A stochastic analysis of subcritical Euclidean fermionic field theories

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Abstract
Building on previous work on the stochastic analysis for Grassmann random variables, we introduce a forward-backward stochastic differential equation (FBSDE) which provides a stochastic quantisation of Grassmann measures. Our method is inspired by the so-called continuous renormalisation group, but avoids the technical difficulties encountered in the direct study of the flow equation for the effective potentials. As an application, we construct a family of weakly coupled subcritical Euclidean fermionic field theories and prove exponential decay of correlations.

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1 Introduction

We consider a class of Grassmann measures constructed as Gibbsian perturbations of a Grassmann Gaussian field on $\mathbb{R}^d$ and prove their existence and clustering properties via stochastic quantisation.

Let $(\omega, \mathcal{M})$ be an algebraic probability space, i.e. a $C^*$-algebra $\mathcal{M}$ endowed with a normalised, positive linear functional $\omega: \mathcal{M} \to \mathbb{C}$ and let $(\psi(f))_{f \in \mathfrak{h}}$ a centered Grassmann Gaussian field on the Hilbert space $\mathfrak{h} := L^2(\mathbb{R}^d, \mathbb{C}^2) \oplus L^2(\mathbb{R}^d, \mathbb{C}^2)$ with covariance

$$ \omega(\psi(f)\psi(g)) = \langle \Theta f, G g \rangle, \quad f, g \in \mathfrak{h}, $$

where $\Theta$ is an anti-linear involution on $\mathfrak{h}$ given by $\Theta(f_1 \oplus f_2) := (\bar{f}_2 \oplus \bar{f}_1)$ and $G$ the bounded operator

$$ G := (1 - \Delta)^{\gamma - d/2} \oplus -(1 - \Delta)^{\gamma - d/2} $$

where $\Delta$ is the Laplacian on $L^2(\mathbb{R}^d, \mathbb{C}^2)$ and $\gamma > 0$. We would like to make sense of the continuous linear functional $\omega^V$ on $\mathcal{M}$ given by

$$ \omega^V(A) := \frac{\omega(A e^V)}{\omega(e^V)}, \quad A \in \mathcal{M}, $$

which we call the Gibbsian Grassmann measure with covariance $G$ and interaction potential $V(\psi)$ given by a function on the Grassmann algebra $\mathcal{M}_\psi$ generated by $\psi$. In particular we will take $V(\psi)$ of the form

$$ V(\psi) := \int_{\mathbb{R}^d} \left[ \frac{\lambda}{4} (\psi_x)^4 + \mu (\psi_x)^2 \right] dx, $$

for some constants $\lambda, \mu = \mu(\lambda)$ and with $\psi_x = \psi(\delta_x)$ and $(\psi_x)^4, (\psi_x)^2$ suitable quartic and quadratic local monomials of the field $\psi$ evaluated at a point $x \in \mathbb{R}^d$. Due to singularities in the covariance $G$ the random variable $\psi_x$ cannot really be defined and the expression of the interaction potential $V$ is at best a suggestive notation for a more concrete family of approximations which avoid both the small scale (or ultraviolet, or UV) singularities implicit in considering local monomials and the large scale (or infrared, or IR) singularities relative to the computation of the integral over all $\mathbb{R}^d$. 

Gibbsian Grassmann measures appear naturally in the study of interacting Euclidean fermionic quantum field theories, see, e.g., [GJ87] for a review, but also in the study of condensed-matter systems and as an emergent description in statistical mechanics, see [Mas08]. Although our Grassmann measures (2) do not describe any relevant field theory nor statistical mechanical systems, they are relevant from a methodological point of view, since they present the same mathematical challenges of subcritical Euclidean fermionic field theories and can thus be considered as a toy model for the latter.

For a rigorous definition of the measure $\omega^V$ we need to regularise the IR and UV singularities and then perform a limit to remove the regularisation parameters. A popular choice is to restrict the randomness in the problem to a finite number of degrees of freedom by working on a finite set of space points. Let $L \in \mathbb{N}$, $\varepsilon \in \{2^{-N} \mid N \in \mathbb{N}\}$ and introduce the $d$-dimensional toroidal lattice $\mathbb{T}^d_{L,\varepsilon} := ((\varepsilon \mathbb{Z})/(L\mathbb{Z}))^d$ and a suitable $\mathbb{T}^d_{L,\varepsilon}$-lattice regularisation\footnote{Because of technical reasons, we will avoid simply replacing $\Delta$ by the lattice Laplacian $\Delta_{L,\varepsilon}$ on $L^2(\mathbb{T}^d_{L,\varepsilon};\mathbb{C}^2)$, see Definition 40 in Section 3.3.} $G^{L,\varepsilon}$ of the continuum covariance $G$ in eq. (1). Moreover let $(\psi_{x}^{L,\varepsilon})_{x \in \mathbb{T}^d_{L,\varepsilon}}$ the Grassmann Gaussian field with covariance $G^{L,\varepsilon}$ (and with the same involution $\Theta$) and discretise the interaction potential as

$$V^{L,\varepsilon}(\psi_{L,\varepsilon}^{x}) := \int_{\mathbb{T}^d_{L,\varepsilon}} \left[ \frac{\lambda}{4} (\psi_{x}^{L,\varepsilon})^4 + \mu(\lambda) (\psi_{x}^{L,\varepsilon})^2 \right] dx,$$

where $\int_{\mathbb{T}^d_{L,\varepsilon}} \cdot dx := \varepsilon^d \sum_{x \in \mathbb{T}^d_{L,\varepsilon}} \cdot$ is a normalised counting measure on $\mathbb{T}^d_{L,\varepsilon}$ and where we choose $\mu(\lambda)$ depending on $\varepsilon, \lambda$ in order to achieve a nontrivial limit as $\varepsilon \to 0$. Indeed due to the singularities in the covariance $G$ we expect to need to consider a diverging family $((\mu(\lambda)))_{\varepsilon > 0}$ in order to compensate for the divergences introduced by the nonlinear factor $e^{V^{L,\varepsilon}(\psi)}$ in the averages. To the regularised potential there corresponds regularised measures $\omega^{V^{L,\varepsilon}}$ and the task is then to establish the existence, uniqueness and the properties of the weak cluster points of the family $((\omega^{V^{L,\varepsilon}}))_{\varepsilon > 0, L > 0}$. This is the problem of taking the IR and UV limits of the renormalised model with interaction (3) and is a well known mathematical question in constructive quantum field theory. Many techniques have been developed both for bosonic and fermionic theories. We will review below some of the relevant literature.

**Stochastic quantisation.** To construct the Grassmann measure $\omega^V$ we use a novel approach based on a non-commutative forward-backward stochastic differential equation (FBSDE), which provides a stochastic quantisation of the measure of interest. We will show how this approach applies to any subcritical Grassmann equivalents of the well-known $\varphi^4$ and $\varphi^3$ bosonic (or commutative) theories.

Stochastic quantisation is here understood in the broad sense of having a map which transport a measure of our choice into a target measure of interest, in this case a Gibbsian measure in the form $\omega^V$. In our particular case the measure of choice is given by a non-commutative probability space $(\hat{\omega}, \mathcal{M})$ endowed with a suitable Gaussian Grassmann field $X$ and another Grassmann field $\Psi$ (non-Gaussian, in general) which will belong to the sub-algebra $\mathcal{M}_X \subseteq \mathcal{M}$ generated by $X$ and such that

$$\text{Law}_{\hat{\omega}}(\Psi)(P) := \hat{\omega}(P(\Psi)) = \omega^V(P(\psi)) = \text{Law}_{\omega^V}(\psi)(P),$$

for a sufficiently large class of functions $P$ on Grassmann fields. The usefulness of this construction is that one can identify sufficiently nice maps $F : X \to \Psi$ which perform the push-forward of the Gaussian law of $X$ under $\hat{\omega}$ to the law of $\psi$ under $\omega$, i.e. $\text{Law}_{\omega^V}(\psi) = F_{\#}\text{Law}_{\hat{\omega}}(X)$. The nice features of the map $F$ together with the Gaussianity of $\text{Law}_{\hat{\omega}}(X)$ are the key to the construction and analysis of $\text{Law}_{\omega^V}(\psi)$.
From a more conceptual viewpoint, the stochastic quantisation, as understood above, is a tool to perform the stochastic analysis of a given measure or process and this paper is one among a series of recent results which try to push forward a program of systematically develop the stochastic analysis of Euclidean quantum field theories or more generally of irregular random fields parametrised by multidimensional continuous index sets like $\mathbb{R}^d$ and possessing particular properties of locality and symmetry. In particular let us note the following works:

- The stochastic analysis of Grassmann random fields, in the language of the present paper has been initiated in [ABDG20] where the infinite volume limit of some weakly interacting Grassmann Gibbsian measures has been established via parabolic stochastic quantisation, i.e. stochastic quantisation based on a Langevin-type equation.

- The stochastic quantisation of the $\Phi^4_3$ bosonic model has been pushed forward in [GH21]. While this work has been able to establish most of the Osterwalder–Schrader [GJ87] axioms, the uniqueness and rotation invariance of the limit is still an open problem.

- A variational method for stochastic quantisation in the bosonic setting has been introduced in [BG20, BG21b, BG21a, Bar22] and used to prove estimates for $\Phi^4_3$ in [CGW20] towards establishing the existence of a phase transition. Our contribution here can be understood as the fermionic version of the variational approach. In the bosonic setting indeed the FBSDE appears as the Euler–Lagrange equation of the optimal control problem for the interacting measure.

- An elliptic variant of stochastic quantisation for bosonic theories has been introduced and studied in [AVG20, ADG21, BD21], including the full construction of an EQFT in two space dimensions and the verification of the Osterwalder–Schrader axioms.

- Stochastic quantisation is a very active field of modern stochastic analysis, with recent advances in the analysis of low dimensional quantum (Euclidean) Yang-Mills theories [CCHS20, CCHS22], large $N$ limit of $O(N)$ vector models in $d = 3$ [SZZ21b] and study of the perturbation expansion of correlations in $\Phi^4_3$ [SZZ21a].

- The point of view on stochastic quantisation as the search for good representation as push-forwards of the target measures is parallel to recent analysis from the point of view of functional inequalities and optimal transport, see e.g. [She22], inspired by a the renormalisation group analysis of [BH20, BB21, BD22] which is very much related to our use of flow equations and renormalisation group ideas as discussed below.

Our analysis builds on the Grassmann probabilistic setting developed in [ABDG20] and pioneered in [OS72]. See also [Lot87] for a similar approach to Fermi fields, in the context of supersymmetric Wiener integrals. In addition to what was done in [ABDG20, OS72], we are here able to remove the ultraviolet cut-off and thus to describe ultraviolet singular superrenormalisable fermionic field theories. Let us point out right now that this is not enough to study some of the well-known fermionic field theories, e.g., the Gross-Neveu model in $d = 2, 3$ [GK85a] and the Yukawa model in $d = 2$ [Les87], which within the formalism of the present paper behave as critical theories which require a more detailed handling of the contributions of small scales to the drift of the stochastic equation. We leave to future work the analysis of these more delicate situations.
The continuous renormalisation group. The stochastic quantisation that we use is introduced via a scale decomposition of the Gaussian process according to the basic ideas of the renormalisation group (RG). In the constructive QFT literature, RG is one of the most successful methods to study Euclidean fermionic theories (i.e. Grassmann measures). Indeed Grassmann measures have been early on understood to be a nice setting where to develop constructive tools and renormalisation group techniques thanks to the intrinsic bounded nature of the relevant Gaussian variables, in striking contrast with the usual (bosonic) Gaussians which have unbounded support.

In general, the renormalisation group is a crucial tool in the construction of Euclidean quantum field theories and in the study of statistical mechanics and critical phenomena. It was pioneered in the work of Wilson [Wil71a, Wil71b], building on the work of Kadanoff [Kad66], where the idea of progressive integration of scales was first introduced for studying critical phenomena. Since then, the RG techniques have witnessed a very intense and diverse development, depending on whether they are used to study, e.g., Euclidean scalar measures [BCG+80, GK85b, FMRS87, BY90, Abd07, BBS19], Grassmann measures [GK85a, FMRS86, Les87, BG90, BGM92], lattice gauge theories [Bal87, Bal88, Dim20], supersymmetric measures [BBS15b, BBS15a, AFP20], or stochastic differential equations [Kup16, Duc21, Duc22] and depending on whether the separation of scales is done in a continuum or discrete fashion. Our list of references is by no means exhaustive and should only be understood as a sample for the interested reader.

The most well-known approach in the case of a continuum of scales is the Polchinski’s flow equation [Pol84], that is, an infinite-dimensional Hamilton–Jacobi–Bellman equation describing the scale-dependent effective potential \( V_t = V_t(\varphi) \), which formally reads:

\[
\frac{\partial}{\partial t} V_t = \frac{1}{2} (D V_t)_{\hat{G}_t}^2 - \frac{1}{2} D^2_{\hat{G}_t} V_t; \tag{4}
\]

where \( \hat{G}_t = \frac{d}{dt} G_t \) with \( G_t \) is some interpolation of the covariance \( G \) of a Gaussian reference measure, and where \( D \) denotes functional derivative, so that \( D^2_{\hat{G}_t} \) is the functional Laplacian associated with the operator \( \hat{G}_t \), and where \( (D V_t)_{\hat{G}_t}^2 \) is the shorthand for \( \langle D V_t, D V_t \rangle_{\hat{G}_t} \) for suitable bilinear form \( \langle \cdot, \cdot \rangle_{\hat{G}_t} \) associated to the infinitesimal covariance \( \hat{G}_t \). While this HJB equation has been an effective tool to estimate the perturbative expansion of bosonic Euclidean field theories [Pol84, Sal99], the rigorous study of eq. (4) at the non-perturbative level is very challenging. An important contribution in this sense was done in [BK87] and later applied to a slightly different equation [BY90, BBS15b, BBS15a], where the basic idea is to show that the solution of the HJB equation is majorised, in a suitable topology, by a quantity solving a simpler first-order equation, without functional Laplacian.

In the context of Grassmann measures the direct study of the flow equation (4) is quite challenging (and essentially an open problem to our knowledge). The validity of the flow equation in relevant examples and without ultraviolet cutoffs has been established by Disertori and Rivasseau [DR00] using a careful analysis of the tree expansion. In most of the literature on the RG for Euclidean Fermionic theories the technical difficulties posed by the continuum equation by studying instead a discrete flow, that is, a sequence \( \langle V_n \rangle_{n \geq 0} \) with the points \( (t_n)_{n \geq 0} \) sufficiently separated. This discrete flow equation, despite more cumbersome, turns out to be controllable in a surprisingly wide range of regimes because of the Pauli principle [GK85a], which later assumed the form of the celebrated determinant bounds [Les87], see also [GMR21] and [FMRS86]. More recently, however, the continuum equation for fermions has received more attention in the spirit of [BK87], see [KM22].

\[2\] This is technically a forward equation. Below, we consider a backward equation and flip the sign of \( V \), so that there will be some apparent incongruences.
A synergy. The key contribution of the present paper, apart from the explicit constructions and bounds, is the idea that the synergy of RG and stochastic quantisation, in the form of a FBSDE, provides a powerful tool to analyse Gibbsian measures. Indeed here we control the solution of the FBSDE by means of a suitable flow equation for the change of the drift with respect to the scale of the decomposition. However, in contrast with other works using an RG approach, and thanks to the presence of the FBSDE, we would only need to solve the Polchinski’s equation in an approximate way, which results in great simplification of the analysis. The approximate flow equation is studied in a space of polynomial Grassmann functionals with controlled locality properties within an analytic setup largely inspired by the work [GMR21]. Note that flow equations have been recently applied to the study of superrenormalisable stochastic partial differential equations [Duc21, Duc22] as well, but without taking full advantage of the presence the SPDE. By transposing the ideas developed in the present paper, and in particular the use of the approximate flow equation in conjunction with the analysis of an associated equation for the fields, the paper [GR22] obtains the stochastic quantisation of the bosonic fractional \( \Phi^4_3 \) theory in the full subcritical regime and in the infinite volume limit.

1.1 Model and main result

Consider a sufficiently large non-commutative probability space \((\mathcal{M}, \omega)\), see Section 2, in which the expectation is denoted by \(\omega(\cdot)\). Introduce a Grassmann Brownian motion (GBM) as the family \((X_{t,\varepsilon}^L, \varepsilon)_{t \geq 0}\) of Gaussian random fields indexed in the Hilbert space\(^3\) \(h_{L,\varepsilon} := \mathcal{L}^2(\mathbb{T}_L^d, \mathbb{C}^2) \oplus \mathcal{L}^2(\mathbb{T}_L^d, \mathbb{C}^2)\). As Grassmann fields they satisfy anti-commutation relations:

\[
X_{t,\varepsilon}^L(f)X_{s,\varepsilon}^L(g) + X_{s,\varepsilon}^L(g)X_{t,\varepsilon}^L(f) = 0, \quad \forall f, g \in h_{L,\varepsilon}, \forall s, t,
\]

and are (centred) Gaussian random fields, namely

\[
\omega(X_{t,\varepsilon}^L(f)X_{s,\varepsilon}^L(g)) = \langle \Theta f, G_{t,\varepsilon}^L g \rangle_0
\]

\[
\omega(X_t(f_1) \cdots X_{t_n}(f_{2n})) = \text{Pf}((\langle \Theta f_i, G_{t,\varepsilon}^L f_j \rangle_0)_{1 \leq i, j \leq 2n}),
\]

where \(G_t^L\) is bounded on \(h\) and \(\Theta\)-antisymmetric, that is, \(\Theta(G_t^L)^*\Theta = -G_t^L\) and where \(\text{Pf}(\cdot)\) denotes the Pfaffian. It is then convenient to label such GBMs in \(\mathbb{T}_L^d\) through a basis of Kronecker’s deltas, \(X_{t,\varepsilon}^L(x) := X_{t,\varepsilon}^L(\delta_x)\). See Section 2 for the detailed definitions and properties of these objects.

From a stochastic perspective, the crucial point to note that the parameter \(t\) is not a physical time nor a stochastic time. Rather, following the core idea of the RG techniques, it is a continuum flow parameter associated with the scale decomposition of the problem. To make this more precise, introduce a differentiable interpolation \((G_t^L)_{t \geq 0}\) such that \(G_0^L = 0\) and \(G_\infty^L = G^L\). This interpolation should also suppress momenta larger than \(2^s\), so that \(G_t^L\) is a bounded operator, uniformly in \(L\) and \(\varepsilon\), if \(t < \infty\). Note that the measure \(\omega^{V_{L,\varepsilon}(\psi^L, \varepsilon)}(\psi^L, \varepsilon)\) is equal to \(\omega^{V_{L,\varepsilon}(X^L_{\infty,\varepsilon})}(X^L_{\infty,\varepsilon})\), so that our problem becomes the analysis of the weak limit

\[
\omega^{V}(\cdot) := \lim_{L \to \infty} \lim_{\varepsilon \to 0} \omega \left( \frac{\cdot e^{V_{L,\varepsilon}(X^L_{\infty,\varepsilon})}}{\omega^{V_{L,\varepsilon}(X^L_{\infty,\varepsilon})}} \right)
\]

\[
\langle f, g \rangle = \sum_{x} \sum_{x \in \mathbb{T}_L^d} e^{tf(x)}g(x).
\]

\(^3\) The space \(\mathcal{L}^2(\mathbb{T}_L^d, \mathbb{C}^2)\) is associated with \(\varepsilon^d\) times the counting measure: if \(f, g \in \mathcal{L}^2(\mathbb{T}_L^d, \mathbb{C}^2)\) we write \(\langle f, g \rangle = \sum_{x} \sum_{x \in \mathbb{T}_L^d} e^{tf(x)}g(x)\).
See Section 3.3 for further details on the precise form of the potential $V^{L,\varepsilon}$. As we already announced, we would like to study the interacting measure (5) via stochastic quantisation, that is, by identifying it as the marginal law of a suitable stochastic process. In particular we will prove that for any nice enough function $P$ the following identity holds true

$$
\omega^V(P(X^{L,\varepsilon}_\infty)) = \omega(P(\Psi^{L,\varepsilon}_\infty)),
$$

provided that the process $(\Psi^{L,\varepsilon}_t)$ solves the following forward-backward stochastic differential equation (FBSDE) on $[0, \infty]$

$$
d\Psi^{L,\varepsilon}_s = \Theta \dot{\Psi}^{L,\varepsilon}_s \omega_s(DV^{L,\varepsilon}_s(\Psi^{L,\varepsilon}_\infty))ds + dX^{L,\varepsilon}_s, \quad \Psi^{L,\varepsilon}_0 = 0,
$$

where $DV^{L,\varepsilon}$ denotes the functional derivative of $V^{L,\varepsilon}$ and where $\omega_t(\cdot)$ denotes the conditional expectation with respect to a filtration such that $(X^{L,\varepsilon}_t)$ is adapted, see Section 2 for the details.

It is important to note that the drift term in (7) satisfies the identity

$$
\omega_s(DV^{L,\varepsilon}(\Psi^{L,\varepsilon}_s)) = DV^{L,\varepsilon}_s(\Psi^{L,\varepsilon}_s),
$$

where $V^{L,\varepsilon}$ is the solution of the flow equation (4). However, we do not want to make this substitution: the FBSDE is a powerful tool that allow us to truncate the flow equation for the effective force $\omega_s(DV^{L,\varepsilon}(\Psi^{L,\varepsilon}_\infty))$ in a suitable way and to control the remainder due to the truncation for any subcritical (that is, superrenormalisable) theory. In fact, we decompose the drift term as

$$
\omega_s(DV^{L,\varepsilon}(\Psi^{L,\varepsilon}_s)) = F^{L,\varepsilon}_s(\Psi_s) + R^{L,\varepsilon}_s,
$$

where the “remainder process” $R^{L,\varepsilon}_s$ solves the following self-consistent equation, for $s > 0$

$$
R^{L,\varepsilon}_s = \int_s^\infty \omega_s(\mathcal{H}_r[F^{L,\varepsilon}_r])dr + \int_s^\infty \omega_s((\Theta \dot{\Psi}^{L,\varepsilon}_r R^{L,\varepsilon}_r D F^{L,\varepsilon}_r(\Psi^{L,\varepsilon}_r))dr,
$$

with

$$
\mathcal{H}_r[F^{L,\varepsilon}_r] := \partial_x F^{L,\varepsilon}_r + \frac{1}{2}D^2 \Theta \dot{\Psi}^{L,\varepsilon}_r F^{L,\varepsilon}_r + \Theta \dot{\Psi}^{L,\varepsilon}_r R^{L,\varepsilon}_r D F^{L,\varepsilon}_r.
$$

We have now the freedom to choose $F^{L,\varepsilon}$ in any convenient way. If we let $F^{L,\varepsilon}_s$ be the solution of the Hamilton-Jacobi-Bellman equation, then $\mathcal{H}_r[F^{L,\varepsilon}_r]$ vanishes and so the remainder process $R^{L,\varepsilon}_s$. More generally, we can take $F^{L,\varepsilon}_s$ to be the solution of a simpler flow equation, provided that we can control Eq. (8). In fact, once we have good estimates for $F^{L,\varepsilon}_s$, we can prove the existence and uniqueness of a pair $(\Psi^{L,\varepsilon}, R^{L,\varepsilon})$ solving Eqs. (7) and (8) by a fixed-point argument, hence the need for a small $\lambda$. Our main result is as follows.

**Theorem 1.** Let $d \in \mathbb{N}$ and assume that $\gamma < \min \{d/4, 1\}$. Then, there exists $\lambda_0 = \lambda_0(\gamma, d)$ and a function $\mu(\lambda)$ such that if $\lambda \leq \lambda_0$, then for any $L \in \mathbb{N}$ and $\varepsilon \in \{2^{-N} | N \in \mathbb{N}\}$ Eq. (7) with $V^{L,\varepsilon}$ as in (3) has a unique global solution $\Psi^{\varepsilon,L}$ and that $\Psi^{\varepsilon,L} \to \Psi$ in a suitable topology, as $\varepsilon \to 0$ and $L \to \infty$.

**Remark 2.** The constraint $\gamma < 1$ is for technical reasons and can be removed provided that further “counter-terms” are included in the potential, e.g., of the form $\int_{\Omega^d} v(\lambda)((\partial_x X^{L,\varepsilon}_t)^2)dx$, where $\partial_x$ is the lattice derivative.
As an application, we prove the exponential clustering of the correlation functions for the interacting measure in (5) via the coupling method. Letting \( \text{Cov}, \nu(v; A; B) := \nu^v(A) - \nu^v(B), \)

we state our result as follows.

**Theorem 3.** Let \( d \in \mathbb{N} \) and assume that \( \gamma < \min \{ d/4, 1 \} \). Consider \( m_1, m_2 \in \mathbb{N} \), and let \( f^{(1,k)}, f^{(2,k')} \in L^2(\mathbb{R}^d; \mathbb{C}^4) \cap W^{[\gamma],1}(\mathbb{R}^d; \mathbb{C}^4), \) for \( k = 1, \ldots, m_1 \) and \( k' = 1, \ldots, m_2 \). Set \( \| f \|_{b,a} := \| f \|_{L^2(\mathbb{R}^d; \mathbb{C}^4)} + \| f \|_{W^{[\gamma],1}(\mathbb{R}^d; \mathbb{C}^4)}. \) Then, there exists \( \lambda_0 = \lambda_0(\gamma, d) \) and a function \( \mu^x(\lambda) \) such that if \( \lambda \leq \lambda_0 \), and \( V^{L,c} \) is as in (3), then

\[
\text{Cov}^\nu \left( \prod_{k=1}^{m_1} X(f^{(1,k)}); \prod_{k=1}^{m_2} X(f^{(2,k)}) \right) \leq d, \gamma, \lambda \left( \prod_{i,k} \| f^{(i,k)} \|_{b,[\gamma]} \right) e^{-c \text{dist}(D_1, D_2)},
\]

for some universal constant \( c > 0 \), where \( D_i := \bigcup_{k=1}^{m_i} \text{supp}(f^{(i,k)}) \).

A corollary of the previous theorem is the exponential decay of the correlation function.

**Corollary 4.** Under the same hypotheses and notations of Theorem 3, if \( m_1 = m_2 = 1 \) and writing \( f_1 = f^{(1,1)} \), \( f_2 = f^{(2,1)} \) then

\[
|\nu^v(X(f_1)X(f_2))| \leq d, \gamma, \lambda, f_1, f_2 e^{-c \text{dist}(D_1, D_2)},
\]

for some universal constant \( c > 0 \), where \( D_i := \text{supp}(f_i) \).

**Comparison with Berezin integration.** In the standard approach, Grassmann measures on a finite dimensional Grassmann algebra are described via the Berezin integration \([Sal99, KTF02, Mas08]\). With analogy to the commutative setting, one can think of the Berezin integral as a “flat” measure for Grassmann variables, i.e. the equivalent of the Lebesgue measure. More in detail, consider the Grassmann algebra generated by \( \{ \psi_x, \sigma \in \mathbb{T}_{L,e}, \sigma = \uparrow, \downarrow \} \) that is, the unital complex algebra whose generators satisfy

\[
\psi^\rho_{x,\sigma} \psi_{x',\sigma'}^{\rho'} + \psi^\rho_{x,\sigma'} \psi_{x',\sigma}^{\rho'} = \delta_{x,x'} \delta_{\rho,\rho'} \delta_{\sigma,\sigma'}, \quad (9)
\]

where \( \sigma = \uparrow, \downarrow \) refers to the spin of the fermions. Let us denote by \( \mathcal{G}_{L,e}^d \) the said Grassmann algebra and note that \( \mathcal{G}_{L,e}^d \) is a finite-dimensional linear space thanks to (9).

By abuse of language we call Grassmann measure any linear functional on \( \mathcal{G}_{L,e}^d \). The Berezin integral \( \omega_{\text{Ber}} \) is the Grassmann measure defined on the linear generators of the algebra as

\[
\omega_{\text{Ber}} \left( \prod_{x \in \mathbb{T}_{L,e}, \sigma = \uparrow, \downarrow} \psi_{x,\sigma}^+ \psi_{x,\sigma}^- \right) = 1,
\]

and equal to zero on any other linearly independent element of \( \mathcal{G}_{L,e}^d \). Note that the order in the product above is unimportant because \( \psi_{x,\sigma}^+ \psi_{x,\sigma}^- \) are commuting elements of \( \mathcal{G}_{L,e}^d \). It is customary to formally write the Berezin integration as

\[
\int d\psi_{x,\uparrow} d\psi_{x,\downarrow} P(\psi) := \omega_{\text{Ber}}(P(\psi)).
\]
This is consistent with interpreting the symbol $\frac{\partial}{\partial \psi_{x,\sigma}}$ on $G^\rho_{L,\varepsilon}$; in fact, the Berezin integration is also written as $\omega_{\text{Ber}}(\cdot) = \prod_{x,\sigma} \frac{\partial}{\partial \psi_{x,\sigma}} \frac{\partial}{\partial \psi_{x,\sigma}^+}$.

Provided that $\varepsilon > 0$ and $L \in \mathbb{N}$, we can always write the law $\omega^{\rho,\Psi}(\cdot) := \text{Law}_\rho(\Psi)$ in the state $\rho$ of any Grassmann random variable $\Psi$ with values in $G^\rho_{L,\varepsilon}$ in terms of the Berezin measure as $\omega^{\rho,\Psi}(\cdot) = \omega_{\text{Ber}}(\nu \cdot) := \int d\nu(\psi) \cdot$ for some “density” $\nu$ taking values in $G^\rho_{L,\varepsilon}$, and, with abuse of notation, refer also to $d\nu$ as a Grassmann measure. Therefore, as long as $L \in \mathbb{N}$ and $\varepsilon > 0$, the law $d\nu$ of $\psi$ under the measure $\omega^V$ as described in (5) can be written in terms of the Berezin integral as follows:

$$d\nu(\psi) \propto \left[ \prod_{x \in T^d_{L,\varepsilon}, \sigma = \uparrow, \downarrow} d\psi_{x,\sigma}^+ d\psi_{x,\sigma}^- \right] e^{-(\psi, (G^{L,\varepsilon})^{-1} \psi) + \int_{T^d_{L,\varepsilon}} \left[ \frac{1}{4} (\psi_x)^4 + \mu^\varepsilon(\lambda)(\psi_x)^2 \right] dx},$$

(10)

where we used the notation $(\psi, A \psi) := \sum_{x, \sigma, \sigma'} \int_{T^d_{L,\varepsilon} \times T^d_{L,\varepsilon}} \psi_{x, \sigma}^+ A_{x, y} \psi_{y, \sigma'}^- dx dy$, and set $(\psi_x)^2 := \sum_{\sigma, \sigma'} \psi_{x, \sigma}^+ \psi_{x, \sigma'}^-$, $(\psi_x)^4 := ((\psi_x)^2)^2$. In other words, (10) makes the Gibbsian Grassmann measure (2) quite concrete in terms of functionals on an abstract finite dimensional Grassmann algebra and it is the customary object of investigation in the mathematical physics literature.

The language of Berezin integration becomes cumbersome when dealing with Grassmann stochastic processes which cannot be faithfully realised on finite dimensional Grassmann algebras. Furthermore, although using an infinite-dimensional abstract Grassmann algebra is possible [KTF02], the useful topology generated by a Gaussian covariance essentially reproduce the analytic setting we realised quite compactly here. See also [ABDG20] for further discussion on similar matters and on a review of previous works which tried to make sense of Grassmann stochastic analysis without using our $C^*$-algebraic topological setting.

We look at the $C^*$-algebra $\mathcal{M}$ as a convenient place where to realise all the Grassmann random variables we need; in this sense, it is a suitable substitute for the finite dimensional Berezin integral. An analogy drawn from the standard commutative probabilistic setting can help the reader to get our point of view: while in finite dimensions all Gaussian measures are absolutely continuous with respect to the Lebesgue measure, in infinite dimensions this is not true anymore and there is not a privileged “base” measure. Similarly, when dealing with infinite dimensional Grassmann Gaussian fields and processes, the appeal of the language of Berezin integration is quite limited. This observation was clear to some researchers in the field, in particular Osterwalder–Schrader [OS72, Ost73, OS73] and Lott [Lot87].

Structure of the paper. In Section 2 we introduce the general formalism of filtered non-commutative probability spaces and characterise in detail Grassmann Brownian martingales. With respect to [ABDG20], we focus on the conditional expectation, which is a crucial ingredient for the derivation of the FBSDE. The latter is the content of Section 3, in which we also describe the model in more detail, the topology on the Grassmann fields and prove some bounds on the covariance of the GBM used in our FBSDE. Finally, in Section 4 we solve the FBSDE by using a truncated flow equation for the drift and, as an application, we prove exponential clustering of the correlation functions.

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2 Grassmann stochastic analysis

In this section, we recall some notions of non-commutative (or algebraic) probability spaces that are necessary for the formulation of the Grassmann FBSDE. In particular, we introduce the concept of Grassmann process, filtration and conditional expectation associated with the latter and describe Grassmann Brownian martingales. We refer to [ABDG20] for further details. The problem of showing that Grassmann random variables can actually be represented as operators acting on suitable Hilbert spaces, see [Ost73, OS73, ABDG20], is postponed to Appendix A.

2.1 Grassmann probability

A non-commutative probability space (NPS) is the pair \((\mathcal{M}, \omega)\), consisting of a \(C^*-\)algebra \(\mathcal{M}\) of operators acting on a separable Hilbert space and \(\omega\) a state on \(\mathcal{M}\), that is, a positive (hence continuous) normalised linear functional.

A NPS is a convenient setting for embedding Grassmann fields and Grassmann-field-valued processes, since we can inherit a topology, which is otherwise missing or not naturally defined on a Grassmann algebra, see [ABDG20], compare with [KTF02].

Recall that for any Hilbert space \(\mathcal{H}\), possibly infinite, we can associate a Grassmann algebra \(\mathcal{H} := \bigoplus_{n \geq 0} \mathcal{H}^n\), where \(\mathcal{H}^n\) is, with abuse of notation, either \(\mathbb{C}\) or \(\mathbb{R}\), depending on whether \(\mathcal{H}\) is a complex or real Hilbert space. Equivalently, we can say that the Grassmann algebra \(\mathcal{H}\) is generated by \(\mathcal{H}\), that is, its generators are indexed in \(\mathcal{H}\). Such generators, can be interpreted as generalised processes or fields, see [Sim74]. This motivates the following definition.

**Definition 5.** (Grassmann field). Let \((\mathcal{M}, \omega)\) be a NPS and \(\mathcal{H}\) a complex separable Hilbert space. A Grassmann field indexed in \(\mathcal{H}\) is a linear map \(\psi: \mathcal{H} \to \mathcal{M}\) such that

\[
\{\psi(f), \psi(g)\} = 0, \quad \forall f, g \in \mathcal{H}.
\]

**Remark 6.**

1. It is clear that a Grassmann field extends to a homomorphism of the Grassmann algebra \(\mathcal{H}\) into \(\mathcal{M}\). by setting \(\psi(f \wedge g) := \psi(f) \psi(g)\). Equivalently, the set of Grassmann fields on \((\mathcal{M}, \omega)\) indexed in \(\mathcal{H}\) can be identified with \(\text{Hom}(\mathcal{H}, \mathcal{M})\).

2. Given two Grassmann fields \(\psi_1\) and \(\psi_2\), in general one has that \(\{\psi_1(f), \psi_2(g)\} \neq 0\), for some \(f, g \in \mathcal{H}\), that is, the generalised random process \(\zeta\) indexed by \(\mathcal{H} \oplus \mathcal{H}\) and defined by \(\zeta(f \oplus g) := \psi_1(f) + \psi_2(g)\), is not a Grassmann field. This motivates Definition 7, see also [ABDG20].

3. Notice that the algebra generated by a Grassmann field \(\psi\) is not self-adjoint. The adjoint field \(\psi^*\) is still a Grassmann field which in general does not satisfy anti-commutation relations with \(\psi\).

**Definition 7.** (Compatibility). Let \(\psi_1\) and \(\psi_2\) be two Grassmann fields indexed by \(\mathcal{H}_1\) and \(\mathcal{H}_2\) respectively. We say that they are compatible or jointly Grassmann fields iff the random field defined by \(\zeta(f \oplus g) := \psi_1(f) + \psi_2(g)\), for any \((f, g) \in (\mathcal{H}_1, \mathcal{H}_2)\) is a Grassmann field indexed by \(\mathcal{H}_1 \oplus \mathcal{H}_2\).
Remark 8. Equivalently, two compatible Grassmann fields anti-commute.

A crucial class of Grassmann fields is given by Gaussian Grassmann fields. More specifically, consider a complex separable Hilbert space $\mathcal{H}$ with a suitable conjugation $\Theta$, that is, an involution such that $\langle f, g \rangle_{\mathcal{H}} = \langle \Theta g, \Theta f \rangle_{\mathcal{H}}$ for any $f, g \in \mathcal{H}$. Because of the antisymmetric nature of the Grassmann fields, Gaussian Grassmann fields are associated with a covariance $G \in \mathcal{B}(\mathcal{H})$ that is $\Theta$-antisymmetric:

$$\Theta G^* \Theta = -G.$$  (11)

Let us define the truncated expectation, or cumulant, of a Grassmann field.

Definition 9. (Cumulant). Let $\mathcal{M}, \omega$ be a NPS and let $\psi$ be a Grassmann field on it indexed in a complex separable Hilbert space $\mathcal{H}$. We define the cumulants $(K_n[\psi])_{n \in \mathbb{N}}$ of $\psi$, as the family of multilinear maps given by $K_1[\psi](f) := \omega(\psi(f))$ and for $n \geq 2$ by recursively solving

$$\omega(\psi(f_1) \cdots \psi(f_n)) = \sum_{\Pi \in \text{Partitions}} (-)^{\pi(\Pi)} \prod_{I \in \Pi} K_{|I|}[\psi](f_{i_1}, \ldots, f_{i_{|I|}}),$$

where $i_1 < \cdots < i_{|I|}$ are the elements of $I$, and where the $\pi(I)$ is the parity of the permutation associated with the partition.

Definition 10. (Gaussian Grassmann field). Let $\mathcal{M}, \omega$ be a NPS and let $\psi$ be a Grassmann field on it indexed in a complex separable Hilbert space $\mathcal{H}$, with conjugation $\Theta$. We say that $\psi$ is a (centred) Gaussian field, if

$$K_2[\psi](f_1, f_2) = \langle \Theta f_1, G f_2 \rangle_{\mathcal{H}}, \quad K_n[\psi](f_1, \ldots, f_n) = 0, \quad \forall n \neq 2,$$

for some $\Theta$-antisymmetric $G \in \mathcal{B}(\mathcal{H})$, see (11).

Remark 11. Equivalently, one can define Gaussianity via the anti-commuting Wick’s rule, which for centred fields implies that

$$\omega(\psi(f_1) \cdots \psi(f_{2n})) = \text{Pf}((\langle \Theta f_i, G f_j \rangle_{\mathcal{H}})_{1 \leq i, j \leq 2n}), \quad \omega(\psi(f_1) \cdots \psi(f_{2n+1})) = 0.$$

2.2 Processes and conditional expectations

In general, we are interested in families of Grassmann fields labelled by a real variable. Such families will be called Grassmann processes or Grassmann-field-valued processes. Of course, our presentation can be generalised to include any ordered labelling set.

Let us begin with defining a filtration on a NPS.

Definition 12. (Filtration). Let $\mathcal{M}, \omega$ be a NPS and let $I \subseteq \mathbb{R}$. A filtration over $I$ is a family $(\mathcal{M}_t)_{t \in I}$ of sub-$C^*$-algebras of $\mathcal{M}$ such that $\mathcal{M}_t \subseteq \mathcal{M}_t'$ for any $t \leq t'$. The triple $(\mathcal{M}, \omega, (\mathcal{M}_t)_{t \in I})$ is called filtered non-commutative probability space (FNPS).

We then define processes and adapted processes as follows.

Definition 13. (Process). Let $\mathcal{M}, \omega$ be a NPS. A process is a family $(\psi_t)_{t \in I}$, of elements of $\mathcal{M}$, for some interval $I \subseteq \mathbb{R}$.

Let $\mathcal{M}, \omega, (\mathcal{M}_t)_{t \in I}$ be a FNPS. A process $(\psi_t)_{t \in I}$ on it is called adapted if $\psi_t \in \mathcal{M}_t$ for any $t \in I$. 

Grassmann stochastic analysis
In particular, we are interested in Grassmann processes.

**Definition 14.** (Grassmann process) Let \((M, \Omega)\) be a NPS and \((\psi_t)_{t \in I}\) a process in it. We say that \((\psi_t)_{t \in I}\) is a Grassmann process (indexed in \(H\)) if for any \(s, t \in I\) \(\psi_s\) and \(\psi_t\) are compatible Grassmann fields (indexed in \(H\)).

Let \((M, \Omega, (M_t))\) be a FNPS. A Grassmann process \((\psi_t)_{t \in I}\) on it is called adapted if \(\psi_t \in M_t\) for any \(t \in I\).

Let us now introduce the conditional expectation with respect to a smaller \(C^*\)-algebra [Tom57]. We follow [Kad04], but note that more general definitions have been provided, e.g., in [AC82].

**Definition 15.** (Conditional expectation) Let \(M\) and \(N \subset M\) be \(C^*\)-algebras. A linear mapping \(\varphi : M \to N\) is a conditional expectation of \(M\) onto \(N\) if it is positive and satisfies
\[
\varphi(1_M) = 1_N, \quad \varphi(N_1 MN_2) = N_1 \varphi(M) N_2,
\]
for any \(N_1, N_2 \in N\) and \(M \in M\).

The following properties hold true, see [Kad04] for a proof and for further details.

**Proposition 16.** Let \(M\) and \(N \subset M\) be \(C^*\)-algebras and let \(\varphi\) be a conditional expectation of \(M\) onto \(N\). Then, for any \(M \in M\):
\[
\varphi(M^*) = \varphi(M)^*, \quad |\varphi(M)|^2 \leq \varphi(|M|^2), \quad \|\varphi(M)\| \leq \|M\|.
\]

Finally, if we have the triple \((M, \omega, (M_t))\) we will denote by \(\omega_t(\cdot)\) the conditional expectation of \(M\) onto \(M_t\).

**Corollary 17.** (Tower property). Let \((M, \omega, (M_t))\) be a FNPS and let \(\omega_t(\cdot)\) be the conditional expectation onto \(M_t\). Then, for any \(s \leq t\) and \(A \in M\) the following property holds true:
\[
\omega_s(\omega_t(A)) = \omega_s(A). \tag{12}
\]

**Definition 18.** (Martingale). Let \((M, \omega, (M_t))\) be a FNPS and let \(\omega_t(\cdot)\) be the conditional expectation. We say that an adapted process \((\psi_t)_{t \in I}\) is a martingale if, for any \(s, t \in I\) such that \(s \leq t\)
\[
\omega_s(\psi_t) = \psi_s.
\]

### 2.3 Grassmann Brownian martingales

We now discuss in detail Grassmann Brownian martingales (GBM). Our exposition is based on abstract definitions, yet one can prove the existence of a FNPS where these objects exist, see also Appendix A.

**Definition 19.** (Grassmann Brownian martingale). Let \(\mathfrak{h}\) be a Hilbert space with conjugation \(\Theta\) and let \((G_t)_{t \in I} \subset \mathcal{B}(\mathfrak{h})\) be \(\Theta\)-antisymmetric. A Grassmann Brownian martingale on \((M, \omega, (M_t))\) indexed in \(\mathfrak{h}\) with covariance \((G_t)_{t \in I}\) is the adapted centred Gaussian Grassmann process \((X_t)_{t \in I}\) with independent increments, that is, \(\omega_s((X_t - X_s)(f)) = 0\) for any \(s \leq t\) and \(f \in \mathfrak{h}\), and such that
\[
\omega(X_t(f)X_s(g)) = (\Theta f, G_{t \wedge s} g)_\mathfrak{h} \quad \forall s, t \in I, \forall f, g \in \mathfrak{h}. \tag{13}
\]
Grassmann Brownian martingales are indeed martingales, as follows from the independence of increments and the fact that $X_t$ is adapted, that is, $\omega(X_t(f)) = X_t(f)$. On the other hand, polynomials of a GBM are not martingales in general. This is however the case if one considers Wick’s products, defined as follows.

**Definition 20.** (Wick products). The Wick’s products of $X_t$ are defined recursively by setting $[1] := 1$ and
\[
[X_t(v_1) \cdots X_t(v_{n+1})] := X_t(v_1)[X_t(v_2) \cdots X_t(v_{n+1})] - \sum_{j=1}^{n} (-)^j \omega(X_t(v_1)X_t(v_j))[X_t(v_2) \cdots X_t(v_{2j-1})X_t(v_{j+1}) \cdots X_t(v_{n+1})].
\]

**Proposition 21.** Wick’s products are martingales, that is, for $s \leq t$
\[
\omega_s([X_t(v_1) \cdots X_t(v_n)]) = [X_s(v_1) \cdots X_s(v_n)].
\]

**Proof.** This can be directly checked via the inductive definition and by exploiting the properties of the GBM, namely that it has independent increments. \qed

We henceforth write $X_{s,t} := X_t - X_s$ and $G_{s,t} := G_t - G_s$ for $s \leq t$. Clearly, $X_{s,t}$ is a Gaussian Grassmann field with covariance $G_{s,t}$.

Finally, since we do not want to rely on any explicit construction, we need a recipe for controlling the norm of the fields. This is not obvious, because $\{X_t(f)| f \in \mathfrak{h}\}$ is not a self-adjoint algebra, see Remark 6. We give the following definition.

**Definition 22.** (Norm-compatibility). Let $(X_t)_t$ be a GBM on $(\mathcal{M}, \omega, (\mathcal{M}_t)_t)$ indexed in $\mathfrak{h}$ (with conjugation $\Theta$) and with differentiable covariance $G_t \in \mathcal{B}(\mathfrak{h})$. Let $\mathcal{G}_t = C_t^2 U_t$, $C_t > 0$, be the polar decomposition of the differential covariance. We say that $X_t$ is norm-compatible if there exists some constant $c_X > 0$ such that
\[
\|X_t(f)\|^2 \leq c_X \int_0^t \left( \|C_s U_s f\|_\mathfrak{h}^2 + \|C_s \Theta f\|_\mathfrak{h}^2 \right) ds, \quad \forall t \in I, \forall f \in \mathfrak{h}.
\]

**Remark 23.** Note that for $X_t$ as above, by the Cauchy-Schwarz inequality we have
\[
|\omega(X_t(f)X_t(g))| \leq \left( \int_0^t \|C_s U_s f\|_\mathfrak{h}^2 ds \right)^{1/2} \left( \int_0^t \|C_s \Theta g\|_\mathfrak{h}^2 ds \right)^{1/2}.
\]

Since $|\omega(X_t(f)X_t(g))| \leq \|X_t(f)\| \|X_t(g)\|$, the requirement in (14) goes in the same direction as the bound above. In passing, we point out that in our setting, we will have $\mathcal{G}_t = C_t^2 U$ with $[C_t, U] = [C_t, \Theta] = 0$. In this case, (14) can be rewritten as $\|X_t(f)\|^2 \leq c_X \int_0^t \|C_s f\|_\mathfrak{h}^2 ds$.

Until now, we have pushed under the rug an important existence problem: in fact, it is a priori not clear whether there exist FNPS where GBMs with the above properties exist. The following theorem provides a positive answer.

**Theorem 24.** Let $\mathfrak{h}$ be a separable Hilbert space with conjugation $\Theta$ and let $(G_t)_t \subset \mathcal{B}(\mathfrak{h})$ be $\Theta$-antisymmetric operators. Then, there exists a FNPS $(\mathcal{M}, \omega, (\mathcal{M}_t)_t)$ where we can represent a norm-compatible GBM indexed in $\mathfrak{h}$ with covariance $G_t$. 

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The proof is by construction and is given in Appendix A.

3 The FBSDE approach

In this section, we derive the forward-backward stochastic differential equation (FBSDE) which realises the stochastic quantisation in (6), for processes indexed in a finite dimensional Hilbert space – which is our setting as long as $\varepsilon \in A$ and $L$ is finite. The derivation of the FBSDE is ultimately a consequence of Itô’s formula for Grassmann processes.

3.1 Flow under the conditional expectation

To begin, note that, because Grassmann fields are labelled by a finite dimensional $\mathfrak{h}$, the set of functions of such Grassmann fields can be identified with the Grassmann algebra $\bigwedge \mathfrak{h} := \bigoplus_{n \geq 0} \mathfrak{h}^\wedge n$. Actually, all elements of $\bigwedge \mathfrak{h}$ are polynomials. In fact, if $(v_j)_{j \in S}$ is a basis for $\bigwedge \mathfrak{h}$, where $S$ is some ordered set, we define $v_\alpha := \bigwedge_{j \in \alpha} v_j$ for all ordered subsets $\alpha \subseteq S$, so that $(v_\alpha)_{\alpha \subseteq S}$ is a linear basis for $\bigwedge \mathfrak{h}$. Accordingly, $F \in \bigwedge \mathfrak{h}$ can be written as $F = \sum_\alpha F_\alpha v_\alpha$ for some $F_\alpha \in \mathbb{C}$, that is, $F$ is a polynomial. For any such $F \in \bigwedge \mathfrak{h}$, if $\psi$ is a Grassmann field indexed by $\mathfrak{h}$, by $F(\psi)$ we mean

$$F(\psi) = \sum_\alpha F_\alpha \psi(v_\alpha) \quad \psi(v_\alpha) := \bigwedge_{j \in \alpha} \psi(v_j).$$

We also set $\bigwedge_{\text{odd}} \mathfrak{h} := \bigoplus_{n \geq 0} \mathfrak{h}^{\wedge (2n+1)}$ and $\bigwedge_{\text{even}} \mathfrak{h} := \bigoplus_{n \geq 0} \mathfrak{h}^{\wedge 2n}$ and say that $F \in \bigwedge \mathfrak{h}$ is an odd (resp. even) if in particular $F \in \bigwedge_{\text{odd}} \mathfrak{h}$ (resp. $F \in \bigwedge_{\text{even}} \mathfrak{h}$). Finally, we say that $F$ is purely Grassmann (or nilpotent) if $F \in \bigoplus_{n \geq 1} \mathfrak{h}^{\wedge n}$.

**Remark 25.** Let $F$ be purely Grassmann and let $f$ be a complex analytic function around $F_0 \in \mathbb{C}$. We can always define $f(F_0 + F) \in \bigwedge \mathfrak{h}$ via the power series expansion, that is, $f(F_0 + F) = \sum_{n \geq 0} f^{(n)}(F_0) F^n$, which is in fact a polynomial. A particular example is the (principal value) logarithm. In particular, if $F$ is purely Grassmann, for any $F_0 \in \mathbb{C}$, $F_0 \neq 0$ we have $\log(F_0 + F) = \log(F_0) + \sum_{n \geq 1} (-)^{n+1} \frac{F_0^{-n} F^n}{n}$. Another example is the exponential function, which is entire. Therefore for any $F \in \bigwedge \mathfrak{h}$ we have $\exp(F) := \sum_{n \geq 0} \frac{F^n}{n!}$.

**Definition 26.** (Functional derivative). If $U = \sum_\alpha U_\alpha v_\alpha \in \bigwedge \mathfrak{h}$ then $DU: \mathfrak{h} \rightarrow \bigwedge \mathfrak{h}$

$$(DU)(v_j) := \sum_\alpha U_\alpha \partial_v j v_\alpha,$$

where $\partial_v j$ is the anti-commuting derivative, that is, linear operators satisfying $\partial_v j 1 = 0$ and $\partial_v j v_i F + v_i \partial_v j F = \langle v_j, v_i \rangle F$ for any $F \in \bigwedge \mathfrak{h}$. If $F: \mathfrak{h} \rightarrow \bigwedge \mathfrak{h}$ is an odd element, we also use the notation:

$$(F, DU) := \sum_{\alpha, j} F(v_j) U_\alpha \partial_v j v_\alpha = \sum_{\alpha, j} F(v_j) U_\alpha \partial_v j v_\alpha. \quad (15)$$

We can also define the functional Laplacian $D^2_{\Theta C}: \bigwedge \mathfrak{h} \rightarrow \bigwedge \mathfrak{h}$ associated with $C \in \mathcal{L}(\mathfrak{h})$ $\Theta$-antisymmetric ($\Theta C^* \Theta = -C$):

$$(\Theta C v_j, v_i) \partial_v j \partial_v j v_\alpha.$$
Remark 27. With abuse of notation, we extend (15) to \( \langle \psi, DU \rangle := \sum_{\alpha, j} U_\alpha \psi(v_j) \partial_{v_j} v_\alpha \), for any compatible Grassmann fields \( \psi, \varphi \) indexed in \( \mathfrak{h} \). Note that \( \langle \Theta C v_j, v_i \rangle \) has to be antisymmetric in \( i, j \) since \( \partial_{v_i} \partial_{v_j} = -\partial_{v_j} \partial_{v_i} \).

Proposition 28. Let \( U \in \wedge \mathfrak{h} \) be even. Let \( X_t \) be a GBM with covariance \( G_t \in \mathcal{B}(\mathfrak{h}) \). Define \( U_t(\psi) := \omega_t(U(\psi + X_t, T)) \quad \text{for} \ t \in [0, T], \) where \( \psi: \mathfrak{h} \to \mathcal{M}_t \) is a Grassmann field compatible with \( X_{t, T} \), see Definition 7. Then, \( U_t \) solves the following Kolmogorov’s equation

\[
\partial_t U_t + \frac{1}{2} \mathcal{D}^2_{\Theta G_t} U_t = 0.
\] (16)

Furthermore, as long as \( U_t(0) \neq 0 \), set \( F_t(\psi) := D \log U_t(\psi) \) according to Remark 25. Then, \( F_t \) solves a Hamilton-Jacobi-Bellman-type equation

\[
\partial_t F_t + \frac{1}{2} \mathcal{D}^2_{\Theta G_t} F_t + \langle \Theta G_t F_t, DF_t \rangle = 0.
\] (17)

Proof. The Kolmogorov’s equation is an immediate consequence of the identity

\[
\omega_t(U(\psi + X_t, T)) = \left( e^{\frac{1}{2} \mathcal{D}^2_{\Theta G_t} T} U \right)(\psi),
\] (18)

which can be made sense of as the power series expansion of the exponential. To prove it, it suffices to note that if \( F \in \wedge \mathfrak{h} \) and \( \psi \) is as above, we have the Taylor formula

\[
F(\psi + X_t, T) = (e^{(X_t, T)D}) F(\psi),
\] (19)

where we used the compatibility of \( \psi \) and \( X_{t, T} \), see also Remark 27 for the notation. By expanding (19), by taking the conditional expectation by using the identity

\[
\omega_t(X_{t, T}(v) P(X_{t, T})) = \omega_t((\Theta v, G_{t, T} DP)(X_{t, T})), \quad \forall v \in \mathfrak{h}, \forall P \in \wedge \mathfrak{h},
\]

which is a consequence of the Gaussianity of \( X_{t, T} \), gives (18).

As long as \( U_t(0) \neq 0 \), \( V_t := \log U_t \) is well-defined and \( U_t = e^{V_t} \). Plugging the latter expression into (16) and using that \( U_t(0) \neq 0 \) proves that \( V_t \) satisfies a Hamilton-Jacobi-Bellman equation:

\[
\partial_t V_t + \frac{1}{2} \mathcal{D}^2_{\Theta G_t} V_t + \frac{1}{2} \langle \Theta D V_t, \dot{G}_t D V_t \rangle = 0.
\]

Taking the functional derivative proves that \( F_t \) satisfies (17). \( \Box \)

Remark 29. Note that since Wick products are martingales, they satisfy the Kolmogorov’s equation, that is, if one defines \( H^f \in \wedge \mathfrak{h} \) by \( H^f \langle X_i \rangle := [X_i(f_1) \cdots X_i(f_n)] \), then \( H^f \) solves Eq. (16).

3.2 Stochastic quantisation: the forward-backward approach

So far we have shown some important functional equations associated with the conditional expectation. The Hamilton-Jacobi-Bellman equation, or its “discrete version”, is the main object of study in the RG techniques. On the other hand, we want to formulate the problem of characterising the interacting measure in (5) purely in terms of the solution of a suitable SDE. Let us then discuss in more detail the type of SDEs we are going to consider.
Definition 30. A drift $F$ on $\wedge \mathfrak{h}$ is a linear mapping $F: \mathfrak{h} \to \wedge_{\text{odd}} \mathfrak{h}$. We can then identify $F(\psi)$ as a mapping from $\mathfrak{h}$ into odd functions of the Grassmann field $\psi$.

We call a family of drifts $(F_t)_{t \in I}$ with $I \subseteq \mathbb{R}$ admissible if it is $L^\infty_{\text{loc}}(I)$, that is, if $F_t = \sum_\alpha (F_t)_\alpha v_\alpha$, then the coefficients $(F_t)_\alpha$ are in $L^\infty_{\text{loc}}(I)$.

Remark 31. Requiring that $F$ is valued in $\wedge_{\text{odd}} \mathfrak{h}$ is natural in the Grassmann context, see below.

Let $(F_t)_t$ be an admissible family of drifts. We consider the additive SDE
\[
\begin{aligned}
d\Psi_t &= F_t(\Psi_t)dt + dX_t, \quad t \in [t_0, T], \\
\Psi_{t_0} &= \varphi,
\end{aligned}
\tag{20}
\]
where $X_t$ is a norm-compatible GBM with differentiable covariance $G_t$ and where the Grassmann field $\varphi: \mathfrak{h} \to M_{t_0}$ is compatible with $X_t$ for any $t \in [t_0, T]$, recall Definition 7. More precisely, (20) is the shorthand for
\[
\Psi_t(f) = \varphi(f) + \int_{t_0}^t (F_s(\Psi_s))(f)ds + X_{t_0,t}(f), \quad t \in [t_0, T], \; \forall f \in \mathfrak{h}.
\tag{21}
\]
A similar equation with time-independent drift and Brownian motion in place of the GBM was already studied in [ABDG20], where the local existence and uniqueness was proven. The same result holds for (20), provided that the time-dependent drift satisfies some mild assumptions.

Proposition 32. (Local existence and uniqueness). Let $(X_t)_t$ be a norm-compatible GBM indexed in a finite Hilbert space $\mathfrak{h}$ with covariance $G_t$, let $(F_t)_t$ be an admissible family of drifts and let $\varphi: \mathfrak{h} \to M_{t_0}$ be a Grassmann field compatible with $X_t$ for any $t \in [t_0, T]$. Then, there exists $T > t_0$, depending on $F_t$ and $G_t$ such that there exists a unique adapted Grassmann process $\Psi_t$ solving (21) for $t \in [t_0, T]$ and compatible with $\varphi$ and with $X_t$ for any $t \geq t_0$.

Proof. Local existence and uniqueness is proven by a Banach fixed point argument in the space $C([t_0, T], \mathcal{M})$ for suitable $T$, see [ABDG20] for details. In particular, the solution can be constructed by Picard’s iteration. Because the $n$-th Picard’s iteration is an adapted Grassmann process compatible with $\varphi$ and with $X_t$ for any $t \geq t_0$, the thesis follows. \qed

Remark 33. To obtain global existence, one needs more information on $F_t$ and $G_t$. This will be done in the next section. In the case of $F_t$ constant in $t$, a simple argument for global existence can be found in [ABDG20].

The fundamental tool that will allow us to put together the functional equations in Proposition 28 with the SDEs is the Itô’s formula.

Lemma 34. (Itô’s lemma). Let $(F_t)_t \subset \wedge \mathfrak{h}$ be a continuously differentiable family and let be $\Psi_t$ solve (21) (same setting) over some interval $[t_0, T]$. Then, for $t \in [t_0, T]$ we have:
\[
\begin{aligned}
\omega_s(F_t(\Psi_t) - F_s(\Psi_s)) &= \int_{t_0}^t \omega_s \left[ \left( \partial_t F_r(\Psi_r) + \frac{1}{2}(D^2_{\mathfrak{h} \mathfrak{h} \mathfrak{h}} F_r)(\Psi_r) + \langle F_r(\Psi_r), DF_r(\Psi_r) \rangle \right) dr \right] s \leq t.
\end{aligned}
\]

Proof. The statement is equivalent to proving that for $s, t \in [t_0, T], |t-s| \ll 1$ we have
\[
\left\| \omega_s \left[ F_t(\Psi_t) - F_s(\Psi_s) - (t-s) \left( \partial_s F_s(\Psi_s) + \frac{1}{2}(D^2_{\mathfrak{h} \mathfrak{h} \mathfrak{h}} F_s)(\Psi_s) \right) - \langle \Psi_{s,t}, DF_s(\Psi_s) \rangle \right] \right\| = o(|t-s|),
\]
where recall that $\Psi_{s,t} := \Psi_t - \Psi_s$. We write

\begin{equation}
F_t(\Psi_t) - F_s(\Psi_s) = F_t(\Psi_t) - F_t(\Psi_s) + F_t(\Psi_s) - F_s(\Psi_s) \\
= F_t(\Psi_t) - F_t(\Psi_s) + (t - s)(\partial_s F_s)(\Psi_s) + o(|t - s|),
\end{equation}

where we used the differentiability of $F_t$. Then, by using that $\Psi_s$ is a Grassmann process, that is, that it satisfies anti-commuting exchange relations at all times, we write

\begin{equation}
F_t(\Psi_t) - F_t(\Psi_s) = ((e(\Psi_{s,t}, D) - 1) F_t)(\Psi_s),
\end{equation}

where the exponential is intended as the (truncated) power series, see also Remark 27 for notation. To bound the right-hand side, we first of all control $s; t$: \vspace{-1em}

\begin{equation}
\sup_j \|\Psi_{s,t}(v_j)\| \\
\leq (t - s) \sup_j \sup_{r \in [s,t]} \|F_r(\Psi_r)(v_j)\| + \|X_{s,t}(v_j)\| \\
\lesssim (t - s)
\left(1 + \sup_j \sup_{r \in [s,t]} \|\Psi_r(v_j)\|\right)^{\text{deg}(F)} \sum_{\alpha} \|F_{\alpha}\|_{L^\infty(I)} + \sup_j \left(\int_s^t \|C_r\|_d^2 dr\right)^{\frac{1}{2}} \\
\lesssim |t - s|^{1/2},
\end{equation}

where we used the local existence of $\Psi_r$ and that $X_t$ is norm-compatible ($\hat{G}_r = C_r^2 U_r$). By the Lagrange’s remainder formula, this implies \vspace{-0.5em}

\begin{equation}
\sup_j \left\|\left(\left[e^{(\Psi_{s,t}, D)} - 1 - \langle\Psi_{s,t}, D\rangle - \frac{1}{2}(\Psi_{s,t}, D)^2\right] F_t\right)(\Psi_s)(v_j)\right\| \\
\lesssim \left(\sup_j \|\Psi_{s,t}(v_j)\|\right)^3 := O(|t - s|^{3/2})
\end{equation}

Concerning the first term in the series in (23). We write:

\begin{equation}
\omega_s((\langle\Psi_{s,t}, D\rangle F_t)(\Psi_s)) = \omega_s((\langle\Psi_{s,t}, D\rangle F_s)(\Psi_s)) + O(|t - s|^{3/2}).
\end{equation}

Regarding the second term, we have instead:

\begin{equation}
\omega_s((\langle\Psi_{s,t}, D\rangle^2 F_t)(\Psi_s)) = (\omega_s((X_{s,t}, D)^2 F_t)(\Psi_s) + o(|t - s|) \\
= \sum_{i,j} (\omega_s(X_{s,t}(v_i)X_{s,t}(v_j))\partial_i \partial_j F_t)(\Psi_s) + o(|t - s|) \\
= (t - s)(D^2_{\hat{G}_s} F_t)(\Psi_s) + o(|t - s|) \\
= (t - s)(D^2_{\hat{G}_s} F_s)(\Psi_s) + o(|t - s|),
\end{equation}

where we used the estimates in (24), the fact that $\Psi_s$ is adapted and that $X_t$ has differentiable covariance. Putting together (22), (25), (26) and (27) gives the claim. \hfill $\Box$

We can finally prove a stochastic quantisation formula. Our first result connects the interacting measure with the solution of an SDE where the drift $(F_t)_{t}$ is given by the solution of a Hamilton-Jacobi-Bellman-type equation.
Proposition 35. Let $0 \leq t \leq T$ and let $V_T \in \bigwedge \mathfrak{h}$, $\mathfrak{h}$ a finite-dimensional Hilbert space with conjugation $\Theta$. Let $(X_t)_t$ be a norm-compatible GBM with differentiable covariance $G_t$ and assume that

$$\omega(e^{V_T(X_t,t)}) \neq 0 \quad \forall s \in [t, T],$$

and set $V_t(\varphi) := \log \omega_t(e^{V_T(\varphi,X_t,t)})$, compare with Theorem 28. Assume that $(\Psi_s)_s$ is the solution of the following SDE on $[t, T]$

$$d\Psi_s = \Theta \dot{G}_s dV_s(\Psi_s)ds + dX_s \quad \Psi_t = 0.$$  \hfill (28)

Then, the following equation holds true for any $P \in \bigwedge \mathfrak{h}$:

$$\omega_t(P(\Psi_s)) = \frac{\omega_t(P(X_{t,s})e^{V_T(X_{t,s})})}{\omega_t(e^{V_T(X_{t,s})})}, \quad t \leq s \leq T.$$ 

**Proof.** Define the following functions for $\alpha \leq S$ and for $s \in [t, T]$:

$$P_{\alpha}(s) := \omega_t(v_{\alpha}(\Psi_s)), \quad \tilde{P}_{\alpha}(s) := \frac{\omega_t((v_{\alpha}U)(X_{t,T}))}{U_t(0)}$$

where $(v_{\alpha})_{\alpha \leq S}$ is our choice of linear basis of the Grassmann algebra $\bigwedge \mathfrak{h}$. By Ito’s lemma and by the fact that $\Psi$ solves the SDE (28), we find that the family $(P_{\alpha}(s))_{\alpha}$ satisfies the following system of ODEs:

$$dP_{\alpha}(s) = \omega_t\left( \frac{1}{2} D_{\Theta G_s}^2 v_{\alpha}(\Psi_s) ds + (d\Psi_s, Dv_{\alpha}(\Psi_s)) \right)$$

$$= \omega_t\left( \frac{1}{2} D_{\Theta G_s}^2 v_{\alpha}(\Psi_s) + (\Theta \dot{G}_s dV_s, Dv_{\alpha}(\Psi_s)) \right) ds$$

$$= \left( \frac{1}{2} D_{\Theta G_s}^2 P_{\alpha}(s) + \sum_{\beta} \sum_{i,j} (\Theta v_i, \dot{G}_s v_j)(V_s)_{\beta} P_{\beta,\alpha \cup i \cup j}(s) \right) ds,$$

where in the second line we used $\omega_t((dX_s, Dv_{\alpha}(\Psi_s))) = \omega_t((\omega_s(dX_s), Dv_{\alpha}(\Psi_s))) = 0$, since $\omega_s(dX_s) = 0$ and since $\Psi_s$ is adapted. Similarly, by Ito’s lemma and by the fact that $U_s(\varphi) := \omega_s(e^{V_T(\varphi,X_s,s)})$ solves the Kolmogorov’s equation, we find that the $(P_{\alpha}(s))_{\alpha}$ satisfy the same ODEs as the $(P_{\alpha}(s))_{\alpha}$:

$$d\tilde{P}_{\alpha}(s) = [U_t(0)]^{-1} \omega_t(dv_{\alpha}(U_s)(X_{t,s}))$$

$$= [U_t(0)]^{-1} \omega_t\left( v_{\alpha} \partial_s U_s(X_{t,s}) + \frac{1}{2} (D_{\Theta G_s}^2 (v_{\alpha} U_s))(X_{t,s}) \right) ds$$

$$= [U_t(0)]^{-1} \omega_t\left( \frac{1}{2} (D_{\Theta G_s}^2 v_{\alpha} U_s)(X_{t,s}) + (\Theta D v_{\alpha}, \dot{G}_s dV_s) U_s(X_{t,s}) \right) ds$$

$$= \left( \frac{1}{2} D_{\Theta G_s}^2 \tilde{P}_{\alpha}(s) + \sum_{\beta} \sum_{i,j} (\Theta v_i, \dot{G}_s v_j)(V_s)_{\beta} \tilde{P}_{\beta,\alpha \cup i \cup j}(s) \right) ds.$$

The existence of $\Psi_s$ over $[t, T]$ implies the existence of $P_{\alpha}(s)$ over the same interval and thus, since $P_{\alpha}(s) = \tilde{P}_{\alpha}(s)$ for $s = t$, we actually have the equality for any $s \in [t, T]$. \hfill \Box

Our ultimate goal is the characterisation of the interacting measure by means of the solution of an SDE that does not require to solve the HJB equation. As it turns out, it suffices to “undo” the HJB flow of the drift $(DV_t)_t$ and thus study the problem by means of a forward–backwards SDE as follows.
Theorem 36. (FBSDE). Let $0 \leq t < T$ and let $V_T \in \bigwedge \mathfrak{h}$, $\mathfrak{h}$ a finite-dimensional Hilbert space with conjugation $\Theta$. Let $(X_t)_t$ be a norm-compatible GBM with differentiable covariance $G_t$. Assume that $\Psi_t$ solves the following equation for $s \in [t, T]$:

$$d\Psi_s = \Theta \hat{G}_s \omega_s (DV_T(\Psi_T)) ds + dX_s, \quad \Psi_t = 0. \tag{29}$$

Then, the following equation holds true for any $P \in \bigwedge \mathfrak{h}$:

$$\omega_t(P(\Psi_s)) = \frac{\omega_t(P(X_{t,s})e^{V_T(X_{t,s})})}{\omega_t(e^{V_T(X_{t,s})})}, \quad t \leq s < T. \tag{30}$$

Remark 37. As should be clear, the crucial difference between the statements in Proposition 35 and Theorem 36 is that the latter relies only on the FBSDE and its solution, whereas the former is formulated in terms of $DV_t$, $V_t$ solving the Hamilton-Jacobi-Bellman equation. This shift in perspective is the key new idea in this paper, it will allow to be able to approximate the flow equation and rely on the solution theory for the FBSDE to carry over the estimate of the error.

Proof. First of all, note that $U_T(0) = e^{V_T(0)} \neq 0$, because $V_T(0) \in \mathbb{C}$. Since $U_s(0)$ is continuous in $s$, we therefore find a $\tilde{t}$ such that $U_s(0) \neq 0$ for any $s \in (\tilde{t}, T]$ and that if $\tilde{t} > -\infty$ then $U_T(0) = 0$. Accordingly, $F_s := DV_s$ is well-defined and solves the Hamilton-Jacobi-Bellman-type equation in (17) with $F_T = DV_T$ by Proposition 28. Then, by Ito’s lemma for $s \in (\tilde{t} \vee t, T]$ we see that $(F_s(\Psi_s))_s$ is a martingale, and in particular

$$\omega_t(DV_T(\Psi_T)) = F_s(\Psi_s).$$

Therefore, for any $\tau \in (\tilde{t} \vee t, T]$ $\Psi_{\tau,s}$ solves also the SDE in (28) with initial condition $\Psi_{\tau,s} = 0$ and therefore, by Proposition 35 we have that

$$\omega_t(P(\Psi_{\tau,s})) = \frac{\omega_t(P(X_{\tau,s})e^{V_T(X_{\tau,s})})}{\omega_t(e^{V_T(X_{\tau,s})})}, \quad \tilde{t} \vee t \leq \tau \leq s \leq T. \tag{31}$$

If $\tilde{t} \leq t$, this concludes the proof. Suppose otherwise that $\tilde{t} > t$. Let us plug $P = e^{-V_T}$ into Eq. (31), obtaining

$$\omega_t(e^{-V_T(\Psi_{\tau,s})}) = 1, \quad \tilde{t} < \tau \leq s \leq T. \tag{32}$$

Because $\tilde{t} > t$, we have that $\sup_{s \in [\tilde{t}, T]} \|\Psi_s\|$ is bounded and thus by Eq. (32) and by Proposition 16 we have

$$\inf_{\tau \in (\tilde{t}, T]} |U_T(0)| = \inf_{\tau \in (\tilde{t}, T]} |\omega_t(e^{V_T(X_{\tau,s})})| \geq e^{-\sup_{s \in [\tilde{t}, T]} \|\Psi_s\|} \omega_t(e^{V_T(X_{\tau,s})}) > 0.$$

This contradicts the definition of $\tilde{t}$, therefore $\tilde{t} \leq t$, and the claim is proven. \qed

Lemma 38. Under the same assumptions of Theorem 36, $\Psi$ is a solution of the FBSDE in (29) iff it is a solution of

$$d\Psi_s = \Theta \hat{G}_s (F_s(\Psi_s) + R_s) ds + dX_s, \quad \Psi_0 = 0,$$

where $(F_t)_{t \in [0,T]} \subset \bigwedge \mathfrak{h}$ is any continuously differentiable interpolating family such that $F_T = DV_T$ and where the remainder process $R_s$ solves the following self-consistent equation:

$$R_s = \int_s^T \omega_s(\mathcal{H}_r[F_r](\Psi_r)) dr + \int_s^T \omega_s((\Theta \hat{G}_r R_r, DF_r(\Psi_r))) dr, \tag{33}$$
Furthermore, if we identify $F_h$ clearly, we let $3.3FBSDE$ for the model since in fact $F_h$ is a martingale, see Proposition 21 and Remark 29. In this case, the control of $F_h$ is simple, but as a consequence the full non-linear term $\langle \Theta \hat{G}_t F_r, DF_r \rangle$ appears in the equation for controlling the remainder process and this is ultimately responsible for the worsening of the estimates. To go beyond the Wick-ordering renormalisation, one needs to partially include the non-linearity into the flow equation for $F_h$, see Section 4.

3.3 FBSDE for the model

We let $A := \{0, 2^{-N} | N \in \mathbb{N}\}$, $N_\infty := \mathbb{N} \cup \{\infty\}$ and let $\mathbb{T}_{L, \varepsilon}^d := ((\varepsilon \mathbb{Z})/(L \mathbb{Z}))^d$ be the toroidal lattice of size $L$ and lattice spacing $\varepsilon$ with the understanding that $\mathbb{T}_{L, 0}^d := \mathbb{T}_{L}^d$ and $\mathbb{T}_{\infty, 0}^d := \mathbb{R}^d$. Because we deal with spin-1/2 fermions, we define for $L \in N_\infty$ and $\varepsilon \in A$, the Hilbert space

$$\mathbb{h}_{L, \varepsilon} := L^2(\mathbb{T}_{L, \varepsilon}^d; \mathbb{C}^2) \oplus L^2(\mathbb{T}_{L, \varepsilon}^d; \mathbb{C}^2),$$

where for $\varepsilon > 0$ the scalar product on it is given by

$$\langle f, g \rangle_{\mathbb{h}_{L, \varepsilon}} = \sum_{\sigma = \uparrow, \downarrow} \sum_{x \in \mathbb{T}_{L, \varepsilon}^d} \varepsilon^d \overline{f_\sigma(x)} g_\sigma(x) =: \int_{\mathbb{T}_{L, \varepsilon}^d} \overline{f(x)} \cdot g(x) d\mathbf{x}.$$

Clearly, $\mathbb{h}_{L, \varepsilon}$ are finite dimensional Hilbert spaces as long as $L$ is finite and $\varepsilon > 0$. We abridge $\mathbb{h}_L := \mathbb{h}_{L, 0}$, $\mathbb{h} := \mathbb{h}_\infty$ and note that for any finite $L$ we can embed $\mathbb{h}_{L, \varepsilon} \hookrightarrow \mathbb{h}_L$ via truncation in the Fourier series. More in detail, we let $(\mathbb{T}_{L, \varepsilon}^d)^* := \{ k \in 2\pi L^{-1} \mathbb{Z}^d | |k|_{\infty} < \pi \varepsilon^{-1} \}$ and $\mathcal{F}_{L, \varepsilon} : \mathbb{h}_{L, \varepsilon} \rightarrow \ell^2((\mathbb{T}_{L, \varepsilon}^d)^*; \mathbb{C}^4)$ be the Fourier transform $(\mathcal{F}_{L, \varepsilon} f)(k) := \int_{\mathbb{T}_{L, \varepsilon}^d} e^{ikx} f(x) d\mathbf{x}$. To any element $f \in \mathbb{h}_{L, \varepsilon}$ we associate the following element of $\mathbb{h}_L$,

$$(\mathcal{F}_{L, 0}^{-1} \mathcal{F}_{L, \varepsilon} f)(x) := L^{-d} \sum_{k \in (\mathbb{T}_{L, \varepsilon}^d)^*} e^{ikx} (\mathcal{F}_{L, \varepsilon} f)(k).$$

Furthermore, if we identify $\mathbb{T}_{L}^d$ with the box $(-L/2, L/2]^d \subset \mathbb{R}^d$, we obtain the embedding

$$\mathbb{h}_{L, \varepsilon} \hookrightarrow \mathbb{h}_L \hookrightarrow \mathbb{h}_\varepsilon.$$
that is, an element of $\mathfrak{h}_L$ is identified with an element of $\mathfrak{h}$ with compact support on $\mathbb{T}^d_L$. This is important to keep in mind because, even though we need to index the GBMs in the finite dimensional Hilbert spaces $\mathfrak{h}_{L, \varepsilon}$ with $L \in \mathbb{N}$ and $\varepsilon > 0$, we want to represent such GBMs in a sufficiently large FNPS that is directly linked to the infinite dimensional space $\mathfrak{h}$.

On $\mathfrak{h}$ we define the conjugation $\Theta(v \oplus w) = (\bar{w} \oplus \bar{v})$ for any $v \oplus w \in \mathfrak{h}$. The operator $G$ in (1) is $\Theta$-antisymmetric and unbounded for $\gamma > 0$ due to its ultraviolet behaviour. We introduce the interpolation, see also [Ste70, Riv91]:

$$G_t := \frac{U}{\Gamma(d/2 - \gamma)} \left( t_{0 \leq t < 1} \int_{2^{-2}}^\infty + t_{t \geq 1} \int_{d/2}^\infty \right) \chi_{\gamma \in d/2 - \gamma} e^{-\gamma + d/2 - 1} e^{(1 - \Delta) d\zeta},$$

where $U := 1 + 1$. It is easy to see that $G_\infty = G$ and that $G_t$ is $\Theta$-antisymmetric and also bounded if $t < \infty$.

Although it is tempting to introduce a regularisation by replacing $\Delta$ with the lattice Laplacian, for technical reasons, we instead make the following choice. We let $\min_\varepsilon$ be a smooth non-decreasing function on $\mathbb{R}_+$ such that $\min_\varepsilon(a) = a$ on $[0, \varepsilon^{-1}]$ and $\min_\varepsilon(a) = \varepsilon^{-1}$ on $[\varepsilon^{-1} + 1, \infty)$. In other words, $\min_\varepsilon(a)$ is a smoothening of $\varepsilon^{-1} \wedge a$.

**Definition 40.** Let $L \in \mathbb{N}_\infty$, $\varepsilon \in A$, $\delta \in (0, 1)$ and let $\varphi_\delta \in C^\infty_c(\mathbb{R})$ be a $\delta^{-1}$-Gevrey function\(^4\) such that $\text{supp}(\varphi_\delta) = [0, \pi]$ and $\varphi_\delta(0, \pi - 1) = 1$. We define $G_{L, \varepsilon}^t$ as the operator on $\mathfrak{h}$ with kernel

$$G_{L, \varepsilon}^t(y; x) := \frac{U}{\Gamma(d/2 - \gamma)} \left( t_{0 \leq t < 1} \int_{2^{-2}}^\infty + t_{t \geq 1} \int_{d/2}^\infty \right) L^{-d} \sum_{k \in (\mathbb{Z}^d)^*} e^{i \mathbf{k} \cdot (y - x)} \chi_{\gamma \in d/2 - \gamma} e^{-\gamma + d/2 - 1} e^{(1 - \Delta) d\zeta},$$

for $y, x \in \mathbb{T}^d_L$ and $G_{L, \varepsilon}^t(y; x) = 0$ otherwise, where the sum should be intended as integral if $L = \infty$. We also let $g_{L, \varepsilon}^t$ be $\Theta$ times the periodisation of $G_{L, \varepsilon}^t$, that is

$$g_{L, \varepsilon}^t(y; x) := \Theta \sum_{m, m' \in \mathbb{Z}^d} G_{L, \varepsilon}^t(y + m L; x + m' L).$$

**Remark 41.** Notice the dependence of $G_{L, \varepsilon}^t$ and $g_{L, \varepsilon}^t$ on $\delta$ when $\varepsilon > 0$. We avoid writing this dependence explicitly to keep our notation as light as possible.

Of course, by definition $g_{L, \varepsilon}^t(y + m L; x + m' L) = g_{L, \varepsilon}^t(y; x)$ for any $m, m' \in \mathbb{Z}^d$. We abridge $g_{L, \varepsilon}^t := g_{\infty, \varepsilon}^t$ and note that, because of translation invariance, the Poisson summation formula implies $g_{L, \varepsilon}^t(y; x) = \sum_{m \in \mathbb{Z}^d} g_{L, \varepsilon}^t(y + L; x)$. We also have

$$G_{L, \varepsilon}^t = 1_{0 \leq t < 1} G_{L, \varepsilon}^1 + \chi_{1 - \log 2 \varepsilon(t)} U \frac{2 (2\gamma - d/2)}{\Gamma(d/2 - \gamma)} L^{-d} \sum_{k \in 2^d \mathbb{Z}^d} e^{-2 \gamma (k^2 + 1)} \varphi_\varepsilon(k),$$

where $\chi_{a \leq t \leq b}$ is a positive smooth approximation to $1_{a \leq t \leq b}$ such that $\chi_{a, b}(t) = 1$ for $a \leq t \leq b$ and $\chi_{a, b}(t) = 0$ for $t < a$ or $t > b + \delta$ for some small fixed $\delta > 0$. We can write $G_{L, \varepsilon}^t = (\mathcal{E}_{L, \varepsilon}^t)^2 U$ and read off the explicit expression for $\mathcal{E}_{L, \varepsilon}^t$ from (37). Note the commutation relations $[\mathcal{E}_{L, \varepsilon}^t, U] = [\mathcal{E}_{L, \varepsilon}^t, \Theta] = 0$. We can finally introduce the following $L, \varepsilon$-dependent family of GBMs.

---

\(^4\) We thus have $\sup_{\varepsilon \in \mathbb{R}_+} |\mathcal{E}_{L, \varepsilon}^t(x)| \leq C^{1 + |\varepsilon| (m)}$, see [GMR21, Rod83].
**Definition 42.** We let \((X_t^{L,e})_{L \in \mathbb{N}, e \in A}\) be the anti-commuting family of norm-compatible GBMs, see Definitions 19 and 22, such that
\[
\omega(X_t^{L,e}(f)X_t^{L,e}(g)) = \left\langle \Theta f, \int_0^t \mathcal{E}_x^{L,e} \mathcal{E}_s^{L,e} U ds \right\rangle_h \quad \forall f, g \in \mathfrak{h}.
\]

**Remark 43.** In Appendix 24, we prove that there exists a sufficiently large FNPS \((\mathcal{M}, \omega, (\mathcal{M}_t)_t)\) where such a family can be constructed. Note that we are requiring that \(X_t^{L,e}\) at different \(L\) and \(e\) are correlated and this is actually natural if they originate from the same “Grassmann white noise”, see Remark 90.

Even though \(X_t^{L,e}(f)\) can be defined for \(f \in \mathfrak{h}\), in view of the application to the stochastic quantisation theorem, see Theorem 36, we think of them as indexed as \(\mathfrak{h}_{L,e} \subset \mathfrak{h}\), when \(L \in \mathbb{N}\) and \(e > 0\). In other words, for fixed \(L \in \mathbb{N}\) and \(e > 0\), we consider only the finite dimensional Grassmann algebra generated by \(X_t^{L,e}(f)\) as \(f \in \mathfrak{h}_{L,e}\). In this regard, it is particularly convenient to switch to the Kronecker’s delta basis and therefore obtain what we call a field on \(\mathbb{R}^d\).

**Definition 44.** A field on \(\mathbb{T}_L^d, e\) is a map \(\psi: \mathbb{T}_L^d, e \to \mathcal{M}^4\).

Define \(\delta_{x,1} := (\delta_{x}, 0) \in L^2(\mathbb{T}_L^d, \mathbb{C}^2)\) and \(\delta_{x,1}^e := (0, \delta_{x}^e) \in L^2(\mathbb{T}_L^d, \mathbb{C}^2)\), where \(\delta_{x}^e\) is the extension of the Kronecker delta to \(\mathbb{T}_L^d\) via Fourier transform \(\delta_{x}^e := \mathcal{F}^{-1}_{L,0} \mathcal{F}_{L,e} \delta_{x}\). The set \((\delta_{x,\sigma})_{x \in \mathbb{T}_L^d, \sigma = \{1, 4\}}\) is an orthogonal basis for \(L^2(\mathbb{T}_L^d, \mathbb{C}^2) \cong L^2(\mathbb{T}_L^d; \mathbb{C}^2)\) (it is not orthonormal though).

**Definition 45.** If \(X_t^{L,e}\) is the GBM of Definition 42, with abuse of notation we let \(X_t^{L,e}: \mathbb{R}^d \to \mathcal{M}^4\) denote the corresponding field on \(\mathbb{R}^d\) \(x \mapsto X_{t,x}^{L,e} := \left( X_{t,x}^{L,e} \right)_{\mu \in \{1, 2\} \times \{1\}}\) where
\[
X_{t,x,(\sigma, +)}^{L,e} := X_{t,x}^{L,e}(\delta_{x,\sigma} + 0), \quad X_{t,x,(\sigma, -)}^{L,e} := X_{t,x}^{L,e}(0 + \delta_{x,\sigma}).
\]
for \(x \in \mathbb{T}_L^d\) and \(X_{t,x+mL}^{L,e} := X_{t,x}^{L,e}\) for any \(m \in \mathbb{Z}^d\).

We introduce the following point-wise symmetric products for Grassmann fields. If \(\psi, \psi'\) and \(\psi''\) are fields we set:
\[
(\psi^{(1)} \cdot \psi^{(2)})(x) := \frac{1}{2} \sum_{\sigma = 1, 4} \sum_{\pi \in S_3} \psi^{(1)}_{x,\pi(\sigma)} \psi^{(2)}_{x,\pi(\sigma)}
\]
\[
(\psi^{(1)} \cdot \psi^{(2)} \cdot \psi^{(3)})(x, \mu) := \frac{1}{6} \sum_{\pi \in S_3} \psi^{(1)}_{x,\pi(\mu)} \psi^{(2)}_{x,\pi(\mu)} \psi^{(3)}_{x,\pi(\mu)}
\]

We also abridge the notation by setting \((\psi_x)^2 := (\psi \cdot \psi)_x\) and \((\psi_x)^3 := (\psi \cdot \psi \cdot \psi)_x,\mu\).

We are finally in position to apply the FBSDE to the construction of the interacting measure. In the FNPS \((\mathcal{M}, \omega, (\mathcal{M}_t)_t)\) we are interested in computing the expectation of suitable observables\(^*\) \(P\) as the weak-limit
\[
\omega^V(P(X)) = \lim_{L \to \infty} \lim_{\varepsilon \to 0} \omega\left( P(X_L^e)e^{V_L^e(X_L^e)} \right) = \lim_{L \to \infty} \omega(P(\Psi^L_e)) = \omega(P(\Psi)),
\]

\(^5\) See also Lemma 79 for the actual applications.
with \( X^{L,\varepsilon} := X^{L,\varepsilon}_\infty \) and \( \Psi^{L,\varepsilon} := \Psi^{L,\varepsilon}_\infty \), where \( X^{L,\varepsilon}_t \) are the GBMs introduced above, where the potential \( V^{L,\varepsilon}_t \) reads

\[
V^{L,\varepsilon}_t(X^{L,\varepsilon}) := \int_{\mathbb{R}^d} \left[ \frac{\lambda}{4} \left( (X^{L,\varepsilon})^2 \right)^2 + \mu T \right] \, dx,
\]

for suitable constant \( \mu T = \mu^2 T(\lambda) \in \mathbb{R} \), and where the lattice field \( \Psi^{L,\varepsilon}_t \) solves for \( T = \infty \)

\[
d\Psi^{L,\varepsilon}_t = \Theta \partial_t \psi_t + \omega_t(DV^{L,\varepsilon}_t(\Psi^{L,\varepsilon}_t)) \, dt + dX^{L,\varepsilon}_t, \quad \Psi^{L,\varepsilon}_0 = 0. \tag{39}
\]

**Remark 46.** Because of the translation invariance of \( \mathcal{L}^{L,\varepsilon} \), we can equivalently solve \( \Psi^{L,\varepsilon}_{t,x,\mu} = \sum_{\mu} \int_0^t \int_{\mathbb{R}^d} \left( \tilde{\mathcal{L}}^{L,\varepsilon}_{\mu}(x; y) \right)_{\mu,\mu} \omega_t(DV^{L,\varepsilon}_t(\Psi^{L,\varepsilon}_t))(y; \mu') \, dy \, ds + X^{L,\varepsilon}_{t,x,\mu} \tag{40}
\]

where \( V^{L,\varepsilon} \) integrated over the whole \( \mathbb{R}^d \). This defines an extension of the solution to the whole \( \mathbb{R}^d \), satisfying \( \Psi^{L,\varepsilon}_{t,x+mL} = \Psi^{L,\varepsilon}_{t,x} \) for any \( m \in \mathbb{Z}^d \).

**Remark 47.** Note that as long as \( T \) is finite, the FBSDE in (39) makes sense for \( \varepsilon \in \mathcal{A} \), since the renormalised potential \( V^{L,\varepsilon}_t \) does not have divergences. On the other hand, if \( T = \infty \), because of the divergence of \( \mu_{L,0}^{L,0} \), (39) should always be understood in the limit \( \varepsilon \to 0 \).

By using Lemma 38, we solve the FBSDE by means of an interpolation scheme, for suitable \( F^{L,\varepsilon}_t \). As it turns out, we can actually take it already in the infinite volume case and likewise replace \( \mathcal{L}^{L,\varepsilon} \) by \( \mathcal{L}^{L,0} \) in (40); in fact, because of translation invariance and because the field \( X^{L,\varepsilon}_t \) is periodic, so is the solution \( \Psi^{L,\varepsilon}_t \) and \( F^{L,\varepsilon}_t \). We make this precise in the following lemma, which will be proven in Section 4.

**Lemma 48.** Let \( (\Psi^{L,\varepsilon}_t, R^{L,\varepsilon}_t) \) solve the system

\[
\Psi^{L,\varepsilon}_t = \int_0^t \tilde{\mathcal{L}}^{L,\varepsilon}(F^{L,\varepsilon}_s(\Psi^{L,\varepsilon}_s) + R^{L,\varepsilon}_s) \, ds + X^{L,\varepsilon}_t,
\]

\[
R^{L,\varepsilon}_t = \int_t^\infty \omega_t(\mathcal{H}_s[F^{L,\varepsilon}_s]) \, ds + \int_t^\infty \omega_t(\langle \tilde{\mathcal{L}}^{L,\varepsilon}_{\mu,\mu} R^{L,\varepsilon}_s, D F^{L,\varepsilon}_s(\Psi^{L,\varepsilon}_s) \rangle) \, ds
\]

with \( \mathcal{H}_s[F^{L,\varepsilon}_s] := \partial_s F^{L,\varepsilon}_s + \frac{1}{2} \tilde{\mathcal{L}}^{L,\varepsilon}_{\mu,\mu} F^{L,\varepsilon}_s + \langle \tilde{\mathcal{L}}^{L,\varepsilon}_{\mu,\mu} F^{L,\varepsilon}_s, D F^{L,\varepsilon}_s \rangle \). Then, \( \Psi^{L,\varepsilon}_t \) is periodic and solves the FBSDE (39).

### 3.4 Sobolev spaces and covariance estimates

Before delving into the details of the solution, we conclude the section by introducing the norms on fields on \( \mathbb{R}^d \) (and on \( \mathbb{R}^+ \times \mathbb{R}^d \)). The crucial property of (regularised) Grassmann field is that they are bounded objects, unlike standard (commutative) Gaussian random variables. This motivates working with the space \( L^\infty(\mathbb{R}^d, \mathcal{M}^4) \), which is introduced in the usual way as the \( L^\infty \) space of functions taking values in a Banach space. This space is equipped with the norm

\[
\| \psi \|_{L^\infty(\mathbb{R}^d, \mathcal{M}^4)} := \sup_{\mu \in \{1, 1\} \times \{0\} \times \{0\}} \text{ess sup}_{x \in \mathbb{R}^d} \| \psi(x; \mu) \|.
\]

Moreover, in order to control the derivatives of the Grassmann fields, we also consider weighted Sobolev spaces

\[
C^n_\mu(\mathbb{R}^d, \mathcal{M}^4) := \{ \psi : \mathbb{R}^d \to \mathcal{M}^4 \| \psi \|_{C^n_\mu(\mathbb{R}^d, \mathcal{M}^4)} < \infty \}
\]
where
\[ \| \varphi \|_{c_t^0(\mathbb{R}^d; \mathcal{M}^{4\times 4})} = \sum_{0 \leq k \leq n} \| \varphi \|_{c_t^k(\mathbb{R}^d; \mathcal{M}^{4\times 4})} \]
\[ \| \varphi \|_{c_t^k(\mathbb{R}^d; \mathcal{M}^{4\times 4})} = \sum_{\nu : |
u| = n} 2^{-nt} \| \partial^{\nu} \varphi \|_{L^\infty(\mathbb{R}^d; \mathcal{M}^{4\times 4})}. \]  
(41)

For simplicity, we abridge the notation to \( \| \cdot \|_{L^\infty}, \| \cdot \|_{c_t^0}, \) and \( \| \cdot \|_{c_t^k}. \) For controlling the infinite volume limit, we shall also consider the following weighted seminorm
\[ \| \varphi \|_{c_t^k(\nu)} := \sum_{\nu : |
u| = n} 2^{-nt} \sup_{\mu \in \{1, 2\} \times \{\pm\}} \sup_{x \in \mathbb{R}^d} \varrho_\nu(x) \| \partial^{\mu} \varphi(x; \mu) \| \]  
where \( \varrho_\nu(x) := (1 + |x|^2)^{-\eta} \) satisfies the compatibility condition \( \varrho_\nu(x) \lesssim \varrho_\nu(y - x), \) as follows straightforwardly by the triangular inequality. Correspondingly, we denote with \( \| \cdot \|_{c_t^k(\nu)} \) the norm in which the summation is over \( \nu : |
u| \leq n \) instead.

Because we study multi-linear operators on the fields, we also set, for any \( m \in \mathbb{N}^k, \ k \in \mathbb{N} \)
\[ \| \varphi \|_{c_t^m((\mathbb{R}^d)^k; \mathcal{M}^{4\times k})} := \sum_{\nu_1, \ldots, \nu_k} 2^{-|m|} \| (\partial^{\nu_1} \otimes \cdots \otimes \partial^{\nu_k}) \varphi \|_{L^\infty((\mathbb{R}^d)^k; \mathcal{M}^{4\times k})}. \]  
(43)
where \( |m| := \sum \nu_i, \) and let
\[ c_t^m((\mathbb{R}^d)^k; \mathcal{M}^{4\times k}) := \{ \varphi : (\mathbb{R}^d)^k \to \mathcal{M}^{4\times k} \| \varphi \|_{c_t^m((\mathbb{R}^d)^k; \mathcal{M}^{4\times k})} < \infty \}. \]

We conclude this section with some estimates on the covariance and on the GBM.

**Lemma 49.** Let \( \varepsilon \in \mathcal{A} \) and let \( \mathcal{g}^\varepsilon_t \) be as in Definition 40. Then,
\[ \sup_{\mu \neq \mu'} \sup_{\nu : |
u| = n} \left| \partial^{\mu'} (\mathcal{g}^\varepsilon_t (x; y))_{\mu, \mu'} \right| \lesssim \varepsilon e^{(2\gamma + n) \theta} e^{-\varepsilon (2^s |x - y|)} \]  
for some universal \( \bar{c}. \) If \( \varepsilon > \varepsilon' \in \mathcal{A}_0 \) and \( \theta > 0, \) then
\[ \sup_{\mu \neq \mu'} \sup_{\nu : |
u| = n} \left| \partial^{\mu'} (\mathcal{g}^\varepsilon_t (x; y) - \mathcal{g}^{\varepsilon'}_t (x; y))_{\mu, \mu'} \right| \lesssim \varepsilon \theta e^{(2\gamma + n + \theta) \theta} e^{-\varepsilon (2^s |x - y|)} \]  
(45)

**Remark 50.** Notice that if \( \varepsilon = 0 \) we can actually choose \( \delta = 1 \) in (44), since \( \varphi_\delta \) does not play any role.

Because of the translational invariance of \( \mathcal{g}^\varepsilon_t, \) we can consider it to be a \( \mathcal{C}^{4 \times 4} \)-valued function on \( \mathbb{R}^d. \) We let \( L^p_{s, \delta, \varepsilon}(\mathbb{R}^d; \mathcal{C}^{4 \times 4}) \) be the \( L^p \) space restricted on \( \mathbb{R}^d \) with respect to the measure \( w_{s, \delta, \varepsilon}(x) \) with
\[ w_{s, \delta, \varepsilon}(x) := e^{c(2^s |x|)} \]  
(46)
for \( c \geq 0 \) and \( \delta \in (0, 1). \) The following corollary is a simple consequence of Lemma 49.

**Corollary 51.** Under the same assumptions of Lemma 49, we have
\[ \sup_{\nu : |
u| = n} \left\| \partial^{\mu} \mathcal{g}^\varepsilon_t \right\|_{L^p_{s, \delta, \varepsilon}(\mathbb{R}^d; \mathcal{C}^{4 \times 4})} \lesssim \varepsilon \left( 2^{(2\gamma + n - d/p)} \right)^s, \]  
\[ \sup_{\nu : |
u| = n} \left\| \partial^{\mu} (\mathcal{g}^\varepsilon_t - \mathcal{g}^{\varepsilon'}_t) \right\|_{L^p_{s, \delta, \varepsilon}(\mathbb{R}^d; \mathcal{C}^{4 \times 4})} \lesssim \varepsilon \left( 2^{(2\gamma + n + \theta - d/p)} \right)^s \]  
with \( \delta \) as in Definition 40, provided that \( c < \bar{c} \)
Proof of Lemma 49. We focus on the case \( s \in [1, -\log_2 \varepsilon] \), since for \( s \in [0, 1) \) the proof is similar. We notice that \((\partial^\nu_y \hat{G}^\varepsilon_s)_{\mu, \nu}\) is proportional to the integral (up to universal factors)

\[
(\partial^\nu_y \hat{G}^\varepsilon_s)_{\mu, \nu}(y; x) \sim 2^{(2\gamma + |\nu|)s} \int_{\mathbb{R}^d} e^{ik \tau (y - x)} e^{-(k^2 + 2^{-2s})(-ik)^\nu \varphi_\delta(2^s \varepsilon |k|)} dk. \tag{48}
\]

We want to prove a bound of the form

\[
|2^{ns} |y - x|^n (\partial^\nu_y \hat{G}^\varepsilon_s)_{\mu, \nu}(y; x)| \lesssim |\nu| C^n(n!)^{\delta - 1} \quad \forall n \in \mathbb{N} \tag{49}
\]

from which (44) for \( \varepsilon > 0 \) follows by optimising over \( n \in \mathbb{N} \). To prove (49), we use integration by parts on (48)

\[
|2^{ns} |y - x|^n (\partial^\nu_y \hat{G}^\varepsilon_s)_{\mu, \nu}(y; x)| \lesssim 2^{(2\gamma + |\nu|)s} \sup_{\mu; |\mu| = n} \int_{\mathbb{R}^d} \left[ \partial^\mu_k e^{-(k^2 + 2^{-2s})(-ik)^\nu \varphi_\delta(2^s \varepsilon |k|)} \right] dk.
\]

By using the Gevrey condition, we have

\[
\sup_{k \in \mathbb{R}^d} |\partial^\mu_k \varphi_\delta(2^s \varepsilon |k|)| \leq C^{1 + |\mu|} (2^s \varepsilon)^{|\mu|} \delta^{-1}
\]

whereas, by using the analyticity in a strip of size one around the real axis we have \(|\partial^\mu_k e^{-(k^2 + 2^{-2s})} \leq C^{1 + |\mu|} (|\mu| !)^{-1} e^{-k^2} \). Since \( 2^s \varepsilon \leq 1 \), (49) follows.

Repeating the same strategy for the difference, we obtain

\[
|2^{ns} |y - x|^n (\partial^\nu_y \hat{G}^\varepsilon_s)_{\mu, \nu}(y; x) - (\partial^\nu_y \hat{G}^\varepsilon'_s)_{\mu, \nu}(y; x)|
\lesssim |\nu| 2^{(2\gamma + |\nu|)s} \delta^{-1} \sup_{\mu; |\mu| = n} \int_{\mathbb{R}^d} \left[ \partial^\mu_k e^{-(k^2 + 2^{-2s})(-ik)^\nu \varphi_\delta(2^s \varepsilon |k|) - \varphi_\delta(2^s \varepsilon' |k|)} \right] dk. \tag{50}
\]

We notice that \( \varphi_\delta(2^s |k|) - \varphi_\delta(2^s \varepsilon' |k|) \) and its derivatives are vanishing unless \(|k| \geq \varepsilon^{-1} 2^{-s}\), therefore

\[
|2^{ns} |y - x|^n (\partial^\nu_y \hat{G}^\varepsilon'_s)_{\mu, \nu}(y; x)| \leq 2^{(2\gamma + |\nu|)s} C^n(n!)^{\delta - 1} e^{-(c2^s)^{-2}}
\leq 2^{(2\gamma + |\nu| + \theta)s} C^n(n!)^{\delta - 1} \sup_{s \in \mathbb{R}_+} 2^{-\theta s_\varepsilon} e^{-\frac{(c2^s)^{-2}}{2}}
\leq 2^{(2\gamma + |\nu| + \theta)s} e^{\theta C^n(n!)^{\delta - 1}},
\]

so that, optimising over \( n \in \mathbb{N} \) gives the stretched exponential decay in (45).

When \( \varepsilon = 0 \), we have \( \varphi_\delta(\varepsilon^{-1} |k|) = 0 \) and thus the integrand is analytic in a strip around the real axis in each of the variables \((k_i)_i \rightarrow 1\). Accordingly, we deform the integration contour by shifting \( k_i \rightarrow k_i + i \text{sign}(y_i - x_i) / 2 \) and obtain

\[
|(\partial^\nu_y \hat{G}^\varepsilon_s)_{\mu, \nu}(2^{-s} y; 2^{-s} x)| \lesssim 2^{(2\gamma + |\nu|)s} e^{-|y_1 - x_1|/2} \int_{\mathbb{R}^d} e^{\text{Re} \left( k^2 - \frac{k^2 + 2^{-2s}}{2} + ik \text{sign}(y_1 - x_1) \right)} |k|^\nu dk
\lesssim 2^{(2\gamma + |\nu|)s} e^{-|y_1 - x_1|/2}.
\]

By rotation invariance this implies the bound (44). \(\square\)

Remark 52. In a similar way and by using the Poisson summation formula to restrict to finite \( L \), one can prove a similar decay estimate for \( \hat{\epsilon}_s^{L,2} \), that is,

\[
\sup_{\mu, |\mu'| = n; |\nu| = n} \left| \partial^\nu_x (\hat{\epsilon}_s^{L,2}(x; y))_{\mu, \mu'} \right| \lesssim 2^{(\gamma + d/2 + n)s} \sum_{m \in \mathbb{Z}^d} e^{-c(2^s |x - y - mL|)^\delta},
\]
with $\delta = 1$ if $\varepsilon = 0$, and for the difference $\partial^\nu_x \left( \mathcal{C}^t_{s,e}(x; y) - \mathcal{C}^t_{s,e'}(x; y) \right)_{\mu, \mu'}$. Notice also the bounds

$$
\sup_{x; |x| = n} \| \partial^\nu_x \mathcal{C}^t_{s,e} \|_{L^\infty_x(T^d; C^{m+4})} \lesssim n, \varepsilon, 2^{(\gamma+d/2+n-d/\rho)]s}.
$$

(51)

Because of the norm-compatibility, (51) and the Young’s inequality imply, for $\gamma < d/2$

$$
\|X^t_{L,\varepsilon}(f)\| \lesssim (d/2 - \gamma)^{-1/2} \| f \|_0.
$$

(52)

We also have the following estimates.

**Corollary 53.** Let $X^t_{L,\varepsilon}$ be the field in Definition 45. Then, uniformly in $L \in \mathbb{N}_\infty$, $\varepsilon \in \mathcal{A}$

$$
\|X^t_{L,\varepsilon}\|_{C^L_t} \lesssim n 2^m.
$$

If $\varepsilon > \varepsilon' \in \mathcal{A}$ and $\theta > 0$, uniformly in $L \in \mathbb{N}_\infty$

$$
\|X^t_{L,\varepsilon} - X^t_{L,\varepsilon'}\|_{C^L_t} \lesssim n, \theta 2^{(\gamma+\theta)\varepsilon/\varepsilon'}.
$$

Finally, for any $L \in \mathbb{N}_\infty$, $\varepsilon \in \mathcal{A}$ and any $\eta > 0$ we have

$$
\|X^t_{L,\varepsilon} - X^t_{L,\varepsilon'}\|_{C^L_t(\eta)} \lesssim n 2^m L^{-\eta}.
$$

**Proof.** The proof is a straightforward consequence of the norm compatibility and the exponential estimates for $\mathcal{C}^t_{s,e}, \mathcal{C}^t_{s,e} - \mathcal{C}^t_{s,e'}$, see Remarks 52, and $\mathcal{C}^t_{s,e} - \mathcal{C}^t_{s,e'}$. In fact,

$$
\| (\partial^\nu_x X^t_{L,\varepsilon} - \partial^\nu_x X^t_{L,\varepsilon'})(x) \| = \left( \int_0^t \int_T |\partial^\nu_x (\mathcal{C}^t_{s,e} - \mathcal{C}^t_{s,e'})(x; z)|^2 dz ds \right)^{1/2} \lesssim |\nu| 2^{(\gamma+|\nu|+\theta)\varepsilon/\varepsilon'},
$$

so that the bound for $\|X^t_{L,\varepsilon} - X^t_{L,\varepsilon'}\|_{C^L_t}$ follows. The bound on $\|X^t_{L,\varepsilon}\|_{C^L_t}$ follows similarly. Concerning the last bound, we have

$$
\sup_{x \in \mathbb{R}^d} \| (\partial^\nu_x X^t_{L,\varepsilon} - \partial^\nu_x X^t_{L,\varepsilon'})(x) \| \lesssim \sup_{x \in \mathbb{R}^d} \left( \int_0^t \int_T \frac{\theta(z)}{\theta(z-x)} |\partial^\nu_x (\mathcal{C}^t_{s,e} - \mathcal{C}^t_{s,e'})(x; z)|^2 dz ds \right)^{1/2} \lesssim \sup_{x \in \mathbb{R}^d} \left( \int_0^t \int_T \frac{\theta(z)}{\theta(z-x)} |\partial^\nu_x \mathcal{C}^t_{s,e}(x; z)|^2 dz ds \right)^{1/2} \lesssim 1 + II + III.
$$

We bound I by using that for $x \in \mathbb{T}_L^d$, $(\mathcal{C}^t_{s,e} - \mathcal{C}^t_{s,e'})(x; z) = \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \mathcal{C}^t_{s,e}(x; z + mL)$ and the estimates for $\partial^\nu_x \mathcal{C}^t_{s,e}$, which control the growing term $\theta^{-1}$. Hence, we have

$$
I \lesssim \left( \int_0^t 2^{(\gamma+2|\nu|)L-2\eta} \sum_{m, m' \in \mathbb{Z}^d} \sum_{L \in \mathbb{N}_\infty} \frac{e^{-|m|+|m'|L}}{2} ds \right)^{1/2} \lesssim 2^{(\gamma+|\nu|)L-\eta}.
$$
We bound II is by using the decay of $\mathcal{C}_s^e$ and $\mathcal{C}_s^{L,e}$ separately to integrate on $\mathbb{T}^d_L$, so that $I \lesssim 2^{(\gamma + |\nu|)L} - \eta$. III is bounded likewise, hence the bound $\|X_t^y - X_t^{L,e}\|_{C^*_p(\eta)}$ is proven. 

4 Solution in the subcritical regime

In this section, we prove Theorems 1 and 3, namely, we prove the existence and uniqueness of a global solution of the FBSDE in (39) and, as an application, we prove the exponential clustering of the correlation functions. We begin by studying the truncated flow equation on some class of polynomials and prove the propagation of suitable norms on such polynomials, see Sections 4.1 and 4.2. In Section 4.3 we prove global existence of the pair $(\Psi, R)$ by a fixed point argument, see Lemma 48. Note that the closer to the critical regime (that is, the closer $\gamma$ to $d/4$), the farther we can truncate the flow equation. Finally, in Section 4.4 we combine this result with the coupling method to obtain the decay of clustered correlations.

4.1 Spaces of Grassmann monomials

We solve the truncated flow equation in the space of polynomials in the Grassmann fields on $\mathbb{R}_+ \times \mathbb{R}^d$. We interpret these polynomials as suitable multilinear operators acting on the fields and depending on the flow parameter. This is in fact natural since the Grassmann fields we consider should belong to the space $\mathcal{C}_s^2$, as it is the case for $X_1^{L,e}$. Furthermore, we work in a translation invariant, inversion invariant and antisymmetric setting, so that these multilinear operators should also possess these symmetries.

Let us lay out our notation in view of Definition 54. We let $k \in \mathbb{N}$, $m = (m_i)_i \in \{0, 2\}^k$ be a word of length $|m| = k$. If $m$ and $m'$ are two words, we let $m \circ m'$ denote the concatenated word obtained by juxtaposition. We write $m' \subset m$, if $m'$ can be formed by removing letters from $m$ and denote by $m' \rightarrow m$ the ways in which $m' \subset m$ can be made into $m$ by letter insertion. Finally, if $m' \subset m$, we denote by $m \setminus m'$ the word obtained by removing from $m$ the letters of $m'$, in the order in which they appear.

If $\mathcal{B}(V; W)$ denotes the Banach space of linear operators from $V$ to $W$, we consider the following spaces of linear operators

$$
\mathbb{L}_t^{(k,m)} := \mathcal{B}(\mathcal{C}_t^m(\mathbb{T}_t^d)^k; \mathcal{M}^4); L^\infty(\mathbb{T}_t^d; \mathcal{M}^4)).
$$

With any operator $K \in \mathbb{L}_t^{(k,m)}$, we can associate an integral kernel, which is generally a distribution. For example, for the word $m = 0^k$ we can write

$$(K_m(\psi(\cdot))) (x; \mu) = \sum_{\mu_1, \ldots, \mu_k \in \{1, \ldots, \infty\}} \int_{\mathbb{T}^d_k} K_m((y, \mu); (x_1, \mu_1); \cdots; (x_k, \mu_k)) \psi(x_1; \mu_1) \cdots \psi(x_k; \mu_k) dx_1 \cdots dx_k.
$$

In our setting, such kernels are a.e. antisymmetric, translation and inversion invariant, that is, if $m = 0^k$, for any permutation $\pi \in S_k$ and any $z \in \mathbb{T}_t^d$

$$
K_m((y, \mu); (x_1, \mu_1); \cdots; (x_k, \mu_k)) = K_m((y - z, \mu); (x_1 - z, \mu_1); \cdots; (x_k - z, \mu_k))
$$

$$
= K_m((-y, \mu); (-x_1, \mu_1); \cdots; (-x_k, \mu_k))
$$

$$
= (-\text{sign}(\pi)) K_m((y, \mu); (x_{\pi(1)}, \mu_{\pi(1)}); \cdots; (x_{\pi(k)}, \mu_{\pi(k)})).
$$

The operators we consider exhibit scale-dependent spatial decay in the separation of points. To keep track of this decay, we introduce the following scale-dependent weight, which is a generalisation of (46),

$$
w_t^{(k,d,c)}(x; \cdots, x_k) := e^{c(2^d \text{St}(x_1; \cdots; x_k))^6},
$$

where $\text{St}(x_1; \cdots; x_k)$ is the Stirling number of the first kind.
for $c \geq 0$, $\delta \in (0, 1)$, where $\text{St}(x_1; \ldots; x_k)$ denotes the Steiner diameter of the set $\{x_1, \ldots, x_k\}$, see, e.g., [GMR21]. If $K \in \mathbb{L}_t^{(k,m)}$, we let $K \cdot w_t^{(k+1)}$ be the operator whose kernel is a.e. the point-wise product of the kernel of $K$ with $w_t^{(k+1)}(x_1; \ldots; x_{k+1})$. We introduce

$$
\mathbb{L}_t^{(k,m)}(s,c) := \left\{ K \in \mathbb{L}_t^{(k,m)} \left\| K \cdot w_t^{(k+1)} \right\|_{\mathbb{L}_t^{(k,m)}} < \infty \right\},
$$

and for $K \in \mathbb{L}_t^{(k,m)}(s,c)$ let $\| K \|_{\mathbb{L}_t^{(k,m)}(s,c)} := \| K \cdot w_t^{(k+1)} \|_{\mathbb{L}_t^{(k,m)}}$. Notice that we dropped the dependence on $\delta$ for simplicity. We can finally define the following spaces of monomials with spatial decay in the separation of points.

**Definition 54.** Let $k \in \mathbb{N}$, $m = (m_i)_i \in \{0,2\}^k$ and let $|m| := \sum_i m_i$. The weighted space of monomials of degree $k \in \mathbb{N}$ at scale $t \geq 0$ is defined as the following subspace of the $l^\infty$-direct sum of $\mathbb{L}_t^{(k,m)}$:

$$
\mathcal{L}_t^{(k)}(s,c) := \left\{ (K_m)_m \in \bigoplus_{m \in \{0,2\}^k} \mathbb{L}_t^{(k,m)}(s,c) \left| \text{antisym., trans. & inv. invariant} \right. \right\}.
$$

The space $\mathcal{L}_t^{(k)}(s,c)$ is equipped with the norm

$$
\| K \|_{\mathcal{L}_t^{(k)}(s,c)} = \sup_{m \in \{0,2\}^k} 2^{|m|(s-t)} \| K_m \|_{\mathbb{L}_t^{(k,m)}(s,c)}.
$$

(53)

Finally, we let $\mathcal{L}_t^{(k)}(s,0) := \mathcal{L}_t^{(k)}(s,0)$ denote the space of monomials of degree $k$, at scale $t$.

**Remark 55.** Notice that for any $s \geq s'$ and $K \in \mathcal{L}_t^{(k)}(s,c)$

$$
\| K \|_{\mathcal{L}_t^{(k)}(s',c)} \leq \| K \|_{\mathcal{L}_t^{(k)}(s,c)}
$$

and thus $\mathcal{L}_t^{(k)}(s,c) \subset \mathcal{L}_t^{(k)}(s',c)$, which follows because $w_t^{(k)}(s',\delta,c) \leq w_t^{(k)}(s,\delta,c)$.

**Remark 56.** Loosely speaking, in the norm $\| \cdot \|_{\mathcal{L}_t^{(k)}(s,c)}$, $t$ is the scale of the fields, whereas $s$ is the scale of the monomial itself. For controlling the flow equation, it is crucial to take $t \leq s$ (except for the local monomial of degree one, see the next section). Having the fields at a scale not bigger than that of the monomial, is what allows us to gain by the renormalisation operator, as can be seen in equation (56) below and motivates the choice of the scale-dependent weight in the definition of $\| \cdot \|_{\mathcal{L}_t^{(k)}(s,c)}$.

We shall now discuss some important operators acting on the spaces of monomials $\mathcal{L}_t^{(k)}$. Let us begin by considering the operation of localisation and renormalisation for the monomials of degree $k = 1$. This operation is crucial for the study of the flow equation carried out in the next section and is well-known in the study of the renormalisation group flow, see, e.g., [Mas08, BBS19]. Ultimately, this operation is necessary because we want to renormalise the interacting measure by adding a suitable local quadratic term only.

**Definition 57.** (Localisation & Renormalisation). The localisation and renormalisation operators are respectively the projections Loc and Ren onto $\mathcal{L}_t^{(1)}$ defined as follows: if $K = (K_0, K_2)$, we let

$$
\text{Loc } K := (\text{Loc}_0 K_0, 0), \quad \text{Ren } K := (0, K_2 + (1 - \text{Loc}_0) K_0),
$$

where

$$
(\text{Loc}_0 K_0)((y; \mu); (x; \mu')) := \delta(y - x) \int_{\mathbb{T}_2^2} K((y, \mu); (z; \mu')) dz.
$$
with \(\delta(y - x)\) denoting the Dirac delta distribution.

**Remark 58.** If \(\varphi \in \mathcal{C}_b^2\) then we can cook up \(\psi^{(m)} := \otimes_l \partial^{m_l} \psi \in \mathcal{C}_b^m\) and set \(K(\psi) = \sum_{m \in \{0, 2\}^k} K_m(\psi^{(m)}).\) Note that even though \(\text{Loc} + \text{Ren} \neq 1\) on \(\mathcal{L}_t^{(1)}\), we still have the identity \(K(\psi) = \text{Loc} K(\psi) + \text{Ren} K(\psi).\) This implies that, in the study of the flow equation, we can use the equivalence relation \(K \sim (\text{Loc} + \text{Ren}) K = (\text{Loc}_0 K_0, K_2 + (1 - \text{Loc}_0) K_0).\)

In view of the application to the flow equation, we provide the following lemma.

**Lemma 59.** For any \(r, s, t \geq 0\) and any \(K \in \mathcal{L}_t^{(1)}\) we have
\[
\|\text{Loc} K\|_{\mathcal{L}_t^{(1)}(s, c)} = \|\text{Loc} K\|_{\mathcal{L}_t^{(1)}(s, c)} \leq \|K\|_{\mathcal{L}_t^{(1)}}.
\]

If \(K \in \mathcal{L}_t^{(1)}(s, c),\) then for any \(c' < c\)
\[
\|\text{Ren} K\|_{\mathcal{L}_t^{(1)}(s, c')} \lesssim_{c - c'} \|K\|_{\mathcal{L}_t^{(1)}(s, c)}.
\]

**Proof.** The identity and the inequality in (54) follow by noting that \((\text{Loc} K)_2 = 0\) and that \(w_{s, \delta, c}(y; y) = 1.\) To prove (55), one should note that, for \(\psi\) smooth enough,
\[
((1 - \text{Loc}_0)K_0(\psi))(y; \mu) = \sum_{\mu'} \int_{T_d^2} K_0((y, \mu); (x; \mu')) \left[ \psi(x; \mu') - \psi(y; \mu') - (x - y) \cdot \partial \psi(y; \mu') \right] dx = \sum_{\mu'} \int_{T_d^2} K_0((y, \mu); (x; \mu')) (x - y)^2 \int_0^1 (1 - t) \partial^2 \psi(y + t(x - y); \mu') dt dx,
\]
where in the second line we used that \(K_0\) is translation and inversion invariant, so that \(\int_{T_d^2} K_0((y, \mu); (x; \mu')) (x - y)^2 dy = 0,\) and where in the last line \((x - y)^2\) and \(\partial^2\) are suitably contracted in the internal indices. We have
\[
\|((1 - \text{Loc}_0)K_0(\psi))\|_{\mathcal{L}_t^{(1)(2)}(s, c')} \leq 2^2 \sup_{y, \mu} \int_{T_d^2/0} \int_0^1 (1 - t) |K_0((y, \mu); (x; \mu')) (x - y)^2 |w_{s, \delta, c}(y; y + t(x - y))| dt dx \leq 2^2 \sup_{y, \mu} \int_{T_d^2/0} \int_0^1 (1 - t) |K_0((y, \mu); (x; \mu')) (x - y)^2 |w_{s, \delta, c}(y; x)| dt dx \leq 2^{2(t - s)} \sup_{y, \mu} \int_{T_d^2} |K_0((y, \mu); (x; \mu')) |w_{s, \delta, c}(y; x)| dx = 2^{2(t - s)} \|K_0\|_{\mathcal{L}_t^{(1)(0)}(s, c)},
\]
where \(\lesssim\) is up to constants depending on \(c - c',\) implying the claim. \(\square\)

Finally, we conclude this section by introducing the Laplacian and the contraction operators. Loosely speaking, these operators implement the action of \(\frac{1}{2} \Delta d\) and of \((\mathcal{G}_s \cdot, \cdot)\) on the space of monomials described above. In the definition below, we think of \(\mathcal{G}_s\) as an element of \(C^\infty(T^2_d \times T^2_d; \mathbb{C}^4 \times \mathbb{C}^4)\) so that we will also write \(\mathcal{G}_s^{(m)}\) for some \(m \in \{0, 2\}^2.\)

**Definition 60.** The Laplacian operator is the linear operator \(\Delta d; \mathcal{L}_t^{(k+2)} \rightarrow \mathcal{L}_t^{(k)}\) for any \(k \geq 1\) that acts as follows: if \(K \in \mathcal{L}_t^{(k+2, m)}\) and \(m' \subset m,\) with \(|m'| = k\),
\[
(\Delta d K)(m)(\psi^{(m')}) := \sum_{m' \rightarrow m} \text{sign}(m' \rightarrow m) K(m)(\psi^{(m')})\mathcal{G}_s^{(m \setminus m')}.
\]
where $\text{sign}(m' \to m)$ is the sign depending on the reshuffling due to $m' \to m$ and due to the non-commutativity of the fields. The contraction operator is the bilinear operator $C_{\mathring{g}^s}$: $\mathcal{L}_t^{(k+1)} \times \mathcal{L}_t^{(k')} \to \mathcal{L}_t^{(k+k')}$, for any $k, k' \geq 1$ that acts as follows: if $K \in \mathbb{L}_t^{(k+1, m)}$, $K' \in \mathbb{L}_t^{(k, m')}$ and $m'' \subset m$, with $|m''| = k$

$$(C_{\mathring{g}^s}(K, K'))_{m'' \circ m'}(\psi(m'')) := \sum_{m' \to m} \text{sign}(m' \to m) \sum_{\mu} \int K_m(\psi(m') \mathring{g}^s(\psi(m''))((y', \mu'), \cdot)) \cdot K_{m''}(\psi(m''))((y', \mu')) dy'. $$

In view of the application to the flow equation the following lemma is crucial.

**Lemma 61.** If $K \in \mathcal{L}_t^{(k+2)}(s, c)$, then $\mathcal{D}_t^{2}(K) \in \mathcal{L}_t^{(k)}(s, c)$ and

$$\|\mathcal{D}_t^{2}(K)\|_{\mathcal{L}_t^{(k)}(s, c)} \leq k^2 \|K\|_{\mathcal{L}_t^{(k+2)}(s, c)} \|\mathring{g}^s\|_{L^\infty}. \quad (57)$$

If $K \in \mathcal{L}_t^{(k+1)}(s, c)$ and $K' \in \mathcal{L}_t^{(k')}(s, c)$, then $C_{\mathring{g}^s}(K, K') \in \mathcal{L}_t^{(k+k')}(s, c)$

$$\|C_{\mathring{g}^s}(K, K')\|_{\mathcal{L}_t^{(k+k')}(s, c)} \leq k \|K\|_{\mathcal{L}_t^{(k+1)}(s, c)} \|K'\|_{\mathcal{L}_t^{(k+k')}(s, c)} \|\mathring{g}^s\|_{L^1}. \quad (58)$$

**Proof.** The bound on the Laplacian operator follows by noting that there are at most $(k+1)(k+2)$ elements in the sum and by noting that

$$\|K_m(\mathring{g}^s(m''))\|_{\mathcal{L}_t^{(k,m')}} = \|K_m(\mathring{g}^s(m'')) \cdot w_{s, \delta, c}^{(k+1)}\|_{\mathcal{L}_t^{(k,m')}} \leq 2^{-|m'|m''} \|\mathring{g}^s(m'')\|_{L^\infty} \|K_m \cdot w_{s, \delta, c}^{(k+3)}\|_{\mathcal{L}_t^{(k+2,m')} \leq 2^{m'/m''} \|\mathring{g}^s\|_{L^\infty} \|K_m\|_{\mathcal{L}_t^{(k+2,m')}}.$$
Corollary 63. If $K \in \mathcal{L}_{s}^{(3)}(s, c)$, then for any $c' < c$, $\text{Ren} \mathcal{D}_{k}^{2}(K) \in \mathcal{L}_{s}^{(1)}(s, c')$ and

$$
\|(\text{Ren} \mathcal{D}_{k}^{2}(K))2\|_{\mathcal{L}_{s}^{(1,2)}(s, c')} \lesssim 2^{-2(s-\ell)} \|K\|_{\mathcal{L}_{s}^{(3)}(s, c)} \|\hat{\mathcal{F}}\|_{L_{s}^{\infty}}.
$$

If $K, K' \in \mathcal{L}_{s}^{(1)}(s, c)$, then for any $c' < c$, $\text{Ren} \mathcal{C}_{k}^{2}(K, K') \in \mathcal{L}_{s}^{(1)}(s, c')$ and

$$
\|(\text{Ren} \mathcal{C}_{k}^{2}(K, K'))2\|_{\mathcal{L}_{s}^{(1,2)}(s, c')} \lesssim 2^{-2(s-\ell)} \|\mathcal{K}\|_{\mathcal{L}_{s}^{(1)}(s, c)} \|\hat{\mathcal{F}}\|_{L_{s}^{\infty}}.
$$

4.2 The truncated flow equation

For technical convenience, we study the flow equation by introducing a further grading with respect to the derivative in the flow parameter. Thus, we decompose the function $\mathcal{F}_{t}^{\ell}$ as follows

$$
\mathcal{F}_{t}^{\ell}(\psi_{t}) = \sum_{\ell \geq 0} \mathcal{F}_{t}^{\ell}^{(0)}(\psi_{t}) = \sum_{\ell \geq 0, k \geq 0} \mathcal{F}_{t}^{(\ell, k)}(\psi_{t}),
$$

where the label $\ell$ is the grading with respect to the operator $\partial_{s}$, whereas $k$ is the grading with respect to the functional derivative $\mathcal{D}$, so that $\mathcal{F}_{t}^{\ell, k} \in \mathcal{L}_{s}^{(k)}$ are monomials of degree $k$. This grading decomposition is natural because the flow equation involves both differential operators. We introduce the following truncated flow equation

$$
\partial_{s} \mathcal{F}_{t}^{\ell}(\psi_{s}) + \frac{1}{2} \mathcal{D}_{k}^{2}(\mathcal{F}_{s}^{\ell}(\psi_{s})) + \Pi_{\leq n} (\mathcal{F}_{t}^{\ell}, D \mathcal{F}_{s}^{\ell}(\psi_{s})) = 0,
$$

where $\Pi_{\leq n}$ is the projection on terms with $\ell \leq n$. The truncation is convenient because it allows us to look for polynomial solutions. Note that we could truncate with respect to both gradings, with no major difference in the analysis. Expanding as in (59), the truncated flow equation reads, for $\ell = 0, \ldots, n$:

$$
\partial_{s} \mathcal{F}_{t}^{\ell}(\psi_{s}) + \frac{1}{2} \mathcal{D}_{k}^{2}(\mathcal{F}_{s}^{\ell}(\psi_{s})) + \sum_{l' = 0}^{\ell} \sum_{k' = 0}^{k-1} \mathcal{C}_{k}^{2}(\mathcal{F}_{s}^{\ell}([k-k']^{(k-1)}), \mathcal{F}_{s}^{\ell, l'}([k'+1])) = 0,
$$

where the operators $\mathcal{D}_{k}^{2}$ and $\mathcal{C}_{k}^{2}$ are as in Definition 60. Note that Eq. (61) is triangular in $\ell$ and this will bring technical simplification to the analysis.

We let

$$
c_{\ell} := (1 - \ell n^{-1})\bar{c}/2,
$$

$\bar{c}$ being the constant in Lemma 49. We make the following Ansatz for the norm of the monomials $\mathcal{F}_{s}^{\ell, k}$, for suitable constants $C > 0$, $\alpha, \beta, \kappa \geq 0$:

$$
\|\mathcal{F}_{t}^{\ell, k}(\psi_{t})\|_{\mathcal{L}_{s}^{(k)}(s, c)} \lesssim C k 1^{2(\alpha - \beta k - \kappa s)}, \quad 0 \leq t \leq s.
$$

First of all, we note that because of the “initial condition” $\mathcal{F}_{0}^{\ell}(\psi_{0}) = \lambda(\psi_{0})^{3}$, the Ansatz (63) can hold only if

$$
\alpha - 3 \beta \geq 0.
$$

Furthermore, as shall see shortly, these estimates can be propagated only if the following conditions are satisfied:

$$
5 \beta - \alpha - 2 \gamma - \kappa > 0, \quad d + 4 \beta - 2 \alpha - 2 \gamma - \kappa > 0,
$$

and

$$
3 \beta + 2 - \alpha - 2 \gamma - \kappa > 0, \quad d + 2 \beta + 2 - 2 \alpha - 2 \gamma - \kappa > 0.
$$
The constraints in (65) makes irrelevant the terms with $k \geq 3$, in the sense of integrability at $s = \infty$ of the Laplacian and contraction operators respectively. On the other hand the constraints in (66) make irrelevant the renormalised term with $k = 1$.

**Remark 64.** Note that the constraints (64), (65) and (66) can be satisfied only provided that

\[
\gamma < \min \{d/4; 1\}, \quad \beta \in (\gamma, \min \{d/2 - \gamma; d/4 + 1/2 - \gamma/2\}), \quad \alpha \in [3\beta, d/2 + \min \{2\beta; \beta + 1\} - \gamma).
\]

The condition $\gamma < d/4$ is what makes the model superrenormalisable. As hinted in the introduction, the constraint $\gamma < 1$ comes from the requirement that the counter-term is a mass term and does not have derivatives fields. Note that the superrenormalisability of the model allows for room in the choice of the parameters $\beta, \alpha$ and $\kappa$.

Given the Ansatz (63), the following norm topology is natural: for any graded sequence $F = (F^\ell[k])_{\ell \geq 0, k \geq 0}$, we introduce the norm

\[
\|F\|_{C, \alpha, \beta, \kappa} := \sup_{\ell \geq 0, k \geq 0} \sup_{0 \leq t \leq s} C^{-k2^{-\alpha k - \kappa k s}} \|F^\ell[k]\|_{\Sigma^k_s(s, c_k)},
\]

where for simplicity we dropped the dependence on $(c_k)$. The following theorem establishes the existence and uniqueness of a global solution of the truncated flow equations.

**Theorem 65.** (Global solution). Let $\varepsilon \in \mathcal{A}$, $n \in \mathbb{N}$, $C > 0$, let $\alpha, \beta, \gamma, \kappa \geq 0$ satisfy the constraints in (64), (65), (66) and let $c_k$ be as in (62). The truncated flow equations (61) with boundary data

\[
F^0_0[k] = K^k_{1_{k \in \{1, 3\}}, \quad \sup_{k \geq 0} \sup_{0 \leq t \leq s} C^{-k2^{-(\alpha - \beta k - \kappa k s)}} \|K^k\|_{\Sigma^k_s(s, c_k)} < \infty,
\]

and for $\ell \geq 1$

\[
F^\ell[k] = 0 \quad \forall k \neq 1, \quad \text{Ren} F^\ell[1](1) = 0, \quad \text{Loc} F^\ell[1](1) = 0
\]

have a unique global solution $F^\varepsilon = (F^\varepsilon[\ell][k])_{\ell \geq 0, k \geq 0}$, such that $F^\varepsilon[\ell][k] \in \Sigma^k_s(s, c_k)$.

**Remark 66.** Note that the monomials $K^k$ have to be local in order to satisfy the said bound.

**Proof.** The proof is by induction in $\ell$. We let $F^\varepsilon[\leq \ell]$ denote the graded sequence

\[
(F^\varepsilon[\ell][k])_{\ell \geq 0, k \geq 0}
\]

and show that $\|F^\varepsilon[\leq \ell]\|_{C, \alpha, \beta, \kappa}$ is finite.

The validity at $\ell = 0$ is obvious because $\partial_s F^\varepsilon[0][0] = 0$ and because of the assumptions. In particular, we have that $\|F^\varepsilon[\leq 0]\|_{C, \alpha, \beta, \kappa} < \infty$. To prove the induction step, we integrate the truncated flow equation (61) for $k \geq 3$ backwards from the final condition at $s = \infty$ with $F^\varepsilon[\ell+1][k] = 0$ for any $\ell \geq 0, k \geq 3$. We have:

\[
F^\varepsilon[\ell+1][k] = \int_s^\infty \left( \frac{1}{2} D_0^\varepsilon (F^\varepsilon[k+2]) + \sum_{\ell' = 0}^{\ell-1} \partial_s C_0^\varepsilon (F^\varepsilon[k-k']) F^\varepsilon[\ell-\ell'][k+1]) ds.
\]

For the term with $k = 1$, we use the equivalence $F^\varepsilon[\ell+1][1] \sim \text{Loc} F_t^\varepsilon[\ell+1][1] + \text{Ren} F_t^\varepsilon[\ell+1][1]$, that is, $F_t^\varepsilon[\ell+1][1] \sim (\text{Loc} F_t^\varepsilon[\ell+1][1])^0 + (\text{Ren} F_t^\varepsilon[\ell+1][1])_2$. We integrate the term with the renormalisation from $s = \infty$, with $(\text{Ren} F_t^\varepsilon[\ell+1][1])_2 = 0$ for any $\ell \geq 0$

\[
(\text{Ren} F_t^\varepsilon[\ell+1][1])_2 = \int_t^\infty \left( \frac{1}{2} (\text{Ren} D_0^\varepsilon (F^\varepsilon[3]) + \sum_{\ell' = 0}^{\ell} (\text{Ren} C_0^\varepsilon (F^\varepsilon[\ell]', F^\varepsilon[\ell']')) ds,
\]
and, finally, we integrate the term with the localisation from 0 to \( t \), with \((\text{Loc} F_{\ell}^{(\ell+1)(1)})_0 = 0\) for any \( \ell \geq 0 \)

\[
(\text{Loc} F_{\ell}^{(\ell+1)(1)})_0 = - \int_0^t \left( \frac{1}{2} (\text{Loc} D_{\ell}^2 (F_{\ell}^{(\ell)(1)}))_0 + \sum_{\ell' = 0}^\ell (\text{Loc} \mathcal{C}_{\ell}^2 (F_{\ell}^{(\ell')(1)}, F_{\ell}^{(\ell'-\ell)(1)}))_0 \right) ds.
\]

Now, recalling the kernel estimates for the propagator \( \| \hat{g}_k^2 \|_{L^p_{x,b,c}} \lesssim 2^{-(d/p-2)c} \), for \( c < \bar{c} \), see Corollary 51, Lemma 61 implies the following estimates:

\[
\| \mathcal{D}^2_{\ell} (F_{\ell}^{(\ell)(k+2)}) \|_{L^2_{(s,c)}} \lesssim k^2 C^{k+2} (\alpha - \beta k) s \lesssim 2^{-(d + \beta - \alpha - 2\gamma) s - \kappa s} \| F_{\ell}^{(\ell)} \|_{C, \alpha, \beta, \kappa},
\]

and for \( \ell' = 0, \ldots, \ell \)

\[
\| \mathcal{C} (F_{\ell}^{(\ell')(k-k')}, F_{\ell}^{(\ell'-\ell')(k+1)}) \|_{L^2_{(s,c)}} \lesssim k C^{k+1} (\alpha - \beta) s \lesssim 2^{-(d + \beta - \alpha - 2\gamma) s - \kappa s} \| F_{\ell}^{(\ell)} \|_{C, \alpha, \beta, \kappa}.
\]

By using these bounds, the constraints (67) and Remark 55, we obtain that for \( k \geq 3 \) and \( r \leq t \) \( F_{\ell}^{(\ell+1)(1)}(k) \in L^r_{(t,c_{\ell+1})} \) and

\[
\| F_{\ell}^{(\ell+1)(1)} \|_{L^r_{(t,c_{\ell+1})}} \leq \int_t^\infty \| \mathcal{D}^2_{\ell} (F_{\ell}^{(\ell)(k+2)}) \|_{L^2_{(s,c)}} ds + \int_t^\infty \sum_{\ell' = 0}^\ell k \| \mathcal{C} (F_{\ell}^{(\ell')(k-k')}, F_{\ell}^{(\ell'-\ell')(k+1)}) \|_{L^2_{(s,c)}} ds \lesssim k^2 C^{k+2} \| F_{\ell}^{(\ell)} \|_{C, \alpha, \beta, \kappa} \int_t^\infty 2^{(\alpha - \beta k) s \lesssim 2^{-(d + \beta - \alpha - 2\gamma) s - \kappa s} ds
\]

\[
+ (\ell + 1) k C^{k+1} \| F_{\ell}^{(\ell)} \|_{C, \alpha, \beta, \kappa} \int_t^\infty 2^{(\alpha - \beta) s \lesssim 2^{-(d + \beta - \alpha - 2\gamma) s - \kappa s} ds \lesssim C^{k+1} \| F_{\ell}^{(\ell)} \|_{C, \alpha, \beta, \kappa} \| F_{\ell}^{(\ell)} \|_{C, \alpha, \beta, \kappa}.
\]

where in the last line \( \lesssim \) is up to constants depending on \( \ell, \alpha, \beta, \gamma, \kappa \). In a similar way, Corollary 51, Corollary 63 with \( c' := c_{\ell+1} \) and \( c := c_{\ell} \), and the constraints (67), we obtain that for \( r \leq t \) \( (\text{Ren} F_{\ell}^{(\ell+1)(1)})_2 \in L^r_{(1,2)}(t, c_{\ell+1}) \) and

\[
\| (\text{Ren} F_{\ell}^{(\ell+1)(1)})_2 \|_{L^r_{(1,2)}(t,c_{\ell+1})} \leq \int_t^\infty \left( \| (\text{Ren} \mathcal{D}^2_{\ell} (F_{\ell}^{(\ell)(1)}))_2 \|_{L^r_{(1,2)}} + \int_t^\infty \sum_{\ell' = 0}^\ell \left( \| (\text{Ren} \mathcal{C}_{\ell}^2 (F_{\ell}^{(\ell')(1)}, F_{\ell}^{(\ell'-\ell)(1)}))_2 \|_{L^r_{(1,2)}} \right) ds \right) \lesssim C^2 \| F_{\ell}^{(\ell)} \|_{C, \alpha, \beta, \kappa} 2^{\alpha - \beta s \lesssim 2^{-(d + \beta - \alpha - 2\gamma) s - \kappa s} ds
\]

\[
+ (\ell + 1) C^2 \| F_{\ell}^{(\ell)} \|_{C, \alpha, \beta, \kappa} 2^{(\alpha - \beta) s \lesssim 2^{-(d + \beta - \alpha - 2\gamma) s - \kappa s} ds \lesssim C^2 \| F_{\ell}^{(\ell)} \|_{C, \alpha, \beta, \kappa} ^2.
\]
Then, we control the term with the localisation operator, for \( r \leq t \)
\[
\begin{align*}
&\| (\text{Loc} \, F^e_t)^{(l+1)} \|_{\mathcal{L}^1_{\text{loc}}(t, t + 1)} \\
&\leq \int_0^t \| \text{Loc} \, D^2_s (F^e_s)^{(l)}(3) \|_{\mathcal{L}^1_{\text{loc}}(s, c_2)} ds \\
&+ \sum_{l' = 0}^l \int_0^t \| \text{Loc} \, C^l_s (F^e_s)^{(l)}(l - l') \|_{\mathcal{L}^1_{\text{loc}}(s, c_2)} ds \\
&\lesssim C^2 \| F^e \|_{C, \alpha, \beta, \kappa} \int_0^t 2^{(\alpha - \beta) s} 2^2 (\gamma - \beta)s 2 - \kappa s ds \\
&+ \left( \ell + 1 \right) C^2 \| F^e \|_{C, \alpha, \beta, \kappa} \int_0^t 2^{(\alpha - \beta) s} 2^{-(d + \beta - \alpha - 2\gamma)s} 2 - \kappa s ds \\
&\lesssim C^2 2^{(\alpha - \beta) t} 2 - \kappa (t + 1) \| F^e \|_{C, \alpha, \beta, \kappa} + \| F^e \|_{C, \alpha, \beta, \kappa}^2,
\end{align*}
\]
(73)
where we used that \( \| (K) \|_{L^1_{\text{loc}}(t, t + 1)} \leq \| K \|_{\mathcal{L}^1_{\text{loc}}(t, t + 1)} \) and the property (54) to set the scale of the field to zero. Putting together (72) and (73), we obtain for \( r \leq t \)
\[
\| (\text{Loc} \, F^e_t)^{(l+1)} \|_{\mathcal{L}^1_{\text{loc}}(t, t + 1)} \lesssim C^2 2^{(\alpha - \beta) t} 2 - \kappa (t + 1) \| F^e \|_{C, \alpha, \beta, \kappa} + \| F^e \|_{C, \alpha, \beta, \kappa}^2
\]
(74)
The bounds (71), (74) imply that \( \| F^e \|_{\ell+1} \|_{C, \alpha, \beta, \kappa} \) is finite. This concludes the proof. \( \Box \)

**Corollary 67.** Let \( F^e \) be the unique solution to the flow equation truncated at \( n \), with the same boundary data as in Theorem 65, but in particular \( F^e_0(\psi_1) = \lambda(\psi_1) \). Let \( H^e_{n, n} \) be defined by \( H^e_{n, n}(\psi_1) = \Pi_{\geq n} (\hat{g} \hat{F}_n^* (\psi_1), DF_n(\psi_1)) \). Then, there exist constants \( C_0 = C_0(\alpha, \beta, \gamma, \kappa, n) \) and \( \lambda_0 = \lambda_0(C) \) independent of \( \epsilon \) and \( L \) such that, if \( C \leq C_0 \) and \( \lambda \leq \lambda_0(C) \) then
\[
\begin{align*}
\| F^e \|_{C, \alpha, \beta, \kappa} &\lesssim 1, \\
\| DF^e \|_{C, \alpha, \beta, \kappa} &\lesssim 1, \\
\| H^e_{n, n} \|_{C, \alpha, \beta, \kappa} &\lesssim \lambda^2,
\end{align*}
\]
uniformly in \( \epsilon \) and \( L \), where the \( \lesssim \) are up to universal constants.

**Proof.** The proof is a corollary of the proof of Theorem 65. In fact, one can read off the bound for \( \| F^e \|_{C, \alpha, \beta, \kappa} \) directly from (71) (which holds for any \( k \geq 0 \)), provided that \( C \) and \( \lambda \) are small enough, the latter depending on \( C \). Then, for some \( C' > C \) we have
\[
\| DF^e \|_{C, \alpha, \beta, \kappa} \leq \sup_{\ell, k \geq 0} \sup_{0 \leq \ell \leq s} kC^\ell 2^{-(\alpha - \beta (k + 1)) s} 2^{\kappa s} \| F^e \|_{(k+1)(s, c_2)} \]
\[
\lesssim \| F^e \|_{C, \alpha, \beta, \kappa}.
\]
Finally, comparing with (61) we have \( H^e_{n, s} \) is \( 1_{n < \ell \leq 2n} \sum_{\ell'} \sum_{k'} C^\ell \left( F^e_{(k') \ell'} \right) \), and therefore, by the proof of Theorem 65 we have the bound
\[
\| H^e_{n, s} \|_{(s, c_2)} \lesssim k(\ell + 1) \| F^e \|_{C, \alpha, \beta, \kappa}^2,
\]
which implies the claim. \( \Box \)

Given the global solution, we can finally define the renormalized chemical potential.

**Definition 68.** The scale-dependent chemical potential \( \mu^e_t \) associated with the global solution \( F^e \) of the truncated flow quations is defined as
\[
\mu^e_t := \sum_{\ell = 0}^n \int_{\mathbb{R}^d} \left( F^e_t \right)_{(\ell)} \, dx,
\]
where \( F^e_t = \left( (F^e_t)^{(1)} \right)_0, (F^e_t)^{(2)}_2, (F^e_t)^{(1)}_0 \) being the local part. The renormalized chemical potential is \( \mu^e := \lim_{t \to \infty} \mu^e_t \).
Remark 69. Notice the bound

\[ |\mu_\ell| \lesssim \sum_{\ell=0}^{n} \|\text{Loc } F_T^{[\ell]}\|_{s} \lesssim \|F\|_{C, \alpha, \beta, \kappa} 2^{(\alpha - \beta - \kappa)T} \]

where by the first constraint in (65) \( \alpha - \beta - \kappa > 2\gamma \).

Next, we prove the convergence \( F^\varepsilon \to F \) in a weaker topology.

**Proposition 70.** (Continuum limit). Let \( \varepsilon > \varepsilon' \in A \). Under the same assumptions of Theorem 65 but with \( \alpha > 3\beta \), let \( F^\varepsilon \) and \( F^\varepsilon' \) be the global solutions of the truncated flow equation and let \( \theta \in \left(0, \frac{\alpha - 3\beta}{2}\right) \). Then

\[ \|F^\varepsilon - F^\varepsilon'\|_{C, \alpha, \beta, \kappa} \lesssim \|F\|_{C, \alpha - \theta, \beta, \kappa} + \|F\|_{C, \alpha - \theta, \beta, \kappa}^{\gamma + 2} \]

**Proof.** Set \( \delta_{\varepsilon, \varepsilon'} F_s := F_s^\varepsilon - F_s^\varepsilon' \) and \( \delta_{\varepsilon, \varepsilon'} \tilde{\mathbf{u}}_s := \tilde{\mathbf{u}}_s^\varepsilon - \tilde{\mathbf{u}}_s^\varepsilon' \). We can write the truncated flow equation for the difference as follows:

\[ \partial_t \delta_{\varepsilon, \varepsilon'} F_s^{[\ell + 1]}(k) = \delta_{\varepsilon, \varepsilon'} \frac{1}{2} \mathbb{D}_{s}^2 (F_s^{[\ell]}(k) + 2) + \sum_{\ell' = 0}^{k} \sum_{k' = 1}^{k} \delta_{\varepsilon, \varepsilon'} C_{\tilde{\mathbf{u}}_s} (F_s^{[\ell']}(k'), F_s^{[\ell']}(k+1-k')) \] (76)

with identically null boundary data \( \delta_{\varepsilon, \varepsilon'} F_0^{[\ell]} = 0 \). Above, the difference operator \( \delta_{\varepsilon, \varepsilon'} \) acts on \( \tilde{\mathbf{u}}_s \) as well,

\[ \delta_{\varepsilon, \varepsilon'} \mathbb{D}_{s}^2 (F_s^{[\ell]}(k)) = \mathbb{D}_{s}^2 (\delta_{\varepsilon, \varepsilon'} F_s^{[\ell]}(k)) + \mathbb{D}_{s}^2 (\delta_{\varepsilon, \varepsilon'} F_s^{[\ell]}(k)) \]

and similar identity holds for \( \delta_{\varepsilon, \varepsilon'} C_{\tilde{\mathbf{u}}_s} \).

To estimate \( \|\delta_{\varepsilon, \varepsilon'} F\|_{C, \alpha, \beta - \theta, \kappa} := \|F^\varepsilon - F^\varepsilon'\|_{C, \alpha, \beta - \theta, \kappa} \), we proceed as in the proof of Theorem 65, the crucial difference being that \( \delta_{\varepsilon, \varepsilon'} F_0^{[\ell]} = 0 \) and the estimate \( \|\delta_{\varepsilon, \varepsilon'} \tilde{\mathbf{u}}_s\|_{L_{\varepsilon}^p, t, \varepsilon} \lesssim 2^{-\left(\frac{d}{p} - 2\gamma - \theta\right) t} \varepsilon^{0} \) see Corollary 51. In particular, one wants to prove inductively that

\[ \|\delta_{\varepsilon, \varepsilon'} F^{[\ell]}\|_{C, \alpha, \beta - \theta, \kappa} \lesssim \|F\|_{C, \alpha - \theta, \beta, \kappa} + \|F\|_{C, \alpha - \theta, \beta, \kappa}^{\ell + 2} \]

Repeating the analysis in the proof of Theorem 65, we obtain the estimates

\[ \|\delta_{\varepsilon, \varepsilon'} F_t^{[\ell + 1]}(k)\|_{\Theta^{[b]}(t, \varepsilon, \varepsilon')} \lesssim \varepsilon^{0} C^{k+1} 2^{(\alpha - \beta - k)T} \lesssim \|F\|_{C, \alpha - \theta, \beta, \kappa}^{\ell + 3} \]

compare with (71), implying the claim. \( \square \)

### 4.3 Control of the remainder: proof of Theorem 1

In this section, we prove the global existence and uniqueness of the pair \( (\Psi_t^L, R_t^L) \) solving the coupled equations

\[ \Psi_t^L = \int_0^t \tilde{\Psi}_s^L (F_s^L (\Psi_s^L)) + R_s^L) \, ds + X_t^L, \]

\[ R_t^L = \int_t^\infty \omega_t (H_{\geq t}^L (\Psi_s^L)) \, ds + \int_t^\infty \omega_t (\tilde{\Psi}_s^L (R_s^L, D_{\Psi_s^L})) \, ds, \] (77)
compare Lemma 48, with input $F^\varepsilon_s$ and $H^\varepsilon_{s;n;\ldots}$ graded sequences satisfying suitable smallness assumptions. Putting together Lemma 48, Theorem 65, Corollary 67 and Proposition 71 below gives the proof of Theorem 1.

To control the global solution of Eq. (77), we use the following topology:

$$\|(\psi, \varphi)\|_{D, a, b} := D^{-1} \sup_{t \in \mathbb{R}_+} 2^{-at} \|\psi_t\|_{C^a_t} + \int_0^\infty 2^{-bt} \|\varphi_t\|_{L^\infty} \, dt,$$

where $D > 0$, $a, b \geq 0$ and where $\|\psi_t\|_{C^a_t} = \sum_{|\nu| \leq 2} 2^{-at} \|\partial^\nu \psi_t\|_{L^\infty}$, see before (41) . We can state our result as follows.

**Proposition 71.** Let $L \in \mathbb{N}_\infty$ and $\varepsilon \in A$. Let $\alpha, \beta, \gamma, \kappa$ satisfy (64), (65) and (66), let $\theta \in \left[0, \frac{\beta - \gamma}{2}\right]$ and let $F^\varepsilon = (F^\varepsilon_{[t]}(k))_{t \geq 0, k \geq 1}$ and $H^\varepsilon_{s;n} = (H^\varepsilon_{s;n,t})_{t \geq 0, k \geq 0}$ be the graded sequences in (77), with $n = |\alpha \kappa^{-1}| + 1$. There exist constants $D$ and $\lambda = 1(d, \alpha, \beta, \gamma)$ such that, if $\|F^\varepsilon\|_{C, \alpha, \beta, \kappa}, \|H^\varepsilon_{s;n}\|_{C, \alpha, \beta, \kappa} \leq \lambda$ for $C < D^{-1}/2$, then (77) has a unique global solution $(\Psi^{L, \varepsilon}, R^{L, \varepsilon})$ satisfying

$$\|(\Psi^{L, \varepsilon}, R^{L, \varepsilon})\|_{D, \gamma + \theta, d - 2\gamma - \alpha + \beta} \leq 1. \quad (78)$$

The following lemma will be used in the proof of Proposition 71.

**Lemma 72.** Let $n, n' \in \mathbb{N}$, let $F = (F^\varepsilon_{[t]}(k))_{t \geq 0, k \geq n'}$ be a graded sequence and let $\psi_t, \varphi_t \in C^2_t$ be Grassmann fields such that $\|\psi_t\|_{C^2_t}, \|\varphi_t\|_{C^2_t} \leq D 2^{kt}$ for some $D, \rho > 0$. Then, provided that $\beta > \rho$ and $C < D^{-1}/2$ the following bounds hold true:

$$\|F_t(\psi_t)\|_{L^\infty} \leq \|F\|_{C, \alpha, \beta, \kappa} 2^{|n + (\rho - \beta)n|} \|\psi_t\|_{C^2_t},$$

$$\|F_t(\psi_t) - F_t(\varphi_t)\|_{L^\infty} \leq \|F\|_{C, \alpha, \beta, \kappa} 2^{|n + (\rho - \beta)n|} \|\psi_t - \varphi_t\|_{C^2_t}.$$

**Proof.** The claim is an immediate consequence of the bounds

$$\|F^\varepsilon_{[t]}(k)(\psi_t)\| \leq \|F^\varepsilon_{[t]}(k)(s, \xi)\|_{C^2_t} \leq \|F\|_{C, \alpha, \beta, \kappa} C^k D 2^{|n| + 2\kappa t} 2^{\beta t},$$

$$\|F^\varepsilon_{[t]}(k)(\psi_t) - F^\varepsilon_{[t]}(k)(\varphi_t)\| \leq \|F\|_{C, \alpha, \beta, \kappa} C^k D 2^{|n| + 2\kappa t} 2^{\beta t} \|\psi_t - \varphi_t\|_{C^2_t}. \quad \square$$

**Proof of Proposition 71.** The proof is an application of the Banach fixed-point theorem, since we can interpret (77) as a fixed-point equation $(\Psi^{L, \varepsilon}, R^{L, \varepsilon}) = \Gamma(\Psi^{L, \varepsilon}, R^{L, \varepsilon})$ where the operator $\Gamma: (\Psi^{L, \varepsilon}, R^{L, \varepsilon}) \mapsto (\hat{\Psi}^{L, \varepsilon}, \hat{R}^{L, \varepsilon})$ is defined in an obvious way.

We first prove that $\Gamma$ maps the closed ball $\|\Psi^{L, \varepsilon}, R^{L, \varepsilon}\|_{D, \gamma + \theta, d - 2\gamma - \alpha + \beta} \leq 1$ into itself. To this end, by using the assumption $\|F^\varepsilon\|_{C, \alpha, \beta, \kappa} \leq \lambda$ and that $\|\Psi^{L, \varepsilon}\|_{C^2_t} \leq D 2^{(\gamma + \theta)s}$ in the said ball, for $C$ small enough Lemma 72 gives

$$\|F^\varepsilon_s(\Psi^{L, \varepsilon})\|_{L^\infty} \leq \lambda 2^{(\alpha - \beta + \gamma + \theta)s},$$

$$\|DF^\varepsilon_s(\Psi^{L, \varepsilon})\|_{L^\infty} \leq \lambda 2^{(\alpha - \beta)s},$$

$$\|H^\varepsilon_{s;n;\ldots}(\Psi^{L, \varepsilon})\|_{L^\infty} \leq \lambda 2^{(\alpha - n\kappa)s}. \quad (79)$$
where we used that $\|DF^\varepsilon\|_{C, \alpha - \beta, \beta, \kappa} \leq \|F^\varepsilon\|_{C', \alpha, \beta, \kappa}$, for some $C' < C$, see (75). Note, in passing, that these estimates would hold true also for $\beta = \gamma$.

To control the equation for the remainder, we need to integrate (79) in $ds$, which can be done provided that $n > \alpha \kappa^{-1}$. Accordingly, we have:

$$
\| \tilde{R}^\varepsilon_t \|_{L^\infty} \lesssim \int_0^\infty \| \omega_t(H^\varepsilon_{\nu n; s}(\Psi^L_s, \varepsilon)) \|_{L^\infty} ds + \int_0^\infty \| \omega_t((\tilde{\phi}^\varepsilon_s \tilde{R}^\varepsilon_s, DF^\varepsilon_s(\Psi^L_s, \varepsilon))) \|_{L^\infty} ds
$$

$$
\lesssim \int_0^\infty \| H^\varepsilon_{\nu n; s}(\Psi^L_s, \varepsilon) \| ds + \int_0^\infty \| \tilde{\phi}^\varepsilon_s \|_{L^\infty} \| R^\varepsilon_s \|_{L^\infty} \| DF^\varepsilon_s(\Psi^L_s, \varepsilon) \|_{L^\infty} ds
$$

$$
\lesssim n \lambda (n \kappa - \alpha)^{-1} + \lambda \int_0^\infty 2^{-(d - 2 \gamma - \alpha + \beta)s} \| R^\varepsilon_s \|_{L^\infty} ds,
$$

where we used (79) and $d - 2 \gamma - \alpha + \beta > 0$. Since in the ball $\| (\Psi^L_s, R^\varepsilon_s) \|_{D, \gamma + \theta, d - 2 \gamma - \alpha + \beta} \leq 1$ we have $\int_0^\infty 2^{-(d - 2 \gamma - \alpha + \beta)s} \| R^\varepsilon_s \|_{L^\infty} ds \leq 1$, the above computations imply:

$$
\int_0^\infty 2^{-(d - 2 \gamma - \alpha + \beta)s} \| R^\varepsilon_s \|_{L^\infty} ds \lesssim n \lambda [ (d - 2 \gamma - \alpha + \beta)^{-1} + (n \kappa - \alpha)^{-1} ] \leq \frac{1}{2}
$$

(80)

for $\lambda$ small enough, depending on $n$ as well. Next, by Lemma 53 we have the estimate $\| X^L \|_{C^2} \leq 2 \gamma / 4$, for $D$ large enough, uniformly in $\varepsilon$. Therefore, we write:

$$
\| \tilde{\Psi}^L_s \|_{C^2} \leq \int_0^t \sum_{\nu, |\nu| \leq 2} \| (2^{-|\nu|} \partial^\nu \tilde{\phi}^\varepsilon_s)(F^\varepsilon_s(\Psi^L_s, \varepsilon) + R^\varepsilon_s) \|_{L^\infty} ds + \| X^L_s \|_{C^2}
$$

$$
\lesssim \int_0^t 2^{(d - 2 \gamma)s} (\| F^\varepsilon_s(\Psi^L_s, \varepsilon) \|_{L^\infty} + \| R^\varepsilon_s \|_{L^\infty}) ds + D 2^{\gamma t} / 4
$$

$$
\lesssim \lambda (d - 2 \gamma - \alpha + \beta)^{-1} 2^{(\gamma + \theta)t} + D 2^{\gamma t} / 4,
$$

where we used (79), the propagator estimates in Corollary 51, that $d - 2 \gamma - \alpha + \beta > 0$ and that $\lambda$ is small enough. Therefore, putting together (80) and (81) we have that the closed unit ball is left invariant by $\Gamma$, that is $\| (\Psi^L_s, \tilde{R}^\varepsilon_s) \|_{D, \gamma + \theta, d - 2 \gamma - \alpha + \beta} \leq 1$.

We now prove that $\Gamma$ is a contraction. To this end, assume that $(\tilde{\Psi}^L_s, \tilde{R}^\varepsilon_s)$ is another solution in the said ball and write the difference of the two solutions

$$
\Psi^L_s - \tilde{\Psi}^L_s = \int_0^t \tilde{\phi}^\varepsilon_s (F^\varepsilon_s(\Psi^L_s, \varepsilon) - F^\varepsilon_s(\tilde{\Psi}^L_s, \varepsilon) + R^\varepsilon_s - \tilde{R}^\varepsilon_s) ds,
$$

$$
R^\varepsilon_s - \tilde{R}^\varepsilon_s = \int_0^\infty \omega_t(H^\varepsilon_{\nu n; s}(\Psi^L_s, \varepsilon) - H^\varepsilon_{\nu n; s}(\tilde{\Psi}^L_s, \varepsilon)) ds
$$

$$
+ \int_0^\infty \omega_t((\tilde{\phi}^\varepsilon_s (R^\varepsilon_s, \varepsilon) - \tilde{R}^\varepsilon_s) \tilde{\phi}^\varepsilon_s (F^\varepsilon_s(\Psi^L_s, \varepsilon))) ds
$$

$$
+ \int_0^\infty \omega_t((\tilde{\phi}^\varepsilon_s \tilde{R}^\varepsilon_s, DF^\varepsilon_s(\Psi^L_s, \varepsilon) - DF^\varepsilon_s(\tilde{\Psi}^L_s, \varepsilon))) ds.
$$

By using the assumption $\| F^\varepsilon \|_{C, \alpha, \beta, \kappa} \leq \lambda$ as well as $\| \Psi^L_s \|_{C^2} \| \tilde{\Psi}^L_s \|_{C^2} \leq D 2^{(\gamma + \theta)s}$, Lemma 72 implies

$$
\| F^\varepsilon_s(\Psi^L_s, \varepsilon) - F^\varepsilon_s(\tilde{\Psi}^L_s, \varepsilon) \|_{L^\infty} \leq \lambda 2^{(\alpha - \beta)s} \| \Psi^L_s - \tilde{\Psi}^L_s \|_{C^2},
$$

$$
\| DF^\varepsilon_s(\Psi^L_s, \varepsilon) - DF^\varepsilon_s(\tilde{\Psi}^L_s, \varepsilon) \|_{L^\infty} \leq \lambda 2^{(\alpha - \beta - \gamma - \theta)s} \| \Psi^L_s - \tilde{\Psi}^L_s \|_{C^2},
$$

$$
\| H^\varepsilon_{\nu n; s}(\Psi^L_s, \varepsilon) - H^\varepsilon_{\nu n; s}(\tilde{\Psi}^L_s, \varepsilon) \|_{L^\infty} \leq \lambda 2^{(\alpha - n \kappa - \gamma - \theta)s} \| \Psi^L_s - \tilde{\Psi}^L_s \|_{C^2}.
$$

(83)
Therefore, plugging these estimates together with (79) into (82) we obtain:

\[
\|\Psi^L_\varepsilon - \tilde{\Psi}^L_\varepsilon\|_{W^{2,\infty}} \lesssim \lambda^2 (\gamma + \theta)^t (d - 2\gamma - \alpha + \beta)^{-1} \max_{s \in \mathbb{R}} \sup_{s \in \mathbb{R}} 2^{-(\gamma + \theta)s} \|\Psi^L_\varepsilon - \tilde{\Psi}^L_\varepsilon\|_{C^2}^2
+ \int_0^\infty 2^{-(d - 2\gamma)s} \|\tilde{R}^L_\varepsilon - \tilde{\tilde{R}}^L_\varepsilon\|_{L^\infty} \, ds,
\]  

(84)

and

\[
\| R^L_\varepsilon - \tilde{\tilde{R}}^L_\varepsilon \|_{L^\infty} \lesssim \lambda(n\kappa - \alpha)^{-1} \max_{s \in \mathbb{R}} \sup_{s \in \mathbb{R}} 2^{-(\gamma + \theta)s} \|\Psi^L_\varepsilon - \tilde{\Psi}^L_\varepsilon\|_{C^2}^2
+ \lambda \int_0^\infty 2^{-(d - 2\gamma - \alpha + \beta)s} \| R^L_\varepsilon - \tilde{\tilde{R}}^L_\varepsilon \|_{L^\infty} \, ds,
\]  

(85)

which for \(\lambda\) small imply the claim.

\[\square\]

**Proof of Lemma 48.** The previous proof implies that Picard’s iteration converges to the solution \(\Psi^L_\varepsilon\). Since \(X^L_\varepsilon(x + mL) = X^L_\varepsilon(x)\), see Definition 45, one can check that every \(n\)-th Picard’s iterate \(\Psi^L_\varepsilon(nL)\) is periodic. In fact, if \(\Psi^L_\varepsilon(nL)\) is periodic, so are \(F^\varepsilon(\Psi^L_\varepsilon(nL))\) and \(\tilde{\Psi}^L_\varepsilon F(\Psi^L_\varepsilon(nL))\) because of translation invariance, implying that also \(\Psi^L_\varepsilon(n + 1)\) is periodic.

\[\square\]

**Corollary 73.** With the same setting as Proposition 71, if \(f \in L^2(\mathbb{R}^d, \mathbb{C}^4) \cap W^{[\gamma],1}(\mathbb{R}^d, \mathbb{C}^4)\), then

\[
\|\Psi^L_\varepsilon(f)\| \lesssim \|\cdot\|_{b,a}
\]

uniformly in \(L \in \mathbb{N}_\infty\), \(\varepsilon \in A\) and \(t \geq 0\), where \(\|\cdot\|_{b,a} := \|\cdot\|_{L^2(\mathbb{R}^d, \mathbb{C}^4)} + \|\cdot\|_{W^{[\gamma],1}(\mathbb{R}^d, \mathbb{C}^4)}\).

**Proof.** It suffices to bound \(\|\Psi^L_\varepsilon(f)\|\) in the fixed point equation by noting that the function \(\Psi^L_\varepsilon f\) satisfies the improved estimate

\[
\|\Psi^L_\varepsilon f\| \lesssim 2^{-(d - \gamma)s} \|f\|_{W^{[\gamma],1}},
\]

and by recalling that \(\|X^L_\varepsilon(f)\| \lesssim \|f\|_{L^2}\), see (52).

\[\square\]

We shall now prove the convergence to the continuum limit, \((\Psi^L_\varepsilon, R^L_\varepsilon) \to (\Psi, R)\).

**Proposition 74.** (Continuum limit). Let \(L \in \mathbb{N}_\infty\) and \(\varepsilon > \varepsilon' \in A\). Under the same assumptions of Proposition 71, but with \(\alpha > 3\beta\), \(0 < \theta \leq \min\{\alpha - 3\beta; \beta - \gamma\}/2\), there exist \(D_\theta\) and \(\lambda = (\lambda(d, \alpha, \beta, \gamma))\) such that, if \(\|F^\varepsilon\|_{C, \alpha - \theta, \beta, \kappa}, \|F^\varepsilon\|_{C, \alpha - \theta, \beta, \kappa}, \|H^{[n]}\|_{C, \alpha - \theta, \beta, \kappa}, \|H^{[n]}\|_{C, \alpha - \theta, \beta, \kappa}, \|H^{[n]}\|_{C, \alpha - \theta, \beta, \kappa} \lesssim \lambda\) for \(C < D_\theta^{-1}/2\), then uniformly in \(L\)

\[
\| (\Psi^L_\varepsilon - \Psi^L_\varepsilon', R^L_\varepsilon - R^L_\varepsilon') \|_{D, \gamma + \theta, d - 2\gamma - \alpha + \beta} \lesssim \varepsilon^\theta.
\]

**Proof.** The equation for the difference is similar to Eq. (82). By following the analysis after (82) and by using Corollary 51, Lemma 53, and Proposition 70 to control the differences \(\|\delta_{\varepsilon, \varepsilon'} \Psi^L_\varepsilon\|_{L^\infty} \lesssim 2^{-(d/2 - 2\gamma - \theta)s} \varepsilon^\theta\), \(\|\delta_{\varepsilon, \varepsilon'} X^L_\varepsilon\|_{C^1} \lesssim 2^{(\gamma + \theta)s} \varepsilon^\theta\) and \(\|\delta_{\varepsilon, \varepsilon'} F\|_{C, \alpha - \theta, \beta, \kappa} \lesssim \varepsilon^\theta\), the claim follows.

\[\square\]

To prove convergence to the infinite-volume-limit \((\Psi^L_\varepsilon, R^L_\varepsilon) \to (\Psi, R)\), we introduce

\[
\|\cdot\|_{L^\infty_{\infty}} := \|\cdot\|_{C^1_{\infty}} + \|\cdot\|_{C^2_{\infty}}d_{\infty} dt,
\]

where \(\|\cdot\|_{L^\infty_{\infty}} := \|\cdot\|_{C^1_{\infty}}\) and \(\|\cdot\|_{C^2_{\infty}}\) were introduced in eq. (42) and below.
Proposition 75. (Infinite-volume limit). Let $L \in \mathbb{N}_\infty$ and $\varepsilon \in \mathcal{A}$. Under the same assumptions of Proposition 71 and for any $\eta > 0$, there exist constants $D_\eta$ and $\lambda = \lambda(d, \alpha, \beta, \gamma)$ such that, if $\| F \|_{C, \alpha, \beta, \gamma} \| H^\varepsilon \|_{C, \alpha, \beta, \gamma} \leq \lambda$, for $C < D_\eta^{-1}/2$, then uniformly in $\varepsilon$ 

$$
\| (\Psi^\varepsilon - \Psi_{L, \varepsilon}^\varepsilon, R^\varepsilon - R_{L, \varepsilon}^\varepsilon) \|_{D, \gamma + \theta, d - 2\gamma - \alpha + \beta, \eta} \leq L^{-\eta}.
$$

We skip the proof of this proposition because the analysis of the next section carries out a similar argument in a more complex setting. See also [ABDG20] for similar arguments in the context of the infinite volume limit of Grassmann SPDEs.

Remark 76. As a consequence of Corollary 73 and the convergence results of Proposition 74 and Proposition 75, under the same hypotheses and notations of Corollary 73, we get

$$
\sup_{t \in \mathbb{R}^+} \| \Psi_t(f) \| \leq \| f \|_{h, a},
$$

$$
\sup_{t \in \mathbb{R}^+} \| \Psi_t(f) - \Psi_t^L(f) \| \leq (\varepsilon^\theta + L^{-\eta}) \| f \|_{h, a}.
$$

4.4 Exponential decay of correlations: proof of Theorem 3

In this section, we prove the exponential decay of correlation functions via a coupling method. We learned this coupling method from the work [Fun91] and more recently explored in [GHR22] in the context of stochastic quantisation. For the sake of simplicity, we shall restrict ourselves to the two-cluster correlation function. The basic idea is to approximate the solution of the FBSDE in suitably disjoint spatial domains by two independent Grassmann processes that solve some other FBSDEs with uncorrelated GBMs as sources. Because the latter are related to a massive operator, one expects that the approximation can be made “exponentially accurate”, depending on the distance of the regions. This technique can be viewed as a sensible replacement of the well-known cluster expansions [Bry84, GJ87] of statistical mechanics, applied in the context of stochastic quantisation in [Dim90].

Let us discuss the coupling method in more detail. Consider $((f_{i,k})_{k=1}^{m_i})_{i=1}^2 \subset f \in L^2(\mathbb{R}^d, \mathcal{C}^d) \cap W^{1,1}(\mathbb{R}^d, \mathcal{C}^d)$ compactly supported, define $D_1 := \bigcup_{i=1}^{m_1} \text{supp}(f_{i,k})$, and assume that $D_1 \cap D_2 = \emptyset$. We let $D_i \supset D_1$ be such that $D_1 \cap D_2 = \emptyset$ and such that $\text{dist}(\partial D_i, D_j)$ is as large as possible. For example one can take (open) half spaces such that $D_1 \cup D_2 = \mathbb{R}^d$. For the sake of brevity, we will here consider the case $L = \infty$ and $\varepsilon = 0$ directly. Recall that $\dot{G}_t = \mathcal{C}^2 U$, with $[\mathcal{C}, U] = [\mathcal{C}, \Theta] = 0$.

Definition 77. We let $(X^{i(j)}_t)_{i=0,1,2}$ be the family of anti-commuting norm-compatible GBMs such that, setting $D_0 := \mathbb{R}^d$,

$$
\omega(X^{i(j)}_t(f), X^{i(j)}_t(g)) = \left\langle \Theta f, \int_0^t \mathcal{C}_t f \mathcal{D}_t \mathcal{C}_t U g ds \right\rangle_h \quad \forall f, g \in \mathfrak{h}.
$$

As anticipated, the process $X^{(0)}_t := X_t$ is the noise of the FBSDEs considered so far, whereas $X^{(i)}_t, i = 1, 2$, are independent local approximations of the latter in the region $D_i$. The independence is read out from (86), which, by the choice of $D_i$ implies $\omega(X^{(1)}_t(f), X^{(2)}_t(g)) = 0$, for any $f, g$. The local approximation is quantified in the following weighted topology

$$
\| \psi \|_{\mathcal{C}^2(D, D', \varepsilon)} := \sum_{n:|n| \leq n} 2^{-nt} \text{ess sup}_{x \in D} \varepsilon^{\text{dist}(x, D')} \| \partial^n \psi(x) \|,
$$

(87)
where \( \psi \) is a lattice field, \( \xi \geq 0 \) and \( D, D' \subset \mathbb{R}^d \) are measurable domains.

**Lemma 78.** Let \((X_t^{(i)})_i\) and \(D_i\) be as above. Then, for any \( t \geq 0 \) and for \( \xi \) small enough

\[
\|X_t - X_t^{(i)}\|_{c_{n}^{\gamma}(D, D', \xi)} \lesssim \text{dist}(D, D'_i)^{-\gamma}.
\]

**Proof.** We note that the derivative of the covariance of \( X_t - X_t^{(i)} \) is \( \mathfrak{C}_t 1_{D'_i} \mathfrak{C}_t U \) and hence, by the norm-compatibility, we can write

\[
\left\| \left( \partial^\nu X_t - \partial^\nu X_t^{(i)} \right)(x) \right\|^2 \lesssim \int_{\mathbb{R}^d} \int_0^t |\partial^\nu_x \mathfrak{C}_s(z - x)|^2 1_{D'_i}(z) dz ds,
\]

where \( \lesssim \) is up to some universal constant. Therefore, by using the estimate \( |\partial^\nu_x \mathfrak{C}_s(x - y)| \lesssim 2(\gamma + d/2 + |\nu|) e^{-\xi^2|x - y|} \), see Remark 52, we have

\[
\sum_{\nu : |\nu| \leq n} 2^{-|\nu|} t \sup_{x \in D_i} \left( \int_{\mathbb{R}^d} \int_0^t |\partial^\nu_x \mathfrak{C}_s(z - x)|^2 1_{D'_i}(z) dz ds \right)^{1/2}
\]

\[
\lesssim \sum_{\nu : |\nu| \leq n} 2^{-|\nu|} t \sup_{x \in D_i} \left[ \int_{\mathbb{R}^d} \int_0^t 2(\gamma + d + |\nu|) e^{-\xi^2|x - y|} 1_{D'_i}(z) dz ds \right]^1 \lesssim \text{dist}(D, D'_i)^{-\gamma},
\]

provided that \( \xi \leq \tilde{c}/3 \). \( \square \)

Let \((\Psi_t^{(i)})_{i=1,2,3}\) be the Grassmann processes that solve the FBSDEs with \((X_t^{(i)})_{i=1,2,3}\) as sources. More precisely, one should think of the pairs \((\Psi_t^{(i)}, R_t^{(i)})\), see Lemma 48, which solve a system of equations that are always well-defined, unlike the FBSDE.

Clearly, also \((\Psi_t^{(i)})_{i=1}^3\) are independent and one should think of \(\Psi_t^{(i)}\) as approximating \(\Psi := \Psi^{(0)}\) in the regions \(D_i\) respectively. This allows for the representation.

**Lemma 79.** Let \( F, G \in \bigoplus_{j=0}^n (L^2(\mathbb{R}^d; \mathbb{C}^4) \cap W^{[\gamma],1}(\mathbb{R}^d; \mathbb{C}^4))^{\wedge j}\), for some \( n \in \mathbb{N} \), be observables and let \( \Psi, \Psi^{(i)} \) be as above. Then

\[
\text{Cov}_\omega(F(X); G(X)) = \text{Cov}_\omega(F(\Psi) - F(\Psi^{(1)}); G(\Psi)) + \text{Cov}_\omega(F(S^{(1)}); G(\Psi) - G(S^{(2)})).
\]

**Proof.** By the stochastic quantisation formula in Theorem 36 and by Remark 76

\[
\text{Cov}_\omega(F(X); G(X)) = \lim_{L \to \infty} \lim_{\varepsilon \to 0} \text{Cov}_\omega(F(X^{L,\varepsilon}); G(X^{L,\varepsilon}))
\]

\[
= \lim_{L \to \infty} \lim_{\varepsilon \to 0} \text{Cov}_\omega(F(\Psi^{L,\varepsilon}); G(\Psi^{L,\varepsilon}))
\]

\[
= \text{Cov}_\omega(F(\Psi); G(\Psi)).
\]
Notice in particular that $F(X^{L,\varepsilon})$ and $G(X^{L,\varepsilon})$ are elements of a finite-dimensional Grassmann algebra. Since $\Psi^{(1)}$ and $\Psi^{(2)}$ are independent, $\text{Cov}_\omega(F(\Psi^{(1)});G(\Psi^{(2)})) = 0$ and the claim follows by simple manipulations.

The lemma reduces the problem to controlling, e.g., $|\text{Cov}_\omega(F(\Psi) - F(\Psi^{(1)});G(\Psi))|$. In the proof of Theorem 3 we also need the following lemma, which is a generalisation of Lemma 72.

**Lemma 80.** In the same setting of Lemma 72 the following bound holds true:

$$||F_1(\psi_1) - F_1(\varphi_1)||_{L^\infty(D,D',\xi)} \lesssim ||F||_{C,\alpha,\beta,\kappa} 2^{[\alpha+\eta+(\rho-\beta)-\kappa]t} ||\psi_1 - \varphi_1||_{C_2^2(D,D',\xi)}.$$

**Proof.** The claim follows from

$$||F_1^{[l(k)]}(\psi_1) - F_1^{[l(k)]}(\varphi_1)||_{L^\infty(D,D',\xi)} \lesssim k||F||_{C,\alpha,\beta,\kappa} C D t^2 2^{[\alpha+\eta+(\rho-\beta)-\kappa]t} ||\psi_1 - \varphi_1||_{C_2^2(D,D',\xi)},$$

compare with the proof of Lemma 72, where we used the triangle inequality to distribute the weight $e^{\xi \text{dist}(y,D')}$, for $\xi$ small enough, on the Steiner tree for any $j = 1, \ldots, k$:

$$e^{\xi \text{dist}(y,D')} F_1^{[l(k)]}(y,\mu; (x_1, \mu_1) \ldots; (x_k, \mu_k)) \leq \left| F_1^{[l(k)]}(y,\mu; (x_1, \mu_1) \ldots; (x_k, \mu_k)) \right| w_{i,j}^{(k+1)} (y;x_1; \ldots; x_k) e^{\xi \text{dist}(x_i,D')}.$$

**Proof of Theorem 3.** Recall the notation $||f||_{b,\gamma} := ||f||_{L^2(R^d;C^\gamma)} + ||f||_{W^{[\gamma],1}(R^d;C^\gamma)}$. $X := X_\infty$ and $\Psi := \Psi_\infty$. By Corollary 73, we have $||\Psi(f^{(i,k)})|| \lesssim ||f^{(i,k)}||_{b,\gamma}$ and $\nabla \Psi^{(i)}(f^{(j,k)}) \lesssim ||f^{(j,k)}||_{b,\gamma}$. Then, by Lemma 79 we can write

$$\left| \text{Cov}_{\omega,\gamma} \left( \prod_{k=1}^{m_1} X(f^{(1,k)}); \prod_{k=1}^{m_2} X(f^{(2,k)}) \right) \right| \lesssim \sum_{i=1,2,j=1,2} \sum_{k=1}^{m_i} ||f^{(j,k')}||_{b,\gamma} \left( \prod_{\ell \neq k}^{m_i} ||(\Psi - \Psi^{(i)})(f^{(i,k)})|| \prod_{\ell \neq k}^{m_i} ||f^{(i,\ell)}||_{b,\gamma} \right)$$

(89)

If $\text{dist}(D_1, D_2) \leq 1$, we write $||\Psi - \Psi^{(i)})(f^{(i,k)})|| \leq ||\Psi(f^{(i,k)})|| + ||\Psi^{(i)}(f^{(i,k)})|| \lesssim ||f^{(i,k)}||_{b,\gamma}$. Otherwise, we will prove that $||\Psi - \Psi^{(i)})(f^{(i,k)})||$ decays exponentially in $\text{dist}(D_1, D_i)$ when $\text{dist}(D_1, D_i) > 0$, and thus in $\text{dist}(D_1, D_2)$, since we can choose $D_1$ so that $\min_i \text{dist}(D_i, D_i)$ is as large as possible. We begin by writing

$$||\Psi - \Psi^{(i)})(f^{(i,k)})|| \lesssim ||f^{(i,k)}||_{b,\gamma} e^{-\xi \text{dist}(D_1, D^\alpha)} ||\Psi - \Psi^{(i)}||_{L^\infty(D,\xi)}$$

(90)

where the weighted $L^\infty(D, \xi)$ norm was defined in (87) and where we used that $||f^{(i,k)}||_{L^1} \lesssim ||f^{(i,k)}||_{b,\gamma}$. By Lemma 72, we obtain the following bounds, compare with (79):

$$||F_1(\Psi_s) - F_1(\Psi^{(i)})||_{L^\infty(D_i,\xi)} \lesssim \lambda 2^{[\alpha-\beta]s} ||\Psi_s - \Psi^{(i)}||_{C_2^2(D_i,\xi)},$$

$$||D F_1(\Psi_s) - D F_1(\Psi^{(i)})||_{L^\infty(D_i,\xi)} \lesssim \lambda 2^{[\alpha-\beta-\gamma]s} ||\Psi_s - \Psi^{(i)}||_{C_2^2(D_i,\xi)},$$

$$||H_{\geq n,\tau} (\Psi_s) - H_{\geq n,\tau} (\Psi^{(i)})||_{L^\infty(D_i,\xi)} \lesssim \lambda 2^{[\alpha-\eta-\gamma]s} ||\Psi_s - \Psi^{(i)}||_{C_2^2(D_i,\xi)},$$

where $\lambda := ||F_1(\Psi_s) - F_1(\Psi^{(i)})||_{L^\infty(D_i,\xi)}$. Compare with (79).
We plug these bounds in the equations for $\Psi_t - \Psi_t^{(i)}$ and $R_t - R_t^{(i)}$, see Lemma 48 to obtain

$$\|\Psi_t - \Psi_t^{(i)}\|_{c_s^2(D_t, D'_t, \xi)} \leq \int_0^t 2^{-(d-2\gamma)s} \left[ \left\| F_s(\Psi_s) - F_s(\Psi_s^{(i)}) \right\|_{L^\infty(D_t, D'_t, \xi)} + \left\| R_s - R_s^{(i)} \right\|_{L^\infty(D_t, D'_t, \xi)} \right] ds$$

$$+ \left\| X_t - X_t^{(i)} \right\|_{c_s^2(D_t, D'_t, \xi)} \lesssim \lambda(d - 2\gamma - \alpha + \beta)^{-1} \sup_{s \in \mathbb{R}_+} \left\| \Psi_s - \Psi_s^{(i)} \right\|_{c_s^2(D_t, D'_t, \xi)}$$

$$+ \int_0^\infty 2^{-(d-2\gamma)s} \left\| R_s - R_s^{(i)} \right\|_{L^\infty(D_t, D'_t, \xi)} ds + \text{dist}(D_t, D'_t)^{-\gamma},$$

and

$$\|R_t - R_t^{(i)}\|_{L^\infty(D_t, D'_t, \xi)} \lesssim \int_0^\infty \left\| H_{n;s}(\Psi_s) - H_{n;s}(\Psi_s^{(i)}) \right\|_{L^\infty(D_t, D'_t, \xi)} ds$$

$$+ \int_0^\infty 2^{-(d-2\gamma)s} \left\| R_s - R_s^{(i)} \right\|_{L^\infty(D_t, D'_t, \xi)} ds$$

$$+ \int_0^\infty 2^{-(d-2\gamma)s} \left\| R_s^{(i)} \right\|_{L^\infty(D_t, D'_t, \xi)} ds$$

$$\lesssim \lambda(n\kappa + \gamma + \theta - \alpha)^{-1} \sup_{s \in \mathbb{R}_+} \left\| \Psi_s - \Psi_s^{(i)} \right\|_{c_s^2(D_t, D'_t, \xi)}$$

$$+ \lambda \int_0^\infty 2^{-(d-2\gamma - \alpha + \beta)s} \left\| R_s - R_s^{(i)} \right\|_{L^\infty(D_t, D'_t, \xi)} ds,$$

which for $\lambda$ small enough imply $\sup_{t \in \mathbb{R}_+} \left\| \Psi_t - \Psi_t^{(i)} \right\|_{c_s^2(D_t, D'_t, \xi)} \lesssim \text{dist}(D_t, D'_t)^{-\gamma}$. Plugging this bound into (90) and choosing $D_t$ so that $\min_i \text{dist}(D'_t, D_i)$ is as large as possible, gives the claim.

**Remark 81.** It is easy to see that the proof extends trivially to the case of $n$-cluster correlation functions. In this case, one finds independent solutions $(\Psi^{(i)})_{i=1}^n$ in each of the clusters, and hence, the exponential clustering follows by simply controlling $\left\| (\Psi - \Psi^{(i)})(f^{(i,k)}) \right\|$ precisely as was done above.

## Appendix A  Proof of Theorem 24

In this section we prove the existence of GBM as described in Theorem 24 and in Definition 42. More precisely, we prove the existence of a NPS where our GBMs are realised as suitable operators acting on a Hilbert space. The construction of Grassmann fields as operators acting on a Hilbert space was already obtained in the early days of Euclidean quantum field theory [OS72, OS73, Ost73], see also [GJ87]. More recently, this algebraic approach has been further developed in [ABDG20] with applications to Grassmann SDE. In addition to what was already done there, here we construct a conditional expectation and discuss GBMs in the framework of FNPST.

The proof of Theorem 24 is by construction and divided into three steps. First of all, we associate a FNPS with the pair $(\mathcal{H}, (P_t)_{t \geq 0})$ consisting of a Hilbert space $\mathcal{H}$, which in our case will be chosen to be $L^2(\mathbb{R}_+; \mathfrak{H})$, and a net of projections $(P_t)_{t \geq 0}$. This association will be called Fock construction for obvious reasons. Then, we construct a conditional expectation associated with the Fock construction. Finally, we provide an explicit representation of the GBM by using the Osterwalder-Schrader construction [OS72, OS73, Ost73].
Proof of Theorem 24

**Definition 82.** (Fock construction). Let $\mathcal{H}$ be a separable Hilbert space and $(P_t)_{t \geq 0}$ an increasing net of projections on it, that is,

\[ P_t P_s = P_{t \wedge s} = P_s P_t. \]  

(91)

The Fock construction associated with the pair $(\mathcal{H}, (P_t)_{t \geq 0})$ is the FNPS $(\mathcal{M}, \omega, (\mathcal{M}_t)_{t \geq 0})$, where $\omega(\cdot) := \langle \Omega, \cdot \Omega \rangle$ with $\Omega$ being the Fock vacuum on the anti-commuting Fock space $\Gamma_0(\mathcal{H})$, and where

\[ \mathcal{M} := \text{CAR}(\mathcal{H}), \quad \mathcal{M}_t := \text{CAR}(P_t \mathcal{H}) \subseteq \mathcal{M}. \]

**Remark 83.** Verifying that the triple $(\mathcal{M}, \omega, (\mathcal{M}_t)_{t \geq 0})$ is a FNPS is a simple check, see the definitions in Section 2.

Although the Fock construction is very simple, there is no general theory of conditional expectations, because the vector $\Omega$, does not have the modular property [Tom67, Tak70, Tak72]. If this were the case, on the other hand, the existence of a conditional expectation of $\mathcal{M}$ onto $\mathcal{M}_t$ would be a consequence of a theorem by Tomiyama [Tom57]. Nonetheless, a conditional expectation can be explicitly defined by means of the fermionic exponential law [Der06]. Set $\mathcal{H}_t := P_t \mathcal{H}$, and write $\mathcal{H} = \mathcal{H}_t \oplus \mathcal{H}_{>t}$, where clearly $\mathcal{H}_{>t} = (1 - P_t) \mathcal{H}$. The exponential law for fermions is the unitary $U_t: \Gamma_0(\mathcal{H}_t \oplus \mathcal{H}_{>t}) \to \Gamma_0(\mathcal{H}_t) \otimes \Gamma_0(\mathcal{H}_{>t})$, such that

\[ U_t a(f) U_t^* = a(P_t f) \otimes 1 + \Xi \otimes a((1 - P_t) f), \quad U_t \Omega = \Omega_t \otimes \Omega_{>t}, \]  

(92)

with $\Xi := \Gamma(-1) = (-)^N$, $N$ being the number operator\(^6\). The exponential law gives another representation of the CAR algebra we consider, as a $\ast$-sub-algebra of the algebra of bounded operators on $\Gamma_0(\mathcal{H}_t) \otimes \Gamma_0(\mathcal{H}_{>t})$. On the tensor product $\mathcal{B}(\Gamma_0(\mathcal{H}_t)) \otimes \mathcal{B}(\Gamma_0(\mathcal{H}_{>t}))$, we introduce the projection map $\tilde{\Omega}_{>t}: \mathcal{B}(\Gamma_0(\mathcal{H}_t)) \otimes \mathcal{B}(\Gamma_0(\mathcal{H}_{>t})) \to \mathcal{B}(\Gamma_0(\mathcal{H}_t)) \otimes \mathcal{B}(\Gamma_0(\mathcal{H}_{>t}))$:

\[ \tilde{\Omega}_{>t}(A \otimes B) := A \otimes 1 \langle \Omega_{>t}, B \Omega_{>t} \rangle. \]

The following properties hold true.

**Lemma 84.** The map $\tilde{\Omega}_{>t}$ is positive and continuous in the operator norm.

**Proof.** Consider a generic element $M = \sum_j A_j \otimes B_j \in \mathcal{B}(\Gamma_0(\mathcal{H}_t)) \otimes \mathcal{B}(\Gamma_0(\mathcal{H}_{>t}))$, the sum being finite. Set $\tilde{A}_j := A_j(\Omega_{>t}, B_j \Omega_{>t})$ and $\tilde{A} := \sum_j \tilde{A}_j$. To prove positivity, we note that an operator of the form $A \otimes 1$ is positive provided that $A$ is positive. We have

\[ \langle \psi, A \psi \rangle_{\Gamma_0(\mathcal{H}_t)} = \sum_j \langle \psi, A_j \psi \rangle_{\Gamma_0(\mathcal{H}_t)}(\Omega_{>t}, B_j \Omega_{>t})_{\Gamma_0(\mathcal{H}_{>t})} \]

\[ = \langle \psi \otimes \Omega_{>t}, M \psi \otimes \Omega_{>t} \rangle_{\Gamma_0(\mathcal{H}_t) \otimes \Gamma_0(\mathcal{H}_{>t})} \geq 0. \]

To prove continuity, we note that $\| A \otimes 1 \|_{\mathcal{B}(\Gamma_0(\mathcal{H}_t))} \otimes \Gamma_0(\mathcal{H}_{>t})) \| \leq \| A \|_{\mathcal{B}(\Gamma_0(\mathcal{H}_t))}$, hence we have the bound:

\[ \| \tilde{\Omega}_{>t}(M) \|_{\mathcal{B}(\Gamma_0(\mathcal{H}_t) \otimes \Gamma_0(\mathcal{H}_{>t}))} = \| A \otimes 1 \|_{\mathcal{B}(\Gamma_0(\mathcal{H}_t)) \otimes \Gamma_0(\mathcal{H}_{>t}))} \]

\[ \leq \| A \|_{\mathcal{B}(\Gamma_0(\mathcal{H}_t))} = \sup_{\| \psi \|_{\Gamma_0(\mathcal{H}_t)} = 1} \left| \sum_j \langle \psi, A_j \psi \rangle(\Omega_{>t}, B_j \Omega_{>t}) \right| \leq \| M \|_{\mathcal{B}(\Gamma_0(\mathcal{H}_t) \otimes \Gamma_0(\mathcal{H}_{>t}))}. \]

As a consequence of Lemma 84, we can extend $\tilde{\Omega}_{>t}$ to the whole CAR algebra of $\Gamma_0(\mathcal{H}_t) \otimes \Gamma_0(\mathcal{H}_{>t})$. We finally define the conditional expectation on $\mathcal{M}_t$.

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\(^6\) Note that $\Xi$ is sometimes called parity operator.
Definition 85. (Conditional Expectation). Let \((\Omega, \mathcal{M}, (\mathcal{M}_t)_{t \geq 0})\) be the Fock construction associated with \((\mathcal{H}, (P_t)_{t \geq 0})\). Let \(U_t\) be the fermionic exponential law and \(\Omega_{>t}\) be the (extension of the) projection defined above. We define the map \(\omega_t: \mathcal{M} \to \mathcal{M}_t\) by:
\[
\omega_t(\cdot) := U_t^* \hat{\Omega}_{>t}(U_t \cdot U_t^*) U_t.
\]

Proposition 86. The map \(\omega_t: \mathcal{M} \to \mathcal{M}_t\) defined above is a conditional expectation.

Proof. The property \(\omega_t(1_M) = 1_{\mathcal{M}_t}\) is a straightforward consequence of the definition. The positivity of \(\omega_t(\cdot)\) follows because unitary conjugation preserves positivity and because the map \(\hat{\Omega}_{>t}\) is positive, see Lemma 84. We are left with checking the property
\[
\omega_t(N_1 MN_2) = N_1 \omega_t(M) N_2,
\]
for any \(M \in \mathcal{M}\) and \(N_1, N_2 \in \mathcal{M}_t\). To this end, it suffices to note that operators such as \(N_1\) and \(N_2\) are mapped to operators of the form \(U_t N_1 U_t^* = \hat{N}_1 \otimes 1\), for some \(\hat{N}_1 \in \mathcal{B}(\Gamma_n(\mathcal{H}_t))\). Therefore, if \(U_t MU_t^* = \sum_j A_j \otimes B_j\), the sum being finite, we have
\[
\omega_t(N_1 MN_2) = U_t^* \hat{\Omega}_{>t}(U_t N_1 MN_2 U_t^*) U_t = U_t^* (\hat{N}_1 \otimes 1) \sum_j (A_j (\hat{\Omega}_{>t} B_j \hat{\Omega}_{>t}) \otimes 1)(\hat{N}_2 \otimes 1) U_t = N_1 \omega_t(M) N_2.
\]
By the continuity of \(\omega_t(\cdot)\) this extends to any \(M \in \mathcal{M}\).

Before providing the construction of the GBM, let us set some notation. If the GBM is indexed in the Hilbert space \(\mathfrak{h}\), with conjugation \(\Theta\), we let \(\mathcal{H} := L^2(\mathbb{R}, \mathfrak{h})\) and \(P_t := P_t \otimes 1\) be an increasing net of projections on it, \(P_t\) be the projection of \(L^2(\mathbb{R})\) onto \(L^2([0, t])\).

Definition 87. Let \(\mathfrak{h}\) be a Hilbert space with conjugation \(\Theta\), let \((\mathcal{H}, (P_t)_{t \geq 0})\) be as above and \((\mathcal{M}, \omega, (\mathcal{M}_t)_{t \geq 0})\) be its Fock construction. Let \((G_t)_{t \in \mathbb{B}(\mathfrak{h})}\) with \(G_0 = 0\), be \(\Theta\)-anti-symmetric and differentiable in \(t\) and write \(\hat{G}_t = C_t^2 U_t\), for some positive \(C_t\) and unitary \(U_t\). On \(\mathcal{M}\) define the process
\[
X_t(f) := a^* \left( \int_0^t \delta_s \otimes C_s U_s f ds \right) + a \left( \int_0^t \delta_s \otimes C_s \Theta f ds \right),
\]
indexed in \(\mathfrak{h}\).

Remark 88. Note that the functions \(g_t := \int_0^t \delta_s \otimes C_s U_s f ds\) and \(\hat{g}_t := \int_0^t \delta_s \otimes C_s \Theta f ds\) are in \(\mathcal{H}\), provided that \(f \in \mathfrak{h}\). In fact, \(\|g_t\|^2 = \int_0^t \|C_s U_s f\|^2 ds\) and likewise for \(\hat{g}_t\).

The following proposition completes the proof of Theorem 24.

Proposition 89. The process \((X_t)_{t \geq 0}\) defined in (93) is a norm-compatible GBM with covariance \(G_t\).

Proof. That \((X_t)_{t \geq 0}\) is adapted follows by direct inspection. We prove that it is a martingale, that is, \(\omega_s(X_t(f)) = X_s(f)\) for any \(0 \leq s \leq t\). For the sake of brevity, we let \(g_{r,t} := \int_r^t \delta_s \otimes C_s U_s f ds\) and \(\hat{g}_{r,t} := \int_r^t \delta_s \otimes C_s \Theta f ds\). We let \(\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_{>s}\) and let \(U_s\) be the corresponding fermionic exponential law. Note that \(P_s g_{0,t} = g_{0,s}\) and \(P_s \hat{g}_{0,t} = \hat{g}_{0,s}\) We have:
\[
U_s X_t(f) U_s^* = a^*(g_{0,s}) \otimes 1 + \Xi \otimes a^*(g_{s,t}) + a(\hat{g}_{0,s}) \otimes 1 + \Xi \otimes a(\hat{g}_{s,t}) .
\]
Therefore
\[ \omega_s(X_t(f)) = U_s^* \bar{\Omega}^s (U_s X_s(f) U_s^*) U_s = U_s^* (a^*(g_{0,s}) \otimes 1 + a(g_{0,s} \otimes 1)) U_s = X_s(f). \]

Next, we prove that it is a centred Grassmann Gaussian process. We begin by checking the anti-commutation relations:
\[
\{ X_t(f), X_{t'}(f') \} = \left\langle \int_0^t \delta_s \otimes C_s \Theta f \, ds, \int_0^{t'} \delta_r \otimes C_r U_r \, f' \, dr \right\rangle_H
  + \left\langle \int_0^{t'} \delta_s \otimes C_s \Theta f' \, ds, \int_0^t \delta_r \otimes C_r U_r \, f \, dr \right\rangle_H
  = \int_0^{t \wedge t'} ((C_s \Theta f, C_s U_s f'_b) + (C_s \Theta f', C_s U_s f)_b) \, ds
  = \int_0^{t \wedge t'} ((G_s^* f, f'_b) + (\Theta f', G_s f)_b) = 0,
\]
where in the last step we used that \( G_s \) is \( \Theta \)-antisymmetric. Then, we show that \( X_t \) is a centred Gaussian. We compute
\[
\left\langle a\left( \int_0^t \delta_r \otimes C_r U_r \, f \, dr \right), X_s(f') \right\rangle = \int_0^{t \wedge s} \langle C_r \Theta f, C_r U_r f' \rangle_b \, dr = \langle \Theta f, G_{t \wedge s} f' \rangle_b,
\]
hence, the claim:
\[
\omega(X_t(f) X_{t_1}(f_1) \cdots X_{t_n}(f_n)) = \omega\left( a\left( \int_0^t \delta_r \otimes C_r U_r \, f \, dr \right) X_{t_1}(f_1) \cdots X_{t_n}(f_n) \right)
  = \sum_{j=1}^n (-1)^j \langle \Theta f, G_{t \wedge t_1} f_j \rangle_b \omega\left( X_{t_1}(f_1) \cdots X_{t_{j-1}} f_{j-1} X_{t_j}(f_j) \cdots X_{t_n}(f_n) \right).
\]
Finally, the norm-compatibility is a direct consequence of \( \| a(f) \|, \| a^*(f) \| \leq \| f \|_b \). This concludes the proof of the proposition. \( \square \)

Note in particular that Proposition 89 implies that Definition 42 is meaningful and that we obtain \( X_t^{L, \varepsilon} \) simply by plugging \( C^L_{s, \varepsilon} \) in place of \( C_s \) in Eq. (93).

**Remark 90.** Let us fix \( b := L^2(\mathbb{R}; \mathbb{C}^n) \) and some unitary \( U \) for concreteness. In this case, we could introduce Grassmann white noise (associated with \( U \) and \( \Theta \)) as the operator-valued distribution
\[
\xi(ds, dz) := a^*(\delta_s ds \otimes U(\cdot, z) dz) + a(\delta_s ds \otimes \Theta(\cdot, z) dz),
\]
so that, if \( C_t \) commutes with \( U \) and \( \Theta \), we can write
\[
X_t(f) = \int_0^t \int_{\mathbb{R}^d} (C_s f)(z) \xi(ds, dz).
\]
This actually holds in our setting, see Section 3.3 and clarifies the meaning of Definition 42.

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