SASAKI–WEYL CONNECTIONS ON CR MANIFOLDS

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Abstract. We introduce and study the notion of Sasaki–Weyl manifold, which is a natural generalization of the notion of Sasaki manifold. We construct a reduction of Sasaki–Weyl manifolds and we show that it commutes with several reductions already existing in the literature.

1. Introduction

The reduction construction is well-known in many areas of differential geometry. It has been initially introduced in symplectic geometry by Marsden and Weinstein and then extended in the context of Kähler, hyper-Kähler, CR and Sasaki geometry. Reductions in conformal geometry have also been studied: a reduction for symplectic conformal manifolds appears in [3] and a reduction for locally conformal Kähler manifolds has been constructed in [1].

The aims of this paper are threefold: (1) to introduce and study the notion of Sasaki–Weyl manifold, which generalizes, in the case when the Weyl connection is closed, the notion of Sasaki manifold, in the same way as the locally conformal Kähler manifolds generalize the Kähler manifolds; (2) to construct and investigate the properties of a reduction for Sasaki–Weyl manifolds, which completes the picture of reductions already existing in the literature; (3) to illustrate our theory with examples and indicate further directions of possible research.

Let \((M, c)\) be a conformal oriented manifold. H. Weyl noticed (see [10]) that there is a one-to-one correspondence between connections on the density line bundle \(L\) of \(M\) (usually called Weyl connections) and torsion-free connections on \(M\) which preserve the conformal structure \(c\). The density line bundle \(L\) is oriented and positive sections of \(L\) correspond to metrics in the conformal class \(c\). If the Weyl connection \(D\) is exact (i.e. there is a global non-vanishing \(D\)-parallel section \(\mu\) of \(L\)) then the corresponding connection on \(M\) is the Levi-Civita connection of the corresponding Riemannian metric \(\mu^{-2}c\). Consider now an oriented (strongly pseudo-convex) CR manifold \((M, H, I)\) and \(L := TM/H\) its co-contact bundle, which inherits an orientation from

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the orientation of $M$ and $H$. Positive sections of $L$ correspond to contact forms of $M$. Tanaka associated (see [3]), in a natural way, to every contact form of $H$ a connection on $M$, the so called Tanaka connection, which preserves the contact form and the CR structure of $M$ and satisfies other natural conditions. Tanaka’s result has been generalized in [3], where to every connection $D$ on $L$ it was associated, in a natural way, a connection on $M$ which coincides with the Tanaka connection of a contact form in the case when $D$ is exact. Therefore, there is an obvious similarity between conformal and CR geometry, Weyl’s theorem from conformal geometry corresponding to the generalized Tanaka’s theorem from CR geometry. By analogy, connections on the co-contact bundle $L$ of a CR manifold $(M, H, I)$ were called in [3] Weyl connections.

In this paper we will push the similarity between conformal and CR geometry further in the realm of Kähler and Sasaki geometry. A Sasaki–Weyl manifold is a CR oriented manifold $(M, H, I)$ together with a Weyl connection $D$ whose Reeb vector field $\psi^D$ satisfies the Sasaki–Weyl condition $L_{\psi^D}(I) = 0$. When $D$ is exact the Sasaki–Weyl condition is precisely the Sasaki condition for the contact form associated to a positive $D$-parallel section of the co-contact bundle $L$ of $(H, I)$. When $D$ is closed, the Sasaki–Weyl condition translates to the Sasaki condition on the universal cover of $M$ with the CR structure induced by $(H, I)$. After briefly recalling some facts we need from CR geometry, we show in Section 3 that every Weyl connection $D$ on an oriented CR manifold $(M, H, I)$ determines, in a natural way, an almost hermitian conformal structure on the cone manifold $L^+$ of oriented elements of the co-contact bundle $L$ of $M$. It turns out that this almost hermitian structure is locally conformal Kähler when $D$ is Sasaki–Weyl with curvature $F^D$ satisfying an additional condition. In Section 4 we perform a reduction of Sasaki–Weyl manifolds and then we study its properties. While closed Sasaki–Weyl connections remain closed through the reduction process, this is not true for exact Sasaki–Weyl connections. Proposition 12 shows that in fact any Sasaki–Weyl manifold whose Weyl connection is closed can be written as a reduction of a Sasaki–Weyl manifold whose Weyl connection is exact. We show the compatibility of the Sasaki–Weyl reduction with the operation of taking universal covers and in Section 5 we show its commutativity with the locally conformal Kähler reduction of Biquard-Gauduchon of the corresponding cone manifolds. We end the paper with examples of closed and non-closed Sasaki–Weyl connections, and an explicit illustration of the Sasaki–Weyl reduction. We hope that our theory (in particular the cone construction we propose in the paper) will provide examples of Vaisman manifolds. Further investigation in this direction is needed.
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2. SASAKI–WEYL MANIFOLDS

We recall that on an oriented (strongly pseudo-convex) CR manifold \((M,H,I)\), \(H\) is a codimension one oriented subbundle of \(TM\), \(I : H \rightarrow H\) is an endomorphism of \(H\) which satisfies \(I^2 = -\text{Id}\) (here and elsewhere “\(\text{Id}\)” denotes the identity endomorphism) and the following integrability condition holds: if \(X\) and \(Y\) are sections of \(H\) then 
\[
[IX,IY] - [X,Y] \text{ is a section of } H \text{ as well and the relation }
\]

\[
[IX,IY] - [X,Y] = I([IX,Y] + [X,IY])
\]

holds. The co-contact bundle \(L := TM/H\) is oriented and a positive section of its dual \(L^{-1}\) is a contact form. To every contact form \(\theta\) (viewed as a 1-form on \(M\) with kernel \(H\)) there is associated a Reeb vector field \(T\), determined by the conditions \(\theta(T) = 1\) and \(i_T d\theta = 0\), and a metric \(g_0\) on \(H\) defined by \(g_0 := \frac{1}{2}d\theta(\cdot,\cdot)\), which is independent of the choice of \(\theta\) and must be positive definite due to the strongly pseudo-convexity condition. If \(X\) is a section of \(H\), so is \(L_T(X)\) and \(L_T(I)\), defined by \(L_T(I)(X) := L_T(IX) - IL_T(X)\), is an endomorphism of \(H\). If \(L_T(I) = 0\) we say that \(T\) (and \(\theta\)) is the Reeb vector field (contact form respectively) of a Sasaki structure.

We now briefly recall some basic facts from the theory of CR-Weyl manifolds, as developed in [3]. A CR-Weyl manifold is an oriented CR manifold \((M,H,I)\) together with a connection \(D\) (called a Weyl connection) on its co-contact bundle \(L\). The curvature \(F^D\) of \(D\) is called the Faraday curvature. The connection \(D\) is closed if \(F^D = 0\) and exact if there is a non-vanishing \(D\)-parallel section of \(L\). Let \(\eta : TM \rightarrow L\) be the natural projection (sometimes called the universal contact form). The Weyl connection \(D\) determines a Reeb vector field \(\psi^D : L \rightarrow TM\), which is a bundle homomorphism (or a section of the bundle \(L^{-1} \otimes TM\)) uniquely determined by the two conditions \(i_{\psi^D} d^D \eta = 0\) and \(\eta \circ \psi^D = \text{Id}\). The terminology “Reeb vector field” for \(\psi^D\) is justified by the following fact: if \(D\) is exact and \(\mu\) is a \(D\)-parallel positive section of \(L\), then \(\psi^D(\mu)\), which is a genuine vector field on \(M\), is the Reeb vector field of the contact form \(\theta := \mu^{-1} \eta\) (for a non-vanishing section \(\mu\) of \(L\) we denote by \(\mu^{-1}\) the section of \(L^{-1}\) induced by \(\mu\), i.e. the natural contraction between \(\mu\) and \(\mu^{-1}\) is the function identically one on \(M\)).
Associated to the Weyl connection $D$ there is also a Weyl-Lie derivative $L_{\psi^D}$, which on vector fields $X$ on $M$ is defined by

$$L_{\psi^D}(X) = \mu^{-1}(L_{\psi^D}(\mu)(X) + D_X(\mu)\psi^D)$$

and which is independent of the chosen (non-vanishing) section $\mu$ of $L$. If $X$ is a section of $H$, then $L_{\psi^D}(X)$ is a section of $L^{-1} \otimes H$. Due to this observation, we can define $L_{\psi^D}(I)$, an endomorphism of $H$ with values in the bundle $L^{-1}$, by the formula $L_{\psi^D}(I)(X) := L_{\psi^D}(IX) - IL_{\psi^D}(X)$.

**Definition 1.** A CR-Weyl manifold $(M, H, I, D)$ whose Weyl connection $D$ satisfies $L_{\psi^D}(I) = 0$ is called Sasaki–Weyl.

**Remark:** We explain now the analogy between Sasaki and Sasaki–Weyl manifolds with closed Weyl connection on one hand, and Kähler and locally conformal Kähler manifolds on the other hand. Recall that an (oriented) locally conformal Kähler manifold (of real dimension greater or equal to 4) is an almost hermitian conformal manifold $(M, c, J)$ with a connection $D^c$ (called the canonical connection) on the density line bundle $L$ of $M$ which satisfies two properties: $D^c$ is closed (i.e. its curvature is zero); the corresponding connection on $M$ (obtained by Weyl’s theorem) preserves $J$. This implies, in particular, that $J$ is integrable. Conversely, if $J$ is integrable and $\Omega$ is the Kähler form of an arbitrary metric $\mu^{-2}c$ in the conformal class $c$, then $(M, c, J)$ is locally conformal Kähler if there is a (unique) closed 1-form $\alpha$ on $M$ such that $d\Omega = \Omega \wedge \alpha$. Moreover, the canonical connection $D^c$ of $(M, c, J)$ satisfies $D^c_X(\mu) = -\frac{1}{2}\alpha(X)\mu$, for every vector field $X$ on $M$. Every local, positive, $D^c$-parallel section of $L$ generates a local Kähler metric in the conformal class $c$. On the other hand, a Sasaki–Weyl manifold $(M, H, I, D)$ whose Weyl connection $D$ is closed enjoys a similar property: every local, positive, $D$-parallel section $\mu$ of the co-contact bundle of $(M, H, I)$ determines a local contact form $\mu^{-1}\eta$ of a Sasaki structure.

In this paper we shall use the following notations and conventions:

Notation: Unless otherwise specified, all our manifolds will be connected and oriented. We shall denote by $X(M)$ the space of vector fields on a manifold $M$. For a CR-Weyl manifold $(M, H, I, D)$ we shall use the following notations: $\pi : L \rightarrow M$ for its co-contact bundle; $\Lambda$ for the image of the Reeb vector field $\psi^D : L \rightarrow TM$, which is a rank one subbundle of $TM$ complementary to the contact bundle $H$; $\bar{D}$ for the connection on the bundle $\pi^*L \rightarrow L$ induced by the Weyl connection $D$; $F^{D,-}$ for the $I$-anti-invariant part of $F^D|_{H \times H}$; $\bar{X}$ for the $D$-horizontal lift to $L$ of $X \in X(M)$.

3. **The cone of a Sasaki–Weyl manifold**

In this Section we fix a CR-Weyl manifold $(M, H, I, D)$. 
Proposition 2. Let \( y \in L \). The connection \( D \) determines a decomposition

\[
T_yL = H_{\pi(y)} \oplus \Lambda_{\pi(y)} \oplus L_{\pi(y)},
\]

and an almost complex structure \( J \) on the manifold \( L \), which, by means of the decomposition \( \Box \), is written as

\[
J|_H = I, \quad J(s) := \psi^D(s),
\]

where \( s \in L_{\pi(y)} \) is non-zero. The almost complex structure \( J \) is integrable if and only if \( D \) is Sasaki–Weyl, \( F^D|_{H \times H} \) is \( I \)-invariant and \( i_{\psi^D}F^D = 0 \).

Proof. Recall that if a section \( s \) of \( L \) is viewed as a vertical vector field on \( L \) and \( X,Y \in \mathcal{X}(M) \), then \([\tilde{X},s] = DX(s)\) and \( u^D([\tilde{X},\tilde{Y}]_y) = \tilde{F}_{\tilde{X},\tilde{Y}}^D \), where \( \tilde{F}^D \) denotes the projection onto the last factor in the decomposition \( \Box \). Let \( X \) and \( Y \) be two sections of \( H \), viewed as horizontal vector fields on \( L \). The integrability tensor \( N^D \) of \( J \), applied to the pair \((X,Y)\), can be calculated as follows:

\[
N^D(X,Y)_y = [J\tilde{X},J\tilde{Y}]_y - J([J\tilde{X},\tilde{Y}]_y + [\tilde{X},J\tilde{Y}]_y) - [\tilde{X},\tilde{Y}]_y
\]

\[
= [\tilde{X}\tilde{Y}] - F^D_{\tilde{X},\tilde{Y}}y - J([\tilde{X}\tilde{Y}] - F^D_{\tilde{X},\tilde{Y}}y - F^D_{\tilde{X},\tilde{Y}}y)
\]

\[
= [X,Y]_y + F^D_{X,Y}y
\]

\[
= -F^D_{\tilde{X},\tilde{Y}}y + J(F^D_{X,Y}y + F^D_{\tilde{X},\tilde{Y}}y) + F^D_{X,Y}y
\]

\[
= -2F^D_{\tilde{X},\tilde{Y}}y + 2F^D_{\tilde{X},\tilde{Y}}\psi^D(y),
\]

where we have used the integrability of the complex structure \( I \) of the bundle \( H \). A similar calculation shows that

\[
N^D(X,\psi^D(s))_y = F^D_{X,\psi^D(s)}y + F^D_{\tilde{X},\tilde{Y}}\psi^D(y) - sL_{\psi^D(I)}(Y),
\]

where \( s \) is a local smooth section of \( L \) non-vanishing at the point \( \pi(y) \). The conclusion follows. \( \Box \)

Lemma 3. Let \( \sigma \) be the natural section of \( \pi^*L \to L \) and \( \omega \) the \( \pi^*\text{L}^2 \)-valued 2-form on \( L \) defined by \( \omega := 1/2d^D(\sigma \pi^*\eta) \). Then \( \omega \) has \( J \)-potential \( \sigma^2 \). Equivalently, \( \omega = 1/4d^D(JD\sigma^2) \).

Proof. Let \( s_0 \) be a positive section of \( L \), \( \gamma := s_0^{-1}Ds_0 \), \( \theta_0 := s_0^{-1}\eta \) and \( T_0 := \psi^D(s_0) \). The section \( s_0 \) determines a trivialization \( L \cong M \times \mathbb{R} \). In the trivialization of \( \pi^*\text{L}^2 \) defined by \( \pi^*s_0^2 \), \( \sigma^2(x,t) = (x,t,t^2) \) and \( D\sigma^2 = (dt^2 + 2t^2\pi^*\gamma)\pi^*s_0^2 \). In order to calculate \( JD\sigma^2 \), we need to determine \( J\pi^*\gamma \) and \( dt^2 \). Note that, for \( X \in H_x \), \( J\tilde{X}(x,t) = IX -
The cone of the CR-Weyl manifold

Definition 5. In particular,

\[ JX = IX + t\gamma(X)T_0 - [t^2\gamma(X)\gamma(T_0) + t\gamma(IX)] \frac{\partial}{\partial t}, \]

for every \( X \in H \). Then

\[ J\pi^*\gamma = \pi^*J_0\gamma - t\gamma(T_0)\pi^*\gamma - \gamma(T_0)dt, \]

where \( J_0 \in \text{End}(TM) \) extends \( I \in \text{End}(H) \) being zero on \( T_0 \), and

\[ dt = -t\pi^*(J_0\gamma) + t^2\gamma(T_0)\pi^*\gamma + \pi^*(\theta_0) + t\gamma(T_0)dt. \]

It follows that \( J(dt^2 + 2t^2\pi^*\gamma) = 2t\pi^*\theta_0 \) or equivalently \( J\bar{D}(\sigma^2) = 2\sigma\pi^*\eta. \)

\[ \square \]

On the total space \( L \) of the co-contact bundle of \( M \) we have an almost complex structure \( J \), provided by Proposition 2 and a conformal 2-form \( \omega \) provided by Lemma 3. Their properties are expressed by the following Lemma, whose proof is an easy calculation.

Lemma 4. The 2-form \( \omega \) is \( J \)-invariant. In the trivialization of \( L \) determined by the positive section \( s_0 \), the bilinear form \( g := \pi^*(s_0^{-2})\omega(\cdot, \cdot, J\cdot) \) has the following expression: \( g(X, Y) = \frac{1}{2}d(s_0^{-1}\eta)(X, Y) \), for \( X, Y \in H \); \( g(\tilde{\psi}D(s_0), \psiD(s_0)) = g \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) = \frac{1}{2} \); \( g(\tilde{X}, \psiD(s_0)) = 0 \), for \( X \in H \). In particular, \( g \) is positive definite (with values in the oriented line bundle \( \pi^*L^2 \)) on the subset \( L^+ \) of positive oriented elements of \( L \). Moreover, for every \( y \in L^+ \) the decomposition

\[ T_y L^+ = H_{\pi(y)} \oplus \Lambda_{\pi(y)} \oplus L_{\pi(y)} \]

is \( g \)-orthogonal.

Definition 5. The cone of the CR-Weyl manifold \((M, H, I, D)\) is the manifold \( L^+ \) together with the almost complex structure \( J \) and the \( \pi^*L^2 \)-valued 2-form \( \omega \).

Theorem 6. Let \((L^+, \omega, J)\) be the cone of the oriented CR-Weyl manifold \((M, H, I, D)\). Then \((L^+, \omega, J)\) is locally conformal Kähler (l.c.K.) if and only if \( D \) is Sasaki–Weyl and \( F^D = kd\eta \), for a section \( k \) of \( L^{-1} \) which satisfies \( D(k) + k^2\eta = 0 \). Moreover, on the subset of \( L^+ \) where \( \pi^*(k) \) is non-vanishing, \((L^+, \omega, J)\) is globally conformal Kähler with Kähler form \( \pi^*(k^2)\omega \).

\[ \text{Proof.} \] With the notation of Lemma 3 let \( \Omega := \pi^*(s_0^{-2})\omega = \frac{1}{2}d(\pi^*\theta_0) + t\pi^*(\gamma \wedge \theta_0) \). The exterior derivative of \( \Omega \) satisfies \( d\Omega = t\pi^*(F^D \wedge \theta_0) - 2\Omega \wedge \pi^*\gamma \). It follows that \((L^+, \omega, J)\) is locally conformal Kähler if \( J \) is integrable and if there is a 1-form \( \alpha := k_0dt + k_1 \) on \( L^+ \), where \( k_0 \in C^\infty(M \times \mathbb{R}^{>0}) \) and \( k_1 \) is a 1-form on \( M \) parametrized by \( t \in \mathbb{R}^{>0} \), such that the following two relations: \( \pi^*(F^D \wedge \theta_0) = \Omega \wedge \alpha \) and

\[ d(t\alpha - 2\pi^*\gamma) = 0 \] hold. The first of these relations is equivalent to the system

\[
\frac{k_0t}{2}\pi^*(d\theta_0) + tk_0\pi^*(\gamma \wedge \theta_0) - \frac{1}{2}k_1 \wedge \pi^*\theta_0 = 0
\]

\[
\pi^*(F^D \wedge \theta_0) - \left[\frac{t}{2}\pi^*(d\theta_0) + t\pi^*(\gamma \wedge \theta_0)\right] \wedge k_1 = 0.
\]

Since \( \theta_0 \) annihilates the vectors in \( H \) and \( d\theta_0 \) is non-degenerate when restricted to \( H \), the first equation implies that \( k_0 = 0 \) and \( k_1 \wedge \pi^*\theta_0 = 0 \). This means that \( \alpha = k_1 = \lambda \pi^*\theta_0 \) for a function \( \lambda \in C^\infty(\mathbb{M} \times \mathbb{R}^{>0}) \) which implies that \( \pi^*(F^D \wedge \theta_0) = \frac{\lambda}{2}\pi^*(d\theta_0 \wedge \theta_0) \), or \( F^D|_{H \times H} = \frac{\lambda k}{2} \pi^*(d\eta|_{H \times H}). \)

Since \( J \) is integrable, \( i_{\psi^D}F^D = 0 \) (see Proposition 2) and we obtain that \( F^D = kd^D\eta \), for \( k := \frac{\lambda k}{2} \) which must be independent of \( t \).

On the other hand, the relation \( d(t\alpha - 2\pi^*\gamma) = 0 \) is equivalent with \( F^D = d(k\eta) \). Since \( F^D = d(k\eta) = kd^D\eta \), the section \( k \) of \( L^{-1} \) must satisfy \( D(k) \wedge \eta = 0 \). Also, \( d(kd^D\eta) = (D(k) + k^2\eta) \wedge d^D\eta = 0 \), which in turn implies, since \( i_{\psi^D}d^D\eta = 0 \), that \( D_{\psi^D}(k) + k^2 = 0 \). The first claim of the Theorem follows. For the second claim, we notice that the canonical Weyl connection \( D^c \) of \((L^+, \omega, J)\) is \( D^c = \pi^*(D - k\eta) \) and satisfies \( D^c(\pi^*k^{-1}) = 0 \).  

\[ \square \]

4. Reductions of Sasaki–Weyl manifolds

We recall that a Lie group \( G \) acts by CR automorphisms on a CR manifold \((\mathbb{M}, \mathbb{H}, I)\) if it preserves \( \mathbb{H} \) and \( I \): \( g_*(H_x) \subset H_{gx} \) and \( g_*I_x = I_{gx} \) for every \( g \in \mathbb{G}, x \in \mathbb{M} \) and \( x \in H_x \). Since \( G \) preserves \( \mathbb{H} \), there is an induced action of \( G \) on the co-contact bundle \( L \) of \( \mathbb{H} \) and on the space \( \Gamma(\mathbb{M}, L) \) of smooth sections of \( L \). We shall usually denote by \( g \cdot y \) the action of \( g \in \mathbb{G} \) on \( y \in L \). The action of \( G \) on \( \Gamma(\mathbb{M}, L) \) is defined by \( (g \cdot s)(x) = g \cdot s(g^{-1}x) \) for every \( g \in \mathbb{G}, \ s \in \Gamma(\mathbb{M}, L) \) and \( x \in \mathbb{M} \). Any vector field \( \xi^a \) on \( \mathbb{M} \), generated by an element \( a \in \mathfrak{g} := \text{Lie}(\mathbb{G}) \), defines a Lie derivative \( L_{\xi^a} \) on \( \Gamma(\mathbb{M}, L) \) by the formula \( L_{\xi^a}(s) := \frac{d}{dt}|_{t=0}\exp(-ta) \cdot s \). The universal contact form \( \eta : TM \rightarrow L \) commutes with the actions of \( G \) on \( M \) and \( L \) and the \( L \)-valued 1-form \( L_{\xi^a}(\eta) \) on \( \mathbb{M} \), defined by \( L_{\xi^a}(\eta)(X) := L_{\xi^a}(\eta(X)) - \eta(L_{\xi^a}X) \) for \( X \in \mathcal{X}(\mathbb{M}) \), is identically zero.

We briefly recall the reduction of CR manifolds as developed in \[7\], \[8\]. Let \( G \) act by CR automorphisms on the CR manifold \((\mathbb{M}, \mathbb{H}, I)\), \( \Theta : \mathbb{M} \rightarrow L \otimes g^* \) (where \( g^* \) denotes the trivial bundle over \( \mathbb{M} \) with fiber \( g^* \) be the moment map, defined by \( \Theta_a := \eta(\xi^a) \) for every \( a \in \mathfrak{g} \) and let \( S := \Theta^{-1}(0) \) (here and elsewhere \( 0 \) will denote the image of the zero section of a vector bundle). The Lie group \( G \) preserves \( S \) and we can consider \( p : S \rightarrow \mathbb{M} := S/G \) the natural projection. If the action of \( G \) on \( S \) is free and proper, \( \mathbb{M} \) is a CR manifold with the
contact bundle $\tilde{H} := p_\ast (TS \cap H)$. If $\theta$ is an arbitrary contact form on $M$ and $g := \frac{1}{2}d\theta(\cdot, I\cdot)$ is the corresponding metric on $H$, then the vector space $H_x$ (for $x \in S$) has the $g$-orthogonal decomposition (independent of the chosen contact form $\theta$)

$$H_x = T_x \oplus I T_x \oplus E_x,$$

where $T_x$ is the tangent space to the orbit of $G$ through $x$. The space $E_x$ is $I$-invariant, is isomorphic via the differential $p_\ast$ of $p$ with $\hat{H}_{p(x)}$ and its complex structure induces the complex structure $\hat{I}$ of $\hat{H}_{p(x)}$.

In this Section we add an horizontal and $G$-invariant Weyl connection to this construction (see Definition below) and we study how it behaves under reduction. The Lie group $G$ is not necessarily connected.

**Definition 7.** Let $G$ be a Lie group acting by CR automorphisms on the CR-Weyl manifold $(M, H, I, D)$. The Lie group $G$ acts by CR-Weyl automorphisms if the following two conditions hold:

1. The connection $D$ is horizontal, i.e. it satisfies

$$D_{\xi^a}(s) = L_{\xi^a}(s),$$

for every $s \in \Gamma(M, L)$ and vector field $\xi^a$ on $M$ generated by $G$.

2. The connection $D$ is $G$-invariant, i.e. it satisfies

$$g \cdot (D_X s) = D_{g \ast X} (g \cdot s),$$

for $g \in G$, $X \in \mathcal{X}(M)$, $s \in \Gamma(M, L)$.

Remark: Before studying the reduction, note that if a Weyl connection $D$ is horizontal and $G$ is connected, then $D$ is $G$-invariant if and only if its Faraday curvature $F^D$ satisfies $F^D(\xi^a, X) = 0$, for every $X \in \mathcal{X}(M)$ and vector field $\xi^a$ on $M$ generated by $G$: indeed, relation (3) implies

$$F^D(\xi^a, X) = L_{\xi^a} D_X s - D_X L_{\xi^a} s - D_{[\xi^a, X]} (s) = \frac{d}{dt}\big|_{t=0} \exp(-ta) \cdot D_{\exp(ta)} (\exp(ta) \cdot s),$$

from where we deduce our claim.

The following Proposition gives a nice description of the tangent space to $S$ in terms of the Weyl connection $D$ (see also [7] for an equivalent proof).

**Proposition 8.** For every $x \in S$, $T_x S = \{X \in T_x M\mid d^D \eta (X, \xi^a) = 0, \forall a \in g\}$.

**Proof.** A vector field $X$ is tangent to $S$ along $S$ if, for every $a \in g$, $(\Theta_a)_\ast (X)$ is tangent to the image of the zero section of $L$. Equivalently, if $v^D (\Theta_a)_\ast (X) = D_X (\Theta_a)$ is zero, where we recall that $v^D$ is the
Lemma 10. The Reeb vector field 

Proof. Proposition 8 implies that projectable onto \( \hat{\text{horizontality}} \) of \( D \). Consider now a section \( \hat{s} \) of \( \hat{\psi} \) where it is tangent to the orbits of \( G \) on \( \hat{L} \). It follows that the sections of the co-contact bundle \( \hat{L} \) of \( \hat{M} \) can be identified with the \( G \)-invariant sections of \( \hat{L} \). Consider now a section \( \hat{s} \) of \( \hat{L} \) and \( \hat{s} \) the (unique, \( G \)-invariant) section of \( \hat{L} \) which projects to \( \hat{s} \). Let \( X \) be a vector field on \( \hat{M} \) and \( X_1, X_2 \) two lifts of \( X \) in \( TS \). Then \( X_1 - X_2 \) is tangent to the orbits of \( G \) in \( S \) and the horizontality of \( D \) implies that \( \hat{D}_{X_1}(\hat{s}) = \hat{D}_{X_2}(\hat{s}) \). Since \( \hat{D} \), \( X \) and \( \hat{s} \) are \( G \)-invariant, so is \( \hat{D}_{X_i}(\hat{s}) \). The section of \( \hat{L} \) induced by \( \hat{D}_{X_i}(\hat{s}) \) will be, by definition, \( \hat{D}_{X_i}(\hat{s}) \). Since \( p^*\hat{D} = \hat{D} \), \( \hat{D} \) is closed if \( D \) is.

Lemma 10. The Reeb vector field \( \psi^D \) is tangent to \( S \) along \( S \) and projects to the Reeb vector field \( \psi^D \) of \( \hat{D} \).

Proof. Proposition 8 implies that \( \psi^D \in \hat{L}^{-1} \otimes TS \) along \( S \), because \( i_{\psi^D}d^D\eta = 0 \). Moreover, since \( \psi^D \) is \( G \)-invariant (because \( \eta : TM \to L \) and \( D \) are) with respect to the action of \( G \) on \( \hat{L}^{-1} \otimes TS \), it is also projectable onto \( \hat{L}^{-1} \otimes T\hat{M} \). Using the natural isomorphism between \( p^*\hat{L} \) and \( L|_S \) it is obvious that \( p^*\hat{\eta} = \eta|_{TS} \) and \( p^*d^D\eta = d^D\eta|_{TS \times TS} \). The conclusion follows.

Theorem 11. Let \( G \) be a Lie group acting by CR-Weyl automorphisms on a Sasaki–Weyl manifold \( (M, H, I, D) \). Let \( \Theta : M \to L \otimes g^* \) defined by \( \Theta_a := \eta(\xi^a) \) (for \( a \in g \)) be the moment map. Suppose that \( G \) acts freely and properly on \( S := \Theta^{-1}(0) \) and let \( (\hat{M} := S/G, \hat{H}, \hat{I}) \) be the CR quotient. Then \( M//G := (\hat{M}, \hat{H}, \hat{I}, \hat{D}) \) is Sasaki–Weyl.
Proof. Let $\tilde{X}$ be a section of $\tilde{H}$ and $X$ its unique lift on $S$ which belongs to $E$ (recall the decomposition (2) of $H$ along $S$). We first claim that $L_{\psi D}(X)$ is a section of $L^{-1} \otimes E$. Indeed, $L_{\psi D}(X)$ belongs to $L^{-1} \otimes H$ because $X$ belongs to $H$ and
\[(d^D\eta)((IL_{\psi D}(X),\xi^a) = (d^D\eta)((L_{\psi D}(IX),\xi^a) = -L_{\psi D}(d^D\eta)(IX,\xi^a) = -i_{\psi D}(F^D \wedge \eta)(IX,\xi^a) = -F^D(IX,\xi^a)\]
which is zero since $D$ is horizontal and $G$-invariant. (In the above calculation we have used the Sasaki–Weyl condition on $D$, $L_{\psi D}(\xi^a) = 0$ because $\psi^D$ is $G$-invariant, $(d^D\eta)(IX,\xi^a) = 0$ because $X$ belongs to $E$, and $\eta(\xi^a) = 0$ along $S$). On the other hand, since $\psi^D$ and $X$ are tangent to $S$, $L_{\psi D}(X)$ is also tangent to $S$ and then, from Proposition 12, $(d^D\eta)(L_{\psi D}(X),\xi^a) = 0$. Our claim follows. It is easy to see now that
$L_{\psi D}(\tilde{I})(\tilde{X}) = p_*L_{\psi D}(I)(X)$ (where $p_* : \tilde{L}^{-1} \otimes TS \to (\tilde{L}^{-1} \otimes TS)/G = \tilde{L}^{-1} \otimes TM$ is the natural projection) and hence $L_{\psi D}(\tilde{I}) = 0$ because $D$ is Sasaki–Weyl.

While $\tilde{D}$ is closed when $D$ is, there are situations when $D$ is exact but $\tilde{D}$ is not. This is illustrated by Proposition 13 which is followed by a criteria (see Proposition 13) which expresses when the exactness property of Sasaki–Weyl connections is preserved through the reduction process.

**Proposition 12.** Let $(M, H, I, D)$ be a Sasaki–Weyl manifold with a closed (not necessarily exact) Weyl connection $D$. Let $\tilde{M}$ be the universal cover of $M$. Then $\tilde{M}$ has an induced Sasaki–Weyl structure $(\tilde{H}, \tilde{I}, \tilde{D})$. Moreover, $\tilde{D}$ is exact and $(M, H, I, D)$ is the Sasaki–Weyl reduction of $(\tilde{M}, \tilde{H}, \tilde{I}, \tilde{D})$ under the action of $\pi_1(M)$ on $\tilde{M}$.

**Proof.** Let $p : \tilde{M} \to M$ be the universal cover with the Deck group $\Gamma$, isomorphic with $\pi_1(M)$. The CR structure $(\tilde{H}, \tilde{I})$ of $\tilde{M}$ is defined such that the map $p : (\tilde{M}, \tilde{H}, \tilde{I}) \to (M, H, I)$ is a CR map. Note that the co-contact bundle $\tilde{L}$ of $(M, \tilde{H}, \tilde{I})$ is isomorphic with $p^*L$ and has the connection $\tilde{D} := p^*D$. It is clear that $(\tilde{M}, \tilde{H}, \tilde{I}, \tilde{D})$ is Sasaki–Weyl (the map $p$ is a local diffeomorphism of CR manifolds which preserves the Weyl connections) and that $\Gamma$ acts by CR automorphisms on $\tilde{M}$. Moreover $\tilde{D}$ is $\Gamma$-invariant: if $g \in \Gamma$, $s \in \Gamma(M, L)$ and $X \in \mathcal{X}(M)$ has the lift $\tilde{X} \in \mathcal{X}(\tilde{M})$, then
\[g \cdot \tilde{D}\tilde{X}(p^*s) = g \cdot p^*D_X(s) = p^*D_X(s) = \tilde{D}\tilde{X}(p^*s)\]
The horizontality of $\tilde{D}$ with respect to the $\Gamma$-action on $\tilde{M}$ is trivially satisfied, since $\Gamma$ is discrete. It follows that $\Gamma$ acts by CR-Weyl automorphisms on $(\tilde{M}, \tilde{I}, \tilde{H}, \tilde{D})$ and that the corresponding Sasaki–Weyl reduction is $(M, H, I, D)$ (the moment map being trivial). \qed
Consider again the situation of Theorem 11. Since \( D \) is \( G \) invariant and \( M \) is connected, the action of \( g \in G \) on any global \( D \)-parallel (non-trivial) section \( \mu \) of \( L \) is of the form \( g \cdot \mu = \rho(g)\mu \), where \( \rho(g) \in \mathbb{R} \) is independent of the chosen \( \mu \). We obtain in this way a group homomorphism \( \rho : G \to \mathbb{R} \setminus \{0\} \) associated to the action of \( G \) on \( M \).

**Proposition 13.** Suppose that the Weyl connection \( D \) is exact and that the Lie group \( G \) is connected. Then the Weyl connection \( \hat{D} \) is also exact if and only if \( \rho(G) = \{1\} \).

**Proof.** We shall use the notations employed in the proof of Proposition 9. Suppose first that \( \rho(G) = \{1\} \) and take a \( D \)-parallel (non-trivial) section \( \mu \) of \( L \). Then its restriction to \( S \) is \( \tilde{D} \)-parallel, \( G \)-invariant and it determines a non-vanishing section \( \tilde{\mu} \) on \( \tilde{L} \), which, from the definition of \( \tilde{D} \), must be \( \tilde{D} \)-parallel. For the converse, suppose now that \( \tilde{D} \) is exact, choose a \( \tilde{D} \)-parallel non-vanishing section \( \tilde{\mu} \) of \( \tilde{L} \), and let \( \tilde{\mu} \) be its corresponding \( \tilde{D} \)-parallel section of \( \tilde{L} \). Let \( x \in S \), \( O_x \subset S \) the orbit of \( G \) through \( x \) and \( g \in G \). Since \( D \) is exact, there is a \( D \)-parallel section \( \mu \) of \( L \) such that \( \mu(x) = \tilde{\mu}(x) \). In particular, the restriction of \( \mu \) to \( S \) is \( \tilde{D} \)-parallel, \( \mu = \tilde{\mu} \) on \( O_x \) (because \( O_x \) is connected) and \( g \cdot \mu = \mu \) on \( O_x \) since \( \tilde{\mu} \) is \( G \)-invariant. On the other hand, since \( G \) preserves \( D \), \( g \cdot \mu \) is also \( D \)-parallel and it must coincide with \( \mu \) on the whole of \( M \) (which is connected). This means that \( g \cdot \mu = \mu \), or \( \rho(g) = 1 \). \( \square \)

**Remark:** With the notations of Proposition 13, notice that \( \theta := \mu^{-1}\eta \) is a contact form of a Sasaki structure and is also \( G \)-invariant. A Sasaki reduction of the Sasaki manifold \((M, H, I, \theta)\) is therefore defined (see [5]), whose underlying CR manifold is \((\tilde{M}, \tilde{H}, \tilde{I})\) and whose contact form is, as it can be readily checked, the form \( \tilde{\mu}^{-1}\tilde{\eta} \).

We end this Section by stating the commutativity between the Sasaki–Weyl reductions and the operation of taking universal covers. This is expressed by the following Proposition, whose proof, lengthy but straightforward, will be omitted. Recall first that an action of a Lie group \( G \) on a manifold \( M \) always lifts to an action of the universal cover \( \tilde{G} \) of \( G \) on the universal cover \( \tilde{M} \) of \( M \), which commutes with the action of \( \pi_1(M) \) on \( \tilde{M} \).

**Proposition 14.** Let \((M, H, I, D)\) be a Sasaki–Weyl manifold and \( G \) a Lie group which acts by CR-Weyl automorphisms on \( M \). Let \( \tilde{G} \) be the universal cover of \( G \). The following statements hold:

1. The group \( \tilde{G} \) acts by CR-Weyl automorphisms on the universal cover \((\tilde{M}, \tilde{H}, \tilde{I}, \tilde{D})\) of \( M \).

2. The action of \( \pi_1(M) \) on \( \tilde{M} \) induces a CR-Weyl action of \( \pi_1(M) \) on \( \tilde{M}/\tilde{G} \), and there is a CR diffeomorphism \( \tilde{M}/\tilde{G} / \pi_1(M) \cong M/G \) which preserves the Weyl connections.
5. The commutativity with the cone construction

In this Section we consider a connected Lie group $G$ acting by orientation preserving CR-Weyl automorphisms on the Sasaki–Weyl manifold $(M, H, I, D)$ whose cone $(L^+, \omega, J)$ is l.c.K. In particular, the curvature $F^D$ of $D$ must satisfy $i_{\xi_a} F^D = 0$ for every vector field $\xi^a$ on $M$ generated by $G$, and $F^D = kD_D \eta$, for a section $k$ of $L^{-1}$. Note that these two conditions on the curvature $F^D$ actually imply that $F^D$ is identically zero on the complement of the zero set $S := \Theta^{-1}(0)$ of the moment map $\Theta$ of the action of $G$ on $(M, H, I)$. Indeed, suppose, on the contrary, that $k$ is non-zero at a point $x \in M \setminus S$ and let $a \in g$ such that $\eta(\xi^a)$ is non-zero on a small neighborhood $V$ of $x$, included in $M \setminus S$. From the proof of Proposition 8 we know that $F^D(\xi^a, X) = kD_D \eta(\xi^a)$, $\forall X \in \mathfrak{X}(M)$. Since $F^D(X, \xi^a) = 0$ we deduce that $\eta(\xi^a)$ is $D$-parallel and non-vanishing on $V$. It follows that $F^D$ is zero on $V$, and hence $k(x) = 0$. We have obtained a contradiction.

For the rest of this section we will suppose that $D$ is closed. In particular, the canonical Weyl connection of the l.c.K. cone $(L^+, \omega, J)$ is $\bar{D} := \pi^* D$. Since the action of $G$ on $M$ is orientation preserving, it induces an action on $L^+$ and on the bundle $\pi^* L$ on $L^+$. We shall denote by $\tilde{\xi}^a$ the vector field on $L^+$ generated by $a \in g$.

**Lemma 15.** The vector fields generated by the action of $G$ on $L^+$ are $D$-horizontal lifts of the vector fields generated by the action of $G$ on $M$.

**Proof.** Let $s \in \Gamma(M, L)$ and define $s_t := \exp(-ta) \cdot s$. Since $D$ is horizontal, we infer that $\frac{d}{dt}|_{t=0} s_t = D_{\xi^a}(s)$. Our claim readily follows: for $x \in M$,

$$\tilde{\xi}^a_{s(x)} = s_*(\xi^a) - D_{\xi^a} \eta(\xi^a)$$

$$= s_* \left( \left. \frac{d}{dt} \right|_{t=0} \exp(ta) x \right) - D_{\xi^a} \eta(\xi^a)$$

$$= \frac{d}{dt} \left|_{t=0} \exp(ta) \cdot s_t(x) - D_{\xi^a} \eta(\xi^a) \right.$$}

$$= \tilde{\xi}^a_{s(x)} + \frac{d}{dt} \left|_{t=0} s_t(x) - D_{\xi^a} \eta(\xi^a) \right.$$}

$$= \tilde{\xi}^a_{s(x)}.$$

**Lemma 16.** The connection $\bar{D}$ on the bundle $\pi^* L \to L^+$ is $G$-invariant and horizontal.
Proposition 17. The Lie group G preserves the hermitian conformal structure of \((L^+, \omega, J)\)

Proof. We first prove that for every \(X \in \mathcal{X}(M)\) and \(g \in G\), \(g_s \tilde{X} = \tilde{g_s X}\): for this, let \(s \in \Gamma(M, L)\) and \(x \in M\). The \(G\)-invariance of \(\tilde{D}\) implies

\[
g_s \tilde{X}_{s(x)} = g_s (s_x X_x - D_{X_x} s) = (g \cdot s) g_s (X_x) - D_{g_s X_x} (g \cdot s) = g_s (X_x) g_s (s_x)\]

which is our claim. It is now easy to see that \(G\) acts by holomorphic transformations on \(L^+\): for every \(X \in H\),

\[
g_s (J \tilde{X}) = g_s (\tilde{I} X) = g_s (\tilde{I} X) = Ig_s (X) = Jg_s (\tilde{X}).\]

In a similar way, \(g_s J(\psi^D (s_0)) = g_s s_0 = J g_s (\psi^D (s_0))\) because \(\psi^D\) is \(G\)-invariant. It follows that \(g_s (J \psi^D) = J (g_s \psi^D)\) and that \(G\) acts by biholomorphic transformations on \(L^+\). Since \(\tilde{D}\) and \(\sigma \pi^* \eta\) are \(G\)-invariant, it is clear that \(\omega = \frac{1}{2} d^D(\sigma \pi^* \eta)\) is also \(G\)-invariant: \(g \cdot \omega(X, Y) = \omega(g_s(X), g_s(Y))\), for every \(g \in G\) and \(X, Y \in \mathcal{X}(L^+)\). The conclusion follows.

According to Proposition 14, there is a l.c.K. reduction (see 14 or 15) of \((L^+, \omega, J)\) under the action of \(G\). It has a distinguished moment map provided by the following Proposition.

Proposition 18. Let \(\Theta : M \to L \otimes \mathfrak{g}^*\) be the moment map for the action of \(G\) on \((M, H, I, D)\). Then the map \(\tilde{\Theta} : L^+ \to \pi^*(L^2 \otimes \mathfrak{g}^*)\) defined by \(\tilde{\Theta}_a := \frac{1}{2} \pi^*(\Theta_a) \sigma\) for \(a \in \mathfrak{g}\), is a moment map for the action on \(G\) on the l.c.K. manifold \((L^+, J, \tilde{\omega})\).

Proof. Let \(X \in \mathcal{X}(L^+)\) be \(\pi\)-projectable and \(Y := \pi_s (X)\). From \(\omega = \frac{1}{2} d^D(\sigma \pi^* \eta)\), we get

\[
\omega(X, \xi^a) = \frac{1}{2} D_X (\sigma \pi^* \Theta_a) - \frac{1}{2} D_{\xi^a} (\sigma \pi^* \eta(Y)) = \frac{\sigma}{2} \pi^* \eta([Y, \xi^a])
\]

\[
= \frac{1}{2} \tilde{D}_X (\sigma \pi^* \Theta_a) - \frac{1}{2} \tilde{D}_{\xi^a} (\sigma \pi^* \eta(Y)) - \frac{\sigma}{2} \tilde{D}_{\xi^a} (\pi^* \eta(Y)) - \frac{\sigma}{2} \pi^* \eta([Y, \xi^a]).
\]

On the other hand, since \(D\) is horizontal and \(\eta\) is \(G\)-invariant,

\[
\tilde{D}_{\xi^a} (\pi^* \eta(Y)) = \pi^* L_{\xi^a} (\eta(Y)) = \pi^* \eta([\xi^a, Y]).
\]

Also, since \(D\) is horizontal and \(\sigma\) is \(G\)-invariant, \(\tilde{D}_{\xi^a} (\sigma) = L_{\xi^a} (\sigma) = 0\). We have proved that \(\omega(X, \xi^a) = \frac{1}{2} \tilde{D}_X (\sigma \pi^* \Theta_a)\) which implies the conclusion.
Theorem 19. Let \((M, H, I, D)\) be a Sasaki–Weyl manifold whose Weyl connection \(D\) is closed. Let \(G\) be a connected Lie group acting by orientation preserving CR-Weyl automorphisms on \(M\) such that the underlying manifold of the Sasaki–Weyl reduction \(M//G = (\hat{M}, \hat{H}, \hat{I}, \hat{D})\) is smooth. Then the l.c.K. reduction of \((L^+, \omega, J)\) under the action of \(G\) coincides with l.c.K. cone of the Sasaki–Weyl reduction \(M//G\).

Proof. We shall use the following maps: \(\pi : L \to M, \hat{\pi} : \hat{L} \to \hat{M}\) for the co-contact bundles of \(M\) and \(M\) respectively, \(\hat{p} : \hat{L}^+ \to \hat{L}^+, p : S \to M\) the natural projections, where \(\hat{L}\) is the restriction of the bundle \(L\) to the zero set \(S := \Theta^{-1}(0)\) of the moment map \(\Theta\) of the action of \(G\) on \(M\). Proposition 18 readily implies that \(\hat{L}^+\) is the zero set of the moment map \(\tilde{\Theta}\) of \(G\) on \(L^+\) and that the underlying manifold of the l.c.K. reduction of \((L^+, \omega, J)\) is \(\hat{L}^+//G\) and coincides with the cone \(\hat{L}^+\) of \((\hat{M}, \hat{H}, \hat{I}, \hat{D})\). We will now check that the complex structure \(J_1\) of \(\hat{L}^+\), induced by the l.c.K. reduction, coincides with the cone complex structure \(J_2\) of \(\hat{L}^+\). For every \(y \in \hat{L}^+\) there is a decomposition

\[
T_yL^+ = L_{\pi(y)} \oplus \tilde{T}_{\pi(y)}D \oplus IT_{\pi(y)}D \oplus \hat{E}_{\pi(y)}D \oplus \Lambda_{\pi(y)}D,
\]

where the superscript “\(D\)” denotes horizontal lift with respect to the connection \(D\) at the point \(y\), \(T_{\pi(y)}\) is the tangent space to the orbit of \(G\) through the point \(\pi(y)\) of \(M\) and \(E_{\pi(y)}\) is the hermitian orthogonal complement of \(T_{\pi(y)} \oplus IT_{\pi(y)}\) in \(H_{\pi(y)}\) (with respect to the metric of \(H_{\pi(y)}\) associated to any contact form of the contact bundle \(H\)). Lemma 15 implies that \(\tilde{T}_{\pi(y)}D\) is the tangent space to the orbit of \(G\) in \(L^+\) through \(y\) and the definition of the cone complex structure \(J\) of \(L^+\) implies that \(IT_{\pi(y)}D = J\tilde{T}_{\pi(y)}D\). The differential \(\hat{p}_* : L_{\pi(y)} \oplus \tilde{E}_{\pi(y)}D \oplus \Lambda_{\pi(y)}D \to T_{\hat{p}(y)}\hat{L}^+\) of \(\hat{p}\) is an isomorphism and the complex structure \(J_1\) of the l.c.K reduction is

\[
J_1(\hat{p}_* \psi_{\pi(y)}^{\tilde{D}}(y)) = -p(y) \in T_{\hat{p}(y)}\hat{L}^+ \\
J_1(\hat{p}_* \hat{X}^{\tilde{D}}) = \hat{p}_* \hat{X}^{\tilde{D}}, \quad \forall X \in E_{\pi(y)}.
\]

On the other hand, since \(p_* E_{\pi(y)} = \hat{H}_{p\pi(y)} = \hat{H}_{\hat{p}\hat{p}(y)}\), the Weyl connection \(\hat{D}\) induces the decomposition

\[
T_{\hat{p}(y)}\hat{L} = \hat{L}_{\hat{p}\hat{p}(y)} \oplus \Lambda_{\hat{p}\hat{p}(y)} \oplus p_* E_{\pi(y)}D,
\]

where the superscript “\(\hat{D}\)” denotes \(\hat{D}\)-horizontal lift and \(\Lambda\) is the image of the Reeb vector field \(\psi^{\hat{D}} : \hat{L} \to T\hat{M}\). Then the cone complex
In order to show that $p$ (recall that $p$), let $\hat{G}$ (which implies (5)). The coincidence of the two metrics on $\hat{G}$-invariant) section $s$ of $\hat{\pi}$. Then $\hat{p} \circ s = \hat{s} \circ p$, $s\pi(y) = y$ and

$$\hat{p}_*(\hat{X}^D_y) = \hat{p}_* \left( s_*(X_{\pi(y)}) - D_{\pi(y)}(s) \right)$$

which implies (5). The coincidence of the two metrics on $\hat{L}^+$ can be proved in a similar way (note that the metric coming from the l.c.K. reduction takes values in a bundle $D$ characterized by $\hat{p}^*D \cong \pi^*L^2|_{\hat{L}^+}$; the cone metric of $\hat{L}^+$ takes values in $\hat{\pi}^*\hat{L}^2$; since $\hat{p}^*\hat{\pi}^*\hat{L}^2 = \pi^*p^*\hat{L}^2 \cong \pi^*(L^2|_s) \cong \pi^*L^2|_{\hat{L}^+}$, it follows that the two metrics take values in the same line bundle).

6. Examples

(1) A class of non-closed Sasaki–Weyl connections can be constructed in the following way: suppose that $D_0$ is an exact Sasaki–Weyl connection on the CR manifold $(M, H, I)$ and let $s_0$ be an arbitrary section of $L^{-1}$. The connection $D := D_0 + s_0\eta$ is non-closed and Sasaki–Weyl: $d^D\eta = d^{D_0}\eta$ implies that $\psi^D = \psi^{D_0}$ and $L_{\psi^D}(I) = L_{\psi^{D_0}}(I) = 0$. It follows that any CR manifold which admits a compatible Sasaki structure admits infinitely many of non-closed Sasaki–Weyl connections, parametrized by smooth sections of the dual of its co-contact bundle. Suppose now that $s_0$ is $D_0$-parallel and non-vanishing. Then $F^D = s_0d^D\eta$, $D(s_0) + s_0^2\eta = 0$ and the cone of $(M, H, I, D)$ is globally conformal Kähler (see Theorem 6), with the Kähler structure defined explicitly as follows: let $\theta := s_0\eta$ and $T_0 := \psi^{D_0}(s_0^{-1})$ be the contact form and Reeb vector field associated to $s_0^{-1}$. Consider the trivialization of $\hat{L}^+$ determined by $s_0^{-1}$. The 1-form $\gamma$ from the proof of Lemma 4 is in this case $s_0\eta$. It vanishes on $H$ and it
takes the value one when applied to $T_0$. The complex structure $J$, written down explicitly in the proof of Lemma 3 becomes

$$JX = IX; \quad J \left( \frac{\partial}{\partial t} \right) = T_0 - t \frac{\partial}{\partial t}. $$

The Kähler form is $\omega = \frac{1}{2} d(t \theta_0)$.

(2) We shall illustrate the Sasaki–Weyl reduction in an explicit situation. Let $S^1 \times \mathbb{Z}$ act on $\hat{M} := \mathbb{C}^n \setminus \{0\} \times \mathbb{R}^> 0$ by

$$(e^{i \theta}, m) \cdot (z_1, \ldots, z_n, t) := (e^{ia_1 \theta} \lambda^m z_1, \ldots, e^{ia_n \theta} \lambda^m z_n, \lambda^{2m} t),$$

where $\lambda > 1$ and $a_1, \ldots, a_n$ are integers, not all of the same sign. The 1-form $\theta := \sum_{p=1}^n x_p dy_p - y_p dx_p - dt$ is preserved, up to homothety, by the action of $S^1 \times \mathbb{Z}$ and determines a Sasaki structure on $\hat{M}$. The Sasaki–Weyl reduction of $\hat{M}$ (with this Sasaki structure), under the action of $S^1 \times \mathbb{Z}$ is defined, and the zero set of its moment map is

$$S := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \setminus \{0\} : \sum_{p=1}^n a_p |z_p|^2 = 0 \} \times \mathbb{R}^> 0.$$

It follows that the underlying manifold $\hat{M}_{\lambda, a_1, \ldots, a_n}$ of the Sasaki–Weyl reduction is diffeomorphic with $(S \cap S^{2n-1})/S^1 \times (\mathbb{R}^> 0 \times \mathbb{R}^> 0)/\mathbb{Z}$, where $S^{2n-1}$ is the unit sphere in $\mathbb{C}^n$, $S^1$ acts on $S^{2n-1}$ by

$$e^{i \theta} \cdot (z_1, \ldots, z_n) := (e^{ia_1 \theta} z_1, \ldots, e^{ia_n \theta} z_n)$$

and $\mathbb{Z}$ acts on $\mathbb{R}^> 0 \times \mathbb{R}^> 0$ by $m \cdot (t_1, t_2) := (\lambda^m t_1, \lambda^{2m} t_2)$. Note that the space $M_{a_1, \ldots, a_n} := (S \cap S^{2n-1})/S^1$ coincides with the Sasaki reduction of $S^{2n-1}$ with its natural Sasaki structure, under the action of $S^1$, and has been identified, for various $(a_1, \ldots, a_n)$, in [6]. We obtain closed, non-exact Sasaki–Weyl connections on products $M_{a_1, \ldots, a_n} \times S^1 \times \mathbb{R}$.

(3) The previous Example can be considerably generalized. Let $(M, g, J)$ be a Kähler manifold with the Kähler form $\omega = d\alpha$, for a 1-form $\alpha$ on $M$. Let $G$ be a discrete group acting freely, proper discontinuously, by holomorphic transformations on $(M, g, J)$ and such that for every $g \in G$, $g^* \alpha = \rho(g) \alpha$ for a positive number $\rho(g)$. The 1-form $\theta := \alpha - dt$ determines a Sasaki structure on $M \times \mathbb{R}^> 0$ and is preserved, up to homothety, by the action $g \cdot (x, t) := (g \cdot x, \rho(g) t)$ of $G$ on $M \times \mathbb{R}^> 0$. Proposition 13 implies that the quotient manifold $(M \times \mathbb{R}^> 0)/G$ has a closed non-exact Sasaki–Weyl connection.

Consider now a Sasaki manifold $(N, H, I, T)$ with Reeb vector field $T$ and $(N \times \mathbb{R}^> 0, \omega, J)$ its Kähler cone, whose Kähler form has potential $t^2$ and whose complex structure $J$ satisfies $J(V) := T$ (where $V := t \frac{\partial}{\partial t}$ is the radial vector field on $\mathbb{R}^> 0$).
and $J|_H = I$ on $H$. The Kähler cone $(N \times \mathbb{R}^>, \omega, J)$ admits an holomorphic $\mathbb{Z}$-action defined by

$$m \cdot (x, t) := (x, \lambda^m t),$$

for a fixed integer $\lambda > 1$, which preserves $\alpha := \frac{1}{4} d^4 t^2$ up to homothety. Therefore, we can take $M := (N \times \mathbb{R}^>, \omega, J)$, $G := \mathbb{Z}$ act on $M$ by (7), and deduce that the quotient $N \times S^1 \times S^1$ of $M \times \mathbb{R}^>$ under the $\mathbb{Z}$-action $m \cdot (x, t_1, t_2) := (x, \lambda^m t_1, \lambda^{2m} t_2)$ has a closed, non-exact, Sasaki–Weyl connection. Moreover, if $K$ is a Lie group which acts by CR automorphisms on $N$ and preserves $T$, then we can couple the action of $K$ on $N$ with the $\mathbb{Z}$-action (7) on $\mathbb{R}^>$, to get an holomorphic action of $K \times \mathbb{Z}$ on $N \times \mathbb{R}^>$ which preserves, up to homothety, the 1-form $\alpha = \frac{1}{4} d^4 t^2$. Again, we can take $M := (N \times \mathbb{R}^>, \omega, J)$ and $G := K \times \mathbb{Z}$. A Sasaki–Weyl reduction $N \times \mathbb{R}^> \times \mathbb{R}^> / K \times \mathbb{Z}$ is defined, which is isomorphic, as a Sasaki–Weyl manifold, with $N//K \times S^1 \times \mathbb{R}$ (where $N//K$ is the Sasaki reduction of $(N, H, I, T)$ under the action of $K$). Example 1 is just a particular case of this construction: $K := S^1$ acts on $N := S^{2n-1}$ by (8) preserving its standard Sasaki structure and $\mathbb{Z}$ acts on $\mathbb{R}^>$ by (7).

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