A review on large $k$ minimal spectral $k$-partitions and Pleijel’s Theorem.

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Abstract

In this survey, we review the properties of minimal spectral $k$-partitions in the two-dimensional case and revisit their connections with Pleijel’s Theorem. We focus on the large $k$ problem (and the hexagonal conjecture) in connection with two recent preprints by J. Bourgain and S. Steinerberger on the Pleijel Theorem. This leads us also to discuss some conjecture by I. Polterovich, in relation with square tilings. We also establish a Pleijel Theorem for Aharonov-Bohm Hamiltonians and deduce from it, via the magnetic characterization of the minimal partitions, some lower bound for the number of critical points of a minimal partition.

1 Introduction

We consider the Dirichlet Laplacian in a bounded regular domain $\Omega \subset \mathbb{R}^2$. In [18] we have started to analyze the relations between the nodal domains of the real-valued eigenfunctions of this Laplacian and the partitions of $\Omega$ by $k$ open sets $D_i$ which are minimal in the sense that the maximum over the $D_i$’s of the ground state energy of the Dirichlet realization of the Laplacian in $D_i$ is minimal. We denote by $\lambda_j(\Omega)$ the increasing sequence of its eigenvalues and by $\phi_j$ some associated orthonormal basis of real-valued eigenfunctions. The groundstate $\phi_1$ can be chosen to be strictly positive in $\Omega$, but the other eigenfunctions $\phi_k$ must have zerosets. For any real-valued continuous function $u$ on $\Omega$, we define the zero set as

$$N(u) = \{x \in \Omega \mid u(x) = 0\}$$ (1.1)

and call the components of $\Omega \setminus N(u)$ the nodal domains of $u$. The number of nodal domains of $u$ is called $\mu(u)$. These $\mu(u)$ nodal domains define a $k$-partition of $\Omega$, with $k = \mu(u)$.

1The ground state energy is the smallest eigenvalue.
We recall that the Courant nodal Theorem [11] (1923) says that, for $k \geq 1$, and if $\lambda_k$ denotes the $k$-th eigenvalue of the Dirichlet Laplacian in $\Omega$ and $E(\lambda_k)$ the eigenspace associated with $\lambda_k$, then, for all real-valued $u \in E(\lambda_k) \setminus \{0\}$, $\mu(u) \leq k$. In dimension 1 the Sturm-Liouville theory says that we have always equality (for Dirichlet in a bounded interval) in the previous theorem (this is what we will call later a Courant-sharp situation).

A theorem due to Pleijel [20] in 1956 says that this cannot be true when the dimension (here we consider the 2D-case) is larger than one. The proof involves lower bounds for the energy of nodal partitions but what is only used is actually that the ground state energy in each of the domain of the partition is the same. This is this link between the proof of Pleijel’s Theorem and the lower bounds for the energy of a partition that we would like to explore in this survey, motivated by two recent contributions of J. Bourgain and S. Steinerberger. We will focus on the large $k$ problem (and the hexagonal conjecture). This leads us also to discuss some conjecture by I. Polterovich [22], which could be the consequence of a “square” conjecture for some still unknown subclass of partitions. We finally discuss some new consequence of the magnetic characterization of minimal partitions [10] for the critical points of this partition.

2 A reminder on minimal spectral partitions

We now introduce for $k \in \mathbb{N}$ ($k \geq 1$), the notion of $k$-partition. We call $k$-partition of $\Omega$ a family $\mathcal{D} = \{D_i\}_{i=1}^k$ of mutually disjoint sets in $\Omega$. We call it open if the $D_i$ are open sets of $\Omega$, connected if the $D_i$ are connected. We denote by $\mathcal{O}_k(\Omega)$ the set of open connected partitions of $\Omega$. We now introduce the notion of spectral minimal partition sequence.

**Definition 2.1**

For any integer $k \geq 1$, and for $\mathcal{D}$ in $\mathcal{O}_k(\Omega)$, we introduce

$$\Lambda(\mathcal{D}) = \max_i \lambda(D_i). \quad (2.1)$$

Then we define

$$\mathcal{L}_k(\Omega) = \inf_{\mathcal{D} \in \mathcal{O}_k} \Lambda(\mathcal{D}). \quad (2.2)$$

and call $\mathcal{D} \in \mathcal{O}_k$ a minimal $k$-partition if $\mathcal{L}_k = \Lambda(\mathcal{D})$.

More generally we can define, for $p \in [1, +\infty)$, $\Lambda^p(\mathcal{D})$ and $\mathcal{L}_{k,p}(\Omega)$ by:

$$\Lambda^p(\mathcal{D}) := \left(\frac{\sum \lambda(D_i)^p}{k}\right)^{\frac{1}{p}}, \quad \mathcal{L}_{k,p}(\Omega) = \inf_{\mathcal{D} \in \mathcal{O}_k} \Lambda^p(\mathcal{D}). \quad (2.3)$$
Note that we can minimize over non necessarily connected partitions and get the connectedness of the minimal partitions as a property (see [18]).

If \( k = 2 \), it is rather well known that \( \mathcal{Q}_2 = \lambda_2 \) and that the associated minimal 2-partition is a **nodal partition**, i.e. a partition whose elements are the nodal domains of some eigenfunction corresponding to \( \lambda_2 \).

A partition \( \mathcal{D} = \{D_i\}_{i=1}^k \) of \( \Omega \) is called **strong** if

\[
\text{Int} \left( \bigcup_{i} D_i \right) \setminus \partial \Omega = \Omega.
\]

(2.4)

Attached to a strong partition, we associate a closed set in \( \overline{\Omega} \), which is called the **boundary set** of the partition :

\[
N(\mathcal{D}) = \bigcup_{i} (\partial D_i \cap \Omega).
\]

(2.5)

\( N(\mathcal{D}) \) plays the role of the nodal set (in the case of a nodal partition).

This suggests the following definition of regularity for a partition:

**Definition 2.2**

We call a partition \( \mathcal{D} \) regular if its associated boundary set \( N(\mathcal{D}) \), has the following properties :

(i) Except for finitely many distinct \( x_i \in \Omega \cap N \) in the neighborhood of which \( N \) is the union of \( \nu_i = \nu(x_i) \) smooth curves \((\nu_i \geq 3)\) with one end at \( x_i \), \( N \) is locally diffeomorphic to a regular curve.

(ii) \( \partial \Omega \cap N \) consists of a (possibly empty) finite set of points \( z_i \). Moreover \( N \) is near \( z_i \) the union of \( \rho_i \) distinct smooth half-curves which hit \( z_i \).

(iii) \( N \) has the **equal angle meeting property**

The \( x_i \) are called the critical points and define the set \( X(N) \). A particular role is played by \( X^{\text{odd}}(N) \) corresponding to the critical points for which \( \nu_i \) is odd. By **equal angle meeting property**, we mean that the half curves meet with equal angle at each critical point of \( N \) and also at the boundary together with the tangent to the boundary.

We say that two elements of the partition \( D_i, D_j \) are **neighbors** and write \( D_i \sim D_j \), if \( D_{ij} := \text{Int} \left( \overline{D_i \cup D_j} \right) \setminus \partial \Omega \) is connected. We associate with each \( \mathcal{D} \) a **graph** \( G(\mathcal{D}) \) by associating with each \( D_i \) a vertex and to each pair \( D_i \sim D_j \) an edge. We will say that the graph is **bipartite** if it can be colored by two colors (two neighbors having two different colors). We recall that the graph associated with a collection of nodal domains of an eigenfunction is always bipartite.
3 Pleijel’s Theorem revisited

Pleijel’s Theorem as stated in the introduction is the consequence of a more precise theorem and the aim of this section is to present a formalized proof of the historical statement permitting to understand recent improvements and to formulate conjectures.

Generally, the classical proof is going through the proposition

**Proposition 3.1**

\[
\limsup_{n \to +\infty} \frac{\mu(\phi_n)}{n} \leq \frac{4\pi}{A(\Omega) \liminf_{k \to +\infty} \frac{\Sigma_k(\Omega)}{k}}.
\]

Here \(\mu(\phi_n)\) is the cardinal of the nodal components of \(\Omega \setminus N(\phi_n)\) and \(A(\Omega)\) denotes the area of \(\Omega\).

Then one establishes a lower bound for \(A(\Omega) \liminf_{k \to +\infty} \frac{\Sigma_k(\Omega)}{k}\), which should be \(> 4\pi\). This property is deduced in [20] from the Faber-Krahn inequality which says:

\[
(\text{Faber – Krahn}) \quad A(D) \lambda(D) \geq \lambda(Disk_1),
\]

for any open set \(D\). Here \(Disk_1\) denotes the disk of area 1.

Behind the statement of Proposition 3.1 we have actually the stronger proposition:

**Proposition 3.2**

\[
\limsup_{n \to +\infty} \frac{\mu(\phi_n)}{n} \leq \frac{4\pi}{A(\Omega) \liminf_{k \to +\infty} \frac{L_k(\Omega)}{k}}.
\]

Here \(L_k(\Omega)\) denotes the smallest eigenvalue (if any) for which there exists an eigenfunction in \(E(L_k)\) with \(k\) nodal domains. If no eigenvalue has this property, we simply write \(L_k(\Omega) = +\infty\).

The proof of Proposition 3.2 is immediate observing first that for any sub-sequence \(n_\ell\), we have

\[
\frac{\lambda_{n_\ell}}{n_\ell} \geq \frac{L_{\mu(\phi_{n_\ell})}}{\mu(\phi_{n_\ell})} = \frac{L_{\mu(\phi_{n_\ell})}}{\mu(\phi_{n_\ell})} \cdot \frac{\mu(\phi_{n_\ell})}{n_\ell}.
\]

If we choose the subsequence \(n_\ell\) such that

\[
\lim_{\ell \to +\infty} \frac{\mu(\phi_{n_\ell})}{n_\ell} = \limsup_{n \to +\infty} \frac{\mu(\phi_n)}{n},
\]

we observe that, by Weyl’s formula, we have:

\[ N(\lambda) \sim \frac{A(\Omega)}{4\pi} \lambda, \]

which implies

\[ \lim_{\ell \to +\infty} \frac{\lambda_{n\ell}}{n\ell} = \frac{4\pi}{A(\Omega)}. \]

We also have

\[ \liminf_{\ell \to +\infty} \frac{L_{\mu(\phi_{n\ell})}}{\mu(\phi_{n\ell})} \geq \liminf_{k \to +\infty} \frac{L_k(\Omega)}{k}. \]

Hence we get the proposition. □

Proposition 3.1 is deduced from Proposition 3.2 by observing that it was established in [18] that

\[ \lambda_k(\Omega) \leq L_k(\Omega) \leq L'_{k}(\Omega). \]

(3.5)

The left hand side inequality is a consequence of the variational characterization of \( \lambda_k \) and the right hand side is an immediate consequence of the definitions. Moreover, and this is a much deeper theorem of [18], the equalities \( L_k(\Omega) = L_k(\Omega) \) or \( L_k(\Omega) = \lambda_k(\Omega) \) imply \( L_k(\Omega) = L_k(\Omega) = \lambda_k(\Omega) \). We say that, in this case, we are in a Courant sharp situation.

If we think that only nodal partitions are involved in Pleijel’s Theorem, it could be natural to introduce \( L'_k(\Omega) \) where we take the infimum over a smaller non-empty class of \( k \)-partitions \( D = (D_1, \cdots, D_k) \). We call \( O'_k \) this yet undefined class, which should contain all the nodal \( k \)-partitions, if any. Natural candidates for \( O'_k \) will be discussed in Section 6.

Definition 3.3

\[ L'_k(\Omega) := \inf_{D\in O'_k} \max_{D_i} \lambda(D_i). \]

(3.6)

Of course we have always

\[ \lambda_k(\Omega) \leq L_k(\Omega) \leq L'_k(\Omega) \leq L_k(\Omega). \]

(3.7)

Hence we have:

**Proposition 3.4**

\[ \limsup_{n \to +\infty} \frac{\mu(\phi_n)}{n} \leq \frac{4\pi}{A(\Omega)} \liminf_{k \to +\infty} \frac{L'_k(\Omega)}{k}. \]

(3.8)
Hence this is the right hand side of (3.8) which seems to be interesting to analyze.
It is clear from (3.7) that all these upper bounds are less than one, which corresponds to a weak asymptotic version of Courant’s Theorem.

We now come back to the proof by Pleijel of his theorem. We apply the Faber-Krahn inequality (3.2) to any element $D_i$ of the minimal $k$-partition $D$, and summing up, we immediately get:

$$A(\Omega) \frac{\mathcal{L}_k(\Omega)}{k} \geq \lambda(Disk_1).$$

(3.9)

Implementing this inequality in Proposition 3.2 we immediately get:

**Theorem 3.5 (Pleijel)**

$$\limsup_{n \to +\infty} \frac{\mu(\phi_n)}{n} \leq \nu_{Pl},$$

(3.10)

with

$$\nu_{Pl} = \frac{4\pi}{\lambda(Disk_1)} \sim 0.691.$$

**Remarks 3.6**

(i) We note that the proof of Pleijel uses only a weak form of (3.9), where $\mathcal{L}_k$ is replaced by $L_k$.

(ii) Note that the same result is true in the Neumann case (Polterovich [22]) under some analyticity assumption on the boundary. Note also that computations for the square were already presented in [20].

(iii) Note that we have the better:

$$A(\Omega) \frac{\mathcal{L}_{k,1}(\Omega)}{k} \geq \lambda(Disk_1).$$

(3.11)

But this improvement has no effect on Pleijel’s Theorem. In particular, we recall that we do not have necessarily $\lambda_k(\Omega) \leq \mathcal{L}_{k,1}(\Omega)$ (take $k = 2$ and use the criterion of Helffer–Hoffmann-Ostenhof [14]). This inequality can be replaced (see [14]) by:

$$\mathcal{L}_{k,1}(\Omega) \geq \frac{1}{k} \sum_{\ell=1}^{k} \lambda_{\ell}(\Omega).$$

(3.12)

Again a Weyl asymptotic shows that this last inequality is strict for $k$ large. We have indeed as $k \to +\infty$

$$\frac{1}{k} \sum_{\ell=1}^{k} \lambda_{\ell}(\Omega) \sim \frac{2\pi k}{A(\Omega)},$$

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to compare with (3.11).

(iv) Pleijel’s Theorem is valid in the case of the Laplace-Beltrami operator on a compact manifold (see some survey inside [3] or [2]).

Around the Hexagonal conjecture

It is rather easy (see [8]) using hexagonal tilings to prove that:

$$A(\Omega) \liminf_{k \to +\infty} \frac{\Sigma_k(\Omega)}{k} \leq A(\Omega) \limsup_{k \to +\infty} \frac{\Sigma_k(\Omega)}{k} \leq \lambda(Hexa_1),$$

(3.13)

where $Hexa_1$ is the regular hexagon of area 1.

Note that any tiling leads to a similar upper bound but $\lambda(Hexa_1)$ gives to our knowledge the smallest lower bound for a fundamental cell of area 1. Here are a few numerical (sometimes exact) values corresponding to the $Hexa_1$, $T_1$, and $Sq_1$ being respectively a regular hexagon, a square of area 1 and an equilateral triangle:

$$\lambda_1(Hexa_1) \sim 18.5901, \lambda_1(Sq_1) = 2\pi^2 \sim 19.7392, \lambda_1(T_1) \sim 22.7929.$$  

(3.14)

In addition it is not known that the regular hexagon with area 1 has the lowest groundstate eigenvalue among all hexagons of the same area. (famous conjecture of Polya and Szegő).

A now well known conjecture (hexagonal conjecture) (Van den Berg, Caffarelli-Lin [10]) was discussed in Helffer–Hoffmann-Ostenhof–Terracini [18], Bonnaillie-Helffer-Vial [8], Bourdin-Bucur-Oudet [6] and reads as follows:

**Conjecture 3.7**

$$A(\Omega) \liminf_{k \to +\infty} \frac{\Sigma_k(\Omega)}{k} = A(\Omega) \limsup_{k \to +\infty} \frac{\Sigma_k(\Omega)}{k} = \lambda(Hexa_1)$$

(3.15)

The minimal partitions corresponding to $\Sigma_{k,1}$ were computed for the torus by Bourdin-Bucur-Oudet [6]. This conjecture would lead to the conjecture that in Pleijel’s estimate we have actually:

**Conjecture 3.8** (Hexagonal conjecture for Pleijel)

$$A(\Omega) \limsup_{n \to +\infty} \frac{\mu(\phi_n)}{n} \leq \nu_{Hex},$$

(3.16)

with

$$\nu_{Hex} = \frac{4\pi}{\lambda(Hexa_1)} \sim 0.677.$$

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We note indeed that
\[ \frac{\nu_{\text{Hex}}}{\nu_{\text{Pl}}} = \frac{\lambda(\text{Disk}_1)}{\lambda(\text{Hexa}_1)} \sim 0.977. \]

We now come back to the enigmatic Proposition 3.4 and consider the asymptotic behavior of \( \frac{\alpha_k^k(\Omega)}{k} \) as \( k \to +\infty \). A first indication that our choice of \( \Omega^\# \) is promising would be to show that the following property holds.

**Property 3.9**
\[ A(\Omega) \liminf_{k \to +\infty} \frac{\alpha_k^k(\Omega)}{k} \geq \lambda(\text{Sq}_1). \] (3.17)
where \( \text{Sq}_1 \) denotes the unit square.

The proof of this property should mimic what was done for \( \Omega_k(\Omega) \) (see for example [8] or [10]), but replacing hexagonal tilings by square tilings.

The philosophy behind the choice of \( \Omega_k^\# \) should be the following: the hexagonal conjecture for \( k \)-partitions should be replaced by the square conjecture when bipartite \( k \)-partitions are involved because square tilings can be colored by two colors with the rule that two neighbors have two different colors.

Note that the existence of classes \( \Omega_k^\# (k \in \mathbb{N}^*) \) such that Property 3.9 is satisfied would give a proof of the conjecture:

**Conjecture 3.10 (Polterovich)**
\[ \limsup_{n \to +\infty} \frac{\mu(\phi_n)}{n} \leq \frac{4\pi}{\lambda(\text{Sq}_1)} = \frac{2}{\pi}. \] (3.18)
This conjecture is due to Iosif Polterovich [22] on the basis of computations for the rectangle of Blum-Gutzman-Smilansky [5]. Due to the computations on the square [20] (see however the discussion in Section 5), together with computations for the rectangle [23], this should be the optimal result.

4 Improving the use of the Faber-Krahn Inequality

4.1 Preliminaries

This section is devoted to reporting on the two previously mentioned results by J. Bourgain and S. Steinerberger. Although not explicitly written in this way, the goal of Bourgain [7] and Steinerberger [24] was to improve Pleijel’s proof by improving the lower bound of lim inf$_{k \to +\infty} L_k(\Omega)$. Bourgain gives a rough estimate of his improvement with a size of $10^{-9}$.

In any case, it is clear from their proof that this will lead to a statement where in (3.10) $\nu_{Pl}$ is replaced by

$$\nu_{Hex} \leq \nu_{Bo} < \nu_{Pl},$$

and

$$\nu_{Hex} \leq \nu_{St} < \nu_{Pl},$$

where $\nu_{Bo}$ and $\nu_{St}$ are the constants obtained respectively by Bourgain and Steinerberger.

4.2 Bourgain’s improvement [7]

One ingredient is a refinement of the Faber-Krahn inequality due to Hansen-Nadirashvili [13]:

**Lemma 4.1 (Hansen-Nadirashvili)**

For a nonempty simply connected bounded domain $\Omega \subset \mathbb{R}^2$, we have

$$A(\Omega) \lambda(\Omega) \geq \left( 1 + \frac{1}{250} \left( 1 - \frac{r_i(\Omega)}{r_0(\Omega)} \right)^2 \right) \lambda(\text{Disk}_1), \quad (4.1)$$

with $r_0(\Omega)$ the radius of the disk of same area as $\Omega$ and $r_i(\Omega)$ the inradius of $\Omega$. 
Actually, J. Bourgain needs (and gives) an additional argument for treating non simply connected domains. In the right hand side of (4.1), not only the inradius occurs but also the smallest area of the components of \( \mathbb{R}^2 \setminus \Omega \).

The other very tricky idea is to use quantitatively that all the open sets of the partition cannot be very close to disks (packing density) (see Blind [4]).

The inequality obtained by Bourgain is the following (see (37) in [7]) as \( k \to +\infty \), is that for any \( \delta \in (0, \delta_0) \)
\[
\frac{\Sigma_k(\Omega)}{k} \geq (1 + o(1))\lambda(\text{Disk}_1)A(\Omega)^{-1} \times b(\delta) \tag{4.2}
\]
where
\[
b(\delta) := (1 + 250\delta^{-3})(\frac{\pi}{\sqrt{12}}(1 - \delta)^{-2} + 250\delta^{-3})^{-1}.
\]
and \( \delta_0 \in (0,1) \) is computed with the help of the packing condition. This condition reads
\[
\frac{\delta_0^2}{250} = \left(\frac{1 - \delta_0}{p}\right)^2 - 1,
\]
where \( p \) is a packing constant determined by Blind [4] (\( p \sim 0.743 \)).

But for \( \delta > 0 \) small enough, we get \( b(\delta) > 1 \) (as a consequence of \( \frac{\pi}{\sqrt{12}} < 1 \)), hence Bourgain has improved what was obtained via Faber-Krahn (see (3.9)).

As also observed by Steinerberger, one gets
\[
\frac{\lambda(\text{Hexa}_1)}{\lambda(\text{Disk}_1)} \geq \sup_{\delta \in (0,\delta_0)} b(\delta) > 1,
\]
which gives a limit for any improvement of the estimate. In any case, we have
\[
\liminf_{k \to +\infty} \frac{\Sigma_k(\Omega)}{k} \geq \lambda(\text{Disk}_1)A(\Omega)^{-1} \times \sup_{\delta \in (0,\delta_0)} b(\delta). \tag{4.3}
\]

\[ \]

4.3 The uncertainty principle by S. Steinerberger

To explain this principle, we associate with a partition \( D = (\Omega_i) \) of \( \Omega \)
\[
D(\Omega_i) = 1 - \frac{\min_j A(\Omega_j)}{A(\Omega_i)}.
\]

We also need to define for an open set \( D \) with finite area, the Fraenkel asymmetry of \( D \):
\[
A(D) = \inf_B A(D \Delta B) \frac{A(\Omega)}{A(D)}.
\]
where the infimum is over the balls of same area and where

$$D \triangle B = (D \setminus B) \cup (B \setminus D).$$

Steinerberger’s uncertainty principle reads:

**Theorem 4.2**

There exists a universal constant $c > 0$, and a $k_0(\Omega)$ such that for each $k$-partition of $\Omega$: $D = (\Omega_i)_{i=1,\ldots,k}$, with $k \geq k_0(\Omega)$,

$$\sum_i (D(\Omega_i) + A(\Omega_i)) \frac{A(\Omega_i)}{A(\Omega)} \geq c. \quad (4.4)$$

### 4.4 Application to equipartitions of energy $\lambda$

Let us show how we recover a lower bound for $\liminf_{k \to +\infty} (\mathcal{L}_k(\Omega)/k)$ improving (3.9) asymptotically. We consider a $k$-equipartition of energy $\lambda$. We recall from [3] that an equipartition is a strong partition for which the ground state energy in each open set $D_i$ is the same. In particular, nodal partitions and minimal partitions for $\mathcal{L}_k$ are typical examples of equipartitions. If we assume that $k \geq k_0(\Omega)$, the uncertainty principle says that its is enough to consider two cases.

We first assume that

$$\sum_i D(\Omega_i) \frac{A(\Omega_i)}{A(\Omega)} \geq \frac{c}{2}.$$ 

We can rewrite this inequality in the form:

$$k \inf_j A(\Omega_j) \leq (1 - \frac{c}{2})A(\Omega).$$

After implementation of Faber-Krahn, we obtain

$$\frac{k}{\lambda} \lambda(Disk_1) \leq (1 - \frac{c}{2})A(\Omega). \quad (4.5)$$

We now assume that

$$\sum_i A(\Omega_i) \frac{A(\Omega_i)}{A(\Omega)} \geq \frac{c}{2}.$$ 

This assumption implies, using that $A(\Omega_i) \leq 2$,

$$A\left(\bigcup\{A(\Omega_i) \geq \frac{c}{2}\}\right) \geq \frac{c}{6} A(\Omega). \quad (4.6)$$

The role of $A$ can be understood from the following inequality due to Brasco, De Philippis, and Velichkov [9]:

There exists $C > 0$ such that, for any open set $\omega$ with finite area,

$$A(\omega)\lambda(\omega) - \lambda(Disk_1) \geq C A(\omega)^2 \lambda(Disk_1). \quad (4.7)$$
If we apply this inequality with $\omega = \Omega$, it reads

$$A(\Omega) \lambda - \lambda(Disk_1) \geq CA(\Omega)^2 \lambda(Disk_1).$$

Hence we get for any $i$ such that $A(\Omega) \geq \frac{c}{6}$, the inequality

$$\lambda(Disk_1)(1 + \frac{Cc^2}{36}) \leq A(\Omega) \lambda,$$

which is an improvement of Faber-Krahn for these $\Omega_i$'s.

Summing over $i$ and using (4.6) leads to

$$\frac{k}{\lambda} \lambda(Disk_1) \leq (1 + \frac{Cc^2}{36})^{-1} A(\Omega) \left(1 + (1 - \frac{c}{6}) \frac{Cc^2}{36}\right),$$

and finally to

$$\frac{k}{\lambda} \lambda(Disk_1) \leq \left(1 - \frac{Cc^3}{216 + 6Cc^2}\right) A(\Omega).$$

Putting (4.5) and (4.8) together, we obtain that for $k \geq k_0(\Omega)$ (as assumed from the beginning) the $k$-partition satisfies

$$\frac{k}{\lambda} \lambda(Disk_1) \leq \max \left(1 - \frac{c}{2}, 1 - \frac{Cc^3}{216 + 6Cc^2}\right) A(\Omega).$$

If we apply this to minimal partitions ($\lambda = \Sigma_k(\Omega)$), this reads

$$\lambda(Disk_1) \leq \max \left(1 - \frac{c}{2}, 1 - \frac{Cc^3}{216 + 6Cc^2}\right) A(\Omega) \liminf_{k \to +\infty} \frac{\Sigma_k(\Omega)}{k}.\quad (4.11)$$

Hence S. Steinerberger recovers Bourgain’s improvement (4.3) with a not explicit constant.

**Remark 4.3** Steinerberger obtains also a similar lower bound to (4.11) for $\liminf_{k \to +\infty} \frac{\Sigma_k(\Omega)}{k}$ using a convexity argument.

## 5 Considerations around rectangles

The detailed analysis of the spectrum of the Dirichlet Laplacian in a rectangle is basic in Pleijel’s paper [20]. As mentioned in [12], other results have been previously obtained in the PHD of A. Stern [25], defended in 1924.

\[\text{At least } C \text{ in (4.7) is not explicit.}\]
Other aspects relative to spectral minimal partitions appear in [18]. Take $\mathcal{R}(a, b) = (0, a\pi) \times (0, b\pi)$. The eigenvalues are given by

$$\hat{\lambda}_{m,n} = \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right),$$

with a corresponding basis of eigenfunctions given by

$$\phi_{m,n}(x, y) = \sin \frac{mx}{a} \sin \frac{ny}{b}.$$

If it is easy to determine the Courant sharp cases when $b^2/a^2$ is irrational (see for example [18]). The rational case is more difficult. Pleijel claims in [20] that in the case of the square it is Courant sharp if and only if $k = 1, 2, 4$. The exclusion of $k = 5, 7, 9$ is however not justified (the author refers indeed to Courant-Hilbert [12] where only pictures are presented, actually taken from the old book (1891) by Pockels [21]). This can actually be controlled by an explicit computation of the nodal sets of each combination ($\theta \in [0, \pi]$):

$$\Phi_{m,n,\theta} := \cos \theta \phi_{m,n} + \sin \theta \phi_{n,m}$$

for $(m, n) = (1, 3), (1, 4),$ and $(2, 3)$.

In this context the following guess could be natural: Suppose that $\hat{\lambda}_{m,n}$ has multiplicity $\mathfrak{m}(m, n)$. Let $\mu_{\text{max}}(u)$ be the maximum of the number of nodal domains of the eigenfunctions in the eigenspace associated with $\hat{\lambda}_{m,n}$.

$$\mu_{\text{max}} = \sup_j (m_j n_j),$$

where the sup is computed over the pairs $(m_j, n_j)$ such that

$$\hat{\lambda}_{m_j, n_j} = \hat{\lambda}_{m,n}.$$

The problem is not easy because one has to consider, in the case of degenerate eigenvalues, linear combinations of the canonical eigenfunctions associated with the $\hat{\lambda}_{m,n}$. Actually, as stated above, the guess is wrong. As observed by Pleijel [20], the eigenfunction $\Phi_{1,3,\pi}$ corresponding to the fifth eigenvalue has four nodal domains delimited by the two diagonals, and $\mu_{\text{max}} = 3$. More generally one can consider $u_k := \Phi_{1,3,\pi}(2^k x, 2^k y)$ to get an eigenfunction associated with the eigenvalue $\lambda_{n(k)} = \hat{\lambda}_{2^k, 3 \cdot 2^k} = 10 \cdot 4^k$ with $4^{k+1}$ nodal domains. The corresponding quotient $\frac{\mu(u_k)}{n(k)}$ is asymptotic to $\frac{8}{\pi}$. This does not contradict the Polterovich conjecture. Note also that for each number $K$, there is an eigenfunction $u$ with $\mu(u) \geq K$. Finally let us mention that counterexamples to a similar guess in the Neumann case can be found in [21].
Pleijel’s constant.
We consider for each $\Omega$ and each orthonormal basis $B_{\Omega} := (u_n)$ of the Dirichlet Laplacian:

$$Pl(\Omega, B_{\Omega}) = \limsup_{n \to +\infty} \frac{\mu_{\Omega,B,n}}{n},$$

(5.1)

where $\mu_{\Omega,B,n}$ denotes the number of nodal domains of $u_n$.

The reference to $B$ is only needed in the case when the Laplacian has an infinite sequence of multiple eigenvalues. We then define:

$$Pl(\Omega) = \sup_{B_{\Omega}} Pl(\Omega, B_{\Omega}).$$

(5.2)

Now the question arises how and whether $Pl(\Omega)$ depends on $\Omega$. Note that $Pl(\Omega) = Pl(T\Omega)$ where $T$ denotes scaling or rotation, reflection, translation. It is not even clear that $\mu_{\Omega,B,n}$ tends for every $\Omega$ to infinity, see [19]. The Pleijel constant could be defined as

$$Pl = \sup_{\Omega} Pl(\Omega),$$

(5.3)

and it is not at all clear that a maximizing pair $(\Omega, B_{\Omega})$ exists. (The square or more generally rectangles might be good candidates as mentioned above.) It would be interesting to find those domains, for which it is possible to calculate $Pl(\Omega)$.

We finally recall (cf [13] and [22]) that

**Proposition 5.1** Let us assume that $b^2/a^2$ is irrational.

$$Pl(\mathcal{R}(a,b)) = \frac{2}{\pi}.$$  

(5.4)

**Proof**
It suffices to consider $\mathcal{R}(\pi, b\pi)$ for irrational $b^2$. Since $b^2$ is irrational the eigenvalues are simple and

$$\mu(\phi_{m,n}) = mn.$$  

(5.5)

Weyl asymptotics tells us that with $\lambda = \hat{\lambda}_{m,n}$:

$$k(m, n) := \# \{ (\bar{m}, \bar{n}) : \hat{\lambda}_{m,n} < \lambda \} = \frac{b\pi}{4} (m^2 + n^2/b^2) + o(\lambda).$$  

(5.6)

We have

$$\lambda_{k(m,n)+1} = \hat{\lambda}_{m,n}.$$  

We observe that $\mu(\phi_{n,m})/\hat{k}(n, m)$ is asymptotically given by

$$P(m, n; b) := \frac{4mn}{\pi(m^2b + n^2/b)} \leq \frac{2}{\pi}.$$  

(5.7)
Next we take a sequence \((m_k, n_k)\) such that \(b = \lim_{k \to +\infty} \frac{n_k}{m_k}\) with \(m_k \to +\infty\). We observe that

\[
\lim_{k \to +\infty} P(m_k, n_k, b) = \frac{2}{\pi}.
\]

This proves the proposition using the sequence of eigenfunctions \(\phi_{m_k,n_k}\).

**Remark 5.2** We have consequently

\[
Pl \geq \frac{2}{\pi},
\]

the conjecture being that one has actually the equality.

The case when \(b^2/a^2 \in \mathbb{Q}\) depends on the discussion at the beginning of the section. We only know that

\[
Pl(R(a, b)) \geq \frac{2}{\pi},
\]

6 **Looking for a class \(\mathcal{O}^\#\).**

We now start the discussion on tentative choices of the classes \(\mathcal{O}^\sharp_k\) (see Definition 3.3).

6.1 **Bipartite partitions**

If we think that only nodal partitions are involved in Pleijel’s theorem, it could be natural to consider as class \(\mathcal{O}^\sharp_k\) the class \(\mathcal{O}^{bp}_k\) of the bipartite strong regular connected \(k\)-partitions \(\mathcal{D} = (D_1, \ldots, D_k)\). Note that there is some arbitrariness in the definition but ”strong” is necessary to define a bipartite partition.

**Definition 6.1**

\[
\mathcal{L}_k^{bp}(\Omega) := \inf_{\mathcal{D} \in \mathcal{O}^{bp, str}_k} \max \lambda(D_i).
\]

Although this definition is natural, all what has been established relatively to \(\mathcal{L}_k(\Omega)\) is unclear or at least unproved in the case of this \(\mathcal{L}_k^{bp}(\Omega)\).

By definition, we know that \(\mathcal{L}_k^{bp}(\Omega) \leq L_k(\Omega)\). If the inequality is strict the infimum cannot by definition correspond to a nodal partition. If we want this notion to be helpful for improving Pleijel’s constant, it is natural to first ask if \(\mathcal{L}_k^{bp}(\Omega) > \mathcal{L}_k(\Omega)\), at least for \(k\) large. However we will show
Proposition 6.2 Suppose that $\Omega$ is simply connected. Then
\[ \mathcal{L}_k^{bp}(\Omega) = \mathcal{L}_k(\Omega). \] (6.2)

Hence this class which could a priori appear to be a natural candidate for $O^\sharp$ does not lead to any improvement of the hexagonal conjecture for Pleijel’s theorem.

6.2 Proof of Proposition 6.2

Particular case.
Suppose that $\Omega \subset \mathbb{R}^2$, $k \geq 2$ and consider a minimal $k$-partition $\mathcal{D} = \{D_1, \ldots, D_k\}$ which is not bipartite. We first prove the proposition in a particular case.

Lemma 6.3 We assume that $\Omega$ is simply connected and that
\[ \#(\bigcup \partial D_i) = 1. \] (6.3)
Then there is a sequence of bipartite $k$-partitions $\hat{\mathcal{D}}_k(\epsilon) = \{\hat{D}_1(\epsilon), \ldots, \hat{D}_k(\epsilon)\}$ of $\Omega$ with the property that
\[ \Lambda(\hat{\mathcal{D}}_k(\epsilon)) \to \mathcal{L}_k(\Omega), \quad j = 1, \ldots, k. \] (6.4)

For $\epsilon > 0$, we define for any element of the partition
\[ D_i(\epsilon) = \{x \in D_i \mid \text{dist}(x, \partial D_i) > \epsilon\}. \] (6.5)

For $\epsilon > 0$ small enough all the $D_i(\epsilon)$ are non empty and connected.

We also define a tubular $\epsilon$-neighborhood of $\partial \Omega \cup N(\mathcal{D})$ in $\Omega$:
\[ S^\epsilon = \{x \in \Omega, d(x, \partial \Omega \cup N(\mathcal{D})) < \epsilon\} \]

$S^\epsilon$ is connected due to Assumption (6.3).

Now as $\epsilon$ tends to zero, $A(S^\epsilon) \to 0$. We consider the $k$-partition $\hat{\mathcal{D}}(\epsilon)$ defined by
\[ \hat{D}_1(\epsilon) = D_1 \cup S^\epsilon, \quad \hat{D}_i(\epsilon) = D_i(\epsilon), \quad \forall i > 1. \]
This gives a connected open $k$-partition of $\Omega$ with the following property:
\[ \lambda(\hat{D}_1(\epsilon)) < \lambda(D_1), \quad \lim_{\epsilon \to 0} \lambda(\hat{D}_i(\epsilon)) = \lambda(D_i), \quad \forall i > 1. \]

□

Figure 2 describes the construction in the case of the disk, assuming (see [18], [15]) that the minimal 3-partition is the Mercedes-star.

General case.
We now give the proof in the general case. Considering the previous discussion, we can distinguish two cases for our minimal $k$-partition $\mathcal{D}$. 

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Figure 2: Scheme of the construction for the Mercedes Star

(i) \( N(D) \) does not meet \( \partial \Omega \)

(ii) \( N(D) \cap \partial \Omega \neq \emptyset \).

In the first case, after relabeling we can call \( D_1 \) the unique element of the partition whose boundary touches \( \partial \Omega \). We follow the previous discussion and define \( S^{(1)}(\epsilon) \) the connected component of the set \( S^\epsilon \) containing \( \partial D_1 \cap \Omega \).

The first element of the approximating \( k \)-partition is then

\[
\hat{D}_1(\epsilon) := D_1 \cup S^{(1)}(\epsilon).
\]

In the second case, after relabeling, we can take as \( D_1 \) one element of the partition such that \( \partial D_1 \cap \partial \Omega \neq \emptyset \) and as before introduce \( \hat{D}_1(\epsilon) \) as before but with \( S^{(1)}(\epsilon) \) the connected component of the set \( S^\epsilon \) containing \( \partial \Omega \).

We now consider the connected components \( \Omega \setminus \overline{D}_1(\epsilon) \). Many of them are simply \( D_j(\epsilon) \) for \( j > 1 \). We keep these open sets as elements of our new partition. Other components contain more than one \( D_\ell \). If we denote by \( \Omega^{(\ell)}(\epsilon) \) such a component, we observe that we are necessarily in case (i) of the previous discussion with \( \Omega^{(\ell)}(\epsilon) \) replacing \( \Omega \). Only one \( D_k(\epsilon) \) inside \( \Omega^{(\ell)}(\epsilon) \) can have its boundary touching \( \overline{D}_1(\epsilon) \). We can iterate inside \( \Omega^{(\ell)}(\epsilon) \) what we have done in \( \Omega \) and the procedure will stop after a finite number of iterations.

Remark 6.4 The case when \( \Omega \) is not simply connected can be handled similarly.

6.3 Almost nodal partitions

Here is a new try for a definition of \( O^* \) in order to have a flexible notion of partitions which are close with nodal partitions. We assume that \( \Omega \) is
regular and simply connected.

We will say that a $k$-partition $\mathcal{D}$ of $\Omega$ of energy $\Lambda(\mathcal{D})$ is almost nodal, if there is a connected open set $\Omega' \subset \Omega$ and a $(k-1)$-subpartition $\mathcal{D}'$ of $\mathcal{D}$ such that $\mathcal{D}'$ is a nodal partition of $\Omega'$ of energy $\Lambda(\mathcal{D})$. Of course a nodal partition is almost nodal. The first useful observation is that for any $k$ there exists always an almost nodal $k$-partition. The proof is obtained using a sufficiently thin ”square” $(k-1)$-partition in $\Omega$ and completing by the complement in $\Omega$ of the closure of the union of the preceding squares. See the right subfigure of Figure 1. Denoting by $O_{a.n}$ the set of the almost nodal partitions, we introduce

$$L_{a.n}(\Omega) = \inf_{D \in O_{a.n}} \Lambda(D).$$

(6.6)

Of course, we have

$$L_k(\Omega) \leq L_{a.n}(\Omega) \leq L_k(\Omega).$$

(6.7)

The next point is to observe, by the same proof giving (3.13) but playing with the square tiling (see Figure 1), that

$$A(\Omega) \limsup_{k \to +\infty} \frac{L_{a.n}(\Omega)}{k} \leq \lambda(Sq_1).$$

(6.8)

Again the question arises about the asymptotic behavior of $\liminf_{k \to +\infty} \frac{L_{a.n}(\Omega)}{k}$. Unfortunately there are good reasons to think that we can improve (6.8) by proving

$$A(\Omega) \limsup_{k \to +\infty} \frac{L_{a.n}(\Omega)}{k} \leq \lambda(Hexa_1).$$

(6.9)

We just give an heuristical hint. For $k$ large, we try to ”almost” fill $\Omega$ with a maximal number of $(k-1)$ adjacent isometric regular hexagons $D_i$ ($i = 1, \ldots, k-1$). For $k$ large, they should have an area of order $A(\Omega)/k$ and an energy of order $k\lambda(Hexa_1)/A(\Omega)$, and we complete the partition by taking as $D_k$ the complement in $\Omega$ of $\bigcup_{i=1}^{k-1} D_i$. We can in addition have the property that $\lambda(D_k) \leq \lambda(D_1)$ (one way is to start with $k$ adjacent regular hexagons and to delete one). Then we construct our set $\Omega'_{k-1}$ by subtracting cracks (edges of some of the hexagons) from $\text{Int}(\bigcup_{i=1}^{k-1} D_i)$ in such a way that $(D_1, \ldots, D_{k-1})$ becomes a nodal $(k-1)$-partition of $\Omega'_{k-1}$ (see Figure 3). Such a construction is detailed in [8], when exploring the consequences of the hexagonal conjecture. This conjecture would actually impose that this is the nodal partition of a Courant sharp eigenfunction but we do not need it at this stage. Then the partition is almost nodal and asymptotically of energy $k\lambda(Hexa_1)/A(\Omega)$.

Actually, starting directly from a minimal $k$-partition $\mathcal{D} = (D_1, \ldots, D_k)$ and proceeding as before with the $(k-1)$ first elements, one can add curved segments belonging to the boundary of the partition such that we get a nodal
Figure 3: Scheme of the construction of the cracks for \((k - 1) = 7\).

partition. Here we use a property observed in [18] (Proof of Proposition 8.3) (see also Corollary 2.11 in [8]). This will directly lead to the stronger equality 
\[
\mathcal{L}_{k,n}^{a,n}(\Omega) = \mathcal{L}_{k}(\Omega).
\]
Of course, one could think that by imposing more regularity on the partition and on \(\Omega'\), one can eliminate this kind of examples. But as in the previous subsection, an approximation of the cracks by fine tubes could probably be used for getting the same inequality. This we have not checked and will be much more technical than for the proof of Proposition 6.2.

Hence the class of almost nodal partitions is probably too large for getting a higher infimum.

6.4 Conclusion

In conclusion, we were looking for smaller classes of partitions containing nodal partitions with the hope to give some justification for the Polterovich conjecture. We have shown that two natural choices do not give a confirmation of this conjecture as initially expected.
7 Pleijel’s Theorem for Aharonov-Bohm operators and application to minimal partitions

7.1 The Aharonov-Bohm approach

Let us recall some definitions and results about the Aharonov-Bohm Hamiltonian in an open set \(\Omega\) (for short \(ABX\)-Hamiltonian) with a singularity at \(X \in \Omega\) as introduced in [17]. We denote by \(X = (x_0, y_0)\) the coordinates of the pole and consider the magnetic potential with flux at \(X\): \(\Phi = \pi\), defined in \(\Omega_X = \Omega \setminus \{X\}\):

\[
A^X(x, y) = (A^X_1(x, y), A^X_2(x, y)) = \frac{1}{2} \left( \frac{-y - y_0}{r^2}, \frac{x - x_0}{r^2} \right). \tag{7.1}
\]

The \(ABX\)-Hamiltonian is defined by considering the Friedrichs extension starting from \(C^\infty_0(\Omega_X)\) and the associated differential operator is

\[
-\Delta^X := (D_x - A^X_1)^2 + (D_y - A^X_2)^2 \quad \text{with} \quad D_x = -i\partial_x \quad \text{and} \quad D_y = -i\partial_y. \tag{7.2}
\]

Let \(K_X\) be the antilinear operator \(K_X = e^{i\theta_X} \Gamma\), with \((x - x_0) + i(y - y_0) = \sqrt{|x - x_0|^2 + |y - y_0|^2} e^{i\theta_X}\), \(\theta_X\) such that \(d\theta_X = 2A^X\), and where \(\Gamma\) is the complex conjugation operator \(\Gamma u = \overline{u}\). A function \(u\) is called \(K_X\)-real, if \(K_X u = u\). The operator \(-\Delta^X\) is preserving the \(K_X\)-real functions and we can consider a basis of \(K_X\)-real eigenfunctions. Hence we only analyze the restriction of the \(ABX\)-Hamiltonian to the \(K_X\)-real space \(L^2_{K_X}(\Omega_X)\) where

\[
L^2_{K_X}(\Omega_X) = \{ u \in L^2(\Omega_X), K_X u = u \}.
\]

It was shown in [17] and [1] that the nodal set of such a \(K_X\) real eigenfunction has the same structure as the nodal set of an eigenfunction of the Laplacian except that an odd number of half-lines meet at \(X\). In particular, for a \(K_X\)-real groundstate (one pole), one can prove [17] that the nodal set consists of one line joining the pole and the boundary.

**Extension to many poles**

We can extend this construction in the case of a configuration with \(\ell\) distinct points \(X_1, \ldots, X_\ell\) (putting a flux \(\pi\) at each of these points). We just take as magnetic potential

\[
A^X = \sum_{j=1}^\ell A^X_j, \quad \text{where} \quad X = (X_1, \ldots, X_\ell).
\]

We can also construct the antilinear operator \(K_X\), where \(\theta_X\) is replaced by a multivalued-function \(\phi_X\) such that \(d\phi_X = 2A^X\). We can then consider the real subspace of the \(K_X\)-real functions in \(L^2_{K_X}(\Omega_X)\). It has been shown that
the $K_X$-real eigenfunctions have a regular nodal set (like the eigenfunctions
of the Dirichlet Laplacian) with the exception that at each singular point
$X_j$ ($j=1,\ldots,\ell$) an odd number of half-lines meet. We recall the following
theorem which is the most interesting part of the magnetic characterization
of the minimal partitions given in [16]:

**Theorem 7.1**

Let $\Omega$ be simply connected. If $D$ is a $k$-minimal partition of $\Omega$, then, by
choosing $X(D,\ell) = X^{\text{odd}}(N(D))$, $D$ is the nodal partition of some $k$-
th $K_X$-real

eigenfunction of the Aharonov-Bohm Laplacian associated with $\Omega_X$.

### 7.2 Analysis of the critical sets in the large limit case

We first consider the case of one pole $X$. We look at a sequence of $K_X$-real

eigenfunctions and follow the proof of Pleijel on the number of nodal do-

tains. We observe that the part devoted to the lower bound works along

the same lines and the way we shall meet $L_k(\Omega)$ is unchanged. When us-

ing the Weyl formula, we observe that only a lower bound of the counting

function is used (see around (3.4)). If the distance of $X$ to the boundary is

larger than $\epsilon$, we introduce a disk $D(X,\epsilon)$ of radius $\epsilon$ around $X$ ($\epsilon > 0$) and

consider the Dirichlet magnetic Laplacian in $\Omega \setminus \bar{D}(X,\epsilon)$. For the $X$ at the
distance less than $\epsilon$ of the boundary, we look at the magnetic Laplacian on

$\Omega$ minus a $(2\epsilon)$-tubular neighborhood of the boundary. In the two cases, we
get an elliptic operator where the main term is the Laplacian $-\Delta$. Hence
we can combine the monotonicity of the Dirichlet problem with respect to
the variation of the domain to the use of the standard Weyl formula (see
around (3.4)) to get (uniformly for $X$ in $\Omega$), an estimate for the counting function

$N_X(\lambda)$ of $-\Delta_A X$ in the following way:

There exists a constant $C > 0$ such that, for any $\epsilon > 0$, as $\lambda \to +\infty$,

$$N_X(\lambda) \geq \frac{1}{4\pi} (1 - C\epsilon) A(\Omega) \lambda + o(\lambda).$$

Hence, for any $\epsilon > 0$, any $X \in \Omega$,

$$\limsup_{n \to +\infty} \mu(\phi^X_n)/n \leq \frac{4\pi}{A(\Omega) \liminf_{k \to +\infty} \frac{\Sigma_X(\Omega)}{k}}.$$ 

Taking the limit $\epsilon \to 0$, we get:

$$\limsup_{n \to +\infty} \mu(\phi^X_n)/n \leq \frac{4\pi}{A(\Omega) \liminf_{k \to +\infty} \frac{\Sigma_X(\Omega)}{k}}. \quad (7.3)$$

\footnote{We recall that $X(N)$ is defined after Definition 2.2}
Till now $X$ was fixed. But everything being uniform with respect to $X$, we can also consider a sequence $\phi^n_X$ corresponding to the $n$-th eigenvalue of $-\Delta A_X^n$.

Suppose that for a subsequence $k_j$, we have a $k_j$-minimal partition with only one pole $X_j$ in $\Omega$. Let $\phi^n_{k_j}$ the corresponding eigenfunction. Hence, we are in a Courant sharp situation. The inequality above leads this time (possibly after extraction of a subsequence) to

$$1 \leq \frac{4\pi}{A(\Omega) \lim \inf_{k \to +\infty} \frac{\Sigma_k(\Omega)}{k}} \leq \nu_{Pl} \sim 0.691.$$ 

Hence a contradiction.

We can play the same game with more than one pole and get as consequence:

**Proposition 7.2** If for $k \in \mathbb{N}$, $D_k$ denotes a minimal $k$-partition, then

$$\lim_{k \to +\infty} \#X_{odd}(N(D_k)) = +\infty.$$  (7.4)

**Proof.**

Suppose indeed that this cardinality does not tend to $+\infty$. We can then extract a subsequence such that this cardinality is finite. After new extractions of a subsequence, we can assume that this cardinality is fixed and that each critical point tends to a limiting point, which could be either at the boundary $\partial \Omega$ or in $\Omega$. We apply Theorem 7.1 and consider the associated Aharonov-Bohm hamiltonians, whose poles are these odd critical points. We can then find a finite number of disks of radius $\epsilon$ centered at these limiting poles such that all the poles are contained in these balls for $k$ large enough.

Then outside of these balls the potential $A_X$ and the derivatives are bounded by a uniform bound (depending on $\epsilon$) and the same construction works and leads to a contradiction.

**Remark 7.3** We recall that an upper bound for $\#X(N(D_k))$ is given in [14] (case with no holes) by using Euler’s formula:

$$\#X_{odd}(N(D_k)) \leq 2k - 4.$$  (7.5)

On the other hand, the hexagonal conjecture for the asymptotic number of odd critical points of a $k$-minimal partition reads as follows:

$$\lim_{k \to +\infty} \frac{\#X_{odd}(N(D_k))}{k} = 2.$$  (7.6)

Hence there are good reasons to believe that the upper bound (7.5) is asymptotically optimal.
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References

[1] B. Alziary, J. Fleckinger-Pellé, P. Takáč. Eigenfunctions and Hardy inequalities for a magnetic Schrödinger operator in \( \mathbb{R}^2 \). Math. Methods Appl. Sci. 26(13), 1093–1136 (2003).

[2] P. Bérard. Inégalités isopérimétriques et applications. Domaines nodaux des fonctions propres. SEDP 1981-1982, Ecole Polytechnique.

[3] P. Bérard and B. Helffer. Remarks on the boundary set of spectral equipartitions, \textcolor{blue}{arXiv:1203.3566}. In press in Philosophical Transactions of the Royal Society (2013).

[4] G. Blind. Über Unterdeckungen der Ebene durch Kreise. Journal für die Reine und Angewandte Mathematik 236 (1969): 14573.

[5] G. Blum, S. Gnutzmann, and U. Smilansky. Nodal domain statistics: A criterion for quantum chaos, Phys. Rev. Lett. 88 (2002), 114101-114104.

[6] B. Bourdin, D. Bucur, and E. Oudet. Optimal partitions for eigenvalues. SIAM J.Sci. Comp. 31(6) 4100-4114, (2009).

[7] J. Bourgain. On Pleijel’s nodal domain theorem. \textcolor{blue}{arXiv:1308.4422v1} [math.SP] 20 Aug 2013.

[8] V. Bonnaillie-Noël, B. Helffer and G. Vial. Numerical simulations for nodal domains and spectral minimal partitions. ESAIM Control Optim. Calc.Var. DOI:10.1051/cocv:2008074 (2008).

[9] L. Brasco, G. De Philippis and B. Velichkov. Faber-Krahn inequalities in sharp quantitative form, \textcolor{blue}{arXiv:1306.0392} (2013).

[10] L.A. Caffarelli and F.H. Lin. An optimal partition problem for eigenvalues. Journal of scientific Computing 31(1/2), DOI: 10.1007/s10915-006-9114.8 (2007).
[11] R. Courant. Ein allgemeiner Satz zur Theorie der Eigenfunktionen selbstadjungierter Differentialausdrücke, Nachr. Ges. Göttingen (1923), 81–84.

[12] R. Courant and D. Hilbert. Methods of Mathematical Physics, Vol. 1. New York (1953).

[13] W. Hansen and N. Nadirashvili. Isoperimetric inequalities in potential theory, Potential Analysis 3, 114 (1994).

[14] B. Helffer, T. Hoffmann-Ostenhof. Remarks on two notions of spectral minimal partitions. Adv. Math. Sci. Appl. 20 (1), 249–263, (2010).

[15] B. Helffer, T. Hoffmann-Ostenhof. On spectral minimal partitions: the case of the disk. CRM proceedings 52, 119–136 (2010).

[16] B. Helffer, T. Hoffmann-Ostenhof. On a magnetic characterization of spectral minimal partitions. JEMS 15, 2081–2092 (2013).

[17] B. Helffer, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, M. P. Owen. Nodal sets for ground states of Schrödinger operators with zero magnetic field in non-simply connected domains. Comm. Math. Phys. 202(3), 629–649 (1999).

[18] B. Helffer, T. Hoffmann-Ostenhof, S. Terracini. Nodal domains and spectral minimal partitions. Ann. Inst. H. Poincaré Anal. Non Linéaire 26, 101–138 (2009).

[19] T. Hoffmann-Ostenhof. Geometric Aspects of Spectral Theory, Open Problem Session (xiv) page 2068-2069, Oberwolfach Reports 33, 2012

[20] A. Pleijel. Remarks on Courant’s nodal theorem. Comm. Pure. Appl. Math. 9, 543–550 (1956).

[21] F. Pockels. Über die partielle Differentialgleichung $\Delta u - k^2 u = 0$ and deren Auftreten in mathematischen Physik. Historical Math. Monographs. Cornell University (2013). (Originally Teubner- Leipzig 1891.)

[22] I. Polterovich. Pleijel’s nodal domain theorem for free membranes. Proceeding of the AMS, Volume 137, Number 3, March 2009, 1021-1024.

[23] U. Smilansky and R. Sankaranarayanan. Nodal domain distribution of rectangular drums. Proceedings of National Conference on Nonlinear Systems and Dynamics (Feb. 24-26, 2005), Aligarh Muslim University, India. [arXiv:lin/0503002] 1-3, March 2005.

[24] S. Steinerberger. Geometric uncertainty principle with an application to Pleijel’s estimate. arXiv:1306.3103v4, 4 Nov 2013.
[25] A. Stern. Bemerkungen über asymptotisches Verhalten von Eigenwerten und Eigenfunctionen. Diss. Göttingen 1925.