ANALYTIC SPREAD AND ASSOCIATED PRIMES OF DIVISORIAL FILTRATIONS

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Abstract. We prove that a classical theorem of McAdam about the analytic spread of an ideal in a Noetherian local ring is true for divisorial filtrations on an excellent local domain $R$ which is either of equicharacteristic zero or of dimension $\leq 3$. In fact, the proof is valid whenever resolution of singularities holds.

1. Introduction

Expositions of the theory of complete ideals, integral closure of ideals and their relation to valuation ideals, Rees valuations, analytic spread and birational morphisms can be found, from different perspectives, in [29], [27], [19] and [20]. The book [27] and the article [20] contain references to original work in this subject. A survey of recent work on symbolic algebras is given in [13]. A different notion of analytic spread for families of ideals is given in [14]. A recent paper exploring ideal theory in 2 dimensional normal local domains using geometric methods is [25].

Let $R$ be a Noetherian local ring with maximal ideal $m_R$. If $I$ is an ideal in $R$, then the Rees algebra of $I$ is the graded $R$-algebra $R[I] = \sum_{n \geq 0} I^n t^n$ and the analytic spread of $I$ is

$$\ell(I) = \dim R[I]/m_R R[I].$$

It is natural to extend this definition to arbitrary filtrations of $R$. Let $\mathcal{I} = \{I_n\}$ be a filtration on $R$. The Rees algebra of the filtration is $R[\mathcal{I}] = \oplus_{n \geq 0} I_n$ and the analytic spread of the filtration $\mathcal{I}$ is defined to be

$$\ell(\mathcal{I}) = \dim R[\mathcal{I}]/m_R R[\mathcal{I}].$$

Thus if $I$ is an ideal in $R$ and $\mathcal{I} = \{I^n\}$ is the $I$-adic filtration, then $\ell(I) = \ell(\mathcal{I})$. The essential new phenomena in the case of arbitrary filtrations $\mathcal{I}$ is that the Rees algebra $R[\mathcal{I}]$ may not be Noetherian.

An essential property of the analytic spread $\ell(I)$ of an ideal is the inequality ([27, Proposition 5.1.6 and Corollary 8.3.9])

$$\text{ht}(I) \leq \ell(I) \leq \dim R.$$  \hspace{1cm} (2)

If $\mathcal{I}$ is a filtration, then $\text{ht}(I_n) = \text{ht}(I_1)$ for all $n$ ([11, equation (7)]) so it is natural to define $\text{ht}(\mathcal{I}) = \text{ht}(I_1)$.

We always have ([11, Lemma 3.6]) that

$$\ell(\mathcal{I}) \leq \dim R$$  \hspace{1cm} (3)

so the second inequality of (2) always holds for filtrations. However, the first inequality of (2), $\text{ht}(\mathcal{I}) \leq \ell(\mathcal{I})$, fails spectacularly, even attaining the condition that $\ell(\mathcal{I}) = 0$ ([11].

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Example 1.2, Example 6.1 and Example 6.6). The last two of these examples are of symbolic algebras of space curves, which are divisorial filtrations.

The condition that a filtration has analytic spread zero has a simple ideal theoretic interpretation ([11, Lemma 3.8]). Suppose that \( \mathcal{I} = \{ I_n \} \) is a filtration in a local ring \( R \). Then the analytic spread \( \ell(\mathcal{I}) = 0 \) if and only if

For all \( n > 0 \) and \( f \in I_n \), there exists \( m > 0 \) such that \( f^m \in m_R I_{mn} \).

Let \( I \) be an ideal in a Noetherian local domain \( R \). The integral closure of the Rees algebra \( R[I[t] = \sum_{n \geq 0} I^n t^n \) in \( R[t] \) is \( \sum_{n \geq 0} T^n t^n \). The ideal \( T^n \) in \( R \) is called the integral closure of \( I^n \) in \( R \). Divisorial valuations of \( R \) are defined later in this introduction. There exist divisorial valuations \( \nu_1, \ldots, \nu_r \) of \( R \) called the Rees valuations of \( I \) and \( a_1, \ldots, a_r \in \mathbb{Z}_{\geq 0} \) such that

\[
T^n = I(\nu_1)_{a_1} \cap \cdots \cap I(\nu_r)_{a_r}
\]

for \( n \in \mathbb{N} \). This expression is irredundant, in the sense that for all \( j \), there exists an \( n \) such that \( T^n \neq \bigcap_{n \neq j} I(\nu_1)_{a_1} \). This is proven (even for a generalization of this statement to arbitrary Noetherian rings) in [20] and [27, Theorem 10.2.2].

We have the following remarkable theorem.

**Theorem 1.1.** ([22], [27, Theorem 5.4.6]) Let \( R \) be a formally equidimensional local ring and \( I \) be an ideal in \( R \). Then \( m_R \in \text{Ass}(R/I^n) \) for some \( n \) if and only if \( \ell(I) = \dim(R) \).

The assumption of being formally equidimensional is not required for the if direction of Theorem 1.1 (this is Burch’s theorem, [3], [27, Proposition 5.4.7]).

We now consider divisorial filtrations. Given an ideal \( I \) in \( R \), the filtration \( \{ T^n \} \) is an example of a divisorial filtration of \( R \). The filtration \( \{ T^n \} \) is Noetherian if \( R \) is universally Nagata. The symbolic powers of an ideal is another example of a divisorial filtration.

Let \( R \) be a Nagata local domain of dimension \( d \) with quotient field \( K \). Let \( \nu \) be a discrete valuation of \( K \) with valuation ring \( V_\nu \) and maximal ideal \( m_\nu \). Suppose that \( R \subset V_\nu \). Then for \( n \in \mathbb{N} \), define valuation ideals

\[
I(\nu)_n = \{ f \in R \mid \nu(f) \geq n \} = m_\nu^n \cap R.
\]

A divisorial valuation of \( R \) ([27, Definition 9.3.1]) is a valuation \( \nu \) of \( K \) such that if \( V_\nu \) is the valuation ring of \( \nu \) with maximal ideal \( m_\nu \), then \( R \subset V_\nu \) and if \( p = m_\nu \cap R \) then 
\[
\text{trdeg}_{K(p)} \kappa(\nu) = \text{ht}(p) - 1,
\]
where \( \kappa(p) \) is the residue field of \( R_p \) and \( \kappa(\nu) \) is the residue field of \( V_\nu \). If \( \nu \) is divisorial valuation of \( R \) such that \( m_R = m_\nu \cap R \), then \( \nu \) is called an \( m_R \)-valuation.

By [27, Theorem 9.3.2], the valuation ring of every divisorial valuation \( \nu \) is Noetherian, hence is a discrete valuation. Suppose that \( R \) is an excellent local domain. Then a valuation \( \nu \) of the quotient field \( K \) of \( R \) which is nonnegative on \( R \) is a divisorial valuation of \( R \) if and only if the valuation ring \( V_\nu \) of \( \nu \) is essentially of finite type over \( R \) ([10, Lemma 5.1]).

In general, the filtration \( \mathcal{I}(\nu) = \{ I(\nu)_n \} \) is not Noetherian; that is, the graded \( R \)-algebra \( \sum_{n \geq 0} I(\nu)_n t^n \) is not a finitely generated \( R \)-algebra. In a two dimensional normal local ring \( R \), the condition that the filtration of valuation ideals \( \mathcal{I}(\nu) \) is Noetherian for all \( m_R \)-valuations \( \nu \) dominating \( R \) is the condition \( (N) \) of Muhly and Sakuma [23]. It is proven in [6] that a complete normal local ring of dimension two satisfies condition \( (N) \) if and only if its divisor class group is a torsion group.

An integral divisorial filtration of \( R \) (which we will also refer to as a divisorial filtration) is a filtration \( \mathcal{I} = \{ I_m \} \) such that there exist divisorial valuations \( \nu_1, \ldots, \nu_s \) and \( a_1, \ldots, a_s \in \mathbb{N} \) such that
\( Z_{\geq 0} \) such that for all \( m \in \mathbb{N} \),

\[ I_m = I(\nu_{1})^{m_{1}} \cap \cdots \cap I(\nu_{s})^{m_{s}}. \]

\( \mathcal{I} \) is called an \( \mathbb{R} \)-divisorial filtration if \( a_{1}, \ldots, a_{s} \in \mathbb{R}_{> 0} \) and \( \mathcal{I} \) is called a \( \mathbb{Q} \)-divisorial filtration if \( a_{1}, \ldots, a_{s} \in \mathbb{Q} \). If \( a_{i} \in \mathbb{R}_{> 0} \), then

\[ I(\nu_{i})^{m_{i}} := \{ f \in R \mid \nu_{i}(f) \geq n_{a_{i}} \} = I(\nu_{i})^{\lceil n_{a_{i}} \rceil}, \]

where \( \lceil x \rceil \) is the round up of a real number.

It is shown in [11, Theorem 4.6] that the “if” statement of Theorem 1.1 is true for \( \mathbb{R} \)-divisorial filtrations of a local domain \( R \). This theorem is stated for divisorial filtrations, but the proof is valid for \( \mathbb{R} \)-divisorial filtrations.

**Theorem 1.2.** ([11, Theorem 4.6]) Suppose that \( R \) is a Noetherian local domain and \( \mathcal{I} = \{ I_{n} \} \) is an \( \mathbb{R} \)-divisorial filtration on \( R \) such that \( \ell(\mathcal{I}) = \dim R \). Then there exists \( n_{0} \in \mathbb{N} \) such that \( m_{R} \in \text{Ass}(R/I^{n_{0}}) \) for all \( n \geq n_{0} \).

An interesting question is if the converse of Theorem 1.1 is also true for divisorial filtrations of a local ring \( R \). We prove this for excellent normal local rings for which resolution of singularities holds in this paper.

**Theorem 1.3.** Let \( R \) be an excellent local ring of equicharacteristic 0, or of dimension \( \leq 3 \). Let \( \mathcal{I} = \{ I_{n} \} \) be a \( \mathbb{Q} \)-divisorial filtration on \( R \). Suppose that \( m_{R} \in \text{Ass}(R/I_{m_{0}}) \) for some \( m_{0} \in \mathbb{Z}_{> 0} \). Then the analytic spread of \( \mathcal{I} \) is \( \ell(\mathcal{I}) = \dim R \). Further, there exists \( n_{0} \in \mathbb{Z}_{> 0} \) such that \( m_{R} \in \text{Ass}(R/I_{n}) \) if \( n \geq n_{0} \).

The following theorem is immediate from Theorems 1.2 and 1.3. Theorem 1.4 generalizes Theorem 1.1 from ideals to filtrations.

**Theorem 1.4.** Let \( R \) be an excellent local ring of equicharacteristic 0, or of dimension \( \leq 3 \). Let \( \mathcal{I} = \{ I_{n} \} \) be a \( \mathbb{Q} \)-divisorial filtration on \( R \). Then the following are equivalent.

1. The analytic spread of \( \mathcal{I} \) is \( \ell(\mathcal{I}) = \dim R \).
2. There exists \( n_{0} \in \mathbb{Z}_{> 0} \) such that \( m_{R} \in \text{Ass}(R/I_{n}) \) if \( n \geq n_{0} \).
3. \( m_{R} \in \text{Ass}(R/I_{m_{0}}) \) for some \( m_{0} \in \mathbb{Z}_{> 0} \).

The conclusions of Theorem 1.3 do not hold for more general filtrations.

**Example 1.5.** There exist \( \mathbb{R} \)-divisorial filtrations \( \mathcal{I} \) such that the conclusions of Theorem 1.3 are false.

**Proof.** Let \( R = k[[x]] \) be a power series ring in one variable over a field \( k \). Let \( I_{n} = (x^{[n\pi]}) \) for \( n \in \mathbb{N} \), and \( \mathcal{I} = \{ I_{n} \} \). For fixed \( n, r \in \mathbb{Z}_{> 0} \). Thus \( f \in I_{n} \) implies \( f^{r} \in m_{R}I_{n} \). Thus \( \ell(\mathcal{I}) = \dim R[\mathcal{I}] = m_{R}R[\mathcal{I}] = 0 < \dim R. \]

Theorem 1.4 is proven for two dimensional normal excellent local rings in [9]. In [9], the technique of Zariski decomposition of divisors on two dimensional nonsingular schemes is used to reduce the analysis of \( \mathbb{Q} \)-divisors on a resolution of singularity of a two dimensional excellent normal domain to the case of numerically effective \( \mathbb{Q} \)-divisors, which have very good properties, allowing the proof of the equivalence of all of the statements of Theorem 1.4 in dimension 2, using these geometric methods. Zariski decomposition does not exist in higher dimensions, even after blowing up ([5], [12], [24, Section IV.2.10], [18, Section 2.3]), so a different method must be used in higher dimensions.
2. Divisorial filtrations on normal excellent local rings

Let $R$ be a normal excellent local ring. Let $\mathcal{I} = \{I_m\}$ where

$$I_m = I(\nu_1)^{a_1} \cap \cdots \cap I(\nu_s)^{a_s},$$

for some divisorial valuations $\nu_1, \ldots, \nu_s$ of $R$ be an $R$-divisorial filtration of $R$, with $a_1, \ldots, a_s \in \mathbb{R}_{>0}$. Then there exists a projective birational morphism $\varphi : X \to \text{Spec}(R)$ such that there exist prime divisors $F_1, \ldots, F_s$ on $X$ such that $V_{\nu_i} = \mathcal{O}_{X,F_i}$ for $1 \leq i \leq s$ ([12] Remark 6.6 to Lemma 6.5). Let $D = a_1 F_1 + \cdots + a_s F_s$, an effective $R$-divisor on $X$ (an effective $R$-Weil divisor). Define $[D] = [a_1] F_1 + \cdots + [a_s] F_s$, an integral divisor. We have coherent sheaves $\mathcal{O}_X([nD])$ on $X$ such that

$$\Gamma(X, \mathcal{O}_X([nD])) = I_n$$

for $n \in \mathbb{N}$. If $X$ is nonsingular then $\mathcal{O}_X([-nD])$ is invertible. The formula (4) is independent of choice of $X$. Further, even on a particular $X$, there are generally many different choices of effective $R$-divisors $G$ on $X$ such that $\Gamma(X, \mathcal{O}_X([-nG])) = I_n$ for all $n \in \mathbb{N}$. Any choice of a divisor $G$ on such an $X$ for which the formula $\Gamma(X, \mathcal{O}_X([-nG])) = I_n$ for all $n \in \mathbb{N}$ holds will be called a representation of the filtration $\mathcal{I}$.

Given an $R$-divisor $D = a_1 F_1 + \cdots + a_s F_s$ on $X$ we have a divisorial filtration $\mathcal{I}(D) = \{I(nD)\}$ where

$$I(nD) = \Gamma(X, \mathcal{O}_X([-nD])) = I(\nu_1)^{[na_1]} \cap \cdots \cap I(\nu_s)^{[na_s]} = I(\nu_1)^{a_1} \cap \cdots \cap I(\nu_s)^{a_s}.$$

We write $R[D] = R[\mathcal{I}(D)]$.

The following result will be used in our proof.

Lemma 2.1. Suppose that $R$ is a universally Nagata domain and $X$ is an integral, projective $R$-scheme. Suppose that $D$ is an ample Cartier divisor on $X$. Then the algebra $\oplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(nD))$ is a finitely generated $R$-algebra.

We give an outline of the proof of this well known result. There exists $m \in \mathbb{Z}_{>0}$ such that $\mathcal{O}_X(mD)$ is very ample. By the argument from the last two lines of page 122 of the proof of [15] Theorem II.5.19 and [15] Remark 5.19.2, the algebra $A = \oplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(nmD))$ is a finitely generated $R$-algebra. There exists $a \in \mathbb{Z}_{>0}$ such that for $0 \leq r < m$, $\Gamma(X, \mathcal{O}_X(rD))$ is generated by global sections. Thus there exist short exact sequences $0 \to \mathcal{K}_r \to \mathcal{O}_X^X \to \mathcal{O}_X(\mathcal{O}_X((r+am)D)) \to 0$ for some $b_r \in \mathbb{Z}_{>0}$ and there exists $c \in \mathbb{Z}_{>0}$ such that we have surjections $\Gamma(X, \mathcal{O}_X(nmD))^{b_r} \to \Gamma(X, \mathcal{O}_X((nm+r+am)D))$ for all $n \geq c$ and $0 \leq r < m$. Thus, since every module $\Gamma(X, \mathcal{O}_X(nD))$ is a finitely generated $R$-module, we have that $\oplus_{n \geq 0} \Gamma(X, \mathcal{O}_X((r+nm)D))$ is a finitely generated $A$-module for $0 \leq r < m$, and so $\oplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(nD))$ is a finitely generated $R$-algebra.

3. Divisors on a resolution of singularities

Let $R$ be a normal excellent local ring with maximal ideal $m_R$. Let $K$ be the quotient field of $R$ and let $k = R/m_R$ be its residue field. Let $\pi : X \to \text{Spec}(R)$ be a birational projective morphism such that $X$ is nonsingular.

A divisor (or integral divisor) $D$ on $X$ is a sum $D = \sum a_i E_i$ where $a_i$ are integers and $E_i$ are prime divisors on $X$. $D$ is a $\mathbb{Q}$-divisor if the $a_i \in \mathbb{Q}$ and $D$ is an $R$-divisor if the $a_i$ are in $R$. The support of $D$ is the algebraic set $\bigcup_{a_i \neq 0} E_i$. $D$ is an effective divisor if all $a_i$ are nonnegative. If $D_1$ and $D_2$ are divisors, then $D_2 \geq D_1$ if $D_2 - D_1$ is an effective divisor.
If \( D_1 \) and \( D_2 \) are \( \mathbb{R} \)-divisors on \( X \), then \( D_1 \) and \( D_2 \) are linearly equivalent, written as \( D_1 \sim D_2 \), if there exists \( f \in K \) such that \( (f) = D_1 - D_2 \). Here \( (f) = \sum \nu_E(f)E \) where the sum is over all prime divisors \( E \) on \( X \) and \( \nu_E \) is the valuation of the valuation ring \( \mathcal{O}_{X,E} \).

Let \( D = \sum a_i E_i \) be an \( \mathbb{R} \)-divisor on \( X \). There is an associated integral divisor \( [D] = \sum [a_i] E_i \) where \([a]\) is the round down of a real number \( a \). Analogously, we define \([-D]\) to be \( \sum [a_i] E_i \) where \([a]\) is the round up of \( a \). We have that \(-[D] = [-D]\).

Associated to \( D \) is an invertible sheaf \( \mathcal{O}_X([D]) \), which is defined by

\[
\mathcal{O}_X([D])|U = \frac{1}{\prod f_i^{[a_i]} \mathcal{O}_X|U}
\]

whenever \( U \) is an open subset of \( X \) such that the \( E_i \) have local equations \( f_i = 0 \) in \( \Gamma(U, \mathcal{O}_X) \).

For \( V \) an open subset of \( X \),

\[
\Gamma(V, \mathcal{O}_X([D])) = \{ f \in K \mid ((f) + [D]) \cap V \geq 0 \} = \{ f \in K \mid ((f) + D) \cap V \geq 0 \}
\]

since for \( f \in K \), \((f) + D \geq 0\) if and only if \((f) + [D] \geq 0\).

If \( D_1 \) and \( D_2 \) are \( \mathbb{R} \)-divisors, then there is a natural map

\[
\mathcal{O}_X([D_1]) \otimes \mathcal{O}_X([D_2]) \to \mathcal{O}_X([D_1 + D_2]).
\]

This multiplication is somewhat subtle if \( D_1 \) and \( D_2 \) are not integral divisors. With the notation of (5), suppose that \( U \) is affine. Suppose that \( D_1 = \sum a_i E_i \) and \( D_2 = \sum b_i E_i \). Then

\[
\mathcal{O}_X([D_1]|U \otimes \mathcal{O}_X([D_2])|U \to \mathcal{O}_X([D_1 + D_2])|U
\]

is the map

\[
ger \prod f_i^{[a_i]} \otimes h \prod f_i^{[b_i]} \mapsto gh \prod f_i^{[a_i + b_i]}
\]

where \( g, h \in \mathcal{O}_X(U) \) and \( 0 \leq c_i = [a_i + b_i] - ([a_i] + [b_i]) \).

Let \( D \) be an integral divisor on \( X \). Write \( D = G - F \) where \( G \) and \( F \) are effective integral divisors. Then the natural inclusion \( \mathcal{O}_X(-G) \to \mathcal{O}_X \) induces an inclusion \( \mathcal{O}_X \to \mathcal{O}_X(G) \), and thus an inclusion \( \mathcal{O}_X(-F) \to \mathcal{O}_X(D) \). Taking global sections, we have an inclusion \( \Gamma(X, \mathcal{O}_X(-F)) \to \Gamma(X, \mathcal{O}_X(D)) \). Now \( \Gamma(X, \mathcal{O}_X(-F)) \) is an intersection of valuation ideals in \( R \) so that \( \Gamma(X, \mathcal{O}_X(D)) \neq 0 \). In particular, we see that there exists an effective integral divisor \( G \) on \( X \) such that \( G \sim D \).

**Lemma 3.1.** Suppose that \( R \) is an excellent local domain such that either \( R \) is of equicharacteristic zero or \( \dim R \leq 3 \) and that \( \nu_1, \ldots, \nu_r \) are divisorial valuations of \( R \). Then there exists a birational projective morphism \( \pi : X \to \text{Spec}(R) \) such that \( X \) is nonsingular and there exist prime divisors \( F_1, \ldots, F_r \) on \( X \) such that \( \mathcal{O}_{X,F_i} \) is the valuation ring of \( \nu_i \) for \( 1 \leq i \leq r \).

**Proof.** By [10, Remark 6.6] to [10, Lemma 6.5], there exists a birational projective morphism \( Y \to \text{Spec}(R) \) such that \( Y \) is normal and there exist prime divisors \( E_1, \ldots, E_r \) on \( Y \) such that \( \mathcal{O}_{Y,E_i} \) is the valuation ring of \( \nu_i \) for \( 1 \leq i \leq r \). There exists a birational projective morphism \( X \to Y \) such that \( X \) is nonsingular by [10, Main Theorem I(n), page 145] or [28] if \( R \) is excellent of equicharacteristic zero, and there exists such a resolution of singularities by [4] if \( \dim R \leq 3 \). \( \square \)
4. The $\gamma_T$ Function

We recall the $\gamma_T$ function defined in [3, Section 3]. The statements and proofs in this section are based on corresponding statements and proofs for the $\sigma_T$ function on pseudo-effective divisors on a projective nonsingular variety in [24, Chapter III, Section 1].

We continue to assume that $R$ is a normal excellent local ring and that $\pi : X \to \text{Spec}(R)$ is a birational projective morphism such that $X$ is nonsingular. Let $G = \sum a_iE_i$ be an effective $\mathbb{R}$-divisor, and $\Gamma$ be a prime divisor on $X$. Let

$$\text{ord}_\Gamma(G) = \begin{cases} a_i & \text{if } \Gamma = E_i \\ 0 & \text{if } \Gamma \not\in \text{Supp}(D). \end{cases}$$

For $D$ an $\mathbb{R}$-divisor, let

$$\tau_T(D) = \inf \{\text{ord}_\Gamma(G) \mid G \geq 0 \text{ and } G \sim D\},$$

and define

$$\gamma_T(D) = \inf \left\{ \frac{\tau_T(mD)}{m} \mid m \in \mathbb{Z}_{>0} \right\}.$$

Since $D$ is linearly equivalent to an effective divisor, there can be only finitely many prime divisors $\Gamma$ such that $\gamma_T(D) > 0$.

If $R$ has dimension 2 and $D$ is an integral divisor on $X$ then $\gamma_T(D)$ is a rational number, but there exists examples where $R$ has dimension 3 and integral divisors $D$ on $X$ such that $\gamma_T(D)$ is an irrational number ([7, Theorem 4.1]).

We have that $\tau_T(D_1 + D_2) \leq \tau_T(D_1) + \tau_T(D_2)$ for $\mathbb{R}$-divisors $D_1$ and $D_2$. If $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ is a function such that $f(k_1 + k_2) \leq f(k_1 + k_2)$ for all $k_1, k_2$, then $\lim_{k \to \infty} \frac{f(k)}{k}$ exists ([24, Lemma III.1.3]). Hence

$$\gamma_T(D) = \lim_{m \to \infty} \frac{\tau_T(mD)}{m}.$$  \hspace{1cm} (7)

Thus if $\alpha \in \mathbb{Q}_{>0}$, we have that

$$\gamma_T(\alpha D) = \alpha \gamma_T(D).$$

We also have that for $\mathbb{R}$-divisors $D_1$ and $D_2$,

$$\gamma_T(D_1 + D_2) \leq \gamma_T(D_1) + \gamma_T(D_2).$$  \hspace{1cm} (8)

If $m \in \mathbb{Z}_{>0}$ and $G \sim mD$ is an effective $\mathbb{R}$-divisor, then $\text{ord}_\Gamma(G) \geq m \gamma_T(D)$.

**Lemma 4.1.** Let $D$ be an $\mathbb{R}$-divisor and $\Gamma$ be a prime divisor on $X$. Then

$$\Gamma(X, \mathcal{O}_X([nD])) = \Gamma(X, \mathcal{O}_X([n(D - \gamma_T(D)\Gamma)]))$$

for all $n \in \mathbb{N}$.

**Proof.** We have that $|n(D - \gamma_T(D)\Gamma)| \leq |nD|$ for all $n \in \mathbb{N}$ since $\gamma_T(D) \geq 0$. Thus there is a natural inclusion $\Gamma(X, \mathcal{O}_X([nD])) \to \Gamma(X, \mathcal{O}_X([nD]))$ for all $n \in \mathbb{N}$.

Suppose that $f \in \Gamma(X, \mathcal{O}_X([nD]))$. Then $(f) + nD \geq 0$ so $\text{ord}_\Gamma((f) + nD) \geq n \gamma_T(D)$. Thus $(f) + nD = U + n \gamma_T(D)$ for some effective $\mathbb{R}$-divisor $U$ which implies that $(f) + [n(D - \gamma_T(D)\Gamma)]$ is effective. Thus $\Gamma(X, \mathcal{O}_X([n(D - \gamma_T(D)\Gamma)])) = \Gamma(X, \mathcal{O}_X([nD]))$. \hspace{1cm} $\square$

**Lemma 4.2.** Let $D$ be an $\mathbb{R}$-divisor and $\Gamma$ be a prime divisor on $X$ such that $\gamma_T(D) > 0$. Let $s \in \mathbb{R}$ be such that $0 \leq s \leq \gamma_T(D)$. Then $\gamma_T(D - s\Gamma) = \gamma_T(D) - s$.  \hspace{1cm} 6
Lemma 4.3. Let $D$ be an $\mathbb{R}$-divisor and $\Gamma$ be a prime divisor on $X$ such that $\gamma_\Gamma(D) > 0$. Then for $s \in \mathbb{R}_{\geq 0}$, $\gamma_\Gamma(D + s\Gamma) = \gamma_\Gamma(D) + s$.

Proof. Let $E = D + s\Gamma$. Let $\gamma = \gamma_\Gamma(D)$. For $0 < c < 1$ we have

$$(1 - c)(D - \gamma\Gamma) + cE = D + (1 - c)\gamma + cs\Gamma).$$

Let $c$ be a rational number with $0 < c < \frac{\gamma}{s + \gamma}$. Then $-\gamma < -(1 - c)\gamma + cs < 0$. By (8),

$$\gamma(D + (1 - c)\gamma + cs\Gamma) \leq \gamma((1 - c)(D - \gamma\Gamma)) + \gamma(cE).$$

By Lemma 4.2 and (7),

$$c(\gamma + s) = \gamma_\Gamma(D) + (1 - c)\gamma + cs \leq (1 - c)\gamma_\Gamma(D - \gamma\Gamma) + c\gamma_\Gamma(E) = c\gamma_\Gamma(E).$$

Thus $\gamma_\Gamma(E) \geq \gamma + s$. By (8), we have that

$$\gamma_\Gamma(E) \leq \gamma_\Gamma(D) + \gamma_\Gamma(s\Gamma) \leq \gamma_\Gamma(D) + s.$$

Lemma 4.4. Let $\Gamma$ be a prime divisor on $X$ and $D$ be a $\mathbb{Q}$-divisor on $X$ such that $\gamma_\Gamma(D) = 0$. Let $A$ be an ample $\mathbb{Q}$-divisor on $X$. Then there exists an effective integral divisor $B$ and $m > 0$ such that $\Gamma$ is not in the support of $B$, $m(D + A)$ is an integral divisor and $B \sim m(D + A)$.

Proof. There exists $n_0 \in \mathbb{Z}$ such that $n_0A$ is an ample integral divisor and $n_0A + \Gamma$ is an ample integral divisor ([15 Exercise II.7.5]). There exists $m \in \mathbb{Z}_{>0}$ and an effective integral divisor $U$ such that $mD$ and $mA$ are integral divisors, $U \sim mD$ and $\alpha := \operatorname{ord}_U < \frac{m}{n_0}$. Necessarily, $\alpha \in \mathbb{N}$. Let $\overline{U} = U - \alpha\Gamma$. $\overline{U}$ is an integral divisor which does not have $\Gamma$ in its support.

$$m(D + A) \sim \overline{U} + \alpha\Gamma + mA = \overline{U} + \alpha(n_0A + \Gamma) + (m - n_0\alpha)A$$

with $m - n_0\alpha > 0$. Further, $\overline{U}$, $\alpha(n_0A + \Gamma)$ and $(m - n_0\alpha)A$ are all three integral divisors. Since $n_0A + \Gamma$ and $A$ are ample $\mathbb{Q}$-divisors, there exists $n \in \mathbb{Z}_{>0}$ and effective integral divisors $D_1$ and $D_2$ which do not contain $\Gamma$ in their supports such that $D_1 \sim n(n_0A + \Gamma)$ and $D_2 \sim n(m - n_0\alpha)A$. Thus

$$n(\overline{U} + \alpha(n_0A + \Gamma) + (m - n_0\alpha)A) \sim n\overline{U} + \alpha D_1 + D_2$$

is an effective divisor which does not contain $\Gamma$ in its support. The divisor

$$B = (n\overline{U} + \alpha D_1 + D_2)$$

satisfies the conclusions of the lemma (with $B \sim nm(D + A)$). \qed
5. Analytic spread and associated primes

In this section we prove Theorem 1.3 in the case that $R$ is normal. We assume that $R$ is normal throughout this section. Let $k = R/m_R$. There exists by Lemma 4.1 a projective birational morphism $\pi : X \to \text{Spec}(R)$ such that $X$ is nonsingular and there exists an effective $\mathbb{Q}$-divisor $D$ on $X$ such that $\mathcal{I} = \mathcal{I}(D)$. Let

$$D = \sum a_i F_i$$

with the $F_i$ distinct prime divisors and $a_i \in \mathbb{Q}_{\geq 0}$.

After reindexing the $F_i$, we may assume that the first $s$ prime divisors $F_1, \ldots, F_s$ are the components of $D$ which contract to $m_R$. For $n \in \mathbb{Z}_{>0}$, we have that $m_R \in \text{Ass}(R/I_n)$ if and only if $I(nD) \neq I(n(\sum_{i=s} F_i))$ which holds if and only if there exists $j$ with $1 \leq j \leq s$ such that $I(nD) \neq I(n(\sum_{i \neq j} F_i))$. Since $m_R \in \text{Ass}(R/I_{m_0})$ for some $m_0$, we have that some $F_j$ contracts to $m_R$ for some $j$ with $1 \leq j \leq s$, $a_j > 0$ and

$$I(m_0 D) \neq I(m_0(\sum_{i \neq j} a_i F_i)) \quad (9)$$

Since $F_j \subset \pi^{-1}(m_R)$, $F_j$ is a projective $k$-variety. Let $L = \Gamma(F_j, \mathcal{O}_{F_j})$. Taking global sections of the exact sequence

$$0 \to \mathcal{O}_X(-F_j) \to \mathcal{O}_X \to \mathcal{O}_{F_j} \to 0,$$

and since $\Gamma(X, \mathcal{O}_X(-F_j)) = m_R$, we obtain that $\Gamma(X, \mathcal{O}_X)/\Gamma(X, \mathcal{O}_X(-F_j)) = R/m_R = k$ and we obtain an inclusion $k = R/m_R \to L$. Since $\pi$ is a projective morphism, we have that $L$ is a finitely generated $R$-module, and since $F_j$ is a projective variety, $L$ is a field. Thus $L$ is a finite extension field of $k$.

Let $D_1 = \sum_{i \neq j} a_i F_i$, so that $-D \leq -D_1 = -D + a_j F_j$. Suppose that $\gamma F_j(-D) > 0$. Then $\gamma F_j(-D_1) = \gamma F_j(-D) + a_j$ by Lemma 4.3. Thus

$$\Gamma(X, \mathcal{O}_X(\lfloor-nD_1\rfloor)) = \Gamma(X, \mathcal{O}_X(\lfloor-nD\rfloor))$$

for all $n \in \mathbb{N}$ by Lemma 4.1, which is a contradiction to (10). Thus $\gamma F_j(-D) = 0$ and $0 \leq \gamma F_j(-D_1) = a_j$.

Let $t \in \mathbb{Q}$ be such that $0 < t < a_j - \gamma F_j(-D_1)$. Let $D_2 = D - t F_j$. Suppose that $\gamma F_j(-D_2) > 0$. Then by Lemmas 4.2 and 4.3

$$0 = \gamma F_j(-D_1 - \gamma F_j(-D_1) F_j) = \gamma F_j(-D_2 + (a_j - \gamma F_j(-D_1) - t)F_j) = \gamma F_j(-D_2) + (a_j - \gamma F_j(-D_1) - t) > 0,$$

a contradiction. Thus $\gamma F_j(-D_2) = 0$.

Since $X$ is the blowup of an ideal $I$ in $R$, there exists an ample integral divisor $A$ on $X$ such that $I \mathcal{O}_X = \mathcal{O}_X(A)$. We may assume that all prime divisors in the support of $A$ appear amongst the $F_i$ in the expansion (8) of $D$ by possibly adding some $F_i$ with $a_i = 0$ to (9). Expand $A = \sum -b_i F_i$ with $b_i \in \mathbb{N}$. The support of $A$ must contain all prime divisors on $X$ which contract to $m_R$, so $b_j > 0$. We have that $F_j = -1/b_j A - \sum_{i \neq j} b_i F_i$, and

$$-D = -D_2 - t F_j = -D_2 + \frac{t}{b_j} A + t \sum_{i \neq j} \frac{b_i}{b_j} F_j$$

Let $\Lambda = -D_2 + t(\sum_{i \neq j} \frac{b_i}{b_j} F_j)$. We have that $\gamma F_j(\Lambda) = 0$ since $\gamma F_j(-D_2) = 0$. Since $\Lambda$ is a $\mathbb{Q}$-divisor, there exists $m \in \mathbb{Z}_{>0}$ and an effective integral divisor $U$ such that $m(\Lambda + \frac{t}{2b_j} A)$ is an integral divisor, $F_j$ is not in the support of $U$ and $m(\Lambda + \frac{t}{2b_j} A) \sim U$ by Lemma 4.4.
Let $\Theta = \Lambda + \frac{1}{2b_j}A$, so that $-D = \Theta + \frac{1}{2b_j}A$. Since $F_j$ is not in the support of $U$, $U$ restricts to an effective Cartier divisor $\overline{U}$ on $F_j$. We have natural isomorphisms

$$ \mathcal{O}_X(m\Theta) \otimes \mathcal{O}_{F_j} \cong \mathcal{O}_X(U) \otimes \mathcal{O}_{F_j} \cong \mathcal{O}_{F_j}(\overline{U}). $$

Let $\tau \in \Gamma(F_j, \mathcal{O}_X(m\Theta) \otimes \mathcal{O}_{F_j})$ be the (nonzero) section corresponding to $\overline{U}$, giving an inclusion $\tau : \mathcal{O}_{F_j} \to \mathcal{O}_X(m\Theta) \otimes \mathcal{O}_{F_j}$.

Since $\frac{1}{2b_j}A$ is an ample $\mathbb{Q}$-divisor, after possibly replacing $m$ with a multiple of $m$, we may further assume that $-\frac{mt}{2b_j}A$ is a very ample integral divisor and by [15, Theorem II.5.2],

$$ H^1(X, \mathcal{O}_X(n\frac{mt}{2b_j}A - F_j)) = 0 \text{ for all } n \geq 1. $$

We have that $\mathcal{O}_X(n\frac{mt}{2b_j}A) \otimes \mathcal{O}_{F_j}$ is a very ample invertible sheaf on the projective $k$-variety $F_j$, so that $\mathcal{S} = \oplus_{n \geq 0} \Gamma(F_j, \mathcal{O}_X(n\frac{mt}{2b_j}A) \otimes \mathcal{O}_{F_j})$ is a finitely generated graded $k$-algebra (by Lemma 6), of dimension $\dim F_j + 1 = \dim R$. Let $\mathcal{S}_{>0} = \oplus_{n > 0} \Gamma(F_j, \mathcal{O}_X(n\frac{mt}{2b_j}A) \otimes \mathcal{O}_{F_j})$, the graded maximal ideal of $\mathcal{S}$. The ring $\oplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(n\frac{mt}{2b_j}A))$ is a finitely generated graded $R$-algebra (by Lemma 2.1), and so its image $k \otimes \mathcal{S}_{>0}$ in $S$ is a finitely generated graded $k$-algebra. Since $L$ is a finite extension field of $k$, $\mathcal{S}$ is finite over $S$ and so $\dim S = \dim R$ (by [2, Corollary A.8]).

The section $\tau$ induces inclusions

$$ \mathcal{O}_X(n\frac{mt}{2b_j}A) \otimes \mathcal{O}_{F_j} \xrightarrow{\tau \otimes \text{id}} \left( \mathcal{O}_X(nm\Theta) \otimes \mathcal{O}_{F_j} \right) \otimes \left( \mathcal{O}_X(n\frac{mt}{2b_j}A) \otimes \mathcal{O}_{F_j} \right) \cong \mathcal{O}_X(-nmD) \otimes \mathcal{O}_{F_j} $$

for all $n \in \mathbb{N}$ (by our choice of $m$, $mD$ is an integral divisor).

For $n \in \mathbb{N}$, let $\Omega_n$ be the image of the natural surjection

$$ \Gamma(X, \mathcal{O}_X([-nD])) \twoheadrightarrow \Gamma(F_j, \mathcal{O}_X([-nD]) \otimes \mathcal{O}_{F_j}). $$

The kernel of this surjection is $\Gamma(X, \mathcal{O}_X([-nD]) \otimes \mathcal{O}_X(-F_j))$ which contains $m\mathcal{R}_\mathcal{G}(X, \mathcal{O}_X([-nD]))$.

Thus $\Omega_n$ is a $k = R/m\mathcal{R}$-module. Further, $\Omega_0 = R/m\mathcal{R}$.

Identifying $S$ with the isomorphic graded $k$-algebra

$$ k \oplus \left( \oplus_{n \geq 0} \Gamma(F_j, \mathcal{O}_X(n\frac{mt}{2b_j}A) \otimes \mathcal{O}_{F_j}) \tau^n \right), $$

we have that $S$ is a finitely generated graded $k$-subalgebra of the graded $k$-algebra $\oplus_{n \geq 0} \Omega_{nm}$. There exists $\beta \in \mathbb{Z}_{>0}$ such that $\mathcal{O}_X(-mD + \beta\frac{mt}{2b_j}A)$ is ample ([15, Exercise II.7.5]) and $\Gamma(F_j, \mathcal{O}_X(\beta\frac{mt}{2b_j}A) \otimes \mathcal{O}_{F_j}) \neq 0$. Let

$$ T = \oplus_{n \geq 0} \Gamma(F_j, \mathcal{O}_X(-nmD + n\beta\frac{mt}{2b_j}A) \otimes \mathcal{O}_{F_j}). $$

Then $T$ is a finitely generated $k$-algebra such that $\dim T = \dim R$ (by Lemma 2.1). A nonzero section $\lambda$ of $\Gamma(F_j, \mathcal{O}_X(\beta\frac{mt}{2b_j}A) \otimes \mathcal{O}_{F_j})$ induces an inclusion

$$ \lambda : \mathcal{O}_{F_j} \to \mathcal{O}_X(\beta\frac{mt}{2b_j}A) \otimes \mathcal{O}_{F_j}, $$

and hence an inclusion of graded $k$-algebras $\oplus_{n \geq 0} \Omega_{nm} \subset T$. 


A $k$-algebra is called subfinite it is is a subalgebra of a finitely generated $k$-algebra. Thus
\[ S \subset \oplus_{n \geq 0} \Omega_{nm} \subset T \]
are subfinite, and so
\[ \dim R = \dim S \leq \dim \oplus_{n \geq 0} \Omega_{nm} \leq \dim T = \dim R \]
by \cite{17} Corollary 4.7. The graded extension $\oplus_{n \geq 0} \Omega_{nm} \to \oplus_{n \geq 0} \Omega_n$ is integral, so
\[ \dim \oplus_{n \geq 0} \Omega_n \geq \dim \oplus_{n \geq 0} \Omega_{nm} = \dim R \]
by the going up theorem (\cite{11} Theorem 5.11). Now
\[ m_R \Gamma(X, \mathcal{O}_X([-lD])) \subset \Gamma(X, \mathcal{O}_X([-lD] - F_j)) \]
for all $l$, so there is natural surjection of graded rings
\[ R[D]/m_R R[D] \to \oplus_{n \geq 0} \Omega_n. \]
Thus the analytic spread $\ell(I) = \dim R[D]/m_R R[D] \geq \dim R$. Since the analytic spread is bounded above by $\dim R$ (\cite{11} Lemma 3.6) we have that $\ell(I) = \dim R$.

6. Proof for excellent domains

In this section we finish the proof of Theorem \cite{13}. That is, we do not assume that $R$ is normal.

There exist $m_R$-valuations $\nu_1, \ldots, \nu_t$ and $a_1, \ldots, a_t \in \mathbb{Q}_{>0}$ such that $I = \{I_n\}$ where $I_n = I(\nu_1)_{a_1^n} \cap \cdots \cap I(\nu_t)_{a_t^n}$ for $n \geq 0$, with $I(\nu_i)_{m} = \{ f \in R | \nu_i(f) \geq m \}$.

Let $S$ be the normalization of $R$ in the quotient field of $R$. Let $m_1, \ldots, m_n$ be the maximal ideals of $S$. Let $J(\nu_i)_m = \{ f \in S | \nu_i(f) \geq m \}$. For $n \in \mathbb{N}$, let
\[ J_n = J(\nu_1)_{a_1^n} \cap \cdots \cap J(\nu_t)_{a_t^n} \]
so that $J_n \cap R = I_n$.

Since we assume that $m_R \in \text{Ass}_R(R/I_{m_0})$, there exists $y \in R/I_{m_0}$ such that $m_R = \text{ann}_R(y)$. We have that $y \in S/J_n$ is nonzero since $R/I_{m_0} \to S/J_{m_0}$ is an inclusion. Thus $\text{Ann}_{S}(y) \neq S$. Since maximal elements in the set of annihilators of elements of $S/J_{m_0}$ are prime ideals (by \cite{21} Theorem 6.1), there exists a prime ideal $Q$ in $S$ which contains $\text{Ann}_{S}(y)$ and is the annihilator of an element $z$ of $S/J_{m_0}$. We have that $Q$ contains $m_RS$ and
\[ \sqrt{m_RS} = \cap m_i \]
so $Q$ is a maximal ideal $m_i$ of $S$. Thus
\[ \ell(\{(J_n)_{m_i}\}) = \dim(S_{m_i}) = \dim R, \]
since Theorem \cite{13} has been proven for normal excellent local rings. Let $B = \oplus_{n \geq 0} J_n$, which is a graded ring. We thus have by \cite{3}, \cite{12} and \cite{11} that $\dim B/m_R B = \dim R$.

Thus there is a chain of distinct prime ideals
\[ C_0 \subset C_1 \subset C_2 \subset \cdots \subset C_d \]
in $B$ which contain $m_R B$, where $d = \dim R$.

There is a natural inclusion of graded rings $R[I] = \oplus_{n \geq 0} I_n \subset B = \oplus_{n \geq 0} J_n$. We will now show that $B$ is integral over $R[I]$. For $a \in \mathbb{Z}_{>0}$, let $R[I]_a$ be the $a$-th truncation of $R[I]$ and $B_a$ be the $a$-th truncation of $B$, so that $R[I]_a$ is the subalgebra of $R[I]$ generated by $\oplus_{n \leq a} I_n$ and $B_a$ is the subalgebra of $B$ generated by $\oplus_{n \leq a} J_n$. It suffices to show that homogeneous elements of $B$ are integral over $R[I]$. Suppose that $f \in J_a$ for
some $a$. Then $f \in B_a$. Let $0 \neq x$ be in the conductor of $S$ over $R$. Then $xJ_n \subset I_n$ for all $n$ since $I_n = J_n \cap R$. Thus $xB_a \subset R[I]_a$, so $f^i \in \frac{1}{x}R[I]_a$ for all $i \in \mathbb{N}$, and so the algebra $R[I]_a[f] \subset \frac{1}{x}R[I]_a$. Since $\frac{1}{x}R[I]_a$ is a finitely generated $R[I]_a$-module and $R[I]_a$ is a Noetherian ring, the ring $R[I]_a[f]$ is a finitely generated $R[I]_a$-module, so that $f$ is integral over $R[I]_a$.

We have a chain of prime ideals

$$Q_0 \subset Q_1 \subset Q_2 \subset \cdots \subset Q_d$$

in $R[I]$ where $Q_i := C_i \cap R[I]$. The $Q_i$ are all distinct since the $C_i$ are all distinct and $B$ is integral over $R[I]$ (by [2, Theorem A.6 (b)]). We have that $m_R R[I] \subset Q_0$, so that $\dim R[I]/m_R R[I] \geq d$. Since this is the maximum possible dimension of $R[I]/m_R R[I]$ by (3), we have that $\ell(I) = d$.

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