Read-Write Memory and $k$-Set Consensus as an Affine Task

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Abstract

The wait-free read-write memory model has been characterized as an iterated Immediate Snapshot (IS) task. The IS task is affine—it can be defined as a (sub)set of simplices of the standard chromatic subdivision. It is known that the task of Weak Symmetry Breaking (WSB) cannot be represented as an affine task. In this paper, we highlight the phenomenon of a “natural” model that can be captured by an iterated affine task and, thus, by a subset of runs of the iterated immediate snapshot model. We show that the read-write memory model in which, additionally, $k$-set-consensus objects can be used is, unlike WSB, “natural” by presenting the corresponding simple affine task captured by a subset of 2-round IS runs. Our results imply the first combinatorial characterization of models equipped with abstractions other than read-write memory that applies to generic tasks.

1 Introduction

A principal challenge in distributed computing is to devise protocols that operate correctly in the presence of failures, given that system components (processes) are asynchronous.

The most extensively studied wait-free model of computation [20] makes no assumptions about the number of failures that can occur, no process should wait for other processes to move for making progress. In particular, in a wait-free solution of a distributed task, a process participating in the computation should be able to produce an output regardless of the behavior of other processes.

Topology of wait-freedom. Wait-free task solvability in the read-write shared-memory model has been characterized in a very elegant way through the existence of a specific continuous map from geometrical structures describing inputs and outputs of the task [23, 21]. A task $T$ is wait-free solvable using reads and writes if and only if there exists a simplicial, chromatic map from a subdivision of the input simplicial complex to the output simplicial complex, satisfying the specification of $T$. Thus, using the iterated standard chromatic subdivision [21] (one such iteration of the standard simplex $s$, denoted by $\text{Chr}s$, is depicted in Figure 1), we obtain a combinatorial representation of the wait-free model. Iterations of this subdivision capture precisely rounds of the iterated immediate snapshot (IIS) model [5, 23].

This characterization can be interpreted as follows: the persistent wait-free read-write model can be captured, regarding task solvability, by an iterated (one-shot) Immediate Snapshot task. Immediate Snapshot is, in turn, captured by the chromatic simplex agreement task [5, 23] on $\text{Chr}s$. 

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Beyond wait-freedom: $k$-concurrency and $k$-set consensus. Unfortunately, very few tasks are solvable *wait-free* in the read-write shared-memory model [3, 23, 29], so a lot of efforts have been applied to characterizing task solvability in various *restrictions* of the *wait-free* model.

A straightforward way to define such a restriction is to bound the *concurrency level* of runs [13]: in the model of $k$-concurrency, at most $k$ processes can be concurrently *active*, i.e., after the invocation of a task and before terminating it. This is a powerful abstraction known, when it comes to solving tasks, to be equivalent to the *wait-free* model in which processes, in addition to read-write shared memory, can access $k$-set consensus objects [12]. Also, the $k$-concurrent task solvability proved to be a good way to measure the power of shared-memory models equipped with failure detectors [8].

Can we represent the model of read-write memory and $k$-set consensus as an iterated task?

**Iterated tasks for $k$-set consensus.** We show that the model of read-write memory and $k$-set consensus objects can be captured by an iterated *affine* task [15]. The task is defined via a simplicial complex $\mathcal{R}_k$, a specific subset of simplices of the second chromatic subdivision $\text{Chr}^2 s$ [21] in which, intuitively, at most $k$ processes concurrently *content*.

(Examples of $\mathcal{R}_1$ and $\mathcal{R}_2$ for the 3-process system are given in Figure 3.) We show that the set of IIS runs corresponding to iterations of this subcomplex $\mathcal{R}_k$, denoted $\mathcal{R}_k^*$ allows for solving precisely the same set of tasks as the model of $k$-set consensus does.

Interestingly, our definition of what it means to solve a task in $\mathcal{R}_k^*$ requires *every* process to output. This contrasts with the conventional definition of task solvability (e.g., using $k$-set-consensus), where a failure may prevent a process from producing an output and, thus, only *correct* processes can be required to output. Indeed, $\mathcal{R}_k^*$ does not account for process failures: every process takes infinitely many steps in every run, but, because of the use of iterated memory, a “slow” process may not be seen by “faster” ones from some point. The requirement that every process outputs is indispensable in an iterated characterization of generic (colored) task solvability that may not allow one process to “adopt” an output of another process. Indeed, even if “fast” processes output, the “slow” ones should be able to continue in order to enable every correct process to output in the corresponding $k$-set-consensus model. For example, task solvability with consensus objects is captured by the “total order” subcomplex $\mathcal{R}_1$ (depicted for 3 processes in the left part of Figure 3) in which, intuitively, *every* subset of processes should be able to solve consensus.

Our result is established by the existence of two algorithms. The first algorithm solves the *simplex agreement task* [23] on $\mathcal{R}_k$ in the model $k$-concurrency. By iterating this solution, we can implement $\mathcal{R}_k^*$ and, thus, solve any task solvable in $\mathcal{R}_k^*$. Then, by showing that the $k$-set consensus model solves every task that the $k$-concurrency model solves, we derive that the former model also solves every task solvable in $\mathcal{R}_k^*$. The second algorithm *simulates* runs of a given algorithm using read-write memory and $k$-set-consensus objects in $\mathcal{R}_k^*$. The simulation is quite interesting in its own right. Compared to simulations in [22, 17, 14, 7, 16], our algorithm ensures that every process
eventually outputs in $R_k^*$, assuming that the simulated algorithm ensures that every correct process eventually outputs.

Thus, a task is solvable using iterations of $R_k$ if and only if it can be solved solvable wait-free using $k$-set-consensus objects or, equivalently, $k$-concurrently. Therefore, the $k$-set-consensus model has a bounded representation as an iterated affine task: processes simply sequentially invoke instances of $R_k$ for a bounded number of times, until they assemble enough knowledge to produce an output for the task they are solving.

Our results suggest a separation between “natural” models that have a matching affine task and, thus, can be captured precisely by a subset of IIS runs and less “natural” ones, like WSB, having a manifold structure that is not affine [18]. We conjecture that such a combinatorial representation can also be found for a large class of restrictions of the wait-freedom, beyond $k$-concurrency and $k$-set consensus. The claim is supported by a recent derivation of the $t$-resilience affine task [30].

Related work. There have been several attempts to extend the topological characterization of [23] to models beyond wait-free [22, 14, 16]. However, these results either only concern the special case of colorless tasks [22], consider weaker forms of solvability [14], or also introduce a new kind of infinite subdivisions [16].

In particular, Gafni et al. [16] characterized task solvability in models represented as subsets of IIS runs via infinite subdivisions of input complexes. This result assumes a limited notion of task solvability in the iterated model that only guarantees outputs to “fast” processes [10, 27, 7] that are “seen” by every other process infinitely often.

In contrast with the earlier work, this paper studies the inherent combinatorial properties of general (colored) tasks and assumes the conventional notion of task solvability. In a sense, our results for the first time truly capture the combinatorial structure of a model of computation beyond the wait-free one.

Roadmap. The rest of the paper is organized as follows. Section 2 gives model definitions, briefly overviews the topological representation of iterated shared-memory models. In Section 3 we present the definition of $R_k$ corresponding to the $k$-concurrency model. In Section 4 we show that $R_k$ can be implemented in the $k$-set-consensus model and that any task solvable in the $k$-set-consensus model can be solved by iterating $R_k$. Section 5 discusses related models and open questions.

2 Preliminaries

Let II be a system composed of $n$ asynchronous processes, $p_1, \ldots, p_n$. We consider two models of communication: (1) atomic snapshots [1] equipped with $k$-set consensus objects and (2) iterated immediate snapshots [5, 23].

Atomic snapshots and $k$-set consensus. The atomic-snapshot (AS) memory is represented as a vector of shared variables, where processes are associated to distinct vector positions, and exports two operations: update and snapshot. An update operation performed by $p_i$ replaces the shared variable at position $i$ with a new value and a snapshot returns the current state of the vector.

The model in which only AS can be invoked is called the AS model. The model in which can be invoked, in addition to AS, also $k$-set consensus objects, for some fixed $k \in \{1, \ldots, n-1\}$, is called the $k$-set consensus model.

Iterated immediate snapshots. In the iterated immediate snapshot (IIS) model [5], processes goes through the ordered sequence of independent memories $M_1, M_2, \ldots$. Each memory $M_r$ is
accessed by a process with a single immediate snapshot operation: the operation performed by \( p_i \) takes a value \( v_i \) and returns a set \( V_{ir} \) of values submitted by other processes (w.l.o.g., we assume that values submitted by different processes are distinct), so that the following properties are satisfied: (self-inclusion) \( v_i \in V_{ir} \); (containment) \( V_{ir} \subseteq V_{jr} \); and (immediacy) \( v_i \in V_{jr} \Rightarrow V_{ir} \subseteq V_{jr} \).

In the IIS communication model, we assume that processes run the full-information protocol: the first value each process writes is its input value. For each \( r > 1 \), the outcome of the immediate snapshot operation on memory \( M_{r-1} \) is submitted as the value for the immediate snapshot operation on memory \( M_r \). After a certain number of such (asynchronous) rounds, a process may gather enough information to decide, i.e., to produce an irrevocable non-\( \perp \) output value. A run of the IIS communication model is thus a sequence \( V_{ir}, i \in \mathbb{N}_n \) and \( r \in \mathbb{N} \), determining the outcome of the immediate-snapshot operation for every process \( i \) and each iterated memory \( M_r \).

**Failures and participation.** In the AS or \( k \)-set consensus model, a process that takes only finitely many steps of the full-information protocol in a given run is called faulty, otherwise it is called correct. A process is called participating if it took at least one step in the computation. We assume that in its first step, a process writes its input in the shared memory. The set of participating processes in a given run is called the participating set. Note that, since every process writes its input value in its first step, the inputs of participating processes are eventually known to every process that takes sufficiently many steps.

In contrast, the IIS model does not have the notion of a faulty process. Instead, a process may appear “slow,” i.e., be late in accessing iterated memories from some point on so that some “faster” processes do not see them.

**Tasks.** In this paper, we focus on distributed tasks. A process invokes a task with an input value and the task returns an output value, so that the inputs and the outputs across the processes which invoked the task, respect the task specification. Formally, a task is defined through a set \( I \) of input vectors (one input value for each process), a set \( O \) of output vectors (one output value for each process), and a total relation \( \Delta : I \rightarrow 2^O \) that associates each input vector with a set of possible output vectors. An input \( \perp \) denotes a not participating process and an output value \( \perp \) denotes an undecided process. Check [21] for more details on the definition.

**Protocols and runs.** A protocol is a distributed automaton that, for each local state of a process, stipulates which shared-memory operation and which state transition the process is allowed to perform in its next step. We assume here deterministic protocols, where only one operation and state transition is allowed in each state. A run of a protocol is defined as a sequence of states and shared-memory operations.

A process is called active at the end of a finite run \( R \) if it participates in \( R \) but did not returned at the end of \( R \). Let \( \text{active}(R) \) denote the set of all processes that are active at the end of \( R \).

A run \( R \) is \( k \)-concurrent \((k = 1, \ldots, n)\) if at most \( k \) processes are concurrently active in \( R \), i.e., \( \max\{|\text{active}(R'); R'\text{ prefix of } R\} \leq k \). The \( k \)-concurrency model is the set of \( k \)-concurrent AS runs.

**Solving a task.** A protocol solves a task \( T = (I, O, \Delta) \) in the \( k \)-set-consensus model (resp., \( k \)-concurrently) if it ensures that in every run of the \( k \)-set-consensus model (resp., every \( k \)-concurrent AS run) in which processes start with an input vector \( I \in I \), (1) all decided values form a vector \( O \in O \) such that \( (I, O) \in \Delta \), and (2) every correct process decides.

1It is known that the \( k \)-concurrency model is equivalent to the \( k \)-set-consensus model \[12\], any task that can be solved \( k \)-concurrently can also be solved in the \( k \)-set-consensus model, and vice versa.
versa.

**Standard chromatic subdivision and IIS.** To give a combinatorial representation of the IIS model, we use the language of simplicial complexes [31, 21]. In short, a simplicial complex is defined as a set of vertices and an inclusion-closed set of vertex subsets, called simplices. The dimension of a simplex $\sigma$ is the number of vertices in it minus one. Any subset of these vertices is called a face of the simplex. A simplicial complex is pure (of dimension $n$) if each its simplices are contained in a simplex of dimension $n$.

A simplicial complex is chromatic if it is equipped with a coloring non-collapsing map $\chi$ from its vertices to the standard $(n-1)$-simplex $s$ of $n$ vertices, in one-to-one correspondence with $n$ colors $1, 2, \ldots, n$. All simplicial complexes we consider here are pure and chromatic.

Refer to Appendix A for more details on the formalism.

For a chromatic complex $C$, we let $\text{Chr} C$ be the subdivision of $C$ obtained by replacing each simplex in $C$ with its chromatic subdivision [23, 24, 25]. The vertices of $\text{Chr} C$ are pairs $(v, \sigma)$, where $p$ is a vertex of $C$ and $\sigma$ is a simplex of $C$ containing $v$. vertices $(v_1, \sigma_1), \ldots, (v_m, \sigma_m)$ form a simplex if all $v_i$ are distinct and all $\sigma_i$ satisfy the properties of immediate snapshots. Subdivision $\text{Chr}^1 s$ for the 2-dimensional simplex $s$ is given in Figure 1. Each vertex represents a local state of one of the three processes $p_1$, $p_2$ and $p_3$ (red for $p_1$, blue for $p_2$ and white for $p_3$) after it takes a single immediate snapshot. Each triangle (2-simplex) represents a possible state of the system. A corner vertex corresponds to a local state in which the corresponding process only sees itself (it took its snapshot before the other two processes moved). An interior vertex corresponds to a local state in which the process sees all three processes. The vertices on the 1-dimensional faces capture the snapshots of size 2.

If we iterate this subdivision $m$ times, each time applying the same subdivision to each of the simplices, we obtain the $m$th chromatic subdivision, $\text{Chr}^m C$. It turns out that $\text{Chr}^m s$ precisely captures the $m$-round (full-information) IIS model, denoted $\text{IS}^m [23]$. Each run of IS$^m$ corresponds to a simplex in $\text{Chr}^m s$. Every vertex $v$ of $\text{Chr}^m s$ is thus defined as $(p, IS^1(p, \sigma), \ldots, IS^m(p, \sigma))$, where each $IS^i(p, \sigma)$ is interpreted as the set of processes appearing in the $i$th IS iteration obtained by $p$ in the corresponding IS$^m$ run. The carrier of vertex $v$ is then defined as the set of all processes seen by $p$ in this run, possibly through the views of other processes: it is the smallest face of $s$ that contains $v$ in its geometric realization [21, Appendix A].

**Simplex agreement.** As we show in this paper, the model of $k$-concurrency can be captured by an iterated simplex agreement task [51, 23].

Let $L$ be a subcomplex of $\text{Chr}^2 s$. In the simplex agreement task, every process starts with the vertex of $s$ of its color as an input and finishes with a vertex of $\text{Chr}^m s$ as an output, so that all outputs constitute a simplex of $\text{Chr}^2 s$ contained in the face of $s$ constituted by the participating processes.

Formally, the task is defined as $(s, L, \Delta)$, where, for every face $t \subseteq s$, $\Delta(t) = L \cap \text{Chr}^2 t$. By running $m$ iterations of this task, we obtain $L^m$, a subcomplex of $\text{Chr}^{2m} s$, corresponding to a subset of IS$^{2m}$ runs (each iteration includes two IS rounds).

### 3 The complex of $k$-set consensus

We now define $\mathcal{R}_k$, a subcomplex of $\text{Chr}^2 s$, that precisely captures the ability of $k$-set consensus (and read-write memory) to solve tasks. The definition of $\mathcal{R}_k$ is expressed via a restriction on the simplices of $\text{Chr}^2 s$ that bounds the size of contention sets. Informally, a contention set of a
simplex $\sigma \in \text{Chr}^2s$ (or, equivalently, of an $IS^2$ run) is a set of processes that “see each other”. When a process $p_i$ starts its $IS^2$ execution after another process $p_j$ terminates, $p_i$ must observe $p_j$’s input, but not vice versa. Thus, a set of processes that see each others’ inputs must have been concurrently active at some point. Note that processes can be active at the same time but the immediate snapshots outputs might not permit to detect it.

Topologically speaking, a contention set of a simplex $\sigma \in \text{Chr}^2s$ is a set of processes in $\sigma$ sharing the same carrier, i.e., a minimal face $t \subseteq s$ that contains their vertices. Thus, for a given simplex $\sigma \in \text{Chr}^2s$, the set of contention sets is defined as follows:

**Definition 1 (Contention sets)**

\[ \text{Cont}(\sigma) = \{ S \subseteq \Pi, \forall p, p' \in S, \text{carrier}(p, \sigma) = \text{carrier}(p', \sigma) \}. \]

Contention sets for simplices of $\text{Chr}^2s$ in a 3-process system are depicted in Figure 2: for each simplex $\sigma \in \text{Chr}^2s$, every face of $\sigma$ that constitutes a red simplex is a contention set of $\sigma$. In an interior simplex, every set of processes are contention sets. Every “total order” simplex (shown in blue in Figure 3a), matching a run in which processes proceed, one by one, in the same order in both $IS^1$ and $IS^2$, has only three singleton as contending sets. All other simplices include a contention set of two processes which consists of the vertices at the boundary.

Now $R_k$ is defined as the set of all simplices in $\text{Chr}^2s$, in which the contention sets of have cardinalities at most $k$:

**Definition 2 (Complex $R_k$)**

\[ R_k = \{ \sigma \in \text{Chr}^2s, \forall S \in \text{Cont}(\sigma), |S| \leq k \}. \]

It is immediate that the set of simplices in $R_k$ constitutes a simplicial complex: every face $\tau$ of $\sigma \in R_k$ is also in $R_k$.

Examples of $R_1$ and $R_2$ for a 3-process system are shown in Figures 3a and 3b, respectively. Obviously, for the unrestricted 3-set consensus case, $R_3 = \text{Chr}^2s$. Note that $R_1$ only contains six “total order” simplices, while $R_2$ consists of all simplices of $\text{Chr}^2s$ that touch the boundary.
4 From $k$-set consensus to $R^*_k$ and back

We show that any task solvable with $k$-set consensus (and read-write shared memory) can be solved in $R^*_k$, and vice versa. The main result is then established via simulations: a run of an algorithm solving a task in one model is simulated in the other.

4.1 From $k$-set consensus to $k$-concurrency

We first show that a $k$-concurrent shared memory system is equivalent, regarding task solvability, to a shared memory system enhanced with $k$-set consensus objects. The result has been stated in a technical report [12], but no explicit proof has been given available in the literature until now, and we fill the gap below. For the sake of completeness and to make referencing simpler, we propose here a direct simulation with proofs.

Simulating a $k$-process shared memory system. We employ generalized state machines (proposed in [13] and extended in [28]) that allow for simulating a $k$-process read-write memory system in the $k$-set-consensus model. To ensure consistency of simulated read and write operations, we use commit-adopt objects [10] that can be implemented using reads and writes. A commit-adopt object exports one operation $propose(v)$ that takes a parameter in an arbitrary range and returns a couple $(\text{flag}, v')$, where $\text{flag}$ can be either commit or adopt and where $v'$ is a previously proposed value. Moreover, if a process returns a commit flag, then every process must return the same value. Further, if no two processes propose different values, then all returned flags must be commit. Liveness of the simulation relies on calls to $k$-simultaneous consensus objects [2]. To access a $k$-simultaneous consensus object, a process proposes a vector of $k$ inputs, one for each of the consensus instances, $1, 2, \ldots, k$, and the object returns a couple $(i, v)$, where index $i$ belongs to $\{1, \ldots, k\}$ and $v$ is a value proposed by some process at index $i$. It ensures that no two processes obtain different values with the same index. Moreover, if $\ell \leq k$ distinct input vectors are proposed then only values at indices $1, \ldots, \ell$ can be output. The $k$-simultaneous consensus object is equivalent to $k$-set-consensus in a read-write shared-memory system [2].

Our simulation is described in Algorithm 1. We use three shared abstractions: an infinite array of $k$-simultaneous consensus objects $KSC$, an infinite array of arrays of $k$ indexed commit-adopt objects $CA$, and a single-writer multi-reader memory $MEM$ with $k$ slots.

In every round, processes use the corresponding $k$-simultaneous consensus object first (line 7).
and then go through the set of $k$ commit-adopt objects (lines 8–10), starting with the index output by the $k$-simultaneous consensus object (line 8). It is guaranteed that at least one process commits, in particular, process $p_j$ that is the first to return from its first commit-adopt invocation in this round (on a commit-object $C$), because any other process with a different proposal must access a different commit-adopt object first and, thus, must invoke $C$ after $p_j$ returns. To ensure that a unique written value is selected, processes replace their current proposal values with the value adopted by the commit-adopt objects (lines 8–10). Note that the processes do not select values corresponding to an older round of simulation, to ensure that processes do not alternate committing and adopting the same value indefinitely.

In the simulation, the simulating processes propose snapshot results for the simulated processes. Once a proposed snapshot has been committed, a process stores in the shared memory the value that the simulated process must write in its next step (based on its simulated algorithm), equipped with the corresponding write counter (line 14). The write counter is then incremented and a new snapshot proposal is computed (line 16). To compute a simulated snapshot, for each process, we select the most recent value available in the memory $MEM$ by comparing the write counters $WC$ (auxiliary function $CurWrite$ at lines 18–22).

**Lemma 1** Algorithm 1 provides a non-blocking simulation of a $k$-process read-write shared-memory system in the $k$-set consensus model. Moreover, if there are $\ell < k$ active processes, then one of the first $\ell$ simulated processes is guaranteed to make progress.
The proof of Lemma 1 can be found in Appendix B. The proof is constructed by showing that:
1) No two different written values are computed for the same simulated process and the same write counter;
2) At every round of the simulation, at least one simulator commits a new simulated operation;
3) Every committed simulated snapshot operation can be linearized at the moment when the actual snapshot operation which served for its computation took place; and
4) Every simulated write operation can be linearized to the linearization time of the first actual write performed by a simulator with the corresponding value.

**Using the extended BG-simulation to simulate a k-concurrent execution.** We have shown that a k-process read-write shared memory system can be simulated in the k-set-consensus model. Now we show that this simulated system can be used to simulate k-concurrency. The idea is to make the obtained k-process system run a BG-simulation protocol [3, 6], so that at most k simulated processes are active at a time.

The BG-simulation technique allows k + 1 processes s_1, ..., s_{k+1}, called BG-simulators, to wait-free simulate a k-resilient execution of any protocol A on m processes p_1, ..., p_m (m > k). The simulation guarantees that each simulated step of every process p_j is either agreed upon by all simulators, or one less simulator participates further in the simulation for each step which is not agreed on (in this, we say that the step simulation is blocked because of the faulty or slow simulator).

The technique was later turned into extended BG-simulation [11]. The core of this technique is the Extended Agreement (EA) algorithm, which ensures safety of consensus but not necessarily liveness: it may block if some process has slowed down in the middle of its execution. Additionally, the EA protocol exports an abort operation that, when applied to a blocked EA instance, re-initializes it so that it can move forward until an output is computed or another process makes it block again.

Our simulation is quite simple. Before running the k-process simulation using Algorithm 1 processes write their input states in the memory. The k-process simulation is used to run an extended BG-simulation that executes the code of the task solution for the simulated initial processes. Once a task output for a simulated process is available in the shared memory, the process stops participating in the k-process simulation. A process that completed its initial write but has not yet been provided with a task output is called active, i.e., this process is both available to be simulated by the BG-simulators and is participating in the simulation of the k BG-simulators.

In our case, we use the extended BG-simulation in a slightly different manner than in the original paper [11]. Instead of running processes in lock-step as much as possible, by selecting the least advanced available process (breadth-first selection), the BG-simulators run the processes with as low concurrency as possible by selecting the most advanced available process (depth-first selection). To prevent simulators from getting blocked on all active processes, a BG-simulator stops participating if the number of active processes is strictly lower than its identifier (the index of the simulated process the BG-simulator is executed on, from 1 to k). If a BG-simulator is blocked on all active simulated processes, but has an identifier lower or equal to their number, it uses the abort mechanism to exclude BG-simulators with large identifiers that should be stopped.

**Lemma 2** All tasks solvable in the k-concurrency model can be solved in the k-set-consensus model.

**Proof.** The BG-simulators select the most advanced available (not blocked by a BG-simulator) process to execute, thus, a process never simulated yet is selected if and only if all currently started simulations of active processes are blocked. But at most m simulated codes can be blocked by m BG-simulators (m ≤ k), thus, at most k active processes can be concurrently simulated and at least of them is not blocked. Moreover, when there are ℓ < k active processes, then progress is
guaranteed to one of the first \( \ell \) BG-simulators, see Lemma\(^1\) The remaining \( k - \ell \) BG-simulators stop participating in the simulation and cannot block the \( \ell \) first ones, as they are eventually excluded using the \textit{abort} mechanism. The \textit{abort} mechanism is used only finitely many times, only once a BG-simulator witnesses that the number of \textit{active} processes has decreased (and there are finitely many processes). Therefore, as long as there are \textit{active correct} processes, the \textit{BG-simulation makes progress} and eventually every \textit{correct} process obtains an output in the task solution. \( \square \)

\section*{4.2 From \( k \)-concurrency to \( R_k \).}

We now show that \( k \)-concurrency can solve \( R_k \), i.e., it can solve the chromatic simplex agreement task on the subcomplex \( R_k \).

\textbf{Lemma 3} A \( k \)-concurrent execution of two rounds of any immediate snapshot algorithm solves the simplex agreement task on \( R_k \).

\textbf{Proof.} Let us consider a set of \( IS^2 \) outputs provided by a \( k \)-concurrent execution of any IS algorithm (e.g., [4]): the set of \( IS^2 \) outputs forms a valid simplex \( \sigma \) in \( Chr^2 s \) [5], as the set of \( k \)-concurrent runs is a subset of the wait-free runs. Let us consider a contention set \( S \) containing two processes \( p \) and \( q \), and let us assume that \( p \) and \( q \) were never executed concurrently during their executions of the two rounds of immediate snapshots. Without loss of generality, we can consider that \( p \)'s computation was terminated before the activation of \( q \), so \( p \) cannot be aware of \( q \)'s input as it did not perform any operation before \( p \) finishes the two rounds of immediate snapshots. Thus, \( p \) cannot see \( q \), which contradicts the contention set definition. Therefore all processes in a contention set were \textit{active} at the same time during the execution, hence a \( k \)-concurrent execution implies that contention sets cannot contain more than \( k \) processes. \( \square \)

It is easy to complete this result by showing that the \( k \)-set consensus model is, regarding task solvability, at least as strong as the \( R_k^* \) model:

\textbf{Theorem 4} Any task solvable by \( R_k^* \) can be solved in the \( k \)-set consensus model.

\textbf{Proof.} As shown in Lemma\(^3\) the simplex agreement task on \( R_k \) is solvable in the \( k \)-concurrency model. Moreover, according to Lemma\(^2\) any task solvable in the \( k \)-concurrency model can be solved in the \( k \)-set consensus model, and hence in particular, the simplex agreement task on \( R_k \). Therefore, by iterating a solution to the simplex agreement task on \( R_k \), a run of \( R_k^* \) can be simulated in the \( k \)-set consensus model and used to solve any task solvable in \( R_k^* \). \( \square \)

\section*{4.3 From \( R_k^* \) to \( k \)-set consensus}

Now we show how to simulate in \( R_k^* \) any algorithm that uses read-write memory and \( k \)-set-consensus objects.

\textit{k-set consensus simulation design.} Making a \textit{non-blocking} simulation of read-write memory can be trivially done in \( R_k^* \), since the set of \( R_k^* \) runs is a subset of \( IS^w \) runs, and there exists several algorithms simulating read-write memory in \( IS^w \), e.g., [17].
Solving \( k \)-set agreement is also not very complicated: every iteration of \( R_k \) provides a set of at most \( k \) leaders, i.e., processes with an \( IS^1 \) output containing at most \( k \) elements, where at least one such leader is visible to every process, i.e., it can be identified as a leader and its input is visible to all. The set of leaders of \( R_2 \) are shown in figure 4a in red, it is easy to observe that every simplex in \( R_2 \) has at most two leaders, and that one is visible to every process (every process with a carrier of size at most 2 is a leader). This property gives a very simple \( k \)-set agreement algorithm: every process decides on the value proposed by one of these \( k \) leaders. We will later show how this property can be derived from the restriction of \( R_k \) on the size of contention sets.

The difficulty of the simulation consists mostly in combining the shared-memory and \( k \)-set agreement simulation, as some processes may be accessing distinct agreement objects while other processes are performing read-write operations. Indeed, liveness of our \( k \)-set-agreement algorithm relies on the participation of visible leaders, i.e., on the fact that the leaders propose values for this instance of \( k \)-set agreement. In this sense, our \( k \)-set agreement algorithm may block if some process is performing a read-write operation or is involved in a different instance of \( k \)-set agreement. Likewise, the read-write memory simulation is only non-blocking, so it can be indefinitely blocked by a process waiting to complete an agreement operation.

The solution we propose consists in (1) synchronizing the two simulations, in order to ensure that, eventually, at least one process will complete its pending operation, and (2) ensuring that the processes collaborate by participating in every simulated operation. In our solution, every process tries to propagate every observed proposed value (for a write operation), and every process tries to reach an agreement in every \( k \)-set-agreement object accessed by some process. For that, we make the processes participate in both simulation protocols (read-write and \( k \)-set agreement) in every round, until they decide.

Even though the simulated algorithm executes only one operation at a time and requires the output of the previous operation to compute the input for the following one, we enrich the simulated process with dummy operations that do not alter the simulation result. Then eventually some undecided process is guaranteed to complete both pending operations, where at most one of them is a dummy one. This scheme provides a non-blocking simulation of any algorithm using read-write shared memory and \( k \)-set agreement objects.

The shared memory simulation from [17] provides progress to the processes with the smallest snapshot output, while our \( k \)-set agreement algorithm provides progress to the leader with the
smallest $R_k$ output, i.e., the smallest $IS^2$ output. We leverage these properties by running the read-write simulation only on the outputs of $R_k$ (i.e., in every second round of immediate snapshots). In the 2-dimensional case, the set of processes with the smallest $IS^2$ outputs are presented as red simplicies in Figure 4b for $R_2$. This way we guarantee that at least the leader with the smallest $R_k$ output will make progress in both simulations. Indeed, the definition of $R_k$ implies that the set of processes with the smallest $R_k$ outputs includes a leader. Figure 4 gives an example of an intersection between the set of processes with the smallest $IS^2$ output and the set of leaders: here every process with the smallest $IS^2$ output has a carrier of size at most 2 and every such process is a leader.

\textit{k-set consensus simulation algorithm.} Algorithm 2 provides a simulation of any algorithm using read-write shared memory (w.l.o.g., atomic snapshots) and $k$-set-agreement objects. The algorithm is based on the shared memory simulation from [17], applied on $IS^2$ outputs of every iteration of $R_k$, combined with a parallel execution of instances of our $k$-set agreement algorithm. The simulation works in rounds that can be decomposed into three stages: communicating through $R_k$, updating local information, and validating progress.

The first stage consists in accessing the new $R_k$ iteration associated with the round, using information on the ongoing operations as an input (see line 6). For memory operations, two objects are contained in $R_k$’s input, an array containing the most recent known write operations for every process, $WriteVal_i$, and a timestamp associated with each process write value, $WriteCount_i$. A single object is used for the agreement operations, $ConsHistory_i$, a list of all adopted proposals for all accessed agreement objects. Finally, a value $State$, set to decided or undecided, is put in $R_k$’s input, to indicate whether the process has completed its simulation.

The second stage consists in updating the local information according to the output obtained from $R_k$ (lines 7–14). The input value of each process observed in the second immediate snapshot of $R_k$ is extracted (line 8). These selected inputs are examined in order to replace the local write values $WriteCount_i$ with the most recent ones, i.e., associated with the largest write counters (lines 9–11). The $ConsHistory$ variable of every leader, i.e., a process with an $IS^1$ output containing at most $k$ undecided process inputs (using the variable $State$), is scanned in order to adopt all its decision estimates (lines 12–14). Moreover, $Leaders_i$ boolean value is used to check if every observed leader transmitted a decision estimate for the pending agreement operation, $ConsId_i$.

The third stage consists in checking whether pending operations can safely be terminated (lines 15–20), and if so, whether the process has completed its simulation (line 21) or if new operations can be initiated (line 21–28).

Informally, it is safe for a process to decide in line 18, as there are at most $k$ Leaders per round, one of which is (1) visible to every process and (2) provides a decision estimate for the pending agreement. Thus every process adopts the decision estimate from a leader of the round, reducing the set of possible distinct decisions to $k$.

A pending memory operation terminates when the round number $r_i$ equals the sum of the currently observed write counters (test at line 15), as in the original algorithm [17]. Indeed, the equality implies that the writes in the estimated snapshot have been observed by every process (line 17). Last, if a process did not terminate, it increments its write counter and, if there is a new operation available, the process selects the operation (see lines 22–28).

If there is a new agreement operation, then the input proposal and the object identifier are selected (line 24) and they are used for the current decision estimate in $ConsHistory_i$ (line 26), unless a value has already been adopted (line 25). If there is a new write operation then the current
write value is simply changed (line 28), a dummy write thus consists in re-writing the same value.  

**Lemma 5** In $R_k^*$, Algorithm 2 provides a non-blocking simulation of any shared memory algorithm with access to $k$-set-agreement objects.

The proof of Lemma 5 is delegated to Appendix B. The main aspects of the proof are taken from the base algorithm from [17], while the liveness of the agreement objects simulation relies on the restriction provided by $R_k^*$ and the maximal size of contention sets.

Lemma 5 implies the following result:

**Theorem 6** Any task solvable in the $k$-set-consensus model can be solved in $R_k^*$.

**Proof.** To solve in $R_k^*$ a task solvable in the $k$-set-consensus model, we can simply use Algorithm 2 simulating any given algorithm solving the task in the $k$-set-consensus model.

The non-blocking simulation provided by Algorithm 2 ensures, at each point, that at least one live process eventually terminates. As there are only finitely many processes, every live process eventually terminates.

Lemma 2, Theorem 4, and Theorem 6 imply the following equivalence result:

**Corollary 7** The $k$-concurrency model, the $k$-set-consensus model, and $R_k^*$ are equivalent regarding task solvability.

5 Concluding remarks: on minimality of $\text{Chr}^2$s for $k$-set consensus

This paper shows that the models of $k$-set consensus and $k$-concurrency are captured by the same affine task $R_k$, defined as a subcomplex of Chr$^2$s. One may wonder if there exists a simpler equivalent affine task, defined as a subcomplex of Chr$s$, the 1-degree of the standard chromatic subdivision. To see that this is in general not possible, consider the case of $k = 1$ (consensus) in a 3-process system. We can immediately see that the corresponding subcomplex of Chr$s$ must contain all “ordered” simplexes depicted in Figure 5. Indeed, we must account for a wait-free 1-concurrent IS$^1$ run in which, say, $p_1$ runs first until it completes (and it must outputs its corner vertex in Chr$s$), then $p_2$ runs alone until it outputs its vertex in the interior of the face ($p_1$, $p_2$) and, finally, $p_3$ must output its interior vertex.

The derived complex is connected. Moreover, any number of its iterations still results in a connected complex. The simple connectivity argument implies that consensus cannot be solved in this iterated model and, thus, the complex cannot capture 1-concurrency.

Interestingly, the complex in Figure 5 precisely captures the model in which, instead of consensus, weaker test-and-set (TS) objects are used: (1) using TS, one easily make sure that at most one process terminates at an IS level, and (2) in IS runs defined by this subcomplex, any pair of

Note that our agreement algorithm is far from efficient for multiple reasons. Progress could be validated at every round and not only when a write is validated. Moreover, processes could also preventively decide the output for objects not yet accessed. Lastly, processes could also adopt proposals from non-leaders when no visible leader has a proposition.
processes can solve consensus using this complex and, thus, a TS object can be implemented. It is not difficult to generalize this observation to \( k \)-TS objects \([26]\): the corresponding complex consists of all simplices of Chr\(s\), contention sets of which are of size at most \( k \). The equivalence (requiring a simple generalization for the backward direction) can be found in \([26, 19]\).

Overall, this raises an intriguing question whether every object, when used in the read-write system, can be captured via a subcomplex of Chr\(^m\)\(s\) for some \( m \in \mathbb{N} \).

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A Simplicial complexes

We review now several notions from topology. For more detailed coverage of the topic please refer to \cite{31, 21}.

A simplicial complex is a set \( V \), together with a collection \( C \) of finite non-empty subsets of \( V \) such that:

1. For any \( v \in V \), the one-element set \( \{v\} \) is in \( C \);
2. If \( \sigma \in C \) and \( \sigma' \subseteq \sigma \), then \( \sigma' \in C \).

The elements of \( V \) are called vertices, and the elements of \( C \) are called simplices. We usually drop \( V \) from the notation, and refer to the simplicial complex as \( C \).

A subset of a simplex is called a face of that simplex.

A subcomplex of \( C \) is a subset of \( C \) that is also a simplicial complex.

The dimension of a simplex \( \sigma \in C \) is its cardinality minus one. The \( k \)-skeleton of a complex \( C \), denoted \( \text{Skel}^k C \), is the subcomplex formed of all simplices of \( C \) of dimension \( k \) or less.

A simplicial complex \( C \) is called pure of dimension \( n \) if \( C \) has no simplices of dimension \( > n \), and every \( k \)-dimensional simplex of \( C \) (for \( k < n \)) is a face of an \( n \)-dimensional simplex of \( C \).

Let \( A \) and \( B \) be simplicial complexes. A map \( f : A \to B \) is called simplicial if it is induced by a map on vertices; that is, \( f \) maps vertices to vertices, and for any \( \sigma \in A \), we have

\[
f(\sigma) = \bigcup_{v \in \sigma} f(\{v\}).
\]

A simplicial map \( f \) is called non-collapsing (or dimension-preserving) if \( \dim f(\sigma) = \dim \sigma \) for all \( \sigma \in A \).

Any simplicial complex \( C \) has an associated geometric realization \( |C| \), defined as follows: Let \( V \) be the set of vertices in \( C \). As a set, we let \( C \) be the subset of \([0,1]^V = \{\alpha : V \to [0,1]\} \) consisting of all functions \( \alpha \) such that \( \{v \in V \mid \alpha(v) > 0\} \subseteq C \) and \( \sum_{v \in V} \alpha(v) = 1 \). For each \( \sigma \in C \), we set \(|\sigma| = \{\alpha \in |C| \mid \alpha(v) \neq 0 \Rightarrow v \in \sigma\} \). Each \(|\sigma| \) is in one-to-one correspondence with a subset of \( \mathcal{R}^n \) of the form \( \{x_1, \ldots, x_n\} \in [0,1]^n \mid \sum x_i = 1 \}. \) We put a metric on \(|C| \) by \( d(\alpha, \beta) = \sum_{v \in V} |\alpha(v) - \beta(v)| \).

A non-empty complex \( C \) is called \( k \)-connected if, for each \( m \leq k \), any continuous map of the \( m \)-sphere into \(|C| \) can be extended to a continuous map over the \((m+1)\)-disk.

A subdivision of a simplicial complex \( C \) is a simplicial complex \( C' \) such that:

1. The vertices of \( C' \) are points of \(|C| \).
2. For any \( \sigma' \in C' \), there exists \( \sigma \in C \) such that \( \sigma' \subseteq |\sigma| \).
3. The piecewise linear map \(|C'| \to |C| \) mapping each vertex of \( C' \) to the corresponding point of \( C \) is a homeomorphism.

Chromatic complexes. We now turn to the chromatic complexes used in distributed computing, and recall some notions from \cite{23}.

Fix \( n \geq 0 \). The standard \( n \)-simplex \( s \) has \( n+1 \) vertices, in one-to-one correspondence with \( n+1 \) colors \( 0, 1, \ldots, n \). A face \( t \) of \( s \) is specified by a collection of vertices from \( \{0, \ldots, n\} \). We view \( s \) as a complex, with its simplices being all possible faces \( t \).
A chromatic complex is a simplicial complex \( C \) together with a non-collapsing simplicial map \( \chi : C \to s \). Note that \( C \) can have dimension at most \( n \). We usually drop \( \chi \) from the notation. We write \( \chi(C) \) for the union of \( \chi(v) \) over all vertices \( v \in C \). Note that if \( C' \subseteq C \) is a subcomplex of a chromatic complex, it inherits a chromatic structure by restriction.

In particular, the standard \( n \)-simplex \( s \) is a chromatic complex, with \( \chi \) being the identity.

Every chromatic complex \( C \) has a standard chromatic subdivision \( \text{Chr}\ C \). Let us first define \( \text{Chr}\ s \) for the standard simplex \( s \). The vertices of \( \text{Chr}\ s \) are pairs \((i, t)\), where \( i \in \{0, 1, \ldots, n\} \) and \( t \) is a face of \( s \) containing \( i \). We let \( \chi(i, t) = i \). Further, \( \text{Chr}\ s \) is characterized by its \( n \)-simplices; these are the \((n + 1)\)-tuples \(((0, t_0), \ldots, (n, t_n))\) such that:

(a) For all \( t_i \) and \( t_j \), one is a face of the other;

(b) If \( j \in t_i \), then \( t_j \subseteq t_i \).

The geometric realization of \( s \) can be taken to be the set \( \{x = (x_0, \ldots, x_n) \in [0, 1]^{n+1} | \sum x_i = 1\} \), with the vertex \( i \) corresponding to the point \( x^i \) with \( i \) coordinates 1 and the other coordinates 0. Then, we can identify a vertex \((i, t)\) of \( \text{Chr}\ s \) with the point

\[
\frac{1}{2k-1}x^i + \frac{2}{2k-1} \left( \sum_{\{j \in t | j \neq i\}} x_j \right) \in |s| \subset \mathbb{R}^{n+1},
\]

where \( k \) is the cardinality of \( t \). Thus, \( \text{Chr}\ s \) becomes a subdivision of \( s \) and the geometric realizations are identical: \(|s| = |\text{Chr}\ s|\).

Next, given a chromatic complex \( C \), we let \( \text{Chr}\ C \) be the subdivision of \( C \) obtained by replacing each simplex in \( C \) with its chromatic subdivision. Thus, the vertices of \( \text{Chr}\ C \) are pairs \((p, \sigma)\), where \( p \) is a vertex of \( C \) and \( \sigma \) is a simplex of \( C \) containing \( p \). If we iterate this process \( m \) times we obtain the \( m^{th} \) chromatic subdivision, \( \text{Chr}^m C \).

Let \( A \) and \( B \) be chromatic complexes. A simplicial map \( f : A \to B \) is called a chromatic map if for all vertices \( v \in A \), we have \( \chi(v) = \chi(f(v)) \). Note that a chromatic map is automatically non-collapsing. A chromatic map has chromatic subdivisions \( \text{Chr}^m f : \text{Chr}^m A \to \text{Chr}^m B \). Under the identifications of topological spaces \(|A| \cong |\text{Chr}^m A|, |B| \cong |\text{Chr}^m B|\), the continuous maps \(|f|\) and \(|\text{Chr}^m f|\) are identical.

### B Omitted proofs

**Lemma 1** Algorithm 7 provides a non-blocking simulation of a \( k \)-process read-write shared-memory system in the \( k \)-set consensus model. Moreover, if there are \( \ell \) < \( k \) active processes, then one of the first \( \ell \) simulated processes is guaranteed to make progress.

**Proof.** We derive correctness of the simulation in Algorithm 7 from the following three properties:

1. Every simulated process follows a unique sequence of operations, (2) simulated snapshots and updates are linearizable, and (3) at least one simulated process, with an associated identifier lower or equal to the number of active processes, takes an infinite number of steps.

Let us first prove two useful simple claims:

1. The write counter of a simulated process never decreases: A write counter can only be modified by incrementing it in line 15 or by adopting it from the result of a commit-adopt

2. Simulated processes are guaranteed to make progress: If there are \( \ell \) simulated processes, at least one of the \( \ell \) processes is active or equal to the number of active processes, taking an infinite number of steps.
operation in lines 8 or 10. Moreover, the write counter and the associated value are only updated when the write counter obtained from a commit-adopt object is not smaller than the current one.

2. If a process has a write counter equal to \( c > 0 \), then at least one write/snapshot operation has been validated, i.e., a simulator passed the test in line 13, for every write counter \( c' \), \( 0 \leq c' < n \): Let us prove this property by induction. Initially, all write counters are set to 0 and, thus, the property is trivially verified. Assume that the property holds for value \( c \), and consider a state in which some process has a write counter equal to \( c + 1 \). Consider the first time a process updates its write counter to \( c + 1 \), it can only be the result of line 15, as adopting the write counter from another process would result in a contradiction (this process must have updated it to \( c + 1 \) first).

Using these claims, let us show our three required properties:

- **For every couple \((s, c)\), where \( s \) is a simulated process and \( c \) is a write counter, all validated writes are identical.** According to the structure of Algorithm 1, an operation is validated if and only if it is returned from a call to a commit-adopt object with a commit flag (line 13). Consider the smallest round \( r \) in which some process obtains a commit flag with a write counter equal to \( c \) from a commit-adopt object associated to \( s \). According to the specification of commit-adopt, every process obtains the same output value at round \( r \) from this commit-adopt object, \((c, val)\), possibly with an adopt flag. According to **Claim 2**, if no process validated a write for \((s, c)\), then no write counter can be greater than \( c \). Thus, every process replaces its current proposal value with the same couple \((c, val)\).

  By contradiction, assume that a process validates a different write value for \((s, c)\). By **Claim 1**, this process must have modified its snapshot estimate without ever increasing its write counter. The first process that modified the estimate could only do it by adopting the proposition from another process, but no other process changed yet its proposal value, which results in a contradiction.

- **Simulated snapshots and updates are linearizable.** A simulated snapshot is the result of applying the auxiliary function \( \text{CurWrites} \) (lines 18–22) on the snapshot result of \( MEM \). The auxiliary function simply selects, for each simulated process, the write value with the greater write counter. By the previous property, all validated writes updated to \( MEM \) with the same counter are identical, and so, only the first completed write value may change the result of the simulated snapshot computation. Indeed, a validated write can only be overwritten by a more recent write value, i.e., a write value associated with a greater write counter, as the write counter strictly increases during a validation and can never decrease afterwards (**Claim 1**).

  Therefore, for a given simulated process, the first write updating \( MEM \) for a given write counter will be selected in any later snapshot until a write associated to a larger write counter is performed. Now we select the linearization point of a simulated write to be the linearization point of the first validated update (line 14) performed with it. A simulated snapshot is linearized to the linearization point of the corresponding validated snapshot operation on \( MEM \). The simulated snapshots and updates ordered by their linearization points constitute a legal sequential history. It is easy to see that a validated snapshot operation, the one which
served for the simulated write computation, is linearized after the preceding write operations of the simulated process. Indeed, a simulated snapshot is computed only by the processes which validated the preceding write (line 16) and only after the update was made to MEM (on line 14), thus, always after the first update was made for the associated write counter.

Finally, we prove liveness of the simulation:

- **A simulated process with an identifier smaller or equal to the number of active processes takes infinitely number of steps.** This result directly follows from the properties of $k$-simultaneous consensus objects. With the number of active processes equal to $m$, processes are provided with an output value $(i, val)$ where $i < \min(m, k)$ and every process with the same index obtains the same value. As processes first access the commit-adopt object associated with the obtained index $i$, some process must obtain a commit flag for its index. Indeed, the first process to obtain an output for its first commit-adopt object in a round, may only have witness other processes with the same proposal. As a commit-adopt object must return a commit flag when every proposed values are identical, this first process must obtain a commit flag associated to its index. Thus, this process validates the corresponding write value and increases its write counter.

By Claim 1, write counters can never decrease. Also, in every round, at least one simulator increases the write counter associated with a simulated process with an identifier smaller or equal to the number of active processes, $m$. Therefore, after an infinite number of rounds, the write counters associated with such a simulated process have been incremented an infinite number of times. But there are only finitely many of them, one per process and simulated process. Thus, the write counter of one of them is incremented an infinite number of times. Using Claim 2, an infinitely incremented write counter implies a simulated process taking an infinite number of simulation steps.

Lemma 5. In $\mathcal{R}_k^*$, Algorithm 2 provides a non-blocking simulation of any shared memory algorithm with access to $k$-set-agreement objects.

Proof. We prove correctness of Algorithm 2 in three steps: (1) The read-write simulation is safe, i.e., the write-snapshot operations are atomic; (2) the $k$-set-agreement algorithm is safe, i.e., processes decide on at most $k$ distinct proposed values for the same agreement object; and (3) the simulation is non-blocking, i.e., there is a non-terminated process which completes an infinite number of simulated operations.

- **The read-write simulation is safe:** The structure of the simulation is taken from an analogous simulation in [17], therefore, this part of the proof is also directly inspired from the one in [17].

In Algorithm 2, memory operations are reduced to a single write/snapshot operation. It is easy to see that any read-write algorithm can be executed in this way by re-writing the same value again to discard a write or by ignoring all or part of the snapshot result to discard read operations. Nevertheless, even if a single write/snapshot operation is provided, it is
not an immediate snapshot, i.e., the write and the snapshot operations cannot be linearized together by batches. We will show that the set of new written values returned in some simulated snapshot during round \(r\), \(W_{\text{proc}}(\text{snap}(r))\), and the set of processes returning this snapshot, \(R_{\text{proc}}(\text{snap}(r))\), can be linearized firstly according to the associated round number and secondly, for operations in the same round, by linearizing the write operations before the read operations.

- **Claim 1:** Write counters can only increase. Indeed, a write counter can only be modified by adopting a strictly greater value from another process (on lines 9–11) or directly incremented 22.

- **Claim 2:** The sum of the write counters of an undecided process is greater or equal to the round number. Let us show this claim with a trivial induction on the round number. The property is true for the first round, \(r = 1\), as the write counter associated to the processes own ids is initially set to 1 while the rest is set to 0. Assume the property is true at round \(r\). Then either the sum result is strictly greater than the round number and thus the property is true for round \(r + 1\), as write counters can only increase (Claim 1), or otherwise the sum is equal to the round number. In the later case the equality test made on line 15 is verified and if the process is still undecided it does not pass the test on line 21 and thus it increments its own write counter on line 22 and the sum becomes, and stays (Claim 1), greater or equal to \(r + 1\).

- **Claim 3:** There is a unique write value associated to each process and write counter, except possibly during the execution of line 11 and lines 21–28. This is a simple observation that the adoption of a write value is always made with its associated write counter on a memory position associated to the corresponding process (see lines 9–11). Otherwise, a write value can only be modified on line 28 by selecting a new write value but this is done only once the associated write counter has been incremented on line 22, a write counter that cannot have been used before, as write counters only increase (Claim 1).

- **Claim 4:** A unique snapshot result can be returned per round. According to the containment property of snapshots operations, the \(IS^2\) output results of a call to an iteration of \(R_k\) can be fully ordered by inclusion, i.e., \(S_1 \subset \ldots \subset S_m\). As the greatest write counter observed in these sets are adopted (see lines 9–11), a process with a larger \(IS^2\) output obtains larger or equal write counters. Therefore the sum of the write counters of a process with a larger \(IS^2\) output is equal to the sum of the write counters of a process with a smaller \(IS^2\) output if and only if the write counters are equal for every corresponding process. But if processes obtain the same sum of write counters, the set of selected write values are identical (Claim 3). As in any round, a snapshot is returned (at line 17) only if the sum of write counters equals the round number (test on line 15), and so all returned snapshot are identical, if any.

- **Claim 5:** A write value can only be replaced with a more recent value. This is a direct corollary of Claims 1 and 3. Indeed, a write value associated with a given process can only replaced by a write value associated to the same process but with a greater associated write counter. Yet, a more recent write value is always associated with a greater write counter than all previously used ones for the corresponding process.
We therefore are provided with a unique sequence of validated snapshots according to rounds, snap(0), . . . , snap(r), snap(r + 1), . . . (Claim 4). We can thus define without ambiguity Wproc(snap(r)) and Rproc(snap(r)), respectively the set of firstly observed write values and the set of processes which returned snap(r) during round r. Moreover, according to Claim 5, a write value can only be replaced by a more recent write value, thus a snapshot returned on a later round can only contain identical or more recent write values. It is obvious that write operations can be safely linearized between the last snapshot which observed an older write value and the first snapshot to return the write value or a more recent one (as processes alternate write and snapshot operations write operations can be linearized in any order in between two snapshots, i.e., there can be at most one write value per process between two consecutively linearized snapshots). Therefore the following is a valid linearization ordering (we leave out the trivial verification that the linearization order respects operations local ordering):

\[ \forall (r, r'), r < r', \text{Lin}(R\text{proc}(\text{snap}(r))) < \text{Lin}(W\text{proc}(\text{snap}(r'))) < \text{Lin}(R\text{proc}(\text{snap}(r'))). \]

- **The k-set-agreement is safe:** The safety of an agreement operation relies on two properties, validity and agreement. We also show an intermediary result later re-used for the liveness property, leader visibility:

  - **Validity:** Every decided value is the input proposal of some process: The estimated decision value in ConsHistory, is always either initialized to the process own proposal (on line 26), or to a value previously adopted from a leader (on line 14). The estimated decisions values are then never dissociated from their corresponding k-set-agreement object identifier, ConsIdi, in the local memory object ConsHistoryi. It can further only be replaced in ConsHistoryi with the current initialized decision estimate, with the same object identifier, from another process (on line 14). Thus deciding on the estimated decision value in ConsHistoryi (at line 18) always decides on some process input proposal.

  - **leader visibility:** For every round, there is an undecided process, with an IS1 output containing at most k inputs from undecided processes, contained in each undecided process Rk output: This property is directly derived from the definition of Rk restricting the contention sets sizes. Indeed, consider the smallest output obtained by an undecided process from an iteration of Rk, denoted Smin. Such a smallest view exists according to the snapshots containment property. By the containment property also, if Smin contains an IS1 output with at most k elements from undecided processes, then every Rk output contains it. Moreover, by the self-inclusion property, Smin contains the IS1 output of an undecided process. Now let us assume that Smin contains only IS1 outputs from undecided processes containing strictly more than k inputs from undecided processes, and let Pmin be one of such a set of undecided processes with inputs contained in an IS1 output. As Smin is the smallest output of an undecided process, all processes in Pmin observed in their Rk output every Rk input from processes in Pmin. This implies that Pmin forms a contention set, which is a contradiction with the assumption that Pmin includes more than k processes.
Agreement: At most $k$ distinct values can be decided: Consider the first round, $r$, at which some process completes an agreement operation for an agreement object (on line 18). A deciding process must have observed only round leaders, i.e., processes with an $IS^i$ output containing at most $k$ undecided processes inputs, with a decision estimate for the considered agreement object ($Leaders_i = true$). Thus according to the leader visibility property, there is a leader observed by every undecided process with a decision estimate for the considered agreement object. Therefore, during this round $r$, every process adopts the decision estimate of a leader for the considered agreement object (at line 14). Thus every process has a decision estimate in its ConsHistory object from one of the at most $k$ round leaders. Moreover, even if processes start a call to this agreement object in a later round, they would discard their own input proposal value directly (see line 25), and therefore the set of decision estimates can only decrease in later rounds.

It is easy to see by combining the self-inclusion and containment properties that at most $k$ processes can obtain a snapshot output containing at most $k$ inputs. Thus there can be at most $k$ leaders in a round, therefore, the set of decided values being bounded by the number of decision estimates adopted at round $r$, at most $k$ distinct values can be decided for a given agreement object.

- The simulation is non-blocking: A non-blocking simulation means that if undecided processes complete an infinite number of steps, then there is an undecided process which completes an infinite number of simulation operations. Showing this property is equivalent to show that there is no reachable configuration where there are undecided processes never completing any simulation steps during infinitely many algorithm rounds (the algorithm does not include waiting statements or infinite loops).

Let us assume we are in such a state. We will first show that the write counter of an undecided process is increased infinitely many times. This simply results from the previously proven Claim 2 from the safety proof of the read write simulation, stating that the sum of write counters of an undecided process is always greater than the round number. Thus, as there is an undecided process completing an infinite number of simulation rounds, there is a write counter increased infinitely many often. This write counter can only be one of an undecided process as a write counter is never increased after termination (see lines 21 and 22). Thus there is a process passing the test on line 15 infinitely often.

If a process passes the test on line 15, then every undecided process with a smaller $R_k$ output, thus a smaller write counter sum, must pass it as well. Let us consider the undecided process, $p_{min}$, which passes this test infinitely often with the smallest $R_k$ output obtained by an undecided process. This process must have a pending active agreement operation as otherwise it would complete its write operation, or terminate, or select a new operation. Without loss of generality, we can take $p_{min}$ to be indefinitely often a leader seen by every undecided process (see the leader visibility property). Therefore every process must eventually obtain a decision estimate for $p_{min}$ pending agreement object (at line 14), and thus $p_{min}$ eventually decides as it eventually only observe leaders with a decision estimate ($Leader = true$). This results in a contradiction, thus the simulation is non-blocking.

\[\square\]
Algorithm 2: k-set consensus simulation in $\mathcal{R}_k^*$: process $i$

1. **Init:** $r_i \leftarrow 0$; $\text{State}_i \leftarrow \text{undecided}$; $\text{ConsId}_i \leftarrow \perp$; $\text{ConsProp}_i \leftarrow \perp$;

2. $\text{WriteVal}_i[i] \leftarrow \text{FirstWrite}_i()$; $\text{WriteCount}_i[i] \leftarrow 1$;

3. $\text{foreach } m \in \{1, \ldots, n\} \setminus \{i\} \text{ do } (\text{WriteCount}_i[m], \text{WriteVal}_i[m]) \leftarrow (0, \perp)$

   $\text{ConsHistory}_i \leftarrow \emptyset$ : List of adopted agreement proposals;

4. **Repeat forever**
   
   5. $r_i \leftarrow r_i + 1$; $\text{Leaders}_i \leftarrow \text{true}$;

   6. $\text{IS}^2_{\text{output}} = \text{IS}^2[r_i](\text{State}_i, (\text{WriteCount}_i, \text{WriteVal}_i), \text{ConsHistory}_i)$;

   **foreach** $(j, \text{View}_j) \in \text{IS}^2_{\text{output}}$ **do**

   7. Let $(\text{State}_j, (\text{WriteCount}_j, \text{WriteVal}_j), \text{ConsHistory}_j) \leftarrow \text{RKInput}(j)$;

   8. **foreach** $m \in \{1, \ldots, n\}$ **do**

   9. if $\text{WriteCount}_j[m] > \text{WriteCount}_i[m]$ then

   10. $\text{WriteCount}_i[m] = \text{WriteCount}_j[m]$, $\text{WriteVal}_i[m] = \text{WriteVal}_j[m]$;

11. if $|\text{Undecided}(\text{View}_j)| \leq k$ then

12. if $\exists (\text{ConsId}_i, \ast) \in \text{ConsHistory}_j$ then $\text{Leaders}_i \leftarrow \text{false}$

   **foreach** $(A_{id}, A_{val}) \in \text{ConsHistory}_j$ **do**

   13. $\text{ReplaceOrAdd} (A_{id}, \ast)$ in $\text{ConsHistory}_i$ with $(A_{id}, A_{val})$;

14. if $(\sum_{m \in \{1, \ldots, n\}} \text{WriteCount}_i[m]) = r_i$ then

15. if $\text{PendingWriteSnapshotOperation}()$ then

16. $\text{TerminateWriteOperation}(\text{WriteVal}_i)$;

17. if $\text{Leaders}_i \land \text{ConsId}_i \neq \perp$ then

18. $\text{ConsProp}_i \leftarrow A_{val}$ where $(\text{ConsId}_i, A_{val}) \in \text{ConsHistory}_i$;

19. $\text{TerminateAgreementOperation}(A_{val})$; $\text{ConsId}_i \leftarrow \perp$;

20. if $\text{Terminated}()$ then $\text{State} \leftarrow \text{decided}$ else

21. $\text{WriteCount}_i[i] \leftarrow \text{WriteCount}_i[i] + 1$;

22. if $\text{NextAgreementOperation}() = \text{Available}$ then

23. $(\text{ConsId}_i, \text{ConsProp}_i) \leftarrow \text{NextAgreement}_i()$;

24. if $\exists (A_{id}, A_{val}) \in \text{ConsHistory}_i$ with $A_{id} = \text{ConsId}_i$ then

25. $\text{Add} (\text{ConsId}_i, \text{ConsProp}_i)$ in $\text{ConsHistory}_i$;

26. if $\text{NextWriteSnapshotOperation}() = \text{Available}$ then

27. $\text{WriteVal}_i[i] \leftarrow \text{NextWrite}_i();$

28. **End repeat:**