A CLASS OF CONSERVED SURFACE LAYER INTEGRALS
FOR CAUSAL VARIATIONAL PRINCIPLES

FELIX FINSTER AND JOHANNES KLEINER

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Abstract. In the theory of causal fermion systems, the physical equations are obtained as the Euler-Lagrange equations of a causal variational principle. Studying families of critical measures of causal variational principles, a class of conserved surface layer integrals is found and analyzed.

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1. INTRODUCTION

In the theory of causal fermion systems, which is a new approach to fundamental physics, the physical equations are obtained by minimizing the so-called causal action under variations of a measure. Causal variational principles were introduced in [3] as a generalization of this causal action principle (see [4] or the survey article [6]). In the meantime, causal variational principles have been studied in various situations, and the mathematical setup has been further extended (see [8] for a recent introduction).

In general terms, in a causal variational principle one minimizes an action $S$ of the form

$$S(\rho) = \int_\mathcal{F} d\rho(x) \int_\mathcal{F} d\rho(y) \mathcal{L}(x, y)$$ (1.1)
under variations of the measure \( \rho \), keeping the total volume \( \rho(\mathcal{F}) \) fixed (for more details see Section 2 below). The support of the measure is referred to as space-time \( M := \text{supp } \rho \).

In this setting, the usual integrals over hypersurfaces in space-time are undefined. Instead, one considers so-called surface layer integrals, being double integrals of the form

\[
\int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) (\cdots) \mathcal{L}(x, y),
\]

where \( \Omega \) is a subset of \( M \) and \( (\cdots) \) stands for a differential operator acting on the Lagrangian. The structure of such surface layer integrals can be understood most easily in the special situation that the Lagrangian is of short range in the sense that \( \mathcal{L}(x, y) \) vanishes unless \( x \) and \( y \) are close together. In this situation, we get a contribution to the double integral (1.2) only if both \( x \) and \( y \) are close to the boundary \( \partial \Omega \). With this in mind, surface layer integrals can be understood as an adaptation of surface integrals to the setting of causal variational principles (for a more detailed explanation see [7, Section 2.3]).

Surface layer integrals were first introduced in [7] in order to formulate Noether-like theorems for causal variational principles. In particular, it was shown that there is a conserved surface layer integral which generalizes the Dirac current in relativistic quantum mechanics (see [7, Section 5]). More recently, in [8] another conserved surface layer integral was discovered which gives rise to a symplectic form on the solutions of the linearized field equations (see [8] Sections 3.3 and 4.3). The present paper is devoted to a systematic study of conserved surface layer integrals. We find a class of conserved surface layer integrals \( I^\Omega_m \) parametrized by a parameter \( m = 1, 2, \ldots \) (see Theorem 3.1). These surface layer integrals are derived by considering families of critical points of the causal variational principle. They can be rewritten as multilinear functionals on the space \( \mathcal{J}_{\text{lin}} \) of solutions of the linearized field equations (see Proposition 3.8). In the case \( m = 1 \), the resulting surface layer integrals have the same structure as those in the Noether-like theorems in [7] (see Corollary 3.9). In the case \( m = 2 \), anti-symmetrizing the bilinear functional in its two arguments \( u, v \in \mathcal{J}_{\text{lin}} \) gives the symplectic form (see Corollary 3.10). Symmetrizing in these arguments, on the other hand, gives rise to a new conserved surface layer integral of the following form (see also Corollary 3.11):

**Theorem 1.1. (symmetric bilinear form on jets)** Given a compact set \( \Omega \subset M \), if the regularity assumptions (r1) and (r2) on page 7 are satisfied, then the bilinear form

\[
(\cdot | \cdot)_\Omega : \mathcal{J}_{\text{lin}} \times \mathcal{J}_{\text{lin}} \to \mathbb{R}
\]

\[
(u | v)_\Omega = \int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \times \left( \nabla_{1,u} \nabla_{1,v} + 2 \nabla_{1,S\Delta_2[u,v]} - \nabla_{2,u} \nabla_{2,v} - 2 \nabla_{2,S\Delta_2[u,v]} \right) \mathcal{L}(x, y)
\]

satisfies the relation

\[
(u | v)_\Omega = \int_{\Omega} \left( \nabla_{1,u} \nabla_{1,v} \frac{\nu}{2} + \nabla_{S\Delta_2[u,v]} \nu \right) d\rho(x).
\]

Here the jets \( u \) and \( v \) are pairs of a real-valued function on \( M \) and a vector field on \( \mathcal{F} \) along \( M \), \( u = (a, u) \in C^\infty(M, \mathbb{R}) \oplus C^\infty(M, T\mathcal{F}) \) and \( v = (b, v) \in C^\infty(M, \mathbb{R}) \oplus C^\infty(M, T\mathcal{F}) \).
$C^\infty(M,T\mathcal{F})$, and $\nabla_u$ denotes a jet derivative, which is a linear combination of multiplication and directional derivative,
\[ \nabla_u \ell(x) := a(x) \ell(x) + \left( D_u \ell \right)(x). \] (1.5)
The indices in $\nabla_1 u$ or $\nabla_2 u$ indicate on which argument of the Lagrangian the jet derivative acts. We always use the convention that these “partial jet derivatives” do not act on jets contained in other derivatives, so that for example for a constant $\nu$ (for details see Section 2),
\[ \nabla_1 u \nabla_1 v \nu = a(x) b(x) \nu. \]
The linearized field equations for a jet $v$ read (for details see Section 3.2)
\[ \nabla_u \left( \int_M \left( \nabla_1 v + \nabla_2 v \right) \mathcal{L}(x, y) \, d\rho(y) - \nabla_v \frac{\nu}{2} \right) = 0, \]
to be satisfied on $M$ for all test jets $u$ in the jet space $\mathcal{J}^{\text{test}}$ (see (2.8) and Definition 3.3 or [8, Section 4.2]). The space of solutions of the linearized field equations is denoted by $\mathcal{J}^{\text{lin}}$. Next, $\Delta_2$ is the quadratic correction to the field equations (3.15), and $S$ is a Green’s operator (see Definition 3.20; a basic introduction is given in Section 2).
The surface layer integral (1.3) corresponds to a surface term, whereas the right side of (1.4) involves an integral over $\Omega$ and is therefore a volume term. If the jets $u, v$ and $S \Delta_2 [u, v]$ have no scalar components, then the volume term vanishes, thereby giving a conservation law for the surface layer integral (see Corollary 3.11).

This paper is structured as follows. In Section 2, we give the preliminaries and collect all the assumptions which enter our constructions. Section 3 contains our main results. In Section 3.1, we prove a general theorem which yields conserved quantities and illustrate it in the example of symmetry transformations. This theorem involves a two-parameter family of solutions of the weak EL equations. In Section 3.2, we compute the surface layer integrals for a two-parameter family constructed from $m$ solutions of the linearized field equations (see Proposition 3.8), using perturbative methods as developed in [5]. In Section 3.4, we discuss the dependence of the bilinear form (1.3) on the choice of the Green’s operator $S$. This freedom changes the bilinear form only modulo a conserved surface layer integral $I^1_\Omega$ (see Example 3.12). More generally, modifying the Green’s operators in $I^\Omega_m$ changes the surface layer integrals only by multiples of $I^\Omega_l$ with $l < m$ (see Theorem 3.13). In Section 4, our constructions are illustrated by computing the surface layer integrals in the example of the lattice system introduced in [8, Section 5].

We finally remark that in [2] the above bilinear form is computed in Minkowski space. It is shown that it is indeed positive definite, giving rise to a scalar product on bosonic and fermionic jets. This scalar product plays an important role when getting the connection between the causal action principle and quantum field theory in Fock spaces [10].

2. Preliminaries

We consider the lower semi-continuous setting as introduced in [8, Section 2]. Thus let $\mathcal{F}$ be a (possibly non-compact) smooth manifold and $\rho$ a Radon measure on $\mathcal{F}$ (i.e. a regular Borel measure with $\rho(K) < \infty$ for any compact $K \subset \mathcal{F}$, where by a measure we always mean a positive measure; for an introduction, see for example [11] or [1]). Moreover, we are given a non-negative function $\mathcal{L} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}_0^+$ (the Lagrangian) with the following properties:
(i) $L$ is symmetric: $L(x, y) = L(y, x)$ for all $x, y \in \mathcal{F}$.
(ii) $L$ is lower semi-continuous, i.e. for all sequences $x_n \to x$ and $y_{n'} \to y$,
$$L(x, y) \leq \liminf_{n, n' \to \infty} L(x_n, y_{n'}).$$

We assume that $\rho$ satisfies the following technical assumption:
(iii) The function $L(x, .)$ is $\rho$-integrable for all $x \in \mathcal{F}$, giving a lower semi-continuous and bounded function on $\mathcal{F}$.

If the total volume $\rho(\mathcal{F})$ is finite, the causal variational principle is to minimize the action \((1.1)\) under variations of the measure $\rho$ (which do not need to satisfy (iii)), keeping the total volume $\rho(\mathcal{F})$ fixed (volume constraint). If $\rho(\mathcal{F})$ is infinite, however, it is not obvious how to implement the volume constraint, making it necessary to proceed as follows: Let $\tilde{\rho}$ be another Borel measure on $\mathcal{F}$ with the properties
\[
|\tilde{\rho} - \rho|_{(\mathcal{F})} < \infty \quad \text{and} \quad (\tilde{\rho} - \rho)(\mathcal{F}) = 0
\]
(where $|.|$ denotes the total variation of a measure; see \cite{11} §28 or \cite{12} Section 6.1). Then the difference of the actions as given by
\[
(S(\tilde{\rho}) - S(\rho)) = \int_{\mathcal{F}} d(\tilde{\rho} - \rho)(x) \int_{\mathcal{F}} d\rho(y) L(x, y)
+ \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d(\tilde{\rho} - \rho)(y) L(x, y) + \int_{\mathcal{F}} d(\tilde{\rho} - \rho)(x) \int_{\mathcal{F}} d(\tilde{\rho} - \rho)(y) L(x, y)
\]
is well-defined (for details see \cite{8} Lemma 2.1). The measure $\rho$ is said to be a minimizer of the causal action if the difference \((2.2)\) is non-negative for all $\tilde{\rho}$ satisfying \((2.1)\),
\[
(S(\tilde{\rho}) - S(\rho)) \geq 0.
\]

We now state the Euler-Lagrange (EL) equations as derived in \cite{8} Lemma 2.3 [by adapting \cite{9} Lemma 3.4 to the non-compact setting].

**Lemma 2.1.** (The Euler-Lagrange equations) Let $\rho$ be a minimizer of the causal action. Then for a suitable value of the real parameter $\nu$, the function $\ell$ defined by
\[
\ell(x) = \int_{\mathcal{F}} L(x, y) \, d\rho(y) - \frac{\nu}{2} : \mathcal{F} \to \mathbb{R}
\]
satisfies the equation
\[
\ell|_{\supp \rho} \equiv \inf_{\mathcal{F}} \ell = 0.
\]

We remark that $\nu$ can be understood as the Lagrange multiplier describing the volume constraint; see \cite{4} §1.4.1.

The EL equations are analyzed most conveniently in the so-called jet formalism, which we now review (for more details see \cite{8}). Let
\[
\Gamma = C^\infty(M, T\mathcal{F})
\]
be the smooth vector fields on $\mathcal{F}$ restricted to $M$ (thus every $u \in \Gamma$ has a smooth extension to $\mathcal{F}$). We define the jet-space on the manifold $\mathcal{F}$ as
\[
\mathcal{J} := \{ u = (a, u) \text{ with } a \in C^\infty(\mathcal{F}, \mathbb{R}) \text{ and } u \in C^\infty(\mathcal{F}, T\mathcal{F}) \}
\]
and denote its restriction to $M$ by
\[
\mathcal{J}|_M := \{ u = (a, u) \text{ with } a \in C^\infty(M, \mathbb{R}) \text{ and } u \in \Gamma \}.
\]
Moreover, let $\Gamma^\text{diff}$ be those vector fields for which the directional derivative of the function $\ell$ exists,

$$\Gamma^\text{diff} = \{ u \in C^\infty(M, T^*F) \mid D_u \ell(x) \text{ exists for all } x \in M \}.$$  

We introduce the space of differentiable one-jets by

$$J^\text{diff} := C^\infty(M, \mathbb{R}) \oplus \Gamma^\text{diff} \subset J^\text{M}$$ (2.7)

and choose a linear subspace $J^\text{test}$ of the form

$$J^\text{test} = C^\infty(M, \mathbb{R}) \oplus \Gamma^\text{test} \subset J^\text{diff}.$$ (2.8)

For a jet $u = (a, u) \in J^\text{diff}$ we define $\nabla u$ as the linear combination of scalar multiplication and directional derivative (1.5). Then the EL equations (2.4) imply the so-called weak EL equations (for details see [8, Section 4.1])

$$\nabla u \ell \mid M = 0 \quad \text{for all } u \in J^\text{test}.$$ (2.9)

Following [5], we also introduce other jet spaces needed below. First, we define a convenient version of a dual of $J^\text{test}$.

**Definition 2.2.** We denote the continuous global one-jets of the cotangent bundle restricted to $M$ by $\mathcal{J}^*$ := $C^0(M, \mathbb{R}) \oplus C^0(M, T^*F)$. We let $(\mathcal{J}^\text{test})^*$ be the quotient space

$$(\mathcal{J}^\text{test})^* := \mathcal{J}^*/\{ (g, \varphi) \in \mathcal{J}^* \mid g(x) a(x) + \langle \varphi(x), u(x) \rangle = 0$$

for all $u = (a, u) \in \mathcal{J}^\text{test}$ and $x \in M \},$$

where $\langle ., . \rangle$ denotes the dual pairing of $T^*_x F$ and $T_x F$.

Here, we have taken the quotient in order to avoid dual jets which are trivial on $\mathcal{J}^\text{test}$. Next, we specify the space of jets which can be used for varying the measure $\rho$. The basic idea is to define differentiability of the jets by corresponding differentiability properties of the Lagrangian:

**Definition 2.3.** For any $\ell \in \mathbb{N}_0 \cup \{\infty\}$, the jet space $\mathcal{J}^\ell \subset \mathcal{J}$ is defined as the vector space of jets with the following properties:

(i) For all $y \in M$ and all $x$ in an open neighborhood of $M$, in suitable charts around $x$ and $y$ the directional derivatives

$$\left( \nabla_{v_1} + \nabla_{v_2} \right) \cdots \left( \nabla_{v_p} + \nabla_{v_p} \right) L(x, y)$$ (2.10)

exist for all $p \in \{1, \ldots, \ell\}$ and all $v_1, \ldots, v_p \in \mathcal{J}^\ell$.

(ii) Integrating the expression (2.10) in $y$ over $M$ with respect to the measure $\rho$, the resulting function (defined on an open neighborhood of $M$) is continuously differentiable in the direction of every jet $u \in \mathcal{J}^\text{test}$.

Throughout the paper, we use the following conventions with respect to partial derivatives and jet derivatives:

- Partial and jet derivatives with an index $i \in \{1, 2\}$, as for example in (2.10), only act on the respective variable of the function $L$. This implies, for example, that the derivatives commute,

$$\nabla_{v_1} \nabla_{v_2} L(x, y) = \nabla_{v_1} \nabla_{v_2} L(x, y).$$ (2.11)
The partial or jet derivatives do not carry an index if they act on functions which only have one free variable. In this case, they act on every appearance of the corresponding variable. This implies, for example, that

\[ \nabla_u \int_{\mathcal{F}} \nabla_1 u \mu(x, y) \, d\nu(y) = \int_{\mathcal{F}} (\nabla_1 u \nabla_1 v + \nabla_1 \nabla_2 \nu) \mu(x, y) \, d\nu(y) , \]

where \( \nabla_u v = (\nabla_u b, \nabla_u v) = (a b + D_u b, a v + (D_u v^i) \partial_i) \) for \( u = (a, u), \, v = (b, v) \) and \((\partial_i)_j\), a local basis of \( T \mathcal{F} \).

We point out that Definition 2.4 depends on the charts of \( \mathcal{F} \), to be chosen in a neighborhood of every point \( x \in M \) (for an explanation of this point see [5, Section 3.3]).

3. A Class of Conserved Surface Layer Integrals

3.1. Conservation Laws for Families of Solutions. In this section we derive a class of conservation laws for families of solutions of the weak EL equations. We let \( \tilde{\rho}_{s,t} \) with \( s, t \in (-\delta, \delta) \) be a two-parameter family of universal measures of the form

\[ \tilde{\rho}_{s,t} = (F_{s,t})_* (f_{s,t} \rho) , \tag{3.1} \]

where \( f_{s,t} \) and \( F_{s,t} \) are smooth,

\[ f \in C^\infty((-\delta, \delta)^2 \times \mathcal{F}, \mathbb{R}^+) \quad \text{and} \quad F \in C^\infty((-\delta, \delta)^2 \times \mathcal{F}, \mathcal{F}) , \tag{3.2} \]

and are trivial at \( s = t = 0 \),

\[ f_{0,0} \equiv 1 , \quad F_{0,0} = 1 \tag{3.3} \]

(the star in (3.1) denotes the push-forward measure, defined for a subset \( U \subset \mathcal{F} \) and a measure \( \mu \) on \( \mathcal{F} \) by \((F_{s,t})_* \mu)(U) = \mu(F_{s,t}^{-1}(U))\); see for example [1, Section 3.6]).

Moreover, we assume that every measure \( \tilde{\rho}_{s,t} \) satisfies the weak EL equations (2.9). Since the jets used for testing (2.8) are defined on the support of the corresponding measure, the variation of \( \rho \) also entails a variation of the test space \( \mathcal{J}_{\text{test}} \). We denote the test space for the measure \( \tilde{\rho}_{s,t} \) by \( \mathcal{J}_{s,t}^{\text{test}} \). Evaluated for the measure \( \tilde{\rho}_{s,t} \) (instead of \( \rho \)), the weak EL equations (2.9) read

\[ \nabla_u \left( \int_{\mathcal{F}} \mathcal{L}(F_{s,t}(x), F_{s,t}(y)) f_{s,t}(y) \, d\rho(y) - \frac{\nu}{2} \right) = 0 \quad \text{for all } u \in \mathcal{J}_{s,t}^{\text{test}} . \tag{3.4} \]

and \( x \in M \). We use the notation

\[ L(x_{s,t}, y_{s,t}) := f_{s,t}(x) \mathcal{L}(F_{s,t}(x), F_{s,t}(y)) f_{s,t}(y) . \]

We note that, following the procedure in [8, Section 4], we could choose \( \mathcal{J}_{s,t}^{\text{test}} \) as the push-forward of \( \mathcal{J}_{0,0}^{\text{test}} \) (see [8 eq. (4.12)]), making it possible to rewrite the weak EL equations (3.4) as (see [8 Lemma 4.2])

\[ \nabla_u \left( \int_{M} L(x_{s,t}, y_{s,t}) \, d\nu(y) - \frac{\nu}{2} f_{s,t}(x) \right) = 0 \quad \text{for all } u \in \mathcal{J}_{s,t}^{\text{test}} , \tag{3.5} \]

valid for every \( s, t \in (-\delta, \delta) \). This has the advantage that the jet space for testing does not depend on \( s \) and \( t \). However, this advantage is not crucial for the following constructions in this section. Therefore, we prefer to work with (3.4) without specifying the jet spaces \( \mathcal{J}_{s,t}^{\text{test}} \).

For technical simplicity, we make the following regularity assumptions:
For all $x \in M$, $p, q \geq 0$ and $r \in \{0,1\}$, the following partial derivatives exist and can be interchanged with the integral:

\[
\int_M \partial_{s'}^r \partial_{s}^p \partial_{t}^q L(x_{s+s',t}, y_{s,t}) \bigg|_{s'=s=t=0} d\rho(y) = \partial_{s'}^r \partial_{s}^p \partial_{t}^q \int_M L(x_{s+s',t}, y_{s,t}) d\rho(y) \bigg|_{s'=s=t=0}.
\]  

(3.6)

(12) The jet $u(t)$ defined by

\[
u(t) = \partial_s L(f_{s,t}, F_{s,t}) \bigg|_{s=0} \text{ is in } \mathcal{J}_{0,t}^{\text{test}} \text{ for all } t \in (-\delta, \delta). \]  

(3.7)

**Theorem 3.1.** Assume that the functions $f$ and $F$ satisfy (3.2) and (3.3) as well as the regularity assumptions (1) and (2). Moreover, assume that the measures $\rho_{s,t}$ given by (3.1) satisfy the weak EL equations (3.4) for all $s$ and $t$ (for jet spaces $\mathcal{J}_{s,t}^{\text{test}}$ chosen in agreement with (2.3)). Then for every compact $\Omega \subset M$ and every $m \in \mathbb{N}$, the following expression vanishes:

\[
S_m^{\Omega} := \int_\Omega d\rho(x) \int_{M \Omega} d\rho(y) (\partial_{1,s} - \partial_{2,s}) (\partial_{1,t} + \partial_{2,t}) L(x_{s,t}, y_{s,t}) \bigg|_{s=t=0} - \frac{\nu}{2} \int_\Omega \partial_s \partial_t^{m-1} f_{s,t}(x) \bigg|_{s=t=0} d\rho(x) = 0.
\]

(3.8)

(3.9)

Here the indices of the symbols $\partial_{1,s}$, $\partial_{2,s}$, ..., again indicate that the derivatives act only on the first or second argument of $L(x_{s,t}, y_{s,t})$, respectively.

Before giving the proof, we briefly explain the structure and significance of this result. The double integral (3.8) is a surface layer integral (1.2) and can thus be interpreted as a surface term. The summand (3.9), on the other hand, is a volume term. Therefore, the general structure of the above equation relating a surface term to a volume term resembles the Gauß divergence theorem. In many applications, the volume term vanishes. Then, considering the situation that $\Omega$ exhausts the region between two surfaces which extend to spatial infinity (as explained in the introduction of [8]; see [8, Figure 1]), the surface layer integral does not depend on the choice of the surfaces, thus giving rise to a conservation law. If the surface term is non-zero, the above theorem still gives quantitative information on how the surface layer integral depends on the choice of the surface. In cases where the volume term has a physical interpretation, this gives insight into the dynamics of the system (similar to the inhomogeneous Maxwell equations in integral form). Moreover, an identity relating a surface to a volume term is very useful for getting estimates similar to energy estimates for hyperbolic PDEs.

**Proof of Theorem 3.1.** Since the scalar component of $\mathcal{J}_{s,t}^{\text{test}}$ comprises all smooth functions (see (2.3)), which are dense in all continuous functions, the term in brackets in (3.4) vanishes. Multiplying by $f_{s,t}(x)$, this means that

\[
\int_M L(x_{s,t}, y_{s,t}) d\rho(y) - \frac{\nu}{2} f_{s,t}(x) = 0
\]

for all $s, t \in (-\delta, \delta)$ and all $x \in M$. Since the right hand side is trivially smooth in $t$ and $s$, we can take $k := m - 1$ derivatives in $t$ and one derivative in $s$ to obtain

\[
\int_M (\partial_{1,s} + \partial_{2,s}) (\partial_{1,t} + \partial_{2,t}) L(x_{s,t}, y_{s,t}) d\rho(y) - \frac{\nu}{2} \partial_s \partial_t^k f_{s,t}(x) \bigg|_{s=t=0} = 0,
\]

(3.10)
where we exchanged differentiation and integration with the help of the regularity assumption (r1). Next, we consider the weak EL equation (3.4), evaluated at \( s = 0 \) with the jet \( u := \partial_s (f_{s,t}, F_{s,t}) \big|_{s=0} \in \mathcal{Y}_{0,t}^{\text{test}} \). Assumption (r2) assures that this jet is in \( \mathcal{Y}_{0,t}^{\text{test}} \). Using that
\[
D_{1,u} \mathcal{L}(F_{0,t}(x), F_{0,t}(y)) = \partial_{1,s} \mathcal{L}(F_{s,t}(x), F_{s,t}(y)) \big|_{s=0},
\]
the weak EL equation (3.4) can be written as
\[
\hat{M} \int_{\Omega} \partial_{1,s} L(x_{s,t}, y_{s,t}) \, d\rho(y) - \frac{\nu}{2} \partial_s f_{s,t}(x) \big|_{s=0} = 0,
\]
valid for all \( t \). We now take \( k \) \( t \)-derivatives of this equation and subtract \( \frac{1}{2} \) times (3.10). We thus obtain
\[
\int_{\Omega} (\partial_{1,s} - \partial_{2,s})(\partial_{1,v} + \partial_{2,v})^k L(x_{s,t}, y_{s,t}) \, d\rho(y) - \frac{\nu}{2} \partial_s \partial_t^k f_{s,t}(x) = 0.
\]
Integrating the last equation over \( \Omega \) gives
\[
\int_{\Omega} \int_{\Omega} (\partial_{1,s} - \partial_{2,s})(\partial_{1,v} + \partial_{2,v})^k L(x_{s,t}, y_{s,t}) \, d\rho(x) \, d\rho(y) = \frac{\nu}{2} \int_{\Omega} \partial_s \partial_t^k f_{s,t}(x) \, d\rho(x).
\]
(3.11)
Since the integrand is anti-symmetric in its arguments \( x \) and \( y \), we also have
\[
\int_{\Omega} \int_{\Omega} (\partial_{1,s} - \partial_{2,s})(\partial_{1,v} + \partial_{2,v})^k L(x_{s,t}, y_{s,t}) \, d\rho(x) = 0.
\]
Subtracting this equation from (3.11) gives the result. \( \square \)

**Example 3.2. (Symmetry transformations)** We now illustrate the above theorem in the example of families of measures obtained by applying symmetry transformations. Thus, specializing the setting in [7, Section 3.1], we let \( F \) be a smooth one-parameter family of diffeomorphisms
\[
F \in C^\infty((-\delta, \delta) \times F \to \mathcal{F}) \quad \text{with} \quad F_0 = 1 \quad \text{and} \quad F_{-s} = F_s^{-1}.
\]
We assume that \( F \) is a symmetry of the Lagrangian in the sense that
\[
\mathcal{L}(x, F_s(y)) = \mathcal{L}(F_{-s}(x), y) \quad \text{for all} \ s \in (-\delta, \delta) \text{ and } x, y \in \mathcal{F}.
\]
Moreover, we let \( \rho \) be a solution of the weak EL equations (2.9) and consider the family of measures \( (\tilde{\rho}_{s,t})_{s,t \in (-\delta, \delta)} \) given by
\[
\tilde{\rho}_{s,t} := (F_s)_* \rho
\]
(note that there is no \( t \)-dependence). The symmetry assumption (3.12) implies that the weak EL equations (3.4) are satisfied for every measure \( \tilde{\rho}_{s,t} \). If the regularity

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1This is a slightly stronger assumption than in [7], where we merely assume that (3.12) holds for \( x, y \in M \). We need the stronger assumption because a necessary criterion for the weak EL equations (3.4) to hold is that term in brackets in (3.3) vanishes identically on \( M \). In [7], however, we merely used that the derivative of the term in brackets vanishes on \( M \).
assumptions (r1) and (r2) hold, Theorem 3.1 applies. Choosing \( m = 1 \), we obtain the conservation law

\[
0 = \int_{\Omega} \int_{M \setminus \Omega} d\rho(x) d\rho(y) \left( \partial_1, s - \partial_2, s \right) \mathcal{L}(F_s(x), F_s(y)) \bigg|_{s=0}
\]

\[
= \frac{d}{ds} \int_{\Omega} \int_{M \setminus \Omega} d\rho(x) d\rho(y) \left( \mathcal{L}(F_s(x), y) - \mathcal{L}(F_{-s}(x), y) \right) \bigg|_{s=0},
\]

which agrees with the Noether-like theorem for a continuously differentiable symmetry of the Lagrangian in [7] (see [7, Theorem 3.3] and later generalizations thereof). We conclude that the Noether-like theorem for a symmetry of the Lagrangian found in [7] is a corollary of the conservation laws found in Theorem 3.1.

\[\Diamond\]

3.2. Formulation in Terms of Linearized Solutions. In order to work out applications of Theorem 3.1, one must construct families of solutions of the weak EL equations (3.4). We now explain how this can be accomplished. Our main result will be to rewrite the conserved surface layer integral of Theorem 3.1 as an \( m \)-multilinear form on solutions of the linearized field equations (see Proposition 3.8). Our results rely on the perturbation expansion for critical measures as developed in [8]. In order to make the present paper self-contained, we recall the basics of the perturbation expansion and work out the combinatorial details in Appendix A.

In preparation, we need to define the notion of solutions of linearized field equations (for details see [8, Section 4.2]). For \( \ell \in \mathbb{N} \) and \( v_1, \ldots, v_\ell \in \mathcal{J} \), we define

\[
\Delta_\ell[v_1, \ldots, v_\ell](x) := \frac{1}{\ell!} \left( \int_{M} \left( \nabla_1, v_1 + \nabla_2, v_1 \right) \cdots \left( \nabla_1, v_\ell + \nabla_2, v_\ell \right) \mathcal{L}(x, y) d\rho(y) \right.
\]

\[
- \nu^2 \left( b_1(x) \cdots b_\ell(x) \right)
\]

(3.13)

where \( x \) can be chosen in a neighborhood of \( M \subset \mathcal{F} \). According to the assumptions in Definition 2.3, the derivatives and the integral in (3.13) exist, and the resulting function is continuously differentiable in the direction of any \( u \in \mathcal{J}^{test} \). Therefore, the function \( \Delta_\ell[v_1, \ldots, v_\ell] \) can be identified with a dual jet \( w^* \in (\mathcal{J}^{test})^* \) (see Definition 2.2), where the dual pairing with a jet \( u \in \mathcal{J}^{test} \) is given by

\[
\langle u, w^* \rangle := \nabla_u \Delta_\ell[v_1, \ldots, v_\ell](x).
\]

(3.14)

Using this identification, the operator \( \Delta_\ell \) in (3.13) gives rise to a mapping

\[
\Delta_\ell : \mathcal{J}^{\ell_1} \times \cdots \times \mathcal{J}^{\ell_\ell} \rightarrow (\mathcal{J}^{test})^*.
\]

(\( \ell \) factors)

Now the linearized field equations can be written as in the next definition (cf. [8, eq. (4.19)]), where we use the abbreviation

\[
\Delta := \Delta_1.
\]

Definition 3.3. A jet \( w \in \mathcal{J}^{1} \) is a solution of the linearized field equations if it satisfies the equation

\[
\langle u, \Delta w \rangle = 0 \quad \text{for all } u \in \mathcal{J}^{\text{test}}.
\]

(3.15)

The vector space of all linearized solutions is denoted by \( \mathcal{J}_{\text{lin}} \subset \mathcal{J}^{1} \).

We remark that, in view of the definitions of \( \mathcal{J}^{1} \) and \( (\mathcal{J}^{\text{test}})^* \), the linearized field equations are equations in space-time \( M \).
The connection to the families of solutions of the EL equations is obtained as follows. Suppose that the family of measures \( \tilde{\rho}_{s,t} \) as defined by (3.1) satisfies the EL equations (3.5) for all \( s \) and \( t \) (thus, as explained before (3.5), we choose the jet spaces \( J^{\text{test}}_{s,t} \) as the push-forward of \( J^{\text{test}} \)). Then differentiating (3.5) with respect to \( s \) or \( t \) at \( s = t = 0 \), one sees that the resulting jets given by

\[
\partial_s(f_{s,t}, F_{s,t})|_{s=t=0} \quad \text{and} \quad \partial_t(f_{s,t}, F_{s,t})|_{s=t=0}
\]

are solutions of the linearized field equations (3.15). In other words, our family is described linearly in \( s \) and \( t \) by two solutions of the linearized field equations. Likewise, the higher \( s \)-and \( t \)-derivatives of (3.5) involve the operator \( \Delta_\ell \) with \( \ell > 1 \). Setting these higher derivatives to zero gives rise to equations which can be solved iteratively using the perturbation theory as developed systematically in [5]. We now recall how this perturbation expansion works. For clarity, we denote the expansion parameter by \( \lambda \). In our situation, \( \lambda \) can be thought of as describing the size of \( s \) and \( t \). This could be made precise by setting \( s = \lambda \hat{s} \) and \( t = \lambda \hat{t} \) with parameters \( \hat{s} \) and \( \hat{t} \) which are of the order one. Then the limit \( s, t \to 0 \) would correspond to taking the limit \( \lambda \to 0 \).

For ease in notation, we avoid the hats and simply take \( \lambda \) as a formal parameter which is used as a book-keeping device to keep track of the different orders in perturbation theory. As we shall see, the conserved surface layer integral \( I_{\Omega}^m \) will involve the \( m \)th order of the perturbation expansion. With this in mind, it is unproblematic to work with formal power expansions in \( \lambda \), and the convergence of the perturbation series is not an issue.

Thus for the function \( f_{s,t} \) we make the power ansatz

\[
f_{s,t}(x) = \sum_{p=0}^{\infty} \lambda^p f_{s,t}^{(p)}(x) \quad \text{with} \quad f_{s,t}^{(0)}(x) = 1. \tag{3.16}
\]

For the expansion of \( F_{s,t} \), we choose a chart around \( x \) and write \( F_{s,t}(x) \) in components as \( (F_{s,t}^{(\alpha)}(x))_{\alpha=1,...,m} \). Then we expand \( F_{s,t} \) componentwise,

\[
F_{s,t}(x)^\alpha = \sum_{p=0}^{\infty} \lambda^p F_{s,t}^{(p)}(x)^\alpha \quad \text{with} \quad F_{s,t}^{(0)}(x)^\alpha = x^\alpha. \tag{3.17}
\]

The choices of \( f_{s,t}^{(0)} \) and \( F_{s,t}^{(0)} \) ensure that for \( \lambda = 0 \), the measure \( \tilde{\rho} \) in (3.1) is equal to the unperturbed measure \( \rho \). For ease in notation, we shall omit the index \( \alpha \) in the expansion of \( F(x) \). But one should keep in mind that the expansion of \( F(x) \) always involves the choice of a chart around \( x \).

It is preferable to write the function \( f_{s,t} \) in the family of measures (3.1) as

\[
f_{s,t} = \exp c_{s,t} \quad \text{with} \quad c \in C^\infty((-\delta, \delta)^2 \times J^m, \mathbb{R}). \tag{3.18}
\]

where the expansion of \( c_{s,t} \) in \( \lambda \) is denoted similar to (3.15) by

\[
c_{s,t}(x) = \sum_{p=0}^{\infty} \lambda^p c_{s,t}^{(p)}(x) \quad \text{with} \quad c_{s,t}^{(0)}(x) = 0.
\]

We write the expansion coefficients again with jets,

\[
(c_{s,t}, F_{s,t})(x) = (0, x) + \sum_{p=1}^{\infty} w_{s,t}^{(p)}(x). \tag{3.19}
\]

In the next lemma we expand the formula in Theorem 3.1 in powers of \( \lambda \).
Lemma 3.4. If (3.18) and (3.19) is the perturbation expansion of a family (3.1) which satisfies the weak EL equations (3.4) as well as the regularity assumptions (r1) and (r2), then for every compact $\Omega \subset M$ and every $p, k \geq 0$, we have

$$0 = \frac{1}{k+1,(p)} := \sum_{\ell=1}^{p} \frac{1}{(\ell-1)!} \sum_{q_1, \ldots, q_\ell \geq 1 \atop q_1 + \cdots + q_\ell = p} \sum_{k_1, \ldots, k_\ell \geq 0 \atop k_1 + \cdots + k_\ell = k} \binom{k}{k_1 \ldots k_\ell} \times \left( \int_{\Omega} dp(x) \int_{M \setminus \Omega} dp(y) \left( \nabla_1 \partial_1^k \partial_1^q_{w_{s,t}}(q_1) - \nabla_2 \partial_2^k \partial_2^q_{w_{s,t}}(q_1) \right) \right. $$

$$ \times \left( \nabla_1 \partial_1^k \partial_1^q_{w_{s,t}}(q_2) + \nabla_2 \partial_2^k \partial_2^q_{w_{s,t}}(q_2) \right) \cdots \left( \nabla_1 \partial_1^k \partial_1^q_{w_{s,t}}(q_\ell) + \nabla_2 \partial_2^k \partial_2^q_{w_{s,t}}(q_\ell) \right) \mathcal{L}(x, y) $$

$$ \left. - \frac{\nu}{2} \int_{\Omega} (\partial_s \partial_t^k c_{s,t}(q_1) \cdots (\partial_s \partial_t^k c_{s,t}(q_\ell) \right) \cdot dp(x) \mid_{s=t=0}. $$

In order not to distract from the main ideas, the proof of this combinatorial lemma is given in Appendix A.

The higher order expansion terms in (3.10) and (3.17) can be expressed in terms of the linearized solutions using the Green’s operator of the linearized field equations, as we now explain.

Definition 3.5. A linear mapping $S : (\mathcal{J}^{\text{test}})^* \rightarrow \mathcal{J}^{\infty}$ is referred to as a Green’s operator if

$$\Delta S \varphi = -\varphi \quad \text{for all } \varphi \in (\mathcal{J}^{\text{test}})^*. \quad (3.20)$$

This is again an equation in space-time $M$. Similar as is the case for hyperbolic partial differential equations or in relativistic physics, the Green’s operators as defined here are not unique, because adding a linearized solution to the Green’s operator gives another Green’s operator. Consequently, there is a freedom in the choice of Green’s operator to every order on perturbation theory. We denote the corresponding choice by $S^{(p)}$.

$$w_{s,t}^{(p)} = S^{(p)} E^{(p)}, \quad (3.21)$$

where

$$E^{(p)}(x) = \sum_{\ell=2}^{p} E^{(p)}_{\ell}(x) \quad (3.22)$$

$$E^{(p)}_{\ell}(x) = \sum_{q_1, \ldots, q_\ell \geq 1 \atop q_1 + \cdots + q_\ell = p} \Delta_{\ell}[w^{(q_1)}_{s,t}, \ldots, w^{(q_\ell)}_{s,t}](x). \quad (3.23)$$

As shown in detail in [5], this perturbation expansion gives rise to families of solutions of the linearized field equations.

We now specify the dependence on the parameters $s$ and $t$. For $w_{s,t}^{(1)}$ we simply take a linear combination of two solutions of the linearized field equations,

$$w_{s,t}^{(1)} := s u + t v \quad \text{with } u, v \in \mathcal{J}^{\infin}. \quad (3.24)$$

Computing the nonlinear corrections $w_{s,t}^{(p)}$ via (3.21), we obtain a two-parameter family of solutions of the weak EL equations. Setting $t = 0$, we denote the $p^{th}$ order correction
to $v$ by $v^{(p)} := w_{s,t}^{(p)}|_{s=0}$. The next lemma gives a connection between $t$-derivatives of $w_{s,t}^{(q)}$ and the jets $v^{(k)}$.

**Lemma 3.6.** For $q \geq 1$, we have

$$
\partial_t^k w_{s,t}^{(q)}|_{s=t=0} = k! v^{(k)} \delta_{q,k} \quad \text{for } k \geq 0.
$$

**Proof.** We proceed inductively in $q$. The statement holds for $q = 1$ by (3.24). For the induction step $q - 1 \to q$, we expand $w^{(q)}$ according to (3.21), (3.22) and (3.23) as

$$
\partial_t^k w_{s,t}^{(q)}|_{s=t=0} = S^{(q)} \sum_{\ell=2} q \sum_{\ell=1} q_{1}, \ldots, q_{\ell} \geq 1 \quad \text{with } q_{1} + \cdots + q_{\ell} = q
$$

where $\Delta_{\ell}$, using the notation

$$
\nabla^{(q)} := \nabla_{1, w_{s,t}^{(q)}} + \nabla_{2, w_{s,t}^{(q)}},
$$

is given by

$$
\Delta_{\ell} \left[ w_{s,t}^{(q_{1})}, \ldots, w_{s,t}^{(q_{\ell})} \right] (x)
$$

and similarly in the second term in (3.26). The inequalities $q_{1}, \ldots, q_{\ell} \geq 1$ imply that in the last sum we get contributions only if $k_{1}, \ldots, k_{\ell} \geq 1$. Combining the above equations proves the induction step. This concludes the proof. \hfill \Box

**Lemma 3.7.** $w_{s,t}^{(q)}$ as a sum, where each summand contains $q$ factors $w_{s,t}^{(1)}$.

**Proof.** For $q = 1$, it is trivially true since we consider $w_{s,t}^{(1)}$ to be given, cf. (3.21). If it is true for $w_{s,t}^{(q-1)}$, expanding $w^{(q)}$ according to (3.21), (3.22) and (3.23) as

$$
w_{s,t}^{(q)} = S^{(q)} \sum_{\ell=2} q \sum_{\ell=1} q_{1}, \ldots, q_{\ell} \geq 1 \quad \text{with } q_{1} + \cdots + q_{\ell} = q
$$

where $\Delta_{\ell}$ is given by (3.25), it is also true for $w_{s,t}^{(q)}$. \hfill \Box

In the following proposition, we again use the notation $v = (b, v)$. 
Proposition 3.8. Let \( w_{s,t}^{(1)} \) be as in (3.24) and \( w_{s,t}^{(q)} \) be the corresponding non-linear correction terms (3.21)-(3.28). If the resulting family satisfies the regularity assumptions (r1) and (r2), then for every compact \( \Omega \subset M \) and every \( m \geq 1 \), we have
\[
0 = I_m \Omega := \sum_{\ell=0}^{m-1} \frac{(m-1)!}{\ell!} \sum_{q_1, \ldots, q_{\ell+1} \geq 1} \frac{1}{(q_1 - 1)!} \times \left( \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left( \nabla_{1,\partial_c(q_1 - 1)} w_{s,t}^{(q_1)} |_{s+t=0} - \nabla_{2,\partial_c(q_1 - 1)} w_{s,t}^{(q_1)} |_{s+t=0} \right) \times \left( \nabla_{1,\partial_c(q_2) - \nabla_{2,\partial_c(q_2)}} \cdots (\nabla_{1,\partial_c(q_\ell) + \nabla_{2,\partial_c(q_\ell)}}) \mathcal{L}(x, y) \right. \right.
\]
\[
= -\frac{\nu}{2} \int_\Omega \left( \partial_\xi \partial_c^{(q_1)} w_{s,t}^{(q_1)} |_{s+t=0} b(q_2) \cdots b(q_\ell) \ dp(x) \right) \right). \tag{3.27}
\]

Proof. Using Lemma 3.6 and Lemma 3.4 gives
\[
I_{k+1,(p)} \Omega = \sum_{\ell=1}^p \frac{k!}{(\ell - 1)!} \sum_{q_1, \ldots, q_\ell \geq 1} \frac{1}{k_1!} \left( \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y) \times \left( \nabla_{1,\partial_c k_1} w_{s,t}^{(q_1)} - \nabla_{2,\partial_c k_1} w_{s,t}^{(q_1)} \right) \times \left( \nabla_{1,\partial_c(q_2) - \nabla_{2,\partial_c(q_2)}} \cdots (\nabla_{1,\partial_c(q_\ell) + \nabla_{2,\partial_c(q_\ell)}}) \mathcal{L}(x, y) \right. \right.
\]
\[
= -\frac{\nu}{2} \int_\Omega \left( \partial_\xi \partial_c^{k_1(q_1)} w_{s,t}^{(q_1)} |_{s+t=0} b(q_2) \cdots b(q_\ell) \ dp(x) \right) \right). \tag{3.27}
\]
The condition of the second summation implies that
\[
k_1 = k - (q_2 + \ldots + q_\ell) = k - p + q_1 \geq 0.
\]
Lemma 3.7 implies that \( \partial_\xi \partial_c^{k_1} w_{s,t}^{(q_1)} \neq 0 \) only if \( q_1 = k_1 + 1 \). Thus, \( I_{k+1,(p)} \Omega \neq 0 \) only if \( m := k + 1 = p \). Changing the summation parameter from \( \ell \) to \( \tilde{\ell} := \ell - 1 \) gives the result. □

3.3. Corollaries and Discussion. We now discuss the implications of Theorem 3.1 and Proposition 3.8 in different situations. We will see that some of the resulting conservation laws were discovered earlier in [7] and [8]. But we will also find new conservation laws, such as Theorem [3.1] in the introduction. As before, we work with jets denoted by \( u = (a, u) \) and \( v = (b, v) \).

We begin with the simplest case \( m = 1 \). In this case, in (3.27) the dependence on \( v \) drops out and, with the help of (3.24), we obtain
\[
I_1 \Omega (u) = \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left( \nabla_{1,\partial_\xi w_{s,t}^{(1)}} - \nabla_{2,\partial_\xi w_{s,t}^{(1)}} \right) \mathcal{L}(x, y)
\]
\[
= -\frac{\nu}{2} \int_\Omega \partial_\xi c_{s,t}^{(1)}(x) \left. d\rho(x) \right|_{s+t=0}
\]
\[
= \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left( \nabla_{1,u} - \nabla_{2,u} \right) \mathcal{L}(x, y) - \frac{\nu}{2} \int_\Omega a(x) \ dp(x) .
\]
The last integral is not a surface layer integral but a volume term. For this reason, \( I_1 \Omega \) in general does not give rise to a conservation law. But it does if the scalar component of \( u \) vanishes:
Corollary 3.9. (The functional $I_0^\Omega$) Assume that the jet $u$ is a solution of the linearized field equations whose scalar component vanishes, i.e.

$$u = (0, u) \in \mathcal{L}_{\text{lin}}.$$

Then, setting $m = 1$, Theorem 3.1 gives rise to the conserved surface layer integral

$$I_0^\Omega(u) = \int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left( D_{1,u} - D_{2,u} \right) \mathcal{L}(x, y).$$

(3.28)

This conservation law is reminiscent of the Noether-like theorems in [7], cf. Example 3.2. Indeed, if $u$ is an infinitesimal generator of a symmetry of the Lagrangian (3.12), the conservation of $I_0^\Omega(u)$ reduces to these Noether-like theorems. In particular, we thus recover current conservation and the conservation of energy-momentum for Dirac wave functions (see [7] Sections 5 and 6). The conservation law (3.28) goes beyond the Noether-like theorems because it holds without any symmetry assumptions for any solution of the linearized field equations whose scalar component vanishes.

Choosing $m = 2$ in (3.27), and denoting $b^{(1)} = b$ and $v^{(1)} = v$ in accordance with (3.24), we obtain

$$I_2^\Omega(u, v) = \int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left( \nabla_{1,\partial_s \partial_t w^{(2)}_{s,t}} - \nabla_{2,\partial_s \partial_t w^{(2)}_{s,t}} \right) \mathcal{L}(x, y) \bigg|_{s=t=0}
- \frac{\nu}{2} \int_{\Omega} \partial_s \partial_t c^{(2)}_{s,t}(x) \bigg|_{s=t=0} d\rho(x)
+ \int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left( \nabla_{1,\partial_s w^{(1)}_{s,t}} - \nabla_{2,\partial_s w^{(1)}_{s,t}} \right) \left( \nabla_{1,v} + \nabla_{2,v} \right) \mathcal{L}(x, y) \bigg|_{s=t=0}
- \frac{\nu}{2} \int_{\Omega} \partial_s c^{(1)}_{s,t}(x) b(x) d\rho(x) \bigg|_{s=t=0}
= \int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left( \nabla_{1,\partial_s \partial_t w^{(2)}_{s,t}} - \nabla_{2,\partial_s \partial_t w^{(2)}_{s,t}} \right) \mathcal{L}(x, y) \bigg|_{s=t=0}
- \frac{\nu}{2} \int_{\Omega} \partial_s \partial_t c^{(2)}_{s,t}(x) \bigg|_{s=t=0} d\rho(x)
+ \int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left( \nabla_{1,u} - \nabla_{2,u} \right) \left( \nabla_{1,v} + \nabla_{2,v} \right) \mathcal{L}(x, y)
- \frac{\nu}{2} \int_{\Omega} a(x) b(x) d\rho(x),$$

(3.29)

where the term $\partial_s \partial_t w^{(2)}_{s,t}$ is given in second order perturbation theory by

$$\partial_s \partial_t w^{(2)}_{s,t} \bigg|_{s=t=0} = S^{(2)} \partial_s \partial_t \Delta_2 \left[ w^{(1)}_{s,t}, w^{(1)}_{s,t} \right] \bigg|_{s=t=0} = 2 S^{(2)} \Delta_2 [u, v],$$

(3.30)

which is symmetric in $u$ and $v$ in light of (2.14). Anti-symmetrizing in $u$ and $v$, the second order term as well as the term involving $\nu$ drop out.

Corollary 3.10. (The functional $I_2^\Omega$ anti-symmetrized) Proposition 3.8 gives rise to the conserved surface layer integral

$$\int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left( \nabla_{1,u} \nabla_{2,v} - \nabla_{1,v} \nabla_{2,u} \right) \mathcal{L}(x, y).$$

(3.31)

This is precisely the symplectic form as found in [8] (cf. [8] Section 3.3)).
We next symmetrize $I^2_\Omega$ in its arguments $u$ and $v$. Then (3.29) simplifies to
\[
\frac{1}{2}(I^2_\Omega(u,v) + I^2_\Omega(v,u)) = \int_\Omega dp(x) \int_{\Omega(M)} dp(y) \left( \nabla_{1,u} \nabla_{1,v} + 2 \nabla_{1,S^{(2)}_\Omega[u,v]} \right)\]
\[
- \nabla_{2,u} \nabla_{2,v} - 2 \nabla_{2,S^{(2)}_\Omega[u,v]} \right) \mathcal{L}(x,y) - \int_\Omega \left( a(x) b(x) + 2 \nabla_{S^{(2)}_\Omega[u,v]} \right) \frac{\nu}{2} dp(x) .
\]

This expression vanishes according to Proposition 3.8. We thus obtain Theorem 1.1 in the introduction, where we replaced the symbol $S^{(2)}_\Omega$ by $S^{(2)}$ and used that, according to (2.11),
\[
(\nabla_{1,u} \nabla_{1,v} \frac{\nu}{2})(x) = a(x) b(x) \frac{\nu}{2} .
\]

As in the above example for $m = 1$, the last integral in (3.32) is a volume term. In order to obtain a conservation law, we need to assume that the scalar component of at least one jet vanishes, i.e.
\[
u = (0, u) \quad \text{or} \quad v = (0, v) .
\]

Moreover, we need to assume that the Green’s operator gives no scalar component, i.e.
\[
S^{(2)}_\Omega[u,v] = (0, w) \quad \text{with} \quad w \in \Gamma .
\]

Under these additional assumptions, we obtain:

**Corollary 3.11.** (The functional $I^2_\Omega$ symmetrized) If (3.33) and (3.34) hold, Proposition 3.8 gives rise to the conserved surface layer integral
\[
\int_\Omega dp(x) \int_{\Omega(M)} dp(y) \left( D_{1,u} D_{1,v} + 2 \nabla_{1,S^{(2)}_\Omega[u,v]} \right)\]
\[
- D_{2,u} D_{2,v} - 2 \nabla_{2,S^{(2)}_\Omega[u,v]} \right) \mathcal{L}(x,y) .
\]

It is worth noting that, in contrast to the symplectic form of Corollary 3.10, the conserved surface layer integral involves a nonlinear correction term.

### 3.4. Dependence on the Green’s Operators.

According to Proposition 3.8 and (3.21), the functionals $I^2_m$ depend on the choice of the Green’s operators $S^{(p)}$. At first sight, this freedom might be surprising, because conserved physical quantities like currents or energy-momentum are canonically defined and do not involve any arbitrariness. In this section, we will analyze what this arbitrariness means. In short, the answer is that different choices of the Green’s operators correspond to taking different linear combinations of conserved quantities. Before working out this statement systematically, we give a simple example.

**Example 3.12.** We return to the functional $I^2_\Omega$ as considered in the above Corollaries 3.10 and 3.11. Clearly, this functional depends on the choice of the Green’s operator $S^{(2)}$, because the jet $w^{(2)}_{s,t}$ in (3.29) is given by
\[
w^{(2)}_{s,t} = (c^{(2)}_{s,t}, w^{(2)}_{s,t}) = S^{(2)}_\Omega(w^{(1)}_{s,t}, w^{(1)}_{s,t}) .
\]

As explained after Definition 3.5, Green’s operators are unique only up to solutions of the linearized field equations. Therefore, we can transform $S^{(2)}$ according to
\[
S^{(2)} \to S^{(2)} + K^{(2)} ,
\]
where \( K^{(2)} : (J^\text{test})^* \to J_{\text{lin}} \). Then, according to (3.29) and (3.30),
\[
I_2^\Omega(u, v) \to I_2^\Omega(u, v) - 2 \int_\Omega \nabla K^{(2)} \Delta_2[u, v] \, \frac{\nu}{2} \, d\rho(x)
\]
\[
+ 2 \int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) \left( \nabla_{1,K^{(2)}} \Delta_2[u, v] - \nabla_{2,K^{(2)}} \Delta_2[u, v] \right) \mathcal{L}(x, y)
\]
\[
= I_2^\Omega(u, v) + 2 I_1^\Omega \left( K^{(2)} \Delta_2[u, v] \right).
\]

Therefore, the functional \( I_2^\Omega \) is modified by the functional \( I_1^\Omega \), evaluated for the linearized solution \( K^{(2)} \Delta_2[u, v] \). In this way, we obtain an interesting and surprising relation between different conservation laws.

We next analyze the dependence on the Green’s operators systematically. In order to make the dependence of \( E^{(p)} \) on \( w^{(1)}_{s,t} \) explicit, we also write the \( p \)th order contribution \( 3.23 \) as
\[
E^{(p)} = E^{(p)} \left[ w^{(1)}_{s,t} \right];
\]
and note that \( E^{(p)} \) is \( p \)-linear in \( w^{(1)}_{s,t} \) (cf. Lemma 3.7).

**Theorem 3.13.** Changing the Green’s operator of order \( q \in \{2, 3, \ldots\} \) by
\[
S^{(q)} \to S^{(q)} + \lambda K^{(q)} \quad \text{with} \quad K^{(q)} : (J^\text{test})^* \to J_{\text{lin}},
\]
the functional \( I_m^\Omega \) in Proposition 3.8 transforms to
\[
I_m^\Omega(u, v) \to \sum_{0 \leq r \leq m-1} \frac{1}{r!} \frac{d^r}{d\lambda^r} I_m^{\Omega(r-q-1)} \left[ \partial_\lambda \left( w^{(1)}_{s,t} + \lambda K^{(q)} E^{(q)}[w^{(1)}_{s,t}] \right) \right],
\]
\[
= \left. w^{(1)}_{s,t} + \lambda K^{(q)} E^{(q)}[w^{(1)}_{s,t}] \right|_{\lambda=s=t=0} \lambda^r. \quad (3.35)
\]

**Proof.** We first point out that each \( \lambda \)-derivative in (3.35) decreases the lower index of \( I^\Omega \) by \( q - 1 \). This can be understood from the fact that for each such derivative, we replace \( q \) factors \( w^{(1)}_{s,t} \) by one linearized solution \( K^{(q)} E^{(q)}[w^{(1)}_{s,t}] \), thus decreasing the number of arguments of \( I^\Omega \) by \( q - 1 \). In order to simplify the notation, we use the convention that \( I^\Omega \) is zero when the lower index is negative or zero. In this way, the sum in (3.35) can be extended to all \( r \in \mathbb{N}_0 \). Similarly, we use the convention that \( E^{(p)} \) vanishes if \( p \) is negative.

Using the product rule and the binomial formula, one sees that the operation in (3.35) is multiplicative, i.e.
\[
\sum_{r=0}^\infty \frac{1}{r!} \frac{d^r}{d\lambda^r} \left( f(\lambda) \, g(\lambda) \right) \bigg|_{\lambda=0} \lambda^r = \left( \sum_{r=0}^\infty \frac{1}{r!} \frac{d^r}{d\lambda^r} f(\lambda) \bigg|_{\lambda=0} \lambda^r \right) \left( \sum_{r=0}^\infty \frac{1}{r!} \frac{d^r}{d\lambda^r} g(\lambda) \bigg|_{\lambda=0} \lambda^r \right).
\]

With this in mind, it suffices to consider the individual terms of the perturbation expansion. In order to further simplify the notation, we use the abbreviation \( K := K^{(q)} E^{(q)} \in J_{\text{lin}} \). Then it remains to show that the contributions \( E^{(p)}_e \) in (3.23) transform as
\[
E^{(p)}_e \left[ w^{(1)}_{s,t} \right] \to \sum_{r=0}^\infty \frac{1}{r!} \frac{d^r}{d\lambda^r} E^{(p-r(q-1))} \left[ w^{(1)}_{s,t} + \lambda K \right] \bigg|_{\lambda=0} \lambda^r. \quad (3.36)
\]
Then, replacing one Green’s function $S(q)$ in (3.23) by a factor $K(q)$ gives
\[
E_{\ell}^{(p)}[\omega_{s,t}^{(1)}] \to \ell \sum_{q_2, \ldots, q_{\ell} \geq 1 \text{ with } q_2 + \cdots + q_{\ell} = p-q} \Delta_{Q}[\nu_{s,t}^{(q_2)} \ldots, \nu_{s,t}^{(q_{\ell})}](x).
\]

If $r$ Green’s functions $S(q)$ are replaced by corresponding factors $K(q)$, one obtains inductively
\[
E_{\ell}^{(p)}[\omega_{s,t}^{(1)}] \to \left(\ell \atop r\right) \sum_{q_{r+1}, \ldots, q_{\ell} \geq 1 \text{ with } q_{r+1} + \cdots + q_{\ell} = p-rq} \Delta_{Q}[\nu_{s,t}^{(q_{r+1})} \ldots, \nu_{s,t}^{(q_{\ell})}](x)
\]
\[
= \frac{1}{r!} \frac{d^r}{d\lambda^r} E_{\ell}^{(p-r(q-1))}[\nu_{s,t}^{(1)} + \lambda K]_{|\lambda=0},
\]
where in the last line we used that the $\lambda$-derivatives give a combinatorial factor $l!/(l-r)!$. The order in perturbation theory $p-r(q-1)$ can be understood from the fact that, being a solution of the linearized field equations, each factor $K$ now counts as a first order perturbation (although it is $q$-linear in $\nu_{s,t}^{(1)}$). Multiplying by $\lambda^r$ and adding all contributions, we obtain (3.36). This concludes the proof. \qed

4. Example: A Lattice System

We now illustrate the previous constructions in a detailed example first considered in [8] Section 5. In this example, the minimizing measure is supported on a two-dimensional lattice in Minkowski space. The dynamical degrees of freedom are an $S^1$-valued field on the lattice. The Euler-Lagrange equations for this field give the discrete wave equation on the lattice.

After recalling the definition of the model, we construct the Green’s operators. Computing the surface layer integrals of Corollaries 3.9 and Theorem 1.1 we find that $I_1$ vanishes identically, whereas $I_2$ gives rise to a non-trivial conserved quantity (the symplectic form in Corollary 3.10 was already computed in [8] Section 5.4).

4.1. Definition. Let $(\mathbb{R}^{1,1}, \langle , \rangle)$ be two-dimensional Minkowski space. Denoting two space-time points by $x = (x^0, x^1)$ and $y = (y^0, y^1)$, the inner product takes the form
\[
\langle x, y \rangle = x^0 y^0 - x^1 y^1.
\]

Let $\mathcal{F}$ be the set
\[
\mathcal{F} = \mathbb{R}^{1,1} \times S^1.
\]

We denote points in $x \in \mathcal{F}$ by $x = (\underline{x}, x^0)$ with $\underline{x} \in \mathbb{R}^{1,1}$ and $x^0 \in [-\pi, \pi]$. Next, we let $A$ be the square
\[
A = (-1, 1)^2 \subset \mathbb{R}^{1,1}.
\]

Moreover, given $\varepsilon \in (0, \frac{1}{4})$, we let $I$ be the the following subset of the interior of the light cones,
\[
I = \{ \underline{x} \in \mathbb{R}^{1,1} \mid \langle x, x \rangle > 0 \text{ and } |x^0| < 1 + \varepsilon \}.
\]

Let $f : \mathbb{R}^{1,1} \to \mathbb{R}$ be the function
\[
f(\underline{x}) = \chi_{B_{\varepsilon}(0,1)}(\underline{x}) + \chi_{B_{\varepsilon}(0,-1)}(\underline{x}) - \chi_{B_{\varepsilon}(1,0)}(\underline{x}) - \chi_{B_{\varepsilon}(-1,0)}(\underline{x}),
\]
where $\chi$ is the characteristic function and $B_\varepsilon(x^0, x^1)$ denotes the open Euclidean ball of radius $\varepsilon$ centred around $(x^0, x^1) \in \mathbb{R}^2 \simeq \mathbb{R}^{1,1}$. Finally, we let $V : S^1 \to \mathbb{R}$ be the function
\[
V(\varphi) = 1 - \cos \varphi .
\]
Given parameters $\delta > 0$, $\lambda_I \geq 2$ and
\[
\lambda_A \geq 2\lambda_I + \varepsilon ,
\]
(4.1)
the Lagrangian $\mathcal{L}$ is defined by
\[
\mathcal{L}(x, y) = \lambda_A \chi_A(x - y) + \lambda_I \chi_I(x - y) + V(x^\varphi - y^\varphi) f(x - y) \\
+ \delta \chi_{B_\varepsilon(0,0)}(x - y) V(x^\varphi - y^\varphi)^2 .
\]
(4.2)
The following results have been proven in [8, Section 5].

**Lemma 4.1.** [8, Lemma 5.1] The function $\mathcal{L}(x, y)$ is non-negative and satisfies the conditions (i) and (ii) on page 4.

We introduce a measure $\rho$ supported on the unit lattice $\mathcal{Z} := \mathbb{Z}^2 \subset \mathbb{R}^{1,1} \subset \mathcal{F}$ by
\[
\rho = \sum_{x \in \mathcal{Z}} \delta_{(x,0)} ,
\]
(4.3)
where $\delta_{(x,x^\varphi)}$ denotes the Dirac measure at $(x, x^\varphi) \in \mathcal{F}$.

**Lemma 4.2.** [8, Lemma 5.2] The measure $\rho$ satisfies the condition (iii) on page 4.

Clearly, the support of $\rho$ is given by
\[
M := \text{supp } \rho = \mathcal{Z} \times \{0\} \subset \mathcal{F} .
\]
(4.4)

**Lemma 4.3.** [8, Lemma 5.3] The measure $\rho$ satisfies the EL equations (2.4) if the parameter $\nu$ in (2.3) is chosen as
\[
\nu = 2\lambda_A + 4\lambda_I .
\]
(4.5)

We remark that the measure $\rho$ is a local minimizer of the causal action as defined in [8, Proposition 4.10]; see [8, Corollary 5.5].

4.2. The Jet Spaces. We next determine the jet spaces. Recall that in (2.5), the smooth vector fields on $M$ were defined as those vector fields which can be extended smoothly to $\mathcal{F}$. In our setting of a discrete lattice (4.4), this is the case for every vector field. Thus the jet space (2.6) can be written as
\[
\mathcal{J}|_M = \{ u = (a, u) \text{ with } a : M \to \mathbb{R} \text{ and } u : M \to T\mathcal{F} \} .
\]
When extending these jets to $\mathcal{F}$, for convenience we always choose extensions which are locally constant on $M$. We denote the vector component by $u = (u^0, u^1, u^\varphi)$. In order to determine the differentiable jets, we recall from the the proof of [8, Lemma 5.3] that
\[
\ell(x, x^\varphi) = \delta V(x^\varphi)^2 \quad \text{if } x \in \mathcal{Z} \\
\ell(x, x^\varphi) \geq \lambda_A + \delta V(x^\varphi)^2 \quad \text{if } x \notin \mathcal{Z} ,
\]
where $\ell$ is defined as in (2.3). Hence the differentiable jets (2.7) are given by
\[
\mathcal{J}^{\text{diff}} = \{ u = (a, u) \text{ with } a : M \to \mathbb{R} \text{ and } u = (0,0, u^\varphi) : M \to T\mathcal{F} \} .
\]
(4.6)
We choose
\[
\mathcal{J}^{\text{test}} = \mathcal{J}^{\text{diff}} ,
\]
which implies that
\[(\mathcal{J}^\text{test})^* = \{ \mathbf{v} = (b, v) \text{ with } b : M \to \mathbb{R} \text{ and } v = (0, 0, v^\varphi) : M \to T^* \mathcal{F} \} .\]

The proof of [8, Proposition 5.6] shows that \( \mathbf{v} = (b, v) \in \mathcal{J}^1 \) as in Definition 2.3 consist of an arbitrary scalar component \( b \) and the vector component
\[ v(x) = \left( v, v^\varphi(x) \right) , \]
where \( v \) is a constant vector in \( \mathbb{R}^{1,1} \) and \( v^\varphi : M \to \mathbb{R} \) is an arbitrary function. Since the derivative in direction of the constant vector vanishes, the resulting term reads
\[ \left( \nabla_{1,v} + \nabla_{2,v} \right) \mathcal{L}(x, y) = \left( b(x) + b(y) + v^\varphi(x) \partial_{x^\varphi} + v^\varphi(y) \partial_{y^\varphi} \right) \mathcal{L}(x, y) . \]

Our convention (2.11) implies that higher derivatives in (2.10) act on \( \mathcal{L}(x, y) \) but not on \( b \) and \( v^\varphi \). Since \( \mathcal{L}(x, y) \) is smooth in \( x^\varphi \) and \( y^\varphi \), it follows that the conditions for \( \mathbf{v} \) being an element of \( \mathcal{J}^\ell \) for \( \ell > 1 \) are not stronger than for the case \( \ell = 1 \). Hence we conclude that
\[ \mathcal{J}^l = \mathcal{J}^1 \quad \text{for all } l \in \mathbb{N}_0 \cup \{ \infty \} . \]

Finally, the solutions of the linearized field equations (3.15) are characterized in the following proposition. We use the notation \( e_t := (1, 0) \in \mathbb{R}^{1,1} \).

**Proposition 4.4.** [8, Proposition 5.6] The linearized solutions \( \mathcal{J}_\text{lin} \) of Definition 2.3 consist of all jets \( \mathbf{v} = (b, v) \in \mathcal{J}^\infty \) with the following properties:

(A) The scalar component \( b : M \to \mathbb{R} \) satisfies the equation
\[ \lambda_A b(x, 0) + \lambda_I \left( b(x + e_t, 0) + b(x - e_t, 0) \right) = 0 . \]

(B) The vector component \( v : M \to T^* \mathcal{F} \) consists of a constant vector \( \mathbf{v} \in \mathbb{R}^{1,1} \) and a function \( v^\varphi : M \to \mathbb{R} \), i.e.
\[ v(x) = \left( v, v^\varphi(x) \right) , \quad (4.7) \]
where the function \( v^\varphi \) satisfies the discrete wave equation on \( Z \),
\[ \sum_{y \in Z} f(x - y) v^\varphi(y, 0) = 0 . \quad (4.8) \]

### 4.3. The Green’s Operators.

In this section, we determine the Green’s operators according to Definition 3.5 and analyze the regularity assumptions (r1) and (r2) on page 7. For \( \mathbf{u} = (a, u) \in \mathcal{J}^\text{test} \) and \( \mathbf{v} = (b, v) \in \mathcal{J}^\infty \), according to (3.13) and (3.14) the linearized field equations read
\[ \langle u, \Delta v \rangle|_x = \left( \lambda_A + 2\lambda_I \right) a(x) b(x) + \lambda_A a(x) b(x) + \lambda_I \left( a(x) b(x + e_t) \right) \]
\[ + \lambda_I \left( a(x) b(x - e_t) \right) - \sum_{y \in Z} v^\varphi(y) f(x - y) - a(x) b(x) \frac{\nu}{2} . \quad (4.9) \]

Hence, denoting the scalar and vector components of \( S \mathbf{u} \) by \( (sb, sv) \) and using (4.5), the defining equation for the Green’s operator (3.20) reads
\[ \lambda_A a(x) sb(x) + \lambda_I \left( a(x) sb(x + e_t) + a(x) sb(x - e_t) \right) \]
\[ - \sum_{y \in Z} sv^\varphi(y) f(x - y) \stackrel{!}{=} -\langle u, \mathbf{v} \rangle|_x = -a(x) b(x) - u^\varphi(x) v^\varphi(x) . \quad (4.10) \]

for all \( \mathbf{u} \in \mathcal{J}^\text{test} \) and \( \mathbf{v} \in (\mathcal{J}^\text{test})^* \). Choosing \( \mathbf{u} = (a, 0) \in \mathcal{J}^\text{test} \) with arbitrary \( a \), we have
\[ \lambda_A sb(x) + \lambda_I \left( sb(x + e_t) + sb(x - e_t) \right) = -b(x) . \quad (4.11) \]
Choosing $u = (0, u) \in J^\text{test}$ with arbitrary $u$, (4.10) yields

$$\sum_{y \in \mathbb{Z}} s v(y) f(x - y) = v(x).$$

(4.12)

Thus, we have found that in our example, the Green’s operator (3.20) does not mix the scalar and vector components of $(J^\text{test})^\ast$. It may give rise to a constant component $(0, S v^0, S v^1, 0) \in J^\infty$ which however drops out of (4.9) since $\Delta$ maps constant components of $J^\infty$ to zero. We use the freedom in the choice of Green’s operators (cf. Section 3.4) to arrange that $S v^0 = S v^1 = 0$.

The vector component of a Green’s operator according to (4.12) is a Green’s operator of the discrete wave equation (as usual, one can choose for example the advanced or retarded Green’s operator). The scalar component of the Green’s operator, on the other hand, can be computed by a discrete Fourier transformation. Since in (4.11) all functions are evaluated for the same value of $x^1$, we can solve this equation for any fixed $x^1$ and omit this variable. For the dependence on $x^0$ we employ the plane-wave ansätze

$$sb(x^0) = a(\omega) e^{-i \omega x^0} + a(\omega) e^{i \omega x^0}, \quad b(x^0) = b(\omega) e^{-i \omega x^0} + b(\omega) e^{i \omega x^0}$$

with frequency $\omega \in [0, \pi]$. We thus obtain the equation

$$a(\omega) = - \frac{b(\omega)}{\lambda A + 2 \lambda I \cos \omega}.$$ 

In view of (4.1), the denominator is strictly positive, giving a unique solution $a(\omega)$. Hence the scalar component of the Green’s operator is given by

$$s(\omega) = - \left( \lambda A + 2 \lambda I \cos \omega \right)^{-1}.$$ 

(4.13)

Next, we evaluate the regularity conditions (r1) and (r2) on page 7.

**Lemma 4.5.** The conditions (r1) and (r2) are satisfied for the families generated by linearized solutions $u, v \in J^\text{lin}$ if $u, v$ have vanishing constant vector component (cf. Proposition 4.4),

$$u = 0 = v$$

(4.14)

and if we choose

$$J^\text{test}_{s, t} = J^\text{diff}_{s, t},$$

(4.15)

for all $s, t$.

**Proof.** Let $u = (a, u), v = (b, v) \in J^\text{lin}$ with $u$ and $v$ as in (4.7) such that (4.14) holds. Define $w_{s, t}^{(1)}$ as in (3.24). Since, by definition of $S$, its image is in $J^\infty$, it follows that all non-linear corrections $w_{s, t}^{(2)}, w_{s, t}^{(3)}$, . . . are in $J^\infty$. Furthermore, by the above choice of $S$ concerning constant components $(v^0, v^1)$, the non-linear correction terms have vanishing constant components as well. It follows that (3.19) consists of a scalar component and a vector component in direction of $v^\varphi$. Thus the resulting family (3.11) consists of functions $f$ and $F$ as described in (3.2), but where $F$ is constant except for the $S^1$ direction. Together with the smoothness of the Lagrangian (4.2) in the $S^1$-directions $x^\varphi$ and $y^\varphi$, this implies that all partial derivatives in (3.6) exist. Furthermore, those partial derivatives are $\rho$-integrable since by the choice of $\rho$ in (4.3) and the bounded support of $\mathcal{L}(x, \cdot), \rho$-integration gives rise to a finite sum. Therefore, the partial derivatives commute with integration, so that (r1) holds.
The condition (r2) holds because, by the property of \( F \) mentioned in the last paragraph, the jet \( u(t) \) as defined in (3.7) is an element of \( J^{\text{diff}} \) (given by (4.6)) and hence, due to the choice (4.15), also is an element of \( J_0^{\text{test}} \).

4.4. The Conserved Quantities. We are now in a position to study the corollaries to our main theorem discussed in Section 3.3.

4.4.1. The Functional \( I_1^\Omega \). We consider Corollary \ref{cor:4.9}. Thus let \( v \in J_{\text{lin}} \) have vanishing scalar component, i.e. \( v = (0, v) \) with \( v \) as in (4.7) and \( v = 0 \). The corresponding derivative reads

\[
D_{1,v} \mathcal{L}(x, y) \bigg|_{x^\varphi = 0 = y^\varphi} = -D_{2,v} \mathcal{L}(x, y) = v^\varphi(x) \frac{d}{dx^\varphi} \left( V(x^\varphi - y^\varphi) f(x - y) \right) + \delta \chi_{B_e(0,0)}(x - y) V(x^\varphi - y^\varphi)^2 \bigg|_{x^\varphi = 0 = y^\varphi} = 0.
\]

Thus \( I_1^\Omega(v) \), as defined in (3.28), vanishes identically for any \( \Omega \subset M \).

4.4.2. The Functional \( I_2^\Omega \) Anti-Symmetrized. Next, we consider Corollary \ref{cor:4.10}. Define

\[
N_t = \{ x \in Z : x^0 = t \} \quad \text{with} \quad t \in \mathbb{Z}
\]

and let \( \Omega_{N_t} \) be the past of \( N_t \), i.e.

\[
\Omega_{N_t} = \{ x \in Z : x^0 \leq t \}.
\]

As noted in before, (3.31) is exactly the symplectic form \( \sigma_{\Omega}(u, v) \) derived in \cite{8}. Thus we have:

**Proposition 4.6.** [8, Proposition 5.8] If we choose \( J^{\text{test}} \) as the jets with spacelike compact support,

\[
J^{\text{test}} = \{ u \in J^{\text{diff}} : \text{supp} u|_{N_t} \text{ is a finite set for all } t \in \mathbb{Z} \},
\]

the conserved functional (3.31) is given by

\[
\sigma_{\Omega_{N_t}}(u, v) = \lambda I \sum_{x \in N_t} \left( a(x) b(x + e_t) - a(x + e_t) b(x) \right) + \sum_{x \in N_t} \left( u^\varphi(x + e_t) v^\varphi(x) - u^\varphi(x) v^\varphi(x + e_t) \right),
\]

where again \( e_t := (1, 0) \in \mathbb{R}^{1,1} \).

Note that the second sum (4.17) is the usual symplectic form associated to a discrete version of the wave equation on \( \mathbb{R}^{1,1} \), cf. Remark 5.9 in \cite{8}.

4.4.3. The Functional \( I_2^\Omega \) Symmetrized. Finally, we consider Corollary \ref{cor:4.11} respectively Theorem \ref{thm:1.1}. We first, we determine \( \Delta_2[u, v] \). For simplicity, we consider the case \( \ref{cor:3.3} \) of vanishing scalar components.

**Proposition 4.7.** If \( u = (0, u), v = (0, v) \in J_{\text{lin}} \), the scalar component of \( \Delta_2[u, v] \) is given by the function

\[
\Delta_2[u, v](x) = \frac{1}{2} \sum_{y \in Z} v^\varphi(y) u^\varphi(y) f(x - y),
\]

whereas the vector component vanishes.
Proof. Equation (3.13) gives
\[
\Delta_2[u, v](x) = \frac{1}{2} \left( \sum_{y \in M} \left( \nabla_{1,u} + \nabla_{2,u} \right) \left( \nabla_{1,v} + \nabla_{2,v} \right) \mathcal{L}(x, y) - \nu \frac{1}{2} a(x) b(x) \right)
\]
\[
= \frac{1}{2} \sum_{y \in M} \left( D_{1,v} + D_{2,v} \right) \left( D_{1,u} + D_{2,u} \right) \mathcal{L}(x, y)
\]
\[
= \frac{1}{2} \sum_{y \in M} \left( v^\varphi(x) \frac{d}{dx^\varphi} + v^\varphi(y) \frac{d}{dy^\varphi} \right) \left( u^\varphi(x) \frac{d}{dx^\varphi} + u^\varphi(y) \frac{d}{dy^\varphi} \right) \mathcal{L}(x, y)
\]
\[
= \frac{1}{2} \sum_{y \in M} \left( v^\varphi(x) u^\varphi(x) \frac{d^2}{dx^\varphi dy^\varphi} + v^\varphi(x) u^\varphi(y) + v^\varphi(y) u^\varphi(x) \frac{d}{dx^\varphi} \frac{d}{dy^\varphi} \right.
\]
\[\left. + v^\varphi(y) u^\varphi(y) \frac{d^2}{dy^\varphi dy^\varphi} \right) \mathcal{L}(x, y),
\]
where we used again that the jets are locally constant. The second derivative of the Lagrangian (4.2) reads
\[
\frac{d^2}{dx^\varphi dy^\varphi} \mathcal{L}(x, y) = f(x - y) \cos(x^\varphi - y^\varphi) + \delta \chi_{B_r(0,0)}(x - y)
\]
\[\cdot \left( 2 \sin(x^\varphi - y^\varphi)^2 + 2 \left( 1 - \cos(x^\varphi - y^\varphi) \right) \cos(x^\varphi - y^\varphi) \right).\]
Evaluated for \(x, y \in M\), this gives
\[
\frac{d^2}{dx^\varphi dy^\varphi} \mathcal{L}(x, y) \bigg|_{x^\varphi = y^\varphi} = f(x - y).
\]
For \(x \in M\), we thus have
\[
\Delta_2[u, v](x) = \frac{1}{2} \sum_{y \in Z} \left( v^\varphi(x) u^\varphi(x) - v^\varphi(x) u^\varphi(y) - v^\varphi(y) u^\varphi(x) + v^\varphi(y) u^\varphi(y) \right) f(x - y)
\]
\[
= \frac{1}{2} \sum_{y \in Z} \left( v^\varphi(y) u^\varphi(y) - v^\varphi(x) u^\varphi(y) - v^\varphi(y) u^\varphi(x) \right) f(x - y),
\]
where in the second step we have used that
\[
\sum_{y \in Z} f(x - y) = 0.
\]
According to Proposition 4.3, \(v^\varphi\) and \(u^\varphi\) satisfy the discrete wave equation (4.8). Thus the last two terms in (4.20) vanish, giving
\[
\Delta_2[u, v](x) = \frac{1}{2} \sum_{y \in Z} v^\varphi(y) u^\varphi(y) f(x - y).
\]
According to (3.14), for \(w = (1, 0) \in \mathcal{J}^{test}\), the scalar component is given by
\[
\nabla_w \Delta_2[u, v](x) = \Delta_2[u, v](x),
\]
proving the first claim of the proposition. In order to determine the vector component, we evaluate (4.19) for \( y \in M \) but \( x \) arbitrary,
\[
\frac{d^2}{d(x^\varphi)^2} \mathcal{L}(x, y) \big|_{0=y^\varphi} = f(x - y) \cos(x^\varphi) + \delta \chi_{B_{x}(0,0)}(x-y) \cdot \left(2 \sin(x^\varphi)^2 + 2(1-\cos(x^\varphi))\cos(x^\varphi)\right),
\]
which implies that
\[
\frac{d^3}{d(x^\varphi)^3} \mathcal{L}(x, y) \big|_{0=y^\varphi} = -f(x - y) \sin(x^\varphi) + \delta \chi_{B_{x}(0,0)}(x-y) \times \left(4 \sin(x^\varphi)\cos(x^\varphi) + 2 \sin(x^\varphi)\cos(x^\varphi) - 2(1-\cos(x^\varphi))\sin(x^\varphi)\right)
\]
and
\[
\frac{d^3}{d(x^\varphi)^3} \mathcal{L}(x, y) \big|_{x^\varphi=0=y^\varphi} = 0.
\]
Since \( \Delta_2 \) takes values in \((J^\text{test})^*\), its vector component merely has a \( S^1\)-component, which we denote as \((\Delta_2[u,v])^\varphi(x)\). It is given by
\[
(\Delta_2[u,v])^\varphi(x) = \frac{d}{dx^\varphi}(\Delta_2[u,v])
\]
\[
= \frac{d}{dx^\varphi} \sum_{y \in M} \left( \frac{d^2}{d(x^\varphi)^2} \frac{d^2}{d(y^\varphi)^2} + \frac{d}{d(x^\varphi)} \frac{d}{d(y^\varphi)} \right) \mathcal{L}(x,y).
\]
Since the jets are locally constant, all derivatives acts only on \( \mathcal{L}(x,y) \). But (4.21) implies the resulting third derivatives of \( \mathcal{L}(x,y) \) vanish on \( M \). Thus we have
\[
(\Delta_2[u,v])^\varphi(x) = 0 \quad \text{for} \ x \in M,
\]
proving the second claim of the proposition. \( \square \)

In the following, we abbreviate the scalar Green’s operator \((4.13)\) applied to the scalar part of \( \Delta_2[u,v] \), given in \((4.18)\), with
\[
s(x) := s(\Delta_2[u,v])(x).
\]

**Proposition 4.8.** Assume that jets \( u, v \in J_{\text{lin}} \) have vanishing scalar component, that the conditions \((4.14)\) and \((4.15)\) hold, and that the regularity assumptions \((r1)\) and \((r2)\) are satisfied. Then, choosing \( \Omega \) as in \((4.16)\), the bilinear form \((.,.)_\Omega\) of Theorem 1.1 is given by
\[
(u|v)_{\Omega_{N_t}} = \sum_{x \in N_{t+1}} \left( u^\varphi(x) v^\varphi(x) - 2 \lambda_I s(x) \right)
- \sum_{x \in N_t} \left( u^\varphi(x) v^\varphi(x) - 2 \lambda_I s(x) \right)
\]
and satisfies the relation
\[
(u|v)_{\Omega_{N_t}} = \nu \sum_{x \in \Omega_{N_t}} s(x).
\]
Proof. According to Section 4.3, the Green’s operators do not mix scalar and vector components. Proposition 4.7 implies that the vector component of \( S \Delta_2 [u, v] \) vanishes, giving for the bilinear form \( (u|v)_\Omega \)

\[
(u|v)_\Omega = \int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) \left( D_{1,u}D_{1,v} - D_{2,u}D_{2,v} + 2s(x) - 2s(y) \right) \mathcal{L}(x, y)
\]

\[
= \sum_{x\in\Omega} \sum_{y\in M\setminus\Omega} \left( u^\varphi(x) v^\varphi(x) \partial_{x^\varphi}^2 - u^\varphi(y) v^\varphi(y) \partial_{y^\varphi}^2 + 2s(x) - 2s(y) \right) \mathcal{L}(x, y)
\]

\[
= \sum_{x\in\Omega} \sum_{y\in M\setminus\Omega} \left( u^\varphi(x) v^\varphi(x) - u^\varphi(y) v^\varphi(y) \right) f(x - y)
\]

\[
+ 2 \sum_{x\in\Omega} \sum_{y\in M\setminus\Omega} \left( s(x) - s(y) \right) \mathcal{L}(x, y),
\]

where we have again used that jets are locally constant. Choosing again \( \Omega = \Omega_N \), as in \((1.16)\), the second to last term gives

\[
\sum_{x\in N_t} \sum_{y\in N_{t+1}} \left( u^\varphi(x) v^\varphi(x) - u^\varphi(y) v^\varphi(y) \right) f(x - y)
\]

\[
= \sum_{x\in N_{t+1}} u^\varphi(x) v^\varphi(x) - \sum_{x\in N_t} u^\varphi(x) v^\varphi(x),
\]

whereas the last term gives

\[
2 \sum_{x\in\Omega_{N_t}} \sum_{y\in M\setminus\Omega_{N_t}} \left( \lambda_A \chi_A(x - y) + \lambda_I \chi_I(x - y) \right)
\]

\[
= 2 \lambda_I \sum_{x\in\Omega_{N_t}} \sum_{y\in M\setminus\Omega_{N_t}} \left( s(x) - s(y) \right) \chi_I(x - y)
\]

\[
= 2 \lambda_I \left( \sum_{x\in N_t} s(x) - \sum_{x\in N_{t+1}} s(x) \right),
\]

giving \((1.22)\). For general compact \( \Omega \), the right hand side of \((1.4)\) is given by

\[
\int_\Omega s(x) \nu d\rho(x) = \nu \sum_{x\in\Omega} s(x),
\]

giving \((4.23)\).

\[\square\]

**Appendix A. Combinatorics of the Perturbation Expansion**

We now give proof of Lemma 3.4. In order to facilitate reading, we give the details of the combinatorics.
Proof of Lemma 3.4. Our task is to expand the functional \( I_m^\Omega \) in Theorem 3.1 in powers of \( \lambda \). We first note that (3.18) implies that

\[
\frac{d^p}{d\lambda^p} c^{(\lambda)} \bigg|_{\lambda=0} = \frac{1}{p!} \frac{d^p}{d\lambda^p} \sum_{\ell=0}^\infty \frac{1}{\ell!} (c^{(\lambda)})^\ell \bigg|_{\lambda=0}
= \frac{1}{p!} \sum_{\ell=1}^p \frac{1}{\ell!} \sum_{q_1, \ldots, q_\ell \geq 1 \text{ with } q_1 + \cdots + q_\ell = p} \left( \prod_{i=1}^\ell \frac{p}{q_i} \right) q_1! \cdots q_\ell! \ c^{(q_1)} \cdots c^{(q_\ell)}
\]

where in the second step we used that \( c^{(0)} = 0 \) in order to truncate the summation over \( \ell \) as well as the general Leibniz rule for differentiable functions.

We next perform the perturbation expansion of the Lagrangian. To this end, we first expand it in a Taylor series in both arguments. In the following formula, \((D_1,F(x)-x)\mathcal{L}(x,y)\) and \((D_2,F(y)-y)\mathcal{L}(x,y)\) again denote the partial derivatives of \(\mathcal{L}(\ldots)\) acting on the first and second argument, in direction of \(F(x)-x\) and \(F(y)-y\), respectively. According to our general convention \(2.11\), these partial derivatives do not act on the arguments \(F(x)-x\) or \(F(y)-y\) of other derivative, but merely on \(\mathcal{L}\). We thus obtain

\[
\mathcal{L}(F(x),F(y)) = \sum_{a=0}^\infty \sum_{b=0}^\infty \frac{1}{a!} \frac{1}{b!} (D_1,F(x)-x)^a (D_2,F(y)-y)^b \mathcal{L}(x,y)
= \sum_{k=0}^\infty \sum_{a+b=k} \frac{1}{a!} \frac{1}{b!} (D_1,F(x)-x)^a (D_2,F(y)-y)^b \mathcal{L}(x,y)
= \sum_{k=0}^\infty \frac{1}{k!} \left( D_1,F(x)-x + D_2,F(y)-y \right)^k \mathcal{L}(x,y),
\]

where the second step consists merely of a reordering of summation indices, and where in the last step we used the generalized binomial theorem. Taking the \(q^{th}\) derivative with respect to \(\lambda\) and evaluating at \(\lambda = 0\), the general Leibniz rule yields

\[
\frac{1}{q!} \frac{d^q}{d\lambda^q} \mathcal{L}(F(x),F(y)) \bigg|_{\lambda=0} = \frac{1}{q!} \sum_{k=0}^q \frac{1}{k!} \sum_{q_1, \ldots, q_k \geq 1 \text{ with } q_1 + \cdots + q_k = q} \left( \prod_{i=1}^k \frac{q}{q_i} \right) q_1! \cdots q_k! \ (D_1,F^{(q_1)}(x)) \cdots (D_1,F^{(q_k)}(x)) \mathcal{L}(x,y)
= \sum_{k=0}^q \frac{1}{k!} \sum_{q_1, \ldots, q_k \geq 1 \text{ with } q_1 + \cdots + q_k = q} (D_1,F^{(q_1)}(x) + D_2,F^{(q_k)}(y)) \cdots (D_1,F^{(q_k)}(x) + D_2,F^{(q_k)}(y)) \mathcal{L}(x,y),
\]

where in the first step we used the equation \( (D_1,F^{(0)}(x)-x)\mathcal{L}(x,y) = 0 \) in order to truncate the summation over \(k\). Moreover, differentiating the expansion (3.17) gives

\[
\frac{d^q}{d\lambda^q} (D_1,F(x)-x + D_2,F(y)-y) = q! \ (D_1,F^{(q)}(x) + D_2,F^{(q)}(y)).
\]
Combing the above formulas, we obtain

\[
\frac{1}{p!} \frac{d^p}{d\lambda^p} f(x) \mathcal{L}(F(x), F(y)) f(y) = \sum_{l, r, q \geq 0 \text{ with } l+r+q=p} f^{(l)}(x) f^{(r)}(y) \frac{1}{q!} \frac{d^q}{d\lambda^q} \mathcal{L}(F(x), F(y)) \big|_{\lambda=0}.
\]

The factors \( f^{(l)}(x) f^{(r)}(y) \) can be calculated similar to \([A, 1]\),

\[
\sum_{l, r \geq 0 \text{ with } l+r=L} \frac{1}{l!} \frac{d^l}{d\lambda^l} c(x) \frac{1}{r!} \frac{d^r}{d\lambda^r} c(y) \big|_{\lambda=0} = \frac{1}{L!} \frac{d^L}{d\lambda^L} c(x) + c(y) \big|_{\lambda=0} = \frac{1}{L!} \frac{d^L}{d\lambda^L} \sum_{a=0}^{\infty} \frac{1}{a!} (c(x) + c(y))^a \big|_{\lambda=0} = \sum_{a=0}^{\infty} \frac{1}{a!}(c(x) + c(y))^a.
\]

Using again the general Leibniz rule, we have

\[
\sum_{l, r, q \geq 0 \text{ with } l+r+q=p} \left( \sum_{l_1, \ldots, l_a \geq 1 \text{ with } l_1 + \cdots + l_a = L} l_1! \cdots l_a! \left( \left( \sum_{k=0}^{q} \frac{1}{k!} \right) \sum_{l, r \geq 0 \text{ with } l+r=L} f^{(l)}(x) f^{(r)}(y) \left( D_{1,F(x)} + D_{2,F(y)} \right) \cdots \left( D_{1,F(y_k)} + D_{2,F(y_k)} \right) \right) \mathcal{L}(x, y) \right)
\]

\[
= \sum_{l, r, q \geq 0 \text{ with } l+r+q=p} \sum_{k=0}^{q} \frac{1}{k!} \sum_{l, r \geq 0 \text{ with } l+r=L} \left( \left( \sum_{a=0}^{\infty} \frac{1}{a!} \right) \frac{d^L}{d\lambda^L} \mathcal{L}(x, y) \right) \left( D_{1,F(x)} + D_{2,F(y)} \right) \cdots \left( D_{1,F(y_k)} + D_{2,F(y_k)} \right)
\]

\[
= \sum_{l, r, q \geq 0 \text{ with } l+r+q=p} \sum_{a=0}^{\infty} \frac{1}{a!} \sum_{k=0}^{q} \frac{1}{k!} \sum_{l_1, \ldots, l_a \geq 1 \text{ with } l_1 + \cdots + l_a = L} \left( \left( \sum_{k=0}^{q} \frac{1}{k!} \right) \sum_{l, r \geq 0 \text{ with } l+r=L} f^{(l)}(x) f^{(r)}(y) \left( D_{1,F(x)} + D_{2,F(y)} \right) \cdots \left( D_{1,F(y_k)} + D_{2,F(y_k)} \right) \right) \mathcal{L}(x, y)
\]
\[ \times \left( c^{(l_1)}(x) + c^{(l_1)}(y) \right) \cdots \left( c^{(l_a)}(x) + c^{(l_a)}(y) \right) \]
\[ \times \left( D_{1,F(q_1)} + D_{2,F(q_1)} \right) \cdots \left( D_{1,F(q_k)} + D_{2,F(q_k)} \right) \mathcal{L}(x, y), \]

where in the last step we argued as follows. Note that for a given value of \((a, k)\), in the second-to-last line, the sum over \(q\) gives \((p - (a + k))\) terms since \(\ell_i, q_i \geq 1\). The sum over these \((p - (a + k))\) terms reads
\[ \sum_{q=k}^{p-a} \sum_{q_1, \ldots, q_k \geq 1 \atop with \, q_1 + \cdots + q_k = q} \sum_{l_1, \ldots, l_a \geq 1 \atop with \, l_1 + \cdots + l_a = p-q} \cdots = \sum_{q=0}^{p} \sum_{q_1, \ldots, q_k \geq 1 \atop with \, q_1 + \cdots + q_k = q} \sum_{l_1, \ldots, l_a \geq 1 \atop with \, l_1 + \cdots + l_a = p-q} \cdots \]

where in the second step we used that extending the range of \(q\) does not change the terms which are summed over, because the conditions \(\ell_i, q_i \geq 1\) exclude those terms for fixed \(k\) and \(a\). It remains to sum over the different values of \((a, k)\), which is most conveniently done via
\[ \sum_{\ell=0}^{p} \sum_{a,k \geq 0 \atop a+k=\ell} \cdots, \]

giving the result of the calculation \((A.2)\).

Next, we use the definition of \(w^{(p)}\) in \((3.19)\). We have
\[ \sum_{q_1, \ldots, q_\ell \geq 1 \atop with \, q_1 + \cdots + q_\ell = p} \nabla_1, m(q_1) \nabla_1, m(q_2) \cdots \nabla_1, m(q_\ell) \mathcal{L}(x, y) \]
\[ = \sum_{q_1, \ldots, q_\ell \geq 1 \atop with \, q_1 + \cdots + q_\ell = p} \left( c^{(q_1)}(x) + D_{1,F(q_1)} \right) \left( c^{(q_2)}(x) + D_{1,F(q_2)} \right) \cdots \left( c^{(q_\ell)}(x) + D_{1,F(q_\ell)} \right) \mathcal{L}(x, y) \]
\[ = \sum_{a, k \geq 0 \atop a+k=\ell} \left( \frac{\ell}{a, k} \right) \sum_{l_1, \ldots, l_a, q_1, \ldots, q_k \geq 1 \atop with \, l_1 + \cdots + l_a + q_1 + \cdots + q_k = p} c^{(l_1)}(x) \cdots c^{(l_a)}(x) D_{1,F(q_1)} \cdots D_{1,F(q_k)} \mathcal{L}(x, y). \]

The last step can be understood as follows. In order to get from the second line to the third line, from every bracket \(c^{(q_i)}(x) + D_{1,F(q_i)}\) we choose either \(c^{(q_i)}(x)\) or \(D_{1,F(q_i)}\). The order of the appearance of the derivatives does not matter because, as explained after \((A.1)\), the derivatives all act on the first argument of \(\mathcal{L}(x, y)\) and not on each other nor on the \(c^{(q_i)}(x)\). Furthermore, the actual value of \(q_i\) does not matter because we sum over all \(q_i\). Thus every term in the second line is characterized by how many \(c^{(q_i)}(x)\) appear (denote this number be \(a\)) and by how many \(D_{1,F(q_i)}\) appear (denote this number be \(k\)), giving rise to the sum over \(a\) and \(k\) in the last line. The binomial coefficient gives the correct combinatorial factor, corresponding to the number of ways that, disregarding ordering, we can choose \(a\) (respectively \(k\)) terms from \(l\) terms.

The same argument as in the last paragraph can be applied to terms \((\nabla_1, m(q_i) + \nabla_2, m(q_i))\) instead of \(\nabla_1, m(q_i)\), giving exactly the terms appearing in the last term in \((A.2)\). Thus the \(p^{th}\) order in the perturbation expansion of the integrand of \((3.8)\) can be
expressed as
\[
\frac{1}{p!} \frac{d^p}{d\lambda^p} f(x) \mathcal{L}(F(x), F(y)) f(y) = \sum_{\ell=0}^p \frac{1}{\ell!} \sum_{q_1, \ldots, q_{\ell} \geq 1 \atop \text{with } q_1 + \cdots + q_{\ell} = p} (\nabla_{1,\omega(q_1)} + \nabla_{2,\omega(q_1)}) \cdots (\nabla_{1,\omega(q_{\ell})} + \nabla_{2,\omega(q_{\ell})}) \mathcal{L}(x, y).
\] (A.3)

The remaining task is to distribute the \( s \)– and \( t \)-derivatives in Theorem 3.1. Since the term for \( \ell = 0 \) in (A.3) only contributes if \( p = 0 \) and thus, according to (3.16) and (3.17), does not have a dependence on \( s \) or \( t \), the summation over \( \ell \) does not need to include the case \( \ell = 0 \). We thus obtain
\[
\left. (\partial_{1,s} - \partial_{2,s}) (\partial_{1,t} + \partial_{2,t}) \frac{1}{p!} \frac{d^p}{d\lambda^p} f_{s,t}(x) \mathcal{L}(F_{s,t}(x), F_{s,t}(y)) f_{s,t}(y) \right|_{s=t=0} = \sum_{\ell=1}^p \frac{1}{\ell!} \sum_{\ell=1}^p (\partial_{1,s} - \partial_{2,s}) (\partial_{1,t} + \partial_{2,t})^k
\times (\nabla_{1,\omega(q_1)} + \nabla_{2,\omega(q_1)}) \cdots (\nabla_{1,\omega(q_{\ell})} + \nabla_{2,\omega(q_{\ell})}) \mathcal{L}(x, y) \right|_{s=t=0}
\times (\nabla_{1,\partial_s^k w_{s,t}} + \nabla_{1,\partial_t^k w_{s,t}}) \cdots (\nabla_{1,\partial_s^k w_{s,t}} + \nabla_{1,\partial_t^k w_{s,t}}) \mathcal{L}(x, y) \right|_{s=t=0}.
\]

Note that the summation over \( k_1, \ldots, k_{\ell} \) needs to include the cases \( k_i = 0 \) since in general \( w_{s,t}^{(p)} \mid_{s=t=0} \neq 0 \) (see (3.19)). For the integrand in (3.9), using (A.1), we obtain for the \( p \)th order
\[
\left. \partial_s \partial_t^k f^{(p)}(x) \right|_{s=t=0} = \sum_{\ell=1}^p \frac{1}{\ell!} \sum_{\ell=1}^p \partial_s \partial_t^k (c^{(q_1)} \cdots c^{(q_{\ell})}) \left|_{s=t=0}\right.
= \sum_{\ell=1}^p \frac{1}{(\ell - 1)!} \sum_{\ell=1}^p \sum_{k_1, \ldots, k_{\ell} \geq 0 \atop \text{with } k_1 + \cdots + k_{\ell} = k} \left( k \right) \left( k_1 \cdots k_{\ell} \right) \partial_s \partial_t^k (c^{(q_1)} \cdots c^{(q_{\ell})}) \left|_{s=t=0}\right.
\]

This gives the result. \( \square \)

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Fakultät für Mathematik, Universität Regensburg, D-93040 Regensburg, Germany
E-mail address: finster@ur.de

Institut für Theoretische Physik, Leibniz Universität, D-30167 Hannover, Germany
E-mail address: johannes.kleiner@itp.uni-hannover.de