Design of Provably Physical-Constraint-Preserving Methods for General Relativistic Hydrodynamics

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The paper develops high-order physical-constraint-preserving (PCP) methods for general relativistic hydrodynamic (GRHD) equations, equipped with a general equation of state. Here the physical constraints, describing the admissible states of GRHD, are referred to the subluminal constraint on the fluid velocity and the positivity of the density, pressure and specific internal energy. Preserving these constraints is very important for robust computations, otherwise violating one of them will lead to the ill-posed problem and numerical instability. To overcome the difficulties arising from the inherent strong nonlinearity contained in the constraints, we derive an equivalent definition of the admissible states. Using this definition, we prove the convexity, scaling invariance and Lax-Friedrichs (LxF) splitting property of the admissible state set $\mathcal{G}$, and discover the dependence of $\mathcal{G}$ on the spacetime metric. Unfortunately, such dependence yields the non-equivalence of $\mathcal{G}$ at different points in curved spacetime, and invalidates the convexity of $\mathcal{G}$ in analyzing PCP schemes. This obstacle is effectively overcame by introducing a new formulation of the GRHD equations. Based on this formulation and the above theories, a first-order LxF scheme is designed on general unstructured mesh and rigorously proved to be PCP under a CFL condition. With two types of PCP limiting procedures, we design high-order, provably (not probably) PCP methods under discretization on the proposed new formulation. These high-order methods include the PCP finite difference, finite volume and discontinuous Galerkin methods.

I. INTRODUCTION

In many cases, high energy physics and astrophysics may involve hydrodynamical problems with special or general relativistic effect, corresponding to that the fluid flow is at nearly speed of light, or the influence of strong gravitational field on the hydrodynamics cannot be neglected. Relativistic hydrodynamics (RHD) is very important in investigating a number of astrophysical scenarios from stellar to galactic scales, e.g. astrophysical jets, gamma-ray bursts, core collapse super-novae, formation of black holes, merging of compact binaries, etc.

The governing equations of RHDs are highly nonlinear, making their analytical treatment extremely difficult. Numerical simulation has become a primary and powerful approach to understand the physical mechanisms in the RHDs. The pioneering numerical work on the RHD equations may date back to the Lagrangian finite difference code via artificial viscosity for the spherically symmetric GRHD equations [23, 24]. Wilson [33] first attempted to solve multi-dimensional RHD equations by using the Eulerian finite difference method with the artificial viscosity technique. Since the 1990s, the numerical study of RHD has attracted considerable attention, and various modern shock-capturing methods based on Riemann solvers have been developed for the RHD equations. The readers are referred to the early review articles [11, 12, 20, 21] and some more recent works e.g. [4, 34, 50] as well as references therein.

Most existing methods do not preserve the positivity of the density, pressure and the specific internal energy as well as the bound of the fluid velocity, although they have been used to solve some RHD problems successfully. There exists a big risk of failure when a numerical scheme is applied to the RHD problems involving large Lorentz factor, low density or pressure, or strong discontinuity. This is because once the negative density/pressure or the superluminal fluid velocity is obtained during numerical simulations, the eigenvalues of the Jacobian matrix become imaginary so that the discrete problem becomes ill-posed. Moreover, the superluminal fluid velocity also yields imaginary Lorentz factor and leads to the violation of the relativistic causality. It is therefore significative to design high-order numerical schemes, whose solutions satisfy the intrinsic physical constraints.

Recent years have witnessed some advances in developing high-order bound-preserving type schemes for hyperbolic conservation laws. Those schemes are mainly built on two types of limiting procedures. One is the simple scaling limiting procedure for the reconstructed or evolved solution polynomials in a finite volume or discontinuous Galerkin (DG) method, see e.g. [10, 33, 43–46, 48]. Another is the flux-corrected limiting procedure, which can be used on high-order finite difference, finite volume and DG methods, see e.g. [6, 7, 13, 17, 18, 40, 41]. A survey of the maximum-principle-satisfying or positivity-preserving high-order schemes based on the first type limiter was presented in [47]. The readers are also referred to [42] for a review of these two approaches. Recently, by extending the above bound-preserving techniques, two types of physical-constraint-preserving (PCP) schemes were developed for the special RHD equations with an ideal equation of state (EOS), i.e., the high-order PCP finite difference WENO (weighted essentially non-oscillatory) schemes [33] and

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the bound-preserving DG methods $^{[20]}$. More recently, the high-order PCP central DG methods were proposed in $^{[38]}$ for special RHD with a general EOS. The extension of PCP schemes to the ideal relativistic magnetohydrodynamics was studied in $^{[37]}$, where the importance of divergence-free magnetic fields in achieving PCP methods was revealed in theory for the first time.

The aim of this paper is to design high-order, provably PCP methods for the GRHD equations with a general EOS, including PCP finite difference, finite volume and DG methods. Developing provably PCP methods for GRHD with a general EOS is very nontrivial and still untouched in literature. The technical challenges mainly come from three aspects: (1). The inherent nonlinear coupling between the GRHD equations due to the Lorentz factor, curved spacetime and general EOS, e.g., the dearth of explicit expression of the primitive variables and flux vectors with respect to the conservative/state vector. (2). One more physical constraint for the fluid velocity in addition to the positivity of density, pressure and specific internal energy. (3). The non-equivalence of the admissible state sets defined at different points in curved spacetime. It is noticed in $^{[27]}$ that Redice, Rezzolla and Galeazzi once attempted to extend the flux-corrected limiter in non-relativistic case $^{[17]}$ to the GRHD equations, but only achieved enforcing the positivity of density. The importance as well as the difficulty of designing completely PCP schemes were also mentioned in $^{[27,28]}$. The work in this paper overcomes the above difficulties, via a new formulation of the GRHD equations and rigourously theoretical analysis on the admissible states of GRHD.

The paper is organized as follows. Sec. II introduces the governing equations of GRHD and the EOS. Sec. III derives several properties of the admissible state set and proposes a new formulation of the GRHD equations, which play pivotal roles in designing provably PCP methods. Sec. IV proves the PCP property of the first-order LxF scheme on general unstructured mesh. High-order, provably PCP methods are presented in Sec. V with detailed implementation procedures, including PCP finite volume and DG methods in Sec. V A and PCP finite difference methods in Sec. V B. Concluding remarks are presented in Sec. VI. For better legibility, all the proofs of the lemmas and theorems are put in Appendix A. Throughout the paper, we use a spacetime signature $(−, +, +, +)$ with Greek indices running from 0 to 3 and Latin indices from 1 to 3. We also employ the Einstein summation convention over repeated indices, and the geometrized unit system so that the speed of light in vacuum and the gravitational constant are equal to one.

## II. GOVERNING EQUATIONS

The general relativistic hydrodynamic (GRHD) equations $^{[12]}$ consist of the local conservation laws of the baryon number density and the stress-energy tensor $T^{\mu\nu}$, $\nabla_\mu(pu^\mu) = 0$, 

$$\nabla_\mu(pu^\mu) = 0, \tag{1}$$

where $\rho$ denotes the rest-mass density, $u^\mu$ represents the fluid four-velocity, and $\nabla_\mu$ stands for the covariant derivative associated with the four-dimensional spacetime metric $g_{\mu\nu}$, i.e., the line element in four-dimensional spacetime is $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. The stress-energy tensor for an ideal fluid is defined by 

$$T^{\mu\nu} = \rho hu^{\mu}u^{\nu} + pg^{\mu\nu},$$

where $p$ denotes the pressure, and $h$ represents the specific enthalpy defined by 

$$h = 1 + e + p/\rho,$$

with $e$ denoting the specific internal energy.

An additional equation for the thermodynamical variables, i.e. the so-called equation of state (EOS), is needed to close the system $^{[11]-[12]}$. In general, the EOS can be expressed as 

$$e = e(p, \rho), \tag{3}$$

or 

$$h = h(p, \rho) = 1 + e(p, \rho) + p/\rho. \tag{4}$$

The relativistic kinetic theory reveals $^{[38]}$ that a general EOS $^{[4]}$ should satisfy 

$$h(p, \rho) \geq \sqrt{1 + p^2/\rho^2} + p/\rho, \tag{5}$$

which is weaker than the condition proposed in $^{[31]}$. This paper focuses on the causal EOS. We also assume that the fluid’s coefficient of thermal expansion is positive, which is valid for most of compressible fluids, e.g. the gases. Then the following inequality holds $^{[38]}$

$$h(p, \rho) \geq 1 + \frac{\Gamma p}{(\Gamma - 1)\rho}. \tag{6}$$

The most commonly used EOS, called the ideal EOS, is 

$$h(p, \rho) = 1 + \frac{\Gamma p}{(\Gamma - 1)\rho}. \tag{7}$$

with $\Gamma \in (1, 2]$ denoting the adiabatic index. The ideal EOS $^{[17]}$ and most of the other EOS reported in numerical RHDs, see e.g. $^{[22,25,24,38]}$, usually satisfy the conditions $^{[9]-[12]}$, and that the function $e(p, \rho)$ is continuously differentiable in $\mathbb{R}^+ \times \mathbb{R}^+$ with 

$$\lim_{p \to 0^+} e(p, \rho) = 0, \quad \lim_{p \to +\infty} e(p, \rho) = +\infty, \tag{8}$$

for any fixed positive $\rho$.

In the “test-fluid” approximation, where the fluid self-gravity is neglected in comparison to the background gravitational field, the dynamics of the system is completely governed by Eqs. $^{[11]}$ and $^{[12]}$, together with the EOS $^{[4]}$. When such an approximation does not hold,
the GRHD equations must be solved in conjunction with the Einstein gravitational field equations, which relate the curvature of spacetime to the distribution of mass-energy.

In this paper, we only focus on the numerical methods for the GRHD equations (11, 2) and (3), assuming that the spacetime metric $g_{\mu\nu}$ and its derivatives $\frac{\partial g_{\mu\nu}}{\partial x^\kappa}$ are given or can be numerically computed by a given solver for the Einstein equations in each numerical time-step. All the following discussions only require that the metric tensor $g_{\mu\nu}$ is real symmetric with signature $(-, +, +, +)$.

In order to solve the GRHD equations by using modern shock-capturing methods, it is more suitable to reformulate the covariant form (1)–(2) into conservative Eulerian form (3)–(4) into admissible state set. For this purpose, we adopt the 3+1 (ADM) formulation (1) to decompose spacetime into a set of non-intersecting spacelike hypersurfaces with normal $(1/\alpha, -\beta^j/\alpha)$, where $\alpha > 0$ is the lapse function and $\beta^j$ is the shift vector. Within this formalism the spacetime metric $g_{\mu\nu}$ is split as

$$ds^2 = -(\alpha^2 - \beta_i^j \beta^j)dt^2 + 2\beta^j dx^i dt + \gamma_{ij} dx^i dx^j,$$

where $\gamma_{ij}$ denotes the 3-metric induced on each spacelike slice and is symmetric positive definite.

Let $g = \det(g_{\mu\nu})$, $\gamma = \det(\gamma_{ij})$ with $\sqrt{-g} = \alpha \sqrt{\gamma}$, and $\Gamma_{\mu\nu}^\lambda$ be the Christoffel symbols. Then, the GRHD equations (11, 2) can be rewritten as a first-order hyperbolic system (3)

$$\frac{1}{\sqrt{-g}} \left( \frac{\partial \sqrt{\gamma} U}{\partial t} + \frac{\partial \sqrt{-g} F_i^j(U)}{\partial x^i} \right) = Q(U),$$

where

$$U = (D, m, E)^T,$$

$$F_i^j = (Dv^i, \alpha \dot{m}, F_{m}^i, v^i)^T,$$

$$Q = \left( 0, T^\mu_{\nu} \frac{\partial g_{\mu\nu}}{\partial x^\kappa} - \Gamma_{\mu\nu}^\kappa \gamma_{ij} \right),$$

with $\dot{v}^i = v^i - \beta^i/\alpha$, the mass density $D = pW$, the momentum density (row) vector $m = \rho hW^2 v$, the energy density $E = \rho hW^2 - p$, and the row vector $e_i$ denoting the $i$-th row of the unit matrix of size 3. Additionally, row vector $v = (v_1, v_2, v_3)$ denotes the 3-velocity of the fluid with the contravariant components defined by

$$v^i = \frac{v_i}{\alpha u^0} + \frac{\beta^i}{\alpha},$$

and the Lorentz factor $W = \alpha u^0 = (1 - v^2)^{-\frac{1}{2}}$ with $v = \sqrt{\gamma_{ij} v^i v^j} = \sqrt{\gamma_{ij} \partial_j f^i}$.

The physical significance of the solution of (9) and the hyperbolicity of (9) require that the constraints

$$\rho > 0, \ p > 0, \ e > 0, \ v < 1,$$

always hold, which can ensure the Jacobian matrix $\partial(\xi_i F^j)/\partial U$ with any $(\xi_1, \xi_2, \xi_3) \neq 0$ has five real eigenvalues, and five independent real eigenvectors, see e.g. [12, 28]. Specifically, these eigenvalues are

$$\lambda^{(2)} = \lambda^{(3)} = \lambda^{(4)} = \xi_j v^j - \frac{\xi_j \beta^j}{\alpha},$$

$$\lambda^{(3\pm2)} = \frac{1}{1 - v^2 c_s^2} \left\{ \xi_j v^j (1 - c_s^2) \pm c_s W^{-1} \right\} \times \sqrt{(1 - v^2 c_s^2)(\xi_j \xi^j) - (1 - c_s^2)(\xi_j v^j)^2} - \frac{\xi_j \beta^j}{\alpha},$$

where $c_s$ is the local sound speed defined by [5, 29]

$$c_s^2 = h^2 \frac{\partial h(p, \rho)}{\partial \rho} \left( \frac{1}{\rho} - \frac{\partial h(p, \rho)}{\partial p} \right).$$

It implies $0 < c_s < 1$ from the condition (6) and $c_s = \sqrt{\frac{e}{\rho h}}$ for ideal EOS (7).

### III. ADMISSIBLE STATE SET

#### A. Definition and equivalent definition

For the GRHD equations (9), it is very natural and intuitive to define the (physically) admissible state set of $U$ as follows.

**Definition 1.** The set of admissible states of the GRHD equations (9) is defined by

$$G = \left\{ U = (D, m, E)^T \, | \, \rho(U) > 0, \right\}$$

$$p(U) > 0, \ e(U) > 0, \ v(U) < 1 \right\}. \tag{11}$$

Unfortunately, it is difficult to verify the four conditions in (11) for the given value of $U$, because there is no explicit expression for the transformation $U \mapsto (\rho, p, e, v)$. This also indicates the difficulty in studying the properties of $G$ and developing the PCP schemes for (9) with the numerical solution in $G$, especially for a general EOS (4). In practice, if giving the value of $U$, then one has to iteratively solve a nonlinear algebraic equation, e.g. an equation for the unknown pressure $p \in \mathbb{R}^+$:

$$E + p = Dh\left(p, \rho^U(p)\right) \left( 1 - \frac{m_j m_j}{(E + p)^2} \right)^{-\frac{1}{2}}, \tag{12}$$

where $\rho^U(p) = D\sqrt{1 - m_j m_j/(E + p)^2}$. Once the positive solution of the above equation is obtained, denoted by $p(U)$, other variables are sequentially calculated by

$$v_j(U) = \frac{m_j}{E + p(U)},$$

$$\rho(U) = D\sqrt{1 - v_j(U)v_j(U)},$$

$$e(U) = e(p(U), p(U)). \tag{13}$$

A equivalent simple definition of $G$ is given as follows, with the proof presented in Appendix A1.
Lemma 1. The admissible state set $\mathcal{G}$ in (11) is equivalent to the following set

$$
\mathcal{G}_\gamma = \left\{ \mathbf{U} = (D, \mathbf{m}, E)^T | D > 0, \ q_\gamma(\mathbf{U}) > 0 \right\},
$$

(14)

where

$$
q_\gamma(\mathbf{U}) := E - \sqrt{D^2 + \mathbf{m}^T \mathbf{m}},
$$

and the matrix $\mathbf{Y} = (\gamma^{ij})_{1 \leq i, j \leq 3}$ is positive definite and usually depends on $(t, x^i)$.

Based on Lemma 1, the admissible state sets $\mathcal{G}$ and $\mathcal{G}_\gamma$ will not be deliberately distinguished henceforth. However, in comparison with $\mathcal{G}$, the constraints in the set $\mathcal{G}_\gamma$ are explicit and directly imposed on the conservative variables, so that they can be very easily verified for given value of $\mathbf{U}$.

B. Mathematical properties

With the help of the equivalence between $\mathcal{G}$ and $\mathcal{G}_\gamma$, the convexity of admissible state set can then be proved, see Lemma 2 with proof displayed in Appendix A2.

Lemma 2. The admissible state set $\mathcal{G}_\gamma$ is an open convex set. Moreover, $\lambda U' + (1 - \lambda)U'' \in \mathcal{G}_\gamma$ for any $U' \in \mathcal{G}_\gamma$, $U'' \in \overline{\mathcal{G}_\gamma}$, and $\lambda \in (0, 1)$, where $\overline{\mathcal{G}_\gamma}$ is the closure of $\mathcal{G}_\gamma$.

The scaling invariance and Lax-Friedrichs (LxF) splitting properties of $\mathcal{G}_\gamma$ can be further obtained.

Lemma 3. Assume that $\mathbf{U} \in \mathcal{G}_\gamma = \mathcal{G}$, then

(i). (Scaling invariance) $\lambda \mathbf{U} \in \mathcal{G}_\gamma$, for any positive $\lambda$.

(ii). (LxF splitting) for any vector $\xi = (\xi_1, \xi_2, \xi_3) \neq 0$,

$$
\mathbf{U} \pm q_\xi^{-1} \xi_j \mathbf{F}^j(\mathbf{U}) \in \mathcal{G}_\gamma,
$$

and

$$
\mathbf{U} \pm \eta^{-1} \xi_j \mathbf{F}^j(\mathbf{U}) \in \mathcal{G}_\gamma, \quad \text{for any } \eta > q_\xi,
$$

where $q_\xi$ is an appropriate upper bound of the spectral radius of the Jacobian matrix $\partial(\xi_j \mathbf{F}^j(\mathbf{U}))/\partial \mathbf{U}$. For general EOS, it can be

$$
q_\xi = \sqrt{\xi_j \xi_i} + |\xi_j \beta|/\alpha.
$$

A smaller/sharper satisfied bound for ideal EOS is

$$
q_\xi = \frac{1}{1 - v^2 c_s^2} \left\{ |\xi_j v^j| (1 - c_s^2) + c_n W^{-1} \right. \nonumber
$$

$$
\times \sqrt{(1 - v^2 c_s^2) (\xi_j \xi_i) - (1 - c_s^2) (\xi_j v^j)^2} + |\xi_j \beta|/\alpha.
$$

(15)

The results in Lemmas 1, 2 and 3 are consistent with the special relativistic case established in [33, 38], if the spacetime is flat or $g_{\mu\nu}$ is the Minkowski metric $\text{diag}\{-1, 1, 1, 1\}$.

However, when $\mathbf{Y}$ is not a constant matrix and changes in spacetime, the admissible state set $\mathcal{G}$ or $\mathcal{G}_\gamma$ becomes dependent on spacetime. In other words, the admissible state sets defined at different points in curved spacetime are inequivalent, i.e. generally $\mathcal{G}_\gamma \neq \mathcal{G}_\gamma$ when $\mathbf{Y} \neq \tilde{\mathbf{Y}}$. This makes it difficult to use the above properties of $\mathcal{G}_\gamma$ to develop PCP methods for GRHD equations [9]. The reason is that most existing techniques for designing bound-preserving type methods, see e.g. [15, 20, 35, 38, 45, 48], highly depend on rewriting the target schemes into some forms of convex combination and then taking advantage of the convexity of the admissible state set. Whereas, unfortunately, in the present case the convexity does not hold between inequivalent admissible state sets defined at different points in curved spacetime, making the related techniques invalidated.

C. Spacetime-independent admissible state set

We find an effective solution to the above “spacetime-dependent” problem via a locally linear map. Specifically, we map the admissible states defined at different points in curved spacetime into a common set

$$
\mathcal{G}_* = \left\{ \mathbf{W} = (w_0, \cdots, w_4)^T | w_0 > 0, \ q(\mathbf{W}) := w_4 - \left(3 \sum_{i=0}^3 w_i^2 \right)^{1/2} > 0 \right\},
$$

(16)

in the sense of

$$
\mathbf{U} \in \mathcal{G}_\gamma \iff \mathbf{W} := \sqrt{\gamma} \Sigma \mathbf{U} \in \mathcal{G}_*,
$$

(17)

where the square matrix $\Sigma$ satisfies $\Sigma^T \Sigma = \text{diag}\{1, \mathbf{Y}^2, 1\}$. One can take $\Sigma$ as $\text{diag}\{1, \mathbf{Y}^2, 1\}$, but a better choice is explicitly defining $\Sigma$ via the Cholesky decomposition of $\mathbf{Y}$ as follows

$$
\Sigma = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & 0 \\
0 & 0 & \Sigma_{22} & \Sigma_{23} & 0 \\
0 & 0 & 0 & \Sigma_{33} & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix},
$$

where

$$
\Sigma_{11} = \sqrt{\gamma_{11}}, \quad \Sigma_{12} = \gamma_{12}/\sqrt{\gamma_{11}}, \quad \Sigma_{13} = \gamma_{13}/\sqrt{\gamma_{11}},
$$

$$
\Sigma_{22} = \sqrt{\gamma_{22} - (\gamma_{12})^2/\gamma_{11}}, \quad \Sigma_{23} = \frac{1}{\Sigma_{22}} \left(\gamma_{23} - \frac{\gamma_{12} \gamma_{13}}{\gamma_{11}}\right),
$$

$$
\Sigma_{33} = \sqrt{\gamma_{33} - (\gamma_{13})^2/\gamma_{11}} - \Sigma_{23}.
$$

It is worth noting that the transformation $\mathbf{U} \mapsto \mathbf{W}$ in (17) is linear in local spacetime.
The set $G_*$ defined in (10) does not depend on spacetime. In fact, $G_*$ is equal to the admissible state set in special relativistic case [35, 38]. Hence it has the following properties, whose proofs are the same as the special RHD case in [35] and omitted here.

**Lemma 4.** The function $q(W)$ defined in (10) is concave and Lipschitz continuous with respect to $W$. The admissible set $G_*$ is an open convex set. Moreover, $\lambda W + (1 - \lambda) W'' \in G_*$ for any vector $W \in G_*$, $W'' \in G_*$, and $\lambda \in (0, 1)$.

**D. $G_*$-associated formulation of GRHD equations**

The above analysis motivates us to develop PCP schemes for GRHD by taking advantages of the convexity of the spacetime-independent set $G_*$. Particularly, we would like to seek a new form of GRHD equations, whose admissible conservative vectors (state vectors) exactly form the set $G_*$. To this end, we multiply Eqs. (9) by the invertible matrix $\Sigma$ from the left, and then obtain the following equivalent form (abbreviated as “W-form” in later text)

$$\frac{\partial W}{\partial t} + \frac{\partial H^i(W)}{\partial x^i} = S(W), \quad (18)$$

where

$$H^i(W) = \sqrt{-g} \Sigma F^i,$$

$$S(W) = \sqrt{-g} \frac{\partial \Sigma}{\partial t} U + \sqrt{-g} \left( U + \frac{\partial \Sigma}{\partial x^j} F^j \right).$$

For convenience, these notations omit the dependence of $H^i$ and $S$ on the metric $g_{\mu\nu}$ and its derivatives $\frac{\partial g_{\mu\nu}}{\partial x^a}$.

Based on the relation (17), the properties of $G_*$ established in Lemma 3 can be directly extended to $G_*$. Let $G_k'$ be the approximation to the cell-average or the centroid-value of $W$ over $I_k$ at $t = t_n$. Approximating the flux in (19) by the LxF flux, and discretizing the time derivative by forward Euler method, one can derive a first-order scheme

$$\frac{d}{dt} \int_{I_k} W dx + \sum_{j \in N_k} \int_{E_{kj}} \xi_{kj} \hat{H}^i dS = \int_{I_k} S dx. \quad (19)$$

Let $\hat{W}_k^n$ be approximated to the cell-average of $W$ over $I_k$ at $t = t_n$. Approximating the flux in (19) by the LxF flux, and discretizing the time derivative by forward Euler method, one can derive a first-order scheme

$$\hat{W}_{k}^{n+1} = \frac{\Delta t_n}{|I_k|} \sum_{j \in N_k} |E_{kj}| \hat{H}_{kj} + \Delta t_n S(W_k^n), \quad (20)$$

where $|I_k|$ and $|E_{kj}|$ respectively denote the volume of $I_k$ and the area of the face $E_{kj}$. The adopted LxF flux is

$$\hat{H}_{kj} = \frac{1}{2} \xi_{kj} \left( H^i(W_k^n) + H^i(W_j^n) \right) - \frac{a_{kj}}{2} (W_j^n - W_k^n),$$

with the numerical viscosity coefficient satisfying

$$a_{kj} \geq \max \left\{ \eta_{kj}(W_k^n), \eta_{kj}(W_j^n) \right\}. \quad (21)$$

The readers are referred to Lemma 5 for the definition of $\eta_k$ for any nonzero vector $\xi \in \mathbb{R}^3$. Here the corresponding cell-centered values of $g_{\mu\nu}$ are used to calculate $H^i(W_k^n)$ and $S(W_k^n)$.

**Theorem 1.** Assume that $W_k^0 \in G_*$ for all $k$. Then the scheme (20) is PCP under the CFL-type condition

$$\Delta t_n \max_k \left( \frac{1}{2 |I_k|} \sum_{j \in N_k} a_{kj} |E_{kj}| + \lambda_S(W_k^n) \right) < 1, \quad (22)$$

where $\lambda_S, 0$ if $q(S(W_k^n)) \geq 0$, otherwise $\lambda_S > 0$ solves

$$q(W_k^n + \lambda_S^{-1} S(W_k^n)) = 0. \quad (23)$$

**IV. A FIRST-ORDER PCP SCHEME**

This section aims to establish the first theoretical result on PCP method for GRHD, i.e., rigorously show the PCP property of the first-order Lax-Friedrichs (LxF) scheme for the GRHD equations in W-form (18) on a general mesh. For convenience, we will also use $x$ to denote $(x^1, x^2, x^3)$ in the following.

Assume that the three-dimensional “spatial” domain is divided into a mesh of cells $\{I_k\}$, such as tetrahedron or hexahedron elements. For generality, the mesh can be unstructured. Let $N_k$ denote the index set of all the neighboring cells of $I_k$. For each $j \in N_k$, let $E_{kj}$ be the face of $I_k$ sharing with its neighboring cell $I_j$, i.e., $E_{kj} = \partial I_k \cap \partial I_j$, and $\xi_{kj} = (\xi_{kj,1}, \xi_{kj,2}, \xi_{kj,3})$ be the unit normal vector of $E_{kj}$ pointing from $I_k$ to $I_j$. The time interval is also divided into mesh $\{t_0 = 0, t_{n+1} = t_n + \Delta t_n, n \geq 0\}$ with the time-step size $\Delta t_n$ determined by the CFL-type condition.

Integrating the W-form (18) over the cell $I_k$ and using the divergence theorem give

$$\frac{d}{dt} \int_{I_k} W dx + \sum_{j \in N_k} \int_{E_{kj}} \xi_{kj} \hat{H}^i dS = \int_{I_k} S dx. \quad (19)$$

Let $\hat{W}_k^n$ be the approximation to the cell-average of $W$ over $I_k$ at $t = t_n$. Approximating the flux in (19) by the LxF flux, and discretizing the time derivative by forward Euler method, one can derive a first-order scheme

$$\hat{W}_{k}^{n+1} = \frac{\Delta t_n}{|I_k|} \sum_{j \in N_k} |E_{kj}| \hat{H}_{kj} + \Delta t_n S(W_k^n), \quad (20)$$

where $|I_k|$ and $|E_{kj}|$ respectively denote the volume of $I_k$ and the area of the face $E_{kj}$. The adopted LxF flux is

$$\hat{H}_{kj} = \frac{1}{2} \xi_{kj} \left( H^i(W_k^n) + H^i(W_j^n) \right) - \frac{a_{kj}}{2} (W_j^n - W_k^n),$$

with the numerical viscosity coefficient satisfying

$$a_{kj} \geq \max \left\{ \eta_{kj}(W_k^n), \eta_{kj}(W_j^n) \right\}. \quad (21)$$

The readers are referred to Lemma 5 for the definition of $\eta_k$ for any nonzero vector $\xi \in \mathbb{R}^3$. Here the corresponding cell-centered values of $g_{\mu\nu}$ are used to calculate $H^i(W_k^n)$ and $S(W_k^n)$.

**Theorem 1.** Assume that $W_k^0 \in G_*$ for all $k$. Then the scheme (20) is PCP under the CFL-type condition

$$\Delta t_n \max_k \left( \frac{1}{2 |I_k|} \sum_{j \in N_k} a_{kj} |E_{kj}| + \lambda_S(W_k^n) \right) < 1, \quad (22)$$

where $\lambda_S, 0$ if $q(S(W_k^n)) \geq 0$, otherwise $\lambda_S > 0$ solves

$$q(W_k^n + \lambda_S^{-1} S(W_k^n)) = 0. \quad (23)$$
V. HIGH-ORDER PCP SCHEMES

This section is devoted to designing high-order, provably PCP schemes for the GRHD equations in W-form [15].

For the sake of convenience, we assume that the spatial domain is divided into a uniform cuboid mesh, with the constant spatial step-size $\Delta \ell$ in $x^\ell$-direction, $\ell = 1, 2, 3$, respectively. And the time interval is divided into mesh $\{ t_0 = 0, t_{n+1} = t_n + \Delta t_n, n \geq 0 \}$, with the time step-size $\Delta t_n$ determined by the CFL-type condition.

To avoid confusing subscripts, in this section we sometimes use the symbol $\mathbf{x}$ or $(x, y, z)$ to replace the independent variables $(x^1, x^2, x^3)$.

A. PCP finite volume and DG schemes

Assume the uniform cuboid mesh is with cells

$$\mathcal{I}_{ijk} = \left[ x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}} \right] \times \left[ y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}} \right] \times \left[ z_{k-\frac{1}{2}}, z_{k+\frac{1}{2}} \right],$$

and $\overline{\mathbf{W}}_{ijk}^n$ be the numerical cell-averaged approximation of the exact solution $\mathbf{W}(t, \mathbf{x})$ over $\mathcal{I}_{ijk}$ at $t = t_n$. We aim at designing PCP finite volume or DG type methods of the GRHD equations [15], whose solution $\overline{\mathbf{W}}_{ijk}^n$ always stays at $\mathcal{G}_*$, if $\mathbf{W}_{ijk} \in \mathcal{G}_*$.

Towards achieving high-order $(K+1)$-th order spatial accuracy, the approximate solution polynomials $\mathbf{W}_{ijk}^n(\mathbf{x})$ of degree $K$ are also built usually, to approximate the exact solution $\mathbf{W}(t_n, \mathbf{x})$ within the cell $\mathcal{I}_{ijk}$. Such polynomial vector $\mathbf{W}_{ijk}^n(\mathbf{x})$ is, either reconstructed in finite volume methods from $\{\overline{\mathbf{W}}_{ijk}^n\}$, or evolved in DG methods. The cell-averaged value of $\mathbf{W}_{ijk}^n(\mathbf{x})$ over the cell $\mathcal{I}_{ijk}$ is required to be $\overline{\mathbf{W}}_{ijk}^n$.

1. Method

For the moment, the forward Euler method is used for time discretization, while high-order time discretization will be considered later. Then, the main implementation procedures of our high-order ($K \geq 1$) PCP finite volume or DG method can be outlined as follows.

Step 0. Initialization. Set $t = 0$ and $n = 0$, and compute $\overline{\mathbf{W}}_{ijk}^n$ and $\mathbf{W}_{ijk}^n(\mathbf{x})$ for each cell $\mathcal{I}_{ijk}$ by using the initial data. Note the convexity of $\mathcal{G}_*$ can ensure $\overline{\mathbf{W}}_{ijk}^n \in \mathcal{G}_*$.

Step 1. Given admissible cell-averages $\{\overline{\mathbf{W}}_{ijk}^n\}$, perform PCP limiting procedure. Use the PCP limiter presented later to modify the polynomials $\{\mathbf{W}_{ijk}(\mathbf{x})\}$ as $\{\mathbf{W}_{ijk}(\mathbf{x})\}$, such that the revised polynomials satisfy

$$\overline{\mathbf{W}}_{ijk}(\mathbf{x}) \in \mathcal{G}_*, \quad \text{for any } \mathbf{x} \in \mathcal{S}_{ijk},$$

where the set $\mathcal{S}_{ijk}$ consists of several important tensor-product quadrature nodes in $\mathcal{I}_{ijk}$. Specifically,

$$\mathcal{S}_{ijk} = (\widehat{\mathbf{S}}_{x}^i \otimes \widehat{\mathbf{S}}_{y}^j \otimes \widehat{\mathbf{S}}_{z}^k) \cup (\nabla \mathbf{S}_{x}^i \otimes \widehat{\mathbf{S}}_{y}^j \otimes \widehat{\mathbf{S}}_{z}^k) \cup (\nabla \mathbf{S}_{x}^i \otimes \widehat{\mathbf{S}}_{y}^j \otimes \nabla \mathbf{S}_{z}^k),$$

where $\widehat{\mathbf{S}}_{x}^i = \{ \mathbf{x}^i_{\mu} \}_{\mu=1}^L$, $\widehat{\mathbf{S}}_{y}^j = \{ \mathbf{y}^j_{\nu} \}_{\nu=1}^L$, $\widehat{\mathbf{S}}_{z}^k = \{ \mathbf{z}^k_{\omega} \}_{\omega=1}^L$ are the L-point Gauss-Lobatto quadrature nodes in the intervals $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $[y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$ and $[z_{k-\frac{1}{2}}, z_{k+\frac{1}{2}}]$, respectively; and $\mathbf{S}_{x}^i = \{ x^i_{\mu} \}_{\mu=1}^Q$, $\mathbf{S}_{y}^j = \{ y^j_{\nu} \}_{\nu=1}^Q$, $\mathbf{S}_{z}^k = \{ z^k_{\omega} \}_{\omega=1}^Q$ are the Q-point Gauss-Legendre quadrature nodes in those three intervals respectively.

For achieving provably PCP property, $L$ is suggested to satisfy $2L - 3 \geq K$. For the accuracy requirement, $Q$ shall satisfy: $2Q \geq K + 1$ for a $(K + 1)$-th order finite volume method, or $Q \geq K + 1$ for a $\mathbb{P}^K$-based DG method [8].

Step 2. For each cell $\mathcal{I}_{ijk}$, evaluate the limiting values of $\overline{\mathbf{W}}_{ijk}^n(\mathbf{x})$ at the Gaussian points on the faces of the cell:

$$\begin{align*}
\mathbf{W}_{\ell,ijk}^n &\leftarrow \overline{\mathbf{W}}_{ijk}^n(\mathbf{x}) \left( \frac{1}{3} \right), \\
\mathbf{W}_{\ell,ijk}^n &\leftarrow \overline{\mathbf{W}}_{ijk}^n(\mathbf{x}) \left( \frac{1}{1} \right), \\
\mathbf{W}_{\ell,ijk}^n &\leftarrow \overline{\mathbf{W}}_{ijk}^n(\mathbf{x}) \left( \frac{2}{0} \right), \\
\mathbf{W}_{\ell,ijk}^n &\leftarrow \overline{\mathbf{W}}_{ijk}^n(\mathbf{x}) \left( \frac{3}{0} \right), \\
\mathbf{W}_{\ell,ijk}^n &\leftarrow \overline{\mathbf{W}}_{ijk}^n(\mathbf{x}) \left( \frac{4}{0} \right), \\
\mathbf{W}_{\ell,ijk}^n &\leftarrow \overline{\mathbf{W}}_{ijk}^n(\mathbf{x}) \left( \frac{5}{0} \right), \\
\mathbf{W}_{\ell,ijk}^n &\leftarrow \overline{\mathbf{W}}_{ijk}^n(\mathbf{x}) \left( \frac{6}{0} \right),
\end{align*}$$

for $\mu, \nu = 1, \ldots, Q$.

Step 3. Compute numerical fluxes. First estimate the upper bound $a_{\ell}^{(e)}$ of the characteristic speed in $x^\ell$-direction by

$$a_{\ell}^{(e)} \geq \max_{i,j,k,\mathbf{x}} \left\{ \eta_{\ell}(\overline{\mathbf{W}}_{ijk}^n(\mathbf{x})) \right\}, \quad (25)$$

with $\xi_{\ell}$ denoting the $\ell$-th row of unit matrix of size 3, $\ell = 1, 2, 3$. Let $\{ \omega_{\mu} \}_{\mu=1}^3$ be the associated weights of the Q-point Gauss-Legendre quadrature and satisfy $\sum_{\mu=1}^3 \omega_{\mu} = 1$. Then for each $i,j,k$, compute the numerical fluxes in $x^\ell$-direction, $\ell = 1, 2, 3$, by

$$\begin{align*}
\hat{H}_{i+\frac{1}{2},j,k}^1 &= \omega_1 \omega_2 \hat{H}_{i+\frac{1}{2},j,k}^1 \left( \mathbf{W}_{\ell,ijk} - \mathbf{W}_{\ell,ijk}^+ \right), \\
\hat{H}_{i+\frac{1}{2},j,k}^2 &= \omega_1 \omega_2 \hat{H}_{i+\frac{1}{2},j,k}^2 \left( \mathbf{W}_{\ell,ijk} - \mathbf{W}_{\ell,ijk}^+ \right), \\
\hat{H}_{i+\frac{1}{2},j,k}^3 &= \omega_1 \omega_2 \hat{H}_{i+\frac{1}{2},j,k}^3 \left( \mathbf{W}_{\ell,ijk} - \mathbf{W}_{\ell,ijk}^+ \right).
\end{align*} \quad (26)$$

with summation convention employed, and the numerical flux $\mathbf{H}^e(\mathbf{W}^-, \mathbf{W}^+)$ taken as the LxF flux

$$\begin{align*}
\mathbf{H}^e(\mathbf{W}^-, \mathbf{W}^+) &= \frac{1}{2} \left( \mathbf{H}^e(\mathbf{W}^-) + \mathbf{H}^e(\mathbf{W}^+) \right) - a_{\ell}^{(e)} (\mathbf{W}^+ - \mathbf{W}^-), \quad \ell = 1, 2, 3.
\end{align*} \quad (27)$$
Numerical fluxes in (26) can be regarded as high-order approximations to
\[
\frac{1}{\Delta x \Delta y} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} H^1(t^n, x_i, y, z) \, dy \, dz,
\]
\[
\frac{1}{\Delta x \Delta y} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} H^2(t^n, x, y_j, z) \, dx \, dz,
\]
\[
\frac{1}{\Delta x \Delta y} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} H^3(t^n, x, y, z_{k+\frac{1}{2}}) \, dx \, dy,
\]
respectively.

**Step 4.** Update the cell-averages by the scheme
\[
\tilde{W}_{ijk}^{n+1} = W_{ijk}^n - \frac{\Delta t_n}{\Delta x} \left( \tilde{H}_{i+j+\frac{1}{2},k}^n - \tilde{H}_{i+j-\frac{1}{2},k}^n \right) - \frac{\Delta t_n}{\Delta y} \left( \tilde{H}_{i,j+k+\frac{1}{2}}^n - \tilde{H}_{i,j+k-\frac{1}{2}}^n \right) + \Delta t_n \tilde{S}_{ijk}^n,
\]
where \( \tilde{S}_{ijk}^n \) denotes an appropriate high-order approximation to the cell-average of \( S \) over the cell \( I_{ijk} \), e.g.
\[
\tilde{S}_{ijk}^n = \omega_{ijk} [\omega_n \left( \tilde{W}_{ijk}^n(x_1^{(6)}, y_j^{(6)}, z_k^{(6)}) \right)],
\]
where Einstein’s summation convention is used. Eq. (28) is the formulation of the finite volume scheme or the discrete equation for cell-averaged values in the DG scheme. *As shown in Theorem 2 later, the PCP limiting procedure in Step 1 can ensure the computed \( \tilde{W}_{ijk}^n \) \( \in \mathcal{G} \), which meets the condition of performing PCP limiting procedure in the next time-forward step, see Step 6.*

**Step 5.** Built the polynomials \( \{ \tilde{W}_{ijk}^{n+1}(x) \} \). For a high-order finite volume scheme, reconstruct the approximate solution polynomial \( \tilde{W}_{ijk}^{n+1}(x) \) from the cell-averages \( \{ \tilde{W}_{ijk}^n \} \); for \( \mathbb{P}^k \)-based DG method (\( k \geq 1 \)), evolve the high-order “moments” of \( \tilde{W}_{ijk}^{n+1}(x) \), similar to (26). The details are omitted here, as these do not affect the PCP property of the proposed schemes.

**Step 6.** Set \( \Delta t_{n+1} = \Delta t_n \). If \( \Delta t_{n+1} < \Delta t_{\text{step}} \), then assign \( n \leftarrow n + 1 \) and go to Step 1, where the admissibility of \( \{ \tilde{W}_{ijk}^n \} \) has been ensured in Step 4. Otherwise, output numerical results and stop.

The main difference between the present PCP method and the traditional method is that the former adds a carefully designed PCP limiting procedure (i.e. Step 1).

2. **PCP limiter**

We now present the PCP limiter used in Step 1, which is a key ingredient of the above high-order PCP method. Without this limiter, the original high-order schemes are generally not PCP, and may easily break down after some time steps in solving some ultra-relativistic problems involving low density or pressure, or very large velocity. The notion of our PCP limiter is extended from the non-relativistic case [26] and special relativistic case [26, 37, 38].

To avoid the effect of the rounding error, we define
\[
\mathcal{G}_\epsilon = \{ W = (W_0, \ldots, W_4) \mid W_0 \geq \epsilon, \ q(W) \geq \epsilon \},
\]
which is a subset of \( \mathcal{G}_\epsilon \) and satisfy \( \lim_{\epsilon \rightarrow 0^+} \mathcal{G}_\epsilon = \mathcal{G}_\epsilon \). Here \( \epsilon \) is a sufficiently small positive number and may be taken as \( \epsilon = 10^{-12} \) in numerical computations.

Under the condition \( \tilde{W}_{ijk} \in \mathcal{G}_\epsilon \) in Step 1, our PCP limiting procedure for each cell \( I_{ijk} \) is divided into the following easily-implemented steps. For simplicity, here we temporarily omit the superscripts \( n \).

- If \( \tilde{W}_{ijk} \notin \mathcal{G}_\epsilon \), then the cell \( I_{ijk} \) is identified as vacuum region approximately. Set \( \tilde{W}_{ijk}(x) = \tilde{W}_{ijk} \) and skip the following steps.
- Enforce the first constraint in \( \mathcal{G}_\epsilon \). Let \( \tilde{W}_{ijk} \) denote the \( \ell \)-th component of \( \tilde{W}_{ijk} \), and \( W_{0,\text{min}} = \min_{x \in S_{ijk}} W_{0,ijk}(x) \). If \( W_{0,\text{min}} < \epsilon \), then \( \tilde{W}_{0,ijk}(x) \) is limited as
  \[
  \tilde{W}_{0,ijk}(x) = \frac{\theta_1 (\tilde{W}_{0,ijk} - \tilde{W}_{0,ijk} - W_{0,\text{min}})}{\theta_1 (\tilde{W}_{0,ijk} - \tilde{W}_{0,ijk})} \tilde{W}_{0,ijk},
  \]
  where \( \theta_1 = (\tilde{W}_{0,ijk} - \epsilon)/(\tilde{W}_{0,ijk} - W_{0,\text{min}}) \). Otherwise, take \( \tilde{W}_{0,ijk}(x) = \tilde{W}_{0,ijk}(x) \). Denote
  \[
  \tilde{W}_{ijk}(x) := (\tilde{W}_{0,ijk}(x), \tilde{W}_{1,ijk}(x), \ldots, \tilde{W}_{4,ijk}(x))^T.
  \]
- Enforce the second constraint in \( \mathcal{G}_\epsilon \). Let \( q_{\text{min}} = \min_{x \in S_{ijk}} q(\tilde{W}_{ijk}(x)) \). If \( q_{\text{min}} < \epsilon \), then \( \tilde{W}_{ijk}(x) \) is limited as
  \[
  \tilde{W}_{ijk}(x) = \frac{\theta_2 (\tilde{W}_{ijk}(x) - \tilde{W}_{ijk})}{\theta_2 (\tilde{W}_{ijk}(x) - q_{\text{min}})} \tilde{W}_{ijk}(x),
  \]
  where \( \theta_2 = (q(\tilde{W}_{ijk}) - q(\tilde{W}_{ijk}))/q(\tilde{W}_{ijk}) \). Otherwise, set \( \tilde{W}_{ijk}(x) = \tilde{W}_{ijk}(x) \).

With the concavity of \( q(W) \), the above PCP limiting procedure yields that the revised/limited polynomial \( \tilde{W}_{ijk}(x) \) satisfy (24).

In the end, we remark several features of the proposed PCP limiter, in addition to its easy implementation. The limiter keeps the conservativity, i.e.
\[
\frac{1}{\Delta t_1 \Delta x \Delta y} \int_{I_{ijk}} \tilde{W}_{ijk}(x) \, dx = W_{ijk}^n.
\]
It also maintains the high-order accuracy when \( \tilde{W}_{ijk}(x) \) approximates a smooth solution without vacuum, similar to [44, 45]. The above PCP limiting procedure is independently performed on each cell, making the PCP limiter easily parallel.

It is worth emphasizing that the PCP limiter does not depend on the reconstructing technique employed.
in Step 5 of a PCP finite volume scheme. Therefore, the proposed PCP finite volume schemes are very friendly, in cooperation with any appropriate reconstructing techniques for $W_{ijk}^n(x)$, e.g., essentially non-oscillatory (ENO) approach \[14\], weighted ENO approach \[10\], piecewise parabolic method \[9\], etc.

3. Provably PCP property

We are now in position to present the theoretical result on the PCP property of the proposed finite volume and DG methods.

Before discussing high-order case ($K \geq 1$), we first present the result for the special case of $K = 0$, i.e., $W_{ijk}^n(x) = W_{ijk}^n$. In this special case, the scheme \[22\] reduces to first-order LxF scheme, and the PCP limiting procedure is not required. As a direct corollary of Theorem \[1\] we immediately have the following consequence.

Corollary 1. When $K = 0$, the scheme \[28\] is PCP under the CFL-type condition

$$\Delta t_n \left( \Delta^{-1}_\ell a_*^{(f)} + \lambda_S \right) < 1,$$

with summation convention employed, where $\lambda_S = 0$ if $q(S_{ijk}) \geq 0$, otherwise $\lambda_S > 0$ is the solution to

$$q(W_{ijk}^n + \lambda_S^{-1}S_{ijk}) = 0. \tag{30}$$

Let $\{\omega_\mu\}_{\mu=1}^L$ be the associated weights of the $L$-point Gauss-Lobatto quadrature, with $\sum_{\mu=1}^L \omega_\mu = 1$ and $\omega_1 = \omega_2 = \frac{1}{1+\gamma}$. We can then rigorously show the PCP property of the proposed methods in high-order case $K \geq 1$, as stated in Theorem \[2\] with the proof displayed in Appendix \[A\].

Theorem 2. Assume $K \geq 1$ and $W_{ijk}^0 \in G_\ell$ for all $i, j, k$. Assume that the condition \[24\] is satisfied by the revised polynomials $\{W_{ijk}^n(x)\}$. Then, under the CFL-type condition

$$\Delta t_n \left( \Delta^{-1}_\ell a_*^{(f)} + \omega_1 \lambda_S \right) < \omega_1, \tag{31}$$

the scheme \[28\] preserves $W_{ijk}^* \in G_\ell$ for all $i, j, k, n$. In other words, the scheme \[28\] with $K \geq 1$ is PCP under the condition \[31\], where $\lambda_S = 0$ if $q(S_{ijk}) \geq 0$, otherwise $\lambda_S$ is the positive solution of Eq. \[30\].

For some high-order finite volume methods, it only needs to reconstruct the limiting values $\{W_{ijk}^{\mu+\nu,\pm,\pm}(x), W_{ijk}^{\mu+\nu,\mp,\pm}(x), W_{ijk}^{\mu+\nu,\pm,\pm}(x)\}$ instead of the polynomial $W_{ijk}^n(x)$. In this case, based on the proof of Theorem \[2\] the condition \[24\] for achieving PCP property can be replaced with the following condition

$$W_{ijk}^{\mu+\nu,\pm,\pm}(x), W_{ijk}^{\mu+\nu,\mp,\pm}(x), W_{ijk}^{\mu+\nu,\pm,\pm}(x) \in G_\ell, \quad \mu, \nu = 1, \cdots, q,$$

$$\frac{1}{1-2\omega_1} \left( \Delta^{-1}_\ell a_*^{(f)}(W_{ijk}^* + W_{ijk+1}^* + W_{ijk+2}^*) + \Delta^2_{\mu}a_\mu^{(2)}(W_{ijk}^*) \right) \in G_\ell, \quad \mu = 1, \cdots, q,$$

for all $i, j, k$, where $\Pi_\ell$ is defined in the proof of Theorem \[2\]. Similar to the discussions in Sec. 5 of \[47\], the previous PCP limiting procedure can be easily revised to meet such condition.

4. Remarks

The scheme \[28\] is only first-order accurate in time. To achieve high-order PCP scheme in time, one can replace the forward Euler time discretization in \[28\] with high-order strong stability preserving (SSP) methods \[13\].

For example, utilizing the third-order SSP Runge-Kutta method gives

$$W_{ijk}^* = \frac{3}{4} W_{ijk}^n + \frac{1}{4} \left( W_{ijk}^n + \Delta t_n L_{ijk}^*(W^*(x)) \right), \tag{32}$$

where $L_{ijk}^*(W^*(x))$ is the numerical spatial operator, and $W_n^*(x)$. $W^*(x)$, $W^{**}(x)$ denote the PCP limited versions of the reconstructed or evolved polynomial vector at each Runge-Kutta stage. Since such SSP method is a convex combination of the forward Euler method, according to the convexity of $G_\ell$, the resulting high-order scheme \[32\] is also PCP under the CFL condition \[31\].

To enforce the condition \[31\] rigorously, we need to get an accurate estimation of $a_*^{(f)}$ for all the Runge-Kutta stages in \[32\] based only on the numerical solution at time level $n$, which is highly nontrivial. Hence, in practical computations, we suggest to take the value of $a_*^{(f)}$ slightly larger. Besides, the time step-size selecting strategy suggested in \[32\] may be adopted to improve computational efficiency.

The high-order SSP multi-step method can also be used for time discretization to achieve high-order PCP schemes c.f. \[28\], and the details are omitted here. The above complication of enforcing the condition \[31\] does not exist if one uses a SSP multi-step time discretization.

B. PCP finite difference scheme

Assume the uniform cuboid mesh with grid points $\{(x_i, y_j, z_k)\}$, and $W_{ijk}^n$ denote the numerical approximation to the value of the exact solution $W(t_n, x_i, y_j, z_k)$.
at the grid point. We would like to design PCP finite difference schemes of the GRHD equations in W-form \((33)\), which preserve \(W_{i,j,k}^{n} \in \mathcal{G}_{s}\) if \(W_{i,j,k}^{0} \in \mathcal{G}_{s}\).

### 1. Method

We also focus on the forward Euler time discretization first, and consider high-order time discretization later. Then, a \(r\)-th order (spatially) accurate, conservative finite difference scheme of the GRHD equations in W-form \((18)\) may be written as

\[
W_{i,j,k}^{n+1} = W_{i,j,k}^{n} + \Delta t_s L_{ijk}(W^n),
\]

where \(\nabla \times (W^{-}, W^+)\) is the LxF flux defined in \((27)\), with

\[
a^{(\ell)}_{s} \geq \max_{i,j,k} \xi_{\ell}(W_{i,j,k}^{n}),
\]

and \(\xi_{\ell}\) denoting the \(\ell\)-th row of the unit matrix of size 3.

#### Step 3. Limit high-order flux. Modify the high-order fluxes \(\hat{H}_{i,j,k}^{1, \text{high}}, \hat{H}_{i,j,k}^{2, \text{high}}\) and \(\hat{H}_{i,j,k}^{3, \text{high}}\) to high-order PCP fluxes defined by

\[
\hat{H}_{i,j,k}^{1, \text{high} + \frac{j}{2}, k} = \hat{H}_{1, i,j,k}^{\text{high}}(\hat{H}_{i,j,k}^{1, \text{high} + \frac{j}{2}, k} - \hat{H}_{i,j,k}^{1, \text{LF} + \frac{j}{2}, k}) + \hat{H}_{i,j,k}^{1, \text{LF} + \frac{j}{2}, k},
\]

\[
\hat{H}_{i,j,k}^{2, \text{high} + \frac{j}{2}, k} = \hat{H}_{2, i,j,k}^{\text{high}}(\hat{H}_{i,j,k}^{2, \text{high} + \frac{j}{2}, k} - \hat{H}_{i,j,k}^{2, \text{LF} + \frac{j}{2}, k}) + \hat{H}_{i,j,k}^{2, \text{LF} + \frac{j}{2}, k},
\]

\[
\hat{H}_{i,j,k}^{3, \text{high} + \frac{j}{2}, k} = \hat{H}_{3, i,j,k}^{\text{high}}(\hat{H}_{i,j,k}^{3, \text{high} + \frac{j}{2}, k} - \hat{H}_{i,j,k}^{3, \text{LF} + \frac{j}{2}, k}) + \hat{H}_{i,j,k}^{3, \text{LF} + \frac{j}{2}, k},
\]

via the PCP flux limiter presented later. In \((36)\), the parameters \(\theta_{1, i,j,k}^{\ell}, \theta_{2, i,j,k}^{\ell}, \theta_{3, i,j,k}^{\ell} \in [0, 1]\), whose computation is the main ingredient of PCP flux limiter.

#### Step 4. Evolve forward by the scheme \((33)-(34)\) with the high-order PCF fluxes in \((36)\).

#### Step 5. Set \(t_{n+1} = t_{n} + \Delta t_{s}\). If \(t_{n+1} < T_{\text{stop}}\), then assign \(n \leftarrow n + 1\) and go to Step 1. Otherwise, output numerical results and stop.

### 2. PCP flux limiter

The PCP flux limiter used in Step 3 is the key point in designing the above PCP finite difference scheme. Its role is to locally modify any appropriate high-order numerical fluxes into high-order PCP fluxes of form \((36)\).

For the sake of convenience, the following notations are introduced. We employ the vector \(\theta_{i,j,k}\) to represent the parameters

\[
\left(\theta_{i-\frac{1}{2}, j,k}, \theta_{i+\frac{1}{2}, j,k}, \theta_{i,j-\frac{1}{2}, k}, \theta_{i,j+\frac{1}{2}, k}, \theta_{i,j,k-\frac{1}{2}}, \theta_{i,j,k+\frac{1}{2}}\right),
\]

and the \(\ell\)-th component of \(\theta_{i,j,k}\) is also denoted by \(\theta_{i,j,k}^{(\ell)}\).

The PCP flux limiter is

\[
W_{i,j,k}(\theta_{i,j,k}) := W_{i,j,k}^{n} + \Delta t_{n} L_{ijk}(W^{n}),
\]

where \(\theta_{i,j,k}^{(\ell)}\) is the PCP flux limiter in \((36)\), with \(\xi_{\ell}\) denoting the \(\ell\)-th row of the unit matrix of size 3.
Our goal is to carefully choose the parameters $\theta_{ijk}$ such that $W_{i,j,k}(\theta_{ijk}) = W_{i,j,k}^{n+1} \in G_s$ provided $W_{i,j,k}^{n} \in G_s$. Specifically, we need to determine the hyperrectangular parameter is as close to 1 as possible while subject to the almost "best" parameters in maintaining the high-order accuracy [41].

Corollary 2. If $W_{i,j,k}^{n} \in G_s$, then $W_{i,j,k}(0) \in G_s$ under the CFL-type condition

$$\Delta t_n \left( \frac{1}{\Delta x} a_0^{(t)} + \lambda_S \right) < 1,$$

(39)

where $\lambda_S = 0$ if $q(S(W_{i,j,k}^{n})) > 0$, otherwise $\lambda_S > 0$ and solves $q(W_{i,j,k}^{n} + \lambda_S^{-1} S(W_{i,j,k}^{n})) = 0$.

However, such an approach (taking $\theta_{ijk} = 0$) evidently destroys the original high-order accuracy and deprives the significance of constructing high-order numerical flux in Step 1. In order to maintain the $r$-th order accuracy of the original numerical flux, each component of the parameters $\theta_{ijk}$ is expected to be $1 - O(\max \{ \Delta x \}^r)$ for smooth solutions.

There exist in the literature two types of positivity-preserving flux limiters, which can be borrowed and extended to the GRHD case, including the cut-off flux limiter [13] and the parametrized flux limiter [6, 17, 18, 40, 41]. The extension of the cut-off limiter to the GRHD case is similar to the special RHD case [39]. In the following, we mainly focus on developing parametrized PCP flux limiter, because the parametrized limiter works well in maintaining the high-order accuracy [41].

The parametrized PCP flux limiter attempts to seek the almost "best" parameters $\theta_{ijk}$, such that each parameter is as close to 1 as possible while subject to $W_{i,j,k}(\theta_{ijk}) \in G_s$. More specifically, such $\theta_{ijk}$ can be computed through the following two sub-steps of Step 3.

**Step 3.1.** For each $i, j, k$, find large parameters $\Lambda_{i,j,k}^{(t)} \in [0, 1], \ell = 1, \ldots, 10$, such that

$$W_{i,j,k}(\theta) \in G_s, \text{ for all } \theta \in \Theta_{i,j,k}^{*} := \bigotimes_{\ell=1}^{6} [0, \Lambda_{i,j,k}^{(t)}],$$

where the symbol "$\otimes$" denotes tensor product.

**Step 3.2.** For each $i, j, k$, set

$$\begin{align*}
\theta_{i+\frac{1}{2}, j, k} &= \min \left\{ \Lambda_{i,j,k}^{(2)}, \Lambda_{i+1,j,k}^{(1)} \right\}, \\
\theta_{i, j+\frac{1}{2}, k} &= \min \left\{ \Lambda_{i,j,k}^{(4)}, \Lambda_{i,j+1,k}^{(3)} \right\}, \\
\theta_{i, j, k+\frac{1}{2}} &= \min \left\{ \Lambda_{i,j,k}^{(6)}, \Lambda_{i,j+1,k+1}^{(5)} \right\}.
\end{align*}$$

In following, we shall present the details of Step 3. Specifically, we need to determine the hyperrectangular $\Theta_{i,j,k}^{*}$ for given values of $W_{i,j,k}(0)$ and $\{C_{\ell}\}_{\ell=1}^{6}$ defined in [35].

To avoid the effect of the rounding error, we introduce a small positive number

$$\epsilon = \min \left\{ 10^{-12}, \min_{i,j,k} \{ W_{0,i,j,k}(0) \}, \min_{i,j,k} \{ q(W_{i,j,k}(0)) \} \right\},$$

where $W_{0,i,j,k}(\theta)$ denotes the first component of $W_{i,j,k}(\theta)$. Under the condition [39], $W_{i,j,k}(0) \in G_s$ implies $\epsilon > 0$, and that $W_{i,j,k}(0)$ belongs to $G_s$ defined in [29]. We then have the following property, whose proof is displayed in Appendix A.6.

Lemma 6. Under the condition [39], the two sets

$$\begin{align*}
\Theta_{0} &= \left\{ \theta \in [0, 1]^{6} \mid W_{0,i,j,k}(\theta) \geq \epsilon \right\}, \\
\Theta &= \left\{ \theta \in [0, 1]^{6} \mid W_{0,i,j,k}(\theta) \geq \epsilon, q(W_{i,j,k}(\theta)) \geq \epsilon \right\},
\end{align*}$$

are both convex.

Based on this lemma, Step 3.1 is divided into the following two sub-steps for each $i, j, k$.

**Step 3.1(a)** Find a big hyperrectangular

$$\Theta_{i,j,k}^{*} := \bigotimes_{\ell=1}^{6} [0, \Lambda_{0}^{(t)}] \subseteq \Theta_{0},$$

within the convex set $\Theta_{0}$ and $\Lambda_{0}^{(t)}$, $\ell = 1, \ldots, 6$, should be as large as possible. Specifically, they are computed by

$$\Lambda_{0}^{(t)} = \begin{cases}
\min \left\{ 1, \frac{W_{0,i,j,k}(0) - \epsilon}{C_{0,\ell}} \right\}, & \text{if } C_{0,\ell} < 0, \\
1, & \text{otherwise},
\end{cases}$$

(40)

where $C_{0,\ell}$ denotes the first component of $C_{\ell}$, the index set $N_{C} = \{ \mu = 1, 2, \ldots, 6 \} \mid C_{0,\mu} < 0$, and $\varepsilon_{0} > 0$ is a small parameter to avoid division by zero and may be taken as $10^{-12}$.

**Step 3.1(b)** Shrink the hyperrectangular $\Theta_{i,j,k}^{*}$ into $\Theta_{i,j,k}^{*}$ such that $\Theta_{i,j,k}^{*} \subseteq \Theta$. Let

$$\mathbf{V}_{\ell_{1}, \ell_{2}, \ell_{3}, \ell_{4}, \ell_{5}, \ell_{6}} = \left( \ell_{1} \Lambda_{0}^{(1)}, \ell_{2} \Lambda_{0}^{(2)}, \ldots, \ell_{6} \Lambda_{0}^{(6)} \right),$$

with $\ell_{1}, \ldots, \ell_{6} \in (0, 1)$, denote 64 vertices of the hyperrectangular $\Theta_{0}$. For each vertex $\mathbf{V}_{\ell_{1}, \ell_{2}, \ell_{3}, \ell_{4}, \ell_{5}, \ell_{6}} =: \mathbf{V}_{\ell}$ do the following:

- If $\mathbf{V}_{\ell} \in \Theta$, then set $\hat{\mathbf{V}}_{\ell} = \mathbf{V}_{\ell}$;
- Otherwise, compute the unique solution of the equation $q(W_{i,j,k}(\lambda \mathbf{V}_{\ell})) = \epsilon$ for the unknown $\lambda \in [0, 1)$, and set $\hat{\mathbf{V}}_{\ell} = \lambda \mathbf{V}_{\ell}$. Here the uniqueness of $\lambda$ is ensured by the concavity of the function $q(W)$ stated in Lemma 4.
This gives \( \hat{\mathbf{v}}_\ell =: (\hat{v}^{(1)}_\ell, \hat{v}^{(2)}_\ell, \cdots, \hat{v}^{(6)}_\ell) \in \Theta \). Finally, compute the parameters \( \{\Lambda^{(\mu)}_{i,j,k}\}_{\mu=1}^6 \) by

\[
\Lambda^{(\mu)}_{i,j,k} = \min_{\ell_1,\cdots,\ell_6 \in \{0,1\}} \hat{v}^{(\mu)}_\ell,
\]

which determines the hyperrectangular

\[
\Theta^*_\ell_{i,j,k} = \bigotimes_{\mu=1}^6 [0, \Lambda^{(\mu)}_{i,j,k}].
\]

3. Provably PCP property

We now study in theory the PCP property of the above high-order finite difference scheme.

Based on the computing approach of the parameters \( \{\Lambda^{(\mu)}_{i,j,k}\}_{\ell=1}^6 \) displayed in Step 3.1(a) and Step 3.1(b), one has \( \Theta^*_\ell_{i,j,k} = \Theta \). This implies

\[
W_{i,j,k}(\theta) \in \mathcal{G}_\ell, \quad \text{for all } \theta \in \Theta^*_\ell_{i,j,k}.
\]

From the definition of \( \theta_{ij} \) in Step 3.2, we obtain

\[
0 \leq \theta_{ij}^{(\ell)} \leq \Lambda^{(\ell)}_{i,j,k}, \quad \ell = 1, \cdots, 6,
\]

Thus \( \theta_{ij} \in \Theta^*_i,j,k \), and \( W_{i,j,k}^{n+1} = W_{i,j,k}(\theta_{ij}) \in \mathcal{G}_\ell \). We then immediately draw the following conclusion.

**Theorem 3.** Assume that the numerical fluxes in (34) are taken as the high-order PCP fluxes in (36), with \( \psi_{i,j+\frac{1}{2},k}, \psi_{i,j+\frac{1}{2},k}, \psi_{i,j,k+\frac{1}{2}} \) computed by the proposed parametrized PCP flux limiter. Then, the resulting scheme (35) is PCP under the CFL-type condition (39).

4. Remarks

The scheme (35) is only first-order accurate in time. High-order SSP methods (13) can be used to replace the forward Euler time discretization in (33), to achieve PCP scheme with high-order accuracy in time. If the SSP Runge-Kutta (resp. multi-step) method is employed, the parameterized PCP flux limiter should be used in each Runge-Kutta stage (resp. each time step).

The proposed PCP flux limiter does not depend on what numerical fluxes one uses, that is to say, any high-order finite difference schemes for the GRHD equations in W-form (18) can be modified into PCP schemes by the proposed parameterized PCP flux limiter.

Although the parameterized PCP flux limiter is presented here for finite difference scheme, it is also applicable for high-order finite volume or DG methods to preserve the admissibility of approximate cell-averages.

VI. CONCLUSIONS

The paper designed high-order, physical-constraint-preserving (PCP) methods for the general relativistic hydrodynamic (GRHD) equations with a general equation of state. It was built on the theoretical analysis of the admissible states of GRHD, and two types of PCP limiting procedures enforcing the admissibility of numerical solutions. To overcome the difficulties arising from the strong nonlinearity contained in the physical constraints, an “explicit” equivalent form of the admissible state set, \( \mathcal{G}_\gamma \), was derived, followed by several pivotal properties of \( \mathcal{G}_\gamma \), including the convexity, scaling invariance and Lax-Friedrichs (LxF) splitting property. It was discovered that the sets \( \mathcal{G}_\gamma \) defined at different points in curved spacetime are inequivalent. This invalidated the convexity of \( \mathcal{G}_\gamma \) in analyzing PCP schemes. To solve this problem, we used a linear transformation to map the different \( \mathcal{G}_\gamma \) into a common set \( \mathcal{G}_\alpha \), which is also convex and exactly the admissible state set in special RHD case. We then proposed a new formulation (called W-form) of the GRHD equations to construct provably PCP schemes by taking advantages of the convexity of \( \mathcal{G}_\alpha \). Under discretization on this W-form, the first-order LxF scheme on general unstructured mesh was proved to be PCP, and high-order PCP finite difference, finite volume and DG methods were designed via two types of PCP limiting procedures. It is of particular significance to conduct more validations and investigations on the proposed PCP methods via ultra-relativistic numerical experiments. This is our further work, which may be explored together with computational astrophysicists.

Appendix A: Proofs

1. Proof of Lemma 1

Proof. The proof consists of two parts.

(1). Show that \( U \in \mathcal{G} \implies U \in \mathcal{G}_\gamma \). When \( U = (D, m, E)^T \in \mathcal{G} \) satisfy the constraints \( \rho(U) > 0, p(U) > 0, e(U) > 0 \) and \( 0 \leq e(U) < 1 \), then

\[
D = \frac{\rho}{\sqrt{1 - v^2}} > 0,
\]

\[
E = \frac{\rho h}{1 - v^2} - p > \rho h - p = \rho(1 + e) > 0.
\]

Using (5) gives

\[
E^2 - (D^2 + m_j m_i)
\]

\[
= \frac{1}{1 - v^2} \left( (\rho h - p)^2 - p^2 - p^2 v^2 \right)
\]

\[
> \frac{1}{1 - v^2} \left( (\rho h - p)^2 - p^2 - p^2 \right) \geq 0,
\]

which, along with \( E > 0 \), further yield that

\[
q_\gamma(U) = E - \sqrt{D^2 + m_j m_i} > 0.
\]
Therefore $\mathbf{U} \in \mathcal{G}_\gamma$.

(2). Show that $\mathbf{U} \in \mathcal{G}_\gamma \implies \mathbf{U} \in \mathcal{G}$. Consider the function

$$
\Psi[p](\mathbf{U}) = D h \left( p, \rho[\mathbf{U}](p) \right) \sqrt{1 - \frac{m \mathbf{Y} \mathbf{m}^\top}{(E + p)^2}} + (E + p) \left( \frac{m \mathbf{Y} \mathbf{m}^\top}{(E + p)^2} - 1 \right), \quad p \in [0, +\infty),
$$

which is related to (12). For given $\mathbf{U} \in \mathcal{G}_\gamma$, we have $\Psi[p](\mathbf{U}) \in C^1(\mathbb{R}^+)$ from $\rho[\mathbf{U}](p) \in C^1(\mathbb{R}^+)$ and $e(p, \rho(p)) \in C^1(\mathbb{R}^+ \times \mathbb{R}^+)$. On the other hand, Eqs. (10) and (11) imply

$$
\lim_{p \to 0^+} h \left( p, \rho[\mathbf{U}](p) \right) = 1, \quad \lim_{p \to +\infty} e \left( p, \rho[\mathbf{U}](p) \right) = +\infty,
$$

which further gives

$$
\lim_{p \to 0^+} \Psi[p](\mathbf{U}) = D \sqrt{1 - \frac{m \mathbf{Y} \mathbf{m}^\top}{E^2}} + \frac{m \mathbf{Y} \mathbf{m}^\top}{E} - E = \left( D - \sqrt{E^2 - m \mathbf{Y} \mathbf{m}^\top} \right) \sqrt{1 - \frac{m \mathbf{Y} \mathbf{m}^\top}{E^2}} < 0,
$$

$$
\lim_{p \to +\infty} \Psi[p](\mathbf{U}) = \lim_{p \to +\infty} D \left[ 1 + e \left( p, \rho[\mathbf{U}](p) \right) \right] \times \sqrt{1 - \frac{m \mathbf{Y} \mathbf{m}^\top}{E + p}} + \frac{m \mathbf{Y} \mathbf{m}^\top}{E + p} - E = +\infty.
$$

According to the intermediate value theorem, $\Psi[p](\mathbf{U})$ has at least one positive zero. In other words, there exist at least one positive solution to the algebraic equation $\Psi[p](\mathbf{U}) = 0$ or (12).

We then indirectly show the uniqueness of positive zero of $\Psi[p](\mathbf{U})$ via the proof by contradiction. Assume that $\Psi[p](\mathbf{U})$ has more than one positive zeros, and the smallest two are respectively denoted by $p_1(\mathbf{U})$ and $p_2(\mathbf{U})$ with $p_2(\mathbf{U}) > p_1(\mathbf{U}) > 0$. Then the equivalence between the equation $\Psi[p](\mathbf{U}) = 0$ and (12) leads to

$$
D h \left( p_i, \rho[\mathbf{U}](p_i) \right) = (E + p_i) \sqrt{1 - \frac{m \mathbf{Y} \mathbf{m}^\top}{(E + p_i)^2}}, \quad (A1)
$$

for $i = 1, 2$. It follows from the constraints in (14) that $h \left( p_i, \rho[\mathbf{U}](p_i) \right) > 0$. Combining (9), we further get

$$
\frac{\partial h}{\partial p} \left( p_i, \rho[\mathbf{U}](p_i) \right) > \frac{1}{\rho[\mathbf{U}](p_i)} > 0. \quad (A2)
$$

In the following, we utilize (10), (A1) and (A2), to evaluate the lower-bound of the derivative $\frac{d \Psi[p](\mathbf{U})}{dp}$ at $p_1$ and $p_2$, which can be expressed as

$$
\frac{d \Psi[p](\mathbf{U})}{dp}(p) = D \left\{ \frac{d h}{d p} \left( p, \rho[\mathbf{U}](p) \right) \sqrt{1 - \frac{m \mathbf{Y} \mathbf{m}^\top}{(E + p)^2}} + \frac{D m \mathbf{Y} \mathbf{m}^\top}{(E + p)^3} \right\} \times \frac{\partial h}{\partial p} \left( p, \rho[\mathbf{U}](p) \right) \right\} + \frac{D m \mathbf{Y} \mathbf{m}^\top}{(E + p)^3} h \left( p, \rho[\mathbf{U}](p) \right) \times \left( 1 - \frac{m \mathbf{Y} \mathbf{m}^\top}{(E + p)^2} \right)^{-\frac{1}{2}} - \frac{m \mathbf{Y} \mathbf{m}^\top}{(E + p)^2} - 1.
$$

Specifically, we have

$$
\frac{d \Psi[p](\mathbf{U})}{dp}(p) \geq \left( D \frac{d h}{d p} \left( p, \rho[\mathbf{U}](p) \right) \right) \left( \sqrt{1 - \frac{m \mathbf{Y} \mathbf{m}^\top}{(E + p)^2}} - \frac{D m \mathbf{Y} \mathbf{m}^\top}{(E + p)^3} h \left( p, \rho[\mathbf{U}](p) \right) \right) \times \left( 1 - \frac{m \mathbf{Y} \mathbf{m}^\top}{(E + p)^2} \right)^{-\frac{1}{2}} - \frac{m \mathbf{Y} \mathbf{m}^\top}{(E + p)^2} - 1.
$$

where (A1) is used in the last equality. In conclusion, $\mathbf{U} \in \mathcal{G}$. The proof is completed.
2. Proof of Lemma 2

Proof. Denote $\lambda U' + (1-\lambda)U''$ by $U_\lambda = (D_\lambda, m_\lambda, E_\lambda)^\top$, then

$$D_\lambda = \lambda D' + (1-\lambda)D'' > 0,$$

and

$$q_\gamma(U_\lambda) = q(\Sigma U_\lambda) = q(\lambda \Sigma U' + (1-\lambda)\Sigma U'')$$

$$\geq \lambda q(\Sigma U') + (1-\lambda)q(\Sigma U'') = \lambda q_\gamma(U') + (1-\lambda)q_\gamma(U'') > 0,$$

where $\Sigma = \text{diag}(1, \tilde{Y}_j^2, 1)$, and the concavity of the function $q(U) = E - \sqrt{D^2 + |m|^2}$ is used. This shows $U_\lambda \in \mathcal{G}_\gamma$, and the convexity of $\mathcal{G}_\gamma$. With the fact that $\mathcal{G}_\gamma$ is open, we complete the proof. $\square$

3. Proof of Lemma 3

Proof. The scaling invariance can be directly verified by the definition of $\mathcal{G}_\gamma$. In the following, we prove the LxF splitting property via two steps.

(1). Show that $U \pm \tilde{\xi}_j F^j(U) \in \overline{\mathcal{G}}_\gamma$. We would like to split it as a form of convex combination

$$U \pm \tilde{\xi}_j F^j(U) = \frac{1}{2} \left( \frac{2}{\tilde{\xi}_j} U \right) \pm \frac{1}{2} \tilde{U} \pm,$$

with

$$U \pm = U \pm \tilde{\xi}_j F^j(U),$$

$$\tilde{U} \pm = \frac{2|\xi_j \beta_j| + (\xi_j \beta_j)}{\alpha \tilde{\xi}_j} U,$$

where $\tilde{U} \in \mathcal{G}_\gamma$, due to the scaling invariance,

$$\xi_j F^j(U) = (D\xi_j v^j, (\xi_j v^j)m + p\xi_j (E + p)\xi_j v^j)^\top,$$

and the positive quantity $\tilde{\xi}_j = \sqrt{\xi_j \xi_j}$ for general EOS, while for ideal EOS with sharper $\xi_j$, we have

$$\tilde{\xi}_j = \frac{1}{1 - v^2 c_s^2} \left\{ \xi_j v^j (1 - c_s^2) + c_s W^{-1} \right\} \sqrt{(1 - v^2 c_s^2) (\xi_j \xi_j) (1 - c_s^2)^2 (\xi_j v^j)^2}.$$

With the help of Lemma 2 and the scaling invariance of $\mathcal{G}_\gamma$, the form in \text{A3} indicates that it suffices to show

$$U \in \mathcal{G}_\gamma.$$ (A4)

To this end, we denote $U \pm = (D \pm, m \pm, E \pm)^\top$, then

$$D \pm = D \left( 1 \pm \frac{\xi_j v^j}{\tilde{\xi}_j} \right),$$

$$E \pm = \rho h W^2 \left( 1 \pm \frac{\xi_j v^j}{\tilde{\xi}_j} \right) - p,$$

and

$$(D \pm)^2 + \xi_j v^j m \pm - (E \pm)^2 = \left( 1 \pm \frac{\xi_j v^j}{\tilde{\xi}_j} \right)^2 W^2 \times \left( \rho^2 + p^2 - (\rho h - p)^2 \right) + p^2 \left( \frac{\xi_j \xi_j}{\tilde{\xi}_j^2} - 1 \right).$$ (A6)

With the formulas (A5)–(A6), we shall prove (A4) in the following for two cases separately, i.e., the general EOS case, and the ideal EOS case with sharper $\tilde{\xi}_j$. We will always employ the Cauchy-Schwarz type inequality

$$(\xi_j v^j)^2 = (\xi_j v^j W^\top) = (\xi_j v^j (v^j Y^j)_\top)^2 \leq (\xi_j v^j \xi_j) (v^j Y^j) W^\top = (\xi_j \xi_j v^j W^\top = v^2 (\xi_j \xi_j).$$ (A7)

First consider general EOS. Using (A5–A7) and (5) gives

$$D \pm \geq D \left( 1 - \frac{|\xi_j v^j|}{\sqrt{\xi_j \xi_j}} \right) \geq D(1 - v) > 0,$$

$$E \pm \geq \rho h W^2 \left( 1 - \frac{|\xi_j v^j|}{\sqrt{\xi_j \xi_j}} \right) - p \geq \frac{\rho h}{1 + v} - p$$

$$> \frac{\rho h}{2} - p \geq 1 \left( \sqrt{\rho^2 + p^2} - p \right) > 0,$$

and

$$(D \pm)^2 + \xi_j v^j m \pm - (E \pm)^2 = \left( 1 \pm \frac{\xi_j v^j}{\tilde{\xi}_j} \right)^2 W^2 \left( \rho^2 + p^2 - (\rho h - p)^2 \right) \leq 0,$$

which immediately imply (A4). Then we focus on the ideal EOS case with sharper $\tilde{\xi}_j$. From (A7) and $0 < c_s < 1$, we derive

$$\sqrt{(1 - v^2 c_s^2) (\xi_j \xi_j) - (1 - c_s^2) (\xi_j v^j)^2}$$

$$= \sqrt{\xi_j \xi_j - (\xi_j v^j)^2 - c_s^2 (v^j (\xi_j \xi_j) - (\xi_j v^j)^2)}$$

$$\geq \sqrt{\xi_j \xi_j - (\xi_j v^j)^2 - (v^j (\xi_j \xi_j) - (\xi_j v^j)^2)}$$

$$= W^{-1} \sqrt{\xi_j \xi_j},$$

and further

$$\tilde{\xi}_j \geq \frac{|\xi_j v^j| (1 - c_s^2) + c_s (1 - v^2) \sqrt{\xi_j \xi_j}}{1 - v^2 c_s^2 \xi_j \xi_j}.$$
This implies

\[ 1 - \frac{|\xi_j v^j|}{\hat{\xi}} \geq 1 - \frac{|\xi_j v^j| (1 - v^2 c_s^2)}{|\xi_j v^j|(1 - c_s^2) + c_s (1 - v^2)\sqrt{\xi_j \xi_j}} \]

\[ = \frac{W^{-2} c_s (\sqrt{\xi_j \xi_j} - |\xi_j v^j| c_s)}{|\xi_j v^j|(1 - c_s^2) + c_s (1 - v^2)\sqrt{\xi_j \xi_j}} \]

\[ \geq \frac{W^{-2} c_s \sqrt{\xi_j \xi_j} - |\xi_j v^j| c_s}{c_s + (|\xi_j v^j| c_s)\sqrt{\xi_j \xi_j}} \]

\[ = \frac{W^{-2} c_s}{c_s + v}, \quad (A8) \]

and further gives

\[ 1 \pm \frac{\xi_j v^j}{\hat{\xi}} \geq 1 - \frac{|\xi_j v^j|}{\hat{\xi}} \geq \frac{W^{-2} c_s}{c_s + v} > \frac{W^{-2} c_s}{c_s + 1} > 0. \]

It follows, along with (A8), that \( D^\pm > 0 \) and

\[ E^\pm > \frac{\rho h c_s}{1 + c_s} - \rho = p \left( \frac{\Gamma}{c_s (1 + c_s)} - 1 \right) > p \left( \frac{\Gamma}{1 + \sqrt{1 - \Gamma}} - 1 \right) \geq 0, \]

where \( 0 < c_s \leq \sqrt{1 - \Gamma} \) and \( \Gamma \in (1, 2) \) are used. Note that \( \hat{\xi}_s \) is a positive solution to the following quadratic equation

\[ (1 - v^2 c_s^2)\hat{\xi}_s^2 - 2 |\xi_j v^j|(1 - c_s^2)\hat{\xi}_s + (1 - c_s^2)(\xi_j v^j)^2 - c_s^2 (1 - v^2)(\xi_j \xi_j) = 0, \]

which is equivalent to

\[ (\xi_j \xi_j - \hat{\xi}_s^2) \xi_j \xi_j = W^2 (\hat{\xi}_s - |\xi_j v^j|)^2 (1 - c_s^2). \]

It implies \( \hat{\xi}_s < \sqrt{\xi_j \xi_j} \). Using (A8) - (A9) for (A6) gives

\[ (D^\pm)^2 + \xi_j m^j m^j = (E^\pm)^2 \geq \frac{\xi_j \xi_j}{\hat{\xi}_s^2}^2 (1 - |\xi_j v^j|^2)^2 W^2 \]

\[ \times \left( p^2 + p^2 - \left( p + \frac{p}{\Gamma - 1} \right)^2 \right) + p^2 \left( \xi_j \xi_j \xi_j \xi_j - 1 \right) \]

\[ \geq \frac{(\xi_j \xi_j)}{\hat{\xi}_s^2} - 1 \left( \frac{c_s^2}{1 - c_s^2} \right) \left( p \left( p - \frac{2 p}{\Gamma - 1} \right) - \frac{p^2}{\Gamma - 1} \right) + p^2 \left( \frac{\xi_j \xi_j \xi_j \xi_j}{\hat{\xi}_s^2} - 1 \right) \]

\[ \leq \frac{(\xi_j \xi_j)}{\hat{\xi}_s^2} - 1 \left( \frac{c_s^2}{1 - c_s^2} \right) \left( p \left( p - \frac{2 p}{\Gamma - 1} \right) - \frac{p^2}{\Gamma - 1} \right) \]

\[ \leq \frac{(\xi_j \xi_j)}{\hat{\xi}_s^2} - 1 \left( \frac{c_s^2}{1 - c_s^2} \right) \left( 1 - c_s^2 \frac{1}{\Gamma - 1} + \frac{2 p}{p} \right) \]

\[ = \frac{(\xi_j \xi_j)}{\hat{\xi}_s^2} - 1 \left( \frac{c_s^2}{1 - c_s^2} \right) \left( 1 - 2 \Gamma + \frac{2 p}{p} \right) \]

\[ = \frac{(\xi_j \xi_j)}{\hat{\xi}_s^2} - 1 \left( \frac{c_s^2}{1 - c_s^2} \right) \frac{1 - 2 \Gamma + \frac{2 p}{p}}{h} < 0, \]

where \( \Gamma \geq 2 \) and \( \Gamma < 1 \) are respectively used in the last two inequalities. In conclusion, (A10) holds for ideal EOS case with sharper \( \hat{\xi}_s \).

The proof of part (1) is completed.

(2). For any \( \eta > \eta_0 \), we have

\[ \mathbf{U} \pm \eta^{-1} \xi_j \mathbf{F}^j (\mathbf{U}) = \left( 1 - \frac{\eta}{\eta} \right) \mathbf{U} + \frac{\eta}{\eta} \left( \mathbf{U} \pm \eta^{-1} \xi_j \mathbf{F}^j (\mathbf{U}) \right). \]

It follows from Lemma 2 and the deduction proved in part (1) that \( \mathbf{U} \pm \eta^{-1} \xi_j \mathbf{F}^j (\mathbf{U}) \in \mathcal{G}_s \).

The proof is completed. Hence, we introduce a remark on Lemma 4. For general EOS, one can also choose

\[ \eta_0 = \frac{\xi_j v^j}{\sqrt{(s + 1)|\xi_j \xi_j| - \xi_j v^j (\xi_j v^j)^2}} + \frac{|\xi_j \xi_j|}{2}, \]

to establish the LxF splitting property in Lemma 4 where \( s = ((\rho h - p)^2 - \rho^2 - p^2)W^2/p^2 \geq 0 \). This choice of \( \hat{\xi}_s \) is smaller/sharper than that in (13), but generally not an upper bound of the spectral radius of \( \partial (\xi_j \mathbf{F}^j (\mathbf{U})) / \partial \mathbf{U} \).

4. Proof of Theorem 1

Before proving Theorem 1, we first introduce a lemma.

Lemma 7. If \( \mathbf{W}_k \in \mathcal{G}_s \) for all \( k \), then for any \( \delta_i \) satisfying

\[ 0 < \max_k \frac{\delta_i}{2 |E_k|} \sum_{j \in N_k} a_{kj} |E_{kj}| < 1, \]

it holds

\[ \mathbf{W}^n = \mathbf{W}_k - \frac{\delta_i}{2 |E_k|} \sum_{j \in N_k} |E_{kj}| \mathbf{H}_{kj} \in \mathcal{G}_s. \]

Proof. Using the identity

\[ \sum_{j \in N_k} |E_{kj}| (\xi_{kj} \cdot \mathbf{Z}) = \int_{E_k} \frac{\partial Z^i}{\partial x^i} d\mathbf{x} \equiv 0, \]

for any constant vector \( \mathbf{Z} = (Z^1, Z^2, Z^3) \), we reformulate \( \mathbf{W}^n_k \) as

\[ \mathbf{W}^n_k = \left( 1 - \frac{\delta_i}{2 |E_k|} \sum_{j \in N_k} a_{kj} |E_{kj}| \right) \mathbf{W}_k \]

\[ + \frac{\delta_i}{2 |E_k|} \sum_{j \in N_k} a_{kj} |E_{kj}| \mathbf{H}_{kj}, \]

with

\[ \mathbf{H}_{kj} = \mathbf{W}^j_k - a_{kj}^{-1} \xi_{kj} \mathbf{F}^j (\mathbf{W}^j_k). \]

Thanks to Lemma 5 and the condition (21), one has \( \mathbf{H}_{kj} \in \mathcal{G}_s \). Thus the form (A11) is a convex combination under the condition (A10). The proof is completed by Lemma 4. Hence, we introduce a remark on Lemma 4. For general EOS, one can also choose

\[ \eta_0 = \frac{\xi_j v^j}{\sqrt{(s + 1)|\xi_j \xi_j| - \xi_j v^j (\xi_j v^j)^2}} + \frac{|\xi_j \xi_j|}{2}, \]

to establish the LxF splitting property in Lemma 4 where \( s = ((\rho h - p)^2 - \rho^2 - p^2)W^2/p^2 \geq 0 \).
Proof. Here the induction argument is used for time level number \( n \). Assume \( \mathcal{W}_k^n \in \mathcal{G}_s \) for all \( k \), then we prove that \( \mathcal{W}_k^{n+1} \) computed by (20) also belongs to \( \mathcal{G}_s \). The scheme (20) can be rewritten as

\[
\mathcal{W}_k^{n+1} = \vartheta \left( \mathcal{W}_k^n - \frac{\delta_t}{L_k} \sum_{j \in N_k} |\mathcal{E}_{kj}| \mathcal{H}_{kj} \right) + \left( 1 - \vartheta \right) \mathcal{W}_k^n + \Delta t_n \mathcal{S}(\mathcal{W}_k^n) = : \vartheta \Xi_H + \Xi_S,
\]

where \( \delta_t = \Delta t_n / \vartheta \), and

\[
\vartheta = \frac{\sum_{j \in N_k} \lambda_{kj} |\mathcal{E}_{kj}|}{\sum_{j \in N_k} \lambda_{kj} |\mathcal{E}_{kj}| + \lambda_S(\mathcal{W}_k^n)} \in (0,1).
\]

Under the condition (22), we know that \( \delta_t \) satisfies (A10) and thus have \( \Xi_H \in \mathcal{G}_s \) by Lemma 7. We then show \( \Xi_S \in \mathcal{G}_s \) as follows.

- If \( q(S(\mathcal{W}_k^n)) \geq 0 \), then \( \lambda_S = 0 \) and \( \vartheta = 1 \), which yields \( \Xi_S = \Delta t_n S(\mathcal{W}_k^n) \). The first component of \( \Xi_S \) is zero, and \( q(\Xi_S) = \Delta t_n q(S(\mathcal{W}_k^n)) \geq 0 \). Hence \( \Xi_S \in \mathcal{G}_s \).

- If \( q(S(\mathcal{W}_k^n)) < 0 \), then \( S(\mathcal{W}_k^n) \notin \mathcal{G}_s \). Thanks to the convexity of \( \mathcal{G}_s \), Eq. (23) has unique positive solution \( \lambda_S \). This implies \( \vartheta \in (0,1) \), and

\[
\mathcal{W}_k^n + \lambda_S(\mathcal{W}_k^n) \in \mathcal{G}_s, \quad \text{for any} \lambda \in [0, \lambda_S^{-1}).
\]

Under the condition (23), \( \Delta t_n (1-\vartheta) < \lambda_S^{-1} \). It follows from the scaling invariance of \( \mathcal{G}_s \) that

\[
\Xi_S = (1 - \vartheta) \left( \mathcal{W}_k^n + \frac{\Delta t_n}{1-\vartheta} S(\mathcal{W}_k^n) \right) \in \mathcal{G}_s \subset \mathcal{G}_s.
\]

Thanks to the scaling invariance, the above deductions imply \( 2\vartheta \Xi_S \in \mathcal{G}_s \), and \( 2\Xi_S \in \mathcal{G}_s \). With Lemma 4 we then have \( \mathcal{W}_k^{n+1} = \frac{1}{2} \cdot 2\vartheta \Xi_H + \frac{1}{2} \cdot 2\Xi_S \in \mathcal{G}_s \). \( \square \)

5. Proof of Theorem 2

Proof. Here the induction argument is used for time level number \( n \). Assume \( \mathcal{W}_{i,j,k}^n \in \mathcal{G}_s \) for all \( i,j,k \), we then show that \( \mathcal{W}_{i,j,k}^{n+1} \) computed by (20) also belongs to \( \mathcal{G}_s \). Define

\[ L_{ijk}^H(\mathcal{W}(x)) := L_{ijk}(\mathcal{W}(x)) - \mathcal{S}_{ijk}, \]

\[ \vartheta := (\Delta t^{-1} a_s^H) / (\Delta t^{-1} a_s^H + \omega_1 \lambda_S) \in (0,1], \]

then

\[ \mathcal{W}_{i,j,k}^{n+1} = \vartheta \Xi_H + \Xi_S, \]

with

\[ \Xi_H = \mathcal{W}_{i,j,k}^n + \vartheta^{-1} \Delta t_n L_{ijk}^H(\mathcal{W}(x)), \]

\[ \Xi_S = (1 - \vartheta) \mathcal{W}_{i,j,k}^n + \Delta t_n \mathcal{S}_{ijk}, \]

The proof of \( \mathcal{W}_{i,j,k}^{n+1} \in \mathcal{G}_s \) is divided into three parts.

(1) First prove \( \Xi_H \in \mathcal{G}_s \). This part will always employ Einstein’s summation convention for indices \( \mu \) and \( \nu \) running from 0 to \( q \). The exactness of the L-point Gauss-Lobatto quadrature rule and the q-point Gauss quadrature rule yields

\[ \mathcal{W}_{i,j,k} = \frac{1}{\Delta_t a_s^H} \int_{I_{ijk}} \mathcal{W}_{i,j,k}(x,y,z) \, dx \, dy \, dz = \sum_{\delta=1}^L \omega_\delta \omega_\mu \omega_\nu \mathcal{W}_{i,j,k}(x^\delta_\mu,y^\delta_\nu,z^\delta_\nu), \]

\[ = \sum_{\delta=1}^{l-1} \omega_\delta \omega_\mu \omega_\nu \mathcal{W}_{i,j,k}(x^\delta_\mu,y^\delta_\nu,z^\delta_\nu), \]

\[ + \omega_1 \omega_\mu \omega_\nu \left( \mathcal{W}_{i-1,j,k}^{1+\mu,\nu} + \mathcal{W}_{i+1,j,k}^{1+\mu,\nu} \right), \] (A12)

where \( \omega_1 = \omega_L \) has been used. Similarly, we have

\[ \mathcal{W}_{i,j,k} = \sum_{\delta=2}^{l-1} \omega_\delta \omega_\mu \omega_\nu \mathcal{W}_{i,j,k}(x^\delta_\mu,y^\delta_\nu,z^\delta_\nu), \]

\[ + \omega_1 \omega_\mu \omega_\nu \left( \mathcal{W}_{i,j,k}^{1+\mu,\nu} + \mathcal{W}_{i,j,k}^{-1+\mu,\nu} \right), \] (A13)

\[ \mathcal{W}_{i,j,k} = \sum_{\delta=2}^{l-1} \omega_\delta \omega_\mu \omega_\nu \mathcal{W}_{i,j,k}(x^\delta_\mu,y^\delta_\nu,z^\delta_\nu), \]

\[ \mathcal{W}_{i,j,k}^{1+\mu,\nu} + \mathcal{W}_{i,j,k}^{-1+\mu,\nu}. \] (A14)

Taking a weighted average of Eqs. (A12)–(A14) gives

\[ \mathcal{W}_{i,j,k} = \frac{1}{\Delta t^{-1} a_s^H} \left( \Delta t^{-1} a_s^H \times \mathcal{W}_{i,j,k} \right) \times \mathcal{X}_H \]

\[ + \Delta t^{-1} a_s^H \times \mathcal{X}_S \]

\[ = \left( 1 - 2 \omega_1 \right) \Pi, + \frac{1}{\Delta t} \omega_\delta \omega_\mu \omega_\nu \left( \Delta t^{-1} a_s^H \mathcal{W}_{i,j,k} \right) \]

\[ + \Delta t^{-1} a_s^H \left( \mathcal{W}_{i,j,k}^{1+\mu,\nu} + \mathcal{W}_{i,j,k}^{-1+\mu,\nu} \right), \]

\[ + \Delta t^{-1} a_s^H \left( \mathcal{W}_{i,j,k}^{1+\mu,\nu} + \mathcal{W}_{i,j,k}^{-1+\mu,\nu} \right), \]

with \( \Pi \), defined by the convex combination

\[ \Pi = \frac{1}{1 - 2 \omega_1} \sum_{\delta=2}^{l-1} \left\{ \omega_\delta \times \omega_\mu \omega_\nu \left( \Delta t^{-1} a_s^H \mathcal{W}_{i,j,k} \right) \times \mathcal{X}_H \right\} \]

\[ + \Delta t^{-1} a_s^H \left( \mathcal{W}_{i,j,k}^{1+\mu,\nu} + \mathcal{W}_{i,j,k}^{-1+\mu,\nu} \right), \]

\[ + \Delta t^{-1} a_s^H \left( \mathcal{W}_{i,j,k}^{1+\mu,\nu} + \mathcal{W}_{i,j,k}^{-1+\mu,\nu} \right), \]

which belongs to \( \mathcal{G}_s \) by the hypothesis and the convexity of \( \mathcal{G}_s \). Furthermore, \( \Xi_H \) can be reformulated as

\[ \Xi_H = (1 - 2 \omega_1) \Pi, + 2 \omega_1 \Pi, \] (A15)
where
\[ \widehat{\Pi} = \frac{\omega_\mu \omega_\nu (\Delta_1^{-1} a_1^{(1)} \Pi_1^{\mu\nu} + \Delta_2^{-1} a_2^{(2)} \Pi_2^{\mu\nu} + \Delta_3^{-1} a_3^{(3)} \Pi_3^{\mu\nu})}{\Delta_1^{-1} a_1^{(1)}}, \]
with
\[ \Pi_1^{\mu\nu} = \frac{1}{2} \left( W_{i-\frac{1}{2},j,k}^{\mu,\nu} + W_{i+\frac{1}{2},j,k}^{\mu,\nu} \right) + \frac{\varpi}{2 a_1^{(1)}} \]
\[ \times \left( \tilde{H}^1 \left( W_{i-j-\frac{1}{2},k}^{\mu,\nu}, W_{i-j+\frac{1}{2},k}^{\mu,\nu} \right) - \tilde{H}^1 \left( W_{i-j-\frac{1}{2},k}^{\mu,\nu}, W_{i-j+\frac{1}{2},k}^{\mu,\nu} \right) \right), \]
\[ \Pi_2^{\mu\nu} = \frac{1}{2} \left( W_{i-\frac{1}{2},j-\frac{1}{2},k}^{\mu,\nu} + W_{i+\frac{1}{2},j-\frac{1}{2},k}^{\mu,\nu} \right) + \frac{\varpi}{2 a_2^{(1)}} \]
\[ \times \left( \tilde{H}^2 \left( W_{i-j-\frac{1}{2},k}^{\mu,\nu}, W_{i-j+\frac{1}{2},k}^{\mu,\nu} \right) - \tilde{H}^2 \left( W_{i-j-\frac{1}{2},k}^{\mu,\nu}, W_{i-j+\frac{1}{2},k}^{\mu,\nu} \right) \right), \]
\[ \Pi_3^{\mu\nu} = \frac{1}{2} \left( W_{i-\frac{1}{2},j-\frac{1}{2},k}^{\mu,\nu} + W_{i+\frac{1}{2},j-\frac{1}{2},k}^{\mu,\nu} \right) + \frac{\varpi}{2 a_3^{(1)}} \]
\[ \times \left( \tilde{H}^1 \left( W_{i-j-\frac{1}{2},k}^{\mu,\nu}, W_{i-j+\frac{1}{2},k}^{\mu,\nu} \right) - \tilde{H}^1 \left( W_{i-j-\frac{1}{2},k}^{\mu,\nu}, W_{i-j+\frac{1}{2},k}^{\mu,\nu} \right) \right), \]
and \( \varpi = \Delta t_n \left( \Delta_1^{-1} a_1^{(1)} + \omega_1 \lambda_S / \omega_1 \right) \in (0, 1) \) under the condition \([iii]\). Note that \( \Pi_1^{\mu\nu} \) can be rewritten as
\[ \Pi_1^{\mu\nu} = \left( 1 - \frac{\varpi}{2} \right) \Pi_{1,1}^{\mu\nu} + \frac{\varpi}{2} \Pi_{1,2}^{\mu\nu}, \tag{A16} \]
where
\[ \Pi_{1,1}^{\mu\nu} = \frac{1}{2} \left( W_{i+\frac{1}{2},j,k}^{\mu,\nu} + \varpi^{-1} H^1 \left( W_{i+\frac{1}{2},j,k}^{\mu,\nu} \right) \right) \]
\[ + \frac{1}{2} \left( W_{i-\frac{1}{2},j,k}^{\mu,\nu} - \varpi^{-1} H^1 \left( W_{i-\frac{1}{2},j,k}^{\mu,\nu} \right) \right), \]
\[ \Pi_{1,2}^{\mu\nu} = \frac{1}{2} \left( W_{i-\frac{1}{2},j+\frac{1}{2},k}^{\mu,\nu} + (a_1^{(1)})^{-1} H^1 \left( W_{i-\frac{1}{2},j+\frac{1}{2},k}^{\mu,\nu} \right) \right) \]
\[ + \frac{1}{2} \left( W_{i+\frac{1}{2},j+\frac{1}{2},k}^{\mu,\nu} - (a_1^{(1)})^{-1} H^1 \left( W_{i+\frac{1}{2},j+\frac{1}{2},k}^{\mu,\nu} \right) \right), \]
with \( \varpi = \varpi_0 = a^{(1)}_1 > a^{(1)}_2 \). With the help of Lemma \([iii]\) and the convexity of \( G_s \), we have \( \Pi_{1,1}^{\mu\nu} \in G_s \) and \( \Pi_{1,2}^{\mu\nu} \in \bar{G}_s \) from \([25]\). These further imply \( \Pi_1^{\mu\nu} \in G_s \) by Lemma \([iii]\) and \([A16]\). Similar arguments yield \( \Pi_2^{\mu\nu}, \Pi_3^{\mu\nu} \in G_s \). Using the convexity of \( G_s \) again, we obtain \( \Pi_* \in G_s \).

From \([A15]\) and \( \Pi_* \in G_s \), we draw the conclusion \( \Xi_H \in G_s \) based on the convexity of \( G_s \).

(2) Then prove \( \Xi_S \in \bar{G}_s \) by separately considering two cases.

- If \( q(S_{ijk}^n) \geq 0 \), then \( \lambda_S = 0 \) and \( \theta = 1 \), which yield \( \Xi_S = \Delta t_n S_{ijk}^n \). Because the first component of \( \Xi_S \) is zero and \( q(\Xi_S) = \Delta t_n q(S_{ijk}^n) \geq 0 \), we thus have \( \Xi_S \in \bar{G}_s \).
- If \( q(S_{ijk}^n) < 0 \), then \( S_{ijk}^n \notin G_s \). Thanks to the convexity of \( G_s \), Eq. \([30]\) has and only has one positive solution, which is \( \lambda_S > 0 \). This further implies \( W_{ijk}^{n+1} + \lambda S_{ijk}^n \in G_s \) for any \( \lambda \in [0, \lambda_S^{-1}] \). Specially, \( W_{ijk}^n + \omega_1 \lambda_S^{-1} S_{ijk}^n \in G_s \). It follows from the scaling invariance of \( G_s \) that
\[ \Xi_S = -\Delta t_n (a_1^{(1)} + \omega_1 \lambda_S) \in G_s \subset \bar{G}_s. \]

Using the deductions proved in parts (1) and (2), we respectively obtain \( 2\delta \Xi_H \in G_s \) and \( 2\Xi_S \in \bar{G}_s \), based on the scaling invariance of \( G_s \). Thanks to Lemma \([iii]\) it holds that
\[ W_{ijk}^{n+1} = \frac{1}{2} \cdot 2\delta \Xi_H + \frac{1}{2} \cdot 2\Xi_S \in G_s. \]

The proof is completed. \( \square \)

6. Proof of Lemma \([iii]\)

Proof. Eq. \([37]\) implies that \( W_{i,j,k}(\theta) \) is linear with respect to \( \theta \). Hence \( \Theta_0 \) is a convex set. Assume that \( \theta_0, \theta_1 \in \Theta_0 \). Then, for any \( \lambda \in [0, 1] \), we have \( \theta_\lambda = (1 - \lambda) \theta_0 + \lambda \theta_1 \in \Theta_0 \). The concavity of the function \( q(W) \) yields
\[ q(W_{i,j,k}(\theta_\lambda)) = q((1 - \lambda) W_{i,j,k}(\theta_0) + \lambda W_{i,j,k}(\theta_1)) \geq (1 - \lambda) q(W_{i,j,k}(\theta_0)) + \lambda q(W_{i,j,k}(\theta_1)) \geq (1 - \lambda) \epsilon + \lambda \epsilon = \epsilon. \]

If follows that \( \theta_\lambda \in \Theta_0, \forall \lambda \in [0, 1] \). Hence \( \Theta \) is convex. \( \square \)

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