Structure of the correlation function at the accumulation points of the logistic map

K. Karamanos,1 I.S. Mistakidis,2 and S.I. Mistakidis3

1 Complex Systems Group, Institute of Nuclear and Particle Physics, NCSR Demokritos, GR 15310, Aghia Paraskevi Attiki, Greece
2 Hellenic Army Academy, Section of Physical Sciences and Applications, Vari Attiki, Greece
3 Zentrum für Optische Quantentechnologien, Universität Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany

(Dated: July 8, 2018)

Abstract

The correlation function of the trajectory exactly at the Feigenbaum point of the logistic map is investigated and checked by numerical experiments. Taking advantage of recent closed analytical results on the symbol-to-symbol correlation function of the generating partition, we are in position to justify the deep algorithmic structure of the correlation function apart from numerical constants. A generalization is given for arbitrary \( m \cdot 2^\infty \) Feigenbaum attractors.

Keywords: correlation function; symbolic dynamics; bifurcation points; Feigenbaum attractors; logistic map.
I. INTRODUCTION

Recently, the study of Complex Systems has gained significant attention. One of the basic aspects of this progress is related with the understanding of correlations in and between such complex systems, which is realized through the use of different complexity measures. Among these, one can mention the transinformation [1–3], the block entropies [4–11] different types of correlation functions [12–17] and number-theoretic notions [6, 18].

One of the Paradigms of Complex Systems is the logistic map. The logistic map has a simple definition but presents complex behavior when fine tuning the control parameter values. In particular, after Feigenbaum’s work, the period-doubling route to chaos has been fairly understood. Also, connections with the theory of second order phase transitions (critical phenomena) have been established and scaling relations have been reported nearby the accumulation point (also called Feigenbaum Point (FP)) with and without the presence of external noise. Furthermore, cantorian fractal structures have been revealed in the transition point connecting the physics of the non-chaotic attractor with self-similarity [19–25]. Recently also, a direct connection with Experimental Mathematics has been established, too [26].

On the other hand, in Non-linear physics, the importance of the study of the correlation function has been realized from the very beginning. Particularly inspiring have been the works of Ruelle [27], Daems and Nicolis [28], and Alonso et al. [12], for the case of resonances of chaotic dynamical systems. In addition, based on the analogies between the period doubling transition and critical phenomena, H. G. Schuster has done a guess on the functional form of the correlation function of the trajectory [17]. Indeed, according to his arguments the correlation function should follow a power law behaviour. In contrast, here, we demonstrate that the correlation function possesses a stratified structure. More recently, using the Feigenbaum renormalization group transformation it has been shown [29] that the correlation function of the trajectory in the one dimensional nonlinear dissipative logistic map is made of a family of power laws with a common scaling factor given by the Feigenbaum constant $\alpha$. In the present work in order to extract the form of the correlation function of the trajectory we propose some more elaborated arguments, using a different approach which is based on the structure of the symbol-to-symbol correlation function [9], that is the correlation function of symbolic dynamics.

After establishing rigorously in a previous work [9] the detailed form of the symbol-to-symbol correlation function we turn now our attention to the structure of the correlation function of the trajectory. To be more concrete, taking advantage from the analytic form of the symbol-to
symbol correlation function and presenting simple arguments we shall show that one can extract up to a good approximation, that is apart from numerical constants, the detailed structure for the correlation function of the trajectory. The above investigation is mainly supported by a detailed numerical study which takes into account a large enough statistical sample of the logistic map. In this manner, we can justify the analytic form of the correlation function of the trajectory from first principles using the Metropolis-Stein and Stein algorithm (MSS algorithm), apart from numerical constants, which depend on the detailed functional form of the map. Furthermore, we make an attempt to generalize these results for an arbitrary \( m \cdot 2^\infty \) accumulation point \([30]\), for \( m = 2, 3, \ldots \), which correspond to the accumulation points of the bifurcation tree \([17, 31]\) (see also Figure 1). Finally, a general form for the correlation function of the trajectory and that obtained from the symbolic dynamics is also suggested. We believe that our results will inspire similar investigations on non-unimodal maps and give further insight providing new complexity measures on real experimental time-series.

The paper is organized as follows. In Sec. II we introduce the logistic map and the definitions of different types of correlation functions that will be used. In Sec. III we present our careful numerical experimentation for the symbol-to-symbol correlation function and for the correlation function of the trajectory at the (first) accumulation point. As it is shown those functions satisfy simple numerical prescriptions, which are explicitly outlined. In addition, we propose some simple arguments which, up to a good approximation, allow for the justification of the functional form of the correlation function of the trajectory from the symbol-to-symbol correlation function apart from arithmetical constants in a systematic basis. We then present analogous results and generalizations for the \( m \cdot 2^\infty \) accumulation points. Finally, in Sec. IV we draw the main conclusions and discuss future plans.

II. THE LOGISTIC MAP

The logistic map is the archetype of a Complex System. Let us elaborate. We introduce the logistic map in its familiar form

\[
x_{n+1} = rx_n(1 - x_n),
\]

where \( r \) is the control parameter value and \( n \) denotes the respective iteration of the map. For the logistic map in this form the generating partition is easily computed, following an argument dating back to the French Mathematician Gaston Julia. To be more specific, for \( f(x) = rx(1 - x) \)
the equation $f'(c) = 0$ gives $c=0.5$, so that the partition of the phase space (which in this case coincides with the unit interval $I=[0,1]$) $L=[0,0.5]$ and $R=(0.5,1]$ is a generating one (see also [32] for a more rigorous definition). Notice that according to Metropolis et al. [33] the information content of the symbolic trajectory is the "minimum distinguishing information". Needless to say, in this representation the logistic map can be viewed as an abstract information generator.

![Bifurcation Diagram](image)

**FIG. 1.** The bifurcation diagram for the logistic map for the superstable $2^n$-cycles. It is shown the control parameter values $r_i$ for the first few bifurcation points and the values $R_i$ for the superstable orbits.

In particular, the period doubling route to chaos has been fairly studied and it is by now well understood. These studies led to the occurrence of the two Feigenbaum constants $\alpha$ and $\delta$ which can be defined by an approximate real space renormalization procedure. Especially, the constant $\delta$ is related with the spacing in the control parameter space of the successive values of occurrence of the superstable periodic orbits and can be roughly estimated by the bifurcation diagram [22, 23]. If we denote as $\{R_n\}$ this set of values, $\delta$ is defined as

$$\delta = \lim_{n \to \infty} \frac{R_n - R_{n-1}}{R_{n+1} - R_n},$$

and for the quadratic map reads

$$\delta \simeq 4.669201609102990... .$$

Moreover, the constant $\alpha$ is related to the rescaling of the period doubling functional composition law and its value for the logistic map reads

$$\alpha = -\lim_{n \to \infty} \frac{d_n}{d_{n+1}} \simeq -2.5029078750095892... .$$
Finally, note that the constants $\alpha$, $\delta$ are related as it can be shown by using renormalization group arguments (see [16, 34] and references therein). The values of the above two constants depend only on the order of the maximum and have long been studied. They are thus, for instance, universal for quadratic maps irrespectively of the exact way one writes down the map.

Figure 1 presents the control parameter values of the bifurcation points denoted as $r_1$, $r_2$, $r_3$, ... while the corresponding values for the superstable orbits are depicted as $R_1$, $R_2$, $R_3$, ... . The values of $d_i$ figuring in the definition of the Feigenbaum constant $\alpha$ are also shown. Note here that Feigenbaum and successors have shown that eq. (2), holds if instead of $R_i$ we use $r_i$.

After the above brief introduction of the logistic map and its properties, we shall next define the (un-normalized) correlation function of the trajectory as

$$ C_{un}(m) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} x_{i+m} x_i, $$

(5)

where the deviation from the real value of the map at the $i-$th iteration is given by $x_i = f^i(x_0) - \overline{x}$ and the corresponding mean value of the map taking into account $N$ iterations (sample) is denoted by $\overline{x} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} f^i(x_0)$. Also, in direct analogy with the above defined un-normalized correlation function one can also introduce here the normalized correlation function

$$ C(m) = \frac{C_{un}(m)}{C_{un}(0)} = \frac{C_{un}(m)}{\sigma^2}, $$

(6)

where $\sigma$ is the mean standard deviation, which normalizes the statistical data.

From the above definitions follows that $C(m)$ (or equally $C_{un}(m)$) yields another measure for the irregularity of the sequence of iterates $x_0$, $f(x_0)$, $f^2(x_0)$,...etc. It tells us how much the deviations of the iterates from their average value, $x_i = x_i - \overline{x}$ that are $m$ steps apart (i.e. $x_{i+m}$ and $x_i$) "know" about each other, on the average. Another remark here is that if $C(m) \to 0$ as $m \to \infty$ then the system does not have the mixing property.

We should here note that the problem of determining the correlation function of an arbitrary dynamical system is difficult to calculate in the general case. This is the reason to resort to other computable observables such as the symbol-to-symbol correlation function [28]. Thus, in direct analogy with the correlation function of the trajectory one can introduce the un-normalized symbol-to-symbol correlation function as

$$ K_{un}(m) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} y_{i+m} y_i, $$

(7)

where the corresponding symbolic functional composition is defined by $y_i = y(f^i(x_0)) - \overline{y}$, with the mean value $\overline{y} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} y(f^i(x_0))$. Here, $f^i(x_0)$ is the real value of the map at the $i -$
th iteration, \( N \) denotes the size of the statistical sample that we take into account and \( m \) is the corresponding distance between two symbols in the symbolic sequence that we examine. In addition, as usual the functional composition \( y_i \) takes the values \( y_i = 0, 1 \) when \( x_i \leq 0.5 \) or \( x_i > 0.5 \) respectively, i.e. it is a step function.

![Graph](image)

FIG. 2. Shown are the symbol-to-symbol correlation function and the correlation function of the trajectory for the logistic map exactly at the accumulation point \( r = FP = 3.56994567... \), with initial condition \( x_0 = 0.5 \). The first \( 10^5 \) iterations have been eliminated from our statistics in order to exclude transients, and the subsequent \( n = 10^8 \) iterations have been taken into account for the numerical calculations. The experimentally determined Lyapunov exponent is \(-5.93 \cdot 10^{-5}\).

Finally, let us also note that one can define a more relevant quantity which is the normalized symbol-to-symbol correlation function

\[
K(m) = \frac{K_{un}(m)}{K_{un}(0)} = \frac{K_{un}(m)}{\sigma'^2},
\]

where \( \sigma' \) denotes the mean standard deviation obtained from the respective symbolic sequence. It has been shown \[35, 36\] that the normalized symbol-to-symbol correlation function does not depend on the choice of the symbols (i.e. \( L \) to "0" and \( R \) to "1") because for binary sequences the correlation functions are unique up to a constant factor, which varies with the choice of these numbers but cancels out via the normalization.

III. THE STRUCTURE OF THE CORRELATION FUNCTION

Motivated by previous works on correlation functions \[27, 29\], we explore here the properties of the symbol-to-symbol correlation function and the correlation function of the trajectory. In order to
cope with the problem of the analytic form of correlation functions, we have studied the correlation function numerically. For the logistic map at the Feigenbaum point, i.e. \( r = 3.56994567 \ldots \), we have calculated both the normalized symbol-to-symbol correlation function (see eq.(8) and Figure 2) and the correlation function of the trajectory (see eq.(6) and Figure 2). To do that we start each time from the initial point \( x_0 = 0.5 \) and take a numerical sample consisting of \( n = 10^8 \) iterations after the elimination of the first \( 10^5 \) iterations (to avoid transients). For reasons of completeness let us note that we have checked that our results, presented below, pertain if we use a different initial condition in the unit interval e.g. \( x_0 = 0.3 \) or \( x_0 = 0.8 \). Also we remark that according to our simulations (omitted here for brevity) the structure of the correlation function (see below) remains the same if instead of a single initial value (e.g. \( x_0 = 0.5, 0.3 \)) we average the correlation function over a uniform ensemble of initial values \( x_0 \in \{ 0.0225, 0.975 \} \) with step \( \delta x_0 = 0.0225 \). At this point we should remind that exactly at the Feigenbaum point the Lyapunov exponent, which is defined in general as 
\[
\lambda = \frac{1}{n} \lim_{n \to \infty} \sum_{i=1}^{n} \ln |f'(x_i)|,
\]
strictly vanishes i.e. \( \lambda = 0 \) (see also Table I), and we are in the presence of the non-chaotic multifractal attractor. Notice also that the same behaviour of the Lyapunov exponent holds for the higher accumulation points. To indicate the behaviour of the map Table I presents the numerically calculated Lyapunov exponent for every accumulation point (using eight decimals for the corresponding control parameter) of the logistic map including and excluding transients from our statistics. We observe that the Lyapunov exponent in each case vanishes, while the transients play no essential role due to the augmented statistics that we use.

![Figure 3](image-url)

**FIG. 3.** Shown are the symbol-to-symbol correlation function and the correlation function of the trajectory for the logistic map with control parameter value \( r = 3.8495 \) (cycle \( 3 \cdot 2^\infty \)) and initial condition \( x_0 = 0.5 \). The first \( 10^5 \) iterations have been eliminated from our statistics in order to exclude transients, and the subsequent \( n = 10^8 \) iterations have been taken into account for the numerical calculations.
On the other hand, as it has already been mentioned, in a previous work [9] providing some theoretical arguments from the viewpoint of the symbolic dynamics we have established the structure of the un-normalized symbol-to-symbol correlation function. These results have also been supported from careful numerical experimentations leading to the compact form

$$K_{un}(m) = A_l \cdot \delta_{m,2^{l-1}(1+2k)},$$

where for a given (fixed) \( l, l \in \{1, 2, 3, \ldots\} \), \( A_l \) is a constant depending only on \( l \), and \( k \) takes all the values from the set of natural numbers \( \{0, 1, 2, 3, \ldots\} \). Here, we have also used the fact that any integer \( m \) can be decomposed in terms of a unique pair of natural numbers \( l, k \) such that \( m = 2^{l-1}(1 + 2k) \). The same holds for any other form of the correlation function that will be presented in the rest of the paper. In [9], on the grounds of the Metropolis-Stein-Stein algorithm, we have established a new theorem, namely that

$$K_{un}(m) = \begin{cases} 
-\frac{1}{9}, & m = 1 + 2 \cdot k \quad (k = 0, 1, 2, \ldots) \\
\frac{1}{18}, & m = 2 + 4 \cdot k \quad (k = 0, 1, 2, \ldots) \\
\frac{10}{72}, & m = 4 + 8 \cdot k \quad (k = 0, 1, 2, \ldots) \\
\frac{13}{72}, & m = 8 + 16 \cdot k \quad (k = 0, 1, 2, \ldots) \\
\frac{29}{144}, & m = 16 + 32 \cdot k \quad (k = 0, 1, 2, \ldots) 
\end{cases}$$

(10)

As it seems, these are the first few numerically obtained coefficients \( A_l \) of eq.(9). We can extend this procedure to infinity, i.e. \( m \to \infty \), and the above constructive scheme guarantees that this

![Diagram](image-url)

**FIG. 4.** Shown are the symbol-to-symbol correlation function and the correlation function of the trajectory for the logistic map with control parameter value \( r = 3.7430 \) (cycle \( 5 \cdot 2^{\infty}(a) \)) and initial condition \( x_0 = 0.5 \). The first \( 10^5 \) iterations have been eliminated from our statistics in order to exclude transients, and the subsequent \( n = 10^8 \) iterations have been taken into account for the numerical calculations.
deep algorithmic structure is kept in all scales. From this infinite stratification, the infinite memory
of the system at the Feigenbaum point is revealed, as this scheme never ends.

In the following, we proceed by extending our numerical experimentation to the structure of
the normalized symbol-to-symbol correlation function taking again into account the first $n = 10^8$
iterations of the logistic map (this scheme is depicted in Figure 2) with initial condition $x_0 = 0.5$.
The corresponding functional structure now reads

$$K(m) = \begin{cases} 
-\frac{1}{2}, & m = 1 + 2 \cdot k \quad (k = 0, 1, 2, \ldots) \\
\frac{1}{4}, & m = 2 + 4 \cdot k \quad (k = 0, 1, 2, \ldots) \\
\frac{1}{1.6}, & m = 4 + 8 \cdot k \quad (k = 0, 1, 2, \ldots) \\
\frac{1}{1.23}, & m = 8 + 16 \cdot k \quad (k = 0, 1, 2, \ldots) \\
\frac{1}{1.10}, & m = 16 + 32 \cdot k \quad (k = 0, 1, 2, \ldots) \\
\sim \frac{1}{1.049}, & m = 32 + 64 \cdot k \quad (k = 0, 1, 2, \ldots) \\
\sim \frac{1}{1.024}, & m = 64 + 128 \cdot k \quad (k = 0, 1, 2, \ldots) \\
\sim \frac{1}{1.012}, & m = 128 + 256 \cdot k \quad (k = 0, 1, 2, \ldots) \\
\sim \frac{1}{1.006}, & m = 256 + 512 \cdot k \quad (k = 0, 1, 2, \ldots) \\
\sim \frac{1}{1.002}, & m = 512 + 1024 \cdot k \quad (k = 0, 1, 2, \ldots) 
\end{cases} \tag{11}$$

which has already been established in [9]. Moreover, proceeding along the same lines one can
calculate the correlation function of the trajectory as it is defined in eqs.(5,6). In this manner, it
can be easily confirmed numerically that its structure has the following simple form

$$C(m) = \begin{cases} 
-\frac{1}{1.0976}, & m = 1 + 2 \cdot k \quad (k = 0, 1, 2, \ldots) \\
\frac{1}{1.2122}, & m = 2 + 4 \cdot k \quad (k = 0, 1, 2, \ldots) \\
\frac{1}{1.0171}, & m = 4 + 8 \cdot k \quad (k = 0, 1, 2, \ldots) \\
\frac{1}{1.0016}, & m = 8 + 16 \cdot k \quad (k = 0, 1, 2, \ldots) \\
\frac{1}{1.00015}, & m = 16 + 32 \cdot k \quad (k = 0, 1, 2, \ldots) \\
\frac{1}{1.000001}, & m = 32 + 64 \cdot k \quad (k = 0, 1, 2, \ldots) \\
\frac{1}{1.000001}, & m = 64 + 128 \cdot k \quad (k = 0, 1, 2, \ldots) \\
\frac{1}{1.0000001}, & m = 128 + 256 \cdot k \quad (k = 0, 1, 2, \ldots) \\
\frac{1}{1.00000001}, & m = 256 + 512 \cdot k \quad (k = 0, 1, 2, \ldots) 
\end{cases} \tag{12}$$

As it can be observed it possesses the known functional form of eq.(9) apart from the numerical
constants $A'_l$, whereas $A'_l \rightarrow 1$ as $m$ becomes large. Therefore, up to now we have established a
general form for the correlation function of the trajectory exactly at the $FP$ point supported from
numerical calculations. In the following, we shall present some arguments in order to show that the correlation functions of the trajectory and that obtained from the symbolic sequence possesses the same time scales. This discussion will lead us to a conceptual understanding for the reason that the correlation function of the trajectory (as it is defined in eq.(5)) contains the same time scales with the symbol-to-symbol correlation function (see eq.(7)), that is

\[
C_{un}(m) = A'_l \cdot \delta_{m,2^l-1-(1+2^l-k)}.
\]  

(13)

Here for a fixed \(l \in \{1, 2, 3, \ldots\}\) the coefficients \(A'_l\) are constants, while \(k\) takes all the values from the set of natural numbers \(\{0, 1, 2, \ldots\}\). Note also that the properties of a single position of the trajectory have already been studied in another context \[37\]-\[39\], where it has been shown that the trajectory exhibits the above mentioned iteration-times property. In the following, we would like to show that when the symbol-to-symbol correlation function saturates, the correlation function of the path is saturated, too, and when the symbol-to-symbol correlation function diminishes, the correlation function of the path diminishes, too. Towards this direction, let us consider the deviation from the critical value \(x_{cr} = 0.5\) as \(0 < \varepsilon_i < 0.5\) with \(\varepsilon_i \varepsilon_j < 0.25\), which remains valid for all cases. As a consequence of the above proposition we can distinguish the following four different cases: a) A contribution in the symbolic correlation function by two terms \(x_k > 0.5\) (so \(y_k = 1\)) and \(x_{k+l} > 0.5\) \((y_{k+l} = 1)\) is +1 and gives a corresponding contribution to the trajectory correlation function of the order of \((0.5 + \varepsilon_i)(0.5 + \varepsilon_j) \simeq 0.25 + \varepsilon + \mathcal{O}(\varepsilon^2)\), that is more than 0.25. b) Secondly, a contribution in the symbolic correlation function by the terms \(x_k < 0.5\) and \(x_{k+l} < 0.5\) is 0 and gives a corresponding contribution to the trajectory correlation function of the order of \((0.5 - \varepsilon_i)(0.5 - \varepsilon_j) \simeq 0.25 - \varepsilon + \mathcal{O}(\varepsilon^2)\), that is less than 0.25. c) On the other hand, a contribution in the symbolic correlation function by two terms \(x_k > 0.5\) and \(x_{k+l} < 0.5\) is 0 and gives a corresponding contribution to the trajectory correlation function of the order of \((0.5 + \varepsilon_i)(0.5 - \varepsilon_j) \simeq 0.25 + \varepsilon + \mathcal{O}(\varepsilon^2)\), i.e. of the order of 0.25. d) Finally, a contribution in the symbolic correlation function by the terms \(x_k < 0.5\) and \(x_{k+l} > 0.5\) is 0 and gives a corresponding contribution to the trajectory correlation function of the order \((0.5 - \varepsilon_i)(0.5 + \varepsilon_j) \simeq 0.25 + \mathcal{O}(\varepsilon^2)\), that is of the order of 0.25.

In order to clarify the meaning of the above approximations let us consider a specific example with \(x_k = 0.7\) and \(x_{k+l} = 0.9\). Then, we have \(\varepsilon_1 = 0.2\) and \(\varepsilon_2 = 0.4\). So, the contribution to the symbolic correlation function is 1 and as a consequence the contribution to the real correlation function is 0.55 = 0.25 + \(\varepsilon > 0.25\). Thus, from the above it is clear that using such a simple argument one can predict correctly the functional form of the correlation function of the trajectory.
from the symbolic one.

TABLE I. In the second column of the table below the values of the control parameter which correspond to the different accumulation points (first column) of the logistic map are presented. In the third and fourth columns we show the Lyapunov exponent obtained from \( n = 10^8 \) iterations, including and excluding transients respectively. We observe no significant differences due to the augmented statistics.

| Accumulation cycle | Accumulation point | Lyapunov exponent \( n = 10^8 \) | Lyapunov exponent \( n = 10^8 \) |
|--------------------|--------------------|-------------------------------|-------------------------------|
| \( 2^\infty \)    | FP                 | \(-5.934 \cdot 10^{-5}\)    | \(-5.934 \cdot 10^{-5}\)    |
| \( 3 \cdot 2^\infty \) | 3.8495             | 0.0237                       | 0.0237                       |
| \( 4 \cdot 2^\infty \) | 3.9612             | 0.0122                       | 0.0122                       |
| \( 5 \cdot 2^\infty (a) \) | 3.7430             | \(-0.0021\)                 | \(-0.0021\)                 |
| \( 5 \cdot 2^\infty (b) \) | 3.9065             | 0.0414                       | 0.0414                       |
| \( 5 \cdot 2^\infty (c) \) | 3.99032            | \(-0.0039\)                 | \(-0.0039\)                 |
| \( 6 \cdot 2^\infty (a) \) | 3.6327             | \(-0.0073\)                 | \(-0.0073\)                 |
| \( 6 \cdot 2^\infty (b) \) | 3.937649           | 0.0127                       | 0.0127                       |
| \( 6 \cdot 2^\infty (c) \) | 3.977800           | 0.0077                       | 0.0077                       |
| \( 6 \cdot 2^\infty (d) \) | 3.997586           | \(-0.0133\)                 | \(-0.0133\)                 |

As the structure of the correlation function of the trajectory for the \( 2^\infty \) scenario has been fairly understood, let us proceed with the next accumulation points. Thus, we further consider the \( 3 \cdot 2^\infty \) scenario which corresponds to the control parameter value \( r = 3.8495 \) of the logistic map (see Table I). In this manner, one can evaluate the symbol-to-symbol correlation for this scenario \( (3 \cdot 2^\infty) \), using the same numerical procedure and statistical sample as previously. From this calculation we can conclude that the normalized symbol-to-symbol correlation function for the \( 3 \cdot 2^\infty \) accumulation point has the following form

\[
C(\tau) = A_l \cdot \delta_{r,3 \cdot 2^l (1+2k)} + B_1 \cdot \delta_{r,1+3 \cdot k} + B_2 \cdot \delta_{r,2+3 \cdot k},
\]

where for a given \( l \in \{1, 2, 3, \ldots\} \), \( A_l, B_1, B_2 \) are constants and \( k \) takes all the values from the set of natural numbers \( \{0, 1, 2, \ldots\} \).

The first few numerical values of the above coefficients for the correlation function of the symbolic sequence are presented in Table II (see second row). As one can easily verify after a straightforward numerical computation the same structure is observed for the correlation function of the trajectory, apart from numerical constants \( A'_l, B'_1, B'_2 \) which depend on the detailed form of the map (see Table III, third row). In the same manner, one can perform the same calculations for the
considerations to the trajectory and that obtained from the symbolic sequence. Indeed, in Tables II and III we extend these
higher accumulation points and find a similar structure for both the correlation function of the trajectory and that obtained from the symbolic sequence. Indeed, in Tables II and III we extend these
considerations to the 4·2∞, 5·2∞(a), 5·2∞(b), 5·2∞(c), 6·2∞(a), 6·2∞(b), 6·2∞(c) and
6·2∞(d) (see also Figures 3, 4) accumulation points and we present the corresponding coefficients

### Table II

| Scenario | mean value | A₀ | A₁ | A₂ | B₁ | B₂ | B₃ | B₄ | B₅ |
|----------|------------|----|----|----|----|----|----|----|----|
| 2∞       | 0.6666     | −1/2 | 1/4 | 1/6 | − | − | − | − | − |
| 3·2∞     | 0.4381     | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4·2∞     | 0.3342     | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5·2∞(a)  | 0.6625     | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5·2∞(b)  | 0.5420     | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5·2∞(c)  | 0.2500     | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6·2∞(a)  | 0.7708     | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6·2∞(b)  | 0.5529     | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6·2∞(c)  | 0.4468     | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6·2∞(d)  | 0.2500     | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

### Table III

| Scenario | mean value | A₀ | A₁ | A₂ | B₁ | B₂ | B₃ | B₄ | B₅ |
|----------|------------|----|----|----|----|----|----|----|----|
| 2∞       | 0.6476     | −1 | −1 | −1 | −1 | −1 | −1 | −1 | −1 |
| 3·2∞     | 0.5313     | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4·2∞     | 0.4181     | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5·2∞(a)  | 0.6311     | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5·2∞(b)  | 0.5472     | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5·2∞(c)  | 0.3382     | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6·2∞(a)  | 0.6609     | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6·2∞(b)  | 0.5504     | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6·2∞(c)  | 0.4610     | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6·2∞(d)  | 0.2826     | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

TABLE II. In the table below we present the first few coefficients of eq.(14) for the normalized symbol-to-
symbol correlation function at each accumulation point. The corresponding mean value for the first n = 10⁸
iterations of the logistic map is also provided.

TABLE III. In the table below we present the first few coefficients of eq.(14) for the normalized correlation
function of the trajectory at each accumulation point. The corresponding mean value for the first n = 10⁸
iterations of the logistic map is also provided.
that may appear in the respective correlation function for each cycle. Notice that the relevant information about the accumulations points and the corresponding patterns have been found in [25]. As for the lower cycles both the correlation function of the trajectory and the symbolic one posses the same time scales. To illustrate the above, Figure 3 shows the symbol-to-symbol versus the correlation function of the trajectory for the $3 \cdot 2^\infty$ cycle. We observe that the correlation function for the trajectory is always larger from the symbolic one. The latter can also be confirmed by a direct comparison of the coefficients $A_i$ or $B_i$ among the different types of correlations presented in Tables II and III. The previous observations also hold for higher accumulation points, e.g. the $5 \cdot 2^\infty(a)$ cycle as shown in Figure 4. Additionally, from the Tables II, III it is clearly seen that for the same type of correlations the coefficients $A_i$ are increasing for higher accumulation points while some of the constants $B_i$ may coincide. This is more rare for the correlation function of the trajectory.

As a result of the above studies we can conclude that the suggested general form for the symbol-to-symbol correlation function at the $FP$ point of the $m \cdot 2^\infty$ attractor reads

\[ C(\tau) = A_l \cdot \delta_{\tau,m \cdot 2^l(1+2k)} + B_1 \cdot \delta_{\tau,1+m \cdot k} + B_2 \cdot \delta_{\tau,2+m \cdot k} + \ldots + B_{(m-1)} \cdot \delta_{\tau,(m-1)+m \cdot k}, \]  

(15)

where as usual for a given $l \in \{1, 2, 3, \ldots\}$ the coefficients $A_l$, $B_1$, $B_2$, ..., $B_{m-1}$ are constants depending only on $l$, $m$ denotes the number of the accumulation point that we consider and $k$ takes all the values from the set of natural numbers $\{0, 1, 2, \ldots\}$. The corresponding form for the correlation function of the trajectory remains the same apart from numerical constants $A'_l$, $B'_1$, $B'_2$, ... which as it has been mentioned previously they depend on the detailed form of the map.

IV. SUMMARY AND CONCLUSIONS

The correlation function is an important quantity measuring correlations in many branches of physics. Obviously, there are also other interesting quantities as for instance the (conditional) block-entropies, the transinformation, the Kolmogorov-Sinai entropy etc. However, it does provide an important measure of correlations by itself.

In the present paper the correlation function of the trajectory at the Feigenbaum point is numerically investigated with careful numerical experimentation. Comparing with the symbol-to-symbol correlation function discussed in the literature theoretically and numerically we observe that it contains the same time scales, that is, it has the same functional form. This result has been also justified up to a good approximation by presenting simple arguments. Moreover, we have
generalized these results for the case of an arbitrary $m : 2^\infty$ Feigenbaum non chaotic multifractal attractor. Finally, we have arrived to an empirical formula summarizing the results.

To recapitulate, we are in position to justify the analytical form of the correlation function of the trajectory from first principles (the MSS algorithm) and in a systematic way, apart from numerical constants which depend on the detailed functional form of the map. Apart from their mathematical beauty such ideas find important practical applications ranging from precursory signals [40] to DNA sequence analysis [3 41], Heart beat rhythms [11] and Linguistics Processes. In this manner, it is still an open problem what information one can extract by using such complexity measures in real experimental time-series and the physical explanation of the correlation function of the trajectory and that obtained from the symbolic sequence for a specific problem. A second path towards this direction would be the generalization of the form of the correlation function for more complex maps as well as non-unimodal maps, see for instance [12].

[1] J. S. Nicolis, “Chaos and Information Processing,” World Scientific, Singapore, 1991.
[2] G. Nicolis, “Introduction to Nonlinear Science,” Cambridge Univ. Press, 1995.
[3] G. Nicolis, and P. Gaspard, Chaos, Solitons and Fractals 4(1), p. 41, 1994.
[4] L. Athanasopoulou, S. Athanasopoulos, K. Karamanos, and Y. Almirantis, Phys. Rev. E 82, 051917, 2010.
[5] P. Grassberger, Int. J. Theor. Phys. 25(9), p. 907, 1986.
[6] K. Karamanos, Lect. Not. Phys. 550, Springer-Verlag, pp. 357–371, 2000.
[7] K. Karamanos, J. Phys. A: Math. Gen. 34, pp. 9231–9241, 2001.
[8] K. Karamanos, and I. Kotsireas, Kybernetes 31(9/10), pp. 1409–1417, 2002.
[9] K. Karamanos, I. S. Mistakidis, and S. I. Mistakidis, Int. J. Bif. Chaos 23(7), 1350118, 2013.
[10] K. Karamanos, and G. Nicolis, Chaos, Solitons and Fractals 10(1), pp. 1135–1150, 1999.
[11] K. Karamanos, S. Nikolopoulos, G. Manis, A. Alexandridi, K. Hizanides, and S. Nikolakeas, Int. J. Bif. Chaos 16(7), pp. 2093–2101, 2006.
[12] D. Alonso, D. Mckernan, P. Gaspard, and G. Nicolis, Phys. Rev. E 54, pp. 2474–2478, 1996.
[13] H. Bai-Lin, “Chaos,” Singapore: World Scientific, 1994.
[14] J. P. Crutchfield, J. D. Farmer, and B. A. Hubermann, Phys. Rep. 92, p. 45, 1982.
[15] R.M. May, Nature 261, pp. 459–467, 1976.
[16] M. Schröder, “Fractals, Chaos, Power Laws,” New York: Freeman, 1991.
[17] H.G. Schuster, “Deterministic Chaos,” Physik-Verlag, Weinheim, 1984.
[18] K. Karamanos, Int. J. Bif. Chaos 11(6), pp. 1683–1694, 2001.
[19] J. P. Crutchfield, M. Nauenberg, and J. Rudnick, Phys. Rev. Lett. 46, p. 933, 1981.
