On nonsupersymmetric $\mathbb{C}^4/Z_N$, tachyons, terminal singularities and flips

K. Narayan

Chennai Mathematical Institute,
SIPCOT IT Park, Padur PO, Siruseri 603103, India.

Abstract

We investigate nonsupersymmetric $\mathbb{C}^4/Z_N$ orbifold singularities using their description in terms of the string worldsheets conformal field theory and its close relation with the toric geometry description of these singularities and their possible resolutions. Analytic and numerical study strongly suggest the absence of nonsupersymmetric Type II terminal singularities (i.e. with no marginal or relevant blowup modes) so that there are always moduli or closed string tachyons that give rise to resolutions of these singularities, although supersymmetric and Type 0 terminal singularities do exist. Using gauged linear sigma models, we analyze the phase structure of these singularities, which often involves 4-dimensional flip transitions, occurring between resolution endpoints of distinct topology. We then discuss 4-dim analogs of unstable conifold-like singularities that exhibit flips, in particular their Type II GSO projection and the phase structure. We also briefly discuss aspects of M2-branes stacked at such singularities and nonsupersymmetric $AdS_4 \times S^7/Z_N$ backgrounds.
1 Introduction and summary

The interplay between string theory and geometry is a fascinating subject, in many cases beautifully elucidated by gauged linear sigma models [1], exhibiting rich structures such as topology change and hints of a quantum completion of classical geometry (see e.g. [2, 3] for reviews). This also turns out to be remarkably useful in cases where spacetime supersymmetry is broken, as in the context of unstable geometries and their resolution via closed string tachyon condensation. It was argued in [4] using the geometry seen by D-brane probes that unstable \( \mathbb{C}^4/\mathbb{Z}_N \) singularities get smoothed out or resolved by the condensation of closed string tachyons localized at the singular tips. The physical picture here was also shown to hold for 2-dim \( \mathbb{C}^2/\mathbb{Z}_N \) singularities in [4] [5] [6] (see [7] [8] for reviews) as well as 3-dim \( \mathbb{C}^3/\mathbb{Z}_N \) and conifold-like singularities [9] [10] [11], where string worldsheet renormalization group flows were used (see also [12]). This is due essentially to the existence of \((2,2)\) worldsheet supersymmetry which protects various observables along renormalization group flows. Although these first order renormalization group flow equations are not quite the same as time evolution in spacetime (being described by second order equations), various qualitative features, such as the directions of evolution of the geometry and the structure of fixed points, are in fact robust. An elegant description of the physics here is obtained from renormalization group flows in closely related...
gauged linear sigma models (GLSMs), which in turn dovetail nicely with the description of singularity resolution in algebraic geometry in the context of nonsupersymmetric 2- and 3-dim singularities.

In this paper, we study nonsupersymmetric $\mathbb{C}^4/\mathbb{Z}_N$ and conifold-like singularities (which are both toric) using the techniques developed for the 3-dim case. This is of interest both from the point of view of understanding string and M-theory backgrounds constructed using such complex 4-dim spaces with singularities, as well as understanding the geometric structure of closed string tachyon condensation in 4-dim spaces, which is somewhat richer than the lower dimensional cases. In recent times, we have seen the emergence of an understanding of the dual field theories to M-theory $AdS_4 \times S^7/\mathbb{Z}_N$ backgrounds obtained from the near horizon limits of M2-branes stacked at supersymmetric $\mathbb{C}^4/\mathbb{Z}_N$ orbifold singularities $[14]$: see also closely related work $[15]$. Various generalizations of this, in particular with M2-branes stacked at other 4-dim singularities, have been studied in e.g. $[16, 17, 18, 19, 20, 23, 24, 25, 26, 27, 28, 29]$ (see also e.g. $[21, 22]$ for some early work on 4-dim singularities). This gives additional perspective to the nonsupersymmetric case we deal with here.

The backgrounds we consider here are of the form M-theory on $\mathbb{R}^{2,1} \times \mathbb{C}^4/\mathbb{Z}_N$, or Type IIA on $\mathbb{R}^{1,1} \times \mathbb{C}^4/\mathbb{Z}_N$ obtained by compactifying one of the directions in $\mathbb{R}^{2,1}$ on a circle. The basic fact we exploit towards obtaining insight into the physical structure of these singularities is that the Type II closed string blowup modes of the orbifold singularity (equivalently the complexified Kahler parameters of its various collapsed cycles) map to metric and 3-form modes when lifted to M theory. To elaborate further, as in the investigation $[30]$ of M-theory on Calabi-Yau 4-folds (see also $[31]$), the classical Kahler parameters arising from variations of the metric combine with scalars dual (in $\mathbb{R}^{2,1}$) to the $U(1)$ gauge fields that arise from compactification of the 3-form on $\mathbb{C}^4/\mathbb{Z}_N$, to give complex scalars representing tachyons or moduli in M-theory on the uncompactified $\mathbb{R}^{2,1}$. Compactifying one of the $\mathbb{R}^{2,1}$ directions on a circle gives Type IIA string theory, with the tachyons or moduli arising as complex scalars from metric variations and the $B$-field compactified on $\mathbb{C}^4/\mathbb{Z}_N$. These complex scalars govern the geometric blowup modes of the $\mathbb{C}^4/\mathbb{Z}_N$ singularity, in string or M-theory: in string theory, these are just NS-NS sector modes, with the RR sector playing no essential role in the resolution of the singularity by tachyon condensation (as in the lower dimensional cases). The resolution structure of these singularities, described by the toric geometry of the singularity, is beautifully captured by gauged linear sigma models. A precise correspondence between these geometric blowup modes and twisted sector string states in the string worldsheet orbifold conformal field theory was established for nonsupersymmetric $\mathbb{C}^2/\mathbb{Z}_N$ $[6]$ and $\mathbb{C}^3/\mathbb{Z}_N$ $[9]$: this is a generalization of the well-known correspondence between e.g. the $N - 1$ blowup modes of a supersymmetric $\mathbb{C}^2/\mathbb{Z}_N$ (i.e. ALE) singularity and twisted sector string states. We describe and use a similar
correspondence here to understand the phase structure of $\mathbb{C}^4/\mathbb{Z}_N$ singularities using in part their orbifold conformal field theory description. Alternatively, assuming the correspondence between twisted sector string states and geometric blowup modes is faithful and complete, our use of the orbifold conformal field theory as a substitute for the toric geometry and resolution structure of the singularity can be used to gain insights into M-theory compactified on such singularities. It would then seem that a similar correspondence might govern the resolution structure of such singularities as described by M-theory, and a more direct M-theory analysis of this would be interesting.

Various aspects of the structure of nonsupersymmetric $\mathbb{C}^4/\mathbb{Z}_N$ orbifold singularities are similar to those of $\mathbb{C}^3/\mathbb{Z}_N$, with some notable new features too (some early work in the mathematics literature on 4-dim quotient singularities appears in e.g. [13]). These singularities, generically incomplete intersections, do not admit any complex structure deformations as for $\mathbb{C}^3/\mathbb{Z}_N$ [32]. They contain localized closed string tachyons or moduli in their twisted sector spectrum governing the blowup modes of the singularity (which are all Kahler): these can be classified into several rings that are chiral and antichiral with respect to the worldsheet supersymmetry, and respect different complex structure with respect to the spacetime coordinates. It is possible to find, for various families of orbifolds, an appropriate Type II GSO projection that ensures that the only tachyons in the system are these localized ones. This GSO projection typically preserves some tachyons in each ring, projecting the others out. As in $\mathbb{C}^3/\mathbb{Z}_N$ orbifolds, condensation due to all tachyons in a given chiral or antichiral ring does not completely resolve a $\mathbb{C}^4/\mathbb{Z}_N$ singularity, unlike the lower dimensional cases: there exist “geometric terminal” singularities, with no Kahler blowup modes (these are what are usually referred to as terminal singularities in the mathematics literature). A new feature in $\mathbb{C}^4/\mathbb{Z}_N$ is the existence of supersymmetric terminal singularities, which do not admit any blowups, Kahler or nonKahler, at the level of the string worldsheet (or equivalently the toric geometry).

It was proven in [9] that nonsupersymmetric Type II $\mathbb{C}^3/\mathbb{Z}_N$ orbifolds always contain a tachyon or modulus in some ring, i.e. they are never (all-ring) terminal, so that such a singularity will always be resolved by some Kahler or nonKahler blowup due to tachyon condensation. In this aspect, the combinatorics of $\mathbb{C}^4/\mathbb{Z}_N$ orbifolds appears much more intricate and an analytical understanding of whether Type II nonsupersymmetric terminal singularities exist appears difficult. However using some analytic techniques and some numerical investigation via a Maple program, it is possible to gain some insight into the structure of these singularities. Our investigation strongly suggests that in fact nonsupersymmetric terminal singularities do not exist, as in $\mathbb{C}^3/\mathbb{Z}_N$, i.e. Type II $\mathbb{C}^4/\mathbb{Z}_N$ orbifolds always contain a tachyon or modulus in their twisted sector spectrum. This implies that the final endpoints of closed string tachyon condensation in unstable 4-dim Type II singularities are either smooth (i.e. completely resolved) or supersym-
metric singularities (which can be terminal, or of lower dimension). It is not clear if there is an obvious physical reason for the non-existence of nonsupersymmetric terminal singularities, so the result, if true, is striking.

Tachyons localized at the singularity signal instabilities of the system which decays via their condensation to more stable endpoints, which generically are also unstable. This cascade process continues and eventually stops when the system has no further instabilities. In the initial stages of the condensation (in the vicinity of the singularity), gravitational backreaction is negligible so that analysing just this process of condensation of the localized tachyonic modes ignoring other string modes is a good approximation. Our analysis of the decay structure of this system using GLSMs ties in closely with the symplectic quotient construction of the resolution of these singularities (we will mostly not describe the equivalent holomorphic quotient construction here). Since multiple decay channels stemming from multiple tachyons exist (there is no canonical resolution as in $\mathbb{C}^3/\mathbb{Z}_N$), the most likely decay channel corresponds to condensation of the most dominant tachyon with the most negative mass-squared in spacetime: on the worldsheet, this is the most relevant twisted sector operator, belonging in some ring. Geometrically, condensation of such a tachyon induces a partial resolution of the $\mathbb{C}^4/\mathbb{Z}_N$ singularity, with a bubble (typically a weighted projective space $w\mathbb{CP}^3$ here) expanding outwards in (RG) time. Typically there are residual singularities on this expanding locus which could be geometric terminal, \textit{i.e.} terminal with respect to the complex structure of the ring containing the condensing tachyon. In this case, a tachyon (or modulus) in some other ring will induce a blowup further resolving the system, consistent with the non-existence of Type II terminal singularities. Systems with multiple tachyons generically exhibit flip transitions, tachyonic analogs of flops: in $\mathbb{C}^3/\mathbb{Z}_N$ orbifolds, this is a blowdown of an unstable 2-cycle accompanied by a blowup of another, more stable, 2-cycle of distinct topology, occurring when a more dominant tachyon condenses during the condensation of some tachyon. In $\mathbb{C}^4/\mathbb{Z}_N$, flips arise from the blowdowns and blowups of cycles of different dimensionality, involving weighted $\mathbb{CP}^2$s and $\mathbb{CP}^1$s. Thus in a sense, the topology change here is stronger.

We also investigate 4-dim conifold-like singularities here, generalizing [11]. These are described by a $U(1)$ action with charges $Q = ( n_1, n_2, n_3, -n_4, -n_5 )$, with $n_i > 0$. They do not have a manifest conformal field theory interpretation but their phase structure can be analysed using GLSMs. Based on the known Type II GSO projections for orbifolds and the phase structure which typically contains residual orbifold singularities, we guess a Type II GSO projection, $\sum_i Q_i = \text{even}$, for these conifold-like singularities. We find a cascade-like decay structure here too, including decays to lower order supersymmetric conifold-like singularities.

Finally we briefly discuss the physics of M2-branes stacked at $\mathbb{C}^4/\mathbb{Z}_N$ singularities and nonsupersymmetric $AdS_4 \times S^7/\mathbb{Z}_N$ backgrounds. The arguments of [33] for a nonperturba-
tive gravitational instability of nonsupersymmetric $AdS_5 \times S^5/Z_N$ backgrounds similar to the bubble-of-nothing \cite{34} decay of a Kaluza Klein vacuum apply to this case also, suggesting a rapid decay of nonsupersymmetric $AdS_4 \times S^7/Z_N$ throat backgrounds. However, it would seem that along the lines of \cite{35}, cutting off the ultraviolet of the throats, by e.g. embedding in a compact space, would yield a finite decay rate. This then suggests the interesting possibility of stable nonsupersymmetric $AdS_4 \times S^7/Z_N$ throat backgrounds in M-theory.

In sec. 2, we describe the twisted sector spectrum of nonsupersymmetric $\mathbb{C}^4/Z_N$ singularities, with a discussion of terminal singularities in sec. 3, and of the phases of 4-dim orbifold and conifold-like singularities in sec. 4. In sec. 5, we discuss M2-branes stacked at $\mathbb{C}^4/Z_N$ and the physics of nonsupersymmetric $AdS_4 \times S^7/Z_N$ backgrounds. Various details are contained in the appendices — Appendix A describes aspects of the orbifold spectrum, Appendix B contains the Maple program we have used while Appendix C reviews aspects of GLSMs as applicable here, and Appendix D elucidates these GLSM techniques in some 4-dim supersymmetric singularities.

2 \quad $\mathbb{C}^4/Z_N$ twisted sector spectrum

The orbifold $\mathbb{C}^4/Z_N(k_1, k_2, k_3, k_4)$ is defined by the action $z_i \to e^{2\pi i k_i/N} z_i$, where $z_i, i = 1, \ldots, 4$, are the four complexified coordinates in $\mathbb{C}^4$, and $j = 1, \ldots, N - 1$, labels the various twisted sectors. Such an orbifold with at least one of the $k_i$ coprime with $N$ satisfies $\{\frac{ik_i}{N}\} = \frac{1}{N}$ for some sector $j_0$ and can be written in canonical form as $\mathbb{C}^4/Z_N(1, p, q, r)$. These include all isolated orbifolds, with $p, q, r$ coprime with $N$. Orbifolds with no $k_i$ coprime w.r.t. $N$ are similar in structure to $\mathbb{C}^3/Z_N$ orbifolds in the sector with some $\{\frac{ik_i}{N}\} = 0$.

To understand the Type II GSO projection for nonsupersymmetric $\mathbb{C}^4/Z_N(1, p, q, r)$ orbifolds, we complexify the fermions (before orbifolding) in each of the four physical 2-planes obtaining the spinor states $\{s_{ij}\} = \{\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}\} \equiv |\pm \pm \pm \pm\rangle$. The projection on these $SO(8)$ spinors to an irreducible chiral spinor requires $\sum s_{ij} = even$, restricting the spinor states to be 8-dimensional. Consider the (Green-Schwarz) orbifold rotation generator:

$$R = \exp \left[\frac{2\pi i}{N}(J_{23} + pJ_{45} + qJ_{67} + rJ_{89})\right].$$

Then $R^N = (-1)^{2(s_{23} + p s_{45} + q s_{67} + r s_{89})}$. For weights $(1, 1, 1, 1)$, this gives $R^N = (-1)^{2(s_{23} + s_{45} + s_{67} + s_{89})}$. So clearly $R^N = 1$ for any of the spinor states: this is always a Type II theory.

Now consider the spinor states $|\pm \pm \pm \pm\rangle$ that are invariant under the orbifold rotation generator $R$: $R|\pm \pm \pm \pm\rangle = (-1)\frac{2\sum s_{ij}}{N}|\pm \pm \pm \pm\rangle$. For $N = 1$, the phase is $(-1)^{2\sum s_{ij}}$, which is trivial for all $2^4 = 16$ spinor states $s_{ij}$: so these preserve all ($N = 8$) susy. For $N = 2$, the phase is $(-1)^{\sum s_{ij}}$: this is trivial for states $|++++\rangle, |--------\rangle, |+++--\rangle$ and the $5$
permutations thereof, which are precisely the 8 states from the chirality projection. Thus this also preserves \( \mathcal{N} = 8 \) supersymmetry. For \( N \geq 3 \), the phase is \((-1)^{(2\sum s_{ij})/N} \) (and permutations) with \( \sum s_{ij} = 0 \) give a trivial phase and are invariant. These are 6 states, giving \( \mathcal{N} = 6 \) supersymmetry.

For a general \( \mathbb{C}^4/\mathbb{Z}_N(1, p, q, r) \) orbifold, the rotation phase is

\[
R| \pm \pm \pm \pm \rangle = (-1)^{\frac{2(p+q+r+1)}{N}}| \pm \pm \pm \pm \rangle = (-1)^{\frac{\pm p + \pm q + \pm r}{N}}| \pm \pm \pm \pm \rangle. \tag{2}
\]

So the phase is trivial if any of the combinations \( \pm 1 \pm p \pm q \pm r = 0 \) \((mod \ 2N)\). If \( e.g. \), \( 1 - p + q + r = 0 \), then there are two states \( |+++-\rangle, |-+--\rangle \) with trivial phase. Thus if no such combination vanishes, then the orbifold completely breaks supersymmetry. This is the family of singularities we deal with in this paper. As we will see below, this dovetails well with the classification of twisted sector states into various chiral and anti-chiral rings.

From (2), we have \( R^N| \pm \pm \pm \pm \rangle = (-1)^{(\pm 1 \pm p \pm q \pm r)}| \pm \pm \pm \pm \rangle \). We require \( R^N = 1 \) (rather than \( R^N = (-1)^F \), \( F \) being the spacetime fermion number), to remove the bulk tachyon retaining a Type II theory with spacetime fermions in the bulk, with possible closed string tachyons localized at the orbifold fixed point. A shift by an even integer \( 2p \) (or \( 2q, 2r \)) does not change the parity of a number, so this gives from the phase

\[
\pm 1 \pm p \pm q \pm r = \text{even} \quad \Rightarrow \quad 1 + p + q + r = \text{even}, \quad i.e. \quad \sum_i k_i = \text{even}, \tag{3}
\]

for a \( \mathbb{C}^4/\mathbb{Z}_N(k_1, k_2, k_3, k_4) \) orbifold, as the Type II GSO projection on the orbifold weights. A detailed RNS formulation of the Type II GSO projection appears in Appendix A.

The spectrum of twisted sector string excitations in a \( \mathbb{C}^4/\mathbb{Z}_N(k_1, k_2, k_3, k_4) \) orbifold conformal field theory, classified using the representations of the \((2, 2)\) superconformal algebra, has a product-like structure (from each of the four complex planes): Appendix A describes various details, generalizing from the discussion of nonsupersymmetric \( \mathbb{C}^3/\mathbb{Z}_N \) singularities [9]. Each complex plane contribution is either chiral \((c_{X_1})\) or antichiral \((a_{X_1})\), giving sixteen chiral and antichiral rings in eight conjugate pairs, labelled \((c_{X_1}, c_{X_2}, c_{X_3}, c_{X_4}), (c_{X_1}, c_{X_2}, c_{X_3}, a_{X_4})\), and so on. These states can be succinctly described by the chiral ring twist field (vertex) operators, having the form \( X_j = \prod_{i=1}^{4} X_i^{j_{jk_i}/N} = \prod_{i=1}^{4} \sigma_{jk_i/N} e^{i(j_{jk_i}/N)(H_i - H_i)}, \) where \( \sigma_a \) is the bosonic twist-a field operator, while the \( H_i \) are bosonized fermions. These operators correspond to either the ground state or the first excited state in each twisted sector. For instance, in the sector where \( \{j_{k_1}/N\}, \{j_{k_2}/N\}, \{j_{k_3}/N\} < \frac{1}{2}, \{j_{k_4}/N\} > \frac{1}{2} \), the ground state is of the form \( \prod_{i=1}^{3} X_i^{j_{jk_i}/N}(X_i^{4 - (j_{k_4}/N)})^*, \) belonging to the \((c_{X_1}, c_{X_2}, c_{X_3}, a_{X_4})\) ring (or simply cccc-ring). Then the \( X_j \) are the first excited states in this sector, obtained by acting with \( \psi_4 \psi_4^* = e^{i(H_i - H_i)} \) on the ground state operator. The conformal dimension of \( X_j \) is \( \Delta_j = \frac{1}{2} \sum_i (\{j_{k_i}/N\}(1 - \{j_{k_i}/N\}) + \frac{1}{2}(\{j_{k_i}/N\})^2) = \frac{1}{2} \sum_i \{j_{k_i}/N\}, \) and
being chiral operators, they satisfy $\Delta_j = \frac{1}{2} R_j$. The worldsheet R-charge $R_j$ and Type II GSO projection for the $X_j$ are\footnote{Note $\{x\} = x - [x]$ denotes the fractional part of $x$, with $[x]$ the integer part of $x$ (the greatest integer $\leq x$). By definition, $0 \leq \{x\} < 1$. Note that, for $m, n > 0$, we have $\lfloor -\frac{m}{n} \rfloor = -\lfloor \frac{m}{n} \rfloor - 1$ and therefore $\lfloor -\frac{m}{n} \rfloor = -\frac{m}{n} - \lfloor -\frac{m}{n} \rfloor = 1 - \left(\frac{m}{n}\right)$.}

\[ R_j \equiv \left( \left\{ \frac{j k_1}{N} \right\}, \left\{ \frac{j k_2}{N} \right\}, \left\{ \frac{j k_3}{N} \right\}, \left\{ \frac{j k_4}{N} \right\} \right) = \sum_i \left\{ \frac{j k_i}{N} \right\}, \quad E_j = \sum_i \left[ \frac{j k_i}{N} \right] = odd, \quad (4) \]

i.e. a GSO-allowed state has $X_j \rightarrow (-1)^{E_j} X_j = -X_j$, the minus sign arising from the ghost contribution to the GSO exponent (in the $(−1, −1)$-picture) ensuring that the total worldsheet $(-1)^F$ is even for a GSO-preserved state. The spacetime masses arising from the mass-shell condition is given by

\[ m_j^2 = \frac{2}{\alpha'} (R_j - 1) . \quad (5) \]

The GSO exponent $E_j$ for a twist field operator $X_j$ depends nontrivially on the twist sector $j$ as well as the specific (anti-)chiral ring that the twist field belongs to. From the $\mathbb{C}^4/\mathbb{Z}_N$ $(k_1, k_2, k_3, k_4)$ worldsheet partition function (see Appendix A), we obtain the GSO exponents $\{42\}$ for the sixteen rings. It is sufficient to discuss eight of these since the others just contain conjugate fields.

We now mention a convenient notation that can be used to study and label twist operators in the various rings. To illustrate this, note that twist operators in e.g. the $(c_{X_1}, c_{X_2}, c_{X_3}, a_{X_4})$-ring can be rewritten as $X_j^{ccca} = \prod_{i=1}^3 X_{\{jk_i/N\}}^i (X_{-\{jk_i/N\}}^i)^* = \prod_{i=1}^3 X_{\{jk_i/N\}}^i (X_{-\{jk_i/N\}}^i)^*$, which resemble twist operators in the $(c_{X_1}, c_{X_2}, c_{X_3}, c_{X_4})$-ring of the orbifold $\mathbb{C}^4/\mathbb{Z}_N(k_1, k_2, k_3, -k_4)$ with $X^4 \rightarrow (X^4)^*$: the R-charges of the operators are identical while the condition on their GSO exponents $E_j = \sum_i \{jk_i/N\} = even$ (see Appendix A) is re-expressed as

\[ X_j^{ccca} = \prod_{i=1}^3 X_{\{jk_i/N\}}^i X_{-\{jk_i/N\}}^{4*} : \quad E_j^{ccca} = \sum_{i=1}^3 \left[ \frac{jk_i}{N} \right] + \left[ -\frac{jk_4}{N} \right] = E_j + 1 + \text{even} = odd , \quad (6) \]

so that as expected for a $ccca$-ring operator, the corresponding GSO exponent $E_j^{ccca}$ is odd. Generalizing, we see that operators in non-$ccca$-rings of the $\mathbb{C}^4/\mathbb{Z}_N (k_1, k_2, k_3, k_4)$ orbifold can be expressed as $cccc$-ring operators of a corresponding orbifold with related weights, with the GSO exponents appearing uniformly odd in this notation, i.e.

\[ X_j^r \rightarrow (-1)^{E_j^r} X_j^r = -X_j^r \quad \text{GSO-allowed if} \quad E_j^r = odd. \]

This rewriting is particularly convenient in our discussion of all-ring terminuality to follow.

It is important to label $\mathbb{C}^4/\mathbb{Z}_N(1, p, q, r)$ orbifolds appropriately in order to have a complete but also simple catalog. We will restrict $p, q, r > 0$, defining thus the chiral $(cccc)$ ring. As discussed above, the $cccc$-ring of this orbifold is then equivalent to the orbifold $\mathbb{C}^4/\mathbb{Z}_N(1, p, q, -r)$.
in the sense that the twisted sector charges (and GSO projections) are the same, and similarly for the other six rings. In all, we then have

\[
\begin{align*}
ccc & \equiv (1, p, q, r), \\
cc & \equiv (1, p, q, -r), \\
cc & \equiv (1, p, -q, r), \\
cc & \equiv (1, -p, q, r), \\
cc & \equiv (1, p, -q, r), \\
cc & \equiv (1, -p, -q, r). 
\end{align*}
\]

Since we include all rings in listing the twisted sector spectrum, it is sufficient to restrict \(0 < p, q, r < N\): for instance, if \(N < r < 2N\), the orbifold \(\mathbb{Z}_N(1, p, q, r) \equiv \mathbb{Z}_N(1, p, q, r - 2N)\) (shifting by \(2N\) maintains the GSO projection), and the \(ccc\)-ring of the latter orbifold is equivalent to the \(ccca\)-ring of \(\mathbb{Z}_N(1, p, q, -(2N - r))\), which is contained within our restricted range since \(0 < 2N - r < N\).

In this description, a supersymmetric orbifold is one where some ring has a vanishing combination \(1 \pm p \pm q \pm r = 0 (\text{mod} \ 2N)\), with GSO-preserved twisted states. For instance, with \(1 - p + q + r = 0 (\text{mod} \ 2N)\), the \(ccca\)-ring is supersymmetric, with spectrum equivalent to \(\mathbb{Z}_N(1, -p, q, r)\). To illustrate this, consider \(\mathbb{Z}_N(1, p, q, r)\) with say the \(ccca\)-ring being supersymmetric, i.e. \(1 + p + q - r = 0\). Then it can be shown that no tachyons or moduli arise in any ring other than the \(ccc\)-ring. For instance, the \(ccc\)-ring R-charges are \(R_{jjcc} = \frac{j}{N} + \{\frac{jr}{N}\} + \{\frac{j}{N}\} + 1 - \{\frac{jr}{N}\} = \frac{j(N+r)}{N} - E_{jjcc} = -E_{jjcc}\), for any \(j\). Now if \(R_{jjcc}^\text{ccca} = 1 = -E_{jjcc}\) for some sector \(j\), then this is a twisted modulus. It is easily seen that the GSO exponents for the \(ccc, ccac\)-rings (and permutations) are even, so these rings do not give any GSO-preserved states in such twist sectors. Also, using \(R_{jjcc}^\text{ccca} = 1\), we have \(e.g. R_{jjcc}^\text{ccca} + 2(\{\frac{jr}{N}\} - \{\frac{j}{N}\}) > 1\) and similarly, \(R_{jjcc}^\text{ccca}, R_{jjcc}^\text{ccac} > 1\) (states in these rings are GSO-preserved). Now in sectors where \(R_{jjcc}^\text{ccca} = -E_{jjcc} = 2\), states in \(ccc, ccac\)-rings (and permutations) are GSO-preserved: but we see using \(R_{jjcc}^\text{ccca} = 2\) that \(R_{jjcc} = 1 + 2(\{\frac{jr}{N}\}, R_{jjcc}^\text{ccca} = 2 + \{\frac{j}{N}\} > 1\), so that these states are irrelevant. Thus only the supersymmetric ring (here \(ccc\)) contributes moduli.

The combinatorics of \(\mathbb{C}^3/\mathbb{Z}_N\) is quite different \([9]\) from \(\mathbb{C}^3/\mathbb{Z}_N\), with no “\(\sum s_{ij}\) cancellation”. To be specific, for \(\mathbb{C}^3/\mathbb{Z}_N\), the weights \((1, 1, 1)\) do not yield a Type II GSO projection (as can be seen from \([3]\), setting say \(k_4 = r = 0\)), so we must shift the weights to \((1, 1, 1 - N)\), which admits a Type GSO projection for \(N\) odd. This latter orbifold is the simplest Type II analog of \(\mathbb{Z}_N(1, 1, 1)\), with R-charges:

\[
\begin{align*}
ccc: R_j &= 2\{\frac{j}{N}\} + \{\frac{j(N-1)}{N}\} = 3\frac{j}{N}, \quad E_j = 2[\frac{j}{N}] + [\frac{j(N-1)}{N}] = -j. \\
cc: R_j &= 2\{\frac{j}{N}\} + \{\frac{j(N-1)}{N}\} = 1 + \frac{j}{N}, \quad E_j = 2[\frac{j}{N}] + [\frac{j(N-1)}{N}] = j - 1. \\
cc: R_j &= \{\frac{j}{N}\} + \{\frac{j}{N}\} + \{\frac{j(N-1)}{N}\} = 1 + \frac{j}{N}, \quad E_j = [\frac{j}{N}] + [\frac{j}{N}] + [\frac{j(N-1)}{N}] = -j - 1.
\end{align*}
\]

So this always has a GSO-preserved tachyon in the \(j = 1\) sector for \(N > 3\) (for \(N = 3\), this is the marginal blowup of the \(\mathbb{Z}_3(1, 1, -2)\) supersymmetric orbifold). For \(\mathbb{C}^2/\mathbb{Z}_N(1, 1, 1)\), the R-charges are: \(cc: R_j = 2\frac{j}{N}, \quad E_j = 2[\frac{j}{N}]\), \(ca: R_j = \{\frac{j}{N}\} + \{\frac{j}{N}\} = 1, \quad E_j = [\frac{j}{N}] = -1.\)
These are GSO-preserved moduli.

3 Nonsupersymmetric terminal singularities

The mass shell condition $m_j^2 = \frac{2}{\alpha'}(R_j - 1)$ above (see (5)) shows that a twisted sector state with $R_j < 1$ is tachyonic ($m_j^2 < 0$), while one with $R_j = 1$ is marginal. States with $R_j > 1$ are massive (irrelevant). An orbifold is terminal (or all-ring terminal) if all twisted sector states from all chiral and anti-chiral rings are irrelevant, i.e. they all have $R_j > 1$. This means that the orbifold admits no geometric blowup modes and cannot be physically resolved by (relevant or marginal) worldsheet string modes.

We recall that for $\mathbb{C}^2/\mathbb{Z}_N$ nonsupersymmetric singularities [6], the Hirzebruch-Jung minimal resolution ensures that a tachyon or modulus always arises in the chiral (or anti-chiral) ring, so that a $\mathbb{C}^2/\mathbb{Z}_N$ singularity is always resolved by twisted sector states in a single chiral (or anti-chiral) ring alone.

$\mathbb{C}^3/\mathbb{Z}_N$ singularities are more complicated: a single chiral (or anti-chiral) ring might be terminal: these are “geometric terminal singularities”, comprising purely Kahler blowup modes, and are often referred to as terminal singularities in the mathematics literature. From the physical point of view, we need to look at all the various rings to understand if an orbifold is terminal, i.e. both Kahler and nonKahler blowup modes (or generic metric blowup modes).

The Type II GSO projection introduces an additional complication by retaining only some states in each chiral ring, so that a possible geometric blowup mode could in fact be physically GSO-disallowed: this might suggest Type II terminal singularities are more likely. However the combinatoric proof in [9] shows that a Type II GSO-preserved tachyon or modulus always arises in one of the $j = 1$ twisted sectors (all rings considered) for a $\mathbb{C}^3/\mathbb{Z}_N(1,p,q)$ orbifold in Type II string theories, while $\mathbb{C}^3/\mathbb{Z}_2(1,1,1)$ is the unique terminal singularity in Type 0 string theories.

For $\mathbb{C}^4/\mathbb{Z}_N$, it turns out that the $j = 1$ twist sectors can be terminal: we must analyze all twisted sectors to see if tachyons or moduli always arise. This makes the system much more complicated and a closed form proof to show the likely absence of terminal singularities is difficult to obtain. However analyzing the $j = 1$ sector gives various constraints on which $\mathbb{C}^4/\mathbb{Z}_N(1,p,q,r)$ singularities can be terminal if at all. It is then possible to perform an “experimental” search using a Maple program (see Appendix B) on these restricted window of possibilities for terminal singularities. This reveals the absence of any nonsupersymmetric Type II terminal singularities as we run the Maple program through $N \leq 400$ for various orbifold weights, as we describe in greater detail below. There are of course Type 0 terminal singularities (as well as supersymmetric ones) as we will see below.
3.1 Type II all-ring terminality

It is straightforward to check that \( \mathbb{C}^4/\mathbb{Z}_N(1,1,1,1) \) singularities are in fact terminal. Since the \( U(4) \) symmetry is unbroken, we need to only check the twisted sector spectrum from the \( cacc, ccca, ccaac, caaa \) rings (the others being permutations thereof).

\( cacc: \) \( E_j = 4\left(\frac{1}{N}\right) = 0 \Rightarrow \) no GSO-preserved state.

\( ccca: \) \( R_j = 3\left(\frac{1}{N}\right) + \left\{ \frac{j-1}{N} \right\} = 3\left(\frac{1}{N}\right) + 1 - \left\{ \frac{j}{N} \right\} = 1 + 2\left(\frac{j}{N}\right) > 1 \) \( (E_j = 3\left[\frac{1}{N}\right] + [\frac{j}{N}] = odd) \).

\( caaa: \) \( R_j = \left\{ \frac{j}{N} \right\} + 3\left\{ \frac{j-1}{N} \right\} = 1 + 2(1 - \left\{ \frac{j}{N} \right\}) > 1 \) \( (E_j = [\frac{j}{N}] + 3[\frac{j}{N}] = odd) \).

\( cca: \) \( R_j = 2\left(\frac{1}{N}\right) + 2\left\{ \frac{j-1}{N} \right\} = 2 \) \( (E_j = 2\left[\frac{j}{N}\right] + 2\left[\frac{j}{N}\right] = even) \).

Thus all GSO-preserved twisted sector states have R-charges \( R_j > 1 \) and are irrelevant for all \( N \): thus these are terminal singularities not resolvable by worldsheet blowups.

Note that there could be geometrically equivalent orbifolds admitting resolutions, essentially because they are different as conformal field theories due to a different GSO projection on the twisted states. For instance, \( \mathbb{C}^4/\mathbb{Z}_4(1,1,1,-3) \), although geometrically equivalent to \( \mathbb{C}^4/\mathbb{Z}_4(1,1,1,1) \) in fact has a marginal blowup mode arising in the \( cccc \) ring \( j = 1 \) sector: \( R_{j=1} = 1 \), with \( E_{j=1} = 3\left(\frac{1}{N}\right) + [\frac{-2}{N}] = -1 \). This is consistent with [36] which discusses a near-horizon supergravity solution for \( M2 \)-branes stacked at a resolved \( \mathbb{C}^4/\mathbb{Z}_4 \) singularity.

Similarly \( \mathbb{C}^4/\mathbb{Z}_N(1,1,p,p) \) singularities are terminal for \( p \) coprime w.r.t. \( N \): we see that

\( cacc: \) \( E_j = 2\left[\frac{p}{N}\right] = even \). \( ccca: \) \( E_j = 2\left[\frac{-p}{N}\right] = even \). \( \Rightarrow \) no GSO-preserved state.

\( cca: \) \( R_j = 2\left(\frac{1}{N}\right) + \left\{ \frac{p}{N} \right\} + \left\{ \frac{-p}{N} \right\} > 1 \) \( (E_j = \left[\frac{1}{N}\right] + \left\{\frac{p}{N}\right\} + \left\{\frac{-p}{N}\right\} > 1) \).

\( caaa: \) \( R_j = \left\{ \frac{1}{N}\right\} + \left\{ \frac{-1}{N} \right\} + \left\{ \frac{p}{N} \right\} + \left\{ \frac{-p}{N} \right\} > 1 \) \( (E_j = \left[\frac{1}{N}\right] + \left\{\frac{-1}{N}\right\} + \left\{\frac{p}{N}\right\} + \left\{\frac{-p}{N}\right\} > 1) \).

Thus all twisted states that are possibly GSO-preserved are irrelevant. We mention that [13] showed that geometric terminal \( \mathbb{C}^4/\mathbb{Z}_N \) singularities must have weights of the form \( \mathbb{Z}_N(1,-1,1,-1) \), with \( N,p \) coprime: this can be recognized as the \( caca \)-ring of the orbifolds in question here. These are in fact supersymmetric singularities as is well known. Note that orbifolds of this kind with \( p \) not coprime w.r.t. \( N \) do in fact contain moduli in their twisted spectrum.

Now we come to nonsupersymmetric \( \mathbb{C}^4/\mathbb{Z}_N(1,p,q,r) \) singularities. Possible twisted sector string states that are GSO-preserved arise from any of the eight (pairs of) rings. In the \( j = 1 \) sector, this requires \( E_{j=1} = \sum \left[\frac{k}{N}\right] = odd \): thus in \( \mathbb{Z}_N(1,p,q,r) \), with \( p,q,r > 0 \), possible GSO-preserved states can only arise from the \( ccca, ccac, cacc, caaa \) rings (for instance, \( E_{j=1} = \left[\frac{1}{N}\right] + \left[\frac{p}{N}\right] + \left[\frac{q}{N}\right] + \left[\frac{r}{N}\right] = 2 \)). Terminality for these states gives the following conditions on their R-charges which must satisfy \( R_j > 1 \):

\[
\begin{align*}
\frac{1}{N} + \frac{p}{N} + \frac{q}{N} + 1 - \frac{r}{N} > 1, & \quad \frac{1}{N} + \frac{p}{N} + 1 - \frac{q}{N} + \frac{r}{N} > 1, \\
\frac{1}{N} + 1 - \frac{p}{N} + \frac{q}{N} + \frac{r}{N} > 1, & \quad \frac{1}{N} + 1 - \frac{p}{N} + 1 - \frac{q}{N} + 1 - \frac{r}{N} > 1, \\
\Rightarrow 1 + p + q > r, & \quad 1 + p + r > q, \quad 1 + q + r > p, \quad p + q + r < 1 + 2N.
\end{align*}
\]
For Type II, \( \sum k_i = \text{even} \), so that \( p + q + r = \text{odd} \). Hence \( r \neq p + q \) and similarly for \( p, q \). Then the inequality \( r > p + q \implies p + q < r < p + q + 1 \) which is not possible for \( r \in \mathbb{Z} \). This has to hold for each of \( p, q, r \): thus we must have (consistent if \( p, q, r > 0 \))

\[
\begin{align*}
r < p + q, & \quad q < p + r, & \quad p < q + r. \\
\end{align*}
\]

Furthermore, we must have \( 1 \pm p \pm q \pm r \neq \nu N, \; \nu \in \mathbb{Z} \), \( i.e. \) the singularity is nonsusy in every ring. In particular, \( 1 \pm p \pm q \pm r \neq 0 \). This means

\[
\begin{align*}
1 + p & \geq q + r, \quad 1 + q \geq p + r, \quad 1 + r \geq p + q. \\
\end{align*}
\]

Now \( 1 + r > p + q \) means \( p + q - 1 < r < p + q \), which is not possible for \( r \in \mathbb{Z} \), and similarly for the other inequalities. Thus we must have

\[
\begin{align*}
r < p + q - 1, & \quad q < p + r - 1, & \quad p < q + r - 1. \\
\end{align*}
\]

Combining (9), (10) and (12) gives (recall \( 0 < p < q < r \))

\[
\begin{align*}
q - p + 1 < r & < p + q - 1, \quad r - p + 1 < q < p + r - 1, \quad r - q + 1 < p < q + r - 1, \\
\end{align*}
\]

as the strongest inequalities. Similar inequalities \( e.g. \) \( p - q - 1 < r < q - 1 + p \) are weaker: the lower bound is \( p - q - 1 < 0 < r \) (since \( p < q \)).

If \( p, q, r \), are not all distinct, we have a non-isolated singularity.

Say \( p, q, r \), are all distinct: then we can take \( 0 < p < q < r \) without loss of generality. Now from the Type II condition \( \sum k_i = \text{even} \), we must have (i) all odd, (ii) all even, (iii) 2 odd and 2 even. If we focus on isolated singularities, then \( p, q, r \), are mutually coprime, so that case (iii), with 2 even, is not allowed, nor is case (ii), all even.

Let us then restrict to \( p, q, r \), all odd, and use the lowest such integers \( p, q, r = 1, 3, 5, 7, 9, 11, 13, \ldots \), restricting to mutually coprime integers for \((1, p, q, r)\).

- \((1, p, q, r) = (1,1,1,3), (1,1,3,5), (1,1,5,7), (1,1,7,9), (1,1,9,11), (1,1,11,13), (1,3,5,7), (1,3,7,11), (1,5,7,11), (1,5,7,13), (1,5,9,13), \ldots \): these are supersymmetric, \( e.g. \) the \textit{ccca}-ring of \( \mathbb{Z}_N(1,1,5,7) \) is equivalent to the supersymmetric \( \mathbb{Z}_N(1,1,5,-7) \).

- \((1,1,1,5), (1,1,1,7), (1,1,1,11), (1,1,3,7), (1,1,5,9), (1,1,5,11), (1,1,5,13), (1,1,7,11), (1,1,7,13), (1,1,9,13), (1,3,7,13), \ldots \): do not satisfy inequalities (13) above (in particular \( r < p+q-1 \)). In other words, a tachyon arises in the \( j = 1 \) sector in some ring.

- \((1,5,7,9), (1,7,9,11), (1,7,9,13), (1,5,11,13), (1,7,11,13), (1,9,11,13), (1,7,11,15) \): satisfy (13) and potentially could be terminal. However the Maple program check for \( N \leq 400 \)
shows no nonsupersymmetric terminal singularity (it does show supersymmetric terminal
singularities e.g. $\mathbb{Z}_5(1, 7, 9, 13) \equiv \mathbb{Z}_5(1, 2, -1, -2)$). More generally, we see that $(1, 2m - 1, 2m + 1, 2m + 3)$ and $(1, 2m - 3, 2m + 1, 2m + 5)$ satisfy (13) for $m > 2$ and $m > 4$
respectively, with $(1, 3, 5, 7)$ and $(1, 5, 9, 13)$ being supersymmetric.

- Miscellaneous Maple checks for orbifolds with $N \leq 30$ and assorted weights show no
terminal singularity.

In general, the Maple output (see Appendix B) expectedly shows the number of tachyons or
moduli increasing as the orbifold order $N$ increases, thus making it less likely to find a terminal
singularity as $N$ increases. Indeed one expects to find a terminal singularity for low orbifold
orders, if at all: the absence thereof in the Maple output is a noteworthy result. Although
our “experimental” search is by no means exhaustive or equivalent to a closed form proof, not
finding any terminal singularity for the above checks and the structure of the Maple output
along with the analysis of the $j = 1$ sector constraints above strongly suggests the non-existence
of nonsupersymmetric Type II terminal singularities.

By a close look at the Maple output, we find that e.g. $\mathbb{Z}_{13}(1, 7, 9, 11)$ is supersymmetric,
with $caaa$ ring, $j = 1$ and $j = 7$ twisted sector moduli: note $\{-\frac{7}{13}, 11\} = \frac{1}{13}$. Also $1 - 7 - 9 - 11 =
-26 = -2N$ here, i.e. saturation of the last inequality in (13). We also point out that e.g.
$\mathbb{Z}_{17}(1, 7, 9, 11)$ has (among others) a $cacc$ ring, $j = 2$ twisted sector tachyon: note $\{\frac{2}{17}, 9\} = \frac{1}{17}$.

In fact one might imagine this to be a general feature, i.e. twisted sectors where $\{\pm j_ak_b\} = \frac{1}{N}$
for some sector $j = j_a$ and ring $k = \{k_b\}$ are likely to always contain tachyons or moduli.
While this is often true, it can be checked that the twisted sector tachyons of e.g. the Type II
orbifold $\mathbb{C}^4/\mathbb{Z}_{41}(1, 7, 9, 11)$ do not arise from any such sector.

3.2 Type 0 terminality

The Type 0 theory has a diagonal GSO projection, the partition function being given in
Appendix A. The spectrum can again be classified in terms of eight chiral and antichiral rings
comprising operators $X_j$, except that all such states exist and are GSO-preserved, i.e. the GSO
exponents are trivial.

For the Type 0 theory to be terminal, as a basic requirement, the $j = 1$ sector should be
terminal: this gives

\[
\frac{1}{N} + \frac{p}{N} + \frac{q}{N} + \frac{r}{N} > 1, \quad \frac{1}{N} + \frac{p}{N} + \frac{q}{N} + 1 - \frac{r}{N} > 1, \\
\frac{1}{N} + \frac{p}{N} + 1 - \frac{q}{N} + \frac{r}{N} > 1, \quad \frac{1}{N} + 1 - \frac{p}{N} + \frac{q}{N} + \frac{r}{N} > 1, \\
\frac{1}{N} + \frac{p}{N} + 1 - \frac{q}{N} + 1 - \frac{r}{N} > 1, \quad \frac{1}{N} + 1 - \frac{p}{N} + \frac{q}{N} + 1 - \frac{r}{N} > 1,
\]
\[
\frac{1}{N} + 1 - \frac{p}{N} + 1 - \frac{q}{N} + \frac{r}{N} > 1, \quad \frac{1}{N} + 1 - \frac{p}{N} + 1 - \frac{q}{N} + 1 - \frac{r}{N} > 1, \quad (14)
\]

which simplify to

\[
1 + p + q + r > N, \quad p + q + 1 > r, \quad p + r + 1 > q, \quad q + r + 1 > p, \quad q + r < p + N, \quad p + r < q + N, \quad p + q < r + N, \quad p + q + r < 1 + 2N, \quad (15)
\]

for Type 0 all-ring terminality in the \( j = 1 \) sector. Some of these can be combined and recast as

\[
N - 1 < p + q + r < 2N + 1, \quad -3 < p + q + r < 3N. \quad (16)
\]

It is possible to check that \( \mathbb{C}^4/\mathbb{Z}_3(1,1,1,2) \) is an all-ring Type 0 terminal singularity: we have the R-charges

\[
\begin{align*}
\text{cccc} : & \ \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3} \right), \quad \left( \frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right), \\
\text{ccac} : & \ \left( \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right), \quad \left( \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3} \right), \\
\text{caac} : & \ \left( \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right), \quad \left( \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right), \\
\text{ccaA} : & \ \left( \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right), \quad \left( \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right), \quad \left( \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3} \right), \quad \left( \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3} \right), \quad (17)
\end{align*}
\]

the other rings (cacc, caca) being permutations of these. These all clearly satisfy \( R_j > 1 \), giving irrelevant states.

Similarly, \( \mathbb{Z}_4(1,1,1,2), \mathbb{Z}_4(1,1,2,3), \mathbb{C}^4/\mathbb{Z}_5(1,1,2,3), \) are also all-ring terminal singularities; \( \mathbb{Z}_4(1,2,3,5), \mathbb{Z}_5(1,3,4,7), \) can also be checked to be terminal, but can be recast as one of the above by shifting some of the weights (but retaining the Type 0 GSO projection).

The Maple output for Type 0 terminality in fact points out the above singularities but does not show any other. This is again of course not exhaustive by any means but suggests that as the orbifold order increases, Type 0 terminality does not occur either.

### 4 \( \mathbb{C}^4/\mathbb{Z}_N \) toric geometry, closed string tachyons and flips

As we have seen, the twisted sector spectrum of nonsupersymmetric \( \mathbb{C}^4/\mathbb{Z}_N \) singularities shows localized closed string tachyonic instabilities: the condensation of these tachyons causes a decay

\footnote{For instance, the \( j = 1 \) sector R-charges for the various rings in \( \mathbb{C}^4/\mathbb{Z}_5(1,1,2,3) \) are

\[
\begin{align*}
\text{cccc} : & \ \left( \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \right), \quad \left( \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \right), \\
\text{ccac} : & \ \left( \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \right), \quad \left( \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \right), \\
\text{caac} : & \ \left( \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \right), \quad \left( \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \right), \quad \left( \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \right), \quad \left( \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \right).
\end{align*}
\]}

the \( j = 2, 3, 4 \) sectors being clearly irrelevant also.
of the system to more stable endpoints, which generically being unstable also, decay. This process eventually stops when the system has no further instabilities, i.e. when all residual endpoints are either fully smooth or supersymmetric singularities (which can be terminal).

Analysing the decay of such an unstable singularity is elegantly done using gauged linear sigma models (GLSMs): a detailed development of GLSMs for supersymmetric toric varieties was performed in [37]. In the present nonsupersymmetric context, they dovetail beautifully with the toric geometry description of the resolution of these singularities (Appendix C reviews various aspects of GLSMs applied to unstable noncompact singularities). These GLSMs are in a sense simplified versions of nonlinear sigma models of strings propagating on these unstable singularities (see e.g. [7, 8] for reviews): localized closed string tachyons are represented as relevant operators that induce renormalization group flows from these unstable fixed points to more stable fixed points typically representing lower order singularities. The endpoints of the RG flows in the GLSM being classical phases coincide with those of the nonlinear sigma model and the GLSM RG flows themselves approximate the nonlinear ones in the low-energy regime.

The GLSMs here, which all have (2, 2) worldsheet supersymmetry, have close connections with their topologically twisted versions (the twisted A-models retain information about Kahler deformations while complex structure information is in general lost): thus various physical observables, in particular those preserving worldsheet supersymmetry (e.g. operators in chiral rings), are protected along the RG flows corresponding to tachyon condensation which give rise to Kahler deformations of the orbifold. Along the flow, only part of the supersymmetry is preserved, that corresponding to the chiral ring containing the condensing tachyon(s) of the parent orbifold. However, at the end of the flow, the (more stable) fixed points being residual orbifolds again have a twisted spectrum comprising all their various chiral rings.

These worldsheet techniques were used to study the condensation of closed string tachyons localized at lower dimensional (\(\mathbb{C}/\mathbb{Z}_N, \mathbb{C}^2/\mathbb{Z}_N\)) orbifold singularities [4, 5, 6]. They were generalized to unstable orbifold \(\mathbb{C}^3/\mathbb{Z}_N\) and conifold-like (\(n_1 n_2 -n_3 -n_4\)) singularities in 3-complex dimensions in [9, 10, 11] (see also e.g. [12]). The fact that single chiral or antichiral rings in 3-dim can be terminal makes the decay phase structure more intricate.

Since there are generically multiple decay channels stemming from multiple tachyons, the most likely decay channel corresponds to condensation of the most relevant tachyon (which belongs in some ring), which induces a partial resolution of the singularity: geometrically this is a weighted \(\mathbb{CP}^2\) expanding in time. Typically there are residual singularities on the expanding locus which could be terminal with respect to the complex structure of the ring containing the condensing tachyon. However since there are no Type II terminal singularities, a tachyon (or modulus) in some other ring will induce a blowup further resolving the system.

Systems with multiple tachyons generically exhibit flip transitions, i.e. a blowdown of a 2-
cycle accompanied by a blowup of a topologically distinct 2-cycle: in $\mathbb{C}^3/\mathbb{Z}_N$ orbifolds, this occurs when a more dominant tachyon condenses during the condensation of some tachyon, the tachyons being twisted sector states in the orbifold conformal field theory. In the context of conifold-like singularities, the instabilities do not appear to have a manifest conformal field theory interpretation, although the GLSM captures the geometric process adequately.

Our use of (first order) worldsheet RG flow to mimic (second order) time evolution in spacetime is clearly an approximation: the RG time of the GLSM agrees qualitatively with time in spacetime in known examples, in the presence of worldsheet supersymmetry, for the special kinds of complex noncompact singular spaces we deal with here. In e.g. [38, 39], it was found that for noncompact singularities the worldsheet beta-function equations show no obstruction to either RG flow (from c-theorems) or time-evolution (since the dilaton can be turned off). Compact tachyons are more intricate – among other things, the dilaton is necessarily turned on. We expect that an uplift to M-theory is consistent with this structure of tachyon condensation and resolution of singularities, using the metric variations and scalar duals (in 3-dims) to the $U(1)$ gauge fields (obtained from the 3-form C-field with some components on the orbifold) as complex Kahler parameters entering in the geometric (GLSM) description of the resolutions of the orbifold singularity.

4.1 $\mathbb{C}^4/\mathbb{Z}_N$ toric geometry

The geometry of such an orbifold can be recovered efficiently using its toric data. Let the toric cone $C(0; e_1, e_2, e_3, e_4)$ of this orbifold be defined by the origin 0 and lattice points $e_1, e_2, e_3, e_4$ in the 4-dimensional toric $\mathbf{N}$ lattice (the box in Figure [1] shows the toric cone for $\mathbb{Z}_{25}(1, -7, 9, 11)$: the points $e_i$ define a 3-dimensional affine “marginality hyperplane” (i.e. tetrahedral cone) $\Delta$ passing through them. The volume of this cone $V(0; e_1, e_2, e_3, e_4) \equiv |\det(e_1, e_2, e_3, e_4)|$ gives the order $N$ of the orbifold singularity (normalizing the cone volume without any additional numerical factors). The specific structure of the orbifold represented by some toric cone $C(0; e_1, e_2, e_3, e_4)$ can be gleaned either using the Smith normal form algorithm [9], or equivalently by realizing relations between the lattice vectors $e_i$ and any vector that is also itself contained in the toric $\mathbf{N}$ lattice: e.g. we see that the cone defined by $e_1 = (N, -p, -q, -r), e_2 = (0, 1, 0, 0), e_3 = (0, 0, 1, 0), e_3 = (0, 0, 0, 1)$, corresponds to $\mathbb{C}^4/\mathbb{Z}_N(1, p, q, r)$ using the relation $(1, 0, 0, 0) = \frac{1}{N}(e_1 + pe_2 + qe_3 + re_4)$ with the lattice point $(1, 0, 0, 0)$. Note that in general this only fixes the orbifold weights up to shifts by the order $N$.

A $\mathbb{C}^4/\mathbb{Z}_N(1, p, q, r)$ orbifold is isolated if $p, q, r$ are coprime w.r.t. $N$: this is equivalent to the condition that there are no lattice points on the walls of the defining toric cone. For example, if $q, N$ have a common factor $n$ with $q = m_1n, N = m_0n$, then the $\{e_1, e_2\}$ and $\{e_1, e_4\}$ walls have the integral lattice points $\frac{1}{n}(N, -p, -q, -r) + \{2\}_n(0, 1, 0, 0) = (m_0, -[2], -m_1, 0)$
and \( \frac{1}{n}(N, -p, -q, -r) + \{ \frac{r}{n} \} \{ 0, 0, 0, 1 \} = (m_0, 0, -m_1, -[\frac{r}{N}]) \) respectively.

Geometric terminal singularities arise if there is no Kähler blowup mode: \( i.e. \) no lattice point in the interior of the toric cone and equivalently no relevant or marginal chiral ring operator in the orbifold spectrum. A physical analysis of the system must include all possible tachyons in all rings, \( i.e. \) both Kähler and non-Kähler blowup modes: this dovetails with our discussion above on the twisted spectrum and the absence of terminal singularities. Note also that \( \mathbb{C}^4/\mathbb{Z}_N \) singularities (as in \( \mathbb{C}^3/\mathbb{Z}_N \)) have no complex structure deformations \(^{32} \); generically these are incomplete intersections.

There is a 1-1 correspondence between the chiral ring operators and points in the \( \mathbf{N} \) lattice toric cone of the orbifold. A given lattice point \( P_j = (x_j, y_j, z_j, w_j) \) can be mapped to a twisted sector chiral ring operator in the orbifold conformal field theory by realizing that this vector can expressed in the \( \{ e_1, e_2, e_3, e_4 \} \) basis as

\[
(x_j, y_j, z_j, w_j) = r_1 e_1 + r_2 e_2 + r_3 e_3 + r_4 e_4 .
\]

If \( 0 < r_i \leq 1 \), then \( P_j \) is in the interior of the cone, and corresponds to an operator \( O_j \) with R-charge \( R_j \equiv (r_1, r_2, r_3, r_4) \). Conversely, it is possible to map an operator \( O_j \) of given R-charge to a lattice point \( P_j \). Concretely, we can check using \(^{158} \) that a \( ccccc \)-ring operator \( X_j \) in twist sector-\( j \) of \( \mathbb{C}^4/\mathbb{Z}_N(1, p, q, r) \) with R-charge \( R_j \) corresponds to a lattice point as

\[
P_j = \left( j, -\left[ \frac{j p}{N} \right], -\left[ \frac{j q}{N} \right], -\left[ \frac{j r}{N} \right] \right) \equiv R_j = \left( \frac{j}{N}, \left\{ \frac{j p}{N} \right\}, \left\{ \frac{j q}{N} \right\}, \left\{ \frac{j r}{N} \right\} \right).
\]

There are always lattice points lying “above” (in the 4-dimensional sense) the “marginality hyperplane” \( \Delta \), corresponding to irrelevant operators: these have \( R_j = \sum_i r_i > 1 \). Interior points lying on \( \Delta \) (\( i.e. \) within the tetrahedral cone) have \( R_j = 1 \) and are marginal operators (moduli), while those “below” (in 4D) the hyperplane \( \Delta \) have \( R_j < 1 \) and correspond to tachyons.

Roughly, the more relevant a tachyon, \( i.e. \) the smaller \( R_j \), the deeper its lattice point is in the interior of the cone. This orbifold toric cone can be subdivided by any of the tachyonic or marginal blowup modes: the irrelevant ones are unimportant physically (see \( e.g. \) \(^{40} \), which reviews such toric subdivisions). Heuristically, since the tachyon is a lattice point in the interior of the cone, a subdivision means removing the “top” subcone \( C(T; e_1, e_2, e_3, e_4) \), retaining the four residual subcones \( C(0; e_1, e_2, e_3, T) \), \( C(0; e_1, e_2, e_4, T) \), \( C(0; e_1, e_3, e_4, T) \), \( C(0; e_2, e_3, e_4, T) \). These are potentially orbifold singularities again, unstable to tachyon condensation. The cumulative volume of the four subcones obtained from a subdivision induced by a lattice point corresponding to a twisted sector operator of R-charge \( R_j \) is \( NR_j \); for a tachyon \( R_j < 1 \), this cumulative

Note that for the \( \mathbb{C}^4/\mathbb{Z}_N(1, p, q, r) \) orbifold, we have the relation \( \frac{x_j (1 + p + q + r)}{N} + y_j + z_j + w_j = r_1 + r_2 + r_3 + r_4 = R_j \), so that for a supersymmetric orbifold \( 1 + p + q + r = 0 (mod 2N) \), we have all \( R_j \) integral since \( x_j, y_j, z_j, w_j \in \mathbb{Z} \), \( i.e. \) there are no tachyonic lattice points.
volume, representing the cumulative order of the residual singularities, is less than the original orbifold order \( N \), indicating a partial resolution of the singularity. For example, condensation of the tachyon \( T \) with \( R_T \equiv (\frac{1}{N}, \frac{p}{N}, \frac{q}{N}, \frac{r}{N}) \) in the \( \mathbb{C}^4/\mathbb{Z}_N(1, p, q, r) \) orbifold, corresponds to the subdivision of the cone \( C(0; e_1, e_2, e_3, e_4) \) by the interior lattice point \( T \equiv (1, 0, 0, 0) \). From the GLSM point of view, this corresponds to RG flow of the single Fayet-Iliopoulos parameter in a GLSM with a \( U(1) \) gauge group and charge matrix \( Q = (1 \ p \ q \ r \ -N) \): this gives the resolved phase as the stable phase. With the \( U(1) \) action on \( \Psi_i \equiv (\phi_1, \phi_2, \phi_3, \phi_4, T) \) being \( \Psi_i \rightarrow e^{2\pi i Q_i \lambda} \Psi_i \), the D-term equation is (equivalently by the symplectic quotient construction)

\[
-D \equiv |\phi_1|^2 + p|\phi_2|^2 + q|\phi_3|^2 + r|\phi_4|^2 - N|T|^2 = r//U(1)
\]

the 1-loop renormalization of \( r \) being \( r = \frac{(1+p+q+r-N)}{2\pi} \log \frac{\mu}{\Lambda} \). The coefficient is \( N(R_T - 1) < 0 \), so that \( r \) flows from \( r < 0 \) in the ultraviolet \( \mu \gg \Lambda \) to \( r > 0 \) in the infrared \( \mu \ll \Lambda \). For \( r < 0 \), \( T \) must have a nonzero expectation value, which with the action \( T \rightarrow e^{-2\pi i N \lambda T} \), Higgses the \( U(1) \) down to a residual \( \mathbb{Z}_N \) acting on the light fields \( \phi_i \) as \( \phi_i \rightarrow e^{2\pi i Q_i \lambda} \phi_i \), giving the orbifold \( \mathbb{Z}_N(1, p, q, r) \). Alternatively for \( r > 0 \), one of the \( \phi_i \) must acquire an expectation value, leaving the light fields \( \{\phi_1, \phi_2, \phi_3, T\} \) and other permutations, which give the coordinate charts describing the blown-up \( w\mathbb{C}P^3 \) (with residual \( \mathbb{Z}_p, \mathbb{Z}_q, \mathbb{Z}_r \), orbifold singularities). The partial resolution in this case typically occurs by the blowup of a weighted \( \mathbb{C}P^3 \) with potentially four residual orbifold singularities, as the GLSM D-term shows\(^\text{4}\). From the holomorphic quotient point of view, introduce coordinates \( x_i, i = 1, \ldots, 5 \), corresponding to the lattice points \( e_i, T \), with the \( \mathbb{C}^* \) quotient action \( x_i \rightarrow \lambda^{Q_i} x_i \) with \( \lambda \in \mathbb{C}^* \). Then the divisors (complex codim one hypersurfaces) \( x_i = 0, i = 1, \ldots, 4 \) are noncompact, while \( x_5 = 0 \) is a compact divisor: on \( x_5 = 0 \), the \( \mathbb{C}^* \) action is \( (x_1, x_2, x_3, x_4, 0) \sim (\lambda x_1, \lambda^p x_2, \lambda^q x_3, \lambda^r x_4, 0) \). For a finite size divisor, we expect a non-degenerate description of the 4-dim space: we must therefore exclude the set \( (x_1, x_2, x_3, x_4) = (0, 0, 0, 0) \). This yields the weighted projective space \( \mathbb{C}P^3_{1, p, q, r} \), described by the coordinate chart \( (x_1, x_2, x_3, x_4) \), equivalent to the symplectic quotient we use here. Systems of multiple tachyons in orbifolds can be analyzed by appropriate generalizations of this GLSM as for \( \mathbb{C}^3/\mathbb{Z}_N \) orbifolds\(^\text{10}\) (Appendix C reviews aspects of GLSMs in this context), and generically exhibit 4-dimensional flip transitions amidst their phases as we will see below.

As is usually the case, the dimensions (or R-charges) of various operators are renormalized under an RG flow induced by some relevant operator (say tachyon \( T_1 \)). An interesting feature of these worldsheet supersymmetric systems is that the R-charges of residual tachyons can be calculated using the combinatorics of the toric fan (as for \( \mathbb{C}^3/\mathbb{Z}_N \), where visualization of the toric cone, being 3-dim, was easier!). Since a residual tachyon \( T_2 \) is contained in the interior (or

\(^4\)See e.g.\(^\text{11}\), which uses the mirror Landau-Ginzburg description of\(^\text{5}\) to show that under condensation of a single tachyon, a \( \mathbb{C}^r/\mathbb{Z}_N \) orbifold decays into \( r \) separated orbifolds.
the “walls”) of a subcone say \( C(0; e_1, e_2, e_3, T_1) \equiv \mathbb{Z}_N \), one of whose defining lattice points is the tachyon \( T_1 \), we have a relation of the form \( T_2 = \frac{1}{N} (r_1 e_1 + r_2 e_2 + r_3 e_3 + r_4 T_4) \) with \( N' < N \). Since the marginality hyperplane \( \Delta' \) of this subcone dips inward relative to \( \Delta \) of the original orbifold, the residual tachyon \( T_2 \) is closer to \( \Delta' \) than it was to \( \Delta \). This means \( T_2 \) must become more irrelevant under the RG flow of \( T_1 \). This is a fairly general statement: a tachyon always becomes less tachyonic (\( i.e. \) more irrelevant) under condensation of some tachyon, a fact that can be checked explicitly and is in fact borne out in the examples below.

There are also important consequences of the GSO projection for the residual orbifold subcones and the lattice points in their interior. Recall that an orbifold \( \mathbb{C}^4/\mathbb{Z}_N(k_1, k_2, k_3, k_4) \) admits a Type II GSO projection if \( \sum_i k_i = \text{even} \) and a twist sector-\( j \) operator \( X_j \) with R-charge \( R_j \) is GSO-preserved if \( E_j = \sum_i [\frac{i k_i}{N}] = \text{odd} \). It can be shown that under condensation of a GSO-preserved tachyon \( T_j = (j, -[\frac{j k_j}{N}], -[\frac{j k_j}{N}], -[\frac{j k_j}{N}]) \equiv (\frac{j}{N}, \{ \frac{i}{N} \}, \{ \frac{i}{N} \}, \{ \frac{i}{N} \}) \), the GSO projection for the residual orbifolds and residual tachyons is consistent with this description, \( i.e. \) each of the four residual orbifolds admits a Type II GSO projection. To show this, recall that the Type II GSO projection requires that \( p + q + r = \text{odd} \) and \( \sum [\frac{j}{N}] + [\frac{j}{N}] + [\frac{j}{N}] = \text{odd} \). The four resulting subcones \( C(0; T_j, e_2, e_3, e_4), C(0; T_j, e_1, e_2, e_4), C(0; T_j, e_1, e_3, e_4), C(0; T_j, e_1, e_2, e_3), \) are orbifolds \( \mathbb{C}^4/\mathbb{Z}_n \) \( (w_1, w_2, w_3, w_4) \), with some weights \( w_i \). The subcone \( C(0; T_j, e_2, e_3, e_4) \), with the defining lattice points being in canonical form, can be recognized as \( \mathbb{C}^4/\mathbb{Z}_j(1, [\frac{j}{N}], [\frac{j}{N}], [\frac{j}{N}]) \), which is manifestly Type II. The lattice relation

\[
(1, 0, 0, 0) = \frac{1}{N[\frac{j}{N}]} (pT_j - [\frac{j}{N}] e_1 + (p[\frac{j}{N}] - q[\frac{j}{N}]) e_3 + (p[\frac{j}{N}] - r[\frac{j}{N}]) e_4)
\]

shows that the subcone \( C(0; T_j, e_1, e_3, e_4) \) is the orbifold

\[
\mathbb{C}^4/\mathbb{Z}_N([\frac{j}{N}]) (p, -[\frac{j}{N}], p[\frac{j}{N}] - q[\frac{j}{N}], p[\frac{j}{N}] - r[\frac{j}{N}]),
\]

the orbifold action being on the coordinates represented by \( T_j, e_1, e_3, e_4 \) respectively. Such a linear combination of lattice vectors giving a vector in the original lattice is only defined up to adding integer multiples of the lattice vectors. (If any of the coefficients of \( T_j, e_1, e_3, e_4 \) vanish, the subcone corresponds to a non-isolated, or lower-dim orbifold.) Now from the weights, we see that

\[
p - [\frac{j}{N}] + p[\frac{j}{N}] - q[\frac{j}{N}] + p[\frac{j}{N}] - r[\frac{j}{N}] = p(1 + [\frac{j}{N}] + [\frac{j}{N}] + [\frac{j}{N}]) - (1 + p + q + r)[\frac{j}{N}] = \text{even},
\]

\( i.e. \) the subcone \( C(0; T_j, e_1, e_3, e_4) \) is a Type II orbifold. Similarly the other subcones can be shown to be Type II orbifolds. Also it can be shown that originally GSO-preserved residual tachyons continue to be GSO-preserved after condensation of a GSO-preserved tachyon for each of the four residual singularities.

While the construction of the toric fan is a straightforward generalization from that of \( \mathbb{C}^3/\mathbb{Z}_N \), visualization is now difficult, especially when trying to understand residual tachyons or moduli within a particular subcone obtained by some tachyon or modulus. Algorithmically
then, it is more convenient to find the twisted sector spectrum of an orbifold, note the most relevant tachyons arising therein, and then analyse the phases of the corresponding gauged linear sigma model (GLSM) to glean the structure of flips and the dynamics of these orbifolds.

For example, in $\mathbb{Z}_{19}(1,5,7,9)$, the most relevant (GSO preserved) tachyon $T_8$ lies in the $ccaa$-ring, the next most relevant tachyons arising in several distinct rings. There are two GSO preserved tachyons $T_8, T_4$ of R-charges $R_8 \equiv (\frac{5}{19}, \frac{2}{19}, \frac{1}{19}, \frac{4}{19}) = \frac{15}{19}$ and $R_4 \equiv (\frac{4}{19}, \frac{1}{19}, \frac{10}{19}, \frac{2}{19}) = \frac{17}{19}$ in the $ccaa$-ring (with spectrum equivalent to the $cccc$-ring of $\mathbb{Z}_{19}(1,5,-7,-9)$, the R-charges being $R_j \equiv (\{\frac{j}{19}\}, \{\frac{5j}{19}\}, \{\frac{-7j}{19}\}, \{\frac{-9j}{19}\})$): the GSO exponents $E_j^{ccaa} = [\frac{j}{19}] + [\frac{5j}{19}] + [\frac{-7j}{19}] + [\frac{-9j}{19}]$ can be checked to be odd, thus preserving the tachyons. The phase structure of the geometry and its blowups induced by the condensation of these tachyons can be analysed by a GLSM with charge matrix

$$Q_i^a = \begin{pmatrix} 4 & 1 & 10 & 2 & -19 & 0 \\ 8 & 2 & 1 & 4 & 0 & -19 \end{pmatrix}.$$  

(20)

The phase boundaries are represented by the rays $\phi_1, \phi_2, \phi_4 \equiv (1,2), \phi_3 \equiv (10,1), \phi_5 \equiv (-19,0), \phi_6 \equiv (0,-19)$. The flow-ray is the vector $F \equiv (1,2)$. The relations $T_4 = \frac{1}{2}(T_8 + e_3)$ and $T_8 = \frac{1}{10}(T_4 + 4e_1 + e_2 + 2e_4)$ show that the $T_4$ lattice point lies on the $T_8, e_3$-wall and is collinear with $T_8, e_3$, while $T_8$ lies in the interior of the subcone $C(0; e_1, e_2, e_4, T_4)$. They also show that $T_4$ becomes marginal after condensation of $T_8$, while $T_8$ acquires the renormalized R-charge $R_8' \equiv (\frac{1}{19}, \frac{4}{19}, \frac{1}{19}, \frac{2}{19}) = \frac{1}{5}$. Analyzing the coordinate charts in the phase diagram shows the four phases to correspond to the unresolved orbifold, the partial blowups induced by condensation of $T_4$ or $T_8$ and the complete blowup induced by condensation of both tachyons $T_8, T_4$. The stable phases correspond to the condensation of $T_8$ alone and of that of $T_8, T_4$. The details can be worked out using the techniques that we will describe below.

**Flip transitions:**

In more interesting cases, there are 4-dimensional flip transitions [10]: these occur when a more relevant tachyon condenses during condensation of some tachyon, causing a transition between two topologically distinct resolution endpoints. For instance, in $\mathbb{Z}_{25}(1,7,9,11)$, the most relevant tachyon $T_3$ with R-charge $R_3 \equiv (\frac{3}{25}, \frac{4}{25}, \frac{2}{25}, \frac{3}{25}) = \frac{17}{25}$ lies in the $cacc$-ring: this has spectrum equivalent to the $cccc$-ring of $\mathbb{Z}_{25}(1,9,7,11)$, so that we can, if we wish, effectively define the orbifold in question here as $C^4/\mathbb{Z}_{25}(1,9,7,11)$. There are two more GSO preserved tachyons $T_7, T_{14}$, in this ring, $T_{14}$ being more relevant with R-charge $R_{14} = \frac{2}{25}$. However the structure of decay of the orbifold induced by $T_3, T_7$ exhibits more features so we focus on these in what follows. The R-charges for this ring are $R_j \equiv (\{\frac{j}{25}\}, \{\frac{-7j}{25}\}, \{\frac{9j}{25}\}, \{\frac{11j}{25}\})$, and the GSO exponents $E_j^{ccca} = [\frac{j}{25}] + [\frac{-7j}{25}] + [\frac{9j}{25}] + [\frac{11j}{25}]$ can be checked to be odd, thus preserving the tachyons. The tachyon $T_7$ has R-charge $R_7 \equiv (\frac{7}{25}, \frac{13}{25}, \frac{2}{25}) = \frac{23}{19}$.
The phase structure of the geometry and its blowups induced by the condensation of these tachyons can be analysed by a GLSM with charge matrix\(^5\)

\[
Q^a_i = \begin{pmatrix} 3 & 4 & 2 & 8 & -25 & 0 \\ 7 & 1 & 13 & 2 & 0 & -25 \\ 14 & 2 & 1 & 4 & 0 & 0 \end{pmatrix}.
\]

The phase boundaries are represented by the rays \(\phi_1 \equiv (3, 7), \phi_2, \phi_4 \equiv (4, 1), \phi_3 \equiv (2, 13), \phi_5 \equiv (-25, 0), \phi_6 \equiv (0, -25).\) The flow-ray is the vector \(F \equiv (4, 1).\) The cone is defined by the bounding vectors \((25, 7, -9, -11), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1),\) with the tachyon lattice points being \(T_3 \equiv (3, 1, -1, 1), T_7 \equiv (7, 2, -2, -3).\) We have the relations \(T_3 = \frac{3}{25}e_1 + \frac{4}{25}e_2 + \frac{2}{25}e_3 + \frac{8}{25}e_4\) and \(T_7 = \frac{7}{25}e_1 + \frac{4}{25}e_2 + \frac{13}{25}e_3 + \frac{2}{25}e_4.\) The relations \(T_7 = \frac{1}{4}e_1 + \frac{1}{4}e_3 + \frac{1}{4}T_3\) and \(T_3 = \frac{1}{13}e_1 + \frac{4}{13}e_2 + \frac{4}{13}e_4 + \frac{2}{13}T_7\) respectively show that the \(T_7\) lattice point lies on the plane containing \(e_1, e_3, T_3\) (rather than in the interior of any subcone defined by \(T_3\) with some three of the four points \(e_i\)), while the \(T_3\) lattice point lies in the interior of the subcone \(C(0; e_1, e_2, e_4, T_7).\) The \(T_7\) relation also shows (using (18)) that after condensation of \(T_3,\) the tachyon \(T_7\) acquires a renormalized R-charge \(R'_7 = 1,\) thus becoming marginal.

The D-term conditions (alternatively the symplectic quotient) are

\[
-D_1 \equiv -D_{\phi_1} \equiv 3|\phi_1|^2 + 4|\phi_2|^2 + 2|\phi_3|^2 + 8|\phi_4|^2 - 25|T_3|^2 = r_1 ,
-D_2 \equiv -D_{\phi_5} \equiv 7|\phi_1|^2 + |\phi_2|^2 + 13|\phi_3|^2 + 2|\phi_4|^2 - 25|T_7|^2 = r_2 ,
\]

with \(r_1, r_2\) being the two Fayet-Iliopoulos parameters representing closed string blowup modes. These have the 1-loop renormalizations \(r_1 = (\frac{7}{25}) \log \frac{r}{\Lambda}\) and \(r_2 = (\frac{7}{25}) \log \frac{r}{\Lambda}.\) By eliminating the appropriate coordinate field, we obtain the auxiliary D-terms useful for gleaning properties of the system crossing phase boundaries:

\[
-D_{\phi_1} \equiv |\phi_2|^2 - |\phi_3|^2 + 2|\phi_4|^2 - 7|T_3|^2 + 3|T_7|^2 = \frac{7r_1 - 3r_2}{25} ,
-D_{\phi_2} \equiv -\frac{D_{\phi_4}}{25} \equiv -|\phi_1|^2 - 2|\phi_3|^2 - |T_3|^2 + 4|T_7|^2 = \frac{r_1 - 4r_2}{25} ,
-D_{\phi_3} \equiv |\phi_1|^2 + |\phi_2|^2 + 4|\phi_3|^2 - 13|T_3|^2 + 2|T_7|^2 = \frac{13r_1 - 2r_2}{25} .
\]

Using these D-term equations and the renormalization group flowlines, we can realize the phase structure of this unstable orbifold (see the phase diagram (Fig.1)). For instance, in the convex

\[\underline{5}\text{Including all three tachyons can be analysed by a 3-parameter GLSM with charge matrix}\]

\[
Q^a_i = \begin{pmatrix} 3 & 4 & 2 & 8 & -25 & 0 & 0 \\ 7 & 1 & 13 & 2 & 0 & -25 & 0 \\ 14 & 2 & 1 & 4 & 0 & 0 & -25 \end{pmatrix}.
\]

The flow-ray for this system is \((4, 1, 2) \equiv \phi_2.\) It is possible to analyse this system using the secondary fan and find the stable phases.
phase boundaries divide the convex hulls by the rays from the origin (0,0). The coordinate charts characterizing the toric variety in the phase given by the convex hull of each column in $Q^a_i$ to realize the results of the above analysis of the D-terms for the phase boundaries and the GLSM phases is the following: read off each column in $Q^a_i$ to realize the results of the above analysis of the D-terms for the phase boundaries and vevs respectively. Note that each of these collections of nonzero vevs are also consistent with the other D-terms. Similarly we can analyse the other convex hulls. A simple operational method to realize the results of the above analysis of the D-terms for the phase boundaries and the GLSM phases is the following: read off each column in $Q^a_i$ given in (21) as a ray drawn out from the origin (0,0) in $(r_1, r_2)$-space, representing a phase boundary. Then the various phases are given by the convex hull bounded by any two of the five phase boundaries represented by the rays $\phi_1 \equiv (3, 7)$, $\phi_2 \equiv \phi_4 \equiv (4, 1)$, $\phi_3 \equiv (2, 13)$, $\phi_5 \equiv (-1, 0)$, $\phi_6 \equiv (0, -1)$. These phase boundaries divide $r$-space into five phase regions, each described, as a convex hull of two phase boundaries, by several possible overlapping coordinate charts obtained by noting all the possible convex hulls that contain it. For instance, the coordinate charts describing the convex hull $\{\phi_2, \phi_6\} \equiv \{\phi_4, \phi_6\}$ are read off as the complementary sets $\{\phi_1, \phi_3, \phi_4, \phi_5\}$, $\{\phi_1, \phi_2, \phi_3, \phi_5\}$. This convex hull is contained in the convex hulls $\{\phi_1, \phi_6\}$, $\{\phi_3, \phi_6\}$: thus the full set of coordinate charts characterizing the toric variety in the phase given by the convex hull $\{\phi_2, \phi_6\} \equiv \{\phi_4, \phi_6\}$ is indeed what we have obtained above using the D-term equations.

The coordinate charts describing the phases of this orbifold, obtained as above, are

\[\{\phi_5, \phi_6\} : (\phi_1, \phi_2, \phi_3, \phi_4), \quad (\phi_3, \phi_5) : (\phi_1, \phi_3, \phi_4, \phi_6), (\phi_1, \phi_2, \phi_3, \phi_6), (\phi_2, \phi_3, \phi_4, \phi_6), (\phi_1, \phi_2, \phi_1, \phi_6),\]

\[\{\phi_2, \phi_6\} : (\phi_1, \phi_3, \phi_4, \phi_5), (\phi_1, \phi_2, \phi_3, \phi_5), (\phi_2, \phi_3, \phi_4, \phi_5), (\phi_1, \phi_2, \phi_4, \phi_5),\]

\[\{\phi_1, \phi_2\} : (\phi_3, \phi_4, \phi_5, \phi_6), (\phi_2, \phi_3, \phi_5, \phi_6), (\phi_2, \phi_3, \phi_4, \phi_5), (\phi_1, \phi_2, \phi_4, \phi_5),\]

\[\{\phi_1, \phi_3\} : (\phi_1, \phi_3, \phi_4, \phi_6), (\phi_1, \phi_2, \phi_3, \phi_6), (\phi_2, \phi_3, \phi_4, \phi_6), (\phi_2, \phi_4, \phi_5, \phi_6),\]

\[\{\phi_1, \phi_4\} : (\phi_1, \phi_4, \phi_5, \phi_6), (\phi_1, \phi_2, \phi_5, \phi_6), (\phi_1, \phi_3, \phi_4, \phi_6), (\phi_1, \phi_2, \phi_3, \phi_6),\]

\[\{\phi_2, \phi_5\} : (\phi_1, \phi_2, \phi_4, \phi_5), (\phi_1, \phi_4, \phi_5, \phi_6), (\phi_1, \phi_2, \phi_5, \phi_6).\]

This shows that the convex hull $\{\phi_5, \phi_6\}$ is the unresolved orbifold phase, while $\{\phi_3, \phi_5\}$ and $\{\phi_2, \phi_6\} \equiv \{\phi_4, \phi_6\}$ correspond to partial blowup by condensation of tachyons $T_7$ and $T_3$ respectively. The convex hulls $\{\phi_1, \phi_2\} \equiv \{\phi_1, \phi_4\}$ and $\{\phi_1, \phi_3\}$ correspond to complete resolutions by condensation of both tachyons $T_3$ and $T_7$, one followed by the other, and are related by a flip.

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6A 2-dimensional convex hull is the interior of a region bounded by two rays emanating out from the origin such that the angle subtended by them is less than $\pi$. 

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From the phase diagram, we see that the stable phases correspond to (i) \{\phi_2, \phi_6\} \equiv \{\phi_4, \phi_6\}, condensation of \(T_3\) alone, and (ii) \{\phi_1, \phi_2\} \equiv \{\phi_1, \phi_4\}, condensation of \(T_3\) followed by a blowup by the now-marginal \(T_7\).

A flip transition itself occurs across the \(\phi_1\)-phase boundary, and in the effective subcone \(C(0; e_2, e_3, e_4, T_3, T_7)\): it is described by the effective D-term equation

\[
|\phi_2|^2 + 2|\phi_4|^2 + 3|T_7|^2 - |\phi_3|^2 - 7|T_3|^2 = \frac{7r_1 - 3r_2}{25} = r_f.
\]  

(24)

The RG flow of this effective FI parameter \(r_f\) is \(r_f = \left(\frac{2}{25}\right) \log \frac{\Lambda}{\Lambda_0}\), showing that the flip proceeds in the direction approaching the region \(7r_1 - 3r_2 > 0\), i.e. the stable phase \(r_1 > \frac{3}{7}r_2\). For \(r_f > 0\), this gives a weighted \(\mathbb{CP}^2\) while for \(r_f < 0\), we have a (weighted) \(\mathbb{CP}^1\). As the phase boundary is crossed, the \(\mathbb{CP}^1\) blows down and the more stable \(w\mathbb{CP}^2\) blows up dynamically.

We also see from the auxiliary D-terms that condensation of the tachyon \(T_7\) in orbifold subcones \(C(0; e_1, e_2, e_3)\) and \(C(0; e_1, e_2, e_3, e_5)\) occurs across the phase boundaries \(\phi_2\) and \(\phi_3\).
interpreted as twisted sector tachyons in one of the orbifold subcos.

4-dim toric lattice. Generically these contain lattice points in their interior, which can be

\[ e_2, e_3, e_4, T_3 \equiv \mathbb{Z}_3(1, 2, 2, -1), \]
\[ C(0; e_1, e_2, e_3, T_3) \equiv \mathbb{Z}_4(-1, 2, 0, -1), \]
\[ C(0; e_1, e_2, e_3, T_3) \equiv \mathbb{Z}_4(3, 4, 2, -1), \]

after shifting the weights to obtain Type II orbifolds. The now-marginal \( T_7 \) lies in the residual \( \mathbb{Z}_4 \) orbifold: its blowup (which is a \( w\mathbb{C}P^2 \), from the D-term \( D_{\phi_2} \)) gives rise to a further resolution, with the resulting space described by the eight coordinate charts mentioned earlier. The geometry of these charts and the way they interlink with each other in the full space is somewhat richer than the corresponding structure in \( \mathbb{C}^3/\mathbb{Z}_N \) orbifolds.

### 4.2 Conifold-like \( (n_1, n_2, n_3 - n_4 - n_5) \) singularities

Consider toric singularities defined by five lattice points \( e_i \in \mathbb{Z}^4 \) satisfying \( \sum_i Q_i e_i = 0 \) with

\[ Q_i = (n_1, n_2, n_3 - n_4 - n_5), \]
\[ n_i > 0 \text{ and } \sum_i n_i \neq 0. \]

These are the 4-dim analogs of the unstable conifold-like singularities studied in \[41\]. The maximally supersymmetric subspace in this family with \( \sum_i Q_i = 0 \) corresponds to toric Calabi-Yau cones, in some sense 4-dim analogs of the 3-dim \( L_{abc} \) Calabi-Yau singularities \[42\]. Some of these (and many other classes of singularities) have been discussed in the context of ABJM-like theories in \textit{e.g.} \[22, 23, 24, 25, 26, 27, 28\].

Let us focus for simplicity on singularities with \( n_1 = 1 \): then the singularity with \( Q_1 = (1, n_2, n_3 - n_4 - n_5) \) can be described as a toric cone defined by the five bounding vectors

\[ e_1 = (-n_2, -n_3, n_4, n_5), e_2 = (1, 0, 0, 0), e_3 = (0, 1, 0, 0), e_4 = (0, 0, 1, 0), e_5 = (0, 0, 0, 1) \]

in a 4-dim toric lattice. Generically these contain lattice points in their interior, which can be interpreted as twisted sector tachyons in one of the orbifold subcones.

The D-term equation for these singularities (without any additional operators added) is

\[ n_1|\phi_1|^2 + n_2|\phi_2|^2 + n_3|\phi_3|^2 - n_4|\phi_4|^2 - n_5|\phi_5|^2 = r, \]  \hspace{1cm} (25)

with the RG flow for the Fayet-Iliopoulos parameter being \( r = (\frac{\sum Q_i}{2\pi}) \log \frac{\Lambda}{\mu} \). For singularities with \( n_1 + n_2 + n_3 > n_4 + n_5 \), \textit{i.e.} \( \sum_i Q_i > 0 \), the geometry has an intrinsic flow from \( r > 0 \) (in the ultraviolet, \( \mu \gg \Lambda \)) to \( r < 0 \) (in the infrared, \( \mu \ll \Lambda \)). The \( r > 0 \) phase is a weighted \( \mathbb{C}P^2 \) blown up while the \( r < 0 \) phase is a (weighted) \( \mathbb{C}P^1 \) blown up. Thus the dynamical evolution of such a singularity naturally gives rise to topology change via the blow-down of a \( w\mathbb{C}P^2 \) and the blowup of a \( \mathbb{C}P^1 \). For singularities with \( n_1 + n_2 + n_3 < n_4 + n_5 \), \textit{i.e.} \( \sum_i Q_i < 0 \), the dynamical
The evolution of the geometry is intrinsically from the $r < 0$ (blown up $\mathbb{CP}^1$) phase to the $r > 0$ (blown up $w\mathbb{CP}^2$) phase.

A more detailed sense for the phases can be obtained from the coordinate charts describing the $r > 0$ and $r < 0$ phases: for instance, if $r > 0$, one of the fields $\phi_1, \phi_2, \phi_3$ must acquire a nonzero vev, leaving behind four light fields generically, and similarly for $r < 0$. These give the coordinate charts for the two phases

$$r > 0: \quad (\phi_2, \phi_3, \phi_4, \phi_5), (\phi_1, \phi_3, \phi_4, \phi_5), (\phi_1, \phi_2, \phi_4, \phi_5),$$

$$r < 0: \quad (\phi_1, \phi_2, \phi_3, \phi_4).$$

The subcones in question are potentially orbifold singularities. For instance, with $n_1 = 1$, using the Smith algorithm of [11] or otherwise, it is possible to see that $C(0; e_2, e_3, e_4, e_5) \equiv \text{flat}$, $C(0; e_1, e_3, e_4, e_5) \equiv \mathbb{Z}_{n_2}(1, n_3, -n_4, -n_5)$, $C(0; e_1, e_2, e_4, e_5) \equiv \mathbb{Z}_{n_3}(1, n_2, -n_4, -n_5)$, $C(0; e_1, e_2, e_3, e_4) \equiv \mathbb{Z}_{n_4}(1, n_2, n_3, -n_5)$, $C(0; e_1, e_2, e_3, e_5) \equiv \mathbb{Z}_{n_5}(1, n_2, n_3, -n_5)$, up to shifts of the orbifold weights by the respective orbifold orders, since these cannot be unambiguously determined. It is reasonable then to guess that the Type II GSO projection for such a non-supersymmetric singularity is

$$\sum_i Q_i = n_1 + n_2 + n_3 - n_4 - n_5 = \text{even},$$

based on the known Type II GSO projection $\sum_i k_i = \text{even}$ for $\mathbb{C}^4/\mathbb{Z}_N(k_1, k_2, k_3, k_4)$ orbifolds, if we make the reasonable assumption that the GSO projection is not broken along the RG flow describing the decay of the system. Setting $n_1 = 1$ again for simplicity, this means $n_2 + n_3 - n_4 - n_5 = \text{odd}$ since $\sum_i Q_i$ is only defined mod 2. For instance, say $n_2 = \text{even}$: then $n_3 - n_4 - n_5 = \text{odd}$, and so $C(0; e_1, e_3, e_4, e_5) \equiv \mathbb{Z}_{n_3}(1, n_3, -n_4, -n_5)$ automatically admits a Type II GSO projection. Now if say $n_3 = \text{odd}$, then $n_4 + n_5 = \text{even}$ and $C(0; e_1, e_2, e_4, e_5) \equiv \mathbb{Z}_{n_4}(1, n_2, -n_4, -n_5 \pm n_3)$ also admits a Type II GSO projection after shifting one of the weights by the (odd) order $n_3$. It is straightforward to show that the other cases are similarly consistent with the Type II GSO projection. Finally note also that this is consistent with the supersymmetric subclass with $\sum_i Q_i = 0$, which do admit Type II descriptions.

In general, there are lattice points in the interior of the cone $C(0; e_1, e_2, e_3, e_4, e_5)$, representing possible blowup modes of the singularity. In many cases, with our definitions of the cone bounding vectors, such interior lattice points can be thought of as defining lower order conifold-like singularities: for instance, a lattice point $e_6 \in C(0; e_1, e_3, e_4, e_5)$ defines the subcone $C(0; e_6, e_2, e_3, e_4, e_5)$ for the lower order conifold-like singularity with some $Q_i^{(2)}$ satisfying $\sum_i Q_i^{(2)} e_i' = 0$, where $e_i' \in \{e_6, e_2, e_3, e_4, e_5\}$. This system including the interior lattice point can thus be described using a GLSM with an enlarged charge matrix $Q_i^a$, $a = 1, 2$, where the
second row is $Q_1^{(2)}$. These lattice points can in general be interpreted as twisted tachyons of one or more orbifold subcones above arising in the decay of the conifold-like singularity. Thus we expect that not all such lattice points will be GSO-preserved, since twisted tachyons of the orbifold subcones have nontrivial GSO projections. If an interior lattice point, e.g. $e_6$ above, has to define a Type II lower order conifold-like singularity, then we must have $\sum_i Q_i^a = even$. Thus in general, we obtain the general GSO projection

$$\sum_i Q_i^a = even, \quad a = 1, 2, \ldots$$

(27)

for such unstable singularities. This is essentially imposing the GSO condition \[ (26) \] on each row of the charge matrix that represents a conifold-like singularity.

One of the simplest examples of such an unstable singularity is $Q_i = (1 1 1 1 1 1 1 1 1)$. This decays in the direction of the $\mathbb{CP}^2$ blowup, the $\mathbb{CP}^1$ blowup being less stable. The toric cone in fact contains no lattice points in its interior so that the final endpoint is flat space, the $\mathbb{CP}^2$ being round. The $\mathbb{CP}^1$ blowup contains the residual (supersymmetric) terminal singularity $\mathbb{Z}_4(1,1,1,1)$.

In general however, the decay structure is more intricate, with multiple decay channels due to multiple interior lattice points that define lower order orbifold or conifold-like singularities. Thus we expect that a high order unstable singularity of this sort will typically have a cascade-like decay structure, containing decays to lower order singularities amongst its phases. This is indeed the case. For instance, consider the singularity $Q = (1 7 8 -5 -13)$. The toric cone is defined by the bounding vectors $e_1 = (-7, -8, 5, 13)$ and $e_2, e_3, e_4, e_5$, as above. Then we see that the lattice point $e_6 = (-1, -1, 1, 2) = \frac{1}{13}(e_1 + e_3 + 2e_4 + e_5)$, lies in the interior of the (orbifold) subcone $C(0; e_1, e_2, e_4, e_5) \equiv \mathbb{Z}_7(1, 8, -5, -13)$. Furthermore using \cite{18}, \cite{19}, we can recognize this interior lattice point as the $j = 1$ twisted sector tachyon $R_{j=1} \equiv (\frac{7}{13}, \frac{1}{13}, \frac{2}{13}, \frac{4}{13})$. Note also the relation $e_6 = (-1, -1, 1, 2) = \frac{1}{13}(2e_1 + e_2 + 3e_3 + 3e_4)$, showing that $e_6$, lying in the interior of $C(0; e_1, e_2, e_3, e_4) \equiv \mathbb{Z}_{13}(1, 7, -5, -5)$, represents the $j = 2$ sector tachyon $R_{j=2} \equiv (\frac{2}{13}, \frac{1}{13}, \frac{3}{13}, \frac{3}{13})$. This suggests that this unstable singularity contains a decay to the supersymmetric singularity $Q = (1 1 1 1 1 -1 2)$ amongst its decay phases (see e.g. \cite{21} for a different description of this singularity, referred to as $Y^{1,2}(\mathbb{CP}^2)$ there, and other supersymmetric singularities): indeed we have the relation $e_6 + e_4 + e_3 - e_4 - 2e_5 = 0$. To realize the detailed phase structure of this system (see Fig\[2\]), we use a GLSM with charge matrix

$$Q_i^a = \begin{pmatrix} 1 & 7 & 8 & -5 & -13 & 0 \\ 0 & 1 & 1 & -1 & -2 & 1 \end{pmatrix}.$$  

(28)

The phase boundaries are represented by the rays $\phi_1 \equiv (1,0)$, $\phi_2 \equiv (7,1)$, $\phi_3 \equiv (8,1)$, $\phi_4 \equiv (-5,-1)$, $\phi_5 \equiv (-13,-2)$, $\phi_6 \equiv (0,1)$. Figure\[2\] shows the various phases and the correspond-
Figure 2: The 3-dim hyperplane of the \((1\ 7\ 8\ -5\ -13)\) flip singularity. The lattice point \(e_6\) (and the subdivisions thereof) depicted here is really the projection onto this 3-dim hyperplane of the actual point (which are in the 4-dim interior of the cone).

ing subdivisions of the toric cone\(^7\). Although this might be slightly difficult to visualise, it helps to note sub-planes defined by three of the six lattice points and the relations of the other lattice points relative to the subplanes, \textit{e.g.} note that \(e_4\) and \(e_6\) lie on the same side of the \(\{e_1, e_2, e_3\}\)-plane, as can be seen from the relation \(e_6 = \frac{1}{13}(2e_1 + e_2 + 3e_3 + 3e_4)\). Ultimately, the phase structure is obtained from the D-term equations

\[
-D_1 \equiv -D_{\phi_6} \equiv |\phi_1|^2 + 7|\phi_2|^2 + 8|\phi_3|^2 - 5|\phi_4|^2 - 13|\phi_5|^2 = r_1 ,
\]

\[
-D_2 \equiv -D_{\phi_1} \equiv |\phi_2|^2 + |\phi_3|^2 + |\phi_6|^2 - |\phi_4|^2 - 2|\phi_5|^2 = r_2 ,
\]

(29)

and the auxiliary D-term equations across the other four phase boundaries (obtained by eliminating the corresponding field)

\[
-D_{\phi_2} \equiv |\phi_1|^2 + |\phi_3|^2 + 2|\phi_4|^2 + |\phi_5|^2 - 7|\phi_6|^2 = r_1 - 7r_2 ,
\]

\(^7\)Note that the subcone \(C(0; e_6, e_2, e_3, e_4, e_5)\) representing the supersymmetric \(Q = (1\ 1\ 1\ -1\ -2)\) singularity has the following toric subdivisions for the \(w\mathbb{CP}^2\) and \(\mathbb{CP}^1\) blowup phases, with

\(r > 0:\ (\phi_2, \phi_4, \phi_5, \phi_6), (\phi_3, \phi_4, \phi_5, \phi_6), (\phi_2, \phi_3, \phi_4, \phi_5), \) \(r < 0:\ (\phi_2, \phi_3, \phi_5, \phi_6), (\phi_2, \phi_3, \phi_4, \phi_6).\)
\[-D_{\phi_3} \equiv |\phi_1|^2 - |\phi_2|^2 + 3|\phi_4|^2 + 3|\phi_5|^2 - 8|\phi_6|^2 = r_1 - 8r_2 ,\]
\[-D_{\phi_4} \equiv |\phi_1|^2 + 2|\phi_2|^2 + 3|\phi_3|^2 - 3|\phi_5|^2 - 5|\phi_6|^2 = r_1 - 5r_2 ,\]
\[-D_{\phi_5} \equiv 2|\phi_1|^2 + |\phi_2|^2 + 3|\phi_3|^2 + 3|\phi_4|^2 - 13|\phi_6|^2 = 2r_1 - 13r_2 .\]  

Using these, or equivalently the operational method described earlier, one finds the various phases with corresponding coordinate charts shown in Figure 2. The 1-loop renormalization of \( r_1 = \frac{-2}{2\pi} \log \frac{\mu}{\Lambda} \), while \( r_2 \), with no renormalization is a modulus. The system thus flows in the direction of the flow-ray \( F \equiv (1, 0) \), which adjoins the stable phases: these include the decay to the supersymmetric \( Q = (1 1 1 - 1 - 2) \) singularity, the stable phases including the \( w\mathbb{C}P^2 \) and \( \mathbb{C}P^1 \) blowups thereof. The occasional (rare) decay of the system precisely along the flow ray (1, 0), \textit{i.e.} along \( r_2 = 0 \), \( r_1 \to \infty \), yields the singular point in the moduli space of the supersymmetric singularity.

The ultraviolet of the system is the direction \((-1, 0)\) opposite the flow-ray, contained in the convex hull \( \{\phi_5, \phi_6\} \): this phase corresponds to the unstable \( \mathbb{C}P^1 \) shrinking in (RG) time. Beginning in this phase, the structure of the D-term equations shows that the RG evolution to the two stable phases goes through one of two possible paths, crossing phase boundaries (i) \( \phi_6, \phi_2, \phi_3 \), or (ii) \( \phi_5, \phi_4 \). The corresponding phase transitions occurring in the process are: (i) a flip occurs across the phase boundaries \( \phi_6, \phi_3, \phi_4 \), (ii) condensation of tachyons corresponding to the lattice points \( e_6 \in C(0; e_1, e_3, e_4, e_5) \) and \( e_6 \in C(0; e_1, e_2, e_3, e_4) \) orbifold subcones occurs across \( \phi_2 \) and \( \phi_5 \) respectively, while (iii) the phase boundary \( \phi_1 \) corresponds to a flop. The final phases correspond to the stable \( w\mathbb{C}P^2 \) expanding outwards, with possible residual singularities on its locus.

5 \textit{M2-branes stacked at} \( \mathbb{C}^4/\mathbb{Z}_N \) \textit{and nonsupersymmetric} \( \text{AdS}_4 \times S^7/\mathbb{Z}_N \) \textit{backgrounds}

In recent times, a Chern-Simons field theory description dual to the near horizon \( \text{AdS}_4 \times S^7/\mathbb{Z}_N \) backgrounds obtained from M2-branes stacked at \( \mathbb{C}^4/\mathbb{Z}_N(1, 1, 1, 1) \) singularities has been found in [14]. Various generalizations of this to M2-branes stacked at diverse singularities have been studied in \textit{e.g.} [16, 17, 18, 19, 20, 23, 24, 25, 26, 27, 28, 29] (see also \textit{e.g.} [21, 22] for some early work on 4-dim singularities). It would seem along these lines that the dual field theories to nonsupersymmetric \( \text{AdS}_4 \times S^7/\mathbb{Z}_N \) backgrounds obtained from M2-branes stacked at nonsupersymmetric \( \mathbb{C}^4/\mathbb{Z}_N \) singularities would also be Chern-Simons theories, but

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\(^8\)Strictly speaking, there is a constant quantum shift in the location of the classical singularity at \( r_2^{(0)} = 0 \): this arises from the 1-loop bosonic potential (see Appendix C) as \( t_2^{eff} = t_2^{(0)} + \frac{1}{2\pi} \sum_i Q_i^2 \log |Q_i^2| = 0 \) defining the singular point \( r_2^{(0)} = r_2^{\prime} \), giving a real codimension-2 singularity after including the effects of the \( \theta \)-angle.
nonsupersymmetric ones. One might imagine that a prescription inspired by [43] for D-branes at orbifold singularities with image M2-branes on a covering space might also work in this case: this has been studied for various supersymmetric orbifolds in e.g. [16, 17, 19]. It would be interesting to explore this further in the nonsupersymmetric context.

We have seen that nonsupersymmetric $\mathbb{C}^4/\mathbb{Z}_N$ orbifold singularities are unstable, with 11-dimensional lifts of closed string tachyons or moduli ensuring the resolutions of these singularities. We have used the 11-dimensional reflections of Type IIA worldsheet string descriptions of these singularities and the phase structure of associated GLSMs to glean the structure of these singularities. One might ask if a similar description could be obtained using an M2-brane probe and Higgsing therein via the couplings of possible Fayet-Iliopoulos parameters to the M-theory background blowup modes, along the lines of a D3-brane probe description of lower dimensional supersymmetric orbifold singularities, as [43, 44] discussed. Indeed a basis of irreducible loops on the quiver can be mapped to the gauge invariant monomials obtained (for toric singularities) in a GLSM description or equivalently the holomorphic quotient construction (see e.g. [45] for a more recent explicit description generalizing [43, 44] to the context of 3-dim Calabi-Yau $L_{abc}$ singularities, and related work [16, 17]). Investigations of this kind for M2-branes in the vicinity of various classes of supersymmetric singularities have been performed in [28]. In the nonsupersymmetric case, it was found already in [3] that in $\mathbb{C}^2/\mathbb{Z}_N$ (and unlike $\mathbb{C}/\mathbb{Z}_n$), the moduli space of the gauge theory was not identical to the geometry and its partial resolutions. It is quite possible that this will be the case for $\mathbb{C}^4/\mathbb{Z}_N$ and M2-brane probes too, once spacetime supersymmetry is broken. However it would be interesting to explore this further, especially in the light of the findings of [48]. Since the Chern-Simons level is related to the orbifold order for the supersymmetric cases [14], it would seem that perhaps tachyon condensation (which induces partial resolutions lowering the order of a singularity) will give rise to flows modifying the Chern-Simons level.

Consider now a stack of $k$ M2-branes placed at a nonsupersymmetric $\mathbb{C}^4/\mathbb{Z}_N(k_1, k_2, k_3, k_4)$ singularity, the full M-theory background being of the form $\mathbb{R}^{2,1} \times \mathbb{C}^4/\mathbb{Z}_N$. The orbifold can be thought of as a cone over $S^7/\mathbb{Z}_N$. Then for a large number of M2-branes, taking the near horizon limit gives a nonsupersymmetric $AdS_4 \times S^7/\mathbb{Z}_N$ background, the radial direction of the orbifold combining with $\mathbb{R}^{2,1}$ to give $AdS_4$. If the group $\mathbb{Z}_N$ acts freely on the $S^7$, the resulting $S^7/\mathbb{Z}_N$ space is smooth. This implies that there are no fixed points on $S^7$ where localized tachyons can arise so that the large flux $AdS_4 \times S^7/\mathbb{Z}_N$ limit is apparently tachyon free and thus potentially a stable nonsupersymmetric background. In the D-brane context for $AdS_5 \times S^5/\Gamma$, various aspects were discussed in [49, 50].

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9Preliminary investigations (with R. Plesser), in the incipient stages of [9] seemed to corroborate this for $\mathbb{C}^3/\mathbb{Z}_N$ singularities.
This however is too quick: there turns out to be a nonperturbative gravitational instability [33] of the sort that plagues a Kaluza-Klein background manifested by Witten’s “bubble of nothing” [34]. Along the same lines, we expect that the $AdS_4 \times S^7 / \mathbb{Z}_N$ can be recast as a Kaluza-Klein compactification over an $S^1$ of $AdS_4 \times w\mathbb{CP}^3$, for some appropriate weighted projective 3-space $w\mathbb{CP}^3$ and $S^1$ periodicity $\sim \frac{1}{N}$. This then will give rise to a similar bubble-of-nothing instability for nonsupersymmetric $AdS_4 \times S^7 / \mathbb{Z}_N$ backgrounds, which are then expected to decay rapidly: as pointed out in the $AdS_5 \times S^5 / \mathbb{Z}_N$ case [33], the decay rate for a conformal theory (no scale) must be zero or infinite, and the integral over the radial coordinate diverges.

An interesting detail here requires understanding the fermion boundary conditions across the $S^1$ for these nonsupersymmetric cases. It would seem that the structure here for general orbifold weights is intricate from the supergravity point of view, since as we run through possible weights, for precisely the supersymmetric values (which are stable), the instanton must not exist. For instance, $\mathbb{Z}_N(1,3,5,7)$ is a supersymmetric singularity, while $\mathbb{Z}_N(1,5,7,9)$ is not, and $\mathbb{Z}_N(1,5,7,11)$ again is supersymmetric. From this point of view, strictly speaking it is not obvious if every nonsupersymmetric $AdS_4 \times S^7 / \mathbb{Z}_N$ background necessarily admits a KK-instanton (manifest in supergravity) that mediates its decay.

The noncompact case apart, as for $AdS_5 \times S^5 / \mathbb{Z}_N$ interestingly pointed out by [35] (and already noted in [33]), the decay rate in the present case is strictly infinite only if the throat is infinitely long corresponding to M2-branes stacked at a noncompact $\mathbb{C}^4 / \mathbb{Z}_N$ singularity. If the throat is instead embedded in an orbifold of an appropriate compact Calabi-Yau 4-fold, then the divergence of the decay rate is regulated by the ultraviolet region, which is the compact Calabi-Yau orbifold. With the instanton action being $B$, cutting off the field theory at a scale $\Lambda$, i.e. at a radial coordinate $r_{UV} \sim R^2 \Lambda$, $R$ being the AdS radius, gives the total decay rate per unit 2 + 1-dim field theory volume as

$$\Gamma \sim e^{-B} \int_{r_0}^\infty dr r^2 \sim e^{-B} \Lambda^3,$$

$$B \sim \frac{r_0^3}{G_{11}} \sim \frac{k^{3/2}}{N^9},$$

where $r_0 \sim \frac{R}{N}$ is the radial coordinate value where the instanton is capped off. Thus for fixed orbifold order $N$ and large number $k$ of M2-branes, the instanton action is large, giving a small decay rate, as for $AdS_5 \times S^5 / \mathbb{Z}_N$ argued in [33]. In this context of a throat cutoff at some $r_{UV}$ with a compact space, perhaps such backgrounds provide useful stable nonsupersymmetric $AdS_4 \times S^7 / \mathbb{Z}_N$ throats in M-theory.

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A Aspects of the $\mathbb{C}^4/\mathbb{Z}_N$ spectrum

The twisted sector spectrum of the $\mathbb{C}^4/\mathbb{Z}_N(k_1, k_2, k_3, k_4)$ orbifold conformal field theory, classified using the representations of the $(2, 2)$ superconformal algebra, has a product-like structure. The worldsheet supercurrents for each complex plane are $G^+_i = \psi^*_i \partial X_i$ or $G^-_i = \psi_i \partial X_i$, with the $U(1)$ currents being $J_i = \psi^*_i \psi_i$. Consider a twist sector $j$, with boundary conditions $X^i(\sigma + 2\pi, \tau) = e^{2\pi ijk_i/N} X^i(\sigma, \tau)$. The worldsheet fermions have half-integral moding, $\psi(z) = \sum_{r \in \mathbb{Z}} \psi^*_r(z^r + (\frac{jk_i}{N}) + \frac{1}{2})$, and $\psi_1(z) = \sum_{r \in \mathbb{Z}} \psi^*_r(z^r + (\frac{jk_i}{N}) + \frac{1}{2})$; thus $\psi_{\frac{jk_i}{N} - 1}(\frac{1}{2})$ (or $\psi^*_{\frac{jk_i}{N} - 1}(\frac{1}{2})$) changes from a creation (or annihilation) operator to an annihilation (or creation) operator as $\{\frac{jk_i}{N}\}$ grows greater than $\frac{1}{2}$, with respect to the twist ground state $|0\rangle_j$, annihilated by all operators with positive moding (see e.g. [51] for a lucid discussion on this). Thus for a complex plane-$i$, $|0\rangle_j$ changes from being a chiral state with $G^+_{-\frac{i}{2}}|0\rangle_j = 0$ to an anti-chiral state with $G^-_{-\frac{i}{2}}|0\rangle_j = 0$ (note that $G^+_{-\frac{i}{2}} = \psi^*_{\frac{jk_i}{N} - 1}(\frac{j}{N}) + \ldots$ and $G^-_{-\frac{i}{2}} = \psi_{\frac{jk_i}{N} - 1}(\frac{j}{N}) + \ldots$, the $\alpha$'s being the operators entering in the worldsheet boson mode expansions). Likewise for $0 < \{\frac{jk_i}{N}\} < \frac{1}{2}$, the first excited state is $\psi_{\frac{jk_i}{N} - 1}(\frac{j}{N}) |0\rangle_j$ and is antichiral, while for $\{\frac{jk_i}{N}\} > \frac{1}{2}$, the first excited state is $\psi^*_{\frac{jk_i}{N} - 1}(\frac{j}{N}) |0\rangle_j$ is chiral. The product structure of the orbifold conformal field theory implies that the spectrum of ground and first excited states can be segregated into various chiral and antichiral rings comprising states that are chiral under either $G_i^+$ or $G_i^-$ for each complex plane: thus e.g. the $(c_{x_1}, c_{x_2}, c_{x_3}, c_{x_4})$ ring consists of states chiral under $\sum_{i=1}^4 G_i^+$, while e.g. the $(a_{x_1}, a_{x_2}, a_{x_3}, a_{x_4})$ ring consists of states chiral under $\sum_{i=1}^4 G_i^+ + G_4^-$. This gives sixteen chiral and anti-chiral rings in eight conjugate pairs.

The zero point energy for the left-moving modes for a single orbifolded complex plane is (with $a_i = \{\frac{jk_i}{N}\}$)

$$E'_0 = \frac{1}{2} \sum_{n=0}^{\infty} (n + a_i) + \frac{1}{2} \sum_{n=0}^{\infty} (n - a_i) - \frac{1}{2} \sum_{n=0}^{\infty} (n + \frac{1}{2} + a_i) - \frac{1}{2} \sum_{n=0}^{\infty} (n + \frac{1}{2} - a_i)$$
$$= \frac{1}{2} a_i - \frac{1}{8}, \quad 0 < a_i < \frac{1}{2}, \quad (32)$$

which, adding up, gives

$$E_0 = \frac{1}{2} \sum_i \{\frac{jk_i}{N}\} - \frac{1}{2}, \quad 0 < \{\frac{jk_i}{N}\} < \frac{1}{2}, \quad (33)$$

where we have used the regularized sum $\sum_{n=0}^{\infty} (n + a) = \frac{1}{24} - \frac{1}{8}(1 - 2a)^2$. If say $\frac{1}{2} < \{\frac{jk_i}{N}\} < 1$, then with the new creation-annihilation operators entering, the zero point energy is modified to

$$E'_0 = \frac{1}{2} \sum_{i \neq 4} \{\frac{jk_i}{N}\} - \frac{1}{2} - \frac{1}{2} \left[ - \left( \frac{1}{2} - \{\frac{jk_4}{N}\} \right) + \left( \{\frac{jk_i}{N}\} - \frac{1}{2} \right) \right] = \frac{1}{2} \sum_{i \neq 4} \{\frac{jk_i}{N}\} - \frac{1}{2} - \frac{1}{2} \{\frac{jk_4}{N}\}, \quad (34)$$
which can be recast as \( E'_0 = \frac{1}{2} \sum_{i} \left( \frac{j k_i}{N} \right) - \frac{1}{2} \), for \( k'_i = (k_1, k_2, k_3, -k_4) \). Thus the conformal weights and R-charges of the twist ground states satisfy

\[
E_0 = \Delta - \frac{1}{2}, \quad \Delta = \pm \frac{1}{2} R.
\]

We now describe the RNS partition functions: these are generalizations of those for non-supersymmetric \( \mathbb{C}^3/\mathbb{Z}_N \) singularities [9]. The Type 0 string on \( \mathbb{C}^4/\mathbb{Z}_N (k_1, k_2, k_3, k_4) \) has a diagonal GSO projection that ties together the left and right movers: it has the partition function

\[
Z = \frac{1}{2N} \sum_{j,l=0}^{N-1} \prod_{i=1}^{4} \frac{\eta(\tau)}{\theta_{1/2+j k_i/N}^{1/2+j k_i/N}(0, \tau)} \left[ \prod_{i=1}^{4} \theta_{i k_i/N}^{j k_i/N} \right]^{2} + \left[ \prod_{i=1}^{4} \theta_{i k_i/N+1/2}^{j k_i/N+1/2} \right]^{2} \pm \left[ \prod_{i=1}^{4} \theta_{i k_i/N+1/2}^{j k_i/N+1/2} \right]^{2},
\]

which exists for any \( k_i, N \). On the other hand, the 1-loop partition function on a \( \mathbb{C}^4/\mathbb{Z}_N (k_1, k_2, k_3, k_4) \) orbifold for a Type II string with separate GSO projections on the left and right movers is given by the sum over twisted sectors as

\[
Z = \frac{1}{4N} \sum_{j,l=0}^{N-1} \prod_{i=1}^{4} \frac{\eta(\tau)}{\theta_{1/2+j k_i/N}^{1/2+j k_i/N}(0, \tau)} \left| \eta^{i}(\tau) \right|^{2},
\]

\[
\zeta_j^i = \prod_{i=1}^{4} \theta_{i k_i/N}^{j k_i/N} - e^{-i \pi \sum_{i} \frac{j k_i}{N}} \prod_{i=1}^{4} \theta_{i k_i/N+1/2}^{j k_i/N+1/2} - e^{-i \pi \sum_{i} \frac{j k_i}{N}} \prod_{i=1}^{4} \theta_{i k_i/N+1/2}^{j k_i/N+1/2},
\]

\( \zeta_j^i \) contains the sum over spin structures for the \( j \)-th twisted sector twisted by \( g^j \) in the “time” direction. The terms in \( Z \) are recognized as the contributions from the twisted bosons in the orbifolded complex dimensions and the fermionic contributions.

This is modular invariant (in particular under the S-transformation) if the phase from the third term above satisfies \( e^{i \pi \sum \frac{j k_i}{N}} = e^{-i \pi \sum \frac{(N-j) k_i}{N}} \), in other words,

\[
\sum_{i} \frac{(N-l) k_i}{N} = - \sum_{i} \frac{l k_i}{N} \quad \text{even} \quad \Rightarrow \quad \sum_{i} k_i = \text{even}.
\]

We can now expand the Type II partition function [37] by expanding the \( \theta \)-functions as \( \theta_{[a]}^{[a]}(0, \tau) = \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}} e^{2\pi i (n+a)b} \), \( q = e^{2\pi i \tau} \), to realize the GSO projection on the twisted states: we obtain the projector

\[
1 - (-1)^{\sum j k_i/N}
\]

for the ground states in the sector where \( \{ j k_i/N \} < \frac{1}{2} \), i.e. the \( \{ c_{X_1}, c_{X_2}, c_{X_3}, c_{X_4} \} \) ring. This is a projector onto twisted states with \( \sum_{i} j k_i/N = E_j = \text{odd} \). In the untwisted \( j = 0 \) sector, this
is in accord with the usual \([1 + (-1)^F] GSO\) projection that removes the (bulk) closed string tachyon (after accounting for a \((-1)^F\) from the ghost contribution to the worldsheet \((-1)^F\)).

Consider now e.g. the sector where \(\{\frac{jk}{N}\} > \frac{1}{2}\) with other \(\{\frac{jk}{N}\} < \frac{1}{2}\). Then we obtain the projector

\[
1 - (-1)^{(\sum_i [jk_i/N] - 1)}
\]

for the ground states (which are in the \((c_{x_1}, c_{x_2}, c_{x_3}, a_{x_4})\) ring), i.e. \(E_j = \sum_i [jk_i/N] = \text{even}\).

The chiral operators \(X_j\) are obtained as the excited state with one extra fermion number from \(\psi_4\) which therefore have the GSO projection \(\sum_i [jk_i/N] = E_j^{ccon} = \text{odd}\), as before. Likewise if two of \(\{\frac{jk}{N}\} > \frac{1}{2}\), we have \(E_j = \text{odd}\) for the ground states so that the \(X_j\), obtained with one extra fermion number in the two sectors, again have \(E_j = \text{odd}\) and so on. Thus the GSO exponent for the chiral operators \(X_j\) is \(E_j = \sum_i [jk_i/N] = \text{odd}\).

Thus we see that the GSO exponents for the various rings are

\[
E_j = \text{odd}, \quad (c_{x_1}, c_{x_2}, c_{x_3}, c_{x_4}), (c_{x_1}, c_{x_2}, a_{x_3}, a_{x_4}), (c_{x_1}, a_{x_2}, c_{x_3}, a_{x_4}), (c_{x_1}, a_{x_2}, a_{x_3}, c_{x_4}),
\]

\[
E_j = \text{even}, \quad (c_{x_1}, c_{x_2}, c_{x_3}, a_{x_4}), (c_{x_1}, c_{x_2}, a_{x_3}, c_{x_4}), (c_{x_1}, a_{x_2}, a_{x_3}, a_{x_4}), (a_{x_1}, a_{x_2}, a_{x_3}, a_{x_4}),
\]

\[
(a_{x_1}, a_{x_2}, a_{x_3}, c_{x_4}), (a_{x_1}, a_{x_2}, c_{x_3}, c_{x_4}), (a_{x_1}, a_{x_2}, c_{x_3}, a_{x_4}), (a_{x_1}, a_{x_2}, a_{x_3}, c_{x_4}),
\]

\[
(a_{x_1}, a_{x_2}, a_{x_3}, a_{x_4}).
\]

(42)

Note that this is consistent with both a twist field and its conjugate field (in the conjugate ring) being GSO-preserved. For instance a \((c_{x_1}, c_{x_2}, c_{x_3}, c_{x_4})\)-ring twist field operator \(X_j = \prod_{i=1}^4 X_{\frac{jk_i}{N}}\) has its conjugate field \(X_j^* = \prod_{i=1}^4 (X_{\frac{jk_i}{N}})^* = \prod_{i=1}^4 (X_{(N-j)\frac{jk_i}{N}})^* = \prod_{i=1}^4 (X_{(N-j)\frac{jk_i}{N}})^*\), lying in the \((N-j)\)-th twist sector in the conjugate ring \((a_{x_1}, a_{x_2}, a_{x_3}, a_{x_4})\). So we see that if \(X_j\) is preserved, i.e. \(E_j = \text{odd}\), then \(E_{N-j} = \sum_i [\frac{(N-j)jk_i}{N}] = -E_j - 4 + \text{even} = \text{odd}\) too, preserving the conjugate field in the conjugate ring too.

We can equivalently understand this by “engineering” a chiral Type II GSO projection for a \(\mathbb{C}^4/\mathbb{Z}_N(k_1, k_2, k_3, k_4)\) orbifold, consistent with that for lower dimensional \(\mathbb{C}^3/\mathbb{Z}_N\) orbifolds \(\mathbb{Z}_2\).

Complexify the eight transverse untwisted fermions as \(\psi_i = e^{iH_i}, \ i = 0, 1, 2, 3,\) and consider a symmetry acting on the untwisted (complex) fermions and the twist fields via \(H_i \to H_i + a_i \pi\),

\[
\psi_i \to \psi_i e^{ia_i \pi} \equiv \psi_i (-1)^{a_i}, \quad X_j \to X_j \exp \left[ i\pi \sum_i a_i \left\{ \frac{jk_i}{N} \right\} \right] \equiv X_j (-1)^{E_j}.
\]

(43)

This defines a \((-1)^F\) \(\mathbb{Z}_2\) action on the untwisted sector thus eliminating the bulk tachyon only if the \(a_i\) are odd integers. The action on the twisted states \(X_j\) is a well-defined \(\mathbb{Z}_2\) if the exponent \(E_j\) is an integer. This GSO exponent can be written as \(E_j = \sum_i a_i \left\{ \frac{jk_i}{N} \right\} = \)
\[ \sum_{i=1}^{N} a_i k_i - \sum_{i=1}^{N} a_i \left( \frac{i k_i}{N} \right). \] Thus \( E_j \) is integral if we have \( a_i = \text{odd} \) satisfying \( \sum_{i=1}^{N} a_i k_i = 0 \pmod{2N} \).

If \( \sum_{i=1}^{N} a_i k_i = \text{odd} \), we see that \( \sum_{i=1}^{N} a_i k_i = \sum_{i=1}^{3} \left( a_i - a_4 \right) k_i + \text{odd} = \text{odd} \), since the first three terms (containing differences of odd integers) are even. Thus no odd \( a_i \) exist satisfying \( \sum_{i=1}^{N} a_i k_i = 0 \pmod{2N} \) if \( \sum_{i=1}^{N} a_i k_i = \text{odd} \) (assuming even \( N \); for odd \( N \), one can shift one of the weights to make \( \sum_{i=1}^{N} a_i k_i = \text{even} \)).

For \( \sum_{i=1}^{N} a_i k_i = \text{even} \), we thus have \( E_j = \sum_{i=1}^{N} a_i \left( \frac{i k_i}{N} \right) \sum_{i=1}^{N} a_i \left( \frac{i k_i}{N} \right), \) and \( \sum_{i=1}^{N} a_i k_i = 0 \pmod{2N} \). Thus for a twisted state \( T \) with R-charge \( R = (r_1, r_2, r_3, r_4) \) in the orbifold \( C^4/Z_N \) \((k_1, k_2, k_3, k_4)\), the GSO exponent is \( E = \sum_{i=1}^{N} a_i r_i \) with \( a_i = \text{odd} \) and \( \sum_{i=1}^{N} a_i k_i = 0 \pmod{2N} \).

Now for a Type II orbifold \( C^4/Z_N (1, p, q, r) \), we have \( p + q + r = \text{odd} \): then \( a_1 = p + q + r, a_2 = a_3 = a_4 = -1, \) satisfy \( \sum_{i=1}^{N} a_i k_i = a_1 + a_2 p + a_3 q + a_4 r = 0 \pmod{2N} \). This gives the GSO exponent \( E_j = \left[ \frac{p}{N} \right] + \left[ \frac{q}{N} \right] + \left[ \frac{r}{N} \right] = \sum_{i=1}^{N} \left( \frac{i k_i}{N} \right). \)

B The Maple program

The Maple code we have used is sufficiently simple and we give it here:

```maple
w := [1,7,9,11];
k := [[w[1],w[2],w[3],w[4]], [w[1],w[2],w[3],-w[4]], [w[1],w[2],-w[3],w[4]],
[w[1],-w[2],w[3],w[4]], [-w[1],w[2],w[3],w[4]], [-w[1],w[2],-w[3],w[4]],
[w[1],-w[2],w[3],-w[4]], [w[1],-w[2],-w[3],w[4]], [w[1],-w[2],-w[3],-w[4]]];
for N from 2 to 400 do
    for j from 1 to 8 do
        for l from 1 to N-1 do
            rl[j,i] := l*k[j,i]/N - floor(l*k[j,i]/N):
            fl[j,i] := floor(l*k[j,i]/N):
        end do:
        Rl := sum('rl[j,i]', 'i=1..4');
        El := sum('fl[j,i]', 'i=1..4');
        if (Rl < 1) then if (type(El,odd)) then
            print(N,j,l,[rl[j,1],rl[j,2],rl[j,3],rl[j,4]],Rl,El,'tachyon')
        end if:
        if (Rl = 1) then if (type(El,odd)) then
            print(N,j,l,[rl[j,1],rl[j,2],rl[j,3],rl[j,4]],Rl,El,'marginal')
        end if:
    end do:
end do:
end do:
```

This is a significantly improved and simplified version of a code written for \( C^3/Z_N \) towards the completion of [9]. Once we input the weights \( w = (w_1, w_2, w_3, w_4) \), the program calculates
the various GSO-preserved twisted sector R-charges in the eight pairs of chiral and antichiral rings of a $\mathbb{C}^4/\mathbb{Z}_N(w_1, w_2, w_3, w_4)$ orbifold for $N \leq 400$, and lists tachyons and moduli that arise as $N$ increases. Thus if a particular orbifold order $N_0$ does not appear in the program output, there are no tachyons or moduli in its spectrum, \textit{i.e.} it is terminal. Various modifications of this can be easily written to accommodate a different range for $N, k_i$, specific rings, or calculate \textit{e.g.} the spectrum of a Type 0 orbifold with a diagonal GSO projection.

C Some aspects of GLSMs

This subsection is essentially a direct generalization (primarily for completeness) of the techniques described in [10, 11] to the 4-dim singularities in question here. The full phase structure of a (noncompact) $\mathbb{C}^4/\mathbb{Z}_N$ orbifold geometry (such as those discussed in this paper) with $n$ tachyons is obtained by studying the Higgs branch of the moduli space of an enlarged gauged linear sigma model (GLSM), admitting $(2, 2)$ worldsheet supersymmetry, with gauge group $U(1)^n$, $4 + n$ chiral superfields $\Psi_i$ and $n$ Fayet-Iliopoulos parameters $r_a$. The action of such a GLSM (in conventions of [1, 37]) is

\[ S = \int d^2z \left[ d^4\theta \left( \bar{\Psi}_i e^{2Q^a_i \Theta_a} \Psi_i - \frac{1}{4e_a^2} \Sigma_a \Sigma_a \right) + \text{Re} \left( it_a \int d^2\theta \Sigma_a \right) \right], \]

(44)

where summation on the index $a = 1, \ldots, n$ is implied. The $t_a = ir_a + \frac{\theta_a}{2\pi}$ are Fayet-Iliopoulos parameters and $\theta$-angles for each of the $n$ gauge fields ($e_a$ being the gauge couplings). The twisted chiral superfields $\Sigma_a$ (whose bosonic components are complex scalars $\sigma_a$) represent field-strengths for the gauge fields. The action of the $U(1)^n$ gauge group on the $\Psi_i$ is given in terms of the $n \times (4 + n)$ charge matrix $Q^a_i$ above as $\Psi_i \rightarrow e^{iQ^a_i \Theta_a} \Psi_i$, $a = 1, \ldots, n$. For the conifold-like singularities, we have $5 + n$ superfields and $n + 1$ FI parameters, with a gauge group $U(1)^{n+1}$ and a $(n + 1) \times (5 + n)$ charge matrix: the $n$ superfields in this case represent lattice points in the interior of one of the subcones in the original singular cone $C(0; e_1, e_2, e_3, e_4, e_5)$. For instance, we have $Q^a_i$ in (21) for the orbifold with $n = 2$ tachyons, and $n = 1$ interior lattice point in (28) for the conifold-like singularity. Such a charge matrix only specifies the $U(1)^n$ action up to a finite group, due to the possibility of a $Q$-linear combination of the rows of the matrix also having integral charges. The specific form of $Q^a_i$ is chosen to conveniently illustrate specific geometric substructures, \textit{e.g.} the tachyons contained within the orbifold, or subcones representing lower order conifold-like singularities. The variations of the $n$ independent FI parameters control the vacuum structure of the theory. The space of classical ground states of this theory can be found from the bosonic potential $U = \sum_a \frac{(D_a)^2}{2e_a^2} + 2 \sum_{a, b} \sigma_a \sigma_b \sum_i Q^a_i Q^b_i |\Psi_i|^2$. Then $U = 0$ requires $D_a = 0$: solving these for $r_a \neq 0$ gives expectation values for the $\Psi_i$, which Higgs the gauge group down to some discrete subgroup and lead to mass terms for the
whose expectation values thus vanish. The classical vacua of the theory are then given in terms of solutions to the D-term equations

$$\frac{-D_a}{e^2} = \sum_i Q_i^a |\Psi_i|^2 - r_a = 0, \quad a = 1, \ldots, n.$$ \hfill (45)

At the generic point in r-space, the $U(1)^n$ gauge group is completely Higgsed, giving collections of coordinate charts that characterize in general distinct toric varieties. In other words, this $n$-parameter system admits several “phases” (convex hulls in r-space, defining the secondary fan) depending on the values of the $r_a$. At boundaries between these phases where some (but not all) of the $r_a$ vanish, some of the $U(1)$s survive giving rise to singularities classically. Each phase is an endpoint since if left unperturbed, the geometry can remain in the corresponding resolution indefinitely (within this noncompact approximation): in this sense, each phase is a fixed point of the GLSM RG flow. However some of these phases are unstable while others are stable, in the sense that fluctuations (e.g. blowups/flips of cycles stemming from instabilities) will cause the system to run away from the unstable phases towards the stable ones. This can be gleaned from the 1-loop renormalization of the FI parameters

$$r_a = \left( \sum_i Q_i^a \right) \log \frac{\mu}{\Lambda}, \hfill (46)$$

where $\mu$ is the RG scale and $\Lambda$ is a cutoff scale where the $r_a$ are defined to vanish. Energy scales here are defined relative to that set by the gauge coupling $e$, which has mass dimension one in 2-dim here. The full GLSM RG flow first goes from free gauge theory in the ultraviolet $\mu \gg e$ through $\mu \ll e$ but with nontrivial dynamics w.r.t. $\Lambda$. Thus in the low energy regime $\mu \ll e$, fluctuations transverse to the moduli space cost energy and the low-lying fluctuations are simply scalars defining the moduli space of the theory: thus the GLSM RG flows approximate the nonlinear ones and a geometric description emerges, given by a nonlinear sigma model on the moduli space. With attention restricted to quasi-topological observables in an appropriate topologically twisted A-model, the gauge coupling $e^2$ itself is not crucial to the discussion.

A generic linear combination of the gauge fields coupling to a linear combination $\sum_a \alpha_a r_a$ of the FI parameters, the $\alpha_a$ being arbitrary real numbers, has a 1-loop running whose coefficient vanishes if

$$\sum_{a=1}^n \sum_{i=1}^{n+4} \alpha_a Q_i^a = 0,$$ \hfill (47)

in which case the linear combination is marginal. This equation defines a codimension-one hyperplane perpendicular to a ray, called the Flow-ray, emanating from the origin and passing through the point $(-\sum_i Q_i^1, -\sum_i Q_i^2, \ldots, -\sum_i Q_i^n)$ in r-space which has real dimension $n$ (for orbifolds; for conifold-like singularities, $a = 1, \ldots, n+1, i = 1, \ldots, 5+n$). Using the
redefinition $Q_i^a' \equiv (\sum_i Q_i^1)Q_i^a - (\sum_i Q_i^n)Q_i^1$, $a \neq 1$, we see that $\sum_i Q_i^{a'} = (\sum_i Q_i^1)(\sum_i Q_i^n) - (\sum_i Q_i^n)(\sum_i Q_i^1) = 0$, for $a \neq 1$, so that the FI parameters coupling to these redefined $n-1$ gauge fields have vanishing 1-loop running. Thus there is a single relevant direction (along the flow-ray) and an $(n-1)$-dimensional hyperplane of the $n-1$ marginal directions in $r$-space. By studying various linear combinations $\sum_a \alpha_a r_a$, we see that the 1-loop RG flows drive the system along the single relevant direction to the (stable) phases in the large $r$ regions of $r$-space, i.e., $r_a \gg 0$ (if none of the $r_a$ is marginal), that are adjacent to the Flow-ray $F \equiv (-\sum_i Q_i^1; -\sum_i Q_i^2; \ldots; -\sum_i Q_i^n)$, or contain it in their interior.

The direction precisely opposite to the Flow-ray, i.e. $-F \equiv (\sum_i Q_i^1; \sum_i Q_i^2; \ldots; \sum_i Q_i^n)$, defines the ultraviolet of the theory. This ray $-F$ will again lie either in the interior of some one convex hull or adjoin multiple convex hulls. It corresponds to the maximally unstable direction which is the unresolved orbifold phase for orbifold geometries or generically the unstable $w\mathbb{CP}^2$ or $w\mathbb{CP}^1$ resolution for the conifold-like singularities.

We restrict attention to the large $r_a$ regions in the phase diagrams (in Figures 1, 2), thus ignoring worldsheet instanton corrections: this is sufficient for understanding the phase structure, and consistent for initial values of $r_a$ whose components in the marginal directions lie far from the center of the marginal $(n-1)$-plane.

The 1-loop renormalization of the FI parameters can be expressed in terms of a perturbatively quantum-corrected twisted chiral superpotential for the $\Sigma_a$ for a general $n$- or $n+1$-parameter system, obtained by considering the large-$\sigma$ region in field space and integrating out those scalars $\Psi_i$ that are massive here (and their expectation values vanish energetically). This leads to the modified potential $U(\sigma) = \frac{s^2}{2} \sum_{a=1}^{n} \left| i\dot{t}_a - \frac{\sum_i Q_i^a}{2\pi} (\log(\sqrt{2} \sum_{b=1}^{n} Q_i^b \sigma_b/\Lambda) + 1) \right|^2$ (for orbifolds). The singularities predicted classically at the locations of the phase boundaries arise from the existence of low-energy states at large $\sigma$. The physics for the nonsupersymmetric cases here is somewhat different from the cases where $\sum_a Q_i^a = 0$ for all $a$, as discussed in general in (1) (and for 3-dim singularities in [10, 11]). Consider the vicinity of such a singularity at a phase boundary but far from the (fully) singular region where all $r_a$ are zero, and focus on the single $U(1)$ (with say charges $Q_i^1$) that is unbroken there (i.e. we integrate out the other $\sigma_a$, $a \neq 1$, by setting them to zero). Now if $\sum_i Q_i^1 = 0$ (i.e. unbroken spacetime supersymmetry), then there is a genuine singularity when $U(\sigma) = \frac{s^2}{2} |i\dot{t}_1 - \frac{1}{2\pi} \sum_i Q_i^1 \log |Q_i^1||^2 = 0$, and if $\sum_i Q_i^a = 0$ for all $a$, this argument can be applied to all of the $U(1)$s. However for the nonsupersymmetric cases here, we have $\sum_i Q_i^a \neq 0$: so if say $\sum_i Q_i^1 \neq 0$ (with the other $Q_i^a$ redefined to $Q_i^a'$ with $\sum_i Q_i^a' = 0$), then along the single relevant direction where $\sum_i Q_i^1 \neq 0$, the potential energy has a $|\log \sigma_1|^2$ growth. Thus the field space accessible to very low-lying states is effectively compact (for finite worldsheet volume) and there is no singularity for any $r_a, \theta_a$, along the RG flow: in other words, the RG flow is smooth along the relevant direction for all values of
$t_1$, and the phase boundaries do not indicate singularities. In these nonsupersymmetric cases, Coulomb branch vacua arise in the IR of the flow $[1352]$.

## D Phases of supersymmetric singularities

Here we apply the techniques we have used here to glean the phase structure of supersymmetric orbifold and conifold-like singularities. We will elucidate two examples.

The orbifold $\mathbb{C}^4/\mathbb{Z}_{11}(1,1,5,7)$ has moduli arising in the $ccca$-ring, with spectrum equivalent to $\mathbb{Z}_{11}(1,1,5,−7)$: these are the GSO-preserved twisted sector states with R-charges $R_1 \equiv (\frac{1}{11}, \frac{1}{11}, \frac{5}{11}, \frac{4}{11}) = 1$ and $R_3 \equiv (\frac{3}{11}, \frac{3}{11}, \frac{4}{11}, \frac{1}{11}) = 1$. It has the toric cone defined by $e_1 = (11, -1, -5, 7), e_2 = (0, 1, 0, 0), e_3 = (0, 0, 1, 0), e_4 = (0, 0, 0, 1)$. Using (18) and (19), the moduli can be seen to correspond to the lattice points $P_1 = (1, 0, 0, 1), P_3 = (3, 0, -1, 2)$, lying on the marginality hyperplane. We have also the relations $P_1 = \frac{1}{3}(P_3 + e_3 + e_4)$ and $P_3 = \frac{1}{4}(e_1 + e_2 + P_1 + e_3)$, which indicate $P_1$ lies on the $\{P_3, e_3, e_4\}$ plane, and $P_3 \in C(0; P_1, e_1, e_2, e_3)$.

From the point of view of the toric cone, the various phases arise from the different ways of blowing up the singularity by these moduli. These phases can be described by a GLSM with charge matrix

$$Q_i = \begin{pmatrix} 1 & 1 & 5 & 4 & -11 & 0 & -11 \\ 3 & 3 & 4 & 1 & 0 & -11 \end{pmatrix}. \quad (48)$$

The structure of this charge matrix is very similar to (21), with five phases, the phase boundaries being $\phi_1 = \phi_2 = (1, 3), \phi_3 = (5, 4), \phi_4 = (4, 1), \phi_5 = (-1, 0), \phi_6 = (0, -1)$. The FI-parameters do not run, being marginal. The phases, related by marginal deformations, correspond to the unresolved orbifold, partial blowup by $R_1$ or $R_3$, and complete blowups by $R_1, R_3$, or $R_3, R_1$, the last two related by a 4-dim flop. After the $R_1$ blowup, the residual singularity $\mathbb{Z}_4(1, 1, -3, 1)$ contains the GSO-preserved modulus $P_3 \equiv (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) = 1$ which completely resolves it, while the residual $\mathbb{Z}_5(1, 1, 4, -11) \equiv \mathbb{Z}_5(1, 1, 1, 1)$ singularity is terminal.

The $Q = (1 \ 5 \ 6 \ -4 \ -8 \ )$ singularity is defined by the toric cone with $e_1 = (-5, -6, 4, 8), e_2 = (1, 0, 0, 0), e_3 = (0, 1, 0, 0), e_4 = (0, 0, 1, 0), e_5 = (0, 0, 0, 1)$, in a 4-dim lattice. The defining relation $-5e_2 = e_1 + 6e_3 - 4e_4 - 8e_5$ shows the subcone $C0; e_1, e_3, e_4, e_5)$ to be the orbifold $\mathbb{Z}_5(1, 1, -4, -8)$. The relation $(-1, -1, 1, 2) \equiv e_6 = \frac{1}{5}(e_1 + e_3 + e_4 + 2e_5)$, shows $e_6$ to be an interior lattice point that lies on the marginality hyperplane of this orbifold: using (18), it can be recognized as the GSO-preserved $j = 1$ twisted sector modulus of $\mathbb{Z}_5(1, 1, -4, -8)$. The phase structure of this system, obtained from a GLSM with charge matrix

$$Q_i^a = \begin{pmatrix} 1 & 5 & 6 & -4 & -8 & 0 \\ 0 & 1 & 1 & -1 & -2 & 1 \end{pmatrix}, \quad (49)$$

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is similar to that of Figure 2 except that there are five phases, the phase boundaries (two of them coincident) being \( \phi_1 = (1,0), \phi_2 = (5,1), \phi_3 = (6,1), \phi_4 = \phi_5 = (-4,-1), \phi_6 = (0,1) \). The FI parameters are marginal and have no RG flow. Being supersymmetric, there are no flips, just marginal flops: crossing the various phase boundaries corresponds to either a flop or condensation of a twisted sector modulus in some residual orbifold subcone.

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