Categories of operators and actions of group operads

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Contents

Introduction 1
1 Group operads and crossed interval groups 4
2 Quotients of the total category 8
3 Associated double categories 15
4 Internal presheaves over the associated double categories 21
5 CoCartesian lifting properties 30
6 The equivalence of notions 36

Introduction

Operads, introduced by Stasheff [18] and May [14], play essential roles in the study of higher algebras. For example, the little disks operads present the levels of higher commutativities. On the other hand, May and Thomason introduced in their paper [15] the notion of categories of operators. In contrast to operads, which are defined in an algebraic way, a category of operators is just a fibration in an appropriate sense over the opposite $\Gamma^{op}$ of the Segal’s category satisfying certain conditions. They pointed out that every topological operad gives rise to a topological category of operators and proved the equivalence of algebras over them. The algebraic operations in operads are presented by certain universal lifting properties of categories of operators. Hence, the relation of operads and categories of operators is compared with that of $G$-spaces and fiber bundles over the classifying space $BG$ for a group $G$. This geometric nature of categories of operators was made use of in the definition of $\infty$-operads by Lurie [12], which is one of the well-used models of homotopy operads.

The original definition of categories of operators, however, only covers the symmetric operads (or multicategories more generally). On the other hand, Zhang introduced group operads [22] as generalizations of the operad $S$ of symmetric groups. As pointed out by Gurski [8], a group operad $G$ may have an
action on multicategories; namely, for a non-symmetric multicategory $M$, the action of $G$ on $M$ is defined to be a right action

$$
\left( \prod_{\sigma \in S_n} M(a_{\sigma(1)} \ldots a_{\sigma(n)}; a) \right) \times G(n) \to \prod_{\sigma \in S_n} M(a_{\sigma(1)} \ldots a_{\sigma(n)}; a)
$$

(0.1)
on the multihom-set for $a, a_1, \ldots, a_n \in \text{Ob} M$ with an appropriate compatibility condition with the compositions. In the author’s previous paper [20], it is called a $G$-symmetric structure and defined in terms of a monoidal structures on the category $\text{GrpOp}$ of group operads. More precisely, a 2-monad $(-) \rtimes G$ on the 2-category $\text{MultCat}$ of multicategories is defined so that a $G$-symmetric structure is nothing but a structure of a (strict) 2-algebra over $(-) \rtimes G$. As a result, we obtain a 2-category $\text{MultCat}_G$ of $G$-symmetric multicategories.

It is then natural to ask for a $G$-symmetric analogue of categories of operators. This is exactly the main theme of this paper. Note that, we already have a non-symmetric analogue: let $\nabla$ be the category of intervals; i.e.

- objects are totally ordered sets of the following form for $n \in \mathbb{N}$:
  $$\langle \langle n \rangle \rangle := \{-\infty, 1, \ldots, n, \infty\};$$

- morphisms are order-preserving maps sending $\pm \infty$ to $\pm \infty$ respectively.

Using the duality $\nabla \cong \Delta^{\text{op}}$ due to Joyal, where $\Delta$ is the simplex category, for a non-symmetric multicategory $M$, a multicategorical version of the categorical wreath product appearing in [2] and [16] gives rise to a fibered category

$$M^\nabla := \Delta \wr M \to \nabla.$$ 

Note that the objects of $M^\nabla$ are finite sequences $\bar{a} = a_1 \ldots a_n$ of objects $a_1, \ldots, a_n \in \text{Ob} M$. Moreover, if $\mu_n : \langle\langle n \rangle\rangle \to \langle\langle 1 \rangle\rangle \in \nabla$ is the morphism which is the dual of the map $[1] \to [n] \in \Delta$ with $0 \mapsto 0$ and $1 \mapsto n$, we have a pullback square

$$
\begin{array}{ccc}
M(a_1 \ldots a_n; a) & \to & M^\nabla(a_1 \ldots a_n, a) \\
\downarrow & & \downarrow \\
\{\mu_n\} & \to & \nabla(\langle\langle n \rangle\rangle, \langle\langle 1 \rangle\rangle)
\end{array}
$$

(0.2)

Actually, the functor $M^\nabla \to \nabla$ is characterized by (0.2) with the universal lifting property with respect to a certain class of morphisms in $\nabla$ as well as Segal condition; i.e. there is a canonical equivalence of categories

$$M^\nabla_{\langle\langle n \rangle\rangle} \simeq (M^\nabla_{\langle\langle 1 \rangle\rangle})^n$$
on the fibers $M^\nabla_{\langle\langle k \rangle\rangle} := M^\nabla \times \nabla \{\langle\langle k \rangle\rangle\}$.

Unfortunately, the category $\nabla$ may not be enough for this purpose. Indeed, the action (0.1) may change the domain of multimorphisms while, in view of (0.2), no fiberwise action on $M^\nabla$ can realize this phenomenon. This is mainly
because \( \nabla \) has no non-trivial isomorphism, so we have to consider an extension of \( \nabla \) so that the symmetry of \( \mathcal{G} \) is taken into account. We can make use of the results obtained in [20]. The category \( \text{GrpOp} \) of group operads can be thought of as a reflective full subcategory of the slice category \( \text{CrsGrp}^{\mathcal{G}/\mathcal{G}} \) of the category of crossed interval groups. Hence, we can consider the total category \( \nabla_{\mathcal{G}} \). Furthermore, quotient categories of \( \nabla_{\mathcal{G}} \) is important. For example, as pointed out by Segal [17], a monoid \( M \) gives rise to an extension \( \hat{M} : \nabla \to \text{Set} \) given by \( \hat{M}(\langle n \rangle) := M^{\times n} \) and, for \( \varphi : \langle m \rangle \to \langle n \rangle \in \nabla \), \( \varphi_\ast := \hat{M}(\varphi) : M^{\times m} \to M^{\times n} \) with

\[
\varphi_\ast(x_1, \ldots, x_m) := \left( \prod_{\varphi(i) = 1} x_i, \ldots, \prod_{\varphi(i) = n} x_i \right),
\]

where the products are taken in the appropriate orders. The Grothendieck construction enables us to regard \( \hat{M} \) as a discrete fibration \( M^\ast \to \nabla \), so it is a discrete version of a category of operators. Notice that a functor \( X : \nabla_{\mathcal{G}} \to \text{Set} \) is equivalent to data

- a functor \( X : \nabla \to \text{Set} \);
- for each \( n \in \mathbb{N} \), a left \( \mathcal{G}(n) \)-action on the set \( X_n := \langle \langle n \rangle \rangle \), say \( \mathcal{G}(n) \times X_n \to X_n; \ (u, x) \mapsto x^u \);

such that, for each \( \varphi : \langle m \rangle \to \langle n \rangle \in \nabla \), \( x \in X(\langle m \rangle) \), and \( v \in \mathcal{G}(n) \),

\[
\varphi_\ast(x)^v = (\varphi_\ast^v)(x^\varphi(v)).
\]

Now, each \( \mathcal{G}(n) \) acts on \( \hat{M}(\langle \langle n \rangle \rangle) = M^{\times n} \) in the obvious way so that it gives rise to an extension \( \hat{M}^\ast_{\mathcal{G}} : \nabla_{\mathcal{G}} \to \text{Set} \). In particular, when \( \mathcal{G} = \mathcal{G} \), we assert that the monoid \( M \) is commutative if and only if the functor \( \hat{M}^\ast \mathcal{G} : \nabla_{\mathcal{G}} \to \text{Set} \) factors through an appropriate quotient category of \( \nabla_{\mathcal{G}} \). For each \( (x_1, \ldots, x_n) \in M^{\times n} \), and for each \( \sigma \in \mathcal{G}_n \), we have

\[
(\mu_n)_\ast((x_1, \ldots, x_n)^\sigma) = (\mu_n)_\ast(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}) = x_{\sigma^{-1}(1)} \ldots x_{\sigma^{-1}(1)}.
\]

It follows that \( M \) is commutative if and only if the two morphisms \( (\mu_n, \sigma), (\mu_n, e_n) : \langle \langle n \rangle \rangle \to \langle \langle 1 \rangle \rangle \in \nabla_{\mathcal{G}} \), here \( e_n \in \mathcal{G}(n) \) is the unit, induce the same map \( \hat{M}^\ast_{\mathcal{G}}(\langle \langle n \rangle \rangle) \to \hat{M}^\ast_{\mathcal{G}}(\langle \langle 1 \rangle \rangle) \) for every \( n \) and \( \sigma \in \mathcal{G}(n) \). In other words, the associated discrete fibration \( \hat{M}^\ast_{\mathcal{G}} : \nabla_{\mathcal{G}} \to \text{Set} \) is a pullback along a quotient \( q : \nabla_{\mathcal{G}} \to \mathcal{Q} \) such that \( q(\mu_n, \sigma) = q(\mu_n, e_n) \). More generally, for a group operad \( \mathcal{G} \), each quotient of the total category \( \nabla_{\mathcal{G}} \) may present a \( \mathcal{G} \)-symmetry on the fibrations over it.

In this point of view, we will establish a \( \mathcal{G} \)-symmetric analogue of categories of operators in the following way. After reviewing the basic results of group operads and crossed interval groups in Section 1, we will attempt to the classification of a sort of quotient categories of \( \nabla_{\mathcal{G}} \) for arbitrary crossed interval group \( G \) in Section 2. It will be seen that the congruence families on \( G \) determine and are determined by quotient categories. For a congruence family \( K \) on \( G \), we write \( \mathcal{Q}_K \) the associated quotient of \( \nabla_{\mathcal{G}} \). Basic constructions and some technical results on congruence families will be discussed. In particular, in the case \( G \) is a group operad \( \mathcal{G} \), two special congruence families \( \text{Kec}^\mathcal{G} \subset \text{Dec}^\mathcal{G} \) will be introduced.
We will see in Section 3 that if we have a pair \((K, L)\) of congruence families satisfying a certain condition, the quotient category \(Q_L\) become a part of an internal category 

\[ Q_L/K \Rightarrow Q_L \]  

in the category \(\text{Cat}\) of small categories. In other words, \((0.3)\) is a double category. For example, if \((K, L) = (\text{Dec}, \text{Kec})\), we write \(G_G := Q_{\text{Kec}}/\text{Dec}\) and \(E_G := Q_{\text{Dec}}\), which are the key to establish \(G\)-symmetric analogue of categories of operators. Moreover, further quotient 

\[ \tilde{Q}_L/K \Rightarrow \tilde{Q}_L \]

will be discussed.

Having an internal category, we are interested in internal presheaves over it. In Section 4, we will see that a \(G\)-symmetric multicategory \(M\) gives rise to an internal presheaf \(M \mapsto \tilde{E}_G\) over the double category \(\tilde{G}_G \Rightarrow \tilde{E}_G\). It will turn out that the construction extends to a 2-functor 

\[ \text{MultCat} \to \text{PSh}(\tilde{G}_G \Rightarrow \tilde{E}_G) \]  

from the 2-category of (non-symmetric) multicategories to that of internal presheaves over \(\tilde{G}_G \Rightarrow \tilde{E}_G\). It will proved that the fibered product \((M \mapsto \tilde{E}_G) \times_{\tilde{E}_G} \tilde{G}_G\) is exactly the image of the free \(G\)-symmetrization \(M \times G\) under the 2-functor \((0.3)\). In other words, it is the required counterpart in the side of fibrations.

In the last two sections, we will compute the essential image of the 2-functor \((0.4)\). We will propose a notion of categories of algebraic \(G\)-operators as analogues of \(\infty\)-operads by Lurie [12]. Namely, they form a 2-subcategory \(\text{Oper}^\text{alg}_G\) of \(\text{PSh}(\tilde{G}_G \Rightarrow \tilde{E}_G)\) consisting of internal presheaves \(X\) over the double category \(\tilde{G}_G \Rightarrow \tilde{E}_G\) such that the canonical functor \(X \to \tilde{E}_G\) satisfies a universal lifting property and Segal condition. Actually, they are alternative models of \(G\)-symmetric multicategories; we will prove the biequivalence \(\text{MultCat}_G \simeq \text{Oper}^\text{alg}_G\) in Section 6.

We finally note that we say the models above are “algebraic” because the actions of \(G\) as in \((0.1)\) are realized in an algebraic way. It is also possible to realize them in a “geometric” way; for example, as fibrations over appropriate categories. It will be established in the author’s future work.

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1 Group operads and crossed interval groups

We review the notion of group operads. It was introduced by Zhang [22] though the axioms were already stated in 1.2.0.2 in the paper [19]. The further theory...
was developed in [5] and [8] while they use different terminology “action oper-
ads.” We give just sketches; for the details, we refer the reader to the literature
above and the author’s previous work [20].

For each \( n \in \mathbb{N} \), set \( S(n) = S_n \) to be the \( n \)-th symmetric group. It turns
out that the family \( S = \{ S(n) \}_{n} \) admits a structure of an operad so that, for
\( \sigma, \tau \in S(n) \), \( \sigma, \tau \in S(k_i) \), we have

\[
\gamma_S(\sigma \tau; \sigma_1 \tau_1, \ldots, \sigma_n \tau_n) = \gamma_S(\sigma; \sigma \tau^{-1}(1), \ldots, \sigma \tau^{-1}(n)) \gamma_S(\tau; \tau_1, \ldots, \tau_n),
\]

where \( \gamma \) is the composition in the operad structure. The group operads are
generalizations of this example.

**Definition.** A **group operad** is an operad \( G \) together with data

- a group structure on each \( G(n) \);
- a map \( G \to S \) of operads so that each \( G(n) \to S(n) \) is a group homomor-
phism, which gives rise to a left \( G(n) \)-action on \( \langle n \rangle \);

which satisfy the identity

\[
\gamma_G(x; x_1 y_1, \ldots, x_n y_n) = \gamma_G(x; x y^{-1}(1), \ldots, x y^{-1}(n)) \gamma_G(y; y_1, \ldots, y_n) \quad (1.1)
\]

for every \( x, y \in G(n) \) and \( x_i, y_i \in G(k_i) \) for \( 1 \leq i \leq n \).

**Example 1.1.** The operad \( S \) is an example of group operads with the identity
map \( S \to S \).

**Example 1.2.** For each \( n \in \mathbb{N} \), set \( B(n) \) to be the braid group on \( n \) strands.
Then, in a similar manner to \( S \), one can find an operad structure on the family
\( B = \{ B(n) \}_{n} \) so that the canonical maps \( B(n) \to S(n) \) define a map of operads.
The canonical map \( B \to S \) together with the group structure on each \( B(n) \)
exhibits \( B \) as a group operad.

**Definition.** A **map of group operads** is a map \( F : G \to H \) of operads satisfying the
following conditions:

(i) each map \( F : G(n) \to H(n) \) is a group homomorphism;

(ii) \( F \) respects the maps into \( S \); i.e. the diagram below commutes:

\[
\begin{array}{ccc}
G & \xrightarrow{F} & H \\
\downarrow & & \downarrow \\
S & \xrightarrow{F} & S
\end{array}
\]

We denote by \( \text{GrpOp} \) the category of group operads and maps of group
operads.

One of the most important features of group operads is that they may act
on multicategories. Recall that a multicategory \( M \) consists of a set \( \text{Ob}M \)
and a set \( M(a_1 \ldots a_n; a) \) for each \( a, a_1, \ldots, a_n \in M \) together with associative
compositions and identities. For a group operad \( G \), we define a multicategory
\( M \rtimes G \) as follows:
• \(\text{Ob}(\mathcal{M} \rtimes \mathcal{G}) = \text{Ob}\mathcal{M}\);

• for \(a, a_1, \ldots, a_n \in \text{Ob}\mathcal{M}\), set

\[
(\mathcal{M} \rtimes \mathcal{G})(a_1 \ldots a_n; a) := \{(f, x) \mid x \in \mathcal{G}(n), f \in \mathcal{M}(a_{x^{-1}(1)} \ldots a_{x^{-1}(n)}; a)\};
\]

• the composition operation is given by

\[
\gamma_{\mathcal{M} \rtimes \mathcal{G}}((f, x); (f_1, x_1), \ldots, (f_n, x_n)) = (\gamma_{\mathcal{M}}(f; f_{x^{-1}(1)}, \ldots, f_{x^{-1}(n)}), \gamma_{\mathcal{G}}(x; x_1, \ldots, x_n)).
\]

The assignment \(\mathcal{M} \mapsto \mathcal{M} \rtimes \mathcal{G}\) extends in a canonical way to an endo-2-functor

\[(-) \rtimes \mathcal{G} : \text{MultCat} \to \text{MultCat}\]

on the 2-category of multicategories, multifunctors, and multinatural transformations. Moreover, there are two identity-on-objects multifunctors

\[
M : \mathcal{M} \rtimes \mathcal{G} \times \mathcal{G} \to \mathcal{M} \rtimes \mathcal{G} ; \quad (f, x, y) \mapsto (f, xy)
\]

\[
H : \mathcal{M} \to \mathcal{M} \rtimes \mathcal{G} ; \quad f \mapsto (f, e),
\]

where \(e\) is the unit of the group \(\mathcal{G}(n)\) for an appropriate \(n \in \mathbb{N}\). Clearly, the multifunctors (1.2) are strictly natural with respect to \(\mathcal{M} \in \text{MultCat}\), and one can see the triple \((-) \rtimes \mathcal{G}, M, H\) forms a (strict) 2-monad on the 2-category \(\text{MultCat}\) in the sense of [3]. The following result is easy to verify.

**Lemma 1.3.** The assignment \(\mathcal{G} \mapsto (-) \rtimes \mathcal{G}\) extends to a functor

\[\text{GrpOp} \to \text{2-Mnd(MultCat)},\]

here the codomain is the category of (strict) 2-monads on \(\text{MultCat}\) and 2-monad transformations.

**Definition** (cf. Definition 5.1 in [3]). Let \(\mathcal{G}\) be a group operad. Then, a \(\mathcal{G}\)-symmetric multicategory is a multicategory \(\mathcal{M}\) equipped with a structure of a (strict) algebra for the 2-monad \((-) \rtimes \mathcal{G}\). More explicitly, a \(\mathcal{G}\)-symmetric multicategory is a multicategory \(\mathcal{M}\) together with a multifunctor \(\mathcal{A} : \mathcal{M} \rtimes \mathcal{G} \to \mathcal{M}\) such that the following diagrams of multifunctors are (strictly) commutative:

\[
\begin{array}{ccc}
\mathcal{M} \rtimes \mathcal{G} \times \mathcal{G} & \xrightarrow{\mathcal{A} \times \mathcal{G}} & \mathcal{M} \times \mathcal{G} \\
\mathcal{M} \times \mathcal{G} & \xrightarrow{\mathcal{A}} & \mathcal{M} \\
\mathcal{M} \rtimes \mathcal{G} & \xrightarrow{H} & \mathcal{M} \rtimes \mathcal{G}
\end{array}
\]

**Remark 1.4.** If \(\mathcal{M}\) be a \(\mathcal{G}\)-symmetric multicategory with \(\mathcal{A} : \mathcal{M} \rtimes \mathcal{G} \to \mathcal{G}\), then \(\mathcal{A}\) is the identity on objects. In this case, by abuse of notations, for \(f \in \mathcal{M}(a_1 \ldots a_n; a)\) and for \(x \in \mathcal{G}(n)\), we will write \(f^x := \mathcal{A}(f, x)\). Note that we have

\[f^x \in \mathcal{M}(a_{x(1)} \ldots a_{x(n)}; a) .\]

**Example 1.5.** A symmetric multicategory (resp. a symmetric operad) in the usual sense is nothing but an \(\mathcal{G}\)-symmetric multicategory (resp. an \(\mathcal{G}\)-symmetric operad) in the sense above.
Example 1.6. Let $*$ be the trivial group operad. Then $*$-symmetric multicategories are nothing but ordinary multicategories in our convention.

Example 1.7. Let $C$ be a monoidal category, and put $C^\otimes$ the associated multicategory; i.e. $\text{Ob} C^\otimes = \text{Ob} C$ and $C^\otimes(X_1 \ldots X_n; X) = C(X_1 \otimes \cdots \otimes X_n; X)$. Hence, for each $X_1, \ldots, X_n \in C$, we have a multimorphism

$$u_{X_1 \ldots X_n} \in C^\otimes(X_1 \ldots X_n; X_1 \otimes \cdots \otimes X_n)$$

corresponding to the identity. If $C^\otimes$ is endowed with a $G$-symmetric structure, then for each $x \in G(n)$, we define an isomorphism

$$\Theta^x_{X_1 \ldots X_n} : X_{x(1)} \otimes \cdots \otimes X_{x(n)} \to X_1 \otimes \cdots \otimes X_n$$

to be the one corresponding to the multimorphism $u^x_{X_1 \ldots X_n}$. It turns out that $\Theta^x$ is a natural isomorphism with $\Theta^x \Theta^y = \Theta^{xy}$. For example, in the case $G = B$ (resp. $S$), the resulting structure on $C$ is nothing but a braided (resp. symmetric) structure on the monoidal structure.

Definition. Let $G$ be a group operad, and let $M$ and $N$ be $G$-symmetric multicategories. Then, a $G$-symmetric multifunctor $M \to N$ is a multifunctor which is a homomorphism of algebras for the 2-monad $(-) \rtimes G$.

We denote by $\text{MultCat}_G$ the 2-category of $G$-symmetric multicategories, $G$-symmetric multifunctors, and multinatural transformations. In view of Lemma 1.3, a map $F : G \to H$ of group operads induces a 2-functor

$$F^* : \text{MultCat}_H \to \text{MultCat}_G.$$ 

In particular, for every group operad $G$, there are canonical 2-functors

$$\text{MultCat}_G \to \text{MultCat}_G \to \text{MultCat}.$$ 

To understand group operads, as pointed out in [20], the notion of crossed interval groups is convenient. We consider the category $\nabla$ given as follows:

- objects are totally ordered sets of the form

$$\langle \langle n \rangle \rangle := \{-\infty, 1, \ldots, n, \infty\}$$

for $n \in \mathbb{N}$;

- morphisms are order-preserving maps which send $\pm \infty$ to $\pm \infty$ respectively.

Then, a crossed interval group is a $\nabla$-set $G$ equipped with data

- a group structure on $G_n = G(\langle \langle n \rangle \rangle)$ for each $n \in \mathbb{N}$ and

- a left group action

$$G_n \times \nabla(\langle \langle m \rangle \rangle, \langle \langle n \rangle \rangle) \to \nabla(\langle \langle m \rangle \rangle, \langle \langle n \rangle \rangle) ; \quad (x, \varphi) \mapsto \varphi^x$$

for each $m, n \in \mathbb{N}$.
satisfying the equations
\[
\varphi^*(xy) = (\varphi^0)^*(x)\varphi^*(y) \\
(\varphi\psi)^* = \varphi^*\psi^*(x)
\]
for \(x, y \in G(n)\), \(\varphi : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle\), and \(\psi : \langle\langle l\rangle\rangle \to \langle\langle m\rangle\rangle\). In addition, for crossed interval groups \(G\) and \(H\), a map \(G \to H\) of \(\nabla\)-sets is called a map of crossed interval groups if it preserves the structures above. We denote by \(\text{CrsGrp}_\nabla\) the category of crossed interval groups and maps of them. By Theorem 2.4 in [21], \(\text{CrsGrp}_\nabla\) is a locally presentable category and has all (small) limits and colimits.

Remark 1.8. The notion of crossed groups was originally introduced by Fiedorowicz and Loday [6] and by Krasauskas [10] in the simplicial case. Although it is easily generalized to arbitrary base categories, crossed interval groups was first studied by Batanin and Markl [1]. Actually, the terminology is due to them.

Example 1.9. The sequence \(\mathcal{S} = \{\mathcal{S}_n\}\) actually admits a structure of a crossed interval groups. Actually, it is a subobject of the terminal object in \(\text{CrsGrp}_\nabla\) computed in [21].

Theorem 1.10 (Theorem 3.3 in [20]). There is a fully faithful functor
\[
\hat{\Psi} : \text{GrpOp} \to \text{CrsGrp}_\nabla/\mathcal{S}
\]
such that \(\hat{\Psi}(G)_n = G(n)\) for each \(n \in \mathbb{N}\).

The \(\nabla\)-set structure on \(\hat{\Psi}(G)\) is described as follows: note that morphisms \(\varphi : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle \in \nabla\) correspond in one-to-one to \((n+2)\)-tuples
\[
\vec{k} = (k_{-\infty}, k_1, \ldots, k_n, k_{\infty})
\]
of non-negative integers with \(\sum k_j = m\) via \(\varphi \mapsto \vec{k}(\varphi)\) with
\[
k_j^{(\varphi)} := \begin{cases} 
\#\varphi^{-1}\{j\} & 1 \leq j \leq n \\
\#\varphi^{-1}\{j\} - 1 & j = \pm\infty 
\end{cases} \quad (1.3)
\]
Then, for \(\varphi : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle \in \nabla\), the induced map is given by
\[
\varphi^* : G(n) \to G(m) : x \mapsto \gamma_G(e_3^{(\varphi)}; \gamma_G(x; e_1^{(\varphi)}, \ldots, e_n^{(\varphi)}), e_{\infty}^{(\varphi)}),
\]
where \(e_j^{(\varphi)} := e_j^{(\varphi)}\).

Example 1.11. Theorem 1.10 justifies the coincidence of the notation \(\mathcal{S}\). Indeed, the functor \(\hat{\Psi}\) sends \(\mathcal{S}\) to \(\mathcal{S}\). Identifying \(\text{GrpOp}\) with its image in \(\text{CrsGrp}_\nabla\), we can hence identify \(\mathcal{S}\).

2 Quotients of the total category

As pointed out in Introduction, we can regard any quotients of \(\nabla G\) may present a kind of symmetries on monoids or higher variants. Hence, the classification of the quotients is an important problem. In particular, in this section, we focus on the quotients of the following form.
**Definition.** Let $G$ be a crossed interval group. Then, a $G$-quotal category is a category $Q$ equipped with a functor $q : \nabla_G \to Q$ satisfying the following conditions:

(i) $q$ is full and bijective on objects, so we may assume $\text{Ob} \ Q = \text{Ob} \nabla$;

(ii) for $\varphi, \varphi' \in \nabla(\langle m \rangle, \langle n \rangle)$ and $x, x' \in G_m$, the equality of morphisms $$q(\varphi, x) = q(\varphi', x') : \langle m \rangle \to \langle n \rangle \in Q$$ in $Q$ implies $\varphi = \varphi'$.

**Lemma 2.1.** Let $G$ be a crossed interval group, and let $Q$ be a $G$-quotal category with $q : \nabla_G \to Q$. Then, the composition $$\nabla \hookrightarrow \nabla_G \xrightarrow{q} Q$$ is faithful and conservative.

**Proof.** Let us denote by $e_m \in G_m$ the unit of the group. Then, the composition (2.1) sends a morphism $\varphi : \langle m \rangle \to \langle n \rangle \in \nabla$ to $q(\varphi, e_m)$. Hence, the condition on $G$-quotal categories directly implies (2.1) is faithful.

Next, suppose $\varphi : \langle m \rangle \to \langle n \rangle \in \nabla$ is a morphism such that $q(\varphi, e_m)$ is an isomorphism. Since $q$ is full, the inverse of $q(\varphi, e_m)$ can be written in the form $q(\psi, y)$ with $\psi : \langle n \rangle \to \langle m \rangle \in \nabla$ and $y \in G_n$. We have

$$\text{id}_{\langle m \rangle} = q(\psi, y) \circ q(\varphi, e_m) = q(\psi \varphi^y, \varphi^*(y))$$

$$\text{id}_{\langle n \rangle} = q(\varphi, e_m) \circ q(\psi, y) = q(\varphi \psi, y).$$

By virtue of the condition on $G$-quotal categories, the former equation implies $\text{id}_{\langle m \rangle} = \psi \varphi^y$ while the latter implies $\text{id}_{\langle n \rangle} = \varphi \psi$. It follows that $\psi$ is an inverse of $\varphi$ in \nabla, so (2.1) is conservative.

Thanks to Lemma 2.1, we can identify morphisms in \nabla with their images in a $G$-quotal category; namely, if $q : \nabla_G \to Q$ is a $G$-quotal category, then by abuse of notation, we write $\varphi = q(\varphi, e_m)$ for every $\varphi : \langle m \rangle \to \langle n \rangle \in \nabla$.

There is a general recipe to construct $G$-quotal categories. For this, we introduce some notions.

**Definition.** Let $G$ be a crossed interval group, and let $\varphi : \langle m \rangle \to \langle n \rangle \in \nabla$ be a morphism. Then, an element $x \in G_m$ is called a right stabilizer of $\varphi$ if for every morphism $\psi : \langle I \rangle \to \langle m \rangle \in \nabla$, we have $\varphi \psi^* = \varphi \psi$. We denote by $\text{RSt}_\varphi G \subset G_m$ the subset of right stabilizers of $\varphi$.

It is obvious that, for $\varphi : \langle m \rangle \to \langle n \rangle \in \nabla$, the subset $\text{RSt}_\varphi G \subset G_m$ is a subgroup.

**Definition.** Let $G$ be a crossed interval group. A congruence family on $G$ is a family $K = \{K_\varphi\}_{\varphi}$ indexed by morphisms in \nabla such that, for every $\varphi : \langle m \rangle \to \langle n \rangle \in \nabla$,

(i) $K_\varphi$ is a subgroup of $\text{RSt}_\varphi G$;

(ii) for every morphism $\chi : \langle n \rangle \to \langle k \rangle \in \nabla$, $K_\varphi \subset K_{\chi \varphi}$. 

1
(iii) for every morphism \( \psi : \langle l \rangle \to \langle m \rangle \in \nabla \), the map \( \psi^* : G_m \to G_l \) restricts to a map \( K_{\varphi} \to K_{\varphi \psi} \);

(iv) for every element \( y \in G_n \), we have
\[
\varphi^*(y) \cdot K_{\varphi} \cdot \varphi^*(y)^{-1} = K_{\varphi \psi}.
\] (2.2)

Remark 2.2. The first three conditions above implies \( K = \{ K_{\varphi} \}_{\varphi} \) forms a crossed group over \( \text{opTw}(\nabla) := \text{Tw}(\nabla)^{\text{op}} \) the opposite of the twisted arrow category of \( \nabla \); \( \text{opTw}(\nabla) \) is the category such that

- the objects are morphisms of \( \nabla \);
- for morphisms \( \varphi_1 : \langle m_1 \rangle \to \langle n_1 \rangle \in \nabla \) for \( i = 1, 2 \), morphisms \( \varphi_1 \to \varphi_2 \) in \( \text{Tw}(\nabla) \) are pairs \((\alpha, \beta)\) of morphisms in commutative squares of the form
\[
\begin{array}{c}
\langle m_1 \rangle \\
\alpha \Downarrow \\
\varphi_1 \\
\langle n_1 \rangle \\
\beta \Downarrow \\
\langle m_2 \rangle
\end{array}
\begin{array}{c}
\langle m_2 \rangle \\
\varphi_2 \\
\langle n_2 \rangle
\end{array} ;
\]

- the composition is given by
\[
(\gamma, \delta) \circ (\alpha, \beta) = (\gamma \alpha, \beta \delta) .
\]

The second and the third conditions imply each morphism \((\alpha, \beta) : \varphi_1 \to \varphi_2 \) in \( \text{opTw}(\nabla) \) induces a map
\[
(\alpha, \beta)^* : K_{\varphi_2} \xrightarrow{\alpha^*} K_{\varphi_2 \alpha} \hookrightarrow K_{\beta \varphi_2 \alpha} = K_{\varphi_1}.
\]

Moreover, for a morphism \((\alpha, \beta) : \varphi_1 \to \varphi_2 \) in \( \text{opTw}(\nabla) \) as above, if \( x \in G_{m_2} \) is a right stabilizer of \( \varphi_2 \) then the pair \((\alpha^*, \beta^*)\) is again a morphism \( \varphi_1 \to \varphi_2 \) in \( \text{opTw}(\nabla) \). Hence, by virtue of the first condition, this defines a left action of \( K_{\varphi_2} \) on the set \( \text{opTw}(\nabla)(\varphi_1, \varphi_2) \). It is easily verified that these data actually form a crossed \( \text{opTw}(\nabla) \)-group structure on the family \( K = \{ K_{\varphi} \}_{\varphi} \).

Example 2.3. The family \( RSt^G = \{ RSt^G_{\varphi} \}_{\varphi} \) is itself a congruence family. Indeed, the first three conditions for congruence families are obvious. To verify (2.2), observe that, for \( \varphi : \langle m \rangle \to \langle n \rangle \in \nabla \), \( \psi : \langle l \rangle \to \langle l \rangle \nabla \), \( x \in RSt^G_{\varphi} \), and \( y \in G_n \), we have
\[
\varphi \psi \varphi^* (y) x \varphi^* (y)^{-1} = (\varphi \psi \varphi^* (y))^{-1} = (\varphi \psi \varphi^* (y)^{-1}) = \varphi \psi .
\]

Note that, in view of Remark 2.2, a congruence family on \( G \) is nothing but a crossed \( \text{opTw}(\nabla) \)-subgroup of \( RSt^G \) satisfying (2.2).

Example 2.4. Let \( \mathcal{G} \) be a group operad. Recall that morphisms \( \varphi : \langle m \rangle \to \langle n \rangle \in \nabla \) correspond to \((n+2)\)-tuples \( k = (k_{-\infty}, k_1, \ldots, k_n, k_{\infty}) \) by the formula (1.3). We set a subset \( \text{Dec}^G_{\varphi} \subset G_m \) to be the image of the map
\[

\begin{align*}
\mathcal{G}(k_{-\infty}) \times \mathcal{G}(k_1^{(x)}) \times \cdots \times \mathcal{G}(k_n^{(x)}) \times \mathcal{G}(k_{\infty}) & \to \mathcal{G}(m) \\
(x_{-\infty}, x_1, \ldots, x_n, x_{\infty}) & \mapsto \gamma(e_{n+2}; x_{-\infty}, x_1, \ldots, x_n, x_{\infty}).
\end{align*}
\] (2.3)
As easily verified, the map $\varphi^*(y) \cdot \gamma(e_{n+2}; x_{\infty}, x_1, \ldots, x_n, x_{\infty})$ induces a composition operation in $G$. Thus, $\text{Dec}_G^G : \text{Dec}_G^G \cdot \varphi^*(y)^{-1} = \text{Dec}_G^G$.

Thus, $\text{Dec}_G^G$ is a congruence family on $G$.

**Example 2.5.** For each $\varphi : \langle m \rangle \rightarrow \langle n \rangle \in \nabla$, we set $\text{Triv}_\varphi$ to consist of a single element $e_m$ which is seen as the unit of $G_m$ for any crossed interval group $G$. Then, clearly $\text{Triv} = \{ \text{Triv}_\varphi \}_\varphi$ is a congruence family on $G$. In view of Remark 2.2, $\text{Triv}$ is the trivial crossed opTw($\nabla$)-group.

**Example 2.6.** Let $K$ be a congruence family on a crossed interval group $G$. For each $\varphi : \langle m \rangle \rightarrow \langle n \rangle \in \nabla$, we have a canonical group homomorphism

$$K_{\varphi} \mapsto \text{RSt}_G^G \mapsto G_m \rightarrow \text{M}_m^\nabla.$$

We put $K_{\varphi}'$ the kernel and claim that $K' = \{ K_{\varphi}' \}_\varphi$ forms a congruence family. Namely, it is an uncrossed opTw($\nabla$)-subgroup of $\text{RSt}_G^G$ since it is the kernel of the map $\text{RSt}_G^G \rightarrow \text{RSt}_G^{\text{M}}$ of crossed opTw($\nabla$)-groups. This observation also leads to the equation (2.2).

We see congruence families on a crossed interval group $G$ are associated to $G$-quotial categories. We use the following lemma.

**Lemma 2.7.** Let $G$ be a crossed interval group, and let $K = \{ K_{\varphi} \}_\varphi$ be a congruence family on $G$. Suppose $\varphi : \langle l \rangle \rightarrow \langle m \rangle$ and $\psi : \langle m \rangle \rightarrow \langle n \rangle$ are morphisms in $\nabla$. Then, for each $y \in G_n$, the composition

$$(K_{\psi} \cdot y) \times G_m \leftrightarrow G_n \times G_m \rightarrow G_m$$

$$(y', x) \mapsto \varphi^*(y') \cdot x$$

induces a map

$$\{ K_{\psi} \cdot y \} \times (K_{\varphi} \setminus G_m) \rightarrow (K_{\psi \varphi}) \setminus G_m.$$

**Proof.** Take $x \in G_m$ and $y \in G_n$. Then, for $u \in K_{\psi}$ and $v \in K_{\varphi}$, we have

$$\varphi^*(vy) \cdot ux = (\varphi^*(y))(v) \cdot (\varphi^*(y)^{-1}) \cdot \varphi^*(y)x. \quad (2.4)$$

By virtue of the conditions on the congruence family $K$, the first term in the right hand side of (2.4) belongs to $K_{\psi \varphi}$, while the second to $K_{\psi \varphi} \subset K_{\psi \varphi}$. Hence, the result follows.

Now, for a congruence family $K$ on a crossed interval group $G$, we define a category $\mathcal{Q}_K$ as follows: the objects are the same as $\nabla$, and for $m, n \in \mathbb{N}$,

$$\mathcal{Q}_K(\langle m \rangle, \langle n \rangle) = \{ (\varphi, [x]) \mid \varphi \in \nabla(\langle m \rangle, \langle n \rangle), [x] \in K_{\varphi} \setminus G_m \}.$$

There is an obvious map $q : \nabla_G(\langle m \rangle, \langle n \rangle) \rightarrow \mathcal{Q}(\langle m \rangle, \langle n \rangle)$. Using Lemma 2.7 and the inclusion $K_{\varphi} \subset \text{RSt}_G^G$, one can see the composition in the total category $\nabla_G$ induces a composition operation in $\mathcal{Q}_K$ so that $q$ is a functor.
Proposition 2.8. For every congruence family $K$ on a crossed interval group $G$, the functor

$$q : \nabla_G \to Q_K$$

given above exhibits $Q_K$ as a $G$-quotal category. Moreover, the assignment $K \mapsto Q_K$ gives a one-to-one correspondence between congruence families on $G$ and (isomorphism classes of) $G$-quotal categories respecting the orders.

Proof. The first statement is obvious. To see the assignment is one-to-one, suppose $Q$ is a $G$-quotal category with $q : \nabla_G \to Q$. For each $\varphi : \langle m \rangle \to \langle n \rangle \in \nabla$, we put

$$K^Q_{\varphi} := \{ x \in G_m | q(\varphi, x) = \varphi \} .$$

We assert $K^Q_{\varphi}$ forms a congruence family on $G$. First, clearly we have $K^Q_{\varphi} \subset K^Q_{\varphi'}$ and $\psi^*(K^Q_{\varphi}) \subset K^Q_{\varphi'}$ whenever the compositions make sense. Next, for $\varphi : \langle m \rangle \to \langle n \rangle$ and $\psi : \langle l \rangle \to \langle m \rangle$, and for each $x \in K^Q_{\varphi}$, we have

$$q(\varphi \psi, x) = q(\varphi \psi, x) ,$$

which implies $K^Q_{\varphi}$ is a congruence family.

It is straightforward that $Q \mapsto K^Q$ is an inverse assignment to $K \mapsto Q_K$. Furthermore, if we have an inclusion $K \subset K'$ between congruence families, i.e. $K^Q_{\varphi} \subset K'^Q_{\varphi}$ for each morphism $\varphi$ in $\nabla$, then there is a functor $Q_K \to Q_{K'}$ which makes the following diagram commute:

\[
\begin{array}{ccc}
\nabla_G & \xrightarrow{g} & Q_K \\
\downarrow & & \downarrow \\
Q_K & \rightarrow & Q_{K'}
\end{array}
\]

In other words, the assignment $K \mapsto K'$ respects the orders, and this completes the proof.

To end the section, we mention a closure operator on the partially ordered set of congruence families. We introduce two classes of morphisms in the category $\nabla$.

Definition. Let $\varphi : \langle m \rangle \to \langle n \rangle \in \nabla$ be a morphism.

1. $\varphi$ is said to be active if $\varphi^{-1}\{\pm \infty\} = \varphi^{-1}\{\pm \infty\}$; equivalently if $k^\varphi_{\pm \infty} = 0$.

2. $\varphi$ is said to be inert if the restriction $\varphi^{-1}(\langle n \rangle) \to \langle n \rangle \subset \langle n \rangle$ is bijective.
Lemma 2.9.  
(1) Every morphism $\varphi$ in $\nabla$ uniquely factors as $\varphi = \mu \rho$ with $\rho$ inert and $\mu$ active.

(2) Every inert morphism admits a unique section.

Remark 2.10. In view of Lemma 2.9, it turns out that the classes $I$ and $A$ of inert morphisms and active morphisms respectively form an orthogonal factorization system $(I, A)$ on $\nabla$; i.e. the following two conditions are satisfied:

(i) the classes $I$ and $A$ are closed under compositions and contains all the isomorphisms;

(ii) every morphisms in $\nabla$ is of the form $\mu \rho$ with $\rho \in I$ and $\mu \in A$;

(iii) for every commutative square

$$
\begin{array}{ccc}
\langle \langle k \rangle \rangle & \xrightarrow{\varphi} & \langle \langle m \rangle \rangle \\
\rho \downarrow & \searrow \chi \nearrow & \mu \\
\langle \langle l \rangle \rangle & \xrightarrow{\psi} & \langle \langle n \rangle \rangle
\end{array}
$$

with $\rho \in I$ and $\mu \in A$, there is a unique diagonal $\chi$ so that $\chi \rho = \varphi$ and $\mu \chi = \psi$.

Note that the notion was first introduced by Freyd and Kelly in [7] under the name factorization. We instead use the name above to emphasize the unique lifting property and to distinguish it from weak factorization systems.

Let $G$ be a crossed interval group, and let $K$ be a congruence family on $G$. Using Lemma 2.9, we construct another congruence family $\overline{K}$ as follows: for each active morphism $\mu$ in $\nabla$, we put $\overline{K}_\mu = K_\mu$. For a general morphism $\varphi$ in $\nabla$, we set $\overline{K}_\varphi \subset \text{RSt}^G_{\varphi}$ to consist $x \in \text{RSt}^G_{\varphi}$ such that, for every morphism $\psi$ with $\varphi \psi$ making sense and active, $\psi^*(x) \in \text{RSt}^G_{\varphi \psi}$ belongs to $\overline{K}_{\varphi \psi}$. Thanks to Lemma 2.9, this extension does not change $\overline{K}_\mu$ for active $\mu$.

Lemma 2.11. In the situation above, the family $\overline{K} = \{ \overline{K}_\varphi \}_\varphi$ forms a congruence family on $G$.

Proof. We first verify $\overline{K}_\varphi \subset \text{RSt}^G_{\varphi}$ forms a subgroup. Suppose $\psi$ is a morphism in $\nabla$ with $\varphi \psi$ making sense and active. For two elements $x, y \in \overline{K}_\varphi$, we have

$$
\psi^*(x^{-1} y) = (\psi^* x^{-1} y)^* (x)^{-1} \psi^*(y) .
$$

(2.6)

Since $x, y \in \text{RSt}^G_{\varphi}$, the composition $\varphi \psi x^{-1} y$ equals to $\varphi \psi$, which is active. Hence, both terms in the right hand side of (2.6) belongs to $K_\varphi$, which implies $x^{-1} y \in \overline{K}_\varphi$.

Using Lemma 2.9 one can verify $\overline{K} = \{ \overline{K}_\varphi \}_\varphi$ forms a crossed opTw($\nabla$)-subgroup of $\text{RSt}^G_{\varphi}$. It remains to verify the formula (2.2). Clearly, the inclusion in one direction will suffice, so we show

$$
\varphi^*(y) \cdot \overline{K}_\varphi \cdot \varphi^*(y)^{-1} \subset \overline{K}_{\varphi^*}
$$

(2.7)
for each \( \varphi : \langle m \rangle \to \langle n \rangle \in \mathcal{N} \) and \( y \in G_n \). Suppose \( \psi : \langle l \rangle \to \langle m \rangle \in \mathcal{N} \) is a morphism with \( \varphi^y \psi \) active. For \( x \in K_{\varphi} \), we have

\[
\psi^*(\varphi^*(y)x\varphi^*(y)^{-1}) = (\varphi \psi \varphi^*(y)^{-1})^*(y) \cdot (\psi \varphi^*(y)^{-1})^*(x) \cdot \psi^*(\varphi^*(y)^{-1})
\]

\[
= (\varphi \psi \varphi^*(y)^{-1})^*(y) \cdot (\psi \varphi^*(y)^{-1})^*(x) \cdot (\varphi^y \psi)^*(y^{-1})
\]

Note that, since \( \varphi^y \psi \varphi^*(y)^{-1} = (\varphi^y \psi)^{y^{-1}} \) is active, the middle term in the right hand side of (2.8) belongs to \( K_{(\varphi^y \psi)^{y^{-1}}} \). Using the formula (2.2) for the congruence family \( K \), one gets

\[
(\varphi^y \psi)^*(y^{-1})^{-1} \cdot K_{(\varphi^y \psi)^{y^{-1}}} \cdot (\varphi^y \psi)^*(y^{-1}) = K_{\varphi^y \psi}.
\]

This implies that \( \psi^*(\varphi^*(y)x\varphi^*(y)^{-1}) \in K_{\varphi^y \psi} \), and the inclusion (2.7) follows. \( \square \)

**Lemma 2.12.** Let \( G \) be a crossed interval group. Then, the assignment \( K \mapsto K_{\varphi} \) defines a closure operator on the ordered set of congruence families on \( G \).

**Proof.** The assignment \( K \mapsto K_{\varphi} \) clearly respects the inclusions. Moreover, since \( K_{\varphi} = K_{\mu} \) for active morphisms \( \mu \), we also have \( K_{\varphi} = K_{\varphi} \) for every morphism \( \varphi \). On the other hand, we have \( K_{\varphi} \subset \text{RSt}_{\varphi}^{G} \), and for every morphism \( \psi \) with \( \varphi \psi \) making sense and active, \( \psi^*(K_{\varphi}) \subset K_{\varphi} \). This implies \( K_{\varphi} \subset K_{\varphi} \). Thus, we obtain the result. \( \square \)

**Definition.** A congruence family \( K \) on a crossed interval group \( G \) is said to be **proper** if it is closed with respect to the closure operator \( (-) \) defined above in the ordered set of congruence families on \( G \); i.e. \( K = K_{\varphi} \).

Every crossed interval group \( G \) admits the minimum proper congruence family; namely the closure \( \text{Triv} \) of the trivial congruence family given in Example 2.3. We write \( \text{Inr}^{G} := \text{Triv} \). Hence, every proper congruence family on \( G \) contains \( \text{Inr}^{G} \). Moreover, it satisfies the following properties.

**Lemma 2.13.** Let \( G \) be a crossed interval group.

1. If a composition \( \varphi \psi \) in \( \mathcal{N} \) is active, then the action of \( \text{Inr}^{G}_{\varphi} \) stabilizes \( \psi \).
2. For every morphism \( \varphi \) in \( \mathcal{N} \), the subgroup \( \text{Inr}^{G}_{\varphi} \subset \text{RSt}_{\varphi}^{G} \) is normal.
3. Let \( K \) be a proper congruence family on \( G \). Then, for an active morphism \( \mu \) and for an inert morphism \( \rho \) with \( \varphi \rho \) making sense, the composition

\[
K_{\mu} \xrightarrow{\rho} K_{\mu \rho} \twoheadrightarrow K_{\mu \rho} / \text{Inr}^{G}_{\mu \rho}
\]

is bijective.

**Proof.** We first show (1). Take the factorization \( \varphi = \mu \rho \) with \( \rho \) inert and \( \mu \) active, and let \( \delta \) be the unique section of \( \rho \). Notice that, in view of Lemma 2.9, \( \delta \) is characterized by the following two properties:

(i) \( \varphi \delta \) is active;
(ii) every morphism $\psi$ with $\varphi\psi$ making sense and active uniquely factors as $\psi = \delta\psi'$ for a morphism $\psi'$.

It follows that $\delta$ is fixed by the action of $RSt^G_\varphi$. Moreover, if $\varphi\psi$ is active, the property above implies there is a morphism $\psi'$ with $\psi = \delta\psi$. Then, for each $x \in \text{Im}^G_\varphi$, we have

$$\psi^x = (\delta\psi')^x = \delta^x\psi^x = \delta\psi' = \psi,$$

so that (1) follows.

Next, suppose $u \in \text{Inr}^G_\varphi$ and $x \in RSt^G_\varphi$. For every morphism $\psi$ with $\varphi\psi$ making sense and active, we have

$$\psi^* (xux^{-1}) = (\psi^x)^* (x) (\psi^x)^* (u) \psi^x (x^{-1}) = (\psi^x)^* (x).$$

Note that $\varphi\psi^x = \varphi\psi$ is active, so $(\psi^x)^* (u)$ is the unit. Moreover, the part (1) implies $\psi^x = \psi^{-1}$. It follows that $\psi^x (xux^{-1})$ vanishes, and we obtain (2).

Finally, we show (3). Let $\delta : \langle m \rangle \to \langle l \rangle \in \nabla$ be the unique section of $\rho$. We assert that the map $\delta^* : K_{\mu\rho} \to K_{\mu}$ induces the inverse of (2.9). Indeed, for each $u \in \text{Im}^G_{\mu\rho}$, the definition of $\text{Im}^G_{\mu\rho}$ and the part (1) imply $\delta^* (u) = e$ and $\delta^u = \delta$. Hence, for every $x \in K_{\mu\rho}$, $\delta^* (xu) = \delta^* (x)$. In other words, $\delta^*$ is $\text{Im}^G_{\mu\rho}$-invariant so that it induces a map

$$\delta^* : K_{\mu\rho} / \text{Im}^G_{\mu\rho} \to K_{\mu}.$$

Since $\delta$ is a section of $\rho$, the map is clearly a left inverse of the map (2.9). To see it is also a right inverse, it is enough to see that, for each $x \in K_{\mu\rho}$, we have $\rho^* (\delta^* (x)) x^{-1} \in \text{Im}^G_{\mu\rho}$. Note that, by virtue of the characterization of $\delta$ above, this holds if and only if the map $\delta^*$ vanishes the element. We have

$$\delta^* (\rho^* (\delta^* (x)) x^{-1}) = (\delta \rho \delta^x)^* (x) \cdot \delta^x (x^{-1}).$$

As mentioned above, $\delta$ is fixed by the action of $RSt^G_{\mu\rho}$, so we have $\delta \rho \delta^x = \delta$ and $\delta^x (x^{-1}) = \delta^* (x^{-1})$. Thus, $\delta^* (\rho^* (\delta^* (x)) x^{-1}) = e$, and we conclude $\delta^*$ is a right inverse of (2.9), which completes the proof.

3 Associated double categories

In the previous section, we see that congruence families are associated with quotal categories by taking the quotients of the total category. Our problem is, on the other hand, higher categorical so we need "higher categorical quotients" in some sense. We realize them in terms of double categories. Recall that a double category is a category internal to the category $\text{Cat}$ of small categories; i.e. a diagram

$$C \quad \xymatrix{ \ar[r]^{s,t} & B }$$

in the category $\text{Cat}$ of small categories together with functors

$$\gamma : C \times_B C \to C \quad \text{and} \quad \iota : B \to C$$

satisfying the appropriate conditions of categories, where the domain of $\gamma$ is the pullback of the cospan $C \xymatrix{ \ar[r] & B \ar[l] }$.
Remark 3.1. In what follows, we will often drop the structure functors $\gamma$ and $\iota$ from the notation and just say, for example, $\mathcal{C} \rightharpoonup \mathcal{B}$ is a double category in the case above.

Let $G$ be a crossed interval group. We construct a double category from a pair $(K, L)$ of proper congruence families satisfying the following conditions:

$(\star 1)$ for each morphism $\varphi : \langle m \rangle \to \langle n \rangle \in \nabla$, the subgroup $K_\varphi \subset G_m$ is contained in the normalizer subgroup $N(L_\varphi)$ of $L_\varphi$; i.e.

$$N(L_\varphi) = \{ x \in G_m \mid xL_\varphi x^{-1} = L_\varphi \} ;$$

$(\star 2)$ if $\psi \varphi$ is a composition of morphisms in $\nabla$, for every $u \in L_\psi$ and for each $x \in K_\varphi$,

$$[\varphi^*(u)x\varphi^*(u)^{-1}] = [x] \in \text{Im}^{G_\varphi}_G \setminus K_{\psi \varphi} .$$

We first define a category $Q_{L//K}$ as follows:

- the objects are the same as $\nabla$;
- for $m, n \in \mathbb{N}$, morphisms $\langle m \rangle \to \langle n \rangle$ in $Q_{L//K}$ are triples $(\varphi, [u], [x])$ with $\varphi \in \nabla(\langle m \rangle, \langle n \rangle)$, $[u] \in \text{Im}^{G_\varphi}_G \setminus K_\varphi$, and $[x] \in L_\varphi \setminus G_m$;
- the composition is given by

$$(\psi, [v], [y]) \circ (\varphi, [u], [x]) = (\psi \varphi^y, [\varphi^*(vy)u\varphi^*(y)^{-1}], [\varphi^*(y)x]) .$$

Note that we have

$$\varphi^*(vy)u\varphi^*(y)^{-1} = (\varphi^y)^*(v) \cdot \varphi^*(y)u\varphi^*(y)^{-1} ,$$

so the conditions on congruence families imply the element belongs to $K_{\psi \varphi \varphi}$. In addition, $[2]$ in Lemma 2.13 and the condition $[\star 2]$ guarantee that the composition does not depend on the choice of representatives. If we are given another morphism $(\chi, [w], [z])$ postcomposable with $(\psi, [v], [y])$, the second component of the composition

$$(\chi, [w], [z]) \circ (\psi, [v], [y]) \circ (\varphi, [u], [x])$$

is represented by the element

$$\varphi^*(\psi^*(wz)v\psi^*(z)^{-1}\psi^*(z)y)u\varphi^*(\psi^*(z)y)^{-1}$$

$$= (\psi^\varphi^y)^*(wz)\varphi^*(vy)u ((\psi^\varphi^y)^*(z)\varphi^*(y))^{-1}$$

$$= (\psi^\varphi^y)^*(wz)\varphi^*(vy)u\varphi^*(y)^{-1}(\psi^\varphi^y)^*(z)^{-1} ,$$

which also represents the second component of the other composition. Thanks to this and the associativity of morphisms in $E_G$, one obtains the associativity of the composition in $G_{L//K}$ so that it is actually a category.

Example 3.2. For every proper congruence family $L$, the pair $(\text{Im}^G_G, L)$ satisfies the conditions $[\star 1]$ and $[\star 2]$. One can verify that there is a canonical isomorphism $Q_{L//\text{Im}^G_G} \cong Q_L$. 

16
Example 3.3. Let $\mathcal{G}$ be a group operad, so we have the congruence family $\text{Dec}^\mathcal{G}$ given in Example 2.6. For each morphism $\varphi : \langle\langle m \rangle\rangle \to \langle\langle n \rangle\rangle \in \nabla$, we set $\text{Kec}_\varphi^\mathcal{G} \subseteq \text{Dec}^\mathcal{G}$ to be the kernel of the composition

$$\text{Dec}^\mathcal{G} \hookrightarrow \mathcal{G}(m) \to \mathcal{G}(n).$$

In view of Example 2.6 the family $\text{Kec}^\mathcal{G} = \{\text{Kec}_\varphi^\mathcal{G}\}$ forms a congruence family on $\mathcal{G}$. Taking the closure in the sense of Lemma 2.12, we obtain proper congruence families $\text{Dec}^\mathcal{G}$ and $\text{Kec}^\mathcal{G}$. One can verify that the pair $(\text{Dec}^\mathcal{G}, \text{Kec}^\mathcal{G})$ satisfies the conditions [•1] and [•2] so that they give rise to a category $\mathcal{G}_{\text{Kec}^\mathcal{G}/\text{Dec}^\mathcal{G}}$.

The category $\mathcal{Q}_{L//K}$ comes equipped with two canonical functors

$$s, t : \mathcal{Q}_{L//K} \rightrightarrows \mathcal{Q}_L,$$  \hspace{1cm} (3.1)

here $\mathcal{Q}_L$ is the $G$-quotal category associated with $L$, such that

- they are the identities on objects;
- for each morphism $(\varphi, [u], [x]) \in \mathcal{Q}_{L//K}(\langle\langle m \rangle\rangle, \langle\langle n \rangle\rangle)$,
  $$s(\varphi, [u], [x]) = (\varphi, [x]), \quad t(\varphi, [u], [x]) = (\varphi, [ux]).$$

Note that the assignment $t$ does not depend on the choice of representatives by virtue of the condition [•1]. Then, the functorialities are easily verified. We assert that the diagram (3.1) canonically admits a structure of a double category: define functors $\gamma : \mathcal{Q}_{L//K} \times \mathcal{Q}_L \to \mathcal{Q}_{L//K}$ and $\iota : \mathcal{Q}_L \to \mathcal{Q}_{L//K}$ by

$$\gamma((\varphi, [u], [x]), (\varphi', [u'], [x])) := (\varphi, [uu'], [x]), \quad \iota(\varphi, [x]) := (\varphi, [e], [x]),$$

where $e$ is the unit in the group $K_\varphi$. These actually define functors thanks to [2] in Lemma 2.13 and the associativity and the unitality are obvious. We call the double category (3.1) the double category associated to the pair $(K, L)$.

Remark 3.4. In the case $L \subseteq K$, the double category $\mathcal{Q}_{L//K} \rightrightarrows \mathcal{Q}_L$ looks like a “homotopy quotient” of the category $\mathcal{Q}_L$ with respect to the congruence family $K$ in the following sense: since the functors (3.1) are the identities on objects, one can see the double category as a 2-category, say $\mathcal{Q}_{L//K}$. For each $m, n \in \mathbb{N}$, the category $\mathcal{Q}_{L//K}(\langle\langle m \rangle\rangle, \langle\langle n \rangle\rangle)$ is a groupoid whose isomorphism classes corresponds in one-to-one to morphisms $\langle\langle m \rangle\rangle \to \langle\langle n \rangle\rangle$ in the $G$-quotal category $\mathcal{Q}_K$ associated with $K$.

We further take a quotient of the double category $\mathcal{Q}_{L//K} \rightrightarrows \mathcal{Q}_L$ using the following general construction.

Proposition 3.5. Let $\mathcal{A}$ be a category, and let $\mathcal{M}$ be a left cancellative class of morphisms in $\mathcal{A}$; i.e. if a composition $\delta \varepsilon$ belongs to $\mathcal{M}$, so does $\delta$. For each $a, b \in \mathcal{A}$, define a relation $\sim_M$ on the set $\mathcal{A}(a, b)$ such that $\alpha \sim_M \alpha'$ if and only if, for each morphism $\beta$ in $\mathcal{A}$ with codomain $a$, one has $\alpha \beta = \alpha' \beta$ as soon as either of the sides belongs to $\mathcal{M}$. Then, the relation $\sim_M$ is a congruence on $\mathcal{A}$ in the sense in II.8 of [13]. Consequently, taking the quotient of each hom-set of $\mathcal{A}$ by $\sim_M$, one gets a quotient category $\mathcal{A} \to \mathcal{A}/\sim_M$.
Proof. It is obvious that $\sim_M$ is an equivalence relation on each hom-set $A(a, b)$. We have to show, for compositions $\alpha\beta\gamma$ and $\alpha\beta'\gamma$ with $\beta \sim_M \beta'$, we have $\alpha\beta\gamma \sim_M \alpha\beta'\gamma$. Suppose a composition $\alpha\beta\gamma\delta$ belongs to $M$. By virtue of the observation above, we have $\beta\gamma\delta \in M$, so $\beta \sim_M \beta'$ implies $\beta\gamma\delta = \beta'\gamma\delta$. Thus, we obtain $\alpha\beta\gamma\delta = \alpha\beta'\gamma\delta$ and conclude $\alpha\beta\gamma \sim_M \alpha\beta'\gamma$.

Remark 3.6. Typical examples of left cancellative class of morphisms come from orthogonal factorization systems (see Remark 2.10). Suppose $(E, M)$ is an orthogonal factorization system on a category $A$. One can see that the class $M$ is left cancellative provided every morphisms in $E$ is an epimorphism. Indeed, if $\varepsilon = \mu\rho$ and $\delta\mu = \nu\sigma$ are factorizations with $\mu, \nu \in M$ and $\rho, \sigma \in E$, then the equation $\delta\varepsilon \circ \text{id} = \nu \circ \sigma\rho$ and the unique lifting property implies $\sigma\rho$ is an isomorphism. Since $\sigma$ is an epimorphism by the assumption on $E$, $\rho$ is an isomorphism so that $\varepsilon \in M$.

We apply Proposition 3.5 to the category $Q_{L//K}$. To obtain a left cancellative classes on it, in view of Remark 3.6, we construct orthogonal factorization system. We define two classes $I_{L//K}$ and $A_{L//}$ of morphisms in $Q_{L//K}$ as follows:

$$I_{L//K} := \{ (\varphi, [u], [x]) \mid \varphi \text{ inert} \},$$

$$A_{L//K} := \{ (\varphi, [u], [x]) \mid \varphi \text{ active} \}.$$

We assert that $(I_{L//K}, A_{L//K})$ forms an orthogonal factorization system on $Q_{L//K}$. In fact, if we have a composition $\mu\rho : \langle\langle m \rangle\rangle \to \langle\langle n \rangle\rangle \in N$ with $\mu$ active and $\rho$ inert, then for each $u \in K_{\mu\rho}$ and $x \in G_m$, one can use (3) in Lemma 2.13 to find a unique element $\pi \in K_{\mu}$ so that

$$(\mu\rho, [u], [x]) = (\mu, [\pi], [e]) \circ (\rho, [e_m], [x]) .$$

As easily verified, the factorization is unique up to a unique isomorphism, and we conclude $(I_{L//K}, A_{L//K})$ is an orthogonal factorization. Since every member of $I_{L//K}$ is a split epimorphism, Lemma 2.9 implies $A_{L//K}$ is left cancellative.

We denote by $Q_{L//K}$ the quotient category of $Q_{L//K}$ obtained by Proposition 3.5 with the class $A_{L//K}$. In particular, as mentioned in Example 3.2, there is an isomorphism $Q_L \cong Q_{L//\text{Inr} G}$. We set $(I_L, A_L)$ the orthogonal factorization system corresponds to $(I_{L//\text{Inr} G}, A_{L//\text{Inr} G})$, so we obtain a quotient category $Q_L \to Q_L$, which corresponds to $Q_{L//\text{Inr} G}$ through the isomorphism.

Remark 3.7. We have a convenient criterion for the congruence $\sim_{A_L/k}$. Suppose $\varphi, \varphi' : \langle\langle m \rangle\rangle \to \langle\langle n \rangle\rangle \in N$, $x, x' \in G_m$, $u \in K_\varphi$, and $u' \in K_{\varphi'}$. Then, we have $\langle\langle x, u \rangle\rangle \sim_{A_L/k} \langle\langle x', u' \rangle\rangle$ if and only if for every $\psi : \langle\langle l \rangle\rangle \to \langle\langle m \rangle\rangle$ with either $\varphi\psi^x$ or $\varphi'\psi'^x$ active, the following equations hold:

$$\varphi\psi^x = \varphi'\psi'^x \in \nabla(\langle\langle l \rangle\rangle, \langle\langle n \rangle\rangle)$$

$$[\psi^*(x)] = [\psi'^*(x')] \in L_{\varphi\psi^x} \setminus G_l$$

$$[(\psi^x)^*(u)] = [(\psi'^x)^*(u')] \in \text{Inr}_{\varphi\psi^x} \setminus K_{\varphi\psi^x} .$$

Furthermore, let $\varphi = \mu\rho$ and $\varphi' = \mu'\rho'$ be the factorization with $\mu, \mu'$ active and $\rho, \rho'$ inert, and say $\delta$ and $\delta'$ are the unique sections of $\rho$ and $\rho'$ respectively. Then, it turns out that we only have to test the conditions above in the cases $\psi = \delta^{-1}_x$ and $\psi = \delta'^{-1}u$. 

18
Example 3.8. Let $G$ be a crossed interval group, and let $(K, L)$ be a pair of proper congruence families satisfying $[\spadesuit 1]$ and $[\spadesuit 2]$. Then, for an active morphism $\mu : \langle m \rangle \to \langle n \rangle$, we have $(\mu, [u], [x]) \sim_{A_{\mu \mid K}} (\mu', [u'], [x'])$ for two morphisms in $\mathcal{Q}_{L/K}$ if and only if they are in fact equal. Indeed, the composition $\mu \circ \text{id}_{\langle m \rangle}$ equals to $\mu$ itself and is active, so we have

\[
\begin{align*}
\mu &= \mu \circ \text{id}_{\langle m \rangle} = \mu' \circ \text{id}_{\langle m \rangle} = \mu' \\
[x] &= [\text{id}_{\langle m \rangle}(x)] = [\text{id}_{\langle m \rangle}(x')] = [x'] \\
[u] &= [(\text{id}_{\langle m \rangle})^*(u)] = [(\text{id}_{\langle m \rangle})^*(u')] = [u'] .
\end{align*}
\]

Example 3.9. Let $G$ be a crossed interval group and $L$ a proper congruence family. For $1 \leq i \leq n$, define $\rho_i : \langle n \rangle \to \langle 1 \rangle$ to be the inert morphism such that

\[
\rho_i(j) = \begin{cases} 
-\infty & j < i \\ 1 & j = i \\ \infty & j > i,
\end{cases}
\]

and put $\delta_i$ the unique section of $\rho_i$. Then, for each $x \in G_n$, we have

\[
(\rho_{x(i)}, [x]) \sim_{A_L} (\rho_i, [\rho_i^* \delta_i^*(x)]) \in \mathcal{Q}_L(\langle n \rangle, \langle 1 \rangle). \tag{3.2}
\]

Indeed, since $\rho_i^* \delta_i^*(x) \in G_n$ acts trivially on active morphisms, we have

\[
\rho_{x(i)} \delta_{x(i)} = \text{id}_{\langle 1 \rangle} = \rho_i \delta_i = \rho_i^* \delta_{x(i)} = \rho_i \rho_i^* \delta_i^*(x) \delta_i^*(x)^{-1} = (\delta_{x(i)}^{-1})^*(x) = \delta_i^* ((\rho_i \delta_i)^*(x)) .
\]

Hence, (3.2) follows from the argument in Remark 3.7.

Lemma 3.10. Let $G$ be a crossed interval group, and let $(K, L)$ be a pair of proper congruence families on $G$ satisfying $[\spadesuit 1]$ and $[\spadesuit 2]$. Then, the quotient functors $\mathcal{Q}_{L/K} \to \mathcal{Q}_{L/K}'$ and $\mathcal{Q}_L \to \mathcal{Q}_L'$ derive functors $s, t : \mathcal{Q}_{L/K} \to \mathcal{Q}_{L/K}'$ so that the diagram below is commutative:

\[
\begin{array}{ccc}
\mathcal{Q}_{L/K} & \xrightarrow{s} & \mathcal{Q}_L \\
\downarrow & & \downarrow \\
\mathcal{Q}_{L/K}' & \xrightarrow{t} & \mathcal{Q}_L'.
\end{array}
\tag{3.3}
\]

Moreover, each square in (3.3) is a pullback of categories.

Proof. The first statement is straightforward. To see the last, we show that when we fix a morphism $\varphi : \langle m \rangle \to \langle n \rangle \in \nabla$ and an element $x \in G_m$, for two elements $u, u' \in K_\varphi$, the following three are all equivalent:

(a) $(\varphi, [u], [x]) \sim_{A_{\varphi \mid K}} (\varphi, [u'], [x]) \in \mathcal{Q}_{L/K}(\langle m \rangle, \langle n \rangle)$;

(b) $(\varphi, [u], [u^{-1}x]) \sim_{A_{\varphi \mid K}} (\varphi, [u'], [u'^{-1}x]) \in \mathcal{Q}_{L/K}(\langle m \rangle, \langle n \rangle)$;

(c) $u^{-1}u' \in \text{In}^G_\varphi$. 

19
Note the equivalence of (a) and (c) implies the left square in (3.3) is a pullback while the equivalence of (b) and (c) implies the other.

Let $\varphi = \mu \rho$ be the factorization with $\mu$ active and $\rho$ inert. In view of Remark 3.7, the condition (a) is satisfied if and only if $\delta^*(u) = \delta^*(u')$ since $\text{Inr}^G_{\varphi,\rho} = \text{Inr}^G_{\rho}$ is trivial. The latter is equivalent to (c) in view of (3) in Lemma 2.13. The same argument also completely goes well for the condition (b) and this completes the proof.

**Proposition 3.11.** Let $G$ be a crossed interval group, and let $K$ be a proper congruence family on $G$. Then, the diagram

$$s, t : \tilde{Q}_L//K \rightrightarrows \tilde{Q}_L$$

admits the structure of a double category inherited from (3.1).

*Proof.* It suffices to see that the vertical composition functor

$$\gamma : Q_L//K \times Q_L \rightarrow Q_L//K$$

induces a functor

$$\tilde{\gamma} : \tilde{Q}_L//K \times \tilde{Q}_L \rightarrow \tilde{Q}_L//K .$$

Note that, in view of the pullback squares in (3.3), we have a canonical isomorphism

$$Q_L//K \times Q_L \cong Q_L \times \tilde{Q}_L \left( \tilde{Q}_L//K \times \tilde{Q}_L \right)$$

of categories, where the right hand side is the limit of the diagram

$$\begin{array}{ccc}
Q_L & \xrightarrow{s} & \tilde{Q}_L // K \\
\downarrow & & \downarrow t \end{array}$$

Thus, we have to show the composition

$$\tilde{\gamma} : Q_L \times \tilde{Q}_L \left( \tilde{Q}_L//K \times \tilde{Q}_L \right) \cong Q_L \times \tilde{Q}_L \rightarrow Q_L//K$$

depends, with respect to the first parameter, only on the images under the functor $Q_L \rightarrow \tilde{Q}_L$. Since $\tilde{\gamma}$ is clearly the identity on objects, we concentrate on morphisms. Note that, for a morphism $(\varphi, [x])$ of $Q_L$ and for morphisms $\varphi$ and $\psi$ of $\tilde{Q}_L//K$ with

$$t(\psi) = s(\varphi) = [\varphi, x] \in \tilde{Q}_L ,$$

the image $\tilde{\gamma}((\varphi, [x]), \psi, \varphi)$ is given as follows: by virtue of Lemma 3.10 63.5 implies there are elements $u, v \in K_\varphi$ so that

$$\varphi = [\varphi, u, u^{-1} x] , \quad \psi = [\varphi, v, x] .$$

Then we have

$$\tilde{\gamma}((\varphi, [x]), \psi, \varphi) = [\varphi, vu, u^{-1} x] .$$

Now, suppose $(\varphi, [x]) \sim_{\mathcal{L}} (\varphi', [x'])$, and take $u', v' \in K_\varphi$ so that

$$\varphi = [\varphi', u', u'^{-1} x'] , \quad \psi = [\varphi', v', x'] .$$
We show the congruence
\[(\varphi, [vu], [u^{-1}x]) \sim_{A_{L/K}} (\varphi', [v'u'], [u'^{-1}x']) \] (3.6)

Let \(\psi\) be a morphism precomposable with \(\varphi\) such that either \(\varphi\psi[\nu]\) or \(\varphi'\psi[\nu']\) is active. Since \(u\) and \(u'\) are right stabilizers of \(\varphi\), this implies either \(\varphi\psi x\) or \(\varphi'\psi x'\) is also active. Then, in view of Remark 3.7, the congruences
\[(\varphi, [u], [u^{-1}x]) \sim_{A_{L/K}} (\varphi', [u'], [u'^{-1}x']) , \quad (\varphi, [v], [x]) \sim_{A_{L/K}} (\varphi', [v'], [x']) \]
imply
\[\varphi\psi u^{-1}x = \varphi'\psi'u'^{-1}x' , \quad [\psi^*u^{-1}x] = [\psi'^*u'^{-1}x'] \]
and
\[[(\psi u^{-1}x)^*(\nu u)] = [(\psi' u'^{-1}x')^*(\nu')] = [(\psi\psi'u'^{-1}x')^*(\nu')] = [(\psi'u'^{-1}x')^*(\nu')].\]

Thus, (3.6) follows, and we obtain \(\tilde{\gamma}((\varphi, [x]), \psi, \varphi) = \tilde{\gamma}((\varphi', [x']), \psi, \varphi)\) as required.

4 Internal presheaves over the associated double categories

We constructed double categories \(Q_{L/K} \Rightarrow Q_L\) and \(\tilde{Q}_{L/K} \Rightarrow \tilde{Q}_L\) for a sort of pairs \((K, L)\) of proper congruence families on crossed interval groups \(G\). Recall that, as they are internal categories in the category \(\mathbf{Cat}\) of small categories, we can consider the following notion on them.

**Definition** (cf. [9], Definition 2.14). Let \(\mathcal{C} \Rightarrow \mathcal{B}\) be a double category. Then an internal presheaf over it consists of the data

- a category \((\mathcal{X} \rightarrow \mathcal{B}) \in \mathbf{Cat}^{/\mathcal{B}}\) over \(\mathcal{B}\);
- a functor \(\mathcal{A}_\mathcal{X} : \mathcal{X} \times_B \mathcal{C} \rightarrow \mathcal{X}\)

in \(\mathbf{Cat}^{/\mathcal{B}}\), where the domain is the pullback of the cospan \(\mathcal{X} \rightarrow \mathcal{B} \leftarrow \mathcal{C}\) and seen as a category over \(\mathcal{B}\) with the composition

\[\mathcal{X} \times_B \mathcal{C} \xrightarrow{\text{proj}} \mathcal{C} \xrightarrow{s} \mathcal{B};\]

such that the diagrams below are commutative:

\[
\begin{array}{ccc}
\mathcal{X} \times_B \mathcal{C} \times_B \mathcal{C} & \xrightarrow{\mathcal{A}_\mathcal{X} \times \text{Id}_\mathcal{B}} & \mathcal{X} \times_B \mathcal{C} \\
\text{Id}_\mathcal{X} \times \gamma_\mathcal{C} & \downarrow & \downarrow \mathcal{A_\mathcal{X}} \\
\mathcal{X} \times_B \mathcal{C} & \xrightarrow{\mathcal{A}_\mathcal{X}} & \mathcal{X} \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{X} \times_B \mathcal{B} & \xrightarrow{\mathcal{A}_\mathcal{X} \times \text{Id}_\mathcal{B}} & \mathcal{X} \times_B \mathcal{C} \\
\text{Id}_\mathcal{X} \times \gamma_\mathcal{B} & \downarrow & \downarrow \mathcal{A_\mathcal{X}} \\
\mathcal{X} \times_B \mathcal{B} & \xrightarrow{\mathcal{A}_\mathcal{X}} & \mathcal{X} \\
\end{array}
\]
A double category $\mathcal{C} \Rightarrow \mathcal{B}$ gives rise to a 2-monad

$$\mathbf{Cat}^/\mathcal{B} \to \mathbf{Cat}^/\mathcal{B} : \mathcal{X} \mapsto \mathcal{X} \times_{\mathcal{B}} \mathcal{C}.$$ 

Actually, internal presheaves over $\mathcal{C} \Rightarrow \mathcal{B}$ are precisely (strict) 2-algebras on it. In particular, they form a 2-category, which we denote by $\mathbf{PSh}(\mathcal{C} \Rightarrow \mathcal{B})$. The 2-morphisms in $\mathbf{PSh}(\mathcal{C} \Rightarrow \mathcal{B})$ are, by definition, natural transformations $\alpha : H \to K : \mathcal{X} \to \mathcal{Y}$ over $\mathcal{B}$ such that the following two horizontal compositions coincide:

$$\begin{array}{c}
\mathcal{X} \times_{\mathcal{B}} \mathcal{C} \xrightarrow{\alpha \times \text{Id}} \mathcal{Y} \times_{\mathcal{B}} \mathcal{C} \\
\mathcal{X} \times_{\mathcal{B}} \mathcal{C} \xrightarrow{\text{Id} \times H} \mathcal{X} \times_{\mathcal{B}} \mathcal{C}
\end{array}$$

One can easily prove the following result.

**Lemma 4.1.** Let $\mathcal{C} \Rightarrow \mathcal{B}$ be a double category such that the functors $s, t : \mathcal{C} \Rightarrow \mathcal{B}$ are the identity on objects. Then, the forgetful functor

$$\mathbf{PSh}(\mathcal{C} \Rightarrow \mathcal{B}) \to \mathbf{Cat}^/\mathcal{B}$$

is locally fully faithful; i.e. for internal presheaves $\mathcal{X}$ and $\mathcal{Y}$, the functor

$$\mathbf{PSh}(\mathcal{C} \Rightarrow \mathcal{B})(\mathcal{X}, \mathcal{Y}) \to \mathbf{Cat}^/\mathcal{B}(\mathcal{X}, \mathcal{Y})$$

is fully faithful.

Suppose we are given a group operad $\mathcal{G}$ and a $\mathcal{G}$-symmetric multicategory $\mathcal{M}$. In view of categories of operators of $\mathcal{M}$ with regard to $\mathcal{G}$, the pair $(\text{Dec}^{\mathcal{G}}, \text{Rec}^{\mathcal{G}})$ plays the fundamental role. We will write

$$\mathcal{G}^{\mathcal{G}} := \mathcal{Q}_{\text{Rec}^{\mathcal{G}}} /_{\text{Dec}^{\mathcal{G}}} , \quad \mathcal{E}^{\mathcal{G}} := \mathcal{Q}_{\text{Rec}^{\mathcal{G}}} , \quad \tilde{\mathcal{G}}^{\mathcal{G}} := \mathcal{Q}_{\text{Rec}^{\mathcal{G}}} /_{\text{Dec}^{\mathcal{G}}} , \quad \tilde{\mathcal{E}}^{\mathcal{G}} := \tilde{\mathcal{Q}}_{\text{Rec}^{\mathcal{G}}} .$$

In this section, we see $\mathcal{M}$ gives rise to internal presheaves over the double categories $\mathcal{G}^{\mathcal{G}} \Rightarrow \mathcal{E}^{\mathcal{G}}$ and $\tilde{\mathcal{G}}^{\mathcal{G}} \Rightarrow \tilde{\mathcal{E}}^{\mathcal{G}}$ given in (3.1) and Proposition 3.11. We need some kind of word calculus, and the following notations are convenient.

**Notation.** Let $S$ be a set and $\bar{a} = a_1 \ldots a_n$ a word in $S$; i.e. $a_i \in S$.

1. If $G$ is a crossed interval group, then for $x \in G_n$, we write

$$x \cdot \bar{a} := a_{x^{-1}(1)} \ldots a_{x^{-1}(n)} .$$

Note that it coincides with the canonical left $G_n$-action on $S^{\times n}$ induced by the map $G_n \to \mathfrak{S}_n$. 

2. Suppose $\varphi : \langle m \rangle \to \langle n \rangle \in \nabla$ is an arbitrary morphism, and say $\varphi^{-1}(j) = \{ i_1 < \cdots < i_{k(\varphi)} \}$ for each $1 \leq j \leq n$. Then, we write

$$\bar{a}^{\varphi}_j = a_{i_1} \ldots a_{i_{k(\varphi)}} .$$

Hence, the concatenated word $\bar{a}_1 \ldots \bar{a}_n$ is a subword of the original $\bar{a}$.
Lemma 4.2. Let $S$ be a set and $\bar{a} = a_1 \ldots a_m$ a word in $S$.

1. Suppose we are given morphisms $\varphi : \langle m \rangle \to \langle n \rangle$ and $\psi : \langle n \rangle \to \langle p \rangle$ in $\nabla$, and say

$$\psi^{-1}\{s\} := \{j_1^s < \cdots < j_r^s\}.$$ Then, we have

$$\bar{a}^\varphi = \bar{a}_{j_1}^\varphi \cdots \bar{a}_{j_r}^\varphi.$$ (1)

2. Let $G$ be a crossed interval group, and write the canonical map $G_n \to \mathcal{W}_n \cong (\mathcal{S}_n \times \mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z}$ in the form

$$y \mapsto (\sigma^y, \varepsilon^y_1, \ldots, \varepsilon^y_n, 0^y).$$ Then, for every morphism $\varphi : \langle m \rangle \to \langle n \rangle \in \nabla$ and every $y \in G_n$,

$$(\varphi^*(y), \bar{a})_j^\varphi = \beta_{y^{-1}(j)} \bar{a}_{y^{-1}(j)}^\varphi.$$ where $\beta$ is the order-reversing permutation.

(2)

3. Let $G$ be a crossed interval group. Suppose we have two morphisms $\varphi, \varphi' : \langle m \rangle \to \langle n \rangle \in \nabla$ and two elements $x, x' \in G_m$. If two morphisms $[\varphi, x], [\varphi', x']$ in $E_G$ coincide with each other, then, for each $1 \leq j \leq n$,

$$(x, \bar{a})_j^\varphi = (x', \bar{a})_j^{\varphi'}.$$ (3)

Proof. The parts [1] is obvious. On the other hand, the part [2] follows from the following characterization of the permutation on $\langle m \rangle$ associated with $\varphi^*(y)$:

(i) the square below is commutative

$$\begin{array}{ccc}
\langle m \rangle & \xrightarrow{\varphi} & \langle n \rangle \\
\psi^*(y) \downarrow & & \downarrow y \\
\langle m \rangle & \xrightarrow{\varphi^y} & \langle n \rangle
\end{array}$$

(ii) for each $j \in \langle n \rangle$, the bijection

$$\varphi^{-1}\{j\} \to (\varphi^y)^{-1}\{y(j)\}$$ restricting the permutation $\varphi^*(y)$ either preserves or reverses the order depending on $\varepsilon^y_j$.

We show [3] Under the identification $\langle k \rangle \cong \nabla(\langle 1 \rangle, \langle k \rangle)$, the data induces maps

$$\langle m \rangle \xrightarrow{\varphi} \langle m \rangle \xrightarrow{\varphi} \langle n \rangle,$$

$$\langle m \rangle \xrightarrow{\varphi'} \langle m \rangle \xrightarrow{\varphi'} \langle n \rangle.$$ (4.1)

It is observed that if $[\varphi, x] = [\varphi', x']$, the two maps (4.1) have the same inverse image of $\langle n \rangle = \{1, \ldots, n\} \subset \langle n \rangle$ where they agree with each other. Hence, the required equation $(x, \bar{a})_j^\varphi = (x', \bar{a})_j^{\varphi'}$ follows for each $1 \leq j \leq n$. \qed
We begin the main construction. For a multicategory \( M \), we define a category \( M \downarrow \text{E}_G \) as follows:

- objects are finite sequences \( \vec{a} = a_1 \ldots a_n \) of objects of \( M \);
- for \( \vec{a} = a_1 \ldots a_m \) and \( \vec{b} = b_1 \ldots b_n \), the hom-set \( (M \downarrow \text{E}_G)(\vec{a}, \vec{b}) \) consists of tuples \((\chi; f_1, \ldots, f_n; [x])\) of
  - \( \chi : \langle m \rangle \to \langle n \rangle \in \nabla \),
  - \([x] \in \text{Kec}_G \setminus G(m)\) represented by \( x \in G(m) \), and
  - \( f_j \in M((x, \vec{a})^\omega_j; b_j) \) for each \( 1 \leq j \leq n \) (see \([3]\) in Lemma \(4.2\) and Lemma \(3.10\));
- for morphisms \((\psi; \vec{f}; [x]) : a_1 \ldots a_l \to b_1 \ldots b_m \) and \((\psi; \vec{g}; [y]) : b_1 \ldots b_m \to c_1 \ldots c_n \), the composition is given by
  \[
  (\psi; \vec{f}; [x]) \circ (\psi; \vec{g}; [y]) = \left( \psi \chi; \gamma(g_1; (y_\ast \vec{f}^\chi_1)^\psi), \ldots, \gamma(g_n; (y_\ast \vec{f}^\chi_n)^\psi); [\psi^* (y) x] \right).
  \]

The composition is in fact associative; indeed, suppose we have another morphism \((\chi; \vec{h}; [z]) : \vec{c} \to d_1 \ldots d_p\), and consider the equation
\[
(\chi; \vec{h}; [z]) \circ (\psi; \vec{f}; [x]) = (\chi; \vec{h}; [z]) \circ (\psi; \vec{g}; [y]) \circ (\chi; \vec{h}; [z]). \tag{4.2}
\]

In terms of the first and the third components of the tuples, the equation clearly holds. If we put \( \chi^{-1} \{ s \} = \{ j_1^s < \cdots < j_p^s \} \), then, in view of Lemma \(4.2\) each term of the second component in the left hand side of \((4.2)\) is given by
\[
\gamma \left( \chi; (z, \vec{g})^\chi; (\psi^* (z), y_\ast \vec{f}^\chi) \right) = \gamma \left( \chi; (g_{z^{-1}(j_1^s)}, \ldots, g_{z^{-1}(j_p^s)}); (y_\ast \vec{f}^\chi)_{z^{-1}(j_1^s)} \cdots (y_\ast \vec{f}^\chi)_{z^{-1}(j_p^s)} \right) = \gamma \left( \chi; (g_{z^{-1}(j_1^s)}, \ldots, g_{z^{-1}(j_p^s)}); (y_\ast \vec{f}^\chi)_{z^{-1}(j_1^s)} \cdots (y_\ast \vec{f}^\chi)_{z^{-1}(j_p^s)} \right).
\]

Clearly, the last term is precisely the one appearing as a component in the left hand side of \((4.2)\), so that the composition is associative. Note that the identity on the object \( a_1 \ldots a_n \in M \downarrow \text{E}_G \) is the tuple
\[
(id_{\langle n \rangle}; id_{a_1}, \ldots, id_{a_n}; [e_n])
\]

**Example 4.3.** In the case \( M = \ast \) is the terminal operad, the resulting category \( \ast \downarrow \text{E}_G \) is nothing but the category \( \text{E}_G \) itself.

We extend the constructions \( M \to M \downarrow \text{E}_G \) to 2-functors. If \( F : M \to \mathcal{N} \) be a \( G \)-symmetric multifunctor, then we define a functor \( F^\circ : M \downarrow \text{E}_G \to \mathcal{N} \downarrow \text{E}_G \) so that

- for each objects \( a_1 \ldots a_m \in M \downarrow \text{E}_G \), we put
  \[
  F^\circ(a_1 \ldots a_m) := F(a_1) \ldots F(a_m);
  \]
- for \( \vec{a} = a_1 \ldots a_m, \vec{b} = b_1 \ldots b_n \in M \downarrow \text{E}_G \), define
  \[
  F^\circ : (M \downarrow \text{E}_G)(\vec{a}, \vec{b}) \to (\mathcal{N} \downarrow \text{E}_G)(F^\circ(\vec{a}), F^\circ(\vec{b}))
  \]
  \[
  (\varphi; f_1, \ldots, f_n; [x]) \mapsto (\varphi; F(f_1), \ldots, F(f_n); [x]).
  \]
The functoriality is easily verified. In addition, if \( \alpha : F \to G : \mathcal{M} \to \mathcal{N} \) is a multinatural transformation of multinatural functors, then one can check that the morphisms

\[
\alpha_{a_1, \ldots, a_m}^\varphi = ([\text{id}]_{\langle\langle m\rangle\rangle}; \alpha_{a_1}, \ldots, \alpha_{a_m}; e_m) : F^\varphi(a_1 \ldots a_m) \to G^\varphi(a_1 \ldots a_m)
\]

for \( a_1 \ldots a_m \in \mathcal{M}(\mathcal{E}_G) \) form a natural transformation \( \alpha^\varphi : F^\varphi \to G^\varphi \). Combining with Example 4.3, we obtain a 2-functor

\[
(-) \downarrow \mathcal{E}_G : \text{MultCat} \to \text{Cat}^{/\mathcal{E}_G}.
\]  

(4.3)

We furthermore consider a quotient of the category \( \mathcal{M} \downarrow \mathcal{E}_G \). For two morphisms

\[
(\varphi; f_1, \ldots, f_n; [x]), (\varphi'; f_1', \ldots, f_n'; [x']) : a_1 \ldots a_m \to b_1 \ldots b_n \in \mathcal{M}(\mathcal{E}_G),
\]

we write \( (\varphi; f_1, \ldots, f_n; [x]) \sim_{\mathcal{E}_G} (\varphi'; f_1', \ldots, f_n'; [x']) \) precisely when we have \( [\varphi, x] = [\varphi', x'] \) in \( \mathcal{E}_G(\langle\langle m\rangle\rangle, \langle\langle n\rangle\rangle) \) and \( f_j = f_j' \) for each \( 1 \leq j \leq n \). Note that, thanks to (3) in Lemma 4.2, the first equation implies

\[
\mathcal{M}(\langle\langle x_* \bar{a} \rangle\rangle^\varphi_j; b_j) = \mathcal{M}(\langle\langle x_* \bar{a} \rangle\rangle_j^\varphi; b_j)
\]

so that the latter comparison makes sense. It is straightforward that the relation \( \sim_{\mathcal{E}_G} \) is a congruence on the category \( \mathcal{M}(\mathcal{E}_G) \). We denote by \( \mathcal{M}(\mathcal{E}_G) \) the resulting quotient category. For each morphism \( (\varphi; f_1, \ldots, f_n; x) \) of \( \mathcal{M}(\mathcal{E}_G) \), we write \( [\varphi; f_1, \ldots, f_n; x] \) its image in \( \mathcal{M}(\mathcal{E}_G) \). It is easily verified that the assignment \( \mathcal{M} \mapsto \mathcal{M}(\mathcal{E}_G) \) also extends to a 2-functor so that the functor \( \mathcal{M}(\mathcal{E}_G) \to \mathcal{M}(\mathcal{E}_G) \) forms a (strict) 2-natural transformation. More explicitly, a multifunctor \( F : \mathcal{M} \to \mathcal{N} \) induces a functor \( \tilde{F}^\varphi : \mathcal{M}(\mathcal{E}_G) \to \mathcal{N}(\mathcal{E}_G) \) such that

- for each object \( \bar{a} = a_1 \ldots a_m \in \mathcal{M}(\mathcal{E}_G) \), \( \tilde{F}^\varphi(\bar{a}) = F(a_1) \ldots F(a_m) \);
- for morphisms, we have

\[
\tilde{F}^\varphi([\varphi; f_1, \ldots, f_n; x]) = [\varphi; F(f_1), \ldots, F(f_n); x].
\]

On the other hand, if \( \alpha : F \to G : \mathcal{M} \to \mathcal{N} \) is a multinatural transformation, we have a natural transformation \( \tilde{\alpha}^\varphi : \tilde{F}^\varphi \to \tilde{G}^\varphi \) with

\[
\tilde{\alpha}_{a_1, \ldots, a_m}^\varphi = ([\text{id}]_{\langle\langle m\rangle\rangle}; \alpha_{a_1}, \ldots, \alpha_{a_m}; e_m)
\]

for each \( a_1 \ldots a_m \in \mathcal{M}(\mathcal{E}_G) \). Observing the canonical identification \( \ast \downarrow \mathcal{E}_G \cong \mathcal{E}_G \), we obtain a 2-functor

\[
(-) \downarrow \mathcal{E}_G : \text{MultCat} \to \text{Cat}^{/\mathcal{E}_G}.
\]  

(4.4)

**Lemma 4.4.** Let \( \mathcal{M} \) be a multicategory. Then, for every group operad \( \mathcal{G} \), the square below is a pullback:

\[
\begin{array}{ccc}
\mathcal{M}(\mathcal{E}_G) & \to & \mathcal{M}(\mathcal{E}_G) \\
\downarrow & & \downarrow \\
\mathcal{E}_G & \to & \mathcal{E}_G
\end{array}
\]
In this case, we set $\varphi_\delta$ and hence the composition

Suppose for a morphism $\varphi_M: M \to E$. Notation. For the construction of an internal presheaf structure, the key is a comparison of $\mathcal{M} \wr \mathcal{G}$ and $\mathcal{M} \wr \tilde{\mathcal{G}}$ by the pullback squares

\[
\begin{array}{ccc}
\mathcal{M} \wr \mathcal{G} & \to & \mathcal{M} \wr \tilde{\mathcal{G}} \\
\downarrow & & \downarrow \\
\mathcal{G} \mathcal{G} & \to & \mathcal{G} \tilde{\mathcal{G}}
\end{array}
\]

Hence, the required internal presheaf structures are functors $\gamma: \mathcal{M} \wr \mathcal{G} \to \mathcal{M} \wr \mathcal{E}$, $\gamma: \mathcal{M} \wr \tilde{\mathcal{G}} \to \mathcal{M} \wr \tilde{\mathcal{E}}$, respectively which satisfy appropriate conditions. Since the latter may be induced from the first, we mainly discuss $\mathcal{M} \wr \mathcal{E}$. Note that the category $\mathcal{M} \wr \mathcal{G}$ is described explicitly as follows: for each objects $\bar{a} = a_1 \ldots a_m, \bar{b} = b_1 \ldots b_n \in \mathcal{M} \wr \mathcal{G}$, the hom-set $(\mathcal{M} \wr \mathcal{G})(\bar{a}, \bar{b})$ consists of tuples $(\varphi, f_1, \ldots, f_n; [u], [x])$ such that

- $[u] \in \text{In}^{\mathcal{G}}_\varphi \setminus \text{Dec}^\mathcal{G}_\varphi$ and $[x] \in \text{Kec}^\mathcal{G}_\varphi \setminus G_m$;
- $(\varphi, f_1, \ldots, f_n; [ux]) : \bar{a} \to \bar{b}$ makes sense as a morphism in $\mathcal{M} \wr \mathcal{E}$;

The composition is given by

\[
(\psi; g_1, \ldots, g_n; [v], [y]) \circ (\varphi, f_1, \ldots, f_n; [u], [x]) = (\psi \varphi^\psi; \gamma(g_1; ((vy)_\bar{f})^\psi), \ldots, \gamma(g_n; (vy)_\bar{f})^\psi; [\varphi^\psi(vy)w\varphi^{-1}(y), [\varphi^\psi(x)]),
\]

and the structure functor $\mathcal{M} \wr \mathcal{G} \to \mathcal{E}$ is an identity-on-object functor with

\[
\begin{array}{ccc}
(\mathcal{M} \wr \mathcal{G})(a_1 \ldots a_m, b_1 \ldots b_n) & \to & \mathcal{E}(\langle m \rangle, \langle n \rangle) \\
(\varphi, f_1, \ldots, f_n; [u], [x]) & \mapsto & (\varphi, [x])
\end{array}
\]

For the construction of an internal presheaf structure, the key is a comparison of the category $\mathcal{M} \wr \mathcal{G}$ with $(\mathcal{M} \times \mathcal{G}) \wr \mathcal{G}$ (see the construction in Section 4).

Notation. For a morphism $\varphi : \langle m \rangle \to \langle n \rangle \in \nabla$, and for each $1 \leq j \leq n$, suppose $\varphi^{-1}(j) = \{i_1 < \cdots < i_{k^j(\varphi)}\}$.

In this case, we set

\[
\delta^\varphi_j : \langle k^j(\varphi) \rangle \to \langle m \rangle ; \quad s \mapsto \begin{cases} -\infty & s = -\infty, \\ i_s & 1 \leq s \leq k^j(\varphi), \\ \infty & s = \infty. \end{cases}
\]

Hence, the composition $\varphi \delta^\varphi_j$ factors through the map $\langle 1 \rangle \to \langle n \rangle$ corresponding to the element $j \in \langle n \rangle$. 

Proof. The result is straightforward from the definition of the category $\tilde{\mathcal{E}}$. □
Remark 4.5. The morphism $\delta_j^{(\varphi)}$ defined above is characterized by the following two properties:

(i) the composition $\varphi \delta_j^{(\varphi)}$ factors through the map $\langle\langle 1 \rangle\rangle \to \langle\langle n \rangle\rangle$ corresponding to the element $j \in \langle\langle n \rangle\rangle$;

(ii) if $\psi$ is a morphism with the previous property, then there is a unique morphism $\psi'$ such that $\psi = \delta_j^{(\varphi)} \psi'$.

Lemma 4.6. Let $G$ be a crossed interval group. Then, for every morphism $\varphi : \langle\langle m \rangle\rangle \to \langle\langle n \rangle\rangle$, the map

$$\tilde{\delta}^{(\varphi)*} : \text{RSt}_\varphi^G \to G_{k_1^{(\varphi)}} \times \cdots \times G_{k_n^{(\varphi)}} ; \ x \mapsto \left( \delta_1^{(\varphi)*}(x), \ldots, \delta_n^{(\varphi)*}(x) \right)$$

is a group homomorphism. Moreover, its kernel contains the subgroup $\text{Inr}_\varphi^G \subset \text{RSt}_\varphi^G$.

Proof. To see each map $\delta_j^{(\varphi)*} : \text{RSt}_\varphi^G \to G_{k_j^{(\varphi)}}$ is a group homomorphism, it suffices to show $\delta_j^{(\varphi)}$ is invariant under the left action of $\text{RSt}_\varphi^G$. This follows from the characterization in Remark 4.5. The last assertion is straightforward. \(\square\)

In the case $G = \mathcal{G}$ is a group operad, if $\varphi = \mu \rho$ is the unique factorization with $\mu$ active and $\rho$ inert, then there are canonical identifications

$$\mathcal{G}(k_1^{(\varphi)}) \times \cdots \times \mathcal{G}(k_n^{(\varphi)}) \cong \text{Dec}_\mu^G = \text{Dec}_\rho^G.$$  

Put $\delta$ the unique section of $\rho$, then one can see the both squares in the diagram below are commutative:

$$\begin{array}{ccc}
\text{Dec}_\mu^G & \xrightarrow{\delta^*} & \text{Dec}_\rho^G \\
\downarrow \quad \downarrow & & \quad \downarrow \approx \\
\text{RSt}_\varphi^G & \xrightarrow{\tilde{\delta}^{(\varphi)*}} & \mathcal{G}(k_1^{(\varphi)}) \times \cdots \times \mathcal{G}(k_n^{(\varphi)})
\end{array} \quad (4.8)
$$

In other words, the composition of the left and the bottom arrows in (4.8) induces the inverse of the map (2.9) in the case $K = \text{Dec}^G$.

Theorem 4.7. Let $\mathcal{G}$ be a group operad, and let $\mathcal{M}$ be a multicategory. Then, the family of maps

$$\Phi : (\mathcal{M} \wr_\mathcal{G} \mathcal{G})(\vec{a}, \vec{b}) \to ((\mathcal{M} \times \mathcal{G}) \ltimes_\mathcal{E} \mathcal{G})(\vec{a}, \vec{b})$$

$$(\varphi; f_1, \ldots, f_n; [u], [x]) \mapsto \left( \varphi; (f_1, \delta_1^{(\varphi)*}(u)), \ldots, (f_n, \delta_n^{(\varphi)*}(u)); [x] \right)$$

for $\vec{a} = a_1 \ldots a_m, \vec{b} = b_1 \ldots b_n \in \mathcal{M} \wr_\mathcal{G} \mathcal{G}$ form an identity-on-objects functor

$$\Phi : \mathcal{M} \wr_\mathcal{G} \mathcal{G} \to (\mathcal{M} \times \mathcal{G}) \ltimes_\mathcal{E} \mathcal{G}.$$  

Moreover, $\Phi$ is an isomorphism of categories which is 2-natural with respect to $\mathcal{M} \in \text{MultCat}$.  

27
Dec makes the square below commute:

\[ \text{Notice that there is a unique active morphism } \delta_j^{(\varphi)}(u) \text{ does not depend on the choice of the representative } u \in \text{Dec}_{\varphi} \text{ for every } 1 \leq j \leq n. \text{ In particular, in view of Lemma 2.13 we may take } u \text{ of the form} \]

\[ u = \gamma(c_{m+2}; c_{\infty}, u_1, \ldots, u_m, c_{\infty}) \in \text{Dec}_{\varphi} \subset G(m) \]  

(4.9)

with \( u_i \in G(k\varphi) \), where \( c_{\infty}(\varphi) := c_{\infty}(\varphi) \). In this case, we have \( \delta_j^{(\varphi)}(u) = u_i \) so that, for each morphism in \( M \times G \) of the form \( (\varphi; f_1, \ldots, f_m; [u], [x]) \), we have

\[ \Phi(\varphi; f_1, \ldots, f_m; [u], [x]) = (\varphi; (f_1, u_1), \ldots, (f_m, u_m); [x]) . \]

Now, for every morphism \( (\psi; g_1, \ldots, g_n; [v], [y]) \) in \( M \times G \) postcomposable with \((\varphi; \tilde{f}; [u], [x])\) above, the explicit formula of the composition operation in \( M \times G \) and the formulas in Lemma 4.2 give the equation

\[ \Phi(\psi; \tilde{g}; [v], [y]) \circ \Phi(\varphi; \tilde{f}; [u], [x]) = \left( \psi \varphi; \left( \gamma_M(g_1; (y_\varphi) \varphi) \cdots (u_\varphi) \varphi) \right) \right) \]

(4.10)

\[ \gamma_M(g_n; (y_\varphi) \varphi) : (\varphi^{\ast}(v) \cdots (\varphi^{\ast}(y)) \right) \]  

The comparison of (4.10) with the formula (4.6) tells us that, in order to have the functoriality of \( \Phi \), we only have to verify the equation

\[ \delta_j^{(\psi \varphi)}(\varphi^{\ast}(v)) = \gamma(\delta_j^{(\varphi)}(v); (y_\varphi) \varphi) \]  

(4.11)

for each \( 1 \leq j \leq n \). By virtue of Lemma 4.6 the left hand side of (4.11) equals

\[ (\psi \varphi \delta_j^{(\psi \varphi)}(v)) \cdot \delta_j^{(\psi \varphi)}(\varphi^{\ast}(v)) \cdot (\varphi^{\ast}(v)) \]  

(4.12)

Notice that there is a unique active morphism \( \varphi_j' : \langle \{ k_j^{(\psi \varphi)} \} \rangle \rightarrow \langle \{ k_j^{(\psi)} \} \rangle \) which makes the square below commute:

\[ \langle \{ k_j^{(\psi \varphi)} \} \rangle \xrightarrow{\varphi_j'} \langle \{ k_j^{(\psi)} \} \rangle \xrightarrow{\delta_j^{(\psi)}} \langle \{ 1 \} \rangle \]

(4.13)

It turns out that each square in (4.13) forms a pullback square of (ordinary) maps, so one has

\[ k_j^{(\varphi)} = k_j^{(\psi \varphi)} \delta_j^{(\psi)}(s) = k_j^{(\psi)} \psi^{-1}(\delta_j^{(\psi)}(s)) \]

for each \( 1 \leq s \leq k_j^{(\psi)} \). Thus, we obtain

\[ (\varphi \delta_j^{(\psi \varphi)})(v) = (\delta_j^{(\psi \varphi)})(v) \]

\[ = \gamma(\delta_j^{(\psi \varphi)}(v); (y_\varphi) \varphi) \]

\[ = \gamma(\delta_j^{(\psi \varphi)}(v); (y_\varphi) \varphi) \]  

(4.14)
where $e_i^{(\psi)} = e_k^{(\psi)}$. On the other hand, in view of the presentation (4.9), we have

$$
\delta_j^{(\psi y)^*} (\varphi^*(y) u \varphi^*(y-1)) = \delta_j^{(\psi y)^*} \left( \gamma_G(e_m+2; e_i^{(\psi)} e_{-\infty}, u_{y^{-1}(1)}, \ldots, u_{y^{-1}(m)}; e_i^{(\psi)}) \right)
$$

$$
= \gamma_G(e_j^{(\psi)}; u_{y^{-1}(\delta_j^{(\psi)(1)})}, \ldots, u_{y^{-1}(\delta_j^{(\psi)}(k_j^{(\psi)})])})
$$

$$
= \gamma_G(e_j^{(\psi)}; (y, \bar{u})^{\psi}_j)
$$

(4.15)

Substituting (4.14) and (4.15) into (4.12), we obtain (4.11), which implies $\Phi$ is actually a functor.

The 2-naturality of $\Phi$ immediately follows from definition. We verify $\Phi$ is an isomorphism of categories. Since it is the identity on objects, it suffices to show $\Phi$ is bijective on each hom-sets. This is actually a consequence of (3) in Lemma 2.13.

**Corollary 4.8.** For every group operad $\mathcal{G}$, the 2-functor $(-) \wr \mathcal{G}$ admits a lift depicted as the dashed arrow in the diagram below:

\[
\begin{array}{ccc}
\text{MultCat} & \xrightarrow{(-) \wr \mathcal{G}} & \text{PSh}(\mathcal{G} \Rightarrow \mathcal{E}_G) \\
\text{forget} & \downarrow & \text{forget} \\
\text{MultCat} & \xrightarrow{\Phi} & \text{Cat/\mathcal{E}_G} \\
\text{forget} & \downarrow & \text{forget} \\
\end{array}
\]

**Proof.** In view of Theorem 4.7, each $\mathcal{G}$-symmetric multicategory $\mathcal{M}$ admits a canonical functor

$$
(M \wr \mathcal{G}) \times_{\mathcal{G}} \mathcal{G} \mathcal{G} = M \wr \mathcal{G} \xrightarrow{\Phi} (M \times \mathcal{G}) \wr \mathcal{G} \rightarrow M \wr \mathcal{G}.
$$

(4.16)

Lemma 4.6 and the direct computation shows that it is in fact a structure of an internal presheaf over the double category $\mathcal{G} \Rightarrow \mathcal{E}_G$. Moreover, since the isomorphism $\Phi$ is 2-natural, the structure functor (4.16) is also 2-natural with respect to $\mathcal{G}$-symmetric multicategories $\mathcal{M}$. Therefore, we obtain the result.

We finally obtain analogues on quotients.

**Theorem 4.9.** Let $\mathcal{G}$ be a group operad, and let $\mathcal{M}$ be a multicategory. Then, there is an isomorphism $\tilde{\Phi} : M \wr \mathcal{G} \xrightarrow{\Phi} (M \times \mathcal{G}) \wr \mathcal{G}$ which is the identity on objects and, on each hom-set, described as

$$
\tilde{\Phi}([\varphi; (f_1, \ldots, f_n; u, x)] = \left[ \varphi; (f_1, \delta_1^{(\varphi)}(u)), \ldots, (f_n, \delta_n^{(\varphi)}(u)); x \right].
$$

(4.17)

Moreover, $\tilde{\Phi}$ is a 2-natural transformation with respect to $\mathcal{M} \in \text{MultCat}$ such that the diagram below is commutative:

\[
\begin{array}{ccc}
\mathcal{M} \wr \mathcal{G} & \xrightarrow{\Phi} & (M \times \mathcal{G}) \wr \mathcal{G} \\
\mathcal{M} \wr \mathcal{G} & \xrightarrow{\tilde{\Phi}} & (M \times \mathcal{G}) \wr \mathcal{G} \\
\end{array}
\]
Proof. We have the following commutative diagram of functors:

\[
\begin{array}{cccc}
  M \ltimes \tilde{G} & \longrightarrow & M \ltimes \tilde{E} & \\
  \downarrow & & \downarrow & \\
  M \ltimes G & \longrightarrow & M \ltimes E & \\
  \downarrow & & \downarrow & \\
  G & \longrightarrow & \tilde{E} & \\
  \uparrow t & & \uparrow t & \\
  \tilde{G} & \longrightarrow & \tilde{E} & \\
\end{array}
\]

(4.18)

Note that, Lemmas 3.10 and 4.4 assert that the bottom and the right faces, as well as the front and the back, are pullbacks. Hence, the “associativity property” of pullbacks (e.g. see Proposition 2.5.9 in [4]) implies the other faces are also pullbacks. In particular, we obtain isomorphisms of categories:

\[
(M \ltimes \tilde{G}) \times \tilde{E} \cong M \ltimes G \circ \Phi \cong (M \ltimes G) \circ \tilde{E} \cong ((M \ltimes G) \circ \tilde{E}) \times \tilde{E}
\]

(4.19)

The explicit computation shows that the isomorphism (4.19) is induced by the identity on \(\tilde{E}\) and an identity-on-object functor \(\Phi : M \ltimes \tilde{G} \to (M \ltimes G) \circ \tilde{E}\) described as (4.17). Moreover, since the functor \(E \to \tilde{E}\) is full and the identity on objects, the pullback along it preserves and reflects fully-faithfulness. Thus, we conclude \(\Phi\) is an isomorphism of categories. The 2-naturality and the compatibility with \(\Phi\) are obvious.

\[\square\]

Corollary 4.10. For every group operad \(G\), the 2-functor \((-) \circ \tilde{E}\) admits a lift depicted as the dashed arrow in the diagram below:

\[
\begin{array}{ccc}
  \text{MultCat}_G & \longrightarrow & \text{PSh}(\tilde{G} \Rightarrow \tilde{E}) \\
  \downarrow \text{forget} & & \downarrow \text{forget} \\
  \text{MultCat} & \longrightarrow & \text{Cat}/\tilde{E}
\end{array}
\]

5 CoCartesian lifting properties

We investigate the image of the functor \(\text{MultCat}_G \to \text{PSh}(\tilde{G} \Rightarrow \tilde{E})\) given in Corollary 4.10.

Definition. For a crossed interval group \(G\), a morphism in \(\tilde{E}_G\) is called active (resp. inert) if it is of the form \([\mu, x]\) for \(\mu : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle \in \nabla\) active (resp. inert) and arbitrary \(x \in G_m\).

In particular, the functor \(\nabla \to \tilde{E}_G\) preserves active morphisms and inert morphisms respectively. Throughout the section, the following inert morphisms in \(\nabla\) play important roles: for each \(1 \leq i \leq n\), we define a morphism \(\rho_i : \langle\langle n\rangle\rangle \to \langle\langle 1\rangle\rangle\) by

\[
\rho_i(j) = \begin{cases} 
-\infty & j < i \\
1 & j = i \\
\infty & j > i
\end{cases}
\]

By abuse of notation, we use the same notation \(\rho_i\) to denote its image in \(\tilde{E}_G\).
Proposition 5.1. Let $G$ be a group operad, and let $M$ be a multicategory. Then, the canonical functor $p_M : M \otimes \tilde{E}_G \to \tilde{E}_G$ satisfies the following properties.

1. Every inert morphism $[\rho, x] : \langle m \rangle \to \langle n \rangle \in \tilde{E}_G$ admits $p_M$-coCartesian lifts along any object in the fiber $(M \otimes \tilde{E}_G)_{\langle n \rangle} := p_M^{-1}(\langle n \rangle)$. More precisely, if $\delta : \langle n \rangle \to \langle m \rangle$ is the section of $\rho$, then for each $\bar{a} = a_1 \ldots a_m \in (M \otimes \tilde{E}_G)_{\langle n \rangle}$, the morphism
   \[
   [\rho, x]_{\bar{a}} := [\rho; \text{id}_{a_{\delta(1)}}, \ldots, \text{id}_{a_{\delta(n)}}; x] : a_1 \ldots a_m \to a_{\delta(1)} \ldots a_{\delta(n)}
   \]
   is $p_M$-coCartesian.

2. For an object $\bar{a} = a_1 \ldots a_n \in (M \otimes \tilde{E}_G)_{\langle n \rangle}$, choose a $p_M$-coCartesian lift $\tilde{\rho}_i : \tilde{a} \to a_i$ along $\bar{a}$ for each $1 \leq i \leq n$. Then, for every object $\tilde{b} \in M \otimes \tilde{E}_G$, the square below is a pullback:
   \[
   \begin{array}{ccc}
   (M \otimes \tilde{E}_G)(\tilde{b}, \tilde{a}) & \xrightarrow{(\tilde{\rho}_1), \ldots, (\tilde{\rho}_n)} & \prod_{i=1}^n (M \otimes \tilde{E}_G)(\tilde{b}, a_i') \\
   p_M \downarrow & & \downarrow p_M \\
   M \otimes \tilde{E}_G(p_M(\tilde{b}), \langle n \rangle) & \xrightarrow{((\rho_1)_!, \ldots, (\rho_n)_!)} & \tilde{E}_G(p_M(\tilde{b}), \langle 1 \rangle)^n
   \end{array}
   \]
   (5.2)

3. For each $1 \leq i \leq n$, take a functor $(\rho_i)_! : (M \otimes \tilde{E}_G)_{\langle n \rangle} \to (M \otimes \tilde{E}_G)_{\langle 1 \rangle}$ together with a natural transformation $\tilde{\rho}_i : \tilde{a} \to (\rho_i)_! \tilde{a}$ which is (componentwisely) $p_M$-coCartesian. Then the functor
   \[
   ((\rho_1)_!, \ldots, (\rho_n)_!) : (M \otimes \tilde{E}_G)_{\langle n \rangle} \to (M \otimes \tilde{E}_G)_{\langle 1 \rangle}^n
   \]
   is an equivalence of categories.

Remark 5.2. The condition [2] actually does not depend on the choice of coCartesian lifts $\tilde{\rho}_i$ of $\rho_i$. Indeed, if one choose another coCartesian lift $\tilde{\rho}'_i : \tilde{a} \to a_i'$, then the uniqueness of the coCartesian lifts implies there is a unique isomorphism $a_i' \cong a_i''$ so that $\tilde{\rho}'_i$ factors through $\tilde{\rho}_i$ followed by the isomorphism. Moreover, it also gives rise to an isomorphism of squares (5.2). Thus, if [2] satisfied for one family of coCartesian lifts, then it is also for the other.

A similar argument shows that the condition [3] does not depend on the choice of the functors $(\rho_i)_!$.

Proof of Proposition 5.1. In order to verify (1) it clearly suffices to consider only the case $x \in G_m$ is the unit. For an inert morphism $\rho : \langle m \rangle \to \langle n \rangle \in \nabla$, set $\delta$ to be the unique section, and suppose we have a morphism in $M \otimes \tilde{E}_G$ of the form
   \[
   [\varphi; f_1, \ldots, f_l; \rho^* y] : a_1 \ldots a_m \to \tilde{b}.
   \]
   We show it uniquely factors through the morphism
   \[
   \tilde{\rho}_{\bar{a}} = [\rho; \text{id}_{a_{\delta(1)}}, \ldots, \text{id}_{a_{\delta(n)}}, e_m] : \tilde{a} \to a_{\delta(1)} \ldots a_{\delta(n)}
   \]
   Thanks to the unique factorization in $\nabla$, we have
   \[
   [\varphi; f_1, \ldots, f_l; \rho^* y] = [\varphi; f_1, \ldots, f_l; y] \circ \tilde{\rho}_{\bar{a}},
   \]
   (5.3)
so there in fact exists a factorization. Moreover, since the morphism \([\varphi; f_1, \ldots, f_n; y]\) is uniquely determined by the underlying morphism \([\varphi, y]\) in \(\tilde{E}_G\) and the tuple \((f_1, \ldots, f_n)\), which is determined by the left hand side. This implies the factorization \((5.3)\) is unique, so \(\tilde{p}_{\varphi}\) is \(p_{\mathcal{M}}\)-coCartesian.

We next see [2]. For an object \(\vec{a} = a_1 \ldots a_n \in (\mathcal{M} \times \tilde{E}_G)_{(\varphi)}\), in view of Remark 5.2 we may assume the lift \(\tilde{\rho}_i = (\tilde{\rho}_i) \vec{a}\) is the one given in the part [1]. Suppose \(\vec{b} = b_1 \ldots b_m \in \mathcal{M} \times \tilde{E}_G\), and \([\varphi, x] : \langle m \rangle \rightarrow \langle n \rangle \in \tilde{E}_G\). Then, if one has a morphism of the form

\[
[\varphi; f_1, \ldots, f_n; x] : \vec{b} \rightarrow \vec{a},
\]

then \(f_i \in \mathcal{M}(((x_i \vec{b})^{\rho_i},a_i)\). On the other hand, we have \((x_i \vec{b})^{\rho_i,\varphi} = (x_i \vec{b})^{\rho_i}\) so that \((5.4)\) makes sense if and only if we have morphisms

\[
[\rho_i \varphi; f_i; x] : \vec{b} \rightarrow a_i
\]

for \(1 \leq i \leq n\). When we fix a morphism \([\varphi, x]\) in \(\tilde{E}_G\), the two data \((5.4)\) and \((5.5)\) clearly correspond in one-to-one to each other. It follows that the square \((5.2)\) is a pullback.

We finally show [3]. Note that, in view of Example 3.8, every morphism in \((\mathcal{M} \times \tilde{E}_G)_{(\varphi)}\) is of the form

\[
[\text{id}_{\langle n \rangle}; f_1, \ldots, f_n; e_n] : a_1 \ldots a_n \rightarrow b_1 \ldots b_n
\]

with \(f_i \in \mathcal{M}(a_i, b_i) = \mathcal{M}(a_i, b_i)\), here \(\mathcal{M}\) is the underlying category of \(\mathcal{M}\). In other words, we have a canonical isomorphism

\[
(\mathcal{M} \times \tilde{E}_G)_{(\varphi)} \cong \mathcal{M}^{\times n} \cong (\mathcal{M} \times \tilde{E}_G)_{(\varphi)}^{\times n}.
\]

Hence, it suffices to show that the functor \((\rho_i)_* : (\mathcal{M} \times \tilde{E}_G)_{(\varphi)} \rightarrow (\mathcal{M} \times \tilde{E}_G)_{(\hat{\varphi})}\) coincides with the projection under the isomorphism \((5.6)\). If \((\rho_i)_*\) is the one induced by the \(p_{\mathcal{M}}\)-coCartesian lifts in the part \([1]\), this follows from the correspondence of \((5.4)\) to \((5.5)\) and the unique factorization \((5.3)\). In view of Remark 5.2 this completes the proof.

We define a 2-subcategory \(\text{Oper}_{\tilde{G}}' \subset \text{Cat}^{\tilde{E}_G}\) as follows:

- objects of \(\text{Oper}_{\tilde{G}}'\) are those categories \(C\) over \(\tilde{E}_G\) that satisfy three properties in Proposition 5.1.

- for \(C, D \in \text{Oper}_{\tilde{G}}'\), the hom-category \(\text{Oper}_{\tilde{G}}'(C, D)\) is the full subcategory of \(\text{Cat}^{\tilde{E}_G}\) spanned by functors \(C \rightarrow D\) over \(\tilde{E}_G\) which preserve coCartesian lifts of inert morphisms in \(\tilde{E}_G\).

Furthermore, we put

\[
\text{Oper}_{\tilde{G}}^{alg} := \text{PSh}(\tilde{G}_G \Rightarrow \tilde{E}_G) \times_{\text{Cat}^{/\tilde{E}_G}} \text{Oper}_{\tilde{G}}',
\]

whose objects are called categories of algebraic \(G\)-operators, and whose morphisms maps of algebraic \(G\)-operators. In other words, a category of operators is an internal presheaf \(\mathcal{X}\) over the double category \(\tilde{G}_G \Rightarrow \tilde{E}_G\) with the functor \(p : \mathcal{X} \rightarrow \tilde{E}_G\) satisfying the following conditions:

32
(i) every inert morphism $\rho : \langle\langle m \rangle\rangle \to \langle\langle n \rangle\rangle \in \tilde{E}G$ admits $p$-coCartesian lifts along any object in the fiber $\mathcal{X}_{\langle\langle m \rangle\rangle} := p^{-1}\{\langle\langle m \rangle\rangle\}$;

(ii) if we are given a $p$-coCartesian morphism $\tilde{\rho}_j : X \to X_i$ covering the inert morphism $\rho_i : \langle\langle n \rangle\rangle \to \langle\langle 1 \rangle\rangle \in \tilde{E}G$ for each $1 \leq i \leq n$, for every object $W \in \mathcal{X}$, the square below is a pullback:

\[
\begin{array}{ccc}
\mathcal{X}(W, X) & \xrightarrow{((\tilde{\rho}_1), \ldots, (\tilde{\rho}_n))} & \prod_{i=1}^n \mathcal{X}(W, X_i) \\
p \downarrow & \simeq & \downarrow p \\
\tilde{E}G(p(W), \langle\langle n \rangle\rangle) & \xrightarrow{((\rho_1), \ldots, (\rho_n))} & \tilde{E}G(p(W), \langle\langle 1 \rangle\rangle)^\times n
\end{array}
\]

(iii) if $(\rho_i)_! : \mathcal{X}_{\langle\langle n \rangle\rangle} \to \mathcal{X}_{\langle\langle 1 \rangle\rangle}$ is a functor induced by the inert morphism $\rho_i : \langle\langle n \rangle\rangle \to \langle\langle 1 \rangle\rangle$ for each $1 \leq i \leq n$, then the functor

\[(\rho_1)_! \times \cdots \times (\rho_n)_! : (\mathcal{M} \times \tilde{E}G)_{\langle\langle n \rangle\rangle} \to (\mathcal{M} \times \tilde{E}G)_{\langle\langle 1 \rangle\rangle}\times n\]

is an equivalence of categories.

Thanks to Corollary 4.10 and Proposition 5.4, the 2-functor $(-) \wr \tilde{E}G$ induces a 2-functor $\text{MultCat}_G \to \text{Oper}_\text{alg} G$, which we also denote by $(-) \wr \tilde{E}G$ by abuse of notation. Thanks to Lemma 4.4, the forgetful functor $\text{Oper}_{\text{alg}} G \to \text{Oper}' G$ is locally fully faithful.

Example 5.3. As we have $* \wr \tilde{E}G \cong \tilde{E}G$, the identity functor $\tilde{E}G \to \tilde{E}G$ exhibits $\tilde{E}G$ as a category of algebraic $G$-operators.

Example 5.4. Recall that every group operad $G$ is itself a $G$-symmetric multicategory with the multiplication map $G \times G \to G$. On the other hand, in view of Theorem 4.9, we have isomorphisms

\[\tilde{G}_G \cong * \wr \tilde{G}_G \cong (\times G) \wr \tilde{E}G \cong G \wr \tilde{E}G .\]

It follows that the functor $s : \tilde{G}_G \to \tilde{E}G$ exhibits $\tilde{G}_G$ as a category of algebraic $G$-operators.

It turns out that there are free $G$-symmetrizations of objects in $\text{Oper}' G$. Indeed, we have the following property on the free construction.

Lemma 5.5. For every $C \in \text{Oper}' G$, the functor

\[C \cong C \times \tilde{G}_G \tilde{E}G \xrightarrow{1d \times 1} C \times \tilde{E}G \tilde{E}G\]

preserves coCartesian lifts of inert morphisms.

Proof. Note that the category $C \times \tilde{G}_G \tilde{G}_G$ is described as follows:

- objects are the same as $C$;
- for $X, Y \in C$, the hom-set $(C \times \tilde{G}_G \tilde{G}_G)(X, Y)$ consists of tuples $[\varphi; f; u, x]$ so that $[\varphi, u, x] : q(X) \to q(Y) \in \tilde{G}_G$ makes sense and $f : X \to Y \in C$ with $q(f) = [\varphi, ux]$.
the composition is given by
\[
[ψ; g; v; y] \circ [φ; f; u; x] = [ψφy; gf; φ∗(vy)uφ∗(y)x]^{-1}, φ∗(y)x
\]
the structure functor \( C × ˜ E \to ˜ G \) is given by
\[
[φ; f; u; x] ↦ [φ, x].
\]

Suppose \([ρ, x]: ⟨⟨m⟩⟩ → ⟨⟨n⟩⟩ ∈ ˜ E \) is an inert morphism, and take a coCartesian lift
\[  \rho;  \pi X : X → X' \] along \( X ∈ C \). The functor (5.9) sends it to
\[
[ρ;  \pi X; e, x] : X → X' ∈ C × ˜ E \to ˜ G \to ˜ E \to ˜ G \to ˜ G.
\]

To see (5.9) is coCartesian, consider a morphism in \( C × ˜ E \to ˜ G \) of the form
\[
[φρ; f; u; x] : X → Y. \]
In view of (3) in Lemma 2.13, there is a unique element \( π ∈ \text{Dec}^ρ_φ \) such that
\[
[φ; u] = [ρ(π)] ∈ \text{Im}_φ^ρ \cap \text{Dec}^ρ_φ, \text{ which implies}
\]
\[
[φρ, ux] = [φ, π] \circ [ρ, x].
\]
On the other hand, since \( \rho;  \pi X; e, x \) is coCartesian, there is a unique factorization
\[
f = f' \circ  \rho;  \pi X; e, x \] with \( f' : X' → Y \) covering \( [φ, π] \). One obtains
\[
[φρ; f; u; x] = [φ; f', \pi; e] \circ [ρ;  \rho;  \pi X; e, x].
\]
Since the morphisms \( f' \) and \( π \) are uniquely determined by the other data, the factorization (5.10) is unique. It follows that the morphism (5.9) is coCartesian.

Proposition 5.6. The free 2-functor
\[
\text{Cat}/ ˜ E \to \text{PSh}( ˜ G \Rightarrow ˜ E) ; \ C ↦ C × ˜ E \to ˜ G
\]
associated to the 2-monad of internal presheaves over the double category \( ˜ G \) restricts to a 2-functor
\[
\text{Oper'}_G \to \text{Oper'}_G^\text{alg}.
\]

Proof. It clearly suffices to show the composition
\[
\text{Oper'}_G \hookrightarrow \text{Cat}/ ˜ E \to \text{PSh}( ˜ G \Rightarrow ˜ E) \to \text{Cat}/ ˜ E
\]
factors through the subcategory \( \text{Oper'}_G \) at the end. We have to verify it regarding objects and 1-morphisms.

Let \( C ∈ \text{Oper'}_G \) with \( q : C \to ˜ E \). We verify the three conditions on categories of algebraic \( G \)-operators for \( C × ˜ E \to ˜ G \). Since the unit \( C → C × ˜ E \to ˜ G \) is the identity on objects, Lemma 5.5 implies \( C × ˜ E \to ˜ G \) admits all the coCartesian lifts of inert morphisms. On the other hand, according to the description of \( C × ˜ E \to ˜ G \) in
the proof of Lemma 5.5 one easily verify the property (iii). To see the property (iii) observe that the category $(\tilde{G})_{\langle n \rangle}$ consists of automorphisms on $\langle n \rangle$ in $\tilde{G}$ of the form
\[ [\text{id}_{\langle n \rangle}, u, e_n] \]
for $u \in \text{Dec}_{G_{\langle n \rangle}}$. It turns out that such morphisms vanished by the functor $t : \tilde{G} \to \tilde{E}_G$, so we obtain isomorphisms
\[ (C \times_{\tilde{E}_G} \tilde{G})_{\langle n \rangle} \cong C \times_{\tilde{E}_G} (\tilde{G})_{\langle n \rangle} \cong C_{\langle n \rangle} \times (\tilde{G})_{\langle n \rangle} \]
Under the identification, it is easily verified that, for each inert morphism $\rho_i : \langle n \rangle \to \langle 1 \rangle$, the induced functor
\[ (\rho_i) : (C \times_{\tilde{E}_G} \tilde{G})_{\langle n \rangle} \to (C \times_{\tilde{E}_G} \tilde{G})_{\langle 1 \rangle} \]
coincides with the one induced by
\[ (\rho_i) : C_{\langle n \rangle} \to C_{\langle 1 \rangle}, \quad (\rho_i) : (\tilde{G})_{\langle n \rangle} \to (\tilde{G})_{\langle 1 \rangle} . \]
Thus, the functor
\[ ((\rho_1)_1, \ldots, (\rho_n)_1) : (C \times_{\tilde{E}_G} \tilde{G})_{\langle n \rangle} \to (C \times_{\tilde{E}_G} \tilde{G})_{\langle 1 \rangle} \]
is an equivalence.

As for 1-morphisms, suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor over $\tilde{E}_G$ for $\mathcal{C}, \mathcal{D} \in \text{Oper}_G'$. We have the following commutative square:
\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\eta \downarrow & & \downarrow \eta \\
C \times_{\tilde{E}_G} \tilde{G} & \xrightarrow{F \times \text{id}} & D \times_{\tilde{E}_G} \tilde{G} \\
\end{array}
\]
In view of Lemma 5.5 all the coCartesian lifts of inert morphisms in $C \times_{\tilde{E}_G} \tilde{G}$ and $D \times_{\tilde{E}_G} \tilde{G}$ are isomorphic to the images of ones in $\mathcal{C}$ and in $\mathcal{D}$ respectively by the vertical functors. It follows that the bottom arrow in (5.11) preserves coCartesian lifts of inert morphisms as soon as so does the top. The required result now follows immediately.

Corollary 5.7. Let $\mathcal{G}$ be a group operad, and let $\mathcal{C}$ be a category of algebraic $\mathcal{G}$-operators. Then, the functor
\[ \mathcal{A}_\mathcal{C} : C \times_{\tilde{E}_G} \tilde{G} \to \mathcal{C} \]
in the internal presheaf structure on $\mathcal{C}$ is a map of algebraic $\mathcal{G}$-operators.

Proof. By virtue of Lemma 5.5 and Proposition 5.6, $\mathcal{A}_\mathcal{C}$ is a 1-morphism in the 2-category $\text{Oper}_G'$. In addition, it is straightforward from the definition of internal presheaves that $\mathcal{A}_\mathcal{C}$ is a map of internal presheaves. Combining them, one obtains the result. \[ \Box \]
The equivalence of notions

The goal of this section is to prove the following result.

**Theorem 6.1.** Let $G$ be a group operad. Then, the 2-functor

$$(-) \mapsto \mathbb{E}_G : \text{MultCat}_G \to \text{Oper}^\text{alg}_G$$

is a biequivalence of 2-categories. In other words, the following hold.

1. It is essentially fully faithful; i.e. for every pair $(\mathcal{M}, \mathcal{N})$ of $G$-symmetric multicategories, the functor

$$\text{MultCat}_G(\mathcal{M}, \mathcal{N}) \to \text{Oper}^\text{alg}_G(\mathcal{M} \mapsto \mathbb{E}_G, \mathcal{N} \mapsto \mathbb{E}_G)$$

is an equivalence of categories.

2. It is essentially surjective; i.e. for every category of algebraic $G$-operators $\mathcal{C}$, there is a $G$-symmetric multicategory $\mathcal{M}$ together with an equivalence $\mathcal{M} \mapsto \mathbb{E}_G \simeq \mathcal{C}$ in $\text{Oper}^\text{alg}_G$.

**Remark 6.2.** It is known that a 2-functor $\mathcal{K} \to \mathcal{L}$ between 2-categories admits a pseudoinverse, i.e. a pseudofunctor $\mathcal{L} \to \mathcal{K}$ which is the inverse up to natural isomorphisms, provided it is essentially fully faithful and essentially surjective in the sense in Theorem 6.1. The reader can find a sketch in Section 3.2 in [11]. Note that, unlike the case of equivalences of ordinary 1-categories, a pseudoinverse might not be a strict one.

In order to prove Theorem 6.1, we need to observe that coCartesian lifts of inert morphisms in $\mathcal{M} \mapsto \mathbb{E}_G$ are preserved coherently by arbitrary maps of algebraic $G$-operators. To simplify the notation, we use the following convention: let $[\rho, x] : \langle\langle m \rangle\rangle \to \langle\langle n \rangle\rangle \in \mathbb{E}_G$ be an inert morphism. Although we may denote by $\hat{\rho} \mapsto [\rho, e_m] : \mathcal{M} \mapsto \mathbb{E}_G$ an arbitrary coCartesian lift $\rho$ along an object $X$ in a general category of algebraic $G$-operators $\mathcal{C}$, we always assume $\hat{\rho} \mapsto [\rho, e_m] \mapsto [\rho, x] \mapsto \mathcal{M} \mapsto \mathbb{E}_G$ in the special case $\mathcal{C} = \mathcal{M} \mapsto \mathbb{E}_G$. In particular, we write

$$\hat{\rho} := [\rho, e_m], \quad \hat{x} := [\mathrm{id}, x].$$

Hence, if $\delta$ is the section of $\rho$, the induced functor

$$\rho_1 : (\mathcal{M} \mapsto \mathbb{E}_G)_{\langle\langle m \rangle\rangle} \to (\mathcal{M} \mapsto \mathbb{E}_G)_{\langle\langle n \rangle\rangle}$$

coincides with the canonical projection so that $\rho_1(a_1 \ldots a_m) = a_{\delta(1)} \ldots a_{\delta(n)}$.

**Lemma 6.3.** Let $G$ be a group operad, and let $\mathcal{M}$ and $\mathcal{N}$ be $G$-symmetric multicategories. Suppose $H : \mathcal{M} \mapsto \mathbb{E}_G \to \mathcal{N} \mapsto \mathbb{E}_G$ is a map of algebraic $G$-operators. Then, for each $\bar{a} = a_1 \ldots a_m$ and each $1 \leq i \leq m$, there is a unique isomorphism

$$\lambda_{\bar{a}, i} : H(a_i) \mapsto (\rho_i)_H(\bar{a}) \in (\mathcal{M} \mapsto \mathbb{E}_G)_{\langle\langle 1 \rangle\rangle} = \mathcal{M}$$

such that $(\hat{\rho})_H(\bar{a}) = \lambda_{\bar{a}, i} \circ H((\hat{\rho})_H(\bar{a})$. Moreover, the family

$$\left\{ \lambda_{\bar{a}} = [\mathrm{id}_{\langle\langle m \rangle\rangle}; \lambda_{\bar{a}, 1}, \ldots, \lambda_{\bar{a}, m}; e_m] \middle| \bar{a} = a_1 \ldots a_m \in \mathcal{M} \mapsto \mathbb{E}_G \right\}$$

is a natural transformation
enjoys the property that, for every inert morphism \([\rho, x] : \langle m \rangle \to \langle n \rangle \in \tilde{E}_G\), say \(\delta\) is the section of \(\rho\), and for each \(\tilde{a} = a_1 \ldots a_m \in M \cap \tilde{E}_G\), the square below commutes in \(M \cap \tilde{E}_G\):

\[
\begin{array}{ccc}
H(a_1) \ldots H(a_m) & \xrightarrow{\lambda_{\tilde{a}}} & H(a_1) \ldots H(a_m) \\
\xrightarrow{[\rho,x] H(a_1) \ldots H(a_m)} & & \xrightarrow{H([\rho,x]_{\tilde{a}})} \\
H(a_{x^{-1}(1)}) \ldots H(a_{x^{-1}(n)}) & \xrightarrow{\lambda_{\tilde{a} x}} & H(a_{x^{-1}(1)}) \ldots H(a_{x^{-1}(n)})
\end{array}
\] (6.2)

Proof. Since the functor \(H : M \cap \tilde{E}_G \to N \cap \tilde{E}_G\) preserves coCartesian lifts of inert morphisms, the morphism \(H((\tilde{p}_i)_\delta) : H(\tilde{a}) \to H(\tilde{a})\) is coCartesian. Thus, the first statement is obvious. As for the second, it turns out that we only have to verify the commutativity of (6.2) for inert morphisms of the forms \([\rho, e_m]\) and \([\text{id}, x]\). In the first case, for each \(1 \leq j \leq n\), we have

\[
(\tilde{p}_j)_{H(\rho, \tilde{a})} \circ H(\tilde{p}_j) \circ \lambda_{\tilde{a}} = \lambda_{\rho, \tilde{a}, j} \circ H((\tilde{p}_j)_{H(\rho, \tilde{a})}) \circ \lambda_{\tilde{a}} = \lambda_{\rho, \tilde{a}, j} \circ H((\tilde{p}_j(\delta))_{\tilde{a}}) \circ \lambda_{\tilde{a}} = \lambda_{\rho, \tilde{a}, j} \circ (\tilde{p}_j(\delta))_{H(\rho, \tilde{a})} \circ \lambda_{\tilde{a}} = \lambda_{\rho, \tilde{a}, j} \circ (\tilde{p}_j(\delta))_{H(\rho, \tilde{a})} \circ \lambda_{\tilde{a}}
\]

In view of (2) in Proposition 5.1 this implies \(H(\tilde{p}_j)_{\lambda_{\tilde{a}}} = \lambda_{\rho, \tilde{a}}\lambda_{\tilde{p}_j H(\rho, \tilde{a})} \circ H(\tilde{a})\), and (6.2) is commutative.

It remains to show the commutativity of (6.2) in the case \(\rho\) is the identity. Similarly to the case above, we have

\[
(\tilde{p}_j)_{H(x, \tilde{a})} \circ H(\tilde{x}_\delta) \circ \lambda_{\tilde{a}} = \lambda_{x, \tilde{a}, j} \circ H((\tilde{p}_j)_{H(x, \tilde{a})} \circ \tilde{x}_\delta) \circ \lambda_{\tilde{a}} = \lambda_{x, \tilde{a}, j} \circ H([\rho_j, \tilde{x}_\delta]_{\tilde{a}}) \circ \lambda_{\tilde{a}}
\] (6.3)

In view of , setting \(\delta_j\) to be the section of \(\rho_j\) for each \(1 \leq j \leq m\), we have

\[
[\rho_j, x] = [\rho_{x^{-1}(j)}, \rho_{x^{-1}(j)}^* \delta_{x^{-1}(j)}(x)] = [\text{id}, \delta_{x^{-1}(j)}(x)] \circ \rho_{x^{-1}(j)} = \rho_{x^{-1}(j)}
\]

as morphisms in \(\tilde{E}_G\) since \(\text{Kec}_{\text{id}(1)}^{G} = \text{Kec}_{\text{id}(1)}^{G} = G(1)\). Substituting it to (6.3), we get

\[
(\tilde{p}_j)_{H(x, \tilde{a})} \circ H(\tilde{x}_\delta) \circ \lambda_{\tilde{a}} = \lambda_{x, \tilde{a}, j} \circ H((\tilde{p}_j_{H(x, \tilde{a})})_{\tilde{a}}) \circ \lambda_{\tilde{a}} = \lambda_{x, \tilde{a}, j} \circ \lambda_{a, x^{-1}(j)} \circ (\tilde{p}_j_{H(a_1)} \circ \lambda_{\tilde{a}} = \lambda_{x, \tilde{a}, j} \circ (\tilde{p}_j_{H(a_1)} \circ \lambda_{\tilde{a}} = \lambda_{x, \tilde{a}, j} \circ (\tilde{p}_j_{H(a_1)} \circ \lambda_{\tilde{a}} = \lambda_{x, \tilde{a}, j} \circ (\tilde{p}_j_{H(a_1)} \circ \lambda_{\tilde{a}} = \lambda_{x, \tilde{a}, j} \circ (\tilde{p}_j_{H(a_1)} \circ \lambda_{\tilde{a}} = \lambda_{x, \tilde{a}, j} \circ \lambda_{\tilde{a}}
\]

Hence, (2) in Proposition 5.1 again implies the commutativity of (6.2). □

37
On the one hand, we noticed how each algebraic category of algebras determined by bilinear maps is recovered from the category $\mathcal{M} \upharpoonright \tilde{E}_G$, here the right hand side is the set of morphisms $a_1 \ldots a_n \to a$ in $\mathcal{M} \upharpoonright \tilde{E}_G$ covering $\mu_n$.

Notation. Given active morphisms $\nu_1 : \langle k_1 \rangle \to \langle l_1 \rangle \in \nabla$ for $1 \leq i \leq n$, we define an active morphism $\nu_1 \cdots \nu_n : \langle k_1 + \cdots + k_n \rangle \to \langle l_1 + \cdots + l_n \rangle \in \nabla$ to be the map

$$\langle k_1 + \cdots + k_n \rangle \cong \{ -\infty \} * \langle k_1 \rangle * \cdots * \langle k_n \rangle * \{ \infty \}$$

$$\begin{align*}
\text{id} \downarrow \coprod_{i=1}^{n} \text{Id}_{\langle l_i \rangle} & \cong \{ -\infty \} * \langle l_1 \rangle * \cdots * \langle l_n \rangle * \{ \infty \} \cong \langle l_1 + \cdots + l_n \rangle,
\end{align*}$$

here $*$ is the join of ordered sets and all maps are order-preserving. In particular, we write

$$\mu_{\tilde{k}} := \mu_{k_1} \cdots \mu_{k_n}.$$

On the other hand, we set $\rho_{\tilde{k}}(\tilde{i}) : \langle k_1 + \cdots + k_n \rangle \to \langle k_i \rangle \in \nabla$ to be the inert morphism with

$$\rho_{\tilde{k}}(\tilde{i})(j) = \begin{cases} -\infty, & j \leq \sum_{s<i} k_s, \\ j - \sum_{s<i} k_s, & \sum_{s<i} k_s < j \leq \sum_{s \leq i} k_s, \\ \infty, & j > \sum_{s \leq i} k_s. \end{cases}$$

We will identify the morphisms above with their images in $\tilde{E}_G$.

Using the notation above, one can immediately see

$$\nu_1 \cdots \nu_n = (\nu_1 \cdots \nu_n) \circ (\nu_1 \cdots \nu_n),$$

for active morphisms $\nu_i : \langle k_i \rangle \to \langle l_i \rangle$ and $\nu'_i : \langle l_i \rangle \to \langle m_i \rangle$ with $\tilde{k} = (k_1, \ldots, k_n)$ and $\tilde{l} = (l_1, \ldots, l_n)$.

Lemma 6.4. Let $\mathcal{C} \in \text{Op}\text{er}_G'$, and suppose we are given active morphisms $\nu_i : \langle k_i \rangle \to \langle l_i \rangle \in \nabla$ for $1 \leq i \leq n$. Put $\tilde{k} = (k_1, \ldots, k_n)$ and $\tilde{l} = (l_1, \ldots, l_n)$, and suppose in addition $(\rho_{\tilde{k}}^{(\tilde{i})})_X : X \to X_i$ and $(\rho_{\tilde{l}}^{(\tilde{i})})_Y : Y \to Y_i$ are coCartesian morphisms in $\mathcal{C}$ covering $\rho_{\tilde{i}}^{(\tilde{i})}$ and $\rho_{\tilde{i}}^{(\tilde{i})}$ respectively. Then, there is a unique bijection

$$\varpi : \prod_{i=1}^{n} \mathcal{C}(X_i, Y_i)_{\nu_i} \to \mathcal{C}(X, Y)_{(\rho_{\tilde{k}}^{(\tilde{i})})_{\nu_1} \cdots \nu_n}$$

such that $(\rho_{\tilde{l}}^{(\tilde{i})})_Y \circ \varpi(f_1, \ldots, f_n) = f_i \circ (\rho_{\tilde{k}}^{(\tilde{i})})_X$, where $\mathcal{C}(V, W)_\nu$ is the set of morphisms $V \to W \in \mathcal{C}$ covering $\nu$. Moreover, if other active morphisms
guaranteed by the property (ii) of objects of \( \text{Oper} \) and the universal property of the coCartesian morphisms. The uniqueness is immediate follows from the equation \( \rho(\vec{m}) \) are given for \( 1 \leq i \leq n \), with \( \vec{m} = (m_1, \ldots, m_n) \), then the square below is commutative:

\[
\prod_{i=1}^{n} \mathcal{C}(Y_i, Z_i)_{\nu_i'} \times \mathcal{C}(X_i, Y_i)_{\nu_i} \xrightarrow{\prod \text{comp}} \prod_{i=1}^{n} \mathcal{C}(X_i, Z_i)_{\nu_i'} \\
\mathcal{C}(Y, Z)_{\nu_1' \circ \cdots \circ \nu_n'} \times \mathcal{C}(X, Y)_{\nu_1 \circ \cdots \circ \nu_n} \xrightarrow{\text{comp}} \mathcal{C}(X, Z)_{(\nu_1' \circ \cdots \circ \nu_n')} \quad \text{(6.6)}
\]

**Proof.** The existence of \( \varpi \) immediately follows from the equation \( \rho_i \mu_i = \mu_i \rho_i(\vec{k}) \) and the universal property of the coCartesian morphisms. The uniqueness is guaranteed by the property (ii) of objects of \( \text{Oper}_G \). To see the last statement, take morphisms \( f_i : X_i \to Y_i \) and \( g_i : Y_i \to Z_i \) covering \( \nu_i \) and \( \nu_i' \) respectively for each \( 1 \leq i \leq n \). Then, we have

\[
(\rho_i(\vec{m}))_Z \circ \varpi(g_1, \ldots, g_n) \circ \varpi(f_1, \ldots, f_n) = g_i \circ (\rho_i(\vec{k}))_Y \circ \varpi(f_1, \ldots, f_n) = g_i f_i \circ (\rho_i(\vec{k}))_X,
\]

so the uniqueness of \( \varpi \) implies

\[
\varpi(g_1 f_1, \ldots, g_n f_n) = \varpi(g_1, \ldots, g_n) \circ \varpi(f_1, \ldots, f_n).
\]

Hence, the commutativity of (6.6) follows. \( \square \)

**Lemma 6.5.** Let \( \mathcal{C} \) be a category of algebraic \( G \)-operators, and let \( X \in \mathcal{C}_{\langle m_i \rangle} \) is an object together with coCartesian morphisms

\[
(\rho_i)_X : X \to X_i
\]

covering the inert morphism \( \rho_i : \langle m_i \rangle \to \langle 1 \rangle \in \nabla \) for \( 1 \leq i \leq m \). For an inert morphism \( [\rho, x] : \langle n \rangle \to \langle n \rangle \in \mathcal{E}_G \), say \( \delta \) is the section of \( \rho \) in \( \nabla \), suppose we are given an object \( X' \in \mathcal{C}_{\langle n \rangle} \) together with coCartesian morphisms

\[
(\rho_j)_{X'} : X' \to X'_{x^{-1}(\delta(n))}
\]

covering the inert morphism \( \rho_j : \langle n \rangle \to \langle 1 \rangle \) for \( 1 \leq j \leq n \). Then, there is a unique isomorphism \( \tilde{[\rho, x]}_X : X \to X' \in \mathcal{C} \) covering the morphism \( [\rho, x] \) which makes the diagram

\[
X \xrightarrow{[\rho, x]_X} X' \xleftarrow{\rho_j}_{X_{x^{-1}(\delta(j))}} \quad \text{(6.7)}
\]

commutes for each \( 1 \leq j \leq n \). Moreover, \( \tilde{[\rho, x]}_X \) is coCartesian.

**Proof.** According to the computation in Example 3.9, we have

\[
\rho_j \circ [\rho, x] = [\rho_{\delta(j)}] \circ x = \rho_{x^{-1}(\delta(j))}
\]
so that the first statement directly follows from the property \( \text{[ii]} \) of categories of algebraic \( G \)-operators. To prove the last, take coCartesian lifts \([\hat{\rho}, x]_X : X \to X''\) of \([\rho, x]\) along \( X \) and \((\hat{\rho}_j)_{X''} : X'' \to X'_j\) of \( \rho_j \) along \( X'' \) for \( 1 \leq j \leq n \).

The computation above shows the composition \((\hat{\rho}_j)_{X''} \circ [\rho, x]_X \) is a coCartesian lift of \( \rho_{x^{-1}(\delta(j))} \), so the uniqueness of coCartesian lifts enables us to assume \( X''_j = X_{x^{-1}(\delta(j))} \) and the following diagram commutes:

\[
\begin{array}{c}
X \\
| \quad | \quad | \quad | \\
(\hat{\rho}_{x^{-1}(\delta(j)))}_X \\
| \quad | \quad | \\
X_{x^{-1}(\delta(j))} \end{array} \quad \begin{array}{c}
X'' \\
| \quad | \\
(\hat{\rho}_j)_{X''} \\
| \\
X_{x^{-1}(\delta(j))}
\end{array}
\]

(6.8)

We have two cones in \( C \) below

\[
\{ (\hat{\rho}_j)_{X'} : X' \to X_{x^{-1}(\delta(j))} \}_{j=1}^n \\
\{ (\hat{\rho}_j)_{X''} : X'' \to X_{x^{-1}(\delta(j))} \}_{j=1}^n 
\]

both of which consist of coCartesian morphisms and lie over the cone \( \{ \rho_j : \langle n \rangle \to \langle 1 \rangle \}_{j=1}^n \) in \( \tilde{E}_G \). Then, the property \( \text{[ii]} \) of categories of algebraic \( G \)-operators implies there is a unique isomorphism \( \theta : X'' \to X' \in C_{\langle n \rangle} \) such that \( (\hat{\rho}_j)_{X'} \theta = (\hat{\rho}_j)_{X''} \).

In view of the uniqueness of the morphism \([\rho, x]_X\), we obtain

\[
\theta \circ [\rho, x]_X = [\rho, x']_X.
\]

In particular, \([\rho, x]_X\) is isomorphic to a coCartesian morphism, so it is itself coCartesian.

When we endow a \( G \)-symmetric structure, it is good to have transfer.

**Lemma 6.6.** Let \( G \) be a group operad. Suppose we are given a multicategory \( M \) and a category of algebraic \( G \)-operators \( C \) together with an equivalence

\[
H : M \otimes \tilde{E}_G \xrightarrow{\sim} C
\]

in the 2-category \( \text{Oper}_G \). Then, there is a unique \( G \)-symmetric structure on \( M \) which makes \( H \) into an equivalence in \( \text{Oper}_G^{\text{alg}} \).

**Proof.** For each \( D \in \text{Oper}_G \) with \( q' : D \to \tilde{E}_G \), for \( X, Y \in D \), and for \([\varphi, x] : q'(X) \to q'(Y) \in \tilde{E}_G \), we write \( D(X, Y)_{[\varphi, x]} \) the set of morphisms of \( D \) lying over \([\varphi, x] \). Hence, in view of Theorem 4.9, we have canonical bijections

\[
M(a_1 \ldots a_n ; a) \cong (M \otimes \tilde{E}_G)(a_1 \ldots a_n, a)_{\mu_n}
\]

\[
\xrightarrow{H} C(H(a_1 \ldots a_n), H(a))_{\mu_n}.
\]

(6.9)
for objects \(a, a_i \in \mathcal{M}\). On the other hand, since \(\tilde{G}_\mathcal{G} \to \tilde{\mathcal{G}}\) is full and the identity on objects, the induced functor \(H_{\tilde{G}_\mathcal{G}} : \mathcal{M} \times \tilde{G}_\mathcal{G} \to C \times \tilde{\mathcal{G}}\) is also an equivalence in \(\mathcal{O}per'_G\) in view of Proposition \textbf{5.10}. Thus, we also have bijections

\[
(M \times \mathcal{G})(a_1 \ldots a_n; a) \cong (M \times \tilde{\mathcal{G}})(a_1 \ldots a_n, a)_{\mu_n}.
\]

(6.10)

Combining with the isomorphisms in (6.9) and (6.10), we now obtain a map

\[
\mathcal{A}_\mathcal{M} : (M \times \mathcal{G})(a_1 \ldots a_n; a) \cong (C \times \tilde{\mathcal{G}})(H(a_1 \ldots a_n), H(a))_{\mu_n}
\]

\[
\cong C(H(a_1 \ldots a_n), H(a))_{\mu_n} \cong M(a_1 \ldots a_n; a).
\]

(6.12)

Note that the internal presheaf structure \(\mathcal{A}_\mathcal{C} : C \times \tilde{\mathcal{G}} \to C\) is the identity on objects; indeed, the following diagram commutes:

\[
\begin{array}{ccc}
C \times \tilde{\mathcal{G}} & \xrightarrow{id \times \mu} & C \\
\downarrow \mathcal{A}_\mathcal{C} & & \downarrow \\
C & & C
\end{array}
\]

(6.11)

We assert that the map (6.12) gives a \(\mathcal{G}\)-symmetric structure on \(\mathcal{M}\). Notice that, the composition operation in \(\mathcal{M}\) is recovered from \(C\) as follows: for each \(a, a_i \in \text{Ob}\mathcal{M}\) and \(\bar{a}^{(i)} = a^{(i)}_1 \ldots a^{(i)}_k\), the composition operation is given by

\[
\mathcal{M}(a_1 \ldots a_n; a) \times \prod_{i=1}^n \mathcal{M}(\bar{a}^{(i)}; a_i)
\]

\[
\cong C(H(a_1 \ldots a_n), H(a))_{\mu_n} \times \prod_{i=1}^n C(H(\bar{a}^{(i)}), H(a_i))_{\mu_{\bar{a}_i}}
\]

\[
\cong C(H(\bar{a}^{(1)} \ldots \bar{a}^{(n)}), H(a))_{\mu_{\bar{a}}}
\]

(6.13)

\[
\cong \mathcal{M}(\bar{a}^{(1)} \ldots \bar{a}^{(n)}; a),
\]

where \(\bar{a}\) is the bijection in Lemma \textbf{6.4} with respect to the image by \(H\) of the standard coCartesian lifts

\[
(\bar{\rho}_i)_{\bar{a}} : \bar{a} \to a_i, \quad (\bar{\rho}_k^{(\bar{a})})_{\bar{a}(1) \ldots \bar{a}(n)} : \bar{a}(1) \ldots \bar{a}(n) \to \bar{a}^{(i)} \in \mathcal{M} \times \tilde{\mathcal{G}}.
\]

Similarly, the composition in \(\mathcal{M} \times \mathcal{G}\) is also recovered from \(C \times \tilde{\mathcal{G}}\). Moreover, thanks to the choice of the coCartesian lifts of \(\rho_i\) and \(\bar{\rho}_i^{(\bar{a})}\), the square below is commutative:

\[
\begin{array}{ccc}
\prod_{i=1}^n (C \times \tilde{\mathcal{G}})(H(\bar{a}^{(i)}), H(a_i))_{\mu_{\bar{a}_i}} & \xrightarrow{id \times \mathcal{A}_\mathcal{C}} & \prod_{i=1}^n (C(H(\bar{a}^{(i)}), H(a_i))_{\mu_{\bar{a}_i}}
\end{array}
\]

\[
\cong (C \times \tilde{\mathcal{G}})(H(\bar{a}^{(1)} \ldots \bar{a}^{(n)}), H(\bar{a}))_{\mu_{\bar{a}}}
\]

\[
\xrightarrow{\mathcal{A}_\mathcal{C}} C(H(\bar{a}^{(1)} \ldots \bar{a}^{(n)}), H(\bar{a}))_{\mu_{\bar{a}}}.
\]
This together with the functoriality of $\mathcal{A}$ implies that the map \((6.12)\) actually defines a multifunctor $\mathcal{A}_M : M \times G \to M$. It furthermore turns out that $\mathcal{A}_M$ is actually a $G$-symmetric structure on $M$; the unitality and the associativity follow from the corresponding axioms for the internal presheaf structure on $C$.

Finally, we see $H$ respects the internal presheaf structures over $\hat{\mathcal{E}}_G \Rightarrow \tilde{\mathcal{E}}_G$. In view of Theorem \([\text{4.1}]\) and the property \([\text{iii}]\) of categories of algebraic $G$-symmetric operators, it suffices to show that, for each morphism $[\varphi; f_1, \ldots, f_n; u, x] : \tilde{b} \to \tilde{a} = b_1 \ldots b_n$ in $M \wr \tilde{\mathcal{E}}_G$, we have

$$H((\tilde{\rho}_j)_{\tilde{b}}) \circ H([\varphi; \mathcal{A}_M(f_1, \delta_{1}^{(\varphi)}(u)), \ldots, \mathcal{A}_M(f_n, \delta_{n}^{(\varphi)}(u)); x]) = H(\mu) \circ \mathcal{A}_C H_{\tilde{\mathcal{E}}_G}([\varphi; f_1, \ldots, f_n; u, x])$$

for each $1 \leq j \leq n$. Let $\varphi = \mu \rho$ be the factorization with $\mu$ active and $\rho$ inert, so $(3)$ in Lemma \(2.13\) allows us to assume $u = \rho^*(\overline{\pi})$ with $\overline{\pi} \in \mathcal{D}_{\pi}$. If we put $\mu = \mu_{\tilde{\rho}}$, then the left hand side of \(6.14\) is computed as

$$H((\tilde{\rho}_j)_{\tilde{b}}) \circ H([\varphi; \mathcal{A}_M(f_1, \delta_{1}^{(\varphi)}(u)), \ldots, \mathcal{A}_M(f_n, \delta_{n}^{(\varphi)}(u)); x]) = H(\mu_{\tilde{\rho}_j}; \mathcal{A}_M(f_j, \delta_{j}^{(\mu)}(\overline{\pi})); e) \circ H(\tilde{\rho}_{\tilde{b}}; [\rho, x]_{\tilde{a}})$$

\(6.15\)

On the other hand, according to \(6.11\), for every standard coCartesian lift $[\rho', x']$ of an inert morphism in $M \wr \tilde{\mathcal{E}}_G$, one has

$$H([\rho', x']) = \mathcal{A}_C H_{\tilde{\mathcal{E}}_G}([\rho', x'])$$

here we identify $[\rho', x']$ with its image in $M \wr \tilde{\mathcal{E}}_G$ using Lemma \(5.5\). Thus, the right hand side of \(6.14\) is given by

$$H((\tilde{\rho}_j)_{\tilde{b}}) \circ \mathcal{A}_C H_{\tilde{\mathcal{E}}_G}([\varphi; f_1, \ldots, f_n; u, x]) = \mathcal{A}_C H_{\tilde{\mathcal{E}}_G}([\varphi; f_1, \ldots, f_n; u, \overline{\pi}, e] \circ [\rho, x]_{\tilde{a}}) = \mathcal{A}_C H_{\tilde{\mathcal{E}}_G}([\mu_{\tilde{\rho}_j}; f_j, \delta_{j}^{(\mu)}(\overline{\pi}); e] \circ H(\tilde{\rho}_{\tilde{b}}; [\rho, x]_{\tilde{a}}).$$

Now, \(6.15\) and \(6.16\) give rise to the equation \(6.14\), and it shows $H$ is a 1-morphism in $\text{Oper}^{G}_{\mathcal{E}}$. The uniqueness is obvious by construction. \(\square\)

**Proof of Theorem \(6.1\)** In order to show \([\text{i}]\) we construct an inverse $\Theta$ of \(6.1\). Fix $G$-symmetric multicategories $M$ and $N$. Suppose $H : M \wr \tilde{\mathcal{E}}_G \to N \wr \tilde{\mathcal{E}}_G$ is a map of algebraic $G$-operators, and take the family

$$\{\lambda_{\tilde{a}} = [\lambda_{\tilde{a}, 1}, \ldots, \lambda_{\tilde{a}, m}; e_m] \in \tilde{a} \in \tilde{a} \ldots \tilde{a}_m\}$$

of morphisms in $N \wr \tilde{\mathcal{E}}_G$ as in Lemma \(6.3\). Note that $H$ induces a functor $H : M \to N$ between underlying categories via the restriction to the fibers over $(1) \in \tilde{\mathcal{E}}_G$. On the other hand, if $f \in M(\tilde{a}; b)$ is a multimorphism, we can take a unique multimorphism $H^{O}(f) \in N(H(\tilde{a}); H(b))$ so that

$$H([\mu_{\tilde{a}}; f; e_m]) = [\mu_{\tilde{a}}; H^{O}(f); e_m].$$
We define a multifunctor $\Theta(H) : \mathcal{M} \to \mathcal{N}$ as follows: for each $a \in \text{Ob}\mathcal{M}$, we set $\Theta(H)(a) := H(a)$. For each $\vec{a} = a_1 \ldots a_m$ and each $b$, define

$$
\Theta(H) : \mathcal{M}(\vec{a}; b) \to \mathcal{N}(\overline{H}(a_1) \ldots \overline{H}(a_m); \overline{H}(b))
$$

$$
f \mapsto \gamma\mathcal{N}(H^f; \lambda_{\vec{a}, 1}, \ldots, \lambda_{\vec{a}, m}) \ .
$$

Since $\lambda_{\vec{a}, 1}$ is the identity for $a \in \mathcal{M}$, $\Theta(H)$ preserves the identities, so we show the multifunctoriality. For $f_i \in \mathcal{M}(a_1^{(i)} \ldots a_i^{(i)}; a_i)$ for $1 \leq i \leq m$, thanks to the commutative square (6.2), we have

$$
(\tilde{\rho}_i)_{H(\vec{a})} \circ H([\mu; f_1, \ldots, f_m; e]) \circ \lambda_{\vec{a}^{(1)} \ldots \vec{a}^{(m)}} = \lambda_{\vec{a}, i} \circ H((\tilde{\rho}_i)_{\vec{a}^{(i)}} \circ [\mu; f_1, \ldots, f_m; e]) \circ \lambda_{\vec{a}^{(1)} \ldots \vec{a}^{(m)}}
$$

$$
= \lambda_{\vec{a}, i} \circ H([\mu_{\vec{k}_i}; f_i; e] \circ (\tilde{\rho}_i^{(1)})_{\vec{a}^{(1)} \ldots \vec{a}^{(m)}}) \circ \lambda_{\vec{a}^{(1)} \ldots \vec{a}^{(m)}}
$$

$$
= \lambda_{\vec{a}, i} \circ [\mu_{\vec{k}_i}; H^f(f_i); e] \circ \lambda_{\vec{a}^{(i)}} \circ (\tilde{\rho}_i)_{H(f_i)} \ldots H(f_m; e)
$$

$$
= \lambda_{\vec{a}, i} \circ [\mu_{\vec{k}_i}; \Theta(H)(f_i); e]
$$

$$
= (\tilde{\rho}_i)_{H(\vec{a})} \circ \lambda_{\vec{a}, i} \circ [\mu_{\vec{k}_i}; \Theta(H)(f_i), \ldots, \Theta(H)(f_m); e],
$$

which, by virtue of the property [2] in Proposition 5.1 implies the square below is commutative:

$$
\begin{array}{c}
H(a_1^{(1)} \ldots H(a_k^{(1)}) \ldots H(a_{m}^{(m)}) \lambda_{\vec{a}^{(1)} \ldots \vec{a}^{(m)}})
\hline
\mu; \Theta(H)(f_1), \ldots, \Theta(H)(f_m); e]
\end{array}
$$

Therefore, we obtain

$$
[\mu_{\vec{k}_1, \ldots, k_m}; \gamma\mathcal{N}(\Theta(H)(f); \Theta(H)(f_1), \ldots, \Theta(H)(f_m); e)]
$$

$$
= [\mu_{\vec{k}_1, \ldots, k_m}; \Theta(H)(f); e] \circ [\mu_{\vec{k}_1}; \Theta(H)(f_1), \ldots, \Theta(H)(f_m); e]
$$

$$
= [\mu_{\vec{k}_1}; H^f(f); e] \circ \lambda_{\vec{a}} \circ [\mu_{\vec{k}_1}; \Theta(H)(f_1), \ldots, \Theta(H)(f_m); e]
$$

$$
= [\mu_{\vec{k}_1}; H^f(f); e] \circ [\mu_{\vec{k}_1}; H(f_1), \ldots, f_m; e] \circ \lambda_{\vec{a}^{(1)} \ldots \vec{a}^{(m)}}
$$

$$
= [\mu_{\vec{k}_1}; \Theta(H)(f); e] \circ \lambda_{\vec{a}^{(1)} \ldots \vec{a}^{(m)}}
$$

$$
= [\mu_{\vec{k}_1}; \Theta(H)(f_1), \ldots, f_m; e] \circ \lambda_{\vec{a}^{(1)} \ldots \vec{a}^{(m)}}
$$

$$
= [\mu_{\vec{k}_1}; \Theta(H)(f_1), \ldots, f_m; e] \circ \lambda_{\vec{a}^{(1)} \ldots \vec{a}^{(m)}}
$$

and the multifunctoriality of $\Theta(H)$ follows. Furthermore, $\Theta(H) : \mathcal{M} \to \mathcal{N}$ is $G$-symmetric. To see this, notice that, for $f \in \mathcal{M}(x, \vec{a}, b)$, we have

$$
H([\mu_m; f; x]) = [\mu_m; H^f(f); e] \circ H(\bar{x}_{\vec{a}})
$$

$$
= [\mu_m; H^f(f); e] \circ \lambda_{x, \vec{a}} \circ \bar{x}_{H(\vec{a})} \circ \lambda_{\vec{a}}^{-1}
$$

$$
= [\mu_m; \Theta(H)(f); x] \circ \lambda_{\vec{a}}^{-1}
$$

$$
= \left[\mu_m; \gamma\mathcal{N}(\Theta(H)(f); \lambda_{\vec{a}, x-1(1)}, \ldots, \lambda_{\vec{a}, x-1(m)}) \circ x\right].
$$

It follows that, for the induced functor $H_\mathcal{G} : \mathcal{M} \times \mathcal{G} \to \mathcal{N} \times \mathcal{G}$, we have

$$
H_\mathcal{G}([\mu_m; f; x, e_m]) = \left[\mu_m; \gamma\mathcal{N}(\Theta(H)(f); \lambda_{\vec{a}, x-1(1)}^{-1}, \ldots, \lambda_{\vec{a}, x-1(m)}^{-1}) \circ x, e_m\right].
$$
Since $H$ is a map of internal presheaves, we obtain

$$H([\mu_m; f^x; e_m]) = \left[\mu_m; \gamma N \left(\Theta(H)(f); \lambda_{\tilde{a},x-1(1)}, \ldots, \lambda_{\tilde{a},x-1(m)}\right)^x; e_m\right]$$

$$= \left[\mu_m; \gamma N \left(\Theta(H)(f)^x; \lambda_{\tilde{a},1}, \ldots, \lambda_{\tilde{a},m}\right); e_m\right]$$

$$= [\mu_m; \Theta(H)(f)^x; e_m] \circ \lambda_{\tilde{a}}^{-1}$$

and so $\Theta(H)(f^x) = \Theta(H)(f)^x$.

We extend $\Theta$ to an actual functor

$$\Theta : \text{Oper}_{\mathcal{G}}^\text{alg}(\mathcal{M} \downarrow \tilde{E}_{\mathcal{G}}, \mathcal{N} \downarrow \tilde{E}_{\mathcal{G}}) \to \text{MultCat}_{\mathcal{G}}(\mathcal{M}, \mathcal{N})$$

as follows: note that, in view of Lemma 4.1, for 1-morphisms $H, K : \mathcal{M} \downarrow \tilde{E}_{\mathcal{G}} \to \mathcal{N} \downarrow \tilde{E}_{\mathcal{G}} \in \text{Oper}_{\mathcal{G}}^\text{alg}$, a 2-morphism $\xi : H \to K$ is nothing but a natural transformation over $\tilde{E}_{\mathcal{G}}$. We set

$$\Theta(\xi) := \{\xi_a : H(a) \to K(a)\}_{a \in \mathcal{M}}.$$  \hspace{1cm} (6.18)

To see (6.18) forms a multinatural transformation $\Theta(H) \to \Theta(K)$, notice that, for each $\tilde{a} = a_1 \ldots a_n \in \mathcal{M} \downarrow \tilde{E}_{\mathcal{G}}$, the naturality of $\xi$ implies the square

$$\begin{array}{ccc}
H(a_1 \ldots a_n) & \xrightarrow{\xi_{\tilde{a}}} & K(a_1 \ldots a_n) \\
\downarrow H(\tilde{a}, a) & & \downarrow K((\tilde{a}, a)) \\
H(a_i) & \xrightarrow{\xi_{a_i}} & K(a_i)
\end{array}$$

is commutative for each $1 \leq i \leq n$. Computing the compositions, one obtains

$$\xi_{\tilde{a}} = \lambda_{\tilde{a}}^{(K)} \circ [\id; \xi_{a_1}, \ldots, \xi_{a_n}; e_n] \circ \lambda_{\tilde{a}}^{(H)^{-1}},$$

where $\lambda^{(H)}$ and $\lambda^{(K)}$ are the ones in Lemma 6.3 for functors $H$ and $K$ respectively. Then, the multinaturality of (6.18) is straightforward.

We verify $\Theta$ is actually an inverse to the functor 6.1. If $F : \mathcal{M} \to \mathcal{N}$ is a $\mathcal{G}$-symmetric multifunctor, then the $\mathcal{G}$-symmetric multifunctor $\Theta(F^\mathcal{G})$ is exactly $F$ itself since $\lambda_{\tilde{a}}$ is trivial in this case. On the other hand, in view of (6.2) and (6.17), the family $\lambda = \{\lambda_{\tilde{a}}\}_{\tilde{a}}$ forms a natural isomorphism $\Theta(H)^{\mathcal{G}} \cong H$. The uniqueness of $\lambda$ implies it is natural with respect to $H$. Hence, (6.1) is an equivalence of categories, and we have finished the proof of the part (1).

Finally, we show the part (2). Let $q : \mathcal{C} \to \tilde{E}_{\mathcal{G}}$ be a category of algebraic $\mathcal{G}$-operators. By virtue of Lemma 6.9 in order to see $\mathcal{C}$ lies in the essential image of $(-) \downarrow \tilde{E}_{\mathcal{G}}$, it suffices to show there is a multicategory $\mathcal{M}$ together with an equivalence $\mathcal{M} \downarrow \tilde{E}_{\mathcal{G}} \cong \mathcal{C}$ in the 2-category $\text{Oper}_{\mathcal{G}}^\text{alg}$. For each finite word $\tilde{W} = W_1 \ldots W_n$ of objects in $\mathcal{C}$ with, say, $q(W_i) = \langle \langle k_i \rangle \rangle$, the property (iii) of categories of algebraic $\mathcal{G}$-operators allows us to take an object $\varpi(\tilde{W}) \in \mathcal{C}(\langle k_1 + \cdots + k_n \rangle)$ together with coCartesian morphisms

$$(\rho_i^{(\tilde{a})})_{\tilde{W}} : \varpi(\tilde{W}) \to W_i$$

covering the inert morphism $\rho_i^{(\tilde{a})} : \langle k_1 + \cdots + k_n \rangle \to \langle k_i \rangle \in \nabla$. In the following argument, we fix such data for each $\tilde{W}$. Note that the coincidence
of the symbol \( \varpi \) here and in Lemma 6.4 is intentional; for \( V = V_1 \ldots V_n \) with \( V_i \in C \) and \( q(V_i) = \langle l_i \rangle \), and for active morphisms \( \nu_i: \langle k_i \rangle \to \langle l_i \rangle \in \nabla \), we have a map

\[
\varpi : \prod_{i=1}^n C(W_i, V_i)_{\nu_i} \to C(\varpi(W_1 \ldots W_n), \varpi(V_1 \ldots V_n))_{\nu_1 \ldots \nu_n}
\]

so that

\[
(\tilde{\rho}_i^{(1)})_\psi \circ \varpi(f_1, \ldots, f_n) = f_i \circ (\tilde{\rho}_i^{(1)})_{\tilde{W}}.
\]

Put \( \mathcal{C} := C_{\langle 1 \rangle} \). Note that, for \( X^{(i)} = X_1^{(i)} \ldots X_{k_i}^{(i)} \) with \( X_j^{(i)} \in \mathcal{C} \), Lemma 6.5 asserts that there is a unique isomorphism

\[
\theta : \varpi(\varpi(X^{(1)}), \ldots, \varpi(X^{(n)})) \cong \varpi(X^{(1)} \ldots X^{(n)}) \in C_{\langle k_1 + \cdots + k_n \rangle}
\]

which makes the square below commutes:

\[
\begin{array}{ccc}
\varpi(\varpi(X^{(1)}), \ldots, \varpi(X^{(n)})) & \xrightarrow{\theta} & \varpi(X^{(1)} \ldots X^{(n)}) \\
\varpi(\varpi(X^{(i)})) & \xrightarrow{\tilde{\rho}_i^{(1)}} & X_j^{(i)} \\
\end{array}
\]

The property (ii) also implies that, for morphisms \( f_j^{(i)} : X_j^{(i)} \to Y_j^{(i)} \in \mathcal{C} \) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq k_n \), the following square is also commutative:

\[
\begin{array}{ccc}
\varpi(\varpi(X^{(1)}), \ldots, \varpi(X^{(n)})) & \xrightarrow{\theta} & \varpi(X^{(1)} \ldots X^{(n)}) \\
\varpi(\varpi(X^{(1)}), \ldots, \varpi(X^{(n)})) & \xrightarrow{\varpi(f_1^{(1)}, \ldots, f_1^{(1)}, \ldots, f_n^{(n)})} & \varpi(Y^{(1)} \ldots Y^{(n)}) \\
\end{array}
\]

(6.19)

In addition, the uniqueness of \( \theta \) guarantees that it makes the diagram below commute:

\[
\begin{array}{ccc}
\varpi(\varpi(X^{(1,1)}), \ldots, \varpi(X^{(1,r_1)})) & \xrightarrow{\varpi(\varpi(X^{(1,1)}), \ldots, \varpi(X^{(1,r_1)}))} & \varpi(\varpi(X^{(n,1)}), \ldots, \varpi(X^{(n,r_n)})) \\
\varpi(X^{(1,1)} \ldots \varpi(X^{(1,r_1)})) & \xrightarrow{\varpi(\varpi(X^{(1,1)}), \ldots, \varpi(X^{(1,r_1)}))} & \varpi(X^{(n,1)} \ldots \varpi(X^{(n,r_n)})) \\
\end{array}
\]

\[
\begin{array}{ccc}
\varpi(\varpi(X^{(1,1)}), \ldots, \varpi(X^{(1,r_1)})) & \xrightarrow{\varpi(\varpi(X^{(1,1)}), \ldots, \varpi(X^{(1,r_1)}))} & \varpi(\varpi(X^{(n,1)}), \ldots, \varpi(X^{(n,r_n)})) \\
\varpi(X^{(1,1)} \ldots \varpi(X^{(1,r_1)})) & \xrightarrow{\varpi(\varpi(X^{(1,1)}), \ldots, \varpi(X^{(1,r_1)}))} & \varpi(X^{(n,1)} \ldots \varpi(X^{(n,r_n)})) \\
\end{array}
\]

(6.20)

We now define a multicategory \( \mathcal{M}_C \) so that

- objects are those in \( \mathcal{C} \);
- for \( X, X_1, \ldots, X_m \in \mathcal{C} \), we set

\[
\mathcal{M}_C(X_1 \ldots X_m; X) := C(\varpi(X_1 \ldots X_m), X)_{\mu_m}.
\]
the composition is given by

\[ \gamma : \mathcal{M}(X_1 \ldots X_n; X) \times \prod_{i=1}^{n} \mathcal{M}(X_i; X_i) \]

\[ \cong C(\varpi(X_1 \ldots X_n), X)_{\mu_n} \times \prod_{i=1}^{n} C(\varpi(X_i), X_i)_{\mu_i} \]

\[ \mapsto C(\varpi(X_1 \ldots X_n), X)_{\mu_n} \times C(\varpi(X_1) \ldots \varpi(X_n)), \varpi(X_1 \ldots X_n)_{\mu_{k_1} + \ldots + k_n} \]

\[ \theta^{-1} \mapsto C(\varpi(X_1) \ldots \varpi(X_n)), X)_{\mu_{k_1} + \ldots + k_n} ; \]

in other words, we have

\[ \gamma(f; f_1, \ldots, f_n) = f \circ \varpi(f_1, \ldots, f_n) \circ \theta^{-1} . \]

The associativity of the composition is verified as follows: take morphisms \( f \in \mathcal{M}(X_1 \ldots X_n; X) \), \( f_i \in \mathcal{M}(X_i^{(1)} \ldots X_i^{(k_i)}; X_i) \), and \( f_j \in \mathcal{M}(X_j^{(1)}; X_j^{(l)}) \) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq k_i \). Then, thanks to the commutative squares \( (6.19) \), \( (6.20) \), we have

\[ \gamma(f; f_1; f_1^{(1)}, \ldots, f_1^{(k_1)}), \ldots, \gamma(f_n; f_n^{(1)}, \ldots, f_n^{(k_n)}) \]

\[ = f \circ \varpi(f_1 \circ \varpi(f_1^{(1)} \ldots f_1^{(k_1)}), \ldots, f_n \circ \varpi(f_n^{(1)}, \ldots, f_n^{(k_n)})) \circ \theta^{-1} \circ \theta^{-1} \]

\[ = f \circ \varpi(f_1, \ldots, f_n) \circ \varpi(f_1^{(1)}, \ldots, f_1^{(k_1)}), \ldots, \varpi(f_n^{(1)}, \ldots, f_n^{(k_n)})) \circ \varpi(\theta, \ldots, \theta)^{-1} \circ \theta^{-1} \]

\[ = \gamma(f; f_1, \ldots, f_n) \circ \varpi(f_1^{(1)}, \ldots, f_1^{(k_1)}), \ldots, \varpi(f_n^{(1)}, \ldots, f_n^{(k_n)})) \circ \theta^{-1} \]

\[ = \gamma(\gamma(f; f_1, \ldots, f_n); f_1^{(1)}, \ldots, f_1^{(k_1)}, \ldots, f_n^{(1)}, \ldots, f_n^{(k_n)}) , \]

which implies the associativity of the composition. The unitality is obvious so that \( \mathcal{M} \) is actually a multicategory.

We define a functor \( P : \mathcal{M} \otimes \mathcal{E}_G \rightarrow \mathcal{C} \) as follows: for each object \( X_1 \ldots X_m \in \mathcal{M} \otimes \mathcal{E}_G \), put

\[ P(X_1 \ldots X_m) := \varpi(X_1 \ldots X_m) . \]

As for a morphism \([\varphi; f_1, \ldots, f_n; x] : X_1 \ldots X_m \rightarrow Y_1 \ldots Y_n\), taking the factorization \( \varphi = \mu \rho \) with \( \mu \) active and \( \rho \) inert, we set

\[ P([\varphi; f_1, \ldots, f_n; x]) := \varpi(f_1, \ldots, f_n) \circ \theta^{-1} \circ [\rho, x]_{X_1 \ldots X_m} , \] (6.21)

where \([\rho, x]_{X_1 \ldots X_m} : \varpi(X_1 \ldots X_m) \rightarrow \varpi(X_{x^{-1}(1)} \ldots X_{x^{-1}(m)})\) is the coCartesian lift of \([\rho, x]\) given in Lemma \( (6.5) \). The functoriality of \( P \) directly follows from the uniqueness of each morphisms in the right hand side of \( (6.21) \). It is clear that \( P \) preserves coCartesian lifts of inert morphisms, so \( P \) is a 1-morphism in \( \text{Oper}_G' \). In addition, the property \([\text{iii}]\) of categories of algebraic \( G \)-operators implies \( P \) is essentially surjective. Hence, in order to see \( P \) is an equivalence, it remains to show it is fully faithful. Consider an arbitrary morphism in \( \mathcal{C} \) of the form

\[ h : \varpi(X_1 \ldots X_m) \rightarrow \varpi(Y_1 \ldots Y_n) \]
covering \([\varphi, x] : \langle\langle m \rangle\rangle \to \langle\langle n \rangle\rangle \in \bar{E}_G\). If \(\varphi = \mu \rho\) is the factorization with \(\mu\) active and \(\rho\) inert, say \(\delta\) is the section of \(\rho\), the universal property of the coCartesian morphism \([\rho, x]_{X_1 \ldots X_m}\) given in Lemma 6.5 implies that there is a unique morphism \(h' : \varpi(X_{x^{-1}(\delta(1))} \ldots X_{x^{-1}(\delta(m))}) \to \varpi(Y_1 \ldots Y_n) \in \mathcal{C}\) covering \(\mu\) such that

\[ h = h' \circ [\rho, x]_{X_1 \ldots X_m} \]

In view of Lemma 6.4, \(h'\) can be uniquely written as \(h' = \varpi(h_1, \ldots, h_m)\). This implies \(P\) is fully faithful, and this completes the proof.

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