Bimetric structure formation: non-Gaussian predictions

João Magueijo$^1$, Johannes Noller$^1$ and Federico Piazza$^2$

$^1$Theoretical Physics, Blackett Laboratory, Imperial College, London, SW7 2BZ, UK
$^2$Perimeter Institute for Theoretical Physics Waterloo, Ontario, N2L 2Y5, Canada

Abstract

The minimal bimetric theory employing a disformal transformation between matter and gravity metrics is known to produce exactly scale-invariant fluctuations. It has a purely equilateral non-Gaussian signal, with an amplitude smaller than that of DBI inflation (with opposite sign) but larger than standard inflation. We consider non-minimal bimetric models, where the coupling $B$ appearing in the disformal transformation $\tilde{g}_{\mu \nu} = g_{\mu \nu} - B \partial_\mu \phi \partial_\nu \phi$ can run with $\phi$. For power-law $B(\phi)$ these models predict tilted spectra. For each value of the spectral index, a distinctive distortion to the equilateral property can be found. The constraint between this distortion and the spectral index can be seen as a “consistency relation” for non-minimal bimetric models.
1 Introduction

As more and more data pours into cosmology (e.g. [1]) the pressure is on theorists and model-builders to predict signatures that would unmistakably falsify cosmological models. No longer is it good enough to face the “zeroth-order challenge”: that of producing scale-invariant scalar fluctuations with the correct amplitude. Deviations from exact scale-invariance will be detected without controversy (if they do exist) in the near future. Fields tantalizingly beyond our reach – such as gravitational waves (tensor modes) or primordial non-Gaussianity [2] – will hopefully become tangible over the next decade. While inflation [3] has dominated the theoretical cosmology scene, interest has never floundered on alternatives, such as pre-big bang cosmology [4, 5], ekpyrotic/cyclic models [6, 7] and a varying speed of light (VSL) [8, 9, 10]. In this paper we take recent work on VSL [11, 12, 13] one step beyond, investigating departures from strict scale-invariance and non-Gaussian signatures.

Inflationary mechanisms where a varying speed of sound $c_s$ plays a relevant role have already been explored [14, 15]. In [15], it was shown that an adiabatic scale invariant spectrum is produced even if the expansion – albeit still inflationary – is far from exponential (the equation of state can be as far as $w \approx -3/4$ from de-Sitter), provided the speed of sound varies appropriately. A class of contracting (“ekpyrotic”) cosmologies where this mechanism can be applied was also found. Non-gaussianities were calculated in the limit of strict scale invariance ($n_s = 1$), they can be large in both the expanding and the contracting cases. The superluminal phase $c_s > 1$ that we are considering here allows to consider expansions that are not even inflationary, as long as the condition $H^2 \propto c_s$ is satisfied (here $H$ is the Hubble parameter) [11].

Perhaps the most elegant formulation of VSL is in the guise of disformal bimetric theories, for which the speed of gravity differs from the speed of light [16, 17]. In general this is achieved by constructing the Einstein-Hilbert action from an “Einstein” metric $g_{\mu \nu}$ (the Einstein frame), whilst minimally coupling the matter fields to a “matter” metric $\hat{g}_{\mu \nu}$ (the matter frame), with:

$$S = \frac{M_{Pl}^2}{2} \int d^4x \sqrt{-g} R[g_{\mu \nu}] + \int d^4x \sqrt{-\hat{g}} \mathcal{L}_m[\hat{g}_{\mu \nu}, \Phi_{Mat}] + S_{\phi}$$  \hspace{1cm} (1.1)

in which $S_{\phi}$ determines the dynamics. The two metrics are related by a non-conformal transformation, such as:

$$\hat{g}_{\mu \nu} = g_{\mu \nu} - B \partial_\mu \phi \partial_\nu \phi,$$  \hspace{1cm} (1.2)

where $\phi$ is the “bi-scalar” field. Here $B$ is chosen to have dimensions of $M^{-4}$, so that $\phi$ has dimensions of $M$. (In this paper we use metrics with signature $- + + +$, and $B$ is defined so that $B > 0$ corresponds to a speed of light larger than the speed of gravity.) In the most general case $B$ is an arbitrary function of $\phi$, but in the minimal theory it’s set to a constant.

In these theories there are two light cones at any point, one for massless matter particles, another for gravitons. More generally the two metrics may be seen as independent representations of the local Lorentz group (or two non-equivalent tetrads [17]), one valid for gravitons
and the other for matter. Thus, different Lorentz transformations must be used to transform among measurements made with matter and gravity (or equivalently, with clocks and rods operated by matter or gravitational phenomena). For this reason causality paradoxes can be skirted \[13, 18\], in contrast to straightforward tachyonic matter \[19\]. This argument makes the bimetric construction important in interpreting superluminal structure formation models.

A number of dynamics \(S_\phi\) for bimetric theories have been considered. It was pointed out in \[13\] that a Klein-Gordon equation for \(\phi\) in the matter frame translates into DBI dynamics in the gravity frame. Its corresponding Lagrangian, however, is not the Klein-Gordon Lagrangian in the matter frame, but simply a cosmological constant. (It was first noted in \[10\] that for bimetric theories a Klein-Gordon action in the matter frame doesn’t translate into a Klein-Gordon equation in that frame). Thus, the simplest bi-scalar dynamics is generated by

\[
S_\phi = \int \sqrt{-\mathring{g}} (-2\Lambda_m),
\]

and \(\Lambda_m < 0\) leads to a speed of light larger than the speed of gravity. If we require the field \(\phi\) to have Klein-Gordon dynamics in the Einstein frame at low energies (when matter and gravity frames coincide), we should consider additionally:

\[
S_\phi = \int \sqrt{-\mathring{g}} \frac{1}{B} - \int \sqrt{-\mathring{g}} \frac{1}{B}
\]

i.e. a positive cosmological constant in the Einstein frame balanced by a negative one in the matter frame, both with magnitude tuned to \(1/(2B)\). This action maps into the DBI action \[20, 21\] in the gravity frame with DBI coupling \(f = -B\), as explained in \[13\]. For a choice of sign where the speed of light in the gravity frame is larger than one \((f = -B < 0)\), this is sometimes labeled “anti-DBI” (although one should note that “flipping” the sign of \(f\) means that this setup cannot be interpreted as portraying a relativistic probe brane embedded in a five dimensional bulk, as usual for DBI; for an earlier study of anti-DBI theories see also \[22\]). Combined with a mass potential in the gravity frame it leads to scaling solutions and scale-invariant fluctuations \[23, 11\], without the need for accelerated expansion or a contracting pre-Big-Bang phase. The investigation of non-Gaussian features in models where \(B\) is allowed to ”run” is the main purpose of this paper.

In the presence of a speed of sound \(c_s \neq 1\) for adiabatic perturbations, the three-point function contains terms proportional to the power spectrum squared and terms which are further multiplied by a factor \(c_s^{-2}\) \[24, 25, 15\]. In the sub-luminal case \(c_s < 1\), the “\(c_s^{-2}\)” terms dominate. This is what enhances non-gaussianities in DBI inflation and makes the three-point function scale dependent \[26, 27\] in the case of a varying speed of sound \[15\] (the combination \(H^2/c_s\) is set to be constant by the scale invariance of the power spectrum and terms that appear with different powers of \(c_s\) will therefore run with the scale). In the opposite limit, the one of infinite speed of sound that we are considering here, the “\(c_s^{-2}\)” terms are suppressed and the remaining terms inherit the scale invariance from the power spectrum.
of the two point function. The dimensionless quantity $f_{NL}$ is of order 1 and has opposite sign to DBI inflation, i.e. $f_{NL} \sim 1 > 0$ with the WMAP sign convention. While such a small signal is surely observationally challenging, our results are also appealing because of a consistency relation between the three- and two- point functions. In fact, in the $c_s \gg 1$ limit the three point function [eq. (4.15) below] becomes independent of the background parameters (such as the equation of state $w$) and only mildly depends on the tilt $n_s - 1$ of the power spectrum.

The structure of this paper is as follows. In Section 2 we review and extend results of cosmological perturbation theory needed for the calculations in this paper. Then, in Section 3 we explain how scale invariance may be achieved in these models and derive the associated non-Gaussian features. The non-minimal model is spelled out in Section 4 with the basic “Gaussian” predictions presented as well as its non-Gaussian properties. Throughout the paper we refer to two appendices, where we explain the more technical aspects of the calculation. Finally in a concluding section we examine our results from a wider perspective.

2 Cosmological Perturbations

Projecting (1.4) onto the Einstein frame leads to the (anti)-DBI action, which belongs to the general class of k-essence models [28]. Cosmological perturbations have been extensively studied for these models. Here we review the main results, extending them wherever needed. The starting point is an action of the form:

$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{2} + P(X, \phi) \right], \quad (2.1)$$

where the pressure $P$ is a general function of the scalar field $\phi$ and the kinetic term $X = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$. The energy density reads

$$\rho = 2XP_X - P, \quad (2.2)$$

while the speed of sound is given by

$$c_s^2 = \frac{P_X}{\rho_X} = \frac{P_X}{P_X + 2XP_{XX}}. \quad (2.3)$$

In a FRW Universe of scale factor $a(t)$ and Hubble rate $H(t) = \dot{a}/a$ we define the slow-roll parameters as follows:

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \quad \epsilon_s \equiv \frac{\dot{c}_s}{c_s H}, \quad \eta \equiv \frac{\dot{\epsilon}}{\epsilon H}, \quad \eta_s \equiv \epsilon_s \frac{\dot{c}_s}{c_s H}. \quad (2.4)$$

In the $n_s = 1$ scale-invariant case, non-Gaussianity has been calculated in [15]. Here we generalize to the case of an arbitrary – albeit small – tilt and negligible running ($\eta \approx$
$0, \eta_s \approx 0$). The calculation of the three-point function also necessitates defining two further parameters derived from $P(X, \phi)$ [24, 25]

\[
\Sigma = XP_{,X} + 2X^2 P_{,XX} = \frac{H^2 \epsilon}{c_s^2},
\]
\[
\lambda = X^2 P_{,XX} + \frac{2}{3} X^3 P_{,XXX}.
\]

At quadratic order, the action for the curvature perturbation $\zeta$ for general speed of sound models is given by [29]

\[
S_2 = \frac{M_{Pl}^2}{2} \int d^3x d\tau \ z^2 \left[ \left( \frac{d\zeta}{d\tau} \right)^2 - c_s^2 (\vec{\nabla} \zeta)^2 \right],
\]

where $\tau$ is conformal time, and $z$ is defined as usual by $z = a\sqrt{2\epsilon/c_s}$. In [15] it was shown that it is convenient to work in terms of the “sound-horizon” time $dy = c_s d\tau$ instead of $\tau$. Explicitly, when $\eta = \eta_s = 0$,

\[
y = \frac{c_s}{(\epsilon + \epsilon_s - 1)aH}.
\]

It is useful to write the behavior in $y$-time of some relevant quantities:

\[
a \sim (-y)^{\frac{1}{\epsilon_s+1}}; \quad c_s \sim (-y)^{\frac{4}{\epsilon_s+1}}; \quad H \sim (-y)^{\frac{4}{\epsilon_s+1}}.
\]

The quadratic action then takes the form

\[
S = \frac{M_{Pl}^2}{2} \int d^3x dy \ q^2 \left[ \zeta'^2 - (\vec{\nabla} \zeta)^2 \right],
\]

where $' \equiv d/dy$, and

\[
q \equiv \sqrt{c_s z} = \frac{a\sqrt{2\epsilon}}{\sqrt{c_s}}.
\]

Upon quantization the perturbations are expressed through creators and annihilators as follows,

\[
\zeta(y, k) = u_k(y)a(k) + u_k^*(y)a^\dagger(-k).
\]

However, in order to make the correct choice of vacuum, it is useful to refer to the canonically-normalized scalar variable $v = M_{Pl}q\zeta$. Then the equations of motion for the Fourier modes are given by

\[
v''_k + \left( k^2 - \frac{q''}{q} \right) v_k = 0.
\]

It is well-known that this results in a scale-invariant spectrum if $q''/q = 2/y^2$. More generally, we have

\[
\frac{q''}{q} = \frac{1}{y^2} \left( \nu^2 - \frac{1}{4} \right),
\]

where $\nu = \nu(y, k)$.
and the solution for $v_k(y)$ corresponding to the Bunch Davis vacuum is

$$v_k(y) = \frac{\sqrt{\pi}}{2} \sqrt{-y} H_\nu^{(1)}(-ky),$$

(2.15)

where $H_\nu^{(1)}$ are Hankel functions of the first kind. The relation between $n_s$, $\nu$ and $\epsilon$ and $\epsilon_s$ is

$$n_s - 1 = 3 - 2\nu = \frac{2\epsilon + \epsilon_s}{\epsilon_s + \epsilon - 1}.$$  

(2.16)

Again, in the limit when $\nu = 3/2$,

$$v_k(y) = -\frac{1}{\sqrt{2k}} \left(1 - \frac{i}{ky}\right) e^{-iky} \quad (\nu = 3/2)$$

(2.17)

and we recover the scale invariant spectrum. Going back to the modes defined in (2.12) we have

$$u_k(y) = -\frac{\sqrt{\pi}}{2} \frac{v_k(y)}{v_s} = \frac{\sqrt{\pi}}{2} \sqrt{-y} H_\nu^{(1)}(-ky).$$

(2.18)

It is useful to adopt an approximate expression for the Hankel functions. By expanding at $|ky| \ll 1$ we obtain

$$H_\nu^{(1)}(-ky) = -i \frac{2\nu \Gamma(\nu)(-ky)^{-\nu}}{\pi} [1 + iky + O(ky)^2] e^{-iky}.$$  

(2.19)

which gives the following approximate expression for $u_k$,

$$u_k(y) \approx -i \frac{H(\epsilon + \epsilon_s - 1)}{2} \frac{(ky)}{2}^{3/2-\nu} (1 + iky) e^{-iky}.$$  

(2.20)

In order to obtain (2.20), eq. (2.8) has been used, together with $\Gamma(\nu \approx 3/2) \approx \sqrt{\pi}/2$. As expected, $u_k(y) \approx \text{const.}$ in the $y \to 0$ limit. To check this explicitly we use (2.9) and note that

$$H = \frac{1}{c_s^{1/2}} \sim (-y)^{\nu - 3/2}. $$  

(2.21)

The derivative of $u_k(y)$ with respect to $y$ is also easily obtained:

$$u_k'(y) \approx -i \frac{H(\epsilon + \epsilon_s - 1)}{2} \frac{(ky)}{2}^{3/2-\nu} k^2 y e^{-iky}.$$  

(2.22)

Finally, the expression for the $\zeta$ Power Spectrum reads

$$P_\zeta \equiv \frac{1}{2\pi^2} k^3 |\zeta_k|^2 = \frac{(\epsilon_s + \epsilon - 1)^2}{2} \frac{2^{\nu - 3}}{(2\pi)^2 \epsilon} \bar{H}^2 \frac{\bar{H}^2}{c_s M_{Pl}^2},$$

(2.23)

where the bar symbol means that the corresponding quantity has to be evaluated, for each mode $k$, at sound horizon exit, i.e., when $y = k^{-1}$.

\footnote{In fact, $\Gamma(\nu) = \sqrt{\pi}/2 [1 + 0.036(\nu - 3/2) + \ldots]$.}
3 The Scale Invariant limit

For the bimetric theories discussed in the introduction, in the Einstein frame the action takes the form:

\[ P(X, \phi) = -f^{-1}(\phi)\sqrt{1 - 2f(\phi)X} + f^{-1}(\phi) - V(\phi) \] (3.1)

where \( X = -\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi \). Scaling solutions of this action have been studied in [30, 15, 19]. In particular, a scale invariant spectrum of primordial perturbations is produced if

\[ f(\phi) = \frac{-3}{V_0} \left[ \frac{1}{4} - \frac{1}{\epsilon^2} \left( \frac{\phi}{M_P} \right)^{-4} \right] \] (3.2)

\[ V(\phi) = V_0 \left( \frac{\phi}{M_{Pl}} \right)^2 \left( 1 - \frac{2\epsilon}{3} \frac{1}{1 + \frac{\epsilon^2 \phi^2}{8M_{Pl}^2}} \right) \]
\[ \simeq V_0 \left[ \left( \frac{\phi}{M_P} \right)^2 - \frac{4}{3} + \ldots \right], \] (3.3)

where we have made use of the slow-roll parameter \( \epsilon = -\dot{H}/H^2 \). In the large \( \phi \) limit (which also corresponds to the large \( c_s \) limit) we recover the form discussed in [13]: \( f(\phi) \simeq -B < 0 \), \( V(\phi) \sim \phi^2 \). This corresponds in general to \( c_s \propto \rho \). In such a strict \( c_s \to \infty \) limit the amplitude of the three point function \( \mathcal{A} \) can be read straightforwardly from (B.12) (we refer the reader to Appendices A and B for the more gruesome technical details). Comparing with the cubic effective action (A.1) we find that only the \( \mathcal{A}_{\zeta^2} \) and \( \mathcal{A}_{(\partial \zeta)^2} \) terms are not subdominant as \( c_s \to \infty \). The resulting total amplitude is independent of the parameters \( (w \text{ or } \epsilon) \) and reads

\[ \mathcal{A}_{c_s \to \infty} = -\frac{1}{8} \sum_i k_i^3 + \frac{1}{K} \sum_{i<j} k_i^2 k_j^2 - \frac{1}{2K^2} \sum_{i \neq j} k_i^2 k_j^3. \] (3.4)

This is precisely the equilateral shape, peaking for \( k_1 = k_2 = k_3 \), that is also obtained in the scaling solutions considered in [15] in the \( \epsilon \to 0 \), \( \alpha \to 0 \) limit. More specifically one obtains

\[ \mathcal{A}_{\epsilon \to 0} = \left( 1 - \frac{1}{c_s^2} \right) \mathcal{A}_{c_s \to \infty} + \mathcal{O}(n_s - 1), \] (3.5)

This shows that in the minimal bimetric model the dimensionless quantity \( f_{NL} \) is of order 1 and has opposite sign to DBI inflation, i.e. \( f_{NL} \sim 1 > 0 \). Thus the model is quite distinct in this respect to standard inflation (for which \( f_{NL} \sim \epsilon \sim 0.1 \) and DBI inflation (for which \( f_{NL} \sim -100 \) is a distinct possibility.) Notice there’s been some confusion [21], both among theorists and observers, regarding the sign of \( f_{NL} \). Here we adopt the convention used by WMAP, where positive \( f_{NL} \) physically corresponds to negative-skewness for the temperature fluctuations: we assign a negative \( f_{NL} \) to DBI inflation, so that \( f_{NL} > 0 \) for the anti-DBI models under consideration.
4 Beyond the minimal model

It could be that the parameter $B$ appearing in the disformal transformation (1.2) is itself a function of $\phi$. In this Section we show that non-minimal theories with power-law $B(\phi)$ lead to tilted spectra, without running. Naturally, more complicated $B(\phi)$ would lead to more complex spectra, so one can’t say that absence of running is a general feature of these models.

First we note that the speed of sound in the gravity frame is:

$$c_s^2 = \frac{K_{,X}}{K_{,XX}} = 1 + 2BX$$  \hspace{1cm} (4.1)

whereas the density and pressure are:

$$\rho = 2XK_{,X} - p = \frac{1}{B} \left( 1 - \frac{1}{c_s} \right) + V$$  \hspace{1cm} (4.2)
$$p = K - V = \frac{1}{B} (c_s - 1) - V$$  \hspace{1cm} (4.3)

Thus scaling solutions may be obtained for a variety of potentials, with the property that

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{3}{2} (1 + w)$$  \hspace{1cm} (4.4)
$$\epsilon_s = \frac{\dot{c}_s}{c_s H}$$  \hspace{1cm} (4.5)

are constant. As explained in the Appendix A of reference [15] the constancy of $\epsilon$ and $\epsilon_s$ permits a simple integration into $c_s = c_s(\phi)$ and $H = H(\phi)$. The Friedmann equations then give a solution for $V(\phi)$ and $f(\phi)$.

$$V = V_0 \left( \frac{\phi}{M_{Pl}} \right)^{-4\epsilon/\epsilon_s} \left( 1 - \frac{2\epsilon}{3} \frac{1}{1 + \epsilon_s^2 \phi^2} \right)$$  \hspace{1cm} (4.6)

$$= V_0 \left( \frac{\phi}{M_{Pl}} \right)^{-4\epsilon/\epsilon_s} \left[ 1 - \frac{16\epsilon^2 M_{Pl}^2}{3\epsilon_s^2 \phi^2} + O \left( \frac{\phi}{M_{Pl}} \right)^{-2} \right]$$  \hspace{1cm} (4.7)

$$f(\phi) = \frac{12}{V_0 \epsilon_s^2} \left( \frac{\phi}{M_{Pl}} \right)^{\frac{4\epsilon}{\epsilon_s} - 2} \left( 1 - \frac{\epsilon_s^4 \phi^4}{64\epsilon^2 M_{Pl}^4} \right)$$  \hspace{1cm} (4.8)

$$= -\frac{3\epsilon_s^2}{16\epsilon V_0} \left( \frac{\phi}{M_{Pl}} \right)^{2 + \frac{4\epsilon}{\epsilon_s}} \left[ 1 + O \left( \frac{\phi}{M_{Pl}} \right)^{-4} \right]$$  \hspace{1cm} (4.9)

Using $B = -f > 0$ and in the limit $c_s \gg 1$ we therefore obtain

$$V = V_0 \left( \frac{\phi}{M_{Pl}} \right)^{-\frac{4\epsilon}{\epsilon_s}}$$  \hspace{1cm} (4.10)
Following the calculation in [11] we find that for these solutions the spectral index is:

\[ n_S - 1 = \frac{\epsilon_s + 2\epsilon}{\epsilon_s + \epsilon - 1}. \]  

(4.12)

Whilst scale-invariance is associated with the universal law

\[ c_s \propto \rho \]  

(4.13)

and also \( f = 1 \) and a quadratic potential for all equations of state, the same doesn’t happen if we depart from scale-invariance. Indeed exact scale invariance requires \( \epsilon_s = 2\epsilon \) so that these parameters fall out of conditions for the spectral index; but this doesn’t happen as soon as \( n_s \neq 1 \). Note, however, that any scaling solution has:

\[ c_s(\phi) = \frac{\epsilon_s^2}{8\epsilon} \phi^2 \]  

(4.14)

an expression that will be essential in evaluating non-Gaussianities.

In Appendix B we present expressions for the Non-Gaussian amplitude \( A \) for general \( c_s \) profiles. Our calculations there consequently apply to both subluminal and superluminal cases, as long as scaling solutions with constant \( \epsilon \) and \( \epsilon_s \) are considered. Once a solution for \( c_s \) is provided, these uniquely determine the Non-Gaussian signature. However, here we exclusively focus on the \( c_s \to \infty \) case relevant in the bimetric context. Considering further the relation (4.14), fixing \( c_s(\phi) \), we have that in the large \( \phi \) limit the appropriate Non-Gaussian amplitude to compute is still \( A_{c_s \to \infty} \).

Combining terms from the general results of Appendix B, eqs. (B.12) and (B.13) and taking the small tilt \( (n_s - 1 \ll 1) \) and \( c_s \to \infty \) limits, we find

\[
A = \left( \frac{k_1 k_2 k_3}{2K^3} \right)^{n_s-1} \left[ -\frac{1}{8} \sum_i k_i^3 + \frac{1}{K} \sum_{i<j} k_i^2 k_j^2 - \frac{1}{2K^2} \sum_{i\neq j} k_i^2 k_j^3 
+ (n_s - 1) \left( -\frac{1}{8} \sum_i k_i^3 - \frac{1}{8} \sum_{i\neq j} k_i^2 k_j^2 + \frac{1}{8} k_1 k_2 k_3 + \frac{1}{2K} \sum_{i<j} k_i^2 k_j^2 - \frac{1}{2K^2} \sum_{i\neq j} k_i^2 k_j^3 \right) 
+ \mathcal{O}\left( \frac{1}{\epsilon_s^2} \right) \right],
\]

(4.15)

where the only dependence on \( \epsilon \) and \( \epsilon_s \) appears either in the “observable” combination \( n_s - 1 \) or in the subleading \( \mathcal{O}(1/\epsilon_s^2) \) terms. Upon approaching scale invariance only the first line inside the square brackets stays relevant, ensuring that \( A \) reduces to the equilateral amplitude (3.4) as required.
Figure 1: We plot the non-Gaussian amplitude from Eq. (4.15) $-\mathcal{A}(1, x_2, x_3)/(x_2 x_3)$ for $n_s = 1$ (left) and $n_s = 0.96$ (right).

The amplitude (4.15) is plotted in Figure 1 and peaks in the equilateral limit $k_1 = k_2 = k_3$. In the local limit $k_1 \ll k_2, k_3$, on the other hand, the first line inside the square brackets of Eq. (4.15) goes to zero. In agreement with the consistency relation [31, 32] we then have

$$\mathcal{A}_{k_1 \ll k_2, k_3} \approx -\frac{1}{2} (n_s - 1) \left( \frac{k_1}{k_2} \right)^{n_s - 1}$$ (4.16)

The predictive power of our result lies in establishing a consistency relationship between $n_s$ and $\mathcal{A}$. In fact, we find a distinctive Non-Gaussian signal for any given spectral index $n_s$. Whilst the overall non-Gaussian amplitude $\mathcal{A}$ still peaks in the equilateral limit $k_1 = k_2 = k_3$ in both red- and blue-tilted cases, its shape is modified when compared with the scale invariant limit. Illustrating this point, Figure 1a shows $\mathcal{A}$ for the exactly scale invariant case $n_s = 1$, whereas in Figure 1b we plot $\mathcal{A}$ for a red-tilted power spectrum with $n_s = 0.96$. Specifically we find an $\mathcal{A}(n_s = 0.96) = \mathcal{A}(n_s = 1) + \Delta \mathcal{A}$, where $\Delta \mathcal{A}$ is approximately one order of magnitude smaller than $\mathcal{A}(n_s = 1)$.

5 Conclusions

Strict scale-invariance has been associated with superluminal bimetric models, where the speed of light is larger than the speed of gravity in the early Universe [11, 13]. Indeed this is a feature of the minimal bimetric model, but in this paper we showed how tilted spectra, red or blue, could be generated by a non-minimal bi-scalar coupling $B(\phi)$. At first this might suggest we’ve fallen into the “theory of anything” trap, but it’s not the case. A unique non-Gaussian shape is predicted for any value of the spectral index, with distinct
distortions away from the scale-invariant equilateral shape appearing for each of the tilted cases. These distortions can be seen as “consistency conditions” for this class of models. This is particularly relevant given the absence of gravitational waves for all bimetric models of this kind. (Note that these models solve the horizon problem for matter but not for gravity, so tensor modes don’t start their lives inside the horizon.)

One might wonder where the proposed running coupling $B(\phi)$ comes from. First note that we don’t need the full (anti-)DBI action (3.1) resulting from (1.4), unless we impose Klein-Gordon dynamics for $\phi$ in the Einstein frame at low energies. This may not be necessary, and if we relax this requirement all we need is (1.3), i.e. a negative cosmological constant $\Lambda_m$ in the matter frame (which, we stress, does not lead to an AdS solution). In fact, if we relax the low-energy requirement, $\Lambda_m$ doesn’t even need to be related to $B$. If, however, we do insist on Klein-Gordon dynamics for $\phi$ in the Einstein frame at low energies, then the negative matter frame cosmological constant should be exactly balanced by a positive Einstein frame cosmological constant, and their common magnitude should be $1/(2B)$.

A number of interesting theoretical connections can be made. In the context of emergent geometry, it’s been pointed out that different emergent metrics may apply to bosons and fermions [33]. The fact that the vacuum energy is negative for fermions and positive for bosons suggests an action of the proposed form, with a speed of light larger than the speed of gravity (i.e. an anti-DBI action in the Einstein frame). Also these models become asymptotically a cuscaton [23] model, a feature that may be used to support the view that they are a UV-complete alternative to inflation. Finally, it is possible that this construction results from an entirely different set up, such as deformed dispersion relations [34]. It is interesting that the dispersion relations needed for scale-invariance are of the same form as those discussed in the context of Horava-Lifshitz theory [35]. More generally a connection with deformed special relativity remains to be fully explored [36, 37]. Absence of exact scale-invariance could then be a major clue into the foundations of these theories.

While work on these theoretical ramifications is an interesting motivation, and should be pursued further, in this paper we focused on the phenomenology of these models. Measuring the shape of the three-point correlator (as opposed to a quantity as muddled as $f_{NL}$) poses an interesting observational challenge. The fact that the matter appears coupled to the measurement of $n_S$ makes these models an interesting target for future experimental work.

6 Acknowledgments

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A Appendix: The cubic action

It is useful to report here the cubic effective action derived in \[24, 25\]. The result is valid outside of the slow-roll approximation and for any time-dependent sound speed:

\[
S_3 = M_{Pl}^2 \int d^3 x \left\{ -a^3 \left[ \Sigma \left( 1 - \frac{1}{c_s^2} \right) + 2 \lambda \right] \frac{\dot{\zeta}^2}{H^3} + a^3 \epsilon (\epsilon - 3 + 3c_s^2) \zeta \dot{\zeta}^2 \\
+ \frac{a \epsilon}{c_s^2} (\epsilon - 2\epsilon_s + 1 - c_s^2) \zeta (\partial \zeta)^2 - 2a \frac{\epsilon}{c_s^2} \dot{\zeta} (\partial \zeta)(\partial \chi) \\
+ \frac{a^3 \epsilon}{2c_s^2} \frac{d}{dt} \left( \frac{\eta}{c_s^2} \right) \zeta^2 \dot{\zeta} + \frac{\epsilon}{2a} (\partial \zeta)(\partial \chi) \partial^2 \chi + \frac{\epsilon}{4a} (\partial^2 \zeta)(\partial \chi)^2 + 2f(\zeta) \frac{\delta L}{\delta \zeta} \right\},
\]

(A.1)

where dots denote derivatives with respect to proper time \(t\), \(\partial\) is a spatial derivative, and \(\chi\) is defined as

\[
\partial^2 \chi = \frac{a^2 \epsilon}{c_s^2} \dot{\zeta}.
\]

(A.2)

Meanwhile, in the last term \(\delta L_{\zeta}^{\chi} \mid_1\) denotes the variation of the quadratic action with respect to the perturbation \(\zeta\):

\[
\frac{\delta L}{\delta \zeta} \mid_1 = a \left( \frac{\partial^2 \chi}{\partial t} + H \partial^2 \chi - \epsilon \partial^2 \zeta \right),
\]

(A.3)

\[
f(\zeta) = \frac{\eta}{4c_s^2} \zeta^2 + \frac{1}{c_s^2 H} \zeta \dot{\zeta} + \frac{1}{4a^2 H^2} [-(\partial \zeta)(\partial \zeta) + \partial^{-2}(\partial_i \partial_j (\partial_i \zeta \partial_j \zeta))] \\
+ \frac{1}{2a^2 H} [(\partial \zeta)(\partial \chi) - \partial^{-2}(\partial_i \partial_j (\partial_i \zeta \partial_j \chi))],
\]

(A.4)

where \(\partial^{-2}\) is the inverse Laplacian. Since \(\delta L_{\zeta}^{\chi} \mid_1\) is proportional to the linearized equations of motion, it can be absorbed by a field redefinition

\[
\zeta \rightarrow \zeta_n + f(\zeta_n).
\]

(A.5)

B Appendix: The three-point function

The three point function can be calculated by following the same method of \[15\] and generalizing it. The standard calculation \[31, 24, 25\], at first order in perturbation theory and in the interaction picture, leads to

\[
\langle \zeta(t, k_1) \zeta(t, k_2) \zeta(t, k_3) \rangle = -i \int_{t_0}^t dt' \langle \zeta(t, k_1) \zeta(t, k_2) \zeta(t, k_3), H_{\text{int}}(t') \rangle,
\]

(B.1)

where \(H_{\text{int}}\) is the Hamiltonian evaluated at third order in the perturbations and is directly derivable from \(\langle A.1 \rangle\) and vacuum expectation values are evaluated w.r.t. the interacting
vacuum $|\Omega\rangle$. By using (2.12) and applying the commutation relations $[a(k), a^\dagger(k')] = (2\pi)^3\delta^3(k - k')$, we can calculate the three point function for each term appearing in the action (A.1). It is useful to follow in detail the calculation for the $^{\zeta\zeta^2n}$ piece. We have:

$$
\langle \zeta(k_1)\zeta(k_2)\zeta(k_3) \rangle_{\zeta^2} = i(2\pi)^3\delta^3(k_1 + k_2 + k_3)u_{k_1}(y_{\text{end}})u_{k_2}(y_{\text{end}})u_{k_3}(y_{\text{end}})
\times \int_{-\infty+i\varepsilon}^{y_{\text{end}}} dy\frac{c_s a^3\epsilon}{a_k^4}(\epsilon - 3 + 3c_s^2)u_{k_1}'(y)\frac{d u_{k_2}'(y)}{d y} + \text{perm. + c.c.} \quad (B.2)
$$

The subscript “end” means that the quantity has to be evaluated at the end of “inflation”. We now substitute (2.20) and use (2.21) to take some time-independent combinations outside the integral and evaluate them at $y = y_{\text{end}}$:

$$
\langle \zeta(k_1)\zeta(k_2)\zeta(k_3) \rangle_{\zeta^2} = i(2\pi)^3\delta^3(k_1 + k_2 + k_3)\frac{H_{\text{end}}^6(1 - \epsilon - \epsilon_s)^62^{\nu - 9} (k_1k_2k_3)^{3-2\nu}}{4^3 M_{\text{Pl}}^4\epsilon^2 c_s^3|y_{\text{end}}|^6(1-\nu)} \Pi_j |k_j^3|
\times \int_{-\infty+i\varepsilon}^{y_{\text{end}}} dy(\epsilon - 3 + 3c_s^2)\frac{a^2}{c_s^3}(1 - ik_1y)k_2^2k_3^2y^2 e^{ik_y} + \text{perm. + c.c.} \quad (B.3)
$$

where we have dropped a factor of $\Pi_j (1 + ik_j y_{\text{end}}) e^{-ik_y}$ as this will be negligibly small in the limit $|k y| << 1$ where the truncated Hankel function expansion (2.19) is valid.

By using (2.9) we can finally express the time dependent quantities inside the integrals as power-laws in $y$. It is useful to report some of the basic results of [15] for integrals of this type. By calling

$$
C = \int_{-\infty+i\varepsilon}^{y_{\text{end}}} dy \left(\frac{y}{y_{\text{end}}}\right)^\gamma (-iy)^n e^{ik_y} \quad (B.4)
$$

For $\gamma + n > -2$ the imaginary part of (B.4) is convergent as $y_{\text{end}} \to 0$. In this case we can approximately extend the upper limit of integration to 0, which amounts to neglecting terms of higher order in $(k|y_{\text{end}}|)$. We thus obtain

$$
\text{Im} C = -(K|y_{\text{end}}|)^{-\gamma}\cos \frac{\gamma\pi}{2}\Gamma(1 + \gamma + n)K^{-n-1} \quad (B.5)
$$

The two types of behavior that we encounter are, in particular,

$$
\frac{a^2 y^2}{c_s} = \frac{c_s}{(1 - \epsilon - \epsilon_s)^2H^2} = \frac{c_s|y_{\text{end}}|}{(1 - \epsilon - \epsilon_s)^2 H_{\text{end}}^2} \left(\frac{y}{y_{\text{end}}}\right)^{\alpha_1} \quad (B.6)
$$

$$
\frac{a^2 y^2}{c_s^3} = \frac{1}{(1 - \epsilon - \epsilon_s)^2H^2c_s} = \frac{1}{(1 - \epsilon - \epsilon_s)^2 H_{\text{end}}^2 c_s|y_{\text{end}}|} \left(\frac{y}{y_{\text{end}}}\right)^{\alpha_2} \quad (B.7)
$$

where

$$
\alpha_1 = n_s - 1 = 3 - 2\nu = \frac{2\epsilon + \epsilon_s}{\epsilon_s + \epsilon - 1} \quad (B.8)
$$

$$
\alpha_2 = \frac{2\epsilon - \epsilon_s}{\epsilon_s + \epsilon - 1} \quad (B.9)
$$

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By using the above formulas and re-expressing everything in terms of quantities calculated at sound horizon crossing \(i.e. \text{when, by convention, } y = K^{-1}\) we finally obtain

\[
\langle \zeta(k_1) \zeta(k_2) \zeta(k_3) \rangle_{\zeta \zeta^2} = (2\pi)^3 \delta^3(k_1 + k_2 + k_3) \frac{\bar{H}^4 (\epsilon + \epsilon_s - 1)^{42^{6\nu - 9}}}{16 M_P^4 \epsilon^2 \ell_s^4 \Pi_j k_j^3} \frac{1}{K^9 - 6\nu} \times \frac{k_1^2 k_2 k_3}{K} \left\{ (\epsilon - 3) \cos \frac{\alpha \pi}{2} \Gamma(1 + \alpha_2) \left[ 1 + (1 + \alpha_2) \frac{k_1}{K} \right] + 3c_s^2 \cos \frac{\alpha_1 \pi}{2} \Gamma(1 + \alpha_1) \left[ 1 + (1 + \alpha_1) \frac{k_1}{K} \right] \right\} + \text{sym.}
\] (B.10)

The three point function is conveniently expressed, after factoring out appropriate powers of the power spectrum, through the amplitude \(\mathcal{A}\),

\[
\langle \zeta(k_1) \zeta(k_2) \zeta(k_3) \rangle = (2\pi)^7 \delta^3(k_1 + k_2 + k_3) P_\zeta \frac{1}{\Pi_j k_j^3} \mathcal{A}.
\] (B.11)

Again, by convention, the power spectrum \(P_\zeta\) in the above formula is calculated for the mode \(K\). By following the same steps above for each of the terms in the action \(A.1\) we obtain

\[
\mathcal{A}_{\zeta \zeta^2} = \frac{1}{4 \epsilon_s^2} \left( \frac{k_1 k_2 k_3}{2 K^3} \right)^{3 - 2\nu} \left[ (\epsilon - 3) I_{\zeta \zeta^2} (\alpha_2) + 3c_s^2 I_{\zeta \zeta^2} (\alpha_1) \right];
\]

\[
\mathcal{A}_{\zeta (\partial \zeta)^2} = \frac{1}{8 \epsilon_s^2} \left( \frac{k_1 k_2 k_3}{2 K^3} \right)^{3 - 2\nu} \left[ (\epsilon - 2\epsilon_s + 1) I_{\zeta (\partial \zeta)^2} (\alpha_2) - c_s^2 I_{\zeta (\partial \zeta)^2} (\alpha_1) \right];
\]

\[
\mathcal{A}_{\zeta \partial \zeta \partial \chi} = \frac{1}{4 \epsilon_s^2} \left( \frac{k_1 k_2 k_3}{2 K^3} \right)^{3 - 2\nu} \left[ -\epsilon I_{\zeta \partial \zeta \partial \chi} (\alpha_2) \right];
\]

\[
\mathcal{A}_{\zeta^2} = \frac{1}{16 \epsilon_s^2} \left( \frac{k_1 k_2 k_3}{2 K^3} \right)^{3 - 2\nu} \left[ \epsilon^2 I_{\zeta^2} (\alpha_2) \right],
\] (B.12)

where

\[
I_{\zeta \zeta^2} (\alpha) = \cos \frac{\alpha \pi}{2} \Gamma(1 + \alpha) \left[ (2 + \alpha) \frac{1}{K} \sum_{i < j} k_i^2 k_j^2 - (1 + \alpha) \frac{1}{K^2} \sum_{i \neq j} k_i^2 k_j^3 \right];
\]

\[
I_{\zeta (\partial \zeta)^2} (\alpha) = - \cos \frac{\alpha \pi}{2} \Gamma(1 + \alpha) \left( \sum_i k_i^2 \right) \left[ \frac{K}{\alpha - 1} + \frac{1}{K} \sum_{i < j} k_i k_j + \frac{1 + \alpha}{K^2} k_1 k_2 k_3 \right] \left[ \frac{K}{\alpha - 1} + \frac{1}{K} \sum_{i < j} k_i k_j + \frac{1 + \alpha}{K^2} k_1 k_2 k_3 \right] \left[ \frac{K}{\alpha - 1} + \frac{1}{K} \sum_{i < j} k_i k_j + \frac{1 + \alpha}{K^2} k_1 k_2 k_3 \right]
\]

\[
= \cos \frac{\alpha \pi}{2} \Gamma(1 + \alpha) \left[ \frac{1}{1 - \alpha} \sum_j k_j^3 + \frac{4 + 2\alpha}{K} \sum_{i < j} k_i^2 k_j^2 - \frac{2 + 2\alpha}{K^2} \sum_{i \neq j} k_i^2 k_j^3 \right] \; \left( \frac{1}{1 - \alpha} \sum_j k_j^3 - \alpha k_1 k_2 k_3 \right);
\]

\[
= \frac{\alpha}{(1 - \alpha)} \sum_{i \neq j} k_i^2 k_j^2 - \alpha k_1 k_2 k_3 \right]\; ;
\]

\[
= \frac{\alpha}{(1 - \alpha)} \sum_{i \neq j} k_i^2 k_j^2 - \alpha k_1 k_2 k_3 \right]\; ;
\]
\[ I_{\zeta\partial\zeta\partial\chi}(\alpha) = \cos \frac{\alpha \pi}{2} \Gamma(1 + \alpha) \left[ \sum_j k_j^3 + \frac{\alpha - 1}{2} \sum_{i \neq j} k_i k_j^2 - \frac{1 + \alpha}{K^2} \sum_{i \neq j} k_i^2 k_j^3 - 2\alpha k_1 k_2 k_3 \right] ; \]

\[ I_{\epsilon^2}(\alpha) = \cos \frac{\alpha \pi}{2} \Gamma(1 + \alpha)(2 + \alpha/2) \left[ \sum_j k_j^3 - \sum_{i \neq j} k_i k_j^2 + 2k_1 k_2 k_3 \right] . \] (B.13)

In the above, \( A_{\epsilon^2} \) accounts for the \( \partial\zeta\partial\chi\partial^2\chi \) and \( (\partial^2\zeta)(\partial\chi)^2 \) terms and the first term in (A.1) has not been considered because it is identically null in the case of a DBI-type action.

One should note that, in contrast to the \( c_s \to \infty \) limit relevant in the bimetric context, in the subluminal case \( (c_s < 1) \), it is in fact the “\( c_s^{-2} \)” terms that dominate.

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