Indecomposable 1-factorizations of the complete multigraph $\lambda K_{2n}$ for every $\lambda \leq 2n^*$

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Abstract

A 1-factorization of the complete multigraph $\lambda K_{2n}$ is said to be indecomposable if it cannot be represented as the union of 1-factorizations of $\lambda_0 K_{2n}$ and $(\lambda - \lambda_0) K_{2n}$, where $\lambda_0 < \lambda$. It is said to be simple if no 1-factor is repeated. For every $n \geq 9$ and for every $(n-2)/3 \leq \lambda \leq 2n$, we construct an indecomposable 1-factorization of $\lambda K_{2n}$ which is not simple. These 1-factorizations provide simple and indecomposable 1-factorizations of $\lambda K_{2s}$ for every $s \geq 18$ and $2 \leq \lambda \leq 2\lfloor s/2 \rfloor - 1$.

We also give a generalization of a result by Colbourn et al. which provides a simple and indecomposable 1-factorization of $\lambda K_{2n}$, where $2n = p^m + 1$, $\lambda = (p^m - 1)/2$, $p$ prime.

Keywords: complete multigraph, indecomposable 1-factorizations, simple 1-factorizations.

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1 Introduction

We refer to [3] for graph theory notation and terminology which are not introduced explicitly here. We recall that the complete multigraph $\lambda K_{2n}$ has $2n$ vertices and each pair of vertices is joined by exactly $\lambda$ edges. A 1-factor of $\lambda K_{2n}$ is a spanning subgraph of $\lambda K_{2n}$ consisting of $n$ edges that are pairwise independent. If $S$ is a set of 1-factors of $\lambda K_{2n}$, then we will denote by $E(S)$ the multiset containing all the edges of the 1-factors of $S$, namely, $E(S) = \bigcup_{F \in S} E(F)$. A 1-factorization $\mathcal{F}$ of $\lambda K_{2n}$ is a partition

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of the edge-set of $\lambda K_{2n}$ into 1-factors. A subfactorization of $\mathcal{F}$ is a subset $\mathcal{F}_0$ of 1-factors belonging to $\mathcal{F}$ that constitute a 1-factorization of $\lambda_0 K_{2n}$, where $\lambda_0 \leq \lambda$. For every $\lambda \geq 1$, it is possible to find a 1-factorization of $\lambda K_{2n}$. Lucas’ construction provides a 1-factorization for the complete graph $K_{2n}$, denoted by $G K_{2n}$ (see [9]). By taking $\lambda$ copies of $G K_{2n}$, we find a 1-factorization of $\lambda K_{2n}$. Obviously, it contains repeated 1-factors. Moreover, we can consider $\lambda_0 < \lambda$ copies of each 1-factor so that it is the union of 1-factorizations of $\lambda_0 K_{2n}$ and $(\lambda - \lambda_0) K_{2n}$. A 1-factorization of $\lambda K_{2n}$ that contains no repeated 1-factors is said to be simple. A 1-factorization of $\lambda K_{2n}$ that can be represented as the union of 1-factorizations of $\lambda_0 K_{2n}$ and $(\lambda - \lambda_0) K_{2n}$, where $\lambda_0 < \lambda$, is said to be indecomposable, otherwise it is called decomposable. An indecomposable 1-factorization might be simple or not.

In this paper, we consider the problem about the existence of indecomposable 1-factorizations of $\lambda K_{2n}$. Obviously, $\lambda > 1$. In order that the complete multigraph $\lambda K_{2n}$ admits an indecomposable 1-factorization, the parameter $\lambda$ cannot be arbitrarily large: we have necessarily $\lambda < 3 \cdot 4 \cdot \cdots (2n - 3)$ or $\lambda < [n(2n-1)]^{n(2n-1)}{(2n^3+n^2-n+1)}{2n^2-n}$, according to whether the 1-factorization is simple or not (see [2]). Moreover, two non-existence results are known. For every $\lambda > 1$ there is no indecomposable 1-factorization of $\lambda K_4$ (see [4]). For every $\lambda \geq 3$ there is no indecomposable 1-factorization of $\lambda K_6$ (see [2]). We recall that in [4] the authors construct simple and indecomposable 1-factorizations of $\lambda K_{2n}$ for $2 \leq \lambda \leq 12$, $\lambda \neq 7, 11$. They also give a simple and indecomposable 1-factorization of $\lambda K_{p+1}$, where $p$ is an odd prime and $\lambda = (p - 1)/2$. In [1] we can find an indecomposable 1-factorization of $(n - p) K_{2n}$, where $p$ is the smallest prime not dividing $n$. This 1-factorization is not simple, but it is used to construct a simple and indecomposable 1-factorization of $(n - p) K_{2s}$ for every $s \geq 2n$. This construction improves the results in [4] for $2 \leq \lambda \leq 12$ (see Theorem 2.5 in [1]). Simple and indecomposable 1-factorizations of $(n - d) K_{2n}$, with $d \geq 2$, $n - d \geq 5$ and $\gcd(n, d) = 1$, are constructed in [8]. Other values of $\lambda$ and $n$ for which the existence of a simple and indecomposable 1-factorization of $\lambda K_{2n}$ is known are the following: $2n = q^2 + 1$, $\lambda = q - 1$, where $q$ is an odd prime power (see [5]); $2n = 2^b + 2$, $\lambda = 2$ (see [7]); $2n = q^2 + 1$, $\lambda = q + 1$, where $q$ is an odd prime power (see [5]); $2n = q^2$, $\lambda = q$, where $q$ is an even prime power (see [5]).

In this paper we prove some theorems about the existence of simple and indecomposable 1-factorizations of $\lambda K_{2n}$, where most of the parameters $\lambda$ and $n$ were not previously considered in literature. We show that for every $n \geq 9$ and for every $(n - 2)/3 \leq \lambda \leq 2n$ there exists an indecomposable 1-factorization of $\lambda K_{2n}$ (see Theorem [1]). We can also exhibit some examples of indecomposable 1-factorizations of $\lambda K_{2n}$ for $n \in \{7, 8\}$, $(n - 2)/3 \leq \lambda \leq n$ (see Proposition 3); and for $n \in \{5, 6\}$, $(n - 2)/3 \leq \lambda \leq n - 2$ (see Proposition 1 and 2). The 1-factorizations in Theorem 1, Proposition 1, 2 and 3 are
not simple. By an embedding result in [4], we can use them to prove the existence of simple and indecomposable 1-factorizations of $\lambda K_{2n}$ for every $s \geq 18$ and for every $2 \leq \lambda \leq 2|s/2| - 1$ (see Theorem 2). We note that for odd values of $s$, the parameter $\lambda$ does not exceed the value $s - 2$. Nevertheless, if $2s = p^n + 1$, where $p$ is a prime, then we can find a simple and indecomposable 1-factorization of $(s - 1)K_{2n}$ (see Theorem 3). By our results we can improve Theorem 2.5 in [1] about the existence of simple and indecomposable 1-factorizations of $\lambda K_{2n}$ for $2 \leq \lambda \leq 12$. We note that in Theorem 2.5 in [1] the existence of a simple and indecomposable 1-factorization of $11K_{2n}$ (respectively, $12K_{2n}$) is known for every $2n \geq 52$ (respectively, $2n \geq 32$). By Theorem 2 a simple and indecomposable 1-factorization of $11K_{2n}$ exists for every $2n \geq 36$. By Theorem 3 there exists a simple and indecomposable 1-factorization of $12K_{2n}$. Moreover, Theorem 3 extends Theorem 2 in [4] to each odd prime power.

2 Basic lemmas.

In Section 3 and 4 we will construct indecomposable 1-factorizations of $\lambda K_{2n}$ for suitable values of $\lambda > 1$. These 1-factorizations contain 1-factor-orbits, that is, sets of 1-factors belonging to the same orbit with respect to a group $G$ of permutations on the vertices of the complete multigraph.

If not differently specified, we use the exponential notation for the action of $G$ and its subgroups on vertices, edges and 1-factors. So, if $e = [x, y]$ is an edge of $\lambda K_{2n}$ and $g \in G$ we set $e^g = [x^g, y^g]$. Analogously, if $F$ is a 1-factor we set $F^g = \{e^g : e \in F\}$. Since we shall treat with sets and multisets, we specify that by an edge-orbit $e^H$, where $H \leq G$, we mean the set $e^H = \{e^h : h \in H\}$ and by a 1-factor-orbit $F^h$ we mean the set $F^h = \{F^h : h \in H\}$. If $h \in H$ leaves $F$ invariant, that is, $F^h = F$, then $h$ is an element of the stabilizer of $F$ in $G$, which will be denoted by $G_F$. The cardinality of $F^h$ is $|H|/|H \cap G_F|$. The following result holds.

Lemma 1. Let $F$ be a 1-factor of $\lambda K_{2n}$ containing exactly $\mu$ edges belonging to the same edge-orbit $e^H$, where $H$ is a subgroup of $G$ having trivial intersection with the stabilizer of $F$ in $G$ and with the stabilizer of $e$ in $G$. The multiset $\cup_{h \in H} E(F^h)$ contains every edge of $e^H$ exactly $\mu$ times.

Proof. We denote by $e_1, \ldots, e_\mu$ the edges in $F \cap e^H$. We show that every edge $f \in e^H$ appears $t_f \geq \mu$ times in the multiset $E(F^H) = \cup_{h \in H} E(F^h)$. For every edge $e_i \in \{e_1, \ldots, e_\mu\}$ there exists an element $h_i \in H$ such that $e_i^h = f$, since $e_i$ and $f$ belong to the same edge-orbit $e^H$. Hence the 1-factor $F^{h_i}$ contains the edge $f$. The 1-factors $F^{h_1}, F^{h_2}, \ldots, F^{h_\mu}$ are pairwise distinct, since $H$ has trivial intersection with $G_F$. Therefore, every edge $f \in e^H$ appears $t_f \geq \mu$ times in the multiset $E(F^H)$. We prove that $t_f = \mu$. In fact, $t_f > \mu$ implies the existence of $h \in H \setminus \{h_1, \ldots, h_\mu\}$ such that
\[ f \in F^h \text{ and then } e_i^h = f = e_i^h \text{ for some } e_i \in \{ e_1, \ldots, e_\mu \}. \] That yields a contradiction, since \( e_i \), as well as \( e \), has trivial stabilizer in \( H \).

To prove the indecomposability of the 1-factorizations in Section 3, we will use the following lemma.

**Lemma 2.** Let \( M \) be a 1-factor of \( \lambda K_{2n} \). Let \( F \) be a 1-factorization of \( \lambda K_{2n} \) containing \( 0 \leq \lambda - t < \lambda \) copies of \( M \) and a subset \( S \) of 1-factors satisfying the following properties:

(i) the multiset \( E(S) \) contains every edge of \( M \) exactly \( t \) times;

(ii) for every \( S' \subset S \), the multiset \( E(S') \) contains \( 0 < \mu < n \) distinct edges of \( M \).

If \( F_0 \subseteq F \) is a 1-factorization of \( \lambda_0 K_{2n} \), where \( \lambda_0 \leq \lambda \), then \( S \subseteq F_0 \) or \( F_0 \) contains no 1-factor of \( S \).

**Proof.** Assume that \( F_0 \) contains \( 0 < s < |S| \) elements of \( S \), say \( F_1, \ldots, F_s \). We denote by \( M' \) the set consisting of the edges of \( M \) that are contained in the multiset \( \cup_{i=1}^s E(F_i) \). By property (i), the set \( M' \) is a non-empty proper subset of \( M \). It is clear from (i) that the 1-factors of \( F \) containing some edges of \( M \) are exactly the \( \lambda - t \) copies of \( M \) together with the 1-factors of \( S \). Therefore, the 1-factorization \( F_0 \) contains \( \lambda_0 \) copies of \( M \), since the edges in \( M \setminus M' \) are not contained in \( \cup_{i=1}^s E(F_i) \). Then the multiset \( E(F_0) \) contains at least \( \lambda_0 + 1 \) copies of each edge in \( M' \), a contradiction. Hence \( s = n \) or \( F_0 \) contains no 1-factor of \( S \).

### 3 Indecomposable 1-factorizations which are not simple.

In what follows, we consider the group \( G \) given by the direct product \( \mathbb{Z}_n \times \mathbb{Z}_2 \) and denote by \( H \) the subgroup of \( G \) isomorphic to \( \mathbb{Z}_n \). We will identify the vertices of the complete multigraph \( \lambda K_{2n} \) with the elements of \( G \), thus obtaining the graph \( \lambda K_G = \left( G, \lambda(G^2) \right) \), where \( G^2 \) is the set of all possible 2-subsets of \( G \) and \( \lambda(G^2) \) is the multiset consisting of \( \lambda \) copies of \( G^2 \).

In \( G \) we will adopt the additive notation and observe that \( G \) is a group of permutations on the vertex-set, that is, each \( g \in G \) is identified with the permutation \( x \rightarrow x + g \), for every \( x \in G \). For the sake of simplicity, we will represent the elements of \( G \) in the form \( a_j \), where \( a \) and \( j \) are integers modulo \( n \) and modulo 2, respectively. The edges of \( \lambda K_G \) are of type \( [a_0, b_1] \) or \( [a_j, b_j] \) and we can observe that each edge \( [a_0, b_1] \) has trivial stabilizer in \( H \). For every \( a \in \mathbb{Z}_n \), we consider the edge-orbit \( M_a = [0, a_1]^H \). Each edge-orbit \( M_a \) is a 1-factor of \( \lambda K_G \). The 1-factors in \( \cup_{a \in \mathbb{Z}_n} M_a \) partition the
edges of type \([a_0, b_1]\). We shall represent the vertices and the 1-factors \(M_a\) as in Figure 1. Observe that, if \(M_a\) contains the edge \([x_0, (x + a)_1]\), then \(M_{n-a}\) contains the edge \([(x + a)_0, x_1]\).

The edges of type \([a_j, b_j]\), with \(j = 0, 1\), can be partitioned by the 1-factors (or, near 1-factors) of a 1-factorization (or, of a near 1-factorization) of \(K_n\). More specifically, for even values of \(n\) we consider the well-known 1-factorization \(GK_n\) defined by Lucas [9]. We recall that in \(GK_n\) the vertex-set of \(K_n\) is \(\mathbb{Z}_{n-1} \cup \{\infty\}\) and \(GK_n = \{L_i : i \in \mathbb{Z}_{n-1}\}\), where \(L_0 = \{[a, -a] : a \in \mathbb{Z}_{n-1} - \{0\}\} \cup \{[0, \infty]\}\) and \(L_i = L_0 + i = \{[a + i, -a + i] : a \in \mathbb{Z}_{n-1} - \{0\}\} \cup \{[i, \infty]\}\). For odd values of \(n\), we consider the 1-factorization \(GK_{n+1}\) and delete the vertex \(\infty\). Each 1-factor \(L_i\) yields a near 1-factor \(L_i^*\) of \(K_n\) where the vertex \(i \in \mathbb{Z}_n\) is unmatched. We denote by \(GK_n^*\) the resulting near 1-factorization of \(K_n\).

For even values of \(n\), we partition the edges \([a_j, b_j]\) of \(\lambda K_{2n}\) into 1-factors of \(\lambda K_{2n}\) as follows. For \(j = 0, 1\), we consider the 1-factorization \(GK_n\) of the complete graph \(K_n\) with vertex-set \(V_j = \{a_j : 0 \leq a \leq n - 1\}\). It is possible to obtain a 1-factor of \(K_{2n}\) by joining, in an arbitrary way, a 1-factor on \(V_0\) to a 1-factor on \(V_1\). We denote by \(\mathcal{F}(GK_n)\) the resulting set of 1-factors of \(K_{2n}\). We denote by \(\mathcal{F}(\lambda K_{2n})\) the multiset consisting of \(\lambda\) copies of \(\mathcal{F}(GK_n)\).

For odd values of \(n\), we partition the edges \([a_j, b_j]\) of \(\lambda K_{2n}\) into 1-factors of \(\lambda K_{2n}\) as follows. For \(j = 0, 1\), we consider the near 1-factorization \(GK_n^*\) of the complete graph \(K_n\) with vertex-set \(V_j = \{a_j : 0 \leq a \leq n - 1\}\). We select an integer \(b \in \mathbb{Z}_n\). For \(i = 0, \ldots, n - 1\), we join the near 1-factor \(L_i^*\) on \(V_0\) to the near 1-factor \(L_{i+b}^*\) on \(V_1\) (subscripts are considered modulo \(n\)) and add the edge \([i_0, (i + b)_{1}]\). We obtain a 1-factor of \(K_{2n}\). We denote by \(\mathcal{F}(GK_n^*, b)\) the resulting set of 1-factors of \(K_{2n}\). We denote by \(\mathcal{F}(\lambda K_{2n}, b)\) the multiset consisting of \(\lambda\) copies of \(\mathcal{F}(GK_n^*, b)\). Observe that the set \([i_0, (i + b)_{1}] : 0 \leq i \leq n - 1\) corresponds to the 1-factor \(M_b\). Hence \(\mathcal{F}(\lambda K_{2n}, b)\) contains every edge of \(M_b\) exactly \(\lambda\) times.

In the following propositions we will construct 1-factorizations of \(\lambda K_G\) which are not simple. They are obtained as described in Lemma 3. Moreover, Lemma 3 will be useful to prove that these 1-factorizations are indecomposable. It is straightforward to prove that the following holds.

**Lemma 3.** Let \(\mathcal{F} = \{F_1, \ldots, F_m\}\) be a set of 1–factors of \(\lambda K_G\) such that each \(F_i\) contains no edge of type \([a_j, b_j]\), has trivial stabilizer in \(H\) and \(F_r \notin F_i^H\) for each pair \((i, r)\) with \(i \neq r\).

Let \(\mathcal{M}\) be the subset of \(\{M_a : a \in \mathbb{Z}_n\}\) containing all the 1–factors \(M_a\) such that \(t(M_a) = \sum_{i=1}^{m} |E(M_a) \cap E(F_i)| > 0\).

If \(|H| = n\) is even and \(t(M_a) \leq \lambda\) for every \(M_a \in \mathcal{M}\), then there exists a 1-factorization of \(\lambda K_G\) whose 1–factors are exactly those of \(F_1^H \cup \cdots \cup F_m^H \cup \mathcal{F}(\lambda K_{2n})\) together with \(\lambda - t(M_a)\) copies of each \(M_a \in \mathcal{M}\) and \(\lambda\) copies of each \(M_a \notin \mathcal{M}\).

If \(|H| = n\) is odd, \(t(M_a) \leq \lambda\) for every \(M_a \in \mathcal{M}\) and there exists at least
Figure 1: The vertices $a_0, b_1$ of $\lambda KG$ are represented on the left and on the right, respectively. Each edge-orbit $M_a$ is a 1-factor of $\lambda KG$. If $M_a$ contains the edge $[0_0, a_1]$, then $M_{n-a}$ contains the edge $[a_0, 0_1]$. 

one 1–factor $M_a \in \{M_a : a \in \mathbb{Z}_n\} \setminus \mathcal{M}$, then there exists a 1–factorization of $\lambda KG$ whose 1–factors are exactly those of $F^{H}_1 \cup \cdots \cup F^{H}_m \cup \mathcal{F}(\lambda KG_n, b)$ together with $\lambda - t(M_a)$ copies of each $M_a \in \mathcal{M}$ and $\lambda$ copies of each $M_a/\in \mathcal{M} \cup \{M_b\}$. 

Lemma 4. Let $\mathcal{F}$ be the 1–factorization of $\lambda KG$ obtained in Lemma 3 starting from $\mathcal{F}' = \{F_1, \ldots, F_m\}$ and the set $\mathcal{M}$. Let $\mathcal{F}_0 \subseteq \mathcal{F}$ be a 1–factorization of $\lambda_0 KG$, $\lambda_0 \leq \lambda$. Let $F_i \in \mathcal{F}'$ and $M_a \in \mathcal{M}$ be such that $F_i$ contains exactly one edge of $M_a$. If one of the following conditions holds:

(i) each 1-factor in $\mathcal{F}' \setminus \{F_i\}$ contains no edge of $M_a$;

(ii) each 1-factor $F \in \mathcal{F}' \setminus \{F_i\}$ containing some edge of $M_a$ is such that either $F^H \subset \mathcal{F}_0$ or $F^H \cap \mathcal{F}_0 = \emptyset$.

then it is either $F^H_i \subset \mathcal{F}_0$ or $F^H_i \cap \mathcal{F}_0 = \emptyset$.

Proof. Assume that $F_i$ satisfies property (i). By Lemma 1, each edge of $M_a$ appears exactly once in the multiset $E(F^H_i)$. Since each 1-factor in $\mathcal{F}' \setminus \{F_i\}$ contains no edge of $M_a$, the 1-factorization $\mathcal{F}$ contains exactly $\lambda - 1$ copies of $M_a$. The assertion follows from Lemma 2 by setting $\mathcal{S} = F^H_i$ and $M = M_a$.

Assume that $F_i$ satisfies property (ii). We can consider the subset $\mathcal{F}_1$ of $\mathcal{F}' \setminus \{F_i\}$ consisting of the 1-factors $F$ containing $s_F \geq 1$ edges of $M_a$ and whose orbit $F^H$ is contained in $\mathcal{F}_0$. The set $\mathcal{F}_1$ might be empty. By Lemma
Proposition 2. Let $M_a$ be the subset of $M$ consisting of the edges of $M_a$ that are contained in the multiset $E(S)$. By the proof of Lemma 1, the set $M'$ consists of $|S| < n$ distinct edges. Each edge of $M_a \setminus M'$ appears exactly $\sum_{F \in F_1} s_F < \lambda_0$ times among the edges of the 1-factors in $S \cup \{\cup_{F \in F_1} F^H\}$. Each edge of $M'$ appears exactly $1 + \sum_{F \in F_1} s_F$ times among the edges of the 1-factors in $S \cup \{\cup_{F \in F_1} F^H\}$. Whence $\sum_{F \in F_1} s_F < \lambda_0$, otherwise the edges of $M'$ would appear at least $\lambda_0 + 1$ times among the edges of the 1-factors in $F_0$. Since the edges of $M_a \setminus M'$ appear $\sum_{F \in F_1} s_F < \lambda_0$ times, the 1-factorization $F_0$ must contain $\lambda_0 - \sum_{F \in F_1} s_F > 0$ copies of $M_a$. Consequently, each edge of $M'$ appears at least $\lambda_0 + 1$ among the edges of the 1-factors in $F_0$. That yields a contradiction. Hence, either $F_0$ contains no 1-factor of $F^H$ or $F^H \subseteq F_0$. \square

Proposition 1. Let $n \geq 5$ and $(n - 2)/3 \leq \lambda \leq n - 2$ such that $n - \lambda$ is even. There exists an indecomposable 1-factorization of $\lambda K_{2n}$ which is not simple.

Proof. Identify $\lambda K_{2n}$ with $\lambda K_G$. If $\lambda < n - 2$, then $n > 5$ and we consider the 1-factor $A$ in Figure 2(a). For $\lambda = n - 2$ we consider the 1-factor $A$ in Figure 3 with $\alpha = 1$. If $\lambda < n - 2$, then $A$ contains exactly $(n - \lambda - 2)/2$ edges of $M_1$ as well as $(n - \lambda - 2)/2$ edges of $M_{n-1}$. It also contains $\lambda$ edges of $M_0$, one edge of $M_2$ and one edge of $M_{n-2}$. If $\lambda = n - 2$, then $A$ contains exactly $\lambda$ edges of $M_0$ as well as one edge of $M_1$ and one edge of $M_{n-1}$. In both cases the stabilizer of $A$ in $H$ is trivial and when $\lambda < n - 2$, the condition $(n - 2)/3 \leq \lambda$ assures that $(n - \lambda - 2)/2 \leq \lambda$. Therefore $F' = \{A\}$ satisfies Lemma 3 and a 1-factorization $F$ of $\lambda K_G$ is constructed as prescribed. We prove that $F$ is indecomposable. Suppose that $F_0 \subseteq F$ is a 1-factorization of $\lambda_0 K_G$, $\lambda_0 < \lambda$. The 1-factor $A$ satisfies condition (i) of Lemma 3 (set $M_a = M_2$ or $M_a = M_1$ according to whether $\lambda < n - 2$ or $\lambda = n - 2$, respectively). Therefore it is either $A^H \subseteq F_0$ or $A^H \cap F_0 = \emptyset$. In the former case, each edge of $M_0$ appears $\lambda$ times in the multiset $E(F_0)$, that is, $\lambda = \lambda_0$, a contradiction. In the latter case, no edge of $M_0$ appears in $E(F_0)$, a contradiction. \square

Proposition 2. Let $n \geq 5$ and $(n + 1)/3 \leq \lambda \leq n - 3$ such that $n - \lambda$ is odd. There exists an indecomposable 1-factorization of $\lambda K_{2n}$ which is not simple.

Proof. The proof is similar to the proof of Proposition 1. \square

Proposition 3. Let $n \geq 7$ and $n - 1 \leq \lambda \leq n$. There exists an indecomposable 1-factorization of $\lambda K_{2n}$ which is not simple.
Figure 2: The 1-factor $A$ in the case: (a) $n - \lambda$ even, $\lambda < n - 2$; (b) $n - \lambda$ odd

Proof. Identify $\lambda K_{2n}$, with $\lambda K_G$ and set $\lambda = n - 1 + r$, where $0 \leq r \leq 1$. We consider the 1-factors $A$ and $B_r$ in Figure 3. In the definition of $A$, we set $\alpha = 3$ if $r = 0$; $\alpha = 2$ if $r = 1$. The 1-factors $A$, $B_r$ have trivial stabilizer in $H$. Moreover, the multiset $E(A) \cup E(B_r)$ is contained in the multiset $E(M)$, where $M = \{M_0, M_1, M_0, M_{n-\alpha}, M_{n+2}\}$. We note that the 1-factors in $M$ are pairwise distinct, since $n \geq 7$. Whence $t(M_a) = |E(M_a) \cap A| + |E(M_a) \cap E(B_r)| \leq \lambda$ for every $M_a \in M$. More specifically, $t(M_0) = (n - 2) + (r + 1) = \lambda$, $t(M_1) = n - r - 2 = \lambda - 1$, $t(M_{a}) = 1$ for every $a \in \{\alpha, n - \alpha, r + 2\}$. By Lemma 3 we construct a 1-factorization $F$ of $\lambda K_G$ that contains $A^H \cup B_r^H$.

We prove that $F$ is indecomposable. Firstly, note that if $\mathcal{F}_0 \subseteq \mathcal{F}$ is a 1-factorization of $\lambda_0 K_G$, $\lambda_0 < \lambda$, then $F^H \subseteq \mathcal{F}_0$ or $F^H \cap \mathcal{F}_0 = \emptyset$ for $F \in \{A, B_r\}$. This follows from Lemma 4 by observing that $A$ and $M_0$ satisfy condition (i). The same can be repeated for $B_r$ and $M_{r+2}$. If $F^H \subseteq \mathcal{F}_0$ and $B_r^H \subseteq \mathcal{F}_0$, then each edge of $M_0$ appears $\lambda$ times in the multiset $E(\mathcal{F}_0)$ and then $\lambda_0 = \lambda$, a contradiction. In the same manner, if $A^H \cap \mathcal{F}_0 = B_r^H \cap \mathcal{F}_0 = \emptyset$, then no edge of $M_0$ appears in the multiset $E(\mathcal{F}_0)$, a contradiction. Therefore, exactly one of the orbits $A^H$, $B_r^H$ is contained in $\mathcal{F}_0$. Without loss of generality, we can assume that $A^H \subseteq \mathcal{F}_0$ and $B_r^H \cap \mathcal{F}_0 = \emptyset$. Each edge of $M_0$ appears at least $n - 2$ times in the multiset $E(\mathcal{F}_0)$, that is, $\lambda_0 \geq n - 2$. Each edge of $M_1$ appears at least $n - 2 - r$ in the multiset $E(\mathcal{F} \setminus \mathcal{F}_0)$, that is, $\lambda - \lambda_0 \geq n - 2 - r$. By summing up these two relations, we have $\lambda \geq 2n - 4 - r$ and since $\lambda \leq n$, this yields $n \leq 5$, a contradiction. $\square$
Figure 3: The 1-factors $A$ and $B_r$, $r = 0, 1$, defined in the proof of Proposition 3

**Proposition 4.** Let $n \geq 9$ and $n + 1 \leq \lambda \leq 2n - 8$. There exists an indecomposable 1-factorization of $\lambda K_{2n}$ which is not simple.

**Proof.** Identify $\lambda K_{2n}$ with $\lambda K_G$. We distinguish the cases $n \neq 11$ and $n = 11$. For $n \neq 11$, we set $\lambda = n + r$, where $1 \leq r \leq n - 8$, and consider the 1-factors $A$ and $B = B_0$ in Figure 3. In the definition of $A$ we set $\alpha = 3$. We also define the 1-factors $C$ and $D$ in Figure 4.

For $n = 11$, we set $\lambda = 9 + r$, where $3 \leq r \leq 5$. We consider the 1-factor $A$ in Figure 3 where $\alpha = 2$ or $\alpha = 3$, according to whether $r = 3, 4$ or $r = 5$, respectively. For $r = 3, 4$ we also consider the 1-factor $B = \{[i_0, i_1] : 1 \leq i \leq r \} \cup \{(i_0, (i + 1)_1) : r + 1 \leq i \leq 10\} \cup \{(0_0, (r + 1)_1)\}$. For $r = 5$, we consider the 1-factor $B = B_0$ in Figure 3 and the 1-factor $C = \{[i_0, i_1] : 1 \leq i \leq 4\} \cup \{(i_0, (i + 1)_1) : 5 \leq i \leq 10, i \neq 6\} \cup \{(0_0, 7_1), (6_0, 5_1)\}$. We can construct a 1-factorization $\mathcal{F}$ of $\lambda K_G$ as described in Lemma 3. By Lemma 4 the 1-factorization $\mathcal{F}$ is indecomposable. The proof is similar to that of Proposition 3.

**Proposition 5.** Let $n \geq 9$ and $\lambda = 2n - 7$. There exists an indecomposable 1-factorization of $\lambda K_{2n}$ which is not simple.

**Proof.** We set $\lambda = n + r$ with $r = n - 7$ and consider the 1-factors in $\mathcal{F}' = \{A, B, C, D\}$, where $A$ and $B = B_0$ are described in Figure 3. In the definition of $A$ we set $\alpha = 3$. The 1-factors $C$ and $D$ are defined in Figure 4. The assertion follows from Lemma 4.

**Proposition 6.** Let $n \geq 9$ and $2n - 6 \leq \lambda \leq 2n - 3$. There exists an indecomposable 1-factorization of $\lambda K_{2n}$ which is not simple.

**Proof.** Identify $\lambda K_{2n}$ with $\lambda K_G$ and set $\lambda = 2n - r$, where $3 \leq r \leq 6$. We consider the 1-factors $A$ and $B = B_1$ in Figure 3. In the definition of the
1-factor $A$, the parameter $\alpha$ assumes the value $\alpha = 2$ if $r \in \{3, 5, 6\}$; $\alpha = 4$ if $r = 4$. We define the 1-factor $C$ as in Figure 5. We also consider the 1-factor $D_r$ in Figure 6 for $r = 3, 4$ and in Figure 7 for $r = 5, 6$. We can apply Lemma 3 and construct a 1-factorization $F$ of $\lambda K_G$ as prescribed. By Lemma 4 we can prove that $F$ is indecomposable.

**Proposition 7.** Let $n \geq 9$ and $\lambda = 2n - 2$. There exists an indecomposable 1-factorization of $\lambda K_{2n}$ which is not simple.

**Proof.** Identify $\lambda K_{2n}$ with $\lambda K_G$. We distinguish the cases $n \geq 11$ and $n = 9, 10$. For $n \geq 11$ we consider the 1-factor $A$ in Figure 6 with $\alpha = 2$ and the 1-factor $B_1 = D$. We also consider the 1-factors $B, C$ in Figure 8.
Figure 6: The 1-factor $D_r$, $r = 3, 4$, defined in the proof of Proposition 6.

Figure 7: The 1-factor $D_r$, $r = 5, 6$, defined in the proof of Proposition 6.
Lemma 3. By Lemma 4, we can prove that \( F \cup \{ C \} \) is indecomposable. For \( n, \lambda \) as described in Lemma 3. By Lemma 4, we can prove that \( F \cup \{ C \} \) is indecomposable.

For \( n = 9, 10 \), we consider two copies of the 1-factor \( A \) in Figure 4. We denote by \( A \) the copy with \( \alpha = 2 \) and by \( B \) the copy with \( \alpha = 3 \) or 4, according to whether \( n = 10 \) or \( n = 9 \), respectively. We consider the 1-factors \( C, D \) and \( R_n \), where \( C = \{(i_0, (i + 1)_1) : 2 \leq i \leq n - 1\} \cup \{(0_0, 2_1), (1_0, 1_1)\}; \)
\( D = \{(i_0, (i + 1)_1) : 2 \leq i \leq n - 3\} \cup \{(0_0, (n - 1)_1), [(n - 1)_0, 0_1], [(n - 2)_0, 2_1], [1_0, 1_1]\}; \)
\( R_9 = \{(i_0, (i + 1)_1) : 0 \leq i \leq 2\} \cup \{(i_0, (i + 2)_1) : 3 \leq i \leq 7\} \cup \{(0_0, 1_1), [0_0, 0_1]\}; \)
\( R_{10} = \{(i_0, (i + 1)_1) : 0 \leq i \leq 2\} \cup \{(i_0, (i + 2)_1) : 3 \leq i \leq 7\} \cup \{(0_0, 1_1), [0_0, 0_1]\}. \)

We can construct a 1-factorization \( F \) as described in Lemma 4. By Lemma 4 we can prove that \( F \) is indecomposable.

**Proposition 8.** Let \( n \geq 9 \) and \( 2n - 1 \leq \lambda \leq 2n \). There exists an indecomposable 1-factorization of \( \lambda K_{2n} \) which is not simple.

**Proof.** Identify \( \lambda K_{2n} \) with \( \lambda K_G \). We consider two copies of the 1-factor \( A \) in Figure 4. We denote by \( A \) the copy with \( \alpha = 2 \) (\( \alpha = 4 \) if \( n = 9 \) and \( \lambda = 18 \)) and by \( B \) the copy with \( \alpha = 3 \). We also consider the 1-factors \( C, D, R \). For \( n \geq 9 \) and \( (n, \lambda) \neq (9, 18) \), the 1-factor \( C \) corresponds to the 1-factor \( B_1 \) in Figure 4. For \( (n, \lambda) = (9, 18) \) it corresponds to the 1-factor \( C \) in Figure 4. For \( n \geq 9 \) and \( \lambda = 2n - 1 \), the 1-factor \( D \) corresponds to the 1-factor \( B_0 \) in Figure 4. For \( n > 9 \) and \( \lambda = 2n \), the 1-factor \( D \) is defined in Figure 4. For \( n = 9 \) and \( \lambda = 2n \), it corresponds to the 1-factor \( B_0 \) in Figure 4. For \( n \geq 9 \) and \( (n, \lambda) \neq (9, 18) \), the 1-factor \( R \) is defined in Figure 4. In the definition of \( R \) we set \( \beta = 3 \) or \( \beta = 4 \) according to whether \( \lambda = 2n - 1 \) or \( \lambda = 2n \), respectively (\( \beta = 5 \) if \( n = 10 \) and \( \lambda = 2n \)). For \( (n, \lambda) = (9, 18) \), we set \( R = \{(i_0, (i + 1)_1) : 0 \leq i \leq 4, i = 8\} \cup \{(i_0, (i + 2)_1) : 5 \leq i \leq 6\} \cup \{(7_0, 6_1)\}. \)

We construct a 1-factorization \( F \) of \( \lambda K_G \) as described in Lemma 3. By Lemma 4 we can prove that \( F \) is indecomposable.

Combining the constructions in the previous propositions, the following result holds.
Theorem 1. Let $n \geq 9$. For every $(n - 2)/3 \leq \lambda \leq 2n$ there exists an indecomposable 1-factorization of $\lambda K_{2n}$ which is not simple.

4 Simple and indecomposable 1-factorizations.

In this section we use Theorem 1 and Corollary 4.1 in [4] to find simple and indecomposable 1-factorizations of $\lambda K_{2n}$. We also generalize the result in [4] about the existence of simple and indecomposable 1-factorizations of $\lambda K_{2n}$, where $2n - 1$ is a prime and $\lambda = (n - 1)/2$. We recall the statement of Corollary 4.1.

Corollary 4.1. [4] If there exists an indecomposable 1-factorization of $\lambda K_{2n}$ with $\lambda \leq 2n-1$, then there exists a simple and indecomposable 1-factorization of $\lambda K_{2s}$ for $s \geq 2n$.

The following results hold.

Theorem 2. Let $s \geq 18$. For every $2 \leq \lambda \leq 2[s/2] - 1$ there exists a simple and indecomposable 1-factorization of $\lambda K_{2s}$.

Proof. For every $n \geq 9$ we set $I_n = \{ \lambda \in \mathbb{Z} : (n - 2)/3 \leq \lambda \leq 2n - 1 \}$ and note that $I_n \cup I_{n+1} = \{ \lambda \in \mathbb{Z} : (n - 2)/3 \leq \lambda \leq 2(n + 1) - 1 \}$. Consider $s \geq 2n \geq 2 \cdot 9$. By Corollary 4.1 of [4], for every $\lambda \in I_n$ there exists a simple and indecomposable 1-factorization of $\lambda K_{2s}$. Since we can consider $9 \leq n \leq [s/2]$, we obtain a simple and indecomposable 1-factorization of $\lambda K_{2s}$ for every $\lambda \in \bigcup_{n=9}^{[s/2]} I_n = \{ \lambda \in \mathbb{Z} : 7/3 \leq \lambda \leq 2[s/2] - 1 \}$. Since $s \geq 2 \cdot 5$, from Proposition 2 and Corollary 4.1 we also obtain a simple and indecomposable 1-factorization of $\lambda K_{2s}$ for $\lambda = 2$. Hence the assertion follows.

Theorem 3. Let $2n - 1$ be a prime power and let $\lambda = n - 1$. There exists a simple and indecomposable 1-factorization of $\lambda K_{2n}$.

Proof. Let $2n - 1 = p^m$, with $p$ an odd prime and $m \geq 1$. Let $GF(p^m)$ be the Galois field of order $p^m$ and let $v$ be a generator of the cyclic multiplicative group $GF(p^m)^* = GF(p^m) - \{0\}$. It is well known that $v$ is a root of an irreducible polynomial over $\mathbb{Z}_p$ of degree $m$, the field $GF(p^m)$ is an algebraic extension of $\mathbb{Z}_p$ and it is $GF(p^m) = \mathbb{Z}_p(v) = \{a_0 + a_1v + a_2v^2 + \cdots + a_{m-1}v^{m-1} | a_i \in \mathbb{Z}_p\}$. Let $V = GF(p^m) \cup \{\infty\}$, $\infty \notin GF(p^m)$, and identify the vertices of the complete multigraph $(n - 1)K_{2n}$ with the elements of $V$, thus the edges are in the multiset $(n - 1)\binom{V}{2}$. The affine linear group $AGL(1,p^m) = \{\phi_{b,a} : a,b \in GF(p^m), b \neq 0\}$ is a permutation group on $V$ where each $\phi_{b,a}$ fixes $\infty$ and maps $x \in V \setminus \{\infty\}$ onto $xb + a$. This action extends to edges and 1-factors. For each edge $e = [x,y]$ and for each 1-factor $F$, we set $e^{\phi_{b,a}} = eb + a = [xb + a, yb + a]$ and $F^{\phi_{b,a}} = Fb + a$.

If $x \neq \infty$ and $y \neq \infty$ we call $\partial e = \{\pm(y-x)\}$ the difference set of $e$. 

13
Consider the following set of edges:

\[ A_0 = \{ [(2i - 1)v + a_1v^2 + \cdots + a_{m-1}v^{m-1}, 2i + a_1v + \cdots + a_{m-1}v^{m-1}], 1 \leq i \leq (p-1)/2, a_r \in \mathbb{Z}_p, 1 \leq r \leq m - 1 \} \]

\[ A_1 = \{ [(2i - 1)v + a_2v^2 + \cdots + a_{m-1}v^{m-1}, (2i)v + a_2v^2 + \cdots + a_{m-1}v^{m-1}], 1 \leq i \leq (p-1)/2, a_r \in \mathbb{Z}_p, 2 \leq r \leq m - 1 \} \]

\[ A_2 = \{ [(2i - 1)v^2 + \cdots + a_{m-1}v^{m-1}, (2i)v^2 + \cdots + a_{m-1}v^{m-1}], 1 \leq i \leq (p-1)/2, a_r \in \mathbb{Z}_p, 3 \leq r \leq m - 1 \} \]

\[ \cdots \]

\[ A_{m-1} = \{ [(2i - 1)v^{m-1}, (2i)v^{m-1}], 1 \leq i \leq (p-1)/2 \} \]

Obviously if \( m = 1 \) we just have \( \mathbb{Z}_p(v) = \mathbb{Z}_p \) and we just take the set \( A_0 \).

Observe that each set \( A_j \), \( j = 0, \ldots, m-1 \), contains exactly \( p^{m-j-1}(p-1)/2 \) edges with difference set \( \{ \pm v^j \} \). Let \( F \) be the 1-factor given by: \( \{ [0, \infty] \} \cup A_0 \cup A_1 \cup \cdots \cup A_{m-1} \). The set \( F = F^{AGL(1,v^m)} \) is a simple and indecomposable 1-factorization of \((n-1)K_{2n}\).

5 Conclusions.

Our methods of construction can be used to obtain indecomposable 1-factorizations of \( \lambda K_{2n} \) for some values of \( \lambda > 2n \). These 1-factorizations are not simple and do not provide simple 1-factorizations, since for these values of \( \lambda \) we cannot apply Corollary 4.1 of [4].

As remarked in Section 1, a necessary condition for the existence of an indecomposable 1-factorization of \( \lambda K_{2n} \) is \( \lambda < [n(2n-1)]^{n(2n-1)}(2n^3+n^2-n+1) \).

It would be interesting to know whether for every \( n \geq 4 \) there exists a parameter \( \lambda(n) < [n(2n-1)]^{n(2n-1)}(2n^3+n^2-n+1) \) depending from \( n \) such that for every \( \lambda > \lambda(n) \) there is no indecomposable 1-factorization of \( \lambda K_{2n} \).

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