1. Introduction and preliminaries

Further we shall follow the terminology of [3, 4, 5, 9, 20]. The set of positive integers is denoted by \( \mathbb{N} \).

A semigroup is a non-empty set with a binary associative operation. A semigroup \( S \) is called \textit{inverse} if for any \( x \in S \) there exists a unique \( y \in S \) such that \( x \cdot y \cdot x = x \) and \( y \cdot x \cdot y = y \). Such an element \( y \) in \( S \) is called \textit{inverse} of \( x \) and denoted by \( x^{-1} \). The map defined on an inverse semigroup \( S \) which maps any element \( x \) of \( S \) its inverse \( x^{-1} \) is called the \textit{inversion}.

If \( S \) is a semigroup, then by \( E(S) \) we denote the subset of idempotents of \( S \), and by \( S^1 \) (resp., \( S^0 \)) we denote the semigroup \( S \) with the adjoined unit (resp., zero). Also if a semigroup \( S \) has zero \( 0_s \), then for any \( A \subseteq S \) we denote \( A^* = A \setminus \{0_s\} \).

If \( E \) is a semilattice, then the semilattice operation on \( E \) determines the partial order \( \leq \) on \( E \):

\[
e \leq f \quad \text{if and only if} \quad ef = fe = e.
\]

This order is called \textit{natural}. An element \( e \) of a partially ordered set \( X \) is called \textit{minimal} if \( f \leq e \) implies \( f = e \) for \( f \in X \). An idempotent \( e \) of a semigroup \( S \) without zero (with zero) is called \textit{primitive} if \( e \) is a minimal element in \( E(S) \) (in \( (E(S))^* \)).

Let \( S \) be a semigroup with zero and \( \lambda \) be a cardinal \( \geq 1 \). On the set \( B_\lambda(S) = (\lambda \times S \times \lambda) \cup \{0\} \) we define the semigroup operation as follows

\[
(\alpha, a, \beta) \cdot (\gamma, b, \delta) = \begin{cases}
(\alpha, ab, \delta), & \text{if } \beta = \gamma; \\
0, & \text{if } \beta \neq \gamma,
\end{cases}
\]

and \( (\alpha, a, \beta) \cdot 0 = 0 \cdot (\alpha, a, \beta) = 0 \cdot 0 = 0 \), for all \( \alpha, \beta, \gamma, \delta \in \lambda \) and \( a, b \in S \). If \( S = S^1 \) then the semigroup \( B_\lambda(S) \) is called the \textit{Brandt} \( \lambda \)-extension of the semigroup \( S \) [12]. Obviously, \( \mathcal{J} = \{0\} \cup \{(\alpha, \emptyset, \beta) \mid \emptyset \text{ is the zero of } S\} \) is an ideal of \( B_\lambda(S) \). We put \( B^0_\lambda(S) = B_\lambda(S)/\mathcal{J} \) and we shall call \( B^0_\lambda(S) \) the \textit{Brandt} \( \lambda^0 \)-extension of the semigroup \( S \) with zero [13]. Further, if \( A \subseteq S \) then we shall denote \( A_{\alpha, \beta} = \{(\alpha, s, \beta) : s \in A\} \) if \( A \) does not contain zero, and \( A_{\alpha, \beta} = \{(\alpha, s, \beta) : s \in A \setminus \{0\}\} \cup \{0\} \) if \( 0 \in A \), for \( \alpha, \beta \in \lambda \). If \( \mathcal{I} \) is a trivial semigroup (i.e., \( \mathcal{I} \) contains only one element), then by \( \mathcal{I}^0 \) we denote the semigroup \( \mathcal{I} \) with the adjoined zero. Obviously, for any \( \lambda \geq 2 \) the Brandt \( \lambda^0 \)-extension of the semigroup \( \mathcal{I}^0 \) is isomorphic to the semigroup of \( \lambda \times \lambda \)-matrix units and any Brandt \( \lambda^0 \)-extension of a semigroup with zero contains the semigroup of \( \lambda \times \lambda \)-matrix units. Further by \( B_\lambda \) we shall denote the semigroup of \( \lambda \times \lambda \)-matrix units and by \( B^0_\lambda(1) \) the subsemigroup of \( \lambda \times \lambda \)-matrix units of the Brandt \( \lambda^0 \)-extension of a monoid \( S \) with zero. A completely 0-simple inverse semigroup is called a
Brandt semigroup [20]. By Theorem II.3.5 [20], a semigroup $S$ is a Brandt semigroup if and only if $S$ is isomorphic to a Brandt $\lambda$-extension $B_\lambda(G)$ of some group $G$.

Let $\{S_i : i \in I\}$ be a disjoint family of semigroups with zero such that $0_i$ is zero in $S_i$ for any $i \in I$. We put $S = \{0\} \cup \bigcup \{S^*_i : i \in I\}$, where $0 \notin \bigcup \{S^*_i : i \in I\}$, and define a semigroup operation on $S$ in the following way

$$s \cdot t = \begin{cases} st, & \text{if } st \in S_i^* \text{ for some } i \in I; \\ 0, & \text{otherwise}. \end{cases}$$

The semigroup $S$ with such defined operation is called the orthogonal sum of the family of semigroups $\{S_i : i \in I\}$ and in this case we shall write $S = \sum_{i \in I} S_i$.

A non-trivial inverse semigroup is called a primitive inverse semigroup if all its non-zero idempotents are primitive [20]. A semigroup $S$ is a primitive inverse semigroup if and only if $S$ is the orthogonal sum of the family of Brandt semigroups [20, Theorem II.4.3]. We shall call a Brandt subsemigroup $T$ of a primitive inverse semigroup $S$ maximal if every Brandt subsemigroup of $S$ which contains $T$, coincides with $T$.

Green’s relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ on a semigroup $S$ are defined by:

- $a \mathcal{L} b$ if and only if $\{a\} \cup Sa = \{b\} \cup Sb$;
- $a \mathcal{R} b$ if and only if $\{a\} \cup aS = \{b\} \cup bS$; and
- $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$,

for $a, b \in S$. For details about Green’s relations see [3 § 2.1] or [11]. We observe that two non-zero elements $(\alpha_1, s, \beta_1)$ and $(\alpha_2, t, \beta_2)$ of a Brandt semigroup $B_\lambda(G)$, $s, t \in G$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \lambda$, are $\mathcal{H}$-equivalent if and only if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ (see [20, p. 93]).

In this paper all topological spaces are Hausdorff. If $Y$ is a subspace of a topological space $X$ and $A \subseteq Y$, then by $\text{cl}_Y(A)$ we denote the topological closure of $A$ in $Y$.

We recall that a topological space $X$ is said to be

- compact if each open cover of $X$ has a finite subcover;
- countably compact if each open countable cover of $X$ has a finite subcover;
- pseudocompact if each locally finite open cover of $X$ is finite.

According to Theorem 3.10.22 of [9], a Tychonoff topological space $X$ is pseudocompact if and only if each continuous real-valued function on $X$ is bounded. Also, a Hausdorff topological space $X$ is pseudocompact if and only if every locally finite family of non-empty open subsets of $X$ is finite. Every compact space and every countably compact space are pseudocompact (see [9]).

We recall that the Stone-Čech compactification of a Tychonoff space $X$ is a compact Hausdorff space $\beta X$ containing $X$ as a dense subspace so that each continuous map $f : X \to Y$ to a compact Hausdorff space $Y$ extends to a continuous map $\overline{f} : \beta X \to Y$ [9].

A topological semigroup is a Hausdorff topological space with a continuous semigroup operation. A topological semigroup which is an inverse semigroup is called an inverse topological semigroup. A topological inverse semigroup is an inverse topological semigroup with continuous inversion. A topological group is a topological space with a continuous group operation and inversion. We observe that the inversion on a topological inverse semigroup is a homeomorphism (see [8, Proposition II.1]). A Hausdorff topology $\tau$ on a (inverse) semigroup $S$ is called (inverse) semigroup if $(S, \tau)$ is a topological (inverse) semigroup.

**Definition 1.1** ([12]). Let $\mathcal{TSG}$ be some category of topological semigroups. Let $\lambda$ be a cardinal $\geq 1$ and $(S, \tau) \in \text{Ob}$ $\mathcal{TSG}$ be a topological monoid. Let $\tau_B$ be a topology on $B_\lambda(S)$ such that

a) $(B_\lambda(S), \tau_B) \in \text{Ob}$ $\mathcal{TSG}$; and

b) for some $\alpha \in \lambda$ the topological subspace $(S_{\alpha, \alpha}, \tau_B|_{S_{\alpha, \alpha}})$ is naturally homeomorphic to $(S, \tau)$.

Then $(B_\lambda(S), \tau_B)$ is called a topological Brandt $\lambda$-extension of $(S, \tau)$ in $\mathcal{TSG}$.

**Definition 1.2** ([13]). Let $\mathcal{TSG}_0$ be some category of topological semigroups with zero. Let $\lambda$ be a cardinal $\geq 1$ and $(S, \tau) \in \text{Ob}$ $\mathcal{TSG}_0$. Let $\tau_B$ be a topology on $B_\lambda^0(S)$ such that

a) $(B_\lambda^0(S), \tau_B) \in \text{Ob}$ $\mathcal{TSG}_0$;
b) for some $\alpha \in \lambda$ the topological subspace $(S_{\alpha,a}, \tau_B|_{S_{\alpha,a}})$ is naturally homeomorphic to $(S, \tau)$. Then $(B^0_\lambda(S), \tau_B)$ is called a topological Brandt $\lambda^0$-extension of $(S, \tau)$ in $\mathfrak{SG}_0$.

We observe that for any topological Brandt $\lambda$-extension $B_\lambda(S)$ of a topological semigroup $S$ in the category of topological semigroups there exist a topological monoid $T$ with zero and a topological Brandt $\lambda^0$-extension $B^0_\lambda(T)$ of $T$ in the category of topological semigroups with zero, such that the semigroups $B_\lambda(S)$ and $B^0_\lambda(T)$ are topologically isomorphic. Algebraic properties of Brandt $\lambda^0$-extensions of monoids with zero, non-trivial homomorphisms between them, and a category which objects are ingredients of the construction of such extensions were described in [17]. Also, in [14] and [17] a category which objects are ingredients in the constructions of finite (resp., compact, countably compact) topological Brandt $\lambda^0$-extensions of topological monoids with zeros was described.

Gutik and Repovš proved that any 0-simple countably compact topological inverse semigroup is topologically isomorphic to a topological Brandt $\lambda$-extension $B_\lambda(H)$ of a countably compact topological group $H$ in the category of topological inverse semigroups for some finite cardinal $\lambda \geq 1$ [16]. Also, every 0-simple pseudocompact topological inverse semigroup is topologically isomorphic to a topological Brandt $\lambda$-extension $B_\lambda(H)$ of a pseudocompact topological group $H$ in the category of topological inverse semigroups for some finite cardinal $\lambda \geq 1$ [15]. Next Gutik and Repovš showed in [16] that the Stone-Čech compactification $\beta(T)$ of a 0-simple countably compact topological inverse semigroup $T$ is a 0-simple compact topological inverse semigroup. It was proved in [15] that the same is true in the case of 0-simple pseudocompact topological inverse semigroups.

In the paper [2] the structure of compact and countably compact primitive topological inverse semigroups was described and showed that any countably compact primitive topological inverse semigroup embeds into a compact primitive topological inverse semigroup.

In this paper we describe the structure of pseudocompact primitive topological inverse semigroups and show that the Tychonoff product of a family of pseudocompact primitive topological inverse semigroups is a pseudocompact topological space. Also we prove that the Stone-Čech compactification of a pseudocompact primitive topological inverse semigroup is a compact primitive topological inverse semigroup.

2. Primitive pseudocompact topological inverse semigroups

**Proposition 2.1.** Let $S$ be a Hausdorff pseudocompact primitive topological inverse semigroup and $S$ be an orthogonal sum of the family $\{B_\lambda(G_i)\}_{i \in \mathcal{I}}$ of topological Brandt semigroups with zeros, i.e. $S = \sum_{i \in \mathcal{I}} B_\lambda(G_i)$. Then the following statements hold:

(i) every non-zero idempotent of $S$ is an isolated point in $E(S)$ and $E(S)$ is a compact semilattice;
(ii) every non-zero $\mathcal{H}$-class in $S$ is a pseudocompact closed-and-open subset of $S$;
(iii) every maximal subgroup in $S$ is a pseudocompact subspace of $S$;
(iv) every maximal Brandt subsemigroup of $S$ is a pseudocompact space and has finitely many idempotents.

**Proof.** (i) First part of the statement follows from Lemma 7 [2]. Then the continuity of the semigroup operation and inversion in $S$ implies that the map $e : S \to E(S)$ defined by the formula $e(x) = x \cdot x^{-1}$ is continuous and hence by Theorem 3.10.24 [9], $E(S)$ is a pseudocompact subspace of $S$ such that every non-zero idempotent in $E(S)$ is an isolated point. Therefore $E(S)$ is compact. Otherwise there exists an open neighbourhood $U(0)$ of the zero 0 of $S$ in $E(S)$ such that the set $E(S) \setminus U(0)$ is infinite. But this contradicts the pseudocompactness of $E(S)$.

(ii) By Corollary 8 from [2] every non-zero $\mathcal{H}$-class in $S$ is a closed-and-open subset of $S$ and hence by Exercise 3.10.F(d) is pseudocompact.

Statement (iii) follows from (ii).

(iv) Let $B_\lambda(G_i)$ be a maximal Brandt subsemigroup of the semigroup $S$. Then statement (i) implies that $E(S)$ is a compact and since every non-zero idempotent of $S$ is an isolated point of $E(S)$ we conclude that $E(B_\lambda(G_i))$ is compact for every $i \in \mathcal{I}$. By Corollary 3.10.27 of [9] the product of a compact space and a pseudocompact space is a pseudocompact space, and hence we have that the
space \( E(B_\lambda(G_i)) \times S \) is pseudocompact. Since \( S \) is a primitive inverse semigroup we conclude that 
\[ B_\lambda(G_i) = E(B_\lambda(G_i)) \cdot S. \]
Now, the continuity of the semigroup operation in \( S \) implies that the map 
\( f: E(B_\lambda(G_i)) \times S \to S \)
defined by the formula \( f(e, s) = e \cdot s \) is continuous, and since the continuous image of a pseudocompact space is pseudocompact we conclude that \( B_\lambda(G_i) \) is pseudocompact. The last statement follows from Theorem 1 of [15].

**Lemma 2.2.** Let \( U \) be an open non-empty subset of a topological group \( G \) and \( A \) be a dense subset of \( G \). Then \( A \cdot U = U \cdot A = G \).

*Proof.* Since \( G \) is a topological group we have that there exists a nonempty open subset \( V \) of \( G \) such that \( V^{-1} = U \). Let \( x \) be an arbitrary point of \( G \). Then \( x \cdot V \) is a nonempty open subset of \( G \), because translations in every topological group are homeomorphisms. Then we have that \( x \cdot V \cap A \neq \emptyset \) and hence \( x \in A \cdot V^{-1} = A \cdot U \). Therefore we get that \( G \subseteq A \cdot U \). The converse inclusion is trivial. Hence \( A \cdot U = G \). The proof of the equality \( U \cdot A = G \) is similar. \( \square \)

**Lemma 2.3.** Let \( \lambda \geq 2 \) be any cardinal and \( U \) be an open non-empty subset of a topological inverse Brandt semigroup \( B_\lambda(G) \) such that \( U \neq \{0\} \). Then \( A \cdot U \cdot A = B_\lambda(G) \) for every dense subset \( A \) of \( B_\lambda(G) \).

*Proof.* By Lemma 7 [2] we have that every non-zero idempotent of the topological inverse semigroup \( B_\lambda(G) \) is an isolated point in \( E(B_\lambda(G)) \). The continuity of the semigroup operation and inversion in \( S \) implies that the map \( \epsilon: S \to E(S) \) defined by the formula \( \epsilon(x) = x \cdot x^{-1} \) is continuous and hence \( G_{\alpha,\beta} \) is an open-and-closed subset of \( B_\lambda(G) \) for all \( \alpha, \beta \in \lambda \). Since \( A \) is a dense subset of \( B_\lambda(G) \) we conclude that \( A \cap G_{\alpha,\beta} \) is a dense subset of \( G_{\alpha,\beta} \) for all \( \alpha, \beta \in \lambda \). Also, since \( \lambda \geq 2 \) we have that \( 0 \in A \cdot U \cdot A \). This implies that it is sufficient to show that \( G_{\alpha,\beta} \subseteq A \cdot U \cdot A \) for all \( \alpha, \beta \in \lambda \).

Since \( G_{\alpha,\beta} \) is an open subset of \( B_\lambda(G) \) for all \( \alpha, \beta \in \lambda \), without loss of generality we assume that \( U \subseteq G_{\alpha_0,\beta_0} \) for some \( \alpha_0, \beta_0 \in \lambda \), i.e., \( U = V_{\alpha_0,\beta_0} \) for some open subset \( V \subseteq G \). Fix arbitrary \( \alpha, \beta \in \lambda \).

Then there exists subsets \( L, R \in G \) such that \( A \cap G_{\alpha_0,\alpha_0} = L_{\alpha_0,\alpha_0} \) and \( A \cap G_{\beta_0,\beta_0} = R_{\beta_0,\beta_0} \). It is obviously that \( L_{\alpha_0,\alpha_0} \) and \( R_{\beta_0,\beta_0} \) are dense subsets of \( G_{\alpha_0,\alpha_0} \) and \( G_{\beta_0,\beta_0} \), respectively. This implies that \( L \) and \( R \) are dense subsets of \( G \). Then by Lemma 2.2 we have that 
\[ G_{\alpha,\beta} = (L \cdot V \cdot R)_{\alpha,\beta} = L_{\alpha_0,\alpha_0} \cdot V_{\alpha_0,\beta_0} \cdot R_{\beta_0,\beta_0} = (A \cap G_{\alpha_0,\alpha_0}) \cdot U \cdot (A \cap G_{\beta_0,\beta_0}) \subseteq A \cdot U \cdot A. \]
This completes the proof of the lemma. \( \square \)

Lemma 2.2 implies the following:

**Proposition 2.4.** Let \( U \) be an open non-empty subset of a topological inverse Brandt semigroup \( B_1(G) \) such that \( U \neq \{0\} \). Then for every dense subset \( A \) of \( B_1(G) \) the following statements hold:

(i) \( A \cdot U \cdot A = B_1(G) \) in the case when \( 0 \) is an isolated point in \( B_1(G) \);

(ii) \( (A \cup \{0\}) \cdot U \cdot (A \cup \{0\}) = B_1(G) \) in the case when \( 0 \) is a non-isolated point in \( B_1(G) \).

Lemma 2.3 and Proposition 2.4 imply the following proposition:

**Proposition 2.5.** Let \( S \) be a Hausdorff primitive inverse topological semigroup such that \( S \) be an orthogonal sum of the family \( \{B_\lambda(G_i)\}_{i \in \mathcal{I}} \) of topological Brandt semigroups with zeros. Let \( |\mathcal{I}| > 1 \) and \( U \) be an open non-empty subset of \( S \) such that \( (U \cap B_\lambda(G_i)) \setminus \{0\} \neq \emptyset \) for any \( i \in \mathcal{I} \). Then \( A \cdot U \cdot A = S \) for every dense subset \( A \) of \( S \).

**Remark 2.6.** Since by Theorem II.4.3 of [20] a primitive inverse semigroup \( S \) is the orthogonal sum of a family of Brandt semigroups, i.e., \( S \) is an orthogonal sum \( \sum_{i \in \mathcal{I}} B_\lambda(G_i) \) of Brandt \( \lambda \)-extensions \( B_\lambda(G_i) \) of groups \( G_i \), we have that Proposition 12 from [2] describes a base of the topology at any non-zero element of \( S \).

Later by \( \mathcal{TISG} \) we denote the category of topological inverse semigroups, where \( \mathcal{ObTISG} \) are all topological inverse semigroups and \( \mathcal{MorTISG} \) are homomorphisms between topological inverse semigroups.

The following theorem describes the structure of primitive pseudocompact topological inverse semigroups.
Theorem 2.7. Every primitive Hausdorff pseudocompact topological inverse semigroup $S$ is topologically isomorphic to the orthogonal sum $\sum_{i \in \mathcal{I}} B_{\lambda_i}(G_i)$ of topological Brandt $\lambda_i$-extensions $B_{\lambda_i}(G_i)$ of pseudocompact topological groups $G_i$ in the category $\mathfrak{TISG}$ for some finite cardinals $\lambda_i \geq 1$. Moreover the family

$$\mathcal{B}(0) = \left\{ S \setminus \left( B_{\lambda_1}(G_{i_1}) \cup B_{\lambda_2}(G_{i_2}) \cup \cdots \cup B_{\lambda_n}(G_{i_n}) \right)^*: i_1, i_2, \ldots, i_n \in \mathcal{I}, n \in \mathbb{N} \right\}$$

determines a base of the topology at zero $0$ of $S$.

Proof. By Theorem II.4.3 of [20] the semigroup $S$ is an orthogonal sum of Brandt semigroups and hence $S$ is isomorphic to the orthogonal sum $\sum_{i \in \mathcal{I}} B_{\lambda_i}(G_i)$ of Brandt $\lambda_i$-extensions $B_{\lambda_i}(G_i)$ of groups $G_i$. We fix any $i_0 \in \mathcal{I}$. Since $S$ is a topological inverse semigroup, Proposition II.2 [3] implies that $B_{\lambda_0}(G_{i_0})$ is a topological inverse semigroup. By Proposition 2.7 $B_{\lambda_0}(G_{i_0})$ is a pseudocompact topological Brandt $\lambda_0$-extension of pseudocompact topological group $G_{i_0}$ in the category $\mathfrak{TISG}$ for some finite cardinal $\lambda_0 \geq 1$. This completes the proof of the first assertion of the theorem.

The second statement of the theorem is trivial in the case when the set of indices $\mathcal{I}$ is finite. Hence later we assume that the set $\mathcal{I}$ is infinite.

Suppose on the contrary that $\mathcal{B}(0)$ is not a base at zero $0$ of $S$. Then, there exists an open neighbourhood $U(0)$ of zero $0$ such that $U(0) \cup \left( B_{\lambda_1}(G_{i_1}) \cup B_{\lambda_2}(G_{i_2}) \cup \cdots \cup B_{\lambda_n}(G_{i_n}) \right)^* \neq S$ for finitely many indices $i_1, i_2, \ldots, i_n \in \mathcal{I}$. Let $V(0) \subseteq U(0)$ be an open neighbourhood of zero in $S$ such that $V(0) \cap V(0) \cdot V(0) \subseteq U(0)$. Then we have that $V(0) \cup \left( B_{\lambda_1}(G_{i_1}) \cup B_{\lambda_2}(G_{i_2}) \cup \cdots \cup B_{\lambda_n}(G_{i_n}) \right)^* \neq S$. We state that there exist a sequence of distinct points $\{ x_k \}_{k \in \mathbb{N}}$ of the semigroup $S$ and a sequence of open subsets $\{ U(x_k) \}_{k \in \mathbb{N}}$ of $S$ such that the following conditions hold:

(i) $x_k \in U(x_k) \subseteq B_{\lambda_i}(G_{i_k})$ for some $i_k \in \mathcal{I}$;

(ii) if $x_k, x_{k_2} \in B_{\lambda_i}(G_{i_k})$ for some $i_k \in \mathcal{I}$, then $k_1 = k_2$;

(iii) $\bigcup_{k \in \mathbb{N}} U(x_k) \subseteq S \setminus V(0)$.

Otherwise we have that $V(0)$ is a dense subset of the subspace

$$S' = S \setminus \bigcup \left( B_{\lambda_1}(G_{i_1}) \cup B_{\lambda_2}(G_{i_2}) \cup \cdots \cup B_{\lambda_n}(G_{i_n}) \right)^*,$$

for some positive integer $n$. Since $S'$ with induced operation from $S$ is a primitive inverse semigroup Proposition 2.5 implies that $V(0) \cdot V(0) \cdot V(0) = S'$ which contradicts the choice of the neighbourhood $U(0)$. The obtained contradiction implies that there exists finitely many indexes $i_1, i_2, \ldots, i_n, \ldots, i_m \in \mathcal{I}$ where $m > n$ such that

$$U(0) \cup \left( B_{\lambda_1}(G_{i_1}) \cup B_{\lambda_2}(G_{i_2}) \cup \cdots \cup B_{\lambda_n}(G_{i_n}) \cup \cdots \cup B_{\lambda_m}(G_{i_m}) \right)^* = S.$$

This completes the proof of the theorem. \hfill \Box

Proposition 2.8. Let $S$ be a primitive Hausdorff pseudocompact topological inverse semigroup which is topologically isomorphic to the orthogonal sum $\sum_{i \in \mathcal{I}} B_{\lambda_i}(G_i)$ of topological Brandt $\lambda_i$-extensions $B_{\lambda_i}(G_i)$ of topological groups $G_i$ in the category $\mathfrak{TISG}$ for some cardinals $\lambda_i \geq 1$. Then the following conditions hold:

(i) the space $S$ is Tychonoff if and only if for every $i \in \mathcal{I}$ the space of the topological group $G_i$ is Tychonoff, i.e., $G_i$ is a $T_0$-space;

(ii) the space $S$ is normal if and only if for every $i \in \mathcal{I}$ the space of the topological group $G_i$ is normal.

Proof. We observe that the $T_0$-topological space of a topological group is Tychonoff (see Theorem 2.6.4 in [19]).

(i) Implication $(\Rightarrow)$ follows from Theorem 2.1.6 of [6].

$(\Leftarrow)$ Suppose that for every $i \in \mathcal{I}$ the space of the topological group $G_i$ is Tychonoff. We fix an arbitrary element $x \in S$. First we consider the case when $x \neq 0$. Then there exists an non-zero $\mathcal{H}$-class $H$ which contains $x$. By Proposition 12 from [2] there exists $i \in \mathcal{I}$ such that the topological space $H$ is homeomorphic to the topological group $G_i$. Then by Proposition 1.5.8 from [6] for every
open neighbourhood \(U(x)\) of \(x\) in \(H\) there exists a continuous map \(f: H \to [0, 1]\) such that \(f(x) = 0\) and \(f(y) = 1\) for all \(y \in H \setminus U(x)\). We define the map \(\tilde{f}: S \to [0, 1]\) in the following way:

\[
\tilde{f}(y) = \begin{cases} 
  f(y), & \text{if } y \in H; \\
  1, & \text{if } y \in S \setminus H.
\end{cases}
\]

Since by Proposition 12 from [2] every non-zero \(\mathcal{H}\)-class is an open-and-closed subset of \(S\) we conclude that such defined map \(\tilde{f}: S \to [0, 1]\) is continuous.

Suppose that \(x = 0\). We fix an arbitrary \(U(0) = S \setminus (B_{\lambda_1}(G_{i_1}) \cup B_{\lambda_2}(G_{i_2}) \cup \cdots \cup B_{\lambda_n}(G_{i_n}))^* \in \mathcal{B}(0)\). Then by Proposition 12 from [2], \(U(0)\) is an open-and-closed subset of \(S\). Thus we have that the map \(f: S \to [0, 1]\) defined by the formula

\[
\tilde{f}(y) = \begin{cases} 
  0, & \text{if } y \in U(0); \\
  1, & \text{if } y \in S \setminus U(0),
\end{cases}
\]

is continuous, and hence by Proposition 1.5.8 from [9] the space \(S\) is Tychoff.

Next we shall prove statement (ii).

\((\Rightarrow)\) Suppose that \(S\) is a normal space. By Lemma 9 of [2] we have that every \(\mathcal{H}\)-class of \(S\) is a closed subset of \(S\). Then by Theorem 2.1.6 from [2] we have that every \(\mathcal{H}\)-class of \(S\) is a normal subspace of \(S\) and hence Definition 1.1 and Proposition 12 of [2] imply that for every \(i \in \mathcal{I}\) the space of the topological group \(G_i\) is normal.

\((\Leftarrow)\) Suppose that for every \(i \in \mathcal{I}\) the space of the topological group \(G_i\) is normal. Let \(F_1\) and \(F_2\) be arbitrary closed disjoint subsets of \(S\).

At first we consider the case when \(0 \notin F_1 \cup F_2\). Then there exists an open neighbourhood \(U(0)\) of zero in \(S\) such that \(F_1 \cup F_2 \subseteq S \setminus U(0)\), i.e., there exist finitely many \(i_1, i_2, \ldots, i_n \in \mathcal{I}\) such that

\[
F_1 \cup F_2 \subseteq (B_{\lambda_1}(G_{i_1}) \cup B_{\lambda_2}(G_{i_2}) \cup \cdots \cup B_{\lambda_n}(G_{i_n})) \setminus \{0\}.
\]

By Corollary 8 of [2] every non-zero \(\mathcal{H}\)-class of \(S\) is open subset in \(S\), and hence we get that the subspace \((B_{\lambda_1}(G_{i_1}) \cup B_{\lambda_2}(G_{i_2}) \cup \cdots \cup B_{\lambda_n}(G_{i_n})) \setminus \{0\}\) of \(S\) is a topological sum of some non-zero \(\mathcal{H}\)-classes of \(S\), and hence it is an open subspace of \(S\). Then by Theorem 2.2.7 from [9] we have that \((B_{\lambda_1}(G_{i_1}) \cup B_{\lambda_2}(G_{i_2}) \cup \cdots \cup B_{\lambda_n}(G_{i_n})) \setminus \{0\}\) is a normal space. Therefore, there exist disjoint open neighbourhoods \(V(F_1)\) and \(V(F_2)\) of \(F_1\) and \(F_2\) in \((B_{\lambda_1}(G_{i_1}) \cup B_{\lambda_2}(G_{i_2}) \cup \cdots \cup B_{\lambda_n}(G_{i_n})) \setminus \{0\}\), and hence in \(S\), respectively.

Suppose that \(0 \in F_1 \cup F_2\). Without loss of generality we can assume that \(0 \in F_1\). Then there exist finitely many \(i_1, i_2, \ldots, i_n \in \mathcal{I}\) such that

\[
F_2 \subseteq (B_{\lambda_1}(G_{i_1}) \cup B_{\lambda_2}(G_{i_2}) \cup \cdots \cup B_{\lambda_n}(G_{i_n})) \setminus \{0\}.
\]

The assumption of the proposition implies that the set \((B_{\lambda_1}(G_{i_1}) \cup B_{\lambda_2}(G_{i_2}) \cup \cdots \cup B_{\lambda_n}(G_{i_n})) \setminus \{0\}\) is closed in \(S\) and hence

\[
\tilde{F}_1 = F_1 \cap \left((B_{\lambda_1}(G_{i_1}) \cup B_{\lambda_2}(G_{i_2}) \cup \cdots \cup B_{\lambda_n}(G_{i_n})) \setminus \{0\}\right)
\]

is a closed subset of \(S\), too. Then the previous arguments of the proof imply that

\[
(B_{\lambda_1}(G_{i_1}) \cup B_{\lambda_2}(G_{i_2}) \cup \cdots \cup B_{\lambda_n}(G_{i_n})) \setminus \{0\}
\]

is a normal space, and hence there exist open disjoint neighbourhoods \(W(\tilde{F}_1)\) and \(U(F_2)\) of the closed sets \(\tilde{F}_1\) and \(F_2\) in \((B_{\lambda_1}(G_{i_1}) \cup B_{\lambda_2}(G_{i_2}) \cup \cdots \cup B_{\lambda_n}(G_{i_n})) \setminus \{0\}\), and hence in \(S\), respectively. We put

\[
U(F_1) = S \setminus (B_{\lambda_1}(G_{i_1}) \cup B_{\lambda_2}(G_{i_2}) \cup \cdots \cup B_{\lambda_n}(G_{i_n}))^* \cup W(\tilde{F}_1).
\]

Then we have that \(U(F_1)\) and \(U(F_2)\) are open disjoint neighbourhoods of \(F_1\) and \(F_2\) in \(S\), respectively. This completes the proof of statement (ii). \(\Box\)

Theorem 2.7 and Proposition 2.8 imply the following:
Corollary 2.9. Every primitive Hausdorff pseudocompact topological inverse semigroup \( S \) is a Tychonoff topological space. Moreover the topological space of \( S \) is normal if and only if every maximal subgroup of \( S \) is a normal subspace.

By Theorem 3.10.21 from [9] every normal pseudocompact space is countably compact, and hence Corollary 2.9 implies the following:

Corollary 2.10. Every primitive Hausdorff pseudocompact topological inverse semigroup \( S \) such that every maximal subgroup of \( S \) is a normal subspace in \( S \) is countably compact.

Proposition 2.11. Every primitive pseudocompact topological inverse semigroup \( S \) is a continuous (non-homomorphic) image of the product \( \tilde{E}_S \times G_S \), where \( \tilde{E}_S \) is a compact semilattice and \( G_S \) is a pseudocompact topological group.

Proof. By Theorem 2.4 the topological semigroup \( S \) is topologically isomorphic to the orthogonal sum \( \sum_{i \in \mathcal{J}} B_{\lambda_i}(G_i) \) of topological Brandt \( \lambda_i \)-extensions \( B_{\lambda_i}(G_i) \) of pseudocompact topological groups \( G_i \) in the category \( \mathfrak{TISG} \) for some finite cardinals \( \lambda_i \geq 1 \) and the family defined by formula (1) determines the base of the topology at zero of \( S \).

Fix an arbitrary \( i \in \mathcal{J} \). Then by Proposition 2.11(iv) the set \( E(B_{\lambda_i}(G_i)) \) is finite. Suppose that \( |E(B_{\lambda_i}(G_i))| = n_i + 1 \) for some integer \( n_i \). Then we have that \( \lambda_i = n_i \geq 1 \). On the set \( E_i = (\lambda_i \times \lambda_i) \cup \{0\} \), where \( 0 \notin \lambda_i \times \lambda_i \) we define the binary operation in the following way

\[
(\alpha, \beta) \cdot (\gamma, \delta) = \begin{cases} 
(\alpha, \beta), & \text{if } (\alpha, \beta) = (\gamma, \delta); \\
0, & \text{otherwise},
\end{cases}
\]

and \( 0 \cdot (\alpha, \beta) = (\alpha, \beta) \cdot 0 = 0 \cdot 0 = 0 \) for all \( \alpha, \beta, \gamma, \delta \in \lambda_i \). Simple verifications show that \( E_i \) with such defined operation is a semilattice and every non-zero idempotent of \( E_i \) is primitive.

By \( \tilde{E}_S \) we denote the orthogonal sum \( \sum_{i \in \mathcal{J}} E_i \). It is obvious that \( \tilde{E}_S \) is a semilattice and every non-zero idempotent of \( \tilde{E}_S \) is primitive. We determine on \( \tilde{E}_S \) the topology of the Alexandroff one-point compactification \( \tau_A \): all non-zero idempotents of \( \tilde{E}_S \) are isolated points in \( \tilde{E}_S \) and the family

\[
\mathcal{B}(0) = \{ U : U \ni 0 \text{ and } \tilde{E}_S \setminus U \text{ is finite} \}
\]

is the base of the topology \( \tau_A \) at zero \( 0 \in \tilde{E}_S \). Simple verifications show that \( \tilde{E}_S \) with the topology \( \tau_A \) is a Hausdorff compact topological semilattice. Later we denote \( (\tilde{E}_S, \tau_A) \) by \( \tilde{E}_S \).

Let \( G_S = \prod_{i \in \mathcal{J}} G_i \) be the direct product of pseudocompact groups \( G_i, i \in \mathcal{J}, \) with the Tychonoff topology. Then by Comfort–Ross Theorem (see Theorem 1.4 in [6]) we get that \( G_S \) is a pseudocompact topological group.

By Corollary 3.10.27 from [9] we have that the product \( \tilde{E}_S \times G_S \) is a pseudocompact space.

Later for every \( i \in \mathcal{J} \) by \( \pi_i : G_S = \prod_{i \in \mathcal{J}} G_i \to G_i \) we denote the projection on the \( i \)-th factor.

Now, for every \( i \in \mathcal{J} \) we define the map \( f_i : E_i \times G_S \to B_{\lambda_i}(G_i) \) by the formulae \( f_i((\alpha, \beta), g) = (\alpha, \pi_i(g), \beta) \) and \( f_i(0, g) = 0_i \) is zero of the semigroup \( B_{\lambda_i}(G_i) \), and put \( f = \bigcup_{i \in \mathcal{J}} f_i \). It is obvious that the map \( f : \tilde{E}_S \times G_S \to S \) is well defined. The definition of the topology \( \tau_A \) on \( \tilde{E}_S \) implies that for every \((\alpha, \beta), g \) \( E_i \times G_i \subseteq \tilde{E}_S \times G_i \) the set \( \{ (\alpha, \beta) \} \times G_i \) is open in \( \tilde{E}_S \times G_S \) and hence the map \( f \) is continuous at the point \((\alpha, \beta), g \). Also for every \( U(0) = S \setminus (B_{\lambda_{i_1}}(G_{i_1}) \cup B_{\lambda_{i_2}}(G_{i_2}) \cup \cdots \cup B_{\lambda_{i_n}}(G_{i_n})) \) the set \( f^{-1}(U(0)) = (\tilde{E}_S \setminus (\lambda_{i_1} \times \lambda_{i_1}) \cup \cdots \cup (\lambda_{i_n} \times \lambda_{i_n})) \times G_S \) is open in \( \tilde{E}_S \times G_S \), and hence the map \( f \) is continuous.

The following theorem is an analogue of Comfort–Ross Theorem for primitive pseudocompact topological inverse semigroup.

Theorem 2.12. Let \( \{ S_i : i \in \mathcal{J} \} \) be a non-empty family of primitive Hausdorff pseudocompact topological inverse semigroups. Then the direct product \( \prod_{i \in \mathcal{J}} S_i \) with the Tychonoff topology is a pseudocompact topological inverse semigroup.
Proof. Since the direct product of the non-empty family of topological inverse semigroups is a topological inverse semigroup, it is sufficient to show that the space $\prod_{j \in J} S_j$ is pseudocompact. Let $\tilde{E}_{S_j} \times G_{S_j}$, and $f_j: \tilde{E}_{S_j} \times G_{S_j} \to S_j$ be the semilattice, the group and the map, respectively, defined in the proof of Proposition 2.11 for any $j \in J$. Since the space $\prod_{j \in J} \left( \tilde{E}_{S_j} \times G_{S_j} \right)$ is homeomorphic to the product $\prod_{j \in J} \tilde{E}_{S_j} \times \prod_{j \in J} G_{S_j}$ we conclude that by Theorem 3.2.4, Corollary 3.10.27 from [9] and Theorem 1.4 from [6] the space $\prod_{j \in J} \left( \tilde{E}_{S_j} \times G_{S_j} \right)$ is pseudocompact. Now, since the map $\prod_{j \in J} f_j: \prod_{j \in J} \left( \tilde{E}_{S_j} \times G_{S_j} \right) \to \prod_{j \in J} S_j$ is continuous we have that $\prod_{j \in J} S_j$ is a pseudocompact topological space. □

Theorem 2.12 implies the following corollary:

**Corollary 2.13.** Let $\{S_i: i \in J\}$ be a non-empty family of Brandt Hausdorff pseudocompact topological inverse semigroups. Then the direct product $\prod_{i \in J} S_i$ with the Tychonoff topology is a pseudocompact topological inverse semigroup.

**Remark 2.14.** E. K. van Douwen [7] showed that Martin’s Axiom implies the existence of two countably compact groups (without non-trivial convergent sequences) such that their product is not countably compact. Hart and van Mill [18] showed that Martin’s Axiom for countable posets implies the existence of a countably compact group which square is not countably compact. Tomita in [21] showed that under $MA_{\text{countable}}$ for each positive integer $k$ there exists a group which $k$-th power is countably compact but its $2k$-th power is not countably compact. In particular, there was proved that for each positive integer $k$ there exists $l = k, \ldots, 2k-1$ and a group which $l$-th power is not countably compact.

In [22] Tomita constructed a topological group under $MA_{\text{countable}}$ which square is countably compact but its cube is not. Also, Tomita in [23] showed that the existence of $2^\gamma$ mutually incomparable selective ultrafilters and $2^\gamma = 2^{2^\gamma}$ implies that there exists a topological group $G$ such that $G^\gamma$ is countably compact for all cardinals $\gamma < \kappa$, but $G^\kappa$ is not countably compact for every cardinal $\kappa \leq 2^\gamma$. Using these results and the construction of finite topological Brandt $\lambda^0$-extensions proposed in [17] we may show that statements similar to aforementioned hold for Hausdorff countably compact Brandt topological inverse semigroups and hence for Hausdorff countably compact primitive topological inverse semigroups.

3. **The Stone-Čech compactification of a pseudocompact primitive topological inverse semigroup**

Let a Tychonoff topological space $X$ be a topological sum of subspaces $A$ and $B$, i.e., $X = A \bigoplus B$. It is obvious that every continuous map $f: A \to K$ from $A$ into a compact space $K$ (resp., $f: B \to K$ from $B$ into a compact space $K$) extends to a continuous map $\hat{f}: X \to K$. This implies the following proposition:

**Proposition 3.1.** If a Tychonoff topological space $X$ is a topological sum of subspaces $A$ and $B$, then $\beta X$ is equivalent to $\beta A \bigoplus \beta B$.

The following theorem describes the structure of the Stone-Čech compactification of a primitive pseudocompact topological inverse semigroup.

**Theorem 3.2.** Let $S$ be a primitive pseudocompact topological inverse semigroup. Then the Stone-Čech compactification of $S$ admits a structure of primitive topological inverse semigroup with respect to which the inclusion mapping of $S$ into $\beta S$ is a topological isomorphism. Moreover, $\beta S$ is topologically isomorphic to the orthogonal sum $\sum_{i \in J} B_{\lambda_i}(\beta G_i)$ of topological Brandt $\lambda_i$-extensions $B_{\lambda_i}(\beta G_i)$ of compact topological groups $\beta G_i$ in the category $\mathcal{T}\mathcal{S}\mathcal{G}\mathcal{E}$ for some finite cardinals $\lambda_i \geq 1$.

**Proof.** By Theorem 2.7, every primitive pseudocompact topological inverse semigroup $S$ is topologically isomorphic to the orthogonal sum $\sum_{i \in J} B_{\lambda_i}(G_i)$ of topological Brandt $\lambda_i$-extensions $B_{\lambda_i}(G_i)$ of...
pseudocompact topological groups $G_i$ in the category $\mathcal{S\mathcal{G}}$ for some finite cardinals $\lambda_i \geq 1$, such that any non-zero $\mathcal{H}$-class of $S$ is an open-and-closed subset of $S$, and the family $\mathcal{B}(0)$ defined by formula (1) determines a base of the topology at zero 0 of $S$.

By Theorem 2.12, $S \times S$ is a pseudocompact topological space. Now by Theorem 1 of [10], we have that $\beta(S \times S)$ is equivalent to $\beta S \times \beta S$, and hence by Theorem 1.3 [1], $S$ is a subsemigroup of the compact topological semigroup $\beta S$.

By Proposition 3.1 for every non-zero $\mathcal{H}$-class $(G_i)_{k,l}$, $k, l \in \lambda_i$, we have that $\text{cl}_{\beta S}((G_i)_{k,l})$ is equivalent to $\beta(G_i)_{k,l}$, and hence it is equivalent to $\beta G_i$. Therefore we get that $\sum_{i \in \mathcal{I}} B_{\lambda_i}(G_i) \subseteq \beta S$. Suppose that $\sum_{i \in \mathcal{I}} B_{\lambda_i}(G_i) \neq \beta S$. We fix an arbitrary $x \in \beta S \setminus \sum_{i \in \mathcal{I}} B_{\lambda_i}(G_i)$. Then Hausdorffness of $\beta S$ implies that there exist open neighbourhoods $V(x)$ and $V(0)$ of the points $x$ and 0 in $\beta S$, respectively, and there exist finitely many $i_1, \ldots, i_n \in \mathcal{I}$ such that $V(0) \cap \beta S \supseteq S \setminus (B_{\lambda_1}(G_{i_1}) \cup B_{\lambda_2}(G_{i_2}) \cup \cdots \cup B_{\lambda_n}(G_{i_n}))^*$. Then we have that $V(x) \cap S \subseteq (B_{\lambda_1}(G_{i_1}) \cup B_{\lambda_2}(G_{i_2}) \cup \cdots \cup B_{\lambda_n}(G_{i_n}))^* \subseteq (B_{\lambda_1}(\beta G_{i_1}) \cup B_{\lambda_2}(\beta G_{i_2}) \cup \cdots \cup B_{\lambda_n}(\beta G_{i_n}))^*$, and since by Theorem 2.7, $\lambda_i$ is finite for every $i \in \mathcal{I}$, we get a contradiction with the initial assumption. This completes the proof of the theorem.

Theorem 3.2 implies the following:

**Corollary 3.3.** Let $S$ be a primitive countably compact topological inverse semigroup. Then the Stone-Čech compactification of $S$ admits a structure of primitive topological inverse semigroup with respect to which the inclusion mapping of $S$ into $\beta S$ is a topological isomorphism.

**Remark 3.4.** Theorem 3.2 and Corollary 3.3 give the positive answer to the Question 1, which we posed in [2].

We define the series of categories as follows:

(i) Let $\text{Ob}(\mathcal{B}^*(\mathcal{CCIG}))$ be all Hausdorff 0-simple countably compact topological inverse semigroups; Let $\text{Ob}(\mathcal{B}^*(\mathcal{PCTG}))$ be all Hausdorff pseudocompact topological inverse Brandt semigroups; Let $\text{Ob}(\mathcal{PPCTG})$ be all primitive Hausdorff pseudocompact topological inverse semigroups; Let $\text{Ob}(\mathcal{PCCPTG})$ be all primitive Hausdorff pseudocompact topological inverse semigroups;

(ii) Let $\text{Mor}(\mathcal{B}^*(\mathcal{CCIG}))$, $\text{Mor}(\mathcal{B}^*(\mathcal{PCTG}))$, $\text{Mor}(\mathcal{PPCTG})$, and $\text{Mor}(\mathcal{PCCPTG})$ be continuous homomorphisms of of corresponding topological inverse semigroups.

Comfort and Ross [3] proved that the Stone-Čech compactification of a pseudocompact topological group is a topological group. Therefore the functor of the Stone-Čech compactification $\beta$ from the category of pseudocompact topological groups back into itself determines a monad. Similar result Gutik and Repovš proved in [17] for the category of all Hausdorff 0-simple countably compact topological inverse semigroups $\mathcal{B}^*(\mathcal{CCIG})$. In the our case by Theorem 3.2 and Corollary 3.3 we get the same:

**Corollary 3.5.** The functor of the Stone-Čech compactification $\beta : \mathcal{B}^*(\mathcal{CCIG}) \to \mathcal{B}^*(\mathcal{CCIG})$ (resp., $\beta : \mathcal{B}^*(\mathcal{PCTG}) \to \mathcal{B}^*(\mathcal{PCTG})$, $\beta : \mathcal{PPCTG} \to \mathcal{PPCTG}$, $\beta : \mathcal{PCCPTG} \to \mathcal{PCCPTG}$) determines a monad.

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