb{C}^Z \otimes \wedge^n \mathbb{C}^Z \). Yet the equivalence \( F_m[n]: \mathcal{V}_t^k \to \text{Rep}^k(\mathfrak{gl}(m|n)) \) for \( m, n \gg 0 \) can be seen as a categorical version of the statement  
\[
\lim_{r, s: r+s \leq k} \mathfrak{F}(r) \otimes \mathfrak{F}^\vee(s) \cong \lim_{r, s: r+s \leq k} \wedge^m \mathbb{C}^Z \otimes \wedge^n \mathbb{C}^Z.
\]
for \( m, n \gg k \).

9.3. Action on tilting objects. In this section we give some results on the action of the functors \( F_a \) on objects \( \mathbf{T}(\lambda) \) in \( \mathcal{V}_t \) for \( t \in \mathbb{Z} \).

Notation 9.3.1. For any \( t \in \mathbb{C} \), denote by \( \tilde{A}_a(t) \) the matrix whose \((\lambda, \mu)\) entry is 1 iff \( \mu \in \lambda + \mathbb{C}_a \cup \lambda - \square_{-(a+t)} \), and zero otherwise.

Let \( a, t \in \mathbb{C} \). Recall that \( F_a \) are exact endofunctors of the category \( \mathcal{V}_t \) which preserve the subcategory \( \mathcal{D}_t \), so each \( F_a(\mathbf{T}(\lambda)) \) decomposes as a direct sum of \( \mathbf{T}(\mu) \).

In general, the formula (1) does not hold. The best approximation is given by the next proposition. Let \( \lambda, \mu \) be two bipartitions. Denote by \( A_{a,\mu}(t) \) the multiplicity of \( \mathbf{T}(\mu) \) in \( F_a(\mathbf{T}(\lambda)) \).

We will denote by \( A_{a}(t) \) the matrix whose entries are \( A_{a,\mu}(t) \) (the entries are indexed by pairs of bipartitions).

Recall that in Section 5 we showed that for \( t \notin \mathbb{Z} \), \( A_{a,\mu}(t) \) is either 1 or 0, and it equals 1 iff \( \mu \in \lambda + \mathbb{C}_a \cup \lambda - \square_{-(a+t)} \).

Proposition 9.3.2. Let \( t \in \mathbb{Z} \). Then  
\[
A_a(t) = D(t) \tilde{A}_a(t) D(t)^{-1}
\]
where \( D(t) \) is the matrix whose entries are \( D_{\mu,\lambda}(t) \) (cf. Section 5).

This agrees with the fact that \( A_a(t) = \tilde{A}_a(t) \) when \( t \notin \mathbb{Z} \), since in this case \( D(t) = \text{Id} \).

Proof. Recall from Theorem 5.0.1 that in the Grothendieck ring of \( \mathcal{V}_t \), we have  
\[
[\mathbf{T}(\lambda)] = \sum_{\mu} D_{\mu,\lambda}(t) [\mathbf{V}(\mu)]
\]
Multiplying both sides by \( [\mathbf{V}_t] \), we have:
\[
[\mathbf{T}(\lambda) \otimes \mathbf{V}_t] = \sum_{\mu} D_{\mu,\lambda}(t) [\mathbf{V}(\mu) \otimes \mathbf{V}_t] = \sum_{\mu,\mu'} D_{\mu,\lambda}(t) \tilde{A}_{\mu',a}(t) [\mathbf{V}(\mu')] = \sum_{\mu,\mu'} D_{\mu,\lambda}(t) \tilde{A}_{\mu',a}(t) [\mathbf{T}(\mu')]
\]
Yet by definition
\[
[\mathbf{T}(\lambda) \otimes \mathbf{V}_t] = \sum_{\mu} A_{\mu,a}(t) [\mathbf{T}(\mu)]
\]
Hence \( A_a(t) = D(t) \tilde{A}_a(t) D(t)^{-1} \).

Although computing explicitly the matrix \( A_a(t) \) is difficult, one can show that part of it is identical to the entries in \( \tilde{A}_a(t) \):

Corollary 9.3.3. For any \( \lambda \), we have:
\[
F_a(\mathbf{T}(\lambda)) = \bigoplus_{\mu \in \lambda + \mathbb{C}_a} \mathbf{T}(\mu) \oplus \bigoplus_{\mu} \mathbf{T}(\mu) \oplus A_{a,\mu}(t).
\]
Proof. Denote by $|\lambda| := |\lambda^*| + |\lambda^0|$ the total size of a bipartition.

Define an order by size on the bipartitions (so that $\lambda \geq \mu$ if $|\lambda| \geq |\mu|$), and write the matrix $D(t)$ in this ordered basis. Then $D(t)$ is an lower-triangular matrix (since $D^\mu_\nu(t) \neq 0$ implies $\lambda \geq \mu$) with 1 on the diagonal. The matrix $D^{-1}(t)$ is then also lower-triangular with 1 on the diagonal.

Meanwhile, the matrix $A_\lambda(t)$ has 1 only in positions $(\lambda, \mu)$ where $|\lambda| = |\mu| \pm 1$. We claim that in the product $A_n(t) = D(t) \tilde{A}_n(t) D(t)^{-1}$ all the entries above the diagonal are equal to the corresponding entry in $\tilde{A}_n(t)$.

Indeed, if $|\lambda| < |\mu|$, then

$$A^\lambda_{\alpha,\mu}(t) = \sum_{\nu,\nu'} D^\lambda_{\nu}(t) \tilde{A}^\nu_{\alpha,\nu'}(t) (D(t)^{-1})_{\nu'}^\mu$$

The above arguments imply that the only summands which are not zero correspond to bipartitions $\nu, \nu'$ such that $|\nu'| = |\nu| \pm 1$, $|\nu'| \geq |\mu|$ and $|\nu| \leq |\lambda|$; these conditions imply that $|\nu| = |\lambda| = |\mu| - 1 = |\nu'| - 1$. Since we require that $D^\lambda_{\nu}(t), (D(t)^{-1})_{\nu'}^\mu \neq 0$, this implies that $\nu = \lambda, \nu' = \mu$, and we are done.

\[\boxdot\]

Remark 9.3.4. Moreover, the entries of $A_n(t) = D(t) \tilde{A}_n(t) D(t)^{-1}$ below the diagonal are non-zero only if $\mu \subset \lambda$ and $(|\lambda^*|, |\lambda^0|) = (|\mu^*|, |\mu^0|) - (i, i + 1)$ for some $i \geq 0$.

10. Tensor product categorification

In this section, we show that $\mathcal{V}_{\lambda}$ is the tensor product of the categorical $\mathfrak{gl}(\infty)$-modules $\text{Pol} \otimes \text{Pol}^t$, where $\text{Pol}$ stands for the category of polynomial representations of $\mathfrak{gl}(\infty)$, and $\text{Pol}^t$ is the same category but with a modified $\mathfrak{sl}_2$ categorical action.

10.1. Category of polynomial representations. The category $\text{Pol}$ has several equivalent descriptions, see [SS, Section 5], [En].

Definition 10.1.1 (Category of polynomial representations). The category $\text{Pol}$ is the full monoidal Karoubian additive subcategory of the category of $\mathfrak{gl}(\infty)$-modules generated by the standard representation $V_\infty := \mathbb{C}^\infty$ of $\mathfrak{gl}(\infty)$.

Equivalently, this is the free Karoubian additive $\mathbb{C}$-linear category on one generator, $V_\infty$. This category is equivalent to the category of Schur functors, see [SS].

This is a semisimple symmetric monoidal category, with simple objects $S^\nu$, $\nu \in \mathcal{P}$ parametrized by the set of all Young diagrams.

Clearly, this can be considered as a lower highest-weight category in the sense of Definition 10.2.1 with subcategories $\text{Pol}^{(k)}$ generated by $S^\nu$ where $|\nu| \leq k$; each of these is a (semisimple) highest-weight category with a finite poset of weights.

On this category, we have a natural type A action (see [HY, 1]), given by the functors $F(M) := V_\infty \otimes M$ and its adjoint $E := i^L(\text{Schur}_{(\infty)} \otimes M)$ (here $\text{Schur}_{(\infty)}$ is the restricted dual of $V_\infty$, $\nu : \text{Pol} \to \text{Mod}_{\mathfrak{gl}(\infty)}$ is the inclusion functor, and $i^L$ its left adjoint). The natural transformation $\tau$ is just the symmetry morphism, as in Section 9 and $x$ is a “limit” version of the natural transformation described in Section 9.

Namely, one can consider $\text{Pol}$ as a certain limit of categories of polynomial representations of algebraic groups $\text{GL}_n$ as $n \to \infty$ (see [HY, 1]). Under this identification, any $M \in \text{Pol}$ corresponds to a sequence $(M_n)_{n \geq 1}$, where $M_n$ is a polynomial $\text{GL}_n$-representation. Then

$$V_\infty \otimes M = (\mathbb{C}^n \otimes M_n)_{n \geq 1}$$

and $x_M : V_\infty \otimes M \longrightarrow V_\infty \otimes M$ is defined as $(x_n := \sum_{1 \leq i, j \leq n} E_{i,j} \otimes E_{i,j})_n$.

This is a categorical version of the isomorphism [1].
On the simple objects, the functors $F_a$, $E_a$ ($a \in \mathbb{Z}$) act by
\[ F_a(S^\nu) = S^{\nu+\square_a}, \quad E_a(S^\nu) = S^{\nu-\square_a} \]
(as usual, if one of the Young diagrams is not defined, the corresponding module is considered to be zero).

This action categorifies the Fock space representation of $\mathfrak{sl}_2$ described in Section 3.3.

\[ Gr(\text{Pol}) \rightarrow \mathfrak{z}, \quad [S^\nu] \rightarrow v_\nu \]

One can easily see that this is the unique categorification of the $\mathfrak{sl}_2$-module $\mathfrak{z}$; indeed, the weight spaces in $\mathfrak{z}$ being one-dimensional, its categorification has to be semisimple, with simple objects parametrized by all Young diagrams. It follows that this categorification is equivariantly equivalent to $\text{Pol}$.

We also consider a twist of this action: namely, let $t \in \mathbb{Z}$, and consider the endofunctors $F' = V_{\infty,T} \otimes (\cdot)$, $E' = V_{\infty,T} \otimes (\cdot)$ on the category $\text{Pol}$, together with the usual symmetry morphism $\tau' = \tau$, and $x' \in \text{End}(E')$ defined by $x' = -x - t$ (this induces the corresponding endomorphism of $F'$). Then $F' = \bigoplus_{a \in \mathbb{Z}} F'_a$, $E' = \bigoplus_{a \in \mathbb{Z}} E'_a$, where
\[ F'_a(S^\nu) = S^{\nu-\square_{-(n+t)}}, \quad E'_a(S^\nu) = S^{\nu+\square_{-(n+t)}} \]

This action categorifies the “shifted dual” Fock space representation $\mathfrak{z}_T^\vee$ of $\mathfrak{sl}_2$ described in Section 3.3.

By abuse of notation, we denote the category of polynomial representations with the usual $\mathfrak{sl}_2$-action by $\text{Pol}$, and the category of polynomial representations with the twisted $\mathfrak{sl}_2$-action by $\text{Pol}_T^\vee$.

10.2. Tensor product categorification. Let $t \in \mathbb{Z}$. We now consider the notion of tensor product categorification $\text{Pol} \otimes \text{Pol}_T^\vee$ in a sense similar to [LV] Remark 3.6.

Definition 10.2.1. A lower highest weight category $C$ is an artinian abelian $C$-linear category together with a poset $(\Lambda, \leq)$ (poset of weights) and a filtration $\Lambda = \bigcup_{k \in \mathbb{Z}^+} \Lambda^k$, such that the following conditions hold:

1. The set $\Lambda$ is in bijection with the set of isomorphism classes of simple objects in $C$.
2. For each $\xi \in \Lambda$, the Serre subcategory $C(\leq \xi)$ generated by simples $\{L(\lambda), \lambda \leq \xi\}$ contains a projective cover $\Delta(\xi)$ of $L(\xi)$, and an injective hull $\nabla(\xi)$ of $\xi$. The objects $\Delta(\xi), \nabla(\xi)$ are called standard and costandard objects in $C$.
3. There exists precisely one isomorphism class of indecomposable objects $T(\xi)$ in $C$ which has $\Delta(\xi)$ as a submodule, $T(\xi)/\Delta(\xi)$ has a filtration with standard subquotients, and $T(\xi)$ also has a filtration with costandard subquotients.

Such objects $T(\xi)$ are the indecomposable tilting objects in $C$.
4. Let $k \geq 0$, and let $C^k$ be the full subcategory of $C$ whose objects are subquotients of finite direct sums of objects $T(\xi), \xi \in \Lambda^k$. Then each $C^k$ is a highest weight category with poset $(\Lambda_k, \leq)$, simple objects $\{L(\xi), \xi \in \Lambda^k\}$, standard objects $\{\Delta(\xi), \xi \in \Lambda^k\}$, costandard objects $\{\nabla(\xi), \xi \in \Lambda^k\}$, and tilting objects $\{T(\xi), \xi \in \Lambda^k\}$. The category $C^k$ also has enough projective and injective objects.
5. The subcategories $C^k$ form a filtration on the category $C$: $C = \bigcup_{k \geq 0} C^k$.

Remark 10.2.2. The main difference between a highest-weight category and a lower highest-weight category is the possible lack of projectives and injectives in $C$.

Definition 10.2.3. An lower highest weight category $C$ with an $\mathfrak{sl}_2$-action $(E, F, x, \tau)$ is a tensor product categorification of $\text{Pol} \otimes \text{Pol}_T^\vee$ if it satisfies the following conditions:

1. Its isomorphism classes of simple objects are in bijection with $P \times P$ (direct product of the sets of simples in $\text{Pol}$, $\text{Pol}_T^\vee$), and the lower highest weight structure on $C$ is compatible with the partial order given by
\[ \lambda \leq \mu \quad \text{if} \quad \omega_{\lambda} - \omega_{\lambda^0} = \omega_{\mu} - \omega_{\mu^0}, \quad \omega_{\lambda} \geq \omega_{\mu} \]
Recall that the partial order on the weights in $\mathcal{P}$ is semisimple, and so it is automatically equivalent to $\text{Pol}_\mathcal{P}$.

We now show that the category $\text{Pol}_\mathcal{P}$ preserves the subcategory of tilting objects. Hence translations of $\text{Pol}_\mathcal{P}$, which are tilting, occur as a subobject of some $T \in \mathcal{C}'$. The corresponding indecomposable direct summand $T(\lambda)$ of $T$ is then the indecomposable tilting object $T(\lambda)$, implying that $\text{C}^{\text{tilt}} \subset \mathcal{C}'$.

We now show that the category $\mathcal{V}_\mathfrak{t}$ with the $\mathfrak{sl}_2$-action satisfies the definition of a tensor product $\text{Pol} \otimes \text{Pol}_\mathcal{P}$, and that such a tensor product is unique.

**Theorem 10.2.6.** The category $\mathcal{V}_\mathfrak{t}$ with the action of $\mathfrak{sl}_2$ is a tensor product categorification $\text{Pol} \otimes \text{Pol}_\mathcal{P}$ in the sense of Definition 10.2.3.

**Proof.** To prove Condition (1), we need to show that $[V(\lambda) : L(\mu)] \neq 0 \Rightarrow \lambda \leq \mu$ in the order given by the lowest highest weight structure on $\mathcal{V}_\mathfrak{t}$. Indeed, recall that by [EHS] Lemma 4.5.2, $[V(\lambda) : L(\mu)] \neq 0$ iff one can obtain $d^\lambda_i$ from $d^\mu_i$ by moving finitely many caps in the cap diagram of $d^\mu_i$ from the left end of a cap to the right end of this cap. This means that $d^\lambda_i, d^\mu_i$ have the same core, and that $\sum_{i \leq a} \lambda_i - i \geq \sum_{i \leq a} \mu_i - i$ for any $a \geq 1$.

By Lemma 4.3.2, $\text{core}(d^\lambda_i) = \text{core}(d^\mu_i)$ implies $\omega^\lambda_0 - \omega^\mu_0 = \omega^\mu_0 - \omega^\mu_0$, and $\sum_{i \leq a} \lambda_i - i \geq \sum_{i \leq a} h_i^\mu - i$ implies $\omega^\lambda_0 \geq \omega^\mu_0$. Hence $\lambda \leq \mu$.

Finally, to prove Condition (3), recall that the statement on translation of standard objects by functors $E_a, F_a$ was proved in Proposition 9.1.1. To prove the analogous statement for translation of costandard objects, recall from [EHS] that $\mathcal{V}_\mathfrak{t}$ possesses a contravariant endofunctor $(\cdot)^\vee : \mathcal{V}_\mathfrak{t}^{\text{op}} \rightarrow \mathcal{V}_\mathfrak{t}$ which is exact, interchanges standard objects with costandard objects (thus preserving mixed tensor power of $\mathcal{V}_\mathfrak{t}$, which are tilting), and which commutes with tensor products. Applying the functor $(\cdot)^\vee$ to the exact sequences in Proposition 9.1.1 we obtain the required results for costandard objects. 

□
Theorem 10.2.7. The tensor product $\text{Pol} \otimes \text{Pol}'$ is unique in the following sense: consider a lower highest-weight category $C$ with an $\mathfrak{sl}_2$-action $(F', E', X', \tau')$ satisfying conditions in Definition 10.2.3, and such that the natural transformation $\tau'_{E'F'} : E'F' \to F'E'$ induced by $\tau'$ is an isomorphism.

Then we have a strongly equivariant equivalence $C \cong V_t$.

Remark 10.2.8. The author has been told by J. Brundan that the requirement on $\tau'_{E'F'}$ can be lifted (i.e. will hold automatically), due to Rouquier’s “$K_0$-control” theorem given in [R]. This will be explained in detail elsewhere.

Proof. Consider the oriented Brauer category $OB$ (see for example [BCNR]). This is the free symmetric monoidal $\mathbb{C}$-linear category generated by a single object $X$ and its dual. The skeleton subcategory $D^0_t$ of the Deligne category $\text{Rep}(GL_t)$, containing the mixed tensor powers of $V_t$, is then the specialization of $OB$ under the relation $\dim(X) = t$, where

$$\dim(X) := ev_X \circ \sigma_{X,X} \circ \text{coev}_X \in \text{End}_{OB}(1)$$

Given any category $C$ with an $\mathfrak{sl}_2$-action $(F', E', X', \tau')$, we have a monoidal functor $\Psi$ from the oriented Brauer category $OB$ to $\text{End}(C)$ taking $X$ to $F$, $X^*$ to $E$, $\sigma_{X,X}$ to $ev_X$ and $\text{coev}_X$ to $\varepsilon, \eta$, the adjunction unit and counit of $E', F'$, and $\sigma_{X,X}$ to $\tau'$. To check that this is indeed a functor, we need to check that $E', F', \varepsilon, \eta, \tau'$ satisfy the relations in [BCNR Theorem 1.1]; this is indeed the case, due to the conditions on $E', F', \tau'$ in the definition of a $\mathfrak{sl}_2$-categorical action, together with the requirement that the natural transformation $\tau'_{E'F'}$ is invertible.

Let $U$ be the simple object in $C$ corresponding to $(\emptyset, \emptyset) \in P \times P$. We define a functor $\Phi : OB \to C$ by setting $\Phi(\cdot) := \Psi(\cdot)U$. Recall that $OB$ has a type A action, defined just as in Definition 6.11 $F = X \otimes (\cdot), E = X^* \otimes (\cdot), \tau = \sigma_{X,X}$ is the symmetry morphism of $X$ and the morphism $x_{X^r \otimes X^s} : X^r \otimes 1 \otimes X^s \otimes 1 \to X^r \otimes X^s$ is given by

$$x_{X^r \otimes X^s} = \sum_{i=1}^r \sigma_{1,i+1} - \sum_{j=1}^s ev_{1,j}$$

where $\sigma_{1,j}$ denotes the symmetry morphism between the first and the $i$-th $X$-factors, and $ev_{1,j}$ denotes the contraction of the first $X$-factor and the $j$-th $X^*$-factor.

We claim that this functor is strongly equivariant, i.e. there exists a natural isomorphism $\zeta : F'\Phi \to \Phi F$ such that

- The induced natural transformation $\zeta' : \Phi E \to E'\Phi$ is an isomorphism,
- $\zeta \circ x'\Phi = \Phi x \circ \zeta : F'\Phi \to \Phi F$,
- $\zeta F \circ F'\zeta = \Phi F \circ F'\Phi : F'^2 \to \Phi F^2$.

The natural isomorphism $\zeta$ is obvious, and the only non-trivial part of this statement is the fact that $x$ is taken to $x'$. Due to the degenerate Hecke algebra relations on $x', \tau'$ and the adjointness of $F', E'$, the natural transformation

$$x'F'^rE'^s : F'^r+1E'^s \to F'^r+1E'^s$$

can be expressed in terms of $\tau'$, adjunction units and counits, and the natural transformation

$$F'^rE'^s x' : F'^rE'^s F \to F'^r+1E'^s F.$$

This means that it is enough to check that $\zeta \circ x'F'F^s|_{F'F^s1} = x'^2 : F'^sU \to F'^sU$ are both zero (the latter can be seen from the action of $\mathfrak{sl}_2$ on $[U] \in Gr(C)$).

Thus $\Phi$ is a strongly equivariant functor. We now claim that factors through the specialization $D^0_t$ of $OB$. This means that the functor $\Phi$ takes the morphism $\dim(X) \in \text{End}_{OB}(1)$ to $t \text{Id}_t$. Indeed, since $U$ is a simple object, $\Phi(\dim(X))$ is a scalar multiple of $\text{Id}_t$.

Let us call this scalar $t'$, and show that $t' = t$. Indeed, consider $x'^2|_{F'^s(U)} : F'F'^s(U) \to F'F'^s(U)$.

Then

$$\zeta \circ x'^2 = \Phi x'^2 \circ \zeta : F'F'u \to F'F'u$$
while
\[ x^2 | \chi^* : X \otimes X^* \to X \otimes X^* = - \dim(X)x. \]
Hence \( x^2 = -t'x' \in \text{End}(F'E'(U)) \). Now, \( x' \) has two generalized eigenvalues on \( F'E'(U) \), which are 0 and \( -t \) (this follows from Condition \([B] \)). This implies \( t' = t \) and \( \Phi(\dim(X)) = t \text{Id}_U \).

This proves the required result, and gives us a strongly equivariant \( \mathbb{C} \)-linear functor \( \Phi : \mathcal{D}_t \to \mathcal{C} \), and extends to a strongly equivariant additive \( \mathbb{C} \)-linear functor \( \Phi : \mathcal{D}_t \to \mathcal{C} \) from the additive envelope of \( \mathcal{D}_t^0 \).

We now show that this functor extends to a strongly equivariant faithful exact functor \( \Phi : \mathcal{V}_t \to \mathcal{C} \).

By the universal property of \( \mathcal{V}_t \) as the abelian envelope of \( \mathcal{D}_t \) (see [EHS, Section 9]), it is enough to show that the functor \( \Phi : \mathcal{D}_t \to \mathcal{C} \) is pre-exact, i.e. that for every morphism \( f \) in \( \mathcal{D}_t \) which is a monomorphism (resp. epimorphism) in \( \mathcal{V}_t \), the morphism \( \Phi(f) \) is again a monomorphism (resp. epimorphism) in \( \mathcal{C} \).

The proof is analogous to arguments in [EHS, Section 9]. We will consider the case of an epimorphism (the case of a monomorphism is similar). In [EHS, Proposition 4.8.1], it was shown that given an epimorphism \( f : A \to B \) in \( \mathcal{D}_t \), there exists a non-zero object \( Z \in \mathcal{D}_t \) such that the epimorphism \( f \otimes \text{Id}_Z : Z \otimes A \to Z \otimes B \) is split.

This implies that \( \Phi(\text{Id}_Z \otimes f) \) is a split epimorphism as well. Write \( Z = e \bigoplus_i V_i \otimes r_i \otimes V_i^* \otimes s_i \), for some idempotent \( e \in \text{End}_\mathbb{C}(\bigoplus_i V_i \otimes r_i \otimes V_i^* \otimes s_i) \). Then the idempotent \( \Phi(e \otimes \text{Id}_A) \in \text{End}_\mathcal{C}(\Phi(\bigoplus_i V_i \otimes r_i \otimes V_i^* \otimes s_i \otimes A)) \) induces an idempotent \( e'_A \in \bigoplus_i F^{r_i} E^{s_i} \Phi A \) (through the isomorphism \( \oplus_i (C^{r_i} \omega^{s_i}) \)).

Similarly, \( \Phi(e \otimes \text{Id}_B) \) induces an idempotent \( e'_B \in \bigoplus_i F^{r_i} E^{s_i} \Phi B \). By the construction above, we have a commutative diagram

\[
\begin{array}{ccc}
\Psi(Z \otimes A) & \xrightarrow{\Phi(\text{Id}_Z \otimes f)} & \Psi(Z \otimes B) \\
\downarrow & & \downarrow \\
e'_A \bigoplus_i F^{r_i} E^{s_i} A & \xrightarrow{\oplus_i F^{r_i} E^{s_i} \Phi(f)} & e'_B \bigoplus_i F^{r_i} E^{s_i} B
\end{array}
\]

where the vertical arrows are isomorphisms. Thus \( \oplus_i F^{r_i} E^{s_i} \Phi(f) \) is a split epimorphism, i.e. \( \oplus_i F^{r_i} E^{s_i} \text{Coker} \Phi(f) = 0 \). Yet \( F^r, E^t \) are exact endofunctors of \( \mathcal{C} \) which do not annihilate any simple object, as can be seen from the action of \( \mathfrak{sl}_2 \) on the Grothendieck group. This implies that \( \text{Coker} \Phi(f) = 0 \), and \( \Phi(f) \) is an epimorphism.

Thus we obtained a strongly equivariant faithful exact functor \( \Phi : \mathcal{V}_t \to \mathcal{C} \), taking \( 1 \) to the simple object \( U \).

**Lemma 10.2.9.** The functor \( \Phi \) takes the standard object \( \mathbf{V}(\lambda) \) to the standard object \( \Delta(\lambda) \in \mathcal{C} \), and similarly for costandard objects.

**Proof.** This follows directly from the facts that both \( \mathcal{V}_t, \mathcal{C} \) satisfy Condition \([B] \) \( \Phi \) is exact, and
\[ \Phi(\mathbf{V}(\varnothing, \varnothing) \cong 1) = U \cong \Delta(\varnothing, \varnothing). \]

The argument for costandard objects is exactly the same.

**Lemma 10.2.10.** The functor \( \Phi \) takes objects in \( \mathcal{D}_t \) to objects in \( \mathcal{C}^{\text{tilt}} \), and induces an equivalence of categories \( \Phi : \mathcal{D}_t \sim \mathcal{C}^{\text{tilt}} \). In particular, \( \mathbf{T}(\lambda) \) is sent to the tilting object \( T(\lambda) \in \mathcal{C} \).

**Proof.** By Lemma 10.2.5 \( \mathcal{C}^{\text{tilt}} \) coincides with the Karoubian additive category generated by translations of \( U \). Since \( \mathcal{D}_t \) is the Karoubian additive envelope of the full subcategory \( \mathcal{D}_t^0 \) generated by mixed tensor powers of \( V_t \), it is enough to check that the \( \Phi \) induces an equivalence of categories between \( \mathcal{D}_t^0 \) and the full subcategory \( \mathcal{C}^{\text{tilt}} \) of \( \mathcal{C} \) whose objects are translations \( G_1 \ldots G_m U \) where \( G_1, \ldots, G_m \in \{ E, F \} \). The fact the \( \Phi(\mathcal{D}_t^0) \subset \mathcal{C}^{\text{tilt}} \) follows directly from the fact
that $\Phi$ is equivariant, and we also conclude that $\Phi : D^b_t \to C^{gl}$ is essentially surjective. Hence we only need to check that $\Phi$ is full (we already know it is faithful); that is, we need to prove that

$$\dim \text{Hom}_D(T, T') = \dim \text{Hom}_{C^{gl}}(\Phi T, \Phi T') \forall T, T' \in D^b_t.$$  

Lemma 10.2.9 together with the fact that $\Phi$ is exact now implies that $[\Phi T : \Delta(\lambda)] = [T : V(\lambda)]$ for any $T \in D^b_t$ and any $\lambda$.

Let $V(\lambda)^\vee \in \mathcal{V}_t$ and $\nabla(\lambda) \in C$ denote the respective costandard objects. Then for any $T, T' \in D^b_t$,

$$\dim \text{Hom}_D(T, T') = \sum_{\lambda}[T : V(\lambda)][T' : V(\lambda)^\vee] = \sum_{\lambda}[\Phi T : \Delta(\lambda)][\Phi T' : \nabla(\lambda)] = \dim \text{Hom}_{C^{gl}}(\Phi T, \Phi T').$$

This completes the proof of the lemma. $\square$

We now show that the functor $\Phi : \mathcal{V}_t \to C$ is fully faithful. For this, we consider the left adjoint functor $\Phi_* : C \to \text{Ind} - \mathcal{V}_t$ and the counit of the adjunction $\varepsilon : \Phi_* \Phi \to \text{Id}$. We need to check that it is an isomorphism. We use the presentation property of $D_t$ in $\mathcal{V}_t$, as described in [LHS]: for any object $M \in \mathcal{V}_t$, there exist objects $T, T' \in D_t$ together with a surjective map $T \to M$ and an injective map $M \to T'$. In such a case, the statement that $\varepsilon_M$ is an isomorphism clearly from the fact that $\varepsilon_T, \varepsilon_{T'}$ are isomorphisms, as shown in Lemma 10.2.10.

It remains to check that the functor $\Phi : \mathcal{V}_t \to C$ is essentially surjective. Indeed, let $M \in \mathcal{C}$.

Since $C$ is a lower highest-weight category, there exists $k \geq 0$ such that $M \in C^k$. Let $P, I \in C^k$ be the projective cover and injective hull of $M$ in $C^k$, respectively. Object an injective object $C^k$, and a map $f : P \to I$ whose image is $M$. Hence it is enough to check that $P, I$ belong to the image of $\Phi$: since the functor $\Phi$ is full and exact, it will follow that the map $f : P \to I$ and $M = \text{Im}(f)$ lie in the image of $\Phi$ as well.

Now, the object $P$ is standardly-filtered, so it has a resolution by tilting objects in $C^k$; similarly, $I$ is costandardly-filtered, so it has a resolution by tilting objects in $C^k$. By Lemma 10.2.10 the functor $\Phi$ induces an equivalence $\Phi : D_t \overset{\sim}{\longrightarrow} C^{gl}$; hence $P, I$ lie in the image of $\Phi$. This completes the proof of the theorem. $\square$

When $t \notin \mathbb{Z}$, the semisimple Deligne category $D_t$ is clearly an (exterior) tensor product of the categorical $\mathfrak{sl}_2 \times \mathfrak{sl}_2$-module $\text{Pol} \boxtimes \text{Pol}^t$ in the sense of [ML] Remark 3.6. The uniqueness of such a tensor product categorification is straightforward, similarly to the uniqueness of the categorification of the $\mathfrak{sl}_2$-representation $\mathfrak{g}$.

11. Future directions

I. Losev suggested ([L1]) two additional problems to be solved in the framework of this project, to be addressed in the future.

11.1. The first problem is to understand the multiplicities in the parabolic category $\text{Par}(\tilde{I})$ (see [Ed] Section 4). Let us give a short description of such a category. Let $t_1, \ldots, t_n \in \mathbb{C}$, and set $\tilde{t} := (t_1, \ldots, t_n)$, $t := \sum t_i$. Consider the tensor categories $\mathcal{V}_{t_i}$ (i = 1, 2, ..., n) with generators $V_{t_i}, V_{t_i}^*$. Denote:

$$\mathfrak{g}_{t_i} := V_{t_i} \otimes V_{t_i}^*$$

The direct product $\mathcal{V}_{\tilde{t}} := \mathcal{V}_{t_1} \boxtimes \mathcal{V}_{t_2} \boxtimes \cdots \boxtimes \mathcal{V}_{t_n}$ is once again a tensor category; given any objects $A_{i_1} \in \mathcal{V}_{t_{i_1}}, \ldots, A_{i_r} \in \mathcal{V}_{t_{i_r}}$, we denote by $A_{i_1} \boxtimes \cdots \boxtimes A_{i_r}$ the object

$$1 \boxtimes \cdots \boxtimes 1 \boxtimes A_{j_1} \boxtimes \cdots \boxtimes 1 \boxtimes A_{j_r} \boxtimes 1 \boxtimes \cdots \boxtimes 1$$

in $\mathcal{V}_{\tilde{t}}$ where $j_1 < j_2 < \cdots < j_r$ is a reordering of $(i_1, \ldots, i_r)$ in increasing order. We will consider objects

$$\mathfrak{g}_{\tilde{t}} := \bigoplus_{1 \leq i,j \leq n} V_{t_i} \boxtimes V_{t_j}^* \quad \mathfrak{n}_+ := \bigoplus_{i<j} V_{t_i} \boxtimes V_{t_j}^*, \quad \mathfrak{n}_- := \bigoplus_{i>j} V_{t_i} \boxtimes V_{t_j}^*, \quad \mathfrak{l} := \bigoplus_{1 \leq i \leq n} \mathfrak{g}_{t_i}$$
Thus $\mathfrak{gl}_t = n_+ \oplus n_- \oplus \mathbb{I}$. This is a Lie algebra object, the image of $\mathfrak{gl}_t$ in $\mathcal{V}_t$ under the exact SM functor $\mathcal{V}_t \rightarrow \mathcal{V}_t^+$ given by

$$V_i \mapsto V_i^+ := V_i \boxtimes \ldots \boxtimes V_n$$

By definition, any object of $\mathcal{V}_t$ carries a natural action of $\mathbb{I}$ (induced by the actions of $\mathfrak{gl}_t$ on objects of $\mathcal{V}_t$, for each $i$).

The parabolic category $O^p_t$ is defined to be the category of $U(\mathfrak{gl}_t)$-modules in $Ind - \mathcal{V}_t$ on which $n_s$ acts locally nilpotently, $\mathfrak{s}t_t := \text{Ker}(Tr : I \rightarrow \mathbb{I})$ acts naturally (via the embedding $\mathfrak{s}t_t \hookrightarrow I$, and the quotient $\mathfrak{z} = I/\mathfrak{s}t_t$ acts semisimply.

Let $s_1, \ldots, s_n \in \mathbb{C}$, and set $\mathfrak{s} := (s_1, \ldots, s_n)$. Let $\lambda^1, \ldots, \lambda^n$ be bipartitions, and consider the simple representation $L_{s_1, \ldots, s_n}(\lambda^1)$ of $\mathfrak{gl}_t$, given by the natural action of $\mathfrak{gl}_t$, twisted by the character $s_i Tr : \mathfrak{gl}_t \rightarrow \mathbb{I}$. In this setting, one can define the parabolic Verma object

$$M_{\lambda, \mathfrak{s}} = \text{Ind}_{\mathfrak{s}t_t \oplus \mathfrak{z}}^{\mathfrak{gl}_t} L_{s_1}(\lambda^1) \boxtimes \ldots \boxtimes L_{s_n}(\lambda^n)$$

in $O^p_t$. This object has a unique irreducible quotient $L_{\lambda, \mathfrak{s}}$ and all simple objects of $O^p_t$ are of this form (see [Et, Proposition 4.5]).

In [Et, Section 4], P. Etingof suggested the problem of computing the Kazhdan-Lusztig coefficients in this category. I. Losev suggested that this should be done similarly to the computation of (the parabolic) super Kazhdan Lusztig coefficients in [BLW], by showing that the functors $F := V_t \otimes (\cdot), E := V_t^s \otimes (\cdot)$ induce an $\mathfrak{s}t(Z)$-categorical action on the category $O^p_t$. The Grothendieck group of $O^p_t$ is then isomorphic, as an $\mathfrak{s}t$-module, to a tensor product of $\mathfrak{s}$ and $\mathfrak{s}^\vee$ (with twisted actions). Computing the Kazhdan Lusztig coefficients would then be possible using techniques analogous to [BLW].

11.2. The second problem is to construct a graded lift of $\mathcal{V}_t$ in the sense of [BLW]. This should allow to compute the graded Kazhdan-Lusztig multiplicities for $\mathcal{V}_t$, similarly to [BLW, Section 5].

References

[BCNR] J. Brundan, J. Comes, D. Nash, A. Reynolds, A basis theorem for the affine oriented Brauer category and its cyclotomic quotients, to appear in Quantum Topology, [arXiv:1401.0574 [math.RT]].
[BLW] J. Brundan, I. Losev, B. Webster, Graded tensor product categorification and the super Kazhdan Lusztig conjecture, Int. Math. Res. Notices 20 (2017), pp. 6329-6410; [arXiv:1310.0349 [math.RT]].
[BS] J. Brundan, C. Stroppel, Highest weight categories arising from Khovanov’s diagram algebra IV: the general linear supergroup, J. Eur. Math. Soc. 14 (2012), pp. 568–609; [arXiv:1108.0652 [math.RT]].
[CO] J. Comes, O. Stroppel, On blocks of Deligne’s category $\text{Rep}(S_t)$, Advances in Mathematics 226 (2011), no. 2, pp. 1331–1377; [arXiv:0910.5695 [math.RT]].
[CW] J. Comes, B. Wilson, Deligne’s category $\text{Rep}(GL_t)$ and representations of general linear supergroups, Represent. Theory 16 (2012), pp. 568–609; [arXiv:1108.0652 [math.RT]].
[DPS] E. Dan-Cohen, I. Penkov, V. Serganova, A Koszul category of representations of finitary Lie algebras, to appear in Advances in Mathematics, [arXiv:1105.3407 [math.RT]].
[D1] P. Deligne, Catégories tannakiennes, The Grothendieck Festschrift, Birkhauser, Boston (1990), pp. 111–195.
[D2] P. Deligne, La catégorie des représentations du groupe symétrique $S_n$, lorsque $n$ n’est pas un entier naturel, Algebraic groups and homogeneous spaces, Tata Inst. Fund. Res. Stud. Math., pp. 209–273, Mumbai (2007); [https://www.math.ias.edu/files/deligne/Symetrique.pdf]
[DM] P. Deligne, J.S. Milne, Tannakian Categories, Hodge Cycles, Motives, and Shimura Varieties, LNM 900 (1982), pp. 101–228.
[En] I. Entova Aizenbud, Notes on restricted inverse limits of categories, [arXiv:1504.01121 [math.RT]].
[EHS] I. Entova Aizenbud, V. Serganova, V. Hinich, Deligne categories and the limit of categories $\text{Rep}(GL(n))$, [arXiv:1511.07699 [math.RT]].
[Et] P. Etingof, Representation theory in complex rank, II, [arXiv:1407.0873 [math.RT]].
