Abstract. We consider the motion of a particle on a Galton–Watson tree, when the probabilities of jumping from a vertex to any one of its neighbours is determined by a random process. Given the tree, positive weights are assigned to the edges in such a way that, viewed along any line of descent, they evolve as a random process. In order to introduce our method for proving transience or recurrence, we first suppose that the weights are i.i.d., reproving a result of Lyons & Pemantle [8]. We then extend the argument to allow a Markovian environment, and finally to a random walk on a Markovian environment that changes the environment. Our approach involves studying the typical behaviour of processes on fixed lines of descent, which we then show determines the behaviour of the process on the whole tree.

Keywords: Random walk in random environment, Galton–Watson, reinforcement.

1 Introduction

We consider the motion of a particle on a Galton–Watson tree, when the probabilities of jumping from a vertex to any one of its neighbours is determined by a random process. The model that we consider is one in which a Galton–Watson tree is sampled first, starting from a root $\rho$. Given the tree, positive weights are assigned to the
edges in such a way that, viewed along any line of descent, they evolve as a random process. In order to introduce our method for proving transience or recurrence, we first suppose that the weights are i.i.d., reproving a result of Lyons & Pemantle [8]. We then extend the argument to allow a Markovian environment, and finally to a random walk on a Markovian environment that changes the environment. Our approach involves studying the typical behaviour of processes on fixed lines of descent, which we then show determines the behaviour of the process on the whole tree. A different proof of Lyons–Pemantle theorem is contained in [10]. See also [7] for the multitype Galton-Watson case. Our proofs develop the approach in [1], combining it with suitable large deviation principles and analysis of the variational formula. The result of Section 4 should be compared to the behaviour of once-reinforced random walk. For this topic see, for example, [1], [2] or [6] and for a general survey of reinforcement see [13]. We emphasize that our model is a strong generalization of the once-reinforced walk, and it exhibits a phase transition (see Theorem 4.4) whereas the once-reinforced walk on the supercritical Galton-Watson tree is always transient (see [1] or [2]).

Let $G$ be an infinite tree with root $\rho$. If two vertices $\nu$ and $\mu$ are the endpoints of the same edge, they are said to be neighbours, and this property is denoted by $\nu \sim \mu$. The distance $|\nu - \mu|$ between any pair of vertices $\nu, \mu$, not necessarily adjacent, is the number of edges in the unique self-avoiding path connecting $\nu$ to $\mu$. We let $|\nu|$ be the distance of $\nu$ from the root $\rho$. We denote by $b(\nu)$ the number of neighbors of $\nu$ at level $|\nu| + 1$, i.e. the number of its offspring, and we use $\nu^{-1}$ to denote the parent of $\nu$. We write $\nu < \mu$ if $\nu$ is an ancestor of $\mu$.

For $\nu$ a vertex of $G$, we write

$$A_\nu = (A_{\nu 1}, A_{\nu 2}, \ldots)$$

to denote the (finite, positive) weights on the edges between $\nu$ and its offspring. The environment $\omega$ for the random walk on the tree is then defined, for any vertex $\nu$ with offspring $\nu i$, $1 \leq i \leq b(\nu)$, by the probabilities

$$\omega(\nu, \nu i) := \frac{A_{\nu i}}{1 + \sum_{1 \leq j \leq b(\nu)} A_{\nu j}}; \quad \omega(\nu, \nu^{-1}) := \frac{1}{1 + \sum_{1 \leq j \leq b(\nu)} A_{\nu j}}.$$  \hspace{1cm} (1.1)

We set $\omega(\nu, \mu) = 0$ if $\mu$ and $\nu$ are not neighbours. Given the environment $\omega$, we define the random walk $X = \{X_n, n \geq 0\}$ that starts at $\rho$ to be the Markov chain with $P_\omega(X_0 = \rho) = 1$, having transition probabilities

$$P_\omega(X_{n+1} = \mu 1 \mid X_n = \mu 0) = \omega(\mu 0, \mu 1).$$
The environment is random in two respects. First, the Galton–Watson tree \( G \) is realized; then, for each vertex \( \nu \in G \), the weights \( A_\nu \) are realized. The combined probability measure from which the environment is realized is denoted by \( \mathbb{P} \) and its expectation by \( \mathbb{E} \), and the semi-direct product \( \mathbb{P} := \mathbb{P} \times \mathbb{P}_\omega \) represents the annealed measure. The details of the probability measures used to construct the environment are given in the subsequent sections.

We use \([\nu, +\infty)\) to denote a generic infinite line of descent from \( \nu \).

2 Random walks in i.i.d. environment.

In this section, we assume that \( G \) is a Galton–Watson tree with offspring mean \( b > 1 \). Given the realization of the tree, we assume that the sets of weights \((A_\nu, \nu \in G)\) are independent, and that, for each \( \nu \), the weights \((A_{\nu i}, 1 \leq i \leq b(\nu))\) are exchangeable, with the distributions of the \( A_{\nu 1}, \nu \in G \), all identical. Under these assumptions, we prove the following theorem, first given by Lyons & Pemantle [8].

Theorem 2.1 Assume that \( G \) and the environment are distributed as above. If 
\[
\inf_{\lambda \in [0,1]} \mathbb{E}[A_{\lambda 1}] > b^{-1},
\]
then \( X \) is transient; that is, with positive probability, \( X \) does not return to the root.

Our proof relies on the Mogulskii large deviations principle.

We augment \( G \) by adjoining a parent \( \rho^{-1} \) to the root \( \rho \). We assume that \( X \) is recurrent and find a contradiction. We then consider the behaviour of the random walk \( X \) observed along any infinite line of descent \( \sigma = [\rho^{-1}, \infty) \). Such lines exist with positive probability, since \( b > 1 \). We call this restricted process \( X^{(\sigma)} \). Note that, by our assumption of recurrence, the process \( X^{(\sigma)} \) has the following transition probabilities:

\[
P_\omega[X_{n+1}^{(\sigma)} = \sigma_{r+1} | X_n^{(\sigma)} = \sigma_r] = \frac{A_{\sigma_{r+1}}}{1 + A_{\sigma_{r+1}}}; \quad P_\omega[X_{n+1}^{(\sigma)} = \sigma_{r-1} | X_n^{(\sigma)} = \sigma_r] = \frac{1}{1 + A_{\sigma_{r+1}}},
\]

where we denote the successive vertices in \( \sigma \) by \( \sigma_j, j \geq -1 \), with \( \sigma_0 := \rho \) and \( \sigma_{-1} := \rho^{-1} \).

We define \( T_{-1} \) to be the first time \( X^{(\sigma)} \) hits \( \rho^{-1} \), and \( T_n \) the first time the process hits \( \sigma_n \). Note that the \( \mathbb{P} \)-distributions of \( T_{-1} \) and \( T_n \) are not affected by the choice of \( \sigma \).
Proposition 2.2 If
\[ \limsup_{n \to \infty} \frac{1}{n} \ln P(T_1 > T_n) > -\ln b, \] (2.1)
then \( X \) is transient.

Proof. We mimic the proof in [1]. By assumption, there exists \( n^* \) such that
\[ b^{n^*} P(T_1 > T_{n^*}) > 1. \] We now construct a branching process as follows. Set
\( \tau := \inf \{ i > 0: X_i = \rho^{-1} \} \). We color green the vertices \( \nu \) at level \( n^* \) which are visited before time \( \tau \). Define
\[ S_\nu = \inf \{ n \geq 0: X_n = \nu \}. \]
Under our assumptions, \( S_\nu < \infty \) a.s. for each \( \nu \). A vertex \( \nu \) at level \( jn^* \), for some integer \( j \geq 2 \), is colored green, if its ancestor \( \mu \) at level \( (j - 1)n^* \) is green, and \( (X_j, j \geq S_\mu) \) hits \( \nu \) before it returns to \( \mu^{-1} \). The green vertices evolve as a Galton–Watson tree, with offspring mean \( b^{n^*} P(T_1 > T_{n^*}) > 1 \). Hence this random tree is supercritical, and thus the probability of there being an infinite number of green vertices is positive. But this contradicts the assumption that \( X \) is recurrent. \( \blacksquare \)

Proof of Theorem 2.1. In view of Proposition 2.2, it is enough to show that (2.1) is satisfied. We use well-known formula for the hitting probability for random walk in random environment, giving
\[ P(T_n < T_1) = \mathbb{E} \left[ \left( \sum_{r=0}^{n} \prod_{j=1}^{r} A_{\sigma_j}^{-1} \right)^{-1} \right]. \]
Denote by \( \lfloor x \rfloor \) the integer part of \( x \). Then it follows directly that
\[ \liminf_{n \to \infty} \frac{1}{n} \ln P(T_n < T_1) = \liminf_{n \to \infty} \frac{1}{n} \ln \mathbb{E} \left[ \left( \sum_{r=0}^{n} \prod_{j=1}^{r} A_{\sigma_j}^{-1} \right)^{-1} \right] \]
\[ = \liminf_{n \to \infty} \frac{1}{n} \ln \mathbb{E} \left[ \min_{r \leq n} \prod_{j=1}^{r} A_{\sigma_j} \right] \]
\[ = \liminf_{n \to \infty} \frac{1}{n} \ln \mathbb{E} \left[ e^{\min_{r \leq n} \sum_{j=1}^{r} \ln A_{\sigma_j}} \right] \]
\[ \geq \liminf_{n \to \infty} \frac{1}{n} \ln \mathbb{E} \left[ e^{\min_{t \in [0,1]} \sum_{j=1}^{\lfloor nt \rfloor} \ln A_{\sigma_j}} \right]. \]

(2.2)
Denote by $D[0,1]$ the space of functions $f: [0,1] \to \mathbb{R}$, which are right-continuous, have limits from the left and $f(0) = 0$. Endow this space with the uniform convergence topology. We write $\mathcal{AC}$ for the subspace of $D[0,1]$ consisting of the absolutely continuous functions. By the Mogulskii theorem (see [4], Theorem 5.1.2), the distribution of $\{(1/n) \sum_{j=1}^{\lfloor nt \rfloor} \ln A_{\sigma j}, \ t \in [0,1]\}$ satisfies a large deviation principle in $D[0,1]$. The rate function for this large deviation principle is

$$I(f) := \int_0^1 \sup_{\lambda} \{f'(t)\lambda - \ln E[A_{\sigma t}^\lambda]\} \, dt,$$

if $f \in \mathcal{AC}$, and $I(f) = +\infty$ if $f \notin \mathcal{AC}$. Note that $I(f)$ is a good rate function, being lower semi-continuous, and having compact $\alpha$-level sets $\{f: I(f) \leq \alpha\}$.

The function $g: \mathcal{AC} \to (-\infty, 0]$ defined by $g(f) = \min_{t \in [0,1]} f(t)$ is continuous in $\mathcal{AC}$. In order to use Varadhan’s lemma (see [4], Theorem 4.3.1), it is sufficient to prove that

$$\limsup_{n \to \infty} \frac{1}{n} \ln E\left[e^{\min_{t \in [0,1]} \sum_{j=1}^{\lfloor nt \rfloor} \ln A_{\sigma j}}\right] < \infty. \quad (2.3)$$

As $\min_{t \in [0,1]} \sum_{j=1}^{\lfloor nt \rfloor} \ln A_{\sigma j} \leq 0$, (2.3) is immediate, and we can apply Varadhan’s lemma to get

$$\lim_{n \to \infty} \frac{1}{n} \ln E\left[e^{\min_{t \in [0,1]} \sum_{j=1}^{\lfloor nt \rfloor} \ln A_{\sigma j}}\right] = \sup_{f \in \mathcal{AC}} \left\{ \min_{t \in [0,1]} f(t) - I(f) \right\}. \quad (2.4)$$

Since the function $\phi(\lambda) := \ln E[A_{\sigma t}^\lambda]$ is convex, it follows from Proposition 5.1 in the Appendix that the solution to the variational formula on the right hand side of (2.4) is given by

$$\sup_{f \in \mathcal{AC}} \left\{ \min_{t \in [0,1]} f(t) - \int_0^1 \sup_{\lambda} \{f'(u)\lambda - \ln E[A_{\sigma t}^\lambda]\} \, du \right\} = \inf_{\lambda \in [0,1]} E[A_{\sigma t}^\lambda]. \quad (2.5)$$

Combining (2.2), (2.4) and (2.5), it follows that (2.1) is satisfied, proving Theorem 2.1.

### 3 Markovian environment

We now show that the proof used in the previous section allows us to treat more general dependence between the weights, provided that we have a suitable large deviation principle.
Let $\sigma$ be an infinite line of descent $[\rho, \infty)$. In this section, we assume that there is a process $\{M_{\sigma_i}, i \geq 1\}$ in a Polish space $\Sigma$, such that the pair $\Gamma_{\sigma_i} := (A_{\sigma_i}, M_{\sigma_i})$, with $i \geq 0$, is a Markov chain on $\Sigma' = (0, \infty) \times \Sigma$, with transition kernel

$$K(x, B) := \mathbb{P}(\Gamma_{\sigma_i} \in B \mid \Gamma_{\sigma_{i-1}} = x, \mathcal{F}_{n-1}),$$

for any $B \in \mathcal{B} := \mathcal{B}(\Sigma')$; here, $\mathcal{F}_n$, $n \geq 1$, is the natural filtration of the process $\Gamma_{\sigma_i}$, $i \geq 1$.

For any vertex $\nu$, recall that the set of vertices which are descendants of $\nu$ consists of those vertices $\mu$ such that $\nu$ lies on the shortest path connecting $\mu$ to the root $\rho$. We deem $\nu$ to be its own descendant. We are motivated by examples where the process $\{A_{\sigma_i}, i \geq 0\}$ is determined as a functional of Markov processes defined on rays.

The following very simple example makes it clear that we need some assumption on $K$ in order to have a uniform behaviour, irrespective of the starting point.

**Example 3.1** Suppose that $A_{\sigma_i}$ is a Markov chain on $(0, \infty)$ defined as follows. Let $\mathcal{D}_+$ be the set of positive dyadic numbers, of the form $m/2^n$ for some $m, n \in \mathbb{N}$. For $x \in \mathcal{D}_+$, define $K(x, \{x/2\}) = 1$. If $x \in (0, \infty) \setminus \mathcal{D}_+$, set

$$K(x, B) := \frac{|B \cap (0, C)|}{C},$$

where $|\cdot|$ denotes Lebesgue measure and $C$ is a positive constant. If $A_\rho \in \mathcal{D}_+$, then the walk $X$ on the binary tree is recurrent, while, if $C$ is large enough and $A_\rho \notin \mathcal{D}_+$, then the walk is transient, by Theorem 2.1.

**Assumption 1.** There exist integers $0 < \ell \leq N$ and a constant $\kappa \geq 1$ such that, for all $x, y \in \Sigma'$ and $B \in \mathcal{B}$, we have

$$K^{(\ell)}(x, B) \leq \frac{\kappa}{N} \sum_{m=1}^{N} K^{(m)}(y, B),$$

where $K^{(\ell)}$ stands for the $\ell$-th convolution of the kernel $K$.

Note that i.i.d. $\{A_{\sigma_i}\}$ satisfy Assumption 1, and so does any finite irreducible Markov chain $(A_{\sigma_i}, M_{\sigma_i})$ for which $A_{\sigma_i}$ takes only positive values. Assumption 1 is needed in order to have a uniform large deviation principle for the empirical mean of the $A_{\sigma_i}$. Since the classical results on large deviations require the finiteness of all moments (see
condition (U), page 95 of [5]), we have to use a suitable truncation in order to apply them. However, we emphasize the fact that we do not assume that the support of the $A_{\sigma_i}$ is either compact or bounded away from zero. We also avoid making any assumptions on the moments of $A_{\sigma_i}$. Instead, setting

$$\eta_{\varepsilon,r} := 1 - \inf_{y \in \Sigma'} \mathbb{P}(\varepsilon < A_{\sigma_1} \leq r \mid \Gamma_{\sigma_0} = y),$$

we require:

**Assumption 2.** For $\eta := \liminf_{\varepsilon \downarrow 0, r \to \infty} \eta_{\varepsilon,r}$, we have $\eta < 1$.

The following example shows that, even when $A_{\sigma_i}$ itself is a Markov chain, Assumption 1 does not in general imply Assumption 2.

**Example 3.2** Suppose that $K(x, \cdot)$ is the mixture $(1 - \alpha) \text{Exp}(1) + \alpha \text{Exp}(\hat{x})$, where $\hat{x} := x \vee 1$, $0 \leq \alpha \leq 1$ and Exp($\lambda$) denotes the exponential distribution with mean $\lambda^{-1}$. Then it is easy to check that $\eta = \alpha$, and that $K^{(2)}(x, \cdot)$ has a density $k^{(2)}(x, \cdot)$ satisfying

$$(1 - \alpha e^{-1})e^{-w} \leq k^{(2)}(x, w) \leq 3e^{-w},$$

uniformly in $x$, so that Assumption 1 is satisfied with $\ell = 2$.

For all $x \in \Sigma'$ and for all $B \in \mathcal{B}(\Sigma')$, define $\Sigma'_{\varepsilon} := (\varepsilon, \infty) \times \Sigma$ and

$$\mathcal{K}_{\varepsilon}(x, B) := \frac{K(x, B \cap \Sigma'_{\varepsilon})}{K(x, (\varepsilon, \infty) \times \Sigma)},$$

note that $\mathcal{K}_{\varepsilon}$ is a probability kernel on $\Sigma'_{\varepsilon}$, and that it satisfies Assumption 1 for all $\varepsilon$ such that $\eta_{\varepsilon,\infty} < 1$. To prove the latter fact, observe that, for all Borel sets $B \in \mathcal{B}(\Sigma'_{\varepsilon})$, we have

$$\mathcal{K}_{\varepsilon}^{(0)}(x, B) \leq (1 - \eta_{\varepsilon,\infty})^{-\ell}(1 - \varepsilon - \eta_{\varepsilon,\infty}^{-\ell})^{-1} K^{(0)}(x, B) \leq \frac{\kappa}{(1 - \eta_{\varepsilon,\infty})^{\ell}N} \sum_{j=1}^{N} K^{(j)}(y, B) \leq \frac{\kappa}{(1 - \eta_{\varepsilon,\infty})^{\ell}N} \sum_{j=1}^{N} \mathcal{K}_{\varepsilon}^{(j)}(y, B).$$

For any $0 < \varepsilon < 1$, and for some $x^* \in [1, \infty) \times \Sigma$, define the measure $\beta_{\varepsilon}$ on $\Sigma'$ by

$$\beta_{\varepsilon}(\cdot) := \mathcal{K}_{\varepsilon}^{(0)}(x^*, \cdot),$$

where $\ell$ is the same as in Assumption 1. Set $\beta(\cdot) = \lim_{\varepsilon \to 0} \beta_{\varepsilon}(\cdot) = K^{(0)}(x^*, \cdot)$. 

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Proposition 3.3 Under Assumptions 1 and 2, if
\[
\liminf_{n \to \infty} \frac{1}{n} \int_{\Sigma'} \ln P(T_{-1} > T_n \mid \Gamma_0 = y) \beta(dy) > - \ln b, \tag{3.6}
\]
then \( X \) is transient.

Proof. Fix \( c > 0 \), to be chosen later. In view of (3.6), and using Jensen’s inequality, we can find \( n^* = n^*(c) \) such that
\[
b^{n^*} \int_{\Sigma'} P(T_{-1} > T_{n^*} \mid \Gamma_0 = y) \beta(dy) > 1/c. \tag{3.7}
\]
Since \( \beta = \lim_{\epsilon \to 0} \beta_\epsilon \), this implies that we can choose \( \epsilon > 0 \) small enough that
\[
b^{n^*} \int_{\Sigma'} P(T_{-1} > T_{n^*} \mid \Gamma_0 = y) \beta_\epsilon(dy) > 1/c. \tag{3.8}
\]
Under the assumption that \( X \) is recurrent, we now construct a random subtree of \( G \), consisting of green vertices, that contains a number of vertices stochastically larger than the number of vertices in a supercritical Galton-Watson tree. These green vertices are such that the random walk \( X \) visits them before it first reaches \( \rho^{-1} \). The fact that this random subtree is infinite with positive probability implies a contradiction, and hence that \( X \) is transient.

A direct calculation shows that, for any \( y \in \Sigma \) and \( 1 \leq j \leq N \),
\[
P \left( \bigcap_{l=1}^{j-1} \{ A_l \geq \epsilon \} \mid \Gamma_j \in B \right) \geq (1 - \eta_{\epsilon, \infty})^N K_{\epsilon}^{(j)}(y, B).
\]
It thus follows, from (3.4) and (3.5), that if \( U \) is uniformly distributed on \( \{1, 2, \ldots, N\} \), independently of \( \Gamma \), then, for all \( y \in \Sigma' \),
\[
P \left( \bigcap_{l=1}^{U-1} \{ A_l \geq \epsilon \} \mid \Gamma_U \in B \right) \geq \kappa^{-1} (1 - \eta_{\epsilon, \infty})^{N+\ell} K_{\epsilon}^{(\ell)}(x^*, B) =: \delta_{\epsilon, \beta}(B). \tag{3.9}
\]
This justifies the following construction. Starting at a vertex \( \nu \) that has an infinite line of descent, consider any line of descent \( \nu, \nu_1, \ldots, \nu_N \) from \( \nu \), and realize the chain \( \Gamma \) along this line of descent. Independently, realize \( U \). Then, whatever the value \( y \in \Sigma' \),
of $\Gamma_0$, by rejecting trajectories of $\Gamma$ with probabilities depending on the values $y, \Gamma_1, \ldots, \Gamma_U$, the distribution $\beta_\varepsilon$ can be obtained as the distribution of $\Gamma_U$ on an event $E_\nu$ of probability $\delta_\varepsilon$, and with $A_j \geq \varepsilon, 1 \leq j \leq U$.

For $\nu$ to be a green vertex, it has to have an infinite line of descent in the tree. It is good if $E_\nu$ occurs, and if $X(\nu, U)$ takes exactly $U$ steps to reach $\nu_U$; the latter event has probability at least $\{\varepsilon/(1 + \varepsilon)\}^N$. Let $\mu$ denote a descendant of $\nu_U$ at distance $n^*$ from it. For the vertex $\mu$ to be green, we require that $X(\nu_U, n^*)$ reaches $\mu$ before it hits $\nu_U^{-1}$, an event of probability

$$
\int_{\Sigma'} P(T_{-1} > T_{n^*} | \Gamma_\nu = y) \beta_\varepsilon(dy),
$$

and that it should have infinite line of descent, an event of conditional probability $1 - q$. Thus the expected number of green ‘offspring’ of a green vertex $\nu$ is at least

$$
\delta_\varepsilon \{\varepsilon/(1 + \varepsilon)\}^N b^n (1 - q) \int_{\Sigma'} P(T_{-1} > T_{n^*} | \Gamma_\nu = y) \beta_\varepsilon(dy).
$$

Choosing $1/c = (1 - q) \delta_\varepsilon \{\varepsilon/(1 + \varepsilon)\}^N$ in (3.8) and taking $n^* = n^*(c)$ gives a mean number of green offspring that exceeds 1. Furthermore, by construction, the distribution of the number of green offspring is the same for all green vertices. Hence the Galton–Watson tree of green vertices is supercritical.

The proofs that follow rely on large deviations results. These cannot be directly applied to $A$, so we need to consider truncations. For this reason, it is convenient to introduce the large deviations results that we shall use applied to a generic process $W := (W_i, i \geq 0)$, which, together with a process $\tilde{M}$ on $\Sigma$, makes $\tilde{\Gamma}$ defined by $\tilde{\Gamma}_i := (W_i, \tilde{M}_i)$ a Markov chain on $\mathbb{R} \times \Sigma$. Let $\tilde{K}$ denote the kernel of this process.

Define

$$\Lambda^{(\tilde{K})}(\lambda) := \limsup_{n \to \infty} \sup_{\tilde{y} \in \mathbb{R} \times \Sigma} \frac{1}{n} \ln \mathbb{E}[e^{\lambda \sum_{i=1}^n W_i | \tilde{\Gamma}_\nu = \tilde{y}}], \quad \Lambda^{(\tilde{K})}_\lambda(x) := \sup \lambda x - \Lambda^{(\tilde{K})}(\lambda),$$

and let

$$S^{(\tilde{K})}_n(t) := \frac{1}{n} \sum_{j=1}^{\lfloor nt \rfloor} W_i, \quad t \in [0, 1].$$

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Theorem 3.4  Fix $0 < C < R < \infty$, and assume that $W_i \in (C, R)$ a.s., for each $i$. If $\tilde{K}$ satisfies 1, then, for any $\Theta \in \mathcal{B}^+$, we have

$$ - \inf_{x \in \Theta} \Lambda_{\tilde{K}}^*(x) \leq \liminf_{n \to \infty} \frac{1}{n} \ln \inf_{y \in \Sigma'} \mathbb{P}(S_n^\tilde{K}(1) \in \Theta | \tilde{\Gamma}_o = y) \leq \limsup_{n \to \infty} \frac{1}{n} \ln \sup_{y \in \Sigma'} \mathbb{P}(S_n^\tilde{K}(1) \in \Theta | \tilde{\Gamma}_o = y) \leq - \inf_{x \in \Theta} \Lambda_{\tilde{K}}^*(x). $$

Proof of Theorem 3.4.  The kernel $\tilde{K}$ satisfies condition $(\hat{U})$, page 95 of [5]. Hence, the theorem is a consequence of the more general Theorem 4.1.14, page 97 of [5], combined with (4.1.24) page 100 of [5], to identify the rate function.

Denote by $\mathcal{AC}$ the space of absolutely continuous functions defined on $[0,1]$, and by $D[0,1]$ the space of functions which are right continuous and have limits from the left, endowed with the uniform convergence topology. The following result is due to Dembo & Zajic [3].

Theorem 3.5  Under the hypotheses of Theorem 3.4, the sequence $\{S_n^\tilde{K}(t), t \in [0,1]\}$ in $\mathcal{D}([0,1])$ satisfies a large deviations principle with the good, convex, rate function

$$ I_{\tilde{K}}^*(f) := \begin{cases} \int_0^1 \Lambda_{\tilde{K}}^* (\dot{f}(u)) \, du, & \text{if } f \in \mathcal{AC} \\ +\infty, & \text{otherwise.} \end{cases} $$

Proof.  In virtue of theorem 3.4, $S_n^\tilde{K}(1)$ satisfies a uniform large deviations principle. We can then use Dembo & Zajic ([3], Theorem 3a) to conclude that $\{S_n^\tilde{K}(t), t \in [0,1]\}$ satisfies an LDP with rate function $I_{\tilde{K}}^*(\cdot)$.

Note that, since $(A_{\sigma_i}, M_{\sigma_i})$ is a Markov chain in $(0, \infty) \times \Sigma$, then $(\ln A_{\sigma_i}, M_{\sigma_i})$ is a Markov chain on $\mathbb{R} \times \Sigma$. Define the kernel

$$ K_{\ln}((\tilde{u}, z), B) := K((e^\tilde{u}, z), E(B)), \quad \tilde{u} \in \mathbb{R}, \ z \in \Sigma, \ B \in \mathcal{B}, $$

where $E(B) := \{(e^\tilde{u}, z): (\tilde{u}, z) \in B\}$. Note that, if $K$ satisfies 1, then so does the kernel $K_{\ln}$.

For $R \in (0, \infty)$ and $C \in [-\infty, 0)$, define the probability kernel $Q_{C,R}$ on $(C, R] \times \Sigma$ by

$$ Q_{C,R}(\tilde{y}, (d\tilde{u}, dz)) := \frac{K_{\ln}(\tilde{y}, (d\tilde{u}, dz))}{K_{\ln}(\tilde{y}, (C, R] \times \Sigma)}, \quad (3.11) $$

and set $Q_R := Q_{-\infty, R}$.
Theorem 3.6 If \( K \) satisfies Assumption 1, then

i) If Assumption 2 holds, the condition

\[
\limsup_{\min\{-C,R\} \to \infty} \inf_{\lambda \in [0,1]} \Lambda^{(Q,C,R)}(\lambda) > -\ln b - \ln(1 - \eta) \tag{3.12}
\]

implies transience of \( X \) on \( G \).

ii) The condition

\[
\inf_{\lambda \in [0,1]} \Lambda^{(K,\ln)}(\lambda) < -\ln b \tag{3.13}
\]

implies recurrence of \( X \) on \( G \).

Remark 3.7 In the case of an i.i.d. environment, (3.12) coincides with the condition in Theorem 2.1.

Proof. We first prove that condition (3.12) implies that (3.6) holds, and hence, by Proposition 3.3, that \( X \) is transient. Observe that, for any \( r > 0 \), we have

\[
\liminf_{n \to \infty} \frac{1}{n} \ln P(T_n < T_{-1} \mid \Gamma_\varrho = y) \beta(dy) = \liminf_{n \to \infty} \frac{1}{n} \ln E \left[ \left( \sum_{l=0}^{n-1} \prod_{j=1}^{l-1} A_j^{-1} \right)^{-1} \mid \Gamma_\varrho = y \right] \beta(dy)
\]

\[
\geq \liminf_{n \to \infty} \frac{1}{n} \ln E \left[ \left( \sum_{l=0}^{n-1} \prod_{j=1}^{l-1} (A_j \wedge r)^{-1} \right)^{-1} \mid \Gamma_\varrho = y \right] \beta(dy) \tag{3.14}
\]

For \( r > 1 > \varepsilon > 0 \), let \( A_{\sigma_i}(\varepsilon, r) = (A_{\sigma_i} \vee \varepsilon) \wedge r \), and set \( C = \ln \varepsilon \) and \( R = \ln r \). Then,
writing \( \tilde{y}_j := (\tilde{u}_j, z_j) \in \mathbb{R} \times \Sigma \) for \( j \geq 1 \) and \( \tilde{y}_0 := (\ln u, z) \) for \((u, z) = y\), we have

\[
\liminf_{n \to \infty} \inf_{y \in \Sigma} \frac{1}{n} \ln \mathbb{E} \left[ e^{\min_{t \in [0, 1]} \sum_{j=1}^{\lfloor nt \rfloor} \ln(A_{j,t})} \mid \Gamma_{\bar{e}} = y \right] \\
\geq \liminf_{n \to \infty} \inf_{y \in \Sigma} \frac{1}{n} \ln \mathbb{E} \left[ e^{\min_{t \in [0, 1]} \sum_{j=1}^{\lfloor nt \rfloor} \ln(A_{j,t})} \prod_{i=1}^{\lfloor nt \rfloor} \mathbb{P}(\mathbb{N}_{\sigma_i} \geq \epsilon) \mid \Gamma_{\bar{e}} = y \right] \\
= \liminf_{n \to \infty} \inf_{y \in \Sigma} \frac{1}{n} \ln \mathbb{E} \left[ e^{\min_{t \in [0, 1]} \sum_{j=1}^{\lfloor nt \rfloor} \tilde{u}_j} \prod_{j=1}^{\lfloor nt \rfloor} K_{\text{ln}}(\tilde{y}_{j-1}, d\tilde{y}_j) \right].
\]

Using \( \inf_{y \in \Sigma} K_{\text{ln}}(y, [\varepsilon, r] \times \Sigma) \geq (1 - \eta_{\varepsilon, r}) > 0 \), which holds for \( \varepsilon \) small enough and \( r \) large enough because \( \eta < 1 \), we have

\[
\liminf_{n \to \infty} \inf_{y \in \Sigma} \frac{1}{n} \ln \mathbb{E} \left[ e^{\min_{t \in [0, 1]} \sum_{j=1}^{\lfloor nt \rfloor} \ln(A_{j,t})} \mid \Gamma_{\bar{e}} = y \right] \\
\geq \liminf_{n \to \infty} \inf_{y \in \Sigma} \frac{1}{n} \ln \left[ (1 - \eta_{\varepsilon, r})^n \mathbb{E} \left[ e^{\min_{t \in [0, 1]} \sum_{j=1}^{\lfloor nt \rfloor} \tilde{u}_j} \prod_{j=1}^{\lfloor nt \rfloor} K_{\text{ln}}(\tilde{y}_{j-1}, d\tilde{y}_j) \right] \right] \\
= \liminf_{n \to \infty} \inf_{y \in \Sigma} \frac{1}{n} \ln \left[ (1 - \eta_{\varepsilon, r})^n \mathbb{E} \left[ e^{\min_{t \in [0, 1]} \sum_{j=1}^{\lfloor nt \rfloor} \tilde{u}_j} \prod_{j=1}^{\lfloor nt \rfloor} \mathbb{P}(\mathbb{N}_{\sigma_j} \geq \epsilon) \mid \Gamma_{\bar{e}} = y \right] \right] \\
= \ln(1 - \eta_{\varepsilon, r}) + \liminf_{n \to \infty} \inf_{y \in \Sigma} \frac{1}{n} \ln \mathbb{E} \left[ e^{\min_{t \in [0, 1]} \sum_{j=1}^{\lfloor nt \rfloor} \tilde{u}_j} \prod_{j=1}^{\lfloor nt \rfloor} \mathbb{P}(\mathbb{N}_{\sigma_j} \geq \epsilon) \mid \Gamma_{\bar{e}} = y \right],
\]

where \( \mathbb{E} \) is the expectation with respect to the Markov chain \( \tilde{G} = (W, \tilde{M}) \) with probability kernel \( Q_{C,R}(\tilde{y}, d\tilde{y}) \) introduced in (3.11).

Next we prove that the kernel \( Q_{C,R} \) satisfies Assumption 1. Note that, for Borel sets \( F \subset (C, R] \) and \( E \in \Sigma \), and for any \( \bar{x}, \bar{y} \in (C, R] \times \Sigma \), we have

\[
Q^{(f)}_{C,R}(\bar{x}, F \times E) \leq (1 - \eta_{\varepsilon, r})^{-f} \cdot \sum_{j=1}^{N} K_{\text{ln}}^{(f)}(\bar{x}, F \times E) \leq (1 - \eta_{\varepsilon, r})^{-f} \sum_{j=1}^{N} K_{\text{ln}}^{(f)}(\bar{y}, F \times E) \\
\leq \frac{M}{(1 - \eta_{\varepsilon, r})^{f} N} \sum_{j=1}^{N} Q_{C,R}^{(f)}(\bar{y}, F \times E).
\]

In the last step, we have used the inequality

\[
K_{\text{ln}}^{(n)}(\bar{y}, F \times E) \leq Q_{C,R}^{(n)}(\bar{y}, F \times E),
\]

\( (3.17) \)
valid for $F \subset (C, R]$ and $n \geq 1$, which is easily proved by induction.

Combining Theorem 3.5 with Varadhan’s lemma, using the uniform large deviations stated in Theorem 3.4, we find that

$$\liminf_{n \to \infty} \inf_{y \in \Sigma'} \frac{1}{n} \ln \mathbb{E} \left[ \exp \left\{ \min_{t \in [0,1]} \sum_{j=1}^{[nt]} (\ln A_{\sigma_j} \land r) \mid \Gamma_{\theta} = y \right\} \right] \geq \ln(1 - \eta_{\varepsilon, r}) + \sup_{f \in A_C} \left\{ \min_{t \in [0,1]} f(t) - I^*_Q C, R (f) \right\},$$

and, since the function $\Lambda_{Q, C, R}$ is convex for any $C$ and $R$, Proposition 5.1 solves the variational formula on the right hand side of (3.18), giving

$$\liminf_{n \to \infty} \inf_{y \in \Sigma'} \frac{1}{n} \ln \mathbb{E} \left[ \exp \left\{ \min_{t \in [0,1]} \sum_{j=1}^{[nt]} (\ln A_{\sigma_j} \land r) \mid \Gamma_{\theta} = y \right\} \right] \geq \ln(1 - \eta_{\varepsilon, r}) + \inf_{t \in [0,1]} \Lambda_{Q, \ln \varepsilon, \ln r}(t).$$

Recalling (3.14), we thus have

$$\liminf_{n \to \infty} \int_{\Sigma'} \frac{1}{n} \ln \mathbb{P} (T_n < T_{-1} \mid \Gamma_{\theta} = y) \beta(dy) \geq \ln(1 - \eta_{\varepsilon, r}) + \inf_{t \in [0,1]} \Lambda_{Q, \ln \varepsilon, \ln r}(t),$$

for any $\varepsilon, r > 0$. By letting $\varepsilon \to 0$ and $r \to \infty$, we get

$$\liminf_{n \to \infty} \int_{\Sigma'} \frac{1}{n} \ln \mathbb{P} (T_n < T_{-1} \mid \Gamma_{\theta} = y) \beta(dy) \geq \ln(1 - \eta) + \limsup_{\min \{1/\varepsilon, r\} \to \infty} \inf_{t \in [0,1]} \Lambda_{Q, \ln \varepsilon, \ln r}(t) > -\ln b,$$

using (3.12), and i) follows from Proposition 3.3.

Next, we prove that if (3.13) holds, then the process is recurrent. In this case we just mimic the proof by Lyons & Pemantle (see [8], proof of Theorem 1.3, page 130). We include the proof for sake of completeness and clarity.

In virtue of (3.13) and the definition of $\Lambda$, we can choose $t_0 \in (0, 1]$ such that

$$\mathbb{E} \left[ \exp \left\{ t_0 \sum_{i=1}^n \ln A_i \right\} \mid \tilde{G}_0 = \tilde{y} \right] < (1/c)^n,$$

for some $c > b$, for all $\tilde{y}$ and all $n$ large enough. Because the branching number of the Galton–Watson tree is $b$, this implies that

$$\mathbb{E} \left( \sum_{\nu: |\nu| = n} \prod_{i=1}^n A_{\nu^{-i}}^{t_0} \right) \leq (b/c)^n,$$
and hence that
\[
\sum_{n \geq 1} \nu \cdot \prod_{i=1}^{n} A_{\nu-i}^{t} < \infty \quad \mathbb{P}\text{-a.s.} \quad (3.21)
\]
Furthermore, (3.20) also implies that, for all \( n \) large enough, \( \mathbb{P}(E_{n}) \leq (b/c)^{n} \), where
\[
E_{n} := \left\{ \sum_{\nu:|\nu|=n} \prod_{i=1}^{n} A_{\nu-i}^{t} \geq 1 \right\}.
\]
Thus a.s. only finitely many of the events \( E_{n} \) occur, and, on \( E_{n}^{c} \), since \( 0 \leq t_{0} \leq 1 \),
\[
\sum_{\nu:|\nu|=n} \prod_{i=1}^{n} A_{\nu-i}^{t} \geq \sum_{\nu:|\nu|=n} \prod_{i=1}^{n} A_{\nu-i}.
\]
(3.22)
It thus follows from (3.21) and (3.22) that the sum of conductances
\[
\sum_{\nu \in \mathcal{G}} \prod_{i=1}^{n} A_{\nu-i} < \infty \quad \text{a.s.,}
\]
and it is well known that the random walk is positive recurrent if and only if this sum finite, proving ii).

4 A walk that changes its environment, once.

In this section, we consider a setting in which the process \( X \) changes the environment. Fix parameters \( L, p > 0 \), and let \( (b_{\sigma}, i \geq 1) \) be a stochastic process, taking values in \([p, +\infty)\), such that the triple \((A_{\sigma}, b_{\sigma}, M_{\sigma})\) is a Markov process along rays. Recalling that
\[
S_{\nu} := \inf\{n \geq 0: X_{n} = \nu\},
\]
define
\[
G(\nu, n) := \begin{cases} A_{\nu} & \text{if } \{A_{\nu} > b_{\nu}\} \cup \{S_{\nu} > n\}; \\ L & \text{if } \{A_{\nu} \leq b_{\nu}\} \cap \{S_{\nu} \leq n\}, \end{cases}
\]
for each vertex \( \nu \) and time \( n \). If \( X_{n} = \nu \), given the environment and \( \mathcal{F}_{n} := \sigma\{X_{1}, X_{2}, \ldots, X_{n}\} \), the probability that \( X_{n+1} = \nu i \) is given by
\[
\frac{G(\nu i, n)}{1 + \sum_{j=1}^{b(\nu)} G(\nu j, n)}, \quad (4.1)
\]
so that the probability of a transition from \( \nu \) to a state \( \nu_i \), which has been visited at least once before and for which \( A_{\nu_i} \leq b_{\nu_i} \), is modified by replacing \( A_{\nu_i} \) by \( L \) in its calculation. Let

\[
D_{\sigma_i} := \begin{cases} L & \text{if } A_{\sigma_i} < b_{\sigma_i} \\ A_{\sigma_i} & \text{if } A_{\sigma_i} \geq b_{\sigma_i} \end{cases}
\]

and denote by \( K^* \) the transition kernel of the Markov chain \( \Gamma^* := (D_{\sigma_i}, b_{\sigma_i}, A_{\sigma_i}, M_{\sigma_i}) \) on \( \mathbb{R}_+ \times \Sigma^* \), where \( \Sigma^* := \mathbb{R}_2^2 \times \Sigma \) is the state space of \( (b_{\sigma_i}, A_{\sigma_i}, M_{\sigma_i}) \), and \( D_{\sigma_i} \) is singled out. As before, define

\[
\eta_{\varepsilon,r} := 1 - \inf_{y \in \mathbb{R}_+ \times \Sigma^*} \mathbb{P}(\varepsilon < A_{\sigma_1} \leq r \mid \Gamma_{\sigma_0}^* = y)
\]

and \( \eta = \lim_{\varepsilon \to 0, r \to \infty} \eta_{\varepsilon,r} \).

**Theorem 4.1** Suppose that \( K^* \) satisfies Assumption 1 and that Assumption 2 also holds. Suppose that \( L, p \geq 1 \). Then the condition

\[
\ln(1 - \eta) > -\ln b \quad (4.2)
\]

implies the transience of \( X \) on \( \mathcal{G} \).

**Corollary 4.2** If \( \eta = 0 \), then the process \( X \) is transient on \( \mathcal{G} \).

**Proof of Theorem 4.1.** Because the process \( X \) can change \( A_{\nu} \) only at the time \( S_\nu \) that \( \nu \) is first visited, the proof of Proposition 3.3 can be used to show that, if \( \eta < 1 \) and

\[
\liminf_{n \to \infty} \frac{1}{n} \int_{\mathbb{R}_+ \times \Sigma^*} \ln \mathbb{P}(T_{n-1} > T_n \mid \Gamma_{\nu}^* = y) \beta(dy) > -\ln b, \quad (4.3)
\]

then \( X \) is transient; here, \( \beta(\cdot) = K^{*(\ell)}(x^*, \cdot) \) for some \( x^* \in \mathbb{R}_+ \times \Sigma^* \), and \( \ell \) is chosen in such a way that there exist \( N \) and \( M \) such that, for all \( x, y \in \mathbb{R}_+ \times \Sigma^* \) and Borel sets \( B \), we have

\[
K^{*(\ell)}(x, B) \leq \frac{M}{N} \sum_{i=1}^N K^{*(\ell)}(y, B).
\]

It remains to determine when (4.3) holds.

For a given ray \( \sigma = [\varrho, \infty) \), let \( Q^D_i := \{\sum_{r=0}^{R_i} \prod_{j=1}^r D_{\sigma_j}^{-1}\}^{-1} \) denote the probability that the random walk starting in \( \varrho \) would hit \( \sigma_i \) before \( \varrho^{-1} \), if the probabilities were
determined solely by the $D_{\sigma_i}$, and, for $i \geq 1$, let $q_i^D := Q_i^D / Q_i^{D-1}$ denote the probability that the same random walk starting in $\sigma_{i-1}$ hits $\sigma_i$ before it hits $\varrho^{-1}$. Then, the probability $q_i^A$ that the walk hits $\sigma_i$ before $\varrho^{-1}$, when started in $\sigma_{i-1}$, is given by

$$A_{\sigma_i} = \frac{1 + D_{\sigma_i}^{-1}(1 - q_i^{D-1})}{1 + A_{\sigma_i}^{-1}(1 - q_i^{D-1})}.$$ 

Now, since $D_{\sigma_i} \geq \theta := p \wedge L \geq 1$ for all $i$, we have

$$1 - q_i^D \leq i^{-1},$$ 

so that, on the event $\bigcap_{i=1}^n \{ A_{\sigma_i} > \varepsilon \}$,

$$\Phi_n \geq \prod_{i=1}^n \frac{1 + i^{-1} \varepsilon^{-1}}{1 + A_{\sigma_i}^{-1}(1 - q_i^{D-1})} \geq k n^{-1/\varepsilon},$$

for a suitable $k$, and

$$\prod_{i=1}^n q_i^D = \prod_{i=1}^n \{1 - (1 - q_i^D)\} \geq 1/n.$$ 

Hence, for (4.3), we have

$$P(T_{-1} > T_n \mid \Gamma^* = y) = \mathbb{E} \left[ \prod_{i=0}^{n-1} P_{\omega}(T_{-1} > T_{i+1} \mid T_{-1} > T_i) \mid \Gamma^*_\varrho = y \right]$$

$$= \mathbb{E} \left[ \prod_{i=1}^n q_i^A \mid \Gamma^*_\varrho = y \right] = \mathbb{E} \left[ \Phi_n \prod_{i=1}^n q_i^D \mid \Gamma^*_\varrho = y \right]$$

$$\geq k n^{-1/\varepsilon} \mathbb{E} \left[ \prod_{i=1}^n q_i^D \mathbb{1}_{A_{\sigma_i} > \varepsilon} \mid \Gamma^*_\varrho = y \right]$$

$$\geq k n^{-(1/\varepsilon)-1} \mathbb{P} \left[ \bigcap_{i=1}^n \{ A_{\sigma_i} > \varepsilon \} \mid \Gamma^*_\varrho = y \right].$$

Hence, from the definition of $\eta_{\varepsilon,\infty}$,

$$\liminf_{n \to \infty} \inf_{y \in \mathbb{R}_+ \times \Sigma^*} \frac{1}{n} \ln P(T_{-1} > T_n \mid \Gamma^*_\varrho = y) \geq \ln(1 - \eta_{\varepsilon,\infty}),$$

and the theorem follows by letting $\varepsilon \to 0$ and using (4.3).
Remark 4.3 Consider a once-reinforced random walk on a Galton–Watson tree, defined as follows. Each edge is initially assigned weight 1. The walk moves to any one of its nearest neighbours, with probability proportional to the weight of the edge traversed. The first time an edge is traversed, its weight becomes $1 + \Delta$, for $\Delta > -1$, and is never changed again. With the choice of $L = 1$ and $A_\nu = 1/(1 + \Delta)$ for all $\nu \in G$, and with $p = \max\{1, 1/(1 + \Delta)\}$, our walk is exactly a once-reinforced random walk. Theorem 4.1 then implies transience for this class of processes, as already proved in [1] or [2].

The next result holds for all choices of $L$ and $p$ such that $L < p$. Define the kernel
\[
K_{ln}^*((\tilde{w}, c, \tilde{\nu}, z), B) := K^*((e^{\tilde{w}}, c, e^{\tilde{\nu}}, z), E^*(B)),
\]
where $\tilde{w} \in [\ln L, \infty)$, $c \in (p, \infty)$, $\tilde{\nu} \in \mathbb{R}$, $z \in \Sigma$, $B \in B$ and $E^*(B) := \{(e^{\tilde{w}}, c, e^{\tilde{\nu}}, z) : (\tilde{w}, c, \tilde{\nu}, z) \in B\}$. Note that, if $K^*$ satisfies Assumption 1, then so does the kernel $K_{ln}^*$.

For $R \in (0, \infty)$ and with $C := \ln L$, define the probability kernel $Q_{C,R}^*$ on $[C, R] \times [p, \infty) \times [C, R] \times \Sigma$ by
\[
Q_{C,R}^*(\tilde{y}, (d\tilde{w}, dc, d\tilde{\nu}, dz)) := \frac{K_{ln}^*(\tilde{y}, (d\tilde{w}, dc, d\tilde{\nu}, dz))}{K_{ln}^*(\tilde{y}, [C, R] \times (p, \infty) \times [C, R] \times \Sigma)}. \tag{4.7}
\]
This kernel describes the distribution of the jumps of the process $(\ln D_{\sigma_i}, b_{\sigma_i}, \ln A_{\sigma_i}, M_{\sigma_i})$ when $\ln A_{\sigma_i}$ is conditioned to be in the interval $[C, R]$. This also implies that $\ln D_{\sigma_i}$ takes values in the same interval. If $K^*$ satisfies Assumption 1, then so does the kernel $Q_{C,R}^*$. Define
\[
\tilde{\Lambda}^{(Q_{C,R}^*)} := \limsup_{n \to \infty} \sup_{\tilde{y}} \frac{1}{n} \ln \mathbb{E}\left[ e^{\lambda \sum_{i=1}^n \ln D_{\sigma_i}} | \Gamma^* = \tilde{y} \right],
\]
where the expected value is taken with respect to the kernel $Q_{C,R}^*$, and the supremum over the set $[C, R] \times [p, \infty) \times [C, R] \times \Sigma$.

Theorem 4.4 Suppose that $K^*$ satisfies Assumption 1 and that $\eta_{L,\infty} < 1$ and $L < p$. Then the condition
\[
\limsup_{R \uparrow \infty} \inf_{\lambda \in [0, 1]} \tilde{\Lambda}^{(Q_{ln}^*,L,R)}(\lambda) > -\ln b - \ln(1 - \eta_{L,\infty}) \tag{4.8}
\]
implies the transience of $X$ on $G$. 

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Remark 4.5 Suppose that $\eta = 0$ and $K^*$ satisfies Assumption 1. In this case, if $L, p \geq 1$, then, no matter what is the initial environment $(A_\nu, \nu \in G)$, the process $X$ is transient, by Corollary 4.2. If instead we assume that $L < p$ and $\eta L, \infty < 1$, then the process can also be recurrent. In this case, (4.8) provides a sufficient condition for transience.

Proof of Theorem 4.4. First, note that

$$
\Phi_n = \prod_{i=1}^{n} \frac{(1 + D_{\sigma_i}^{-1}(1 - q_{i-1}^D))}{1 + A_{\sigma_i}^{-1}(1 - q_i^D)} \mathbb{I}_{\{A_{\sigma_i} \geq L\}} \geq \prod_{i=1}^{n} \mathbb{I}_{\{A_{\sigma_i} \geq L\}},
$$

and hence that

$$
\mathbb{P}(T_{-1} < T_n \mid \Gamma^* = y) \geq \mathbb{E}\left[ \prod_{i=1}^{n} q_i^P \mathbb{I}_{\gamma_i^{-1} \{A_{\sigma_i} \geq L\}} \left\vert \Gamma^* = y \right\} \right]
$$

$$
\geq \mathbb{E}\left[ (\sum_{r=0}^{n} \prod_{j=1}^{r-1} D_{\sigma_n}^{-1})^{-1} \mathbb{I}_{\gamma_i^{-1} \{A_{\sigma_i} \geq L\}} \mid \Gamma^* = y \right].
$$

Now the last line of (4.9) is at most

$$
\mathbb{E}\left[ (n + 1) \max_{r \leq n} \prod_{j=1}^{r-1} D_{\sigma_n}^{-1} \mathbb{I}_{\gamma_i^{-1} \{A_{\sigma_i} \geq L\}} \mid \Gamma^* = y \right]
$$

$$
\geq \mathbb{E}\left[ (n + 1) \max_{r \leq n} \prod_{j=1}^{r-1} (D_{\sigma_n} \wedge R)^{-1} \mathbb{I}_{\gamma_i^{-1} \{A_{\sigma_i} \geq L\}} \mid \Gamma^* = y \right].
$$

This, in turn, implies that

$$
\liminf_{n \to \infty} \inf_{y \in \Sigma'} \frac{1}{n} \ln \mathbb{P}(T_{-1} > T_n \mid \Gamma^* = y)
$$

$$
\geq \liminf_{n \to \infty} \inf_{y \in \Sigma'} \frac{1}{n} \ln \mathbb{E}\left[ e^{-\min_{\varepsilon \in (0,1)} \sum_{i=1}^{[n\varepsilon]} \ln D_{\sigma_i} \wedge R \mathbb{I}_{\gamma_i^{-1} \{A_{\sigma_i} \geq L\}}} \mid \Gamma^* = y \right].
$$

We now argue much as for (3.16) in the proof of the first part of Theorem 3.6, proving that

$$
\liminf_{n \to \infty} \inf_{y \in \Sigma'} \frac{1}{n} \ln \mathbb{P}(T_{-1} > T_n \mid \Gamma^* = y) \geq \limsup_{R \to \infty} \inf_{\Lambda \in [0,1]} \Lambda^{(Q^*_{\varepsilon} L, R)}(\lambda) + \ln(1 - \eta L) > - \ln b.
$$
Hence, (4.3) holds, and this ends the proof.

As an example, we consider the case where $b_\nu = p$ is constant for all $\nu \in \mathcal{G}$. Suppose that $L^{-1} = p^{-1} + \varepsilon$ and that $A_{\sigma_i} \in (L, C)$ a.s., for all $\nu \in \mathcal{G}$ and for some constant $C$. Note that then $L < p$ and $\eta_{L, \infty} = 0$.

We prove that $X$ is transient if $\inf_{\lambda \in [0, 1]} \Lambda^{(K_{ln})} > - \ln b$, and recurrent if $\inf_{\lambda \in [0, 1]} \Lambda^{(K_{ln})} < - \ln b - \ln (1 + \varepsilon)$. Transience is a consequence of Theorem 4.4 with $Q_{ln} L, ln C = K_{ln}$. Next, we turn to the proof of recurrence under the assumption $\inf_{\lambda \in [0, 1]} \Lambda^{(K_{ln})} < - \ln b - \ln (1 + \varepsilon)$. In this case, we have that

$$\Phi_n = \prod_{i=1}^{n} \left(1 + \frac{(L^{-1} - A_{\sigma_i}^{-1})(1 - q_i^D)}{1 + A_{\sigma_i}^{-1}(1 - q_i^D)}\right)$$

$$\leq \prod_{i=1}^{n} (1 + L^{-1} - A_{\sigma_i}^{-1}) \leq \prod_{i=1}^{n} (1 + \varepsilon) \leq (1 + \varepsilon)^n.$$

Hence

$$\mathbb{P}(T_{-1} > T_n \mid \Gamma^* = y) = \mathbb{E}[\Phi_n \prod_{i=1}^{n} q_i^D \mid \Gamma^* = y] \leq (1 + \varepsilon)^n \cdot \mathbb{E}\left[\prod_{i=1}^{n} q_i^D \mid \Gamma^* = y\right].$$

This, by Theorem 3.4, using Varadhan's lemma and Proposition 5.1, implies that

$$\limsup_{n \to \infty} \sup_{y \in \Sigma'} \frac{1}{n} \ln \mathbb{P}(T_{-1} > T_n \mid \Gamma^* = y) \leq \inf_{\lambda \in [0, 1]} \Lambda^{(K_{ln})}(\lambda) + \ln (1 + \varepsilon) < - \ln b.$$ 

The expected number of vertices at level $n$ which is visited before the first return to the origin is bounded above by $b^n \sup_{y \in \Sigma'} \mathbb{P}(T_{-1} > T_n \mid \Gamma^* = y)$. Hence the expected time needed to return to the origin is bounded by

$$1 + \sum_{n=1}^{\infty} b^n \sup_{y \in \Sigma'} \mathbb{P}(T_{-1} > T_n \mid \Gamma^* = y) < \infty.$$

The latter proves positive recurrence.
Proposition 5.1 Suppose that $\phi: \mathbb{R} \rightarrow [-\infty, +\infty]$ is a convex function, with $\phi(0) = 0$. Then
\[
\sup_{f \in \mathcal{AC}} \left\{ \min_{t \in [0,1]} f(t) - \int_0^1 \sup_{\lambda} \{ f'(u)\lambda - \phi(\lambda) \} \, du \right\} = \inf_{\lambda \in [0,1]} \phi(\lambda). \tag{5.1}
\]

Proof. We first prove that the right-hand side of (5.1) is a lower bound. Let $\phi$ be finite on $F \subset \mathbb{R}$, and let $t^* \in [0,1] \cap \overline{F}$ be such that
\[
\lim_{t \to t^* \atop t \in F} \phi(t) = \inf_{0 \leq t \leq 1} \phi(t).
\]
Such a $t^*$ exists, in virtue of the convexity of $\phi$. Then, by convexity, $\phi$ has a (non-empty) sub-derivative $\text{SD}(\phi)\{t^*\}$ at $t^*$. Recall that $c \in \text{SD}(\phi)\{a\}$ means that $\phi(t) \geq \phi(a) + c(t - a)$ for all $t$.

If $t^* \in (0,1)$, then $0 \in \text{SD}(\phi)\{t^*\}$, and we choose $f(t) = 0$ for all $t$ to get
\[
\inf_{\lambda \in \mathbb{R}} \phi(\lambda) = \inf_{\lambda \in [0,1]} \phi(\lambda)
\]
as a lower bound for the left hand side of (5.1).

If $t^* = 0$, then there is a $c \geq 0$ with $c \in \text{SD}(\phi)\{0\}$, so that $\phi(t) \geq \phi(0) + ct$ for all $t$. Take $f(t) = ct$ for all $t$. Since $c \geq 0$, we have $\min_{0 \leq t \leq 1} f(t) = 0$, and we get
\[
-c - \sup_t \{ ct - \phi(t) \} \geq - \sup_t \{ ct - \phi(0) - ct \} = \phi(0) = \inf_{\lambda \in [0,1]} \phi(\lambda)
\]
as a lower bound for the left hand side of (5.1).

If $t^* = 1$, then there is a $c \leq 0$ with $c \in \text{SD}(\phi)\{1\}$, so that $\phi(t) \geq \phi(1) + ct - 1$ for all $t$. Take $f(t) = ct$. As $c \leq 0$, we have $\min_{0 \leq t \leq 1} f(t) = c$, and we get
\[
c - \sup_t \{ ct - \phi(t) \} \geq c - \sup_t \{ ct - \phi(1) - c(t - 1) \} = c + \phi(1) - c = \inf_{\lambda \in [0,1]} \phi(\lambda)
\]
as a lower bound for the left hand side of (5.1).

Next we turn to the proof of the upper bound. Fix any $t^* \in [0,1]$. Notice that, for any $f \in \mathcal{A}C$, we have $\min_{t \in [0,1]} f(t) \leq 0$ and $(f(1) - \min_{t \in [0,1]} f(t)) \geq 0$. Hence, taking
\( \lambda = t^* \) for all \( u \in [0,1] \), the left-hand side of (5.1) is bounded above by

\[
\sup_{f \in \mathcal{AC}} \left\{ \min_{t \in [0,1]} f(t) - f(1)t^* + \phi(t^*) \right\} = \sup_{f \in \mathcal{AC}} \left\{ \min_{t \in [0,1]} f(t)(1 - t^*) - (f(1) - \min_{t \in [0,1]} f(t))t^* + \phi(t^*) \right\} \leq \phi(t^*).
\]

By taking the infimum over \( t^* \in [0,1] \) we have the upper bound. \( \square \)

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