ON THE ERGODIC THEOREM FOR AFFINE ACTIONS ON HILBERT SPACE

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Abstract. The note establishes a new weak mean ergodic theorem (Theorem A) for 1-cocycles associated to weakly mixing representations of amenable groups.

Introduction

In a groundbreaking paper [15], Shalom discovered several deep connections between the representation theory of an amenable group and aspects of its large-scale geometry. One motivation for his work, among several others, was the development of a “spectral” approach to Gromov’s celebrated theorem on the virtual nilpotency of groups of polynomial growth [8]. More precisely, Shalom established, Theorem 1.11 in [15], that if it could be shown that any group of polynomial growth $G$ possessed property $H_{FD}$ (see Definition 3.2), then this would suffice to establish that $G$ would have a finite-index subgroup with infinite abelianization—the key step in Gromov’s proof which involves the use of Hilbert’s 5th problem. As a means of establishing property $H_{FD}$, Shalom conjectured that for a group of polynomial growth, a sequence of almost fixed points for any affine action with weakly mixing linear part could be obtained by averaging the associated 1-cocycle over an appropriate subsequence of $n$-balls centered at the identity: see section 6.7 in [15]. In their paper [4], Cornulier, Tessera, and Valette made decisive contributions to Shalom’s program. In particular, their investigation of averaging properties of groups over controlled Følner sequences has directly influenced the approach taken in this paper.

Statement of results. We establish the following weak mean ergodic theorem for affine actions of finitely-generated amenable groups on Hilbert space. We will say that a sequence $(\mu_n)$ of regular Borel probability measures on a countable, discrete group $G$ forms a Reiter sequence if $\|\mu_n - g \ast \mu_n\| \to 0$ for all $g \in G$, where $g \ast \mu_n(h) = \mu_n(g^{-1}h)$. A countable discrete group is said to be amenable if it admits a Reiter sequence.

Theorem A (Weak Mean Ergodic Theorem). Let $\pi : G \to \mathcal{B}(\mathcal{H})$ be an ergodic orthogonal representation of a finitely generated amenable group $G$, and let $b : G \to \mathcal{H}$ be a 1-cocycle...
associated to $\pi$. Let $S$ be a finite symmetric generating set for $G$, and let $| \cdot |$ denote the word length in $S$. If $(\mu_n)$ is a Reiter sequence for $G$, then
\[ \int \frac{1}{|g|} b(g) \, d\mu_n(g) \to 0 \]
in the weak topology on $H$. If $\pi$ is weakly mixing, then
\[ \int \frac{1}{|g|} |\langle b(g)\xi, \xi \rangle| \, d\mu_n(g) \to 0 \]
for all $\xi \in H$.

We note that while $\frac{1}{|e|}$ is technically undefined, by convention it will be understood to denote 0 here and throughout.

In particular, if $b : \mathbb{Z} \to H$ is a 1-cocycle, then $b$ is completely determined by $\xi = b(1)$, so that for $n \geq 1$ we have $\frac{1}{n} b(n) = A_n(\xi) := \frac{1}{n} \sum_{k=0}^{n-1} \pi(k)\xi$: a similar formula holds for $-n$ via the identity $b(-n) = -\pi(-n)b(n)$. So, in this case the result reduces to the fact that the Cesàro sums $C_n(\xi, \eta) = \frac{1}{n} \sum_{k=1}^{n} \langle A_k(\xi), \eta \rangle$ and $C'_n(\xi, \eta) = \frac{1}{n} \sum_{k=1}^{n} |\langle A_k(\xi), \eta \rangle|$ converge to 0 for all $\xi, \eta \in H$. The stronger summation holds for all ergodic representations and is equivalent to the (weak) mean ergodic theorem of von Neumann.

In fact, for the class of abelian groups, we will give a new “geometric” proof of the mean ergodic theorem as a consequence of the following theorem in combination with Theorem A.

**Theorem B.** Let $G$ be finitely generated amenable group admitting a controlled Følner sequence (see Definition 1.7). Let $\pi : G \to \mathcal{O}(H)$ be an orthogonal representation, and let $b : G \to H$ be a 1-cocycle associated to $\pi$. Suppose that
\[ \int \frac{1}{|g|} \langle b(g^{-1}), \xi \rangle \, d\mu_n(g) \to 0 \]
for all $\xi \in H$ and all Reiter sequences $(\mu_n)$. Then the affine action $G \rtimes H$ associated to $b$ admits a sequence of almost fixed points.

To see how this implies the mean ergodic theorem for $\mathbb{Z}$, we point out that by an observation of Cornulier–Tessera–Valette (Proposition 3.1 in [4]) a consequence of $G \rtimes H$ admitting almost fixed points is that
\[ \frac{1}{|g|} \|b(g)\| \to 0 \]
as $|g| \to \infty$; in other words, the 1-cocycle $b$ has sublinear growth. In fact, sublinearity of a 1-cocycle is actually equivalent in general to the mean ergodic theorem, i.e., the statement that
\[ \int \frac{1}{|g|} \|b(g)\| \, d\mu_n(g) \to 0 \]
for all Reiter sequences $(\mu_n)$ (see Proposition 1.16).
sequence produce the desired sequence of almost fixed points. In the weak mixing case, however, one already knows that the 1-cocycle must be “almost weakly sublinear” in the sense that for any $\epsilon > 0$ and $\xi \in \mathcal{H}$, the subset consisting of all elements $g \in G$ such that $|\langle b(g), \xi \rangle| \geq \epsilon |g|$ has measure 0 for all left invariant means on $G$. In the case of a compact representation, this would be sufficient to conclude weak sublinearity. It seems plausible that under additional structural assumptions on the 1-cocycle or the group, one may be able to derive weak sublinearity in the general case.

Remarks on the proofs. The paper is an application of the authors’ philosophy of investigating the “large scale” properties of affine actions of groups on Hilbert space. Though the above results are stated for affine actions, even in this case the proofs crucially rely on a coarsening of the notion of an affine action, the concept of an array—the terminology is meant to evoke the geometric regularity of the orbits of affine actions—formalized by the authors in [3]. The main reason is that starting from an affine action $G \actson \mathcal{H}$, from the associated 1-cocycle $b$, one can construct an array $\alpha : G \rightarrow \mathcal{V}$ into a $G$-invariant positive cone $\mathcal{V} \subset \mathcal{H} \otimes \mathcal{H}$. Such a map cannot lie within a uniformly bounded distance of an unbounded 1-cocycle, since the equation $b(g) = -\pi_g b(g^{-1})$ holds for all $g \in G$ for any 1-cocycle $b$. Some potential implications of the positivity of this map will be discussed in Section 3.

The notion of an array is best viewed from a geometric, rather than algebraic, perspective. Indeed, a length function on a discrete group $G$ may be viewed as a positive array associated with the trivial representation. In general, an array can be thought of as a Hilbert-space valued length function on $G$ which is compatible with some orthogonal $G$-representation $\pi$. The presence of an array then becomes a tool through which properties of the representation can be used to impose large scale conditions on the group, and vice versa. For example, it is shown in [3], Proposition 1.7.3, that a non-amenable group admitting a proper array into its left-regular representation, e.g., non-elementary Gromov hyperbolic groups, cannot be decomposed as a direct product of infinite groups. Turning to the topic at hand, the presence of a controlled Følner sequence imposes a strong large-scale “finite dimensionality” condition on the group $G$—for the case of weak polynomial growth, a point already well made in [8]. Viewed in this light, the content of Theorem B is that this forces any geometric realization of the group which is uniformly distributed throughout an infinite-dimensional Hilbert space to be essentially degenerate.

1. Geometry and Representation Theory

In this section we will introduce the main definitions, concepts, and perspective to provide a context for the exposition.

Notation 1.1. Let $X$ be a set and let $f, g : X \rightarrow \mathbb{R}_{\geq 0}$ be maps. We write $f = O(g)$ or $f \ll g$ if there exists a finite set $F \subset X$ and a constant $C > 0$ such that $f(x) \leq C \cdot g(x)$ for all $x \in X \setminus F$. We will write $f \lesssim g$ if $f \ll g$ for a constant $C \leq 1$. We write $f = o(g)$ if $f \lesssim \epsilon \cdot g$ for all $\epsilon > 0$.

1.1. Isometric actions on Hilbert space.
Definition 1.2. An orthogonal representation $\pi : G \to \mathcal{O}(H)$ is said to be ergodic if for any $\xi \in H$ we have that $\pi_g(\xi) = \xi$ for all $g \in G$ if and only if $\xi = 0$, i.e., $\pi$ has no non-zero invariant vectors. The representation $\pi$ is said to be weakly mixing if the diagonal representation $\pi \otimes \pi : G \to \mathcal{O}(H \otimes H)$ is ergodic. In particular weakly mixing representations are ergodic.

If $\pi : G \to \mathcal{O}(H)$ is an orthogonal representation, a map $b : G \to H$ is said to be a 1-cocycle associated to $\pi$ if it satisfies the Leibniz identity

$$b(gh) = \pi_g(b(h)) + b(g),$$

for all $g, h \in G$. It is essentially a consequence of the Mazur–Ulam theorem that any isometric action $G \curvearrowright^T H$ may be written as $T_g(\xi) = \pi_g(\xi) + b(g)$ for some orthogonal representation $\pi$ and an associated 1-cocycle $b(g)$ and conversely. The representation $\pi$ is known as the linear part of $T$.

Definition 1.3. An isometric action $G \curvearrowright^T H$ is said to admit almost fixed points if there exists a sequence $(\xi_n)$ of vectors in $H$ such that

$$\|T_g(\xi_n) - \xi_n\| \to 0$$

for all $g \in G$.

Definition 1.4. We will say that a 1-cocycle $b$ associated to an orthogonal representation $\pi : G \to \mathcal{O}(H)$ is almost inner if the associated affine isometric action $G \curvearrowright H$ admits almost fixed points.

1.2. Geometric group theory. Throughout the paper $G$ will be a countable discrete group, often finitely generated. Recall that a length function $|\cdot| : G \to \mathbb{R}_{\geq 0}$ is a map satisfying: (1) $|g| = 0$ if and only if $g = e$ is the identity; (2) $|g^{-1}| = |g|$, for all $g \in G$; and (3) $|gh| \leq |g| + |h|$, for all $g, h \in G$. A length function is proper if the map $g \mapsto |g|$ is proper, i.e., all sets of bounded length are finite. If $|\cdot|$ is a length function, then we denote

$$B(n) = \{g \in G : |g| \leq n\},$$

the ball of radius $n$ centered at the identity, and

$$S(n) = \{g \in G : |g| = n\},$$

the sphere of radius $n$ centered at the identity. If $G$ is generated by a finite set $S$, then the function which assigns to each $g \in G$ the least integer $k$ such that $g$ can be written as a product of $k$ elements from $S \cup S^{-1}$ is a proper length function, known as a word length function.

Notation 1.5. Let $G$ be a finitely generated, discrete group with a fixed finite, symmetric, generating set $S$. Let $F \subset G$ be a finite subset. We set

$$\partial F := \bigcup_{g \in S} gF \Delta F,$$

where “$\Delta$” denotes the symmetric difference.
Definition 1.6. A sequence \((F_n)_{n \in \mathbb{N}}\) of finite subsets of G is said to form a Følner sequence if
\[
\frac{|gF_n \Delta F_n|}{|F_n|} \to 0
\]
for all \(g \in G\).

Definition 1.7. Let G be a finitely generated, discrete group with a fixed finite, symmetric, generating set S. For a constant \(K > 0\), a sequence \((F_n)_{n \in \mathbb{N}}\) of finite subsets of G is said to be a K-controlled Følner sequence if
\[
\frac{\partial F_n}{|F_n|} \leq \frac{K}{\text{diam } F_n},
\]
where \(\text{diam } F_n\) is defined to be the least integer \(m\) such that \(F_n \subset B(m)\). The group admits a controlled Følner sequence if it admits a K-controlled Følner sequence for some \(K\).

Definition 1.8. A finitely generated group G is said to have polynomial growth if for some (equivalently, for any) proper word length function we have that
\[
\limsup_n \frac{\log |B(n)|}{\log n} < \infty.
\]
The group G is said to be of weak polynomial growth if
\[
\liminf_n \frac{\log |B(n)|}{\log n} < \infty
\]
for any proper word length.

The following observation is due to Shalom.

Proposition 1.9 (Shalom, Lemma 6.7.3 in [15]). If G is a finitely generated group of polynomial growth of degree \(d\), then for any proper word length, there is a subsequence \(S \subset \mathbb{N}\) such that the sequence of balls \((B(n))_{n \in S}\) form a K-controlled Følner sequence for \(K > 10d\).

In fact, a group G which satisfies a doubling condition \(|B(2n)| \leq C \cdot |B(n)|\) for some subsequence admits a controlled Følner sequence by an observation of Tessera, [17], Remark 4.10. Gromov’s “Regularity lemma” ([8], section 3) shows that groups of weak polynomial growth have the doubling condition. By the work of Tessera several large classes of groups are known to admit controlled Følner sequences.

Proposition 1.10 (Tessera, Theorem 11 in [17]). The following classes of groups admit controlled Følner sequences:

1. polycyclic groups;
2. wreath products \(D \wr \mathbb{Z}\) with \(D\) finite;
3. semi-direct products \(\mathbb{Z}[\frac{1}{mn}] \rtimes \mathbb{Z}/n\) \(\mathbb{Z}\), with \(m, n\) coprime and \(|mn| \geq 2\).

By results of Mal’cev and Auslander, it is known that a group G is polycyclic if and only if G is realizable as a solvable subgroup of \(\text{GL}(n, \mathbb{Z})\), cf. [6], section III.A.5.

The full extent of the class of amenable groups admitting a controlled Følner sequence is unknown. An interesting problem would be to determine exactly which solvable groups with finite Hirsch number belong to this class or at least have property \(H_{FD}\).
(To recall, let \( G \) be a solvable group with derived series \( G > G^{(1)} > G^{(2)} > \cdots > G^{(n)} > G^{(n+1)} = \{1\} \). The \textit{Hirsch number} is then defined to be the sum of the torsion-free ranks of the abelian groups \( G^{(i)}/G^{(i+1)} \), \( i = 1, \ldots, n \). See section 6.6 in [15] for a discussion on this problem.) We pose the following, more concrete question:

**Question 1.11.** If \( \Gamma \) is a solvable subgroup of \( \text{GL}(n, \mathbb{Z}[1/p]) \), does \( \Gamma \) admit a controlled Følner sequence?

We remark that \( \mathbb{Z}[1/p] \) can probably not be replaced with \( \mathbb{Z}[\tau] \) for some non-algebraic number \( \tau \), since \( \text{GL}(2, \mathbb{Z}[\tau]) \) contains a copy of \( \mathbb{Z}\mathbb{Z} \) which ought not to admit a controlled Følner sequence.

### 1.3. Arrays

The definition of an array was formally introduced in [3] as a means for unifying the concepts of length functions and 1-cocycles into orthogonal representations. Arrays fit naturally into the theory of the large scale geometry of discrete groups, being closely related to in particular Guoliang Yu’s Property A, cf. [13], and Ozawa’s class \( S \), cf. [2]. (See section 1 of [3] for an in-depth discussion of the relationship between arrays with values into the left-regular representation and “negative curvature” in geometric group theory.) We now recall the definition.

**Definition 1.12.** Let \( \pi : G \to \mathcal{O}(\mathcal{H}) \) be an orthogonal representation of a countable discrete group \( G \). A map \( \alpha : G \to \mathcal{H} \) is called an \textit{array} if for every finite subset \( F \subset G \) there exists \( K \geq 0 \) such that

\[
\|\pi_g(\alpha(h)) - \alpha(gh)\| \leq K,
\]

for all \( g \in F, h \in G \) (i.e., \( \alpha \) is \textit{boundedly equivariant}). It is an easy exercise to show that for any array \( \alpha \) on a finitely generated group \( G \) there exists a proper word length function on \( G \), a scalar multiple of which bounds \( \|\alpha(g)\| \) from above.

**Lemma 1.13.** Let \( G \) be a finitely generated group equipped with some proper word length associated to a finite, symmetric, generating set \( S \). If \( \alpha : G \to \mathcal{O}(\mathcal{H}) \) is an array into an orthogonal representation \( \pi \), then \( \bar{\alpha}(g) := \frac{1}{|g|} \alpha(g) \otimes \alpha(g) \), with \( \bar{\alpha}(e) := 0 \), is an array into \( \pi \otimes \pi \).

**Proof.** The proof is very similar to the proof of Proposition 1.4 of [3]: we include it here only for the sake of completeness. First, for every \( g \in G \), we denote by \( B_g := \sup_{h \in \mathbb{G}} \|\alpha(gh) - \pi_g(\alpha(h))\| \) and from the assumptions we have \( B_g < \infty \). Using the triangle inequality together with the bounded equivariance property, for all \( k \in \mathbb{G} \) we have \( \|\alpha(k)\| \leq D|k| \), where \( D = \max_{s \in S} B_s \). This further implies that for every \( \ell \in \mathbb{G} \) we have the following inequality

\[
\sup_{k \neq e, \ell^{-1}} \frac{\|\alpha(k)\|}{|k|} = \sup_{k \neq e, \ell^{-1}} \frac{\|\alpha(k)\|}{|k|} \cdot \frac{|k|}{|\ell|} \leq D(|\ell| + 1).
\]

To check the bounded equivariance for \( \bar{\alpha} \), we fix \( g, h \in G \) where \( h \neq e, g^{-1} \). Applying the triangle inequality and using successively the bounded equivariance property, the
basic inequality \( \|gh| - |h| \leq |g| \), and the inequality (1.2), we have
\[
\|\tilde{\alpha}(gh) - (\pi \otimes \pi)g\tilde{\alpha}(h)\| \leq \frac{\|\alpha(gh) - \pi_g\alpha(h)\otimes\alpha(gh)\|}{|gh|} + \frac{\|\pi_g\alpha(h)\otimes(\alpha(gh) - \pi_g\alpha(h))\|}{|gh|} + \frac{1}{|gh|} - \frac{1}{|h|} \\
\leq B_g \frac{\|\alpha(gh)\|}{|gh|} + B_g \frac{\|\alpha(h)\|}{|gh|} + |gh| - |h| \frac{\|\alpha(h)\|}{|h|} \frac{\|\alpha(h)\|}{|gh|} \\
\leq B_g D(|g| + 2) + D^2 |g|(|g| + 1).
\]
This implies that for every \( g, h \in G \) we have
\[
\|\tilde{\alpha}(gh) - (\pi \otimes \pi)g\tilde{\alpha}(h)\| \leq \max\{B_g D(|g| + 2) + D^2 |g|(|g| + 1), \|\tilde{\alpha}(g^{-1})\|, \|\tilde{\alpha}(g)\|\},
\]
which concludes our proof as the right hand expression depends only on \( g \). \(\square\)

1.4. **Large scale lipschitz maps.** Let \( \mathcal{V} \) be a normed vector space. We will say a map \( f : G \to \mathcal{V} \) is large scale lipschitz if there exists a map \( C : G \to \mathbb{R}_{>0} \) such that for all \( g \in G \), \( \|f(g) - f(gs)\| \leq C(s) \). An array can be viewed in some sense as the formal “adjoint” of some large scale lipschitz map \( f : G \to \mathcal{H} \) with respect to the representation \( \pi \), viz.,

**Proposition 1.14.** If \( \alpha : G \to \mathcal{H} \) is an array associated to \( \pi \), then \( \alpha^*(g) := \pi(g)\alpha(g^{-1}) \) is large scale lipschitz. Conversely, if \( f : G \to \mathcal{H} \) is large scale lipschitz, then \( f^*(g) := \pi(g)f(g^{-1}) \) is an array associated to \( \pi \).

The proof consists of a straightforward check that the respective identities are satisfied.

Given a finite, symmetric generating set \( S \) for \( G \), for any map \( f : G \to \mathbb{R} \) we define the variation function \( \delta f : G \to \mathbb{R}^S \) by \( \delta f(g)(s) := f(g) - f(gs) \).

**Definition 1.15.** A bounded function \( f : G \to \mathbb{R} \) is said to be higson if \( \|\delta f\| \in C_0(G) \), where \( \|\cdot\| \) is the euclidean norm on \( \mathbb{R}^S \).

Note that if \( f : G \to \mathcal{V} \) is a large scale lipschitz map into a normed vector space \( \mathcal{V} \), then \( g \mapsto \frac{1}{|g|} f(g) \) is higson.

We define \( \mathcal{H}^\infty(G) \) to be Banach space of all higson functions. For all \( 1 \leq p < \infty \), we also define \( \mathcal{H}^p(G) \) to be the Banach space of all higson functions \( f \) such that \( \|\delta f\| \in f^p(G) \). Note that the definition of \( \mathcal{H}^p(G) \) for all \( 1 \leq p \leq \infty \) does not depend on the choice of finite generating set.

Our interest in higson functions stems from the following “rigidity” phenomenon which can be observed under the assumption of ergodicity.

**Proposition 1.16.** If \( f \in \mathcal{H}^\infty(G) \) is a function such that
\[
\int f(g^{-1})d\mu_n(g) \to 0
\]
for all Reiter sequences \( (\mu_n) \), then \( f \in C_0(G) \).

**Proof.** Suppose by contradiction that \( f \) does not belong to \( C_0(G) \). Without loss of generality, we would have that there would exist \( c > 0 \) and a sequence \( (g_n) \) of elements in \( G \) such that \( f(g_n) \geq c \) for all \( n \in \mathbb{N} \). Since \( f \in \mathcal{H}^\infty(G) \), for any finite subset \( F \subset G \)
there exists \( n \in \mathbb{N} \) sufficiently large so that \( f(h) \geq c/2 \) for all \( h \in g_n F \). Hence, passing to a subsequence of \( (g_n) \), there is a Folner sequence \( (F_k) \) with the property that \( f(h) \geq c/2 \) for all \( h \in g_n F_k^{-1} \) for all \( k \in \mathbb{N} \). Taking \( \mu_k \) to be the uniform probability measure on the set \( F_k g_n^{-1} \), we would then have constructed a Reiter sequence such that \( \lim \inf_k \int f(g^{-1}) \, d\mu_k(g) \geq c/2 > 0 \), a contradiction. \( \square \)

**Definition 1.17.** Let \( G \) be an amenable group, and let \( f : G \to \mathcal{V} \) be a large scale lipchitz map. We say that \( f \) has **sublinear growth** if \( \limsup |g| \geq n \| f(g) \| / |g| = 0 \). We say that \( f \) has **almost sublinear growth** if \( \int \frac{1}{|g|} \| f(g) \| \, d\mu_n(g) \to 0 \) for all Reiter sequences \( (\mu_n) \).

**Proposition 1.18.** Let \( G \) be an amenable group. Let \( f : G \to \mathcal{H} \) be a large scale lipschitz map in to Hilbert space. If \( f \) is symmetric, i.e., \( \| f(g) \| \equiv \| f(g^{-1}) \| \), then the following statements are equivalent:

1. \( f \) has sublinear growth;
2. \( f \) has almost sublinear growth;
3. \( f_\xi(g) := \langle f(g), \xi \rangle \) has sublinear growth for all \( \xi, \in \mathcal{H} \) and the set \( V := \left\{ \frac{1}{|g|} f(g) \right\}_{g \in G} \) is precompact;
4. \( f_\xi \) has almost sublinear growth for all \( \xi, \in \mathcal{H} \) and \( V \) is precompact.

**Proof.** The implications (1) \( \Rightarrow \) (2), (1) \( \Rightarrow \) (3), and (3) \( \Rightarrow \) (4) are trivial, while the implication (2) \( \Rightarrow \) (1) follows directly by Proposition 1.16 applied to the function \( \frac{1}{|g|} \| f(g) \| \). Therefore, we only need prove the implication (4) \( \Rightarrow \) (1).

To this end, note that if \( V \) is precompact, then for any \( \epsilon > 0 \) we can find a set of vectors \( \xi_1, \ldots, \xi_n \in \mathcal{H} \) so that

\[
\int \left( \frac{1}{|g|} \| f(g) \| \right)^2 \, d\mu(g) \leq C \sum_{i=1}^n \int \frac{1}{|g|} \| \langle f(g), \xi_i \rangle \| \, d\mu(g) + \epsilon
\]

holds for any probability measure \( \mu \), where \( C := \sup_{g \in G} \| f(g) \| / |g| < \infty \). Thus, by almost sublinear growth of each \( f_\xi \) and the Cauchy–Schwarz inequality, we have that \( \int \frac{1}{|g|} \| f(g) \| \, d\mu_n(g) \to 0 \) along any Reiter sequence. By symmetry, the result then obtains by Proposition 1.16. \( \square \)

2. **Main Results**

2.1. **Arrays and the weak mean ergodic theorem.** In this section we present the proof of Theorem A. Though the theorem was stated explicitly for cocycles, the natural context for the theorem is actually the class of arrays. This is essentially due to the fact that there is no well-defined product of cocycles, while such a product exists for the class of arrays. This allows us to exploit the weak mixingness in order to derive the strong form of the theorem in that case.

**Theorem 2.1** (Theorem A). Let \( \pi : G \to \mathcal{O}(\mathcal{H}) \) be an ergodic orthogonal representation of a finitely generated amenable group \( G \), and let \( \alpha : G \to \mathcal{H} \) be an array. Let \( S \) be a finite, symmetric, generating set for \( G \), and let \( | \cdot | \) denote the word length in \( S \). If \( (\mu_n)_{n \in \mathbb{N}} \) is a Reiter
sequence for \( G \), then

\[
\int \frac{1}{|g|} \alpha(g) \, d\mu_n(g) \to 0
\]

in the weak topology. If \( \pi \) is weakly mixing, then

\[
\int \frac{1}{|g|} |\langle \alpha(g), \xi \rangle| \, d\mu_n(g) \to 0
\]

for all \( \xi \in \mathcal{H} \).

Before we begin the proof, we pause to introduce some convenient notation to be used here as well as in the sequel.

**Notation 2.2.** Let \( \alpha : G \to H \) be an array. We set

\[
\alpha^\flat(g) = \frac{1}{|g|} \alpha(g),
\]

where by convention \( \alpha^\flat(e) = 0 \). \( \mathcal{H} \otimes \mathcal{H} \) will be denoted as \( \tilde{\mathcal{H}} \). The representation \( \pi \otimes \pi : G \to \mathcal{O}(\tilde{\mathcal{H}}) \) will be denoted as \( \tilde{\pi} \). The array \( \tilde{\alpha} : G \to \tilde{\mathcal{H}} \) is defined as

\[
\tilde{\alpha}(g) = \frac{1}{|g|} \alpha(g) \otimes \alpha(g),
\]

where \( \tilde{\alpha}(e) = 0 \) by convention.

**Proof of Theorem 2.1.** The proofs of these formulas are inspired by the standard approach to the (weak) mean ergodic theorem for amenable groups. We begin by proving (2.1). To this end, we fix \( \epsilon > 0 \), \( n \in \mathbb{N} \) and note that there exists a finite subset \( F_n \subset G \) such that

\[
\|\alpha^\flat(gh) - \pi(g)\alpha^\flat(h)\| \leq \epsilon
\]

whenever \( g \in B(n) \) and \( h \in G \setminus F_n \). Let \( \xi \in \mathcal{H} \) be a vector of the form \( \xi = (1 - \pi(g^{-1}))\eta \) for some \( g \in B(n) \), \( \eta \in \mathcal{H} \). We then have that

\[
\left| \int \langle \alpha^\flat(h), \xi \rangle \, d\mu_N[h] \right|
= \left| \int \langle \alpha^\flat(h) - \pi(g)\alpha^\flat(h), \eta \rangle \, d\mu_N[h] \right|
\leq \left| \int \langle \alpha^\flat(gh) - \pi(g)\alpha^\flat(h), \eta \rangle \, d\mu_N[h] \right| + \int |\langle \alpha^\flat(k), \eta \rangle| \, d|\mu_N(g^{-1}k) - \mu_N(k)|
\leq ||\eta|| \int \|\alpha^\flat(gh) - \pi(g)\alpha^\flat(h)\| \, d\mu_N[h] + \sup_k ||\alpha^\flat(k)|| \cdot ||\mu_N - g * \mu_N||_1 \lesssim 2||\eta||\epsilon,
\]

since \( \lim_{N} \mu_N(F_n) = 0 \) and \( ||\alpha^\flat|| \) is bounded. By inspection, the estimate holds for the span \( V := \text{span}\{\xi : \exists g \in G, \eta \in \mathcal{H}(\xi = (1 - \pi(g))\eta)\} \), establishing the theorem in that case. Since \( \int ||\alpha^\flat(g)|| \, dv_n(g) \) is uniformly bounded, the result then extends to the closure of \( V \), which by ergodicity is all of \( \mathcal{H} \). This concludes the proof of (2.1).
For the proof of the second part, formula (2.2), we note that if \( \alpha : G \to H \) is an array for \( \pi \), then \( \tilde{\alpha}(g) \) is an array for \( \tilde{\pi} \) by Lemma 1.13. Applying this, we see that

\[
\left| \int \langle \tilde{\alpha}^\circ(h), \xi \otimes \xi \rangle \, d\mu_N(h) \right| = \int |\langle \alpha^\circ(h), \xi \rangle|^2 \, d\mu_N(h) \to 0
\]

by the proof of (2.1). By the Cauchy–Schwarz inequality, we have that

\[
\int |\langle \alpha^\circ(h), \xi \rangle| \, d\mu_N(h) \leq \left( \int |\langle \alpha^\circ(h), \xi \rangle|^2 \, d\mu_N(h) \right)^{1/2},
\]

and we are done. \( \square \)

In the case the 1-cocycle is proper, there is a sharpening of the above result. The proof is identical to the proof of the previous theorem, using Proposition 1.4 from [3] instead of Lemma 1.13.

**Proposition 2.3.** Let \( \pi : G \to H \) be a weakly mixing orthogonal representation. If \( b : G \to H \) is a proper 1-cocycle, then

\[
\int \frac{1}{\|b(g)\|} |\langle b(g), \xi \rangle| \, d\mu_n(g) \to 0
\]

for all Reiter sequences \( (\mu_n) \).

### 2.2. Theorem B and the mean ergodic theorem

We begin with the main technical theorem in this section, the formulation and proof of which are inspired by Lemma 3.4 in [4].

**Theorem 2.4.** Let \( G \) be a finitely generated, discrete group in the class \( CF \). Let \( b : G \to H \) be a 1-cocycle associated to an orthogonal representation \( \pi \). Assume that

\[
\frac{1}{|g|} |\langle b(g), \xi \rangle| \in C_0(G)
\]

for all \( \xi \in H \) (i.e., \( b \) is weakly sublinear). Let \( (F_n)_{n \in \mathbb{N}} \) be a K-controlled Følner sequence. Let \( \nu_n \) be the uniform measure on \( F_n \). There exists a sequence \( (\mu_k) \) of finitely supported measures \( \mu_k \in \text{co}(\nu_n : n \in \mathbb{N}) \) such that \( \xi_k := \int b(g) \, d\mu_k(g) \) form a sequence of almost fixed points for the affine action \( G \curvearrowright T^1 H \) associated to \( b \).

**Proof.** Fix a word length \( | \cdot | \) coming from some finite, symmetric generating set \( S \subset G \). Let \( d_n = \text{diam} F_n \). We set \( F_n(g) = g F_n \Delta F_n \subset \partial F_n \subset B(d_n + 1) \), for each \( g \in S \). Let \( \eta_n = \int b(g) \, d\nu_n(g) \).

For all \( n \in \mathbb{N} \) we have the *a priori* estimate

\[
\|T_g(\eta_n) - \eta_n\| = \left\| \int b(h) \, d\nu_n(g^{-1} h) - \int b(h) \, d\nu_n(h) \right\|
\]

\[
\leq \frac{1}{|F_n|} \int_{F_n(g)} \|b(h)\| \, dh
\]

\[
\leq C(d_n + 1) \cdot \frac{|\partial F_n|}{|F_n|} \leq 2CK,
\]

where \( C = \sup_{s \in S} \|b(s)\| \).
Therefore, we need only show that for any $\xi \in \mathcal{H}$ and $g \in S$, we have that
\begin{equation}
\lim_{n} |\langle T_{g}(\eta_{n}) - \eta_{n}, \xi \rangle| = 0.
\end{equation}
Indeed, the sequence $(T_{g}(\eta_{n}) - \eta_{n})_{n \in \mathbb{N}}$ would then have 0 as a weak limit point for any $g \in S$. Thus, the sequence $\bigoplus_{g \in S} (T_{g}(\eta_{n}) - \eta_{n}) \subset \bigoplus_{g \in S} \mathcal{H}$ converges weakly to 0, so that by passing to the convex hull, the theorem obtains.

We now fix $\xi \in \mathcal{H}$. By assumption 2.7 for every $\epsilon > 0$ there exists a finite set $E_{\epsilon} \subset G$ such that
\begin{equation}
|\langle b(g), \xi \rangle| < \epsilon|g|
\end{equation}
for all $g \in G \setminus E_{\epsilon}$.

Since $\lim_{n} \nu_{n}(E_{\epsilon}) = 0$, we have that for any $g \in S$,
\begin{align}
|\langle T_{g}(\eta_{n}) - \eta_{n}, \xi \rangle| &= \left| \int b(h) \, dv_{n}(g^{-1}h) - \int \langle b(h), \xi \rangle \, dv_{n}(h) \right| \\
&= \frac{1}{|F_{n}|} \left| \int_{F_{n}(g)} \langle b(h), \xi \rangle \, dh \right| \\
&\leq \frac{1}{|F_{n}|} \int_{F_{n}(g)} |\langle b(h), \xi \rangle| \, dh \\
&\lesssim 2\epsilon(d_{n} + 1) \frac{\partial F_{n}}{|F_{n}|} \leq 4K\epsilon,
\end{align}
and we are done. \qed

Hence, if $G$ is a group in the class $\mathcal{CF}$, then for any 1-cocycle $b : G \to \mathcal{H}$ associated to some orthogonal representation, $b$ is not almost inner only if there exists a vector $\xi \in \mathcal{H}$ so that $f(g) := \langle b(g), \xi \rangle$ does not have sublinear growth.

**Question 2.5.** By Proposition 3.1 in [4] we know that any almost inner 1-cocycle has sublinear growth. For a general amenable group, is it the case that any weakly sublinear 1-cocycle is in fact (strongly) sublinear?

**Theorem 2.6** (Theorem B). Let $G$ be finitely generated group in the class $\mathcal{CF}$. Let $\pi : G \to \mathfrak{O}(\mathcal{H})$ be an orthogonal representation, and let $b : G \to \mathcal{H}$ be a 1-cocycle associated to $\pi$. Suppose that
\begin{equation}
\int \frac{1}{|g|} (b(g^{-1}), \xi) \, d\mu_{n}(g) \to 0
\end{equation}
for all $\xi \in \mathcal{H}$ and all Reiter sequences $(\mu_{n})$. Then the affine action $G \curvearrowright^{T} \mathcal{H}$ associated to $b$ admits a sequence of almost fixed points.

**Proof.** The proof follows directly from Proposition 1.16 combined with Theorem 2.4. \qed

**Definition 2.7.** Let $G$ be a finitely generated group and let $\mu$ be a probability measure on $G$. A function $u : G \to \mathcal{V}$ into a vector space is said to be $\mu$-**harmonic** if
\begin{equation}
u(g) = \int u(gs) \, d\mu(s)
\end{equation}
for all $g \in G$. 
Let \( \mu \) be a probability measure with finite second moment, i.e., \( \int |g|^2 d\mu(g) < \infty \).
We know, cf. Theorem 6.1 in [14] and Theorem 6.1 in [5], that every group \( G \) without property (T) of Kazhdan admits at least one \( \mu \)-harmonic 1-cocycle for some (irreducible) representation.

**Proposition 2.8.** Let \( G \) be a group in the class \( CF \), \( \pi : G \to \mathcal{O}(\mathcal{H}) \) be an orthogonal representation, and \( b : G \to \mathcal{H} \) be a \( \mu \)-harmonic 1-cocycle with \( \mu \) having finite second moment. Let \( \pi_0 \) be the restriction of \( \pi \) to the (invariant) subspace \( \mathcal{H}_0 \) spanned by the image of \( b \). If \( V := \left\{ \frac{1}{|g|} b(g) \right\} \) is precompact, then \( \pi_0 \) is compact.

**Proof.** Suppose by contradiction that \( \mathcal{H}_0 \) contains an non-zero invariant subspace \( \mathcal{K} \) on which the restriction of \( \pi \) is weakly mixing. Setting \( b' : G \to \mathcal{K} \) defined by \( b'(g) := \text{proj}_\mathcal{K} b(g) \), we then would have that \( b' \) is a harmonic 1-cocycle into a weakly mixing representation such that \( V' := \text{proj}_\mathcal{K} V = \left\{ \frac{1}{|g|} b'(g) \right\} \) is precompact. Proposition 1.18 then implies that \( b'(g) \) has sublinear growth; hence, by Theorem 2.4 it is almost inner. However, no non-zero harmonic 1-cocycle into an orthogonal representation can be almost inner, cf. Theorem 6.1 in [5]. Therefore, \( b' \equiv 0 \) which contradicts the fact that the span of \( V' \) is dense in \( \mathcal{K} \). Thus, we have shown that \( \pi_0 \) contains no non-zero, weakly mixing subrepresentation which implies that \( \pi_0 \) is compact. \( \square \)

### 3. Final Remarks and Open Problems

#### 3.1. On the growth of harmonic functions.

**Conjecture 3.1.** Let \( G \) be an amenable group, and let \( \mu \) be a probability measure with finite second moment and trivial Poisson boundary (cf. [11]). If \( u : G \to \mathbb{R} \) is a lipschitz \( \mu \)-harmonic function such that

\[
\int \frac{1}{|g|} |u(g)| d\mu_n(g) \to 0
\]

for all Reiter sequences \((\mu_n)\), then \( u \) has sublinear growth.

Notice that if \( u \) is harmonic, then \( |u| \) is subharmonic, i.e., \( |u|(g) \leq \frac{1}{|S|} \sum_{s \in S} |u|(gs) \) for all \( g \in G \), so the conjecture may be posed in this generality.

**Definition 3.2.** A group \( G \) has property \( H_{FD} \) of Shalom if any affine action \( G \ltimes^T \mathcal{H} \) on Hilbert space with weakly mixing linear part admits almost fixed points.

**Proposition 3.3.** Suppose that Conjecture 3.1 holds for a group \( G \) which admits a controlled Følner sequence. Then \( G \) has property \( H_{FD} \).

The proof is an easy consequence of Theorem 2.4.

**Remark 3.4.** A result of Hebisch and Saloff-Coste, Theorem 6.1 in [9], shows that there a no non-constant real-valued harmonic functions of sublinear growth on a group of polynomial growth. It would be interesting if a variant of this argument could be made to apply to harmonic functions of almost sublinear growth.
A stated in Proposition 1.10, the known classes of amenable groups which admit controlled Følner sequences are: groups of (weak) polynomial growth; polycyclic groups, i.e., lattices in solvable Lie groups; wreath products \( D \wr Z \) with \( D \) finite; semi-direct products \( \mathbb{Z}_m \rtimes \mathbb{Z}_n \), with \( m, n \) coprime and \( |mn| \geq 2 \). The latter three classes are the work of Tessera, Theorem 11 in [17]. Each of these classes is known have property \( H_{F_D} \) by the seminal work of Shalom, Theorems 1.13 and 1.14 in [15]. The advantage to the approach suggested here is that it may potentially offer a broad, conceptually unified way of deriving property \( H_{F_D} \).

It follows from an argument given in [15] (Theorem 6.7.2) that the conjecture implies Gromov’s theorem. The approach presented here is conceptually quite different from other approaches to Gromov’s theorem [1, 8, 10, 12, 16].

We also point out that another consequence of Conjecture 3.1 would be that there are solvable groups, e.g., \( \mathbb{Z} \wr \mathbb{Z} \), which do not admit controlled Følner sequences, cf. Theorem 1.15 in [15].

3.2. On the space \( \mathcal{H}^p(G) \). As a last remark, we develop another line of thought towards establishing the mean ergodic theorem for affine actions of groups of polynomial growth independently of Gromov’s theorem.

**Theorem 3.5.** Let \( G \) be a one-ended group with a finite, symmetric, generating set \( S \). If \( f \in \mathcal{H}^1(G) \), then \( f \in C_0(G) + C1 \).

**Proof.** For every \( \epsilon > 0 \), choose \( r \) sufficiently large so that

\[
K_r := \sum_{g \in G \setminus B_r} \sum_{s \in S} |f(g) - f(gs)| < \epsilon.
\]

Since \( G \) is one-ended \( G \setminus B_r \) contains exactly one infinite connected component \( U_r \). For every pair of elements \( g, h \in U_r \) there exists a sequence of elements \( x_1, \ldots, x_n \) in \( U_r \) such that \( g = x_1, h = x_n \) and \( x_i \) is in \( S \) for all \( i = 1, \ldots, n - 1 \). Hence it follows by the triangle inequality that

\[
|f(g) - f(h)| \leq K_r
\]

which proves the claim.

In fact, in the case that \( f \) is positive, a slightly weaker condition will suffice:

**Theorem 3.6.** For \( f \in \ell^\infty(G) \) and \( F \in \ell^\infty(G \times S) \), let \( f \cdot F(g, s) := f(g)F(g, s) \). Let \( G \) be a one-ended group with a finite, symmetric, generating set \( S \). Suppose that \( f \in \ell^\infty(G) \), \( f \geq 0 \). If \( \|f \cdot \partial f\| \in \ell^1(G) \), then \( f \in C_0(G) + C1 \).

Note that since \( f \geq 0 \), we have that \( \|f \cdot \partial f\| \leq \|\partial(f^2)\| \); hence, by the boundedness of \( f \) and standard estimation techniques it follows that if \( f^p \in \mathcal{H}^1(G) \) for any \( 1 \leq p < \infty \), then it holds that \( f \in C_0(G) + C1 \).

**Proof.** Let \( \Gamma = \Gamma(G, S) \) be the Cayley graph of \( G \) with respect to the generating set \( S \). We produce a new graph \( \Gamma' \) by subdividing each edge in \( \Gamma \) so the the vertex set of \( \Gamma' \) may be identified with \( V(\Gamma) \cup E(\Gamma) \) and \( \Gamma' \) is again one-ended. We define a map \( f' : V(\Gamma') \to \mathbb{R} \) by \( f'(g) := f(g)^2 \) for \( g \in V(\Gamma) \) and \( f'(e) := f(g)f(gs) \) for \( e = (g, gs) \in E(\Gamma) \). Now by
assumptions we can see that $\|\partial f\| \in \ell^1(V(G'))$, so by Theorem 3.5, we can conclude that $f^2 \in C_0(G) + C1$. By the positivity of $f$, this suffices to show the result. □

**Proposition 3.7.** Let $G$ be a one-ended group in the class $\mathcal{CF}$. If $b$ is a $1$-cocycle associated to an ergodic representation $\pi : G \to \mathcal{O}(H)$ such that
\[
\frac{1}{|g|} |(b(g), \xi)| \in \mathcal{H}^1(G).
\]
for all $\xi \in H$, then $b$ is almost inner. The same holds assuming that $\pi$ is weakly mixing and
\[
\frac{1}{|g|} |(b(g), \xi)| \in \mathcal{H}^1(G).
\]

**Proof.** The proof follows directly from Theorem 3.5 and Theorem 2.4. □

**Proposition 3.8.** If $G$ is a group of polynomial growth, then there exists $1 \leq p < \infty$ such that for any $1$-cocycle $b : G \to H$ we have that
\[
\frac{1}{|g|} (b(g), \xi) \in \mathcal{H}^p(G)
\]
for all $\xi \in H$.

**Proof.** Fixing a finite generating set $S$, we have that $\sum_{s \in S} \|\frac{1}{|g|} b(g) - \frac{1}{|gs|} b(gs)\| \ll \frac{1}{|g|}$.

Choosing an integer $p$ such that $R^{p-2} \gg |B(R)|$, we have that
\[
\sum_{g \in G} \sum_{s \in S} \left( \frac{1}{|g|} b(g) - \frac{1}{|gs|} b(gs) \right)^p \ll \sum_{g \in G} |g|^{-p} \ll \sum_{n \in \mathbb{N}} n^{-2}
\]
from which the result easily obtains. □

We formulate the following conjecture as an alternative approach to Gromov’s theorem.

**Conjecture 3.9.** If $G$ is a one-ended group of polynomial growth, then for any $1 \leq p < \infty$ any positive function $f \in \mathcal{H}^p(G)$ belongs to $C_0(G) + C1$.

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