ON THE EXISTENCE OF OVERCOMPLETE SETS IN SOME CLASSICAL NONSEPARABLE BANACH SPACES

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Abstract. For a Banach space $X$ its subset $Y \subseteq X$ is called overcomplete if $|Y| = \text{dens}(X)$ and $Z$ is linearly dense in $X$ for every $Z \subseteq Y$ with $|Z| = |Y|$. In the context of nonseparable Banach spaces this notion was introduced recently by T. Russo and J. Somaglia but overcomplete sets have been considered in separable Banach spaces since the 1950ties.

We prove some absolute and consistency results concerning the existence and the nonexistence of overcomplete sets in some classical nonseparable Banach spaces. For example: $c_0(\omega_1)$, $C([0, \omega_1])$, $L_1(\{0, 1\}^{\omega_1})$, $\ell_p(\omega_1)$, $L_p(\{0, 1\}^{\omega_1})$ for $p \in (1, \infty)$ or in general WLD Banach spaces of density $\omega_1$ admit overcomplete sets (in ZFC). The spaces $\ell_\infty$, $\ell_\infty/c_0$, spaces of the form $C(K)$ for $K$ extremally disconnected, superspaces of $l_1(\omega_1)$ of density $\omega_1$ do not admit overcomplete sets (in ZFC). Whether the Johnson-Lindenstrauss space generated in $\ell_\infty$ by $c_0$ and the characteristic functions of elements of an almost disjoint family of subsets of $\mathbb{N}$ of cardinality $\omega_1$ admits an overcomplete set is undecidable. The same refers to all nonseparable Banach spaces with the dual balls of density $\omega_1$ which are separable in the weak* topology. The results proved refer to wider classes of Banach spaces but several natural open questions remain open.

1. Introduction

All Banach spaces considered in this paper are infinite dimensional and over the reals. The density $\text{dens}(X)$ of a Banach space $X$ is the minimal cardinality of a norm dense subset of $X$. Other terminology and notation used in the introduction can be found in Section 2.1.

Definition 1 ([37]). Let $X$ be an infinite dimensional Banach space. A set $Y \subseteq X$ is called overcomplete (in $X$) if $|Y| = \text{dens}(X)$ and $Z$ is linearly dense in $X$ for every $Z \subseteq Y$ with $|Z| = |Y|$.

Overcomplete sets have been investigated for separable Banach spaces as overcomplete sequences, hypercomplete sequences or overfilling sequences. They appear in research related to basic sequences in general Banach spaces (e.g. [48, 45, 32, 39]). In particular, Terenzi has proved in [48] a remarkable dichotomy concerning sequences with no basic subsequences which involves sequences overcomplete in their span. The structure of overcomplete sequences has been investigated in [44]. For other aspects of overcomplete sequences see [11, 10, 14, 17]. The following existence, nonexistence and consistency results have been obtained so far:

The author would like to thank the anonymous referee for many helpful suggestions improving the presentation and for the idea of how to fix a gap in the previous version of the proof of Proposition 18.

The author would like to thank G. Plebanek for helpful comments.
Every separable Banach space admits an overcomplete set ([25]).

\((\mathsf{CH})\) \(X\) admits an overcomplete set if \(\text{dens}(X^*) = \omega_1\) ([37]).

A Banach space \(X\) does not admit an overcomplete set if

- \(X = \ell_1(\omega_1)\) ([37]),
- \(\text{cf}(\text{dens}(X)) > \omega\) ([37]),
- \(\text{dens}(X) > \omega_1\) and \(X\) admits a fundamental biorthogonal system, in particular, if \(X\) admits a Markushevich basis or is WLD ([37]).

\((\neg \mathsf{CH})\) \(\ell_\infty\) does not admit an overcomplete set ([37]).

The purpose of this paper is to present further existence and nonexistence results which can be divided into positive, negative, consistency and independence results.

Among positive results in Theorem 11 we prove in ZFC (i.e., without any extra set-theoretic assumptions) that the following Banach spaces admit overcomplete sets:

- Every WLD Banach space of density \(\omega_1\), in particular
  1. \(\ell_p(\omega_1), L_p(\{0, 1\}^{\omega_1})\) for \(p \in (1, \infty)\).
  2. \(L_1(\{0, 1\}^{\omega_1})\).
  3. \(c_0(\omega_1)\).
  4. \(C(K)\)s for \(K\) a Corson compact where all Radon measure have separable supports.
- \(C([0, \omega_1])\).
- \(C(K)\), where \(K\) is the one point compactification of a refinement of the order topology on \([0, \omega_1)\) obtained by isolating all points of some subset of \([0, \omega_1)\).

Note that these are the first results showing the existence in ZFC of nonseparable Banach spaces admitting overcomplete sets. Also it follows that it is consistent that there are Banach spaces \(X\) with \(\text{dens}(X^*) > \omega_1\) which admit overcomplete sets. Namely assuming the negation of CH, consider \(L_1(\{0, 1\}^{\omega_1})\) or \(\ell_2(\omega_1) \oplus \ell_1\). On the other hand we extend the list from [37] of Banach spaces which do not admit overcomplete sets in ZFC (i.e., without any extra set-theoretic assumptions) and include there the following:

- \(Xs\) of density \(\omega_1\) which contain \(\ell_1(\omega_1)\) (Theorem 29).
- \(C(K)s\) for \(K\) compact, Hausdorff, infinite and extremally disconnected (Theorem 29).
- \(\ell_\infty, \ell_\infty/c_0, L_\infty(\{0, 1\}^{\omega_1})\) (Theorem 29).
- \(C([0, 1]^{\omega_1}), C(\{0, 1\}^{\omega_1})\), (Theorem 29).
- \(C(K)\) which is Grothendieck space of density \(\omega_1\) (Theorem 30).
- Banach space \(X\) of density \(\kappa\), where \(\kappa\) is a cardinal satisfying \(\text{cf}(\kappa) > \omega_1\) and
  - \(X\) contains an isomorphic copy of \(\ell_1(\omega_1)\) (Theorem 37).
  - \(X^*\) contains a nonseparable WLD subspace (Theorem 38).
  - \(X\) is a nonreflexive Grothendieck space (Theorem 39).
  - \(X = C(K)\) for \(K\) compact, Hausdorff and scattered (Theorem 40).

It should be noted that the nonexistence of overcomplete sets in Banach spaces \(X\) which contain \(\ell_1(\text{dens}(X))\) for \(\text{dens}(X) \geq \omega_2\) can be directly concluded from

\footnote{Recall that ZFC is a standard axiomatization of mathematics (see e.g. [21, 26]). A statement \(\phi\) is said to be consistent (with ZFC) if adding it to ZFC does not lead to a contradiction and is said to be independent from ZFC or undecidable if both \(\phi\) and the negation of \(\phi\) are consistent.}
Theorem 3.6 of of [37] (see the remarks after the proof of Theorem 29)). This argument covers all Banach spaces of the form $C(K)$ for $K$ Hausdorff, compact, infinite and extremally disconnected of densities bigger or equal to $\omega_2$, $C([0,1]^\kappa)$, $C([0,1]^\kappa)$ for $\kappa \geq \omega_2$, $\ell_\infty(\lambda)$, $\ell_\infty(\lambda)/c_0(\lambda)$, $L_\infty([0,1]^\lambda)$ for $\lambda \geq \omega_1$. Note, for example, that the case of $\ell_\infty$ having the density equal to continuum does not follow from Theorem 3.6 of of [37] if CH holds. Our Theorem 29 has a uniform proof for all densities of uncountable cofinality, in particular for the density of continuum whose value depend on CH.

We also obtain the following consistency results:

- (MA+$\neg$CH) A Banach space $X$ does not admit an overcomplete set if
  - $\omega < \text{cf}(\text{dens}(X)) \leq \text{dens}(X) < c$ and $B_{X^*}$ is separable in the weak* topology (Theorem 19).
  - $\text{dens}(X) = \omega_1$ and $B_{X^*}$ is not monolithic in the weak* topology (Theorem 20).

- It is consistent with MA for partial orders having precaliber $\omega_1$ and the negation of CH that every Banach space whose dual has density $\omega_1$ admits an overcomplete set (Theorem 29).

- The statement that every Banach space whose dual has density $\omega_1$ admits an overcomplete set is consistent with any size of the continuum (Theorem 24).

- ($p = c > \omega_1$) No nonreflexive Grothendieck space of regular density (in particular equal to $c$ under the above assumption $p = c$) admits an overcomplete set (Corollary 34).

Based on the above we conclude a couple of independence results:

- The existence of overcomplete sets is independent for all Banach spaces $X$ satisfying: $\text{dens}(X) = \text{dens}(X^*) = \omega_1$ and $B_{X^*}$ is not monolithic in the weak* topology, in particular such that $B_{X^*}$ is weakly* separable.

- The existence of a Banach space $X$ admitting an overcomplete set and satisfying: $\text{dens}(X) = \omega_1$ and $L_1([0,1]^{\omega_1}) \subseteq X^*$ is independent (Corollary 33).

A classical example of Johnson and Lindenstrauss of a Banach space $X$ satisfying: $\text{dens}(X) = \text{dens}(X^*) = \omega_1$ and $B_{X^*}$ is weakly* separable is the Banach space generated in $\ell_\infty$ by $c_0$ and $\{1_A : A \in \mathcal{A}\}$, where $\mathcal{A}$ is an almost disjoint family of subsets of $\mathbb{N}$ of cardinality $\omega_1$. So in particular, the existence of overcomplete sets in such spaces is independent by Corollary 21 and the CH result of 37.

Corollaries of the above results include:

- A WLD Banach space $X$ admits an overcomplete set if and only if the density of $X$ is less or equal to $\omega_1$ (Corollary 12).

- A Banach space $X$ of density $\omega_1$ with an unconditional basis admits an overcomplete set if and only if $X$ is WLD (Corollary 33).

- If $X$ is a Banach spaces such that $\text{cf}(\text{dens}(X)) > \omega$, $\text{dens}(X) > \omega_1$ and $L_1([0,1]^{\text{dens}(X)}) \subseteq X^*$, then $X$ does not admit an overcomplete set (Corollary 32).

- If $\kappa$ is an infinite cardinal, then $C([0,\kappa])$ admits an overcomplete set if and only if $\kappa \leq \omega_1$ (Theorem 11 (2) and result of 37 i.e., Theorem 5 (3) in this paper).
We explain briefly the structure of the paper and the methods used. In Section 2 we establish terminology, remind known results and prove some general facts. Section 3 is devoted to positive results. They are obtained in Theorem [11] which is proved by stepping-up the original construction of Klee with the help of a sequence of coherent injections from countable ordinals into \( \mathbb{N} \).

Section 4 contains consistency results involving Martin’s axiom and simple finite support iterations as well as the Cohen model. The main ingredient is Proposition [18] where it is proved under \( \text{MA}+\neg \text{CH} \) that if \( D = \{ x_\xi : \xi < \kappa \} \subseteq X \), where \( X \) is a Banach space with weakly\(^*\) separable dual ball and \( \kappa < \omega \) and \( x_\xi \notin \overline{\text{lin}}\{x_\eta : \eta < \xi \} \) for any \( \xi < \kappa \), then \( D \) can be covered by countably many hyperplanes. We need the hypothesis on \( D \) as in ZFC in any separable \( X \) there is \( D \subseteq X \) of cardinality \( \omega_1 \) which cannot be covered by countably many hyperplanes. Indeed, using the original method of Klee (see the proof of Theorem [3]) in a separable Banach space one can construct a set of cardinality \( \omega \) where every infinite subset is linearly dense.

Section 5 is devoted to negative results which follow from the existence of linearly independent functionals \( \phi, \psi \in X^* \) which assume single values \( r, s \in \mathbb{R} \) on big subsets of a given linearly dense set. Then \( s\phi - r\psi \) defines a hyperplane including a big subset of a linearly dense set. This is Lemma [20] which is the main tool of that section. Its hypothesis is that the dual sphere \( S_X^* \) has many points of character (with respect to the weak\(^*\) topology) equal to the density of \( X \). Characters of functionals as points have nice interpretations for \( C(K) \) spaces as types of uniform regularity of Radon measures ([34], [27]). In fact our proof of Lemma [20] is inspired by the methods of [27]. To make the main conclusions in Theorem [29] we need a dense range linear operator from the space into a space where all characters are big, this is achieved in Lemma [28] in Section 6. We use counting arguments (e.g. like in Lemma [3]) to obtain negative results for Banach spaces \( X \) such that \( \text{cf}(\text{dens}(X)) > \omega_1 \). The last section discusses unanswered questions. Some of them are quite fundamental.

2. Preliminaries

2.1. Notation and terminology. The notation and terminology should be fairly standard.

\( f|A \) denotes the restriction of a function \( f \) to the set \( A \). \( 1_A \) will denote the characteristic function of a set \( A \) (relative to some superset given in the context). \( \mathbb{N} \) stands for the non-negative integers. Sometimes \( n \in \mathbb{N} \) is identified with the set \( \{0, \ldots, n-1\} \). For \( n \in \mathbb{N} \) by \( \omega_n \) we denote the \( n \)-th infinite cardinal, \( \varepsilon \) stands for the cardinality of the continuum, i.e., \( \mathbb{R} \). \( \text{cf}(\xi) \) denotes the cofinality of an ordinal \( \xi \). \( \mathbb{R} \) denotes the reals, \( \mathbb{Q} \) denotes the rationals and \( \mathbb{Q}_+ \) denotes the positive rationals. For a set \( A \) by \( [A]^2 \) we mean the collection of all two-element subsets of \( A \).

All Banach spaces considered in this paper are infinite dimensional and over the reals. \( X^* \) stands for the dual space of \( X \). \( B_X \) and \( S_X \) stand for the unit ball and the unit sphere in \( X \) respectively. \( \text{lin}(X) \) denotes the linear span of \( X \) and \( \overline{\text{lin}}(X) \) its closure. \( \ker(x^*) \) is the kernel of \( x^* \in X^* \). The density \( \text{dens}(X) \) of a Banach space \( X \) is the minimal cardinality of a norm dense subset of \( X \). We write \( X \equiv Y \) if Banach spaces \( X \) and \( Y \) are isometrically isomorphic. If \( X \) is a Banach space and \( I \) a set, then \( (x_i, x^*_i)_{i \in I} \) is called biorthogonal if \( x^*_i(x_j) = \delta_{i,j} \) for any \( i, j \in I \). Such a biorthogonal system is called fundamental if \( \{x_i : i \in I\} \) is linearly dense in \( X \) and it is called total if the span of \( \{x_i : i \in I\} \) is dense in the weak\(^*\) topology in
$X^*$. Moreover, it is called a Markushevich basis if it is both fundamental and total. Recall that a Banach space $X$ is called injective if given any pair of Banach spaces $Y \subseteq Z$ and any linear bounded operator $T : Y \to X$ there is $S : Z \to X$ which extends $T$. Examples of injective Banach spaces are spaces of the form $C(K)$ for $K$ extremally disconnected, which are exactly the Stone spaces of complete Boolean algebras (see e.g. [38]).

For a compact Hausdorff space $K$ by $C(K)$ we mean the Banach space of real-valued continuous functions on $K$ with the supremum norm. For $x \in K$ an element $\delta_x \in C(K)^*$ is given by $\delta_x(f) = f(x)$ for all $f \in C(K)$. All topological spaces considered in the paper are Hausdorff. $\chi(x, X)$ is the character of a point $x$ in the space $X$, i.e., the minimal cardinality of a neighbourhood base at $x$. The pseudocharacter of a point in the space $X$ is the minimal cardinality of a family of open sets whose intersection is \{x\}. It is well known that the pseudocharacter of a point in a compact Hausdorff space is equal to its character. $\text{Clop}(K)$ stands for the Boolean algebra of clopen subsets of a space $K$.

A hyperplane is a one-codimensional subspace. By $L_p(\{0, 1\}^\kappa)$ for $p \in [1, \infty]$ and $\kappa$ a cardinal we mean $L_p(\mu)$, where $\mu$ is the homogeneous probability product measure on $\{0, 1\}^\kappa$. The class of WLD (weakly Lindelöf determined) Banach spaces has many nice characterizations, the most convenient for this paper is the one as the class of Banach spaces $X$ which admit a linearly dense set $D \subseteq X$ such that \{d \in D : x^*(d) \neq 0\} is countable for each $x^* \in X^*$ ([17]). $X$ is a Grothendieck Banach space if and only in $X^*$ weakly* convergent sequences coincide with weakly convergent sequences.

The continuum hypothesis abbreviated as CH denotes the statement ‘$\aleph = \omega_1$”. The terminology concerning Martin’s axiom, dense sets, filters in partial orders and forcing can be found in [28]. Definitions of cardinal invariants like $p$, $\text{add}$, $\text{cov}$, etc., and the information on the Cichoń and the van Douwen diagrams can be found in [8]. A subset $A$ of a partial order $\mathbb{P}$ is said to be centred if for every finite $B \subseteq A$ there is $p \in \mathbb{P}$ such that $p \leq q$ for every $q \in B$. A partial order $\mathbb{P}$ is said to have precaliber $\omega_1$ if given an uncountable $A \subseteq \mathbb{P}$ there is a centred and uncountable $B \subseteq A$.

2.2. Some previous results. The following two simple lemmas were implicitly used in [37].

**Lemma 2.** Suppose that $X$ and $Y$ are two Banach spaces of the same density and $T : X \to Y$ is a bounded linear operator whose range is dense in $Y$. If $Y$ does not admit an overcomplete set, then $X$ does not admit an overcomplete set.

**Proof.** Let $\kappa$ be a cardinal such that the densities of $X$ and of $Y$ are $\kappa$. Suppose that $D = \{d_\xi : \xi < \kappa\}$ is an overcomplete set in $X$. Let $A \subseteq \kappa$ be of cardinality $\kappa$, and let $y \in Y$ and $\varepsilon > 0$. There is $x \in X$ such that $\|T(x) - y\| < \varepsilon/2$. Since $D = \{d_\xi : \xi < \kappa\}$ is overcomplete, there is a finite linear combination $x' \in X$ of elements of $\{d_\xi : \xi \in A\}$ such that $\|x' - x\| < \varepsilon/2\|T\|$. So there is a finite linear combination $y' = T(x')$ of elements of $\{T(d_\xi) : \xi \in A\}$ satisfying $\|y' - y\| \leq \|y' - T(x')\| + \|T(x') - y\| < \varepsilon$. This shows that every subset of $T[D]$ of cardinality $\kappa$ is dense in $Y$. Since the density of $Y$ is $\kappa$ we conclude that $T[D]$ is overcomplete in $Y$.

$\square$
Theorem 5. Suppose that \( \lambda < cf(\kappa) \) are uncountable cardinals and \( X \) is a Banach space of density \( \kappa \) such that \( X = \bigcup_{\xi < \lambda} X_\xi \), where \( X_\xi \)s are proper closed subspaces of \( X \). Then \( X \) does not admit an overcomplete set.

Proof. Suppose that \( D \subseteq X \) has cardinality \( \kappa \). As \( D = \bigcup_{\xi < \lambda} (D \cap X_\xi) \) and \( \lambda < cf(\kappa) \), there is \( \xi < \lambda \) such that \( D \cap X_\xi \subseteq X_\xi \) has cardinality \( \kappa \). As \( X_\xi \) is a proper closed subspace of \( X \), the set \( D \) is not overcomplete in \( X \). \( \square \)

Theorem 4 (\cite{25}). Suppose that \( X \) is a Banach space, \( \emptyset \neq B \subseteq \mathbb{N} \), \( \{x_n : n \in B\} \subseteq X \) consists of norm one vectors and \( \lambda_k \) for \( k \in \mathbb{N} \) are distinct elements of the interval \((0, 1/2)\). Let \( y_k = \sum_{n \in B} \lambda_n^y x_n \) for each \( k \in \mathbb{N} \). Then for every infinite \( C \subseteq \mathbb{N} \) we have \( \lim (\{y_k : k \in C\}) = \lim (\{x_n : n \in B\}) \). Consequently every infinite dimensional separable Banach space admits an overcomplete set.

Proof. Let \( x^* \) be a norm one linear bounded functional on \( \text{lin}(\{x_n : n \in B\}) \). It is enough to show that there is \( k \in C \) such that \( x^*(y_k) \neq 0 \). Define \( \sigma_n = x^*(x_n) \) for \( n \in B \) and \( \sigma_n = 0 \) for \( n \in \mathbb{N} \setminus B \). We have \( \lim \sup_{n \to \infty} \sqrt{\sigma_n} \leq \sup_{n \in B} \sqrt{|x^*(x_n)|} \leq 1 \) and so the formula

\[
f(\lambda) = \sum_{n \in B} x^*(x_n)\lambda_n^y
\]

defines an analytic function on \((-1, 1)\). This function is zero on \((-1, 1)\) only if \( x^*(x_n) = 0 \) for each \( n \in B \), which is not the case since \( x^* \) is not the zero functional on \( \text{lin}(\{x_n : n \in B\}) \). So \( f \) cannot have infinitely many zeros in \((0, 1/2)\), which means that for some \( k \in C \) we have \( 0 \neq f(\lambda_k) = x^*(\sum_{n \in B} \lambda_n^y x_n) = x^*(y_k) \) as required.

Proof. Suppose that \( X \) is a Banach space.

1. (CH) If the density of \( X^* \) is \( \omega_1 \), then \( X \) admits an overcomplete set.
2. (\neg CH) \( \ell_\infty \) does not admit an overcomplete set.
3. If \( X \) admits a fundamental biorthogonal system and has density bigger than \( \omega_1 \), then \( X \) does not admit an overcomplete set.
4. If the cofinality of the density of \( X \) is bigger than \( \mathfrak{c} \), then \( X \) does not admit an overcomplete set.
5. \( \ell_1(\omega_1) \) does not admit an overcomplete set.

2.3. General facts.

Definition 6. Suppose that \( X \) is a Banach space and \( Y \) is its closed subspace. For \( y^* \in S_{Y^*} \) and \( x \in X \) we define

1. \( E(y^*) = \{x^* \in S_{X^*} : x^*[Y] = y^*\} \).
2. \( [y^*](x) = \{x^*(x) : x^* \in E(y^*)\} \).

Lemma 7. Let \( \kappa \) be an infinite cardinal and \( X \) be a Banach space and \( x^* \in S_{X^*} \). Then \( \chi(x^*,B_{X^*}) \leq \kappa \) (with respect to the weak* topology) if and only if there is a closed subspace \( Y \) of \( X \) of density \( \leq \kappa \) such that \( E(x^*[Y]) = \{x^*\} \).

Proof. If \( \chi(x^*,B_{X^*}) \leq \kappa \), then there is its open sub-basis of cardinality not bigger than \( \kappa \) consisting of sub-basic open sets of the form \( U(y, \varepsilon) = \{y^* \in X^* : |(x^* - y^*)(y)| < \varepsilon \} \) for \( y \in X \) and \( \varepsilon > 0 \). If \( Y \) is a subspace of \( X \) generated by all such \( y^* \)s, it has density not bigger than \( \kappa \). Moreover if \( y^*(y) = x^*(y) \) for any \( y \in Y \), then \( y^* = x^* \), that is \( E(x^*[Y]) = \{x^*\} \).
For the reverse implication let \( D \subseteq Y \) be a norm dense set of cardinality not bigger than \( \kappa \) such that \( E(x^*[Y]) = \{x^*\} \). We claim that all finite intersections of the sets of the form \( U(d, \varepsilon) \) for \( d \in D \) form a neighbourhood basis of \( x^* \). Since \( B_{X^*} \) is compact in the weak* topology, the character of points is equal to their pseudocharacter, that is, it is enough to prove that \( x^* \) is the only point of the intersection of such \( U(d, \varepsilon) \) for \( d \in D \). But if \( y^* \in Y^* \setminus \{x^*[Y]\} \), then there is \( d \in D \) such that \( x^*(d) \neq y^*(d) \) and so there is \( \varepsilon \) such that \( y^* \not\in U(d, \varepsilon) \), as required. \( \square \)

**Lemma 8.** Suppose that \( X \) is a Banach space and \( Y \) is its closed subspace and that \( y^* \in S_{Y^*} \) and \( x \in X \). Then \( E(y^*) \) is a nonempty convex and closed in the weak* topology subset of \( S_{X^*} \). In particular, \( [y^*](x) \subseteq \mathbb{R} \) is convex.

Proof. Note that \( E(y^*) = \{x^* \in B_{X^*} : x^*[Y] = y^*\} \) since for every \( x^* \in E(y^*) \) already \( Y \) contains witnesses for \( \|x^*\| \geq 1 \). It is clear that \( (tx_1^* + (1-t)x_2^*)|Y = y^* \) for any \( x_1^*, x_2^* \in E(y^*) \) and \( 0 \leq t \leq 1 \). Also if \( x^* \in B_{X^*} \setminus E(y^*) \), then there is \( y \in Y \) and \( \varepsilon > 0 \) such that \( |x^*(y) - y^*(y)| > \varepsilon \) and so \( \{z^* \in B_{X^*} : z^*(y) \in (x^*(y) - \varepsilon, x^*(y) + \varepsilon)\} \) is a weak* open neighbourhood of \( x^* \) disjoint form \( E(y^*) \) which proves that \( E(y^*) \) is closed. The nonemptyness follows from the Hahn-Banach theorem. \( \square \)

### 3. Positive results

In this section we step-up the argument of Klee from \[25\] to some nonseparable Banach spaces using a coherent sequence of injections from \( \alpha < \omega_1 \) into \( \mathbb{N} \). The main applications of this are Theorem \[11\] and Corollary \[12\]

**Lemma 9.** There is a sequence \( (e_\alpha : \alpha < \omega_1) \) such that

1. \( e_\alpha : \alpha \rightarrow \mathbb{N} \) is injective for every \( \alpha < \omega_1 \),
2. \( \{\beta < \alpha_1, \alpha_2 : e_{\alpha_1}(\beta) \neq e_{\alpha_2}(\beta)\} \) is finite for every \( \alpha_1, \alpha_2 < \omega_1 \).

Consequently for every uncountable \( A \subseteq \omega_1 \) and every \( \gamma < \omega_1 \) there is an uncountable \( A' \subseteq A \setminus \gamma \) such that \( e_{\alpha_1}|(\gamma + 1) = e_{\alpha_2}|(\gamma + 1) \) for every \( \alpha_1, \alpha_2 \in A' \).

Proof. The construction of \( (e_\alpha : \alpha < \omega_1) \) is by transfinite recursion and is standard (see Ex. 28.1. of \[21\]).

To prove the second part of the lemma note that there is an uncountable \( A_1 \subseteq A \setminus \gamma \) and a finite \( F \subseteq \gamma + 1 \) such that for all \( \alpha \in A_1 \) \( e_\alpha(\beta) = e_{\gamma+1}(\beta) \) for all \( \beta \in \gamma \setminus F \). There is an uncountable \( A' \subseteq A_1 \) such that \( e_{\alpha_1}|F = e_{\alpha_2}|F \) for all \( \alpha_1, \alpha_2 \in A' \). It follows that for all \( \alpha \in A' \) we have the same \( e_{\alpha}|(\gamma + 1) \). \( \square \)

**Theorem 10.** Suppose that \( X \) is a Banach space which admits a linearly dense set \( \{x_\alpha : \alpha < \omega_1\} \) such that there is a norm closed subspace \( Y \subseteq X^* \) of finite codimension \( n \in \mathbb{N} \) such that \( \{\alpha < \omega_1 : y^*(x_\alpha) \neq 0\} \) is at most countable for each \( y^* \in Y \). Then there is \( k \in \mathbb{N} \) with \( k \leq n \) such that every subspace of \( X \) of codimension \( k \in \mathbb{N} \) admits an overcomplete set.

Proof. Let \( (e_\alpha : \alpha < \omega_1) \) be as in Lemma \[9\] and let \( B_\alpha \subseteq \mathbb{N} \) be the range of \( e_\alpha \). Let \( r_\alpha \) for \( \alpha < \omega_1 \) be distinct elements of \( (0, 1/2) \). For \( \alpha < \omega_1 \) define \( y_\alpha \in X \) by

\[
y_\alpha = \sum_{n \in B_\alpha} r_\alpha^n x_{e_\alpha^{-1}(n)}.
\]

First we will prove that whenever \( y^* \in Y \setminus \{0\} \) and \( A \subseteq \omega_1 \) is uncountable then there is \( \alpha \in A \) such that \( y^*(y_\alpha) \neq 0 \).
Let γ < ω₁ be such that y*(xₐ) = 0 for γ < ω < ω₁. By Lemma 9 there is an uncountable A' ⊆ A such that for all α ∈ A' we have the same (injective) eₐ(γ + 1). Let us call it g : γ + 1 → N. In particular γ < min(A'). By Theorem 11 for B ⊆ N being the range of g we obtain that

\[ \overline{lin}\{(z_α : α ∈ A')\} = \overline{lin}\{(x_β : β ≤ γ)\} \]

where

\[ z_α = \sum_{n ∈ B} r^n_α x_{g⁻¹(n)}. \]

In particular this means that there is α ∈ A' such that y*(zᵦ) ≠ 0. But

\[ y*(yₐ) = y*(z_α + \sum_{n ∈ B_α \setminus B} r^n_α x_{g⁻¹(n)}) = y*(z_α) ≠ 0 \]

since e⁻¹_α[B \ B_α] = (γ, α) ⊆ (γ, ω₁).

Now we will show that the codimension of \( \overline{lin}\{(y_α : α ∈ A)\} \) is at most n for every uncountable A ⊆ ω₁. Otherwise there are linearly independent \( z^*_₁, ..., z^*_{n+1} ∈ X^* \) such that

\[ \{y_α : α ∈ A\} ⊆ \bigcap \{ker(z^*_i) : 1 ≤ i ≤ n+1\}. \]

We will derive a contradiction from this hypothesis. Let \( X^* = Y ⊕ W \) where W is n-dimensional. Let \( z^*_i = y^*_i + w^*_i \) where \( y^*_i ∈ Y \) and \( w^*_i ∈ W \) for \( 1 ≤ i ≤ n+1 \). There is \( (r₁, ..., r_{n+1}) ∈ ℝ^{n+1}\setminus\{0\} \) such that \( \sum_{1 ≤ i ≤ n+1} r_i w^*_i = 0 \), \( \sum_{1 ≤ i ≤ n+1} r_i z^*_i ∈ Y\setminus\{0\} \) as \( z^*_₁, ..., z^*_{n+1} \) are linearly independent. But this means that a nonzero element of Y is zero on all the elements of \( \{y_α : α ∈ A\} \) which contradicts our previous findings.

To conclude the theorem we consider uncountable A ⊆ ω₁ such that \( \overline{lin}\{(y_α : α ∈ A)\} \) has the biggest possible codimension in X. Then \( \overline{lin}\{(y_α : α ∈ A')\} \) is dense in \( \overline{lin}\{(y_α : α ∈ A)\} \) for any uncountable A' ⊆ A and so \( \{y_α : α ∈ A\} \) is overcomplete in \( \overline{lin}\{(y_α : α ∈ A)\} \) which is of some codimension k ∈ N for some k ≤ n. As all subspaces of a fixed finite codimension of a Banach space are mutually isomorphic this shows that overcomplete sets are present in all subspaces of X of codimension k.

□

Theorem 11. The following Banach spaces admit overcomplete sets:

1. Every WLD Banach space of density ω₁, in particular
   (a) \( \ell_p(ω₁) \), \( L_p([0, 1]^{ω₁}) \) for \( p ∈ (1, ∞) \),
   (b) \( L_1([0, 1]^{ω₁}) \),
   (c) \( c_0(ω₁) \),
   (d) \( C(K) \)s for K a Corson compact where all Radon measure have separable supports.

2. \( C([0, ω₁]) \),
3. \( C(K) \), where K is the one point compactification of a refinement of the order topology on \([0, ω₁])\) obtained be isolating all points of some subset of \([0, ω₁])\).

Proof. The proof will consist of showing that the above spaces satisfy the hypothesis of Theorem 10.

For (1) we apply Theorem 10 for \( n = 0 \) as a Banach space is WLD if and only if it admits a linearly dense set such that every functional is countably supported by it (Theorem 7 of [17]).
For (2) and (3) we apply Theorem 10 for \( n = 1 \). We identify the compactification point with \( \{ \omega_1 \} \). The dual spaces to the spaces from (2) and (3) are \( \ell_1([0,\omega_1]) \) as the spaces are scattered \( (36) \). As \( Y \subseteq C(K)^* \) we consider
\[
Y = \{ \mu \in \ell_1([0,\omega_1]) : \mu(\{\omega_1\}) = 0 \}.
\]
As the linearly dense set we consider\[
D = \{ 1_{\{\alpha\}} : \alpha \text{ is isolated in } K \} \cup \{ 1_{[0,\omega_1]} \} \cup \{ 1_{(\alpha,\omega_1]} : \alpha < \omega_1 \}.
\]
It is clear that any \( \mu \in Y \) is zero on all but countably many elements of \( D \). Also \( D \) is linearly dense as \( 1_{(\alpha,\beta]} = 1_{(\alpha,\omega_1]} - 1_{(\beta,\omega_1]} \) for any \( \alpha < \beta < \omega_1 \) and \( 1_{[0,\alpha]} = 1_{\{0\}} + 1_{(0,\alpha]} \). Moreover all clopen sets of \( K \) are finite unions of intervals and characteristic functions of clopen sets generate \( C(K) \) as \( K \) is totally disconnected since it is scattered and compact.

So Theorem 10 implies that either \( C(K) \) admits an overcomplete set or hyperplanes of \( C(K) \) admit overcomplete sets. But hyperplanes of such \( C(K) \) are isomorphic to the entire \( C(K) \) since \( K \) admits nontrivial convergent sequences as it is a scattered compact space.

\[ \square \]

**Corollary 12.** A WLD Banach space \( X \) admits an overcomplete set if and only if the density of \( X \) is less or equal to \( \omega_1 \).

**Proof.** The existence follows from Theorem 11 (1) and the nonexistence from the results of\( (37) \) (in this paper Theorem 5 (2)). \[ \square \]

### 4. Consistency results

The purpose of this section is to prove the consistency negative results in Theorems 19 and 20 and in Corollary 21. The general case of a nonseparable Banach space \( X \) of density smaller than \( \kappa \) with its dual ball \( B_X^* \) separable or even nonmonolithic in the weak∗ topology can be reduced to the case of a Banach space of the form \( C(K) \) for \( K \) separable with a countable dense set \( D \) and nonmetrizable of weight \( \kappa < \kappa \) in Theorems 19 and 20. To exclude the existence of overcomplete sets in such Banach spaces we prove that certain sets can be covered by countably many hyperplanes and any linearly dense set contains such a subset of cardinality \( \kappa \) (Proposition 17). Having this, if \( cf(\kappa) > \omega \), then one hyperplane contains \( \kappa \) many elements of the set, so the set cannot be overcomplete. In the proof of Proposition 17 we use Martin’s axiom MA and the negation of \( CH \) to build linear bounded functionals \( y \in \ell_1(D) \) whose kernels are the covering hyperplanes mentioned above. For this, in Definition 13, we define a partial order \( P \) of finite approximations to such \( y \)s and prove density lemmas and the countable chain condition of \( P \) in Lemmas 14, 15 and 16. In fact, the proof of the c.c.c. of \( P \) is relatively hard and the rest of the Section is related to the necessity of such a complicated \( P \). In Lemma 22 we prove that simpler partial orders (having precaliber \( \omega_1 \)) cannot do the job of \( P \). It follows in Theorem 23 that weaker versions of Martin’s axiom are consistent with the existence of overcomplete sets in Banach spaces like in Theorems 19 and 20 and that the results obtained in\( (37) \) under \( CH \) are consistent with arbitrary size of the continuum (Theorem 24). Understanding forcing is required only when reading the proofs of the latter results i.e., Lemma 22 and Theorems 23 and 24. For the first part of this section we assume from the reader some familiarity with Martin’s axiom and the way one applies it, for this the reader may consult\( (24, 28) \).
**Definition 13.** Let $K$ be a compact Hausdorff space with a dense subset $\{d_n : n \in \mathbb{N}\}$ and $\kappa$ an infinite cardinal. Let $\{x_\xi : \xi < \kappa\} \subseteq K$ be distinct nonisolated points and $\{f_\xi : \xi < \kappa\} \subseteq C(K)$ satisfy $f_\xi(x_\xi) = 1$, $f_\xi(x_\eta) = 0$ for all $\xi < \eta < \kappa$ and $\|f_\xi\| \leq M$ for all $\xi < \kappa$ and a rational $M > 2$.

We define a partial order $\mathbb{P}$ consisting of conditions $p = (n_p, y_p, X_p, \varepsilon_p)$ such that

(a) $\varepsilon_p \in \mathbb{Q}^+$, $n_p \in \mathbb{N}$, $n_p > 0$,
(b) $y_p : n_p \rightarrow \mathbb{Q}$, $y_p(0) \neq 0$,
(c) $X_p$ is a finite subset of $\kappa$,
(d) $1 - \sum_{n < n_p} |y_p(n)| = \delta_p \geq \varepsilon_p 2^M|M|^{X_p} + 1$,
(e) $|\sum_{n < n_p} y_p(n)f_\xi(d_n)| < \varepsilon_p$ for every $\xi \in X_p$.

We declare $p \leq q$ if

(i) $\varepsilon_p \leq \varepsilon_q$,
(ii) $y_p \supseteq y_q$,
(iii) $n_p$ $\geq n_q$,
(iv) $X_p \supseteq X_q$.

It is easy to see that $\mathbb{P}$ with $\leq$ is a partial order.

**Lemma 14.** Let $\mathbb{P}$ and the corresponding objects be as in Definition 13. For every $\xi < \kappa$ there is $p \in \mathbb{P}$ such that $\xi \in X_p$.

**Proof.** We define $p = (n_p, y_p, X_p, \varepsilon_p)$ by putting $n_p = 1$, $X_p = \{\xi\}$. The value of $y_p(0)$ is chosen so that $0 < y_p(0) \leq 1/2$ and $|y_p(0)f_\xi(d_0)| < 1/(4M^4)$ hold, $\varepsilon_p = 1/(4M^4)$, $\delta_p = \varepsilon_p 2^M = 1/2 < 1 - |y_p(0)|$.

The density of the set $E$ in the following Lemma will be crucial in the proof of the c.c.c. of $\mathbb{P}$ in Lemma 13 and is the most technical part of this section.

**Lemma 15.** Let $\mathbb{P}$ and the corresponding objects be as in Definition 13. Let $n, k \in \mathbb{N}$. The following sets are dense in $\mathbb{P}$.

- $C_n = \{p \in \mathbb{P} : n_p \geq n\}$,
- $D_k = \{p \in \mathbb{P} : \varepsilon_p \leq 1/k\}$,
- $E = \{p \in \mathbb{P} : \delta_p \geq \varepsilon_p 2^M |M|^{X_p} + 1\}$.

**Proof.** The density of $C_n$s is clear as $(n, y, X_p, \varepsilon_p) \leq p$ for any $n \geq n_p$ and $y : n \rightarrow \mathbb{Q}$ such that $y[p, n] = 0$. So given $k \in \mathbb{N}$ and $q \in \mathbb{P}$ let us focus on finding $p \leq q$ in $D_k \cap E$ which will finish the proof of the lemma. Let $X_q = \{\xi_0, ..., \xi_{m-1}\}$ for $\xi_0 < ... < \xi_{m-1} < \omega_1$ and some $m \in \mathbb{N}$. First we will find $y : m \rightarrow \mathbb{R}$ such that

1. $|\sum_{n \leq n_q} y_q(n)f_{\xi_j}(d_n) + \sum_{i < m} y(i)f_{\xi_j}(x_{\xi_i})| = 0$ for every $j < m$,
2. $|y(j)| \leq \varepsilon_q + M \sum_{i < j} |y(i)|$ for every $j < m$,

We do it by induction on $0 \leq j < m$. Suppose that we are done for $i < j < m$. Define

$$y(j) = -\sum_{n \leq n_q} y_q(n)f_{\xi_j}(d_n) - \sum_{i < j} y(i)f_{\xi_j}(x_{\xi_i}).$$

As $f_{\xi_j}(x_{\xi_i}) = 1$ and $f_{\xi_j}(x_{\xi_i}) = 0$ for $j < i < m$ we obtain (1). Note that we keep (2) as $|y(j)| \leq \varepsilon_q + M \sum_{i < j} |y(i)|$ since $\|f_\xi\| \leq M$ for every $\xi < \kappa$ and by Definition 13 (e). Now we note that

3. $|y(j)| \leq \varepsilon_q M^3$ for all $j < m$,
4. $\sum_{i < m} |y(i)| \leq \delta_q/2$. 

□
To prove (3) by induction on \( j < m \) we use (2) and the fact that \( \sum_{i<j} M^j < M^i \) for every \( j \in \mathbb{N} \) since \( M > 2 \):

\[
|y(j)| \leq \varepsilon_q + M \sum_{i<j} |y(i)| \leq \varepsilon_q + \varepsilon_q M \sum_{i<j} M^{3i} \leq \varepsilon_q(1 + MM^{3(j-1)+1}) = \varepsilon_q(1 + M^{3j-1}) \leq \varepsilon_q M^{3j},
\]

so \( \sum_{i<n} |g(i)| \leq \varepsilon_q M^{3m+1} \leq \delta_q/2 \) by Definition 13 (d) which gives (4) and completes the proof of the properties of \( y \).

Now we are ready to start defining \( p \leq q \) such that \( p \in D_k \cap E \). Let \( \theta \in \mathbb{Q}_+ \) satisfy the following:

\[
\begin{align*}
(5) & \quad m\theta \leq \delta_q/4, \\
(6) & \quad m\theta(\theta + M + 1)2M^{6m+1} \leq \delta_q/4 \\
(7) & \quad m\theta(\theta + M + 1) \leq \min(1/k, \varepsilon_q).
\end{align*}
\]

Let \( y_i \in \mathbb{Q} \) for \( i < m \) be such that \( |y_i - y(i)| < \theta \) for every \( i < m \) and let \( n_i \in \mathbb{N} \) for \( i < m \) be distinct and such that \( n_i > n_q \) and \( |f_{\xi_j}(x_{\xi_i}) - f_{\xi_j}(d_{n_i})| < \theta \) for every \( i, j < m \). This can be achieved because \( \mathbb{Q} \) is dense in \( \mathbb{R} \) and \( \{d_n : n \in \mathbb{N}\} \) is dense in \( K \), \( f_{\xi_j} \)s are continuous, and \( x_{\xi_i} \)s are nonisolated. Define \( p = (n_p, y_p, X_p, \varepsilon_p) \) as follows: \( n_p = \max(\{n_i : i < m\}) + 1 \), for \( n < n_p \) define

\[
y_p(n) = \begin{cases} y_q(n) & \text{if } n < n_q, \\
y_i & \text{if } n = n_i, i < m \\
0 & \text{otherwise for } n < n_p.
\end{cases}
\]

\( X_p = X_n, \varepsilon_p = m\theta(\theta + M + 1) \). First let us check that \( p \in \mathbb{P} \). Condition (a) - (c) of Definition 13 are clear. To prove condition (d) note that by (4) and by the choice of \( y_i \) we have \( \sum_{i<m} |y_i| \leq \delta_q/2 + m\theta \), and so using (5) and (6) we conclude that

\[
(8) \quad \delta_p = \delta_q - \sum_{i<m} |y_i| \geq \delta_q/2 - m\theta \geq \delta_q/4 \geq \varepsilon_p 2M^{6m+1} = \varepsilon_p 2M^{6|X_p|+1}
\]

as required in Definition 13 (d). To prove (e) note that by (1) and by the choice of \( n_i \)s and the fact that \( |y(i)| \leq 1 \) for each \( i < m \) (which follows (4) and 13 (d) for \( q \)) we conclude that

\[
|\sum_{n<n_p} y_p(n)f_{\xi_j}(d_{n_i})| = |\sum_{n<n_q} y_p(n)f_{\xi_j}(d_{n_i}) + \sum_{i<m} y_p(n_i)f_{\xi_j}(x_{\xi_i})| =
\]

\[
= |\sum_{i<m} y_p(n_i)f_{\xi_j}(d_{n_i}) - \sum_{i<m} y(i)f_{\xi_j}(x_{\xi_i})| \leq
\]

\[
\leq M \sum_{i<m} |y_p(n_i) - y(i)| + \sum_{i<m} |f_{\xi_j}(d_{n_i}) - f_{\xi_j}(x_{\xi_i})| \leq m\theta(M + 1) \leq m\theta(\theta + M + 1) = \varepsilon_p
\]

for every \( j < m \) which is condition (e) of Definition 13. We have \( p \leq q \) by (7). Also \( p \in D_k \) by (7). Finally \( p \in E \) by (8).

\[
\square
\]

**Lemma 16.** \( \mathbb{P} \) satisfies c.c.c.
Proof. Given \( p_t = (n_{p_t}, \pi_{p_t}, X_{p_t}, \varepsilon_{p_t}) \in P \) for \( \xi < \omega_1 \) by Lemma 15 we may assume that \( p_t \in E \) for each \( \xi < \omega_1 \) and by passing to an uncountable set we may assume that \( n_{p_t} = n, \pi_{p_t} = y, \varepsilon_{p_t} = \varepsilon \) for some \( n \in \mathbb{N}, y : n \to \mathbb{Q} \) and \( \varepsilon \in \mathbb{Q}_+ \). We claim that then \( (n, y, X_{p_t} \cup X_{p_t}, \varepsilon) \leq p_{\xi, \eta} \). The only nonclear part of Definition 13 to check is (d), but it follows from the fact that the conditions are in \( E \) of Lemma 15.

Lemma 17. (\( \text{MA} + \neg \text{CH} \)) Let \( K \) be a compact Hausdorff space with a dense subset \( \{d_i : i \in \mathbb{N}\} \) and \( \kappa \) an uncountable cardinal satisfying \( \kappa < \xi \). Suppose that \( \{x_\xi : \xi < \kappa\} \subseteq K \) are distinct nonisolated points and \( \{f_\xi : \xi < \kappa\} \subseteq C(K) \) satisfy \( f_\xi(x_\xi) = 1, f_\xi(x_\eta) = 0 \) for all \( \xi < \eta < \kappa \) and \( \|f_\xi\| \leq M \) for all \( \xi < \kappa \) and some \( M > 0 \). Then there are sets \( B_m \subseteq \kappa \) for \( m \in \mathbb{N} \) such that \( \bigcup_{m \in \mathbb{N}} B_m = \kappa \) and \( y_m \in \ell_1 \setminus \{0\} \) for \( m \in \mathbb{N} \) such that

\[
\sum_{i \in \mathbb{N}} y_m(i)f_\xi(d_i) = 0
\]

for all \( \xi \in B_m \) and all \( m \in \mathbb{N} \).

Proof. If \( M \leq 2 \), we replace it with some \( M > 2 \). In any case we may assume that it is a rational. Let \( \mathbb{P} \) be the partial order from Definition 13. We consider the countable power \( \mathcal{S} \) with finite supports of partial order \( \mathbb{P} \) with coordinatewise order. By Lemma 16 and MA + \( \neg \text{CH} \) we know that finite products of \( \mathbb{P} \) satisfy the c.c.c. and so \( \mathcal{S} \) satisfies the c.c.c. Applying MA let \( G \subseteq \mathcal{S} \) be a filter in \( \mathbb{S} \) meeting the following dense sets for \( \xi \in \kappa \) and \( k, n, m \in \mathbb{N} \):

\[
F_\xi = \{s \in \mathcal{S} : \exists k \in \mathbb{N} \xi \in X_{s(k)}\}
\]

\[
C_{n,m} = \{s \in \mathcal{S} : s(m) \in C_n\}
\]

\[
D_{k,m} = \{s \in \mathcal{S} : s(m) \in D_k\}
\]

The density of these sets follows from Lemmas 14 and 15 and the fact that the supports of the conditions of the product are finite. In particular if \( s \in \mathcal{S} \) and \( \xi \in \kappa \) we find \( k \in \mathbb{N} \) not belonging to the support of \( s \) and define \( s' \leq s \) with \( s' \in F_\xi \) using Lemma 14 on the coordinate \( k \).

For \( m \in \mathbb{N} \) define \( B_m = \bigcup\{X_{s(m)} : s \in G\} \). By the density of each \( F_\xi \) for each \( \xi < \kappa \) we have \( \bigcup_{m \in \mathbb{N}} B_m = \kappa \). Let \( y_m = \bigcup\{y_{s(m)} : s \in G\} \). It follows from the conditions (b) and (d) of Definition 13 and the density of the sets \( C_{n,m} \) for \( n, m \in \mathbb{N} \) that \( y_m \in \ell_1 \setminus \{0\} \) for each \( m \in \mathbb{N} \). The final condition of the lemma follows from the density of the sets \( C_{n,m} \) and \( D_{k,m} \) for \( k, n, m \in \mathbb{N} \) and the condition (e) of Definition 13.

Proposition 18. (\( \text{MA} + \neg \text{CH} \)) Suppose that \( X \) is a Banach space whose dual unit ball \( B_{X^*} \) is separable in the weak* topology. Let \( \kappa < \xi \) be a cardinal and \( \{x_\xi : \xi < \kappa\} \subseteq X \) be a set satisfying \( x_\xi \not\in \overline{\text{lin}}\{x_\eta : \xi < \kappa\} \) for every \( \xi < \kappa \). Then \( \{x_\xi : \xi < \kappa\} \) can be covered by countably many hyperplanes of \( X \).

Proof. Let \( \{d_i^* : i \in \mathbb{N}\} \) be a countable set dense in \( B_{X^*} \) with the weak* topology. We may assume that \( \kappa \) is uncountable, in particular that \( X^* \) is nonseparable in the norm. First we will argue that we may assume that \( \sum_{i \in \mathbb{N}} \alpha_id_i^* = 0 \) if and only if \( \alpha_i = 0 \) for every \( i \in \mathbb{N} \).

For this let \( Y \) be the norm closure of \( \{d_i^* : i \in \mathbb{N}\} \) in \( X^* \). As \( X^* \) is nonseparable and \( Y \) is separable the quotient \( X^*/Y \) is infinite dimensional and so there is an
infinite biorthogonal system \((e_i, \phi_i)_{i \in \mathbb{N}}\) in \(X^*/Y \times (X^*/Y)^*\). Let \(\pi : X^* \to X^*/Y\) be the quotient map. Let \(\varphi_i = \phi_i \circ \pi \in X^*\) and let \(e_i^* \in X^*\) be such that \(\pi(e_i^*) = e_i\).

Now let us prove that \(\{e_i^* : i \in \mathbb{N}\}\) has the desired properties, where \(c_i = (1 - 1/(i + 1))d_i^* + e_i^*/(i + 1)\|e_i^*\|\) for each \(i \in \mathbb{N}\) i.e., that \(\{c_i^* : i \in \mathbb{N}\}\) is dense in \(B_{X^*}\) with the weak* topology and \(\sum_{i \in \mathbb{N}} \alpha_i c_i^* = 0\) if and only if \(\alpha_i = 0\) for every \(i \in \mathbb{N}\). The density is clear. Also \(\psi_j(\sum_{i \in \mathbb{N}} \alpha_i c_i^*) = \alpha_j/(j+1)\|e_j^*\|\) so we obtain the second property as well and so may assume that \(\sum_{i \in \mathbb{N}} \alpha_i d_i^* = 0\) if and only if \(\alpha_i = 0\) for every \(i \in \mathbb{N}\).

For every \(\xi < \kappa\) there is a norm one functional \(x_\xi^* \in X^*\) such that \(x_\xi^*\) is zero on \(\overline{\text{lin}}(x_\eta : \eta < \xi)\) and \(x_\xi^*(x_\xi) \neq 0\). By multiplying \(x_\xi\)s we may assume that \(x_\xi^*(x_\xi) = 1\) for each \(\xi < \kappa\). We can divide \(\kappa\) into countably many sets \(A_n \subseteq \kappa\) such that each \(\{x_\xi : \xi \in A_n\}\) is norm bounded. Now consider \(K = B_{X^*}\) with the weak* topology. For \(\eta < \kappa\) define continuous functions \(f_\eta : K \to \mathbb{R}\) by \(f_\eta(x_\eta) = x_\eta^*(x_\eta)\) and note that \(f_\eta(x_\eta) = 0\) if \(\eta < \xi < \kappa\) and \(f_\xi(x_\xi) = 1\). It follows from Lemma 17 that for each \(n \in \mathbb{N}\) we can find \(B_m^\eta \subseteq A_n\) for \(m \in \mathbb{N}\) such that \(\bigcup_{m \in \mathbb{N}} B_m^\eta = A_n\) for each \(n \in \mathbb{N}\) and \(y_m^\eta \in \ell_1 \setminus \{0\}\) satisfying for each \(\xi \in B_m^\eta\)

\[
(\sum_{i \in \mathbb{N}} y_m^\eta(i)d_i^*)(x_\xi) = \sum_{i \in \mathbb{N}} y_m^\eta(i)f_\xi(d_i^*) = 0.
\]

Note that \(\sum_{i \in \mathbb{N}} y_m^\eta(i)d_i^* \in X^* \setminus \{0\}\) by the property of \(\{d_i : i \in \mathbb{N}\}\) from the first part of the proof and by the fact that \(y_m^\eta \in \ell_1 \setminus \{0\}\). Hence

\[
H_m^\eta = \{x \in X : (\sum_{i \in \mathbb{N}} y_m^\eta(i)d_i^*)(x) = 0\}
\]

is a hyperplane of \(X\). So each set \(\{x_\xi : \xi \in B_m^\eta\}\) is included in a hyperplane, as required. \(\square\)

**Theorem 19.** (MA+−CH) Suppose that the density of a Banach space \(X\) is smaller than \(\mathfrak{c}\) and has uncountable cofinality and that the dual unit ball \(B_{X^*}\) is separable in the weak* topology. Then \(X\) does not admit an overcomplete set.

**Proof.** Suppose that \(D\) is a linearly dense subset of \(X\). We will show that it is not overcomplete. As the density of \(X\) is \(\kappa\) we can construct \(\{x_\xi : \xi < \kappa\} \subseteq D\) satisfying \(x_\xi \notin \overline{\text{lin}}(x_\eta : \eta < \xi)\) for every \(\xi < \kappa\). Then \(\{x_\xi : \xi < \kappa\}\) can be covered by countably many hyperplanes of \(X\) by Proposition 13. Since the cofinality of \(\kappa\) is uncountable, one of these hyperplanes contains \(\kappa\) many vectors \(x_\xi\) which shows that \(D\) is not overcomplete. \(\square\)

Recall that a topological space is called monolithic if the closures of countable sets are metrizable.

**Theorem 20.** (MA+−CH) Suppose that \(X\) is a Banach space of density \(\omega_1\) whose dual ball is not monolithic in the weak* topology. Then \(X\) does not admit an overcomplete set.

**Proof.** Let \(D = \{d_i : i \in \mathbb{N}\} \subseteq B_{X^*}\) be a countable set whose weak*-closure in \(X^*\) is nonmetrizable. We will construct a linear bounded operator \(T : X \to Y\) whose range is dense in \(Y\) and \(Y\) is a Banach space of density \(\omega_1\) whose dual ball is weak* separable. This will be enough by Lemma 2 and Theorem 19.

Define a linear bounded operator \(T : X \to \ell_\infty\) by \(T(x) = (d_i(x))_{i \in \mathbb{N}}\). Let \(Y\) be the norm closure of \(T[X]\) in \(\ell_\infty\). Let \(\delta_i \in (\ell_\infty)^*\) be defined by \(\delta_i(f) = f(i)\) for
Let \( f \in \ell_\infty \) and \( i \in \mathbb{N} \). By the Krein-Millman theorem the rational convex combinations of \( \delta_i \)s are weak* dense in the dual ball of \( \ell_\infty \). In particular the dual ball \( B_{\ell_\infty^*} \) is separable in the weak* topology. The dual ball of \( Y \) with the weak* topology is a continuous image of \( B_{\ell_\infty^*} \) (taking restrictions of functionals is a continuous map which is onto by the Hahn-Banach theorem). So \( B_{Y^*} \) is separable in the weak* topology as required.

To show that \( Y \) is nonseparable in the norm it is enough to show that \( B_{Y^*} \) is nonmetrizable in the weak* topology. The dual ball \( B_{Y^*} \) maps continuously by \( T^* \) onto \( T^*[B_{Y^*}] \subseteq X^* \) which contains \( \{ T^*(\delta_i) : i \in \mathbb{N} \} = \{ d_i : i \in \mathbb{N} \} \). As \( T^*[B_{Y^*}] \) is compact, it contains the closure of \( \{ d_i : i \in \mathbb{N} \} \) which is nonmetrizable, so \( B_{Y^*} \) is nonmetrizable since continuous images of compact metrizable spaces are metrizable. 

\[ \square \]

**Corollary 21.** (MA+¬CH) Suppose that \( \mathcal{A} \) is an almost disjoint family of subsets of \( \mathbb{N} \) of cardinality \( \kappa < \mathfrak{c} \) of uncountable cofinality. Then the Banach space generated in \( \ell_\infty \) by \( c_0 \) and \( \{ 1_A : A \in \mathcal{A} \} \) does not admit an overcomplete set.

**Proof.** As a nonseparable subspace of \( \ell_\infty \) the space satisfies the hypothesis of Theorem [19].

It is well-known that the space above is isometric to \( C_0(K_{\mathcal{A}}) \) where \( K_{\mathcal{A}} \) is locally compact scattered space of weight \( \kappa \) and of Cantor-Bendixson height two known as \( \Psi \)-space, Mrówka-Isebell space or Alexandroff-Urysohn space. One can see that the dual of the space above has density \( \kappa \) as well.

The remaining part of this section is devoted to results showing that the positive CH results of [37] are consistent with any size of the continuum. The first result, Theorem [23], also shows that the a relatively complex Definition [13] and a relatively delicate argument in Lemma [14] are unavoidable.

**Lemma 22.** Suppose that \( X \) is a Banach space of density \( \omega_1 \) which admits an overcomplete set and that \( \mathbb{P} \) is a partial order which has precaliber \( \omega_1 \). Then \( \mathbb{P} \) forces that the completion of \( X \) admits an overcomplete set.

**Proof.** Let \( D = \{ x_\alpha : \alpha < \omega_1 \} \) be an overcomplete set in \( X \). Let \( \dot{X} \) stand for a \( \mathbb{P} \)-name for the completion of \( X \) in the generic extension by \( \mathbb{P} \). We claim that \( \mathbb{P} \) forces that \( \dot{D} \) is an overcomplete set in \( \dot{X} \).

Let \( \dot{A} \) be a \( \mathbb{P} \)-name for an uncountable subset of \( \omega_1 \), \( \varepsilon > 0 \) and \( \dot{x} \) be a \( \mathbb{P} \)-name for an element of the completion of \( X \) and let \( \{ \dot{\alpha}_\xi : \xi < \omega_1 \} \) be \( \mathbb{P} \)-names such that \( \mathbb{P} \vDash \dot{A} = \{ \dot{\alpha}_\xi : \xi < \omega_1 \} \). By the density of \( X \) in its completion we can find \( p \in \mathbb{P} \) and \( x \in X \) such that \( p \vDash \| \dot{x} - \dot{x} \| < \varepsilon / 2 \). For each \( \xi < \omega_1 \) find \( p_\xi \vDash \dot{\alpha}_\xi = \dot{\alpha}_\xi \).

Since \( \mathbb{P} \) has precaliber \( \omega_1 \), there is an uncountable \( B \subseteq \omega_1 \) such that any finite subset of \( \{ p_\alpha : \xi \in B \} \) has a lower bound in \( \mathbb{P} \).

Since \( D \) is overcomplete in \( X \) we have \( \xi_1, ..., \xi_k \in B \) and \( r_i \in \mathbb{R} \) for \( 1 \leq i \leq k \) for some \( k \in \mathbb{N} \) such that \( \| x - \sum_{1 \leq i \leq k} r_i x_\alpha_{\xi_i} \| < \varepsilon / 2 \). Then

\[ p \vDash \| \dot{x} - \sum_{1 \leq i \leq k} r_i \dot{x}_\alpha_{\xi_i} \| < \varepsilon. \]

Let \( q \leq p_{\alpha_1}, ..., p_{\alpha_k} \). Then

\[ q \vDash \{ \alpha_{\xi_1}, ..., \alpha_{\xi_k} \} \subseteq \dot{A}. \]
This shows that \( q \) forces that the distance of \( x \) from the closure of the linear span of \( \{ x_\alpha : \alpha \in A \} \) is smaller than \( \varepsilon \). Since \( \varepsilon \) was arbitrary it shows that \( \mathbb{P} \) forces that \( \{ x_\alpha : \alpha \in A \} \) is linearly dense in the completion of \( X \). Since \( A \) was an arbitrary \( \mathbb{P} \)-name for an uncountable subset of \( \omega_1 \) this proves that \( D \) remains an overcomplete set in the completion of \( X \).

\[ \square \]

**Theorem 23.** It is consistent with \( \text{MA} \) for partial orders having precaliber \( \omega_1 \) and the negation of \( \text{CH} \) that every Banach spaces whose dual has density \( \omega_1 \) admits an overcomplete set.

**Proof.** Let \( V \) be a model of \( \text{ZFC} \) and \( \text{GCH} \). Let \( (\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha)_{\alpha \leq \omega_2} \) be a finite support iteration of forcings of cardinality \( \omega_1 \), having precaliber \( \omega_1 \) such that \( V[G_{\omega_2}] \) satisfies \( 2^{\omega_2} = \omega_2 \) and Martin’s axiom for partial orders having precaliber \( \omega_1 \) and where \( G_{\omega_2} \) is \( \mathbb{P}_{\omega_2} \)-generic over \( V \). Let \( G_\alpha = G_{\omega_2} \cap \mathbb{P}_\alpha \) for any \( \alpha \leq \omega_2 \).

Let \( X \) be any Banach space in \( V[G_{\omega_2}] \) whose dual has density \( \omega_1 \). Let \( E \subseteq X \) be a dense linear (non-closed) subspace over \( \mathbb{Q} \) of \( X \) of cardinality \( \omega_1 \). Without loss of generality we may assume that \( E = \omega_1 \). So some functions \( + : \omega_1 \times \omega_1 \to \omega_1 \) and \( \cdot : \mathbb{Q} \times \omega_1 \to \omega_1 \) represent linear operations in \( E \) and \( \parallel \cdot \parallel : \omega_1 \to \mathbb{R} \) represents the norm on \( E \). So \( X \) is the completion of \( E \) in \( V[G_{\omega_2}] \). Using the \( \text{c.c.c.} \) of \( \mathbb{P}_{\omega_2} \) and applying the standard arguments we can find \( \alpha < \omega_2 \) such that \( \mathbb{P}_{\omega_2} \) forces that \( +, \cdot, \parallel \cdot \parallel \) are in \( V[G_\alpha] \). As \( \mathbb{P}_\alpha \) is a finite support iteration of \( \text{c.c.c.} \) forcings of cardinality \( \omega_1 \) and \( \alpha < \omega_2 \) the model \( V[G_\alpha] \) satisfies \( \text{CH} \). It follows that the completion \( X_\alpha \) of \( E \) in \( V[G_\alpha] \) admits an overcomplete set \( D \subseteq X_\alpha \) because the dual \( X_\alpha^* \) in \( V[G_\alpha] \) must have density \( \omega_1 \), as otherwise, a norm discrete subset \( \{ \phi_\alpha : \alpha < \omega_2 \} \subseteq X_\alpha^* \) by the Hahn-Banach theorem would produce a norm discrete subset of \( X^* \) of cardinality \( \omega_2 \) in \( V[G_{\omega_2}] \) contradicting the choice of \( X \).

By the standard argument (see e.g., 1.5.A of [7]) the iteration \( \mathbb{P}_{\omega_2} \) is equivalent to the iteration \( \mathbb{P}_\alpha \ast \dot{\mathbb{S}}^\alpha \) where \( \mathbb{P}_\alpha \) forces that \( \dot{\mathbb{S}}^\alpha \) is a finite support iteration of forcings having precaliber \( \omega_1 \). But such an iteration has precaliber \( \omega_1 \) (e.g., Theorem 1.5.13 of [7]). So we are in the position to apply Lemma 22 in \( V[G_\alpha] \) to conclude that \( D \) stays overcomplete in \( X \) in \( V[G_{\omega_2}] \).

\[ \square \]

**Theorem 24.** The statement that every Banach space whose dual has density \( \omega_1 \) admits an overcomplete set is consistent with any size of the continuum.

**Proof.** Let \( V \) be a model of \( \text{ZFC} \) which satisfies \( \text{GCH} \) and let \( \kappa \) be any cardinal of uncountable cofinality and for an infinite \( A \subseteq \kappa \) let \( \mathbb{P}_A \) stands for the partial order for adding Cohen reals labelled by elements of \( A \), that is \( \mathbb{P}_A \) consist of finite partial functions from \( A \) into \( \{ 0, 1 \} \) and is considered with the inverse inclusion as the order. Let \( G_\kappa \subseteq \mathbb{P}_\kappa \) be \( \mathbb{P}_\kappa \)-generic over \( V \). Let \( G_A = G_\kappa \cap \mathbb{P}_A \). As is well know ([28, 24]) the continuum of the model \( V[G_\kappa] \) assumes value \( \kappa \). We will show that any Banach space in \( V[G_\kappa] \) whose dual has density \( \omega_1 \) admits in \( V[G] \) an overcomplete set.

Let \( X \) be any Banach space in \( V[G_\kappa] \) whose dual has density \( \omega_1 \). Let \( E \subseteq X \) be a dense linear (non-closed) subspace over \( \mathbb{Q} \) of \( X \) of cardinality \( \omega_1 \). Without loss of generality we may assume that \( E = \omega_1 \). So some functions \( + : \omega_1 \times \omega_1 \to \omega_1 \) and \( \cdot : \mathbb{Q} \times \omega_1 \to \omega_1 \) represent linear operations in \( E \) and \( \parallel \cdot \parallel : \omega_1 \to \mathbb{R} \) represents the norm on \( E \). So \( X \) is the completion of \( E \) in \( V[G_\kappa] \). Using the \( \text{c.c.c.} \) of \( \mathbb{P}_\kappa \) and applying the standard arguments we can find \( A \subseteq \kappa \) in \( V \) of cardinality \( \omega_1 \) such
that \( \mathbb{P}_\kappa \) forces that \( +, \cdot, \| \| \) are in \( V[G_A] \). As \( \mathbb{P}_A \) adds only \( \omega_1 \) Cohen reals due to the fact that \( A \) has cardinality \( \omega_1 \) the model \( V[G_A] \) satisfies CH. It follows that the completion \( X_A \) of \( E \) in \( V[G_A] \) admits an overcomplete set \( D \subseteq X_A \) because the dual \( X_A^* \) in \( V[G_A] \) must have density \( \omega_1 \), as otherwise, a norm discrete subset \( \{ \phi_\alpha : \alpha < \omega_2 \} \subseteq X_A^* \) by the Hahn-Banach theorem would produce a norm discrete subset of \( X^* \) of cardinality \( \omega_2 \) in \( V[G] \) contradicting the choice of \( X \).

By the standard argument \( \mathbb{P}_\kappa \) is isomorphic with \( \mathbb{P}_A \times \mathbb{P}_{\kappa \setminus A} \) and so by the product lemma \( V[G] = V[G_A][G_{\kappa \setminus A}] \). Since \( \mathbb{P}_{\kappa \setminus A} \) has precaliber \( \omega_1 \) in \( V[G_A] \) we are in the position to apply Lemma 22 to conclude that \( D \) stays overcomplete in \( X \) in \( V[G] \).

\[ \square \]

5. Negative results

Recall the definitions of \( E(y^*) \) and \( [y^*](x) \) from Section 2.

**Lemma 25.** Suppose that \( \kappa \) is an infinite cardinal and that \( X \) is a Banach space of density \( \kappa \), its subspace \( Y \subseteq X \) has density smaller than \( \kappa \) and \( D \subseteq X \) is linearly dense in \( X \). Let \( y^* \in S_{X^*} \) be such that \( \chi(x^*, B_{X^*}) = \kappa \) for every \( x^* \in E(y^*) \). Then, for every subspace \( W \) with \( Y \subseteq W \subseteq X \), \( \text{dists}(W) < \kappa \) and every \( w^* \in S_{W^*} \) with \( w^*|Y = y^* \), there is \( d \in D \) such that \( [w^*](d) \) contains a nondegenerate interval in \( \mathbb{R} \).

**Proof.** By the hypothesis and Lemma 8 and Lemma 7 the set \( E(w^*) \) is a convex closed subset of \( S_X \), which contains at least two distinct points \( x_1^*, x_2^* \). The set \( \{ x \in X : x_1^*(x) = x_2^*(x) \} \) is a closed proper subspace of \( X \) and hence there is \( d \in D \) which does not belong to it, i.e., without loss of generality we have \( x_1^*(d) < x_2^*(d) \). So

\[ (x_1^*(d), x_2^*(d)) \subseteq \{ (tx_1^* + (1 - t)x_2^*)(d) : 0 \leq t \leq 1 \} \subseteq [w^*](d). \]

\[ \square \]

**Lemma 26.** Suppose that \( \kappa \) is a cardinal of uncountable cofinality, \( X \) is a Banach space of density \( \kappa \), \( Y \) is a subspace of \( X \) of density smaller than \( \kappa \) and \( y^* \in S_{Y^*} \) is such that \( \chi(x^*, B_{Y^*}) = \kappa \) for all points \( x^* \in E(y^*) \). Suppose that \( D \subseteq X \) is of cardinality \( \kappa \) and such that \( D \setminus E \subseteq X \) is linearly dense in \( X \) for every \( E \subseteq D \) of cardinality less than \( \kappa \).

Then the set of all \( x^* \in S_X \), for which there is \( D' \subseteq D \) of cardinality \( \kappa \) such that the set \( \{ x^*(d) : d \in D' \} \) is a singleton is weakly* dense in \( E(y^*) \). In particular, \( X \) does not admit an overcomplete set.

**Proof.** Let us first conclude the last part of the lemma from the main part. Suppose that \( D \subseteq X \) is overcomplete. So using the main part of the lemma find an \( x_1^* \in E(y^*) \) and \( D' \subseteq D \) of cardinality \( \kappa \) such that \( \{ x_1^*(d) : d \in D' \} = \{ r \} \) for an \( r \in \mathbb{R} \). If \( r = 0 \) we conclude that \( D' \) is a subset of a hyperplane which contradicts the hypothesis that \( D \) is overcomplete. If \( r \neq 0 \), use the fact that \( D' \) is linearly dense, since \( D \) is overcomplete and use again the main part of the lemma finding \( D'' \subseteq D' \) of cardinality \( \kappa \) and a \( x_2^* \in E(y^*) \) such that \( \{ x_2^*(d) : d \in D'' \} = \{ s \} \) for an \( s \in \mathbb{R} \). The functional \( x_2^* \) can be taken different from \( \pm x_1^* \) (using the fact that \( E(y^*) \) cannot contain both of the \( x_1^* \) and \(-x_1^* \)) as they both cannot extend \( y^* \) and using the fact that \( E(y^*) \) is not a singleton by Lemma 7 and so linearly independent from \( x_1^* \).

So for \( z^* = \frac{s}{r} x_1^* - x_2^* \neq 0 \) we have \( D'' \subseteq \{ x \in X : z^*(x) = 0 \} \) which contradicts the hypothesis that \( D \) is overcomplete.
So now, let us turn to the proof of the main part of the lemma. Let \( D = (d_\alpha : \alpha < \kappa) \) be an enumeration of \( D \). Let \( U = \{ x^* \in X^* : x^*(x_i) \in I_i, 1 \leq i \leq k \} \cap E(y^*) \) be a nonempty weakly* open subset of \( E(y^*) \) where \( x_1, ..., x_k \in X, k \in \mathbb{N} \) and \( I_i \)'s are nonempty open intervals in \( \mathbb{R} \).

First let us prove that there is a closed subspace \( Y \subseteq W \subseteq X \) of density less than \( \kappa \) with \( x_1, ..., x_k \in W \) and a functional \( w^* \in W^* \) of norm one satisfying \( w^*|Y = y^* \) and \( w^*(x_i) \in I_i \) for all \( 1 \leq i \leq k \) and there is a nondegenerate open interval \( I \subseteq \mathbb{R} \) such that

1. for every closed \( Z \) of density smaller than \( \kappa \) satisfying \( W \subseteq Z \subseteq X \) and
2. for every \( z^* \in S_{Z^*} \) satisfying \( z^*|W = w^* \) and
3. for every \( A \subseteq \kappa \) satisfying \( |A| < \kappa \) there is \( \beta \in \kappa \setminus A \) such that

\[ I \subseteq [z^*|(d_\beta)]. \]

Indeed, if this was not the case, then for every closed subspace \( Y \subseteq W \subseteq X \) of density smaller than \( \kappa \) such that \( x_1, ..., x_k \in W \) and every norm one functional \( w^* \in S_{W^*} \) satisfying \( w^*|Y = y^* \) and \( w^*(x_i) \in I_i \) for all \( 1 \leq i \leq k \) and every nondegenerate interval \( I \) with rational endpoints there is a closed \( Z \) of density smaller than \( \kappa \) satisfying \( W \subseteq Z \subseteq X \) and \( z^* \in S_{Z^*} \) satisfying \( z^*|W = w^* \) and \( A \subseteq \kappa \) satisfying \( |A| < \kappa \) such that for every \( \beta \in \kappa \setminus A \) we have \( I \not\subseteq [z^*|(d_\beta)] \).

Let \( Y_1 \) be the subspace of \( X \) generated by \( Y \) and \( x_1, ..., x_k \) and let \( y_1 \in E(y^*) \cap U \). Enumerating all nondegenerate intervals with rational endpoints as \( (J_n)_{n \in \mathbb{N}} \) we could recursively construct increasing sequence \( (W_n)_{n \in \mathbb{N}} \) of closed subspaces of \( X \) of densities smaller than \( \kappa \) and increasing subsets \( (A_n)_{n \in \mathbb{N}} \) of \( \kappa \) of cardinals smaller than \( \kappa \) and \( (w_n^*)_{n \in \mathbb{N}} \) satisfying \( W_0 \supseteq Y_1 \), \( w_0|Y_1 = y_1|Y_1 \) and \( w_n \in S_{W_n} \) and \( w_{n+1}|W_n = w_n \) for every \( n \in \mathbb{N} \) and \( w_n(x_i) \in I_i \) for all \( 1 \leq i \leq k \) and \( J_n \not\subseteq [w_n^*|(d_\beta)] \) for every \( \beta \in \kappa \setminus A_n \). Take \( W \) to be the closure of \( \bigcup_{n \in \mathbb{N}} W_n \) and \( w^* \in S_{W^*} \) to be the unique functional satisfying \( w^*|W_n = y_n \) for each \( n \in \mathbb{N} \) and put \( A = \bigcup_{n \in \mathbb{N}} A_n \). \( W \) has density smaller than \( \kappa \) and \( A \) has cardinality smaller than \( \kappa \) by the uncountable cofinality of \( \kappa \). So \( (d_\xi : \xi \in \kappa \setminus A) \) is linearly dense by the hypothesis of the lemma and by Lemma \[2\] there is \( n \in \mathbb{N} \) such that \( J_n \subseteq [w^*|(d_\beta)] \subseteq [w_n^*|(d_\beta)] \) for some \( \beta \in \kappa \setminus A \subseteq \kappa \setminus A_n \). But this contradicts the choice of \( w_n^* \) and completes the proof of the existence of \( W, w^*, I \) as in \( (1) - (3) \).

So let \( W, w^*, I \) be as in \( (1) - (3) \). Let \( r \in I \). Now by transfinite recursion we can construct an increasing sequence \( (Z_\xi)_{\xi < \kappa} \) of closed subspaces of \( X \) and a sequence \( (z_\xi)_{\xi < \kappa} \) and a sequence \( (\alpha_\xi)_{\xi < \kappa} \) of distinct elements of \( \kappa \) such that

- \( Z_0 = W, z_0^* = w^* \),
- \( Z_\xi \) has density not bigger than the maximum of the density of \( W \) and the cardinality of \( \xi \),
- \( z_\xi^* \in S_{Z_\xi} \),
- \( z_\xi^*|Z_\eta = z_\eta^* \) for every \( \eta < \xi < \kappa \),
- \( z_{\xi+1}(d_\alpha) = r \).

Given \( Z_\xi, z_\xi^* \) and \( \{ \alpha_\eta : \eta < \xi \} \) as above, use \( (1) - (3) \) to find \( \alpha_\xi \in \kappa \setminus \{ \alpha_\eta : \eta < \xi \} \) and \( Z \supseteq Z_\xi \) and \( z^* \in S_{Z^*} \) such that \( z^*(d_\alpha) = r \in I \). Now define \( Z_{\xi+1} \) as the subspace of \( X \) generated by \( Z_\xi \) and \( d_\alpha \) and \( z_{\xi+1}^* \in S_{Z_{\xi+1}} \) such that \( z_{\xi+1}^* = z^*|Z_{\xi+1} \).

Then we also have \( z_{\xi+1}^*(d_\alpha) = r \). At a limit stage \( \lambda < \kappa \) define \( Z_\lambda = \bigcup_{\xi < \lambda} Z_\xi \) and \( z_\lambda^* \) to be a norm one extension of \( \bigcup_{\xi < \lambda} z_\xi^* \) to \( Z_\lambda \).
Let $Z = \bigcup_{\xi < \kappa} Z_\xi$ and $z^* \in Z^*$ be such that $z^*|Z_\xi = z^*_\xi$ for every $\xi < \kappa$. By extending $z^*$ to $X$ we have $x^* \in X^*$ such that $x^*(d_\alpha) = r$ for all $\xi < \kappa$, moreover such an $x^*$ is in $U \cap E(y^*)$ since $x^*|Y_1 = w^*|Y_1 = y^*_1|Y_1$ as required.

As a corollary we obtain the following:

**Proposition 27.** Let $X$ be a Banach space of density $\kappa$ of uncountable cofinality. If $\chi(x^*, B_{X^*}) = \kappa$ for every $x^* \in S_{X^*}$, then $X$ does not admit an overcomplete set.

**Lemma 28.** Suppose that $\kappa$ is an uncountable cardinal. There is an injective Banach space $X_\kappa$ and its subspace $Y_\kappa$ of density $\kappa$ such that $\chi(z^*, B_{Z^*}) \geq \kappa$ for any $z^* \in S_{Z^*}$ and any subspace $Z$ satisfying $Y_\kappa \subseteq Z \subseteq X_\kappa$.

**Proof.** Consider the Boolean algebra $B$ of all clopen subsets of $\{0, 1\}^\kappa$ with the product topology and the Boolean completion $A$ of $B$, that is a complete Boolean algebra, where $B$ is dense, i.e., such that for every $a \in A \setminus \{0\}$ there is $b \in B \setminus \{0\}$ satisfying $b \leq a$, where $\leq$ is the Boolean order. $B$ is generated as a Boolean algebra by sets $b_\alpha = \{x \in \{0, 1\}^\kappa : x(\alpha) = 1\}$ for $\alpha < \kappa$. By the density of $B$ in $A$ every element $a \in A$ is the supremum of a maximal pairwise disjoint set of elements of $B$. As $B$ is c.c.c. its pairwise disjoint sets of elements are at most countable. For every $A \subseteq \kappa$ consider the Boolean algebra $B_A$ generated by $\{b_\alpha : \alpha \in A\}$ and the Boolean algebra $A_A \subseteq A$ of all Boolean suprema in $A$ of all sets of elements of $B_A$. It is easy to check that $A_A$ is a (complete) subalgebra of $A$ for every $A \subseteq \kappa$. So by the previous observation we have

$$A = \bigcup \{A_A : A \subseteq \kappa, |A| \leq \omega\}.$$  

Let $K$ be the Stone space of $A$, i.e., a compact Hausdorff space whose algebra of clopen sets is isomorphic to $A$ (we will identify it with $A$). As $A$ is complete, $K$ is extremely disconnected and in particular it is totally disconnected and $C(K)$ is an injective Banach space. Put $X_\kappa = C(K)$. As $K$ is totally disconnected, 

$$X_\kappa = C(K) = \overline{\text{lin}}(\left\{1_U : U \text{ clopen in } K\right\}) = \overline{\text{lin}}(\left\{1_a : a \in A\right\}).$$

We can define some subspaces of $X_\kappa$:

$$Y_\kappa = \overline{\text{lin}}(\left\{1_a : a \in B\right\}),$$

$$X_A = \overline{\text{lin}}(\left\{1_a : a \in A_A\right\}).$$

As the cardinality of $B$ is $\kappa$, it is clear that the density of $Y_\kappa$ is $\kappa$ as required. Again, since $K$ is totally disconnected, every element of $C(K)$ can be approximated by a sequence of linear combinations of characteristic functions of clopen sets, so

$$X_\kappa = \bigcup \{X_A : A \subseteq \kappa, |A| \leq \omega\}.$$  

and consequently every subspace $W \subseteq X_\kappa$ of density less than $\kappa$ is included in a subspace of the form $X_A \subseteq X_\kappa$ where $|A| < \kappa$.

Now fix a subspace $Z \subseteq X_\kappa$ such that $Y_\kappa \subseteq Z$ and let us prove that $\chi(z^*, B_{Z^*}) \geq \kappa$ for every $z^* \in S_{Z^*}$. Using Lemma 7 it is enough to prove that given a subspace $W \subseteq Z$ of density less than $\kappa$ and $w^* \in S_{W^*}$ there are $x_1^*, x_2^* \in S_{X^*_\kappa}$ such that $x_1^*|W = w^* = x_2^*|W$ and there is $\alpha \in \kappa$ such that $x_1^*(1_{b_\alpha}) \neq x_2^*(1_{b_\alpha})$ (since $b_\alpha \in Y_\kappa \subseteq Z$).

Let $A \subseteq \kappa$ be such that $|A| < \kappa$ and $W \subseteq X_A$. By the Hahn-Banach theorem extend $w^*$ to an element $v^*$ of $S_{X^*_A}$. Let $\alpha$ be any element of $\kappa \setminus A$. Note that $X_A$ is
isometric to $C(K_A)$, where $K_A$ is the Stone space of the Boolean algebra $A$ as it is generated by the characteristic functions of elements of $A$. Consider the Boolean subalgebra $C$ of $A$ generated by $A \cup \{b_\alpha\}$. Elements of $A$ are suprema of elements of $B$ and $\alpha \notin A$, so for each nonzero element $a$ of $A$ we have $a \cap b_\alpha \neq \emptyset \neq a \setminus b_\alpha$.

It follows that each ultrafilter of $A$ (i.e., a point of $K_A$) has exactly two extensions to ultrafilters in $C$, one containing $b_\alpha$ and one containing its complement. It follows that the Stone space $L$ of $C$ is homeomorphic to $K_A \times \{0, 1\}$. We will identify it with $K_A \times \{0, 1\}$. Note that under this identification $b_\alpha = K_A \times \{1\}$ and $L \setminus b_\alpha = K_A \times \{0\}$ and the embedding $\iota$ of $C(K_A)$ into $C(L)$ is given by $\iota(f) = f \circ \pi$, where $\pi : K_A \times \{0, 1\} \to K_A$ is the canonical projection.

By the Riesz representation theorem let $\mu$ be a Radon measure on $K_A$ corresponding to $v^*$. Consider two measures $\nu_0, \nu_1$ on $L = K_A \times \{0, 1\}$ given by

$$\nu_i(V_0 \times \{0\} \cup V_1 \times \{1\}) = \mu(V_i)$$

for Borel subsets $V_0, V_1$ of $K_A$ and $i = 0, 1$. They satisfy $\mu(f) = \nu_0(f \circ \pi) = \nu_1(f \circ \pi)$ for $f \in C(K_A)$, $\nu_i(C(K_A)) = \nu_i |X_A = v^*$ for $i = 0, 1$. Also the variation norm of each $\nu_i$ is the same as for $\mu$, hence it is one. Moreover $\nu_i(b_\alpha) = i$. So by the Hahn-Banach theorem there are two norm one extensions $x_1^*, x_2^* \in S_{X^*}$ of $\nu_1$ and $\nu_1$ respectively. They are extensions of $v^*$ satisfying $x_1^*(1_{b_\alpha}) \neq x_2^*(1_{b_\alpha})$ which completes the proof of the required property of $v^*$ and the proof of the lemma.

\[\square\]

**Theorem 29.** Let $\kappa$ be a cardinal of uncountable cofinality. If $X$ is a Banach space of density $\kappa$ which contains an isomorphic copy of $\ell_1(\kappa)$, then $X$ does not admit an overcomplete set. Consequently the following Banach spaces do not admit overcomplete sets:

1. $C(K)$ for any infinite extremally disconnected compact Hausdorff $K$.
2. $\ell_\infty(\lambda), \ell_\infty(\lambda)/c_0(\lambda), L_\infty([0, 1]^{\lambda})$ for any infinite cardinal $\lambda$.
3. $C([0, 1]^\kappa), C([0, 1]^{\kappa})$.

**Proof.** Let $Y_\kappa \subseteq X_\kappa$ be as in Lemma $28$. The universal property of $\ell_1(\kappa)$ implies that there is a surjective bounded linear operator $T : \ell_1(\kappa) \to Y_\kappa$. By the injectivity of $X_\kappa$, there is a bounded linear extension $R : X \to X_\kappa$ of $T$. Applying Lemma $28$ for $R[X]$ we conclude that $\chi(x^*, B_{R[X]}) = \kappa$ for every $x^* \in S_{R[X]}$. So Proposition $27$ implies that $R[X]$ does not admit an overcomplete set. Now Lemma $2$ yields that $X$ does not admit an overcomplete set. To conclude the second part of the theorem we will note that the Banach spaces in question contain appropriate nonseparable versions of $\ell_1$ and will use the first part of the Theorem.

1. $K$ is extremally disconnected if and only if the Boolean algebra $Clop(K)$ of clopen subsets of $K$ is complete. By Balcar-Franek theorem (5) $Clop(K)$ contains an independent family $\mathcal{F}$ of cardinality equal to $|Clop(K)|$. $\{A \setminus K \setminus A : A \in \mathcal{F}\}$ generates a copy of $\ell_1(|Clop(K)|)$ in $C(K)$. As $K$ is totally disconnected, we have $|Clop(K)| = \text{dens}(C(K))$, so we have $\ell_1(\text{dens}(C(K))) \subseteq C(K)$. To use the first part of the theorem it is now enough to note that $cf(\text{dens}(C(K))) > \omega$. This is because $|Clop(K)|^{\omega} = |Clop(K)|$ by a theorem of Pierce (31) and $cf(\kappa^{\omega}) > \omega$ for any cardinal $\kappa$ by the König Theorem (5.13 of [24]).

2. The spaces $\ell_\infty(\lambda)$ are isomorphic to the spaces $C(\beta \lambda)$ respectively and $\beta \lambda$ is extremally disconnected, so apply (1). The spaces $L_\infty([0, 1]^{\lambda})$ are isomorphic to the spaces $C(HY_\lambda)$ respectively, where $HY_\lambda$ is the Hewitt-Yosida space, i.e. the
Stone space of the homogeneous measure algebra of Maharam type $\lambda$. $HY_{\lambda}$ is extremally disconnected, so apply (1).

To prove the nonexistence of overcomplete sets in the spaces $X = \ell_{\infty}(\lambda)/c_0(\lambda)$ we note that the quotient map is an isometry on the copy of $\ell_1(\kappa)$ for $\kappa = 2^\lambda$ of the form $\{1_A - 1_{K\setminus A} : A \in \mathcal{F}\}$ from the proof of (1). This is because the intersections in infinite independent families must be infinite and the only characteristic functions of clopen sets which are in $c_0(\lambda)$ are characteristic functions of finite sets.

(3) The coordinate functions in $C([-1,1]^\kappa)$ or $C([-1,1]^\kappa)$ generate a copy of $\ell_1(\kappa)$ and obviously these spaces are isometric to $C([0,1]^\kappa)$ or $C([0,1]^\kappa)$ respectively. □

A nice characterization of Banach spaces containing $\ell_1(\kappa)$ for $\kappa$ of uncountable cofinality can be found in [12].

We observe that the nonexistence of overcomplete sets in all Banach spaces $X$ which contain $\ell_1(\kappa)$ of density $\omega_1$ can be directly concluded from Theorem 3.6 of [37] (In this paper Theorem 11.7 of [4]) which says that a nonseparable Banach space has a fundamental biorthogonal system if it has a weakly compactly generated quotient of the same density. This argument covers all Banach spaces considered in Theorem 29 of densities bigger or equal to $\omega_2$. Theorem 29 provides new results for densities $\omega_1$ but also provides a uniform ZFC proof for spaces like $\ell_\infty$, $\ell_\infty/c_0$, $L_\infty([0,1]^{\omega_1})$ whose densities may be equal to $\omega_1$ or may be bigger than $\omega_1$ depending on CH.

Lemma 30 ([30] 35). Let $\kappa$ be an infinite cardinal. If $X$ is a Banach space containing an isomorphic copy of $\ell_1(\kappa)$, then $X$ admits $\ell_2(\kappa)$ as its quotient.

Proof. If a Banach space $X$ contains an isomorphic copy of $\ell_1(\kappa)$, then $X^*$ contains an isomorphic copy of $(C([0,1]^{\kappa}))^*$ by Proposition 3.3. of [30], hence $X^*$ contains an isomorphic copy of $L_1([0,1]^{\kappa})$. Now, following Proposition 1.5 of [35], we prove that $L_1([0,1]^{\kappa})$ contains an isomorphic copy of $\ell_2(\kappa)$. For this we need to note that $\{0,1\}^\kappa$ is a compact abelian group and the product measure is its Haar measure and that $\{\gamma_\xi : \xi < \kappa\}$ forms a set of independent characters in the sense of Definition 1.4 of [35], where $\gamma_\xi(x) = x(\xi)$ for $x \in \{0,1\}^\kappa$ and $\xi < \kappa$. Let $T : \ell_2(\kappa) \to X^*$ be the obtained isomorphic embedding. Let $J : X \to X^{**}$ be the canonical embedding. So $T^* \circ J : X \to (\ell_2(\kappa))^{**} \cong \ell_2(\kappa)$. To complete the proof it is enough to show that $T^* \circ J$ is surjective. For this it is enough to show that $(T^* \circ J)^*$ is injective. But as $T^{**}$ is $T$ since $\ell_2(\kappa)$ is reflexive, we have that $(T^* \circ J)^* = J^* \circ T^{**} = J^* \circ T = T$ which is injective. □

Corollary 31. Let $X$ be a Banach space of density $\omega_1$ with an unconditional basis. $X$ admits an overcomplete set if and only if $X$ is WLD.

Proof. If $X$ is WLD and of density $\omega_1$, then $X$ admits an overcomplete set by Theorem 11.11 By Theorem 1.7 of [4] a Banach spaces with an unconditional basis is WLD if and only if $\ell_1(\omega_1)$ does not isomorphically embed into $X$. So if $X$ is not WLD we have a copy of $\ell_1(\omega_1)$ in $X$ and may conclude that $X$ does not admit an overcomplete set using Theorem 29. □

Corollary 32. Suppose that $\kappa > \omega_1$ is a cardinal of uncountable cofinality and that $X$ is a Banach space of density $\kappa$ whose dual contains an isomorphic copy of $L_1([0,1]^{\kappa})$. Then $X$ does not admit an overcomplete set.
Corollary 33. Whether every Banach space of density \( \omega_1 \) whose dual contains \( L_1(\{0,1\}^{\omega_1}) \) admits an overcomplete set is undecidable.

Proof. By Argyros’ solution of Pelczynski’s conjecture ([H]) if \( \kappa > \omega_1 \) and the dual of a Banach space \( X \) contains \( L_1(\{0,1\}^\kappa) \), then \( X \) contains \( \ell_1(\kappa) \). Now apply Theorem 29.

Corollary 34. \((p = c > \omega_1)\) No nonreflexive Grothendieck space of regular density (in particular equal to \( c \)) admits an overcomplete set.

Proof. Assume \( p = c > \omega_1 \). The cardinal \( p \) is a regular cardinal (Theorem 3.1. of [13]). It is proved in [21] that under the assumption \( p = c > \omega_1 \) every nonreflexive Grothendieck space has \( \ell_\infty \) as a quotient (in fact, it is concluded from the existence of an isomorphic copy of \( \ell_1(\kappa) \) in the space). If the density of \( X \) is \( c \), then the above result and Lemma 2 and Theorem 29 imply that \( X \) does not admit an overcomplete set. If the density of \( X \) is regular and bigger than \( c \), then the statement of the corollary follows from results of [37] (which is Theorem 3 of [5] in this paper).

In fact, Theorem 30 excludes in ZFC densities of any cofinality bigger than \(\omega_1\). So the role of the hypothesis \( p = c > \omega_1 \) above is limited to excluding the possibility of the existence of a nonreflexive Grothendieck space of density \( \omega_1 \) (Note that nonreflexive Grothendieck spaces of density \( \omega_1 < c \) consistently exist ([23]). See also the discussion after Question 44).

The following lemma will be used in the proof of Theorem 30.

Lemma 35. Let \( \kappa \) be an infinite cardinal and \( K \) be a Hausdorff compact space. Let \( P(K) \) denote the set of all Radon probability measures on \( K \). Then \( \chi(x^*, B_{C(K)^*}) \geq \kappa \) for all \( x^* \in S_{C(K)^*} \) if and only if \( \chi(p, P(K)) \geq \kappa \) for every \( p \in P(K) \).

Proof. We will interpret Radon measures on \( K \) as functionals on \( C(K) \). Also note that \( P(K) \) is a closed, and so compact subset of \( B_{C(K)^*} \). This is because if \( \mu \in B_{C(K)^*} \setminus P(K) \), then \( \mu(1_K) < 1 \) or \( \mu(f) < 0 \) for some \( f \geq 0 \) and \( f \in C(K) \) and both of these conditions define weak* open sets. It follows that the pseudocharacter of points in \( P(K) \) relative to \( P(K) \) is equal to their character relative to \( P(K) \).

The forward implication follows from the fact that \( P(K) \) is a \( G_\delta \) subset of \( B_{C(K)^*} \) being equal to the intersection of all sets \( \{ \mu \in B_{C(K)^*} : \mu(1_K) > 1 - 1/n \} \) for \( n \in \mathbb{N} \). Hence \( \mu \in P(K) \) satisfying \( \chi(p, P(K)) < \kappa \) would also have a pseudobasis in \( B_{C(K)^*} \) of cardinality smaller than \( \kappa \).

Now we will focus on proving the backward implication. Assume that \( \chi(p, P(K)) \geq \kappa \) for every \( p \in P(K) \). Suppose that \( \mu \in B_{C(K)^*} \). Any intersection of less than \( \kappa \)-many weakly* open sets containing \( \mu \) includes an intersection of the form
\[
\bigcap_{x \in Y} \bigcap_{n \in \mathbb{N}} \{ \nu \in X^* : |\nu(x) - \mu(x)| < 1/n \} = \bigcap_{x \in Y} \{ \nu \in X^* : \nu(x) = \mu(x) \},
\]
where \( Y \subseteq X \) is of cardinality less than \( \kappa \). So to prove that \( \chi(\mu, B_{C(K)}) \geq \kappa \), which is equivalent to proving that the pseudocharacter of \( \mu \) relative to \( B_{C(K)} \) is bigger or equal to \( \kappa \), it is enough to prove that the intersections of the above form always contain some \( \nu \in B_{C(K)} \setminus \{ \mu \} \).

To do so fix \( Y \subseteq X \) of cardinality less than \( \kappa \) and decompose \( \mu \) as \( \mu = \mu_+ - \mu_- \), where \( \mu_+, \mu_- \) are positive Radon measures on \( K \) with disjoint supports. By extending \( Y \) we may assume that \( 1_K \in Y \). Note that \( \mu_+ / \|\mu_+\|, \mu_- / \|\mu_-\| \in P(K) \).

As the pseudocharacter of \( \mu_+ / \|\mu_+\| \) relative to \( P(K) \) is not smaller than \( \kappa \), there is \( \nu \in P(K) \setminus \{ \mu_+ / \|\mu_+\| \} \) which is in

\[
\bigcap_{x \in Y} \bigcap_{n \in \mathbb{N}} \{ \nu \in X^* : |\nu(x) - (\mu_+ / \|\mu_+\|)(x)| < 1/n \} = \bigcap_{x \in Y} \{ \nu \in X^* : \|\mu_+\| \nu(x) = \mu_+(x) \}.
\]

In particular

\[
\|\mu_+\| \nu(1_K) + \mu_-(1_K) = \mu_+(1_K) - \mu_-(1_K) = \mu(1_K) \leq 1
\]

and \( \|\mu_+\| \nu + \mu_- \) is positive as the sum of two positive measures, so \( (\|\mu_+\| \nu - \mu_-) \in B_{C(K)} \). Moreover \( (\|\mu_+\| \nu - \mu_-)(x) = \mu(x) \) for all \( x \in Y \) and \( \|\mu_+\| \nu - \mu_- \neq \mu \) as \( \nu \neq \mu_+ / \|\mu_+\| \), as required.

\[\square\]

**Theorem 36.** Suppose that \( K \) is an infinite compact Hausdorff space such that \( C(K) \) is Grothendieck space of density \( \omega_1 \). Then \( C(K) \) does not admit an overcomplete set.

**Proof.** As is well know, if \( C(K) \) is Grothendieck, the \( K \) has no nontrivial convergent sequence and so \( K \) is not scattered, in particular there is a perfect \( L \subseteq K \). As an infinite closed subset of \( K \), it must be nonmetrizable, again by the nonexistence of nontrivial convergent sequences. So \( C(L) \) is a quotient of \( C(K) \) of density \( \omega_1 \) and is Grothendieck as this property is preserved by taking quotients. We will prove that \( C(L) \) does not admit an overcomplete set, which is enough by Lemma 2.

Theorem 3.5 and Proposition 5.3. of [27] imply that if \( L \) has no isolated points and \( C(L) \) is Grothendieck, then no probability Radon measure on \( L \) is a \( G_\delta \) point in the space \( P(L) \) of all probability Radon measures on \( L \). Lemma 35 implies that \( \chi(x^*, B_{C(L)}) = \omega_1 \) for every \( x^* \in B_{C(L)} \) and Proposition 24 implies that \( C(L) \) does not admit an overcomplete set, as required.

\[\square\]

Note that it is possible (consistently) that a Grothendieck space \( C(K) \) does not contain a copy of \( \ell_1(\omega_1) \). Such an example was constructed by Talagrand in [41] under CH.

6. **Negative results for densities of cofinality bigger than \( \omega_1 \)**

**Theorem 37.** Let \( \kappa \) be a cardinal satisfying \( \text{cf}(\kappa) > \omega_1 \). If \( X \) is a Banach space of density \( \kappa \) containing an isomorphic copy of \( \ell_1(\omega_1) \), then \( X \) does not admit an overcomplete set.

**Proof.** Let \( T : \ell_1(\omega_1) \to L_\infty(\{0,1\}^{<\omega}) \) be a linear bounded operator such that \( T(1_{\alpha}) = x_\alpha \) for all \( \alpha < \omega_1 \), where \( x_\alpha \) is the \( \alpha \)-th coordinate function. It exists by the universal property of \( \ell_1(\omega_1) \). Let \( S : X \to L_\infty(\{0,1\}^{<\omega}) \) be an extension of \( T \) obtained using the injectivity of the space \( L_\infty(\{0,1\}^{<\omega}) \). For each \( \alpha < \omega_1 \) consider the subspace \( Y_\alpha \) of \( L_\infty(\{0,1\}^{<\omega}) \) consisting of all elements which depend on coordinates in below \( \alpha \). The union \( \bigcup_{\alpha < \omega_1} Y_\alpha \) is the entire space and \( x_\alpha \notin Y_\alpha \) for any \( \alpha < \omega_1 \). It follows that \( S^{-1}[Y_\alpha] \) for \( \alpha < \omega_1 \) form a strictly increasing
sequence of proper subspaces of $X$. So Lemma $\text{3}$ implies that $X$ does not admit an overcomplete set.

\begin{proof}
Let $\kappa$ be a cardinal satisfying $cf(\kappa) > \omega_1$. Suppose that $X$ is a Banach space of density $\kappa$ such that $X^*$ contains a nonseparable WLD subspace. Then $X$ does not admit an overcomplete set.

Proof. Let $Y$ be a nonseparable WLD space and let $T : Y \to X^*$ be an isomorphism onto its image. We may assume that the density of $Y$ is $\omega_1$ as subspaces of WLD spaces are WLD (Corollary 9 of [17]). Let $\{y_\alpha : \alpha < \omega_1\}$ be a linearly dense subset of $Y$ such that each element $y^* \in Y^*$ is countably supported by $\{y_\alpha : \alpha < \omega_1\}$ i.e., $s(y^*) = \{\alpha < \omega_1 : y^*(y_\alpha) \neq 0\}$ is at most countable. The existence of such a linearly dense set is equivalent to being WLD by Theorem 7 of [17].

Let $J : X \to X^{**}$ be the canonical isometric embedding and $S = T^* \circ J : X \to Y^*$.

Note that $Y^* = \bigcup_{\alpha < \omega_1} Z_\alpha$ where

$$Z_\alpha = \{y^* \in Y^* : s(y^*) \subseteq \alpha\} = \bigcap_{\beta \geq \alpha} \ker(J(y_\beta))$$

is a norm-closed subspace of $Y^*$ for each $\alpha < \omega_1$. So to use Lemma $\text{3}$ and conclude that $X$ does not admit an overcomplete set it is enough to note that the subspaces $S^{-1}[Z_\alpha]$ are proper for $\alpha < \omega_1$. To do so choose $x_\alpha \in X$ such that $T(y_\alpha)(x_\alpha) \neq 0$. We get $S(x_\alpha)(y_\alpha) = T(y_\alpha)(x_\alpha) \neq 0$, so $s(S(x_\alpha)) \not\subseteq \alpha$ and so $x_\alpha \not\in S^{-1}[Z_\alpha]$ as required.

\end{proof}

\begin{theorem}
Let $\kappa$ be a cardinal satisfying $cf(\kappa) > \omega_1$. If $X$ is a nonreflexive Grothendieck space of density $\kappa$, then it does not admit an overcomplete set.
\end{theorem}

Proof. In [20] R. Haydon proved that if $X$ is a nonreflexive Grothendieck space, then $X^*$ contains an isomorphic copy of $L_1(\{0,1\}^\mathbb{N})$. As $p \geq \omega_1$, it is a nonseparable WLD subspace. So Theorem $\text{35}$ can be applied.

\begin{proof}
Let $\kappa$ be a cardinal satisfying $cf(\kappa) > \omega_1$. Let $K$ be a scattered compact space of cardinality $\kappa$ (equivalently $C(K)$ has density $\kappa$). Then the Banach space $C(K)$ does not admit an overcomplete set.

Proof. Let $\{f_\xi : \xi < \kappa\} \subseteq C(K)$. We will show that there is $A \subseteq \kappa$ such that $|A| = \kappa$ and $\{f_\xi : \xi \in A\}$ does not separate points of $K$. Let $X \subseteq K$ be of cardinality $\omega_1$. A continuous image of a compact scattered space is scattered. So $f_\xi[X] \subseteq \mathbb{R}$ is countable. It follows that for every $\xi < \kappa$ there is a pair $\{x,y\} \in [X]^2$ such that $f_\xi(x) = f_\xi(y)$. As there are $\omega_1$ pairs in $[X]^2$ and $\kappa$ has cofinality bigger than $\omega_1$, we conclude that there are $x,y \in X$ such that $A = \{\xi < \kappa : f_\xi(x) = f_\xi(y)\}$ has cardinality $\kappa$. Then $\{f_\xi : \xi \in A\}$ does not separate points of $K$, as required.

\end{proof}

\begin{remark}
Note that in the result above we show that for every $D \subseteq C(K)$ which is linearly dense there is $D' \subseteq D$ of the same cardinality which does not generate $C(K)$ even as an algebra. This is a stronger property than not being overcomplete. One notes that this property behaves differently than the property of not being overcomplete. For example, under CH the algebra $\ell_\infty$ contains $D$ such that $D' \subseteq D$
generates $\ell_\infty$ as an algebra for every uncountable $D' \subseteq D$. For this represent $\ell_\infty$ as an increasing sequence of algebras $C(K_\alpha)$ for $\alpha < \omega_1$, where $K_\alpha$s are totally disconnected and metrizable. Choose $f_\alpha \in \ell_\infty$ which separates all points of $K_\alpha$, then $D = \{ f_\alpha : \alpha < \omega_1 \}$ works. On the other hand it is consistent that for any set $\{ T_\alpha : \alpha < \xi \} \subseteq B(\ell_2)$ which generates a subalgebra of $B(\ell_2)$ of density $\xi$ there is a subset $A \subseteq \xi$ of cardinality $\xi$ such that $T_\alpha$ is not in the algebra generated by $\{ T_\beta : \beta \in A \setminus \{ \alpha \} \}$ for any $\alpha \in A$ (22). This applies to $\ell_\infty \subseteq B(\ell_2)$.

After this paper has been completed and submitted it was proved in [16] that it is consistent (with any possible size of the continuum) that no Banach space of density $\kappa$ with $\text{cf}(\kappa) > \omega_1$ admits an overcomplete set (Theorem 8 of [16]).

### 7. Final remarks and questions

The main topic of this paper is to determine which nonseparable Banach spaces admit overcomplete sets. The first natural question would be to determine the densities of Banach spaces which admit overcomplete sets as we now know that there are such nonseparable spaces. Here the most restrictive possibility would be to prove the positive answer to the following:

**Question 41.** Can one prove in ZFC that if a Banach space admits an overcomplete set, then $\text{dens}(X) \leq \omega_1$?

Under CH the results of [37] (in this paper Theorem 5 (4)) imply that no Banach space of density $\omega_1$ for $n > 1$ can admit an overcomplete set. As mentioned at the end of section 6 after this paper has been completed and submitted it was proved in Theorem 3 of [16] that it is consistent (with any possible size of the continuum) that every nonseparable Banach space is a union of $\omega_1$ of its hyperplanes, consequently it is consistent that no Banach space of density $\kappa$ with $\text{cf}(\kappa) > \omega_1$ admits an overcomplete set (Theorem 8 of [16]). No consistent example of an overcomplete set in a Banach space of density $\omega_2$ is known. On the other hand it is also left open if in ZFC there can be overcomplete sets of singular cardinalities (cf. Question 1.2 (ii) of [37]). In the light of the results of Section 6 this is especially interesting for cardinals of cofinalities $\omega$ or $\omega_1$:

**Question 42.** Is there (in ZFC or consistently) a Banach space of singular density which admits an overcomplete set?

Note that it is not uncommon that completely distinct phenomena take place in Banach spaces of some singular densities (see [4], [3]). Recall that the matter of densities of Banach spaces which admit overcomplete sets was completely settled for WLD spaces by Corollary 12.

For the lack of examples of Banach spaces with overcomplete sets of densities above $\omega_1$ the next natural question is to characterize Banach spaces of density $\omega_1$ which admit an overcomplete set. Such characterization as being WLD was obtained in Corollary 31 for Banach spaces of density $\omega_1$ with an unconditional basis. By Corollary 21 such a characterization for the general class of Banach spaces of density $\omega_1$ cannot be in terms of properties which do not change when we pass from one model of set theory to another. However we can also aim at characterizations under additional set-theoretic hypotheses. For example under CH having the dual of cardinality not bigger than $\omega_1$ can serve as such a characterization by the results of [37] if we have the positive answer to the following:
Question 43. Does CH imply that no Banach space whose dual has density bigger than \( \omega_1 \) admits an overcomplete set?

As noted in the Introduction under the negation of CH there are nonseparable Banach spaces whose duals have cardinality \( c > \omega_1 \) which admit overcomplete sets. But we do not know what is ZFC answer to the following

Question 44. Is it true that no Banach space whose dual has density bigger than \( c \) admits an overcomplete set?

On the other hand taking into account Theorem 20 one could hope for another consistent characterization of Banach spaces of density \( \omega_1 \) which admit overcomplete sets as spaces with monolithic dual balls in the weak* topology:

Question 45. Does MA and the negation of CH imply that every Banach space whose dual ball is monolithic in the weak* topology admit an overcomplete set?

Actually the above question seems open even in ZFC. Here the key case could be the following

Question 46. Does \( C(K) \) admit an overcomplete set if \( K \) is the ladder system space of \( [33] \)?

The results of section 4 shed some light on the relation between the existence of overcomplete sets and the cardinal characteristics of the continuum in the sense of [8]. Recall that MA for partial orders having precaliber \( \omega_1 \) implies that \( p = c \) and \( \text{add}(\mathcal{M}) = c \) (2.15 and 2.20 of [28]). So by Theorem 23 the statement that every Banach space whose dual has density \( \omega_1 \) admits an overcomplete set is consistent with all cardinal invariants in van Douwen’s diagram being \( c \) and all cardinal invariants in the Cichoń’s diagram above or equal to \( \text{add}(\mathcal{M}) \) being \( c \). This is somewhat surprising because, clearly overcomplete sets cannot exist in Banach spaces of density \( \omega_1 \) if we can cover their sets of cardinality \( \omega_1 \) by countably many hyperplanes. Moreover, hyperplanes are examples of nowhere dense sets but \( \text{cov}(\mathcal{M}) \) is the minimal cardinality of a subset of \( \mathbb{R} \) which cannot be covered by the union of countably many nowhere dense sets and has value \( c > \omega_1 \) under the above version of Martin’s axiom.

On the other hand Pawlikowski proved that MA for partial orders having precaliber \( \omega_1 \) is consistent with \( \text{cov}(\mathcal{N}) = \omega_1 < c = \omega_2 \) (29). Moreover, the main technical result of Section 4 that is Lemma 15 consists of a construction of an element of \( \ell_1 \) and it is known that the structure of \( \ell_1 \) is related to cardinal characteristics of the measure rather than category (e.g. Theorem 2.3.9 of [7], [6]). So it is natural to ask the following:

Question 47. Is it true that Banach spaces \( X \) with separable dual balls in the weak* topology satisfying \( \text{dens}(X) < \text{cov}(\mathcal{N}) \) and \( \text{cf}(\text{dens}(X)) > \omega \) do not admit overcomplete sets?

There are also some natural open questions left which concern more particular classes for Banach spaces like the following three questions:

Question 48. Can one prove in ZFC that the Banach spaces \( C([0, \xi]) \) for all ordinals \( \xi < \omega_2 \) admit overcomplete sets?

Partial results related to Question 48 are Theorem 11 (2) and result of [37] (Theorem 3 in this paper). They cover all remaining ordinals as \( C([0, \xi]) \) admits a fundamental biorthogonal system \( (1_{[0, \eta]}, \delta_\eta - \delta_{\eta+1})_{\eta < \xi} \) and \( 1_{[0, \xi]}, \delta_\xi \).
**Question 49.** Can one prove in ZFC that no nonreflexive Grothendieck space admits an overcomplete set?

Partial results related to Question 49 are Corollary 34, Theorem 36, and Theorem 39. Although the above negative ZFC results do not imply the positive answer to Question 49 the exotic $C(K)$s with the Grothendieck property which we know from the literature are covered by our results. For example examples of Brech ([9]), Fajardo ([23]) and Sobota and Zdomskyy ([40]) contain $\ell_1(\text{dens}(C(K)))$ so do not admit an overcomplete set by Theorem 29. Talagrand’s example from [41] does not contain $\ell_1(\omega_1)$ but is covered by Theorem 36. Haydon’s example of [19] is induced by a Boolean algebra which satisfies the subsequential completeness property and so has the weak subsequential separation property of [26]. Consequently by the results of [26] it contains an independent family of size $\mathfrak{c}$, which yields $\ell_1(\mathfrak{c})$ and implies that there is no overcomplete set by Theorem 29.

**Question 50.** Can one prove in ZFC that no Banach space of the form $C(K)$, where $K$ is scattered compact Hausdorff space of cardinality bigger than $\omega_1$ admits an overcomplete set?

This would improve Theorem 40.

Looking at our results, it seems also intriguing if the property of admitting an overcomplete set behaves well with respect to canonical operations on Banach spaces. For example:

**Question 51.** Is the admitting overcomplete sets a hereditary property with respect to closed subspaces of the same density?

This would be a generalization of the main part of Theorem 29. We also do not know the answer to the following:

**Question 52.** Does the direct sum of two Banach spaces that admit overcomplete sets admit an overcomplete set? In particular does $X \oplus \mathbb{R}$ admit an overcomplete set if $X$ does so?

A positive answer to the above question would simplify the conclusion of Theorem 19. At the end we note that admitting an overcomplete set is not (at least consistently) a three space property: the space $C(K)$ of Corollary 21 satisfies $C(K)/c_0 \equiv c_0(\omega_1)$ but consistently does not admit an overcomplete set, while $c_0$ and $c_0(\omega_1)$ admit overcomplete sets by Theorem 4 and Theorem 11 (1) (c).

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