CLASSICAL LIMIT IN SCALAR QFT AT HIGH TEMPERATURE

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It is shown that in a scalar quantum field theory at high temperatures we can compute even time dependent observables from an effective theory, which can be interpreted as a (nonlocal) classical statistical field theory. We examine the retarded Greens function in the local approximation, and define the classical self energy. Its imaginary part agrees with the quantum case in the leading order.

1 Introduction

The complete description of a high temperature plasma concerns also the computation of dynamical quantities (damping rates, topological transitions etc.). Since there does not exist a direct nonperturbative method to perform these calculations, it is worth to concentrate on the most relevant degrees of freedom of the system and to use an effective theory for them. At high temperatures the most relevant long wavelength modes are highly excited, thus behave classically. Therefore a good candidate for the effective theory in this regime is the classical theory.

The classical theory, however, is plagued by UV divergences. In the case of static observables the dimensional reduction program solves this problem. There the classical model is interpreted as a bare theory such that we get finite results for the physical observables. The bare parameters can be calculated by eliminating the UV modes of the system. The generalization to time dependent quantities is, however, not straightforward. A natural conjecture (c.f. Ref.) is to use $H_{\text{eff}} = \int \frac{1}{2} \pi^2 + L_{\text{DR}}(\phi)$ as the effective Hamiltonian where $L_{\text{DR}}$ effective Lagrangian comes from the dimensional reduction. Using this conjecture it has been shown that the Greens functions of the scalar theory are finite up to two loop order.

To better establish this idea we have to examine the properties of the classical limit in hot quantum field theories. Firstly we have to find some dimensionless quantity, which could govern this limit and contains $\hbar$. In the free case it is $\hbar \beta \omega_k$, so we may try to use this combination also in the perturbation theory. Then we can work in $\hbar = 1$ units. Secondly the time evolution of the classical fields is deterministic, they are functions of the time and the initial conditions. To arrive to a classical description we have to integrate over the initial conditions, in the quantum language the initial states.
2 Effective theory

Putting together these two requirements we will rewrite the thermal average
\[ \langle T \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_1) \rangle = \frac{1}{Z} \text{Tr} \left( e^{-\beta \hat{H}} T \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_1) \right). \] (1)

As initial states we use coherent states \(|\eta\rangle\), the eigenstates of the annihilation operator at \(t = t_0\). We choose a separation scale \(\Lambda (m \ll \Lambda \ll T)\). In the sense of the Introduction the small parameter \(\beta \Lambda\) stands for \(\bar{\hbar}\). We introduce the notations
\[ \text{Tr}_{UV} e^{-\frac{\beta}{2} \hat{H}} O_{\text{eff}} = \text{Tr}_{UV} e^{-\frac{\beta}{2} \hat{H}} \hat{O} e^{-\frac{\beta}{2} \hat{H}} \text{Tr}_{UV} e^{-\beta \hat{H}}. \] (4)

That is we have reduced the 4D path integration into two 3D path integrations over a phase space ("dimensional reduction"), and we can identify the effective action and the effective operator in this reduced effective model.

The integration over the UV degrees of freedom is, as usual, supposed to be perturbative. The omission of the IR initial conditions mean a slight modification in the usual perturbation theory. For the scalar field theory we find after some algebra
\[ \text{Tr}_{UV} e^{-\beta \hat{H}} \hat{O} e^{-\beta \hat{H}} e^{-i \int H_I(i \frac{\partial}{\partial t}) O \left( \frac{\delta}{i \delta j} \right) e^{-\frac{\beta}{2} \int j G^j \int j j \hat{\phi}_0} \bigg|_{j=0}, \] (5)

where the integration contour is symmetric (\(c : i/2\beta \rightarrow 0 \rightarrow \tau_f \rightarrow 0 \rightarrow -i/2\beta\), \(\tau_f\) is larger than any explicit time arguments), \(H_I\) is the interaction part of the Hamiltonian, changes sign on the anti-time ordered part of the contour, and
\[ \hat{\phi}_0(t, k) = \Theta(\Lambda > |k|) e^{-\beta \omega_k} \left( \phi_k \cos \omega_k t + \pi_k \frac{\sin \omega_k t}{\omega_k} \right) \]
\[ G(t_1, t_2, k) = \frac{1}{2\omega_k} \left[ \Theta_c(t_1 > t_2) e^{-i\omega_k(t_1 - t_2)} + \Theta_c(t_2 > t_1) e^{-i\omega_k(t_2 - t_1)} \right] + \Theta(|k| > \Lambda) \frac{n(\omega_k)}{\omega_k} \cos \omega_k(t_1 - t_2), \]  

(6)  

\( n(\omega) = [e^{-\beta\omega} - 1]^{-1} \). The perturbation theory for the evaluation of the UV traces is like ordinary real time perturbation theory, but in a background field \( \tilde{\phi}_0 \) which follows a free time evolution, and with the propagator \( G \), which forbids the propagation of the IR thermal modes.

3 IR and classical (local) approximation

Similar to the \( \hbar \) expansion in quantum mechanics we can now expand with respect to \( \beta \Lambda \). We concentrate on the leading "classical" order. We can omit the \( e^{-\frac{2\pi i k_0}{\beta}} \) factor in the background field, its imaginary time evolution can also be neglected (because \( e^{-\beta} = 1 + O(\Lambda \beta) \)). Another consequence is the loss of quantum coherence, i.e. we can write for the UV thermal expectation values

\[ \langle T\hat{\Phi}(x_1)\ldots\hat{\Phi}(x_n) \rangle_{UV} \rightarrow \langle \hat{\Phi}(x_1) \rangle_{UV} \ldots \langle \hat{\Phi}(x_n) \rangle_{UV}. \]

This is because the IR dominant part of the complete propagator comes from the effective theory, therefore in the computation of the UV thermal expectation values all IR propagators can be cut (which does not lead to vacuum diagrams). The complete proof will be published elsewhere \[^5\]. Therefore the effective operator of the time ordered n-point function can be substituted by \( O_{eff} \rightarrow \Phi(x_1)\ldots\Phi(x_n) \), where \( \Phi(x) \) is the effective one point function (c.f. eq. \[4\]).

Now we can use perturbation theory to determine \( H_{eff} \) and \( \Phi(x) \). In \( H_{eff} \) there is no time dependence, therefore \( t_f = 0 \), i.e., the Matsubara contour can be taken. The background is the integration variable \( \phi \), the conjugate momentum dependence remains free. The rest of the computations is equivalent to dimensional reduction, so finally we arrive at a form which agrees with the conjecture mentioned in the Introduction.

For \( \Phi(x) \) we can derive a gap equation, which can be written in a differential equation form. The initial conditions \( \Phi(k, t = 0) = \phi_k, \ \dot{\Phi}(k, t = 0) = \pi_k \), and, symbolically,

\[ (\partial^2_t + \omega_k^2) \Phi = -\frac{\lambda}{\beta} \Phi - \Phi \Phi - \Phi \Phi \Phi. \]  

(7)  

The first correction modifies the mass term to \( m(T, \Lambda) \) in the same way as in the effective Hamiltonian. The second term, however, is nonlocal, we can denote
it as $\lambda(x - x')/6$. We can define its time-local part $\lambda_{\text{loc}} = \lambda(t = t')$, it is time dependent, and it is the same as the coupling of the effective Hamiltonian. The nonlocal part after partial integration, with $N(\omega) = \Theta(\omega - \Lambda)n(\omega)$

$$
\lambda_{\text{nonl}} = \int \frac{d^3q}{(2\pi)^3} \frac{1}{\omega_q \omega_{k-q}} \left[ \frac{1 + N(\omega_q) + N(\omega_{k-q})}{\omega_q + \omega_{k-q}} \cos(\omega_q + \omega_{k-q})(t - t') + \frac{N(\omega_{k-q}) - N(\omega_q)}{\omega_q - \omega_{k-q}} \cos(\omega_q - \omega_{k-q})(t - t') \right] \frac{\partial}{\partial t'} + O(\Lambda \beta).
$$

(8)

The local time evolution is determined by the effective Hamiltonian, just as in the classical statistical mechanics. Therefore a local approximation provides at the same time the classical limit. The quantum effects, as we could see, induce nonlocalities in the time evolution.

We can also reintroduce $\hbar$ by substituting

$$
\beta \rightarrow \hbar \beta \quad \lambda \rightarrow \hbar \lambda.
$$

The neglected order is $O(\Lambda \beta \hbar) \sim O(\hbar)$, are the terms which are suppressed by some power of $\hbar$. There are other quantum corrections, however: some of them are powers of $1/\hbar$ (e.g. the one loop mass correction), others are just logarithmically suppressed (as the nonlocal contribution to the damping rate $\sim \sqrt{\hbar} \ln \hbar$). The IR effective theory contains both corrections, the local (classical) approximation only the dominant ones.

4 The retarded Greens function

Let us illustrate this with the calculation of the classical retarded Greens function. The details can be found in Refs. We use the linear response theory:

$$
H \rightarrow H + \int j \phi, \quad \text{which gives a current dependent solution } \Phi(x; j).
$$

The classical retarded Greens function is

$$
D_{R}^{cl}(x - x') = \frac{1}{Z} \int D\phi D\pi e^{-\beta \Phi(x; j)} \left. \frac{\delta \Phi(x; j)}{\delta j(x')} \right|_{j=0}.
$$

(9)

We can define the self energy from the Schwinger-Dyson equation

$$
D_{R}^{cl}(x - x') = D_{R}^{(0)}(x - x') + \int d^3y \int d^3y' D_{R}^{(0)}(x - y) \Pi^{cl}(y - y') D_{R}^{cl}(y' - x'),
$$

(10)

where $D_{R}^{(0)}(k, t) = -\Theta(t) \sin \omega_k t/\omega_k$. Its imaginary part

$$
\text{Im } \Pi^{cl}(\omega, p) = \frac{\pi}{2} \lambda^2 T^2 \int \prod_{i=1}^{3} \frac{d^3p_i}{(2\pi)^3} \frac{(2\pi)^3 \delta(S \mathbf{p}_i - \mathbf{p})}{\omega_1 \omega_2 \omega_3} \frac{1}{\omega_1 \omega_2 \omega_3} \left[ \omega_1 \delta(\omega - \omega_1 - \omega_2 - \omega_3) + \omega \delta(\omega + \omega_1 - \omega_2 - \omega_3) \right].
$$

(11)
The imaginary part of the quantum self energy agrees with this result at the leading order, if the effective parameters are used. The first correction is a relative $O\left(\sqrt{\lambda} \ln \lambda \right) = O\left(\sqrt{\bar{\hbar}} \ln \bar{\hbar} \right)$, which can be reproduced from the nonlocal time evolution described earlier. On the other hand the classical result can be thought of as the leading term in the high temperature expansion of the quantum result. The corrections then come from the subleading terms in the high temperature expansion.

The quantum correction, being relatively suppressed as $\sqrt{\lambda} \ln \lambda$, overwhelms the first classical correction term. That limits the use of the classical theory: we have to keep only the leading (nontrivial) order, and may average with respect to the free Hamiltonian.

5 Summary

We have constructed an effective theory for computing time dependent observables in scalar field theory. The form of the effective Hamiltonian (after IR approximation) confirms the conjecture described in the Introduction; the time evolution is, however, nonlocal. The local approximation provides the usual classical limit, where the time evolution is generated by the effective Hamiltonian. We have calculated the classical retarded Greens function and defined the classical self energy. Its imaginary part also agrees off-shell with the corresponding quantum result. The first correction, however, contains $\ln \bar{\hbar}$, it is a genuine quantum effect. Hence, the use of the classical theory beyond leading order does not appear meaningful.
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