COMPARING \( GL_n \)-REPRESENTATIONS BY CHARACTERISTIC-FREE ISOMORPHISMS BETWEEN GENERALIZED SCHUR ALGEBRAS

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With an appendix by STEPHENDONKIN

Abstract. Isomorphisms are constructed between generalized Schur algebras in different degrees. The construction covers both the classical case (of general linear groups over infinite fields of arbitrary characteristic) and the quantized case (in type \( A \), for any non-zero value of the quantum parameter \( q \)). The construction does not depend on the characteristic of the underlying field or the choice of \( q \neq 0 \). The proof combines a combinatorial construction with comodule structures and Ringel duality. Applications range from equivalences of categories to results on the structure and cohomology of Schur algebras to identities of decomposition numbers and also of \( p \)-Kostka numbers, in both cases reproving and generalizing row and column removal rules.

1. Introduction

1.1. General linear groups and Schur algebras. Let \( k \) be an infinite field of arbitrary characteristic. The homogeneous polynomial representations of the general linear group \( GL_n(k) \) of a fixed degree \( r \) are precisely the modules over (classical) Schur algebras \( S_k(n,r) \). Classical and quantized Schur algebras and their representations have been used in a variety of contexts, both on structural and on numerical level. Among these is the representation theory of the algebraic group \( GL_n \) and of the finite group \( GL_n \) in describing and in cross characteristic, representation theory of symmetric groups and of their Hecke algebras, polynomial functors and group cohomology. This article deals with classical Schur algebras, whose representations are precisely the polynomial representations of the algebraic group \( GL_n(k) \) over any infinite field \( k \); simultaneously we deal with the quantized Schur algebras.

1.2. The main result. Let \( GL_n = GL_n(k) \) for a fixed infinite field \( k \). The simple \( GL_n \)-representations of degree \( r \) are parametrized by partitions \( \lambda \in \Lambda^+(n,r) \) of \( r \) into not more than \( n \) parts. A saturated subset of \( \Lambda^+(n,r) \) is a subset \( \Pi \) such that \( x < y \in \Pi \) whenever \( x < y \in \Pi \). Choose any saturated subset \( \Pi \subset \Lambda^+(n,r) \). Moreover, choose any \( m \in \mathbb{N} \) such that the rows of the partitions in \( \Pi \) have not more than \( m \) boxes, that is, each partition \( \lambda \in \Pi \) fits into an \( n \times m \) rectangle. Let \( \hat{\lambda} \) be the complement of \( \lambda \) within this rectangle; then \( \hat{\lambda} \) is a partition of \( mn - r \). Our Main Theorem 6.3 implies equivalences of categories as follows:

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The abelian category of $GL_n$-modules of degree $r$ with simple composition factors indexed by $\lambda \in \Pi$ is equivalent to the category of $GL_n$-modules of degree $mn - r$ with simple composition factors indexed by $\hat{\lambda}$. This equivalence implies identities between $GL_n$-decomposition numbers in degrees $r$ and $mn - r$, and isomorphisms in cohomology between representations in degrees $r$ and $nm - r$. The equivalence preserves vector space dimensions.

Stronger and more explicitly, the Main Theorem 6.3 states the existence of an isomorphism between quotient algebras of Schur algebras (modulo ideals in heredity chains) as follows: For any natural numbers $n$ and $r$ and for any saturated set $\Pi \subset \Lambda^+(n, r)$, and for any natural number $m$ such that the partitions in $\Pi$ have rows of length not more than $m$, there is an isomorphism between (classical or quantized) generalized Schur algebras:

$$S(n, r)/I_{\Pi} \simeq S(n, nm - r)/I_{\bar{\Pi}}.$$  

Here, taking complements of partitions defines a bijection between the index sets $\Pi$ and $\bar{\Pi}$, which determine the ideals $I_{\Pi}$ and $I_{\bar{\Pi}}$ in the heredity chain. The isomorphisms of algebras will be constructed explicitly.

Quotients as in the main theorem sometimes are called generalized Schur algebras. Behind these isomorphisms is another, more general, set of isomorphisms on centralizer subalgebras of Ringel duals of Schur algebras.

The above statement about equivalences of categories follows from these isomorphisms. The correspondence between projective modules of the two algebras is again given by sending a partition $\lambda$ to its complement $\hat{\lambda}$, defined above.

The Main Theorem 6.3 covers not only the case of classical Schur algebras, but also that of quantized Schur algebras (again for general linear groups). In this case, Theorem 6.3 generalizes a result of Beilinson, Lusztig and MacPherson [1], who constructed isomorphisms between a quantized Schur algebra $S(n, r)$ and a quotient of $S(n, r + n)$, in order to describe quantized enveloping algebras geometrically (by transferring a description via convolution of ‘pairs of flags’ from quantized Schur algebras to an inverse limit of $q$-Schur algebras, which contains the quantized enveloping algebra). These isomorphisms can be recovered as products of two maps in the family of isomorphisms provided by Theorem 6.3.

1.3. Outline of the article. For background on Schur algebras, the reader is referred to Green [9] (classical case) and Donkin [7] (quantized case). Additional information and references can be found in [18]. Here is an outline of the article: In Section 2 we collect basic information about Schur algebras, and we fix some notation. Sections 3 to 6 are devoted to the proof of the Main Theorem 6.3. The ingredients of the proof are:

(a) combinatorics of a construction sending a partition $\lambda$ to its ‘complement’ $\hat{\lambda}$ (Section 3);
(b) a result (Theorem 4.3), identifying homomorphisms of certain representations (tensor products of exterior powers) of classical or quantized $GL_n$ in different degrees by using equations given by comodule structures (Section 4);
(c) Ringel duality for quasi-hereditary algebras (Section 5), using that the representations occurring in Section 5 are so-called tilting modules.

In Section 6, the Main Theorem is stated in full generality and its proof is completed by putting together the earlier results. In Sections 7 to 9 we collect some applications to decomposition numbers, to cohomology, to group algebras of symmetric groups and Hecke algebras and in particular to $p$-Kostka numbers. We also recover the isomorphisms constructed in [1], implying James' column removal rule. Moreover, we construct a Morita equivalence between centralizer subalgebras of generalized Schur algebras, which implies James' row removal rule.

The results of this article, which are characteristic independent, complement the characteristic dependent isomorphisms of centralizer subalgebras of (classical) Schur algebras constructed in [13] and also in [15].

The appendix, written by Stephen Donkin, provides an alternative approach to and a generalization of the Main Theorem.

2. Setup and notation

2.1. Notation. Throughout, let $k$ be an infinite field. For a non-zero natural number $n$ define $\underline{n} = \{1, \ldots, n\}$. Let $r$ be another non-zero natural number. We define sets of multi-indices of length $r$ with entries in $\underline{n}$ as follows:

\[
I(n, r) = \{ \underline{i} = (i_1, \ldots, i_r) \mid i_p \in \underline{n} \},
\]

\[
I^-(n, r) = \{ \underline{i} \in I(n, r) \mid i_1 > \ldots > i_r \}.
\]

Denote by $\Lambda^+(n, r)$ the set of partitions of $r$ with at most $n$ parts and by $P(n, r)$ the set of sequences of non-negative integers $(a_1, a_2, \ldots)$ such that $0 \leq a_i \leq n$ and $a_1 + a_2 + \ldots = r$. Let $P^+(n, r)$ be the subset of partitions in $P(n, r)$, and $P^+(n, r)_m$ the subset of $P^+(n, r)$ such that $a_{m+1} = a_{m+2} = \ldots = 0$. The ordering considered on both $\Lambda^+(n, r)$ and $P^+(n, r)$ is the dominance ordering, namely,

\[
\lambda \geq \mu \iff \lambda_1 + \ldots + \lambda_s \geq \mu_1 + \ldots + \mu_s
\]

for all natural numbers $s \geq 1$ and $|\lambda| = |\mu|$. Given a partition $\alpha \in P^+(n, r)$, we define the following sets of tableaux:

\[
Tab(\alpha) = \{ \alpha' - tableaux with entries in \underline{n} \},
\]

\[
Tab^+(\alpha) = \{ S \in Tab(\alpha) \mid entries are strictly increasing down columns \},
\]

\[
Tab^-(\alpha) = \{ S \in Tab(\alpha) \mid entries are strictly decreasing down columns \},
\]

\[
STab(\alpha) = \{ S \in Tab^-(\alpha) \mid entries are weakly decreasing along its rows \}.
\]

The set $STab$ is in one-to-one correspondence with the set of all semi-standard tableaux $S \in Tab(\alpha)$, that is by definition the set of all tableaux $S$ whose entries are strictly increasing down columns and weakly increasing along its rows. By abuse of notation, we call a tableau in $STab$ also semi-standard. Note that we are always using the conjugate partition $\alpha'$ in these definitions.
2.2. Schur algebras. Let $G = GL_n$ and denote by $k^G$ the set of all the $k$-valued functions on $G$. Define maps

$$\Delta : k^G \to k^{G \times G} \text{ and } \varepsilon : k^G \to k$$

by setting $\Delta(f)(g_1, g_2) = f(g_1 g_2)$ and $\varepsilon(f) = f(1)$ for any $f \in k^G$ and $g_1, g_2 \in G$. Here $1$ denotes the identity matrix of size $n \times n$. We will identify $k^G \otimes k^G$ with a subspace of $k^{G \times G}$ by

$$(f_1 \otimes f_2)(g, h) = f_1(g) f_2(h).$$

For each pair $(\mu, \nu) \in \mathbb{N} \times \mathbb{N}$ let $c_{\mu \nu}$ be the function which associates to each $g \in G$ its $(\mu, \nu)$-entry $g_{\mu \nu}$. Denote by $A_k(n)$ the $k$-subalgebra of $k^G$ generated by $c_{\mu \nu}$ $(\mu, \nu \in \mathbb{N})$. Since $k$ is infinite, this is precisely the polynomial ring over $k$ in $n^2$ indeterminates $c_{\mu \nu}$. Let $A_k(n, r)$ be the homogenous subspace in $A_k(n)$ of degree $r$ in the indeterminates $c_{\mu \nu}$. Then $A_k(n, r)$ as a $k$-vector space is spanned by

$$c_{\mu \nu} = c_{i_1 j_1} c_{i_2 j_2} \cdots c_{i_r j_r},$$

for all $i, j \in I(n, r)$. As in Green [9], we separate in this notation multi-indices $i, j$ by a comma, while we do not separate natural numbers $i_1 j_1$. The maps $\Delta$ and $\varepsilon$ defined above, when applied to the functions $c_{\mu \nu}$, satisfy the following equations:

$$\Delta(c_{i j}) = \sum_{k \in I(n, r)} c_{i k} \otimes c_{k j}, \quad \varepsilon(c_{i j}) = \delta_{i j}$$

where $\delta_{i j} = 1$ or 0, if $i = j$ or $i \neq j$. The vector space $A_k(n, r)$ forms a coalgebra with comultiplication $\Delta$ and counit $\varepsilon$. The (classical) Schur algebra, denoted by $S_k(n, r)$, is the dual of the coalgebra $A_k(n, r)$. In the following we will omit the subscript $k$ (to indicate the underlying field) on coalgebras and algebras.

Let $W$ be a representation of $G$. Suppose $W$ has basis $\{w_a\}$. Then we call $W$ a homogeneous polynomial representation of degree $r$ (in defining characteristic) if

$$g \cdot w_b = \sum_a r_{ab}(g) w_a$$

where all $r_{ab} \in A(n, r)$. By [9], homogeneous polynomial representations of degree $r$ are precisely the modules for the Schur algebra $S(n, r)$. Alternatively, they are precisely the comodules for the coalgebra $A(n, r)$. Note that the natural representation of $G$ is polynomial of degree one. Moreover, tensor products, symmetric powers and exterior powers of polynomial representations are again polynomial. More details about exterior powers are given in Section 2.2.3 where their structure maps as comodules are given.

In the quantized case, the Schur algebra is defined in a similar way. For details, the reader is refered to Donkin [7]. In this paper, groups and algebras are always considered over a field $k$, and the quantum parameter $q \in k$ always will be assumed to be non-zero; in particular this implies that $q$ is invertible.

2.3. Exterior powers. For non-zero natural numbers $n$ and $r$, denote by $S(n, r)$ either a classical Schur algebra as discussed by Green in [9] for the algebraic group $GL_n$ (see Section 2.2.3), or a $q$-Schur algebra as defined in Donkin’s book [7]. We assume that the quantum parameter $q$ is different from zero. Let $M$ be the classical or quantum
monoid which affords precisely the polynomial representations of classical or quantum $GL_n$ respectively. By $E$ or $V$ we denote the natural left or right $M$-module respectively – that means right or left $k[M]$-comodule; note the switch of sides of the actions when translating from $M$-modules to $k[M]$-comodules. Let $E$ have $k$-basis $e_1,\ldots,e_n$ and suppose $V$ has $k$-basis $v_1,\ldots,v_n$. Viewed as a comodule over the coordinate ring $k[M]$, then $E$ and $V$ have structure maps:

$$\tau_{1,E}(e_i) = \sum_{j=1}^{n} e_j \otimes c_{ji}, \quad \tau_{1,V}(v_i) = \sum_{j=1}^{n} c_{ij} \otimes v_j$$

where $1 \leq i \leq n$.

The exterior powers of $E$ and $V$ are again left and right $M$-modules. Using the notation for sets of multi-indices introduced in Section 2.1 as $q \neq 0$, the exterior power $\wedge^r V$ has $k$-basis $\{\hat{e}_i \mid i \in \Lambda^-(n,r)\}$ and $\wedge^r E$ has $k$-basis $\{\hat{e}_i \mid i \in \Lambda^-(n,r)\}$ where $\hat{e}_i = v_1 \wedge \ldots \wedge v_r$ and $\hat{e}_i = e_i \wedge \ldots \wedge e_i$. Note that $\wedge^r V = 0 = \wedge^r E$ if $r > n$. The structure maps of the exterior powers are given by (see [7], 1.3.1):

$$\tau_{r,E}(\hat{e}_i) = \sum_{j \in \Lambda^-(n,r)} \hat{e}_j \otimes (\hat{i} : \hat{j}), \quad \tau_{r,V}(\hat{v}_i) = \sum_{j \in \Lambda^-(n,r)} (\hat{i} : \hat{j}) \otimes \hat{v}_j,$$

with bideterminants

$$(1) \quad \langle \hat{i} : \hat{j} \rangle = \sum_{\pi \in \Sigma_r} (-q)^{l(\pi)} c_{i\pi,j}, \quad \langle \hat{i} : \hat{j} \rangle = \sum_{\pi \in \Sigma_r} (-1)^{l(\pi)} c_{i\pi,j}.$$  

Here $l(\pi)$ denotes the length of the permutation $\pi$. For any $\alpha \in P^+(n,r)$ and $S \in Tab(\alpha)$, define

$$\hat{e}_S = e_{S(1,1)} \wedge \ldots \wedge e_{S(1,1)} \otimes e_{S(1,2)} \wedge \ldots \wedge e_{S(\alpha_2,2)} \otimes \ldots,$$

$$\hat{v}_S = v_{S(1,1)} \wedge \ldots \wedge v_{S(1,1)} \otimes v_{S(1,2)} \wedge \ldots \wedge v_{S(\alpha_2,2)} \otimes \ldots.$$  

Here $S(i,j)$ denotes the entry in row $i$ and column $j$ of tableau $S$. Note that $\hat{v}_S = 0 = \hat{e}_S$ if $S$ has a repetition in the entries of one of its columns. The tensor product of the exterior powers $\Lambda^\alpha V$, denoted by $\Lambda^\alpha V$, has $k$-basis $\{\hat{v}_S \mid S \in Tab^{-}(\alpha)\}$, and $\Lambda^\alpha E$ has $k$-basis $\{\hat{e}_S \mid S \in Tab^{-}(\alpha)\}$. We denote by $\tau_{\alpha,E}$ and $\tau_{\alpha,V}$ the structure maps of $\Lambda^\alpha E$ and $\Lambda^\alpha V$ respectively; we write shortly $\tau_{\alpha}$ when no confusion arises about the vector space used. Define bideterminants

$$(S : T) = \prod_{i} (S_i : T_i) \quad \text{and} \quad \langle S : T \rangle = \prod_{i} \langle S_i : T_i \rangle.$$  

Here we consider the columns (read from top to bottom) of a tableau $S$ as multi-indices and write $S_i$ for the $i$-th column of the tableau $S$. Note that, compared to Lemma 1.3.1 in [7], there is a switch of rows and columns here; this is due to the different definition of the set $Tab^{-}$.

For later use, we next collect a few technical results; in particular, property (3) of the following Lemma can be taken as the definition of the quantum determinant.

**Lemma 2.1.** With the notation as above, the following statements hold true:
3. Combinatorics

3.1. Complements of Partitions. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ in $\Lambda^+(n,r)$ and a positive integer $m \geq \lambda_1$, we define the complement partition $T_m(\lambda)$ or $\hat{\lambda}$ of $\lambda$ with respect to $m$ as follows:

$$T_m(\lambda) = \hat{\lambda} = (m - \lambda_n, m - \lambda_{n-1}, \ldots, m - \lambda_1).$$

So the function $T_m$ takes the complement of $\lambda$ within an $n \times m$-rectangle. Note that it really depends from both $m$ and $n$. Since we only apply the function when $n$ is fixed, we write shortly $T_m$. When both $n$ and $m$ are understood, we write $\hat{\lambda}$. For example, let $\lambda = (4,2,1)$ in $\Lambda^+(4,7)$, so $n = 4$. Then $T_4(\lambda) = (4 - \lambda_4, 4 - \lambda_3, 4 - \lambda_2, 4 - \lambda_1) = (4,3,2,0)$ whereas $T_3(\lambda) = (5,4,3,1)$. Observe that for $m = 5 \geq \lambda_1 = 4$ the coloured partition in the picture is $T_5(\lambda)$; the remaining white boxes of the $n \times m$ rectangle are a skew-diagram $\iota(\lambda)$, obtained by reflecting $\lambda$ at its top left corner.

$$\begin{array}{c}
(5,4,3,1) = T_5(4,2,1) = T_m(\lambda) \longrightarrow \\
\iota(\lambda)
\end{array}$$

In later sections of this paper, we will also work with the conjugate situation: instead of taking a complement of a partition $\lambda \in \Lambda^+(n,r)$ with $\lambda_1 \leq m$ in an $n \times m$-rectangle, we take the complement of the conjugate partition $\lambda' = \alpha \in P^+(n,r)_m$ in an $m \times n$-rectangle; in the situations occuring, $m$ is fixed and we write $n - \alpha$ for the complement in the conjugate situation; in particular $\lambda = T_m(\lambda) = (n - \alpha)'$.

We collect some simple properties of forming the complement of a partition. The proof is left to the reader.

Lemma 3.1. For partitions $\lambda, \mu \in \Lambda^+(n,r)$ and a natural number $m \geq \lambda_1$ we have: $T_m^2(\lambda) = \lambda$, and $\lambda \geq \mu$ implies $T_m(\lambda) \geq T_m(\mu)$. 

Proof. The first two statements can be found in [7], Lemma 1.3.1, the third statement in [2], Theorem 4.1.7 (see also [7], Section 0.20). In case $\alpha = (n, \ldots, n)$, note that there is only one element, say $A$, in $\mathcal{A}^-(\alpha)$, namely the tableau whose columns are all filled decreasingly with $n$ to $1$. In this case $(A : A) = \prod_{i=1}^{m} (\mathbb{I} : \mathbb{I})$ with $\mathbb{I}$ as above. 

(1) Let $S \in Tab^-(\alpha)$, then the structure maps $\tau_{\alpha,E}$ and $\tau_{\alpha,V}$ are given by

$$\tau_{\alpha,E}(\hat{e}_S) = \sum_{T \in Tab^-(\alpha)} \hat{e}_T \otimes (T : S),$$

$$\tau_{\alpha,V}(\hat{v}_S) = \sum_{T \in Tab^-(\alpha)} (S : T) \otimes \hat{v}_T.$$ 

(2) For any $S,T \in Tab^-(\alpha)$ we have $(S : T) = (S : T)$.

(3) We have $\det = (\mathbb{I} : \mathbb{I}) = (\mathbb{I} : \mathbb{I})$, where $\mathbb{I} = (1,2,\ldots,n)$ and $\mathbb{I} = (n,\ldots,2,1)$. In particular, if $\alpha = (n,\ldots,n) \in \Lambda^+(n,m)$ then $\tau_\alpha$ is the map given by multiplication with $(\det)^m$. 

3.2. Complements of tableaux. If \( n \geq r \), then for any multi-index \( \hat{i} \) in \( I^-(n, r) \), we define the complement \( \hat{\hat{i}} \) of \( \hat{i} \) by \( \hat{\hat{i}} = (i_{r+1}, \ldots, i_n) \) with \( i_{r+1} > \cdots > i_n \) such that \( \hat{i} \cup \hat{\hat{i}} = n \) and \( \hat{i} \cap \hat{\hat{i}} = \emptyset \); here, by abuse of notation, we consider multi-indices \( \hat{i} = (j_1, \ldots, j_r) \) of some length \( r \) with pairwise different entries as sets with elements \( j_\rho \), for \( 1 \leq \rho \leq r \).

As before, we consider the columns, read from top to bottom, of a tableau \( S \) as multi-indices. If \( S_i \) is the \( i \)-th column of the tableau \( S \) and has no repeated entries, then \( \hat{S}_i \) is the complement of the multi-index \( S_i \), as defined above.

For a partition \( \lambda \) with \( m \geq \lambda_1 \), and a \( \lambda \)-tableau \( S \) with entries from the set \( \underline{n} \), whose entries are strictly decreasing down columns, we shall define the complement \( T_m(\lambda) \)-tableau \( T_m(\lambda) \) by

\[
T_m(\lambda)_i = \hat{S}_{m+1-i}.
\]

If \( m \) is fixed, we will write \( \hat{S} \) for \( T_m(\lambda) \). For example, let \( \lambda = (4, 2, 1) \) in \( \Lambda^+(4, 7) \) and \( m = 5 \) as before and let \( S \), filled with entries in \( \{1, \ldots, 4\} \), be given by

\[
\begin{array}{cccc}
4 & 3 & 3 & 2 \\
3 & 2 \\
1 \\
\end{array}
\]

Define the skew-tableau \( \iota(S) \) by reflecting the tableau \( S \) at its top left corner. Then \( T_5(S) \) is obtained by placing \( \iota(S) \) in the right-hand bottom corner of an \( n = 4 \times m = 5 \) box and by then filling the complement of each column into the empty boxes of that column, for example:

\[
T_5(S) =
\begin{array}{cccc}
4 & 4 & 4 & 4 \\
3 & 3 & 2 & 1 \\
2 & 1 & 1 \\
1 \\
\end{array}
\]

with \( \iota(S) =
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
\end{array}
\)

\[\text{Theorem 3.2. For any } \lambda \in \Lambda^+(n, r) \text{ and integer } m \geq \lambda_1, \text{ } S \text{ is a semi-standard } \lambda \text{-tableau if and only if } T_m(S) \text{ is a semi-standard } T_m(\lambda) \text{-tableau.}\]

\[\text{Proof. Without loss of generality, it is enough to prove the claim in the case where } \lambda \text{ has only two columns and } m = 2. \text{ We need a partial ordering defined on the set of all multi-indices. Namely, for } \hat{i} = (i_1, \ldots, i_v) \in I(n, v) \text{ and } \hat{j} = (j_1, \ldots, j_w) \in I(n, w), \text{ write } \hat{j} \preceq \hat{i} \text{ if } w \leq v \text{ and } j_1 \leq i_1, \ldots, j_w \leq i_w.\]

Let \( \lambda \) be a two-column partition in \( \Lambda^+(n, r) \). Consider a semi-standard tableau \( S \) of shape \( \lambda \) with columns \( S_1 \) and \( S_2 \), filled by entries in \( \underline{n} \). Since \( S \) is semi-standard, both its columns can be considered as multi-indices, say

\[
S_1 = \hat{i} = (i_1, \ldots, i_v) \quad \text{and} \quad S_2 = \hat{j} = (j_1, \ldots, j_w)
\]

where \( \hat{i} \) and \( \hat{j} \) are ordered by size and each have entries in \( \underline{n} \) without repetitions. By the definition in Section 2.1, \( S \in STab \) means that entries in rows of \( S \) are weakly decreasing and entries in columns of \( S \) are strictly decreasing. Since \( S \) is semi-standard, then \( w \leq v \) and by definition \( \hat{j} \preceq \hat{i} \).

Let \( \hat{a} \) and \( \hat{b} \) be any multi-indices and define the operation \( P_{i,j} \) on the multi-indices \( \hat{a} \) and \( \hat{b} \) for \( i, j \in \{0, 1, \ldots, n\} \) as follows: \( P_{i,j} \) bumps out \( i \) from \( \hat{a} \) and \( j \) from \( \hat{b} \). If \( i \notin \hat{a} \)
(resp. \( j \notin b \), then \( P_{i,j} \) does nothing to \( a \) (resp. \( b \)). For example, if \( i = a_s \) and \( j = b_t \), then the \( \rho \)th entries of the multi-indices \( P_{i,j}a \) and \( P_{i,j}b \) are

\[
[P_{i,j}a]_\rho = \begin{cases} 
  a_\rho & \rho < s, \\
  a_{\rho+1} & \rho \geq s,
\end{cases} 
[P_{i,j}b]_\rho = \begin{cases} 
  b_\rho & \rho < t, \\
  b_{\rho+1} & \rho \geq t.
\end{cases}
\]

The length of the multi-indices \( P_{i,j}a \) and \( P_{i,j}b \) are now both one shorter than that of \( a \) and \( b \) respectively.

\[
S = \begin{array}{|c|}
\hline
i \\
\hline
\end{array} \quad \text{and} \quad T_2(S) = \begin{array}{|c|}
\hline
\hat{i} \\
\hline
\end{array}
\]

We now fix \( a = (n, \ldots, 1) = b \) as multi-indices containing the numbers \( 1, \ldots, n \) in strictly decreasing order. With the operations \( P_{i,j} \) we can write \( \hat{i} \) and \( \hat{j} \) as follows:

\[
(2) \quad \hat{i} = P_{w,0} \cdots P_{w+1,0} \cdot P_{w,jw} \cdots P_{t_1,j_1}(a),
(3) \quad \hat{j} = P_{w,jw} \cdots P_{t_1,j_1}(b).
\]

Note that the two columns of \( \hat{S} = T_2(S) \) are exactly \( \hat{j} \) and \( \hat{i} \). So to prove \( T_2(S) \) is semi-standard, we only need to show that \( \hat{i} \preceq \hat{j} \). We do this by induction on the number of the above operations \( P_{i,j} \) that we take. More precisely, we start with the multi-indices \( a = (n, \ldots, 1) = b \), which certainly satisfy \( a \preceq b \). Then we successively apply operations \( P_{i,j} \) with pairs of indices \((i, j)\) where \( i \) and \( j \) are from the sets \( \hat{i} \) and \( \hat{j} \cup \{0\} \) respectively. In all \( P_{i,j} \) used in the above Equations \((2) \) and \((3) \), we always have \( i \geq j \). And we never repeat any index \( i \) or \( j \) except possibly \( j = 0 \). Suppose after \( \rho \leq w \) such operations, we get multi-indices \( \hat{k} \) and \( \hat{l} \) with \( \hat{k} \preceq \hat{l} \). Then for \( p \in \hat{k}, q \in \hat{j} \) with \( p \geq q \), we must have \( p = k_s, q = l_t \) with \( s \leq t \). Therefore,

\[
[P_{p,q}(\hat{k})]_\rho = [\hat{k}]_\rho, \quad 1 \leq \rho \leq s - 1, \\
[P_{p,q}(\hat{l})]_\rho = [\hat{l}]_\rho, \quad 1 \leq \rho \leq t - 1.
\]

Hence,

\[
[P_{p,q}(\hat{k})]_\rho \leq [P_{p,q}(\hat{l})]_\rho, \quad 1 \leq \rho \leq s - 1, \\
[P_{p,q}(\hat{k})]_\rho = [\hat{k}]_{\rho+1} \leq [\hat{l}]_{\rho+1} = [P_{p,q}(\hat{l})]_\rho, \quad s \leq \rho \leq t - 1, \\
[P_{p,q}(\hat{k})]_\rho = [\hat{k}]_{\rho+1} \leq [\hat{l}]_{\rho+1} = [P_{p,q}(\hat{l})]_\rho, \quad t \leq \rho \leq n.
\]

Hence \( P_{p,q}(\hat{k}) \preceq P_{p,q}(\hat{l}) \), which is also true in case \( q = 0 \). Inductively,

\[
P_{w,jw} \cdots P_{t_1,j_1}(a) \preceq P_{w,jw} \cdots P_{t_1,j_1}(b)
\]

which implies immediately that

\[
\hat{j} = P_{w,0} \cdots P_{w+1,0} \cdot P_{w,jw} \cdots P_{t_1,j_1}(a) \preceq P_{w,jw} \cdots P_{t_1,j_1}(b) = \hat{i}
\]

Since the property of being semi-standard is determined by comparing any two neighbouring columns, the proof generalises to any semi-standard tableau \( S \) of shape \( \lambda \in \Lambda^+(n, r) \) which is filled by entries in \( n \). \( \square \)
3.3. Signs of tableaux. Using earlier notations, we finally introduce the sign of a tableau, needed for the explicit construction of the isomorphisms between quantized Schur algebras in the next section. In Section 3.2, we defined for \( n \geq r \) and any multi-index \( \underline{i} \in I^{-}(n,r) \) its complement \( \underline{j} \in I^{-}(n,n-r) \) to be the complement of \( \underline{i} \) in \( \underline{n} \). Let \( \omega = v_{n} \wedge \ldots \wedge v_{1} \), a generator of \( \wedge^{n} V \), and define the sign of \( \underline{i} \in I^{-}(n,r) \) by the equation
\[
v_{\underline{i}} \wedge v_{\underline{j}} = \text{sgn}(\underline{i})\omega.
\]
We generalize the definition of a sign for a multi-index to that of a sign of a tableau \( S \in \text{Tab}^{-}(\alpha) \) with \( m \) columns \( S_{1}, \ldots, S_{m} \) as follows:
\[
\text{sgn}(S) = \text{sgn}(S_{1}) \cdot \text{sgn}(S_{2}) \cdots \text{sgn}(S_{m})
\]
Moreover, we define
\[
|\underline{i}| = i_{1} + \ldots + i_{r}
\]
and
\[
\epsilon(S) = (m-1)|\widehat{S}_{1}| + (m-2)|\widehat{S}_{2}| + \ldots + 2|\widehat{S}_{m-2}| + |\widehat{S}_{m-1}|
\]
where \( \widehat{S}_{i} \) is the complement of the multi-index \( S_{i} \).

4. Homomorphisms between exterior powers

Fix a natural number \( m \). Then for any \( \alpha = (\alpha_{1}, \ldots, \alpha_{m}) \in \mathbb{P}^{+}(n,r)_{m} \), we denote by \( n - \alpha \) the partition \( (n - \alpha_{m}, \ldots, n - \alpha_{1}) \). Note that this depends on the choice of \( m \), compare with \( T_{m} \) in Section 3. In this section, we compare morphism spaces between tensor products of exterior powers \( \wedge^{i} \) and \( \wedge^{n-i} \) respectively. The main result identifies such two morphism spaces in different degrees. We will use the notation for exterior powers introduced in Section 2.3. In particular, \( V \) is an \( n \)-dimensional vector space. Then we have for \( 1 \leq r \leq n \) the following result – which is a special case of [7], Lemma 1.3.3:

**Lemma 4.1.** The linear map \( \mu = \mu_{r} : \wedge^{n} V \otimes \wedge^{r} V \longrightarrow \wedge^{r} V \otimes \wedge^{n} V \) defined by
\[
\mu(\omega \otimes \check{v}_{\underline{i}}) = q^{-|\underline{i}|} \check{v}_{\underline{i}} \otimes \omega
\]
for \( \underline{i} \in I^{-}(n,r) \) is an \( M \)-module homomorphism.

**Proof.** By [2] Theorem 4.1.9 or [7] Section 0.20, we have \( q^{i} \cdot c_{ij} \cdot \det = q^{i} \cdot \det \cdot c_{ij} \) for \( 1 \leq i, j \leq n \), where \( \det = (l : l) \) with \( l = (n, \ldots, 2, 1) \), see Lemma 2.1. Next use the definition of bideterminants \( (\underline{i} : \underline{j}) \) given in Section 2.3 Equation (1) and sum over all \( \underline{j} \in I^{-}(n,r) \) to obtain:
\[
q^{-|\underline{i}|} \sum_{\underline{j} \in I^{-}(n,r)} (\underline{i} : \underline{j}) \cdot \det = \sum_{\underline{j} \in I^{-}(n,r)} q^{-|\underline{j}|} \cdot \det \cdot (\underline{j} : \underline{i})
\]
for any \( \underline{i} \in I^{-}(n,r) \). Let \( \tau_{n,r} \) and \( \tau_{r,n} \) be the structure maps of the comodules \( \wedge^{n} V \otimes \wedge^{r} V \) and \( \wedge^{r} V \otimes \wedge^{n} V \) respectively. Applying these structure maps to an element \( \omega \otimes \check{v}_{\underline{i}} \), we get as coefficients of an element \( \check{v}_{\underline{j}} \otimes \omega \) precisely the left-hand and right-hand side of Equation (1). Then Equation (1) implies that
\[
((id \otimes \mu_{r}) \circ \tau_{n,r})(\omega \otimes \check{v}_{\underline{i}}) = (\tau_{r,n} \circ \mu_{r})(\omega \otimes \check{v}_{\underline{i}})
\]
where \( \underline{i} \in I^{-}(n,r) \). So \( \phi \) is a \( k[M] \)-comodule homomorphism. \( \square \)
Lemma 4.2. Fix a natural number \( m \) and let \( \alpha \in P^+(n,r)_m \). Then the linear map 
\[
\phi_\alpha : \wedge^\alpha V \otimes \wedge^{n-\alpha} V \longrightarrow (\wedge^n V)^{\otimes m}
\]
defined by 
\[
\phi_\alpha(\hat{v}_S \otimes \hat{v}_T) = \text{sgn}(S)\delta_{S,T}q^{-\epsilon(S)}\omega^{\otimes m}
\]
for any \( S,T \in \text{Tab}^-(\alpha) \) is an \( M \)-module homomorphism.

Proof. To show the claim, we proceed by induction on \( m \). The multiplication map 
\[
\wedge^\alpha V \otimes \wedge^{n-\alpha} V \longrightarrow \wedge^{s+t} V
\]
is an \( M \)-module homomorphism. If \( m = 1 \), then \( \phi_\alpha \) is the 
multiplication map \( \wedge^{\alpha_1} V \otimes \wedge^{n-\alpha_1} V \longrightarrow \wedge^n V \) which, by definition, maps 
\[
\hat{v}_i \otimes \hat{v}_j \mapsto \delta_{ij} \text{sgn}(i)\omega,
\]
and here \( S = i \) and \( T = j \) are elements in \( \text{Tab}^-(\alpha) = I^-(n,\alpha_1) \).

Now suppose \( m > 1 \). Given a partition \( \alpha = (\alpha_1, \ldots, \alpha_m) \in P^+(n,r)_m \), assume that 
for \( (\alpha_2, \ldots, \alpha_m) \in P^+(n,r-\alpha_1)_{m-1} \) the induction assumption holds. Split any tableau
\( S \in \text{Tab}^-(\alpha) \) into the first column \( S_1 \) and a tableau \( S_e \in P^+(n,r-\alpha_1)_{m-1} \) with columns
\( S_2, \ldots, S_m \). Then \( \epsilon(S) = (m - 1)|S_1| + \epsilon(S_e) \), and \( \phi_\alpha \) is the composite map
\[
\wedge^\alpha V \otimes \wedge^{n-\alpha} V \longrightarrow \wedge^{\alpha_1} V \otimes (\wedge^{\alpha_2} V \otimes \cdots \otimes \wedge^{\alpha_m} V) \otimes \wedge^{n-\alpha_1} V \\
\longrightarrow \wedge^{\alpha_1} V \otimes (\wedge^n V)^{\otimes m-1} \otimes \wedge^{n-\alpha_1} V \\
\longrightarrow \wedge^{\alpha_1} V \otimes \wedge^{n-\alpha_1} V \otimes (\wedge^n V)^{\otimes m-1} \\
\longrightarrow (\wedge^n V)^{\otimes m}.
\]
In fact, if \( S \neq T \), then both \( \phi_\alpha \) and the composite map return zero. Suppose next that 
\( S = T \). By induction and Lemma 4.1, the map \( \phi_\alpha \) is an \( M \)-module homomorphism. \( \square \)

Theorem 4.3. Fix a natural number \( m \) and let \( \alpha, \beta \) be partitions in \( P^+(n,r)_m \).
There is a vector space isomorphism 
\[
\theta_\alpha : \wedge^\alpha V \longrightarrow \wedge^{n-\alpha} E
\]
defined by 
\[
\theta_\alpha(\hat{v}_S) = \text{sgn}(S)q^{-\epsilon(S)}\hat{e}_S
\]
for \( S \in \text{Tab}^-(\alpha) \). Sending \( \varphi \) to \( \theta_\beta \circ \varphi \circ \theta_\alpha^{-1} \) defines an isomorphism
\[
(5) \quad \text{Hom}_{S(n,r)}(\wedge^\alpha V, \wedge^\beta V) \simeq \text{Hom}_{S(n, nm-r)}(\wedge^{n-\alpha} E, \wedge^{n-\beta} E).
\]

In other words, the following commutative diagram provides a bijection between \( M \)-morphisms \( \varphi \) in degree \( r \) and \( M \)-morphisms \( \psi \) in degree \( nm - r \).

\[
\begin{array}{ccc}
\wedge^\alpha V & \longrightarrow & \wedge^{n-\alpha} E \\
\varphi \downarrow & & \downarrow \psi := \theta_\beta \circ \varphi \circ \theta_\alpha^{-1} \\
\wedge^\beta V & \longrightarrow & \wedge^{n-\beta} E \\
\theta_\beta \downarrow & & \theta_\alpha \downarrow \\
\end{array}
\]

Proof. The map \( \theta_\alpha \) is an isomorphism of vector spaces; this is immediate from the 
above description of bases for \( \wedge^\alpha V \) and \( \wedge^{n-\alpha} E \). Clearly, the inverse of the map \( \varphi \mapsto \psi \) is
given by \( \psi \mapsto \theta_\beta^{-1} \circ \psi \circ \theta_\alpha \). So to prove the main statement, that is Equation (5), it
suffices to verify that \( \psi := \theta_\beta \circ \varphi \circ \theta_\alpha^{-1} \) is an \( M \)-module homomorphism if \( \varphi \) is so. As
usual, we are viewing \( M \)-modules as \( k[M] \)-comodules.
Notation: Throughout we will be using that indexing over a set of tableaux is equivalent to indexing over the set of complement tableaux, as defined in Section 2. Suppose that

\[ \text{Tab}^-(\alpha) = \{ A_1, \ldots, A_a \} \] and \[ \text{Tab}^- (\beta) = \{ B_1, \ldots, B_b \}. \]

Write \( A \) for the \( a \times a \) matrix with \((i,j)^{th}\) entry \((A_i: A_j)\), and \( B \) for the \( b \times b \) matrix with \((i,j)^{th}\) entry \((B_i: B_j)\). Moreover, define the following matrices:

\[
\begin{align*}
\text{sgn}(A) &= \text{diag}(\text{sgn}(A_1), \ldots, \text{sgn}(A_a)), \\
\text{sgn}(B) &= \text{diag}(\text{sgn}(B_1), \ldots, \text{sgn}(B_b)), \\
q^A &= \text{diag}(q^{(A_1)}, \ldots, q^{(A_a)}), \\
q^B &= \text{diag}(q^{(B_1)}, \ldots, q^{(B_b)}).
\end{align*}
\]

Here \( \varepsilon(-) \) is defined as in Section 3.3. Since \( q \neq 0 \), the matrices \( q^A \) and \( q^B \) have inverses, denoted by \( q^{-A} \) and \( q^{-B} \) respectively. Note that

\[ \text{Tab}^-(n-\alpha) = \{ \hat{A}_1, \ldots, \hat{A}_a \} \] and \[ \text{Tab}^-(n-\beta) = \{ \hat{B}_1, \ldots, \hat{B}_b \}. \]

Thus we can define matrices \( \hat{A}, \hat{B}, \text{sgn}(\hat{A}) \) etc. for partitions \( n-\alpha \) and \( n-\beta \) in the same way as we defined matrices \( A, B, \text{sgn}(A) \) etc. for \( \alpha \) and \( \beta \).

**Step 1:** Suppose \( \varphi \) is a \( M \)-module homomorphism \( \wedge^\alpha V \to \wedge^\beta V \) given by

\[
\varphi(\hat{v}_{A_i}) = \sum_{B_j \in \text{Tab}^-(\beta)} x_{A_iB_j} \hat{v}_{B_j},
\]

for \( A_i \in \text{Tab}^-(\alpha) \). Then by definition, \( \psi \) is given by

\[
\psi(e_{\hat{A_i}}) = \sum_{B_j \in \text{Tab}^-(\beta)} \text{sgn}(A_i)q^{(A_i)}x_{A_iB_j} \text{sgn}(B_j)q^{-\epsilon(B_j)} e_{\hat{B}_j}.
\]

Let \( X \) be the \( a \times b \) matrix with \((i,j)^{th}\) entry \( x_{A_iB_j} \) given by Equation (6).

**Step 2:** By definition, the map \( \psi \) is a \( M \)-module homomorphism if and only if \((\tau_{n-\beta} \circ \psi) = ((\psi \otimes 1) \circ \tau_{n-\alpha})\); here \( \tau_{n-\alpha} \) and \( \tau_{n-\beta} \) denote the comodule structure maps of \( \wedge^{n-\alpha} E \) and \( \wedge^{n-\beta} E \) respectively, see Section 2.3. In the following we spell out this latter equation. Let \( e_{\hat{A}_i} \) be a basis element in \( \wedge^{n-\alpha} E \). By Equation (7) and Lemma 2.1 we get

\[
\sum_{j=1}^b \sum_{l=1}^b \text{sgn}(A_i)q^{(A_i)}x_{A_iB_l} \text{sgn}(B_l)q^{-\epsilon(B_l)} e_{\hat{B}_l} \otimes (\hat{B}_l : \hat{B}_l)
\]

\[
= (\tau_{n-\beta} \circ \psi)(e_{\hat{A}_i})
\]

\[
= ((\psi \otimes 1) \circ \tau_{n-\alpha})(e_{\hat{A}_i})
\]

\[
= \sum_{j=1}^b \sum_{l=1}^a \text{sgn}(A_u)q^{(A_u)}x_{A_uB_l} \text{sgn}(B_l)q^{-\epsilon(B_l)} e_{\hat{B}_l} \otimes (\hat{A}_u : \hat{A}_u).
\]

This equation holds precisely if equality holds for the coefficients of each basis element \( e_{\hat{B}_j} \). Hence the map \( \psi \) is an \( M \)-module homomorphism if and only if the following
matrix identity holds:

\[
\text{sgn}(A)q^AX\text{sgn}(B)q^{-B} \hat{B}^{tr} = \left( \sum_{t=1}^{b} \text{sgn}(A_i)q^{\ell(A_i)}x_{A_iB_t}\text{sgn}(B_t)q^{-\epsilon(B_i)}\langle \hat{B}_j : \hat{B}_i \rangle \right)_{i=1...a, j=1...b} \\
= \left( \sum_{u=1}^{a} \text{sgn}(A_u)q^{\ell(A_u)}x_{A_uB_j}\text{sgn}(B_j)q^{-\epsilon(B_j)}\langle \hat{A}_u : \hat{A}_i \rangle \right)_{i=1...a, j=1...b} \\
= \hat{A}^{tr}\text{sgn}(A)q^AX\text{sgn}(B)q^{-B}
\]

Note that the matrix \(X\) in the middle of the expressions determines what are rows and what are columns; thus the matrices \(\hat{A}\) and \(\hat{B}\) have to be transposed. Multiplying this equation from the left by \(\text{sgn}(A)\) and on the right by \(\text{sgn}(B)\) gives the equivalent equation

\[
(8) \quad \Pi := q^AX\text{sgn}(B)q^{-B} \hat{B}^{tr} \text{sgn}(B) = \text{sgn}(A)\hat{A}^{tr}\text{sgn}(A)q^AXq^{-B} =: \Gamma.
\]

Hence the map \(\psi\) is an \(M\)-module homomorphism if and only if \(\Pi = \Gamma\).

**Step 3:** The map \(\varphi\) given by Equation (6) is an \(M\)-module homomorphism by assumption. Hence, by definition, \(\tau_\beta \circ \varphi = (id \otimes \varphi) \circ \tau_\alpha\). Let \(\hat{v}_{A_i}\) be a basis element of \(\wedge^\alpha V\). By Lemma 2.1 and Equation (6) we have

\[
\sum_{t=1}^{b} \sum_{j=1}^{b} x_{A_iB_t}(B_t : B_j) \otimes \hat{v}_{B_j} = (\tau_\beta \circ \varphi)(\hat{v}_{A_i}) \\
= ((id \otimes \varphi) \circ \tau_\alpha)(\hat{v}_{A_i}) \\
= \sum_{u=1}^{a} \sum_{j=1}^{b} (A_i : A_u)x_{A_uB_j} \otimes \hat{v}_{B_j}.
\]

Comparing the coefficients of the basis elements \(\hat{v}_{B_j}\), we get that the following matrix equation is satisfied:

\[
(9) \quad XB = \left( \sum_{t=1}^{b} x_{A_iB_t}(B_t : B_j) \right)_{i,j} = \left( \sum_{u=1}^{a} (A_i : A_u)x_{A_uB_j} \right)_{i,j} = AX.
\]

**Step 4:** On \((\wedge^n V)^{\otimes m}\), the structure map \(\tau_{n^{\otimes m}} = \tau_{(n,...,n)}\) is multiplication by \((det)^m\), see Lemma 2.1(3). We use the map \(\phi_\alpha\) defined in Lemma 4.2. Let \(\hat{v}_{A_i} \otimes \hat{\wedge}_j\) be a basis element in \(\wedge^\alpha V \otimes \wedge^{n-\alpha} V\). Then, since \(\phi_\alpha\) is an \(M\)-module homomorphism, and using
Lemma 2.1

\[ \sum_{s=1}^{a} (A_i : A_s)(\hat{A}_j : \hat{A}_s) \sgn(A_s)q^{-\epsilon(A_s)} \omega^{\otimes m} \]
\[ = \sum_{s=1}^{a} \sum_{t=1}^{a} (A_i : A_s)(\hat{A}_j : \hat{A}_t) \delta_{A_s,A_t} \sgn(A_s)q^{-\epsilon(A_s)} \omega^{\otimes m} \]
\[ = ((id \otimes \phi_\alpha) \circ (\tau_\alpha \otimes \tau_{n-\alpha}))(\hat{\upsilon}_{A_i} \otimes \hat{\upsilon}_{A_j}) \]
\[ = (\tau_{\alpha} \otimes \phi_\alpha)(\hat{\upsilon}_{A_i} \otimes \hat{\upsilon}_{A_j}) \]
\[ = \delta_{A_i,A_j} \sgn(A_i)q^{-\epsilon(A_i)}(det)^m \omega^{\otimes m}. \]

Comparing the coefficients of $\omega^{\otimes m}$ in this last equation, we obtain the matrix equation:

\[ A \sgn(A)q^{-A} \hat{A}^\text{tr} = \left( \sum_{s=1}^{a} (A_i : A_s)(\hat{A}_j : \hat{A}_s) \sgn(A_s)q^{-\epsilon(A_s)} \right)_{i,j} \]
\[ = \left( \delta_{A_i,A_j} \sgn(A_i)q^{-\epsilon(A_i)}(det)^m \right)_{i,j} \]
\[ = \sgn(A)q^{-A}(det)^m I. \]

Since the quantum parameter $q$ is non-zero, the matrix on the right-hand side of the last equation is invertible in $k[M][\det^{-1}]$. This implies that the matrix $A$ is right invertible and the matrix $\hat{A}$ is left invertible in $k[M][\det^{-1}]$. Reversing the roles of $A$ and $\hat{A}$ and using that $\hat{A} = A$, we get that $A$ is invertible on either side in $k[M][\det^{-1}]$. We next multiply this equation from the left by $q^A$ and from the right by $\sgn(A)$, obtaining:

\begin{align*}
q^A A \sgn(A)q^{-A} \hat{A}^\text{tr} \sgn(A) & = (det)^m I, \quad \text{(10)} \\
q^B B \sgn(B)q^{-B} \hat{B}^\text{tr} \sgn(B) & = (det)^m I, \quad \text{(11)}
\end{align*}

where the second equation is obtained similarly by using the map $\phi_B$ instead of $\phi_\alpha$.

**Step 5:** Our aim is to prove that $\psi$ is an $M$-module homomorphism. By Step 2 it is enough to show that Equation (5) holds, that is $\Gamma = \Pi$. In order to show the latter, we next combine the matrix equalities obtained in previous steps:

\[ q^A A q^{-A} \Gamma = [q^A A q^{-A} \sgn(A) \hat{A}^\text{tr} \sgn(A)] q^A X q^{-B} \]
\[ = (det)^m q^A X q^{-B} \quad \text{(using Equation (10))} \]
\[ = q^A X q^{-B} (det)^m \]
\[ = q^A X q^{-B} q^B B q^{-B} \sgn(B) \hat{B}^\text{tr} \sgn(B) \quad \text{(using Equation (11))} \]
\[ = q^A A q^{-A} [q^A X \sgn(B)q^{-B} \hat{B}^\text{tr} \sgn(B)] \quad \text{(using Equation (9))} \]
\[ = q^A A q^{-A} \Pi. \]

Hence $0 = q^A A q^{-A}(\Gamma - \Pi)$, and since $A$ is invertible by Step 4, this implies the validity of Equation (5) in $k[M][\det^{-1}]$, and hence in the subcoalgebra $k[M]$. \qed
An obvious, but crucial, corollary of Theorem 4.3 provides isomorphisms of rings. Here, as always we assume \( q \neq 0 \); hence we do not have to distinguish between \( V \) and \( E \), which are related by an \( M \)-pairing, see [7].

**Corollary 4.4.** For any subset \( X \subset P^+(n,r)_m \), there is an isomorphism of algebras:

\[
\text{End}_{S(n,r)} \left( \bigoplus_{\alpha \in X} \Lambda^\alpha V \right) \simeq \text{End}_{S(n,nm-r)} \left( \bigoplus_{\alpha \in X} \Lambda^{n-\alpha} V \right).
\]

**Remark:** This corollary implies equivalences of categories, as stated in the introduction. These equivalences may be obtained more directly. The category of all representations of the (algebraic or quantised) group \( G \) has an autoequivalence induced by inverting group elements (that is, by the Hopf algebra antipode) and transposing matrices. Other autoequivalences can be obtained by tensoring with powers of the determinant. Appropriately composing these two kinds of equivalences and restricting to homogeneous polynomial representations produces the equivalences implied by the Corollary. Adding combinatorial arguments, as developed in Section 3, would then prove the existence of isomorphisms as in the previous Corollary, without explicitly constructing them.

We also note that the set \( X \) in the Corollary can be chosen arbitrarily. We will only use a special choice of \( X \), which is compatible with the quasi-hereditary structure.

## 5. Ringel Duality

### 5.1. General results on Ringel duality.

Tensor products of exterior powers, the representations used in the previous section, have a special structural property: they are direct summands of the characteristic tilting module of the Schur algebra in question. The present section uses this property, and known results on tilting modules, to turn Corollary 4.4 into a statement on Schur algebras. Let us recall some basics on quasi-hereditary algebras and Ringel duality. A convenient reference in our context is the appendix of [7].

**Definition 5.1.** Let \( \{ L(\lambda) : \lambda \in \Lambda \} \) be a complete set of simple modules of a finite dimensional algebra \( A \), \( P(\lambda) \) the projective cover of \( L(\lambda) \) and let \( (\Lambda, \leq) \) be a partial ordering on \( \Lambda \). Then \( (A, \Lambda, \leq) \) is called a quasi-hereditary algebra if for each \( \lambda \in \Lambda \), there exists a quotient module \( \Delta(\lambda) \) of \( P(\lambda) \), called standard module, such that

1. The kernel of the canonical map \( P(\lambda) \to \Delta(\lambda) \) is filtered by \( \Delta(\mu) \) with \( \mu > \lambda \);
2. The kernel of the canonical map \( \Delta(\lambda) \to L(\lambda) \) is filtered by \( L(\mu) \) with \( \mu < \lambda \).

An equivalent ring-theoretic definition for quasi-hereditary algebras implies that \( A \) is quasi-hereditary if and only if \( A^{\text{op}} \) is so. If we denote the standard module of \( A^{\text{op}} \) by \( \Delta_A^{\text{op}}(\lambda) \), then \( \nabla(\lambda) = \nabla_A(\lambda) = \Delta_A^{\text{op}}(\lambda)^* \) is called a costandard module of \( (A, \leq) \). Here \( * \) denotes the usual vector space dual.

Let \( \mathcal{F}(\Delta) \) be the full subcategory of \( A\text{-mod} \) consisting of modules filtered by \( \Delta \)'s and let \( \mathcal{F}(\nabla) \) be the full subcategory of \( A\text{-mod} \) consisting of modules filtered by \( \nabla \)'s. Ringel
[20] proved that for each $\lambda \in \Lambda$ there is an indecomposable module $T(\lambda)$ such that 
$$\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) = \text{add}(\oplus_{\lambda \in \Lambda} T(\lambda)).$$

The module $T = \oplus_{\lambda \in \Lambda} T(\lambda)$ is called the characteristic tilting module of $(A, \leq)$. The endomorphism ring $R = \text{End}_A(T)$, and any algebra Morita equivalent to it, is called Ringel dual of $A$. Ringel has shown that $R$ is again a quasi-hereditary algebra. Moreover, the Ringel dual of $R$ is Morita equivalent to $A$ (see [20]).

We need one more definition in this context: A subset $Y$ of $\Lambda$ is called saturated if $y_2 < y_1$ and $y_1 \in Y$ implies $y_2 \in Y$. The following statements are well-known and easy to prove.

**Lemma 5.2.** Let $(A, \Lambda, \leq)$ be a quasi-hereditary algebra and let $T = \oplus_{\lambda \in \Lambda} T(\lambda)$ be its characteristic tilting module. For any saturated subset $Y$ of $\Lambda$,

1. the primitive idempotents $e_{\lambda}$ with $\lambda \notin Y$ generate an ideal $J$ in the heredity chain of $A$;
2. the module $T(Y) = \oplus_{\lambda \in Y} T(\lambda)$ is exactly the characteristic tilting module of the quasi-hereditary algebra $A/J$;
3. the endomorphism ring $\text{End}_A(T(Y)) = \text{End}_{A/J}(T(Y))$ is of the form $eRe$, where $R$ is the Ringel dual of $A$ and $e$ is an idempotent in $R$.

The (classical or quantized) Schur algebras $S(n, r)$ are quasi-hereditary algebras with respect to $\Lambda = \Lambda^+(n, r)$ and the dominance ordering $\leq$.

**Theorem 5.3** (Donkin [6], [7], Section 3.3). The indecomposable tilting modules for the Schur algebra $S(n, r)$ are precisely the direct summands of the modules $\wedge^\alpha E$, for $\alpha \in \Lambda^+(n, r)$. The indecomposable tilting module $T(\alpha)$ occurs exactly once as a direct summand of $\wedge^\alpha E$, and if $T(\lambda)$ is a direct summand of $\wedge^\alpha E$ then $\lambda \leq \alpha$.

5.2. **Morita equivalences between generalised Schur algebras.** Given a Schur algebra $S(n, r)$ and a saturated set $\Pi \subset \Lambda^+(n, r)$, define $I_\Pi$ to be the ideal in the heredity chain of $S(n, r)$, generated by the primitive idempotents $e_{\alpha}$ with $\alpha \notin \Pi$. Moreover, after fixing a natural number $m$ such that the partitions in $\Pi$ have rows of length not more than $m$, define $\hat{\Pi}$ as the set with elements $\hat{\alpha} := T_m(\alpha)$ for $\alpha \in \Pi$. Note that $\hat{\Pi}$ is again saturated. Combining Corollary 4.3 with Lemma 5.2 and Theorem 5.3 we can show:

**Theorem 5.4.** Fix a natural number $m$. For any natural numbers $n$ and $r$ and any saturated set $\Pi \subset \Lambda^+(n, r)$, such that the partitions in $\Pi$ have rows of length not more than $m$, there is a Morita equivalence:

$$S(n, r)/I_\Pi \simeq S(n, nm - r)/I_{\hat{\Pi}}.$$

**Proof.** Consider the following modules over $S(n, r)$ and $S(n, nm - r)$ respectively:

$$T = \bigoplus_{\alpha \in \Pi} \wedge^\alpha E \oplus \bigoplus_{\beta \notin \Pi} T(\beta),$$

$$\hat{T} = \bigoplus_{\alpha \in \Pi} \wedge^{(n - \alpha')} E \oplus \bigoplus_{\beta \notin \hat{\Pi}} T(\beta).$$
which are full tilting modules by Theorem 5.3. Note that the exterior powers here are defined using conjugate partitions $\alpha'$ for $\alpha \in \Pi$. Let

$$A = \text{End}_{S(n,r)}(T)$$

and

$$B = \text{End}_{S(n,nm-r)}(\tilde{T})$$

be the Ringel duals of $S(n,r)$ and $S(n,nm-r)$ respectively. By Corollary 4.4, we have

$$e_Ae = \text{End}_{S(n,r)}(\bigoplus_{\alpha \in \Pi} \wedge^{\alpha'} E) \simeq \text{End}_{S(n,nm-r)}(\bigoplus_{\alpha \in \Pi} \wedge^{(n-\alpha')} E) = f_Bf$$

where $e$ and $f$ are idempotents in $A$ and $B$. By Lemma 5.2 and Theorem 5.3, the direct sum $\bigoplus_{\alpha \in \Pi} \wedge^{\alpha'} E$ is a full tilting module over $S(n,r)$, generated by the primitive idempotents $e_\alpha$ with $\alpha \notin \Pi$. Moreover $\bigoplus_{\alpha \in \Pi} \wedge^{n-\alpha'} E$ is a full tilting module over $S(n,nm-r)$, generated by the primitive idempotents $e_\beta$ with $\beta \notin \hat{\Pi}$. Hence, using Ringel duality, we get the following Morita equivalence:

$$S(n,r)/\Pi \simeq S(n,nm-r)/\Pi.$$  

$$\square$$

6. Isomorphisms between generalized Schur algebras

It just remains to be shown that the Morita equivalences in Theorem 5.4 indeed are isomorphisms between generalized Schur algebras.

Lemma 6.1. Let $(A, \leq_A)$ and $(B, \leq_B)$ be two quasi-hereditary $k$-algebras, which are Morita equivalent such that the Morita equivalence induces an isomorphism between $\leq_A$ and $\leq_B$. Then this Morita equivalence sends standard modules to standard modules, say $\Delta_A(i)$ to $\Delta_B(i)$. Furthermore, suppose for all $i$, the modules $\Delta_A(i)$ and $\Delta_B(i)$ have the same $k$-dimension. Then the algebras $A$ and $B$ are isomorphic.

Proof. Standard modules are relatively projective, namely projective in truncated categories, that is as modules over quasi-hereditary quotients. Hence the Morita equivalence preserves the property of being standard. The Morita equivalence also sends simple modules to simple modules, thus it identifies the decomposition matrices $([\Delta(i) : L(j)])_{i,j}$ of $A$ and of $B$. By induction it follows that the $A$-simple $L_A(i)$ has the same $k$-dimension as the $B$-simple $S_B(i)$. Hence the Morita equivalence does not change multiplicities of projectives in the regular representation and therefore it provides an algebra isomorphism. $\square$

We keep the notation of the previous section for classical or quantized Schur algebras. Let $\alpha \in \Lambda^+(n,r)$ with $\alpha_1 \leq m$.

Lemma 6.2. The two Weyl modules $\Delta(\alpha)$ and $\Delta(\hat{\alpha})$ have the same $k$-dimension.

Proof. It is known (see [9, 7]) that the $k$-dimension of the Weyl module $\Delta(\alpha)$ is precisely the number of semi-standard $\alpha$-tableaux: $\dim_k \Delta(\alpha) = |\text{Stab}(\alpha)|$. Note that in the literature semistandard tableaux are often defined by requiring entries to increase along rows and to increase strictly along columns, whereas we have required entries to (weakly/strictly) decrease. Replacing an entry $i$ by $n + 1 - i$ provides a bijection
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between the two kinds of semistandard tableaux. By Theorem 3.2 we have $|STab(\alpha)| = |STab(\hat{\alpha})|$. □

An alternative proof of this Lemma could be based on the isomorphism $\Delta(\hat{\alpha}) \simeq \nabla(\alpha)^* \otimes \text{def}^\otimes m$. This can be shown by checking that both sides have the same highest weight and also the same dimension, by Weyl’s character formula.

Recall the definition of the ideals $I_\Pi$ and $I_{\hat{\Pi}}$ given in Theorem 5.4. We get the main theorem:

**Theorem 6.3.** For any natural numbers $n$ and $r$ and for any saturated set $\Pi \subset \Lambda^+(n,r)$, and for any natural number $m$ such that the partitions in $\Pi$ have rows of length not more than $m$, there is an isomorphism between (classical or quantized) generalized Schur algebras:

$$S(n,r)/I_\Pi \simeq S(n,nm-r)/I_{\hat{\Pi}}.$$ 

The resulting equivalence of categories of polynomial $G$-modules sends $M(\alpha)$ to $M(\hat{\alpha})$ where $M \in \{L, \Delta, P, T\}$.

**Proof.** This is a consequence of Theorem 5.4, Lemma 6.1 and Lemma 6.2. □

We note that the isomorphisms in Theorem 6.3 in the classical case do not depend on the choice of the ground field or on its characteristic, and in the quantized case they do not depend on the value of the quantum parameter $q$, provided $q \neq 0$.

The embedding of $S(n,r)$-mod into $M$-mod preserves cohomology, and so does for any ideal $I$ in a heredity chain the embedding $S(n,r)/I$-mod into $S(n,r)$-mod. Therefore Theorem 6.3 provides us with isomorphisms of cohomology groups; using the above notation we get for example:

$$Ext^*(L(\alpha), L(\beta)) \simeq Ext^*(L(\hat{\alpha}), L(\hat{\beta})).$$

Here, the extension groups may be taken over $M$ or over the respective (classical or quantized) Schur algebras.

7. Applications: Decomposition numbers and structure of Schur algebras

We list different applications of the isomorphisms and equivalences of categories provided by Theorem 6.3: identities of decomposition numbers, isomorphisms of cohomology groups, and a factorisation of the isomorphisms provided earlier by [1]. Unexplained terminology, background information and references can be found, for example, in [18]. We keep the notation of the previous sections.

7.1. Decomposition numbers of Schur algebras.

**Corollary 7.1.** The following identity holds for the decomposition numbers of classical and quantum $GL_n$: Fix a natural number $m$. Then

$$[\Delta(\lambda) : L(\mu)] = [\Delta(\hat{\lambda}) : L(\hat{\mu})]$$
for any positive integer $r$ and for any $\lambda$ and $\mu$ with $\lambda', \mu' \in P^+(n,r)_m$.

Note that by varying $m$ and iterating the process of taking complements, we can get many (in general, infinitely many) different complement partitions $\lambda$ and $\mu$, thus equating the decomposition number $[\Delta(\lambda) : L(\mu)]$ with many other decomposition numbers in various degrees.

**Example.** Denote by $D_p(n,r)$ the decomposition matrix of the classical Schur algebra $S(n,r)$ over a field of prime characteristic. Label the columns and rows of these decomposition matrices in the same order, in both cases starting in the top left corner. Moreover, denote the decomposition number zero by a dot. Choose $p = 2$ and $p = 5$ respectively, and consider the Schur algebras $S(3,5)$ and $S(3,10)$.

\[
\begin{array}{c|ccc}
D_2(3,5) & \ast & \ast & \ast \\
(5,0,0) & 1 & 1 & 1 \\
(4,1,0) & . & 1 & . \\
(3,2,0) & . & 1 & 1 \\
(3,1,1) & . & 1 & 1 \\
(2,2,1) & . & . & 1 \\
\end{array}
\quad
\begin{array}{c|cc}
D_5(3,5) & \ast & \ast \\
(5,0,0) & 1 & . \\
(4,1,0) & . & 1 \\
(3,2,0) & . & 1 \\
(3,1,1) & . & 1 \\
(2,2,1) & . & . \\
\end{array}
\quad
\text{and}
\begin{array}{c|cccc}
D_2(3,10) & \ast & \ast & \ast & \ast \\
(10,0,0) & 1 & 1 & 1 & 1 \\
(9,1,0) & . & 1 & 1 & 1 \\
(8,2,0) & . & . & 1 & 1 \\
(7,3,0) & . & . & . & 1 \\
(6,4,0) & . & . & . & . \\
(5,5,0)^* & . & . & . & 1 \\
(8,1,1) & . & . & . & 1 \\
(7,2,1) & . & . & . & 1 \\
(6,3,1) & . & . & . & 1 \\
(5,4,1)^* & . & . & . & . \\
(6,2,2) & . & . & . & 1 \\
(5,3,2)^* & . & . & . & . \\
(4,4,2)^* & . & . & . & 1 \\
(4,3,3)^* & . & . & . & . \\
\end{array}
\]

We notice that the decomposition matrix $D_2(3,5)$ occurs as a submatrix of $D_2(3,10)$; in the latter matrix, we consider the submatrix defined by the rows and corresponding columns marked with a star. Similarly for prime 5, the decomposition matrices $D_5(3,5)$ is contained in $D_5(3,10)$ in the same way as for $p = 2$. 
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In fact, the Schur algebra $S(3, 5)$ is isomorphic to a quotient of the Schur algebra $S(3, 10)$, and this is true over any (infinite) field of any characteristic: Choose $m = 5$ and $\Pi = \Lambda^+(3, 5)$. Then

\[ \hat{\Pi} := \{(5, 5, 0), (5, 4, 1), (5, 3, 2), (4, 4, 2), (4, 3, 3)\} \subseteq \Lambda^+(3, 10), \]

which is precisely the set of partitions indexed by a star in the above decomposition matrices.

Finally we note that on a computational level, Weyl modules associated to a partition or to its complement have already been compared by Pittaluga and Strickland \cite{19}: explicit bases of these Weyl modules have been compared, in order to get improved bounds for dimensions of simple modules. Theorem 6.3 explains on a structural level why this comparison of bases is natural.

7.2. Factorisation of maps constructed by Beilinson, Lusztig and MacPherson. In their geometric study of quantum groups of type $A$, Beilinson, Lusztig and MacPherson \cite{11} constructed surjective maps

\[ S(n, r + n) \twoheadrightarrow S(n, r), \]

whose kernels $J$ have been shown to be ideals in the heredity chain of the Schur algebra $S(n, r + n)$. Hence these maps can be viewed as isomorphisms between generalized Schur algebras: $S(n, r + n)/J \simeq S(n, r)$. For more information on these maps see also \cite{8, 10, 11}.

Indeed, these maps can be written as products of two isomorphisms provided by Theorem 6.3 as follows: Start with $\Lambda^+(n, r + n)$ and $m = r + 1$. Let $\lambda$ be a hook partition with first row containing $m = r + 1$ boxes and first column containing $n$ boxes. Define $\Pi$ to be the saturated set of all partitions $\mu$ less than or equal to $\lambda$ in the dominance ordering. These are precisely the partitions in $\Lambda^+(n, r + n)$ whose first column has exactly $n$ boxes. Applying the complement construction to $\lambda$ yields a partition $\lambda = n - \lambda \in \Lambda^+(n, (n-1)r)$, which has the form of an $n \times r$ rectangle. The partitions in $\hat{\Pi}$ are precisely those partitions $\mu$ of $(n-1)r$ into not more than $n$ parts, which are less
than or equal to $\hat{\lambda}$, since the complement construction preserves the dominance order, by Lemma 3.1. Applying the complement construction for a second time we now use $m' = r = m - 1$. This sends $\lambda$ to $\hat{\lambda}$, which has just one row with $r$ boxes. The partitions in $\hat{\Pi}$ are being sent to the partitions $\hat{\mu}$ of $r$ into not more than $n$ parts, which are less than or equal to $\hat{\lambda}$; this set is all of $\Lambda^+(n,r)$.

![Diagram](image)

Altogether we have factored the isomorphism from $[\Pi]$ as follows:

$$S(n, r + n)/J \simeq S(n, (n - 1)r)/J' \simeq S(n, r)$$

where $J$ and $J'$ are ideals in the respective heredity chains. So in particular, James’ column removal formula is recovered:

**Column Removal:** For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with non-zero $\lambda_n$, define $\hat{\lambda} = (\lambda_1 - 1, \ldots, \lambda_n - 1)$. Let $\lambda, \mu$ be partitions of $r$ such that $\lambda_n$ and $\mu_n$ are non-zero. Then

$$[\Delta(\lambda) : L(\mu)] = [\Delta(\hat{\lambda}) : L(\hat{\mu})].$$

Moreover, iterating this construction will relate Schur algebras $S(n, r)$ with generalized Schur algebras $S(n, r + ln)/J$, for all natural numbers $l$ (see $[\Pi]$).

8. Application: Row removal in Schur algebras

In this section, we reprove James’ row removal formula for decomposition numbers (see $[10]$) as an application of Theorem 6.3 combined with Green $[9]$, Section 6.5, showing in particular, that this is not just a numerical result but reflects an isomorphism between generalized Schur algebras.

**Row Removal:** For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, define $\hat{\lambda} = (\lambda_2, \ldots, \lambda_n)$. Let $\lambda, \mu$ be partitions of $r$ such that $\lambda_1 = \mu_1$. Then

$$[\Delta(\lambda) : L(\mu)] = [\Delta(\hat{\lambda}) : L(\hat{\mu})].$$

Before constructing the underlying isomorphism of generalized Schur algebras which in particular also equates decomposition numbers, we recall how to obtain row removal numerically, using the above strategy. Compare with Donkin’s proof of James’s row removal (see $[4]$); also note that row removal is just a special case of Donkin’s horizontal cut principal, given in $[5]$, Theorem 1 and $[7]$, Equation 4.2(9).
8.1. **Row removal numerically.** Decomposition number \([\Delta(\lambda) : L(\mu)]\) equals zero whenever \(\mu \not\leq \lambda\). Suppose that \(\mu \leq \lambda\) are partitions in \(\Lambda^+(n, r)\) such that \(\lambda_1 = \mu_1\). Define the saturated set \(\Pi\) to be the set of all partitions which are less than or equal to \(\lambda\) in the dominance ordering. In particular \(\mu \in \Pi\). We now apply the complement construction twice. The first time we choose \(m = \lambda_1 = \mu_1\). Then we obtain partitions with the \(n\)th part equal to zero. We then change from \(GL_n\) to \(GL_{n-1}\) (see [9], Section 6.5) dropping the \(n\)th part, and apply again the complement construction with the same \(m\) as before. Finally we change back from \(GL_{n-1}\) to \(GL_n\) by adding an \(n\)th component zero:

\[
[\Delta(\lambda) : L(\mu)] = [\Delta(\lambda_1, \ldots, \lambda_n) : L(\mu_1, \ldots, \mu_n)] = [\Delta(m - \lambda_n, \ldots, m - \lambda_1) : L(m - \mu_n, \ldots, m - \mu_1)] = [\Delta(m - \lambda_n, \ldots, m - \lambda_2, 0) : L(m - \mu_n, \ldots, m - \mu_2, 0)] = [\Delta(m - \lambda_2, \ldots, \lambda_n) : L(\mu_2, \ldots, \mu_n)] = [\Delta(\lambda_2, \ldots, \lambda_n, 0) : L(\mu_2, \ldots, \mu_n, 0)]
\]

Before we construct the isomorphism between generalised Schur algebras underlying row removal, we first collect two remarks used in the construction.

8.2. **Cosaturated sets and idempotents.** We collect first some general facts. Let \(A\) be an algebra. Given an idempotent \(e \in A\) and an ideal \(J\) of \(A\); by abuse of notation we write \(e\) for the image of \(e\) under the natural epimorphism \(A \to A/J\). Then

\[
e(A/J)e = (eAe)/(eJe).
\]

Let \((A, \Lambda)\) be a quasi-hereditary algebra. For a subset \(\Lambda^* \subseteq \Lambda\), we will say that \(e\) is an idempotent corresponding to \(\Lambda^*\), if \(e\) is the sum of primitive orthogonal idempotents (possibly with higher multiplicities) corresponding to the labels in \(\Lambda^*\). A subset \(\Lambda^* \subseteq \Lambda\) is called cosaturated if \(\Lambda \setminus \Lambda^*\) is saturated.

Let \(e\) be the idempotent corresponding to a cosaturated set \(\Lambda^* \subseteq \Lambda\). In this situation, \(eAe\) is again quasi-hereditary with respect to \(\Lambda^*\). We call such an idempotent \(e\) cosaturated. So suppose \(e\) is cosaturated in a quasi-hereditary algebra \((A, \Lambda)\) and \(J\) is an ideal in the hereditary chain of \(A\); this means \(A/J\) is quasi-hereditary with respect to some saturated subset \(\Lambda' \subseteq \Lambda\). Then \(e\) is cosaturated in \(A/J\). This means \(e(A/J)e\) is again quasi-hereditary with indexing set \(\Lambda' \cap \Lambda^*\).

8.3. **Relating \(GL_n\)-representations and \(GL_{n-1}\)-representations.** Notations and details not given here can be found in Green [9], in particular we follow Section 6.5. Given \(N \geq n\). Then Green considers \(I(n, r)\) as a subset of \(I(N, r)\) in the natural way. With this convention, \(S(n, r)\) can be considered as a subalgebra of \(S(N, r)\). Define an injective map from \(\Lambda(n, r)\) into \(\Lambda(N, r)\) by adding \(N - n\) empty rows:

\[
\lambda \mapsto \lambda^* = (\lambda_1, \ldots, \lambda_n, 0, \ldots, 0).
\]
Denote the image of $\Lambda(n, r)$ in $\Lambda(N, r)$ by $\Lambda(n, r)^\ast$. Using the notation as in Green [9], define the idempotent

$$e = \sum_{\beta \in \Lambda(n, r)^\ast} \xi_\beta$$

in $S(N, r)$. Denote by $\hat{\beta}$ the partition associated to the composition $\beta$. Then the idempotent $\xi_\beta$ is associated to $\xi_{\hat{\beta}}$. Moreover, $\xi_\beta$ is a sum of primitive orthogonal idempotents corresponding to weights greater than or equal to $\bar{\beta}$, see [9], Section 4.7 (a). Note that the set $\Lambda^+(n-1, r)^\ast$ of partitions in $\Lambda(n-1, r)^\ast$ is a cosaturated subset of $\Lambda(N, r)$. Hence $e$ is cosaturated and $eS(N, r)e$ is quasi-hereditary with respect to $\Lambda^+(n, r)^\ast$ and the dominance ordering. Moreover, Green shows that

$$S(n, r) \simeq eS(N, r)e.$$  

As there is an isomorphism between the indexing sets of these two quasi-hereditary algebras, this isomorphism identifies the quasi-hereditary structures of $eS(N, r)e$ with the quasi-hereditary structure of $S(n, r)$. In particular, $eL(\lambda^\ast) = L(\lambda)$ and $eL(\mu) = 0$ for $\mu \in \Lambda(N, r)\setminus \Lambda(n, r)^\ast$, and $e\Delta(\lambda^\ast) = \Delta(\lambda)$.

8.4. Row removal algebraically. We finally construct the isomorphism between generalised Schur algebras, along the lines of the calculation in Section 8.1. The construction is in four steps:

(i) We first apply Theorem 6.3 combined with the arguments given in Sections 8.2 and 8.3. Let $\Lambda$ be the saturated subset of $\Lambda^+(n, r)$ consisting of all partitions with first rows of length smaller than or equal to $m$. Then by Theorem 6.3

$$S(n, r)/I_\Lambda \simeq S(n, nm - r)/I_{\hat{\Lambda}}.$$  

Here the indexing set $\hat{\Lambda}$ is a saturated subset of $\Lambda^+(n, nm-r)$, consisting of all partitions of $nm-r$ fitting into an $n \times m$ rectangle.

(ii) The argument in Section 8.3 provides an isomorphism

$$eS(n, nm-r)e \simeq S(n-1, nm-r)$$

of quasi-hereditary algebras, where $e$ is defined like in Equation (12): the indexing set corresponding to $e$ is $\Lambda^+(n-1, nm-r)^\ast$. Let $\Pi \subseteq \Lambda^+(n-1, nm-r)$ be such that $\Pi^\ast = \hat{\Lambda} \cap \Lambda^+(n-1, nm-r)^\ast \subseteq \Lambda^+(n, nm-r)$. Then $\Pi$ is a cosaturated subset of $\Lambda^+(n-1, nm-r)$. Hence we have the following isomorphism of quasi-hereditary algebras:

$$e(S(n, r)/I_\Lambda)e \simeq e(S(n, nm-r)/I_{\hat{\Lambda}})e \quad \text{by Equation (14)},$$

$$\simeq (eS(n, nm-r)e)/(eI_{\hat{\Lambda}}e) \quad \text{by Section 8.2},$$

$$\simeq S(n-1, nm-r)/(eI_{\hat{\Lambda}}e) \quad \text{by Equation (15)},$$

$$\simeq S(n-1, nm-r)/I_\Pi \quad \text{by Equation (14)}.$$  

Note that in the first isomorphism, by abuse of notation, we write $e$ for the preimage of $e$ under the isomorphism in Equation (14). The indexing set of the left-hand side now
consists precisely of the partitions of \( r \) with first row of length \( m \), fitting into an \( n \times m \)-rectangle. The indexing set of the right-hand side is \( \Pi \). The above isomorphism induces on the indexing sets the following identification: given \( \lambda \in \Lambda^+(n, r) \) with \( \lambda_1 = m \), then
\[
\lambda = (\lambda_1, \ldots, \lambda_n) \rightarrow (m - \lambda_n, \ldots, m - \lambda_2, m - \lambda_1) \\
\rightarrow (m - \lambda_n, \ldots, m - \lambda_2, 0) \\
\rightarrow (m - \lambda_n, \ldots, m - \lambda_2).
\]

(iii) We next apply Theorem 6.3 with respect to \( \Pi \) and with \( m \) as above. This means we take the complement of a partition with \( nm - r \) boxes inside an \((n - 1) \times m\) rectangle, and as such the complement has \((n - 1)m - (nm - r) = r - m\) boxes. Hence, by Theorem 6.3, we have
\[
S(n - 1, nm - r)/I_{\Pi} \cong S(n - 1, r - m)/I^{\hat{\Pi}}.
\]

On the indexing set the following identification happens:
\[
(m - \lambda_n, \ldots, m - \lambda_2) \mapsto (\lambda_2, \ldots, \lambda_n).
\]

(iv) We finally apply again Green’s isomorphism (see Section 8.3): there exists an idempotent \( f \) defined like in Equation (12) such that
\[
S(n - 1, r - m) \cong fS(n, r - m)f.
\]
The indexing set corresponding to \( f \) is \( \Lambda^+(n - 1, r - m) \). By (iii), the ideal \( I^{\hat{\Pi}} \) is an ideal in the hereditary chain of \( S(n - 1, r - m) \). Hence
\[
S(n - 1, r - m)/I^{\hat{\Pi}} \cong (fS(n, r - m)f)/I^{\hat{\Pi}_{\hat{\Pi}}},
\]
where \( I_{\hat{\Pi}_{\hat{\Pi}}} \) is the image of \( I^{\hat{\Pi}} \) under the isomorphism in Equation (16). In this last step, the identification on the indexing sets is:
\[
(\lambda_2, \ldots, \lambda_n) \mapsto (\lambda_2, \ldots, \lambda_n, 0).
\]
In total we constructed an isomorphism of quasi-hereditary algebras
\[
e(S(n, r)/I_{\Lambda})e \cong (fS(n, r - m)f)/I^{\hat{\Pi}}.
\]
Here the indexing set on the left-hand side consists of partitions \( \lambda \) of \( r \) with at most \( n \) parts such that \( \lambda_1 = m \). This guarantees that the partitions of the indexing set fit into an \( n \times m \)-rectangle. The indexing set on the right-hand side consists of partitions of \( r - m \) into \( n \) parts where the \( n \)-th part is zero and the first part is smaller than or equal to \( m \). On the indexing sets the above isomorphism induces the identification of \( \lambda = (\lambda_1, \ldots, \lambda_n) \) with \( (\lambda_2, \ldots, \lambda_n, 0) \).

9. APPLICATION: EQUATING \( p \)-KOSTKA NUMBERS

Fix a natural number \( r \) and partitions \( \lambda \) and \( \mu \) of \( r \). The permutation module \( M^\lambda \) over \( \Sigma_r \) is the module obtained by inducing the trivial representation from the Young subgroup \( \Sigma_\lambda \) to the symmetric group \( \Sigma_r \). The Young module \( Y^\mu \) is the unique indecomposable direct summand of \( M^\mu \) which contains the Specht module \( S^\mu \) as a submodule. Every permutation module \( M^\lambda \) is a direct sum of Young modules \( Y^\mu \) (with multiplicities) where the indices satisfy \( \mu \geq \lambda \). The \( p \)-Kostka number \( (M^\lambda : Y^\mu) \) is defined to be
the multiplicity of the Young module $Y^\mu$ occurring, up to isomorphism, in a direct sum decomposition of the permutation module $M^\lambda$. Thus we have:

$$M^\lambda = \bigoplus_{\mu \succeq \lambda} (M^\lambda : Y^\mu) Y^\mu$$

If $\lambda$ and $\mu$ are partitions of $r$ with not more than $n$ parts, then we can reinterpret the $p$-Kostka number in terms of the Schur algebra $S(n, r)$. For each partition $\lambda$, there is an idempotent $\xi_\lambda$ in the Schur algebra, as defined in \cite{9, 7}. The space $\xi_\lambda E^{\otimes r}$, when viewed as right $k\Sigma_r$-module, is isomorphic to the permutation module $M^{\lambda}$. Then $\xi_\lambda \in S(n, r) = \text{End}_{k\Sigma_r} (E^{\otimes r})$ is the identity in $\text{End}_{k\Sigma_r} (M^{\lambda})$, that is, the projection onto $M^{\lambda}$. The identity on the indecomposable direct summand $Y^{\lambda}$ of $M^{\lambda}$ is a primitive idempotent $e_\lambda \in S(n, r)$ contained in $\xi_\lambda$. We can write $\sum_\lambda \xi_\lambda = \sum_\lambda \epsilon_\lambda$, where $\epsilon_\lambda$ is the sum of all primitive idempotents equivalent to $e_\lambda$, which occur in $\sum_\lambda \xi_\lambda$. Hence $\xi_\lambda = \sum_{\mu} \xi_\lambda \epsilon_\mu = \sum_{\mu \succeq \lambda} \xi_\lambda \epsilon_\mu$. (Note that $\sum_\lambda \xi_\lambda$ is not the unit element of $S(n, r)$, since we take the sum over partitions, not over compositions.) This gives us:

$$M^\lambda = \xi_\lambda E^{\otimes r} = \sum_{\mu} \xi_\lambda \epsilon_\mu E^{\otimes r}.$$

In particular we can rewrite the $p$-Kostka number $(M^\lambda : Y^\mu)$ as a multiplicity $[\xi_\lambda : \epsilon_\mu]$ of the primitive idempotents equivalent to $e_\mu$ in $\xi_\lambda$. Note that this is also the multiplicity of $e_\mu$ occurring, up to equivalence, in $\xi_\lambda \cdot \epsilon_\mu$.

Appropriate sums of the idempotents $\xi_\lambda$ generate the ideals in a heredity chain of the Schur algebra. Suppose $J$ is an ideal in the heredity chain of $A$. Let $\bar{\cdot} : A \to A/J$ be the quotient map. Then the primitive idempotent $e_\mu$ is sent either to zero, or to a primitive idempotent of $A/J$. It is sent to zero if and only if $\xi_\mu \in J$. Moreover the image is non-zero, if and only if the image of any primitive idempotent equivalent to $e_\mu$ is non-zero, too; then the multiplicity of $e_\mu$ in $\xi_\lambda$ is preserved under the quotient map. Hence

$$[\xi_\lambda : \epsilon_\mu] = [\xi_\lambda : \bar{\epsilon}_\mu].$$

The isomorphism in Theorem \ref{6.3} preserves heredity chains and thus sends the equivalence class of $\bar{\xi}_\lambda$ to the equivalence class of $\bar{\xi}_\lambda$ for $\lambda \in \Pi$. This proves the following corollary (which also follows by rewriting the multiplicity $(M^\lambda : Y^\mu)$ as the dimension of the weight space $L(\mu)^{\lambda}$ and then using the isomorphisms $L(\mu + \omega) \simeq \text{det} \otimes L(\mu)$ and $L(\mu^n) \simeq L(\mu)^n$, see also \cite{7}, 4.4(1)(v)):

**Corollary 9.1.** Suppose $\lambda$ and $\mu$ are partitions in $\Lambda^+(n, r)$. Choose a natural number $m$ such that $\lambda_1, \mu_1 \leq m$. Then

$$(M^\lambda : Y^\mu) = (M^{\lambda} : Y^{\bar{\mu}}).$$

Consequently, the column and row removal formulas for $p$-Kostka numbers hold true.

**Proof.** We have already shown the multiplicity formula. Note that we have independence of $n$ in the following sense: The partitions $\lambda$ and $\mu$ may as well be considered as elements of $\Lambda^+(N, r)$ for any $N \geq n$ (by formally adding zeroes at the end). This does not change the multiplicities. For instance, we have

$$(M^\lambda : Y^\mu) = (M^{(\lambda,0)} : Y^{(\mu,0)}).$$
Now the row removal formula follows by applying the complement construction twice as described in Section 8.

The column removal rule for $p$-Kostka numbers follows by applying the complement construction twice as described in Section 7.2. □

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Appendix by Stephen Donkin

We give here an alternative approach which gives substantial generalizations of the main result, Theorem 5.3 above.

We first consider the classical case $q = 1$. The Schur algebra $S(n, r)$, as defined by Green, [7], is the dual algebra of the subcoalgebra $A(n, r)$ of the coordinate algebra of a the general linear group $GL_n$ and a similar construction of ”generalized Schur algebras” is given in [2], [3] in the context of arbitrary reductive groups. It therefore seems natural to proceed directly in terms of coalgebras. The dual of an isomorphism of coalgebras is an isomorphism of the dual algebras and so we, for the most part, restrict ourselves to producing coalgebra isomorphisms.

We fix a connected reductive group $G$ defined over a field $k$ which we assume algebraically closed (but see Remark 4 below). We regard $k[G]$ as a Hopf algebra over $k$ with structure maps $\delta : k[G] \to k[G] \otimes k[G]$, $\epsilon : k[G] \to k$, $\sigma : k[G] \to k[G]$ (as in [6; 0.7] for example)

For an algebraic group $H$ over $k$ we write $X(H)$ for the additive abelian group of algebraic group homomorphisms from $H$ to the multiplicative group of $k$. We assume the usual notation for reductive groups and their representations, as in for example [2], [3]. In particular, we have, inside the weight lattice $X = X(T)$, the set of dominant weights $X^+(T)$. For each $\lambda \in X^+(T)$ we have the irreducible module $L(\lambda)$ of highest weight $\lambda$, and the module $\nabla(\lambda)$ induced from a one dimensional $B$-module on which $T$ acts via $\lambda$. Here $B$ is a Borel subgroup whose set of roots forms a system of negative roots in the root system of $G$, with respect to $T$. We write $\Phi^+$ (resp. $\Phi^-$) for the set of positive (resp. negative) roots and $\Phi$ for $\Phi^+ \cup \Phi^-$, the set of roots. There is a natural partial order on $X(T)$: we write $\lambda \leq \mu$ if $\mu - \lambda$ is a sum of positive roots. For a $G$-module $V$ we write $\text{cf}(V)$ for the coefficient space of $V$, i.e. the subspace of $k[G]$ spanned by the coefficient functions $f_{ij}$, $i \in I$, where $v_i$, $i \in I$, is a basis of $V$ and the $f_{ij}$ are defined by the equations

$$g v_i = \sum_{j \in I} f_{ji}(g) v_j$$

for all $i \in I$, $g \in G$. Recall that for $G$-modules $V_1, V_2$ we have $\text{cf}(V_1 \otimes V_2) = \text{cf}(V_1) \cdot \text{cf}(V_2)$.

We recall the construction of the generalized Schur algebras. Let $\Pi$ be a subset of $X^+(T)$. A (rational) $G$-module $V$ is said to belong to $\Pi$ if every composition factor of $V$ belongs to $\{L(\lambda) | \lambda \in \Pi\}$. Among all submodules belonging to $\Pi$ of an arbitrary $G$-module $V$ there is a unique maximal one, denoted $O_{\Pi}(V)$. In this way we get a left exact functor $O_{\Pi}$ from the category of rational $G$-modules to itself. The set $\Pi$ is said to be saturated if it has the property that whenever $\lambda \in \Pi$ and $\mu$ is a dominant weight less such that $\mu \leq \lambda$ then $\mu \in \Pi$. Regarding $k[G]$ as a rational left $G$-module we have the submodule $A(\Pi) = O_{\Pi}(k[G])$, and indeed $A(\Pi)$ is a subcoalgebra of $k[G]$. In fact $A(\Pi)$ is the sum of all spaces $\text{cf}(V)$, as $V$ ranges over all $G$-modules belonging to $\Pi$.

It is shown in [2] that if $\Pi$ is finite and saturated then $A(\Pi)$ is finite dimensional. However, saturation is not important for what follows so we now allow $\Pi$ to be any finite
subset or $X^+(T)$. As $A(\Pi) \subset A(\Gamma)$, whenever $\Pi \subseteq \Gamma \subseteq X^+(T)$ we have that $A(\Pi)$ is finite dimensional since we may take for $\Gamma$ the smallest saturated subset containing $\Pi$, and this is finite. The generalized Schur algebra $S(\Pi)$ is defined as the dual algebra of the coalgebra $A(\Pi)$.

We shall describe two natural situations in which one gets an isomorphism of Schur coalgebras $A(\Gamma) \rightarrow A(\Pi)$ (and hence of Schur algebras $S(\Pi) \rightarrow S(\Gamma)$). First suppose that $L$ is a one dimensional $G$-module and that $\mu \in X(T)$ is the representation of $T$ afforded by $L$. Then $L = L(\mu)$. We let $d_\mu$ be the element of $k[G]$ such that $gx = d_\mu(g)x$, for all $g \in G$, $x \in L$. Notice that the dual module $L(\mu)^*$ is isomorphic to $L(-\mu)$. It follows that $d_\mu d_{-\mu} = 1$, in particular $d_\mu \in k[G]$ is a unit.

Let $\Pi$ be a subset of $X^+(T)$. We have $L \otimes L(\lambda) \cong L(\lambda + \mu)$, for $\lambda \in X^+(T)$. Hence a $G$-module $V$ has composition factor $L(\lambda)$ if and only if $L \otimes V$ has composition factor $L(\lambda + \mu)$. If $V$ is a $G$-module belonging to $\Pi$ then $L \otimes V$ belongs to $\mu + \Pi = \{\lambda + \mu | \lambda \in \Pi\}$ and hence $d_\mu \text{cf}(V) = \text{cf}(L \otimes V) \subseteq A(\mu + \Pi)$. Thus we get $d_\mu A(\Pi) \subseteq A(\mu + \Pi)$. But, by the same token, we have $d_{-\mu} A(\mu + \Pi) \subset A(\Pi)$ and hence $d_\mu A(\Pi) = A(\mu + \Pi)$. Since $d_\mu(gh) = d_\mu(g)d_\mu(h)$, for all $g, h \in G$, we have $\delta(d_\mu) = d_\mu \otimes d_\mu$ and we get our first source of isomorphisms of Schur coalgebras.

**Principle 1** Suppose that $\mu \in X(T)$ is such that $L(\mu)$ is one dimensional and $d_\mu \in k[G]$ is given by $gx = d_\mu(g)x$ for all $g \in G$, $x \in L(\mu)$. Then the map $A(\Pi) \rightarrow A(\mu + \Pi)$, given by multiplication by $d_\mu$, is a coalgebra isomorphism.

**Remark 1** We thus get an isomorphism of Schur coalgebras for each element of $X(G/G')$, where $G'$ is the derived group of $G$. As we have $G = TG'$ we may identify $X(G/G')$ with $X(T/T')$, where $T' = G' \cap T$. In other words we have specifically an isomorphism $A(\Pi) \rightarrow A(\mu + \Pi)$ for each $\mu \in X(T)$ which has $T'$ in its kernel.

**Remark 2** The arguments above are formal and hold quite generally. Let $(H, \delta, \epsilon)$ be a Hopf algebra. Let $\text{Gp}(H)$ be the set of group-like elements, i.e. an elements $g \in H$ such $d(g) = g \otimes g$ and $\epsilon(g) = 1$. Then $\text{Gp}(H)$ is a subgroup of the group of units of $H$. Let $\{L_\lambda | \lambda \in X\}$ be a complete set of pairwise non-isomorphic right $H$-comodules. There is a natural action of $\text{Gp}(H)$ on $X$ defined as follows. If $g \in \text{Gp}(H)$ then $kg$ is a one dimensional submodule of $H$ and, for $\lambda \in X$, the comodule $kg \otimes L_\lambda$ is simple. We define $g\lambda \in X$ by the condition $kg \otimes L_\lambda \cong L_{g\lambda}$.

For a subset $\Pi$ of $X$, we may define a functor $O_\Pi$, form the category of right $H$-comodules to itself. Then viewing $H$ itself as a right $H$-comodule (with structure map $\delta : H \rightarrow H \otimes H$) we may form the subcoalgebra $A(\Pi) = O_\Pi(H)$. The above argument shows that multiplication by $g$ determines an isomorphism of coalgebras from $A(\Pi)$ to $A(g\Pi)$.

Our second source of isomorphisms of Schur algebras is from automorphisms of $G$. Let $\phi : G \rightarrow G$ be an automorphism of our reductive group $G$. For a $G$-module $V$ affording the representation $\pi : G \rightarrow \text{GL}(V)$ we form the module $V^\phi$ with the same underlying $k$-space $V$ on which $G$ acts according to the representation $\pi \circ \phi$. We
note that \((V^\phi)^\psi \cong V^{\phi \circ \psi}\), for automorphisms \(\phi, \psi\), and that \(V \cong V^\phi\) if \(\phi\) is inner. Moreover, we have (from the definitions) \(\text{cf}(V^\phi) = \phi^\sharp(\text{cf}(V))\), for a \(G\)-module \(V\), where \(\phi^\sharp : k[G] \to k[G]\) is the comorphism of \(\phi\). Hence \(\phi^\sharp(A(\Pi)) = A(\Pi)\) if \(\phi\) is an inner automorphism. Let \(\phi\) be a general automorphism. Then by the conjugacy theorems for Borel subgroups and maximal tori, there exists an element \(g \in G\) such that \(\phi(B) = B^g\) and \(\phi(T) = T^g\). Thus we may write \(\phi = \gamma \circ \psi\), where \(\psi\) is an automorphism stabilizing \(B\) and \(T\) and \(\gamma\) is an inner automorphism. Now \(\psi\) restricts to an isomorphism \(\psi_0 : T \to T\) which induces an isomorphism \(f : X(T) \to X(T)\) (the restriction of the comorphism \(\psi_0^\sharp : k[T] \to k[T]\)). Moreover, since \(\psi\) preserves \(B\), the isomorphism \(f\) induces a bijection on negative roots, and hence also on positive roots. The map \(f\) is an automorphism of the root datum \((X, \Phi, \check{X}, \check{\Phi})\) and conversely, for any automorphism of the root datum \(f'\) there is an automorphism \(\psi' : G \to G\) stabilizing \(B\) and \(T\) and inducing \(f\) on \(X(T)\) (see [9; 11.4.3 Theorem]).

Now let \(\Pi\) be a finite subset of \(X^+(T)\) and let \(V\) be a module belonging to \(\Pi\). Then the weights of \(V^\phi\) are the weights of the form \(f(\mu)\), for \(\mu\) a weight of \(V\). Hence \(V^\phi\) belongs to \(f(\Pi)\) and \(\text{cf}(V^\phi) = \phi^\sharp(\text{cf}(V)) \leq A(f(\Pi))\). Thus we get \(\phi^\sharp(A(\Pi)) \leq A(f(\Pi))\) and hence (applying the same principle with the inverse of \(\psi\) in place of \(\psi\)) we have \(\phi^\sharp(A(\Pi)) = A(f(\Pi))\).

**Principle 2** Let \(\Pi\) be a finite subset of \(X^+(T)\). Let \(\phi\) be an isomorphism of \(G\) inducing the isomorphism \(f\) on the root datum as above. Then the restriction of \(\phi^\sharp\) is an isomorphism from \(A(\Pi)\) to \(A(f(\Pi))\). Conversely, for any automorphism of the root datum \(f\) there is an isomorphism \(A(\Pi) \to A(f(\Pi))\), namely the restriction of \(\phi^\sharp\) where \(\phi\) is an automorphism of \(G\) stabilizing \(B\) and \(T\) and inducing \(f\) on \(X(T)\).

**Remark 3** We may define \(f : X \to X\) by \(f(\mu) = -w_0\mu\). For \(\lambda\) a dominant weight we set \(\lambda' = -w_0\lambda = f(\lambda)\), where \(w_0\) is the longest element of the Weyl group \(W = N_G(T)/T\), and for a subset \(\Pi\) of \(X^+(T)\) set \(\Pi = f(\Pi) = \{-w_0\lambda | \lambda \in \Pi\}\). Thus, for each finite subset \(\Pi\) of \(X^+(T)\), we get a natural isomorphism \(A(\Pi) \to A(\Pi^*)\).

**Remark 4** Suppose \(G\) is defined and split over an arbitrary field \(k\). The first principle is still valid and the proof goes through without change to give isomorphisms of the \(k\)-coalgebras \(A(\Pi)\) and \(A(\mu + \Pi)\) (defined as subcoalgebras of \(k[G]\)). Moreover, for each automorphism \(f\) of the root datum there is a \(k\)-isomorphism \(\phi : G \to G\) inducing \(f\) (see e.g. [8;II, 1.15 Proposition]) so that the second principle is valid also in this case and we have an isomorphism of the coalgebras \(A(\Pi)\) and \(A(f(\Pi))\) (defined as subcoalgebras of \(k[G]\)).

**Remark 5** We shall not attempt to give general versions of principles of quantum groups of Principles 1 and 2, though it is clear that it would be possible to do so within the framework of, for example, [1].

We now turn our attention to the quantum general linear group \(G\), of degree \(n\), over an arbitrary field \(k\), with parameter \(0 \neq q \in k\), as in [6]. We have the set of dominant
weights $X^+(n)$ and the set of polynomial dominant weights $\Lambda^+(n)$ and, for $r \geq 0$, the set $P^+(n, r)$ of polynomial dominant weights $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda_1 + \cdots + \lambda_n = r$.

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in X^+(n)$ we put $\lambda^* = (-\lambda_n, \ldots, -\lambda_1)$ (i.e. $-w_0\lambda$, where $w_0$ is the longest element of the Weyl group). For $\Pi \subset X^+(T)$, we put $\Pi^* = \{\lambda^* | \lambda \in \Pi\}$.

For $\Pi$ finite (and not necessarily saturated) we have the Schur coalgebra $A(\Pi)$ and its dual algebra $S(\Pi)$ defined as above (see also [5]). We shall produce an isomorphism $A(\Pi) \to A(\Pi^*)$, for $\Pi$ finite. (Note that in the case $q = 1$ we could directly invoke Principle 2.)

Note that multiplication by the determinant $d \in k[G]$ gives an isomorphism $A(\Pi) \to dA(\Pi) = A(\omega + \Pi)$ as in Principle 1, where $\omega = (1, 1, \ldots, 1)$.

We now show that $A(\Pi)$ is isomorphic to $A(\Pi^*)$. Let $\sigma : k[G] \to k[G]$ be the antipode. Then the relationship between the coefficient spaces of a finite dimensional left $G$-module $V$ and the dual left $G$-module $V^*$ is $\text{cf}(V^*) = \sigma(\text{cf}(V))$. Now if $V$ is such that $\text{cf}(V) = A(\Pi)$ (e.g. $V$ is $A(\Pi)$ itself as a left $G$-module) then we get $\sigma(A(\Pi)) = \sigma(\text{cf}(V)) = \text{cf}(V^*) \leq A(\Pi^*)$. Thus we get $\sigma^2(A(\Pi)) \leq \sigma(A(\Pi^*)) \leq A(\Pi)$ and now by dimensions (and the fact that $\sigma$ is injective) we have $\sigma(A(\Pi)) = A(\Pi^*)$.

Moreover $\sigma : k[G] \to k[G]$ is an antimorphism of coalgebras so we get that $\sigma$ induces an isomorphism $S(\Pi^*) \to S(\Pi)^{op}$ (where $^{op}$ indicates the opposite algebra). [Note that this is a general argument for Hopf algebras and subcoalgebras defined by restricting composition factors.] It remains to prove that $S(\Pi)$ is isomorphic to $S(\Pi)^{op}$ for any finite saturated subset $\Pi$. We can replace $S(\Pi)$ by the isomorphic algebra $S(\Pi + m\omega)$ and choosing $m$ large we can assume that $\Pi$ consists of polynomial weights. Then $A(\Pi) = \oplus_{r \geq 0} A(\Pi(r))$, where $\Pi(r) = \Pi \cap P^+(n, r)$, so we may assume $\Pi$ homogeneous of some degree. But there is an anti-automorphism $J$ of $S(n, r)$ (see p82 of [5]) which fixes each idempotent $\xi_\alpha$. By means of $J$ one generalizes to general $q$ the contravariant dual $M^0$ of a finite dimensional $S(n, r)$-module $M$ (as discussed by Green, [7], in the case $q = 1$) - the action is $(aa)(m) = \alpha(j(a)m)$, for $\alpha \in M^0 = \text{Hom}_k(M, k)$, $m \in M$, $\alpha \in M^0$. Moreover, $M$ and $M^0$ have the same character hence the same composition factors. Now if $a$ annihilates $M$ then $J(a)$ annihilates $M^0$, so $J(I_\Pi) \leq I_\Pi$ and $J$ induces an isomorphism $S(\Pi) \to S(\Pi)^{op}$.

This brings us to our third principle, which is a combination of principles 1 and 2 in the situation of quantum general linear groups.

**Principle 3** For any finite subset $\Pi$ of $X^+$ and $m \in \mathbb{Z}$ we have $A(\Pi) \cong A(m\omega + \Pi^*)$ and hence $S(\Pi) \cong S(m\omega + \Pi^*)$.

**Remark 6** We fix $r$ and $m$ and let $\Pi$ be any subset of $P^+(n, r)$ consisting of partitions $\lambda = (\lambda_1, \ldots, \lambda_n) \in P^+(n, r)$ with $\lambda_1 \leq m$. Then $\Pi$ (as in Theorem 6.3) is $m\omega + \Pi^*$. Hence we get $S(\Pi) \cong S(\Pi)$. If $\Gamma \subset \Sigma \subset X^+$ then we get (from the definitions) that $A(\Gamma) \subset A(\Sigma)$ and the kernel $I_{\Gamma, \Sigma}$, say, of the surjective algebra homomorphism $S(\Sigma) \to S(\Gamma)$ consists of all $x \in S(\Sigma)$ which annihilate all modules belonging to $\Gamma$.

Since $\Pi \subset P^+(n, r)$ and $\Pi \subset P^+(n, nm - r)$ we get $S(n, r)/I_{\Pi} \cong S(n, nm - r)/I_{\Pi}$, where $I_{\Pi} = I_{\Pi, P^+(n, r)}$ and $I_{\Pi} = I_{\Pi, P^+(n, nm - r)}$. This gives the main result of this paper, Theorem 6.3, but without the restriction of saturation.
Remark 7 Finally, we remark that the isomorphism above are defined integrally for \( \Pi \) saturated. If \( G \) is a general linear group or Chevalley group, then \( G \) is defined over \( \mathbb{Z} \) in the usual way. This amounts to giving a suitable \( \mathbb{Z} \)-form of the coordinate algebra of the complex group and gives rise to integral Schur coalgebras \( A(\Pi)_\mathbb{Z} \) (see [2; Section 4] and [4]) and the Schur algebra over an arbitrary field is obtained by base change from the integral one. Then the above arguments are valid over \( \mathbb{Z} \) and give isomorphisms \( S_\mathbb{Z}(\Pi) \to S_\mathbb{Z}(\Pi + \lambda), S(\Pi) \to S(\Pi^*) \) etc. So one obtains \( S_\mathbb{Z}(n,r)/I_{\Pi,\mathbb{Z}} \to \ldots \) (where \( I_{\Pi,\mathbb{Z}} = I_{\Pi,\mathbb{C}} \cap S_\mathbb{Z}(n,r) \)), and in the quantum case \( S(n,r)_{\mathbb{Z}[t,t^{-1}]}/I_{\mathbb{Z}[t,t^{-1}],\Pi} \to \ldots \) which specializes to Theorem 6.3 over a field by base change.

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