INTERPOLATION AND PEAK FUNCTIONS FOR THE NEVANLINNA AND SMIRNOV CLASSES

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ABSTRACT. It is known (implicit in [HMNT]) that when $\Lambda$ is an interpolating sequence for the Nevanlinna or the Smirnov class then there exist functions $f_\lambda$ in these spaces, with uniform control of their growth and attaining values 1 on $\lambda$ and 0 in all other $\lambda' \neq \lambda$. We provide an example showing that, contrary to what happens in other algebras of holomorphic functions, the existence of such functions does not imply that $\Lambda$ is an interpolating sequence.

1. INTRODUCTION

Consider the Nevanlinna class

$$N = \{ f \in H(\mathbb{D}) : \lim_{r \to 1} \int_{0}^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} < +\infty \},$$

which is a complete metric space with the distance defined by

$$d(f, g) = \lim_{r \to 1} \int_{0}^{2\pi} \log(1 + |f(re^{i\theta}) - g(re^{i\theta})|) \frac{d\theta}{2\pi}.$$

Definition. A sequence $\Lambda \subset \mathbb{D}$ is a (free) interpolating sequence for $N$ if the space of traces $N|\Lambda$ is ideal, that is, whenever $f \in N$ and $\{\omega_\lambda\}_{\lambda \in \Lambda}$ is a bounded sequence there exists $g \in N$ such that $g(\lambda) = \omega_\lambda f(\lambda)$, $\lambda \in \Lambda$. We shall write $\Lambda \in \text{Int} N$.

Since $N$ is an algebra, it is easily seen that $\Lambda \in \text{Int} N$ if and only if for every bounded sequence $\{v_\lambda\}_{\lambda \in \Lambda}$ there exists $f \in N$ such that $f(\lambda) = v_\lambda$, $\lambda \in \Lambda$. In particular, if $\Lambda \in \text{Int} N$ there exist functions $f_\lambda \in N$ interpolating the values

$$\delta_{\lambda, \lambda'} = \begin{cases} 1 & \text{if } \lambda' = \lambda \\ 0 & \text{if } \lambda' \neq \lambda. \end{cases}$$

Moreover this can be achieved with functions $f_\lambda$ such that $\sup_{\lambda \in \Lambda} d(f_\lambda, 0) < \infty$, as can be seen by going through the details of the proof of [HMNT] Theorem 1.2

Note that for other algebras of holomorphic functions the analogous size control of these $f_\lambda$ is an immediate consequence of the open mapping theorem applied to the restriction operator. This is the case for $H^\infty$, the algebra of bounded holomorphic functions, the Korenblum algebra $A^{-\infty}$, or the Smirnov class $N^+$ (see below). However, $N$ is not even a topological vector space [ShSh], so no open mapping theorem can be used here.

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Conversely, for $H^\infty$ and $A^{-\infty}$ (and for other Banach spaces), the existence of functions $f_\lambda$ with uniform control of their size and interpolating the values $\delta_{\lambda,\lambda'}$ implies that $\Lambda$ is an interpolating sequence (see [Gar07 Chap.VII], [Ma, Lemma 2.3]). We provide an example showing that this is not the case for $N$ nor $N^+$. For the Nevanlinna class, we have:

**Theorem 1.1.** Let $\Lambda$ be a sequence in $D$. Then,

(a) If $\Lambda \in \text{Int } N$, then there exist $C > 0$ and functions $f_\lambda \in N$ such that
   (i) $d(f_\lambda, 0) \leq C$,
   (ii) $f_\lambda(\lambda') = \delta_{\lambda,\lambda'}$, $\lambda' \in \Lambda$.

(b) The converse fails: there is a sequence $\Lambda \notin \text{Int } N$ for which there exist $C > 0$ and $f_\lambda \in N$ satisfying (i) and (ii).

On the other hand, the Smirnov class $N^+$ is defined by

$$N^+ = \{ f \in N : \lim_{r \to 1} \int_{0}^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} = \int_{0}^{2\pi} \log^+ |f^*(e^{i\theta})| \frac{d\theta}{2\pi} \} ,$$

where $f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$ (which exists a.e. $\theta \in [0, 2\pi]$).

Since $N^+$ is an $F$-space ([Ya, Lemma 1]), an application of the open mapping theorem for such spaces (see [Ru73 2.11, p.47]) shows that there exist $C > 0$ and functions $f_\lambda \in N^+$ satisfying (i) and (ii) from Theorem 1.1. But more can be said. Denote by $\mathcal{F}$ the class of convex, increasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\lim_{t \to +\infty} \psi(t)/t = +\infty$. It is known that if $f \in N^+$, there exists $\psi \in \mathcal{F}$, depending on $f$, such that

$$\int_{0}^{2\pi} \psi \left[ \log(1 + |f^*(e^{i\theta})|) \right] \frac{d\theta}{2\pi} < +\infty .$$

**Theorem 1.2.** Let $\Lambda$ be a sequence in $D$. Then,

(a) If $\Lambda \in \text{Int } N^+$, then there exist $\psi \in \mathcal{F}$, $C > 0$ and functions $f_\lambda \in N^+$ such that
   (i) $\int_{0}^{2\pi} \psi \left[ \log(1 + |f^*_\lambda(e^{i\theta})|) \right] \frac{d\theta}{2\pi} \leq C$,
   (ii) $f_\lambda(\lambda') = \delta_{\lambda,\lambda'}$, $\lambda' \in \Lambda$.

(b) The converse fails: there is a sequence $\Lambda \notin \text{Int } N^+$ for which there exist $\psi \in \mathcal{F}$, $C > 0$ and $f_\lambda \in N^+$ satisfying (i) and (ii).

As in the Nevanlinna case, part (a) is implicit in the proof of [HMNT Theorem 1.3], while part (b) will follow from an explicit example.

2. Preliminaries. Interpolation in the Nevanlinna and Smirnov Classes

A complete description of the interpolating sequences for $N$ and $N^+$, including a characterisation of the traces, was given in [HMNT Theorems 1.2 and 1.3]. In particular, we will make use of the following geometric characterisation.
Let \( b_\lambda(z) = \frac{z - \lambda}{1 - \lambda z} \) be a Blaschke factor and \( B_\lambda(z) := \prod_{\lambda' \neq \lambda} \left( -\frac{\lambda'}{\lambda} \right) b_{\lambda'}(z) \) the Blaschke product with one factor omitted. Given a finite measure \( \mu \) in \( \mathbb{T} \), let
\[
P[\mu](z) = \int_0^{2\pi} \frac{1 - |z|^2}{|z - e^{i\theta}|^2} d\mu(\theta)
\]
denote its Poisson transform.

**Theorem A.** [HMNT] Let \( \Lambda \subset \mathbb{D} \).

(a) \( \Lambda \in \text{Int } N \) if and only if there exists \( \mu \) positive measure with finite mass on \( \mathbb{T} \) such that
\[
|B_\lambda(\Lambda)| \geq e^{-P[\mu](\lambda)}, \quad \lambda \in \Lambda.
\]

(b) \( \Lambda \in \text{Int } N^+ \) if and only if there exists \( w \geq 0, w \in L^1(\mathbb{T}) \), such that
\[
|B_\lambda(\Lambda)| \geq e^{-P[w](\lambda)}, \quad \lambda \in \Lambda.
\]

In classical terminology, when \( w \) is a positive function in \( L^1(\mathbb{T}) \), the harmonic function \( u = P[w] \) is called quasi-bounded. According to [ArGa, Theorem 1.3.9, p.10], for any such functions there exists \( \psi \in \mathcal{F} \) such that
\[
\sup_{\psi(r < 1)} \int_0^{2\pi} \psi[u(re^{i\theta})] \frac{d\theta}{2\pi} = \int_0^{2\pi} \psi[w(e^{i\theta})] \frac{d\theta}{2\pi} < +\infty.
\]

3. PROOF. Necessity.

As said before, the necessity of conditions (i) and (ii) in Theorems 1.1 and 1.2 is implicit in [HMNT]. We briefly recall how this goes.

Assume \( \Lambda \in \text{Int } N \) and let \( \mu \) be the measure given by Theorem A (a). Consider the function
\[
g(z) = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta).
\]
Since \( \text{Re } g(z) = P[\mu](z) \geq 0 \) we see that \( g \in N^+ \) and also \( e^g \in N \). Letting \( H = (2 + g)^2 \) we have \( H \in N^+ \) and
\[
|H(\lambda)| \geq \left( 2 + \text{Re } g(\lambda) \right)^2 \geq \left( 2 + \log \frac{1}{|B_\lambda(\Lambda)|} \right)^2 = \left( 1 + \log \frac{e}{|B_\lambda(\Lambda)|} \right)^2,
\]
whence letting \( \phi(t) = (1 + t)^{-2} \) we obtain, for any \( \lambda' \in \Lambda \),
\[
|\delta_{\lambda\lambda'}| \leq |\lambda| \phi \left( \frac{e}{|B_\lambda(\Lambda)|} \right) e^{P[\mu](\lambda)} |H(\lambda)|.
\]

**Theorem.** [Gar77, Theorem 4] Let \( \phi : [0, \infty) \rightarrow [0, \infty) \) be a decreasing function such that \( \int_0^\infty \phi(t) \, dt < \infty \). There exists \( C > 0 \) such that if \( \{v_\lambda\}_{\lambda} \) is a sequence with
\[
|v_\lambda| \leq |B_\lambda(\Lambda)| \phi \left( \frac{e}{|B_\lambda(\Lambda)|} \right) \lambda \in \Lambda,
\]
then there exists \( F \in H^\infty \) with \( F(\lambda) = v_\lambda, \lambda \in \Lambda \), and \( \|F\| \leq C \int_0^\infty \phi(t) \, dt \).
Applying Garnett’s theorem to the sequence \( \{ \frac{\delta_{\lambda'}}{e^{P[\mu](\lambda)}|H(\lambda)|} \}_{\lambda \in \Lambda} \) we obtain a constant \( C(\phi) \) and functions \( F_\lambda \in H^\infty \) with \(\|F_\lambda\|_\infty \leq C(\phi) \) and
\[
F_\lambda(\lambda') = \frac{\delta_{\lambda'}}{e^{P[\mu](\lambda)}|H(\lambda)|}.
\]
Defining 
\[
f_\lambda(z) = F_\lambda(z)e^{g(z)}H(z)
\]
we finally have (ii) and
\[
\log^+ |f_\lambda(z)| \leq \log C(\phi) + P[\mu](z) + \log^+ |H(z)|,
\]
which implies (i).

The same proof for the Smirnov case provides interpolating functions \( f_\lambda \in N^+ \) with
\[
\log^+ |f_\lambda(z)| \leq \log C(\phi) + P[w](z) + \log^+ |H(z)|,
\]
where \( w \) is given by Theorem A(b). Since \( H \in N^+ \), the subharmonic function \( \log^+ |H(z)| \) has a quasi-bounded harmonic majorant, and (i) in Theorem 1.2 follows from (I).

4. PROOF. LACK OF SUFFICIENCY.

In order to construct examples of non-interpolating sequences satisfying (i) and (ii) in Theorems 1.1 and 1.2, consider the dyadic intervals on \( \mathbb{T} \):
\[
I_{n,k} = \{ e^{2\pi i \theta} : k2^{-n} \leq \theta < (k+1)2^{-n} \}, \quad n \in \mathbb{N}, \ k = 0, \ldots, 2^n - 1.
\]

Let us prove first Theorem 1.1 (b).

Consider the sequence \( A \) defined in the following way: on the ray terminating at an end of a dyadic interval of the \( n \)-th generation (and which is not and end of an interval in a previous generation) consider the dyadic sequence with radii \( 1 - 2^{-m} \), starting at \( m = 2n \). Explicitly
\[
A = \{ a_{m}^{n,k} \}_{n \in \mathbb{N}, 0 \leq k < 2^n - 1, k \text{ odd}} \quad a_{m}^{n,k} = (1 - 2^{-m})e^{2\pi ik2^{-n}}.
\]

Now, to each \( a \in A \) associate a point \( b \) on the same ray and so that
\[
(2) \quad g(a, b) := \left| \frac{a - b}{1 - \bar{a}b} \right| \simeq \exp \left( -\frac{1}{1 - |a|} \right).
\]
We thus obtain a sequence \( B \), which can be explicitly given by
\[
B = \{ b_{m}^{n,k} \}_{n \in \mathbb{N}, 0 \leq k < 2^n - 1, k \text{ odd}} \quad b_{m}^{n,k} = (1 - e^{-2^m})a_{m}^{n,k}.
\]

Lemma 4.1. The sequences \( A \) and \( B \) are both interpolating for \( H^\infty \).

Recall that \( A \in \text{Int } H^\infty \) means that there exists \( C > 0 \) (the interpolation constant) such that for every bounded sequence \( \{ v_a \}_{a \in A} \) there is \( F \in H^\infty \) with \( \|F\|_\infty \leq C\|\{ v_a \}\|_\infty \) and \( F(a) = v_a \), \( a \in A \).
Figure 1. Representation of the points of $A$ “above” the interval $I_{n,k}$

**Proof.** Let us see that $A \in \text{Int } H^\infty$. By Carleson’s theorem ([Gar07, Theorem 1.1, Chap.VII]) it is enough to see that $A$ is separated in the pseudo-hyperbolic metric $\varrho$, which is obvious from the definition, and that $\nu = \sum_{a \in A} (1 - |a|) \delta_a$ is a Carleson measure.

Let us see first that $A$ is a Blaschke sequence:

$$\sum_{a \in A} 1 - |a| = \sum_{n=1}^{\infty} \sum_{k=0}^{2^n-1} \sum_{m \geq 2n} 2^{-m} \approx \sum_{n=1}^{\infty} 2^n 2^{-2n} < +\infty$$

In order to see that $\nu$ is a Carleson measure we have to prove that there exists $C > 0$ such that $\nu(Q(I)) \leq C|I|$ for all $I$ interval in $\mathbb{T}$, where $Q(I) = \{re^{i\theta} : 0 < r < |I|, e^{i\theta} \in I\}$ is the associated Carleson box. It is enough to consider the case where $I$ a dyadic interval. Thus let $I = I_{n,k}$ and

$$Q(I_{n,k}) = \{re^{i\theta} : r > 1 - 2^{-n}, \theta \in 2\pi[k2^{-n}, (k+1)2^{-n})\}.$$  

By construction, in $Q(I_{n,k})$ there are $2^j$ rays of the $(n+j)$-th generation. Hence

$$\sum_{a \in A \cap Q(I_{n,k})} 1 - |a| \approx \sum_{j=1}^{\infty} 2^j \sum_{m \geq 2(n+j)} 2^{-m} \approx \sum_{j=1}^{\infty} 2^{-2n-j} \lesssim 2^{-n} = |I_{n,k}|.$$  

Define $\Lambda = A \cup B$. Let us see first that there exist $C > 0$ and $f_\lambda$ satisfying (i) and (ii) in Theorem 1.1.

Fix $\lambda = a_{m,n}^k$ and denote by $\tilde{\lambda} = b_{m,n}^k$ its “twin”. As just seen, there exist $C > 0$ and $P_\lambda^A \in H^\infty$ such that
\[ \| P_A^\lambda \|_\infty \leq C, \]
\[ P_A^\lambda (\lambda) = 1 \quad \text{and} \quad P_A^\lambda (a_{m',k'}) = 0 \quad \forall (n', k', m') \neq (n, k, m). \]

As in the proof of Lemma 4.1, we can see that \((B \setminus \{\tilde{\lambda}\}) \cup \{\lambda\}\) is also in \(\text{Int } H_\infty\), and with interpolation constant \(C > 0\) independent of \(\lambda\). Therefore there exist \(P_B^\lambda \in H_\infty\) such that
\[ \| P_B^\lambda \|_\infty \leq C, \]
\[ P_B^\lambda (\lambda) = 1 \quad \text{and} \quad P_B^\lambda (b_{n',k'}^\lambda) = 0 \quad \forall (n', k', m') \neq (n, k, m). \]

Define finally
\[ f_\lambda := c_\lambda P_A^\lambda P_B^\lambda b_\lambda e^{g_\lambda}, \]
where
\[ g_\lambda (z) = \frac{\lambda^* + z}{\lambda^* - z} \quad (\lambda^* = \lambda/|\lambda|) \]
and \(c_\lambda\) is chosen so that \(f_\lambda (\lambda) = 1\).

Notice that, by construction, (ii) in Theorem 1.1 holds. In order to see (i) notice that (2) gives
\[ |c_\lambda| = \frac{1 \exp \left(-1 + |\lambda| \right)}{|b_\lambda (\lambda)|} \simeq \exp \left(\frac{1}{1 - |\lambda|} - \frac{1 + |\lambda|}{1 - |\lambda|}\right) \lesssim 1. \]

Then
\[ \log |f_\lambda (z)| = \log |c_\lambda| + \log |P_A^\lambda (z)| + \log |P_B^\lambda (z)| + \log |b_\lambda (z)| + \text{Re } g_\lambda (z) \]
and therefore
\[ \log^+ |f_\lambda (z)| \leq \log \|c\|_\infty + 2 \log C + \text{Re } g_\lambda (z). \]

Since
\[ \sup_{r < 1} \int_0^{2\pi} \text{Re } g_\lambda (r e^{i\theta}) \frac{d\theta}{2\pi} = \sup_{r < 1} \int_0^{2\pi} \frac{1 - r^2}{|r e^{i\theta} - \lambda^*|^2} \frac{d\theta}{2\pi} = 1, \]
we have (i), as desired.

Let us see now that \(\Lambda \notin \text{Int } N\) by seeing that there is no \(\mu\) satisfying the condition of Theorem A(a). Since
\[ \log \frac{1}{|B_\lambda (\lambda)|} \simeq \frac{1}{1 - |\lambda|} \quad \lambda \in \Lambda, \]
such \(\mu\) should satisfy in particular (fixed any \(n, k\))
\[ 1 \lesssim \frac{(1 - |a_n^{n,k}|) P[\mu] (a_n^{n,k})}{\mu(e^{2\pi ik2^{-n}})} \quad \forall m \geq 2n. \]
This would force the measure \(\mu\) to satisfy \(1 \lesssim \mu\{e^{2\pi ik2^{-n}}\}\) for all \(n, k\) (ShSh, Theorem 2.2), and therefore \(\mu(\mathbb{T})\) could not be finite.

Let us prove now Theorem 1.2 (b). In the same construction done for \(N\) consider a sequence made of the “first” couple of points of each ray, and with a slightly bigger separation. More precisely, let \(\tilde{\Lambda} = \tilde{A} \cup \tilde{B}\), where
\[ \tilde{A} = \{a_{n,k}\}_{n \in \mathbb{N}, 0 \leq k < 2^n - 1, k \text{ odd}} \quad a_{n,k} = (1 - 2^{-2n})e^{2\pi ik2^{-n}}. \]
and $\tilde{B} = \{b_{n,k}\}_{n,k}$ is so that $a_{n,k}$ and $b_{n,k}$ are on the same ray and

$$g(a_{n,k}, b_{n,k}) = \exp\left( -\frac{1}{(1 - |a_{n,k}|) \log_2(\frac{1}{1 - |a_{n,k}|})} \right) = \frac{2^{2n}}{2n}. \tag{4}$$

As in Lemma 4.1, $\tilde{A}$ and $\tilde{B}$ are $H^\infty$-interpolating sequences, so there exist bounded peak functions $P_\lambda^A$, $P_\lambda^B$ with the same properties as before.

Given $\lambda \in \tilde{A}$, let $I_\lambda$ denote the Privalov “shadow” of $\lambda$ on $\mathbb{T}$, that is

$$I_\lambda = \{e^{i\theta} : |e^{i\theta} - \lambda| \leq 2(1 - |\lambda|)\}.$$

Let

$$w_\lambda(\theta) = \frac{C_0}{(1 - |\lambda|) \log_2(\frac{1}{1 - |\lambda|})} \chi_{I_\lambda}(e^{i\theta}),$$

where $c$ is a universal constant to be chosen later, and consider

$$g_\lambda(z) = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} w_\lambda(\theta) \frac{d\theta}{2\pi}.$$

Notice that $\|w_\lambda\|_{L^1(\mathbb{T})} \simeq c(\log_2(\frac{1}{1 - |\lambda|}))^{-1} \lesssim 1$ and

$$\text{Re } g_\lambda(\lambda) = P[w_\lambda](\lambda) \simeq \frac{1}{1 - |\lambda|} \int_{I_\lambda} w_\lambda(\theta) \, d\theta \geq \frac{C_1 C_0}{(1 - |\lambda|) \log_2(\frac{1}{1 - |\lambda|})}.$$

Now define $f_\lambda$ as in (3), with these new $g_\lambda$. Again, it is clear that (ii) in Theorem 1.2 holds. Also, $\{c_\lambda\}_{\lambda \in \tilde{A}}$ is bounded if $C_0$ is chosen appropriately: if $\lambda = a_{n,k}^m$,

$$\log |c_\lambda| = \log \frac{1}{g(a_{n,k}, b_{n,k})} - \text{Re } g_{a_{n,k}}(a_{n,k}) \leq \frac{1 - C_1 C_0}{(1 - |a_{n,k}|) \log_2(\frac{1}{1 - |a_{n,k}|})} \leq 0.$$

In order to see (i) notice that, as before, there exists $\tilde{C} > 0$ such that

$$\log^+ |f_\lambda| \leq \log \|c\|_\infty + 2 \log C + P[w_\lambda] \leq \tilde{C} + P[w_\lambda].$$

Therefore, by (1), for $\lambda = a_{n,k}^m$ and taking $\psi(t) = (1 + t) \log(1 + t) \in \mathcal{F}$ we get (i):

$$\int_0^{2\pi} \psi[\log^+ |f_\lambda^m(e^{i\theta})|] \frac{d\theta}{2\pi} \leq \int_0^{2\pi} \psi[\tilde{C} + \frac{C_0 \chi_{I_\lambda}(\theta)}{(1 - |\lambda|) \log_2(\frac{1}{1 - |\lambda|})}] \frac{d\theta}{2\pi}$$

$$= \int_{\theta \notin I_\lambda} \psi(\tilde{C}) \frac{d\theta}{2\pi} + \int_{I_\lambda} \psi[\tilde{C} + \frac{C_0}{(1 - |\lambda|) \log_2(\frac{1}{1 - |\lambda|})}] \frac{d\theta}{2\pi}$$

$$\lesssim 1 + (1 - |\lambda|) \psi \left[ \frac{2C_0}{(1 - |\lambda|) \log_2(\frac{1}{1 - |\lambda|})} \right]$$

$$\lesssim 1 + \frac{1}{\log_2(\frac{1}{1 - |\lambda|})} \log_2 \left( \frac{2C_0}{(1 - |\lambda|) \log_2(\frac{1}{1 - |\lambda|})} \right) \lesssim 1.$$
Let us finish by proving that $\Lambda \notin \operatorname{Int} N^+$. Assume that there is $w \in L^1(\mathbb{T})$ satisfying Theorem A (b). Then
\[
\frac{1}{(1 - |a_{n,k}|) \log_2 \left( \frac{1}{1 - |a_{n,k}|} \right)} = \log \frac{1}{\varrho(a_{n,k}, b_{n,k})} \leq \log \frac{1}{|B_{a_{n,k}}(a_{n,k})|} \leq P[w](a_{n,k}) = \int_0^{2\pi} \frac{1 - |a_{n,k}|^2}{|a_{n,k} - e^{i\theta}|^2} w(\theta) \frac{d\theta}{2\pi},
\]
and therefore
\[
\sum_{n \geq 1} \sum_{k=0}^{2^n-1} \frac{1}{2n} \approx \sum_{n \geq 1} \sum_{k=0}^{2^n-1} \frac{1}{\log_2 \left( \frac{1}{1 - |a_{n,k}|} \right)} \leq \int_0^{2\pi} \sum_{n,k} \frac{1 - |a_{n,k}|^2}{|a_{n,k} - e^{i\theta}|^2} w(\theta) \frac{d\theta}{2\pi}.
\]
We will have a contradiction as soon as we prove that
\[
\sup_{\theta \in [0,2\pi]} \sum_{n,k} \frac{(1 - |a_{n,k}|^2)^2}{|a_{n,k} - e^{i\theta}|^2} < \infty.
\]
With no loss of generality assume that $e^{i\theta} = 1$ and that the dyadic intervals $I_{n,k}$, $0 \leq k < 2^n - 1$, are ordered so that $e^{i\theta} \in I_{n,0}$. Then we have
\[
|e^{i\theta} - a_{n,k}|^2 = |1 - a_{n,k}|^2 \approx (1 - |a_{n,k}|)^2 + |e^{2\pi ik 2^{-n}} - 1|^2 
\approx (2^{-2n})^2 + (k2^{-n})^2 = 2^{-2n} (2^{-2n} + k^2).
\]
Since for each $e^{i\theta}$ there is only one $a_{n,k}$ with $1 - |a_{n,k}| \simeq |e^{i\theta} - a_{n,k}|$, we have then
\[
\sum_{n,k} \frac{(1 - |a_{n,k}|^2)^2}{|a_{n,k} - e^{i\theta}|^2} \approx \sum_{n \geq 1} \sum_{k=1}^{2^{n-1}} \frac{2^{-4n}}{2^{-2n}k^2} \approx \sum_{n \geq 1} 2^{-2n}.
\]

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