Quantum Origin of (Newtonian) Mass and Symmetry for Lorentz Covariant Physics

Otto C. W. Kong, and Hock King Ting

Department of Physics and Center for High Energy and High Field Physics,
National Central University,
Chung-lí, Taiwan 32054

Abstract

The Galilean symmetry and the Poincaré symmetry are usually taken as the fundamental (relativity) symmetries for ‘nonrelativistic’ and ‘relativistic’ physics, respectively, quantum or classical. Our fully group theoretical formulation approach to the theories, together with its natural companion of mechanics from symplectic geometry, ask for different perspectives. We present a sketch of the full picture here, emphasizing aspects which are different from the more familiar picture. The letter summarizes our earlier presented formulation while focusing on the part beyond, with an adjusted, or corrected, identification of the basic representations having the (Newtonian) mass as a Casimir invariant. Discussion on the limitations of the Poincaré symmetry for the purpose is particularly elaborated.

Keywords: Particle Mass, Casimir Invariants, (Quantum) Relativity Symmetry, Lorentz Covariant Quantum Mechanics,

PACS numbers:
I. INTRODUCTION

We give in this letter a sketch of the full group theoretical formulation framework for the ‘nonrelativistic’ and ‘relativistic’ theories of particle dynamics focusing mostly on the quantum part, especially highlighting what is different from most of the usual discussions of the subject matter in textbooks and beyond. This supplements our detailed analyses for the cases of a single spin zero particle [1, 2] and completes the story.

Symmetry consideration is one of the most important theoretical guiding principles in modern physics. A group theoretical formulation of fundamental theories can go a very long way ‘explaining’ the fundamental structures. In Refs. [1, 2] in particular, a (relativity) symmetry group theoretical formulation of quantum (particle) mechanics, with the Lorentz covariant case at the top level retrieving both the ‘nonrelativistic’ quantum and the ‘relativistic’ followed by the ‘nonrelativistic’ classical mechanics as approximations from contractions of the symmetries, has been presented. The formulation, at the various levels, gives the full theories for a single particle. The spacetime, or space and time, models, or models for the particle phase space, the observables algebras, are all described starting with a single irreducible representation of the identified fundamental relativity symmetry. The Lie algebra/group structures also give the natural symplectic structures that make the phase spaces what they are, with the dynamics more or less dictated from the covariant Hamiltonian perspective. The current letter presents an important improvement and an extension on that. Firstly, we give a new perspective about the fundamental notion of (Newtonian) particle mass $m$ as a Casimir invariant of the quantum symmetry. The proper way to look at that is actually more transparent in relation to the extension of the group theoretical formulation to a composite system from which we retrieve the fact that mass is an additive notion as well as a natural notion of center of mass, directly from the representation theory. Apparently, none of that has been exactly appreciated before, at least not in the full group representational point of view.

Casimir invariants are intrinsic characteristics of the physical system with symmetry properties corresponding to any irreducible representation. Having particle mass from a Casimir invariant, as Einstein rest mass $m_k$ in the case of the Poincaré symmetry in particular, has been well known. However, against the common idea, the Poincaré symmetry cannot truly serve the role for a theory of Lorentz covariant quantum mechanics well. One needs a bigger
symmetry, denoted by $H_R(1,3)$ [2], incorporating Lorentz covariant Heisenberg commutation relations. We will present below a careful analysis of the issue involved and make the case clear. Moreover, Einstein rest mass is not quite the same conceptual notion as the Newtonian mass. And it has a fundamental role that is somewhat questionable (see for example the books [3, 4] and references therein). The so-called on-shell mass condition, having the Casimir invariant $-m_E^2c^2$ as the square of the particle energy-momentum four-vector is, after all, generally not enforced for a generic state of a quantum field theory, which is supposed to come from the Poincaré symmetry. We will show that the (Newtonian) particle mass $m$ arising from an irreducible representation of $H_R(1,3)$ actually can essentially be taken as the same as $m_E$ for the cases where the on-shell mass condition has been verified to hold experimentally. The truth is we do not even have any meaningful notion of ‘rest mass’ for a particle moving under a Lorentz force from a background electromagnetic field. That is a simple and clear result under any Hamiltonian formulation which seems however not to have been otherwise notified.

In the next section, we present issues about the Newtonian mass within just the Galilean setting. That works directly only at the quantum level, which then maintains at the classical level as the latter theory is an approximation to the quantum one. Hence, we see a quantum origin of the Newtonian mass. The role of $m$ in the quantization of Newtonian mechanics has been appreciated, in the language of projective representation [6]. One only has to take the mathematically more proper and natural language of $U(1)$ central extension, incorporating the Heisenberg commutation relation at the relativity symmetry level [6], that would provide the setting to identify $m$ as a linear Casimir invariant from the eigenvalue of representation of the central charge. Otherwise, most of the mathematics has been given in Ref.[7]. The story is to be completed with the analysis for a composite system, of two particles, in Sec.III. We emphasize important issues related to all Casimir invariants more directly and explicitly, bringing attention to issues which are worth more careful studies. All that provides also a firm background for the corresponding aspects of the Lorentz covariant case, which we present in Sec.IV followed by the matching against the usual Poincaré symmetry picture in Sec.V after which we conclude. A key perspective is that the Poincaré symmetry should not be taken as the full relativity symmetry.
II. THE QUANTUM GALILEAN SYMMETRY FOR A SINGLE PARTICLE

The $U(1)$ central extension of the classical Galilean symmetry $\tilde{G}(3)$ can be taken as having the eleven generators below with the nontrivial Lie products given by

\[ [J_{ij}, J_{hk}] = i\hbar(\delta_{jk}J_{ih} + \delta_{ih}J_{jk} - \delta_{ik}J_{jh} - \delta_{jh}J_{ik}) , \]
\[ [J_{ij}, K_k] = i\hbar(\delta_{ik}K_j - \delta_{jk}K_i) , \]
\[ [J_{ij}, P_k] = i\hbar(\delta_{ik}P_j - \delta_{jk}P_i) , \]
\[ [K_i, H] = i\hbar P_i , \]
\[ [K_i, P_j] = i\hbar \delta_{ij}M , \]

(1)

\[ i, j, \cdot \text{ goes from 1 to 3 with } J_{ij} = -J_{ji} . \]

The central charge generator $M$ is essentially a Casimir invariant. For a unitary representation, it is represented by $m\hat{1}$, a real multiple of the identity operator. We can take $m > 0$. $K_i$ and $P_i$ we consider to be represented by Hermitian operators $m\hat{X}_i$ and $\hat{P}_i$. The relations give a representation of the basic quantum symmetry $H(3)$, the Heisenberg-Weyl group here generated by $M$, $K_i$ and $P_i$, and we have the theory of quantum mechanics for a particle of mass $m$ conveniently taken with the position and momentum operators $\hat{X}_i$ and $\hat{P}_i$ as the basic observables. Taking that as a representation for $H(3)$ as a subgroup, we check how to extend it to a representation of the full group. The other Casimir invariants are given by $2MH - P_i P^i$ and $\frac{1}{2}T_{ij}T^{ij}$ with $T_{ij} \equiv MJ_{ij} - (K_i P_j - P_i K_j)$, which actually ‘commute’ with $K_i$, $P_i$, and $H$ (Einstein summation convention assumed). Note that $T_{ij}$, like the quadratic Casimir invariants, do not exist as elements of the Lie algebra. They have to be taken as elements of the universal enveloping algebra. Of course at the representation level, all the operator sum/product combinations are well defined. Introducing intrinsic angular momentum operators $\hat{S}_{ij} \equiv \frac{1}{m}\hat{T}_{ij}$, we have the familiar $\hat{J}_{ij} = \hat{L}_{ij} + \hat{S}_{ij}$ for the orbital angular momentum operator $\hat{L}_{ij} = \hat{X}_i\hat{P}_j - \hat{P}_i\hat{X}_j$. Hence, finding admissible $\hat{H}$ and $\hat{S}_{ij}$ completes all such unitary representations. Obviously, we have the energy observable

\[ \hat{H} = \frac{1}{2m}\hat{P}_i\hat{P}^i + \mathcal{V} . \]

(2)

for some real eigenvalue of the corresponding Casimir invariant written as $2m\mathcal{V}$. $\mathcal{V}$ is an internal (potential) energy, but constant for a single particle hence physically irrelevant. What is particularly noteworthy is that $\hat{S}_{ij}$ can be seen as representation of generators of an $SO(3)$ group, or its double cover, as the original $\hat{J}_{ij}$ except that the rotational group commute with the subgroup of $H(3)$ extended with $\hat{H}$ the unitary irreducible representations of which
we have just depicted. As a result, we can see that any (unitary irreducible) representation of $\tilde{G}(3)$ is simply given by a direct product of such a representation of the former and one of $SO(3)$. The last part is of course given by the $2s + 1$ dimensional representations of intrinsic angular momentum $s(s + 1)\hbar^2$, for integral (or half-integral) values of $s \geq 0$.

The direct product structure says that each of the $2s + 1$ components of a state is exactly described by that of the $s = 0$ state, like a basic Schrödinger wavefunction. Each irreducible representation is hence characterized by the triple \{\(m, \mathcal{V}, s\)\}.

A few words about the positive mass assumption are in order. As illustrated in Ref.\[7\], \{\(m, \mathcal{V}, s\)\} and \{-\(m, -\mathcal{V}, \bar{s}\)\} are anti-unitary equivalent representations. Here $\bar{s}$ denotes an $SO(3)$ representation conjugates to that of $s$, hence essentially the same. Naively, there is no difficulty taking things to the $m \to 0$ limit \[7\]. However, zero central charge really means the collapse of the Heisenberg commutation relation formally. Formulating physics of something like the photon of course is really a more complicated matter involving the Lorentz covariant as well as the extra gauge symmetry, not to say that we do not have available a first principle group theoretical formulation for gauge field theories other than patchwork as presented in the quantum field theory textbook. While we have formulated quantization/de-quantization as symmetry deformation/contraction, the notion of second quantization in quantum field theory also lacks a direct group theoretical formulation at the same level.

### III. TWO PARTICLE SYSTEM AND NONTRIVIAL DYNAMICS

Let us consider a composite system as a product of two irreducible representations of \{\(m_a, \mathcal{V}_a, s_a\)\} and \{\(m_b, \mathcal{V}_b, s_b\)\}. The first question is what are the admissible values of the Casimir invariants. A closely related question is if the representation divides into a number of irreducible components.

Firstly, we have

$$\hat{M} = \hat{M}_a \otimes \hat{I} + \hat{I} \otimes \hat{M}_b ,$$

as well as

$$\hat{P}_i = \hat{P}_{ia} \otimes \hat{I} + \hat{I} \otimes \hat{P}_{ib} .$$
The naive expressions give mass and momentum as additive quantities. In particular, we have a (total) mass $m = m_a + m_b$ for the composite system. The three $\hat{X}_i$ however are not the kind of simple sum of those of the individual particles. $\hat{X}_i$ do not represent generators directly. We have instead

$$\hat{X}_i \equiv \frac{1}{m} \hat{K}_i = \frac{1}{m} (m_a \hat{X}_{ia} \otimes \hat{I} + \hat{I} \otimes m_b \hat{X}_{ib}) .$$

(5)

Hence, we have the notion of the center of mass position operator $\hat{s}$ dictated by the representation theory. Physicists are understandably uncomfortable about taking $\hat{K}_i$ and $M$ in the place of the usually used $X_i$ and $I$ as generators for the Heisenberg-Weyl symmetry as an abstract symmetry. However, looking at the logic of the formulation from abstract mathematics to its application to physics as presented here, without the bias from what has become too familiar, one will be able to agree with the author that this is the better way of seeing the symmetry structure behind the physics.

From this point onwards, we simplify the kind of expressions and write in the form as $\hat{P}_i = \hat{P}_{ia} + \hat{P}_{ib}$ and $\hat{X}_i = \frac{1}{m} (m_a \hat{X}_{ia} + m_b \hat{X}_{ib})$. To go on with the analysis of the composite system, it is convenient to introduce the complementary observables given by the operators $\hat{R}_i \equiv \hat{X}_{ia} - \hat{X}_{ib}$ and $\hat{Q}_i \equiv \frac{1}{m} (m_b \hat{P}_{ia} - m_a \hat{P}_{ib})$. The operators all commute with $\hat{X}_i$ and $\hat{P}_i$ while $[\hat{R}_i, \hat{Q}_j] = i\hbar \delta_{ij}$. For example, with the Hilbert space tensor product as spanned by simultaneous momentum eigenstates $|p_{ia}, p_{ib}\rangle$, a more convenient basis to work with would simply be given by $|p_i, q_i\rangle$, simultaneous eigenstates of $\hat{P}_i$ and $\hat{Q}_i$. $\hat{X}_i$ and $\hat{P}_i$ are the basic dynamic observables for the center of mass degrees of freedom, while $\hat{R}_i$ and $\hat{Q}_i$ those for the degrees of freedom of the relative motion between the particles which is internal to the composite system. One can go as far as seeing the two sets as sets of canonical noncommutative coordinates for the phase space of the system [8, 9]. With a bit of calculation, one obtains for the product representation

$$\hat{J}_{ij} = (\hat{X}_i \hat{P}_j - \hat{P}_i \hat{X}_j) + (\hat{R}_i \hat{Q}_j - \hat{Q}_i \hat{R}_j) + \hat{S}_{ija} + \hat{S}_{ijb} ,$$

(6)

giving

$$\hat{S}_{ij} = \hat{R}_i \hat{Q}_j - \hat{Q}_i \hat{R}_j + \hat{S}_{ija} + \hat{S}_{ijb} .$$

(7)

The interpretation is that the relative motion between the two particles gives rise to an ‘orbital’ angular momentum which is, however, intrinsic to the system as a composite. The
latter behaves exactly like part of the spin of the composite taken as a particle. The ‘spin’ of the full \( \hat{S}_{ij} \), \( i.e. \) \( \frac{1}{2} \hat{S}_{ij} \hat{S}^{ij} \), is effectively a Casimir invariant. This ‘spin’ \( s \) for the composite would come from a standard angular momentum addition picture as the sum of the three parts. The different \( s \) values correspond to different irreducible representations as components of the product representation which is reducible. For example, with the \( |p_i, q_i\rangle \) basis, each set of vectors with fixed \( q^2 = q_i q^i \) span a subspace invariant under the rotations generated by \( \hat{S}_{ij} \). It is important to emphasize that neither \( \frac{1}{2} \hat{J}_{ij} \hat{J}^{ij} \) nor \( \frac{1}{2} \hat{L}_{ij} \hat{L}^{ij} \), \( \hat{L}_{ij} \equiv \hat{X}_i \hat{P}_j - \hat{P}_i \hat{X}_j \), serves as Casimir invariant, only \( \frac{1}{2} \hat{S}_{ij} \hat{S}^{ij} \), which for a composite system of spin zero particles has \( \hat{S}_{ij} \) as ‘orbital’ angular momentum \( \hat{L}_s \equiv \hat{R}_i \hat{Q}_j - \hat{Q}_i \hat{R}_j \), does. The \( |p_i, q_i\rangle \) basis, even restricted to fixed \( q^2 = q_i q^i \), is however generally not the best basis for the description of a particular two-particle system. The irreducible component of the product representation has, for the degree of freedom for the relative motion, a fixed angular momentum which contains both the ‘orbital’ part and the part of the spins of the particles. Apart from the \( |p_i\rangle \) for the center of mass degree of freedom, the the degree of freedom for the relative motion is to be described in a Hilbert space of finite dimension \( 2\ell_s + 1 \), with \( \ell_s(\ell_s + 1)\hbar^2 \) being the eigenvalue of the Casimir invariant. All that are independent of the actual interaction dynamics between the particles. Full implications of that are worth being studied very carefully in practical systems.

We still have \( \hat{H} = \hat{H}_a + \hat{H}_b \), the additive energy, which is then given as

\[
\hat{H} = \frac{1}{2m} \hat{P}_i \hat{P}^i + \frac{1}{2\mu} \hat{Q}_i \hat{Q}^i + \mathcal{V}_a + \mathcal{V}_b ,
\]  

(8)

where we have introduced the reduced mass \( \mu = \frac{m_a m_b}{m} \). Hence, we have the effective Casimir invariant

\[
\mathcal{V} = \frac{q^2}{2\mu} + \mathcal{V}_a + \mathcal{V}_b ,
\]  

(9)

for each of the subspace invariant under \( \hat{S}_{ij} \), which is also invariant under \( \hat{H} \). That gives an irreducible component of the product representation under the full \( \tilde{G}(3) \) group. Physically, each such component as an irreducible representation gives a possible symmetry structure of a composite system of the two particles. \( \frac{q^2}{2m} \) then contributes a potential energy term to the composite that depends on the dynamical properties of the relative motion between the two particles. That is, however, not true interaction with nontrivial dynamics.

It is obvious that \( \hat{H} = \hat{H}_a + \hat{H}_b \) can only be the physical Hamiltonian when there is no interaction between the particles. In the presence of interaction, the physical Hamiltonian, or
the energy observables would have an additional dynamical potential term which depends typically only on \( \hat{R}_i \hat{R}^i \), which is connected to having \( \frac{1}{2} \hat{S}_{ij} \hat{S}^{ij} \) as a Casimir invariant. In any case, intuitively, one can see that the interaction potential has to be independent of \( \hat{X}_i \) and \( \hat{P}_i \). When one writes down a nontrivial dynamical description of a particle, it is really the dynamics of \( \hat{R}_i - \hat{Q}_i \), that one is writing. With the interaction paradigm, nontrivial dynamics is present only when we have more than one particle in a system in which the individual particles are not dynamically independent. The conclusion is that \( \hat{H} \) or the generator \( H \) it represents is really irrelevant so long as nontrivial dynamics are concerned. The \( H_{\mathcal{R}}(3) \) symmetry \([1]\), the ten-generator subgroup with \( H \) in \( \tilde{G}(3) \) taken out, gives us all the required structure except Eq. (2). The latter is about the dropping out of \((2m)\mathcal{V}\) as a Casimir invariant. In summary, for nontrivial dynamics, we have a fully symmetry description with only \( H_{\mathcal{R}}(3) \) as the relativity symmetry, which falls short of dictating the physical Hamiltonian. However, the natural symplectic structure the Lie group gives on the phase space at least gives all possible dynamics under admissible generic Hamiltonians as symmetries \([1]\). We will see below how the Lorentz covariant version gives the limiting ‘nonrelativistic’ energy observable with the right kinetic part, though the corresponding symmetry \( H_{\mathcal{R}}(1,3) \) does not have a generator as the (physical) free particle Hamiltonian either. However, the physical Hamiltonian for free particles or any nontrivial dynamics as a function of the basic observables can always be consistently identified. From this perspective, the proper origin of the generator \( H \) in the quantum and classical dynamics of a Newtonian particle has to be retrieved from the dynamics of the Lorentz covariant system to which it is an approximation. Otherwise, it may not be seen as an integral part of the ‘nonrelativistic’ system. Without \( H \) has a generator of the fundamental (relativity) symmetry, the Newtonian time has no connection to the symmetry. It can only be retrieved from the proper limit of the particle phase space from the Lorentz covariant theory with \( H_{\mathcal{R}}(1,3) \) symmetry. Otherwise, for the theory with \( H_{\mathcal{R}}(3) \) symmetry only, it is no more than the evolution parameter of the physical Hamiltonian which the mathematics cannot exactly identify.

For nontrivial dynamics of the composite system of two spin zero particles, for example, we expect the physical Hamiltonian to be of the form

\[
\hat{H}_{\text{phys}} = \frac{1}{2m} \hat{P}_i \hat{P}^i + \frac{1}{2\mu} \hat{Q}_i \hat{Q}^i + V(\hat{R}_i \hat{R}^i) .
\]  

(10)
It is invariant under the rotations generated by $\hat{J}_{ij} = \hat{L}_{ij}$ and $\hat{S}_{ij} = \hat{L}_{ij}$. The practical Hilbert space is the representation space of an irreducible component of the product representation fixed the initial value of $\ell_s$, which is spanned by $|m_{\ell_s}\rangle$, $m_{\ell_s}\hbar$ being the eigenvalue of $\hat{L}_{12}$.

IV. THE LORENTZ COVARIANT VERSION AND ITS $c \to \infty$ LIMIT

The relevant Lie algebra for Lorentz covariant quantum mechanics is $H_R(1,3)$, as given by

$$[J'_{\mu\nu}, J'_{\rho\sigma}] = i\hbar c \left(\eta_{\nu\sigma} J'_{\mu\rho} + \eta_{\mu\rho} J'_{\nu\sigma} - \eta_{\mu\sigma} J'_{\nu\rho} - \eta_{\nu\rho} J'_{\mu\sigma}\right),$$

$$[J'_{\mu\nu}, Y_\rho] = i\hbar c \left(\eta_{\mu\rho} Y_\nu - \eta_{\nu\rho} Y_\mu\right),$$

$$[J'_{\mu\nu}, E_\rho] = i\hbar c \left(\eta_{\mu\rho} E_\nu - \eta_{\nu\rho} E_\mu\right),$$

$$[Y_\mu, E_\nu] = i\hbar c \eta_{\mu\nu} M,$$  \hspace{1cm} (11)

with the Minkowski four-vector indices going from 0 to 3, where $\eta_{\mu\nu} = \text{diag}\{-1,1,1,1\}$. We start here with an irreducible representation of the Heisenberg-Weyl subgroup $H(1,3)$, based on (pseudo-)Hermitian operators $\hat{X}_\mu \equiv \frac{1}{m}\hat{Y}_\mu$ and $\hat{P}_\mu \equiv \frac{1}{c}\hat{E}_\mu$, with $[\hat{X}_\mu, \hat{P}_\nu] = i\hbar\eta_{\mu\nu}\hat{I}$ for $m$ being the eigenvalue of the Casimir invariant $M$. The representation is essentially as the one presented in Ref.[2], with only necessary easy to trace adjustments involving the identification of $M$ as the central charge generator. The pseudo-Hermiticity is really proper self-adjointness with respect to the pseudo-unitary inner product which is a well defined Lorentz invariant one with no match from unitary one upon careful analysis. It naturally gives a Minkowskian metric operator on the representation space. Similar to the $G(3)$ or $H_R(3)$ case, an irreducible representation of $H_R(1,3)$ is given by as a direct product of one for the Heisenberg-Weyl group $H(1,3)$ and one for an $SO(1,3)$, or its double cover $SL(2, C)$, generated by $\hat{S}_{\mu\nu} = \hat{J}_{\mu\nu} - \hat{L}_{\mu\nu}$, where $\hat{J}_{\mu\nu} = \frac{1}{c}\hat{J}'_{\mu\nu}$ and $\hat{L}_{\mu\nu} \equiv \hat{X}_\mu \hat{P}_\nu - \hat{P}_\mu \hat{X}_\nu$. The corresponding two Casimir invariants, $\frac{1}{2}\hat{S}_{\mu\nu} \hat{S}^{\mu\nu}$ and $\frac{1}{2}\epsilon^{\mu\nu\sigma\rho} \hat{S}_{\mu\nu} \hat{S}_{\sigma\rho}$ for the finite dimensional pseudo-unitary representations exactly matching to the (pseudo-)Hermitian $\hat{X}_\mu$ and $\hat{P}_\mu$, have values $2\hbar^2[s_L(s_L + 1) + s_R(s_R + 1)]$ and $2\hbar^2[s_L(s_L + 1) - s_R(s_R + 1)]$, respectively, for the labeling with the left-handed and right-handed spins $\{m, s_L, s_R\}$ of dimension $(2s_L + 1)(2s_R + 1)$. Explicitly, we have $S^{ij}_{\ell_s} = \frac{1}{2}(S_{ij} \pm iS_{oij})$.

The full picture discussed here is an improved version of that presented in Ref.[2] which fails to appreciate the role of the central charge as the particle mass and does not address
nonzero spin. We can also have a picture for the composite system along the line as discussed for the $\tilde{G}(3)$ or $H_R(3)$ case. With $Y_i$ replacing $K_i$, changing or actually extending the three vector indices $i$ and $j$ to Minkowski four-vector indices $\mu$ and $\nu$ for most of the results there gives the right results which we skip to present explicitly. The only results for the $\tilde{G}(3)$ case which do not go along are the part for $\hat{H}$ and $\mathcal{V}$ as they are irrelevant. As discussed above, we need only the $H_R(3)$, which does not have $H$ as a generator and $\mathcal{V}$ as a Casimir invariant as the case of the full $\tilde{G}(3)$.

To get to the approximation with Lorentz boosts replaced by Galilean ones, we take the Lie algebra and the representation of interest to the $c \to \infty$ limit of $K_i = \frac{1}{c^2} J'_{i0}$, $P_i = \frac{1}{c} E_i$, $U = -\frac{1}{c} Y_0$, and $J_{ij} = \frac{1}{c} J'_{ij}$ with the renaming $H \equiv -E_0$. Mathematically, it is to be formulated as a symmetry contraction [10–12], taken from the Lie algebra to the group, the proper extension of the universal enveloping algebra or group $C^*$-algebra and their relevant representation as a single representation [1, 2] for the full dynamical theory, including the symplectic structure. In this case, it is easy to see that the generator set \{ $J_{ij}, K_i, P_i, H$ \} gives exactly a Galilean group $G(3)$ without the central extension. The set \{ $J_{ij}, Y_i, P_i, M$ \} gives an $H_R(3)$ as the symmetry for the Galilean quantum particles. We have

$$[U, H] = -i\hbar M, \quad [K_i, Y_j] = -i\hbar \delta_{ij} U,$$

as the extra nontrivial Lie products among the full set. For the irreducible representation \{ $m, s_L, s_R$ \}, we have $\hat{P}_i$ directly represents $P_i$, $\hat{H} = -c \hat{P}_0$ which apparently goes to infinite limit, and $\hat{T} = \frac{1}{m} \hat{U} = \frac{1}{c^2} \hat{X}_0$ which goes to vanishing limit. $\hat{X}_i$ and $\hat{S}_{ij}$ are unchanged. $\hat{J}_{ij} = (\hat{X}_i \hat{P}_j - \hat{P}_i \hat{X}_j) + \hat{S}_{ij}$ is retrieved. $\hat{K}_i = \frac{1}{c^2} \hat{S}_{i0} - \frac{1}{c^2} \hat{X}_i \hat{H} + \hat{P}_i \hat{T}$ which has the limit as $\hat{P}_i \hat{T}$. If one put the parallel Lorentz boost to Galilean boost contraction onto the Lie algebra of the $\hat{S}_{\mu\nu}$, one can easily see that the the left- and right-handed spin generators both become essentially just $\hat{S}_{ij}$ (with a factor $\frac{1}{2}$, which makes no practical difference). Of course the particle mass as the key Casimir invariant maintains. Hence, one obtains \{ $m, s$ \} from \{ $m, s_L, s_R$ \}, while missing $\mathcal{V}$. That is, we have only labels from Casimir invariants of $H_R(3)$. From Eq.(8), we can say we have formally a diverging $\mathcal{V}$. However, all the logic is in favor of simply dropping $\mathcal{V}$ from consideration.

Simply taking the operators with their limiting forms here is somewhat too naive. In the spirit of the contraction as a process to get to the approximation of the dynamical theory, one should check how the operators representing the generator of the contracted symmetry
act as on each part of the original irreducible representation space which corresponds to an irreducible representation of the contracted symmetry. Such an analysis has been performed for the coherent state representation in Ref. [2], though with a somewhat different picture of the Heisenberg-Weyl symmetry. The results are that: the spin zero representation space as the span of \{|p^\mu, x^\mu\rangle\} gives as irreducible components each as the coherent state representation of \(H_R(3)\) spanned \{|p^\rho\rangle\} at infinite \(p^\rho\) (and \(e = c p^\rho\)) and a fixed \(t\); \(\hat{H}\) and \(\hat{T}\) no longer act as operator on such a quantum phase space; and \(\hat{K}_i\) can be taken as \(t\hat{P}_i\) for a classical time \(t\), hence generates \(t\)-dependent translations (in \(x^\rho\)). It is easy to check that the results stand. Or one can think about the spin zero part of the original representation space as spanned by the energy-momentum eigenstates \{\(|p^\mu\rangle\}\}, which at the contraction limit is left with the subspace at \(p^\rho \rightarrow \infty\), hence essentially a Hilbert space spanned by \{\(|p^\rho\rangle\}\} as \(H_R(3)\) case, for which the time ‘observable’ is classical. Ref. [2] has also illustrated how the observable algebra as an associated irreducible representation of universal enveloping algebra or (some proper extension) of the group algebra of \(H_R(1,3)\) reduces to that of \(H_R(3)\), as well as how the dynamics as dictated by the symplectic structure the theory naturally puts on the quantum phase space works in the same fashion. The free particle Hamiltonian can be taken as a natural Lorentz covariant extension of the Galilean \(\hat{H}\), i.e. as \(\frac{\hat{P}_\mu \hat{P}^\mu}{2m}\).

V. CRITIQUE ON POINCARÉ SYMMETRY AS RELATIVITY SYMMETRY FOR LORENTZ COVARIANT PARTICLE DYNAMICS

Poincaré symmetry is the widely accepted relativity symmetry for any Lorentz covariant system. However, at least from the point of view of a serious group theoretical construction of the physical theories as presented in Ref. [1, 2], it is definitely inadequate for quantum theories. How far it works for the classical case is illustrated in Ref. [13]. But our perspective is to see a classical theory only as an approximation from the quantum one to be formulated from the symmetry contraction of the latter [1, 2]. Notice that Poincaré symmetry taken as generated by \{\(J_{\mu\nu}', E_\mu\} is a subgroup of our \(H_R(1,3)\). However, an irreducible representation restricted to a subgroup is generally reducible. In addition, a Casimir invariant of a subgroup may not survive as such for the full group. It is the mostly unaddressed presence of the full \(H_R(1,3)\) symmetry which hides many of the symmetry theoretical problems in those ‘relativistic’ theories presented as one of Poincaré symmetry. It is interesting to note
that earlier group theoretical studies of the subject matter [14, 15] actually shows strong indications of preferring $H_\mu(1, 3)$ over the Poincaré symmetry, though not exactly completely clear and definite as in our language. Textbooks in group theory or quantum field theory, however, typically address only Poincaré symmetry, following textbooks on Einstein special relativity.

It has been reasonably well appreciated that the Poincaré symmetry does not contain any Heisenberg commutation relation and does not have any nontrivial $U(1)$ central extension. An abuse of the almost equivalent language of projective representation often used, instead of the proper one of $U(1)$ central extension, in the discussion of quantum physics adds another source of confusion. A projective representation for a group is a unitary representation of its $U(1)$ central extension. We can and, in our opinion, should always take the latter seriously as the symmetry. A trivial extension means taking a direct product with an otherwise independent $U(1)$ group. The Heisenberg commutation relations and the related phase transformations are then taken not as part of the fundamental symmetry while their role in the theory of course is, if anything, only more fundamental than the part as the Poincaré symmetry. For example, take the group theoretical presentation picture in the widely used textbook by Tung [12], it is much a parallel of what we have presented above, however, ascribed to the representation theory of the Poincaré symmetry. Starting with a (unitary) representation of the subgroup of translation, generated by $\{P_\mu\}$, specifically a Hilbert space spanned by the eigenvector $\{|p\rangle\}$, for a particle of rest mass $m > 0$, an irreducible representation of spin zero is claimed with the condition $-p_\mu p^\mu = m^2 c^2$. In fact, one has $P_\mu P^\mu$ as a Casimir invariant and $-m^2 c^2$ taken as its eigenvalue. However, that representation picture really fails to match to the formulation of ‘relativistic’ quantum mechanics or quantum field theory (for the scalar particle), i.e. the dynamics with the Klein-Gordon equation

$$-(\hbar^2)\partial_\mu \partial^\mu \phi(x^\mu) = m^2 c^2 \phi(x^\mu).$$

The spacetime wavefunction $\phi(x^\mu) = \langle x^\mu | \phi \rangle$ is supposed to be the Fourier transform of the momentum space one of $\phi(p^\mu) = \langle p^\mu | \phi \rangle$. The functional value of the wavefunction is really the coordinate of the state in the Hilbert space on the corresponding eigenstate basis. For our irreducible representation at hand, the basis set obtained is really only $\{|p^j\rangle\}$ as $p^0$ is fixed to the single value of $\sqrt{p_\mu p^\mu + m^2 c^2}$. We really have only $\phi(p^j)$ and hence also $\phi(x^j)$, the Hilbert space for the ‘nonrelativistic’ case! That is also why we do not have a Lorentz
covariant Hamiltonian formulation for such a theory. To complete the representation of the Hilbert space of \(|p^\mu\rangle\) or \(|p^i\rangle\), one needs to know how \(\hat{J}_{\mu\nu}\) act on the basis vectors, or the wavefunctions. There, the familiar picture of orbital angular momentum \(L_{\mu\nu} \equiv \hat{X}_\mu \hat{P}_\nu - \hat{P}_\mu \hat{X}_\nu\) is essentially assumed with say \(\hat{X}_\mu = i\hbar \frac{\partial}{\partial p_\mu}\), [or the equivalent \(\hat{X}_\mu = x_\mu, \hat{X}_\mu = -i\hbar \frac{\partial}{\partial x_\mu}\) for \(\phi(x^\mu)\)]. That is indeed explicitly presented in the article by Bargmann and Wigner \[16\], which is a key source of most later presentations. Hence, the basis quantum structure of the Heisenberg commutation relation is put in independent of the representation theory of Poincaré symmetry. That is essentially promoting the representation to one of \(H_R(1,3)\), only not done in name and in a fully consistent manner.

Furthermore, while we have illustrated how an irreducible representation of \(H_R(1,3)\) can be identified with one as a product of an irreducible representation of \(H(1,3)\) and one of \(SO(1,3)\), with the latter as trivial for the spin zero case, the feature is not there for the Poincaré symmetry. Our \(S_{\mu\nu}\) cannot be obtained from combinations of generators of the latter. Ref.\[16\], for example, defines \(S_{\mu\nu}\) as \(J_{\mu\nu} - L_{\mu\nu}\) based on \(\hat{X}_\mu = i\hbar \frac{\partial}{\partial p_\mu}\), as discussed above, and use the \(S_{\mu\nu}\) to rewrite the Pauli-Lubáński vector or its magnitude square as the other Casimir invariant before using that to obtain the representations of various spin. Again, it is really representations of \(H_R(1,3)\) that are being dealt with, but in the name of only its subgroup of Poincaré symmetry. The simple picture of a quantum field/wavefunction for a case of nontrivial spin as consisting of the finite number of spin components each as the single one for the spin zero case, quite often used in the group theoretical picture of quantum field theories (for example as presented in Maggiore \[17\]) cannot only be justified through adopting our language of the \(H_R(1,3)\) symmetry. And as such, it is direct and straightforward.

The shortcoming of the Poincaré symmetry is even more transparent in the observable algebra construction. The universal enveloping algebra or the group algebra can only has like function of \(\hat{J}_{\mu\nu}\) and \(\hat{P}_\mu\). There is no \(\hat{X}_\mu\) available, not even the spatial position operators \(\hat{X}_i\)! In spirit, it is exactly the same story, in order to promote a representation of the translational subgroup to one for the Poincaré symmetry that can be used in quantum theories, we have to assume the Heisenberg commutation relation, hence turn it into a representation of \(H_R(1,3)\). Only in the latter case can we obtain the natural description of the observable algebra and its action on the states from the group theory \[2\]. It is the same in spirit for the classical case. Poincaré symmetry cannot give a Lorentz covariant particle phase space, nor an observable
algebra containing the representation of the corresponding Lie algebra as in the Galilean case.

With $H_R(1,3)$, one loses the Einstein rest mass-squared as the Casimir invariant. The identification of the central charge as particle mass gives though the particle a mass, and the notion of center of mass for a system of particles, which maintains at the Galilean limit. The latter feature is not there for Poincaré symmetry, which hence gives no notion of additive mass for particle composites. A priori, $m$ and $m_E$ are independent notions. Note that the notion of constant $m_E$, or the so-called on-shell mass condition, is really given up in the more applicable dynamically nontrivial case of quantum field theory. Well, for a (Lorentz covariant) physical Hamiltonian for a free particle, the natural choice is obviously $\hat{H}_s = \frac{\hat{P}_\mu \hat{P}^\mu}{2m}$ with the Hamilton’s (or Heisenberg) equations \cite{2,8,9} giving $\frac{dX^\mu}{ds} = \frac{1}{m} \hat{P}^\mu$ and $\frac{dp^\mu}{ds} = 0$, for $s$ being the evolution parameter of the Hamiltonian flow generated by $\hat{H}_s$. One of course wants to look at the dynamics for the $\hat{H}_s$ eigenstates. The eigenvalue is clearly $-\frac{m^2}{2m}$ with constant $m_E$. One can see that with the usual notion of Einstein particle proper time $\tau$ here is a simple rescaling of the evolution parameter, as $s = \tau \frac{m}{m_E}$. We can extend that to the case of a charge particle moving inside a background electromagnetic field, when we have

$$\hat{H}_s = \frac{1}{2m} \left( \hat{P}_\mu - \frac{e}{c} A_\mu \right) \left( \hat{P}^\mu - \frac{e}{c} A^\mu \right)$$

for which

$$\frac{dX^\mu}{ds} = \frac{1}{m} \left( \hat{P}^\mu - \frac{e}{c} A^\mu \right) \equiv \frac{1}{m} \pi^\mu, \quad \frac{d}{ds} \left( \hat{P}^\mu - \frac{e}{c} A^\mu \right) = 0.$$  

Note that $\hat{P}^\mu$ being the observable in the Heisenberg commutation relation, as essentially the Poisson bracket between canonical variables \cite{8}, is the only canonical momentum which is now not $m_E \frac{dx^\mu}{d\tau}$. The same feature can be seen in corresponding classical theory. We can have the same relation $s = \tau \frac{m}{m_E}$ gives actually $\pi_\mu \pi^\mu = -m_E^2 c^2$ for the ‘momentum’ $\pi_\mu \equiv m_E \frac{dx^\mu}{d\tau}$. Moreover, the classical Hamiltonian $H_s = \frac{1}{2m} \left( p_\mu - \frac{e}{c} A_\mu \right) \left( p^\mu - \frac{e}{c} A^\mu \right)$ gives exactly the Newtonian form of equation of motion under the Lorentz force for $s = \tau$, and hence $m_E = m$. The $H_R(1,3)$ based theories, quantum or classical, hence agree with the established picture. It is important to emphasize that only the canonical momentum $p^\mu$ and its quantum analog $\hat{P}^\mu$, should be taken as the energy-momentum four-vector, the basic additive observable quantity the total sum of which conserves for any close system. As generally true for the Hamiltonian formulation, the momentum may not be mass $\times$ velocity, and when that does not hold, the latter product is not necessarily a quantity of
any physical importance. We may not have the Newtonian form of the equation of motion as \( \text{force} = \text{mass} \times \text{acceleration} \), but rate of change of (canonical) momentum equals to the force as negative the potential gradient maintains. That has been generally accepted as correct. For a more general dynamical situation of a particle, there is no reason to insist that the on-shell mass condition as constant \( p_\mu p^\mu \) has to be satisfied. Einstein himself did not support that. In a 1935 paper \([18]\), he raised exactly the same concern that the ‘momentum’ \( \pi^\mu \) may not be truly important momentum, \( \text{i.e.} \) the canonical \( p^\mu \). The simple classical electrodynamics given by \( H_s \) (with \( s \) as \( \tau \)), and the Newtonian equation of motion under a Lorentz force, gives only the relation \( \pi_\mu \pi^\mu = -m_E c^2 \) as a trivial extension of the constant magnitude of Einstein four-velocity as \( \Delta x^\mu / \Delta \tau \), which is not the on-shell mass condition. \( p_\mu p^\mu \) is not even a constant of motion for a generic given electromagnetic field. \( m_E \) describes only the magnitude of the four-vector \( \pi^\mu \scriptstyle (/c) \), or the value of \( \pi^0 \scriptstyle (/c) \) in the rest frame, by definition. Even the notion of ‘rest mass’ may really be meaningless, because for a particle under an accelerating force the particle rest frame is not an inertial frame, hence the notion of rest mass being a fundamental property of the particle not justified. For the same reason, the particle proper time as the time coordinate in the particle rest frame should not be a useful evolution parameter. Besides, the accelerating charge particle would radiate and hence experience radiation reaction. Newtonian form of the equation of motion with Lorentz force would not be exactly correct in general.

For a composite system, parallel analysis as shown above for the \( H_R(3) \) or \( \tilde{G}(3) \) symmetric ‘nonrelativistic’ case obviously goes through, with additive mass \( m \), notion of center of mass and \( \frac{1}{2} \mathbf{S}_{\mu\nu} \mathbf{S}^{\mu\nu} \), here together with and \( \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_{\mu\nu} \mathbf{S}_{\rho\sigma} \), as effective Casimir invariants are obtained group theoretically. Again \( \hat{J}_{\mu\nu} \) and \( \hat{L}_{\mu\nu} \) have no such roles. \( \hat{S}_{\mu\nu} \) contains the ‘orbital’ angular momentum \( \hat{R}_\mu \hat{Q}_\nu - \hat{Q}_\mu \hat{R}_\nu \). Note that the physical implication of those Casimir invariants in a composite system is not quite the same as in an elementary system of a single particle here. From the symmetry perspective, there is no reason at all to expect the need of anything beyond a single irreducible representation to be necessary for the description of an elementary system taken as one that cannot be seen as composed of different parts. A reducible representation as the sum of irreducible parts gives exactly a description of a system with parts. The Casimir invariants are the symmetry characters of an elementary system. A composite system constructed from the product of irreducible representations has parts of different Casimir invariants in themselves. The Casimir invariants of the system
mathematically characterize different subspaces of the phase space of the composite system which are invariant under all observables constructed from the symmetry. If one starts with a state within one such invariant subspace, the description of any dynamical features of the state surely needs not go beyond that subspace. However, at least in a quantum theory, there is no trivial answer to the question if we can prepare an initial state that is a linear combination of states in different such invariant subspaces. Here, explicitly, it is the question of if we can prepare a state as a nontrivial linear combination of states of different ‘spin’ for \( \hat{S}_{\mu\nu} \). The question is particularly interesting in relation to the notion of quantum frames of reference which has been catching popularity lately (interested readers are referred to Ref.\[19\] and references therein). In particular, a carefully detailed analysis has been presented by Loveridge et.al.\[20\] addressing symmetry issues with a practical consideration of the relative nature of observed quantities, as well as the validity of superselection rules.

The key concluding statement of “observable quantities are invariant under symmetry and that, in quantum mechanical laboratory experiments, the measured statistics pertain not to some absolute quantity, but rather to an observable, relative quantity, corresponding to the system and apparatus combined, along with the appropriate high localisation limit on the side of the apparatus” is to be taken as the background to understand our discussion of the observables as apparent ‘absolute quantities’. Here, the “apparatus” can be replaced by, or embodies, the physical object as the frame of reference to give physical definition to the observables and “high localisation limit” essentially corresponds to objects which can be well described as classical. For a composite of two particles with one being essentially classical, of large mass, that conclusion says that the \( R-Q \) variables as relative observables to the center of mass describe well the physics of the light, quantum, particle as that given by the absolute quantity description picture of the single particle system we presented. Though that is what one would expect, the solid confirmation of that is as important as it is interesting. To truly look at a system of quantum particles, however, or quantum particles observed from a physical frame of reference, the quantum nature of it which cannot be neglected, more studies, probably along the lines of Refs.\[19\] and \[21\], would be needed.
VI. SUMMARY

It has been well appreciated that ‘nonrelativistic’ quantum mechanics can be seen as a representation theory of the Heisenberg-Weyl symmetry $H(3)$. It is obvious that it should be extended to $H_R(3)$ through incorporating the rotations making $X_i$ and $P_i$ three-vectors. The symmetry is essentially $\tilde{G}(3)$, the quantum version of the Galilean symmetry, with the ‘time-translation’ part taken out. We have illustrated why it is not a good idea to have the latter part so long as we want to describe more than free particles. In an interacting system, the ‘time’ as built into the Galilean symmetry cannot be the Newtonian time of the dynamical evolution. That is true even in the classical case. Looking at the mathematical structure as it is, the obvious Lorentz covariant counterpart is $H_R(1,3)$.

The perspective of taking the position operators not directly as the representation of symmetry generators but as $\frac{1}{m}$ times that with the (Newtonian) mass $m$ as the eigenvalue of the central charge generator fully reconciles the Heisenberg-Weyl symmetry picture with that from the Galilean one. An extended analysis to essentially includes are aspects of the dynamical theory of a spin zero particle has been presented in Ref. [1], including the retrieval of the classical approximate theory. We have given above the simple extension to the cases of finite spin from finite dimensional representations of Lorentz symmetry. The full framework has a natural parallel for the full Lorentz covariant setting (see Ref. [2] for details of spin zero case). Apart from giving a symmetry origin of the notion of particle (Newtonian) mass, the formulation also dictates the notion of center of mass. The fundamental aspects are otherwise not incorporated into the symmetry formulations.

The Einstein rest mass $m_E$ is a different notion, which may be taken as the same as $m$ for a free particle or one moving under a Lorentz force. When a particle is driven by a force into acceleration, the particle rest frame is not inertial, the significance of $m_E$ then is questionable. Constant $-m_E^2$ as the magnitude square of the energy-momentum four-vector, the on-shell condition, is dictated by the Poincaré symmetry. However, we have discussed how stories of quantum theory as representations of Poincaré symmetry are really about representations of the larger $H_R(1,3)$ for which one does not have $-m_E^2$ as Casimir invariant. We have also seen that even for a classical particle moving under a Lorentz force, the notion of constant rest mass is really an empty one and the magnitude of the true energy-momentum four-vector is not a constant of motion. Taking the $H_R(1,3)$ seriously,
however, leads to a very different theory of quantum mechanics, with a notion of Minkowski metric operator on the vector space of states, which is no longer a Hilbert space \cite{Bet}. The latter is fully compatible with a noncommutative geometric picture having the position and momentum observables $\hat{X}_\mu$ and $\hat{P}_\mu$ as noncommutative coordinates of the quantum phase space \cite{Kong}. The symmetry theoretical picture can be seen as giving that phase space as the model of the physical spacetime. Poincaré symmetry may be seen as the natural isometry group of the Minkowski spacetime, like the Euclidean group of the Newtonian space. We see quantum theories as asking for a picture beyond that to truly respect the notion of the position observables, for example, as really the position coordinate of a particle/physical object in the quantum physical space(time) and our symmetry formulation gives a fully consistent story of all that.

The authors are partially supported by research grants number 109-2119-M-008-016 and 110-2119-M-008-016 of the MOST of Taiwan.

\begin{thebibliography}{99}
\bibitem{Chew} C.S. Chew, O.C.W. Kong, and J. Payne, J. High Energy Phys. Gravit. Cosmol. 5 (2019) 553.
\bibitem{Bet} S. Bedić, O.C.W. Kong, and H.K. Ting, Symmetry 13 (2021) 22.
\bibitem{Trump} M.A. Trump and W.C. Schieve, \textit{Classical Relativistic Many-Body Dynamics}, Springer Science+Business Media 1999.
\bibitem{Horwitz} L.P. Horwitz, \textit{Relativistic Quantum Mechanics}; Springer Science+Business Media: Dordrecht, The Netherlands, 2015.
\bibitem{Currie} D.C. Currie, T.F. Jordan, and E.G.C. Sudarshan, Rev. Mod. Phys. 35 (1963) 350.
\bibitem{deAz} J.A. de Azcárraga and J.M. Izquierdo, \textit{Lie Groups, Lie Algebras, Cohomology and Some Applications in Physics}, Cambridge University Press, New York, 1995.
\bibitem{Levy} J.-M. Lévy-Leblond, J. Math. Phys. 4 (1963) 776.
\bibitem{Kong} O.C.W. Kong, Results Phys. 19 (2020) 103606.
\bibitem{Kong2} O.C.W. Kong and W.-Y. Liu, Chin. J. Phys. 71 (2021) 418.
\bibitem{Inonu} E. İnönü, E.P. Wigner, Proc. Natl. Acad. Sci. USA 39 (1953) 510.
\bibitem{Gilmore} R. Gilmore, \textit{Lie Groups, Lie Algebras, and Some of Their Applications}, Dover, 2005.
\bibitem{Tung} W.K. Tung, \textit{Group Theory in Physics}, World Scientific, 1985.
\end{thebibliography}
[13] O.C.W. Kong and J. Payne, Symmetry 13 (2021) 1925.

[14] J.S. Zmuidzinas, J. Math. Phys. 7 (1966) 764.

[15] J.E. Johnson, Phys. Rev. 181 (1969) 1755.

[16] Y. Bargmann and Wigner, E. P., Proc. Natl. Acad. Sci. USA 34, 211 (1948).

[17] M. Maggiore, *A Modern Introduction to Quantum Field Theory*, Oxford University Press, 2005.

[18] A. Einstein, Bull. Am. Math. Soc., 223 April (1935).

[19] F. Giacomini, E. Castro-Ruiz, and Č. Brukner, Nature Commun. 10 (2019) 494.

[20] L. Loveridge, T. Miyadera, and P. Busch, Found. Phys. 48 (2018) 135.

[21] O.C.W. Kong, Results in Physics, 31, 105033 (2021).