Finite, fiber- and orientation-preserving group actions on orientable Seifert manifolds with orientable base space

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Abstract

In this paper we consider the finite groups that act fiber- and orientation-preservingly on orientable Seifert manifolds that fiber over an orientable base space without boundary. We establish a method of constructing such group actions and then show that if an action satisfies a condition on the obstruction term of the Seifert manifold, it can be derived from the given construction. The obstruction condition is refined and the general structure of the finite groups that act via the construction is provided.

Keywords geometry; topology; 3-manifolds; finite group actions; Seifert fiberings

1 Introduction

The main question asked in this paper is: “What are the possible finite, fiber- and orientation-preserving group actions on an orientable Seifert manifold with orientable base space and without boundary?” We consider this by first providing a construction of an orientation-preserving group action on a given Seifert manifold. This construction is founded upon the way a Seifert manifold is put together as Dehn fillings of $S^1 \times F$. Here $F$ is a surface with boundary. The construction is - in a general sense - to take a product action on $S^1 \times F$ and extend across the Dehn fillings.

We analyze, refine, and rework a result of Peter Scott and William Meeks. This result establishes that if a finite action on $S^1 \times F$ respects the product structure on the boundary, then there is a product structure that agrees with the original product structure on the boundary and is left invariant by the action. This result allows us to consider when finite actions can be constructed via the given method.

The main result then shows that given a finite, orientation and fiber-preserving action, the action can be constructed via the given method - provided it satisfies a condition on the obstruction term of the Seifert manifold. This obstruction condition is then refined and the general structure of such a group is provided.

We now give some preliminary definitions. Let $M$ be an oriented smooth manifold of dimension 3 without boundary and $G$ be a finite group. We let $Diff(M)$ be the group of self-diffeomorphisms of $M$, and then define a $G$-action on $M$ to be an injection $\varphi : G \to Diff(M)$. We use the notation $Diff_+(M)$ for the group of orientation-preserving self-diffeomorphisms of $M$.

$M$ will be assumed to be an orientable Seifert-fibered manifold. We use the original Seifert definition. That is, a Seifert manifold is a 3-manifold such that $M$ can be decomposed into disjoint fibers where each fiber is a simple closed curve. Then for each fiber $\gamma$, there exists a fibered neighborhood (that is, a subset consisting of fibers and containing $\gamma$) which can be mapped under a fiber-preserving map onto a solid fibered torus. [1]

We call a Seifert bundle a Seifert manifold $M$ along with a continuous map $p : M \to B$ where $p$ identifies each fiber to a point. For clarity, we denote the underlying space of $B$ as $B_U$. [1]
Following [2], we use \((n_1, \ldots, n_k; m_1, \ldots, m_l)\) as a data set for a 2-orbifold \(B\) with \(k\) cone points of orders \(n_1, \ldots, n_k\), and \(l\) corner reflectors of orders \(m_1, \ldots, m_l\).

A \(G\)-action is said to be fiber-preserving on a Seifert manifold \(M\) if for any fiber \(\gamma\) and any \(g \in G\), \(\varphi(g)(\gamma)\) is some fiber of \(M\). We use the notation \(Diff^f(M)\) for the group of fiber-preserving self-diffeomorphisms of \(M\) (given some Seifert fibration). For distinction, we use the notation \(Diff^{f,f}(M)\) to refer to I-fiber-preserving diffeomorphisms (given some I-fibration). Given a fiber-preserving \(G\)-action, there is an induced action \(\varphi_{B_u} : G_{B_u} \to Diff(B_u)\) on the underlying space \(B_u\) of the base space \(B\).

Given a finite action \(\varphi : G \to Diff^f(M)\), we define the orbit number of a fiber \(\gamma\) under the action to be \(\# \text{Orb}_\varphi(\gamma) = \# \{ \alpha | \varphi(g)(\gamma) = \gamma \text{ for some } g \in G \}\).

If we have a manifold \(M\), then a product structure on \(M\) is a diffeomorphism \(k : A \times B \to M\) for some manifolds \(A\) and \(B\). [3] If a Seifert-fibered manifold \(M\) has a product structure \(k : S^1 \times F \to M\) for some surface \(F\) and \(k(S^1 \times \{x\})\) are the fibers of \(M\) for each \(x \in F\), then we say that \(k : S^1 \times F \to M\) is a fibered product structure of \(M\).

We note here that a fibering product structure on \(M\) is equivalent to the existence of a foliation of \(M\) by both circles and by surfaces diffeomorphic to \(F\) so that any circle intersects each foliated surface exactly once.

Given that the first homology group (equivalently the first fundamental group) of a torus is \(\mathbb{Z} \times \mathbb{Z}\) generated by two elements represented by any two nontrivial loops that cross at a single point, we can use the meridian-longitude framing from a product structure as representatives of two generators. If we have a diffeomorphism \(f : T_1 \to T_2\) and product structures \(k_i : S^1 \times S^1 \to T_i\), then we can express the induced map on the first homology groups by a matrix that uses bases for \(H_1(T_i)\) derived from the meridian-longitude framings that arise from \(k_i : S^1 \times S^1 \to T_i\). We denote this matrix as \[ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^k_i \]

We say that a \(G\)-action \(\varphi : G \to Diff(A \times B)\) is a product action if for each \(g \in G\), the diffeomorphism \(\varphi(g) : A \times B \to A \times B\) can be expressed as \((\varphi_1(g), \varphi_2(g))\) where \(\varphi_1(g) : A \to A\) and \(\varphi_2(g) : B \to B\). Here \(\varphi_1 : G \to Diff(A)\) and \(\varphi_2 : G \to Diff(B)\) are not necessarily injections.

Given an action \(\varphi : G \to Diff(M)\) and a product structure \(k : A \times B \to M\), we say that \(\varphi\) leaves the product structure \(k : A \times B \to M\) invariant if \(\psi(g) = k^{-1} \circ \varphi(g) \circ k\) defines a product action \(\psi : G \to Diff(A \times B)\).

If we have a manifold \(\tilde{M}\) with torus boundary components and each of those boundary tori \(T_i\) have a product structure \(k_i : S^1 \times S^1 \to T_i\), then we say a \(G\)-action \(\varphi : G \to Diff(M)\) respects the product structures on the boundary tori if \(k_j^{-1} \circ \varphi(g) \circ k_i : S^1 \times S^1 \to S^1 \times S^1\) can be expressed as \((\varphi_1(g), \varphi_2(g))\) where \(\varphi_1 : G \to Diff(S^1)\) and \(\varphi_2 : G \to Diff(S^1)\). These again are not necessarily injections.

Suppose that we now have a fibered product structure \(k : S^1 \times F \to M\). We then say that each boundary torus is positively oriented if the fibers are given an arbitrary orientation and then each boundary component of \(k([u] \times F)\) is oriented by taking the normal vector to the surface according the orientation of the fibers.

## 2 Dehn fillings and Seifert manifolds.

We first establish some background work on Dehn fillings and Seifert manifolds.

This section broadly follows the construction from [4]. We use the notation for a closed orientable Seifert manifold \(M\) with orientable base space:

\[(g, o_1((q_1, p_1), \ldots, (q_n, p_n)), q_i > 0)\]

This notation implies that \(M\) is a manifold that can be decomposed into a manifold \(\tilde{M} \cong S^1 \times F\) that is trivially fibered with boundary \(\partial \tilde{M} = T_1 \cup \ldots \cup T_n\), and \(X = V_1 \cup \ldots \cup V_n\), a disjoint collection of fibered solid tori (the notation specifies the fibration). \(M\) is reobtained by a gluing map \(d : \partial X \to \partial \tilde{M}\). This is defined as follows:

Take a given fibered product structure \(k_{\tilde{M}} : S^1 \times F \to \tilde{M}\) on \(\tilde{M}\), and some particular product structure \(k_X : S^1 \times (D_1 \cup \ldots \cup D_n) \to X\) where each \(D_i\) is a disk. Then define product structures \(k_{\partial V_i} : S^1 \times S^1 \to \partial V_i\) and \(k_{T_i} : S^1 \times S^1 \to T_i\) by parameterizing each component of \(\partial F\) and \(\partial D_i\) with a positive orientation by some diffeomorphisms \(\rho_i : S^1 \to (\partial F)_i\) and \(\sigma_i : S^1 \to \partial D_i\), and then taking \(k_{\partial V_i}(u, v) = k_X(u, \sigma_i(v))\) and \(k_{T_i}(u, v) = k_{\tilde{M}}(u, \rho_i(v))\).
\( d : \partial X \to \partial \hat{M} \) is then a diffeomorphism such that \( d(\partial V_i) = T_i \) and

\[
(k_{T_i}^{-1} \circ d|_{\partial V_i} \circ k_{\partial V_i})(u, v) = (u^{x_i}u^{p_i}, u^{y_i}v^{q_i})
\]

Where \( x_iq_i - y_ip_i = -1 \) and \(|y_i| < q_i\).

We note therefore that the induced fibration on each solid torus \( V_i \), is a \((-q_i, y_i)\) fibration (according to \( k_{\partial V_i} \)).

We now quote Theorem 1.1. from [5] regarding Seifert invariants:

**Theorem 2.1.** Let \( M \) and \( M' \) be two orientable Seifert manifolds with associated Seifert invariants

\[
(g, o_1 | (\alpha_1, \beta_1), \ldots, (\alpha_s, \beta_s)) \quad \text{and} \quad (g, o_1 | (\alpha'_1, \beta'_1), \ldots, (\alpha'_s, \beta'_s))
\]

respectively. Then \( M \) and \( M' \) are orientation-preservingly diffeomorphic by a fiber-preserving diffeomorphism if and only if, after reindexing the Seifert pairs if necessary, there exists an \( n \) such that:

i) \( \alpha_i = \alpha'_i \) for \( i = 1, \ldots, n \) and \( \alpha_i = \alpha'_j = 1 \) for \( i, j > n \)

ii) \( \beta_i \equiv \beta'_i \mod \alpha_i \) for \( i = 1, \ldots, n \)

iii) \[
\sum_{i=1}^{s} \beta_i \alpha_i = \sum_{i=1}^{t} \beta'_i \alpha'_i
\]

The consequence of this theorem is that we can perform the following "moves" on the Seifert invariants:

i) Permute the indices

ii) Add or delete a Seifert pair \((1,0)\)

iii) Replace \((\alpha_1, \beta_1), (\alpha_2, \beta_2)\) by \((\alpha_1, \beta_1 + m\alpha_1), (\alpha_2, \beta_2 - m\alpha_2)\) for some integer \( m \).

From this we yield the Corollary:

**Corollary 2.2.** Let \( M \) and \( M' \) be two orientable Seifert manifolds with associated Seifert invariants

\[
(g, o_1 | (\alpha_1, \beta_1), \ldots, (\alpha_s, \beta_s)) \quad \text{and} \quad (g, o_1 | (\alpha_1, \beta_1 + m_1\alpha_1), \ldots, (\alpha_s, \beta_s + m_s\alpha_s))
\]

respectively. Then \( M \) and \( M' \) are orientation-preservingly diffeomorphic by a fiber-preserving diffeomorphism if and only if

\[
\sum_{i=1}^{s} m_i = 0
\]

**Proof.** By Theorem 2.1, we need only consider the third condition. The first two conditions hold trivially. So, the two manifolds are diffeomorphic if and only if:

\[
\sum_{i=1}^{s} \beta_i \alpha_i = \sum_{i=1}^{s} \beta_i + m_i\alpha_i = \sum_{i=1}^{s} \beta_i \alpha_i + \sum_{i=1}^{s} m_i
\]

Hence, if and only if

\[
\sum_{i=1}^{s} m_i = 0
\]

We can now define normalized Seifert invariants so that any orientable Seifert manifold over an orientable base space can be expressed as:

\[
(g, o_1 | (q_1, p_1), \ldots, (q_n, p_n), (1, b))
\]

Where \( 0 < p_i < q_i \) and \( b \) is some integer called the obstruction term.

The constant:

\[
e = -(b + \sum_{i=1}^{n} \frac{p_i}{q_i})
\]
is known as the Euler class of the Seifert bundle and is zero if and only if the Seifert bundle is covered by the trivial bundle. Alternatively, it is zero if the manifold $M$ has the geometry of either $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, or $E^3$. For more details, refer to [6].

3 Construction of a finite, orientation and fiber-preserving action

We now present a construction for a finite, orientation and fiber-preserving action on a Seifert manifold $M = (g, a_1\{|(q_1, p_1), \ldots, (q_n, p_n)\})$. Here the Seifert invariants are not necessarily normalized.

We first note that according to Section 2, we can decompose $M$ into $\hat{M}$ and $X$ where $\hat{M} \cong S^1 \times F$ is trivially fibered and $X$ is a disjoint union of $n$ solid tori. We then have a gluing map $d : \partial X \rightarrow \partial \hat{M}$, so that for a fiber preserving product structure $k_{\hat{M}} : S^1 \times F \rightarrow \hat{M}$, there is some $k_X : S^1 \times (D_1 \cup \ldots \cup D_n) \rightarrow X$ and restricted positively oriented product structures $k_{\partial V_i} : S^1 \times S^1 \rightarrow \partial V_i$ and $k_{T_i} : S^1 \times S^1 \rightarrow T_i$ such that $(k_{T_i}^{-1} \circ d|_{\partial V_i} \circ k_{\partial V_i})(u, v) = (u^{\hat{x}}, v^{\hat{y}}, u^y v^x)$.

3.1 Constructing a finite, fiber-preserving action on $\hat{M}$.

We pick a finite, fiber-preserving group action on $\hat{M}$ by first choosing some (not-necessarily effective) group action $\varphi_1 : G \rightarrow Diff(S^1)$. This will necessarily be of the form:

$$\varphi_1(g)(u) = \theta_1(g)u^\alpha(g)$$

Here $\theta_1 : G \rightarrow S^1$ and $\alpha : G \rightarrow \{-1, 1\}$. The precise nature of these maps is shown in Section 3.5.

Then we choose a (not-necessarily effective) group action $\varphi_2 : G \rightarrow Diff(F)$ such that if we parameterize each component of $\partial F$ in the same way as in Section 2 and then express $\partial F = \{(v, i)|v \in S^1, i \in \{1, \ldots, n\}\}$, we can write:

$$\varphi_2(g)|_{\partial F}(v, i) = (\theta_2(i, g)u^{\alpha(g)}, \beta(g)(i))$$

Here $\theta_2 : \{1, \ldots, n\} \times G \rightarrow S^1$, and $\beta : G \rightarrow perm(\{1, \ldots, n\})$ are such that $\beta(g)(i) = j$ only if $(q_i, p_i) = (q_j, p_j)$.

Then we define our group action $\varphi : G \rightarrow Diff(\hat{M})$ by:

$$(k_{\hat{M}}^{-1} \circ \varphi(g) \circ k_{\hat{M}})(u, x) = (\varphi_1(g)(u), \varphi_2(g)(x))$$

So now we can fully express $\varphi : G \rightarrow Diff(\hat{M})$ on the boundary of $\hat{M}$ by:

$$(k_{T_{\beta(g)(i)}}^{-1} \circ \varphi(g) \circ k_{T_i})(u, v) = (\theta_1(g)u^{\alpha(g)}, \theta_2(i, g)u^{\alpha(g)})$$

We note here that (according to the set framing of each boundary torus), each element $g \in G$ acts on a boundary tori $T_i$ by mapping it to $T_{\beta(g)(i)}$ with:

- a rotation by $\theta_1(g)$ in the longitudinal direction.
- a rotation by $\theta_2(i, g)$ in the meridional direction.
- a reflection in the meridian and longitude if $\alpha(g) = -1$.

3.2 Inducing a finite, fiber-preserving action on $\partial X$.

So we can now induce an action on $\partial X$ by:

$$\psi : G \rightarrow Diff(\partial X), \psi(g) = d^{-1} \circ \varphi(g)|_{\partial \hat{M}} \circ d$$

This we can fully express (after simplification) as:

$$(k_{\partial V_{\beta(g)(i)}}^{-1} \circ \psi(g) \circ k_{\partial V_i})(u, v) = (\theta_1(g)^{-u_2(i, g)p_i}u^{\alpha(g)}, \theta_1(g)^{\alpha(g)}\theta_2(i, g)^{x_i\alpha(g)})$$
Therefore (according to the set framing of each boundary torus), each element $g \in G$ acts on a $\partial V_i$ by mapping it to $\partial V_{\beta(g)(i)}$ with:

- a rotation by $\theta_1(g)^{-q_i}\theta_2(i, g)\theta_1(g)^{p_i}$ in the longitudinal direction.
- a rotation by $\theta_1(g)^{p_i}\theta_2(i, g)^{-q_i}$ in the meridional direction.
- a reflection in the meridian and longitude if $\alpha(g) = -1$.

Alternatively, we could view this action by each element $g \in G$ mapping $\partial V_i$ to $\partial V_{\beta(g)(i)}$ with:

- a rotation by $\theta_1(g)$ along $(-q_j, y_j)$ curves (along the fibers).
- a rotation by $\theta_2(i, g)$ along $(p_j, -x_j)$ curves.
- a reflection in the meridian and longitude if $\alpha(g) = -1$.

### 3.3 Extending the induced action to $X$.

We here note that:

$$k_X^{-1}(X) = \{(u, v, i) | u \in S^1, v \in D, i \in \{1, \ldots, n\}\}$$

Where $D$ is the unit disc. Hence the action $\psi : G \to Diff(X)$ straightforwardly extends by coning inwards. This works as the product structure on $X$ is such that the fibration is normalized. Hence, the extended action is fiber-preserving.

### 3.4 The final action.

So now we have defined finite, fiber-preserving actions on $\tilde{M}$ and $X$ such that they agree under the gluing map $d : \partial X \to \partial \tilde{M}$. This completes the construction.

**Remark 1.** Note that in these examples $\varphi_1 : G \to Diff(S^1)$ and $\varphi_2 : G \to Diff(F)$ are not injections in all cases and so not necessarily effective actions.

### 3.5 Conditions for $\varphi_1 : G \to Diff(S^1)$ and $\varphi_2 : G \to Diff(F)$

We here establish some necessary and sufficient conditions in the construction of $\varphi_1 : G \to Diff(S^1)$ and $\varphi_2 : G \to Diff(F)$.

**Proposition 3.1.** The following are necessary and sufficient conditions on $\theta_1 : G \to S^1$ and $\alpha : G \to \{-1, 1\}$ for $\varphi_1 : G \to Diff(S^1)$ to be a homomorphism:

**i)** $\alpha : G \to \{-1, 1\}$ is a homomorphism,

**ii)** $\theta_1(g_1g_2) = \theta_1(g_1)\theta_1(g_2)^{\alpha(g_1)}$

**Proof.** We calculate $\varphi_1(g_1g_2)(u) = \theta_1(g_1g_2)u^{\alpha(g_1g_2)}$ and:

$$\varphi_1(g_1) \circ \varphi_1(g_2)(u) = \theta_1(g_1)(\theta_1(g_2)u^{\alpha(g_2)})^{\alpha(g_1)} = \theta_1(g_1)\theta_1(g_2)^{\alpha(g_1)}u^{\alpha(g_2)\alpha(g_1)}$$

These are equal for all values of $u$. Hence for $u = 1$ we have that $\theta_1(g_1g_2) = \theta_1(g_1)\theta_1(g_2)^{\alpha(g_1)}$.

This establishes part ii) and then implies that $u^{\alpha(g_1g_2)} = u^{\alpha(g_1)\alpha(g_2)}$ which establishes part i). $\square$

**Proposition 3.2.** The following are necessary conditions on $\theta_2 : \{1, \ldots, n\} \times G \to S^1$, $\alpha : G \to \{-1, 1\}$, and $\beta : G \to perm(\{1, \ldots, n\})$ if $\varphi_2 : G \to Diff(F)$ is a homomorphism:

**i)** $\alpha : G \to \{-1, 1\}$ is a homomorphism,

**ii)** $\beta : G \to perm(\{1, \ldots, n\})$ is a homomorphism,

**iii)** $\theta_2(i, g_1g_2) = \theta_2(\beta(g_2)(i), g_1)\theta_2(i, g_2)^{\alpha(g_1)}$
We then let $f$ with boundary and $\hat{\beta}$ These are again equal for all values of $v$.

Now, for $v = 1$ we have that $\theta_2(i, g_1 g_2) = \theta_2(\beta(g_2)(i), g_1)\theta_2(i, g_2)^{\alpha(g_1)}$.

This establishes part iii) and leaves $v^{\alpha(g_1 g_2)} = v^{\alpha(g_1 \alpha(g_2)}$ which establishes part i).

\begin{proof}

Proof. We first calculate $\varphi_2(g_1 g_2)(v, i) = (\theta_2(i, g_1 g_2)v^{\alpha(g_1 g_2)}, \beta(g_1 g_2)(i))$. Then calculate:

$$
\varphi_2(g_1) \circ \varphi_2(g_2)(v, i) = \varphi_2(g_1)(\theta_2(i, g_2)v^{\alpha(g_2)}, \beta(g_2)(i))
= (\theta_2(\beta(g_2)(i), g_1)(\theta_2(i, g_2)v^{\alpha(g_2)})^{\alpha(g_1)}, \beta(g_2) \circ \beta(g_1)(i))
= (\theta_2(\beta(g_2)(i), g_1)\theta_2(i, g_2)^{\alpha(g_1)}v^{\alpha(g_1 \alpha(g_2)}), \beta(g_2) \circ \beta(g_1)(i))
$$

These are again equal for all values of $v$ and $i$. We immediately have that $\beta(g_1 g_2) = \beta(g_1) \circ \beta(g_2)$ and part ii) follows.

\end{proof}

\section{Actions on $\hat{M}$}

In order to find out to what extent finite, fiber-preserving actions can be derived from the construction set out in Section 3, we first need to establish a result regarding actions on $\hat{M}$. In this section we always take $F$ to be an orientable surface with boundary and $\hat{M}$ to be the fibered manifold that has boundary made up of tori described earlier.

The main result we prove in this section is an adaptation of Theorem 2.3 in [7]. It will state that if $\hat{M}$ has a product structure, then provided the restricted product structures on each boundary component are respected by the action, then there is another product structure on $\hat{M}$ that is left invariant by the group action. Moreover, the two product structures foliate the boundary tori identically.

We first state some preliminary results.

\begin{lemma}

Let $\varphi : G \to Diff(F)$ be a finite group action with $F$ not a disc. Then $F$ contains a $\varphi$-equivariant essential simple arc.

\end{lemma}

\begin{proof}

$F/\varphi$ is a 2-orbifold. We can then pick an essential simple arc in the underlying space of $F/\varphi$ that doesn’t intersect the exceptional points and then lift this to a $\varphi$-equivariant essential simple arc in $F$.

\end{proof}

\begin{lemma}

Let $\psi : G \to Diff(T)$ be a finite group action on a Seifert fibered torus. Suppose that there exists a fibering product structure $k : S^1 \times S^1 \to T$. Then $\psi : G \to Diff(T)$ is equivalent to a fiber-preserving group action that leaves the product structure $k : S^1 \times S^1 \to T$ invariant. Moreover, the conjugating map is fiber-preserving and isotopic to the identity.

\end{lemma}

\begin{proof}

First note that necessarily, $\psi(g)_* = \pm \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix}$. This follows from the fact that $\pm \begin{bmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has finite order only if $c = 0$.

We then note that by [2], the only possible quotient types are a torus or $S^2(2, 2, 2)$. By [8] these refer respectively to actions of groups $Z_m \times Z_n$ and $Dih(Z_m \times Z_n)$ where $Z_m \times Z_n$ acts by preserving the orientation of the fibers and the dihedral $Z_2$ subgroup of $Dih(Z_m \times Z_n)$ acts by reversing the orientation of the fibers.

We first consider the torus case. This will receive an induced fibration from $T$. We then can pick a fibering product structure on $T/\psi$. This product structure can be lifted to an invariant fibering product structure $k' : S^1 \times S^1 \to T$. According to this product structure, the group acts as rotations along the fibers or along loops $k'(\{u\} \times S^1)$. As such, it preserves any fibration up to isotopy. So we can assume that $k' : S^1 \times S^1 \to T$ is in fact isotopic to the original product structure $k : S^1 \times S^1 \to T$.

We then let $f = k' \circ k^{-1}$. So that $k^{-1} \circ f^{-1} \circ \psi(g) \circ f \circ k = k^{-1} \circ \psi(g) \circ k$.

This is a product. It also follows that $f$ is fiber-preserving and isotopic to the identity.

If the action has quotient of $S^2(2, 2, 2)$, then we note that as the fiber orientation-preserving subgroup $Z_m \times Z_n$ is a normal subgroup, we can consider the induced $Z_2$-action on the quotient of the $Z_m \times Z_n$-action. This is necessarily a “spin” action by [8] and we can pick a fibering product structure on $T/(\hat{Z}_m \times \hat{Z}_n)$ as above but that is further left invariant under the “spin” action.

\end{proof}
Lemma 4.3. Let \( k : S^1 \times F \to \hat{M} \) and \( k' : S^1 \times F \to \hat{M} \) be fibering product structures so that they foliate the boundary tori identically. Then \( k(\{1\} \times F) \) is freely isotopic to \( k'(\{1\} \times F) \).

Proof. Consider, \( k'^{-1} \circ k : S^1 \times F \to S^1 \times F \). Necessarily, this can be expressed in the form \( (k'^{-1} \circ k)(u, x) = (k_1(u, x), k_2(x)). \)

So now by composing with the diffeomorphism \( l : S^1 \times F \to S^1 \times F \) given by \( l(u, x) = (u, k_2^{-1}(x)) \), we have that \( (k'^{-1} \circ k \circ l)(u, x) = (k_1(u, x), x) \).

Consider \( (k \circ l)(S^1 \times \{x\}) \) and \( (k')(S^1 \times \{x\}) \). These are the same fiber. Hence \( (k \circ l)(\{1\} \times F) \) and \( (k')(\{1\} \times F) \) are freely isotopic by isotoping along the fibers.

The proof of the theorem follows that of [7] in an adapted and expanded form.

Theorem 4.4. Let \( k : S^1 \times F \to \hat{M} \) be a fibering product structure such that the finite group action \( \varphi : G \to Diff^+_{fr}(\hat{M}) \) respects the restricted product structures on each boundary torus. Then there exists an isotopic fibering product structure \( k' : S^1 \times F \to \hat{M} \) such that the group action \( \psi : G \to Diff(S^1 \times F) \) given by \( \psi(g) = k'^{-1} \circ \varphi(g) \circ k' \) for each \( g \in G \) is a product action and foliates the boundary identically to \( k \).

Proof. We proceed by induction on the Euler Characteristic of \( F \).

Base Case: \( \chi(F) = 1 \)

We therefore have \( \hat{M} \) as a trivially fibered solid torus with \( k : S^1 \times F \to \hat{M} \), a fibering product structure. By the product structure on the boundary, we have a foliation by meridianal circles that each bound a disc and the usual longitudinal Seifert fibration by circles. So any of the meridianal circles are necessarily \( \varphi \)-equivariant. Then taking such a circle, we apply the equivariant Dehn’s Lemma to yield a \( \varphi \)-equivariant disc \( D \) whose boundary agrees with the product structure on the boundary of the solid torus. We now decompose along \( Orb(D) = \{D_1, \ldots, D_s\} \) to yield a collection \( B_1, \ldots, B_s \) of balls, each which are homeomorphic to \( I \times D \) and fibered by arcs.

So starting with \( B_1 \) we have the action \( \varphi_1 : Stab(B_1) \to Diff(B_1) \) given by \( \varphi_1(g) = \varphi(g)|_{B_1} \).

Note that the quotient orbifold \( B_1 / \varphi_1 \) necessarily has boundary either \( S^2(n, n) \) or \( S^2(2, 2, n) \). This follows from [9], that show that these are the only orientable quotients of \( S^2 \) where the action fixes one point or exchanges two points (corresponding to the two discs \( D_1, D_2 \)).

We here use the proof of the Smith conjecture (see ball orbifolds in [10]) to see that \( B_1 / \varphi_1 \) has the following possible forms with induced (orbifold) foliations on part of the boundary shown by Figure 1.

On the part of the boundary that lifts into \( \partial \hat{M} \), the first two are foliated simply by circles, and the third is foliated by circles and one 1-orbifold with cone points of order 2 on either end.

This first can then clearly be foliated by discs that agree with the foliation by circles on the boundary. The second can be foliated by discs with a cone point of order \( n \) with the discs agreeing with the foliation by circles on the boundary.

The third can be foliated by discs with cone points order \( n \) - with the discs having boundaries given by the circles - and a 2-orbifold of the form shown in Figure 2.

Each of these can be taken to hit each induced orbifold \( I \)-fiber once and will lift to an invariant foliation of \( B_1 \) by discs that each hit each \( I \)-fiber once.

Figure 1: Possible quotients with induced orbifold foliations on part of the boundary
We therefore have a product structure $k_1 : I \times F \to B_1$ left invariant by the action $\varphi : Stab(B_1) \to Diff(B_1)$ whose foliation (by arcs and circles) on the part of its boundary that intersects with the boundary of $\hat{M}$ is equal to the restricted foliation from $k : S^1 \times F \to \hat{M}$.

Now for each $B_i$, there is some $g_i \in G$ such that $\varphi(g_i)(B_1) = B_i$.

We can then define product structures $k_i : I \times F \to B_i$ by $k_i = \varphi(g_i) \circ k_1$. Note that as each $\varphi(g_i)$ leaves the original product structure $k : S^1 \times F \to \hat{M}$ invariant on the boundary of $\hat{M}$ then each $k_i : I \times F \to B_i$ foliates $B_i$ (by arcs and circles) on the part of its boundary that intersects with the boundary of $\hat{M}$ the same way as the restricted foliation from $k : S^1 \times F \to \hat{M}$.

Then for any $g \in G$ such that $\varphi(g)(B_i) = B_j$ we have $g = g_i h g_i^{-1}$ for some $h \in Stab(B_1)$ and can calculate $k_j^{-1} \circ \varphi(g) \circ k_i = k_j^{-1} \circ \varphi(h) \circ k_1$. This is a product by above.

So now we have a collection of product structures on each $B_1, \ldots, B_s$ that are left invariant under the action. We view these now as invariant foliations by arcs and discs. By construction, we yield invariant foliations of $\hat{M}$ by circles and discs. This is possible as each of the invariant foliations of $B_i$ are equal to the restricted foliation from $k : S^1 \times F \to \hat{M}$ on the part of its’ boundary that intersects with the boundary of $\hat{M}$.

These invariant foliations give our required $k' : S^1 \times F \to \hat{M}$.

Inductive Step:

We suppose the result holds for $\chi(F) > c$ and now consider $\chi(F) = c$.

We induce the action $\varphi_F : G_F \to Diff(F)$ on the base space of the fibration and then apply Lemma 4.1 to yield a $\varphi_F$-equivariant essential simple arc in $F$. We call this arc $\lambda$ and define $A_1$ to be the annulus made up of fibers that project to $\lambda$. As $\varphi : G \to Diff(M)$ is fiber-preserving, this is necessarily $\varphi$-equivariant.

Cutting along the collection of annuli $Orb(A_1)$ will yield a disjoint collection $\{\hat{M}_1, \ldots, \hat{M}_n\}$ of manifolds with boundary which fiber over surfaces $\{F_1, \ldots, F_n\}$. Necessarily, each of these have greater Euler number than $F$.

Now pick $\hat{M}_1$. Then pick any boundary torus $T$ of $\hat{M}_1$ that contains $A_1$. This consists of annuli that were originally contained in a boundary tori of $\hat{M}$ before being cut open - we refer to these as $A_1', \ldots, A_m'$ - or some annuli in the collection $Orb(A_1)$ - we refer to these as $A_1, \ldots, A_m$. Each of $A_1', \ldots, A_m'$ inherit product structures $k_{A_i'} : S^1 \times I \to A_i'$ that are respected under the restricted action of $Stab(T)$.

Now consider $T/\text{Stab}(T)$. This will necessarily be either another torus consisting of two glued annuli - one referring to the projection of $A_1$ and the other referring to the projection of $A_i'$ - or an $S^2(2,2,2,2)$ consisting of two glued together $D(2,2)$ - again, one referring to the projection of $A_1$ and the other referring to the projection of $A_i'$. This follows from 2.

Case 1: $T/\text{Stab}(T)$ is a torus.

The annulus covered by $A_1'$ has an induced Seifert fibration and foliation by arcs. The annulus covered by $A_1$ has an induced Seifert fibration and can by foliated by arcs so that $T/\text{Stab}(T)$ is foliated by circles that cross each fiber once.

Case 2: $T/\text{Stab}(T)$ is $S^2(2,2,2,2)$

The $D(2,2)$ covered by $A_1'$ has an induced orbifold Seifert fibration and orbifold foliation as shown below in Figure 3. The $D(2,2)$ covered by $A_1$ has an induced orbifold Seifert fibration and can be orbifold foliated so that $T/\text{Stab}(T)$ is orbifold foliated so that each leaf crosses each fiber once.
Moreover these orbifold foliations can be chosen so that they lift to give \( T \) a foliation that is invariant under \( Stab(T) \); agrees with the foliation by arcs given by \( k_{A_1^i} : S^1 \times I \to A_1^i \); and is isotopic to the induced foliation of \( T \) from the original \( k : S^1 \times F \to \hat{M} \). This follows from Lemma 4.2.

This then defines a product structure \( k_T : S^1 \times S^1 \to T \) invariant under the action of \( Stab(T) \) which restricts to a product structure \( k_{A_1^i} : S^1 \times I \to A_1^i \) invariant under \( Stab(A_1^i) \).

Now, for each \( T_i \in Orb_{Stab(\hat{M})}(T) \), there is some \( g_i \in G \) such that \( \varphi(g_i)(T) = T_i \).

We then define product structures \( k_{T_i} : S^1 \times S^1 \to T_i \) by \( k_{T_i} = \varphi(g_i) \circ k_{T_1} \).

So now for any \( g \in G \) with \( \varphi(g)(T_i) = T_j \) for some \( i, j \), we have that \( g = g_j g_i^{-1} \) for some \( g' \in Stab(T_1) \). So then \( k_{T_j}^{-1} \circ \varphi(g) \circ k_{T_i} = k_{T_i}^{-1} \circ \varphi(g') \circ k_{T_i} \).

Hence it is a product and the product structures on each of the tori \( T_i \) are respected under \( Stab(\hat{M}_1) \).

We do this for each orbit of boundary components of \( \hat{M}_1 \) to yield product structures on each boundary tori that are respected under \( Stab(\hat{M}_1) \) and that agree with the inherited product structure from the original boundary of \( \hat{M} \).

We can now translate these product structures to the boundaries of each \( \hat{M}_i \).

Now we can reconstruct \( \hat{M} \) and can assume that we have respected product structures on each of the connected components of the union of \( \partial \hat{M} \) and \( Orb(A_1) \). Pick the first connected component \( C \) that yielded \( T \) when we cut as shown in Figure 4.

The product structure on this connected component is necessarily isotopic to the original product structure by construction. Suppose that the product structure on some other connected component \( C' \) was defined by translating by \( \varphi(g) \). We now note that \( k : S^1 \times F \to \hat{M} \) and \( \varphi(g) \circ k : S^1 \times F \to \hat{M} \) satisfy the requirements of Lemma 4.3. Hence applying the lemma, we yield that the restricted product structure on \( C' \) from \( \varphi(g) \circ k : S^1 \times F \to \hat{M} \) is isotopic to the original product structure \( k : S^1 \times F \to \hat{M} \).

Hence, in regular neighborhoods of each of the connected components, we adjust the product structure \( k : S^1 \times F \to \hat{M} \) to equal the invariant product structures on the connected components.

It then follows that the respected product structures on each of the boundary tori of \( \hat{M}_1 \) extend within.
We can then apply the inductive hypothesis to assume that $k_{\hat{M}_1}: S^1 \times F_1 \rightarrow \hat{M}_1$ is in fact left invariant under the action of $Stab(\hat{M}_1)$.

We translate this product structure to each $\hat{M}_i$ to yield the required invariant product structure. □

**Remark 2.** We remark here that it is not sufficient simply that there are product structures on the boundary tori that are respected by the action. It is required also that the product structures can be extended within. We give the following example to illustrate this:

**Example 4.1.** We let $F$ be an annulus and $k : S^1 \times F \rightarrow \hat{M}$ be a fiber-preserving product structure. Let $G = \mathbb{Z}_m$ act on $\hat{M}$ by simply rotating by $\frac{2\pi}{m}$ along the fibers. This action will preserve any fiber-preserving product structure (up to isotopy) on each boundary torus.

So now pick meridians on the first torus to be the loops that are $(0, 1)$ curves according to $k : S^1 \times F \rightarrow \hat{M}$ and meridians on the second torus to be loops that are $(1, 1)$ curves according to $k : S^1 \times F \rightarrow \hat{M}$. These are both left invariant, but there is no product structure on $\hat{M}$ that restricts to these on the boundary.

5 Main Result.

We now prove the main result, whic states that given a condition on the obstruction term, any finite, orientation and fiber-preserving action on an orientable Seifert 3-manifold that fibers over an orientable base space can be derived via the presented construction in Section 3.

To prove this, we first state Theorem 2.8.2 of [11]:

**Theorem 5.1.** Suppose that each of $(M_1, m_1)$ and $(M_2, m_2)$ is a Seifert-fibered space with nonempty boundary and with fixed admissible fibration, but that neither $(M_1, m_1)$ is a solid torus with $m_1 = \emptyset$. Let $f : (M_1, m_1) \rightarrow (M_2, m_2)$ be an admissible diffeomorphism, and suppose that for some regular fiber $\gamma$ in $M_1$, $f(\gamma)$ is homotopic in $M_2$ to a regular fiber. Then $f$ is admissibly isotopic to a fiber-preserving diffeomorphism. If $f$ is already fiber-preserving on some union $U$ of elements of $m_1$, then the isotopy may be chosen to be relative to $U$.

This then leads us to what we will require:

**Lemma 5.2.** Let $W$ be a Seifert fibered torus and let $h : T \rightarrow T$ be a fiber-preserving diffeomorphism with $h_* = id$. Then $h : T \rightarrow T$ can be extended to a fiber-preserving diffeomorphism $\tilde{h} : T \times I \rightarrow T \times I$ with $\tilde{h}(x, 1) = (h(x), 1), \tilde{h}(x, 0) = (x, 0)$. Here $T \times I$ is fibered as a unique extended fibration.

**Proof.** We note first that an isotopy to the identity exists. We then need only check that such an isotopy can be taken to fiber-preserving.

So there exists a diffeomorphism $H : W \times I \rightarrow T$, such that $H(x, 1) = h(x)$ and $H(x, 0) = x$ with $H_t : T \rightarrow T$ a diffeomorphism for each $t \in I$.

We now define the diffeomorphism $\tilde{H} : T \times I \rightarrow T \times I$ by $\tilde{H}(x, t) = (H(x, t), t)$. This diffeomorphism is fiber-preserving on the boundary of $T \times I$.

We now give $T \times I$ the boundary pattern consisting of the union of its’ two boundary tori. Then certainly $\tilde{H}$ is an admissible diffeomorphism and moreover it is the identity on one boundary component, so the image of a fiber being homotopic to a fiber condition is trivially satisfied.

So we apply Theorem 5.1 to yield an isotopic map $\tilde{h}$ that is fiber-preserving and agrees with $\tilde{H}$ on the boundary. So that $\tilde{h}(x, 1) = \tilde{H}(x, 1) = (H(x, 1), 1) = (h(x), 1)$ and $\tilde{h}(x, 0) = \tilde{H}(x, 0) = (H(x, 0), 0) = (x, 0)$. □

We now prove the main result:

**Theorem 5.3.** Let $M$ be an orientable Seifert 3-manifold that fibers over an orientable base space. Let $\varphi : G \rightarrow Diff^+_*(M)$ be a finite group action on $M$ such that the obstruction term can expressed as

$$b = \sum_{i=1}^{m} (b_i \cdot \#Orb_{\varphi}(\alpha_i))$$

for a collection of fibers $\{\alpha_1, \ldots, \alpha_m\}$ and integers $\{b_1, \ldots, b_m\}$. Then $\varphi$ can be derived via the construction set out in Section 3.
**Proof.** We let $M$ be the Seifert 3-manifold with normalized invariants:

$$M = (g, o_1((q_1, p_1), \ldots, (q_n, p_n), (1, b)))$$

Firstly, without loss of generality, we can assume that the orbits of each $\{\alpha_1, \ldots, \alpha_m\}$ are distinct. If $\alpha_i, \alpha_j$ were in the same orbit, then we note that $b_i \cdot \#\text{Orb}_\varphi(\alpha_i) + b_j \cdot \#\text{Orb}_\varphi(\alpha_j) = (b_i + b_j) \cdot \#\text{Orb}_\varphi(\alpha_i)$ so that we do not require $\alpha_j$ for the property to still hold.

Secondly, we can suppose without loss of generality that the first $t$ of the fibers $\{\alpha_1, \ldots, \alpha_t\}$ are regular and each critical fiber $\gamma_1, \ldots, \gamma_n$ is in the orbit of one of $\{\alpha_{t+1}, \ldots, \alpha_m\}$. If one is not, it can be added into the collection with a coefficient of zero. This will not change the sum.

Now let

$$A = \sum_{i=1}^{t} \#\text{Orb}_\varphi(\alpha_i)$$

Then rewrite the Seifert invariants as:

$$M = (g, o_1((q_1, p_1), \ldots, (q_n, p_n), (1, b), (1, 0)_1, \ldots, (1, 0)_A)$$

So now each $(1, 0)_i$ refers to a regular fiber which is in the orbit of some fiber in the collection $\{\alpha_1, \ldots, \alpha_t\}$. Call this collection of fibers $\{\beta_1, \ldots, \beta_A\}$.

Now let $\{\beta_{A+1}, \ldots, \beta_{n+A}\} = \{\gamma_1, \ldots, \gamma_n\}$. Note that $\{\beta_1, \ldots, \beta_{n+A}\} = \text{Orb}_\varphi(\{\alpha_1, \ldots, \alpha_m\})$.

Define a function: $h : \{1, \ldots, n + A\} \rightarrow \mathbb{Z}$ by $h(j) = b_i$ if $\beta_j \in \text{Orb}_\varphi(\alpha_i)$.

Take closed, fibered regular neighborhoods $N(\alpha_1), \ldots, N(\alpha_m)$ and then define:

$$X = \text{Orb}_\varphi(N(\alpha_1) \cup \ldots \cup N(\alpha_m)), \hat{M} = M \setminus X$$

So $X$ is a collection of fibered solid tori and $M$ can be reobtained by some (fiber-preserving) gluing map $d : \partial X \rightarrow \partial \hat{M}$. This gluing map corresponds to the presentation:

$$M = (g, o_1((q_1, p_1 + h(1)q_1), \ldots, (q_n, p_n + h(n)q_n), (1, h(n + 1)), \ldots, (1, h(n + A)))$$

This is possible by Corollary 2.2. as

$$\sum_{j=1}^{n+1} h(j) = \sum_{i=1}^{m} b_i \cdot \#\text{Orb}_\varphi(\alpha_i) = b$$

For convenience, denote:

$$(g, o_1((q_1, p_1 + h(1)q_1), \ldots, (q_n, p_n + h(n)q_n), (1, h(n + 1)), \ldots, (1, h(n + A)))$$

$$= (g, o_1((q'_1, p'_1), \ldots, (q'_n, p'_n), (q'_{n+1}, p'_{n+1}), \ldots, (q'_{n+A}, p'_{n+A}))$$

From Section 2, this gives us a fibering product structure $\hat{M} : S^1 \times F \rightarrow \hat{M}$ and a product structure $k_X : S^1 \times (D_1 \cup \ldots \cup D_{n+A}) \rightarrow X$ so that according to it, each $V_i$ in $X$ has a normalized fibration. We then have that

$$(d|_{V_i})_* = \begin{bmatrix} x'_i & p'_i & k_{v_i} \\ y'_i & q'_i & k_{t_i} \end{bmatrix}$$

for the nontrivially fibered solid tori according to these product structures. So now the fibrations on these $V_i$ are a $(-q_i, y'_i)$ fibration. The action can only send some $V_i$ to a $V_j$ if they have the same fibration. Hence $(-q_i, y'_i) = (-q_j, y'_j)$.

We now show that the action can only send some $V_i$ to $V_j$ if they have the same associated fillings.

Now, we have $x'_i q_i - y'_i (p_i + h(i)q_i) = -1$ and $x'_j q_i - y'_i (p_j + h(i)q_i) = -1$. Hence:
We can then apply Lemma 4.2 to get
\[ x'_i q_i (p_j + h(i)q_i) - y'_i p_i (p_j + h(i)q_i) = -(p_j + h(i)q_i) \]
\[ x'_j q_i (p_i + h(i)q_i) - y'_i p_j (p_i + h(i)q_i) = -(p_i + h(i)q_i) \]
So that
\[ q_i (x'_i (p_j + h(i)q_i) - x'_j (p_i + h(i)q_i)) = p_i - p_j. \]
But now, \(-q_i < p_i - p_j < q_i\), hence \(-1 < (x'_i (p_j + h(i)q_i) - x'_j (p_i + h(i)q_i)) < 1\), and so
\[ x'_i (p_j + h(i)q_i) = x'_j (p_i + h(i)q_i). \]
But \(x'_i, (p_j + h(i)q_i)\) are coprime and so are \(x'_j, (p_j + h(i)q_i)\), hence \(x'_i = x'_j\) and \((p_i + h(i)q_i) = (p_j + h(i)q_i)\) so that \(p_i = p_j\) as well as \(p'_i = p'_j\).
We can henceforth assume that if the action sends some \(V_i\) to a \(V_j\), then the fillings must be the same. Note that this is true also for the fillings of trivially fibered tori by construction.
So then \(\hat{M}\) is a Seifert fibered 3-manifold with boundary such that there is a fiber-preserving restricted action given by:
\[ \hat{\varphi} : G \to Diff^\text{f}(\hat{M}), \hat{\varphi}(g) = \varphi(g)|_{\hat{M}} \]
We now proceed to show that there is a product structure on \(\hat{M}\) such that \(\hat{\varphi}\) respects the restricted product structures on the boundary tori. We do so to employ Theorem 4.4.
Now take \(T_i\) arbitrarily and consider the action given by \(\hat{\varphi}(g)|_{T_i}\) for each \(g \in Stab(T_i)\).
By restricting \(k_{\hat{M}} : S^1 \times F \to \hat{M}\) and \(k_X : S^1 \times (D_1 \cup \ldots \cup D_{n+4}) \to X\) as in Section 2 to \(k_{T_i} : S^1 \times S^1 \to T_i\) and \(k_{\partial V_i} : S^1 \times S^1 \to \partial V_i\) we have the following homological diagram:
\[ \begin{array}{ccc}
H_1(T_i) & \xleftarrow{\delta} & H_1(\partial V_i) \\
(\hat{\varphi}(g)|_{T_i})_* & \downarrow & (d|_{\partial V_i})_* \\
H_1(T_i) & \xleftarrow{\delta} & H_1(\partial V_i) \\
(\hat{\varphi}(g)|_{T_i})_* & \downarrow & (d|_{\partial V_i})_*
\end{array} \]
Now, as the action extends into \(V_i\) and is finite, we must have that \((d|_{\partial V_i})_* \circ \hat{\varphi}(g)|_{T_i} \circ d|_{\partial V_i})_* = \pm id.\) Hence \((\hat{\varphi}(g)|_{T_i})_* = \pm id\) for all \(g \in Stab(T_i)\).
We can then apply Lemma 4.2 to get \(f_i : T_i \to T_i\) such that \(f_i\) is fiber-preserving, isotopic to the identity, and \(k_{T_i}^{-1} \circ f_i^{-1} \circ \hat{\varphi}(g)|_{T_i} \circ f_i \circ k_{T_i}\) is a product map for each \(g \in Stab(T_i)\).
Now pick \(g_j \in G\) for each \(T_j \in Orb(T_i)\) such that \(\hat{\varphi}(g_j)|_{T_i} = T_j\).
We translate the conjugating map \(f_i : T_i \to T_i\) to each \(T_j \in Orb(T_i)\) by defining \(f_j = \hat{\varphi}(g_j)|_{T_i} \circ f_i \circ k_{T_i} \circ h_j \circ k_{T_j}^{-1}\) where:
\[ h_j(u,v) = \begin{cases} (u,v) & \text{if } \hat{\varphi}(g_j) \text{ preserves the orientation of the fibers} \\ (u^{-1},v^{-1}) & \text{if } \hat{\varphi}(g_j) \text{ reverses the orientation of the fibers} \end{cases} \]
Each \(f_j\) is certainly fiber-preserving, we check that they are isotopic to the identity.
Note that we have the diagram:
\[ \begin{array}{ccc}
H_1(T_i) & \xleftarrow{\delta} & H_1(\partial V_i) \\
\hat{\varphi}(g_j)_* & \downarrow & \pm id \\
H_1(T_j) & \xleftarrow{\delta} & H_1(\partial V_j) \\
\hat{\varphi}(g_j)_* & \downarrow & \pm id
\end{array} \]
So that necessarily \( \dot{\varphi}(g_j)_* = \pm id \) depending on whether the orientation on the fibers are reversed or not. Consequently, \( f_j \) is isotopic to the identity.

Then for any \( g \in G, g = g_2 h g_1^{-1} \), for some \( h \in \text{Stab}(T_i) \) and some \( T_j, T_j \in \text{Orb}(T_i) \). We calculate: \( k^{-1}_{T_j} \circ f^{-1}_{j_2} \circ \dot{\varphi}(g)|_{T_j} \circ f_{j_2} \circ k_{T_j} = h^{-1}_2 \circ (k^{-1}_1 \circ f^{-1}_1 \circ \dot{\varphi}(g)|_{T_1} \circ f_1 \circ k_{T_1}) \circ h_1 \). So that \( k^{-1}_{T_j} \circ f^{-1}_{j_2} \circ \dot{\varphi}(g)|_{T_j} \circ f_{j_2} \circ k_{T_j} \) is also a product map, and the product structures \( f_j \circ k_{T_j} : S^1 \times S^1 \to T_j \) for \( T_j \in \text{Orb}(T_i) \) are left invariant under \( \dot{\varphi} \).

We can now do this for each of the distinct orbits of boundary tori.

As each \( f_j \) is isotopic to the identity and fiber-preserving, we can employ Lemma 5.2, to define \( f \in \text{Diff}^+_S(M) \) so that \( f|_{T_1} = f_j \) and \( f \) is the identity outside of a regular neighborhood of each boundary torus. \( f \) is necessarily isotopic to the identity.

So now, the product structure \( f \circ k_{\hat{M}} : S^1 \× F \to \hat{M} \) is such that \( f \circ k_{T_j} : S^1 \times S^1 \to T_j \) for each \( T_j \) are respected under \( \dot{\varphi} \) and moreover is isotopic to \( k_{\hat{M}} \).

Then we have what we require to employ Theorem 4.4: a product structure on \( \hat{M} \) such that \( \dot{\varphi} \) respects the restricted product structures on the boundary tori. So we yield a product structure \( k'_{\hat{M}} : S^1 \times F \to \hat{M} \) such that each \( k^{-1}_{\hat{M}} \circ \dot{\varphi}(g) \circ k'_{\hat{M}} \) is a product map. We can assume that each component of \( k^{-1}_{\hat{M}} \circ \dot{\varphi}(g) \circ k'_{\hat{M}} \) are isometries under some metrics.

Therefore, we must have that on each boundary component \( T_j \):

\[
(k^{-1}_{\hat{T}_j}(g)) \circ \dot{\varphi}(g) \circ k'_{\hat{T}_j}(u,v) = (\theta_1(g) u^{\alpha_1(g)}, \theta_2(\cdot, g) v^{\alpha_2(g)})
\]

But now \( \alpha_1(g) = \alpha_2(g) \) as the action is orientation-preserving.

It remains to show that we can pick a product structure on \( X \) that is left invariant. We know that there is a product structure \( k'_X : S^1 \times (D_1 \cup \ldots \cup D_i) \to X \) so that according to the product structure \( k'_{\hat{M}} : S^1 \times F \to \hat{M} \) we have:

\[
(k_{\hat{T}_j}^{-1} \circ d|_{\hat{T}_j} \circ k_{\theta V_i})(u,v) = (u^{\phi_1}, v^{\phi_2}, u^{\theta_1}, v^{\theta_2})
\]

If we let \( \varphi_X \) be the action restricted to \( X \), we have that according to this product structure, the action on the boundary of \( X \) looks like:

\[
(k_{\theta V_i}^{-1}(g)) \circ \varphi_X(\cdot, g) \circ k_{\theta V_i}(u,v) = (\theta_1(g)^{-u} \theta_2(\cdot, g) u^\alpha, \theta_1(g)^y \theta_2(\cdot, g) u^{-1} v^\alpha)
\]

That is, it respects the restricted product structures. Hence we can consider \( \text{Stab}(V_i) \) for each \( V_i \) to apply Theorem 4.4 and translate in a similar way to above and in the proof of Theorem 4.4.

This completes the proof.

We now state some of the immediate corollaries of Theorem 5.3:

**Corollary 5.4.** Let \( M \) be an orientable Seifert 3-manifold that fibers over an orientable base space. Let \( \varphi : G \to \text{Diff}^+_S(M) \) be a finite group action on \( M \) such that a fiber is left invariant. Then \( \varphi \) can be derived via the construction set out in Section 3.

**Proof.** Let \( \alpha \) be the fiber left invariant. Then \#\( \text{Orb}_\varphi(\alpha) \) = 1 and so \( b = b \cdot \#\text{Orb}_\varphi(\alpha) \).

**Corollary 5.5.** Let \( M \) be an orientable Seifert 3-manifold that fibers over an orientable base space with only one cone point of order \( q \). Let \( \varphi : G \to \text{Diff}^+_S(M) \) be a finite group action on \( M \). Then \( \varphi \) can be derived via the construction set out in Section 3.

**Proof.** Let \( \alpha \) be the fiber that refers to the cone point of order \( q \). Then \#\( \text{Orb}_\varphi(\alpha) \) = 1 and so \( \dot{\varphi} = b \cdot \#\text{Orb}_\varphi(\alpha) \).

**Corollary 5.6.** Let \( M \) be an orientable Seifert 3-manifold that fibers over an orientable base space. Let \( \varphi : G \to \text{Diff}^+_S(M) \) be a finite group action on \( M \) so that there are two fibers \( \alpha, \beta \) with \#\( \text{Orb}_\varphi(\alpha) \), \#\( \text{Orb}_\varphi(\beta) \) coprime. Then \( \varphi \) can be derived via the construction set out in Section 3.
We now proceed to refine the obstruction condition. First, two lemmas are established and then a proposition which

These corollaries give some simple situations under which the conditions of Theorem 5.1 are satisfied. We give an example of the use of these corollaries:

**Example 5.1.** We consider a Seifert manifold $M$ which fibers over an orientable base space $B$ which has the cone points $2, 2, 3, 3, 3$. Now any action on $B$ would necessarily only be able to exchange the two cone points of order 2 and permute the cone points of order 3. Hence a critical fiber $\alpha$ referring to one of the cone points of order 2, must have that $\#\text{Orb}_{\varphi}(\alpha)$ is 1 or 2. Similarly, there is a critical fiber $\beta$ referring to one of the cone points of order 3, that must have either $\#\text{Orb}_{\varphi}(\beta)$ as 1 or 3. If either $\#\text{Orb}_{\varphi}(\alpha)$ or $\#\text{Orb}_{\varphi}(\beta)$ is 1, then we can apply Corollary 5.4. If $\#\text{Orb}_{\varphi}(\alpha) = 2$ and $\#\text{Orb}_{\varphi}(\beta) = 3$, then we can apply Corollary 5.6.

In all cases any finite, orientation and fiber-preserving action on $M$ must be derived via the construction set out in Section 3. This is regardless of the obstruction term.

### 6 Obstruction condition.

If $\varphi : G \to Diff_{+}^f(M)$ is a finite group action, we henceforth call satisfaction of

$$ b = \sum_{i=1}^{s} (b_{i} \cdot \#\text{Orb}_{\varphi}(\alpha_{i})) $$

for some fibers $\{\alpha_{1}, \ldots, \alpha_{s}\}$ and integers $\{b_{1}, \ldots, b_{s}\}$, satisfying the obstruction condition.

**Remark 3.** We note that the obstruction condition is not always satisfied. We give the following example:

**Example 6.1.** Construct by a Seifert 3-manifold $M$ fibering over an even genus $g$ surface with no exceptional fibers and odd obstruction $b$ by taking two trivially fibered manifolds $M_1 = S^3 \times F_1$ and $M_2 = S^1 \times F_2$ where $F_1, F_2$ are genus $\frac{g}{2}$ surfaces with a disc removed, and then gluing according to the map $d(u_1, v_1) = (u_1^{-1}v_2, v_2)$ between boundary tori.

Define the rotation $\text{rot}_2 : F_i \to F_i$ to be an order 2 rotation that leaves the boundary invariant.

Define an orientation-preserving finite and fiber-preserving action on $M_1$ and $M_2$ by: $f_1(u_1, x_1) = (u_1, \text{rot}_2(x_1))$, $f_1(u_2, x_2) = (-u_2, \text{rot}_2(x_2))$ and $f_2(u_1, x_1) = (u_2, x_2), f_2(u_2, x_2) = (u_1, x_1)$

It can be checked that these agree over the gluing torus.

So then the projected action on the genus $g$ surface is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-action and all orbit numbers are even. Hence, it cannot be that

$$ b = \sum_{i=1}^{s} (b_{i} \cdot \#\text{Orb}_{\varphi}(\alpha_{i})) $$

We now proceed to refine the obstruction condition. First, two lemmas are established and then a proposition which provides a convenient equivalent statement for the obstruction condition that can be used to apply our results.

**Lemma 6.1.** Let $\varphi : G \to Diff(S)$ be a finite group action on a surface $S$. Suppose that the orbifold $S/\varphi$ has data set $(n_1, \ldots, n_k; m_1, \ldots, m_l)$. Then the possible orbit numbers under $\varphi$ are $|G|/n_1, \ldots, |G|/n_k, |G|/2m_1, \ldots, |G|/2m_l$ and $|G|$.

**Proof.** $S$ is an order $|G|$ orbifold cover of $S/\varphi$. Therefore any regular point of $S/\varphi$ lifts to $|G|$ points of $S$, any of these points have orbit number $|G|$. Any neighborhood of a cone point of order $n_i$ is covered by a collection of discs in $S$, each disc is an $n_i$-fold cover of the neighborhood. Hence the number of discs that cover the neighborhood is $\frac{|G|}{n_i}$. Thus the center of each disc has orbit number $\frac{|G|}{n_i}$.

Any neighborhood of a corner reflector of order $m_i$ is covered by a collection of discs in $S$, each disc is an $2m_i$-fold cover of the neighborhood. Hence the number of discs that cover the neighborhood is $\frac{|G|}{2m_i}$. Thus the center of each disc has orbit number $\frac{|G|}{2m_i}$. \hfill $\square$

**Lemma 6.2.** Let $n_1, \ldots, n_k$ be factors of $N$. Then $\frac{N}{\text{lcm}(n_1, \ldots, n_k)} = \text{gcd}(\frac{N}{n_1}, \ldots, \frac{N}{n_k})$. 

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We begin with the if statement. So by Lemma 6.2, this implies:

\[
gcd\left(\frac{N}{n_1}, \frac{N}{n_2}\right) \text{lcm}(n_1, n_2) = \frac{N^2 \text{lcm}(n_1, n_2)}{n_1 n_2 \text{lcm}\left(\frac{N}{n_1}, \frac{N}{n_2}\right)}
\]

\[
= \frac{N^2 \text{lcm}(n_1, n_2)}{\text{lcm}(n_2 N, n_1 N)}
\]

\[
= \frac{N^2 \text{lcm}(n_1, n_2)}{N \text{lcm}(n_2, n_1)}
\]

For the inductive step, we work in a similar fashion:

\[
gcd\left(\frac{N}{n_1}, \ldots, \frac{N}{n_k}\right) = gcd\left(gcd\left(\frac{N}{n_1}, \ldots, \frac{N}{n_{k-1}}\right), \frac{N}{n_k}\right)
\]

\[
= gcd\left(\frac{N}{\text{lcm}(n_1, \ldots, n_{k-1})}, \frac{N}{n_k}\right)
\]

\[
= \frac{N^2}{n_k \text{lcm}(n_1, \ldots, n_{k-1}) \text{lcm}\left(\frac{N}{\text{lcm}(n_1, \ldots, n_{k-1})}, \frac{N}{n_k}\right)}
\]

\[
= \frac{N^2}{\text{lcm}(N n_k, N \text{lcm}(n_1, \ldots, n_{k-1}))}
\]

\[
= \frac{N}{\text{lcm}(n_1, \ldots, n_k)}
\]

\[
= N
\]

**Proposition 6.3.** Let \(\varphi : G \to Diff^f_+(M)\) be a finite group action and \(\varphi_{B_U} : G_{B_U} \to Diff(B_U)\) the induced action on the underlying space of the base space \(B\) which has branching data \((n_1, \ldots, n_k; m_1, \ldots, m_l)\). Then \(\varphi : G \to Diff^f_+(M)\) satisfies the obstruction condition if and only if \(\frac{|G_{B_U}|}{lcm(n_1, \ldots, n_k; 2m_1, \ldots, 2m_l)}\) divides \(b\).

**Proof.** We first note that there exists fibers \(\{\alpha_1, \ldots, \alpha_s\}\) and integers \(\{b_1, \ldots, b_s\}\) such that

\[
b = \sum_{i=1}^{s} (b_i \cdot \#Orb_{\varphi}(\alpha_i))
\]

if and only if there exists points \(\{x_1, \ldots, x_s\} \subset B_U\) and integers \(\{b_1, \ldots, b_s\}\) such that

\[
b = \sum_{i=1}^{s} (b_i \cdot \#Orb_{\varphi_{B_U}}(x_i))
\]

We begin with the if statement. So by Lemma 6.2, \(\frac{|G_{B_U}|}{lcm(n_1, \ldots, n_k; 2m_1, \ldots, 2m_l)}\) divides \(b\).

Hence there exist \(\{b_1, \ldots, b_{k+l}\}\) such that

\[
b = \sum_{i=1}^{k} b_i \cdot \frac{|G_{B_U}|}{n_i} + \sum_{i=1}^{l} b_i \cdot \frac{|G_{B_U}|}{2m_i}
\]

by Euclid’s algorithm.

So by Lemma 6.1, there are \(\{x_1, \ldots, x_k, x_{k+1}, \ldots, x_{k+l}\} \subset B_U\) such that \(\#Orb_{\varphi_{B_U}}(x_i) = \frac{|G_{B_U}|}{n_i}\) and \(\#Orb_{\varphi_{B_U}}(x_i) = \frac{|G_{B_U}|}{2m_i}\). Thus,

\[
b = \sum_{i=1}^{k+l} b_i \cdot \#Orb_{\varphi_{B_U}}(x_i)
\]
For the only if, suppose that there exist points \( \{x_1, \ldots, x_s\} \subset B_U \) and integers \( \{b_1, \ldots, b_s\} \) such that

\[
b = \sum_{i=1}^{s} (b_i \cdot \# Orb_{\varphi_{Bu}}(x_i))
\]

Without loss of generality, we can assume that the orbit numbers of all the \( x_i \) are different, that \( s = k + l \) (set \( b_i = 0 \) if necessary), and that the branching data of each \( x_i \) is \( n_i \) for \( i = 1, \ldots, k \) and \( 2m_i \) for \( i = k + 1, \ldots, l \).

Hence, by Lemma 6.1,

\[
b = \sum_{i=1}^{k} b_i \cdot \frac{|G_{Bu}|}{n_i} + \sum_{i=1}^{l} b_i \cdot \frac{|G_{Bu}|}{2m_i}
\]

and so

\[
gcd(|G_{Bu}|/n_1, \ldots, |G_{Bu}|/n_k, |G_{Bu}|/2m_1, \ldots, |G_{Bu}|/2m_l)
\]

divides \( b \).

So by Lemma 6.1, \( \text{gcd}(|G_{Bu}|/n_1, \ldots, |G_{Bu}|/n_k, |G_{Bu}|/2m_1, \ldots, |G_{Bu}|/2m_l) \) divides \( b \). \( \square \)

This result then allows us to quickly establish whether the obstruction condition is satisfied based on the order of the induced action on the base space and the least common multiple of the data from the orbifold quotient of the induced action. This is a convenient way to establish results based on possible quotient types.

7 Group structures.

We now establish the possible structures of the groups that can act orientation-preservingly on a Seifert manifold (satisfying the obstruction condition).

**Proposition 7.1.** Suppose that \( \varphi : G \to Diff(S^1) \times Diff(F) \) is a finite group action with \( \varphi(g)(u, x) = (\varphi_{S^1}(g)(u), \varphi_{F}(g)(x)) \) such that \( \varphi_{S^1}(g) \) is orientation-preserving if and only if \( \varphi_{F}(g) \) is orientation-preserving. Suppose that there exists \( g_+ \in G \) such that \( \varphi_{S^1}(g_+) \) is orientation-reversing and \( g_+^2 = 1 \). Then \( G \) is isomorphic to a subgroup of a semidirect product of \( \mathbb{Z}_n \times \varphi_{F}(G)_+ \) and \( \mathbb{Z}_2 \).

**Proof.** First let \( \varphi(G)^{fop} \) be the subgroup of \( \varphi(G) \) where each element is orientation-preserving on both components.

We now consider the structure of \( \varphi(G)^{fop} \). We note that \( \varphi(G)^{fop} \) is a finite subgroup of \( \varphi_{S^1}(G)_+ \times \varphi_{F}(G)_+ \). Now \( \varphi_{S^1}(G)_+ \cong \mathbb{Z}_n \) for some \( n \) and so \( \varphi(G)^{fop} \) is a finite subgroup of \( \mathbb{Z}_n \times \varphi_{F}(G)_+ \).

We then consider the short-exact sequence \( 1 \to \varphi(G)^{fop} \to \varphi(G) \to \mathbb{Z}_2 \to 1 \).

This splits if there is an element in \( \varphi(G) \) of order 2 that is not in \( \varphi(G)^{fop} \). By assumption, \( \varphi(g_+) \) is such an element. The result then follows. \( \square \)

8 Summary

We have shown that provided that the obstruction term is satisfied, then a finite, fiber- and orientation-preserving action can be constructed via our method. The final section above gives some form to the kinds of finite groups that act this way. We note that there is the restriction that \( G \) contains an order 2 element that reverses the orientation of the fibers and therefore reverses the orientation on the base space. In the particular case of the base space being \( S^2 \) this is not a restriction as any finite group that acts is a subgroup of a finite group that has this property. In particular, we will establish in a future paper that the finite groups that act fiber- and orientation-preservingly on Seifert manifolds fibering over \( S^2 \) (and satisfying the obstruction condition) are of the form \( (\mathbb{Z}_n \times H) \circlearrowright_{-1} \mathbb{Z}_2 \) where \( \mathbb{Z}_2 \) acts by anticommuting with each element of \( \mathbb{Z}_n \times H \) and \( H \) is a finite group that acts orientation-preservingly on \( S^2 \).
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