Asymmetric Orbifolds, Noncommutative Geometry and Type I String Vacua

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Abstract
We investigate the D-brane contents of asymmetric orbifolds. Using T-duality we find that the consistent description of open strings in asymmetric orbifolds requires to turn on background gauge fields on the D-branes. We derive the corresponding noncommutative geometry arising on such D-branes with mixed Neumann-Dirichlet boundary conditions directly by applying an asymmetric rotation to open strings with pure Dirichlet or Neumann boundary conditions. As a concrete application of our results we construct asymmetric type I vacua requiring open strings with mixed boundary conditions for tadpole cancellation.
1. Introduction

As is known since the work of Connes, Douglas and Schwarz [1], matrix theory compactifications on tori with background three-form flux lead to noncommutative geometry. Starting with the early work [2] one has subsequently realized that open strings moving in backgrounds with non-zero two-form flux or non-zero gauge fields have mixed boundary conditions leading to a noncommutative geometry on the boundary of the string worldsheet [3-14]. As pointed out in [9], also the effective theory on the D-branes becomes a noncommutative Yang-Mills theory.

We know from the discovery of D-branes, that Dirichlet branes made their first appearance by studying the realization of T-duality on a circle in the open string sector [15]. For instance, starting with a D9 brane, the application of T-duality leads to a D8-brane where the ninth direction changes from a Neumann boundary condition to a Dirichlet boundary condition. Thus, one may pose the question how D-branes with mixed Neumann-Dirichlet boundary conditions fit into this picture. Does there exist a transformation relating pure Dirichlet or Neumann boundary conditions to mixed Neumann-Dirichlet boundary conditions?

At first sight unrelated, there exists the so far unresolved problem of what the D-brane content of asymmetric orbifolds is. The simplest asymmetric orbifold is defined by modding out by T-duality itself, which is indeed a symmetry as long as one chooses the circle at the self-dual radius. Thus, as was argued in [16] and applied to type I compactifications in [17], in this special case D9- and D5-branes are identified under the asymmetric orbifold action. However, the general T-duality group for compactifications on higher dimensional tori contains more general asymmetric operations. For instance, the root lattice of SU(3) allows an asymmetric $\hat{\mathbb{Z}}_3$ action. (A left-right asymmetric $\mathbb{Z}_N$ symmetry is denoted by $\hat{\mathbb{Z}}_N$.) The closed string sector can very well live with such non-geometric symmetries [18] but what about the open string sector? Since all type II string theories contain open strings in the non-perturbative D-brane sector, in order for asymmetric orbifolds to be non-perturbatively consistent, one has to find a realization of such non-geometric symmetries in the open string sector, as well. Thus, the question arises what the image of a D9-brane under an asymmetric $\hat{\mathbb{Z}}_N$ action is.

The third motivation for the investigation performed in this paper is due to recently introduced orientifolds with D-branes at angles [19-23]. We investigated orientifold models for which the world-sheet parity transformation, $\Omega$, is combined with a complex conjugation, $\mathcal{R}$, of the compact coordinates. After dividing by a further left-right symmetric $\mathbb{Z}_N$
space-time symmetry the cancellation of tadpoles required the introduction of so-called twisted open string sectors. These sectors were realized by open strings stretching between D-branes intersecting at non-trivial angles. As was pointed out in [20], these models are related to ordinary Ω orientifolds by T-duality. However, under this T-duality the former left-right symmetric $\mathbb{Z}_N$ action is turned into an asymmetric $\hat{\mathbb{Z}}_N$ action in the dual model. Thus, we are led to the problem of describing asymmetric orientifolds in a D-brane language. Note, that using pure conformal field theory methods asymmetric orientifolds were discussed recently in [24].

In this paper, we study the three conceptually important problems mentioned above, for simplicity, in the case of compactifications on direct products of two-dimensional tori. It turns out that all three problems are deeply related. The upshot is that asymmetric rotations turn Neumann boundary conditions into mixed Neumann-Dirichlet boundary conditions. This statement is the solution to the first problem and allows us to rederive the noncommutative geometry arising on D-branes with background gauge fields simply by applying asymmetric rotations to ordinary D-branes. The solution to the second problem is that asymmetric orbifolds necessarily contain open strings with mixed boundary conditions. In other words: D-branes manage to incorporate asymmetric symmetries by turning on background gauge fluxes, which renders their world-volume geometry noncommutative. Gauging the asymmetric symmetry can then lead to an identification of commutative and noncommutative geometries. In this sense asymmetric type II orbifolds are deeply related to noncommutative geometry. Apparently, the same holds for asymmetric orientifolds, orbifolds of type I. Via T-duality the whole plethora of $\Omega\mathcal{R}$ orientifold models of [20-22] is translated into a set of asymmetric orientifolds with D-branes of different commutative and noncommutative types in the background. We will further present a D-brane interpretation of some of the non-geometric models studied in [24] and some generalizations thereof.

In section 2 we describe a special class of asymmetric orbifolds on $T^2$. Employing T-duality we first determine the tori allowing an asymmetric $\hat{\mathbb{Z}}_N$ action, where we discuss the $\hat{\mathbb{Z}}_3$ example in some detail. Afterwards we study D-branes in such models and also determine the zero-mode spectrum for some special values of the background gauge flux. In section 3 we apply asymmetric rotations to give an alternative derivation of the propagator on the disc with mixed Neumann-Dirichlet boundary conditions. Moreover, we compute the commutator of the coordinate fields confirming the well known results in the literature. In the final section we apply all our techniques to the explicit construction of a $\mathbb{Z}_3 \times \hat{\mathbb{Z}}_3$ orientifold containing D-branes with mixed boundary conditions.
2. D-branes in asymmetric orbifolds

In this section we investigate in which way open strings manage to implement asymmetric symmetries. Naively, one might think that asymmetric symmetries are an issue only in the closed string sector, as open strings can be obtained by projecting onto the left-right symmetric part of the space-time. However, historically just requiring the asymmetric symmetry under T-duality on a circle led to the discovery of D-branes. This T-duality acts on the space-time coordinates as

\[(X_L, X_R) \rightarrow (-X_L, X_R). \tag{2.1}\]

Thus, the open string sector deals with T-duality by giving rise to a new kind of boundary condition leading in this case to the well known Dirichlet boundary condition. Compactifying on a higher dimensional torus \(T^d\), in general with non-zero \(B\)-fields, the T-duality group gets enlarged, so that one may ask what the image of Neumann boundary conditions under these actions actually is.

In the course of this paper we restrict ourselves to the two-dimensional torus \(T^2\) and direct products thereof. For concreteness consider type IIB compactified on a \(T^2\) with complex coordinate \(Z = X_1 + iX_2\) allowing a discrete \(\mathbb{Z}_N\) symmetry acting as

\[\Theta : (Z_L, Z_R) \rightarrow (e^{i\theta} Z_L, e^{i\theta} Z_R) \tag{2.2}\]

with \(\theta = 2\pi/N\). The essential observation is that performing a usual T-duality operation in the \(x_1\)-direction

\[T : (Z_L, Z_R) \rightarrow (-Z_L, Z_R) \tag{2.3}\]

yields an asymmetric action on the T-dual torus \(\hat{T}^2\)

\[\hat{\Theta} = T\Theta T^{-1} : (Z_L, Z_R) \rightarrow (e^{-i\theta} Z_L, e^{i\theta} Z_R). \tag{2.4}\]

The aim of this paper is to investigate the properties of asymmetric orbifolds defined by actions like \((2.4)\).

\[
\begin{array}{ccc}
\text{TYPE IIA} & \xrightarrow{R \leftrightarrow \alpha'/R} & \text{TYPE IIB} \\
\downarrow \hat{\Theta} & & \downarrow \Theta \\
\text{TYPE IIA/} \hat{\mathbb{Z}}_N & \xrightarrow{R \leftrightarrow \alpha'/R} & \text{TYPE IIB/} \mathbb{Z}_N
\end{array}
\]

Diagram 1: Duality relation
The strategy we will follow is depicted in the commuting diagram 1. In order to obtain the features of the asymmetric orbifold, concerning some questions it is appropriate to directly apply the asymmetric rotation $\hat{\Theta}$. For other questions it turns that it is better to first apply a T-duality and then perform the symmetric rotation $\Theta$ and translate the result back via a second T-duality. If not explicitly present in the equations, we have set $\alpha' = 1$ both for the closed and the open string.

2.1. Definition of the T-dual torus

The first step is to define the T-dual torus $\hat{T}^2$ allowing indeed an asymmetric action (2.4). Let the torus $T^2$ be defined by the following two vectors

\[ e_1 = R_1, \quad e_2 = R_2 e^{i\alpha}, \]  

(2.5)

so that the complex and Kähler structures are given by

\[ U = \frac{e_2}{e_1} = \frac{R_2}{R_1} e^{i\alpha}, \]
\[ T = b + iR_1 R_2 \sin \alpha. \]  

(2.6)

The left and right moving zero-modes, i.e. Kaluza-Klein and winding modes, can be written in the following form

\[ p_L = \frac{1}{i\sqrt{U_2 T_2}} [U m_1 - m_2 - T(n_1 + U n_2)], \]
\[ p_R = \frac{1}{i\sqrt{U_2 T_2}} [U m_1 - m_2 - T(n_1 + U n_2)]. \]  

(2.7)

Applying T-duality in the $x_1$-direction exchanges the complex and the Kähler modulus yielding the torus $\hat{T}^2$ defined by the vectors

\[ \hat{e}_1 = \frac{1}{R_1}, \quad \hat{e}_2 = \frac{b}{R_1} + iR_2 \sin \alpha \]  

(2.8)

and the two-form flux

\[ \hat{b} = \frac{R_2}{R_1} \cos \alpha. \]  

(2.9)

For the Kaluza-Klein and winding modes we get

\[ p_L = -\frac{1}{i\sqrt{U_2 T_2}} [\hat{U} n_1 + m_2 - \hat{T}(m_1 + \hat{U} n_2)], \]
\[ p_R = -\frac{1}{i\sqrt{U_2 T_2}} [\hat{U} n_1 + m_2 - \hat{T}(m_1 + \hat{U} n_2)]. \]  

(2.10)
from which we deduce the relation of the Kaluza-Klein and winding quantum numbers

\[ \hat{m}_1 = -n_1, \quad \hat{m}_2 = m_2, \quad \hat{n}_1 = -m_1, \quad \hat{n}_2 = n_2. \] (2.11)

If the original lattice of \( T^2 \) allows a crystallographic action of a \( \mathbb{Z}_N \) symmetry, then the T-dual Narain-lattice of \( \hat{T}^2 \) does allow a crystallographic action of the corresponding asymmetric \( \hat{\mathbb{Z}}_N \) symmetry. In view of the orientifold model studied in section 4, we present the \( \mathbb{Z}_3 \) case as an easy example.

2.2. The \( \hat{\mathbb{Z}}_3 \) torus

One starts with the \( \mathbb{Z}_3 \) lattice defined by the basis vectors

\[ e_1^A = R, \quad e_2^A = R \left( \frac{1}{2} + i \frac{\sqrt{3}}{2} \right) \] (2.12)

and arbitrary \( b \)-field. The complex and Kähler moduli are

\[ U^A = \frac{1}{2} + i \frac{\sqrt{3}}{2}, \]
\[ T^A = b + i R^2 \frac{\sqrt{3}}{2}. \] (2.13)

This lattice has the additional property that it allows a crystallographic action of the reflection at the \( x_2 \)-axis, \( \mathcal{R} \). This was important for the study of \( \Omega \mathcal{R} \) orientifolds in [20]. We call this lattice of type \( \mathbf{A} \). Recall from [20], that under \( \Omega \mathcal{R} \) all three \( \mathbb{Z}_3 \) fixed points are left invariant. For zero \( b \)-field one obtains for instance for the T-dual \( \mathbf{A} \) lattice

\[ \hat{e}_1^A = \frac{1}{R}, \quad \hat{e}_2^A = i R \frac{\sqrt{3}}{2} \] (2.14)

and \( \hat{b}^A = 1/2 \). That this rectangular lattice features an asymmetric \( \hat{\mathbb{Z}}_3 \) symmetry and that all three “fixed points” of the \( \hat{\mathbb{Z}}_3 \) are left invariant under \( \Omega \) is not obvious at all. This shows already how T-duality can give rise to fairly non-trivial results.

As we have already shown in [20] there exists a second \( \mathbb{Z}_3 \) lattice, called type \( \mathbf{B} \), allowing a crystallographic action of the reflection \( \mathcal{R} \). The basis vectors are given by

\[ e_1^B = R, \quad e_2^B = \frac{R}{2} + i \frac{R}{2 \sqrt{3}} \] (2.15)
with arbitrary $b$-field leading to the complex and Kähler moduli

$$U^B = \frac{1}{2} + i \frac{1}{2\sqrt{3}},$$

$$T^B = b + i \frac{R^2}{2\sqrt{3}}. \quad (2.16)$$

For the $B$ lattice only one $\mathbb{Z}_3$ fixed point is invariant under $\Omega \mathcal{R}$, the remaining two are interchanged. For $b = 0$ the T-dual lattice is defined by

$$\hat{e}_1^B = \frac{1}{R}, \quad \hat{e}_2^B = i \frac{R}{2\sqrt{3}} \quad (2.17)$$

with $\hat{b}^B = 1/2$. It is a non-trivial consequence of T-duality that only one of the three $\hat{\mathbb{Z}}_3$ “fixed points” is left invariant under $\Omega$.

If one requires the lattices to allow simultaneously a symmetric $\mathbb{Z}_3$ and an asymmetric $\hat{\mathbb{Z}}_3$ action one is stuck at the self-dual point $U = T$ yielding $R = 1$ and $b = 1/2$. Note, that this is precisely the root lattice of the $SU(3)$ Lie algebra. Since now we are equipped with lattices indeed allowing a crystallographic action of asymmetric $\hat{\mathbb{Z}}_N$ operations, we can move forward to discuss their D-brane contents.

2.3. Asymmetric rotations of D-branes

In order to divide a string theory by some discrete group we first have to make sure that the theory is indeed invariant. For the open string sector this means that the D-branes also have to be arranged in such a way that they reflect the discrete symmetry. Thus, for instance we would like to know what the image of a D0-brane under an asymmetric rotation is. In the compact case we can ask this question for the discrete $\hat{\mathbb{Z}}_N$ rotations defined in the last subsection, but we can also pose it quite generally in the non-compact case using a continuous asymmetric rotation

$$
\begin{pmatrix}
X_{1,L}' \\
X_{2,L}'
\end{pmatrix} =
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
X_{1,L} \\
X_{2,L}
\end{pmatrix}, \quad
\begin{pmatrix}
X_{1,R}' \\
X_{2,R}'
\end{pmatrix} =
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
X_{1,R} \\
X_{2,R}
\end{pmatrix}.
\quad (2.18)
$$

As outlined already in the beginning of section 2 (see diagram 1), instead of acting with the asymmetric rotation on the Dirichlet boundary conditions of the D0-brane, it is equivalent to go to the T-dual picture, apply first a symmetric rotation on the branes and then perform a T-duality transformation in the $x_1$-direction. In the T-dual picture the D0-brane becomes a D1-brane filling only the $x_1$-direction. Thus, the open strings are of Neumann type in the
$x_1$-direction and of Dirichlet type in the $x_2$-direction. The asymmetric rotation becomes a symmetric rotation, which simply rotates the D1-brane by an angle $\theta$ in the $x_1$-$x_2$ plane. Thus, after the rotation the D1 boundary conditions in these two directions read

$$\partial_\sigma X_1 + \tan \theta \partial_\sigma X_2 = 0,$$

$$\partial_\tau X_2 - \tan \theta \partial_\tau X_1 = 0.$$ (2.19)

If we are on the torus $T^2$ there is a distinction between values of $\theta$, for which the rotated D1-brane intersects a lattice point, and values of $\theta$, for which the D1-brane densely covers the entire $T^2$. In the first case, one still obtains quantized Kaluza-Klein and winding modes as computed in [5].

If the D1-brane runs $n$-times around the $e_1$ circle and $m$ times around the $e_2$ circle until it intersects a lattice point, the relation

$$\cot \theta = \cot \alpha + \frac{n}{mU_2}$$ (2.20)

holds. As an example we show in figure 1 a rotated D1-brane with $n = 2$ and $m = 1$.

![Figure 1](image)

In the following we will mostly consider D-branes of the first kind, which we will call rational D-branes. Finally, T-duality in the $x_1$-direction has the effect of exchanging $\partial_\sigma X_1 \leftrightarrow -\partial_\tau X_1$, leading to the boundary conditions [25]

$$\partial_\sigma X_1 + \cot \theta \partial_\tau X_2 = 0,$$

$$\partial_\sigma X_2 - \cot \theta \partial_\tau X_1 = 0.$$ (2.21)

As emphasized already, one could also perform the asymmetric rotation directly on the Dirichlet boundary conditions for the D0-brane and derive the same result. Thus, we conclude that an asymmetric rotation turns a D0-brane into a D2-brane with mixed boundary.
conditions. The last statement is the main result of this paper. As has been discussed intensively in the last year, mixed boundary conditions arise from open strings travelling in a background with non-trivial two-form flux, \( B \), or non-trivial gauge flux, \( F \),

\[
\partial_\sigma X_1 + (B + F) \partial_\tau X_2 = 0, \\
\partial_\sigma X_2 - (B + F) \partial_\tau X_1 = 0.
\] (2.22)

Thus, we can generally identify

\[
cot \theta = F = B + F,
\] (2.23)

which in the rational case becomes (note that \( \cot \theta \) is not necessarily rational)

\[
cot \theta = \cot \alpha + \frac{n}{mU_2} = B + F.
\] (2.24)

Since the \( B \) field is related to the shape of the torus \( T^2 \) and the \( F \) field to the D-branes, from (2.24) we extract the following identifications

\[
B = \cot \alpha, \quad F = \frac{n}{mU_2}.
\] (2.25)

In section 3 we will further elaborate the relation between asymmetric rotations and D-branes with mixed boundary conditions and will present an alternative derivation of some of the noncommutativity properties known for such boundary conditions. In the remainder of this section we will focus our attention on the zero mode spectrum for open strings stretched between D-branes with mixed boundary conditions. In particular, we will demonstrate that in the compact case open strings stretched between identical rational D-branes do have a non-trivial zero mode spectrum. This is in sharp contrast to some statements in the literature [26] saying that Neumann boundary conditions allow Kaluza-Klein momentum, Dirichlet boundary conditions allow non-trivial winding but general mixed D-branes do have neither of them.

2.4. Kaluza-Klein and winding modes

Since we can not easily visualize a D-brane with mixed boundary conditions, we first determine the zero-mode spectrum in the closed string tree channel and then transform the result into the open string loop channel. Thus, we are looking for boundary states (see also [27]) in the closed string theory satisfying the following boundary state conditions

\[
[\partial_\tau X_{1,cl} + \cot \theta \partial_\sigma X_{2,cl}] |B\rangle = 0, \\
[\partial_\tau X_{2,cl} - \cot \theta \partial_\sigma X_{1,cl}] |B\rangle = 0.
\] (2.26)
Rewriting (2.26) in terms of the complex coordinate the boundary condition reads

\[
[\partial_\tau Z_{cl} - i \cot \theta \partial_\sigma Z_{cl}] |B\rangle = 0. \tag{2.27}
\]

Using the mode expansion

\[
Z_{cl} = \frac{z_0}{2} + \frac{1}{2} (p_L + p_R) \tau + \frac{1}{2} (p_L - p_R) \sigma + \frac{i}{\sqrt{2}} \sum_{n \neq 0} \left( \frac{\alpha_n}{n} e^{-i n (\tau + \sigma)} + \bar{\alpha}_n \frac{1}{n} e^{-i n (\tau - \sigma)} \right),
\]

\[
\overline{Z}_{cl} = \frac{\overline{z}_0}{2} + \frac{1}{2} (\overline{p}_L + \overline{p}_R) \tau + \frac{1}{2} (\overline{p}_L - \overline{p}_R) \sigma - \frac{i}{\sqrt{2}} \sum_{n \neq 0} \left( \frac{\overline{\alpha}_n}{n} e^{-i n (\tau + \sigma)} + \bar{\overline{\alpha}}_n \frac{1}{n} e^{-i n (\tau - \sigma)} \right)
\]

one obtains

\[
[ (p_L + p_R) - i \cot \theta (p_L - p_R) ] |B\rangle = 0,
\]

\[
[ \alpha_n + e^{2i \theta} \bar{\alpha}_{-n} ] |B\rangle = 0
\]

with similar conditions for the fermionic modes. Inserting (2.10) and (2.11) into the first equation of (2.29) one can solve for the Kaluza-Klein and winding modes

\[
\hat{m}_1 = -\frac{n}{m} \hat{n}_2, \quad \hat{m}_2 = \frac{n}{m} \hat{n}_1 \tag{2.30}
\]

giving rise to the following zero-mode spectrum

\[
M_{cl}^2 = \frac{|r + s \hat{U}|^2}{\hat{U}_2} \frac{|n + m \hat{T}|^2}{\hat{T}_2} \tag{2.31}
\]

with \( r, s \in \mathbb{Z} \). We observe that this agrees with the spectrum derived in [5] by employing T-duality. Note, that the formula (2.31) is explicitly \( SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z}) \) invariant. Thus, the bosonic part of a boundary state satisfying (2.29) is given by

\[
|B\rangle_{(n, m)} = \sum_{r, s \in \mathbb{Z}} \exp \left( \sum_{n \in \mathbb{Z}} \frac{1}{n} e^{2i \theta} \alpha_{-n} \bar{\alpha}_{-n} \right) |r, s\rangle_{(n, m)}. \tag{2.32}
\]

Using this boundary state we compute the tree channel annulus partition function. Transforming the result via a modular transformation into the loop channel, we can extract the zero-mode contribution and conclude that open strings stretching between identical rational D-branes carry non-vanishing zero modes giving rise to masses

\[
M_o^2 = \frac{|r + s \hat{U}|^2}{\hat{U}_2} \frac{\hat{T}_2}{|n + m \hat{T}|^2}. \tag{2.33}
\]
It would be interesting to derive this quantization condition directly in the open string sector. Again making use of T-duality in the $x_1$-direction allows us to extract also the individual Kaluza-Klein and winding contributions

$$
\hat{P} = \frac{1}{2} (\hat{p}_L + \hat{p}_R) = \sqrt{\frac{T_2}{U_2}} \frac{m\hat{T}_2}{|n + m \hat{T}|^2} \left( s\hat{U}_2 + i(r + s\hat{U}_1) \right),
$$

$$
\hat{L} = \frac{1}{2} (\hat{p}_L - \hat{p}_R) = -\sqrt{\frac{T_2}{U_2}} \frac{n + m\hat{T}_1}{|n + m \hat{T}|^2} \left( (r + s\hat{U}_1) - is\hat{U}_2 \right).
$$

(2.34)

Note, that the zero modes indeed satisfy the boundary condition (2.21). Summarizing, we now have the means to compute annulus amplitudes for open strings stretched between different kinds of D-branes with rational mixed boundary conditions. As an example, we discuss the $\mathbb{Z}_3$ case in some more detail.

### 2.5. D-branes in the asymmetric $\mathbb{Z}_3$ orbifold

Consider the $\mathbb{Z}_3$ lattice of type $A$ and start with a D$_1$-brane with pure Dirichlet boundary conditions ($\theta = 0$)

$$
\partial_\tau X_1 = 0,
$$

$$
\partial_\tau X_2 = 0.
$$

(2.35)

Successively applying the asymmetric $\mathbb{Z}_3$ this D-brane is mapped to a mixed D$_2$-brane with boundary conditions ($\theta = 2\pi/3$)

$$
\partial_\sigma X_1 - \frac{1}{\sqrt{3}}\partial_\tau X_2 = 0,
$$

$$
\partial_\sigma X_2 + \frac{1}{\sqrt{3}}\partial_\tau X_1 = 0
$$

(2.36)

and a mixed D$_3$-brane with boundary conditions ($\theta = -2\pi/3$)

$$
\partial_\sigma X_1 + \frac{1}{\sqrt{3}}\partial_\tau X_2 = 0,
$$

$$
\partial_\sigma X_2 - \frac{1}{\sqrt{3}}\partial_\tau X_1 = 0.
$$

(2.37)

In the orbifold theory these three kinds of D-branes are identified. This reflects that their background fields are being identified according to

$$
\mathcal{F} \equiv \mathcal{F} + \frac{1}{\sqrt{3}}
$$

(2.38)
or equivalently
\[ \theta \equiv \theta + \frac{2\pi}{3}. \tag{2.39} \]
The two coordinates \( X_1 \) and \( X_2 \) yield the following contribution to the annulus partition function for open strings stretched between identical D-branes
\[
A^{i\beta}_{ii} = \frac{\eta^{[\alpha}_{\beta]} e^{-2\pi t^{2} R^2}}{\eta^{3}} \left( \sum_{r \in \mathbb{Z}} e^{-2\pi t^{2} r^{2} R^2} \right) \left( \sum_{s \in \mathbb{Z}} e^{-2\pi t \frac{s^2}{4 R^2}} \right) \tag{2.40}
\]
independent of \( i \in \{1, 2, 3\} \). Open strings stretched between different kinds of D-branes give rise to shifted moding and yield the partition function
\[
A_{i,i+1} = n_{i,i+1} \frac{\eta^{[\frac{1}{2}+\alpha}_{\beta]} e^{-2\pi t^{2} R^2}}{\eta^{[\frac{1}{2}+\alpha}_{1}} \eta^{[\frac{1}{2}+\alpha}_{2]} \tag{2.41}
\]
which looks like a twisted open string sector. As we know from [20–22] here we have to take into account extra multiplicities, \( n_{i,i+1} \), which have a natural geometric interpretation as multiple intersection points of D-branes at angles in the T-dual picture. By this reasoning we find that for the \( A \) type lattice the extra factor is one. However, for the three D-branes generated by \( \hat{\mathbb{Z}}_3 \) when one starts with a D-brane with pure Neumann boundary conditions, \( \theta \in \{\pi/2, \pi/6, -\pi/6\} \), T-duality tells us that there must appear an extra factor of three in front of the corresponding annulus amplitude (2.41). In the orientifold construction presented in section 4 these multiplicities are important to give consistent models.

3. Asymmetric rotations and noncommutative geometry

In section 2 we have pointed out that on \( T^2 \) or \( \mathbb{R}^2 \) D-branes with mixed boundary conditions can be generated by simply applying an asymmetric rotation to an ordinary D-brane with pure Neumann or Dirichlet boundary conditions. Thus, it should be possible to rederive earlier results for the two-point function on the disc
\[
\langle X_i(z) X_j(z') \rangle, \tag{3.1}
\]
for the operator product expansion (OPE) between vertex operators on the boundary
\[
e^{i p X(\tau)} e^{i q X(\tau')} \tag{3.2}
\]
or for the commutator of the coordinate fields
\[
[X_i(\tau, \sigma), X_j(\tau, \sigma')] \tag{3.3}
\]
by applying an asymmetric rotation on the corresponding quantities for open strings ending on D0-branes in flat space-time.
3.1. Two-point function on the disc

The two-point function on the disc for both \( X_1 \) and \( X_2 \) of Dirichlet type reads

\[
\langle X_i(z) X_j(z') \rangle = -\alpha' \delta_{ij} \left( \ln |z - z'| - \ln |z - \bar{z}'| \right)
\]

\[
= -\alpha' \delta_{ij} \frac{1}{2} \left( \ln(z - z') + \ln(\bar{z} - \bar{z'}) - \ln(z - \bar{z}') - \ln(\bar{z} - z') \right)
\]

(3.4)

from which, formally using

\[
X_i(z) = X_{i,L}(z) + X_{i,R}(\bar{z}),
\]

(3.5)

we can directly read off the individual contributions from the left- and right-movers. Performing the asymmetric rotation

\[
X_L \to AX_L, \quad X_R \to A^T X_R,
\]

(3.6)

where \( A \) denotes an element of \( SO(2) \), leads to the following expression for the propagator in the rotated coordinates

\[
\langle X_i(z) X_j(z') \rangle = -\alpha' \delta_{ij} \ln |z - z'| - \alpha' \delta_{ij} \left( \sin^2 \theta - \cos^2 \theta \right) \ln |z - \bar{z}'| - \alpha' \epsilon_{ij} \sin \theta \cos \theta \ln \left| \frac{z - \bar{z}'}{\bar{z} - z'} \right|.
\]

(3.7)

This expression agrees precisely with the propagator derived in [2] with the identification

\[
\mathcal{F} = \begin{pmatrix}
0 & \cot \theta \\
-\cot \theta & 0
\end{pmatrix}.
\]

(3.8)

Thus, by applying an asymmetric rotation we have found an elegant and short way of deriving this propagator without explicit reference to the boundary conditions or the background fields. Moreover, since the commutative D0-brane is related in this smooth way to a noncommutative D2-brane, it is suggesting that also both effective theories arising on such branes are related by some smooth transformation. Such an explicit map between the commuting and the noncommuting effective gauge theories has been determined in [9].

3.2. The OPE of vertex operators

In this subsection we apply an asymmetric rotation also to the operator product expansion of tachyon vertex operators \( \mathcal{O}(z) = e^{ipX}(z) \) on the boundary. Of course this OPE is a direct consequence of the correlator (3.7) restricted to the boundary, but nevertheless we
would like to see whether we can generate the noncommutative $\ast$-product directly via an asymmetric rotation. Taking care of the left- and right-moving contributions in the OPE between vertex operators living on a pure Dirichlet boundary we can write for $|z| > |z'|$

$$e^{ipX(z)} e^{iqX(z')} = \frac{(z - z')^{\alpha'}_{PL} q_{PL} (z - z')^{\alpha'}_{PR} q_{PR}}{(z - \bar{z'})^{\alpha'}_{PL} q_{PR} (\bar{z} - z')^{\alpha'}_{PR} q_{PL}} e^{i(p+q)X(z')} + \ldots \quad (3.9)$$

Now we apply an asymmetric rotation (3.6) together with

$$p_L \to A p_L, \quad p_R \to A^T p_R;$$
$$q_L \to A q_L, \quad q_R \to A^T q_R,$$

and, after all, identifying $p_L = p_R, q_L = q_R$ we obtain

$$e^{ipX(z)} e^{iqX(z')} = \frac{[\zeta - z'] \zeta'_{PL} p_{PL} (\zeta - z')^{\alpha'}_{PR} q_{PR}}{[\zeta - \bar{z}'] (\zeta - z')^{\alpha'}_{PL} q_{PR} (\bar{z} - z')^{\alpha'}_{PR} q_{PL}} e^{i(p+q)X(z')} + \ldots \quad (3.10)$$

Restricting (3.11) to the boundary and choosing the same branch cut as in [9] we finally arrive at

$$e^{ipX(\tau)} e^{iqX(\tau')} = (\tau - \tau')^{\alpha'}_{PL} p_{PL} (1 + \sin^2 \theta - \cos^2 \theta) \exp (i\pi \alpha' \sin \theta \cos \epsilon_{ij} p_i q_j) e^{i(p+q)X(\tau')} + \ldots \quad (3.11)$$

This is precisely the OPE derived in [7,9]. It shows that it is indeed possible to derive the $\ast$-product $e^{ipX(\tau)} e^{iqX(\tau')} \sim e^{ipX} \ast e^{iqX}$ directly via an asymmetric rotation, where the noncommutative algebra $\mathcal{A}$ of functions $f$ and $g$ is defined as

$$f \ast g = fg - i\pi \alpha' \sin \theta \cos \theta \epsilon_{ij} \partial_i f \partial_j g + \ldots \quad (3.12)$$

3.3. The commutator of the coordinates

While the two-point function derived above already implies that the commutator of the coordinate fields is non-vanishing, i.e. the geometry on the D-brane non-commutative, we would like to rederive this result directly via studying D-branes with mixed boundary conditions, as well. This is done by the quantization of the bosonic coordinate fields of
the open string. We start with the T-dual situation with two D-branes intersecting at an arbitrary angle $\theta_2 - \theta_1$ (see figure 2).

The open string boundary condition at $\sigma = 0$ are

\[
\begin{align*}
\partial_\sigma X_1 + \tan \theta_1 \partial_\sigma X_2 &= 0, \\
\partial_\tau X_2 - \tan \theta_1 \partial_\tau X_1 &= 0,
\end{align*}
\]

and at $\sigma = \pi$ we require

\[
\begin{align*}
\partial_\sigma X_1 + \tan \theta_2 \partial_\sigma X_2, &= 0, \\
\partial_\tau X_2 - \tan \theta_2 \partial_\tau X_1 &= 0.
\end{align*}
\]

The mode expansion satisfying these two boundary conditions looks like

\[
X_1 = x_1 + i\sqrt{\alpha'} \sum_{n \in \mathbb{Z}} \frac{\alpha_{n+\nu}}{n+\nu} e^{-i(n+\nu)\tau} \cos[(n + \nu)\sigma + \theta_1] +
\]

\[
i\sqrt{\alpha'} \sum_{m \in \mathbb{Z}} \frac{\alpha_{m-\nu}}{m-\nu} e^{-i(m-\nu)\tau} \cos[(m - \nu)\sigma - \theta_1],
\]

\[
X_2 = x_2 + i\sqrt{\alpha'} \sum_{n \in \mathbb{Z}} \frac{\alpha_{n+\nu}}{n+\nu} e^{-i(n+\nu)\tau} \sin[(n + \nu)\sigma + \theta_1] -
\]

\[
i\sqrt{\alpha'} \sum_{m \in \mathbb{Z}} \frac{\alpha_{m-\nu}}{m-\nu} e^{-i(m-\nu)\tau} \sin[(m - \nu)\sigma - \theta_1],
\]

with $\nu = (\theta_2 - \theta_1)/\pi$. Using the usual commutation relation

\[
[\alpha_{n+\nu}, \alpha_{m-\nu}] = (n + \nu) \delta_{m+n,0}
\]

and the vanishing of the commutator of the center of mass coordinates $x_1$ and $x_2$ one can easily show that for D-branes at angles the general equal time commutator vanishes

\[
[X_i(\tau, \sigma), X_j(\tau, \sigma')] = 0.
\]
Therefore, the geometry of D-branes at angles, but without background gauge fields, is always commutative.

Performing T-duality in the $x_1$ direction one gets the two mixed boundary conditions for the open strings
\[
\begin{align*}
\partial_\sigma X_1 + \cot \theta_1 \partial_\tau X_2 &= 0, \\
\partial_\sigma X_2 - \cot \theta_1 \partial_\tau X_1 &= 0
\end{align*}
\tag{3.19}
\]
at $\sigma = 0$ and
\[
\begin{align*}
\partial_\sigma X_1 + \cot \theta_2 \partial_\tau X_2 &= 0, \\
\partial_\sigma X_2 - \cot \theta_2 \partial_\tau X_1 &= 0
\end{align*}
\tag{3.20}
\]
at $\sigma = \pi$. The mode expansion satisfying these boundary conditions is
\[
\begin{align*}
X_1 &= x_1 - \sqrt{\alpha'} \sum_{n \in \mathbb{Z}} \frac{\alpha_{n+\nu}}{n+\nu} e^{-i(n+\nu)\tau} \sin[(n+\nu)\sigma + \theta_1] - \\
& \quad \sqrt{\alpha'} \sum_{m \in \mathbb{Z}} \frac{\alpha_{m-\nu}}{m-\nu} e^{-i(m-\nu)\tau} \sin[(m-\nu)\sigma - \theta_1], \\
X_2 &= x_2 + i\sqrt{\alpha'} \sum_{n \in \mathbb{Z}} \frac{\alpha_{n+\nu}}{n+\nu} e^{-i(n+\nu)\tau} \sin[(n+\nu)\sigma + \theta_1] - \\
& \quad i\sqrt{\alpha'} \sum_{m \in \mathbb{Z}} \frac{\alpha_{m-\nu}}{m-\nu} e^{-i(m-\nu)\tau} \sin[(m-\nu)\sigma - \theta_1],
\end{align*}
\tag{3.21}
\]
which one can also derive performing an asymmetric rotation on the mode expansion for pure Dirichlet type branes. Now one can compute the commutator
\[
[X_1(\tau, \sigma), X_2(\tau, \sigma')] = [x_1, x_2] + i \sum_{n \in \mathbb{Z}} \frac{2\alpha'}{n+\nu} \sin[(n+\nu)\sigma + \theta_1] \sin[(n+\nu)\sigma' + \theta_1].
\tag{3.22}
\]
For $\sigma$ and $\sigma'$ not both equal to zero or $\pi$ the second term in (3.22) is constant, so that the whole expression can be set to zero by choosing
\[
[x_1, x_2] = \frac{2\pi i\alpha'}{F_2 - F_1}.
\tag{3.23}
\]
Note, that this value also can be obtained directly from the canonical quantization \[3\]. However, as has been shown in \[11,14\], for $\sigma = \sigma' = 0$ the evaluation of the sum in (3.22) yields
\[
[X_1(\tau, 0), X_2(\tau, 0)] = \frac{2\pi i\alpha' F_1}{1 + F_1^2}
\tag{3.24}
\]
and for $\sigma = \sigma' = \pi$ one analogously obtains
\[
[X_1(\tau, \pi), X_2(\tau, \pi)] = \frac{2\pi i\alpha' F_2}{1 + F_2^2}.
\tag{3.25}
\]
Thus, the coordinates only noncommute at the boundary of the word-sheet, where the commutator can be expressed entirely in terms of the gauge field on the local D-brane. If both ends of the open string end on the same D-brane with \( \theta_1 = \theta_2 \), both terms in (3.22) become singular, but the sum of them give rise to the same expression (3.24) and (3.25) for the commutator at the ends of the open string. Thus, only for \( \theta_1 = \theta_2 \in \{0, \pi/2\} \) the \( X_1 \) and \( X_2 \) coordinates commute on the D-brane, in all other cases the end points see a noncommuting space-time. Moreover, in the compact case for rational D-branes the noncommutative theory on the D-branes is mapped via T-duality to a commutative theory on D-branes at angles. This is only a special example of the more general rule pointed out in [9] that for rational points the noncommutative torus is T-dual to a commutative one.

At the end of this section let us briefly comment on the algebraic structure of the noncommutative torus we have obtained by the asymmetric rotation on the D-branes. As shown in the previous section, the tachyon vertex operator \( \mathcal{O} = e^{ipX(\tau)} \) leads to a noncommutative algebra \( \mathcal{A} \), defined in eq.(3.13). As explained in [9], the algebra \( \mathcal{A} \) of tachyon vertex operators can be taken at either end of the open string. Therefore the open string states form a bimodule \( \mathcal{A} \times \mathcal{A}' \), where \( \mathcal{A} \) is acting on the boundary \( \sigma = 0 \) and \( \mathcal{A}' \) on the boundary \( \sigma = \pi \) of the open string. Specifically, for an open string whose first boundary \( \sigma = 0 \) is related to a D-brane with parameter \( \theta_1 \) and whose second boundary \( \sigma = \pi \) is attached to a D-brane with parameter \( \theta_2 \), the algebra \( \mathcal{A} \) of functions on the noncommutative torus is generated by

\[
U_1 = \exp(iy_1 - \frac{2\pi^2 \alpha' F_1}{1 + F_1^2} (\partial/\partial y_2)),
\]

\[
U_2 = \exp(iy_2 + \frac{2\pi^2 \alpha' F_1}{1 + F_1^2} (\partial/\partial y_1)),
\]

which obey

\[
U_1 U_2 = \exp(-2\pi i \frac{2\pi \alpha' F_1}{1 + F_1^2}) U_2 U_1.
\]

On the other hand, the algebra \( \mathcal{A}' \) is generated by

\[
\tilde{U}_1 = \exp(iy_1 + \frac{2\pi^2 \alpha' F_2}{1 + F_2^2} (\partial/\partial y_2)),
\]

\[
\tilde{U}_2 = \exp(iy_2 - \frac{2\pi^2 \alpha' F_2}{1 + F_2^2} (\partial/\partial y_1)),
\]

obeying

\[
\tilde{U}_1 \tilde{U}_2 = \exp(2\pi i \frac{2\pi \alpha' F_2}{1 + F_2^2}) \tilde{U}_2 \tilde{U}_1.
\]
4. Asymmetric orientifolds

Another motivation for studying such asymmetric orbifolds arises in the construction of type I vacua. In \cite{20-22} we have considered so-called supersymmetric orientifolds with D-branes at angles in six and four space-time dimensions which in the six-dimensional case were defined as

$$\text{Type IIB on } T^4 \{\Omega R, \Theta\}$$

with $R: z_i \to -z_i$, the $z_i$ being the complex coordinates of the $T^4$. Upon T-dualities in the directions of their real parts one obtains an ordinary orientifold where, however, the space-time symmetry becomes asymmetric

$$\text{Type IIB on } \hat{T}^4 \{\Omega, \hat{\Theta}\}.$$  \hfill (4.2)

In the entire derivation in section 2 we have identified the two constructions explicitly via T-duality, relating branes with background fields to branes at angles. While in the $\Omega R$ orientifolds $\Theta$ identified branes at different locations on the tori, $\hat{\Theta}$ now maps branes with different values of their background gauge flux upon each other. As the background fields determine the parameter which rules the noncommutative geometry, branes with different geometries are identified according to (2.38). In this manner asymmetric orbifolds and orientifolds provide extremely exotic string backgrounds. In the particular example of the $\hat{\mathbb{Z}}_3$ orientifolds, as obtained from the $\mathbb{Z}_3 \Omega R$ orientifold in \cite{20} via T-duality, all background fields are equivalent to a vanishing field, all geometries equivalent to a commutative one. But in any of the models, where the orbifold group contains an element of order 2, i.e. with even $N$, the background fields can only be “gauged away” on one half of the D-branes, the other half stays noncommutative.

From the above mentioned identification it is now clear that the $\mathcal{N} = (0, 1)$ supersymmetric asymmetric $\hat{\mathbb{Z}}_N$ orientifolds \cite{1.2} have the same one loop partition functions as the corresponding symmetric $\mathbb{Z}_N$ orientifolds \cite{4.1}. The only difference is that instead of D7-branes at angles, we introduce D9-branes with appropriate background fields. Thus, a whole class of asymmetric orientifolds has already been studied in the T-dual picture involving D-branes at angles. One could repeat the whole computation for the asymmetric orientifolds \cite{1.2}, getting of course identical results. Note, the model \cite{4.2} is really a type I vacuum, as $\Omega$ itself is gauged. Thus, in principle there exist the possibility that heterotic
dual models exist. Of course, in six dimensions most models have more than one tensor-multiplet so that no perturbative heterotic dual model can exist. It would be interesting to look for heterotic duals for the four dimensional models discussed in [22].

In the following we will construct the even more general six-dimensional $\mathbb{Z}_3 \times \hat{\mathbb{Z}}_3$ orientifold

$$\text{Type IIB on } \hat{T}^4 \{\Omega, \Theta, \hat{\Theta}\}$$

(4.3)

which is T-dual to

$$\text{Type IIB on } T^4 \{\Omega \mathcal{R}, \Theta, \hat{\Theta}\},$$

(4.4)

where in fact, as shown in section 2.2, the two tori are identical $T^4 = \hat{T}^4 = SU(3)^2$. The freedom to choose their complex structures gives rise to a variety of three distinct models, which are denoted by $AA, AB, BB$ as in [20]. Note, that the same orbifold group is generated by a pure left-moving $\mathbb{Z}_3L, \Theta_L = \hat{\Theta} \Theta^{-1}$, and a pure right-moving $\mathbb{Z}_3R, \Theta_R = \hat{\Theta} \Theta$. As was also shown in [24] this model actually has $\mathcal{N} = (1, 1)$ supersymmetry, but one can get $\mathcal{N} = (0, 1)$ supersymmetry by turning on non-trivial discrete torsion.

4.1. Tadpole cancellation

The computation of the various one-loop amplitudes is straightforward. For the loop channel Klein bottle amplitude we obtain

$$K^{(ab)} = \frac{8c}{12} \int_0^\infty \frac{dt}{t^4}\left[\rho_{00} \Lambda^a \Lambda^b + \rho_{01} + \rho_{02} + n_{\hat{\Theta}, \Omega}^{(ab)} \rho_{10} + n_{\hat{\Theta}, \Omega \mathcal{R}}^{(ab)} \epsilon \rho_{11} + n_{\hat{\Theta}, \Omega \mathcal{R}}^{(ab)} \epsilon \rho_{12} + n_{\hat{\Theta}, \Omega \mathcal{R}}^{(ab)} \rho_{20} + n_{\hat{\Theta}, \Omega \mathcal{R}}^{(ab)} \epsilon \rho_{21} + n_{\hat{\Theta}, \Omega \mathcal{R}}^{(ab)} \epsilon \rho_{22}\right],$$

(4.5)

where $c \equiv V_6/(8\pi^2 \alpha')^3$ and $\epsilon$ is a phase factor defining the discrete torsion. Further we have introduced a similar notation as in [24]

$$\rho_{00} = \sum_{\alpha, \beta = 0, \frac{1}{2}} (-1)^{2\alpha + 2\beta + 4\alpha \beta} \frac{\vartheta[\alpha - \frac{1}{2} \beta]}{\eta^2},$$

$$\rho_{0h} = \sum_{\alpha, \beta = 0, \frac{1}{2}} (-1)^{2\alpha + 2\beta + 4\alpha \beta} \frac{\vartheta[\alpha - \frac{1}{2} \beta]}{\eta^6} \prod_{i=1}^2 2 \sin(\pi h_i) \frac{\vartheta[\alpha + h_i]}{\vartheta[\frac{1}{2} + h_i]}, \quad h \neq 0,$$

(4.6)

$$\rho_{gh} = \sum_{\alpha, \beta = 0, \frac{1}{2}} (-1)^{2\alpha + 2\beta + 4\alpha \beta} \frac{\vartheta[\alpha - \frac{1}{2} \beta]}{\eta^6} \prod_{i=1}^2 \frac{\vartheta[\alpha + g_i]}{\vartheta[\frac{1}{2} + g_i]}, \quad g, h \neq 0.$$
with \( g, h \in \{(1/3, -1/3), (2/3, -2/3)\} \) for which we use the shorter notation \( g, h \in \{1, 2\} \). The index \( (ab) \) denotes the three possible choices of lattices, AA, AB and BB, and \( \Lambda^a \) are the zero mode contributions (2.33) to the partition function

\[
\Lambda^A = \sum_{m_1, m_2} e^{-\pi t \left[ m_1^2 + \frac{4}{3} \left( \frac{m_1}{3} - m_2 \right)^2 \right]},
\]

\[
\Lambda^B = \sum_{m_1, m_2} e^{-\pi t \left[ m_1^2 + 12 \left( \frac{m_1}{3} - m_2 \right)^2 \right]}.
\]

Finally, \( n^{(ab)}_{\hat{\Theta}, \Omega} \) denotes the trace of the action of \( \Sigma_2 \) on the fixed points in the \( \Sigma_1 \) twisted sector. Taking into account that the origin is the only common fixed point of \( \mathbb{Z}_3 \) and \( \hat{\mathbb{Z}}_3 \), they can be determined to be

\[
n^{(ab)}_{\hat{\Theta}, \Omega} = \begin{cases} 
9 & \text{for (AA)} \\
3 & \text{for (AB)} \\
1 & \text{for (BB)} 
\end{cases}
\]

and

\[
n^{(ab)}_{\hat{\Theta}, \Omega} = n^{(ab)}_{\hat{\Theta}_2, \Omega_2} = \begin{cases} 
-3 & \text{for (AA)} \\
\sqrt{3} & \text{for (AB)} \\
1 & \text{for (BB)} 
\end{cases}
\]

The remaining numbers are given by complex conjugation of (4.9). Applying a modular transformation to (4.3) yields the tree channel Klein bottle amplitude

\[
\tilde{K}^{(ab)} = \frac{32c}{3} \int_0^\infty dt \left[ n_0^a n_0^b \tilde{\Lambda}^a \tilde{\Lambda}^b + \frac{1}{3} n^{(ab)}_{\hat{\Theta}, \Omega} \rho_{00} \rho_{01} + \frac{1}{3} n^{(ab)}_{\hat{\Theta}, \Omega} \rho_{02} + 3 \rho_{10} - n^{(ab)}_{\hat{\Theta}_2, \Omega_2} \epsilon \rho_{11} - n^{(ab)}_{\hat{\Theta}, \Omega_2} \epsilon \rho_{12} + 3 \rho_{20} - n^{(ab)}_{\hat{\Theta}_2, \Omega_2} \epsilon \rho_{21} - n^{(ab)}_{\hat{\Theta}, \Omega_2} \epsilon \rho_{22} \right].
\]

with \( n_0^A = \sqrt{3} \) and \( n_0^B = 1/\sqrt{3} \). The lattice contributions are

\[
\tilde{\Lambda}^A = \sum_{m_1, m_2} e^{-3\pi t \left[ m_1^2 + \frac{4}{3} \left( \frac{m_1}{3} - m_2 \right)^2 \right]},
\]

\[
\tilde{\Lambda}^B = \sum_{m_1, m_2} e^{-\pi t \left[ \frac{4}{3} m_1^2 + 4 \left( \frac{m_1}{3} - m_2 \right)^2 \right]}.
\]

In order to cancel these tadpoles we now introduce D-branes with mixed boundary conditions. For both the A and the B lattice we choose three kinds of D-branes with \( \theta \in \{\pi/2, \pi/6, -\pi/6\} \). The asymmetric \( \hat{\mathbb{Z}}_3 \) cyclically permutes these three branes, whereas
the symmetric $Z_3$ leaves every brane invariant and acts with a $\gamma_{\Theta,i}$ matrix on the Chan-Paton factors on each brane. Since $\tilde{Z}_3$ permutes the branes, all three $\gamma_{\Theta,i}$ actions must be the same. The computation of the annulus amplitude gives

$$A^{(ab)} = \frac{c}{12} \int_0^\infty \frac{dt}{t^4} \left[ M^2 \rho_{00} \Lambda^a \Lambda^b + (\text{Tr} \gamma_{\Theta})^2 \rho_{01} + (\text{Tr} \gamma_{\Theta^2})^2 \rho_{02} + M^2 n_{\Theta,1}^{(ab)} \rho_{10} + (\text{Tr} \gamma_{\Theta})^2 n_{\Theta,\Theta}^{(ab)} \epsilon \rho_{11} + (\text{Tr} \gamma_{\Theta^2})^2 n_{\Theta,\Theta^2}^{(ab)} \epsilon \rho_{12} + M^2 n_{\Theta^2,1}^{(ab)} \rho_{20} + (\text{Tr} \gamma_{\Theta})^2 n_{\Theta^2,\Theta}^{(ab)} \epsilon \rho_{21} + (\text{Tr} \gamma_{\Theta^2})^2 n_{\Theta^2,\Theta^2}^{(ab)} \epsilon \rho_{22} \right].$$

where the $\hat{\Theta}$ twisted sector is given by open strings stretched between D-branes with $\theta_i$ and $\theta_{i+1}$. Thus, $n_{\Theta,1}^{(ab)}$ denotes the intersection number of two such branes and $n_{\Theta,\Theta}^{(ab)}$ the number of intersection points invariant under $\Theta$. The actual numbers turn out to be the same as the multiplicities of the closed string twisted sectors in (4.8) and (4.9). For the tree channel amplitude we obtain

$$\tilde{A}^{(ab)} = \frac{c}{6} \int_0^\infty dl \left[ M^2 \left( n_{00} a_n^b n_{00} \tilde{\Lambda}^a \tilde{\Lambda}^b + \frac{1}{3} n_{\Theta,1}^{(ab)} \rho_{01} + \frac{1}{3} n_{\Theta,1}^{(ab)} \rho_{02} \right) + (\text{Tr} \gamma_{\Theta})^2 \left( 3 \rho_{10} - n_{\Theta,\Theta}^{(ab)} \tilde{\epsilon} \rho_{11} - n_{\Theta,\Theta}^{(ab)} \epsilon \rho_{12} \right) + (\text{Tr} \gamma_{\Theta^2})^2 \left( 3 \rho_{20} - n_{\Theta^2,\Theta^2}^{(ab)} \epsilon \rho_{21} - n_{\Theta^2,\Theta^2}^{(ab)} \epsilon \rho_{22} \right) \right].$$

Finally, one has to compute the Möbius amplitude

$$M^{(ab)} = -\frac{c}{12} \int_0^\infty dt \int_0^\infty d\ell \left[ M \left( n_{00} a_n^b n_{00} \tilde{\Lambda}^a \tilde{\Lambda}^b + \text{Tr}(\gamma_{\Theta,\Theta^{-1}}) \rho_{01} + \text{Tr}(\gamma_{\Theta,\Theta^2}) \rho_{02} + M n_{\Theta,\Theta}^{(ab)} \rho_{11} + \text{Tr}(\gamma_{\Theta,\Theta^{-1}}) n_{\Theta,\Theta}^{(ab)} \epsilon \rho_{12} + \text{Tr}(\gamma_{\Theta,\Theta^2}) n_{\Theta,\Theta^2}^{(ab)} \epsilon \rho_{10} + M n_{\Theta^2,\Theta}^{(ab)} \rho_{22} + \text{Tr}(\gamma_{\Theta,\Theta^{-1}}) n_{\Theta^2,\Theta}^{(ab)} \epsilon \rho_{21} + \text{Tr}(\gamma_{\Theta,\Theta^2}) n_{\Theta^2,\Theta^2}^{(ab)} \epsilon \rho_{22} \right) \right],$$

with argument $q = -\exp(-2\pi t)$. Transformation into tree channel leads to the expression

$$\tilde{M}^{(ab)} = -\frac{8c}{3} \int_0^\infty dl \left[ M \left( n_{00} a_n^b n_{00} \tilde{\Lambda}^a \tilde{\Lambda}^b + \frac{1}{3} n_{\Theta,\Theta}^{(ab)} \rho_{01} + \frac{1}{3} n_{\Theta,\Theta}^{(ab)} \rho_{02} \right) + \text{Tr}(\gamma_{\Theta,\Theta^{-1}}) \left( 3 \rho_{11} - n_{\Theta,\Theta^2}^{(ab)} \epsilon \rho_{12} - n_{\Theta,\Theta^2}^{(ab)} \epsilon \rho_{10} \right) + \text{Tr}(\gamma_{\Theta,\Theta^2}) \left( 3 \rho_{22} - n_{\Theta,\Theta}^{(ab)} \epsilon \rho_{20} - n_{\Theta^2,\Theta}^{(ab)} \epsilon \rho_{21} \right) \right].$$
The three tree channel amplitudes give rise to two independent tadpole cancellation conditions

\[ M^2 - 16M + 64 = 0, \]
\[ (\text{Tr} \gamma \Theta)^2 - 16 \text{Tr}(\gamma^T \Omega \gamma^{-1}) + 64 = 0. \]  

(4.16)

Thus, we have \( M = 8 \) D9-branes of each kind and the action of \( \mathbb{Z}_3 \) on the Chan-Paton labels has to satisfy \( \text{Tr} \gamma \Theta = 8 \) implying that we have the simple solution that \( \gamma \Theta \) is the identity matrix.

4.2. The massless spectrum

Having solved the tadpole cancellation conditions we can move forward and compute the massless spectrum of the effective commutative field theory in the non-compact space-time. In computing the massless spectra we have to take into account the actions of the operations on the various fixed points. In the closed string sector we find the spectra shown in table 1

| \( \epsilon \) | (ab) | spectrum |
|---|---|---|
| 1 | – | \((1, 1)\) Sugra + 4 \(\times V_{1,1}\) |
| \( e^{\pm 2\pi i/3} \) | AA | \((0, 1)\) Sugra + 6 \(\times T + 15 \times H\) |
| | AB | \((0, 1)\) Sugra + 9 \(\times T + 12 \times H\) |
| | BB | \((0, 1)\) Sugra + 10 \(\times T + 11 \times H\) |

**Table 1:** closed string spectra

The computation of the massless spectra in the open string sector is also straightforward and yields the result in table 2

| \( \epsilon \) | (ab) | spectrum |
|---|---|---|
| 1 | – | \(V_{1,1}\) in \(SO(8)\) |
| \( e^{\pm 2\pi i/3} \) | AA | \(V\) in \(SO(8) + 4 \times H\) in \(28\) |
| | AB | \(V\) in \(SO(8) + 1 \times H\) in \(28\) |
| | BB | \(V\) in \(SO(8)\) |

**Table 2:** open string spectra

All the spectra shown in table 1 and table 2 satisfy the cancellation of the non-factorizable anomaly. Note, that the configurations \(AB\) and \(BB\) were not analyzed in [24]. Thus,
we have successfully applied the techniques derived in section 2 and section 3 to the construction of asymmetric orientifolds.

5. Conclusions

In this article we have pointed out a relationship between the realization of asymmetric operations in the open string sector and noncommutative geometry arising at the boundary of the world-sheet of open strings. More concretely, we have shown that a left-right asymmetric rotation transforms an ordinary Neumann or Dirichlet boundary condition into a mixed Neumann-Dirichlet boundary condition. We have employed this observation to rederive the noncommutativity relations for the open string. Moreover, we have solved the problem of how the open string sector manages to incorporate asymmetric symmetries. It simply turns on background gauge fluxes. In asymmetric orbifolds different values of background gauge fields on the D-branes get identified and correspondingly different geometries, commutative or noncommutative, as well. Finally, we have considered a concrete asymmetric type I vacuum, where D-branes with mixed boundary conditions were introduced to cancel all tadpoles.

We have restricted ourselves to the case of products of two-dimensional tori. It would be interesting to generalize these ideas to more general asymmetric elements of the T-duality group and to discuss the dual heterotic description. Furthermore, it would be interesting to see whether via the asymmetric rotation one can gain further insight into the relation between the effective noncommutative and commutative gauge theories on the branes.

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