AN EFFICIENT WAY TO COMPUTE HARDER-NARASIMHAN FILTRATIONS FOR REPRESENTATIONS OF BIPARTITE QUIVERS

CHI-YU CHENG

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Abstract. We establish a deterministic polynomial time algorithm for computing the Harder-Narasimhan filtrations of representations of bipartite quivers.

1. INTRODUCTION

Our goal in this paper is to provide a constructive approach to the Harder-Narasimhan filtrations of quiver representations. Specifically, we first establish in Theorem A a link between Harder-Narasimhan filtrations and discrepancies of representations of acyclic quivers. The later notion, introduced in [CK21], allows us to bring methods from algebraic complexity to the study of Harder-Narasimhan filtrations of representations of acyclic quivers. Combining algebraic complexity tools from [CK21] and Theorem A, we are able to show in Theorem B that the Harder-Narasimhan filtration of a representation of a bipartite quiver can be computed efficiently.

The paper was originally motivated by the relations between subrepresentations of quivers that contradict various stability conditions maximally in their numerical senses. Stability plays an important role in algebraic geometry for constructing moduli spaces of algebro-geometric objects. In the case of representations of quivers, two commonly used stability conditions are weight stability and slope stability. Weight stability was introduced by King in [Kin94] to give a Geometric Invariant Theory approach to the construction of moduli of representations of finite dimensional algebras. On the other hand, slope stability was introduced in [HilPn02] to construct Harder-Narasimhan filtrations for representations of quivers.
To better explain how these stability conditions are related, we let $Q$ be a finite acyclic quiver with the set of vertices $Q_0$. For a representation $M$ of $Q$, we let $\dim M \in \mathbb{N}^{Q_0}$ be its dimension vector. We recall that given an integral weight $\theta : \mathbb{Z}^{Q_0} \to \mathbb{Z}$, a representation $M$ is $\theta$-semistable if and only if $\theta(M) := \theta(\dim M) = 0$ and $\theta(M') \leq 0$ for every subrepresentation $M'$. Hence $M$ is $\theta$-unstable if and only if there is a subrepresentation $M'$ such that $\theta(M') > \theta(M) = 0$.

In [CK21], the discrepancy of any $M$ with respect to $\theta$ is defined to be the number $\text{disc}(M, \theta) = \max_{M' \subseteq M} \theta(M')$ where $M' \subseteq M$ is a subrepresentation of $M$. We see that if $M$ is $\theta$-unstable, then the subrepresentation that witnesses the discrepancy $\text{disc}(M, \theta)$ is the most contradicting to the $\theta$-semistability of $M$ in its obvious numerical sense.

A more intricate measure of $\theta$-stability comes from the Hilbert-Mumford criterion, which states that for every $\theta$-unstable representation, there is a one parameter subgroup that contradicts $\theta$-stability. Kempf showed in [Kem78] that among all destabilizing one parameter subgroups, there is a unique indivisible one (up to its conjugacy class of a parabolic subgroup) that maximally contradicts $\theta$-stability in a more delicate numerical sense. The maximally destabilizing one parameter subgroups of a representation induce a filtration of subrepresentations that we shall refer to as Kempf’s filtration. This is where the Harder-Narasimhan filtration from slope stability can be related as well.

In [HdlPn02], the Harder-Narsimhan filtration of a representation with respect to a slope is constructed by inductively finding the unique subrepresentation which is maximal among all subrepresentations having the highest slope. Such a subrepresentation is called the strongly contradicting semistability (abbreviated as sccs) subrepresentation.

With these in mind the following objects are naturally associated to an unstable representation $M$:

1. The discrepancy of $M$ and its witnessing subrepresentations (not necessarily unique).
2. The subrepresentation that is sccs in $M$ and the Harder-Narasimhan filtration of $M$.
3. The one parameter subgroups that maximally contradict the semistability of $M$ and Kempf’s filtration induced by those one parameter subgroups.

The relation between (2) and (3) is already unraveled by the paper [Zam14]. The main result is that for an unstable representation, its Harder-Narasimhan filtration coincides with its Kempf filtration. The first of our two main results Theorem A establishes a link between (1) and (2). The second main result Theorem B ensures that the Harder-Narasimhan filtrations can be found efficiently.

1.1. The main results. Let $Q$ be a finite acyclic quiver and let $\theta, \kappa : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ be $\mathbb{Z}$-linear functions where $\kappa(M) > 0$ for each non-zero representation $M$ of $Q$. We let $\mu = \theta / \kappa$ be the corresponding slope.

**Theorem A.** Let $M$ be a representation of $Q$ with $\theta(M) = 0$. Let $0 \subseteq M_1 \subseteq \cdots \subseteq M_{r-1} \subseteq M_r = M$
be the Harder-Narasimhan filtration of $M$. Then there is an $M_l$ in the filtration that attains the discrepancy $\text{disc}(M, \theta)$. Namely, $\theta(M_l) \geq \theta(M')$ for any subrepresentation $M' \subset M$. In particular, if $M$ is unstable, the filtration induced by the maximally destabilizing one parameter subgroups of $M$ contains a witness to the discrepancy of $M$.

It is noteworthy that the discrepancy of an unstable representation depends only on $\theta$ but its Harder-Narasimhan filtration depends on $\mu$, which has an extra piece $\kappa$. While the Harder-Narasimhan filtration may change due to different choices of $\kappa$, Theorem A implies that the Harder-Narasimhan filtration of a representation $M$ with $\theta(M) = 0$ always contains a term that witnesses the discrepancy.

The main result of [CK21] is the existence of efficient ways to obtain a witness to the discrepancy for bipartite quivers. Since Theorem A affirms that the Harder-Narasimhan filtration contains a witness to the discrepancy, the next natural question we ask is that is there an efficient way to compute Harder-Narasimhan filtrations? We have an affirmative answer for bipartite quivers. However, during a discussions [Chi21], the author was informed that there exists a deterministic polynomial time algorithm to obtain a witness to the discrepancy for arbitrary acyclic quivers due to Derksen and Huszar. Once the algorithm is published, Theorem B below can be generalizes to arbitrary acyclic quivers.

**Theorem B.** There exist deterministic polynomial time algorithms to compute the Harder-Narasimhan filtration for representations of bipartite quivers over the field of real numbers.

Theorem B is inviting for several purposes. First, computing Kempf’s one parameter subgroups in general is very hard. If a version of Theorem B exists over the field of complex numbers, then the result of [Zam14] ensures that Kempf’s one parameter subgroups can be computed efficiently for representations of bipartite quivers. What is more, [Hos14] also shows that the stratification of the space of representations of a quiver of a fixed dimension by Harder-Narasimhan types coincides with the stratification by Kempf’s one parameter subgroups. Hence an investigation of Theorem B over the field of complex numbers also sheds new light on computing either stratification.

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2. **Stability**

2.1. **Set up.** Let $k$ be a field of arbitrary characteristic, not necessarily algebraically closed. A quiver $Q = (Q_0, Q_1, t, h)$ consists of $Q_0$ (vertices) and $Q_1$ (arrows) together with two maps $t, h : Q_1 \to Q_0$. The image of an edge under $t$ (resp. $h$) is the tail (resp. head) of the edge. We represent $Q$ as a directed graph with vertices $Q_0$ and directed edges $a : ta \to ha$ for $a \in Q_1$.

A representation of $Q$ consists of a collection of finite dimensional $k$-vector spaces $W_v$, for each $v \in Q_0$ together with a collection of $k$-linear maps $\psi_a : W_{ta} \to W_{ha}$, for each $a \in Q_1$. A morphism between two representations $(W, \psi_a), (U, \phi_a)$ is a collection of $k$-linear maps $\varphi_v : W_v \to U_v$ for each $v \in Q_0$, such that
Lemma 2.2. Theorem 2.4, and HdlPn02, from [Kin94]. There is a unique filtration $0 = Q_0 \subseteq M$ of $M = (W_v, \psi_a)$ whenever each $W_v$ is a subspace of $W_a$ and each $\psi_a$ restricts to $\psi_v$. We will write $M' \subseteq M$ if $M'$ is a subrepresentation of $M$. We also note that the collection of representations of $Q$ forms an abelian category.

The dimension vector of a representation $M = (W_v, \psi_a)$ of $Q$ is the tuple $\text{dim} M \in \mathbb{Z}^{Q_0}$ where $(\text{dim} M)_v = \dim W_v$ as a $k$-vector space for each $v \in Q_0$. A weight is a $\mathbb{Z}$-linear function $\mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}$. If $\theta$ is a weight, we write $\theta(\text{dim} M)$ as $\theta(M)$.

Finally, we let $Q$ be an acyclic quiver throughout the rest of the paper. That is, $Q$ does not have an oriented cycle.

2.2. Slope stability. In this section we recall semistability of representations of quivers with respect to a slope, along with some important properties Lemma 2.2, Lemma 2.3, and Theorem 2.4 from [HdlPn02]. Fix two weights $\theta, \kappa : \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}$ where we require $\kappa(M) > 0$ for all non-trivial representations $M$. We set $\mu(M) = \theta(M)/\kappa(M)$ for any non-trivial representation $M$.

Definition 2.1. We say a representation $M$ of $Q$ is $\mu$-semistable if $\mu(N) \leq \mu(M)$ for every non-zero subrepresentation $N$.

Lemma 2.2. Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence of representations of $Q$. Then the following conditions are equivalent:

1. $\mu(L) \leq \mu(M)$,
2. $\mu(L) \leq \mu(N)$,
3. $\mu(M) \leq \mu(N)$.

A fundamental result is the following

Lemma 2.3. Let $M$ be a representation of $Q$. There is a unique subrepresentation $N$ such that

1. $\mu(N)$ is maximal among subrepresentations of $M$, and
2. if $\mu(N') = \mu(N)$, then $N' \subseteq N$.

We call such a representation $N$ the strongly contradicting semistability (abbreviated as scss) subrepresentation of $M$.

Theorem 2.4 (Harder-Narasimhan filtration). Let $M$ be a representation of $Q$. There is a unique filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r = M$ such that

1. $\mu(M_i/M_{i-1}) > \mu(M_{i+1}/M_i)$ for $i = 1, \ldots, r - 1$,
2. the representation $M_i/M_{i-1}$ are $\mu$-semistable for $i = 1, \ldots, r$.

The above filtration is called the Harder-Narasimhan filtration of $M$. It is constructed by setting $M_1$ to be the scss subrepresentation of $M$, then by inductively setting $M_i/M_{i-1}$ as the scss subrepresentation of $M/M_{i-1}$.

2.3. Weight stability. Another stability condition on representations of $Q$ comes from [Kin94] in the sense of Geometric Invariant Theory when $k$ is algebraically closed. Fix a dimension vector $d \in \mathbb{Z}^{Q_0}$. A weight $\theta$ corresponds to a character of the linear algebraic group $\prod_{v \in Q_0} \text{GL}(d_v)$ acting on the set of representations of $Q$ with dimension vector $d$. King showed that in this setting, a representation $M$ is $\theta$-semistable if and only if $\theta(M) = 0$ and $\theta(N) \leq 0$ for all subrepresentations $N \subseteq M$. We therefore adopt the following stability definition for arbitrary fields.
Definition 2.5. Let \( d \in \mathbb{Z}_{\geq 0}\) be a dimension vector and let \( \theta \) be a weight with \( \theta(d) = 0 \). A representation \( M \) with dimension vector \( d \) is \( \theta \)-semistable if \( \theta(N) \leq 0 \) for all subrepresentations \( N \subseteq M \).

The notion of discrepancy (Definition 2.6) was introduced in [CK21] to describe the optimal solutions to simultaneous robust subspace recovery. This very notion allows us to compute Harder-Narasimhan filtrations of bipartite quivers using tools from algebraic complexity, as we shall see in the next section.

Definition 2.6. Let \( M \) be a representation and let \( \theta \) be a weight. The discrepancy of \( M \) with respect to \( \theta \) is written as

\[
\text{disc}(M, \theta) = \max_{N \subseteq M} \theta(N).
\]

We say a subrepresentation \( N \) is \( \theta \)-optimal in \( M \) if \( \theta(N) = \text{disc}(M, \theta) \).

We recall the following way to go between weight stability and slope stability:

Lemma 2.7. Let \( \theta \) and \( \kappa \) be two weights with \( \kappa(N) > 0 \) for every nonzero representation \( N \). Let \( \mu \) be the corresponding slope \( \theta(-)/\kappa(-) \). For every dimension vector \( d \in \mathbb{Z}_{\geq 0} \), define the new weight

\[
\theta_d(N) = \theta(N)\kappa(d) - \theta(d)\kappa(N)
\]

Then a representation \( M \) with dimension vector \( d \) is \( \mu \)-semistable if and only if \( M \) is \( \theta_d \)-semistable. Moreover, the new slope \( \mu_d(-) = \theta_d(-)/\kappa(-) \) defines the same slope stability condition so that the Harder-Narasimhan filtrations of a representation with respect to \( \mu \) and \( \mu_d \) coincide.

Proof. Let \( N \) be any representation. We then have

\[
\mu(N) = \frac{\theta(N)}{\kappa(N)} \leq \mu(M) = \frac{\theta(d)}{\kappa(d)} \Leftrightarrow \theta(N)\kappa(d) - \theta(d)\kappa(N) \leq 0
\]

Since \( \theta_d(N) = \theta(N)\kappa(d) - \theta(d)\kappa(N) \) and \( \theta_d(M) = 0 \) by construction, the first statement follows.

For the second statement, simply note that

\[
\mu_d(N) = \kappa(d) \cdot \mu(N) - \theta(d).
\]

Hence \( \mu_d \) is a positive scalar multiple of \( \mu \), followed by a translation. \( \square \)

3. The main results

We fix two weights \( \theta, \kappa \) on \( \mathbb{Z}_{\geq 0} \) where \( \kappa(N) > 0 \) for each non-zero representation \( N \). Let \( \mu(-) = \theta(-)/\kappa(-) \) be the slope.

Lemma 3.1. Let \( M \) be a \( \mu \)-unstable representation and let \( 0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M \) be the Harder-Narasimhan filtration for \( M \). Suppose \( \theta(M_1) > 0 \) and \( M' \) is \( \theta \)-optimal in \( M \). Then \( M' \) contains \( M_1 \). In this case, \( M' \) is \( \mu \)-semistable if and only if \( M' \) is the soss subrepresentation of \( M \).

Proof. Note that \( \theta(M_1) > 0 \) implies \( \text{disc}(M, \theta) = \theta(M') > 0 \). Suppose on the contrary that \( M_1 \not\subseteq M' \). We then have a proper inclusion \( M' \nsubseteq M_1 + M' \). In particular, \( \kappa(M') < \kappa(M_1 + M') \). On the other hand, \( \theta(M') \geq \theta(M_1 + M') \) as \( M' \) attains the discrepancy. If \( \theta(M_1 + M') \leq 0 \), we automatically have \( \mu(M') \geq \mu(M_1 + M') \). If \( \theta(M_1 + M') > 0 \), we have

\[
\frac{\theta(M')}{\kappa(M')} \geq \frac{\theta(M_1 + M')}{\kappa(M')} > \frac{\theta(M_1 + M')}{\kappa(M_1 + M')}.
\]
In any case we deduce that $\mu(M') > \mu(M_1 + M')$. By lemma 2.2, the following short exact sequence

$$0 \to M' \to M_1 + M' \to (M_1 + M')/M' \simeq M_1/(M_1 \cap M') \to 0$$

implies $\mu(M') > \mu(M_1 + M') > \mu(M_1/(M_1 \cap M'))$.

On the other hand, since $M_1$ is an scss subrepresentation of $M$, we have $\mu(M_1 \cap M') \leq \mu(M_1)$. Now Lemma 2.2 applied to the following short exact sequence

$$0 \to M_1 \cap M' \to M_1 \to M_1/(M_1 \cap M') \to 0$$

imply $\mu(M_1/(M_1 \cap M')) \geq \mu(M_1)$. In sum, we obtained

$$\mu(M') > \mu(M_1 + M') > \mu(M_1/(M_1 \cap M')) \geq \mu(M_1).$$

However, $M_1$ is an scss subrepresentation of $M$. We arrive at a contradiction.

For the second implication of the lemma, if $M'$ is the scss subrepresentation of $M$, it is obviously $\mu$-semistable. Conversely, if $M'$ is $\mu$-semistable, since it contains $M_1$, we must have $\mu(M_1) \leq \mu(M')$. This implies $\mu(M_1) = \mu(M')$ so that $M' = M_1$.

Now we can make sense of the following proposition.

**Proposition 3.2. Let $M$ be $\mu$-unstable and let $M_1$ be the scss subrepresentation of $M$ with $\theta(M_1) > 0$. If $M' \subset M$ is $\theta$-optimal in $M$, we then have

$$\theta(M'/M_1) = \text{disc}(M'/M_1, \theta).$$

Namely, $M'/M_1$ is $\theta$-optimal in $M/M_1$.

**Proof.** We obviously have $\theta(M'/M_1) \leq \text{disc}(M/M_1, \theta)$. Suppose $N/M_1$ is $\theta$-optimal in $M/M_1$. By assumption $M'$ is $\theta$-optimal in $M$ so that $\theta(M') \geq \theta(N)$. However, this inequality is equivalent to

$$\theta(M'/M_1) = \theta(M') - \theta(M_1) \geq \theta(N) - \theta(M_1) = \theta(N/M_1) = \text{disc}(M/M_1, \theta).$$

Hence we must have $\theta(M'/M_1) = \text{disc}(M/M_1, \theta)$. The proposition is proved. □

**Theorem 3.3.** Let $M$ be a $\mu$-unstable representation and let

$$0 \subseteq M_1 \subseteq \cdots \subseteq M_r \subseteq M = M'$$

be its Harder-Narasimhan filtration. Suppose there is an integer $l$ such that $\mu(M_l/M_{l-1}) > 0$ but $\mu(M_{l+1}/M_l) \leq 0$. Then the term $M_l$ in the filtration is $\theta$-optimal in $M$. Namely, $\theta(M_l) \geq \theta(M')$ for any subrepresentation $M' \subset M$. In addition, $M_l$ has the highest slope among all $\theta$-optimal subrepresentations.

**Remark 3.4.** We note that if $M$ is $\mu$-unstable and $\theta(M) = 0$, such an integer $l$ stated in the theorem exists. To see this, first note that $\theta(M_l)$ would have to be positive to contradict $\mu$-semistability of $M$. Second, if $\mu(M_i/M_{i-1}) > 0$ for all $i$, we would obtain the chain of inequalities $\theta(M) > \theta(M_{l-1}) > \cdots > \theta(M_1) > 0$. Hence there must be a smallest integer $l$ for which $\mu(M_{l+1}/M_l) \leq 0$.

**Proof of Theorem 3.3.** We will prove the theorem by induction on $l$. Let $l = 1$. We need to show that $\theta(M_1) \geq \theta(M')$ for every $M' \subset M$. We break this down into several steps.

1. For any $M'_1 \subset M_1$, we show that $\theta(M'_1) \leq \theta(M_1)$.
2. Next, assuming on the contrary that there is an $M' \subset M$ with $\theta(M') > \theta(M_1)$, we prove that such an $M'$ induces the proper inclusion $M_1 \subset M_1 + M'$.
Proposition 3.2. \( CK \) dictates that Lemma 3.1 (equivalently \( \mu_{\text{dim}} \)).

For step (1), suppose on the contrary that there is an \( M_1 \subseteq M_1 \) such that \( \theta(M_i') > \theta(M_1) \). We would then have

\[
\mu(M_i') = \frac{\theta(M_i')}{\kappa(M_i')} \geq \frac{\theta(M_1)}{\kappa(M_1)} = \frac{\theta(M_i)}{\kappa(M_i)} > \theta(M_1)
\]

since \( \kappa(M_i') < \kappa(M_1) \) and \( \theta(M_i'), \theta(M_1) \) are both positive. This contradicts the fact that \( M_1 \) is an \( \text{scss} \) subrepresentation of \( M \). Step (1) is completed.

For step (2), since \( M_1 \) is \( \text{scss} \) in \( M \), we get \( \mu(M') \leq \mu(M_1) \). This together with the assumption that \( \theta(M') > \theta(M_1) \) imply \( \kappa(M') > \kappa(M_1) \). This clearly implies that \( M' \) is not contained in \( M_1 \subseteq M_1 + M' \), finishing step (2).

For step (3), simply note that \( \theta(M') > \theta(M_1) \geq \theta(M_1 \cap M') \) by step (1). We then have

\[
\mu((M_1 + M')/M_1) = \mu(M'/M_1 \cap M') = \frac{\theta(M') - \theta(M_1 \cap M')}{\kappa(M') - \kappa(M_1 \cap M')}
\]

\[
> \frac{\theta(M_1)}{\kappa(M_1)} = \mu(M_1)
\]

These complete all three steps and therefore the base case of the induction.

If \( \mu(M_{i+1}/M_1) > 0 \) but \( \mu(M_{i+2}/M_{i+1}) \leq 0 \), consider the Harder-Narasimhan filtration for \( M/M_1 \):

\[
0 \subseteq M_2/M_1 \subseteq \cdots \subseteq M/M_1.
\]

By the induction hypothesis, \( \theta(M_{i+1}/M_1) = \text{disc}(M/M_1, \theta) \). Moreover, Proposition 3.2 implies \( \text{disc}(M/M_1, \theta) = \text{disc}(M, \theta) - \theta(M_1) \). Combining these two, we get

\[
\theta(M_{i+1}/M_1) = \theta(M_{i+1}) - \theta(M_1) = \text{disc}(M, \theta) - \theta(M_1).
\]

Therefore, we get \( \theta(M_{i+1}) = \text{disc}(M, \theta) \), as desired.

Finally, the maximality of slope of \( M_1 \) can also be proved by induction on \( l \), together with Lemma 3.1.

We now specialize to the field \( \mathbf{R} \) of real numbers. This is where efficient algorithms are readily available. In [CK21], the authors showed two algorithms within deterministic polynomial time, called Algorithm G and Algorithm P for any bipartite quiver \( Q \). Given a representation \( M \) with \( \theta(M) = 0 \), Algorithm G detects if \( M \) is \( \theta \)-semistable. If \( M \) is \( \theta \)-unstable, Algorithm P produces a subrepresentation that is \( \theta \)-optimal. We present a way to compute the Harder-Narasimhan filtration of representations, using Algorithm G and Algorithm P.

Given a \( \mu \)-unstable representation \( M^0 \) of dimension \( d_0 \), apply Algorithm G in [CK21] that produces a subrepresentation \( M^1 \) that is \( \theta_{d_1} \)-optimal. Suppose \( \dim \mathbf{M}^1 = d_1 \). Now apply Algorithm G [CK21] to detect if \( M^1 \) is \( \theta_{d_1} \)-semistable (equivalently \( \mu \)-semistable). If it is, then Lemma 3.1 dictates that \( M^1 \) is the first term in the Harder-Narasimhan-filtration of \( M \). If not, apply Algorithm P to produce a subrepresentation \( M^2 \) of \( M^1 \) that is \( \theta_{d_2} \)-optimal. We now have a filtration

\[
M^2 \subsetneq M^1 \subsetneq M^0
\]

with the following two properties for each \( i = 1, 2 \):

1. \( \mu(M_i) > \mu(M_{i-1}) \), and
2. \( M_i \) is \( \theta_{d_{i-1}} \)-optimal in \( M_{i-1} \).
The above process produces a filtration $M^r \subseteq \ldots \subseteq M^0$ where the successive terms satisfy the above two properties. This process cannot continue indefinitely as we are dealing with finite dimensional representations. In the end the last term $M^r$ of the filtration will have to be $\mu$-semistable. We now show that $M^r$ is the scss subrepresentation of $M$.

**Proposition 3.5.** Let $0 \subseteq M^r \subseteq \ldots \subseteq M^0$ be a filtration of representations of $Q$ and let $\dim M^i = d_i \in \mathbb{Z}^{2n}$. Suppose the following two conditions hold for each $i$:

1. $\mu(M^{i+1}) > \mu(M^i)$ for $i = 0, \ldots, r-1$,
2. $M^i$ is $\theta_{d_i-1}$-optimal in $M^{i-1}$.

Then $M^r$ is $\mu$-semistable if and only if $M^r$ is the scss subrepresentation of $M^i$ for $i = 0, \ldots, r-1$.

**Proof.** We let $M^i_1$ be the scss subrepresentation of $M^i$. The if part is trivial. The proof for the only if part can be carried out by induction. Suppose $M^r$ is $\mu$-semistable. Then Lemma 3.1 applied to $\theta_{d_{r-1}}$ implies $M^r = M^i_1$. Now suppose $M^r = M^i_1$, we will show that $M^r = M^i_{1-1}$. For this, since $M^r$ is $\theta_{d_{r-1}}$-optimal in $M^{r-1}$, Lemma 3.1 implies $M^l$ contains $M^i_{1-1}$. This immediately implies $M^i_{1-1} = M^i_1$. By the induction hypothesis, $M^r = M^i_1$ so that $M^r = M^i_{1-1}$, finishing the induction. \(\square\)

We just showed that one obtains the first term of the Harder-Narasimhan filtration by applying Algorithm G and Algorithm P finitely many times. Since the Harder-Narasimhan filtration is obtained by finding the scss subrepresentation inductively, we have established the

**Theorem 3.6.** There exist deterministic polynomial time algorithms to compute Harder-Narasimhan filtrations of representations of bipartite quivers over the field of real numbers.

According to [Chi21], Algorithm G and Algorithm P exist for arbitrary acyclic quivers as well due to Derksen and Huszar. Once the algorithms for acyclic quiver are published, the above theorem generalizes to acyclic quivers.

### 4. A Short Example

In this example we demonstrate that not all $\theta$-optimal subrepresentations occur in the Harder-Narasimhan filtration. The set up is motivated by [CK21]. Let us consider the following representation $M$ of the bipartite quiver over the field $\mathbb{C}$ of complex numbers:

![Diagram](attachment:image.png)

Here $v_1 = e_2, v_2 = e_1, v_3 = 2e_1, v_4 = e_3$ in $\mathbb{C}^4$. Label the four vertices on the left by $x_1, x_2, x_3$ and $x_4$ from top to bottom. Label the single vertex on the right as $y$. We let $\theta$ and $\kappa$ be two weights where $\theta(x_1) = 4, \theta(y) = -4$, and $\kappa(x_1) = \kappa(y) = 1$ for each $i$. We also let $\mu(-) = \theta(-)/\kappa(-)$ be the corresponding slope. It is then easy to check that $\theta(M) = 0$. The linear span $\text{Sp}(e_1)$ in $\mathbb{C}^4$ comes from
Similarly $\operatorname{Sp}(e_1, e_2)$ comes from $v_1, v_2, v_3$ and $\operatorname{Sp}(e_1, e_3)$ comes from $v_2, v_3, v_4$. Finally $\operatorname{Sp}(e_1, e_2, e_3)$ comes from $v_1, v_2, e_3, v_4$. Abusing the notations, we let $\operatorname{Sp}(e_1), \operatorname{Sp}(e_1, e_2), \operatorname{Sp}(e_1, e_3)$, and $\operatorname{Sp}(e_1, e_2, e_3)$ be their corresponding subrepresentations of $M$.

We have computed that

1. the four subrepresentations $\operatorname{Sp}(e_1)$, $\operatorname{Sp}(e_1, e_2)$, $\operatorname{Sp}(e_1, e_3)$, $\operatorname{Sp}(e_1, e_2, e_3)$ are $\theta$-optimal in $M$ with $\theta$ value 4, and
2. $0 \subsetneq \operatorname{Sp}(e_1) \subsetneq \operatorname{Sp}(e_1, e_2, e_3) \subsetneq M$ is the Harder-Narasimhan filtration of $M$.

For the first statement, notice that for any subrepresentation $M'$ of $M$,

$$\theta(M') = 4 \cdot (\#\{v_i \mid v_i \in M'\} - \dim(\operatorname{Sp}\{v_i \mid v_i \in M'\})).$$

Therefore, for $\theta(M')$ to be positive, $v_2$ and $v_3$ must be included in $M'$. The only proper subrepresentations that contain $v_2, v_3$ are exactly the four subrepresentations listed in statement (1). Therefore, they are also $\theta$-optimal.

Statement (2) may also be checked easily with the information used in step (1). However, we sketch an approach using geometric invariant theory. To begin with, the group $G = \mathbb{C}^\times \times \operatorname{GL}(4)$ acts on the space of representations $N$ of the bipartite quiver with $\dim N = \dim M$. Since $\theta(M) = 0$ and $\theta(\operatorname{Sp}(e_1)) = 4 > 0$, $M$ is both $\mu$ and $\theta$-unstable.

There is the diagonal torus $D \simeq (\mathbb{C}^\times)^4$ in $\operatorname{GL}(4)$. Let $T = (\mathbb{C}^\times)^4 \times D \subset G$ be the maximal torus in $G$. Then the set of one parameter subgroups of $T$ is naturally identified with the lattice $\mathbb{Z}^4$. Using the standard norm on the set of one parameter subgroups of $T$, we computed that Kempf’s one parameter subgroup is represented by $(1, 4, 4, 1, 4, 1, 1, -8)$. Therefore, the filtration induced by this one parameter subgroup is exactly $0 \subsetneq \operatorname{Sp}(e_1) \subsetneq \operatorname{Sp}(e_1, e_2, e_3) \subsetneq M$. Applying the main result of [Zam14], we know the above filtration is precisely the Harder-Narasimhan filtration of $M$.

We see that the two $\theta$-optimal solutions $\operatorname{Sp}(e_1, e_2)$ and $\operatorname{Sp}(e_1, e_3)$ are not included in the filtration. This is not surprising as

$$\mu(\operatorname{Sp}(e_1, e_2)/\operatorname{Sp}(e_1)) = \mu(\operatorname{Sp}(e_1, e_3)/\operatorname{Sp}(e_1)) = \mu(\operatorname{Sp}(e_1, e_2, e_3)/\operatorname{Sp}(e_1)) = 0,$$

implying that $\operatorname{Sp}(e_1, e_2)$ and $\operatorname{Sp}(e_1, e_3)$ are not the ssq subrepresentations of $M/\operatorname{Sp}(e_1)$.

Also note that since $\mu(\operatorname{Sp}(e_1, e_2, e_3)/\operatorname{Sp}(e_1)) = 0$ while $\mu(\operatorname{Sp}(e_1)) = 4/3 > 0$, the first term $\operatorname{Sp}(e_1)$ in the Harder-Narasimhan filtration is a $\theta$-optimal subrepresentation, agreeing with the description of Theorem 3.3.

References

[Chi21] Calin Chindris. Personal communication. 2021.
[CK21] Calin Chindris and Daniel Kline. Simultaneous robust subspace recovery and semi-stability of quiver representations. J. Algebra, 577:210–236, 2021.
[HdlPn02] Lutz Hille and José Antonio de la Peña. Stable representations of quivers. J. Pure Appl. Algebra, 172(2-3):205–224, 2002.
[Hos14] Victoria Hoskins. Stratifications associated to reductive group actions on affine spaces. Q. J. Math., 65(3):1011–1047, 2014.
[Kem78] George R. Kempf. Instability in invariant theory. Ann. of Math. (2), 108(2):299–316, 1978.
[Kin94] A. D. King. Moduli of representations of finite-dimensional algebras. Quart. J. Math. Oxford Ser. (2), 45(180):515–530, 1994.
[Zam14] Alfonso Zamora. On the Harder-Narasimhan filtration for finite dimensional representations of quivers. Geom. Dedicata, 170:185–194, 2014.