Gravitational instabilities and faster evolving density perturbations

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(01/10/98)

Abstract

The evolution of inhomogeneities in a spherical collapse model is studied by expanding the Einstein equation in powers of inverse radial parameter. In the linear regime, the density contrast is obtained for flat, closed and open universes. In addition to the usual modes, an infinite number of new growing modes are contained in the solutions for pressureless open and closed universes. In the nonlinear regime, we obtain the leading growing modes in closed forms for a flat universe and also, in the limits of small and large times, for an open universe.

I. INTRODUCTION

The Newtonian and the relativistic theories of small perturbations form the theoretical basis for our present understanding of the formation of structures in the universe [1–3]. In the Newtonian regime, which is appropriate for pressureless universes, the hydrodynamic equations, incorporating the expansion of the universe, are perturbed by small fluctuations. Subsequently, a second-order differential equation is obtained whose solutions give the decaying and the growing modes of the density contrast. This quantity measures the evolution
rate of the perturbation with respect to the background density and thus determines whether structures can evolve from small initial perturbations or not.

In the relativistic regime, one expands the Einstein metric \( g_{\mu\nu} \) around the Friedmann-Robertson-Walker metric by a local perturbation, \( h_{\mu\nu} \), which arises naturally in an imperfect fluid model of the universe. Consequently, the evolution of the perturbations is formulated by means of the energy conservation equation, which can be written in terms of the time derivatives only, and a combination of the linearized Einstein equations (see e.g. [1], chapter 15). Space derivatives are eliminated in this procedure and a second-order ordinary differential equation in time is obtained for the density contrast.

The procedure in the relativistic regime can be summarized as follows. Fixing the gauge such that the perturbation vanishes in the \((i0)\) and \((00)\) directions, one obtains, for the space-space component of the Einstein equation,

\[
\nabla^2 h_{ij} - \frac{\partial^2 h_{ik}}{\partial x^j \partial x^k} - \frac{\partial^2 h_{jk}}{\partial x^i \partial x^k} + \frac{\partial^2 h_{kk}}{\partial x^i \partial x^j} - R^2 \dot{h}_{ij} + R \ddot{R} (h_{ij} - \delta_{ij} h_{kk}) + 2 \dot{R}^2 (-2h_{ij} + \delta_{ij} h_{kk}) = -8\pi G \left( \rho - p \right) R^2 h_{ij} - 8\pi G R^4 \delta_{ij} \left( \delta \rho - \delta p \right) + \text{(imperf. fl. correc.)},
\]

where the last term corresponds to the imperfect fluid corrections which are given in terms of a velocity field. The differential equation contains space derivatives of the metric and, therefore, is extremely difficult to solve. In general, matter distribution constrained to a small region has large space derivative as compared to the metric itself. This is clear for a well-localized Gaussian distribution, \( \rho(\vec{x}) \sim e^{-\lambda(\vec{x} - \vec{x}_0)^2} \), for large \( \lambda \), which cannot be expanded in inverse powers of the distance \( \frac{1}{|\vec{x} - \vec{x}_0|} \). The off-diagonal components of the Einstein equations are not amenable either, once again, due to the space derivatives. However, the time-time component contains no space derivative and reads

\[
\ddot{h}_{kk} - 2 \frac{\ddot{R}}{R} \dot{h}_{kk} + 2 \left( \frac{\dot{R}}{R} \right)^2 h_{kk} = -8\pi G \left( \delta \rho + 3 \delta p \right) R^2.
\]

Additional informations can also be obtained from the energy-momentum conservation equa-
\[ \dot{\delta} + 3 \frac{\dot{R}}{R} (\delta \rho - \delta p) = (\rho - p) \frac{\partial}{\partial \tau} \left( \frac{h_{kk}}{2r^2} \right) + \text{(imperf. fl. correc.)} \]  

(3)

which, together with a few general hypothesis, leads to the simple relation \( \delta(t) \equiv \delta \rho / (\rho + p) = -h_{kk} / 2R^2 \). This relation, when used in (2), gives rise to a second-order differential equation whose solutions give the growing and the decaying modes of the density contrast \( \delta(t) \).

In the nonlinear regime, evolution of mass density fluctuations has been studied in the matter-dominated Einstein-de-Sitter universe by means of the power spectrum, which is obtained from the Fourier transform of the density contrast \( \delta \).

Although the standard procedure is formulated in such a way as to enable us to describe the evolution of a general kind of perturbation, the hypothesis of an imperfect fluid with a velocity field which is at the root of this scheme cannot be applied to certain kinds of fluctuations. In the standard scheme, the fluctuation seeds, which are believed to give origin to the galaxies, are described by Poisson-like perturbations. These are well-localized functions with large space derivatives.

Contrary to the standard scheme, we study soft perturbations, which are almost homogeneous at large scales. These perturbations are especially interesting for the purpose of studying inhomogeneities at very large scales. They are also important for studying totally inhomogeneous universes. Although perturbative expansion around a comoving metric is a subtle procedure, smooth perturbations at large scales can be incorporated into perturbative schemes.

We consider the homogeneous Friedmann-Robertson-Walker metric as a background metric and expand the perturbation in powers of the inverse radial parameter, \( i.e. \) in powers of \( 1/r \). The advantage of this procedure, which is valid for very large scales, is that we can control the space-dependence of the perturbation at each order, and, as we shall see, the Einstein equations will provide us with more informations on the evolution of the perturbations.

\[ ^1 \text{It is worth commenting that the inclusion of higher derivatives may introduce zero modes.} \]
In this approach, equation (1) simplifies and all the space derivatives disappear in the first approximation (the zero-th order terms, being homogeneous, are zero modes of space derivatives). In particular, even though (3) is still valid, we no longer need higher-order time derivatives to determine the density contrast.

Our aim is to use the aforementioned expansion in $1/r$ for spherically symmetric configurations, i.e., for the S-waves, and verify the presence of the growing modes for the density contrast.

We parameterize the metric as

$$ds^2 = dt^2 - R_p(t, r)^2 \left( \frac{dt^2}{1 - kr^2} + r^2 d\Omega^2 \right)$$

where the scale factor $R_p(t, r)$ is a function of the radial parameter as well as time. Using this metric in the Einstein equation we obtain

$$2 \frac{\ddot{R}_p}{R_p} + (1 + 3\omega) \left( \frac{\dot{R}_p}{R_p} \right)^2 + 2kr\omega \frac{R'_p}{R_p^3} - \frac{(1 + 3\omega)k}{R_p^2} \left( \frac{2}{r} (1 + 2\omega) \frac{R'_p}{R_p^3} + (1 - \omega) \frac{R''_p}{R_p^4} + 2\omega \frac{R''_p}{R_p^3} \right) (1 - kr^2) = 0.$$  

where “dot” and “prime” stand for time and space derivatives, $k = 0, 1, -1$ for flat, closed and open universes respectively and the parameter $\omega$ is the pressure-density proportionality constant, i.e. $P = \omega \rho$. The density is given by the time-time component of the Einstein equation

$$3 \left( \frac{\dot{R}_p}{R_p} \right)^2 + \left( -\frac{4 R'_p}{r R_p^3} + \frac{R''_p}{R_p^4} - 2 \frac{R''_p}{R_p^3} \right) (1 - kr^2) + \frac{3k}{R_p^2} + 2kr \frac{R''_p}{R_p^3} = 8\pi G \rho_p(r, t).$$

To solve the above equations we expand the scale factor and the density around the Friedmann background by small fluctuations in powers of inverse radial parameter. That is,

$$R_p(t, r) = R + \sum_{n=0}^{\infty} \delta R_n(t)r^{-n},$$

$$\rho_p(t, r) = \rho + \sum_{n=0}^{\infty} \delta \rho_n(t)r^{-n},$$

where $R$ and $\rho$ are the Friedmann scale factor and density respectively. It is worth commenting that these solutions are not just gauge modes of the metric but represent genuine
perturbations\(^\text{2}\). It is easy to see that a solution \(\zeta_\mu = \zeta_\mu,\nu + \zeta_\nu,\mu\) where \(\delta g_{\mu\nu}\) is the metric perturbation, such that \(\delta g_{0r} = 0\) and \(\delta g_{rr} \sim t^a r^{-n}\), cannot be obtained.

In the following sections, we solve the perturbed Einstein equations for different values of \(k\) and \(\omega\) and find the density contrast which is defined as:

\[
\delta(t, r) = \frac{\delta \rho(t, r)}{\rho(t)} \quad (9)
\]

where \(\delta \rho\) is given by the summation term in equation (8).

II. MATTER-DOMINATED OPEN UNIVERSE

The scale factor in the Einstein equation (5) is expanded as in (7) and the coefficients at the \(r^{-N}\) order are collected. For \(\omega = 0\) and \(k = -1\), these coefficients satisfy the recursive relation

\[
\sum_{n,m,l} \left(2\hat{\delta} R_n \delta R_m \delta R_l \delta R_{N-n-m-l} + \delta R_n \delta R_m \delta R_l \delta R_{N-n-m-l} \right) + \\
+ \sum_n n(4 - N + n)\delta R_n \delta R_{2-n} = 0, \quad (10)
\]

where \(\delta R_n\) represents \(R + \delta R_0\) for \(n = 0\) and the indices \(n, m\) and \(l\) run from zero to infinity. The above equation can be solved recursively for each \(N\). In general, using the parametric expressions \(R = \cosh \psi - 1\) and \(t = \sinh \psi - \psi\), we obtain an inhomogeneous second-order differential equation, whose non-homogeneous terms are given in terms of \(\delta R_{j<n}(\psi)\). Although, it is not possible to solve such an equation in the general case, the homogeneous part of equation (10),

\[
\frac{d^2 \delta R_n}{d\psi^2} + \left(n - \frac{1}{2 \sinh^2 \frac{\psi}{2}} \right) \delta R_n = 0, \quad (11)
\]

can be solved recursively for each \(n\). The corresponding solutions are

\(^2\)For a comprehensive study of the gauge freedom arising in the context of density perturbations, we refer the reader to reference [11].
\[(\delta R_n)_+ \sim \frac{1}{\sqrt{n \sinh(\psi/2)}} \left( \cosh(\psi/2) \sin \sqrt{n}\psi - 2\sqrt{n}\sinh(\psi/2) \cos \sqrt{n}\psi \right),\]

\[(\delta R_n)_- \sim \frac{1}{\sinh(\psi/2)} \left( \cosh(\psi/2) \cos \sqrt{n}\psi + 2\sqrt{n}\sinh(\psi/2) \sin \sqrt{n}\psi \right), \quad (12)\]

Using this result in (6), we obtain the density and subsequently the density contrast by using expression (9). This is given by the growing mode

\[\delta_+(\psi) \sim \frac{\sin (\sqrt{n}\psi)}{\sqrt{n}} \left[ \frac{3 \sinh \psi}{(1 - \cosh \psi)^2} + \frac{2n \sinh \psi}{1 - \cosh \psi} \right] + \cos \left( \sqrt{n}\psi \right) \frac{5 + \cosh \psi}{1 - \cosh \psi}, \quad (13)\]

and the decaying mode

\[\delta_-(\psi) \sim \cos \left( \sqrt{n}\psi \right) \left[ \frac{3 \sinh \psi}{(1 - \cosh \psi)^2} + \frac{2n \sinh \psi}{1 - \cosh \psi} \right] - \sqrt{n} \sin \left( \sqrt{n}\psi \right) \frac{5 + \cosh \psi}{1 - \cosh \psi}. \quad (14)\]

The \(n = 0\) \(i.e.\) the \(r\)-independent perturbation) expressions reproduce the standard growing mode

\[\delta^{n=0}_+(\psi) \sim \frac{3\psi \sinh \psi}{(1 - \cosh \psi)^2} + \frac{5 + \cosh \psi}{1 - \cosh \psi}, \quad (15)\]

and the standard decaying mode

\[\delta^{n=0}_-(\psi) \sim \frac{3 \sinh \psi}{(1 - \cosh \psi)^2} \quad (16)\]

for the density contrast \([1,2]\). The solutions \((13)\) and \((14)\) show that in addition to the standard growing mode, infinitely many growing modes exist at all higher orders in the perturbation expansion in the linear regime.

All the above results can be used for the closed universe by making the substitution \(\psi \equiv -i\theta\).

Non-linear growing modes would be expected if one could solve the inhomogeneous differential equation \((10)\). In Section V, we obtain these modes in the two extreme limits of small and large times. In the small-time limit, open and close universes behave as a flat universe. In the asymptotic, \(i.e.\) large time, limit the scale factor varies linearly with time and the inhomogeneous equation \((10)\) can once again be solved at higher orders. We obtain a new growing mode in this limit.
III. MATTER-DOMINATED FLAT UNIVERSE

For the flat universe, the series expansion (3) of the Einstein equation (1) leads to the recursive equation

$$\sum_{n,m,l,p} \left( 2 \ddot{\delta R}_n \delta R_m \delta R_l \delta R_p + \dot{\delta R}_n \dddot{\delta R}_m \delta R_l \delta R_p \right) r^{-(n+m+l+p)}$$

$$+ \sum_{n,m} \left( n(2 - m) \delta R_n \delta R_m \right) r^{-(n+m+2)} = 0$$

(17)

where the indices \( n, m, l \) and \( p \) run from zero to infinity. Unlike, the differential equation (10) for the open universe which could not be solved without discarding the inhomogeneous terms, the above inhomogeneous equation for the flat universe can be solved recursively to find the perturbation coefficients \( R[n](t) \).

The first few coefficients are

$$\delta R_0 = c_0 t^{\frac{2}{3}} ,$$

(18)

$$\delta R_1 = c_1 t^{\frac{2}{3}} + c_2 t^{-\frac{1}{4}} ,$$

(19)

$$\delta R_2 = c_1 t^{\frac{2}{3}} + c_3 t^{-\frac{1}{4}} - \frac{1}{4} c_2^2 t^{-\frac{5}{4}} ,$$

(20)

$$\delta R_3 = \frac{9}{10} c_1 t^{\frac{2}{3}} + c_4 t^{\frac{2}{3}} + \frac{9}{2} c_2 t^{\frac{4}{3}}$$

$$+ c_5 t^{-\frac{1}{4}} + \left( \frac{1}{4} c_1 c_2^2 - \frac{1}{2} c_2 c_3 \right) t^{-\frac{1}{6}} + \frac{1}{6} c_2^3 t^{-\frac{7}{6}} ,$$

(21)

$$\delta R_4 = \left( \frac{9}{4} c_1^2 - \frac{9}{5} \right) t^{\frac{4}{3}} + c_4 t^{\frac{2}{3}} + c_6 t^{-\frac{4}{3}} + \frac{27}{8} c_2^3 t^{-\frac{4}{3}}$$

$$+ \frac{C_{dd}}{t^{\frac{4}{3}}} + \frac{C_a}{t^{\frac{7}{3}}} + \frac{C_b}{t^{\frac{10}{3}}} ,$$

(22)

$$\delta R_5 = \frac{243}{280} c_1 t^2 + \left( \frac{27}{10} c_4 - \frac{81}{20} c_1^3 + \frac{36}{5} c_1^2 \right) t^{\frac{1}{3}} - \frac{243}{8} c_2 t + c_7 t^{\frac{2}{3}} +$$

$$\left( - \frac{153}{5} c_1 c_2 - \frac{333}{10} c_1 c_3 + \frac{27}{2} c_5 + \frac{513}{10} c_1^2 c_2 \right) t^{\frac{1}{3}} + c_8 t^{-\frac{1}{3}} +$$

$$\left( - \frac{693}{40} c_1 c_2^2 + \frac{45}{4} c_2 c_3 \right) t^{-\frac{4}{3}} + C_c t^{-\frac{4}{3}} + C_d t^{-\frac{7}{3}} + C_f t^{-\frac{10}{3}} ,$$

(23)

The results have been reproduced by a MapleV-5, program [13].
where $C$'s are functions of either preceding $c$'s or are new constants themselves. In general, the leading growing modes can be written in a closed form as

$$\delta R_+(r, t) = \sum_{n=0}^{\infty} c'_n r^{2n+3} t^{(n+2)/3}.$$  \hspace{1cm} (24)

From the time-time component of the Einstein equation (3), we recover the mass distribution and subsequently the density contrast $\delta(t, r) = \sum_{n=0}^{\infty} \delta_n(t) r^{-n}$, whose first few terms, up to an overall constant factor of $3/4$, are

$$\begin{align*}
\delta_1 &= -4c_2 t^{-1}, \\
\delta_2 &= 4(c_1c_2 - c_3) t^{-1} + 9c_2^2 t^{-2}, \\
\delta_3 &= -\frac{12}{5} c_1 t^{\frac{3}{2}} - 6c_2 t^{-\frac{3}{2}} + 4 \left[ c_1 c_3 + c_1 c_2 + c_1^2 c_2 - c_5 \right] t^{-1} + 18 \left[ c_2 c_3 - c_1^2 c_4 \right] t^{-2} - 18 c_2^3 t^{-3}, \\
\delta_4 &= \left( -\frac{14}{5} c_1 + \frac{47}{5} c_1^2 \right) t^{\frac{3}{2}} + \left[ \frac{184}{5} c_1 c_2 - 16c_3 \right] t^{-\frac{3}{2}} + 17c_2^2 t^{-\frac{5}{2}} + 4 \left[ c_2 c_4 + c_1 c_5 + c_1^3 c_2 - c_6 + c_1 c_3 - 2c_1^2 c_2 - c_1^2 c_3 \right] t^{-1} + 9 \left[ -4c_1 c_2 c_3 + 2c_2 c_5 - 2c_1 c_2^2 + 3c_1^2 c_2^2 + c_3^3 \right] t^{-2} + 54 \left[ c_1 c_2^3 - c_2^2 c_3 \right] t^{-3} + \frac{135}{4} c_2^4 t^{-4}, \\
\delta_5 &= \frac{162}{5} c_1 t^{\frac{3}{2}} + \left[ -54c_2 - \frac{81}{2} c_2 \right] t^{\frac{5}{2}} + \left[ -12c_4 - 4c_3^2 + 16c_2^2 + \frac{132}{5} c_2^2 c_1 - \frac{96}{5} c_2^3 - \frac{36}{5} c_4 \right] t^{\frac{7}{2}} + C_{gg} t^{-\frac{7}{2}} + C_{kk} t^{-\frac{7}{2}} + C_{h} t^{-\frac{7}{2}} + C_{it} t^{-1} + C_{jt} t^{-2} + C_{kt} t^{-3} + C_{lt} t^{-4} + C_{mt}^{-5}.
\end{align*}$$  \hspace{1cm} (25, 26, 27, 28, 29)

In general, the leading growing modes can be represented in the closed form

$$\delta_+ = \sum_{n=0}^{\infty} c''_n r^{2n+3} t^{2(n+1)/3},$$  \hspace{1cm} (30)

These results show that up to the third order in the perturbation expansion, the density contrast does not grow. This is due to the fact, mentioned in the introduction, that the perturbation coefficients at these orders are not sensitive to the space-dependence of the perturbed scale factor $R_p(t, r)$. However, the third-order perturbation term contains the usual growing mode in the linear regime\footnote{It is worth making the comparison with the open universe where the standard zero modes} namely $\delta \sim t^{2/3}$.

(This term emerges because of...
the $t^{4/3}$ growing mode in (21). (The nonlinear modes are given by $n \neq 0$ terms in expression (30) and fully agree with the results of reference [4].

Once again, we see that in addition to the usual growing modes arising in the linear regime, many more growing modes appear at higher orders in the perturbation series. The higher growing modes can have interesting consequences for structure formation since different perturbative regimes can correspond to different orders in the perturbation series. For very large structures, such as superclusters, the dominating terms are the first few, which vary with a smaller powers of time and so are expected to appear later. On the other hand, small fluctuations are described by higher-order terms in the perturbative series and thus can form earlier. This simple argument points toward a bottom-up scenario for the formation of structures and thus a universe dominated by Cold dark matter [17].

IV. RADIATION-DOMINATED FLAT UNIVERSES

In this case, we solve, using a MapleV-5 program [17], a recursive equation similar to (17) for the coefficients of the scale factor. As in the previous cases, the leading growing modes can be written in the following closed form:

$$\delta R_+ (r, t) = \sum_{n=0}^{\infty} c_{2n+3} \left( \frac{\sqrt{t}}{r} \right)^{2n+3}.$$  (31)

Substituting this solution in the time-time component of the Einstein equation (6) and subsequently using (9) we obtain the expression

$$\delta_+ (r, t) = \sum_{n=0}^{\infty} c_2^{n+3} \frac{t^{n+1}}{r^{2n+3}}.$$  (32)

for the leading growing modes of the density contrast.

Once again, in addition to the standard growing mode, $\delta \sim t$, which arises at the third order of the series, we obtain an infinite number of higher-order growing modes.

already appear at the zero-th order.
V. NONLINEAR MODES IN THE OPEN UNIVERSE

In Section II, we obtained the density contrast for a pressureless open universe in the linear regime. We then commented that the nonlinear modes can be obtained in the asymptotic limit and in the limit of small times. In this section, we study an open universe in these two extreme limits. Our analyses are not restricted to a dust universe.

It is a non-trivial task to define the asymptotic domain in the open Friedman universe. Indeed, in an asymptotic limit compatible with the Einstein equation and with any linear equation of state, the scale factor increases linearly with time. This limit can be used, formally, to describe the asymptotic behaviour of the scale factor in the open universe, keeping in mind that, physically, the notion of large times is only meaningful when used relative to the Hubble time. On the other hand, observations are performed for small times, namely for the primordial universe, when open and closed universes behave as a flat universe.

In this section, we discuss these two limits of our solutions, i.e. $R \sim t$, and $R \sim t^{2/3}$. In the former case, the suppression of the growing modes implies that the inhomogeneities cease to grow after a sufficiently long time. Indeed, solving the series equation (10) in the asymptotic limit gives (33)

\begin{align}
\delta R_0 &= t \\
\delta R_1 &= c_1 \cos \ln t + c_2 \sin \ln t \\
\delta R_2 &= c_2 \cos \sqrt{2} \ln t + c_4 \sin \sqrt{2} \ln t + \\
&\quad \frac{15}{272} c_2^2 t^{-1} \sin \sqrt{2} \ln t \cos \left(2 - \sqrt{2}\right) \ln t - \frac{25}{272} c_1^2 t^{-1} \cos \sqrt{2} \ln t \cos \left(2 + \sqrt{2}\right) \ln t - \\
&\quad \frac{7}{68} c_1 c_2 t^{-1} \left(\sin \sqrt{2} \ln t \cos \left(2 - \sqrt{2}\right) \ln t - \cos \sqrt{2} \ln t \cos \left(2 + \sqrt{2\right) \ln t \right) + \text{few similar terms.}
\end{align}

Substituting the above solutions in the time-time component of the Einstein equation (8), we recover the mass distribution and subsequently the following density contrast:
\[ \delta_1 = 2c_1 \cos \ln t + 2c_2 \sin \ln t - 6c_1 \sin \ln t + 6c_2 \cos \ln t , \]  
\[ \delta_2 = -6\sqrt{2}c_3 \sin \sqrt{2} \ln t + 6\sqrt{2}c_4 \cos \sqrt{2} \ln t + \text{few similar terms}. \]  
\[ (37) \]

The terms \( \cos \ln t \) and \( \sin \ln t \) do not represent any growing modes and only a decaying mode of \( 1/t \) is indicated by the above solutions. In the radiation-dominated era, however, we do find a growing mode. In this era, the first few corrections to the scale factor are

\[ \delta R_0 = t , \]  
\[ \delta R_1 = \frac{1}{\sqrt{t}} c_1 \sin \frac{\sqrt{7}}{2} \ln t + \frac{1}{\sqrt{t}} c_2 \cos \frac{\sqrt{39}}{6} \ln t , \]  
\[ \delta R_2 = -\frac{1}{\sqrt{t}} c_3 \cos \frac{\sqrt{7}}{2} \ln t + \frac{1}{\sqrt{t}} c_4 \sin \frac{\sqrt{7}}{2} \ln t + \mathcal{O}(t^{-3}) \]  
\[ (39) \]

and consequently to the density contrast are

\[ \delta_1 = -[c_1 + \sqrt{39}c_2] \sqrt{t} \sin[\frac{\sqrt{39}}{6} \ln t] + [-c_2 + c_1 \sqrt{t} \sqrt{39}] \cos[\frac{\sqrt{39}}{6} \ln t] , \]  
\[ \delta_2 = -\frac{\sqrt{7}}{14} f_3(t) \sin[\frac{\sqrt{7}}{2} \ln t] - 3c_3 \sqrt{t} \cos[\frac{\sqrt{7}}{2} \ln t] - 3c_4 \sqrt{t} \sin[\frac{\sqrt{7}}{2} \ln t] + \frac{\sqrt{7}}{14} f_4(t) \cos[\frac{\sqrt{7}}{2} \ln t] \]  
\[-3c_1 c_2 \sin[\frac{\sqrt{39}}{6} \ln t] \cos[\frac{\sqrt{39}}{6} \ln t] + \frac{7 + 13}{4} t c_2^2 \cos^2[\frac{\sqrt{39}}{6} \ln t] + \frac{7 + 13}{4} t c_2^2 \sin^2[\frac{\sqrt{39}}{6} \ln t] \]  
\[ + \frac{5}{2} \sqrt{39} t (c_2^2 - c_1^2) \cos[\frac{\sqrt{39}}{6} \ln t] \sin[\frac{\sqrt{39}}{6} \ln t] \]  
\[ + \frac{1}{2} \sqrt{t} f_4(t) \sin[\frac{\sqrt{7}}{2} \ln t] + \frac{1}{2} \sqrt{t} f_3(t) \cos[\frac{\sqrt{7}}{2} \ln t] \]  
\[-3 \sqrt{t} \left(c_3 \sin \frac{\sqrt{7}}{2} \ln t - c_4 \cos \frac{\sqrt{7}}{2} \ln t \right) \]  
\[ (43) \]

where \( f_3 \) and \( f_4 \) are functions depending on integrals of the functions already appearing above. That is,

\[ f_3 = \int 25t^{-5/2} c_1^2 \cos \left(\frac{\sqrt{2}}{2} \ln t\right) - \frac{38}{t^{5/2}} c_1^2 \cos \left(\frac{\sqrt{2}}{2} \ln t\right) \cos^2 \left(\frac{\sqrt{39}}{6} \ln t\right) + \cdots \]  
\[ (44) \]

\[ f_4 = \int 25t^{-5/2} c_1^2 \sin \left(\frac{\sqrt{2}}{2} \ln t\right) - \frac{38}{t^{5/2}} c_1^2 \sin \left(\frac{\sqrt{2}}{2} \ln t\right) \cos^2 \left(\frac{\sqrt{39}}{6} \ln t\right) + \cdots \]  
\[ (45) \]

We find the growing mode \( \delta \sim \sqrt{t} \) already at the first order. Thus, asymptotically, the inhomogeneities can grow in an open universe during radiation-dominated era but stop
growing during the matter-dominated era. We interpret these results as a freezing of the present mass distribution.

At small times, the parametric solutions for the Friedmann scale factor in the matter-dominated open universe approach the solution for the flat universe, i.e. \( R \sim t^{2/3} \). However, in the above perturbation scheme we cannot simply abandon the constant \( k \) and approach the result from the point of view of the Einstein equations for a flat universe, since there are important contributions from the space-dependent derivatives through \( k \) (e.g. in equation (3)). Using again a MapleV-5 program we find

\[
\delta R_0 = c_0 t^{2/3},
\]

\[
\delta R_1 = c_1 t^{2/3} + \cdots,
\]

\[
\delta R_2 = c_2 t^{2/3} + \cdots,
\]

\[
\delta R_3 = c_3 t^{2/3} + \cdots.
\]

Substituting these in the time-time component of the Einstein equation (3), we recover the mass distribution and subsequently the following perturbative contributions to the density contrast:

\[
\delta_0 = C_0 t^{2/3},
\]

\[
\delta_1 = C_1 t^{-1} + C_2 t^{2/3} + \cdots,
\]

\[
\delta_2 = C_3 t^{-1} + C_4 t^{-2} + C_5 t^{2/3} + \cdots,
\]

\[
\delta_3 = C_6 t^{-1} + C_7 t^{-2} + C_8 t^{-3} + C_9 t^{2/3} + C_{10} t^{-4} + C_{11} t^2 + \cdots,
\]

where \( C \)'s are the constant coefficients. Two points are worth mentioning. In the open universe, growing modes appear earlier due to the \( r \) factors in the time-time component of the Einstein equation (3). In addition to the usual growing mode, namely \( t^{2/3} \), there exist new

\[5\] Interestingly, the seizure of the inhomogeneities to grow after a long time is compatible with the numerical results of reference for a self-similar Newtonian universe.
growing modes which appear at lower orders as compared to the flat universe (see equation (30)).

Moreover, later in the expansion of the scale factor the appearance of new growing modes due to the inhomogeneous part of the Einstein equation as written in (10) for the matter-dominated era proceeds as in the flat universe, but now with enhanced consequences for the density contrast (namely they appear at lower orders). This profile could not be studied directly in the parametric solution for the matter-dominated open universe due to the complications arising from the inhomogeneous part of the differential equation (10).

In addition to the usual growing modes, the density contrast has an infinite number of higher-order growing modes. These are not expected in the usual linear perturbative schemes [1–3,13]. The higher growing modes were overlooked in the exact solutions for the open universe (12) because only the homogeneous part of the differential equation was considered, while the new growing modes seem to be due to a back reaction of the system to the perturbation itself.

The modes are divided in families, once they appear at some order of the perturbation, they continue appearing at higher orders together with their descendants which are one order smaller in time.

VI. SMALL-DISTANCE PERTURBATION

In the preceding sections, we have considered the perturbative expansion appropriate for large distances. We now analyze the perturbative expansion in positive powers of $r$ which is more appropriate for small distances and thus more relevant to the formation of galaxies and other smaller structures. We start with the expansion

$$R_p(t, r) = R(t) + \sum_{n=0}^{\infty} \delta R_n(t) r^n,$$

which is substituted into the Einstein equation (5). At order $r^{-2}$, the equation is trivially satisfied and the order $r^{-1}$ terms imply that $\delta R_1$ vanishes. The next term, however, involves both $R$ and $\delta R_2$. If we insist on using the homogeneous result for $R$ then the whole series...
disappears. That is to say \( \delta R_n = 0 \) for \( n \neq 0 \). This problem can be understood once one realizes that there exists the simple separable solution

\[
R_p(t, r) = \frac{R(t)}{1 + g(t)r^2},
\]

which implies that \( \delta R_2 \) is non-vanishing and proportional to \( R \), namely \( \delta R_2 = -Rg(t) \). However, we cannot find the full solution perturbatively. A further exact solution can also be found and reads

\[
R_p(t, r) = \frac{R(t)}{r}.
\]

In this case, the equation for \( R(t) \) depends on the equation of state. Furthermore, imposing the condition that \( p_r = 0 \) we recover the scale factor of the flat universe, namely \( R(t) \sim t^{\frac{2}{3}} \), while requiring \( p_\theta = 0 \) we recover the result of the closed universe, namely \( R(t) \sim \theta - \sin \theta \).

The structure of the small-time perturbative expansion in the open universe is such that new terms corresponding to new growing modes are introduced into the recursive relations. For the flat universe we clearly find a dominant term in \( t \), of the type \( \delta R_N(t) \approx \frac{1}{N} \frac{R^{(N+1)}}{r^{2N+1}} \) for large values of \( N \), foreseeing a logarithmic correction to the scale factor. Moreover, the correction term itself grows at relatively small values of time. Since \( R(t) = At^{\frac{2}{3}} \), the Hubble constant is proportional to \( A^{3/2} \), and we obtain \( \delta R_N \sim A^{N+1} \frac{R^{(N+1)}}{r^{2N+1}} \), and \( \delta_N \sim A^{N} \frac{R^{N}}{r^{2N+1}} \). The higher-order terms are, therefore, important for \( t^{\frac{2}{3}} \sim Ar^2 \sim H_0r^3 \), in the case of the flat universe.

VII. CONCLUSIONS

We analyzed the perturbation of a homogeneous Friedmann-Robertson-Walker space-time as a power series in the inverse of the radial parameter for the S-wave modes. The perturbative solutions depend not only on whether the universe is open, closed or flat, but also on the epoch when the perturbation starts, namely if it starts at early or at late times compared to the age of the universe. The standard expressions for the density contrast in both linear and non-linear regimes are contained in our results.
For the open and closed universes the full expression for the density contrast cannot be obtained, since one needs to solve a set of inhomogeneous recursive differential equations for the perturbation functions, where at a given order the lower-order solutions become the inhomogeneous part of the differential equation. However, the homogeneous part of the equations can be solved in closed form in all cases, and contains, in addition to the usual modes, infinitely many growing modes. The inhomogeneous part of the equation contains further valuable informations as far as the nonlinear modes are concerned. We have solved the full inhomogenous equation for the open matter-dominated universe in two limits: small and large times. At small times, the open universe behaves as a flat universe, with the difference that the nonlinear growing modes appear at lower orders in the perturbation series. The analysis of the large-time solutions, which covers both radiation and matter-dominated eras, indicates the disappearance of the growing modes at future times.

For the flat universe, the differential equation simplifies since the scale factor depends explicitly on time and not parametrically through trigonometric expressions as in the closed and open universes. We have solved the inhomogeneous differential equation for the flat universe and have obtained the full expression for the linear and nonlinear growing modes (which fully agrees with the standard results). These modes grow faster as one goes to higher orders in the perturbation series. This points towards a bottom-up scenario for the structure formation and a universe dominated by cold dark matters.

In order to verify the observational consequences of the modes growing faster than the standard modes of the linear regime, it is necessary to rewrite the results in terms of observable quantities such as the luminosity distance rather than the variables \((t, r)\). This is presently under investigation [16].

We thank F. Brandt for his help with the Maple programs. R.M. thanks A. Albrecht for helpful discussions. This work was supported by Conselho Federal de Desenvolvimento Científico e Tecnológico (CNPq-Brazil) and Fundação de Amparo a Pesquisa do Estado de São Paulo (FAPESP).
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