Modeling and Numerical Investigation of Aerodynamic Characteristics of a Propeller Circling on a Whirling Arm*

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A whirling arm is an effective device to measure propeller characteristics at low advance ratios. In the authors’ preceding research, an apparent advance ratio was adopted to determine propeller characteristics. The apparent advance ratio is determined only by the relative airspeed of the rotating arm and propeller in still air, and is used in wind tunnel measurements. However, in the case of whirling arm measurements, the advance ratio should be determined appropriately considering the additional airspeed induced by propeller wakes and swirling flow generated by the rotating arm. To address the above issue, we propose an airspeed model of blades considering velocity fields induced by the surrounding vortices. We also propose a procedure to calculate an appropriate advance ratio and the steady characteristics based on the proposed model. The validity of the proposed airspeed model is evaluated by making a comparison with the results of wind tunnel experiments. The corrected propeller characteristics using our airspeed model shows good consistency with the data referred from an existing propeller characteristics database.

Key Words: Aerodynamic Modeling, Aerodynamic Characteristics of Propellers, Whirling Arm, Aerodynamics

Nomenclature

\[ B: \text{number of blades} \]
\[ C_Q: \text{torque coefficient} \]
\[ C_T: \text{thrust coefficient} \]
\[ C_l: \text{lift coefficient of airfoils, } 1/\text{rad} \]
\[ C_d: \text{drag coefficient of airfoils, } 1/\text{rad} \]
\[ c_p: \text{chord length of blade elements, m} \]
\[ D: \text{diameter, m} \]
\[ E: \text{tip loss factor} \]
\[ J: \text{advance ratio} \]
\[ l: \text{length, m} \]
\[ Q: \text{torque, N} \cdot \text{m} \]
\[ T: \text{thrust, N} \]
\[ t: \text{time, s} \]
\[ U: \text{airspeed, m/s} \]
\[ u: \text{airspeed induced by vortices, m/s} \]
\[ v: \text{induced airspeed of blade elements, m/s} \]
\[ \alpha: \text{angle of attack of blade elements, rad} \]
\[ \eta: \text{propulsion efficiency} \]
\[ \Gamma: \text{circulation, m}^2/\text{s} \]
\[ \theta_p: \text{pitch angle of blade elements, rad} \]
\[ \nu: \text{dynamic viscosity, m}^2/\text{s} \]
\[ \rho: \text{density of the air, kg/m}^3 \]
\[ \sigma: \text{blade solidity} \]
\[ \phi: \text{incident angle, rad} \]
\[ \psi: \text{rotation angle, rad} \]
\[ \omega: \text{angular rate, rad/s} \]

\[ (r_w, \psi_w, z_w): \text{a coordinate system for an arm} \]
\[ (r_b, \psi_b, z_b): \text{a coordinate system for vortex filaments} \]
\[ (r_p, \psi_p, z_p): \text{a coordinate system for a propeller} \]

1. Introduction

Investigating propeller characteristics is one of the most crucial parts for modeling the dynamics of multi-rotor unmanned aerial vehicles (UAVs). The characteristics at low advance ratios are particularly important to understand the dynamic properties of multi-rotor UAVs because they usually fly at low airspeed or hover.

The characteristics of propellers have been measured through wind tunnel experiments.\(^1\)\(^–\)\(^3\) However, wind tunnels have a lower speed limit for steady airflows, and the slower the airflow, the greater turbulence. This property is not suitable for measuring the propeller characteristics at low advance ratios.

On the other hand, whirling arms do not have a lower speed limit for airflows. The whirling arm is a horizontal arm connected to a vertical shaft driven by a servo motor. A specimen (i.e., a propeller and a DC brushless motor) is attached to one end of the arm. We can independently control the rotational speed of the propeller and the arm without any lower speed limit. This feature is useful for measuring the characteristics at low advance ratios.

The authors investigated the characteristics of small propellers used for multi-rotor UAVs by using a whirling arm in preceding works.\(^4\)\(^–\)\(^5\) The results demonstrated that the
whirling arm can measure the characteristics even if the advance ratio is near zero or negative.

In these works,\(^4,5\) the authors adopted the ratio of the rotational speed of the propeller to that of the arm as the advance ratio of the propeller. This ratio is called an apparent advance ratio throughout this paper. However, surrounding vortices, such as propeller wakes and the swirling flow generated by the arm, induce velocity fields around blade elements. This fact shows that the appropriate advance ratio in whirling arm measurements is not determined only by the mechanical motion of the propeller and the arm.

Although whirling arms have the above advantage, the validity of measurement results\(^4,5\) has not been proven because how to determine the appropriate advance ratio taking account of the velocity fields induced by the surrounding vortices is yet unknown. Evaluating the measured characteristics according to the appropriate advance ratio is the key issue for proving the validity of the results of whirling arm measurements.

In this paper, the authors consider the aerodynamic characteristics of a propeller circling on a whirling arm. The velocity field of the propeller wakes is modeled as the moving helical vortex tubes. Furthermore, the velocity field of the swirling flow induced by the rotating arm is modeled as the boundary layer flow on a rotating disc. The aerodynamic characteristics taking into account these velocity fields are determined using the hybrid blade element momentum theory (HBEMT).

Furthermore, the authors numerically investigate the characteristics of a small propeller at the advance ratio from 0.05 to approximately 0.60 using the proposed model. The validity of the proposed model is evaluated from the viewpoint of consistency with existing wind tunnel experiment data.

2. Theoretical Model

We firstly discuss modeling of the velocity fields of a propeller circling on a whirling arm. Next, the propeller characteristics are determined using the HBEMT with the proposed airspeed model.

2.1. Problem settings

2.1.1. Whirling arm and coordinate systems

Figure 1 shows the structure of the whirling arm. The rotary shaft is vertical and rotates with a constant angular velocity \(\omega_{rw}\). The whirling arm is a cantilever that is horizontally attached to the rotary shaft.\(^5\) The propeller (diameter \(D_p\) and number of blades \(B\)) is attached at the far end of the arm. This propeller rotates with a constant angular velocity \(\omega_p\), and generates thrust \(T\) and torque \(Q\). The rotation axis of the propeller is aligned to the tangent line of the circling orbit of the center of the propeller. The distance from the center of the propeller to the \(zw\) axis (i.e., a radius of the propeller circling orbit) is \(lw\).

We use three coordinate systems to model the velocity fields in this paper.

Firstly, the blade coordinate systems \(O_p-\left(\vec{l}_{p}, \vec{w}_{p}, \vec{z}_{p}\right)\) shown in Fig. 1 are defined to represent positions of the blade elements of the \(i\)th blade, where \(i = 1, 2, \ldots, B\) is the index of the blade. The \(\vec{l}_{p}\) axis is set along the \(i\)th blade and rotates with the blade. The \(\vec{z}_{p}\) axis is set to coincide with the rotation axis of the propeller. The angle \(\psi_{p}\) denotes the rotation of the \(i\)th blade from the vertically upward position. We refer to \(O_p\) as the “propeller position.”

Secondly, an arm coordinate system \(O_w - (r_w, \psi_w, zw)\) is defined as shown in Fig. 1. The \(zw\) axis is set to coincide with the rotation axis of the rotary shaft. Therefore, it is perpendicular to the rotation surface of the arm. The origin \(O_w\) is the center of the circling orbit of \(O_p\). The angle \(\psi_w\) is the rotation angle of the arm. The \(r_w\) axis is defined from \(O_w\) toward \(O_p\).

Thirdly, a vortex filament coordinate system is defined. Let us assume that the propeller wake consists of \(b\) helical vortex filaments. The vortex filament coordinate systems \(iO_{m}^{(j,k)} - (\vec{l}_{h}^{(j,k)}, \vec{w}_{h}^{(j,k)}, \vec{z}_{h}^{(j,k)})\) are defined for the \(k\)th filament in the \(j\)th wake generated by the \(i\)th blade, as shown in Fig. 2. The details of the discretized wake model and the numbering of wakes and filaments are explained in Section 2.1.2.

We refer to \(iO_{m}^{(j,k)}\) as the “position of the \(k\)th filament in the \(j\)th wake to the \(i\)th blade.” The position vector from \(iO_{m}^{(j,k)}\) to a line element on the vortex filament belonging to this coordinate system is denoted as \(X_{m}^{(j,k)}(\vec{l}_{h}^{(j,k)}, \vec{w}_{h}^{(j,k)}, \vec{z}_{h}^{(j,k)})\). A vector \(e_{b}(X_{m}^{(j,k)})\) denotes the unit tangent vector of the helical vortex filament at \(X_{m}^{(j,k)}\). The position vector from \(iO_{m}^{(j,k)}\) to the blade element at the radius \(r_p\) of the \(i\)th blade is repre-
sent as \( i_x^{(j,k)}(t_p) \). Denote the time when the \( j \)-th wake is generated as \( t_j \). At \( t = t_j \), \( O_m^{(j)} = O_h^{(j)} \) for all \( k \). Here, \( O_h^{(j)} \) denotes the common initial position of all vortex filaments in the \( j \)-th wake.

All vortex filaments in the \( j \)-th wake move along the \( i_x^{(j,k)} \) axis with individual velocities. Each filament has a constant axial velocity \( i_w^{(j,k)} \) and a constant tangential velocity \( i_w^{(j,k)} \). These velocities are set to be identical to the induced velocities at the position of the \( k \)-th blade element at the time when the \( j \)-th wake is generated, namely, \( i_w^{(j,k)} = v_i(t_p) \) \( i_w^{(j,k)} \) also moves along the \( i_x^{(j,k)} \) axis in \( i_w^{(j,k)} \). The rotation angle of the \( i \)-th blade from its position at \( t = t_j \) is \( \psi^{(j,k)}_h \).

2.1.2. Velocity fields induced by surrounding vortices

The flow induced by the arm’s rotation is modeled as a boundary layer flow of a thin rotating disk, shown in Fig. 3. The velocity field of the boundary layer flow is denoted as \( u_w = [u_{w_1}, u_{w_2}, u_{w_3}] \).

Induced velocity models considering the shape of the arm can be complicated and requires high computational cost. Therefore, the authors adopted a simple rotating disk model in this work. Investigating the effect of the arm’s shape will be a future task.

The wake of the circling propeller forms an outward spiral from the circling orbit. In this work, the spiral wake is modeled as finite length straight wakes which are tangentially moving away from the circling orbit with constant velocities. Figure 4 shows the evolution process of this “discrete wake model.”

Note that if \( \omega_w \) is sufficiently smaller than the axial velocity of the wake, as is often the case when measuring propeller characteristics at low advance ratios, the propeller does not plunge into the separated wake during the previous round.

For a given time step \( \Delta t \), the time is discretized as \( t_i = \Delta t \cdot s \), where \( s = 0, 1, \cdots \) is called “step.” Let us assume that the wakes are generated and separated from the blades on every \( \Delta t \) as the propeller circles.

The first wake is generated at \( t = 0 \). At \( t = t_j, (s + 1) \) wakes exist, and the rotation angle of the arm is \( \psi_w = \omega_w \cdot s \cdot \Delta t \). As shown in Fig. 4, generated wakes are indexed as \( j = 1, 2, \cdots, s \). Helical vortex filaments are also indexed as \( k = 1, 2, \cdots, b \), numbered from the center outward, as shown in Fig. 5.

Figure 6 shows the positional relationship between the blade element at \( i_x^{(j,k)}(t_p) \) and the line element at \( i_x^{(j,k)} \) in the filament coordinate system for the \( k \)-th vortex filament in the \( j \)-th wake generated by the \( i \)-th blade. The angle resulting from the circling of the propeller and the advancing of the wake during \( \Delta t \) is defined as,

\[
\Delta \psi^{(j,k)}_w = \arccos \left( \frac{d_{h}^{(j,k)}}{l_w} \right),
\]

where \( d_{h}^{(j,k)} = i_w^{(j,k)} \cdot (s - j) \cdot \Delta t \) denotes the travel distance of the \( k \)-th vortex filament in \((t - t_j)\) (i.e., elapsed time since generation of the \( j \)-th wake). The angle \( \psi^{(j,k)}_w \) denotes the rotation angle of the arm when the \( \psi_w \) wake was generated.

The \( k \)-th vortex filament belonging to the \( j \)-th wake induced by the \( i \)-th blade induces a velocity field \( u_h^{(j,k)}(t_p) = [u_{h\psi}, \psi^{(j,k)}_i, i^{(j,k)}] \). The velocity field induced by the \( j \)-th
2.2. Airspeed modeling

The airspeed of the blade element of the \( i \)th blade is denoted as \( U_i(r_p) = \left[ U_a(r_p), U_t(r_p) \right] \). According to the discussion in Section 2.1.2,

\[
U_a(r_p) = (l_w + r_p \cdot \sin^i \psi_p) \omega_w + u_{a,0}(r_p) + u_{a,i}(r_p),
\]

(2)

\[
U_t(r_p) = r_p \cdot \omega_p + u_{t,0}(r_p) + u_{t,i}(r_p).
\]

(3)

The first term of the right-hand side of each equation represents airspeed components determined by the mechanical rotation of the propeller and the arm. The other terms represent the components induced by surrounding vortices.

2.2.1. Modeling of \( u_{a,i} \) and \( u_{t,i} \)

The basic idea of the proposed modeling is similar to a vortex panel method, which has been successfully applied to the modeling of the velocity field induced the propeller wake advancing in the axial direction.\(^{5,7}\)

When a propeller is advancing in the axial direction, a straight wake is generated behind the propeller. On the other hand, the wake generated behind a circling propeller forms an outward spiral wake from the circling orbit as the arm rotates.

This means a straight application of the vortex panel method to our problem requires an enormous calculation effort because of the aerodynamic interaction between the multiple surrounding wakes. The aerodynamic interaction between the helical vortex tubes is thus ignored in the presented modeling. The validity of this simplification is confirmed in Section 3.5.2, via a comparison of the data from wind tunnel experiments.

In this paper, a discrete wake model is adopted as explained in Section 2.1.2. The spiral wake is approximated by using multiple straight wakes (see Fig. 4). The characteristics of the orbiting propeller should be determined by taking into account the induced velocity field of these wakes.

Assuming that the propeller wake never diffuses nor connects with other wakes, the propeller wake is modeled as a cylindrical semi-infinite helical vortex tube.\(^{8}\) Okulov et al.\(^{9}\) proposed an induced velocity model of a helical vortex tube. Based on their ideas, we model the velocity field which multiple helical vortex tubes induce to the blade elements of circling blades.

The induced velocity around the blade element at the radius \( r_p \) of the \( i \)th blade is represented as \( u_{i}^{(j,k)}(r_p) \). This is obtained through integrating the differential induced velocities of the line element over the helical vortex filament, \( d^i u_{i}^{(j,k)} = \left[ d^i u_{i}^{(j,k)}_a, d^i u_{i}^{(j,k)}_t, d^i u_{i}^{(j,k)}_n \right] \).

The length of the line element of a cylindrical filament is denoted as \( dl \). Assume that the line element has a differential vector potential \( dA_{i}^{(j,k)} \). According to the Biot-Savart law, \( dA_{i}^{(j,k)} \) is given by

\[
d^iA_{i}^{(j,k)}(x_{i}^{(j,k)}) = \frac{i}{4\pi} \int_{l} \frac{\left| \epsilon \right|}{\left| x_{i}^{(j,k)} - x_{l}^{(j,k)} \right|} dl,
\]

(4)

where \( i \Gamma_{i}^{(j,k)} \) denotes circulation around the central axis of the line filament. According to the standard Ossen model,\(^{10}\)

\[
i \Gamma_{i}^{(j,k)} = \int_{0}^{1} \left[ 1 - \exp \left( -\frac{\left| x_{i}^{(j,k)} - x_{0}^{(j,k)} \right|^2}{4\nu \left( t - t_{i} \right)} \right) \right] dl,
\]

(5)

where \( i \Gamma_{0}^{(j,k)} \) is constant and its value is equivalent to the circulation of the 4th filament in the \( j \)th wake generated by the \( i \)th blade at \( t = t_{j} \).

The differential velocity field \( d^i u_{i}^{(j,k)} \) is calculated as follows,

\[
d^i u_{i}^{(j,k)} = \nabla \times dA_{i}^{(j,k)},
\]

(6)

where \( \nabla \) denotes a differential operator in the cylindrical coordinate system. Therefore, the velocity field induced by a filament is given by

\[
\left. d^i u_{i}^{(j,k)}(r_p) \right|_{r_p} = \int_{l} \left[ d^i u_{i}^{(j,k)}(x_{i}^{(j,k)}(r_p)) \right] dl.
\]

(7)

A Kapteyn series model of Eq. (7) is utilized for the numerical investigation. See Okulov et al.\(^{9}\) for the details.

By using the components of \( u_{i}^{(j,k)} \) in the vortex filament coordinate system, the axial and tangential components of
the induced velocities at the \( i \)th blade element are expressed as follows,
\[
\begin{align*}
\vec{V}^{(j,k)}_{a_i}(r_p) &= i \vec{u}^{(j,k)}_{a_i} \cdot \cos \Delta \psi^{(j,k)}_w + \left[ i \vec{u}^{(j,k)}_{a_h} \cdot \cos \phi^{(j,k)}_h \\
&\quad + i \vec{u}^{(j,k)}_{a_h} \cdot \sin \phi^{(j,k)}_h \right] \cdot \sin \Delta \psi^{(j,k)}_w,
\end{align*}
\]
(8)
\[
\begin{align*}
\vec{V}^{(j,k)}_{b_i}(r_p) &= \left[ i \vec{u}^{(j,k)}_{\theta_h} \cdot \cos \phi^{(j,k)}_h - i \vec{u}^{(j,k)}_{\theta_h} \cdot \sin \phi^{(j,k)}_h \right] \cdot \cos \Delta \psi^{(j,k)}_w \\
&\quad + \left[ i \vec{u}^{(j,k)}_{\theta_h} \cdot \sin \phi^{(j,k)}_h \right] \cdot \cos \Delta \psi^{(j,k)}_w \cdot \sin \psi_p,
\end{align*}
\]
(9)
where
\[
i \phi^{(j,k)}_h(r_p) = \arcsin \left[ \frac{\psi_p \cdot \cos \psi_p}{\sqrt{\gamma_0 \omega_w}} \right].
\]
(10)
Based on Eqs. (8) and (9), the airspeed induced by the propeller wakes is obtained as follows,
\[
\begin{align*}
\vec{u}_{a_i}(r_p) &= \sum_{l=1}^b \sum_{m=1}^{l-1} \sum_{n=1}^b [\vec{V}^{(j,m)}_{a_i}(r_p)],
\end{align*}
\]
(11)
\[
\begin{align*}
\vec{u}_{b_i}(r_p) &= \sum_{l=1}^b \sum_{m=1}^{l-1} \sum_{n=1}^b [\vec{V}^{(j,m)}_{b_i}(r_p)].
\end{align*}
\]
(12)

2.2.2. Modeling of \( \vec{u}_{a_i} \) and \( \vec{u}_{b_i} \)

From the assumption in Section 2.1.2, the model of the boundary layer flow induced by the rotating disc is applied to the modeling of the velocity field induced by the swirling flow of the rotating arm. The velocity model of Imayama\(^{13} \)is used for this study.

In order to numerically solve the governing Navier-Stokes equations in the cylindrical coordinate system, the equations are made dimensionless by introducing the non-dimensional velocities \( (F(\xi), G(\xi), \text{and } H(\xi)) \) and the non-dimensional pressure \( (P(\xi)) \) defined as follows,
\[
\begin{align*}
F(\xi) &= \frac{\vec{u}_r}{\omega_0 \omega_w}, \quad G(\xi) &= \frac{\vec{u}_\theta}{\omega_0 \omega_w}, \quad H(\xi) &= \frac{\vec{u}_z}{\sqrt{\gamma_0 \omega_w}},
\end{align*}
\]
(13)
\[
\begin{align*}
P(\xi) &= \frac{P(r_p)}{\rho \omega_0 \omega_w}.
\end{align*}
\]
In the above equations, \( \xi = (r_p \cdot \cos \theta_p) \cdot \sqrt{\omega_0 / V} \) is the non-dimensional height of the blade elements from the rotating disc.

The governing equations are expressed as the following non-dimensional forms using Eq. (13),
\[
F^2(\xi) + H(\xi) \frac{\partial F(\xi)}{\partial \xi} - G^2(\xi) - \frac{\partial^2 F(\xi)}{\partial \xi^2} = 0,
\]
(14)
\[
2 \cdot F(\xi) \cdot H(\xi) + H(\xi) \frac{\partial G(\xi)}{\partial \xi} - \frac{\partial^2 G(\xi)}{\partial \xi^2} = 0,
\]
(15)
\[
H(\xi) \frac{\partial H(\xi)}{\partial \xi} + \frac{\partial G(\xi)}{\partial \xi} - \frac{\partial^2 H(\xi)}{\partial \xi^2} = 0.
\]
(16)
To solve Eqs. (14)–(16), the following boundary conditions are applied,
\[
F(0) = 0, \quad G(0) = 1, \quad H(0) = 0, \quad F(\infty) = 0, \quad G(\infty) = 0.
\]
(17)
These conditions represent no-slip conditions at the rotating surface of the arm.

Figure 7 shows the non-dimensional velocities numerically obtained from Eqs. (14)–(17). A fourth-order Runge-Kutta integration method is used for integration. From this result, all of the non-dimensional velocities can be assumed as constants for \( \xi > 8 \).

Using \( \vec{u}_r(\xi) \), \( \vec{u}_\theta(\xi) \), and \( \vec{u}_z(\xi) \), the velocity components \( \vec{u}_{a_i}(r_p) \) and \( \vec{u}_{b_i}(r_p) \) are expressed as follows,
\[
\vec{u}_{a_i}(r_p) = \vec{u}_{\psi_i}(r_p),
\]
(18)
\[
\vec{u}_{b_i}(r_p) = \vec{u}_r(\xi) \cdot \cos \psi_p - \vec{u}_z(\xi) \cdot \sin \psi_p,
\]
(19)

2.3. Aerodynamic characteristics of the propeller

Thrust coefficient \( C_T \), torque coefficient \( C_Q \), and propulsion efficiency \( \eta \) are defined as follows,
\[
C_T = \frac{T}{\rho \cdot (n_p)^2 \cdot (D_p)^5},
\]
(20)
\[
C_Q = \frac{Q}{\rho \cdot (n_p)^2 \cdot (D_p)^5},
\]
(21)
\[
\eta = \frac{1}{2\pi} \cdot J \cdot \frac{C_T}{C_Q},
\]
(22)
where \( J \) denotes the advance ratio. This paper assumes that all propeller blades have an identical shape and are evenly spaced. Let us define an instantaneous advance ratio of the \( i \)th blade as
\[ i \bar{J} = \frac{U_d(i \bar{r}_p)}{U_i(i \bar{r}_p)}, \]  \hspace{1cm} (23)

where \( \bar{r}_p \) denotes a representative radius. A suitable selection of \( \bar{r}_p \) is discussed later in Section 3.5.1.

Note that \( i \bar{J} \) is not constant even for the constant speed rotation. The fluctuation of \( i \bar{J} \) is mainly caused by the periodically changing component of \( U_a \) (i.e., the first term on the right-hand side of Eq. (2)). This change results from the fact that the axial airspeeds of the inner and outer blades are different for the circling propeller on the arm.

The maximum relative error for the above component of \( U_a \) during one revolution of a blade is

\[ \frac{(\bar{r}_p + r_p)\omega_u - (\bar{r}_p - r_p)\omega_u}{(\bar{r}_p - r_p)\omega_u} = 1 + \frac{\kappa}{1 - \kappa} - 1, \]  \hspace{1cm} (24)

where \( \kappa = r_p/\bar{r}_p \). Therefore, the fluctuation becomes negligible for \( \bar{r}_p \gg r_p \).

According to the definition in Eq. (23), the instantaneous advance ratio of the propeller is defined as

\[ \bar{J} = \frac{1}{B} \sum_{i=1}^{B} [i \bar{J}]. \]  \hspace{1cm} (25)

If the induced velocity fields around the propeller are ignored (i.e., assuming that the propeller is advancing in still air), \( U_i = \bar{r}_p \cdot \omega_p \) for any \( i \). Therefore, \( U_i \) does not depend on \( i \psi_p \). An average of \( U_a \) for the rotation period of the propeller is equal to \( \bar{r}_p \cdot \omega_p \). This reduces Eq. (25) to the constant \( \bar{J} = (\bar{r}_p \cdot \omega_p)/(\bar{r}_p \cdot \omega_p) = J_{app} \). Let us call \( J_{app} \) an apparent advance ratio. In our previous works,\(^4,5\) \( J_{app} \) was adopted to investigate characteristics.

In reality, the induced velocity fields have non-negligible effects on the propeller characteristics. Therefore, the influence of the velocity fields on the \( \bar{J} \) in Eq. (25) is taken into consideration through the local airspeed in Eq. (23).

From the viewpoint of measuring propeller characteristics, it is required to determine a representative value of the advance ratio (i.e., constant \( \bar{J} \)). Here, the authors propose using an average of \( \bar{J} \) for the rotation period of the propeller as a representative advance ratio in whirling arm experiments,

\[ \bar{J}_f = \frac{1}{f_p} \int_{\tau = \bar{r}_p}^{f_p} \bar{J}(\tau) \, d\tau, \]  \hspace{1cm} (26)

where \( f_p = 2\pi/\omega_p \). Further discussion about \( \bar{J}_f \) in numerical computation is described in Section 3.3.

Definitions in Eqs. (20) and (21) are usually applied to wind tunnel experiments. On the other hand, the hybrid blade element momentum theory is applied to determine the appropriate \( C_T \) and \( C_Q \) for our problem (see Section 2.4).

### 2.4. Propeller modeling

In this work, the HBEMT is used for the propeller modeling. This theory is based on blade element momentum theory (BEMT) and blade element theory (BET). The HBEMT is useful for analyzing the propeller aerodynamics in non-hover flights. Khan et al.\(^{12} \) applied the HBEMT to analyze the propeller aerodynamics in forward flight. This paper uses the HBEMT to determine the axial and tangential induced velocities that vary with respect to \( r_p \).

Figure 8 shows airspeed components and differential aerodynamic forces of a blade element. The unknowns are non-dimensional induced velocities, \( i \tilde{v}_a = (\bar{r}_p/(n_p \cdot D_p)) \cdot i \tilde{v}_b \) and \( i \tilde{v}_t = (\bar{r}_p/(n_p \cdot D_p)) \).

Let us define a nonlinear optimization problem using the non-dimensional radius of the i-th blade, \( \tilde{r}_p = (r_p/R_p) \), \( R_p = (D_p/2) \). Denote the root position as \( \tilde{r}_{p,\text{root}} \), and the tip position as \( \tilde{r}_{p,\text{tip}} \). Then, \( i \tilde{v}_a[k] = i \tilde{v}_b[k] \cdot i \tilde{v}_t[k] = i \tilde{v}_t[k] \cdot \tilde{r}_p[k] \), \( k = 1, 2, \ldots, b \) are determined by solving the following problem for each blade \( i = 1, \ldots, B \).

**Problem 1.**

For given \( i \tilde{r}_p[k], i \tilde{r}_p[1] < \ldots < i \tilde{r}_p[b], i \tilde{r}_p[1] = i \tilde{r}_{p,\text{root}}, \) and, \( i \tilde{r}_p[b] = i \tilde{r}_{p,\text{tip}}, \) find \( i \tilde{v}_a[k], i \tilde{v}_b[k], i \tilde{v}_t[k] \), s.t.

\[ \begin{align*}
&\frac{d^2 C_T}{d\tilde{r}_p^2} \Bigg|_{\text{BEMT}} \cdot \left( i \tilde{v}_a[k] , i \tilde{v}_b[k] \right) = \frac{d^2 C_T}{d\tilde{r}_p^2} \Bigg|_{\text{BET}} \cdot \left( i \tilde{v}_a[k] , i \tilde{v}_b[k] \right), \\
&\frac{d^2 C_Q}{d\tilde{r}_p^2} \Bigg|_{\text{BEMT}} \cdot \left( i \tilde{v}_a[k] , i \tilde{v}_b[k] \right) = \frac{d^2 C_Q}{d\tilde{r}_p^2} \Bigg|_{\text{BET}} \cdot \left( i \tilde{v}_a[k] , i \tilde{v}_b[k] \right). \end{align*} \]  \hspace{1cm} (27) \hspace{1cm} (28)

The left sides of Eqs. (27) and (28) are non-dimensional aerodynamic distributions derived from the BEMT (see Adkins et al.\(^{13} \)). Using the parameters shown in Fig. 8, these distributions are expressed as follows,

\[ \begin{align*}
\frac{d^2 C_T}{d\tilde{r}_p^2} \Bigg|_{\text{BEMT}} &= \pi \cdot \tilde{r}_p[k] \cdot \left\{ \tilde{U}_a(i \tilde{r}_p[k]) + i \tilde{v}_a[k] \right\} \\
&\cdot \left( i \tilde{v}_a[k] \right) \cdot E(i \tilde{r}_p[k]), \tag{29}
\end{align*} \]

\[ \begin{align*}
\frac{d^2 C_Q}{d\tilde{r}_p^2} \Bigg|_{\text{BEMT}} &= \frac{\pi}{2} \cdot \left( i \tilde{r}_p[k] \right)^2 \cdot \left\{ \tilde{U}_a(i \tilde{r}_p[k]) + i \tilde{v}_a[k] \right\} \\
&\cdot \left( i \tilde{v}_a[k] \right) \cdot E(i \tilde{r}_p[k]), \tag{30}
\end{align*} \]

where \( \tilde{U}_a = U_a/(n_p \cdot D_p) \) denotes the non-dimensional axial airspeed.

In Eqs. (29) and (30), \( E \) denotes the tip loss factor of Prandtl, which depends on operating conditions of the blade. Note that the incident angle at the tip of a blade (\( \psi_{\text{tip}} \)) varies with respect to \( i \psi_p \). Taking into account this situation, the tip loss factor proposed by McCormick\(^{14} \) is adopted, which is defined as
\[ E(i \hat{r}_p[k]) = \frac{2}{\pi} \arccos \left[ \exp \left( -\frac{B(1-i \hat{r}_p[k])}{2 \sin \phi_{tip}} \right) \right]. \] (31)

The right sides of Eqs. (27) and (28) are the non-dimensional aerodynamic distributions derived from the BET (see McCormick). Using the parameters shown in Fig. 8, these distributions are expressed as follows,

\[
\frac{d^i C_T}{d^i r_p}_{BET} = \frac{\pi}{8} \sigma \cdot \left[ \left( \hat{U}_d(i \hat{r}_p[k]) + i \hat{v}_d[k] \right)^2 + \left( \hat{U}_t(i \hat{r}_p[k]) - i \hat{v}_t[k] \right)^2 \right] \left[ C_i \cdot \cos \left( i \phi_e(i \hat{r}_p[k]) \right) - C_i \cdot \sin \left( i \phi_e(i \hat{r}_p[k]) \right) \right],
\] (32)

\[
\frac{d^i C_Q}{d^i r_p}_{BET} = \frac{\pi}{16} \cdot \sigma \cdot \left[ \left( \hat{U}_d(i \hat{r}_p[k]) + i \hat{v}_d[k] \right)^2 + \left( \hat{U}_t(i \hat{r}_p[k]) - i \hat{v}_t[k] \right)^2 \right] \left[ C_i \cdot \frac{\sin \left( i \phi_e(i \hat{r}_p[k]) \right)}{\cos \left( i \phi_e(i \hat{r}_p[k]) \right)} \right] \cdot i \hat{r}_p[k],
\] (33)

where \( \hat{U}_t = U_t/(n_p D_p) \) denotes the non-dimensional tangential airspeed. In Eqs. (32) and (33), \( \hat{\psi}_e(i \hat{r}_p[k]) \) and \( \hat{\psi}_e(i \hat{r}_p[k]) \) are expressed as follows,

\[
\hat{\psi}_e(i \hat{r}_p[k]) = \hat{\psi}_e(i \hat{r}_p[k]) + \hat{\alpha}_s(i \hat{r}_p[k]),
\] (34)

\[
\hat{\psi}_e(i \hat{r}_p[k]) = \arctan \left( \frac{\hat{U}_d(i \hat{r}_p[k]) + i \hat{v}_d[k]}{\hat{U}_t(i \hat{r}_p[k]) - i \hat{v}_t[k]} \right).
\] (35)

Once suitable \( \hat{\psi}_d(i \hat{r}_p) \) and \( \hat{\psi}_t(i \hat{r}_p) \) are found by solving Problem 1, the instantaneous value of aerodynamic coefficients of the \( i \)th blade are determined as follows,

\[
iC_T = \int_{j \hat{r}_p}^{i \hat{r}_p} \left[ \frac{d^i C_T(i \hat{r}_p)}{d^i r_p}_{BET} \right] d^i \hat{r}_p,
\] (36)

\[
iC_Q = \int_{j \hat{r}_p}^{i \hat{r}_p} \left[ \frac{d^i C_Q(i \hat{r}_p)}{d^i r_p}_{BET} \right] d^i \hat{r}_p.
\] (37)

The instantaneous value of the aerodynamic coefficients of the propeller is calculated using Eqs. (36) and (37).

\[
C_T(t) = \sum_{i=1}^{B} [iC_T], \quad C_Q(t) = \sum_{i=1}^{B} [iC_Q].
\] (38)

Representative values of Eq. (38), \( C_T \) and \( C_Q \), are defined as follows,

\[
C_T(t) = \frac{1}{f_p} \int_{j \hat{r}_p}^{i \hat{r}_p} C_T(\tau) d \tau, \quad C_Q(t) = \frac{1}{f_p} \int_{j \hat{r}_p}^{i \hat{r}_p} C_Q(\tau) d \tau.
\] (39)

where \( f_p \) denotes the rotation period of the propeller.

3. Numerical Investigation

The numerical investigations of the propeller characteris-
For $s \leq 0$, $J_0 = 0$, $J_0 = 0$, $u_0 = 0$, $u_0 = 0$.

(41)

Firstly, calculate the induced velocities ($u_0$, $u_0$, $u_0$, and $u_0$) using Eqs. (11), (12), (18), and (19) for every element of all blades. Secondly, obtain $U_u$ and $U_i$ from Eqs. (2) and (3), respectively. Thirdly, solve Problem 1 to determine a suitable $\delta u_0$ and $\delta u_i$. Next, calculate $J_0$, $C_T$, and $C_Q$ from Eqs. (23), (36), and (37), respectively, by numerical integration. Finally, calculate $J_0$, $C_T$, $C_Q$, and $\eta$ from Eqs. (25), (38), and Eq. (22), respectively.

If the termination condition in Eq. (43) is not satisfied, update $s$ and repeat the calculation.

### 3.3. Representative values and termination condition

The investigation should be executed until the convergence of the advance ratio is confirmed. This is because the convergence of the advance ratio also means the convergence of other characteristics depending on the advance ratio.

The discretized version of Eq. (26) is defined as,

$$J_r = \frac{1}{s_p} \sum_{p=1}^{s_p} J[p],$$

where $s_p = \lfloor(2\pi)/(\omega_0 \cdot \Delta t)\rfloor$. $J_r$ has some fluctuation caused by the discretized wake model in Fig. 4. Therefore, the relative error between $J_r[s]$ and $J_r[s-1]$ is not useful to check whether or not the iteration has reached a steady state.

Let us introduce the average of $J_r$ for the rotation period of the arm. For a small $\delta > 0$, define the termination condition as

$$J_r[s] = \frac{1}{s_w} \sum_{p=1}^{s_w} J_r[p], \quad \left| \frac{J_r[s] - J_r[s-1]}{J_r[s]} \right| < \delta, \quad (43)$$

where $s_w = \lfloor(2\pi)/(\omega_0 \cdot \Delta t)\rfloor$. Furthermore, $J_r$ in Eq. (43) is utilized as representative value of the advance ratio in the numerical investigation.

A discretized version of Eq. (39) is defined as follows,

$$\tilde{C}_{T_r}[s] = \frac{1}{s_p} \sum_{q=1}^{s_p} C_T[q], \quad \tilde{C}_{Q_r}[s] = \frac{1}{s_p} \sum_{q=1}^{s_p} C_Q[q].$$

$\tilde{C}_{T_r}$ and $\tilde{C}_{Q_r}$ in Eq. (44) are also utilized as representative values in the numerical investigation.

$$\tilde{C}_{T_r}[s] = \frac{1}{s_w} \sum_{q=1}^{s_w} \tilde{C}_{T_r}[q], \quad \tilde{C}_{Q_r}[s] = \frac{1}{s_w} \sum_{q=1}^{s_w} \tilde{C}_{Q_r}[q] \quad (44)$$

The propulsion efficiency $\eta$ in Eq. (22) is calculated using $J_r$, $\tilde{C}_{T_r}$, and $\tilde{C}_{Q_r}$.

### 3.4. Target propeller and settings

#### 3.4.1. Target propellers

For the numerical investigation, a propeller designed for small UAVs is selected. Table 1 shows the specifications.

Figure 11 shows radial distribution of the non-dimensional chord $\tilde{C}_p = (c_p/R_p)$, and Fig. 12 that of the pitch angle $\theta_p$. These figures are reproductions from open-access data of propeller geometry on the UIUC Propeller Data Site.\(^\text{17}\)

#### 3.4.2. Operating conditions

Table 2 shows the operating conditions of the propeller and the whirling arm in the numerical investigation.

#### 3.4.3. Calculation time steps

Table 3 shows the values of $\Delta t$ selected for each case of

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\(^{17}\)This integration is feasible only after $s \geq 2\pi/(\omega_0 \cdot \Delta t)$.
In order to accurately grasp the convergence of the velocity field, $\Delta t$ should be set to be sufficiently smaller than the rotation period of the propeller. Here we set the values of $\Delta t$ to less than $1/3$ of $f_p$. We also set $\delta$ in Eq. (43) to $\delta = 3.5 \times 10^{-5}$ for all cases.

3.5. Results and comparison with wind tunnel experiments

In this subsection, the results of the investigation are shown. At first, time histories of the airspeed and the characteristics for a selected case are presented. Next, the obtained propeller characteristic curves are presented in comparison with the results from wind tunnel experiments.

3.5.1. Time histories

Prior to show the main part of the investigation, a typical transient behavior of the calculated values for a selected case ($\Delta t = 0.0050$ [s], $J_{app} = 0.60$, $\omega_p = 418.900$ [rad/s], $\omega_w = 6.773$ [rad/s]) is presented.

Figures 13–16 show the time histories of the airspeed. The very fast change in $U_a$ mainly results from the rotation of the propeller. On the other hand, the slow change of $U_a$ results from the evolution of the discretized wakes. The range of variation of $U_t$ is 10 times smaller than that of $U_a$ because the range of variation of $u_{th}$ is much smaller than that of $u_{ah}$.

The influence of $u_{aw}$ and $u_{tw}$ on the propeller characteristics seems minor because the ranges of variation in $u_{aw}$ and $u_{tw}$ are very small. In this case, $u_{aw}$ and $u_{tw}$ are in persistent oscillation for $t > 4.7$ [s] (or when the number of laps of the arm exceeds five). This suggests that the wake of the cir-
clinging propeller is fully established and steady.

Figures 17–19 show the time histories of the propeller characteristics. The red and blue lines show the characteristics of each blade, respectively. The black lines show the characteristics of the entire propeller. Tables 4 and 5 show the average and standard deviations, respectively, of the characteristics during the last round of the arm. These results show that the change in characteristics mainly depends on the change in axial airspeed.

Figure 20 shows the time history of the relative error of \( \hat{J}_r \), and Fig. 21 shows that of \( \hat{C}_{Tr} \). The relative error of \( \hat{J}_r \) largely fluctuates at all steps when compared to that of \( \hat{C}_{Tr} \). The termination condition in Eq. (43) is satisfied at \( t = 6.36 \text{[s]} \) in this case. We can confirm \( \hat{C}_{Tr} \) and \( \hat{C}_Q \) are in steady oscillation after this time, from Figs. 18 and 19. This demonstrates that \( \hat{J}_r \) is an appropriate index to evaluate the convergence.

Figures 22 and 23 show the radial distribution of differential aerodynamic forces of each blade. These results confirm that the aerodynamic forces at \( \hat{r}_p = 0.75 \) are dominant for both blades. These results justify selecting the common representative radius for all blades.

3.5.2. Comparison to wind tunnel experiments

Figures 24–32 show the propeller characteristics obtained from the numerical investigation. The plots named “Model corrected” show the plots of the characteristics with respect to \( \hat{J}_r \) (i.e., results from the proposed model). The plots named “Not corrected” show the plots of the characteristics

Table 4. Values of \( \hat{J}_r \), \( \hat{C}_{Tr} \), and \( \hat{C}_Q \).

| Items  | \( \hat{J}_r \) \((\times 10^{-2})\) | \( \hat{C}_{Tr} \) \((\times 10^{-3})\) | \( \hat{C}_Q \) \((\times 10^{-3})\) |
|--------|-----------------------------------|-----------------------------------|-----------------------------------|
| Blade 1 | 0.6208                            | 0.0167                            | 0.0025                            |
| Blade 2 | 0.6179                            | 0.0171                            | 0.0025                            |
| Propeller | 0.6193                            | 0.0340                            | 0.0052                            |

Table 5. Standard deviations of \( \hat{J}_r \), \( \hat{C}_{Tr} \), and \( \hat{C}_Q \).

| Items  | \( \hat{J}_r \) \((\times 10^{-2})\) | \( \hat{C}_{Tr} \) \((\times 10^{-3})\) | \( \hat{C}_Q \) \((\times 10^{-3})\) |
|--------|-----------------------------------|-----------------------------------|-----------------------------------|
| Blade 1 | 5.7278                            | 5.3955                            | 4.4672                            |
| Blade 2 | 5.0776                            | 4.6126                            | 3.6125                            |
| Propeller | 1.0229                            | 2.0097                            | 1.9586                            |
with respect to \( J_{\text{app}} \) (i.e., results ignoring the effect of the surrounding wakes). The horizontal error bars show the standard deviation of \( J_r \) during the last rotation of the arm. The vertical error bars show the standard deviation of the characteristics during the last rotation of the arm. For comparison, these plots also show the results from the wind tunnel experiments by UIUC (Brandt et al.17)). We can see the characteristics obtained through the proposed model keep better consistency with the results of UIUC as compared to the results without correcting the advanced ratio. This tendency was confirmed for all propellers investigated in this study.

The results suggest that the proposed model has a potential to appropriately correct the advance ratio for whirling arm experiments.

3.5.3. Effect of the induced flow models on the propeller characteristic correction

In this work, the induced airspeeds, \( u_h = [u_{\text{va}}, u_{\text{vb}}] \) and \( u_w = [u_{\text{va}}, u_{\text{vb}}] \), are introduced into \( U_h, U_l \) in Eqs. (2) and (3) to correct the propeller characteristics based on the apparent advance ratio.

To make clear the effect of \( u_h \) and \( u_w \), a comparison of calculated propeller characteristics with different types of corrections and wind tunnel experiment data from UIUC\(^{17}\) is shown in Figs. 33–35 (i.e., \( \omega_p = 418.9 \text{[rad/s]}, \Delta t = 4.167 \times 10^{-4} \text{[s]}, \delta = 1.000 \times 10^{-6}; \) the other operating conditions are selected as listed in Table 1 and Table 2).

We can see the effect of \( u_h \) is dominant for these cases. These results are expected from average magnitude of \( u_{\text{va}}, u_{\text{vb}} \) as it is much greater than that of \( u_{\text{va}}, u_{\text{vb}} \) (see Figs. 13–16). The influence of \( u_w \) is significant only when the nondimensional height \( \zeta \) of a blade element is small (e.g., \( \zeta < 7 \) from Fig. 7). Although, \( u_w \) still has a non-negligible effect for these cases.

4. Conclusion

In this work, the authors proposed a model of the velocity field induced by propeller wakes and the swirling flow of the arm using the vortex method. The authors also proposed a procedure for calculating the characteristics of a circling propeller using the proposed model. The appropriate (i.e., non-apparent) advance ratio and the aerodynamic characteristics were determined by using the hybrid blade element momentum theory (HBEMT).

A numerical investigation was performed to evaluate the validity of the model. The results from the proposed model keep better consistency with the wind tunnel data from UIUC experiments as compared to the results without considering the surrounding vortices. This fact suggests that the proposed model will be useful to correct the data of whirling arm experiments and compensate the non-negligible effects resulting from the circling of the propeller.

In future works, the authors will tackle correction of the experimental data using the proposed model.
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