Soft Gluons in Logarithmic Summations

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Abstract

We demonstrate that all the known single- and double-logarithm summations for a parton distribution function can be unified in the Collins-Soper resummation technique by applying soft approximations appropriate in different kinematic regions to real gluon emissions. Neglecting the gluon longitudinal momentum, we obtain the $k_T$ (double-logarithm) resummation for two-scale QCD processes, and the Balitsky-Fadin-Kuraev-Lipatov (single-logarithm) equation for one-scale processes. Neglecting the transverse momentum, we obtain the threshold (double-logarithm) resummation for two-scale processes, and the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (single-logarithm) equation for one-scale processes. If keeping the longitudinal and transverse momenta simultaneously, we derive a unified resummation for large Bjorken variable $x$, and a unified evolution equation appropriate for both intermediate and small $x$. 
1. Introduction

It is known that radiative corrections in perturbative QCD (PQCD) produce large logarithms at each order of the coupling constant. Double logarithms appear in processes involving two scales, such as $\ln^2(p^+b)$ with $p^+$ the large longitudinal momentum of a parton and $1/b$ the small inverse impact parameter, where $b$ is conjugate to the parton transverse momentum $k_T$. In the kinematic end-point region with large Bjorken variable $x$, there exist $\ln^2(1/N)$ from the Mellin transformation of $\ln(1-x)/(1-x)_+$, for which the two scales are the large $p^+$ and the small infrared cutoff $(1-x)p^+$ for gluon emissions from the parton. Single logarithms are generated in processes involving only one scale, such as $\ln p^+$ and $\ln(1/x)$, for which the relevant scales are the large $p^+$ and the small $xp^+$, respectively. These logarithmic corrections to a parton distribution function have been summed to all orders by various methods, which are the $k_T$ resummation for $\ln^2(p^+b)$ [1], the threshold resummation for $\ln^2(1/N)$ [2, 3, 4], the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equation for $\ln p^+$ [5], and the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation for $\ln(1/x)$ [6].

In this paper we shall demonstrate that all the above single- and double-logarithm summations can be derived in the Collins-Soper (CS) resummation technique [1]. The point is the soft approximation for real gluon emissions, with which a parton distribution function $\phi(x + l^+/p^+, |k_T + l_T|)$ is associated. The arguments of $\phi$ indicate that the parton, emerging from a hadron, carries the longitudinal momentum $xp^+ + l^+$ and the transverse momentum $k_T + l_T$ in order to radiate a real gluon with the momentum $l$. If neglecting the $l^+$ dependence,

$$\phi(x + l^+/p^+, |k_T + l_T|) \approx \phi(x, |k_T + l_T|),$$  \hspace{1cm} (1)

the scale $(1 - x)p^+$ will not appear. Hence, Eq. (1) is inappropriate for the region with large $x \rightarrow 1$. Under this soft approximation, we derive the $k_T$ resummation for intermediate $x$, which involves two scales: the large $xp^+$ and the small $k_T$, and the BFKL equation for small $x$, which involves only one scale $xp^+ \approx k_T$. If neglecting the $l_T$ dependence,

$$\phi(x + l^+/p^+, |k_T + l_T|) \approx \phi(x + l^+/p^+, k_T),$$  \hspace{1cm} (2)

the transverse degrees of freedom of a parton can be integrated out, and $k_T$ will not be a relevant scale. Therefore, Eq. (2) is inappropriate for small $x$,
where the scale $k_T$ is of order $xp^+$, and not negligible. Under this soft approximation, we derive the threshold resummation for large $x$, which involves two scales: the large $xp^+$ and the small $(1-x)p^+$, and the DGLAP equation for intermediate $x$, which involves only one scale $xp^+ \sim (1-x)p^+$.

In the regions where Eqs. (1) and (2) are inappropriate, we should keep both the $l^+$ and $l_T$ dependences, and employ $\phi(x + l^+/p^+, |k_T + l_T|)$ for real gluon emissions directly. In this case the three scales $xp^+$, $(1-x)p^+$ and $k_T$ exist simultaneously. We shall derive a unified resummation (a unification of the $k_T$ and threshold resummations) for large $x$, and a unified evolution equation (a unification of the DGLAP and BFKL equations), which is suitable for both intermediate and small $x$. In conclusion, we are able to reproduce all the logarithmic summations and derive their unifications simply by employing appropriate soft approximations for real gluon emissions in the CS technique. The results are summarized in Table I.

2. Master Equation

Consider a parton distribution function $\phi(x, k_T, p^+)$ for a hadron with the light-like momentum $p^\mu = p^+ \delta^\mu$, which describes the probability that a parton carries the longitudinal momentum $xp^+$ and the transverse momentum $k_T$. If the parton is a quark, $\phi$ is written, in the minimal subtraction scheme, as

$$\phi(x, k_T, p^+) = \int \frac{dy^-}{2\pi} \int \frac{d^2 y_T}{4\pi^2} e^{-ixp^+y^-+i k_T \cdot y_T} \langle p | \bar{q}(y^-, y_T) | p \rangle, \quad (3)$$

where $\gamma^+$ is a Dirac matrix, and $|p\rangle$ denotes the hadron. Averages over spin and color are understood. If the parton is a gluon, the operator in the hadronic matrix element is replaced by $F_{\mu}^+(y^-, y_T) F^{\mu+}(0)$. The above definition is given in the axial gauge $n \cdot A = 0$ with the gauge vector $n^\mu = \delta^\mu_-$ lying on the light cone. To implement the CS technique, we allow $n$ to vary arbitrarily away from the light cone ($n^2 \neq 0$) \[1\], and the parton distribution function becomes gauge dependent. However, it will be observed that the kernels for various logarithmic summations turn out to be $n$-independent. This is natural, since it has been shown that parton distribution functions defined for different $n$ possess the same infrared structure, and thus the same evolution behavior, though different ultraviolet structure \[7\]. After the
derivation, we bring \( n \) back to the light cone, and the gauge invariance of the parton distribution function is restored. That is, the arbitrary vector \( n \) appears only at the intermediate stage of the derivation, and acts as an auxiliary tool.

The master equation in the CS technique is a differential equation of \( \phi \) in \( p^+ \) \[1\] \[8\]. Because of the scale invariance of \( \phi \) in \( n \) as indicated by the gluon propagator, \(-iN^{\mu \nu}(l)/l^2\), with

\[
N^{\mu \nu} = g^{\mu \nu} - \frac{n^{\mu}l^{\nu} + n^{\nu}l^{\mu}}{n \cdot l} + n^2 \frac{l^{\mu}l^{\nu}}{(n \cdot l)^2}, \tag{4}
\]

\( \phi \) depends on \( p^+ \) via the ratio \((p \cdot n)^2/n^2\). Hence, we have the chain rule relating the derivative \( dp^+ \) to \( dn_\alpha \),

\[
p^+ \frac{d}{dp^+} \phi = -\frac{n^2}{v \cdot n} \frac{d}{dn_\alpha} \phi , \tag{5}
\]

with \( v \) a dimensionless vector along \( p \). The operator \( d/dn_\alpha \) applies to gluon propagators, leading to

\[
\frac{d}{dn_\alpha} N^{\mu \nu} = -\frac{1}{n \cdot l}(l^{\mu}N^{\alpha \nu} + l^{\nu}N^{\mu \alpha}) . \tag{6}
\]

The loop momentum \( l^\mu \) \((l^\nu)\) contracts with the vertex the differentiated gluon attaches, which is then replaced by a special vertex,

\[
\hat{v}_\alpha = \frac{n^2 v_\alpha}{v \cdot n n \cdot l} . \tag{7}
\]

This special vertex can be read off from the combination of Eqs. \[3\] and \[3\].

Employing Ward identities \[3\], a diagram with the contraction of \( l^\mu \) can be expressed as the difference of the diagram, in which the particle (quark or gluon) propagator after the contraction is removed, and the diagram, in which the particle propagator before the contraction is removed. Hence, a pair cancellation exists between the diagrams with adjacent contractions of \( l^\mu \).

The summation of all the diagrams with different differentiated gluons then reduces to a single new diagram, where the most external particle propagator is removed. That is, the special vertex appears at the outer end of the parton line in this new diagram. We obtain the master equation \[1\] \[8\],

\[
p^+ \frac{d}{dp^+} \phi(x, k_T, p^+) = 2\bar{\phi}(x, k_T, p^+) , \tag{8}
\]
shown in Fig. 1(a), where the new diagram denoted by $\tilde{\phi}$ contains the special vertex represented by a square. The coefficient 2 comes from the equality of $\tilde{\phi}$ with the special vertex on either of the two parton lines.

The collinear region of the loop momentum $l$ is not important because of the factor $1/(n \cdot l)$ in $\hat{v}_\alpha$ with nonvanishing $n^2$. Therefore, the important regions of $l$ are soft and hard, in which the subdiagram containing the special vertex can be factorized from $\tilde{\phi}$ according to Figs. 1(b) and 1(c) at lowest order, respectively. The second subdiagram in Fig. 1(c), as a soft subtraction, guarantees a hard momentum flow. The remaining part is the original distribution function $\phi$. Therefore, $\tilde{\phi}$ is factorized into the convolution of the subdiagram containing the special vertex with $\phi$.

The soft contribution from Fig. 1(b) is written as
\[ \bar{\phi}_s(x, k_T, p^+) = \bar{\phi}_{sv}(x, k_T, p^+) + \bar{\phi}_{sr}(x, k_T, p^+) , \] (9)

with
\[
\bar{\phi}_{sv} = \left[ ig^2 C_F \mu^\epsilon \int \frac{d^{4-\epsilon} l}{(2\pi)^{4-\epsilon}} N_{\nu\beta}(l) \hat{v}^\beta v^\nu \frac{v \cdot l}{v \cdot ll^2} - \delta K \right] \phi(x, k_T, p^+) ,
\] (10)
\[
\bar{\phi}_{sr} = ig^2 C_F \mu^\epsilon \int \frac{d^{4-\epsilon} l}{(2\pi)^{4-\epsilon}} N_{\nu\beta}(l) \hat{v}^\beta v^\nu \frac{v \cdot l}{v \cdot ll^2} 2\pi i \delta(l^2)
\]
\[ \times \phi(x + l^+/p^+, |k_T + l_T|, p^+) , \] (11)
corresponding to the virtual and real gluon emissions, respectively. The color factor $C_F = 4/3$ should be replaced by $N_c = 3$ in the case with the parton being a gluon. The additive counterterm $\delta K$ is specified in the modified minimal subtraction scheme. The hard contribution from Fig. 1(c) is given by
\[ \bar{\phi}_h(x, k_T, p^+) = G(xp^+ / \mu, \alpha_s(\mu))\phi(x, k_T, p^+) , \] (12)
with the hard function
\[
G = -ig^2 C_F \mu^\epsilon \int \frac{d^{4-\epsilon} l}{(2\pi)^{4-\epsilon}} N_{\nu\beta}(l) \hat{v}^\beta \left[ \frac{x \cdot \hat{p} - l}{(xp - l)^2} \gamma^\nu + \frac{v^\nu}{v \cdot l} \right] - \delta G ,
\]
\[ = -\frac{\alpha_s(\mu)}{\pi} C_F \ln \frac{xp^+ \nu}{\mu} , \] (13)
where $\delta G$ is an additive counterterm. In the case with the parton being a gluon, the expression of $G$ can be written down straightforwardly. The gauge
factor $\nu = \sqrt{(v \cdot n)^2/|n|^2}$ confirms our argument that $\phi$ depends on $p^+$ via the ratio $(p \cdot n)^2/|n|^2$.

3. $k_T$ Resummation and BFKL Equation

We first discuss the soft approximation in Eq. (1) for $\phi$ associated with the real gluon emission. Fourier transforming Eq. (11) into the impact parameter $b$ space in order to decouple the $l_T$ integration, we derive

$$\tilde{\phi}_s(x, b, p^+) = K(1/(b\mu), \alpha_s(\mu))\phi(x, b, p^+),$$

(14)

with the soft function

$$K = ig^2 C_F \mu^\epsilon \int \frac{d^{4-\epsilon} l}{(2\pi)^{4-\epsilon}} N_{\nu \beta}(l) \frac{\delta^d l}{v \cdot l} \left[ \frac{1}{l^2} + 2\pi i \delta(l^2)e^{i l \cdot b} \right] - \delta K,$$

(15)

Hence, in the intermediate $x$ region $\phi$ involves two scales, the large $x p^+$ that characterizes the hard function $G$ in Eq. (13) and the small $1/b$ that characterizes the soft function $K$.

Using $\tilde{\phi} = \tilde{\phi}_s + \tilde{\phi}_h$, the master equation (8) becomes

$$p^+ \frac{d}{dp^+} \phi(x, b, p^+) = 2 \left[ K(1/(b\mu), \alpha_s(\mu)) + G(x p^+ / \mu, \alpha_s(\mu)) \right] \phi(x, b, p^+).$$

(16)

Since both the ultraviolet divergences in $K$ and $G$ come from the virtual gluon contribution $\tilde{\phi}_sv$, they cancel each other, such that $K + G$ is renormalization-group (RG) invariant. The single logarithms $\ln(b\mu)$ and $\ln(x p^+ / \mu)$, contained in $K$ and $G$, respectively, are organized by the RG equations

$$\mu \frac{d}{d\mu} K = -\gamma_K = -\mu \frac{d}{d\mu} G.$$

(17)

The anomalous dimension of $K$, $\lambda_K = \mu d \delta K / d\mu$, is given, up to two loops, by [10]

$$\gamma_K = \frac{\alpha_s}{\pi} C_F + \left( \frac{\alpha_s}{\pi} \right)^2 C_F \left[ C_A \left( \frac{67}{36} - \frac{\pi^2}{12} \right) - \frac{5}{18} n_f \right],$$

(18)
with $n_f$ the number of quark flavors and $C_A = 3$ a color factor. The solution of Eq. (17) gives

$$K(1/(b\mu), \alpha_s(\mu)) + G(xp^+/\mu, \alpha_s(\mu)) = -\int_{1/b}^{xp^+} \frac{d\mu}{\mu} \gamma_K(\alpha_s(\mu)),$$

(19)

where the initial conditions $K(1, \alpha_s(1/b))$ and $G(1, \alpha_s(xp^+))$ that contribute only to the single-logarithm summation have been dropped. Solving the differential equation (16) with the above expression inserted, we obtain the $k_T$ resummation [8],

$$\phi(x, b, p^+) = \Delta_k(b, xp^+)\phi^{(0)}(x),$$

(20)

with the (Sudakov) exponential

$$\Delta_k(b, xp^+) = \exp \left[-2 \int_{1/b}^{xp^+} \frac{dp}{p} \int_{1/b}^{p} \frac{d\mu}{\mu} \gamma_K(\alpha_s(\mu)) \right].$$

(21)

In the small $x$ region with $xp^+ \sim k_T$, or $xp^+ \sim 1/b$ in the $b$ space, the above two-scale case reduces to the one-scale case. The source of double logarithms, i.e., the integral containing $\gamma_K$ in Eq. (19), is less important. Instead of applying the RG equation (17), we simply add Eqs. (10)-(12), or equivalently, Figs. 1(b) and 1(c). The result can be understood in the way that the function $G$ introduces an ultraviolet cutoff of order $xp^+ \sim k_T$, which comes from the first subdiagram of Fig. 1(c), to the virtual soft gluon contribution. Without Fourier transformation, $\tilde{\phi}$ can be reexpressed, according to Eqs. (10) and (11), as

$$\tilde{\phi}(x, k_T, p^+) = g^2N_c \int \frac{d^4l}{(2\pi)^4} \langle 0 | v^\beta v^\nu | l \rangle \left[ \theta(k_T^2 - l^2) \phi(x, k_T, p^+) \right.$$

$$+2\pi i \delta(l^2) \phi(x, |k_T + l_T|, p^+) \bigg] ,$$

(22)

where the color factor has been replaced by $N_c$, because we consider the gluon distribution function in the small $x$ region. The $\theta$ function, defining the ultraviolet cutoff $k_T$, is the consequence of the inclusion of $G$.

To make a variation in $x$ via a variation in $p^+$, we assume a fixed parton momentum. Under this assumption, the momentum fraction $x$ is proportional to $1/p^+$, and we have [9]

$$p^+ \frac{d}{dp^+} \phi(x, k_T, p^+) = -x \frac{d}{dx} \phi(x, k_T, p^+) .$$

(23)
Performing the integrations over $l^+$ and $l^-$ in Eq. (22) and using Eq. (23), the master equation (8) reduces to the BFKL equation [11],

$$
\frac{d\phi(x, k_T, p^+)}{d \ln(1/x)} = \tilde{\alpha}_s \int \frac{d^2l_T}{\pi l_T^2} \left[ \phi(x, |k_T + l_T|, p^+) - \theta(k_T^2 - l_T^2) \phi(x, k_T, p^+) \right],
$$

(24)

with the coupling constant $\tilde{\alpha}_s = N_c \alpha_s / \pi$.

### 4. Threshold Resummation and DGLAP Equation

We then consider the soft approximation in Eq. (4). In this case the dependence on $k_T$ can be integrated out from both sides of Eqs. (10)-(12), and the scale $(1-x)p^+$ enters. We employ the Mellin transformation to bring $\tilde{\phi}_{sr}$ from the momentum fraction $x$ space to the moment $N$ space,

$$
\tilde{\phi}_{sr}(N, p^+) = \int_0^1 dx x^{N-1} \tilde{\phi}_{sr}(x, p^+),
$$

(25)

under which the $l^+$ integration decouples. Combined with the soft virtual contribution in Eq. (14), we derive

$$
\tilde{\phi}_s(N, p^+) = K(p^+/(N\mu), \alpha_s(\mu)) \phi(N, p^+),
$$

(26)

with the soft function

$$
K = ig^2 C_F \mu^\epsilon \int_0^1 dz \int \frac{d^{4-\epsilon}l}{(2\pi)^{4-\epsilon}} N_{\nu\beta}(l) \frac{\delta(1-z)}{l^2}
+ 2\pi i \delta(l^2) \delta \left(1 - z - \frac{l^+}{p^+} \right) z^{N-1} - \delta K,
$$

$$
= \frac{\alpha_s(\mu)}{\pi} C_F \ln \frac{p^+\nu}{N\mu},
$$

(27)

and the counterterm $\delta K$ the same as that in Eq. (13). Therefore, in the large $x$ region $\phi$ involves two scales, the large $xp^+ \sim p^+$ from the hard function $G$ in Eq. (13) and the small $(1-x)p^+ \sim p^+/N$ from the soft function $K$.

Similarly, Eqs. (16)-(19) hold but with $1/b$ replaced by $p^+/N$. To sum $\ln(1/N)$, we regard the derivative $p^+ d\phi/dp^+$ as

$$
p^+ \frac{d\phi}{dp^+} = \frac{p^+}{N} \frac{\partial \phi}{\partial(p^+/N)},
$$

(28)
which leads to the threshold resummation,
\[ \phi(N, p^+) = \Delta_t(N, p^+) \phi^{(0)} \]  
with the exponential
\[ \Delta_t(N, p^+) = \exp \left[ -2 \int_{p^+/N}^{p^+} \frac{dp}{p} \int_{p^+}^{p^+} \frac{d\mu}{\mu} \gamma K(\alpha_s(\mu)) \right]. \]

In the intermediate \( x \) region the above two-scale case reduces to the one-scale case because of \( xp^+ \sim (1 - x)p^+ \), and the source of double logarithms becomes less important. Without the Mellin transformation, the addition of Eqs. (10)-(12), with the soft approximation in Eq. (2) inserted, leads to the DGLAP equation [9],
\[ p^+ \frac{d}{dp^+} \phi(x, p^+) = \int_x^1 \frac{d\xi}{\xi} P(x/\xi, p^+) \phi(\xi, p^+), \]
with the kernel
\[ P(z, p^+) = \frac{\alpha_s(p^+)}{\pi} C_F \frac{2}{(1 - z)_+}, \]
where the variable change \( \xi = x + l^+/p^+ \) has been employed. The argument of \( \alpha_s \) has been chosen as the single scale \( xp^+ \sim (1 - x)p^+ \), which is of order \( p^+ \). Note that the kernel \( P \) differs from the splitting function \( P_{qq} = (\alpha_s C_F/\pi)(1 + z^2)/(1 - z)_+ \) by the term \((z^2 - 1)/(1 - z)_+\), which is finite in the \( z \to 1 \) limit. The reason is that the real gluon emission is evaluated under soft approximation as deriving \( P \), while it is calculated exactly as deriving \( P_{qq} \).

5. Unified Logarithmic Summations

In this section we study the case, in which both the \( l^+ \) and \( l_T \) dependences of \( \phi \) in Eq. (11) are retained. It will be shown that a unified resummation for large \( x \) and a unified evolution equation for intermediate and small \( x \) are derived. Obviously, we should apply both the Fourier and Mellin transformations to Eq. (11), and obtain
\[ \bar{\phi}_s(N, b, p^+) = K(p^+/N\mu, 1/(b\mu), \alpha_s(\mu)) \phi(N, b, p^+), \]
with the soft function

$$K = i g^2 C_F \mu^\epsilon \int_0^1 dz \int \frac{d^{4-\epsilon}}{(2\pi)^{4-\epsilon}} N_{\nu\beta}(l) \frac{\hat{\delta}^\beta \nu^\nu}{v \cdot l} \left[ \frac{\delta(1-z)}{l^2} \right]$$

$$+ 2\pi i \delta(l^2) \delta \left( \frac{1 - z - \frac{l^+}{p^+}}{l^2} \right) z^{-1} e^{i l^+ \cdot b} - \delta K ,$$

$$= \frac{\alpha_s(\mu)}{\pi} C_F \left[ \ln \frac{1}{b\mu} - K_0 \left( \frac{2\nu p^+ b}{N} \right) \right] ,$$

(34)

$K_0$ being the modified Bessel function. It is easy to examine the large $b$ and $N$ limits of the above expression: for $p^+ b \gg N$, we have $K_0 \to 0$ and the soft function $K$ approaches Eq. (15) for the $k_T$ resummation. For $N \gg p^+ b$, we have $K_0 \approx -\ln(\nu p^+ b/N)$ and $K$ approaches Eq. (27) for the threshold resummation.

Equation (34) implies a characteristic scale of order

$$\frac{1}{b} \exp \left[ -K_0 \left( \frac{p^+ b}{N} \right) \right] .$$

(35)

Following the similar procedures from Eqs. (19)-(21), we derive the unified resummation,

$$\phi(N, b, p^+) = \Delta_u(N, b, p^+) \phi(0) ,$$

(36)

with the exponential

$$\Delta_u(N, b, p^+) \exp \left[ -2 \int_{\exp[-K_0(p^+ b/N)/b]}^{p^+} \frac{dp}{p} \int_{\exp[-K_0(p^+ b)/b]}^{p} \frac{d\mu}{\mu} \gamma_K(\alpha_s(\mu)) \right] ,$$

(37)

which is appropriate for arbitrary $b$ and $N$. The lower bound of $p$ corresponds to Eq. (33), while the lower bound of $\mu$ is motivated by the $b \to \infty$ and $b \to 0$ limits, at which Eq. (37) approaches Eq. (27) and Eq. (30), respectively.

In the intermediate and small $x$ regions, it is not necessary to resum the double logarithms $\ln^2(1/N)$. After extracting the $k_T$ resummation, the remaining single-logarithm summation corresponds to a unification of the DGLAP and BFKL equations, since both the $l^+$ and $l_T$ dependences have been kept. We reexpress the function $\phi$ in the integrand of $\tilde{\phi}_{sr}$, under the Fourier transformation, as

$$\phi(x + l^+ / p^+, b, p^+) = \theta((1 - x)p^+ - l^+) \phi(x, b, p^+)$$

$$+ [\phi(x + l^+ / p^+, b, p^+) - \theta((1 - x)p^+ - l^+) \phi(x, b, p^+)] .$$

(38)
The contribution from the first term is combined with \( \bar{\phi}_{sv} \), giving the soft function \( K \) for the \( k_T \) resummation. The RG solution of \( K + G \) is given by

\[
K + G = \bar{\alpha}_s(x p^+) \left[ \ln(1 - x) + \ln(p^+ b) \right] - \int_{1/b}^{x p^+} \frac{d\mu}{\mu} \gamma_K(\alpha_s(\mu)) ,
\]

(39)

where the first term on the right-hand side comes from the extra \( \theta \) function in Eq. (38). The color factor has been replaced by \( N_c \), since we are considering the gluon distribution function. The contribution from the second term is written as

\[
i N_c g^2 \int \frac{d^4 l}{(2\pi)^4} N_{\nu\beta}(l) \frac{\partial^\nu \bar{v}}{v \cdot l} 2\pi i \delta(l^2) e^{il \cdot b} \\
\times \left[ \phi(x + l^+/p^+, b, p^+) - \theta((1 - x)p^+ - l^+) \phi(x, b, p^+) \right] ,
\]

(40)

which will generate the splitting function below.

The master equation (8) then becomes

\[
p^+ \frac{d}{dp^+} \phi(x, b, p^+) = -2 \left[ \int_{1/b}^{x p^+} \frac{d\mu}{\mu} \gamma_K(\alpha_s(\mu)) - \bar{\alpha}_s(x p^+) \ln(p^+ b) \right] \phi(x, b, p^+) \\
+ 2\bar{\alpha}_s(x p^+) \int_1^1 dz P_{gg}(z) \phi(x/z, b, p^+) ,
\]

(41)

with the splitting function

\[
P_{gg} = \left[ \frac{1}{(1 - z)_+} + \frac{1}{z} - 2 + z(1 - z) \right] ,
\]

(42)

obtained from Eq. (40). The term \(-2 + z(1 - z)\) finite as \( z \to 0 \) and \( z \to 1 \) has been added. The term proportional to \( \ln(1 - x) \) in Eq. (39) has been absorbed into \( P_{gg} \) to arrive at the plus distribution \( 1/(1 - z)_+ \). We first extract the exponential \( \Delta \) from the \( k_T \) resummation,

\[
\Delta(x, b, Q_0, p^+) = \exp \left( -2 \int_{x Q_0}^{x p^+} \frac{dp}{p} \left[ \int_{1/b}^{p} \frac{d\mu}{\mu} \gamma_K(\alpha_s(\mu)) - \bar{\alpha}_s(p) \ln(pb) \right] \right) ,
\]

(43)

where \( Q_0 \) is an arbitrary low energy scale. It is easy to justify by substitution that the gluon distribution function is given by

\[
\phi(x, b, Q) = \Delta(x, b, Q_0, Q) \phi^{(0)} \\
+ 2 \int_1^1 dz \int_{Q_0}^{Q} \frac{d\mu}{\mu} \bar{\alpha}_s(x \mu) \Delta_k(x, b, \mu, Q) P_{gg}(z) \phi(x/z, b, \mu) ,
\]

(44)
with $\phi^{(0)}$ the initial condition of $\phi$. Equation (44) is the unified evolution equation, which can be regarded as a modified version of the Ciafaloni-Catani-Fiorani-Marchesini equation [12].

6. Conclusion

In this paper we have demonstrated that all the known single- and double-logarithm summations, including their unifications, can be derived in the CS technique. The point is the treatment of the real gluon contributions. Simply adopting soft approximations appropriate in different kinematic regions, i.e., neglecting the $l^+$ or $l_T$ dependence in the parton distribution function associated with the real gluon emission, the CS technique reduces to the various logarithmic summations. If keeping both the $l^+$ and $l_T$ dependences, a unified resummation for large $x$ and a unified evolution equation for intermediate and small $x$ are obtained. Our conclusion has been summarized in Table I.

The $k_T$ and threshold resummations, and the DGLAP and BFKL equations have been widely studied and applied to many QCD processes. The unified resummation is appropriate for the analysis of dijet production [13], in which the transverse energy of one jet (the trigger jet) is measured, while the other jet (the probe jet) has large rapidity up to 3.0, that corresponds to high $x$ values. The unified evolution equation, because of its extra $Q$ dependence at small $x$, is appropriate for the explanation of the HERA data of the proton structure function $F_2(x, Q^2)$ [14]. These subjects will be discussed elsewhere.

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Table I. Logarithmic summations derived from the Collins-Soper technique under different soft approximations at different Bjornen variables $x$. 

|               | small $x$             | intermediate $x$ | large $x$            |
|---------------|-----------------------|------------------|----------------------|
| neglect $l^+$ | BFKL equation         | $k_T$ resummation|
| neglect $l_T$ | DGLAP equation        | threshold resummation |
| no neglect    | unified               | unified equation | unified resummation  |

**Figure Captions**

**FIG. 1.** (a) The derivative $p^+ d\phi / dp^+$ in the axial gauge. (b) The soft structure and (c) the ultraviolet structure of the $O(\alpha_s)$ subdiagram containing the special vertex.
\[ p^+ \frac{d}{dp^+} = 2 \]

(a)

(b)

(c)

FIG. 1