Geometric and Differential Features of Scators as Induced by Fundamental Embedding

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Abstract: The scator space, introduced by Fernández-Guasti and Zaldívar, is endowed with a product related to the Lorentz rule of addition of velocities. The scator structure abounds with definitions calculationally inconvenient for algebraic operations, like lack of the distributivity. It occurs that situation may be partially rectified introducing an embedding of the scator space into a higher-dimensional space, that behaves in a much more tractable way. We use this opportunity to comment on the geometry of automorphisms of this higher dimensional space in generic setting. In parallel, we develop commutative-hypercomplex analogue of differential calculus in a certain, specific low-dimensional case, as also leaned upon the notion of fundamental embedding, therefore treating the map as the main building block in completing the theory of scators.

Keywords: scators; non-distributive algebras; Lorentz velocity addition formula; hypercomplex numbers; fundamental embedding

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1. Introduction

Derivative objects of vectors have been of considerable concern in mathematics and physics for a long time [1,2]. An interesting development came from considering not just direct sums of algebraic entities of varying dimensions, but essentially from the introduction of spaces of joint algebraic elements of different tensor rank (see e.g., [3,4]). While initial concepts settled around Cartesian matrices, vectors and scalars, the next stages of development were more sophisticated.

The first mentions of scators appeared a long time ago, see [1,4], but here, we focus on the new approach proposed recently by Fernández-Guasti and Zaldívar, see [5–8]. The main motivation came from physics [5]. In the so-called hyperbolic case, scators are connected to Lorentz rule of addition of velocities and certain deformations of the Lorentz metric in the Minkowski space-time, see [6,7]. Thus, the hyperbolic case is associated with a generalization of the special relativity. The elliptic case can be considered as a new extension of the complex numbers to higher dimensions [8,9]. Algebraic properties of scators suffer from many inconvenient perplexities including, most importantly, lack of distributivity. In [10], we introduced so-called fundamental embedding (see also [11]), here engaging the map in making the scators more tractable in calculation and interpretable, and, last but not least, making them not as prone to mistakes due to the counter-intuitive algebraic features of these objects.

The whole paper might be thought as a new approach to calculational rules for scators. It is organized as follows. Section 2 contains preliminary information about scators and introduces the fundamental embedding. Section 3 is concerned with extended scator space automorphisms, as generated by its basis. We give a systematic study of dualities [10] of arbitrary dimension and parity, briefly characterizing their influence in causal realms appearing in the scator space. Sections 4 and 5
are centered around the derivative of scator function with respect to a scalar argument and derivation of formulas, enabling faster calculations of such derivatives in the $1 + 2$ dimensional case. Section 6 is devoted to scator-holomorphic functions, as falling from the previous considerations. Finally, Section 7 holds place for some conclusions.

2. Preliminaries

Let $S = E^{1 + n}$ be a vector space of objects of the form

$$\mathbf{a} = (a_0; a_1, \ldots, a_n) = a_0 + \sum_{i=1}^{n} a_i \mathbf{e}_i,$$

(1)
called scators. We call $S$ a base scator space. The addition of objects and multiplication by scalars act component-wise. Hence, as far as the additive structure is concerned, it is a typical vector space of dimension $1 + n$.

The 0th component is called the scalar component and the remaining components are known as director components of a scator [6,12]. In this paper, we confine ourselves to scators with non-vanishing scalar component. This subset will be denoted by $S'$. We consider only the hyperbolic case [12]. The product in $S'$ is defined in the following way [6]:

$$\mathbf{a} \mathbf{b} = a_0 b_0 \prod_{i=1}^{n} \left(1 + \frac{a_i b_j}{a_0 b_0}\right) + a_0 b_0 \sum_{i=1}^{n} \left[ \prod_{j \neq i}^{n} \left(1 + \frac{a_j b_i}{a_0 b_0}\right) \right] \left( \frac{a_i}{a_0} + \frac{b_i}{b_0} \right) \mathbf{e}_i,$$

(2)

where

$$\mathbf{a} = (a_0; a_1, \ldots, a_n), \quad \mathbf{b} = (b_0; b_1, \ldots, b_n).$$

(3)
The above defined product is symmetric, but suffers from lack of distributivity with respect to addition and is not associative in general; additionally it obeys

$$c(\mathbf{a} \mathbf{b}) = (c\mathbf{a}) \mathbf{b} = \mathbf{a} (c\mathbf{b}).$$

(4)

Excluding zero divisors from the multiplicative algebra of scators, we obtain a well defined, unambiguous and associative product on $S$. This condition comes together with the exclusion of scators with scalar component being zero.

We implement the operation of hypercomplex conjugation

$$\mathbf{a}^* = (a_0; -a_1, \ldots, -a_n),$$

(5)

and for hypercomplex scator entities, we have the norm of a scator $\mathbf{a} = (a_0; a_1, \ldots, a_n)$ as given by

$$||\mathbf{a}||^2 = a_0^2 \prod_{i=1}^{n} (1 - \frac{a_i^2}{a_0^2})$$

(6)
of course—using definition of the product and a notion of hypercomplex conjugation, it can be shown that

$$||\mathbf{a}||^2 = \mathbf{a}^* \mathbf{a}.$$

(7)

Definition 1. Scators with positive square modulus will be referred to as time-like, scators with negative square modulus will be referred to as space-like, while scators with zero square modulus will be referred to as light-like [10].
Since zero divisors are excluded from considerations, we have
\[ ||\alpha\beta||^2 = ||\alpha||^2 ||\beta||^2. \]  \tag{8}

\textbf{Remark 1.} As a consequence of these conditions and above stated remark we have also
\begin{itemize}
  \item Product of time-like scator with time-like scator is still time-like.
  \item Product of two space-like scators is time-like.
  \item Product of space-like scator with time-like scator is space-like.
  \item Product of light-like scator with any scator is light-like.
\end{itemize}

It was shown in [13] that known subluminal particles described by time-like scators undergo the so-called restricted space conditions, as well as superluminal particles described with space-like scators would undergo so-called super-restricted space conditions. Restricted space conditions generate a closed subset under multiplication, but scators obeying super-restricted space conditions do not constitute a closed set under multiplication [13].

The main ideas developed in the present paper follow directly from the distributive interpretation of the scator algebra introduced in [10]. Concepts appearing further will be of both geometric and differential nature.

\textbf{Definition 2.} Let \( S \) be a scator space, \( A \) be the space generated by products \( e_{i_1} \ldots e_{i_k} \), \( i_1, \ldots, i_k = 1, \ldots, n \), and \( S' \subset S \) contains scators with non-vanishing scalar component. We define a map

\[ F : S' \to A, \] \tag{9}

such that

\[ F(\alpha) = a_0 + \sum_{j=1}^{n} \sum_{\{i_k\}} a_{i_1} \ldots a_{i_k} e_{i_1} \ldots e_{i_k}. \] \tag{10}

In other words,

\[ F(\alpha) = a_0 \prod_{k=1}^{n} \left( 1 + \frac{a_k}{a_0} e_k \right). \] \tag{11}

We call \( F \) the fundamental embedding and we denote

\[ \tilde{S} := F(S). \] \tag{12}

Note that the space \( \tilde{S} \) constitutes a closed subset under multiplication, although addition spoils this property. The space \( A \) can be viewed as a commutative analogue of the Clifford algebra, while \( \tilde{S} \) resembles the Clifford or Lipschitz group [14–16].

The fundamental embedding can be extended on some elements from outside \( S' \). In particular, we have

\textbf{Remark 2.} We find, as developed in [11], that

\[ F(e_i) = e_i \] \tag{13}

setting clear notational distinction between bases of \( S \) (domain) and \( A \) (image).

Let \( A \) be a multivector space generated by products \( e_{i_1} \ldots e_{i_k} \) \((1 \leq i_1 < \ldots < i_k \leq n)\), of elements given above by the action of fundamental embedding. We denote set of such multivectors by

\[ B = \{ e_1, \ldots, e_n, e_1 e_2, \ldots, e_1 e_n, \ldots, e_1 \ldots e_n \} \] \tag{14}
and call it a basis in \( A \)-space, while we will refer to this space itself as extended scator space.

Because of closure under multiplication in \( A \), fundamental embedding is multiplicative homomorphism between \( S' \) and \( A \), i.e.

\[
F(\hat{a})F(\hat{\beta}) = F(\hat{a}\hat{\beta}). \tag{15}
\]

**Remark 3.** We denote by \( \pi \) the natural projection of \( A \) onto the scator space \( S \). Despite the fact we have

\[
\pi \circ F = id_{S'}, \quad F \circ \pi = id_{\tilde{S}}, \tag{16}
\]

these two maps are not mutually inverse to each other, since

\[
F : S' \to A, \quad \pi : A \to S. \tag{17}
\]

Hypercomplex conjugation may act on the level we want, i.e.

\[
F(\hat{a}^*) = F(\hat{a})^*. \tag{18}
\]

It can be easily checked, that

\[
F(c\hat{a}) = cF(\hat{a}), \tag{19}
\]

where \( c \) is a constant.

Reciprocal of fundamental embedding acts like a projection from \( A \) to \( S \) and it is additive, but not multiplicative homomorphism.

### 3. Geometry of Duality Automorphisms

Content of this section applies to any \( 1+n \) dimensionality of the scator space. Note that the basis of extended scator space generates group of automorphisms of itself, so that

\[
B = \text{Aut}(B). \tag{20}
\]

Basis \( B \) in \( A \)-space gives rise to objects related directly with any chosen scator \( \hat{a} \). For example, we have

**Definition 3.** If we have scator space \( S = E^{1+n} \), we call a multivector of the form

\[
e_{\text{max}} = e_1 \ldots e_n \tag{21}
\]

the maximal form acting on \( A \).

This motivates the next definition.

**Definition 4.** If we have a scator \( \hat{a} = (a_0; a_1, a_2, \ldots, a_n) \), we call an object given by

\[
\hat{\bar{a}} = F^{-1}(e_{\text{max}}F(\hat{a})) \tag{22}
\]

its dual, or ordinary dual scator.

Then we also define its \( J \)-dual as

\[
\hat{\bar{a}}_J = F^{-1}(e_JF(\hat{a})), \tag{23}
\]

where \( J \) is multi-index \( ij \ldots k \), where \( 1 \leq i < j < \ldots < k \leq n \) and \( e_J := e_ie_j \ldots e_k \). Here \( i, j, k \) and other possible labels take the values between 1 and \( n \).
Note that we need to perform all calculations on an extended space level, since otherwise we would get 0 instead, in the place of all duals, if we apply the reciprocal of fundamental embedding before the calculation triggers.

Because of \( F \) being multiplicative homomorphism, there also holds

\[
||\hat{\alpha}||^2 = F(\hat{\alpha})F(\hat{\alpha}^*). \tag{24}
\]

A piece of terminology is in order. All dualities are idempotent and preserve scator product. We call an odd dual the dual given in terms of a multivector that is a product obtained of odd number of vectors \( e_i \) \((i = 1, \ldots, n)\), and we call an even dual, the dual that is given in terms of product obtained from an even number of such vectors.

**Theorem 1.** Even dualities commute with hypercomplex conjugation, while odd dualities anti-commute with hypercomplex conjugation.

**Proof.** To show this, we begin with some even duality

\[
\hat{\alpha}^*_{\text{even}} = F^{-1}(F(\hat{\alpha}^*)) = F^{-1}(F(\hat{\alpha}))^* = F^{-1}((\mathbf{e}_{\text{even}}F(\hat{\alpha}))^*), \tag{25}
\]

where the subscript even stands for an even combination of indices. Then, the conjugation triggers

\[
F^{-1}(\mathbf{e}_{\text{even}}F(\hat{\alpha}^*)) = F^{-1}(\mathbf{e}_{\text{even}}F(\hat{\alpha})^*), \tag{26}
\]

since the considered multivector is even, and hypercomplex conjugation causes an even number of additional minus signs, so that we have

\[
F^{-1}(\mathbf{e}_{\text{even}}F(\hat{\alpha}^*)) = \mathbf{e}_{\text{even}}^{\hat{\alpha}^*}. \tag{27}
\]

In a similar manner, we obtain

\[
\hat{\alpha}^*_{\text{odd}} = F^{-1}(F(\hat{\alpha})) = F^{-1}((\mathbf{e}_{\text{odd}}F(\hat{\alpha}))^*) = -F^{-1}(\mathbf{e}_{\text{odd}}F(\hat{\alpha}^*)) = -\mathbf{e}_{\text{odd}}^{\hat{\alpha}^*}, \tag{28}
\]

where the subscript odd stands for an odd combination of indices. Thus, the proof is completed. \( \square \)

A transformation that exchanges time-like events with space-like events (and conversely) and leaves light-like events invariant (in a sense of being light-like) is called a causality swap (or a pseudo-isometry). Then, we see from introduced above notions that even dualities are isometries and odd dualities are causality swaps.

**Lemma 1.** Operation of inversion commutes with all dualities.

**Proof.** It follows naturally in the case of even dualities. With odd dualities, it is counterintuitive, but we should remember, that one minus sign in inverse of a scator arises because of hypercomplex conjugation, and another, because of a causality swap. Hence, we have finished the proof. \( \square \)

### 4. Derivative with Respect to Scalar

Next, we need to recall this simple fact, that scator set \( S = E^{1+n} \) can be treated as usual, \( 1+n \) dimensional vector space with ordinary rules for addition, since we can straightforwardly introduce the notion of a limit of a scator function of single scalar argument \( s \)

\[
\hat{\alpha}(s) = (a_0(s); a_1(s), \ldots, a_n(s)), \tag{29}
\]
and we can indirectly introduce the limit of $\overset{0}{\alpha}(s)$ at a point $s_0$ a scator of the form

$$
\overset{0}{\alpha}_0 = \lim_{s \to s_0} (a_0(s); a_1(s), \ldots, a_n(s))
$$

(30)

if, of course, each component as a scalar function is continuous in an ordinary sense.

Note that the above definition must be understood as a limit in Heine’s sense, and not as a limit in Cauchy’s sense, since we have an indefinite norm. Contrary, in elliptic case above definition could be stated in Cauchy sense, since norm is semi-definite.

**Definition 5.** We call $\overset{0}{\alpha}(s)$ a continuous scator map, when there exists limit

$$
\overset{0}{\alpha}(s_0) = \lim_{s \to s_0} (a_0(s); a_1(s), \ldots, a_n(s))
$$

(31)

for all $s_0$ in $s$ axis (or its subset, given as a domain on which this map is defined), and, in addition, we can treat each component itself as a continuous real-valued function.

Now recall that scalars constitute a closed subset in scator space, with their usual arithmetics and algebraic properties of field. Moreover, the multiplication of a scator by a scalar acts component-wise. Hence, having thus defined the notion of smoothness, we can proceed with derivative

$$
\overset{0}{\alpha}'(s_0) = \lim_{s \to s_0} \overset{0}{\alpha}(s) - \overset{0}{\alpha}(s_0) \over s - s_0
$$

(32)

which we will refer the to as a derivative of a scator $\overset{0}{\alpha}(s)$ at a point $s_0$. This exists only if the scator function fulfills requirement of continuity.

Since defined derivative is taken with respect to a scalar parameter, it acts component-wise, so that

$$
\overset{0}{\alpha}'(s_0) = (a'_0(s_0); a'_1(s_0), \ldots, a'_n(s_0))
$$

(33)

what follows instantly from the rule of multiplication of a scator by a scalar and the definition of the derivative.

We shall denote this derivative in an algebraic manner as

$$
\overset{0}{\alpha}'(s_0) = \frac{d\overset{0}{\alpha}}{ds}(s_0)
$$

(34)

and call it a velocity scator of $\overset{0}{\alpha}(s)$, understood as a trajectory.

Now, we head towards treating outcoming stuff more systematically and more formally: we use fundamental embedding as introduced in [10] to obtain some explicit results that would rather be obscured by non-distributive multiplication in $S$. Since we are using $F$ embedding, we need the assumption that temporal components of considered scators are non-zero (restriction to $S'$).

From now on, we will denote a derivative with respect to scalar variable $s$ with a subscript bearing its name.

When we are given a scator function of the scalar parameter $s$, from the direct calculation, it is clear that

$$
F(\overset{0}{\alpha}_s) \neq [F(\overset{0}{\alpha})]_s
$$

(35)

but there is an obvious fact, that

$$
F^{-1}(F(\overset{0}{\alpha}_s)) = \pi([F(\overset{0}{\alpha})]_s).
$$

(36)
This might seem not too useful, but it will prove its worth soon. We should recall now that \( F^{-1} \) is not a multiplicative homomorphism. Hence, we have

\[
||\overset{\circ}{\alpha}_s||^2 = \overset{\circ}{\alpha}_s \overset{\circ}{\alpha}_s^* = F(\overset{\circ}{\alpha}_s)F(\overset{\circ}{\alpha}_s^*),
\]

(37)
since a squared modulus is just a scalar. However, then, from (35)

\[
||\overset{\circ}{\alpha}_s||^2 \neq [F(\overset{\circ}{\alpha})]_s[F(\overset{\circ}{\alpha})^*_s]_s
\]

(38)
since it is norm of different element, what is more, it is not even belonging to \( S' \) set, on which we base our intuitions.

Recall two important equations from [10]: firstly, we have

\[
F(\overset{\circ}{\alpha} \overset{\circ}{\beta}) = F(\overset{\circ}{\alpha}) F(\overset{\circ}{\beta}),
\]

(39)
i.e., \( F \) is multiplicative homomorphism. However, \( F \) is not an additive homomorphism. For instance, in the 1 + 2-dimensional case, we have

\[
F(\overset{\circ}{\alpha} + \overset{\circ}{\beta}) = F(\overset{\circ}{\alpha}) + F(\overset{\circ}{\beta}) + \kappa(\overset{\circ}{\alpha}, \overset{\circ}{\beta}) e_{max},
\]

(40)
where

\[
\kappa(\overset{\circ}{\alpha}, \overset{\circ}{\beta}) = \frac{(a_1 + b_1)(a_2 + b_2)}{a_0 + b_0} - \frac{a_1 a_2}{a_0} - \frac{b_1 b_2}{b_0},
\]

(41)
i.e., \( F(\overset{\circ}{\alpha} + \overset{\circ}{\beta}) \neq F(\overset{\circ}{\alpha}) + F(\overset{\circ}{\beta}) \).

**Remark 4.** There is

\[
F(\lim_{\delta \rightarrow s_0} \overset{\circ}{\alpha}(s)) = \lim_{\delta \rightarrow s_0} F(\overset{\circ}{\alpha}(s)).
\]

(42)

Now, using the above equalities and remark, we can prove

**Lemma 2.** If we are given continuous scator function \( \overset{\circ}{\alpha}(s) \in E^{1+2} \) of a scalar \( s \) and the fundamental embedding \( F \), then

\[
F(\overset{\circ}{\alpha}_s) = [F(\overset{\circ}{\alpha})]_s + h(\overset{\circ}{\alpha}(s)) e_{max},
\]

(43)
where

\[
h(\overset{\circ}{\alpha}(s)) = \frac{a_1 a_2 s}{a_0 s} - \left(\frac{a_1 a_2}{a_0}\right)_s.
\]

(44)

**Proof.** It could be shown by direct calculation, but we will do it in a little more instructive way.

We start from the definition of derivative and previously proven lemma

\[
F(\overset{\circ}{\alpha}_s) = F \left( \lim_{\delta \rightarrow s_0} \frac{\overset{\circ}{\alpha}(s) - \overset{\circ}{\alpha}(s_0)}{s - s_0} \right) = \lim_{\delta \rightarrow s_0} F \left( \frac{\overset{\circ}{\alpha}(s) - \overset{\circ}{\alpha}(s_0)}{s - s_0} \right).
\]

(45)

Then, let us recall Equation (40), and simple property \( F(c \overset{\circ}{\alpha}) = c F(\overset{\circ}{\alpha}) \), where \( c \) is some number. From these, we obtain

\[
\lim_{\delta \rightarrow s_0} \frac{F(\overset{\circ}{\alpha}(s) - \overset{\circ}{\alpha}(s_0))}{s - s_0} = \lim_{\delta \rightarrow s_0} \frac{F(\overset{\circ}{\alpha}(s)) - F(\overset{\circ}{\alpha}(s_0)) + \kappa(\overset{\circ}{\alpha}(s), -\overset{\circ}{\alpha}(s_0))}{s - s_0}.
\]

(46)
It is obvious that the first two terms in the above considered limit just gives rise to $[F(\hat{\alpha})]_s$. Then, we will take care below of function $\kappa$ defined in (41), which in this case, takes the form

$$\kappa(\hat{\alpha}(s), -\hat{\alpha}(s_0)) = \frac{(a_1(s) - a_1(s_0))(a_2(s) - a_2(s_0))}{a_0(s) - a_0(s_0)} - \frac{a_1 a_2}{a_0}(s) + \frac{a_1 a_2}{a_0}(s_0).$$  \hspace*{1cm} (47)

Since this expression is divided by $s - s_0$ in (46), which we are most interested in, and moreover, it is under the limit, it all looks like a bunch of derivatives for few scalar functions; we instantly obtain

$$\lim_{s \to s_0} \frac{1}{s - s_0} \left( \frac{(a_1(s) - a_1(s_0))(a_2(s) - a_2(s_0))}{a_0(s) - a_0(s_0)} - \frac{a_1 a_2}{a_0}(s) + \frac{a_1 a_2}{a_0}(s_0) \right),$$  \hspace*{1cm} (48)

or, in other words

$$\frac{a_1 a_2}{a_0} - \frac{a_1 a_2}{a_0} = h(\hat{\alpha}(s)),$$  \hspace*{1cm} (49)

hence, we have completed the proof. \qed

Here is some reason for the differentiation of scator with respect to a scalar variable to seem so not-straightforward: to obtain the differential $d\hat{\alpha}(s)$ standing in the numerator of algebraically treated derivative, we need to subtract two infinitely close scators; and subtraction, just as addition, is not consistent with the nature of fundamental embedding.

There are a few possible things to see, for example

**Remark 5.** We have

$$F((\hat{\alpha} + \hat{\beta})_s) = (F(\hat{\alpha}) + F(\hat{\beta}))_s + e_{max}(\kappa(\hat{\alpha}_s, \hat{\beta}_s) + h(\hat{\alpha}(s)) + h(\hat{\beta}(s))).$$  \hspace*{1cm} (50)

**Motivation:** It is pretty straightforward to see:

$$F((\hat{\alpha} + \hat{\beta})_s) = F(\hat{\alpha}_s + \hat{\beta}_s) = F(\hat{\alpha}_s) + F(\hat{\beta}_s) + e_{max}\kappa(\hat{\alpha}_s, \hat{\beta}_s),$$  \hspace*{1cm} (51)

and then, from Lemma 2, we get

$$F((\hat{\alpha} + \hat{\beta})_s) = [F(\hat{\alpha})]_s + [F(\hat{\beta})]_s + e_{max}(\kappa(\hat{\alpha}_s, \hat{\beta}_s) + h(\hat{\alpha}(s)) + h(\hat{\beta}(s))),$$  \hspace*{1cm} (52)

as stated.

We could go the other way round when it comes to performing the above given calculation; this second approach leads to

**Corollary 1.** Functions expressing the behavior of $F$ satisfy identity

$$\kappa(\hat{\alpha}(s), \hat{\beta}(s)) + h((\hat{\alpha} + \hat{\beta})(s)) = \kappa(\hat{\alpha}_s, \hat{\beta}_s) + h(\hat{\alpha}(s)) + h(\hat{\beta}(s)).$$  \hspace*{1cm} (53)

**Motivation:** It may be shown by direct calculation, although we will do it in a slightly wiser way

$$F((\hat{\alpha} + \hat{\beta})_s) = [F((\hat{\alpha} + \hat{\beta}))]_s + e_{max}h((\hat{\alpha} + \hat{\beta})(s)),$$  \hspace*{1cm} (54)

and then, from (40), we have

$$F((\hat{\alpha} + \hat{\beta})_s) = [F(\hat{\alpha}) + F(\hat{\beta}) + \kappa(\hat{\alpha}, \hat{\beta})e_{max}]_s + e_{max}h((\hat{\alpha} + \hat{\beta})(s)),$$  \hspace*{1cm} (55)
so that, when we compare with previous remark
\[ c_s(\dot{a}(s), \dot{\beta}(s)) + h((\dot{a} + \dot{\beta})(s)) = c(\dot{a}_s, \dot{\beta}_s) + h(\dot{a}(s)) + h(\dot{\beta}(s)), \] (56)
as it was aimed.

5. Modified Leibniz Rule for Scators

In this section, we confine ourselves to the 1 + 2 dimensional scator space, introducing the prototypical version of differential calculus, as introduced by fundamental embedding.

It is obvious from straightforward calculation
\[ (\dot{a}(s)\dot{\beta}(s))_s \neq \dot{a}_s\dot{\beta}_s, \] (57)
i.e., the usual Leibniz rule is not obeyed.

Below, we will see that there is a systematic way to obtain the generalization of this usual result to scator space \( S \).

**Theorem 2.** Scator algebra obeys the modified Leibniz rule
\[ (\dot{a}(s)\dot{\beta}(s))_s = \dot{a}_s\dot{\beta}_s - h(\dot{a}(s))\dot{\beta}(s) - h(\dot{\beta}(s))\dot{a}(s), \] (58)
with respect to differentiation.

**Proof.** Again, the fundamental embedding is of primary importance (we will omit writing down the argument of all functions)
\[ (\dot{a}\dot{\beta})_s = F^{-1}(F(\dot{a}\dot{\beta})_s), \] (59)
where we can use Lemma 2, to obtain
\[ (\dot{a}\dot{\beta})_s = F^{-1}(F(\dot{a}\dot{\beta})_s + e_{\text{max}} h(\dot{a}\dot{\beta})) = F^{-1}(F(\dot{a}\dot{\beta})_s), \] (60)
from that the reciprocal of fundamental embedding is additive homomorphism.

So, since the algebra of \( A \)-space is distributive, we obtain
\[ F^{-1}(F(\dot{a}\dot{\beta})_s) = F^{-1}(F(\dot{a})_s F(\dot{\beta}) + F(\dot{a}) F(\dot{\beta})_s), \] (61)
and here we use (35); this combined with, again, additive nature of \( F^{-1} \), suggests that we need to go back with the differentiation under the action of \( F \), luckily, due to Lemma 2, we know how to make it.

Hence, we obtain
\[ F^{-1}(F(\dot{a}\dot{\beta})_s) = F^{-1}((F(\dot{a}) - e_{\text{max}} h(\dot{a}))(\dot{\beta} + F(\dot{a})(\dot{\beta})_s - e_{\text{max}} h(\dot{\beta}))). \] (62)

This shows us that indeed, rightly, we have recalled definitions for the dual of a scator. We have then
\[ F^{-1}(F(\dot{a}_s\dot{\beta}_s) - h(\dot{a})\dot{\beta}) + F(\dot{a}_s) F(\dot{\beta}_s) - h(\dot{\beta}) F(\dot{a}_s)), \] (63)
or, in other words
\[ F^{-1}(F(\dot{a}_s\dot{\beta}_s) - h(\dot{a})\dot{\beta} + F(\dot{a}_s) F(\dot{\beta}_s) - h(\dot{\beta}) F(\dot{a}_s)), \] (64)
so that we get

\[ \langle \hat{\alpha}(s) \hat{\beta}(s) \rangle_s = \hat{\alpha}_s \hat{\beta} + \hat{\alpha}_s \hat{\beta} - h(\hat{\alpha}(s)) \hat{\beta}(s) - h(\hat{\beta}(s)) \hat{\alpha}(s), \quad (65) \]

which is exactly the thesis stated. \( \square \)

### 6. Hypercomplex Holomorphic Functions

The notion of scator-holomorphic function was recently introduced by Fernández-Gausti [17]. Here, we propose a different approach to the subject, as following from previously stated remarks and conjectures.

We start as Fernández-Gausti, expecting that the value of derivative will not depend on the direction chosen. Hence, we consider derivatives along target coordinate axes. The striking property of such an approach is that then the derivative with respect to the scator object becomes the derivative with respect to scalar parameter, as developed in the former section.

We denote:

\[ \hat{\alpha} = a_0 + a_1 \hat{e}_1 + a_2 \hat{e}_2 \]

and

\[ \hat{z} = t \hat{e}_1 + y \hat{e}_2. \]

We assume that components of \( \hat{\alpha} \) depend on \( t, x \) and \( y \), and that there exists the derivative of \( \hat{\alpha} \) with respect to \( \hat{z} \), i.e., the derivative does not depend on the direction in the space \( (t, x, y) \) (in full analogy with the case of standard holomorphic functions).

Computing the derivative along \( x \), we have:

\[ \frac{d \hat{\alpha}}{d \hat{z}} = \lim_{h \to 0} \frac{\hat{\alpha}(x + h \hat{e}_1) - \hat{\alpha}(x)}{h \hat{e}_1} = \hat{e}_1 \frac{\partial \hat{\alpha}}{\partial x} = \pi F(\hat{e}_1 \frac{\partial \hat{\alpha}}{\partial x}) = \pi (\hat{e}_1 F(\frac{\partial \hat{\alpha}}{\partial x})), \quad (66) \]

and then it follows

\[ \frac{d \hat{\alpha}}{d \hat{z}} = \pi (\hat{e}_1 F(\hat{\alpha})_x + F(\hat{e}_1 h(\hat{\alpha}) \hat{e}_{max})), \quad (67) \]

where \( h(\hat{\alpha}) \) contains \( x \)-derivatives. Finally, we have

\[ \frac{d \hat{\alpha}}{d \hat{z}} = a_{1,x} + a_{0,x} \hat{e}_1 + a_{1,y}a_{2,x} \hat{e}_2, \quad (68) \]

Similarly, considering other basis directions, we have

\[ \frac{d \hat{\alpha}}{d \hat{z}} = a_{2,y} + a_{0,y} \hat{e}_2 + \frac{a_{1,y}a_{2,y}}{a_{0,y}} \hat{e}_1 \quad (69) \]

and

\[ \frac{d \hat{\alpha}}{d \hat{z}} = a_{0,t} + a_{1,t} \hat{e}_1 + a_{2,t} \hat{e}_2. \quad (70) \]

A comparison of all these three disparate results gives the system

\[ a_{1,x} = a_{2,y} = a_{0,t}, \]

\[ a_{0,x} = \frac{a_{1,y}a_{2,y}}{a_{0,y}} = a_{1,t}, \]

\[ a_{1,y}a_{2,x} = a_{0,y} = a_{2,t} \quad (71) \]

which can be considered as a hyperbolic analogue of the Cauchy–Riemann equations in the \( 1 + 2 \) dimensional case. The set of solutions to this system is not as rich as the set of holomorphic functions. In a recent paper, all solutions to this system were found and classified; see [18].
7. Concluding Remarks

The most developed physical approach to anti-commutative Clifford algebras seems to be geometric algebra \[14,15\], yielding the picture so deeply built into the nature of physical world. This kind of elaboration apparently does not fit in with commutative scator space structure. However, with fundamental embedding, we probably could extend the potential for some real-world applications beyond simple relativistic kinematics \[5\].

The main result of the paper is an alternative way to considering scators as compared with the traditional approach \[3\] and the original approach by Fernández-Guasti \[7\]. Scators as firstly introduced in \[9\] cannot be easily treated, especially when we consider calculational grounds. Therefore, it seems to us natural to develop formal tools enabling the introduction of ordinary algebraic and differential rules for objects as closely related to scators as their images through fundamental embedding.

The scator product has counter-intuitive non-disctributive properties. The fundamental embedding (11) maps scators into objects in a higher dimensional space denoted by \( A \), where the multiplication can be carried out in a straightforward, distributive way. Projecting the result on the scator space, we rederive the definition of the scator product (2) (in fact, this can be seen as an independent motivation for this definition). One has to remember that the fundamental embedding does not preserve the additive structure of the scator space. Therefore, dealing with differentiation, we had to derive and take into account all needed corrections, compare (40) and (50).

Using the fundamental embedding, it was possible to complete the task of defining new calculational tools, reaching even an early prototype of differential calculus (although it was also found in \[17\], in a different way) with the modified, scator-corresponding Leibniz rule (this element is novel). We introduced also new geometric notions which enable faster calculation without the risk of falling into a trap of non-distributive algebra.

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