SUSPENSION SPLITTING AND COHOMOTOPY SETS OF SIMPLY CONNECTED 7-MANIFOLDS

PENGCHENG LI AND RUIZHI HUANG

Abstract. Let $M$ be a closed simply connected 7-manifold. In this paper we give homotopy decompositions of the reduced suspension space $\Sigma M$ into a wedge sum of simple spaces when localized at a set of primes. As an application, we combine them to investigate the cohomotopy sets $\pi^k(M)$ or the $p$-local cohomotopy sets $\pi^k(M; \mathbb{Z}_p)$.

1. Introduction

Seven dimensional manifolds are important in geometry and topology. For instance, in differential geometry, among others there has been a steadily increasing interest in $G_2$-manifolds. Following the seminal work of Donaldson-Thomas [DT98], in the new century the gauge theory of $G_2$-manifolds [KLL20] was developed rapidly. Passing to algebraic topology of gauge theory, there is a rough principle [The10, So19, Hua22] that the homotopy of gauge groups is largely controlled by the homotopy type of the underlying manifolds. In geometric topology, Kreck [Kre18] and Crowley-Nordström [CN19] independently gave a complete classification of closed smooth 2-connected 7-manifolds in terms of a system of algebraic invariants. Based on his famous work [Kre99], Kreck [Kre18] also made further contributions to the classification of simply connected 7-manifolds with torsion free second homology.

In recent years the suspension homotopy of manifolds has attracted a lot of interest of many topologists. For instance, So and Theriault [ST], Li [Li24] determined the homotopy type of suspended 4-manifolds, Huang [Hua21], Li and Zhu [LZ24], Amelotte, Cutler and So [ACS] studied the problem for certain 5-manifolds, while Huang [Hua23] and Cutler and So [CS22] studied the homotopy type of 6-manifolds. In addition, Membrillo-Solis [MS19] studied the problem for $S^3$-bundle over $S^4$ at large primes. All of these works can be applied to characterize important invariants in geometry and mathematical physics, such as reduced $K$-groups and gauge groups.

In this paper, we are interested in the suspension homotopy of 7-manifolds. Let $M$ be a closed simply connected 7-manifold whose reduced integral homology groups $H_*(M)$ are

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given by the following table:

| $i$ | 2 | 3 | 4 | 5 | 0,7 otherwise |
|-----|---|---|---|---|---------------|
| $H_* (M)$ | $\mathbb{Z}^l \oplus H$ | $\mathbb{Z}^k \oplus T$ | $\mathbb{Z}^k \oplus H$ | $\mathbb{Z}^l \oplus \mathbb{Z}$ | $0$ |

where $r, d \geq 0$, and $H$ and $T$ are two finitely generated abelian torsion groups. Note that for Kreck’s manifolds with torsion free second homology, we have $H = 0$. Let $P^n (A)$ be the Moore space such that the reduced cohomology $\tilde{H}^*(P^n (A); \mathbb{Z})$ is isomorphic to the abelian group $A$ if $* = n$ and $0$ otherwise [Nei10]. For each $k \geq 1$, there is a canonical homotopy cofibration sequence for $P^n (k) = P^n (\mathbb{Z}/k)$:

$$S^{n-1} \xrightarrow{k} S^{n-1} \xrightarrow{i_{n-1}} P^n (k) \xrightarrow{q_n} S^n,$$

where $k$ is the degree $k$ map, $i_{n-1}$ and $q_n$ are the canonical inclusion and pinch maps, respectively. For based CW-complexes $X, Y$, we denote $X \simeq (\frac{1}{3}) Y$ if $X$ is homotopy equivalent to $Y$ when localized away from 2. By Lemma 2.6, for each $r$ there is a lift $\overline{\alpha}_r : S^6 \rightarrow P^3 (3^r)$ such that $q_3 \circ \overline{\alpha}_r = \alpha$, where $\alpha \in \pi_6 (S^3)$ is a generator of order 3. Recall the Steenrod reduced power operation $P^1 = P^1 : H^*(\cdot; \mathbb{Z}/3) \rightarrow H^{*+4}(\cdot; \mathbb{Z}/3)$. By naturality, the fact that $\alpha$ is detected by $P^1$ [Har02] implies that the maps $\overline{\alpha}_r, i_3 \alpha$ and their suspensions are also detected by $P^1$.

**Theorem 1.1.** Let $M$ be a closed simply-connected 7-manifold with $H_4 (M)$ given by (1.1).

1. If $P^1 : H^3 (M; \mathbb{Z}/3) \rightarrow H^7 (M; \mathbb{Z}/3)$ is trivial, then there is a homotopy equivalence

$$\Sigma M \simeq (\frac{1}{4}) \bigvee_{i=1}^l (S^3 \vee S^6) \vee \bigvee_{i=1}^k (S^4 \vee S^5) \vee P^4 (H) \vee P^5 (T) \vee P^6 (H) \vee S^8.$$

2. If $P^1 : H^3 (M; \mathbb{Z}/3) \rightarrow H^7 (M; \mathbb{Z}/3)$ is nontrivial, then the homotopy type of $\Sigma M$ can be described as follows.

(a) If there exists $x \in H^3 (M; \mathbb{Z}/3)$ such that $P^1 (x) \neq 0$ and $x \in \text{im}(\beta_r)$ for some $r$, then there is a homotopy equivalence

$$\Sigma M \simeq (\frac{1}{4}) \bigvee_{i=1}^l (S^3 \vee S^6) \vee \bigvee_{i=1}^k (S^4 \vee S^5) \vee P^4 \left(\frac{H}{\mathbb{Z}/3^r}\right) \vee P^5 (T) \vee P^6 (H) \vee X^8 (\overline{\alpha}_r),$$

where $X^8 (\overline{\alpha}_r) = P^4 (3^r) \cup \overline{\alpha}_r e^8$ and $r_u$ is the minimum of $r$ such that $x \in \text{im}(\beta_r)$ and $P^1 (x) \neq 0$.

(b) If for any $x \in H^3 (M; \mathbb{Z}/3)$ with $P^1 (x) \neq 0$, there hold $x \notin \text{im}(\beta_r)$ and $\beta_s (x) = 0$ for any $r, s \geq 1$, then there is a homotopy equivalence

$$\Sigma M \simeq (\frac{1}{4}) \bigvee_{i=1}^l (S^3 \vee S^6) \vee \bigvee_{i=1}^{k-1} (S^4 \vee S^5) \vee P^4 (H) \vee P^5 (T) \vee P^6 (H) \vee S^5 \vee (S^4 \cup \overline{\alpha}_e e^8).$$
homotopy type of $\Sigma M$ is closely related to the cohomotopy sets $\pi^*$ maps from a smooth closed manifold $M$ to $S^k$ that depends on $M$, $b = 0$ if $Sq^2$ acts trivially on $H^3(M; \mathbb{Z}/2)$. Assume that the Steenrod square $Sq^2$ acts trivially on $H^2(M; \mathbb{Z}/2)$, then localized at 2, $\Sigma M$ is homotopy equivalent to one of the following complexes:

\[ \begin{align*}
(i) \quad & \bigvee_{i=1}^{l} (S^3 \vee S^6) \vee \bigvee_{i=1}^{k} (S^4 \vee S^5) \vee \bigvee_{i=1}^{b} C^6_\eta \vee \bigvee (S^3 \cup_{-\nu} \Sigma \nu' e^8), \varepsilon \in \{0, 1\}; \\
(ii) \quad & \bigvee_{i=1}^{l} (S^3 \vee S^6) \vee \bigvee_{i=1}^{k} (S^4 \vee S^5) \vee \bigvee_{i=1}^{b} C^6_\eta \vee \bigvee (S^4 \cup_{\nu} \Sigma \nu' e^8); \\
(iii) \quad & \bigvee_{i=1}^{l} (S^3 \vee S^6) \vee \bigvee_{i=1}^{k} (S^4 \vee S^5) \vee \bigvee_{i=1}^{b} C^6_\eta \vee \bigvee (S^3 \cup_{\nu} S^4 \cup_{\nu} \Sigma \nu' + 2 \Sigma \nu' e^8), \varepsilon \in \{0, 1\}; \\
(iv) \quad & \bigvee_{i=1}^{l} (S^3 \vee S^6) \vee \bigvee_{i=1}^{k} (S^4 \vee S^5) \vee \bigvee_{i=1}^{b} C^6_\eta \vee \bigvee (S^3 \cup_{\nu} C^6_\eta \cup_{\nu} \Sigma \nu' + i_4 \Sigma \nu' e^8), \varepsilon \in \{0, 1\}. 
\end{align*} \]

Here $C^6_\eta = \Sigma^2 CP^2$, $\eta = \eta_3 \in \pi_4(S^3)$ and $\nu' \in \pi_6(S^3) \cong \mathbb{Z}/4$ are generators, and $i_4: S^4 \rightarrow C^6_\eta$ is the inclusion map of the bottom cell.

As can be seen from [ST, Hua21, CS22], the homotopy type of the suspension space $\Sigma M$ determines homotopy type of gauge groups of principle bundles over $M$. Theorem 1.1 and Theorem 1.2 can be directly applied to characterize the $p$-local homotopy type of gauge groups; the detailed discussion is similar to that in [CS22, Section 5.2] and are left to readers.

Due to the Pontryagin-Thom construction (cf. [Kos93, Chapter IX, Theorem 5.5]), the characterization of the cohomotopy set $\pi^k(M) = [M, S^k]$ of homotopy classes of based maps from a smooth closed manifold $M$ to $S^k$ is a hot topic in geometry and topology, see [Tay12, KMT12] for a complete characterization of $\pi^k(M)$ of 4-manifolds $M$. The homotopy type of $\Sigma M$ is closely related to the cohomotopy sets $\pi^k(M)$ or the $p$-local cohomotopy sets $\pi^k(M; \mathbb{Z}(p)) = [M, S^k(p)]$, where $p$ is a prime. On the one hand, there holds the suspension isomorphism $\pi^k(M) \cong \pi^{k+1}(\Sigma M)$ in the stable range $\dim(M) \leq 2k - 2$. On the other hand, unstable homotopy theory emerges as a powerful tool for characterizing
the cohomotopy sets $\pi^k(M)$ in unstable range, see [LPW23, LZ24, ACS] for instance. In the second part we apply the suspension space of the cell structure of $M$ and Theorem 1.1 to study the cohomotopy sets $\pi^*(M)$ of a closed simply connected 7-manifold $M$. Especially, we get the following theorem.

**Theorem 1.3.** Let $M$ be a closed simply-connected 7-manifold with $H_*(M)$ given by (1.1).

1. $\pi^7(M) \cong H^7(M) \cong \mathbb{Z}$, $\pi^k(M) = 0$ for $k = 1$ or $k \geq 8$.
2. $\pi^2(M)$ is a left $\pi^3(M)$-torsor, that is, the natural left action of $\pi^3(M)$ on $\pi^2(M)$ is transitive and free.
3. Localized away from 2, we have $\pi^5(M) \cong H^5(M)$; if $M$ is smooth, spin and contains no 2-torsion in homology, then there is a short exact sequence

$$0 \to \mathbb{Z}/2 \to \pi^5(M) \to H^5(M) \to 0,$$

which splits if the Steenrod square $Sq^2$ acts trivially on $H^3(M;\mathbb{Z}/2)$.
4. The suspension $E_{\pi^5}$: $\pi^3(M) \to \pi^4(M)$ is an isomorphism if $E_{\pi^5}$ is localized away from 2, or if $M$ is nonspin.
5. For each prime $p \geq 5$, the $p$-local cohomotopy Hurewicz map $h_{(p)}: \pi^4(M;\mathbb{Z}_{(p)}) \to H^4(M;\mathbb{Z}_{(p)})$ is surjective. For $\beta \in H^4(M;\mathbb{Z}_{(p)})$, let $e \in \pi^4(M;\mathbb{Z}_{(p)})$ be some element with $h_{(p)}(e) = \beta$. There is a bijection between $\iota_{(p)}^{-1}(\beta)$ and the cokernel of the homomorphism

$$\psi_e: H^3(M;\mathbb{Z}_{(p)}) \to H^7(M;\mathbb{Z}_{(p)}), \quad \psi_e(\gamma) = \gamma \circ \delta_e,$$

where $\delta_e = e^*(i) \in H^4(M;\mathbb{Z}_{(p)})$ with $i \in H^4(S^4_{(p)};\mathbb{Z}_{(p)})$ the fundamental class.

For a simply connected 7-manifold $M$, the well-known Wu’s formula $Sq^2(x) = x \circ w_2(M)$ [Wu54], where $w_2(M)$ is the second Stiefel-Whitney class of $M$, implies that $Sq^2$ acts trivially on $H^3(M;\mathbb{Z}/2)$ if and only $w_2(M) = 0$, that is, $M$ is spin. Localized away from 2, the concrete group structure of $\pi^3(M) \cong \pi^4(\Sigma M)$ is also computed, see Corollary 5.7. When $M$ is spin, the suspension $E_{\pi^5}$ in (4) is not surjective, see Example 5.8. Theorem 1.3 (5) is also generalized to characterize the $p$-local cohomotopy set $\pi^4(M;\mathbb{Z}_{(p)})$ of a CW-complex $M$ of dimension at most 9, see Theorem 5.12.

The paper is organized as follows. In Section 2 we compute certain homotopy groups of odd primary Moore spaces and some suspensions of $\mathbb{C}P^2$, and provide a comprehensive listing of globally utilized lemmas. Section 3 focuses on the application of homology decomposition techniques to analyze the reduced suspension space of a simply connected 7-manifold $M$, assuming that $M$ exhibits no 2-torsion in its homology. Moving forward, Section 4 delves into the homotopy type of the suspension $\Sigma M$ when localized away from 2, culminating in the proof of Theorem 1.1. At the end of this section we sketch the proof of Theorem 1.2. Section 5 is dedicated to the study of ($p$-local) cohomotopy sets for a simply-connected 7-manifold, leading to the proof of Theorem 1.3.

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2. Preliminaries

In this section, we provide the various technical results used in the paper, including some homotopy theory of odd primary Moore spaces and a useful criterion for suspension spaces (Lemma 2.11).

Throughout the paper all spaces are based CW-complexes, all maps are base-point-preserving and are identified with their homotopy classes in notation. For an abelian group $G$, denote by

$$G \cong C_1(x_1) \oplus \cdots \oplus C_n(x_n)$$

if $G$ has generators $x_1, \cdots, x_n$, where $C_i$ are cyclic groups. We shall frequently use the homotopy groups (cf. [Tod62]):

- $\pi_3(S^2) \cong \mathbb{Z}/\eta$, $\pi_{n+1}(S^n) \cong \mathbb{Z}/2\langle \eta_2 \rangle$ for $n \geq 3$, where $\eta_n = \Sigma^{n-2}\eta$ for $n \geq 2$;
- $\pi_{n+2}(S^n) \cong \mathbb{Z}/2\langle \eta_1^2 \rangle$, where $\eta_1^2 = \eta_n\eta_{n+1}$;
- $\pi_6(S^3) \cong \mathbb{Z}/12\langle \nu \rangle$, $\pi_7(S^4) \cong \mathbb{Z}/2\langle \nu_4 \rangle \oplus \mathbb{Z}/12\langle \Sigma\nu \rangle$, and $\pi_{n+3}(S^n) \cong \mathbb{Z}/24\langle \nu_n \rangle$ for $n \geq 5$. We have $\eta_3 = 6\Sigma\nu$, $\Sigma^2\nu = 2\nu_5$.

Let $n \geq 3$. For any (finitely generated) abelian group $A$, let $P^n(A)$ be the Moore space such that $\tilde{H}^*(P^n(A);\mathbb{Z}) \cong A$ if $* = n$ and 0 otherwise [Nei10]. Recall that there holds a homotopy equivalence

$$P^n(A \oplus B) \simeq P^n(A) \vee P^n(B)$$

for any abelian groups $A$ and $B$. For each $k \geq 1$, denote $P^n(k) = P^n(\mathbb{Z}/k)$ the mod $k$ Moore space. For any spaces $X$, we abuse the notations $i_n: S^n \to X$ and $q_m: X \to S^m$ to denote the canonical inclusion and pinch map, respectively. For example, we have the canonical cofibration sequence for $P^n(k)$:

$$S^{n-1} \xrightarrow{k} S^{n-1} \xrightarrow{i_{n-1}} P^n(k) \xrightarrow{q_n} S^n.$$ 

**Lemma 2.1** (cf. [Nei10]). Let $m, n \geq 3$, $r, s \geq 1$ be integers and let $p$ be an odd prime. There is a homotopy equivalence

$$P^m(p^r) \wedge P^n(p^s) \simeq P^{m+n}(p^{\min(r,s)}) \vee P^{m+n-1}(p^{\min(r,s)}).$$

Further, if $q_1$ and $q_2$ are distinct primes, then $P^n(q_1^r) \wedge P^n(q_2^s)$ is contractible.

We have the following stable homotopy groups of Moore space.

**Lemma 2.2** (cf. [Bau85] or [BH91]). Let $p$ be an odd prime and let $m, n \geq 3$ and $r, s \geq 1$ be integers.

1. $[S^n, P^{n+1}(p^r)] \cong \mathbb{Z}/p^r\langle i_n \rangle$.
2. $[S^{n+1}, P^{n+1}(p^r)] = [P^{n+1+i}(p^r), S^n] = 0$ for $i = 0, 1$.
3. $[P^{n+1}(p^r), P^n(p^s)] = 0$ and $[P^n(p^r), P^m(q^s)] = 0$ for coprime $p, q$. 

SUSPENSION SPLITTING AND COHOMOTOPY SETS OF SIMPLY CONNECTED 7-MANIFOLDS 5
Lemma 2.3 (cf. [CMN79, Nei81, CMN87]). Let $p$ be an odd prime.

1. There is a p-local homotopy equivalence

\[
\Phi: T^{2n+1}\{p^r\} \times \Omega\left(\bigvee_{\alpha} P^n_{n_\alpha}(p^r)\right) \xrightarrow{\approx(p)} \Omega P^{2n+1}(p^r),
\]

where $n_\alpha \geq 4n$ and for each $n$, there exists exactly one $\alpha$ such that $n_\alpha = 4n$; the space $T^{2n+1}\{p^r\}$ is defined by the p-local homotopy fibration sequence

\[
S^{2n-1} \times \prod_{k=1}^{\infty} S^{2p^k n-1}(p^{r+1}) \xrightarrow{\partial_r} T^{2n+1}\{p^r\} \to \Omega S^{2n+1}.
\]

2. There is a p-local homotopy equivalence

\[
\Psi: S^{2n-1} \times \prod_{k=1}^{\infty} S^{2p^k n-1}(p^{r+1}) \times \Omega\Sigma\left(\bigvee_{\alpha} P^n_{n_\alpha}(p^r)\right) \xrightarrow{\approx(p)} \Omega \Sigma P^{2n+1}(p^r),
\]

where $\bigvee P^n_{n_\alpha}(p^r)$ is an infinite bouquet of mod $p^r$ Moore spaces, with only finitely many Moore spaces in each dimension and the least value of $n_\alpha$ being $4n - 1$.

Lemma 2.4. For any odd prime $p$ and positive integer $r$, there holds

\[
\pi_7(P^5(p^r)) \cong \mathbb{Z}/p^r/\langle [1_P, 1_P] \circ v_7 \rangle \oplus \mathbb{Z}/(p, 3)\langle i_4(\Sigma\alpha) \rangle,
\]

where $[1_P, 1_P]$ denotes the Whitehead product of the identity $1_P$ on $P^4(p^r)$ and $v_7 \in \pi_7(\Sigma P^4(p^r) \wedge P^4(p^r))$ is the generator corresponding to the generator $i_7 \in \pi_7(P^8(p^r))$.

Proof. Taking $n = 2$ in (2.2), we compute that

\[
\pi_7(T^5(p^r)) \cong \pi_6(\Omega P^5(p^r)) \cong \pi_6(T^5\{p^r\}) \oplus \mathbb{Z}/p^r,
\]

\[
\pi_6(T^5\{p^r\}) \cong \pi_6(S^3)(p) \cong \mathbb{Z}/(3, p).
\]

Thus $\pi_7(T^5(p^r)) \cong \mathbb{Z}/p^r \oplus \mathbb{Z}/(3, p)$.

Let $\Phi_1$ and $\Phi_2$ be the restriction of the homotopy equivalence $\Phi$ (2.2) to $T^5\{p^r\}$ and $\Omega P^8(p^r)$, respectively. By the construction of $\Phi$, there is a homotopy commutative diagram

\[
\xymatrix{
\pi_6(P^4(p^r) \wedge P^4(p^r)) \ar[r]^{\langle E, E \rangle_2} & \pi_6(\Omega\Sigma P^4(p^r)) & \pi_6(\Omega\Sigma P^4(p^r) \wedge P^4(p^r)) \ar[l]_{\langle \Phi_2 \rangle_1} \ar[u]_{E_2}.
}
\]
where $\langle E, E \rangle$ denotes the Samelson product of the suspension $E: P^I(p') \to \Omega \Sigma P^I(p')$, and the vertical $E_\ast$ is the suspension isomorphism, the Freudenthal suspension theorem. Since the adjoint of $\langle E, E \rangle$ is the Whitehead product $[1_p, 1_p]$, we get $\Phi_2 \simeq [1_p, 1_p] \circ \tau_7$, which generates the direct summand $\mathbb{Z}/p'$ of $\pi_7(P^5(p'))$.

When $p = 3$, the composition

$$\Phi_1: S^3 \hookrightarrow S^3 \times \prod_{i=1}^\infty S^{2^i 3^{r+i-1}} \xrightarrow{\partial_r} T^3 \{3^r \} \xrightarrow{\Phi_1} \Omega P^5(3^r)$$

is homotopic equivalent to the inclusion of the bottom cell; hence we have

$$\Phi_1 \simeq \Omega(\alpha) E: S^3 \to \Omega S^4 \to \Omega P^5(3^r).$$

By the group structure computed above, we get a split monomorphism

$$\pi_6(S^3) = \mathbb{Z}/3(\alpha) \xrightarrow{\partial_r} \pi_7(S^4) \xrightarrow{(i_4)_*} \pi_7(P^5(3^r)).$$

The proof of the Lemma is finished. □

**Lemma 2.5.** For any $n \geq 6$, we have

$$\pi_{n+2}(P^n(p')) = 0 \quad \text{for} \quad p \geq 5, \quad \pi_{n+2}(P^n(3^r)) \cong \mathbb{Z}/3(i_{n-1}(\Sigma^{n-4}\alpha)).$$

**Proof.** The statement for $p \geq 5$ is proved by [Hua23, Lemma 6.5]. Let $p = 3$. By the Freudenthal suspension theorem, it suffices to show $\pi_9(P^7(3^r)) = \mathbb{Z}/3(i_6(\Sigma^3\alpha)).$

Consider the cofibration $S^6 \xrightarrow{3^r} S^6 \xrightarrow{i_6} P^7(3^r)$. By the Blakers-Massey theorem (cf. [BM52]), it is a homotopy fibration up to degree 10. Hence, there is an exact sequence

$$\mathbb{Z}/3 \cong \pi_9(S^6) \xrightarrow{3^r} \pi_9(S^6) \cong \pi_9(P^7(3^r)) \to \pi_8(S^6) = 0,$$

where $\times 3^r$ is trivial. Hence $\pi_9(P^7(3^r)) \cong \pi_9(S^6) \cong \mathbb{Z}/3(\Sigma^3\alpha)$. Since $P^7(3^r)$ is contractible at any prime $p \neq 3$ and rationally, $\pi_9(P^7(3^r)) \cong \pi_9(P^7(3^r)) \cong \mathbb{Z}/3(\Sigma^3\alpha)$ as required. This completes the proof of the lemma. □

**Lemma 2.6.** $\pi_6(P^3(3^r)) \cong \mathbb{Z}/3^r \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3$. Moreover, one direct summand $\mathbb{Z}/3$ is generated by $\tilde{\alpha}_r$, which satisfies the formula

$$q_3 \tilde{\alpha}_r = \alpha, \quad B(\chi_s^r) \tilde{\alpha}_r = \tilde{\alpha}_s \text{ for } s \geq r.$$  

**Proof.** By (2.2) and (2.3) we compute that

$$\pi_5(\Omega P^3(3^r)) \cong \pi_5(T^3 \{3^r \}) \oplus \pi_5(\Omega P^4(3^r)),
\pi_5(T^3 \{3^r \}) \cong \pi_6(S^3) \cong \mathbb{Z}/3.$$

By [CS22, Lemma 3.1], $\pi_5(\Omega P^4(3^r)) \cong \mathbb{Z}/3^r \oplus \mathbb{Z}/3$. Hence $\pi_6(P^3(3^r)) \cong \mathbb{Z}/3^r \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3$.

On the other hand, by Lemma 2.3 (2.4) we have

$$\pi_5(\Omega F^3 \{3^r \}) \cong \pi_5(S^5 \{3^r \}) \oplus \pi_5(\Omega P^4(3^r)) \cong \pi_5(\Omega P^4(3^r)) \cong \mathbb{Z}/3^r \oplus \mathbb{Z}/3,
\pi_4(\Omega F^3 \{3^r \}) \cong \pi_4(S^5 \{3^r \}) \oplus \pi_4(\Omega P^4(3^r)) = 0.$$
It follows that there is a split short exact sequence
\[(\varepsilon) \quad 0 \to \pi_5(\Omega F^3(3^r)) \xrightarrow{(\Omega \beta_x)_3} \pi_5(\Omega P^3(3^r)) \xrightarrow{(\Omega q)_3} \pi_5(\Omega S^3(3^r)) \to \pi_4(\Omega F^3(3^r)) = 0,\]

Let \( \tilde{\alpha}_r \) be the generator such that \( q_3 \tilde{\alpha}_r = \alpha \). Set \( B(\chi_s^r)\tilde{\alpha}_r = x \cdot \tilde{\alpha}_s \) for some \( x \in \mathbb{Z}/3 \). Then by (2.1), when \( s \geq r \) there hold
\[
\alpha = q_3 \tilde{\alpha}_r = (q_3 B(\chi_s^r))\tilde{\alpha}_r = q_3(x \cdot \tilde{\alpha}_s) = x \cdot \alpha.
\]
Hence \( x = 1 \) and the Lemma is proved. \( \square \)

**Lemma 2.7.** \( \pi_7(P^4(3^r)) \cong \mathbb{Z}/3(\Sigma \tilde{\alpha}_r) \) and \( \pi_7(P^4(p^r)) = 0 \) for \( p \geq 5 \).

**Proof.** For an odd prime \( p \), by [CMN79, Nei81] there is a \( p \)-local homotopy equivalence
\[
\Omega P^4(p^r) \simeq_{(p)} S^3(p^r) \times \Omega(\bigvee_{k=0}^{\infty} P^{2k+7}(p^r)).
\]
It follows that
\[
\pi_7(P^4(p^r)) \cong \pi_6(S^3(p^r)) \oplus \pi_7(P^7(p^r)) = \pi_6(S^3(p^r)) \cong \mathbb{Z}/(3, p).
\]

Let \( p = 3 \) and consider the suspension homomorphism \( E_4 \colon \pi_6(P^3(3^r)) \to \pi_7(P^4(3^r)) \), where \( \pi_6(P^3(3^r)) \) is given by the split exact sequence \((\varepsilon)\). By construction (cf. [Nei81, page 69]), the restriction \( \Psi_4 \) of the homotopy equivalence \( \Psi \) to the factor \( \Omega \Sigma P^3(3^r) \) is a multiplicative extension of certain relative Samelson product \( P^3(3^r) \to \Omega F^3(3^r) \). It follows that the suspension \( E_4 \) sends the direct summand \( \pi_6(\Omega F^3(3^r)) \cong \pi_5(\Omega \Sigma P^3(3^r)) \cong \mathbb{Z}/3^r \oplus \mathbb{Z}/3 \) of \( \pi_6(P^3(3^r)) \) to zero. On the other hand, the following commutative square
\[
\begin{array}{ccc}
\pi_6(P^3(3^r)) & \xrightarrow{(q_3)_4} & \pi_6(S^3(3^r)) \\
\downarrow E_4 & & \downarrow E_4 \\
\pi_7(P^4(3^r)) & \xrightarrow{(q_4)_4} & \pi_7(S^4(3^r))
\end{array}
\]
implies that \( (q_4)_4 E_4 \), and hence \( E_5 \colon \pi_6(P^3(3^r)) \to \pi_7(P^4(3^r)) \) is nontrivial. Therefore, \( \Sigma \tilde{\alpha}_r \) is a generator of \( \pi_7(P^4(3^r)) \cong \mathbb{Z}/3 \). \( \square \)

For each \( n \geq 3 \), there is a canonical homotopy cofibration sequence
\[
S^{n+1} \xrightarrow{\pi_n\phi_n} S^n \xrightarrow{i_n} C^{m+2}_n = \Sigma^{n-2} CP^2 \to S^{n+2}.
\]

**Lemma 2.8.** Let \( n \geq 3 \) and let \( 1_n \) be the identity map on \( S^n \). There hold
\[
\pi_n(C^{m+2}_n) \cong \mathbb{Z}\langle i_n \rangle, \quad \pi_{n+1}(C^{m+2}_n) = 0, \quad \pi_{n+2}(C^{m+2}_n) \cong \mathbb{Z}\langle \tilde{\zeta} \rangle,
\]
\[
[C^{m+2}_n, S^{n+2}] \cong \mathbb{Z}\langle q_{n+2} \rangle, \quad [C^{m+2}_n, S^{n+1}] = 0, \quad [C^{m+2}_n, S^n] \cong \mathbb{Z}\langle \tilde{\zeta} \rangle,
\]
where \( \tilde{\zeta} \) and \( \tilde{\zeta} \) satisfy the formulae
\[
q_{n+2}\tilde{\zeta} = 2 \cdot 1_{n+2}, \quad \tilde{\zeta} i_n = 2 \cdot 1_n.
\]

**Proof.** See [AT66, Section 8] or [Li]. \( \square \)
Lemma 2.9. There hold isomorphisms
\[ \pi_7(C_7^\eta) \cong \mathbb{Z}\langle i_4\nu_4 \rangle \oplus \mathbb{Z}/6\langle i_4(\Sigma\nu') \rangle, \quad \pi_7(C_6^\eta) \cong \mathbb{Z}\langle \xi \rangle, \]
where \( \xi \) satisfies the formula \( \Sigma\xi = i_4\nu_4\eta_7 \).

Proof. See [Muk82, Proposition 8.3 and Lemma 8.5]. \( \square \)

The following two lemmas will be frequently used in the subsequent sections.

Theorem 2.10. [Hat02, Theorem 4H.3] Let \( X \) be a simply connected CW-complex. Denote \( H_i(X;\mathbb{Z}) \) by \( H_i \). Then there is a sequence of complexes \( X_i \) (which are called the \( i \)-th homology section of \( X \)) such that

- \( H_j(X_i;\mathbb{Z}) \cong H_j(X;\mathbb{Z}) \) for \( j \leq i \) and \( H_j(X_i;\mathbb{Z}) = 0 \) for \( j > i \);
- \( X_2 = P^3(H_2) \), and \( X_i \) is defined by a homotopy cofibration sequence
  \[ P^i(H_i) \xrightarrow{f_{i-1}} X_{i-1} \xrightarrow{\iota_{i-1}} X_i, \]
  where \( f_{i-1} \) is homologically trivial, that is, \( f_{i-1} \) induces trivial homomorphisms of integral homology groups;
- \( X \cong \hocolim \{ X_2 \xrightarrow{\iota_2} \cdots \xrightarrow{\iota_i} X_{i-1} \xrightarrow{\iota_i} X_i \xrightarrow{\iota_i} \cdots \} \).

Lemma 2.11 (see also Lemma 5.6 of [ST]). Consider a homotopy cofibration sequence of simply connected CW-complexes:

\[ S \xrightarrow{f} \bigvee_i A_i \vee B \xrightarrow{g} \Sigma C. \]

If the composition \( S \xrightarrow{f} \bigvee_i A_i \vee B \xrightarrow{g_j} A_j \), where \( g_j \) is the canonical pinch map, is null homotopic for each \( j \), then there is a homotopy equivalence

\[ \Sigma C \cong \bigvee_i A_i \vee D, \]

where \( D \) is the homotopy cofibre of the composition \( S \xrightarrow{f} \bigvee_i A_i \vee B \xrightarrow{g_B} B \) with \( g_B \) the projection .

Proof. Consider the diagram of homotopy cofibration sequences

\[ \begin{array}{ccc}
S & \xrightarrow{f} & \bigvee_i A_i \vee B \xrightarrow{g} \Sigma C \\
\downarrow f_i & & \downarrow g_i \\
S & \xrightarrow{f} & A_i \xrightarrow{g_i} F_i,
\end{array} \]

where \( g_i \) is the projection onto \( A_i \), \( F_i \) is the homotopy cofibre of \( f_i = g_i \circ f \), and \( g_i \) and \( h_i \) are the induced maps. By the condition \( f_i \) is null homotopic, and thus \( F_i \cong A_i \vee \Sigma S \). In
particular, \( g_i \) has a left homotopy inverse \( g_i \circ h_i \). Define the composition

\[
g : \Sigma C \xrightarrow{\mu'} \bigvee_i \Sigma C \xrightarrow{\digamma h_i} \bigvee_i F_i \xrightarrow{\digamma \varphi_i} \bigvee_i A_i,
\]

where \( \mu' \) is the iterated comultiplication of \( \Sigma C \). From the above, we see that \( g \) is a left homotopy inverse of \( g_A : \bigvee_i A_i \xrightarrow{i_A} \bigvee_i A_i \lor B \xrightarrow{g} \Sigma C \), where \( i_A \) is the inclusion map.

Consider the homotopy commutative diagram with homotopy cofibration rows:

\[
\begin{array}{ccc}
S & \xrightarrow{f} & \bigvee_i A_i \lor B \xrightarrow{g} \Sigma C \\
& \xrightarrow{i_A} & \\
& \downarrow & \downarrow \phi \\
& & \Sigma C \xrightarrow{h_D'} D' \\
& \xrightarrow{q_B \circ f} & B \xrightarrow{g_D} D
\end{array}
\]

where \( D \) is the homotopy cofibre of \( q_B \circ f \) by definition with the induced maps \( g_D \) and \( h_D \), \( D' \) is the homotopy cofibre of \( g_A \) with the induced map \( h_D' \), and the homotopy extension \( \phi \) of \( h_D \) exists since \( q_B \circ i_A \) is null homotopic. From the diagram, it is easy to see that \( \phi \) induces an isomorphism on homology, and hence is a homotopy equivalence by the Whitehead theorem. Further, since \( g \circ g_A \) is homotopic to the identity map, the top homotopy cofibration sequence splits to give a homotopy equivalence

\[
\Sigma C \simeq \bigvee_i A_i \lor D' \simeq \bigvee_i A_i \lor D.
\]

The lemma is proved. \( \square \)

### 3. Homology sections of \( \Sigma M \)

Let \( M \) be a closed simply connected 7-manifold with \( H_*(M) \) given by (1.1). By Lemma 2.10 and (1.1), the homology sections \( M_i \) of \( M \) are given by the following homotopy cofibration sequences:

\[
\begin{align*}
\bigvee_{i=1}^k S^2 \lor P^3(T) & \xrightarrow{f_2} M_2 \to M_3, \text{ where } M_2 \simeq \bigvee_{i=1}^l S^2 \lor P^3(H); \\
\bigvee_{i=1}^k S^3 \lor P^4(H) & \xrightarrow{f_3} M_3 \to M_4, \bigvee_{i=1}^l S^4 \xrightarrow{f_4} M_4 \to M_5 = M_6,
\end{align*}
\]

where the attaching maps \( f_2, f_3, f_4 \) are homologically trivial. It is clear that the suspension of the above homotopy cofibration sequences give the homology decomposition of \( \Sigma M \).
In the remainder of this paper we assume that the torsion groups $H$ and $T$ are both 2-torsion-free. Note that this assumption does affect the homotopy type of the suspension $\Sigma M_i$ when localized away from 2.

**Lemma 3.1.** There is a homotopy equivalence

$$\Sigma M_3 \simeq \bigvee_{i=1}^l S^3 \vee \bigvee_{i=1}^k S^4 \vee P^4(H) \vee P^5(T).$$

**Proof.** By Theorem 2.10 and (1.1), there is a homotopy cofibration sequence

$$\bigvee_{i=1}^k S^3 \vee P^4(T) \xrightarrow{\Sigma f_2} \Sigma M_2 \to \Sigma M_3,$$

where the map $\Sigma f_2$ is homologically trivial and $\Sigma M_2 \simeq \bigvee_{i=1}^l S^3 \vee P^4(H)$. Consider the following homologically trivial compositions

$$S^3 \hookrightarrow \bigvee_{i=1}^k S^3 \vee P^4(T) \xrightarrow{\Sigma f_2} \Sigma M_2 \twoheadrightarrow X, \ X \in \{S^3, P^4(p^\ast)\};$$

$$P^4(p^\ast) \hookrightarrow P^4(T) \hookrightarrow \bigvee_{i=1}^k S^3 \vee P^4(T) \xrightarrow{\Sigma f_2} \Sigma M_2 \twoheadrightarrow Y, \ Y \in \{S^3, P^4(p^\ast)\},$$

where the unlabeled arrows “$\hookrightarrow$” and “$\twoheadrightarrow$” denote the canonical inclusion and pinch maps, respectively. Then by the Hurewicz theorem and Lemma 2.2, we see that all the above compositions are null homotopic. The Lemma then follows by Lemma 2.11. □

**Lemma 3.2.** There is a homotopy equivalence

$$\Sigma M_4 \simeq P^4(H) \vee P^5(T) \vee P^6(H) \vee \bigvee_{i=1}^{l-a} S^3 \vee \bigvee_{i=1}^k S^4 \vee \bigvee_{i=1}^{k-a} S^5 \vee \bigvee_{i=1}^a C^5,$$

where $0 \leq a \leq \min\{r,d\}$ is some integer that depends on $M$, and $a = 0$ if and only if the Steenrod square $Sq^{2}$ acts trivially on $H^2(M; \mathbb{Z}/2)$.

**Proof.** By Theorem 2.10, (1.1) and Lemma 3.1, there is a homotopy cofibration sequence

$$\bigvee_{i=1}^k S^4 \vee P^5(H) \xrightarrow{\Sigma f_3} \Sigma M_3 \to \Sigma M_4,$$
where $\Sigma f_3$ is homologically trivial and $\Sigma M_3 \simeq \bigvee_{i=1}^{l} S^3 \vee \bigvee_{i=1}^{k} S^4 \vee P^4(H) \vee P^5(T)$. By the Hurewicz theorem and Lemma 2.2, the following compositions

$$S^4 \hookrightarrow \bigvee_{i=1}^{k} S^4 \vee P^5(H) \xrightarrow{\Sigma f_3} \Sigma M_3 \to P^4(H) \to X, \; X \in \{S^4, P^4(p^\ast)\};$$

$$P^5(p^\ast) \hookrightarrow P^5(H) \hookrightarrow \bigvee_{i=1}^{k} S^4 \vee P^5(H) \xrightarrow{\Sigma f_3} \Sigma M_3 \to Y, \; Y \in \{S^3, S^4, P^4(p^+, P^5(p^f))\}$$

are null homotopic, where the unlabelled maps are the canonical inclusions and pinch maps, respectively. It follows by Lemma 2.11 that there is a homotopy equivalence

$$\Sigma M_4 \simeq \bigvee_{i=1}^{k} S^4 \vee P^4(H) \vee P^5(T) \vee P^6(T) \vee C_{g_1},$$

where $C_{g_1}$ is the homotopy cofibre of some map $g_1: \bigvee_{i=1}^{l} S^4 \to \bigvee_{i=1}^{l} S^3$. It is clear that the map $g_1$ can be represented by a matrix $M_{g_1}$ whose entries are zero or the Hopf map $\eta: S^1 \to S^3$. By the elementary matrix operation we may assume that $M_{g_1}$ is a (rectangular) diagonal matrix with a diagonal elements $\eta$; consequently, there is a homotopy equivalence

$$C_{g_1} \simeq \bigvee_{i=1}^{a} C_\eta^5 \vee \bigvee_{i=1}^{l-a} S^3 \vee \bigvee_{i=1}^{k-a} S^5.$$

The proof of the Lemma is finished. \hfill \Box

It is well-known that $\eta^2 \in \pi_{n+2}(S^n)$ is detected by the secondary operation $\Theta$ based on the relation (cf. [Har02])

$$\text{Sq}^1(\text{Sq}^2 \text{Sq}^4) + \text{Sq}^2 \text{Sq}^2 = 0.$$ 

Note that if $M$ is a smooth spin manifold, then similar arguments to that on [MM79, page 32] show that the secondary operation $\Theta$ acts trivially on $H^*(M; \mathbb{Z}/2)$.

**Lemma 3.3.** If the secondary operation $\Theta$ acts trivially on $H^2(M; \mathbb{Z}/2)$ (say $M$ is smooth and spin), then there is a homotopy equivalence

$$\Sigma M_5 \simeq P^4(H) \vee P^5(T) \vee P^6(H) \vee \bigvee_{i=1}^{l-a} S^3 \vee \bigvee_{i=1}^{k-b} S^4 \vee \bigvee_{i=1}^{k-a} S^5 \vee \bigvee_{i=1}^{l-b} S^6 \vee \bigvee_{i=1}^{a} C_\eta^5 \vee \bigvee_{i=1}^{b} C_\eta^6,$$

where $0 \leq a, b \leq \min\{k, l\}$; $a = 0$ if $\text{Sq}^2$ acts trivially on $H^2(M; \mathbb{Z}/2)$, and $b = 0$ if $\text{Sq}^2$ acts trivially on $H^3(M; \mathbb{Z}/2)$.

**Proof.** By Theorem 2.10 and (1.1), there is a homotopy cofibration sequence

$$\bigvee_{i=1}^{l} S^5 \xrightarrow{\Sigma f_l} \Sigma M_4 \to \Sigma M_5,$$
SUSPENSION SPLITTING AND COHOMOTOPY SETS OF SIMPLY CONNECTED 7-MANIFOLDS

where $\Sigma f_4$ is homologically trivial and $\Sigma M_4$ is given by Lemma 3.2. It follows by Lemma 2.8 that the homologically trivial composition

$$S^5 \hookrightarrow \bigvee_{i=1}^l S^5 \xrightarrow{\Sigma f_4} \Sigma M_4 \hookrightarrow \bigvee_{i=1}^a C_\eta^5 \to C_\eta^5$$

is null homotopic. By Lemma 2.11 and 2.2, we see that there is a homotopy equivalence

$$\Sigma M_5 \simeq \bigvee_{i=1}^{k-a} S^5 \vee P^4(H) \vee P^5(T) \vee P^6(H) \vee \bigvee_{i=1}^a C_\eta^5 \vee C_{g_2},$$

where $C_{g_2}$ is the homotopy cofibre of some map

$$g_2: \bigvee_{i=1}^l S^0 \to \bigvee_{i=1}^{l-a} S^3 \vee \bigvee_{i=1}^k S^4.$$

The assumption of the Lemma implies that the space $\bigvee_{i=1}^{l-a} S^3$ retracts off $C_{g_2}$. By similar arguments to that in the proof of Lemma 3.2, we see that there is a homotopy equivalence

$$C_{g_2} \simeq \bigvee_{i=1}^{l-a} S^3 \vee \bigvee_{i=1}^{k-b} S^4 \vee \bigvee_{i=1}^{l-b} S^6 \vee \bigvee_{i=1}^b C_\eta^6$$

for some integer $0 \leq b \leq \min\{k, l\}$. The Lemma follows immediately. \qed

4. HOMOTOPY DECOMPOSITIONS OF $\Sigma M$ WHEN LOCALIZED AWAY FROM 2

Let $M$ be a closed simply connected 7-manifold with $H_*(M)$ given by (1.1). In this section we discuss the homotopy type of the reduced suspension $\Sigma M$ after localization away from 2.

Firstly we have the following immediate consequence of Lemma 3.3.

**Lemma 4.1.** There is a homotopy equivalence

$$\Sigma M_5 \simeq (\frac{1}{k}) \bigvee_{i=1}^l (S^3 \vee S^6) \vee \bigvee_{i=1}^k (S^4 \vee S^5) \vee P^4(H) \vee P^5(T) \vee P^6(H).$$

Set $H = H_3 \oplus H_{\neq 3}$ and $T = T_3 \oplus T_{\neq 3}$ with

$$H_3 = \bigoplus_{i=1}^m \mathbb{Z}/3^{r_i}, \quad T_3 = \bigoplus_{i=1}^t \mathbb{Z}/3^{s_i}.$$

**Lemma 4.2.** There is a homotopy equivalence

$$\Sigma M \simeq (\frac{1}{k}) \bigvee_{i=1}^l (S^3 \vee S^6) \vee \bigvee_{i=1}^k S^5 \vee P^6(H) \vee P^4(H_{\neq 3}) \vee P^5(T_{\neq 3}) \vee C_h,$$

where $h: S^7 \to \bigvee_{i=1}^k S^4 \vee \bigvee_{i=1}^m P^4(3^{r_i}) \vee \bigvee_{j=1}^l P^5(3^{s_j}).$
Proof. Consider the cofibration \( S^7 \xrightarrow{\Sigma f_6} \Sigma M_5 \to \Sigma M \). We have \( \pi_7(S^3) \cong \mathbb{Z}/2 \) [Tod62], \( \pi_7(S^5) \cong \pi_7(S^6) \cong \mathbb{Z}/2 \) and \( \pi_7(P^6(H))((\frac{1}{2})) = 0 \). Hence the compositions

\[
S^7 \xrightarrow{\Sigma f_6} \Sigma M_5 \to \bigvee_{i=1}^l (S^3 \vee S^6) \vee \bigvee_{i=1}^k S^5 \vee P^6(H) \to X, \ X \in \{S^3, S^6, S^5, P^6(H)\}
\]

are null homotopic when localized away from 2, where the first unlabelled map is the pinch map given by Lemma 4.1. By Lemma 2.4 and 2.7, we have \( \pi_7(P^4(p^r)) = \pi_7(P^5(p^r)) = 0 \) for \( p \geq 5 \). Thus the wedge sum \( \bigvee_{i=1}^l (S^3 \vee S^6) \vee \bigvee_{i=1}^k S^5 \vee P^6(H) \vee P^4(H_{\#3}) \vee P^5(T_{\#3}) \) retracts off \( \Sigma M \), by Lemma 2.11. \( \square \)

The map \( h: S^7 \to V := \bigvee_{i=1}^k S^4 \vee \bigvee_{i=1}^m P^4(3^i) \vee \bigvee_{j=1}^t P^5(3^j) \) in Lemma 4.2 can be expressed as

\[
h = \sum_{i=1}^k h_{S,i} + \sum_{i=1}^m h_{H,i} + \sum_{j=1}^t h_{T,j} + \theta_h,
\]

where \( \theta_h \) is a sum of Whitehead products and

\[
h_{S,i}: S^7 \xrightarrow{h} V \to S^4, \ S^4_{i} = S^4, \ i = 1, \ldots, k;
\]

\[
h_{H,i}: S^7 \xrightarrow{h} V \to P^4(3^i), \ i = 1, \ldots, m;
\]

\[
h_{T,j}: S^7 \xrightarrow{h} V \to P^5(3^j), \ j = 1, \ldots, t.
\]

Observe that the cohomology ring \( H^*(\Sigma M; R) \) has trivial cup products for any a principle integral domain \( R \) implying that the cup products in \( H^*(C_{h_S}; \mathbb{Z}) \), \( H^*(C_{h_H}; \mathbb{Z}/3^i) \) and \( H^*(C_{h_T}; \mathbb{Z}/3^i) \) are all trivial. Since \( \pi_7(S^4)((\frac{1}{2})) \cong \mathbb{Z}((\frac{1}{2})\langle \nu_4 \rangle) \oplus \mathbb{Z}/3\langle \Sigma \alpha \rangle \) and the Hopf map \( \nu_4 \) has Hopf invariant 1, (from [CS22, Lemma 2.5]) we must have

\[
h_{S,i} = x_i \cdot \Sigma \alpha, \ x_i \in \mathbb{Z}/3.
\]

By Lemma 2.7, \( h_{H,i} = y_i \cdot \Sigma \alpha \), \( y_i \in \mathbb{Z}/3 \). By Lemma 2.4 and [CS22, Lemma 2.4], we have

\[
h_{T,j} = z_j \cdot i_4(\Sigma \alpha), \ z_j \in \mathbb{Z}/3.
\]

By the Hilton-Milnor theorem, the Whitehead product component \( \theta_h \) is determined by groups of the following form:

\[
\pi_7(\Sigma S^3 \wedge S^3), \ \pi_7(\Sigma S^3 \wedge P^4(3^i)), \ \pi_7(\Sigma P^3(3^i) \wedge P^4(3^j)), \ \pi_7(\Sigma P^4(3^i) \wedge P^4(3^j)).
\]

Using the convention \( S^n = P^{n+1}(3\infty) \), these groups can be written uniformly as

\[
\pi_7(\Sigma P^{a}(3^i) \wedge P^{b}(3^j)), \text{ where } a + b = 7 \text{ or } a + b = 8.
\]

All cup products in \( H^*(\Sigma M; R) \) are trivial also implying that all cup products in the \( H^*(C_{h_S}; R) \) are trivial; this also holds when \( \theta_h \) is replaced by its components. Then applying [CS22, Lemma 2.4] we get that \( \theta_h = 0 \); that is, the map \( h \) contains no Whitehead products.
Lemma 4.3. The map $h: S^7 \to \bigvee_{i=1}^k S^4 \lor \bigvee_{i=1}^m P^4(3^r_i) \lor \bigvee_{j=1}^l P^5(3^s_j)$ in Lemma 4.2 is given by the equation

\begin{equation}
(4.1) \quad h = \sum_{i=1}^k x_i \cdot \Sigma \alpha + \sum_{i=1}^m y_i \cdot \Sigma \alpha r_i + \sum_{j=1}^t z_j \cdot i_4(\Sigma \alpha),
\end{equation}

where $x_i, y_i, z_j \in \{0, \pm 1\}$. In particular, the homotopy cofibre $C_h$ is a suspension.

Lemma 4.4. Let $X$ be the homotopy cofibre of $\Sigma \alpha$ or $i_4(\Sigma \alpha)$ or $\Sigma \alpha r_i$ given by Theorem 1.1. Then there is an isomorphism

$$P^1: H^4(X; \mathbb{Z}/3) \to H^8(X; \mathbb{Z}/3).$$

Proposition 4.5. If $P^1$ acts trivially on $H^4(C_h; \mathbb{Z}/3)$, then there is a homotopy equivalence

$$C_h \simeq \bigvee_{i=1}^k S^4 \lor \bigvee_{u \neq i=1}^m P^4(3^r_i) \lor \bigvee_{j=1}^t P^5(3^s_j) \lor S^8.$$ 

Proof. By Lemma 4.4, the assumption compels that the coefficients $x_i, y_i, z_j$ in (4.1) are zero for any $i, j$. Hence the map $h$ is null homotopic and the Lemma follows. \hfill \Box

Proposition 4.6. Suppose that $P^1$ acts nontrivially on $H^4(C_h; \mathbb{Z}/3)$. The following hold:

1. If there exists $x \in H^4(C_h; \mathbb{Z}/3)$ such that $P^1(x) \neq 0$ and $x \in \text{im}(\beta_r)$ for some $r$, then there is a homotopy equivalence

$$C_h \simeq \bigvee_{i=1}^k S^4 \lor \bigvee_{u \neq i=1}^m P^4(3^r_i) \lor \bigvee_{j=1}^t P^5(3^s_j) \lor (P^4(3^r_u \cup \alpha_{r_u} e^8),$$

where $r_u$ is the minimum of $r$ such that $x \in \text{im}(\beta_r)$ and $P^1(x) \neq 0$.

2. If for any $x \in H^4(C_h; \mathbb{Z}/3)$ with $P^1(x) \neq 0$, there hold $x \notin \text{im}(\beta_r)$ and $\beta_s(x) = 0$ for any $r, s \geq 1$, then there is a homotopy equivalence

$$C_h \simeq \bigvee_{i=1}^{k-1} S^4 \lor \bigvee_{i=1}^m P^4(3^r_i) \lor \bigvee_{j=1}^t P^5(3^s_j) \lor (S^4 \cup \alpha_{c^8}).$$

3. If for any $x \in H^4(C_h; \mathbb{Z}/3)$ with $P^1(x) \neq 0$, there hold $x \notin \text{im}(\beta_r)$ for any $r \geq 1$, while $\beta_s(x) \neq 0$ for some $s \geq 1$, then there is a homotopy equivalence

$$C_h \simeq \bigvee_{i=1}^k S^4 \lor \bigvee_{u \neq i=1}^m P^4(3^r_u \cup \alpha_{\Sigma e^8}).$$

where $s_u$ is the maximum of $s$ such that $\beta_s(x) \neq 0$. 

Summarizing the above discussion we get

Theorem 4.7. The map $h: S^7 \to \bigvee_{i=1}^k S^4 \lor \bigvee_{i=1}^m P^4(3^r_i) \lor \bigvee_{j=1}^l P^5(3^s_j)$ in Lemma 4.2 is given by the equation

\begin{equation}
(4.1) \quad h = \sum_{i=1}^k x_i \cdot \Sigma \alpha + \sum_{i=1}^m y_i \cdot \Sigma \alpha r_i + \sum_{j=1}^t z_j \cdot i_4(\Sigma \alpha),
\end{equation}

where $x_i, y_i, z_j \in \{0, \pm 1\}$. In particular, the homotopy cofibre $C_h$ is a suspension.
1. There is a homotopy equivalence map

Thus, after applying elementary row matrix operations, which means composing suitable
method in [Hua22, Section 3] or [LZ24, Section 2] to reduce the coefficients of the equation
(4.1).

(1) The assumption of (1) implies that \( y_i = 1 \) for some \( 1 \leq i \leq m \). The formulae (2.5) imply the following homotopy equalities:

\[
\begin{bmatrix}
1_p & 0 \\
-q_4 & 1_p
\end{bmatrix}
\begin{bmatrix}
\Sigma \tilde{\alpha}_r \\
\Sigma \alpha
\end{bmatrix}
= \begin{bmatrix}
\Sigma \tilde{\alpha}_r \\
0
\end{bmatrix}
: S^7 \rightarrow P^4(3^r) \vee S^4,
\]

\[
\begin{bmatrix}
1_p & 0 \\
-B(\chi^*_i) & 1_p
\end{bmatrix}
\begin{bmatrix}
\Sigma \tilde{\alpha}_r \\
\Sigma \alpha_s
\end{bmatrix}
= \begin{bmatrix}
\Sigma \tilde{\alpha}_r \\
0
\end{bmatrix}
: S^7 \rightarrow P^4(3^r) \vee P^4(3^s) \text{ for } r \leq s,
\]

\[
\begin{bmatrix}
1_p & 0 \\
-q_4 & 1_p
\end{bmatrix}
\begin{bmatrix}
\Sigma \tilde{\alpha}_r \\
i_4 \Sigma \alpha
\end{bmatrix}
= \begin{bmatrix}
\Sigma \tilde{\alpha}_r \\
0
\end{bmatrix}
: S^7 \rightarrow P^4(3^r) \vee P^5(3^s).
\]

Thus, after applying elementary row matrix operations, which means composing suitable
self-homotopy equivalences of the codomain \( V \) of \( h \), to the representation matrix of the
map \( h \), we may assume that in the equation (4.1), \( x_i = z_j = 0 \) for all \( 1 \leq i \leq k \) and
\( 1 \leq j \leq t \), and there exists a unique nonzero \( y_i \), say \( y_u \) as chosen in the statement. The
homotopy equivalence in (1) is proved.

(2) The assumption of (2) implies that \( y_i = z_j = 0 \) for any \( 1 \leq i \leq m \) and \( 1 \leq j \leq t \),
and \( y_i = 1 \) for some \( 1 \leq i \leq k \). After composing a self-homotopy equivalence of \( V \), we
may assume that \( x_1 = 1 \) and \( x_2 = \cdots = x_k = 0 \). The statement (2) is proved.

(3) The assumption of (3) implies that \( x_i = y_j = 0 \) for any \( 1 \leq i \leq k \) and \( 1 \leq j \leq m \),
and \( z_j = 1 \) for at least one \( j \). By the homotopy equality

\[
\begin{bmatrix}
1_p & 0 \\
-B(\chi^*_i) & 1_p
\end{bmatrix}
\begin{bmatrix}
i_4 (\Sigma \alpha) \\
i_4 (\Sigma \alpha)
\end{bmatrix}
\]

we may assume that \( z_j = 1 \) for exactly one \( j \), say \( z_v = 1 \) as chosen in the statement. The
proof of (3) is finished.

\[ \square \]

Proof of Theorem 1.1. By the naturality of \( P^1 \), \( P^1 \) acts trivially on \( H^3(M; \mathbb{Z}/3) \) if and
only if \( P^1 \) acts trivially on \( H^4(C^h; \mathbb{Z}/3) \). Combining Lemma 4.2, Proposition 4.5 and 4.6
we then get the Theorem.

\[ \square \]

We end this section by giving the 2-local homotopy type of \( \Sigma M_{(2)} \) in a special case.

Proposition 4.7. Let \( M \) be a smooth spin simply connected 7-manifold with \( H_*(M) \) given
by (1.1), where \( H \) and \( T \) are both 2-torsion-free.

1. There is a homotopy equivalence

\[
\Sigma M_5 \simeq_{(2)} \bigvee_{i=1}^{l-a} S^3 \vee \bigvee_{i=1}^{k-b} S^4 \vee \bigvee_{i=1}^{k-a} S^5 \vee \bigvee_{i=1}^{l-b} S^6 \vee \bigvee_{i=1}^{a} C^5_{\eta} \vee \bigvee_{i=1}^{b} C^6_{\eta},
\]

where \( 0 \leq a, b \leq \min\{k, l\} \); \( a = 0 \) if \( \text{Sq}^2 \) acts trivially on \( H^2(M; \mathbb{Z}/2) \), and \( b = 0 \) if \( \text{Sq}^2 \) acts trivially on \( H^3(M; \mathbb{Z}/2) \).
2. If $\text{Sq}^2$ acts trivially on $H^2(M; \mathbb{Z}/2)$, i.e., $a = 0$, then localized at 2, $\Sigma M$ is homotopy equivalent to one of the following complexes:

(i) \[ \bigvee_{i=2}^{l} S^3 \vee S^4 \vee S^5 \vee S^6 \bigvee_{i=1}^{k-b} S^5 \bigvee_{i=1}^{l-b} S^6 \bigvee_{i=1}^{b} C_6^0 \bigvee (S^3 \cup_{\varepsilon \cdot \eta} e^8), \varepsilon \in \{0,1\}; \]

(ii) \[ \bigvee_{i=1}^{l} S^3 \vee S^4 \vee S^5 \vee S^6 \bigvee_{i=1}^{k-b} S^5 \bigvee_{i=1}^{l-b} S^6 \bigvee_{i=1}^{b} C_6^0 \bigvee (S^4 \cup_{\eta \cdot \varepsilon} e^8); \]

(iii) \[ \bigvee_{i=2}^{l} S^3 \vee S^4 \vee S^5 \vee S^6 \bigvee_{i=1}^{k-b} S^5 \bigvee_{i=1}^{l-b} S^6 \bigvee_{i=1}^{b} C_6^0 \bigvee (S^3 \vee S^4 \cup_{\varepsilon \cdot \eta} (\sum_{\text{even}}) + 2 \varepsilon \cdot \eta \cdot e^8), \varepsilon \in \{0,1\}; \]

(iv) \[ \bigvee_{i=2}^{l} S^3 \vee S^4 \vee S^5 \vee S^6 \bigvee_{i=1}^{k-b} S^5 \bigvee_{i=1}^{l-b} S^6 \bigvee_{i=1}^{b} C_6^0 \bigvee (S^3 \vee C^0_6 \cup_{\varepsilon \cdot \eta} (\sum_{\text{even}}) + i_4 \cdot e^8), \varepsilon \in \{0,1\}. \]

Proof. (1) The homotopy equivalence follows by Lemma 3.3.

(2) The assumption that $\text{Sq}^2(H^2(M; \mathbb{Z}/2)) = 0$ implies $a = 0$ in (4.2) and that $M$ is smooth spin implying that the composition

\[ S^7 \xrightarrow{\Sigma f_5} \Sigma M_5 \to \bigvee_{i=1}^{l} S^5 \bigvee_{i=1}^{b} S^6 \]

is null homotopic, hence we have $\Sigma M \simeq_{(2)} \bigvee_{i=1}^{l} S^5 \bigvee_{i=1}^{b} S^6 \bigvee C_h$ for some map

\[ h: S^7 \to \bigvee_{i=1}^{l} S^3 \bigvee_{i=1}^{b} S^4 \bigvee_{i=1}^{b} C_6^0. \]

Similar arguments to that between Lemma 4.2 and 4.3 show that the map $h$ contains no Whitehead products and the component

\[ S^7 \xrightarrow{h} \bigvee_{i=1}^{l} S^3 \bigvee_{i=1}^{b} S^4 \bigvee_{i=1}^{b} C_6^0 \]

is generated by $i_4 \nu_4 \in \pi_7(C^6_6)$ (Lemma 2.9). Thus we may put

\[ h = (2) \sum_{i=1}^{l} x_i \cdot \eta \Sigma \nu' + \sum_{i=1}^{k-b} y_i \cdot \Sigma \nu' + \sum_{i=1}^{b} z_i \cdot i_4(\Sigma \nu'), \]

where $x_i, z_i \in \mathbb{Z}/2$ and $y_i \in \mathbb{Z}/4$. Since every component of $h$ is a suspension, $h$ can be represented by a column vector and after applying suitable elementary row operations, we may assume that

\[ x_1 \in \{0,1\}, \ x_2 = \cdots = x_l = 0; \]

\[ y_1 \in \{0,1,2\}, \ y_2 = \cdots = y_{k-b} = 0; \]

\[ z_1 \in \{0,1\}, \ z_2 = \cdots = z_b = 0. \]
The case $y_1 = z_1 = 0$ yields the first complex listed in (2). Assume that $y_1 \neq 0$ or $z_1 \neq 0$. There hold homotopy equalities:

\[
\begin{bmatrix}
14 & 0 \\
\eta & 1_3
\end{bmatrix}
\begin{bmatrix}
\Sigma \nu' \\
\eta \Sigma \nu'
\end{bmatrix}
= \begin{bmatrix}
\Sigma \nu' \\
0
\end{bmatrix} : \mathbb{S}^7 \to \mathbb{S}^4 \vee \mathbb{S}^3,
\]

\[
\begin{bmatrix}
1_4 & 0 \\
i_4 & 1_\eta
\end{bmatrix}
\begin{bmatrix}
\Sigma \nu' \\
i_4 \Sigma \nu'
\end{bmatrix}
= \begin{bmatrix}
\Sigma \nu' \\
0
\end{bmatrix} : \mathbb{S}^7 \to \mathbb{S}^4 \vee C_{\eta}^6,
\]

\[
\begin{bmatrix}
1_4 & 0 \\
\zeta & 1_4
\end{bmatrix}
\begin{bmatrix}
i_4 \Sigma \nu' \\
2 \Sigma \nu'
\end{bmatrix}
= \begin{bmatrix}
i_4 \Sigma \nu' \\
0
\end{bmatrix} : \mathbb{S}^7 \to C_{\eta}^6 \vee \mathbb{S}^4,
\]

where $1_\eta$ denotes the identity map on $C_{\eta}^6$. It follows that $(y_1, z_1) \in \{(1, 0), (2, 0), (0, 1)\}; y_1 = 1$ also implying $x_1 = 0$. The remaining complexes in (2) follow immediately. 

\section{5. Cohomotopy sets}

Let $M$ be a closed simply connected 7-manifold with $H_*(M)$ given (1.1). This section is devoted to study the classical cohomotopy sets $\pi^k(M) = [M, S^k]$ or the $p$-local cohomotopy sets $\pi^k(M; \mathbb{Z}(p)) = [M, S^k(\mathbb{Z}(p))]$, where $p$ is an odd prime. Recall from [HMR75, Theorem 5.1, 5.3] that for any $k \geq 2$, the cohomotopy set $\pi^k(M)$ is the pullback of the canonical diagram

$$
\pi^k(M; \mathbb{Z}(\frac{1}{p})) \to \pi^k(M; \mathbb{Z}(0)) \leftarrow \pi^k(M; \mathbb{Z}(2)),
$$

and the induced map $\pi^k(M) \to \pi^k(M; \mathbb{Z}(2))$ and $\pi^k(M) \to \pi^k(M; \mathbb{Z}(\frac{1}{p}))$ are injective. Hence $\pi^k(M)$ is closely related with $\pi^k(M; \mathbb{Z}(p))$. Furthermore, let $P$ be a subset of the set of all primes and 0, then for any connected $H$-space $X$, there hold isomorphisms (cf. Corollary 6.5 of [HMR75])

$$\left[M, X\right]_P \cong \left[M, X\right]_P(\mathbb{Z}(p)) \cong \left[M, X\right]_P(\mathbb{Z}(2)),$$

where $X\left(P\right)$ and $\left[M, X\right]_P$ are the $P$-localization of the space $X$ and the group $[M, X]$, respectively.

Recall that for each $k$, there are the ($p$-local) cohomotopy Hurewicz maps

$$h^k : \pi^k(M) \to H^k(M), \quad h^k(p) : \pi^k(M; \mathbb{Z}(p)) \to H^k(M; \mathbb{Z}(p)).$$

It is well-known that the cohomotopy Hurewicz map (or the degree map) gives an isomorphism $\pi^7(M) \cong H^7(M) \cong \mathbb{Z}$. Since $H^6(M) = 0$, from [Tay12, Section 6.1] we get

$$\pi^6(M) \cong H^7(M; \mathbb{Z}/2) / \text{Sq}^2 \left(H^5(M; \mathbb{Z}/2)\right) = H^7(M; \mathbb{Z}/2) / \text{Sq}^2 \left(H^5(M; \mathbb{Z}/2)\right),$$

where the integral Steenrod square $\text{Sq}^2 = \text{Sq}^2 \rho_2$ with $\rho_2$ the mod 2 reduction has the same image with the usual $\text{Sq}^2$. It is also well-known that $M$ is spin if and only if $\text{Sq}^2$ acts trivially on $H^5(M; \mathbb{Z}/2)$. Hence we have the following.

**Lemma 5.1.** $\pi^k(M) = 0$ for $k = 1$ or $k \geq 8; \pi^7(M) \cong H^7(M) \cong \mathbb{Z}; \pi^6(M) \cong \mathbb{Z}/2$ if $M$ is spin, otherwise $\pi^6(M) = 0$. 
The Hopf fibration \( S^1 \to S^3 \xrightarrow{\eta} S^2 \) induces a transitive left action of \( \pi^3(M) \) on \( \pi^2(M) \). Since \( \pi^1(M) = 0 \), this action is also free. Then applying [KMT12, Theorem 3] we get

**Lemma 5.2.** \( \pi^2(M) \) is a left \( \pi^3(M) \)-torsor.

In the remainder of this section we shall investigate the cohomotopy sets \( \pi^k(M) \) with \( k = 3, 4, 5 \) from the perspective of \( p \)-localization.

5.1. \( \pi^5(M) \). By Freudenthal’s suspension theorem, \( \pi^5(M) \) is an abelian group.

**Proposition 5.3.** Localized away from 2, there is an isomorphism \( \pi^5(M) \cong H^5(M) \).

**Proof.** The homotopy cofibration sequence \( S^7 \xrightarrow{\Sigma f_6} \Sigma M_5 \xrightarrow{\Sigma q} \Sigma M \xrightarrow{\Sigma q} S^8 \) implies an exact sequence of groups

\[
\left[ S^2 M_5, S^6 \right] \xrightarrow{(\Sigma^2 f_6)^p} \left[ S^8, S^6 \right] \xrightarrow{(\Sigma^2 q)^p} \left[ \Sigma M, S^6 \right] \xrightarrow{(\Sigma q)^p} \left[ \Sigma M_5, S^6 \right] \xrightarrow{(\Sigma q)^p} \left[ S^7, S^6 \right].
\]

When localized away from 2, by (5.1), (5.2) and Lemma 4.1 we compute that

\[
\left[ \Sigma M, S^6 \right]_{(\frac{1}{2})} \cong \left[ \Sigma M_5, S^6 \right]_{(\frac{1}{2})} \cong \bigoplus_{i=1}^l \left[ S^6, S^6 \right]_{(\frac{1}{2})} \oplus \left[ P^6(H), S^6 \right]_{(\frac{1}{2})}
\cong \bigoplus_{i=1}^l \mathbb{Z}(\frac{1}{2}) \oplus (H \otimes \mathbb{Z}(\frac{1}{2})) \cong H^5(M) \otimes \mathbb{Z}(\frac{1}{2}).
\]

The Lemma then follows by Freudenthal’s suspension theorem. \( \square \)

It is clear that \( \pi^5(M) \cong H^5(M) \) when localized at 0. Then one can also get the above proposition by applying [LPW23, Theorem 1.1] with \( n = 5, p \geq 3 \).

**Proposition 5.4.** Let \( M \) be a smooth spin simply connected 7-manifold with 2-torsion-free homology. There is a short exact sequence of groups

\[
0 \to \mathbb{Z}/2 \to \pi^5(M) \to H^5(M) \to 0,
\]

which is split if \( \text{Sq}^2 \) acts trivially on \( H^3(M; \mathbb{Z}/2) \).

**Proof.** By Proposition 5.3, it suffices to prove the short exact sequence in the Lemma when all spaces and groups are localized at 2.

The assumption of the Proposition implies that the attaching map \( \Sigma f_6 : S^7 \to \Sigma M_5 \) with \( \Sigma M_5 \) given by (4.2) doesn’t contain the component \( \eta \) and \( \eta^2 \), hence by Lemma 2.11, the attaching map \( \Sigma f_6 \) factors as the composition

\[
\Sigma f_6 : S^7 \xrightarrow{\Sigma f_6} \bigvee_{i=1}^{l-a} S^3 \vee \bigvee_{i=1}^{k-b} S^4 \vee \bigvee_{i=1}^{a} C^5_{\eta} \vee \bigvee_{i=1}^{b} C^6_{\eta} \hookrightarrow \Sigma M_5,
\]

and the wedge sum \( \bigvee_{i=1}^{k-a} S^5 \vee \bigvee_{i=1}^{l-b} S^6 \) retracts off \( \Sigma M \). Recall that the Hopf map \( \nu_4 \in \pi_7(S^4) \) has Hopf invariant 1 and that \( \pi_7(S^3) \cong \mathbb{Z}/2(\nu \eta_6) \). It follows that when localized
Combining the equations (5.3), we have

\[ q_\cdot (\Sigma^2 f)_6 = (2) \sum_{i=1}^{l-a} z_i \cdot i_4(\Sigma^2 f)_6, \]

where \( x_i, w_i, z_i' \in \mathbb{Z}/2, y_i \in \mathbb{Z}/4 \) and \( z_i \in \mathbb{Z}/2(2) \).

By (4.2), we compute that

\[ \mathbb{Z}/2(\Sigma f)_6 \cong \bigoplus_{i=1}^{l-b} \mathbb{Z}/2[\Sigma M, S^6] \oplus \bigoplus_{i=1}^{b} \mathbb{Z}/2[\Sigma^2 M, S^6], \]

where \( a = 0, q_6 i_4 : S^4 \rightarrow C^6 \rightarrow S^6 \). It follows that in the exact sequence (5.2), the induced homomorphisms \((\Sigma f)_6\) and \((\Sigma^2 f)_6\) are both trivial, and hence there is a short exact sequence

\[ 0 \rightarrow \mathbb{Z}/2(\eta^2) \xrightarrow{\Sigma f}_6 \rightarrow \mathbb{Z}/2[\Sigma M, S^6] \xrightarrow{(\Sigma^2 f)_6} \mathbb{Z}/2[\Sigma M, S^6] \rightarrow 0. \]

Note that the wedge sum \( \bigvee_{i=1}^b S^6 \) retracts off \( \Sigma M \) implying that the direct summand \( \bigoplus_{i=1}^b \mathbb{Z}/2(1) \) of \( [\Sigma M, S^6] \) splits off \( [\Sigma M, S^6] \). Thus the above short exact sequence is split when \( b = 0 \), or equivalently \( \text{Sq}^2 \) acts trivially on \( H^3(M; \mathbb{Z}/2) \). We complete the proof by combining the isomorphisms

\[ [M, S^6] \cong [\Sigma M, S^6], \quad [\Sigma M, S^6] \cong (2) \bigoplus_{i=1}^I \mathbb{Z} \cong (2) H^5(M). \]
5.2. \( \pi^3(M) \). Consider the EHP fibration sequence (cf. [Nei10, Corollary 4.4.3])
\[
\Omega^2 S^4 \xrightarrow{\Omega H} \Omega^2 S^7 \xrightarrow{P} S^3 \xrightarrow{E} \Omega S^4 \xrightarrow{H} \Omega S^7.
\]

**Proposition 5.5.** The suspension \( E_2: \pi^3(M) \rightarrow \pi^4(M) \) is an isomorphism if either \( E_2 \) is localized away from 2 or \( M \) is nonspin.

**Proof.** Since \( E \) is an \( H \)-space, the suspension map \( E: S^3 \rightarrow \Omega S^4 \) has a left homotopy inverse; consequently, the suspension \( E_2: \pi^3(M) \rightarrow \pi^4(\Sigma M) \) is injective.

For the surjectivity, consider the EHP exact sequence of sets (localized away from 2)
\[
\pi^7(\Sigma^2 M) \xrightarrow{P} \pi^3(M) \xrightarrow{E} \pi^4(\Sigma M) \xrightarrow{H} \pi^7(\Sigma M).
\]

Here we use the identification \([X, \Omega Y] = [\Sigma X, Y]\). By Lemma 5.1, \( \pi^7(\Sigma M) \cong \mathbb{Z}/2 \) for spin \( M \) and \( \pi^7(\Sigma M) = 0 \) for nonspin \( M \). Hence the suspension \( E_2 \) is surjective when localized away from 2, or if \( M \) is nonspin.

The obstruction for \( E: S^3 \rightarrow \Omega S^4 \) to be an \( H \)-map is given by a map \( S^3 \wedge S^3 \rightarrow \Omega S^4 \). Then Lemma 5.1 implies that any composition \( M \rightarrow S^6 \rightarrow \Omega S^4 \) is trivial when \( E_2 \) is localized away from 2, or if \( M \) is nonspin. It follows that the suspension \( E_2 \) is a homomorphism, and hence an isomorphism in these two cases. \( \square \)

**Lemma 5.6.** \([S^4 \cup_{\Sigma a} e^8, S^4] \cong \mathbb{Z}, [X^8, S^4] = 0 \) and \([X^8(\Sigma \alpha), S^4] \cong \mathbb{Z}/3^{r-1}\).

**Proof.** It is easy to compute that \([P^5(p'), S^4] \cong [P^6(p'), S^4] = 0 \) for any odd prime \( p \). Then the groups in the Lemma can be obtained by the following exact sequences
\[
0 \rightarrow [S^4 \cup_{\Sigma a} e^8, S^4] \rightarrow [S^4, S^4] \xrightarrow{(\Sigma a)^p} [S^7, S^4], \quad 0 \rightarrow [X^8, S^4] \rightarrow [P^5(3^r), S^4] = 0,
\]
\[
0 \rightarrow [X^8(\Sigma \alpha), S^4] \rightarrow [P^4(3^r), S^4] \xrightarrow{(\Sigma \alpha)^p} [S^7, S^4]_{(3)}.
\]

Combining Theorem 1.1 and Lemma 5.6, we have the concrete group structure of \( \pi^3(M) \) after localization away from 2; the details are omitted.

**Corollary 5.7.** Let \( M \) be given by Theorem 1.1. If the condition of (2a) holds, then after localization away from 2 there hold an isomorphism
\[
\pi^3(M) \cong \big( \bigoplus_{i=1}^{k} \mathbb{Z} \big) \oplus (H/\mathbb{Z}/3^{r\alpha}) \oplus \mathbb{Z}/3^{r\alpha-1};
\]
otherwise we have
\[
\pi^3(M) \cong \big( \bigoplus_{i=1}^{k} \mathbb{Z} \big) \oplus H \cong (\bigoplus_{i=1}^{k}) H^3(M).
\]

The following Example 5.8 shows that the suspension \( E_2: \pi^3(M) \rightarrow \pi^4(\Sigma M) \) is generally not surjective in the spin case.
Example 5.8. Let $M$ be a smooth spin simply connected 7-manifold whose homology groups $H_*(M)$ is given by (1.1) with $k = l = 1$ and $H = T = 0$. Assume further that $Sq^2$ acts trivially on $H^3(M; \mathbb{Z}/2)$ and that $Sq^2$ acts nontrivially on $H^2(M; \mathbb{Z}/2)$. Then the suspension $E_2: \pi^3(M) \to \pi^3(\Sigma M)$ is not surjective.

**Proof.** By (4.2) and the above assumption, there is a homotopy equivalence

\[ \Sigma M_5 \simeq S^4 \vee C_7^5 \vee S^6. \]

By (5.3), we can put

\[ \Sigma f_6 = x \cdot \Sigma \nu' + y \cdot \xi, \quad x \in \mathbb{Z}/12, \quad y \in \mathbb{Z}. \]

Consider the induced exact sequence

\[ [\Sigma^2 M_5, S^4] \overset{(\Sigma^2 f_6)^\sharp}{\longrightarrow} [S^8, S^4] \overset{(\Sigma \eta)^\sharp}{\longrightarrow} [\Sigma M, S^4], \]

where $\pi^8(S^4) \cong \mathbb{Z}/2\langle \nu_4 \eta \rangle \oplus \mathbb{Z}/2\langle (\Sigma \nu') \eta \rangle$ [Tod62] and

\[ [\Sigma^2 M_5, S^4] \cong [S^6, S^4] \oplus [C_7^5, S^4] \oplus [S^7, S^4] \cong \mathbb{Z}/2 \langle \eta_4 \rangle \oplus \mathbb{Z} \langle \tilde{\eta} \rangle \oplus (\mathbb{Z} \langle \nu_4 \rangle \oplus \mathbb{Z}/12 \langle \nu' \rangle). \]

We have

\[ (\Sigma^2 f_6)^\sharp(\eta_4) = x \cdot \eta_4 (\Sigma^2 \nu') = x \cdot (\Sigma \nu') \eta, \]

\[ (\Sigma^2 f_6)^\sharp(\tilde{\eta}) = y \cdot \tilde{\eta} (i_4 \nu \eta \xi) = y \cdot 2 \nu \eta = 0, \]

\[ (\Sigma^2 f_6)^\sharp(\nu_4) = (\Sigma^2 f_6)^\sharp(\Sigma \nu') = 0. \]

It follows that the cokernel of $(\Sigma^2 f_6)^\sharp$ always contains the direct summand $\mathbb{Z}/2 \langle \nu_4 \eta \rangle$, and hence the composite $\nu_4 \eta \eta (\Sigma q) \in [\Sigma M, S^4]$ is essential. Note that

\[ H(\nu_4 \eta \eta (\Sigma q)) = H(\nu_4) \eta \eta (\Sigma q) = \eta \eta (\Sigma q) \]

is a generator of $[\Sigma M, S^7] \cong \mathbb{Z}/2$, where $H$ is the second James-Hopf invariant, we see that the element $\nu_4 \eta \eta (\Sigma q) \in [\Sigma M, S^4]$ is not a suspension. The proof of the assertion in the Example is finished.

\qed

5.3. $\pi^4(M; \mathbb{Z}(p))$. In this subsection, we follow [Tay12, Section 6.3] to study the $p$-local cohomotopy set $\pi^4(M; \mathbb{Z}(p)) = [M, S^4(\mathbb{Z}(p))]$ for primes $p \geq 5$.

The quaternionic Hopf fibration sequence

\[ S^3 \hookrightarrow S^7 \xrightarrow{\nu} S^4 \xrightarrow{i} \mathbb{B}S^3 = \mathbb{H}P^\infty \]

induces an exact sequence of sets

\[ \pi^3(M) \xrightarrow{i_*} \pi^7(M) \xrightarrow{(\nu_4)} \pi^4(M) \xrightarrow{\nu_*} [M, \mathbb{H}P^\infty] \to 0, \]

where $\nu_4$ is surjective because the canonical inclusion map $\nu: S^4 \to \mathbb{H}P^\infty$ is 7-connected. We have $\pi^7(M) \cong H^7(M)$ and $\pi^3(M; \mathbb{Z}(p)) \cong H^3(M; \mathbb{Z}(p))$ for $p \neq 3$ by Corollary 5.7. Note that for each prime $p \geq 5$, $S^7(p)$ is a homotopy associative and homotopy commutative $H$-space, implying that the fibration sequence (5.4) extends from the right by $\mathbb{B}S^3(\mathbb{Z}(p)) \xrightarrow{\omega} \mathbb{B}S^7(\mathbb{Z}(p))$. Since $\pi_i(\mathbb{B}S^i(\mathbb{Z}(p))) = 0$ for $5 \leq i \leq 10$ and $\pi_{11}(\mathbb{B}S^3(\mathbb{Z}(p))) \cong \mathbb{Z}/5 \otimes \mathbb{Z}(p)$ (cf. [Tod62]), the
map $BS^3_{(p)} \to K(\mathbb{Z}_{(p)}, 4)$ giving a generator of $H^4(\mathbb{B}S^3_{(p)}) \cong \mathbb{Z}_{(p)}$ is 11-connected, implying that $[M, BS^3_{(p)}] \cong H^4(M; \mathbb{Z}_{(p)})$ for $p \geq 5$. Now replacing $BS^7_{(p)}$ by $\Omega^3 BS^7_{(p)}$ and applying [Tay12, Theorem 5.2 and Remark 5.3], we have the following.

**Lemma 5.9.** Let $p \geq 5$ be a prime. There is an exact sequence of sets

$$H^3(M; \mathbb{Z}_{(p)}) \to H^7(M; \mathbb{Z}_{(p)}) \xrightarrow{(\psi_e)_e} \pi^4(M; \mathbb{Z}_{(p)}) \xrightarrow{h_{(p)}} H^4(M; \mathbb{Z}_{(p)}) \to 0.$$  

For $\beta \in H^4(M; \mathbb{Z}_{(p)})$, let $e \in [M, S^4_{(p)}]$ be some element with $h_{(p)}(e) = \iota_e(e) = \beta$. There is a bijection

$$\text{coker}(\psi_e) \xrightarrow{1:1} \iota_e^{-1}(\beta),$$

where $\psi_e : H^3(M; \mathbb{Z}_{(p)}) \to H^7(M; \mathbb{Z}_{(p)})$ is the homomorphism defined by the composition

$$\psi_e(\gamma) : M \xrightarrow{\Delta} M \wedge M \xrightarrow{\gamma \wedge e} S^3_{(p)} \wedge S^4_{(p)} \xrightarrow{\delta} \Omega^2 \Sigma BS^7_{(p)},$$

where $\delta$ is characterized by [Tay12, Corollary 5.5].

**Proposition 5.10.** Let $M$ be a closed simply connected 7-manifold. Let $p \geq 5$ be a prime.

1. The cohomotopy Hurewicz map $h_{(p)} : \pi^4(M; \mathbb{Z}_{(p)}) \to H^4(M; \mathbb{Z}_{(p)})$ is surjective.
2. For $\beta \in H^4(M; \mathbb{Z}_{(p)})$, let $e \in [M, S^4_{(p)}]$ be some element with $h_{(p)}(e) = \iota_e(e) = \beta$. There is a bijection between $\iota_e^{-1}(\beta)$ and the cokernel of the homomorphism

$$\psi_e : H^3(M; \mathbb{Z}_{(p)}) \to H^7(M; \mathbb{Z}_{(p)}), \quad \psi_e(\gamma) = \gamma \circ \delta_e,$$

where $\delta_e = e^*(i) \in H^4(M; \mathbb{Z}_{(p)})$ with $i \in H^4(S^7_{(p)}; \mathbb{Z}_{(p)})$ the fundamental class.

**Proof.** By Lemma 5.9, it suffices to understand the homomorphism $\psi_e$. Let $p \geq 5$ be a prime. Since $\mathcal{D}' : S^7_{(p)} \to \Omega^2 \Sigma BS^7_{(p)}$, $\mathcal{D}'$ factors through the degree $c_e$-map on $S^7_{(p)}$. It follows that $\psi_e$ factors as the composition

$$(5.5) \quad H^3(M; \mathbb{Z}_{(p)}) \xrightarrow{\bar{\psi}} H^7(M; \mathbb{Z}_{(p)}) \xrightarrow{(c_e)_e} H^7(M; \mathbb{Z}_{(p)}),$$

where the induced homomorphism $(c_e)_e$ is just the multiplication by $c_e$. By the construction of $\psi_e$ given by Lemma 5.9, the homomorphism $\bar{\psi}$ in (5.5) just sends $\gamma$ to $\gamma \circ \delta_e$, where $\delta_e = e^*(i) \in H^4(M; \mathbb{Z}_{(p)})$ with $i \in H^4(S^7_{(p)}; \mathbb{Z}_{(p)})$ the fundamental class. By Lemma 5.11 below and [Tay12, Corollary 5.5], the homomorphisms $\psi_e$ and $\bar{\psi}$ have isomorphic cokernels. $\square$

Let $\mathcal{L}(-) = \text{map}(S^1, -)$ be the free loop space functor on the category of topological spaces.

**Lemma 5.11.** The homomorphism $(\mathcal{L}e)_* : H_7(\mathbb{B}S^3_{(p)}) \to H_7(\mathbb{B}S^7_{(p)})$ is an isomorphism for each prime $p \geq 5$.

**Proof.** Let $m = 3$ or 7. $S^m_{(p)}$ is an $H$-space implying that there is a homotopy equivalence

$$S^m_{(p)} \times BS^m_{(p)} \xrightarrow{\cong} \mathcal{L}BS^m_{(p)}.$$
The homology algebra $H_*(BS^m(p))$ is the polynomial algebra $\mathbb{Z}_p[x_{m+1}]$, where $x_{m+1}$ has dimension $m + 1$ (cf. [RS65, Section 4]). It follows that there is an isomorphism

$$H_*(BS^m(p)) \cong H_*(S^m(p)) \otimes H_*(BS^m(p)) \cong \Lambda_{\mathbb{Z}_p}(e_m) \otimes \mathbb{Z}_p[x_{m+1}].$$

We have $H_6(LS^4(p)) \cong \mathbb{Z}/2 \otimes \mathbb{Z}_p = 0$ and $H_7(LS^4(p)) = 0$, see [Zil77, page 21]. The Blakers-Massey theorem (cf. [Ark11, Theorem 6.4.2]) for the homotopy fibration $\omega \colon LS^4(p) \rightarrow BS^7(p)$ tells that the canonical map $C_\omega \colon BS^7(p)$ is 10-connected, where $C_\omega$ is the homotopy cofibre of $\omega$. Hence there is an exact sequence of homology groups:

$$0 = H_7(LS^4(p)) \rightarrow H_7(BS^7(p)) \xrightarrow{(\omega)_*} H_7(BS^7(p)) \rightarrow H_6(LS^4(p)) = 0,$$

which implies that $(\omega)_*$ is an isomorphism.

**Proof of Theorem 1.3.** The Theorem is a summarization of Lemma 5.1, 5.2 and Proposition 5.3, 5.4, 5.5, 5.10. \qed

Note that the above arguments are reasonable for CW-complexes of dimension at most 9. We have the following generalized result.

**Theorem 5.12.** Let $M$ be a CW-complex of dimension $\leq 9$ and let $p \geq 5$ be a prime.

1. The cohomotopy Hurewicz map $h_{(p)} : \pi^4(M;\mathbb{Z}_p) \rightarrow H^4(M;\mathbb{Z}_p)$ is onto the subset of classes $\beta$ of $H^4(M;\mathbb{Z}_p)$ such that $\beta^2 = \beta \sim \beta = 0$.

2. For $\beta \in H^4(M;\mathbb{Z}_p)$ with $\beta^2 = 0$, there is a bijection between $i_{\beta}^{-1}(\beta)$ and the cokernel of the homomorphism

$$\psi_e : H^3(M;\mathbb{Z}_p) \rightarrow H^7(M;\mathbb{Z}_p), \quad \psi_e(\gamma) = \gamma \sim \delta_e,$$

where $e \in [M,S^4(p)]$ be some element with $h_{(p)}(e) = i_{\beta}(e) = \beta$, and $\delta_e = e^*(e) \in H^4(M;\mathbb{Z}_p)$ with $e \in H^4(S^4(p);\mathbb{Z}_p)$ the fundamental class.

**Proof.** Localized at $p \geq 5$, the induced map $BS^3(p) \rightarrow BS^7(p)$ is the standard quotient map which takes the 8-cells of $BS^3(p)$ homeomorphically to 8-cells of $BS^7(p)$. By dimensional reason, we can identify $[M,BS^3(p)]$ with $H^4(M;\mathbb{Z}_0)$ and $[M,BS^7(p)]$ with $H^8(M;\mathbb{Z}_0)$. It follows that the induced map is just the cup product square, and hence the $p$-local cohomotopy map $h_{(p)} : \pi^4(M;\mathbb{Z}_p) \rightarrow H^4(M;\mathbb{Z}_p)$ is onto the subset of classes $\beta$ of $H^4(M;\mathbb{Z}_p)$ such that $\beta^2 = \beta \sim \beta = 0$.

Applying [LPW23, Theorem 1.1] with $n = 3$ and $p \geq 5$, we see that there is an exact sequence of groups

$$H^2(M;\mathbb{Z}_p) \xrightarrow{\partial_3} H^{2p}(M;\mathbb{Z}_p) \rightarrow \pi^3(M;\mathbb{Z}_p) \xrightarrow{h_{(p)}} H^3(M;\mathbb{Z}_p) \rightarrow 0.$$

Note that the condition "$n \geq 2p - 1$" is redundant because $\pi^3(\ast;\mathbb{Z}_p)$ is an abelian group for $p \geq 5$. It follows that for complexes $X$ of dimension at most 9, there holds an isomorphism

$$h_{(p)} : \pi^3(M;\mathbb{Z}_p) \xrightarrow{\cong} H^3(M;\mathbb{Z}_p).$$
Similarly, we have $h_{(\rho)} : \pi^7(M; \mathbb{Z}_{(\rho)}) \xrightarrow{\cong} H^7(M; \mathbb{Z}_{(\rho)})$.

The remainder of the proof is totally parallel to that of Proposition 5.10. □

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