Quantitative unique continuation for parabolic equations with Neumann boundary conditions

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Abstract. In this paper, we establish a globally quantitative estimate of unique continuation at one time point for solutions of parabolic equations with Neumann boundary conditions in bounded domains. Our proof is mainly based on Carleman commutator estimates and a global frequency function argument, which is motivated from a recent work [5]. As an application, we obtain an observability inequality from measurable sets in time for all solutions of the above equations.

Keywords. Unique continuation, Observability, Neumann boundary condition

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1 Introduction

Let $T > 0$ and let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a bounded connected open set with boundary $\partial \Omega$ of class $C^\infty$. Let the principal part $A(\cdot)$ be a $N \times N$ symmetric matrix with $C^2(\overline{\Omega})$ coefficients and satisfy the uniform ellipticity condition, i.e., there is a constant $\lambda \geq 1$ so that

$$\lambda^{-1}|\xi|^2 \leq A(x)\xi \cdot \xi \leq \lambda|\xi|^2 \text{ for all } x \in \mathbb{R}^N \text{ and } \xi \in \mathbb{R}^N. \quad (1.1)$$

We consider the following linear parabolic equation with the homogeneous Neumann boundary condition:

$$\begin{cases}
\partial_t u - \text{div}(A\nabla u) + B \cdot \nabla u + au = 0 & \text{in } \Omega \times (0,T), \\
A\nabla u \cdot \vec{n} = 0 & \text{on } \partial \Omega \times (0,T), \\
u(\cdot,0) \in L^2(\Omega). 
\end{cases} \quad (1.2)$$

Here, the potentials $a \in L^\infty(\Omega \times (0,T)), B \in (L^\infty(\Omega \times (0,T)))^N$, and $\vec{n}$ is the unit outward normal vector to $\partial \Omega$. According to [4, Theorem 10.9] and [2, Theorem 4.3], the equation (1.2) has a unique solution $u \in L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$. Moreover, for each $\delta \in (0,T)$, $u \in C([\delta,T];H^1(\Omega))$.

Throughout the paper, we denote the usual inner product and norm in $(L^2(\Omega))^k (k \geq 1)$ by $(\cdot, \cdot)$ and $\| \cdot \|$, respectively; $\|a\|_{L^\infty(\Omega \times (0,T))} := \|a\|_{L^\infty(\Omega \times (0,T))}$; $\|B\|_{L^\infty(\Omega \times (0,T))} := \|B\|_{L^\infty(\Omega \times (0,T))}$; $C(\cdot)$ denotes a generic positive constant depending on what are enclosed in the brackets. We shall occasionally use the sum index convention.

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The main theorems of this paper concerning the quantitative estimate of unique continuation, as well as the observability inequality, for all solutions of (1.2) can be stated as follows.

**Theorem 1.1.** Let \( \tilde{\omega} \) be a non-empty open subset in \( \Omega \). Then, there are constants \( K := K(A, \Omega, \tilde{\omega}) > 0 \) and \( \beta := \beta(A, \Omega, \tilde{\omega}) \in (0, 1) \) so that for any \( t \in (0, T] \) and any \( u(\cdot, 0) \in L^2(\Omega) \), the corresponding solution to (1.2) satisfies

\[
\int_{\Omega} |u(x, t)|^2 \, dx \leq e^{Ke^1[1+\frac{1}{2}+\|a\|_\infty^2+\|B\|_\infty^3+\|\bar{a}\|_\infty^2+\|\bar{B}\|_\infty^3]} \left( \int_{\Omega} |u(x, 0)|^2 \, dx \right)^{\beta} \left( \int_{\tilde{\omega}} |u(x, t)|^2 \, dx \right)^{1-\beta}.
\]

**Theorem 1.2.** Let \( E \) be a subset of positive Lebesgue measure in \( (0, T) \), and let \( \omega \) be a non-empty open subset of \( \Omega \). Then there exist two constants \( \tilde{K}_1 := \tilde{K}_1(E, A, \Omega, \omega) > 0 \) and \( \tilde{K}_2 := \tilde{K}_2(A, \Omega, \omega) > 0 \) so that for any \( u(\cdot, 0) \in L^2(\Omega) \), the corresponding solution of (1.2) satisfies

\[
\left( \int_{\Omega} |u(x, T)|^2 \, dx \right)^{1/2} \leq e^{K_2 |E|_\infty + \|B\|_\infty^3 + \|\bar{a}\|_\infty^2 + T(\|a\|_\infty^2 + \|B\|_\infty^3)} \int_{\omega \times E} |u(x, t)| \, dx \, dt.
\]

Several remarks are given in order.

**Remark 1.1.** When the principal part \( A(\cdot) \equiv I \) (the \( N \times N \) order identity matrix) and the potential \( B = 0 \) in the equation (1.2), a similar result to Theorem 1.1 has been recently established in [5]. Here we extend the result to the more general case by adapting the method used in [5].

**Remark 1.2.** In the particular case that \( E = (0, T) \), the constant \( \tilde{K}_1(E, A, \Omega, \omega) \) appearing in Theorem 1.2 could be taken the form of \( \tilde{K}_1(A, \Omega, \omega) / T \). Indeed, the observability constant obtained here has the same optimal dependence on the \( L^\infty \)-norm of potentials \( a \) and \( B \) as those in the nice work [6].

**Remark 1.3.** For the more general case that the principle part \( A(\cdot) \) in the equation (1.2) depend on both the time and spatial variables, similar results could be obtained by using the method of the proof we developed here, as well as the argument in [3, Theorem 4.1].

The interpolation inequality at one time point as stated in Theorem 1.1 is a globally quantitative version of unique continuation for solutions of parabolic equations. The investigate of unique continuation properties for solutions of parabolic-type partial differential equations has a very long history. Giving an exhaustive bibliography on the subject is by far beyond the scope of the present paper, but the reader can consult the survey article [13] and the literature cited there.

It is worth to mention that the same type of interpolation inequality as in Theorem 1.1 has been obtained in [3, 9, 10, 11] for solutions of parabolic equations with the zero Dirichlet boundary condition, by using a (local)-parabolic-type frequency function method developed in [7]. To the best of our knowledge, the arguments in the local nature of these works are not valid in our present case because of different boundary conditions.

The observability inequality on measurable subsets for parabolic equations was previously studied in a large number of publications. When \( E \) is the whole time interval and the observation region \( \omega \) is a non-empty open subset, we refer the reader to the classical monograph [8]. The approach is mainly based on the method of a globally Carleman estimate, which involves with an exponential weight function. When \( E \) is only a subset of positive Lebesgue measure in the time interval and the observation region \( \omega \) is a non-empty open subset, we refer the reader to a series of recent works [1, 10, 11, 12] for the observability inequality for parabolic-type evolution systems.
We would like to stress that the observability estimate from measurable sets in the time variable established as in Theorem 1.2 has several applications in control theory. In particular, it implies bang-bang properties of minimal norm and minimal time optimal control problems (see, for instance, [10]).

The first effort of this paper is to prove a quantitative estimate of unique continuation for parabolic equations by using a frequency function method, which is borrowed from [5]. Secondly, we utilize a telescoping method to prove an observability inequality for the same equations.

The rest of this paper is organized as follows. In Section 2, we show several auxiliary lemmas, whose proofs are provided in Appendix. Sections 3 and 4 prove Theorems 1.1 and 1.2, respectively.

2 Preliminaries

First of all, we give two standard energy estimates for solutions of (1.2). For the sake of completeness we provide their detailed proofs in the appendix.

Lemma 2.1. For all \( t \in [0, T] \), the solution \( u \) of (1.2) satisfies
\[
\int_{\Omega} |u(x,t)|^2 dx + \lambda^{-1} \int_{0}^{t} \int_{\Omega} |\nabla u|^2 dx ds \leq e^{t(2\|a\|_{\infty} + \lambda\|B\|_{\infty})^2} \int_{\Omega} |u(x,0)|^2 dx. \tag{2.1}
\]

Lemma 2.2. For all \( t \in (0, T] \), the solution \( u \) of (1.2) satisfies
\[
\int_{\Omega} |\nabla u(x,t)|^2 dx \leq 2\lambda^3 t e^{t(2\|a\|_{\infty} + 2\lambda\|B\|_{\infty})} \int_{\Omega} |u(x,0)|^2 dx. \tag{2.2}
\]

The next result is concerned with some identities linked to the Carleman commutator.

Proposition 2.3. Let
\[
\Phi(x,t) := \frac{s\varphi(x)}{\Gamma(t)}, \quad s > 0, \quad \Gamma(t) := T - t + h, \quad h > 0, \quad \varphi \in C^\infty(\Omega).
\]
For any \( f \in H^2(\Omega) \), let
\[
\begin{align*}
A_\varphi f &:= -A\nabla \Phi \cdot \nabla f - \frac{1}{2} \text{div}(A\nabla \Phi)f, \\
S_\varphi f &:= -\text{div}(A\nabla f) - \eta f, \quad \text{where} \quad \eta := \frac{1}{2} \partial_t \Phi + \frac{1}{4} A\nabla \Phi \cdot \nabla \Phi, \\
S'_\varphi f &:= -\partial_t \eta f. \tag{2.3}
\end{align*}
\]

Then, we have
\[
(i)
\int_{\Omega} A_\varphi f f dx = -\frac{1}{2} \int_{\partial\Omega} A\nabla \Phi \cdot \nu f^2 dS;
\]

\[
(ii)
\int_{\Omega} S_\varphi f f dx = \int_{\Omega} A\nabla f \cdot \nabla f dx - \int_{\Omega} \eta f^2 dx - \int_{\partial\Omega} A\nabla f \cdot \nu f dS;
\]

\[
(iii)
\int_{\Omega} S'_\varphi f f dx + 2 \int_{\Omega} S_\varphi f A_\varphi f dx
\]
\[
2 
\int_{\partial \Omega} \left( \nabla f \cdot \vec{n} \right) \left( \nabla \Phi \cdot \nabla f \right) dS = \int_{\partial \Omega} \left( \nabla \Phi \cdot \vec{n} \right) \left( \nabla f \cdot \nabla f \right) dS
+ \int_{\partial \Omega} \left( \nabla f \cdot \vec{n} \right) \text{div} \left( \nabla \Phi \right) f dS + \int_{\partial \Omega} \left( \nabla \Phi \cdot \vec{n} \right) \eta f^2 dS
- 2 \int_{\Omega} A_{ij} \partial_{x_j} f \partial_{x_i} A_{kl} \partial_{x_k} f \partial_{x_l} \Phi d\xi
- 2 \int_{\Omega} A_{ij} \partial_{x_j} f \partial_{x_i} A_{kl} \partial_{x_k} f \partial_{x_l} \Phi d\xi
- 2 \int_{\Omega} A_{ij} \partial_{x_j} f \partial_{x_i} A_{kl} \partial_{x_k} f \partial_{x_l} \Phi d\xi
- \frac{2}{\Omega} \int_{\Omega} \left( \eta + \frac{1}{4} A_{ij} \partial_{x_j} \phi \partial_{x_i} A_{kl} \partial_{x_k} \phi \partial_{x_l} \Phi + \frac{8}{3} A_{ij}^2 \phi A_{ij} \phi \cdot \nabla \Phi \right) f^2 d\xi.
\]

Proof. By integrations by parts, we can easily check (i) and (ii). We next turn to the proof of (iii). According to the definitions of \(S_{\varphi f} f\) and \(A_{\varphi f} f\), it is clear that

\[
2 \langle S_{\varphi f} f, A_{\varphi f} f \rangle = \int_{\Omega} \left[ \text{div}(\nabla f) + \eta f \right] \left[ 2 \nabla \Phi \cdot \nabla f + \text{div}(\nabla \Phi) f \right] d\xi. \tag{2.4}
\]

Firstly, by integrations by parts, we have that

\[
2 \int_{\Omega} \text{div}(\nabla f) A_{ij} \partial_{x_i} \phi \text{div} f d\xi
= 2 \int_{\partial \Omega} \left( \nabla f \cdot \vec{n} \right) \left( \nabla \Phi \cdot \nabla f \right) dS - 2 \int_{\Omega} \nabla f \cdot \nabla \left( \nabla \Phi \cdot \nabla f \right) d\xi
\]

and

\[
-2 \int_{\Omega} A_{ij} \partial_{x_j} f \partial_{x_i} A_{kl} \partial_{x_k} f \partial_{x_l} \Phi d\xi
= - \int_{\partial \Omega} \left( \nabla \Phi \cdot \vec{n} \right) \left( \nabla f \cdot \nabla f \right) dS + \int_{\Omega} \partial_{x_1} A_{ij} \partial_{x_j} f A_{kl} \partial_{x_k} f \partial_{x_l} \Phi d\xi
+ \int_{\Omega} \text{div}(\nabla \Phi) A_{ij} \partial_{x_i} \phi \text{div} f d\xi.
\]

The above two equalities imply that

\[
2 \int_{\Omega} \text{div}(\nabla f) A_{ij} \partial_{x_i} \phi \text{div} f d\xi
= 2 \int_{\partial \Omega} \left( \nabla f \cdot \vec{n} \right) \left( \nabla \Phi \cdot \nabla f \right) dS - \int_{\partial \Omega} \left( \nabla \Phi \cdot \vec{n} \right) \left( \nabla f \cdot \nabla f \right) dS
- 2 \int_{\Omega} A_{ij} \partial_{x_j} f \partial_{x_i} A_{kl} \partial_{x_k} f \partial_{x_l} \Phi d\xi + \int_{\Omega} \partial_{x_1} A_{ij} \partial_{x_j} f A_{kl} \partial_{x_k} f \partial_{x_l} \Phi d\xi
- 2 \int_{\Omega} A_{ij} \partial_{x_j} f \partial_{x_i} A_{kl} \partial_{x_k} f \partial_{x_l} \Phi d\xi
- \int_{\Omega} \text{div}(\nabla \Phi) A_{ij} \partial_{x_i} \phi \text{div} f d\xi. \tag{2.5}
\]
Secondly, by integrations by parts, we get that

\[
\begin{align*}
\int_{\Omega} \text{div}(A \nabla f) \text{div}(A \nabla \Phi) f \, dx &= \int_{\partial \Omega} (A \nabla f \cdot \vec{n}) \text{div}(A \nabla \Phi) f \, dS - \int_{\Omega} A \nabla f \cdot \nabla \left[ \text{div}(A \nabla \Phi) \right] f \, dx \\
&= \int_{\partial \Omega} (A \nabla f \cdot \vec{n}) \text{div}(A \nabla \Phi) f \, dS - \int_{\Omega} A \nabla f \cdot \nabla \left[ \text{div}(A \nabla \Phi) \right] f \, dx \\
&\quad - \int_{\Omega} \text{div}(A \nabla \Phi) A \nabla f \cdot \nabla f \, dx,
\end{align*}
\]

and

\[
2 \int_{\Omega} A \nabla \Phi \cdot \nabla \eta f \, dx = \int_{\partial \Omega} (A \nabla \Phi \cdot \vec{n}) \eta f^2 \, dS - \int_{\Omega} A \nabla \Phi \cdot \nabla \eta f^2 \, dx \\
&\quad - \int_{\Omega} \eta \text{div}(A \nabla \Phi) f^2 \, dx.
\]

It follows from the definition of \( S'_{\varphi f} \) and (2.4)-(2.7) that

\[
\begin{align*}
\int_{\Omega} S'_{\varphi f} f \, dx + 2 \int_{\Omega} S_{\varphi f} A_{\varphi f} \, dx &= 2 \int_{\partial \Omega} (A \nabla f \cdot \vec{n}) (A \nabla \Phi \cdot \nabla f) \, dS - \int_{\partial \Omega} (A \nabla \Phi \cdot \vec{n}) (A \nabla f \cdot \nabla f) \, dS \\
&\quad + \int_{\partial \Omega} (A \nabla f \cdot \vec{n}) \text{div}(A \nabla \Phi) \, dS + \int_{\partial \Omega} (A \nabla \Phi \cdot \vec{n}) \eta f^2 \, dS \\
&\quad - 2 A_{ij} \partial_{x_j} f \partial_{x_i} A_{kl} \partial_{x_k} f \partial_{x_k} \Phi \, dx + \int_{\Omega} \partial_{x_i} A_{ij} \partial_{x_j} f A_{kl} \partial_{x_k} f \partial_{x_k} \Phi \, dx \\
&\quad - 2 \int_{\Omega} A \nabla^2 \Phi \nabla f \cdot \nabla f \, dx - \int_{\Omega} A \nabla f \cdot \nabla \left[ \text{div}(A \nabla \Phi) \right] f \, dx \\
&\quad - \int_{\Omega} (\partial_{x_i} + A \nabla \Phi \cdot \nabla f) f^2 \, dx.
\end{align*}
\]

Finally, using \( \partial_t \Phi = \frac{1}{\Gamma} \Phi \) and \( \partial^2_t \Phi = \frac{2}{\Gamma} \partial_t \Phi \), we obtain that

\[
\begin{align*}
&- \partial_{x_i} \eta + A \nabla \eta \cdot \nabla \Phi \\
&= \frac{1}{2} \partial^2_x \Phi - A \nabla \Phi \cdot \nabla \Phi_t - \frac{1}{4} A \nabla \Phi \cdot \nabla \Phi - \frac{1}{4} A_{ij} \partial_{x_j} \Phi \partial_{x_i} A_{kl} \partial_{x_k} \Phi - \frac{1}{2} A \nabla^2 \Phi A \nabla \Phi \cdot \nabla \Phi \\
&= \frac{2}{\Gamma} \partial_t \Phi - \frac{1}{2} A \nabla \Phi \cdot \nabla \Phi - \frac{s}{4 \Gamma} A_{ij} \partial_{x_j} \varphi \partial_{x_i} A_{kl} \partial_{x_k} \Phi + \frac{s}{2 \Gamma} A \nabla^2 \varphi A \nabla \Phi \cdot \nabla \Phi.
\end{align*}
\]

This, along with (2.8), yields (iii). \( \Box \)

The following three results will be useful in the proof of Theorem 1.1.
Proposition 2.4. Let $h > 0, T > 0$ and $F_1(\cdot), F_2(\cdot) \in C([0, T])$. Consider two positive functions $y(\cdot), N(\cdot) \in C^1([0, T])$ such that

\[
\begin{align*}
\left| \frac{1}{2} y'(t) + N(t)y(t) \right| &\leq \left( \frac{1}{2} N(t) + \frac{S_0}{T - t + h} + S_1 \right) y(t) + F_1(t)y(t), \\
N'(t) &\leq \left( \frac{1}{2} N(t) + S_1 \right) N(t) + F_2(t),
\end{align*}
\] (2.9)

where $S_0, S_1 \geq 0$. Then for any $0 \leq t_1 < t_2 < t_3 \leq T$, one has

\[
y(t_2)^{1+M} \leq y(t_3)^{1+M} y(t_1)^{M} e^{D} \left( \frac{T - t_1 + h}{T - t_3 + h} \right)^{3S_0(1+M)}, \quad \forall \ M \geq M_0,
\]

with

\[
M_0 := 3\int_{t_2}^{t_3} \frac{e^{S_1s}F_1}{(T-t_2+s)^2} ds
\]

and

\[
D := 3(M + 1) \left[ \left( \int_{t_1}^{t_2} |F_2| dt + S_1 \right) (t_3 - t_1) + \int_{t_1}^{t_2} |F_2| dt \right].
\]

The proof of Proposition 2.4 is a slight modification for that of Lemma 4.3 in [3]. For the sake of completeness, we give its detailed proof below.

Proof. By the second inequality of (2.9), we have that

\[
\left[ (T - t + h)^{1+S_0} e^{-tS_1} N(t) \right] \leq (T - t + h)^{1+S_0} e^{-tS_1} F_2(t).
\] (2.11)

On one hand, integrating (2.11) over $(t, t_2)$ with $t \in (t_1, t_2)$, we obtain that

\[
\left( \frac{T - t_2 + h}{T - t + h} \right)^{1+S_0} e^{-S_1(t_2-t)} N(t_2) - \int_{t_1}^{t_2} |F_2(s)| ds \leq N(t).
\] (2.12)

It follows from the first inequality of (2.9) and (2.12) that

\[
y'(t) + \left[ \frac{T - t_2 + h}{T - t + h} \right]^{1+S_0} e^{-S_1(t_2-t)} N(t_2) - \frac{2S_0}{T - t + h} - 2S_1 - 2 \int_{t_1}^{t_2} |F_2(s)| ds - 2|F_1(t)| \right] y(t) \leq 0,
\]

where $t \in [t_1, t_2]$. This implies that

\[
\begin{align*}
N(t_2) \int_{t_1}^{t_2} &\left( \frac{T - t_2 + h}{T - t + h} \right)^{1+S_0} e^{-S_1(t_2-t)} dt \\
&\leq \frac{y(t_1)}{y(t_2)} \left( \frac{T - t_1 + h}{T - t_2 + h} \right)^{2S_0} \left( S_1 + \int_{t_1}^{t_2} |F_2(s)| ds + 2 \int_{t_1}^{t_2} |F_1(s)| ds \right).
\end{align*}
\] (2.13)

On the other hand, integrating (2.11) over $(t_2, t)$ with $t \in (t_2, t_3)$, we get that

\[
N(t) \leq e^{S_1(t-t_2)} \left( \frac{T - t_2 + h}{T - t + h} \right)^{1+S_0} \left( N(t_2) + \int_{t_2}^{t_3} |F_2(s)| ds \right).
\] (2.14)
It follows from the first inequality of (2.9) and (2.14) that
\[
y'(t) + 3 \left[ \left( \frac{T - t + h}{T - t + h} \right)^{1 + S_0} e^{S_1(t - t_2)} \left( N(t_2) + \int_{t_2}^{t_3} |F_2(s)|ds \right) + \frac{S_0}{T - t + h} + S_1 + |F_1(t)| \right] y(t) \geq 0,
\]
where \( t \in [t_2, t_3] \). This yields that
\[
y(t_2) \leq e^{3 \left( N(t_2) + \int_{t_2}^{t_3} |F_2(s)|ds \right) \int_{t_2}^{t_3} \left( \frac{T - t + h}{T - t + h} \right)^{1 + S_0} e^{S_1(t - t_2)} dt} \times y(t_3) \frac{\left( T - t_2 + h \right)}{\left( T - t_3 + h \right)}^{3S_0} e^{3S_1(t_3 - t_2)} + 3 \int_{t_2}^{t_3} |F_1(s)|ds.
\]
(2.15)

According to (2.15) and (2.13), it is clear that
\[
y(t_2) \leq y(t_3) \left( \frac{y(t_1)}{y(t_2)} \right) \frac{\left( T - t_1 + h \right)}{\left( T - t_2 + h \right)}^{2S_0} e^{2(t_2 - t_1)} \left( s_1 + \int_{t_1}^{t_2} |F_2(s)|ds \right) + 2 \int_{t_1}^{t_2} |F_1(s)|ds \right) M_0 \times e^{3S_0} 3S_1(t_3 - t_2) + 3 \int_{t_2}^{t_3} |F_1(s)|ds
\]
and
\[
y(t_1) \frac{\left( T - t_1 + h \right)}{\left( T - t_2 + h \right)}^{2S_0} e^{2(t_2 - t_1)} \left( s_1 + \int_{t_1}^{t_2} |F_2(s)|ds \right) + 2 \int_{t_1}^{t_2} |F_1(s)|ds \right) \geq 1,
\]
where \( M_0 \) is given by (2.10). The above two inequalities imply that for any \( M \geq M_0 \),
\[
y(t_2) \leq y(t_3) \left( \frac{y(t_1)}{y(t_2)} \right) \frac{\left( T - t_1 + h \right)}{\left( T - t_2 + h \right)}^{2S_0} e^{2(t_2 - t_1)} \left( s_1 + \int_{t_1}^{t_2} |F_2(s)|ds \right) + 2 \int_{t_1}^{t_2} |F_1(s)|ds \right) M \times e^{3S_0} 3S_1(t_3 - t_2) + 3 \int_{t_2}^{t_3} |F_1(s)|ds.
\]
(2.16)

The result follows from (2.16) immediately.

In summary, we finish the proof of Proposition 2.4. \( \square \)

**Proposition 2.5** ([5]). Let \( \Omega \subset \mathbb{R}^N (N \geq 1) \) be a bounded connected open set with boundary \( \partial \Omega \) of class \( C^\infty \), and let \( \tilde{\omega} \) be a nonempty open subset of \( \Omega \). Then there exists a positive integer \( d \), some points \( p_1, p_2, \cdots, p_d \in \tilde{\omega} \), and \( (\psi_1, \psi_2, \cdots, \psi_d) \in (C^\infty(\Omega))^d \) so that for all \( 1 \leq i \leq d \),

(i) \( \psi_i > 0 \) in \( \Omega \), \( \psi_i = 0 \) on \( \partial \Omega \);

(ii) \( \{ x \in \Omega : |\nabla \psi_i(x)| = 0 \} = \{ p_j : 1 \leq j \leq d \} \);
(iii) $p_i$ is the unique global maximum of $\psi_i$;
(iv) the critical points of $\psi_i$ are nondegenerate;
(v) for any $1 \leq j \leq d$, $\max_{\partial \Omega} \psi_j = \max_{\partial \Omega} \psi_i$.

Let
\[
\begin{align*}
\varphi_{i,1} := & \psi_i - \max_{\partial \Omega} \psi_i, 1 \leq i \leq d, \\
\varphi_{i,2} := & -\psi_i - \max_{\partial \Omega} \psi_i, 1 \leq i \leq d.
\end{align*}
\] (2.17)

For all $1 \leq i \leq d$, it is clear that $\varphi_{i,1} = \varphi_{i,2}$ on $\partial \Omega$, and $\nabla \varphi_{i,1} + \nabla \varphi_{i,2} = 0$ on $\partial \Omega$. (2.18)

**Proposition 2.6 ([5]).** Under the assumptions of Proposition 2.5, there exist positive constants $c_1 := c_1(\Omega, \tilde{\omega}), \ldots, c_6 := c_6(\Omega, \tilde{\omega})$ so that for each $1 \leq i \leq d$, there are subsets $B_i, C_i, \vartheta_i \subset \Omega$ with $B_i \cap C_i = \emptyset (1 \leq i \leq d)$ and $\vartheta_i$ being a neighbourhood of $\partial \Omega$ satisfying

(i) In $D_i$,
\[
c_1 |\nabla \varphi_{i,1}|^2 \leq |\varphi_{i,1}| \leq c_2 |\nabla \varphi_{i,1}|^2,
\]
where $D_i := \Omega \setminus (B_i \cup C_i)$;

(ii) In $B_i$,
\[
c_1 |\nabla \varphi_{i,1}|^2 \leq |\varphi_{i,1}| \leq c_2 |\nabla \varphi_{i,1}|^2;
\]

(iii) In $C_i$,
\[
c_1 |\nabla \varphi_{i,1}|^2 \leq |\varphi_{i,1}|;
\]

(iv) There is $1 \leq j \leq d$ with $j \neq i$ so that
\[
\varphi_{i,1} - \varphi_{j,1} \leq -c_3 \text{ in } C_i;
\]

(v) $c_4 |\nabla \varphi_{i,2}|^2 \leq |\varphi_{i,2}|$ in $\Omega$ and $|\varphi_{i,2}| \leq c_5 |\nabla \varphi_{i,2}|^2$ in $\vartheta_i$;

(vi) $\varphi_{i,2} - \varphi_{i,1} \leq -c_6 \text{ in } \Omega \setminus \vartheta_i$.

### 3 Proof of Theorem 1.1

We start this section by introducing some notations. For each $1 \leq i \leq d$, we set
\[
\begin{align*}
\varphi_i(x) := & \varphi_{i,1}(x), \ x \in \Omega, \\
\varphi_{d+i}(x) := & \varphi_{i,2}(x), \ x \in \Omega
\end{align*}
\]
and
\[
\begin{align*}
\Phi_i(x, t) := & \frac{s}{\Gamma(t)} \varphi_{i,1}(x), \ (x, t) \in \Omega \times [0, T], \\
\Phi_{d+i}(x, t) := & \frac{s}{\Gamma(t)} \varphi_{i,2}(x), \ (x, t) \in \Omega \times [0, T],
\end{align*}
\] (3.1)
with \( s \in (0, 1], \Gamma(t) = T - t + h \) and \( h \in (0, 1] \). For each \( 1 \leq i \leq 2d \), we define

\[
    f_i := we^{\Phi_i/2} \quad \text{and} \quad \eta_i := \frac{1}{2} \partial_t \Phi_i + \frac{1}{4} A \nabla \Phi_i \cdot \nabla \Phi_i, \quad (x, t) \in \Omega \times (0, T),
\]

where \( u \) is the solution to (1.2). By (1.2), we find that

\[
    \partial_t f_i - \text{div}(A \nabla f_i) = -a f_i + \frac{1}{2} f_i B \cdot \nabla \Phi_i + \eta_i f_i - \frac{1}{2} \text{div}(A \nabla \Phi_i) f_i - A \nabla \Phi_i \cdot \nabla f_i - B \cdot \nabla f_i \quad (3.2)
\]

and

\[
    A \nabla f_i \cdot \vec{n} - \frac{1}{2} f_i A \nabla \Phi_i \cdot \vec{n} = 0 \quad \text{on} \quad \partial \Omega \times (0, T). \quad (3.3)
\]

Let \( f := (f_i)_{1 \leq i \leq 2d} \), \( Sf := (S_{\varphi i} f_i)_{1 \leq i \leq 2d} \), \( Af := (A_{\varphi i} f_i)_{1 \leq i \leq 2d} \), \( F := (-a f_i + \frac{1}{2} f_i B \cdot \nabla \Phi_i - B \cdot \nabla f_i)_{1 \leq i \leq 2d} \) and \( S'f := (S'_{\varphi i} f_i)_{1 \leq i \leq 2d} \). Then by (3.2) and (3.3), we have that

\[
    \partial_t f + Sf = Af + F \quad \text{in} \quad \Omega \times (0, T). \quad (3.4)
\]

The proof of Theorem 1.1 will be carried out by the following five stages.

**Stage 1.** We claim that

\[
    \begin{cases}
        \langle Af, f \rangle = 0, \\
        \langle Sf, f \rangle = \sum_{i=1}^{2d} \int_{\Omega} A \nabla f_i \cdot \nabla f_i dx - \sum_{i=1}^{2d} \int_{\Omega} \eta_i f_i^2 dx, \\
        \frac{d}{dt} \langle Sf, f \rangle = 2 \langle Sf, f \rangle + \langle S'f, f \rangle.
    \end{cases} \quad (3.5)
\]

For this purpose, firstly, we observe that

\[
    \langle Af, f \rangle = -\sum_{i=1}^{2d} \int_{\Omega} A \nabla \Phi_i \cdot \nabla f_i f_i dx - \frac{1}{2} \sum_{i=1}^{2d} \int_{\Omega} \text{div}(A \nabla \Phi_i) f_i^2 dx \\
    = -\sum_{i=1}^{2d} \int_{\Omega} A \nabla \Phi_i \cdot \nabla f_i f_i dx + \frac{1}{2} \sum_{i=1}^{2d} \int_{\Omega} A \nabla \Phi_i \cdot \nabla f_i f_i dx - \frac{1}{2} \sum_{i=1}^{2d} \int_{\partial \Omega} A \nabla \Phi_i \cdot \vec{n} f_i^2 dS \\
    = -\frac{1}{2} \sum_{i=1}^{2d} \int_{\Omega} A \nabla \Phi_i \cdot \vec{n} f_i^2 dS,
\]

which, combined with (3.1) and (2.18), indicates that

\[
    \langle Af, f \rangle = -\frac{1}{2} \left( \sum_{i=1}^{d} \int_{\partial \Omega} A \nabla \Phi_i \cdot \vec{n} f_i^2 dS + \sum_{i=1}^{d} \int_{\partial \Omega} A \nabla \Phi_{d+i} \cdot \vec{n} f_{d+i}^2 dS \right)
\]
Finally, it follows from the latter that

$$\langle Sf, f \rangle = \sum_{i=1}^{2d} \int_{\Omega} A \nabla f_i \cdot \nabla f_i \, dx - \sum_{i=1}^{2d} \int_{\Omega} \eta_i f_i^2 \, dx - \sum_{i=1}^{2d} \int_{\partial\Omega} A \nabla f_i \cdot \vec{n} f_i \, dS$$

$$= 2d \int_{\Omega} A \nabla f_i \cdot \nabla f_i \, dx - \sum_{i=1}^{2d} \int_{\Omega} \eta_i f_i^2 \, dx - \sum_{i=1}^{2d} \int_{\partial\Omega} A \nabla f_i \cdot \vec{n} f_i \, dS$$

$$= 2d \int_{\Omega} A \nabla f_i \cdot \nabla f_i \, dx - \sum_{i=1}^{2d} \int_{\Omega} \eta_i f_i^2 \, dx.$$ (3.7)

By (3.7), (3.3) and similar arguments as those to get (3.6), we have that

$$\frac{d}{dt} \langle Sf, f \rangle = 2\langle Sf, f_i \rangle - \sum_{i=1}^{2d} \int_{\Omega} \partial_t \eta_i f_i^2 \, dx + \sum_{i=1}^{2d} \int_{\partial\Omega} A \nabla f_i \cdot \vec{n} \partial_t f_i \, dS$$

$$= 2\langle Sf, f_i \rangle + \langle S' f, f \rangle.$$ (3.8)

**Stage 2.** We show the following two inequalities:

$$\left\{ \begin{array}{l}
\frac{1}{2} \frac{d}{dt} \|f\|^2 + N(t) \|f\|^2 \\ N'(t) \leq \frac{2\langle Sf, Af \rangle}{\|f\|^2} + \frac{\|f\|^2}{2} + \|A\|_\infty \left( \max_{1 \leq i \leq 2d} \|\nabla f_i\|_\infty \|B\|_\infty \right)^2 + 2\|B\|_\infty^2 \frac{\|\nabla f\|_2^2}{\|f\|^2},
\end{array} \right.$$ (3.9)

where $N(t) = \langle Sf, f \rangle/\|f\|^2$.

Indeed, on one hand, according to (3.4) and the first equality in (3.5), we get that

$$\frac{1}{2} \frac{d}{dt} \|f\|^2 + \langle Sf, f \rangle = \langle F, f \rangle.$$ (3.9)
On the other hand, by \( (3.4) \) and \( (3.5) \), we have that
\[
\mathbf{N}'(t)\|\mathbf{f}\|^4 = (2\langle S\mathbf{f}, \mathbf{f}_t \rangle + \langle S'\mathbf{f}, \mathbf{f} \rangle)\|\mathbf{f}\|^2 - \langle S\mathbf{f}, \mathbf{f} \rangle(-2\langle S\mathbf{f}, \mathbf{f} \rangle + 2\langle \mathbf{F}, \mathbf{f} \rangle) \\
= (2\langle S\mathbf{f}, \mathbf{A}\mathbf{f} \rangle + \langle S'\mathbf{f}, \mathbf{f} \rangle)\|\mathbf{f}\|^2 - 2\|S\mathbf{f}\|^2\|\mathbf{f}\|^2 + 2\langle S\mathbf{f}, \mathbf{F} \rangle\|\mathbf{f}\|^2 \\
+ 2\langle S\mathbf{f}, \mathbf{f} \rangle^2 - 2\langle S\mathbf{f}, \mathbf{f} \rangle\langle \mathbf{F}, \mathbf{f} \rangle \\
= (2\langle S\mathbf{f}, \mathbf{A}\mathbf{f} \rangle + \langle S'\mathbf{f}, \mathbf{f} \rangle)\|\mathbf{f}\|^2 - 2\|S\mathbf{f} - \frac{1}{2}\mathbf{F}\|\mathbf{f}\|^2 \\
+ \frac{1}{2}\|\mathbf{F}\|^2\|\mathbf{f}\|^2 + 2\langle S\mathbf{f} - \frac{1}{2}\mathbf{F}, \mathbf{f} \rangle^2 - \frac{1}{2}\langle \mathbf{F}, \mathbf{f} \rangle^2 \\
\leq (2\langle S\mathbf{f}, \mathbf{A}\mathbf{f} \rangle + \langle S'\mathbf{f}, \mathbf{f} \rangle)\|\mathbf{f}\|^2 + \|\mathbf{F}\|^2\|\mathbf{f}\|^2.
\]  
(3.10)

Note that
\[
\|\mathbf{F}\| = \|(-af_i + \frac{1}{2}f_i B \cdot \nabla \Phi_i - B \cdot \nabla f_i)_{1 \leq i \leq 2d}\| \\
\leq \left( \|a\|_\infty + \max_{1 \leq i \leq 2d} \|\nabla \Phi_i\|B\|_\infty \right)\|\mathbf{f}\| + \|B\|\|\nabla \mathbf{f}\|.
\]

This, along with \( (3.9) \) and \( (3.10) \), yields \( (3.8) \).

**Stage 3.** We claim that for \( s := s(A, \Omega, \bar{\omega}) \in (0, 1] \) sufficiently small,
\[
\eta_i \leq 0, \quad \langle S\mathbf{f}, \mathbf{f} \rangle \geq 0, 
\]
and
\[
\langle S'\mathbf{f}, \mathbf{f} \rangle + 2\langle S\mathbf{f}, \mathbf{A}\mathbf{f} \rangle \leq \frac{1 + C_0(A, \Omega, \bar{\omega})}{\Gamma} \langle S\mathbf{f}, \mathbf{f} \rangle + \frac{C(A, \Omega, \bar{\omega})}{h^2}\|\mathbf{f}\|^2,
\]
where \( C_0(A, \Omega, \bar{\omega}) \in (0, 1) \) and \( C(A, \Omega, \bar{\omega}) > 0 \).

To this end, by \( (3.1) \) and \( (2.17) \), we firstly observe that
\[
\eta_i := \frac{1}{2} \partial_i \Phi_i + \frac{1}{4} A \nabla \Phi_i \cdot \nabla \Phi_i = \begin{cases} \frac{s}{4l^2} (-2|\varphi_{i,1}| + sA \nabla \varphi_{i,1} \cdot \nabla \varphi_{i,1}), & 1 \leq i \leq d, \\
\frac{s}{4l^2} (-2|\varphi_{i-d,2}| + sA \nabla \varphi_{i-d,2} \cdot \nabla \varphi_{i-d,2}), & d + 1 \leq i \leq 2d. 
\end{cases}
\]

(3.13)

By \((i), (ii), (iii) and (v)\) in Proposition 2.6, we have that
\[
|\nabla \varphi_{i,1}|^2 \leq \frac{1}{c_1}|\varphi_{i,1}| \text{ and } |\nabla \varphi_{i,2}|^2 \leq \frac{1}{c_4}|\varphi_{i,2}|
\]
for each \( 1 \leq i \leq d \). Thus, for \( s := s(A, \Omega, \bar{\omega}) \in (0, 1] \) sufficiently small, we obtain that
\[
-2|\varphi_{i,1}| + sA \nabla \varphi_{i,1} \cdot \nabla \varphi_{i,1} \\
\leq -2|\varphi_{i,1}| + s\lambda|\nabla \varphi_{i,1}|^2 \leq 0, \quad \forall \ 1 \leq i \leq d
\]
(3.14)

and
\[
-2|\varphi_{i-d,2}| + sA \nabla \varphi_{i-d,2} \cdot \nabla \varphi_{i-d,2} \\
\leq -2|\varphi_{i-d,2}| + s\lambda|\nabla \varphi_{i-d,2}|^2 \leq 0, \quad \forall \ d + 1 \leq i \leq 2d.
\]
(3.15)
Then (3.11) follows from (3.13)-(3.15) and (3.5).

Next, we turn to the proof of (3.12). According to (iii) in Proposition 2.3, it is clear that

\[
\langle S' f, f \rangle + 2 \langle S f, Af \rangle = (I_1) + (I_2) - \frac{2}{\Gamma} \sum_{m=1}^{2d} \int_{\Omega} \left( \eta_m + \frac{1}{4} A \nabla \Phi_m \cdot \nabla \Phi_m + \frac{s}{8} A_{ij} \partial_{x_j} \varphi_m \partial_{x_i} A_{kl} \partial_{x_l} \Phi_m \partial_{x_k} \Phi_m \right) f_m^2 \, dx
\]

\[
- \frac{2}{\Gamma} \sum_{m=1}^{2d} \int_{\Omega} \left( \frac{s}{8} A \nabla^2 \varphi_m A \nabla \Phi_m \cdot \nabla \Phi_m \right) f_m^2 \, dx,
\]

(3.16)

where

\[
(I_1) = -2 \sum_{m=1}^{2d} \int_{\Omega} A_{ij} \partial_{x_j} f_m \partial_{x_i} A_{kl} \partial_{x_l} f_m \partial_{x_k} \Phi_m \, dx + \sum_{m=1}^{2d} \int_{\Omega} \partial_{x_j} A_{ij} \partial_{x_j} f_m A_{kl} \partial_{x_l} f_m \partial_{x_k} \Phi_m \, dx
\]

and

\[
(I_2) = 2 \sum_{m=1}^{2d} \int_{\partial \Omega} (A \nabla f_m \cdot \vec{n})(A \nabla \Phi_m \cdot \nabla f_m) \, dS - \sum_{m=1}^{2d} \int_{\partial \Omega} (A \nabla \Phi_m \cdot \vec{n})(A \nabla f_m \cdot \nabla f_m) \, dS
\]

\[+ \sum_{m=1}^{2d} \int_{\partial \Omega} (A \nabla \Phi_m \cdot \vec{n}) \text{div}(A \nabla \Phi_m) f_m \, dS + \sum_{m=1}^{2d} \int_{\partial \Omega} (A \nabla \Phi_m \cdot \vec{n}) \Phi_m f_m^2 \, dS.
\]

We will estimate the right hand sum in (3.16). This will be done by four steps as follows.

**Step 1.** We show that

\[
(I_1) \leq \frac{s C(A, \Omega, \bar{\omega})}{\Gamma} \sum_{m=1}^{2d} \int_{\Omega} |\nabla f_m|^2 \, dx + \frac{s C(A, \Omega, \bar{\omega})}{h} \int_{\Omega} |f|^2 \, dx.
\]

(3.17)

Indeed, after some simple calculations, we can directly check that

\[
(I_1) \leq \frac{s C(A, \Omega, \bar{\omega})}{\Gamma} \sum_{m=1}^{2d} \int_{\Omega} |\nabla f_m|^2 \, dx + \frac{s C(A, \Omega, \bar{\omega})}{h} \sum_{m=1}^{2d} \int_{\Omega} |\nabla f_m| |f_m| \, dx
\]

\[
\leq \frac{s C(A, \Omega, \bar{\omega})}{\Gamma} \sum_{m=1}^{2d} \int_{\Omega} |\nabla f_m|^2 \, dx + \frac{s C(A, \Omega, \bar{\omega})}{h} \int_{\Omega} |f|^2 \, dx,
\]

which indicates (3.17).

**Step 2.** We claim that

\[
(I_2) \leq \frac{s^2 C(A, \Omega, \bar{\omega})}{h^2} \sum_{m=1}^{d} \int_{\Omega} |f_m|^2 \, dx
\]

\[+ \frac{s C(A, \Omega, \bar{\omega})}{\Gamma} \sum_{m=1}^{2d} \int_{\Omega} |\nabla \Phi_m|^2 |f_m|^2 \, dx + \frac{s C(A, \Omega, \bar{\omega})}{\Gamma} \sum_{m=1}^{2d} \int_{\Omega} |\nabla f_m|^2 \, dx.
\]

(3.18)
Since $\psi_i = 0$ on $\partial \Omega$ ($1 \leq i \leq d$), it holds that $\nabla \psi_i = \partial_n \psi_i \vec{n}$ on $\partial \Omega \times (0, T)$. This yields that

$$\nabla \Phi_m = \partial_n \Phi_m \vec{n} \quad \text{on} \quad \partial \Omega \times (0, T), \quad \forall \ 1 \leq m \leq 2d. \quad (3.19)$$

Firstly, by (3.3), (3.19), (3.1) and (2.18), we have that

$$2 \sum_{m=1}^{2d} \int_{\partial \Omega} (A \nabla f_m \cdot \vec{n}) (A \nabla \Phi_m \cdot \nabla f_m) dS \quad (3.20)$$

$$= 2 \sum_{m=1}^{d} \int_{\partial \Omega} \left( \frac{1}{2} A \nabla \Phi_m \cdot \vec{n} f_m \right)^2 \partial_n \Phi_m dS + 2 \sum_{m=1}^{d} \int_{\partial \Omega} \left( \frac{1}{2} A \nabla \Phi_{d+m} \cdot \vec{n} f_{d+m} \right)^2 \partial_n \Phi_{d+m} dS$$

$$= 2 \sum_{m=1}^{d} \int_{\partial \Omega} \left( \frac{1}{2} A \nabla \Phi_m \cdot \vec{n} f_m \right)^2 \partial_n \Phi_m dS - 2 \sum_{m=1}^{d} \int_{\partial \Omega} \left( \frac{1}{2} A \nabla \Phi_m \cdot \vec{n} f_m \right)^2 \partial_n \Phi_m dS$$

$$= 0.$$

Secondly, it follows from (2.18), (3.1), (3.19) and (1.2) that

$$- \sum_{m=1}^{2d} \int_{\partial \Omega} (A \nabla \Phi_m \cdot \vec{n})(A \nabla f_m \cdot \nabla f_m) dS \quad (3.21)$$

$$= - \sum_{m=1}^{d} \int_{\partial \Omega} (A \nabla \Phi_m \cdot \vec{n})(A \nabla f_m \cdot \nabla f_m - A \nabla f_{d+m} \cdot \nabla f_{d+m}) dS$$

$$= - \sum_{m=1}^{d} \int_{\partial \Omega} (A \nabla \Phi_m \cdot \vec{n}) e^{\Phi_m} \left[ A \nabla u \cdot \nabla u + u A \nabla u \cdot \nabla \Phi_m + \frac{1}{4} u^2 A \nabla \Phi_m \cdot \nabla \Phi_m \right] dS$$

$$+ \sum_{m=1}^{d} \int_{\partial \Omega} (A \nabla \Phi_m \cdot \vec{n}) e^{\Phi_m} \left[ A \nabla u \cdot \nabla u - u A \nabla u \cdot \nabla \Phi_m + \frac{1}{4} u^2 A \nabla \Phi_m \cdot \nabla \Phi_m \right] dS$$

$$= -2 \sum_{m=1}^{d} \int_{\partial \Omega} (A \nabla \Phi_m \cdot \vec{n}) e^{\Phi_m} u A \nabla u \cdot \vec{n} \partial_n \Phi_m dS$$

$$= 0.$$

Thirdly, by (3.1) and (2.18), we have that

$$\sum_{m=1}^{2d} \int_{\partial \Omega} (A \nabla \Phi_m \cdot \vec{n}) \eta_m f_m^2 dS \quad (3.22)$$

$$= \sum_{m=1}^{d} \int_{\partial \Omega} \left( \frac{1}{2} \partial_t \Phi_m + \frac{1}{4} A \nabla \Phi_m \cdot \nabla \Phi_m \right) (A \nabla \Phi_m \cdot \vec{n}) u^2 e^{\Phi_m} dS$$

$$+ \sum_{m=1}^{d} \int_{\partial \Omega} \left( \frac{1}{2} \partial_t \Phi_{m+d} + \frac{1}{4} A \nabla \Phi_{m+d} \cdot \nabla \Phi_{m+d} \right) (A \nabla \Phi_{m+d} \cdot \vec{n}) u^2 e^{\Phi_{m+d}} dS$$

$$= \sum_{m=1}^{d} \int_{\partial \Omega} \left( \frac{1}{2} \partial_t \Phi_m + \frac{1}{4} A \nabla \Phi_m \cdot \nabla \Phi_m \right) (A \nabla \Phi_m \cdot \vec{n}) u^2 e^{\Phi_m} dS$$

$$+ \sum_{m=1}^{d} \int_{\partial \Omega} \left( \frac{1}{2} \partial_t \Phi_{m+d} + \frac{1}{4} A \nabla \Phi_{m+d} \cdot \nabla \Phi_{m+d} \right) (A \nabla \Phi_{m+d} \cdot \vec{n}) u^2 e^{\Phi_{m+d}} dS$$

$$= \sum_{m=1}^{d} \int_{\partial \Omega} \left( \frac{1}{2} \partial_t \Phi_m + \frac{1}{4} A \nabla \Phi_m \cdot \nabla \Phi_m \right) [A \nabla \Phi_m + \nabla \Phi_{m+d}] \cdot \vec{n} u^2 e^{\Phi_m} dS$$

$$= 0.$$
Finally, it follows from (3.1), (1.2) and (2.18) that
\[
\sum_{m=1}^{2d} \int_{\partial \Omega} (\nabla f_m \cdot \vec{n}) \text{div} (A \nabla \Phi_m) f_m dS = \sum_{m=1}^{d} \int_{\partial \Omega} A \nabla \Phi_m \cdot \vec{n} \text{div} (A \nabla \Phi_m) f_m^2 dS
\]
\[
= -\sum_{m=1}^{d} \int_{\Omega} |\text{div} (A \nabla \Phi_m)|^2 f_m^2 dx + \sum_{m=1}^{d} \int_{\Omega} A \nabla \Phi_m \cdot \nabla [\text{div} (A \nabla \Phi_m)] f_m^2 dx
\]
\[
\leq \frac{s^2 C(A, \Omega, \vec{\omega})}{h^2} \sum_{m=1}^{d} \int_{\Omega} f_m^2 dx + \frac{s C(A, \Omega, \vec{\omega})}{\Gamma} \sum_{m=1}^{d} \int_{\Omega} |\nabla \Phi_m|^2 f_m^2 dx + \frac{s C(A, \Omega, \vec{\omega})}{\Gamma} \sum_{m=1}^{d} \int_{\Omega} |\nabla f_m|^2 dx.
\]
This, along with (3.20)-(3.22), yields (3.18).

**Step 3.** We show that for \( s := s(A, \Omega, \vec{\omega}) \in (0, 1] \) sufficiently small,
\[
-\frac{2}{\Gamma} \sum_{m=1}^{2d} \int_{\Omega} \left( \eta_m + \frac{1}{8\lambda} |\nabla \Phi_m|^2 \right) f_m^2 dx \leq \frac{C(A, \Omega, \vec{\omega})}{h^2} ||\mathbf{f}||^2 + \frac{2 - s/(c\lambda)}{\Gamma} \sum_{m=1}^{2d} \int_{\Omega} (-\eta_m) f_m^2 dx,
\]
where \( c := 2(c_2 + c_5), c_2 \) and \( c_5 \) are given by Proposition 2.6.

Firstly, by (i) and (ii) in Proposition 2.6, we have that \( |\varphi_{m,1}| \leq c_2 |\nabla \varphi_{m,1}| \) in \( B_m \cup D_m \) for each \( 1 \leq m \leq d \). This implies that
\[
-|\nabla \Phi_m|^2 = -\frac{s^2}{c^2} |\nabla \varphi_{m,1}|^2 \leq -\frac{s^2}{c^2} |\varphi_{m,1}|
\]
\[
= \frac{2s}{c_2} \left( \frac{s}{2c^2} |\varphi_{m,1}| \right) \leq \frac{2s}{c_2} \eta_m \text{ in } B_m \cup D_m, \forall 1 \leq m \leq d.
\]
Here, we used (3.1) and (2.17). The latter estimate yields that
\[
-\frac{2}{\Gamma} \sum_{m=1}^{d} \int_{B_m \cup D_m} \left( \eta_m + \frac{1}{8\lambda} |\nabla \Phi_m|^2 \right) f_m^2 dx \leq \frac{2 - s/(2c_2 \lambda)}{\Gamma} \sum_{m=1}^{d} \int_{B_m \cup D_m} (-\eta_m) f_m^2 dx.
\]

Secondly, according to (iv) in Proposition 2.6, there is \( c_3 > 0 \) so that for each \( 1 \leq m \leq d \), there exists \( 1 \leq p_m \leq d \) with \( p_m \neq m \) satisfying
\[
\varphi_{m,1} - \varphi_{p_m,1} \leq -c_3 \text{ in } C_m.
\]
Note that
\[
|\eta_m + \frac{1}{8\lambda} |\nabla \Phi_m|^2 | \leq \frac{s C(A, \Omega, \vec{\omega})}{\Gamma^2} \text{ and } f_m^2 = e^{s(\varphi_{m,1} - \varphi_{p_m,1})} f_{p_m}^2, \forall 1 \leq m \leq d.
\]
It follows from (3.25) and (3.26) that
\[ -\frac{2}{\Gamma} \sum_{m=1}^{d} \int_{\Omega} \left( \eta_m + \frac{1}{8\lambda} |\nabla \Phi_m|^2 \right) f_m^2 \, dx \leq \frac{s C(A, \Omega, \overline{w})}{\Gamma^3} \sum_{m=1}^{d} \int_{\Omega} f_m^2 \, dx \]
\[ \leq \frac{s C(A, \Omega, \overline{w})}{\Gamma^3} e^{-c_3 \phi} \sum_{m=1}^{d} \int_{\Omega} f_m^2 \, dx \]
\[ \leq \frac{s C(A, \Omega, \overline{w})}{\Gamma^3} e^{-c_3 \phi} \|f\|^2 \]
\[ \leq \frac{C(A, \Omega, \overline{w})}{\Gamma^3} \|f\|^2. \]

(3.27)

Thirdly, by (v) in Proposition 2.6, we get that
\[ |\varphi_{m,2}| \leq c_5 |\nabla \varphi_{m,2}| \quad \text{in} \quad \partial_m, \quad \forall \ 1 \leq m \leq d. \]

This implies that
\[ -|\nabla \Phi_m|^2 = -|\nabla \Phi_{m-d+d}|^2 = -\frac{s^2}{\Gamma^2} |\nabla \varphi_{m-d,2}|^2 \]
\[ \leq -\frac{s^2}{c_5 \Gamma^2} |\varphi_{m-d,2}| = \frac{2s}{c_5} \left( -\frac{s}{2\Gamma^2} |\varphi_{m-d,2}| \right) \leq \frac{2s}{c_5} \eta_m \quad \text{in} \quad \partial_{m-d}, \quad \forall \ d+1 \leq m \leq 2d, \]

which indicates
\[ -\frac{2}{\Gamma} \sum_{m=d+1}^{2d} \int_{\partial_{m-d}} \left( \eta_m + \frac{1}{8\lambda} |\nabla \Phi_m|^2 \right) f_m^2 \, dx \leq \frac{2 - s/(2c_5 \lambda)}{\Gamma} \sum_{m=d+1}^{2d} \int_{\partial_{m-d}} (-\eta_m) f_m^2 \, dx. \]

(3.28)

Finally, according to (vi) in Proposition 2.6, there is \( c_6 > 0 \) so that for each \( 1 \leq m \leq d, \)
\[ \varphi_{m,2} - \varphi_{m,1} \leq -c_6 \quad \text{in} \quad \Omega \setminus \vartheta_m. \]

(3.29)

Observe that
\[ \left| \eta_{d+m} + \frac{1}{8\lambda} |\nabla \Phi_{d+m}|^2 \right| \leq \frac{s C(A, \Omega, \overline{w})}{\Gamma^2} \quad \text{and} \quad f_{d+m}^2 = e^{s(\varphi_{m,2} - \varphi_{m,1}) \phi} f_m^2, \quad \forall \ 1 \leq m \leq d. \]

(3.30)

It follows from (3.29) and (3.30) that
\[ -\frac{2}{\Gamma} \sum_{m=d+1}^{2d} \int_{\Omega \setminus \vartheta_m} \left( \eta_m + \frac{1}{8\lambda} |\nabla \Phi_m|^2 \right) f_m^2 \, dx \]
\[ \leq \frac{s C(A, \Omega, \overline{w})}{\Gamma^3} e^{-c_6 \phi} \sum_{m=1}^{d} \int_{\Omega \setminus \vartheta_m} f_m^2 \, dx \]
\[ \leq \frac{s C(A, \Omega, \overline{w})}{\Gamma^3} e^{-c_6 \phi} \|f\|^2 \]
\[ \leq \frac{C(A, \Omega, \overline{w})}{\Gamma^3} \|f\|^2. \]

This, along with (3.28), (3.24), (3.27) and the first inequality in (3.11), implies (3.23).
Step 4. We end the proof of (3.12).
It follows from (3.16), (3.17) and (3.18) that
\[\begin{align*}
\langle S'f, f \rangle + 2\langle Sf, Af \rangle & \leq \frac{sC(A, \Omega, \bar{\omega})}{\Gamma} \sum_{m=1}^{2d} \int_{\Omega} |\nabla \Phi_m|^2 f_m^2 \, dx + \frac{sC(A, \Omega, \bar{\omega})}{\Gamma} \sum_{m=1}^{2d} \int_{\Omega} |\nabla f_m|^2 \, dx + \frac{sC(A, \Omega, \bar{\omega})}{h^2} \|f\|^2 \\
& \quad - \frac{2}{\Gamma} \sum_{m=1}^{2d} \int_{\Omega} \left( \eta_m + \frac{1}{4} A \nabla \Phi_m \cdot \nabla \Phi_m + \frac{s}{8} A_i \partial_{x_i} \varphi_m \partial_{x_i} \Phi_m \partial_{x_k} \Phi_m \right) f_m^2 \, dx \\
& \quad - \frac{2}{\Gamma} \sum_{m=1}^{2d} \int_{\Omega} \left( \frac{s}{4} A \nabla^2 \varphi_m A \nabla \Phi_m \cdot \nabla \Phi_m \right) f_m^2 \, dx.
\end{align*}\] (3.31)

Note that
\[\lambda^{-1} \nabla \Phi_m \cdot \nabla \Phi_m \leq A \nabla \Phi_m \cdot \nabla \Phi_m \leq \lambda \nabla \Phi_m \cdot \nabla \Phi_m,
\]
and
\[\frac{8}{s} A_i \partial_{x_i} \varphi_m \partial_{x_i} \Phi_m \partial_{x_k} \Phi_m \leq sC(A, \Omega, \bar{\omega})|\nabla \Phi_m|^2.
\]
Then by (3.31), we have that for \( s := s(A, \Omega, \bar{\omega}) \in (0, 1] \) sufficiently small,
\[\begin{align*}
\langle S'f, f \rangle + 2\langle Sf, Af \rangle & \leq \frac{sC(A, \Omega, \bar{\omega})}{\Gamma} \sum_{m=1}^{2d} \int_{\Omega} |\nabla f_m|^2 \, dx + \frac{C(A, \Omega, \bar{\omega})}{\Gamma} \|f\|^2 \\
& \quad - \frac{2}{\Gamma} \sum_{m=1}^{2d} \int_{\Omega} \left( \eta_m + \frac{1}{4} \lambda |\nabla \Phi_m|^2 \right) f_m^2 \, dx \\
& \quad - \frac{1}{\Gamma} \sum_{m=1}^{2d} \int_{\Omega} \left( \frac{s}{8} A_i \partial_{x_i} \varphi_m \partial_{x_i} \Phi_m \partial_{x_k} \Phi_m \right) f_m^2 \, dx.
\end{align*}\] (3.32)

It follows from (3.32), (3.23) and the second equality in (3.5) that for \( s := s(A, \Omega, \bar{\omega}) \in (0, 1] \) sufficiently small,
\[\begin{align*}
\langle S'f, f \rangle + 2\langle Sf, Af \rangle & \leq \frac{sC(A, \Omega, \bar{\omega})}{\Gamma} \sum_{m=1}^{2d} \int_{\Omega} A \nabla f_m \cdot \nabla f_m \, dx + \frac{C(A, \Omega, \bar{\omega})}{\Gamma} \|f\|^2 \\
& \quad + \frac{2}{\Gamma} \sum_{m=1}^{2d} \int_{\Omega} \left( -\eta_m \right) f_m^2 \, dx \\
& \quad \leq 1 + \frac{C_0(A, \Omega, \bar{\omega})}{\Gamma} \langle Sf, f \rangle + \frac{C(A, \Omega, \bar{\omega})}{\Gamma} \|f\|^2,
\end{align*}\]
where \( C_0(A, \Omega, \bar{\omega}) \in (0, 1) \). Hence, (3.12) holds.

Stage 4. We show that for \( s := s(A, \Omega, \bar{\omega}) \in (0, 1] \) sufficiently small,
\[\|f(col, T - lh)\|^{2(1+M)} \leq \|f(col, T)\|^2 \|f(col, T - 2lh)\|^{2M} e^{(M+1)C(A, \Omega, \bar{\omega}, l)(1+h^2\|u\|_\infty + \|B\|_\infty)} \quad \forall \; M \geq M_l, \] (3.33)
where \( l > 1, 0 < h \leq \min\{T/(4l), 1\} \), and \( M_l \) will be precised later (see (3.37) below).
Indeed, by (3.8), the second equality in (3.5), (1.1), the first inequality in (3.11) and (3.12), we have that for $s = s(A, \Omega, \bar{\omega}) \in (0, 1]$ sufficiently small,
\[
\left| \frac{1}{2} \frac{d}{dt} \| f \|^2 + N(t) \| f \|^2 \right| 
\leq \left( \| a \|_\infty + \max_{1 \leq i \leq 2d} \| \nabla \Phi_i \| \| B \|_\infty + \frac{\lambda}{2} \| B \|_\infty^2 \right) \| f \|^2 + \frac{1}{2\lambda} \| \nabla f \|^2
\] (3.34)
\[
\leq \left( \| a \|_\infty + \max_{1 \leq i \leq 2d} \| \nabla \Phi_i \| \| B \|_\infty + \frac{\lambda}{2} \| B \|_\infty^2 \right) \| f \|^2 + \frac{1}{2} N(t) \| f \|^2
\]
and
\[
N'(t) \leq \left( \frac{2\lambda \| B \|_\infty^2}{h^2} + \frac{1 + C_0(A, \Omega, \bar{\omega})}{T - t + h} \right) N(t) + 2 \left( \| a \|_\infty + \max_{1 \leq i \leq 2d} \| \nabla \Phi_i \| \| B \|_\infty \right)^2 .
\] (3.35)

Let $l > 1$ and $0 < h \leq \min\{T/(4l), 1\}$. According to (3.34), (3.35) and Proposition 2.4, where $t_1 := T - 2lh, t_2 := T - lh, t_3 := T, S_0 := C_0(A, \Omega, \bar{\omega}), S_1 := 2\lambda \| B \|_\infty^2, y(t) := \| f(\cdot, t) \|^2, N(t) := N(t)$,
\[
F_1(t) := \| a \|_\infty + \max_{1 \leq i \leq 2d} \| \nabla \Phi_i \| \| B \|_\infty + \frac{\lambda}{2} \| B \|_\infty^2
\]
and
\[
F_2(t) := \frac{C(A, \Omega, \bar{\omega})}{h^2} + 2 \left( \| a \|_\infty + \max_{1 \leq i \leq 2d} \| \nabla \Phi_i \| \| B \|_\infty \right)^2 ,
\]
it is clear that
\[
\| f(\cdot, T - lh) \|^2(1 + M) \leq \| f(\cdot, T) \|^2 \| f(\cdot, T - 2lh) \|^{2M} K_{l,M}, \quad \forall M \geq M_l .
\] (3.36)

Here,
\[
M_l := 3 \int_{T - 2lh}^{T - lh} \frac{\| \nabla \Phi_i \| \| B \|_\infty}{(T - t + h)^{1 + C_0(A, \Omega, \bar{\omega})}} \, dt .
\] (3.37)
\[
D_{l,M} := 12l^2(M + 1)C(A, \Omega, \bar{\omega}) + 12lh(M + 1) \int_{T - 2lh}^{T} \left( \| a \|_\infty + \max_{1 \leq i \leq 2d} \| \nabla \Phi_i \| \| B \|_\infty \right)^2 \, dt
\] (3.38)
\[
+ 6lh(M + 1) \| a \|_\infty + 3 \| B \|_\infty (M + 1) \int_{T - 2lh}^{T} \left( \max_{1 \leq i \leq 2d} \| \nabla \Phi_i \| \right) dt + 15\lambda h(M + 1) \| B \|_\infty^2 ,
\]
and
\[
K_{l,M} := e^{D_{l,M} + 2l^3 + \frac{3C_0(A, \Omega, \bar{\omega})}{(1 + M)} .}
\] (3.39)

It follows from (3.38) and (3.39) that
\[
D_{l,M} \leq (M + 1)C(A, \Omega, \bar{\omega}, l) \left( 1 + h^2 \| a \|_\infty^2 + \| B \|_\infty^2 \right)
\]
and
\[
K_{l,M} \leq e^{(M + 1)C(A, \Omega, \bar{\omega}, l) \left( 1 + h^2 \| a \|_\infty^2 + \| B \|_\infty^2 \right) .}
\] (3.40)
This, along with (3.36), implies (3.33).

Stage 5. We end the proof of Theorem 1.1.

It will be split into three steps as follows.

Step 1. We show that for \( s := s(A, \Omega, \overline{\omega}) \in (0, 1) \) sufficiently small, there are four constants \( \overline{M} := \overline{M}(A, \Omega, \overline{\omega}) > 0 \), \( \overline{C}_1 := \overline{C}_1(A, \Omega, \overline{\omega}) > 1 \), \( l_0 := l_0(A, \Omega, \overline{\omega}) > 1 \) and \( \mu := \mu(\Omega, \overline{\omega}) > 0 \) so that

\[
\begin{align*}
\left( \int_\Omega |u(x, T)|^2 dx \right)^{1+\overline{M}} & \leq e^{(1+\overline{M})\overline{C}_1[1+\|a\|_{\infty}^{2/3}+\|B\|_{\infty}^{2/3}+T(\|a\|_{\infty}^{2/3}+\|B\|_{\infty}^{2/3})]} \left( \int_\Omega |u(x, 0)|^2 dx \right)^{\overline{M}} \\
& \times \left( e^{\frac{\overline{M}}{x}} \int_\Omega |u(x, T)|^2 dx + e^{-\frac{\overline{M}}{x}} \int_\Omega |u(x, 0)|^2 dx \right),
\end{align*}
\]

where \( 0 < h \leq \min\{T/(4l_0), 1\} \), \( \|a\|_{\infty}^{2/3}h < 1 \) and \( l_0h\|B\|_{\infty}^2 < 1 \).

To this end, firstly, on one hand, by Lemma 2.1, we have that for any \( 0 \leq t_1 \leq t_2 \leq T \),

\[
\|u(\cdot, t_2)\| \leq e^{(t_2-t_1)(\|a\|_{\infty}^{2/3}+\|B\|_{\infty}^2)}\|u(\cdot, t_1)\|.
\]

This yields that

\[
\begin{align*}
\int_\Omega |u(x, T)|^2 dx & \leq e^{2lh(\|a\|_{\infty}^{2/3}+\|B\|_{\infty}^2)} \int_\Omega |u(x, T-lh)|^2 e^{-(1-lh)\varphi^{1.1} e^{-\frac{\overline{M}}{x}} \varphi^{1.1} dx} \int_\Omega f_1(x, T-lh)^2 dx, \\
& \leq e^{T(\|a\|_{\infty}^{2/3}+\|B\|_{\infty}^2)} e^{-\frac{\overline{M}T}{x}} \int_\Omega f_1(x, T-lh)^2 dx,
\end{align*}
\]

where \( l > 1 \), \( 0 < h \leq \min\{T/(4l), 1\} \) and \( C_1 := C_1(\Omega, \overline{\omega}) > 0 \). On the other hand, since \( \varphi_{i,2} \leq \varphi_{i,1} \) in \( \Omega \) for each \( 1 \leq i \leq d \) (see (2.17)), it follows from (3.36) that for \( s := s(A, \Omega, \overline{\omega}) \in (0, 1) \) sufficiently small,

\[
\|f_i(\cdot, T-lh)\|^{2(1+M)} \leq \sum_{i=1}^d \|f_i(\cdot, T)\|^2 \left( \sum_{i=1}^d \|f_i(\cdot, T-lh)\|^2 \right)^M K_{i,M}, \quad \forall M \geq M_i,
\]

where \( l > 1 \) and \( 0 < h \leq \min\{T/(4l), 1\} \). Moreover, noting that \( \Phi_i \leq 0 \) in \( \Omega \times [0, T] \) for each \( 1 \leq i \leq d \) (see (3.1) and (2.17)), by (3.42), we get that

\[
\|f_i(\cdot, T-2lh)\|^{2} \leq e^{2T(\|a\|_{\infty}^{2/3}+\|B\|_{\infty}^2)}\|u(\cdot, 0)\|^{2}.
\]

Secondly, according to Proposition 2.5, there exist two positive constants \( r := r(\Omega, \overline{\omega}) \) and \( \mu := \mu(\Omega, \overline{\omega}) \in (0, 1) \) so that

\[
\omega_{i,r} := \{x : |x-p_i| < r\} \subset \overline{\omega} \quad \text{and} \quad \varphi_{i,1} \leq -\mu \quad \text{in} \quad \Omega \setminus \omega_{i,r}, \quad \forall 1 \leq i \leq d.
\]

These, along with (3.1) and Lemma 2.1, imply that

\[
\begin{align*}
\|f_i(\cdot, T)\|^2 & = \int_{\omega_{i,r}} |u(x, T)|^2 e^{\varphi_{i,1}} dx + \int_{\Omega \setminus \omega_{i,r}} |u(x, T)|^2 e^{\varphi_{i,1}} dx \\
& \leq \int_{\omega_{i,r}} |u(x, T)|^2 dx + e^{-\mu} e^{2T(\|a\|_{\infty}^{2/3}+\|B\|_{\infty}^2)} \int_\Omega |u(x, 0)|^2 dx.
\end{align*}
\]

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for each $1 \leq i \leq d$. Here we used the fact that $\varphi_{i,1} \leq 0$ in $\Omega$ for each $1 \leq i \leq d$ (see (2.17)).

Finally, it follows from (3.43)-(3.46) that for $s := s(A, \Omega, \bar{\omega}) \in (0,1]$ sufficiently small,

$$
\left( \int_{\Omega} |u(x,T)|^2 dx \right)^{1+M} 
\leq e^{(1+M)T(\|a\|_{\infty} + \frac{1}{2}\|B\|_2^2)} e^{sc_1(1+M)} \left( \int_{\Omega} |f_i(x,T-lh)|^2 dx \right)^{1+M} 
\leq K_{l,M} e^{(1+M)T(\|a\|_{\infty} + \frac{1}{2}\|B\|_2^2)} e^{sc_1(1+M)} \left( \sum_{i=1}^{d} \|f_i(\cdot,T-2lh)\|^2 \right)^{M} \left( \sum_{i=1}^{d} \|f_i(\cdot,T)\|^2 \right)^{M} 
\leq dK_{l,M} e^{(1+M)T(\|a\|_{\infty} + \frac{1}{2}\|B\|_2^2)} e^{sc_1(1+M)} \left( \int_{\Omega} |u(x,0)|^2 dx \right)^{M} 
\times \left( \int_{\Omega} |u(x,T)|^2 dx + e^{-\frac{3\mu}{2}} e^{2T(\|a\|_{\infty} + \frac{1}{2}\|B\|_2^2)} \int_{\Omega} |u(x,0)|^2 dx \right), \quad \forall \ M \geq M_t.
$$

(3.47)

On one hand, when $0 < h \leq \min\{T/(4l),1\}$ and $lh\|B\|_2^2 < 1$, we obtain from (3.37) that

$$
M_t \leq 3e^{4\lambda} \frac{(1 + l)C_0(A,\Omega,\bar{\omega})}{1 - (\frac{1}{4})C_0(A,\Omega,\bar{\omega})} := \bar{M}_t.
$$

(3.48)

Obviously, there exists a constant $l_0 := l_0(A,\Omega,\bar{\omega}) > 1$ so that

$$
\frac{sc_1(1 + \bar{M}_t)}{2(1 + l)h} - \frac{s\mu}{h} \leq \frac{\bar{M}_t - s\mu}{2h} \quad \text{for each } l \geq l_0.
$$

(3.49)

Here, we used the fact that $C_0(A,\Omega,\bar{\omega}) \in (0,1)$. On the other hand, when $0 < h \leq \min\{T/(4l),1\}$ and $\|a\|_{\infty} h < 1$, we get from (3.40) that

$$
K_{l,M} \leq e^{(M+1)C_0(A,\Omega,\bar{\omega})} \left(1 + \|a\|_{\infty} h^{2/3} + \|B\|_2^2\right).
$$

(3.50)

It follows from (3.48)-(3.50) that when $0 < h \leq \min\{T/(4l),1\}$, $\|a\|_{\infty} h < 1$ and $lh\|B\|_2^2 < 1$,

$$
M_{l_0} \leq \bar{M}_{l_0}, \quad \frac{sc_1(1 + \bar{M}_{l_0})}{2(1 + l_0)h} - \frac{s\mu}{h} \leq \frac{\bar{M}_{l_0} - s\mu}{2h} \quad \text{and} \quad K_{l_0,\bar{M}_{l_0}} \leq e^{C_0(A,\Omega,\bar{\omega})(1 + \|a\|_{\infty} h^{2/3} + \|B\|_2^2)}.
$$

These, along with (3.47) (where $l$ and $M$ are replaced by $l_0$ and $\bar{M}_{l_0}$, respectively), imply (3.41).

**Step 2.** We show that for any $h \in (0,1]$ such that $h \geq T/(4l_0)$ or $\|a\|_{\infty} h \geq 1$ or $lh\|B\|_2^2 \geq 1$,

$$
\int_{\Omega} |u(x,T)|^2 dx \leq e^{C_2(\frac{1}{2} + \|a\|_{\infty} h^{2/3} + \|B\|_2^2)} e^{2T(\|a\|_{\infty} h^{2/3} + \|B\|_2^2)} \int_{\Omega} |u(x,0)|^2 dx
$$

(3.51)

where $C_2 := \bar{C}_2(A,\Omega,\bar{\omega}) > 0$ and $s \in (0,1]$.

Indeed, by (3.42), we can directly check that

$$
\int_{\Omega} |u(x,T)|^2 dx \leq e^{2T(\|a\|_{\infty} h^{2/3} + \|B\|_2^2)} \int_{\Omega} |u(x,0)|^2 dx
\leq e^{2T(\|a\|_{\infty} h^{2/3} + \|B\|_2^2)} e^{\frac{3\mu}{2} (\|a\|_{\infty} h^{2/3} + \|B\|_2^2)} \int_{\Omega} |u(x,0)|^2 dx,
$$

(3.52)
where $h \in (0, 1]$ satisfies $h \geq T/(4l_0)$ or $\|a\|_{\infty}^{2/3}h \geq 1$ or $l_0h\|B\|_{\infty}^{2} \geq 1$. Then (3.51) follows from (3.52) immediately.

**Step 3.** We complete the proof of Theorem 1.1.

By (3.41) and (3.51), one can conclude that for $s := s(A, \Omega, \tilde{\omega}) \in (0, 1]$ sufficiently small and for any $h \in (0, 1]$, 
\[
\left( \int_{\Omega} |u(x, T)|^2 \, dx \right)^{1+\tilde{M}} \leq e^{(1+\tilde{M})\tilde{C}_3[1+\frac{1}{\alpha}+\|a\|_{\infty}^2/\alpha + \|B\|_{\infty}^2 + T(\|a\|_{\infty} + \frac{1}{\alpha} \|B\|_{\infty}^2)]} \left( \int_{\Omega} |u(x, 0)|^2 \, dx \right)^{\tilde{M}}
\]
\[
\times \left( e^{\frac{\lambda}{2\beta}} \int_{\Omega} |u(x, T)|^2 \, dx + e^{-\frac{\lambda}{2\beta}} \int_{\Omega} |u(x, 0)|^2 \, dx \right)
\]
where $\tilde{C}_3 := \tilde{C}_3(A, \Omega, \tilde{\omega}) > 0$. Now, choose $h \in (0, 1]$ such that 
\[
\frac{1}{2} \left( \int_{\Omega} |u(x, T)|^2 \, dx \right)^{\tilde{M}+1} = e^{-\frac{\lambda}{2\beta}} e^{(1+\tilde{M})\tilde{C}_3[1+\frac{1}{\alpha}+\|a\|_{\infty}^2/\alpha + \|B\|_{\infty}^2 + T(\|a\|_{\infty} + \frac{1}{\alpha} \|B\|_{\infty}^2)]} \left( \int_{\Omega} |u(x, 0)|^2 \, dx \right)^{\tilde{M}+1}.
\]
It follows from (3.53) and (3.54) that 
\[
\left( \int_{\Omega} |u(x, T)|^2 \, dx \right)^{1+\tilde{M}} \leq e^{(1+\tilde{M})\tilde{C}_4[1+\frac{1}{\alpha}+\|a\|_{\infty}^2/\alpha + \|B\|_{\infty}^2 + T(\|a\|_{\infty} + \frac{1}{\alpha} \|B\|_{\infty}^2)]} \left( \int_{\Omega} |u(x, 0)|^2 \, dx \right)^{\tilde{M}} \int_{\Omega} |u(x, T)|^2 \, dx,
\]
where $\tilde{M} := 2\tilde{M} + 1$ and $\tilde{C}_4 := \tilde{C}_4(A, \Omega, \tilde{\omega}) > 0$. Hence, the result follows from the latter inequality immediately.

In summary, we finish the proof of Theorem 1.1. \qed

4 \hspace{1em} Proof of Theorem 1.2

Now, we are able to present the proof of Theorem 1.2 by using a telescoping series method. For the convenience of the reader, we provide here the detailed computation, although it is more or less similar to that of [11, Theorem 4].

**Proof of Theorem 1.2.** According to Theorem 1.1 and Young’s inequality, it is clear that for any $0 \leq t_1 < t_2 \leq T$,
\[
\|u(\cdot, t_2)\| \leq \varepsilon \|u(\cdot, t_1)\| + \frac{K_1 \varepsilon^{-\alpha}}{\varepsilon^a} \|u(\cdot, t_2)\|_{L^2(\omega)}, \quad \forall \varepsilon > 0.
\]
(4.1)

Here, $\tilde{\omega} \in \omega \cap \Omega$, $K_1 := e^{\frac{\alpha}{1-\alpha} K_2}$, and $\alpha := \frac{\beta}{1-\beta}$ with $K, \beta$ being the same constants appearing in Theorem 1.1. By Nash’s inequality and Poincaré’s inequality, we have that 
\[
\|g\|_{L^2(\omega)} \leq K_3 \|g\|_{L^1(\omega)} \|\nabla g\|^{1-\theta}, \quad \forall \ g \in H^1_0(\Omega),
\]
(4.2)
where \( \bar{\omega} \in \omega \), \( K_3 := K_3(\Omega, \omega, \bar{\omega}) > 0 \) and \( \theta := 2/(2 + N) \). Let \( \eta \in C_0^\infty(\Omega) \) be such that \( 0 \leq \eta \leq 1 \) and \( \eta \equiv 1 \) on \( \bar{\omega} \). It follows from (4.2) (where \( g \) is replaced by \( \eta u(\cdot, t_2) \)) and Lemma 2.1 that

\[
\|u(\cdot, t_2)\|_{L^2(\omega)} = \|\eta u(\cdot, t_2)\|_{L^2(\omega)} \leq K_3 \|u(\cdot, t_2)\|_{L^1(\omega)}^\theta \|
abla (\eta u(\cdot, t_2))\|^{1-\theta}
\leq K_3 \|u(\cdot, t_2)\|_{L^1(\omega)}^\theta (K_4 \|u(\cdot, t_2)\| + \|\nabla u(\cdot, t_2)\|)^{1-\theta}
\leq K_3(K_4K_5)^{1-\theta} \|u(\cdot, t_2)\|_{L^1(\omega)} (\|u(\cdot, t_1)\| + \|\nabla u(\cdot, t_2)\|)^{1-\theta},
\]

where \( K_4 := K_4(\Omega, \omega, \bar{\omega}) > 1 \) and \( K_5 := e^{T(2\|a\|_{\infty} + \lambda \|B\|_{\infty})} \). This, along with Young’s inequality, implies that

\[
\|u(\cdot, t_2)\|_{L^2(\omega)} \leq \delta (\|u(\cdot, t_1)\| + \|\nabla u(\cdot, t_2)\|) + \frac{K_6}{\delta^{N/2}} \|u(\cdot, t_2)\|_{L^1(\omega)}, \quad \forall \delta > 0. \tag{4.3}
\]

Here, \( K_6 := K_3(K_3K_4K_5)^{N/2} \). By Lemma 2.2, we have that

\[
\|\nabla u(\cdot, t_2)\| \leq \frac{K_7}{(t_2 - t_1)^{1/2}} \|u(\cdot, t_1)\|, \tag{4.4}
\]

where \( K_7 := 2\lambda^2 e^{T\|a\|_{\infty} + \lambda \|B\|_{\infty}} \). Hence, from (4.1), (4.3) and (4.4) with

\[
\frac{K_1}{\varepsilon^\alpha} e^{\frac{\delta}{\varepsilon^{\alpha+1}}} \delta \left(1 + \frac{K_7}{(t_2 - t_1)^{1/2}}\right) = \varepsilon,
\]

we get that

\[
\|u(\cdot, t_2)\| \leq 2\varepsilon \|u(\cdot, t_1)\| + \frac{K_1K_6}{\varepsilon^\alpha} e^{\frac{\delta}{\varepsilon^{\alpha+1}}} \left[\frac{K_1}{\varepsilon^\alpha} e^{T(2\|a\|_{\infty} + \lambda \|B\|_{\infty})} \left(1 + \frac{K_7}{(t_2 - t_1)^{1/2}}\right)\right]^{N/2} \|u(\cdot, t_2)\|_{L^1(\omega)}
\leq 2\varepsilon \|u(\cdot, t_1)\| + \frac{2^{N/2}K_1^{1+N/2}K_6^{k-1/2}K_7^{k-1/2}}{e^{\alpha(\alpha+1)N/2}K_7^{k-1/2}} e^{\frac{\Delta T \lambda \|B\|_{\infty}}{2\varepsilon}} \|u(\cdot, t_2)\|_{L^1(\omega)}, \tag{4.5}
\]

where \( \gamma := \alpha + (\alpha + 1)N/2 \), \( K_8 := 2\alpha^{1+N/2} + N^{1+N/2}K_1^{1+N/2}K_6^{k-1/2}K_7^{k-1/2} \) and \( K_9 := (1 + 3N/4)K_2 \).

On the other hand, let \( E \) be a subset of positive measure in \((0, T)\). Let \( \ell_0 \) be a density point of \( E \). According to Proposition 2.1 in [10], for each \( \kappa > 1 \), there exists \( \ell_1 \in (\ell_0, T) \), depending on \( \kappa \) and \( E \), so that the sequence \( \{\ell_m\}_{m \geq 1} \), given by

\[
\ell_{m+1} := \ell_0 + \frac{1}{\kappa^m}(\ell_1 - \ell_0),
\]

satisfies that

\[
\ell_m - \ell_{m+1} \leq 3|E \cap (\ell_{m+1}, \ell_m)|. \tag{4.6}
\]

Next, let \( 0 < \ell_{m+2} < \ell_{m+1} \leq t < \ell_m < \ell_1 < T \). It follows from (4.5) that

\[
\|u(\cdot, t)\| \leq \varepsilon \|u(\cdot, \ell_m+2)\| + \frac{K_8}{\varepsilon^\gamma} e^{-\frac{\kappa_9}{\varepsilon^\gamma}} \|u(\cdot, t)\|_{L^1(\omega)}, \quad \forall \varepsilon > 0. \tag{4.7}
\]

By Lemma 2.1, we have that

\[
\|u(\cdot, \ell_m)\| \leq e^{T(\|a\|_{\infty} + \frac{\lambda \|B\|_{\infty}}{2})} \|u(\cdot, t)\|.
\]
This, along with (4.7), implies that
\[
\|u(\cdot, \ell_m)\| \leq e^{T\|a\|_\infty + \frac{\gamma}{2} \|B\|_\infty^2} \left( e^{\|u(\cdot, \ell_{m+2})\|} + \frac{K_8}{\varepsilon^\gamma} e^{\frac{\gamma}{\varepsilon^{m+2}} \|u(\cdot, t)\|_{L^1(\omega)}} \right), \quad \forall \varepsilon > 0,
\]
which indicates that
\[
\|u(\cdot, \ell_m)\| \leq e^{\|u(\cdot, \ell_{m+2})\|} + \frac{K_{10}}{\varepsilon^\gamma} e^{\frac{\gamma}{\varepsilon^{m+2}} \|u(\cdot, t)\|_{L^1(\omega)}}, \quad \forall \varepsilon > 0,
\]
where
\[
K_{10} := \left[ e^{T\|a\|_\infty + \frac{\gamma}{2} \|B\|_\infty^2} \right]^{1+\gamma} K_8.
\]
Integrating the latter inequality over \( E \cap (\ell_{m+1}, \ell_m) \), we get that
\[
|E \cap (\ell_{m+1}, \ell_m)||u(\cdot, \ell_m)|| \leq \varepsilon |E \cap (\ell_{m+1}, \ell_m)||u(\cdot, \ell_{m+2})| + \frac{K_{10}}{\varepsilon^\gamma} e^{\frac{\gamma}{\varepsilon^{m+2}} \int_{\ell_{m+1}}^{\ell_m} \int_{\ell_m}^{\ell_{m+1}} \chi_E \|u(\cdot, t)\|_{L^1(\omega)} dt} \leq \frac{3 \kappa^m}{(\ell_1 - \ell_0)(\kappa - 1)} \frac{K_{10}}{\varepsilon^\gamma} e^{\frac{\gamma}{\varepsilon^{m+2}} \int_{\ell_{m+1}}^{\ell_m} \int_{\ell_m}^{\ell_{m+1}} \chi_E \|u(\cdot, t)\|_{L^1(\omega)} dt} + \varepsilon \|u(\cdot, \ell_{m+2})||
\]
for each \( \varepsilon > 0 \). This yields that
\[
\frac{3 \kappa^m}{(\ell_1 - \ell_0)(\kappa - 1)} \frac{K_{10}}{\varepsilon^\gamma} e^{\frac{\gamma}{\varepsilon^{m+2}} \int_{\ell_{m+1}}^{\ell_m} \int_{\ell_m}^{\ell_{m+1}} \chi_E \|u(\cdot, t)\|_{L^1(\omega)} dt} + \varepsilon \|u(\cdot, \ell_{m+2})||
\]
for each \( \varepsilon > 0 \). Denote \( \tilde{d} := \frac{3 \kappa^m}{(\ell_1 - \ell_0)(\kappa - 1)} \). It follows from (4.9) that
\[
e^{-\tilde{d} \varepsilon^{m+2}} \|u(\cdot, \ell_m)|| - e^{1+\gamma} e^{-\tilde{d} \varepsilon^{m+2}} \|u(\cdot, \ell_{m+2})|| \leq \frac{3 \kappa^m}{(\ell_1 - \ell_0)(\kappa - 1)} \int_{\ell_{m+1}}^{\ell_m} \chi_E \|u(\cdot, t)\|_{L^1(\omega)} dt
\]
for each \( \varepsilon > 0 \). Choosing \( \varepsilon = e^{-\tilde{d} \varepsilon^{m+2}} \) in the latter inequality, we observe that
\[
e^{-(1+\gamma)\tilde{d} \varepsilon^{m+2}} \|u(\cdot, \ell_m)|| - e^{-(2+\gamma)\tilde{d} \varepsilon^{m+2}} \|u(\cdot, \ell_{m+2})|| \leq \frac{3 \kappa^m}{(\ell_1 - \ell_0)(\kappa - 1)} \int_{\ell_{m+1}}^{\ell_m} \chi_E \|u(\cdot, t)\|_{L^1(\omega)} dt.
\]
Take \( \kappa := \sqrt{\frac{\kappa^m}{\tilde{d} \varepsilon^{m+2}}} \) in (4.10). Then we have that
\[
e^{-(2+\gamma)\tilde{d} \varepsilon^{m+2}} \|u(\cdot, \ell_m)|| - e^{-(2+\gamma)\tilde{d} \varepsilon^{m+2}} \|u(\cdot, \ell_{m+2})|| \leq \frac{3 \kappa^m}{(\ell_1 - \ell_0)(\kappa - 1)} \int_{\ell_{m+1}}^{\ell_m} \chi_E \|u(\cdot, t)\|_{L^1(\omega)} dt.
\]
Changing \( m \) to \( 2m' \) and summing the above inequality from \( m' = 1 \) to infinity give the desired result. Indeed,
\[
e^{-T(\|a\|_\infty + \frac{\gamma}{2} \|B\|_\infty^2) e^{-(2+\gamma)\tilde{d} \varepsilon^{m+2}} \|u(\cdot, T)||
\]
\[ \leq e^{-\gamma t} \left| p(t, \ell_2) \right| \]
\[ \leq \sum_{m' = 1}^{+\infty} \left( e^{-\gamma t} \left| p(t, \ell_{2m'}) \right| - e^{-\gamma t} \left| p(t, \ell_{2m' + 2}) \right| \right) \]
\[ \leq \frac{3K_{10}}{\kappa K_9} \sum_{m' = 1}^{+\infty} \int_{\ell_{2m' + 1}}^{\ell_{2m'}} \chi_E \| u(t) \|_{L^1(\omega)} \, dt \]
\[ \leq \frac{3K_{10}}{\kappa K_9} \int_0^T \chi_E \| u(t) \|_{L^1(\omega)} \, dt. \]

In summary, we finish the proof of Theorem 1.2. \qed

5 Appendix

Proof of Lemma 2.1. Multiplying the first equation of (1.2) by \( u \) and integrating it over \( \Omega \times (0, t) \) (with any \( t \in (0, T) \)), we have that
\[
\int_\Omega |u(x, t)|^2 \, dx - \int_\Omega |u(x, 0)|^2 \, dx + 2 \int_0^t \int_\Omega A \nabla u \cdot \nabla u \, dx \, ds = -2 \int_0^t \int_\Omega B \cdot \nabla u \, dx \, ds - 2 \int_0^t \int_\Omega au^2 \, dx \, ds.
\]
By (1.1) and Young’s inequality, we get that
\[
\int_\Omega |u(x, t)|^2 \, dx + \frac{2}{\lambda} \int_0^t \int_\Omega |\nabla u|^2 \, dx \, ds
\]
\[ \leq \int_\Omega |u(x, 0)|^2 \, dx + 2\|B\|_{\infty} \int_0^t \int_\Omega |\nabla u|^2 \, dx \, ds + 2\|a\|_{\infty} \int_0^t \int_\Omega |u|^2 \, dx \, ds
\]
\[ \leq \int_\Omega |u(x, 0)|^2 \, dx + \lambda^{-1} \int_0^t \int_\Omega |\nabla u|^2 \, dx \, ds + (2\|a\|_{\infty} + \lambda\|B\|_{\infty}^2) \int_0^t \int_\Omega |u|^2 \, dx \, ds,
\]
which indicates
\[
\int_\Omega |u(x, t)|^2 \, dx + \lambda^{-1} \int_0^t \int_\Omega |\nabla u|^2 \, dx \, ds
\]
\[ \leq \int_\Omega |u(x, 0)|^2 \, dx + (2\|a\|_{\infty} + \lambda\|B\|_{\infty}^2) \int_0^t \int_\Omega |u|^2 \, dx \, ds
\]
\[ \leq \int_\Omega |u(x, 0)|^2 \, dx + \left( 2\|a\|_{\infty} + \lambda\|B\|_{\infty}^2 \right) \int_0^t \left( \int_\Omega |u(x, s)|^2 \, dx + \lambda^{-1} \int_0^s \int_\Omega |\nabla u(x, \tau)|^2 \, dx \, d\tau \right) \, ds.
\]
This, along with Gronwall’s inequality, implies (2.1). \qed

Proof of Lemma 2.2. Multiplying the first equation of (1.2) by \(-t\text{div}(A \nabla u)\) and integrating it over \( \Omega \times (0, t) \), we obtain that
\[
t \int_\Omega A \nabla u(x, t) \cdot \nabla u(x, t) \, dx + 2 \int_0^t \int_\Omega s \text{div}(A \nabla u)^2 \, dx \, ds - \int_0^t \int_\Omega A \nabla u \cdot \nabla u \, dx \, ds
\]
\[ = 2 \int_0^t \int_\Omega B \cdot \nabla u \text{div}(A \nabla u) \, dx \, ds + 2 \int_0^t \int_\Omega s a u \text{div}(A \nabla u) \, dx \, ds.
\]
This, along with Young’s inequality, implies that

\[
    t \int_\Omega A \nabla u(x,t) \cdot \nabla u(x,t) \, dx - \int_0^t \int_\Omega A \nabla u \cdot \nabla u \, dx \, ds \\
    \leq \|B\|_\infty^2 \int_0^t \int_\Omega s |\nabla u|^2 \, dx \, ds + \|a\|_\infty^2 \int_0^t \int_\Omega s |u|^2 \, dx \, ds. \tag{5.1}
\]

By (1.1) and (2.1), we get that

\[
    \int_0^t \int_\Omega A \nabla u \cdot \nabla u \, dx \, ds \leq \lambda^2 e^{t(2\|a\|_\infty + \lambda\|B\|_\infty^2)} \int_\Omega |u(x,0)|^2 \, dx \tag{5.2}
\]

and

\[
    \|a\|_\infty^2 \int_0^t \int_\Omega s |u|^2 \, dx \, ds \leq \|a\|_\infty^2 \int_0^t e^{s(2\|a\|_\infty + \lambda\|B\|_\infty^2)} \int_\Omega |u(x,0)|^2 \, dx \\leq \|a\|_\infty \int_0^t e^{s(3\|a\|_\infty + \lambda\|B\|_\infty^2)} \int_\Omega |u(x,0)|^2 \, dx \tag{5.3}
\]

It follows from (1.1) and (5.1)-(5.3) that

\[
    t \int_\Omega |\nabla u(x,t)|^2 \, dx \lesssim 2\lambda^3 e^{t(3\|a\|_\infty + \lambda\|B\|_\infty^2)} \int_\Omega |u(x,0)|^2 \, dx + \lambda\|B\|_\infty^2 \int_0^t s \int_\Omega |\nabla u|^2 \, dx \, ds,
\]

which, combined with Gronwall’s inequality, indicates (2.2). \qed

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