Monotonicity of skew information and its applications in quantum resource theory

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Abstract
An alternative proof about the monotonicity of skew information via operator algebra approach is formulated. Furthermore, the strong monotonicity of skew information under particular quantum Trace-Preserving and Completely Positive maps is confirmed. A family of new quantum resource measures is introduced if the resource can be characterized by a resource destroying map and the free operation should be also modified. The proposed measure is easy-calculating and applicable to the quantum coherence resource theory as well as the quantum asymmetry theory. The operational interpretation needs to be further investigated.

Keywords Monotonicity of skew information · Quantum resource theory · Resource destroying map · Resource measure

1 Introduction

Wigner–Yanase skew information [1,2] was introduced originally to express the amount of information in density operator not commuting with the observables. For now there are a lot of generalizations and applications of skew information in quantum information sciences [3–5]. On the other hand, quantum resource theories (QRTs) [6] offer a highly versatile and powerful framework for studying different phenomena in quantum physics, and typical examples are quantum entanglement [7], quantum
coherence [8], quantum reference frames [9] and quantum thermodynamics [10]. A general quantum resource theory consists of a class of “free” states along with a class of “free” or allowable operations as well as a reasonable resource quantifier [11]. The essential resource theoretic condition is that the set of free states is closed under the set of free operations. Hence, any state that is not free is a resource since it cannot be obtained using the allowable operations [12]. From a resource theoretic perspective, the crucial property of resource measures is that they are monotonic under the specified class of operations.

Among various type of resources, one can specify the resource using the resource destroying maps [13]. Recently, Shunlong Luo et al. [5] formulated a generalization of skew information [3] from the viewpoint of state–channel interaction. So it is natural to consider whether we can use this state–channel interaction to quantifying the resource given the resource destroying channel.

In this paper, we first give an alternative proof of the monotonicity of skew information via operator algebra approach and then prove that the skew information is of strong monotonicity, which is an open problem in [4]. Second, we formulate a family of generalized skew information which acted as proper resource measures in the framework of resource destroying map, and the free operation should be also modified in the sense that it does not disturb the resource destroying map.

2 Monotonicity of skew information

Wigner and Yanase [1,2] introduced the notion of skew information of a density operator $\rho$ with respect to a self-adjoint observable $H$,

$$I(\rho, H) = -\frac{1}{2} \text{Tr} \left[ \rho^p, H \right] \left[ \rho^{1-p}, H \right],$$

for $p = \frac{1}{2}$ and Dyson suggested extending this to $p \in (0, 1)$. Recently Shunlong Luo et al. [5] gave some new generalizations. Given a Hilbert space $\mathcal{H}$, let $\mathcal{B}(\mathcal{H})$ denote the set of all operator on $\mathcal{H}$. For any operator $K \in \mathcal{B}(\mathcal{H})$, which needs not to be Hermitian, the skew information $I(\rho, K)$ of $\rho$ with respect to $K$ is defined as [5]:

$$I(\rho, K) = \left\| \left[ \sqrt{\rho}, K \right] \right\|^2 = \text{Tr} \left[ \sqrt{\rho}, K \right] [ \sqrt{\rho}, K \right] = \text{Tr} \left( \rho K^\dagger K \right) + \text{Tr} \left( \rho K K^\dagger \right) - 2 \text{Tr} \left( \sqrt{\rho} K^\dagger \sqrt{\rho} K \right),$$

where $[X, Y] = XY - YX$ for all $X, Y \in \mathcal{B}(\mathcal{H})$ and $\|X\|^2 = \text{Tr}(X^\dagger X)$ is the norm induced by the Hilbert–Schmidt inner product $\langle X, Y \rangle = \text{Tr} X^\dagger Y$. And if $\lambda$ is a Trace-Preserving and Completely Positive (TPCP) map with Kraus operators $K_i$, i.e., $\lambda(\rho) = \sum_i K_i \rho K_i^\dagger$, it is natural to define the state–channel interaction [5] by

$$I(\rho, \lambda) = \sum_i I(\rho, K_i).$$
The quantity \( I(\rho, \lambda) \) enjoys some pleasant properties such as nonnegativity, convexity, monotonicity, and \( I(\rho, \lambda) \) is independent of the choice of Kraus operators of \( \lambda \), see [5] for more details. Among various properties of \( I(\rho, \lambda) \), it is crucial that \( I(\rho, \lambda) \) is monotone under some TPCP map \( \mathcal{E}(\rho) = \sum_n M_n \rho M_n^\dagger \), i.e.,

\[
I(\rho, \lambda) \geq I(\mathcal{E}(\rho), \lambda)
\]

The original proof of monotonicity in [5] is based on Landau–von Neumann equation. In this section, we give an alternative proof of the monotonicity of the skew information via the operator algebra method. And furthermore, we show that \( I(\rho, \lambda) \) is of strong monotonicity, i.e.,

\[
I(\rho, \lambda) \geq \sum_n p_n I(\rho_n, \lambda),
\]

where \( p_n = \text{Tr} M_n \rho M_n^\dagger \) and \( \rho_n = M_n \rho M_n^\dagger / p_n \). To this end, we first recall some lemmas.

**Lemma 1** [14] If \( \varphi \) is a unital positive map, then for every normal element \( a \) in its domain, we have \( \varphi(a^\dagger a) \geq \varphi(a^\dagger) \varphi(a) \) and \( \varphi(a^\dagger a) \geq \varphi(a) \varphi(a^\dagger) \).

**Lemma 2** [15] Assume that \( f : \mathbb{R}^+ \to \mathbb{R} \) is an operator monotone function with \( f(0) \geq 0 \) and \( \alpha : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) is a unital Schwarz mapping, that is, \( \alpha(X^\dagger X) \geq \alpha(X^\dagger) \alpha(X) \) for all \( X \in \mathcal{B}(\mathcal{H}) \), then

\[
S_A^f \left( \alpha^\dagger (\rho_1) || \alpha^\dagger (\rho_2) \right) \geq S_A^{f(\alpha)}(\rho_1, \rho_2)
\]

for \( A \in \mathcal{B}(\mathcal{H}) \) and for invertible density operators \( \rho_1, \rho_2 \in \mathcal{B}(\mathcal{H}) \), where

\[
S_A^f(\rho_1 || \rho_2)
= \langle A \rho_2^{1/2}, f(\Delta(\rho_1/\rho_2)) \left( A \rho_2^{1/2} \right) \rangle
= \text{Tr} \rho_2^{1/2} A^\dagger f(\Delta(\rho_1/\rho_2)) \left( A \rho_2^{1/2} \right),
\]

and \( \Delta(\rho_1/\rho_2) : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) is the linear mapping defined by

\[
\Delta(\rho_1/\rho_2)(X) = \rho_1 X \rho_2^{-1}.
\]

**Lemma 3** For a TPCP map \( \mathcal{E} \) acted as \( \mathcal{E}(\rho) = \sum_n M_n \rho M_n^\dagger \) and an operator \( K \in \mathcal{B}(\mathcal{H}) \), \( [K, M_n] = 0 \) if and only if \( \mathcal{E}^\dagger(K) = K \) and \( \mathcal{E}^\dagger(K^\dagger K) = K^\dagger K \); \( [K^\dagger, M_n] = 0 \) if and only if \( \mathcal{E}^\dagger(K) = K \) and \( \mathcal{E}^\dagger(K K^\dagger) = K K^\dagger \), where \( \mathcal{E}^\dagger \) is the adjoint of \( \mathcal{E} \) in the sense that \( \langle X, \mathcal{E}(Y) \rangle = \langle \mathcal{E}^\dagger(X), Y \rangle \).

**Proof** The necessity is very simple since if \( [M_n, K] = 0 \) for all \( n \) then \( M_n K = K M_n \) and then \( K^\dagger M_n^\dagger = M_n^\dagger K^\dagger \), the conclusion follows.
For the sufficiency, we only need to consider the equality

\[
\sum_n [M_n, K]^\dagger [M_n, K] = K^\dagger K - \mathcal{E}^\dagger \left( K^\dagger K \right) - K^\dagger \mathcal{E}^\dagger (K) + \mathcal{E}^\dagger \left( K^\dagger K \right).
\]

The second statement is followed similarly.

We now in a position to give a proof of the monotonicity as well as strong monotonicity of \( I(\rho, \lambda) \).

\[\square\]

**Theorem 4** For a TPCP map \( \lambda(\rho) = \sum_i K_i \rho K_i^\dagger \), \( I(\rho, \lambda) \) is well defined:

\[ I(\rho, \lambda) = \sum_i I(\rho, K_i). \]

If TPCP map \( \mathcal{E}(\rho) = \sum_n M_n \rho M_n^\dagger \) satisfies \( \mathcal{E}^\dagger (K_i) = K_i \) and \( \mathcal{E}^\dagger (K_i^\dagger K_i) = K_i^\dagger K_i \) as well as \( \mathcal{E}^\dagger (K_i K_i^\dagger) = K_i K_i^\dagger \), then \( I(\rho, \lambda) \) is monotone and strong monotone under the action of \( \mathcal{E} \):

\[ I(\rho, \lambda) \geq I(\mathcal{E}(\rho), \lambda); \]

\[ I(\rho, \lambda) \geq \sum_n p_n I(\rho_n, \lambda), \]

where \( p_n = \text{Tr}M_n \rho M_n^\dagger \) and \( \rho_n = M_n \rho M_n^\dagger / p_n \).

**Proof** Since \( \mathcal{E} \) is a TPCP map, then by Lemma 1, \( \mathcal{E}^\dagger \) is a unital completely positive map and satisfies the Schwarz inequality: \( \mathcal{E}(X^\dagger X) \geq (\mathcal{E}(X^\dagger))^\dagger \mathcal{E}(X) \). For a fixed Kraus operator \( K_i \) of \( \lambda \), \( \mathcal{E}^\dagger (K_i) = K_i \) and \( \rho = \rho_1 = \rho_2 \), consider the function \( f(x) = \sqrt{x} \), and it is easy to see that \( f(x) \) is operator concave as well as operator monotone \[16\], by Lemma 2, we have

\[ \text{Tr} \left( K_i \sqrt{\rho} K_i^\dagger \sqrt{\rho} \right) \leq \text{Tr} \left( K_i \sqrt{\mathcal{E}(\rho)} K_i^\dagger \sqrt{\mathcal{E}(\rho)} \right). \]

Since \( \mathcal{E}^\dagger (K_i^\dagger K_i) = K_i^\dagger K_i \) and \( \mathcal{E}^\dagger (K_i K_i^\dagger) = K_i K_i^\dagger \), by taking inner product with respect to \( \rho \), we have

\[ \text{Tr} \left( K_i \rho K_i^\dagger \right) = \text{Tr} \left( K_i \mathcal{E}(\rho) K_i^\dagger \right), \]

\[ \text{Tr} \left( \rho K_i K_i^\dagger \right) = \text{Tr} \left( \mathcal{E}(\rho) K_i K_i^\dagger \right). \]

Since both these two terms are linear on \( \rho \), we can conclude that
\[ I(\rho, K_i) = \text{Tr} \left( \rho K_i^\dagger K_i \right) + \text{Tr} \left( \rho K_i K_i^\dagger \right) - 2\text{Tr} \left( K_i \sqrt{\rho} K_i^\dagger \sqrt{\rho} \right) \geq I(\mathcal{E}(\rho), K_i), \]

by summing up all \( i \) it must hold that

\[ I(\rho, \lambda) \geq I(\mathcal{E}(\rho), \lambda). \]

Thus, we reproduce the monotonicity result of \( I(\rho, \lambda) \) in [5] via the operator algebra viewpoint. Next, we consider the strong monotonicity of \( I(\rho, \lambda) \).

For the free operation \( \mathcal{E}(\rho) = \sum_n M_n \rho M_n^\dagger \), consider the channel

\[ \tilde{\mathcal{E}}(\rho \otimes |0\rangle \langle 0|) = \sum_n M_n \otimes U_n (\rho \otimes |0\rangle \langle 0|) M_n^\dagger \otimes U_n^\dagger \]

where \( U_n \) is a unitary operator \( \sum_{n=0}^{N-1} |k + n \text{ (mod } N)\rangle \langle k| \) and \( N \) is the number of Kraus operators \( M_n \). It is easy to verify that

\[ \tilde{\mathcal{E}}^\dagger(K_i \otimes I) = K_i \otimes I \]

and

\[ \tilde{\mathcal{E}}^\dagger(K_i^\dagger K_i \otimes I) = K_i^\dagger K_i \otimes I, \]

\[ \tilde{\mathcal{E}}^\dagger(K_i K_i^\dagger \otimes I) = K_i K_i^\dagger \otimes I, \]

by the monotonicity of \( I(\rho, \lambda) \), and therefore,

\[ I(\rho, \lambda) \]
\[ = I(\rho \otimes |0\rangle \langle 0|, \lambda \otimes I) \]
\[ \geq I \left( \sum_n M_n \rho M_n^\dagger \otimes |n\rangle \langle n|, \lambda \otimes I \right) \]
\[ = \sum_n p_n I(\rho_n, \lambda). \]

This finished the proof. \( \square \)

In [4], the authors showed that for Hermitian operator \( H \), \( I(\rho, H) \) is monotone under a TPCP map \( \mathcal{E}(\rho) = \sum_n M_n \rho M_n^\dagger \):

\[ I(\rho, H) \geq I(\mathcal{E}(\rho), H) \]
as long as $E$ satisfies $E^\dagger(H) = H$ and $E^\dagger(H^2) = H^2$, or equivalently, $[H, M_n] = 0$ for all $n$, but they only proved that $I(\rho, H)$ satisfies the strong monotonicity in two-dimensional case. In fact, by same line reasoning in Theorem 4, we can show that $I(\rho, H)$ satisfies the strong monotonicity in general. Virtually,

$$I(\rho, H) = I(\rho \otimes |0\rangle\langle 0|, H \otimes I) \geq I \left( \tilde{E}(\rho \otimes |0\rangle\langle 0|, H \otimes I) \right) = I \left( \sum_n M_n \rho M_n^\dagger \otimes |n\rangle\langle n|, H \otimes I \right) = \sum_n p_n I(\rho_n, H).$$

Hence, we solve the open problem left in [4].

We can even do further. In [17], the authors formulated a family of functions

$$g_p(x) = \begin{cases} \frac{1}{p(1-p)}(x - x^p) & p \neq 1 \\ x \log x & p = 1. \end{cases}$$

where $x > 0$ and $p \in (0, 2]$. For strictly positive $A, B$, they define [17]

$$J_p(K, A, B) = \left( K\sqrt{B} , g_p(\Delta(A/B)) \left( K\sqrt{B} \right) \right) = \begin{cases} \frac{1}{p(1-p)}(\text{Tr}K^\dagger AK - \text{Tr} \left( K^\dagger A^p KB^{1-p} \right)) & p \in (0, 1) \cup (1, 2), \\ \text{Tr}K K^\dagger A \log A - \text{Tr}K^\dagger AK \log B & p = 1, \\ -\frac{1}{2} \left( \text{Tr}K^\dagger AK - \text{Tr}AKB^{-1}K^\dagger A \right) & p = 2. \end{cases}$$

One can see that this definition generalizes the relative entropy as well as the skew information. In fact, when $p = 1$ and $K = I$, $J_p(K, A, B)$ reduces to the usual relative entropy, i.e.,

$$J_1(I, A, B) = \text{Tr} \left( A \log A - A \log B \right).$$

And when $p = \frac{1}{2}$, $K = K^\dagger$ and $A = B$,

$$J_{\frac{1}{2}}(K, A, A) = -2\text{Tr} \left[ K, A^\frac{1}{2} \right] \left[ K, A^\frac{1}{2} \right],$$

which yields the original skew information up to a constant.

Since $J_p(K, A, B)$ enjoys some pleasant properties like skew information, it is natural to extend our previous discussion by utilizing this unified entropy.
Define
\[ I_p(\rho, K) = J_p(K, \rho, \rho) \]
for operator \( K \) and density operator \( \rho \) and \( p \in \left[ \frac{1}{2}, 1 \right] \), and when a TPCP map \( \lambda \) written as \( \lambda(\rho) = \sum_i K_i \rho K_i^\dagger \), we define
\[ I_p(\rho, \lambda) = \sum_i I_p(\rho, K_i) \]
for \( p \in \left[ \frac{1}{2}, 1 \right] \). If \( p = \frac{1}{2} \), \( I_p(\rho, \lambda) \) is very similar to \( I(\rho, \lambda) \) in Theorem 4, but there are several subtleties due to the linear term. As a consequence, we only need to require that \( E^\dagger(K K^\dagger) = K K^\dagger \), or equivalently, \([K, M_n^\dagger] = 0 \) for all \( n \). We must show that \( I_p(\rho, \lambda) \) share the same properties as \( I(\rho, \lambda) \) when \( p \in \left[ \frac{1}{2}, 1 \right] \). We first show that \( I_p(\rho, \lambda) \) is independent of the Kraus operator representations and thus \( I_p(\rho, \lambda) \) is indeed a quantity about \( \rho \) and channel \( \lambda \). In fact, if \( \lambda(\rho) = \sum_i E_i \rho E_i^\dagger = \sum_j K_i \rho K_j^\dagger \)
for all density operator \( \rho \), then there exists a unitary matrix \( U = (u_{ij}) \) such that \( E_i = \sum_j u_{ij} K_j \). By some direct calculation, one can see that \( I_p(\rho, \lambda) = \sum_i I_p(\rho, E_i) = \sum_j I_p(\rho, K_j) \), which is independent of the Kraus operators of \( \lambda \). The nonnegativity and convexity of \( I_p(\rho, \lambda) \) can be found in [17]. For \( p \in (0, 1) \), the function \( f(x) = x^p \) is operator monotone as well as operator concave [16]. So the monotonicity and strong monotonicity of \( I_p(\rho, \lambda) \) under TPCP map \( E(\rho) = \sum_n M_n \rho M_n^\dagger \) follow from the same line in Theorem 4. However, there does not exist any discussion about the monotonicity when \( 1 \leq p \leq 2 \). Since in this case, the function \( f(x) = x^p \) is a operator convex function, the method in Lemma 2 is not available anymore. We leave it as an open problem. For now, we can conclude the following theorem.

**Theorem 5** For a TPCP map \( \lambda(\rho) = \sum_i K_i \rho K_i^\dagger \) with Kraus operators \( K_i \), the other TPCP map \( E(\rho) = \sum_n M_n \rho M_n^\dagger \) satisfies \( E^\dagger(K_i K_i^\dagger) = K_i K_i^\dagger \) and \( E^\dagger(K_i K_i^\dagger) = K_i K_i^\dagger \) for all \( i \) and \( p \in (0, 1) \), the quantity
\[ I_p(\rho, \lambda) = \sum_i I_p(\rho, K_i) \]
is nonnegative, convex on \( \rho \) and monotonicity under the action of \( E \). That is, \( I(\rho, \lambda) \geq 0 \) with equality if and only if \( \lambda(\rho) = \rho \), \( I(\rho, \lambda) \) is convex on \( \rho \) and more crucially,
\[ I_p(\rho, \lambda) \geq I_p(E(\rho), \lambda). \]

Furthermore, it is of strong monotonicity:
\[ I_p(\rho, \lambda) \geq \sum_n p_n I_p(\rho_n, \lambda), \]
where \( p_n = \text{Tr} M_n \rho M_n^\dagger \) and \( \rho_n = M_n \rho M_n^\dagger / p_n \).
3 Applications in resource framework

Given a Hilbert space $\mathcal{H}$, we say that a TPCP map $\lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a resource destroying map [13] if it satisfies the following two properties:

1. It maps any free state $\rho \in \mathcal{F}(\mathcal{H})$ to itself, i.e., $\lambda(\rho) = \rho$.
2. It maps any (possibly not free) density operator $\rho$ to a free state; i.e., $\lambda(\rho) \in \mathcal{F}(\mathcal{H})$.

From its definition, it is not clear at all that a resource destroying map exists for a given QRT. However, the full necessary and sufficient conditions for the existence of a resource destroying map were derived in [18].

In this section, we show that the skew information $I(\rho, \lambda)$ and its generalization $I_p(\rho, \lambda)$ can be acted as a proper resource measure with respect to resource destroying map $\lambda$. The typical example is in quantum asymmetry theory. The resource theory of asymmetry with respect to a given representation of a symmetry group $G$ has been used extensively to distinguish and quantify the symmetry-breaking properties of both the states and the operations [19,20]. The asymmetry theory can be described by a resource destroying map [13], namely

$$\lambda(\rho) = \int d\mu U_g \rho U_g^\dagger,$$

where $U_g$ is the unitary representation of $g \in G$ and $d\mu$ is the Haar measure with respect to $G$. For simplicity, we confine our focus on finite group $G$ but note that the conclusions hold also for compact Lie group. The symmetric state of the free state is that invariant under $\lambda$, i.e.,

$$\mathcal{F} = \{ \rho \mid \lambda(\rho) = \rho \}.$$

An equivalently characterization is that [19]

$$\mathcal{F} = \left\{ \rho \mid U_g \rho U_g^\dagger = \rho \text{ for all } g \in G \right\}.$$

The free operation we consider in this paper is the TPCP map $\mathcal{E}(\rho) = \sum_n M_n \rho M_n^\dagger$ such that $\mathcal{E}^\dagger(U_g \rho U_g^\dagger) = U_g \mathcal{E}^\dagger(U_g \rho U_g^\dagger) = U_g$ for all $g \in G$. Since $U_g$ are unitary and thus normal, the second type of condition $\mathcal{E}^\dagger(U_g \mathcal{E}^\dagger(U_g \rho U_g^\dagger) = U_g \mathcal{E}^\dagger(U_g \rho U_g^\dagger) = I$ is satisfied automatically. By Lemma 3, it holds that $[M_n, U_g] = 0$ for all $n$ and all $g \in G$. Therefore, the generalized skew information between state and channel $I_p(\rho, \lambda)$ can be served as a proper resource measure of asymmetry due to Theorem 5.

Another typical example is the resource theory of coherence [8,21–23]. In this case, the resource destroying map is the dephasing map

$$\Delta(\rho) = \sum_i |i\rangle\langle i| \rho |i\rangle\langle i|$$

for a fixed basis $\{|i\rangle\}$. The Kraus operators are $K_i = |i\rangle\langle i|$, and for $p = \frac{1}{2}$, $I_{\frac{1}{2}}(\rho, \Delta)$ reproduces the result in [23] up to a constant.
4 Discussion

In this paper, we formulated a family of resource measure extended by skew information when the resource theory can be characterized by a resource destroying map and for a class of particular free operations. The new measure acted well in asymmetry theory and coherence theory. There are also some unsolved problems. The first one is how to generalize our measure to meet a wider range of free operations. After all, the requirement for commute relation between Kraus operators seems to be severe. Second problem is to consider the monotonicity of $I_p(\rho, \lambda)$ under the TPCP maps when $p \in (1, 2]$. Recently, the author in [24] demonstrated the monotonicity of quantum quasi-entropy under partial trace, but it is different from our present open problem. It is expected that a similar operator algebra approach should be applied. The last question is to figure out the relationship between our measure and the measure introduced in [25].

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