WILLMORE TYPE INEQUALITY USING MONOTONICITY FORMULAS

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Abstract. Simon type monotonicity formulas for the Willmore functional \( \int |H|^2 \) in the hyperbolic space \( \mathbb{H}^n \) and \( \mathbb{S}^n \) are obtained. The formula gives a lower bound of \( \int_\Sigma |H|^2 \) where \( \Sigma \) is any closed surface in \( \mathbb{H}^n \).

1. Introduction

We study the Willmore functional
\[
\int_\Sigma |H|^2
\]
for a 2-surface \( \Sigma \) in the standard hyperbolic \( n \)-space \( \mathbb{H}^n \) of constant sectional curvature -1 and standard \( n \)-sphere \( \mathbb{S}^n \). Here \( H \) is the mean curvature vector of \( \Sigma \) in the respective ambients. Willmore established the following classic result in the Euclidean space \( \mathbb{R}^3 \).

Theorem 1. (See for example [Wil71]) Given any closed smooth 2-surface \( \Sigma \subset \mathbb{R}^3 \),
\[
\int_\Sigma |H|^2 \geq 16\pi
\]
with equality occurring if and only if \( \Sigma \) is a standard sphere of any radius.

There are many proofs of the inequality (1). For instance, one can invoke classical differential geometry methods (see for example [Küh05, Theorem 4.46]), and Simon [Sim93] (cf. Gilbarg and Trudinger [GT83, Eq. (16.31)]) obtained the inequality using a monotonicity formula.

By a stereographic projection from \( \mathbb{S}^3 \setminus \{0,0,0,1\} \subset \mathbb{R}^4 \) to \( \mathbb{R}^3 \times \{0\} \), one obtains an analog result for \( \mathbb{S}^3 \) from the result of the Euclidean 3-space.

Theorem 2. (See Introduction in [MNT14]) Given any closed smooth surface \( \Sigma \subset \mathbb{S}^3 \),
\[
\frac{1}{4} \int_\Sigma |H|^2 \geq 4\pi - |\Sigma|
\]
with equality if and only if \( \Sigma \) is a geodesic sphere in \( \mathbb{S}^3 \).

Chen [Che74] observed the conformal invariance properties of the Willmore functional, so his proof worked well for both the hyperbolic \( n \)-space and the \( n \)-sphere.
Theorem 3. (See [Che74]) Give any closed smooth 2-surface $\Sigma^2 \subset \mathbb{H}^n$, 

$$ \frac{1}{4} \int_{\Sigma} |H|^2 \geq 4\pi + |\Sigma| $$

with equality occurring if and only if $\Sigma$ lies in a $\mathbb{H}^3$ subspace as a geodesic sphere.

We are concerned with the hyperbolic case mainly, and with the notations not clarified now we have the following result.

Theorem 4. Give any closed 2-surface $\Sigma^2 \subset \mathbb{H}^n$, if $o \in \Sigma$ is a point of multiplicity $k \geq 1$,

$$ |\Sigma| + 4k\pi = - \int_{\Sigma} \left| \frac{1}{w} X^\perp + \frac{1}{2} H \right|^2 + \frac{1}{4} \int_{\Sigma} |H|^2. $$

It easily implies Chen’s result [Che74] as a corollary. Also, based on a similar idea, we prove for a surface $\Sigma^2 \subset \mathbb{S}^n$ the following theorem.

Theorem 5. Given any closed $\Sigma^2 \subset \mathbb{S}^n$ and $0 < \sigma < \rho < \pi$, if $o \in \Sigma$ is a point of multiplicity $k \geq 1$, then

$$ -2 \frac{1}{w(\rho)} \int_{\Sigma_\rho} \cos r + 4k\pi - |\Sigma_\rho| $$

$$ = \frac{1}{w(\rho)} \int_{\Sigma_\rho} X^\perp \cdot H - \int_{\Sigma_\rho} \left| \frac{1}{w} X^\perp + \frac{1}{2} H \right|^2 + \frac{1}{4} \int_{\Sigma_\rho} |H|^2. $$

Since our proofs are based on monotonicity formulas, one can follow the same philosophy in [HS74] and generalize Theorem 4 and 5 to general Riemannian manifolds with upper sectional curvature bounds. However, for the purpose of a clear exposition, we deal only with the two special cases $\mathbb{H}^n$ and $\mathbb{S}^n$.

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2. Preliminaries

First, we recall a basic comparison theorem of sectional curvature.

Lemma 1. (See Theorem 27 of [Pet98, Chapter 5]) Assume that $(M^n, g)$ satisfies $\sec \leq K$, the metric $g$ written in geodesic polar coordinates centered at $x \in M$ is $dr^2 + g_r$, then

$$ \text{Hess}_M r \geq \frac{\sn_K'(r)}{\sn_K(r)} g_r. $$
Here, \( sn_K \) is defined to be
\[
    sn_K(r) = \frac{1}{\sqrt{K}} \sin(\sqrt{K}r) \quad \text{if } K > 0;
\]
\[
    sn_K(r) = r \quad \text{if } K = 0;
\]
\[
    sn_K(r) = \frac{1}{\sqrt{-K}} \sinh(\sqrt{-K}r) \quad \text{if } K < 0.
\]

If \( K > 0 \), the estimate on \( \text{Hess}_{\mathcal{M}} r \) is only valid with \( r < \frac{\pi}{\sqrt{K}} \).

Suppose that the Levi-Civita connection on \((\mathcal{M}, g)\) is \( \nabla \), then \( r(y) = \text{dist}_{\mathcal{M}}(x, y) \) gives a vector field \( \nabla r \). We define \( X \) to be the following,
\[
    X = sn_K(r) \nabla r.
\]

Given any 2-surface \( \Sigma \) in \((\mathcal{M}, g)\), we are concerned with an estimate of the quantity \( \text{div}_\Sigma X \).

**Lemma 2.** Given any \( x \in \Sigma^2 \subset M^n \) with \( \sec \leq K \),
\[
    \text{div}_\Sigma X \geq 2 sn'_K(r).
\]

**Proof.** Let \( \{e_i\} \), \( i = 1, 2 \) be a chosen orthonormal frame spanning \( T\Sigma \), we use the convention of summation over repeated indices, and apply Theorem 1,
\[
    \text{div}_\Sigma X = \langle \nabla_{e_i} X, e_i \rangle
    = \langle \nabla_{e_i} (sn_K(r) \nabla r), e_i \rangle
    = \langle sn_K(r) \langle \nabla_{e_i} \nabla r, e_i \rangle + sn'_K(r) |\nabla e_i r|^2 \rangle
    = \langle sn_K(r) \text{Hess}_{\mathcal{M}} r(e_i, e_i) + sn'_K(r) |\nabla e_i r|^2 \rangle
    \geq sn'_K(r) \langle g_{\mathcal{M}}(e_i, e_i) + |\nabla e_i r|^2 \rangle
    = sn'_K(r) \langle g_{\mathcal{M}}(e_i, e_i) + |\nabla e_i r|^2 \rangle
    = 2 sn'_K(r).
\]
Hence the proof is concluded. \( \square \)

Although one can obtain the same estimate by Jacobi fields or exponential maps similar to that of [HS74], using Lemma 1 is much more direct and convenient in our settings.

### 3. Willmore type inequalities

Our main result Theorem 4 is a finer version of the Willmore type inequality [2]. In its proof we established a monotonicity formula [5].

Let \( \phi(r)^{-1} = w(r) = \int_0^r \sinh td t \). Note that \( w(r) \) is a constant multiple of the volume of a \( \mathbb{H}^2 \)-geodesic ball. We see that \( \phi'(r) = -\sinh r/w^2 \).
Proof. Let $\nabla$ and $\nabla^\Sigma$ be respectively the connections on $\mathbb{H}^n$ and $\Sigma$. Given any number $\sigma > 0$, we define a cutoff version of $\phi$ as $\phi_\sigma(r) = \phi(\max\{\sigma, r\})$.

Let $r(x) = \text{dist}_{\mathbb{H}^n}(o, x)$ be the geodesic distance from $o$ to any point $x \in \mathbb{H}^n$ and $\Sigma_\rho = \{x \in \Sigma : r(x) < \rho\}$. We choose the Lipschitz vector field $Y(x) = (\phi_\sigma(r) - \phi(\rho))^+ X$ where $0 < \sigma < \rho < \infty$. Let $V(x) = \cosh r(x) = \text{sn}_1(r)$, we see that $\nabla V = X$. $V$ is called a static potential in general relativity literatures, see Chrusciel and Herzlich [CH03]. We often write $\phi$ instead of $\phi(r)$ and similarly for other quantities.

We make use of the first variation formula,

\begin{equation}
\int_{\Sigma} \text{div}_\Sigma Y = - \int_{\Sigma} \langle Y, H \rangle,
\end{equation}

where $H$ is the mean curvature vector.

We calculate $\text{div}_\Sigma Y$ first. By Lemma 2,

\begin{equation}
\text{div}_\Sigma Y = 2(\phi_\sigma - \phi(\rho)) V + \sinh r |\nabla^\Sigma r|^2 ((\phi_\sigma - \phi(\rho))^+)'.
\end{equation}

Note that $|\nabla^\perp r|$, the length of $\nabla r$ along normal direction of $\Sigma$ is $|\nabla^\perp r|^2 = 1 - |\nabla r|^2$. Integrating $\text{div}_\Sigma Y$ over $\Sigma$ then gives

\begin{equation}
\int_{\Sigma} \text{div}_\Sigma Y = -2\phi(\rho) \int_{\Sigma_\rho} V + 2\phi(\sigma) \int_{\Sigma_\sigma} V + 2 \int_{\Sigma_{\rho, \sigma}} \phi V \\
+ \int_{\Sigma_\mu \backslash \Sigma_\sigma} \phi'(r) \sinh r (1 - |\nabla^\perp r|^2) \\
= -2\phi(\rho) \int_{\Sigma_\rho} V + 2\phi(\sigma) \int_{\Sigma_\sigma} V \\
+ |\Sigma_\rho \backslash \Sigma_\sigma| - \int_{\Sigma_\rho \backslash \Sigma_\sigma} \phi'(r) \sinh r |\nabla^\perp r|^2,
\end{equation}

where we have used a consequence of a simple calculus,

\begin{equation}
2\phi V + \phi' \sinh r = 1.
\end{equation}

Since $H$ is a vector normal to $\Sigma$,

\begin{equation}
- \int_{\Sigma} \langle Y, H \rangle = - \int_{\Sigma_\rho} (\phi_\sigma(r) - \phi(\rho))^+ \sinh r \nabla r \cdot H \\
= \phi(\rho) \int_{\Sigma_\rho} \sinh r |\nabla^\perp r| \cdot H - \phi(\sigma) \int_{\Sigma_\sigma} \sinh r |\nabla^\perp r| \cdot H \\
- \int_{\Sigma_\rho \backslash \Sigma_\sigma} \phi \sinh r |\nabla^\perp r| \cdot H.
\end{equation}
We then have
\[- \phi \sinh r \nabla^\perp r \cdot H + \phi' \sinh r |\nabla^\perp r|^2 \]
\[= - \frac{1}{w} X^\perp \cdot H - \frac{|X^\perp|^2}{w^2} \]
\[= - \left( \frac{1}{w} X^\perp + \frac{1}{2} H \right)^2 + \frac{1}{4} |H|^2, \]
(9)
where \(X^\perp = \sinh r \nabla^\perp r\) is the normal component to \(\Sigma\) of the vector field \(X\).

To collect (5), (8) and (9), one has
\[-2 \phi(\rho) \int_{\Sigma_\rho} V + 2 \phi(\sigma) \int_{\Sigma_\sigma} V + |\Sigma_\rho \setminus \Sigma_\sigma| \]
\[= \phi(\rho) \int_{\Sigma_\rho} \sinh r \nabla^\perp r \cdot H - \phi(\sigma) \int_{\Sigma_\sigma} \sinh r \nabla^\perp r \cdot H \]
\[+ \int_{\Sigma_\rho \setminus \Sigma_\sigma} \left| \frac{1}{w} X^\perp + \frac{1}{2} H \right|^2 + \frac{1}{4} \int_{\Sigma_\rho \setminus \Sigma_\sigma} |H|^2. \]
(10)

Then (10) is the monotonicity formula we desired. Since \(\Sigma\) is closed, letting \(\rho \to +\infty\) and \(\sigma \to 0\), the above greatly simplifies as
\[|\Sigma| + 2 \lim_{\sigma \to 0} \phi(\sigma) \int_{\Sigma_\sigma} V = - \int_{\Sigma} \left| \frac{1}{w} X^\perp + \frac{1}{2} H \right|^2 + \frac{1}{4} \int_{\Sigma} |H|^2. \]

Since \(\Sigma_\sigma\) is locally Euclidean with multiplicity \(k\) at the point \(o \in \Sigma\) and \(V(0) = 1\), the fact that the limit
\[\lim_{\sigma \to 0} \phi(\sigma) \int_{\Sigma_\sigma} V = \lim_{\sigma \to 0} \frac{\pi \sigma^2}{\int_0^\infty \sinh t dt} \cdot \left( \frac{1}{\pi \sigma^2} \int_{\Sigma_\sigma} V \right) \]
\[= 2\pi \lim_{\sigma \to 0} \frac{|\Sigma_\sigma|}{\pi \sigma^2} = 2\pi \]
exists will give finally (3). \(\square\)

If \(\Sigma\) has multiplicity greater than 1 somewhere, (2) is similar to that of Li and Yau [LY82] when a point in \(\Sigma\) is covered multiple times by the immersion. In fact, we have the following interesting result analogous to [LY82 Theorem 6],

**Corollary 1.** Give any closed 2-surface \(\Sigma^2 \subset \mathbb{H}^n\),
\[\frac{1}{4} \int_{\Sigma} |H|^2 < |\Sigma| + 8\pi, \]
(12)
then \(\Sigma\) is embedded in \(\mathbb{H}^n\).

**Proof.** If \(\Sigma\) is of multiplicity at least two somewhere \(x \in \Sigma\), then by (11),
\[\frac{1}{4} \int_{\Sigma} |H|^2 \geq |\Sigma| + 8\pi, \]
but this contradicts with (12). Hence, \(\Sigma\) has to be embedded. \(\square\)
When $\Sigma^2 \subset \mathbb{H}^n$ is a surface with boundary, one should apply the first variation formula with boundary,
\[
\int_{\Sigma} \text{div}_\Sigma Y = -\int_{\Sigma} \langle Y, H \rangle + \int_{\partial \Sigma} \langle Y, \eta \rangle,
\]
where $\eta$ is the outward pointing normal of $\partial \Sigma$ to $\Sigma$. We prove similarly,

**Corollary 2.** Suppose that $\Sigma$ is a 2-surface with nonempty boundary, if $o \in \Sigma$ is an interior point of $\Sigma$, then
\[
|\Sigma| + 4\pi \leq \int_{\partial \Sigma} \left( \frac{1}{w} X, \eta \right) - \int_{\Sigma} \left| \frac{1}{w} X^\perp + \frac{1}{2} H \right|^2 + \frac{1}{4} \int_{\Sigma} |H|^2;
\]
and if $o \in \partial \Sigma$ is a boundary point of $\Sigma$, then
\[
|\Sigma| + 2\pi \leq \int_{\partial \Sigma} \left( \frac{1}{w} X, \eta \right) - \int_{\Sigma} \left| \frac{1}{w} X^\perp + \frac{1}{2} H \right|^2 + \frac{1}{4} \int_{\Sigma} |H|^2.
\]

We now give an estimate of $\int \frac{1}{4} |H|^2$ in the same fashion as [GT83, Eq. (16.31)] and arrive the following.

**Theorem 6.** Given any closed 2-surface in $\mathbb{H}^n$ and a real number $\rho > 0$, then
\[
4\pi + |\Sigma_{\rho}| \leq \frac{1}{w(\rho)} \int_{\Sigma_{\rho}} V + \frac{1}{4} \int_{\Sigma} |H|^2.
\]

*Proof.* The proof goes as before with some modifications. From (4), (7), (5) and (6), and also using the shorthand $X = \sinh r \nabla r$ we have
\[
-2\phi(\rho) \int_{\Sigma_{\rho}} V + 2\phi(\sigma) \int_{\Sigma_{\sigma}} V + |\Sigma_{\rho} \setminus \Sigma_{\sigma}|
\]
\[
= -\int_{\Sigma_{\rho}} \left( \phi_{\sigma} - \phi(\rho) \right)_+ X \cdot H - \int_{\Sigma_{\rho} \setminus \Sigma_{\sigma}} \left| X^\perp \right|^2 \frac{1}{w^2}.
\]

For any $x \in \Sigma_{\rho} \setminus \Sigma_{\sigma}$, we claim the following
\[
-(\phi_{\sigma} - \phi(\rho))_+ X \cdot H - \frac{|X^\perp|^2}{w^2} \leq \frac{1}{4} |H|^2.
\]
Then from (14) and (15), we have
\[
-2\phi(\rho) \int_{\Sigma_{\rho}} V + 2\phi(\sigma) \int_{\Sigma_{\sigma}} V + |\Sigma_{\rho} \setminus \Sigma_{\sigma}|
\]
\[
= -\int_{\Sigma_{\sigma}} \left( \phi_{\sigma} - \phi(\rho) \right)_+ X \cdot H + \frac{1}{4} \int_{\Sigma_{\rho} \setminus \Sigma_{\sigma}} |H|^2
\]
\[
\leq -\int_{\Sigma_{\sigma}} \left( \phi_{\sigma} - \phi(\rho) \right)_+ X \cdot H + \frac{1}{4} \int_{\Sigma} |H|^2.
\]

By letting $\sigma \to 0$, we obtain (13) immediately.
Now we turn to the proof of the claimed estimate (15). Indeed, when \(X \cdot H \geq 0\), the inequality is trivial. When \(X \cdot H \leq 0\), we have similar to (9) that
\[
-(\phi_\sigma - \phi(\rho))_+ X \cdot H - \frac{|X|^2}{w^2} \leq -\left| \frac{1}{w} X^\perp + \frac{1}{2} H \right|^2 + \frac{1}{4} |H|^2 + \phi(\rho) X \cdot H
\]
Hence the proof is concluded. \(\square\)

Now we turn to the case of surfaces in an \(n\)-sphere \(S^n\).

Proof of Theorem 5. Let \(r(x) = \text{dist}_{S^n}(x, o)\), \(X = \sin r \nabla r\), \(\phi(r)^{-1} = w(r) = \int_0^r \sin t dt\). We proceed similarly as the proof of Theorem 4 noting the relation
\[
2\phi \cos r + \phi' \sin r = -1.
\]
\(\square\)

Similar to the proof of Theorem 6, we have

Theorem 7. Given any closed 2-surface in \(S^n\) and a real number \(0 < \rho < \pi\), then
\[
4\pi - |\Sigma_\rho| \leq \frac{1}{w(\rho)} \int_{\Sigma_\rho} \cos r + \frac{1}{4} \int_{\Sigma} |H|^2.
\]

Remark 1. The reason that we do not take a limit \(\rho \to \infty\) is the existence of conjugate points in \(S^n\).

4. Equality case of Corollary 3

Now we turn to discuss the equality case of Corollary 3. We recall some basics of the hyperboloid model of the hyperbolic space \(\mathbb{H}^n\). The readers can find relevant materials in [Pet98, Chapter 3]. \(\mathbb{H}^n\) can be realized as a pseudo-sphere in Minkowski space i.e.
\[
\mathbb{H}^n = \{ x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{1,n} : \langle x, x \rangle = -1, x_0 > 0 \},
\]
where the bilinear form \(\langle \cdot, \cdot \rangle\) is defined to be
\[
\langle x, y \rangle = -x_0 y_0 + x_1 y_1 + \cdots + x_n y_n.
\]
The notation \(\langle \cdot, \cdot \rangle\) is still used because of no confusion caused. By differentiating the relation \(\langle x, x \rangle = -1\), we find that the tangent space \(T_x \mathbb{H}^n\) at \(x \in \mathbb{H}^n\) is \(\{ y \in \mathbb{R}^{1,n} : \langle x, y \rangle = 0 \}\). By restricting \(\langle \cdot, \cdot \rangle\) to \(T_x \mathbb{H}^n\), we get the standard Riemannian metric \(\langle \cdot, \cdot \rangle_{T_x \mathbb{H}^n}\) on \(\mathbb{H}^n\). The geodesic passing through \(x \in \mathbb{H}^n\) with unit velocity \(z \in T_x \mathbb{H}^n\) is
\[
x(\rho) = x \cosh \rho + z \sinh \rho, \quad \rho \in (-\infty, \infty).
\]
If we know the endpoints \( x \neq y \) of a geodesic segment, we can solve \( \rho = \rho(x, y) \) and \( z := z(x, y) \in T_x \mathbb{H}^n \),

\[
(x, y) = -\cosh \rho, z = \sinh^{-1} \rho(y + x(x, y)).
\]

**Theorem 8.** The equality holds in (3) if and only if \( \Sigma \) lies in a \( \mathbb{H}^3 \) subspace as a geodesic 2-sphere.

**Proof.** One easily verifies the equality of (3) for geodesic 2-sphere \( s \). If equality holds in (3), then \( \Sigma \) is of multiplicity one at every point of \( x \in \Sigma \) by Corollary [1]. Also, there is some point \( y \in \Sigma \) such that \( H(y) \neq 0 \). Most importantly, the mean curvature vector \( H \) at \( y \in \Sigma \) can be evaluated in terms of

\[
-\frac{1}{2} H(y) = \frac{\sinh \rho}{w(\rho)} \tau^\perp,
\]

where \( \rho \) and \( \tau \in T_y \mathbb{H}^n \) are respectively the length and the velocity vector at \( y \) of the geodesic segment from from any \( x \in \Sigma \) to \( y \). Let \( \{e_1, e_2\} \) span \( T_y \Sigma \), note that (13) says that \( \tau \) is a linear combination of \( \{e_1, e_2, H(y)\} \), and hence every point \( x \in \Sigma \) lies in a \( \mathbb{H}^3 \) subspace of \( \mathbb{H}^n \). \( \mathbb{H}^3 \) is totally geodesic in \( \mathbb{H}^n \), we can then consider \( n = 3 \) only. Now we identify every point and the tangent space as elements in \( \mathbb{R}^{1,3} \), and from (16),

\[
\tau = x \sinh \rho + z \cosh \rho.
\]

Let \( \nu \) be the outward pointing normal of \( \Sigma \) in \( \mathbb{H}^3 \) at \( y \), by (17),

\[
-\frac{1}{2} H(y) = \frac{\sinh \rho}{w(\rho)} (x \sinh \rho + z \cosh \rho, \nu) \nu
\]

\[
= \frac{\sinh^2 \rho}{w(\rho)} (x, \nu) \nu + \frac{\cosh \rho}{w(\rho)} (y + x(x, y), \nu) \nu
\]

\[
= \frac{\sinh^2 \rho}{w(\rho)} (x, \nu) \nu + \frac{\cosh \rho}{w(\rho)} (x, \nu) (x, y) \nu
\]

\[
= \frac{\sinh^2 \rho}{w(\rho)} (x, \nu) \nu - \frac{\cosh^2 \rho}{w(\rho)} (x, \nu) \nu
\]

\[
= -\frac{1}{w(\rho)} (x, \nu) \nu.
\]

Since \( H = -H \nu \), the mean curvature \( H \) at \( y \) of \( \Sigma \) immersed in \( \mathbb{H}^3 \) is

\[
\frac{1}{2} H(y) = -\frac{(x, \nu)}{w(\rho)} = \frac{(x, \nu)}{1 - \cosh \rho} = \frac{(x, \nu)}{1 + (x, y)}.
\]

We prove now that \( H(y) \) can not be less than 0. Assume on the contrary that \( H(y) = -2 \coth t < 0 \) where \( t > 0 \), we fix the coordinates of \( \mathbb{R}^{1,3} \) now by setting the point \( \exp_y(-tv) \) to be \( o = (1, 0, \ldots, 0) \in \mathbb{R}^{1,n} \) where \( \exp_y \) is the exponential map of \( \mathbb{H}^3 \) at \( y \). Note that \( o \) is the origin under polar
coordinates
\[ [0, \infty) \times S^2 \to \mathbb{R}^{1,n} \]
\[ (s, \theta) \mapsto (\cosh s, \theta \sinh s), \]

where \( \theta \in S^2 \subset \mathbb{R}^3 \). We assume that \( y = (\cosh t, \theta \sinh t) \) since the distance from \( o \) to \( y \) is \( t \), and \( x = (\cosh \bar{t}, \bar{\theta} \sinh \bar{t}), \bar{t} > 0 \). \( \nu \) is then \( (\sinh t, \theta \cosh t) \).

By inserting the values of \( y, \nu \) and \( x \) to the identity (19), we get
\[
-\coth t = -\frac{\cosh \bar{t} \sinh t + \sinh \bar{t} \cosh t \theta \cdot \bar{\theta}}{1 - \cosh t \cosh \bar{t} + \sinh t \sinh \bar{t} \theta \cdot \bar{\theta}}.
\]
Here \( \theta \cdot \bar{\theta} \) is the standard \( \mathbb{R}^3 \) inner product. This readily reduces to
\[
0 = 2 \sinh \bar{t} \sinh t \cosh \theta \cdot \bar{\theta} - \cosh \bar{t} \sinh^2 t \\
+ \cosh t - \cosh^2 t \cosh \bar{t} \\
= \sinh \bar{t} \sinh(2t) \theta \cdot \bar{\theta} - \cosh \bar{t} \cosh(2t)
\]
which is however not possible since \( \theta \cdot \bar{\theta} \leq 1 \).

So \( H(y) > 0 \). We can set instead \( H(y) = 2 \coth t > 0 \) with \( t > 0 \). We use this \( t \) and do the same thing as before, we arrive
\[
\coth t = \frac{-\cosh \bar{t} \sinh t + \sinh \bar{t} \cosh t \theta \cdot \bar{\theta}}{1 - \cosh t \cosh \bar{t} + \sinh t \sinh \bar{t} \theta \cdot \bar{\theta}},
\]
and finally \( t = \bar{t} \). Then \( \Sigma \) has to be a geodesic sphere of radius \( t \). \( \square \)

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