A HOMOLOGICAL BRIDGE BETWEEN FINITE AND INFINITE DIMENSIONAL REPRESENTATIONS OF ALGEBRAS

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Abstract. Given a finite dimensional algebra Λ, we show that a frequently satisfied finiteness condition for the category \( P^\infty(\Lambda\text{-mod}) \) of all finitely generated (left) \( \Lambda \)-modules of finite projective dimension, namely contravariant finiteness of \( P^\infty(\Lambda\text{-mod}) \) in \( \Lambda\text{-mod} \), forces arbitrary modules of finite projective dimension to be direct limits of objects in \( P^\infty(\Lambda\text{-mod}) \). Among numerous applications, this yields an encompassing sufficient condition for the validity of the first finitistic dimension conjecture, that is, for the little finitistic dimension of \( \Lambda \) to coincide with the big (this is well-known to fail over finite dimensional algebras in general).

1. Introduction

Let \( \Lambda \) be an Artin algebra – the reader may, for example, think of a finite dimensional algebra over a field – and let \( \Lambda\text{-mod} \) be the category of all finitely generated left \( \Lambda \)-modules, \( \Lambda\text{-Mod} \) the category of all left \( \Lambda \)-modules. The full subcategories consisting of the objects of finite projective dimension in \( \Lambda\text{-mod} \) and \( \Lambda\text{-Mod} \) are denoted by \( P^\infty(\Lambda\text{-mod}) \) and \( P^\infty(\Lambda\text{-Mod}) \), respectively. Our interest is focused on the following two homological dimensions of \( \Lambda \): the left little finitistic dimension, \( \ell \text{fin.dim} \Lambda \), which is the supremum of the projective dimensions attained on the objects in \( P^\infty(\Lambda\text{-mod}) \), and the left big finitistic dimension, \( \ell \text{Fin.dim} \Lambda \), which is defined correspondingly based on \( P^\infty(\Lambda\text{-Mod}) \). For several decades, the smoothest possible connection between \( P^\infty(\Lambda\text{-mod}) \) and \( P^\infty(\Lambda\text{-Mod}) \), namely equality of the little and big finitistic dimensions, had been conjectured to hold, at least when \( \Lambda \) is finite dimensional over a field. However, in 1991 this equality was shown to fail, even over finite dimensional monomial relation algebras [13]. Our goal here is to prove that a certain, very frequently satisfied, finiteness condition on \( P^\infty(\Lambda\text{-mod}) \) entails that each object in the big category \( P^\infty(\Lambda\text{-Mod}) \) is a direct limit of objects in the small category \( P^\infty(\Lambda\text{-mod}) \); in particular, this condition implies that the two finitistic dimensions coincide.

Before we state this result with greater precision, we give a brief overview over previously established connections between \( \Lambda\text{-mod} \) and \( \Lambda\text{-Mod} \). In the late seventies, Ringel used the thorough understanding of \( \Lambda\text{-mod} \) for tame hereditary algebras \( \Lambda \) which was available at that point to give detailed descriptions of representations in \( \Lambda\text{-Mod} \) in this situation [19]. We quote from his introduction: “The recent progress in the representation theory of finite dimensional algebras was limited mainly to the modules of finite length and one would be interested to know in which way the structure of the modules of finite length determines the behavior of arbitrary modules. Two results of this type are known...”. The results he subsequently refers to are as follows: If \( \Lambda \) has finite representation type, then all objects in \( \Lambda\text{-Mod} \) are direct sums of objects from \( \Lambda\text{-mod} \) (see [1] and [20]); if, on the other hand, \( \Lambda \)

The work of the first author was partially supported by an NSF grant, while that of the second author was partially supported by a Fulbright grant.
fails to have finite representation type, \( \Lambda \text{-Mod} \setminus \Lambda \text{-mod} \) contains indecomposable objects \([2]\). In the meantime, Crawley-Boevey added several striking results to this list by showing that, for a finite dimensional algebra \( \Lambda \) over an algebraically closed base field, certain non-finitely generated modules, called generic modules, completely determine the representation type of \( \Lambda \); they are the non-finitely generated indecomposable modules which have finite length over their endomorphism rings (see \([6\) and \([7]\)). In particular, he proved that \( \Lambda \) has infinite representation type if and only if there exist generic objects in \( \Lambda \text{-Mod} \), tame representation type if and only if, for each dimension \( d \), there exist only finitely many generic \( \Lambda \)-modules having length \( d \) over their endomorphism rings; finally, \( \Lambda \) has domestic type precisely when there are only finitely many isomorphism classes of generic modules in \( \Lambda \text{-Mod} \) altogether.

In 1979, Auslander and Smalø introduced the concept of contravariant finiteness of a full subcategory \( A \) of \( \Lambda \text{-mod} \) \([4]\). \( A \) is said to be contravariantly finite in \( \Lambda \text{-mod} \) in case each object \( M \) in \( \Lambda \text{-mod} \) has an \( A \)-approximation in the following sense: there exists a homomorphism \( f : A \to M \) with \( A \in A \) such that each map in \( \text{Hom}_\Lambda(B, M) \) with \( B \in A \) factors through \( f \). One of the reasons why this concept is of major importance to the homology of a finite dimensional algebra lies in the fact that, if \( P^\infty(\Lambda \text{-mod}) \) is contravariantly finite in \( \Lambda \text{-mod} \), then the minimal \( P^\infty(\Lambda \text{-mod}) \)-approximations of the simple left \( \Lambda \)-modules – \( A_1, \ldots, A_n \) say – constitute the basic building blocks for arbitrary representations in \( P^\infty(\Lambda \text{-mod}) \). In fact, due to Auslander and Reiten \([3]\), a module \( M \) belongs to \( P^\infty(\Lambda \text{-mod}) \) if and only if \( M \) is a direct summand of a module \( N \) having a filtration \( N = N_0 \supset N_1 \supset \cdots \supset N_m = 0 \) such that all consecutive factors \( N_i/N_{i+1} \) belong to \( \{A_1, \ldots, A_n\} \). In particular, this structure result implies of course that \( \ell \text{fin.dim } \Lambda = \sup \{p \text{dim } A_i \mid i = 1, \ldots, n\} \).

Our main result (Theorem 4.4) states that contravariant finiteness of \( P^\infty(\Lambda \text{-mod}) \) in \( \Lambda \text{-mod} \) forces each left \( \Lambda \)-module of finite projective dimension to be a direct limit of finitely generated modules of finite projective dimension. More strongly, each object in \( P^\infty(\Lambda \text{-mod}) \) is then a direct limit of finitely generated modules having filtrations with consecutive factors in \( \{A_1, \ldots, A_n\} \). As a consequence, contravariant finiteness of \( P^\infty(\Lambda \text{-mod}) \) entails the equality \( \ell \text{fin.dim } \Lambda = \ell \text{Fin.dim } \Lambda \). This answers in the positive a question left open in \([10]\). As an extra bonus, the result tells us where to look if we wish to determine the big finitistic dimension, a notorious quandary in general. In view of the preceding paragraph, we can obviously conclude that \( \ell \text{Fin.dim } \Lambda = \sup \{p \text{dim } A_i \mid i = 1, \ldots, n\} \) in the contravariantly finite case. We illustrate a ‘typical’ computation of this ilk with an example. Another application of our main theorem, combining it with a result of Crawley-Boevey, was pointed out to us by H. Krause. Namely, contravariant finiteness of \( P^\infty(\Lambda \text{-mod}) \) in \( \Lambda \text{-mod} \) implies covariant finiteness and thus secures the existence of almost split sequences in \( P^\infty(\Lambda \text{-mod}) \). On the other hand, the conclusion of our result is not left-right symmetric. In the final section of this paper, we present a finite dimensional monomial relation algebra \( \Lambda \) such that \( P^\infty \) is contravariantly finite on one side, while the big and little finitistic dimensions of \( \Lambda \) differ on the other.

Some of our tools should be of independent interest. In Section 2, we show that, over a left noetherian ring, each left module \( M \) is the directed union of those countably generated submodules which have projective dimensions bounded above by that of \( M \). As a consequence, the left big finitistic dimension of a left artinian ring equals the supremum of those finite projective dimensions which are attained on countably generated left modules. In Section 3, we collect some closure properties of the category Add \( M \) consisting of all direct sums of
direct summands of a fixed module $M$. In particular, we explore closure of such categories under direct limits.

**Acknowledgements.** This paper was written while the second author was visiting the University of California at Santa Barbara, and he would like to thank the members of the mathematics department of UCSB, and in particular his coauthor, for the hospitality he received during his stay there. Both authors are indebted to Hans Sverre Smalø for an excellent job of typesetting and graphing.

2. The big finitistic dimension equals the countably generated finitistic dimension

The primary goal of this section is to show that, for each left artinian ring $R$, the big finitistic dimension, $\ell \text{Fin.dim} R$, coincides with the supremum of those finite projective dimensions which are attained on countably generated left $R$-modules. The argument is an off-spring of Kaplansky’s classical theorem [16] stating that every projective $R$-module is a direct sum of countably generated components.

Recall that a module $X$ is said to be countably presented if there exists an exact sequence

$$R^{(K)} \to R^{(L)} \to X \to 0$$

with $|K|, |L| \leq \aleph_0$. Clearly, every left noetherian ring has the property that each of its countably generated left modules is countably presented; the same is true if the base ring $R$ is countable or a countably generated algebra over a field.

**Proposition 2.1.** Let $R$ be a ring with the property that each countably generated left $R$-module is countably presented. Then each left $R$-module $M$ is the directed union of those countably generated submodules which have projective dimensions bounded above by that of $M$.

**Remarks.** 1. This proposition is akin to Corollary 3.2.5 of [18], but gives more structural information about the module $M$. We thank H. Krause for bringing this reference to our attention.

2. Observe that, in the conclusion of Proposition 2.1, the attribute ‘countably generated’ cannot be replaced by ‘finitely generated’. In fact, not even over a finite dimensional algebra, need modules of finite projective dimension be direct limits of finitely generated modules of finite projective dimension. This is an immediate consequence of the fact that the big finitistic dimension may exceed the little.

**Proof of the proposition:** We may assume that $\text{p dim } M < \infty$, say $\text{p dim } M = m$. It clearly suffices to show that each countable subset of $M$ is contained in a countably generated submodule $M'$ with $\text{p dim } M' \leq m$. Let

$$0 \to P_m \xrightarrow{f_m} P_{m-1} \xrightarrow{f_{m-1}} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

be a projective resolution of $M$. By Kaplansky’s theorem [16], each $P_i$ is of the form $P_i = \bigoplus_{j \in A_i} P_{ij}$, where all of the $P_{ij}$’s are countably generated.
Given a countable subset $U$ of $M$, choose a countable subset $B_0^{(1)} \subseteq A_0$ such that $f_0(\bigoplus_{j \in B_0^{(1)}} P_{0j})$ contains $U$. Then the kernel of the restriction of $f_0$ to $\bigoplus_{j \in B_0^{(1)}} P_{0j}$ is in turn countably generated by hypothesis, which permits us to pick a countable set $B_1^{(1)} \subseteq A_1$, with the property that
\[
\text{Ker}(f_0) \cap \left( \bigoplus_{j \in B_0^{(1)}} P_{0j} \right) \subseteq f_1 \left( \bigoplus_{j \in B_1^{(1)}} P_{1j} \right).
\]

An obvious induction further yields countable subsets $B_i^{(1)} \subseteq A_i$ for $1 \leq i \leq m$ such that
\[
\text{Ker}(f_{i-1}) \cap \left( \bigoplus_{j \in B_i^{(1)}} P_{i-1,j} \right) \subseteq f_i \left( \bigoplus_{j \in B_i^{(1)}} P_{ij} \right)
\]
for $1 \leq i \leq m$.

Now set $B_i^{(2)} = B_i^{(1)}$. In view of the fact that the image
\[
f_m \left( \bigoplus_{j \in B_i^{(2)}} P_{mj} \right) \subseteq \text{Ker} f_{m-1}
\]
is countably generated, we can find a countable subset $B_{m-1}^{(2)} \subseteq A_{m-1}$ containing $B_{m-1}^{(1)}$ such that
\[
f_m \left( \bigoplus_{j \in B_{m-1}^{(2)}} P_{mj} \right) \subseteq \text{Ker} f_{m-1} \cap \left( \bigoplus_{j \in B_{m-1}^{(2)}} P_{m-1,j} \right).
\]
An induction analogous to the preceding one then yields countable subsets $B_i^{(2)} \subseteq A_i$, $0 \leq i \leq m$ such that $B_i^{(1)} \subseteq B_i^{(2)}$ and
\[
f_i^{+1} \left( \bigoplus_{j \in B_i^{(2)}} P_{ij} \right) \subseteq \text{Ker}(f_i) \cap \left( \bigoplus_{j \in B_i^{(2)}} P_{ij} \right)
\]
for $0 \leq i \leq m - 1$.

Next set $B_0^{(3)} = B_0^{(2)}$, and continue. We iterate these $m$-step inductions moving back and forth along the given projective resolution of $M$, and an induction on this level will supply us, for each $k \in \mathbb{N}$, with countable sets $B_0^{(k)}, \ldots, B_m^{(k)}$ such that $B_i^{(k-1)} \subseteq B_i^{(k)} \subseteq A_i$ for $0 \leq i \leq m$, having the additional properties that
\[
\text{Ker}(f_{i-1}) \cap \left( \bigoplus_{j \in B_i^{(k)}} P_{i-1,j} \right) \subseteq f_i \left( \bigoplus_{j \in B_i^{(k)}} P_{ij} \right)
\]
for $1 \leq i \leq m$ and odd $k$, as well as
\[
f_i^{+1} \left( \bigoplus_{j \in B_i^{(k)}} P_{i+1,j} \right) \subseteq \text{Ker}(f_i) \cap \left( \bigoplus_{j \in B_i^{(k)}} P_{ij} \right)
\]
for $0 \leq i \leq m - 1$ and even $k$.

Finally, we set $B_i = \bigcup_{k \in \mathbb{N}} B_i^{(k)} \subseteq A_i$ for $0 \leq i \leq m$, define $M'$ to be the countably generated submodule $f_0(\bigoplus_{j \in B_0} P_{0j})$ of $M$, and observe that
Corollary 2.2. If \( R \) is a left artinian ring, then \( \ell \operatorname{Fin.dim} \) equals \( \sup \{ p \dim M \mid M \text{ a countably generated left } R\text{-module with } p \dim M < \infty \} \).

**Proof:** Let \( N \) be any left \( R\)-module of finite projective dimension. Since \( R \) satisfies the hypothesis of Proposition 2.1, we have \( N = \lim_{i \in I} N_i \), where \( \{N_i\}_{i \in I} \) is a directed family of countably generated modules with \( p \dim N_i \leq p \dim N \) for \( i \in I \). In view of the fact that the functor Tor commutes with direct limits, this implies that the flat dimension of \( N \) is bounded above by the supremum of the flat dimensions of the \( N_i \). But our hypothesis on \( R \) makes the flat dimension equal to the projective dimension and thus completes the argument. \( \square \)

### 3. Remarks on the category \( \text{Add}(M) \)

Given a module \( M \), we denote by \( \text{Add}(M) \) the full subcategory of \( R\)-Mod having as objects the modules which are isomorphic to arbitrary direct sums of direct summands of \( M \). The main purpose of this section is to assemble a number of properties of \( \text{Add}(M) \) for a \( \Sigma \)-pure injective module \( M \), to be used towards our main theorem. (That \( M \) be \( \Sigma \)-pure injective means that all direct sums of copies of \( M \) are pure injective.) However, due to the fact that this category holds interest in its own right, we will go a little beyond the requirements of the following section.

Start by recalling that every finitely generated module over an Artin algebra is \( \Sigma \)-pure injective, since of finite length over its endomorphism ring. On the other hand, \( \Sigma \)-pure injectives are far from being finitely generated over the base ring, in general; in particular, any generic module has this property. More generally, recall that the results in \[22\] and \[9\] give an equivalent characterization of \( \Sigma \)-pure injectivity of an \( R\)-module \( M \) in terms of a descending chain condition for certain submodules of \( M \) over its endomorphism ring, a condition which is obviously inherited by pure \( R \)-submodules of \( M \); as a consequence, \( \Sigma \)-pure injectivity is passed on to pure submodules.

Our first observations address closure of the category \( \text{Add}(M) \) under direct limits (we follow the convention of reserving the term ‘direct limit’ for colimits over directed index sets). A twin sibling of the following fact was proved by Lenzing in \[17\], where it is shown that, for any finitely presented module \( M \) of finite length, the category \( \text{Add}(M) \) has the mentioned closure property.

**Observation 3.1.** Let \( R \) be any ring and \( M \) a \( \Sigma \)-pure injective left \( R \)-module. Then every directed system of objects in \( \text{Add}(M) \) attains its direct limit in \( \text{Add}(M) \).

**Proof:** Let \( (A_i, f_{ij})_{i,j \in I, i \leq j} \) be any directed system of objects \( A_i \) in \( \text{Add}(M) \), and denote by \( A \) its direct limit. Moreover, let \( \iota_j : A_j \to \bigoplus_{i \in I} A_i \) be the canonical embeddings. It is well known that the short exact sequence

\[
0 \to \bigoplus_{i,j \in I, i < j} (\iota_j f_{ij} - \iota_i)(A_i) \to \bigoplus_{i \in I} A_i \to A \to 0
\]
is pure. Its middle term being \( \Sigma \)-pure injective by hypothesis, the sequence therefore splits. Now each \( \Sigma \)-pure injective module is a direct summand of components, each of which has a local endomorphism ring \([22], p. 1100\), and consequently the Krull-Remak-Schmidt-Azumaya Theorem guarantees that all terms of the above exact sequence in turn belong to Add(\( M \)). \( \square \)

Interestingly, the question whether Add(\( M \)) is closed under direct limits can be decided by testing solely well-ordered chains in general. We record this fact as

**Observation 3.2.** Given any left \( R \)-module \( M \), the category Add(\( M \)) is closed under arbitrary direct limits, provided that it is closed under direct limits over well-ordered index sets.

**Proof:** Again we let \((A_i, f_{ij})_{i,j \in I, i \leq j}\) be a directed system of objects in Add(\( M \)) and denote its direct limit by \( A \). If the index set \( I \) is finite, our claim is trivial, so let us suppose that \(|I| = \aleph_\alpha\) for some ordinal number \( \alpha \). We will prove the observation by a transfinite induction on \( \alpha \). In case \( \alpha = 0 \), we may clearly assume that \( I = \{i_1, i_2, i_3, \ldots\} \) and that \( i_j < i_k \) in the partial order of \( I \) whenever \( j < k \). Our hypothesis therefore guarantees that \( A \) is in turn an object of Add(\( M \)).

Now we suppose that \( \alpha > 0 \) and that for all ordinal numbers \( \beta < \alpha \) our claim is true; in other words, we assume that all direct limits of directed systems of objects in Add(\( M \)) extending over index sets of cardinalities less than \( \aleph_\alpha \) belong to Add(\( M \)). We write \( \Omega_\alpha \) for the first ordinal number of cardinality \( \aleph_\alpha \) and index the set \( I \) by the ordinals below \( \Omega_\alpha \), i.e., we write \( I = \{i_\beta \mid \beta < \Omega_\alpha\} \). It will be convenient to identify an ordinal number \( \gamma \) with the set of all ordinal numbers strictly smaller than \( \gamma \). Our aim is to show that \( A \) is the direct limit of a directed system \((C_\beta, g_{\beta, \gamma})_{\beta, \gamma < \Omega_\alpha, \beta \leq \gamma}\) with the property that all of the objects \( C_\beta \) belong to Add(\( M \)). Since the new index set, \( \Omega_\alpha \), is totally ordered, we can then again invoke the hypothesis to complete the proof.

For that purpose, we observe that, given any infinite subset \( I' \subseteq I \), there exists a directed subset \( I'' \) of \( I \) which contains \( I' \) and has the same cardinality as \( I' \). This allows us to express \( I \) as the directed union of a chain of subsets \( I_\gamma, \gamma < \Omega_\alpha \), such that \( I_\gamma \subseteq I_\delta \) whenever \( \gamma < \delta \), that \( I_\gamma \) is finite whenever \( \gamma \) is finite, and that \( I_\gamma \) has cardinality less than or equal to that of \( \gamma \) whenever \( \gamma \) is an infinite ordinal number. Indeed, set \( I_0 = \{i_0\} \), let \( \delta > 0 \), and suppose that, for all \( \gamma < \delta \), directed subsets \( I_\gamma \) satisfying the above requirements and having the property that \( \{i_\beta \mid \beta < \gamma\} \subseteq I_\gamma \) have already been constructed. Set \( L_\delta = \bigcup_{\gamma < \delta} I_\gamma \cup \{i_\delta\} \). By construction \( L_\delta \) is either finite or else has a cardinality bounded above by that of \( \delta \). If \( L_\delta \) is finite, we can clearly choose a finite directed subset \( I_\delta \) of \( I \) containing \( L_\delta \); if, on the other hand, \( L_\delta \) is infinite, the initial remark of this paragraph permits us to choose a directed subset \( I_\delta \) of \( I \) which contains the set \( L_\delta \) and has the same cardinality as the latter. A subsidiary induction thus gives us the desired family of subsets \( I_\gamma \) for \( \gamma < \Omega_\alpha \). The hypothesis of the principal induction now yields that for each \( \gamma \), the direct limit \( C_\gamma \) of the directed subsystem \((A_i, f_{ij})_{i,j \in I_\gamma, i \leq j}\) of our original system is an object of Add(\( M \)). Letting \( g_{\beta, \gamma} : C_\beta \to C_\gamma \) for \( \beta < \gamma \) be the natural map resulting from the fact that \( I_\beta \subseteq I_\gamma \), we are thus in a position to apply the hypothesis to the system \((C_\beta, g_{\beta, \gamma})_{\beta, \gamma < \Omega_\alpha, \beta \leq \gamma}\) to conclude that \( A = \lim \rightarrow C_\beta \) in turn belongs to Add(\( M \)). \( \square \)

An additional remark on the category Add(\( M \)) for a \( \Sigma \)-pure injective module \( M \) will turn out helpful in Section 4, namely
Observation 3.3. Let \( M \) be \( \Sigma \)-pure injective, and suppose that all monomorphisms in \( \text{add}(M) \) split. Then the same is true for monomorphisms in \( \text{Add}(M) \); in other words, given any monomorphism \( f : A \to B \) with \( A, B \) in \( \text{Add}(M) \), there exists a submodule \( C \) of \( B \) such that \( B = \text{Im}(f) \bigoplus C \), and both \( \text{Im}(f) \) and \( C \) again belong to \( \text{Add}(M) \).

Proof: Let \( f \) be as specified in the claim, and write \( B = \bigoplus_{i \in I} B_i \), where each \( B_i \) is isomorphic to a direct summand of \( M \). Then \( f \) is necessarily pure, since the image \( f(F) \) of any finitely generated submodule \( F \) of \( A \) is contained in a finite direct sum of the \( B_i \) and hence is a direct summand of \( B \) by hypothesis. As mentioned in the beginning of this section, \( \Sigma \)-pure injectivity is passed on to pure submodules, and therefore \( \text{Im}(f) \) is a direct summand of \( B \). Finally we use once more the fact that each \( \Sigma \)-pure injective module is a direct sum of submodules with local endomorphism rings, in order to deduce that both \( \text{Im}(f) \) and any complement of \( \text{Im}(f) \) in \( B \) are in turn direct sums of direct summands of \( M \). \( \square \)

4. \( \text{Fin}. \ \text{dim} \) versus \( \text{fin}. \ \text{dim} \) in case \( \mathcal{P}^\infty(\Lambda\text{-mod}) \) is contravariantly finite

This section is devoted to proving our main result, namely that, for any Artin algebra \( \Lambda \), the left big and little finitistic dimensions coincide, provided that the full subcategory \( \mathcal{P}^\infty(\Lambda\text{-mod}) \) of finitely generated left \( \Lambda \)-modules of finite projective dimension is contravariantly finite in the category of all finitely generated left \( \Lambda \)-modules.

Recall that \( \mathcal{P}^\infty(\Lambda\text{-mod}) \) is contravariantly finite in \( \Lambda\text{-mod} \) if, for each finitely generated left \( \Lambda \)-module \( M \), there exists an object \( A \in \mathcal{P}^\infty(\Lambda\text{-mod}) \) and a homomorphism \( f : A \to M \) such that the induced sequence of functors from \( \mathcal{P}^\infty(\Lambda\text{-mod}) \) to the category of abelian groups,

\[
\text{Hom}_\Lambda(-, A)|_{\mathcal{P}^\infty(\Lambda\text{-mod})} \to \text{Hom}_\Lambda(-, M)|_{\mathcal{P}^\infty(\Lambda\text{-mod})} \to 0,
\]

is exact; see [4]. In that case, \( A \) is called a (right) \( \mathcal{P}^\infty(\Lambda\text{-mod}) \)-approximation of \( M \). It is well-known that, existence provided, the \( \mathcal{P}^\infty(\Lambda\text{-mod}) \)-approximations of minimal length of a given module \( M \in \Lambda\text{-mod} \) are all isomorphic, and hence it makes sense to refer to the minimal \( \mathcal{P}^\infty(\Lambda\text{-mod}) \)-approximation of \( M \) in that case.

It is known that for any left serial algebra \( \Lambda \), the category \( \mathcal{P}^\infty(\Lambda\text{-mod}) \) is contravariantly finite in \( \Lambda\text{-mod} \) [3]; moreover, the minimal \( \mathcal{P}^\infty(\Lambda\text{-mod}) \)-approximations of the simple left \( \Lambda \)-modules can be explicitly described in that case. Initial criteria for contravariant finiteness in more general situations were developed in [10].

Throughout this section, we will abbreviate \( \ell \text{fin}. \ \text{dim} \Lambda \) by \( \text{fin}. \ \text{dim} \Lambda \) and \( \ell \text{Fin}. \ \text{dim} \Lambda \) by \( \text{Fin}. \ \text{dim} \Lambda \).

Theorem 4.1. Suppose that \( \mathcal{P}^\infty(\Lambda\text{-mod}) \) is contravariantly finite in \( \Lambda\text{-mod} \). Then

\[
\text{fin}. \ \text{dim} \Lambda = \text{Fin}. \ \text{dim} \Lambda.
\]

Proof: Since \( \text{Fin}. \ \text{dim} \Lambda \) is the supremum of those projective dimensions which are attained on countably generated left \( \Lambda \)-modules of finite projective dimension by Corollary 2.2, it is enough to focus on a countably generated module \( M \) with \( \text{p dim} M < \infty \), say \( \text{p dim} M = n \). Let \( M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots \subseteq M_r \subseteq \cdots \subseteq M \) be a chain of finitely generated submodules such that \( M = \cup_{i \geq 1} M_i \). For each \( i \), we fix the beginning of a finitely generated projective resolution of \( M_i \), say

\[
0 \to \Omega_{n,i} \to P_{n-1,i} \to \cdots \to P_{i,i} \to P_{0,i} \to M_i \to 0
\]
and, over each of the inclusions $M_i \to M_{i+1}$, we choose a chain morphism consisting of maps $f_{k,i,i+1} : P_{k,i} \to P_{k,i+1}$ for $0 \leq k \leq n-1$, and $f_{n,i,i+1} : \Omega_{n,i} \to \Omega_{n,i+1}$. Defining $f_{k,i,j} = f_{k,j-1,j} \circ \cdots \circ f_{k,i,i+1}$ for all $j > i$ and $0 \leq k \leq n$, we thus obtain a directed system of exact sequences indexed by the natural numbers. Passing to the direct limit of this system gives us a projective resolution of the module $M$ which we label

$$0 \to \lim_{\longrightarrow} \Omega_{n,i} \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

In particular, $P_n := \lim_{\longrightarrow} \Omega_{n,i}$ is projective due to the fact that $\text{pdim } M = n$. Clearly, all $P_k$ are countably generated since our directed system extends over $\mathbb{N}$. Therefore, we can decompose $P_n$ in the form $P_n = \bigoplus_{j \in \mathbb{N}} Q_{n,j}$, where all the $Q_{n,j}$ are finitely generated. Denoting by $g_i : \Omega_{n,i} \to P_n$ the canonical morphisms, we obtain an increasing sequence $i_1, i_2, i_3, \ldots$ of natural numbers such that $\bigoplus_{j \leq m} Q_{n,j} \subseteq \text{Im } g_{i_m}$ for $m \in \mathbb{N}$. It is clearly harmless to pass to a suitable cofinal subsystem of the $\Omega_{n,i}$, which permits us to assume that $i_k = k$ for all $k$. Next we observe that each $\Omega_{n,i}$ contains a submodule $P_{n,i} = \bigoplus_{j \leq i} P_{n,j,i}$ isomorphic to the finite direct sum $\bigoplus_{j \leq i} Q_{n,j}$ with the property that $g_i$ restricts to an isomorphism $P_{n,j,i} \to Q_{n,j}$ for all $j \leq i$; just keep in mind that the restriction of the map $g_i$ to the preimage of $\bigoplus_{j \leq i} Q_{n,j}$ splits. Since $g_{i+1} f_{n,i,i+1}(P_{n,i}) = g_i(P_{n,i})$, the map $f_{n,i,i+1}$ induces a split monomorphism $P_{n,i} \to g_{i+1}^{-1}(\bigoplus_{j \leq i+1} Q_{n,j})$. Consequently, an obvious induction on $i$ allows us to choose the $P_{n,j,i}$ in such a way that the squares

$$\begin{array}{ccc}
P_{n,i+1} & \xrightarrow{g_{i+1}} & \bigoplus_{j \leq i+1} Q_{n,j} \\
f_{n,i,i+1} & \downarrow & \\
P_{n,i} & \xrightarrow{g_i} & \bigoplus_{j \leq i} Q_{n,j} \\
\end{array}$$

commute. In other words, this process yields a directed subsystem

$$\begin{array}{ccc}
P_{n,i+1} & \xrightarrow{g_{i+1}} & \Omega_{n,i+1} \\
f_{n,i,i+1} & \downarrow & \\
P_{n,i} & \xrightarrow{g_i} & \Omega_{n,i} \\
\end{array}$$

of the system $(\Omega_{n,i}, f_{n,i,j})_{i,j \in \mathbb{N}, i \leq j}$ such that $\lim_{\longrightarrow} P_{n,i} = \lim_{\longrightarrow} \Omega_{n,i} = P_n$.

At this point, we interrupt the argument with a lemma which will allow us to supplement our original directed system of resolutions

$$(S_1) \quad P_{n-1,i} \to P_{n-2,i} \to \cdots \to P_{0,i} \to M_i \to 0$$
by a system
\[(T_i) \quad P'_{n-1,i} \to P'_{n-2,i} \to \cdots \to P'_{0,i} \to N_i \to 0,\]
where \(N_i \in \mathcal{P}(\Lambda\text{-mod})\), together with an epimorphism \((T_i) \to (S_i)\) of directed systems, to the effect that \(N = \lim_{\to} N_i\) has a projective dimension bounded above by \(\text{fin.dim } \Lambda\) and the kernel of the induced epimorphism \(N \to M\) is ‘under control’.

The following lemma is based on the fact that contravariant finiteness of \(\mathcal{P}(\Lambda\text{-mod})\) in \(\Lambda\text{-mod}\) implies the existence of an injective cogenerator inside the category \(\mathcal{P}(\Lambda\text{-mod})\) (see [4]); indeed, if \(I\) is the minimal right \(\mathcal{P}(\Lambda\text{-mod})\)-approximation of the minimal injective cogenerator for \(\Lambda\text{-mod}\), then each object of \(\mathcal{P}(\Lambda\text{-mod})\) embeds into an object of \(\text{add } I\) and every inclusion \(I' \hookrightarrow X\) with \(I' \in \text{add } I\) and \(X \in \mathcal{P}(\Lambda\text{-mod})\) splits. In particular, this entails that every monomorphism in \(\text{add}(I)\) splits, i.e., the hypotheses of Observation 3.3 are satisfied for \(M = I\).

**Lemma 4.2.** Let \(I\) be a relative injective cogenerator for \(\mathcal{P}(\Lambda\text{-mod})\) as above, and suppose that we are given an exact commutative diagram of the form

\[
\begin{array}{ccc}
0 & \to & X \\
\uparrow g & & \uparrow \beta \\
Q & \to & P \\
\uparrow h_Q & & \uparrow f_P \\
Q' & \to & P' \\
\uparrow g' & & \uparrow f_Y \\
0 & \to & Y \\
\end{array}
\]

with \(Q, Q', P,\) and \(P' \in \mathcal{P}(\Lambda\text{-mod})\). Then there exist modules \(I_0\) and \(I'_0\) in \(\text{add } I\), together with homomorphisms \(\gamma : Q \to I_0\) and \(\gamma' : Q' \to I'_0\), as well as a homomorphism \(h : I'_0 \to I_0\) such that the following diagram has exact rows and commutes.

\[
\begin{array}{ccc}
0 & \to & X \\
\uparrow g & & \uparrow (\beta \gamma) \\
Q & \to & P \oplus I_0 \\
\uparrow h_Q & & \uparrow (f_P \circ h) \\
0 & \to & Z \\
\uparrow g' & & \uparrow (\beta' \gamma') \\
Q' & \to & P' \oplus I'_0 \\
\uparrow f_X & & \uparrow (1,0) \\
0 & \to & Y \\
\end{array}
\]

Here \(Z = \text{Coker } (\beta \gamma)\), \(Z' = \text{Coker } (\beta' \gamma')\) are in \(\mathcal{P}(\Lambda\text{-mod})\), and the remaining maps are induced by the cokernels.
Proof of the lemma: Since $I$ is a cogenerator for $\mathcal{P}^\infty(\Lambda\text{-mod})$, we can choose monomorphisms $\gamma : Q \to I_0$ and $\gamma' : Q' \to I'_0$. So if we introduce maps $Q \to P \oplus I_0$ and $Q' \to P \oplus I'_0$ as in the above diagram and denote by $Z$ and $Z'$ their cokernels, respectively, the two squares in each of the top and bottom planes commute. Moreover, $Z, Z'$ in turn belong to $\mathcal{P}^\infty(\Lambda\text{-mod})$ by the hypothesis on $Q$ and $Q'$. Next we use the relative injectivity of $I$ to obtain $h : I'_0 \to I_0$ such that $\gamma h_Q = h \gamma'$. It is now straightforward to check that the entire diagram commutes, which completes the proof of the lemma. \[\square\]

We return to the proof of Theorem 4.1. At this point, we label the maps in the projective resolutions of the $M_i$, say $g_{k,i} : P_k,i \to P_{k-1,i}$. By applying the lemma, first to the diagrams

\[
\begin{array}{cccccccccc}
0 & \rightarrow & P_{n-1,i+1} & / & \Omega_{n,i+1} & \rightarrow & g_{n-1,i+1} & P_{n-2,i+1} & \rightarrow & g_{n-2,i+1} & \rightarrow & \text{Im}(g_{n-2,i+1}) & \rightarrow & 0 \\
& & & & f_{n-1,i,i+1} & & f_{n-2,i,i+1} & & & & & & \\
& & & & P_{n-1,i+1} & / & P_{n,i+1} & & & & & & \\
0 & \rightarrow & P_{n-1,i} & / & \Omega_{n,i} & \rightarrow & g_{n-1,i} & P_{n-2,i} & \rightarrow & g_{n-2,i} & \rightarrow & \text{Im}(g_{n-2,i}) & \rightarrow & 0 \\
& & & & P_{n-1,i} & / & P_{n,i} & & & & & & \\
\end{array}
\]

for $i \in \mathbb{N}$, and by then moving inductively along the sequences

\[P_{n-1,i} \rightarrow P_{n-2,i} \rightarrow \cdots \rightarrow P_0,i \rightarrow M_i \rightarrow 0,\]

we obtain the following directed system of diagrams in $\Lambda\text{-mod}$:

\[
\begin{array}{cccccccccc}
0 & \rightarrow & \Omega_{n,i} & \rightarrow & P_{n-1,i} & \rightarrow & g_{n-1,i} & P_{n-2,i} & \rightarrow & g_{n-2,i} & \rightarrow & P_{1,i} & \rightarrow & P_0,i & \rightarrow & M_i & \rightarrow & 0 \\
& & & & (1,1) & & (1,0) & & (1,0) & & (1,0) & & & & & & \\
0 & \rightarrow & P_{n,i} & \rightarrow & P_{n-1,i} & \rightarrow & P_{n-2,i} \oplus I_{n-2,i} & \rightarrow & P_{1,i} \oplus I_{1,i} & \rightarrow & P_0,i \oplus I_0,i & \rightarrow & N_i & \rightarrow & 0 \\
& & & & \left(\begin{array}{c}
1 \\
0 \\
\end{array}\right) & & \left(\begin{array}{c}
1 \\
0 \\
\end{array}\right) & & \left(\begin{array}{c}
1 \\
0 \\
\end{array}\right) & & \left(\begin{array}{c}
1 \\
0 \\
\end{array}\right) & & \left(\begin{array}{c}
1 \\
0 \\
\end{array}\right) & & & & & & \\
0 & \rightarrow & I_{n-2,i} & \rightarrow & I_{1,i} & \rightarrow & I_0,i & \rightarrow & K_i & \rightarrow & 0 \\
& & & & (1) & & (1) & & (1) & & & & & & \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & \end{array}
\]

Here the objects $I_{k,i}$ all belong to $\text{add}(I)$. The upper two horizontal sequences of each of these diagrams are exact by construction, whereas the induced kernel sequence in the third row will not be exact in general; in fact, the homology in the term labeled $n-2$ of that sequence is isomorphic to $\Omega_{n,i}/P_{n,i}$. On the other hand, the direct limit of the inclusions $P_{n,i} \hookrightarrow \Omega_{n,i}$
is the identity $P_n \to P_n$, and hence the snake lemma ensures that, post-limits, we arrive at an *exact* commutative diagram of the form

\[
\begin{array}{ccccccc}
0 & \to & P_n & \to & P_{n-1} & \to & \cdots & \to & P_1 & \to & P_0 & \to & M & \to & 0 \\
\uparrow & & \uparrow^{(1,0)} & & \uparrow & & \cdots & & \uparrow & & \uparrow & & \uparrow & & 0 \\
0 & \to & P_n & \to & P_{n-1} & \to & \cdots & \to & P_1 & \oplus & I_{n-2} & \to & P_0 & \oplus & I_0 & \to & N & \to & 0 \\
\uparrow & & \uparrow^{(0)} & & \uparrow & & \cdots & & \uparrow & & \cdots & & \uparrow & & \uparrow & & \uparrow & & 0 \\
0 & \to & I_{n-2} & \to & \cdots & \to & I_1 & \to & I_0 & \to & K & \to & 0 \\
\end{array}
\]

Now Observation 3.1 yields $I_j \in \text{Add}(I)$ and, as a consequence, Observation 3.3 gives us $K \in \text{Add}(I)$. Since the $N_i$ belong to $\mathcal{P}^\infty(\Lambda\text{-mod})$ by construction, the relative cogenerating property of $I$ furthermore permits us to choose a directed system of embeddings $N_i \hookrightarrow I_i'$ with $I_i' \in \text{add}(I)$, which shows that $N = \varprojlim N_i$ in turn embeds into an object of $\text{Add}(I)$ by 3.1. In view of 3.3, this implies that the exact sequence $0 \to K \to N \to M \to 0$ splits, and therefore $p\dim M \leq p\dim N \leq \sup\{p\dim N_i \mid i \in \mathbb{N}\} \leq \text{fin.dim} \Lambda$. □

In the light of Theorem 4.1, the following is an immediate consequence of the result of Auslander and Reiten quoted in the introduction; just keep in mind that each $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation of a module $M$ contains a minimal one as a direct summand [3].

**Corollary 4.3.** If $\mathcal{P}^\infty(\Lambda\text{-mod})$ is contravariantly finite in $\Lambda\text{-mod}$, and $C_1, \ldots, C_n$ are arbitrary $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximations of the simple left $\Lambda$-modules, then

\[
\text{Fin.dim} \Lambda = \sup\{p\dim C_i \mid 1 \leq n\}.
\]

□

We can actually strengthen the conclusion of Theorem 4.1, so as to provide information on the structure of the non-finitely generated objects in $\mathcal{P}^\infty(\Lambda\text{-Mod})$ as follows: Given modules $M_1, \ldots, M_n$ in $\Lambda\text{-mod}$ we denote by $\text{filt}(M_1, \ldots, M_n)$ the full subcategory of $\Lambda\text{-mod}$ having as objects all finitely generated modules $X$ that possess filtrations with consecutive factors in $\{M_1, \ldots, M_n\}$; in other words, $\text{filt}(M_1, \ldots, M_n)$ consists of those modules $X$ which contain chains of the form $X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_r = 0$ such that each factor $X_i/X_{i+1}$ is isomorphic to some $M_j$. Moreover, $\text{filt}(M_1, \ldots, M_n)$ will stand for the full subcategory of $\Lambda\text{-Mod}$ the objects of which are the direct limits of modules in $\text{filt}(M_1, \ldots, M_n)$. Due to the above-mentioned result of Auslander and Reiten, we know: In case $\mathcal{P}^\infty(\Lambda\text{-mod})$ is contravariantly finite in $\Lambda\text{-mod}$ and $A_1, \ldots, A_n$ are the minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximations of the simple left $\Lambda$-modules, $\mathcal{P}^\infty(\Lambda\text{-mod})$ consists of the direct summands of modules in $\text{filt}(A_1, \ldots, A_n)$. In view of the proof of Theorem 4.1, this description of the finitely generated modules of finite projective dimension extends to non-finitely generated candidates as follows.
Theorem 4.4. If $\mathcal{P}^\infty(\Lambda\text{-mod})$ is contravariantly finite in $\Lambda\text{-mod}$ and $A_1, \ldots, A_n$ are as above, then

$$\mathcal{P}^\infty(\Lambda\text{-Mod}) = \overrightarrow{\text{filt}}(A_1, \ldots, A_n).$$

In particular, each object of $\mathcal{P}^\infty(\Lambda\text{-Mod})$ is a direct limit of modules in $\mathcal{P}^\infty(\Lambda\text{-mod}).$

Proof: That $\overrightarrow{\text{filt}}(A_1, \ldots, A_n)$ is contained in $\mathcal{P}^\infty(\Lambda\text{-Mod})$ is clear. For the other inclusion, start by noting that each full subcategory $C$ of $\Lambda\text{-Mod}$ which is closed under direct limits is also closed under direct summands. This is well known (see e.g. [8], Lemma 1), but we include the easy argument: If $B$ is a direct summand of an object $C$ in $C$ and $\pi: C \to C$ a projection onto $B$, then $B$ is the direct limit of the system

$$C \xrightarrow{\pi} C \xrightarrow{\pi} C \xrightarrow{\pi} \cdots$$

By Proposition 2.1, each object in $\mathcal{P}^\infty(\Lambda\text{-Mod})$ is the direct union of countably generated objects in $\mathcal{P}^\infty(\Lambda\text{-mod})$, whence it suffices to show that each countably generated module $M$ of finite projective dimension belongs to $\overrightarrow{\text{filt}}(A_1, \ldots, A_n)$. But the proof of Theorem 4.1 shows that $M$ is a direct summand of a direct limit of modules in $\mathcal{P}^\infty(\Lambda\text{-mod})$ and thus belongs to the closure of $\mathcal{P}^\infty(\Lambda\text{-mod})$ under direct limits as explained above. That the latter category is contained in $\overrightarrow{\text{filt}}(A_1, \ldots, A_n)$, finally, follows from [3] and another application of our initial remark. □

Note that our arguments for Theorems 4.1 and 4.4 only use the existence of a $\mathcal{P}^\infty(\Lambda\text{-mod})$ approximation of $D(\Lambda)$, where $D$ is the standard duality $\text{mod-}\Lambda \to \Lambda\text{-mod}$. This observation does not lead to any significant generalization of our results however. Indeed, by [11], in the presence of the inequality $\text{fin.dim } \Lambda < \infty$, the existence of a $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation of $D(\Lambda)$ forces $\mathcal{P}^\infty(\Lambda\text{-mod})$ to be contravariantly finite.

The final consequence of our main results was observed by Henning Krause, who pointed out to us that a result of Crawley-Boevey applies to our context.

Corollary 4.5. If the subcategory $\mathcal{P}^\infty(\Lambda\text{-mod})$ of $\Lambda\text{-mod}$ is contravariantly finite, then it is also covariantly finite. In particular, $\mathcal{P}^\infty(\Lambda\text{-mod})$ has almost split sequences in that case.

Proof: Suppose that $\mathcal{P}^\infty(\Lambda\text{-mod})$ is contravariantly finite in $\Lambda\text{-mod}$. As we know, $\ell(\text{Fin.dim } \Lambda$ is finite in that case, equal to $d$ say. In particular, $\mathcal{P}^\infty(\Lambda\text{-Mod})$ is closed under direct products – just keep in mind that direct products of projectives are projective. By Theorem 4.4, this is the same as to say that the closure of $\mathcal{P}^\infty(\Lambda\text{-mod})$ under direct limits is also closed under direct products. Hence Theorem 4.2 of [8] tells us that $\mathcal{P}^\infty(\Lambda\text{-mod})$ is indeed covariantly finite. □

On the other hand, it is not true in general that covariant finiteness of $\mathcal{P}^\infty(\Lambda\text{-mod})$ in $\Lambda\text{-mod}$ implies contravariant finiteness. Indeed, since the category of modules of projective dimension $\leq 1$ is always covariantly finite in $\Lambda\text{-mod}$ by [3], the examples in [15] and [21] exhibit covariantly finite categories $\mathcal{P}^\infty(\Lambda\text{-mod})$ which fail to be contravariantly finite. A further discussion of the two properties can be found in [14].

5. Examples
Throughout this section, \( \Lambda = K\Gamma/I \) will be a finite dimensional path algebra modulo relations over a field \( K \). We will denote by \( J \) its Jacobson radical, by \( e_i \) the primitive idempotents of \( \Lambda \) going with the vertices of the quiver \( \Gamma \), and by \( S_i = \Lambda e_i / J e_i \) representatives of the simple left \( \Lambda \)-modules. Moreover, given a left \( \Lambda \)-module \( M \) and a primitive idempotent \( e_i \), we will call an element \( x \in M \) a top element of type \( e_i \) in case \( x \in M \setminus J M \) and \( e_i x = x \).

We start with an example which illustrates the applicability of corollary 4.3.

**Example 5.1.** Let \( \Lambda \) be the monomial relation algebra \( K\Gamma/I \) where \( \Gamma \) is the quiver

![Quiver Diagram](image)

and \( I \subseteq K\Gamma \) is the unique ideal of the path algebra with the property that the indecomposable projective left \( \Lambda \)-modules have the following graphs. For our graphing conventions, we refer the reader to [12].

![Graphs](image)

We will sketch a proof for the fact that \( \mathcal{P}^\infty(\Lambda\text{-mod}) \) is contravariantly finite in \( \Lambda\text{-mod} \) by exhibiting (minimal) \( \mathcal{P}^\infty(\Lambda\text{-mod}) \)-approximations of the simple left \( \Lambda \)-modules and by partly verifying that they have the claimed properties. Due to [3], securing approximations for the simple modules guarantees contravariant finiteness. The following module \( A_1 \) is a
\(P^\infty(\Lambda\text{-mod})\)-approximation of \(S_1\). Namely, \(A_1 = B_1 \oplus C_1\), where \(B_1\) corresponds to the representation

\[
\begin{array}{c}
K \\
\downarrow \phi \\
K^2 \\
\downarrow g \\
K \\
\downarrow f \\
0 \\
\end{array}
\]

of \(\Gamma\) modulo \(I\). The module \(C_1\) is defined similarly, with the vertex 10 taking over the role of 9 and the arrow \(\gamma\) taking over that of \(\beta\).

Note first that \(\Omega^1(A_1) = (\Lambda e_2/\Lambda \nu)^2 \oplus (\Lambda e_4)^2\), hence \(p \dim A_1 = 3\). In particular, \(A_1 \in \mathcal{P}^\infty(\Lambda\text{-mod})\). Moreover, denote by \(K\) the kernel of the homomorphism \((\psi, \rho) : B_1 \oplus C_1 \to S_1\), where \(\psi : B_1 \to S_1\) and \(\rho : C_1 \to S_1\) are the canonical epimorphisms (unique up to a nonzero scalar factor). It can be checked that \(\text{Ext}^1_\Lambda(M, K) = 0\) for each object \(M \in \mathcal{P}^\infty(\Lambda\text{-mod})\), which shows that the map \(\text{Hom}_\Lambda(M, A_1) \to \text{Hom}_\Lambda(M, S_1)\) induced by \((\psi, \rho)\) is onto, as required. Along the same line, one can verify that the minimal \(\mathcal{P}^\infty(\Lambda\text{-mod})\)-approximations \(A_i\) of the simple modules \(S_i\) with \(i \geq 2\) are as follows: \(A_2 = \Lambda e_2/\Lambda \nu\); \(A_3 = \Lambda e_3\); \(A_4\) corresponds to the representation

\[
\begin{array}{c}
K \\
\downarrow f_4 \\
K^3 \\
\downarrow f_4 \\
K \\
\downarrow f_4 \\
0 \\
\end{array}
\]

where \(f_4\) has matrix \(\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\); moreover, \(A_5 = \Lambda e_5\); \(A_6 = \Lambda e_6/\Lambda \sigma\); \(A_7 = S_7\); \(A_8 = S_8\); \(A_9\) is \(B_1\) as defined above; \(A_{10}\) is \(C_1\) as defined above; \(A_{11} = A_{12} = \Lambda e_1 \oplus \Lambda e_2/\Lambda(\varepsilon, \chi)\). Thus Corollary 4.3 is applicable. It yields

\[\ell \text{fin.dim } \Lambda = \ell \text{Fin.dim } \Lambda = \sup\{p \dim A_i \mid 1 \leq i \leq 12\} = p \dim A_1 = 3.\]
Our final example exhibits a class of finite dimensional monomial relation algebras $\Lambda$ such that $\ell(\text{fin.dim}\, \Lambda) < \ell(\text{Fin.dim}\, \Lambda)$, while the subcategory $P^\infty(\text{mod-}\Lambda) \subseteq \text{mod-}\Lambda$ of the category of all finitely generated right $\Lambda$-modules is contravariantly finite.

**Example 5.2.** Given a positive integer $n$, let $\Lambda = K\Gamma/I$ be the monomial relation algebra defined in [13], proof of Theorem E, where $\Gamma$ is the quiver

![Quiver Diagram](image)

The relations are such that the indecomposable projective right $\Lambda$-modules have the following graphs (in particular, $e_2\Lambda \cong S_2$).

![Graphs](image)

In view of the fact that there are unique arrows $1 \to a_0, a_i \to a_{i+1}$ for $i \geq 0$, and $b \to b, b \to a_0, c_i \to c_i, 1 \to c_i$ for $i = 1, 2$, the above graphs determine $\Lambda$ up to isomorphism.
In [13], it was shown that \( \ell \text{fin.dim} \Lambda = n + 1 \), whereas \( \ell \text{Fin.dim} \Lambda = n + 2 \). On the other hand, the subcategory \( \mathcal{P}^\infty(\text{mod-}\Lambda) \) of finitely generated right \( \Lambda \)-modules of finite projective dimension is contravariantly finite in \( \text{mod-}\Lambda \). The minimal \( \mathcal{P}^\infty(\text{mod-}\Lambda) \)-approximations \( A(i) \) of the simple right \( \Lambda \)-modules \( e_i \Lambda / e_i J \) for \( i \in \Gamma_0 \) are listed below.

\[
A(1) \quad A(2) \quad A(a_0) \quad A(a_1) \quad A(a_2) \quad A(a_3) \quad \cdots \quad A(a_{n+1})
\]

\[
\begin{array}{cccccccc}
\sigma & 1 & \tau & \bullet & a_0 & a_1 & a_2 & a_3 & \cdots & a_{n+1} \\
1 & 1 & \quad & a_1 & a_2 & \cdots & a_n \\
\tau & \quad & \quad & a_0 & \quad & \quad & \quad
\end{array}
\]

\[
A(b) \quad A(c_1) \quad A(c_2)
\]

\[
\begin{array}{cccccccc}
b & a_1 & c_1 & c_2 & 1 & 1 & 1 \\
b & c_1 & 1 & c_2 & 1 & 1 & 1 \\
a_0 & \quad & \quad & \quad & \quad & \quad & \quad
\end{array}
\]

We give a bit of detail to back our claim for \( A(1) \) and \( A(b) \). Clearly, \( p \text{ dim } A(1) = 1 \). Let \( M \) be any object in \( \mathcal{P}^\infty(\text{mod-}\Lambda) \) containing a top element \( x \) of type \( e_1 \). Since \( p \text{ dim } e_2 \Lambda / e_2 J < \infty \) and since the vertex 2 is a source of \( \Gamma \), we may assume that \( Me_2 = 0 \). Note moreover that \( \Omega^1(M)e_1 J^2e_1 = 0 \), because \( \Omega^1(M) \) is contained in the radical of a projective module. This implies that \( \Omega^1(M) \) cannot contain a top element of type \( e_1 \), for otherwise all higher syzygies \( \Omega^j(M) \) would contain such a top element as well. Consequently, the graph of \( \Lambda x \) coincides with that of \( A(1) \) and \( \Lambda x \) is a direct summand of \( M \). This guarantees that each homomorphism \( M \to S_1 \) factors through \( A(1) \).

We are still briefer in justifying our claim for \( A(b) \). We just point to the fact that all homomorphisms \( M \to e_b \Lambda / e_b J \) where \( M \in \mathcal{P}^\infty(\text{mod-}\Lambda) \) has a graph belonging to the series.
factor through $A(b)$. Another application of Corollary 4.3 therefore yields that

$$r \text{ Fin} \dim \Lambda = \sup \{ p \dim A_i \mid i \in \Gamma_0 \} = p \dim A(a_0) = p \dim A(b) = 2.$$  

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