Consistency of the $\pi\Delta$ interaction in chiral perturbation theory

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Abstract

We analyze the constraint structure of a spin-3/2 particle interacting with a pseudoscalar. Requiring the self consistency of the considered effective field theory imposes restrictions on the possible interaction terms. In the present case we derive two constraints among the three lowest-order $\pi\Delta$ interaction terms. From these constraints we find that the total Lagrangian is invariant under the so-called point transformation. On the other hand, demanding the invariance under the point transformation alone is less stringent and produces only classes of relations among the coupling constants.

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I. INTRODUCTION

The $\Delta(1232)$ resonance plays an important role in the phenomenological description of low- and medium-energy processes such as pion-nucleon scattering, electromagnetic pion production, Compton scattering etc. This is due to the rather small mass gap between the $\Delta(1232)$ and the nucleon, the strong coupling of the $\Delta(1232)$ to the $\pi N$ channel, and its relatively large photon decay amplitudes. In an effective-field-theory (EFT) approach to, say, the single-nucleon sector one encounters two possibilities. In conventional baryon chiral perturbation theory (BChPT) (see, e.g., Refs. [2, 3, 4] for an introduction) one is restricted to the threshold regime (of pion production) and the dynamical effects due to the $\Delta(1232)$ are encoded in the values of the low-energy constants of the most general effective Lagrangian. Alternatively, one can try to include the $\Delta(1232)$ as an explicit dynamical degree of freedom. In doing so, one hopes to improve the convergence behavior of the chiral expansion by reordering important terms which in an ordinary chiral expansion would show up at higher orders. Moreover, if one succeeds in defining a suitable power-counting scheme one may even be able to perform calculations of processes which involve center-of-mass energies covering the resonance region. Clearly, there is a strong motivation for taking the $\Delta(1232)$ as an explicit dynamical degree of freedom into account and this issue has already attracted considerable attention for quite some time. So far, most of the calculation have been performed in the heavy-baryon framework [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. On the other hand, more recently several methods have been devised to obtain renormalization schemes leading to a simple and consistent power counting in a manifestly Lorentz-invariant approach of BChPT [18, 19, 20, 21, 22, 23, 24, 25]. It is thus natural to investigate a consistent Lorentz-invariant formulation including spin-3/2 degrees of freedom (see, e.g., Refs. [26, 27, 28, 29, 30, 31, 32, 33, 34, 35]).

BChPT with explicit $\Delta$ degrees of freedom ($\Delta$ChPT) is a field theory of a system with constraints. Therefore, one encounters the highly non-trivial problem of a consistent interaction of higher-spin fields (see, e.g., Refs. [36, 37, 38, 39]). In a Lorentz-invariant formulation of a field theory involving particles of higher spin ($s \geq 1$), one necessarily introduces unphysical degrees of freedom [40, 41]. Therefore one has to impose constraints which specify the physical degrees of freedom. To write down interaction terms which lead to the correct number of physical degrees of freedom has proven to be a difficult problem. There are various suggestions for constructing consistent interactions involving spin-3/2 particles (see, e.g., Refs. [42, 43, 44, 45, 46, 47, 48, 49, 50]). In this context we note that the problems showing up only for large field configurations are not relevant to low-energy EFTs because these deal with small fluctuations of field variables around the vacuum. For larger field configurations the higher-order terms (infinite in number) generate contributions to physical quantities which are no longer suppressed by powers of small expansion parameters. Therefore, for large fluctuations the conclusions drawn from an analysis of a finite number of terms of the effective Lagrangian cannot be trusted. On the other hand, for small fluctuations around the vacuum one requires that the theory describes the right number of degrees of freedom in a self-consistent way. The interaction terms can be analyzed order by order in a small parameter expansion. Such an analysis leads to non-trivial constraints on the possible interactions.

In the present paper we consider the leading-order interaction terms of the pion with the $\Delta(1232)$ in low-energy effective field theory—$\Delta$ChPT. We derive consistency conditions for the $\pi\Delta$-interaction terms by analyzing the structure of the constraints using the canonical
(Hamilton) formalism. For reasons of simplicity we suppress the isospin degree of freedom.

II. PROPERTIES OF THE LAGRANGIAN FOR A SPIN-3/2 SYSTEM

A. The free Lagrangian of a spin-3/2 system

Fields with spin 3/2 can be described via the Rarita-Schwinger formalism, where the field is represented by a vector spinor $\psi^\mu$ \[40]. The most general free Lagrangian reads \[41\]

$$\mathcal{L}^{3/2} = \bar{\psi}^\alpha \Lambda^{A}_{\alpha\beta} \psi^\beta,$$

where

$$\Lambda^{A}_{\alpha\beta} = -[(i \not\partial - m) g_{\alpha\beta} + i A (\gamma_\alpha \partial_\beta + \gamma_\beta \partial_\alpha) + \frac{i}{2}(3A^2 + 2A + 1) \gamma_\alpha \not\partial \gamma_\beta + m(3A^2 + 3A + 1) \gamma_\alpha \gamma_\beta],$$

with $A \neq -1/2$ an arbitrary real parameter.

The generalization for an arbitrary space-time dimension $n$ is (see, e.g., Ref. \[49\])

$$\mathcal{L}^{A,n}_{3/2} = \bar{\psi}^\alpha \Lambda^{A,n}_{\alpha\beta} \psi^\beta,$$

where

$$\Lambda^{A,n} = -\left\{ (i \not\partial - m) g_{\alpha\beta} + i A (\gamma_\alpha \partial_\beta + \gamma_\beta \partial_\alpha) + \frac{i}{n-2} \left[ (n - 1)A^2 + 2A + 1 \right] \gamma_\alpha \not\partial \gamma_\beta \\
+ \frac{m}{(n-2)^2} \left[ n(n-1)A^2 + 4(n-1)A + n \right] \gamma_\alpha \gamma_\beta \right\}, \quad n \neq 2.$$

In the special case of $A = -1$, Eq. \[3\] does not explicitly depend on $n$.

B. Point invariance

The free Lagrangian of Eq. \[3\] is invariant under the set of transformations

$$\begin{align*}
\psi_\mu & \rightarrow \psi_\mu + \frac{4a}{n} \gamma_\mu \psi^\nu, \\
A & \rightarrow \frac{An - 8a}{n(1 + 4a)}, \quad a \neq -\frac{1}{4},
\end{align*}$$

which are often referred to as a point transformation. The invariance under the point transformation guarantees that the physical quantities do not depend on $A$ \[42, 43\], provided that the interaction terms are also invariant under the point transformation. We will not impose the last condition but will rather obtain it as a consequence of consistency in the sense of having the right number of degrees of freedom.
III. GENERAL CONSIDERATIONS

In this section we outline the method of analyzing systems with constraints of the second class starting with a finite number of degrees of freedom. For a more detailed description see, e.g., Refs. [51, 52, 53]. Let us consider a classical system with $N$ degrees of freedom $q_i$ and velocities $\dot{q}_i = dq_i/dt$ described by the Lagrangian $L(q, \dot{q})$. Here, we assume that $L$ contains the $\dot{q}$’s at the most quadratically. The Hamiltonian is obtained using the Legendre transform

$$H(q, p) = \sum_{i=1}^{N} p_i \dot{q}_i - L(q, \dot{q}),$$

where the $p_i$ are the canonical momenta defined by

$$p_i \equiv \frac{\partial L(q, \dot{q})}{\partial \dot{q}_i}, \quad i = 1, \ldots, N.$$

Since the Hamiltonian is a function of $q$ and $p$, the velocities $\dot{q}_i$ have to be replaced using Eq. (8). If, according to Eq. (8), this is not possible because

$$\det A = 0,$$

with

$$A_{ij} = \frac{\partial p_i}{\partial \dot{q}_j},$$

we are dealing with a singular system. With a suitable change of coordinates, the Lagrangian can be written as a linear function of the unsolvable new velocities $\dot{q}_i'$. In the following the new coordinates are again denoted by $q_i$. Let the unsolvable $\dot{q}_i$ be the first $n$ velocities $\dot{q}_1, \ldots, \dot{q}_n$. The so-called primary constraints occur as follows. The Lagrangian $L$ can be written as

$$L(q, \dot{q}) = \sum_{i=1}^{n} F_i(q)\dot{q}_i + G(q, \dot{q}_{n+1}, \ldots, \dot{q}_N)$$

from which we obtain as the canonical momenta

$$p_i = \begin{cases} F_i(q) & \text{for } i = 1, \ldots, n, \\ \frac{\partial G(q, \dot{q}_{n+1}, \ldots, \dot{q}_N)}{\partial \dot{q}_i} & \text{for } i = n + 1, \ldots, N. \end{cases}$$

The first part of Eq. (11) can be reexpressed in terms of the relations

$$\theta_i(q, p) = p_i - F_i(q) = 0, \quad i = 1, \ldots, n,$$

which are referred to as the primary constraints. Using Eq. (7), we consider the Hamiltonian

$$H(q, p) = \sum_{j=n+1}^{N} p_j \dot{q}_j(p, q) - G(q, \dot{q}_{n+1}(p, q), \ldots, \dot{q}_N(p, q)) + \sum_{i=1}^{n} \lambda_i \theta_i(q, p),$$

where the $\lambda$’s are Lagrange multipliers taking care of the primary constraints and the $\dot{q}_i(p, q)$ are the solutions to Eq. (11) for $i = n + 1, \ldots, N$. We determine the $\lambda$’s by using the condition that the constraints have to be conserved in time. The time evolution of the primary constraints $\theta_i$ is given by the Poisson bracket with the Hamiltonian so that the condition for conservation in time reads

$$\{\theta_i, H\} = 0.$$
Either all the $\lambda$’s can be determined from these equations, or new constraints arise. The number of these secondary constraints corresponds to the number of the $\lambda$’s (or linear combinations thereof) which could not be determined. Again we demand the conservation in time of these (new) constraints and try to solve the remaining $\lambda$’s from these equations, etc. The number of physical degrees of freedom is given by the initial number of degrees of freedom minus the number of constraints. In order for a theory to be consistent, the chain of new constraints has to terminate such that at the end of the procedure the correct number of degrees of freedom has been generated.

IV. CONSISTENCY CONDITIONS FOR THE $\pi\Delta$ INTERACTION TERMS

In this section we will determine the Hamiltonian for pions and deltas in analogy to the discussion above. The consistency of the interaction terms in the Lagrangian is only guaranteed, if the correct number of degrees of freedom is generated. Taking the fields $\psi^\mu$ and $\psi^\mu_\dagger$ and the canonical momenta $\pi^\mu_\psi$ and $\pi^\mu_\psi_\dagger$ as the independent variables we have in total $2 \times 2 \times 4 \times 4 = 64$ components, so 48 constraints are needed to obtain the right number of degrees of freedom for a spin-3/2 system. This will generate conditions among the coupling constants $g_1$, $g_2$, and $g_3$ of the original Lagrangian, as will be shown in the following.

A. Constraint analysis in four dimensions

The total leading-order Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{3/2} + \mathcal{L}_{\pi\Delta},$$

(15)

where $\mathcal{L}_0$ denotes the free-pion Lagrangian

$$\mathcal{L}_0 = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} M^2 \phi^2,$$

(16)

$\mathcal{L}_{3/2}$ is given in Eq. (1), and the leading-order $\pi\Delta$-interaction terms read \[13\]

$$\mathcal{L}_{\pi\Delta} = -\bar{\psi}^{\mu} \left[ g_1 \gamma^\alpha \gamma_5 \phi_{\alpha} + \frac{g_2}{2} (\gamma_{\mu} \partial_{\nu} \phi + \partial_{\mu} \phi_{\gamma_5}) \gamma_5 + \frac{g_3}{2} \gamma_{\mu} \gamma^\alpha \gamma_5 \phi_{\alpha} \right] \psi_\nu.$$  

(17)

The values of the coupling constants $g_i$ are not restricted by symmetries imposed on the most general effective Lagrangian. For convenience we will use a manifestly hermitian expression for $\mathcal{L}_{3/2}$ differing from Eq. (1) by an irrelevant total derivative:

$$\mathcal{L}_{3/2} = -\bar{\psi}^{\alpha} \left[ \left( \frac{i}{2} \left( \gamma^\alpha \partial - \partial \gamma^\alpha \right) - m \right) g_{\alpha\beta} + \frac{i A}{2} \left( \gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha \right) \right] \psi_\beta + \frac{i}{4} (3A^2 + 2A + 1) \left( \gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha \right) + m(3A^2 + 3A + 1) \gamma_\alpha \gamma_\beta \psi. $$

(18)

Reference \[13\] discusses the most general chiral spin-3/2 Lagrangian which, in particular, implies that one deals with pion and $\Delta$ isospin triplets and quadruplets, respectively. We have simplified the discussion by neglecting the isospin degree of freedom, because the present results do not depend on the isospin structure. Moreover, the present coupling constants $g_i$ correspond to $-g_i/F$ in the full chiral case, where $F$ denotes the pion-decay constant in the chiral limit.
We will express the Hamiltonian corresponding to Eq. (15) as the sum of four terms,
\[ H = H_1 + H_2 + H_3 + H_4. \] (19)

To that end we first calculate the canonical momenta. The canonical momentum field \( \pi \) corresponding to the pion field is given by
\[ \pi = \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L_0}{\partial \dot{\phi}} + \frac{\partial L_{\pi \Delta}}{\partial \dot{\phi}} = \dot{\phi} + F, \] (20)

where
\[ F = \left( -\frac{g_1}{2} - g_2 + \frac{g_3}{2} \right) \bar{\psi}_0 \gamma_0 \gamma_5 \psi_0 + \left( \frac{g_2}{2} - \frac{g_3}{2} \right) \bar{\psi}_i \gamma_i \gamma_5 \psi_0 \]
\[ + \frac{g_1}{2} \bar{\psi}_i \gamma_0 \gamma_5 \psi_i + \left( \frac{g_2}{2} - \frac{g_3}{2} \right) \bar{\psi}_0 \gamma_i \gamma_5 \psi_i + \frac{g_3}{2} \bar{\psi}_i \gamma_0 \gamma_5 \gamma_j \psi_j. \] (21)

We define \( H_1 \) as
\[ H_1 = \pi \dot{\phi} - L_0 = \frac{1}{2} (\pi^2 - F^2) + \frac{1}{2} (\partial_i \phi)^2 + \frac{1}{2} M^2 \phi^2. \] (22)

The second term, \( H_2 \), is defined by
\[ H_2 = -L_{\pi \Delta} \]
\[ = -F(\pi - F) + \frac{g_1}{2} \left( \bar{\psi}_j \gamma_i \gamma_5 \partial_\phi \psi_j - \bar{\psi}_0 \gamma_i \gamma_5 \partial_\phi \psi_0 \right) \]
\[ + \frac{g_2}{2} \left( -\bar{\psi}_0 \gamma_0 \partial_\phi \gamma_5 \psi_1 + \bar{\psi}_j \gamma_j \partial_\phi \gamma_5 \psi_i - \bar{\psi}_i \gamma_0 \partial_\phi \gamma_5 \psi_0 + \bar{\psi}_1 \gamma_j \partial_\phi \gamma_5 \psi_j \right) \]
\[ + \frac{g_3}{2} \left( -\bar{\psi}_0 \gamma_0 \gamma_5 \partial_\phi \psi_0 + \bar{\psi}_0 \gamma_0 \gamma_5 \gamma_j \partial_\phi \psi_j + \bar{\psi}_j \gamma_j \gamma_5 \gamma_0 \partial_\phi \psi_0 + \bar{\psi}_j \gamma_j \gamma_5 \gamma_k \partial_\phi \psi_k \right). \] (23)

Because of the fermionic nature of the spin-3/2 field, \( \psi^\mu \) and \( \psi^{\mu \dagger} \) are Grassmann fields. We define the corresponding momenta as
\[ \pi^\mu_\psi = \frac{\partial^R L}{\partial (\partial_0 \psi^\mu)}, \]
\[ \pi^{\mu \dagger}_\psi = \frac{\partial^L L}{\partial (\partial_0 \psi^{\mu \dagger})}; \] (24)

where \( \partial^R \) and \( \partial^L \) denote the right and the left derivatives. We define
\[ H_3 = \pi^\mu_\psi \dot{\psi}^\mu + \dot{\psi}^{\mu \dagger} \pi^{\mu \dagger}_\psi - \frac{L_{\pi \Delta}}{2}, \] (25)

where the time and space components of \( \pi^\mu_\psi \) are given by
\[ \pi^0_\psi = \frac{i}{4} (3A^2 + 4A + 1) \bar{\psi}_i \gamma_i - \frac{3}{4} i (A^2 + 2A + 1) \bar{\psi}_0 \gamma_0, \] (26)
\[ \pi^i_\psi = \frac{i}{2} \bar{\psi}_i \gamma_0 + \frac{i}{4} (3A^2 + 4A + 1) \bar{\psi}_0 \gamma_i - \frac{i}{4} (3A^2 + 2A + 1) \bar{\psi}_j \gamma_j \gamma_0 \gamma_i. \] (27)
Analogously,

\[
\pi_0^0 = -\frac{i}{4}(3A^2 + 4A + 1)\gamma_0\gamma_i\psi_i + \frac{3}{4}i(A^2 + 2A + 1)\psi_0, \tag{28}
\]

\[
\pi_i^i = \frac{i}{2}\psi_i - \frac{i}{4}(3A^2 + 4A + 1)\gamma_0\gamma_i\psi_0 - \frac{i}{4}(3A^2 + 2A + 1)\gamma_i\gamma_j\psi_j. \tag{29}
\]

Since \(\dot{\psi}_\mu\) and \(\psi_\mu^\dagger\) do not appear in these equations, constraints are generated, as described in the previous section. These primary constraints are

\[
\theta_0^0 = \pi_0^0 - i\frac{1}{4}(3A^2 + 4A + 1)\bar{\psi}_i\gamma_i + \frac{3}{4}i(A^2 + 2A + 1)\bar{\psi}_0\gamma_0 = 0,
\]

\[
\theta_i^i = \pi_i^i - \frac{i}{2}\bar{\psi}_i\gamma_0 - \frac{i}{4}(3A^2 + 4A + 1)\bar{\psi}_0\gamma_i + \frac{i}{4}(3A^2 + 2A + 1)\bar{\psi}_j\gamma_j\gamma_0\gamma_i = 0, \tag{30}
\]

and analogous constraints \(\theta_0^0\) and \(\theta_i^i\). Finally, we define the last piece in terms of the Lagrange multipliers and constraints as

\[
\mathcal{H}_4 = \theta_\mu^\mu \lambda_\mu + \lambda_\mu^\dagger \theta_\mu^0. \tag{31}
\]

From the condition that \(\theta_\psi^0\) etc. have to be zero throughout all time we obtain a set of linear equations for the eight Lagrange multipliers \(\lambda\) and \(\lambda^\dagger\), where each component \(\lambda_\mu\) and \(\lambda_\mu^\dagger\) is a four-component object.

To this end we define the Poisson brackets

\[
\{\psi_0(\vec{x}), \pi_0^0(\vec{y})\} = \{\psi_0^\dagger(\vec{x}), \pi_0^0(\vec{y})\} = \delta^3(\vec{x} - \vec{y}),
\]

\[
\{\psi_i(\vec{x}), \pi_i^0(\vec{y})\} = \{\psi_i^\dagger(\vec{x}), \pi_i^0(\vec{y})\} = \delta_{ij}\delta^3(\vec{x} - \vec{y}),
\]

\[
\{\phi(\vec{x}), P(\vec{y})\} = \delta^3(\vec{x} - \vec{y}), \tag{32}
\]

where we have omitted the Dirac–spinor indices. In addition, we have

\[
\{A, B\} = -(-)^{P(A)P(B)}\{B, A\}, \tag{33}
\]

where \(P(X)\) takes the value 1 for Grassmann fields and 0 otherwise. The remaining Poisson brackets vanish. Since the time evolution is governed by the Poisson brackets, we obtain from demanding the time independence of the primary constraints

\[
0 = \{\theta_\psi^0, H\}
\]

\[
= (\mathcal{F} - \pi) \left[\left(\frac{-g_1}{2} - g_2 + \frac{g_3}{2}\right)\bar{\psi}_0\gamma_0\gamma_5 + \left(\frac{g_2}{2} - \frac{g_3}{2}\right)\bar{\psi}_i\gamma_i\gamma_5\right]
\]

\[
- \left(\frac{g_1}{2} + \frac{g_3}{2}\right)\bar{\psi}_0\gamma_0\gamma_5\partial_i\phi - \frac{g_2}{2}\bar{\psi}_i\gamma_0\gamma_5\partial_i\phi + \frac{g_3}{2}\bar{\psi}_j\gamma_j\gamma_0\gamma_i\partial_i\phi
\]

\[
+ iA\partial_i\bar{\psi}_i\gamma_0 - \frac{i}{2}(3A^2 + 2A - 1)\partial_i\bar{\psi}_0\gamma_i - \frac{i}{2}(3A^2 + 2A + 1)\partial_i\bar{\psi}_j\gamma_j\gamma_i
\]

\[
+ m\left[3A(A + 1)\bar{\psi}_0 - (3A^2 + 3A + 1)\bar{\psi}_i\gamma_i\right]
\]

\[
+ \frac{i}{2}(3A^2 + 4A + 1)\lambda_\mu^\dagger\gamma_0\gamma_i - \frac{3}{2}i(A^2 + 2A + 1)\lambda_0^\dagger,
\]

\[
0 = \{\theta_i^i, H\}. \tag{34}
\]
\[ F - \pi \left( \frac{g_1}{2} \bar{\psi}_i \gamma_5 + \frac{g_2 - g_3}{2} \bar{\psi}_0 \gamma_i \gamma_5 + \frac{g_3}{2} \bar{\psi}_j \gamma_0 \gamma_j \gamma_5 \gamma_i \right) \\
+ \frac{g_1}{2} \bar{\psi}_i \gamma_5 \gamma_j \phi + \frac{g_2}{2} \left( -\bar{\psi}_0 \gamma_0 \gamma_5 \partial_i \phi + \bar{\psi}_j \gamma_5 \gamma_0 \partial_i \phi + \bar{\psi}_j \gamma_5 \gamma_0 \partial_j \phi \right) \\
+ \frac{g_3}{2} \left( \bar{\psi}_0 \gamma_0 \gamma_j \gamma_5 \gamma_0 \partial_j \phi - \bar{\psi}_j \gamma_5 \gamma_0 \gamma_5 \partial_i \phi \right) \\
- i \partial_k \bar{\psi}_i \gamma_k - i A \left( \partial_i \bar{\psi}_0 \gamma_0 + \partial_i \bar{\psi}_k \gamma_k + \partial_k \bar{\psi}_k \gamma_i \right) \\
+ \frac{i}{2} \left( 3A^2 + 2A + 1 \right) \left( -\partial_k \bar{\psi}_0 \gamma_0 \gamma_j \gamma_0 \gamma_i + \partial_k \bar{\psi}_j \gamma_j \gamma_0 \gamma_0 \gamma_i \right) \\
-m(3A^2 + 3A + 1) \left( \bar{\psi}_0 \gamma_0 \gamma_i - \bar{\psi}_k \gamma_k \gamma_i \right) + m \bar{\psi}_i \\
+i \lambda_i^I + \frac{i}{2} \left( 3A^2 + 4A + 1 \right) \lambda_0^I \gamma_0 \gamma_i + \frac{i}{2} \left( 3A^2 + 2A + 1 \right) \lambda_j^I \gamma_j \gamma_i , \]  
and analogous equations from \( \{ \theta_0^I, H \} = 0 = \{ \theta_i^I, H \} \). The set of linear equations (34) and (35) may then be formulated as

\[ \lambda^I M^I = X^I , \]  
and analogously

\[ M \lambda = X , \]  
where \( M \) and \( M^I \) are 16 \times 16 matrices. The expressions for \( M \) and \( X \) are given in the appendix. In the following we only consider the equations for \( \lambda \) explicitly, the equations for \( \lambda^I \) can be treated analogously. Equation (37) cannot be solved for all the \( \lambda \)'s, since

\[ \det M = 0 . \]  
But this was to be expected, because until now only 32 constraints have been generated [Eqs. (30) plus the analogous constraints \( \theta_0^I \) and \( \theta_i^I \)]. To get the correct number of degrees of freedom, in total 48 constraints are needed. The condition of solvability of Eq. (37) supplies further (secondary) constraints. To determine these we diagonalize \( M \)

\[ S^{-1}MS = \begin{pmatrix} 2 \cdot 1_{4 \times 4} & 0 & 0 & 0 \\ 0 & 2 \cdot 1_{4 \times 4} & 0 & 0 \\ 0 & 0 & -4(1 + 3A + 3A^2) \cdot 1_{4 \times 4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \]  
where the nonsingular matrix \( S \) is given in the appendix. Thus Eq. (37) transforms to

\[ S^{-1}MS \lambda^I = S^{-1}X^I , \]  
where

\[ \lambda^I = S^{-1} \lambda . \]  
As can be seen from \( S^{-1}MS \) in Eq. (39), the so far unknown Lagrange multiplies \( \lambda_0, \lambda_1, \) and \( \lambda_2 \) can be determined from Eq. (40), but it cannot be solved for \( \lambda_3 \). On the other hand, secondary constraints are generated, which read

\[ \theta_0^I \lambda^I = (S^{-1}X)_3 = 0 . \]  

\[ ^2 \text{Recall that we count the components from 0 to 3.} \]
In total we obtain eight more constraints [four from Eq. (42) and four from the analogous procedure for $\lambda^\dagger$], so we are still missing eight constraints. To obtain these, it is necessary that the time conservation of the secondary constraints does not lead to a determination of $\lambda^\dagger_3$. Equation (42) reads in full

$$\theta^4_{\psi^\dagger} = (S^{-1}X)_3 = \frac{1 + A}{4(1 + 3A + 3A^2)} \gamma_0 \gamma_1 \gamma_3 [(1 + 3A)\gamma_0 X_0 + (1 + A)\gamma_i X_i] = 0. \quad (43)$$

This condition is equivalent to

$$\theta^4_{\psi^\dagger} = (1 + 3A)\gamma_0 X_0 + (1 + A)\gamma_i X_i = 0. \quad (44)$$

Substituting Eqs. (A4)–(A5) for $X$, Eq. (44) can be written as

$$\theta^4_{\psi^\dagger} = (F - \pi)\xi_1 + \xi_2 + \xi_3 = 0, \quad (45)$$

where

$$\xi_1 = i\gamma_5 \{[-(1 + 3A)(g_1 + 2g_2 - g_3) + 3(1 + A)(g_2 - g_3)]\gamma_0 \psi_0 + \gamma_i \gamma_j \psi_j \partial_i \phi, \quad (47)$$

$$\xi_2 = i\gamma_5 \{-[(1 + 3A)(g_1 + g_3) - (1 + A)g_2 + 3(1 + A)g_3]\gamma_i \psi_0 + [2g_2 + 2(1 + A)g_1] \gamma_i \gamma_j \partial_j \psi_0 - 4A(2A + 1)\gamma_0 \gamma_i \gamma_j \partial_j \psi_j \partial_i \phi \}, \quad (48)$$

$$\xi_3 = 4(2A + 1)\gamma_0 \partial_i \psi_i - 4(2A^2 + 3A + 1)\gamma_i \partial_i \psi_0 - 4A(2A + 1)\gamma_0 \gamma_i \gamma_j \partial_i \psi_j + 2im \{3(1 + A)(1 + 2A)\psi_0 - (6A^2 + 5A + 1)\gamma_0 \gamma_i \psi_i \}. \quad (50)$$

To conserve this constraint in time, the Poisson bracket of $\theta^4_{\psi^\dagger}$ with the Hamiltonian must be equal to zero:

$$\{\theta^4_{\psi^\dagger}, H\} \equiv 0. \quad (46)$$

As explained above, to obtain further constraints, we cannot allow for $\lambda^\dagger_3$ to be solvable from Eq. (46). To satisfy this condition we require that $\psi_3^\dagger = (S^{-1}\psi)_3$ must not appear in Eq. (45). The following conclusions can be drawn from this condition:

1. Since $F$ contains the combination $\psi_3^\dagger$, $\xi_1$ of Eq. (45) has to disappear. This results in

$$-(1 + 3A)(g_1 + 2g_2 - g_3) + 3(1 + A)(g_2 - g_3) = 0, \quad (47)$$

$$(1 + 3A)(g_2 - g_3) + (1 + A)g_1 + 3(1 + A)g_3 = 0, \quad (48)$$

which is fulfilled, if and only if

$$g_2 = Ag_1, \quad (49)$$

$$g_3 = -\frac{1}{2}(1 + 2A + 3A^2)g_1. \quad (50)$$

2. Secondly, all terms containing $\psi_3^\dagger$ in $\xi_2$ have to disappear. But this is automatically fulfilled, if Eqs. (47) and (48) hold.

3. The remaining terms do not contain the combination $\psi_3^\dagger$, so no more conditions occur.
Thus, if Eqs. (49) and (50) hold, we get eight more constraints, namely

\[ \theta^5_{\psi} = \{ \theta^4_{\psi}, H \} = 0, \quad (51) \]

and analogous expression for \( \theta^5_{\psi} \). Demanding the time independence of \( \theta^5_{\psi} \) (and \( \theta^5_{\psi} \)) no more constraints are generated and all remaining \( \lambda'_3 \) multipliers are determined. Thus the chain of constraints terminates at this point and the correct number of physical degrees of freedom has been generated.

It is interesting to note that, after inserting Eqs. (49) and (50), the total Lagrangian of Eq. (15) fulfills the point invariance of Eqs. (5) and (6). Thus, a suitable field redefinition in the form of Eq. (5) transforms the Lagrangian from a general \( A \) to a fixed \( A \), say, \( A = -1 \).

### B. Constraint analysis in an arbitrary dimension \( n \)

The calculation for \( n \) dimensions and \( A = -1 \) is analogous to the one of the previous subsection. In fact, it is simpler, since the resulting sets of linear equations corresponding to Eqs. (36) and (37) already have diagonal form. For \( A = -1 \) the Lagrangian does not explicitly depend on the space-time dimension \( n \). Thus we obtain the same result as we get when substituting \( A = -1 \) into Eqs. (49) and (50), namely,

\[ g_2 = g_3 = -g_1. \quad (52) \]

With these conditions the chain of constraints terminates at the correct number of degrees of freedom. To obtain the equivalent conditions for a general \( A \) and an arbitrary dimension \( n \), we use the following transformation

\[ \psi^{\mu} \rightarrow (g^{\mu\nu} - \frac{A + 1}{n - 2} \gamma^{\mu} \gamma^{\nu}) \psi_{\nu}. \quad (53) \]

This leads to the following relations among the coupling constants

\[ g_2(A) = Ag_1, \quad (54) \]
\[ g_3(A) = \frac{1 + 2A + A^2(n - 1)}{n - 2}. \quad (55) \]

Of course, for \( n = 4 \) we reproduce the results of the previous subsection.

The requirement of the consistency of the theory automatically results in the invariance under the point transformation. On the other hand, demanding the invariance under the point transformation alone is not sufficient to obtain the relations of Eqs. (49) and (50), as will be shown in the next section.

### V. CONDITIONS FOR POINT INVARIANCE

To determine the general conditions for the coupling constants, that follow from requiring the point invariance of the Lagrangian, we apply the point transformation in four dimensions to the interaction term of the Lagrangian:

\[ \psi^{\mu} \rightarrow \psi^{\mu} + a \gamma_{\mu} \gamma_{3} \psi^{3}, \]
\[ A \rightarrow A' = \frac{A - 2a}{1 + 4a}, \quad a \neq -\frac{1}{4}, \quad (56) \]
The change of the Lagrangian under this transformation, \( \Delta L = \mathcal{L}' - \mathcal{L} \), is then set to zero.\(^3\)

We obtain for \( \Delta L \)

\[
\Delta L = \Delta L_1 + \Delta L_2 + \Delta L_3,
\]

(57)

where

\[
\begin{align*}
\Delta L_1 &= -g_1 a \bar{\psi}^\mu [(u_\mu \gamma_\nu + u_\nu \gamma_\mu) \gamma_5 + (1 + a) \gamma_\mu \not\! \gamma_5 \gamma_\mu] \psi^\nu, \\
\Delta L_2 &= -\frac{1}{2} \bar{\psi}^\mu \{ [g_2(A')(1 + 4a) - g_2(A)] (\gamma_\mu u_\nu + u_\mu \gamma_\nu) \gamma_5 - 2a(1 + 4a)g_2(A') \gamma_\mu \not\! \gamma_5 \gamma_\mu \} \psi^\nu, \\
\Delta L_3 &= -\frac{1}{2} [g_3(A')(1 + 4a)^2 - g_3(A)] \bar{\psi}^\mu \gamma_\mu \not\! \gamma_5 \gamma_\nu \psi^\nu.
\end{align*}
\]

To fulfill \( \Delta L = 0 \) we obtain the following conditions

\[
\begin{align*}
g_1 a + \frac{1}{2} [g_2(A')(1 + 4a) - g_2(A)] &= 0, \\
g_1 a(1 + a) - a(1 + 4a)g_2(A') + \frac{1}{2} [g_3(A')(1 + 4a)^2 - g_3(A)] &= 0.
\end{align*}
\]

(58)

(59)

These functional equations for \( g_2(A) \) and \( g_3(A) \) can be solved. The solutions are

\[
\begin{align*}
g_2(A) &= g_1 [2z_2 + (1 + 4z_2)A], \\
g_3(A) &= 2g_1 \left[ z_3 + \left( \frac{1}{2} - z_2 + 4z_3 \right) A + \left( \frac{1}{4} + 4z_3 - 2z_2 \right) A^2 \right],
\end{align*}
\]

(60)

(61)

where \( z_2 \) and \( z_3 \) are arbitrary (see also Ref. [43]). We compare these with the relations that follow from consistency, Eqs. (49) and (50). As we can see, the latter are more stringent, because the parameters that are arbitrary in Eqs. (60) and (61), are fixed to

\[
\begin{align*}
z_2 &= 0, \\
z_3 &= -\frac{1}{4},
\end{align*}
\]

(62)

(63)

when requiring consistency of the theory.

VI. SUMMARY

We have considered the Hamilton formalism for a system of spin-3/2 particles interacting with pseudoscalars. The Lorentz-invariant formulation of a field theory for spin-3/2 particles necessarily leads to the introduction of unphysical degrees of freedom. In order to obtain the right number of degrees of freedom some constraints have to be imposed.

In the present work we have considered the Rarita-Schwinger formulation for spin-3/2 particles and have analyzed the constraint structure for the lowest-order \( \pi \Delta \) interaction terms. The requirement of the consistency of the corresponding effective field theory in the sense of having the right number of degrees of freedom has led to non-trivial constraints among the three coupling constants of the lowest-order \( \pi \Delta \) interaction. As a result of these

\(^3\) In principle \( \Delta L \) could also be a total derivative, but this does not apply in the present case.
constraints the total Lagrangian is invariant under the so-called point transformation, guaranteeing that the physical quantities are independent of the off-shell parameter \( A \). On the other hand, demanding the invariance under the point transformation alone is less stringent and produces only a class of relations among the coupling constants. We conclude that the analysis of the constraint structure is an important ingredient in the construction of the most general effective field theory including particles with spin \( S \geq 1 \). In particular, as a rule it leads to a reduction in the number of free parameters of the Lagrangian.

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APPENDIX A: M, X AND S

The left-hand side of Eq. (37) is given by

\[
(M \lambda)_0 = -\left(3A^2 + 6A + 3\right)\lambda_0 + \left(3A^2 + 4A + 1\right)\gamma_0 \gamma_i \lambda_i, \tag{A1}
\]

\[
(M \lambda)_i = 2\lambda_i + \left(3A^2 + 4A + 1\right)\gamma_0 \gamma_i \lambda_0 + \left(3A^2 + 2A + 1\right)\gamma_i \gamma_j \lambda_j. \tag{A2}
\]

Thus the matrix \( M \) reads

\[
M = \begin{pmatrix}
-B & C\gamma_0 \gamma_1 & C\gamma_0 \gamma_2 & C\gamma_0 \gamma_3 \\
C\gamma_0 \gamma_1 & 2 - D & D\gamma_1 \gamma_2 & D\gamma_1 \gamma_3 \\
C\gamma_0 \gamma_2 & D\gamma_2 \gamma_1 & 2 - D & D\gamma_2 \gamma_3 \\
C\gamma_0 \gamma_3 & D\gamma_3 \gamma_1 & D\gamma_3 \gamma_2 & 2 - D
\end{pmatrix}. \tag{A3}
\]

The components of the vector \( X \) on the right-hand side of Eq. (37) are given by

\[
X_0 = i(F - \pi)\gamma_5 \left[ (g_1 + 2g_2 - g_3)\psi_0 - (g_2 - g_3)\gamma_0 \gamma_i \psi_i \right] + i\lambda_i \phi_0 \gamma_5 \left[ (g_1 + g_3)\gamma_0 \gamma_i \psi_0 + g_2 \psi_i + g_3 \gamma_i \gamma_j \psi_j \right] - 2A \lambda_i \psi_i + (3A^2 + 2A - 1)\gamma_0 \gamma_i \lambda_i \partial_i \psi_0 + (3A^2 + 2A + 1)\gamma_i \gamma_j \psi_j - 6imA(A + 1)\gamma_0 \psi_0 + 2im(3A^2 + 3A + 1)\gamma_0 \psi_1. \tag{A4}
\]

\[
X_i = -i(F - \pi)\left[ g_1 \gamma_5 \psi_i - (g_2 - g_3)\gamma_0 \gamma_5 \gamma_i \psi_0 + g_3 \gamma_i \gamma_j \psi_j \right] + i\lambda_j \phi_0 \gamma_5 \gamma_i \psi_i + ig_2 \left( \partial_i \phi_0 \gamma_5 \psi_0 + \partial_i \phi_0 \gamma_5 \gamma_j \psi_j + \partial_j \phi_0 \gamma_5 \gamma_i \psi_j \right) + 2\gamma_0 \gamma_k \partial_k \psi_i + 2A \left( -\partial_i \psi_0 + \gamma_0 \gamma_j \partial_j \psi_k + \gamma_0 \gamma_i \partial_k \psi_k \right) - (3A^2 + 2A + 1) \left( -\gamma_i \gamma_k \partial_k \psi_0 + \gamma_0 \gamma_i \gamma_j \partial_j \psi_j \right) - 2im(3A^2 + 3A + 1) \left( \gamma_i \psi_0 + \gamma_0 \gamma_i \gamma_k \psi_k \right) - 2im(3A^2 + 3A + 1) \left( \gamma_i \psi_0 + \gamma_0 \gamma_i \gamma_k \psi_k \right) - 2im(3A^2 + 3A + 1) \left( \gamma_i \psi_0 + \gamma_0 \gamma_i \gamma_k \psi_k \right) . \tag{A5}
\]

The matrix \( S \) of Eq. (39) is given by

\[
S = \begin{pmatrix}
0 & 0 & \frac{3(1+A)}{1+3A} & \frac{1+3A}{1+3A} \\
\gamma_0 \gamma_0 & \gamma_2 \gamma_0 & \gamma_0 \gamma_3 & \gamma_0 \gamma_3 \\
0 & \gamma_1 \gamma_0 & \gamma_0 \gamma_1 \gamma_2 \gamma_3 & \gamma_0 \gamma_1 \gamma_2 \gamma_3 \\
\gamma_1 \gamma_0 & 0 & \gamma_1 \gamma_0 & \gamma_1 \gamma_0
\end{pmatrix}. \tag{A6}
\]
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