SO(n, n + 1)-SURFACE GROUP REPRESENTATIONS AND HIGGS BUNDLES

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Abstract. We study the character variety of representations of the fundamental group of a closed surface of genus \( g \geq 2 \) into the Lie group \( \text{SO}(n, n + 1) \) using Higgs bundles. For each integer \( 0 < d \leq n(2g - 2) \), we show there is a smooth connected component of the character variety which is diffeomorphic to the product of a certain vector bundle over a symmetric product of a Riemann surface with the vector space of holomorphic differentials of degree \( 2, 4, \ldots, 2n - 2 \). In particular, when \( d = n(2g - 2) \), this recovers Hitchin’s parameterization of the Hitchin component. We also exhibit \( 2^{2g+1} - 1 \) additional connected components of the \( \text{SO}(n, n + 1) \)-character variety and compute their topology. Moreover, representations in all of these new components cannot be continuously deformed to representations with compact Zariski closure. Using recent work of Guichard-Wienhard on positivity, it is shown that each of the representations which define singularities (i.e. those which are not irreducible) in these \( 2^{2g+1} - 1 \) connected components are positive Anosov representations.

1. Introduction

Since Higgs bundles were introduced, they have found application in parameterizing connected components of the moduli space of reductive surface group representations into a reductive Lie group \( G \). In particular, for a closed surface \( S \) with genus \( g \geq 2 \), Hitchin gave an explicit parameterization of all but one of the connected components of the space of conjugacy classes of reductive representations of the fundamental group of \( S \) into the Lie group \( \text{PSL}(2, \mathbb{R}) \) [23]. Namely, he showed that each component with nonzero Euler class is diffeomorphic to the total space of a smooth vector bundle over an appropriate symmetric product of the surface. When the Euler class is maximal, this recovers a parameterization of the Teichmüller space of \( S \) as a vector space of complex dimension \( 3g - 3 \).

Hitchin later showed that for \( G \) a connected split real form, such as \( \text{PSL}(n, \mathbb{R}) \) or \( \text{SO}(n, n + 1) \), there is a connected component of this moduli space of representations which directly generalizes Teichmüller space [24]. Moreover, Hitchin parameterized this connected component, now called the Hitchin component, by a vector space of holomorphic differentials on the surface \( S \) equipped with a Riemann surface structure. In this paper, we use Higgs bundle techniques to generalize both of these results for the group \( \text{SO}(n, n + 1) \).

Let \( \Gamma = \pi_1(S) \) be the fundamental group of a closed surface \( S \) of genus \( g \geq 2 \). For a real reductive algebraic Lie group \( G \), we will refer to the space of conjugacy classes of representations \( \rho : \Gamma \rightarrow G \) of \( \Gamma \) into \( G \) whose images have reductive Zariski closure as the \( G \)-character variety; it will be denoted by \( \mathcal{X}(G) \). For connected reductive Lie groups, topological \( G \) bundles on \( S \) are classified by a characteristic class \( \omega \in \mathcal{H}^2(S, \pi_1(G)) \cong \pi_1(G) \). Thus, the \( G \)-character variety decomposes as

\[
\mathcal{X}(G) = \bigcup_{\omega \in \pi_1(G)} \mathcal{X}_\omega(G),
\]

where \( \mathcal{X}_\omega(G) \) is the component of the \( G \)-character variety associated with the characteristic class \( \omega \).
where the equivalence class of a reductive representation \( \rho : \Gamma \rightarrow G \) lies in \( \mathcal{X}^\omega(G) \) if and only if the flat \( G \) bundle determined by \( \rho \) has topological type determined by \( \omega \in \pi_1(G) \).

The space \( \mathcal{X}^\omega(G) \) is nonempty and connected for each \( \omega \in \pi_1(G) \) when \( G \) is compact and semisimple [34] and also when \( G \) is complex and semisimple [30]. Since \( G \) is homotopic to its maximal compact subgroup, \( \mathcal{X}^\omega(G) \) is connected if every representation in \( \mathcal{X}^\omega(G) \) can be continuously deformed to one with compact Zariski closure. Connectedness of \( \mathcal{X}^\omega(G) \) has been proven for many real forms using this technique, see [33, 6].

There are exactly two known families of Lie groups for which the space \( \mathcal{X}^\omega(G) \) is not connected. When \( G \) is a split real form, the Hitchin component is not distinguished by an invariant \( \omega \in \pi_1(G) \). Similarly, when \( G \) is a group of Hermitian type, the connected components of \textit{maximal representations} are usually not labeled by topological invariants \( \omega \in \pi_1(G) \). Both Hitchin representations and maximal representations define an important class of representations: they are the only known components of \( \mathcal{X}(G) \) which consist entirely of Anosov representations [29, 7].

1.1. \textbf{New components for} \( G = \text{SO}(n, n+1) \). The group \( \text{SO}(n, n+1) \) has two connected components, we will denote the connected component of the identity by \( \text{SO}_0(n, n+1) \). For \( n \geq 3 \), the group \( \text{SO}(n, n+1) \) is a split group, but not of Hermitian type. Nevertheless, we show that the \( \text{SO}_0(n, n+1) \)-character variety has many non-Hitchin connected components which are not distinguished by a topological invariant \( \omega \in \pi_1(\text{SO}_0(n, n+1)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \).

\textbf{Theorem 4.11.} Let \( \Gamma \) be the fundamental group of a closed surface \( S \) of genus \( g \geq 2 \) and let \( \mathcal{X}(\text{SO}(n, n+1)) \) be the \( \text{SO}(n, n+1) \)-character variety of \( \Gamma \). For each integer \( d \in (0, n(2g-2)) \), there is a smooth connected component \( \mathcal{X}_d(\text{SO}(n, n+1)) \) of \( \mathcal{X}(\text{SO}(n, n+1)) \) which does not contain representations with compact Zariski closure. Furthermore, for each choice of Riemann surface structure \( X \) on \( S \), the space \( \mathcal{X}_d(\text{SO}(n, n+1)) \) is diffeomorphic to the product

\[
\mathcal{X}_d(\text{SO}(n, n+1)) \cong F_d \times \bigoplus_{j=1}^{n-1} H^0(K^{2j}),
\]

where \( F_d \) is the total space of a rank \( d + (2n - 1)(g-1) \) vector bundle over the symmetric product \( \text{Sym}^n(2g-2)^{d}(X) \) and \( H^0(K^{2j}) \) is the vector space of holomorphic differentials of degree \( 2j \).

In fact, the representations \( \rho \in \mathcal{X}_d(\text{SO}(n, n+1)) \) factor through the connected component of the identity \( \text{SO}_0(n, n+1) \subset \text{SO}(n, n+1) \).

\textbf{Remark 1.1.} As a direct corollary, the connected components \( \mathcal{X}_d(\text{SO}(n, n+1)) \) deformation retract onto the symmetric product \( \text{Sym}^n(2g-2)^{d}(X) \). In particular, the cohomology ring of \( \mathcal{X}_d(\text{SO}(n, n+1)) \) is the same as the cohomology ring of the symmetric product \( \text{Sym}^n(2g-2)^{d}(X) \) which was computed in [31].

Using the isomorphism \( \text{PSL}(2, \mathbb{R}) \cong \text{SO}_0(1, 2) \). Theorem 4.11 recovers Hitchin’s parameterization of the nonzero Euler class components of \( \mathcal{X}(\text{PSL}(2, \mathbb{R})) \) mentioned above. Also, when the label \( d \) in Theorem 4.11 is maximal, the vector bundle \( F_{n(2g-2)} \) is the rank \( (4n-1)(g-1) \) vector space of holomorphic differentials of degree \( 2n \). Thus, we recover the parameterization of the \( \text{SO}(n, n+1) \)-Hitchin component as a vector space of holomorphic differentials. When \( n = 2 \), Theorem 4.11 gives a parameterization of an \( \text{SO}_0(2, 3) = \text{PSp}(4, \mathbb{R}) \)-version of \( \text{Sp}(4, \mathbb{R}) \) components discovered in [16]. For \( n > 2 \) and \( 0 < d < n(2g-2) \) the components are new.

There is also a connected component associated to \( d = 0 \) which has non-orbifold singularities. We briefly describe it here. Let \( X \) be a Riemann surface structure on
S and let $\operatorname{Pic}(X)$ be the Picard group of holomorphic line bundles on $X$. Consider the space $\tilde{F}_0$ defined by

$$\tilde{F}_0 = \{ (M, \mu, \nu) \mid M \in \operatorname{Pic}^0(X), \mu \in H^0(M^{-1}K^n), \nu \in H^0(MK^n) \}.$$ 

Recall that the group of matrices $\left( \begin{array}{cc} \lambda & 0 \\ 0 & \lambda^{-1} \end{array} \right)$ and $\left( \begin{array}{cc} 0 & \lambda \\ \lambda^{-1} & 0 \end{array} \right)$ for $\lambda \in \mathbb{C}^*$ is isomorphic to $\mathbb{O}(2, \mathbb{C})$. There is a natural action of $\mathbb{O}(2, \mathbb{C})$ on $\tilde{F}_0$ given by:

$$g \cdot (M, \mu, \nu) = \begin{cases} (M, \lambda^{-1} \mu, \lambda \nu) & \text{if } g = \left( \begin{array}{cc} \lambda & 0 \\ 0 & \lambda^{-1} \end{array} \right) \\ (M^{-1}, \lambda^{-1} \mu, \lambda \nu) & \text{if } g = \left( \begin{array}{cc} 0 & \lambda \\ \lambda^{-1} & 0 \end{array} \right). \end{cases}$$

**Theorem 5.1.** Let $\Gamma$ be the fundamental group of a closed surface $S$ of genus $g \geq 2$ and let $\mathcal{X}(\mathbb{O}(n, n+1))$ be the $\mathbb{O}(n, n+1)$-character variety of $\Gamma$. For each $n \geq 2$, there is a connected component $X_0(\mathbb{O}(n, n+1))$ of $\mathcal{X}(\mathbb{O}(n, n+1))$ which does not contain representations with compact Zariski closure. Furthermore, for each Riemann surface structure on $S$, the space $X_0(\mathbb{O}(n, n+1))$ is homeomorphic to

$$X_0(\mathbb{O}(n, n+1)) \cong F_0 \times \bigoplus_{j=1}^{n-1} H^0(K^{2j}),$$

where $F_0$ is the GIT quotient $\tilde{F}_0/\mathbb{O}(2, \mathbb{C})$ of the $\mathbb{O}(2, \mathbb{C})$-space $\tilde{F}_0$ described above and $H^0(K^{2j})$ is the vector space of holomorphic differentials of degree $2j$.

In fact, the representations $\rho \in X_0(\mathbb{O}(n, n+1))$ factor through the connected component of the identity $\mathbb{O}(n, n+1) \subset \mathbb{O}(n, n+1)$.

**Remark 1.2.** In Section 5, we provide a parameterization of the singular space $X_0(\mathbb{O}(n, n+1))$. This is a direct analogy of the associated component of the set of maximal $\mathbb{O}(2, 3)$-representations provided in [1]. Unlike the maximal $\mathbb{O}(2, 3)$ case, the component $X_0(\mathbb{O}(n, n+1))$ does not arise from a known topological invariant for $n \geq 3$. Thus, to show that $X_0(\mathbb{O}(n, n+1))$ does indeed define a connected component, it is necessary to analyze the local structure around the singularities (see Lemma 5.15). Also analogous to the computations in [1], $X_0(\mathbb{O}(n, n+1))$ deformation retracts onto the quotient of $\operatorname{Pic}^0(X)$ by the $\mathbb{Z}_2$ action of inversion. In particular, the rational cohomology of $X_0(\mathbb{O}(n, n+1))$ is given by

$$H^j(X_0(\mathbb{O}(n, n+1)), \mathbb{Q}) = \begin{cases} H^j(S^n)^{2j}, \mathbb{Q} & \text{if } j \text{ is even} \\ 0 & \text{otherwise} \end{cases}.$$ 

Each nonzero cohomology class $sw_1 \in H^1(S, \mathbb{Z}_2)$ corresponds to a connected principal $\mathbb{Z}_2$ bundle (i.e., orientation double cover) $X_{sw_1} \rightarrow X$. If $\iota$ is the covering involution of the covering, then the Prym variety $\operatorname{Prym}(X_{sw_1}, X)$ is defined as the kernel of $\operatorname{Id} + \iota^* : \operatorname{Pic}^0(X_{sw_1}) \rightarrow \operatorname{Pic}^0(X_{sw_1})$. That is,

$$\operatorname{Prym}(X_{sw_1}, X) = \{ M \in \operatorname{Pic}^0(X_{sw_1}) \mid \iota^* M = M^{-1} \}.$$ 

The Prym variety of an orientation double cover of a Riemann surface has two connected components determined by an invariant $sw_2 \in H^2(X, \mathbb{Z}_2)$. Let $K_{X_{sw_1}}$ denote the canonical bundle of the double cover $X_{sw_1}$.

**Theorem 5.3.** Let $\Gamma$ be the fundamental group of a closed surface $S$ of genus $g \geq 2$ and let $\mathcal{X}(\mathbb{O}(n, n+1))$ be the $\mathbb{O}(n, n+1)$-character variety of $\Gamma$. For each $n \geq 2$ and each $(sw_1, sw_2) \in (H^1(S, \mathbb{Z}_2) \setminus \{0\}) \times H^2(X, \mathbb{Z}_2)$, there is a connected component $X_{sw_1}\mathcal{X}^{sw_2}(\mathbb{O}(n, n+1))$ of $\mathcal{X}(\mathbb{O}(n, n+1))$ which does not contain representations with compact Zariski closure. Furthermore, for each Riemann surface structure $X$
on $S$, the space $X_{sw^2}(\text{SO}(n,n+1))$ is a smooth orbifold diffeomorphic to

$$F_{sw^2}^\infty / (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \times \bigoplus_{j=1}^{n-1} H^0(K^j_{\mathcal{X}})$$

where $F_{sw^2}^\infty \to \text{Prym}_{sw^2}(X_{sw^1}, X)$ is the rank $(4n - 2)(2g - 2)$ vector bundle over the connected component of the Prym variety associated to $sw_2$ with $\pi^{-1}(\mathcal{M}) = H^0(MK^0_{\mathcal{X}_{sw^1}})$. Here the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ action is generated by $(M, \mu) \mapsto (M, -\mu)$ and $(M, \mu) \mapsto (i \ast M, i \ast \mu)$, where $i$ is the covering involution of $X_{sw^1}$.

When $n$ is even, the representations in the components $X_{sw^2}(\text{SO}(n,n+1))$ factor through the connected component of the identity $\text{SO}_0(n,n+1)$, however when $n$ is odd, they do not.

**Remark 1.3.** Since the space $F_{sw^2}^\infty$ deformation retracts onto $\text{Prym}_{sw^2}(X_{sw^1})$, the homotopy type of each component $X_{sw^2}(\text{SO}(n,n+1))$ is the same as the quotient of $(S^1)^{2g-2}$ be the $\mathbb{Z}_2$ action of inversion. In particular, its cohomology is given by

$$H^j(X_0(\text{SO}(n,n+1)), \mathbb{Q}) = \begin{cases} H^j((S^1)^{2g-2}, \mathbb{Q}) & \text{if } j \text{ is even} \\ 0 & \text{otherwise} \end{cases}.$$  

Again, for $n = 2$, the connected components $X_{sw^2}(\text{SO}(2,3))$ of Theorem 5.3 were described by Alessandrini and the author in [1].

**Remark 1.4.** The components of Theorems 4.11 and 5.1 are labeled by an integer invariant $d \in [0, n(2g - 2)]$ and the components of Theorem 5.3 are labeled by $\mathbb{Z}_2$-invariants $(sw_1, sw_2) \in H^0(S, \mathbb{Z}_2) \times \{0\} \times H^0(S, \mathbb{Z}_2)$. Similar types of invariants have recently been associated to the spectral data of certain $\text{SO}(n,n+1)$ Higgs bundles by Schaposnik and Baraglia-Schaposnik in [35] and [3]. It would be very interesting to relate the spectral data invariants to the component description above.

### 1.2. Generalizations of low dimensional isomorphisms.

The group $\text{SO}_0(2,3)$ is isomorphic to $\text{PSp}(4, \mathbb{R})$, and the connected components of Theorem 4.11 are a $\text{PSp}(4, \mathbb{R})$ version of the $\text{Sp}(4, \mathbb{R})$ components discovered by Gothen in [16]. The groups $\text{Sp}(2n, \mathbb{R})$ and $\text{SO}_0(2, n)$ provide two families of Hermitian groups which generalize $\text{SO}_0(2,3)$. However, the space of maximal $\text{Sp}(2n, \mathbb{R})$ representations behaves differently for $n = 2$ and $n \geq 3$, and the space maximal $\text{SO}_0(2, n)$ representations behaves differently for $n = 3$ and $n \geq 4$.

For $\text{Sp}(4, \mathbb{R})$, there are $3 \cdot 2^{2g} + 2g - 4$ connected components of the space of maximal $\text{Sp}(4, \mathbb{R})$-representations [16], while the space of maximal $\text{Sp}(2n, \mathbb{R})$ representations has $3 \cdot 2^{2g}$ connected components for $n \geq 3$ [13]. In particular, for maximal $\text{Sp}(4, \mathbb{R})$ representations, $2g - 4$ of the connected components consist entirely of Zariski dense representations [5]. Similarly, for maximal $\text{SO}_0(2,3)$ representations, the $4g - 5$ of the connected components from Theorem 4.11 with $d \in (0, 4g - 4)$ consist entirely of Zariski dense representations [1]. The remaining connected components of maximal $\text{Sp}(4, \mathbb{R})$ and $\text{SO}_0(2,3)$ representations contain representations which factor through a Fuchsian representation $\rho_{\text{Fuch}} : \Gamma \to \text{SL}(2, \mathbb{R})$ [5, 19].

For $n \geq 3$, each connected component of maximal $\text{Sp}(2n, \mathbb{R})$ representations contains representations which factor through a Fuchsian representation [19]. Similarly, there are $2^{2g+1}$ connected components of maximal $\text{SO}_0(2, n)$ representations for $n \geq 4$, and each component contains representations which factor through a Fuchsian representation.

Theorems 4.11 gives an explanation of this difference as a consequence of the low dimensional isomorphism $\text{SO}_0(2,3) \cong \text{PSp}(4, \mathbb{R})$. Namely, the extra maximal components appearing for $\text{Sp}(4, \mathbb{R})$ and $\text{SO}_0(2,3)$ are an $\text{SO}(n,n+1)$ phenomenon.
For $n \geq 3$, the group $SO(n, n + 1)$ is a split group but not of Hermitian type. As a result, Theorems 4.11, 5.1 and 5.3 provide the first examples of connected components of $\mathcal{X}(\pi_1, G)$ which are not maximal, not Hitchin, and are not distinguished by a topological invariant in $\pi_1(G)$.

The component count of $\mathcal{X}(SO_0(n, n + 1))$ was established by Goldman [15] for $n = 1$ and by combining the work Bradlow-Garcia-Prada-Gothen [4] and Gothen-Oliveira [17] for $n = 2$. For $n \geq 3$, Theorems 4.11, 5.1 and 5.3 provide a new lower bound for the number of components of the space $\mathcal{X}(SO(n, n + 1))$. Namely,

$$(1.1) \quad |\pi_0(\mathcal{X}(SO(n, n + 1)))| \geq 2^{2g+2} + 1 + n(2g-2) + 2(2^{2g} - 1).$$

Here, the first $2^{2g+2}$ components contain representations with Zariski closure in $SO_0(n, n + 1)$) and the remaining components come from Theorems 4.11, 5.1 and 5.3. In [2], the connected components of $\mathcal{X}(SO(n, m))$ are counted, and for $m = n + 1$ it is shown that the lower bound in (1.1) is indeed an equality.

### 1.3. Positive Anosov representations

We now turn to the geometry of the representations in the components described by Theorem 5.1 and Theorem 5.3. Anosov representations were introduced by Labourie [29] and have many interesting geometric and dynamic properties which generalize convex cocompact representations into rank one Lie groups. Important examples of Anosov representations include quasi-Fuchsian representations, Hitchin representations into split groups and maximal representations into groups of Hermitian type.

Recently, Guichard and Wienhard [22] introduced the notion of a $P_\Theta$-positive Anosov representation which refines the notion of an Anosov representation. In particular, the spaces of Hitchin representations are positive with respect to the Borel subgroup [11, 29] and, for a Hermitian group $G$ of tube type, maximal representations are positive with respect to the parabolic subgroup which gives rise to the Shilov boundary of the Riemannian symmetric space of $G$ [7].

Since the Lie group $SO(n, n + 1)$ is split, it admits a notion of positivity with respect to the Borel subgroup. Interestingly, $SO(n, n + 1)$ also admits a notion of positivity with respect to the generalized flag variety $SO(n, n + 1)/P_\Theta$ consisting of flags $V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{R}^{2n+1}$ where $V_j$ is an isotropic (with respect to a signature $(n, n + 1)$ inner product) $j$-plane. We will call this $P_\Theta$-positivity.

**Remark 1.5.** The set of positive Anosov representations is open in the character variety. In [20], it is conjectured that positive Anosov representations are also closed in the character variety. In fact, it can be shown that the set of positive Anosov representations is closed in the set of irreducible representations [40]. Namely, let $\rho_j : \Gamma \to SO(n, n + 1)$ be a sequence of $P_\Theta$-positive Anosov representation which converge to $\rho_\infty : \Gamma \to SO(n, n + 1)$. If the action of each $\rho_j$ on $\mathbb{R}^{2n+1}$ via the standard representations of $SO(n, n + 1)$ is irreducible and $\rho_\infty$ is also irreducible, then $\rho_\infty$ is $P_\Theta$ Anosov.

Here we prove that the set on non-irreducible representations in the connected components $\mathcal{X}_0(SO(n, n + 1))$ and $\mathcal{X}_{sw}^2(SO(n, n + 1))$ are $P_\Theta$ positive Anosov.

**Theorem 7.13.** Let $SO(n, n + 1)/P_\Theta$ be the generalized flag variety of flags

$$V_1 \subset \cdots \subset V_{n-1} \subset V_{n-1}^\perp \subset \cdots \subset V_1^\perp \subset \mathbb{R}^{2n+1},$$

where $V_j \subset \mathbb{R}^{2n+1}$ is an isotropic $j$-plane. If $n \geq 2$, then the set of representations $\rho$ in $\mathcal{X}_0(SO(n, n + 1))$ or $\mathcal{X}_{sw}^2(SO(n, n + 1))$ for which the action of $\rho$ on $\mathbb{R}^{2n+1}$ is reducible is a nonempty set consisting entirely of $P_\Theta$-positive Anosov representation.

**Remark 1.6.** Assuming the results mentioned in Remark 1.5, Theorem 7.13 can be significantly strengthened to the statement that the components $\mathcal{X}_0(SO(n, n + 1))$
and \( \mathcal{X}_{\text{an}}^{sw}(\text{SO}(n, n+1)) \) consist entirely of Anosov representations. The argument is as follows: Let \( \rho_0 \) be a reducible representation in \( \mathcal{X}_0(\text{SO}(n, n+1)) \) or \( \mathcal{X}_{\text{an}}^{sw}(\text{SO}(n, n+1)) \). Since positive representations define an open set in the character variety, there is an open neighborhood \( U_{\rho_0} \) of \( \rho_0 \) consisting of \( P_\Theta \)-positive representations. In particular, there exists \( \rho \in U_{\rho_0} \) which is irreducible. Since positivity is closed in the set of irreducible representations, all irreducible representations \( \rho \in \mathcal{X}_0(\text{SO}(n, n+1)) \) are \( P_\Theta \)-positive. Thus, by Theorem 7.13 all representations in \( \mathcal{X}_0(\text{SO}(n, n+1)) \) and \( \mathcal{X}_{\text{an}}^{sw}(\text{SO}(n, n+1)) \) are Anosov.

For \( n = 2 \), Theorem 7.13 follows from maximality of the corresponding representations. For \( n \geq 3 \), the proof relies heavily on the work of Guichard-Wienhard and Guichard-Labourie-Wienhard on positive representations \[22, 20\] and establishing that the representations which correspond to the singularities of \( \mathcal{X}_{\text{an}}^{sw}(\text{SO}(n, n+1)) \) and \( \mathcal{X}_0(\text{SO}(n, n+1)) \) are products of Hitchin representations in \( \text{SO}(n-1, n) \) with an \( \text{SO}(2) \) representation or \( \text{SO}(n, n) \)-Hitchin representations.

For \( 0 < d < n(2g-2) \), the spaces \( \mathcal{X}_d(\text{SO}(n, n+1)) \) from Theorem 4.11 are smooth; hence all the representations in these components are irreducible. Thus, if there exists a representation \( \rho \in \mathcal{X}_d(\text{SO}(n, n+1)) \) which is positive Anosov, then, by Remark 1.5, \( \mathcal{X}_d(\text{SO}(n, n+1)) \) would consist entirely of positive Anosov representations. There are however no obvious model representations to consider in the components \( \mathcal{X}_d(\text{SO}(n, n+1)) \). In particular, for \( n = 2 \), all representations in these components are Zariski dense. We conjecture this holds for the components \( \mathcal{X}_d(\text{SO}(n, n+1)) \) for \( 0 < d < n(2g-2) \).

**Conjecture 1.7.** For \( 0 < d < n(2g-2) \), all representations in the component \( \mathcal{X}_d(\text{SO}(n, n+1)) \) from Theorem 4.11 are Zariski dense.

**Organization of Paper:** In Sections 2 and 3, we recall the necessary features of Higgs bundles and character varieties. In Section 4, we prove Theorem 4.11 and in Section 5 we prove Theorems 5.1 and 5.3. In Section 6, we prove results about the Zariski closures of representations in the new components of \( \mathcal{X}(\text{SO}(n, n+1)) \). Finally, in Section 7, the notion of positive Anosov representations is recalled and we use the results on Zariski closures to prove Theorem 7.13.

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2. **Higgs bundles and surface group representations**

We start by recalling the necessary facts about Higgs bundles and surface group representations which, in subsequent sections, will be used for \( G = \text{SO}(n, n+1) \).

2.1. **G Higgs bundles.** Let \( G \) be a real algebraic semisimple Lie group\(^1\) with Lie algebra \( g \), and fix \( H \subset G \) a maximal compact subgroup with Lie algebra \( h \). Let \( g = h \oplus m \) be the corresponding Cartan decomposition of the Lie algebra \( g \). Here

\(^1\)In fact, with slight modifications, everything works for real reductive Lie groups. We will not need this more general setting.
$m$ is the orthogonal complement of $h$ with respect to the Killing form of $g$, and the
splitting $h \oplus m$ consists of the $\pm 1$-eigenspaces of an involution $\theta : g \to g$. Thus,
$[m, m] \subset h$ and $[h, m] \subset m$, and the splitting $g = h \oplus m$ is invariant with respect to
the adjoint action of $H$ on $g$. Complexifying everything, we have an $Ad_{H_C}$ invariant
decomposition
\[ g_C = h_C \oplus m_C. \]

Let $X$ be a closed Riemann surface of genus $g \geq 2$ and canonical bundle $K$. For
any group $G$, if $P$ is a principal $G$ bundle and $\alpha : G \to \text{GL}(V)$ is a linear representation,
denote the associated vector bundle $P \times_G V$ by $P[V]$.

**Definition 2.1.** Fix a smooth principal $H_C$ bundle $P \to X$. A $G$ Higgs bundle
structure on $P$ is a pair $(\mathcal{P}, \varphi)$ where $\mathcal{P}$ is a holomorphic principal $H_C$ bundle with
underlying smooth bundle $P$ and $\varphi \in H^0(X, \mathcal{P}[m_C] \otimes K)$ is a holomorphic section
of the associated $m_C$ bundle twisted by $K$. The section $\varphi$ is called the Higgs field.

**Example 2.2.** If $G$ is compact, then $H_C = G_C$ and $m_C = \{0\}$. Thus for compact
groups, a Higgs bundle is the same as a holomorphic principal $G_C$ bundle. When
$G$ is a complex semisimple Lie group, we have $H_C = G$ and $m_C \cong g$. In this case,
a $G$ Higgs bundle consists of a holomorphic $G$ bundle together with a holomorphic
$K$-twisted section of the adjoint bundle.

If $\alpha : H_C \to \text{GL}(V)$ is a linear representation of $H_C$, the data of a $G$ Higgs bundle
can be described by the vector bundle associated to $\alpha$ and a section of another
associated bundle. For instance, if $\alpha : \text{GL}(n, \mathbb{C}) \to \text{GL}(\mathbb{C}^n)$ is the standard rep-
resentation, then a $\text{GL}(n, \mathbb{C})$ Higgs bundle is equivalent to a rank $n$ holomorphic
vector bundle $\mathcal{E} \to X$ and a holomorphic section $\Phi$ of $\text{End}(\mathcal{E}) \otimes K$. Similarly, using
the standard representation, an $\text{SL}(n, \mathbb{C})$ Higgs bundle is equivalent to a $\text{GL}(n, \mathbb{C})$
Higgs bundle $(\mathcal{E}, \Phi)$ with $\Lambda^n \mathcal{E} = \mathcal{O}$ and $\text{Tr}(\Phi) = 0$.

To form the moduli space of Higgs bundles, we need the notion of stability.

**Definition 2.3.** A $\text{GL}(n, \mathbb{C})$ Higgs bundle $(\mathcal{E}, \Phi)$ is called stable if for all $\Phi$-invariant subbundles $\mathcal{F} \subset \mathcal{E}$ we have $\frac{\text{deg}(\mathcal{F})}{\text{rk}(\mathcal{F})} < \frac{\text{deg}(\mathcal{E})}{\text{rk}(\mathcal{E})}$. An $\text{SL}(n, \mathbb{C})$ Higgs bundle $(\mathcal{E}, \Phi)$ is

- **stable** if all $\Phi$-invariant subbundles $\mathcal{F} \subset \mathcal{E}$ satisfy $\text{deg}(\mathcal{F}) < 0$,
- **polystable** if $(\mathcal{E}, \Phi) = \bigoplus (\mathcal{E}_j, \Phi_j)$ where each $(\mathcal{E}_j, \Phi_j)$ is a stable $\text{GL}(n_j, \mathbb{C})$
  Higgs bundle with $\text{deg}(\mathcal{E}_j) = 0$ for all $j$.

There are appropriate notions of stability and polystability for $G$ Higgs bundles.
With respect to these notions, the moduli space of $G$ Higgs bundles is defined as
a polystable quotient. Rather than recalling the definition of polystability for $G$
Higgs bundles, we will use the following result (see [12]).

**Proposition 2.4.** Let $G$ be a real form of an irreducible subgroup of $\text{SL}(n, \mathbb{C})$. A $G$
Higgs bundle $(\mathcal{P}, \varphi)$ is polystable if and only if the associated $\text{SL}(n, \mathbb{C})$ Higgs bundle
is polystable.

The gauge group $G_{H_C}$ of smooth bundle automorphisms of a smooth $H_C$ bundle
$P_{H_C}$ acts on the set of Higgs bundle structures $(\mathcal{P}, \varphi) = (\delta_{\mathcal{P}}, \varphi)$ by the adjoint
action.

**Definition 2.5.** Fix a smooth principal $H_C$ bundle $P_{H_C}$ on $X$. The moduli space
of $G$ Higgs bundle structures on $P_{H_C}$ consists of isomorphism classes of polystable
Higgs bundles with underlying smooth bundle $P_{H_C}$,
\[ \mathcal{M}(P_{H_C}, G) = \{\text{polystable } G\text{-Higgs bundle structures on } P_{H_C}\}/G_{H_C}. \]

The union over the set of isomorphism classes of smooth principal $H_C$ bundles on $X$
of the spaces $\mathcal{M}(P_{H_C}, G)$ will be referred to as the moduli space of $G$ Higgs bundles
and denoted by $\mathcal{M}(G)$. 

The space $\mathcal{M}(G)$ can in fact be given the structure of an complex analytic variety of expected dimension $\text{dim}(G)/(g-1)$ [23, 39, 36]. Since $H_{\mathbb{C}}$ and $G$ are both homotopy equivalent to $H$, the set of equivalence classes of topological $H_{\mathbb{C}}$ bundles on $X$ is the same as the set of equivalence classes of topological $G$ bundles on $X$. Denote this set by $\text{Bun}_X(G)$. If the group $G$ is connected, then

$$\text{Bun}_X(G) \cong H^2(X, \pi_1(G)) \cong \pi_1(G).$$

If $G$ is not connected, the description is slightly more complicated, see [33, Section 3.1]. This gives a decomposition of the Higgs bundle moduli space:

$$\mathcal{M}(G) = \bigsqcup_{a \in \text{Bun}_X(G)} \mathcal{M}^a(G),$$

where $a \in \text{Bun}_X(G)$ is the topological type of the underlying $H_{\mathbb{C}}$ bundle of the Higgs bundles in $\mathcal{M}^a(G)$.

The automorphism group $\text{Aut}(\bar{\partial}_P, \varphi)$ of a polystable $G$ Higgs bundle $(P, \varphi)$ is defined by

$$\text{Aut}(\bar{\partial}_P, \varphi) = \{g \in \mathcal{G}_C| (Ad_g \bar{\partial}_P, Ad_g \varphi) = (\bar{\partial}_P, \varphi)\}.$$

The center $Z(G_C)$ of $G_C$ is the intersection of the center of $H_{\mathbb{C}}$ and the kernel of the representation $Ad : H_{\mathbb{C}} \to \text{GL}(m_C)$. Thus, we always have $Z(G_C) \subset \text{Aut}(\bar{\partial}_P, \varphi)$.

Using our definition of polystability from Proposition 2.4, we use the following (nonstandard) definition of stability of a $G$ Higgs bundle.

**Definition 2.6.** Let $G$ be a semisimple Lie group which is a real form of an irreducible subgroup of $\text{SL}(n, \mathbb{C})$. A polystable $G$ Higgs bundle $(P, \varphi)$ is stable if $\text{Aut}(P, \varphi)$ is finite.

Given a polystable $G$ Higgs bundle $(P, \varphi)$, consider the complex of sheaves

$$C^\bullet(P, \varphi) : P[h_C] \xrightarrow{ad_\varphi} P[m_C] \otimes K.$$

This gives a long exact sequence in hypercohomology:

\begin{equation}
0 \longrightarrow \mathbb{H}^0(C^\bullet(P, \varphi)) \longrightarrow H^0(P[h_C]) \xrightarrow{ad_\varphi} H^0(P[m_C] \otimes K) \longrightarrow \mathbb{H}^1(C^\bullet(P, \varphi)) \longrightarrow \mathbb{H}^2(C^\bullet(P, \varphi)) \longrightarrow 0.\end{equation}

Note that the automorphism group $\text{Aut}(\bar{\partial}_P, \varphi)$ acts on $\mathbb{H}^1(C^\bullet(P, \varphi))$. Using standard slice methods of Kuranishi (see [27, Chapter 7.3] for details for the moduli space of holomorphic bundles), a neighborhood of the isomorphism class of a polystable Higgs bundle $(P, \varphi)$ in $\mathcal{M}(G)$ is given by

\begin{equation}
\kappa^{-1}(0)/\text{Aut}(P, \varphi)\end{equation}

where $\kappa : \mathbb{H}^1(C^\bullet(P, \varphi)) \to \mathbb{H}^2(C^\bullet(P, \varphi))$ is the so called Kuranishi map.

When $\mathbb{H}^1(C^\bullet(P, \varphi)) = 0$, this simplifies considerably. Namely, in this case, a neighborhood of the isomorphism class of a polystable Higgs bundle $(P, \varphi)$ in $\mathcal{M}(G)$ is given by

$$\mathbb{H}^1(C^\bullet(P, \varphi))/\text{Aut}(P, \varphi).$$

**Remark 2.7.** For all of the $SO(n, n+1)$ Higgs bundles considered in the subsequent sections we will prove that the relevant $\mathbb{H}^2$ always vanishes. For this reason, we will not recall the construction of the Kuranishi map.

When the automorphism group $\text{Aut}(\bar{\partial}, \varphi)$ is finite, the GIT quotient above simplifies to a regular quotient. This gives the following characterizations of smooth points and orbifold points of $\mathcal{M}(G)$. 
Proposition 2.8. Let $G$ be a semisimple real Lie group. If $(\mathcal{P}, \varphi)$ is a polystable $G$ Higgs bundle with $\mathbb{H}^2(C^* (\mathcal{P}, \varphi)) = 0$ and $\text{Aut}(\mathcal{P}, \varphi)$ finite, then the isomorphism class of $(\mathcal{P}, \varphi)$ is an orbifold point of $\mathcal{M}(G)$ of type $\text{Aut}(\mathcal{P}, \varphi)/\mathcal{Z}(G)$. In particular, if $\text{Aut}(\mathcal{P}, \varphi) = \mathcal{Z}(G)$, then $(\mathcal{P}, \varphi)$ defines a smooth point of $\mathcal{M}(G)$.

Let $p_1, \ldots, p_{n-1}$ be a basis of $\mathfrak{sl}(n, \mathbb{C})$ invariant homogeneous polynomials on $\mathfrak{s}(n, \mathbb{C})$ with $\deg(p_j) = j + 1$. Given an $\mathfrak{sl}(n, \mathbb{C})$ Higgs bundle $(E, \Phi)$, the tensor $p_j(\Phi)$ is a holomorphic differential of degree equal to the degree of $p_j$. The map

$$(E, \Phi) \mapsto (p_1(\Phi), \ldots, p_{n-1}(\Phi))$$

from the set of Higgs bundles to the vector space $\bigoplus_{j=2}^{n} H^0(K^j)$ descends to a map

$$h : \mathcal{M}(\mathfrak{sl}(n, \mathbb{C})) \to \bigoplus_{j=2}^{n} H^0(K^j).$$

The map $h$ will be referred to as the Hitchin fibration. In [25], Hitchin showed that $h$ is a proper map. The properness of the Hitchin fibration will play a key role in Sections 4 and 5.

Finally, we have the notion of reducing the structure group of a $G$ Higgs bundle. This will be important in Sections 6 and 7.

Definition 2.9. Let $G$ and $G'$ be semisimple Lie groups with maximal compact subgroups $H$ and $H'$ and Cartan decompositions $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}'$. Suppose $i : G' \to G$ is an embedding such that $i(\mathfrak{h}') \subset \mathfrak{h}$ and $di(\mathfrak{m}') \subset \mathfrak{m}$. A $G$ Higgs bundle $(\mathcal{P}, \varphi)$ reduces to a $G'$ Higgs bundle $(\mathcal{P}', \varphi')$ if the holomorphic $H_{\mathbb{C}}$ bundle $\mathcal{P}$ admits a holomorphic reduction of structure group to the $H'_{\mathbb{C}}$ bundle $\mathcal{P}'$ and, with respect to this reduction, $\varphi \in H^0((\mathcal{P}' \times_{H'_{\mathbb{C}}} m'_{\mathbb{C}}) \otimes K) \subset H^0((\mathcal{P} \times_{H_{\mathbb{C}}} m_{\mathbb{C}}) \otimes K)$.

Remark 2.10. Note that a polystable $G$-Higgs bundle $(\mathcal{P}, \varphi)$ reduces to its maximal compact subgroup if and only if the Higgs field $\varphi$ vanishes.

2.2. Relation to surface group representations. Let $\Gamma$ be the fundamental group of a closed oriented surface $S$ of genus $g \geq 2$ and let $G$ be a real algebraic semisimple Lie group.

Definition 2.11. A representation $\rho : \Gamma \to G$ is reductive if the Zariski closure of the image $\rho(\Gamma) \subset G$ is a reductive subgroup.

The conjugation action of $G$ on $\text{Hom}(\Gamma, G)$ does not in general have a Hausdorff quotient. However, if we restrict to the set of reductive representations, the quotient will be Hausdorff.

Definition 2.12. The $G$-character variety $\mathcal{X}(G)$ of a surface group $\Gamma$ is the space of conjugacy classes of reductive representations of $\Gamma$ in $G$:

$$\mathcal{X}(G) = \text{Hom}^{\text{red}}(\Gamma, G)/G.$$

Example 2.13. The set of Fuchsian representations $\text{Fuch}(\Gamma) \subset \mathcal{X}(\Gamma, \text{SO}(1, 2))$ is defined to be the subset of conjugacy classes of faithful representations with discrete image. The space $\text{Fuch}(\Gamma)$ defines one connected components of $\mathcal{X}(\Gamma, \text{SO}(1, 2))$ [15] and is in one to one correspondence with the Teichmüller space of isotopy classes of marked Riemann surface structures on the surface $S$.

Each representation $\rho \in \mathcal{X}(G)$ defines a flat $G$ bundle

$$E_\rho = (\tilde{S} \times G)/\Gamma.$$
This gives a decomposition of the $G$ character variety:

$$\mathcal{X}(G) = \bigsqcup_{a \in \text{Bun}_S(G)} \mathcal{X}^a(G),$$

where $a \in \text{Bun}_S(G)$ is the topological type of the flat $G$ bundle of the representations in $\mathcal{X}^a(G)$.

We will rely heavily on the following theorem which was proven by Hitchin [23], Donaldson [10], Corlette [9] and Simpson [38] in various generalities. For the proof of the general statement below, see [12].

**Theorem 2.14.** Let $S$ be a closed oriented surface of genus $g \geq 2$ and $G$ be a real algebraic semisimple Lie group. For each Riemann surface structure $X$ on $S$ there is a homeomorphism between the moduli space $\mathcal{M}(G)$ of $G$ Higgs bundles on $X$ and the $G$-character variety $\mathcal{X}(G)$. Furthermore, this homeomorphism is a diffeomorphism when restricted to the smooth loci. Moreover, for each $a \in \text{Bun}_S(G)$, this homeomorphism identifies the spaces $\mathcal{M}^a(G)$ and $\mathcal{X}^a(G)$.

In Sections 6 and 7, it will be important to determine when a representation has smaller Zariski closure. This leads to the definition of a representation factoring through a reductive subgroup.

**Definition 2.15.** Let $G$ and $G'$ be reductive Lie groups and $i : G' \to G$ be an embedding. A representation $\rho : \Gamma \to G$ factors through $G'$ if there exists a representations $\rho' : \Gamma \to G'$ such that $\rho = i \circ \rho'$.

**Remark 2.16.** The group $SO(1, 2)$ is the set of isometries of the hyperbolic plane and $SO_0(1, 2)$ is the set of orientation preserving isometries. Note that since the surface $S$ is assumed to be orientable, all Fuchsian representations $\rho$ from Example 2.13 factor through the connected component of the identity $SO_0(1, 2)$.

The following proposition is immediate from Theorem 2.14.

**Proposition 2.17.** Let $G'$ be a reductive Lie subgroup of a semisimple Lie group $G$. A reductive representation $\rho : \Gamma \to G$ factors through $G'$ if and only if the corresponding polystable $G$ Higgs bundle $(P, \varphi)$ reduces to a $G'$ Higgs bundle. In particular, $\rho$ has compact Zariski closure if and only if $\varphi = 0$.

**Definition 2.18.** Let $G$ be a real form of a subgroup of $SL(n, \mathbb{C})$. A representation $\rho : \Gamma \to G$ is called irreducible if the induced representation $\Gamma \to SL(n, \mathbb{C})$ has no nonzero proper invariant subspaces.

For $G = SL(n, \mathbb{C})$, Theorem 2.14 gives a one to one correspondence between irreducible representations and stable $SL(n, \mathbb{C})$ Higgs bundles [23, 39]. This implies the following proposition which will play a key role in Sections 6 and 7.

**Proposition 2.19.** Suppose $G$ is a real form of an irreducible subgroup of $SL(n, \mathbb{C})$. Let $\rho : \Gamma \to G$ be a reductive representation and let $(\mathcal{P}, \varphi)$ be the corresponding $G$ Higgs bundle given by Theorem 2.14. The representation $\rho$ is irreducible if and only if the $SL(n, \mathbb{C})$ Higgs bundle associated to $(\mathcal{P}, \varphi)$ is stable.

### 3. $SO(n, n + 1)$ Higgs Bundles

In this section we specialize to the group $SO(n, n + 1)$ of orientation preserving automorphisms of $\mathbb{R}^{2n+1}$ which preserve a nondegenerate symmetric quadratic form of signature $(n, n + 1)$. The group $SO(n, n + 1)$ has two connected components, denote the connected component of the identity by $SO_0(n, n + 1)$. If $Q_n$ and $Q_{n+1}$
are positive definite symmetric $n \times n$ and $(n+1) \times (n+1)$ matrices, then the Lie algebra $\mathfrak{so}(n, n+1)$ is defined by the matrices

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} Q_n & -Q_{n+1} \\ -Q_{n+1} & -Q_n \end{pmatrix} + \begin{pmatrix} Q_n & -Q_{n+1} \\ -Q_{n+1} & -Q_n \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 0,
\]

where $A$ is an $n \times n$ matrix, $B$ is an $n \times (n+1)$ matrix, $C$ is an $(n+1) \times n$ matrix and $D$ is an $(n+1) \times (n+1)$ matrix. Thus,

\[
(3.1) \quad A^T Q_n + Q_n A = 0, \quad D^T Q_{n+1} + Q_{n+1} D = 0 \quad \text{and} \quad B = -Q_n^{-1} C^T Q_{n+1}.
\]

The maximal compact subgroup of $SO(n, n+1)$ is $S(O(n) \times O(n+1))$. Using (3.1), the complexified Cartan decomposition of the Lie algebra $so(n, n+1) \otimes \mathbb{C}$ is

\[
so(2n+1, \mathbb{C}) = (so(n, \mathbb{C}) \oplus so(n+1, \mathbb{C})) \oplus Hom(V, W)
\]

where $V$ and $W$ are the standard representations of $O(n, \mathbb{C})$ and $O(n+1, \mathbb{C})$. Using Definition 2.1, an $SO(n, n+1)$ Higgs bundle on $X$ is a pair $(\mathcal{P}, \varphi)$ where $\mathcal{P} \to X$ is a holomorphic $S(O(n, \mathbb{C}) \times O(n+1, \mathbb{C}))$-principal bundle and $\varphi$ is a holomorphic section of $\mathcal{P}[Hom(V, W)] \otimes K.$

Given a holomorphic principal $O(n, \mathbb{C})$ bundle, the rank $n$ vector bundle $V$ associated to the standard representation satisfies $det(V)^2 = (\Lambda^n V)^2 \cong \mathcal{O}$. Furthermore, $V$ admits an orthogonal structure $Q_V \in H^0(\mathcal{O}(1))$. An orthogonal structure $Q_V$ will be interpreted as a holomorphic symmetric isomorphism $Q_V : V \to V^\ast$. We take the following vector bundle definition of an $SO(n, n+1)$ Higgs bundle.

**Definition 3.1.** An $SO(n, n+1)$ **Higgs bundle** on Riemann surface $X$ is a triple $(V, W, \eta)$ where

- $V$ and $W$ are respectively rank $n$ and $(n+1)$ holomorphic vector bundles on $X$ equipped with holomorphic orthogonal structures $Q_V$ and $Q_W$ such that $det(V) = det(W)$.
- $\eta \in H^0(Hom(V, W) \otimes K)$.

**Remark 3.2.** An $SO(n, n+1)$ Higgs bundle $(V, W, \eta)$ reduces to an $SO_{q}(n, n+1)$ Higgs bundle if and only if $\Lambda^n V = \mathcal{O}$.

The $S(O(n, \mathbb{C}) \times O(n+1, \mathbb{C}))$-gauge group consists of pairs $(g_V, g_W)$ where $g_V$ and $g_W$ are smooth automorphisms of $V$ and $W$ such that

\[
g_V^T Q_V g_V = Q_V, \quad g_W^T Q_W g_W = Q_W \quad \text{and} \quad Id = det(g_V) \otimes det(g_W) : \Lambda^n V \otimes \Lambda^{n+1} W \to \Lambda^n V \otimes \Lambda^{n+1} W.
\]

Such a gauge transformation acts on the data $(V, W, \eta)$ by

\[
(g_V, g_W) \cdot (\partial_V, \partial_W, \eta) = (g_V \partial_V g_V^\ast, g_W \partial_W g_W^\ast, g_W \eta g_W^{-1})
\]

The $SL(2n+1, \mathbb{C})$ Higgs bundle $(E, \Phi)$ associated to an $SO(n, n+1)$ Higgs bundle $(V, W, \eta)$ is given by

\[
(3.2) \quad (E, \Phi) = \begin{pmatrix} V \oplus W, & \begin{pmatrix} 0 & \eta^* \\ \eta & 0 \end{pmatrix} \end{pmatrix}
\]

where $\eta^*$ is defined by $\eta^* = -Q_V^{-1} \circ \eta^T \circ Q_W : W \to V \otimes K$. Such a Higgs bundle will be represented schematically as

\[
\begin{array}{c}
V \xrightarrow{\eta^*} \\
\eta \\
W
\end{array}
\]

where we have suppressed the twisting by $K$ from the notation.
Note that, when restricted to $\text{SO}(n,n+1)$-Higgs bundles, the Hitchin fibration (2.3) maps to the space of even holomorphic differentials. Indeed, $Tr(\Phi^j)$ form a basis of invariant polynomials and for Higgs fields $\Phi$ of the form (3.2), $Tr(\Phi^j) = 0$ for $j$ odd and $Tr(\Phi^j) = 2Tr((\eta^* \otimes \eta)^j)$. The expected dimension of the moduli space $\mathcal{M}(\text{SO}(n,n+1))$ is

$$\dim(\text{SO}(n,n+1))(2g-2) = n(2n+1)(2g-2) = \dim \left( \bigoplus_{j=1}^{n} H^0(K^{2j}) \right).$$

We will use the following proposition to conclude the hypercohomology group $H^2(C^*(V,W,\eta))$ vanishes in some nice cases (see [12, Proposition 3.17]).

**Proposition 3.3.** If $(V,W,\eta)$ is a polystable $\text{SO}(n,n+1)$ Higgs bundle such that the associated $\text{SL}(2n+1,\mathbb{C})$ Higgs bundle given by (3.2) is stable, then

$$H^2(C^*(V,W,\eta)) \equiv 0.$$ 

By Proposition 2.8, we have the following corollary.

**Corollary 3.4.** If $(V,W,\eta)$ is a polystable $\text{SO}(n,n+1)$ Higgs bundle such that the associated $\text{SL}(2n+1,\mathbb{C})$ Higgs bundle given by (3.2) is stable, then the isomorphism class of $(V,W,\eta)$ defines an $\text{Aut}(V,W,\eta)$-orbifold point of $\mathcal{M}(\text{SO}(n,n+1))$. In particular, since the center of $\text{SO}(n,\mathbb{C}) \times \mathbb{O}(n+1,\mathbb{C})$ is trivial, $(V,W,\eta)$ defines a smooth point of $\mathcal{M}(\text{SO}(n,n+1))$ if and only if $\text{Aut}(V,W,\eta)$ is trivial.

3.1. **Topological classes of $\text{SO}(n,n+1)$ bundles on $X$.** Recall that the set of equivalences classes of $\text{SO}(n,n+1)$ bundles $\text{Bun}_X(\text{SO}(n,n+1))$ on $X$ gives a decomposition of the moduli space of $\text{SO}(n,n+1)$ bundles. Recall also that $\text{Bun}_X(\text{SO}(n,n+1)) = \text{Bun}_X(\text{SO}(n) \times \text{O}(n+1))$ since $\text{SO}(n,n+1)$ is homotopy equivalent to its maximal compact subgroup.

An $\text{O}(n)$ bundle $V \to X$ has a first and second Stiefel-Whitney class

$$sw_1(V) = sw_1(\Lambda^n V) \in H^1(X,\pi_0(\text{O}(n))) \quad \text{and} \quad sw_2(V) \in H^2(X,\mathbb{Z}_2).$$

When $n \geq 3$, $\pi_1(\text{O}(n)) = \mathbb{Z}_2$ and these characteristic classes are in bijective correspondence with $\text{Bun}_X(\text{O}(n))$.

**Proposition 3.5.** For $n \geq 3$, we have

$$\text{Bun}_X(\text{SO}(n,n+1)) \cong H^1(X,\mathbb{Z}_2) \times H^2(X,\mathbb{Z}_2).$$

**Proof.** An $\text{SO}(n) \times \text{O}(n+1)$ bundle is equivalent to a pair $(V,W)$ where $V$ is an $\text{O}(n)$ bundle and $W$ is an $\text{O}(n+1)$ bundle with $\text{det}(V) = \text{det}(W)$. The Stiefel-Whitney classes of $V$ and $W$ determine the topological class of an $\text{SO}(n,n+1)$ bundle, but, since $\text{det}(V) = \text{det}(W)$ we have $sw_1(V) = sw_1(W)$. \qed

For $n \geq 3$, the moduli space of $\text{SO}(n,n+1)$ Higgs bundles thus decomposes as

$$\mathcal{M}(\text{SO}(n,n+1)) = \bigsqcup_{(a,b,c) \in H^1(X,\mathbb{Z}_2) \times H^2(X,\mathbb{Z}_2) \times H^2(X,\mathbb{Z}_2)} \mathcal{M}^{a,b,c}(\text{SO}(n,n+1)).$$

Moreover, an $\text{SO}(n,n+1)$ Higgs bundle in $\mathcal{M}^{a,b,c}(\text{SO}(n,n+1))$ reduces to an $\text{SO}(n,n+1)$ Higgs bundle if and only if $a = 0.$
3.2. $\text{SO}(1, 2) = \text{PGL}(2, \mathbb{R})$ Higgs bundles. For $\text{SO}(1, 2)$, we can explicitly describe the Higgs moduli space. Moreover, in this case, the connected component description is deduced from topological invariants of orthogonal bundles. Although these results are not new, we include the arguments here since the methods will be generalized in subsequent sections. One important difference of the $\text{SO}(n, n + 1)$ generalizations is that they are not distinguished by a known topological invariant for $n \geq 3$.

Using Definition 3.1, an $\text{SO}(1, 2)$ Higgs bundle $(V, W, \eta)$ is given by $(\Lambda^2 W, W, \eta)$ where $W$ is a rank two holomorphic vector bundle with an orthogonal structure $Q_W$. The $\text{SL}(3, \mathbb{C})$ Higgs bundle associated to $(\Lambda^2 W, W, \eta)$ is represented by

$$\Lambda^2 W \xrightarrow{\eta} W.$$  

As above, rank 2 orthogonal bundles on $X$ have first and second Stiefel-Whitney classes $(sw_1, sw_2) \in H^1(X, \mathbb{Z}_2) \oplus H^2(X, \mathbb{Z}_2)$. If $\mathcal{M}^{\text{sw}_1}_W(\text{SO}(1, 2))$ is the moduli space of $\text{SO}(1, 2)$ Higgs bundles consisting of triple $(\Lambda^2 W, W, \eta)$ where the first and second Stiefel-Whitney classes $W$ are $(sw_1, sw_2)$, then

$$(3.6) \quad \mathcal{M}^{\text{so}(1, 2)} = \coprod_{\substack{(sw_1, sw_2) \in H^1(X, \mathbb{Z}_2) \oplus H^2(X, \mathbb{Z}_2)}} \mathcal{M}^{\text{sw}_1}_W(\text{SO}(1, 2)).$$

If the first Stiefel-Whitney class of $W$ vanishes, then the structure group of $W$ reduces to $\text{SO}(2, \mathbb{C})$. Since $\text{SO}(2, \mathbb{C}) \cong \mathbb{C}^*$, a holomorphic orthogonal bundle $(W, Q_W)$ is isomorphic to

$$(W, Q_W) = \left( M \oplus M^{-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right),$$

where $M \in \text{Pic}^d(X)$ is a degree $d$ holomorphic line bundle. In this case, $\Lambda^2 W \cong \mathcal{O}$, the second Stiefel-Whitney class is given by the degree of $M$ mod 2, and the Higgs field $\eta$ decomposes as $\eta = (\mu, \nu) \in H^0(M^{-1}K) \oplus H^0(MK)$. The associated $\text{SL}(3, \mathbb{C})$ Higgs bundle given by

$$\mu \quad \nu \quad \mu^{-1} \quad \nu^{-1} \quad \mathcal{O} \quad M$$

$$(3.7) \quad M \xrightarrow{\mu} \mathcal{O} \xrightarrow{\nu} M^{-1} \xrightarrow{\mu^{-1}} \nu^{-1} \xrightarrow{\mu} M.$$

If $\text{deg}(M) > 0$, then the $\text{SO}(1, 2)$ Higgs bundle (3.7) is polystable if and only if $\mu \neq 0 \in H^0(M^{-1}K)$. Thus $\text{deg}(M) \leq 2g - 2$. Note that the $\text{S}(\mathcal{O}(1, \mathbb{C}) \times \mathcal{O}(2, \mathbb{C}))$ gauge transformation

$$\left( \begin{array}{ccc} -1 & 0 \\ 0 & -1 \end{array} \right): M \oplus O \oplus M^{-1} \to M^{-1} \oplus O \oplus M$$

gives an isomorphism between the data $(M, \mu, \nu)$ and $(M^{-1}, \nu, \mu)$. Thus we may assume $\text{deg}(M) \geq 0$.

Let $\mathcal{M}_d(\text{SO}(1, 2))$ denote the moduli space of polystable $\text{SO}(1, 2)$ Higgs bundles of the form (3.5) with vanishing first Stiefel-Whitney class and $\text{deg}(M) = d$. The

$\text{Note that this switching isomorphism is in the } \text{S}(\mathcal{O}(1, \mathbb{C}) \times \mathcal{O}(2, \mathbb{C}))-\text{gauge group but not the } \text{SO}(1, \mathbb{C}) \times \text{SO}(2, \mathbb{C}))-\text{gauge group. In fact, the moduli space } \mathcal{M}(\text{SO}(1, 2)) \text{ is a double cover of } \mathcal{M}_{sw_1}(\text{SO}(1, 2)). \text{ The fiber of the map } \mathcal{M}(\text{SO}(1, 2)) \to \mathcal{M}_d(\text{SO}(1, 2)) \text{ is connected when } d = 0 \text{ and consists of two isomorphic components if } d \neq 0.$
moduli space $M_{sw_1=0}(SO(1,2))$ decomposes as

$$M_{sw_1=0}(SO(1,2)) = \bigsqcup_{0 \leq d \leq 2g-2} M_d(SO(1,2)).$$

Hitchin proved the following theorem for $PSL(2,\mathbb{R}) = SO_0(1,2)$.

**Theorem 3.6.** ([23, Theorem 10.8]) For each integer $d \in (0, 2g-2)$, the moduli space $M_d(SO(1,2))$ is smooth and diffeomorphic to a rank $(d+g-1)$-vector bundle $\mathcal{F}_d$ over the $(2g-2-d)$-symmetric product $\text{Sym}^{2g-2-d}(X)$.

**Proof.** Let $\tilde{\mathcal{F}}_d = \{(M, \mu, \nu) \mid M \in \text{Pic}^d(X), \mu \in H^0(M^{-1}K), \nu \in H^0(MK)\}$. By the above discussion, there is a surjective map $\tilde{\mathcal{F}}_d \to M_d(SO(1,2))$ defined by sending $(M, \mu, \nu)$ to the isomorphism class of the Higgs bundle (3.5). It is straightforward to check that the $SO(1,2)$ Higgs bundles associated to two points $(M, \mu, \nu)$ and $(M', \mu', \nu')$ lie in the same gauge orbit if and only if $M' = M$, $\mu' = \lambda \mu$ and $\nu' = \lambda^{-1} \nu$ for $\lambda \in \mathbb{C}^*$.

This gives a diffeomorphism between the quotient space $\mathcal{F}_d = \tilde{\mathcal{F}}_d/\mathbb{C}^*$ and the moduli space $M_d(SO(1,2))$. The map $\pi_d : \mathcal{F}_d \to \text{Sym}^{2g-2-d}(X)$ defined by taking the projective class of $\mu$ is surjective. For a divisor $D \in \text{Sym}^{2g-2-d}(X)$, the fiber $\pi_d^{-1}(D)$ is (non-canonically) identified with $H^0(\mathcal{O}(-D)K^2) \cong \mathbb{C}^{d+g-1}$ where $\mathcal{O}(-D)$ is the inverse of the line bundle associated to $D$. □

**Remark 3.7.** Note that when the integer invariant $d = 2g-2$, the connected component $M_d(SO(1,2))$ is diffeomorphic to the vector space $H^0(K^2)$ of holomorphic differentials on $X$. These are the Higgs which correspond to the Fuchsian representations from Example 2.13. In particular, we recover the classical result that the Teichmüller space of $S$ is diffeomorphic to a vector space of complex dimension $3g-3$. Moreover, the Fuchsian representation which corresponds to zero in $H^0(K^2)$ uniformizes the Riemann surface $X$.

**Theorem 3.8.** The space $M_0(SO(1,2))$ deformation retracts onto $\text{Pic}^0(X)/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts by inversion. In particular, $M_0(SO(1,2))$ is homotopy equivalent to the quotient of a 2g-dimensional torus by inversion.

**Proof.** Let $(M, \mu, \nu)$ be an $SO_0(1,2)$ Higgs bundle with $\text{deg}(M) = 0$. The associated $\text{SL}(3,\mathbb{C})$ Higgs bundle is given by (3.5). Since $\text{deg}(M) = 0$, the bundle $M \oplus M^{-1} \oplus \mathcal{O}$ is polystable as a holomorphic vector bundle. Thus, the family $(M, t\mu, t\nu)$ is a family of polystable $SO(1,2)$ Higgs bundles which converge to $(M, 0, 0)$. Finally, the $S(O(2) \times O(1))$-gauge transformation (3.8) defines an isomorphism between the $SO(1,2)$ Higgs bundles associated to $(M, 0, 0)$ and $(M^{-1}, 0, 0)$. □

So far we have assumed that the first Stiefel-Whitney class of the $O(2,\mathbb{C})$ bundle $W$ is zero. Equivalently, we have only considered $SO(1,2)$ Higgs bundles which reduce to $SO_0(1,2)$ Higgs bundles. We now recall Mumford’s description of holomorphic $O(2,\mathbb{C})$ bundles [32].

**Proposition 3.9.** Let $sw_1 \in H^1(X,\mathbb{Z}_2) \setminus \{0\}$ with corresponding unramified double cover $\pi : X_{sw_1} \to X$, and denote the covering involution by $\iota : X_{sw_1} \to X_{sw_1}$. Consider the following space:

$$(3.9) \quad \text{Prym}(X_{sw_1}, X) = \{ M \in \text{Pic}^0(X_{sw_1}) \mid \iota^* M = M^{-1}\}.$$

There is a bijection between $\text{Prym}(X_{sw_1}, X)$ and holomorphic $O(2,\mathbb{C})$ bundles on $X$ with first Stiefel-Whitney class $sw_1$ given by

$$M \longmapsto (W, Q_W) = (\pi_* M, \pi_* \iota^*) \quad .$$
Theorem 3.11. For $M$ component $S$ isomorphism between $(\mathcal{X}, \pi^*W)$ equivalent to the quotient of a $\pi^*Q_W$.

Given $M \in \text{Pic}^0(X_{sw_1})$ with $\pi^*M = M^{-1}$ we get an orthogonal bundle $(W, Q_W) = (\pi^*S, \pi^*\eta^*)$. Since $X_{sw_1} \to X$ is unramified, $\pi^*\pi_*(M) = M \oplus \pi^*M$, and the above construction gives a bijection.

□

Remark 3.10. The space $\text{Prym}(X_{sw_1}, X)$ has two connected components. For $M \in \text{Prym}(X_{sw_1}, X)$, the second Stiefel-Whitney class of the orthogonal bundle $\pi_*M$ distinguishes the connected component which contains $M$ [32]. We will write

\[(3.10) \quad \text{Prym}(X_{sw_1}, X) = \bigsqcup_{\text{sw}_2 \in H^2(X, \mathbb{Z}_2)} \text{Prym}^\text{sw}_2(X_{sw_1}, X).
\]

The connected component of the identity, $\text{Prym}^0(X_{sw_1}, X)$, is an $g - 1$ dimensional abelian variety called the Prym variety of the covering $X_{sw_1} \to X$. Moreover, $\text{Prym}^1(X_{sw_1}, X)$ is a $\text{Prym}^0(X_{sw_1}, X)$ torsor.

As in the proof of Theorem 3.8, the bundle $\Lambda^2W \oplus W$ is a polystable vector bundle for an $SO(1, 2)$ Higgs bundle $(W, \eta)$ which defines a point of $\mathcal{M}_{sw_2}(SO(1, 2))$. Thus, the family of polystable Higgs bundles $(W, \tau \eta)$ converges to $(W, 0)$. Furthermore, the $SO(1, 1) \times O(2, \mathbb{C})$-gauge transformation $(g_{\Lambda^2W}, g_W) = (\det(Q_W), Q_W)$ defines an isomorphism between $(W, \eta)$ and $(W^*, \eta^*)$. Thus, we have proven the following:

Theorem 3.11. For $(sw_1, sw_2) \in (H^1(X, \mathbb{Z}_2) \setminus \{0\}) \times H^2(X, \mathbb{Z}_2)$, the connected component $\mathcal{M}_{sw_2}^\text{sw}(SO(1, 2))$ from (3.6) deformation retracts onto the moduli space $\mathcal{M}_{sw_2}^\text{sw}(SO(1) \times O(2))$. Since $\mathcal{M}_{sw_2}^\text{sw}(SO(1) \times O(2))$ is given by the quotient of the torus $\text{Prym}^\text{sw}_2(X_{sw_1}, X)$ by inversion, the space $\mathcal{M}_{sw_2}^\text{sw}(SO(1, 2))$ is homotopy equivalent to the quotient of a $(2g - 2)$-dimensional torus by inversion.

4. Parameterizing the smooth components $\mathcal{M}_d(SO(n, n + 1))$

In this section we will prove Theorems 4.11. We start by recalling Hitchin’s parameterization of the $SO_0(n, n + 1)$-Hitchin component.

Recall from Example 2.13 that the set of Fuchsian representations $\text{Fuch}(\Gamma) \subset \mathcal{X}(SO(1, 2))$ defines a particularly interesting class of representations. Recall that the second symmetric product of the standard representation of $GL(2, \mathbb{R})$ on $\mathbb{R}^2$ is the standard representation of $SO(1, 2)$ on $\mathbb{R}^3$. The $2n^{th}$-symmetric product of the standard representation of $GL(2, \mathbb{R})$ defines an irreducible representation $SO(1, 2) \to SL(2n + 1, \mathbb{R})$ which preserves a signature $(n, n + 1)$ quadratic form on $\mathbb{R}^{2n+1}$. Thus we have an irreducible representation

\[i : SO(1, 2) \to SO(n, n + 1).\]

This defines a map $\iota : \mathcal{X}(SO(1, 2)) \to \mathcal{X}(SO(n, n + 1))$, where $\iota(\rho) = i \circ \rho$.

Definition 4.1. The $SO(n, n + 1)$-Hitchin component $\text{Hit}(SO(n, n + 1))$ is the connected component of $\mathcal{X}(SO(n, n + 1))$ that contains $\iota(\text{Fuch}(\Gamma))$.

Remark 4.2. The map $i : SO(1, 2) \to SO(n, n + 1)$ is an example of a principal embedding of $PSL(2, \mathbb{R}) \cong SO_0(1, 2)$ into a split real Lie group $G$ of adjoint type. The Hitchin component for a split group $G$ is defined as the deformation space of the image of $i(\text{Fuch}(\Gamma))$ in $\mathcal{X}(\Gamma, G)$. See [28] and [24] for more details.
Theorem 4.3. [24, Theorem 7.5] The Hitchin component \( \text{Hit}(\SO(n, n + 1)) \) is diffeomorphic to the vector spaces of holomorphic differentials \( \bigoplus_{j=1}^{n} H^0(K^{2j}) \).

For \( \text{Hit}(\SO(n, n + 1)) \), the map \( \bigoplus_{j=1}^{n} H^0(K^{2j}) \to \mathcal{M}(\SO(n, n + 1)) \) can be defined by sending a tuple of differentials \((q_2, q_4, \ldots, q_{2n})\) to the Higgs bundle \((V, W, \eta)\) where

\[
V = K^{n-1} \oplus K^{n-3} \oplus \cdots \oplus K^{3-n} \oplus K^{1-n},
\]

\[
W = K^n \oplus K^{n-2} \oplus \cdots \oplus K^{2-n} \oplus K^{-n},
\]

\[
\eta = \begin{pmatrix} q_2 & q_4 & q_6 & \cdots & q_{2n-2} & q_{2n} \\ 1 & q_2 & q_4 & \cdots & q_{2n-4} & q_{2n-2} \\ 0 & 1 & q_2 & q_4 & \cdots & q_{2n-4} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 1 & q_2 & \cdots & q_{2n-6} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} : V \to W \otimes K
\]

The orthogonal structures on \( V \) and \( W \) are the standard ones:

\[
Q_V = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} : V \to V^* \quad \text{and} \quad Q_W = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} : W \to W^*.
\]

Remark 4.4. Since the Hitchin component is smooth, the automorphism group of \((V, W, \eta)\) of the form (4.1) is trivial. Also, since \( \Lambda^n(V) = \mathcal{O} \) and \( \Lambda^{n+1}W = \mathcal{O} \), all Higgs bundles in \( \text{Hit}(\SO(n, n + 1)) \) reduce to \( \SO_0(n, n + 1) \) Higgs bundles.

Recall from Remark 3.7 that the \( \SO(2, 1) \) Higgs bundles which give rise to the Fuchsian representations which uniformizes the Riemann surface \( X \) has \( \SL(3, \mathbb{C}) \) Higgs bundle given by

\[
(E, \Phi) = \begin{pmatrix} K \oplus \mathcal{O} \oplus K^{-1}, & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\]

Moreover, the \( \SL(2n + 1, \mathbb{C}) \) Higgs bundle \( (V \oplus W, \begin{pmatrix} 0 & \eta^* \\ \eta & 0 \end{pmatrix}) \) associated to the locus where the differential \( q_2, \ldots, q_{2n} \) are all zero is the \( n^{th} \) symmetric product of the Higgs bundle \((E, \Phi)\) from (4.2). Thus, Theorem 4.3 really does parameterizes the Hitchin component from Definition 4.1.

4.1. The components \( \mathcal{M}_d(\SO(n, n + 1)) \). We will now show that the connected components \( \mathcal{M}_d(\SO(1, 2)) \) from Theorem 3.6 generalize to \( \mathcal{M}(\SO(n, n + 1)) \). We start with some preliminary lemmas.

Lemma 4.5. For each integer \( d \in (0, n(2q - 2)] \), define the space \( \tilde{\mathcal{F}}_d \) by

\[
\tilde{\mathcal{F}}_d = \{(M, \mu, \nu) \mid M \in \text{Pic}^d(X), \ \mu \in H^0(M K^n) \setminus \{0\}, \ \text{and} \ \nu \in H^0(M K^n)\}.
\]

There is a well defined smooth map

\[
\tilde{\Psi}_d : \tilde{\mathcal{F}}_d \times \bigoplus_{j=1}^{n-1} H^0(K^{2j}) \to \{\text{stable } \SO(n, n + 1)\text{-Higgs bundles }\}
\]

defined by \( \tilde{\Psi}_d(M, \mu, \nu, q_2, \cdots q_{2p-2}) = (V, W, \eta) \) where

\[
(V, Q_V) = (K^{n-1} \oplus K^{n-3} \oplus \cdots \oplus K^{3-n} \oplus K^{1-n}, \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix})
\]
Let \( \Psi_d : \mathcal{F}_d \times \bigoplus_{j=1}^{n-1} H^0(K^{2j}) \to \{ \text{polystable } SO(n, n+1) \text{ Higgs bundles} \} \) be given by (4.4). If \( (M, \mu, \nu, q_2, \cdots, q_{2n-2}) \) and \( (M', \mu', \nu', q_2', \cdots, q_{2n-2}') \) are two points in \( \mathcal{F}_d \times \bigoplus_{j=1}^{n-1} H^0(K^{2j}) \), then the Higgs bundles \( \Psi_d(M, \mu, \nu, q_2, \cdots, q_{2n-2}) \) and \( \Psi_d(M', \mu', \nu', q_2', \cdots, q_{2n-2}') \) lie in the same \( \mathcal{S}(O(n, \mathbb{C}) \times O(n+1, \mathbb{C})) \) gauge orbit if and only if for \( \lambda \in \mathbb{C}^* \) and all \( j \),

\[
M = M', \quad \mu = \lambda \mu', \quad \nu = \lambda^{-1} \nu' \quad \text{and} \quad q_{2j} = q_{2j},
\]

\( ^3 \) In [37], Simpson works with parabolic bundles, however the proof for the non parabolic case is identical.
Proof. For \((M, \mu, \nu, q_2, \ldots, q_{2n-2}) \in \tilde{F}_d \otimes \bigoplus_{j=1}^{n-1} H^0(K^{2j})\), let \((V, W, \eta)\) be the \(SO(n, n+1)\) Higgs bundle defined by \(\widetilde{\Phi}_d(M, \mu, \nu, q_2, \ldots, q_{2n-2})\). It is given by (4.5). Write
\[
(4.9) \quad W = M \oplus W_0 \oplus M^{-1} \quad \text{where} \quad W_0 = K^{n-2} \oplus \cdots \oplus K^{2-n}.
\]
Recall that the action of an element \((g_V, g_W) \in G_{lc}(V, W, \eta)\) in the gauge group is given by
\[
(g_V, g_W) \cdot (\bar{\partial}_V, \bar{\partial}_W, \eta) = (g_V \bar{\partial}_V g_V^{-1}, g_W \bar{\partial}_W g_W^{-1}, g_W \eta g_W^{-1})
\]
where \(g_V\) and \(g_W\) are smooth orthogonal gauge transformations with \(\det(g_V) \cdot \det(g_W) = 1\). With respect to the decomposition (4.9), a gauge transformation \(g_W\) decomposes as
\[
(4.10) \quad g_W = \begin{pmatrix} b_0 & A & c_0 \\ B & g_{W_0} & C \\ b_n & D & c_n \end{pmatrix} \quad \text{and} \quad \eta = \begin{pmatrix} \alpha \\ \eta_0 \\ \beta \end{pmatrix},
\]
where \(g_{W_0}\) is an orthogonal gauge transformation of \(W_0\) and
\[
A = (a_1 \cdots a_{n-1}) , \quad D = (d_1 \cdots d_{n-1}) , \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} , \quad C = \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \end{pmatrix},
\]
\[
\alpha = (0 \cdots 0 \nu) , \quad \beta = (0 \cdots 0 \mu) \quad \text{and} \quad \eta_0 = \begin{pmatrix} 1 & q_2 & \cdots & q_{2n-2} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & q_2 \end{pmatrix}.
\]
For the Higgs field, \(g_W \eta g_W^{-1}\) is given by
\[
(4.11) \quad g_W \eta g_W^{-1} = \begin{pmatrix} b_0 \alpha + A \eta_0 + c_0 \beta \\ B \alpha + g_{W_0} \eta_0 + D \beta \\ c_0 \alpha + C \eta_0 + c_n \beta \end{pmatrix} g_V^{-1} = \begin{pmatrix} \alpha' \\ \eta_0' \\ \beta' \end{pmatrix}.
\]
For \(\alpha' = (0 \cdots 0 \nu')\), \(\beta' = (0 \cdots 0 \mu')\), \(\eta_0' = \begin{pmatrix} 1 & q_2' & \cdots & q_{2n-2}' \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & q_2' \end{pmatrix}\), the goal is to show that, if
\[
(g_V, g_W) \cdot \begin{pmatrix} \alpha \\ \eta_0 \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha' \\ \eta_0' \\ \beta' \end{pmatrix},
\]
then \(g_V = I_d V\) and \(g_W = \begin{pmatrix} \lambda & 0 \\ 0 & I_d W_0 \end{pmatrix} \lambda^{-1}\).

We will do this by first showing \(A, B, C, D, c_0, b_n\) all vanish. Since \(g_V\) and \(g_W\) are holomorphic and negative degree line bundles do not have nonzero holomorphic sections, both \(g_V\) and \(g_W\) are upper triangular and \(b_n = 0\). The term \(\alpha' = (0 \cdots 0 \nu')\) is given by \((b_0 \alpha + A \eta_0 + c_0 \beta) \cdot g_V^{-1}\) from (4.11). A computation shows \((b_0 \alpha + A \eta_0 + c_0 \beta)\) is given by
\[
(4.12) \quad \begin{pmatrix} a_1 & a_1 q_2 + a_2 & a_1 q_4 + a_2 q_2 + a_3 & \cdots & b_0 \nu + \sum_{j=1}^{n-1} a_j q_{2n-2j} + c_0 \mu \end{pmatrix}.
\]
Since \(g_V^{-1}\) is invertible and upper triangular and \(\alpha' = (0 \cdots 0 \nu')\), we conclude that \(a_j = 0\) for \(1 \leq j \leq n-1\). Hence the matrix \(A\) in (4.10) vanishes. By a similar computation, the matrix \(D\) from (4.10) also vanishes.
Recall that the gauge transformation $g_W$ is orthogonal with respect to the orthogonal structure $Q_W$, i.e., $g_W^T Q_W g_W = Q_W$. If $Q_{W_0}$ is the restriction of the orthogonal structure $Q_W$ to the subbundle $W_0$, then, using the decomposition (4.10),

$$g_W^T Q_W g_W = \begin{pmatrix} b_0 & B^T & 0 \\ B & c_0 & C \\ C^T & c_n & 1 \end{pmatrix} \begin{pmatrix} Q_{W_0} \\ B \\ 0 \end{pmatrix} \begin{pmatrix} b_0 & 0 & c_0 \\ B & g_{W_0} & C \\ 0 & 0 & c_n \end{pmatrix} = \begin{pmatrix} Q_{W_0} \\ B \\ 0 \end{pmatrix} .$$

Thus,

$$g_W^T Q_W g_W = \begin{pmatrix} b_0 & B^T & 0 \\ B & c_0 & C \\ C^T & c_n & 1 \end{pmatrix} \begin{pmatrix} Q_{W_0} \\ B \\ 0 \end{pmatrix} \begin{pmatrix} b_0 & 0 & c_0 \\ B & g_{W_0} & C \\ 0 & 0 & c_n \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & Q_{W_0} & 0 \\ 1 & 0 & 0 \end{pmatrix} .$$

The term $C^T Q_{W_0}$ is given by $(c_{n-1}, c_{n-2}, \ldots, c_1)$. Since $C^T Q_{W_0} g_{W_0} = 0$ and $g_{W_0}$ is invertible and upper triangular, we conclude that $C = 0$. Similarly, the term $B$ also vanishes. This forces $0 \neq b_0 = c_n^{-1}$ and thus $c_0 = 0$.

Finally, by Hitchin’s parameterization of $\text{Hit}(\text{SO}(n-1, n))$, we conclude $g_{W_0}$ and $g_{W_0}$ are either both the identity or minus the identity. However, since $\det(g_{W_0}) = \det(g_{W_0}) = \det(g_{W_0})$ we conclude, both $g_{W_0}$ and $g_{W_0}$ are the identity.

Using Corollary 3.4, we have:

**Corollary 4.7.** If $(V, W, \eta)$ is an $\text{SO}(n, n+1)$ Higgs bundle of the form (4.5), then the automorphism group $\text{Aut}(V, W, \eta)$ is trivial. In particular, the isomorphism class of such a $(V, W, \eta)$ defines a smooth point of $\mathcal{M}(\text{SO}(n, n+1))$.

**Remark 4.8.** Note that the only time we used the fact that the degree of $M$ is nonzero was to conclude that $b_n$ and hence $c_0$ both vanish.

For $d \in (0, n(2g-2)]$, let $\tilde{\mathcal{F}}_{-d} = \{(M, \nu, \mu) \mid M \in \text{Pic}^{-d}(X), \nu \in H^0(MK^n) \setminus \{0\}, \text{and} \mu \in H^0(M^{-1}K^n)\}$, and define a map

$$\tilde{\Psi}_{-d} : \tilde{\mathcal{F}}_{-d} \times \bigoplus_{j=1}^{n-1} H^0(K_{2j}) \to \{\text{polystable } \text{SO}(n, n+1)-\text{Higgs bundles}\}$$

by (4.5). We have the following proposition relating the images of $\tilde{\Psi}_d$ and $\tilde{\Psi}_{-d}$.

**Proposition 4.9.** For $d \in (0, n(2g-2)]$, and $(M, \mu, \nu) \in \tilde{\mathcal{F}}_d$, the stable $\text{SO}(n, n+1)$ Higgs bundles $\tilde{\Psi}_d(M, \mu, \nu, q_2, \ldots, q_{2n-2})$ and $\tilde{\Psi}_{-d}(M^{-1}, \nu, \mu, q_2, \ldots, q_{2n-2})$ are in the same $\text{S}(\text{O}(n, \mathbb{C}) \times \text{O}(n+1, \mathbb{C}))$-gauge orbit.

**Proof.** By Lemma 4.6, if $d \in (0, n(2g-2)]$ and $M \in \text{Pic}^d(X)$, then the $\text{SO}(n, n+1)$ Higgs bundles $(V_{\pm ; d}, W_{\pm ; d}, \eta_{\pm ; d})$ which are given by $\tilde{\Psi}_d(M, \mu, \nu, q_2, \ldots, q_{2n-2})$ and $\tilde{\Psi}_{-d}(M^{-1}, \nu, \mu, q_2, \ldots, q_{2n-2})$ have

$$V_{-d} = K^{-1} \oplus K^{-3} \oplus \cdots \oplus K^{3-n} \oplus K^{1-n} \quad \quad W_{-d} = M \oplus W_0 \oplus M^{-1}$$

$$V_{d} = K^{-1} \oplus K^{-3} \oplus \cdots \oplus K^{3-n} \oplus K^{1-n} \quad \quad W_{d} = M^{-1} \oplus W_0 \oplus M$$

where $W_0 = K^{-2} \oplus K^{-4} \oplus \cdots \oplus K^{2-n}$ and Higgs field $\eta_{\pm ; d}$ given by (4.5).

Consider the following orthogonal gauge transformation

$$(4.14) \quad g_W = \begin{pmatrix} -id_{W_0} & -1 \\ -1 & -1 \end{pmatrix} : M \oplus W_0 \oplus M^{-1} \to M \oplus W_0 \oplus M .$$

A simple calculation shows that $(-id_W g_W)$ defines an $O(n, \mathbb{C}) \times O(n+1, \mathbb{C})$-gauge transformation which provides the desired isomorphism.
Remark 4.10. Note that if the gauge transformation (4.14) defines an \( \text{SO}(n, \mathbb{C}) \times \text{SO}(n+1, \mathbb{C}) \) gauge transformation if and only if \( n \) is even.

We are now set up to prove the \( \Psi_d \) maps onto a connected component of \( \mathcal{M}(\text{SO}(n, n+1)) \), and hence onto a connected component of the \( \text{SO}(n, n+1) \) character variety \( \mathcal{X}(\text{SO}(n, n+1)) \).

Theorem 4.11. Let \( \Gamma \) be the fundamental group of a closed surface \( S \) of genus \( g \geq 2 \) and let \( \mathcal{X}(\text{SO}(n, n+1)) \) be the \( \text{SO}(n, n+1) \)-character variety of \( \Gamma \). For each integer \( d \in (0, n(2g-2)] \), there is a smooth connected component \( \mathcal{X}_d(\text{SO}(n, n+1)) \) of \( \mathcal{X}(\text{SO}(n, n+1)) \) which does not contain representations with compact Zariski closure. Furthermore, for each choice of Riemann surface structure \( X \) on \( S \), the space \( \mathcal{X}_d(\text{SO}(n, n+1)) \) is diffeomorphic to the product

\[
\mathcal{X}_d(\text{SO}(n, n+1)) \cong \mathcal{F}_d \times \bigoplus_{j=1}^{n-1} H^0(K^2),
\]

where \( \mathcal{F}_d \) is the total space of a rank \( d + (2n-1)(g-1) \) vector bundle over the symmetric product \( \text{Sym}^{n(2g-2)-d}(X) \) and \( H^0(K^2) \) is the vector space of holomorphic differentials of degree \( 2j \).

Proof. Let \( \tilde{\mathcal{F}}_d \) be as in (4.3). There is a free \( \mathbb{C}^* \)-action on \( \tilde{\mathcal{F}}_d \) given by

\[
\lambda \cdot (M, \mu, \nu) = (M, \lambda \mu, \lambda^{-1} \nu).
\]

Let \( \mathcal{F}_d \) be the quotient, \( \mathcal{F}_d = \tilde{\mathcal{F}}_d / \mathbb{C}^* \). By Lemma 4.5, Lemma 4.6 and Corollary 4.7, there is a smooth map

\[
\Psi_d : \mathcal{F}_d \times \bigoplus_{j=1}^{n-1} H^0(K^2) \longrightarrow \mathcal{M}(\text{SO}(n, n+1))
\]

defined by

\[
\Psi_d([M, \mu, \nu], q_2, \cdots, q_{2n-2}) = [\tilde{\Psi}_d(M, \mu, \nu, q_2, \cdots, q_{2n-2})] \in \mathcal{M}(\text{SO}(n, n+1)).
\]

which is a diffeomorphism onto its image.

Just as in Hitchin’s proof of Theorem 3.6, there is a map from \( \mathcal{F}_d \) to the \((n(2g-2)-d)\)th symmetric product of \( X \) defined by taking the projective class of the nonzero section \( \mu \in H^0(M^{n-1}K^n) \setminus \{0\} \)

\[
\pi_d : \mathcal{F}_d \longrightarrow \text{Sym}^{n(2g-2)-d}(X).
\]

Given a divisor \( D \in \text{Sym}^{n(2g-2)-d}(X) \), the fiber \( \pi_d^{-1}(D) \) is non-canonically identified with \( H^0(\mathcal{O}(-D) \otimes K^n) \cong \mathbb{C}^{d+2(2n-1)(g-1)} \), where \( \mathcal{O}(D) \) is the inverse of the line bundle associated to \( D \). Thus, the space \( \mathcal{F}_d \) is a rank \( d + (2n-1)(g-1) \)-vector bundle over the compact space \( \text{Sym}^{n(2g-2)-d}(X) \).

Let \((E, \Phi)\) be the \( \text{SL}(2n+1, \mathbb{C}) \) Higgs bundle associated to the \( \text{SO}(n+1) \) Higgs bundle \( \tilde{\Psi}_d(M, \mu, \nu, q_2, \cdots, q_{2n-2}) \). Let \( h : \mathcal{M}(\text{SL}(2n+1, \mathbb{C})) \to \bigoplus_{j=2}^{2n+1} H^0(K^j) \) denote the Hitchin fibration defined by the basis of invariant polynomials \((p_1, \cdots, p_n)\) so that

\[
p_j(\Phi) = \begin{cases} q_{2j} & 1 \leq j \leq n-1 \\ \mu \otimes \nu & j = n \end{cases}.
\]

To show the image of \( \Psi_d \) is closed, consider a divergent sequence \( \{x_i\} \) in the image of \( \Psi_d \). Denote, the inverse image of \( \{x_i\} \) by \( \Psi_d^{-1}(x_i) = (y_i, q_2, \cdots, q_{2n-2}) \) for
$y_i \in \mathcal{F}_d$; this sequence diverges in $\mathcal{F}_d \times \bigoplus_{j=1}^{n-1} H^0(K^{2j})$. Thus, either there exists $j$ so that $q_{2j}$ goes to infinity in $H^0(K^{2j})$ or, since $\text{Sym}^{n(2g-2)-d}(X)$ is compact, there is a subsequence, $x_{i_k}$ so that $\pi_d(y_{i_k})$ converges to $y_\infty \in \text{Sym}^{n(2g-2)-d}(X)$ and $y_{i_k}$ goes to infinity in the fiber direction. In either case, (4.15) and the properness of the Hitchin fibration imply the sequence $\{x_i\} = \{\Psi_d(y_i, q_2^{i}, \ldots, q_{2n-2}^{i})\}$ diverges in $\mathcal{M}(\text{SO}(n, n+1))$. Thus, the image of $\Psi_d$ is a closed subset.

By a simple calculation, the dimension of $\mathcal{F}_d \times \bigoplus_{j=1}^{n-1} H^0(K^{2j})$ is the expected dimension of the moduli space $\mathcal{M}(\text{SO}(n, n+1))$. Hence, since the $\mathcal{F}_d \times \bigoplus_{j=1}^{n-1} H^0(K^{2j})$ is a manifold without boundary and the image of $\Psi_d$ is closed, we conclude that the image of $\Psi_d$ is also open.

We have established that for each integer $d \in (0, n(2g-2)]$ the image of $\Psi_d$ defines a smooth connected component $\mathcal{M}_d(\text{SO}(n, n+1))$ of $\mathcal{M}(\text{SO}(n, n+1))$. Recall that the correspondence between the $G$ Higgs bundle moduli space and the $G$ character variety is a diffeomorphism on the smooth locus. Since, the connected components $\mathcal{M}_d(\text{SO}(n, n+1))$ are smooth, we conclude that for each integer $d \in (0, n(2g-2)]$ there is a smooth connected component $\mathcal{X}_d(\text{SO}(n, n+1))$ of the $\text{SO}(n, n+1)$-character variety which is diffeomorphic to $\mathcal{F}_d \times \bigoplus_{j=1}^{n-1} H^0(K^{2j})$. Finally, since the Higgs field is non-vanishing for Higgs bundles in $\mathcal{M}_d(\text{SO}(n, n+1))$, no representation in $\mathcal{X}_d(\text{SO}(n, n+1))$ has compact Zariski closure by Proposition 2.17.

**Remark 4.12.** For the maximal value $d = n(2g-2)$, the space $\mathcal{X}_{n(2g-2)}(\text{SO}(n, n+1))$ is the $\text{SO}(n, n+1)$-Hitchin component $\text{Hit}(\text{SO}(n, n+1))$.

**Corollary 4.13.** The component $\mathcal{X}_d(\text{SO}(n, n+1))$ deformation retracts onto the symmetric product $\text{Sym}^{n(2g-2)-d}(X)$. In particular,

$$H^*(\mathcal{X}_d(\text{SO}(n, n+1)), \mathbb{Z}) \cong H^*(\text{Sym}^{n(2g-2)-d}(X), \mathbb{Z}) .$$

Recall from (3.4) that the moduli space $\mathcal{M}(\text{SO}(n, n+1))$ decomposes into a disjoint union of spaces $\mathcal{M}^{a,b,c}(\text{SO}(n, n+1))$ where the isomorphism class of a Higgs bundle $(V, W, \eta)$ lies in $\mathcal{M}^{a,b,c}(\text{SO}(n, n+1))$ if and only if the first Stiefel-Whitney classes of $V$ and $W$ are given by

$$a = sw_1(V) = sw_1(W) , \quad b = sw_2(V) \quad \text{and} \quad c = sw_2(W) .$$

Thus, we have $\mathcal{M}_d(\text{SO}(n, n+1)) \subseteq \mathcal{M}^{0,0,d \mod 2}(\text{SO}(n, n+1))$. Moreover, if $(V, W, \eta)$ is a polystable $\text{SO}(n, n+1)$ Higgs bundle, then the corresponding representation $\rho \in \mathcal{X}(\text{SO}(n, n+1))$ lifts to the split real form $\text{Spin}(n, n+1) \subseteq \text{Spin}(2n+1, \mathbb{C})$ if and only if the second Stiefel-Whitney classes of $V$ and $W$ are the same.

**Corollary 4.14.** A representation in the component $\mathcal{X}_d(\text{SO}(n, n+1))$ lifts to $\text{Spin}(n, n+1)$ if and only if $d \equiv 0 \mod 2$.

5. **The singular components** $\mathcal{M}_0(\text{SO}(n, n+1))$ and $\mathcal{M}_{sw_1}(\text{SO}(n, n+1))$

We now show the components of $\mathcal{M}(\text{SO}(1, 2))$ from Theorems 3.8 and 3.11 also generalize to $\mathcal{M}(\text{SO}(n, n+1))$. These components are more difficult to describe because they are singular.

Consider the space $\tilde{\mathcal{F}}_0$ defined by

$$(5.1) \quad \tilde{\mathcal{F}}_0 = \{ (M, \mu, \nu) \mid M \in \text{Pic}^0(X), \mu \in H^0(M^{-1}K^n), \nu \in H^0(MK^n) \} .$$
The group $O(2, \mathbb{C})$ is isomorphic to the group of $2 \times 2$ matrices generated by $\left( \begin{array}{cc} \lambda & 0 \\ 0 & \lambda^{-1} \end{array} \right)$ and $\left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right)$ for $\lambda \in \mathbb{C}^\ast$. There is a natural action of $O(2, \mathbb{C})$ on $\mathcal{F}_0$ given by:

$$\left( \begin{array}{cc} \lambda & 0 \\ 0 & \lambda^{-1} \end{array} \right) \cdot (M, \mu, \nu) = (M, \lambda^{-1} \mu, \lambda \nu)$$

and

$$\left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right) \cdot (M, \mu, \nu) = (M^{-1}, \nu, \mu).$$

**Theorem 5.1.** Let $\Gamma$ be the fundamental group of a closed surface $S$ of genus $g \geq 2$ and let $\mathcal{X}(SO(n, n+1))$ be the $SO(n, n+1)$-character variety of $\Gamma$. For each $n \geq 2$, there is a connected component $\mathcal{X}_0(SO(n, n+1))$ of $\mathcal{X}(SO(n, n+1))$ which does not contain representations with compact Zariski closure. Furthermore, for each Riemann surface structure on $S$, the space $\mathcal{X}_0(SO(n, n+1))$ is homeomorphic to

$$\mathcal{X}_0(SO(n, n+1)) \cong \mathcal{F}_0 \times \bigoplus_{j=1}^{n-1} H^0(K_j^2),$$

where $\mathcal{F}_0$ is the GIT quotient $\tilde{\mathcal{F}}_0/\!/O(2, \mathbb{C})$ of the $O(2, \mathbb{C})$-space $\tilde{\mathcal{F}}_0$ from (5.1) and $H^0(K_j^2)$ is the vector space of holomorphic differentials of degree $2j$.

**Corollary 5.2.** Since $\mathcal{F}_0$ deformation retracts onto $Pic^0(X)/\mathbb{Z}_2$, the connected component $\mathcal{X}_0(SO(n, n+1))$ is homotopic to the quotient of $(S^1)^{2g}$ by the $\mathbb{Z}_2$ action given by inversion. In particular, its rational cohomology is given by

$$H^j(\mathcal{X}_0(SO(n, n+1)), \mathbb{Q}) = \begin{cases} H^j((S^1)^{2g}, \mathbb{Q}) & \text{if } j \text{ is even} \\ 0 & \text{otherwise} \end{cases}.$$ 

For $sw_1 \in H^1(S, \mathbb{Z}_2) \setminus \{0\}$, let $X_{sw_1} \to X$ be the associated orientation double cover. Denote the covering involution by $\iota$ and set

$$\text{Prym}(X_{sw_1}, X) = \{ M \in Pic^0(X_{sw_1}) \mid \iota^* M = M^{-1} \}.$$ 

As in (3.10), $\text{Prym}(X_{sw_1}, X)$ has two connected components $\text{Prym}^{sw_2}(X_{sw_1}, X)$ which are labeled by an invariant $sw_2 \in H^2(X, \mathbb{Z}_2)$.

Consider the following space

$$\mathcal{F}^{sw_2}_{sw_1} = \{ (M, \mu) : M \in \text{Prym}^{sw_2}(X_{sw_1}, X) \text{ and } \mu \in H^0(\mathbb{Z}_2 \otimes \mathbb{Z}_2) \}.$$ 

For $n > 1$, $H^0(\mathbb{Z}_2 \otimes \mathbb{Z}_2) = \mathbb{C}^{2n-1}(9g-1)$; thus, $\mathcal{F}^{sw_2}_{sw_1}$ is a vector bundle. The group $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ acts on $\mathcal{F}^{sw_2}_{sw_1}$ by

$$(M, \mu) \to (M, -\mu) \quad \text{and} \quad (M, \mu) \to (\iota^* M, \iota^* \mu).$$

The quotient space $\mathcal{F}^{sw_2}_{sw_1}/\mathbb{Z}_2 \otimes \mathbb{Z}_2$ is an orbifold. Here the orbifold points correspond to pairs $(M, \mu)$ with $M = M^{-1}$ and $\iota^* \mu = \pm \mu$.

**Theorem 5.3.** Let $\Gamma$ be the fundamental group of a closed surface $S$ of genus $g \geq 2$ and let $\mathcal{X}(SO(n, n+1))$ be the $SO(n, n+1)$-character variety of $\Gamma$. For each $n \geq 2$ and each $(sw_1, sw_2) \in (H^1(X, \mathbb{Z}_2) \setminus \{0\}) \times H^2(X, \mathbb{Z}_2)$, there is a connected component $\mathcal{X}^{sw_2}_{sw_1}(SO(n, n+1))$ of $\mathcal{X}(SO(n, n+1))$ which does not contain representations with compact Zariski closure. Furthermore, for each Riemann surface structure $X$ on $S$, the space $\mathcal{X}^{sw_2}_{sw_1}(SO(n, n+1))$ is a smooth orbifold diffeomorphic to

$$\mathcal{F}^{sw_2}_{sw_1}/(\mathbb{Z}_2 \otimes \mathbb{Z}_2) \times \bigoplus_{j=1}^{n-1} H^0(K_j^2),$$

where $\mathcal{F}^{sw_2}_{sw_1} \to \text{Prym}^{sw_2}(X_{sw_1}, X)$ is the rank $(4n - 2)(2g - 2)$ vector bundle from (5.3), and the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ action is given by (5.4).

Recall from Remark 3.10 that $\text{Prym}^{sw_2}(X_{sw_1}, X)$ is topologically a $(2g - 2)$-dimensional torus.
Corollary 5.4. Since \( F_{\text{sw}}^u / \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) deformation retracts onto \( \text{Prym}^{\text{sw}}(X) / \mathbb{Z}_2 \), the connected component \( \mathcal{X}_{\text{sw}}^u(\text{SO}(n, n+1)) \) is homotopic to the quotient of \((S^1)^{2g-2}\) by the \( \mathbb{Z}_2 \) action given by inversion. In particular, its rational cohomology is

\[
H^j(\mathcal{X}_{\text{sw}}^u(\text{SO}(n, n+1)), \mathbb{Q}) = \begin{cases} 
H^j((S^1)^{2g-2}, \mathbb{Q}) & \text{if } j \text{ is even} \\
0 & \text{otherwise}
\end{cases}.
\]

Remark 5.5. For \( n = 2 \), these results were proven in [1]. In the \( n = 2 \) case, the extra invariants arise from topological invariants of the Cayley partner of a maximal \( \text{SO}_0(2, 3) \) Higgs bundle. Since \( \text{SO}(n, n+1) \) is not a group of Hermitian type for \( n \geq 3 \), the proofs of the above theorems require a more technical analysis.

Corollary 5.6. For \( n \geq 3 \), the character variety \( \mathcal{X}(\text{SO}(n, n+1)) \) of a genus \( g \geq 2 \) closed surface has at least \( 2^{2g+2} + 2^{2g+1} - 1 + n(2g-2) \) connected components.

Proof. The topological invariants of a flat \( \text{SO}(n, n+1) \) bundle are a first Stiefel-Whitney class and two second Stiefel-Whitney classes. This gives \( 2^{2g+2} \) topological invariants. For each value of these invariants, there is a connected component of the character variety \( \mathcal{X}(\text{SO}(n, n+1)) \) which contains representations \( \rho : \Gamma \to \text{SO}(n, n+1) \) with compact Zariski closure [14]. The \( n(2g-2) \) components from Theorem 4.11, the connected component from Theorem 5.1 and the \( 2(2^{2g-1} - 1) \) components from Theorem 5.3 do not contain any representations with compact Zariski closures. This gives \( 2^{2g+2} + 2^{2g+1} - 1 + n(2g-2) \) connected components. \( \square \)

Remark 5.7. In [2], the connected components of \( \mathcal{X}(\text{SO}(n, m)) \) are computed. In particular, we show there are exactly \( 2^{2g+2} + 2^{2g+1} - 1 + n(2g-2) \) connected components of the character variety \( \mathcal{X}(\text{SO}(n, n+1)) \).

5.1. Proof of Theorem 5.1. To prove Theorem 5.1 we start with a sequence of lemmas.

Lemma 5.8. Consider the space \( \tilde{F}_0 \) from (5.1) with the \( \text{O}(2, \mathbb{C}) \) action given by (5.2). Define the subspace \( \tilde{F}_{0}^{ps} \subset \tilde{F}_0 \) by

\[
\tilde{F}_{0}^{ps} = \{(M, \mu, \nu) \in \tilde{F}_0 \mid \mu = 0 \text{ if and only if } \nu = 0 \}.
\]

For each \( x \in \tilde{F}_0 \), the orbit \( \text{O}(2, \mathbb{C}) \cdot x \) is closed if and only if \( x \in \tilde{F}_{0}^{ps} \). In particular,

\[
\mathcal{F}_0 = \tilde{F}_0 / \text{O}(2, \mathbb{C}) = \tilde{F}_{0}^{ps} / \text{O}(2, \mathbb{C})
\]

Proof. If \( x \in \tilde{F}_0 \setminus \tilde{F}_{0}^{ps} \), then either \( \mu = 0 \) or \( \nu = 0 \) but not both. Suppose \( \mu \neq 0 \) and \( \nu = 0 \). The orbit through \( x \) is not closed since, for \( \lambda \in \mathbb{R}^2 \), we have

\[
\lim_{\lambda \to 0} \left( \begin{array}{c} \lambda \\
0 \lambda^{-1} \end{array} \right) \cdot (M, \mu, 0) = (M, 0, 0).
\]

It is clear that the orbit through a point \( (M, 0, 0) \in \tilde{F}_{0}^{ps} \) is closed. Similarly, if \( (M, \mu, \nu) \in \tilde{F}_{0}^{ps} \) with \( \mu \neq 0 \) and \( \nu \neq 0 \), then the \( \text{O}(2, \mathbb{C}) \)-orbit through \( (M, \mu, \nu) \) is closed. \( \square \)

It is straightforward to compute the \( \text{O}(2, \mathbb{C}) \)-stabilizers of points of \( \tilde{F}_{0}^{ps} \).

Lemma 5.9. Let \( \tilde{F}_{0}^{ps} \) be the space from (5.5), the \( \text{O}(2, \mathbb{C}) \)-stabilizer of a point \( (M, \mu, \nu) \in \tilde{F}_{0}^{ps} \) is

\[
\text{Stab}_{\text{O}(2, \mathbb{C})}(M, \mu, \nu) = \begin{cases} 
\text{O}(2, \mathbb{C}) & \text{if } M = M^{-1} \text{ and } \mu = \nu = 0 \\
\text{SO}(2, \mathbb{C}) & \text{if } M \neq M^{-1} \text{ and } \mu = \nu = 0 \\
\mathbb{Z}_2 & \text{if } M = M^{-1} \text{ and } \mu \neq 0 \text{ and } \mu = \lambda \nu \text{ for } \lambda \in \mathbb{C}^* \\
\{1\} & \text{otherwise}
\end{cases}.
\]
For each point of \((M, \mu, \nu, q_2, \cdots, q_{2p-2}) \in \tilde{F}_0 \times \bigoplus_{j=1}^{n-1} H^0(K^{2j})\) define the \(\SO(n, n+1)\)-Higgs bundle \(\tilde{\Psi}_0(M, \mu, \nu, q_2, \cdots, q_{2p-2}) = (V, W, \eta)\) by

\[
V = K^{n-1} \oplus K^{n-3} \oplus \cdots K^{3-n} \oplus K^{1-n},
\]

\[
W = M \oplus K^{n-2} \oplus K^{n-4} \oplus \cdots K^{4-n} \oplus K^{2-n} \oplus M^{-1},
\]

\[
(5.6) \quad \eta = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & \nu \\
1 & q_2 & q_4 & \cdots & q_{2n-4} & q_{2n-2} \\
0 & 1 & q_2 & q_4 & \cdots & q_{2n-4} \\
0 & 0 & 1 & q_2 & \cdots & q_{2n-6} \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
0 & 1 & q_2 & \cdots & \cdots & 0 \\
\end{pmatrix} : V \to W \oplus K.
\]

Lemma 5.10. For two points \((M, \mu, \nu, q_2, \cdots, q_{2n-2})\) and \((M', \mu', \nu', q'_2, \cdots, q'_{2n-2})\) in \(\tilde{F}_0 \times \bigoplus_{j=1}^{n-1} H^0(K^{2j})\) the associated \(\SO(n, n+1)\) Higgs bundles from (5.6) are gauge equivalent if and only if \(q_{2j} = q'_{2j}\) for all \(j\) and \((M, \mu, \nu)\) and \((M', \mu', \nu')\) are in the same \(\SO(2, \mathbb{C})\)-orbit.

Proof. As in the proof of Lemma 4.6, for \((M, \mu, \nu, q_2, \cdots, q_{2n-2}) \in \tilde{F}_0 \times \bigoplus_{j=1}^{n-1} H^0(K^{2j})\), let \((V, W, \eta)\) be the \(\SO(n, n+1)\) Higgs bundle defined by \(\tilde{\Psi}_0(M, \mu, \nu, q_2, \cdots, q_{2n-2})\). It is given by (5.6). Write

\[
(5.7) \quad W = M \oplus W_0 \oplus M^{-1} \quad \text{for} \quad W_0 = K^{n-2} \oplus \cdots \oplus K^{2-n}.
\]

With respect to the decomposition (5.7), a gauge transformation \(g_W\) of \(W\) and the Higgs field decomposes as

\[
(5.8) \quad g_W = \begin{pmatrix}
b_0 & A & c_0 \\
B & g_{W_0} & C \\
b_n & D & c_n
\end{pmatrix}
\quad \text{and} \quad \eta = \begin{pmatrix}
\alpha \\
\gamma_0 \\
\beta
\end{pmatrix}.
\]

Recall that in the proof of Lemma 4.6, we only used the positivity assumption on the degree of the line bundle \(M\) to show that \(c_0\) and \(b_n\) where zero. Thus, using the same arguments as the proof of Lemma 4.6, we have

\[
(5.9) \quad g_W = \begin{pmatrix}
b_0 & 0 & c_0 \\
0 & g_{W_0} & 0 \\
b_n & 0 & c_n
\end{pmatrix}.
\]

Using (4.13), we have \(b_0b_n = 0, c_0c_n = 0, c_0b_n + b_0c_n = 1\). Thus, either \(b_n = 0, c_0 = 0\) and \(b_0 = c_n^{-1}\) or \(b_0 = 0, c_n = 0\) and \(b_n = c_0^{-1}\). Furthermore, by Hitchin’s parameterization of Hit(\(\SO(n, n-1)\)), we have

\[
(g_{W_0}, g_V) = (Id_{W_0}, Id_V) \quad \text{or} \quad (g_{W_0}, g_V) = (-Id_{W_0}, -Id_V)
\]

However, since \(\det(g_W) \cdot \det(g_V) = 1\), we must have

- \(g_V = Id_V\) and \(g_{W_0} = Id_{W_0}\) if \(b_0 = 0, c_0 = 0\) and \(b_0 = c_n^{-1}\),
- \(g_V = Id_V\) and \(g_{W_0} = Id_{W_0}\) if \(n\) is odd and \(b_0 = 0, c_0 = 0\) and \(b_n = c_0^{-1}\),
- \(g_V = -Id_V\) and \(g_{W_0} = -Id_{W_0}\) if \(n\) is even and \(b_0 = 0, c_n = 0\) and \(b_n = c_0^{-1}\).
To conclude, the subgroup $\mathcal{G}_{\Phi_0}$ of the $S(O(n, \mathbb{C}) \times O(n+1, \mathbb{C}))$ gauge group which preserves the image of $\tilde{\Psi}_0$ is given by gauge transformations $(g_V, g_V)$ of the form
\[
\left( \begin{array}{ccc} \lambda & 0 & 0 \\ 0 & Id_{W_0} & 0 \\ 0 & 0 & \lambda^{-1} \end{array} \right), \quad \text{Id}_V, \quad \text{and} \quad \left( \begin{array}{ccc} 0 & 0 & \lambda \\ 0 & (-1)^{n+1} Id_{W_0} & 0 \\ \lambda^{-1} & 0 & 0 \end{array} \right), (-1)^{n+1} \text{Id}_V,
\]
where $\lambda \in \mathbb{C}^*$. In both cases, the group $\mathcal{G}_{\Phi_0}$ is isomorphic to $O(2, \mathbb{C})$.

Define an $O(2, \mathbb{C})$ action on $\tilde{\mathcal{F}}_0 \times \bigoplus_{j=1}^{n-1} H^0(K^{2j})$ by trivially extending the action on $\tilde{\mathcal{F}}_0$ from (5.2). We will now show that the Higgs bundle $\tilde{\Psi}_0(x)$ is polystable if and only if the $x \in \tilde{\mathcal{F}}_0$.

**Lemma 5.11.** For $x \in \tilde{\mathcal{F}}_0 \times \bigoplus_{j=1}^{n-1} H^0(K^{2j})$, the Higgs bundle $\tilde{\Psi}_0(x)$ from (5.6) is polystable if and only if $x \in \tilde{\mathcal{F}}_0^s \times \bigoplus_{j=1}^{n-1} H^0(K^{2j})$. The Higgs bundle $\tilde{\Psi}_0(x)$ is stable if and only if $x \in \tilde{\mathcal{F}}_0^s \times \bigoplus_{j=1}^{n-1} H^0(K^{2j})$, where $\tilde{\mathcal{F}}_0^s = \{(M, \mu, \nu) \in \tilde{\mathcal{F}}_0^s \mid \mu \neq 0 \text{ and } \nu \neq 0\}$.

**Proof.** Let $(V, W, \eta)$ be the Higgs bundle $\tilde{\Psi}_0(M, \nu, q_2, \cdots, q_{2n-2})$, it is given by (5.6). First note that if $\mu = 0$, then $M$ is an $\Phi$-invariant degree zero subbundle of the associated $SL(2n+1, \mathbb{C})$ Higgs bundle $(E, \Phi) = \left(V \oplus W, \left( \begin{array}{cc} 0 & \eta' \\ \eta & 0 \end{array} \right) \right)$. Similarly, if $\nu = 0$, then $M^{-1}$ is an degree zero $\Phi$-invariant subbundle. If $x \in (\tilde{\mathcal{F}}_0 \setminus \tilde{\mathcal{F}}_0^s) \times \bigoplus_{j=1}^{n-1} H^0(K^{2j})$, then the Higgs bundle $(E, \Phi)$ has a degree zero invariant subbundle but is not polystable. Thus, the associated Higgs bundle $\tilde{\Psi}_0(x)$ is not polystable.

For $x \in \tilde{\mathcal{F}}_0^s \setminus (\tilde{\mathcal{F}}_0 \times \bigoplus_{j=1}^{n-1} H^0(K^{2j}))$, the $SL(2n+1, \mathbb{C})$ Higgs bundle $(E, \Phi)$ associated to $\tilde{\Psi}(x)$ is the direct sum of the polystable $SL(2, \mathbb{C})$ Higgs bundle $(M \oplus M^{-1}, \Phi = 0)$ and a Higgs bundle in $\text{Hit}(SO(n-1, \mathbb{C}))$. Thus, $\tilde{\Psi}_0(x)$ is polystable. By Lemma 5.9, the automorphism group of such a Higgs bundle is not finite, hence, $\tilde{\Psi}_0(x)$ is polystable but not stable.

The rest of the proof is similar to Lemmas 4.5 and 4.6. The $SL(2n+1, \mathbb{C})$ Higgs bundle associated to $x = (M, \mu, \nu, 0, \cdots, 0) \in \tilde{\mathcal{F}}_0^s \times \bigoplus_{j=1}^{n-1} H^0(K^{2j})$ can be written schematically as
\[
\frac{K^{n-1}}{M \oplus M^{-1}} \xrightarrow{(\mu, \nu)} \frac{K^{n-2}}{(\nu, \mu)^*} \cdots \xrightarrow{1} \frac{K^{2-n}}{K^{1-n}}.
\]
Such a Higgs bundle is not fixed by the $\mathbb{C}^*$-action, but is fixed by the subgroup of $2\mu$th-roots of unity and the above summands are each eigen-bundles of a holomorphic gauge transformation. For such cyclic Higgs bundles, checking polystability reduces to checking for destabilizing subbundles in each bundle in the chain (see Proposition 6.3 [37]). Since none of the line bundles in the chain are invariant, it suffices to check $(M \oplus M^{-1})$. As $M$ and $M^{-1}$ both have degree zero, $M \oplus M^{-1}$ has no positive degree subbundles.

Note that the only isotropic subbundles of $M \oplus M^{-1}$ are the summands $M$ and $M^{-1}$. Since neither of these summands are invariant, if $N \subset M \oplus M^{-1}$ is a degree zero invariant line subbundle, then $N$ is an orthogonal subbundle. Thus, we can
take its orthogonal complement and split the Higgs bundle as a stable $\mathsf{SL}(2n, \mathbb{C})$ Higgs bundle plus an invariant degree zero line bundle. This implies the Higgs bundle (5.10) above is polystable. Moreover, by Lemmas 5.9 and 5.13 the the stabilizer of $\Psi_0(x)$ is finite. Hence, by Definition 2.6, the $\mathsf{SO}(n, n + 1)$ Higgs bundle $\Psi_0(x)$ is stable.

As in the proof of Lemma 4.5, stability is an open condition. Thus, there is an open neighborhood $U$ of $(\mu, \nu, 0, \cdots, 0)$ such that the Higgs bundles (5.6) are stable for $(\mu, \nu, q_2, \cdots, q_{2n-2}) \in U$. Using the gauge transformations (4.7), the Higgs bundle $(V, W, \lambda \eta)$ is gauge equivalent to $\Psi_0(M, \mu, \lambda^{2n\nu}, \lambda^2 q_2, \lambda^2 q_4, \cdots, \lambda^{2n-2} q_{2n-2})$.

Since stability is preserved by scaling the Higgs field, $\tilde{\Psi}_0(F_0^s \times \bigoplus_{j=1}^{n-1} H^0(K^2) )$ consists of stable $\mathsf{SO}(n, n + 1)$ Higgs bundles.

Putting together the above lemmas, we have the following proposition.

**Proposition 5.12.** The following spaces are homeomorphic

$$
\left( \tilde{\mathcal{F}}_0 / \mathcal{O}(2, \mathbb{C}) \right) \times \bigoplus_{j=1}^{n-1} H^0(K^2) \cong \left( \tilde{\mathcal{F}}_0^{ps} / \mathcal{O}(2, \mathbb{C}) \right) \times \bigoplus_{j=1}^{n-1} H^0(K^2)
$$

$$
\tilde{\Psi}_0 \left( \tilde{\mathcal{F}}_0^{ps} \times \bigoplus_{j=1}^{n-1} H^0(K^2) \right) \big/ \mathcal{O}(2, \mathbb{C}) \cong \tilde{\Psi}_0 \left( \tilde{\mathcal{F}}_0^{ps} \times \bigoplus_{j=1}^{n-1} H^0(K^2) \right) \big/ \mathcal{O}(2, \mathbb{C})
$$

In particular, if $\tilde{\mathcal{F}}_0^{ps} / \mathcal{O}(2, \mathbb{C}) = \mathcal{F}_0^{ps}$, then we have a continuous map

$$
(5.11) \quad \Psi_0 : \mathcal{F}_0^{ps} \times \bigoplus_{j=1}^{n-1} H^0(K^2) \longrightarrow \mathcal{M}(\mathsf{SO}(n, n + 1))
$$

which is a homeomorphism onto its image.

We will show that the image of $\Psi_0$ is open and closed. However, since the image of $\Psi_0$ is singular, this is more complicated than the proof of Theorem 4.11. We start by analyzing the stable locus.

**Lemma 5.13.** Consider the map $\Psi_0$ from (5.11) and the spaces $\tilde{\mathcal{F}}_0^s \subset \tilde{\mathcal{F}}_0^{ps}$ from Lemma 5.11. Denote the quotients of these spaces by $\mathcal{Q}(2, \mathbb{C})$ by $F_0^s$ and $F_0^{ps}$ respectively. The image $\Psi_0(F_0^s \times \bigoplus_{j=1}^{n-1} H^0(K^2))$ is open in $\mathcal{M}(\mathsf{SO}(n, n + 1))$, and the closure of the image is given by

$$
\Psi_0(F_0^s \times \bigoplus_{j=1}^{n-1} H^0(K^2)) = \Psi_0(F_0^{ps} \times \bigoplus_{j=1}^{n-1} H^0(K^2))
$$

**Proof.** As in (4.15), we can choose a basis of invariant polynomials $(p_1, \cdots, p_n)$ so that

$$
p_j(\Psi(M, \mu, \nu, q_2, \cdots, q_{2n-2})) = \begin{cases} q_j & 1 \leq j \leq n - 1 \\ \mu \otimes \nu & j = n \end{cases}
$$

Let $h : \mathcal{M}(\mathsf{SO}(n, n + 1)) \rightarrow \bigoplus_{j=1}^{n} H^0(K^2)$ denote the Hitchin fibration. Note that image $h(\Psi_0(F_0^s \times \bigoplus_{j=1}^{n-1} H^0(K^2)))$ is $H^0(K^{2n}) \setminus \{0\} \times \bigoplus_{j=1}^{n-1} H^0(K^2)$.

Since all of the Higgs bundles in the image are stable, the image is a smooth orbifold. Moreover, the dimension of $\Psi_0(F_0^s \times \bigoplus_{j=1}^{n-1} H^0(K^2))$ is the expected dimension
of the moduli space $\mathcal{M}(\text{SO}(n, n + 1))$. If $U$ is a sufficiently small neighborhood of $x \in F_0^n \times \bigoplus_{j=1}^{n-1} H^0(K^{2j})$, then $h(\Psi_0(U))$ is clearly open in the Hitchin base. Since $\Psi_0(x)$ is either a smooth point or an orbifold point, $h^{-1}(h(\Psi_0(U))) = \Psi_0(U)$. Thus, the image $\Psi_0(F_0^n \times \bigoplus_{j=1}^{n-1} H^0(K^{2j}))$ is open.

As in the proof of closedness for Theorem 4.11, to compute the closure of the image we use the properness of the Hitchin fibration. Let $x_i = (y_i, q_2, \cdots, q_{2n-2})$ be a sequence in $F_0^n \times \bigoplus_{j=1}^{n-1} H^0(K^{2j})$ which diverges and such that the image $\Psi_0(x_i)$ converges to $\mathcal{M}(\text{SO}(n, n + 1))$. By the properness of the Hitchin fibration, the sequence of differentials $q_{2j}$ must converge for all $j$ and $\lim_{i \to \infty} y_i \in F_0^{ps} \setminus F_0^n$. Thus, the closure of $\Psi_0(F_0^n \times \bigoplus_{j=1}^{n-1} H^0(K^{2j}))$ in $\mathcal{M}(\text{SO}(n, n + 1))$ is $\Psi_0(F_0^n \times \bigoplus_{j=1}^{n-1} H^0(K^{2j})).$ \hfill $\Box$

To show that the image of $\Psi_0$ defines a connected component of the moduli space $\mathcal{M}(\text{SO}(n, n + 1))$, it remains to show that $\Psi_0(F_0^{ps} \times \bigoplus_{j=1}^{n-1} H^0(K^{2j}))$ is open (see Lemma 5.17). To do this, the local structure of points in the boundary of the closure from Lemma 5.13 must be examined. We will show that a local neighborhood of such a point in $\mathcal{M}(\text{SO}(n, n + 1))$ is homeomorphic the corresponding open neighborhood in $F_0^{ps} \times \bigoplus_{j=1}^{n-1} H^0(K^{2j})$. This amounts to studying the complex (2.1).

**Remark 5.14.** Let $x = ([M, 0, 0], q_2, \cdots, q_{2n-2})$ be a point in $F_0^{ps} \setminus F_0^n \times \bigoplus_{j=1}^{n-1} H^0(K^{2j})$. Using the $\mathbb{C}^*$ action on the Higgs bundle moduli space, a sufficiently small open neighborhood of $\Psi_0(x)$ can be brought into an open of $\Psi_0([M, 0, 0], 0, \cdots, 0)$. Thus, it suffices to prove that an open neighborhood of $\Psi_0([M, 0, 0], 0, \cdots, 0)$ is homeomorphic to an open neighborhood of $([M, 0, 0], 0, \cdots, 0)$ in $F_0^{ps} \times \bigoplus_{j=1}^{n-1} H^0(K^{2j})$.

For $M \in \text{Pic}^0(X)$, consider the $\text{SO}(n, n + 1)$ Higgs bundle $(V, W, \eta)$ given by

$$(V, Q_V) = \left( K^{n-1} \oplus K^{n-3} \oplus \cdots \oplus K^{3-n} \oplus K^{1-n}, \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right),$$

$$(W, Q_W) = \left( M \oplus K^{n-2} \oplus K^{n-4} \oplus \cdots \oplus K^{2-n} \oplus M^{-1}, \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right)$$

and $\eta$ given by (5.6) with $q_2 = \cdots = q_{2n-2} = \mu = \nu = 0$. Similar to (4.6), the $\text{SL}(2n + 1, \mathbb{C})$ can be represented schematically by

$$K^{n-1} \longrightarrow K^{n-2} \longrightarrow \cdots \longrightarrow 0 \longrightarrow M \longrightarrow M^{-1} \longrightarrow K^{1-n}.$$  

The Lie algebra bundle with fiber $\mathfrak{so}(n, \mathbb{C}) \oplus \mathfrak{so}(n + 1, \mathbb{C})$ consists of $Q_V$ and $Q_W$ skew symmetric endomorphisms of $V$ and $W$ respectively, we will use the notation $\Lambda^2_V V \oplus \Lambda^2_W W \subset \text{End}(V) \oplus \text{End}(W)$.

Write the line bundle decompositions of $V$ and $W$ from (5.6) as follows

$$V = V_{1-n} \oplus V_{3-n} \oplus \cdots \oplus V_{n-3} \oplus V_{n-1},$$
$W = W_{2-n} \oplus W_{4-n} \oplus \cdots \oplus W_0 \oplus \cdots W_{n-4} \oplus W_{n-2}$,

where $V_j = K^{-j}$, $W_j = K^{-j}$ if $j \neq 0$ and $W_0 = M \oplus M^{-1}$ if $n$ is odd and

$W_0 = M \oplus C \oplus M^{-1}$ if $n$ is even. In terms of the above splittings, sections of $\Lambda^2_Q V$

consist of $n \times n$ matrices which are antisymmetric with respect to reflecting about the anti-diagonal. This gives a grading

$$\Lambda^2_Q V \oplus \Lambda^2_Q W \cong \bigoplus_{k=4-2n}^{2n-4} (\Lambda^2_Q V \oplus \Lambda^2_Q W)_k .$$

One computes that $(\Lambda^2_Q V)_k = 0$ for $k$ odd

$$\text{(5.13)} \quad (\Lambda^2_Q V)_{2k} \cong \bigoplus_{j=[\frac{k}{2}]}^{n-k-2} \text{Hom}(V_{1-n+2j}, V_{1-n+2j+2k}) \quad \text{for } 0 \leq 2k \leq 2n - 4 .$$

Similarly (but changing the indexing scheme),

$$\text{(5.14)} \quad (\Lambda^2_Q W)_{2k} \cong \bigoplus_{j=[\frac{k}{2}]}^{n-k-2} \text{Hom}(W_{2-n+2j}, W_{2-n+2j+2k}) \quad \text{for } 0 \leq 2k \leq 2n - 6 .$$

For $k = 0$, we have

$$\text{(5.15)} \quad (\Lambda^2_Q W)_0 \cong \Lambda^2 W_0 \oplus \bigoplus_{j=[\frac{1}{2}]}^{n-2} \text{Hom}(W_{2-n+2j}, W_{2-n+2j}) .$$

For $n$ even, $(\Lambda^2_Q W)_{2k-1} = 0$ for all $k$. But when $n$ is odd, we have

$$\text{(5.16)} \quad (\Lambda^2_Q W)_{1-2k} \cong (\Lambda^2_Q W)_{2k-1} \cong \begin{cases} \text{Hom}(W_{1-2k}, W_0) & \text{for } 0 < 2k - 1 \leq 2 - n \\ 0 & \text{otherwise} \end{cases}$$

Similarly, the bundle $\text{Hom}(V, W) \otimes K$ acquires a grading

$$\text{(5.17)} \quad \text{Hom}(V, W)_{2k+1} \otimes K = \begin{cases} \bigoplus_{j=0}^{n-k-2} \text{Hom}(V_{1-n+2j}, W_{2-n+2k+2j}) \otimes K & \text{for } k \geq 0 \\ \bigoplus_{j=0}^{n+k-1} \text{Hom}(V_{n-1-2j}, W_{n-2j+2k}) \otimes K & \text{for } k < 0 \end{cases} ,$$

and

$$\text{(5.18)} \quad \text{Hom}(V, W)_{2k} \otimes K = \begin{cases} 0 & \text{if } n \text{ is even} \\ \text{Hom}(V_{-2k}, W_0) \otimes K & \text{if } n \text{ is odd} \end{cases} .$$

Moreover, the Higgs field $\eta$ is a holomorphic section of $\text{Hom}(V, W)_{1} \otimes K$. Thus, $ad_\eta$ maps $(\Lambda^2_Q V \oplus \Lambda^2_Q W)_k$ to $\text{Hom}(V, W)_{k+1} \otimes K$, and we have a graded complex

$$C^*_k = C^*(V, W, \eta)_k : \quad (\Lambda^2_Q V \oplus \Lambda^2_Q W)_k \xrightarrow{ad_\eta} \text{Hom}(V, W)_{k+1} \otimes K .$$

$$(gv, gw) \xrightarrow{\eta \circ gv - gw \circ \eta} .$$
In the hypercohomology sequence from (2.1) we have

\[ 
0 \longrightarrow \mathbb{H}^0(C^*_k) \longrightarrow H^0((\Lambda^2_Q V \oplus \Lambda^2_Q W)_k) \xrightarrow{\text{ad}_\eta} H^0(\text{Hom}(V,W)_{k+1} \otimes K) \\
\longrightarrow \mathbb{H}^1(C^*_k) \longrightarrow H^1((\Lambda^2_Q V \oplus \Lambda^2_Q W)_k) \xrightarrow{\text{ad}_\gamma} H^1(\text{Hom}(V,W)_{k+1} \otimes K) \\
\longrightarrow \mathbb{H}^2(C^*_k) \longrightarrow 0 .
\]

Lemma 5.15. If \((V,W,\eta)\) is an \(\text{SO}(n,n+1)\) Higgs bundle given by (5.12), then the second hypercohomology groups \(\mathbb{H}^2(C^*_k)\) in the above sequences vanish for all \(k\).

Proof. Recall that \(V_j = K^{-j}\) for all \(j\), \(W_j = K^{-j}\) for \(j \neq 0\) and \(W_0 = M \oplus M^{-1}\) if \(n\) is odd and \(W_0 = M \oplus M^{-1} \oplus \mathcal{O}\) if \(n\) is even.

If \(k \leq -2\), then, using the decompositions (5.17) and (5.18), the holomorphic bundle \(\text{Hom}(V,W)_{k+1} \otimes K\) is a direct sum of line bundles with degree at least \(4g - 4\). Thus, \(H^1(\text{Hom}(V,W)_{k+1} \otimes K) = 0\), and so \(\mathbb{H}^2(C^*_k(V,W,\eta)) = 0\).

For \(k \geq -1\), we will show that the map

\[ \text{ad}_\eta : H^1((\Lambda^2_Q V \oplus \Lambda^2_Q W)_k) \rightarrow H^1(\text{Hom}(V,W)_{k+1} \otimes K) \]

is surjective. First assume \(2k \geq 2\). In this case, \((\Lambda^2_Q V \oplus \Lambda^2_Q W)_{2k}\) is given by (5.13) and (5.14) and \(\text{Hom}(V,W)_{2k+1} \otimes K\) is given by (5.17). We claim that \(\text{ad}_\eta\) defines an isomorphism between these two sheaves. Indeed, for \(0 \leq j \leq \left\lfloor \frac{n-k}{2} \right\rfloor - 1\) and each \(\alpha \in \text{Hom}(V_{-n+2j},V_{-n+2j+2k})\) we have

\[ V_{-n+2j} \xrightarrow{\alpha} V_{-n+2j+2k} \xrightarrow{1} W_{2-n+2j+2k} \otimes K . \]

Similarly, for \(\left\lfloor \frac{n-k}{2} \right\rfloor \leq j \leq -n-k+2\) and each \(\beta \in \text{Hom}(W_{2-n+2j},W_{2-n+2j+2k})\) we have

\[ V_{-n+2j} \xrightarrow{1} W_{2-n+2j} \otimes K \xrightarrow{\beta} W_{2-n+2j+2k} \otimes K . \]

Thus we have an isomorphism

\[ (\Lambda^2_Q V)_{2k} \oplus (\Lambda^2_Q W)_{2k} \xrightarrow{\text{ad}_\eta} \text{Hom}(V,W)_{2k+1} \otimes K . \]

If \(n\) is even, we are done. If \(n\) is odd, then we have \((\Lambda^2_Q V \oplus \Lambda^2_Q W)_{2k-1} \cong \text{Hom}(W_{-2k+1},W_0)\) and \(\text{Hom}(V,W)_{2k} \otimes K \cong \text{Hom}(V_{-2k},W_0) \otimes K\). The Higgs field again defines an isomorphism since, for any \(\gamma \in \text{Hom}(W_{-2k+1},W_0)\) we have

\[ V_{-2k} \xrightarrow{1} W_{-2k+1} \otimes K \xrightarrow{\gamma} W_0 \otimes K . \]

For \(k = -1\), first note that if \(n\) is even, then \(\text{Hom}(V,W)_0 \otimes K = 0\). If \(n\) is odd, then \((\Lambda^2_Q V \oplus \Lambda^2_Q W)^{-1} \cong \text{Hom}(K^{-1},W_0)\) and \(\text{Hom}(V,W)_0 \otimes K = \text{Hom}(\mathcal{O},W_0) \otimes K\). Again, the Higgs field gives an isomorphism since, for any \(\delta \in \text{Hom}(K^{-1},W_0)\)

\[ \mathcal{O} \xrightarrow{1} K^{-1} \xrightarrow{\delta} W_0 . \]

Finally, for \(k = 0\), \((\Lambda^2_Q V \oplus \Lambda^2_Q W)_0\) is given by (5.13) and (5.15) and \(\text{Hom}(V,W)_1 \otimes K\) is given by (5.17). Recall that

\[ \Lambda^2 W_0 \cong \begin{cases} 
\text{Hom}(M,M) & \text{if } n \text{ is odd} \\
\text{Hom}(\mathcal{O},M) \oplus \text{Hom}(\mathcal{O},M^{-1}) \oplus \text{Hom}(M,M) & \text{if } n \text{ is even}
\end{cases} .
\]

First note that \(\text{Hom}(M,M)\) is in the kernel of \((\Lambda^2_Q V \oplus \Lambda^2_Q W)_0 \rightarrow \text{Hom}(V,W)_1 \otimes K\) since \(M \oplus M^{-1}\) is invariant by the Higgs field. We claim that the map is surjective
and the kernel is exactly the summand \( \text{Hom}(M, M) \). In particular, the induced map on \( H^1 \) is surjective. The Higgs field defines isomorphisms

\[
\begin{aligned}
\text{Hom}(V_{1-n+2j}, V_{1-n+2j}) &\cong \text{Hom}(V_{1-n+2j}, W_{2-n+2j}) \otimes K \quad 2-n + 2j < 0 \\
\text{Hom}(W_{2-n+2j}, W_{2-n+2j}) &\cong \text{Hom}(V_{1-n+2j}, W_{2-n+2j}) \otimes K \quad \left\lfloor \frac{n}{2} \right\rfloor \leq j \leq n - 2
\end{aligned}
\]

since for each \( \epsilon \in \text{Hom}(V_{1-n+2j}, V_{1-n+2j}) \) and \( \epsilon' \in \text{Hom}(W_{n-2-2j}, W_{n-2-2j}) \)

\[
V_{1-n+2j} \xrightarrow{\epsilon} V_{1-n+2j} \xrightarrow{1} W_{2-n+2j} \otimes K \quad \text{and}
\]

\[
V_{1-n+2j} \xrightarrow{1} W_{2-n+2j} \otimes K \xrightarrow{\epsilon'} W_{2-n+2j} \otimes K .
\]

If \( n \) is odd, we are done, but if \( n \) is even we need to consider the map

\[
\text{Hom}(V_{-1}, V_{-1}) \oplus \Lambda^2 W_0 \to \text{Hom}(V_{-1}, W_0) \otimes K .
\]

In this case, \( \Lambda^2 W_0 \cong \text{Hom}(O, M) \oplus \text{Hom}(O, M^{-1}) \oplus \text{Hom}(M, M) \) and

\[
\text{Hom}(V_{-1}, W_0) \otimes K = (\text{Hom}(V_{-1}, O) \oplus \text{Hom}(V_{-1}, M) \oplus \text{Hom}(V_{-1}, M^{-1})) \otimes K .
\]

As above, the Higgs field defines isomorphisms

\[
\text{Hom}(V_{-1}, V_{-1}) \cong \text{Hom}(V_{-1}, O) \otimes K , \quad \text{Hom}(O, M) \cong \text{Hom}(V_{-1}, M) \otimes K
\]

and

\[
\text{Hom}(O, M^{-1}) \cong \text{Hom}(V_{-1}, M^{-1}) \otimes K .
\]

In particular, \( \text{ad}_\eta : H^1((\Lambda^2 QV \oplus \Lambda^2 W)_{m}) \to H^1(\text{Hom}(V, W)_{1} \otimes K) \) is surjective. \( \square \)

**Lemma 5.16.** The hypercohomology group \( \mathbb{H}^1(C_k^*(V, W, \eta)) \) for a Higgs bundle of type (5.12) satisfy the following properties:

- \( \mathbb{H}^1(C_k^*(V, W, \eta)) \equiv 0 \) for \( k > 0 \),
- \( \mathbb{H}^1(C_0^*(V, W, \eta)) \cong H^1(\text{Hom}(M, M)) \),
- for \( k < 0 \) and \( k \neq -n \), \( \mathbb{H}^1(C_k^*) \cong \begin{cases} H^0(K^k) & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases} \),
- \( \mathbb{H}^1(C_{-n}(V, W, \eta)) \cong \begin{cases} H^0(MK^n) \oplus H^0(M^{-1}K^n) \oplus H^0(K^n) & \text{if } n \text{ is even} \\ H^0(MK^n) \oplus H^0(M^{-1}K^n) & \text{if } n \text{ is odd} \end{cases} \).

**Proof.** In the proof of Lemma 5.15 it was shown that the map

\[
\text{ad}_\eta : (\Lambda^2 QV \oplus \Lambda^2 W)_k \to \text{Hom}(V, W)_{k+1} \otimes K
\]

is an isomorphism for \( k > 0 \). Thus, \( \mathbb{H}^1(C_k^*(V, W, \eta)) \equiv 0 \) for \( k > 0 \).

Also in the proof of Lemma 5.15, it was shown that \( (\Lambda^2 QV \oplus \Lambda^2 W)_{0} \cong \text{Hom}(M, M) \oplus \text{Hom}(V, W)_{1} \otimes K \) and that the map \( \text{ad}_\eta \) is given by

\[
\text{ad}_\eta : \text{Hom}(M, M) \oplus \text{Hom}(V, W)_{1} \otimes K \xrightarrow{(0, \text{Id})} \text{Hom}(V, W)_{1} \otimes K.
\]

Thus, \( \mathbb{H}^1(C_k^*(V, W, \eta)) \cong H^1(\text{Hom}(M, M)) \).

Recall from (5.13), (5.14), (5.17) that for \( k < 0 \), \( (\Lambda^2 QV \oplus \Lambda^2 W)_{2k} \) is given by

\[
\bigoplus_{j=0}^{n+k-1} \text{Hom}(V_{n-1-2j}, V_{n-1-2j+2k}) \oplus \bigoplus_{j=0}^{n+k} \text{Hom}(W_{n-2-2j}, W_{n-2-2j+2k})
\]

and

\[
\text{Hom}(V, W)_{2k+1} \otimes K = \bigoplus_{j=0}^{n+k-1} \text{Hom}(V_{n-1-2j}, W_{n-2j+2k}) \otimes K .
\]
First note that $H^1(\Lambda^2_Q(V \oplus W)_{-2k}) = 0$, since it is direct sum of line bundles with degree at least $4g-4$. A simple computation similar to those in the proof of Lemma 5.15 shows that if $2k \neq -n$, then the Higgs field gives isomorphisms

\[ \text{Hom}(V_{n-1-2j}, V_{n-1-2j+2k}) \cong \text{Hom}(V_{n-1-2j}, W_{n-2j+2k}) \otimes K \]

for $0 \leq j \leq \left\lfloor \frac{n+k}{2} \right\rfloor - 1$ and

\[ \text{Hom}(W_{n-2-2j}, W_{n-2-2j+2k}) \cong \text{Hom}(V_{n-1-2j}, W_{n-2j+2k}) \otimes K \]

for $\left\lfloor \frac{n+k}{2} \right\rfloor \leq j \leq n + k - 2$. In particular, this proves that, for $k < 0$ and $k \neq n$,

\[ \mathbb{H}^1(C^*_n) \cong H^0(\text{Hom}(V_{n-2k+1}, W_{-n+2}) \otimes K) \cong H^0(K^{2k}) . \]

When $-n = 2k$, the isomorphism (5.19) holds for $0 \leq j \leq \left\lfloor \frac{n+k}{2} \right\rfloor - 1$. For $j = 0$, we have

\[ \text{Hom}(V_{-1}, W_0) \otimes K \cong \text{Hom}(V_{-1}, K) \oplus \text{Hom}(V_{-1}, M K) \oplus \text{Hom}(V_{-1}, M^{-1} K) \]

and, with respect to this splitting, the map induced by the Higgs field is given by

\[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} : \text{Hom}(V_{-1}, V_{-1}) \rightarrow \text{Hom}(V_{-1}, W_0) \otimes K . \]

Since $\text{Hom}(V_1, W_{-n+2}) \cong K^n$, we conclude

\[ \mathbb{H}^1(C^*_n) \cong H^0(M K^n) \oplus H^0(M^{-1} K^n) \oplus H^0(K^n) . \]

If $n$ is even, we are done. When $n$ is odd, then from (5.16) and (5.18) we have

\[ (\Lambda^2_0 V \oplus \Lambda^2_0 W)_{2k+1} = \begin{cases} \text{Hom}(W_{-2k-1}, M \oplus M^{-1}) & n < 2k+1 < 0, \\ 0 & \text{otherwise} \end{cases} \]

and

\[ \text{Hom}(V, W)_{2k+2} \otimes K = \begin{cases} \text{Hom}(V_{-2k-2}, M \oplus M^{-1}) \otimes K & 0 \leq 2k+2 \leq n, \\ 0 & \text{otherwise} \end{cases} . \]

For $n < 2k+1 < 0$, the Higgs field defines an isomorphism

\[ \text{Hom}(W_{-2k-1}, M \oplus M^{-1}) \cong \text{Hom}(V_{-2k-2}, M \oplus M^{-1}) \otimes K , \]

and so \( \mathbb{H}^1(C^*_{2k+1}) = \begin{cases} H^0(M K^n) \oplus H^0(M^{-1} K^n) & -n = 2k + 1, \\ 0 & \text{otherwise} \end{cases} \). \( \square \)

The following lemma completes the proof of Theorem 5.1.

**Lemma 5.17.** The image of the map $\Psi_0 : F^p_0 \times \bigoplus_{j=1}^{n-1} H^0(K^{2j}) \rightarrow \mathcal{M}(SO(n, n+1))$ from (5.11) is open and closed.

**Proof.** By Lemma 5.13, the image of $\Psi_0$ is closed in $\mathcal{M}(SO(n, n+1))$. Also, by Lemma 5.13, for any $x \in F^p_0 \times \bigoplus_{j=1}^{n-1} H^0(K^{2j})$ there is an open neighborhood of $x$ which is contained in the image of $\Psi_0$.

Now suppose $x \in F^p_0 \setminus F^p_0 \times \bigoplus_{j=1}^{n-1} H^0(K^{2j})$, and recall that $x$ can be written as

\[ x = ([M, 0, 0], q_2, \ldots, q_{2n-2}) \]

for $M \in \text{Pic}^0(X)$ and $q_{2j} \in H^0(K^{2j})$. By Remark 5.14, it suffices to consider points of the form $x = ([M, 0, 0], 0, \ldots, 0)$ in $F^p_0 \times \bigoplus_{j=1}^{n-1} H^0(K^{2j})$. By Lemma 5.15, we
have $H^2(C^\bullet(\Psi_0(x))) = 0$, thus, by (2.2), an open neighborhood of the Higgs bundle $\Psi_0(x)$ is given by

$$H^1(C^\bullet(\Psi_0(x))/\text{Aut}(\Psi_0(x)) = \begin{cases} H^1(C^\bullet(\Psi_0(x)))/\mathcal{O}(2, \mathbb{C}) & \text{if } M^2 = \mathcal{O} \\ H^1(C^\bullet(\Psi_0(x)))/\text{SO}(2, \mathbb{C}) & \text{if } M^2 \neq \mathcal{O} \end{cases}.$$ 

By Lemma 5.16, we have

$$H^1(C^\bullet(\Psi_0(x))) \cong H^0(\mathcal{M}^n) \times H^0(M^{-1}K^n) \times H^1(\mathcal{O}) \times \bigoplus_{j=1}^{n-1} H^0(K^2).$$

Here, $H^1(\mathcal{O}) = H^1(\Lambda^2(M \oplus M^{-1}))$ and $\delta \in H^1(\mathcal{O})$ is given by \((\delta_{-\delta}) \in H^1(\text{End}(M \oplus M^{-1}))\).

If $M^2 \neq \mathcal{O}$, then by Lemmas 5.9 and 5.10, $\text{Aut}(\Psi_0(x))$ is generated by the orthogonal gauge transformations

$$(g_{V*}, g_{W}) = \left( Id_Y, \left( \lambda^{-1} Id_{W_0} \right) \right).$$

Such a gauge transformation acts on \((\mu, \nu, \delta, q_2, \cdots, q_{2n-2}) \in H^1(C^\bullet(V, W, \eta))\) by \((\delta, \mu, \nu, q_2, \cdots, q_{2n-2}) \mapsto (\delta, \lambda^{-1} \mu, \lambda \nu, q_2, \cdots, q_{2n-2}).$$

If $M^2 = \mathcal{O}$, then by Lemmas 5.9 and 5.10, $\text{Aut}(\Psi_0(x))$ is generated by the orthogonal gauge transformations

$$\left( Id_Y, \left( \lambda^{-1} Id_{W_0} \right) \right) \text{ and } \left( (-1)^{n+1} Id_Y, (-1)^{n+1} \left( \lambda^{-1} Id_{W_0} \right) \right).$$

The second gauge transformation acts on \((\delta, \mu, \nu, q_2, \cdots, q_{2n-2}) \in H^1(C^\bullet(V, W, \eta))\) by \((\delta, \mu, \nu, q_2, \cdots, q_{2n-2}) \mapsto (-\delta, \lambda^{-1} \nu, \lambda \mu, q_2, \cdots, q_{2n-2}).$$

Since an open neighborhood of $M \in \text{Pic}^0(X)$ is given by an open neighborhood of zero in $H^1(\mathcal{O})$, an open neighborhood of a lift $\tilde{x} \in \mathcal{F}_0 \times \bigoplus_{j=1}^{n-1} H^0(K^2)$ of $x$ is also given by a neighborhood of zero in

$$H^1(\mathcal{O}) \times H^0(\mathcal{M}^n) \times H^0(M^{-1}K^n) \times \bigoplus_{j=1}^{n-1} H^0(K^2).$$

Since the map $\Psi_0 : \mathcal{F}^{ps}_0 \times \bigoplus_{j=1}^{n-1} H^0(K^2) \longrightarrow \mathcal{M}(\text{SO}(n, n+1))$ is a homeomorphism onto its image, the map $\Psi_0$ in open at $x$.

5.2. **Proof of Theorem 5.3.** As in previous sections, for each nonzero $sw_1 \in H^1(X, \mathbb{Z}_2)$, let $\pi : X_{sw_1} \rightarrow X$ be the corresponding connected orientation double cover. If $\iota$ denotes the covering involution, then consider the space

$$\text{Prym}(X_{sw_1}, X) = \{ M \in \text{Pic}^0(X_{sw_1}) | \iota^* M = M^{-1} \}.$$

Recall that Proposition 3.9 defines a one to one correspondence between holomorphic rank two orthogonal bundles $(W, Q_W)$ with first Stiefel-Whitney class $sw_1$ and $\text{Prym}(X_{sw_1}, X)$ given by $M \mapsto (\pi_\ast M, \pi_\ast \iota^*).$ Recall also that $\text{Prym}(X_{sw_1}, X)$ has two connected components $\text{Prym}^{sw_2}(X_{sw_1}, X)$ labeled by the second Stiefel-Whitney class $sw_2$ of the orthogonal bundle $(\pi_\ast M, \pi_\ast \iota^*).$

**Lemma 5.18.** Define the space $\mathcal{F}^{sw_2}_{sw_1}$ by

$$\mathcal{F}^{sw_2}_{sw_1} = \{ (M, \mu) | M \in \text{Prym}^{sw_2}(X_{sw_1}, X), \mu \in H^0(M^{-1}K^n_{X_{sw_1}}) \}$$

where $\iota : \mathcal{X}_{sw_1} \rightarrow \mathcal{X}_{sw_1}$ is the morphism associated to the orientation double cover $X_{sw_1} \rightarrow X$.
There is a well defined smooth map

\[(5.21) \tilde{\Psi}^{S_{\text{wu}}} : \mathcal{F}^{S_{\text{wu}}} \times \bigoplus_{j=1}^{n-1} H^0(K^{2j}) \to \{ \text{stable SO}(n, n+1)\text{-Higgs bundles} \} \]

defined by \(\tilde{\Psi}^{S_{\text{wu}}}((M, \mu, q_2, \ldots, q_{2n-2}) = (V, W, \eta))\), where

\[V = K^{n-1} \oplus K^{n-3} \oplus \cdots \oplus K^{3-n} \oplus K^{1-n}, \]
\[W = \pi_* M \oplus K^{n-2} \oplus K^{n-4} \oplus \cdots \oplus K^{4-n} \oplus K^{2-n}, \]
\[\eta = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \pi_* \mu \\ 1 & q_2 & q_4 & \cdots & q_{2n-4} & q_{2n-2} \\ 0 & 1 & q_2 & q_4 & \cdots & q_{2n-4} \\ 0 & 0 & 1 & q_2 & \cdots & q_{2n-6} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 1 & q_2 \end{pmatrix} : V \to W \otimes K. \]

Moreover, \(\tilde{\Psi}^{S_{\text{wu}}}((M, \mu, q_2, \ldots, q_{2n-2}) \text{ and } \tilde{\Psi}^{S_{\text{wu}}}((M', \mu', q'_2, \ldots, q'_{2n-2}) \text{ lie in the same} \)
\[S(O(n, C) \times O(n+1, C)) \text{ gauge orbit if and only if, for all } j \]
\[M' = M \quad \mu' = \pm \mu \quad q'_{2j} = q_{2j} \quad \text{or} \quad M' = \iota^* M \quad \mu' = \pm \iota^* \mu \quad q'_{2j} = q_{2j}. \]

**Proof.** Before checking the image of \(\tilde{\Psi}^{S_{\text{wu}}} \) consists of stable Higgs bundles, we first prove the statement about gauge orbits. Two Higgs bundles \((V, W, \eta)\) and \((V', W', \eta')\) in the image of \(\tilde{\Psi}^{S_{\text{wu}}} \) lie in the same \(S(O(n) \times O(n+1)) \) gauge orbit if and only if \(\pi^*(V, W, \eta)\) to \(\pi^*(V', W', \eta')\) are gauge equivalent on \(X_{\text{wu}}\) via an \(\iota^*\)-invariant gauge transformation. Since \(\pi^* K_X = K_{X_{\text{wu}}} \), the \(\text{SO}(n, n+1)\) Higgs bundle \(\pi^* \tilde{\Psi}^{S_{\text{wu}}}((M, \mu, q_2, \ldots, q_{2n-2}) \text{ on } X_{\text{wu}}\) are in the image of \(\tilde{\Psi}_0\) from (5.6), with \(M \in \text{Prym}(X_{\text{wu}})\) and \(\nu = \iota^* \mu\). By Lemma 5.22, the \(\text{SO}(n, n+1)\) Higgs bundles \(\tilde{\Psi}_0(M, \mu, \nu, q_2, \ldots, q_{2n-2}) \text{ and } \tilde{\Psi}_0(M', \mu', \nu', q_2, \ldots, q_{2n-2}) \text{ on } X_{\text{wu}}\) are in the same gauge orbit if and only

\[(M', \mu', \nu', q'_{2j}, \ldots, q'_{2n-2}) = (M, \lambda \mu, \lambda^{-1} \nu, q_2, \ldots, q_{2n-2}) \quad \text{or} \quad (M', \mu', \nu', q'_{2j}, \ldots, q'_{2n-2}) = (M^{-1}, \lambda^{-1} \nu, \lambda \mu, q_2, \ldots, q_{2n-2}) \]

for \(\lambda \in \mathbb{C}^*\). The corresponding gauge transformations are given by (5.9), and are \(\iota^*\)-invariant if and only if \(\lambda = \lambda^{-1}, \text{i.e. } \lambda = \pm 1\).

Polystability of the Higgs bundle (5.22) follows almost immediately from the proof of the Lemma 5.11. Namely, for the zero locus of the holomorphic differentials \((q_2, \ldots, q_{2n-2})\) the corresponding \(\text{SL}(2n+1, \mathbb{C})\) Higgs bundles are cyclic and can be represented schematically as:

\[K^{p-1} \xrightarrow{1} \cdots \xrightarrow{1} K^{1-p}. \]

To check that this Higgs bundle is polystable it suffices to show \(\pi_* M\) has no positive invariant subbundles. In fact, \(\pi_* M\) does not have any positive degree subbundles. Indeed, if \(0 \to L \to \pi_* M\) is a holomorphic subbundle if and only if there is a positive degree \(\iota^*\)-invariant subbundle \(0 \to L \to \pi^* \pi_* M = M \oplus M^{-1}\). But \(M \oplus M^{-1}\) has no positive subbundles.
If $\mu = 0$, then the Higgs bundle is a direct sum of a stable $SL(2n - 1, \mathbb{C})$ Higgs bundle with a degree zero stable rank 2 bundle. Since $sw_1 \neq 0$, the orthogonal bundle $\pi_* M$ does not have any isotropic line subbundle. Thus, if $\mu \neq 0$ and $N \subset \pi_* M$ is a degree zero invariant line subbundle, then $N$ is an orthogonal subbundle. Thus, we can take its orthogonal complement and split the Higgs bundle as a stable $SL(2n, \mathbb{C})$ Higgs bundle plus an invariant degree zero line bundle. This implies the Higgs bundles in (5.22) are polystable for $q_{2j} = 0$. Since the automorphism group of such a Higgs bundles is finite, the Higgs bundle is stable. Using the openness of stability, as in Lemma 5.11 we conclude that $\Psi^{sw_2}$ is well defined. □

There is a $Z_2 \oplus Z_2$ action on $\mathcal{F}^{sw_2}_{sw_1}$ generated by

$$(M, \mu) \to (M, -\mu) \quad \text{and} \quad (M, \mu) \to (\pi^* M, \pi^* \mu).$$

Moreover, if we extend this action trivially to $\mathcal{F}^{sw_2}_{sw_1} \times \bigoplus_{j=1}^{n-1} H^0(K^2 j)$ then the map $\tilde{\Psi}^{sw_2}$ from (5.21) is $Z_2 \oplus Z_2$-equivariant. This gives a well defined continuous map

$$(5.23) \quad \Psi^{sw_2}_{sw_1} : \mathcal{F}^{sw_2}_{sw_1} / (Z_2 \oplus Z_2) \times \bigoplus_{j=1}^{n-1} H^0(K^2 j) \longrightarrow \mathcal{M}(SO(n, n + 1))$$

which is a homeomorphism onto its image. The following lemma completes the proof of Theorem 5.3.

**Lemma 5.19.** For each $(sw_1, sw_2) \in H^1(X, Z_2) \setminus \{0\} \times H^2(X, Z_2)$, the image of the map $\Psi^{sw_2}_{sw_1}$ from (5.23) is open and closed in $\mathcal{M}(SO(n, n + 1))$.

**Proof.** The proof is almost equivalent to the proof of Theorem 4.11. Let $(E, \Phi)$ be the $SL(2n + 1, \mathbb{C})$ Higgs bundle associated to $\Psi^{sw_2}_{sw_1}(M, \mu, q_2, \cdots, q_{2n-2})$. As in (4.15), we can choose a basis of invariant polynomials $(p_1, \cdots, p_n)$ so that

$$p_j(\Psi^{sw_2}_{sw_1}(M, \mu, \nu, q_2, \cdots, q_{2n-2})) = \begin{cases} q_{2j} & 1 \leq j \leq n - 1 \\ \pi_* \mu^T \otimes \pi_* \mu & j = n \end{cases}.$$ 

By properness of the Hitchin fibration, the image of any divergent sequence in $\mathcal{F}^{sw_2}_{sw_1} \times \bigoplus_{j=1}^{n-1} H^0(K^2 j)$ also diverges in $\mathcal{M}(SO(n, n + 1))$. Thus, the image of $\Psi^{sw_2}_{sw_1}$ is closed.

A simple calculation shows that the dimension of the image of $\Psi^{sw_2}_{sw_1}$ is the expected dimension of the moduli space $\mathcal{M}(SO(n, n + 1))$. Since every point in the image is a smooth point or an orbifold point, the image of $\Psi^{sw_2}_{sw_1}$ is open by the same argument for openness in Lemma 5.13. □

### 6. Zariski closures of reducible representations

Recall from Proposition 2.19 that a representation $\rho : \Gamma \to SO(n, n + 1)$ is reducible if and only if the corresponding $SL(2n + 1, \mathbb{C})$ Higgs bundle is strictly polystable. Moreover, a representation $\rho$ has Zariski closure $G' \subset SO(n, n + 1)$ if and only if the structure group of the corresponding $SO(n, n + 1)$ Higgs bundle reduces to $G'$ (see Proposition 2.17).

#### 6.1. A few important subgroups of $SO(n, n + 1)$

Recall that $SO(n, n + 1)$ is the group of orientation preserving linear automorphisms of $\mathbb{R}^{2n+1}$ which preserve a signature $(n, n+1)$-inner product. More generally, the group $O(n, m)$ is the group of linear automorphism of $\mathbb{R}^{n+m}$ which preserve a signature $(n, m)$-inner product.
If $Q_n$ and $Q_m$ are positive definite symmetric $n \times n$ and $m \times m$ matrices, then $O(n,m)$ consists of elements of $g \in GL(\mathbb{R}^{n+m})$ so that
\[
g^T \begin{pmatrix} Q_n & -Q_m \\ -Q_m & Q_n \end{pmatrix} g = \begin{pmatrix} Q_n & -Q_m \\ -Q_m & Q_n \end{pmatrix}.
\]
The group has $O(n,m)$ has four connected components which we will denote by $O^{\pm,\pm}(n,m)$. If $\mathbb{R}^{n,0} \subset \mathbb{R}^{n+m}$ is a positive definite subspace of maximal dimension and $\mathbb{R}^{0,m} \subset \mathbb{R}^{n,m}$ is a negative definite subspace with maximal dimension, then an element $g \in O(n,m)$ is in $O^{+,+}(n,m)$ if it preserves an orientation of $\mathbb{R}^{n,0}$ and reverses an orientation of $\mathbb{R}^{0,m}$. The components $O^{+,+}(n,m)$, $O^{-,+}(n,m)$ and $O^{-,-}(n,m)$ are defined similarly. The group $O^{+,\pm}(n,m)$ consists of elements which preserve the orientation an $\mathbb{R}^{n,0}$.

**Proposition 6.1.** If the quadratic form $Q_{n+1}$ is given by $Q_{n+1} = \begin{pmatrix} Q_n & Q_v \\ Q_v & Q_m \end{pmatrix}$, then matrices of the form $\begin{pmatrix} A & B \\ B & C \end{pmatrix}$ define subgroups of $SO(n,n+1)$ isomorphic to
- $SO(n,n-1) \times SO(2)$ if $a = n-1$, $A \in SO(n,n-1)$ and $B \in SO(2)$,
- $SO(n,n)$ if $a = n$, $A \in SO(n,n)$ and $B = 1$,
- $O^{\pm,\pm}(n,n)$ if $a = n$, $A \in O^{\pm,\pm}(n,n)$ and $B = det(A)$,
- $S(O^{\pm,\pm}(n,n-1) \times SO(2))$ if $a = n-1$, $A \in O^{\pm,\pm}(n,n-1)$, $B \in SO(2)$ and $det(A) = det(B)$.

The definition of an $O(n,m)$ Higgs bundle is similar to that of an $SO(n,n+1)$ Higgs bundle.

**Definition 6.2.** An $O(n,m)$ Higgs bundle over a Riemann surface $X$ is a triple $(V,W,\eta)$ where
- $V$ and $W$ are respectively rank $n$ and $m$ holomorphic vector bundles on $X$ equipped with holomorphic orthogonal structures $Q_V$ and $Q_W$.
- $\eta \in H^0(\text{Hom}(V,W) \otimes K)$.

An $O(n,m)$ Higgs bundle $(V,W,\eta)$ is an $O^{+,\pm}(n,m)$ Higgs bundle if $det(V) = O$.

For $G' \subset SO(n,n+1)$ one of the subgroups from Proposition 6.1, the following characterizes when an $SO(n,n+1)$ Higgs bundle reduces to a $G'$ Higgs bundle.

**Proposition 6.3.** Let $G'$ be one of the subgroups from Proposition 6.1. An $SO(n,n+1)$ Higgs bundle $(V,W,\eta)$ on $X$ reduces to a $G'$ Higgs bundle if
- $G' = SO(n,n-1) \times SO(2)$, $(W,Q_W) = \left(W_0 \oplus M \oplus M^{-1}, \begin{pmatrix} Q_{w_0} & 0 \\ 0 & 1 \end{pmatrix} \right)$, where $(W_0,Q_{w_0})$ is a rank $(n-1)$ holomorphic orthogonal bundle with trivial determinant, $M \in \text{Pic}^0(X)$ and
  \[\eta = \begin{pmatrix} 0 \\ 0 \end{pmatrix} : V \rightarrow (W_0 \oplus M \oplus M^{-1}) \otimes K.\]
- $G' = SO(n,n)$, $(W,Q_W) = \left(W_0 \oplus O, \begin{pmatrix} Q_{w_0} & 0 \\ 0 & 1 \end{pmatrix} \right)$, where $(W_0,Q_{w_0})$ is a rank $n$ holomorphic orthogonal bundle with trivial determinant and
  \[\eta = \begin{pmatrix} 0 \\ 0 \end{pmatrix} : V \rightarrow (W_0 \oplus O) \otimes K.\]
- $G' = O^{+,\pm}(n,n)$, $(W,Q_W) = \left(W_0 \oplus M, \begin{pmatrix} Q_{w_0} & 0 \\ 0 & 1 \end{pmatrix} \right)$, where $(W_0,Q_{w_0})$ is a rank $n$ holomorphic orthogonal bundle, $M \in \text{Pic}^0(X)$ such that $det(W_0) = M$ and
  \[\eta = \begin{pmatrix} 0 \\ 0 \end{pmatrix} : V \rightarrow (W_0 \oplus M) \otimes K.\]
- $G' = S(O^{+,+}(n,n-1) \times O(2))$, $(W,Q_W) = \left(W_0 \oplus W', \begin{pmatrix} Q_{w_0} & Q_{w_0}' \\ Q_{w_0}' & Q_{w_0} \end{pmatrix} \right)$, where $(W_0,Q_{w_0})$ is a rank $(n-1)$ holomorphic orthogonal bundle, $(W',Q_{w_0}')$ is a rank 2 holomorphic orthogonal bundle with $det(W_0) = det(W')$ and
  \[\eta = \begin{pmatrix} 0 \\ 0 \end{pmatrix} : V \rightarrow (W_0 \oplus W') \otimes K.\]
Proposition 6.5.

With the above notation, a Higgs bundle in \( M \) if and only if given by a tuple that \( \text{SO} \)

The map \( \bigoplus_{j=1}^{n-1} H^0(K^{2j}) \oplus H^0(K^n) \) is defined by sending a tuple of holomorphic differentials \((q_2, q_4, \cdots, q_{2n-2}, q_n)\) to the Higgs bundle \((V, W, \eta)\) where

\[
V = K^{n-2} \oplus K^{n-4} \oplus \cdots \oplus K^{2-n} \oplus \mathcal{O},
\]

\[
W = K^{n-1} \oplus K^{n-3} \oplus \cdots \oplus K^{3-n} \oplus K^{1-n},
\]

\[
(6.1) \quad \eta = \begin{pmatrix}
q_2 & q_4 & q_6 & \cdots & q_{2n-2} & 0 \\
1 & q_2 & q_4 & \cdots & q_{2n-4} & 0 \\
0 & 1 & q_2 & \cdots & q_{2n-6} & 0 \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
0 & 1 & q_2 & 0 & & \\
0 & 0 & 1 & 0 & & \\
0 & 0 & 0 & q_n
\end{pmatrix} : V \rightarrow W \otimes K.
\]

The orthogonal structures on \( V \) and \( W \) are given by

\[
Q_V = \begin{pmatrix} 1 \\ \cdots \\ 1 \end{pmatrix} \quad \text{and} \quad Q_W = \begin{pmatrix} 1 \\ \cdots \\ 1 \end{pmatrix}.
\]

Remark 6.4. Note that in \((6.1)\), if \( q_n = 0 \), then the Higgs bundle reduces to \( \text{SO}(n, n-1) \subset \text{SO}(n, n) \).

6.2. Zariski closures of reducible representations. Recall from \((5.6)\) that a Higgs bundle in \( \mathcal{M}_0(\text{SO}(n, n+1)) \) is determined by a tuple \((M, \mu, \nu, q_2, \cdots, q_{2n-2})\) where \( M \in \text{Pic}^0(X) \), \( \mu \in H^0(M^{-1}K^n) \), \( \nu \in H^0(MK^n) \) and \( q_{2j} \in H^0(K^{2j}) \) such that \( \mu = 0 \) if and only if \( \nu = 0 \). Moreover, by Lemma 5.9, the isomorphism class associated to a tuple \((M, \mu, \nu, q_2, \cdots, q_{2n-2})\) is a singular point of \( \mathcal{M}_0(\text{SO}(n, n+1)) \) if and only if \( \mu = \nu = 0 \) or \( M = M^{-1} \) and \( \mu = \lambda \nu \) for \( \lambda \in \mathbb{C}^* \).

Proposition 6.5. With the above notation, a Higgs bundle in \( \mathcal{M}_0(\text{SO}(n, n+1)) \) given by a tuple \((M, \mu, \nu, q_2, \cdots, q_{2n-2})\)

- reduces to an \( \text{SO}(n, n-1) \times \text{SO}(2) \) Higgs bundle whose \( \text{SO}(n, n-1) \)-factor is in the Hitchin component if \( \mu = \nu = 0 \),
- reduces to an \( \text{SO}(n, n) \) Higgs bundle in the Hitchin component if \( M = \mathcal{O} \) and \( \mu = \lambda \nu \) for \( \lambda \in \mathbb{C}^* \),
- reduces to an \( \mathcal{O}^+ \)-bundle if \( \mathcal{M}^2 = \mathcal{O} \) and \( \mu = \lambda \nu \) for \( \lambda \in \mathbb{C}^* \).

Proof. Let \((V, W, \eta)\) denote the \( \text{SO}(n, n+1) \) Higgs bundle corresponding to a tuple \((M, \mu, \nu, q_2, \cdots, q_{2n-2})\). It is given by \((5.6)\). The bundle \( W \) can be written as \( W = M \oplus M^{-1} \) where \( W_0 = K^{n-2} \oplus K^{n-4} \oplus \cdots \oplus K^{2-n} \). If \( \mu = \nu = 0 \), then with respect to this splitting the Higgs field decomposes as

\[
\eta = \begin{pmatrix} \eta_0 \\ 0 \end{pmatrix} : V \rightarrow (W_0 \oplus M \oplus M^{-1}) \otimes K,
\]

where \( \eta_0 : V \rightarrow W_0 \otimes K \) is the Higgs field in the \( \text{SO}(n, n-1) \)-Hitchin component associated to the holomorphic differentials \((q_2, \cdots, q_{2n-2})\). Thus, by Proposition 6.3, the structure group reduces to \( \text{SO}(n, n-1) \times \text{SO}(2) \).
If \( M = M^{-1} \) and \( \mu = \lambda \nu \), consider the portions of \( \eta \) and \( \eta^* \) given by

\[
\eta_\mu = \begin{pmatrix} \lambda \mu \\ \mu \end{pmatrix} : K^{-n} \rightarrow M \oplus M^{-1} \quad \text{and} \quad \eta^*_\mu = (\mu \lambda \mu) : M \oplus M^{-1} \rightarrow K^n.
\]

The kernel of \( \eta^*_\mu \) is an orthogonal subbundle of \( M \oplus M^{-1} \) which is isomorphic to \( M \). Moreover, the image of \( \eta_\mu \) is exactly the orthogonal complement of \( \ker(\eta^*_\mu) \). Thus the orthogonal bundle \( (W, Q_W) \) can be written as

\[
(W, Q_W) = \left( M \oplus K^{n-2} \oplus \cdots \oplus K^{2-n} \oplus M \left( \begin{array}{c} 1 \\ \vdots \\ 1 \\ 1 \end{array} \right) \right)
\]

with Higgs field given by

\[
\eta = \begin{pmatrix} 0 & \cdots & 0 & \mu \\ 1 & q_2 & \cdots & q_{2n-2} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 1 & q_2 & 0 \end{pmatrix} : V \rightarrow W \otimes K.
\]

Thus, the Higgs bundle \( (V, Q_V, W, Q_W, \eta) \) decomposes a direct sum of \( M \) (with zero Higgs field) and \( (V, Q_V, W_0, Q_{W_0}, \eta_0) \) where \( W_0 = K^{n-2} \oplus \cdots \oplus K^{2-n} \oplus M \). The determinant of \( W_0 \) is \( M \), thus the structure group of the Higgs bundle reduces to \( \Omega^{+,(n,n)} \subset SO(n, n + 1) \) by Proposition 6.3. When \( M = \mathcal{O} \), the structure group reduces to \( SO(n, n) \) and \( \mu \in H^0(K^n) \). Thus, the Higgs field \( \eta' : V \rightarrow W_0 \) is in the \( SO(n, n) \)-Hitchin component (6.1).

\[\square\]

**Theorem 6.6.** If \( \rho : \Gamma \rightarrow SO(n, n + 1) \) is a reducible representation which defines a point in \( \mathcal{X}_0(SO(n, n + 1)) \) or \( \mathcal{X}_{sw}^1(SO(n, n + 1)) \), then there is a finite index subgroup \( \hat{\Gamma} \subset \Gamma \) such that the restriction of \( \rho \) to \( \hat{\Gamma} \) either factors through \( SO(n, n - 1) \times SO(2) \) with \( SO(n, n - 1) \) factor in the Hitchin component, or factors through an \( SO(n, n) \)-Hitchin representation.

**Proof.** If \( \rho \in \mathcal{X}_0(SO(n, n + 1)) \) is reducible, then the associated \( SO(n, n + 1) \) Higgs bundle is determined by a tuple \( (M, \mu, \nu, q_2, \cdots, q_{2n-2}) \) with \( \mu = \nu = 0 \) or \( M = M^{-1} \) and \( \mu = \lambda \nu \) for \( \lambda \in \mathbb{C}^* \). Indeed, if \( 0 \neq \mu \neq \lambda \nu \), then the corresponding \( SL(n, \mathbb{C}) \) Higgs bundle is stable, and hence the representation \( \rho \) is irreducible.

By Proposition 6.3, if \( \mu = \nu = 0 \) the Higgs bundle reduces to an \( SO(n, n - 1) \times SO(2) \) Higgs bundle whose \( SO(n, n - 1) \)-factor is in the Hitchin component. Similarly, if \( M = \mathcal{O} \) and \( \mu = \lambda \mu \), the Higgs bundle reduces to an \( SO(n, n) \) Higgs bundle in the Hitchin component. In both of these cases, the corresponding representation either factors through \( SO(n, n - 1) \times SO(2) \) with \( SO(n, n - 1) \) factor in the Hitchin component or an \( SO(n, n) \)-Hitchin representation. If \( M^2 \cong \mathcal{O}, \mathcal{O} \neq \mathcal{O} \) and \( \mu = \lambda \nu \), the first Stiefel-Whitney class \( sw_1 \) of the orthogonal bundle \( M \) is nonzero. Let \( \pi : X_{sw_1} \rightarrow X \) be the associated connected orientation double cover. Since \( \pi^* M = \mathcal{O} \) and \( \pi^* K = K_{X_{sw_1}} \), the pull back of the Higgs bundle to \( X_{sw_1} \) reduces to an \( SO(n, n) \) Higgs bundle in the Hitchin component. Thus, the restriction of the representation \( \rho \) to the index two subgroup \( \pi_1(X_{sw_1}) = \hat{\Gamma} \) factors through an \( SO(n, n) \)-Hitchin representation.

For \( \rho \in \mathcal{X}_{sw}^1(SO(n, n + 1)) \), let \( X_{sw_1} \rightarrow X \) be the connected orientation double cover associated to \( sw_1 \in H^1(X, \mathbb{Z}_2) \setminus \{0\} \) and let \( \pi_1(X_{sw_1}) = \hat{\Gamma} \subset \Gamma \) be the associated index two subgroup. By construction, the restriction of \( \rho \) to \( \hat{\Gamma} \) defines a representation in the connected component \( \mathcal{X}_0(SO(n, n + 1)) \) of the character \( SO(n, n + 1) \)-character variety of \( \hat{\Gamma} \). Thus, there is a finite index subgroup of \( \hat{\Gamma} \subset \hat{\Gamma} \).
such that the restriction of $\rho$ factors through $SO(n, n-1) \times SO(2)$ with $SO(n, n-1)$ factor in the Hitchin component or an $SO(n, n)$-Hitchin representation. □

6.3. **Zariski closure of representations in $X_d(SO(n, n+1))$ for $d > 0$.** Recall that the connected components $X_d(SO(n, n+1))$ from Theorem 4.11 are smooth for $d \in (0, n(2g-2))$. In particular, every representation in such a component is irreducible. Recall also that every representations in the components $X_d(SO(n, n+1))$ factors through the connected component of the identity $SO_0(n, n+1)$. Note that the Zariski closure of $SO_0(n, n+1) \subset SO(n, n+1)$ is the full group $SO(n, n+1)$.

**Conjecture 6.7.** For $0 < d < n(2g-2)$, all representations in $X_d(SO(n, n+1))$ are Zariski dense.

For $d = n(2g-2)$ the component $X_{n(2g-2)}(SO(n, n+1))$ is the $SO(n, n+1)$-Hitchin component. Thus, by the definition of the Hitchin component (Definition 4.1), $X_{n(2g-2)}(SO(n, n+1))$ contains representations which are not Zariski dense. In [1], it is shown that, for $n = 2$ and $d \in (0, 4g-4)$, every representation in the components $X_d(SO(2, 3))$ is Zariski dense. The proof relies on the fact that $SO_0(2, 3)$ is a group of Hermitian type and that the representations in $X_d(SO(2, 3))$ are maximal representations. Thus, by results of [8], the Zariski closure of such a representation is a tightly embedded subgroup of Hermitian type. Using the Higgs bundles, one can rule out the handful of proper subgroups of $SO_0(2, 3)$ which are tightly embedded.

For $n > 2$, the group $SO_0(n, n+1)$ is not of Hermitian type, so the above methods do not apply. However, since the components $X_d(SO(n, n+1))$ are smooth, the only way a representation $\rho \in X_d(SO(n, n+1))$ can have a Zariski closure $G'$ smaller than $SO_0(n, n+1)$ is if $G'$ is a simple Lie group and there is a faithful irreducible representation representation $\psi : G' \to GL(\mathbb{R}^{2n+1})$ which preserves a signature $(n, n+1)$ inner product and $\rho$ factors through $G'$:

$$
\Gamma \xrightarrow{\rho} SO(n, n+1) \xrightarrow{\psi} G'.
$$

As an example, the signature of the Killing form for $SU(p, p)$ is $(2p^2, 2p^2-1)$, thus the adjoint representation of $SU(p, p)$ provides such an irreducible representation. This doesn’t occur until $SO(7, 8)$. Also, there is an irreducible seven dimensional representation of $G_2$ which preserves a signature $(3, 4)$ inner product. However, one can show directly that the Higgs bundles in the components $M_d(SO(3, 4))$ do not reduce to $G_2$.

Using the software *Atlas*, one can list the irreducible representations of a fixed Lie group $G'$ which admit an irreducible representations which preserves a signature $(n, n+1)$ inner product. In particular, for $3 < n < 7$, there are no simple Lie groups $G'$ which admits a faithful irreducible representation $\psi : G' \to GL(\mathbb{R}^{2n+1})$ which preserves an signature $(n, n+1)$ inner product.

7. **Positive Anosov representations**

In this section we show that all reducible representations in the connected components of $X(SO(n, n+1))$ described in Theorems 5.1 and 5.3 are positive Anosov representations. We first recall the notion of an Anosov representation, then review the work of Guichard-Wienhard [22] and Guichard-Labourie-Wienhard [20] on positive representations. After describing the positive structures for the groups $SO(n, n)$ and $SO(n, n+1)$, Theorem 7.13 is proven.
Anosov representations were introduced by Labourie [29] and have many interesting geometric and dynamic properties which generalize convex cocompact representations into rank one Lie groups. Important examples of Anosov representations include Hitchin representations into split real groups and maximal representations into Lie groups of Hermitian type. We will describe the main properties of Anosov representations which will be useful for our setting, and refer the reader to [29, 21, 18, 26] for more details.

Let $G$ be a semisimple Lie group and $P \subset G$ be a parabolic subgroup. Let $L \subset P$ be the Levi factor (the maximal reductive subgroup) of $P$. If $P^{pp}$ denotes the opposite parabolic of $G$, then $L = P \cap P^{pp}$. We will mostly be interested in $G = \text{SO}(n, n + 1)$, in this case all parabolic subgroups are conjugate to their opposites. We will assume all parabolic subgroups are conjugate to their opposite from now on.

The homogeneous space $G/L$ is the unique open $G$ orbit in $G/P \times G/P$. A pair of distinct generalized flags $(x, y) \in G/P \times G/P$ are called transverse if they are in the unique open $G$-orbit $G/L$.

**Definition 7.1.** Let $\Gamma$ be the fundamental group of a closed surface of genus $g \geq 2$. Let $\partial_{\infty} \Gamma$ be the Gromov boundary of the group $\Gamma$, topologically $\partial_{\infty} \Gamma \cong \mathbb{R}P^1$. A representation $\rho : \Gamma \to G$ is $P$ Anosov if and only if there exists a unique continuous boundary map $\xi_{\rho} : \partial_{\infty} \Gamma \to G/P$ which satisfies

- Equivariance: $\xi(\gamma \cdot x) = \rho(\gamma) \cdot \xi(x)$ for all $\gamma \in \Gamma$ and all $x \in \partial_{\infty} \Gamma$.
- Transversality: for all distinct $x, y \in \partial_{\infty} \Gamma$ the generalized flags $\xi(x)$ and $\xi(y)$ are transverse.
- Dynamics preserving: see [29, 21, 18, 26] for the precise notion.

The map $\xi_{\rho}$ will be called the $P$ Anosov boundary curve.

**Remark 7.2.** The following facts about Anosov representations will be important:

- Openness: Let $\rho : \Gamma \to G$ be a $P$ Anosov representation, there is an open neighborhood of $\rho$ in $\mathcal{X}(G)$ consisting of $P$ Anosov representations.
- Action of centralizer: The centralizer of $\rho$ acts trivially on $\xi(\partial_{\infty} \Gamma)$.
- Finite index subgroups: A representation $\rho$ is a $P$ Anosov representation if and only if the restriction of $\rho$ to any finite index subgroup $\hat{\Gamma} \subset \Gamma$ is $P$ Anosov.

### 7.1. Positive Anosov representations

The important cases of Hitchin representations and maximal representations define connected components of Anosov representations. Both Hitchin representations and maximal representations satisfy an additional “positivity” property which is a closed condition. For Hitchin representations this was proven by Labourie [29] and Fock-Goncharov [11], and for maximal representations by Burger-Iozzi-Wienhard [7]. These notions of positivity have recently been unified by Guichard-Wienhard [22]. The generalized notion of positivity defined below is conjectured to be a closed condition.

For a parabolic subgroup $P \subset G$, denote the Levi factor of $P$ by $L$ and the unipotent subgroup by $U \subset P$. The Lie algebra $\mathfrak{p}$ of $P$ admits an $\text{Ad}_L$-invariant decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ where $\mathfrak{l}$ and $\mathfrak{u}$ are the Lie algebras of $L$ and $U$ respectively. Moreover, the unipotent Lie algebra $\mathfrak{u}$ decomposes as

$$\mathfrak{u} = \bigoplus \mathfrak{u}_\beta,$$

where $\mathfrak{u}_\beta$ is an irreducible $L$-representation. Recall that a parabolic subgroup $P$ is determined by fixing a simple restricted root system $\Delta$ of a maximal $\mathbb{R}$-split torus
of $G$, and choosing a subset $\Theta \subset \Delta$ of simple roots. To each simple root $\beta_j \in \Theta$ there is a corresponding irreducible $L$-representation space $u_{\beta_j}$.

**Definition 7.3.** (22, Definition 4.2) A pair $(G, P_\Theta)$ admits a positive structure if for all $\beta_j \in \Theta$, the $L_\Theta$-representation space $u_{\beta_j}$ has an $L_\Theta^0$-invariant acute convex cone $c_{\beta_j}^0$, where $L_\Theta^0$ denotes the identity component of $L_\Theta$.

**Remark 7.4.** When $G$ is a split real form and $\Theta = \Delta$, the corresponding parabolic is a Borel subgroup of $G$. In this case, the connected component of the identity of the Levi factor is $L_\Delta^0 \cong (\mathbb{R}^+)^{r_k(\Delta)}$ and each simple root space $u_{\beta_i}$ is one dimensional. The $L_\Delta^0$-invariant acute convex cone in each simple root space $u_{\beta_i}$ is isomorphic to $\mathbb{R}^+$. When $G$ is a group of Hermitian type and $P$ is the maximal parabolic associated to the Shilov boundary of the Riemannian symmetric space of $G$, the pair $(G, P)$ also admits a notion of positivity [8].

Recall that the Weyl group $W$ of a root system is generated by reflections $s_\alpha$ associated to the simple roots $\alpha \in \Delta$. In [22], it is shown that, if $(G, P_\Theta)$ admits a notion of positivity, then there is at most one simple root $\beta_0 \in \Theta$ which, in the Dynkin diagram of $\Delta$, is connected to $\Delta \setminus \Theta$. Denote the longest word in the Weyl group of $\Delta \setminus \Theta$ by $\omega^0(\Delta \setminus \Theta)$.

**Definition 7.5.** If $(G, P_\Theta)$ admits a positive structure, define $W(\Theta) \subset W$ as the subgroup generated by \{ $\sigma_\beta \mid \beta \in \Theta$ \} where

$$\sigma_\beta = s_\beta \text{ if } \beta \in \Theta \setminus \{\beta_0\} \quad \text{and} \quad \sigma_{\beta_0} = \omega^0(\Delta \setminus \Theta).$$

If $(G, P_\Theta)$ admits a positive structure, then exponentiating certain combinations of elements in the $L_\Theta^0$-invariant acute convex cones give rise to a semigroup $U_{\Theta}^0 \subset U$.

**Definition 7.6.** Suppose $(G, P_\Theta)$ admits a positive structure and, for each $\beta \in \Theta$, let $c_{\beta}^0$ be the corresponding $L_\Theta^0$-invariant acute convex cone in $u_\beta$ given in Definition 7.3. Denote the longest word in the group $W(\Theta)$ by $\omega^0_\Theta$, and suppose:

$$\omega^0_\Theta = \sigma_{\beta_{1}} \cdots \sigma_{\beta_{\ell}}$$

is a reduced expression. Define the semigroup $U_{\Theta}^{\omega^0_\Theta} \subset U_\Theta$ to be the image of

$$F_{\beta_1} \cdots F_{\beta_\ell} : c_{\beta_{1}}^0 \times \cdots c_{\beta_{\ell}}^0 \rightarrow U_{\Theta}.$$  

$$(v_{\beta_1}, \ldots, v_{\beta_\ell}) \rightarrow \exp(v_{\beta_1}) \cdots \exp(v_{\beta_\ell})$$

The semigroup $U_{\Theta}^{\omega^0_\Theta}$ will be called the positive semigroup.

**Theorem 7.7.** (Theorem 4.5 [22]) The positive subsemigroup $U_{\Theta}^{\omega^0_\Theta}$ from Definition 7.6 is independent of the reduced expression of the longest word $\omega^0_\Theta$ of $W(\Theta)$.

The positive semigroup $U_{\Theta}^{\omega^0_\Theta}$ allows one to define a notion of positively ordered triples in the generalized flag variety $G/P_\Theta$. Since the group $G$ acts transitively on the space of transverse generalized flags, any two generalized flags $x, y \in G/P_\Theta$ can be mapped to the generalized flags $(x_+, x_-)$ associated to $P_\Theta$ and $P_\Theta^{opp}$ respectively.

**Definition 7.8.** (22, Definition 4.6) Let $x_+ \in G/P_\Theta$ be the generalized flag associated to $P_\Theta$ and $x_- \in G/P_\Theta$ be the generalized flag associated to $P_\Theta^{opp}$. Any flag $x_0$ which is transverse to $x_+$ is the image of $x_-$ under a unique element $u_0 \in U_\Theta$. The triple $(x_+, x_0, x_-)$ is positive if $u_0$ is in the positive subsemigroup $U_{\Theta}^{\omega^0_\Theta}$.

With respect to the orientation on $\partial_\infty \Gamma$, we say that a triple of pairwise distinct points $(a, b, c)$ is a positive triple if the points appear in this order.

**Definition 7.9.** (22, Definition 5.3) If the pair $(G, P_\Theta)$ admits a positive structure, then a $P_\Theta$ Anosov representation $\rho : \Gamma \rightarrow G$ is called a positive if the Anosov boundary curve $\xi : \partial_\infty \Gamma \rightarrow G/P_\Theta$ sends positive triples in $\partial_\infty \Gamma$ to positive triples in $G/P_\Theta$. 

7.2. **Positive structures for \( SO(n, n) \) and \( SO(n, n+1) \).** We now discuss the positive structures for the groups \( SO(n, n) \) and \( SO(n, n+1) \) and discuss how the embeddings

\[
SO(n, n - 1) \subset SO(n, n) \subset SO(n, n + 1)
\]

preserve these notions of positivity. For \( j \in \{2n - 1, 2n, 2n + 1\} \) and \( x = (x_1, \cdots, x_j) \in \mathbb{R}^j \), the inner product

\[
\langle x, x \rangle_{n,n-1} = 2x_1x_{2n-1} + \cdots + 2x_{n-1}x_{n+1} + x_n^2
\]

has signature \((n, n - 1)\), the inner product

\[
\langle x, x \rangle_{n,n} = 2x_1x_{2n} + \cdots + 2x_nx_{n+1}
\]

has signature \((n, n)\), and the inner product

\[
\langle x, x \rangle_{n,n+1} = 2x_1x_{2n+1} + \cdots + 2x_nx_{n+2} - x_{n+1}^2
\]

has signature \((n, n + 1)\).

Consider the following isometric embeddings

\[
(\mathbb{R}^{2n-1}, \langle \cdot, \cdot \rangle_{n,n-1}) \rightarrow (\mathbb{R}^{2n}, \langle \cdot, \cdot \rangle_{n,n})
\]

\[
(x_1, \cdots, x_{2n-1}) \mapsto (x_1, \cdots, x_{n-1}, \frac{x_n}{\sqrt{2}}, \frac{x_n}{\sqrt{2}}, x_{n+1}, \cdots, x_{2n-1})
\]

\[
(\mathbb{R}^{2n}, \langle \cdot, \cdot \rangle_{n,n}) \rightarrow (\mathbb{R}^{2n+1}, \langle \cdot, \cdot \rangle_{n,n+1})
\]

\[
(x_1, \cdots, x_{2n}) \mapsto (x_1, \cdots, x_n, 0, x_{n+1}, \cdots, x_{2n})
\]

Let \( \iota_{n,n-1} : SO(n, n - 1) \rightarrow SO(n, n) \) and \( \iota_{n,n} : SO(n, n) \rightarrow SO(n, n + 1) \) be the embeddings induced by the isometric embeddings (7.5).

\( G = SO(n, n - 1), \Theta = \Delta \): The group \( SO(n, n - 1) \) consists of \((2n - 1) \times (2n - 1)\)-matrices \( A \in SL(2n - 1, \mathbb{R}) \) which preserve the inner product (7.2). The set of diagonal matrices

\[
T = \{ diag(t_1, t_2, \cdots, t_{n-1}, 1, t_{n-1}^{-1}, \cdots, t_1^{-1}) \mid t_i \in \mathbb{R}^* \}
\]

is a maximal split torus of \( SO(n, n - 1) \). The Lie algebra \( t \) of \( T \) is given by

\[
t = \{ diag(x_1, \cdots, x_{n-1}, 0, -x_{n-1}, \cdots, -x_1) \mid x_i \in \mathbb{R} \}.
\]

Consider the simple root system \( \Delta = \{ \beta_1, \cdots, \beta_{n-2}, \beta_{n-1} \} \) with

\[
\beta_j(diag(x_1, \cdots, x_{n-1}, 0, -x_{n-1}, \cdots, -x_1)) = \begin{cases} 
 x_j - x_{j+1} & 1 \leq j \leq n - 2 \\
 x_{n-1} & j = n - 1
\end{cases}.
\]

The parabolic \( P_\Delta \) associated to \( \Delta \) has Levi factor \( L_\Delta = T \). The decomposition of the unipotent Lie algebra \( U_\Delta \) into irreducible \( L_\Delta \) representations is the same as the decomposition into positive root spaces. Let \( E_{ij} \) be the elementary matrix with a 1 in the \((i, j)\) entry and zero elsewhere. The root spaces of the simple roots are

\[
\mathfrak{g}_{\beta_j} = \langle E_{i,j+1} - E_{2n-1-i, 2n-i} \rangle.
\]

The identity component of the Levi factor \( L_\Delta^0 \) consists of diagonal matrices of the form (7.6) with positive entries. An element \((t_1, \cdots, t_{n-1}, 1, t_{n-1}^{-1}, \cdots, t_1^{-1})\) acts on the simple root space \( \mathfrak{g}_{\beta_j} \) by \( t_it_{i+1}^{-1} \) for \( 1 \leq i \leq n - 2 \) and by \( t_{n-1} \) on \( \mathfrak{g}_{\beta_{n-1}} \). The invariant affine cone in the simple root space \( \mathfrak{g}_{\beta_j} \) is

\[
C^\Delta_{\beta_j} = \mathbb{R}^+ \cdot \langle E_{i,j+1} - E_{2n-1-i, 2n-i} \rangle.
\]
The group \( W(\Theta) \) from Definition 7.5 is the whole Weyl group \( W \), and is generated by reflections \( s_\beta \). A reduced expression for the longest word \( \omega^0_\Delta(\text{SO}(n,n-1)) \) in the Weyl group for \( \text{SO}(n,n-1) \) is given by

\[
\omega^0_\Delta(\text{SO}(n,n-1)) = b_1 b_2 \cdots b_{n-1} \cdot
\]

where
\[
b_j = s_{\beta_{n-j}} s_{\beta_{n-j+1}} \cdots s_{\beta_{n-2}} s_{\beta_{n-1}} s_{\beta_{n-2}} \cdots s_{\beta_{n-j}} \cdot
\]

Define \( B^\Delta_j : c^\Delta_{\beta_{n-j}} \times \cdots \times c^\Delta_{\beta_{n-2}} \times c^\Delta_{\beta_{n-1}} \times c^\Delta_{\beta_{n-2}} \cdots \times c^\Delta_{\beta_{n-j}} \to U_\Delta \) by

\[
B^\Delta_j(u_{n-j}, \cdots, u_{n-2}, u_{n-1}, u_{n-2}, \cdots, u_{n-j}) = \exp(u_{n-j}) \cdots \exp(u_{n-2}) \cdot \exp(u_{n-1}) \cdot \exp(u_{n-2}) \cdots \exp(u_{n-j})
\]

The positive semisubgroup \( U^>_{\Delta} \subset U_\Delta \) from Definition 7.6 is given by the image of

\[
B^\Delta_1 \cdots B^\Delta_{n-1}.
\]

\( G = \text{SO}(n,n) \), \( \Theta = \Delta \): The group \( \text{SO}(n,n) \) consists of \( 2n \times 2n \)-matrices \( A \in \text{SL}(2n,\mathbb{R}) \) which preserve the inner product (7.3). The set of diagonal matrices

\[
\mathcal{T} = \{ A = \text{diag}(t_1, t_2, \cdots, t_n, t_n^{-1}, \cdots, t_1^{-1}) \mid t_i \in \mathbb{R}^* \}
\]

is a maximal split torus of \( \text{SO}(n,n) \). The Lie algebra \( \mathfrak{t} \) of \( \mathcal{T} \) is given by

\[
\mathfrak{t} = \{ X = \text{diag}(x_1, \cdots, x_n, -x_n, \cdots, -x_1) \mid x_i \in \mathbb{R} \}.
\]

Consider the simple root system \( \Delta = \{ \delta_1, \cdots, \delta_n \} \) with

\[
\delta_j(X) = \begin{cases} x_j - x_{j+1} & 1 \leq j \leq n - 1 \\ x_{n-1} + x_n & \end{cases}
\]

The parabolic \( P_\Delta \) associated to \( \Delta \) has Levi factor \( L_\Delta = \mathcal{T} \). The decomposition of the unipotent Lie algebra \( \mathfrak{u}_\Delta \) into irreducible \( L_\Delta \) representations is the same as the decomposition into positive root spaces \( \mathfrak{u}^\Delta_+ = \bigoplus_{\delta \in \mathbb{R}^+} \mathfrak{u}_\delta \). The root spaces of the simple roots are given by

\[
\mathfrak{g}_\delta = \begin{cases} \langle E_{i,i+1} - E_{2n+1-i,2n-i} \rangle & 1 \leq i \leq n - 1 \\ \langle E_{1+n,n-1} - E_{n+2,n} \rangle & i = n \\ \end{cases}
\]

The identity component of the Levi factor \( L^0_\Delta \) consists of diagonal matrices of the form (7.9) with positive entries. An element \( (t_1, \cdots, t_n, t_n^{-1}, \cdots, t_1^{-1}) \) acts on the simple root space \( \mathfrak{g}_\delta \) by \( t_i t_{i+1}^{-1} \) for \( 1 \leq i \leq n - 1 \) and by \( t_{n-1} t_n \) on \( \mathfrak{g}_{\delta_n} \). The invariant acute cone \( c^\Delta_\delta \) in the simple root space \( \mathfrak{g}_\delta \) is given by

\[
c^\Delta_\delta = \begin{cases} \mathbb{R}^+ \cdot \langle E_{i+1,i} - E_{2n-i,2n-1-i} \rangle & 1 \leq i \leq n - 1 \\ \mathbb{R}^+ \cdot \langle E_{n-1,n+1} - E_{n,n+2} \rangle & i = n \\ \end{cases}
\]

Since \( \Theta = \Delta \), the group \( \mathcal{W}(\Theta) \) from Definition 7.5 is the whole Weyl group \( \mathcal{W} \); it is generated by reflections \( s_\beta \). A reduced expression for the longest word \( \omega^0_\Delta(\text{SO}(n,n)) \) in the Weyl group for \( \text{SO}(n,n) \) is given by

\[
\omega^0_\Delta(\text{SO}(n,n)) = d_1 d_2 \cdots d_n,
\]

where

\[
d_j = \begin{cases} s_{\delta_{n+1-j}} & j \leq 2 \\ s_{\delta_{n+1-j}} \cdot s_{\delta_{n-j}} \cdots s_{\delta_{n-2}} \cdot s_{\delta_{n-1}} \cdot s_{\delta_{n-2}} \cdots s_{\delta_{n-j}} \cdot s_{\delta_{n+1-j}} & 3 \leq j \leq n \end{cases}
\]
For \( j = 1, 2 \) define
\[
D_j : c_{\delta_{n+1-j}}^\Delta \to U_\Delta
\]
and, for \( 3 \leq j \leq n \) define
\[
D_j : c_{\delta_{n+1-j}}^\Delta \times \cdots \times c_{\delta_{n-2}}^\Delta \times c_{\delta_{n-1}}^\Delta \times c_{\delta_{n-2}}^\Delta \times \cdots \times c_{\delta_{n+1-j}}^\Delta \to U_\Delta
\]
by
\[
D_j(u_{n+1-j}, \ldots, u_{n-2}, v_{n-1}, v_n, w_{n-2}, \ldots, w_{n+1-j}) = \exp(u_{n+1-j}) \cdots \exp(u_{n-2}) \cdot \exp(v_{n-1}) \cdot \exp(v_n) \cdot \exp(w_{n-2}) \cdots \exp(w_{n+1-j}).
\]

The positive semisubgroup \( U_\Delta^{\geq 0} \subset U_\Delta \) from Definition 7.6 is given by the image of
\[
D_1 \cdot D_2 \cdot D_3 \cdots D_n.
\]

Proposition 7.10. The embedding \( \iota_{n,n-1} : SO(n, n-1) \to SO(n, n) \) induced by the isometric embedding (7.5) maps the positive semigroup of \((SO(n, n-1), \Delta)\) into the positive semigroup of \((SO(n, n), \Delta)\).

Proof. Let \( E_{ij} \) be the elementary matrix with a 1 in the \((i, j)\) entry and zero elsewhere. The isometric embedding \( \iota_{n,n-1} \) from (7.5) induces a map \( \iota : \mathfrak{gl}(2n - 1, \mathbb{R}) \to \mathfrak{gl}(2n, \mathbb{R}) \) given by
\[
\iota(E_{ij}) = \begin{cases} 
E_{ij} & 1 \leq i < n \text{ and } 1 \leq j < n \\
E_{in} + E_{n,n+1} & j = n \\
E_{i+1,j} & n + 1 \leq i \leq 2n - 1 \text{ and } 1 \leq j < n \\
E_{i,j+1} & 1 \leq i < n \text{ and } n + 1 \leq j \leq 2n - 1 \\
E_{i+1,j+1} & n + 1 \leq i \leq 2n - 1 \text{ and } n + 1 \leq j \leq 2n - 1 
\end{cases}
\]

In particular, for \( 1 \leq j \leq n - 2 \), the restriction of \( \iota \) to the simple root space \( u_\beta \) is the identity, \( \iota(u_\beta) = Id : u_\beta \to u_\beta \). The restriction of \( \iota \) to the simple root space \( u_\beta \) maps \( u_\beta \) into the direct sum of the simple root spaces \( u_{\beta_{n-1}} \oplus u_{\beta_1} \) via the diagonal map \( x \to (x, x) \). Thus, the inclusion \( \iota : \mathfrak{so}(n, n-1) \to \mathfrak{so}(n, n) \) maps the product of positive cones \( c_{\beta_{n-1}}^\Delta \) into the product of positive cones \( c_{\beta_1}^\Delta \).

Let \( U_\Delta^{\geq 0}(n, n-1) \) be the positive subsemigroup for \( SO(n, n-1) \). If \( g \in U_\Delta^{\geq 0}(n, n-1) \), then
\[
g = g_1 : g_2 \cdots g_{n-1},
\]
where \( g_j = B_{\Delta}(u_{n-j}, \ldots, u_{n-2}, v_{n-1}, w_{n-2}, \ldots, w_{n-j}) \) is defined by equation (7.8).

Recall that \( u_{n-i}, w_{n-i} \in u_{\beta_{n-i}} \text{ and } v_{n-i} \in u_{\beta_{n-i}} \). The image \( \iota(g_j) \) is given by
\[
\iota(g_j) = \exp(\iota(u_{n-j})) \cdots \exp(\iota(u_{n-2})) \cdot \exp(\iota(v_{n-1})) \cdot \exp(\iota(w_{n-2})) \cdots \exp(\iota(w_{n-j})).
\]

Recall the definition of \( D_j \) from (7.11). By the definition of \( \iota \) we have
\[
\iota(g_j) = \begin{cases} 
D_1 \left( \frac{v_{n-1}}{\sqrt{2}} \right) \cdot D_2 \left( \frac{v_{n-1}}{\sqrt{2}} \right) & j = 1 \\
D_{j+1}(u_{n-j}, \ldots, u_{n-2}, \frac{v_{n-1}}{\sqrt{2}}, \frac{v_{n-1}}{\sqrt{2}}, w_{n-2}, \ldots, w_{n-j}) & 2 \leq j
\end{cases}
\]

Hence, \( \iota(g) = \iota(g_1) \cdots \iota(g_{n-1}) \) is in the positive semigroup \( U_\Delta^{\geq 0}(n, n) \) of \( SO(n, n) \).

\[
G = SO(n, n + 1), \Theta = \{ \beta_1, \ldots, \beta_{n-1} \} : The group \( SO(n, n+1) \) consists of \((2n + 1) \times (2n + 1)\)-matrices \( A \in SL(2n + 1, \mathbb{R}) \) which preserve the inner product (7.4). The maximal torus \( T \) and the simple root system \( \Delta \) analogous to \( SO(n, n-1) \). Namely,
\[
T = \{ A = diag(t_1, t_2, \ldots, t_n, 1, t_n^{-1}, \ldots, t_1^{-1}) \mid |t_i| \in \mathbb{R}^* \},
\]
Consider the subset \( \Theta = \{ \beta_1, \cdots, \beta_{n-1} \} \subset \Delta \), the parabolic \( P\Theta \) has Levi factor \( L\Theta \) consisting of matrices of the form

\[
\begin{pmatrix}
t_1 & \cdots & t_{n-1} \\
t_{n-1} & \cdots & t_1^{-1}
\end{pmatrix}
\]

where \( t_i \in \mathbb{R}^+ \) and \( A \in \text{SO}(1,2) \).

For \( 1 \leq j \leq n-2 \), the irreducible \( L\Theta \) representations \( u_{\beta_j} \) associated to the simple roots \( \beta_j \in \Delta \) are one dimensional. The irreducible \( L\Theta \) representation \( u_{\beta_{n-1}} \) is the three dimension vector space spanned by the root spaces for the positive roots \( \beta_{n-1}, \beta_{n-1} + \beta_n \) and \( \beta_{n-1} + 2\beta_n \).

Thus, the vector space \( u_{\beta_{n-1}} \) is given by

\[
u_{\beta_{n-1}} = \langle E_{n-1,n} - E_{n+2,n+3}, E_{n-1,n+1} - E_{n+1,n+3}, E_{n-1,n+2} - E_{n,n+3} \rangle.
\]

For \( (t_1, \cdots, t_{n-1}, A) \) in the identity component of the Levi factor \( L\Theta \), the action on \( u_{\beta_{i}} \) is by

\[
(t_1, \cdots, t_{n-1}, A) \cdot x = \begin{cases} 
  t_i t_{i+1}^{-1} x & \text{for } x \in u_{\beta_i} \text{ and } 1 \leq i \leq n-2 \\
  t_{n-1}^{-1} x A^{-1} & \text{for } x \in u_{\beta_{n-1}} \text{ and } A \in \text{SO}(1,2).
\end{cases}
\]

As before, for \( 1 \leq i \leq n-2 \), the positive reals define an invariant acute convex cone \( c^\Theta_{\beta_{n-1}} \subset u_{\beta_i} \). For \( i = n-1 \), the interior of the light cone in \( \mathbb{R}^{1,2} \) is the invariant acute convex cone, namely in the basis (7.13),

\[
c^\Theta_{\beta_{n-1}} = \{(x, y, z) \mid 2xz - y^2 > 0\}.
\]

The element \( \beta_\Theta = \beta_{n-1} \in \Theta \) is the unique element of \( \Theta \) which, in the Dynkin diagram of \( \Delta \), is connected to \( \Delta \setminus \Theta \). The group \( W(\Theta) \) from Definition 7.5 is generated by \( \{\sigma_1, \cdots, \sigma_{n-1}\} \), where

- \( \sigma_j = \sigma_{\beta_j} \) for \( 1 \leq j \leq n-2 \),
- \( \sigma_{n-1} \) is the longest word in the Weyl group of \( \beta_\Theta \cup \Delta \setminus \Theta = \{ \beta_{n-1}, \beta_n \} \).

The group \( W(\Theta) \) is isomorphic to the Weyl group of type \( B_{n-1} \) with its standard generators. Thus, the longest word \( \omega^\Theta_{\Theta}(SO(n, n+1)) \) has reduced expression

\[
\omega^\Theta_{\Theta}(SO(n, n+1)) = b_1 \cdots b_{n-1},
\]

where

\[
b_j = s_{\beta_{n-j}} \cdot s_{\beta_{n-j-1}} \cdot \cdots \cdot s_{\beta_{n-2}} \cdot s_{\beta_{n-1}} \cdot s_{\beta_{n-2}} \cdots s_{\beta_{n-j-1}} \cdot s_{\beta_{n-j}}.
\]

Define \( B^\Theta_j : c^\Theta_{\beta_{n-j}} \times \cdots \times c^\Theta_{\beta_{n-2}} \times c^\Theta_{\beta_{n-1}} \times c^\Theta_{\beta_{n-2}} \cdots c^\Theta_{\beta_{n-j}} \rightarrow U\Theta \) as in (7.8):

\[
B^\Theta_j(u_{n-j}, \cdots, u_{n-2}, v_{n-1}, w_{n-2}, \cdots, w_{n-j}) = \exp(u_{n-j}) \cdot \exp(u_{n-2}) \cdot \exp(v_{n-1}) \cdot \exp(w_{n-2}) \cdots \exp(w_{n-j}).
\]

The positive semigroup \( U^\Theta_+ \subset U\Theta \) from Definition 7.6 is given by the image of \( B^\Theta_1 \cdots B^\Theta_{n-1} \).
Proposition 7.11. The embedding $\iota_{n,n+1} : \text{SO}(n,n) \rightarrow \text{SO}(n,n+1)$ induced by the isometric embedding (7.5) maps the positive semigroup of $(\text{SO}(n,n), \Delta)$ into the positive semigroup of $(\text{SO}(n,n+1), \Theta)$.

Proof. Let $E_{ij}$ be the elementary matrix with a 1 in the $(i,j)$ entry and zero elsewhere. The isometric embedding $\iota_{n,n}$ from (7.5) induces a map $\iota : \mathfrak{gl}(2n, \mathbb{R}) \rightarrow \mathfrak{gl}(2n+1, \mathbb{R})$ given by

$$
\iota(E_{ij}) = \begin{cases}
E_{ij} & 1 \leq i \leq n \text{ and } 1 \leq j \leq n \\
E_{i+1,j} & n+1 \leq i \leq 2n \text{ and } 1 \leq j < n \\
E_{i,j+1} & 1 \leq i < n \text{ and } n+1 \leq j \leq 2n \\
E_{i+1,j+1} & n+1 \leq i \leq 2n \text{ and } n+1 \leq j \leq 2n
\end{cases}.
$$

In particular, for $1 \leq j \leq n-2$, the restriction of the $\iota$ to the simple $u_{\beta_j}$ is the identity $\iota|_{u_{\beta_j}} = Id : u_{\beta_j} \rightarrow u_{\beta_j}$. Thus, for $1 \leq j \leq n-2$, $\iota$ maps the positive cone $c^\Delta_j$ identically onto the positive cone $c^\Theta_j$. For $j = n-1$ and $j = n$, the restriction of $\iota$ to $u_{\beta_j}$ is the identity:

$$
\iota|_{u_{\beta_{n-1}}} = Id : u_{\beta_{n-1}} \rightarrow \mathfrak{g}_{\beta_{n-1}} \subset u_{\beta_{n-1}}^\Theta \quad \text{and} \quad \iota|_{u_{\beta_n}} = Id : u_{\beta_n} \rightarrow \mathfrak{g}_{\beta_n+2\beta_n} \subset u_{\beta_{n-1}}.
$$

Recall that the space $u_{\beta_{n-1}}^\Theta$ is a direct sum of root spaces $u_{\beta_{n-1}}^\Theta = \mathfrak{g}_{\beta_{n-1}} \oplus \mathfrak{g}_{\beta_{n-1}+\beta_n} \oplus \mathfrak{g}_{\beta_{n-1}+2\beta_n}$. The map $\iota$ is given by

$$
\iota : u_{\beta_{n-1}} \oplus u_{\beta_n} \rightarrow \mathfrak{g}_{\beta_{n-1}} \oplus \mathfrak{g}_{\beta_{n-1}+\beta_n} \oplus \mathfrak{g}_{\beta_{n-1}+2\beta_n} \quad \text{by} \quad (a, b) \rightarrow (a, 0, b).
$$

Moreover, since $a \in c^\Delta_{\beta_{n-1}}$ and $b \in c^\Delta_{\beta_n}$ implies $2ab > 0$, $\iota((c^\Delta_{\beta_{n-1}} \times c^\Delta_{\beta_n}) \subset c^\Theta_{\beta_{n-1}}$.

Let $U^\Delta_{\Theta}(n,n)$ be the positive subsemigroup for $\text{SO}(n,n)$. If $g \in U^\Delta_{\Theta}(n,n)$, then

$$
g = g_1 \cdot g_2 \cdot \ldots \cdot g_n,
$$

where

$$
g_j = \begin{cases}
D_1(v_n) & j = 1 \\
D_2(v_{n-1}) & j = 2 \\
D_j(u_{n+1-j}, \ldots, u_{n-2}, v_{n-1}, v_1, u_1, \ldots, u_{n+1-j}) & \text{if } 3 \leq j \leq n
\end{cases}
$$

are defined by equation (7.11). Recall that $u_{n-i}, v_{n-1}, v_1 \in u_{\beta_{n-i}}$. The image $\iota(g_j)$ is given by $\exp(\iota(u_{n+1-j})) \cdots \exp(\iota(u_{n-2})) \cdot \exp(\iota(v_{n-1})) \cdot \exp(\iota(v_1)) \cdot \exp(\iota(w_{n-2})) \cdots \exp(\iota(w_{n+1-j})).$

Recall the definition of $B^\Theta_{\beta_j}$ from (7.15). By the definition of $\iota$ we have

$$
\iota(g_1) \cdot \iota(g_2) = B^\Theta_{\beta_1}(i(v_1 + v_{n-1})) \quad \text{and for } 3 \leq j \leq n,
$$

$$
\iota(g_j) = B^\Theta_{\beta_j}(u_{n+1-j}, \cdots, u_{n-2}, i(v_1 + v_{n-1}), w_{n-2}, \cdots, w_{n+1-j}).
$$

Hence, $\iota(g) = \iota(g_1) \cdots \iota(g_n)$ is in the positive semigroup $U^\Delta_{\Theta}$ of $\text{SO}(n,n+1)$. \hfill \Box

7.3. **Positive $\text{SO}(n,n+1)$-representations.** We are now ready to prove the reducible representations in the components $\mathcal{X}_0(\text{SO}(n,n+1))$ and $\mathcal{X}^{\text{as}_{\mathbb{R}}}(\text{SO}(n,n+1))$ are $\mathcal{P}_\Theta$ positive Anosov representations. For $\Theta = \{\beta_1, \cdots, \beta_{n-1}\}$, the generalized flag variety $\text{SO}(n,n+1)/\mathcal{P}_\Theta$ consists of the set flags

$$
V_1 \subset \cdots \subset V_{n-1} \subset V_{n-1}^+ \subset \cdots \subset V_1^+ \subset \mathbb{R}^{2n+1},
$$

where $V_j \subset \mathbb{R}^{2n+1}$ is an isotropic $j$-plane. We start with the following proposition.
Proposition 7.12. Let $\rho : \Gamma \to SO(n, n + 1)$ be a representation. If there is a finite index subgroup $\hat{\Gamma} \subset \Gamma$ such that the restriction of $\rho$ to $\hat{\Gamma}$ factors through an $SO(n, n - 1) \times SO(2)$-representation with $SO(n, n - 1)$ factor a Hitchin representation or $\rho$ factors through an $SO(n, n)$-Hitchin representation, then $\rho$ is a positive $P_\Theta$ Anosov representation.

Proof. Let $\rho_0 : \Gamma \to SO(n, n - 1)$ and $\rho'_0 : \Gamma \to SO(n, n)$ be Hitchin representations. By [11, 29], there are $\rho_0$ and $\rho'_0$ equivariant positive Anosov boundary curves

$$\xi_{\rho_0} : \partial_\infty \Gamma \to SO(n, n - 1)/B_{n, n - 1} \quad \text{and} \quad \xi_{\rho'_0} : \partial_\infty \Gamma \to SO(n, n)/B_{n, n} ,$$

where $B_{n, n - 1} \subset SO(n, n - 1)$ and $B_{n, n} \subset SO(n, n)$ are Borel subgroups. The embeddings

$$SO(n, n - 1) \xrightarrow{\iota_{n, n - 1}} SO(n, n) \xrightarrow{\iota_{n, n}} SO(n, n + 1)$$

induced by (7.5) induce maps

$$SO(n, n - 1)/B_{n, n - 1} \xrightarrow{\iota_{n, n - 1}} SO(n, n)/B_{n, n} \xrightarrow{\iota_{n, n}} SO(n, n + 1)/P_\Theta .$$

By Propositions 7.10 and 7.11 the Anosov boundary curves

$$\iota_{n, n} \circ \iota_{n, n - 1} \circ \xi_{\rho_0} : \partial_\infty \Gamma \longrightarrow SO(n, n + 1)/P_\Theta \quad \text{and} \quad \iota_{n, n} \circ \xi_{\rho'_0} : \partial_\infty \Gamma \longrightarrow SO(n, n + 1)/P_\Theta$$

are $\Theta$-positive. The centralizer of $\iota_{n, n-1}(SO(n, n - 1))$ in $SO(n, n + 1)$ contains $SO(n, n - 1) \times SO(2)$. Thus, by Remark 7.2, if $\rho : \Gamma \to SO(n, n + 1)$ is a representation and there exists a finite order subgroup $\hat{\Gamma} \subset \Gamma$ such that the restriction of $\rho$ to $\hat{\Gamma}$ factors through an $SO(n, n - 1) \times SO(2)$-representation with $SO(n, n - 1)$ factor a Hitchin representation or $\rho$ factors through an $SO(n, n)$-Hitchin representation, then $\rho$ is a positive $P_\Theta$ Anosov representation. \[ \square \]

Theorem 7.13. Let $SO(n, n + 1)/P_\Theta$ be the generalized flag variety of flags

$$V_1 \subset \cdots \subset V_{n-1} \subset V_{n-1}^\perp \subset \cdots \subset V_1^\perp \subset \mathbb{R}^{2n+1} ,$$

where $V_j \subset \mathbb{R}^{2n+1}$ is an isotropic j-plane. If $n \geq 2$, then the set of representations in $X_0(SO(n, n + 1))$ or $X_{sw_1}^{\Theta}(SO(n, n + 1))$ which are not irreducible is a nonempty set which consists of positive $P_\Theta$ Anosov representations.

Proof. By Theorem 6.6, every reducible representation in the connected components $X_0(SO(n, n + 1))$ and $X_{sw_1}^{\Theta}(SO(n, n + 1))$ satisfy the hypothesis of Proposition 7.12. Thus, all reducible representations in $X_0(SO(n, n + 1))$ or $X_{sw_1}^{\Theta}(SO(n, n + 1))$ is a positive $P_\Theta$ Anosov representation. \[ \square \]

Remark 7.14. In [20], it is shown that the set of positive representations is open, and it is conjectured to be closed. In fact, it can be shown that the set of positive Anosov representations is closed in the set of irreducible representations [40]. Namely, let $\rho_j : \Gamma \to SO(n, n + 1)$ is a sequence of positive $P_\Theta$ Anosov representation which converge to $\rho_\infty : \Gamma \to SO(n, n + 1)$. If each $\rho_j$ is irreducible (see Definition 2.18) and $\rho_\infty$ is also irreducible, then $\rho_\infty$ is a positive $P_\Theta$ Anosov representation.

Assuming these results, Theorem 7.13 can be significantly strengthened to the statement that the components $X_0(SO(n, n + 1))$ and $X_{sw_1}^{\Theta}(SO(n, n + 1))$ consist entirely of Anosov representations.

The argument is as follows: Let $\rho$ be a reducible representation in $X_0(SO(n, n + 1))$ or $X_{sw_1}^{\Theta}(SO(n, n + 1))$. Since positive representations define an open set in the character variety, there is an open neighborhood $U_\rho$ of $\rho$ consisting of $\Theta$-positive representations. In particular, there exists $\rho \in U_\rho$ which is irreducible. Since
positivity is closed in the set of irreducible representations, all irreducible representations \( \rho \in X_0(\text{SO}(n,n+1)) \) are \( \Theta \)-positive. By Theorem 7.13 all representations in \( X_0(\text{SO}(n,n+1)) \) and \( X^{\text{sw}}_{\text{sw}}(\text{SO}(n,n+1)) \) are positive \( P_\Theta \) Anosov representations.

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