Automaticity of spacetime diagrams generated by cellular automata on commutative monoids

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Abstract

It is well-known that the spacetime diagrams of some cellular automata have a fractal structure: for instance Pascal’s triangle modulo 2 generates a Sierpiński triangle. It has been shown that such patterns can occur when the alphabet is endowed with the structure of an Abelian group, provided the cellular automaton is a morphism with respect to this structure. The spacetime diagram then has a property related to $k$-automaticity. We show that this condition can be relaxed from an Abelian group to a commutative monoid, and that in this case the spacetime diagrams still exhibit the same regularity.

Introduction

This works inscribes itself in a series of articles aiming at classifying cellular automata into meaningful subsets. Our starting point here is the well-known fact that Pascal’s triangle modulo 2, which can be computed by a simple cellular automaton performing a XOR, produces a spacetime diagram that converges to a Sierpinski triangle. From there a series of questions emerges. Why? How does it work? Can we characterize a class of cellular automata that exhibit similar behaviors?

Some have studied the graphical limit sets of cellular automata with very lax algebraic structures, or no structure at all — see for instance [vHPS93, vHPS01a, vHPS01b, MJ15]. We shall impose a strong algebraic constraint on the transition rule of the cellular automaton, and look at what can be deduced about its spacetime diagram. Our long term objective is to discuss “summable cellular automata”, for which it makes sense to isolate the influence of a single cell, and where the global transition function can be reconstructed by “summing” all these influences. Predicting the state of a cell in such a cellular automaton is expected to be an easy task, since no interaction is allowed to take place, but actually finding a description of the spacetime diagram is a nontrivial task.

Let us denote $\Sigma$ the alphabet. Instead of the usual local transition function $\Sigma \rightarrow \Sigma$, a summable cellular automaton is naturally defined by a function $\Sigma \rightarrow \Sigma^I$ that describes the influence of each cell on its neighborhood. Arguably, the minimal algebraic structure allowing us to do that is to endow $\Sigma$ with a binary operation $\cdot$ that makes $(\Sigma, \cdot)$ a commutative monoid, and to require that the cellular automaton of interest be an endomorphism of $(\Sigma, \cdot)^{\mathbb{Z}}$. A more in-depth discussion of this algebraic requirement can be found in [GNW10], section 1.1.

Let then $(\Sigma, \cdot)$ be a finite commutative monoid, $I$ some finite subset of $\mathbb{Z}$, and $(f_i)_{i \in I}$ a family of endomorphisms of $\Sigma$. We can then define the following endomorphism $F$ of $\Sigma^\mathbb{Z}$:
The monoid operation is denoted multiplicatively, as it will be throughout this paper, with the exception of Section $\Pi$ $F$ is a cellular automaton on the alphabet $\Sigma$, with neighborhood included in $-I$. Conversely, if $F$ is a cellular automaton over the alphabet $\Sigma$ that is also an endomorphism of $\Sigma^2$, then one can choose a neighborhood $\mathcal{N}$ of $F$, and define, for $i \in I = -\mathcal{N}$ and $s \in \Sigma$,

$$f_i(s) = F(\tilde{s})_i,$$

where $\tilde{s}$ is the word of $\Sigma^2$ defined by $\tilde{s}_n = \begin{cases} s & \text{if } n = 0 \\ e & \text{otherwise} \end{cases}$, $e$ denoting the identity element of $\Sigma$.

The support of a configuration $c \in \Sigma^2$ is defined by $\text{supp}(c) = \{ n \in \mathbb{Z} ; c_n \neq e \}$. We say a configuration is finite if it has finite support. We will consider only finite initial configurations. $\mathbb{Z}_m$ denotes the finite cyclic group of order $m$.

- The case $(\Sigma, \cdot) = (\mathbb{Z}_2, +)$ was treated by Willson in [Wil84]. It includes Pascal’s triangle modulo 2, and describes the fractal structure of the limit spacetime diagram in terms of matrix substitution systems.
- The case $(\Sigma, \cdot) = (\mathbb{Z}_p, +)$ was treated by Takahashi in [Tak92]. It is a generalisation of Willson’s article.
- The case $(\Sigma, \cdot) = (\mathbb{Z}_m, +)$ was treated by Allouche et al. in [AvHP97]. It describes the spacetime diagram in terms of $k$-automaticity, which is another name for matrix substitution systems, and sorts out for which $k$ the spacetime diagram is $k$-automatic and for which $k$ it is not. We will also adopt the language of $k$-automaticity in this paper.
- The case when $(\Sigma, \cdot)$ is a (finite) abelian group was treated in [GNW10]. It uses an example as a cellular automaton already studied by Macfarlane in [Mac04].

Let us introduce briefly the notions used in the statement of the main theorem. A cellular automaton $F$, when running on an initial configuration $c \in \Sigma^2$, produces a spacetime diagram $(F^i(c))_{i \in \mathbb{Z} \times \mathbb{N}}$, that is a double sequence with values in $\Sigma$. The regularity of such double sequences will be described in terms of $k$-automaticity. The reference for all things $k$-automaticity is [AS03], in our case particularly its chapter 14, since we are concerned with double sequences. Actually, we are concerned more specifically with sequences indexed by $(x, y) \in \mathbb{Z} \times \mathbb{N}$ that are, in the language of [AS03] and [RY20], $[-k, k]$-automatic. Since this is the only kind of automaticity we will care about, we will use “$k$-automatic” in lieu of “$[-k, k]$-automatic”.

**Definition 1.** Let $k \geq 2$ be a integer. A double sequence $(U(x, y))_{(x, y) \in \mathbb{Z} \times \mathbb{N}}$ is $k$-

- $U(x, y)$ is a function of $e(x, y)$;
- for $s, t \in [0, k - 1]$, $x \in \mathbb{Z}$ and $y \in \mathbb{N}$, $e(kx + s, ky + t)$ is a function of $s, t$ and $e(x, y)$. 

$$F : \left\{ \begin{array}{c}
\Sigma^2 \\ (r_n)_{n \in \mathbb{Z}} \\
(\prod_{i \in I} f_i(r_{n-i}))_{n \in \mathbb{Z}} \end{array} \right\}.$$
In other words, \((U(x, y))\) is \(k\)-automatic if it can be produced by a deterministic finite automaton with output reading the digits of \(x\) and \(y\) in base \(k\) (see Theorem 14.2.1 of [AS03]). In this fashion, the main result of [GNW10] is:

**Theorem 1.** If \((\Sigma, \cdot)\) is an abelian \(p\)-group, then the double sequence generated by a cellular automaton starting on a finite configuration is \(p\)-automatic.

Moreover, the proof is constructive, in that a finite automaton generating the double sequence from the \(p\)-ary representation of the timespace coordinates of the cell is algorithmically derived from the cellular automaton’s local transition rule.

In order to state our new result, let us introduce a few more notations.

**Definition 2.** Let \(A\) be a nonempty subset of \(\{n \in \mathbb{N}; n \geq 2\}\). A sequence \((U(x, y))\) is \(A\)-automatic if there exists, for each \(k \in A\), a \(k\)-automatic sequence \((V(x, y))\), such that \(U(x, y)\) is a function of \((V(x, y))\), such \(U(x, y)\) is \(\mathbb{N}\)-automatic if it takes values in a finite set \(\mathbb{B}\) and the preimage of every element of \(\mathbb{B}\) is a semilinear subset of \(\mathbb{Z} \times \mathbb{N}\).

In other words, when \(A\) is nonempty, \((U(x, y))\) is \(A\)-automatic if it can be produced by a deterministic finite automaton with output reading \(x\) and \(y\) in all bases \(k \in A\). Note that, if \(A \subseteq B\), \(A\)-automaticity implies \(B\)-automaticity (this includes the case \(A = \emptyset\)). We will use the fact that the semilinear subsets of \(\mathbb{Z} \times \mathbb{N}\) are its rational subsets.

For every element \(x\) of a finite semigroup \((S, \cdot)\), there are least positive integers \(i\) and \(p\) such that \(x^{i+p} = x^i\); these are called respectively the index and the period of \(x\). We denote \(\pi(S)\) the set of prime divisors of periods of elements of \(S\). Note that when \(S\) is a group, by Cauchy’s theorem, \(\pi(S)\) is the set of prime divisors of its order \(|S|\).

We can then state the following corollary of theorem 1.

**Proposition 1.** If \((\Sigma, \cdot)\) is an abelian group, then the double sequence generated by a one-dimensional cellular automaton on a finite initial configuration is \(\pi(\Sigma)\)-automatic.

It is perhaps surprising that, in this statement, \(\pi(\Sigma)\) is always a set of primes. This means that, for instance, by Cobham-Semënov theorem, there is no way to get a 6-automatic spacetime diagram, unless it is actually \(\emptyset\)-automatic.

**Proof.** For each prime number \(p\), let \(\Sigma_p\) the subgroup of \(\Sigma\) of elements of order a power of \(p\); then \(\Sigma\) is isomorphic to \(\prod_p \Sigma_p\), and every endomorphism of \(\Sigma\) factorizes into a product of endomorphisms of the \(\Sigma_p\)-s.

Our main result is then simply that the same statement that is true of groups in Proposition, is true of monoids in general.

**Theorem 2.** If \((\Sigma, \cdot)\) is a commutative monoid, then the double sequence generated by a one-dimensional cellular automaton on a finite initial configuration is \(\pi(\Sigma)\)-automatic.

This paper is organized as follows. Section 1 is an erratum of a lemma in [GNW10], which treated the case of abelian groups: it can be skipped. Section 2 treats the case “orthogonal” to groups, namely aperiodic monoids. It is then shown in Section 3 how these two base cases can be brought together to treat the general case.
1 Groups

This section is an erratum of a Proposition of \cite{GNW10}. However, the main theorem of \cite{GNW10}, reenunciated as Theorem \cite{1} of the present paper, stands and its proof is basically correct, so the reader may skip to Section \ref{section2}. That being said, Proposition 4 of \cite{GNW10} is wrong. What is stated in that paper is the following:

\begin{center}
\begin{tabular}{|c|}
\hline
Let $R$ be a finite commutative ring, $M$ a finite $R$-module, $k$ and $m$ a positive integers, $\Lambda$ a finite set of indices, and for $i \in \Lambda$, $f_i : \llbracket m, +\infty \rrbracket \to \mathbb{Z}$ and $g_i : \llbracket m, +\infty \rrbracket \to \mathbb{Z}$ such that for all $y \in \llbracket m; +\infty \rrbracket$ and $t \in [0, k - 1]$, \\
\begin{itemize}
    \item $g_i(y) < y$;
    \item $f_i(ky + t) = kf_i(y)$ and $g_i(ky + t) = kg_i(y) + t$.
\end{itemize}
\end{tabular}
\end{center}

For $x \in \mathbb{Z} \times \mathbb{N}$, let $\Xi^y_x \in M$ be such that when $y \geq m$,

\[ \Xi^y_x = \sum_{i \in \Lambda} \mu_i \Xi^{g_i(y)}_{x + f_i(y)}. \]

Then there exists a finite set $E$ and a function $e : \mathbb{Z} \times \mathbb{N} \to E$ such that

\begin{itemize}
    \item $\Xi^y_x$ is a function of $e(x, y)$;
    \item for $s, t \in [0; k - 1]$, $e(kx + s, ky + t)$ is a function of $s, t$ and $e(x, y)$.
\end{itemize}

This is false because it implies that the sequence $(e(n, 0))_{n \in \mathbb{N}} \in M^{\mathbb{N}}$ is $p$-automatic, when it can be arbitrary. The proposition can be fixed by assuming that the $\Xi^y_x$-s, for $y \leq m$, are almost all equal to 0. The proof remains essentially the same, and this mistake does not impact other statements of \cite{GNW10}, because the proposition is only applied to cases where the added assumption is true. Let us seize this opportunity to fix the proposition, generalize it from $R$-modules to commutative monoids, and simplify its proof. In this proposition, we will use the additive notation for the monoid operation.

**Proposition 2.** Let $(M, +, 0)$ be a finite commutative monoid, $m$ a positive integer, $k \geq 2$ an integer, $\Lambda$ a finite set of indices, and for $i \in \Lambda$, $f_i : \llbracket m, +\infty \rrbracket \to \mathbb{Z}$ and $g_i : \llbracket m, +\infty \rrbracket \to \mathbb{Z}$ such that for all $y \in \llbracket m; +\infty \rrbracket$ and $t \in [0, k - 1]$, \\

\begin{itemize}
    \item $g_i(y) < y$;
    \item $f_i(ky + t) = kf_i(y)$ and $g_i(ky + t) = kg_i(y) + t$.
\end{itemize}

Let $(\phi_i)_{i \in \Lambda}$ be a family of endomorphisms of $M$, and $\Xi : \mathbb{Z} \times \mathbb{N} \to M$ be such that \{(x, y) \in \mathbb{Z} \times \llbracket 0; m - 1 \rrbracket; \Xi(x, y) \neq 0 \} is finite and when $y \geq m$,

\[ \Xi(x, y) = \sum_{i \in \Lambda} \phi_i \Xi(x + f_i(y), g_i(y)). \]

Then $\Xi$ is $k$-automatic.

**Proof.** Let us recursively define, for $j, y \leq 0$ and $x \in \mathbb{Z}$, the following endomorphisms $\alpha_j(x, y)$ of $M$:
• if \( y < m \) then \( a_j(x, y) = \begin{cases} 
\text{id} & \text{if } (0, j) = (x, y) \\
0 & \text{otherwise} 
\end{cases} \),

• if \( y \geq m \) then \( a_j(x, y) = \sum_{i \in \Lambda} q_i \circ a_j(x + f_i(y), g_i(y)) \)

We can then recursively apply Equation (2) to get:

\[
\Xi(x, y) = \sum_i \sum_{j=0}^{m-1} a_j(x - i, y) \circ \Xi(i, j) \tag{3}
\]

Let \( I \) be the finite set \( \{ i \in \mathbb{Z} : \exists j < m \quad \Xi(i, j) \neq 0 \} \). According to Eq. (3), \( \Xi(x, y) \) is a function of the \( a_j(x - i, y) \)-s for \( (i, j) \in F \).

We now prove by recursion on \( y \) that, for all \( x \in \mathbb{Z}, \ y \in \mathbb{N}, \ j < m \) and \( 0 \leq s, t < k : \)

\[
a_j(kx + s, ky + t) = \sum_{j'} \sum_{j''=0}^{m-1} a_j'(x - i', y) \circ a_j(ki' + s, kj' + t) \tag{4}
\]

• if \( y < m \), then \( a_j'(x - i', y) = \begin{cases} 
\text{id} & \text{if } (0, j') = (x - i', y) \\
0 & \text{otherwise} 
\end{cases} \), therefore

\[
\sum_{j'} \sum_{j''=0}^{m-1} a_j'(x - i', y) \circ a_j(ki' + s, kj' + t) = a_j(kx + s, ky + t).
\]

• if \( y \geq m \) and Eq. (4) is true for strictly smaller values of \( y \), then:

\[
\begin{align*}
\alpha_j(kx + s, ky + t) \\
= \sum_{i \in \Lambda} q_i \circ a_j(kx + s + f_i(ky + t), g_i(ky + t)) \\
= \sum_{i \in \Lambda} q_i \circ a_j(k(x + f_i(y)) + s, kg_i(y) + t) \\
= \sum_{i \in \Lambda} \biggl( \sum_{i' \neq j''=0}^{m-1} a_j(x + f_i(y) - i', g_i(y)) \circ a_j(ki' + s, kj' + t) \biggr) \\
= \sum_{i' \neq j''=0}^{m-1} \biggl( \sum_{i \in \Lambda} q_i \circ a_j(x + f_i(y) - i', g_i(y)) \biggr) \circ a_j(ki' + s, kj' + t) \\
= \sum_{i' \neq j''=0}^{m-1} a_j(x - i', y) \circ a_j(ki' + s, kj' + t)
\end{align*}
\]

Since almost all \( a_j(ki' + s, kj' + t) \)-s for \( j, j' < m, s, t < k \) and \( i', j' \in \mathbb{Z} \), are equal to 0, there exists therefore a finite set of indices \( I'' \supseteq I \) such that, for each \( s, t \in [0; k - 1] \), \( a_j(kx + s - i, ky + t) \) is a function of \( (a_j(x - i, y))_{i \in I', j < m} \). Let us say \( I'' \subseteq \lfloor d_{\min}, d_{\max} \rfloor \).

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Let \( \delta_{\text{max}} \) be such that \( \delta_{\text{max}} \geq \max \left( d_{\text{max}}, d_{\text{max}} + \frac{\delta_{\text{max}}}{k} \right) \) and \( \delta_{\text{min}} \) such that \( \delta_{\text{min}} \leq \min \left( d_{\text{min}}, d_{\text{min}} - 1 + \left( \frac{\delta_{\text{min}} + 1}{k} \right) \right) \). Let \( J = \{ \delta_{\text{min}}, \delta_{\text{max}} \} \) and \( \beta(x, y) = (\alpha_j(x - i, y))_{i \in J, j \in \mathbb{Z}} \). Notice that \( J \supseteq I' \supseteq I \), so that \( \Xi(x, y) \) is a function of \( \beta(x, y) \). Moreover, for each \( s, t < m \), \( \beta(kx + s, ky + t) = (\alpha_j(kx + s - i, ky + t))_{i \in J, j \in \mathbb{Z}} \) depends only on \( \Xi(x - (t' - \left\lfloor \frac{s - i}{k} \right\rfloor))_{i \in J, j \in \mathbb{Z}} \). Indeed, when \( i \in J \) and \( i' \in I \),

\[
\begin{align*}
d_{\text{min}} & \leq t' - \left\lfloor \frac{s - i}{k} \right\rfloor - d_{\text{max}} - 0 \\
d_{\text{min}} - 1 + \left( \frac{\delta_{\text{min}} + 1}{k} \right) & \leq t' - \left\lfloor \frac{s - i}{k} \right\rfloor - d_{\text{max}} - \frac{\delta_{\text{max}}}{k} \\
\delta_{\text{min}} & \leq t' - \left\lfloor \frac{s - i}{k} \right\rfloor - \frac{\delta_{\text{max}}}{k}
\end{align*}
\]

We can then conclude the proof by choosing, in Definition \( E = \text{End}(M)^{I\times[0,m-1]} \) and \( e(x, y)(i, j) = \alpha_j(x - i, y) \).

### 2 Aperiodic Monoids

A monoid \( M \) is aperiodic if the period of all of its elements is 1: for every \( a \in M \), there exists \( n > 0 \) such that \( a^{n+1} = a^n \). On any commutative monoid \( M \), one can define a quasiorer \( : x \leq y \) iff there exists \( z \in M \) such that \( x = yz \). Let \( 1 \) be the identity element of \( M \); for every \( x \in M \), \( x \leq 1 \).

Suppose \( M \) is a commutative aperiodic monoid. Let \( a, b \in M \) be such that \( a \leq b \leq a \). Let \( x, y \in M \) be such that \( a = bx \) and \( b = ay \). Then \( a = \alpha(xy) = \alpha(xy)^n \) for every \( n \geq 0 \). Let \( n \) be such that \( y^n+1 = y^n \). Then \( a = \alpha(xy)^n = \alpha x^n y^n = \alpha x^n y^{n+1} = \alpha(xy)^n y = \alpha y = b \). Therefore \( \leq \) is a preorder on \( M \). In the following proposition, "min" refers to this preorder. Note that min \( 0 = \max M = 1 \).

**Proposition 3.** Let \( M \) be a finite commutative aperiodic monoid. Then there exists \( \omega \in \mathbb{N} \) such that, for every \( n \in \mathbb{N} \) and every finite sequence \( (x_i) \in M^n \),

\[
\prod_{i=1}^{n} x_i = \min \left\{ \prod_{i \in A} x_i; A \subseteq [1, n], |A| \leq \omega \right\} \quad (5)
\]

**Proof.** Let us rewrite \( \prod_{i=1}^{n} x_i = \prod_{x \in M} x^{a_x} \), where \( a_x \) is the number of occurrences of \( x \) in \( (x_i)_{i=1}^{n} \). Let \( N > 0 \) be such that, for every \( x \in M \), \( x^{N+1} = x^N \). Then \( \prod_{i=1}^{n} x_i = \prod_{x a_x \geq 0} x^{\min(a_x, N)} \). The proposition is therefore true for \( \omega = N \times |M| \). This bound is extremely crude, but as long as we do not care for efficiency, it will do.

**Proposition 4.** Let \( \Sigma \) be a finite commutative aperiodic monoid, \( I \) a finite subset of \( \mathbb{Z} \) and \( (f_i)_{i \in I} \) a family of endomorphisms of \( \Sigma \). Let \( F : \Sigma^\mathbb{Z} \to \Sigma^\mathbb{Z} \) be the cellular automaton defined by
\[ F(r) = \prod_{i \in I} f_i(r_{n-i}) \]

Then, on any finite initial configuration, the spacetime diagram generated by \( F \) is \( \emptyset \)-automatic.

**Proof.** Let \( \epsilon \) be the initial configuration with finite support. For every \( (i, j) \in \mathbb{Z} \times \mathbb{N} \),

\[ F^i(\epsilon) = \prod_{x_0, \ldots, x_{i-1}} f_{x_i} f_{x_{i-1}} \cdots f_{x_1}(c_{x_0}). \tag{6} \]

Now, let us define a deterministic finite automaton with output with the following characteristics:

- Its set of states is \( \{ q_0 \} \cup \Sigma^0 \times \mathcal{P}\left([1, \omega]^{[2]}\right) \), where \( \mathcal{P}(X) \) denotes the set of unordered pairs of \( X \) and \( \mathcal{P}(X) \) the power set of \( X \).
- Its alphabet is the disjoint union \( (\text{supp}(\epsilon))^\omega \cup I^\omega \).
- Its transition function is defined in the following way:
  - for each \( (x_1, \ldots, x_m) \in \text{supp}(\epsilon)^\omega \),
    \[ \delta(q_0, (x_1, \ldots, x_m)) = \left( (c_{x_1}, \ldots, c_{x_m}), \{ (i, j); x_i \neq x_j \} \right); \]
  - for each \( (a_1, \ldots, a_m) \in \Sigma^\omega \), \( A \subseteq [1, \omega]^{[2]} \) and \( (x_1, \ldots, x_m) \in I^\omega \),
    \[ \delta \left( ((a_1, \ldots, a_m), A), (x_1, \ldots, x_m) \right) = \left( (f_{x_1}(a_1), \ldots, f_{x_m}(c_{a_m})), A \cup \{ (i, j); x_i \neq x_j \} \right); \]
  - in other cases, \( \delta \) is undefined.

- Its output set is \( \Sigma \). The output of state \( ((a_1, \ldots, a_m), A) \) is \( \prod_{i \in B} a_i \), where \( B \) is a maximal subset of \( [1, \omega] \) such that \( B^{[2]} \subseteq A \).

The idea is that this automaton, instead of following one "path of computation" of the cellular automaton, follows \( \omega \) at once, and keeps track of which pairs of branches are distinct—that is the role of the set of ordered pairs. The output of a state is then obtained by choosing a maximal subsequence of pairwise distinct paths, and multiplying their outputs. This has no meaning unless the paths end up in the same cell of the spacetime diagram. We shall then consider the monoid morphism \( \varphi : (\text{supp}(\epsilon)^\omega \cup I^\omega)^\ast \rightarrow \mathbb{Z}^\omega \times \mathbb{N} \) defined by \( \varphi(\hat{x}) = (\hat{x}, 0) \) for every \( \hat{x} \in \text{supp}(\epsilon)^\omega \) and \( \varphi(\hat{x}) = (\hat{x}, 1) \) for every \( \hat{x} \in I^\omega \).

For each \( a \in \Sigma \), let \( \mathcal{L}_a \) be the rational set of words over \( \text{supp}(\epsilon)^\omega \cup I^\omega \) whose output by this finite automaton is \( a \). Let \( \Delta_a \subseteq \mathbb{Z} \times \mathbb{N} \) be the diagonal of \( \varphi(\mathcal{L}_a) \), i.e.

\[ (i, j) \in \Delta_a \iff (i, \ldots, i, k) \in \varphi(\mathcal{L}_a). \]

Eq. (5) becomes

\[ F^i(\epsilon) = \min \left\{ a \in \Sigma; (i, j) \in \Delta_a \right\}. \tag{7} \]

Since \( \Delta_a \) is a rational subset of \( \mathbb{Z} \times \mathbb{N} \) for every \( a \in M \), it follows that \( (F^i(\epsilon))_{i,j} \) is \( \emptyset \)-automatic.
3 General Case

Let $(\Sigma, \cdot)$ be a finite commutative monoid. An element $x$ of $\Sigma$ is idempotent if $x^2 = x$. Let $(e_{\lambda})_{\lambda \in \Lambda}$ be an enumeration of the idempotents of $\Sigma$. For each $\lambda \in \Lambda$, let

$$S_{\lambda} = \{ x \in \Sigma; \exists n > 0 \ x^n = e_{\lambda} \}.$$ 

$\Sigma = \bigsqcup_{\lambda \in \Lambda} S_{\lambda}$ is a partition of $\Sigma$ into subsemigroups: this is the decomposition of $\Sigma$ into a semilattice of Archimedean semigroups that has been known since [1K54].

Let $\lambda \in \Lambda$ and $x \in S_{\lambda}$ an element of index $i$ and period $p$. $(x^i, x^{i+1}, \ldots, x^{i+p-1})$ is a subgroup of $\langle x \rangle$ that is isomorphic to $\mathbb{Z}/p\mathbb{Z}$; let $f$ be its neutral element. Then there exists $k > 0$ such that $x^k = f$. There exists also $n > 0$ such that $x^n = e_{\lambda}$. We thus get $x^{nk} = (x^n)^k = e_{\lambda}^k = e_{\lambda}$ and $x^{nk} = (x^k)^n = f^n = f$, so $e_{\lambda}$ is the only neutral element of $S_{\lambda}$.

Let $K_{\lambda}$ be the set of elements of $S_{\lambda}$ of index 1, ie

$$K_{\lambda} = \{ x \in S_{\lambda}; \exists p \ x = x^{1+p} \}.$$ 

$K_{\lambda}$ is a subgroup of $S_{\lambda}$, called its kernel. Let $\sim$ be the binary relation defined on $\Sigma$ by: $x \sim y$ iff $\exists \lambda \ x, y \in K_{\lambda}$. $\sim$ is an equivalence relation in $\Sigma$; moreover, it is a congruence on $\Sigma$: if $x \sim y$ and $a \sim b$ then $ax \sim by$. This means $\Sigma/\sim$ has a natural monoid structure, in our case that of a finite commutative aperiodic monoid.

Let $f \in \text{End}(\Sigma), \lambda \in \Lambda$ and $x \in K_{\lambda}$. Let $p$ be such $x = x^{1+p}$. Then $f(x)^{1+p} = f(x^{1+p}) = f(x)$, so $f(x) \in K_{\mu}$ for some $\mu \in \Lambda$. Let $n \geq 0$ be such that $x^n = e_{\lambda}$. Then $f(e_{\lambda})$ is idempotent and $f(e_{\lambda}) = f(x)^n \in K_{\mu}$, so $f(e_{\lambda}) = e_{\mu}$. Therefore, if $x \sim y$, then $f(x) \sim f(y)$, which means we can quotient $f$ to get $\tilde{f} \in \text{End}(\Sigma/\sim)$.

Let $I$ be a finite subset of $\mathbb{Z}$, $(f_i)_{i \in I}$ a family of endomorphisms of $\Sigma$, $c \in \Sigma^I$ a finite configuration, and $F : \Sigma^I \to \Sigma^I$ the cellular automaton defined by

$$F(c) = \prod_{i \in I} f_i(c_{i-1}).$$

We can quotient everything and get a cellular automaton $\tilde{F} : \left(\Sigma/\sim\right)^I \to \left(\Sigma/\sim\right)^I$ such that, for every $(i, j) \in \mathbb{Z} \times \mathbb{N}$, $\tilde{F}_j(c)_i = F_j(c)_i$. So, according to Proposition 4, $(\tilde{F}(c), i, j)$ is $\varnothing$-automatic.

Let us define $\psi$ on $\Sigma$ by: for every $x \in S_{\lambda}, \psi(x) = xe_{\lambda} \in K_{\lambda}$. $\psi$ is an endomorphism of $\Sigma$. Let $K = \prod_{\lambda \in \Lambda} K_{\lambda}$, and let us denote $a_{\lambda}$ the canonical embedding of $K_{\lambda}$ into $K$.

For any endomorphism $f$ of $\Sigma$, there is a unique endomorphism $f_\psi$ of $K$ such that $f_\psi \psi = \psi f$: it is such that for every $x \in S_{\lambda}$, if $f(x) \in S_{\mu}$, $f_\psi(a_\lambda(x)) = a_\mu(f(x))$.

We can thus define a cellular automaton $F_\psi : K^I \to K^I$ such that, for every $(i, j) \in \mathbb{Z} \times \mathbb{N}$, $F_\psi(c)_i = \psi(F_j(c)_i)$. So, according to Theorem 1, $(\psi(F_j(c)), i, j)$ is $\pi(K)$-automatic, or rather $\pi(\Sigma)$-automatic, since $\pi(K) = \pi(\Sigma)$.

But since any element $x \in \Sigma$ can be recovered from $\tilde{x} \in \Sigma/\sim$ and $\psi(x), F_j(c)$ is a function of $F_j(c)$ and $\psi(F_j(c))$; therefore, $(F_j(c), i, j)$ is $\pi(\Sigma)$-automatic.
Conclusion

It is perhaps worth mentioning two things. First, it is possible to separate quite easily a spacetime diagram into its $p$-automatic components, simply by writing the group $K$ from Section 3 as the product of its $p$-subgroups. Second, the proof is constructive: there is an algorithm that, from descriptions of $(\Sigma, \cdot)$, the transition rule and the initial configuration, produces a description of the spacetime diagram in terms of $k$-automatic sequences.

The next natural step to generalize this result would be to remove the identity element, and suppose $(\Sigma, \cdot)$ is a mere finite commutative semigroup. A monoid is just a semigroup with an identity, there isn’t really anything more to it, so we should except the same kind of regularity, but of course the added difficulty is that there is no notion of finite configuration, so one would have to assume that the initial configuration has some regularity itself. According to [RY20], this is true when $\Sigma = \mathbb{Z}_p$: if the initial configuration is $p$-automatic, then so is the spacetime diagram.

Lastly, let us add that it feels like Proposition [2] and/or Theorem [1] should be an easy consequence of some generalization of Christol theorem [Chr79], like the two-dimensional one proven by Salon [Sal86], although it is not yet clear to the author how this would work.

References

[AS03] Jean-Paul Allouche and Jeffrey Shallit. Automatic Sequences: Theory, Applications, Generalizations. Cambridge University Press, 2003.

[AvHP+97] Jean-Paul Allouche, Fritz von Haeseler, Heinz-Otto Peitgen, Antje Petersen, and Guentcho Skordev. Automaticity of double sequences generated by one-dimensional linear cellular automata. Theoretical Computer Science, 188(1-2):195–209, 1997.

[Chr79] Gilles Christol. Ensembles presque périodiques k-reconnaissables. Theoretical Computer Science, 9(1):141–145, 1979.

[GNW10] Johannes Gütschow, Vincent Nesme, and Reinhard F. Werner. The fractal structure of cellular automata on abelian groups. Discrete Mathematics & Theoretical Computer Science, DMTCS Proceedings vol. AL, Automata 2010 - 16th Intl. Workshop on CA and DCS, January 2010.

[Mac04] A J Macfarlane. Linear reversible second-order cellular automata and their first-order matrix equivalents. Journal of Physics A: Mathematical and General, 37(45):10791–10814, oct 2004.

[MJ15] Johannes Müller and Hengrui Jiang. Graphical limit sets for general cellular automata. Theoretical Computer Science, 580:14–27, 2015.

[RY20] Eric Rowland and Reem Yassawi. Automaticity and invariant measures of linear cellular automata. Canadian Journal of Mathematics, 72(6):1691–1726, 2020.

[Sal86] Olivier Salon. Suites automatiques à multi-indices. Séminaire de théorie des nombres de Bordeaux, pages 1–36A, 1986.
[Tak92] Satoshi Takahashi. Self-similarity of linear cellular automata. *Journal of Computer and System Sciences*, 44(1):114–140, 1992.

[TK54] Takayuki Tamura and Naoki Kimura. On decompositions of a commutative semigroup. *Kodai Mathematical Seminar Reports*, 6(4):109–112, 1954.

[vHPS93] Fritz von Haeseler, Heinz-Otto Peitgen, and Guentcho Skordev. Cellular automata, matrix substitutions and fractals. *Annals of Mathematics and Artificial Intelligence*, 8(3):345–362, 1993.

[vHPS01a] Fritz von Haeseler, Heinz-Otto Peitgen, and Guentcho Skordev. Self-similar structure of rescaled evolution sets of cellular automata I. *International Journal of Bifurcation and Chaos*, 11:913–926, 2001.

[vHPS01b] Fritz von Haeseler, Heinz-Otto Peitgen, and Guentcho Skordev. Self-similar structure of rescaled evolution sets of cellular automata II. *International Journal of Bifurcation and Chaos*, 11(4):927–942, 2001.

[Wil84] Stephen J Willson. Cellular automata can generate fractals. *Discrete Applied Mathematics*, 8(1):91–99, 1984.