NUMERICAL SIMULATION OF THE NONLINEAR FRACTIONAL REGULARIZED LONG-WAVE MODEL ARISING IN ION ACOUSTIC PLASMA WAVES

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Abstract. This paper aimed at obtaining the traveling-wave solution of the nonlinear time fractional regularized long-wave equation. In this approach, firstly, the time fractional derivative is accomplished using a finite difference with convergence order $O(\delta t^{2-\alpha})$ for $0 < \alpha < 1$ and the nonlinear term is linearized by a linearization technique. Then, the spatial terms are approximated with the help of the radial basis function (RBF). Numerical stability of the method is analyzed by applying the Von-Neumann linear stability analysis. Three invariant quantities corresponding to mass, momentum, and energy are evaluated for further validation. Numerical results demonstrate the accuracy and validity of the proposed method.

1. Introduction. Nonlinear models play important role in various filed of physics such as field theory, hydrodynamics, and condensed matter physics [19, 1, 4]. Solitary waves are wave pulses or packets, which propagate in nonlinear dispersive media. Due to dynamical balance between the dispersive and nonlinear effects, these waves maintain a stable waveform. The generalized regularized long-wave equation is one of the useful mathematical model to explain the development of the undular bore in numerous science and engineering fields. The classical model is given by:

$$\frac{\partial u(x,t)}{\partial t} + \gamma \frac{\partial u(x,t)}{\partial x} + \epsilon u^p(x,t) \frac{\partial u(x,t)}{\partial x} - \mu \frac{\partial^3 u(x,t)}{\partial x^3 \partial t} = 0,$$

where $p$ is a positive integer, and $\gamma$ and $\mu$ are positive constants, here $\mu$ is called dissipative, that require the boundary conditions $u \to 0$ as $x \to \infty$. Peregrine
[19] first introduced the RLWE in order to describe the propagation of unidirectional weakly dispersive and weakly nonlinear water waves. Benjamin et al. [1] proposed that this equation be applied to a large number of real-world problems in place of the standard Korteweg de Vries equation (KdVE). A crucial role is played by RLWE in investigating nonlinear dispersive waves. These waves have widespread application in various fields such as rotating flows in tubes, pressure waves in mixtures of gas bubbles in liquids, dispersive longitudinal waves in elastic rods, magnetohydrodynamic waves in plasma, and ion acoustic waves in plasma. Bona et al. [4] adopted the integer-ordered formulation of the equation in order to describe the propagation of surface water waves in a channel. It is worth to mention that the main characteristic of the trajectory of the fractional order derivatives is non-local as the memory effect [20]. Many authors derived that fractional differential equations (FDEs) are more suitable than integer order ones, because fractional derivatives describe the memory and hereditary properties of diverse materials and processes [12, 13, 17, 8]. Recently, FDEs have gained much interest in many research areas such as engineering, physics, chemistry, economics, and other branches of science [20, 25, 7, 17, 8, 16]. Consequently, the RLWE related to the derivatives of fractional order generalizes of the RLWE (1) to interpret the water waves. Taking account to the great significance of fractional ordered derivatives, Kumar et al. [12, 13] formulated the time fractional RLWE (TFRLWE) by substituting the first-order time derivative by fractional derivative termed in Caputo sense of order \(0 < \alpha \leq 1\) into the standard RLWE as

\[
\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \gamma \frac{\partial u(x,t)}{\partial x} + \epsilon u^p(x,t) \frac{\partial u(x,t)}{\partial x} - \mu \frac{\partial^3 u(x,t)}{\partial x^2 \partial t} = f(x,t), \quad (x, t) \in \Omega \times (0, T],
\]

with initial condition

\[
u(x,0) = f(x), \quad x \in \Omega,
\]

and the boundary conditions

\[
u(a,t) = g_1(t), \quad \nu(b,t) = g_2(t), \quad t > 0,
\]

where \(\Omega = (a,b)\), the symbol \(T\) is final time, \(f(x,t)\) denotes a source term, and the notation \(\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\) represents the Caputo fractional derivative [20], which can be defined as follows:

\[
\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} 
\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\xi)}{\partial \xi} \frac{1}{(t-\xi)^\alpha} d\xi, & 0 < \alpha < 1, \\
\frac{\partial u(x,t)}{\partial t}, & \alpha = 1.
\end{cases}
\]

The RLWE and TFRLWE have been studied by means of various techniques. Bona and Bryant [3] proved the existence and uniqueness of the solution for the RLWE. The corresponding analytical solution was determined subject to restrictions in the boundary and initial conditions [3]. Gardner et al. [6] utilized least-squares technique with linear space-time finite element (FE) method to approximate the RLWE. Peregrine [19] introduced a finite-difference (FD) scheme with first-order accurate approximation for solving RLWE. Guo and Cao [9] developed a new Fourier pseudospectral method with a restrain operator which is applied to RLWE. Bahardwaj and Shankar [2] developed a numerical approach based on FD scheme using the quintic B-splines. Raslan [21] applied a collocation-based cubic B-spline scheme for approximating the RLWE. Esen and Kutluay [5] implemented the lumped Galerkin
technique based on a quadratic B-spline to solve the RLWE. Dag et al. [22] adopted a quintic B-spline Galerkin FE method to solve the RLWE. Oruç et al. [18] used the Haar wavelet involving FD method to approximate the RLWE. Mokhtari and Mohammadi [14] employed the collocation method using Sinc basis to solve the RLWE. Shahriari et al. [23] proposed a hybrid method adopting the $\theta$-weighted scheme and the Galerkin method based on Alpert multiwavelets. They proved that the method used in this paper is convergent and unconditionally stable. In our knowledge, only a few numerical scheme have been presented for approximating the TFRLWE. Kumar et al. [12] adopted the $q$-homotopy analysis transform method for solving the TFRLWE. Kumar et al. [13] introduced a new fractional extension of RLWE. Also, they proved existence and uniqueness of the solution by applying the fixed-point theorem.

Our goal in this paper is to introduce the numerical approach based on the radial basis function (RBF) for the approximation solution of the nonlinear TFRLWE. The context of the remaining part of this paper has been arranged as follows. Section 2 describes the time-discrete scheme of the governing problem in the time variable via a finite difference technique and implements the RBF collocation methods to obtain a full-discrete scheme in the spatial direction. Section 3 investigates the numerical stability of the proposed methods by employing a linearized stability analysis. Section 4 presents on a test problem for illustrating the effectiveness and accuracy of the scheme when compared with other numerical methods reported in the literature. Section 5 summarizes the main conclusions.

2. Numerical implementation. In this section, we firstly propose a finite difference method for time semi discretization, followed by the RBF based on global and local collocation schemes for spatial discretization to approximate the nonlinear TFRLWE. To introduce the numerical scheme for the solution of Eq. (2), we need some preliminaries. Let us define

\[ t^n = n\tau, \quad n = 0, 1, \ldots, M, \]
\[ x_j = x_0 + jh = a + jh, \quad j = 1, 2, \ldots, N, \]

where $\tau = \frac{T}{N}$ and $h = \frac{b-a}{N}$ denote the step size of time and spatial variables, respectively.

2.1. Time fractional derivative approximation. In this section, we apply the FD scheme to discretize the time fractional derivative term appeared in Eq. (2) by the following approximation:

\[
\frac{\partial^\alpha u(x,t^{n+1})}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^{t^{n+1}} \frac{\partial u(x,\xi)}{\partial \xi} \frac{1}{(t^{n+1} - \xi)^\alpha} d\xi \\
= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \int_{k\tau}^{(k+1)\tau} \frac{\partial u(x,\xi)}{\partial \xi} \frac{1}{(t^{n+1} - \xi)^\alpha} d\xi \\
\approx \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \int_{k\tau}^{(k+1)\tau} \frac{\partial u(x,\xi_k)}{\partial \xi} \frac{1}{(t^{n+1} - \xi)^\alpha} d\xi,
\]

where the first-order time derivative is discretized by

\[
\frac{\partial u(x,\xi_k)}{\partial \xi} = \frac{u(x,t^{k+1}) - u(x,t^k)}{\tau} + O(\tau), \quad \xi_k \in [t^k, t^{k+1}],
\]
and we can conclude that
\[
\frac{\partial^n u(x, t^{n+1})}{\partial t^n} = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \int_{k}^{(k+1)\tau} \frac{u(x, t^{k+1}) - u(x, t^k)}{\tau} \, O(\tau) \frac{1}{(n+1-\xi)^\alpha} \, d\xi
\]
\[
= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \frac{u(x, t^{k+1}) - u(x, t^k)}{\tau} \int_{k}^{(k+1)\tau} \frac{1}{\tau^\alpha} \, dr
\]
\[
= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n} \frac{u(x, t^{n+k}) - u(x, t^{n-k})}{\tau} \left[ (n-k+1)^{1-\alpha} - (n-k)^{1-\alpha} \right]
\]
\[
= \begin{cases} 
    a_\alpha (u^{n+1} - u^n) + a_\alpha \sum_{k=1}^{n} b_k (u^{n+1-k} - u^{n-k}), & n \geq 1, \\
    a_\alpha (u^1 - u^0), & n = 0,
\end{cases} + O(\tau^{2-\alpha}). \quad (5)
\]

where \(a_\alpha = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}, \ b_k = (k+1)^{1-\alpha} - k^{1-\alpha} \ k = 0, 1, \ldots, n.\)

The time derivative of nonlinear TRLWE is discretized by means of the common FD formula, which consists of space derivatives using the \(\theta\)-weighted \((0 \leq \theta \leq 1)\) scheme between two successive times \(n\) and \(n+1\) as
\[
\frac{\partial^n u(x, t^{n+1})}{\partial t^n} + \theta (\gamma u_x + \tau \mu u_{xx})^n + (1-\theta) (\gamma u_x + \tau \mu u_{xx})^n - \frac{\mu}{\tau} (u_x^{n+1} - u_x^n) = f^{n+1},
\]
where \(u(x, t^{n+1}) = u^{n+1}, f(x, t^{n+1}) = f^{n+1} \) and \(t^{n+1} = t^n + \tau.\)

**Lemma 2.1.** (See [14].) The nonlinear term \((u^p u_x)^n\) in the above relation can be linearized by using Taylor’s series expansion
\[
(u^p u_x)^n \approx (u^p)^n u_x^{n+1} + p(u^{p-1}) u_x^n u_x^{n+1} - p(u^p) u_x^n + O(\tau^2), \text{ for } p = 1, 2, \ldots . \]  
(7)

Substituting from Eqs. (5) and (7) into Eq. (6), we obtain the following recursive formulas
\[
\left( a_\alpha \tau + \theta \tau \epsilon (u^0)^{p-1} u_x^0 \right) u_x^1 + \theta \tau \left( 1 + \epsilon (u^0)^p \right) u_x^1 = a_\alpha \tau u_x^0 + \tau \left[ \epsilon \left( (p+1)\theta - 1 \right) (u^0)^p - (1-\theta) \right] u_x^0 - \mu u_{xx}^0 + \tau f^0, \quad (8)
\]
and
\[
\left( a_\alpha \tau + \theta \tau \epsilon (u^n)^{p-1} u_x^n \right) u_x^{n+1} + \theta \tau \left( 1 + \epsilon (u^n)^p \right) u_x^{n+1} = a_\alpha \tau u_x^n + \tau \left[ \epsilon \left( (p+1)\theta - 1 \right) (u^n)^p - (1-\theta) \right] u_x^n - \mu u_{xx}^n + a_\alpha \tau \sum_{k=1}^{n} b_k (u^{n+1-k} - u^{n-k}) + \tau f^{n+1}, \quad (9)
\]
at \(n = 0\) and \(n \geq 1\), respectively. The time semi-discretization equation for \(\theta = 1\) leads to backward Euler method and for \(\theta = \frac{1}{2}\) leads to the famous Crank-Nicolson
method. The spatial discretization scheme will be accomplished using the meshless methods based on the RBF in the next two subsection in details.

2.2. Spatial discretization: Global RBF collocation method. The collocation methods based on RBF are gaining popularity in the geosciences due to their competitive accuracy, functionality on unstructured meshes, and natural extension into higher dimensions. The RBF interpolation method uses linear combinations of translates of one function \( \phi \) of a single real variable. The approximation solution \( u(\mathbf{x}, t^n) \) at a point of interest \( \mathbf{x}_i \) is expanded in the following form,

\[
u(\mathbf{x}_i, t^n) = u^n_i \simeq \sum_{j=1}^{N} \lambda^n_j \phi(r_{ij}), \tag{10}\]

where \( \{\lambda^n_j\} \) are unknown coefficients of the \( n^{th} \) time layer, \( \phi \) is a radial basis function, and \( r_{ij} = |\mathbf{x}_i - \mathbf{x}_j| \). Globally supported RBFs are generally given as follows:

- Generalized Multiquadric (GMQ) \((c^2 + r^2)^{\beta}\),
- Inverse Multiquadric (IMQ) \(\frac{1}{\sqrt{c^2 + r^2}}\),
- Inverse Quadratic (IQ) \((c^2 + r^2)^{-1}\),
- Multiquadric (MQ) \(\sqrt{c^2 + r^2}\),

where constant \( c \) is commonly known as the RBF shape parameter. It is used to control the shape of the functions and is determined empirically for each radial basis function. The proper selection of the shape parameter is a major role in function approximation by RBF and has always interested scientists. The unknown coefficient vector \( \lambda^n_j : j = 1, \ldots, N \) will be obtained by the collocation method. The system (10) can be put in a matrix form as below:

\[
u^n = A \lambda^n. \tag{11}\]

One can split the matrix \( A \) into two matrices, namely \( A_b \) (matrix-associated boundary) and \( A_d \) (matrix-associated internal) which correspond to two boundary points and \( N - 2 \) interior points as follows:

\[
A = A_d + A_b,
\]

in which

\[
A_d = [\phi(r_{ij}) : 2 \leq i \leq N - 1, 1 \leq j \leq N \text{ and } 0 \text{ elsewhere}],
A_b = [\phi(r_{ij}) : i = 1, N, 1 \leq j \leq N \text{ and } 0 \text{ elsewhere}].
\]

By putting Eq. (10) into Eq. (19), we can transcribe the following equations for the interior points of the domain set \([a, b]\),

\[
\begin{align*}
(a_\alpha \tau + \theta \tau \epsilon p & ) \left( \sum_{j=1}^{N} \lambda^n_j \phi_j(x_i) \right)^p - \sum_{j=1}^{N} \lambda^n_j \phi_j'(x_i) \left( \sum_{j=1}^{N} \lambda^n_j \phi_j(x_i) \right)^{p-1} \\
+ \left( 1 + \theta \tau \epsilon \left( \sum_{j=1}^{N} \lambda^n_j \phi_j(x_i) \right)^p \right) \sum_{j=1}^{N} \lambda^n_j \phi_j'(x_i) - \mu \sum_{j=1}^{N} \lambda^n_j \phi_j''(x_i)
\end{align*}
\]
where $\phi$ O. NIKAN, S.M. MOLAVI-ARABSHAI AND H. JAFARI

\[
\begin{align*}
\text{G} & \in \text{matrix form, we have the following algebraic system of equations:} \\
& \text{since all nodes in the domain are required. Combining Eqs. (12) and (13) in a} \\
& \text{matrix form, we have the following algebraic system of equations:} \\
& \text{The expressions (12) and (13) represent the global RBF (GRBF) approximation} \\
& \text{to every element of the} \ i \\n& \text{to obtaining the values of the unknown vector} \ \lambda \\
& \text{Select the values} \ \lambda \\
& \text{Generate} \\
& \text{Step 1:} \\
& \text{algorithm works in the following manner:} \\
& \text{Find the initial solution} \\
& \text{Calculate the approximate solution} \\
& \text{Step 4:} \\
& \text{The coefficients} \ \lambda \\
& \text{Step 5:} \\
& \text{Calculate the approximate solution} \ u^{n+1} \ at \ the \ successive \ time \ steps \ from} \\
& \text{Step 4 and Eq. (11).}
\end{align*}
\]

$$
\begin{align*}
& = a_\alpha \tau \sum_{j=1}^{N} \lambda_j^n \phi_j(x_i) + \tau \left[ \left( (p + 1)\theta - 1 \right) \sum_{j=1}^{N} \lambda_j^n \phi_j(x_i) \right]^p - (1 - \theta) \sum_{j=1}^{N} \lambda_j^n \phi_j(x_i) \\
& - \mu \sum_{j=1}^{N} \lambda_j^n \phi_j''(x_i) - a_\alpha \tau \sum_{k=1}^{n} b_k (u^{n+1-k} - u^{n-k}) + \tau f^{n+1},
\end{align*}
$$

\[
\text{where} \ \phi' (r_{ij}) = \frac{d}{dx} \phi(|x-x_j|) |_{x=x_i}, \ \phi'' (r_{ij}) = \frac{d^2}{dx^2} \phi(|x-x_j|) |_{x=x_i}, \ \ i = 2, 3, \ldots, N-1.
\]

Taking into account Eqs. (3) and (10), we have the following equation for the boundary points,

\[
\begin{align*}
& \sum_{j=1}^{N} \lambda_j^{n+1} \phi_j(x_1) = g_1(t^{n+1}), \\
& \sum_{j=1}^{N} \lambda_j^{n+1} \phi_j(x_N) = g_2(t^{n+1}).
\end{align*}
\]

The expressions (12) and (13) represent the global RBF (GRBF) approximation since all nodes in the domain are required. Combining Eqs. (12) and (13) in a matrix form, we have the following algebraic system of equations:

\[
\begin{align*}
& \left[ a_\alpha \tau A_d + A_b - \mu C + \theta \tau (B + (\epsilon(D + E))) \right] \lambda^{n+1} = \left[ a_\alpha \tau A_d - \mu C \\
& + \tau \{ \epsilon((p + 1)\theta - 1)E - (1 - \theta)B \} \right] \lambda^{n} + G^{n+1},
\end{align*}
\]

in which

\[
\begin{align*}
& u^n_x = B\lambda^n, \ D = p(u^n)^{p-1} * u^n_x * A_d, \ E = (u^n)^p * B, \\
& B = [\phi_j (x_i) : i = 2, \ldots, N-1, j = 1, \ldots, N \ \text{and 0 elsewhere}]_{N \times N}, \\
& C = [\phi_j'' (x_i) : i = 2, \ldots, N-1, j = 1, \ldots, N \ \text{and 0 elsewhere}]_{N \times N}, \\
& G^{n+1} = G_1^{n+1} + G_2^{n+1} + F^{n+1},
\end{align*}
\]

where

\[
\begin{align*}
& G_1^{n+1} = [g_1(t^{n+1}), 0, \ldots, 0, g_2(t^{n+1})]^T, \\
& G_2^{n+1} = \left\{-a_\alpha \tau \sum_{k=1}^{n} b_k (u^{n+1-k} - u^{n-k}) \right\}^T, \\
& F^{n+1} = [0, \tau f^{n+1}_1, \ldots, \tau f^{n+1}_{N-1}, 0].
\end{align*}
\]

The symbol "*" means that the $i$th component of the vector $u^n$ is multiplied to every element of the $i$th row of matrix. The relation (14) demonstrates a system of $N$ linear equations in $N$ unknown parameters $\lambda_j$. The approximate solution $u^{n+1}$ will be achieved from Eq. (10) at any node in the domain $[a, b]$ posterior to obtaining the values of the unknown vector $\lambda_j, j = 1, 2, \ldots, N$ at each time step. The results of this section can be obtained in the following algorithm. The algorithm works in the following manner:

Step 1: Generate $N$ collocation nodes on the bounded domain $[a, b]$. 
Step 2: Select the values $\delta t$ and $\theta$ such that $(0 \leq \theta \leq 1)$. 
Step 3: Find the initial solution $u^0$ from Eq. (3) and then employ Eq. (11) to calculate $\lambda^n = A^{-1}u^n$. 
Step 4: The coefficients $\lambda^n_{j+1}$ are computed from Eq. (14). 
Step 5: Calculate the approximate solution $u^{n+1}$ at the successive time steps from Step 4 and Eq. (11).
2.3. Spatial discretization: The RBF-FD collocation method. This local RBF technique can be viewed as a generalized form of the traditional finite difference technique; hence, it is also known as the RBF-FD technique [24, 15]. The weights in the FD technique are calculated by means of polynomial interpolation, whereas they are calculated by enforcing an RBF interpolant through a grid point and a number of neighboring points in the RBF-FD technique. In the GRBF method, an ill-conditioned and large linear system is created that can lead to uncertain results. On the other hand, the local RB-FD method for each point (center) uses only a limited number of points, instead of applying the entire points. The linear system resulting from the RBF-FD generates sparse system matrices and causes to some savings in the computational time. Therefore, the RBF-FD is able to deal with the ill-conditioning of the global method. Since the global version of RBF has the disadvantage of ill-conditioning, one can use the RBF-FD instead [24].

For the convenience of marking, we suppose that $S^t = \{x^{(i)}_1, \ldots, x^{(i)}_{N_I}\} \subseteq \Xi$ be the local support domain (stencil), and every point $x^{(i)}_k$ corresponds to a point $x_i$ in the collection points set $\Xi = \{x_1, \ldots, x_N\}$ that be an uniform partition of $[a, b]$ where $x_1 = a$ and $x_N = b$.

The distributed nodes in the computational domain

![Figure 1. The distributed nodes in the computational domain with a stencil.](image)

The first and second derivatives of $u^k(x)$ in time semi-discretization can be evaluated approximately using only function values in the stencil of $x_i$, i.e. as follows:

$$\left. \frac{\partial u^k(x)}{\partial x} \right|_{x=x_i} = \sum_{j=1}^{N_I} w_{i,j}^{x,1} u^k(x^i_j), \quad (15)$$

$$\left. \frac{\partial^2 u^k(x)}{\partial x^2} \right|_{x=x_i} = \sum_{j=1}^{N_I} w_{i,j}^{x,2} u^k(x^i_j), \quad (16)$$

where $N_I$ is the number of nodes in the stencil of $i$th node, $w_{i,j}^{x,1}$ and $w_{i,j}^{x,2}$ are the RBF-FD coefficients corresponding to the first and second order derivatives, respectively. It is worth to point out that a RBF-FD stencil of size $N_I$ requires the $N_I - 1$ nearest neighbors (see Figure 1).

By replacing relations (15) and (16) in Eq. (19) and substituting collocation nodes for time steps $k = n, n + 1$ yields in the following time semi-discretization equation
Writing Eq. (17) together with the boundary conditions in a matrix form, we obtain

$$Au^{n+1} = Bu^n + F,$$

(18)

where the elements of $A = [a_{ij}]_{1 \leq i,j \leq N}$, $B = [b_{ij}]_{1 \leq i,j \leq N}$ and $F = [F_i^{n+1}]_{1 \leq i \leq N}$ are

$$a_{ii} = \left( a_\alpha \tau + \theta \tau \epsilon (u^n_{i})^{p-1} \sum_{j=1}^{N_j} w_{ij}^{x,1} u^n_{i,j} \right) u^{n+1}_{i,i} + \theta \tau \left( 1 + \epsilon (u^n_{i})^{p} \right) \sum_{j=1}^{N_j} w_{ij}^{x,1} u^n_{i,j},$$

$$a_{ij} = \theta \tau \left( 1 + \epsilon (u^n_{i})^{p} \right) w_{ij}^{x,1} - \mu w_{ij}^{x,2}, \quad i \neq j,$$

$$b_{ii} = a_\alpha \tau u^n_{i} + \tau \left[ \epsilon (p+1)\theta - 1 \right] (u^n_{i})^{p} \sum_{j=1}^{N_j} w_{ij}^{x,1} u^n_{i,j} - \mu w_{ij}^{x,2},$$

$$b_{ij} = \theta \tau \left[ \epsilon (p+1)\theta - 1 \right] (u^n_{i})^{p} \sum_{j=1}^{N_j} w_{ij}^{x,1} u^n_{i,j} - \mu w_{ij}^{x,2}, \quad i \neq j,$$

$$F_i^{n+1} = \left\{ -a_\alpha \tau \sum_{k=1}^{n} b_{kn}(k)(u^n_{i,k+1} - u^n_{i,k-1}) + \tau f_i^{n+1} \right\}.$$

If boundary node $x_k$ is in the support domain of node $x_i$, then $F$ is updated as follows:

$$F_i^{n+1} = -a_\alpha \tau \sum_{k=1}^{n} b_{kn}(k)(u^n_{i,k+1} - u^n_{i,k-1}) + \tau f_i^{n+1} - \theta \tau \left( 1 + \epsilon (u^n_{i})^{p} \right) w_{ij}^{x,1} + \mu w_{ij}^{x,2}.$$

3. Numerical stability analysis of the method. The purpose of the current section is to describe the stability of the proposed numerical solution. In the nonlinear convective term, we must first freeze one variable locally, then use the standard Fourier analysis to determine the condition for stability to be imposed on the time step $\tau$. By applying the proposed method for the locally constant equation in the case $f \equiv 0$, we have

$$\left( a_\alpha \tau + \theta \tau \epsilon A^{p-1} B \right) u^{n+1} + \theta \tau \left( 1 + \epsilon B^p \right) u^{n+1}_{xx} - \mu u^{n+1}_{xx} = a_\alpha \tau u^n + \tau \left[ \epsilon (p+1)\theta - 1 \right] B^p - \left( 1 - \theta \right) u^n_x.$$
- \mu u^n_{xx} - a_\alpha \tau \sum_{k=1}^{n} b_k (u^{n+1-k} - u^{n-k}),
\tag{19}
\end{align}

where \(A = u^n, B = u^n_x\).

In view of the Von Neumann’s method for each \(j\) by taking \(u^n_j = \xi^n e^{j\varphi_j}\) and substituting in Eq.\(19\), it yields that,

\[
\begin{align*}
(a_\alpha \tau + \theta \tau \epsilon p A^{p-1} B) \xi^{n+1} e^{j\varphi_j} + \theta \tau \left(1 + \epsilon B^p\right) \xi^{n+1} \xi \varphi e^{j\varphi_j} - \mu \xi^{n+1} \varphi^2 e^{j\varphi_j} \\
= a_\alpha \tau \xi^n e^{j\varphi_j} + \tau \left[ \epsilon \left(p + 1\right) - 1 \right] B^p - (1 - \theta) \xi^n \varphi e^{j\varphi_j} \\
- \mu \xi^{n+1} \varphi^2 e^{j\varphi_j} - a_\alpha \tau \sum_{k=1}^{n} b_k \left(\xi^{n+1-k} e^{j\varphi_j} - \xi^{n-k} e^{j\varphi_j}\right),
\end{align*}
\tag{20}
\]

where \(\ell\) denotes the imaginary unit and \(\varphi\) is real. After simplifying we obtain,

\[
\xi = \frac{\alpha_1 + \ell \beta_1}{\alpha_2 + \ell \beta_2},
\tag{21}
\]

where

\[
\begin{align*}
\alpha_1 &= \left(a_\alpha \tau + \theta \tau \epsilon p A^{p-1} B\right) + \mu \eta^2, \\
\beta_1 &= \theta \tau \left(1 + \epsilon B^p\right) \varphi, \\
\alpha_2 &= a_\alpha \tau + \mu \eta^2 - a_\alpha \tau \sum_{k=1}^{n} b_k \left(\xi^{1-k} - \xi^{-k}\right), \\
\beta_2 &= \tau \left[ \epsilon \left(p + 1\right) - 1 \right] B^p - (1 - \theta) \eta.
\end{align*}
\]

Now, we have

\[
\begin{align*}
|\xi|^2 = \frac{\alpha_1^2 + \beta_1^2}{\alpha_2^2 + \beta_2^2} = \frac{N}{D} = \frac{N}{N + (D - N)},
\tag{22}
\end{align*}
\]

In the aforesaid relation if \((D - N) \geq 0\) then it concludes that \(|\xi| \leq 1\). Rearranging the terms and simplifying, we get

\[
D - N \geq 0.
\tag{23}
\]

Therefore \(|\xi| \leq 1\) and the necessary condition for the stability is provided and we can say that our method is convergent.

In what follows, the stability of \((17)\) is analyzed by the Von-Neumann analysis \cite{11}. It is well known that stability analysis is only applied to partial differential equations with constant coefficients. For the stencils with three uniform nodes \(x = \{x_{m-1}, x_m, x_{m+1}\}, f \equiv 0\), and \(\theta = \frac{1}{2}\), we obtain

\[
\begin{align*}
(a_\alpha \tau + \theta \tau \epsilon p A^{p-1} B) \left(u^{n+1}_{m-1} + u^{n+1}_m + u^{n+1}_{m+1}\right) + \tau \left(1 + \epsilon A^p\right) \left(u^{x,1}_{m-1} u^{n+1}_{m-1} + u^{x,1}_m u^{n+1}_m + u^{x,1}_{m+1} u^{n+1}_{m+1}\right) \\
+ u^{x,1}_{m-1} u^{n+1}_{m+1}) - 2 \mu (u^{x,2}_{m-1} u^{n+1}_{m-1} + u^{x,2}_m u^{n+1}_m + u^{x,2}_{m+1} u^{n+1}_{m+1}) \\
= \tau \left[ \epsilon \left(p + 1\right) - 2 \right] \omega^p - 1 - \left(u^{x,1}_{m-1} u^{n}_{m-1} + u^{x,1}_m u^{n}_m + u^{x,1}_{m+1} u^{n}_{m+1}\right)
\end{align*}
\]
Suppose that the solutions of (24) to be as below:

\[ u^n(x, m = m, m + 1) \]

where \( \ell \) is the imaginary unit and \( \varphi \) is real. Firstly, replacing the Fourier mode (25) into the recurrence Eq. (24), it yields that

\[
\left( a_\alpha \tau + \theta \tau \omega^p A^p - 1 \right) \varepsilon^{n+1} (\varepsilon^{-(m-1)\varphi} + \varepsilon^{\ell m \varphi} + \varepsilon^{\ell (m+1) \varphi}) \\
+ \tau(1 + \varepsilon^A) \varepsilon^{n+1} (w_{m-1}^{x,1} e^{-\ell (m-1) \varphi} + w_m^{x,1} e^{\ell m \varphi} e^{\ell (m+1) \varphi}) \\
- 2\mu \varepsilon^{n+1} (w_{m-1}^{x,2} e^{-\ell (m-1) \varphi} + w_m^{x,2} e^{\ell m \varphi} + w_{m+1}^{x,2} e^{\ell (m+1) \varphi}) \\
= \varepsilon \left[ \varepsilon ((p + 1 - 2) \omega^p - 1) \right] \varepsilon^n (w_{m-1}^{x,1} e^{-\ell (m-1) \varphi} + w_m^{x,1} e^{\ell m \varphi} + w_{m+1}^{x,1} e^{\ell (m+1) \varphi}) \\
- 2\mu \varepsilon^n (w_{m-1}^{x,2} e^{-\ell (m-1) \varphi} + w_m^{x,2} e^{\ell m \varphi} + w_{m+1}^{x,2} e^{\ell (m+1) \varphi}) \\
- 2a_\alpha \tau \sum_{k=0}^{n} a_\alpha b_k (\varepsilon^{n-k+1} - \varepsilon^{n-k}) (e^{-\ell (m-1) \varphi} + e^{\ell m \varphi} + e^{\ell (m+1) \varphi}).
\]

Next, let \( \varepsilon^{n+1} = \zeta^n \) and suppose that \( \zeta = \zeta(\varphi) \) is independent of time. Then we can write:

\[
\left( a_\alpha \tau + \theta \tau \omega^p A^p - 1 \right) \zeta (e^{-\ell \varphi} + 1 + e^{\ell \varphi}) + \tau(1 + \varepsilon^A) \zeta \left( w_{m-1}^{x,1} e^{-\ell \varphi} + w_m^{x,1} \\
+ w_{m+1}^{x,1} e^{\ell \varphi} \right) - 2\mu \zeta (w_{m-1}^{x,2} e^{-\ell \varphi} + w_m^{x,2} + w_{m+1}^{x,2} e^{\ell \varphi}) \\
= \varepsilon \left[ \varepsilon ((p + 1 - 2) \omega^p - 1) \right] \left( w_{m-1}^{x,1} e^{-\ell \varphi} + w_m^{x,1} + w_{m+1}^{x,1} e^{\ell \varphi} \right) \\
- 2\mu \left( w_{m-1}^{x,2} e^{-\ell \varphi} + w_m^{x,2} + w_{m+1}^{x,2} e^{\ell \varphi} \right) \\
- 2a_\alpha \tau \sum_{k=0}^{n} a_\alpha b_k (\zeta^{k+1} - \zeta^{k}) (e^{-\ell \varphi} + 1 + e^{\ell \varphi}).
\]

After some algebraic manipulation, let us represent Eq. (26) as follows:

\[
|\zeta| = \left| \frac{X_1 + \ell X_2}{Y_1 + \ell Y_2} \right|,
\]

in which

\[
X_1 = \varepsilon \left[ \varepsilon ((p + 1 - 2) \omega^p - 1) \right] \left( w_{m-1}^{x,1} \cos(\varphi) + w_m^{x,1} + w_{m+1}^{x,1} \cos(\varphi) \right) \\
- 2\mu \left( w_{m-1}^{x,2} \cos(\varphi) + w_m^{x,2} + w_{m+1}^{x,2} \cos(\varphi) \right) \\
- 2a_\alpha \tau \sum_{k=0}^{n} a_\alpha b_k (\zeta^{k+1} - \zeta^{k}) (2 \cos(\varphi) + 1),
\]

\[
Y_1 = 1.
\]
\[ X_2 = \tau \left[ \epsilon \left( (p + 1) - 2 \right) A^p - 1 \right] \]

\[ \left( -w^{x,1}_{m-1} \sin(\varphi) + w^{x,1}_{m+1} \sin(\varphi) \right) - 2\mu \left( -w^{x,2}_{m-1} \sin(\varphi) + w^{x,2}_{m+1} \sin(\varphi) \right), \]

\[ Y_1 = (a_n \tau + \theta \tau \mu A^{p-1} B) \left( 2 \cos(\varphi) + 1 \right) + \tau (1 + \epsilon B^p) \left( w^{x,1}_{m-1} \cos(\varphi) + w^{x,1}_{m+1} \cos(\varphi) \right) \]

\[ + w^{x,1}_{m+1} \cos(\varphi) - 2\mu \left( w^{x,2}_{m-1} \cos(\varphi) + w^{x,2}_{m+1} \cos(\varphi) \right), \]

\[ Y_2 = (\tau (1 + \epsilon B^p) \left( -w^{x,1}_{m-1} \sin(\varphi) + w^{x,1}_{m+1} \sin(\varphi) \right) \]

\[ - 2\mu \left( -w^{x,2}_{m-1} \sin(\varphi) + w^{x,2}_{m+1} \sin(\varphi) \right). \]

Therefore, we have

\[ |\zeta|^2 = \frac{X_2^2 + X_2^2}{Y_1^2 + Y_2^2}. \] (28)

If condition \( |\zeta| \leq 1 \) is satisfied, then we can conclude the RBF-FD is unconditionally stable.

4. **Numerical results.** This section presents a numerical example to demonstrate the efficiency and the validation of the proposed numerical method when used to approximate numerically the TFRLWE. To illustrate the accuracy of method, we compute the following error norms:

\[ L_{\infty} = \max_{1 \leq j \leq N-1} |u(x_j, T) - U(x_j, T)|, \]

\[ L_2 = \left( \sum_{j=1}^{N} (u(x_j, T) - U(x_j, T))^2 \right)^{\frac{1}{2}}. \]

where \( u \) and \( U \) are the exact and approximate solutions, respectively. The conservation property belonging to the TFRLWE will be assessed by calculating quantities corresponding to mass, momentum and energy

\[ I_1 = \int_a^b udx \simeq h \sum_{i=1}^{N} U_i, \]

\[ I_2 = \int_a^b \left( u^2 + \mu \left( \frac{\partial u}{\partial x} \right)^2 \right) dx \simeq h \sum_{i=1}^{N} \left( (U_i)^2 + \mu(U_x)_i^2 \right), \] (29)

\[ I_3 = \int_a^b \left( u^3 + 3(u)^2 \right) dx \simeq h \sum_{i=1}^{N} \left( (U_i)^3 + 3(U_x)^2 \right), \]

respectively. It can be seen from Eq. (29) that integrals are approximated by utilizing the trapezium rule.

Let us consider the nonlinear TFRLWE [12, 13]

\[ \frac{\partial^p u(x,t)}{\partial t^p} + \frac{\partial u(x,t)}{\partial x} + u(x,t) \frac{\partial u(x,t)}{\partial x} - \frac{\partial^3 u(x,t)}{\partial x^2 \partial t} = 0, \] (30)
with the boundary and initial conditions given as follows:

\[ u(x,0) = 3d \text{sech}^2(\sigma x), \]
\[ u(a,t) = u(b,t) = 0, \quad x \geq 0, \]

where \( d \) is an arbitrary constant and \( \sigma = \frac{1}{2} \sqrt{\frac{d}{1+d}} \). When \( \alpha = 1 \), the equation (30) has the following single solitary wave solution

\[ u(x,t) = 3d \text{sech}^2(\sigma x - \omega t + \psi_0), \quad \omega = \frac{1}{2} \sqrt{d(1+d)}, \]

where \( 3d, \sigma, \omega/\sigma, \) and \( \psi_0 \) represent the amplitude, width, velocity, and an arbitrary constant, respectively.

We solve the single solitary wave by the method presented in this paper with several values of \( h, \tau \) and \( c \). Tables 1 and 2 display the errors in \( L_\infty \) and \( L_2 \) norms and the three invariants \( I_1, I_2 \) and \( I_3 \) for the two cases taken at various times. Based on the comprehensive comparisons in these tables, it is observed that the coefficient matrix of RBF-FD collocation scheme is more well-posed than the GRBF collocation scheme and the conservation quantities are almost constant during the simulation when time increases. Table 3 compares the error norms \( L_\infty \) and \( L_2 \) the conservation quantities \( I_1, I_2 \) and \( I_3 \) of present method with the results of the existing methods in the literature. Tables 4 and 5 compare the numerical results of the MQ-RBF method and those from [12] and [13]. Table 6 lists the values of the three invariants for different times throughout the numerical simulation when \( \alpha = 0.5 \).

One can see that the conservation quantities are almost constant during the simulation by increasing time. Figure 2 represents the approximate solution \( u(x,t) \) with respect to time variable \( t \) for various values of \( \alpha \) and \( d \) by the MQ-RBF approach with \( c = 0.55 \). As shown in Figure 2, we see that as \( \alpha \) approaches 1, the approximate solutions of the fractional partial differential equation obtained by the MQ-RBF scheme converge to that of integer-order partial differential equation. From Figure 2, it can be mentioned that the approximate solution \( u(x,t) \) decreases by increasing the value of \( \alpha \) and the approximate solution \( u(x,t) \) increases by increasing the value of \( d \). Figure 3 displays the graphs of the approximate solution \( u(x,t) \) versus spatial direction \( x \) and temporal direction \( t \) for the values of \( \alpha \in \{0.5, 0.75\} \) using the MQ-RBF approach that the characteristics of this figure are consistent with [12, Figures 2 and 3]. Figure 4 represents the plots of the approximate solutions \( u(x,t) \) and their computational errors \( L_\infty \) by the MQ-RBF-FD at \( T = 20 \) by letting amplitudes 0.3 and 0.09.

5. **Conclusion.** The present paper adopted two meshless techniques for numerically solving the TFRLWE using the RBF-FD and GRBF based collocation schemes. The obtained numerical results and the comparisons between them and several other techniques indicate the considerable accuracy of these techniques. The results obtained from the RBF-FD technique are somewhat similar to those obtained from the GRBF technique. The system matrix corresponding to the GRBF technique is ill-conditioned and dense. Contrarily, a well-conditioned and sparse matrix is observed in the RBF technique. Therefore, the quantity of nodes in the RBF-FD technique may be increased to a certain level. Both of these techniques can be applied to high-dimensional problems. It has been shown that the linearized scheme of the proposed approach is unconditionally stable using the linearized stability analysis. Three invariant constants of motion were studied and the obtained results
Figure 2. The behavior of approximate solution for distinct values of $d$ ($\alpha = 0.9$) (left panel) and $\alpha$ ($d = 0.03$) (right panel).

Figure 3. The behavior of approximate solution by letting $d = 0.03$, $\nu = 1$, $h = 0.1$, $\tau = 0.001$, and $c = 1$ for $\alpha \in \{0.5, 0.75\}$. 
demonstrated that the method can be considered as conservative. These meshless techniques can be extended as a beneficial numerical method for solving a broad class of nonlinear wave equations of these forms.

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Table 4. The approximate solution \( u(x, t) \) for \( d = 0.03 \) and \( x = 2 \) by taking \( \tau = 0.0001 \), \( h = 0.1 \) and \( c = 1 \).

\[
\begin{array}{cccc}
T & \alpha = 0.8 \\
\hline \\
& Method of [12] & Method of [13] & Present Method \\
0.01 & 0.08519576412 & 0.07279537400 & 0.08537651205 \\
0.02 & 0.08357213920 & 0.07279537400 & 0.08264519947 \\
0.03 & 0.08214278398 & 0.07128811205 & 0.08181623410 \\
0.04 & 0.08083623113 & 0.07066631588 & 0.08090025798 \\
0.05 & 0.07962032856 & 0.07198119193 & 0.07973865076 \\
0.06 & 0.07847675139 & 0.06956555612 & 0.07781678939 \\
0.07 & 0.07739366105 & 0.06906896549 & 0.07695183111 \\
0.08 & 0.07630277983 & 0.06860131956 & 0.07614922701 \\
0.09 & 0.07590726375 & 0.06815922842 & 0.07520979105 \\
0.10 & 0.07443460092 & 0.06774010956 & 0.07487357774 \\
\end{array}
\]

Table 5. The approximate solution \( u(x, t) \) for \( d = 0.03 \) and \( x = 2 \) by letting \( \tau = 0.0001 \), \( h = 0.1 \) and \( c = 1 \).

\[
\begin{array}{cccc}
T & \alpha = 0.9 \\
\hline \\
& Method of [12] & Method of [13] & Present Method \\
0.01 & 0.08603543519 & 0.07872750691 & 0.0864043159 \\
0.02 & 0.08484592054 & 0.07786464651 & 0.08489204894 \\
0.03 & 0.08373410924 & 0.07706362124 & 0.08360036158 \\
0.04 & 0.08267550004 & 0.07630395167 & 0.08286912350 \\
0.05 & 0.08165864444 & 0.07557650967 & 0.08227835554 \\
0.06 & 0.08067840291 & 0.07487606663 & 0.08184132412 \\
0.07 & 0.07972566456 & 0.07419921448 & 0.07986788665 \\
0.08 & 0.07880196853 & 0.07354354582 & 0.07819674713 \\
0.09 & 0.07790338981 & 0.07290726375 & 0.07660742243 \\
0.10 & 0.07702808602 & 0.07228897179 & 0.07661724331 \\
\end{array}
\]

Table 6. Invariants for single solitary wave by taking \( d = 0.03 \), \( h = 0.1 \), and \( c = 0.95 \) when \( \alpha = 0.5 \).

\[
\begin{array}{cccc}
T & I_1 & I_2 & I_3 \\
\hline \\
0.00 & 0.197708647779586 & 0.128293299043990 & 0.387537096937904 \\
0.01 & 0.19770939335031 & 0.126849746867847 & 0.38716678533068 \\
0.02 & 0.19770939335031 & 0.126832805997773 & 0.38711399310940 \\
0.03 & 0.197705314048835 & 0.126802946718958 & 0.387058367254051 \\
0.04 & 0.197698761277219 & 0.126780218019371 & 0.387001260827619 \\
0.05 & 0.197690552584066 & 0.126757804752181 & 0.386943215828569 \\
0.06 & 0.197681450241377 & 0.126735633568625 & 0.386884508751169 \\
0.07 & 0.197671429969532 & 0.126713659452037 & 0.386825304512479 \\
0.08 & 0.197660770590404 & 0.126691840813316 & 0.386765710882986 \\
0.09 & 0.197649536587529 & 0.126670152526776 & 0.386705802887836 \\
0.10 & 0.197637994751921 & 0.126648574139578 & 0.386645635247663 \\
\end{array}
\]
Figure 4. The plots of single solitary wave solution and their computational errors by letting $\alpha = 1$, $\tau = 0.01$, $h = 0.125$, and $c = 0.8$ for $d = 0.1$ (up) and $d = 0.03$ (down) at time $T = 20$.

REFERENCES

[1] T. B. Benjamin, J. L. Bona and J. J. Mahony, Model equations for long waves in nonlinear dispersive systems, *Phil. Trans. R. Soc. Lond. A.*, 272 (1972), 47–78.

[2] D. Bhardwaj and R. Shankar, A computational method for regularized long wave equation, *Comput. Math. Appl.*, 40 (2000), 1397–1404.

[3] J. L. Bona and P. J. Bryant, A mathematical model for long waves generated by wavemakers in non-linear dispersive systems, *Proc. Camb. Phil. Soc.*, 73 (1973), 391–405.

[4] J. L. Bona, W. G. Pritchard and L. R. Scott, An evaluation of a model equation for water waves, *Phil. Trans. R. Soc. Lond. A.*, 302 (1981), 457–510.
[5] A. Esen and S. Kutluay, Application of a lumped Galerkin method to the regularized long wave equation, *Appl. Math. Comput.*, **174** (2006), 833–845.

[6] L. R. T. Gardner, G. A. Gardner and A. Dogan, A least-squares finite element scheme for the RLW equation, *Comm. Numer. Meth. Eng.*, **12** (1996), 795–804.

[7] A. Golbabai and O. Nikan, A computational method based on the moving least-squares approach for pricing double barrier options in a time-fractional Black–Scholes model, *Comput. Econ.*, **55** (2020), 119–141.

[8] A. Golbabai, O. Nikan and T. Nikazad, Numerical investigation of the time fractional mobile-immobile advection-dispersion model arising from solute transport in porous media, *Int. J. Appl. Comput. Math.*, **5** (2019), 50, 22 pp.

[9] B. Y. Guo and W. M. Cao, The Fourier pseudospectral method with a restrain operator for the RLW equation, *J. Comput. Phys.*, **74** (1988), 110–126.

[10] A. Houwe, J. Sabi’u, Z. Hammouch and S. Y. Doka, Solitary pulses of the conformable derivative nonlinear differential equation governing wave propagation in low-pass electrical transmission line, *Phys. Scr.*, 2019.

[11] D. Kaya, S. G"ulbahar, A. Yok"us and M. Gülbahar, Solutions of the fractional combined KdV–mKdV equation with collocation method using radial basis function and their geometrical obstructions, *Adv. Difference Equ.*, **2018** (2018), 77, 16 pp.

[12] D. Kumar, J. Singh and D. Baleanu, A new analysis for fractional model of regularized long-wave equation arising in ion acoustic plasma waves, *Math. Methods Appl. Sci.*, **40** (2017), 5642–5653.

[13] D. Kumar, J. Singh, D. Baleanu and Sushila, Analysis of regularized long-wave equation associated with a new fractional operator with Mittag-Leffler type kernel, *Phys. A.*, **492** (2018), 155–167.

[14] R. Mokhtari and M. Mohammadi, Numerical solution of GR/LW equation using Sinc-collocation method, *Comput. Phys. Commun.*, **181** (2010), 1266–1274.

[15] O. Nikan, A. Golbabai and T. Nikazad, Solitary wave solution of the nonlinear KdV-Benjamin-Bona-Mahony-Burgers model via two meshless methods, *Eur. Phys. J. Plus.*, **134** (2019), 367.

[16] O. Nikan, H. Jafari and A. Golbabai, Numerical analysis of the fractional evolution model for heat flow in materials with memory, *Alex. Eng. J.*, **59** (2020), 2627–2637.

[17] O. Nikan, J. A. Machado, A. Golbabai and T. Nikazad, Numerical approach for modeling fractal mobile-immobile transport model in porous and fractured media, *Int. Commun. Heat Mass Transf.*, **111** (2020), 104443.

[18] O. Oruç, F. Bulat and A. Esen, Numerical solutions of regularized long wave equation by Haar wavelet method, *Mediterr. J. Math.*, **13** (2016), 3235–3253.

[19] D. H. Peregrine, Calculations of the development of an undular bore, *J. Fluid. Mech.*, **25** (1966), 321–330.

[20] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[21] K. R. Raslan, A computational method for the regularized long wave (RLW) equation, *Appl. Math. Comput.*, **167** (2005), 1101–1118.

[22] B. Saka, I. Dağ and A. Doğan, Galerkin method for the numerical solution of the RLW equation using quadratic B-splines, *Int. J. Comput. Math.*, **81** (2004), 727–739.

[23] M. Shahriari, B. N. Saray, M. Lakestani and J. Manafian, Numerical treatment of the Benjamin-Bona-Mahony equation using alpert multiwavelets, *Eur. Phys. J. Plus*, **133** (2018), 201.

[24] A. I. Tolyšt’ykh and D. A. Shirobokov, On using radial basis functions in a “finite difference mode” with applications to elasticity problems, *Comput. Mech.*, **33** (2003), 68–79.

[25] N. Valliammal, C. Ravichandran, Z. Hammouch and H. M. Baskonus, A new investigation on fractional-ordered neutral differential systems with state-dependent delay, *Int. J. Nonlin. Sci. Num.*, **20** (2019), 803–809.

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