Nonhomogeneous Quadratic Duality and Curvature

L. E. Positselski

Introduction

A quadratic algebra is a graded algebra with generators of degree 1 and relations of degree 2. Let \( A \) be a quadratic algebra with the space of generators \( V \) and the space of relations \( I \subset V \otimes V \). The classical quadratic duality assigns the quadratic algebra \( A! \) with generators from \( V^* \) and the relations \( I^\perp \subset V^* \otimes V^* \) to the algebra \( A \). According to the classical results of Priddy and Löfwall [1, 3], \( A! \) is isomorphic to the subalgebra of \( \text{Ext}_A^*(k, k) \) generated by \( \text{Ext}_A^1(k, k) \). Priddy called an algebra \( A \) a Koszul algebra if this subalgebra coincides with the whole of \( \text{Ext}_A^*(k, k) \). Koszul algebras constitute a wonderful class of quadratic algebras, which is closed under a large set of operations, contains the main examples, and perhaps admits a finite classification.

In this paper, we propose an extension of the quadratic duality to the nonhomogeneous case. Roughly speaking, a nonhomogeneous quadratic algebra (or a quadratic-linear-scalar algebra, a QLS-algebra) is an algebra defined by (generators and) nonhomogeneous relations of degree 2. A quadratic-linear algebra (QL-algebra) is an algebra defined by nonhomogeneous quadratic relations without the scalar parts; in other words, it is an augmented QLS-algebra. The precise definition takes into account the fact that a collection of nonhomogeneous relations does not necessarily “make sense” (its coefficients must satisfy some equations; the Jacobi identity is a classical example).

The dual object for a QL-algebra is [6] a quadratic DG-algebra [7]. The dual object for a QLS-algebra is a set of data which we call a quadratic CDG-algebra (“curved”), defined up to an equivalence. The classical Poincaré–Birkhoff–Witt theorem on the universal enveloping algebra structure [9] finds its natural place in this context as a particular case of the fact that every Koszul CDG-algebra corresponds to a QLS-algebra.

Remarkable examples of nonhomogeneous quadratic duality are provided by Differential Geometry. The algebra of differential operators on a manifold may be considered as a QL-algebra defined by the commutation relations for the vector fields. The dual object for this algebra is the de Rham complex; the corresponding equivalence of the categories of modules is constructed in [4]. The algebra of differential operators in a vector bundle is a QLS-algebra. The dual object is the algebra of differential forms with coefficients in linear operators in this bundle and with the exterior differential defined by means of a connection. Its square is not zero—it is equal to the commutator with the curvature; the equivalence relation mentioned in the previous paragraph corresponds to changing the connection. Thus, the curvature corresponds to the scalar part of the relations.

A question arises about the obstructions to existence of a QL-algebra structure (i.e., of “a flat connection”) on a given QLS-algebra. In the present paper we construct obstructions of this kind generalizing the Chern classes of vector bundles [8] and (which is less evident) the Chern–Weil classes of principal \( G \)-bundles. Our analogues of the secondary characteristic classes [5] form the Chern–Simons functor on the category of CDG-algebras.
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Conventions and notation. An algebra is an associative algebra with a unit $e$ over a fixed ground field $k$; all graded and filtered algebras are assumed to be locally finite-dimensional. The symbol $[a, b]$ denotes the supercommutator $ab - (-1)^{\tilde{a}\tilde{b}}ba$.

§1. Definitions

Definition 1. A weak QLS-algebra is an algebra $A$ together with a subspace of generators $W \subset A$, $e \in W$, satisfying the following conditions. Let $V$ be a hyperplane in $W$ complementary to $k \cdot e$, $T(V)$ the tensor algebra, $T_n(V)$ the subspace of elements of degree $\leq n$, and $J$ the kernel of the natural projection $T(V) \to A$. It is required that

1) an algebra $A$ be generated by its subspace $W$;
2) the ideal $J$ be generated by its subspace $J_2 = J \cap T_2(V)$.

(Clearly, this condition does not depend on the choice of $V$.) In this case, the underlying quadratic algebra $A^{(0)}$ is defined by the generators $V \simeq W/k \cdot e$ and the relations $I = J_2 \mod J_1 \subset V \otimes V$.

Definition 2. A QLS-algebra is an algebra $A$ together with a filtration $F$: $0 \subset F_0 A \subset F_1 A \subset F_2 A \subset \cdots \subset A$, $\bigcup F_i A = A$, $F_0 A = k$, $F_i A \cdot F_j A \subset F_{i+j} A$, such that the associated graded algebra $\text{Gr}_F A$ is quadratic.

A (weak) QLS-algebra $A$ is said to be Koszul if $A^{(0)}$ is a Koszul algebra. The second of our two definitions implies the first one for $W = F_1 A$. We shall show in 3.3 that in the Koszul case these definitions are equivalent.

Definition 3. A (weak) QL-algebra is a (weak) QLS-algebra together with an augmentation (a ring homomorphism) $\varepsilon: A \to k$; let $A_+ = \text{Ker} \varepsilon$ denote the augmentation ideal.

A morphism of weak QLS-algebras is an algebra homomorphism preserving the subspace $W$. A morphism of QLS-algebras is an algebra homomorphism preserving the filtration. A morphism of weak QL-algebras is an algebra homomorphism preserving $W$ and the augmentation. This defines the category $\mathcal{WQLS}$ of weak QLS-algebras and its full subcategory $\mathcal{QLS}$, as well as the category $\mathcal{WQL}$ of weak QL-algebras and its full subcategory $\mathcal{QL}$. The categories of Koszul QLS- and QL-algebras are denoted by $\mathcal{KLS}$ and $\mathcal{KL}$, respectively.

Now we define the dual objects.

Definition 4. A DG-algebra $B^\bullet$ is a graded algebra (with upper indices) together with a derivation $d: B \to B$ of degree +1 ($d(a \cdot b) = d(a) \cdot b + (-1)^{\tilde{a}}a \cdot d(b)$) such that $d^2 = 0$. A morphism of DG-algebras is a homomorphism of graded algebras commuting with the derivations. We assume below that $B^i = 0$ for $i < 0$. A DG-algebra $B^\bullet$ is said to be quadratic (Koszul) if the algebra $B$ is quadratic (Koszul). Let $\mathcal{QDG}$ and $\mathcal{KDG}$ be the categories of quadratic and Koszul DG-algebras, respectively.
**Definition 5.** A CDG-algebra is a triple $\Psi = (B, d, h)$, where $B$ is a graded algebra, $d$ is a derivation of $B$ of degree $+1$, and $h \in B^2$, such that

1) $d^2 = [h, \cdot]$, 
2) $d(h) = 0$.

In the sequel we assume that $B^i = 0$ for $i < 0$. A CDG-algebra $\Psi$ is called *quadratic* (Koszul) if the algebra $B$ is quadratic (Koszul).

**Definition 6.** A *morphism of CDG-algebras* $\varphi: \Psi \to \Psi'$ is a pair $\varphi = (f, \alpha)$, where $f: B \to B'$ is a homomorphism of graded algebras and $\alpha \in B'^1$, satisfying the conditions

1) $d'f(x) = f(dx) + [\alpha, f(x)]$, 
2) $h' = f(h) + d'\alpha - \alpha^2$.

The composition $(f, \alpha) \circ (g, \beta)$ is the morphism $(f \circ g, \alpha + f(\beta))$. The *identity morphism* is $(\text{id}, 0)$. This defines the *category* $\mathcal{CDG}$ of CDG-algebras and its full subcategories $\mathcal{QCDG}$ and $\mathcal{KCDG}$ of quadratic and Koszul CDG-algebras, respectively. Two CDG-algebras $\Psi$ and $\Psi'$ are called *equivalent* if $B = B'$ and there exists a morphism of the form $(\text{id}, \alpha): \Psi \to \Psi'$, in other words, if $d' = d + [\alpha, \cdot]$ and $h' = h + d\alpha + \alpha^2$; in this case we write $\Psi' = \Psi(\alpha)$.

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**§2. Duality Functor**

**2.1.** Let $A$ be a weak QLS-algebra with the space of generators $W \subset A$. Set $B = A^{(0)}!$; we will denote by upper indices the grading on $B$. Choose a hyperplane $V \subset W$ complementary to $k \cdot e$ in $W$; we have $J_2 \subset k \oplus V \oplus V$. Note that $J_2 \cap (k \oplus V) = 0$ and $J_2 \mod (k \oplus V) = I$. Thus, $J_2$ can be represented as the graph of a linear map $I \to k \oplus V$, which we will denote by $(-h, -\varphi)$, where $h \in I^* \cong B^2$ and $\varphi: I \to V$. Let $d_1 = \varphi^*: B^1 \to B^2$; then the relations in the algebra $A$ can be written in the form

$$p + \varphi(p) + h(p) = 0, \quad p \in I \subset V \otimes V, \quad \varphi = d_1^*.$$  \hfill (*)

**2.2. Proposition.** The map $d_1: B^1 \to B^2$ can be extended to a derivation $d$ of the algebra $B$. The triple $(B, d, h)$ is a CDG-algebra.

**Proof.** Tensoring the relation (*) by $V$ on the left and on the right, we obtain

$$q + \varphi_{12}(q) + h_{12}(q) = 0 = q + \varphi_{23}(q) + h_{23}(q) \quad (\text{mod } J)$$

for any $q \in V \otimes I \cap I \otimes V$, whence $-\varphi_{12}(q) + \varphi_{23}(q) = h_{12}(q) - h_{23}(q) \mod J$. The latter equation implies

$$\varphi_{12}(q) - \varphi_{23}(q) \in I$$  \hfill (1)

and

$$\varphi(-\varphi_{12}(q) + \varphi_{23}(q)) + h(-\varphi_{12}(q) + \varphi_{23}(q)) + h_{12}(q) - h_{23}(q) = 0 \quad \text{(mod } J).$$

Since the second summand lies in $k$, while the first and the third one belong to $V$, we have

$$\varphi(\varphi_{12}(q) - \varphi_{23}(q)) = h_{12}(q) - h_{23}(q), \quad \hfill (2)$$

$$h(\varphi_{12}(q) - \varphi_{23}(q)) = 0. \quad \hfill (3)$$
Dualizing (1), (2), and (3) and taking into account the fact that the operator \( d_2 = (\varphi_{12} - \varphi_{23})^* \) continues \( d_1 \) by the Leibniz rule and that \( (h_{12} - h_{23})^* = [h, \cdot] \), we obtain, respectively, the equations (mod \( I^\perp \otimes B^1 + B^1 \otimes I^\perp \)):

\[
d_2(I^\perp) = 0, \quad d_2 \circ d_1 = [h, \cdot], \quad d_2(h) = 0.
\]

The first equation means that \( d_1 \) can be extended to \( B \); the second and the third one are equivalent to the CDG-algebra axioms. □

2.3. Proposition. Let \( V \) and \( V' \) be two direct complements to \( k \cdot e \) in \( W \), and let \( \Psi \) and \( \Psi' \) be the corresponding CDG-algebras. Then \( \Psi' = \Psi(\alpha) \) for a uniquely defined element \( \alpha \in B^1 \).

Proof. Suppose that for \( \alpha \in B^1 = V^* \) we have \( V' = \{v + \alpha(v), \ v \in V\} \), and let \( \varphi \) and \( h \) correspond to the complementary hyperplane \( V \). Then for any \( p \in I \subset V \otimes V \) we have

\[
0 = p + \varphi(p) + h(p) = [p + \alpha_1(p) + \alpha_2(p) + \alpha \otimes \alpha(p)] + [\varphi(p) - \alpha_1(p) - \alpha_2(p) + \alpha(\varphi(p) - \alpha_1(p) - \alpha_2(p))] + [h(p) - \alpha(\varphi(p)) + \alpha \otimes \alpha(p)].
\]

Thus, the operators

\[
\varphi' = \varphi - \alpha_1 - \alpha_2, \quad h' = h - \alpha \circ \varphi + \alpha \otimes \alpha
\]

correspond to the choice of the direct complement \( V' \). Dualizing and using the fact that \( (\alpha_1 + \alpha_2)^* = [\alpha, \cdot] \) and \( \alpha \circ \varphi = d(\alpha) \), we obtain \( d_1' = d_1 - [\alpha, \cdot] \) and \( h' = h - d(\alpha) + \alpha^2 \). □

2.4. Proposition. The construction of Subsection 2.1 defines a fully faithful contravariant functor \( D = D_{QLS} : WQLS \rightarrow QCDG \).

Proof. To define the functor on objects, choose the subspace \( V \) arbitrarily for every weak QLS-algebra. The natural isomorphism

\[
\{f \in \text{Hom}(k \oplus V, k \oplus V') : f|_k = \text{id}\} \simeq \text{Hom}(V, V') \oplus V^*,
\]

together with Proposition 2.3, allows to define it on morphisms. It is obviously fully faithful. □

2.5. Let \( A \) be a weak QL-algebra with the augmentation ideal \( A_+ \). Set \( V = A_+ \cap F_1 \). Then it is easy to see that \( h = 0 \), and we obtain a DG-algebra \( (B, d) \).

Proposition. This defines a fully faithful contravariant functor \( D = D_{QL} : WQL \rightarrow QDG \).

2.6. Conversely, let \( \Psi \) be a quadratic CDG-algebra. Set \( V = B^*_1, \ I = B^*_2 \hookrightarrow V \otimes V \), let \( A = A(\Psi) \) be the algebra with generators from \( V \) and relations \( (\ast) \), let \( W \) be the image of \( k \oplus V \) in \( A \), and put \( F_n A = W^n \). It is easy to see that if \( (A(\Psi), W) \) is a weak QLS-algebra and \( A^{(0)!} \simeq B \), then \( D(A, W) = \Psi \).
2.7. Examples. 1. Let $\mathfrak{g}$ be a Lie algebra. Then the enveloping algebra $U\mathfrak{g}$ is a QL-algebra, and any QL-algebra $A$ for which $\text{Gr}_F A$ is a symmetric algebra can be obtained in this way. The dual DG-algebra $(A^*\mathfrak{g}^*, d)$ is the standard cohomological complex of the Lie algebra $\mathfrak{g}$. The QLS-algebras $A$ for which $\text{Gr}_F A$ is a symmetric algebra correspond to central extensions of Lie algebras: the algebra $A = U\mathfrak{g}^*/(1-c)$ is assigned to a central extension $0 \to k \cdot c \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$. The dual object is $\Psi = (A^*\mathfrak{g}^*, d, h)$, where $h$ is the cocycle of the central extension.

2. The Clifford algebra $\{vw + wv = Q(v, w), v, w \in V\}$ is a QLS-algebra, and all QLS-algebras with $\text{Gr}_F A = A^*V$ are Clifford algebras. The dual object is $\Psi = (S^*V, 0, Q)$. All QL-algebras with $\text{Gr}_F A = A^*V$ have the form $\{vw + wv = \lambda(v)w + \lambda(w)v\}$, $\lambda \in V^*$.

3. Let $A = k \oplus A_+$ be a (finite-dimensional) augmented algebra. Let us endow $A$ with the structure of a QL-algebra by setting $F_i A = A$ for $i \geq 1$. Then the dual DG-algebra is the reduced cobar-construction for $A$, $D(A, F) = C^*(A) = \sum_{n=0}^{\infty} A_+^{\otimes n}$.

2.8. Remark. Under the quadratic duality, commutative algebras correspond to universal enveloping algebras of Lie algebras. In particular, we have:

a) a duality between commutative QLS-algebras and quadratic Lie CDG-algebras, and

b) a duality between Lie QL-algebras and supercommutative quadratic DG-algebras.

§3. Bar Construction

3.1. Bar-complex for CDG-algebras. The following construction is due to A. E. Polishchuk. Let $\Psi = (B, d, h)$ be a CDG-algebra, $B_i = 0$ for $i < 0$, $B_0 = k$. Put $B(\Psi) = \sum_{n=0}^{\infty} B_+^{\otimes n}$, where $B_+ = \sum_{i=1}^{\infty} B_i$, and, denoting by $(b_1 | b_2 | \ldots | b_n)$ the element $b_1 \otimes b_2 \otimes \ldots \otimes b_n \in B(\Psi)$, endow $B(\Psi)$ with a coalgebra structure:

$$\Delta(b_1 | b_2 | \ldots | b_n) = \sum_{k=0}^{n}(b_1 | \ldots | b_k) \otimes (b_{k+1} | \ldots | b_n).$$

There are two gradings on $B(\Psi)$, namely, the internal and the homological ones:

$$\deg_i(b_{i_1} | b_{i_2} | \ldots | b_{i_n}) = i_1 + i_2 + \ldots + i_n, \quad \deg_h(b_{i_1} | b_{i_2} | \ldots | b_{i_n}) = n$$

for $b_{i_j} \in B_{i_j}$; set $B^k = \{b : \deg_i b - \deg_h b = k\}$. Let us define the differentials $\partial$, $d$, and $\delta$ on $B^*$ (of bidegrees $(0, -1)$, $(1, 0)$, and $(2, 1)$, respectively) by the formulas

$$\partial(b_{i_1} | \ldots | b_{i_n}) = \sum_{k=1}^{n-1}(-1)^{i_1+\ldots+i_k+k-1}(b_{i_1} | \ldots | b_{i_k} b_{i_{k+1}} | \ldots | b_{i_n}),$$

$$d(b_{i_1} | \ldots | b_{i_n}) = \sum_{k=1}^{n}(-1)^{i_1+\ldots+i_{k-1}+k-1}(b_{i_1} | \ldots | d(b_{i_k}) | \ldots | b_{i_n}),$$

$$\delta(b_{i_1} | \ldots | b_{i_n}) = \sum_{k=1}^{n+1}(-1)^{i_1+\ldots+i_{k-1}+k-1}(b_{i_1} | \ldots | b_{i_{k-1}} | h | b_{i_k} | \ldots | b_{i_n}).$$
It is straightforward to check that $\partial$, $d$, and $\delta$ are superderivations of the coalgebra $\mathcal{B}$ and $(d + \partial + \delta)^2 = 0$. Let $(\mathcal{C}_\bullet(\Psi), D)$ be the dual DG-algebra to the DG-coalgebra $\mathcal{B}^\bullet(\Psi)$:

$$\mathcal{C}_\bullet(\Psi) = \sum_{n=1}^{\infty} \left( \sum_{i=1}^{\infty} B_i \right)^{\otimes n}, \quad D = (\partial + d + \delta)^*. $$

### 3.2. Definition.

The bar-cohomology algebra of a CDG-algebra $\Psi$ is the homology algebra of the DG-algebra $\mathcal{C}_\bullet(\Psi)$, $H_{\text{bar}}(\Psi) = H_{\bullet}(\mathcal{C}(\Psi), D)$.

**Proposition** (Löfwall’s subalgebra theorem). If $\Psi$ is quadratic, then $H_{\text{bar}}^0(\Psi)$ is isomorphic to the algebra $A(\Psi)$ constructed in 2.6. The filtration $F_n = F_1^n$ on $A$ is induced by the deg$_i$-filtration of the cobar-complex $\mathcal{C}_\bullet(\Psi)$.

The proof is immediate. □

**Corollary.** If $A$ is a weak QLS-algebra, then $H_{\text{bar}}^0(D(A)) = A$.

### 3.3. Poincaré–Birkhoff–Witt theorem.

**Theorem.** Let $\Psi = (B, d, h)$ be a Koszul CDG-algebra. Then there is an isomorphism $\text{Gr} A(\Psi) \simeq B^!$.

**Proof.** (We shall see that the Koszul condition can be weakened to the requirement that $\text{Ext}^i_B(k, k)_{i+1} = 0$ for all $i$, or, equivalently $\text{Ext}^3_B(k, k) = 0$ for $i \geq 4$.) There is a spectral sequence $E^1_{p,q} = \text{Ext}_{B^!}(k, k)_p \Rightarrow H_{p+q}^b(\Psi)$ induced by the deg$_i$-filtration on $\mathcal{C}_\bullet(\Psi)$. Since $B$ is Koszul, we have $E^1_{p,q} = 0$ for $p + q \neq 0$, and $E^r_{p,q}$ degenerates at the term $E^1$. Therefore, $\text{Gr} H_{\bullet}(B, d, h) = \text{Ext}_B(k, k) = B^!$. □

**Corollary.** Any Koszul weak QLS-(QL-)algebra $A$ is a QLS-(QL-)algebra, and $H^b(D(A)) = A$. The duality functors

$$D_{\text{QL}}: \mathcal{K}L \to \mathcal{K}DG \text{ and } D_{\text{QLS}}: \mathcal{K}LS \to \mathcal{K}CDG$$

are antiequivalences of categories.

**Proof.** Apply Subsection 2.6, the proof of the theorem, and the fact that the algebras $B$ and $B^!$ are Koszul simultaneously. □

### 3.4. Without the Koszul condition the statement of Theorem 3.3 fails. A counterexample [6]: the relations

$$xy = x + y, \quad x^2 + yz = z$$

imply $yz = zy$, although they have the form $(\ast)$ for a certain DG-algebra.

§4. An Example: $D$-$\Omega$-Duality

Strictly speaking, these examples do not keep within our scheme, and we shall only show that they are similar to it (however, the scheme can be extended to include them).
4.1. Let $M$ be a smooth manifold, $\mathcal{O}(M)$ the ring of smooth functions on $M$, $\mathcal{E}$ a vector bundle on $M$, $D(M, \mathcal{E})$ the ring of differential operators in $\mathcal{E}$, and $F_n D(M, \mathcal{E})$ the subspace of operators of degree at most $n$. The equation

$$\text{Gr}_F D(M, \mathcal{E}) = \text{End} \mathcal{E} \otimes_{\mathcal{O}(M)} S^*_\mathcal{O}(M)(\text{Vect}(M))$$

allows us to consider $\text{Gr}_F D(M, \mathcal{E})$ as a “quadratic algebra over $\text{End} \mathcal{E}$” and $D(M, \mathcal{E})$ as a QLS-algebra; then $\text{Gr}_F D(M, \mathcal{E})^1 = \Omega^*(M, \text{End} \mathcal{E})$ is the algebra of differential forms on $M$ with coefficients in $\text{End} \mathcal{E}$.

In order to construct a direct complement $V$ to $\text{End} \mathcal{E}$ in $F_1 D(M, \mathcal{E})$, we choose a connection $\nabla$ on $\mathcal{E}$, define an embedding

$$i: \text{End} \mathcal{E} \otimes \text{Vect} M \to F_1 D(M, \mathcal{E}), \quad i(a \otimes v) = a \circ \nabla_v,$$

and put $V = \text{Im} i$. It is easy to see that all left $\text{End} \mathcal{E}$-invariant direct complements to $\text{End} \mathcal{E}$ in $F_1 D(M, \mathcal{E})$ can be obtained in this way.

Let $d^\nabla$ be the de Rham differential on $\Omega^*(M, \text{End} \mathcal{E})$ defined by means of the connection on $\text{End} \mathcal{E}$ induced by $\nabla$, and let $h^\nabla \in \Omega^*(M, \text{End} \mathcal{E})$ be the curvature of the connection $\nabla$. Comparing the relation

$$\nabla_X \nabla_Y - \nabla_Y \nabla_X = \nabla_{[X,Y]} + h^\nabla(X,Y)$$

with the formula (*) and taking into account the relationship between $[\cdot, \cdot]$ and $d_1$, we conclude that $D(D(M, \mathcal{E})^{\text{opp}}, F) = (\Omega^*(M, \text{End} \mathcal{E}), d^\nabla, h^\nabla)$.

4.2. Principal bundles.

4.2.1. Let $M$ be a manifold, $G$ a Lie group, $P$ a (right) principal $G$-bundle over $M$, and $\pi: P \to M$ the corresponding projection. Let $g_p$ be the bundle of Lie algebras over $M$ associated with $P$ by means of the adjoint representation of $G$ (in other words, the sections of $g_p$ are $G$-equivariant vector fields on $P$), and let $U_p$ be the corresponding bundle of enveloping algebras.

**Definition.** The ring of differential operators on a principal $G$-bundle is the ring of $G$-equivariant differential operators on its total space, $D(M, P) = D(P)^G$. The filtration $F$ “by the order along the base” on $D(M, P)$ is defined as follows:

$$F_n D(M, P) = \{ D \in D(M, P) : \text{ad}^{n+1}(\pi^* f)(D) = 0 \ \forall f \in \mathcal{O}(M) \}.$$

4.2.2. **Proposition.** $\text{Gr}_F D(M, P) \simeq U_p \otimes_{\mathcal{O}(M)} S^*_{\mathcal{O}(M)}(\text{Vect}(M))$.

**Proof.** First one has to show that $F_0 D(M, P) \simeq \Gamma(U_p)$. Then the isomorphism is defined using the highest symbol operator

$$\sigma_n : F_n D(M, P) \to \Gamma(M, U_p \otimes S^n T(M)),$$

$$\sigma_n(D)(\xi) = \text{ad}^n(\pi^* f)(D)_m \in U_{p,m} \text{ for } m \in M, \xi \in T^*_m(M), f \in \mathcal{O}(M), \text{ and } d_m f = \xi. \quad \square$$
4.2.3. Let us choose a connection $\nabla$ on the principal $G$-bundle $P$ and construct a direct complement $V$ to $F_0D(M, P)$ in $F_1D(M, P)$ as follows: $V = \langle u \cdot H_\nabla(v), u \in \Gamma(U_p), v \in \text{Vect}(M) \rangle$, where $H_\nabla(v)$ is the horizontal (with respect to $\nabla$) lifting of the vector $v$ to $P$.

Then $D_{QLS}(D(M, P), F) = (\Omega^*(M, U_p), d^\nabla, h^\nabla)$, where $d^\nabla$ is defined by means of the connection $\nabla^u$ on $U_p$ associated with $\nabla$, and $h^\nabla \in \Omega^2(M, g_p) \subset \Omega^2(M, U_p)$ is the curvature of $\nabla$.

§5. Characteristic Classes

In this section we suppose that the characteristic of the ground field is equal to 0.

5.1. Let $\Psi_0 = (B, \delta_0, h_0)$ be a CDG-algebra, $[B, B]$ be the linear subspace generated by the supercommutators in $B$, $C = B/[B, B]$, and $T: B \to C$ be the projection. It is clear from the Leibniz identity that the operator $\delta_C$ on $C$ induced by $\delta_0$ is well-defined. Notice that $\delta_C^2 = 0$ and $\delta_C$ does not change when a CDG-algebra $\Psi$ is replaced by an equivalent one. We put $h(\alpha) = h_0 + \delta_0\alpha + \alpha^2$ and $\delta(\alpha) = \delta_0 + [\alpha, \cdot]$ for any $\alpha \in B^1$.

5.2. Main lemma. There exist naturally defined differential forms $\omega^{(i)}_n$ on the vector space $B^1$, $n = 1, 2, \ldots, i = 0, \ldots, n$, satisfying the following conditions:

(i) $\omega^{(i)}_n$ is a differential $i$-form with values in $B^{2n-i}$;
(ii) $\omega^{(0)}_n = h^n$;
(iii) $d\omega^{(i)}_n = \delta\omega^{(i+1)}_n$, $i = 0, \ldots, n-1$, where $d$ is the de Rham differential;
(iv) $d\omega^{(n)}_n = 0$.

Proof. We put $\omega^{(i)}_n = \sum (d\alpha)^i h^{n-i}$, where $d\alpha$ is the tautological 1-form on $B^1$ with values in $B^1$ and $\sum$ denotes the summation over all rearrangements of the factors. Verification is based on the identities $\delta(h) = 0$, $dh = \delta d\alpha$. □

5.3. Chern classes. Set $c_n = T(h^n_0) \in C^{2n}$. The Chern classes are the cohomology classes of the elements $c_n$.

Theorem. a) $\delta_C(c_n) = 0$.

b) The cohomology class of $c_n$ does not change when a CDG-algebra $\Psi$ is replaced by an equivalent one.

Proof. a) Moreover, $\delta_0(h^n_0) = 0$.

b) follows from the equation $d(h^n) = \delta(\omega^{(1)}_n)$. □

In characteristic $p$ the theorem remains true for $2n < p$.

5.4. If $\Psi_0 = (\Omega^*(M, \text{End} \mathcal{E}), d^\nabla, h^\nabla)$, then $C^* = \Omega^*(M)$, the map $T$ is the (matrix) trace, and we obtain the usual Chern classes.

5.5. Chern classes: the case of a principal bundle.

5.5.1. Lemma. Let $A$ be a graded algebra generated by its graded vector subspace $W$. Then $[A, A] = [W, A]$.  

8
**Proof.** Proceed by induction using the identity

\[ [ab, c] = [a, bc] + (-1)^{(\mathbb{b} + \mathbb{c})\mathbb{a}}[b, ca]. \] □

**Proposition.** Let \( \mathfrak{g} \) be a Lie algebra. Then

\[ (U\mathfrak{g}/[U\mathfrak{g}, U\mathfrak{g}]) \simeq (S^*\mathfrak{g})_\mathfrak{g}. \]

**Proof.** The map \( S^*\mathfrak{g} \to U\mathfrak{g}, \ x^n \mapsto x^n \), is an isomorphism of \( \mathfrak{g} \)-modules, and \( U\mathfrak{g}/[U\mathfrak{g}, U\mathfrak{g}] \simeq (U\mathfrak{g})_\mathfrak{g} \) by the lemma. □

### 5.5.2. Proposition.

- **a)** \( \Omega^*(M, U_p)/[\Omega^*(M, U_p), \Omega^*(M, U_p)] \simeq \Omega^*(M, U_p/[U_p, U_p]) \).
- **b)** The vector bundle \( U_p/[U_p, U_p] \) is trivial with the fiber \( (S^*\mathfrak{g})_\mathfrak{g} \).
- **c)** The form \( c_n \) lies in the space \( \Omega^{2n}(M, (S^*\mathfrak{g})_\mathfrak{g}) \). Let \( P \) be an invariant polynomial on \( \mathfrak{g} \) of degree \( n \). Then \( \langle P, c_n \rangle \) is equal to the Chern–Weil characteristic form \( P(h^\nabla) \) corresponding to \( P \) [8], where angle brackets \( \langle , \rangle \) denote the pairing

\[ (S^*\mathfrak{g}^*)_0 \times (S^*\mathfrak{g})_\mathfrak{g} \to \mathbb{R}. \]

**Proof.**

- a) If \( A \) is a supercommutative algebra, then

\[ A \otimes B/[A \otimes B, A \otimes B] = A \otimes B/[B, B]. \]

- b) The adjoint action of \( G \) is trivial in \( U_p/[U_p, U_p] \).

- c) follows from the definition of the isomorphism in Proposition 5.5.1.

### 5.6. The Chern–Simons functor.

**Definition.** The category \( \text{C}2 \) of two-term complexes is defined as follows. Its objects are pairs \( (C; c) \), where \( C = (\delta: C_1 \to C_0) \) is a morphism of vector spaces and \( c \in C_0 \). Morphisms from \( (C; c) \) to \( (C', c') \) are pairs \( (f; c'_1) \), where \( f = (f_0, f_1) \), \( f_i: C_i \to C'_i \) is a pair of morphisms forming a commutative square with \( \delta \) and \( \delta' \), and \( c'_1 \in C'_1 \) is an element for which \( c' - f_0(c) = \delta'c'_1 \). The composition of morphisms is defined by the formula

\[ (f, c''_1) \circ (g, c'_1) = (f \circ g, c''_1 + f(c'_1)). \]

**Construction.** The Chern–Simons functor \( CS_n: \text{CDG} \to \text{C}2 \) is constructed as follows. On objects:

\[ CS_n(\Psi) = (\delta_C: C^{2n-1}/\delta_C C^{2n-2} \to C^{2n} \cap \text{Ker} \delta_C; c_n), \]

where \( c_n = T(h^\nabla_0) \). On morphisms:

\[ CS_n(f, \alpha) = (f_*, c^{(1)}_n), \quad c^{(1)}_n = c^{(1)}_n(f, \alpha) = \int_\gamma \omega^{(1)}_n, \]

where \( f: \Psi \to \Psi', \omega^{(1)}_n \) is the 1-form corresponding to the algebra \( \Psi'(-\alpha) \), and \( \gamma \) is a smooth path in \( B^{t1} \) joining the points 0 and \( \alpha \).
5.7. Chern–Simons classes. Let $E = (E, \partial)$ be a DG-algebra, and let $\phi: \Psi \to E$ be a morphism of CDG-algebras. Then $c_n^{(1)}(\phi) \in E^{2n-1}/([E, E] + \partial E)$, $\partial c_n^{(1)} = -f_*(c_n)$, and when the algebra $\Psi$ is replaced by an equivalent one the chain $c_n^{(1)}$ changes by an element from $f(B^{2n-1})$. Thus, the class

$$c_n^{(1)} \in E^{2n-1}/(f(B) + [E, E] + \partial E)$$

is an invariant of the morphism $\phi$.

Let $P$ be a principal $G$-bundle over $M$. When $\Psi = (\Omega^*(M, U_p), d^\nabla, h^\nabla)$, $(E, \partial) = (\Omega^*(P, U_g), d)$, $f = \pi^*$, and $\alpha$ is the connection form, one obtains the usual Chern–Simons classes.

Corrections made twenty years later:
1. The Chern classes of 5.4 are more precisely described as the components of the Chern character (up to factorial factors). In other words, they correspond to power sums of symmetric variables rather than to the elementary symmetric polynomials.
2. Proposition 5.5.2(b) only holds as stated when the Lie group $G$ is connected. One has to make a separated consideration of invariant polynomials for nonconnected Lie groups in this case.

Notes added twenty years later:
1. The “extended scheme including the examples” promised in §4 was indeed worked out (even if not in the most detailed or easily accessible form) in the auxiliary material to the author’s monograph “Homological algebra of semimodules: Semi-infinite homological algebra of associative algebraic structures”, Sections 0.4.3–0.4.4 and 11.5–11.6.
2. The most important aspect of the CDG-ring theory that was overlooked in the original 1992-93 paper is that CDG-rings actually form a 2-category rather than just a 1-category. While CDG-rings themselves describe the curvatures and their 1-morphisms are responsible for changing connections, the 2-morphisms correspond to the gauge transformations.

Let $\Psi = (B, d, h)$ and $\Psi' = (B', d', h')$ be two CDG-algebras, and let $(f, \alpha): \Psi \to \Psi'$ and $(g, \beta): \Psi \to \Psi'$ be two CDG-morphisms between them. A 2-morphism $(f, \alpha) \to (g, \beta)$ is an invertible element of degree zero $z \in B'^0$ satisfying the conditions

1) $g(x) = zf(x)z^{-1}$ for all $x \in B$,
2) $\beta = z\alpha z^{-1} + d'(z)z^{-1}$.

If a pair $(f, \alpha)$ is a morphism of CDG-algebras and $z$ is an invertible element in $B'^0$, then the pair $(g, \beta)$ defined by the above formulas is also a morphism of CDG-algebras. Notice the difference between DG- and CDG-morphisms: while invertible cocycles of degree zero act by adjunctions on DG-morphisms, invertible cochains of degree zero act by adjunction on CDG-morphisms.

The 2-category structure on CDG-rings may be possibly used to defined (quasi-coherent) stacks of CDG-algebras, extending the definitions of quasi-coherent sheaves of CDG-algebras given in Appendix B.1 to the author’s memoir “Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence” and Section 1.2 to
the preprint “Cohenent analogous of matrix factorizations and relative singularity categories”. Under the $D$-$\Omega$ duality, these would correspond to a certain kind of twisted differential operators (e.g., in the étale or analytic topology).

Finding a 2-category version of the Chern–Simons functor construction of Subsection 5.6 would be also interesting.

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