The relationship between two flavors of oblivious transfer at the quantum level

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Though all-or-nothing oblivious transfer and one-out-of-two oblivious transfer are equivalent in classical cryptography, we here show that due to the nature of quantum cryptography, a protocol built upon secure quantum all-or-nothing oblivious transfer cannot satisfy the rigorous definition of quantum one-out-of-two oblivious transfer.

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I. INTRODUCTION

Mystery of quantum cryptography has long intrigued scientists. On one hand, several cryptographic tasks such as the quantum conjugate coding[1] and the well-known quantum key distribution[2, 3, 4] have made great successes. They achieved theoretically unbreakable security which can never be reached by their classical counterparts. But, on the other hand, some no-go theorems were established, indicating that quantum cryptography is not always powerful for any task. In particular, the MLC no-go theorem[5, 6] rules out the possibility of non-relativistic unconditionally secure quantum bit commitment (QBC), and the Lo's insecurity proof of one-sided two-party quantum secure computations[7] indicates that one-out-of-two oblivious transfer is impossible either.

Oblivious transfer (OT) is an important concept found to be very useful in designing multi-party cryptography protocols[8]. There are two major flavours of OTs. The original one[1, 9] is simply known as oblivious transfer, while sometimes can also be called all-or-nothing OT. Another related notion was proposed later, which is called one-out-of-two OT[10]. In classical cryptography, it was shown that these two are computationally equivalent[11]. Essentially, a protocol was presented in Ref.[11] to illustrate that secure all-or-nothing OT can lead to secure one-out-of-two OT. Furthermore, it was believed that secure one-out-of-two OT can lead to secure BC[7]. This standard classical reduction chain reveals the connection between the security of OT and BC protocols in the classical level.

Very recently, a quantum all-or-nothing OT protocol was developed[12]. This OT does not rigorously satisfy the requirement of one-sided two-party quantum secure computation protocols, on which the Lo's insecurity proof was based. Thus it could remain unconditionally secure against the cheating strategy in the Lo's proof. Nevertheless, at the first glance, this result would conflict with the Lo's conclusion and in turn with the MLC no-go theorem (i.e., secure quantum one-out-of-two OT and QBC would be possible) if the mentioned standard classical reduction were justified.

More intriguingly, it has also been realized that “reductions and relations between classical cryptographic tasks need not necessarily apply to their quantum equivalents”[13]. Indeed, it will be shown in this paper that once we intend to build an one-out-of-two OT protocol on a secure quantum all-or-nothing OT protocol with the method developed in Ref.[11], it is impossible that the resultant protocol can satisfy the rigorous definition of one-out-of-two OT on which the Lo’s proof was based. In this sense, secure quantum all-or-nothing OT does not imply secure quantum one-out-of-two OT, i.e. the above classical reduction chain is broken in the present quantum cryptography case. As a result, there exists no logic conflict between the existence of secure quantum all-or-nothing OT protocol and the MLC no-go theorem of QBC.

The paper is organized as follows. In the next two sections, the definitions of two flavors of OTs will be stated precisely and a brief review on their classical equivalence will be presented. The relationship between these OTs in the quantum level will be revealed in the section IV, and how it is related to the cheating strategy in the Lo’s proof will be studied in the section V. In the section VI, it will be indicated that the breaking of the reduction chain is not simply a matter of the definition, rather it is originated from the nature of quantum cryptography itself.

II. DEFINITIONS

Let us first state precisely the definitions of different OTs on which the discussion in this paper is based. In Ref.[11] where the classical equivalence between these OTs was proven, the definitions of all-or-nothing OT and one-out-of-two OT were summarized as:

Definition A: all-or-nothing OT
(A-i) Alice knows one bit b.
(A-ii) Bob gets bit b from Alice with probability 1/2.
(A-iii) Bob knows whether he got b or not.
(A-iv) Alice does not know whether Bob got b or not.
Definition B: one-out-of-two OT
(B-i) Alice knows two bits $b_0$ and $b_1$.
(B-ii) Bob gets bit $b_j$ and not $b_j$ with $Pr(j = 0) = Pr(j = 1) = 1/2$.
(B-iii) Bob knows which of $b_0$ or $b_1$ he got.
(B-iv) Alice does not know which $b_j$ Bob got.

In the Lo’s insecurity proof of one-sided two-party quantum secure computations\[9\], a more rigorous definition of one-out-of-two OT was specifically introduced as:

Definition C: rigorous one-out-of-two OT
(C-i) Alice inputs $i$, which is a pair of messages $(m_0, m_1)$.
(C-ii) Bob inputs $j = 0$ or 1.
(C-iii) At the end of the protocol, Bob learns about the message $m_j$, but not the other message $m_{\bar{j}}$, i.e., the protocol is an one-sided two-party secure computation $f(m_0, m_1, j = 0) = m_0$ and $f(m_0, m_1, j = 1) = m_1$.
(C-iv) Alice does not know which $m_j$ Bob got.

Meanwhile, the definition of one-sided two-party quantum secure computations used in the Lo’s proof reads

Definition D: one-sided two-party secure computation
Suppose Alice has a private (i.e. secret) input $i \in \{1, 2, ..., n\}$ and Bob has a private input $j \in \{1, 2, ..., m\}$. Alice helps Bob to compute a prescribed function $f(i, j) \in \{1, 2, ..., p\}$ in such a way that, at the end of the protocol:
(a) Bob learns $f(i, j)$ unambiguously;
(b) Alice learns nothing [about j or $f(i, j)$];
(c) Bob knows nothing about $i$ more than what logically follows from the values of $j$ and $f(i, j)$.

Obviously, Definition C is a special case of Definition D. In Ref.\[9\] it is proven that any protocol satisfying Definition D is insecure. Therefore as a corollary, there should not exist a secure quantum one-out-of-two OT protocol which satisfies Definition C rigorously.

III. CLASSICAL EQUIVALENCE

The proof of the classical equivalence between the two flavors of OTs is provided in Ref.\[10\]. The major part of the proof is the following procedure, showing how secure one-out-of-two OT can be implemented upon secure all-or-nothing OT.

Protocol P:
(1) Alice and Bob agree on a security parameter $s$;
(2) Alice chooses at random $K$s bits $r_1, r_2, ..., r_{Ks}$;
(3) For each of these $K$s bits Alice uses the all-or-nothing OT protocol to disclose the bit $r_k$ to Bob:
(4) Bob selects $U = \{i_1, i_2, ..., i_{\alpha_s}\}$ and $V = \{i_{\alpha_s+1}, i_{\alpha_s+2}, ..., i_{2\alpha_s}\}$ where $\alpha_s = Ks/3$ with $U \cap V = \emptyset$ and such that he knows $r_k$ for each $k \in U$;
(5) Bob sends $(X, Y) = (U, V)$ or $(X, Y) = (V, U)$ to Alice according to a random bit $j$;
(6) Alice computes $c_0 = \bigoplus_{x \in X} r_x$ and $c_1 = \bigoplus_{y \in Y} r_y$;
(7) Alice returns to Bob $b_0 \oplus c_0$ and $b_1 \oplus c_1$;
(8) Bob computes $\bigoplus_{u \in U} r_u \in \{0, 1\}$ and uses it to get his secret bit $b_j$.

IV. RELATIONSHIP AT THE QUANTUM LEVEL

Though the two definitions of one-out-of-two OT (Definitions B and C) seem to be consistent with each other, we here will show that, in the quantum level, if a secure quantum all-or-nothing OT protocol satisfies Definition A and can be used as a “black box”, a Protocol P built upon it via the above procedure does not satisfy Definition C rigorously, though it satisfies Definition B.

The deviation from Definition C lies in (C-ii) and (C-iii). Consider Alice’s input $i$ in Protocol P. In the step (7) of the protocol, we can see that $i$ includes not only the secret bits $b_0$ and $b_1$, but also $c_0$ and $c_1$. The steps (5) and (6) shows that $c_0$ and $c_1$ not only depend on Alice’s input $r_1, r_2, ..., r_{Ks}$, but also depend on how Bob selects $X, Y, U$ and $V$, i.e. they depend on Bob’s input $j$. Therefore, Protocol P cannot be viewed as a “black box” function $f(i(m_0, m_1, j))$, where $i$ and $j$ are the private inputs of Alice and Bob respectively. Instead, it has the form $f(i(m_0, m_1, j), j)$, where Alice’ input will be varied according to Bob’s input, and its value is not determined until Bob’s input has been completed. That is, Protocol P does not rigorously satisfy Definition C, nor Definition D as the description of the function $f$ is different.

Though the difference seems tiny at the first glance, its consequences are significant at the quantum level. This can be seen from two aspects:
(I) The con side: Protocol P cannot be used as a black box since the sequence of the participants’ inputs is important, i.e. we have to deal with the details of the protocol when it is used to build other protocols. As argued in the introduction of Ref.\[9\], to ensure that the standard classical reduction can apply to quantum cryptographic protocols, “one must be allowed to use a quantum cryptographic protocol as a ‘black box’ primitive in building up more sophisticated protocols and to analyze the security of those new protocols with classical probability theory\[10\]. Therefore the above character of Protocol P make it unsuitable to be used as a rigorous quantum one-out-of-two OT to connect the reduction chain between quantum all-or-nothing OT and QBC. Other applications of Protocol P in quantum cryptography may also have a limited power.

(II) The pro side: Protocol P is not covered by the cheating strategy in Ref.\[2\] for the following reason. According to the strategy, Bob can change the value of $j$ from $j_1$ to $j_2$ by applying a unitary transformation to his own quantum machine. Therefore he can
learn \( f(i(m_0, m_1), j_1) \) and \( f(i(m_0, m_1), j_2) \) simultaneously without being found by Alice. However, for the function \( f(i(m_0, m_1, j), j) \), the value \( f(i(m_0, m_1, j_1), j_2) \) is meaningless. Without the help of Alice, Bob cannot change \( i \) from \((m_0, m_1, j_1)\) to \((m_0, m_1, j_2)\). Hence he cannot learn \( f(i(m_0, m_1, j_1), j_1) \) and \( f(i(m_0, m_1, j_2), j_2) \) simultaneously by himself. Namely, though the cheating strategy works for any protocol satisfying Definition D, it does not work for Protocol P.

On the other hand, though \( c_0 \) and \( c_1 \) depend on Bob’s input \( j \), from the protocol it can be seen clearly that they are insufficient for Alice to learn the value of \( j \). Thus Protocol P is still secure against Alice. In this sense, the relaxed definition of one-out-of-two OT (Definition B) is satisfied.

V. DEFEATING THE CHEATING STRATEGY

In this section, the above conclusion (II) will be rigorously proven. For convenience, let us first recall the cheating strategy in the Lo’s proof in more details. According to the section III of Ref. [5], in any protocol satisfying Definition D, Alice and Bob’s actions on their quantum machines can be summarized as an overall unitary transformation \( U \) applied to the initial state \( |u\rangle_{in} \in H_A \otimes H_B \), i.e.

\[
|u\rangle_{fin} = U |u\rangle_{in}. 
\]

When both parties are honest, \( |u^h\rangle_{in} = |i\rangle_A \otimes |j\rangle_B \) and

\[
|u^h\rangle_{fin} = |v_{ij}\rangle \equiv U(|i\rangle_A \otimes |j\rangle_B). 
\]

Therefore the density matrix that Bob has at the end of protocol is

\[
\rho^{ij} = Tr_A |v_{ij}\rangle \langle v_{ij}|. 
\]

Bob can cheat in this protocol, because given \( j_1, j_2 \in \{1, 2, ..., m\} \), there exists a unitary transformation \( U^{j_1, j_2} \) such that

\[
U^{j_1, j_2} \rho^{i, j_1} (U^{j_1, j_2})^{-1} = \rho^{i, j_2} 
\]

for all \( i \). It means that Bob can change the value of \( j \) from \( j_1 \) to \( j_2 \) by applying a unitary transformation independent of \( i \) to the state of his quantum machine. This equation is proven as follows.

Alice may entangles the state of her quantum machine \( A \) with her quantum dice \( D \) and prepares the initial state

\[
\frac{1}{\sqrt n} \sum_i |i\rangle_D \otimes |i\rangle_A. 
\]

She keeps \( D \) for herself and uses the second register \( A \) to execute the protocol. Suppose Bob’s input is \( j_1 \). The initial state is

\[
|u\rangle_{in} = \frac{1}{\sqrt n} \sum_i |i\rangle_D \otimes |i\rangle_A \otimes |j_1\rangle_B. 
\]

At the end of the protocol, it follows from Eqs. (1) and (3) that the total wave function of the combined system \( D, A, \) and \( B \) is

\[
|v_{ij}\rangle_{in} = \frac{1}{\sqrt n} \sum_i |i\rangle_D \otimes U(|i\rangle_A \otimes |j_1\rangle_B). 
\]

Similarly, if Bob’s input is \( j_2 \), the total wave function at the end will be

\[
|v_{j_2}\rangle_{in} = \frac{1}{\sqrt n} \sum_i |i\rangle_D \otimes U(|i\rangle_A \otimes |j_2\rangle_B). 
\]

Due to the requirement (b) in Definition D, the reduced density matrices in Alice’s hand for the two cases \( j = j_1 \) and \( j = j_2 \) must be the same, i.e.

\[
\rho_{j_1}^{Alice} = Tr_B |v_{j_1}\rangle \langle v_{j_1}| = Tr_B |v_{j_2}\rangle \langle v_{j_2}| = \rho_{j_2}^{Alice}. 
\]

Equivalently, \( |v_{j_1}\rangle \) and \( |v_{j_2}\rangle \) have the same Schmidt decomposition

\[
|v_{j_1}\rangle = \sum_k a_k |\alpha_k\rangle_A \otimes |\beta_k\rangle_B 
\]

and

\[
|v_{j_2}\rangle = \sum_k a_k |\alpha_k\rangle_A \otimes |\beta'_k\rangle_B. 
\]

Now consider the unitary transformation \( U^{j_1, j_2} \) that rotates \( |\beta_k\rangle_B \) to \( |\beta'_k\rangle_B \). Notice that it acts on \( H_B \) alone and yet, as can be seen from Eqs. (10) and (11), it rotates \( |v_{j_1}\rangle \) to \( |v_{j_2}\rangle \), i.e.

\[
|v_{j_2}\rangle = U^{j_1, j_2} |v_{j_1}\rangle. 
\]

Since

\[
D \langle i | v_{j_2} \rangle = \frac{1}{\sqrt n} \langle v_{j_2} | \langle v_{j_2}| 
\]

[see Eqs. (2), (4), and (5)], by multiplying Eq. (12) by \( D \langle i | \) on the left, one finds that

\[
|v_{j_2}\rangle = U^{j_1, j_2} |v_{j_1}\rangle = U^{j_2, j_2} |v_{j_1}\rangle. 
\]

Taking the trace of \( |v_{j_2}\rangle \langle v_{j_2}| \) over \( H_A \) and using Eq. (14), Eq. (13) can be obtained.

Note that all these equations are just those presented in the Lo’s proof [7]. We now consider Protocol P, where Alice’s input \( i \) is dependent of Bob’s input \( j \). In the above proof, all \( i \) in the equations should be replaced by \( i(j) \) from the very beginning. Consequently, Eq. (13) becomes

\[
D \langle i(j) | v_{j_2} \rangle = \frac{1}{\sqrt n} \langle v_{j_2} | \langle v_{j_2}| 
\]

In this case multiplying Eq. (12) by \( D \langle i_2 | (i_2 = i(j_2) \) for short) on the left cannot give Eq. (14) any more. Instead, the result is

\[
|v_{i_2 j_2}\rangle = U^{j_1, j_2} U^{i_1, j_2} |v_{i_1 j_1}\rangle, 
\]
where $U^{i_1,i_2} \equiv_D |i_2\rangle \langle i_1|_D$. Then Eq.\(\eqref{11}\) is replaced by

$$U^{i_1,j_2}U^{i_1,i_2}\rho_{i_1,j_1}(U^{i_1,j_2}U^{i_1,i_2})^{-1} = \rho^{i_2,j_2}. \quad \text{(17)}$$

Note that $U^{i_1,i_2}$ is the unitary operation on Alice’s side. This implies that without Alice’s help, Bob cannot change the density matrix he has from side. This implies that without Alice’s help, Bob cannot.

That is why Bob’s cheating strategy fails in Protocol P.

VI. ORIGIN OF THE INEQUIVALENCE

It is valuable to find out the underlying reason why Protocol P does not satisfy the rigorous Definition C. An illusion is naturally aroused that the reason is due to a relaxed Definition A of all-or-nothing OT used in the work. However, it is not true. In fact, we never need to deal with the details of the all-or-nothing OT in the section IV; we simply use it as a black box. Even when the most rigorous definition of all-or-nothing OT is used, the discussion in that section is still valid. Thus it is not a matter of definition that the classical equivalence between the two flavours of OTs cannot rigorously apply to the present quantum case.

The real origin of this result can be found in the equations in the previous section. By comparing Eqs.\(\eqref{13}\) and \(\eqref{15}\), we can see that if there does not exist a system $D$, Protocol P will become insecure too. That is, if Alice does not introduce the quantum system $D$ in Eq.\(\eqref{15}\), Protocol P will show no difference from the protocols satisfying Definition D. In classical cryptography, Alice surely does not have such a system. That is why the two flavors of OTs seem equivalent. In quantum cryptography, if Alice does not make full use of the computational power but simply executes the protocol with the quantum system $A$ alone, she cannot defeat Bob’s cheating either. The difference between Protocol P and a rigorous one-out-of-two OT can only be manifested when the protocol is indeed executed at the quantum level. In this sense, the underlying origin is the nature of quantum cryptography itself.

VII. DISCUSSIONS AND SUMMARY

It has been shown that though one-out-of-two OT can be built upon all-or-nothing OT in classical cryptography, a Protocol P built upon a secure quantum all-or-nothing OT protocol via the same method cannot satisfy the rigorous Definition C of quantum one-out-of-two OT. Considering that a secure quantum all-or-nothing OT protocol was already established \[12\], which is not denied by the Lo’s insecurity proof of the one-sided two-party secure computations \[7\] because it does not satisfy the requirement on which the proof is based, it seems unlikely that such a protocol can lead to another protocol satisfying the requirement. Furthermore, if a secure protocol satisfying the rigorous definition of quantum one-out-of-two OT existed, it would be used as a black box primitive to implement secure QBC according to Ref.\[7\], conflicting with the MLC no-go theorem. On the contrary, it is more logically consistent that no other method is available to build a rigorous quantum one-out-of-two OT protocol upon quantum all-or-nothing OT. That is, the two flavors of OTs should not be rigorously equivalent in quantum cryptography.

Though the profound understanding of the exact relationship between the two flavors of OTs at the quantum level is still awaited, at least, one thing is clearly elaborated in this work: the classical equivalence between these OTs cannot be directly applied to quantum cryptography. This finding provides yet an intriguing example demonstrating that reductions and relations between classical cryptographic tasks need careful re-examination in quantum cases.

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