An Example of the Curvature Tensor for a Quantum Space

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Abstract

The paper is constructed in two parts. In the first part we introduce the concept of the algebra $\mathbb{R}^2_Q$ of $Q$-meromorphic functions on the quantum plane. The $A_1(q)$-algebra of $Q$-analytic functions considered in [6] is seen as a proper subalgebra.

In the second part we find a formula for the curvature tensor on $\mathbb{R}^2_Q$. It is seen that when the quantization parameter tends to 1, then this formula gives the flatness of the usual $\mathbb{R}^2$.

1 Introduction

Non-commutative geometry and quantum groups are applied to problems of physics in different ways. Classical and quantum mechanics on the Manin quantum plane have been studied in [1,4,5].

A non-commutative framework for calculus and differential geometry from point of view of discrete calculus has been introduced by Kauffman [2]. In his work he modeled the notions of derivations and derivatives with respect to different parameters by commutators.

Our main objection is to transfer classical mechanics on a Poisson algebra to its functional quantization in the sense of [6].

More precisely we want to define an analogue of the poisson bracket on $\mathbb{R}^2_Q$ and develop an appropriate classical mechanics on $\mathbb{R}^2_Q$ parallel to that on $\mathbb{R}^2$. 
On the first step, following the new interpretation of the Manin quantum plane in [6] and by a completion of [7], we provide the necessary tools which enable us to introduce the notions of covariant derivative and curvature tensor on $\mathbb{R}^2_Q$.

In order that we can apply Kauffman modeling of derivations we have to deal with a sufficiently large class of functions. This is done in part two of this paper, where the formula for the curvature tensor on $\mathbb{R}^2_Q$ is obtained showing that when the quantization parameter goes to 1, this gives us the flatness of $\mathbb{R}^2$. Also a generalized formula for the Poisson bracket of two elements in $\mathbb{R}^2_Q$ is given and properties are studied.

2 The algebra of the Q-Meromorphic Functions

Let $D=\{q \in \mathbb{C} : |q| \leq 1\}$ be the unit disk in $\mathbb{C}$. Recalling from [6], $A_1(q)$ will be the $\mathbb{C}$-algebra of all absolutely convergent power series $\sum_{i=0}^{\infty} a_i q^i$ in $D$ with values in $\mathbb{C}$. Also we denote by $A_0(q)$ the $\mathbb{C}$-algebra of all absolutely convergent power series $\sum_{i=0}^{\infty} c_i q^i$ on $D - \{0\}$ with values in $\mathbb{C}$. We can generalize the concept of $Q$-analytic functions on the 2-intervals of $\mathbb{R}^2$ with values in $A_1(q)$ to the algebra of $Q$-analytic functions on the 2-interval of $\mathbb{R}^2$ with values in $A_0(q)$ without any difficulty. Assume now that $\Omega = \mathbb{R} - \{0\} \times \mathbb{R} - \{0\}$ and let

$$f = \sum_{i,j,k>_{-\infty}} a_{ijk} q^i t_1^j t_2^k$$  \hspace{1cm} (2-1)

be an absolutely convergent power series on $D - \{0\} \times \Omega$ with values in $\mathbb{C}$. (The sign $>$ under the second $\sum$ indicates $j, k$ are bounded below). Clearly we can consider $f$ as a function from $\Omega$ into $A_0(q)$ admitting the absolutely convergent Laurent expansion

$$f = \sum_{i,j,k>_{-\infty}} a_{ij}(q) t_1^i t_2^j$$  \hspace{1cm} (2-2)

on $\Omega$. Since the above series is absolutely convergent on $\Omega$, we can also write it as

$$f = \sum_{i,j=0}^{\infty} t_1^{-i} \alpha_{ij}(t_1, t_2) t_2^{-j}$$  \hspace{1cm} (2-3)

where the $\alpha_{ij}$'s are absolutely convergent power series on $\mathbb{R}^2$ with values in $A_0(q)$ and the sing - over the $\sum$ means that the indices are bounded above.

**Definition 2-1:** With the above notations and conventions let

$$\hat{f} = \sum_{i,j=0}^{\infty} x^{-i} \hat{\alpha}_{ij}(x, p) p^{-j}$$  \hspace{1cm} (2-4)

be obtained from $f$ by the correspondence
\[ t^i_1 t^j_2 = t^j_2 t^i_1 \rightarrow x^i p^j. \]

We call \( \hat{f} \) a Q-meromorphic function on \( \Omega \) with values in \( A_0(q) \) or simply a Q-meromorphic function on \( \Omega \).

The two functions \( \frac{1}{x} \) and \( \frac{1}{p} \) are Q-meromorphic functions on \( \Omega \) satisfying the following commutation relations

\[
\begin{align*}
\frac{1}{x} &= \frac{1}{x} = 1, \quad \frac{1}{p} = \frac{1}{p}p = 1 \\
\frac{1}{p} &= q^{-1} \frac{1}{x}p, \quad \frac{1}{x} = q^{\frac{1}{2}}x \\
\frac{1}{p} &= q^{\frac{1}{2}} \frac{1}{p}, \quad \left( \frac{1}{x} \right)^i = \frac{1}{x}, \quad \left( \frac{1}{p} \right)^j = \frac{1}{p}.
\end{align*}
\]

By using these commutation relations we always follow the order \((x^ip^j)_{i,j,i,j,i,j} \rightarrow -\infty \) in writing the Q-meromorphic functions as above.

**Remark 2-1:** If \( \hat{f}(x,p) \) is a Q-analytic on \( \Omega \) with values in \( A_0(q) \), then for \( k,l \in \mathbb{Z} \), \( \hat{f}(q^k x, q^l p) \) is also a Q-analytic function on \( \Omega \) with values in \( A_0(q) \).

**Definition 2-2:** The product of two Q-meromorphic functions

\[
\hat{f} = \sum_{i_1, i_2 = 0}^{\infty} x^{-i_1} \hat{a}_{i_1 i_2}(x, p) p^{-i_2}
\]

\[
\hat{g} = \sum_{j_1, j_2 = 0}^{\infty} x^{-j_1} \hat{b}_{j_1 j_2}(x, p) p^{-j_2}
\]

on \( \Omega \) will be defined by

\[
\hat{f} \cdot \hat{g} = \sum_{i_1, i_2 = 0}^{\infty} \sum_{j_1, j_2 = 0}^{\infty} q^{i_2 j_1} x^{-i_1 - j_1} (\hat{a}_{i_1 i_2}(x, q^{-j_1} p) \cdot \hat{b}_{j_1 j_2}(q^{-i_2} x, p)) p^{-i_2 - j_2}
\]

where the above product between \( \hat{a}_{i_1 i_2} \) and \( \hat{b}_{j_1 j_2} \) is the product of two Q-analytic functions with values in \( A_0(q) \) in the sense of [6].

**Lemma 2-1:** With the above notations the product of two Q-meromorphic functions \( \hat{f} \) and \( \hat{g} \) on is Q-meromorphic function on \( \Omega \)

**proof.** The proof is easily seen from the fact that \( \hat{a}_{i_1 i_2}(x, q^{-j_1} p) \cdot \hat{b}_{j_1 j_2}(q^{-i_2} x, p) \) is a Q-analytic function on the quantum plane with values in \( A_0(q) \).

From the above lemma we can see that the set of all Q-meromorphic functions on \( \Omega \) with values in \( A_0(q) \) is a non-commutative, associative, unital \( A_0(q) \)-algebra. This algebra which we denote hereafter by \( \mathbb{R}_Q^2 \), contains \( A_Q \), the \( A_1(q) \)-algebra of Q-analytic functions on the
quantum plane with values in $A_1(q)$, as its subalgebra. It is clear that $\mathbb{R}_Q^2$ is the (1, D-0, $A_0(q)$) functional quantization of $M$: the $\mathbb{C}$-algebra of all absolutely convergent power series \[ \sum_{i,j>\infty} a_{ij} t_i t_j \] on $\Omega$ with values in $\mathbb{C}$ in the sense of [6], and if we denote by $A$ the $\mathbb{C}$-algebra of all entire functions of the form \[ \sum_{i,j=0} a_{ij} t_i t_j^2 \] on $\mathbb{R}^2$ with values in $\mathbb{C}$, then $i_A \Phi_A = \Phi_M i_{AQ}$

Where $\Phi_A$ and $\Phi_M$ are the quantization maps defined in [6], $A_Q \longrightarrow \mathbb{R}_Q^2$ and $A \longrightarrow M$ are the canonical injections and $i_A : A \rightarrow M$ and $i_{AQ} : A_Q \rightarrow \mathbb{R}_Q^2$ are the inclusions.

3 Derivative and the curvature tensor

Following Wess and Zumino [3,8] we can generalize the differential calculus by defining differential operators $\partial_x$ and $\partial_y$ as follows:

\[ \frac{\partial}{\partial x} x = 1 + q^2 x \frac{\partial}{\partial x} + (q^i - 1) y \frac{\partial}{\partial y} \]

\[ \frac{\partial}{\partial x} y = q y \frac{\partial}{\partial x} \]

\[ \frac{\partial}{\partial y} x = q^{-1} x \frac{\partial}{\partial y} \]

\[ \frac{\partial}{\partial y} y = 1 + q^2 y \frac{\partial}{\partial y} \]

from above relations, it’s easy to see that:

\[ \frac{\partial}{\partial y} (y^n x^m) = y^{n-1} x^m \frac{1 - q^{2n}}{1 - q} \]

**Definition 3-1:** For a fixed element $H \in \mathbb{R}_Q^2$, the derivative of an arbitrary element $f \in \mathbb{R}_Q^2$ with respect to the time parameter $t$ is defined by $\frac{df}{dt} := [f, H]$. From this definition it is seen that $H$ is independent of $t$.

(for a mechanical system, $H$ can be considered as the Hamiltonian function of the system.)

In classical differential geometry the Levi-civita connection on $\mathbb{R}^2$ gives us the following covariant derivatives on functions on $\mathbb{R}^2$:

\[ \nabla_x f := \frac{\partial f}{\partial x} = \partial_x f \]

\[ \nabla_y f := \frac{\partial f}{\partial y} = \partial_y f \]
**Definition 3-2:** for each \( f \in \mathbb{R}_Q^2 \) we define the following covariant derivatives with respect to \( x \) and \( y \) by:

\[
\nabla_x f := [f, H] \frac{1}{[x,H]}
\]

\[
\nabla_y f := [f, H] \frac{1}{[y,H]}
\]

\[
\nabla_{xy} f := [f, H] \frac{1}{[xy,H]}
\]

**Proposition 3-1:** the following properties of the covariant derivative is obtained:

a) \( \nabla_{\lambda x} f = \frac{1}{\lambda} \nabla_x f \)

b) \( \nabla_{[x,y]} f = (1 - q)^{-1} \nabla_{xy} f \)

c) \( \frac{1}{\nabla_{xy}} = \frac{x}{\nabla_y} + \frac{y}{\nabla_x} \)

**proof:**

a) It’s obvious by using definition 3-2.

b) Now, we compute \( \nabla_{[x,y]} f \) as follow:

\[
\nabla_{[x,y]} f = \nabla_{xy} - yx f = \nabla_{xy} - qyx f = \nabla_{xy(1-q)} f
\]

\[
= (1 - q)^{-1} \nabla_{xy} f
\]

c) \( \nabla_{xy} f = [f, H] \frac{1}{x[y,H] + [x,H]y} \)

\[
\Rightarrow \frac{1}{\nabla_{xy}} = (x[y,H] + [x,H]y) \frac{1}{[H,H]}
\]

\[
x \frac{1}{\nabla_y} + \frac{1}{\nabla_x} * y
\]

And by abuse of notation we can write it as

\[
\frac{1}{\nabla_{xy}} = \frac{x}{\nabla_y} + \frac{y}{\nabla_x}
\]

**Definition 3-3:** The curvature tensor of \( \mathbb{R}_Q^2 \) is defined as follows:
$R(x, y) : \mathbb{R}^2_Q \rightarrow \mathbb{R}$

$R(x, y)f = [\nabla_x, \nabla_y]f - \nabla_{[x,y]}f$ \hspace{1cm} (3-11)

Now, we compute both $[\nabla_x, \nabla_y]f$ and $\nabla_{[x,y]}f$ separately:

$[\nabla_x, \nabla_y] = [\partial_x, \partial_y] = \partial_x \partial_y - \partial_y \partial_x$

$= \partial_x \partial_y - q \partial_y \partial_x$

$= (1 - q) \partial_x \partial_y$ \hspace{1cm} (3-12)

Now, let $f \in \mathbb{R}^2_Q$ and $f = \sum_{i,j} a_{ij}(q)x^i y^j$ then

$[\partial_x, \partial_y]f = (1 - q) \partial_x \partial_y(\sum_{i,j} a_{ij}(q)x^i y^j)$

$(1 - q) \sum_{i,j} a_{ij}(q) \partial_x \partial_y(x^i y^j)$ \hspace{1cm} (3-13)

$[\partial_x, \partial_y]f = \frac{1}{(1+q)^2(1-q)} \sum_{i,j} q^{i+j+1}(1 - q^{2j})(1 - q^{2i})a_{ij}(q)y^{i-1}x^{j-1}$ \hspace{1cm} (3-14)

In this way we obtain the following formula for the curvature of $\mathbb{R}^2_Q$

$R(x, y)f = \frac{1}{(1+q)^2(1-q)} \sum_{i,j} q^{i+j+1}(1 - q^{2j})(1 - q^{2i})a_{ij}(q)y^{i-1}x^{j-1} - (1 - q)^{-1}(\frac{x}{\nabla y} + \frac{y}{\nabla x})$

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