Point classification of the second order ODE’s and its application to Painleve equations

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Abstract. The first part of this work is a review of the point classification of second order ODEs done by Ruslan Sharipov. His works were published in 1997-1998 in the Electronic Archive at LANL. The second part is an application of this classification to Painlevé equations. In particular, it allows us to solve the equivalence problem for Painlevé equations in an algorithmic form.

1 Introduction

It is a well-known fact that the following class of the second order ODEs

\[ y'' = P(x,y) + 3Q(x,y)y' + 3R(x,y)y'^2 + S(x,y)y'^3 \]  

(1)

is closed under the generic point transformations

\[ \tilde{x} = \tilde{x}(x,y), \quad \tilde{y} = \tilde{y}(x,y). \]  

(2)

It means that the transformed equation is again given by (1), but with some other coefficients:

\[ \tilde{y}'' = \tilde{P}(\tilde{x},\tilde{y}) + 3\tilde{Q}(\tilde{x},\tilde{y})\tilde{y}' + 3\tilde{R}(\tilde{x},\tilde{y})\tilde{y}'^2 + \tilde{S}(\tilde{x},\tilde{y})\tilde{y}'^3. \]  

(3)

Suppose we are given two arbitrary equations (1) and (3). The problem of existence of the change of variables (2) that transforms equations (1) and (3) one into the other is called the Equivalence Problem. If we apply transformation (2) for equation (1), we get the explicit formulas for the coefficients \( \tilde{P}(\tilde{x}(x,y),\tilde{y}(x,y)), \tilde{Q}(\tilde{x}(x,y),\tilde{y}(x,y)), \tilde{R}(\tilde{x}(x,y),\tilde{y}(x,y)), \) and \( \tilde{S}(\tilde{x}(x,y),\tilde{y}(x,y)) \) in terms of \( P(x,y), Q(x,y), R(x,y), S(x,y) \) and the partial derivations of the unknown functions \( \tilde{x}(x,y) \) and \( \tilde{y}(x,y) \) on \( x \) and \( y \) up to the third order. These formulas are rather complicated, and in general situation the equivalence problem can not be solved explicitly.

The main approach usually employed is to find invariants of equations (1). Invariant is a function that is preserved by transformations (2), i.e., \( I(x,y) = I(\tilde{x}(x,y),\tilde{y}(x,y)) \). Invariants Theory of
equations \([1]\) was initiated in the works of R.Liouville \([14]\), S.Lie \([13]\), A.Tresse \([17, 18]\), E.Cartan \([4, 16]\) (Late 19th- and Early 20th-Century) and continued in the Late 20th-Century in works \([6, 9, 7, 3, 1]\) and others. Background is described in papers of L. Bordag \([1, 2]\).

However, only advanced computer software for symbolic calculations gave an opportunity to make a substantial progress. In the series of papers \([19, 20, 21]\) Ruslan Sharipov succeeded to construct the system of (pseudo)invariants which he calculated explicitly in the terms of the coefficients of equations \([1]\). On the basis of this system he classified equations \([1]\). This classification is more general than all previous ones. The relation between the (pseudo)invariants from works \([20, 21]\) and the semiinvariants from works \([4, 14]\) (as they were presented in \([2]\)) was shown in paper \([11]\) and here in Section 7. Moreover, in all possible cases the set of the invariants can be broadened. By employing this technique, in \([10]\), \([11]\) and \([12]\) the equivalence problem for some equations was solved.

The first part of the present paper is a survey of \([19, 20, 21]\). We also add additional subcases (see Subsection 5.8) not mentioned in the cited works. The second part is an application of this classification for studying Painlevé equations.

## 2 Classification

**Pseudoinvariant of weight** \(m\) is a function transformed under transformations \([2]\) with the factor \(\det T\) (the Jacobi determinant) in the power \(m\),

\[
J(x, y) = (\det T)^m \cdot J(\tilde{x}(x, y), \tilde{y}(x, y)), \quad T = \begin{pmatrix}
\frac{\partial \tilde{x}}{\partial x} & \frac{\partial \tilde{x}}{\partial y} \\
\frac{\partial \tilde{y}}{\partial x} & \frac{\partial \tilde{y}}{\partial y}
\end{pmatrix}.
\]

**Pseudotensorial field of weight** \(m\) and **valence** \(r, s\) is an indexed set transformed under change of variables \([2]\) by the rule

\[
F_{j_1 \ldots j_s}^{i_1 \ldots i_r} = (\det T)^m \sum_{p_1 \ldots p_r q_1 \ldots q_s} S_{p_1}^{i_1} \cdots S_{p_r}^{i_r} T_{j_1}^{q_1} \cdots T_{j_s}^{q_s} \tilde{F}_{q_1 \ldots q_s}, \quad \text{where} \quad S = T^{-1}.
\]

Given the coefficients \(P, Q, R,\) and \(S\) of equation \([1]\), we introduce a three-dimensional array by the rule

\[
\Theta_{111} = P, \quad \Theta_{121} = \Theta_{211} = \Theta_{112} = Q, \quad \Theta_{122} = \Theta_{212} = \Theta_{221} = R, \quad \Theta_{222} = S.
\]

As “Gramian matrices” we take the following two,

\[
d^{ij} = d_{ij} = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}, \quad d^{ij} \text{ is a pseudotensorial field of weight } 1, \quad d_{ij} \text{ is of weight } -1.
\]

We raise the first index

\[
\Theta_{ij}^k = \sum_{r=1}^2 d^{kr} \Theta_{rj}, \quad (4)
\]
Under the change of variables (2) the quantities $\Theta^k_{ij}$ are transformed “almost” as an affine connection (for transformation rule see [19]).

Using $\Theta^k_{ij}$ as the affine connection, we construct the “curvature tensor”

$$\Omega^r_{ij} = \frac{\partial \Theta^k_{ir}}{\partial u^j} - \frac{\partial \Theta^k_{jr}}{\partial u^i} + \sum_{q=1}^{2} \Theta^q_{ir} \Theta^r_{jq} - \sum_{q=1}^{2} \Theta^q_{jr} \Theta^r_{iq},$$

here $u^1 = x$, $u^2 = y$, and the “Ricci tensor” $\Omega^r_{ij} = \sum_{k=1}^{2} \Omega^k_{rkj}$. Both these objects are not tensors. On the contrary, the three-dimensional array

$$W^r_{ijk} = \nabla_i \Omega^r_{jk} - \nabla_j \Omega^r_{ik}$$

is a tensor. Here we employ $\Theta^k_{ij}$ in covariant differentiation instead of the affine connection.

Using the tensor $W^r_{ijk}$, we introduce two extra pseudocovectorial fields,

$$\alpha_i = \frac{1}{2} \sum_{j=1}^{2} \sum_{k=1}^{2} W^r_{ijk} d^j$$

is a pseudocovectorial field of weight 1,

$$\beta^i = 3 \nabla_i \alpha_k d^k \alpha_r + \nabla_r \alpha_k d^k \alpha_i$$

is a pseudocovectorial field of weight 3.

The pseudovectorial fields are $\alpha^j = d^j \alpha_k$ of weight 2 and $\beta^i = d^i \beta_i$ of weight 4.

There are only three possible cases:

1. maximal degeneration case, in which $\alpha=0$;
2. intermediate degeneration case, in which $3F^5 = \alpha^i \beta_i = 0$, i.e. the fields $\alpha$ and $\beta$ are collinear;
3. general case, in which $3F^5 = \alpha^i \beta_i \neq 0$, i.e. the fields $\alpha$ and $\beta$ are non-collinear.

3 Maximal degeneration case

The coordinates of the pseudovectorial field $\alpha$ are $\alpha^1 = B$, $\alpha^2 = -A$, where

$$A = P_{0.2} - 2Q_{1.1} + R_{2.0} + 2PS_{1.0} + SP_{1.0} - 3PR_{0.1} - 3RP_{0.1} - 3QR_{1.0} + 6QQ_{0.1},$$
$$B = S_{2.0} - 2R_{1.1} + Q_{0.2} - 2SP_{0.1} - PS_{1.0} + 3QS_{1.0} + 3QS_{1.0} + 3QR_{0.1} - 6RR_{1.0}.$$  \hfill (5)

Hereinafter symbol $K_{i,j}$ denotes the partial differentiation: $K_{i,j} = \partial^i \partial_j K / \partial x^i \partial y^j$. In this case both the conditions $A = 0$ and $B = 0$ hold.

**Theorem 1** (Lie). All equations (1) with $A = 0$ and $B = 0$, where $A$ and $B$ given by (5), are equivalent to $\tilde{y}'' = 0$ by point transformation (2). They have an 8-dimension point symmetries algebra.

For the details see papers [14], [6], and others.
4 General case

The pseudovectorial fields $\alpha = (B, -A)$ and $\beta = (G, H)$ are non-collinear, so their scalar product is non-zero. The pseudoinvariant $F$ defined as

$$3F^5 = AG + BH,$$

where $A$ and $B$ are from (1),

$$G = -BB_{1,0} - 3AB_{0,1} + 4BA_{0,1} + 3SA^2 - 6RBA + 3QB^2,$$

$$H = -AA_{0,1} - 3BA_{1,0} + 4AB_{1,0} - 3PB^2 + 6QAB - 3RA^2,$$

is of weight 5. Since $F \neq 0$, the functions $\varphi_1 = -\partial \ln F / \partial x$ and $\varphi_2 = -\partial \ln F / \partial y$ are well-defined.

Employing $\Theta^k_i$ in relations (1), we construct an affine connection $\Gamma^k_{ij}$ and two non-collinear vectorial fields $X$ and $Y$

$$\Gamma^k_{ij} = \Theta^k_{ij} - \frac{\varphi_i \delta^k_j + \varphi_j \delta^k_i}{3}, \quad X = \frac{\alpha}{F^2}, \quad Y = \frac{\beta}{F^2}.$$

Their covariant derivatives are linear combinations of the basis fields $X$ and $Y$,

$$\nabla_X X = \hat{\Gamma}^1_{11} X + \hat{\Gamma}^2_{11} Y, \quad \nabla_X Y = \hat{\Gamma}^1_{12} X + \hat{\Gamma}^2_{12} Y,$$

$$\nabla_Y X = \hat{\Gamma}^1_{21} X + \hat{\Gamma}^2_{21} Y, \quad \nabla_Y Y = \hat{\Gamma}^1_{22} X + \hat{\Gamma}^2_{22} Y.$$

Here $\hat{\Gamma}^k_{ij}$ are scalar invariants of equation (1). In paper [20] they were denoted by

$$I_3 = \hat{\Gamma}^1_{12}, \quad I_6 = \hat{\Gamma}^2_{21}, \quad I_7 = \hat{\Gamma}^3_{22}, \quad I_8 = \hat{\Gamma}^4_{22}.$$

Differentiating these invariants along vector fields $X$ and $Y$ produces more invariants

$$XI_k = I_{k+8}, \quad YI_k = I_{k+16}.$$

Repeating the procedure of differentiation along $X$ and $Y$, we can construct an infinite sequence of invariants. The explicit formulas for the basic four invariants read as

$$I_3 = \frac{B(HG_{1,0} - GH_{1,0})}{3F^9} - \frac{A(HG_{0,1} - GH_{0,1})}{3F^9} + \frac{HF_{0,1} + GF_{1,0}}{3F^5} +$$

$$\frac{BG^2 P}{3F^9} - \frac{(AG^2 - 2HBG)Q}{3F^9} + \frac{(BH^2 - 2HAG)R}{3F^9} - \frac{AH^2 S}{3F^9},$$

$$I_6 = \frac{H(AB_{0,1} - BA_{0,1})}{3F^9} + \frac{G(AB_{1,0} - BA_{1,0})}{4F^7} - \frac{(AF_{0,1} - BF_{1,0})}{3F^3},$$

$$I_7 = \frac{GB^2 P}{3F^9} - \frac{(HB^2 - 2GBA)Q}{3F^9} - \frac{(GA^2 - 2HBA)R}{3F^9} - \frac{HA^2 S}{3F^9},$$

$$I_8 = \frac{G(AG_{1,0} - BH_{1,0})}{3F^9} + \frac{H(AG_{0,1} - BH_{0,1})}{3F^9} - \frac{10(HF_{0,1} + GF_{1,0})}{3F^5} -$$

$$\frac{BG^2 P}{3F^9} + \frac{(AG^2 - 2HBG)Q}{3F^9} - \frac{(BH^2 - 2HAG)R}{3F^9} + \frac{AH^2 S}{3F^9}.$$

The case of general position splits into three subcases:

1. in the infinite sequence of invariants $I_k$ there exist two functionally independent ones; in this case the dimension of the point symmetries algebra is $\text{dim}(Z) = 0$;
2. invariants $I_k$ are functionally dependent, but not all of them are constants; in this case $\dim(Z) = 1$;

3. all invariants in the sequence $I_k$ are constants; here $\dim(Z) = 2$.

**Example.** For equation (6.54) in the handbook by E. Kamke [8]

$$y'' = y^2 + 4yy' + y^2y^2 \quad \text{we have} \quad \dim(Z) = 1.$$  

5 **Intermediate degeneration case**

In this case $F = 0$, but $A \neq 0$ or $B \neq 0$, and the pseudovectorial fields $\alpha$ and $\beta$ are collinear.

In the case $A \neq 0$ by $\varphi_1$ and $\varphi_2$ we redenote the functions

$$\varphi_1 = -3\frac{B + A_1}{5A} + \frac{3}{5}Q, \quad \varphi_2 = 3\frac{B + A_1}{5A^2} - 3\frac{B + A_1 + 3BQ}{5A^2} + \frac{6}{5}R, \quad (7)$$

and in the case $B \neq 0$ we let

$$\varphi_1 = -3A \frac{AS - B_1}{5B^2} - 3\frac{A_0 + B_1}{5B} - \frac{6}{5}Q, \quad \varphi_2 = 3\frac{AS - B_1}{5B} - \frac{3}{5}R. \quad (8)$$

Employing the introduced functions and $\Theta^5_{ij}$ from [4], we construct the affine connection $\Gamma^k_{ij}$ and a pseudoinvariant $\Omega$ of weight 1,

$$\Gamma^k_{ij} = \Theta^k_{ij} - \frac{\varphi_i\delta^k_j + \varphi_j\delta^k_i}{3}, \quad \Omega = \frac{5}{3} \left( \frac{\partial \varphi_1}{\partial y} + \frac{\partial \varphi_2}{\partial x} \right). \quad (9)$$

As $A \neq 0$, the explicit formula for the pseudoinvariant $\Omega$ reads as

$$\Omega = 2\frac{BA_1(BP + A_1)}{A^3} - \frac{(2B_1 + 3BQ)A_1}{A^2} + \frac{(A_0 - 2B_1)BP}{A^2} - \frac{BA_2 + B^2P_1}{A^2} + \frac{B_2}{A} + \frac{3B_1Q + 3BQ_1B - B_1P - B_1P}{A} + Q_01_ - 2R_1_0. \quad (10)$$

And in the case $B \neq 0$ the similar formula is

$$\Omega = 2\frac{AB_0_1(AS - B_0_1)}{B^3} - \frac{(2A_0 - 3AR)B_0_1}{B^2} + \frac{(B_1 - 2A_0)AS}{B^2} + \frac{AB_0 + A^2S_0_1}{B} - \frac{A_0_2}{B} + \frac{3A_1R + 3AR_0_1 - A_1S - AS_1}{B} + R_1_0 = -2Q_0_1. \quad (11)$$

The rule of covariant differentiation of the pseudotensorial field was given in [19].

$$\nabla_k F_{j_1\ldots j_r} = \frac{\partial F_{j_1\ldots j_r}}{\partial u^k} + \sum_{r=1}^s \sum_{r=1}^s \Gamma^k_{j_kj_r} F_{j_1\ldots j_r} - \sum_{r=1}^s \sum_{r=1}^s \Gamma^k_{j_kj_r} F_{j_1\ldots j_r} + m\varphi_k F_{j_1\ldots j_r}.$$  

If a pseudotensorial field $F$ has valence $(r, s)$ and weight $m$, the pseudotensorial field $\nabla F$ has valence $(r, s + 1)$ and weight $m$.

The pseudovectorial fields $\alpha$ and $\beta$ are collinear, hence there exists a coefficient $N$ such that $\beta = 3N\alpha$. $N$ is a pseudoinvariant of weight 2.

We let

$$\xi^i = d^i j j, \quad M = -\alpha_i \xi^i, \quad \gamma = -\xi - 2\Omega \alpha, \quad (12)$$

5
Here $\xi$ is a pseudovectorial field of weight 3, $M$ is a pseudoinvariant of weight 4, $\gamma$ is a pseudovectorial field of weight 3.

In the cases $A \neq 0$ and $B \neq 0$ the pseudoinvariant $N$ is given by the formulas

$$N = \frac{H}{3A}, \quad N = \frac{G}{3B},$$  \hspace{1cm} (13)

respectively. The pseudovectorial field $M$ in the case $A \neq 0$ reads as

$$M = -\frac{12BN(BP + A_{1.0})}{5A} + BN_{1.0} + \frac{24}{5}BNQ + \frac{6}{5}NB_{1.0} + \frac{6}{5}NA_{0.1} - AN_{0.1} - \frac{12}{5}ANR,$$  \hspace{1cm} (14)

and in the case $B \neq 0$ it is given by the formula

$$M = -\frac{12AN(AS - B_{0.1})}{5B} - AN_{0.1} + \frac{24}{5}ANR - \frac{6}{5}NA_{0.1} - \frac{6}{5}NB_{1.0} + BN_{1.0} - \frac{12}{5}BNQ.$$  \hspace{1cm} (15)

In the case $A \neq 0$ the field $\gamma$ is

$$\gamma^1 = -\frac{6BN(BP + A_{1.0})}{5A^2} + \frac{18NBQ}{5A} + \frac{6N(B_{1.0} + A_{0.1})}{5A} - N_{0.1} - \frac{12}{5}NR - 2\Omega B,$$
$$\gamma^2 = -\frac{6N(BP + A_{1.0})}{5A} + N_{1.0} + \frac{6}{5}NQ + 2\Omega A.$$  \hspace{1cm} (16)

In the case $B \neq 0$ the field $\gamma$ is

$$\gamma^1 = -\frac{6N(AN - B_{0.1})}{5B} - N_{0.1} + \frac{6}{5}NR - 2\Omega B,$$
$$\gamma^2 = -\frac{6AN(AS - B_{0.1})}{5B^2} + \frac{18NAR}{5B} - \frac{6N(A_{0.1} + B_{1.0})}{5B} + N_{1.0} - \frac{12}{5}NQ + 2\Omega A.$$  \hspace{1cm} (17)

### 5.1 First case of intermediate degeneration: $M \neq 0$

If in (14), (15) $M \neq 0$, the pseudovectorial fields $\alpha$ in (5) and $\gamma$ in (10), (17) are non-collinear. Moreover, it means that $N \neq 0$ in (13). Consider the expansion $\nabla \gamma = \hat{\Gamma}_{22}^1 \alpha + \hat{\Gamma}_{22}^2 \gamma$. The basic invariants are the following ones,

$$I_1 = \frac{M}{N^2}, \quad I_2 = \frac{\Omega^2}{N}, \quad I_3 = \frac{\hat{\Gamma}_{22}^1}{M}.$$  \hspace{1cm} (18)

Here $M$, $N$, and $\Omega$ are from (14), (15), (13), (10), (11). The explicit formula for $\hat{\Gamma}_{22}^1$ is

$$\hat{\Gamma}_{22}^1 = \frac{\gamma^1 \gamma^2 (\gamma^1_0 - \gamma^2_1) + (\gamma^2)^2 \gamma^1_0 - (\gamma^1)^2 \gamma^2_0 + 3Q(\gamma^1)^2 \gamma^2 + P(\gamma^1)^3 + 3Q(\gamma^1)^2 \gamma^2 + 3R(\gamma^1)^2 (\gamma^2)^2 + S(\gamma^2)^3}{M}.$$  \hspace{1cm} (19)

By differentiating the invariants $I_1$, $I_2$ and $I_3$ along fields $\alpha$ and $\gamma$ we get new invariants

$$I_{k+3} = \frac{\nabla_\alpha I_k}{N}, \quad I_{k+6} = \frac{(\nabla_\gamma I_k)^2}{N^3}.$$  \hspace{1cm} (19)

The first case of intermediate degeneration splits into three subcases

1. In the infinite sequence of invariants $I_k$ there exist two functionally independent ones; in this case the dimension of the point symmetries algebra is $\text{dim}(Z) = 0$;

2. invariants $I_k$ are functionally dependent but not all of them are constants; here we have $\text{dim}(Z) = 1$;
3. all invariants in the sequence $I_k$ are constants; here $\dim(Z) = 2$.

Example. For equation (6.45) in the handbook by E. Kamke [8]

$$y'' = ay^2 + by, \quad ab \neq 0,$$

one has $\dim(Z) = 1$.

5.2 Second case of intermediate degeneration

If in (14), (15) $M = 0$, the pseudovectorial fields $\alpha$ in (5) and $\gamma$ in (16), (17) are collinear. Hence, there exists a coefficient $\Lambda$ such that $\gamma = \Lambda \alpha$. Here $\Lambda$ is a pseudoinvariant of weight 1, in the cases $A \neq 0$ and $B \neq 0$ being respectively

$$\Lambda = -\frac{\gamma^2}{A}, \quad A \neq 0, \quad \text{or} \quad \Lambda = \frac{\gamma^1}{B}, \quad B \neq 0.$$

The explicit formulas for $\Lambda$ are

$$\Lambda = -\frac{6N(AS - B_{0,1})}{5B^2} - \frac{N_{0,1}}{B} + \frac{6NR}{5B} - 2\Omega.$$  \hfill (20)

$$\Lambda = \frac{6N(BP + B_{1,0})}{5A^2} - \frac{N_{1,0}}{A} - \frac{6NQ}{5A} - 2\Omega.$$  \hfill (21)

Let us calculate the curvature tensor using the connections (9):

$$R_{qij} = \frac{\partial \Gamma_{ik}^{q}}{\partial u^l} - \frac{\partial \Gamma_{ik}^{q}}{\partial u^j} + \sum_{s=1}^{2} \Gamma_{is}^{q} \Gamma_{jq}^{s} - \sum_{s=1}^{2} \Gamma_{js}^{q} \Gamma_{iq}^{s}, \quad u^1 = x, \quad u^2 = y,$$

and the pseudotensorial field of the weight 1:

$$R_q^k = \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} R_{qij} d^{ij},$$

where $\lambda_1$ and $\lambda_2$ are its eigenvalues. Now we construct the pseudocovectorial field of the weight -1.

If $A \neq 0$, we let

$$\omega_1 = -\frac{R_1^2}{A}, \quad \omega_2 = \frac{\lambda_2 - R_2^2}{A},$$  \hfill (22)

where

$$\begin{align*}
\omega_1 &= \frac{12PR}{5A} + \frac{54Q^2}{25A} - \frac{P_{0,1}}{A} + \frac{6Q_{1,0}}{5A} - \frac{PA_{0,1} + BP_{1,0} + A_2}{5A^2} - \frac{2B_{1,0}P}{5A^2} + \frac{3QA_{1,0} - 12PBQ}{25A^2} + \frac{6B^2P^2 + 12A_{1,0}BP + 6A_2}{25A^3}, \\
\omega_2 &= \frac{6A + 3\Omega}{5A} + \frac{-5BP_{0,1} + 6BQ_{1,0} + 12RP}{25A^2} + \frac{54BQ^2}{25A^2} - \frac{12B^2PQ}{25A^3} + \frac{3BQA_{1,0}}{25A^3} - \frac{2B_{1,0}P + BA_{0,1}P + B^2P_{1,0} + BA_2}{5A^3} + \frac{6BA_{1,0}^2 + 6B^3P^2 + 12B^2A_{1,0}P}{25A^4}.
\end{align*}$$

And if $B \neq 0$,

$$\omega_1 = \frac{R_1^2 - \lambda_2}{B}, \quad \omega_2 = \frac{R_2}{B},$$  \hfill (23)
We introduce one more pseudocovectorial field of weight 1,
\[ w = N\omega + \nabla\Lambda + \frac{1}{3}\nabla\Omega. \]

It is collinear to the pseudovectorial field \( \alpha \) in (5), and thus there exists \( K \) such that \( w = K\alpha \).

\[
K = \frac{\Lambda_{1,0} + \Lambda \varphi_1}{A} + \frac{\Omega_{1,0} + \Omega \varphi_1}{3A} + \frac{N\omega_1}{A}, \quad A \neq 0. 
\]

\[
K = \frac{\Lambda_{0,1} + \Lambda \varphi_2}{B} + \frac{\Omega_{0,1} + \Omega \varphi_2}{3B} + \frac{N\omega_2}{B}, \quad B \neq 0. 
\]

By \( \varepsilon \) we denote a pseudocovectorial field of weight 1,
\[ \varepsilon = N\omega + \nabla\Lambda. \]

Raising indices by matrix \( d^{ij} \), we get the pseudovectorial field \( \varepsilon \) of weight 2,
\[ \varepsilon^1 = N\omega_2 + \Lambda_{0,1} + \varphi_2\Lambda, \quad \varepsilon^2 = -N\omega_1 - \Lambda_{1,0} + \varphi_1\Lambda. \]

The fields \( \varepsilon \) in (26) and \( \alpha \) in (5) are non-collinear, and we can write
\[ \nabla_\varepsilon \varepsilon = \Gamma^1_{122}\alpha + \Gamma^1_{222}\varepsilon. \]

\[
\Gamma^1_{12} = \frac{5\varepsilon^1\varepsilon^2(\varepsilon^1_{1,0} - \varepsilon^1_{0,1})}{3N\Omega} + \frac{5(\varepsilon^2)^2\varepsilon^1_{1,0} - 5(\varepsilon^1)^2\varepsilon^1_{0,1}}{3N\Omega} + \\
\frac{5P(\varepsilon^1)^3 + 15Q(\varepsilon^1)^2\varepsilon^2 + 15R(\varepsilon^1)^2 + 5S(\varepsilon^2)^3}{3N\Omega}. 
\]

We introduce pseudoscalar fields
\[ L = KN + \frac{5}{9}N + 3A\Omega + \frac{7}{9}\Omega^2 + 2\Lambda^2. \]

\[ E = \Gamma^1_{22} - \frac{\nabla_\varepsilon L}{N} + \frac{4\Lambda\nabla_\varepsilon \Lambda}{N} + \frac{17\Omega\nabla_\varepsilon \Lambda}{6N} + \frac{12L^2}{5N} + \frac{53L\Lambda}{5N} - \frac{48L^2}{5N} - \\
\frac{62L^2\Omega^2}{15N} + \frac{8L}{3} + \frac{48L^4}{5N} + \frac{106L^3\Omega}{5N} + \frac{16L^2}{3} + \\
\frac{1163L^2\Omega^2}{60N} + \frac{137L\Omega^3}{9} + \frac{50L\Omega^2}{108} + \frac{203L^2}{135N} + \frac{77L^4}{27}. \]

Employing the above objects, we can define invariants
\[ I_1 = \frac{\Lambda^{12}}{\Omega^2 N^2}, \quad I_2 = \frac{L^4}{N^2 \Omega^4}, \quad I_3 = \frac{E^8 N^4}{\Omega^4}. \]

Here \( \Lambda \) is from (20), (21), \( \Omega \) is from (10), (11), \( N \) is from (13), \( L \) is from (35), \( E \) is from (28): In the second case of intermediate degeneration the algebra of the point symmetries of equation (1) is 1-dimensional if and only if all invariants \( I_1, I_2, I_3 \) are identically constant. In other cases it is trivial.
5.3 Third case of intermediate degeneration

In this case $N \neq 0$ in (13), $M = 0$ in (14), (15), $\Omega = 0$ in (10), (11), $\Lambda \neq 0$ in (20), (21). Consider again the pseudocovectorial field $\omega$ of the weight -1 from (22), (23). Raising indices by the matrix $d^{ij}$, we get the vector field $\omega$, $\omega^1 = \omega_2$, $\omega^2 = -\omega_1$. Since $\Lambda \neq 0$, $\omega$, and $\alpha$ are non-collinear, we obtain the following relation

$$\nabla \omega = \hat{\Gamma}^1_{12} \alpha + \hat{\Gamma}^2_{22} \omega,$$

$$\hat{\Gamma}^1_{22} = \frac{5\omega^1 \omega^2 (\omega^1_0 - \omega^2_0)}{\Lambda} + \frac{5(\omega^2)^2 \omega^1_0 - 5(\omega^1)^2 \omega^2_0}{\Lambda} + \frac{5P(\omega^1)^3 + 15Q(\omega^1)^2 \omega^2 + 15R\omega^1(\omega^2)^2 + 5S(\omega^2)^3}{6\Lambda}.$$

In this case we define new $L$ and $E$,

$$L = K + \frac{5}{9} + \frac{2\Lambda^2}{N}, \quad \text{with } K \text{ from (24), (25)} \quad (29)$$

$$E = \frac{\nabla \omega L}{N} + \frac{9L^2}{5N} - \frac{2L}{N} = \frac{12\Lambda^2}{3N^2} + \frac{7\Lambda^2}{N} + \frac{5}{9N} + \frac{63\Lambda^4}{20N^3}.$$

Let us construct the invariants:

$$I_1 = \frac{L^8 N^6}{\Lambda^{12}}, \quad I_2 = \frac{E N^3}{\Lambda^4}.$$

Here $L$, $E$ are from (29), $N$ is from (13), $\Lambda$ is from (20), (21).

In the third case of intermediate degeneration the algebra of the point symmetries of equation (1) is 1-dimensional if and only if both the invariants $I_1$, $I_2$ are identically constant. In other cases it is trivial.

Example. For Emden-Fowler equation (6.11) with $n = -3$ in the handbook by E. Kamke [8]

$$y'' = \frac{-ax^m}{y^3}, \quad a \neq 0, \quad \text{one has } \dim(Z) = 1.$$

5.4 Fourth case of intermediate degeneration

In this case $N \neq 0$ in (13), $M = 0$ in (14), (15), $\Omega = 0$ in (10), (11), $\Lambda = 0$ in (20), (21), $K \neq -5/9$ in (21), (25). Consider again the vectorial field $\omega$, $\omega^1 = \omega_2$, $\omega^2 = -\omega_1$, from (22), (23). Since $\Lambda = 0$, $\omega$, and $\alpha$ are collinear, we can define a new scalar field $\Theta$ by the relationship $\omega = \Theta \alpha$,

$$\Theta = \frac{\omega^1}{A}, \quad A \neq 0, \quad \Theta = \frac{\omega^2}{B}, \quad B \neq 0. \quad (30)$$

The covariant differential $\theta = \nabla \Theta$ is a pseudocovectorial field of weight -2,

$$\theta_1 = \Theta_{1,0} - 2\phi_1 \Theta, \quad \theta_2 = \Theta_{0,1} - 2\phi_2 \Theta. \quad (31)$$

The corresponding pseudovectorial field of weight -1 is $\theta^1 = \theta_2$, $\theta^2 = -\theta_1$. Let us calculate its convolution with $\alpha$ from (25),

$$L = \frac{5}{9} \sum_{i=1}^{2} \alpha_i \theta^i. \quad (32)$$

And the relation $L = K + 5/9$ holds true, where $K$ is from (24), (25).
Since $L \neq 0$, fields $\theta$ (31) and $\alpha$ (5) are non-collinear,

\[ \nabla_\theta \theta = \hat{\Gamma}_{22}^1 \alpha + \hat{\Gamma}_{22}^2 \theta, \]

\[ \hat{\Gamma}_{22}^1 = -\frac{5\theta^1 \theta^2 (\theta^1_{0,0} - \theta^2_{0,1})}{9L} - \frac{5(\theta^2)^2 \theta^2_{0,1} - 5(\theta^1)^2 \theta^1_{0,1}}{9L} - \frac{5P(\theta^1)^3 + 15Q(\theta^1)^2 \theta^2 + 15R(\theta^1)^2 + 5S(\theta^2)^3}{9L}. \]

We introduce one more pseudoscalar field,

\[ E = \hat{\Gamma}_{22}^1 + \frac{27N}{5} \left( \Theta + \frac{5}{9N} \right)^3 - \frac{3}{4} \left( \Theta + \frac{5}{9N} \right)^2 \]

that gives rise to the invariant

\[ I_1 = \frac{\bar{E}^{0} N^{12}}{L^{20}}, \]

where $E$ is from (33), $N$ is from (13), $L$ is from (32).

In the fourth case of intermediate degeneration the algebra of the point symmetries of equation (1) is 1-dimensional if and only if the invariant $I_1$ is identically constant. Otherwise it is trivial.

5.5 Fifth case of intermediate degeneration

In this case $N \neq 0$ in (13), $M = 0$ in (14), (15), $\Omega = 0$ in (10), (11), $\Lambda = 0$ in (20), (21), $K = -5/9$ in (24), (25). All equations (1) are equivalent to

\[ y'' = \frac{1}{y'}, \text{ or another form } y'' = -\frac{5}{4x}y' + \frac{4}{3}x^2y^3. \]

The algebra of point symmetries is 3-dimensional, see also [15].

5.6 Sixth case of intermediate degeneration

In this case $N = 0$ in (13), $\Omega \neq 0$ in (10), (11). The pseudovectorial fields $\omega$, $\omega^1 = \omega_2$, $\omega^2 = -\omega_1$ from (22), (23) and $\alpha$ from (5) are non-collinear,

\[ \nabla_\omega \omega = \hat{\Gamma}_{22}^1 \alpha + \hat{\Gamma}_{22}^2 \omega, \]

\[ \hat{\Gamma}_{22}^1 = -\frac{5\omega^1 \omega^2 (\omega^1_{0,0} - \omega^2_{0,1})}{9\Omega} - \frac{5(\omega^2)^2 \omega^1_{0,1} - 5(\omega^1)^2 \omega^2_{0,1}}{9\Omega} - \frac{5P(\omega^1)^3 + 15Q(\omega^1)^2 \omega^2 + 15R(\omega^1)^2 + 5S(\omega^2)^3}{9\Omega}. \]

The corresponding invariants are

\[ I_1 = L = \nabla_\omega K - \frac{21}{25}K^2 - K, \]

\[ I_2 = \Omega^2 \hat{\Gamma}_{22}^1 - \nabla_\omega L - \frac{72}{625}K^3 + \frac{63}{50}K^2 + \frac{12}{25}KL - KL - L. \]

where $K$ is from (24), (25), $\Omega$ is from (10), (11).

In the sixth case of intermediate degeneration the algebra of point symmetries of equation (1) is 1-dimensional if and only if both invariants $I_1, I_2$ are identically constant. In other cases it is trivial.
5.7 Seventh case of intermediate degeneration

In this case \( N = 0 \) in (13), \( \Omega = 0 \) in (10), (11). The pseudovectorial fields \( \theta \) from (31) and \( \alpha \) from (5) are non-collinear,

\[
\nabla_\theta \theta = \hat{\Gamma}^{1}_{22} \theta + \hat{\Gamma}^{2}_{22} \theta,
\]

\[
\hat{\Gamma}^{1}_{22} = \theta^1 \theta^2 (\theta^1_{0,0} - \theta^2_{0,1}) - (\theta^2)^2 \theta^1_{0,1} + (\theta^1)^2 \theta^2_{1,0} - P(\theta^1)^3 - 3Q(\theta^1)^2 \theta^2 - 3R(\theta^1)^2 - S(\theta)^3.
\]

We define a pseudoscalar field and an invariant,

\[
L = \hat{\Gamma}^{1}_{22} - \frac{1}{2} \Theta^2, \quad I_1 = \frac{L_1}{L^5}, \quad I_2 = \frac{\Theta^2}{L},
\]

where \( \Theta \) is from (30), and a pseudoinvariant

\[
L_1 = \nabla_\theta L = L_{1,0} \theta^1 + L_{0,1} \theta^2 - 4L(\varphi_1 \theta^1 + \varphi_2 \theta^2)
\]

where we have employed (34), (30) and (7), (8).

In the seventh case of intermediate degeneration the algebra of the point symmetries of equation (1) is 2-dimensional if and only if \( L = 0 \); and is 1-dimensional if and only if \( L \neq 0 \) and if \( I_1 \) is identically constant. In other cases the algebra is trivial.

**Example.** For equation (6.5) in the handbook by E. Kamke [8]

\[
y'' = ay^2 + bx + c, \quad a \neq 0 \quad \text{one has } \dim(Z) = 2 \quad \text{if } \quad b^2 = ac \quad \text{otherwise } \dim(Z) = 1.
\]

5.8 Additional subcases of intermediate degeneration

We define a pseudovectorial field \( \eta \), where \( \eta^i = d^{ij} \nabla_j M \) and its scalar product with the field \( \xi \) from (12):

\[
Z = d_{ij} \eta^i \xi^j.
\]

Here \( Z \) is a pseudoinvariant of weight 7. Then the first case of the intermediate degeneration splits into four subcases.

**Subcase 1.1.** \( M \neq 0, \Omega \neq 0, Z \neq 0 \).

**Subcase 1.2.** \( M \neq 0, \Omega \neq 0, Z = 0 \).

**Subcase 1.3.** \( M \neq 0, \Omega = 0, Z \neq 0 \).

**Subcase 1.4.** \( M \neq 0, \Omega = 0, Z = 0 \).

Subject to the pseudoinvariant \( \Theta \) from (30), the seventh case of the intermediate degeneration splits into the two subcases.

**Subcase 7.1.** \( N = 0, \Omega = 0, \Theta \neq 0 \).

**Subcase 7.2.** \( N = 0, \Omega = 0, \Theta = 0 \).

5.9 Tree of intermediate degeneration cases

The following diagram illustrates the cases of the intermediate degeneration.
6 Classification of Painlevé equations

Let us show how Painlevé equations are included into the proposed classification scheme.

1. Equation Painlevé I is in Case 7.1 of intermediate degeneration.

2. Equations Painlevé III-VI (except special cases!) are in Case 1.3 of intermediate degeneration.

3. Special cases.

   (a) Equation Painlevé II is in Case 1.4 of intermediate degeneration.

   (b) Equation Painlevé III with 3 zero parameters is in Case 1.4 of intermediate degeneration.

   (c) Equation Painlevé III \((0, b, 0, d)\) or \((a, 0, c, 0)\) (they are equivalent) is in Case 1.4 of intermediate degeneration.

   (d) Equation Painlevé V \((a, b, 0, 0)\) is in Case 1.4 of intermediate degeneration.

   (e) Equation Painlevé III \((0, 0, 0, 0)\) is in Case of maximal degeneration.

   (f) Equation Painlevé V \((0, 0, 0, 0)\) is in Case of maximal degeneration.

   (g) Equation Painlevé VI \((0, 0, 0, \frac{1}{2})\) is in Case of maximal degeneration.
7 Relation between the semiinvariants

In work of E. Cartan [4] the notations $P = -a_4, Q = -a_3, R = -a_2, S = -a_1, A = -L_1, B = -L_2$ were adopted, where $L_1$ and $L_2$ are the components of the projective curvature tensor. In the work of R.Liouville [14] there were provided the semiinvariants $\nu_5, w_1, i_2$, and the parameter $R_1$ (see review [2]). Relations between them and the pseudoinvariants $F, \Omega, N$ and the component $H$ are as follows,

$$F^5 = \nu_5, \quad H = L_1(L_2)_x - L_2(L_1)_x + 3R_1, \quad \Omega = -w_1 - \frac{\nu_5 a_4}{L_1^4} - 4\frac{(L_1)_x R_1}{L_1^3}, \quad N = i_2^3/3.$$

Other pseudovectorial fields and pseudoinvariants appeared firstly in [19, 20, 21].

8 Solution of equivalence problem for some Painlevé equations

8.1 Painlevé I equation

The equivalence problem for this equation was effectively solved in paper [10].

Theorem 2. Equation (1) is equivalent to Painlevé I equation

$$\ddot{y} = 6\dot{y}^2 + \ddot{x}$$

under the point transformations [3] if and only if the following conditions hold: $F = 0$ in (6), $A \neq 0$ or $B \neq 0$ in (3), $\Omega = 0$ in (11), (1), $N = 0$ in (10), $W = 0$ in (37), $V = 0$ in (38), $\Theta \neq 0$ in (30), $L_1 \neq 0$ in (32). Invariants $I_1$ and $I_2$ in (34) are functionally independent. The point transformation is

$$\tilde{x} = 1/\sqrt{12I_1}, \quad \tilde{y} = \pm\sqrt{I_2}(\sqrt{12}/\sqrt{12})$$

and the pseudoinvariant $V$ is introduced by (35), (30) and (7), (8),

$$V = \nabla_\alpha L_1 = (L_1)_{1,0} B - (L_1)_{0,1} A - 5L_1(B\varphi_1 - A\varphi_2).$$

Example. The equation

$$y'' = -\sin^3 y(6x\cos^2 y + \sin y) + \frac{1}{x}(-18x^3 \cos^3 y \sin^2 y - 3x^2 \sin^3 y \cos y - 2)y' -$$

$$- (18x^3 \cos^4 y \sin y + 3x^2 \sin^2 y \cos^2 y)y')y^2 - (6x^4 \cos^5 y + x^3 \sin y \cos^3 y + x)y')y^3$$

is equivalent to Painlevé I equation. The corresponding invariants and the change of variables are

$$I_1 = \frac{1}{12} \frac{1}{x^5 \sin^3 y}, \quad I_2 = \frac{12x \cos^2 y}{\sin y}, \quad \tilde{y} = x \cos y, \quad \tilde{x} = x \sin y.$$

Example. The equation

$$y'' = 6y^2 + f(x)$$
is equivalent to Painlevé I equation if and only if \( f(x) = mx + n \), where \( m, n \) are the constants, \( m \neq 0 \). Let us check the conditions of Theorem 2:

\[
A = 12, \quad B = 0, \quad F = 0, \quad \Omega = 0, \quad N = 0, \quad W = \frac{f''(x)}{248832}, \quad V = 0,
\]

\[
\Theta = \frac{y}{12}, \quad L_1 = -\frac{f'(x)}{20736}, \quad I_1 = \frac{f'(x)}{12f''(x)}, \quad I_2 = \frac{12y^2}{f'(x)}.
\]

**Examples.** Only these equations in the handbook by E. Kamke [8]

6.3 \( y'' = 6y^2 + x \), Painlevé I

6.5 \( y'' = -ay^2 - bx - c \), with \( a, b \neq 0 \)

are equivalent to Painlevé I equation.

**8.2 Painlevé II equation**

The equivalence problem for this equation was effectively solved in papers [10] and [11].

**Theorem 3.** Equation (1) is equivalent to Painlevé II equation

\[
\tilde{y}'' = 2\tilde{y}^3 + \tilde{x}\tilde{y} + \alpha
\]

with the parameter \( \alpha = \pm J \) with \( J = (4 + 10I_6 - 60I_3)/(50\sqrt{I_9}) \) if and only if the following conditions hold: \( F = 0 \) in (6), \( A \neq 0 \) or \( B \neq 0 \) in (5), \( \Omega = 0 \) in (10), (11), \( M \neq 0 \) in (14), (15), \( I_1 = 18/5 \) in (18), \( I_3 \neq 0 \) in (19), invariant \( J \) is constant. Among the invariants \( I_3, I_6, \) and \( I_9 \) from (18), (19, \( k = 3 \)) one can find two functionally independent. The point transformation is \( \tilde{y} = 1/\sqrt{2500I_9}, \tilde{x} = 5I_6/\sqrt{2500I_9} - 3J \sqrt{2500I_9}/2 \).

**Example.** The equation

\[
y'' = (-2x^3 - xy + a)y^3
\]

is equivalent to Painlevé II equation with \( \tilde{a} = a \). Let us check the conditions of Theorem 3,

\[
A = 0, \quad B = -12x, \quad F = 0, \quad \Omega = 0, \quad M = \frac{288}{5}, \quad I_1 = \frac{18}{5}, \quad J = \pm a,
\]

\[
I_3 = \frac{2x^3 + xy - a}{30x^3}, \quad I_6 = \frac{2xy - 3a}{10x^3}, \quad I_9 = \frac{1}{2500x^6}, \quad \tilde{y} = x, \quad \tilde{x} = y.
\]

**Example.** The equation

\[
y'' = y^3 + f(x)y + g(x)
\]

is equivalent to Painlevé II equation if and only if \( g(x) = c = \text{const} \), \( f(x) = mx + n \), where \( m, n \) are the constants, \( m \neq 0 \). Let us check the conditions of Theorem 3,

\[
A = 6y, \quad B = 0, \quad F = 0, \quad \Omega = 0, \quad M = \frac{72}{5}, \quad I_1 = \frac{18}{5}, \quad J = \frac{g(x)y}{\sqrt{2(f'(x)y + g'(x))}},
\]

\[
I_3 = \frac{y^3 + f(x)y + g(x)}{15y^3}, \quad I_6 = \frac{2f(x)y + 3g(x)}{5y^3}, \quad I_9 = \frac{2(f'(x)y + g'(x))^2}{625y^8}.
\]
Examples. Only these equations in the handbook by E. Kamke [8]

\[
6.6 \quad y'' = 2y^3 + xy + a, \quad \text{Painleve II,}
\]

\[
6.8 \quad y'' = 2a^2y^3 - 2abxy + b, \quad a, b \neq 0, \quad \text{where } \tilde{a} = \frac{1}{2},
\]

\[
6.9 \quad y'' = -ay^3 - bxy - cy - d, \quad a, b \neq 0, \quad \text{where } \tilde{a} = \frac{d\sqrt{c}}{b\sqrt{-2}}.
\]

\[
6.142 \quad y'' = \frac{y'^2}{2y} + 4y^2 + 2xy, \quad \text{where } \tilde{a} = 0,
\]

\[
6.145 \quad y'' = \frac{y'^2}{2y} - \frac{ay^2}{2} - \frac{bxy}{2}, \quad a, b \neq 0, \quad \text{where } \tilde{a} = 0,
\]

\[
6.27 \quad y'' = -ay' - bx^m y^n, \quad n = 3, \quad m, b \neq 0, \quad \text{where } \tilde{a} = 0.
\]

are equivalent to Painleve II equation.

8.3 Painlevé III equation with 3 zero parameters

A general form of the Painlevé equations III reads as

\[
y'' = \frac{1}{y}(y')^2 - \frac{1}{x}y' + \frac{1}{x}(ay^2 + b) + cy^3 + \frac{d}{y}.
\]

It is a 4-parameter family of equations, which we denote by \(\text{PIII}(a, b, c, d)\). If three of four parameters vanish, all these equations are equivalent one to another. Referring to work [7], we write the change of variables: 1), 3): \(x = \tilde{x}, \quad y = \frac{1}{\tilde{y}}\), 2): \(x = \tilde{x}^2/2, \quad y = \tilde{y}^2\).

\[
\text{PIII}(0, b, 0, 0) \overset{1)}{\rightarrow} \text{PIII}(-b, 0, 0, 0) \overset{2)}{\rightarrow} \text{PIII}(0, 0, -b, 0) \overset{3)}{\rightarrow} \text{PIII}(0, 0, 0, b),
\]

The equivalence problem for this equation was effectively solved in paper [11].

**Theorem 4.** Equation (1) is equivalent to Painlevé III equation with 3 zero parameters if and only if the following conditions hold: \(F \neq 0\) in (2), \(A \neq 0\) or \(B \neq 0\) in (5), \(\Omega = 0\) in (10), (11), \(M \neq 0\) in (14), (15), \(I_1 = 3/5, \quad I_3 = 1/15\) from (18).

**Example.** The equation 6.75 in the handbook by E. Kamke [8]

\[
y'' = \frac{2}{x}y' - ey
\]

is not equivalent to Painleve III equation with 3 zero parameters. Let us check the conditions of Theorem 4,

\[
A = -e^y, \quad B = 0, \quad F = 0, \quad \Omega = 0, \quad M = \frac{e^{2y}}{15}, \quad I_1 = 3 \frac{3}{5}, \quad I_3 = \frac{1}{15} - \frac{4}{15e^{2y}}.
\]

**Example.** The equation

\[
y'' = f(x)y' - ey
\]

is equivalent to Painleve III equation with 3 zero parameters if and only if function \(f(x)\) is the solution of equation \(f^2(x) - f'(x) = 0\), hence \(f(x) = 1/(c - x)\), where \(c\) is the constant. Let us check the conditions of Theorem 4,

\[
A = -e^y, \quad B = 0, \quad F = 0, \quad \Omega = 0, \quad M = \frac{e^{2y}}{15}, \quad I_1 = 3 \frac{3}{5}, \quad I_3 = \frac{1}{15} - \frac{2(f^2(x) - f'(x))}{15e^y}.
\]
Examples. Only these equations in the handbook by E. Kamke [8]

6.14 \( y'' = e^y \),

6.28 \( y'' = -ay' - be^y + 2a, \ b \neq 0, \ a = 0 \) or \( a = -1, \)

6.76 \( y'' = -\frac{a}{x} y' - be^y, \ b \neq 0, \ a = 0 \) or \( a = 1, \)

6.77 \( y'' = \frac{a}{x} y' - bx^{4-2a}e^y, \ b \neq 0, \ a = 1, \)

6.83 \( y'' = -\frac{a(e^y - 1)}{x^2}, \ a = -2, \)

6.110(111) \( y'' = \frac{y'^2}{y} \pm \frac{1}{y}, \)

6.118 \( y'' = \frac{y'^2}{y} - ay' - by^2 + 2ay, \ b \neq 0, \ a = 0 \) or \( a = -1, \)

6.127 \( y'' = \frac{y'^2}{y} - by^2, \ b \neq 0, \)

6.172 \( y'' = \frac{y'^2}{y} - \frac{a}{x} y' - by^2, \ b \neq 0, \ a = 0 \) or \( a = 1 \)

are equivalent to to Painlevé III equation with 3 zero parameters.

8.4 Equations PIII(0,0,0,0), PV(0,0,0,0), PVI(0,0,0,1/2)

\[
\text{PIII}(0,0,0,0) : y'' = \frac{1}{y} y'^2 - \frac{1}{x} y', \quad \text{PV}(0,0,0,0) : y'' = \left( \frac{1}{2y} + \frac{1}{y - 1} \right) y'^2 - \frac{1}{x} y', \\
\text{PVI}(0,0,0,\frac{1}{2}) : y'' = \left( \frac{1}{y} + \frac{1}{y - 1} + \frac{1}{y - x} \right) \frac{y'^2}{2} - \left( \frac{1}{x - 1} + \frac{1}{y - 1} + \frac{1}{y - x} \right) y' + \frac{y(y - 1)}{2x(x - 1)(y - x)}. 
\]

For the equations PIII(0,0,0,0), PV(0,0,0,0), PVI(0,0,0,\frac{1}{2}) the conditions \( A = 0 \) and \( B = 0 \) hold where \( A \) and \( B \) are from [5] then according to Theorem 1 (Lie) they are equivalent to \( y'' = 0 \).

8.5 Algorithm for Painlevé III-VI equations

Equations Painlevé III-VI are 4-parameter families, they depend on parameters \( a, b, c, \) and \( d \). As they are in the first case of intermediate degeneration, we should use formulas (18), (19) to calculate the invariants. It is a well-known fact that \( I_2 = 0 \) for all Painlevé equations, and therefore only two sequences generated by \( I_1 \) and \( I_3 \) are nontrivial.

The next useful fact is that all invariants are rational functions of the variables “\( x \)” and “\( y \)”. Let us take the invariant \( I_1 \) (its formula is the simplest). First we regard the symbol \( I_1 \) as a parameter in order to convert rational function into polynomial. This polynomial depend on variables “\( x \)” and “\( y \)”, parameters \( a, b, c, d \) (they are parameters of the equation), and the new parameter \( I_1 \). For example, invariant \( I_1 \) for the Painlevé III equation is

\[
I_1 = \frac{3}{5} \frac{64(c^2 y^8 - 22cy^4 d + d^2)x^2 - 4(cy^4a + db + 49(cy^4b + day^2))yx + a^2y^6 - 22ay^4b + b^2y^2}{(8cy^4x - 8dx - yb + ay^2)^2} 
\]

The associated polynomial is the following one,

\[
P_1(x, y; a, b, c, d, I_1) = 5I_1(8cy^4x - 8dx - yb + ay^2)^2 - (64(c^2 y^8 - 22cy^4d + d^2)x^2 - 4(cy^4a + db + 49(cy^4b + day^2))yx + a^2y^6 - 22ay^4b + b^2y^2) = 0
\]
We shall call \( y \) a "higher" variable. In the same way we construct a polynomial \( P_4(x, y; a, b, c, d, I_4) \) from the formula for invariant \( I_4 \). Using Buchberger algorithm (see [5]), we reduce polynomials \( P_1(x, y; a, b, c, d, I_1) \) and \( P_4(x, y; a, b, c, d, I_4) \) with respect to "higher"variable \( y \). As a result we get a polynomial \( Q_4(x; a, b, c, d, I_4) \) and a formula for variable \( y = R_1(x; a, b, c, d, I_1, I_4) \), where \( R_1 \) is a rational function. In the same way we construct a polynomial \( P_7(x, y; a, b, c, d, I_7) \) by the invariant \( I_7 \). Then we reduce it together with the polynomial \( P_1(x, y; a, b, c, d, I_1) \) and get a new polynomial \( Q_2(x; a, b, c, d, I_1) \).

Now we reduce polynomials \( Q_1(x; a, b, c, d, I_4) \) and \( Q_2(x; a, b, c, d, I_7) \) with respect to the variable \( x \). We get a quantity \( K(a, b, c, d, I_1, I_4, I_7) \) and a formula for variable \( x = R_2(a, b, c, d, I_1, I_4, I_7) \), where \( R_2 \) is also a rational function.

Repeating this procedure as many times as necessary, we obtain a relation between the invariants \( K(I_1, I_4, \ldots) = 0 \) that is a necessary condition of the equivalence as well as the formulas for the parameters \( a, b, c, \) and \( d \) and for the variables \( x \) and \( y \) via invariants. These formulas form the sufficient conditions and complete the solution.

The main difficulty of this method is a bulky form of these polynomials, and this is why at present the equivalence problem is successfully solved only for Painlevé IV equation, see [12]. But the final formulas are too complicated, so here we present only the necessary conditions of the equivalence.

8.5.1 Necessary conditions for Painlevé IV equation

Equation Painlevé IV depends on two parameters \( a \) and \( b \),

\[
\text{PIV}(a, b) : \quad y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - a)y + \frac{b}{y}.
\]

We introduce additional invariants using formulas (18), (19),

\[
J_1 = \frac{5I_1}{72}, \quad J_4 = \frac{I_4}{2160}, \quad J_{10} = \frac{I_{10}}{12960}, \quad I_{10} = \frac{\nabla_\alpha I_4}{N}.
\]

Theorem 5. If equation (38) is equivalent to Painlevé IV equation under the transformations (3), then the following necessary conditions hold: \( F = 0 \) in (3), \( A \neq 0 \) or \( B \neq 0 \) in (3), \( \Omega = 0 \) in (10), (11), \( M \neq 0 \) in (14), (15), \( Z \neq 0 \) in (36), a) \( K_0 = 0 \) in (39) for PIV(a,0) equation; b) \( K_n = 0 \) in (40) for PIV(a,b) equation, \( b \neq 0 \).\n
\[
K_0 = 4608J_4^4 - 3248J_1J_4^3 + 808J_4^2 + 48000J_4J_1^2 - 16500J_4J_1 - 83J_1 + 1125J_4 + 125000J_1^2 + 3,
\]

(39)
and
\[ K = 22^{22} \cdot 3^9 J_0^0 - 218^3 \cdot 4^4 \cdot 7229 J_1^0 + 214^3 \cdot 2^2 (20412 \cdot 10^3 J_4 + 795377 J_7^0) + \\
+ 210^3 \cdot 5 (11664000 J_{10} - 293875200 J_4 - 31700411) J_0^0 + \\
+ 2^9 \cdot 3 \cdot (47628 \cdot 10^3 J_4^2 + 347502500 J_4 - 33816 \cdot 10^3 J_{11} + 15747999 J_5^0 + \\
+ 2^8 (550148750 J_{10} + 1701 \cdot 10^7 J_{10} J_4 - 15275925 \cdot 10^4 J_4^2 - 31879206254 - \\
- 7217838) J_4 + 2^9 (5312667 + 437746 \cdot 10^4 J_4 + 405 \cdot 10^7 J_{10} J_4 + 46305 \cdot 10^6 J_5^0 + \\
+ 479194 \cdot 10^6 J_4^2 - 1168733750 J_{10} - 129705 \cdot 10^6 J_{10} J_4) J_4^2 + 2^2 (-2157057 - \\
- 337746700 J_4 + 12948575 \cdot 10^2 J_{10} J_4 + 6615 \cdot 10^9 J_{10} J_4^2 + 33184 \cdot 10^7 J_{11} J_4 - \\
- 697457 \cdot 10^6 J_4^3 - 219765 \cdot 10^5 J_3^2 J_4^2 + 2^2 \cdot 5 (9675 \cdot 10^5 J_5^0 + \\
- 17852625 J_{10} + 33823650 J_4 - 847425 \cdot 10^4 J_{10} J_4 - 615125 \cdot 10^6 J_{10} J_4^2 + \\
+ 8080625 \cdot 10^5 J_{10}^2 + 11864525 \cdot 10^3 J_4^2 + 9261 + 7875 \cdot 10^7 J_{10}^2 J_4 J_4 + \\
+ 5^8 (15435 J_{10} - 21609 J_4 - 12027400 J_4^2 - 16033 \cdot 10^5 J_4^2 + 11606 \cdot 10^3 J_{10} J_4 + \\
+ 5 \cdot 10^7 J_3^2 J_4 + 343 \cdot 10^8 J_4^3 + 20875 \cdot 10^5 J_{10} J_4^2 - 58 \cdot 10^7 J_{10}^2 J_4 - 175 \cdot 10^4 J_{10}^2). \] (40)

Example. The equation No. 34 from the book [22] named "Painleve 34" equation

\[ X \text{XXXIV. } y'' = \frac{y^2}{2y} + 4ay^2 - xy - \frac{1}{2y}, \hspace{1em} a = \text{const} \neq 0 \]

is not equivalent to Painleve IV equation, although \( K_n = 0 \) from [33]. Let us check the conditions of Theorem 5,

\[ A = 6a - \frac{3}{2y^3}, \hspace{1em} B = 0, \hspace{1em} F = 0, \hspace{1em} \Omega = 0, \hspace{1em} M = \frac{9a(35 + 4ay^3)}{10y^4}, \hspace{1em} Z = 0, \hspace{1em} K_n = 0. \]

9 Cases of Painlevé equation with non-trivial algebra of point symmetries

In a general situation Painlevé equations have the trivial algebra of point symmetries. But in some special cases the dimension of the point symmetries algebra is 8, 2, or 1.

1. For equations \( PIII(0,0,0,0), PV(0,0,0,0), PV I(0,0,0, \frac{1}{2}) \) we have \( \text{dim}(Z) = 8 \).

2. For equation \( PIII \) with 3 zero parameters we have \( \text{dim}(Z) = 2 \), and the operators are

\[ X_1 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \hspace{1em} X_2 = (x \ln x - 2x) \frac{\partial}{\partial x} + y \ln x \frac{\partial}{\partial y}. \]

3. For equation \( PIII(0,b,0,d) \) or \( PIII(-b,0,-d,0) \)

\[ PIII(0,b,0,d) : y'' = \frac{y'^2}{y} - \frac{y'}{x} + \frac{b}{x} + \frac{d}{y}, \hspace{1em} X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}. \]

\[ PV(a,b,0,0) : y'' = \left( \frac{1}{2y} + \frac{1}{y - 1} \right) y'^2 - \frac{y'}{x} + \frac{(y - 1)^2}{x^2} \left( ay + b \right), \hspace{1em} X = x \frac{\partial}{\partial x}, \]

(they are equivalent under the transformations \( x = \hat{x}, y = 1/\hat{y} \), see [7] and equation \( PV(a,b,0,0) \) we have \( \text{dim}(Z) = 1 \).
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