Composite spherically symmetric configurations in Jordan-Brans-Dicke theory

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Abstract

In this article, a study of the scalar field shells in relativistic spherically symmetric configurations has been performed. We construct the composite solution of Jordan-Brans-Dicke field equation by matching the conformal Brans solutions at each junction surfaces. This approach allows us to associate rigorously with all solutions as a single glued "space", which is a unique differentiable manifold $M^4$.

1 Introduction

The field equations in relativistic field theories are non linear in nature. For a classical field, the differential equations consist of purely geometric requirements imposed by the idea that space and time can be represented by a Riemannian (Lorentzian) manifold, together with the description of the interaction of matter and gravitation contained in Einstein’s equations.

This equation involved a match between a purely geometrical object so called Einstein tensor $G$, and an object which depends on the properties of matter fields the energy-momentum tensor $T$ which contains quantities like the density and pressure. Hence, the geometry of 4D spacetime is governed by the matter it contains. However, this split is artificial.

The variables $x$, used in Einstein’s equations, represent co-ordinates of points of abstract four-dimensional manifold $4$. In geometrical sense the different solutions of Einstein equations generate different 4D pseudo-Riemannian space-time manifolds $V_4$ ($g$). In other words, the emergent geometry is not a priori assumed but defined from the solutions. Co-ordinates in each of such spaces have the specific properties differing from their properties in other spaces [1].

Einstein’s equations determine the solution of a given physical problem up to four arbitrary functions, i.e., up to a choice of gauge transformations. Evidently, a structure of spacetimes is mathematically represented by Einstein’s equations and four co-ordinate conditions [2].

$$G_{ik} = T_{ik},$$

(1)

$$C(\mu)g_{ik} = 0.$$  

(2)

where $g_{ik}$ metric tensor and $C(\mu)$ - some algebraic or differential operators. Thereby for any four of components $g_{ik}$ emerge the relations with remaining six and, probably, any others, known functions. Certainly, equations [2] cannot be covariant for the arbitrary transformations of independent variables, and similarly should not contradict Einstein’s equations or to be their consequence. Moreover, in every class of co-ordinates each time is postulated new subsystem [2]. Usually, four of ten field equations will not be transformed according to any rules, but simply replaced by hand with the new. Generally speaking, physics looks different in two different classes of co-ordinates.

It is well known, since the pioneering paper of Jordan [3] that the action of a scalar tensor theory is invariant under local transformations of units that are under general conformal transformations. This method of conformal transformation pointed out first by Pauli. The method provides a clear and powerful technique, free from mathematical ambiguity, but nevertheless requires careful consideration from the

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It is exactly the non-triviality of unambiguously field-decomposition into a physical part, which represents the true gravitational effect, and a pure geometric part, which represents the feigned gravitational effect coupled with choice classes of co-ordinates. Moreover, scalar tensor theories are mathematically equivalent to general relativity and there are convincing arguments that they are also physically equivalent to it \([4], [5]\). In other terms the conformally invariant scalar may be represented in disguise in infinite number of ways as commonly viewed as special kinds of matter. Clearly, there is no trace of concerning what conformal frame we live in, and on what physical grounds we are able to select which.

2 Field equations.

A general Lagrangian for a scalar-tensor gravity theory,

\[
L = \sqrt{\text{g}} \left[ f(\phi) R - \frac{\omega(\phi)}{\phi} g^{ik} \phi, i \phi, k + 2\phi V(\phi) \right] + L_m
\]

where \(f(\phi) > 0\) and \(L_m\) is the Lagrangian density describing "ordinary" matter as opposed to the scalar field \(\phi\), which effectively plays the role of a form of non-conventional matter in the field equations. Here \(g_{ik}\) is the metric tensor with determinant \(g\), \(f(\phi)\) and \(\omega(\phi)\) are arbitrary coupling functions, \(\phi\) is the scalar field with potential \(V(\phi)\). The Lagrangian \((3)\) is invariant under the conformal transformations

\[
g^{*}_{ik} = \phi^{2\zeta} g_{ik}, \quad \zeta \neq 1, \quad \phi^{*} = \phi^{1-\zeta}
\]

this is equivalent to the transformation applied to a line element,

\[
ds^{*2} = \phi^{2\zeta} ds^2.
\]

The case \(\zeta = 1\), represents just a self-interacting scalar \(\phi\) minimally coupled to gravity and is designated as Einstein conformal frame, with

\[
g^{*}_{ik} = e^{2\zeta \phi} g_{ik}, \quad \phi^{*} = \kappa e^{2\zeta \phi}
\]

In the literature the Jordan frame \((3)\) and the Einstein frame \((6)\) are those discussed most frequently.

The Jordan frame in which the theory is formulated is the set of dynamical variables \(g_{ik}, \phi\) describing the gravitational field Lagrangian can then take, among others, the equivalent forms

\[
L = \sqrt{g} \left[ f(\phi^{*}) R^{*} + V^{*}(\phi^{*}) \right] = \sqrt{-g} \left[ R(g) - g_{ik} \phi, i \phi, k - V(\phi) \right]
\]

Thus the theory can be expressed in terms of infinite number of conformally related frames; in such a system one can always make a field redefinition (a change of variables) to another conformal frame via a conformal mapping. Here we restrict out attention to Jordan-Brans-Dicke (JBD) theory, where \(\omega(\phi) = \text{const}, V(\phi) = 0\). From the above action we can find the JBD field equations

\[
R_{ik} - \frac{1}{2} R g_{ik} = \frac{1}{\phi} T^{m}_{ik} + T^{JBD}_{ik},
\]

where

\[
T^{JBD}_{ik} = \frac{\omega}{\phi^2} \left( \nabla_i \phi \nabla_k \phi - \frac{1}{2} g_{ik} \nabla^j \phi \nabla^j \phi \right) +
+ \frac{1}{\phi} \left( \nabla_i \nabla_k \phi - g_{ik} \nabla^j \phi \right) \]

and

\[
\nabla_j \nabla^j \phi = \frac{\nabla^m \nabla^k g_{ik}}{3 + 2\omega}.
\]
where $T^{JBD}_{ik}$ is often identified with an effective stress-energy tensor of the scalar field $\phi$. The metric tensor satisfies to those or other co-ordinate conditions if some of quantities $g_{ik}$ are linked by some relations, - whether it be in any point, on a surface or in four-dimensional domain $\Omega \subset \mathbb{R}^4$. By definition all co-ordinate systems in manifold $M^4$ at least locally are equivalent; on the other hand if in $M^4$ the metric is introduced, properties of functions $g_{ik}$ in different co-ordinates become different.

In relativistic scalar tensor theory, a covariant presentation of the matching conditions, across a separating hypersurface, requires the continuity of the first and second fundamental forms. Let us consider two distinct manifolds $M^4_+ \text{ and } M^4_-$. The metric in these manifolds generated by set of solutions of field equations $\mathbb{S}$ given by $g^+_{ik}(x^+)$ and $g^-_{ik}(x^-)$, in terms of independently defined coordinate systems $x^+$ and $x^-$. The manifolds glued at the boundary hypersurfaces $\Sigma_+$ and $\Sigma_-$ using independently defining co-ordinates systems $x^\pm$. A common manifold $M = M^+ \cup M^-$ is obtained by assuming the continuity of four-dimensional coordinates $x^\pm$ across $\Sigma$, then $g^+_{ik} = g^-_{ik}$ is required, which together with the continuous derivatives of the metric components $\partial g_{ik}/\partial x^j |_+ = \partial g_{ik}/\partial x^j |_-$, provide the Lichnerowicz conditions $\mathbb{L}$.

The resulting manifold $M$ is geodesically complete and possesses two regions connected by a hypersurface $\Sigma$. The extrinsic curvature, or the second fundamental form, is defined as

$$K^\eta_{k\pm} = \frac{1}{2}g_{ik} \frac{\partial g_{ki}}{\partial \eta} |_{\eta = \pm 0}$$

where $\eta$ the proper distance away from the $\Sigma$.

### 3 Composite solution in Jordan-Brans-Dicke theory.

In the present article we will assume the static spherically symmetric configurations involved the matching of conformally invariant interior "vacuum" solutions. The solution will be given in terms of explicit closed-form functions of the radial coordinate for the three metric coefficients. The physical and geometrical meaning of the coordinate $r$ is not defined by the spherical symmetry of the problem and is unknown a priori $\mathbb{H}$. Note, that its choice has been discussed from physical point of view by Eddington as early as in $\mathbb{S}$. According to the widespread common opinion, the most common form of line element of a spherically symmetric spacetime in comoving coordinates can be written as

$$ds^2 = -g_{tt}(r,t)dt^2 + g_{rr}(r,t)dr^2 + 2g_{rt}(r,t)drdt + \rho(r,t)^2(d\theta^2 + \sin^2(\theta)d\varphi^2).$$

(11)

We are free to reset our clocks by defining a new time coordinate

$$t = t' + u(r),$$

with $u(r)$ an arbitrary function of $r$. This allows us to eliminate the off-diagonal element $g_{rt}$. Therefore we shall consider the matching of two static and spherically symmetric spacetimes given by the following line elements

$$ds^2_\pm = -g_{tt}(r,t)_\pm dt^2 + g_{rr}(r,t)_\pm dr^2 + \rho(r,t)_\pm^2(d\theta^2 + \sin^2(\theta)d\varphi^2).$$

(12)

of $M^4_\pm$, respectively, where $g_{tt}(r,t)$, $g_{rr}(r,t)$ and $\rho(r,t)$ are of class $C^2$.

In the static spherically symmetric case the choice of spherical coordinates and static metric dictates the form of three of the gauge fixing coefficients $\mathbb{12}$:

$$\Gamma_t = 0, \Gamma_\theta = -\cot \theta, \Gamma_\varphi = 0,$$

(13)

where

$$\Gamma_i = -\frac{1}{\sqrt{g}}g_{ik}\partial_j(\sqrt{g}g^{jk}),$$

(14)

but the form of the $\rho(r)$ are still not fixed. In the literature one can find different choices of the function $\rho(r)$ but the isotropic class of coordinates is those discussed most frequently. The static spherically symmetric matter free solution of JBD theory in isotropic coordinates is given by $\mathbb{9}$. 


It is well known, that among the four classes of the Brans static spherically symmetric solution of the vacuum JBD theory of gravity only two are really independent \[10\]. It should be noted that the JBD action has a conformal invariance \[3\] characterized by a constant gauge parameter \(\zeta\). Arbitrary value of \(\zeta\) can actually lead to shift from the value of Brans solutions and change scalar field and metric coefficients.

The general solutions for static spherically symmetric configurations in isotropic class of coordinates

\[
ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}[dr^2 + \rho(r)^2(d\theta^2 + \sin^2(\theta)d\phi^2)].
\]

then looks like \[11\]:

\[
\begin{align*}
\alpha(r) &= \alpha_0 \left(1 - \frac{B}{r}\right)^{\frac{1+\zeta C}{\lambda}}, \\
\beta(r) &= \beta_0 \left(1 + \frac{B}{r}\right)^2 \left(1 - \frac{B}{1 + \frac{B}{r}}\right)^{\frac{\lambda C(1+\zeta C)}{\lambda - 1}}, \\
\phi(r) &= \phi_0 \left(1 - \frac{B}{1 + \frac{B}{r}}\right)^{\frac{\lambda C(1-2\zeta)}{\lambda - 1}},
\end{align*}
\]

and

\[
\begin{align*}
\alpha(r) &= \alpha_0 e^{2(1+\zeta C)\arctan(\frac{r}{B})}, \\
\beta(r) &= \beta_0 \left(1 + \frac{B^2}{r^2}\right) e^{2(1+\zeta C)\arctan(\frac{r}{B})} \left(1 - \frac{B}{1 + \frac{B}{r}}\right)^{\frac{\lambda C(1-2\zeta)}{\lambda - 1}}, \\
\phi(r) &= \phi_0 e^{2(1+\zeta C)\arctan(\frac{r}{B})},
\end{align*}
\]

where the constants \(\lambda, C, \zeta\) are still connected via the JBD field equations by

\[
\lambda^2 = 1 + C + \frac{1}{2}C^2 \left[2 + \omega - 2\zeta(3 + 2\omega) + \zeta^2(6 + 4\omega)\right].
\]

For our advance to the composite spherical configurations, we look at the families of scalar field shells. The scalar layers generated in this approach are obtained by matching of scalar field and introducing a discontinuity in its derivative. In this case the boundary surface entails via the field equations a jump in second derivations of metric coefficient, but metric coefficient and its first derivatives remains continuous.

Now in order to justify calling the geometry a composite configuration we need an explicit definition for the constants in the solutions \[10\]-\[17\]. At the each boundary surface we have five equations for five unknown variables \(B, C, \alpha_0, \beta_0, \phi_0\). Thus, from the junction conditions, the each interior metric and scalar field parameters can be determined at the boundary surface in terms of the respective exterior metric and scalar field parameters \(\zeta\). Consequently, once the constants of the ambient space are fixed, the property in the interiors layers region roughly corresponds to the values inferred from observations.

The brief computation of yields:

\[
B_{\text{int}} = \pm \sqrt{r_s^4[X + Y + YC_{\text{int}}(X - Y)\zeta_{\text{int}}C_{\text{int}}] \over 2 + 2C_{\text{int}}\zeta_{\text{int}} + r_s[X + Y + YC_{\text{int}}(X - Y)\zeta_{\text{int}}C_{\text{int}}]},
\]

where

\[
\begin{align*}
X &= \frac{2B_{\text{ext}}(1 - B_{\text{ext}}\lambda_{\text{ext}} + rs(1 + C_{\text{ext}} - C_{\text{ext}}\lambda_{\text{ext}}))}{rs(B_{\text{ext}}^2 - r_s^2)\lambda_{\text{ext}}}, \\
Y &= \frac{2B_{\text{ext}}(1 + C_{\text{ext}}\lambda_{\text{ext}})}{(B_{\text{ext}}^2 - r_s^2)\lambda_{\text{ext}}}. \\
\end{align*}
\]

All terms in \[19\], \[20\] with index \textit{int} contribute to the internal and \textit{ext} external scalar layers, \(r_s\) denotes radius of boundary hypersurface.
In this case we deal with non-linear equations for the unknown variables; the situation with finding constant of integration $C_{\text{int}}$ is more complicated. It seems that numerical integration is the "simplest" way to treat this problem. To this end, following this approach we regard exterior region of JBD composite spherically symmetric configurations by the Schwarzschild metric or a flat space. As a particular but interesting example, we consider the case where gauge parameter $\zeta = 1/C$. This value of parameter makes possible to find a field configurations in which the inside of the shell is conformal Brans I or Brans II space while the outside is a flat space. It easy to show that the arbitrary constants are determined by using matching conditions and the constant $B$ determined for conformal Brans I as follows:

$$B = r_s \sqrt{\frac{2}{2 + \omega}},$$

(21)

and for conformal Brans II

$$B = -r_s \sqrt{\frac{2}{-2 - \omega}},$$

(22)

Moreover, we can tune the scalar field of the outside of the shell; while of the inside will be flat space. This solution is asymptotically flat and hence that connects two flat regions of space. An interesting result in this case is that the flat space produces Keplerian mass of the scalar configurations. Otherwise, for $\zeta = 1/C$ we might treat each of solutions (16), (17) as independently derived solution matched to a flat space. One now has three connected regions, that is, one-side conformal Brans I, a both-side flat region and another one-side Brans I with different arbitrary constant or Brans II region.

We stress the fact that for all above examples the Darmois-Israel junction conditions [12] are fulfilled.

4 A solutions with throat.

The metric coefficients in conformal Brans solutions are required to satisfy some constraints, enumerated in [13], in order that they have a throat. The important point is that the metric coefficients in Eqs. (16) and (17) depend on three parameters $\omega, \zeta, C$ and satisfying the inequality

$$1 + C + \frac{1}{2} C^2 \left[2 + \omega - 2\zeta(3 + 2\omega) + \zeta^2 (6 + 4\omega) \right] > 0.$$ (23)

Confronting the isotropic Brans I solution with the conformal Brans I metric, the parameter $\lambda$ is defined as

$$\lambda^2 = 1 + C + \frac{1}{2} C^2 (2 + \omega),$$ (24)

so that in the case of conformal solutions we have additional degree of freedom. This implies that the range of $\omega$ is dictated by the range of $C$ and $\zeta$ which, in turn, is to be dictated by the requirements of throat geometry. In this context, the violation of the weak energy condition combined with an adequate choice of $C$ and could provide a viability of "wormhole" spacetime and less restrictive interval for $\omega$ from the case of $\frac{3}{2} < \omega < \frac{4}{3}$ considered in [14].

The new results include matching between solutions with throat and other conformal Brans and after that Schwarzschild solutions. First of all we point out that in this model the stars acquires features of a many-component objects (shells of "scalar gas" with different properties) whose distribution in the observed 3-dimensional volume can has. Moreover, such a picture can represents a Schwarzschild background, while the interior should be considered as vacuum solution of JBD which defined a Keplerian mass of this object.

In scheme presented in this report, studies of possible "wormhole" solutions in alternative gravitation was thought of as a way of understanding the role of different fields as the carrier of exoticity together with the aim of finding phenomena for which different qualitative behaviors to those of standard General Relativity model may arise.
5 Summary

The contents of the paper may be summarized as under:

1. The conformal invariance of the vacuum JBD theory shows that the solutions in it are not unique. A conformal transformation between Brans solutions can be interpreted as a change of local units of length \[9\]. On the other hand, scalar fields are commonly viewed as an abnormal kind of matter originating exactly from this energy-momentum tensor. One can in principle assume gauge-dependence of right-hand-side of equation\[4\] as a variety of matter fields with different equations of state. Due to gauge invariance of JBD field equations, it is possible to generate and infinite set of solutions simply by assigning arbitrary values to the gauge parameter \(\zeta\) and the resulting sets of solutions form families of spacetimes having the same physical content.

2. As we have seen, even restricting to static spherically symmetry JBD theory has number of conformal solutions. Then one consequence of this is the possibility to use these conformal solutions as reservoirs shells of ”scalar gas” of a ”star” match it each other and finally with Schwarzschild metric or a flat spacetime.

3. The JBD theory contains some regions of the parameter space in which the scalar field may play the role of exotic matter, implying that it might be possible to build a wormholelike spacetime with the presence of ”scalar gas” at the throat. Varying the real gauge parameter \(\zeta\), one can obtain value of \(\lambda\) from any given \(\omega\) on either side of the divide \(\omega = -3/2\) but not across it since it is the fixed point of the relation \[23\].

It should also be noted that because Jordan-Brans-Dicke is a highly non-linear theory, it is not always easy to understand what qualitative features solutions might possess, and here the composite class of solutions can used an a guide.

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