A Numerical Approach to Stability of Multiclass Queueing Networks

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Abstract

Multiclass queueing networks (McQNs) provide a natural mathematical framework for modeling a wide range of stochastic systems, e.g., manufacturing lines, computer grids, and telecommunication systems. They differ from the classical Jacksonian network model in that the same (physical) item entering the system may require multiple service stages at the same station, with different service and routing characteristics. This distinguishing feature has a significant impact on the assessment of stability of such networks; more specifically, while for Jackson networks stability is equivalent to sub-criticality, for certain McQNs such an equivalence does not hold. As such, analytical stability conditions are not available in general and one needs to rely on numerical methods for assessing stability. We propose a numerical, simulation-based method for determining the stability region of a McQN with respect to its arrival rate(s). We exploit (stochastic) monotonicity of McQNs in order to design a stochastic search scheme in the parameter space; stability is then a monotone property and the stability region can be recovered from its boundary points, which can be evaluated via stochastic approximation schemes. Numerical experiments show that the monotonicity condition, which we prove for Jackson networks, carries over to all networks that we considered, including relatively complex networks for which the stability condition is not known. An extensive set of simulation experiments shows the effectiveness of our method.

1 Introduction

A Multi-class Queueing Network (McQN) is an open network of queues in which jobs of different classes are processed and routed through the system. Each class is assigned to a pre-specified server (station) and has its own routing and service characteristics. Upon service completion a job either changes its class and moves to the corresponding queue, or leaves the system.

McQNs provide a broad modeling framework. They arise as mathematical models for manufacturing systems (assembly lines), where raw pieces enter the system at random times and are successively processed by a number of servers, each providing some specific type(s) of service, visiting possibly multiple times a certain server; in this context, a class encodes the type of piece and its processing stage. Other McQN models arise in communication networks (packet transmission) and in computer science (distributed systems).

A key characteristic of the McQN is that pieces of the same type, visiting the same station during different processing stages, may have different service demands; this reflects the effect that any particular piece may undergo some physical transformations when going through the process. It is this feature that distinguishes a general McQN from a usual Jackson network, and this fact has significant implications in terms of the stability of the network. More specifically, it is known that for Jackson networks stability is equivalent to sub-criticality (i.e., the requirement that at every node the traffic rate, or service utilization, be smaller than 1); as a consequence, solving the traffic equations, one can easily derive analytical (equivalent) conditions for stability. For general McQNs, however, such a procedure is not valid, and one may not derive closed-form stability conditions; in fact, stability always implies sub-criticality, but the converse is not generally true, so one may only derive necessary conditions for stability.

Background – In the last decades much research has been done on establishing stability conditions for McQNs. While it was widely believed that sub-criticality (sub-unitary traffic rate at each station) and
stability are equivalent, as is it the case of Jackson networks, it turned out that in general stability only implies sub-criticality, but the converse fails. A plethora of intriguingly simple examples of subcritical but unstable networks were provided in the early 90’s, of both deterministic (3) and stochastic (1, 4, 8, 10) nature; for an overview of such examples we refer to [9]. Given that sub-criticality could not be used as a stability criterion, a substantial research effort was put into developing alternative approaches. Deterministic (non-random) models have been studied in [11], [5], [3] and it has been established that deterministic re-entrant lines featuring an overall FBFS or LBFS discipline are stable under the sub-criticality condition. In general, the most popular methods for proving stability of McQNs are based on the asymptotic analysis of the fluid-limit model associated with the McQN. Such approaches were introduced in [1], further developed in [10], [12] and [6] and allow one to derive sufficient conditions for stability. Several important results have been established in this way; for instance, stability (under any queue-policy) follows under the stronger condition that the cumulated traffic rate at all stations is sub-unitary [9]. Furthermore, stability of the fluid limit model can be linked to the existence of a positive solution to some linear or quadratic program, the advantage of such methods being that stronger types of stability are established (in that the positive solution corresponds to a Lyapunov function); see [14] and [7]. For an excellent survey of stability of McQNs, including some classical examples of unstable, sub-critical networks, we refer to [9].

Motivation – Consider an automatic manufacturing system in which (raw) pieces are processed, and which is modeled as an McQN. Inserting a larger amount of raw pieces into the system will typically result in a higher congestion. This motivates why an engineer would like to gain insight into the maximal input rate for which the network remains stable. As argued above, answering this stability question is by no means trivial; in fact, even the monotonicity assumption (that the congestion increases with respect to the input rate, which is true for Jackson networks) is not straightforward for general McQNs and defines a research topic itself. In addition, current methods can establish sufficient conditions for stability, e.g., stability of the fluid model or the existence of a positive solution to some linear program, but they have limitations: the fluid model may have multiple solutions and/or instability of the network is not necessarily implied by instability of the corresponding fluid model. In fact, when stability is not equivalent to sub-criticality, no analytical characterization of stability exists in general.

Objective – Knowledge of the stability region of McQNs is of crucial interest in a broad range of engineering settings. Motivated by the lack of analytical tools for assessing stability of an McQN, the aim of this paper is to develop a numerical procedure for networks in which jobs arrive according to independent (among classes) Poisson processes, and their corresponding service times are exponentially distributed. The reason for considering this framework is that the queue-configuration defines a Markov process, for which ‘stability’ amounts positive (Harris) recurrence (and ergodicity) and this allows for using a highly powerful mathematical machinery.

Approach – A key observation underlying our analysis is that for the networks under study, stability amounts to boundedness (over time) in probability of the associated queue-process. This fact allows one to represent the stability region as the support of a specific class of continuous test-functionals associated with the queue-process and infer information on the geometry of the stability set from relevant properties of these functionals. Furthermore, provided that the network fulfills the above-mentioned (stochastic) monotonicity condition, the stability region is a star-shaped domain in the parameter space and the test-functional is monotone, with ‘small’ values lying at the boundary, hence indicating the ‘proximity’ of unstable parameters; in particular, the section of the stability region along any positive direction (ray) is an open segment connecting the origin to some ‘stability threshold’, to be evaluated numerically. Interpolation of such points (along various rays) yields an approximation of the stability region.

Contributions – The main contribution of this paper is the development of an automated, numerical method for determining the stability region associated with a monotone McQN. To the best of our knowledge, the method presented in this paper is among the first simulation-based methods capable of reliably detecting the stability region of a broad class of queueing networks. Our methodology can be used to determine the stability region for McQNs for which no explicit results are available. In addition, we provide a general, unified simulation methodology for McQNs, which illustrates the ease of implementation of our method. An extensive set of numerical experiments shows that:
• the method recovers the stability region for networks for which stability is equivalent to sub-
criticality, i.e., for which analytic stability conditions are available;

• the required monotonicity condition, which is satisfied by Jackson networks, carries over to all
networks that we considered, including relatively complex networks for which stability conditions
are not known;

• numerical stability conditions can be established, in general, for stochastically monotone McQNs.

Organization of the paper – In Section 2 we introduce our numerical method for determining the
stability region; in particular, we briefly introduce the mathematical concept of McQNs and formulate the
main result of this paper. Furthermore, in Section 3 we present a general method for simulating McQNs
(needed to implement the method introduced in Section 2), illustrated by a number of examples. Finally,
in Section 4 we perform numerical experiments showing that the McQNs meet the above-mentioned
monotonicity condition, and that the method introduced in Section 2 provides (within certain error
margin) the correct stability threshold(s).

2 Description of the Approach

In this section we develop a numerical method for determining the stability region (with respect to arrival
rates) of McQNs for which the associated queue-process is Markov. In Section 2.1 we introduce the
mathematical concept of multi-class queueing networks and provide a brief account of stability concepts
related to such networks. Furthermore, in Section 2.2 we show that for networks satisfying a certain
(stochastic) monotonicity condition, the stability region defines a star-shaped domain in the parameter
space, which can be recovered from the critical thresholds along positive (one-dimensional) directions.
Finally, in Section 2.3 we design a one-dimensional stochastic approximation scheme for stochastically
monotone McQNs.

2.1 Mathematical Description of McQNs

A multi-class queueing network consists of \( \aleph \geq 1 \) stations, labeled \( i = 1, \ldots, \aleph \) and \( d \geq 1 \) classes, labeled
\( k = 1, \ldots, d \), such that jobs/customers of class \( k \) require service to some fixed station \( \mathcal{S}(k) \). We assume
w.l.o.g. the mapping \( k \mapsto \mathcal{S}(k) \) is surjective, hence any server will be used; in particular, this means
that \( \aleph \leq d \). In this setup, classes should be understood as processing stages of a given (physical)
piece of raw-material (or any unprocessed item); the multi-class nature of the network is then motivated by
the fact that the same piece may require multiple visits to a given server within the process, but at
each visiting stage may have different service and routing characteristics due to the presumed physical
transformations. When the mapping \( \mathcal{S} \) is bijective, each station acts as a (single-class) \( M/M/1 \) queue
and one recovers the topology of a Jackson network. Finally, we denote by \( \mathcal{K}_i : = \mathcal{S}^{-1} \{ \{i\} \} \) the set of
classes which are served at station \( i \); by assumption, \( \mathcal{K}_i \) is non-empty, for each \( i \).

The dynamics of an McQN can be described as follows. Jobs of class \( k \) enter the network according
to a Poisson process with rate \( \theta_k \geq 0 \); by assumption, \( \theta_k = 0 \) corresponds to a void arrival process,
meaning that class \( k \) does not have external input. Upon arrival, a job of class \( k \) is assigned to station
\( \mathcal{S}(k) \); depending on the station service/queue policy, it either starts receiving service immediately, or it
is enqueued in a waiting line. The service time of a job of class \( k \) is exponentially distributed with rate
\( \beta_k > 0 \) and assumed independent of everything else. After finishing service at station \( \mathcal{S}(k) \), a job of
class \( k \) turns into a job of class \( l \), with probability \( R_{kl} \) and moves to station \( \mathcal{S}(l) \) (where it follows the
corresponding queueing routine) or leaves the network with probability

\[
R_{k0} := 1 - \sum_{l=1}^{d} R_{kl}.
\]
To ensure that the network is open, we assume that the matrix $R := \{R_{i,j}\}_{i,j=1,\ldots,d}$ is sub-stochastic, i.e., we have $(I - R)^{-1} = I + R + (R')^2 + \ldots$. This condition entails that any job will eventually leave the network (in finite time). Furthermore, we define the vector of effective arrival rates by $\lambda := (I - R)^{-1}\theta$.

An McQN satisfying $R_{k,k+1} = 1$, for $k = 1, \ldots, d - 1$ and $R_{d,0} = 1$, such that only class 1 has non-trivial external input, i.e., $\theta_2 = \ldots = \theta_d = 0$, is called a reentrant line. Reentrant lines are the most popular instances of an McQN, as they provide mathematical models for specific manufacturing systems. Furthermore, if $k \mapsto \mathcal{S}(k)$ is injective, we recover the Jackson network model, whereas if all classes using server $i$ have the same service rate, i.e., $\beta_k$ only depends on $\mathcal{S}(k)$, (but possibly different routing) the network is said to be of Kelly type.

We define the traffic rate of station (node) $i$ as

$$\rho_i := \sum_{k \in \mathcal{K}_i} \frac{\lambda_k}{\beta_k} = \sum_{k \in \mathcal{K}_i} \lambda_k M_k,$$

where $M_k = 1/\beta_k$ denotes the expected service time of class $k$. Station $i$ is called sub-critical if $\rho_i < 1$ and the network is called sub-critical if every node is.

In what follows, we only consider non-idling service policies, i.e., the server works as long as there are jobs in the queue. Moreover, we shall consider two classes of service-policies: head-of-the-queue (HQ) and processor-sharing (PS); the former means that only the first job in the queue may receive service, while the latter means that several jobs in the queue may receive service simultaneously, each receiving a certain fraction of the server-capacity which only depends on its class and on the queue composition (in that the order of arrival is irrelevant). Moreover, since the service times are exponentially distributed, it suffices to assume that for PS service policies only the first job from each class may receive service. In fact, from a probabilistic perspective, if $\nu$ jobs of the same class are served simultaneously, each receiving a given fraction $w$ of server-capacity, it is the same as if only one of them (by convention, the first one) is served, receiving a fraction $\nu w$ of server-capacity. Indeed, if $\nu_j$ jobs of class $k_j$, for $j = 1, \ldots, J$, receive simultaneously service at station $i$, each job of class $k_j$ being assigned a fraction $w_j$ of server-capacity, i.e., the service rate of any job of class $k_j$ is $w_j \beta_{k_j}$, then the minimum of all service times follows the same distribution as the minimum of $J$ exponential r.v.'s $S_j$ with rates $\nu_j w_j \beta_{k_j}$, for $j = 1, \ldots, J$, respectively; namely, an exponential distribution with rate $\sum_j \nu_j w_j \beta_{k_j}$. In addition, the probability of having a job of class $k_{j_0}$ finishing service first is given by

$$\frac{\nu_{j_0} w_j \beta_{j_0}}{\sum_{j=1}^J \nu_j w_j \beta_{k_j}} = \Pr\{S_{j_0} = \min_{j=1,\ldots,J} S_j\},$$

which shows the probabilistic equivalence. Therefore, we shall assume that for any given queue-configuration, the server selects a subset of classes present in the queue and assigns a weight (of server-capacity) to the first representatives (with respect to the order of arrival) of each of the selected classes.

### 2.2 Numerical Stability of McQNs

Informally, a queueing network is said to be stable if the number of jobs in the system remains bounded over time; in practice, this is a desirable property as it ensures that the network is able to (eventually) process all incoming workload. Other forms of stability require that the departure rate (at each station) matches the input rate; this is called rate-stability. In principle, any sensible concept of stability related to queueing networks formalizes the fact that ‘jobs do not pile up’ in queues indefinitely over time. When the associated queue-process is Markovian (under the assumptions made in Section 2.1), all these concepts are equivalent to the stronger concept of positive Harris recurrence.

For a wide class of queueing networks, e.g., Jackson networks, stability amounts to sub-criticality. However, this is not always the case and this equivalence fails even for simple networks, as has been illustrated by several examples in the early 90’s; below we provide such a (counter) example.
domain in the positive quadrant \( \{ \theta \in \mathbb{R}^d : \theta \geq 0 \} \), having the origin as a ‘vantage point’; that is, if \( \theta \in \Theta \), then the whole segment connecting the origin to \( \theta \) will be contained in \( \Theta \). Such a region can be approximated by evaluating and interpolating its boundary points along a given set of rays; see Figure 2 for a graphical description of such an approximation method. In such situations, determining the stability region of a McQN can be reduced to a one-dimensional problem, in which the arrival rate vector varies along the ray \( \Theta = \{ \theta v : \theta \geq 0 \} \), where \( v = (v_1, \ldots, v_d) \neq 0 \) is a positive direction, i.e., satisfying \( v_k \geq 0 \), for \( k = 1, \ldots, d \) and \( \|v\| = 1 \); this is the case, for instance, for reentrant lines, where \( v = (1, 0, \ldots, 0) \). Furthermore, the stability region along the ray \( \Theta \) will be, in fact, a half-open segment of the form \( (0, \theta^\ast \bar{v}) \), and hence the problem reduces to evaluating the ‘instability threshold’ \( \theta^\ast \). By the above considerations, in what follows we shall identify our parameter space with \( \Theta := [0, \infty) \) and aim to develop a search procedure for determining the critical threshold \( \theta^\ast > 0 \) above which the network becomes unstable.

Example 1. Consider a re-entrant line with two servers and six classes, with the routing indicated in Figure 1. Both servers employ an FCFS service discipline, with \( \theta \) denoting the (Poisson) arrival rate and \( M_1, \ldots, M_6 \) denoting the expected (exponential) service times of the respective classes. We have

\[
\rho_1 = \theta(M_1 + M_6), \quad \rho_2 = \theta(M_2 + M_3 + M_4 + M_5).
\]

However, for \( \theta = 1 \), \( M_1 = M_3 = M_4 = M_5 = 0.001 \), \( M_2 = 0.897 \) and \( M_6 = 0.899 \) we have \( \rho_1 = \rho_2 = 0.9 \) (sub-criticality) but the network is unstable; see [10]. In fact, stability implies sub-criticality but the converse holds only in some particular cases; see [9]. An interesting question is therefore: under which conditions on the external input is the network stable? More specifically, one faces the task of determining the set

\[
\Theta_s := \{ \theta : \text{the network is stable under } P_{\theta} \},
\]

where \( \theta = (\theta_1, \ldots, \theta_d) \) is a vector of arrival rates and \( P_{\theta} \) denotes the corresponding probability measure. When analytical conditions for stability (e.g., sub-criticality) are known, one can express \( \Theta_s \) analytically:

\[
\Theta_s = \{ \theta : \rho_i = \|CM(I - R')^{-1}\theta\|_i < 1, \ i = 1, \ldots, N \},
\]

where \( C \) denotes the \( 8 \times d \) incidence matrix defined by \( C_{ik} := 1 \{ k \in K_i \} \) and \( M \) denotes the \( d \times d \) diagonal matrix with \( M_{kk} = M_k \). When analytical tools for deciding stability are lacking, one can simply simulate the dynamics of the network of interest and verify its stability, i.e., study the long-term behavior of the queues for a given set of numerical parameters; however, for determining the whole stability region (or its intersection with lower-dimensional manifolds) one needs to infer further properties of the McQN under consideration, since simulating the network for all sets of numerical parameters (arrival rates) is obviously infeasible. One such property is stochastic monotonicity which corresponds to the intuition that increasing the external arrival rate(s) will result in a higher network congestion, hence a larger (expected) number of jobs in the network; the concept of stochastic monotonicity will be detailed in Section 2.4.

Provided that the McQN of interest is monotone, the stability region appears as an open star-shaped set of numerical parameters (arrival rates) will result in a higher network congestion, hence a larger (expected) number of jobs in the network; the concept of stochastic monotonicity will be detailed in Section 2.4.

Figure 1: A first-came-first-served reentrant line (Bramson – Dai).
2.3 Stochastic Approximation of the Stability Threshold

In this section, we consider McQNs satisfying the assumptions in Section 2.1. We throughout denote by \( \mathcal{X} = \{X_t : t \geq 0\} \) the Markov process (defined on a suitable state-space \( \mathbb{X} \) of network configurations – see Section 3.2 for technical details) describing the network dynamics. In addition, we shall assume (unless otherwise specified) that the network starts empty. We say that an McQN is **stochastically monotone** if there exists a bounded mapping \( \varphi : \mathcal{X} \to \mathbb{R} \) vanishing at infinity, satisfying the following conditions:

(I) the mapping \( (t, \theta) \mapsto \varphi_t(\theta) := \mathbb{E}_\theta[\varphi(X_t)] \) is non-increasing in both arguments;

(II) the limit \( \varphi := \lim_{t \to \infty} \varphi_t \) is continuous and strictly decreasing on the stability set \( \Theta_s \);

note that, provided that (I) holds true, the limit in (II) always exists (point-wise) and \( \varphi = \inf \varphi_t \).

Stochastic monotonicity is a quite common property of processes arising in queueing theory. For instance, for Jackson networks it holds true, for a wide class of functions \( \varphi \), e.g., indicator functions of compact decreasing sets or negative exponentials of the total number of jobs in the network. Indeed, property (I) follows by the standard stochastic ordering theory for continuous-time Markov chains (see [13]), while (II) can be directly verified by noting that \( \varphi \) can be expressed as an expectation with respect to the stationary distribution of the queue-process, which is available in closed-form; see the Appendix.

For general McQNs, however, it has not been proven that such monotonicity properties hold; developing a unified approach to stochastic monotonicity of McQNs requires a thorough analysis of the Markovian structure of the underlying queue-process and constitutes an interesting (and challenging) topic of research on its own. In Section 4 we provide the output of our numerical simulations which show that, for all McQNs that we empirically studied, monotonicity holds.

Recall that, under the monotonicity assumptions, the stability set can be written as

\[ \Theta_s = [0, \theta_*) = \{\theta \geq 0 : \varphi(\theta) > 0\}, \]

where \( \theta_* := \sup \Theta_s \) is called in the sequel stability threshold; consequently, the stability set appears as the support of a continuous function. Since \( \varphi(\theta_*) = 0 \), one can approximate \( \theta_* \) by evaluating the root \( \theta_\varepsilon \) of the equation \( \varphi(\theta) = \varepsilon \), for some small \( \varepsilon > 0 \); note that the root exists and is unique, by (II) and \( \theta_\varepsilon \uparrow \theta_* \), for \( \varepsilon \downarrow 0 \). The interpretation of \( \theta_\varepsilon \) is as follows: since \( \varphi_t(\theta) = \mathbb{E}_\theta[\varphi(X_t)] \), by Markov’s Inequality,

\[
\forall \theta > \theta_\varepsilon : \mathbb{P}_\theta \{\phi(X_t) \geq c\} \leq \frac{1}{c} \varphi_t(\theta) \leq \frac{\varepsilon}{c};
\]

that is, for any \( c > 0 \) and (small) probability threshold \( \varphi \in (0, 1) \), choosing \( \varepsilon \in (0, c\varphi] \), there exists some \( t_0 > 0 \) such that at any time \( t \geq t_0 \), the probability that \( X_t \) lies in the set \( \phi^{-1}([c, 1]) \) is bounded by \( \varphi \). In many situations \( \phi^{-1}([c, 1]) \) appears as a reasonably large compact set, hence from a practical standpoint,
this can be regarded as unstable behavior. In fact, should a (non-degenerate) stationary distribution exist under $\mathbb{P}_\theta$, it assigns at most mass $\varphi$ to the set $\phi^{-1}([c,1])$.

Furthermore, for estimating $\theta_*$ we use a stochastic approximation scheme of Robbins-Monro (RM) type, which requires that the values of $\varphi$ (for various parameters $\theta$) are evaluated by simulation; more specifically, an RM approximation scheme is an iterative method which constructs a sequence of parameter updates such that at every update an unbiased unbiased estimate of $\varphi(\theta)$ is used to generate a new parameter. The main difficulty when applying a RM scheme in this setting arises from the fact that one needs to construct a sample from $\varphi$, which appears as a stationary (limiting) measure of the process $\mathcal{X}$. There are two possible approaches to deal with this issue:

1. direct simulation $\varphi(\theta)$ via regenerative ratios, which in turn requires simulating the queue process along a regenerative cycle; see [15].

2. simulating instead $\varphi_1(\theta)$, for an increasing sequence of time-horizons $t > 0$ and invoking an approximation argument; see [15].

Method (1) seems more forthright. Note however that recurrence times are random and they may become arbitrarily large as the input parameter $\theta$ approaches the threshold $\theta_*$; since it is expected that the approximation scheme will stabilize somewhere in the neighborhood of $\theta_*$ (hence close to $\theta_*$), such a method seems rather demanding in terms of computational effort. Method (2) avoids this inconvenience and allows for a better control of the simulation time-horizon, hence over the computational complexity.

In the light of the above discussion, we shall focus in what follows on method (2). To this end, we note that the search space can be reduced to $(0, \bar{\theta})$, where $\bar{\theta}$ denotes the subcritical threshold, given by

$$\bar{\theta} := \min_1 \delta_i; \quad \delta_i := \left[ (CM(I - R')^{-1} \varphi) \right]^{-1}. \quad (2)$$

Recall that stability implies sub-criticality (hence the stability region is a subset of the sub-criticality region) and note that each $\delta_i$ is calculated as the maximal $\theta$ on the direction $\vec{v}$ for which station $i$ is sub-critical; when $\Theta$ is restricted to one-dimensional manifolds, this inclusion reduces to $\theta_* \leq \bar{\theta}$.

We choose some arbitrary increasing sequence $\{t_n\}_{n \geq 1}$ of positive numbers satisfying $t_n \uparrow \infty$ and let $\mathcal{D}_n(\theta)$ denote the distribution of $(X_{t_n})$ under $\mathbb{P}_\theta$, for $\theta \in [0, \bar{\theta}]$ and $n \geq 1$. Furthermore, we fix some sequence $\{b_n\}_{n \geq 1}$ of decreasing positive numbers satisfying $b_n \downarrow 0$ and $\varepsilon > 0$ and define the sequence of iterates (for $n \geq 1$)

$$\vartheta_{n+1} = \min \{ \bar{\theta}, \max \{ \vartheta_n + b_n \cdot (Z_n - \varepsilon), 0 \} \}; \quad (3)$$

here $\vartheta_1 \in (0, \bar{\theta})$ is arbitrarily chosen, and for each $n$ the r.v. $Z_n$ follows the conditional distribution $\mathcal{L}[Z_n|\vartheta_n] = \mathcal{D}_n(\vartheta_n)$, given $\vartheta_n$. Note that, by construction, $\vartheta_n \in [0, \bar{\theta}]$ for $n \geq 1$. Our next result establishes the convergence of the iterates $\vartheta_n$ in $[\bar{\theta}]$ towards $\theta_\varepsilon$; for $n \to \infty$ and, under slightly more restrictive conditions, it provides the magnitude of the approximation error; the complete proof is given in the Appendix.

**Theorem 1.** For any $\varepsilon > 0$, $\vartheta_1 \in [0, \bar{\theta}]$ and positive sequences $\{b_n, t_n\}_{n \geq 1}$, satisfying $\lim_n t_n = \infty$, $\sum_n b_n = \infty$ and $\sum_n b_n^2 < \infty$, the random iterates $\{\vartheta_n : n \geq 1\}$ defined by (3) satisfy $\vartheta_n \longrightarrow \theta_\varepsilon$, a.s.

Furthermore, assume that the family of derivatives $\varphi'_i$ converges uniformly on $(0, \theta_*), \text{ for } t \longrightarrow \infty,$ and that $\inf |\varphi'(\theta)| > 0$. If $b_n = c/n^{\omega}$, for $\omega \in (1/2, 1]$ and $c > (\omega - 1/2)/\left(\inf |\varphi'_i(\theta)|\right)$ and

$$\sup_{\theta \in [0, \theta_\varepsilon]} |\varphi_i(\theta) - \varphi(\theta)| = o(n^{-\kappa}),$$

for some $\kappa > \omega - 1/2$, then $(\vartheta_n - \theta_\varepsilon) = O(n^{-(\omega-1/2)})$, in probability.

We conclude this section by discussing the approximation error of the proposed method. Recall that $\theta_\varepsilon = \varphi^{-1}(\varepsilon)$ approximates $\theta_*$ (for $\varepsilon \downarrow 0$), while $\theta_\varepsilon$ is approximated by iterates of the form $\vartheta_n$, given by (3), according to Theorem 1. In practice, we fix some large $n \geq 1$ and use the estimate $\vartheta_n$ to approximate $\theta_*$. The absolute error consists of a random component and a deterministic one; that is,

$$\Delta_n^\varepsilon := |\vartheta_n - \theta_*| \leq |\vartheta_n - \theta_\varepsilon| + (\theta_* - \theta_\varepsilon). \quad (4)$$
While the behavior of the random component $|\theta_n - \theta|_t$ in \( \Theta \) is given in Theorem \( \Theta \) to analyze the
deterministic part in \( \Theta \) we note that for small $\varepsilon$ we have
\[
\varepsilon = \varphi(\theta_*) - \varphi(\theta_t) \approx -\varphi'(\theta_*)(\theta_* - \theta_t);
\]
that is, $\theta_* - \theta_t \approx \varepsilon/|\varphi'(\theta_*)|$. In particular, if $\varphi'(\theta)$ is bounded away from 0 (close to $\theta_*$), then one obtains
$\theta_t \to \theta_*$ at a linear rate. Nevertheless, if $\lim_{\theta \to \theta_*} \varphi'(\theta) = 0$ then convergence is slower, as we shall see in
our numerical experiments in Section \( \Theta \).

We conclude that the accuracy of the method is related to the behavior of the derivative $\varphi'$ in the
neighborhood of $\theta_*$; namely, defining $c := \lim_{\theta \to \theta_*} |\varphi'(\theta)|$, we see that the larger $c$ is, the better the approximation accuracy.

### 3 Simulation of McQNs

In this section, we provide the technical details for simulating the queue-process associated with an
McQN. In that sense, we adapt a general simulation scheme for continuous-time Markov chains (CTMC)
to our specific type of process.

#### 3.1 Simulation of CTMC with Fixed Transitions

Let $\mathcal{X} := \{X_t\}_{t \geq 0}$ denote a CTMC on the denumerable state-space $\mathcal{X}$. As is well-known, provided that
$\mathcal{X}$ is uniformizable, i.e., spends a strictly positive amount of time in each state, it has the same finite-
dimensional distributions as the embedded Markov chain $\tilde{X} := \{\tilde{X}_n\}_{n \geq 0}$ obtained by sampling $\mathcal{X}$ at
jump-times.

In many situations, the process of interest admits a fixed set of (possible) transitions $\mathcal{T}$ from any
given state $\xi \in \mathcal{X}$; intuitively, a transition $\tau$ from $\xi$ must specify a jump rate and a (new) state. Formally,
a transition $\tau$ is specified by a pair $(h_\tau, f_\tau)$ such that

- $h_\tau : \mathcal{X} \to [0, \infty)$ specifies the rate at which the transition $\tau$ occurs from $\xi$; note that $h_\tau(\xi) = 0$
means that a transition $\tau$ is not possible from $\xi$.
- $f_\tau : \mathcal{X} \to \mathcal{X}$ specifies the new state where the current state is mapped to by a $\tau$-transition; we
assume that $f_\tau(\xi) = \xi$, provided that $h_\tau(\xi) = 0$.

In practice, the evolution of such a process can be described as follows: provided that the process is
in a state $\xi \in \mathcal{X}$, one defines the (finite) quantity
\[
\ell(\xi) := \sum_{\tau \in \mathcal{T}} h_\tau(\xi).
\]  
Provided that $\ell(\xi) > 0$, the process spends in state $\xi$ a (finite) holding time $T$ which is exponentially
distributed with rate $\ell(\xi)$, after which it jumps to a new state $\xi = f_\tau(\xi)$, where $j$ is an r.v. on $\mathcal{T}$ with
distribution $J_\xi$ given by
\[
\forall \tau \in \mathcal{T} : \ P\{j = \tau\} = \frac{h_\tau(\xi)}{\ell(\xi)}.
\]  
Denoting by $\{T_n : n \geq 1\}$ the sequence of (independent) holding times and letting $J_n := \sum_{m \leq n} T_m$,
for $n \geq 1$, denote the jump epochs of $\mathcal{X}$, a typical sample path of $\mathcal{X}$ can be described as follows:

- $X_t = \tilde{X}_n$ for $t \in [J_n, J_{n+1})$, where $J_0 := 0$.
- $T_{n+1}$ is exponentially distributed with rate $\ell(\xi)$ in \( \Theta \), given $\tilde{X}_n = \xi$;
- $\tilde{X}_{n+1} = f_j(\tilde{X}_n)$, where $j \in \mathcal{T}$ is an r.v. with distribution $J_\xi$ in \( \Theta \), given $\tilde{X}_n = \xi$.

This leads to a straightforward simulation scheme, provided that a a suitable state-space together with the
set of transitions $\{(h_\tau, f_\tau) : \tau \in \mathcal{T}\}$ is given.
3.2 The Markovian Structure of McQNs

To formalize the dynamics of an McQN as a continuous-time Markov process, one must specify the state space of the process and the types of possible transitions of the process, together with the associated rates. In general, we define \( X := Q_1 \times \cdots \times Q_N \), where each \( Q_i \) denotes the space of queue-configurations at station \( i = 1, \ldots, N \); the structure of each individual \( Q_i \) depends on the corresponding server policies and will be discussed later, in Section 3.3. Furthermore, the set of all possible transitions is

\[
\mathcal{T} := \{0,1,\ldots,d\} \times \{0,1,\ldots,d\} \setminus \{(0,0)\}
\]

One distinguishes the following types of transitions:

- \( \tau = (0,l) \), with \( l = 1,\ldots,d \) corresponds to an external arrival to class \( l \);
- \( \tau = (k,l) \), with \( k,l = 1,\ldots,d \) corresponds to a switch from class \( k \) to class \( l \);
- \( \tau = (k,0) \), with \( k = 1,\ldots,d \) corresponds to a job of class \( k \) leaving the system.

Note that transitions of type \( (0,l) \), with \( l = 1,\ldots,d \) (external arrivals), are always possible, whereas transitions of the type \( (k,l) \), with \( k = 1,\ldots,d \) are only possible when jobs of class \( k \) are in service.

Provided that \( \xi = [p_1,\ldots,p_N] \in X \), for any \( k \in K_i \) and \( l = 0,1,\ldots,d \) it holds that

\[
h_{(k,l)}(\xi) = h_{(k,l)}(p_i) = W_k(p_i)R_{kl},
\]

where \( W_k(p_i) \), for \( k \in K_i \), encodes the likelihood that the first upcoming event in the network dynamics corresponds to a job of class \( k \) finishing service. Of course, \( W_k(p_i) > 0 \) if and only if class \( k \) receives service in the queue-configuration \( p_i \) and the vector \( (W_k(p_i) : k \in K_i) \) defines a probability distribution on \( K_i \).

Furthermore, transitions in the network consist either of insertion/deletion of jobs in/from the network configuration, or the composition of the two operations, i.e., \( f_{(k,l)} = f_{(0,l)} \circ f_{(k,0)} \), for \( k,l = 1,\ldots,d \), whenever \( W_k(p_i) > 0 \). Finally, we note that, provided that \( k \in K_i \), insertion and deletion of a job of class \( k \) only acts on the \( i \)-component (it only affects the configuration of station \( i \)); more specifically, if \( \xi = [p_1,\ldots,p_N] \), \( p'_i := I_k(p_i) \) and \( p''_i := D_k(p_i) \) then we have

\[
f_{(0,k)}(\xi) := [p_1,\ldots,p'_i,\ldots,p_N], \text{ resp. } f_{(k,0)}(\xi) := [p_1,\ldots,p''_i,\ldots,p_N].
\]

We conclude that the Markovian structure of a McQN is specified by the following elements:

1. a service policy encoded by the quantities \( W_k(p_i) \), for \( i = 1,\ldots,N \) and \( k \in K_i \);
2. a queue policy encoded by the operators \( I_k,D_k : Q_i \rightarrow Q_i \), for \( i = 1,\ldots,N \) and \( k \in K_i \).

The service policy indicates the fractions of service capacity allocated to each class, in a given configuration, while the queue policy specifies how the queue-configuration at the station is modified when a new job arrives to (or leaves) the queue. A few standard service policies can be distinguished; in a head-of-the-queue (HQ) policy the server processes the first job in the queue only, whereas in a processor-sharing (PS) policy one or more jobs (from different classes) can be executed simultaneously and the server capacity will be divided (for instance equally) among them. Standard queue policies are: \textit{first-came-first-serve} (FCFS) and \textit{static buffer priority} (SBP); the latter assumes that classes are ordered w.r.t. priority, i.e., a lower index corresponds to a higher priority ranking.

\textbf{Remark 1.} The key distinction between HQ and PS policies is that, while the former keeps track of the order of arrival of jobs, the latter does only take into consideration the structure of the buffer, hence queue-policies are meaningful only in combination with a HQ service policy. In fact, the following "duality" holds: for HQ policies the service allocation is fixed but queue-insertion is flexible, whereas for PS policies insertion is fixed and service allocation is flexible.
Given an McQN with \( \aleph \) stations, the associated (Markov) queue-process can be constructed as follows:

1. For \( i = 1, \ldots, \aleph \) one defines the space of queue-configurations at station \( i \), together with insertion/deletion operators \( I_k, D_k : \mathbb{Q}_i \to \mathbb{Q}_i \) and the service allocation mappings \( W_k \), for \( k \in \mathcal{K}_i \).

2. Let \( \mathbb{X} := \mathbb{Q}_1 \times \ldots \times \mathbb{Q}_\aleph \) and for a given \( \xi = [p_1, \ldots, p_\aleph] \) define the following types of transitions (below, \( k \in \mathcal{K}_i \) and \( l = 1, \ldots, d \)):
   
   - external arrival: \( h_{(0,k)}(\xi) = \theta_k \) and
   
   \[ f_{(0,k)}(\xi) := [p_1, \ldots, I_k(p_i), \ldots, p_\aleph]; \]

   - departure: \( h_{(k,0)}(\xi) = W_k(p_i)R_\alpha \) and

   \[ f_{(k,0)}(\xi) = [p_1, \ldots, D_k(p_i), \ldots, p_\aleph]; \]

   - class change: \( h_{(l,k)}(\xi) = W_k(p_i)\beta_k R_\delta \) and

   \[ f_{(l,k)}(\xi) = [f_{(0,l)} \circ f_{(k,0)}](\xi). \]

By convention, we set \( f_{(k,l)}(\xi) = \xi \), for \( l = 0, 1, \ldots, d \), provided that \( W_k(p_i) > 0 \). Note that step (2) in the above modeling scheme is standard, while in step (1) one has to encode each server-configuration; this will be detailed in Section 3.3.

### 3.3 Encoding Queue-configurations

Since combinations of service and queue policies can be chosen independently for any station, in what follows we restrict our analysis to a multi-class station, with classes \( k \in \mathcal{K} \), for which we shall illustrate the modeling elements for a few standard server policies.

For convenience, we shall introduce the following notations: let \( \mathbb{N}^\mathcal{K} := \{ \mathbf{x} = (x_k : k \in \mathcal{K}) : x_k \in \mathbb{N} \} \); for any \( \mathbf{x} \in \mathbb{N}^\mathcal{K} \) we let \( \sigma[\mathbf{x}] := \{ k \in \mathcal{K} : x_k \neq 0 \} \), resp. \( ||\mathbf{x}|| := \sum_{k \in \mathcal{K}} x_k \), denote the support, resp. the norm, of \( \mathbf{x} \). In addition, when \( \mathcal{K} \) is (totally) ordered we define \( \kappa(\mathbf{x}) := \min \sigma[\mathbf{x}] \). Finally, for any \( k \in \mathcal{K} \), \( \mathbf{e}_k \) denotes the \( k \)-unit vector in \( \mathbb{N}^\mathcal{K} \).

For a systematic approach, we shall distinguish below between HQ and PS policies. For HQ policies the service allocation is fixed and one can choose the insertion operators (queue discipline), whereas for PS policies the insertion operators are fixed and one has the freedom to choose the service allocation.

#### 3.3.1 HQ Policies

Under an HQ policy, a (non-empty) queue-configuration \( p \in \mathbb{Q} \) must always specify the head of the queue, which will be denoted by \( \kappa(p) \). In such cases, \( W_k(p) = 1 \{ \kappa(p) = k \} \), for \( k \in \mathcal{K} \) and \( p \neq \emptyset \), and the deletion operators satisfy \( D_k(p) = p \) for \( \kappa(p) \neq k \), hence one only needs to define \( D_k(p) \) for configurations \( p \) satisfying \( \kappa(p) = k \). Furthermore, depending on the underlying queue policy, we have:

- Under an FCFS policy, we define

\[
\mathbb{Q}[\mathcal{K}] = \{ p = (k_1, \ldots, k_n) : k_1, \ldots, k_n \in \mathcal{K} \},
\]

i.e., the set of ordered sequences with elements (digits) in \( \mathcal{K} \). In addition, \( \kappa(k_1, \ldots, k_n) = k_1 \) and insertion/deletion operators are defined as follows: \( I_k(k_1, \ldots, k_n) = (k_1, \ldots, k_n, k) \) and

\[ D_k(k_1, \ldots, k_n) = \begin{cases} (k_2, \ldots, k_n), & k = k_1; \\ (k_1, \ldots, k_n), & \text{otherwise.} \end{cases} \]
• Under a (non-preemptive) SBP policy, classes are ordered according to their priority ranking. Once the current job finishes, the highest-ranked job in the queue is executed. In this case, the space of (non-empty) queue configurations is given by 
\[ Q = \{ p = (k, x) : k \in K, x \in \mathbb{N}^K \} \]; note that \( k = \kappa(p) \) and \( x \) specifies the composition of the waiting buffer. Furthermore, the insertion/deletion operators act as follows:

\[
I_k(p) := \begin{cases} (k, 0), & p = \emptyset; \\
(l, x + e_k), & p = (l, x), \end{cases}
\]

respectively,

\[
D_k(p) := \begin{cases} (\kappa(x), x - e_{\kappa(x)}), & x \neq 0; \\
\emptyset, & x = 0. \end{cases}
\]

3.3.2 PS Policies

In this case, we define \( Q := \mathbb{N}^K \), \( I_k(x) := x + e_k \) and \( D_k(x) := x - e_k \), for \( k \in \sigma[x] \). Furthermore, usual service-allocation mappings are:

• equalitarian allocation, specified by \( W_k(x) = \frac{1}{\# \sigma[x]} \)

• proportional allocation, specified by \( W_k(x) = \frac{x_k}{\| x \|} \)

• preferential allocation, specified by \( W_k(x) = 1 \{ \kappa(x) = k \} \).

4 Numerical Results

In this section we illustrate the implementation of the numerical method introduced in Section 2 to approximate the critical threshold for an McQN. To this end, we choose some small \( \varepsilon > 0 \) and average out a large number of Robbins-Monro iterates to construct an estimator

\[
\hat{\theta}_{\varepsilon} := \frac{1}{n} \sum_{m=1}^{n} \vartheta_{m},
\]

for the solution \( \theta_{\varepsilon} \) of \( \varphi(\theta) = \varepsilon \); the averaging method is less sensitive to initial jumps, cf. [2].

For the numerical experiments in this section, we take \( b_n = c_1/\omega^2 \) and \( t_n = c_2/n \), for \( c_1, c_2 > 0 \) and \( \omega \in (1/2, 1] \). The choice of the parameters depends on the type of network under consideration. Moreover, \( \varepsilon \) is chosen to achieve a given level of accuracy. It turns out to be convenient to choose \( c_1 \) depending on \( \varepsilon \) in order to keep the magnitude of the parameter update within certain bounds; for illustrative purposes, we shall also analyze the effect of varying the value of \( \varepsilon \).

In this section we include results that correspond to a set of meaningful examples. As mentioned above, the validity of the monotonicity conditions has not been proven in general, and will therefore be tested numerically when necessary, for validating our method.

In Section 4.1 we consider Jackson networks (for which the stability region is known) and we test our method against the true values of \( \theta_{\ast} \) along various (positive) rays, for a simple network with two nodes. In Section 4.2 we consider multi-class reentrant lines; in particular, we revisit the networks in Figures 1 and 3 for which we approximate the critical value \( \theta_{\ast} \) in some situations when it does not coincide with \( \bar{\theta} \), but we also test our method in some cases when \( \theta_{\ast} = \bar{\theta} \).
$n = 10000$, $\wp \bar{\theta}$ is, For Jackson networks stability is equivalent to sub-criticality, hence stability conditions are known: that is, $\theta_* = \bar{\theta}$. Moreover, monotonicity conditions (I) and (II) can be proved; see the Appendix.

For illustrative purposes, consider an open Jackson network consisting of two nodes, with input rates $\theta_1$, resp. $\theta_2$, and service rates $\beta_1$, resp. $\beta_2$. We further assume that any job finishing service at the first station moves to the second one with probability $\varphi \in [0, 1]$ or leaves the system with probability $1 - \varphi$.

Following the formalism of Section 2.1, one obtains $\lambda_1 = \theta_1$ and $\lambda_2 = \varphi \theta_1 + \theta_2$, whence

$$\Theta_* = \{ \bar{\theta} = (\theta_1, \theta_2) \in \Theta : \theta_1 < \beta_1, \varphi \theta_1 + \theta_2 < \beta_2 \}.$$ 

Let $\bar{v} = (1, v)$, with $v \geq 0$ and note that $\delta_1 = \beta_1$ and $\delta_2 = \beta_2/(v + \varphi)$. Moreover, for any $0 < \theta < \delta_1 \wedge \delta_2$,

$$\varphi(\theta) = \frac{(\delta_1 - \theta)(\delta_2 - \theta)}{(\delta_1 - \theta) - \bar{\delta} \cdot e^{-1}}.$$  \hspace{1cm} (7)

For $\theta \to \delta_1 \wedge \delta_2$, one obtains the approximation

$$\varphi'(\theta) \approx -\frac{1}{1 - e^{-1}} \cdot \frac{\delta_1 \vee \delta_2}{\delta_1 \wedge \delta_2} \cdot \frac{(\delta_1 \vee \delta_2) - (\delta_1 \wedge \delta_2)}{(\delta_1 \vee \delta_2) - (\delta_1 \wedge \delta_2) \cdot e^{-1}},$$

which shows that derivative $\varphi'(\theta)$ vanishes close to $\theta_* = \delta_1 \wedge \delta_2$ if and only if $\delta_1 = \delta_2$; in particular, one expects a slower rate of convergence of $\theta$ towards $\theta_*$ when $\delta_1$ and $\delta_2$ are close to each other.

In the following, we fix the service rates $\beta_1 = 2$ and $\beta_2 = 1.6$ and apply our procedure, with parameters $n = 10000$, $c_1 = \sqrt{1/\varepsilon}$, $c_2 = 1000$ and $\omega = 0.75$, to determine $\theta_*$ for various values of $v$ and $\varepsilon$; we fix $\varphi = 0.2$. A summary of these results (along with the true instability threshold) is presented in Table 1; the values of $v$ in the first column correspond to slopes of $0$, $15$, $30$, $45$, $60$ and $75$ degrees, respectively. The convergence is slow for $v = 0.577$, which is due to the fact that in this case $\delta_1 = 2$ and $\delta_2 \simeq 2.06$ are quite close, hence $\varphi'(\theta)$ is small in magnitude close to $\theta_*$.  

| $\varepsilon$ | $\theta = \bar{\theta}$ |
|---------------|------------------|
| $10^{-2}$     | 1.9730           |
| $10^{-4}$     | 1.9999           |
| $10^{-6}$     | 2.0000           |
| $v = 0.00$    | 1.9809           |
| $v = 0.27$    | 1.9998           |
| $v = 0.58$    | 1.9904           |
| $v = 1.00$    | 1.3199           |
| $v = 1.73$    | 0.8271           |
| $v = 3.73$    | 0.4061           |  

Table 1: Critical threshold estimates for the Jackson network.

4.1 Jackson Networks.

For Jackson networks stability is equivalent to sub-criticality, hence stability conditions are known: that is, $\theta_* = \bar{\theta}$. Moreover, monotonicity conditions (I) and (II) can be proved; see the Appendix.

For illustrative purposes, consider an open Jackson network consisting of two nodes, with input rates $\theta_1$, resp. $\theta_2$, and service rates $\beta_1$, resp. $\beta_2$. We further assume that any job finishing service at the first station moves to the second one with probability $\varphi \in [0, 1]$ or leaves the system with probability $1 - \varphi$.

Following the formalism of Section 2.1, one obtains $\lambda_1 = \theta_1$ and $\lambda_2 = \varphi \theta_1 + \theta_2$, whence

$$\Theta_* = \{ \bar{\theta} = (\theta_1, \theta_2) \in \Theta : \theta_1 < \beta_1, \varphi \theta_1 + \theta_2 < \beta_2 \}.$$ 

Let $\bar{v} = (1, v)$, with $v \geq 0$ and note that $\delta_1 = \beta_1$ and $\delta_2 = \beta_2/(v + \varphi)$. Moreover, for any $0 < \theta < \delta_1 \wedge \delta_2$,

$$\varphi(\theta) = \frac{(\delta_1 - \theta)(\delta_2 - \theta)}{(\delta_1 - \theta) - \bar{\delta} \cdot e^{-1}}.$$  \hspace{1cm} (7)

For $\theta \to \delta_1 \wedge \delta_2$, one obtains the approximation

$$\varphi'(\theta) \approx -\frac{1}{1 - e^{-1}} \cdot \frac{\delta_1 \vee \delta_2}{\delta_1 \wedge \delta_2} \cdot \frac{(\delta_1 \vee \delta_2) - (\delta_1 \wedge \delta_2)}{(\delta_1 \vee \delta_2) - (\delta_1 \wedge \delta_2) \cdot e^{-1}},$$

which shows that derivative $\varphi'(\theta)$ vanishes close to $\theta_* = \delta_1 \wedge \delta_2$ if and only if $\delta_1 = \delta_2$; in particular, one expects a slower rate of convergence of $\theta$ towards $\theta_*$ when $\delta_1$ and $\delta_2$ are close to each other.

In the following, we fix the service rates $\beta_1 = 2$ and $\beta_2 = 1.6$ and apply our procedure, with parameters $n = 10000$, $c_1 = \sqrt{1/\varepsilon}$, $c_2 = 1000$ and $\omega = 0.75$, to determine $\theta_*$ for various values of $v$ and $\varepsilon$; we fix $\varphi = 0.2$. A summary of these results (along with the true instability threshold) is presented in Table 1; the values of $v$ in the first column correspond to slopes of $0$, $15$, $30$, $45$, $60$ and $75$ degrees, respectively. The convergence is slow for $v = 0.577$, which is due to the fact that in this case $\delta_1 = 2$ and $\delta_2 \simeq 2.06$ are quite close, hence $\varphi'(\theta)$ is small in magnitude close to $\theta_*$.  

4.2 Multi-class Reentrant Lines.

In this section we apply our method to derive the stability set for reentrant lines, with the networks in Figures 4 and 3 being our leading examples. In some cases, stability is equivalent to sub-criticality; this occurs, for instance, when at all stations:

- all classes have the same service rate (Kelly);
- classes corresponding to later (resp. earlier) visits have priority over earlier (resp. later) ones.

In general, however, $\theta_* \leq \bar{\theta} = \min \{\delta_1, \delta_2\}$. In addition, stochastic monotonicity for such networks, although expected, has not formally been established. To cope with this issue, we shall perform numerical simulations suggesting that monotonicity assumption (I) holds for the networks under consideration.

| $\varepsilon$ | $\theta = \bar{\theta}$ |
|---------------|------------------|
| $10^{-2}$     | 1.9730           |
| $10^{-4}$     | 1.9999           |
| $10^{-6}$     | 2.0000           |
| $v = 0.00$    | 1.9809           |
| $v = 0.27$    | 1.9998           |
| $v = 0.58$    | 1.9904           |
| $v = 1.00$    | 1.3199           |
| $v = 1.73$    | 0.8271           |
| $v = 3.73$    | 0.4061           |
Consider the reentrant line in Figure 1; for this network, $\delta_1 = (M_1 + M_6)^{-1}$, $\delta_2 = (M_2 + M_3 + M_4 + M_5)^{-1}$. We apply our method to approximate the (unknown) critical value $\theta^*$ for the network in Example 1.

We derive a sequence of estimates $\hat{\theta}_\varepsilon$, for several values of $\varepsilon$; see the top part of Table 2 for an overview of these results. The parameters are $n = 20000$, $c_1 = \varepsilon - 0.8$, $c_2 = 1000$ and $\omega = 0.84$. Note that convergence is rather slow in this case, indicating that $\varphi'(\theta)$ approaches 0 as $\theta \to \theta^*$; a possible explanation might be the fact that $\delta_1 = \delta_2 = 1.111$. Finally, the bottom part of Table 2 shows that the stochastic monotonicity condition (I) holds true for this network.

### 4.2.2 An SBP reentrant line

Consider the network in Figure 3, in which both stations employ a (non-preemptive) SBP policy; we have $\delta_1 = (M_1 + M_4)^{-1}$ and $\delta_2 = (M_2 + M_3)^{-1}$. We choose $\beta_1 = 2$, $\beta_2 = 1.25$, $\beta_3 = 8$ and $\beta_4 = 2.5$.

We construct estimates $\hat{\theta}_\varepsilon$ for the instability threshold $\theta^*$ (which is not known in this case – we compare our results to $\bar{\theta}$ instead), for various values of $\varepsilon$; an overview of our results is given in the top part of Table 3, where the algorithm parameters are $n = 10000$, $c_1 = \varepsilon^{-0.8}$, $c_2 = 1000$ and $\omega = 0.8$; the monotonicity condition is illustrated in the bottom of Table 3.
Figure 3: An SBP reentrant line (Lu-Kumar).

$$\theta = 0.6, \bar{\theta} = 0.6.$$ Table 4 displays the estimates $\hat{\theta}_\varepsilon$ along with the true threshold (the algorithm parameters are the same as those in the previous paragraph); the bottom table reflects the monotonicity assumption.

Table 4: Critical threshold estimates (top) and monotonicity (bottom) for the network of Figure 3 (‘Kelly version’).

We finally consider the situation that $\beta_1 = \beta_4 = 1.6, \beta_2 = \beta_3 = 1.2$ (which is actually a Kelly network), hence $\theta_* = \bar{\theta} = 0.6$. Table 4 displays the estimates $\hat{\theta}_\varepsilon$ along with the true threshold (the algorithm parameters are the same as those in the previous paragraph); the bottom table reflects the monotonicity assumption.

5 Concluding remarks

In this paper we have constructed a simulation-based numerical method for determining the stability region (in terms of arrival rates) associated with Markovian McQNs. Our method is designed based on specific stochastic monotonicity properties, and identifies parameter thresholds at which the queue sizes ‘explode’. In this way the stability region can be found for systems for which no explicit stability criterion is available; for instance, our experiments suggest that the stability threshold in Example 1 is $\theta_* \approx 0.92$.

As we have demonstrated, the approach can be implemented into an automated computational tool in a relatively straightforward way. The computation time for a Robbins-Monro procedure, i.e., generating a sequence of iterates $\{\theta_m : 1 \leq m \leq n\}$ of type (13), is less than 300 seconds/10000 iterates for the networks considered in this paper (on a standard computer) and does not depend (significantly) on the value of $\varepsilon$.

The applicability of our method relies on a particular stochastic monotonicity condition which needs to be fulfilled by the network under consideration. Such a condition has been proven for Jackson (single-class) networks, whereas our numerical experiments provide strong numerical evidence that it carries over to more general multi-class networks. A thorough formal investigation of such stochastic monotonicity properties is an interesting (but highly challenging) topic for future research. In addition, the extension to non-Markovian systems could be explored.
Appendix

Proof of Theorem 1

The proof is based on Theorems 1 and 2 in [15]. Namely, for every $n \geq 1$ and $\theta \geq 0$, let us define:

$$U_n(\theta) := \mathbb{E}[(\varepsilon - Z_n) | \theta], \quad V_n(\theta) := \text{Var}[(\varepsilon - Z_n) | \theta].$$

By the stochastic monotonicity assumption (I), $U_n(\theta)$ is continuous and increasing w.r.t. $\theta \in (0, \bar{\theta})$ and $n$. In addition, we have $V_n(\theta) \leq 1$, for any $n$ and $\theta$.

For the convergence part we apply Theorem 1 in [15]; to this end, we verify the following set of conditions:

(i) $U_n, V_n : [0, \infty) \rightarrow \mathbb{R}$ are measurable, s.t.

$$\sup_{(n, \theta)} \frac{|U_n(\theta)|}{1 + |\theta|} < \infty, \quad \sup_{(n, \theta)} V_n(\theta) < \infty;$$

(ii) for any $\delta > 0$ there exists $n_\delta \geq 1$ s.t. $|\theta - \theta_{\varepsilon}| > \delta$ entails $(\theta - \theta_{\varepsilon})U_n(\theta) > 0$, for $n \geq n_\delta$;

(iii) $\sum_n b_n^2 < \infty$ and for any $\delta_1, \delta_2 > 0$, satisfying $\delta_1 < \delta_2$, it holds that

$$\sum_{n \geq 1} b_n \left( \inf_{(\theta - \theta_{\varepsilon}) \in (\delta_1, \delta_2)} |U_n(\theta)| \right) = \infty.$$

Condition (i) is immediate since $U_n(\theta) \in (-1, 1)$ and $V_n(\theta) \in [0, 1]$, for any $n \geq 1$. Set $U(\theta) := \varepsilon - \varphi(\theta)$ and note that $\theta_{\varepsilon}$ appears as the (unique) root of the equation $U(\theta) = 0$, with $U$ being strictly increasing. Let $\delta > 0$; since $U(\theta_{\varepsilon} + \delta) > 0$ and $U_\varepsilon(\theta) \uparrow U(\theta)$, for $n \rightarrow \infty$, it follows that there exists some $n_\delta \geq 1$ such that $n \geq n_\delta$ entails $U_n(\theta_{\varepsilon} + \delta) > 0$, hence for any $\theta > \theta_{\varepsilon} + \delta$ it holds that

$$(\theta - \theta_{\varepsilon})U_n(\theta) \geq (\theta - \theta_{\varepsilon})U_n(\theta_{\varepsilon} + \delta) > 0.$$

On the other hand, since $\theta \leq \theta_{\varepsilon} - \delta$ entails $U_n(\theta) \leq U(\theta) < U(\theta_{\varepsilon}) = 0$, for any $n \geq 1$, (ii) follows true.

Finally, to verify (iii) we let $\delta_1 < \delta_2$ and (as before) we choose $n_1 \geq 1$ (depending only on $\delta_1$), such that $U_{n_1}(\theta) > 0$ for $\theta > \theta_{\varepsilon} + \delta_1$ and every $n \geq n_1$. Since $U_n(\theta) < 0$ for $\theta \leq \theta_{\varepsilon} - \delta_1$, for any $n \geq 1$, we obtain for $n \geq n_1$ and $|\theta - \theta_{\varepsilon}| \in (\delta_1, \delta_2)$

$$|U_n(\theta)| = \min \{ U_n(\theta_{\varepsilon} + \delta_1), -U_n(\theta_{\varepsilon} - \delta_2) \} \geq \min \{ \varepsilon - \varphi_{n_1}(\theta_{\varepsilon} + \delta_1), \varphi(\theta_{\varepsilon} - \delta_2) - \varepsilon \};$$

using $\lim_n \min\{u_n, v\} = \min\{\lim_n u_n, v\}$ yields

$$\inf_{\theta} |U_n(\theta)| \geq \min \{ \varepsilon - \varphi(\theta_{\varepsilon} + \delta_1), \varphi(\theta_{\varepsilon} - \delta_2) - \varepsilon \} > 0,$$

where the infimum is taken w.r.t. $|\theta - \theta_{\varepsilon}| \in (\delta_1, \delta_2)$. Hence, the assumptions $\sum_n b_n = \infty$ and $\sum_n b_n^2 < \infty$ guarantee (iii), which proves the first claim.

For the second part, we invoke Theorem 2 in [15]; to this end, we verify the following set of conditions:

(i) For any $n$, the function $U_n(\theta) = \mathbb{E}[(\varepsilon - Z_n) | \theta]$ is strictly increasing in $\theta$; in particular, there exists a unique root $\theta_{n, \varepsilon}$ for $U_n(\theta) = 0$.

(ii) The function sequence $\{T_n\}_{n \geq 1}$, defined as

$$T_n(\theta) := \begin{cases} 
U_n(\theta)/(\theta - \theta_{n, \varepsilon}), & \theta \neq \theta_{n, \varepsilon}; \\
-\varphi'(\theta_{\varepsilon}), & \theta = \theta_{n, \varepsilon},
\end{cases}$$

satisfies $T_n(\theta) \in [A_1, A_2]$, for all $n, \theta$, with $A_1 > 0$ and $T_n(\theta_{n}) \rightarrow -\varphi'(\theta_{\varepsilon})$ for $\theta_n \rightarrow \theta_{\varepsilon}$.  

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(iii) There exists constants $0 \leq A_3 < A_4$ such that $A_3 \leq V_n(\theta) = \text{Var}[Z_n|\theta] \leq A_4$, for all $n, \theta$, and $\vartheta_n \to \vartheta_\varepsilon$ entails $V_n(\vartheta_n) \to \sigma^2 > 0$.

(iv) there exist $\kappa, \omega$, s.t. $(\omega - 1/2) \in (0, \kappa)$ and

$$(\theta_{n,\varepsilon} - \vartheta_\varepsilon) = o(n^{-\kappa}), \ n^{\omega} b_n \to c > (\omega - 1/2)/A_1.$$ 

Condition (i) is immediate since $U_n(\theta) = \varepsilon - \varphi_{t_n}(\theta)$ and, for any $t > 0$, $\varphi_t$ is strictly decreasing, satisfying $\varphi_t(0) = 1$ and vanishing at infinity.

To verify (ii), we note that since $\varphi'_t$ is continuous and non-vanishing on $[0, \theta_*]$, hence it is bounded away from both infinity and 0, for any $t > 0$; moreover, since $\varphi'_{t_n}$ converges uniformly to $\varphi'_t$, which is continuous, non-vanishing on $(0, \theta_*)$, it follows that $\varphi'_{t_n}$ is uniformly bounded away from both 0 and infinity. Furthermore, if $\vartheta_n \to \vartheta_\varepsilon$, such that $\vartheta_n \neq \theta_{n,\varepsilon}$, for all $n$, we obtain $T_n(\vartheta_n) = -\varphi'_{t_n}(x_n)$, for some $x_n$ satisfying $|x_n - \vartheta_\varepsilon| < \delta$, for small $\delta > 0$. The convergence $\varphi'_{t_n}(x_n) \to \varphi'(\vartheta_\varepsilon)$ follows from the uniform convergence of the derivatives (and is not affected if $\vartheta_n = \theta_{n,\varepsilon}$, for some $n$).

The conditional variance $V_n(\theta)$ converges uniformly, since (by assumption)

$$V_n(\theta) = \mathbb{E}_0^0[\phi^2(X_{t_n})] - \mathbb{E}_0^0[\phi(X_{t_n})]^2 \longrightarrow V(\theta).$$

Therefore, if $\vartheta_n \to \vartheta_\varepsilon$ then $V_n(\vartheta_n) \longrightarrow V(\vartheta_\varepsilon) > 0$, which proves (iii).

Finally, let $C_\delta := \inf_{\theta - \theta_* < \delta} |\varphi'(\theta)|$, for $\delta > 0$; since $\varphi'(\vartheta_\varepsilon) < 0$, for small $\delta$ we have $C_\delta > 0$. On the other hand, for every $n \geq 1$ it holds that

$$\varphi(\theta_{n,\varepsilon}) - \varphi_{t_n}(\theta_{n,\varepsilon}) = \varphi(\theta_{n,\varepsilon}) - \varphi(\theta_\varepsilon) = -\varphi'(z_n)(\theta_{n,\varepsilon} - \theta_\varepsilon),$$

for some $z_n \in (\theta_\varepsilon, \theta_{n,\varepsilon})$; for the first equality we used the fact that $\varphi(\vartheta_\varepsilon) = \varepsilon = \varphi_{t_n}(\theta_{n,\varepsilon})$, while the second one follows by the mean value theorem. Consequently, one obtains

$$(\theta_{n,\varepsilon} - \theta_\varepsilon) \leq C_\delta^{-1} \sup_{\theta \in [0, \theta_*)} |\varphi_{t_n}(\theta) - \varphi(\theta)| = o(n^{-\kappa}),$$

for large $n \geq 1$, satisfying $|\theta_{n,\varepsilon} - \theta_\varepsilon| < \delta$; this proves the claim and concludes the proof. \square

**Proof of Stochastic Monotonicity for Jackson Networks**

To verify condition (I), we apply the results in [13]. The associated queue-process defines a CTMC on $\mathbb{R}^d$ endowed with the componentwise ordering. Letting $f^+_k(x) := x \pm e_k$, resp., and $B_k := \{x : k \in \sigma[x]\}$ for $k = 1, \ldots, d$, its generator can be written as

$$Q_\theta \phi = \sum_{k=1}^d \beta_k \left( \phi \circ f^+_k - \phi \right) + \sum_{k=1}^d \beta_k \left[ R_{k0} \left( \phi \circ f^-_k - \phi \right) + \sum_{l=1}^d R_{kl} \left( \phi \circ f^+_l \circ f^-_k - \phi \right) \right] 1_{B_k}.$$ 

$Q_\theta$ defines (for any $\theta \geq 0$) a (strongly) monotone generator; this follows by Theorems 4.1 (ii) and 5.4. In addition, (strong) monotonicity with respect to $\theta$ follows from the fact that $\theta \leq \vartheta$ entails $Q_\theta \leq_{st} Q_{\vartheta}$ (see Theorem 3.4) while monotonicity with respect to time follows from Theorem 3.7, by noting that $\{Q_\theta \phi(0) \leq 0$, for any non-increasing $\phi$.

Finally, to verify (II) recall that, under equilibrium, the queue-lengths at stations $k = 1, \ldots, d$ are independent and each one follows a geometric distribution with success probability $1 - (\lambda_k/\beta_k)$, respectively, provided that $\lambda \leq \beta$ (stability condition), with $\beta = (I - R^T)^{-1} \theta$. Taking $\phi(x) = \exp(-\alpha \|x\|)$, for some $\alpha > 0$, yields

$$\varphi(\theta) = \lim_{t \to \infty} \mathbb{E}_\theta[\phi(X_t)] = \frac{\lambda_k}{\beta_k - \lambda_k};$$

one can easily verify now that $\varphi$ is continuous and that $\nabla_\phi \varphi < 0$ (componentwise); this justifies (II). \square
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References

[1] A.N. Rybko and A.L. Stolyar. Ergodicity of Stochastic Processes Describing the Operation of Open Queueing Networks. Problemy Peredachi Informatsii 28, pp. 3–26, 1993.

[2] B.T. Polyak and A.B. Juditski. Acceleration of Stochastic Approximation by Averaging. SIAM Journal on Control and Optimization 30, pp. 838–855, 1992.

[3] S.H. Lu and P.R. Kumar. Distributed Scheduling Based on Due Dates and Buffer Priorities. IEEE Transactions on Automatic Control 36, pp. 1406–1416, 1991.

[4] T.I. Seidman. ‘First come, first serve’ can be unstable!. IEEE Transactions on Automatic Control 39, pp. 2166–2171, 1994.

[5] P.R. Kumar and T.I. Seidman. Dynamic Instabilities and Stabilization Methods in Distributed Real-time Scheduling of Manufacturing Systems. IEEE Transactions on Automatic Control 35, pp. 289–298, 1990.

[6] D. Bertsimas, D. Gamarnik and J.N. Tsitsiklis. Stability Conditions for Multiclass Fluid Queueing Networks. IEEE Transactions on Automatic Control 41, pp. 1618–1631, 1996.

[7] P.R. Kumar and S.P. Meyn. Stability of Queueing Networks and Scheduling Policies. IEEE Transactions on Automatic Control 40, pp. 251–260, 1995.

[8] M. Bramson. Instability of FIFO Queueing Networks. Annals of Applied Probability 4, pp. 414–431, 1994.

[9] M. Bramson. Stability of Queueing Networks. Probability Surveys 5, pp. 169–345, 1994.

[10] J.G. Dai. On Positive Harris Recurrence of Multiclass Queueing Networks: A Unified Approach via Fluid Limit Models. Annals of Applied Probability 5, pp. 49–77, 1995.

[11] P.R. Kumar. Re-entrant Lines. Queueing Systems: Theory and Applications 13, pp. 87–110, 1993.

[12] J.G. Dai and G. Weiss. Stability and Instability of Fluid Models for Re-Entrant Lines. Mathematics of Operations Research 21, pp. 115–134, 1996.

[13] W.A. Massey. Stochastic Orderings for Markov Processes on Partially Ordered Spaces. Mathematics of Operations Research 12, pp. 350–367, 1987.

[14] D. Down and S.P. Meyn. Piecewise Linear Test Functions for Stability of Queueing Networks. Queueing Systems 27, pp. 205–226, 1994.

[15] D.L. Burkholder. On a Class of Stochastic Approximation Processes. Annals of Mathematical Statistics 27, pp. 1044–1059, 1956.

[16] S. Asmussen and P.W. Glynn. Stochastic Simulation: Algorithms and Analysis. Springer Science, NY, 2007.