The Shape of the Electron and Generalization of Heisenberg Quantization Based on Established Experiments

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In experiments of improved measurement of the shape of the electron by measuring the electron’s electric dipole moment, a non-zero result has not been found. However, the generalized Heisenberg quantization explores the existence of the non-zero position-position minimal uncertainty, which leads to electrons having a non-zero size, and their shape being a sphere with a radius determined by the position-position minimal uncertainty. Heisenberg quantization was proposed in the empty space. By using established experiments the generalization of Heisenberg quantization is obtained in the physical space (the empty space plus the cosmic magnetic field). In the generalized Heisenberg quantization, the position-momentum commutator of Heisenberg quantization is maintained; However, both the momentum-momentum commutator and the position-position commutator are noncommutative. The momentum-momentum noncommutative parameter is determined by the cosmic magnetic field; The maintenance of the Bose-Einstein statistics ascertains
the existence of the non-zero position-position noncommutative parameter and the relevant non-zero position-position minimal uncertainty. Thus, the longstanding question of whether electrons are size-less is clarified. The generalized Heisenberg quantization recovers all predictions of Heisenberg quantization.

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Heisenberg quantization (HQ). Quantum mechanics based on HQ is one of the most successful physics theories. It has obtained highly accurate test of all experiments. HQ was proposed in the empty space, i.e. the true-vacuum. It reads (In this paper, we consider a two dimensional space)

\[
[x_i, p_j] = i\hbar\delta_{ij}, \quad [x_i, x_j] = 0, \quad [p_i, p_j] = 0, \quad (i, j = 1, 2),
\]

where it is supposed both position-position and momentum-momentum are commuting.

Eqs. (1) is the foundation of quantum mechanics in the empty space.

Generalized Heisenberg quantization. In the real universe, space and a cosmic magnetic field are simultaneously exist in a very large cosmic space scale since the early Universe. In the reality, the empty space does not exist; The physical space is the empty space plus the cosmic magnetic field, in other word, the cosmic magnetic field is the intrinsic magnetic field of the physical space. The present intergalactic magnetic field \(B^c\) has an intensity of

\[
B^c \sim 10^{-8}\text{Gauss} = 10^{-12}\text{T}
\]

and its dominant scale-length \(\sim 10\) kpc [1]. In the following, we mean that the cosmic magnetic field is this \(B^c\).

Although \(B^c\) is so weak, its role in quantum mechanics is much more important than our previous understanding. In the traditional way, \(B^c\) is treated as perturbation in the Schrödinger equation. However, there is another way, this intrinsic magnetic field \(B^c\) of the physical space, according to the minimal coupling of the electromagnetic interaction, should be included in the definition of momentum. This way exhibits important effects of \(B^c\), which is not noticed before.

When the cosmic magnetic field \(B^c\) is considered, HQ is generalized as:

\[
[p_i, \tilde{p}_j] = i\hbar\delta_{ij}, \quad [\tilde{p}_i, \tilde{p}_j] = i\xi^2\eta_{\epsilon ij}, \quad [\tilde{x}_i, \tilde{x}_j] = i\xi^2\theta_{\epsilon ij}, \quad (i, j = 1, 2),
\]

where \((\tilde{p}, \tilde{x})\) are expressed by \((x, p)\):

\[
\tilde{p}_i = \xi[p_i + \eta_{\epsilon ij}x_j/(2\hbar)], \quad \tilde{x}_i = \xi[x_i - \theta_{\epsilon ij}p_j/(2\hbar)].
\]
In the above, $\epsilon_{ij}$ is an antisymmetric unit tensor, $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$; $\xi = (1 + \theta_c \eta_c / 4\hbar^2)^{-1/2}$ is the scaling factor, which is demanded by the consistence of Eqs.(3).

In the tilde system ($\tilde{\mathbf{p}}, \tilde{\mathbf{x}}$), the first equation of Eqs.(1) is maintained. However, commutators of the momentum-momentum and the position-position are non-commuting, their noncommutative parameters are, respectively, $\eta_c$ and $\theta_c$. Therefore, the physical space is noncommutative.

Eqs.(3) is the foundation of quantum mechanics in the physical space (PSQM).

Because HQ has obtained highly accurate test of all experiments, both $\eta_c$ and $\theta_c$ should be extremely small, they do not change effects of HQ. All predictions of HQ are recovered by PSQM.

In the following, we demonstrate how Eqs.(1) is generalized to Eqs.(3). When $B^c$ is considered, the minimal coupling of the electromagnetic interaction demands that the canonical momentum $\mathbf{p}$ should be replaced by $\tilde{\mathbf{p}}_i \sim p_i - qA^c_i(x)$, where $\mathbf{A}^c$ is the vector potential of $B^c$. At the micro-scale, $B^c$ can be considered as a constant field. Taking the direction of $B^c$ as the $z$ direction, it follows that $A^c_i = -B^c \epsilon_{ij} x_j / 2$. Thus, we obtain the expression of $\tilde{\mathbf{p}}_i$ by $p_i$ and $x_j$ in Eqs.(4), there the momentum-momentum noncommutative parameter

$$\eta_c = \hbar q B^c. \quad (5)$$

In the system described by $\tilde{\mathbf{p}}_i$, the position $x_i$ should be replaced by $\tilde{x}_i$ accordingly. To find the expression of $\tilde{x}_i$, we consider the following fundamental physics conditions: the system ($\tilde{\mathbf{p}}_i, \tilde{x}_i$) must satisfy

(I) Maintaining the first equation of HQ Eqs.(1);

(II) Maintaining Bose-Einstein statistics.

When the state vector space of identical bosons is constructed by generalizing one-particle quantum mechanics, for two dimensional harmonic oscillator, the annihilation-creation operators ($\tilde{a}_i, \tilde{a}_i^\dagger$) are:

$$\tilde{a}_i^\dagger = \sqrt{\frac{\mu \omega}{2\hbar}} \left( \tilde{x}_i - \frac{i}{\mu \omega} \tilde{p}_i \right), \ (i = 1, 2). \quad (6)$$

where $\mu$ is mass of the considered particle and $\omega$ is its characteristic frequency.
To maintain Bose-Einstein statistics the condition is that $\tilde{a}_i^\dagger$ and $\tilde{a}_j^\dagger$ should be commuting: $[\tilde{a}_i^\dagger, \tilde{a}_j^\dagger] = 0$. This leads to two results:

(i) The consistency expression of the $\tilde{x}_i$ by $x_i$ and $p_j$ in Eqs. (1) is obtained.

(ii) The position-position noncommutative parameter $\theta_c$ must be a non-zero constant.

Using Eq. (6) we obtain $[\tilde{a}_1^\dagger, \tilde{a}_2^\dagger] = i \xi \mu \omega [\theta_c - \eta_c/(\mu \omega)^2]/(2\hbar)$. If $\theta_c$ was zero, $[\tilde{a}_1^\dagger, \tilde{a}_2^\dagger] = -i \xi \mu \omega [\eta_c/(\mu \omega)^2]/(2\hbar) \neq 0$, thus Bose-Einstein statistics could not be maintained. Therefore, the maintenance of Bose-Einstein statistics concludes that $\theta_c$ must be a non-zero constant,

$$\theta_c \neq 0. \quad (7)$$

Although Eq. (7) is obtained in an example, it is enough to ascertain the existence of the non-zero $\theta_c$ in general cases.

End of deriving Eqs. (3).

The position-position minimal uncertainty of PSQM. In PSQM Eqs. (3) there are two new minimal uncertainties: The second equation in Eqs. (3) indicates that the momentum-momentum minimal uncertainty $(\Delta \tilde{p})_{min} \equiv (\Delta \tilde{p}_1)_{min} = (\Delta \tilde{p}_2)_{min}$ of $\tilde{p}$ reads (Neglect the high order term $\eta_c \theta_c$, thus $\xi \sim 1$)

$$(\Delta \tilde{p})_{min} = \sqrt{\eta_c/2}; \quad (8)$$

From the third equation in Eqs. (3), it follows the position-position minimal uncertainty

$$(\Delta \tilde{x})_{min} = \sqrt{\theta_c/2}. \quad (9)$$

Because both $\eta_c$ and $\theta_c$ are extremely small, the possibility $\eta_c \theta_c = \text{const}$, i.e. $\theta_c \propto 1/\eta_c$ should be excluded. This explains that $(\Delta \tilde{x})_{min} \cdot (\Delta \tilde{p})_{min}$ does not satisfies the position-momentum minimal uncertainty $(\Delta \tilde{x} \cdot \Delta \tilde{p})_{min} = \hbar/2$ of the first equation in Eqs. (3):

$$(\Delta \tilde{x})_{min} \cdot (\Delta \tilde{p})_{min} = \sqrt{\eta_c \theta_c/2} \neq \hbar/2.$$  

From Eq. (5), it follows (Fundamental physical constants are taken from [2]) $\eta_c = \hbar q B_c \sim 10^{-65} \text{kg}^2 \text{m}^2 \text{s}^{-2}$.

There are experiments of measurement of the shape of the electron by measuring the electron’s electric dipole moment $d$ [3–6]. These experiments have obtained only upper limits $d < 1.1 \times 10^{-27} \text{e} \cdot \text{m}$ to $d < 8.7 \times 10^{-29} \text{e} \cdot \text{m}$, which mean that the electron is a sphere, its radius $r_e < 10^{-27} \text{m}$ to $r_e < 10^{-28} \text{m}$, correspondingly, $\theta_c < 10^{-54} \text{m}^2$ to $\theta_c < 10^{-56} \text{m}^2$. 


$(\Delta \tilde{x})_{\text{min}}$ provides a fundamental minimal length scale, which needs to be considered in relevant fields of physics. Now we clarify a longstanding question: whether electrons have a finite size, and what is its shape?

The existence of the non-zero $(\Delta \tilde{x})_{\text{min}}$ Eq. (9) in PSQM indicates that the procedure of dividing a line segment is not infinitive, it stops at the position-position minimal uncertainty $\sqrt{\theta_c}/2$. For a three dimensional dynamical system, for example, the system of electrons, the procedure of dividing a finite volume of space is not infinitive, it stops at a minimal volume, in which the lengths of all three dimensions are the position-position minimal uncertainty $\sqrt{\theta_c}/2$. The minimal volume is a sphere of a radius $\sqrt{\theta_c}/8$. Thus, base on the existence of the non-zero $\theta_c$ in general cases, PSQM concludes that electrons have a non-zero size: a sphere with a radius $\sqrt{\theta_c}/8$.

**Discussions.** The established PSQM Eqs. (3) works in the physical space. The variables of PSQM are $(\tilde{p}, \tilde{x})$. Commutators of $\tilde{p} - \tilde{p}$ and $\tilde{x} - \tilde{x}$ are noncommutative, thus the physical space is a noncommutative space. The cosmic magnetic field $B^c$ is included in $\tilde{p}$. The $\tilde{p} - \tilde{p}$ noncommutative parameter $\eta_c$ is determined by $B^c$. By using Eqs. (4), all calculations of PSQM are realized by using variables $(x, p)$, PSQM recovers all predictions of HQ.

Furthermore, PSQM exhibits important new futures: The maintenance of the Bose-Einstein statistics explores the existence of a non-zero position-position noncommutative parameter $\theta_c$ Eq. (7) and the relevant non-zero position-position minimal uncertainty $\sqrt{\theta_c}/2$ Eq. (9). The existence of the non-zero $\theta_c$ plays the key role in clarifying that electrons are not size-less. In HQ, by using point particle model, all predictions at quantum mechanics level and quantum electrodynamics level have obtained experimental highly accurate tests. It indicates that point particle model is very accuracy. The success of point particle model does not mean that electrons are size-less. On the experimental side, experiments

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2. $\theta_c$ is dynamical system dependent. Because in the Lagrangian formalism the definition of the canonical momentum is dynamical dependent, so the dynamical parameter dependence of $\theta_c$ can be understood.

3. In [7], the Heisenberg-Weyl algebra of quantum mechanics in non-commutative space was proposed, which works in the empty space. It has two free parameters, the position-position non-commutative parameter $\theta$ and the momentum-momentum non-commutative parameter $\eta$. Up to now, experiments only established upper-limits on $\theta$ [8, 9] and $\eta$ [10].
of improved measurement of the shape of the electron by measuring the electron’s electric
dipole moment have obtained only vanishingly small upper limits, i.e. a non-zero result
has not been found; However, from the existence of the non-zero position-position minimal
uncertainty $\sqrt{\theta_c/2}$, PSQM ascertains that, in the reality, electrons are not a size-less point
particle, they have a non-zero size, and their shape is a sphere with a radius determined
by the position-position minimal uncertainty.

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