On the Genus Expansion in the Topological String Theory

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ABSTRACT

A systematic formulation of the higher genus expansion in topological string theory is considered. We also develop a simple way of evaluating genus zero correlation functions. At higher genera we derive some interesting formulas for the free energy in the $A_1$ and $A_2$ models. We present some evidence that topological minimal models associated with Lie algebras other than the A-D-E type do not have a consistent higher genus expansion beyond genus one. We also present some new results on the $CP^1$ model at higher genera.
1. Introduction

The topological field theory approach to two-dimensional gravity \([1, 2, 3]\) has uncovered a fascinating interplay among the twisted \(N = 2\) supersymmetry \([4, 5]\), the theory of KP hierarchy and the topological Landau-Ginzburg description \([6, 7]\). It is now well-understood that the integrable structure of the topological strings at genus zero is governed by the dispersionless KP hierarchy \([8, 9, 10]\). Physical spectra and recursion relations among correlation functions of topological strings are described in a simple manner using the Landau-Ginzburg formulation \([7, 11, 12]\).

There remain at least two issues which should be better understood in the topological field theory approach to two-dimensional gravity. One is how to take into account the full phase space coupling constants in the Landau-Ginzburg description of topological strings (for an attempt, see \([13]\)). The other is the explicit and hopefully systematic evaluation of the higher genus expansion. Studying the latter issue is our purpose in the present paper. In particular explicit calculations may be useful since there has been no satisfactory formulation so far to take into account the higher genera in topological Landau-Ginzburg approach.

We are motivated by the remarkable expression for the genus one free energy \(F_1\). It has been known that

\[
F_1 = \frac{1}{24} \log \det u_{\alpha \beta},
\]

where \(u_{\alpha \beta} = \partial^3 F_0 / \partial t_0 \partial t_\alpha \partial t_\beta\) with \(t_\alpha\) and \(F_0\) being the small phase space couplings and the genus zero free energy, respectively \([2, 14]\). It is remarkable that (1.1) is valid for topological gravity coupled to any topological minimal matter. Moreover (1.1) gives rise to the genus one correlation functions expressed entirely in terms of genus zero quantities. In what follows we shall study if this kind of structure emerges also in the higher genus free energies.

The paper is organized as follows. In sect. 2 we briefly review the genus zero theory. Some useful formulas for the genus zero free energy are obtained. Sect. 3 is
devoted to the higher genus calculations. The explicit higher genus results for the $A_1$ and $A_2$ models are presented. In sect.4 we shall examine topological gravity coupled to topological matter other than the $A$-type models. We present some evidence against the consistency of models associated with Lie algebras other than the A-D-E type at higher genus. We also present some new results on the higher genus expansion of the $CP^1$ model. Finally in sect.5 we discuss the vector model in the light of our strategy. We relegate to appendices A, B and C the explicit forms of constitutive relations for the $A_1$, $A_2$ and $D_4$ models, respectively. In appendix D the free energy of pure gravity theory up to genus five is obtained.

2. Topological Strings at Genus Zero

In this section topological gravity coupled to topological matter at genus zero is considered. We start with reviewing the basic ingredients of the genus zero theory on the basis of [2]. We then discuss how to calculate explicitly genus zero correlation functions in the full phase space.

Consider a topological matter theory whose BRST invariant observables are denoted as $O_\alpha$ with $\alpha \in I$; a certain set of integers. In particular $O_0$ stands for the identity operator. In the minimal topological matter, for instance, the relation between the set $I$ and the exponents of the ADE Lie algebra is well known. Two-point functions are given by

$$\langle O_\alpha O_\beta \rangle = \eta_{\alpha\beta}, \quad (2.1)$$

where $\eta_{\alpha\beta}$ is a topological metric which is non-degenerate $\eta_{\alpha\beta} \eta^{\beta\gamma} = \delta^\gamma_\alpha$. The primary fields $O_\alpha$ generate the commutative associative algebra

$$O_\alpha O_\beta = C_{\alpha\beta}^\gamma O_\gamma, \quad (2.2)$$

It is clear that

$$\langle O_\alpha O_\beta O_\gamma \rangle = C_{\alpha\beta\gamma}, \quad (2.3)$$

where $C_{\alpha\beta\gamma} = C_{\alpha\beta}^\rho \eta_{\rho\gamma}$ and $C_{\alpha\beta0} = \eta_{\alpha\beta}$. 

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Let us turn on topological gravity. Coupling to topological gravity makes the physical spectrum of the theory quite rich. In addition to the original observables $O_\alpha$ which we now call gravitational primary fields, there appear gravitational descendant fields $\sigma_n(O_\alpha)$ for $n = 1, 2, \ldots$. For the sake of convenience we write $\sigma_0(O_\alpha) = O_\alpha$. Note that the identity operator $O_0$ is identified with the puncture operator $P$. Then, in the presence of gravity (2.1) should read

$$\langle PO_\alpha O_\beta \rangle = \eta_{\alpha\beta}. \quad (2.4)$$

In the case of pure gravity the only primary field is the puncture operator $P$. The puncture operator, together with its first descendant $\sigma_1(P)$ (dilaton field), plays a distinguished role in topological string theory.

To each $\sigma_n(O_\alpha)$ we associate a coupling constant $t_{n,\alpha}$. For short we write $t_\alpha = t_{0,\alpha}$ which are called primary couplings henceforth. The coupling constant space spanned by all $t_{n,\alpha}$ is referred to as the full phase space, whereas its subspace with $t_\alpha \neq 0, t_{n,\alpha} = 0 \ (n \geq 1)$ is called the small phase space. When all $t_{n,\alpha}$ vanish the system is on critical. Switching on $t_{n,\alpha}$ is equivalent to put the system in non-trivial backgrounds. Evolution of the topological string in such backgrounds is of our main interest.

We employ the flat coordinate system to describe the phase space [15,7]. The flat coordinates are advantageous since in the small phase space $\langle PO_\alpha O_\beta \rangle$ turn out to be independent of $t_\alpha$ and generic three-point functions depending upon $t_\alpha$ are evaluated from the free energy $F_0$

$$C_{\alpha\beta\gamma}(t) = \frac{\partial^3 F_0}{\partial t_\alpha \partial t_\beta \partial t_\gamma}. \quad (2.5)$$

The two-point functions

$$u_\alpha = \langle PO_\alpha \rangle \quad (2.6)$$

will play a role of the fundamental order parameters in our topological string theory.
In the small phase space we have

\[ u_\alpha = \eta_{\alpha\beta} t_\beta. \quad (2.7) \]

This simple relation is invalidated in the full phase space where \( u_\alpha \) depend on \( t_{n,\alpha} \) in a complicated way. In spite of this fact there exists the following remarkable observation. Suppose that a two-point function \( \langle XY \rangle \) with \( X, Y \in \{ \sigma_n(O_\alpha) \} \) is given as a function of \( t_\alpha \) in the small phase space. Note also that the relation (2.7) is invertible. Then we obtain \( \langle XY \rangle \) expressed in terms of \( u_\alpha \), i.e.

\[ \langle XY \rangle = R_{XY}(u). \quad (2.8) \]

It was shown in [2] that the function \( R_{XY}(u) \) is universal in the sense that (2.8) continues to be valid even in the full phase space. The relation (2.8) is thus quite fundamental, and called the constitutive relation. The proof is based on the topological recursion relation

\[ \langle \sigma_n(O_\alpha)XY \rangle = \langle \sigma_{n-1}(O_\alpha)O_\beta \rangle \langle O_\beta X Y \rangle \quad (2.9) \]

with \( O_\beta = \eta^{\beta\gamma}O_\gamma \). We note that our normalization of \( t_{n,\alpha} \) is different from [2], and hence there is no factor \( n \) on the RHS of (2.9).

The other important recursion relation is the puncture equation [1,2]

\[ \langle P \sigma_{n_1}(O_{\alpha_1}) \cdots \sigma_{n_s}(O_{\alpha_s}) \rangle = \sum_{i=1}^{s} \prod_{k=1}^{i} \sigma_{n_k-\delta_{ik}}(O_{\alpha_k}) \quad (2.10) \]

which is valid in the small phase space. Using the full phase space free energy \( F_0 \) we can cast (2.10) into the differential equation

\[ \frac{\partial}{\partial t_0} F_0 = \frac{1}{2} t_\alpha \eta_{\alpha\beta} t_\beta + \sum_{n=0}^{\infty} t_{n+1,\beta} \frac{\partial}{\partial t_{n,\beta}} F_0. \quad (2.11) \]
Taking a derivative with respect to $t_\alpha$ one finds

$$u_\alpha = \eta_{\alpha\beta} t_\beta + \sum_{n=0}^{\infty} t_{n+1,\beta} \langle \sigma_n(\mathcal{O}_\beta) \mathcal{O}_\alpha \rangle,$$  \hspace{1cm} (2.12)

where

$$u_\alpha = \frac{\partial^2 F_0}{\partial t_0 \partial t_\alpha}, \hspace{1cm} \langle \sigma_n(\mathcal{O}_\beta) \mathcal{O}_\alpha \rangle = \frac{\partial^2 F_0}{\partial t_{n,\beta} \partial t_\alpha}. \hspace{1cm} (2.13)$$

This is the string equation at genus zero [2]. Since the RHS is written in terms of $u_\alpha$ (constitutive relation), (2.12) is a kind of mean field theoretic equation to determine the order parameters $u_\alpha$. The puncture equation (2.10) in fact holds for arbitrary genus. Thus we have the string equation for all genera to which we turn in later sections.

The genus zero free energy is known to take the universal form which is now described as

**Proposition 1.** The free energy at $g = 0$ is given by

$$F_0(t) = \frac{1}{2} \sum_{n,m=0}^{\infty} \langle \sigma_n(\mathcal{O}_\alpha) \sigma_m(\mathcal{O}_\beta) \rangle \tilde{t}_{n,\alpha} \tilde{t}_{m,\beta},$$  \hspace{1cm} (2.14)

where $\tilde{t}_{n,\alpha} = t_{n,\alpha} - \delta_{n1} \delta_{\alpha0}$ [8,9,10].

This formula is not that obvious since two-point functions have the non-trivial $t_{n,\alpha}$ dependence.

[proof] It is sufficient if one can show

$$\sum_{n,m=0}^{\infty} \left( \frac{\partial}{\partial t_{k,\gamma}} \langle \sigma_n(\mathcal{O}_\alpha) \sigma_m(\mathcal{O}_\beta) \rangle \right) \tilde{t}_{n,\alpha} \tilde{t}_{m,\beta} = 0. \hspace{1cm} (2.15)$$

From the constitutive relation we can make use of the chain rule

$$\frac{\partial}{\partial t_{k,\gamma}} = \frac{\partial u_\rho}{\partial t_{k,\gamma}} \frac{\partial}{\partial u_\rho}. \hspace{1cm} (2.16)$$
To calculate the $u_\rho$-derivative we first note that

\[
\frac{\partial}{\partial t_\tau} \langle \sigma_n(\mathcal{O}_\alpha) \sigma_m(\mathcal{O}_\beta) \rangle = \langle \sigma_n(\mathcal{O}_\alpha) \sigma_m(\mathcal{O}_\beta) \mathcal{O}_\tau \rangle \\
= \langle \sigma_{n-1}(\mathcal{O}_\alpha) \mathcal{O}_\sigma \rangle \langle \sigma_{m-1}(\mathcal{O}_\beta) \mathcal{O}_\delta \rangle \langle \mathcal{O}^\sigma \mathcal{O}^\delta \mathcal{O}_\tau \rangle,
\]

where we have used (2.9) twice. On the other hand the LHS of the above may be evaluated by using the chain rule (2.16). Thus we get

\[
\frac{\partial}{\partial u_\rho} \langle \sigma_n(\mathcal{O}_\alpha) \sigma_m(\mathcal{O}_\beta) \rangle = \langle \sigma_{n-1}(\mathcal{O}_\alpha) \mathcal{O}_\sigma \rangle \langle \sigma_{m-1}(\mathcal{O}_\beta) \mathcal{O}_\delta \rangle C^{\sigma\delta\rho},
\]

where

\[
C^{\sigma\delta\rho} = g^\rho\tau \langle \mathcal{O}^\sigma \mathcal{O}^\delta \mathcal{O}_\tau \rangle
\]

and $g_{\alpha\beta}g^{\beta\gamma} = \delta^\gamma_\alpha$ with $g_{\alpha\beta} = \partial^3 F_0/\partial t_0 \partial t_\alpha \partial t_\beta$. Plugging (2.18) into the LHS of (2.15) and using the string equation one achieves the desired result.]

In a mean field theory the variation of the free energy with respect to the order parameters gives rise to the self-consistent mean field equation. What do we obtain if we take a variation of the free energy (2.14) with respect to $u_\alpha$? To see this define $u^\rho = \eta^{\rho\gamma} u_\gamma$ and evaluate $\partial F_0/\partial u^\rho$. Using (2.18) we get

\[
\frac{\partial F_0}{\partial u^\rho} = \frac{1}{2} C^{\rho\alpha\beta} (u) X_\alpha X_\beta
\]

with

\[
X_\alpha = \left( -u_\alpha + \eta_{\alpha\gamma} t_\gamma + \sum_{n=0}^\infty t_{n+1,\gamma} \langle \sigma_n(\mathcal{O}_\gamma) \mathcal{O}_\alpha \rangle \right).
\]

Thus under the extremum condition $\partial F_0/\partial u^\rho = 0$ we obtain (string equation)$^2=0$. For pure gravity this was noted earlier in [16,17].

Let us next turn to explicit calculations of genus zero correlation functions. We first prove that
Proposition 2. The quantities

\[ u_{\alpha_1 \cdots \alpha_n} \equiv \frac{\partial^{n+1} F_0}{\partial t_0 \partial t_{\alpha_1} \cdots \partial t_{\alpha_n}} \]  

(2.22)

can be expressed as a sum of tree graphs with \( n \)-external lines. The propagator and vertices are given in the proof.

[proof] The genus zero string equation (2.12) is written as

\[ u_\alpha = \eta_{\alpha \beta} t_\beta + f_\alpha(t, u), \]
\[ f_\alpha(t, u) = \sum_{n=0}^{\infty} t_{n+1, \beta} \langle O_\beta O_\alpha \rangle. \]  

(2.23)

Taking the derivatives of this equation with respect to \( t_\beta \) and using the fact that \( f_\alpha \) depend on primary couplings only through the order parameters, one obtains

\[ u_{\alpha \beta} = \eta_{\alpha \beta} + f_\alpha \gamma u_\beta^\gamma, \]
\[ u_{\alpha \beta \gamma} = f_{\mu \nu \rho} u_\alpha^\mu u_\beta^\nu u_\gamma^\rho, \]
\[ u_{\alpha \beta \gamma \delta} = f_{\mu \nu \rho \sigma} u_\alpha^\mu u_\beta^\nu u_\gamma^\rho u_\delta^\sigma + [f_{\mu \nu \lambda} u_\lambda^\tau f_{\tau \rho \sigma} u_\alpha^\mu u_\beta^\nu u_\gamma^\rho u_\delta^\sigma + \text{(2 more terms)}], \]
\[ \cdots, \]

(2.24)

where

\[ f_{\alpha \alpha_1 \cdots \alpha_n} = \frac{\partial^n f_\alpha}{\partial u^{\alpha_1} \cdots \partial u^{\alpha_n}}. \]  

(2.25)

We have derived the second line in (2.24) by noticing that the first line is written as

\[ \delta_\alpha^\rho - f_\alpha \gamma \eta^{\gamma \rho} = [u^{-1}]^\rho_{\alpha}, \]  

(2.26)

where \( u_{\alpha \beta} [u^{-1}]^\beta_\gamma = \delta_\alpha^\gamma \). The rest of the formulas is obtained in a similar manner. (2.24) are the desired expressions for \( u_{\alpha_1 \cdots \alpha_n} \) where \( u_{\alpha \beta} \) is the propagator and (2.25) are the vertices (the indices of \( u \) are raised and lowered by the metric \( \eta \)).
Since the vertices are easily obtained by using the Gauss-Manin relation as we shall see below, (2.24) provides an efficient way of calculating the genus zero correlation functions. Let us consider the $A_1$ model and the $A_2$ model as examples.

The $A_1$ Model (Pure Topological Gravity)

The $g = 0$ string equation is simply given by

$$u(t) = \sum_{n=0}^{\infty} t_n \frac{u(t)^n}{n!}, \quad (2.27)$$

where we have set $t_n \equiv t_{0,n}$. Along the line explained above one evaluates $u^{(n)} = \frac{\partial^n}{\partial t_0^n} u$ as follows

$$u' = \frac{1}{M},$$
$$u'' = \frac{G[2]}{M^3},$$
$$u''' = \frac{3G[2]^2}{M^5} + \frac{G[3]}{M^4},$$
$$u^{(4)} = \frac{15G[2]^3}{M^7} + \frac{10G[2]G[3]}{M^6} + \frac{G[4]}{M^5},$$
$$u^{(5)} = \frac{105G[2]^4}{M^9} + \frac{105G[2]^2G[3]}{M^8} + \frac{10G[3]^2}{M^6} + \frac{15G[2]G[4]}{M^7} + \frac{G[5]}{M^6},$$

where $M = 1 - G[1]$ and

$$G[k] = \sum_{n=0}^{\infty} t_{n+k} \frac{u(t)^n}{n!}. \quad (2.29)$$

The variables $G[n]$’s were first introduced in [16].

The $A_2$ Model

In the $A_2$ model we have two primaries, $O_0 = P$ and $O_1 = Q$. Let us denote $\langle PP \rangle = u$ and $\langle PQ \rangle = v$. Following the $A_1$ case we introduce the $G$-type variables
as

\[
G[k] = t_{k,1} + t_{k+1,0}u + t_{k+1,1}v + t_{k+2,0}uv + t_{k+2,1}\left(\frac{u^3}{6} + \frac{v^2}{2}\right) + \cdots,
\]

\[
H[k] = t_{k,0} + t_{k+1,0}v + t_{k+1,1}\frac{u^2}{2} + t_{k+2,0}\left(\frac{u^3}{3} + \frac{v^2}{2}\right) + t_{k+2,1}\frac{u^2v}{2} + \cdots.
\] (2.30)

These variables are characterized by the relation

\[
\frac{\partial}{\partial v}\left(\frac{G[n]}{H[n]}\right) = \left(\frac{G[n+1]}{H[n+1]}\right), \quad \frac{\partial}{\partial u}\left(\frac{G[n]}{H[n]}\right) = \left(\frac{H[n+1]}{uG[n+1]}\right).
\] (2.31)

The string equation is then written as

\[ u = G[0], \quad v = H[0]. \] (2.32)

The propagators, the three- and four-vertices given in (2.24) and (2.25) turn out to be

\[
\begin{pmatrix}
  u_{00} & u_{01} \\
  u_{10} & u_{11}
\end{pmatrix} = \frac{1}{(1 - H[1])^2 - uG[1]^2} \begin{pmatrix}
  G[1] & 1 - H[1] \\
  1 - H[1] & uG[1]
\end{pmatrix},
\] (2.33)

\[
f_{000} = G[2], \quad f_{001} = H[2], \quad f_{011} = uG[2], \quad f_{111} = G[1] + uH[2],
\]

\[
f_{0000} = G[3], \quad f_{0001} = H[3], \quad f_{0011} = uG[3], \quad f_{0111} = G[2] + uH[3],
\]

\[
f_{1111} = 2H[2] + u^2G[3].
\] (2.34)

**General Case**

Let us define the $G$-type variables as

\[
G_\alpha[k] = \eta_{\alpha\beta}t_{k,\beta} + \sum_{n=0}^\infty t_{n+1+k,\beta}\langle \sigma_n(O_\beta)O_\alpha \rangle.
\] (2.35)

Then the string equation (2.12) becomes

\[ u_\alpha = G_\alpha[0]. \] (2.36)
Note that $G_\alpha[n]$ is characterized by the Gauss-Manin relation

$$\frac{\partial}{\partial u^\beta} G_\alpha[n] = C_{\alpha,\beta}^\gamma (u) G_\gamma [n + 1],$$  \hspace{1cm} (2.37)

which can be derived from the constitutive relation and (2.9).

The relation between $F_0$ and tree graphs is seen from another point of view. From the free energy in the form (2.14) one has

$$\frac{\partial F_0}{\partial t_0} = \sum_{m=0}^{\infty} \langle P \sigma_m (O_{\beta}) \rangle \bar{t}_{m,\beta}. \hspace{1cm} (2.38)$$

Considering this as a function of $t$ and $u$, say $S(t, u)$, it is easy to show that

$$\frac{\partial S}{\partial u^\alpha} = 0 \iff \text{string equation}, \hspace{1cm} (2.39)$$

which may be regarded as the action principle (at $g = 0$) [18,19,14]. Moreover, having the solution $u(t)$ of (2.39) it follows that

$$S(t, u(t)) = \frac{\partial F_0}{\partial t_0}. \hspace{1cm} (2.40)$$

Hence, $\partial F_0/\partial t_0$ can be evaluated by the tree level Feynman diagrams with the action $S$.

In the $A_1$ and $A_2$ cases it turns out that the explicit form of the action is very simple. We find

$$S_{A_1} = -\frac{1}{2} u^2 + \sum_{n=0}^{\infty} t_n \frac{u^{n+1}}{(n + 1)!},$$

$$S_{A_2} = -uv + \sum_{n,m=0}^{\infty} (m - 2)!! t_{3n+2m-1} \frac{v^n u^m}{n! m!}. \hspace{1cm} (2.41)$$

where, in $S_{A_2}$, $t_n$ ($n \not\equiv 0 \mod 3$) is defined by $t_{3n+\alpha+1} = t_{n,\alpha}$ and $t_n = 0$ if $n \equiv 0 \mod 3$ or $n < 0$.  

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Finally we remark that the following observations will be important in subsequent sections.

**Lemma 1.** The relations, (2.35) and (2.36), defining the transformation from the couplings \( t_{n,\alpha} \) to the moments \( G_\alpha[n] \) are invertible as formal power series, that is

\[
 t_{n,\alpha} \in \mathbb{Q}\{[G_\beta[m] \mid \beta \in I; m \geq 0]\},
\]

where \( \mathbb{Q}\{[x_1, x_2, \cdots] \} \) means the ring of the formal power series of \( x_1, x_2, \cdots \) with coefficients in \( \mathbb{Q} \).

**[proof]** (2.35) says

\[
 t_{n,\alpha} = n_{\alpha\beta} \left\{ G_\beta[n] - \sum_{m=n+1}^{\infty} t_{m,\beta} \text{polynomial in } u_\gamma \left( = G_\gamma[0] \right) \right\}. \tag{2.43}
\]

These relations are of triangular form in \( n \), and hence by iterative substitution of these, one obtains the results.]

Notice that, by solving (2.18) we can express \( f_{\alpha_1 \cdots \alpha_n} \)'s (and hence \( G_\alpha[n] \)'s) as polynomials in \( u_\alpha, u_{\alpha\beta}, [u^{-1}]_{\alpha\beta}, u_{\alpha\beta\gamma}, \cdots \). Thus we get

**Lemma 2.**

\[
 t_{n,\alpha} \in \mathbb{Q}[[u_\alpha, u_{\alpha\beta}, [u^{-1}]_{\alpha\beta}, u_{\alpha\beta\gamma}, \cdots \mid \alpha, \beta \cdots \in I]]. \tag{2.44}
\]
3. Topological Strings at Higher Genera

Let $F_g$ be the genus $g$ free energy of topological gravity coupled to topological matter associated with a simply laced Lie algebra $\mathcal{G}$. The genus expansion of the free energy then reads
\[
\log Z = F = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g
\]
with $\lambda$ being the string coupling constant. The partition function $Z$ is characterized by the following conditions [20,21]:

**Condition 1**: $Z$ is a $\tau$-function of the Drinfeld-Sokolov hierarchies associated to the Lie algebra $\mathcal{G}$.

**Condition 2**: $Z$ satisfies the $W$-constraints associated with the Lie algebra $\mathcal{G}$ (twisted by the Coxeter element):
\[
W_n^{(s+1)} Z = 0, \quad (s \in I, \ n \geq -s).
\] (3.1)

Among the $W$-constraints the $L_{-1}$ condition is equivalent to the puncture equation
\[
\frac{\partial F_g}{\partial t_{0,0}} = \frac{1}{2} \eta_{\alpha\beta} t_{0,\alpha} t_{0,\beta} \delta_{g,0} + \sum_{n=1}^{\infty} t_{n+1,\beta} \frac{\partial F_g}{\partial t_{n,\beta}}.
\] (3.2)

The $L_0$ condition reduces to the dilaton equation
\[
\frac{\partial F_g}{\partial t_{1,0}} = (2g - 2) F_g + \sum_{n=0}^{\infty} t_{n,\beta} \frac{\partial F_g}{\partial t_{n,\beta}}.
\] (3.3)

The conditions 1 and 2 are highly over determined system and many of them are redundant. In fact, in the case of $A_1$, $A_2$ models, one can prove that

**Lemma 3.** $F_g$ can be determined uniquely by the puncture equation, the dilaton equation and the flow equations with respect to the primary couplings $\{t_{0,\alpha}\}$ and the dilaton coupling $t_{1,0}$.
Consider a correlation function of primary operators

\[ \langle O_{\alpha_1} \cdots O_{\alpha_N} \rangle_g. \] (3.4)

When all the primary couplings are taken to be zero, these correlators are sum of finite terms of the form

\[ (1 - t_{1,0})^{-\left[N + \sum_{(n,\alpha) \in I_+} k_{n,\alpha} + 2g - 2\right]} \prod_{(n,\alpha) \in I_+} t_{n,\alpha}^{k_{n,\alpha}}, \] (3.5)

where \( I_+ = \{(n, \alpha) | (n, \alpha) \neq (1,0), (0, \alpha)\} \). The coefficients of these terms can be determined by using the primary and dilaton flow equations, and the puncture equation via induction on \( g \) and \( N \). []

It seems likely that this uniqueness lemma also holds for all ADE-type theories. In the following we derive some explicit formulas for \( F_g \) in the \( A_1 \) and \( A_2 \) cases making use of lemma 3.

### 3.1. The \( A_1 \) model

The flow in the \( A_1 \) model is governed by the KdV equation. Some data on the \( A_1 \) potentials are available in appendix A. By simple degree counting, \( F_g \) is homogeneous of degree \( 3(1 - g) \) under the assignment \( \deg(t_i) = 1 - i \). The first few terms of \( F_g \) are

\[
F_0 = \frac{t_0^3}{3!} + t_1 \frac{t_0^3}{3!} + t_2 \frac{t_0^4}{4!} + 2 \frac{t_1^2 t_0^3}{2! 3!} + \cdots,
\]

\[
F_1 = \frac{1}{24} \left( t_1 + \frac{t_1^2}{2!} + t_0 t_2 + \cdots \right),
\]

\[
F_2 = \frac{1}{5760} (5 t_4 + \cdots).
\] (3.6)

For \( g = 1 \) the closed expression for \( F_1 \) has been obtained [2,16,22]. The result is

\[
F_1 = \frac{1}{24} \log u',
\] (3.7)

where \( u = F_0'' \). For \( g \geq 2 \) one can prove that
Theorem 1: There exists a formula for $F_g$ ($g > 1$) of the form:

$$F_g = \sum_{\sum (k-1)l_k=3g-3} a_{l_2 \ldots l_{3g-2}} \frac{[u^n]l_2 \ldots [u(3g-2)]l_{3g-2}}{[u']^{2(1-g)+\sum kl_k}} + \sum_{\sum (k-1)l_k=3g-3} b_{l_2 \ldots l_{3g-2}} \frac{[G[2]]l_2 \ldots [G[3g-2]]l_{3g-2}}{[u']^{2(1-g)+\sum l_k}}. \quad (3.8)$$

[proof] We will prove the first line of the formula. The second one is obtained with the aid of (2.28). As we have shown in the previous section, the coupling $t_i$ is solved as a formal power series in $u, u', \frac{1}{u}, u'', u''', \ldots$. Then one has an expression of the form

$$F_g \in \mathbb{Q}[[u, u', \frac{1}{u}, u'', u''', \ldots]]. \quad (3.9)$$

We rewrite the puncture equation (3.2) as

$$\mathcal{L}_{-1}F_g \equiv \left( \frac{\partial}{\partial t_0} - \sum_{n=0}^{\infty} t_{n+1} \frac{\partial}{\partial t_n} \right) F_g = \delta_{g,0} \frac{t_0^2}{2}. \quad (3.10)$$

Especially for $g = 0$ one obtains

$$\mathcal{L}_{-1}u^{(k)} = \delta_{k,0}. \quad (3.11)$$

These two equations imply that $F_g$ ($g > 0$) cannot depend on $u$. Since there is no variable of positive degree other than $u$, the number of available variables is finite. The highest derivative variable is $u^{3g-2}$. The power of degree zero factor $u'$ is fixed since each term of $F_g$ should have a total number of $(2g - 2)$ $t_0$-derivatives. This proves the theorem.]

Thus we are left with a finite number of unknown coefficients at each genus. These coefficients can be fixed by using the first nontrivial constitutive relation

$$\langle P\sigma_1 \rangle = \frac{u^2}{2} + \frac{u''}{12}. \quad (3.12)$$

We mention here that each term in $F_g$ can be associated with a $g$-loop graph with $g - 1$ external lines, in which $1/u'$ and $u^{(n+1)}$ play the role of the propagator.
and $n$-vertex. This correspondence was first noticed in [2]. The second expression in (3.8) was obtained earlier in [16]. Furthermore they point out that the coefficients $b_{t_2 \cdots t_{3g-2}}$ yield the intersection numbers on moduli space of Riemann surfaces.

In appendix D we will list the explicit formulas up to $g = 4$. For $g \geq 2$ the higher derivative terms in $F_g$ are given by

$$F_g = a_1 \frac{u^{(3g-2)}}{u'^g} + a_2 \frac{u''u^{(3g-3)}}{u'^{g+1}} + a_3 \frac{u'''u^{(3g-4)}}{u'^{g+1}} + a_4 \frac{u''^2u^{(3g-4)}}{u'^{g+2}} + \cdots, \quad (3.13)$$

where

\begin{align*}
a_1 &= \frac{1}{24g!}, \\
a_2 &= \frac{-21}{24g(g-2)!}, \\
a_3 &= \frac{3(65-57g)}{7024g(g-2)!}, \\
a_4 &= \frac{-3(1226 + 325g - 1029g^2)}{144024g(g-2)!}. \\
\end{align*}

The coefficient $a_1$ has already been obtained in [16].

3.2. The $A_2$ Model

In the $A_2$ case, the degrees of couplings and the free energy are $\text{deg}(t_{n,\alpha}) = 3(1-n) - \alpha$, and $\text{deg}(F_g) = 8(1-g)$. Constitutive relations and two-point functions are given explicitly in appendix B. The puncture and dilaton equations for the $A_2$ model read

\begin{align*}
\frac{\partial F_g}{\partial t_{0,0}} &= t_{0,0}t_{0,1}\delta_{g,0} + \sum_{n=0}^{\infty} t_{n+1,\alpha} \frac{\partial F_g}{\partial t_{n,\alpha}}, \\
\frac{\partial F_g}{\partial t_{1,0}} &= (2g-2)F_g + \sum_{n=0}^{\infty} t_{n,\alpha} \frac{\partial F_g}{\partial t_{n,\alpha}}. \\
\end{align*}

The flow equations can be read off from the constitutive relations in appendix B.
Now, we study the generalization of the $A_1$ result (Theorem 1). Let us write the genus zero variables as

$$u = \frac{\partial^2 F_0}{\partial t_{0,0} \partial t_{0,0}}, \quad v = \frac{\partial^2 F_0}{\partial t_{0,0} \partial t_{0,1}}.$$  \hfill (3.16)

By the same argument as in the previous $A_1$ case, one has an expansion of the form

$$F_g \in \mathbb{Q}[\left[ \frac{1}{uu'^2 - v'^2}, u, v, u', v', u'', v'', \cdots \right]], \quad (g > 1).$$  \hfill (3.17)

The degrees are $\deg(u^{(n)}) = 2 - 3n$, $\deg(v^{(n)}) = 3 - 3n$. It is proved that $v$ can be eliminated from the expansion while $u$ still survives. Since $u$ has positive degree the expansion of $F_g$ will be an infinite series.

In spite of this difficulty we can show that

**Theorem 2:**

$$F_1 = \frac{1}{24} \log \det u_{\alpha\beta} = \frac{1}{24} \log(uu'^2 - v'^2),$$

$$F_2 = \frac{1}{1152} Q_1 - \frac{1}{360} Q_2 - \frac{1}{1152} Q_3 + \frac{1}{360} Q_4,$$  \hfill (3.18)

where we have used $\langle PQQ \rangle_0 = uu'$ (see appendix B) and

$$Q_1 = u_{0\alpha\beta\gamma\delta}(u^{-1})^{\alpha\beta}(u^{-1})^{\gamma\delta},$$

$$Q_2 = u_{0\alpha\beta\gamma}(u^{-1})^{\alpha\delta}(u^{-1})^{\beta\mu}(u^{-1})^{\gamma\nu},$$

$$Q_3 = u_{0\alpha\beta\gamma}(u^{-1})^{\alpha\gamma}(u^{-1})^{\beta\delta}(u^{-1})^{\mu\nu},$$

$$Q_4 = u_{0\alpha\beta\gamma}(u^{-1})^{\alpha\gamma}(u^{-1})^{\beta\rho}(u^{-1})^{\mu\nu}. \hfill (3.19)$$

[proof] By construction, these formulas satisfy the puncture and dilaton equations. The remaining task is to check the flow equations (up to dilaton), which is straightforward (by using the computer).  

These results suggest that for the $A_2$ model...
Ansatz: The genus \( g > 0 \) free energy \( F_g \) can be expressed as a sum of diagrams with \( g \)-loops and \( (g - 1) \)-external lines with the puncture index 0, in which the inverse propagator is \( u_{\alpha \beta} \) and the \( n \)-vertex \( u_{\alpha_1 \cdots \alpha_n} \) is given by

\[
 u_{\alpha_1 \cdots \alpha_n} = \frac{\partial^{n+1} F_0}{\partial t_0 \partial t_{\alpha_1} \cdots \partial t_{\alpha_n}} \quad (3.20)
\]

We have partial results for \( g = 3 \) (there exist about one hundred terms),

\[
 F_3 = \frac{1}{82944} P_1 - \left[ \frac{1}{27648} P_2 + \frac{1}{8640} P_3 \right] - \left[ \frac{1}{27648} P_4 + \frac{29}{16128} P_5 + \frac{1}{82944} P_6 + \frac{73}{725760} P_7 \right] + \cdots \quad (3.21)
\]

where \( P_i \) are given by

\[
 P_1 = u_{00\alpha\beta\gamma\delta \mu \nu} [u^{-1}]^{\alpha \beta} [u^{-1}]^{\gamma \delta} [u^{-1}]^{\mu \nu},
\]
\[
 P_2 = u_{00\alpha\beta\gamma\delta \mu} u_{0\rho\sigma} [u^{-1}]^{\alpha \rho} [u^{-1}]^{\beta \sigma} [u^{-1}]^{\gamma \delta} [u^{-1}]^{\mu \sigma},
\]
\[
 P_3 = u_{00\alpha\beta\gamma \mu \nu} u_{\nu \rho \sigma} [u^{-1}]^{\alpha \rho} [u^{-1}]^{\gamma \nu} [u^{-1}]^{\delta \rho} [u^{-1}]^{\mu \sigma},
\]
\[
 P_4 = u_{00\alpha\beta \mu \nu \rho \sigma} [u^{-1}]^{\alpha \beta} [u^{-1}]^{\gamma \mu} [u^{-1}]^{\delta \nu} [u^{-1}]^{\rho \sigma},
\]
\[
 P_5 = u_{00\alpha\beta \delta \mu} u_{\nu \rho \sigma} [u^{-1}]^{\alpha \beta} [u^{-1}]^{\gamma \nu} [u^{-1}]^{\delta \rho} [u^{-1}]^{\mu \sigma},
\]
\[
 P_6 = u_{00\alpha \beta \gamma \mu \nu \rho \sigma} [u^{-1}]^{\alpha \gamma} [u^{-1}]^{\beta \delta} [u^{-1}]^{\mu \nu} [u^{-1}]^{\rho \sigma},
\]
\[
 P_7 = u_{00\alpha \beta \gamma \delta \mu} u_{\mu \nu \rho \sigma} [u^{-1}]^{\alpha \mu} [u^{-1}]^{\beta \nu} [u^{-1}]^{\gamma \rho} [u^{-1}]^{\delta \sigma}.
\]

Furthermore, for any \( g > 1 \), we can show that the highest derivative term becomes

\[
 F_g = \frac{1}{24 g!} u_{\underbrace{00 \cdots 0}_{g}}^{\alpha_1 \beta_1 \cdots \alpha_g \beta_g} [u^{-1}]^{\alpha_1} [u^{-1}]^{\beta_1} \cdots [u^{-1}]^{\alpha_g} [u^{-1}]^{\beta_g} + \cdots \quad (3.23)
\]

It is interesting to note that all these diagrammatic formulas for \( A_2 \) coincide with that of \( A_1 \). Then we have made some computations to see if the simple diagrammatic structure observed for the \( A_1, A_2 \) models exists also for other models. According to our preliminary results on the \( A_3 \) and \( D_4 \) models more elaborate structure seems to be required to represent the higher genus terms.
4. General Models

In this section we wish to examine to what extent the observation made so far is valid for more general models. In [1,2,14] the general structure of topological gravity coupled to topological matter is described. The main result there is the “topological recursion relation” at $g = 0$ and $g = 1$. They read

$$\langle \sigma_n(\mathcal{O}_\alpha)_{XY} \rangle_0 = \langle \sigma_{n-1}(\mathcal{O}_\alpha)\mathcal{O}_\beta \rangle_0 \langle \mathcal{O}^\beta_{XY} \rangle_0$$  \hspace{1cm} (4.1)

and

$$\langle \sigma_n(\mathcal{O}_\alpha) \rangle_1 = \frac{1}{24} \langle \sigma_{n-1}(\mathcal{O}_\alpha)\mathcal{O}_\beta \mathcal{O}_\beta \rangle_0 + \langle \sigma_{n-1}(\mathcal{O}_\alpha)\mathcal{O}_\beta \rangle_0 \langle \mathcal{O}_\beta \rangle_1.$$  \hspace{1cm} (4.2)

The $g = 1$ formula (1.1) is the direct consequence of these equations [2,14]. Hence, the problem of checking the validity of the $g = 1$ formula reduces to the consistency check between the flow equations and the above recursion relations. This has been done at least for $A_p$ models using the direct relation between the KP and dispersionless KP formalism [2,23].

Recently Dubrovin has constructed topological matter theory associated to each Coxeter group (not only ADE but also BC and others) based on the WDVV equations [24]. The existence as well as the uniqueness of higher genus extension of these models is an interesting open question to which we now address ourselves.

4.1. Formulation of the problem

To clarify the points we first state the problem clearly. Let $F_0$ be any solution of the WDVV equation [1,7]

$$C_{\alpha\beta\mu}\eta^{\mu\nu}C_{\nu\gamma\delta} = C_{\alpha\gamma\mu}\eta^{\mu\nu}C_{\nu\beta\delta},$$  \hspace{1cm} (4.3)

where the notation is the same as in sect.2. The flow equations consistent with
(4.3) should take the form

$$\frac{\partial u_\beta}{\partial t_\alpha} = [\langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle_0 + \lambda^2 \text{ (terms with two derivatives)} + O(\lambda^4)]' \quad (4.4)$$

where $' = \partial/\partial t_0$, $\lambda$ is a parameter and terms of order $\lambda^{2n}$ consist of differential polynomials in $u$'s with a total number of $2n$ derivatives and suitable degree, whose coefficients will be determined by the commutativity of the flows. We now ask if such flows exist. If so, are they unique?

For the ADE models it is believed that the answer is affirmative and the unique flow corresponds to the Drinfeld-Sokolov hierarchy. We will check this in the $D_4$ case explicitly.

There are four primaries $P, Q, R$ and $S$ among which $S$ has the $\mathbb{Z}_2$ parity odd and the others are even. The order parameters are

$$u = \langle PP \rangle, \ v = \langle PQ \rangle, \ w = \langle PR \rangle, \ f = \langle PS \rangle. \quad (4.5)$$

The flow equations (or the constitutive relation) are determined by their commutativity. We start with the dispersionless ($\lambda = 0$) terms which are provided from the Landau-Ginzburg description using the flat coordinate [7]. It is remarkable that the commutativity condition determines the full flows completely up to one parameter $\lambda$ which is identified as the string coupling constant. In appendix C we list the results of full genus flow up to the dilaton flow equation. We also checked that the $g = 1$ terms are consistent with (1.1).

### 4.2. Two-primary models

To get some insight into the non-simply laced cases, let us consider a two-primary model whose $g = 0$ correlation functions are defined by

$$\langle PPQ \rangle_0 = \langle Q^5 \rangle_0 = 1. \quad (4.6)$$
The flow equations consistent with the topological recursion relation should be

\[
\frac{\partial u}{\partial t_{0,1}} = [v]',
\]

\[
\frac{\partial v}{\partial t_{0,1}} = \left[ \frac{u^3}{6} + \lambda^2 \left( \frac{u'^2}{24} + \frac{uu''}{12} \right) + a_1 \lambda^4 u^{(4)} \right]',
\]

\[
\frac{\partial u}{\partial t_{1,0}} = \left[ uv + \frac{\lambda^2 v''}{6} + a_2 \lambda^4 u^{(4)} \right]',
\]

\[
\frac{\partial v}{\partial t_{1,0}} = \left[ \frac{u^4}{8} + \frac{v^2}{2} + \lambda^2 \left( \frac{uu'^2}{6} + \frac{u^2v''}{6} \right) + \lambda^4 \left( a_5 u'^2 + a_4 u' u^{(3)} + a_3 uu^{(4)} + a_6 v^{(4)} \right) \right]',
\]

where \(a_1, \ldots, a_6\) are numerical coefficients yet to be determined.

In the RHS of (4.7) the \(g = 0\) terms are obtained from the constitutive relations (see sect. 2.3 in [2]). The \(g = 1\) terms are determined by imposing the mutual commutativity of flows. We find that the result is unique up to an overall constant which is absorbed into \(\lambda^2\). This \(g = 1\) result is also consistent with the genus one free energy (1.1), which is seen by expanding, for instance, \(u = \sum_{g=0}^{\infty} \lambda^{2g} u_g\) and evaluating \(\partial u_1/\partial t_{1,0} = \partial^3 F_1/\partial t_{1,0} \partial t_{0,0}^2\) with the aid of the \(g = 0\) flow equations.

At \(g = 2\) the commutativity of the flows demands that

\[
0 = \left[ \frac{\partial}{\partial t_{0,1}}, \frac{\partial}{\partial t_{1,0}} \right] u = \lambda^4 \left( \frac{7}{72} - a_4 - 2a_5 \right) u^{(3)^2} + \left( \frac{11}{72} - a_3 - 2a_4 - 2a_5 \right) u'' u^{(4)}
\]

\[
+ \left( \frac{5}{72} + a_1 - 2a_3 - a_4 \right) u' u^{(5)}
\]

\[
+ \left( \frac{1}{72} + a_1 - a_3 \right) uu^{(6)} + (a_2 - a_6) v^{(6)} \right) + O(\lambda^6),
\]

\[
0 = \left[ \frac{\partial}{\partial t_{0,1}}, \frac{\partial}{\partial t_{1,0}} \right] v = \lambda^4 \left( 15a_6 u''^3 + \left( -\frac{1}{72} - 20a_1 + a_4 + 2a_5 \right) u^{(3)^2} v^{(3)} + 15a_6 u'^2 u^{(4)}
\]

\[
+ (-15a_1 + a_3 + a_4) v'' u^{(4)} + \left( -\frac{1}{36} - 15a_1 + a_4 + 2a_5 \right) u'' v^{(4)}
\]

\[
+ (-a_2 - a_6) u'^2 v^{(6)} + u' (60a_6 u'' u^{(3)} + \left( -\frac{1}{36} - 6a_1 + a_3 + a_4 \right) v^{(5)}
\]

\[
+ \left( -\frac{a_2}{2} + a_6 \right) u^2 u^{(6)} + u (10a_6 u'^{3}) + 15a_6 u'^2 u^{(4)} + (a_2 - 6a_6) u' v^{(5)}
\]

\[
+ (-\frac{1}{72} + a_1 + a_3) v^{(6)} \right) + O(\lambda^6).
\]

(4.8)
Unfortunately these two equations are inconsistent (no solution exist for \(a_1, \ldots, a_6\)). Thus the model (4.6) cannot be extended to higher genus beyond \(g = 1\).

In general, for a two-primary model having \(g = 0\) correlation functions

\[
\langle PPQ \rangle_0 = \langle Q^{s+2} \rangle_0 = 1,
\]

(4.9)

the flow equations up to \(g \leq 1\) take the form

\[
\begin{align*}
\frac{\partial}{\partial t_{0,1}} u &= [v]', \\
\frac{\partial}{\partial t_{0,1}} v &= \left[ \frac{u^s}{s!} + \lambda^2 \left( \frac{u^{s-3} u^2}{(s-3)! \cdot 24} + \frac{u^{s-2} u''}{(s-2)! \cdot 12} \right) \right]', \\
\frac{\partial}{\partial t_{1,0}} u &= \left[ u v + \lambda^2 v''/6 \right]', \\
\frac{\partial}{\partial t_{1,0}} v &= \left[ s \frac{u^{s+1}}{(s+1)!} + \frac{v^2}{2} + \lambda^2 (s+1) \left( \frac{u^{s-2} u^2}{(s-2)! \cdot 24} + \frac{u^{s-1} u''}{(s-1)! \cdot 12} \right) \right]'.
\end{align*}
\]

(4.10)

These \(g = 1\) terms are again consistent with (1.1). Except for the case of \(s = 2\) (the Boussinesq equations), however, one cannot generalize these equations for \(g > 1\), which we have checked explicitly for \(s = 3, 4, 5\).

The spectrum of this two-primary model is characterized by the exponents \(I = \{1, s\}\) and the Coxeter number \(s + 1\). Hence the case \(s = 3\) (or \(s = 5\)) corresponds to the \(B_2(= C_2)\) model (or \(G_2\) model) of the Dubrovin’s table [24]. Of course, there exist integrable hierarchies corresponding to these algebras. The above results show that these hierarchies are not consistent with the topological recursion relation. This result could be related to the fact that one needs more than one \(\tau\)-functions in the Hirota formalism of these hierarchies. It is rather remarkable that the consistency of these models fails at the two-loop level.
4.3. THE $CP^1$ MODEL

We now discuss the $CP^1$ topological sigma model coupled to gravity [1]. The spectrum of the model consists of two primaries, $P$ and $Q$, and their descendants. Classically $P$ and $Q$ correspond to the zero form and the Kähler form which generate the cohomology of $CP^1$. We shall see that the $CP^1$ flow equations have higher genus extension.

The constitutive relation for $\langle QQ \rangle$ is given by

$$\langle QQ \rangle = e^u$$

with $u = \langle PP \rangle$ and $v = \langle PQ \rangle$ [2]. This characteristic form of $\langle QQ \rangle$ realizes the idea of a quantum deformation of the classical cohomology ring due to the instanton effect [1]. The other genus zero functions $\langle \sigma_n(X)Y \rangle$ are obtained using the topological recursion relations [2].

To consider the flow equations in the $CP^1$ model let us concentrate on the primary and dilaton flow equations. We start with the dispersionless flows

$$\frac{\partial u}{\partial t_{0,Q}} = [v]', \quad \frac{\partial v}{\partial t_{0,Q}} = [e^u]',$$

$$\frac{\partial u}{\partial t_{1,P}} = [uv]', \quad \frac{\partial v}{\partial t_{1,P}} = [(u - 1)e^u]'$$

where the relations $\langle \sigma_1(P)P \rangle = uv$ and $\langle \sigma_1(P)Q \rangle = (u - 1)e^u$ have been utilized. As in the previous section we want to determine higher genus terms by imposing the commutativity among flows. In contrast to the Dubrovin type models, it is difficult to enumerate candidate higher terms in the $CP^1$ model since $u$ and $v$ do not possess definite degrees. However, it turns out that we can still fix the higher
genus terms consistently. The result reads

\[
\frac{\partial u}{\partial t_{0,Q}} = [v]',
\]

\[
\frac{\partial v}{\partial t_{0,Q}} = [e^u + \lambda^2 G_1 e^u + \lambda^4 G_2 e^u + \lambda^6 G_3 e^u + O(\lambda^8)]',
\]

\[
\frac{\partial u}{\partial t_{1,P}} = [uv + \lambda^2 H_1 + \lambda^4 H_2 + \lambda^6 H_3 + O(\lambda^8)]',
\]

\[
\frac{\partial v}{\partial t_{1,P}} = [(u - 1)e^u + \frac{1}{2}v^2 + \lambda^2 K_1 e^u + \lambda^4 K_2 e^u + \lambda^6 K_3 e^u + O(\lambda^8)]',
\]

where

\[
G_1 = \frac{1}{24} u'^2 + \frac{1}{12} u'',
\]

\[
G_2 = \frac{1}{360} u''' + \frac{1}{180} u'u''' + \frac{7}{1440} u'^2u'' + \frac{1}{1920} u'^4 + \frac{1}{240} u'^2u'^2,
\]

\[
G_3 = \frac{1}{20160} u^{(6)} + \frac{1}{6720} u^{(5)}u' + \frac{1}{60480} u^{(4)}u'' + \frac{1}{5040} u^{(4)}u'^2 + \frac{1}{13520} u'^3
\]

\[
+ \frac{23}{120960} u''u'^2 + \frac{41}{60480} u'''u''u' + \frac{1}{6720} u'''u''' + \frac{1}{181440} u'''u^3 + \frac{29}{13520} u''^3
\]

\[
+ \frac{37}{120960} u'^2u'^2 + \frac{11}{161280} u'^4 + \frac{1}{322560} u'^6,
\]

\[
H_1 = \frac{1}{6} v'',
\]

\[
H_2 = -\frac{1}{360} v'''',
\]

\[
H_3 = \frac{1}{15120} v^{(6)},
\]
\[ K_1 = \frac{1}{24}(u + 3)u'^2 + \frac{1}{12}(u + 2)u'', \]
\[ K_2 = \frac{1}{360}(u + 4)u'' + \frac{1}{180}(u + 5)u'u'' + \frac{7}{1440}(u + 6)u'^2u'' + \frac{1}{1920}(u + 7)u^4 + \frac{1}{240}(u + 5)u''^2, \]
\[ K_3 = \frac{1}{20160}(u + 6)u(6) + \frac{1}{6720}(u + 7)u(5)u' + \frac{19}{60480}(u + 7)u(4)u'' + \frac{1}{5040}(u + 8)u(4)u'^2 + \frac{23}{120960}(u + 7)u''u'^2u' + \frac{41}{60480}(u + 8)u''u''u' + \frac{1}{6720}(u + 9)u''u'^3 + \frac{29}{181440}(u + 8)u'^3 + \frac{37}{120960}(u + 9)u'^2u'^2 + \frac{11}{161280}(u + 10)u''u'^4 + \frac{1}{322560}(u + 11)u^6. \]

From the order \( \lambda^2 \) terms we have confirmed that the genus one free energy again takes the form of (1.1).

These computations show that the \( CP^1 \) model has in fact a consistent higher genus extension which would provide valuable information on the holomorphic maps from higher-genus Riemann surfaces onto \( CP^1 \). One of us (T.E.) has recently obtained a preliminary result that the underlying integrable hierarchy of the \( CP^1 \) model is the Toda lattice hierarchy as suggested in [10] (with a constraint that two kinds of Toda times \( \{t_n\}, \{\bar{t}_n\} \) coincide \( t_n = \bar{t}_n, (n = 1, 2, \cdots) \)) and the times \( t_{n,Q} \) associated with the \( Q \) operator are identified with the Toda times \( t_{n,Q} = t_{n+1}, (n = 0, 1, 2, \cdots) \). Details of the \( CP^1 \) model will be discussed elsewhere.

5. Discussions

What we have been trying to do in the present paper is somewhat similar to the perturbation series in quantum field theories. In the Feynman diagram expansion of conventional field theories the amplitudes associated with multi-loop diagrams are written in terms of the tree level propagators and vertices according to the Feynman rule. In topological string theory our intention has been to organize the genus expansion in such a way that the \( g \geq 1 \) free energy at each genus is expressed in terms of the genus zero quantities. We have partially succeeded in pursuing this
program through explicit but tedious calculations. The full clarification remains still open.

In order to embody an idea it is instructive to examine a simpler geometrical model. Taking the vector model as such an example we present the calculations in a way parallel to those for gravity models.

The Vector Model

The O(N) vector model has been considered by several groups [25,26,27] to enhance our understanding of the matrix model for two-dimensional gravity. We follow closely ref.[25] and perform the calculations in the light of our spirit.

The O(N) model describes geometrical critical phenomena of randomly branched polymers. In the double scaling limit the free energy \( F \) is defined from the partition function \( Z(t) \) as \( F(t) = \log Z(t) \) where we turn on infinitely many coupling constants \( t_n \ (n = 0, 1, 2, \cdots) \) which are conjugate to the scaling operators \( \sigma_n \). In particular \( \sigma_0 \) is the marking operator \( P \) in the theory. There exists a “polymer” coupling constant \( \lambda \) with respect to which we are going to define the “genus” expansion \( F = \sum_{g=0}^{\infty} \lambda^{g-1} F_g \).

Introduce an one-point function \( \langle P \rangle = F' \equiv f/\lambda \) with \( ' = \partial/\partial t_0 \) which is the basic order parameter, and hence the counterpart of \( u_\alpha \) in gravity theory. The “polymer” equation, i.e. the analog of the string equation, now reads [25]

\[
t_0 + \sum_{n=1}^{\infty} t_n R_{n-1} = 0, \tag{5.1}
\]

where \( R_n/\lambda = \langle \sigma_n \rangle = \partial F/\partial t_n \) (i.e. \( R_0 = f \)). We demand that the flows parametrized by \( t_n \) commute. Then one has

\[
\frac{\partial}{\partial t_n} f = [R_n]', \tag{5.2}
\]

thereby \( R_n \) become the flow potentials. These flow potentials obey

\[
(n + 1)R_n = \lambda[R_{n-1}]' + f R_{n-1}. \tag{5.3}
\]
\( R_n \) has an expansion of the form
\[
R_n = \frac{f^{n+1}}{(n+1)!} + \frac{1}{2} \frac{f^{n-1}}{(n-1)!} f' + \lambda^2 \left[ \frac{1}{8} \frac{f^{n-3}}{(n-3)!} f'' + \frac{1}{6} \frac{f^{n-2}}{(n-2)!} f''' \right] + \ldots. \tag{5.4}
\]

Explicitly one finds
\[
R_1 = \frac{1}{2} (f^2 + \lambda f'), \\
R_2 = \frac{1}{6} (f^3 + 3\lambda f f' + \lambda^2 f''), \\
R_3 = \frac{1}{24} (f^4 + 6\lambda f^2 f' + 4\lambda^2 f f'' + 3\lambda^2 f'^2 + \lambda^3 f'''). 	ag{5.5}
\]

Let us expand \( f = \sum_{g=0}^{\infty} \lambda^g f_g \), then it is convenient to introduce the moments \( I_k \) as
\[
I_k = \sum_{n=0}^{\infty} t_{n+k} \frac{f^n}{n!}. \tag{5.6}
\]

The following relations are useful
\[
f'_0 = -\frac{1}{I_1}, \quad I'_n = -\frac{I_{n+1}}{I_1}. \tag{5.7}
\]

It is now straightforward to evaluate \( f_g \) using \( I_k \). Having (5.5) we first expand the polymer equation as follows
\[
0 = I_0 + \lambda \left( \frac{I_1 f_1 + \frac{1}{2} f'_0}{I_1} \right) + \lambda^2 \left( \frac{I_1 f_2 + \frac{1}{2} I_2 f_1^2 + \frac{1}{2} I_3 f_1 f'_0 + \frac{1}{2} I_2 f_1' + \frac{1}{2} I_4 f_0^2 + \frac{1}{2} I_3 f''_0}{I_1} \right) + \ldots \tag{5.8}
\]
from which one obtains
\[
0 = I_0 = t_0 + \sum_{n=1}^{\infty} t_n \frac{f^n}{n!}, \\
f_1 = \frac{I_2}{2I_1^2}, \tag{5.9}
\]
\[
f_2 = \frac{5I_2^3}{8I_1^5} + \frac{2I_2 I_3}{3I_1^4} - \frac{I_4}{8I_1^3},
\]

where the first equation is the \( g = 0 \) polymer equation. With the aid of (5.7) these
expressions are integrated, yielding

\[ F_0 = \sum_{n=0}^{\infty} t_n \frac{f_0^{n+1}}{(n+1)!}, \]
\[ F_1 = -\frac{1}{2} \log I_1, \] (5.10)
\[ F_2 = -\frac{5 I_2^2}{24 I_1^3} + \frac{I_3}{8 I_1^2}. \]

The results suggest that \(1/I_1\) can be interpreted as a propagator and \(I_k\) as a \((k+1)\)-point vertex. Thus our "genus" expansion is nothing but the loop expansion and \(\lambda\) plays a role of the Planck constant. In fact the explicit form of \(Z\) is obtained in [25] which admits the loop expansion in agreement with ours. We shall now turn to this issue.

It is seen from (5.3) that the polymer equation is written as [25]

\[ t_0 + \sum_{n=1}^{\infty} \frac{t_n}{n!} \left( \lambda \frac{\partial}{\partial t_0} + f \right)^n \cdot 1 = 0. \] (5.11)

Substituting \(f = \lambda(\log Z)'\) this turns out to be the linear equation

\[ \left[ t_0 + \sum_{n=1}^{\infty} \frac{t_n}{n!} \left( \lambda \frac{\partial}{\partial t_0} \right)^n \right] Z = 0, \] (5.12)

which can be easily integrated. We obtain [25]

\[ Z = \int dz \exp \left( \frac{1}{\lambda} \sum_{n=0}^{\infty} t_n \frac{z^{n+1}}{(n+1)!} \right) = \int dz \exp \frac{1}{\lambda} S_{\text{eff}}(z). \] (5.13)

(5.12) is then recognized as the Schwinger-Dyson equation for this integral. It is amusing to note here the similarity of the polymer action \(S_{\text{eff}}(z)\) to the gravity action \(S_{A_1}\) in (2.41). Furthermore the saddle point equation for \(Z\) becomes the
\( g = 0 \) polymer equation in (5.9). Then the perturbation expansion around the classical solution, \( \xi = z - z_{cl} \) with \( z_{cl} = f_0 \), yields

\[
Z = e^{\frac{1}{\lambda}S_{\text{eff}}(z_{cl})} \int d\xi \exp \left[ \frac{1}{\lambda} \left( \frac{1}{2} I_1 \xi^2 + \frac{1}{6} I_2 \xi^3 + \frac{1}{24} I_3 \xi^4 + \cdots \right) \right].
\] (5.14)

Note that \( S_{\text{eff}}(z_{cl}) = F_0 \) in (5.10). Using the Gaussian integration formula

\[
\int d\xi \exp \left( \frac{1}{2\lambda} I_1 \xi^2 \right) \xi^{2n} = \left( \frac{-2\pi\lambda}{I_1} \right)^{1/2} (2n-1)! \left( \frac{-\lambda}{I_1} \right)^n,
\] (5.15)

one obtains

\[
Z = \left( \frac{-2\pi\lambda}{I_1} \right)^{1/2} e^{\frac{1}{\lambda}S_{\text{eff}}(z_{cl})} \left[ 1 + \lambda \left( \frac{-5 I_2^2}{24 I_1^3} + \frac{I_3}{8 I_1^2} \right) \right.
\]
\[
+ \lambda^2 \left( \frac{385 I_2^4}{1152 I_1^6} - \frac{35 I_2^2 I_3}{64 I_1^5} + \frac{35 I_3^2}{384 I_1^4} + \frac{7 I_2 I_4}{48 I_1^4} - \frac{I_5}{48 I_1^3} \right) \right.
\]
\[
+ \lambda^3 \left( \frac{-85085 I_2^6}{82944 I_1^9} + \frac{25025 I_2^4 I_3}{9216 I_1^8} - \frac{5005 I_2^2 I_3^2}{3072 I_1^7} + \frac{385 I_3^3}{3072 I_1^6} \right.
\]
\[
- \frac{1001 I_2^3 I_4}{1152 I_1^7} + \frac{77 I_2 I_3 I_4}{128 I_1^6} - \frac{21 I_4^2}{640 I_1^5} + \frac{77 I_2^2 I_5}{384 I_1^6} \right.
\]
\[
- \frac{7 I_3 I_5}{128 I_1^5} - \frac{I_2 I_6}{32 I_1^5} + \frac{I_7}{384 I_1^4} \right) + \cdots.
\] (5.16)

Finally, up to an additive constant \( \left( \frac{1}{2} \log(-2\pi\lambda) \right) \), the free energy is given by

\[
F = \frac{1}{\lambda} S_{\text{eff}}(z_{cl}) - \frac{1}{2} \log I_1 + \lambda \left( \frac{-5 I_2^2}{24 I_1^3} + \frac{I_3}{8 I_1^2} \right) \right.
\]
\[
+ \lambda^2 \left( \frac{5 I_2^4}{16 I_1^6} - \frac{25 I_2^2 I_3}{48 I_1^5} + \frac{I_3^2}{12 I_1^4} + \frac{7 I_2 I_4}{48 I_1^4} - \frac{I_5}{48 I_1^3} \right) \right.
\]
\[
+ \lambda^3 \left( \frac{-1105 I_2^6}{1152 I_1^9} + \frac{985 I_2^4 I_3}{384 I_1^8} - \frac{445 I_2^2 I_3^2}{288 I_1^7} + \frac{11 I_3^3}{96 I_1^6} - \frac{161 I_2^3 I_4}{192 I_1^7} \right.
\]
\[
+ \frac{7 I_2 I_3 I_4}{12 I_1^6} - \frac{21 I_4^2}{640 I_1^5} + \frac{113 I_2^2 I_5}{576 I_1^6} - \frac{5 I_3 I_5}{96 I_1^5} - \frac{I_2 I_6}{32 I_1^5} + \frac{I_7}{384 I_1^4} \right) + \cdots.
\] (5.17)

The result indeed agrees with (5.10).
Now, a natural question arises: is it possible to find an analogous integral representation for the partition function in topological string theories? If so, what is the physical meaning of such a representation? In branched polymers $S_{\text{eff}}(z)$ is the Landau-Ginzburg action containing precisely the effect of the fluctuations around the mean field. Thus what could be imagined in the case of gravity will be the Landau-Ginzburg type effective theory for gravity. For pure gravity Yoneya has proposed an interesting formula for the partition function [17]. To write down this formula explicitly one has to solve the highest weight condition of the Virasoro algebra, which unfortunately is a difficult task. At present we do not know how to deal with this problem, though we expect that deeper understanding of this issue may shed a new light toward the construction of topological string field theory.

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APPENDIX A

A_1 potentials (KdV equations)

Some constitutive relations for A_1 are

\[ \langle PP \rangle = R_1 = u, \]
\[ \langle P\sigma_1 \rangle = R_2 = \frac{u^2}{2} + \frac{\lambda^2 u''}{12}, \]
\[ \langle P\sigma_2 \rangle = R_3 = \frac{u^3}{6} + \frac{\lambda^2 u^2}{24} + \frac{uu''}{12} + \frac{\lambda^4 u^{(4)}}{4} \]
\[ \langle P\sigma_3 \rangle = R_4 = \frac{u^4}{24} + \frac{\lambda^2 uu'^2}{24} + \frac{u^2 u''}{24} + \lambda^4 \frac{u'^2}{120} + \frac{uu^{(3)}}{120} + \frac{uu^{(4)}}{240} + \frac{\lambda^6 u^{(6)}}{1440}, \]
\[ \langle P\sigma_4 \rangle = R_5 = \frac{u^5}{120} + \lambda^4 \frac{11 uu'^2}{1440} + \frac{uu'^{2}}{160} + \frac{uu'^{(3)}}{240} + \frac{u^2 u^{(4)}}{480}, \]
\[ \langle \sigma_1 \sigma_1 \rangle = \frac{u^3}{3} + \frac{\lambda^2 uu'^2}{24} + \frac{uu''}{6} + \frac{\lambda^4 u^{(4)}}{144}, \]
\[ \langle \sigma_1 \sigma_2 \rangle = \frac{u^4}{8} + \lambda^2 \frac{uu'^2}{12} + \lambda^4 \frac{23 uu'^2}{1440} + \frac{uu'^{(3)}}{60} + \frac{uu^{(4)}}{90}, \]
\[ \langle \sigma_2 \sigma_2 \rangle = \frac{u^5}{20} + \lambda^2 \frac{u^2 u'^2}{12} + \frac{u^3 u''}{12} + \lambda^4 \frac{7 uu'^2}{240} + \frac{23 uu'^2}{720} + \frac{uu'u^{(3)}}{30} + \frac{uu^{(4)}}{90}, \]
\[ + \lambda^6 \frac{13 uu^{(4)}}{5760} + \frac{uu^{(4)}}{240} + \frac{uu^{(5)}}{576} + \frac{uu^{(6)}}{1440} + \frac{\lambda^8 u^{(8)}}{57600}. \]

(A.1)

In general, R_n is determined by the Gelfand-Dikii recursion relation [28]

\[(2n + 1)D R_{n+1} = \left[ \frac{\lambda^2}{4} D^3 + 2uD + u' \right] R_n, \] (A.2)
and we have the genus expansion of $R_n$

$$R_n = \sum_{g=0}^{\infty} \lambda^{2g} R_{n,g}, \quad R_{n,g} = \sum_{k=g+1}^{3g} \frac{u^{n-k}}{(n-k)!} P_{n,g}(u', \ldots, u^{(2g)}), \quad (A.3)$$

where $P_{n,g}$ is a polynomial in $u', u'', \ldots, u^{(2g)}$.

Explicit formulas up to $g = 5$ are given as follows:

$$R_{n,0} = \frac{u^n}{n!}, \quad (A.4)$$

$$R_{n,1} = \frac{1}{12} \frac{u^{n-2}}{(n-2)!} u'' + \frac{1}{24} \frac{u^{n-3}}{(n-3)!} u'^2, \quad (A.5)$$

$$R_{n,2} = \frac{u^{n-6}}{(n-6)!} \frac{u^4}{1152} + \frac{u^{n-5}}{(n-5)!} \frac{11u'^2 u''}{1440}$$
$$+ \frac{u^{n-4}}{(n-4)!} \left[ \frac{u'^2}{160} + \frac{u' u^{(3)}}{120} \right] + \frac{u^{n-3}}{(n-3)!} \frac{u^{(4)}}{240}, \quad (A.6)$$

$$R_{n,3} = \frac{u^{n-9}}{(n-9)!} \frac{u^6}{82944} + \frac{u^{n-8}}{(n-8)!} \frac{17u'^4 u''}{69120} + \frac{u^{n-7}}{(n-7)!} \frac{83u'^2 u^{(2)}}{80640} + \frac{u'^3 u^{(3)}}{2016}$$
$$+ \frac{u^{n-6}}{(n-6)!} \frac{61u'^3}{120960} + \frac{43u' u^{(3)} u''}{20160} + \frac{5u'^2 u^{(4)}}{8064} \bigg] + \frac{u^{n-5}}{(n-5)!} \left[ \frac{23u^{(3)}^2}{40320} + \frac{19u'' u^{(4)}}{20160} + \frac{u' u^{(5)}}{2240} \bigg] + \frac{u^{n-4}}{(n-4)!} \frac{u^{(6)}}{6720}. \quad (A.7)$$
\[
R_{n,A} = \frac{u^{n-12}}{(n-12)!} \frac{u^8}{7962624} + \frac{u^{n-11}}{(n-11)!} \frac{23u^6u''}{4976640} + \frac{u^{n-10}}{(n-10)!} \left[ \frac{893u'^4u''^2}{19353600} + \frac{13u^5u^{(3)}}{967680} \right] \\
+ \frac{u^{n-9}}{(n-9)!} \left[ \frac{259u'^2u''^2}{2073600} + \frac{439u'^3u''(3)}{2419200} + \frac{17u'^4u^{(4)}}{645120} \right] \\
+ \frac{u^{n-8}}{(n-8)!} \left[ \frac{1261u'^4}{29030400} + \frac{227u'u''^2u^{(3)}}{604800} + \frac{659u'^2u^{(3)}(3)^2}{4838400} + \frac{527u'^2u''u^{(4)}}{2419200} + \frac{17u'^3u^{(5)}}{483840} \right] \\
+ \frac{u^{n-7}}{(n-7)!} \left[ \frac{31u'^6u^{(3)^2}}{161280} + \frac{5u'^2u^{(4)}}{32256} + \frac{u'u^{(3)}u^{(4)}}{4480} + \frac{u'u''u^{(5)}}{6720} + \frac{u'^2u^{(6)}}{32256} \right] \\
+ \frac{u^{n-6}}{(n-6)!} \left[ \frac{23u^{(4)^2}}{483840} + \frac{19u^{(3)}u^{(5)}}{241920} + \frac{11u''u^{(6)}}{241920} + \frac{u'^7}{60480} \right] \\
+ \frac{u^{n-5}}{(n-5)!} \frac{u^{(8)}}{241920}. 
\]

(A.8)
\[ R_{n,5} = \frac{u^{n-15}}{(n-15)!} \frac{u^{10}}{955514880} + \frac{u^{n-14}}{(n-14)!} \frac{29u^8 u''}{477757440} + \frac{u^{n-13}}{(n-13)!} \frac{19u^6 u''^2}{17203200} + \frac{u^7 u^{(3)}}{4354560} \\
+ \frac{u^{n-12}}{(n-12)!} \left\{ \frac{187u^4 u''^3}{25804800} + \frac{7u^5 u'' u^{(3)}}{1105920} + \frac{17u^6 u^{(4)}}{27869184} \right\} \\
+ \frac{u^{n-11}}{(n-11)!} \left\{ \frac{4097u^2 u^r u^r}{283852800} + \frac{4471u^3 u'' u^{(3)}}{10644800} + \frac{1303u^4 u^{(3)}}{170311680} + \frac{23u^4 u'' u^{(4)}}{1892352} + \frac{299u^5 u^{(5)}}{255467520} \right\} \\
+ \frac{u^{n-10}}{(n-10)!} \left\{ \frac{79u^n u^{(5)}}{120275200} + \frac{2579u'u'' u^{(3)}}{45619200} + \frac{1877u^2 u'' u^{(3)}}{30412800} + \frac{1493u^2 u'' u^{(4)}}{30412800} \right\} \\
+ \frac{127u^3 u^{(3)} u^{(4)}}{5322240} + \frac{839u^3 u'' u^{(5)}}{5322240} + \frac{139u^4 u^{(6)}}{8515840} \\
+ \frac{u^{n-9}}{(n-9)!} \left\{ \frac{3851u^2 u^r u^{(3)}}{91238400} + \frac{467u^3 u^{(3)}}{22809600} + \frac{7171u^4 u^{(4)}}{319334400} + \frac{15629u'u'' u^{(3)} u^{(4)}}{159667200} \right\} \\
+ \frac{1817u^2 u^{(4)}}{127733760} + \frac{383u^2 u^r u^{(5)}}{11827200} + \frac{7543u^2 u^{(3)} u^{(5)}}{319334400} + \frac{4283u^2 u'' u^{(6)}}{319334400} + \frac{13u^3 u^{(7)}}{7983360} \\
+ \frac{u^{n-8}}{(n-8)!} \left\{ \frac{7939u^{(3)} u^{(4)}}{319334400} + \frac{6353u'' u^{(4)}}{319334400} + \frac{13u'' u^{(3)} u^{(5)}}{394240} + \frac{3067u'u^{(4)} u^{(5)}}{159667200} \right\} \\
+ \frac{3001u'' u^{(6)}}{319334400} + \frac{13u'u^{(3)} u^{(6)}}{950400} + \frac{109u'u'' u^{(7)}}{159667200} + \frac{71u^2 u^{(8)}}{63866880} \\
+ \frac{u^{n-7}}{(n-7)!} \left\{ \frac{71u^{(5)} u^{(5)}}{21288960} + \frac{61u^{(4)} u^{(6)}}{10644480} + \frac{19u^{(3)} u^{(7)}}{5322240} + \frac{17u'' u^{(8)}}{10644480} + \frac{u'u^{(9)}}{2128896} \right\} \\
+ \frac{u^{n-6} u^{(10)}}{(n-6)! 10644480} \\
\]
APPENDIX B

\( A_2 \) potentials (Boussinesq equation)

The Boussinesq potentials \((R_n, S_n)\) are obtained by the following recursion relation:

\[
(n + 3)D R_{n+3} = (3vD + 2v')R_n + \left( \frac{2\lambda^2}{3} D^3 + 2uD + u' \right) S_n,
\]

\[
(n + 3)D S_{n+3} = \left[ \frac{\lambda^4}{18} D^5 + \frac{5\lambda^2}{6} uD^3 + \frac{5\lambda^2}{4} u'D^2 + \left( \frac{3\lambda^2}{4} u'' + 2u^2 \right) D + \left( \frac{\lambda^2}{6} u''' + 2uu' \right) \right] R_n + (3vD + v') S_n,
\]  

where \( \cdot' = \partial/\partial t_{0,0} = D \).

Initial data are

\[
\langle PP \rangle = R_1 = u, \quad \langle PQ \rangle = S_1 = v, \quad \langle QP \rangle = R_2 = v,
\]

\[
\langle QQ \rangle = S_2 = \frac{u^2}{2} + \frac{\lambda^2 u''}{12}.
\]

Hence we have

\[
\langle \sigma_1 (P) P \rangle = R_4 = uv + \frac{\lambda^2 v''}{6},
\]

\[
\langle \sigma_1 (P) Q \rangle = S_4 = \frac{u^2}{3} + \frac{v^2}{2} + \frac{\lambda^2}{8} \left[ \frac{u'^2}{8} + \frac{uu''}{4} \right] + \frac{\lambda^4 u^{(4)}}{72},
\]

\[
\langle \sigma_1 (Q) P \rangle = R_5 = \frac{v^3}{6} + \frac{v^2}{2} + \frac{\lambda^2}{8} \left[ \frac{u'^2}{8} + \frac{uu''}{6} \right] + \frac{\lambda^4 u^{(4)}}{90},
\]

\[
\langle \sigma_1 (Q) Q \rangle = S_5 = \frac{u^2 v}{2} + \frac{\lambda^2}{12} \left[ \frac{u' u'}{12} + \frac{uu'}{12} + \frac{uu''}{6} \right] + \frac{\lambda^4 v^{(4)}}{90},
\]

\[
\langle \sigma_2 (P) P \rangle = R_7 = \frac{u^4}{12} + \frac{u'^2}{2} + \frac{\lambda^2}{24} \left[ \frac{5uu'^2}{24} + \frac{v'^2}{12} + \frac{u''^2}{6} + \frac{v''}{6} \right]
\]

\[
+ \frac{\lambda^4}{72} \left[ \frac{7u'^2}{144} + \frac{5uu''}{72} + \frac{uu''}{36} \right] + \frac{\lambda^6 u^{(6)}}{756},
\]

\[35\]
\[ \langle \sigma_2(P)Q \rangle = S_7 = \frac{v^3}{3} + \frac{v^3}{6} + \lambda^2 \left[ \frac{v v u'^2}{8} + \frac{u u' v'}{6} + \frac{6 v v u''}{4} + \frac{6 v^2 u''}{6} \right] \\
+ \lambda^4 \left[ \frac{u u'' v''}{18} + \frac{v u'^{(3)}}{36} + \frac{u' v'^{(3)}}{24} + \frac{v u^{(4)}}{72} + \frac{4 u v^{(4)}}{36} \right] + \lambda^6 \frac{v(6)}{756}. \]  

\[ \langle \sigma_2(Q)P \rangle = R_8 = \frac{w^3 v}{6} + \frac{w^3}{6} + \lambda^2 \left[ \frac{w v u'^2}{8} + \frac{6 w u' v'}{6} + \frac{6 w v u''}{4} + \frac{12 w^2 v''}{12} \right] \\
+ \lambda^4 \left[ \frac{13 w u'' v''}{360} + \frac{7 w u'^{(3)}}{360} + \frac{w u'^{(3)}}{30} + \frac{w u^{(4)}}{90} + \frac{6 w v^{(4)}}{60} \right] + \lambda^6 \frac{v(6)}{1080}, \]  

and

\[ \langle \sigma_2(Q)Q \rangle = S_8 = \frac{w^5}{30} + \frac{w^2 v^2}{4} + \lambda^2 \left[ \frac{7 w^2 u'^2}{48} + \frac{6 w v'^{2}}{12} + \frac{7 w^3 u''}{72} + \frac{6 w^2 u''}{24} + \frac{uv v''}{6} \right] \\
+ \lambda^4 \left[ \frac{11 w^2 u''}{160} + \frac{17 w u'^{2}}{240} + \frac{v'^{2}}{45} + \frac{17 w u'^{(3)}}{180} + \frac{w v'^{(3)}}{45} + \frac{17 w^2 u^{(4)}}{720} + \frac{w v^{(4)}}{90} \right] \\
+ \lambda^6 \left[ \frac{w^2 u^{(3)}}{720} + \frac{67 w u'' v''}{4320} + \frac{w u'^{(5)}}{144} + \frac{w u^{(6)}}{432} \right] + \lambda^8 \frac{u(8)}{12960}. \]  

Next, some two-point functions of descendants read as follows.

\[ \langle \sigma_1(P)\sigma_1(P) \rangle = \frac{w^4}{4} + uv^2 + \lambda^2 \left[ \frac{u u'^2}{3} + \frac{v'^2}{12} + \frac{6 u''}{12} + \frac{13 u''^2}{144} + \frac{6 v v''}{3} \right] \]  

\[ + \lambda^4 \left[ \frac{13 u''^2}{144} + \frac{u' u'^{(3)}}{9} + \frac{u u^{(4)}}{18} + \frac{4 u^{(6)}}{432} \right], \]  

\[ \langle \sigma_1(Q)\sigma_1(P) \rangle = \frac{w^3 v}{2} + \frac{w^3}{3} + \lambda^2 \left[ \frac{w v u'^2}{4} + \frac{w u' v'}{4} + \frac{6 w v u''}{12} + \frac{6 w^2 v''}{4} \right] \\
+ \lambda^4 \left[ \frac{29 w u'' v''}{360} + \frac{11 w v'^{(3)}}{360} + \frac{17 w v'^{(3)}}{120} + \frac{w u^{(4)}}{40} + \frac{6 w v^{(4)}}{180} \right] + \lambda^6 \frac{v(6)}{540}, \]  

(B.12)
\[ \langle \sigma_2(P)\sigma_1(P) \rangle = \frac{u^4 v}{3} + \frac{w^3}{2} \]
\[ + \lambda^2 \left[ \frac{13 uvu''^2}{24} + \frac{5 u^2 u' v'}{12} + \frac{v v'^2}{6} + \frac{7 u^2 v v''}{12} + \frac{2 u^3 v''}{9} + \frac{v^2 v''}{4} \right] \]
\[ + \lambda^4 \left[ \frac{17 u' v' u''}{72} + \frac{5uvu''^2}{36} + \frac{7 u'^2 v''}{48} + \frac{19 uv u'' v''}{72} + \frac{13 v u' u''}{72} \right] \]
\[ + \lambda^6 \left[ \frac{29 u''(3) u(3)}{1008} + \frac{67 v''(4) u(4)}{3024} + \frac{11 u''(4) v(4)}{378} + \frac{v' u(5)}{126} + \frac{5 u' v(5)}{336} \right] \]
\[ + \lambda^8 \frac{11 v(6)}{3024} + \frac{w(6)}{168} + \lambda^8 \frac{v(8)}{4536}. \]

(B.13)
There are four primaries $P, Q, R$ and $S$ in the $D_4$ model. The order parameters are defined by

\[ u = \langle PP \rangle, \quad v = \langle PQ \rangle, \quad w = \langle PR \rangle, \quad f = \langle PS \rangle. \]

We determine the constitutive relations by requiring the commutativity of flow equations. Starting with the dispersionless ($\lambda = 0$) terms we find
\[ \langle PQ \rangle = v, \]
\[ \langle QQ \rangle = \frac{u^3}{3} + uv + w + \lambda^2 \left[ \frac{uu''}{6} + \frac{v''}{6} \right] + \frac{\lambda^4 u^{(4)}}{180}, \]
\[ \langle RQ \rangle = \frac{f^2}{2} + u^2 v + \frac{v^2}{2} + \lambda^2 \left[ \frac{u'u'}{2} + \frac{vv''}{3} + \frac{uv''}{2} \right] + \frac{\lambda^4 v^{(4)}}{20}, \]
\[ \langle SQ \rangle = -fu - \frac{\lambda^2 f''}{6}, \]
\[ \langle PR \rangle = w, \]
\[ \langle QR \rangle = \frac{f^2}{2} + u^2 v + \frac{v^2}{2} + \lambda^2 \left[ \frac{u'u'}{2} + \frac{vv''}{3} + \frac{uv''}{2} \right] + \frac{\lambda^4 v^{(4)}}{20}, \]
\[ \langle RR \rangle = -f^2 u + \frac{u^5}{5} + uv^2 + \lambda^2 \left[ -\frac{f'^2}{12} + u^2 u'^2 + \frac{v'^2}{12} + \frac{u'u'}{6} - \frac{f f''}{3} + \frac{2u^3 u''}{3} + \frac{vv''}{3} + \frac{uw''}{3} \right] \quad (C.2) \]
\[ + \lambda^4 \left[ \frac{11u^2 u''}{18} + \frac{5uu^2}{12} + \frac{5uu'u'(3)}{9} + \frac{5u^2 u^{(4)}}{36} + \frac{2w^{(4)}}{45} \right] \]
\[ + \lambda^6 \left[ \frac{7u'(3)^2}{216} + \frac{13u''u^{(4)}}{216} + \frac{u'u^{(5)}}{30} + \frac{uu^{(6)}}{90} \right] + \frac{\lambda^8 u^{(8)}}{3600}, \]
\[ \langle SR \rangle = fu^2 - fv + \lambda^2 \left[ \frac{f'u'}{2} + \frac{uf''}{2} + \frac{fu''}{3} \right] + \frac{\lambda^4 f^{(4)}}{20}, \]
\[ \langle PS \rangle = f, \]
\[ \langle QS \rangle = -fu - \frac{\lambda^2 f''}{6}, \]
\[ \langle RS \rangle = fu^2 - fv + \lambda^2 \left[ \frac{f'u'}{2} + \frac{uf''}{2} + \frac{fu''}{3} \right] + \frac{\lambda^4 f^{(4)}}{20}, \]
\[ \langle SS \rangle = -\frac{u^3}{3} + uv - w + \lambda^2 \left[ -\frac{wu''}{6} + \frac{v''}{6} \right] - \frac{\lambda^4 u^{(4)}}{180}. \]
\[ \langle P_1(P) \rangle = -\frac{f^2}{2} + \frac{v^2}{2} + uw + \lambda^2 \left[ \frac{uu^2}{2} + \frac{u^2 u''}{6} + \frac{w''}{3} \right] \\
+ \lambda^4 \left[ \frac{u''^2}{36} + \frac{u'u'(3)}{12} + \frac{uu(4)}{30} \right] + \frac{\lambda^6 u(6)}{840}, \]

\[ \langle Q_1(P) \rangle = f^2 u + u^3 v + u^2 v + vw \\
+ \lambda^2 \left[ \frac{f^2}{4} + \frac{vu^2}{2} + 2uu'v' + \frac{v^2}{4} + \frac{ff''}{2} + \frac{7uvu''}{6} + \frac{u^2 v'' + vv''}{2} \right] \\
+ \lambda^4 \left[ \frac{2u''v''}{3} + \frac{5v'u'(3)}{12} + \frac{u'(3)}{2} + \frac{7uv(4)}{60} + \frac{uv(4)}{4} \right] + \frac{\lambda^6 v(6)}{56}, \]

\[ \langle R_1(P) \rangle = -\frac{3f^2u^2}{2} + \frac{u^6}{6} + f^2v + \frac{3u^2v^2}{2} + \frac{v^3}{3} + \frac{w^2}{2} \\
+ \lambda^2 \left[ -\frac{uf^2}{2} + \frac{3ff'u'}{2} + 2uu'v' + \frac{v^2}{2} + \frac{uw^2}{2} + \frac{uu'w'}{2} - \frac{3ff''}{2} \right] \\
- \frac{2f^2u''}{3} + u^4u'' + \frac{2v^2u''}{3} + \frac{3uvu''}{2} + \frac{u^2w''}{2} \\
+ \lambda^4 \left[ \frac{7u^4}{12} - \frac{23f'^2u''}{120} + 5uu'^2u'' + \frac{131u^2u''^2}{72} + \frac{23u'^2}{120} + \frac{49u''u''}{180} \right] \\
- \frac{41f'^3}{180} + \frac{29u^2u'u'(3)}{12} + \frac{4w'u'(3)}{45} + \frac{41v'u'(3)}{180} + \frac{17u'u'(3)}{60} - \frac{29ff'(4)}{180} \\
+ \frac{13u^3u(4)}{36} + \frac{29vv(4)}{180} + \frac{17uu(4)}{90} \right] \\
+ \lambda^6 \left[ \frac{1799u^6}{3240} + \frac{2491u'u''u(3)}{1080} + \frac{529uu(3)}{2} \\[45] + \frac{29u'^2u(4)}{45} + \frac{877uu'u(4)}{1080} + \frac{53uu'u(5)}{135} + \frac{31u^2u(6)}{540} + \frac{121w(6)}{7560} \right] \\
+ \lambda^8 \left[ \frac{301u(4)^2}{7200} + \frac{2351u(3)u(5)}{32400} + \frac{1499uu(6)}{32400} \right] \\
+ \frac{197u'u(7)}{10800} + \frac{43uu(8)}{10800} + \lambda^{10} u(10) \right], \]

\[ \langle S_1(P) \rangle = fu^3 - 2fuv + fw \\
+ \lambda^2 \left[ 2uf'u' + \frac{f'u^2}{2} - \frac{f'v'}{2} + u^2 f'' - \frac{vf''}{2} + \frac{7fuu''}{6} - \frac{fv''}{2} \right] \\
+ \lambda^4 \left[ \frac{2f''u''}{3} + \frac{uf(3)}{2} + \frac{5f'u(3)}{12} + \frac{uf(4)}{4} + \frac{7fu(4)}{60} \right] + \frac{\lambda^6 f(6)}{56}. \]

(C.3)
APPENDIX D

A$_1$ free energy (1)

\[ F_1 = \frac{1}{24} \log u', \]
\[ F_2 = \frac{u'^3}{360u^4} - \frac{7u''u^{(3)}}{192u^5} + \frac{u^{(4)}}{1152u^6}, \]
\[ F_3 = -\frac{5u'^6}{648u^8} + \frac{59u'u^{(4)} - 83u'^2u^{(3)} + 59u^{(3)}}{3024u^{10}} - \frac{1273u''u^{(3)}u^{(4)} - 103u^{(4)} + 353u'^2u^{(5)}}{7168u^{10}} + \frac{2211840}{648} \]
\[ - \frac{15120u'^6}{161280u^{14}} + \frac{322560u^6}{7u''u^{(6)}} - \frac{483840u'^4}{322560u^8} + \frac{u^{(7)}}{2211840} \]
\[ F_4 = \frac{463u''^9}{4860u^{12}} - \frac{193u'u^{(3)}u^{(4)} + 14903u'^5u^{(3)}u^{(4)}}{34560u^{10}} - \frac{305129u'u^3u^{(3)}u^{(3)}}{1658880u^9} + \frac{22809u''u^{(3)}u^{(4)}}{1146880u^8} + \frac{619u''u^{(3)}u^{(4)}}{6075u^{10}} + \frac{10153u''u^{(3)}u^{(4)}}{518400u^9} + \frac{13138507u''u^2u^{(3)}u^{(4)}}{154828800u^8} + \frac{2211840u'^6}{7u''u^{(9)}} \]
\[ + \frac{46080u'^6}{90720u^{10}} + \frac{4423680u'^4}{15482880u^{12}} + \frac{2243u'^5u^{(5)}}{103680u^9} + \frac{415273u'^3u^{(3)}u^{(5)}}{1327104u^8} - \frac{12035u'u^{(3)}u^{(5)}}{1548288u^{10}} - \frac{171343u'^2u^{(4)}u^{(5)}}{3096576u^9} + \frac{949u(u^{(4)}u^{(5)}}{884736u^8} + \frac{9221u''u^{(5)}u^{(5)}}{3096576u^{10}} + \frac{12907u''u^{(5)}u^{(6)}}{3628800u^8} + \frac{60941u'^2u^{(3)}u^{(6)}}{17203200u^7} + \frac{59u'u^{(4)}u^{(6)}}{1720320u^6} + \frac{15179u'u^{(4)}u^{(6)}}{3096576u^8} - \frac{197u(u^{(6)}}{6193152u^5} - \frac{212267u'^3u^{(7)}}{464486400u^6} + \frac{20639u''u^{(3)}u^{(7)}}{77414400u^6} - \frac{2069u^{(4)}u^{(7)}}{92897280u^5} + \frac{2323u'^2u^{(8)}}{51609600u^6} - \frac{163u^{(3)}u^{(8)}}{15482880u^5} \]
\[ - \frac{7u''u^{(9)}}{u^{(10)}}, \]

where

\[ u^{(n)} = \frac{\partial^n}{\partial t^n} u, \quad u = \sum_{n=0}^{\infty} t_n \frac{u^n}{n!}. \]
\[ A_1 \text{ free energy (2)} \]

\[
F_1 = -\frac{1}{24} \log M
\]

\[
F_2 = \frac{7G[2]^3}{1440M^8} + \frac{29G[2]^2G[3]}{5760M^4} + \frac{G[4]}{1152M^3},
\]

\[
F_3 = \frac{245G[2]^6}{20736M^{10}} + \frac{193G[2]^4G[3]}{6912M^9} + \frac{205G[2]^2G[3]^2}{13824M^8} + \frac{583G[3]^3}{580608M^7}
\]

\[
+ \frac{53G[2]^3G[4]}{6912M^8} + \frac{1121G[2]G[3]G[4]}{241920M^7} + \frac{607G[4]^2}{2903040M^6} + \frac{17G[2]^2G[5]}{11520M^4}
\]

\[
+ \frac{503G[3]G[5]}{1451520M^6} + \frac{77G[2]G[6]}{414720M^5} + \frac{G[7]}{82944M^4},
\]

\[
F_4 = \frac{259553G[2]^9}{2488320M^{15}} + \frac{475181G[2]^7G[3]}{1244160M^{14}} + \frac{145693G[2]^5G[3]^2}{331776M^{13}} + \frac{43201G[2]^3G[3]^3}{248832M^{12}}
\]

\[
+ \frac{134233G[2]G[3]^4}{7962624M^{11}} + \frac{14147G[2]^6G[4]}{124416M^{13}} + \frac{83851G[2]^4G[3]G[4]}{414720M^{12}} + \frac{26017G[2]^2G[3]^2G[4]}{331776M^{11}}
\]

\[
+ \frac{185251G[3]^3G[4]}{4976640M^{11}} + \frac{5609G[2]^3G[4]^2}{276480M^{11}} + \frac{177G[2]G[3]G[4]^2}{20480M^{10}} + \frac{175G[4]^3}{995328M^9}
\]

\[
+ \frac{21329G[2]^5G[5]}{829440M^{12}} + \frac{13783G[2]^3G[3]G[5]}{414720M^{11}} + \frac{1837G[2]G[3]^2G[5]}{259200M^{10}} + \frac{7597G[2]^2G[4]G[5]}{1382400M^{10}}
\]

\[
+ \frac{719G[3]G[4]G[5]}{829440M^9} + \frac{533G[2]G[5]^2}{1935360M^9} + \frac{2471G[2]^4G[6]}{552960M^{11}} + \frac{7897G[2]^2G[3]G[6]}{2073600M^{10}}
\]

\[
+ \frac{1997G[3]^2G[6]}{6635520M^{9}} + \frac{1081G[2]^2G[4]G[6]}{2322432M^9} + \frac{487G[5]G[6]}{18579456M^8} + \frac{4907G[2]^3G[7]}{8294400M^{10}}
\]

\[
+ \frac{16243G[2]G[3]G[7]}{58060800M^9} + \frac{1781G[4]G[7]}{92897280M^8} + \frac{53G[2]^2G[8]}{921600M^9} + \frac{947G[3]G[8]}{92897280M^8}
\]

\[
+ \frac{149G[2]G[9]}{39813120M^8} + \frac{G[10]}{7962624M^7},
\]

where

\[
M = 1 - G[1],
\]

\[
G[k] = \sum_{n=0}^{\infty} t^{n+k} u(t)^n \frac{1}{n!}.
\]

The results are in agreement with those obtained previously in \[16,22\].
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