The equidistribution of Fourier coefficients of half-integral weight modular forms on the plane

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Abstract Let $f = \sum_{n=1}^{\infty} a(n)q^n \in S_{k+1/2}(N, \chi_0)$ be a non-zero cuspidal Hecke eigenform of weight $k + \frac{1}{2}$ and the trivial nebentypus $\chi_0$ where the Fourier coefficients $a(n)$ are real. Bruinier and Kohnen conjectured that the signs of $a(n)$ are equidistributed. This conjecture was proved to be true by Inam, Wiese and Arias-de-Reyna for the subfamilies $\{a(tn^2)\}_n$ where $t$ is a squarefree integer such that $a(t) \neq 0$. Let $q$ and $d$ be natural numbers such that $(d, q) = 1$. In this work, we show that $\{a(tn^2)\}_n$ is equidistributed over any arithmetic progression $n \equiv d \mod q$.

Keywords Shimura lift · Fourier coefficients · Half-integral weight · Sato-Tate equidistribution

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1 Introduction

Let $k \geq 2$, $4 \mid N$ be integers, $\chi \pmod{N}$ a Dirichlet character, and let $f = \sum_{n=1}^{\infty} a(n)q^n \in S_{k+1/2}(N, \chi)$ be a non-zero cuspidal Hecke eigenform of weight $k + \frac{1}{2}$. Applying the Shimura lift to $f$ for a fixed squarefree $t$ such that $a(t) \neq 0$, we get $F_t = \sum_{n=1}^{\infty} A_t(n)q^n \in S_{2k}(N/2, \chi^2)$ the Hecke eigenform of weight $2k$.

When $\chi = 1$, Bruinier and Kohnen suggested in [3] that half of the coefficients $a(n)$ are positive among all non-zero Fourier coefficients. This suggestion was formulated later explicitly as a conjecture in [7]. Assuming some error term for the convergence of the Sato-Tate distribution for integral weight modular forms in [5], Inam and Wise showed when $F_t$ has no CM that half of the coefficients $a(tn^2)$ are positive. They formulated this result in terms of Dedekind-Dirichlet density. They
also showed with Arias-de-Reyna in [11], that \((a(tn^2))_{n \in \mathbb{N}}\) are equidistributed when \(F_t\) has CM and the equidistribution was reformulated in both CM and not CM cases using Dedekind-Dirichlet and natural densities. Later, those results were obtained in [6] by removing the error term assumption.

The present work gives an improvement of the Bruinier-Kohnen conjecture. Indeed, under the error term hypothesis that we will explain below, our main result is the following theorem.

**Theorem 1** Assume the setting of the introduction and suppose that \(F_t\) does not have complex multiplication. Let \(q\) be a natural number. Suppose that for all Dirichlet characters \(\varepsilon \pmod{q}\) and all roots of unity \(\xi\) such that \(\xi \in \text{Im } \varepsilon\), there are \(C_{\varepsilon, \xi} > 0\) and \(\alpha_{\varepsilon, \xi} > 0\) such that

\[
\left| \frac{\# \left\{ p \leq x \text{ prime } | p \nmid N, \varepsilon(p) = \xi, \frac{A_{\varepsilon}(p)}{2\varepsilon(p)^{1/2}} \chi(p) \in [a, b] \right\}}{\pi(x)} - \frac{\mu([a, b])}{\# \text{Im } \varepsilon} \right| \leq C_{\varepsilon, \xi} x^{\alpha_{\varepsilon, \xi}}.
\]

Then for all integers \(d, (d, q) = 1\), the sets

\[
\left\{ n \in \mathbb{N} \mid (n, N) = 1, n \equiv d \pmod{q}, a(tn^2) \chi(n) > 0 \right\} \quad \text{and} \quad \left\{ n \in \mathbb{N} \mid (n, N) = 1, n \equiv d \pmod{q}, a(tn^2) \chi(n) < 0 \right\}
\]

have equal positive natural densities and both are half of the natural density of

\[
\left\{ n \in \mathbb{N} \mid (n, N) = 1, n \equiv d \pmod{q}, a(tn^2) \chi(n) \neq 0 \right\}.
\]

We discuss here two aspects of this theorem. Consider first the case when \(\chi = 1\) and the coefficients \(a(n)\) are real. Then for all natural numbers \(q\) and \(d\) such that \((d, q) = 1\), we have

\[
\lim_{x \to +\infty} \frac{\# \left\{ n \leq x \mid n \equiv d \pmod{q}, a(tn^2) \geq 0 \right\}}{\# \left\{ n \leq x \mid n \equiv d \pmod{q}, a(tn^2) \neq 0 \right\}} = \frac{1}{2}
\]

This extends the results obtained in [5, 11], and therefore, one can ask if the Bruinier-Kohnen conjecture remains true over arithmetic progressions. We have no numerical experiments yet to support this hypothesis.

Consider now the general case \(f \in S_{k+1/2}(N, \chi)\). Let \(q\) be a natural number, \(\varepsilon \pmod{q}\) a Dirichlet character and \(\xi \in \text{Im } \varepsilon\). From the main theorem above and since the density of the set (3) is independent of \(d\) by Proposition 4 and Remark 2, the sets

\[
\left\{ n \in \mathbb{N} \mid (n, N) = 1, \varepsilon(n) = \xi, a(tn^2) \chi(n) > 0 \right\} \quad \text{and} \quad \left\{ n \in \mathbb{N} \mid (n, N) = 1, \varepsilon(n) = \xi, a(tn^2) \chi(n) < 0 \right\}
\]
have equal positive natural densities and both are half of the natural density of
\[ \left\{ n \in \mathbb{N} \mid (n, N) = 1, \varepsilon(n) = \xi, \frac{a(tn^2)}{\chi(n)} \neq 0 \right\}. \]
In the particular case \( q = N \) and \( \varepsilon = \chi \), we deduce that when \( \xi \neq \pm i \), the sets
\[ \left\{ n \in \mathbb{N} \mid \chi(n) = \xi, \Re \left( a(tn^2) \right) > 0 \right\} \quad \text{and} \quad \left\{ n \in \mathbb{N} \mid \chi(n) = \xi, \Re \left( a(tn^2) \right) < 0 \right\} \]
have equal positive natural densities and both are half of the natural density of
\[ \left\{ n \in \mathbb{N} \mid \chi(n) = \xi, a(tn^2) \neq 0 \right\}. \]

Geometrically, the coefficients \( a(tn^2) \) with \( \chi(n) = \xi \) belong to the same line and they are equidistributed over it. When \( \xi = \pm i \), we obtain a similar result and the coefficients \( a(tn^2) \) with \( \chi(n) = i \) or \( -i \) are equidistributed over the vertical line that passes through \( i \) and \( -i \). Once again, one can ask more generally if the Fourier coefficients \( a(n) \) with \( (n, N) = 1 \), that belong to the same line, are equidistributed geometrically as above.

## 2 Notions of Density

Recall that the set of primes (resp. the set of natural numbers) \( S \subseteq \mathbb{P} \) (resp. \( A \subseteq \mathbb{N} \)) has a natural density \( d(S) \) (resp. \( d(A) \)) if the limit \( d(S) = \lim_{x \to +\infty} \frac{\pi_S(x)}{x} \) (resp. \( d(A) = \lim_{x \to +\infty} \frac{\#\{n \leq x \mid n \in A\}}{x} \)) exists, where \( \pi_S(x) \) and \( \pi(x) \) are defined by
\[ \pi(x) = \#\{p \leq x \mid p \in \mathbb{P}\} \quad \text{and} \quad \pi_S(x) = \#\{p \leq x \mid p \in S\}. \]
The set of primes (resp. of natural numbers) \( S \) (resp. \( A \)) is said to have Dirichlet density \( \delta(S) \) (resp. Dedekind-Dirichlet density \( \delta(A) \)) if the limit
\[ \delta(S) = \lim_{x \to 1^+} \frac{\sum_{n \leq x} \frac{\pi_S(n)}{n}}{\log \left( \frac{1}{1-x} \right)} \quad \text{and} \quad \delta(A) = \lim_{x \to 1^+} \left( z - 1 \right) \sum_{n \in A} \frac{1}{n^z} \]
exists. Recall that if the set \( A \) of natural numbers has natural density \( d(A) \), then it also has Dedekind-Dirichlet density \( \delta(A) \) with \( d(A) = \delta(A) \). Further, the set of primes \( S \) is said to be regular if there is a holomorphic function \( g(z) \) on \( \Re(z) \geq 1 \) such that
\[ \sum_{p \in S} \frac{1}{p^z} = \delta(S) \log \left( \frac{1}{z-1} \right) + g(z). \]
We need the following technical lemma (see [5, Lemma 2.1]).

**Lemma 1** Let \( S_1 \) and \( S_2 \) be two regular sets of primes such that \( \delta(S_1) = \delta(S_2) \). Then the function \( \sum_{p \in S_1} \frac{1}{p^z} - \sum_{q \in S_2} \frac{1}{q^z} \) is analytic on \( \Re(z) \geq 1 \).

The following proposition said that the set of primes \( S \) is regular if it has a natural density that satisfies certain error term (see [5, Proposition 2.2]).

**Proposition 1** Let \( S \subseteq \mathbb{P} \) be a set of primes that have natural density \( d(S) \). Define \( E(x) = \frac{\pi_S(x)}{\pi(x)} - d(S) \) to be the error function. Suppose that there are \( \alpha > 0 \), \( C > 0 \), and \( M > 0 \) such that for all \( x > M \) we have \( |E(x)| \leq Cx^{-\alpha} \). Then \( S \) is a regular set of primes.
3 The Chebotarev-Sato-Tate equidistribution

We recall now some properties of the Shimura lift (see [12]). The Fourier coefficients of \( f \) and \( F_t \) are related by the following formula

\[
A_t(n) = \sum_{d|n} \chi_{t,N}(d) d^{k-1} a \left( \frac{n^2}{d^2} \right),
\]

(6)

where \( \chi_{t,N} \) denotes the character \( \chi_{t,N}(d) := \chi(d) \left( \frac{-1}{d} N^2 \right) \). Since \( f \) is the Hecke eigenform for the Hecke operator \( T_{p^2} \), \( F_t \) is an eigenform for the Hecke operator \( T_p \), for all primes \( p \nmid N \). Further, we have \( F_t = a(t) F \), where \( F \) is a normalised Hecke eigenform independant of \( t \).

Applying the Ramanujan-Petersson bound to the Fourier coefficients of \( F_t \), then \( |A_t(n)| \leq \frac{2}{\sqrt{n}} \). Since \( F_t \in S_{2k}(N/2, \chi^2) \), then \( A_t(p) = \chi^2(p) A_t(p) \). Therefore \( \frac{A_t(p)}{\chi(p)} \in \mathbb{R} \) and define

\[
B_t(p) := \frac{A_t(p)}{2a(t)p} \chi(p) \in [-1, 1].
\]

Notice that \( a(t) \in \mathbb{R} \), since \( a(t) = \frac{A_t(1)}{\chi(1)} \).

Recall that the Sato-Tate measure \( \mu \) is the measure on \([-1, 1]\) given by \( \frac{2}{\sqrt{1 - t^2}} dt \).

We state the important Sato-Tate equidistribution theorem for \( \Gamma_0(N) \) (see Theorem B of [2]).

**Theorem 2** (Barnet-Lamb, Geraghty, Harris, Taylor). Let \( k \geq 1 \) and let \( F_t = \sum_{n \geq 1} A(n)q^n \in S_{2k}(N/2, \chi^2) \) be a cuspidal Hecke eigenform of weight \( 2k \) for \( \Gamma_0(N) \). Suppose that \( F_t \) is without multiplication. Denote by \( \text{Im} \chi \) the image of \( \chi \) and let \( \xi \in \text{Im} \chi \). Then, when \( p \) runs through the primes \( p \nmid N \) such that \( \chi(p) = \xi \), the numbers \( B(p) = \frac{A_t(p)}{2a(t)p} \chi(p) \in [-1, 1] \) are \( \mu \)-equidistributed in \([-1, 1]\).

Inam et al. (see [5], [11], [6]) obtained the equidistribution of the coefficients \( a(tn^2) \) by using Theorem 2. In order to prove the geometric equidistribution on the plan as it was explained in the introduction, we need the following hybrid Chebotarev-Sato-Tate equidistribution proved for elliptic curves in [10] for the first time, and it has been generalized recently by Wong (see [13]) particularly to non-CM Hecke eigenforms.

**Proposition 2** (Wong) Let \( q \) be a natural number and \( d \) an integer with \( (d, q) = 1 \). Let \([a, b] \subset [-1, 1]\) and put \( S_{[a,b]} := \{ p \text{ prime } | \ p \equiv d \pmod{q}, B_t(p) \in [a, b] \} \). The set \( S_{[a,b]} \) has natural density equal to \( \frac{2}{\sqrt{1 - t^2}} dt \).

Using Dirichlet’s theorem on arithmetic progressions, this proposition could be rewritten as follows.

**Proposition 3** Let \( q \) be a natural number, \( \varepsilon \pmod{q} \) a Dirichlet character and \( \xi \) a root of unity such that \( \xi \in \text{Im} \varepsilon \). Let \([a, b] \subset [-1, 1]\) and put \( S_{[a,b]} := \{ p \text{ prime } | \varepsilon(p) = \xi, B_t(p) \in [a, b] \} \). The set \( S_{[a,b]} \) has natural density equal to \( \frac{1}{\#\text{Im} \varepsilon} \int_a^b \sqrt{1 - t^2} dt \), where \( \#\text{Im} \varepsilon \) is the cardinality of the image of \( \varepsilon \).
We will use frequently throughout the paper the following lemma (see [9]).

**Lemma 2** Under the hypothesis fixed in the introduction, let \( n \) be an integer such that \((n, N) = 1\). Then \( \frac{a(n^2)}{\chi(n)} \in \mathbb{R} \).

### 4 Preliminaries Results

We next show that the Chebotarev-Sato-Tate theorem (see [13, Proposition 2.2]) gives the equidistribution of the coefficients \( a(tp^2) \) when a primes \( p \) run over arithmetic progressions.

**Theorem 3** We use the assumptions fixed in the introduction and suppose that \( F_t \) has no CM. Let \( q \) be a natural number, \( \varepsilon \) (mod \( q \)) a Dirichlet character and \( \xi \) a root of unity such that \( \xi \in \text{Im } \varepsilon \). Define the set of primes

\[
P_{\varepsilon, \xi, >} := \left\{ p \in \mathbb{P} \mid \varepsilon(p) = \xi, \frac{a(tp^2)}{\chi(p)} > 0 \right\},
\]

and similarly \( P_{\varepsilon, \xi, <}, P_{\varepsilon, \xi, \geq}, P_{\varepsilon, \xi, \leq}, \) and \( P_{\varepsilon, \xi, =0} \). Let \( d \) be an integer such that \((d, q) = 1\). Define also

\[
P_{d, q, >} := \left\{ p \in \mathbb{P} \mid p \equiv d \mod q, \frac{a(tp^2)}{\chi(p)} > 0 \right\},
\]

and similarly \( P_{d, q, <}, P_{d, q, \geq}, P_{d, q, \leq}, P_{d, q, =0} \).

The sets \( P_{d, q, >}, P_{d, q, <}, P_{d, q, \geq}, P_{d, q, \leq}, P_{d, q, =0} \) have natural density \( \frac{1}{\phi(q)} \) and \( P_{d, q, =0} \) has natural density 0. Further, the sets \( P_{\varepsilon, \xi, >}, P_{\varepsilon, \xi, <}, P_{\varepsilon, \xi, \geq}, P_{\varepsilon, \xi, \leq}, P_{\varepsilon, \xi, =0} \) have natural density \( \frac{1}{2 \# \text{Im } \varepsilon} \) and \( P_{\varepsilon, \xi, =0} \) has natural density 0, where \( \# \text{Im } \varepsilon \) is the cardinality of the image of \( \varepsilon \).

**Proof** Define the sets \( \pi_{d,q, >}(x) := \# \{ p \leq x \mid p \equiv d \mod q, \frac{a(tp^2)}{\chi(p)} > 0 \} \), and similarly, \( \pi_{d,q}(x), \pi_{d,q,<}(x), \pi_{d,q,\geq}(x), \pi_{d,q,\leq}(x), \) and \( \pi_{d,q,=0}(x) \). Without loss of generality, we can assume that \( F_t \) is normalised and thus \( a(t) = 1 \). Denote the character \((\frac{-1}{N^2 t})\) by \( \chi_1(.) = (\frac{-1}{N^2 t}) \). The formula (6) yields

\[
\frac{a(tp^2)}{\chi(p)} > 0 \iff B_t(p) > \frac{\chi_1(p)}{2\sqrt{p}}.
\]

Let \( \epsilon > 0 \). Since for all \( p > \frac{1}{4\epsilon^2} \), we have \( \frac{\chi_1(p)}{2\sqrt{p}} = \frac{1}{2\sqrt{p}} < \epsilon \), then

\[
\pi_{d,q, >}(x) + \# \{ p \leq x \text{ prime } \mid p \equiv d \mod q, p \leq \frac{1}{4\epsilon^2} \} \geq \# \{ p \leq x \text{ prime } \mid p \equiv d \mod q, B_t(p) > \epsilon \}. \tag{7}
\]

Applying Proposition\ref{lemma2} we get

\[
\lim_{x \to \infty} \frac{\# \{ p \leq x \text{ prime } \mid p \equiv d \mod q, B_t(p) > \epsilon \}}{\pi(x)} = \frac{\mu([\epsilon, 1])}{\phi(q)}
\]
Theorem 4

The Chebotarev-Sato-Tate theorem satisfies certain error term. The proof is closely similarly by using Proposition 3.

Suppose further that \( \alpha > \frac{\pi}{a} \)

\[
\lim_{x \to \infty} \frac{\pi_{d,q}(x)}{\pi(x)} = \mu([0, 1]).
\]

It follows that \( \liminf_{x \to \infty} \frac{\pi_{d,q}(x)}{\pi(x)} \geq \mu([0, 1]) \) for all \( \epsilon > 0 \), hence \( \liminf_{x \to \infty} \frac{\pi_{d,q}(x)}{\pi(x)} \geq \mu([0, 1]) = \frac{1}{2} \).

Similarly, we have

\[
\liminf_{x \to \infty} \frac{\pi_{d,q}(x)}{\pi(x)} \geq \mu([0, 1]) = \frac{1}{2}.
\]

Since \( \pi_{d,q}(x) = \pi_{d,q}(x) - \pi_{d,q}(\pi) \), then \( \limsup_{x \to \infty} \frac{\pi_{d,q}(x)}{\pi(x)} = \frac{1}{2} \). Using the same method, we obtain the densities of \( \mathbb{P}_{d,q,<}, \mathbb{P}_{d,q,\geq}, \) and \( \mathbb{P}_{d,q,\leq} \). Finally, since \( \pi_{d,q,=0}(x) = \pi_{d,q,\geq}(x) - \pi_{d,q,\geq}(x) \), then the density of \( \mathbb{P}_{d,q,=0}(x) \) is zero.

The densities of the sets \( \mathbb{P}_{\varepsilon,\xi,>, \mathbb{P}_{\varepsilon,\xi,\geq, \mathbb{P}_{\varepsilon,\xi,\leq, \mathbb{P}_{\varepsilon,\xi,=0} \) and \( \mathbb{P}_{\varepsilon,\xi,=0} \) are obtained similarly by using Proposition 3.

The following theorem said that the set of primes of Theorem 3 is regular if the Chebotarev-Sato-Tate theorem satisfies certain error term. The proof is closely similar to that of [3, Theorem 4.2].

Theorem 4

Assuming the assumptions of Theorem 3 and suppose there are \( C > 0 \) and \( \alpha > 0 \) such that

\[
\left| \left\{ p \leq x \text{ prime} \mid \varepsilon(p) = \xi, \frac{A_t(p)}{\chi(p)} \in [a, b] \right\} \right| = \frac{\mu([a, b])}{\# \Im \varepsilon} \leq \frac{C}{x^\alpha}.
\]

Then, the sets \( \mathbb{P}_{\varepsilon,\xi,\geq, \mathbb{P}_{\varepsilon,\xi,\leq, \mathbb{P}_{\varepsilon,\xi,=0} \) are regular sets of primes.

Remark 1

Let \( \xi \) be a \( q \)th root of unity. The previous error term is weaker than the one conjectured by Akiyama and Tanigawa (see [1]) and it can be obtained by [3, Theorem 1.3] if GRH is assumed and also, if \( L(\varepsilon, \text{Sym}^m F_q \otimes \eta) \) is automorphic over \( \mathbb{Q} \) for every \( m \) and for all irreducible characters \( \eta \) of \( G(\mathbb{Q}(\xi)/\mathbb{Q}) \).

To proceed with our proof, we establish the following two lemmas.

Lemma 3

Assuming the assumptions fixed in the introduction and suppose that \( F_t \) has no CM. Let \( q \) be a natural number. Suppose that for all \( \varepsilon \) (mod \( q \)) Dirichlet characters and all roots of unity \( \xi \) such that \( \xi \in \Im \varepsilon \), there are \( C_{\varepsilon,\xi} > 0 \) and \( \alpha_{\varepsilon,\xi} > 0 \) such that

\[
\left| \left\{ p \leq x \text{ prime} \mid p \nmid N, \varepsilon(p) = \xi, \frac{A_t(p)}{\chi(p)} \in [a, b] \right\} \right| - \frac{\mu([a, b])}{\# \Im \varepsilon} \leq \frac{C_{\varepsilon,\xi}}{x^{\alpha_{\varepsilon,\xi}}}.
\]

Suppose further that \( a(t) > 0 \). Define the multiplicative function, \( \forall n \in \mathbb{N}, \)
The equidistribution of Fourier

\[ f(n) = \begin{cases} 
1, & \text{if } \frac{a(tn^2)}{\chi(n)} > 0 \text{ and } (n, N) = 1, \\
-1, & \text{if } \frac{a(tn^2)}{\chi(n)} < 0 \text{ and } (n, N) = 1, \\
0, & \text{if } a(tn^2) = 0 \text{ and } (n, N) = 1, \\
0, & \text{if } (n, N) \neq 1.
\]

Let \( d \) be an integer with \((d, q) = 1\). Then the Dirichlet series

\[ F(z) = \sum_{n \geq 1 \atop n \equiv d \mod q} f(n) \frac{z^n}{n^z} \]

is holomorphic on \( \Re(z) \geq 1 \).

**Proof** We have

\[
\sum_{n \geq 1 \atop n \equiv d \mod q} f(n) \frac{z^n}{n^z} = \frac{1}{\varphi(q)} \sum_{n=1}^{\infty} f(n) \frac{z^n}{n^z} \times \left( \sum_{\varepsilon \mod q} \varepsilon(n) \varepsilon(d) \right)
\]

\[
= \frac{1}{\varphi(q)} \sum_{\varepsilon \mod q} \left( \sum_{n=1}^{\infty} f(n) \frac{z^n}{n^z} \right) \times \varepsilon(d).
\]

Since the first sum is finite, it suffices to show that \( G_{\varepsilon}(z) = \sum_{n=1}^{\infty} f(n) \frac{z^n}{n^z} \) is holomorphic on \( \Re(z) \geq 1 \).

Since \( a(t) > 0 \), and \( \forall m, n \in \mathbb{N}, (m, N) = 1, (n, N) = 1 \),

\[
\frac{a(tm^2) a(tn^2)}{\chi(m) \chi(n)} = a(t) \frac{a(tm^2n^2)}{\chi(mn)}
\]

then \( f(n) \) is multiplicative.

Applying [11, Lemma 2.1.2], we obtain

\[
\log G_{\varepsilon}(z) = \sum_{p \in \mathfrak{P}} \frac{f(p) \varepsilon(p)}{p^z} + g(z),
\]

where \( g(z) \) is a function that is holomorphic on \( \Re(z) > \frac{1}{2} \). Hence

\[
\log G_{\varepsilon}(z) = \sum_{p \in \mathfrak{P}} \frac{f(p) \varepsilon(p)}{p^z} + g(z)
\]

\[
= \sum_{\xi \in \mathfrak{I}(\varepsilon)} \xi \sum_{p \in \mathfrak{P}_{\xi, \varepsilon}} \frac{f(p)}{p^z} + g(z)
\]

\[
= \sum_{\xi \in \mathfrak{I}(\varepsilon)} \xi \left( \sum_{p \in \mathfrak{P}_{\xi, \varepsilon, >}} \frac{1}{p^z} - \sum_{p \in \mathfrak{P}_{\xi, \varepsilon, <}} \frac{1}{p^z} \right) + g(z).
\]
The sets $\mathbb{P}_{\varepsilon,\xi,>}$ and $\mathbb{P}_{\varepsilon,\xi,<}$ are regular sets of primes, and they have the same density $\frac{1}{2\# \Im \varepsilon}$ by Theorem 3. Therefore by Lemma 1, $\log G_{\varepsilon}(z)$ is holomorphic on $R(z) \geq 1$, and consequently $G_{\varepsilon}(z)$ is also holomorphic.

**Lemma 4** We use the assumptions fixed in the introduction and suppose that $F_{z}$ has no CM. Let $q$ be a natural number. Suppose that for all Dirichlet characters $\varepsilon \pmod{q}$ and all roots of unity $\xi$ such that $\xi \in \Im \varepsilon$, there are $C_{\varepsilon,\xi} > 0$ and $\alpha_{\varepsilon,\xi} > 0$ such that

\[
\left| \# \left\{ p \leq x \text{ prime} \mid p \nmid N, \varepsilon(p) = \xi, \frac{A_{\varepsilon}(p)}{2\alpha_{\varepsilon}(p)^{2} \chi(p)} \in [a, b] \right\} \right| - \frac{\mu([a, b])}{\# \Im \varepsilon} \leq C_{\varepsilon,\xi} x^{\alpha_{\varepsilon,\xi}}.
\]

(9)

Then for all integers $d$, $(d, q) = 1$, the set

\[
\{ n \in \mathbb{N} \mid (n, N) = 1, n \equiv d \pmod{q}, a(tn^{2}) \neq 0 \}
\]

has natural density.

**Proof**

We have

\[
\sum_{n \equiv d \pmod{q}} \frac{f(n)^{2}}{n^{2}} = \frac{1}{\varphi(q)} \sum_{\varepsilon \pmod{q}} \left( \sum_{n=1}^{\infty} \frac{f(n)^{2} \varepsilon(n)}{n^{2}} \right) \times \varepsilon(d).
\]

We shall define $H_{\varepsilon}(z) = \sum_{n=1}^{\infty} \frac{f(n)^{2} \varepsilon(n)}{n^{2}}$. Applying [11, Lemma 2.1.2] to get

\[
\log H_{\varepsilon}(z) := \sum_{p \in \mathbb{P}} \frac{f(p)^{2} \varepsilon(p)}{p^{2}} + g_{\varepsilon}(z)
\]

\[
= \sum_{\xi \in \Im \varepsilon} \xi \left( \sum_{p \in \mathbb{P}_{\varepsilon,\xi,>} \cup \mathbb{P}_{\varepsilon,\xi,<}} \frac{1}{p^{2}} \right) + g_{\varepsilon}(z),
\]

where $g_{\varepsilon}(z)$ is a function that is holomorphic on $Re(z) > \frac{1}{2}$. Applying Theorem 4, the sets $\mathbb{P}_{\varepsilon,\xi,>}$ and $\mathbb{P}_{\varepsilon,\xi,<}$ are regular sets of primes of natural density $\frac{1}{2\# \Im \varepsilon}$. Then

\[
\sum_{p \in \mathbb{P}_{\varepsilon,\xi,>} \cup \mathbb{P}_{\varepsilon,\xi,<}} \frac{1}{p^{2}} = \frac{1}{\# \Im \varepsilon} \log \left( \frac{1}{z - 1} \right) + h_{\xi}(z),
\]

where $h_{\xi}$ is a holomorphic function on $Re(z) \geq 1$. It follows that

\[
\log H_{\varepsilon}(z) := \sum_{\xi \in \Im \varepsilon} \xi \left( \sum_{p \in \mathbb{P}_{\varepsilon,\xi,>} \cup \mathbb{P}_{\varepsilon,\xi,<}} \frac{1}{p^{2}} \right) + g_{\varepsilon}(z) + \sum_{\xi \in \Im \varepsilon} \xi h_{\xi}(z) + g_{\varepsilon}(z).
\]
Thus \( \log H_{\varepsilon_0}(z) = \log \left( \frac{1}{z} \right) + h_1(z) + g_{\varepsilon_0}(z) \) where \( \varepsilon_0 \) is the principal Dirichlet character modulo \( q \), and \( \log H_{\varepsilon}(z) = \sum_{\xi \in \text{Im } \varepsilon} h_{\xi}(z) + g_{\varepsilon}(z) \) when \( \varepsilon \neq \varepsilon_0 \). From this we see that in all cases, there is \( b_{\varepsilon} \in \mathbb{C} \) satisfying

\[
H_{\varepsilon}(z) = \frac{b_{\varepsilon}}{z - 1} + k_{\varepsilon}(z),
\]

where \( k_{\varepsilon} \) is holomorphic on \( \text{Re}(z) \geq 1 \). Therefore

\[
\sum_{n \equiv \varepsilon \mod q} \frac{f(n)^2}{n^z} = \frac{b}{z - 1} + k(z),
\]

where \( b \in \mathbb{C} \) and \( k \) is holomorphic on \( \text{Re}(z) \geq 1 \). We can now apply Wiener-Ikehara’s theorem (see [8]) to deduce the result.

**Remark 2** Notice that the natural density of the set

\[
\{ n \in \mathbb{N} \mid (n, N) = 1, n \equiv d \mod q, a(tn^2) \neq 0 \}
\]

is independent of the choice of \( d \). Indeed, from Wiener-Ikehara’s theorem we know that this density is equal to \( \frac{h_1(1) + g_{\varepsilon_0}(1)}{\varphi(q)} \).

5 Proof of Theorem 1

Before starting the proof, recall the theorem of Delange (see [4]).

**Theorem 5** Let \( g : \mathbb{N} \rightarrow \mathbb{C} \) be a multiplicative arithmetic function for which:

1. \( \forall n \in \mathbb{N}, \, |g(n)| \leq 1 \).
2. There exists \( a \in \mathbb{C} \) such that \( a \neq 1 \) and satisfying

\[
\lim_{x \rightarrow +\infty} \frac{\sum_{p \text{ prime}} g(p)}{\pi(x)} = a.
\]

Then we have

\[
\lim_{x \rightarrow +\infty} \frac{\sum_{n \leq x} g(n)}{x} = 0.
\]

We can now piece together the previous lemmas to prove Theorem 1.

**Proof** We have

\[
\sum_{\substack{1 \leq n \leq x \\mod q \atop n \equiv \varepsilon \mod q}} f(n) = \frac{1}{\varphi(q)} \sum_{\varepsilon \mod q} \left( \sum_{1 \leq n \leq x} f(n) \varepsilon(n) \right) \times \overline{\varepsilon(d)}.
\]

(10)

For a Dirichlet character \( \varepsilon \) modulo \( q \), we have

\[
\lim_{x \rightarrow +\infty} \frac{\sum_{1 \leq p \leq x} f(p) \varepsilon(p)}{\pi(x)} = \lim_{x \rightarrow +\infty} \sum_{\xi \in \text{Im } \varepsilon} \xi \frac{\# \{ p \leq x \mid p \in \mathbb{P}, \xi(p) > 0 \}}{\pi(x)} - \xi \frac{\# \{ p \leq x \mid p \in \mathbb{P}, \xi(p) < 0 \}}{\pi(x)} = 0,
\]
since $P_{\varepsilon, >}$ and $P_{\varepsilon, <}$ have the same natural density $\frac{1}{\varphi(q)}$. Applying Delange’s theorem, we get
$$\lim_{x \to +\infty} \frac{\sum_{1 \leq n \leq x} f(n)\varepsilon(n)}{x} = 0,$$
and consequently
$$\lim_{x \to +\infty} \frac{\sum_{n \equiv d \mod q} f(n)}{x} = 0.$$

From which we have
$$\lim_{x \to +\infty} \frac{\# \left\{ n \leq x \mid (n, N) = 1, n \equiv d \mod q, a(tn^2) > 0 \right\}}{x} - \frac{\# \left\{ n \leq x \mid (n, N) = 1, n \equiv d \mod q, a(tn^2) < 0 \right\}}{x} = 0. \quad (11)$$

By Lemma 4, there is $b > 0$ such that
$$\lim_{x \to +\infty} \frac{\# \left\{ n \leq x \mid (n, N) = 1, n \equiv d \mod q, a(tn^2) > 0 \right\}}{x} + \frac{\# \left\{ n \leq x \mid (n, N) = 1, n \equiv d \mod q, a(tn^2) < 0 \right\}}{x} = b. \quad (12)$$

The result follows from (11) and (12).

We show finally by another method how the natural density of the set defined in Lemma 4 is independent of $d$.

**Proposition 4** Assuming the assumptions of the main theorem. Then, the natural density of the set
$$\{ n \in \mathbb{N} \mid (n, N) = 1, n \equiv d \mod q, a(t_n^2) \neq 0 \}$$
is equal to
$$\frac{1}{\varphi(q)} \lim_{z \to 1^+} (z - 1) \sum_{(n, q) = 1}^{\infty} \frac{f(n)^2}{n^z}.$$

**Proof** Since $\{ n \in \mathbb{N} \mid (n, N) = 1, n \equiv d \mod q, a(t_n^2) \neq 0 \}$ has natural density by Lemma 4 then it suffices to prove that the Dedekind-Dirichlet density of this set is equal to $\frac{1}{\varphi(q)} \lim_{z \to 1^+} (z - 1) \sum_{(n, q) = 1}^{\infty} \frac{f(n)^2}{n^z}$.

We shall define $B(z) = \sum_{n \equiv d \mod q} f(n)^2$ and $C_\varepsilon(z) = \sum_{n=1}^{\infty} \frac{f(n)^2 \varepsilon(n)}{n^z}$ where $\varepsilon$ runs over Dirichlet characters modulo $q$. We must now compute $\lim_{z \to 1^+} (z - 1)B(z)$. By the same computations as in the previous theorem, it suffices to compute $\lim_{z \to 1^+} (z - 1)C_\varepsilon(z)$. We have
\[
\frac{C_{\xi}(z)}{L(z, \varepsilon)} = \prod_{p \in \mathbb{P}} \sum_{k=0}^{\infty} f(p^k) \varepsilon(p^k) p^{-kz} \times \prod_{p \in \mathbb{P}} (1 - \frac{\varepsilon(p)}{p^z}) \\
= \prod_{p \in \mathbb{P}} (1 - \frac{\varepsilon(p)}{p^z}) \times \prod_{p \in \mathbb{P}} \left( 1 + \sum_{k=1}^{\infty} \frac{\varepsilon(p^k)}{p^{kz}} \right) \\
= \prod_{p \in \mathbb{P}} \left( \begin{array}{c} (1 - \frac{\varepsilon(p)}{p^z}) \\ \frac{1}{p^z} + \sum_{k=2}^{\infty} \frac{\varepsilon(p^k)}{p^{kz}} \end{array} \right) \\
\times \prod_{p \in \mathbb{P}} \left( \begin{array}{c} (1 - \frac{\varepsilon(p^2)}{p^{2z}} + h_1(z, p)) \\ \frac{1}{p^{2z}} + h_2(z, p) \end{array} \right),
\]

where \(h_1(z, p)\) and \(h_2(z, p)\) are the remaining terms. Applying logarithm to \(\frac{C_{\xi}(z)}{L(z, \varepsilon)}\) and notice that \(\sum_{p \in \mathbb{P}} \log \left( 1 - \frac{\varepsilon(p^2)}{p^{2z}} + h_1(z, p) \right)\) is holomorphic on \(\text{Re}(z) \geq 1\). On the other hand, we have \(\sum_{p \in \mathbb{P}} \log \left( 1 - \frac{\varepsilon(p)}{p^z} + h_2(z, p) \right) = \sum_{p \in \mathbb{P}} \frac{\varepsilon(p)}{p^z} + h_3(z, p)\) where \(h_3(z, p)\) is holomorphic on \(\text{Re}(z) \geq 1\). Further, since for all roots of unity \(\xi\) such that \(\xi \in \text{Im} \varepsilon\), the set \(\mathbb{P}_{\varepsilon, \xi} = 0\) is a regular set of primes of density 0 by Theorem 3, then

\[
\sum_{p \in \mathbb{P}} \frac{\varepsilon(p)}{p^z} = \sum_{\xi \in \text{Im} \varepsilon} \sum_{p \in \mathbb{P}_{\varepsilon, \xi}} \frac{1}{p^z}
\]

is also holomorphic on \(\text{Re}(z) \geq 1\). Thus \(\frac{C_{\xi}(z)}{L(z, \varepsilon)}\) is holomorphic on \(\text{Re}(z) \geq 1\) and by taking exponential we see that \(\frac{C_{\xi}(z)}{L(z, \varepsilon)}\) is also holomorphic on \(\text{Re}(z) \geq 1\). Then the limit \(\lim_{z \to 1^+} (z - 1)C_{\varepsilon_0}(z)\) exists, where \(\varepsilon_0\) is the principal character modulo \(q\), and \(\lim_{z \to 1^+} (z - 1)C_{\xi}(z) = 0\) when \(\xi \neq \varepsilon_0\).

\[
\lim_{z \to 1^+} (z - 1)B(z) = \frac{1}{\varphi(q)} \lim_{z \to 1^+} (z - 1)C_{\varepsilon_0}(z) \\
= \frac{1}{\varphi(q)} \lim_{z \to 1^+} (z - 1) \sum_{n=1}^{\infty} \frac{f(n)^2}{n^z}. 
\]

We conclude with some related remarks.
Remark 3 When \( q = N \) or \( (q, N) = 1 \), the Dedekind-Dirichlet density of the set 
\( \{ n \in \mathbb{N} \mid (n, N) = 1, n \equiv d \mod q, \alpha(tn^2) = 0 \} \) exists. Indeed, we have

\[
\lim_{z \to 1^+} (z - 1) \sum_{n \geq 1 \atop n \equiv d \mod q} \frac{1}{n^z} = \frac{1}{q}.
\]

By Lemma 3, it follows that

\[
\lim_{z \to 1^+} (z - 1) \left( 2 \sum_{(n, N) = 1 \atop n \equiv d \mod q} \frac{1}{n^z} + \sum_{(n, N) = 1 \atop \alpha(tn^2) = 0} \frac{1}{n^z} + \sum_{(n, N) \neq 1 \atop n \equiv d \mod q} \frac{1}{n^z} \right) = \frac{1}{q}. \tag{13}
\]

Let \( \chi_0 \) be a principal character modulo \( N \). We have

\[
\sum_{(n, N) = 1 \atop n \equiv d \mod q} \frac{1}{n^z} = \frac{\sum_{n \equiv d \mod q} \chi_0(n)}{n^z} \frac{1}{\varphi(q)} \sum_{n \geq 0} \frac{\chi_0(n)}{n} \sum_{\varepsilon \mod q} \sum_{d \geq 0} \chi_0(n) \varepsilon(n) \frac{1}{n^z}.
\]

Following our hypothesis, if \( q = N \) we consider \( \chi_0 \) as a character modulo \( N \); if \( (q, N) = 1 \) we consider it as a character modulo \( qN \). Therefore \( \lim_{z \to 1^+} \sum_{(n, N) = 1 \atop n \equiv d \mod q} \frac{1}{n^z} \) exists and thus \( \lim_{z \to 1^+} \sum_{(n, N) \neq 1 \atop n \equiv d \mod q} \frac{1}{n^z} \) also exists. Replacing this in (13) and the result follows.

Remark 4 The weaker version of Theorem 1 could be obtained using Proposition 4. Indeed, in the proof of the previous proposition there is \( b > 0 \) such that \( \lim_{z \to 1^+} (z - 1) B(z) = b \). Hence \( \{ n \in \mathbb{N} \mid (n, N) = 1, n \equiv d \mod q, \alpha(tn^2) \neq 0 \} \) has a Dedekind-Dirichlet density equal to \( b \). It follows from (13) that

\[
\lim_{z \to 1^+} (z - 1) \left( \sum_{(n, N) = 1 \atop n \equiv d \mod q} \frac{1}{n^z} + \sum_{(n, N) \neq 1 \atop n \equiv d \mod q} \frac{1}{n^z} \right) = \frac{1}{q} - b.
\]

Replace this in (13) to get

\[
\lim_{z \to 1^+} (z - 1) \sum_{(n, N) = 1 \atop n \equiv d \mod q \atop \chi(n^2) > 0} \frac{1}{n^z} = \frac{b}{2}.
\]
The equidistribution obtained here is in terms of the Dedekind-Dirichlet density only.

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