Ergodicity of Random Walks on Random DFA

Borja Balle

bballe@cs.mcgill.ca
Reasoning and Learning Laboratory
School of Computer Science
McGill University

Abstract. Given a DFA we consider the random walk that starts at the initial state and at each time step moves to a new state by taking a random transition from the current state. This paper shows that for typical DFA this random walk induces an ergodic Markov chain. The notion of typical DFA is formalized by showing that ergodicity holds with high probability when a DFA is sampled uniformly at random from the set of all automata with a fixed number of states. We also show the same result applies to DFA obtained by minimizing typical DFA.

1 Introduction

Deterministic finite automata (DFA) is a well-known computational model which has been used in computer science for a long time. In the context of learning theory, DFA’s ability to succinctly represent regular languages makes them an interesting hypothesis class for learning regular languages and other simpler concepts. Unfortunately, it is known that learning DFA under quite general learning models is a formidable problem [1,2,3]. Empirical investigations, however, suggest that most DFA may not be as hard to learn as these worst-case (conditional) lower bounds indicate [4,5]. These seemingly contradicting facts raise the question of whether existing hardness results are excessively pessimistic and, in fact, typical DFA are easy to learn.

A common approach to characterize the nature of typical objects inside a class is to draw elements from the class uniformly at random and study which properties hold with high probability. For finite classes this amounts to showing that a certain property holds for all but a negligible fraction of objects in the class. Approaches of this sort have been recently used for showing that typical decision trees and DNF formulas can be learned from examples drawn uniformly at random from \( \{0,1\}^n \) in polynomial time [6,7]. In contrast, it was recently showed in [8] that random decision trees and random DNF formulas are hard to learn from statistical queries under arbitrary distributions over \( \{0,1\}^n \).

The results in [8] also show that learning random DFA under arbitrary distributions is hard in the statistical query model. However, the question of whether random DFA can be learned under the uniform distribution is a long-standing open problem on which very little progress has been made. The main obstruction for studying the learnability (and other properties) of typical DFA seems to
be our poor understanding of the structural regularities exhibited by DFA constructed at random – it is interesting to note how this contrasts with the vast amount of information known about random undirected graphs [9]. Indeed, just very few results about the structure of random DFA are known; see Section 2.5 for details.

In this paper we provide some new insights about the structure of typical DFA by studying the behavior of random walks on random DFA. We show that with high probability, a random walk starting at the initial state of a randomly constructed DFA and taking transitions at random induces an ergodic Markov chain; that is, the state distribution in the random walk will converge to a stationary distribution. We also show that the same holds if one considers DFA obtained by minimizing randomly generated DFA. This is relevant to the learnability of typical DFA under the uniform distribution because the state reached by each one of these examples corresponds to the state reached by a particular realization of the random walk we just described. Hence, trying to understand the distribution over states reached by uniformly generated examples seems to be a good starting point for understanding how these examples can provide useful information for learning the function computed by a DFA. In addition, since the function computed by a DFA is invariant under minimization, showing that ergodicity is conserved after minimizing the DFA is also important.

The rest of the paper is structured as follows. Section 2 defines our notation and describes previous results. Our first result showing that random walks on random DFA are ergodic is proved in Section 3. Then Section 4 shows that ergodicity of DFA is conserved under minimization. We conclude the paper in Section 5.

2 Preliminaries and Related Work

Given a finite alphabet \( \Sigma \) we use \( \Sigma^* \) to denote the set of all strings over \( \Sigma \). We use \( \lambda \) to denote the empty string and write \( \Sigma^+ = \Sigma^* \setminus \{ \lambda \} \). Given a predicate \( P \) we use \( 1_{P} \) to denote the indicator variable that takes value 1 if \( P \) is true and value 0 otherwise. Given a set \( X \) we write \( \mathcal{P}(X) \) to denote the powerset of \( X \) containing all of its subsets. For any positive integer \( k \) we write \( [k] = \{1, \ldots, k\} \).

2.1 Finite Automata

A non-deterministic finite automaton (NFA) is a tuple \( A = (\Sigma, Q, q_0, \tau, \phi) \), where \( \Sigma \) is a finite alphabet, \( Q \) is a finite set of states, \( q_0 \in Q \) is a distinguished initial state, \( \tau : Q \times \Sigma \rightarrow \mathcal{P}(Q) \) is the transition function, and \( \phi : Q \rightarrow \{0,1\} \) is the termination function. The transition function can be inductively extended to a function \( \tau : Q \times \Sigma^* \rightarrow \mathcal{P}(Q) \) by setting \( \tau(q, \lambda) = \{q\} \) and \( \tau(q, x\sigma) = \bigcup_{q' \in \tau(q, x)} \tau(q', \sigma) \) for all \( q \in Q \), \( x \in \Sigma^* \), and \( \sigma \in \Sigma \). The characteristic function of \( A \) is \( f_A : \Sigma^* \rightarrow \{0,1\} \) defined as \( f_A(x) = \lor_{q \in \tau(q_0, x)} \phi(q) \). The language accepted by \( A \) is the set \( L_A = f_A^{-1}(1) \subseteq \Sigma^* \). For any \( q \in Q \) we denote by \( A_q = (\Sigma, Q, q, \tau, \phi) \) the NFA obtained by letting \( q \) be the initial state of \( A \).
We also define the following extension of \( \tau \) over sets of states and strings. Given \( Q' \subseteq Q \) and \( X \subseteq \Sigma^* \), let
\[
\tau(Q', X) = \bigcup_{q \in Q'} \bigcup_{x \in X} \tau(q, x).
\]

We introduce special notations for the following choices of \( X \): \( \tau_1(Q') = \tau(Q', \Sigma) \) and \( \tau_\ast(Q') = \tau(Q', \Sigma^*) \).

A state \( q' \in Q \) is accessible from another state \( q \in Q \) if there exists a string \( x \in \Sigma^* \) such that \( q' \in \tau(q, x) \); that is, if \( q' \in \tau_\ast(q) \). The set of states \( \tau_\ast(q_0) \) accessible from the initial state are called reachable. If \( q \) is also accessible from \( q' \) then we say that \( q \) and \( q' \) communicate. Communication is an equivalence relation that induces a partition of \( Q \) into communicating classes. A communicating class \( Q' \subseteq Q \) is closed if \( \tau_\ast(Q') = Q' \). Because \( \tau_\ast(Q') \subseteq Q \) for every \( Q' \subseteq Q \), each NFA must have at least one closed communicating class. Let \( k > 1 \). A closed communicating class \( Q' \) is \( k \)-periodic if there exists a partition of \( Q' \) into \( k \) parts \( Q'_0, \ldots, Q'_{k-1} \) such that for any \( 0 \leq i \leq k - 1 \) we have \( \tau_1(Q'_i) = Q'_{i+1 \mod k} \). If a closed communicating class \( Q' \) is not \( k \)-periodic for any \( k > 1 \) then we say it is aperiodic. Every state belonging to a closed communicating class is called recurrent; the rest of states are called transient. We shall sometimes write \( Q = Q^r \cup Q^t \) to denote the partition of \( Q \) into recurrent and transient states. The following is a useful fact about accessibility of recurrent states.

**Fact 1.** For all \( q \in Q \) we have \( \tau_\ast(q) \cap Q^t \neq \emptyset \). In addition, if \( Q^t \) contains a single closed communicating class, then \( \tau_\ast(q) \cap Q^t = Q^t \).

We note that these definitions only depend on the transition structure defined by \( \tau \). In particular, they are independent of the termination function \( \phi \).

Two states \( q, q' \in Q \) are called undistinguishable if \( A_q \) and \( A_{q'} \) define the same language: \( L_A = L_{A'} \). Undistinguishability defines an equivalence relation known as Myhill–Nerode equivalence. An NFA is minimal if no pair of states are indistinguishable; that is, if each Myhill–Nerode equivalence class contains only one state.

A deterministic finite automaton (DFA) is a NFA \( A = (\Sigma, Q, q_0, \tau, \phi) \) such that for every \( q \in Q \) and \( \sigma \in \Sigma \) we have \(|\tau(q, \sigma)| = 1 \). In this case we can – and shall – identify the transition function \( \tau : Q \times \Sigma \to \mathcal{P}(Q) \) with a function of type \( \tau : Q \times \Sigma \to Q \). All definitions made for NFA are also valid for DFA.

We shall use \( n \) to denote the number of states \( |Q| \) and \( r \) to denote the size of the alphabet \( |\Sigma| \) when convenient. In this case we may also identify \( Q \) with \([n]\) and \( \Sigma \) with \([r]\).

### 2.2 State-Merging Operations

Let \( A = (\Sigma, Q, q_0, \tau, \phi) \) and \( \tilde{A} = (\Sigma, \tilde{Q}, \tilde{q}_0, \tilde{\tau}, \tilde{\phi}) \) be two NFA over the same alphabet. We say that \( \tilde{A} \) is obtained from \( A \) by a merge operation if there exists a function \( \Psi : Q \to \tilde{Q} \) satisfying the following:
1. $\Psi$ is exhaustive,
2. $\Psi(q_0) = \tilde{q}_0$,
3. for every $q \in Q$ we have $\phi(q) = \tilde{\phi}(\Psi(q))$,
4. for every $q \in Q$ and $\sigma \in \Sigma$, if $q' \in \tau(q, \sigma)$ then $\Psi(q') \in \tilde{\tau}(\Psi(q), \sigma)$.

If $\tilde{q} \in \tilde{Q}$ is such that $|\Psi^{-1}(\tilde{q})| > 1$, then we say that $\tilde{q}$ is obtained by merging all the states in $\Psi^{-1}(\tilde{q})$. A merge operation is elementary if $|\tilde{Q}| = |Q| - 1$, implying that only two states $q, q' \in Q$ are merged by $\Psi$. In this case the restriction $\Psi|_{Q \setminus \{q, q'\}}$ is a bijection onto $\tilde{Q} \setminus \{\Psi(q)\}$. It is immediate to verify by induction on the length of $x$ that the following holds for any merging operation $\Psi$: for all $q \in Q$ and $x \in \Sigma^*$, $q' \in \tau(q, x)$ implies $\Psi(q') \in \tilde{\tau}(\Psi(q), x)$.

2.3 DFA Minimization

DFA minimization is an operation that starts with a DFA $A$ recognizing a language $L$ and yields a new DFA $A'$ with minimal size among those that recognize $L$. Minimization algorithms for DFA have been extensively studied in the literature, see [10] for a comprehensive review. Here we describe a simple minimization algorithm based on state-merging operations. We will use this algorithm to study how the structure of a DFA is modified by the minimization procedure.

What follows is a high-level description of the algorithm, making special emphasis on the steps that actually modify the structure of the DFA. The algorithm uses a subroutine called $\text{MyhillNerodeClasses}$ to partition the set of states $Q$ into equivalence classes of undistinguishable states. This can be done in several ways – e.g. using Hopcroft’s algorithm [11] based on partition refinement – but the details are not relevant to us. Given a DFA $A$ the algorithm works as follows. First, remove all unreachable states. Second, partition the remaining states into undistinguishable equivalence classes. And third, apply a sequence of elementary merge operations to collapse each set in the partition into a single state. This merging process will start and end with a DFA but may produce an NFA in its intermediate steps. Pseudocode for this minimization algorithm is given in Figure 1.

2.4 Random Walks on Finite Automata

Given an NFA $A$, a random walk on $A$ is a realization of the following Markov chain over the state space $Q$: starting at the initial state $q_0$, for $t \geq 0$ we choose a next state $q_{t+1}$ from $Q$ at random according to a distribution that assigns probability

$$P[q_{t+1} = q | q_t] = \frac{\sum_{\sigma} \mathbb{1}_{q \in \tau(q, \sigma)} | \tau(q, \sigma) |}{\sum_{\sigma} | \tau(q, \sigma) |}$$

to each state $q \in Q$. Note that this corresponds to choosing one of the transitions from $q$ uniformly at random. In the case of a DFA this is equivalent to choosing $\sigma \in \Sigma$ uniformly at random and letting $q_{t+1} = \tau(q_t, \sigma)$.

In order to study the evolution of this random walk it is useful to look at the state distribution of the associated Markov chain. Identifying $Q$ with $[n]$,
we define the *transition matrix* \( P \in \mathbb{R}^{n \times n} \) of the Markov chain associated with \( A \) as \( P(i,j) = \mathbb{P}[q_{t+1} = j \mid q_t = i] \). Note that \( P \) is a row stochastic matrix. A *distribution over states* is a vector \( p \in \mathbb{R}^n \) such that: \( p(i) \geq 0 \) and \( \sum_i p(i) = 1 \). A distribution \( p \) is *stationary* with respect to the Markov chain given by \( P \) if \( pP = p \). If the distribution over states in the Markov chain at time \( t \) is given by \( p_t \), the distribution at time \( t + 1 \) can be computed as \( p_{t+1} = p_tP \). Thus, given an initial state distribution \( p_0 \), the state distribution after \( t \) steps can be computed as \( p_t = p_0P^t \). Note that in the case of a random walk on an NFA the initial distribution corresponds to the indicator vector \( e \) such that \( e(q_0) = 1 \) and \( e(q) = 0 \) for \( q \in Q \setminus \{q_0\} \). A Markov chain is said to be *ergodic* if there exists a stationary distribution \( p \) such that for every initial distribution \( p' \) one has \( \lim_{t \to \infty} p'P^t = p \).

It is well-known that several properties of Markov chains can be characterized in terms of the structure of a directed graph obtained by considering all transitions that occur with positive probability \[12\]. In the case of the Markov chain associated with a random walk on an NFA, this directed graph is the one corresponding to the transition structure given by \( \tau \): it has \( n \) nodes and contains an arc from \( q \) to \( q' \) if and only if there exists \( \sigma \in \Sigma \) such that \( q' \in \tau(q, \sigma) \). Using this point of view, the ergodicity of the Markov chain associated with a random walk on an NFA can be characterized in terms of the structure of its closed communicating classes.

**Theorem 1** \([12]\). *If an NFA \( A \) contains a unique closed communicating class \( Q' \), and \( Q' \) is aperiodic, then the Markov chain associated with the random walk on \( A \) is ergodic.*
2.5 Random DFA

In order to study the properties of typical DFA we need to define a random process for obtaining automata sampled from the uniform distribution over the class of all DFA with a given number of states over a fixed alphabet. Let $\Sigma$ be an alphabet of size $r$ and $Q$ a set of $n$ states. We construct a random DFA over $\Sigma$ and $Q$ as follows. The initial state $q_0$ is chosen uniformly at random from $Q$. For any $q \in Q$ and $\sigma \in \Sigma$ we determine the endpoint of transition $\tau(q, \sigma)$ by drawing a state uniformly at random from $Q$. Finally, for every $q \in Q$ we assign a value to $\phi(q)$ chosen uniformly at random from $\{0, 1\}$. All these random choices are mutually independent. Hereafter we refer to the outcome of this process as a random DFA.

Investigating the structure of random DFA entails identifying properties that hold with high probability with respect to this sampling process. In particular, for a fixed alphabet size, we look for properties that occur almost surely as the number of states in the random DFA grows; that is, properties that hold with probability $1 - o(1)$ when the number of states $n \to \infty$. Noticeably enough, just a few results of this type can be found in the literature. The first examples we are aware of appear in an early book on automata synthesis [13]. Since then, just a few more formal results of this type have been proven, most of them motivated by either learning theory [13], Černý’s conjecture [15], average behavior of DFA minimizations algorithms [16,17,18], or by pure mathematical interest in the structure of random DFA [19,20]. Among these, Grusho’s result was the first to establish an interesting fact about closed communicating classes in random DFA: with high probability they are unique and large.

**Theorem 2 ([19]).** When $n \to \infty$, a random DFA satisfies the following with probability $1 - o(1)$:

1. the DFA contains a single closed communicating class,
2. the size $M$ of this closed communicating class satisfies $|M - cn| \leq f(n)$ for some function $f(n) = o(n)$ and some constant $c$,
3. the constant $c$ above is the positive solution of $c = 1 - e^{-\sigma r}$.

Table 1 displays the approximate value of $c$ for several alphabet sizes. We see that already for small alphabet sizes the communicating class contains almost all states. This result was established by Grusho in the form of a central limit theorem. He also established a similar result for the number of reachable states in a random DFA. Using different techniques, a concentration inequality equivalent to Grusho’s result on the number of reachable states was proved in [20]. This paper also shows that with high probability the number of reachable states in a random DFA is almost the same after miminizing the automaton. Our results provide additional information about the structure of closed communicating classes in random DFA and their minimized versions. In particular, we show that Grusho’s closed communicating class is aperiodic, and that minimizing a DFA with a unique closed and aperiodic communicating class yields a DFA with a unique closed and aperiodic communicating class.
Table 1. Values of $c(r)$ from Theorem 2 truncated to the third digit

| $r$ | 2   | 3   | 4   | 5   | 6   | 7   |
|-----|-----|-----|-----|-----|-----|-----|
| $c$ | 0.796| 0.940| 0.980| 0.993| 0.997| 0.999|

3 Random Walks on Random DFA are Ergodic

In this section we state and prove our first result. It basically states that the closed communicating class identified in Grusho’s theorem is aperiodic. As described in Section 2.4, a direct consequence of this result is that, with high probability, random walks on random DFA induce ergodic Markov chains.

**Theorem 3.** When $n \to \infty$, with probability $1 - o(1)$ a random DFA has a single closed communicating class which is aperiodic and whose size $M$ satisfies $|M - cn| = o(n)$ with $c$ as in Theorem 2.

The main idea of the proof is to bound the probability that a random DFA contains a $k$-periodic closed communicating class for some $k \geq 2$. We begin with two technical lemmas.

**Lemma 1.** For any $s \geq 1$ and $0 \leq x \leq 1$, one has

$$\frac{x^s}{(1-x)^{(1-x)/x}} \leq 1.2.$$ 

**Proof.** Since $x^s \leq x$ for $s \geq 1$ and $0 \leq x \leq 1$, it is enough to consider the case $s = 1$ given by $f(x) = x(1-x)^{1-x}$. We start by showing that $f$ is concave. A rutinary computation shows that

$$f''(x) = \frac{-2x^3 - 2x^2 + (1-x)\ln^2(1-x) - 2x(1-x)\ln(1-x)}{x^3(1-x)^{1/x}},$$

where clearly $h(x) \geq 0$ for $x \in [0,1]$. Furthermore, $g(0) = 0$ and $g'(x) = 6x^2 + 4x\ln(1-x) + \ln^2(1-x)$. Since for $x \geq 0$ we have $4x\ln(1-x) + \ln^2(1-x) \geq -3x^2 - 2x^3$, we see that $g'(x) \geq 3x^2 - 2x^3 \geq 0$ for $x \in [0,1]$. Thus $f''(x) \leq 0$ in $[0,1]$ and $f$ is concave. Now, by concavity of $f$ and monotonicity of degree one polynomials, the following holds for all $x, y \in [0,1]$:

$$f(x) \leq \max\{f(y) + f'(y)(1-y), f(y) + f'(y)(0-y)\}.$$ 

Taking $y = 0.795$ we get $f(x) \leq 1.2$. $\square$

**Lemma 2.** There exists a positive constant $C$ such for any $2 \leq k \leq m \leq n$ and $r \geq 2$ the following holds:

$$\binom{n}{m} \frac{(m-1)}{m} \frac{m!}{(m/k)^r} \left(\frac{m}{kn}\right)^{mr} \leq C \cdot \min\{m^k, 2^m\} \cdot \left(\frac{1.2}{k^{r-1}}\right)^m.$$
Proof. First note that the following bounds can be easily derived from Stirling’s approximation and common bounds for binomial coefficients:

\[
\binom{n}{m} \leq \frac{C \cdot n^m}{m! \cdot e^m \left(\frac{n}{n-m}\right)^{n-m}},
\]

\[
\binom{m-1}{k-1} \leq \min\{m^k, 2^m\},
\]

\[
\frac{1}{\Gamma(m/k)^k} \leq \left(\frac{e \cdot m}{k} + 1\right)^m \leq \left(\frac{ek}{m}\right)^m,
\]

where \(C\) is a positive constant. Combining these bounds in the obvious way one obtains:

\[
\binom{n}{m} \binom{m-1}{k-1} \frac{m!}{\Gamma(m/k)^k} \left(\frac{m}{kn}\right)^{mr}
\]

\[
\leq C \cdot \min\{m^k, 2^m\} \cdot \left(\frac{m}{kn}\right)^{r-1} \left(\frac{n}{n-m}\right)^{(n-m)/m}^m.
\]

Finally, invoking Lemma \[\text{I}\] with \(x = m/n\) we get:

\[
\left(\frac{m}{kn}\right)^{r-1} \left(\frac{n}{n-m}\right)^{(n-m)/m} \leq \frac{1.2}{k^{r-1}}.
\]

\[\square\]

Now let \(m \leq n\) and \(2 \leq k \leq m\). A DFA contains a \(k\)-periodic closed communicating class of size \(m\) if and only if there exists a subset of states \(Q' \subseteq Q\) with \(|Q'| = m\) that can be partitioned into \(k\) disjoint subsets \((Q'_0, \ldots, Q'_{k-1})\) such that \(\tau(Q_i) = Q_{i+1} \mod k\) for all \(0 \leq i < k-1\). We use \(E_{m,k}\) to denote the event that a random DFA contains a \(k\)-periodic closed communicating class of size \(m\). The following lemma bounds the probability of \(E_{m,k}\).

**Lemma 3.** There exists a constant \(\alpha > 0\) such that for any \(m \leq n\) and any \(2 \leq k \leq m\) one has \(\mathbb{P}[E_{m,k}] = O(e^{-\alpha m})\).

**Proof.** Let \(Q' \subseteq Q\) be a fixed subset with \(m\) states and \((Q'_0, \ldots, Q'_{k-1})\) a fixed partition of \(Q'\) into \(k\) parts. Let us write \(m_i = |Q'_i|\). When assigning the transitions of a random DFA, the probability that \(Q'\) is a \(k\)-periodic closed communicating class with this particular partition is at most

\[
\left(\frac{m_1}{n}\right)^{m_0r} \left(\frac{m_2}{n}\right)^{m_1r} \cdots \left(\frac{m_0}{n}\right)^{m_{k-1}r} = \left(\frac{m_0^{m_0} m_1^{m_1} \cdots m_{k-1}^{m_{k-1}}}{n^m}\right)^r.
\]

Note that the function \(f(x_0, \ldots, x_{k-1}) = x_1^{x_0} \cdots x_{k-1}^{x_{k-2}} x_0^{x_{k-1}}\) under the constraints \(x_i > 0\) and \(x_0 + \cdots + x_{k-1} = m\) is maximized for \(x_i = m/k\).

To count the number of partitions of a set of \(m\) states into an ordered tuple of \(k\) sets of states, imagine that we first choose the sizes \((m_0, \ldots, m_{k-1})\) such
that \( m_i > 0 \) and \( m_0 + \cdots + m_{k-1} = m \), and then we choose each \( Q'_i \) of size \( m_i \). Let \( s(m, k) \) denote the number of tuples of sizes \( (m_0, \ldots, m_{k-1}) \) satisfying the conditions. Using \( s(m, 1) = 1 \), \( s(m, m) = 1 \), and \( s(m, k) = \sum_{j=1}^{n-(k-1)} s(m-j, k-1) \), it is easy to show that \( s(m, k) \leq \binom{m}{k-1} \). Furthermore, once the sizes are chosen, the number of ways in which the sets in the partition can be chosen is given by the multinomial coefficient

\[
\binom{m}{m_0, \ldots, m_{k-1}} = \frac{m!}{\Gamma(m_0 + 1) \cdots \Gamma(m_{k-1} + 1)},
\]

which is maximized by the (non-necessarily integer) choice \( m_i = m/k \). Combining the above observations we get

\[
P[E_{m,k}] \leq \binom{n}{m} \binom{m-1}{k-1} \frac{m!}{\Gamma(m/k)^k} \left( \frac{1}{kn} \right)^m,
\]

which by Lemma 2 implies that

\[
P[E_{m,k}] \leq C \cdot \min\{m^k, 2^m\} \cdot \left( \frac{1.2}{k^{n-1}} \right)^m.
\]

Now note that since \( r \geq 2 \), for \( k = 2 \) we have \( P[E_{m,2}] \leq C \cdot m^2 \cdot 0.6^m \), and for \( k \geq 3 \) we have \( P[E_{m,k}] \leq C \cdot 0.8^m \). Therefore we can conclude that for any \( 2 \leq k \leq m \) one has \( P[E_{m,k}] = O(e^{-\alpha m}) \) for some \( \alpha > 0 \). \( \square \)

Now we can use this lemma to give a proof for Theorem 3 using a union bound argument.

**Proof (of Theorem 3).** Let \( E \) denote the event that a random DFA has a periodic closed communicating class and \( E_m \) the event that a random DFA has a periodic closed communicating class of size \( m \). Let \( [n] = U \cup L \) denote a partition of \([n]\) into the sets of unlikely and likely sizes of a closed communicating class of a random DFA. According to Theorem 2 we can take \( L = [cn - f(n), cn + f(n)] \) and \( U = [1, cn - f(n)) \cup (cn + f(n), n] \) where \( f(n) = o(n) \). Note that we have \( |L| = 2f(n) = o(n) \) and every \( m \in L \) satisfies \( m = n(c + o(1)) \). Furthermore, if \( E_U \) denotes the event that a random DFA contains a closed communicating class whose size belongs to \( U \), by Theorem 2 we have \( P[E_U] = o(1) \).

Using these facts we can now conclude that the probability that a random DFA has a periodic closed communicating class is

\[
P[E] \leq P[E_U] + \sum_{m \in L} P[E_m] \leq o(1) + \sum_{m \in L} \sum_{k=2}^m P[E_{m,k}]
\]

\[
\leq o(1) + o(n) \cdot n(c + o(1)) \cdot O(e^{-\alpha n(c+o(1))}) = o(1).
\]

This implies that with probability \( 1 - o(1) \) all closed communicating classes of a random DFA are aperiodic. Since the event in Theorem 2 together with the event that all closed communicating classes are aperiodic imply the event in Theorem 3 we conclude that this last event holds with probability \( 1 - o(1) \). \( \square \)
4 Effect of DFA Minimization on Aperiodic Closed Communicating Classes

Now we present our second result which studies the effect of DFA minimization on the structure of aperiodic closed communicating classes. In particular, we show that minimizing a DFA with a single closed communicating which is also aperiodic yields a DFA with that same property. When combined with Theorem 3 this implies that ergodicity of random walks holds even if one considers minimized versions of randomly generated DFA.

Theorem 4. Let $A$ be a DFA and $\tilde{A}$ the output of the algorithm in Figure on input $A$. If $A$ contains a single closed and aperiodic communicating class, then $\tilde{A}$ also contains a single closed and aperiodic communicating class.

We begin with a simple lemma about merges of NFA containing a single closed communicating class.

Lemma 4. Let $A = (\Sigma, Q_0, r, \phi)$ be an NFA, $\Psi$ a merge operation, and $\tilde{A} = \Psi(A) = (\Sigma, \dot{Q}, \dot{q}_0, \dot{\tau}, \dot{\phi})$. Let $Q^r$ and $\dot{Q}^r$ denote the sets of recurrent states in $A$ and $\tilde{A}$ respectively. If $Q^r$ contains a single closed communicating class, then $\Psi(Q^r) \subseteq \dot{Q}^r$.

Proof. Let $q \in Q^r$ be recurrent in $A$. To show that $\Psi(q)$ is recurrent in $\tilde{A}$ we must show that we have $\Psi(q) \in \tau_\ast(\dot{q})$ for every $\dot{q} \in \tau_\ast(\Psi(q))$. Let $q'$ be an arbitrary state in $\Psi^{-1}(\dot{q})$. Then, since $Q^r$ forms a single communicating class, by Fact we have $q \in \tau_\ast(q')$, which yields $\Psi(q) \in \tau_\ast(\dot{q})$.

The next lemma shows that having no unreachable states is a property of NFA conserved by merge operations.

Lemma 5. Let $A = (\Sigma, Q_0, r, \phi)$ be an NFA and $\Psi$ a merge operation. If $A$ has no unreachable states, then $\Psi(A)$ has no unreachable states.

Proof. Let $\tilde{A} = \Psi(A) = (\Sigma, \dot{Q}, \dot{q}_0, \dot{\tau}, \dot{\phi})$. Let $\dot{q} \in \dot{Q}$ and choose some $q \in \Psi^{-1}(\dot{q})$. By hypothesis we have $q \in \tau_\ast(q_0)$, which implies $\dot{q} \in \tau_\ast(\dot{q}_0)$.

Now we are ready to prove the first half of Theorem 4 that having a single closed communicating class is a property invariant under merge operations.

Lemma 6. Let $A = (\Sigma, Q_0, r, \phi)$ be an NFA with no unaccessible states and $\Psi$ a merge operation. If $A$ contains a single closed communicating class, then $\Psi(A)$ also contains a single closed communicating class.

Proof. Let $\tilde{A} = \Psi(A) = (\Sigma, \dot{Q}, \dot{q}_0, \dot{\tau}, \dot{\phi})$ and write $\dot{Q} = \dot{Q}^t \cup \dot{Q}^r$ for the partition of $\dot{Q}$ into transient and recurrent states. Since every closed communicating class will be contained in $\dot{Q}^r$, it suffices to show that for every pair of states $\dot{q}, \dot{q}' \in \dot{Q}^r$ we have $\dot{q}' \in \tau_\ast(\dot{q})$. Start by choosing arbitrary states $q \in \Psi^{-1}(\dot{q})$ and $q' \in \Psi^{-1}(\dot{q}')$. Because $A$ has a single closed communicating class and no unaccessible states, we must have $Q^r \subseteq \tau_\ast(q) \cap \tau_\ast(q')$ by Fact 1. Now choose an arbitrary $q'' \in Q^r$...
and note that by Lemma 4 we must have $q'' = \Psi(q'') \in \tilde{Q}$. Finally, to build a path from $\tilde{q}$ to $\tilde{q}'$ we observe the following: $q'' \in \tau_\star(\tilde{q})$ because $q'' \in \tau_\star(q)$, and $\tilde{q}' \in \tau_\star(\tilde{q}')$ because $\tilde{q}' \in \tau_\star(\tilde{q}')$ and both are recurrent.

The last ingredient is given by the following lemma which states that aperiodicity is also invariant under merge operations.

**Lemma 7.** Let $A = (\Sigma, Q, q_0, \tau, \phi)$ be an NFA with no unreachable states containing a single closed communicating class and $\Psi$ a merge operation. If the closed communicating class in $A$ is aperiodic, then $\Psi(A)$ contains a single closed communicating class which is also aperiodic.

**Proof.** Let $\tilde{A} = \Psi(A) = (\Sigma, \tilde{Q}, \tilde{q}_0, \tilde{\tau}, \tilde{\phi})$ and write $\tilde{Q} = \tilde{Q}^\text{t} \cup \tilde{Q}^\text{r}$ for the partition of $\tilde{Q}$ into transient and recurrent states. By Lemma 4 we know that $\tilde{Q}^\text{r}$ contains a single closed communicating class. Suppose that $\tilde{Q}^\text{r}$ is $k$-periodic for some $k > 1$. This means there exists a partition $\tilde{Q}_0, \ldots, \tilde{Q}_{k-1}$ of $\tilde{Q}^\text{r}$ such that $\tilde{\tau}_1(\tilde{Q}_i) = \tilde{Q}_{i+1}$ for $0 \leq i \leq k-1$. We claim that then the sets $Q_i = \Psi^{-1}(\tilde{Q}_i) \cap Q^\text{r}$ induce a $k$-periodic partition of the closed communicating class $Q^\text{r}$ of $A$. Note that the sets $Q_i$ are disjoint by construction. In addition, by Lemma 4 we necessarily have $Q^\text{r} = Q_0 \cup \cdots \cup Q_{k-1}$. To prove that this partition is $k$-periodic, we will show that, for any $0 \leq i \leq k-1$, the two inclusions in $\tau_1(Q_i) = Q_{i+1}$ hold. First note that since $Q_i \subseteq Q^\text{r}$, by construction we have $\tau_1(Q_i) \subseteq Q^\text{r}$. Now suppose that for some $q \in Q_i$ and $\sigma \in \Sigma$ there exists a state $q' \in \tau(q, \sigma) \cap Q_j$ with $j \neq i + 1$. Then we have $\Psi(q') \in \tilde{Q}_j$ and $\Psi(q') \in \tilde{\tau}(\Psi(q) \sigma) \subseteq \tilde{Q}_{j+1}$, which is impossible by the choice of $j$ and the assumption that $\tilde{Q}^\text{r}$ is $k$-periodic. Hence, necessarily $\tau_1(Q_i) \subseteq Q_{i+1}$. Now suppose there exists $q \in Q_{i+1} \setminus \tau_1(Q_i)$. Then, since $q$ is recurrent, there must exist $\sigma \in \Sigma$ and $q' \in Q_j$ with $j \neq i$ such that $q \in \tau(q', \sigma)$. In this case, we have $\Psi(q) \in \tilde{Q}_{i+1}$ and $\Psi(q) \in \tilde{\tau}(\Psi(q') \sigma) \subseteq \tilde{Q}_{j+1}$, which again is impossible. Thus, we get $Q_{i+1} \subseteq \tau_1(Q_i)$. $\square$

Theorem 4 now follows immediately from Lemma 7 because $\tilde{A}$ is obtained from $A$ by removing all unreachable states and applying a sequence of merge operations.

## 5 Conclusion

We have shown that random walks on typical DFA will converge to a unique stationary distribution by studying the structure of closed communicating classes in randomly constructed DFA. However, our results do not provide any bounds as to the speed at which this convergence takes place. Usual methods for establishing rapid mixing of Markov chains do not apply in our case because in general a random walk on a DFA is neither lazy nor reversible. As future work we plan to investigate whether a more precise study of the structure of closed communicating classes in random DFA can be used to bound the mixing speed for these random walks.

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1. All subindex calculations throughout this proof are performed modulo $k$. 
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