Duality for multidimensional ruin problem

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Abstract

We consider a $d$-dimensional insurance network, with initial capital $a \in \mathbb{R}^d_+$, operating under a risk diversifying treaty; this is described in terms of a regulated random walk $\{Z_n^{(a)}\}$ via Skorokhod problem in $\mathbb{R}^d_+$ with reflection matrix $R$; $\{Y_n^{(a)}\}$ denotes the corresponding pushing process. Ruin (in a strong sense) of $\{Z_n^{(a)}\}$ is defined as the marginal deficit of each company being positive (and hence zero surplus) at some time $n$. A dual storage network is introduced through time reversal at sample path level over finite time horizon; the stochastic analogue is again a regulated random walk $\{W_n\}$ in $\mathbb{R}^d_+$ starting at 0. It is shown that ruin for $\{Z_n^{(a)}\}$ corresponds to $\{W_n\}$ hitting open upper orthant determined by $R^{-1}a$ before hitting the boundary of $\mathbb{R}^d_+$, even at the sample path level. Under natural hypotheses, we show that $\mathbb{P}(\text{ruin of } \{Z_n^{(a)}\} \text{ in finite time}) = \lim_{n \to \infty} \mathbb{P}(W_n \gg R^{-1}a : n < \text{boundary hitting time of storage process}) = \lim_{n \to \infty} \mathbb{P}(Y_n^{(0)} \gg R^{-1}a : \Delta Y_n^{(0)} \gg 0)$. A notion of $d$-dimensional ladder height distribution is defined, and a Pollaczek-Khinchine formula derived; an expression for the ladder height distribution is presented. Our method is applicable to ruin problem for a continuous time $d$-dimensional Cramer-Lundberg type network, where the companies act independently in the absence of treaty.

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1 Introduction

Connection between ruin probabilities of actuarial risk theory and asymptotic behaviour of storage processes in queuing theory is well known in the one dimensional context for more than fifty years; for example, see [20, 29]. Such a connection has been inspired by the so called duality results for random walks in \( \mathbb{R} \); see [30], and especially Chapter XII of [13] and [4]. A comprehensive exposition is given in [1].

In recent years there has been considerable interest in multidimensional insurance models, including the associated ruin problems; see [2, 3, 9, 7] for example. Notions of ruin in these relate to the vector current surplus process hitting a preassigned open set; the latter is generally taken to be the complement of the nonnegative orthant (corresponding to at least one insurance company in the network getting ruined), or the negative orthant (corresponding to all the companies getting ruined at the same time), or a preassigned half space (corresponding to the current total surplus of all companies going below a certain level).

As pointed out by Buhlman, in spite of the unfortunate terminology, the term ‘ruin of a company’ does not imply that the concerned company is crashing out of business, but only highlights a “need for additional capital”; see p.133 of [6]. It is also referred to as “capital injection by the shareholders of the company” in [11]. Well known optimality properties of one and higher dimensional Skorokhod problem (see [14, 26, 8, 22]) suggest an optimal way of going about it.

A few years back we had proposed a multidimensional insurance model in terms of Skorokhod problem (SP, for short) in an orthant, describing the joint dynamics of \( d \) insurance companies operating under a risk diversifying treaty. According to the treaty, when a company in the network needs an amount to prevent its surplus from getting wiped out, the required capital injection is obtained from other companies in the network, as well as from the shareholders in pre-agreed proportions; and the optimal way to go about is provided by the SP; see the discussion just after Theorem 2.1 below. The reflection matrix will not be diagonal in general, that is, we need to consider oblique reflection. It has been argued in [23, 24] that it results in a reasonable model. So the regulated/ reflected part of the solution to the SP gives the optimal (vector) current surplus, and the pushing part of the solution gives the optimal (vector) capital injection (for averting ruin), while operating under the risk diversifying treaty.
In this paper we consider the ruin problem for multidimensional insurance models that are described in terms of regulated random walks in a $d-$dimensional orthant. The reflection matrix is taken to be constant matrix. Clearly, the vector 0 has a special status, and this leads to canonical notion(s) of ruin of the network; see also [25]. We give 3 natural, but closely related, notions of ruin: ss-ruin corresponding to each company needing positive capital injection, that is, each company having nonzero marginal deficit (and hence zero surplus), at time $n$, s-ruin corresponding to each company having zero surplus with at least one having nonzero marginal deficit as well at $n$, and ruin corresponding to each company having zero surplus at $n$, for some $n$. Note that all are connected to the surplus process hitting the state 0 in finite time. Under minimal conditions, these three notions coincide with probability 1; besides, in the one-dimensional case, these coincide with the classical notion of ruin.

In our setup, the SP for the sequence of partial sums can be built out of a sequence of Linear Complementarity problems for a sequence of vectors, of course, corresponding to the same reflection matrix $R$. When the (discrete) time horizon is finite, through time reversal, we are led naturally to a dual discrete time regulated random walk in the $d-$dimensional orthant, which is referred to as a storage network. This storage network admits a reasonable interpretation. In finite time horizon, ruin of the insurance network is characterized in terms of the dual storage network crossing a certain threshold, at the sample path level; the matrix $R^{-1}$ plays a major role. For considering the stochastic setup in infinite time horizon, we introduce various hypotheses, including the coordinatewise net profit condition. It is shown that the ss-ruin probability of the insurance network can be expressed as the probability of the storage network exceeding a certain threshold (given in terms of the initial capital) before hitting the boundary of the orthant. Moreover, it turns out that the asymptotic behaviour of the dual storage network before hitting the boundary, and the asymptotic behaviour of the pushing process (when it is strictly increasing) associated with the insurance network (with initial capital 0) are closely related. We also introduce an appropriate notion of $d-$dimensional ladder height distribution, and obtain a Pollaczek-Khinchine formula for ss-ruin probability; we are able to express the ladder height distribution in terms of the given data.

We now indicate a class of examples covered by our analysis. Suppose that, in the absence of the risk diversifying treaty, the joint dynamics of the companies is a continuous time $d-$dimensional renewal risk process given by
(3.43) in Example 3.13. The scalar i.i.d. interarrival times, the random mechanism governing which among the companies would take the claim at an arrival time, and the i.i.d. $d$-dimensional claim size vectors form independent families of random variables. To study the ruin problem in this case it is enough to consider the process at claim arrival times; and the process observed only at claim arrival times constitutes a random walk in $\mathbb{R}^d$. So our method is applicable to study the ruin problem for such processes. An important special case is that of a Cramer-Lundberg type network; in the absence of the treaty, the joint dynamics is that of $d$ independent one dimensional Cramer-Lundberg processes.

To our knowledge [5] seems to be the only other paper to have considered duality and multidimensional risk models. However, the approach and emphasis seem to be quite different from ours. For example, in [5], the queueing process and the dual risk process may be based on spaces whose dimensions widely differ, with the latter being set-valued in general; also only normal reflection has been considered.

We now briefly outline the organisation of the paper. Section 2 deals with the deterministic setup, while Section 3 concerns the stochastic setup. In Section 2, we introduce insurance networks described in terms of regulated random walks in an orthant, and the notion of ruin for such networks. The dual discrete time storage network for an insurance network is then presented, over a finite time horizon. This section concludes with sample-path characterization of ruin in terms of dual storage network. Stochastic analogues are considered in Section 3 along with appropriate hypotheses. Duality results, in the sense of equality in distribution, are derived. A Pollaczek-Khinchine formula for ss-ruin probability is obtained, and the ladder height distribution is identified using duality. A detailed discussion concerning ruin problem for renewal risk type network is also given.

We now conclude Section 1 with the list of all hypotheses needed in the sequel.

1.1 Hypotheses

Notation: We shall denote by SP and LCP, respectively, the Skorokhod problem and the linear complementarity problem. For $x \in \mathbb{R}^d$, $(x)_i$ denotes the $i$-th component of $x$. For $x, y \in \mathbb{R}^d$, we shall write: $x \geq y$ if $(x)_i \geq (y)_i$ for all $1 \leq i \leq d$; $x > y$ if $x \geq y$ with $(x)_i > (y)_i$ at least for some $i$; $x \gg y$ if $(x)_i > (y)_i$ for all $1 \leq i \leq d$. Also for vectors $x, y$, $x \ll y$ is the same as
\( y \gg x \); similar comments apply to \( x \leq y, x < y \).

\( G \triangleq \mathbb{R}_+^d \) denotes the \( d \)-dimensional nonnegative orthant, and \( G \) denotes its interior \( \{ x \in \mathbb{R}^d : x \gg 0 \} \). All random variables and processes are defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \); \( \mathbb{E} \) denotes expectation w.r.t. \( \mathbb{P} \).

Vectors will be denoted by lower case alphabets, while random variables by capital letters.

**\( \text{(H1)} \)** \( R = ((R_{ij})) = I - P^t \) is a \( d \times d \) real matrix such that \( P_{ii} = 0, P_{ij} \geq 0, i \neq j \), for all \( 1 \leq i, j \leq d \), and spectral radius of \( P \) is strictly less than 1. Here \( R \) denotes reflection matrix.

**\( \text{(H2)} \)** There exists \( k \in \{1, 2, \ldots, d\} \) such that \( (R^{-1})_{ik} > 0 \) for all \( 1 \leq i \leq d \); that is, at least one column vector of \( R^{-1} \) has strictly positive entries.

**\( \text{(H3)} \)** \( A_i, i = 1, 2, \ldots \) denote one dimensional i.i.d. random variables such that \( A_i > 0 \); these are (scalar) interarrival times.

**\( \text{(H4)} \)** \( X_\ell, \ell = 1, 2, \ldots \) are i.i.d. \( \mathbb{R}_+^d \)-valued random variables; these are vector claim sizes.

**\( \text{(H5)} \)** \( \{ A_i : i \geq 1 \}, \{ X_\ell : \ell \geq 1 \} \) are independent families of random variables.

**\( \text{(H6)} \)** For each \( \ell = 1, 2, \ldots \) and \( i = 1, 2, \ldots, d \), \( \mathbb{P}((X_\ell)_i > x) > 0, \quad \forall x \geq 0 \); that is, marginal claim sizes have unbounded support.

**\( \text{(H7)} \)** For each \( \ell = 1, 2, \ldots \) and \( i = 1, 2, \ldots, d \), \( \mathbb{P}((X_\ell)_i = x) = 0, \quad \forall x > 0 \); that is, \( (X_\ell)_i \) has no atoms in \( (0, \infty) \); however, there can be an atom at 0.

**\( \text{(H8)} \)** \( c = ((c), \ldots, (c)_d) \gg 0 \) with \( (c)_i \) denoting constant premium rates. \( A_1, (X_1)_i, 1 \leq i \leq d \) have finite expectations, and \( \mathbb{E}[(c)_i A_1 - (X_1)_i] > 0, \quad 1 \leq i \leq d \); this is coordinatewise net profit condition.

Note that \( \text{(H1)}, \text{(H2)} \) concern only the reflection matrix \( R \), and involve no probabilistic assumptions. Our analysis on deterministic set up in Section 2 will involve only \( \text{(H1)} \).

**Remark 1.1** (i) By the spectral radius condition in \( \text{(H1)} \) note that

\[ R^{-1} = I + P^t + (P^t)^2 + (P^t)^3 + \cdots \quad (1.1) \]
is a matrix with nonnegative entries, with diagonal entries $\geq 1$.

(ii) In the context of insurance models, in addition to (H1), it is natural to assume that $\sum_{j \neq i} P_{ij} \leq 1$ for all $i$, that is $P$ is a substochastic matrix.

(iii) Note that (H2) holds if $P$ is irreducible; see [28]. It also holds in the feedforward case.

2 Deterministic setup

In this section we introduce the deterministic analogues of insurance and storage networks described in terms of regulated random walks in an orthant. We establish duality results in a finite discrete time horizon at sample path level.

2.1 SP and LCP

We now describe Skorokhod problem (SP, for short) in an orthant for partial sums in the deterministic set up; this basically involves solving a sequence of linear complementarity problems (LCP). Required references on SP will be given at appropriate places, while [10] is an encyclopaedic work on LCP; [19] gives an exposition on the connection between SP and LCP.

Let $R$ be a reflection matrix satisfying (H1). Let $a = ((a)_1, \ldots, (a)_d) \in G$. Let $\{u_n, n \geq 1\}$ denote a sequence in $\mathbb{R}^d$. A pair $\{y^{(a)}_n, n \geq 0\}, \{z^{(a)}_n, n \geq 0\}$ of sequences in $\mathbb{R}^d$ is said to be a solution to the deterministic Skorokhod problem $SP(\{a + \sum u_n\}, R)$ if the following hold:

(s0) $y^{(a)}_0 = 0, \quad z^{(a)}_0 = a$.

(s1) For $1 \leq i \leq d$ Skorokhod equation holds, that is,

$$(z^{(a)}_n)_i = (a)_i + \sum_{\ell=1}^n (u_\ell)_i + (y^{(a)}_n)_i + \sum_{j \neq i} R_{ij} (y^{(a)}_n)_j, \quad n \geq 1; \quad (2.1)$$

or equivalently in vector notation

$$z^{(a)}_n = a + \sum_{\ell=1}^n u_\ell + R y^{(a)}_n$$
$$= z^{(a)}_{n-1} + u_n + R \Delta y^{(a)}_n, \quad n \geq 1, \quad (2.2)$$

where $\Delta y^{(a)}_n = y^{(a)}_n - y^{(a)}_{n-1}$. 

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(s2) $z_n^{(a)} \in \overline{G}$ for all $n \geq 1$.

(s3) $y_n^{(a)} \geq y_{n-1}^{(a)}$, $n \geq 1$ as vectors; moreover $y_n^{(a)}$ can increase only when $(z_n^{(a)})_i = 0$, that is

$$\langle z_n^{(a)}, \Delta y_n^{(a)} \rangle = 0, \quad n \geq 1. \quad (2.3)$$

Note that (s2) is a constraint, while (2.3) in (s3) is a minimality condition.

We refer to $\{y_n^{(a)}\}, \{z_n^{(a)}\}$ respectively as the pushing part, regulated/ reflected part of the solution to $SP(\{a + \sum u_n\}, R)$.

To describe the linear complementarity problem, let $\eta \in \mathbb{R}^d$ and $R$ as above. We say a pair $\xi, \zeta \in \mathbb{R}^d$ is a solution to $LCP(\eta, R)$ if $\zeta = \eta + R\xi$, $\xi \geq 0, \zeta \geq 0$ as vectors, and $\langle \xi, \zeta \rangle = 0$. We denote $\xi = \Phi(\eta, R)$, $\zeta = \Psi(\eta, R)$ and call them respectively the pushing part, regulated part of the solution.

A compilation of required results on deterministic (or equivalently sample path) SP for partial sums is given below. For details and proofs see [15, 26, 19, 8, 18, 21, 22] and references therein.

**Theorem 2.1** Let $R$ satisfy (H1); let $u_n \in \mathbb{R}^d$, $n \geq 1$. Then the following hold.

(i) There is a unique solution pair to $LCP(\eta, R)$ for any $\eta \in \mathbb{R}^d$.

(ii) There is a unique solution pair $\{y_n^{(a)}\}, \{z_n^{(a)}\}$ to $SP(\{a + \sum u_n\}, R)$ for any $a \in \overline{G}$.

(iii) $\{y_n^{(a)}\}, \{z_n^{(a)}\}$ is the solution pair to $SP(\{a + \sum u_n\}, R)$ if and only if for $n = 1, 2, \ldots$ $(\Delta y_n^{(a)}, z_n^{(a)})$ is the solution to $LCP(z_{n-1}^{(a)} + u_n, R)$.

(iv) If $a, b \in \overline{G}$ with $a \leq b$, then for $n \geq 1$,

$$\Delta y_n^{(a)} \geq \Delta y_n^{(b)}, \quad (2.4)$$

$$z_n^{(a)} \leq z_n^{(b)}, \quad (2.5)$$

$$0 \leq y_n^{(a)} - y_n^{(b)} \leq R^{-1}(b - a). \quad (2.6)$$

(v) For $n \geq 1$, put $h_n^{(a)} = ((h_n^{(a)})_1, \ldots, (h_n^{(a)})_d)$ where

$$(h_n^{(a)})_i = \sup_{k \leq n} \max \{0, -((a)_i + \sum_{\ell=1}^k (u_\ell)_i)\}.$$

Then

$$y_n^{(a)} \leq R^{-1}h_n^{(a)}, \quad n \geq 1. \quad (2.7)$$
See Theorem 6 of [18], and Proposition 3.2 and Theorem 4.1 of [22] for proofs of (2.4) - (2.7).

The framework above can be given the following interpretation in the context of insurance models. We consider $d$ insurance companies operating under a risk diversification treaty specified in terms of $R$, with $P$ being a substochastic matrix. Claims are assumed to arrive at regular intervals $k = 1, 2, \cdots$, and are settled instantaneously. According to the treaty, if Company $i$ requires an amount $(\Delta \theta_k)_i$ at time $k$ to avert ruin, then for $j \neq i$, Company $j$ gives $P_{ij}(\Delta \theta_k)_i = |R_{ji}|(\Delta \theta_k)_i$ from its surplus; any shortfall has to be provided by the shareholders of Company $i$ as capital injection; for $j \neq i$, if Company $j$ is not able give from its surplus, then Company $j$ may also have to get capital injection. So the surplus of any company is required to be nonnegative. The spectral radius condition in (H1) means that the network is 'open', in the sense that capital injection for the entire network is also possible; this makes the Skorokhod problem well posed. With each company striving to minimize its liability, Skorokhod problem provides the optimal way of operating under the treaty. So under optimality, a company can invoke the treaty only when it is in the red, and the amount it gets from all sources is just enough to keep it afloat. See [23, 24]. In view of the above, note that

$(a)_i = \text{initial capital of Company } i$;
$(u_k)_i = \text{(premium income for Company } i \text{ during } (k - 1, k]) \text{ minus (claim amount for Company } i \text{ due to } k-\text{th claim})$;
$(z^{(a)}_k)_i = \text{current surplus of Company } i \text{ at time } k, \text{ under optimality}$;
$(y^{(a)}_k)_i = \text{cumulative amount obtained by Company } i \text{ from all sources till time } k \text{ for the purpose of averting ruin, under optimality; so } \Delta y^{(a)}_k)_i = \text{marginal deficit of Company } i \text{ at time } k, \text{ under optimality}$.

Thus the regulated/ reflected part of the solution to $SP\left(\{a + \sum u_n\}, R\right)$ gives the optimal joint dynamics of $d$ companies operating under the treaty. Many notions introduced in the sequel are related to insurance models.

### 2.2 3 notions of ruin

Clearly the state 0 has a special place in our set up. In [25] we had defined ruin as the event that the regulated process hitting the origin; this definition works well when dealing with continuous random variables. However this definition may not be strong enough if the process can hit 0 without any
sector being in deficit. Therefore we now define three natural closely related notions of ruin; as we shall see later, these notions coincide under certain probabilistic assumptions.

Let \( a, R, \Delta y \) be as above. We say ruin occurs for \( \{z^{(a)}_n\} \) if \( z^{(a)}_k = 0 \) for some \( k \geq 1 \); similarly s-ruin occurs for \( \{z^{(a)}_n\} \) if \( z^{(a)}_k = 0, \Delta y^{(a)}_k > 0 \) for some \( k \geq 1 \); and ss-ruin occurs for \( \{z^{(a)}_n\} \) if \( \Delta y^{(a)}_k \gg 0 \) for some \( k \geq 1 \).

Because of the minimality condition (2.3), note that ss-ruin implies s-ruin which in turn implies ruin. Note that ss-ruin denotes each company having nonzero marginal deficit (and hence zero surplus) at time \( n \), while s-ruin corresponds to each company having zero surplus with at least one having nonzero marginal deficit as well at time \( n \), and ruin means that each company having zero surplus at time \( n \), for some \( n \).

Let \( n \geq 1 \) be fixed. Using (2.3) for \( n \), the Skorokhod equation (2.2) successively for \( k \leq n \), and uniqueness of the solution to \( LCP(z^{(a)}_{n-1} + u_n, R) \), we get

\[
\Delta y^{(a)}_n \gg 0 \iff -R^{-1}u_n \gg R^{-1}z^{(a)}_{n-1}
\iff -R^{-1}u_n - R^{-1}u_{n-1} \gg R^{-1}z^{(a)}_{n-2} + \Delta y^{(a)}_{n-1}
\iff \sum_{\ell=n-k}^{n} (-R^{-1}u_{\ell}) \gg R^{-1}z^{(a)}_{n-(k+1)} + [y^{(a)}_{n-1} - y^{(a)}_{n-(k+1)}]
\iff \sum_{\ell=2}^{n} (-R^{-1}u_{\ell}) \gg R^{-1}z^{(a)}_{1} + [y^{(a)}_{n-1} - y^{(a)}_{1}]
\iff -R^{-1}a + \sum_{\ell=1}^{n} (-R^{-1}u_{\ell}) \gg y^{(a)}_{n-1} \tag{2.8}
\]

(Note that the difference between the two sides of the last inequality in the string (2.8) is \( \Phi(z^{(a)}_{n-1} + u_n, R) \), by uniqueness of solution to LCP.) Thus we have

**Proposition 2.2** Let \( R \) satisfy (H1); let \( n \geq 1 \) be fixed. Then the following
\[ \Delta y_n^{(a)} \gg 0 \iff -R^{-1} u_n \gg R^{-1} z_{n-1}^{(a)} \]
\[ \iff -R^{-1} a + \sum_{\ell=1}^{n} (-R^{-1} u_{\ell}) \gg y_{n-1}^{(a)}. \]  
(2.9)

\[ z_n^{(a)} = 0, \, \Delta y_n^{(a)} > 0 \iff -R^{-1} u_n > R^{-1} z_{n-1}^{(a)} \]
\[ \iff -R^{-1} a + \sum_{\ell=1}^{n} (-R^{-1} u_{\ell}) > y_{n-1}^{(a)}. \]  
(2.10)

Moreover, in all three cases, if left side holds then
\[ y_n^{(a)} = -R^{-1} a + \sum_{\ell=1}^{n} (-R^{-1} u_{\ell}). \]  
(2.12)

Proof: (2.9) has been established prior to the statement of the proposition. Replacing $\gg$ by $>$, $\geq$, respectively, in the proof of (2.9), one can prove (2.10), (2.11). Now (2.12) is an easy consequence.

2.3 Storage network in finite discrete time horizon

We begin with an elementary observation.

**Proposition 2.3** Let $R$ satisfy (H1). Then for any $\eta \in \mathbb{R}^d$,
\[ \chi = \Phi(\eta, R), \, \varpi = \Psi(\eta, R) \]
\[ \iff \varpi = \Phi(-R^{-1} \eta, R^{-1}), \, \chi = \Psi(-R^{-1} \eta, R^{-1}). \]  
(2.13)

In particular, $LCP(\theta, R^{-1})$ has a unique solution pair for any $\theta \in \mathbb{R}^d$.

Proof: Clearly
\[ \varpi = \eta + R\chi \iff \chi = -R^{-1} \eta + R^{-1} \varpi. \]
Hence, whenever $\chi \geq 0, \varpi \geq 0, \langle \chi, \varpi \rangle = 0$ hold, (2.13) would also hold. As $R$ is invertible, uniqueness of LCP corresponding to $R^{-1}$ follows from that of LCP corresponding to $R$.  

We now consider Skorokhod problem for a collection of partial sums related to the earlier one through time reversal. To describe the sample path (or equivalently the deterministic) set up, we need to look at a finite discrete time horizon.

Assume that \( R \) satisfies (H1). Let \( n \geq 1 \) be fixed. Let \( u_k \in \mathbb{R}^d, 1 \leq k \leq n \). Set \( \hat{u}_1 = -R^{-1}u_n, \hat{u}_2 = -R^{-1}u_{n-1}, \ldots, \hat{u}_n = -R^{-1}u_1 \); so \( \hat{u}_k = -R^{-1}u_{n+1-k}, 1 \leq k \leq n \). Put \( w_0 = 0, v_0 = 0 \). For \( 1 \leq k \leq n \), let \( \Delta v_k, w_k \) be the unique solution guaranteed by Proposition 2.3 to \( LCP(w_{k-1} + \hat{u}_k, R^{-1}) \). So \( \Delta v_k \geq 0, w_k \geq 0, \langle w_k, \Delta v_k \rangle = 0 \), and

\[
\begin{align*}
w_k &= \sum_{\ell=1}^{k} \hat{u}_\ell + R^{-1}v_k \\
&= w_{k-1} + \hat{u}_k + R^{-1}\Delta v_k, 1 \leq k \leq n \tag{2.14}
\end{align*}
\]

where \( v_k = v_0 + \sum_{\ell=1}^{k} \Delta v_\ell \). That is, in the spirit of Theorem 2.1, \( v_k, w_k, 1 \leq k \leq n \), solve the Skorokhod problem \( SP(\{\sum_{\ell=1}^{k} \hat{u}_\ell, 1 \leq k \leq n\}, R^{-1}) \). Note that uniqueness of the solution to \( SP(\{\sum_{\ell=1}^{k} \hat{u}_\ell, 1 \leq k \leq n\}, R^{-1}) \) follows from Proposition 2.3; see also [19]. We refer to this set up as a deterministic storage network in finite discrete time horizon; here, \( v_k, w_k \) are, respectively, pushing and regulated parts of the storage network.

We now give an interpretation of the storage network. Suppose there are \( d \) storage depots of infinite capacity; let the initial stock be 0 at each depot. While demands might be continuously made, fresh stocks and reinforcements arrive only at the end of periods \( k = 1, 2 \ldots \); readings only at the end of the periods are available. Following assumptions are made.

(a) All demand at a depot during a certain period is met at the end of the same period, if necessary by bringing in reinforcement.

(b) A need for reinforcement at Depot \( i \) at the end of period \( k \), indicates that available stock at the end of period \( k \), including the arrival (and possible inflow as given in this paragraph later) at the end of period \( k \), has not been sufficient to fulfil the demand. This can trigger increased demand at Depots \( j \neq i \) during subsequent periods. So reinforcements are sent to Depots \( j \neq i \) as well at the end of the same period \( k \); such an inflow at Depot \( j \) can also be used to take care of possible unfulfilled demand at that depot at the end of period \( k \). (Such a mechanism may be motivated by a desire to avoid wider customer dissatisfaction in a cooperative setting, or as an attractive business opportunity in a competitive setting.)
(c) Reinforcement supplied to Depot $i$ at the end of period $k$ (due to unfulfilled demand at that depot) is just enough to fulfil the shortfall at the end period $k$; that is, reinforcement is ‘minimal’.

The above interpretation leads to the following meanings.

$(\hat{u}_k)_i = (\text{amount of fresh supply arriving at Depot } i \text{ at the end of period } k) \text{ minus (demand at Depot } i \text{ during the period } (k - 1, k)];$

$(w_k)_i = \text{current stock at Depot } i \text{ at the end of period } k, \text{ after taking into account all reinforcements to Depot } i \text{ till the end of period } k; \text{ so } (w_k)_i \geq 0 \text{ for all } i, k;$

$(R^{-1})_{ii}(\Delta v_k)_i = \text{amount of reinforcement sent to Depot } i \text{ at the end of period } k, \text{ due to unfulfilled demand after taking into account existing stock, fresh supply and inflow to Depot } i \text{ due to shortfall at other depots at the end period } k;$

$(R^{-1})_{ij}(\Delta v_k)_j = [(R^{-1})_{ij}/(R^{-1})_{jj}](R^{-1})_{jj}(\Delta v_k)_j = \text{amount of reinforcement (inflow) sent to Depot } i \text{ due to shortfall at Depot } j, \text{ for } j \neq i, \text{ at the end of period } k.$

Therefore note that

$$(\Delta v_k)_i > 0 \iff (w_{k-1})_i + (\hat{u}_k)_i + \sum_{j \neq i} (R^{-1})_{ij}(\Delta v_k)_j < 0$$

$$\iff -(\hat{u}_k)_i > (w_{k-1})_i + \sum_{j \neq i} (R^{-1})_{ij}(\Delta v_k)_j.$$

In such a case $(w_k)_i = 0,$ that is,

$$(R^{-1})_{ii}(\Delta v_k)_i = -[(w_{k-1})_i + (\hat{u}_k)_i + \sum_{j \neq i} (R^{-1})_{ij}(\Delta v_k)_j]. \quad (2.15)$$

Note: The storage network described above might be suitable when the depots are viewed upon as different banks in a small geographical region. Reinforcement at one bank can result in (defensive) inflow at other banks; of course, it is assumed that the exact quantum of reinforcement at one bank is known (or made known) to other banks without delay. The setup can also be looked upon as different branches of the same bank, with reinforcements coming only from a central node (which is not considered part of the network).
2.4 A connection

Let \( n \geq 1 \) be fixed; let \( u_k \in \mathbb{R}^d \), \( 1 \leq k \leq n \), \( a \in \Gamma \), and \( R \) be a matrix as before. We consider \( \{ y_k^{(a)} , z_k^{(a)} : 1 \leq k \leq n \} \) and \( \{ v_k, w_k : 1 \leq k \leq n \} \) defined earlier.

Define

\[
\sigma_{bd} = \inf \{ k \geq 1 : w_k \in \partial G \}, \quad \quad (2.16)
\]

\[
\vartheta_{R^{-1}a} = \inf \{ k \geq 1 : w_k \gg R^{-1}a \}; \quad \quad (2.17)
\]

l.h.s. is taken as \( +\infty \) if no infimum exists in the above two definitions. Note that \( \sigma_{bd} \) is the first hitting time of the boundary, while \( \vartheta_{R^{-1}a} \) is the first entrance time into the open upper orthant \( \{ x \gg R^{-1}a \} \) for \( \{ w_k : 1 \leq k \leq n \} \).

Lemma 2.4 Let \( R \) satisfy (H1). Let \( n \geq 1 \) be fixed, and \( a \in \Gamma \). If \( \Delta y_n^{(a)} \gg 0 \), then \( \vartheta_{R^{-1}a} \leq n < \sigma_{bd} \), and \( w_n \gg R^{-1}a \).

Proof: By the string (2.8 ), \( \sum_{\ell=1}^k \hat{u}_\ell = \sum_{\ell=1}^k (-R^{-1}u_{n+1-\ell}) \gg 0 \), for \( 1 \leq k \leq n \), and \( \sum_{\ell=1}^n \hat{u}_\ell = \sum_{\ell=1}^n (-R^{-1}u_{\ell}) \gg R^{-1}a \). So by definition of LCP(\( w_{k-1} + \hat{u}_k, R^{-1} \)) and (2.14 ), we now get \( \Delta v_k = 0 \), \( w_k \gg 0 \) for \( 1 \leq k \leq n \), and \( w_n \gg R^{-1}a \). Result now follows by definitions (2.16 ),(2.17 ).

Our next objective is to prove a converse of Lemma 2.4. If \( \{ w_k : 1 \leq k \leq n \} \) does not hit \( \partial G \), and \( w_n \gg R^{-1}a \), in the phraseology of storage network, note the following. At \( k = n - 1 \), \( w_{n-1} \) is more than sufficient to meet any potential reinforcement required due to \( (\hat{u}_n - R^{-1}a) \), (in the sense that \( w_{n-1} \) is enough to meet any reinforcement that may be required due to \( \hat{u}_n \) and still be left with a stock of at least \( R^{-1}a \).) And at \( k = 1, 2, \ldots, (n - 2) \), \( w_k \) is more than sufficient to meet any potential reinforcement required due to \( \hat{u}_{k+1}, \ldots, \hat{u}_n, (\hat{u}_n - R^{-1}a) \). For fixed \( 1 \leq k \leq (n - 1) \), note that part of the potential reinforcement required due to \( \hat{u}_{k+m} \) can be met from \( \hat{u}_{k+1}, \ldots, \hat{u}_{k+m-1} \).

The above comments lead us to the following finite auxiliary sequence of LCP’s. Let \( \Delta \xi^{(a)}_1 = \Phi((-R^{-1}a - R^{-1}u_1), R^{-1}) \), \( \zeta^{(a)}_1 = \Psi((-R^{-1}a - R^{-1}u_1), R^{-1}) \), and \( \Delta \xi^{(a)}_k = \Phi((-R^{-1}a - R^{-1}u_k), R^{-1}) \), \( \zeta^{(a)}_k = \Psi((-R^{-1}a - R^{-1}u_k), R^{-1}) \), for \( 2 \leq k \leq n \). Therefore we have

\[
\zeta^{(a)}_1 = -R^{-1}a - R^{-1}u_1 + R^{-1}\Delta \xi^{(a)}_1, \quad \quad (2.18)
\]

\[
\zeta^{(a)}_k = -R^{-1}\Delta \xi^{(a)}_{k-1} - R^{-1}u_k + R^{-1}\Delta \zeta^{(a)}_k, \quad 2 \leq k \leq n, \quad (2.19)
\]
subject to \( \zeta_k^{(a)} \geq 0, \Delta \xi_k^{(a)} \geq 0, \langle \zeta_k^{(a)}, \Delta \xi_k^{(a)} \rangle = 0, 1 \leq k \leq n \). It may be noted that the above auxiliary sequence of LCP’s does not form an SP. However, we have the following.

**Lemma 2.5** Let \( R \) satisfy (H1). Then \( \Delta \xi_k^{(a)} = z_k^{(a)}, \zeta_k^{(a)} = \Delta y_k^{(a)} \) for \( 1 \leq k \leq n \).

**Proof:** For \( k = 1 \), the result is immediate from Proposition 2.3. For \( k \geq 2 \), by repeated use of Proposition 2.3, we get

\[
\Delta \xi_k^{(a)} = \Phi(-R^{-1}\Delta \xi_{k-1}^{(a)} - R^{-1}u_k, R^{-1}) = \Psi(-R(-R^{-1}\Delta \xi_{k-1}^{(a)} - R^{-1}u_k), R) = \Psi(\Delta \xi_{k-1}^{(a)} + u_k, R) = z_k^{(a)},
\]

as required. The other assertion is similarly proved. ■

**Lemma 2.6** Let \( R \) satisfy (H1). Let \( n \geq 1 \) be fixed, and \( a \in \overline{G} \). If \( \vartheta_{R^{-1}a} \leq n < \sigma_{bd} \) and \( w_n \gg R^{-1}a \), then \( \Delta y_n^{(a)} \gg 0 \).

**Proof:** The discussion following Lemma 2.4 indicates that we first look at \( k = n - 1 \). To find the potential reinforcement required due to \( \hat{u}_n - R^{-1}a = (-R^{-1}u_1 - R^{-1}a) \) one needs to solve \( LCP((-R^{-1}a - R^{-1}u_1), R^{-1}) \). From (2.18) it is clear that \( R^{-1}\Delta \xi_1^{(a)} \) is the potential reinforcement required due to \( (-R^{-1}a - R^{-1}u_1) \). So by the hypothesis, it follows that \( w_{n-1} \gg R^{-1}\Delta \xi_1^{(a)} \).

Proceeding analogously for \( 1 \leq k \leq n \), by (2.19) we see that at time \( k \), the potential reinforcement required due to \( \hat{u}_{k+1}, \cdots, \hat{u}_{n-1}, (\hat{u}_n - R^{-1}a) \) is \( R^{-1}\Delta \xi_{n-k}^{(a)} \). Hence by our hypothesis it now follows that \( w_k \gg R^{-1}\Delta \xi_{n-k}^{(a)}, k = n - 1, n - 2, \cdots, 2, 1 \). In particular \( w_1 \gg R^{-1}\Delta \xi_{n-1}^{(a)} \). As \( w_1 = \hat{u}_1 = -R^{-1}u_n \) by Lemma 2.5 it now follows that \( -R^{-1}u_n \gg R^{-1}z_{n-1}^{(a)} \).

Hence it follows by Proposition 2.2 that \( \Delta y_n^{(a)} \gg 0 \).

Combining Lemma 2.4 and Lemma 2.6 we get first part of the next result.

**Theorem 2.7** (i) Let \( R \) satisfy (H1). Let \( n \geq 1 \) be fixed, and \( a \in \overline{G} \). Then \( \Delta y_n^{(a)} \gg 0 \) if and only if \( \vartheta_{R^{-1}a} \leq n < \sigma_{bd} \), \( w_n \gg R^{-1}a \). In such a case \( \nu_n = 0, z_n^{(a)} = 0, y_n^{(a)} = -R^{-1}a + \sum_{\ell=1}^{n}(-R^{-1}u_\ell) \).
(ii) With $R, n$ as in (i), $[y_n^{(0)} : \Delta y_n^{(0)} \gg 0] = [w_n : n < \sigma_{bd}]$ holds; (here $[\gamma : A]$ denotes the value of $\gamma$ subject to the constraint $A$.) In such a case

$$y_n^{(0)} = w_n = \sum_{\ell=1}^{n} (-R^{-1}u_\ell).$$

(2.20)

Proof: To prove part (ii), take $a = 0$ in part (i). Clearly (2.20) holds in this case.

The above discussion can be extended to quantities related to other two notions of ruin as well. For this define

$$\sigma_0 = \inf\{k \geq 1 : w_k = 0\}, \quad (2.21)$$

$$\theta_{R^{-1}a} = \inf\{k \geq 1 : w_k > R^{-1}a\}. \quad (2.22)$$

Note that $\sigma_0$ is the first hitting time of the origin zero, while $\theta_{R^{-1}a}$ is the first entrance time into the set $\{x \geq R^{-1}a\}$ for $\{w_k : 1 \leq k \leq n\}$. We have

**Theorem 2.8** (i) Let $R$ satisfy (H1). Let $n \geq 1$ be fixed, and $a \in \bar{G}$. Then $z_n^{(a)} = 0$, $\Delta y_n^{(a)} > 0$ if and only if $\theta_{R^{-1}a} \leq n < \sigma_0$, $v_n = 0$, $w_n > R^{-1}a$. In such a case $v_n = 0$, $z_n^{(a)} = 0$, $y_n^{(a)} = -R^{-1}a + \sum_{\ell=1}^{n}(-R^{-1}u_\ell)$.

(ii) With $R, n$ as in (i), $[y_n^{(0)} : z_n^{(0)} = 0, \Delta y_n^{(0)} > 0] = [w_n : v_n = 0, n < \sigma_0]$ holds; in such a case also (2.20) holds.

Proof: As $\Delta y_n^{(a)} > 0$ does not necessarily imply $z_n^{(a)} = 0$, we need to specify it as well; similarly $v_n = 0$ has to be spelt out. With these modifications, replacing $\gg$ by $>$ at appropriate places in the earlier discussion/results, the theorem can be established. □

For $\{w_k : 1 \leq k \leq n\}$, denote the first entrance time into the closed upper orthant $\{x \geq R^{-1}a\}$ by

$$\bar{\theta}_{R^{-1}a} = \inf\{k \geq 1 : w_k \geq R^{-1}a\}. \quad (2.23)$$

Similar analysis, replacing $\gg$ by $\geq$ leads to

**Theorem 2.9** (i) Let $R$ satisfy (H1). Let $n \geq 1$ be fixed, and $a \in \bar{G}$. Then $z_n^{(a)} = 0$ if and only if $\bar{\theta}_{R^{-1}a} \leq n$, $v_n = 0$, $w_n \geq R^{-1}a$. In such a case $v_n = 0$, $z_n^{(a)} = 0$, $y_n^{(a)} = -R^{-1}a + \sum_{\ell=1}^{n}(-R^{-1}u_\ell)$.

(ii) With $R, n$ as in (i), $[y_n^{(0)} : z_n^{(0)} = 0] = [w_n : v_n = 0]$ holds; in such a case also (2.20) holds.
An interesting corollary of the above is

**Corollary 2.10** Notation as above. Then

(i) \( \Delta y_n^{(a)} \gg 0 \) is equivalent to \( \Delta y_n^{(0)} \gg 0, \ y_n^{(0)} \gg R^{-1}a; \)

(ii) \( z_n^{(a)} = 0 \) is equivalent to \( z_n^{(0)} = 0, \ y_n^{(0)} \geq R^{-1}a. \)

In such a case \( y_k^{(a)} = y_k^{(0)} - R^{-1}a, \ z_k^{(a)} = z_k^{(0)} \) for \( k \geq n. \)

### 3 Stochastic setup

In Section 2 we had derived some sample path duality results. We now consider the corresponding situation in the stochastic setup. The connection among the ruin probability of an insurance network with initial capital \( a \), certain asymptotic behaviour of the storage network starting at 0, and an appropriate asymptotic functional of the pushing process of the insurance network with initial capital 0, is our goal. We make the assumptions (H1)-(H8) at various stages.

#### 3.1 Two related regulated random walks

Assume (H1). Put \( U_k(\omega) = A_k(\omega)c - X_k(\omega), \ \omega \in \Omega, k = 1, 2, \cdots. \) Let \( a \in \overline{G} \) denote the vector of initial capitals. Solving the deterministic problem \( SP(\{a + \sum U_\ell(\omega)\}, R) \) path-by-path, that is, taking \( u_n = U_n(\omega), \ n \geq 1 \) for an arbitrary but fixed \( \omega \in \Omega, \) we get the pushing process \( \{Y_n^{(a)}: n \geq 0\} \), and regulated process \( \{Z_n^{(a)}: n \geq 0\} \) satisfying

(S0) \( Y_0^{(a)} = 0, \ Z_0^{(a)} = a. \)

(S1) Skorokhod equation holds, that is,

\[
Z_n^{(a)}(\omega) = a + \sum_{\ell=1}^{n} U_\ell(\omega) + RY_n^{(a)}(\omega)
\]

\[
= Z_{n-1}^{(a)}(\omega) + U_n(\omega) + R\Delta Y_n^{(a)}(\omega), \ n \geq 1, \quad (3.1)
\]

where \( \Delta Y_n^{(a)}(\omega) = Y_n^{(a)}(\omega) - Y_{n-1}^{(a)}(\omega). \)

(S2) \( Z_n^{(a)}(\omega) \in \overline{G}, \ n \geq 1. \)
\((\text{S3})\) \(\Delta Y_n^{(a)}(\omega) \geq 0\) as vectors; also
\[
\langle Z_n^{(a)}(\omega), \Delta Y_n^{(a)}(\omega) \rangle = 0, \quad n \geq 1. \tag{3.2}
\]

So the pair of processes \(\{Y_n^{(a)}, Z_n^{(a)} : n \geq 0\}\) solves the Skorokhod problem \(SP(\{a + \sum U_k\}, R)\).

Suppose \((\text{H3})-(\text{H5})\) hold in addition to \((\text{H1})\). Then the partial sums \((a + \sum_{\ell=1}^n U_\ell), \ n \geq 1\) form a random walk in \(\mathbb{R}^d\). So \(Z_n^{(a)}, \ n \geq 0\) is a regulated random walk starting at \(a\) with \(Y_n^{(a)}\) being the corresponding pushing process. This set up represents a discrete time insurance network operating under a risk diversifying treaty.

Next, put \(\hat{U}_k(\omega) = -R^{-1}U_k(\omega), \ \omega \in \Omega, k = 1, 2, \cdots\). Take \(W_0 = 0, V_0 = 0\). Solving the linear complementarity problem \(LCP((W_{k-1}(\omega) + \hat{U}_k(\omega)), R^{-1}), \ k = 1, 2, \cdots\) recursively we get the pushing process \(\{V_n : n \geq 0\}\), and regulated process \(\{W_n : n \geq 0\}\) satisfying

\((\text{DS0})\) \(V_0 = 0, \ W_0 = 0\).

\((\text{DS1})\) Skorokhod equation holds, that is,
\[
\begin{align*}
W_n(\omega) &= \sum_{\ell=1}^n \hat{U}_\ell(\omega) + R^{-1}V_n(\omega) \\
&= W_{n-1}(\omega) + \hat{U}_n(\omega) + R^{-1}\Delta V_n(\omega), \quad n \geq 1, \tag{3.3}
\end{align*}
\]
where \(\Delta V_n(\omega) = V_n(\omega) - V_{n-1}(\omega)\).

\((\text{DS2})\) \(W_n(\omega) \in \overline{\mathcal{C}}, \ n \geq 1\).

\((\text{DS3})\) \(\Delta V_n(\omega) \geq 0\) as vectors; also
\[
\langle W_n(\omega), \Delta V_n(\omega) \rangle = 0, \quad n \geq 1. \tag{3.4}
\]

Thus the pair of processes \(\{V_n, W_n : n \geq 0\}\) solves the Skorokhod problem \(SP(\{\sum \hat{U}_k\}, R^{-1})\). Under the hypotheses \((\text{H1}),(\text{H3})-(\text{H5})\), as before, \(W_n, \ n \geq 0\) is a regulated random walk in the orthant starting at 0. This is the stochastic analogue of the storage network considered earlier, but now over an infinite time horizon.

The next two results give implications of the coordinatewise net profit condition; see also the proof of Proposition 2.2 of [25].
Proposition 3.1 Let (H1), (H3)-(H5), (H8) hold. Then there is a $\overline{G}$-valued random variable $H_0$ such that

$$\mathbb{P}(Y_n^{(0)} \ll H_0, \forall n \geq 0) = 1.$$  \hfill (3.5)

Moreover, for a.e. $\omega$, there exists an integer $k(\omega)$ such that $(\Delta Y_k^{(0)}(\omega))_i = 0, 1 \leq i \leq d$ for all $k \geq k(\omega)$.

Proof: By (H8) note that $\mathbb{E}((U_1)_i) > 0, 1 \leq i \leq d$. Also by (H3)-(H5), \{(U_\ell)_i, \ell \geq 1\} is a sequence of i.i.d. random variables. So by the strong law of large numbers there is $B \in \mathcal{F}$ with $\mathbb{P}(B) = 1$ such that $\sum_{\ell=1}^n(U_\ell(\omega))_i \to +\infty$ as $n \to \infty$, for all $1 \leq i \leq d$, for $\omega \in B$. Hence for $\omega \in B$ there is an integer $n_0(\omega)$ such that

$$\sup_{k \leq n} \max_{1 \leq i \leq d} \{0, -(\sum_{\ell=1}^k(U_\ell(\omega))_i)\} = (h_0(\omega))_i \equiv \sup_{k \leq n_0(\omega)} \max_{1 \leq i \leq d} \{0, -(\sum_{\ell=1}^k(U_\ell(\omega))_i)\},$$

for $n \geq n_0(\omega)$. Put $h_0(\omega) = ((h_0(\omega))_1, \ldots, (h_0(\omega))_d)$, and take $H_0(\omega) = R^{-1}h_0(\omega), \omega \in B$. Observe that (3.5) now follows from (2.7). From the above and (2.7) we also get that $\Delta Y_n^{(0)}(\omega) = 0$ for $n > n_0(\omega), \omega \in B$. 

Analogous to (2.16), (2.21) respectively, define

$$\sigma_{bd}(\omega) = \inf\{k \geq 1 : W_k(\omega) \in \partial G\}, \quad (3.6)$$

$$\sigma_0(\omega) = \inf\{k \geq 1 : W_k(\omega) = 0\}. \quad (3.7)$$

Proposition 3.2 Let (H1), (H3)-(H5), (H8) hold. Then

$$\mathbb{P}(\sigma_{bd} < \infty) = 1.$$  \hfill (3.8)

Proof: Let $B$ be as in the proof of Proposition 3.1. By (1.1), $R^{-1}$ has only nonnegative entries with diagonal entries $\geq 1$. Hence $\sum_{\ell=1}^n(-R^{-1}U_\ell(\omega))_i \to (-\infty)$ as $n \to \infty$, for all $1 \leq i \leq d$, for $\omega \in B$. So for $\omega \in B$ there is an integer $n_1(\omega)$ such that $\sum_{\ell=1}^n(-R^{-1}U_\ell(\omega))_i < 0$ for all $n \geq n_1(\omega), 1 \leq i \leq d$. Required conclusion (3.8) now follows.

Remark 3.3 Suppose $n \geq 1$ is arbitrary but fixed. Note that the random variable $(U_n, U_{n-1}, \ldots, U_1)$ has the same distribution as $(U_1, \ldots, U_{n-1}, U_n)$. Hence the pathwise discussion in Section 2 for $w_1, w_2, \ldots, w_{n-1}, w_n$ is applicable to random variables $W_1, W_2, \ldots, W_{n-1}, W_n$ in the sense of results holding in law, that is, equality in law; ($=^d$ shall denote equality in law.)
3.2 Ruin of insurance network

We now define ruin times corresponding to the notions of ruin introduced to earlier. For $a \in \overline{G}, \omega \in \Omega$ define

\[ \varrho_s^{(a)}(\omega) = \inf\{k \geq 1 : \Delta Y_k^{(a)}(\omega) \gg 0\}, \quad (3.9) \]
\[ \varrho_s^{(a)}(\omega) = \inf\{k \geq 1 : Z_k^{(a)}(\omega) = 0, \Delta Y_k^{(a)}(\omega) > 0\}, \quad (3.10) \]
\[ \varrho_r^{(a)}(\omega) = \inf\{k \geq 1 : Z_k^{(a)}(\omega) = 0\}. \quad (3.11) \]

$\varrho_s^{(a)}(\omega)$ is taken to be $+\infty$ if there is no $k \geq 1$ satisfying the requirement; similar comment applies to the other cases. Clearly $\varrho_r^{(a)} \leq \varrho_s^{(a)} \leq \varrho_s^{(a)}$. Note that $\varrho_s^{(a)}, \varrho_s^{(a)}, \varrho_r^{(a)}$ are ruin times corresponding to, respectively, ss-ruin, s-ruin, ruin.

The next result indicates when the three notions may coincide.

**Proposition 3.4** Let (H1), (H3)-(H5), (H7) hold, and $c \gg 0$. Then for any $a \in \overline{G}$,

\[ \mathbb{P}(\varrho_s^{(a)} = \varrho_s^{(a)} = \varrho_s^{(a)}) = 1. \quad (3.12) \]

**Proof:** For fixed integer $k \geq 1$, we need to show that $Z_k^{(a)} = 0$ implies $\Delta Y_k^{(a)} \gg 0$ with probability 1. As $c \gg 0$ and $A_i > 0$, note that $(R^{-1}(Z_{k-1}^{(a)} + A_k c))_i > 0$ for all $1 \leq i \leq d$. By (H7) $(R^{-1}X_k)_i \geq 0$, and has no atoms on $(0, \infty)$. Also by (H3)-(H5), $R^{-1}(Z_{k-1}^{(a)} + A_k c)$ and $R^{-1}X_k$ are independent random variables. Consequently $\mathbb{P}((R^{-1}(Z_{k-1}^{(a)} + A_k c - X_k))_i = 0) = 0$ for all $1 \leq i \leq d$. If $Z_k^{(a)}(\omega) = 0$, by (3.1) note that $-(R^{-1}(Z_{k-1}^{(a)} + A_k c - X_k))_i(\omega) = (\Delta Y_k^{(a)}))_i(\omega) \geq 0$. The required conclusion now follows.

The next result implies that various events associated with ss-ruin, like stochastic analogues of the string (2.8 ), have positive probability.

**Proposition 3.5** Let (H1)-(H6) hold, and let $c \gg 0$. Then the following hold.

(i) $\delta_0 = \mathbb{P}(-R^{-1}U_\ell \gg 0) > 0$ for any $\ell \geq 1$. Moreover $\mathbb{P}(-R^{-1}U_\ell \gg 0$ infinitely often $) = 1$.

(ii) $\mathbb{P}(-R^{-1}U_\ell \gg b) > 0$ for any fixed $\ell \geq 1, b \in \overline{G}$.

**Proof:** Let $\ell \geq 1$. Let $k$ be as in (H2). Take $\beta = \max\{(R^{-1})_{ij}/(R^{-1})_{jk} : 1 \leq j \leq d\}$. By (H2) we get $0 < \beta < \infty$. By (H6) support of $(X_\ell)_k = [0, \infty)$. 

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Consequently, as $A_\ell > 0$ and independent of $(X_\ell)_k$, we now get $\mathbb{P}((X_\ell)_k > \beta A_\ell) > 0$. So by the definition of $\beta$ note that $\mathbb{P}(-(R^{-1}U_\ell)_i > 0, \forall 1 \leq i \leq d) > 0$. The first assertion now follows. As $U_\ell$ are i.i.d. random variables, note that $\delta_0$ does not depend on $\ell$. An application of the second Borel-Cantelli lemma now gives the second assertion. This proves (i).

To prove (ii), let $\beta_0 = \max\{(b_j)/(R^{-1})_{jk} : 1 \leq j \leq d\}$. Proceeding as in (i), we get that $\mathbb{P}((X_\ell)_k > \beta A_\ell + \beta_0) > 0$. Required conclusion is obtained as above.

### 3.3 Pollaczek-Khinchine formula

We will now confine ourselves to the ss-ruin problem. Besides being the appropriate $d$-dimensional analogue of the classical ruin problem, it seems to be more amenable to analysis. Of course, Proposition 3.4 gives sufficient conditions for the three notions of ruin to be equivalent with probability one.

We begin with a duality result.

**Theorem 3.6** Assume (H1)-(H6),(H8). Let $a \in \overline{G}$. Then

$$0 < \mathbb{P}(\varrho_{ss}^{(a)} < \infty) = \mathbb{P}(\vartheta_{R^{-1}a} < \sigma_{bd}) < 1.$$  \hspace{1cm} (3.13)

where $\vartheta_{R^{-1}a}(\omega) = \inf\{k \geq 1 : W_k(\omega) \gg R^{-1}a\}$, and $\varrho_{ss}^{(a)}, \sigma_{bd}$ are given respectively by (3.9),(3.6). Moreover $\mathbb{P}(\Delta Y_n^{(a)} \gg 0) > 0$ and hence $\mathbb{P}(\sigma_{bd} > n) > 0$ for any $n \geq 1$. Also

$$\lim_{|a| \to \infty, a \in G} \mathbb{P}(\varrho_{ss}^{(a)} < \infty) = 0.$$ \hspace{1cm} (3.14)

**Proof:** Note that the equality in (3.13) follows by part (i) of Theorem 2.7. Now taking $b = R^{-1}a$ in part (ii) of Proposition 3.5, we see that $\mathbb{P}(\varrho_{ss}^{(a)} = 1) > 0$; so the first inequality in (3.13) follows. By (H8), $\mathbb{E}((R^{-1}U_1)_i) > 0$, and hence $\mathbb{P}((R^{-1}U_1)_i > 0) > 0$ for all $i$. Hence $\mathbb{P}(W_1 \in \partial G) > 0$; that is, $\mathbb{P}(\sigma_{bd} = 1) > 0$. Consequently the second inequality in (3.13) now follows.

Next fix $n \geq 1$; we proceed as in the proof of part (i) of Proposition 3.5. Note that $R^{-1}X_n$ is independent of $R^{-1}Z^{(a)} + A_n R^{-1}c$; so by (H2),(H6) we get $\mathbb{P}(R^{-1}X_n \gg R^{-1}Z^{(a)}_{n-1} + A_n R^{-1}c) > 0$; that is, $\mathbb{P}(-R^{-1}U_n \gg R^{-1}Z^{(a)}_{n-1}) > 0$, which is equivalent to ss-ruin Proposition 2.2. So by part (i) of Theorem 2.7, the second assertion follows. Finally, by Proposition 3.2, it follows that $\lim_{n \to \infty} \mathbb{P}(\sigma_{bd} > n) = 0$. Hence (3.13) now implies (3.14).
Now assume (H1)-(H6),(H8). Note that the \(d\)-dimensional insurance network \(\{ (Y_n^{(0)}, Z_n^{(0)}) : n = 0, 1, 2, \ldots \}\), with initial capital 0, is a strong Markov process starting at \((0, 0)\). As there is no dispersion, and as drift, reflection are constants, \(\{ Z_n^{(0)} : n = 0, 1, 2, \ldots \}\) is also a strong Markov process starting at 0. Take \(\tau_0 \equiv 0\). For \(n \geq 1\), define
\[
\tau_n(\omega) = \inf\{ k \geq \tau_{n-1}(\omega) + 1 : \Delta Y_k^{(0)}(\omega) \gg 0 \}, \quad (3.15)
\]
if the set \(\{ k \geq \tau_{n-1}(\omega) + 1 : \Delta Y_k^{(0)}(\omega) \gg 0 \} \neq \emptyset\); put \(\tau_n(\omega) = +\infty\) if there is no \(k \geq \tau_{n-1}(\omega) + 1\) such that \(\Delta Y_k^{(0)}(\omega) \gg 0\). Note that \(\tau_k, k \geq 0\) are stopping times w.r.t. the natural filtration.

For convenience, write \(Y^{(0)}(k, \omega) = Y_k^{(0)}(\omega), Z^{(0)}(k, \omega) = Z_k^{(0)}(\omega)\); so \(Y^{(0)}(\tau_j, \omega) = Y^{(0)}(\tau_j(\omega), \omega), Z^{(0)}(\tau_j, \omega) = Z^{(0)}(\tau_j(\omega), \omega)\). For \(n \geq 1\), define
\[
L_n(\omega) = Y^{(0)}(\tau_n, \omega) - Y^{(0)}(\tau_{n-1}, \omega), \quad \text{if} \ \tau_n(\omega) < +\infty; \quad (3.16)
\]
\[
L_n^+(\cdot) = L_n(\cdot) \text{ restricted to } \{ \tau_n < +\infty \}; \quad (3.17)
\]
in the above note that \(Y^{(0)}(\tau_0) \equiv 0\). Clearly \(L_n\) takes value in \(\{0\} \cup G\), and \(L_n^+\) in \(G\). We shall call \(L_n^+\) the \(d\)-dimensional first strictly ascending ladder height random variable, and \(L_n^+\) the \(d\)-dimensional \(k\)-th strictly ascending ladder height random variable.

Now by the second assertion in Proposition 3.1, note that for a.e. \(\omega \in \Omega\), there is \(n_0(\omega)\) such that \(\tau_n(\omega) = +\infty\), and hence \(L_n(\omega) = 0\) for all \(n \geq n_0(\omega)\). Define
\[
\beta(\omega) = \inf\{ k \geq 1 : \tau_k(\omega) = +\infty \} = \inf\{ k \geq 1 : L_k(\omega) = 0 \}. \quad (3.18)
\]
From the above it is clear that \(\beta < +\infty\) with probability 1.

Note that \(Z^{(0)}(\tau_n, \omega) = 0\) if \(\tau_n(\omega) < +\infty\). Consequently, by the strong Markov property, conditional on \(\tau_n < +\infty\),
\[
\{(Y^{(0)}(\tau_k + j) - Y^{(0)}(t_k)), Z^{(0)}(\tau_k + j)) : j = 0, 1, 2, \ldots \}, \quad k = 0, 1, 2, \ldots, n
\]
are \((n + 1)\) independent stochastic processes; also the first \(n\) of these, that is, corresponding to \(k = 0, 1, \ldots, n - 1\) are identically distributed as well. In particular, conditional on \(\tau_n < +\infty\), while the \((n + 1)\) random variables \(\tau_1, \tau_2 - \tau_1, \ldots, \tau_n - \tau_{n-1}, \tau_{n+1} - \tau_n\) are independent, the first \(n\) of these are i.i.d. finite valued random variables; (here \(\tau_{n+1} - \tau_n\) is taken as \(+\infty\) when \(\tau_{n+1} = \)
Hence, conditional on $\tau_n < \infty$, random vectors $L_1, L_2, \cdots, L_n, L_{n+1}$ are independent, and $L_1, \cdots, L_n$ are i.i.d. random vectors taking value in $G$, with $L_k = L_k^+, 1 \leq k \leq n$; also $L_{n+1} = 0$ if and only if $\tau_{n+1} = +\infty$. For any Borel set $B \subseteq G$ define

$$\alpha_+(B) = \mathbb{P}(L_1^+ \in B), \ B \subseteq G; \quad (3.19)$$

$$\alpha_0(B) = \frac{1}{\alpha_+(G)} \alpha_+(B), \ B \subseteq G. \quad (3.20)$$

Note that $\alpha_+$ is a defective distribution, while $\alpha_0$ is the corresponding normalized probability distribution both concentrated on $G$. Take $M_0 \equiv 0$, and for $n \geq 1$, define

$$M_n(\omega) = \sum_{j=1}^{n} L_j(\omega); \quad (3.21)$$

$$M(\omega) = \sum_{j=1}^{\infty} L_j(\omega);$$

note that $M_n(\omega) = Y(0)(\tau_n(\omega), \omega)$, if $\tau_n(\omega) < \infty$ and $M_n(\omega) = M_{n-1}(\omega)$ if $\tau_n(\omega) = \infty$.

**Theorem 3.7** Assume (H1)-(H6),(H8). Let $\vartheta_{ss}, \sigma_{bd}, \vartheta_{R^{-1}a}$ be as in Theorem 3.6. Denote $p \equiv \mathbb{P}(\bar{U}_1 \in G) = \mathbb{P}(-R^{-1}U_1 \in G)$; note that $0 < p < 1$. Then $(\beta - 1)$ has a geometric distribution with parameter $(1 - p)$, $\alpha_+(G) = p$, and the distribution of $M$ is the geometric compound

$$\nu_{M}(B) = (1 - p)\delta_0(B) + \sum_{k=1}^{\infty} (1 - p)p^k \alpha_0^{(k)}(B), \quad (3.22)$$

for any Borel set $B \subseteq \{0\} \cup G$. Moreover ruin probability for the insurance network is given by

$$\mathbb{P}(\vartheta_{ss}^{(a)} < \infty) = \mathbb{P}(M \gg R^{-1}a) = (1 - p) \sum_{n=1}^{\infty} \alpha_+^{(n)}(\{x \gg R^{-1}a\}), \quad a \in \mathbb{G}. \quad (3.23)$$

**Proof:** By hypotheses (H5),(H6),(H1) and (1.1) we get $p > 0$; by the coordinatewise net profit condition (H8) it is clear that $p < 1$. As $0 = R^{-1}0$, we have
\[ \vartheta_0 = \inf\{k \geq 1 : W_k \in G\} \] Clearly \{\sigma_{bd} \leq \vartheta_0\} = \{\sigma_{bd} < \vartheta_0\} = \{\sigma_{bd} = 1\} as events. Consequently by Theorem 3.6

\[ P(\beta = 1) = P(\tau_1 = +\infty) = P(\Delta Y_k^{(0)} \gg 0 \text{ never happens}) = P(\vartheta_k^{(0)} = \infty) = P(\sigma_{bd} < \vartheta_0) = P(\sigma_{bd} = 1) = (1 - p). \quad (3.24) \]

From the discussion preceding the theorem, conditional on \(\tau_n < \infty\), note that \(L_1, \cdots, L_n\) are i.i.d. random vectors with distribution \(\alpha_0\) for each \(n\); also

\[ P(L_{n+1} = 0 \mid \tau_n < \infty) = P(\tau_{n+1} = \infty \mid \tau_n < \infty) = P(\tau_1 = \infty) = P(L_1 = 0) = (1 - p). \quad (3.25) \]

Since \(\{\tau_j < \infty, j \leq k\} = \{\tau_k < \infty\}, k \geq 1\), proceeding recursively and using (3.24),(3.25), we have

\[ P(\beta = n + 1) = P(\tau_{n+1} = \infty, \tau_j < \infty, 1 \leq j \leq n) = P(\tau_{n+1} = \infty \mid \tau_n < \infty)P(\tau_n < \infty) = (1 - p)P(\tau_n < \infty) = (1 - p)P(\tau_{n-1} < \infty) = (1 - p)pP(\tau_{n-1} < \infty) = (1 - p)p^n, \quad n = 0, 1, \cdots \quad (3.26) \]

Thus \(\beta - 1\) has a geometric distribution with parameter \(1 - p\). From (3.25) it also follows that \(\alpha_+(G) = p\).

Clearly

\[ M(\omega) = \lim_{n \to \infty} M_n(\omega) = \sup_{n \geq 0} M_n(\omega) = \sum_{k=1}^{\beta(\omega)-1} L_k(\omega), \quad \text{if } \beta(\omega) \geq 2; \quad (3.27) \]

\[ = 0, \quad \text{if } \beta(\omega) = 1. \]

By (3.27),(3.24), we get \(P(M = 0) = (1 - p)\). Now by (3.21), conditional on \(\beta = n + 1\), it is seen that \(M_n\) is distributed as \(\alpha_0^{*}(n)\). Hence, using
(3.26), (3.19), for any Borel set $B \subseteq G$,

$$
\mathbb{P}(M \in B) = \sum_{n=1}^{\infty} \mathbb{P}(M \in B \mid \beta - 1 = n)\mathbb{P}(\beta - 1 = n)
$$

$$
= \sum_{n=1}^{\infty} \mathbb{P}(M_n \in B \mid \beta = n + 1)\mathbb{P}(\beta = n + 1)
$$

$$
= \sum_{n=1}^{\infty} (1 - p)p^n \alpha_0^{(n)}(B)
$$

$$
= (1 - p)\sum_{n=1}^{\infty} \alpha_+^{(n)}(B). \quad (3.28)
$$

By (3.21), note that

$$
M(\omega) = \lim\{Y^0_k(\omega) : (\Delta Y^0_k(\omega) \gg 0)\}. \quad (3.29)
$$

Therefore by the definition of ruin, Corollary 2.10, Theorem 3.6 we get for any $a \in \mathcal{G}$,

$$
\mathbb{P}(\rho_{ss}^{(a)} < \infty) = \mathbb{P}(\Delta Y^0_k \gg 0 \text{ for some } k \geq 1)
$$

$$
= \mathbb{P}(\Delta Y^0_k \gg 0, Y^0_k \gg R^{-1}a \text{ for some } k \geq 1)
$$

$$
= \mathbb{P}(M \gg R^{-1}a) \quad \text{(3.30)}
$$

From (3.28), (3.30), required conclusions (3.22), (3.23) follow.

**Theorem 3.8** Assume (H1)-(H6), (H8). Let $\rho_{ss}^{(a)}, \rho_{bd}, \rho_{R^{-1}a}, p, \beta, \alpha_+, \alpha_0, M, \nu_M$ be as in Theorem 3.7. Define

$$
\hat{U}_k^+ = \hat{U}_k \text{ restricted to } \{\hat{U}_k \in G\}, \quad (3.31)
$$

$$
\mu_+(B) = \mathbb{P}(\hat{U}_1^+ \in B) = \mathbb{P}(\hat{U}_1 \in B), \quad B \subseteq G. \quad (3.32)
$$

Note that $\mu_+$ is a defective distribution concentrated on $G$, with $0 < p = \mu_+(G) < 1$. Set $\mu_0(\cdot) = \frac{1}{p}\mu_+ (\cdot)$. Define the compound geometric $\nu(\cdot)$ by

$$
\nu(B) = (1 - p)\delta_0(B) + \sum_{k=1}^{\infty} (1 - p)p^k \mu^{(k)}_0(B), \quad B \in \mathcal{B}(\mathbb{R}^d). \quad (3.33)
$$

Then the following hold:
(i) \( (\sigma_{bd} - 1) \) has a geometric distribution with parameter \( (1-p) \), and hence \( \sigma_{bd} = d \beta \).  
(ii) \( \nu \) is a probability measure concentrated on \( \{0\} \cup G \), such that
\[
P(\max_{k<\sigma_{bd}} W_k \in B) = \nu(B), \quad B \subseteq \{0\} \cup G; \tag{3.34}
\]
here \( (\max_{k<\sigma_{bd}} W_k)(\omega) = (\max_{k<\sigma_{bd}(\omega)} (W_k(\omega)), \ldots, \max_{k<\sigma_{bd}(\omega)} (W_k(\omega))) \).

Also
\[
\nu(\{x \gg z\}) = \lim_{n \to \infty} P(W_n \gg z, \sigma_{bd} > n), \quad z \in \overline{G}; \tag{3.35}
\]
that is, on \( [0, \sigma_{bd}) \), \( W_n \) converges in distribution to \( W(\sigma_{bd} - 1) \).

(iii) \( M = d \max_{k<\sigma_{bd}} W_k \), and hence \( \nu_M = \nu \).

(iv) \( \mu_0 = \alpha_0, \ \hat{U}^+ = d L^+_1 \); that is, \( \mu_+ \) is the \( d \)-dimensional ladder height distribution, in other words
\[
P(L_1^+ \in B) = P(-R^{-1}(cA_1 - X_1) \in B), \quad B \subseteq G. \tag{3.36}
\]

(v) For any \( a \in \overline{G} \),
\[
P(\rho(a) < \infty) = P(\max_{k<\sigma_{bd}} W_k \gg R^{-1}a) = \nu(\{x \gg R^{-1}a\}) = \sum_{k=1}^{\infty} (1-p) \mu_+^{(k)}(\{x \gg R^{-1}a\}). \tag{3.37}
\]

**Proof:** We already have \( 0 < p < 1 \); also \( \mu_+(G^c) = 0 \), by (3.32). Thus \( \mu_+ \) is a defective distribution and \( \mu_0 \) a probability distribution, both concentrated on \( G \). For \( k = 1, 2, \ldots \), clearly \( \mu_0^{(k)} \) is concentrated on \( G \); consequently the geometric compound \( \nu \) is a probability measure concentrated on \( \{0\} \cup G \).

By (3.6) and Proposition 3.2, \( 1 \leq \sigma_{bd} < \infty \) with probability 1. Also by (3.24), \( P(\sigma_{bd} - 1 = 0) = P(\beta - 1 = 0) = 1 - p \).

By Theorem 2.7 and Remark 3.3, we get for \( k = 1, 2, \ldots \)
\[
[W_k : \{k < \sigma_{bd}\}] = d \ [Y_k^{(0)} : \{\Delta Y_k^{(0)} \gg 0\}], \tag{3.38}
\]
where \( [\gamma : A] \) stands for the random variable \( \gamma \) restricted to the set \( A \). As \( [Y_j^{(0)} : \{\Delta Y_j^{(0)} \gg 0\}] \gg Y_{j-1}^{(0)} \forall j, \ {\sigma_{bd} > k} = \cap_{k=1}^{N} \{\sigma_{bd} > \ell\}, \) and
\[ W_j = \sum_{\ell=1}^{j} \hat{U}_\ell, \quad 1 \leq j \leq k \] on \( \{ \sigma_{bd} > k \} \), by (3.38)

\[
\mathbb{P}(\sigma_{bd} > k) = \mathbb{P}(W_k \gg W_{k-1} \gg \cdots \gg W_1 \gg 0)
= \mathbb{P}(\hat{U}_j \gg 0, \ 1 \leq j \leq k) = p^k.
\tag{3.39}
\]

Thus \( \sigma_{bd} - 1 \) has a geometric distribution with parameter \( (1 - p) \).

Clearly \( W(\sigma_{bd} - 1) = W(0) = 0 \) if \( \sigma_{bd} - 1 = 0 \). By the arguments given in the derivation of (3.39 ), and (3.31 ) it follows that on \( \{ \sigma_{bd} - 1 = k \} \),

\[
W(\sigma_{bd} - 1) = d_{\max} k W_k
= d_{\sum_{j=1}^k \hat{U}_j^+}
= d_{\max_{j<\sigma_{bd}} W_j}),
\tag{3.40}
\]

for \( k = 1, 2, \ldots \)

From (3.39 ),(3.40 ), for any Borel set \( B \subseteq (\{0\} \cup G) \), we get

\[
\mathbb{P}(\max_{j<\sigma_{bd}} W_j) \in B) = \mathbb{P}(W(\sigma_{bd} - 1) \in B)
= \sum_{k=0}^{\infty} \mathbb{P}(W(\sigma_{bd} - 1) \in B \mid \sigma_{bd} - 1 = k)\mathbb{P}(\sigma_{bd} - 1 = k)
= (1 - p)\delta_0(B) + \sum_{k=1}^{\infty} (1 - p)p^k \mathbb{P}(W_k \in B \mid \sigma_{bd} - 1 = k)
= (1 - p)\delta_0(B) + \sum_{k=1}^{\infty} (1 - p)p^k \mu_0^*(k) (B)
= \nu(B)
\tag{3.41}
\]

Note that (3.35 ) is clear from the above arguments.

To prove \( M = d_{\max_{k<\sigma_{bd}} W_k} \), it is enough to consider the case \( \sigma_{bd} > 1 \), equivalently \( \beta > 1 \). Now by Theorem 2.7 and Remark 3.3, we have

\[
[W_k : \{ k < \sigma_{bd} \}] = d_{[Y_k^{(0)} : \{ \Delta Y_k^{(0)} \gg 0 \}] \leq M_k,
\]

where \([\gamma : A]\) stands for the random variable \( \gamma \) restricted to the set \( A \). Hence by (3.29 ),

\[
\mathbb{P}(\max_{k<\sigma_{bd}} W_k) \gg x) \leq \mathbb{P}(M \gg x), \ x \in \overline{G}.
\]

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Next, since $\beta < \infty$ with probability 1, for a.e. $\omega$ we have $M(\omega) = Y_k^{(0)}(\omega)$ with $\Delta Y_k^{(0)}(\omega) \gg 0$, for some $k$ that may depend on $\omega$. So again by Theorem 2.7 and Remark 3.3 as above, $M = dW_k$ with $k < \sigma_{bd}$, when $M = Y_k^{(0)}$. Hence it follows that

$$\mathbb{P}(M \gg x) \leq \mathbb{P}(\max_{k<\sigma_{bd}} W_k \gg x), \quad x \in \overline{G}.$$ 

Thus $M = d \max_{k<\sigma_{bd}} W_k$.

Therefore Theorem 3.7 and (3.41) imply $\nu = \nu_M$. As $\beta = d \sigma_{bd}$, in view of the expressions (3.22),(3.33), an elementary argument using characteristic functions gives $\alpha_0 = \mu_0, \alpha_+ = \mu_+, L_1^+ = d \hat{U}_1^+$.

Finally (3.37) in assertion (v) follows by (3.23), (3.30), (3.33) and assertion (iv). This completes the proof.

**Remark 3.9** In the classical one-dimensional renewal risk (Sparre Andersen) model, recall that "ruin" is defined as the event that the surplus goes strictly below zero level in finite time. This is the same as $\Delta y_n \gg 0$ for some $n \geq 1$ in our framework in terms of the one-dimensional Skorokhod problem with $G = (0, \infty)$ and normal reflection; so $R = R^{-1} = 1$. Also this is true even at the sample path level, that is, in the deterministic setup. In the classical model, Pollaczek-Khinchine formula for ruin problem is generally expressed as a compound geometric distribution involving the one-dimensional ladder height distribution; for definition of ladder height distribution in the classical model (without any reference to Skorokhod problem), relevant proofs and more information, see Chapter 6 of [27]. In view of Theorems 3.7, 3.8, it is clear that our definition and the classical notion of ladder height distribution coincide in the one-dimensional case. So $\alpha_+$ given by (3.19), or equivalently $\mu_+$ given by (3.32), can be regarded as the $d-$dimensional analogue of ladder height distribution; moreover $\mu_+$ gives an explicit expression for the ladder height distribution in the $d-$dimensional renewal risk set up. In fact, when $d = 1$, and $A_1, X_1$ both have exponential distributions, (that is, in the classical Cramer-Lundberg model with exponential claim sizes,) it is easy to verify that r.h.s. of (3.32) (or r.h.s. of (3.36)) is an appropriate multiple of the integrated tail distribution of claim sizes. For the general one dimensional Cramer-Lundberg model, it is known that the ladder height distribution is the same as the integrated tail of claim sizes; see [27].

**Note:** (3.22) (or equivalently (3.23)), (3.33) (or equivalently (3.37)) may be considered *Pollaczek-Khinchine formula* for multidimensional ruin
problem. Because of (3.32), all the quantities on r.h.s. of (3.33), (3.37) are in terms of the given data of the model.

Following corollary is a version of the duality theorem in one dimension; see [1].

**Corollary 3.10** Let \( d = 1 \); define \( \sigma_0(\omega) = \inf\{k \geq 1 : W_k(\omega) = 0\} \), and \( Y^{(0)}(\omega) = \lim_{n \to \infty} Y^{(0)}_n(\omega) \). Then \( Y^{(0)}(\omega) = M \), \( W_n \) converges in distribution to \( W(\sigma_0 - 1) \), and hence to \( Y^{(0)}_\infty \); also for any \( a \geq 0 \),

\[
\mathbb{P}(\xi^{(a)} < \infty) = \mathbb{P}(W(\sigma_0 - 1) \gg a) = \lim_{n \to \infty} \mathbb{P}(W_n \gg a) = \mathbb{P}(Y^{(0)}_\infty \gg a).
\]

(3.42)

**Proof:** As \( d = 1 \) note that \( \partial G = \{0\} \), and hence by Proposition 3.2, \( \sigma_{bd} = \sigma_0 < \infty \) with probability 1. Also \( Y^{(0)}(\omega) = M \), as \( \Delta Y_j^{(0)} = 0 \) whenever it is not strictly positive. Clearly \( W(\sigma_0) = 0 \). Now put \( \gamma_0 = 0, \gamma_1 = \sigma_0, \) and \( \gamma_n = \inf\{k \geq \gamma_{n-1} + 1 : W_k = 0\}, \) \( n \geq 1 \), denoting successive times of visit to the origin. As \( W(\gamma_j) = 0 \) for all \( j \geq 1 \), by the strong Markov property of \( \{W_\ell\} \), it follows that between successive visits to the origin, the process behaves like independent copies of \( \{W_0, W_1, \ldots, W(\sigma_0 - 1)\} \). Therefore by the proof of Theorem 3.8, we now get \( W_n \) converges in distribution to \( W(\sigma_0 - 1) \); (3.42) is also a consequence of Theorem 3.8.

One-dimensional duality theorem is given in Corollary 3.2 on p.49 of [1], or Theorem 5.1.2 on p.151 of [27]. Note that \( W_k = d \max_{j \leq k} W_j \) if \( k < \sigma_0 \), one-dimensional analogue of a portion of (3.40), is implicit in the proofs given in these references, incidentally, using the one-dimensional Skorokhod reflection map.

**Note:** For \( d \geq 2 \) note that \( M = 0 \) does not imply \( Y^{(0)}_n(\omega) = 0 \) for all \( n \). However, for \( d = 1 \) the implication holds.

**Note:** Let \( d \geq 2 \). Note that \( \mathbb{P}(\sigma_{bd} = \sigma_0) > 0 \); because of the coordinate-wise net profit condition, this probability could even be substantial. It is not clear what conditions ensure \( \mathbb{P}(\sigma_{bd} = \sigma_0) = 1 \). It is also not clear when the process \( \{W_n\} \) has a limiting distribution.

**Remark 3.11** Observe that the hypotheses (H2), (H6) ensure that the various events associated with ruin have positive probability even if claim size vector has only one nonzero component, provided that the nonzero component is sufficiently large, as proved in Proposition 3.5. In fact, these two conditions can be replaced with
(H9) For each $\ell \geq 1$, $\mathbb{P}(R^{-1}X_\ell \gg x) > 0$ for any $x \gg 0$; that is, $R^{-1}X_\ell$ is supported on an unbounded upper orthant.

If claim size vector itself has an unbounded upper orthant as support then (H9) holds as $(R^{-1})_{ii} \geq 1, \forall i$ by (1.1). Note that Theorems 3.6, 3.7, 3.8 continue to hold even with (H2),(H6) replaced by (H9). However, in our exposition we persist with (H2),(H6) as we want to emphasize that Example 3.13 below is covered by our analysis.

**Remark 3.12** [16, 17] consider storage networks driven by fairly general Levy inputs. When the basic driving process is of same kind, it is interesting to note the similarity and the difference between the storage networks considered in these papers and the one considered here. Note that $\mathbb{E}(\hat{U}_1) \ll 0$ by (H8); its analogue is assumed in [17]. While the reflection matrix is $R$ in [17], in our case it is $R^{-1}$ for the storage network. Thus in the storage network considered in [17] both the drift and the reflection vectors are inward looking, while in our storage network the drift vector is inward looking, but the reflection vector on each face of $\partial G$ is outward looking. For the storage network considered in [17] the limiting distribution has an atom at the origin; while it is not clear if our storage network $W_n$ has a limiting distribution in general, note that $W(\sigma_{bd} - 1)$ has an atom at 0 in our set up.

### 3.4 Examples

We give a few classes of examples for which our analysis can be applied.

**Example 3.13** Renewal risk type network: All processes are defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})$. Let $A_\ell, \ell \geq 1$, denote i.i.d. interarrival times corresponding to a renewal counting process $\{N(t) : t \geq 0\}$. Let $p_i, 1 \leq i \leq d$ be positive numbers such that $\sum_{i=1}^d p_i = 1$. Let $J^{(1)}, \ldots, J^{(d)}$ be independent one dimensional random variables taking values in $[0, \infty)$; these are independent of $\{A_\ell\}$. Here $J^{(i)}$ represents claim size distribution for Company $i$. Let $J$ be a $d-$dimensional random variable such that $J = (0, \ldots, 0, J^{(i)}, 0, \ldots, 0)$ with probability $p_i$, for $1 \leq i \leq d$; (here coordinates other than $i-$th are zero). So $J$ takes values on the boundary of the orthant $G$. Clearly $J$ is not absolutely continuous even if $J^{(i)}$ are. Moreover, the marginals $(J)_i, 1 \leq i \leq d$ of $J$ are not independent even though $J^{(i)}$ are. Let $F^{(i)}$ denote the distribution function of $J^{(i)}$, and let $(F)_i$ denote the
$i$—th marginal distribution function of $J$, for $1 \leq i \leq d$. It can be seen that

$$(F)_{i}(u) = [p_{i}F^{(i)}(u) + (1 - p_{i})]I_{[0, \infty)}(u), \ u \in \mathbb{R}.$$ 

So, $(F)_{i} \neq F^{(i)}$, and $(F)_{i}$ has an atom at $u = 0$ even if $F^{(i)}$ is continuous.

Now let $X_{\ell}, \ell \geq 1$, be i.i.d. random variables having the same distribution as $J$; these represent vector claim sizes. If $J^{(i)}$ are continuous, note that (H7) is satisfied. Let

$$H^{(a)}(t) = a + tc + \sum_{\ell=1}^{N(t)} X_{\ell}, \ t \geq 0,$$  

(3.43) denote the joint dynamics of the $d$ insurance companies in the absence of the risk diversifying treaty; here $a$ denotes the initial capital of the companies, while $c$ denotes the vector of premium rates. At an arrival time, it is assumed that an independent mechanism governed by the probability vector $(p_{1}, \ldots, p_{d})$ determines the company $i$ that has to take the claim; interarrival times $\{A_{\ell}\}$, claim sizes having law $J^{(i)}, 1 \leq i \leq d$, and the random mechanism governed by $(p_{1}, \ldots, p_{d})$ are taken to be independent of each other. Thus $H^{(a)}(\cdot)$ is a $d$—dimensional renewal risk process.

An important special case is that of Cramer-Lundberg type network. In the absence of treaty, the joint dynamics is that of $d$ independent Cramer-Lundberg processes $H^{(i)}, 1 \leq i \leq d$, with respective initial capital $a_{i}$, premium rate $(c)_{i}$, claim number process $\{N^{(i)}(t)\}$, which is a Poisson process with rate $\lambda_{i} > 0$, and claim size $J^{(i)}$. In such a case $N(t) = N^{(1)}(t) + \cdots + N^{(d)}(t), t \geq 0$ is also a Poisson process with rate $\lambda = \lambda_{1} + \cdots + \lambda_{d}$; so $A_{\ell}$ are i.i.d. random variables having exponential distribution with parameter $\lambda$. Take $p_{i} = \lambda_{i}/\lambda$ in the above set up. Then it can be seen that $H^{(a)}(t) =^{d} (H^{(1)}(t), \ldots, H^{(d)}(t))$ as processes.

To describe the joint dynamics under risk diversifying treaty, one can use continuous time Skorokhod problem $SP(H^{(a)}(\cdot), R)$; see [23, 25]. Accordingly, we seek $d$—dimensional r.c.l.l. processes $\{Y^{(a)}(t), Z^{(a)}(t), t \geq 0\}$ satisfying the Skorokhod equation

$$Z^{(a)}(t) = H^{(a)}(t) + RY^{(a)}(t), \ t \geq 0,$$  

(3.44) such that $Y^{(a)}(0) = 0, Z^{(a)}(0) = a, Z^{(a)}(t) \in \overline{G}, t \geq 0$, each component of $Y^{(a)}(\cdot)$ is nondecreasing, and $(Y^{(a)})_{i}(\cdot)$ can increase only when $(Z^{(a)})_{i}(\cdot) = 0$, in the sense that

$$(Y^{(a)})_{i}(t) - (Y^{(a)})_{i}(s) = \int_{(s,t]} I_{[0,\{0\}]}((Z^{(a)})_{i}(u))d(Y^{(a)})_{i}(u),$$  

(3.45)
for $0 \leq s < t$. Thanks to (H1), this Skorokhod problem has a unique solution; the solution also has desired optimal property as in the discrete time.

Put $\Delta Y^a(t) = Y^a(t) - Y^a(t-), \ t \geq 0$. We say that ss-ruin occurs for the process $\{Z^a(t)\}$ if $\Delta Y^a(t) \gg 0$ for some $t > 0$; by (3.45), note that ss-ruin means that every company has zero surplus and nonzero marginal deficit at some time $t$. Other two notions of ruin can be similarly defined as in Subsection 3.2. As $c \geq 0$ and $A_t > 0$, note that each coordinate of $H^a(\cdot)$, and hence that of $Z^a(\cdot)$ is strictly increasing between claim arrivals; so $(Y^a)_i$ can change only at a claim arrival time, for any $i$. In particular, only at a claim arrival time ruin can occur. Therefore, as in the one dimensional case, for studying ruin problem, it is sufficient to consider these processes at claim arrival times. For $n = 1, 2, \cdots$ let $T_n = A_1 + A_2 + \cdots + A_n$ denote the arrival times, and let $T_0 = 0$; set $H^a_n = H^a(T_n), n \geq 0$. Then $H^a_n, n \geq 0$ is a random walk in $\mathbb{R}^d$ starting at $a$. Put $Y^a_n = Y^a(T_n), Z^a_n = Z^a(T_n), n = 0, 1, 2, \cdots$ Then $\{Z^a_n\}$ is the associated regulated/ reflected random walk with $\{Y^a_n\}$ as the corresponding pushing process. Note that $\Delta Y^a_n = \Delta Y^a(T_n), n \geq 1$. So, ss-ruin for the process $\{Z^a(t), t > 0\}$ occurs when and only when ss-ruin for the regulated random walk $\{Z^a_n, n \geq 1\}$ occurs. Moreover, the respective probabilities of ss-ruin in finite time also coincide. Similar comments apply also to other notions of ruin. Another consequence of the above discussion is

$$
M = \lim_{n \to \infty} \{Y^0_n: \Delta Y^0_n \gg 0\}
= \lim_{t \to \infty} \{Y^0(t): \Delta Y^0(t) \gg 0\}.
$$

Assume now that (H1)-(H6),(H8) hold. Therefore Theorem 3.8 can be used to conclude that

$$
\mathbb{P}(\ell_{ss}^a < \infty) = \mathbb{P}(M \gg R^{-1}a),
$$

giving the probability of ruin in finite time for the regulated/ reflected process $\{Z^a(t): t \geq 0\}$ corresponding to initial capital vector $a$, in terms of an asymptotic functional of the pushing process $\{Y^0(t): t \geq 0\}$ corresponding to zero initial capital exceeding threshold $R^{-1}a$.

**Example 3.14** 1–dimensional problem revisited: Take $d = 1$. The classical renewal risk process is then given by the one-dimensional analogue $H^a(\cdot)$ of (3.43). In this case the scalar $R = 1$. Consider the one-dimensional Skorokhod problem in the half line $[0, \infty)$ for $H^a(\cdot)$; let $Z^a(\cdot), Y^a(\cdot)$ denote,
respectively, the regulated/ reflected process and the pushing process. If the
claim size distribution is continuous, then it is easily seen that the events
\( \{ H^{(a)}(t) < 0 \text{ for some } t > 0 \} \) and \( \{ \Delta Y^{(a)}(t) > 0 \text{ for some } t > 0 \} \) coincide with probability 1. So the classical notion of ruin probability and all 3 no-
tions of ruin probability discussed here are the same with probability 1. In
particular, by Corollary 3.10 and (3.47 ), it follows that
\[
\mathbb{P}(Y^{(a)}(0) \gg R^{-1}a) = \mathbb{P}(H^{(a)}(t) < 0 \text{ for some } t > 0).
\]

For example, if claim size distribution is Pareto(\( \alpha \)) then \( Y^{(a)}(0) \) has Pareto(\( \alpha - 1 \)) distribution. Thus the rich reservoir of results from ruin theory (see [1, 12, 27] can be used to get information on the asymptotics of the pushing process of
Skorokhod problem.

Example 3.15 Suppose \( \mathbb{P}(X_{\ell} \gg x) > 0, \forall x \in \Omega, \ell = 1, 2, \cdots \) Take \( R = I \),
the identity matrix. Then (H9) given in Remark 3.11 is satisfied; clearly (H1)
holds. If (H3)-(H5),(H7),(H8) also hold, then by Remark 3.11, our analysis is
applicable in this case as well. This situation corresponds to the companies
operating without the risk diversifying treaty, claim arrival times being the
same for all companies, but claim sizes may be dependent; that is, the \( d \)
companies jointly take care of different (possibly dependent) components
of the vector claims, or the companies take care of each claim in non-zero
proportions. Capital injection needed by a company is completely taken care
of by its shareholders. In this set up, ruin of the network is the same as
the \( d \)–dimensional renewal risk process \( H^{(a)}(\cdot) \) given by (3.43 ) operating
without the risk diversifying treaty hitting the negative orthant.

References

[1] S. Asmussen and H. Albrecher: Ruin Probabilities, (Second edition).
World Scientific, Singapore, 2010.

[2] F. Avram, Z. Palmowski and M. Pistorius: Exit problem of a two-
dimensional risk process from the quadrant: exact and asymptotic re-
sults. The Annals of Applied Probability 18 (2008) 2421 – 2449.

[3] N. Bauerle and R. Grubel: Multivariate risk processes with interacting
intensities. Advances in Applied Probability 40 (2008) 578 – 601.
[4] N.H. Bingham: Random Walk and Fluctuation Theory. In Handbook of Statistics, Vol.19 (eds. D.N. Shanbhag and C.R. Rao), Elsevier, Amsterdam, 2001.

[5] B. Blaszczyszyn and K. Sigman: Risk and duality in multidimensions. Stoc. Proc. Appl. 83 (1999) 331-356.

[6] H. Buhlman: Mathematical Methods in Risk Theory. Springer-Verlag, Berlin-Heidelberg, 1970.

[7] W.-S. Chan, H. Yang and L. Zhang: Some results on ruin probabilities in a two-dimensional risk model. Ins. Math. Econ. 32 (2003) 345-358

[8] H. Chen and A. Mandelbaum: Leontief systems, RBV’s and RBM’s. In Proceedings of Imperial College Workshop on Applied Stochastic Processes, (ed. M.H.A. Davis and R.J. Elliott), pp. 1-43. Gordon and Breach, New York, 1991.

[9] J. Collamore: First passage times of general sequences of random vectors: a large deviations approach. Stochastic Processes and Applications 78 (1998) 97-130.

[10] R.W. Cottle, J.S. Pang and R.E. Stone: The Linear Complementarity Problem. Academic Press, New York, 1992.

[11] D.C.M. Dickson and H.R. Waters: Some optimal dividends problem. ASTIN Bulletin 34 (2004) 49-74.

[12] P. Embrechts, C. Kluppelberg and T. Mikosch: Modelling Extremal Events for Insurance and Finance. Springer, Heidelberg, 1997.

[13] W. Feller: An Introduction to Probability Theory and its Applications, Vol. II. Wiley-Eastern, New Delhi, 1969.

[14] J.M. Harrison: Brownian Motion and Stochastic Flow Systems. Wiley, New York, 1985.

[15] J.M. Harrison and M.I. Reiman: Reflected Brownian motion on an orthant. Ann. Probab. 9 (1981) 302-308.

[16] O. Kella: Stability and nonproduct form of stochastic fluid networks with Levy inputs. Ann. Appl. Prob. 6 (1996) 186-199.
[17] O. Kella: Stochastic storage networks: stationarity and the feedforward case. *J. Appl. Prob.* 34 (1997) 498-507.

[18] O. Kella and W. Whitt: Stability and structural properties of stochastic fluid networks. *J. Appl. Prob.* 33 (1996) 1169-1180.

[19] A. Mandelbaum: The dynamic complementarity problem. *Lecture Notes*, Technion, Oct. 1989.

[20] N.U. Prabhu: On the ruin problem of collective risk theory. *Ann. Math. Statis.* 32 (1961) 757-764.

[21] K. Ramanan: Reflected diffusions defined via the extended Skorokhod map. *Elec. J. Probab.* 11 (2006) 934-992.

[22] S. Ramasubramanian: A subsidy-surplus model and the Skorokhod problem in an orthant. *Math. Oper. Res.* 25 (2000) 509-538.

[23] S. Ramasubramanian: An insurance network: Nash equilibrium. *Ins. Math. Econ.* 38 (2006) 374-390.

[24] S. Ramasubramanian: Multidimensional insurance model with risk-reducing treaty. *Stoc. Models* 27 (2011) 363-387.

[25] S. Ramasubramanian: A multidimensional ruin problem. *Comm. Stoc. Anal.* 6 (2012) 33-47.

[26] M.I. Reiman: Open queueing networks in heavy traffic. *Math. Oper. Res.* 9 (1984) 441-458.

[27] T. Rolski, H. Schmidli, V. Schmidt and J.L. Teugels: *Stochastic Processes for Insurance and Finance*. Wiley, Chichester, 1999.

[28] E. Seneta: *Non-negative matrices and Markov chains*, (Second edition). Springer-Verlag, New York, 1981.

[29] D. Siegmund: The equivalence of absorbing and reflecting barrier problems for stochastically monotone Markov processes. *Ann. Probab.* 4 (1976) 914-924.

[30] F. Spitzer: A combinatorial lemma and its application to probability theory. *Trans. Amer. Math. Soc.* 82 (1956) 323-339.