On mutually inverse transforms of functions on a half-line

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Abstract

Two transforms of functions on a half-line are considered. It is proved that their composition gives a concave majorant for every non-negative function. In particular, this composition is the identity transform on the class of nonnegative concave functions. Applications of this result to some problems of mathematical physics are indicated. Several open questions are formulated.

The main result

The following transforms of functions on the positive half-line $\mathbb{R}$ naturally arise in problems of mathematical physics (see Section 3):

$$F[f](x) = \sup_{t>0} \frac{f(xt)}{t + 1}, \quad G[f](x) = \inf_{t>0} f(xt) \left(1 + \frac{1}{t}\right).$$

Throughout this paper, we assume that these transforms are applied to functions on which they are well defined, i.e., $F[f](x)$ and $G[f](x)$ are finite for any $x$. For example, for $F[f]$, it suffices that the function $f(t)$ be at most linearly growing, i.e.,

$$\sup_{t \in [0,a]} f(t) < C(a + 1), \quad a > 0.$$  

The composition is denoted by $G[F[f]] = GF[f]$. For concave functions $f$ of a certain class, it was shown in [2] that the composition is an identity transform.

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We will prove that \( GF[f] = f \) if and only if \( f \) is any monotone concave function satisfying some conditions. Moreover, for an arbitrary nonnegative function \( f \), the composition \( GF[f] \) is the smallest concave majorant of \( f \), i.e., it coincides with the function \( \tilde{f}(t) = \inf \{ \varphi(t) \mid \varphi \geq f, \varphi \text{ concave} \} \). This property of \( F \) and \( G \) resembles the Legendre–Young duality transform \( [1] \), but no relations to it have been found.

Consider two classes of concave functions on \( \mathbb{R}_+ \). The class \( A \) consists of concave nondecreasing nonnegative functions. The graph of each function \( \varphi \in A \) consists of three parts (from left to right): the segment from the point \((0,0)\) to some point \( A_1 \), the arc of the concave function from \( A_1 \) to \( A_2 \), and the horizontal ray from \( A_2 \) to \(+\infty\). Some of these parts may be degenerate, which corresponds to the cases \( A_1 = (0, \varphi(0)) \), \( A_2 = A_1 \) or \( A_2 = +\infty \). For example, the function \( f \equiv 1 \) belongs to \( A \) and for it \( A_1 = A_2 = (0,1) \). Any concave increasing function satisfying the condition \( \varphi(0) > 0 \) belongs to \( A \), and, for it \( A_1 = (0, \varphi(0)) \), \( A_2 = +\infty \). The class \( B \) consists of concave nonincreasing nonpositive functions with the asymptote \( y = kt \), \( t \to +\infty \). The graph of such a function consists of three parts (from left to right): a horizontal segment to some point \( B_1 \), the arc of the concave function from \( B_1 \) to \( B_2 \), and the ray from \( B_1 \) to \(+\infty\), whose extension beyond the point \( B_2 \) passes through zero. Some of these parts may be degenerate. For example, for \( f \equiv -1 \) we have \( B_1 = B_2 = +\infty \); while for the function \( \varphi(t) = -\frac{1}{t+1} - t \) we have \( B_1 = (0, -1), B_2 = +\infty \).

The classes \( A \) and \( B \) are convex and closed in the topology of uniform convergence on each compact set. They intersect only on the zero function. The \( A \)-hull of a function \( f \) is the smallest majorant by functions from \( A \), i.e. \( f_A(t) = \inf \{ \varphi(t) \mid \varphi \geq f, \varphi \in A \} \). Similarly we define the \( B \)-hull: \( f_B(t) = \inf \{ \varphi(t) \mid \varphi \geq f, \varphi \in B \} \). Note that the \( A \)-hull is defined for any function \( f \) with at most linear growth, while the \( B \)-hull is defined only for non-positive \( f \).

**Theorem 1** If a function \( f \) takes at least one positive value then \( GF[f] = f_A \), otherwise \( GF[f] = f_B \).

**Corollary 1** \( GF[f] = f \) if and only if \( f \in A \cup B \).

**Corollary 2** If \( f \) is a nonnegative function then \( GF[f] \) is the concave majorant of \( f \). For nonnegative functions, the equality \( GF[f] = f \) holds if and only if \( f \) is concave.

The proof of Theorem 1 is geometric. We begin with an auxiliary assertion. Suppose that the function \( f : \mathbb{R}_+ \to \mathbb{R} \) grows at most linearly. Through
a given point $A = (-a, 0)$, $a > 0$ we draw a minimum-slope straight line $\ell$ lying above the graph of $f$, i.e. $\ell(t) \geq f(t)$, $t \in \mathbb{R}_+$ (here and below, a linear function is denoted the same as the corresponding line). We call a left supporting line of the graph of $f$. In contrast to a usual supporting line $\ell$, for which $\inf_{t \in \mathbb{R}_+} (\ell(t) - f(t)) = 0$, the left supporting line must intersect the negative horizontal half-line $\{(a, 0) \mid a \geq 0\}$.

**Lemma 1** For any $x > 0$, the straight line passing through the points $(-x, 0)$ and $(0, F[f](x))$ is a left supporting line to the graph of the function $f$.

**Proof.** At the point with abscissa $z$, this line has the ordinate

$$y = \frac{(x + z)F[f](x)}{x} = (1 + t)F[f](x),$$

where $t = z/x$. Hence,

$$\inf_{z \geq 0} (y(z) - f(z)) = \inf_{t \geq 0} ((1 + t)F[f](x) - f(xt)) \geq 0.$$ 

Therefore, the line is above the graph of $f$ and its slope cannot be reduced, since $\inf_{t \geq 0} (F[f](x) - \frac{1}{1+t} f(xt)) = 0$. 

**Proof of the theorem** Denote $z = xt$. Then $G[f](x) = \inf_{z > 0} f(z) \frac{x + z}{z}$. 

Hence,

$$GF[f](x) = \inf_{z > 0} F[f](z) \frac{x + z}{z}. \quad (1)$$

From the point $A = (-z, 0)$, we draw a supporting line to the graph of $f$. It intersects the vertical axis at the point $M = (0, F[f](z))$. Let also $B = (x, 0)$. Drop a perpendicular to the horizontal axis at the point $B$; it crosses the supporting line at the point $N$. Since the triangles $AMO$ and $ANB$, where $O$ is the origin, are similar, we obtain. $BN/OM = AB/AO = (z + x)/z$. Thus, $BN = F[f](z)(z + x)/z$. Combining this with (1), we conclude that $GF[f](x)$ is equal to the smallest value of $\ell(x)$ over all left supporting lines $\ell$ to the graph of $f$. Let denote $\hat{f}$ the smallest concave majorant of $f$. It is the pointwise minimum of all supporting lines to the graph of $f$.

Assume that $f(t) > 0$ at least at one point $t$. In the graph of $\hat{f}$ there are points $A_1$, $A_2$ (which may coincide or grow to $+\infty$) such that the supporting line to the graph of $\hat{f}$ at the point $A_1$ passes through the origin (we use the rightmost of these points), while the supporting line at the point $A_2$ is horizontal (we use the leftmost point). Thus, $A_2$ is a maximizer of $\hat{f}$ and, by assumption, this maximum is positive. For any point between $A_1$ and $A_2$, the supporting line to $f$ intersects the negative horizontal half-line; hence, this is a left supporting line. Therefore, the graph of $GF[f]$ coincides with
2. Comments

Remark 1 After interchanging the factors in the composition, the resulting operator \( FG \) is not an identity one in the classes \( A \) and \( B \). We do not know, for which functions \( f \), it holds that \( FG[f] = f \). One condition can be obtained by applying the following argument. Let denote \( \ell_z \) the line passing through the points \((-z,0)\) and \((0,f(z))\). It is easy to see that \( G[f](t) = \inf_{z>0} \ell_z(t) \). Then the equality \( FG[f] = f \) is equivalent to the fact that each line of the family \( \{\ell_z \mid z > 0\} \) is a right supporting line to the graph of \( G[f] \).

If \( f \) is nonnegative, this condition is equivalent to this requirement for \( f \).

Denote by \( H \) the family of functions \( \varphi(t) = \frac{at}{1+bt}, \ a,b > 0 \). The equality \( FG[f] = f \) holds if and only if \( G[f] \) is not identically zero and, for any pair of points in the graph of \( f \), the arc of a function from the family \( H \) that connects these points lies above the graph of \( f \). Finding more adequate conditions for the equality \( FG[f] = f \) remains an open problem.

3. Some applications

Denote by \( T \) the convex cone in a linear space. Let \( \alpha, \beta \) and \( \gamma \) be nonnegative linear functions on \( T \) such that \( \alpha(t) > 0 \) for all \( t \) from \( T \) not equal to zero. Consider the nonnegative functions

\[
f(x) = \sup \{\gamma(t) \mid \alpha(t) \leq 1, \beta(t) \leq x\}, \quad g(x) = \sup \{\gamma(t) \mid \alpha(t)+x^{-1}\beta(t) \leq 1\}
\]
on \( \mathbb{R}_+ \). Corollary 1 allows to prove the following

Proposition 1 The above functions are connected by the relations \( g = F[f] \) and \( f = G[g] \), where \( F \) and \( G \) are the transforms defined in Section 1.

Proof. It is easy to see that the first function is concave. Therefore, by Corollary 1, it suffices to show that

\[
g(x) = \sup_{r \in (0,1)} rf \left( x \frac{1-r}{r} \right) = \sup_{t>0} \frac{f(xt)}{1+t}, \tag{2}
\]

To prove (2) note first that \( g(x) \geq rf \left( x \frac{1-r}{r} \right) \) for all \( r \in (0,1] \). For any \( \varepsilon > 0 \) there is \( t_\varepsilon \in T \) such that \( \alpha(t_\varepsilon) \leq 1, \beta(t_\varepsilon) \leq x \frac{1-r}{r} \) and \( f \left( x \frac{1-r}{r} \right) \leq
\[ \gamma(t_\varepsilon) + \varepsilon. \] Then \( rt_\varepsilon \in \mathbb{T} \) and \( \alpha(rt_\varepsilon) + x^{-1} \beta(rt_\varepsilon) \leq r + (1-r) \leq 1. \) Hence \( g(x) \geq \gamma(rt_\varepsilon) \geq rf \left( x \frac{1-r}{r} \right) - r \varepsilon. \) Since \( \varepsilon \) is arbitrary, we obtain the required inequality.

For any \( \varepsilon > 0 \) there is \( t_\varepsilon \in \mathbb{T} \setminus \{0\} \) such that \( \alpha(t_\varepsilon) + x^{-1} \beta(t_\varepsilon) \leq 1 \) and \( g(x) \leq \gamma(t_\varepsilon) + \varepsilon. \) Since \( r = \alpha(t_\varepsilon) \in (0,1], \) \( \alpha(t_\varepsilon/r) = 1. \) Then \( \beta(t_\varepsilon/r) \leq x(1-r)/r \) and hence \( \gamma(t_\varepsilon/r) \leq f \left( x \frac{1-r}{r} \right). \) Thus,

\[ g(x) \leq \gamma(t_\varepsilon) + \varepsilon \leq rf \left( x \frac{1-r}{r} \right) + r \varepsilon. \]

It suffices to note that the condition \( r \in (0,1) \) in (2) can be replaced by the condition \( r \in (0,1] \) by the concavity of \( f. \) \( \square \)

**Example 1.** Let \( \mathbb{T} \) be the cone of positive operators on a Hilbert space \( H \) with the inner product \( \langle \cdot, \cdot \rangle \) and the norm \( | \cdot | \). To avoid the technical difficulties, we assume that \( H \) is finite-dimensional.

Given any positive operator \( G \) on \( H \) with a zero lower spectral bound and any \( E > 0, \) the space \( H \) can be equipped with the inner product

\[ \langle \varphi, \psi \rangle^G_E = \langle \varphi, \psi \rangle + \langle \varphi, G \psi \rangle/E, \quad \varphi, \psi \in H \]

and the corresponding norm \( | \varphi \rangle^G_E = \sqrt{\langle \varphi, \varphi \rangle^G_E} \) of any vector \( \varphi \in H. \) Then, for any linear operator \( A \) on \( H, \) the quantity

\[ |||A|||_G^E = \sup \{|A\varphi| \mid \varphi \in H, |\varphi\rangle^G_E \leq 1 \} \]

is the operator norm of \( A \) regarded as an operator from the space \( H \) with the inner product \( \langle \cdot, \cdot \rangle^G_E \) to the space \( H \) with the inner product \( \langle \cdot, \cdot \rangle. \) One can show (see [2]) that

\[ |||A|||_G^E = \sup \{ \sqrt{\text{Tr}AXA^*} \mid X \in \mathbb{T}, \text{Tr}X + E^{-1} \text{Tr}GX \leq 1 \}. \]

For any linear operator \( A \) on \( H, \) we can define the family of norms

\[ ||A||^G_E = \sup \{|A\varphi| \mid \varphi \in H, |\varphi| \leq 1, \langle \varphi, G \varphi \rangle \leq E \}, \quad E > 0, \]

which are useful for a number of problems of mathematical physics [2]. It is proved in [2] that

\[ ||A||^G_E = \sup \{ \sqrt{\text{Tr}AXA^*} \mid X \in \mathbb{T}, \text{Tr}X \leq 1, \text{Tr}GX \leq E \}. \]

\(^1\)Specific properties of these norms are exhibited in the case of an unbounded operator \( G \) and an infinite-dimensional Hilbert space \( H, \) in which these norms define topologies weaker than the topology of the operator norm.
This representation shows that the function $E \to [\|A\|_E^G]^2$ is concave on $\mathbb{R}_+$ and tends to the ordinary operator norm of $A$ as $E \to +\infty$. By Proposition 1 for each operator $A$ on $H$ the nonnegative functions $f_A(E) = [\|A\|_E^G]^2$ and $g_A(E) = [\|||A|||_E^G]^2$ are related by the transforms $F$ and $G$:

$$g_A = F[f_A], \quad f_A = G[g_A].$$

(3)

These relations were obtained in [2, Theorem 3] by applying a rather complicated method involving special properties of the functions $f_A$ and $g_A$. Proposition 1 shows that these relations are a special case of the more general property presented in Corollary 1.

The norms $\| \cdot \|_E^G$ (naturally defined for operators on separable Hilbert spaces in the case of an unbounded operator $G$) were found useful for some problems in the theory of open quantum systems. They were used to prove a generalization of the Kretschmann–Schlingemann–Werner theorem [2, Theorem 2], which made it possible to obtain a Stinespring representation for strongly converging sequences of quantum channels. These norms also arise in describing strongly continuous quantum dynamic semigroups in the case of energy constraints imposed on the states of the quantum system [3].

The norms $||| \cdot |||_E^G$ naturally arise in the theory of relatively bounded operators on infinite-dimensional Hilbert spaces [4, 5].

Relation (3) defines a one-to-one correspondence between the norms $\| \cdot \|_E^G$ and $||| \cdot |||_E^G$ as functions on $\mathbb{R}_+$, which is of interest for both theoretical considerations and physical applications (see [2], the example after Theorem 3).

References

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