ESTIMATION OF NORMAL DISTRIBUTION PARAMETERS AND ITS APPLICATION TO CARBONATION DEPTH OF CONCRETE GIRDER BRIDGES

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ABSTRACT. Taking carbonation depth uncertainty into account is key to approach durability analysis of concrete girder bridges in a probabilistic way. The Normal distribution has been widely used to represent the probability distribution of carbonation depth. In this study, two new methods such as Least Squares method and Bayesian Quantile method, are used to estimate the parameters of the Normal distribution. These two considered methods are also compared with the commonly used Maximum Likelihood method via an extensive numerical simulation and three real carbonation depth data examples based on performance measures such as, K-S test, RMSE and $R^2$. The numerical study reveals that the Least Squares method is the best one for estimating the parameters of the Normal distribution. Statistical analysis of real carbonation depth data sets are presented to demonstrate the applicability and the conclusion of the simulation results.

1. Introduction. In reinforced concrete bridge structures, the carbonation depth is a key deterioration factor to determine the durability of the concrete structures. The characterization of carbonation depth is essential for the carbonation reliability analysis of concrete girder bridges. Carbonation depth are usually employed in carbonation service life prediction of existing concrete girder bridges as deterministic coefficients. Nevertheless, several studies have demonstrated that these type of carbonation depth measurements are characterized by significant dispersion [9, 12]. A basic task in carbonation service life prediction problem is to fit probability distribution to samples of carbonation depth.

The carbonation depth variation is usually described using the so-called Normal distribution that is the most widely used and accepted in the specialized literature. The fact that the carbonation depth follow a Normal distribution have been proved by [9] and [12]. In recent years, Normal distribution has been commonly used, accepted and recommended distribution in literature to express the carbonation depth probability distribution, considering its flexibility and simplicity, as well as it being a good fit to carbonation depth data [5, 6, 14–18, 20, 21, 28–31].

The parameter estimation of Normal distribution for application of carbonation depth is crucial because the distribution function fitting more accurately the carbonation depth data will reduce the uncertainties in carbonation service life

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prediction of concrete girder bridges. The most common method used for the estimation of parameters of Normal distribution is the Maximum Likelihood (ML), see [3]. Besides, Bayesian Estimation method [1,2,8,11,13,22,23] and Least Squares method [4,7,10,26] are also frequently used for estimating parameters, and numerous applications can be found in the literature. These two methods remain relatively unknown in parameter estimation of Normal distribution, though they provide satisfactory and frequently superior results compared with the most commonly used estimation method.

The main scope of this study is to identify the most efficient estimators among three estimators for Normal carbonation depth distribution applications. The remainder of the paper is organized as follows. In section 2, a short introduction of the Normal distribution is described. In section 3, analytic presentation of the theoretical background of the examined parameter estimation methods is presented. In section 4, the performances of all methods are compared via detailed simulation study. In section 5, mentioned parameter estimation methods are applied to three real data sets of carbonation depth of concrete girder bridges. Finally, the main conclusions of this study are summarized in the last section.

2. Normal distribution. Several researchers have devoted to develop an adequate statistical model to describe carbonation depth distribution. The Normal function is commonly used for fitting the measured carbonation depth probability distribution. [9] and [12] claim that Normal probability distribution function is the best one to describe the variation in carbonation depth.

A random variable $x$ has a Normal distribution if its probability density function is defined by Eq. (1). The probability density function (pdf) of Normal distribution can be written as

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2} \left(\frac{x-\mu}{\sigma}\right)^2\right]$$  \hspace{1cm} (1)

where $\mu$ is the location parameter and $\sigma$ is the scale parameter.

The cumulative distribution function (cdf) of the Normal distribution is given by

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2} \left(\frac{t-\mu}{\sigma}\right)^2\right] dt$$  \hspace{1cm} (2)

The unknown parameters can be determined with several different methods.

3. Estimation methods. In this section the three considered estimation methods were described to obtain the estimates of the parameters $\mu$ and $\sigma$ of the Normal distribution.

3.1. Maximum likelihood method. Let $x_1, x_2, \ldots, x_n$ be a random sample of size $n$ from the Normal distribution with parameters $\mu$ and $\sigma$ with pdf given by Eq. (1).

The maximum likelihood function is given by

$$L(\mu, \sigma) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2} \left(\frac{x_i-\mu}{\sigma}\right)^2\right]$$  \hspace{1cm} (3)
and its logarithmic form is
\[ \ln L(\mu, \sigma) = -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 \] (4)

The maximum likelihood estimates for \( \mu \) and \( \sigma \) can be obtained from the likelihood function by solving the following equations
\[ \frac{\partial \ln L(\mu, \sigma)}{\partial \mu} = 0 \] (5)
and
\[ \frac{\partial \ln L(\mu, \sigma)}{\partial \sigma} = 0 \] (6)

Thus, the maximum likelihood estimates \( \hat{\mu} \) and \( \hat{\sigma} \), of \( \mu \) and \( \sigma \), respectively, are given by
\[ \hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} x_i \] (7)
and
\[ \hat{\sigma} = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2} \] (8)
where \( \bar{x} \) is the sample mean.

3.2. **Bayes Quantile method.** The Bayes Quantile method is another technique commonly used in parameter estimation. In the Bayesian setting, we assume that the prior probability density function is the Jeffrey’s non-informative prior \( \pi(\mu, \sigma) = \frac{1}{\sigma} \) under the mean square error loss function. We are interested in obtaining Bayesian estimates of a quantile \( x_\alpha \), subject to a desired or specified percentage. If two quantile values at the two different percentages are determined, parameters \( \mu \) and \( \sigma \) can be easily obtained. The Bayesian estimates method are presented next.

3.2.1. **Quantile model.** By taking inverse function of the cumulative distribution function of standard Normal distribution and solving for \( x \) we obtain the expression for the quantile \( x_\alpha \) under the desired percentage \( \alpha \) given by
\[ x_\alpha = \mu + \Phi^{-1}(\alpha)\sigma \] (9)
where \( \Phi^{-1}(\bullet) \) is the inverse of the cumulative standard Normal probability distribution function.

3.2.2. **MCMC method.** To begin with, a Markov chain is constructed by sampling. The limiting distribution of the Markov chain is the actual distribution \( f(\theta) \) of a parameter \( \theta \). With the convergent Markov chain, parameters of the quantile model are then estimated. The Gibbs sampling, which is one of the most popular MCMC algorithms in the Bayesian analysis [27], is adopted herein to facilitate the calculation. The greatest advantage of MCMC method is the combination of the Monte Carlo integration with the effective sampling technique. This method makes use of dynamic computer simulation technique rather than complex calculation to acquire the optimal solution.
We now consider the parameter estimation procedure by illustrating the principle of the Gibbs sampling algorithm. The Gibbs sampling algorithm may be summarized as follows.

Set \( \theta = (\mu, \sigma) \) for the Normal model, and set the initial values \( \theta^{(0)} \). Then, for \( t = 1, \ldots, T \).

**Step 1.** Let \( \theta = \theta^{(t-1)} \).

**Step 2.** For \( j = 1, \ldots, J \), update \( \theta_j \) from \( \theta_j \sim f(\theta_j|\theta_{\neq j},X) \), where \( \theta_{\neq j} = (\theta_1, \ldots, \theta_{j-1}, \theta_{j+1}, \ldots, \theta_J) \).

**Step 3.** Set \( \theta^{(t)} = \theta \) for saving the generated set of values at \( t + 1 \) iteration of the algorithm.

Hence, given a particular state of the Markov chain \( \theta^{(t)} \), we generate the new parameter values in Step 2 by

\[
\begin{align*}
\theta_1^{(t)} & \sim f(\theta_1|\theta_2^{(t-1)}, \theta_3^{(t-1)}, \ldots, \theta_J^{(t-1)}, X), \\
\theta_2^{(t)} & \sim f(\theta_2|\theta_1^{(t-1)}, \theta_3^{(t-1)}, \ldots, \theta_J^{(t-1)}, X), \\
& \quad \ldots \\
\theta_J^{(t)} & \sim f(\theta_J|\theta_1^{(t-1)}, \theta_2^{(t-1)}, \ldots, \theta_{J-1}^{(t-1)}, X),
\end{align*}
\]

So far, an iteration of the Gibbs algorithm has been completed, and the parameter set \( \theta^{(t-1)} \) has been updated as \( \theta^{(t)} = (\theta_1^{(t)}, \theta_2^{(t)}, \ldots, \theta_J^{(t)}) \). The first \( t \) randomized samples can be removed by a large number of iterations \( T \), and consequently a Gibbs system sample \( \theta^{(t+1)}, \theta^{(t+2)}, \ldots, \theta^{(T)} \) can be established. The Gibbs system sample depends on the actual distribution \( f(\theta) \), which can be used to calculate the parameter estimates.

Thus, the quantile \( x_\alpha \) for the Normal distribution is given by

\[
x_\alpha = \mu + \Phi^{-1}(\alpha)\sigma = f(\mu, \sigma)
\]

where \( \theta_1 = \mu \) and \( \theta_2 = \sigma \).

3.2.3. **Parameter estimation.** For two different percentages \( \alpha_1 \) and \( \alpha_2 \), the corresponding quantile can be calculated by

\[
\begin{align*}
x_{\alpha_1} &= \mu + \Phi^{-1}(\alpha_1)\sigma \\
x_{\alpha_2} &= \mu + \Phi^{-1}(\alpha_2)\sigma
\end{align*}
\]

where \( x_{\alpha_1} \) and \( x_{\alpha_2} \) are the quantiles related to percentages \( \alpha_1 \) and \( \alpha_2 \), respectively. The parameters \( \mu \) and \( \sigma \) may be derived from Eqs. (11) and (12) as

\[
\begin{align*}
\mu &= \frac{x_{\alpha_2} \cdot \Phi^{-1}(\alpha_2) - x_{\alpha_1} \cdot \Phi^{-1}(\alpha_1)}{\Phi^{-1}(\alpha_2) - \Phi^{-1}(\alpha_1)} \\
\sigma &= \frac{x_{\alpha_2} - x_{\alpha_1}}{\Phi^{-1}(\alpha_2) - \Phi^{-1}(\alpha_1)}
\end{align*}
\]

If two quantile \( x_{\alpha_1} \) and \( x_{\alpha_2} \) at the two different percentages \( \alpha_1 \) and \( \alpha_2 \) are determined, parameters \( \mu \) and \( \sigma \) can be easily obtained from Eqs. (13) and (14), respectively.

3.3. **Least squares method.** There are various method to estimate the parameters. The Least Squares method based on relationship between the empirical cumulative distribution function and the order statistics is frequently used to estimate parameters of distributions. The Least Squares method that we consider for the parameter of Normal distribution are described in the following subsections.
Let \( x_1 \leq x_2 \leq \cdots \leq x_n \) be the order statistics of a size \( n \) random sample. The following cumulative distribution function \( F_n(x) \) is calculated:

\[
F_n(x) = \begin{cases} 
0, & x < x_1, \\
\frac{k}{n}, & x_k \leq x < x_{k+1}, \quad k = 1, 2, \ldots, n-1, \\
1, & x > x_n
\end{cases}
\] (15)

The cdf Eq. (2) \( F(x) \) can be expanded in Fourier sine series:

\[
F(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \exp\left(-\frac{k^2 \omega^2}{2}\right) \sin(k\omega \frac{x-\mu}{\sigma})
\] (16)

For the Normal distribution, the Least Squares method estimates \( \hat{\mu} \) and \( \hat{\sigma} \) of the parameters \( \mu \) and \( \sigma \), respectively, are obtained by minimizing the function:

\[
\Pi = \min_{\mu, \sigma} \sum_{i=1}^{n} (F(x_i) - F_n(x_i))^2
\] (17)

where \( F(x_i) \) and \( F_n(x_i) \) are obtained by Eqs. (15) and (16), respectively.

This is achieved by partially differentiating Eq. (24) with respect to each parameter:

\[
\begin{align*}
\frac{\partial \Pi}{\partial \mu} &= \sum_{i=1}^{n} [F_n(x_i) - F(x_i)] \cdot \frac{\partial F(x_i)}{\partial \mu} = 0, \\
\frac{\partial \Pi}{\partial \sigma} &= \sum_{i=1}^{n} [F_n(x_i) - F(x_i)] \cdot \frac{\partial F(x_i)}{\partial \sigma} = 0
\end{align*}
\] (18)

where

\[
\begin{align*}
\frac{\partial F(x_i)}{\partial \mu} &= -\frac{2\omega}{\pi \sigma} \sum_{k=1}^{\infty} \exp(-k^2 \omega^2/2) \cos\left(k\omega \frac{x-\mu}{\sigma}\right), \\
\frac{\partial F(x_i)}{\partial \sigma} &= -\frac{2\omega(x-\mu)}{\pi \sigma^2} \sum_{k=1}^{\infty} \exp(-k^2 \omega^2/2) \cos\left(k\omega \frac{x-\mu}{\sigma}\right)
\end{align*}
\] (19)

Since no closed-form solutions are available, solving the above system with the well-known Gauss-Newton iterative algorithm using the programming package MATLAB software. The initial estimates were chosen as the Maximum Likelihood estimates [25].

4. Simulation study. In this section, a simulation study was performed to compare the performance of the different method discussed in the previous section. 2000 random samples of sizes \( n = 10, 20, 30, 50, 100, 200, 300, 500, 1000 \) were generated from the Normal distribution. Since any Normal distribution data can be standardized to have a location parameter of 0 and scale parameter of 1, only samples with parameters \( \mu = 0 \) and \( \sigma = 1 \) were generated. The accuracy of the estimates is compared by using the following performance measures root mean square error (RMSE) [19], Kolmogorov-Smirnov test (KS) [24] and coefficient of determination (R^2) [32]. The simulations were performed on the programming package MATLAB software.

The results of simulation study are presented in Table 1. The following conclusions can be drawn:
Table 1. Comparison of the estimation methods

| n   | Parameter | Maximum likelihood method | Bayesian Quantile method | Least Squares method |
|-----|-----------|---------------------------|--------------------------|---------------------|
|     |           | \( \mu \) | \( \sigma \) | \( \mu \) | \( \sigma \) | \( \mu \) | \( \sigma \) |
| 10  | mean      | 0.12214 | 1.15672 | 0.11672 | 1.16318 | 0.09491 | 1.08113 |
|     | RMSE      | 0.26513 | 0.35772 | 0.25617 | 0.36147 | 0.19817 | 0.27136 |
|     | KS        | 0.35337 | 0.32109 | 0.24578 |            |            |            |
|     | \( R^2 \) | 0.83298 | 0.84576 |            |            | 0.88978 |            |
| 20  | mean      | 0.07571 | 1.10291 | 0.06984 | 1.08983 | 0.05116 | 1.05886 |
|     | RMSE      | 0.18364 | 0.24536 | 0.19225 | 0.22139 | 0.14281 | 0.18775 |
|     | KS        | 0.26355 | 0.28776 |            |            | 0.19771 |            |
|     | \( R^2 \) | 0.90137 | 0.89516 |            |            | 0.92335 |            |
| 30  | mean      | 0.05319 | 1.06572 | 0.05187 | 1.07102 | 0.04785 | 1.04213 |
|     | RMSE      | 0.15361 | 0.21369 | 0.14793 | 0.20398 | 0.11251 | 0.15720 |
|     | KS        | 0.15367 | 0.13476 |            |            | 0.09877 |            |
|     | \( R^2 \) | 0.95226 | 0.96237 |            |            | 0.97562 |            |
| 50  | mean      | 0.04367 | 1.05318 | 0.04412 | 1.05277 | 0.03918 | 1.03889 |
|     | RMSE      | 0.11623 | 0.15617 | 0.10987 | 0.15726 | 0.08273 | 0.12918 |
|     | KS        | 0.12981 | 0.13287 |            |            | 0.08726 |            |
|     | \( R^2 \) | 0.96314 | 0.96512 |            |            | 0.98715 |            |
| 100 | mean      | 0.03647 | 1.04981 | 0.03265 | 1.04912 | 0.02797 | 1.03276 |
|     | RMSE      | 0.07629 | 0.13912 | 0.07292 | 0.14021 | 0.05172 | 0.09885 |
|     | KS        | 0.08398 | 0.08203 |            |            | 0.06512 |            |
|     | \( R^2 \) | 0.97651 | 0.97261 |            |            | 0.99143 |            |
| 200 | mean      | 0.02674 | 1.03628 | 0.02556 | 1.03719 | 0.02102 | 1.01493 |
|     | RMSE      | 0.05728 | 0.07635 | 0.05276 | 0.07682 | 0.04729 | 0.05112 |
|     | KS        | 0.06729 | 0.07102 |            |            | 0.05112 |            |
|     | \( R^2 \) | 0.98112 | 0.98372 |            |            | 0.99557 |            |
| 300 | mean      | 0.01839 | 1.02987 | 0.01821 | 1.02898 | 0.01315 | 1.01011 |
|     | RMSE      | 0.03672 | 0.05729 | 0.03629 | 0.05827 | 0.02791 | 0.03174 |
|     | KS        | 0.05237 | 0.05311 |            |            | 0.04986 |            |
|     | \( R^2 \) | 0.99108 | 0.99203 |            |            | 0.99778 |            |
| 500 | mean      | 0.00587 | 1.00532 | 0.00526 | 1.00516 | 0.00338 | 1.00201 |
|     | RMSE      | 0.02392 | 0.03738 | 0.02371 | 0.03276 | 0.01818 | 0.01679 |
|     | KS        | 0.03129 | 0.03063 |            |            | 0.02701 |            |
|     | \( R^2 \) | 0.99536 | 0.99277 |            |            | 0.99913 |            |
| 1000| mean      | 0.00161 | 1.00114 | 0.00108 | 1.00112 | 0.00036 | 1.00008 |
|     | RMSE      | 0.01307 | 0.02119 | 0.01298 | 0.02101 | 0.00737 | 0.00082 |
|     | KS        | 0.01112 | 0.01134 |            |            | 0.00601 |            |
|     | \( R^2 \) | 0.99821 | 0.99903 |            |            | 0.99996 |            |

(1) All estimators of the parameters are positively biased, i.e. the estimators exceed the true value of the parameters. The biases of all estimators of the parameters tend to zero for large \( n \), i.e. the estimators are asymptotically
unbiased for the parameters. The Least Squares method has the smallest bias among the considered three estimators.

(2) As the sample size increases, the estimates of $\mu$ and $\delta$ generally approach to their true values. An increase in the sample size of the simulated Normal distribution data generally results in the improvement of the three methods. Overall, the RMSE and KS value decreases as the sample size increases while the $R^2$ increases as the sample size increases.

(3) The difference between the Maximum Likelihood method and the Bayesian Quantile method is very little for large sample sizes, but it is slightly more for smaller sample sizes.

Therefore, all the methods show identical performance for estimating the location and scale parameters of Normal distribution unless the sample size is small. However, the Least Squares method performs best for all sample sizes than other methods considered here such as, Maximum Likelihood method and Bayesian Quantile method.

5. Application to carbonation depth. In this section, the parameter estimation methods defined in the previous sections are applied to the real data—carbonation depth. Three real data sets of carbonation depth are analyzed for comparing the considered three estimation methods for the Normal distribution.

The first data set represents 12 measurements of the carbonation depth of a reinforced concrete girder bridge [21]: 12.5, 13.2, 13.9, 14.1, 14.3, 14.6, 14.9, 15.0, 15.3, 15.7, 16.4, 17.1 mm.

The second data set represents 18 measurements of the carbonation depth of the Chorng-ching Viaduct [9]: 8, 11, 15, 15, 17, 18, 20, 22, 22, 26, 28, 30, 30, 31, 33, 38, 38, 40 mm.

The third data set represents 27 measurements of the carbonation depth of pier of a reinforced concrete girder bridge [6]: 2.0, 2.1, 2.2, 2.3, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 3.0, 3.2, 3.2, 3.3, 3.3, 3.3, 3.4, 3.4, 3.5, 3.5, 3.6, 3.7, 3.8, 3.9 mm.

Table 2. Parameter estimates, RMSE, KS and $R^2$ for the first data set

| Method                               | Estimated parameters | RMSE   | KS    | $R^2$  |
|--------------------------------------|----------------------|--------|-------|--------|
| Maximum Likelihood method            | $\mu = 14.7500$, $\sigma = 1.2923$ | 0.2677 | 0.1912 | 0.8826 |
| Bayesian Quantile method             | $\mu = 14.6534$, $\sigma = 1.4505$ | 0.2301 | 0.2171 | 0.8755 |
| Least Squares method                 | $\mu = 14.5703$, $\sigma = 1.2197$ | 0.1329 | 0.1162 | 0.9283 |

Table 3. Parameter estimates, RMSE, KS and $R^2$ for the second data set

| Method                               | Estimated parameters | RMSE   | KS    | $R^2$  |
|--------------------------------------|----------------------|--------|-------|--------|
| Maximum Likelihood method            | $\mu = 24.5556$, $\sigma = 9.5808$ | 1.0122 | 0.1175 | 0.9218 |
| Bayesian Quantile method             | $\mu = 24.6528$, $\sigma = 10.3198$ | 0.9526 | 0.1013 | 0.9427 |
| Least Squares method                 | $\mu = 23.5642$, $\sigma = 10.6848$ | 0.7128 | 0.0816 | 0.9577 |
Table 4. Parameter estimates, RMSE, KS and $R^2$ for the third data set

| Method                     | Estimated parameters | $\mu$  | $\sigma$ | RMSE  | KS   | $R^2$ |
|----------------------------|----------------------|--------|----------|-------|------|-------|
| Maximum Likelihood method  |                      | 2.9852 | 0.5702   | 0.0441| 0.0966| 0.9761|
| Bayesian Quantile method   |                      | 3.0127 | 0.5985   | 0.0412| 0.0843| 0.9788|
| Least Squares method       |                      | 2.9697 | 0.6770   | 0.0391| 0.0498| 0.9916|

Table 2, Table 3 and Table 4 show the estimators of the shape and scale parameters of the Normal distribution with values of RMSE, KS, and $R^2$ on carbonation depth real data. According to the root mean square error, the Least Squares method has a smaller RMSE value than that of other methods. According to the K-S test, the Least Squares method yields a smaller KS value than that of other methods. According to the coefficient of determination, the Least Squares method yields a larger $R^2$ value than that of other methods. The results indicate that the Least Squares method is better than other methods in terms of RMSE, KS, and $R^2$ values.

Hence, for the real given data sets of carbonation depth, we concluded that the Least Squares estimation method is the best among the three considered estimation methods.

6. Conclusions. In this study, we introduce two new parameter estimation methods such as, the Least Squares method and the Bayesian Quantile method for Normal distribution. It is a flexible and simple distribution that may helpful for modeling the carbonation depth data. Mathematical expressions of the estimators are derived for the Least Squares method, the Bayesian Quantile method and the Maximum Likelihood method. We compare the performance of three methods for Normal distribution through a simulation study and three real data sets of carbonation depth. Therefore, it is concluded from both simulated and real data sets that all the methods show identical performance for estimating the location and scale parameters of Normal distribution unless the sample size is small. However, the Least Squares method performs better than other methods considered here such as, the Bayesian Quantile method and the Maximum Likelihood method.

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REFERENCES

[1] P. Biswabrata and K. Debasis, Bayes estimation and prediction of the two-parameter gamma distribution, *Journal of Statistical Computation & Simulation*, 81 (2011), 1187–1198.
[2] P. Biswabrata and K. Debasis, Bayes estimation for the Block and Basu bivariate and multivariate Weibull distributions, *Journal of Statistical Computation and Simulation*, 86 (2016), 170–182.
[3] G. Canavos, *Applied Probability Statistical Methods*, New York: Little & Brown Company, 1998.
[4] M. J. Diamantopoulou, R. Özcelik and F. Crecente-Campo, Estimation of Weibull function parameters for modelling tree diameter distribution using least squares and artificial neural networks methods, *Biosystems Engineering*, 133 (2015), 33–45.
[5] H. L. Gan and X. L. Xie, Carbonation life prediction of service reinforced concrete bridge based on reliability theory of durability, *Concrete*, 3 (2013), 48–51. (in Chinese)
[6] X. Guan, D. T. Niu and J. B. Wang, Carbonation service life prediction of coal boardwalks bridges based on durability testing, *Journal of Xi’an University of Architecture and Technology*, 47 (2015), 71–76. (in Chinese)
[7] H. P. Hong, S. H. Li and T. G. Mara, Performance of the generalized least-squares method for the Gumbel distribution and its application to annual maximum wind speeds, *Journal of Wind Engineering and Industrial Aerodynamics*, 119 (2013), 121–132.
[8] S. Y. Huang, Wavelet based empirical Bayes estimation for the uniform distribution, *Statistics & Probability Letters*, 32 (1997), 141–146.
[9] M. T. Liang, R. Huang and S. A. Fang, Carbonation service life prediction of existing concrete viaduct/bridge using time-dependent reliability analysis, *Journal of Marine Science and Technology*, 21 (2013), 94–104.
[10] H. L. Lu and S. H. Tao, The estimation of Pareto distribution by a weighted least square method, *Quality & Quantity*, 41 (2007), 913–926.
[11] B. Miladinovic and C. P. Tsokos, Ordinary, Bayes, empirical Bayes, and non-parametric reliability analysis for the modified Gumbel failure model, *Nonlinear Analysis*, 71 (2009), 1426–1436.
[12] U. J. Na, S. J. Kwon, S. R. Chaudhuri, et al., Stochastic model for life prediction of RC structures exposed to carbonation using random field simulation, *KSCE Journal of Civil Engineering*, 16 (2012), 133–143.
[13] J. Nabakumar, K. Somesh and C. Kashinath, Bayes estimation for exponential distributions with common location parameter and applications to multi-state reliability models, *Journal of Applied Statistics*, 43 (2016), 2697–2712.
[14] D. T. Niu, Y. Q. Chen and S. Yu, Model and reliability analysis for carbonation of concrete structures, *Journal of Xi’an University of Architecture and Technology*, 27 (1995a), 365–369. (in Chinese)
[15] D. T. Niu, Y. C. Shi and Y. S. Lei, Reliability analysis and probability model of concrete carbonation, *Journal of Xi’an University of Architecture and Technology*, 27 (1995b), 252–256. (in Chinese)
[16] D. T. Niu, Z. P. Dong and Y. X. Pu, Fuzzy prediction on carbonation life of concrete structures, *Proceedings of the Ninth Conference of Civil Engineering Society*, Hanzhou, (1999a), 367–370. (in Chinese)
[17] D. T. Niu, Z. P. Dong and Y. X. Pu, Random model of predicting the carbonated concrete depth, *Industrial Construction*, 29 (1999b), 41–45. (in Chinese)
[18] D. T. Niu, C. F. Yuan and C. F. Wang, et al., Carbonation service life prediction of reinforced concrete railway bridges based on durability testing, *Journal of Xi’an University of Architecture and Technology*, 43 (2011), 160–165. (in Chinese)
[19] T. B. M. J. Ouarda, C. Charron and J. Y. Shin, et al., Probability distributions of wind speed in the UAE, *Energy Conversion & Management*, 93 (2015), 414–434.
[20] J. X. Peng and J. R. Zhang, Incremental process based carbonation depth prediction model of concrete structures and its probability analysis, *Journal of Highway and Transportation Research and Development*, 29 (2012), 54–83. (in Chinese)
[21] F. Ren, J. Y. Liu and X. Y. Pei, et al., Reliability analysis of bridge durability based on concrete carbonation, *Journal of Highway and Transportation Research and Development*, 21 (2004), 71–80. (in Chinese)
[22] P. K. Singh, S. K. Singh and U. Singh, Bayes estimator of Inverse Gaussian parameters under general entropy loss function using Lindley’s approximation, *Communications in Statistics - Simulation and Computation*, 37 (2008), 1750–1762.
[23] A. A. Soliman, Comparison of linex and quadratic Bayes estimators for the Rayleigh distribution, *Communications in Statistics-theory and Methods*, 29 (2000), 95–107.
[24] M. Y. Sulaiman, A. M. Akaak and M. A. Wahab, et al., Wind characteristics of Oman, *Energy*, 27 (2002), 35–46.
[25] F. J. Torres, Estimation of parameters of the shifted Gompertz distribution using least squares, maximum likelihood and moments methods, *Journal of Computational & Applied Mathematics*, 255 (2014), 867–877.
[26] J. W. Wu, W. L. Hung and C. H. Tsai, Estimation of parameters of the Gompertz distribution using the least squares method, *Applied Mathematics and Computation*, 158 (2004), 133–147.
[27] W. Xia, X. X. Dai and Y. Feng, Bayesian-MCMC-based parameter estimation of stealth aircraft RCS models, Chinese Physics, 24 (2015), 129501.
[28] S. H. Xu, D. T. Niu and Q. L. Wang, The determination of concrete cover depth under atmospheric condition, China Civil Engineering Journal, 38 (2005), 45–68. (in Chinese)
[29] Z. T. Yu and D. J. Han, Carbonation reliability assessment of existing reinforced concrete girder bridges, Journal of South China University of Technology, 32 (2004), 50–66. (in Chinese)
[30] C. F. Yuan, D. T. Niu and Q. S. Gai, et al., Durability testing and carbonation life prediction of Songhu River Bridge, Bridge Construction, 2 (2010), 21–24. (in Chinese)
[31] C. F. Yuan, D. T. Niu and C. T. Sun, Carbonation depth prediction of Songhu River Highway Bridge, Concrete, 6 (2009), 46–48. (in Chinese)
[32] J. Z. Zhou, E. Erdem and G. Li, et al., Comprehensive evaluation of wind speed distribution models: A case study for North Dakota sites, Energy Conversion and Management, 51 (2010), 1449–1458.

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