Collective oscillations of a trapped Fermi gas near the unitary limit

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We calculate the oscillation frequencies of trapped Fermi condensate with particular emphasis on the equation of state of the interacting Fermi system. We confirm Stringari’s finding that the frequencies are independent of the interaction in the unitary limit, and we extend the theory away from that limit, where the interaction does affect the frequencies of the compressional modes only.

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The remarkable advances in producing and measuring properties of atomic condensates give a strong impetus to develop theory to meet the challenges of interpreting the experiments. In the case of fermion condensates, one is now at the early stages of coming to an understanding of the first experimental results \cite{1,2}. One of the characteristic properties is the frequencies of normal modes of vibration. Stringari \cite{3} has developed the theory at the unitary limit, finding that the oscillation frequencies are independent of the details of the interaction. Here we extend the theory away from the unitary limit where the interaction has some effect.

The "unitary limit" is a term to describe a two-component Fermi gas with a short-range interaction, characterized by a scattering length that is large compared to the length scale set by the particle density. This limit was discussed in 1999, when one of us (GFB) formulated as a challenge to many-body theorists to clarify the structure of the ground state of a fictitious neutron matter, interacting with an infinite scattering length \cite{4}. At the time when the challenge has been issued it was not really clear even if such matter is stable in principle, as even though estimates of this number have been extracted by us from the numerical results provided by the authors \cite{2}. We shall discuss the normal modes of oscillation for a Fermi system in a harmonic trap, whose energy per particle is given by Eq. (2).

\begin{equation}
\varepsilon(n) = \frac{3}{5} \frac{\hbar^2 k_F^2}{2m} F\left(\frac{1}{k_F a}\right)
= \frac{3}{5} \frac{\hbar^2 k_F^2}{2m} \left[ \xi - \frac{\xi}{k_F a} - \frac{5\nu}{3k_F^2 a^2} + O\left(\frac{1}{(k_F a)^3}\right) \right], \tag{2}
\end{equation}

where $F(x)$ is a universal function and the constants $\xi \approx 0.44$, $\zeta \approx 1$ and $\nu \approx 1$ (the last two values have been extracted by us from the numerical results provided by the authors \cite{2}). We note that the last term is independent of density and can be dropped from the present analysis.

We assume throughout that the system behaves hydrodynamically, i.e. that the pressure tensor is isotropic. If the system is superfluid, then as long as the oscillation frequency is below the gap frequency needed to break-up a Cooper pair, this condition is expected to be fulfilled. The ground state density satisfies the equation

\begin{equation}
\nabla^2 P(n_0) + \nabla [n_0 \cdot \nabla U] = 0, \tag{3}
\end{equation}

where $n_0$ is the ground state density, $U$ is the trapping potential, and $P$ is the pressure. $P$ is related to the energy per particle $\varepsilon(n)$ by

\begin{equation}
P(n) = n^2 \frac{d\varepsilon(n)}{dn}. \tag{4}
\end{equation}
The slow and small-amplitude normal modes satisfy the following equation
\[ -m \nabla^2 [c^2(n_0)n_1] - \nabla (n_1 \cdot \nabla U) = m \omega^2 n_1 \]  
(5)

where \( n_1 \) is the oscillating density and
\[ c^2(n) = \frac{1}{m} \frac{dP(n)}{dn} \]  
(6)
gives the speed of sound \( c \) in a uniform condensate at density \( n \). It is straightforward to show that Kohn’s generalized theorem [13] (stating that the frequencies of the dipole modes are the trap frequencies) is satisfied. It is convenient to make a change of variable in this equation, replacing \( n_1 \) by the variable \( f_1 \) defined by
\[ n_1 = n_0 f_1 / c^2(n_0). \]  
(7)

After some algebra making use of Eq. (3) we find
\[ \nabla \cdot (n_0 \nabla f_1) = -\omega^2 \frac{n_0}{c^2(n_0)} f_1. \]  
(8)

This equation has the formal advantage of being Hermitian, and thus easier to study in perturbation theory.

As noted before [14], the equations of motion admit simple scaling solutions for gases in harmonic traps that satisfy polytropic equations of state. Before discussing a perturbative treatment, we shall present results of an analysis under the assumption of a polytropic equation of state. As we were completing this work, we learned of a similar analysis by Heiselberg, who presents a general solution in terms of hypergeometric functions (which reduce to polynomials in this case) [15]. The polytropic equation of state is
\[ P(n) = a n^\gamma, \]  
(9)

where the constant \( \gamma \) is the adiabatic index of the gas. The ground state density is given by
\[ n_0(r) = \left[ \frac{\gamma - 1}{\gamma} - \frac{\mu(r)}{\gamma \alpha} \right]^\nu. \]  
(10)

Here we introduced
\[ \mu(r) = \mu_0 - U(r), \quad \gamma = 1 + \frac{1}{\nu}, \]  
(11)

where \( \mu_0 \) is the chemical potential and the harmonic trap potential is given by
\[ U(r) = \frac{m \omega_0^2 (x^2 + y^2 + \lambda^2 z^2)}{2}. \]  
(12)

The corresponding local sound speed is given by
\[ c^2(r) = \frac{\mu(r)}{\nu m}. \]  
(13)

Using these expressions, Eq. (8) can be written in the form
\[ \frac{\mu(r)}{\nu m} \Delta f_1 + \nabla \frac{\mu(r)}{\nu m} \cdot \nabla f_1 = -\omega^2 f_1. \]  
(14)

Due to the particular polynomial form of \( \mu(r) \) the eigenfunctions \( f_1 \) have a polynomial character as well. From Eq. (9) one can extract an effective adiabatic index for a Fermi gas in the vicinity of a Feshbach resonance
\[ \gamma = \frac{d \ln P}{d \ln n} = \frac{5}{3} \left[ 1 + \frac{\zeta}{10 \xi k_F a} + \mathcal{O} \left( \frac{1}{(k_F a)^2} \right) \right]. \]  
(15)

Note that the quadrupole mode in the spherical condensate and the transverse quadrupole modes in the deformed condensate do not depend on the equation of state. That is because the flow in these modes is incompressible and the internal energy does not change during the oscillation cycle. The frequencies of the monopole and of the two compressional modes in the deformed condensate have a dependence on \( \gamma \), and we can use that to estimate the frequency shift.

Eq. (15) shows that the effective adiabatic index is larger than 5/3 on the BCS side of the Feshbach resonance (when \( a > 0 \)). This behavior implies that the frequency of the radial oscillations should increase as well when going from the BCS to the BEC side of the Feshbach resonance. This conclusion agrees with the conjecture made by Stringari [3], except that now the frequency shift has been evaluated explicitly in terms of the properties of the system. In the unitary limit, when \( \gamma = 5/3 \), these results also agree with previous results for the non-interacting Fermi systems in traps [16] and the results for a superfluid Fermi system away from a Feshbach resonance in a spherical trap [17]. The quadrupole frequencies obtained using scaling solutions [14] and the sum-rule approach [15], in the limit of a non-interacting Fermi gas, are different, and indeed correspond to the diabatic limit or collisionless regime [19]. In this limit the sphericity of the Fermi surface is lost during oscillations and the cloud behaves like a normal Fermi gas in Landau’s zero sound regime.

The polytropic analysis is useful to show the basic dependence on the system parameters, but the parameter

**TABLE I:** Results for a polytropic gas. For \( \lambda \ll 1 \) only leading terms are shown and \( c_{1,2,3} \) are some constants.

| Trap type | Mode | \( f_1 \) | \( \omega^2 / \omega_0^2 \) |
|-----------|------|----------|---------------------------|
| spherical | \( L = 1 \) | \( x, y, z \) | 1 |
| \( \lambda = 1 \) | \( L = 2 \) | \( xy, \text{ etc.} \) | 2 |
| \( L = 0 \) | \( \lambda \ll 1 \) | \( xy, x^2 - y^2 \) | 2 |
| axial | \( M = \pm 2 \) | \( xx, yz \) | 1 + \( \lambda^2 \) |
| radial | \( M = \pm 1 \) | \( x^2 + y^2 + c_1 \lambda^2 x^2 + c_2 \) | 2\( \gamma \) |
| axial | \( z^2 + c_3 \) | \( \lambda^2 (3\gamma - 1) / \gamma \) |
Eq. (12) for the frequency shift becomes
\[
\frac{\delta \omega^2}{\omega^2} = \frac{\zeta}{\xi^{1/2}} \frac{\hbar}{m n \omega R a} K = \frac{\zeta}{\xi} \frac{1}{k_F(0) a} K,
\]  
(24)
where the dimensionless factor \( K \) is given by
\[
K = \frac{6 \int |\nabla f|^2 d^3 \tilde{r}}{5 \int |\nabla f|^2 d^3 \tilde{r}} - \frac{4}{5} \int f_1^2 d^3 \tilde{r},
\]  
(25)
and \( k_F(0) \) is the value of the local Fermi momentum at the center of the trap. The prefactor in Eq. (24) displays the scaling of the frequency shift with respect to the physical parameters of the condensate. As expected, the shift is inversely proportional to the combination \( k_F a \).

Finally, it has a nontrivial dependence on \( \xi \) and \( \zeta \), the universal parameters defining the energy per particle in the vicinity of a Feshbach resonance.

All the needed radial integrals have the form
\[
I_{m,n} = \int_0^1 \tilde{r}^m (1 - \tilde{r}^2)^n d\tilde{r},
\]  
(26)
\[
J_{m,n} = \int_0^1 \tilde{r}^m (1 - \tilde{r}^2)^n (1 - b \tilde{r}^2 + c \tilde{r}^4) d\tilde{r},
\]  
(27)
after integration over angular variables. Since in Eq. (24) both denominators and numerators have the same angular dependence, the specific values of the angular integrals cancels out in the case of incompressible modes.

We present the results for the various cases in Table II. One sees that the shift of the dipole mode vanishes, as required by the generalized Kohn’s theorem. The shift also vanishes for the pure quadrupole modes, for reasons noted earlier. The cases of most interest are the monopole mode in spherical traps (\( \lambda = 1 \)) and the \( M = 0 \) modes in axially deformed (essentially cylindrical) traps with \( \lambda \ll 1 \). The \( K \) factors are non-vanishing in these cases, but they are rather small, for example, \( K_{radial} \approx 0.12 \). This has the same order of magnitude as the factor 1/10 in Eq. (15) for the change in the effective adiabatic index. We also note that the sum of factors \( K \) determining the shifts for the radial and axial modes equals the factor \( K \) for the pure monopole mode in the spherical case.

The two experimental results available so far \cite{11,12} are still in noticeable disagreement with each other to permit a detailed comparison with theory. Nevertheless, both experiments show distinctly a qualitative agreement with theory as far as the character of the frequency shift is concerned, in the vicinity of the Feshbach resonance. The fact that both experiments seem to favor the adiabatic character of the oscillations should not be interpreted yet as a confirmation of the existence of superfluidity in these systems, since the sphericity of the Fermi surface can be maintained by collisions. In this respect the situation here is to some extent similar to the expansion of a cold Fermi gas \cite{24,21}. It is also important to determine experimentally the frequencies of the transversal...
TABLE II: Results for $K$.

| trap type  | mode       | $f_1$     | $\omega$ | $K$    |
|------------|------------|-----------|----------|--------|
| spherical  | dipole     | $z$       | $\omega_0$ | 0      |
|            | monopole   | $1 - 2r^2$ | $2\omega_0$ | $\frac{256}{225\pi}$ |
|            | quadrupole | $xy$      | $\sqrt{2}\omega_0$ | 0      |
| axial      | $M = \pm 2$ | $xy, x^2 - y^2$ | $\sqrt{2}\omega_0$ | 0      |
|            | $M = \pm 1$ | $xz, yz$  | $\omega_0$ | 0      |
| radial     | $x^2 + y^2 + \frac{2}{3} \lambda^2 z^2 - \frac{2}{3}$ | $\frac{10}{15} \lambda \omega_0$ | $\frac{1024}{2025\pi}$ |
| axial      | $1 - 6\lambda^2 z^2$ | $\frac{12}{25} \lambda \omega_0$ | $\frac{256}{2025\pi}$ |

quadrupole modes, since a shift in these frequencies can point to a complex structure of the cloud, similar to that discussed in Ref. 22.

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Note added. After submitting this manuscript we learned of a few another studies of Fermi systems using the polytropic equation of state [23].

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