SECOND ORDER REGULARITY FOR DEGENERATE NONLINEAR ELLIPTIC EQUATIONS

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Abstract. We investigate the second order regularity of solutions to degenerate nonlinear elliptic equations.

1. Introduction and results. We are interested in establishing Hessian summability for weak solutions of degenerate nonlinear elliptic equations in divergence form. In particular we consider weak solutions to

\[- \operatorname{div}(A(|\nabla u|)\nabla u) + b(x)|\nabla u|^q = f(x) \quad \text{in } \Omega,\]

in an open set \(\Omega \subseteq \mathbb{R}^N\). The real valued function \(A: \mathbb{R}^+ \to \mathbb{R}^+\) is of class \(C^1\), with

\[
\limsup_{t \to 0^+} tA(t) < \infty
\]

and

\[
-1 < \inf_{t > 0} \frac{tA'(t)}{A(t)} := m_A \leq M_A := \sup_{t > 0} \frac{tA'(t)}{A(t)} < \infty.
\]

We also assume that

\[
A(s) \geq Ks^{\tilde{\vartheta}} \quad \text{for some } \tilde{\vartheta} \geq 0 \quad \forall s > 0.
\]

When \(b(x)\) is not identically zero, we assume that \(q > \frac{\tilde{\vartheta}+1}{2}\).

We shall consider solutions of class \(C^{1,\alpha}\). This natural in general according to [3, 5, 4, 12, 13]. Therefore we give the following

Definition 1.1. We say that \(u \in C^{1,\alpha}(\Omega)\) is a weak distributional solution to (1.1), if

\[
\int_{\Omega} A(|\nabla u|)(\nabla u, \nabla \phi) \, dx + \int_{\Omega} b(x)|\nabla u|^q \phi \, dx = \int_{\Omega} f \phi \, dx
\]

for every \(\phi \in C_c^\infty(\Omega)\).

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We will frequently exploit the fact that the equation is no longer degenerate outside the critical set \( Z_u \),
\[
Z_u := \{ \nabla u = 0 \}.
\]
Consequently it is also natural to assume that the solution is of class \( C^2 \) outside the critical set.

**Remark 1.2.** We will use the notation \( u_i := u_{x_i}, \) \( i = 1 \ldots, N \), to indicate the partial derivative of \( u \) with respect to \( x_i \). This are the classic derivatives since \( u \) is of class \( C^1 \). The second derivatives will be indicated with \( u_{ij}, \) \( i, j = 1 \ldots, N \). In this case, since \( u \) is of class \( C^2 \) only far from the singular set \( Z_u \), we agree that \( u_{ij} \) coincides with the second derivatives far from the singular set \( Z_u \), while \( u_{ij} = 0 \) on the singular set \( Z_u \). It is important to note that, at this stage and without the needed regularity information, this is just a notation inspired by the Stampacchia’s theorem. The rigorous proof of the fact that \( u_{ij} \) actually represent the second distributional derivatives will be a consequence of our Theorem 1.5.

Our aim is to study the summability of the second derivatives of the solutions. When \( A(t) = t^{p-2} \) the operator reduces to standard \( p \)-Laplacian. In this case, from [2, 10, 11] (see also [1, 6]), it follows that \( u \in W^{2,3}_\text{loc}(\Omega) \) if \( 1 < p < 3 \), and that if \( p \geq 3 \) and the source term \( f \) is strictly positive then \( u \in W^{2,q}_\text{loc}(\Omega) \) for \( q < \frac{3}{p-2} \). We may look at this type of regularity as an issue in the context of the Calderón-Zygmund theory for nonlinear degenerate problems. We refer the reader to [7, 8] and the references therein.

Here we shall extend the results in [2, 10, 11]. The setting described above is really more general than the case of the \( p \)-Laplacian. Then the proofs in [2, 10, 11] and the results as well needs appropriate modifications. We start with the following

**Theorem 1.3.** Let \( u \in C^{1,\alpha}(\Omega) \cap C^2(\Omega \setminus Z_u) \) be a solution to (1.1) with \( f, b \in W^{1,\infty}(\Omega) \). Assume that \( B_{2\rho}(x_0) \subset \Omega \) and \( y \in \Omega \). Then, for \( 0 \leq \beta < 1 \) and \( \gamma < N - 2 \) for \( N \geq 3 \) while \( \gamma = 0 \) if \( N = 2 \), we have
\[
\int_{B_{\rho}(x_0)} \frac{A(|\nabla u|)|\nabla u_i|^2}{|x-y|^{\gamma}|u_i|^\beta} \leq C \quad \forall i = 1, \ldots, N
\]
with \( C = C(\gamma, \beta, q, ||f||_{W^{1,\infty}(\Omega)}, b, ||u||_\infty, \text{dist}(B_{\rho}(x_0), \Omega^c)) \).

The local regularity result of Theorem 1.3 holds without sign assumption on the source term \( f \). If else a sign assumption on \( f \) is imposed, than we can prove a summability result regarding the inverse of the weight \( A(|\nabla u|) \). We have

**Theorem 1.4.** Let \( u \in C^{1,\alpha}(\Omega) \cap C^2(\Omega \setminus Z_u) \) be a solution to (1.1) with \( f, b \in W^{1,\infty}(\Omega) \) and \( f(x) \geq c(\rho, x_0) > 0 \), in \( B_{2\rho}(x_0) \subset \Omega \) for some \( \rho = \rho(x_0) > 0 \). Then we have
\[
\int_{B_{\rho}(x_0)} \frac{1}{A(|\nabla u|)^\alpha} \frac{1}{|x-y|^\gamma} \leq C
\]
with \( 1 < \alpha < 1 + \frac{1}{p} \), \( \gamma < N - 2 \), if \( N \geq 3 \) and \( \gamma = 0 \) if \( N = 2 \) and \( C = C(\gamma, m_A, M_A, q, f, b, ||\nabla u||_{\infty}, \rho, x_0, \alpha) \). The same result follows if we assume that \( f(x) \leq c(\rho, x_0) < 0 \) in \( B_{2\rho}(x_0) \subset \Omega \).

In particular \( \mathcal{L}(\{A(|\nabla u|) = 0\}) = 0 \).

Theorem 1.4 is actually an estimate on the way the operator degenerate near the critical set. It might have future applications in the study of the qualitative properties of the solutions. Here, as a consequence, we shall point out a further
correlated regularity result, see Theorem 1.5 below. Before we start observing that
the estimates in Theorem 1.4 and in Theorem 1.3 (namely (1.6) and (1.7)), hold
in a general compact set of $\Omega$. The same regularity holds all over the domain once
we assume that there are no critical points of the solutions up to the boundary,
namely $Z_u \cap \partial \Omega = \emptyset$. This an abstract assumption always verified each time we
may exploit the Hopf boundary lemma, see [9]. The global regularity results follow
via a covering argument and the proofs and the statements are postponed in Section
4, see Theorem 4.1 and Theorem 4.2.

As mentioned here above, the estimates on the second derivatives and the esti-
mates of the summability of the weight can be exploited jointly in order to obtain the
following

**Theorem 1.5.** Let $\Omega \subset \mathbb{R}^N$ a bounded smooth domain and let $u \in C^{1,\alpha}(\Omega) \cap
C^2(\Omega \setminus Z_u)$ be a solution to (1.1) with $f, b \in W^{1,\infty}(\Omega)$. Assume that $f$ is positive
in $\Omega$ (possibly vanishing on the boundary). Then
\[
 u \in W^{2,s}(K) \quad \text{for any } s < \min\{2; 1 + \tilde{\theta}^{-1}\} \tag{1.8}
\]
for every compact set $K \subset \Omega$, with $u_{ij} = 0$ on $Z_u$ for $ij = 1, \ldots, N$. If we further
assume that $Z_u \cap \partial \Omega = \emptyset$, then
\[
 u \in W^{2,s}(\Omega) \quad \text{for any } s < \min\{2; 1 + \tilde{\theta}^{-1}\}. \tag{1.9}
\]

The paper is organized as follows: We prove the local regularity of the second
derivatives in Section 2, while the summability of the weight is studied in Section 3.
The covering argument needed to obtain the global results is developed in Section
4, where e also prove Theorem 1.5.

2. Local regularity. We start noticing that, if $u \in C^{1,\alpha}(\Omega) \cap C^2(\Omega \setminus Z_u)$ is a
solution of (1.1), then the derivatives of the solution are solutions to the linearized
equation, i. e.
\[
 L_u(u_i, \phi) = \int_\Omega A(|\nabla u|)(\nabla u_i, \nabla \phi)dx + \int_\Omega \frac{A'(|\nabla u|)}{|\nabla u|}(\nabla u, \nabla u_i)(\nabla u, \nabla \phi) \, dx 
\]
\[
 + \int_\Omega b_i(x)|\nabla u|^q \phi \, dx + q \int_\Omega b(x)|\nabla u|^{q-2}(\nabla u, \nabla u_i)\phi \, dx 
\]
\[
 - \int_\Omega f_i \phi \, dx = 0 \tag{2.1}
\]
for every $\phi \in C^\infty_c(\Omega \setminus Z_u)$. This follows just putting $\phi_i$ as test function in (1.1)
and integrating by parts.

To exploit such equation we will use a regularization argument. For $\varepsilon > 0$ we
consider $G_\varepsilon(t) = (2t - 2\varepsilon)\chi_{[\varepsilon, 2\varepsilon]}(t) + t\chi_{[2\varepsilon, \infty]}(t)$ for $t > 0$, while $G_\varepsilon(-t)$
for $t \leq 0$ ($\chi_{[a,b]}(\cdot)$ denoting the characteristic function of a set). We will assume
that the ball $B_{2\varepsilon}(x_0)$ is contained in $\Omega$ and we will consider a cut-off function
$\varphi_\rho = \varphi \in C^\infty_c(B_{2\rho}(x_0))$ such that
\[
 \varphi = 1, \text{ in } B_{\rho}(x_0) \quad \text{and } \quad |\nabla \varphi| \leq \frac{2}{\rho}. \tag{2.2}
\]
Also, for $\beta \in [0, 1)$ and $\gamma < N - 2$ if $N \geq 3$ ($\gamma = 0$ for $N = 2$) fixed, we set
\[
 T_\gamma(t) = \frac{G_\varepsilon(t)}{|t|^{\beta}}, \quad H_\gamma(t) = \frac{G_\delta(t)}{|t|^{\gamma+1}}. \tag{2.3}
\]
Proof of Theorem 1.3. Let us consider the test function
\[ \phi = T_\varepsilon(u_i) H_\delta(|x - y|) \varphi^2 = T_\varepsilon(u_i) H_\delta \varphi^2. \] (2.4)
According to (2.3), it follows that such a test function can be plugged in the linearized equation (2.1), since it vanishes in a neighbourhood of the critical set \( Z_u \).
Consequently by (2.1) we get
\[
\begin{align*}
\int \Omega A(\nabla u)|\nabla u_i|^2 T_\varepsilon(u_i) H_\delta \varphi^2 &+ \int \Omega \frac{A'(\nabla u)}{|\nabla u|}(\nabla u, \nabla u_i)^2 T_\varepsilon(u_i) H_\delta \varphi^2 \\
+ \int \Omega A(\nabla u)(\nabla u_i, \nabla_x H_\delta) T_\varepsilon(u_i) \varphi^2 &+ \int \Omega \frac{A'(\nabla u)}{|\nabla u|}(\nabla u, \nabla u_i)(\nabla u, \nabla_x H_\delta) T_\varepsilon(u_i) \varphi^2 \\
+ 2 \int \Omega A(\nabla u)(\nabla u_i, \nabla \varphi) T_\varepsilon(u_i) H_\delta \varphi &+ 2 \int \Omega \frac{A'(\nabla u)}{|\nabla u|}(\nabla u, \nabla u_i)(\nabla u, \nabla \varphi) T_\varepsilon(u_i) H_\delta \varphi \\
+ q \int \Omega b(x)|\nabla u|^q - 2(\nabla u, \nabla u_i) \cdot T_\varepsilon(u_i) \cdot H_\delta \cdot \varphi^2 &+ \int \Omega b_i(x)|\nabla u|^q \cdot T_\varepsilon(u_i) \cdot H_\delta \cdot \varphi^2 \\
= \int \Omega f_i \cdot T_\varepsilon(u_i) \cdot H_\delta \cdot \varphi^2.
\end{align*}
\] (2.5)
It is convenient to set:
\[
\begin{align*}
I_1(\varepsilon, \delta) &= \int \Omega A(\nabla u)|\nabla u_i|^2 T_\varepsilon(u_i) H_\delta \varphi^2 \\
I_2(\varepsilon, \delta) &= \int \Omega \frac{A'(\nabla u)}{|\nabla u|}(\nabla u, \nabla u_i)^2 T_\varepsilon(u_i) H_\delta \varphi^2
\end{align*}
\] (2.6)
and
\[
\begin{align*}
I_3(\varepsilon, \delta) &= \int \Omega A(\nabla u)(\nabla u_i, \nabla_x H_\delta) T_\varepsilon(u_i) \varphi^2 \\
I_4(\varepsilon, \delta) &= \int \Omega \frac{A'(\nabla u)}{|\nabla u|}(\nabla u, \nabla u_i)(\nabla u, \nabla_x H_\delta) T_\varepsilon(u_i) \varphi^2 \\
I_5(\varepsilon, \delta) &= 2 \int \Omega A(\nabla u)(\nabla u_i, \nabla \varphi) T_\varepsilon(u_i) H_\delta \varphi \\
I_6(\varepsilon, \delta) &= 2 \int \Omega \frac{A'(\nabla u)}{|\nabla u|}(\nabla u, \nabla u_i)(\nabla u, \nabla \varphi) T_\varepsilon(u_i) H_\delta \varphi \\
I_7(\varepsilon, \delta) &= q \int \Omega b(x)|\nabla u|^q - 2(\nabla u, \nabla u_i) \cdot T_\varepsilon(u_i) \cdot H_\delta \cdot \varphi^2 \\
I_8(\varepsilon, \delta) &= \int \Omega b_i(x)|\nabla u|^q \cdot T_\varepsilon(u_i) \cdot H_\delta \cdot \varphi^2 \\
I_9(\varepsilon, \delta) &= \int \Omega f_i \cdot T_\varepsilon(u_i) \cdot H_\delta \cdot \varphi^2.
\end{align*}
\]
Regarding the terms \( I_1 \) and \( I_2 \), exploiting (1.3), we note that
\[
I_1 + I_2 \geq \int \Omega A(\nabla u)|\nabla u_i|^2 T_\varepsilon(u_i) H_\delta \varphi^2 \tag{2.7}
\]
when \( A'(\nabla u) \) is nonnegative, while
\[
I_1 + I_2 \geq \int \Omega A(\nabla u)|\nabla u_i|^2 T_\varepsilon(u_i) H_\delta \varphi^2 - \int \Omega |A'(\nabla u)||\nabla u||\nabla u_i|^2 T_\varepsilon(u_i) H_\delta \varphi^2
\]
when \( A'(\nabla u) \) is negative. Therefore
\[
I_1 + I_2 \geq (1 + m_A) I_1. \tag{2.8}
\]
Thence, from (2.5), we get

\[(1 + m_A) \int_{\Omega} A(|\nabla u|)|\nabla u_i|^2 T_\varepsilon(u_i) H_\delta \varphi^2 \leq |I_3| + \ldots + |I_9|.\]  

(2.9)

Now we estimate the right hand side of (2.9) letting \( \delta \to 0 \). Indeed, using the fact that \(|T_\varepsilon(t)| \leq t^{1-\beta}\) and the Young inequality \(ab \leq \varrho a^2 + \frac{b^2}{4\varrho}\), we deduce that

\[
\limsup_{\delta \to 0} (|I_3| + |I_4|) \leq \gamma \int_{\Omega} [A(|\nabla u|) + |A'|(|\nabla u|)||\nabla u|||\nabla u_i|| T_\varepsilon(u_i)] \frac{|T_\varepsilon(u_i)|}{|x - y|^{\gamma + 1}} \varphi^2
\]

\[
\leq \gamma (1 + |M_A|) \int_{\Omega} A(|\nabla u|)|\nabla u_i|| T_\varepsilon(u_i)| \frac{|T_\varepsilon(u_i)|}{|x - y|^{\gamma + 1}} \varphi^2
\]

\[
\leq \gamma (1 + |M_A|) \int_{\Omega} \frac{\sqrt{A(|\nabla u|)|\nabla u_i|| \chi_{\{u_i \leq \varepsilon\}}}}{|x - y|^{\gamma + 1/2}} \varphi \sqrt{\frac{A(|\nabla u|)}{|x - y|^{\gamma + 1}}} u_i \varphi^2
\]

\[
\leq \varrho \int_{\Omega} \frac{A(|\nabla u||\nabla u_i|^2}{|x - y|^{\gamma + 1/2}} \chi_{\{u_i \leq \varepsilon\}} \varphi^2 + \frac{\gamma^2 (1 + |M_A|)^2}{4\varrho} \int_{\Omega} \frac{A(|\nabla u|)|u_1|^2 - \beta}{|x - y|^{\gamma + 1}} \varphi^2
\]

\[
\leq \varrho \int_{\Omega} \frac{A(|\nabla u||\nabla u_i|^2}{|x - y|^{\gamma + 1/2}} \chi_{\{u_i \leq \varepsilon\}} \varphi^2 + \frac{\gamma^2 (1 + |M_A|)^2}{4\varrho} \int_{\Omega} \frac{A(|\nabla u|)|u_1|^2 - \beta}{|x - y|^{\gamma}} \varphi^2
\]

\[
\leq \varrho \int_{\Omega} \frac{A(|\nabla u||\nabla u_i|^2}{|x - y|^{\gamma + 1/2}} \chi_{\{u_i \leq \varepsilon\}} \varphi^2 + \frac{\gamma^2 (1 + |M_A|)^2}{4\varrho} \int_{\Omega} \frac{A(|\nabla u|)|u_1|^2 - \beta}{|x - y|^{\gamma}} \varphi^2
\]

\[(2.10)\]

where we also used the fact that \(A(t)t\) is locally bounded and we have set

\[M = M(\rho, x_0, \gamma, \Omega) = \max \left\{ \sup_{y \in \Omega} \int_{B_{2\rho}(x_0)} \frac{1}{|x - y|^{\gamma}} \, dx, \sup_{y \in \Omega} \int_{B_{2\rho}(x_0)} \frac{1}{|x - y|^{\gamma + 1}} \, dx, \sup_{y \in \Omega} \int_{B_{2\rho}(x_0)} \frac{1}{|x - y|^{\gamma + 1}} \, dx \right\} \]

\[L = L(\rho, x_0) = \sup_{x \in B_{2\rho}(x_0)} |\nabla u|. \]

(2.11)

Exploiting the fact that \(|\nabla \varphi| \leq \frac{2}{\varrho}, |T_\varepsilon(t)| \leq t^{1-\beta}\) and the Young inequality, we also get that

\[
\limsup_{\delta \to 0} (|I_5| + |I_6|) \leq 2 \int_{\Omega} [A(|\nabla u|) + |A'|(|\nabla u|)||\nabla u|||\nabla u_i|| T_\varepsilon(u_i)] \frac{|T_\varepsilon(u_i)|}{|x - y|^{\gamma}} \varphi^2
\]

\[
\leq 4(1 + |M_A|) \int_{\Omega} \frac{1}{\rho} \frac{\sqrt{A(|\nabla u|)|\nabla u_i|| \chi_{\{u_i \leq \varepsilon\}}}}{|x - y|^{\gamma + 1/2}} \varphi \sqrt{\frac{A(|\nabla u|)}{|x - y|^{\gamma + 1}}} u_i \varphi^2
\]

\[
\leq \varrho \int_{\Omega} \frac{1}{|x - y|^{\gamma + 1/2}} \chi_{\{u_i \leq \varepsilon\}} \varphi^2 + \frac{4(1 + |M_A|)^2}{\varrho \rho^2} \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)|u_1|^2 - \beta}{|x - y|^{\gamma}} \varphi^2
\]

\[
\leq \varrho \int_{\Omega} \frac{1}{|x - y|^{\gamma + 1/2}} \chi_{\{u_i \leq \varepsilon\}} \varphi^2 + \frac{4(1 + |M_A|)^2}{\varrho \rho^2} \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)|u_1|^2 - \beta}{|x - y|^{\gamma}} \varphi^2
\]

(2.12)

Now we set \(B = \sup_{x \in B_{2\rho}(x_0)} |b(x)|\) and we get

\[
\limsup_{\delta \to 0} |I_7| \leq q B \int_{\Omega} \frac{|\nabla u|^q - 1|\nabla u_i|| T_\varepsilon(u_i)|}{|x - y|^{\gamma}} \varphi^2
\]

\[
\leq \int_{\Omega} \frac{\sqrt{A(|\nabla u|)|\nabla u_i|}}{|x - y|^{\gamma + 1/2}} \chi_{\{u_i \leq \varepsilon\}} \varphi \frac{|q B|}{\sqrt{A(|\nabla u|)}} |\nabla u|^q \frac{|u_i|^2}{|x - y|^{\gamma}} \varphi
\]
\[
\begin{align*}
&\leq \vartheta \int_\Omega A(|\nabla u|)|\nabla u_i|^2 \frac{\varphi^2}{|x-y|^\gamma} + q^2 B^2 \frac{1}{4\vartheta} \int_\Omega A(|\nabla u|)|x-y|^\gamma \varphi^2 \\
&\leq \vartheta \int_\Omega A(|\nabla u|)|\nabla u_i|^2 \frac{\varphi^2}{|x-y|^\gamma} + q^2 B^2 \frac{1}{4K\vartheta} \int_\Omega |\nabla u|^{1-\beta} \varphi^2 \\
&\leq \vartheta \int_\Omega A(|\nabla u|)|\nabla u_i|^2 \frac{\varphi^2}{|x-y|^\gamma} + q^2 B^2 M\hat{C}_3(L) \frac{1}{4K\vartheta}.
\end{align*}
\] (2.13)

Setting \( B_I = \sup_{x \in B_{2\rho}(x_0)} \sum_{i=1}^N |b_i(x)| \) we also deduce that
\[
\limsup_{\delta \to 0} |I_8| \leq B_I \int_\Omega |\nabla u|^q \frac{|u_i|^{1-\beta}}{|x-y|^\gamma} \varphi^2 \leq B_I \int_{B_{2\rho}(x_0)} |\nabla u|^{q+1-\beta} \leq B_I M\hat{C}_4(L). 
\] (2.14)

Finally, setting \( F = \sup_{x \in B_{2\rho}(x_0)} \sum_{i=1}^N |f_i(x)| \), we get that
\[
\limsup_{\delta \to 0} |I_9| \leq F \int_\Omega \frac{|u_i|^{1-\beta}}{|x-y|^\gamma} \varphi^2 \leq F M\hat{C}_5(L). 
\] (2.15)

Taking into account (2.9), letting \( \delta \to 0 \), exploiting the above estimates and evaluating \( T'_\epsilon \), we get
\[
(1 + m_A) \int_\Omega A(|\nabla u|)|\nabla u_i|^2 \frac{G_\epsilon'(u_i) - \beta G_\epsilon(u_i)}{|u_i|^{1+\beta}} \frac{\varphi^2}{|x-y|^\gamma} - 3\vartheta \int_\Omega A(|\nabla u|)|\nabla u_i|^2 \frac{\varphi^2}{|x-y|^\gamma} + 4(1 + |M_\alpha|)^2 M\hat{C}_3(L) + 4(1 + |M_\alpha|)^2 M\hat{C}_2(L) + q^2 B^2 M\hat{C}_3(L) \\
\leq \frac{\gamma^2(1 + |M_\alpha|)^2 M\hat{C}_3(L) + 4(1 + |M_\alpha|)^2 M\hat{C}_2(L) + q^2 B^2 M\hat{C}_3(L)}{4\vartheta} + B_I M\hat{C}_4(L) + F M\hat{C}_5(L). 
\] (2.16)

Now we fix \( \vartheta \) sufficiently small such that
\[
(1 + m_A) \left( G_\epsilon'(u_i) - \beta \frac{G_\epsilon(u_i)}{|u_i|} \right) - 3\vartheta \chi_{\{|u_i| \geq \varepsilon\}} > 0 
\] (2.17)
so that
\[
\int_{B_\rho(x_0)} \frac{A(|\nabla u|)|\nabla u_i|^2}{|x-y|^\gamma |u_i|^{\beta}} \leq \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)|\nabla u_i|^2}{|x-y|^\gamma |u_i|^{\beta}} \varphi^2 \leq C 
\] (2.18)
where \( C = C(\gamma, \beta, q, f, b, ||\nabla u||_\infty, \rho, x_0) \).

3. Local summability of the weight. We exploit here the summability properties of the second derivatives of the solution proved in Theorem 1.3 to obtain information on the summability of \((A(|\nabla u|))^{-1}\).

**Proof of Theorem 1.4.** Consider
\[
\phi = \frac{1}{(\varepsilon + A(|\nabla u|))^{\alpha}} H_\delta |x-y|^\gamma \varphi^2 = \frac{1}{(\varepsilon + A(|\nabla u|))^{\alpha}} H_\delta \varphi^2 
\] (3.1)
with \( \varepsilon > 0 \), \( \varphi \) and \( H_\delta \) defined as in (2.2) and (2.3). Note that \( \phi \) can be used as test function in (1.5), so that
\[
\int_{B_{2\rho}(x_0)} f(x) \phi = \int_{B_{2\rho}(x_0)} f(x) \frac{1}{(\varepsilon + A(|\nabla u|))^{\alpha}} H_\delta \varphi^2 
\]
is the one given by Theorem 1.3. Now we proceed further observing that

\[ C \] with \( \beta \)

It is convenient to set:

\[ I_a = \int_{B_{2\rho}(x_0)} \frac{1}{(\varepsilon + A(|\nabla u|))^{\alpha} |x-y|^\gamma} \varphi^2 \]

\[ I_a = -\alpha \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)(\nabla u, \nabla |\nabla u|A'(|\nabla u|))}{(\varepsilon + A(|\nabla u|))^{\alpha+1}} H_\delta \varphi^2 \]

\[ I_b = \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)(\nabla u, \nabla \nabla H_\delta)}{(\varepsilon + A(|\nabla u|))^{\alpha}} \varphi^2 \]

\[ I_c = 2 \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)(\nabla u, \nabla \varphi)}{(\varepsilon + A(|\nabla u|))^{\alpha}} H_\delta \varphi \]

\[ I_d = \int_{B_{2\rho}(x_0)} \frac{b(x)|\nabla u|^q}{(\varepsilon + A(|\nabla u|))^{\alpha}} H_\delta \varphi^2. \]

Recalling that we are assuming that the source term \( f \) is positive, we deduce that

\[ c(\rho, x_0) \int_{B_{2\rho}(x_0)} \frac{1}{(\varepsilon + A(|\nabla u|))^{\alpha}} H_\delta \varphi^2 \leq |I_a| + |I_b| + |I_c| + |I_d|. \]

In the following we let \( \delta \to 0 \) and exploit the Young inequality \( ab \leq \vartheta a^2 + \frac{\varphi^2}{4\vartheta} \) and Theorem 1.3, obtaining

\[ \limsup_{\delta \to 0} |I_a| \leq \alpha \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)A'(|\nabla u|)|\nabla u|}{(\varepsilon + A(|\nabla u|))^{\alpha+1}} \sum_{i=1}^{N} |\nabla u_i| \frac{1}{|x-y|^\gamma} \varphi^2 \]

\[ = \alpha \int_{B_{2\rho}(x_0)} \frac{1}{(\varepsilon + A(|\nabla u|))^{\alpha/2}} \frac{A(|\nabla u|)A'(|\nabla u|)|\nabla u|}{(\varepsilon + A(|\nabla u|))^{\alpha+1/2}} \sum_{i=1}^{N} |\nabla u_i| \frac{1}{|x-y|^{\gamma/2}} \varphi^2 \]

\[ \leq \alpha \partial I_o + \frac{M_A^2 N}{4\vartheta} \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)A'(|\nabla u|)N|\nabla u|^2}{(\varepsilon + A(|\nabla u|))^{\alpha+2}} \left( \sum_{i=1}^{N} |\nabla u_i| \right)^2 \frac{1}{|x-y|^{\gamma/2}} \varphi^2 \]

\[ \leq \alpha \partial I_o + \frac{M_A^2 N}{4\vartheta} \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)N|\nabla u|^2}{(\varepsilon + A(|\nabla u|))^{\alpha}} \sum_{i=1}^{N} |\nabla u_i|^2 \frac{1}{|x-y|^{\gamma}} \varphi^2 \]

\[ \leq \alpha \partial I_o + \frac{M_A^2 N}{4\vartheta} \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)N|\nabla u|^2}{|\nabla u|^2} \frac{1}{|x-y|^{\gamma}} \varphi^2 \]

\[ \leq \alpha \partial I_o + \frac{M_A^2 N}{4\vartheta} \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)N|\nabla u|^2}{|\nabla u|^2} \frac{1}{|x-y|^{\gamma}} \varphi^2 \]

with \( \beta := (\alpha - 1)\vartheta \) and observing that \( 0 \leq \beta < 1 \) since \( 1 < \alpha < 1 + \frac{1}{\beta} \). The constant \( C \) is the one given by Theorem 1.3. Now we proceed further observing that

\[ \limsup_{\delta \to 0} |I_b| \leq \gamma \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)|\nabla u|}{(\varepsilon + A(|\nabla u|))^{\alpha+1}} \varphi^2 \]
\[
\begin{align*}
&\leq \gamma \int_{B_{2\rho}(x_0)} \frac{|\nabla u|}{A(|\nabla u|)^{\alpha-1}} \frac{1}{|x-y|^{\gamma+1}} \varphi^2 \\
&\leq \frac{\gamma}{K^{\alpha-1}} \int_{B_{2\rho}(x_0)} \frac{|\nabla u|}{|\nabla u|^{\beta(\alpha-1)}} \frac{1}{|x-y|^{\gamma+1}} \varphi^2 \\
&\leq \frac{M\hat{C}_6(L)}{K^{\alpha-1}}.
\end{align*}
\]

where \( M \) and \( L \) are defined as in (2.11) and we are also using the fact that \( \hat{\vartheta}(\alpha-1) < 1 \). Similarly, recalling that \( |\nabla \varphi| \leq \frac{2}{\rho} \), we get

\[
\begin{align*}
\limsup_{\delta \to 0} |I_c| &\leq 2 \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)|\nabla u||\nabla \varphi|}{(\varepsilon + A(|\nabla u|))^{\alpha}} \frac{1}{|x-y|^{\gamma}} \varphi^2 \\
&\leq \frac{4}{\rho K^{\alpha-1}} \int_{B_{2\rho}(x_0)} \frac{|\nabla u|}{|\nabla u|^{\beta(\alpha-1)}} \frac{1}{|x-y|^{\gamma}} \varphi^2 \\
&\leq \frac{4M\hat{C}_7(L)}{\rho K^{\alpha-1}}.
\end{align*}
\]

As above we use the notation \( B = \sup_{x \in B_{2\rho}(x_0)} |b(x)| \). We have

\[
\begin{align*}
\limsup_{\delta \to 0} |I_d| &\leq \int_{B_{2\rho}(x_0)} \frac{|b(x)||\nabla u|^q}{(\varepsilon + A(|\nabla u|))^{\alpha}} \frac{1}{|x-y|^{\gamma}} \varphi^2 \\
&\leq \int_{B_{2\rho}(x_0)} \frac{1}{(\varepsilon + A(|\nabla u|))^{\alpha}} \frac{B}{|x-y|^{\gamma}} \varphi^2 \\
&\leq \partial I_o + \frac{B^2}{4\vartheta} \int_{B_{2\rho}(x_0)} \frac{|\nabla u|^{2q}}{|\nabla u|^{\beta(\alpha-1)}} \frac{1}{|x-y|^{\gamma}} \varphi^2 \\
&\leq \partial I_o + \frac{B^2}{4\vartheta K^{\alpha}} \int_{B_{2\rho}(x_0)} \frac{|\nabla u|^{2q}}{|\nabla u|^{\beta(\alpha-1)}} \frac{1}{|x-y|^{\gamma}} \varphi^2 \\
&\leq \partial I_o + \frac{B^2 M\hat{C}_6(L)}{4\vartheta K^{\alpha}}.
\end{align*}
\]

observing that \( 2q > \alpha \hat{\vartheta} \) since \( q > \frac{\alpha+1}{2} \) and \( 1 < \alpha < 1 + \frac{1}{\beta} \).

Collecting the previous estimates, by (3.4), and letting \( \delta \to 0 \) we have

\[
\begin{align*}
(c(\rho, x_0) - (\alpha + 1)\hat{\vartheta}) \int_{B_{2\rho}(x_0)} \frac{1}{(\varepsilon + A(|\nabla u|))^{\alpha}} &\frac{1}{|x-y|^{\gamma}} \varphi^2 \\
&\leq \frac{M^2}{4\vartheta} \frac{N^2 C}{\alpha} + \gamma \frac{M\hat{C}_6(L)}{K^{\alpha-1}} + \frac{4M\hat{C}_7(L)}{\rho K^{\alpha-1}} + \frac{B^2 M\hat{C}_8(L)}{4\vartheta K^{\alpha}}.
\end{align*}
\]

For \( \hat{\vartheta} \) sufficient small such that \( (c(\rho, x_0) - (\alpha + 1)\hat{\vartheta}) > 0 \), letting \( \varepsilon \to 0 \), we get the thesis

\[
\begin{align*}
\int_{B_{\rho}(x_0)} \frac{1}{(A(|\nabla u|))^{\alpha}} &\frac{1}{|x-y|^{\gamma}} \leq \int_{B_{2\rho}(x_0)} \frac{1}{(A(|\nabla u|))^{\alpha}} |x-y|^{\gamma} \varphi^2 \leq C
\end{align*}
\]

where \( C = C(\gamma, \beta, q, f, b, ||\nabla u||_{\infty}, \rho, x_0, \alpha) \). \( \square \)

4. Global results. In this section we deduce global regularity information, starting from the local regularity results already proved. Let us define the neighborhood
Therefore \( I_3(\partial \Omega) = \{ x \in \Omega | d(x, \partial \Omega) \leq \delta \} \). Without loss of generality, in all the section, we will assume that

\[
\Omega \setminus I_3(\partial \Omega) \subset \bigcup_{i=1}^{S} B_{\rho}(x_i)
\]

and \( x_i \in \overline{\Omega \setminus I_3(\partial \Omega)} \) and \( \rho < \delta \). We will state our results under the general assumption

\[
Z_u \cap \partial \Omega = \emptyset.
\]

This assumption is actually verified in all the situations when the Hopf boundary Lemma holds, we refer therefore to [9].

**Theorem 4.1.** Let \( \Omega \subset \mathbb{R}^N \) a bounded smooth domain and let \( u \in C^{1,\alpha}(\bar{\Omega}) \cap C^2(\Omega \setminus Z_u) \) be a solution to (1.1) with \( f, b \in W^{1,\infty}(\Omega) \). Then for every \( i = 1, ..., N, 0 \leq \beta < 1 \) and \( \gamma < N - 2 \) (\( \gamma = 0 \) if \( N = 2 \)), we have

\[
\int_{K} \frac{A(\nabla u) \nabla u_i^2}{|x-y|^\gamma |u_i|^\beta} \, dx \leq C^*(K) \quad \forall y \in \Omega,
\]

for any compact set \( K \subset \Omega \). If we also assume that \( Z_u \cap \partial \Omega = \emptyset \), then

\[
\int_{\Omega} \frac{A(\nabla u) \nabla u_i^2}{|x-y|^\gamma |u_i|^\beta} \, dx \leq C^* \quad \forall y \in \Omega.
\]

**Proof.** The proof follows via a covering argument. We prove directly the estimate in (4.2), since the estimate in (4.1) follows with the same proof more easily. In all the proof the reader should take into account that we are integrating with respect the \( x \)-variable, and the center of the kernel \( y \) is varying all over the domain. Under our assumptions, we can take \( \delta > 0 \) such that there are no critical points of the solution in the neighborhood \( I_3(\partial \Omega) \). It follows therefore in this case that \( A(\nabla u) > 0 \) in \( I_3(\partial \Omega) \) and \( u \in C^2(3\delta(\partial \Omega)) \). We set

\[
\begin{align*}
\overline{M} &= \max \left\{ \sup_{y \in \Omega} \int \frac{1}{|x - y|^\gamma} \, dx; \sup_{y \in \Omega} \int \frac{1}{|x - y|^{\gamma+1}} \, dx; \sup_{y \in \Omega} \int \frac{1}{|x - y|^{\gamma+2}} \, dx \right\} \\
\mathcal{L} &= \sup_{x \in \Omega} |\nabla u| \\
\overline{B} &= \sup_{x \in \Omega} |b(x)|, \quad \overline{B}_I = \sup_{x \in \Omega} \sum_{i=1}^{N} |b_i(x)| \\
\overline{F} &= \sup_{x \in \Omega} \sum_{i=1}^{N} |f_i(x)|
\end{align*}
\]

and, repeating verbatim the argument of the proof of Theorem 1.3 with the new notations, we get that

\[
\int_{B_\rho(x_i)} \frac{A(\nabla u) \nabla u_i^2}{|x-y|^\gamma |u_i|^\beta} \leq \tilde{C}(\gamma, \beta, q, m, M, \overline{M}, \mathcal{L}, \overline{B}, \overline{F}).
\]

Therefore

\[
\int_{\Omega \setminus I_3(\partial \Omega)} \frac{A(\nabla u) \nabla u_i^2}{|x-y|^\gamma |u_i|^\beta} \leq \sum_{i=1}^{S} \int_{B_\rho(x_i)} \frac{A(\nabla u) \nabla u_i^2}{|x-y|^\gamma |u_i|^\beta} \leq S \tilde{C}.
\]
we also have that
\[ 
\mathcal{A} = \sup_{x \in I_\delta} A(|\nabla u|) |\nabla u|^\beta < \infty 
\]
\[ 
\mathcal{B} = \sup_{x \in I_\delta(\partial \Omega)} \sum_{i,j} |u_{ij}|^2 
\]
we also have that
\[ 
\int_{I_\delta(\partial \Omega)} A(|\nabla u|) |\nabla u|^2 \leq \int_{I_\delta(\partial \Omega)} \frac{A(|\nabla u|) |\nabla u|^{-\beta} |\nabla u|^2}{|x-y|} \leq \mathcal{A} \mathcal{D} \mathcal{M}. 
\] (4.6)

Finally, by (4.5) and (4.7), we deduce that
\[ 
\int_\Omega A(|\nabla u|) |\nabla u| \leq \int_{\Omega \setminus I_{3\delta}(\partial \Omega)} A(|\nabla u|) |\nabla u|^2 + \int_{I_{3\delta}(\partial \Omega)} \frac{A(|\nabla u|) |\nabla u|^2}{|x-y|^{\gamma} |u_i|^2} \leq S\mathcal{C} + \mathcal{A} \mathcal{D} \mathcal{M} = \mathcal{C}^*. 
\] (4.8)

We prove here a global summability result for \((A|\nabla u|)^{-1}\) using Theorem 4.1.

Let as above \(I_\delta\) be the neighborhood of \(\partial \Omega\) of radius \(\delta\) and consider the same covering
\[ 
\Omega \setminus I_{3\delta}(\partial \Omega) \subset \bigcup_{i=1}^S B_{\rho}(x_i) 
\]
with \(x_i \in \Omega \setminus I_{3\delta}(\partial \Omega)\) and \(\rho < \delta\). We set
\[ 
\bar{\lambda} = \inf_{x \in I_\delta} A(|\nabla u|) 
\]
\[ 
\mu = \inf_{x \in \Omega \setminus I_\delta} f(x) . 
\] (4.9)

**Theorem 4.2.** Let \(\Omega \subset \mathbb{R}^N\) a bounded smooth domain and let \(u \in C^{1,\alpha}(\Omega) \cap C^2(\Omega \setminus Z_u)\) be a solution to (1.1) with \(f, b \in W^{1,\infty}(\Omega)\). Assume that \(f\) is positive in \(\Omega\) (possibly vanishing on the boundary). Then, for every compact set \(K \subset \Omega\), we have
\[ 
\int_K \frac{1}{(A(|\nabla u|))^{\alpha}} \frac{1}{|x-y|^\gamma} \leq \mathcal{C}^*(K) 
\] (4.10)
with \(1 < \alpha < 1 + \frac{1}{\beta},\) \(\gamma < N - 2,\) if \(N \geq 3\) and \(\gamma = 0\) if \(N = 2\) and \(\mathcal{C}^* = \mathcal{C}^*(K, \gamma, \mu, \bar{\lambda}, m_A, M_A, \alpha, f, ||\nabla u||_{\infty})\). If we further assume that \(Z_u \cap \partial \Omega = \emptyset\), then
\[ 
\int_{\Omega} \frac{1}{(A(|\nabla u|))^{\alpha}} \frac{1}{|x-y|^\gamma} \leq \mathcal{C}^* . 
\] (4.11)

**Proof.** We deal directly with the more difficult case, namely we prove (4.11). In all the proof the reader should take into account that we are integrating with respect the \(x\)-variable, and the center of the kernel \(y\) is varying all over the domain. Under our assumptions, we can take \(\delta > 0\) such that there are no critical points of the solution in the neighborhood \(I_{3\delta}(\partial \Omega)\). It follows therefore that \(\bar{\lambda} > 0\) in \(I_{3\delta}\) and \(u \in C^2(3\delta(\partial \Omega))\). Furthermore, since \(f\) is positive in the interior of \(\Omega\), we can also assume that \(\mu > 0\). By Theorem 4.2, since \(\mu > 0\), we get that
\[ 
\int_{B_{\rho}(x_i)} \frac{1}{(A(|\nabla u|))^{\alpha}} \frac{1}{|x-y|^\gamma} \leq \hat{\mathcal{C}}(\gamma, \alpha, \mu, q, m_A, M_A, \bar{\lambda}, \bar{B}, \bar{F}) . 
\] (4.12)
where \( \overline{M}, \overline{L} \) and \( \overline{B} \) are as in (4.3). Since \( \hat{\lambda} > 0 \) we also have that
\[
\int_{I_{\partial \Omega}} \frac{1}{(A(|\nabla u|))^{\alpha}} \frac{1}{|x-y|^\gamma} \leq \frac{1}{\lambda^\alpha} \overline{M}.
\] (4.13)

Consequently the thesis follows now via a standard covering argument, since we have
\[
\int_{\Omega} \frac{1}{(A(|\nabla u|))^{\alpha}} \frac{1}{|x-y|^\gamma} \leq \int_{I_{\partial \Omega}} \frac{1}{(A(|\nabla u|))^{\alpha}} \frac{1}{|x-y|^\gamma} + \int_{\Omega \setminus I_{\partial \Omega}} \frac{1}{(A(|\nabla u|))^{\alpha}} \frac{1}{|x-y|^\gamma} \leq \frac{1}{\lambda^\alpha} \overline{M} + \sum_{i=1}^{S} \int_{B_{\delta}(x_i)} \frac{1}{(A(|\nabla u|))^{\alpha}} \frac{1}{|x-y|^\gamma} \leq \frac{1}{\lambda^\alpha} \overline{M} + S\hat{C} = \overline{C}^*.
\] (4.14)

We are now ready to end the paper with the proof of Theorem 1.5.

**Proof of Theorem 1.5.** For any compact set \( K \subset \Omega \), we have
\[
\int_{K} |\nabla u_i|^s \, dx = \int_{K} \frac{A(|\nabla u|)^{\frac{s}{2}} |\nabla u_i|^s}{|u_i|^\beta} \frac{|u_i|^{\beta - \frac{s}{2}}}{A(|\nabla u|)^{\frac{s}{2}}} \, dx
\]
\[
\leq \left( \int_{K} \frac{A(|\nabla u|)|\nabla u_i|^2}{|u_i|^\beta} \, dx \right)^{\frac{s}{2}} \left( \int_{K} \frac{|u_i|^{\beta - \frac{s}{2}}}{A(|\nabla u|)^{\frac{s}{2}}} \, dx \right)^{\frac{2-s}{2}}
\]
\[
\leq (\overline{C}^*(K))^{\frac{s}{2}} K^{-\beta \frac{s}{2}} \left( \int_{K} \frac{A(|\nabla u|)^{\frac{\beta - s}{2}}}{A(|\nabla u|)^{\frac{s}{2}}} \, dx \right)^{\frac{2-s}{2}}.
\] (4.15)

Here we exploit (1.4) and \( \overline{C}^*(K) \) is the constant arising from Theorem 4.1. Under our assumption, namely for \( s < 1 + \frac{1}{2} \), we can fix \( \beta < 1 \) with \( 1 - \beta \) small such that
\[
\frac{s}{2-s} - \frac{s}{(2-s)\beta} > 1 + \frac{1}{\hat{\theta}}.
\]

This allows to use Theorem 4.2 to deduce that
\[
\int_{K} |\nabla u_i|^s \, dx \leq (\overline{C}^*(K))^{\frac{s}{2}} K^{-\beta \frac{s}{2}} (\overline{C}^*(K))^{\frac{2-s}{2}}.
\] (4.16)

It remains to show that \( u_{ij} \), which are defined as in Remark 1.2, are actually the distributional derivatives. To do this we note that, for \( \varphi \in C_c^\infty(K) \), we have:
\[
\int_{K} u_{ij} \varphi \frac{G_x(|\nabla u|)}{|\nabla u|} = - \int_{K} u_{ij} \varphi x \frac{G_x(|\nabla u|)}{|\nabla u|} - \int_{K \setminus \{x \in |\nabla u| < 2x \}} |\nabla u| |\nabla u_j| \varphi \frac{1}{x},
\] (4.17)

since \( u \) is smooth outside the critical set \( Z_u \) and \( \frac{G_x(|\nabla u|)}{|\nabla u|} \) vanishes in a neighborhood of \( Z_u \). Taking into account (4.16) we can pass to the limit via the dominated convergence theorem. Since \( \frac{G_x(|\nabla u|)}{|\nabla u|} \to \chi(K \setminus Z_u) \) in \( K \) as \( \varepsilon \) tends to zero, we get the claim.

The same proof works to prove that \( u \in W^{2,s}(\Omega) \) for any \( s < \min\{2; 1 + \hat{\theta}^{-1}\} \) once that (4.2) and (4.11) are in force.
\[\square\]
REFERENCES

[1] A. Canino, P. Le and B. Sciunzi, Local $W^{2, m}_{\text{loc}}$ regularity for $p(\cdot)$-Laplace equations, *Manuscripta Mathematica*, **140** (2013), 481–496.

[2] L. Damascelli and B. Sciunzi, Regularity, monotonicity and symmetry of positive solutions of $m$-Laplace equations, *J. Differential Equations*, **206** (2004), 483–515.

[3] E. Di Benedetto, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, *Nonlinear Anal.*, **7** (1983), 827–850.

[4] T. Kuusi and G. Mingione, Universal potential estimates, *J. Funct. Anal.*, **262** (2012), 4205–4269.

[5] G. M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, *Nonlinear Anal.*, **12** (1988), 1203–1219.

[6] C. Mercuri, G. Riey and B. Sciunzi, A regularity result for the $p$-Laplacian near uniform ellipticity, *Siam J. Math. Anal.*, **48** (2016), 2059–2075.

[7] G. Mingione, Regularity of minima: An invitation to the dark side of the calculus of variations, *Applications of Mathematics*, **51** (2006), 355–426.

[8] G. Mingione, The Calderon-Zygmund theory for elliptic problems with measure data, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.(5)*, **6** (2007), 195–261.

[9] P. Pucci and J. Serrin, *The Maximum Principle*, Birkhauser, Boston, 2007.

[10] B. Sciunzi, Some results on the qualitative properties of positive solutions of quasilinear elliptic equations, *NoDEA. Nonlinear Differential Equations and Applications*, **14** (2007), 315–334.

[11] B. Sciunzi, Regularity and comparison principles for $p$-Laplace equations with vanishing source term, *Comm. Cont. Math.*, **16** (2014), 1450013, 20pp.

[12] E. Teixeira, Regularity for quasilinear equations on degenerate singular sets, *Math. Ann.*, **358** (2014), 241–256.

[13] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, *J. Differential Equations*, **51** (1984), 126–150.

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