The large $N$ limit of four dimensional Yang-Mills field coupled to adjoint fermions on a single site lattice

A. Hietanen and R. Narayanan

Department of Physics, Florida International University, Miami, FL 33199, USA
E-mail: rajamani.narayanan@fiu.edu

ABSTRACT: We consider the large $N$ limit of four dimensional $SU(N)$ Yang-Mills field coupled to adjoint fermions on a single site lattice. We use perturbative techniques to show that the $Z_N^4$ center-symmetries are broken with naive fermions but they are not broken with overlap fermions. We use numerical techniques to support this result. Furthermore, we present evidence for a non-zero chiral condensate for one and two Majorana flavors at one value of the lattice gauge coupling.

KEYWORDS: $1/N$ Expansion, Eguchi-Kawai reduction, Adjoint fermions, Lattice Gauge Field Theories.
1. Introduction

Continuum reduction [1] holds in the 't Hooft limit of large $N$ gauge theories in $d > 2$ enabling one to obtain physical results in the infinite volume limit and zero temperature by working on a $l^d$ torus with $l$ of the order of one or two fermi. Reducing $l$ below a certain critical value will force the theory to go into the deconfined phase. This in contrast to the theory in $d = 2$ where one can set $l = 0$ and work on a single site lattice. This is referred to as Eguchi-Kawai reduction [2] and it works in $d = 2$ since the physical theory is in the confined phase for all temperatures.

A weak coupling analysis [3] on a single site lattice shows that the large $N$ limit of $SU(N)$ Yang-Mills theory breaks all $Z_N^d$ symmetries associated with the Polyakov loops if $d > 2$. Fermions in the fundamental representation will not affect this argument but fermions in the adjoint representation can modify the results of the weak coupling analysis. A continuum analysis of the theory with adjoint fermions on $\mathbb{R}^3 \times S^1$ with periodic boundary conditions for fermions in the compact direction shows that the $Z_N$ symmetry is not broken in that direction [4]. An analysis on $S^3 \times S^1$ also shows a region where the $Z_N$ symmetry is not broken [5]. A lattice analysis of the same theory with Wilson fermions indicates
that one can reduce the compact direction to a single site on the lattice and still maintain the $Z_N$ symmetry [6, 7, 8]. 

The natural question that follows is if the $Z_d^d$ symmetries remain unbroken on a single site lattice in $d > 2$ dimensions. Recent numerical analysis with Wilson fermions in $d = 4$ has shown this is most likely the case for a wide range of quark masses [10].

In this paper we address the same question as above in $d = 4$ using naïve fermions and overlap fermions. The reason to study naïve fermions is the apparent absence of doublers on a single site lattice since we only have the zero momentum mode with periodic boundary conditions. But, we will show using weak coupling analysis and numerical analysis that naïve fermions break the $Z_4$ symmetries but overlap fermions do not. The reason for this difference is a certain $Z_2$ symmetry that is broken by naïve fermions.

Fermions play a dynamical role and we opted to use the hybrid Monte Carlo algorithm for generating gauge field configurations. Since we are on a single site lattice, the size of the fermion matrix is $4(N^2 - 1) \times 4(N^2 - 1)$ and we compute the fermionic force exactly. We perform a full inversion of the overlap Dirac matrix using an exact diagonalization of its kernel, the Wilson-Dirac matrix. We present the details of the underlying hybrid Monte Carlo algorithm whose computational cost scales like $N^6$. We will show in this paper that one can use this algorithm to compute physics quantities like the chiral condensate. Detailed physics results will be presented in [11].

2. The single site model

The partition function on a single site lattice is given by

$$Z = \int [dU] e^{-S} = \int [dU] e^{-S_g - S_f},$$

(2.1)

where

$$S_g = -bN \sum_{\mu \neq \nu = 1}^4 \text{Tr} \left[ U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger \right],$$

(2.2)

is the standard Wilson gauge action. With $f$ flavors of Dirac fermions, the contribution to the action from fermions is

$$S_{n,o} = -f \log \det H_{n,o},$$

(2.3)

where $\gamma_5 H_{n,o}$ is the naïve (n) or overlap (o) Dirac operator. The $N \times N$ matrices $U_\mu$, $\mu = 1, 2, 3, 4$ belong to $SU(N)$. The lattice gauge coupling constant is $b = \frac{1}{g^2 N}$. The fermions are in the adjoint representation and they couple to the gauge fields, $V_\mu$, given by

$$V_{\mu}^{ab} = \frac{1}{2} \text{Tr} \left[ T^a U_\mu T^b U_\mu^\dagger \right],$$

(2.4)

where $T^a$, $a = 1, \ldots, (N^2 - 1)$, are traceless hermitian matrices that generate the $su(N)$ lie algebra. We have chosen the generators, $T^a$, to satisfy

$$\text{Tr} T^a T^b = 2 \delta^{ab}.$$  

(2.5)
The fully anti-symmetric structure constants are defined using

\[ [T^a, T^b] = \sum_c i f_{ab}^c T^c. \]  

(2.6)

Both \( H_n \) and \( H_o \) are \( 4(N^2 - 1) \times 4(N^2 - 1) \) hermitian matrices and correspond to naïve Dirac fermions and overlap Dirac fermions respectively. The determinant of \( H_{n,o} \) is positive definite and therefore the logarithm is well defined. Furthermore, we will show in the following two subsections that there exists a hermitian matrix \( \Sigma \) such that

\[ \Sigma H_{n,o} \Sigma = H_{n,o}^*, \]  

(2.7)

which implies that all eigenvalues of \( H_{n,o} \) are doubly degenerate reflecting the adjoint nature of the fermions. In addition, both naïve and overlap fermions obey chiral symmetry and therefore the eigenvalues of \( H_{n,o} \) will come in \( \pm \) pairs. Therefore the factor, \( f \), in front of \( S_f \) can be an integer (single Dirac flavor) or half-integer (single Majorana flavor) for all values of \( N \).\(^2\) Since \( f \) appears only as a multiplicative factor in the fermion action, we may take the number of fermions flavors, \( f \), to be any real number and we will do so in this paper.

2.1 Naïve fermions

The matrix \( H_n \) is given by

\[ H_n = \begin{pmatrix} \mu & C \\ C^{\dagger} & -\mu \end{pmatrix} \]  

(2.8)

where \( \mu \) is the fermion mass. The chiral matrix, \( C \), is

\[ C = \frac{1}{2} \sum_\mu \sigma_\mu \left( V_\mu - V_\mu^{t} \right); \]  

(2.9)

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \sigma_4 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}. \]  

(2.10)

Note that

\[ \sigma_2 \sigma_\mu^* \sigma_2 = -\sigma_\mu. \]  

(2.11)

Since \( V_\mu \) are real matrices, it follows from (2.11) that

\[ C^* = -\sigma_2 C \sigma_2. \]  

(2.12)

If we define

\[ \Sigma = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}; \quad \Sigma^\dagger = \Sigma; \quad \Sigma^2 = 1, \]  

(2.13)

then (2.7) follows from (2.12).

\(^2\)Note that one should have written \( \gamma_5 H_{n,o} \) in (2.3) but this is the same as writing \( H_{n,o} \) as long as \( f \) is an integer multiple of \( \frac{1}{2} \).
2.2 Overlap fermions

The hermitian Wilson Dirac operator is given by

\[
H = \begin{pmatrix}
4 - m - \frac{1}{2} \sum_{\mu} (V_{\mu} + V_{\mu}^{\dagger}) & \frac{1}{2} \sum_{\mu} \sigma_{\mu} (V_{\mu} - V_{\mu}^{\dagger}) \\
-\frac{1}{2} \sum_{\mu} \sigma_{\mu}^{\dagger} (V_{\mu} - V_{\mu}^{\dagger}) & -4 + m + \frac{1}{2} \sum_{\mu} (V_{\mu} + V_{\mu}^{\dagger})
\end{pmatrix}
\]

with \( m \) being the Wilson mass parameter and

\[
w_{\mu} = \frac{1}{2} \begin{pmatrix} 1 & -\sigma_{\mu} \\ \sigma_{\mu}^{\dagger} & -1 \end{pmatrix}.
\]

Using the definition of \( \Sigma \) from (2.13), we have

\[
\Sigma w_{\mu} \Sigma = w_{\mu}^{\ast}, \quad \Sigma \gamma_{5} = \gamma_{5},
\]

and therefore it follows from (2.14) that

\[
\Sigma H \Sigma = H^{\ast}.
\]

The hermitian massive overlap Dirac matrix, \( H_{0} \), is defined by

\[
H_{0} = \frac{1}{2} \left[ (1 + \mu) \gamma_{5} + (1 - \mu) \epsilon (H) \right],
\]

where \( \mu \in [0,1] \) is the bare mass. (2.7) for overlap fermions follows from (2.17).

2.3 Fermion boundary conditions

We have assumed periodic boundary conditions for fermions in the previous two subsections. Other choices of boundary conditions that do not generate a \( U(1) \) anomaly amount to replacing \( V_{\mu} \) by \( V_{\mu} e^{i 2\pi k_{\mu}/N} \) with integer valued \( k_{\mu} \) [12]. Physical results are expected to depend on the choice of boundary conditions. This is in contrast to the case of large \( N \) gauge theories coupled to fundamental fermions. In that case, the \( e^{i 2\pi k_{\mu}/N} \) factor can be absorbed by a change of gauge fields that only changes the Polyakov loop and not the action. If the \( Z_{N} \) symmetries are not broken as is the case in the confined phase, this change will not affect physical results.

3. Weak coupling analysis

Our aim is to study whether the \( Z_{N}^{4} \) symmetries are broken in the weak coupling limit. We follow [3] and perform the weak coupling analysis by decomposing \( U_{\mu} \) according to

\[
U_{\mu} = e^{ia_{\mu}} D_{\mu} e^{-ia_{\mu}}; \quad D_{\mu}^{ij} = e^{i\theta_{\mu}^{ij}} \delta_{ij}.
\]

Keeping \( \theta_{\mu}^{ij} \) fixed, we expand in powers of \( a_{\mu} \). The lowest contribution to \( S_{g} \) comes from the quadratic term in \( a_{\mu} \) [3] and the lowest contribution to \( S_{f} \) comes from setting \( a_{\mu} = 0 \). Each \( V_{\mu} \) has \( \frac{N(N-1)}{2} \) two by two blocks of the form

\[
\begin{pmatrix}
\cos(\theta_{\mu}^{ij} - \theta_{\mu}^{ij}) & \sin(\theta_{\mu}^{ij} - \theta_{\mu}^{ij}) \\
-\sin(\theta_{\mu}^{ij} - \theta_{\mu}^{ij}) & \cos(\theta_{\mu}^{ij} - \theta_{\mu}^{ij})
\end{pmatrix}
\]

with $1 \leq i < j \leq N$. The remaining $(N-1) \times (N-1)$ matrix is a unit matrix. Therefore, the gauge field effectively has $(N-1)$ zero momentum modes and $N(N-1)$ non-zero momentum modes of the form $e^{i(\theta^i_\mu - \theta^j_\mu)}$ with $1 \leq i \neq j \leq N$.

The computation of the fermion determinant reduces to a free field calculation at this order and the result is

$$S_{n,o} = -4 f \sum_{i \neq j} \ln \lambda_{n,o}(\theta^i - \theta^j + \phi) - 4(N-1) f \ln \lambda_{n,o}(\phi)$$

(3.3)

where $e^{i\phi_\mu}$, $\phi_\mu = \frac{2\pi k_\mu}{N}$, is the phase associated with the boundary condition in the $\mu$ direction. The eigenvalues, $\pm \lambda(p)$, are two fold degenerate and the explicit expressions are

$$\lambda_n(p) = \sqrt{\mu^2 + \sum_\mu \sin^2 p_\mu}$$

(3.4)

for naive fermions and

$$\lambda_o(p) = \sqrt{\frac{1 + \mu^2}{2} + \frac{1 - \mu^2}{2} \frac{2 \sum_\mu \sin^2 p_\mu - m}{\left(2 \sum_\mu \sin^2 \frac{p_\mu}{2} - m\right)^2 + \sum_\mu \sin^2 p_\mu}}$$

(3.5)

for overlap fermions. The complete result from fermions and gauge fields is

$$S = \sum_{i \neq j} \left\{ \ln \left[ \sum_\mu \sin^2 \frac{1}{2} (\theta^i_\mu - \theta^j_\mu) \right] - 4 f \ln \lambda_{n,o}(\theta^i - \theta^j + \phi) \right\} - 4(N-1) f \ln \lambda_{n,o}(\phi).$$

(3.6)

Independent of the actual values of $\theta^i_\mu$, the fermion eigenvalues will have $(N-1)$ zero modes with periodic boundary conditions when the mass is set to zero. If all the $\theta^i_\mu$ are different for each $\mu$, then the fermions should not have exact zero modes when we set $p_\mu$ equal to $(\theta^i_\mu - \theta^j_\mu)$ with $i \neq j$. If the fermion spectrum has more than $(N-1)$ zero modes, we will refer to the extra ones as doubler zero modes.

In order to find the minimum of $S$, we consider the Hamiltonian

$$H = \frac{1}{2} \sum_{\mu,i} (\pi^i_\mu)^2 + \beta S.$$ 

(3.7)

For large $\beta$, the Boltzmann measure $e^{-H}$ will be dominated by the minimum of $S$. We can perform a HMC update of the $\pi, \theta$ system to find this minimum. The equations of motion for our $H$ are

$$\dot{\theta}^i_\mu = \pi^i_\mu$$

(3.8)

and

$$\dot{\pi}^i_\mu = -\beta \frac{\partial S}{\partial \theta^i_\mu} = -\beta \left( F^i_\mu - 2 f F^i_{n,o} \right).$$

(3.9)

The gauge contribution to the force is

$$F^i_\mu = \sum_{j \neq i} \frac{\sin \left( \theta^i_\mu - \theta^j_\mu \right)}{\sum_\nu \sin^2 \frac{1}{2} \left( \theta^i_\nu - \theta^j_\nu \right)}.$$ 

(3.10)
The naïve fermion contribution to the force is

\[ F_n^i = \sum_{a=\pm} \sum_{j \neq i} \sin \left[ \frac{2p_{\mu a}^{ij}}{\mu^2 + c_{\mu}^i} \right], \quad (3.11) \]

and the overlap fermion contribution to the force is

\[ F_o^i = \frac{1 - \mu^2}{2} \sum_{a=\pm} \sum_{j \neq i} \sin \frac{p_{\mu a}^{ij} c_{\mu}^i}{\frac{1}{2} \left( d_{\mu}^j \right)^{3/2} + \frac{1 - \mu^2}{2} b_{\mu}^j d_{\mu}^j}, \quad (3.12) \]

where

\[ c_{\mu}^i = \sum_{\nu} \sin^2 \frac{1}{2} \left( \theta_{\mu}^i - \theta_{\mu}^j \right) \]

\[ b_{\mu}^i = 2 \sum_{\nu} \sin^2 \frac{1}{2} \left( \theta_{\mu}^i - \theta_{\mu}^j \right) - \frac{1}{2} \sum_{\nu} \sin^2 \frac{1}{2} \left( \theta_{\mu}^i - \theta_{\mu}^j \right) \]

\[ d_{\mu}^j = \left[ b_{\mu}^j \right]^2 + c_{\mu}^j \]

\[ p_{\mu}^{ij} = \theta_{\mu}^i - \theta_{\mu}^j \pm \phi_{\mu}. \quad (3.13) \]

It is clear from (3.9–3.12) that

\[ \sum_i \pi_{\mu}^i = 0. \quad (3.14) \]

If we start with

\[ \sum_i \pi_{\mu}^i = 0, \quad (3.15) \]

then it will remain zero. Then it follows from (3.8) that

\[ \sum_i \dot{\theta}_{\mu}^i = 0. \quad (3.16) \]

If we start with

\[ \sum_i \theta_{\mu}^i = 0, \quad (3.17) \]

then it will remain zero. These are just the conditions for remaining in SU(N).

A choice for the order parameters associated with the \( Z_N^4 \) symmetries is [3]

\[ P_{\mu} = \frac{1}{2} \left( 1 - \frac{1}{N^2} |\text{Tr} U_{\mu}|^2 \right) = \frac{1}{N^2} \sum_{i,j} \sin^2 \frac{1}{2} \left( \theta_{\mu}^i - \theta_{\mu}^j \right) \quad (3.18) \]

If \( P_{\mu} = \frac{1}{2} \), then the \( Z_N \) symmetry in that direction is not broken. If \( \theta_{\mu}^i \) are uniformly distributed in a width \( \alpha \leq 2\pi \), then

\[ \lim_{N \to \infty} P_{\mu} = \frac{1}{2} \left[ 1 - \left( \frac{2}{\alpha} \sin \frac{\alpha}{2} \right)^2 \right]. \quad (3.19) \]
3.1 Naïve fermions break the $Z_N^4$ symmetries

We assume periodic boundary conditions and set $\phi_\mu = 0$ in (3.6). We pick one value of $N$ and $\beta$ and calculate $P_\mu$ as a function of $f$. In order to clearly see symmetry breaking we use rotational symmetry on the lattice and choose to label our directions such that $P_1 < P_2 < P_3 < P_4$ for each configuration in our thermalized ensemble. We set $\mu = 0.01$ to avoid potential singularities that could occur for the massless case. The plot for three different choices of $(\beta, N)$ is shown in Fig. 1. The value is well below $\frac{1}{2}$ and it seems to approach a value for large $f$ that is consistent with $\alpha = \pi$ in (3.19). Since all three choices for $(\beta, N)$ give consistent values, we have plotted the results for $\beta = 4$ and $N = 23$ in Fig. 2. The small deviations one sees between the four different $P_\mu$ are simply due to our ordering scheme combined with finite $N$ effects.

With $N = 23$, we expect 22 exact zero modes for $\lambda(p)$ as explained in section 3. We plot the average of the smallest eigenvalue, $\lambda(p)$ with $p_\mu = (\theta_\mu^i - \theta_\mu^j)$ and $i \neq j$, in Fig. 3. It is clear that this eigenvalue is non-zero for all values of $f$ indicating that there are no doubler zero modes. The reason for the breaking of the $Z_N^4$ symmetries can be understood by looking at the total action obtained from (3.4) and (3.6):

$$S = \sum_{i \neq j} \ln \left[ \sum_{\mu} \sin^2 \frac{1}{2} (\theta_\mu^i - \theta_\mu^j) \right] - 2f \sum_{i \neq j} \ln \left[ \mu^2 + \sum_{\mu} \sin^2 (\theta_\mu^i - \theta_\mu^j) \right].$$

(3.20)

The fermionic contribution cannot separate $\theta_\mu^i = \theta_\mu^j$ from $\theta_\mu^i = \theta_\mu^j + \pi$ implying that the fermion contribution alone will result in a distribution of eigenvalues restricted to a width of $\pi$.

3.2 Overlap fermions do not break the $Z_N^4$ symmetries

Contrary to naïve fermions, overlap fermions do separate $\theta_\mu^i = \theta_\mu^j$ from $\theta_\mu^i = \theta_\mu^j + \pi$ as is evident from the presence of the Wilson factors in (3.5). Therefore, we do not expect overlap fermions to break the $Z_N^4$ symmetries. A plot of $P_\mu$ for several values of $f$ at $N = 23$, $\beta = 1$, and $m = 5$ with $\mu = 0.01$ in Fig. 4 shows this to be the case. The very small deviation close to $f = \frac{1}{2}$ is a consequence of finite $N$ effects. Since we do not expect to go beyond $N = 23$ in the full simulation of the model, this plot will serve as a guide to what one can expect in a full simulation.

Since there are no doubler zero modes, we do not have any restriction on the values for the Wilson mass, $m$, used in the Wilson-Dirac kernel as described in section 2.2. But, we cannot make it arbitrarily large since one can see by a direct computation that the large $m$ limit of overlap fermions is naïve fermions [13]. A plot of $P_\mu$ as a function of $m$ is shown for $f = \frac{1}{2}$ and $f = 1$ in Fig. 5. It indicates that $3 \leq m \leq 8$ is an appropriate range of values of $m$ where the $Z_N^4$ symmetries are not broken for $f = \frac{1}{2}$ and that range only gets bigger as $f$ increases. A plot of the lowest positive eigenvalue of $H_m$ in Fig. 6 and the smallest eigenvalue, $\lambda_m(p)$ with $p_\mu = (\theta_\mu^i - \theta_\mu^j)$, $i \neq j$, in Fig. 7 shows that there are no doubler zero modes in this range of $m$. We can use this range of $m$ for our full numerical simulation with overlap fermions. One cannot take the Wilson mass parameter to zero in
Figure 1: Naïve fermions: Plot of $P_\mu$ as a function of $f$ for three different choices of $(\beta, N)$ and $\mu = 0.01$.

the weak coupling limit. This is intimately tied to the fact that the $Z_N$ symmetries are not broken and $U_\mu \neq 1$ in the weak coupling limit.

3.3 Effect of boundary conditions

In order to study the effect of boundary conditions on $Z_N^4$ symmetry breaking, we focus on $f = \frac{1}{2}$ and $f = 1$. We set $\phi_\mu = 0$ for $\mu = 2, 3, 4$ and varied $\phi_1$ by setting it to equal to $\frac{2\pi k}{N}$ with $k$ an integer in the range $0 \leq k < N/2$. The plot of $P_\mu$ as a function of $\phi_1$ is shown in Fig. 8. We see that the $Z_N$ symmetry in the $\mu = 1$ direction is broken if $\phi_1 > \frac{\pi}{2}$. This seems to be the case in the limit of large $N$ and seems to be roughly independent of $f$. Furthermore, the $Z_N$ symmetries in the other three directions with periodic boundary conditions are not broken. This result could help us force feed momentum for quarks in the adjoint representation. In order to pursue this, we need to study the effect of $\phi_1$ on the chiral condensate and see if chiral symmetry is restored when the $Z_N$ symmetry is broken and if the chiral condensate is independent of the value of $\phi_1$ when the $Z_N$ symmetry is not broken.

4. Hybrid Monte Carlo algorithm

One could use the standard Metropolis algorithm to study the single site model but the computational cost would scale like $N^8$ [10]. We will use the Hybrid Monte Carlo algorithm [14] to numerically study the model on a single site lattice. Fermion matrix inversions
Figure 2: Naïve fermions: Plot of $P_\mu$ as a function of $f$ at $N = 23$, $\beta = 4$, and $\mu = 0.01$.

dominate the computational cost of this algorithm and will therefore scale like $N^6$. We set $H^{ij}_\mu$ as momentum variables conjugate to $U^{ij}_\mu$ with the condition that $H_\mu$ is a traceless hermitian matrix. Therefore, we can write $H_\mu$ as

$$H_\mu = \sum_a H_{\mu a} T^a. \quad (4.1)$$

The Hamiltonian for the Hybrid Monte Carlo algorithm is

$$\mathcal{H} = \frac{1}{2} \sum_\mu \text{Tr} H^2_\mu + S. \quad (4.2)$$

Since the algorithm has not been used in the past for gauge theories with adjoint fermions on a single site lattice and since we do not use the conventional pseudo-fermion algorithm for dealing with the fermionic determinant, we present the necessary details in this section.

The equations of motion for $U_\mu$ are

$$\frac{dU_\mu}{d\tau} = iH_\mu U_\mu. \quad (4.3)$$

Setting $\frac{d\mathcal{H}}{d\tau} = 0$ results in

$$\sum_\mu \text{Tr} \left[ H_\mu \frac{dH_\mu}{d\tau} \right] + \frac{dS_2}{d\tau} + \frac{dS_{n,o}}{d\tau} = 0, \quad (4.4)$$
Figure 3: Naïve fermions: Plot of smallest eigenvalue, $\lambda(p)$ with $p_\mu = (\theta^i_\mu - \theta^j_\mu)$ and $i \neq j$, for $N = 23$, $\beta = 4$, and $\mu = 0.01$.

where we can write

$$
\frac{dS_g}{d\tau} = \sum_\mu \text{Tr} [H_\mu D^g_\mu] \\
\frac{dS_{n,o}}{d\tau} = \sum_\mu \text{Tr} [H_\mu D^{n,o}_\mu]. \tag{4.5}
$$

A simple calculation results in

$$
D^g_\mu = -ibN \sum_\nu \left[ U_{\mu} U_{\nu} U^\dagger_{\nu} U^\dagger_{\mu} + U_{\mu} U^\dagger_{\nu} U^\dagger_{\mu} U_{\nu} - U^\dagger_{\nu} U_{\mu} U^\dagger_{\nu} U^\dagger_{\mu} - U^\dagger_{\mu} U_{\nu} U^\dagger_{\nu} U^\dagger_{\mu} \right] \tag{4.6}
$$

The derivative of $V_\mu$ is

$$
\frac{dV^{ab}_\mu}{d\tau} = - \sum_{cd} H_{\mu c} f^a_{d} V^{db}_\mu = - \sum_d \bar{H}^{a}_{\mu d} V^{db}_\mu \tag{4.7}
$$

where

$$
\bar{H}^{a}_{\mu d} = \sum_c H_{\mu c} f^a_{d}, \tag{4.8}
$$

and we have used (2.6) and (4.1). Using the anti-symmetry of the structure constants, it follows that $\bar{H}_\mu$ are anti-symmetric real matrices. In analogy with (4.3), we can write

$$
\frac{dV_\mu}{d\tau} = - \bar{H}_\mu V_\mu. \tag{4.9}
$$
Figure 4: Overlap fermions: Plot of $P_\mu$ as a function of $f$ at $N = 23$, $\beta = 1$, $m = 5$, and $\mu = 0.01$.

4.1 Derivation of $D^n_\mu$ for naïve fermions:

We start by noting that

\[
\frac{dS_n}{d\tau} = -f \text{Tr} \left( \frac{1}{m^2 + CC^\dagger} \left( \frac{dC}{d\tau} C^\dagger + C \frac{dC^\dagger}{d\tau} \right) \right),
\]

(4.10)

and

\[
\frac{dC}{d\tau} = -\frac{1}{2} \sum_\mu \sigma_\mu \left( \bar{H}_\mu V_\mu + V_\mu^t \bar{H}_\mu \right).
\]

(4.11)

Let us define

\[
M_{\mu \nu} = \frac{1}{4} \text{tr} \left[ \frac{1}{m^2 + CC^\dagger} \sigma_\mu \sigma_\nu^t \right]; \quad M_{\mu \nu}^\dagger = M_{\nu \mu},
\]

(4.12)

where tr stands for trace only over the spin index. Using (2.11) and (2.12) we see that $M_{\mu \nu}$ are real matrices. We can rewrite (4.10) as

\[
\frac{dS_n}{d\tau} = -f \sum_\mu \text{Tr} \bar{H}_\mu \bar{A}_\mu,
\]

(4.13)

where

\[
\bar{A}_\mu = \left[ V_\mu B_\mu + B_\mu V_\mu^t \right] - \text{transpose},
\]

(4.14)

are real anti-symmetric matrices and

\[
B_\mu = \sum_\nu (V_\nu - V_\nu^t) M_{\mu \nu}.
\]

(4.15)
Figure 5: Overlap fermions: Plot of $P_\mu$ as a function of $m$ at $N = 23$, $\beta = 1$, and $\mu = 0.01$ for two different values of $f$.

This brings us down to the form we need, namely,

$$ \frac{dS_n}{d\tau} = \sum_{\mu} \text{Tr} H_\mu D_\mu^n $$

(4.16)

where

$$ D_\mu^n = i\frac{f}{2} \sum_{ab} \bar{A}_{\mu b} [T^a, T^b]. $$

(4.17)

4.2 Derivation of $D_\mu^n$ for overlap fermions:

Let the eigenvalues of the Wilson-Dirac operator, $H$, given in (2.14) be

$$ H |\lambda_i\rangle = \lambda_i |\lambda_i\rangle; \quad \lambda_i > 0 $$

$$ H |\omega_i\rangle = \omega_i |\omega_i\rangle; \quad \omega_i < 0 $$

$$ \langle \lambda_i | \lambda_j \rangle = \delta_{ij}; \quad \langle \omega_i | \omega_j \rangle = \delta_{ij}; \quad \langle \lambda_i | \omega_j \rangle = 0. $$

(4.18)

The dimension of $H$ is $4(N^2 - 1) \times 4(N^2 - 1)$ and we will assume that there are an equal number of positive and negative eigenvalues.$^3$ If follows from (2.17) that all eigenvalues of $H$ are doubly degenerate since the pair of eigenvectors $(\Sigma |\lambda_i\rangle, |\lambda_i\rangle)$ have the same eigenvalue and the same is the case for the pair $(\Sigma |\omega_i\rangle, |\omega_i\rangle)$.

The fermion action is defined by

$$ S_o = -f \text{Tr} \log H_o, $$

(4.19)

$^3$This is a reasonable assumption since we can restrict ourselves to a fixed topological sector.
**Figure 6:** Overlap fermions: Plot of the lowest positive eigenvalue of $H$ as a function of $m$ at $N = 23$, $\beta = 1$, and $\mu = 0.01$ for two different values of $f$.

and

$$
\frac{dS_o}{d\tau} = -f \text{Tr} \frac{1}{H_o} \frac{dH_o}{d\tau}.
$$  (4.20)

It follows from (2.18) that

$$
\frac{dH_o}{d\tau} = \frac{1 - \mu d\epsilon(H)}{2}.
$$  (4.21)

We can write

$$
\epsilon(H) = \sum_i |\lambda_i\rangle \langle \lambda_i| - \sum_i |\omega_i\rangle \langle \omega_i|.
$$  (4.22)

Therefore,

$$
\frac{de(H)}{d\tau} = \sum_i \frac{d|\lambda_i\rangle}{d\tau} \langle \lambda_i| - \sum_i \frac{d|\lambda_i\rangle}{d\tau} \langle \lambda_i| - \sum_i \frac{d|\omega_i\rangle}{d\tau} \langle \omega_i| - \sum_i \frac{d|\omega_i\rangle}{d\tau} \langle \omega_i|,
$$  (4.23)

and

$$
\frac{d|\lambda_i\rangle}{d\tau} = \sum_{j \neq i} \langle \lambda_j | \frac{dH}{d\tau} | \lambda_i \rangle \lambda_i - \lambda_j \langle \lambda_j | + \sum_j \frac{\langle \omega_j | \frac{dH}{d\tau} | \lambda_i \rangle}{\lambda_i - \omega_j},
$$  (4.24)

$$
\frac{d|\omega_i\rangle}{d\tau} = \sum_j \langle \omega_j | \frac{dH}{d\tau} | \omega_i \rangle \omega_i - \lambda_j \langle \omega_j | + \sum_{j \neq i} \langle \omega_j | \frac{dH}{d\tau} | \omega_i \rangle \omega_i - \omega_j \langle \omega_j |.
$$  (4.25)

The above equations imply

$$
\frac{de(H)}{d\tau} |\lambda_k\rangle = 2 \sum_j \frac{\langle \omega_j | \frac{dH}{d\tau} | \lambda_k \rangle}{\lambda_k - \omega_j} |\lambda_k\rangle - \sum_j \frac{\langle \omega_j | \frac{dH}{d\tau} | \lambda_k \rangle}{\lambda_k - \omega_j} |\omega_j\rangle.
$$
**Figure 7**: Overlap fermions: Plot of smallest eigenvalue, $\lambda(p)$ with $p_\mu = (\theta^i_\mu - \theta^j_\mu)$ and $i \neq j$, for $N = 23$, $\beta = 1$, and $\mu = 0.01$.

\[
\frac{d\varepsilon(H)}{d\tau} |_{\omega_{k}} = 2 \sum_j \frac{|\lambda_j \rangle \langle \lambda_j| \frac{dH}{d\tau} |\omega_{k}\rangle}{\lambda_j - \omega_{k}}. \tag{4.26}
\]

Inserting (4.21) into (4.20) and using the above equations, we get

\[
\frac{dS_o}{d\tau} = f(1 - \mu) \text{Tr} \left[ \frac{dH}{d\tau} (G + G^\dagger) \right], \tag{4.27}
\]

with

\[
G = \sum_{j,k} \frac{|\lambda_j \rangle \langle \lambda_j| \frac{1}{\Pi_\mu} |\omega_{k}\rangle \langle \omega_{k}|}{\omega_{k} - \lambda_j}. \tag{4.28}
\]

The symmetries in (2.17) and (2.7) imply that

\[
\Sigma G \Sigma = G^*. \tag{4.29}
\]

Equations (4.27), (2.14) and (4.9) can be combined to give

\[
\left[ \frac{dS_o}{d\tau} \right] = f(1 - \mu) \text{Tr} \bar{H}_\mu \left[ V_\mu \left( G + G^\dagger \right) \omega_\mu - \omega_\mu^i \left( G + G^\dagger \right) V_\mu^i \right]. \tag{4.30}
\]

Let

\[
\bar{B}_\mu = \text{tr} \left( G + G^\dagger \right) \omega_\mu, \tag{4.31}
\]
where tr stands for sum over the spin indices. Using (2.16) and (4.29) we can see that $B_\mu$ are real. With the definition of $B_\mu$, (4.30) becomes

$$\frac{dS_\phi}{d\tau} = f(1-\mu)\text{Tr} \tilde{H}_\mu \left[ V_\mu B_\mu - B_\mu^T V_\mu^T \right] = f(1-\mu)\text{Tr} \tilde{H}_\mu \tilde{A}_\mu, \tag{4.32}$$

where

$$\tilde{A}_\mu = V_\mu B_\mu - B_\mu^T V_\mu^T \tag{4.33}$$

are real anti-symmetric matrices. The rest of the steps are identical to that of naïve fermions.

5. Numerical results

In this section we present numerical results from our Hybrid Monte Carlo algorithm and show that they support the conclusions reached in section 3.

Fig. 9 shows a plot of $P_\mu$ as a function of $f$ obtained using naïve fermions. We have set $b = 5$, $N = 11$ and $\mu = 0.01$. Naïve fermions break the $Z_N^4$ symmetries as expected from the weak coupling analysis. The values of $P_\mu$ in Fig. 9 are consistent with the distribution of $\theta^i_\mu$ uniform with a width less than $\pi$ and this is evident for $f = 1$ in Fig. 10.

Turning our attention to overlap fermions, we plot $P_\mu$ as a function of $m$ in Fig. 11 for $b = 7$, $N = 11$, $f = \frac{1}{2}$ and $\mu = 0.01$. As expected the $Z_N^4$ symmetries are not broken for a wide range of $m$. A plot of the lowest eigenvalue of $H$ as a function of $m$ in Fig. 12 shows that there are no doubler modes in the same wide range of $m$. 

Figure 8: Overlap fermions: Plot of $P_\mu$ as a function of $\phi_1$ for three different values of $(f, N)$ with $\beta = 1$ and $\mu = 0.01$. 

---

**Figure 8:** Overlap fermions: Plot of $P_\mu$ as a function of $\phi_1$ for three different values of $(f, N)$ with $\beta = 1$ and $\mu = 0.01$. 

---
If chiral symmetry is broken, we expect the low lying eigenvalues of the overlap Dirac matrix to scale like $\frac{1}{N^2-1}$. Furthermore, we expect agreement with chiral random matrix theory predictions for the distribution of the low lying eigenvalues in accordance with the symplectic ensemble [15]. Instead of using the analytical formulas in [15] we found it convenient to numerically generate the eigenvalue distribution governed by the symplectic ensemble using Hybrid Monte Carlo techniques and compare with the numerical results obtained for this model. The chiral Random Matrix theory ensemble for a symplectic matrix, $C = \sum_\mu \sigma_\mu C_\mu$, is

$$Z = \int [dC_\mu] e^{-\sum_\mu \sum_\mu [c_\mu^\dagger c_\mu]^2} [\det H_{\text{rmnt}}]^f ; \quad H_{\text{rmnt}} = \left( \mu \ C^\dagger - \mu \right)$$

with $C_\mu$ being a real square matrix. We expect one scale factor to relate the distributions of the eigenvalues of $H_{\text{rmnt}}$ and $H_o$. Since we want to eliminate the scale set by the chiral condensate, we focus on

$$r = \left\langle \frac{\lambda_1}{\lambda_2} \right\rangle,$$

where $0 < \lambda_1 < \lambda_2$ are the first two non-degenerate eigenvalues. Fig. 13 shows a plot of $r$ as a function of $f$ obtained at $N = 11$ and $b = 5$ with $m = 5$ and $\mu = 0.01$. The results are also compared with the numbers one gets from chiral random matrix theory. There is agreement with chiral random matrix theory for $f = \frac{1}{2}$ and $f = 1$ but not for $f = 2$ or $f = 3$. The case of $f = \frac{1}{2}$ corresponds to the large $N$ limit of super Yang-Mills and this is one case where chiral symmetry assures supersymmetry.

Figure 9: Naïve fermions: Plot of $P_\mu$ as a function of $f$ for $N = 11$, $b = 5$ and $\mu = 0.01$. 

\[ \begin{array}{c|c|c|c|c|c} \mu=1 & \mu=2 & \mu=3 & \mu=4 \\ \hline 0 & 0.05 & 0.1 & 0.15 & 0.2 & 0.25 & 0.3 \\ \end{array} \]
Figure 10: Naïve fermions: Distribution of $\theta_\mu^i$ for $N = 11, b = 5$ and $\mu = 0.01$ at $f = 2$.

One can try to understand the above behavior by considering the running of the coupling constant, $\alpha = \frac{1}{4\pi b}$ with the scale set by the lattice spacing, $a$ [18]. Using the two-loop beta function [16, 17], we have

$$\frac{a d\alpha}{da} = 2 \left( \frac{11}{3} - \frac{4}{3} f \right) \alpha^2 + 2 \left( \frac{34}{3} - \frac{32}{3} f \right) \alpha^3 + \cdots,$$

where the first two coefficients listed are independent of the the renormalization scheme. Asymptotic freedom is lost when $f > \frac{11}{4}$. The two-loop expression predicts an infrared fixed point [19] if $\frac{17}{16} < f < \frac{11}{4}$ with the fixed point coupling becoming smaller as $f$ increases in this range. Our current data (Fig. 13) is consistent with the presence of a chiral condensate in the absence of an infra-red fixed point. One should keep in mind that one has to take the large $N$ limit at a fixed $b$ and then take the continuum limit, $b \to \infty$, in order to arrive at a physically relevant conclusion.

6. Future outlook

We have provided compelling arguments for the preservation of $Z_N^4$ symmetries in the large $N$ limit of Yang-Mills theories coupled to adjoint overlap fermions on a single site lattice.
Figure 11: Overlap fermions: Plot of $P_\mu$ as a function of $m$ for $N = 11$, $b = 7$, $f = \frac{1}{2}$ and $\mu = 0.01$.

We have provided all the necessary details to perform a Hybrid Monte Carlo algorithm and we have shown numerical feasibility. We plan to study several different values of $N$ in the range of 11 to 18 and we also plan to study several different couplings in the range of 0.35 to 5.0 [11]. One can use a physical observable like the Wilson loop to study the beta function on the lattice and investigate the presence/absence of an infrared fixed point. In this context, it would be useful to perform a weak coupling expansion of the beta function on the single site lattice and confirm that one obtains the same result as on an infinite lattice. Perturbative computations on a single site lattice do not have the standard divergences and a perturbative computation of the Wilson loop will clarify the role of perimeter divergence on a single site lattice. Finally, one should extend the numerical results to other choices of boundary conditions that are not periodic.

Acknowledgments

We acknowledge discussions with Barak Bringoltz, Michael Buchoff, Aleksey Cherman, Aleksi Kurkela, Joyce Myers, Herbert Neuberger, Erich Poppitz, and Mithat Ünsal. The authors acknowledge partial support by the NSF under grant number PHY-0854744. A.H also acknowledges partial support by the U.S. DOE grant under Contract DE-FG02-01ER41172.

References

[1] R. Narayanan and H. Neuberger, Phys. Rev. Lett. 91, 081601 (2003) [arXiv:hep-lat/0303023].
**Figure 12:** Overlap fermions: Plot of the lowest eigenvalue of $H$ as a function of $m$ for $N = 11$, $b = 7$, $f = \frac{1}{2}$ and $\mu = 0.01$.

[2] T. Eguchi and H. Kawai, Phys. Rev. Lett. **48**, 1063 (1982).
[3] G. Bhanot, U. M. Heller and H. Neuberger, Phys. Lett. B **113**, 47 (1982).
[4] P. Kovtun, M. Unsal and L. G. Yaffe, JHEP **0706**, 019 (2007) [arXiv:hep-th/0702021].
[5] T. J. Hollowood and J. C. Myers, arXiv:0907.3665 [hep-th].
[6] B. Bringoltz, JHEP **0906**, 091 (2009) [arXiv:0905.2406 [hep-lat]].
[7] B. Bringoltz, arXiv:0911.0352 [hep-lat].
[8] E. Poppitz and M. Unsal, arXiv:0911.0358 [hep-th].
[9] P. F. Bedaque, M. I. Buchoff, A. Cherman and R. P. Springer, arXiv:0904.0277 [hep-th].
[10] B. Bringoltz and S. R. Sharpe, Phys. Rev. D **80**, 065031 (2009) [arXiv:0906.3538 [hep-lat]].
[11] A. Hietanen and R. Narayanan, in preparation.
[12] E. Poppitz and M. Unsal, JHEP **0903**, 027 (2009) [arXiv:0812.2085 [hep-th]].
[13] R. Narayanan and H. Neuberger, Nucl. Phys. B **443**, 305 (1995) [arXiv:hep-th/9411108].
[14] T. DeGrand and C. E. Detar, *New Jersey, USA: World Scientific (2006) 345 p*
[15] P. H. Damgaard and S. M. Nishigaki, Phys. Rev. D **63**, 045012 (2001) [arXiv:hep-th/0006111].
[16] W. E. Caswell, Phys. Rev. Lett. **33**, 244 (1974).
[17] D. R. T. Jones, Nucl. Phys. B **75**, 531 (1974).
Figure 13: Overlap fermions: Plot of the ratio, $r$, as a function of $f$ at $N = 11$, $b = 5$, $m = 5$ and $\mu = 0.01$ is compared with chiral random matrix theory predictions.

[18] M. Creutz, Cambridge, Uk: Univ. Pr. (1983) 169 P. (Cambridge Monographs On Mathematical Physics)

[19] T. Banks and A. Zaks, Nucl. Phys. B 196, 189 (1982).