PARAMETRIC INEQUALITIES AND WEYL LAW FOR THE VOLUME SPECTRUM

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ABSTRACT. We show that the Weyl law for the volume spectrum in a compact Riemannian manifold conjectured by Gromov can be derived from parametric generalizations of two famous inequalities: isoperimetric inequality and coarea inequality. We prove two such generalizations in low dimensions and obtain the Weyl law for 1-cycles in 3-manifolds. We also give a new proof of the Almgren isomorphism theorem.

1. INTRODUCTION

Consider the space of \( k \)-dimensional Lipschitz cycles in a Riemannian manifold \( M \) with coefficients in an abelian group \( G \). One can define the flat semi-norm \( F(x) \) on this space as the infimum of volumes of Lipschitz chains with boundary \( x \); define flat distance as \( F(x, y) = F(x - y) \) \([7, 15\text{, Appendix 1}]\). If we consider the quotient of this space by the equivalence relation \( x \sim y \) if \( F(x, y) = 0 \) and then take its completion, then we obtain the space of flat \( k \)-cycles \( Z_k(M; G) \) with flat distance \( F \). This is a natural space to consider when trying to find submanifolds that solve a certain calculus of variations problem (like minimal surfaces) without a priori specifying the topological type of the solution.

In the 1960’s Almgren initiated a program of developing Morse theory on the space \( Z_k(M; G) \). It follows from Almgren’s work that there is a non-trivial cohomology class \( \alpha \in H^{n-k}(Z_k(M; Z_2); Z_2) \) corresponding to a sweepout of \( M \) by an \((n - k)\)-parameter family of \( k \)-cycles \( F : X \to Z_k(M; Z_2) \) with \( F^*(\alpha) \neq 0 \in H^{n-k}(X; Z_2) \). Moreover, all cup powers \( \alpha^p \) are also non-trivial and the corresponding families of cycles we call \( p \)-sweepouts. Corresponding to each \( p \) we can define the \( p \)-width

\[
\omega_p^k(M) = \inf \left\{ \sup_{x \in X} \text{Vol}_k(F(x)) : F \text{ is a } p \text{-sweepout} \right\}
\]

The \( p \)-widths correspond to volumes of certain generalized minimal submanifolds that arise via a min-max argument. In dimension \( k = n - 1, 3 \leq n \leq 7 \), they are smooth minimal hypersurfaces and for \( k = 1 \) they are stationary geodesic nets. The study of \( p \)-widths led to many remarkable breakthroughs in recent years, including the resolution of the Willmore Conjecture \([23]\) and Yau’s conjecture on existence of
infinitely many minimal surfaces [16], [30]. (See also [37], [3], [25] and references therein).

Gromov suggested to view these widths, or “volume spectrum”, as non-linear analogs of the eigenvalues of the Laplacian [11], [12], [13]. This framework is very useful for obtaining new results about widths and some other non-linear min-max geometric quantities (see, for example, [9], [19], [27]). In line with this analogy Gromov conjectured that \( \omega^k_p \)’s satisfy an asymptotic Weyl law:

\[
\lim_{p \to \infty} \frac{\omega^k_p(M)}{p^{n-k}} = a(n, k) \text{Vol}(M)^{\frac{k}{n}}
\]

For \( k = n - 1 \) this was proved in [21].

In this paper we show that for \( k < n - 1 \) Gromov’s conjecture can be reduced to proving parametric versions of two famous inequalities in geometry: the isoperimetric inequality and coarea inequality. We prove these parametric inequalities in low dimensions and as a consequence obtain the Weyl law for 1-cycles in 3-manifolds.

**Theorem 1.1.** For every compact Riemannian 3-manifold \( M \)

\[
\lim_{p \to \infty} \frac{\omega^1_p(M)}{p^{\frac{3}{2}}} = a(3, 1) \text{Vol}(M)^{\frac{1}{n}}
\]

We do not know the value of constant \( a(3, 1) \). In [4] Chodosh and Mantoulidis proved that optimal sweepouts of the round 2-sphere are realized by zero sets of homogeneous polynomials and that \( a(2, 1) = \sqrt{\pi} \), but for all other values of \( n > k \geq 1 \) constants \( a(n, k) \) remain unknown.

1.1. Some applications of the Weyl law. In [16] Irie, Marques and Neves used the Weyl law to prove that for a generic Riemannian metric on an \( n \)-dimensional manifold, \( 3 \leq n \leq 7 \), the union of embedded minimal hypersurfaces forms a dense set. Marques, Neves and Song proved a stronger equidistribution property of minimal hypersurfaces in [26]; their proof is based on the idea that one can “differentiate” both sides of [1] with respect to some cleverly chosen variations of metric. Similar results were proved by Gaspar and Guaraco using the Weyl law in the Allen-Cahn setting [8] (see also [3]). Li used the Weyl law to prove existence of infinitely many minimal hypersurfaces for generic metrics in higher dimensions, \( n > 7 \) [18]. In [30] Song proved some Weyl law-type asymptotic estimates for certain non-compact manifolds and used them to prove that in dimensions \( 3 \leq n \leq 7 \) for all Riemannian metrics on a compact manifold (not only generic ones) there exist infinitely many minimal hypersurfaces (see also [31] where similar generalizations were applied to prove a “scarring” result for minimal hypersurfaces).
In [29] Song showed that the density result for generic metrics can be obtained without the full strength of the Weyl law. This observation was used in [22] (together with a bumpy metrics theorem for stationary geodesic nets proved in [32]) to prove that for a generic Riemannian metric on a compact manifold $M^n$, $n \geq 2$, stationary geodesic nets form a dense set, even though the Weyl law for 1-cycles in dimensions $n \geq 4$ is not known. In [17] Li and Staffa used Theorem 1.1 to prove an equidistribution result for stationary geodesic nets in 3-manifolds, analogous to that of [26] for minimal hypersurfaces.

1.2. Parametric isoperimetric inequality. The isoperimetric inequality of Federer-Fleming asserts that given a $k$-dimensional Lipschitz cycle $z$ in $\mathbb{R}^n$, there exists a $(k+1)$-dimensional Lipschitz chain $\tau$, such that $\partial \tau = z$ and

\begin{equation}
Vol_{k+1}(\tau) \leq c(n) Vol_k(z)^{\frac{k+1}{k}}
\end{equation}

This inequality is useful when $z$ lies in a ball of large radius. If $z \subset B_R$ with $R \leq Vol_k(z)^{\frac{1}{k}}$, then the inequality simply follows by taking a cone over $z$.

In this paper we will be interested in the situation when a mod 2 cycle $z$ is contained in a small cube (say, of side length 1) and has very large volume. In this case, we can subdivide the cube into smaller cubes of side length $Vol(z)^{-\frac{1}{n-k}}$. Applying the Federer-Fleming deformation in this lattice we can find a “deformation chain” that pushes $z$ into the $k$-dimensional skeleton of this lattice. Since the total volume of the $(k+1)$-skeleton of the lattice is $\sim \left(\frac{1}{Vol(z)^{\frac{1}{n-k}}}\right)^{k+1} Vol(z)^{\frac{n}{n-k}}$ we can then obtain a filling $\tau$ of $z$ with

\begin{equation}
Vol_{k+1}(\tau) \leq c(n) Vol_k(z)^{\frac{n-k-1}{n-k}}
\end{equation}

Note that when $Vol(z) \gg 1$ this is a much better bound than (2) or the cone inequality.

We would like to prove a parametric version of inequality (3), namely, given a family of relative cycles $F : X \to Z_k([0, 1]^n, \partial[0, 1]^n; G)$ we would like to find a continuous family of $(k+1)$-chains $H : X \to I_{k+1}([0, 1]^n; G)$ with $\partial H(x) = F(x)$ in $\partial[0, 1]^n$ and so that the mass of $H(x)$ is controlled in terms of the mass of $F(x)$ as in (3).

First, observe that we need to assume that map $F$ is contractible, since otherwise a continuous family of fillings $H$ does not exist. Secondly, existence of a family of fillings with controlled mass depends on the choice of coefficients $G$. In Section 3 we show that for every integer $N$ there exists a contractible 1-parameter family of 0-cycles in $Z_0(S^1, Z)$ of mass $\leq 2$, such that every continuous family of fillings must contain a chain of mass $> N$. This example can be generalized to higher dimensions and codimensions.
However, for $G = \mathbb{Z}_2$ we can optimistically conjecture:

**Optimistic Conjecture 1.2.** Let $F : X \to \mathbb{Z}_k([0, 1]^n, \partial[0, 1]^n; \mathbb{Z}_2)$ be a contractible family. Then there exists a family of $(k + 1)$-chains $H : X \to I_{k+1}([0, 1]^n; \mathbb{Z}_2)$, such that $\partial H(x) - F(x)$ is supported in $\partial[0, 1]^n$ and

$$M(H(x)) \leq c(n) \max\{1, M(F(x))^{\frac{n-k-1}{n-k}}\}$$

The actual conjecture that we prove in the case of low dimensions and use for the proof of the Weyl law is somewhat more technical. We will be interested in the situation where maximal mass $M(F(x)) \sim p^{\frac{n-k}{n}}$. In this case we have

$$p^{\frac{n-k-1}{n}} + M(F(x))p^{-\frac{1}{n}} \sim M(F(x))^{\frac{n-k-1}{n}}$$

We also require that the family $F(x)$ is $\delta$-localized, a regularity condition that can always be guaranteed after a small perturbation (see Section 2.3). Finally, it will be convenient to state the conjecture for more general domains in $\mathbb{R}^n$ with piecewise smooth boundary satisfying certain regularity condition on the boundary (see Section 2.1). In particular, it is satisfied if $\partial \Omega$ is smooth.

**Conjecture 1.3.** Let $\Omega \subset \mathbb{R}^n$ be a connected domain, $\partial \Omega$ piecewise smooth boundary with $\theta$-corners, $\theta \in (0, \pi)$, $0 \leq k < n$. There exist constants $c(\Omega) > 0$ and $\delta(\Omega, L, p) > 0$ with the following property. Let $F : X^p \to \mathbb{Z}_k(\Omega, \partial \Omega; \mathbb{Z}_2)$ be a continuous $\delta(\Omega, L, p)$-localized contractible $p$-dimensional family with $M(F(x)) \leq L$. Then there exists map $H : X \to I_{k+1}(\Omega; \mathbb{Z}_2)$, such that

- $\partial H(x) - F(x) \subset \partial \Omega$ for all $x$;
- $M(H(x)) \leq c(\Omega)(p^{\frac{n-k-1}{n}} + M(F(x))p^{-\frac{1}{n}})$

We also want to state a parametric isoperimetric inequality conjecture for cycles in $\mathbb{R}^n$. It doesn’t have immediate applications to the Weyl law, but is interesting on its own. Unlike conjectures above we expect it to be true for coefficients in every abelian group $G$. For simplicity we state it for $G = \mathbb{Z}$.

**Conjecture 1.4.** Let $F : X^p \to \mathbb{Z}_k(\mathbb{R}^n; \mathbb{Z})$ be a continuous family of cycles. There exists a family of chains $H : X^p \to I_{k+1}(\mathbb{R}^n; \mathbb{Z})$, such that $\partial H(x) = F(x)$ and

$$M(H(x)) \leq c(n) M(F(x))^{\frac{k+1}{k}}$$

The main difficulty in this conjecture seems to be proving a bound for $M(H(x))$ that is independent of $p$. An inequality of this type with constant $c(n, p)$ that depends on $p$ subexponentially would also be of interest. In another direction, one can ask if it is possible to replace $c(n)$ with $c(k)$ and extend this result to infinite-dimensional Banach spaces as in [10] (see also [35]) or prove a similar inequality for the Hausdorff content as in [20].
1.3. Parametric coarea inequality. One can similarly formulate a parametric version of the coarea inequality. Let $M$ be a Riemannian manifold with boundary and let $M_\varepsilon = M \setminus cl(N_\varepsilon(\partial M))$, $M$ minus closed $\varepsilon$-neighborhood of its boundary. If $z$ is a Lipschitz relative cycle in $M$ and $r_0 > 0$, then the coarea inequality applied to the distance function implies the existence of some $r \in [0, r_0]$, such that $\text{Vol}(\partial(z \upharpoonright M_r)) \leq \frac{\text{Vol}(z)}{r_0}$. Very optimistically, one could conjecture that given a family of relative cycles $\{z_x\}_{x \in X}$ and $r_0 > 0$ there exists $r \in [0, r_0]$, such that $\text{Vol}(\partial(z_x \upharpoonright M_r)) \leq \frac{\text{Vol}(z_x)}{r_0}$ for all $x \in X$.

This is not always the case. Consider a 1-parameter family $\{z_x\}_{x \in [0,1]}$ of relative cycles in $M = [0,1]^2$ with a small but tightly wound spiral moving from the center to the boundary of $[0,1]^2$ (Fig. 1). No matter how you choose $r$, for some value of $x \in [0,1]$ the spiral will hit the boundary of $M_r$ and the intersection may have arbitrarily many points. Letting $r$ vary as a continuous function of $x$ doesn’t help.

However, what can help is if we allow ourselves to modify family $\{z_x\}$ in the boundary $\partial M_r$, while only increasing its mass by a small amount. It seems natural to conjecture that the amount of extra mass that we may have to add should be less than or comparable to the size of an optimal $p$-sweepout of the boundary by $k$-cycles, that is $\lesssim p^{\frac{n-k-1}{n-k}}$. Note that this is much smaller than the mass of an optimal $p$-sweepout of the $n$-dimensional interior by $k$-cycles ($\gtrsim p^{\frac{n-k}{n}}$).

Also, observe the following: if the family of relative cycles is a $p$-sweepout and if it can be represented by a family of chains, such that their boundaries are a continuous family in $\partial M_r$, then the family of boundaries is a $p$-sweepout of $\partial M_r$. On the other hand, we know that an optimal $p$-sweepout of $[0,1]^2$ by 1-cycles has maximal mass.
of \(const \sqrt{r}\), while every \(p\)-sweepout of \(\partial[0,1]^2\) must have 0-cycles of mass \(\geq p\). It follows that if we want the family of boundary cycles to be continuous, then \(\frac{M(z_z)}{r_0}\) bound is not sufficient.

Motivated by this we state the following optimistic conjecture:

**Optimistic Conjecture 1.5.** Let \(n > k \geq 1\). Let \(F : X^p \to Z_k([0,1]^n, \partial[0,1]^n; \mathbb{Z}_2)\) be a continuous family of cycles and \(r \in (0, \frac{1}{2})\). Then there exists a family \(F' : X \to I_k([r_0,1-r_0]^n; \mathbb{Z}_2)\), such that

1. \(F'(x) \cap (r,1-r)^n = F(x) \cap (r,1-r)^n\);
2. \(M(F'(x)) \leq M(F(x)) + c(n)p^{\frac{1}{n-1}}\);
3. \(\partial F'(x)\) is a continuous family of \((k-1)\)-cycles in \(\partial[r_0,1-r_0]^n\);
4. \(M(\partial F'(x)) \leq c(n)\left(\frac{M(F'(x))}{r} + p^{\frac{k}{n-1}}\right)\)

In our more down-to-earth conjecture that we use to prove the Weyl law we allow the \(F'(x)\) to be a small perturbation of \(F(x)\) in the interior of \((r,1-r)^n\). In addition, we require that the family satisfies a certain technical assumption that the mass does not concentrate at a point. We also state the result for more general domains with piecewise smooth boundary with \(\theta\)-corners (their definition is given in Section 2.1).

**Conjecture 1.6.** Let \(\Omega \subset \mathbb{R}^n\) be a connected domain, \(\partial\Omega\) piecewise smooth boundary with \(\theta\)-corners, \(\theta \in (0,\pi)\). Fix \(\eta > 0\). For all \(p \geq p_0(\Omega)\), \(r \in (0, r_0(\Omega))\) and \(\delta > 0\) the following holds. Let \(F : X^p \to Z_k(\Omega, \partial\Omega; \mathbb{Z}_2)\) be a continuous map with no concentration of mass. There exists a map \(F' : X \to I_k(\partial(\Omega_\delta); \mathbb{Z}_2)\), such that

1. \(\partial F'(x)\) is a continuous \(\delta\)-localized family in \(Z_{k-1}(\partial\Omega_\delta; \mathbb{Z}_2)\);
2. \(\mathcal{F}(F(x) \cap \Omega_\delta, F'(x)) < \eta\);
3. \(M(F'(x)) \leq M(F(x)) + \frac{M(F'(x))}{r}p^{\frac{1}{n-1}} + C(\Omega)M(\partial(\Omega_\delta))p^{\frac{k}{n-1}}\);
4. \(M(\partial F'(x)) \leq c(\Omega)\left(\frac{M(F'(x))}{r} + p^{\frac{k}{n-1}}\right)\).

Moreover, if \(F\) is a \(p\)-sweepout of \(\Omega\), then \(\partial F' : X \to \partial(\Omega_\delta)\) is a \(p\)-sweepout of \(\partial(\Omega_\delta)\) by \((k-1)\)-cycles.

We note that the parametric coarea inequality Conjecture for \(k\)-cycles in \(n\)-domains together with parametric isoperimetric inequality for \((k-1)\)-cycles in \((n-1)\)-domains can be used to prove parametric isoperimetric inequality for \(k\)-cycles in \(n\)-domain.

1.4. Weyl law. In the proof of the Weyl law for \((n-1)\)-cycles in a compact \(n\)-manifold in [21] manifold \(M\) is subdivided into small domains \(U_i\) and then a family of hypersurfaces \(\{\Sigma_i\}\) is defined that restricted to each \(U_i\) is a \(p\)-sweepout \(\{\Sigma_i' = \Sigma_i \cap U_i\}\) of \(U_i\). However, these hypersurfaces have boundary in \(\partial U_i\) and do not form a family of cycles in \(M\) (only a family of chains). A crucial step in the proof is to glue this family of chains into a family of cycles in \(M\) by adding a family of \((n-1)\)-chains
inside \( \bigcup \partial U_i \). Since we are using \( \mathbb{Z}_2 \) coefficients their total volume is bounded by the volume of \( \bigcup \partial U_i \) and is negligible compared to \( p^{\frac{1}{n}} \) for \( p \) very large.

In the case of higher codimension we can similarly define a family of \( k \)-chains, which restrict to a \( p \)-sweepout of \( U_i \) by relative cycles. However, to carry out the second step we need to find a family of chains in \( \bigcup \partial U_i \) of controlled volume that we can add to \( \Sigma_i \) to turn it into a family of cycles. The problem is twofold:

1. The volumes of \( \partial \Sigma_i \) can be arbitrarily large;
2. Even if we find a way to control the volume of \( \partial \Sigma_i \), we also need to construct a continuous family of chains filling \( \partial \Sigma_i \) of volume that is negligible compared to the maximal volume of \( \Sigma_i \).

Together the parametric coarea inequality and parametric isoperimetric inequality solve these two problems.

1.5. Organization. In Section 2 we prove some important results on how families of cycles can be approximated with families that have some nice special properties that we call \( \delta \)-localized families. In Section 3 we use these results to give a new simpler proof of the Almgren isomorphism theorem. In Section 4 we prove a parametric coarea inequality for 1-cycles in 3-dimensional domains. In Section 5 we prove a parametric isoperimetric inequality for 0-cycles in 2-dimensional domains. In Section 6 we show how the Weyl law can be proved using the parametric coarea and isoperimetric inequality conjectures.

Remark 1.7. After an earlier draft of this paper appeared on the arxiv Bruno Staffa proved Conjectures 1.6 and 1.3 for \( k = 1 \) and all \( n \) \[33\], \[34\]. Consequently, Staffa obtained the Weyl law Theorem 6.1 for 1-cycles in all dimensions.

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2. Approximation results for families of cycles

2.1. Piecewise smooth boundaries with corners. It will be convenient to work with manifolds that have piecewise smooth boundaries with no cusps. In this subsection we give a more precise definition of the “no cusps” property.

Let \( M \) be a Riemannian manifold with piecewise smooth boundary. Given a subset \( Y \subset M \) let \( N_r(Y) = \{x : dist(x,Y) < r\} \) denote the \( r \)-neighbourhood of \( Y \).
Given $\theta \in (0, \frac{\pi}{2})$ we will say that $\partial M$ is a boundary with $\theta$-corners if for every $p \in \partial M$ and every $\varepsilon > 0$ there exists an open ball $B \ni p$ and a $(1 + \varepsilon)$-bilipschitz diffeomorphism $\Phi : B \to \mathbb{R}^n$, such that $\Phi(p) = 0$ and $\Phi(\partial M \cap B)$ lies in a union of at most $n$ hyperplanes $P_1, \ldots, P_j$ in general position (that is, the intersection of any $k$ hyperplanes is a linear subspace of codimension $k + 1$), s.t. the dihedral angle $\angle(P_i, P_{i'}) \geq \theta$ for all $i \neq i'$.

We will need the following useful lemma.

**Lemma 2.1.** For every $\varepsilon > 0$ there exist $r_0 > 0$ and $(1 + \varepsilon)$-Lipschitz maps $E : \partial M \times [0, r_0] \to U$, $N_{\frac{r_0}{2}}(\partial M) \subset U \subset N_{11r_0}(\partial M)$, such that $E$ is the identity map on $\partial M$.

**Proof.** If $\partial M$ is smooth, the we can choose $E$ to be the exponential map $E = \exp_{\partial M}$.

Otherwise, consider a stratification of $\partial M$ into $0-$, $1-$, ..., $(n - 1)$-dimensional smooth strata $\{S_i\}$. Inductively, we can define a continuous vectorfield $V$ on $\partial M$, such that

- $V$ is smooth and $|V(x)| = 1$ on $S_i$ for each $i$;
- $\angle(V(x), \partial M) \geq \frac{\theta}{3}$.

We can extend $V$ to a smooth vectorfield in the interior of $M$ and define $E : \partial M \times [0, r] \to N_r(\partial M)$ to be a smooth map with $\frac{\partial E}{\partial t}(x, t) = V(x)$. Then for $r$ sufficiently small this map will be the desired bilipschitz diffeomorphism. □

Let $r_0 > 0$ and $E$ be as in the lemma above. For any $r \in (0, r_0)$ let $M_r = M \setminus E(\partial M \times [0, r])$.

Define a projection map $\Phi_r^* : M \to M_r$ given by $\Phi_r^*(E(x, t)) = E(x, r)$ if $x \in M \setminus M_r$ and $\Phi_r^*(x) = x$ otherwise. Observe that $\Phi_r^*$ is $(1 + \varepsilon)$-Lipschitz.

We have the following consequence of the coarea formula for flat chains.

**Theorem 2.2.** Let $\tau \in I_k(M; G)$ then there exists a constant $c(\varepsilon)$ with $\lim_{\varepsilon \to 0} c(\varepsilon) = 1$, so that for every $\varepsilon > 0$ there exists $r \in (0, \varepsilon)$, such that $\tau \cap M_r \in I_k(M_r; G)$ and $M(\tau \cap \partial M_r) \leq c(\varepsilon) \frac{M(\tau)}{\varepsilon}$.

Moreover, if $\tau \in Z_k(M, \partial M; G)$, then $\tau \cap M_r \in Z_k(M_r, \partial M_r; G)$ and $\partial(\tau \cap M_r) \in Z_{k-1}(\partial M_r; G)$.

### 2.2. Cubical complexes.

We will be considering families of flat chains parametrized by a cubical complex. We start with some definitions that will be convenient when working with these complexes, following [21].

Given $m \in \mathbb{N}$ let $I^m$ denote the $m$-dimensional cube $[0, 1]^m$. For each $j \in \mathbb{N}$, $I(1, j)$ denotes the cube complex on $I^1$ whose 1-cells and 0-cells (vertices) are, respectively,

$[0, 3^{-j}], [3^{-j}, 2 \cdot 3^{-j}], \ldots, [1 - 3^{-j}, 1]$ and $[0], [3^{-j}], \ldots, [1 - 3^{-j}], [1]$.  

We denote by $I(m,j)$ the cell complex on $I^m$:

$$I(m,j) = I(1,j) \otimes \cdots \otimes I(1,j) \ (m \text{ times}).$$

Then $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_m$ is a $k$-cell of $I(m,j)$ if and only if $\alpha_i$ is a cell of $I(1,j)$ for each $i$, and $\sum_{i=1}^m \dim(\alpha_i) = k$. We will abuse notation by identifying a $k$-cell $\alpha$ with its support: $\alpha_1 \times \cdots \times \alpha_m \subseteq I^m$.

Let $X$ be a cubical subcomplex of $I(m,j)$. For $i \geq j$ let $X(i)$ denote a subcomplex of $I(m,i)$, whose cells lie in a cell of $X$. In other words, $X(i)$ is obtained from $X$ by subdividing each $k$-cell of $X$ into $3^q(i-j)$ smaller cells. We use the notation $X(i)_k$ to denote the set of all $k$-cells in $X(i)$. If $E$ is a cell of $X(i)$, to simplify notation we will write $E_k$ to denote the $k$-skeleton of the cell $E$ (dropping $(i)$) and $E(i')_k$ to denote the $k$-skeleton of $X(i + i') \cap E$. We write $X^p$ to denote a cubical subcomplex with cells of maximal dimension $p$.

2.3. **Approximation theorem.** Let $G = \mathbb{Z}$ or $\mathbb{Z}_p$. Assume that $M$ is a manifold with piecewise smooth boundary $\partial M$ with $\theta$-corners for some $\theta \in (0, \frac{\pi}{2})$. Let $Z_k(M, \partial M; G)$ denote the space of relative cycles with coefficients in group $G$. Let $F$ denote the flat distance in $Z_k(M, \partial M; G)$.

Each individual cycle in a continuous family $\{F(x)\}$ of flat cycles can be well-approximated by a polyhedral cycle. However, close to every nice looking cycle in the family there could be a sequence of some wilder and wilder looking wiggly cycles converging to it. For example their supports may converge in Hausdorff topology to the whole space. This makes it hard to apply some continuous deformations or surgeries to the family. It is desirable to have an approximation theorem which tells us that we can pick a sufficiently fine discrete subset of the parameter space, and the values of $F$ on that discrete subset uniquely determine the homotopy class of the family, and, moreover, if we apply some continuous deformation to this discrete family so that it satisfies a certain property, we could interpolate to obtain a continuous family, which also satisfies this property.

This gives rise to discrete and continuous settings in Min-Max Theory (see e.g. [28, 23, 24]).

The main result of this section is the following approximation theorem.

**Theorem 2.3.** (1) There exists $c = c(M,p) > 0$, such that the following holds. For every $\eta > 0$ there exists $\varepsilon_0 > 0$, such that if $F : X^p(q)_0 \to Z_k(M, \partial M; G)$ is a map with $F(F(a), F(b)) < \varepsilon_0$ for every pair of adjacent vertices $a$ and $b$, then there exists an extension of $F$ to a continuous map $F : X \to Z_k(M, \partial M; G)$, such that for each cell $C$ in $X(q)$ and $x, y \in C$ we have $F(F(x), F(y)) < c\eta \sup_{w \in X(q)_0} M(F(w))$.

(2) There exists $\varepsilon(M,p) > 0$, such that if $F_0 : X \to Z_k(M, \partial M; G)$ and $F_1 : X \to Z_k(M, \partial M; G)$ are two maps with $F(F_0(x), F_1(x)) < \varepsilon$ for all $x$ and $\sup_{x \in X} M(F(x)) < \infty$, $i = 0, 1$, then $F_0$ and $F_1$ are homotopic.
Informally, Theorem 2.3 means that we can replace a continuous family of cycles with a sufficiently dense (in the parameter space) discrete family. We can then perturb each of the cycles in the discrete family and complete them to a new continuous family. This new continuous family will be homotopic to the original family. In general, our procedure may increase the mass of cycles by a constant \( c(p,M) \). In the last subsection we show that if we assume that the family of cycles satisfies no concentration of mass condition (and without this condition in the case of 0-cycle), then the mass will only increase by an amount that goes to 0 as \( \varepsilon \) goes to 0.

Our method is to first modify the family so that for a small ball \( B \) in the parameter space and any two points \( x, y \in B \), the difference \( F(x) - F(y) \) lies in some fixed (and depending only on \( B \)) collection of disjoint convex sets. We will call such families “\( \delta \)-localized”. Then we can interpolate between \( F(x) \) and \( F(y) \) using radial contraction inside these convex sets.

### 2.4. Admissible collections of open sets in \( M \)

Pick \( r_0 < \text{injrad}(M) \). If \( M \) has smooth boundary let \( E_{\partial M} : \partial M \times [0,r_0] \to M \) denote the normal exponential map from the boundary of \( M \). More generally, let \( E_{\partial M} \) denote map \( E \) from Lemma 2.1.

We say that a collection of open set \( U_i \subset M \) is \( r \)-admissible if they are all disjoint and:

1) if \( U_i \) is disjoint from \( \partial M \), then \( U_i \) is a ball of radius \( r_i \);
2) if \( U_i \) is not disjoint from \( \partial M \), then \( U_i = E_{\partial M}(B_i \times [0, r_i]) \);
3) \( \sum r_i < r \).

We need the following two elementary lemmas about collections of \( r \)-admissible sets.

**Lemma 2.4.** Let \( \mathcal{B}_1 = \{\beta_i\} \) be a finite set of balls in \( \partial M \) and \( \mathcal{B}_2 = \{B_j\} \) be a finite set of disjoint balls contained in the interior of \( M \). There exists an \( r \)-admissible collection \( \{U_i\} \), such that \( \cup U_i \supset (\cup E_{\partial M}(\beta_i \times [0, \text{rad}(\beta_i)]) \cup \cup B_j) \) and \( r \leq \sum \text{rad}(\beta_i) + 3\sum \text{rad}(B_j) \).

**Proof.** If \( \beta_{i_1} \) and \( \beta_{i_2} \) intersect, then we can find a ball \( \beta \subset \partial M \) of radius \( \leq \text{rad}(\beta_{i_1}) + \text{rad}(\beta_{i_2}) \) that contains their union. Hence, we may replace \( \{\beta_i\} \) with a collection of disjoint balls containing the union of original balls without increasing the sum of their radii. Let’s call this replacement operation (*)

If \( B_j \) intersects \( E_{\partial M}(\beta_i \times [0, \text{rad}(\beta_i)]) \), then there exists \( \beta' \) of radius \( \leq \text{rad}(\beta_i) + 3\text{rad}(B_j) \), so that \( B_j \subset E(\beta' \times [0, \text{rad}(\beta')]) \). (Note that instead of factor 3 we could’ve used \( 2 + \varepsilon \) for \( \varepsilon \to 0 \) as \( \text{rad}(B_j) \to 0 \).) We define new sets \( \mathcal{B}_1' = \mathcal{B}_1 \setminus \{\beta_i\} \cup \{\beta'\} \) and \( \mathcal{B}_2' = \mathcal{B}_2 \setminus \{B_j\} \). By (*) we can replace \( \mathcal{B}_1' \) with a collection of disjoint ball \( \mathcal{B}_1'' \). Let (**) denote the operation of replacing \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) with \( \mathcal{B}_1'' \) and \( \mathcal{B}_2'' \). Note that (**) decreases the number of balls in \( \mathcal{B}_2 \) by 1, decreases the sum of radii of balls in \( \mathcal{B}_2 \) by \( \text{rad}(B_j) \) and increases the sum of radii of balls in \( \mathcal{B}_1 \) by \( 3\text{rad}(B_j) \).
We perform (***) repeatedly, until $E_{\partial M}(\beta \times [0, rad(\beta)])$ and $B_j$ are all disjoint. The process terminates since the number of balls in $B_2$ was finite. □

**Lemma 2.5.** Let $\{U^1_{i}\}_{i=1}^{v_1}$ be an $r_1$-admissible and $\{U^2_{i}\}_{i=1}^{v_2}$ be an $r_2$-admissible collection of open sets. Then there exists an $r$-admissible collection $\{U_i\}_{i=1}^{v}$ with $v \leq v_1 + v_2$, $\sqcup U^1_i \cup \sqcup U^2_i \subset \sqcup U_i$ and $r \leq 3(r_1 + r_2)$.

**Proof.** We replace the collection of all $\{U^1_{i}\}$ and $U^2_{i}$, which are balls in the interior of $M$, with a collection $\{B_i\}$ of disjoint balls that contain their union. We can do this with $\sum \text{rad}(B_i) \leq \sum \text{rad}(U^1_i) + \sum \text{rad}(U^2_i)$.

Some of balls $B_i$ may intersect $\partial M$. We replace them with $\exp(\beta \times [0, r']) \supset B_i$ for $\beta \subset \partial M$ and $r' \leq 3\text{rad}(B_i)$.

Now we are in the situation where we can apply Lemma 2.4 □

2.5. **Localized families.** Let $V \subset X(q)$. We will say that a map $F : V \rightarrow Z_k(M, \partial M; G)$ is $\varepsilon$-fine if for every cell $C$ in $X(q)$ and $a, b \in C \cap V$ we have $F(F(a), F(b)) \leq \varepsilon$. We will say that a map $F : V \rightarrow Z_k(M, \partial M; G)$ is $\delta$-localized if for every cell $C$ of $X(q)$ there exists a $\delta$-admissible collection of open sets $\{U_i^C\}$, such that for every pair $a, b \in C \cap V$ we have the support $\text{supp}(F(a) - F(b)) \subset \bigcup U_i^C$. Given a cell $C$ in $X(q)$ let $C_0 = C \cap X(q)_0$.

**Proposition 2.6.** There exists $\delta_0(M, p) > 0$, $c(n, k, p) > 0$ and $\varepsilon_0(M, p) > 0$ with the following property. Let $F : X^p(q)_0 \rightarrow Z_k(M, \partial M; G)$ be a $\delta_0$-localized map, $\delta < \delta_0$.

I. There exists an extension map $F : X(q) \rightarrow Z_k(M, \partial M; G)$, such that

1. the extension is $\delta$-localized;
2. for every cell $C$ and $x, y \in C$ we have

$$F(F(x), F(y)) < c\delta \max_{w \in C_0} M(F(w) \cup U_i^C)$$

3. for every cell $C$ and $x \in C$ we have

$$M(F(x)) \leq \max_{w \in C_0} M(F(w)) + c \max_{w \in C_0} M(F(w) \cup U_i^C)$$

II. Suppose in addition that $F$ is defined on some subcomplex $Y(q) \subset X(q)$ and $F : Y \rightarrow Z_k(M, \partial M; G)$ is $\varepsilon$-fine, $\delta < \delta_0, \varepsilon < \varepsilon_0$. Then there exists an extension map $F : X(q) \rightarrow Z_k(M, \partial M; G)$, such that

1. the extension is $\delta\varepsilon$-localized;
2. for every cell $C$ in $X(q)$ and $x, y \in C$ we have

$$F(F(x), F(y)) < \delta \varepsilon \sup_{w \in C_0 \cup (C \cap Y)} M(F(w))$$

and $M(F(x)) \leq c \max_{w \in C_0 \cup (C \cap Y)} M(F(w))$. 
Proof. We prove I. Part II follows with some straightforward modifications.

We start by extending to 1-skeleton.

Let $C$ be an $m$-dimensional cell. Assume we have extended $F$ for all $x \in \partial C$. For each $(m - 1)$-dimensional cell $C' \subset \partial C$ assume, inductively, that there exists a $c(m - 1)\delta$-admissible collection $\mathcal{B}(C') = \{U_1^{C'}, ..., U_{i(C')}^{C'}\}$ with $\text{supp}(F(x) - F(y)) \subset \cup U_i^{C'}$ for all $x, y \in C'$.

By Lemma 2.5 there exists $c(m) > 0$ and a $c(m)\delta$-admissible collection of open sets $\mathcal{B}(C) = \{U_1^C, ..., U_{i(C)}^C\}$, such that

$$\cup_{i,C' \subset \partial C} U_i^{C'} \subset \cup_i U_i^C$$

For each $i$ we define contraction maps $R_i^C$ as follows. If $U_i^C = B_{r_i}(p)$ is a ball centered at $p$ we let $R_i^C : B_{r_i}(p_i) \to B_{tr_i}(p_i)$ be the radial contraction map. If $U_i^C = E(\beta_{r_i}(p_i) \times [0, r_i])$ we define $R_i^C(E(x, r)) = E(tx, tr_i)$

Let $v(C)$ denote the vertex in 0-skeleton of $C$ that is closest to 0 (thinking of $C$ as the subset of the unit cube).

Define function $\sigma(x, t)$ on $\partial C \times [0, 1]$ by

$$\sigma(x, t) = F(x) + \sum_i R_i^C((F(v(C)) - F(x)) \cup U_i^C)$$

Since $\sigma(x, 1) = F(v(C))$ for all $x \in \partial C$ this gives a well-defined extension of $F$ to $C$.

Now we would like to verify the bounds on $\mathcal{F}$ and the mass. By taking a cone over $F(x) - F(y) \subset \cup U_i^C$ we verify $\mathcal{F}(F(x), F(y)) \leq c(p)\delta \max_{w \in C_0} M(F(w) \cup \cup U_i^C)$ + $M(F(y) \cup \cup U_i^C) \leq c(p)\delta \max_{w \in C_0} M(F(w) \cup U_i^C)$.

Inductively we can check that

$$M(F(x)) \leq \max_{w \in C_0} M(F(w)) + c(p)\max_{w \in C_0} M(F(w) \cup U_i^C)$$

for sufficiently large $c(p)$.

The following proposition is the key technical result necessary for the proof of Theorem 2.3. It asserts that every sufficiently fine (in the flat topology) discrete family of cycles can be approximated by a $\delta$-localized discrete family of cycles.

**Proposition 2.7.** Given $\delta > 0$, there exists $\varepsilon_0(p, M, \delta) > 0$ with the following property. Let $f : X^p(q)_0 \to \mathcal{Z}_k(M, \partial M; G)$ be a map with $\mathcal{F}(f(x), f(y)) < \varepsilon$ for any two adjacent vertices $x$ and $y$ and $\varepsilon \in (0, \varepsilon_0)$.

Then there exists $\tilde{q}(p, M, \delta)$ and a map $F : X(q + \tilde{q})_0 \to \mathcal{Z}_k(M, \partial M; G)$ that coincides with $f$ on $X(q)_0$, such that for every cell $C$ in $X(q)$ we have

1. $\mathcal{F}(f(x), F(y)) < c(M, p, \delta)\varepsilon$ for all $x \in C \cap X(q)_0$ and $y \in C \cap X(q + \tilde{q})_0$;
2. $M(F(y)) \leq \sup_{w \in C \cap X(q)_0} M(f(w)) + c(M, p, \delta)\varepsilon$ for $y \in C \cap X(q + \tilde{q})_0$.
(3) $F$ is $\delta$-localized, moreover, if $E \subset C$ is a cell of $X(q + \tilde{q})$, then for the corresponding $\delta$-admissible collection $\{U_i^E\}_{i=1}^{n(p)}$ we have $M(F(y) \cup U_i^E) \leq \sup_{w \in C \cap X(q)_{0}} M(f(w) \cup U_i^E)$.

Proof. First we define $F$ on $X(q)_{0}$ setting $F(x) = f(x)$.

We will define a sequence of numbers $q = q_0 < ... < q_p = \tilde{q}$ and maps $F_j : X(q_j) \cap X(q_{j+1}) \rightarrow Z_k(M, \partial M; G)$ for $j = 0, ..., p$. $F_p : X(q + \tilde{q})_0 \rightarrow Z_k(M, \partial M; G)$ will be the desired localized extension.

We organize our proof as follows. First, we collect all definitions that will be needed in the proof. Then we prove a key lemma constructing a localized family of chains filling a given localized family of cycles. Finally, we give a proof of the inductive step of the construction.

**Definitions and notation**

- Let $r_1(M)$ be sufficiently small, so that every ball of radius $r_1$ that lies in $M$ is $1.01$-bilipschitz diffeomorphic to an $r_1$-ball in $\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$, and for every ball $\beta(r_1)$ in $\partial M$ we have that $E_{\partial M}$ is a $1.01$-bilipschitz diffeomorphism on $\beta(r_1) \times [0, r]$.
- Let $r_0(M, \delta) = \min\{\frac{1}{2} r_1, \delta\}$ and consider a collection of sets $\{B_1, ..., B_N\}$ covering $M$, such that each $B_i$ is either a closed ball of radius $r_0$, or $B_i = E_{\partial M}(\beta_i, [0, r_0])$, where $\beta_i$ is a closed ball of radius $r_0$ in $\partial M$. We will call such sets “generalized balls”. Note that we can choose covering $\{B_i\}$, so that the number $N$ of generalized balls in the covering satisfies $N \leq c(M)\frac{1}{\delta^3}$. Let $tB_i$, $t > 0$, denote the concentric generalized ball with the same center point and the radius $tr_0$.

Given a chain $\tau$ by the coarea inequality Theorem 2.2 there exist generalized balls $B_i'$, $B_i \subset B_i' \subset 2B_i$, such that

$$M(\tau \cup \partial B_i') \leq \frac{2}{r_0} M(\tau \cup 2B_i) \tag{4}$$

It will be convenient to use $\{B_i'\}$ to define a collection of sets with disjoint interiors as follows: we let $U_1 = B_1'$ and let $U_i$ be the closure of $B_i' \setminus \bigcup_{j=1}^{i-1} B_j'$ for $i > 1$. We will say that $\{U_i\}$ is a coarea covering for $\tau$. Note that $U_i \subset B_i' \subset 2B_i$ for each $i$, and

$$M(\tau \cup \bigcup_{i=1}^N \partial U_i) \leq M(\tau \cup \bigcup_{i=1}^N \partial B_i') \leq \frac{2N}{r_0} M(\tau)$$

More generally, given a finite collection $\{\tau_j\}_{j=1}^L$, applying coarea inequality to $\sum \tau_j$, there exists a collection of generalized balls $B_i'$, $B_i \subset B_i' \subset 2B_i$, such that for each $j$
we have
\begin{equation}
M(\tau_j \cup \partial B'_i) \leq \frac{2L}{r_0} \max_{i=1,\ldots,L} M(\tau_i \cup 2B_i)
\end{equation}

As above we set \(U_1 = B'_1\) and \(U_i = B'_i \setminus \bigcup_{j=1}^{i-1} B'_j\) for \(i > 1\) and call \(\{U_i\}\) a coarea covering for \(\{\tau_j\}_{i=1}^L\). We then have
\begin{equation}
M(\tau_j \cup \partial \bigcup_{i=1}^N U_i) \leq \frac{2NL}{r_0} \max_{j=1,\ldots,L} M(\tau_j)
\end{equation}

- It will be convenient to define a way of chopping away a chain \(\tau_j\) by intersecting it with sets from the coarea covering \(\{U_i\}\) of \(\tau_j\). Given a collection of chains \(\{\tau_j\}_{j=1}^L\) and a coarea covering \(\{U_i\}_{i=1}^N\) for \(\{\tau_j\}\) we let
\begin{equation}
d_i(\tau_j, \{U_i\}) = \tau_j - \tau_j \cup \bigcup_{j=1}^i U_j
\end{equation}

We then have the following mass bound for the boundary of \(\partial d_i(\tau_j)\):
\begin{equation}
M(\partial d_i(\tau_j)) \leq M(\partial \tau_j) - \sum_{i=1}^L M(\partial \tau_i \cup U_i) + \frac{2NL}{r_0} M(\tau_j)
\end{equation}

For \(i > N\) define \(d_i(\tau)\) to be the empty chain.
- For each cell \(C \subset X(q)\) let \(x(C) \in C \cap X(q)_0\) denote the vertex of \(C\) that is closest to \((0, \ldots, 0)\) (thinking of \(X(q)\) as contained in the ambient unit cube of large dimension).
- \(\text{dist}_\infty(x, y) = \min_j \{|x_j - y_j|\}\)
- For a face \(E^d \subset X(q)_1\) we define \(\text{Center}(E)\) to be the collection of all points \(x \in E\) with \(\text{dist}_\infty(x, \partial E) > 1/3^{q+1}\).
- It will be useful to define maps between 0-skeleta of subdivisions of cells with nice properties. These constructions are similar to [23][Appendix C].

Given a cell \(E\) of \(X(q)\) and \(x \in E\) we let \([x]_{q'}\) denote the closest point of \(E(q')_0\) to \(x\); if there is more than one vertex of \(E(q')_0\) that minimizes \(d_\infty(x, \cdot)\), then we set \([x]_{q'}\) to be the vertex closest to \((0, \ldots, 0)\) in the ambient cube \(I^m \supset X \supset E\).
- We will say that a map \(G : Y(q')_0 \to Y(q'')_0\), \(q' \geq q''\), is an adjacency-preserving extension if
  1. \(G(x) = x\) for all \(x \in Y(q'')_0\);
  2. for any two \(x, y \in Y_1(q')_0\) that lie in the same face of \(Y_1(q')\) we have that \(G(x), G(y)\) are in the same face of \(Y_2(q'')\).
• Given a cell $E$ of $Y(q)$ and $q' > q''$ define $H_{q',q''} : E(q')_0 \to E_0(q'')_0$ by $H_{q',q''}(x) = [x]_{q''}$. We can extend this definition to a map $H_{q',q''} : X(q+q')_0 \to X(q+q'')_0$ by inductively applying $H_{q',q''}$ on each face. We observe that $H_{q',q''}$ is an adjacency-preserving extension. It will be convenient to write $H_q$ for $H_{q',q''}$.

• Let $L_t(E) = \{ x \in E : d_\infty(x, \partial E) = t \}$ and let $\phi_t : L_t(E) \to \partial E$ be a bijective homothety from the center point of $E$. We define maps

\[
P : E(q' + 1)_0 \setminus \text{Center}(E) \to \partial E(q')_0 \times I^1(q')_0
\]

\[
Q : \partial E(q')_0 \times I^1(q')_0 \to E(q')_0 \setminus \text{Center}(E)
\]

by setting

\[
P_q(x) = ([L_{d_\infty(x, \partial E)}(x)]_{q'}, 3^{q+1}d_\infty(x, \partial E))
\]

\[
Q_q(x, t) = [L_{t/3^{q+1}}(x)]_{q'}
\]

Note that the composition map

\[
Q \circ P(x) : E(q' + 1)_0 \setminus \text{Center}(E) \to E(q')_0 \setminus \text{Center}(E)
\]

is an adjacency-preserving extension. Moreover, we have $Q \circ P(x) = H_{q'+1,q}(x)$ for $x \in \partial E(q')_0$.

**Lemma 2.8.** There exists $\varepsilon_0(k, j, M, \delta) > 0$ with the following property. Let $F : Y^j(q)_0 \to Z_k(M, \partial M; G)$ be a $\delta$-localized map with $F(F(x, 0) < \varepsilon$ for all $x \in Y^j(q)_0$ and some $\varepsilon < \varepsilon_0(k, j, M, \delta)$). Then there exists $\overline{q}(k, j, M, \delta), c_1(k, j, M), c_2(k, j, M, \delta)$ and map $\tau : Y^j(q + \overline{q})_0 \to I_{k+1}(M; G)$, such that $\tau$ is $c_1\delta$-localized, $M(\tau(x)) < c_2\varepsilon$ and $\partial \tau(x) = F \circ H_{\tau}(x)$.

**Proof.** The proof is by induction on $n - k$. If $n = k$, then taking $\tau(x) = 0$ gives the desired result.

Assume the lemma holds for families of $(k + 1)$-cycles. To prove the result for families of $k$-cycles we now proceed by induction on $j$.

If $j = 0$ the result is immediate. Assume the lemma holds for $(j - 1)$-dimensional families. Applying it to the $(j-1)$-skeleton $Y(q)_{j-1}$ of $Y(q)$ we obtain $c_1(k, j-1, M)\delta$-localized family

\[
\tau_{j-1} : Y(q)_{j} \cap Y(q + q_{j-1})_0 \to I_{k+1}(M; G)
\]

with $\partial \tau(x) = F \circ H_{\tau_{j-1}}(x)$ and a mass bound

\[
M(\tau_{j-1}(x)) \leq c_2(k, j-1, M, \delta)\varepsilon
\]

Let $C^j$ be a $j$-dimensional face of $Y(q)$. Let

\[
Q : \partial C^j(q_{j-1})_0 \times I^1(q_{j-1})_0 \to C(q_{j-1})_0 \setminus \text{Center}(C)
\]
be the map defined as in (10). We will now construct a map
\[ \tau_{\partial C \times I} : \partial C^j(q_{j-1})_0 \times I^1(q_{j-1})_0 \to I_{k+1}(M; G) \]
We set \( \tau_{\partial C \times I}(x, 0) = \tau_{j-1} \circ Q(x) \).

Since \( F \) is \( \delta \)-localized there exists a \( \delta \)-admissible collection of sets \( \{U_t^C\} \) with \( F(x) - F(x(C)) \) supported in \( \bigcup U_t^C \) for all \( x \in C_0 \). Hence, there exists a \((k+1)\)-chain \( fill(x) \) supported in \( \bigcup U_t^C \) with \( M(fill(x)) < 2\varepsilon \) and \( \partial fill(x) = F(x) - F(x(C)) \) for all \( x \in C_0 \). We let
\[ \tau_{\partial C \times I}(x, 1) = \tau_{j-1}(x(C)) + fill(H_{q_{j-1}}(x)) \]
We define
\[ \tilde{\tau}(x) = \tau_{\partial C \times I}(x, 1) - \tau_{\partial C \times I}(x, 0) \]
From the definition we have that \( \{\tilde{\tau}(x)\}_{x \in \partial C(q_{j-1})_0} \) is a family of relative cycles.

Since \( \{\tau_{\partial C \times I}(x)\}_{x \in \partial C(q_{j-1})_0} \) is \( c(k, j-1, M) \)-\( \delta \)-localized, applying Lemma 2.5 we have that \( \{\tilde{\tau}(x)\} \) is \( c' \)-\( \delta \)-localized for \( c' < 2c_1(j-1, k, M) \). We also have the mass bounds
\[ M(\tilde{\tau}(x)) \leq M(\tau_{\partial C \times I}(x, 0)) + M(\tau_{\partial C \times I}(x, 1)) \leq (2c_2(j - 1, k, M) + 1)\varepsilon \]
Hence, we can apply the inductive assumption for families of \((k+1)\)-cycles. We obtain that there exists a \( c_1(k+1, j-1, M)c' \)-\( \delta \)-localized family \( \{\sigma(x)\}_{x \in \partial C(q_{j-1}+q')_0} \) of \((k+2)\)-chains, such that \( \partial \sigma(x) = \tilde{\tau} \circ H_{q_{j-1}+q', q_{j-1}}(x) \). Let \( \{U_t\} \) be a coarea covering for the family \( \{\sigma(x)\}_{x \in \partial C(q_{j-1}+q')_0} \).

Increasing \( q' \) if necessary we may assume that \( q_{j-1} + q' \geq \log_3(N) + 1 \). Let
\[ n(t) = 3^{q_{j-1}+q'+1} \text{dist}_\infty(t, \partial C \times \{1\}) \]
For \( t \in I(q_{j-1}+q')_0 \) we have
\[ \tau'(x, t) = \tau_{\partial C \times I}(x, 0) + \partial d_{n(t)}(\sigma(x), \{U_t\}) \]
Observe the following properties of \( \tau' \):
\begin{enumerate}
  \item \( \partial \tau'(x, t) = F_{j-1} \circ H_{q_{j-1}+q'}(x) \) for \( t < 1 \);
  \item \( \tau'(x, 0) = \tau_{\partial C \times I}(x) \);
  \item \( \tau'(x, 1) = \tau_{\partial C \times I}(H_{q_{j-1}+q', q_{j-1}}(x), 1) = \tau_{j-1}(x(C)) + fill(H_{q_{j-1}+q'}(x)) \).
\end{enumerate}
By Lemma 2.5 we also have that this family is \( c_1(k, j, M) \)-\( \delta \)-localized for \( c_1(k, j, m) \) bounded in terms of \( c_1(k, j-1, M) \), \( c_1(k+1, j-1, M) \) and our choice of \( r_0 \). From (8) we have that
\[ M(\partial \tau'(x, t)) \leq c_2(k, j-1, M, \delta)\varepsilon + c_2(k+1, j-1, M, \delta) \frac{2N3^{q'}}{r_0} \varepsilon \]
Since \( r_0 \) was determined by \( \delta \) and \( M \) and our choice of \( q' \) depended only on \( k, j \) and \( M \), we can bound this expression by \( c_2(k, j, M, \delta)\varepsilon \).

Hence, we’ve obtained a \( c_1 \)-\( \delta \)-localized family
\[ \{\tau'(x, t)\}_{(x, t) \in \partial C(q_{j-1}+q')_0 \times I(q_{j-1}+q')_0} \]
We let $\overline{q} = q_{j-1} + q' + 1$. We can then define a family $\tau$ on $C(\overline{q})_0 \setminus \text{Center}(C)$ by setting $\tau(x) = \tau' \circ P$ for the map $P : C(\overline{q})_0 \to \partial C(q_{j-1} + q')_0 \times I(q_{j-1} + q')_0$. For $x \in \text{Center}(C) \cap C(\overline{q})_0$ we set $\tau(x) = \tau_{j-1}(x(C)) + \text{fill}(H_{\overline{q}}(x))$. It follows from the above estimates that $\{\tau(x)\}$ is a $c_1(k, j, M)\delta$-localized family with $M(\tau(x)) \leq c_2(k, j, M, \delta)\varepsilon$. Defining $\tau$ in this way on each $j$-face we obtain the desired family. \qed

We can now prove Proposition 2.7. In calculations below let $c_1$ denote a constant on $M$, $p$ and $k$. The value of $c_1$ may change from line to line.

We will define a sequence of $c_1\delta$-localized maps $F_j : X(q)_j \cap X(q_j)_0 \to Z_k(M, \delta M; G)$ with $q_0 = q$ and satisfying $F_j(x) = F_{j-1} \circ H_{q_j, q_{j-1}}(x)$ for $x \in X(q)_j \cap X(q + q_{j-1})_0$ and such that for every cell $E^i$ of $X(q)$, $l \leq j - 1$, we have

i. $F(f(x), F(y)) < c(M, l - 1, \delta)\varepsilon$ for $x \in E \cap X(q)_0$ and $y \in E \cap X(q + q_{j-1})_0$;

ii. $M(F(y)) \leq \sup_{w \in E \cap X(q)_0} M(f(w)) + c(M, p, \delta)\varepsilon$ for $y \in E \cap X(q + q_{j-1})_0$;

iii. if $D \in E$ is a cell of $X(q + q_{j-1})$ and $\{U^D_i\}$ is the corresponding $c_1\delta$-admissible collection, then $M(F(y) \cup U^D_i) \leq \sup_{w \in E \cap X(q)_0} M(f(w) \cup U^D_i)$ for $y \in E \cap X(q + q_{j-1})_0$.

First we set $F_0(x) = F(x)$ for $x \in X(q)_0$. Assume we have defined $F_{j-1}$ on $X(q)_{j-1} \cap X(q_{j-1})_0$. For each $j$-dimensional face $C$ of $X$, by Lemma 2.8 there exists a $c_1\delta$-localized family $\{\tau(x)\}_{x \in \partial C(q_{j-1} + q)_0}$, such that $\partial \tau(x) = F(x(C)) - F_{j-1} \circ H_{q_j, q_{j-1}}(x)$. We let $q_j = q_{j-1} + \overline{q} + 1$ and define

\[ F_j(x) = F_{j-1} \circ H_{q_j, q_{j-1}}(x) = F \circ H_{q_j}(x) \]

on $X(q)_{j-1} \cap X(q_j)_0$. We now fix $j$-dimensional cell $C$ and describe the extension of $F_j$ from $\partial C \cap X(q_j)_0$ to $C \cap X(q_j)_0$.

Let $\{U_i\}$ be a coarea covering for the family $\{\tau(x)\}_{x \in \partial C(q_{j-1} + q)_0}$. To obtain the desired mass bound for $F_j$ we will need the following construction. We defined $F_j$ by composing $F_{j-1}$ with an adjacency preserving extension on $\partial C$ and now we'd like to extend to the interior of $C$ by “chopping away” $\tau(x)$ using [7], taking the boundary of the resulting $(k + 1)$-chain and adding it to the value of $F_j$ on the boundary, until we deform the family of $\{F_j(x)\}_{x \in \partial C(q_j)_0}$ to the constant family equal to $F(x(C))$ on Center($C$). The problem with this approach is that the restriction $F_j(x)|\cup U_i$ for $x \in \partial C(q_j)_0$ can be large for some values of $i$, while $F(x(C))|\cup U_i$ can be large for other values of $i$, so the interpolation may increase the total mass to more than $M(F(x(C))) + M(F(y))$ (for some $y \in C_0$), violating mass bounds [ii] and [iii] above. Instead, we will make a mass-minimizing choice in each $U_i$ that will guarantee that the mass bounds are satisfied.

For each $i = 1, ..., N$ we let $m_i = \tau(x(i))|\cup U_i$, where $x(i) \in \partial C(q_{j-1} + q)_0$ is such that

\[ M((\partial \tau(x(i))|\cup U_i - F(x(C))|\cup U_i) \leq M((\partial \tau(x)|\cup U_i - F(x(C))|\cup U_i) \]
for all $x \in \partial C(q_{j-1} + \bar{q})_0$. Define
\[
b(\{\tau(x)\}, F(x(C)), \{U_i\}) = \sum_{i=1}^{N} m_i
\]
From estimate (8) we have the following mass bounds for $b(\{\tau(x)\}, F(x(C)), \{U_i\})$ and its boundary:
\[
\begin{align*}
M(b(\{\tau(x)\}, F(x(C)), \{U_i\})) &\leq \sum_{x \in C(q_{j-1} + \bar{q})_0} M(\tau(x)) \\
M(\partial b(\{\tau_j\}, F(x(C)), \{U_i\}) - F(x(C)) &\leq \min_{x \in C(q_{j-1} + \bar{q})_0} M(\partial \tau(x) - F(x(C)) \\
(13) &+ \frac{2jN3^{q_{j-1}+\bar{q}}}{\tau_0} \max_{x \in C(q_{j-1} + \bar{q})_0} M(\tau(x))
\end{align*}
\]
We define a map
\[
\tau' : \partial C(q_{j-1} + \bar{q})_0 \times I(q_{j-1} + \bar{q})_0 \to I_{k+1}(M; G)
\]
as follows. Let $n(t) = 3^{q_{j-1}+\bar{q}}(1 - t)$ (without any loss of generality we may assume $N < 3^{q_{j-1}+\bar{q}}$). We set
\[
\tau'(x, t) = \tau \circ H_{q_j, q_{j-1}}(x) - d_{n(t)}(\tau \circ H_{q_j, q_{j-1}}(x) - b(\{\tau(x)\}, F(x(C)), \{U_i\}))
\]
Finally, we let
\[
F_j(x) = F(x(C)) - \partial \tau' \circ P_{q_{j-1}}(x)
\]
for $x \in C(q_j)_0 \setminus Center(C)$. Observe that this definition coincides with (11) on $\partial C(q_j)_0$. For $x \in C(q_j)_0 \cap Center(C)$ we define
\[
F_j(x) = F(x(C)) - \partial b(\{\tau(x)\}, F(x(C)), \{U_i\})
\]
The flat distance bound (1) follows from (12); the mass bounds (ii) and (iii) follow from (13) and our construction of $b$. \qed

Remark 2.9. Notice that if $M$ is a PL manifold, then we can perform the above construction starting with a polyhedral approximation $F(x)$ for each cycle $f(x)$, $x \in X(q)_0$. The construction in the proof of Proposition 2.7 then gives a discrete family $F : X(q')_0 \to Z^D_k(M, \partial M; G)$ of polyhedral cycles and Proposition 2.6 a continuous family $F : X \to Z^D_k(M, \partial M; G)$ of polyhedral cycles. Hence, whenever it is convenient, we can replace a Riemannian metric with a PL approximation and a family of flat cycles with an approximating family of polyhedral cycles. All relative $k$-dimensional cycles in our construction may be chosen so that $F(x) \cap \partial M = \partial F(x)$ and is a $(k - 1)$-dimensional polyhedral cycle in $\partial M$.

Combining Proposition 2.6 and 2.7 we obtain the following Proposition.
Proposition 2.10. For all $\delta > 0$ sufficiently small there exists $\varepsilon_1 > 0$, such that the following holds. Given a map $F : X(q)_0 \to Z_k(M, \partial M; G)$, which is $\varepsilon_1$-fine, there exists $q'$ and a continuous extension $F : X(q + q') \to Z_k(M, \partial M; G)$, such that:
1. $F : X(q + q') \to Z_k(M, \partial M; G)$ is $\delta$-localized;
2. for every cell $C$ of $X(q)$, $x \in C_0$ and $y \in C \cap X(q + q')_0$, we have $\mathcal{F}(F(x), F(y)) < c(M, p)\max_{w \in X(q)_0} M(F(w))$;
3. $M(F(x)) \leq c(M, p)(1 + \delta) \max_{w \in C_0} M(F(w))$.

Proof. Follows by applying Propositions 2.7 and 2.6.

Theorem 2.3 (1) directly follows from Proposition 2.10. To prove Theorem 2.3 (2) we need the following lemma.

Lemma 2.11. For every sufficiently small $\delta > 0$ there exists $\varepsilon_2(M, p, \delta) > 0$ such that the following holds. Let $F_0 : X(q) \to Z_k(M, \partial M; G)$ and $F_1 : X(q) \to Z_k(M, \partial M; G)$ be two $\delta$-localized maps. Suppose $\mathcal{F}(F_0(x), F_1(x)) \leq \varepsilon < \varepsilon_2$. Suppose
$$m = \max\{\sup_{w \in X} M(F_0(w)), \sup_{w \in X} M(F_1(w))\} < \infty$$

Then there exists a homotopy $F : X(q) \times I(1, q) \to Z_k(M, \partial M; G)$ between $F_0$ and $F_1$, such that
1. $\mathcal{F}(F(x, t), F_0(x)) < c(p, M)m\delta + \varepsilon_2$ for all $x \in X$;
2. $M(F(x, t)) \leq c(p, M)m$ for all $(x, t) \in X \times I$.

Proof. First we will define $F$ restricted to $X(q) \times \{0\} \cup X(q) \times I(1, q) \cup X(q) \times \{1\}$ for some $\tilde{q} \geq q$ using the construction from Proposition 2.7. Then we will apply Proposition 2.6 II to extend $F$ to a continuous map defined everywhere.

Let $F|_{X(q)_0 \times 0} = F_0$ and $F|_{X(q)_0 \times 1} = F_1$. Applying the inductive step extension argument from the proof of Proposition 2.7 we define $c\delta$-localized map $F^1$ on $v \times I(q)_0$ for each $v \in X(q)_0$, such that $F^1(v, t) = F(v, t) = F_1(v)$ for $t = 0, 1$. Note that $\mathcal{F}(F(v, t), F_0(v)) \leq \varepsilon_2$. Using the same extension argument we can then extend $F^1$ to a $\delta$-localized map on $(X(q + q_1)_0 \times I(q)_0) \cap (X(q) \times I(q_1) \times \{0, 1\})$. If we define $F^1(x) = F(x)$ on $X(q) \times \{0, 1\}$, then by Lemma 2.5 $F^1$ will still be $c\delta$-localized. This defines $F^1$ on $(X(q) \times I(q)_1) \cup (X(q_1) \times I(q))_0$. We apply the inductive step from the proof of Proposition 2.7 to define $F^i, i = 2, \ldots, p$, until we obtain a $c\delta$-localized map on $X(q)_0 \times I(q)$. We can then apply Proposition 2.6 II to extend $F$ to a continuous map defined everywhere. The mass and flat norm estimates follow from the corresponding estimates in Proposition 2.6 and the inductive step in the proof of Proposition 2.7.

Let $F_0$ and $F_1$ be two maps as in Theorem 2.3 (2). Let $m = \sup_{w \in X} M(F_0(w)) + \sup_{w \in X} M(F_1(w))$. 

\[ \square \]
By Proposition 2.10 for every sufficiently small \( \delta > 0 \) we can define \( \delta \)-localized maps \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \), which agree with \( F_0 \) and \( F_1 \) on the 0-skeleton of some subdivision \( X(q')_0 \). By Proposition 2.10 we then have

\[
\mathcal{F}(\mathcal{F}_0(x), \mathcal{F}_1(x)) \leq c(M, p)m\delta + \varepsilon
\]

For \( \delta > 0 \) and \( \varepsilon > 0 \) sufficiently small we have that \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \) are homotopic by Lemma 2.11.

Now we would like to construct a homotopy between \( F_i \) and \( F_i \) for \( i = 0, 1 \).

Let \( \delta_i \) be a decreasing sequence of positive numbers converging to 0 and \( \eta_i = \min\{\delta_i, \varepsilon_2(\delta_i)\} \), where \( \varepsilon_2(\delta_i) > 0 \) is from Proposition 2.10. Let \( q_i \geq q \) be an increasing sequence of integers, so that \( F_0 : X(q_i) \to Z_k(M, \partial M; G) \) is \( \varepsilon_i \)-fine. Applying Proposition 2.10 we obtain a sequence of \( \delta_i \)-localized maps \( G_i : X(q'_i) \to Z_k(M, \partial M; G) \). We claim that \( G_i \to F_0 \). Given \( x \in X \) let \( x_i \) denote the closest point of \( X(q_i) \) to \( x \).

\[
(15) \quad \mathcal{F}(G_i(x), F_0(x)) \leq \mathcal{F}(G_i(x), G_i(x_i)) + \mathcal{F}(G_i(x_i), F_0(x_i)) + \mathcal{F}(F_0(x_i), F_0(x))
\]

\[
\leq c(M, p)m\delta_i + \varepsilon_i
\]

The convergence follows by standard compactness results for flat cycles [6, 4.2.17].

To finish the construction we need to define a homotopy between \( G_i \) and \( G_{i+1} \). Observe that if \( G_i \) is \( \delta_i \)-localized with domain \( X(q'_i) \), then it is also \( \delta_i \)-localized with domain \( X(q'_{i+1}) \) for \( q'_{i+1} > q'_i \). It follows from inequality (15) that \( \mathcal{F}(G_{i+1}(x), G_i(x)) \to 0 \). Assuming that \( \delta_i \) is small enough we can apply Lemma 2.11 to define homotopy \( H_i(x, t) \) between maps \( G_i : X(q'_i) \to Z_k(M, \partial M; G) \) and \( G_{i+1} : X(q'_{i+1}) \to Z_k(M, \partial M; G) \) for all \( i > 0 \). Moreover, the flat norm estimates from Proposition 2.10 and Lemma 2.11 imply that for each \( x \) and \( t \) the sequence \( H_i(x, t) \) is Cauchy in \( \mathcal{F} \). This implies continuity of homotopy from \( F_0 \) to \( F_0 \). The same construction works for the homotopy from \( F_1 \) to \( F_1 \).

2.6. Homotopies with better estimates for the mass. We need to prove an analogue of Theorem 2.3 with better estimate for the mass of cycles when the family satisfies some additional assumptions.

In case of 0-cycles we prove approximation theorem with an optimal bound.

Proposition 2.12. There exists \( \varepsilon_0(M, p) > 0 \) and \( c(M, p) > 0 \), such that the following holds.

Let \( F : X(q)_0 \to Z_0(M, \partial M; \mathbb{Z}_2) \) be a map with \( \mathcal{F}(F(a), F(b)) < \varepsilon \leq \varepsilon_0 \) for every pair of adjacent vertices \( a \) and \( b \), then there exists an extension of \( F \) to a continuous map \( \tilde{F} : X \to Z_0(M, \partial M; \mathbb{Z}_2) \) with the following properties. For each cell \( C \) in \( X(q) \) and \( x, y \in C \) we have

(a) \( \mathcal{F}(F(x), F(y)) < c \sup_{w \in X_0} M(F(w)) \varepsilon \)

(b) \( M(F(x)) \leq \sup_{w \in X_0} M(F(w)) \).
Proof. We observe that every $\varepsilon$-fine family of 0-cycle $F : X(q)_0 \to Z_0(M, \partial M; \mathbb{Z}_2)$ is $\delta$-localized for some $\delta \leq c(p)\varepsilon$. Hence, there is no need for Proposition 2.7.

We can prove an analogue of Proposition 2.6 with a better bound for the mass. We do it as follows.

First we extend to 1-skeleton as follows. Let $E \subset X(q)_1$ and let $\{U^E_j\}$ be a $\delta$-admissible collection of open sets containing $F(x) - F(y)$, $\partial E = x - y$. Therefore, we must have $M(F(x) \cap U^E_j) = M(F(y) \cap U^E_j) \mod 2$. Let $z_j = 0$ if $M(F(x) \cap U^E_j) = 0 \mod 2$ and $z_j = \{c_j\}$ if $M(F(x) \cap U^E_j) = 1 \mod 2$, where $c_j$ is a center point of $U^E_j$.

Let e denote the midpoint of E. We define

$$F(e) = F(x) - \sum F(x) \cap U^E_j + \sum z_j = F(y) - \sum F(y) \cap U^E_j + \sum z_j$$

We homotop $F(x)$ to the midpoint of $E$ by radially contracting $F(x) \cap U^E_j$ in each $U^E_j$. We do the same for $F(y)$.

Similarly, given a cell $C$ and a $\delta$-admissible collection $\{U^C_j\}$, so that $F(x) - F(y) \subset \cup U^C_j$ for any $x, y \in \partial C$, we define cycles $z_j$ and radially contract family $\{F(x)\}_{x \in \partial C}$ to $F(e) = F(x) - \sum F(x) \cap U^C_j + \sum z_j$, where e is the center point of $C$. \[\square\]

For higher dimensional cycles we obtain a better bound for families that have no concentration of mass.

Map $F : X \to Z_k(M, \partial M; G)$ is said to have no concentration of mass if

$$\limsup_{r \to 0} \{M(F(x) \cap B_r(p)) : p \in M, x \in X\} = 0$$

for every $x \in X$ (see [24, 3.7]).

**Proposition 2.13.** Let $F : X(q) \to Z_k(M, \partial M; G)$ be a map with no concentration of mass and $\sup_{x \in X} M(F(x)) < \infty$. For every $\varepsilon > 0$ and $\delta > 0$ there exists a $\delta$-localized map $F' : X^p(q + \bar{q}) \to Z_k(M, \partial M; G)$, such that

(a) $F(F(x), F'(x)) < \varepsilon$;

(b) $M(F'(x)) \leq \sup_{x \in X_0} M(F(x)) + \varepsilon$.

**Proof.** Let $m = \sup_{x \in X} M(F(x))$. Let $c_0 > 0$ be larger than the constant $c(n, p)$ appearing in Proposition 2.6. We will choose $\delta' > 0$ and $\varepsilon' > 0$ satisfying the following conditions.

Conditions for $\delta'$:

1. $\delta' < \frac{\delta}{c_0}$;

2. If $\{U^p_i\}_{l=1}^{n(p)}$ is a $\delta'$-admissible collection of open sets constructed in Proposition 2.7 then (using no concentration of mass condition) we have

$$\sup_{x \in X} \sum_{l=1}^{n(p)} M(F(x) \cap U_i) < \varepsilon'$$

Then (using Proposition 2.7) we have

$$\sup_{x \in X} \sum_{l=1}^{n(p)} M(F(x) \cap U_i) < \frac{\varepsilon}{10c_0}$$

for every $x \in X$. \[\square\]
Let \( X^p \) be a cubical subcomplex that is also a \( p \)-dimensional manifold and \( G = \mathbb{Z}_2 \) or \( \mathbb{Z} \). If \( G = \mathbb{Z} \) we demand that \( X \) is oriented. Let \( M \) be a Riemannian manifold and \( \partial M \) be piecewise smooth with \( \theta \)-corners. Fix a PL structure on \( M \) that is a bilipschitz approximation of \( M \). To a \( \delta \)-localized map \( F : X(q_0) \to Z_k^\text{PL}(M, \partial M; G) \) we associate a polyhedral \((p+k)\)-cycle \( A_F \in Z^\text{PL}_{p+k}(X \times M; G) \) constructed as follows. For each \( l \)-cell \( C^l \subset X(q)_l \) we will define a polyhedral \((l+k)\)-chain \( A(C) \in I^\text{PL}_{l+k}(X \times M; G) \). We will then define \( A_F = \sum_C A(C) \), where the sum is over all \( p \)-dimensional cells of \( X(q) \). The definition is by induction on \( l \). For each \( x \in X \) we let \( i_x : M \to X \times M \) be the inclusion \( i_x(a) = (x, a) \). For \( l = 0 \) we define \( A(x) = i_x(F(x)) \) for each \( x \in X(q) \). For every \((l-1)\)-cell \( C^{l-1} \subset X(q)_{l-1} \) assume that we have already defined \( A(C) \). Given a set \( U \subset M \) and a bilipschitz diffeomorphism \( f : U \to \mathbb{R}^n \) with \( f(U) \) convex, a point \( x \in U \) and a relative cycle \( B \in Z_L(C \times U, C \times (U \cap \partial M); G) \) let \( \text{cone}_x(B) = g^{-1}(\text{Cone}_{g(pC,x)}(g(B))) \)
where $p_C$ is the center point of cell $C$, $g = (id_C, f) : C \times U \to C \times \mathbb{R}^n$ and $\text{Cone}_g(p_C, x)(g(B))$ denotes the cone over $g(B)$ in $C \times f(U)$ with vertex $(p_C, f(x))$. We will choose $f$ to be the exponential map from point $p$ if $p$ lies in the interior of $M$, or map $E$ from Section 2.1 if $p \in \partial M$.

Define $A(\partial C) = \sum_{E \in \partial C} (-1)^i A(E)$ where the sign $(-1)^i$ is defined in the standard way for the boundary operator on a cubical complex, $\partial C = \sum_{E \in \partial C} (-1)^i E$. Let $\{U_i^C\}$ be an admissible collection of sets corresponding to $\delta$-localized family $\{F(x)\}_{x \in C_0}$. We will choose $x_i \in U_i^C$ for each $i$; if $U_i^C$ intersects $\partial M$, then we pick $x_i \in \partial U_i^C$. We fix a vertex $v \in C_0$ and set

$$A(C) = A(v) \times C + \sum_i \text{cone}_{x_i}([A(\partial C) - A(v) \times \partial C] \cap C \times U_i^C)$$

Observe that by induction and our definition of $\delta$-localized families $A(\partial C) - A(v) \times \partial C$ is an $(l - 1)k$-dimensional polyhedral relative cycle in $\bigsqcup_i C \times U_i^C$, so $A(C)$ is well-defined.

It follows from the construction that $A$ has the following properties:

1. for an open and dense subset of points $x \in C$ and for an open and dense subset of points $x \in \partial C$ we have that

$$\text{proj}_M(\text{proj}_X^{-1}(x) \cap A(C))$$

is a polyhedral $k$-cycle;

2. there exists a sequence $q = q_0 < q_1 < q_2 < ...$ and $\varepsilon_i \to 0$, s.t. discrete families of cycles $F^i : C(q_i) \to \mathbb{Z}_k(M, \partial M; G)$ given be $F^i(x) = \text{proj}_M(\text{proj}_X^{-1}(x) \cap A(C))$ are $c(l - 1)\delta$-localized (with respect to a fixed admissible collection $\{U_i^C\}$ that depends only on $C$) and $\varepsilon_i$-fine.

Observe that if follows from the definition that if $F_1$ and $F_2$ are two $\delta$-localized families with $\mathcal{F}(F_1, F_2) < \varepsilon$ for all $x \in X(q_0)$, then for all sufficiently small $\varepsilon > 0$ the corresponding $(p + k)$-cycles $A_{F_1}$ and $A_{F_2}$ are homologous (cf. [14, Section 1]). In particular, given a map $F : X \to \mathbb{Z}_k(M, \partial M; G)$ we can define $\alpha_F \in H_{p+k}(X \times M, X \times \partial M; G)$ corresponding to cycle $A_{F'}$, where $F'$ is a $\delta$-localized approximation of $F$ obtained by Theorem 2.7 and $\alpha_F$ is independent of the choice of $F'$.

**Theorem 3.1.** Two maps $F_0 : X \to \mathbb{Z}_k(M, \partial M; G)$ and $F_1 : X \to \mathbb{Z}_k(M, \partial M; G)$ are homotopic if and only if $\alpha_{F_0} = \alpha_{F_1} \in H_{p+k}(X \times M, X \times \partial M; G)$.

**Proof.** One direction follows immediately from the construction: if $F_0$ and $F_1$ are homotopic, then $\alpha_{F_0} = \alpha_{F_1}$. Indeed, let $F : [0, 1] \times X \to \mathbb{Z}_k(M, \partial M; G)$ be the homotopy. By Theorem 2.7 there exists a $\delta$-localized approximation $F'$ of $F$ and the restrictions $F'|_{i \times X}$ are $\delta$-localized approximations of $F_i$ for $i = 0, 1$. Applying our construction to $F'$ we obtain a $(p + 1 + k)$-chain $B$ with $\partial B = A_{F'|_{i \times X}} - A_{F'\mid_{0 \times X}}$. 
Now we will prove the other direction. Let $F_0' : X(q)_0 \to \mathbb{Z}_k(M, \partial M; G)$ and $F_1' : X(q)_0 \to \mathbb{Z}_k(M, \partial M; G)$ be two $\varepsilon$-fine $\delta$-localized maps that $\varepsilon$-approximate $F_0$ and $F_1$ correspondingly.

Define $A_{F_1}$ and $A_{F_2}$ as described above. Assume that they represent the same homology class. We would like to construct a homotopy from $F_1$ to $F_2$. It is enough to construct a sequence of maps $F_i : X(q)'_0 \to \mathbb{Z}_k(M, \partial M; G)$, $q' \geq q$, which are

1) $\varepsilon$-fine;
2) satisfy $\mathcal{F}(F_i(x), F_{i+1}(x)) \leq \varepsilon$ for all $x \in X(q)'_0$;
3) $\mathcal{F}(F_0(x), F_1(y) - F_0(y)) \leq c\varepsilon$ for $x, y$ in the same cell of $X(q)$;
4) $F^N(x)$ is the empty cycle for all $x$.

By Theorem 2.3 such a sequence of maps would guarantee existence of a continuous homotopy contracting $F_1 - F_0$.

We will define sequence $F_i$ by defining a sequence of $(p + k)$-cycle $A_i$ in $X \times M$ and intersecting them with fibers of $\text{proj}_X$.

We have that cycles $A_{F_1}$ and $A_{F_2}$ are constructed in such a way that for every $\varepsilon > 0$ picking a sufficiently fine set of discrete points $X(q'')_0$ (i.e. $q''$ sufficiently large) we can guarantee that family $\{\text{proj}^{-1}(x) \cap A_{F_j}\}$ is $\varepsilon$-fine. We observe that every $(p + k)$-cycle in general position will have this property. This follows from PL version of transversality, which we explain in more detail below.

Fix a fine PL structure on of $M$ that is $(1 + \varepsilon)$-bilipschitz to the original metric. Fix a cell $E^n_X \times E^n_M$, $E_X \subset X$, $E_M \subset M$. Let $C$ be a linear $m$-cell contained in $E^n_X \times E^n_M$, $m > 0$. Let $\theta(C)$ denote the angle that $C$ makes with fibers of $\text{proj}_X : E_X \times E_M \to E_X$, defined as follows. Fix a point $a$ in the interior of $C$. Let $P_M$ denote the $n$-plane that passes through $a$ and contains a fiber of $\text{proj}_X$. Let $P_C$ denote the $m$-plane that passes through $a$ and contains $C$. If $m > p$ let $\beta = \{v_1, \ldots, v_{m-p}\}$ be a set of linearly independent vectors contained in $P_C \cap P_M$, otherwise set $\beta = \{0\}$. Define $\theta(C)$ to be the minimum over all angles between non-zero vectors $v_1 \in P_C$ perpendicular to $\text{span}(\beta)$ and $v_2 \in P_M$ perpendicular to $\text{span}(\beta)$. Note that this definition is independent of the choice of $a$.

We will say that a polyhedral chain (resp. relative cycle) $A^m$ in $X^n \times M^n$, $m \geq p$, is in general position if the following conditions are satisfied:

1) $A \cap X \times \partial M$ is an $(m - 1)$-dimensional polyhedral chain (resp. cycle);
2) Each cell $C_i^a$ of $A$ is contained in the interior of a cell $E^j$ of $X \times M$ with $j \geq i$;
3) For each cell $C_i^i$, $i > 0$, of $A$ we have $\theta(C) > 0$.

**Lemma 3.2.** If $A^m$ is a polyhedral relative cycle in general position, then there exists an open and dense subset $G \subset X$, such that the following holds:

(a) for every $x \in G$ we have that $\text{proj}_X^{-1}(x) \cap A$ is a polyhedral $(m-p)$-dimensional relative cycle and $\text{proj}_X^{-1}(x) \cap C \cap X \times \partial M$ is a polyhedral $(m - p - 1)$-dimensional cycle.
(b) for every $\varepsilon > 0$ there exists $\delta > 0$, so that if $x, y \in G$ and $\text{dist}(x, y) < \delta$ then $F(proj_M(proj_X^{-1}(x)), proj_M(proj_X^{-1}(y))) < \varepsilon$.

**Proof.** Let $(A)_0$ denote the 0-skeleton of $A$ and let $G \subset X$ be the union of interiors of all $n$-dimensional faces of $X$ minus $proj_X((A)_0)$. By [30, Theorem 1.3.1] and our angle condition $proj_X^{-1}(x)$ is a $(m - p)$-dimensional polyhedral cycle for each point $x \in G$, and given $y_i \in G$ and a piecewise linear segment $L_i$ connecting $y_i$ to $x$, $proj_X^{-1}(L_i)$ is a polyhedral $(m - p + 1)$-chain with $\text{Vol}(L_i) \to 0$ as $y_i \to x$. □

Let $B$ denote a polyhedral $(p + k + 1)$-chain with $\partial B = A_{F_1} - A_{F_2}$. There exists a fine subdivision of $B$, so that for any $(p + k)$-cell $C$ of $B$ and almost every $x \in X$ we have that $z_C(x) = proj_X^{-1}(x) \cap C$ is a polyhedral $k$-chain of volume less than $\varepsilon$.

Let $N$ denote the number of $(p + k + 1)$-cells in $B$. We can pick a large enough $q' \geq q$, so that for every two adjacent vertices $x_1, x_2 \in X(q'_0)$ and every $(p + k)$-cell $C$ of $B$ we have

$$F(proj_M(proj_X^{-1}(x_1) \cap C), proj_M(proj_X^{-1}(x_2) \cap C)) < \varepsilon/N
$$

Let $\{C_i\}_{i=1}^N$ be $(p + k + 1)$-cells of $B$. Define a sequence of maps $F_i : X(q'_0) \to \mathcal{Z}_k(M, \partial M; G)$ as follows:

$$F_i(x) = proj_M(proj_X^{-1}(x) \cap \partial(B \setminus \bigcup_{j=1}^i C_j))$$

By construction this defines a sequence of $\varepsilon$-close $\varepsilon$-fine discrete maps and by Theorem 2.3 we can construct a homotopy from $F : X \to \mathcal{Z}_k(M, \partial M; G)$ given by $F(x) = F_1(x) - F_0(x)$ to the 0-map. □

Let $F_{A} : \pi_m(\mathcal{Z}_k(M^n; G)) \to H_{m+k}(M^n \times S^m; G)$ be a map defined as follows. Given a map $f : S^n \to \mathcal{Z}_k(M^n; G)$ representing class $[f] \in \pi_m(\mathcal{Z}_k(M^n; G))$ we set $F_{A}([f]) = \alpha_f \in H_{m+k}(M^n \times S^m; G)$.

**Theorem 3.3.** $F_{A}$ is a bijective groupoid homomorphism.

**Proof.** By Theorem 3.1 if $f : S^n \to \mathcal{Z}_k(M; G)$ and $g : S^n \to \mathcal{Z}_k(M; G)$ are two homotopic maps, then cycles $A_f$ and $A_g$ are homologous, so the map is well defined. It also follows from the construction of $A_f$ that $F_{A}$ is a homomorphism and injectivity of $F_{A}$ follows by Theorem 3.1.

Every homology class in $H_{m+k}(M^n \times S^m; G)$ can be represented by a polyhedral chain $A$ in general position, so that for a sufficiently large $q$ discrete family $\{proj_M(proj_X^{-1}(x) \cap A)\}_{x \in X(q)_0}$ represents a unique homotopy class in $\pi_m(\mathcal{Z}_k(M^n; G))$. Hence, $F_{A}$ is also surjective. □

By the Kunneth formula we have $H_{m+k}(M^n \times S^m; G) \cong H_{m+k}(M^n; G) \oplus H_k(M^n; G)$. If we restrict the domain of $F_{A}$ to maps $f : S^n \to \mathcal{Z}_k(M^n; G)$ with a fixed a base point cycle $z \in \mathcal{Z}_k(M; G)$ we obtain the following isomorphism.
Corollary 3.4 (Almgren isomorphism theorem). The homotopy group \( \pi_m(\mathcal{Z}_k(M^n;G);\{[z]\}) \) is isomorphic to the homology group \( H_{m+k}(M^n;G) \).

For future applications we will also need the following Proposition, which allows us to replace a contractible family of relative cycles with a contractible family of absolute cycles.

Proposition 3.5. Let \( F : X(q) \to \mathcal{Z}_k(M, \partial M;G) \) be a contractible map. For every \( \varepsilon > 0 \) there exists a map \( G : X \to \mathcal{Z}_k(M,G) \), such that

\[
\mathcal{F}(F(x) \cup \text{int}(M), G(x) \cup \text{int}(M)) < \varepsilon
\]

Proof. Let \( A_F \) denote a \( (p+k) \)-dimensional relative cycle in \( X \times M \) corresponding to \( F \). Since \( F \) is contractible, by Theorem 3.1 there exists a cycle \( B \) with \( \partial B - A_F \subset X \times \partial M \). For \( \partial B \) in general position and \( q \) sufficiently large we can define an \( \varepsilon/2 \)-fine map \( G : X(q)_0 \to \mathcal{Z}_k(M,G) \) defined by \( G(x) = \text{proj}_M(\text{proj}_X^{-1}(x)) \cap \partial B \).

By Theorem 2.3 we can complete \( G \) to a continuous family of absolute cycles approximating \( F \). \( \square \)

We finish this section with a counterexample to parametric isoperimetric inequality for contractible families of cycles with integer coefficients.

Proposition 3.6. Let \( N > 0 \). There exists a contractible family of 0-cycles \( F_N : S^1 \to \mathcal{Z}_0(S^1;\mathbb{Z}) \), such that

- \( \mathcal{M}(F_N(x)) \leq 2 \) for all \( x \in S^1 \);
- any family of fillings \( H : S^1 \to I_1(S^1;\mathbb{Z}) \) with \( \partial H(x) = F_N(x) \) satisfies \( \mathcal{M}(H(x_0)) > N \) for some \( x_0 \in S^1 \).

Proof. Consider \([0,1] \times S^1\) and let \( \{\gamma_i = \{t_i\} \times S^1\} \) be a collection of \( 2N \) disjoint closed curves in \([0,1] \times S^1\) all oriented the same way and with \( t_i \in (0, \frac{1}{2}) \); let \( \{\beta_i = \{s_i\} \times S^1\} \) be a collection of \( 2N \) disjoint closed curves all oriented in the opposite way to \( \gamma_i \) and with \( s_i \in (\frac{1}{2}, 1) \). Observe that we can make a PL perturbation of curves \( \gamma_i \) and \( \beta_i \), so that fibers of the projection map \( p : [0,1] \times S^1 \to [0,1] \) intersect \( \bigcup \gamma_i \cup \beta_i \) in at most 2 points (and when the intersection has exactly two points they have opposite orientation).

Identifying the endpoints of \([0,1]\) we can think of curves \( \gamma_i \) and \( \beta_i \) as lying in \( S^1 \times S^1 \). Let \( X = S^1 \), then for sufficiently large \( q \) we have that \( \{F(x) = p^{-1}(x) \cap (\bigcup \gamma_i \cup \beta_i) : x \in X(q)_0\} \) is an \( \varepsilon \)-fine discrete family of cycles. Since \( Z = \bigcup \gamma_i \cup \beta_i \) is null-homologous as a 1-cycle in \( S^1 \times S^1 \) the family \( F(x) \) is contractible. Every family of fillings \( H \) of \( F \) then corresponds to a filling \( \tau \) of \( Z \). It is easy to see, using a winding number argument, that either \( p^{-1}(0) \cap \tau \) or \( p^{-1}(1) \cap \tau \) is a circle of multiplicity \( \geq N \). It follows that \( \mathcal{M}(H(x_0)) \geq N \) for some \( x_0 \in S^1 \). \( \square \)
4. Parametric coarea inequality

Let \( \Omega \subset \mathbb{R}^3 \) be a domain with \( \partial \Omega \) piecewise-smooth boundary with \( \theta \)-corners. Let \( \Omega_\varepsilon = \Omega \setminus E(\partial \Omega \times [0, \varepsilon]) \), where \( E \) is the “exponential map” from Section 2.1 and \( \Sigma = \partial \Omega_\varepsilon \). In this section we will prove a somewhat stronger version of Conjecture 1.6 for \( n = 3 \) and \( k = 1 \).

**Theorem 4.1.** Fix \( \eta > 0 \). For all \( \varepsilon \in (0, \varepsilon_0) \), \( \delta_1 > 0 \) and \( p \geq p_0(\Omega, \varepsilon) \) the following holds. Let \( F : X^p \to Z_1(\Omega, \partial \Omega; \mathbb{Z}_2) \) be a continuous map with no concentration of mass. There exists a map \( F' : X \to I_1(cl(\Omega_\varepsilon); \mathbb{Z}_2) \), such that

1. \( \partial F' \) is a continuous family of 0-cycles in \( \partial \Omega_\varepsilon \) that is \( \delta_1 \)-localized on \( X(q') \) for sufficiently large \( q' \);
2. \( F(F(x), \Omega_\varepsilon, F(x)) < \eta \);
3. \( M(F'(x)) \leq M(F(x) + c(n) M(\partial \Omega)(p^{\frac{1}{2}} + \frac{M(F(x))}{\varepsilon^{\sqrt{p}}}) \);
4. \( M(\partial F'(x)) \leq c(n) \ M(\partial \Omega)p \).

Moreover, if \( F \) is a p-sweepout of \( \Omega \), then \( \partial F' : X \to Z_0(\partial \Omega_\varepsilon; \mathbb{Z}_2) \) is a p-sweepout of \( \partial \Omega_\varepsilon \).

**Proof.** By Proposition 2.13, we can replace family \( F \) with a \( \delta \)-localized family, which is arbitrarily close to \( F \) in flat norm, while increasing the mass by an arbitrarily small amount. Hence, without any loss of generality we may assume that \( F \) is \( \delta \)-localized for some very small (compared to \( \eta \) and \( \varepsilon \)) \( \delta > 0 \).

**4.1. Proof strategy.** We give a brief informal overview of the proof strategy. For each value of \( x \in X(q)_0 \) we can use coarea inequality to find \( s(x) \), so that the restriction \( F(x)|_{\Omega_\varepsilon - s(x)} \) satisfies a good bound on boundary mass. We then project the part sticking outside of \( \Omega_\varepsilon \) onto \( \Sigma = \partial \Omega_\varepsilon \) and denote the resulting chain (that will be a relative cycle in \( \Omega_\varepsilon \)) by \( C_{s(x)}(F(x)) \) (see precise definitions in subsection 4.2). This gives us a way to define the family on the 0-skeleton \( X(q)_0 \). The challenge is to extend this family to the higher dimensional skeleta of \( X(q) \).

For two adjacent vertices \( x, \ y \) we have that \( I = C_{s(x)}(F(x)) - C_{s(y)}(F(y)) \) is a chain supported in \( \Sigma \). Recall that \( F \) is \( \delta \)-localized and so we can assume that for any two adjacent vertices the values of \( s(x) \) and \( s(y) \) are chosen so that \( F(x) - F(y) \) is supported away from \( \partial \Omega_\varepsilon - s(x) \) and \( \partial \Omega_\varepsilon - s(y) \). It follows that we can deform \( I \) into a chain \( I' = C_{s(x)}(F(x)) - C_{s(y)}(F(x)) \) and that we have a good bound on the mass of \( \partial I' \). To interpolate between \( C_{s(x)}(F(x)) \) and \( C_{s(y)}(F(y)) \) we would like to contract \( I \), while controlling its mass and boundary mass. Note that the mass of \( I \) may be as large as that of \( F(x) \) or \( F(y) \). So to do this we subdivide \( \Sigma \) into small triangles \( \{Q_i\} \) with the goal of contracting \( I \) radially triangle by triangle. However, we have no control over how large the mass of the intersection of \( I \) with \( \partial Q_i \) is (but can assume that \( F \) was slightly perturbed so that the intersection has finite mass), so contracting
\( (I)_i = I \cup Q_i \) may increase the boundary mass by an arbitrary amount. To solve this problem we add to each \( C_{s(x)}(F(x)) \) a chain \( \sum_i f_i(x) \), where each \( f_i(x) \) is supported in \( \partial Q_i \) and is a kind of “filling” of the 0-chain \( C_{s(x)}(F(x)) \cup \partial Q_i \), reducing its mass to \( \leq 1 \).

With this in mind we will define \( F'(x) = \sum_i C_{s(x)}(F(x)) \cup Q_i + f_i(x) + \text{Cone}(\partial F(x) \cup Q_i) \) on the 0-skeleton \( X(q)_0 \). The last term is the cone over the boundary of \( C_{s(x)}(F(x)) \) in \( Q_i \) with vertex \( q_i \in Q_i \). Adding the cone makes some of the interpolation formulas simpler as we extend \( F' \) to higher dimensional faces of \( X(q) \). The interpolation process is illustrated on Fig 3.

Let’s ignore, for simplicity, the difference between values of \( F \) on adjacent vertices, since they are contained in a collection of tiny balls away from where we “cut” \( F(x) \). Interpolation procedure then can be thought of as radial scaling of terms like \( [C_{s(x)}(F(x)) - C_{s(x_j)}(F(x))] \cap Q_i + f_i(x_i) - f_i(x_j) \) for different vertices \( x_i \) and \( x_j \) in the 0-skeleton of a cell in \( X(q) \). On Fig 3 on the left we see \( (F'(x_4))_i = (C_{s(x_4)}(F(x)))_i + f_i(x_4) + \text{Cone} \), where \( s(x_4) \) corresponds to the cut that is closest to \( \partial \Omega \). Then we have cuts corresponding to \( s(x_1), s(x_2) \) and \( s(x_3) \) indicated on the same picture. The cut corresponding to \( s(x_2) \) has mass 1 and the other two cuts have mass 3. The interpolation will involve scaling the difference between these cuts as on the figure on the right and adding a portion of the cone over \( q_i \) to connect the boundaries. Inductively we will assume that on the boundary of a cell \( C \) the family of chains is of this form and then linearly change the scaling factors to homotop the

\[ F(x_1) \text{ and } F(x_2) \text{ are two cycles with } x_1, x_2 \in C \cap X(q)_0. \]

We use coarea inequality to cut them in \( \Omega \setminus \Omega_\varepsilon \). Then we project the part sticking out outside of \( \Omega_\varepsilon \) onto \( \Sigma = \partial \Omega_\varepsilon \). 

\[ (I)_i = I \cup Q_i \] may increase the boundary mass by an arbitrary amount. To solve this problem we add to each \( C_{s(x)}(F(x)) \) a chain \( \sum_i f_i(x) \), where each \( f_i(x) \) is supported in \( \partial Q_i \) and is a kind of “filling” of the 0-chain \( C_{s(x)}(F(x)) \cup \partial Q_i \), reducing its mass to \( \leq 1 \).

With this in mind we will define \( F'(x) = \sum_i C_{s(x)}(F(x)) \cup Q_i + f_i(x) + \text{Cone}(\partial F(x) \cup Q_i) \) on the 0-skeleton \( X(q)_0 \). The last term is the cone over the boundary of \( C_{s(x)}(F(x)) \) in \( Q_i \) with vertex \( q_i \in Q_i \). Adding the cone makes some of the interpolation formulas simpler as we extend \( F' \) to higher dimensional faces of \( X(q) \). The interpolation process is illustrated on Fig 3.

Let’s ignore, for simplicity, the difference between values of \( F \) on adjacent vertices, since they are contained in a collection of tiny balls away from where we “cut” \( F(x) \). Interpolation procedure then can be thought of as radial scaling of terms like \( [C_{s(x)}(F(x)) - C_{s(x_j)}(F(x))] \cap Q_i + f_i(x_i) - f_i(x_j) \) for different vertices \( x_i \) and \( x_j \) in the 0-skeleton of a cell in \( X(q) \). On Fig 3 on the left we see \( (F'(x_4))_i = (C_{s(x_4)}(F(x)))_i + f_i(x_4) + \text{Cone} \), where \( s(x_4) \) corresponds to the cut that is closest to \( \partial \Omega \). Then we have cuts corresponding to \( s(x_1), s(x_2) \) and \( s(x_3) \) indicated on the same picture. The cut corresponding to \( s(x_2) \) has mass 1 and the other two cuts have mass 3. The interpolation will involve scaling the difference between these cuts as on the figure on the right and adding a portion of the cone over \( q_i \) to connect the boundaries. Inductively we will assume that on the boundary of a cell \( C \) the family of chains is of this form and then linearly change the scaling factors to homotop the
family in $Q_i$ to a cycle $F'(x_j) \cap Q_i$ that has the least boundary mass (in the case of the figure this is $x_j = x_2$ or $x_j = x_4$).

4.2. **Notation.** We need to make the following definitions:

- Let $\pi : \Omega \setminus \Omega_\varepsilon \to \Sigma$ be the projection map.
- Given a $k$-chain $\tau \in I_1(\Omega; \mathbb{Z}_2)$ and $s \in [0, \varepsilon]$ let $L_s(\tau) = \pi(\tau \cap (\Omega_\varepsilon \setminus \Omega_\varepsilon)) \subset \Sigma$ and $C_s(\tau) = (\tau \cap \Omega_\varepsilon) + L_s(\tau) \subset \text{cl}(\Omega_\varepsilon)$.
- Let $[\tau]_{s_1, s_2} = C_{s_2}(\tau) - C_{s_1}(\tau)$.
- For $x \in X(q)$ let $B(x)$ denote the cell of $X(q)$ of smallest dimension, such that $x$ lies in its interior; if $x \in X(q)_0$, then $B(x) = x$.
- For each vertex $x \in X(q)_0$ we apply applying coarea inequality Theorem 2.2 (here we assume that $\varepsilon \leq \varepsilon(\Omega)$ is sufficiently small) to find $s(x) \in [\varepsilon, \varepsilon_0]$ such that $M(\partial C_{s(x)}(F(x))) \leq \frac{1.5}{\varepsilon - 4\delta} M(F(x) \cap \text{cl}(\Omega \setminus \Omega_\varepsilon)) \leq \frac{2}{\varepsilon} M(F(x) \cap \text{cl}(\Omega \setminus \Omega_\varepsilon))$

and, moreover, we have that $\partial \Omega_{\varepsilon - s(x)}$ is disjoint from a $\delta$-admissible family $\{U_i\}$.

In particular, for any $x, y \in X(q)_0$ that lie in a common face of of $X(q)$ we have that $C_{s(x)}(F(x) - F(y))$ is an absolute cycle.

- For $p > p_0(\Omega, \varepsilon)$ we can triangulate $\Sigma$ by subdividing it into $N'$ subsets $Q_i$, $N' \leq \text{const} \ p M(\partial \Omega)$, with disjoint interiors and piecewise smooth boundaries, such that there exists a $(1 + \varepsilon^2)$-bilipschitz diffeomorphism $P_i$ from $Q_i$ to a convex simplex $Q'_i \subset \mathbb{R}^2$, $\text{diam}(Q'_i) \leq p^{-\frac{1}{2}}$ and $M(\partial Q'_i) \leq p^{-\frac{3}{2}}$. 

\textbf{Figure 3.}
For each triangle $Q_i$ fix a point $q_i$ in the interior of $Q_i$. Given a cycle $z$ let $\text{Cone}_i(z) = P_i^{-1}(\text{Cone}_{P_i(q_i)}(P_i(z \cup Q_i)))$ denote the cone over $z \cup Q_i$ with vertex $q_i$.

Let $\Psi^i_\rho : Q_i \to Q_i$ be the radial scaling (towards $q_i$) map $\Psi^i_\rho(x) = P_i^{-1}(P_i(q_i) + \rho(P_i(x) - P_i(q_i)))$, $\rho \in [0, 1]$.

Given a $k$-chain $\tau$, $(k-1)$-cycle $x$ and $\rho \in [0, 1]$ define the following “conical collar” map:

$$\Phi^i_\rho(\tau, x) = \text{Cone}_i(x) - \text{Cone}_i(\Psi^i_\rho(x)) + \Psi^i_\rho(\tau)$$

We will drop superscript $i$ whenever it is clear to which region the map is being applied.

In the special case when $x = \partial \tau$ we have that $\Phi^i_\rho(\tau, x)$ fixes the boundary of $\tau$, and replaces the chain with a sum of a shrunk copy of itself and a piece of the cone over $\partial \tau$. More generally, we have $\partial \Phi^i_\rho(\tau, x) = x - \Psi^i_\rho(x) + \partial \Psi^i_\rho(\tau)$.

We will use the following notation: given a chain $I$ we let $(I)_i = I \cup Q_i$.

Let $k \geq 2$. Given chains $\tau_1, \ldots, \tau_k$ and $0 \leq \rho_1 \leq \cdots \leq \rho_{k-1} \leq 1$ define a chain $\Delta_i((\tau_1, \ldots, \tau_k), (\rho_1, \ldots, \rho_{k-1}))$ in $Q_i$:

$$\Delta_i((\tau_j)_{j=1}^k, (\rho_j)_{j=1}^{k-1}) = (\tau_1)_i + \sum_{j=1}^{k-1} \Phi^i_{\rho_1 \cdots \rho_j}((\tau_{j+1} - \tau_j)_i, (\partial \tau_j)_i) + \text{Cone}_i(\Psi^i_{\rho_1 \cdots \rho_{k-1}}((\partial \tau_k)_i))$$

For $k = 1$ we can write

$$\Delta_i(\tau) = (\tau)_i + \text{Cone}_i((\partial \tau)_i) = \Delta_i((\tau, \tau'), (0))$$

where $\tau'$ can be any chain.

For each $x \in X(q)_0$ we define $f_i(x)$ to be a 1-chain in $I_i(\partial Q_i; \mathbb{Z}_2)$ with the following property. Let $V_i$ denote the set of vertices of $Q_i$, then $\partial f_i(x) - (C_{s(x)}(F(x)))_i \cup \partial Q_i \subset V_i$.

4.3. **Definition of $F'$ on $X(q)_0$.** After an arbitrarily small perturbation of the family we may assume that $C_{s(x)}(F(x))$ intersects $\bigcup \partial Q_i$ transversely and $\partial C_{s(x)}(F(x))_i \cup \partial Q_i$ is empty for each $x \in X(q)_0$.

We define

$$F'(x) = F(x)_i \cup \Omega_\varepsilon + \sum_i \Delta_i(C_{s(x)}(F(x))) + f_i(x) \quad (16)$$

Observe that with this definition if $x$ and $y$ are vertices in a cell $D$ of $X(q)$, then there exists a cycle $e(x, y)$ of length $\leq \delta$ and contained in a $\delta$-admissible collection
of open sets in $\text{cl}(\Omega_\varepsilon)$, s.t.
\[
F'(x) = F(y) \cap \Omega_\varepsilon + \sum_i \Delta_i(C_{s(x)}(F(y))) + f_i(x) + e(x, y)
\]

4.4. **Inductive property.** Assume that we defined $F'$ on the $k$-skeleton of $X(q)$, so that it satisfies the following property:

(Inductive property for $k$-skeleton $X(q)_k$) For every $p'$-dimensional cell $C^{p'}$ of $X(q)$, $k \leq p' \leq p$ and point $y \in C$ the following holds. Let $(x_j)_{j=1}^{2^{p'}}$ denote the set of points in $C \cap X(q)_0$ and assume that they are numbered so that $s(x_{j_1}) \leq s(x_{j_2})$ if $j_1 < j_2$. Then there exist functions

\[
\rho^*_j : C \cap X(q)_k \rightarrow [0, 1], j \in \{1, ..., 2^{p'} - 1\}, i \in \{1, ..., N'\}. \text{ Let}
\]

\[
n_i(x) = \#\left\{ \prod_{j=1}^l \rho^*_j(x) : l \in \{1, ..., 2^{p'} - 1\} \right\} - 1
\]

($n_i(x)$ will correspond to the number of distinct collars in $Q_i$), then the following properties hold:

i. $\sum_{i=1}^{N'} n_i(x) \leq \text{dim}(B(x)) \leq k$; moreover, for all $x_j \notin B(x)_0$ we have $\rho^*_j(x) = 1$ if $s(x_j) < \min_{x \in B(x)_0} \{s(x)\}$ and $\rho^*_j(x) = 0$ if $s(x_j) = \max_{x \in B(x)_0} \{s(x)\}$;

ii. $\sum_i (1 - \rho^*_i(x)) \mathbf{M}((\partial(C_{s(x)}(F(y)))_i))$

\[
+ \sum_i \sum_j (\rho^*_i(x)\rho^*_j(x) - \rho^*_i(x)\rho^*_j(x)^{-1})(x) \mathbf{M}((\partial(C_{s(x)}(F(y)))_i))
\]

\[
+ \sum_i \rho^*_i(x)\rho^*_i(x)^{-1}(x) \mathbf{M}((\partial(C_{s(x)}(F(y)))_i)) \leq \frac{2 \mathbf{M}(F(x))}{\varepsilon}
\]

• $e : C \cap X(q)_k \rightarrow \mathbb{Z}_1(\text{cl}(\Omega_\varepsilon), \Sigma; \mathbb{Z}_2)$ of relative cycles of length $\leq c(k)\delta$ and contained in a $c(k)\delta$-admissible collection of sets, such that

\[
F'(x) = F(y) \cap \Omega_\varepsilon + e(x) + \sum_i \left( \Delta_i((C_{s(x)}(F(y)))_{j=1}^{2^{p'}}, (\rho_j(x))_{j=1}^{2^{p'}-1} \right)
\]

\[
+ f_i(x_1) + \sum_{j=1}^{2^{p'}-1} \Psi^i_{\rho^*_j(x)} f_i(x_{j+1} - f_i(x_j))
\]

for all $x \in C \cap X(q)_k$. 

We claim that our definition \([16]\) for \(k = 0\) satisfies the inductive assumption. Indeed, let \(x \in X(q)_0\) and \(y\) lie in a \(p'\)-dimensional cell \(D\) of \(X(q)\) that contains \(x\). Setting \(\rho_j^i(x) = 1\) for all \(j\), such that \(s(x_j) < s(x)\), and \(\rho_j^i(x) = 0\) for all \(j\), such that \(s(x_j) \geq s(x)\), we can write \([16]\) as

\[
F'(x) = F(x) \cup \Omega_\varepsilon + \sum_i \left( \Delta_i((C_{s(x_j)}(F(x)))_{j=1}^{2p'}, (\rho_j^i(x))_{j=1}^{2p'-1})
+ f_i(x_1) + \sum_{j=1}^{2p'-1} \Psi^i_{\rho^i(x)\rho^i_j(x)}(f_i(x_{j+1}) - f_i(x_j)) \right)
\]

Let \(G(x, y)\) denote the cycle obtained by replacing \(F(x)\) in the expression above with \(F(y)\):

\[
G(x, y) = F(y) \cup \Omega_\varepsilon + \sum_i \left( \Delta_i((C_{s(x_j)}(F(y)))_{j=1}^{2p'}, (\rho_j^i(x))_{j=1}^{2p'-1})
+ f_i(x_1) + \sum_{j=1}^{2p'-1} \Psi^i_{\rho^i(x)\rho^i_j(x)}(f_i(x_{j+1}) - f_i(x_j)) \right)
\]

Then using the fact that \(F(x)\) is \(\delta\)-localized and our choices of \(s(x)\) (that guarantee \(\partial C_{s(x)}(F(x)) = \partial C_{s(x)}(F(y))\) if \(x\) and \(y\) lie in the same cell of \(X(q)\)) we obtain that \(e(x) = F'(x) - G(x, y)\) is a cycle of length less than \(\delta\). Property (i) of functions \(\rho_j^i\) is immediate and property (ii) follows since \(M(\partial C_{s(x)}(F(x))) \leq \frac{2M(F(x))}{\varepsilon}\).

Next we show that the inductive property implies the desired mass and boundary mass bounds.

**Lemma 4.2.** Suppose the inductive property is satisfied for \(k = p' = p\) and \(y \in C^p \subset X(q)\). Then

1. \(F(F(y) \cup \Omega_\varepsilon, F'(y)) < \eta\);
2. \(M(F'(y)) \leq M(F(y)) + c(n)(\frac{M(F(y))}{\varepsilon \frac{1}{\sqrt{p}}} + M(\Omega)\sqrt{p})\);
3. \(M(\partial F'(y)) \leq c(n) M(\partial \Omega)^p\).

**Proof.** The first inequality follows immediately from \([17]\).

Now we prove the second inequality. Observe that

\[
M \left( \Phi^i_p((C_{s(x_j+1)}(F(y)) - C_{s(x_j)}(F(y)))_{j=1}^{p}) \right)
\leq \rho(M((F(y))_{i=1}^{s(x_j)+1})) + (1 - \rho) M((\partial C_{s(x_j)}(F(y)))_{j=1}^{p})
\]

and

\[
M(\Psi^i_p(f_i(x_{j+1}) - f_i(x_j))) \leq \rho M(\partial Q_i)
\]
Unravelling the definition of $\Delta_i$ and using property (ii) of functions $\rho_j^i(y)$ we obtain:

\[
M(F'(y)) \leq M(F(y)) + \sum_{i} \sum_{j=1}^{2p-1} \left( M(Cone_i(\Psi_j^i(y)) \cdots \rho_j^i(y))(\partial C_s(x_j))_i \right)
- M(Cone_i(\Psi_j^i(y)) \cdots \rho_{j+1}^i(y))(\partial C_s(x_j))_i + M(\Psi_j^i(y) \cdots \rho_j^i(y)(f_i(x_{j+1}) - f_i(x_j)))
\leq M(F(y)) + \frac{2M(F(y))}{\varepsilon} \frac{1}{\sqrt{p}} + N' \frac{p}{\sqrt{p}}
\leq M(F(y)) + \frac{2M(F(y))}{\varepsilon} \frac{1}{\sqrt{p}} + c(n) M(\partial \Omega) \sqrt{p}
\]

To prove inequality (3) observe that

\[
M((\partial C_s(x_j)(F(x)) + \Delta_i(C_s(x_j)(F(y)), C_s(x_{j+1})(F(y)), \rho) +
+ \partial \Psi_j^i(f_i(x_{j+1}) - f_i(x_j)))_i) \leq 4
\]

where we may have one boundary point at the tip of the cone $q_i \in Q_i$ due to an odd number of boundary points in $(\partial C_s(x_{j+1})(F(y)))$ and at most 3 boundary points $[F(y)]_{s(x_j), s(x_{j+1})} \partial Q_i + f_i F(y)$ at the vertices of triangle $Q_i$. The total number of vertices in triangulation of $\Sigma$ is bounded by $c'(n)pM(\partial \Omega)$. Hence, using the first property of functions $\rho_j^i F(y)$, we obtain

\[
M(\partial F'(y)) \leq \sum_{i}^{N'} M((\partial F'(x))_i)
\leq 4N' + \sum_{i,j} M(\partial \Psi_j^i(x) \cdots \rho_j^i(x)(f_i(x_{j+1}) - f_i(x_j)))
\leq 4c'(n)pM(\partial \Omega) + \sum_{i} n_i(y) \leq c(n)pM(\partial \Omega)
\]

\[
\square
\]

Now we assume that $F'$ is defined and satisfies the inductive property on the $k$-skeleton $X(q)k$. We will define $F''$ on the $(k + 1)$-skeleton and prove that it satisfies the inductive property on $X(q)k+1$. By Lemma 4.2 this will finish the proof of the theorem.

4.5. Inductive step. Let $D$ be a $(k + 1)$-cell in $X(q)$. We will define map $F_D : \partial D \times [0, N']$ such that $F_D = F''$ on $\partial D \times \{0\}$ and $F_D(x, N')$ is a constant map for all $x \in D$. This immediately gives the desired extension of $F''$ to $D$ by identifying $D$ and $\partial D \times [0, N']/\partial D \times \{N'\}$. We will show that it satisfies the (Inductive property) and, hence, the mass and boundary mass bounds.
Let $D_0 = \{x_1, \ldots, x_{2k+1}\}$ denote the collection of vertices in $D \cap X(q)_0$. Assume that the vertices are numbered so that $s(x_{j_1}) \leq s(x_{j_2})$ if $j_1 < j_2$. Let $y_i(D) \in D_0$ be a vertex with

$$M((\partial C_{s(y_i(D))}(F(x_1)))_i) = \min\{M((\partial C_{s(y)}(F(x_1)))_i) : x \in D_0\}$$

(Recall that $\partial C_{s(y)}(F(x_1)) = \partial C_{s(y_i)}(F(x))$ for all $x \in D_0$.) By the inductive property for all $x \in \partial D$ we can write

$$F'(x) = F(x) + \sum_{i} \Delta_i \left( \left( (C_{s(x_j)}(F(x_1)))_i \right)_{j=1}^{2k+1}, \left( \rho^i_j(x) \right)_{j=1}^{2k+1-1} \right)$$

$$+ e(x) + \sum_{i} \sum_{j=1}^{2k+1-1} \Psi^i_{\rho^i_1(x) \ldots \rho^i_j(x)}(f_i(x_{j+1}) - f_i(x_j))$$

Now we will define functions $\rho^i_j(x, t), t \in [0, N']$ and $x \in \partial D$, as follows:

1. For $t < i - 1$ we set $\rho^i_j(x, t) = \rho^i_j(x)$;
2. For $t \in [i - 1, i]$ 

$$\rho^i_j(x, t) = \begin{cases} 
(t - i + 1) + (1 - (t - i + 1))\rho^i_j(x) & \text{if } s(x_j) < s(y_i(D)) \\
(1 - (t - i + 1))\rho^i_j(x) & \text{if } s(x_j) \geq s(y_i(D)) 
\end{cases}$$

3. For $t > i$ we set 

$$\rho^i_j(x, t) = \begin{cases} 
1 & \text{if } s(x_j) < s(y_i(D)) \\
0 & \text{if } s(x_j) \geq s(y_i(D)) 
\end{cases}$$

We define

$$F_D(x, t) = C_{s(x_1)}(F(x_1)) + \sum_{i} \Delta_i ((C_{s(x_j)}(F(y)))_{j=1}^{2k+1}, (\rho^i_j(x, t))_{j=1}^{2k+1-1})$$

$$+ e(x) + f_i(x_1) + \sum_{i} \sum_{j=1}^{2k+1-1} \Psi^i_{\rho^i_1(x, t) \ldots \rho^i_j(x, t)}(x_{j+1} - x_j)$$

(18)

From $F_D$ we obtain the definitions of $\rho^i_j$ and $F'$ on $D$ by identifying $D$ and $\partial D \times [0, N']$ to $D \times \{N'\}$. We claim that with this definition the inductive property will be satisfied for $F'$ on $D$. Fix a point $y \in X(q)_0$ in a cell $C_{p'}$ of $X(q)$ that intersects $D$, $k + 1 \leq p' \leq p$. Let $\{z_l\}$ denote the vertices of $C$ arranged, as usual, so that $(s(z_l))$ is an increasing sequence. We define $\tilde{\rho}^i_l(x) = \rho^i_j(x)$ if $z_l = x_j$. For the values of $l$ that
correspond to vertices \( z_i \) that do not lie in \( D_0 \) and every \( x \in D \cap C \) we set \( \tilde{p}_i(x) = 1 \) if \( s(z_i) < s(x_{2k+1}) \) and \( \tilde{p}_i(x) = 0 \) if \( s(z_i) \geq s(x_{2k+1}) \). For \( x \in D \cap C \) consider

\[
G(x, y) = F(y) \cup x + \sum_i \Delta_i((C_{s(z_i)}(F(y)))_{l=1}^{2p'}, (\tilde{p}_i(x))_{l=1}^{2p'-1}) + \tilde{f}_i(y_1) + \sum_i \sum_l \tilde{\Psi}_i(x) (\tilde{f}_i(z_{l+1}) - \tilde{f}_i(z_l))
\]

It follows from the definition \([18]\) and the inductive assumption of \( e(x) \) that \( e'(x) = F'(x) - G(x, y) \) is a family of cycles of length \( \leq c(k + 1)\delta \) contained in a \( c(k + 1)\delta \)-admissible collection of sets.

We would like to show that properties (i)-(ii) of functions \( \tilde{p}_i(x) \) hold. The first property follows from the inductive assumption and our definitions of \( \tilde{p}_i(x, t) \) and \( \tilde{p}_i(x) \).

To see that (ii) holds observe that the choice of \( y_i(D) \) as the vertex with minimal \( M((\partial C_{s(y_i(D))}(F(x_1)))) \) implies that for \( x' = (x, t) \) as defined in \([18]\) functions \( \tilde{p}_i(x') \) satisfy

\[
(1 - \tilde{p}_1(x, t)) M((\partial (C_{s(z_1)}(F(y))))_i) + \sum_l (\tilde{p}_1(x, t) - \tilde{p}_1(x, t) - \tilde{p}_{l+1}(x, t)) M((\partial (C_{s(z_l)}(F(y))))_i)
\]

\[
+ \tilde{p}_1(x, t) \tilde{p}_{2p'-1}(x, t) M((\partial (C_{s(z_{2p'}})(F(y))))_i)
\]

\[
\leq (1 - \tilde{p}_1(x, 0)) M((\partial (C_{s(z_1)}(F(y))))_i) + \sum_l (\tilde{p}_1(x, 0) - \tilde{p}_1(x, 0) - \tilde{p}_{l+1}(x, 0)) M((\partial (C_{s(z_l)}(F(y))))_i)
\]

\[
+ \tilde{p}_1(x, 0) \tilde{p}_{2p'-1}(x, 0) M((\partial (C_{s(z_{2p'}})(F(y))))_i)
\]

Finally, we want to prove that for a given \( \delta_1 > 0 \) and some sufficiently large \( q' \) map \( \partial F^' : X(q') \rightarrow \mathbb{Z}_q(S, \mathbb{Z}_2) \) is \( \delta_1 \)-localized on \( X(q') \). Observe that in the construction above, given any two \( x, y \) in a cell \( C \) of \( X(q) \) and \( Q_i \subset \Sigma \) we have that \( (\partial F^'(x) - \partial F'(y)) \cap Q_i \) is supported in at most \( k = \sup_{x \in C} M(\partial F^'(x) \cap Q_i) \) balls of radius bounded by \( c(p) \text{dist}_{\infty}(x, y) \). Hence, choosing \( q'(p, \delta_1, \sup_{x \in X(q)} M(\partial F^'(x)) \geq \) \( q \) sufficiently large we have that map \( F' \) is \( \delta_1 \)-localized on \( X(q') \).

This finishes the proof of Theorem 4.1. \( \square \)

5. Parametric isoperimetric inequality

In this section we prove a parametric isoperimetric inequality Conjecture \([3]\) for 0-cycles in a disc. One can think about this result as a quantitative version of the Dold-Thom Theorem.

Let \( Q \) denote a 2-dimensional disc of radius 1.
\textbf{Theorem 5.1.} There exists constant $c(n) > 0$ with the following property. Let $\varepsilon > 0$, $\delta > 0$ and $\Phi : X^p \to Z_0(Q, \partial Q; \mathbb{Z}_2)$ be a continuous contractible map. There exists a $\delta$-localized map $\tilde{\Phi} : X^p \to Z_0(Q; \mathbb{Z}_2)$ and $\Psi : X \to I_1(Q; \mathbb{Z}_2)$, such that

- $F(\Phi(x) \cup \text{int}(Q), \tilde{\Phi}(x) \cup \text{int}(Q)) < \varepsilon$ for all $x$;
- $\partial \Psi(x) = \tilde{\Phi}(x)$ for all $x$;
- $M(\Psi(x)) \leq c(M(\Phi(x))p^{-\frac{1}{2}} + p^{\frac{1}{2}})$;
- $M(\tilde{\Phi}(x)) \leq 2M(\Phi(x)) + c\varepsilon$.

We will need the following lemma:

\textbf{Lemma 5.2.} Let $F : X^p(q)_0 \to Z_0(Q; \mathbb{Z}_2)$ be an $\varepsilon'$-fine family and let $I \subset \partial Q$ be a small interval of length $L$. There exists a continuous family $F' : X(q) \to Z_0(Q; \mathbb{Z}_2)$ with the following properties:

1. $F'$ is $\varepsilon'$-fine for $\varepsilon' = c(p)(\varepsilon + \sup_{x \in X(q)_0} M(F(x))L)$;
2. $F'(x) \cup \text{int}(Q) = F(x) \cup \text{int}(D)$ for all $x \in X(q)_0$;
3. $M(F'(x) \cup I) \leq 2p$.

\textbf{Proof.} Let $z$ denote the cycle of mass $1$ supported on the midpoint $e$ of $I$ and let $B = B_L(e)$.

For every $x \in X(q)_0$ define $F'(x) = F(x) - F(x) \cup B + z$ if $M(F(x) \cup I)$ is odd and $F'(x) = F(x) - F(x) \cup B$ if $M(F(x) \cup I)$ is even.

Inductively we extend $F'$ to the $k$-skeleton of $X(q)$.

Fix edge $E \subset X(q)_1$ with $\partial E = x - y$ and let $T$ be a 1-chain, $M(T) \leq \varepsilon$, with $\partial T = F(x) - F(y)$. We may assume that $T$ is a finite collection of disjoint linear arcs. We remove all arcs of $T$ whose endpoints are contained in $I$ and replace each arc of $T$ connecting a point $a \in Q$ to $b \in I$ with an arc connecting $a$ to the midpoint $e$ of $I$. We call the resulting 1-chain $T'$. Note that by triangle inequality $T'$ satisfies

$$M(T') \leq M(T) + \max\{M(F'(x)), M(F'(y))\}L$$

We have that $\partial T' = F'(x) - F'(y)$. We contract each arc of $T'$ one by one to obtain a family of 1-chains $T'_t$ and define $F'(t) = F'(x) + \partial T'_t$ for $t \in E$. Observe that $M(F'(t)) \leq \max\{M(F'(x)), M(F'(y))\} + 2$.

Now we assume that we have extended to the $k$-skeleton with $M(F'(x) \cup I) \leq 2k$ and so that the family is $c(k)\varepsilon$-localized. Let $C$ be a $(k+1)$-face. By the assumption that the family is localized there exists a ball $B_r(e)$, $r \in L, L + c(k)\varepsilon$, such that $F'(x) \cup \partial B_r(e) = \emptyset$ for all $x \in \partial C$. Let $R_t : B_r(e) \to B_{r-t}(e)$ denote a 1-Lipschitz map defined as (in polar coordinates) $R_t(\rho, \theta) = \left(\frac{r-t}{r-L}\rho + \frac{t}{r-L}, \theta\right)$ for $\rho > L$ and $R_t(\rho, \theta) = (\rho, \theta)$. Note that $R_t$ shrinks annulus $A(L, r, c)$ onto $A(L, r - t, r)$. For $t \in [0, r - L]$ define $F'(x, t) = F'(x) - F'(x) \cup B_r(e) + R_t(F'(x) \cup B_r(e))$. For $t = r - L$ we are
guaranteed that the family $F'(x, t) \cup B_r(e)$ satisfies $M(F'(x, t) \cup B_r(e) \cup \partial Q^2) \leq 2k + 2$. Then we can apply radial contraction to the centerpoint $e$.

This contracts the family $F'(x) \cup B_r(e), \ x \in \partial C$, to a single cycle. We contract $F'(x)$ outside of $B_r(e)$ in the usual way (see Proposition 2.12).

Lemma 5.3. Suppose $F : X^p \to \mathbb{Z}_0(Q^2; \mathbb{Z}_2)$, and there exists $I \subset \partial Q^2$, such that $M(F(t) \cup I) \leq K$. There exists a continuous family of 1-chains $\{\tau_t\}$, such that
1. $\partial \tau_t - F(t) \subset \partial Q$;
2. $M(\tau_t) \leq 2(M(F(t) \cup \text{int}(Q)) + K)$
3. $M(\partial \tau_t) \leq 2M(F(t) \cup \text{int}(Q)) + 2K$

Proof. Pick a point $a \in \mathbb{R}^2$ outside of disc $Q$, such that two tangent lines from $a$ to $\partial Q$ touch $\partial Q$ at the endpoints of interval $I$.

Given a point $q \in Q$ there is a unique line $L_q$ passing through $a$ and $q$. Let $H(q) = L_q \cap (\partial Q \setminus I)$. We define $\tau_t = \text{Cone}_a(H(F(t))) - \text{Cone}_a(F(t))$. □

Now we can prove Theorem 5.1. By Proposition 3.5 we can replace $\Phi$ with an $\varepsilon$-close family of absolute cycles $\Phi' : X \to \mathbb{Z}_0(Q; \mathbb{Z}_2)$ and by Lemmas 5.3 and 5.2 we may assume that there exists a family of chains $\tau : X \to I_1(Q; \mathbb{Z}_2)$, such that $\partial \tau(x) = \Phi'(x)$. Note that the family will have the desired bound for the mass in the boundary.

Observe that for each $x$ the chain $\tau$ consists of finitely many interval segments each lying on a ray from a fixed point $a$ (point $a$ lies outside of $Q$). We subdivide $Q$ into $p$ regions $\{Q_i\}$ of boundary length and diameter $\sim \frac{1}{\sqrt{p}}$.

We perform a “bend-and-cancel” construction (14, 15) to reduce the length of chains $\tau_t$ by pushing them into the 1-skeleton of the subdivision $\cup Q_i$.

For each $Q_i \subset Q$ let $B_{2\eta}(p_i) \subset Q_i$ be a ball chosen so that every line passing through the point $a$ (from the proof of Lemma 5.3) intersects at most one ball $B_{2\eta}(p_i)$. Let $\text{proj}_{p_i} : Q_i \setminus p_i \to \partial Q_i$ denote the projection map from the point $p_i$ and define a piecewise linear map $P_i : Q_i \to Q_i$, such that $P_i = \text{proj}_{p_i}$ on $Q_i \setminus B_{2\eta}(p_i)$ (in particular, $P_i(Q_i \setminus B_{2\eta}(p_i)) = \partial Q_i$) and $P_i(y) = y$ for all $y \in B_{\eta}(p_i)$. Note that $P_i$ can be chosen so that for any line $l$ passing through $Q_i$ we have $M(P_i(l \cap Q_i)) \leq 2\text{diam}(Q_i) \leq \frac{2}{\sqrt{p}}$.

Since $P_i$ is the identity map on $\partial Q_i$ we can define a map $P : Q \to Q$ by $P(x) = P_i(x)$ for $x \in Q_i$.

Given a 0-cycle $z \subset Q$ define a continuous map $l_i : \mathbb{Z}_0(Q; \mathbb{Z}_2) \to I_1(Q_i; \mathbb{Z}_2)$ by setting $l_i(z)$ to be the union of linear arcs connecting each point of $z \cap Q_i$ to the corresponding point of $P_i(z \cap Q_i)$. Let $l(z) = \sum_i l_i(z)$.

Define $\Psi(x) = P(\tau(x)) + l(\Phi'(x))$
We observe that $\partial \Psi(x) \cap \text{int}(Q) = \Phi'(x)$ and

$$M(\Psi(x)) \leq M(P(\tau(x)) + M(l(\Phi'(x)))$$

$$\leq M(\bigcup \partial Q_i) + \frac{2 M(\Phi'(x))}{\sqrt{p}} + M(\Phi'(x))$$

$$\leq c(\sqrt{p} + \frac{M(\Phi'(x))}{\sqrt{p}})$$

This finishes the proof of Theorem 5.1.

We now prove Conjecture 1.3 for $k = 0$ and $n = 2$.

**Theorem 5.4.** Let $\Omega$ be a 2-dimensional connected simply connected manifold, $\partial \Omega$ piecewise smooth boundary with $\theta$-corners, $\theta \in (0, \pi)$, $L > 0$, $p \in \mathbb{N}$. There exist constants $c(\Omega) > 0$ and $\delta(L, p, \Omega) > 0$ with the following property. Let $F : X^p \to Z_0(\Omega, \partial \Omega; \mathbb{Z}_2)$ be a continuous contractible $\delta$-localized $p$-dimensional family with $\sup_{x \in X} M(F(x)) \leq L$. Then there exists map $H : X \to I_1(\Omega; \mathbb{Z}_2)$, such that

- $\partial H(x) - F(x) \subset \partial \Omega$ for all $x$;
- $M(H(x)) \leq c(\Omega)(p^{\frac{1}{2}} + M(F(x))p^{-\frac{1}{2}})$.

**Proof.** Let $F : X \to Z_0(\Omega, \mathbb{Z}_2)$. Composing with a bilipschitz homeomorphism $\Pi$ between $\Omega$ and disc $Q$ we obtain a family $F'$ of 0-cycles in $Q$. We apply Theorem 5.1 with $\varepsilon_i \to 0$ to obtain a sequence of $\delta$-localized maps $\tilde{F}_i$ converging uniformly to $F'$ and a corresponding family of fillings $\Psi_i$ of $\tilde{F}_i$.

Pick $i$ large enough so that $F(\tilde{F}_i(x), F'(x)) < \frac{\delta}{2}$. We will define a family $\tilde{H}(x)$ with

$$\partial \tilde{H}(x) = F'(x) - \tilde{F}_i(x)$$

and $M(\tilde{H}(x)) \leq c_0(p)L\delta$. We define $\tilde{H}(x)$ inductively over the skeleton of $X$. On each vertex $x \in X_0$ define $\tilde{H}(x)$ to be the area minimizing filling of $F'(x) - \tilde{F}_i(x)$. By the flat distance bound there exists a $\delta$-admissible collection of open sets $\{U^x_i\}$, such that $\tilde{H}(x)$ is supported in $\cup U^x_i$.

Now we prove the inductive step. Suppose we defined $\tilde{H}$ on the $k$-skeleton of $X$ and for each $k$-cell $C$ the family of chains $\{\tilde{H}(x)\}_{x \in C}$ is supported in a $c'(k)\delta$-admissible collection of sets $\{U^C_i\}$ and $M(\tilde{H}(x)) \leq c_0(k)\delta \sup_{y \in X} M(\tilde{H}(y))$ for some constants $c_0$ and $c'$ that depend only on $k$. Let $D$ be a $(k+1)$-cell in $X$. By Lemma 2.5 there exists a constant $c'(k+1) > 0$ and a $c'(k+1)\delta$-admissible collection of sets $\{U^D_i\}$, such that family of cycles $\{F'(x) - \tilde{F}_i(x)\}_{x \in D}$ and family of chains $\{H(x)\}_{x \in \partial D}$ are supported in $\cup U^D_i$. Observe that for each $i$ we can define a continuous family of conical fillings $\tau_i(x), x \in D$, of $F'(x) - \tilde{F}_i(x)$ in $U^D_i$. (“Conical fillings” means a collection of linear segments from the support of $(F'(x) - \tilde{F}_i(x))_{U^D_i}$ to the center point of $U^D_i$.) It will be convenient to fix polar coordinates $(y, t), y \in \partial D, t \in [0, 1], y(1, 0, y(0, 0)$ the center point of cell $D$. Let $h_i(y, t)$ denote the radial contraction towards the center point of $U^D_i$ of relative cycle $\tau_i(y) - \tilde{H}(y)_{U^D_i}$, so
that \( h_i(y, 1) = \tau_i(y) - \tilde{H}(y)\mathcal{U}_i' \) and \( h_i(y, 0) = 0 \). Define \( \tilde{H}(y, t) = \sum_i \tau_i(y) - h_i(y, t) \). It is straightforward to check, using the inductive assumption, that it follows from this definition that \( \tilde{H}(x) \) is continuous on \( \mathcal{D} \), \( \partial \tilde{H}(x) = F'(x) - \tilde{F}_i(x) \) and

\[
\mathbf{M}(\tilde{H}(y, t)) \leq 2 \mathbf{M}(\tilde{H}(y)) + c'(k + 1) \mathbf{M}(F'(x)) \delta \leq c_0(k + 1)L\delta
\]

for sufficiently large \( c_0(k + 1) \). This finishes the construction of \( \tilde{H} \).

Composing \( H' = \Psi_i + \tilde{H} \) with \( \Pi^{-1} \) we obtain the desired family \( H \) of fillings in \( \Omega \). The upper bound for \( \mathbf{M}(H(x)) \) follows from the upper bound for \( \mathbf{M}(\Psi_i(x)) \) and choosing \( \delta = \delta(p, L, \Omega) \) sufficiently small.

Observe, that from Theorems 5.4 and 4.1 one can deduce the proof of Conjecture 1.3 for \( k = 1 \) and \( n = 3 \). We omit the details as this result is not used in the proof of the Weyl law.

6. Proof of the Weyl law

**Theorem 6.1.** Assume that Conjecture 1.6 holds for all families of \( k \)-cycles in domains in \( \mathbb{R}^n \) and Conjecture 1.3 holds for all families of \((k - 1)\)-cycles in domains in \( \mathbb{R}^{n-1} \). Then the Weyl law for \( k \)-cycles in \( n \)-manifolds holds:

For every compact Riemannian manifold \( M \) (possibly with boundary)

\[
\lim_{p \to \infty} \frac{\omega_p^k(M)}{p^{n-k}} = a(n, k) \text{Vol}(M)^{\frac{k}{n}}
\]

Combined with Theorems 4.1 and 5.1 we obtain

**Corollary 6.2.** Weyl law holds for 1-cycles in 3-manifolds:

For every compact Riemannian 3-manifold \( M \) (possibly with boundary)

\[
\lim_{p \to \infty} \frac{\omega_p^1(M)}{p^{\frac{3}{2}}} = a(3, 1) \text{Vol}(M)^{\frac{1}{2}}
\]

**6.1. Proof of Theorem 6.1.** In [21] it was proved that for any compact contractible domain \( U \subset \mathbb{R}^n \) we have

\[
\lim_{p \to \infty} \frac{\omega_p^k(U)}{p^{n-k}} = a(n, k)\text{Vol}(U)^{\frac{k}{n}}
\]

where \( a(n, k) \) is a constant that depends only on \( n \) and \( k \).

Let \( M \) be a compact \( n \)-manifold and consider sequence \( \{\omega_p^k\} \) of \( k \)-dimensional \( p \)-widths of \( M \). It is known that for some constants \( a_1(n) \leq a_2(n) \) we have

\[
0 < a_1 \text{Vol}(M)^{\frac{k}{n}} = \liminf_{p \to \infty} \frac{\omega_p^k}{p^{n-k}} \leq \limsup_{p \to \infty} \frac{\omega_p^k}{p^{n-k}} = a_2 \text{Vol}(M)^{\frac{k}{n}} < \infty
\]

We will show that \( a_2 \leq a(n, k) \leq a_1 \) which implies Theorem 6.1.
The inequality \( a_1 \geq a(n, k) \) was proved in [21, Theorem 4.1]. It remains to prove \( a_2 \leq a(n, k) \).

Fix \( \varepsilon > 0 \). Fix a triangulation \( T \) of \( M \), so that each \( n \)-dimensional simplex \( U_i \), \( i = 1, \ldots, N \), in the triangulation is \((1 + \varepsilon)\)-bilipschitz homeomorphic to a domain with \( \theta \)-corners in \( \mathbb{R}^n \), for some fixed \( \theta \in (0, \frac{\pi}{2}) \).

Let \( \tilde{U}_i \subset \mathbb{R}^n \) denote the images of \( U_i \) under \((1 + \varepsilon)\)-bilipschitz homeomorphism and assume that they lie at a distance greater than 1 from each other. We connect sets \( \tilde{U}_i \) by tubes of very small volume to obtain a connected set \( V \subset \mathbb{R}^n \). By Weyl law for domains in \( \mathbb{R}^n \) for all sufficiently large \( p \) there exists a \( p \)-sweepout \( \Phi : X^p \to Z_k(V; \mathbb{Z}_2) \) of \( V \) by \( k \)-cycles of mass bounded by \( a(n, k)((1 + 2\varepsilon)Vol(M))^\frac{k}{n}p^{\frac{n-k}{n-1}} \). Let \( \tilde{\Phi}_i : X \to Z_k(\tilde{U}_i; \mathbb{Z}_2) \) denote the restriction of \( \Phi \) to \( \tilde{U}_i \) and let \( \Phi_i : X \to Z_k(U_i; \mathbb{Z}_2) \) denote the map obtained by composing with bilipschitz homeomorphism from \( \tilde{U}_i \) to \( U_i \).

Given \( U \subset M \) with \( \theta \)-corners and a sufficiently small \( \eta > 0 \), we can define a \((1 + c(U)\eta)\)-bilipschitz homeomorphism \( \Delta^\eta_U : U_\eta \to U \). (Recall that \( U_\eta \) is the region obtained by removing a small neighbourhood of \( \partial U \) from \( U \), \( U_\eta = U \setminus E([0, \eta] \times \partial U) \), where \( E \) is as in Lemma 2.1). Choose a sequence of small positive numbers \( \eta_1 > \ldots > \eta_{N-1} > 0 \), so that the maps \( \Delta^V_{\eta_{m-1}, \eta_{m}} : (V_m)_{\eta_{m}} \to V_m \) are well-defined, where \( V_m = \bigcup_{j=1}^{m} U_j \). Since \( p \) can be chosen arbitrarily large, we assume that \( p^{\frac{1}{n-1}} < \eta_{N-1} \).

We will define a sequence of maps \( F_m : X \to Z_k(V_m; \mathbb{Z}_2) \), satisfying

\[
M(F_m(x)) \leq (1 + c_m\eta_1) \sum_{i=1}^{m} M(\Phi_i(x)) \\
+ c_m p^{\frac{n-k+1}{n-1}} + \frac{c_m}{\eta_{m-1}} \sum_{i=1}^{m} M(\Phi_i(x)) p^{-\frac{1}{n-1}}
\]

(set \( \eta_0 = 1 \)).

For \( m = 1 \) we set \( F_1 = \Phi_1 \). Assume by induction that we defined \( F_{m-1} \) satisfying the mass bound 19. If \( V_{m-1} \) and \( U_m \) have disjoint boundaries, then we set \( F_m = F_{m-1} + \Phi_m \).

Otherwise, let \( S = \partial V_{m-1} \cap \partial U_m \).

We apply the coarea inequality Conjecture 1.6 to maps \( F_{m-1} \) and \( \Phi_m \) to obtain \( \delta \)-localized maps \( \Phi'_m : X \to Z_k((U_m)_{\eta_{m-1}}; \mathbb{Z}_2) \) and \( F'_{m-1} : X \to Z_k((V_{m-1})_{\eta_{m-1}}; \mathbb{Z}_2) \),
such that, assuming that $p$ is sufficiently large,

$$
M(\Phi'_m(x)) \leq M(\Phi_m(x)) + c(U_m)p^{\frac{n-k-1}{n-1}} + c(U_m)\frac{M(\Phi_m(x))}{\eta_{m-1}}p^{-\frac{1}{n-1}}
$$

$$
M(F'_{m-1}(x)) \leq M(F_{m-1}(x)) + c(V_{m-1})p^{\frac{n-k-1}{n-1}} + c(V_{m-1})\frac{M(F_{m-1}(x))}{\eta_{m-1}}p^{-\frac{1}{n-1}}
$$

$$
\leq \sum_{i=1}^{m-1} M(\Phi_i(x)) + c'_m p^{\frac{n-k-1}{n-1}} + c'_m \sum_{i=1}^{m-1} M(\Phi_i(x)) p^{-\frac{1}{n-1}}
$$

$$
M(\partial \Phi'_m(x)) \leq c(U_m)(\frac{M(\Phi_m(x))}{\eta_{m-1}} + p^{\frac{n-1}{n}})
$$

$$
M(\partial F'_{m-1}(x)) \leq c'_m (\sum_{i=1}^{m-1} \frac{M(\Phi_i(x))}{\eta_{m-1}} + p^{\frac{n-1}{n}})
$$

Here $c'_m > c_m$ is a suitably chosen constant. Note that we used $p^{-\frac{1}{n-1}} < \eta_{m-1}$ to simplify the bounds for mass and boundary mass of $F'_{m-1}(x)$.

Moreover, by Conjecture 1.6 we have that families $\{\partial F'_{m-1}(x)\}$ and $\{\Phi'_{m}(x)\}$ are $p$-sweepouts of boundaries of the corresponding regions. Define $F'_{m-1} = \Delta_{\eta_{m-1}}(F_{m-1}(x))$ and $\Phi'_{m}(x) = \Delta_{U_{m}}(\Phi_{m}(x))$.

Observe that since both $\{\Phi'_{m}(x)\}_{S}$ and $\{\Phi_{m}(x)\}_{S}$ are $p$-sweepouts of $S$ by relative $(k-1)$-cycles, the family

$$
\{f_S(x) = (\partial \Phi_{m}(x) + \partial F_{m-1}(x))\}_{S}
$$

is a contractible family of relative $(k-1)$-cycles in $S$.

By the isoperimetric inequality Conjecture 1.3 applied to $f_S$ there exists a family $\Psi_S : X \to I_k(S;\mathbb{Z}_2)$ with $\partial \Psi_S(x) - f_S(x)$ supported in $\partial S$ and satisfying

$$
M(\Psi_S(x)) \leq c(S)p^{\frac{n-k-1}{n-1}} + (M(\partial F'_{m-1}(x)) + M(\partial \Phi'_{m}(x))) p^{-\frac{1}{n-1}}
$$

$$
\leq c'_m (p^{\frac{n-k-1}{n-1}} + \sum_{i=1}^{m-1} \frac{M(\partial \Phi_i(x))}{\eta_{m-1}} p^{-\frac{1}{n-1}})
$$

Here $c'_m$ is defined in terms of $c'_m$ and $c(S)$.

We define

$$
F_m(x) = F_{m-1}(x) + \Phi_{m}(x) + \Psi_S(x)
$$

Observe that $F_m$ is a continuous family of relative cycles and a $p$-sweepout of $V_m$.

(That fact that this is a $p$-sweepout follows by restricting family to a small ball in $U_m$ and observing that in that small ball it is homotopic to the restriction of $\Phi_m$). It
is straightforward to check that the inductive assumption for the mass bounds will be satisfied for some sufficiently large $c_m \geq c'_m$.

For $m = N$ we obtain a $p$-sweepout $F_N$ of $M$ with

$$\frac{M(F_N(x))}{p^{a-k}} \leq (1 + c(M)\eta_1^k)(\frac{M(\Phi(x))}{\eta_{N-1}p^{a-k}} + c(M)\frac{M(\Phi(x))}{\eta_{N-1}p^{a-k} + \frac{1}{n-1}})$$

$$\leq a(n,k)(1 + c(M)\epsilon^k)(1 + c(M)\eta_{N-1}^k)Vol(M)\frac{k}{n} + c(\epsilon, \eta_{N-1}, M)\left(p^{-\frac{1}{n-1}}\right)$$

Observe that as $p \to \infty$ the second term goes to 0. Since $\eta_1$ and $\epsilon$ can be chosen to be arbitrarily small we conclude that $a_2 \leq a(n,k)$. This finishes the proof of Theorem 6.1.

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