Pseudo-Formal Linearization Method for Nonlinear Systems and Its Application to an Electric Power System

Kazuo Komatsu¹ and Hitoshi Takata²

¹National Institute of Technology, Kumamoto College, 2659-2 Suya, Koshi, Kumamoto 861-1102, Japan
²Kagoshima University, 1-21-40 Korimoto, Kagoshima 890-0065, Japan
E-mail: ¹kaz@kumamoto-nct.ac.jp

Abstract This paper presents a pseudo-formal linearization method based on the polynomial approximation for nonlinear systems. The given nonlinear system which is expressed by the ordinary differential equation is piecewisely linearized with respect to an augmented linearization function which consists of polynomials by the formal linearization approach. Then each linearized system is smoothly united into a single linear one by an automatic choosing function. This method is making use of Taylor expansion as the basis and Chebyshev interpolation as a faster calculation of linearization. As an application of this method, a nonlinear observer is designed to estimate the states of an electric power system. Numerical experiments show the effectiveness of this method.

Keywords: nonlinear system, pseudo-formal linearization, polynomial expansion, nonlinear observer, linearization function, Taylor expansion, Chebyshev interpolation, electric power system

1. Introduction

Most of systems have inherently nonlinear characteristics. They are most naturally described by nonlinear differential equations. It is not easy to find and implement the optimal estimation and control to them directly because of their nonlinearity. The tools of linear analysis are quite well developed, while those being able to deal with nonlinear phenomena have been feeble and few. So one wishes to have linearizing techniques of nonlinear systems so as to apply the linear theories or tools [1], [2], [5]-[18].

One of the earliest treatment of the problem of linearization was made by Poincaré, and then by Sternberg [1], et al. For several decades, this problem has been studied using geometric methods by many researchers [6], [8], [9], [12], [15]. Though many interesting results have been developed, they are, in general, not so easily applicable to practical systems. Therefore, it is necessary to expand to the study how to compute the linearization that enables easy implementation. One of the easy design techniques is a formal linearization method, and we have formerly been studied [7], [10], [11], and then it has been developing into a pseudo-formal linearization method in order to seek more precise accuracy of the linearization [13], [14], [16]-[18].

In this paper, we consider the pseudo-formal linearization method based on the polynomial expansions for nonlinear systems. To improve the accuracy of the linearization, some state regions of the given nonlinear system are divided into some subdomains considering the nonlinearity. And a linearization function which consists of polynomials is introduced to transform a nonlinear system into a linear system with respect to this function piecewisely. Finally, linearized systems on each subdomain are smoothly united into a single linear system by applying an automatic choosing function. This presented method is making use of Taylor expansion as the basis [13] and Chebyshev interpolation as a faster calculation of the coefficients of the linearized systems [18]. As an application of the method, we synthesize a nonlinear observer.

To show the effectiveness of the method, numerical experiments of the linearization are indicated. We carry out numerical simulations of a state estimation problem to an electric power system. Results of numerical experiments show that the pseudo-formal linearization method can effectively improve its performance.
2. Statement of Problem

Nonlinear dynamic and measurement equations are described by

$$\Sigma: \dot{x}(t) = f(x(t)), \quad x(0) = x_0 \in D \tag{1}$$
$$\eta(t) = h(x(t)) \tag{2}$$

where \( t \) denotes time, \( \cdot = \frac{d}{dt} \), \( x = [x_1, \cdots, x_n]^T \) is an \( n \)-dimensional state vector, \( D \subseteq \mathbb{R}^n \) is a domain of \( x, \eta = [\eta_1, \cdots, \eta_m]^T \) is a \( m \)-dimensional measurement vector, \( f \in \mathbb{R}^n \) and \( h \in \mathbb{R}^m \) are sufficiently smooth nonlinear vector-valued functions. Problems are to transform nonlinear dynamic systems into pseudo-formal linear systems and to determine the state of the nonlinear dynamic system from the given measurement data \( \eta \).

3. Pseudo-Formal Linearization of Taylor Type

We consider a pseudo-formal linearization method which is based on Taylor expansion putting to use up to the higher order terms [13], as a basic theory.

First we introduce a vector-valued separable function

$$C : D \rightarrow \mathbb{R}^L \tag{3}$$

which is continuously differentiable. For example, it might be \( C = [I : 0] \) \([L : L \times L \text{ unit matrix}]\) for simplicity. Considering the nonlinearity of the given nonlinear system, this \( C(x) \) can be determined. Let \( D \) be a domain of \( C^{-1} \), and the domain \( D \) is divided into \((M+1)\) subdomains:

$$D = \bigcup_{k=0}^{M} D_k \tag{4}$$

where

$$D_M = D - \bigcup_{k=0}^{M-1} D_k$$

and \( C^{-1}(D_k) \neq \emptyset \) (see Fig. 1). \( D_k(0 \leq k \leq M-1) \) endowed with a lexicographic order is the Cartesian product

$$D_k = \prod_{j=1}^{L} [a_{kj}, b_{kj}) , \quad (a_{kj} < b_{kj})$$

We here introduce an automatic choosing function of the sigmoid type,

$$I_k(\zeta) = \prod_{j=1}^{L} \left( 1 - \frac{1}{1 + \exp (2\mu (\zeta_j - a_{kj}))} \right), \quad (0 \leq k \leq M-1) \tag{5}$$

$$I_M(\zeta) = 1 - \sum_{k=0}^{M-1} I_k(\zeta)$$

so that

$$\sum_{k=0}^{M} I_k(\zeta) = 1 \tag{6}$$

where

$$\zeta = [\zeta_1, \cdots, \zeta_L]^T = C(x)$$

and \( \mu \) is a positive real value. \( I_k(\zeta) \) is analytic and almost unity on \( D_k \), otherwise it is almost zero (see Fig. 1). Using this automatic choosing function, we present a pseudo-formal linearization for nonlinear systems to improve the accuracy of a conventional formal linearization method [11] by using Taylor expansion considering terms up to a higher order as follows.

We define an \( N \)th order formal linearization function that consists of polynomials defined by

$$\phi(x) \triangleq [x_1, x_2, \cdots, x_n, x_1^2, x_1 x_2, \cdots, x_n^2, \cdots, x_1^r_1 x_2^r_2 \cdots x_n^r_n, r_1 \leq r_2 \leq \cdots \leq r_n]$$

and \( \phi(x) \) is a domain of \( \{r_1, \cdots, r_n\} \) under the condition \( r_1 + \cdots + r_n \leq N \). Deriving the derivative of each element of \( \phi \) along with the solution of the given nonlinear system (Eq. (1)), we obtain

$$\dot{\phi}(r_1, \cdots, r_n)(x) = \frac{\partial}{\partial x^T} \phi(r_1, \cdots, r_n)(x) : \dot{x}$$

$$= \frac{\partial}{\partial x^T} \phi(r_1, \cdots, r_n)(x) f(x) \triangleq f_{(r_1, \cdots, r_n)}(x) \tag{8}$$

To each element of the linearization function, applying Taylor expansion up to the \( N \)th order about a nominal operating point \( x = \hat{x}_k \), where

$$\hat{x}_k = [\hat{x}_{1k}, \cdots, \hat{x}_{nk}]^T \in D_k \tag{9}$$

Figure 1 Pseudo-formal linearization using automatic choosing function

$$f(x)$$

$$\sum_{i=0}^{M} \left( A^{(i)} \phi(x) + b^{(i)} \right) I_i(\zeta)$$

$$A^{(i)} \phi(x) + b^{(i)}$$

$$x_0, b_0, a_1, \hat{x}_1, b_1, a_2, \hat{x}_2, b_2, a_3, \hat{x}_3, x, \zeta$$
then we have
\[
\dot{\phi}(r_1, \ldots, r_n)(x) = f^*_r(r_1, \ldots, r_n)(x)
\]
\[
= \sum_{j_1=0}^{(j_1+\cdots+j_n)=N} \sum_{j_n=0}^{N} \frac{\partial}{\partial x_1^j \cdots \partial x_n^j} f^*_r(r_1, \ldots, r_n)\bigg|_{x=x_k} \times \sum_{i=1}^{n} \left( x_i - \hat{x}_k^i \right)^{j_i} + R^e_N(x_1, \ldots, x_n) \tag{10}
\]
where
\[
A^{(k)}(r_1, \ldots, r_n) = \left[ \begin{array}{c} A^{(k)(0 \cdots 0)}_{(r_1 \cdots r_n)} \quad A^{(k)(01 \cdots 0)}_{(r_1 \cdots r_n)} \quad \cdots \quad A^{(k)(ij \cdots j_n)}_{(r_1 \cdots r_n)} \\ \vdots \\ A^{(k)(0 \cdots 0)}_{(r_1 \cdots r_n)} \end{array} \right],
\]
\[
R^e_N(r_1, \ldots, r_n)(x) = \frac{1}{(N+1)!} \left( x_1 - \hat{x}_k^1 \right) \frac{\partial}{\partial x_1} + \cdots + \left( x_n - \hat{x}_k^n \right) \frac{\partial}{\partial x_n} \tag{11}
\]
if \( \dot{f} \in C^{N+1} \). Therefore,
\[
\dot{\phi}(x)(r_1, \ldots, r_n) = \left[ A^{(k)(0 \cdots 0)}_{(r_1 \cdots r_n)} \quad A^{(k)(01 \cdots 0)}_{(r_1 \cdots r_n)} \quad \cdots \quad A^{(k)(ij \cdots j_n)}_{(r_1 \cdots r_n)} \right] \dot{\phi}(x) + A^{(k)(0 \cdots 0)}_{(r_1 \cdots r_n)} + R^e_N(r_1, \ldots, r_n)(x) \tag{12}
\]
Thus, it follows that on the subdomain \( D_k \),
\[
\dot{\phi}(x) = A^{(k)} \dot{\phi}(x) + b^{(k)} + R^e_N(r_1, \ldots, r_n)(x) \tag{13}
\]
where
\[
A^{(k)} = \left[ A^{(k)(ij \cdots j_n)}_{(r_1 \cdots r_n)} \right], \quad b^{(k)} = \left[ A^{(k)(0 \cdots 0)}_{(r_1 \cdots r_n)} \right]
\]
\[
R^e_N(r_1, \ldots, r_n)(x) = \left[ R^e_N(r_1, \ldots, r_n)(x) \right]
\]
We unite \( (M+1) \) linearized systems (Eq. (13)) on the subdomains into a single linear system on the whole domain by using Eq. (5) as
\[
\dot{\phi}(x) = \sum_{k=0}^{M} \dot{\phi}(x)I_k(\zeta)
\]
\[
= \sum_{k=0}^{M} (A^{(k)} \dot{\phi}(x) + b^{(k)} + R^e_N(r_1, \ldots, r_n)(x))I_k(\zeta)
\]
\[
= \dot{A}(\zeta)\dot{\phi}(x) + \dot{b}(\zeta) + \dot{R}_{N+1}(x, \zeta) \tag{14}
\]
where
\[
\dot{A}(\zeta) = \sum_{k=0}^{M} A^{(k)}I_k(\zeta), \quad \dot{b}(\zeta) = \sum_{k=0}^{M} b^{(k)}I_k(\zeta)
\]
\[
\dot{R}_{N+1}(x, \zeta) = \sum_{k=0}^{M} R^e_N(r_1, \ldots, r_n)(x)I_k(\zeta)
\]
Finally a pseudo-formal linearization system is defined as
\[
\Sigma_2 : \dot{z}(t) = \dot{A}(\zeta)z(t) + \dot{b}(\zeta), \quad z(0) = \phi(x(0)) \tag{15}
\]
From Eq. (7), its inversion is carried out by using
\[
\dot{x}(t) = [I, 0, \ldots, 0]z(t) \tag{16}
\]
as the approximated value of \( x(t) \), where \( I \) is the \( n \times n \) unit matrix.

This pseudo-formal linearization method has the following error bounds. Let \( \| \cdot \| \) be the Euclidean norm.

**Theorem 1**
An error bound when a nonlinear system is approximated by the pseudo-formal linearization is
\[
\varepsilon_{N+1} = \max_{k \leq M} \left\{ \sup_{x \in x_k} \| R^e_N(x_1, \ldots, x_n)(x) : x \in D_k \right\}, 0 \leq k \leq M \tag{17}
\]

**Theorem 2**
An error bound of the pseudo-formal linearization for a nonlinear dynamic system is
\[
\| \dot{x}(t) - \dot{x}(t)\| \leq \| \phi(x(t)) - z(t)\|
\]
\[
\leq e \| A \|_{\text{max}} \| \phi(x(0)) - z(0)\| + \frac{\varepsilon_{N+1} e \| A \|_{\text{max}}^t - 1}{\| A \|_{\text{max}}} \tag{18}
\]

(Proof. See Ref. [14])

Equation (18) indicates that the first term originates from the initial error and the second term originates from the functional approximation error in this approach. We could expand this pseudo-formal linearization of Taylor type to another Chebyshev type.

4. **Pseudo-Formal Linearization of Chebyshev Type**

We here consider the pseudo-formal linearization method which is based on Chebyshev interpolation with both higher accuracy and faster calculation (See Appendix and Ref. [18]).
A nonlinear dynamic system is the same as Eq. (1), and the domain is assumed to be denoted by the Cartesian product for simplicity as

$$D = \prod_{i=1}^{n} [m_i - p_i, m_i + p_i] \quad (m_i \in R, p_i > 0) \quad (19)$$

The domain $D$ of $D$ is divided into $(M + 1)$ subdomains as showed in Section 3 by the vector-valued separable function $C(x)$ in Eq. (3).

To make use of Chebyshev interpolation, the state vector $x$ is changed into $y$, so that $y$ has the basic domain of the Chebyshev polynomials

$$D_0 = \prod_{i=1}^{n} [-1, 1] \quad (20)$$

and $y$ is rewritten as

$$y = P(k)^{-1} (x - M(k)) \in D_0 \quad (21)$$

where

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_L \\ y_{L+1} \\ \vdots \\ y_n \end{bmatrix}, \quad M(k) = \begin{bmatrix} m_1(k) \\ \vdots \\ m_L(k) \\ m_{L+1} \\ \vdots \\ m_n \end{bmatrix}$$

$$P(k) = \begin{bmatrix} p_1(k) \\ \vdots \\ p_L(k) \\ p_{L+1} \\ \vdots \\ p_n \end{bmatrix}$$

$$m_i(k) = \frac{1}{2} (a_{ki} + b_{ki}), \quad p_i(k) = \frac{1}{2} (b_{ki} - a_{ki})$$

Then the given dynamic system (Eq. (1)) becomes

$$\dot{y}(t) = P(k)^{-1} f(P(k)y(t) + M(k)) \quad (22)$$

To this nonlinear system, we apply a pseudo-formal linearization method by using Chebyshev interpolation [18].

An $N$th order formal linearization function that consists of polynomials is defined by

$$\phi(x) \triangleq \left[ x_1, x_2, \ldots, x_n, x_1^2, \frac{x_1 x_2}{1!}, \ldots, \frac{x_1 x_n}{2!}, \ldots, \frac{x_1^2 x_2}{2!}, \ldots, \frac{x_1^2 x_n}{2!}, \ldots, \frac{x_1^N x_2}{N!}, \ldots, \frac{x_1^N x_n}{N!} \right]^T$$

$$= [\phi_{10 \cdots 0}(x), \cdots, \phi_{11 \cdots n}(x), \cdots, \phi_{NN \cdots N}(x)]^T \quad (23)$$

Then a pseudo-formal linearization algorithm of Chebyshev type is derived as follows.

**Pseudo-Formal Linearization Algorithm of Chebyshev Type**

(L-1) Given

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0 \in D$$

(L-2) Set

$$L, C(x), M, \mu, N, D_k, \mathcal{M}^{(k)}, P^{(k)} \quad (k = 1, \cdots, M)$$

(L-3)

(L-3.1)

$$I_k(\zeta) = \prod_{j=1}^{L} \left(1 - \frac{1}{1 + \exp (2\mu(\zeta - a_kj))} \right) - \frac{1}{1 + \exp (-2\mu(\zeta - b_kj))} \quad (0 \leq k \leq M - 1)$$

(L-3.2)

$$T_q(y_{1j}) = \cos(q \cdot \cos^{-1} y_{1j}), \quad y_{1j} = \cos \frac{2j_1 + 1}{2N + 2} \pi \quad (i = 1, \cdots, n, j_i = 0, \cdots, N)$$

(L-3.3)

$$d_q^s = \begin{cases} \prod_{j=0}^{q-2s-1} \frac{q - s - j}{s!} (2s < q) \\ (-1)^s \frac{q - s}{s!} \quad (q = 2s) \end{cases}$$

(L-3.4)

$$C_{(r_1 \cdots r_n)}^{(k)}(y) = \frac{\partial}{\partial y^s} P(k)^{-1} \phi_{(r_1 \cdots r_n)}(P(k)y + M(k)) \times f(P(k)y + M(k))$$

(L-3.5)

$$C_{(q_1 \cdots q_n)}^{(k)} = \frac{2^{n-\gamma}}{n!} \sum_{j_1=0}^{N} \cdots \sum_{j_n=0}^{N} \prod_{i=1}^{n} (N + 1) \sum_{j_1=0}^{N} \cdots \sum_{j_n=0}^{N} \sum_{j_n=0}^{N} G_{(r_1 \cdots r_n)}^{(k)}(y_{1j_1}, y_{2j_2}, \cdots, y_{nj_n})$$

$$\times T_{q_1}(y_{1j_1}) T_{q_2}(y_{2j_2}) \cdots T_{q_n}(y_{nj_n})$$

$$\gamma = \{ \text{the number of } q_i = 0 : 1 \leq i \leq n \}$$

(L-3.6)

$$A_{(r_1 \cdots r_n)}^{(k)} = \sum_{q_1=0}^{N} \cdots \sum_{q_n=0}^{N} \sum_{s_1=0}^{|q_1|/2} \cdots \sum_{s_n=0}^{|q_n|/2} C_{(r_1 \cdots r_n)}^{(k)}(A_{(r_1 \cdots r_n)}^{(k)})$$
\[
\frac{d^{(q_1)} y_1}{dt^{q_1}} \cdots \frac{d^{(q_n)} y_n}{dt^{q_n}} \left( q_1 = 2s_1 \right) \left( -M^{(k)}_1 \right) q_1 - s_1 - j_1 \\
\cdots \left( q_n = 2s_n \right) \left( -M^{(k)}_n \right) q_n - 2s_n - j_n
\]

(A-4)

\[
A^{(k)} = \left[ A^{(k)}_{(r_1 \cdots r_n)} \right], \quad b^{(k)} = \left[ A^{(k)}_{(r_1 \cdots r_n)} \right] (0, 0)
\]

(k = 1, \cdots, M),

\[
\tilde{A}(\zeta) = \sum_{k=0}^{M} A^{(k)} I_k(\zeta), \quad \tilde{b}(\zeta) = \sum_{k=0}^{M} b^{(k)} I_k(\zeta)
\]

(L-5) Solve

\[
\dot{z}(t) = \tilde{A}(\zeta)z(t) + \tilde{b}(\zeta), \quad z(0) = \phi(x(0))
\]

(L-6)

\[
\dot{x}(t) = [I, 0, \cdots, 0]z(t)
\]

5. Nonlinear Observer

In this section, we synthesize a nonlinear observer as an application of the pseudo-formal linearization of Taylor type. The nonlinear dynamic system in Eq. (1) is transformed into the linear system in Eq. (15) by the pseudo-formal linearization of Taylor type.

Next we linearize the measurement equation (Eq. (2)). We apply Taylor expansion up to the \( N \)th order to each \( \eta_k \) on the same subdomain \( D_k \) as in a state space about a nominal operating point \( x = \hat{x}_k \), then we have

\[
\eta_k = h_r(x)
\]

\[
\approx \sum_{j_1=0}^{(j_1+\cdots+j_n)=N} \cdots \sum_{j_n=0}^{(j_1+\cdots+j_n)=N} \frac{\partial^{(j_1+\cdots+j_n)}}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}} h_r(x) \bigg|_{x=\hat{x}_k}
\]

\[
\times \prod_{i=1}^{n} \left( x_i - \hat{x}_{ik} \right)^{j_i} \Delta \hat{h}_r^{(k)}(x)
\]

\[
= \sum_{j_1=0}^{(j_1+\cdots+j_n)=N} \cdots \sum_{j_n=0}^{(j_1+\cdots+j_n)=N} H_r^{(k)(j_1, \cdots, j_n)} \phi(j_1, \cdots, j_n)(x)
\]

where

\[
H_r^{(k)(j_1, \cdots, j_n)} = \frac{\partial^{(j_1+\cdots+j_n)}}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}} h_r^{(k)}(x) \bigg|_{x=0}
\]

Thus, \( \{ \eta_k \} \) on a subdomain \( D_k \) is approximated by the formal linearization function as

\[
\eta_k = h_r(x) \approx \left[ H_r^{(k)(0, \cdots, 0)}, H_r^{(k)(1, \cdots, 0)}, \cdots, H_r^{(k)(j_1, \cdots, j_n)}, \cdots, H_r^{(k)(0, \cdots, N)} \right] \phi(x) + H_r^{(k)(0, \cdots, 0)}
\]

and a linear measurement equation with respect to \( \phi \) is obtained as

\[
\eta \approx \left[ H_r^{(k)(j_1, \cdots, j_n)} \right] \phi(x) + \left[ H_r^{(k)(0, \cdots, 0)} \right]
\]

Applying Eq. (5) to Eq. (24) yields

\[
\eta \approx \sum_{k=0}^{M} \left( H_r^{(k)} \phi(x) + d^{(k)} \right) I_k(\zeta)
\]

\[
= \sum_{k=0}^{M} H_r^{(k)} I_k(\zeta) \phi(x) + \sum_{k=0}^{M} d^{(k)} I_k(\zeta)
\]

(25)

Therefore, a pseudo-formal linearization system for the measurement equation is approximately derived as

\[
\dot{\eta}(t) = \tilde{H}(\zeta)z(t) + d(\zeta)
\]

(26)

To the linearized systems in Eqs. (15) and (26), the linear observer theory \([3]\) is applied and an identity observer is obtained as

\[
\dot{\hat{z}}(t) = \tilde{A}(\zeta) \hat{z}(t) + \tilde{b}(\zeta) + K(t)(\eta(t) - \tilde{H}(\zeta) \hat{z}(t) - d(\zeta))
\]

\[
= \sum_{k=0}^{M} \left\{ (A^{(k)} \hat{z}(t) + b^{(k)}) + K^{(k)}(\eta(t) - H^{(k)} \hat{z}(t) - d^{(k)}) \right\} I_k(\zeta)
\]

(27)

where \( \zeta = \hat{C}(\hat{z}) \) and \( K^{(k)}(t) \) is the observer gain on a subdomain \( D_k \) given by

\[
K^{(k)}(t) = \frac{1}{2} P^{(k)(t)} H^{(k)^T} S^{(k)}(t)
\]

(28)

\( P^{(k)(t)} \) satisfies the matrix Riccati differential equation

\[
\dot{P}^{(k)(t)} = A^{(k)} P^{(k)(t)} + P^{(k)(t)} A^{(k)^T} + Q^{(k)(t)}
\]

\[
- P^{(k)(t)} H^{(k)^T} S^{(k)}(t) H^{(k)} P^{(k)(t)}
\]

where \( Q^{(k)(t)} \), \( S^{(k)}(t) \) and \( P^{(k)(0)} \) are arbitrary real symmetric positive definite matrices.

From Eq. (16), the estimate \( \hat{x}(t) \) of a nonlinear observer becomes

\[
\dot{\hat{x}}(t) = [I, 0, \cdots, 0] \hat{z}(t)
\]

(29)

6. Numerical Experiments

6.1 Pseudo-formal linearization

Consider the simple nonlinear dynamic system

\[
\dot{x} = x - x^2, \quad D = [-\frac{1}{4}, \frac{5}{4}] \subset R
\]

(30)
The parameters are set as $M = 2$, $\mu = 50$, $\zeta = x$ in Eqs. (4) and (5). $D$ is divided into $D = \bigcup_{k=0}^{2} D_k$ where

$$D_0 = \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], D_1 = \left[ \begin{array}{cc} 1 & 3 \\ 3 & 4 \end{array} \right], D_2 = \left[ \begin{array}{cc} 3 & 5 \\ 5 & 4 \end{array} \right]$$ (31)

Taylor expansion points are set at

$$\hat{x}_0 = 0, \hat{x}_1 = 0.5, \hat{x}_2 = 1$$

for the method of Taylor type in Eq. (9), and other parameters are

$$\mathcal{M}^{(0)} = 0, \mathcal{M}^{(1)} = 0.5, \mathcal{M}^{(2)} = 1$$

$$\mathcal{P}^{(k)} = 0.25 (k = 0, 1, 2)$$

for the one of Chebyshev type in Eq. (21)

Figure 2 shows the true value $x$, which is a solution of the given system (Eq. (30)), an approximated value $\hat{x}$ (Taylor, $N = 3$) obtained by the method of Taylor type in Eq. (16) when the order of the linearization function $N$ is 3, and $\hat{x}$ (Chebyshev, $N = 3$) obtained by the method of Chebyshev type (L-6). $\hat{x}$ (old, $N = 3$) denotes a result obtained by a formal linearization method [11] when the order of the linearization function $N$ is 3, and $\hat{x}$ (Taylor) is a solution based on the Taylor expansion method truncated at the first order [2] for comparison. To clarify the difference in the approximation errors in Fig. 2, Fig. 3 shows the integral square errors of the estimation

$$J(t) = \int_{0}^{t} (x(\tau) - \hat{x}(\tau))^2 d\tau$$ (32)

for the various orders in these cases.

![Figure 2](image_url) True value $x$ and estimates $\hat{x}$ compared with those of previous methods

Figures 2 and 3 indicate that the pseudo-formal linearization method has higher accuracy than that of the Taylor method [2] and the formal linearization method [11]. These linearization methods accurately improve as the order of the linearization function increases.

### 6.2 Nonlinear observer for an electric power system

As an application of the pseudo-formal linearization, we consider a nonlinear observer in Section 5 for an electric power system [7]. When one synchronous machine is directly connected to an infinite bus power system, the dynamic equation under certain assumptions is described by

$$M_d \ddot{\delta}(t) + D_d \dot{\delta}(t) + \frac{e_s e_f'}{x_d} \sin \delta(t) = P_m$$ (33)

where $\delta(t)$ is a load angle, $M_d$ is a moment of inertia, $D_d$ is a damping coefficient, $e_s$ is an infinite bus voltage, $e_f'$ is an excitation voltage, $x_d$ is a synchronous reactance, and $P_m$ is a mechanical input power. Putting the state variables as

$$\begin{cases} x_1(t) = \delta(t) - \sin^{-1}(P_m \frac{x_d}{e_s e_f'}) \\ x_2(t) = \dot{\delta}(t) \end{cases}$$

the electric power system is described by

$$\begin{cases} \dot{x}_1 = x_2 \triangleq f_1(x) \\ \dot{x}_2 = -\frac{e_s e_f'}{M_d x_d} \sin x_1 - \frac{D_d}{M_o} x_2 + \frac{P_m}{M_o} \triangleq f_2(x) \end{cases}$$ (34)

If the reactive power is measured, the measurement equation is described by

$$\eta(t) = \frac{e_s}{x_d} (e_s - e_f' \cos \delta(t))$$ (35)

and it becomes by the state variable as

$$\eta = \frac{e_f'^2}{x_d} - \frac{e_s e_f'}{x_d} \cos x_1 \triangleq h(x)$$ (36)

A domain $D$ is considered to be

$$D = [-0.75, 0.75] \times [-3, 3]$$
In order to apply the pseudo-formal linearization method in Section 3, we set \( C(x) = x_1 \) and \( L = 1 \) because of its highest nonlinearity. We divide this whole domain into three subdomains \( (M = 2) \) as
\[
D_0 = [-0.75, -0.25], \quad D_1 = [-0.25, 0.25] \\
D_2 = [0.25, 0.75]
\]
The nominal operating points are set at the center point of each subdomain as
\[
\{\hat{x}_{10}, \hat{x}_{11}, \hat{x}_{12}\} = \{-0.5, 0, 0.5\}
\]
The initial value of the system is
\[
x(0) = [0.6727, 0]^T
\]
and the parameter of the automatic choosing function in Eq. (5) is set as \( \mu = 100. \)

\[\begin{align*}
\hat{x}_1(Taylor) & \\
\hat{x}_1\,(old) & \\
\hat{x}_1(N=2) & \\
\hat{x}_2(Taylor) & \\
\hat{x}_2\,(old) & \\
\hat{x}_2(N=2) & \\
\end{align*}\]

Figure 4 True value \( x_1 \) and estimates \( \hat{x}_1 \) of nonlinear observers for an electric power system

To these nonlinear systems in Eqs. (34) and (36), we apply the nonlinear observer. The system parameters of the power system are set as
\[
M_o = 0.0265, \quad D_a = 0.005, \quad P_{in} = 0.8
\]

\[
\hat{x}(0) = [0.4, 0.4]^T
\]
and the parameters for the observer are set as
\[
Q^{(k)}(t) = 5I, \quad S^{(k)}(t) = 10, \quad P^{(k)}(0) = 5I \quad (k = 0, 1, 2)
\]

\[\begin{align*}
J(t)(N=2) & \\
J(t)(old) & \\
J(t)(Taylor) & \\
\end{align*}\]

Figure 5 True value \( x_2 \) and estimates \( \hat{x}_2 \) of nonlinear observers for an electric power system

To these nonlinear systems in Eqs. (34) and (36), we apply the nonlinear observer. The system parameters of the power system are set as
\[
M_o = 0.0265, \quad D_a = 0.005, \quad P_{in} = 0.8
\]

\[
e_s = c_f = x_d = 1
\]
The initial value of the observer is
\[
\hat{x}(0) = [0.4, 0.4]^T
\]
and the parameters for the observer are set as
\[
Q^{(k)}(t) = 5I, \quad S^{(k)}(t) = 10, \quad P^{(k)}(0) = 5I \quad (k = 0, 1, 2)
\]

\[\begin{align*}
J(t)(N=2) & \\
J(t)(old) & \\
J(t)(Taylor) & \\
\end{align*}\]

Figure 6 Integral square errors of nonlinear observers

Figures 4 and 5 show the true value \( x \) and the estimate \( \hat{x} \) in Eq. (29) when the order of the linearization function \( N \) is 2. \( \hat{x}(old, N = 2) \) refers to a result obtained by the previous method [11] when the order of the linearization function \( N \) is 2, and \( \hat{x}(Taylor) \) is a result by the conventional observer based on the linearization using the first order Taylor expansion [5] for comparison. Figure 6 shows the integral square errors of the estimation
\[
J(t) = \int_0^t (x(\tau) - \hat{x}(\tau))^T(x(\tau) - \hat{x}(\tau))d\tau
\]
in these cases.

From Figs. 4, 5 and 6, the nonlinear observer by the pseudo-formal linearization has better performance than the previous methods.

7. Conclusions

We have stated two pseudo-formal linearization methods for nonlinear systems based on the polynomial expansion considering terms up to a higher order to improve the accuracy of the linearization. One is based on Taylor expansion as the basis, and another is based on Chebyshev interpolation as a faster calculation of the coefficients of the linearized systems. We have synthesized a nonlinear observer as an application of the method and applied it to a problem of transient state of an electric power system. From the results of numerical experiments, the pseudo-formal linearization method can effectively improve its performance.
When we apply the pseudo-formal linearization to practical systems, there are still many problems to settle such as how to divide a domain into some subdomains optimally, how to set parameters of a nonlinear observer, and so on.

Appendix

Chebyshev Approximation

(A.1) Least-squares approach

The Chebyshev orthogonal polynomial approximation is one of the most popular and important approaches for nonlinear functions [4]. From the property of the Fourier series for orthogonal functions, even if truncated at any finite order \( N \), the resulting equation is always optimal in the sense of minimizing

\[
\int_{D_0} \left| \sum_{q_1=0}^{N} \cdots \sum_{q_n=0}^{N} C^{(k)}(r_1, \ldots, r_n) \prod_{i=1}^{n} T_{q_i}(y_i) \right|^2 dy
\]

This yields

\[
C^{(k)}(r_1, \ldots, r_n) = \frac{2^{n-1}}{\pi^n} \int_{D_0} G^{(k)}(r_1, \ldots, r_n)(y) \prod_{i=1}^{n} T_{q_i}(y_i) dy
\]

and

\[
A^{(k)}(r_1, \ldots, r_n) = \frac{\partial^{(j_1, \ldots, j_n)}}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}} \left. G^{(k)}(r_1, \ldots, r_n) \left( p^{(k)-1}(x - M^{(k)}) \right) \right|_{x=0}
\]

instead of (L-3.5) and (L-3.6).

Since these equations include the integration and derivation, computation time might be longer than a simple summation.

(A.2) Min-max approach

Chebyshev interpolation is based on the min-max technique and yields (L-3.5) and (L-3.6) in Section 4 (See Ref. [4]). These equations are given by simple summations, so the computation time is shorter than by the above (A.1).

References

[1] S. Sternberg: Local contractions and a theorem of Poincaré, Am. J. Math., Vol. 79, No. 4, pp. 809-824, 1957.
[2] Y. N. Yu, K. Vongouriya and L. N. Weisman: Application of an optimal control theory to a power system, IEEE Trans. Power Appl. Syst., Vol. PAS-89, No. 1, pp. 55-62, 1970.
[3] G. W. Johnson: A deterministic theory of estimation and control, IEEE Trans. on Autom. Control, Vol. 14, pp. 380-384, 1974.
[4] T. Akasaka: Numerical Computation, Corona Pub., 1974 (in Japanese).
[5] A. P. Sage and C. C. White: III: Optimum Systems Control, 2nd ed., Prentice-Hall, Inc., 1977.
[6] R. W. Brockett: Feedback invariants for nonlinear systems, Proc. IFAC Congress, pp. 1115-1120, 1978.
[7] H. Takata: Transformation of a nonlinear system into an augmented linear system, IEEE Trans. on Autom. Control, Vol. AC-24, No. 5, pp. 736-741, 1979.
[8] R. Su: On the linear equivalents of nonlinear systems, Syst. Control Lett., Vol. 2, No. 1, pp. 48-52, 1982.
[9] A. J. Krener: Approximate linearization by state feedback and coordinate change, Syst. Control Lett., Vol. 5, pp. 181-185, 1984.
[10] K. Komatsu and H. Takata: A formal linearization by the Chebyshev interpolation and its applications, Proc. IEEE CDC, Vol. 1, pp. 70-75, 1996.
[11] K. Komatsu and H. Takata: Design of nonlinear observer using augmented linear system based on formal linearization of polynomial type, Int. J. Comput. Electr. Autom. Control Inf. Eng., Vol. 3, No. 11, pp. 2523-2526, 2009.
[12] J. Lei and H. K. Khalil: Feedback linearization for nonlinear systems with time-varying input and output delays by using high-gain predictors, IEEE Trans. Autom. Control, Vol. 61, No. 8, pp. 2262-2268, 2016.
[13] K. Komatsu and T. Takata: A nonlinear observer via pseudo-formal linearization for both state and measurement equations of nonlinear scalar-measurement systems, J. Signal Process., Vol. 21, No. 6, pp. 291-296, 2017.
[14] H. Takata and K. Komatsu: A pseudo-formal linearization of polynomial type for nonlinear systems and its applications, J. Signal Process., Vol. 22, No. 1, pp. 9-16, 2018.
[15] S. Yang, P. Wang and Y. Tang: Feedback linearization-based current control strategy for modular multilevel converters, IEEE Trans. Power Electron., Vol. 33, No. 1, pp. 161-174, 2018.
[16] K. Komatsu and T. Takata: A pseudo-formal linearization using Chebyshev expansion and its application to nonlinear observer for nonlinear scalar-measurement systems, J. Signal Process., Vol. 23, No. 2, pp. 49-54, 2019.
[17] H. Takata and K. Komatsu: On a pseudo-formal linearization method via the orthogonal polynomial approximation, J. Signal Process., Vol. 23, No. 6, pp. 230-239, 2019.
[18] K. Komatsu and T. Takata: Computer algorithms of pseudo-formal linearization and nonlinear observer for nonlinear scalar-measurement systems by Chebyshev interpolation, J. Signal Process., Vol. 24, No. 2, pp. 51-59, 2020.
Kazuo Komatsu received his B.S. degree in computer science and Dr. Eng. degree in electrical engineering from Kyushu Institute of Technology in 1985 and 1995, respectively. He is currently a Professor at the Department of Human-Oriented Information Systems Engineering in National Institute of Technology, Kumamoto College. His research interests include formal linearization for nonlinear systems and its applications. He is a member of the RISP.

Hitoshi Takata received his B.S. degree in electrical engineering from Kyushu Institute of Technology in 1968 and his M.S. and Dr. Eng. degrees in electrical engineering from Kyushu University in 1970 and 1974, respectively. He is currently a Professor Emeritus at Kagoshima University. His research interests include the control, linearization and identification of nonlinear systems.

(Received November 2, 2020)