LOCAL WELL-POSEDNESS FOR THE ZAKHAROV SYSTEM IN DIMENSION \( d \leq 3 \)

AKANSHA SANWAL

Abstract. The Zakharov system in dimension \( d \leq 3 \) is shown to be locally well-posed in Sobolev spaces \( H^s \times H^l \), extending the previously known result. We construct new solution spaces by modifying the \( X^{s,b} \) spaces, specifically by introducing temporal weights. We use contraction mapping principle to prove local well-posedness in the same. The result obtained is sharp up to endpoints.

1. Introduction

We consider the Cauchy problem for the Zakharov system

\[
\begin{align*}
 i\partial_t u + \Delta u &= nu \\
 \partial_t^n n - \Delta n &= \Delta |u|^2
\end{align*}
\]

with initial data

\[ u(x,0) = u_0(x), \quad n(x,0) = n_0(x), \quad \partial_t n(x,0) = n_1(x), \]

where \((u_0, n_0, n_1) \in H^s \times H^l \times H^{l-1} \).

Here \( u : \mathbb{R}^{d+1} \to \mathbb{C} \) and \( n : \mathbb{R}^{d+1} \to \mathbb{R} \) denote the slowly changing amplitude of a high frequency electric field and the deviation of ion density from its equilibrium, respectively. This system was derived by Zakharov [23] to model the propagation of Langmuir waves in a plasma.

1.1. Known results. Local well-posedness in \( H^2 \times H^1 \times L^2 \) was shown in [18] along with some smoothing for the Schrödinger part for \( d \leq 3 \). The result was improved by Bourgain and Colliander [7] who proved well-posedness of the two and three dimensional system in the energy space \((H^1 \times L^2)\) using Fourier restriction norms and contraction mapping. Their result also includes small data global well-posedness if the \( H^1 \) norm of the Schrödinger data is sufficiently small. In [11], local well-posedness for dimension \( d \geq 1 \) was shown for a range of indices \((s,l)\) depending on \( d \) by refining the contraction mapping argument in Fourier restriction spaces. In particular, for the two and three dimensional system, local well-posedness is proved in \( H^s \times H^l \) for \((s,l)\) satisfying \( l \leq s \leq l + 1, \ l \geq 0 \) and \( 2s - l \geq 1 \). For the 1D system, the result covers the range \(-\frac{1}{2} \leq s - l \leq 1, 2s - l \geq \frac{1}{2} \). The best available result for local well-posedness of the system in three dimensions is proved in [2] which covers the range \( l > \frac{1}{2} \), \( l \leq s \leq l + 1 \) and \( 2s - l > \frac{1}{2} \) (see Figure 1) while that in two space dimensions is proved in [3] for \((s,l) = (0, -\frac{1}{2})\). Both the aforementioned results exploit the transversality of the characteristic hypersurfaces in the resonant interactions to obtain sharp multilinear estimates. These estimates, referred to as generalisations of the Loomis Whitney inequality [16, 4, 5], shall play a vital role in proving the non-linear estimates in sections 3 and 4. For the 1D system, in [19], global well-posedness without any smallness assumption on the initial data is proved in \( H^2 \times L^2 \) for \( \frac{3}{2} < s < 1 \). In [20], local well-posedness is shown in spaces \( H^s \times H^l \) for \( 1 < p < 2 \). Here, the regularity of the initial data can be lowered so that \( s < 0 \) and \( l < -\frac{1}{2} \) by using modified \( X^{s,b} \) spaces. Global well-posedness using mass conservation is proved in [9] for \((s,l) = (0, -\frac{1}{2})\) which is the largest space where the system is locally well-posed. In [1], global well-posedness and scattering results are proved.

We throw some light on ill-posedness results now. In [14], for the 1D system, it is shown that the constraint \( 2s - l \geq \frac{1}{2} \) is optimal in the sense that norm inflation occurs otherwise. For \( l < -\frac{3}{2} \) and \( s = 0 \), it is shown that the data to solution map fails to be uniformly continuous from \( H^s(\mathbb{R}) \times H^l(\mathbb{R}) \times H^{l-1}(\mathbb{R}) \) to \( C([0,T]; H^s(\mathbb{R})) \times C([0,T]; H^l(\mathbb{R})) \times C([0,T]; H^{l-1}(\mathbb{R})) \). This nearly matches the 1D result of [6] where the authors prove ill-posedness for \( s < 0 \) and \( l \leq -\frac{3}{2} \). The failure of \( C^2 \) differentiability of the data to solution map is proved for \( l < -\frac{1}{2} \). In [8, section 9], the authors prove that for a general dimension \( d \geq 1 \), the flow map is not \( C^2 \) for \( s, l \) and \( d \) satisfying \( l < \frac{5}{2} - 2, \ s - l > 2, \ 2s - l < \frac{d-2}{2} \) or \( s - l < -1 \). For \( d = 2 \), stronger counterexamples in [3, section 6] prove that the flow map is not \( C^2 \) for \( 2s - l < \frac{1}{4} \) or \( l < -\frac{5}{2} \). In [11, Proposition 3.2], using the close relation of the system (1.1) – (1.2) to the cubic non-linear Schrödinger equation, the authors prove that the data to solution map for the system fails to be Lipschitz continuous at the point \((s,l) = (\frac{5}{4}, -\frac{1}{4})\). Using power series expansion for the solutions, as done in [15] for the Schrödinger equation, norm inflation results have been proved for the Zakharov system in dimension \( d \geq 1 \) in [12]. In particular, for \( d = 3 \) it is proved that all the boundaries but \( l = -\frac{1}{2} \) are sharp. The interested reader can look [10] for a summarised account of ill-posedness results for the
system (1.1) – (1.2) in dimension \( d \geq 1 \).

In this article, we provide a unified local well-posedness result for the system (1.1) – (1.2) in \( d \leq 3 \) which holds in the region (1.3) (see Figures 1 and 2) using the contraction mapping principle in modified \( X^{s,b} \) spaces. Our results read

**Theorem 1.1.** The system (1.1) with initial conditions (1.2) is locally well-posed for \( d \leq 3 \) provided

\[
l > -\frac{1}{2} \quad \max \left\{ l - \frac{1}{2}, \frac{1}{4} \right\} < s < l + 2.
\]

**Remark 1.2.**

(i) From the existing literature, it follows that \((s,l) = (0, -\frac{1}{2})\) is the optimal point at which local well-posedness can be proved by a contraction mapping argument. This also corresponds to the scaling critical regularity for the three dimensional system, [11, Section 2, pp 7-8]. Our result does not cover this because of the choice of our function spaces and losses which arise from the application of the estimates in lemma 2.4 and 2.6. Hence the endpoint remains an open problem for \( d = 3 \).

(ii) The regularity \((s,l) = (1, 0)\) which corresponds to the energy space for the system (1.1) – (1.2) lies on the boundary in the results covered by [11, 2]. The inclusion of the same as an interior point of the region of well-posedness is noteworthy.

(iii) Using the normal form approach of [13], one can improve the local well-posedness result for the system (1.1) – (1.2). This was achieved in [21] for \( d \geq 2 \). However, the result does not cover the negative regularity region for the wave solution.

**Figure 1.** Region of well-posedness for \( d = 3 \) in [2]

The strips extend infinitely to the upper right in both the regions.

**Figure 2.** New region of well-posedness for \( d \leq 3 \)

1.1.1. **Outline.** In section 2, we introduce the notation and previously known estimates. We also define our function spaces and state the required properties of \( X^{s,b} \) spaces therein. Known multilinear estimates are also enlisted. Sections 3 and 4 are devoted to the proof of the crucial multilinear estimates for the Schrödinger and wave non-linearity, respectively. Finally, in section 5, we give a short proof of theorem 1.1.

2. **Notation and preliminaries**

2.1. **Reduced system.** For \(|\nabla|:= \sqrt{-\Delta}\), using the transformation \( v = n - i|\nabla|^{-1} \partial_t n \), the system (1.1) formally reduces to the following first order system:

\[
\begin{align*}
    i \partial_t u + \Delta u &= u \text{Re}(v) \\
    i \partial_t v + |\nabla|v &= -|\nabla||u|^2.
\end{align*}
\]

We observe that \((u, v)\) solves (2.1) iff \((u, \text{Re}(v))\) solves (1.1). Henceforth, we shall consider the system (2.1) for our analysis.
2.2. Fourier multipliers. Let \( \chi \in C_0^\infty(\mathbb{R}) \) be non-negative such that \( \chi(r) = 1 \) for \( |r| \leq 1 \), \( \chi(r) = 0 \) for \( |r| \geq 2 \). Set \( \phi(r) := \chi(r) - \chi(2r) \) and \( \phi_\lambda(r) := \phi(r/\lambda) \). Then
\[
\sum_{\lambda \in 2^\mathbb{N}, \lambda \geq 1} \phi_\lambda(r) = 1, \quad \phi_1 = \chi.
\]
For \( \lambda \in 2^\mathbb{N}, \lambda > 1 \), we define the Fourier multipliers
\[
P_\lambda = \phi_\lambda(|\nabla|), \quad P_\lambda^{(i)} = \phi_\lambda(|\partial_i|), \quad C_\lambda = \phi_\lambda(|i\partial_t + \Delta|), \quad Q_\lambda = \phi_\lambda(|i\partial_t |\pm |\nabla||),
\]
and for \( \lambda = 1 \), we define
\[
P_1 = \chi(|\nabla|), \quad P_1^{(i)} = \chi(|\partial_i|), \quad C_1 = \chi(|i\partial_t + \Delta|), \quad Q_1 = \chi(|i\partial_t |\pm |\nabla||).
\]
Thus, \( P_\lambda \) is a Fourier multiplier which localises the spatial frequencies to the set \( \{ \frac{1}{2} \leq |\xi| \leq 2\lambda \} \) while \( P_\lambda^{(i)} \) localises the temporal frequencies to the set \( \{ \frac{1}{2} \leq |\tau| \leq 2\lambda \} \). In a similar spirit, \( C_\lambda \) and \( Q_\lambda \) localise the space-time Fourier support to distances \( \sim \lambda \) from the paraboloid and the cone, respectively. More precisely, for \( L \in 2^\mathbb{N} \) the space-time Fourier supports of \( C_L, C_1, Q_L \) and \( Q_1 \) can be written as
\[
S_L = \left\{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d : \frac{L}{2} \leq |\tau + |\xi|^2| \leq 2L \right\}, \quad S_1 = \left\{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d : |\tau + |\xi|^2| \leq 2 \right\}, \quad W_L^\pm = \left\{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d : \frac{L}{2} \leq |\tau \pm |\xi|| \leq 2L \right\}, \quad W_1^\pm = \left\{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d : |\tau \pm |\xi|| \leq 2 \right\}.
\]
To restrict the Fourier support to larger sets, we use the notation
\[
P_{\leq \lambda} = \sum_{\mu \in 2^\mathbb{N}, \mu \leq \lambda} \phi_\lambda(|\nabla|), \quad P_{\leq \lambda}^{(i)} = \sum_{\mu \in 2^\mathbb{N}, \mu \leq \lambda} \phi_\lambda(|\partial_i|), \quad C_{\leq \lambda} = \sum_{\mu \in 2^\mathbb{N}, \mu \leq \lambda} \phi_\lambda(|i\partial_t + \Delta|), \quad Q_{\leq \lambda} = \sum_{\mu \in 2^\mathbb{N}, \mu \leq \lambda} \phi_\lambda(|i\partial_t |\pm |\nabla||).
\]
For \( \lambda \in 2^\mathbb{N} \), we use the shorthand \( f_\lambda = P_\lambda f \).
We use the notation \( A \lesssim B \) to indicate \( A \leq cB \), for a constant \( c > 0 \) and \( A \sim B \) when both \( A \lesssim B \) and \( B \lesssim A \) hold. We use \( A \ll B \) to denote \( A \leq cB \) for a constant \( c \) much smaller than 1.

2.3. Function spaces. The space-time Fourier transform of a function \( f \) is denoted by \( \mathcal{F}(f) \), where \((t, x)\) are the physical space variables and \((\tau, \xi)\) the corresponding Fourier variables. We use angled brackets to denote \(|\langle x \rangle| := (1 + x^2)^{\frac{1}{2}}\), and \(x^+\) denotes a number slightly bigger than \(x\) in the following.

We set \( \theta := \frac{1}{2} + . \) Given \( s, l \in \mathbb{R} \), we define the parameters \( 0 \leq a, b < 2 - 2\theta, \theta', s' \) and \( \beta \) as follows:
\[
a := \begin{cases} s - l + 3 + \theta, & s - l \geq 1 \\ s - l < 1 \end{cases}, \quad b := \begin{cases} 0, & s - l > 0 \\ (l - s) + 2\theta - 1, & s - l \leq 0 \end{cases}, \quad \theta' := \begin{cases} \theta, & s - l > 0 \\ 1, & s - l < 0 \end{cases},
\]
\[
s' := \begin{cases} s, & s - l \geq 0 \\ s + 2\theta - 2, & s - l < 0 \end{cases} \quad \text{and} \quad \beta := \begin{cases} l + a + 2\theta - 1, & s - l \geq 1 \\ l, & s - l < 1 \end{cases} .
\]
(2.2)

For \( \lambda \in 2^\mathbb{N} \), \( s, a, b, \theta, \theta' \in \mathbb{R} \), we control the frequency localised Schrödinger component of the Zakharov evolution by the following
\[
\|u_\lambda\|_{L^\infty_{t,1}L^{\lambda} L^2} + \lambda^p\|\tau + |\xi|^2\|^p \mathcal{F}(C_{\lambda,2} u_\lambda)\|_{L^2_{t,\xi}} + \lambda^s \|\xi|^{-2a+b}\|\tau + |\xi|^2\|^{\theta} \mathcal{F}(\lambda^s \partial_t u_\lambda)\|_{L^2_{t,\xi}}.
\]
To control the frequency localised Schrödinger non-linearity, we define
\[
\|F_\lambda\|_{L^\infty_{t,1}L^{2\theta} L^2} + \lambda^s \|\xi|^{2\theta-2a+b}\|\tau + |\xi|^2\|^{\theta-1} \mathcal{F}(\lambda^s \partial_t F_\lambda)\|_{L^2_{t,\xi}} + \lambda^s \|\xi|^{-2a+b}\|\tau + |\xi|^2\|^{\theta-1} \mathcal{F}(\lambda^s \partial_t F_\lambda)\|_{L^2_{t,\xi}}.
\]
For \( l, \beta, \alpha, a, \theta \in \mathbb{R} \), the evolution of the frequency localized wave component is controlled by the following norm
\[
\|u_\lambda\|_{W^{l,\beta,a}_{t,1} L^{2\theta} L^2} \leq \lambda^{l+2\theta-3} \|P_{\leq \lambda}^{(i)} F_\lambda\|_{L^\infty_{t,1}L^{2\theta} L^2} + \lambda^s \|\tau - |\xi|^2\|^\theta \mathcal{F}(\lambda^s \partial_t Q_{\lambda,2} u_\lambda)\|_{L^2_{t,\xi}} + \lambda^\beta \|F_\lambda\|_{L^\infty_{t,1}L^{2\theta} L^2}.
\]
and the RHS of the wave equation is controlled by
\[
\|G_\lambda\|_{B^{l,-a}_{t,1} L^{2\theta} L^2} \leq \lambda^{l+2\theta-3} \|G_\lambda\|_{L^\infty_{t,1}L^{2\theta} L^2} + \lambda^{l-\alpha} \|\tau - |\xi|^2\|^{\alpha} \mathcal{F}(\lambda^\alpha \partial_t Q_{\lambda,2} G_\lambda)\|_{L^2_{t,\xi}} + \lambda^{\beta-1} \|F_\lambda\|_{L^\infty_{t,1}L^{2\theta} L^2}.
\]
The parameters \(a, b, s', \beta\) and \(\theta'\) are required to prove the bilinear estimates in the full region \((1,3). \) \(a\) measures the loss of regularity for the Schrödinger component in the low \((\ll \lambda^2)\) temporal frequency region as can be seen from the weight \(m_S(\tau) := \left(\frac{1 + |\tau|}{\lambda^3}\right)^a, 0 \leq a < 1.\) Note that
\[
m_S(\tau) \sim \begin{cases} 
\lambda^{-a}, & |\tau| \lesssim \lambda, \\
1, & |\tau| \sim \lambda^2,
\end{cases} \quad \text{and } m_S(\tau) \gg 1, |\tau| \gg \lambda^2,
\]
while
\[
m_W(\tau) := \left(\frac{\lambda + |\tau|}{\lambda}\right)^{a} \gtrsim 1.
\]
The parameter \(b\) gives a gain in regularity to the Schrödinger evolution in the high modulation regime and helps to achieve bilinear estimates for the wave non-linearity in the region \((1,3).\) \(a\) and \(b\) are not non-zero simultaneously and hence their sum is always less than 1. It is this restriction on the upper bounds for the parameters \(a\) and \(b\) that does not allow us to achieve the boundaries \((s - l = 2, -1)\) of the region described by \((1,3).\)

In the region \(l + 1 \leq s < l + 2,\) where the Schrödinger component is more regular, we require \(a > 0.\) In the “balanced” region \(l < s < l + 1,\) we choose \(a = 0 = b.\) In the final regime i.e. \(l - 1 < s \leq l,\) where the wave component is more regular, we choose \(a = 0, b > 0.\) Similarly, for the high modulation wave regularity \(\beta,\) we have \(\beta \sim s - 1\) when \(1 \leq s - l < 2,\) which is greater than or equal to the “ideal” regularity \(l.\) In the balanced regime, it is chosen to be \(l\), while in the thin strip \(-1 < s - l \leq -\frac{1}{2},\) where the wave is more regular, we choose \(\beta\) to be \(\sim s + \frac{1}{2}\) which is less than \(l.\) For the regularity \(s'\) of the high modulation Schrödinger norm, we see that the change in the exponent of the modulation weight viz \(\theta',\) compensates for the loss. Note that in the low modulation, the \(S_\lambda\) and \(W_\lambda\) norms are exactly the \(X^{s,b}_\lambda\) norms with \(\theta' > 2.\)

We also observe that the choice of the parameter \(\theta' (\geq \frac{1}{2})\) enables us to use various properties of the standard \(X^{s,b}_\lambda\) spaces and the bilinear estimates from \([3, 2]\) (see Section 2.6) appropriately.

**Remark 2.1.** Equation (2.2) provides us with one possible choice for the parameters. We reckon other choices might be plausible too.

The evolution of the full Schrödinger solution and non-linearity is controlled by the following norms
\[
\|u\|_{S^{s,t,\theta}} = \left(\sum_{\lambda \in 2^d} \|u_\lambda\|_{S^{s,t,\theta}_\lambda}^2\right)^{\frac{1}{2}}, \quad \|F\|_{N^{s,t,\theta-1}} = \left(\sum_{\lambda \in 2^d} \|F_\lambda\|_{N^{s,t,\theta-1}_\lambda}^2\right)^{\frac{1}{2}},
\]
and that of the wave solution and wave non-linearity is controlled by
\[
\|v\|_{W^{s,t,\theta}} = \left(\sum_{\lambda \in 2^d} \|v_\lambda\|_{W^{s,t,\theta}_\lambda}^2\right)^{\frac{1}{2}}, \quad \|G\|_{R^{s,t,\theta-1}} = \left(\sum_{\lambda \in 2^d} \|G_\lambda\|_{R^{s,t,\theta-1}_\lambda}^2\right)^{\frac{1}{2}}.
\]

For \(0 < T \leq 1,\) we localise the norms in time by defining
\[
\|u\|_{S^{s,t,\theta}(T)} = \inf\{\|\tilde{u}\|_{S^{s,t,\theta}} : \tilde{u} \in S^{s,t,\theta}, \tilde{u}|_{(0,T) \times \mathbb{R}^d} = u\}
\]
The norms \(N^{s,l,\theta-1}(T), W^{l,s,\theta}(T)\) and \(R^{s,l,\theta-1}(T)\) are defined similarly.

### 2.4. Properties of \(X^{s,b}_\lambda\) spaces

Since our function spaces are a variant of the standard \(X^{s,b}_\lambda\) spaces, we note some properties of the same in the following. The proofs can be found in [11, Lemma 2.1, 2.2], [22, section 2.6].

**(i)** Let \(u\) be a solution to the problem \(\partial_t u = L(u) + F\) with \(L = ih(\nabla / i)\) for some real valued polynomial \(h\) with \(h(0) = f.\) For a smooth time cutoff \(\chi\) such that \(\chi(t) = 1, |t| \leq 1, \chi(t) = 0, |t| > 2\) and \(\chi(T(t)) := \chi(t),\) \(0 < T \leq 1\) and \(s, b \in \mathbb{R},\) we have
\[
\|\chi(t)e^{iTf}f\|_{X^{s,b}_{\chi(t)e^{iTf}}} \lesssim \|f\|_{H^s}.
\]  

**(ii)** If \(v\) is a solution to the problem \(\partial_t v = L(v) + F,\) with \(L = ih(\nabla / i)\) \(v(0) = 0,\) we have for \(-\frac{1}{2} < b' \leq b \leq b' + 1,\)
\[
\|\chi(t)v\|_{X^{s,b}_{\chi(t)+b'}(\xi)} \lesssim T^{1+b''-b}\|F\|_{X^{s',b''}_{\chi(t)+b''}(\xi)}.
\]  

**(iii)** For \(\frac{2}{q} + \frac{d}{r} = \frac{d}{r}, 2 \leq r < \infty,\) the following Strichartz estimate holds
\[
\|u\|_{L^r_t L^q_x} \lesssim \|u\|_{X^{s,b}_{\Delta}}^{\frac{1}{r} + \frac{1}{q}},
\]
where the \(\Delta\) in the subscript denotes that the space in consideration is for the Schrödinger equation. For the wave part, we only use
\[
\|u\|_{L^{r}_t L^q_x} \lesssim \|u\|_{X^{s,b}_{\Delta}}^{\frac{1}{r} + \frac{1}{q}}.
\]
(iv) For $b > \frac{1}{2}$, $X^{s,b}(\mathbb{R} \times \mathbb{R}^d) \hookrightarrow C(\mathbb{R}; H^s(\mathbb{R}^d))$, $X^{s,b}((0, T) \times \mathbb{R}^d) \hookrightarrow C((0, T); H^s(\mathbb{R}^d))$.

(v) For $2 \leq p < \infty$, $b \geq \frac{1}{2} - \frac{1}{p}$, we have
$$\|u\|_{L^p_t L^2_x} \lesssim \|u\|_{X^{0,b}}. \quad (2.7)$$

(vi) For any $s \in \mathbb{R}$ and $-\frac{1}{4} < b' < \frac{1}{4}$, we have
$$\|u\|_{X^{s,b'}(T)} \lesssim T^{b-b'}\|u\|_{X^{s,b}(T)}. \quad (2.8)$$

(vii) Duality and complex conjugation:
$$(X^{s,b}_\tau = h(\xi))' = X^{-s,-b}_{\tau = -h(-\xi)} \quad \|\mathcal{F}\|_{X^{s,b}_\tau = h(\xi)} = \|u\|_{X^{s,b}_\tau = h(\xi)}.$$

2.5. Duhamel formulae and energy inequalities. Let $I_S$ and $I_W$ be the solution operators for the inhomogeneous Schrödinger equation and the half wave equation, respectively, i.e.
$$I_S[F](t) = -i \int_0^t e^{i(t-s)\Delta} F(s)ds, \quad I_W[G](t) = -i \int_0^t e^{i(t-s)\|\nabla\|} G(s)ds.$$

Then we have the following estimates.

**Lemma 2.2.** (Energy inequality for the Schrödinger equation) Let $a, b, s', \theta' \in \mathbb{R}$ be as defined in (2.2). For any $\lambda \in 2\mathbb{N}$ and a smooth time cutoff $\chi$, we have:
$$\|\chi(t)e^{it\lambda u_0}\|_{L^{s',0}_x} \lesssim \lambda^s\|u_0\|_{L^2_x}, \quad \|\chi(t)I_S[F_\lambda]\|_{L^{s',0}_x} \lesssim \|F_\lambda\|_{L^{\theta',s-1}_x}.$$

**Proof.** We first note the following property of the linear group wrt temporal frequency and modulation localisation, which we shall use repeatedly
$$C_x f = e^{t \lambda F_x(t)}(e^{-t \lambda f}) \quad (2.9).$$

The norm for the free solution is given by
$$\|\chi(t)e^{it\lambda u_0}\|_{L^{s',0}_x} = \lambda^s\|\chi(t)e^{it\lambda u_0}\|_{L^{s',0}_x} + \lambda^s\|\tau + |\xi|^2\theta F(C_{\ll \lambda^2} \chi(t)e^{it\lambda u_0})\|_{L^{s',0}_x} + \lambda^{s'-2a+b}\|\tau + |\xi|^2\theta' F((\lambda + |\partial_\xi|)^a C_{\gg \lambda^2} \chi(t)e^{it\lambda u_0})\|_{L^{s',0}_x}.$$

Using (2.5) and (2.3), we have
$$\lambda^s\|\chi(t)e^{it\lambda u_0}\|_{L^{s',0}_x} \lesssim \lambda^s\|\tau + |\xi|^2\theta F(\chi(t)e^{it\lambda u_0})\|_{L^{s',0}_x} \lesssim \lambda^s\|u_0\|_{L^2_x},$$

and for the second term using (2.9), we have
$$\lambda^s\|\tau + |\xi|^2\theta F(C_{\ll \lambda^2} \chi(t)e^{it\lambda u_0})\|_{L^{s',0}_x} = \lambda^s\|\tau + |\xi|^2\theta F(e^{it\lambda P_{\ll \lambda^2}}(\chi(t)u_0))\|_{L^{s',0}_x} \lesssim \lambda^s\|u_0\|_{L^2_x}.$$

A similar computation gives
$$\lambda^{s'-2a+b}\|\tau + |\xi|^2\theta' F((\lambda + |\partial_\xi|)^a C_{\gg \lambda^2} \chi(t)e^{it\lambda u_0})\|_{L^{s',0}_x} \lesssim \lambda^{s'-2a+b}\|P_{\ll \lambda^2}^{\theta'}(\chi(t)u_0)\|_{L^{s',0}_x}.$$

Now consider the $S_\lambda$ norm of the Duhamel integral. The $L^\infty_t L^2_x$ term is decomposed as follows
$$\lambda^s\|\chi(t)e^{i(t-s)\lambda} C_{\ll \lambda^2} F_\lambda(s)ds\|_{L^{s',0}_x} \lesssim \lambda^s\|\chi(t)e^{i(t-s)\lambda} C_{\ll \lambda^2} F_\lambda(s)ds\|_{L^{s',0}_x}.$$

Using properties (2.5) and (2.4) respectively, the first term can be bounded by $\lambda^s\|\tau + |\xi|^2\theta F(C_{\ll \lambda^2} F_\lambda)\|_{L^{s',0}_x}$. The second term can be further written as
$$\lesssim \lambda^s\|\tau + |\xi|^2\theta F(C_{\ll \lambda^2} F_\lambda(s)ds\|_{L^{s',0}_x} \lesssim \lambda^s\|\tau + |\xi|^2\theta F(C_{\ll \lambda^2} F_\lambda(s)ds\|_{L^{s',0}_x}.$$

Using $\|I_S[C_{\gg \lambda^2} G]\|_{L^{s',0}_x} \lesssim \mu^{-1}\|C_{\gg \lambda^2} G\|_{L^{s',0}_x}$, the first term can be bounded by $\lambda^{s'-2a+b}\|\tau + |\xi|^2\theta' F((\lambda + |\partial_\xi|)^a C_{\gg \lambda^2} F_\lambda)\|_{L^{s',0}_x}$ by noting that the temporal weight $m_\lambda(\tau) \gtrsim 1$ and $b > 0$.

The low modulation norm of the Duhamel integral is also decomposed into
$$\lambda^s\|\tau + |\xi|^2\theta F(C_{\ll \lambda^2} \chi(t)e^{i(t-s)\Delta} C_{\ll \lambda^2} F_\lambda(s)ds\|_{L^{s',0}_x} = (I) + (II)$$
An application of (2.4) provides the correct bound for (I). To handle (II), we consider the following cases pertaining to the high modulation norm for the Schrödinger non-linearity:

(i) $a = 0 = b$: A straightforward application of (2.4) gives

\[
(II) \lesssim \lambda^s \| (\tau + |\xi|^2)^{\theta - 1} F(\xi) \|_{L^2_{x,t}}.
\]

(ii) $a > 0, b = 0$: We note that we can bound (II) by $\lambda^{s-2\theta} \| (\tau + |\xi|^2)^{\theta - 1} F((\lambda + |\partial_t|)^a C_A c_A \tilde{F}_A) \|_{L^2_{x,t}}$ using (2.4), provided the temporal frequencies of $\tilde{F}_A$ are $\gtrsim \lambda^2$. We consider the case when the temporal frequencies of $F_A$ are $\ll \lambda^2$. Note that this implies that $F_A$ has a modulation of size $\sim \lambda^2$. We use (2.9) and Sobolev embedding to obtain

\[
(II) = \lambda^s \| P(t) (\chi(t)) \int_0^t P(s) e^{-i\lambda^2 s} F_A(s) \|_{H^s \| L^2_t} \lesssim \lambda^s \| P(t) (\chi(t)) \int_0^t P(s) e^{-i\lambda^2 s} F_A(s) \|_{W^{1,p}_{1,2}, p = \frac{2}{3 - 2\theta}} \lesssim \lambda^s \left( \| P(t) (\chi(t)) \int_0^t P(s) e^{-i\lambda^2 s} F_A(s) \|_{L^1_t \| L^2} \right.
\]

where the prime denotes derivative wrt time.

Using Hölder’s inequality, unitarity of $e^{it\Delta}$ and $\| T_s [C_A \mu A] \|_{L^{\infty}_t L^2_x} \lesssim \mu^{-1} \| C_A \mu A \|_{L^{\infty}_t L^2_x}$, we have

\[
(II.1) \lesssim \lambda^s \| \chi(t) \|_{L^1_t \| L^\infty_x} \| \int_0^t P(s) e^{-i\lambda^2 s} F_A(s) \|_{L^1_t \| L^2_x} \lesssim \lambda^{s-2} \| P(t) (\chi(t)) \|_{L^\infty_t \| L^2_x}.
\]

(II.2) is further written as

\[
(II.2) \lesssim \lambda^s \left( \| P(t) (\chi(t)) \int_0^t e^{-i\lambda^2 s} F_A(s) \|_{L^1_t \| L^2_x} + \| P(t) (\chi(t)) \|_{L^\infty_t \| L^2_x} \right) \lesssim \lambda^s \| \chi(t) \|_{L^1_t \| L^\infty_x} \| \int_0^t P(s) e^{-i\lambda^2 s} F_A(s) \|_{L^1_t \| L^2_x}.
\]

The first term above can be handled like (II.1). For the second term, we use Bernstein’s inequality and decompose the time cutoff $\chi(t)$ to obtain

\[
\lesssim \lambda^{s-2\theta-1} \| P(t) (\chi(t)) P(t) e^{-i\lambda^2 s} F_A \|_{L^1_t \| L^2_x} \lesssim \lambda^{s-2\theta-1} \| P(t) (\chi(t)) P(t) e^{-i\lambda^2 s} F_A \|_{L^1_t \| L^2_x}.
\]

The first term above does not contribute for the second, while the last inequality we use that the $L^1$ norm of a time cut-off at high temporal frequencies ($\gtrsim \lambda^2$) is $\lesssim \lambda^{s-2}$, see [17, Lemma 2.4] for a proof.

(iii) $b > 0, a = 0$: As for case (i), we have

\[
(III) \lesssim \lambda^s \| (\tau + |\xi|^2)^{\theta - 1} F(\xi) \|_{L^2_{x,t}} \lesssim \lambda^{s-2\theta-2} \| F(\xi) \|_{L^2_{x,t}} \lesssim \| F(\xi) \|_{N^{s,\theta - 1}_A}.
\]

where in the last inequality we use that the $L^1$ norm of a time cut-off at high temporal frequencies ($\gtrsim \lambda^2$) is $\lesssim \lambda^{s-2}$, see [17, Lemma 2.4] for a proof.

(iv) $b > 0, a = 0$: As for case (i), we have

\[
(IV) \lesssim \lambda^s \| (\tau + |\xi|^2)^{\theta - 1} F(\xi) \|_{L^2_{x,t}} \lesssim \| F(\xi) \|_{N^{s,\theta - 1}_A}.
\]

For (III), (2.4) suffices. For (IV), we again consider three cases:

(i) $a = 0 = b$: From the definition of the norm and (2.4),

\[
(IV) \lesssim \lambda^s \| (\tau + |\xi|^2)^{\theta - 1} F(\xi) \|_{L^2_{x,t}} \lesssim \| F(\xi) \|_{N^{s,\theta - 1}_A}.
\]
(ii) $a > 0, b = 0$ : We prove the estimate for $a = 1$. Interpolation with the case $a = 0$ then leads us to the desired result. From the definition of the $S_\lambda$ norm, we have for $a = 1$, using (2.9) and Sobolev embedding

$$
(IV) = \lambda^{s-2} \left\| (\tau + |\xi|^2)^b \mathcal{F}(\lambda + |\partial_t|)C_{s,\lambda^2} \chi(t) \int_0^t e^{(t-s)\Delta}C_{s,\lambda^2}F_\lambda(s)ds \right\|_{L^2_{t,x}}^2
$$

$$
= \lambda^{s-2} \| (\lambda + |\partial_t|)(P^{(t)}_{s,\lambda^2}\chi(t) \int_0^t P^{(s)}_{s,\lambda^2} e^{-is\Delta}F_\lambda(s)ds) \|_{H^p_L^2}
\leq \lambda^{s-2} \| (\lambda + |\partial_t|)(P^{(t)}_{s,\lambda^2}\chi(t) \int_0^t P^{(s)}_{s,\lambda^2} e^{-is\Delta}F_\lambda(s)ds) \|_{W^{2,p}_{t,L^2}}
\leq \lambda^{s-2} \| (\lambda + |\partial_t|)(P^{(t)}_{s,\lambda^2}\chi(t) \int_0^t P^{(s)}_{s,\lambda^2} e^{-is\Delta}F_\lambda(s)ds) \|_{L^2_{t,L^2}}
$$

$$
+ \lambda^{s-2} \| (\lambda + |\partial_t|)(P^{(t)}_{s,\lambda^2}\chi(t) \int_0^t P^{(s)}_{s,\lambda^2} e^{-is\Delta}F_\lambda(s)ds)' \|_{L^2_{t,L^2}} =: (IV.1)+(IV.2),
$$

where the prime denotes derivative wrt $t$.

Using the product estimate for $a = 1, \frac{1}{p} = \frac{1}{2} + \frac{1}{7}$, we have

$$
(IV.1) \leq \lambda^{s-3} \| (\lambda + |\partial_t|)(\tau + |\xi|^2)^b F_{s,\lambda^2} \|_{L^2_{t,x}} \leq \lambda^3 \| (\tau + |\xi|^2)^b F_{s,\lambda^2} \|_{L^2_{t,x}}.
$$

On applying the product rule to (IV.2), we find that the first term is similar to (IV.1). Using the product estimate for $\frac{1}{p} = \frac{1}{4} + \frac{1}{7}$, the second term can be bounded by

$$
\lambda^{s-2} \| (\lambda + |\partial_t|)(P^{(t)}_{s,\lambda^2}\chi(t) \int_0^t P^{(s)}_{s,\lambda^2} e^{-is\Delta}F_\lambda(s)ds) \|_{L^2_{t,L^2}}
\leq \lambda^{s-3} \| (\lambda + |\partial_t|)(\tau + |\xi|^2)^b F_{s,\lambda^2} \|_{L^2_{t,x}} \leq \lambda^{s-1} \| C_{s,\lambda^2}F_\lambda \|_{L^2_{t,x}} \leq \| F_\lambda \|_{N^{s,\theta}_{t,x}^{-1}}.
$$

(iii) $b > 0, a = 0$ : From the definition of the $S_\lambda$ norm in the case $b > 0$ and using (2.9), with the prime denoting derivative wrt time, we have

$$
(IV) \leq \lambda^{s+20-2b} \left( \left\| P^{(t)}_{s,\lambda^2}\chi(t) \int_0^t P^{(s)}_{s,\lambda^2} e^{-is\Delta}F_\lambda(s)ds \right\|_{L^2_{t,x}} + \left\| (P^{(t)}_{s,\lambda^2}\chi(t) \int_0^t P^{(s)}_{s,\lambda^2} e^{-is\Delta}F_\lambda(s)ds)' \right\|_{L^2_{t,x}} \right)
$$

$$
=: (IV.3)+(IV.4).
$$

Unitarity of $e^{it\Delta}$ and an application of (2.4) gives

$$
(IV.3) \leq \lambda^{s+20-2b} \left\| e^{-it\Delta}C_{s,\lambda^2}F_\lambda(s)ds \right\|_{L^2_{t,x}} \leq \lambda^{s+20-2b} \| (\tau + |\xi|^2)^{-1} F_{s,\lambda^2} \|_{L^2_{t,x}}
\leq \lambda^{s+20-2b} \| (\tau + |\xi|^2)^{-1} F_{s,\lambda^2} \|_{L^2_{t,x}} \leq \lambda^{s} \| (\tau + |\xi|^2)^{-1} F_{s,\lambda^2} \|_{L^2_{t,x}}.
$$

Using the product rule, we have

$$
(IV.4) \leq \lambda^{s+b+20-2} \left( \left\| P^{(t)}_{s,\lambda^2}\chi(t) \int_0^t P^{(s)}_{s,\lambda^2} e^{-is\Delta}F_\lambda(s)ds \right\|_{L^2_{t,x}} + \| P^{(t)}_{s,\lambda^2}\chi(t) \int_0^t P^{(s)}_{s,\lambda^2} e^{-it\Delta}F_\lambda(s)ds \|_{L^2_{t,x}} \right)
$$

$$
=: (IV.41)+(IV.42)
$$

(IV.41) can be handled exactly in the same way as (IV.3) while for (IV.42), we decompose the time cutoff

$$
(IV.42) \leq \lambda^{s+2b+20-2} \left( \left\| P^{(t)}_{s,\lambda^2}\chi(t) \int_0^t P^{(s)}_{s,\lambda^2} e^{-it\Delta}F_\lambda(s)ds \right\|_{L^2_{t,x}} + \| P^{(t)}_{s,\lambda^2}\chi(t) \int_0^t P^{(s)}_{s,\lambda^2} e^{-it\Delta}F_\lambda(s)ds \|_{L^2_{t,x}} \right).
$$

The first term does not contribute while the second term by

$$
\lambda^{s+b+20-2} \left( \left\| (P^{(t)}_{s,\lambda^2}\chi(t) - \chi(t)) \int_0^t P^{(s)}_{s,\lambda^2} e^{-it\Delta}F_\lambda(s)ds \right\|_{L^2_{t,x}} + \| (\chi(t) \int_0^t P^{(s)}_{s,\lambda^2} e^{-it\Delta}F_\lambda(s)ds \|_{L^2_{t,x}} \right)
\leq \lambda^{s+b+20-4} \| P^{(t)}_{s,\lambda^2} e^{-it\Delta}F_\lambda \|_{L^2_{t,x}} \leq \lambda^{s+2b-3} \| e^{-it\Delta}C_{s,\lambda^2}F_\lambda \|_{L^2_{t,x}} \leq \lambda^s \| (\tau + |\xi|^2)^{-1} F_{s,\lambda^2} \|_{L^2_{t,x}},
$$

noting that the second term in the first display above vanishes.

\[\square\]

**Lemma 2.3.** (Energy inequality for the wave equation) Let $l, \beta, \theta, a \in \mathbb{R}$ be as defined in (2.2). For any $\lambda \in 2\mathbb{N}$ and a smooth time cutoff $\chi$, we have:

\[
\| \chi(t) e^{it\nabla} v_0 \|_{W^{l+\beta,\theta}} \leq \lambda^l \| v_0 \|_{L^2_t},
\]

\[
\| \chi(t) \mathcal{L}_W [G_\lambda] \|_{W^{l+\beta,\theta}} \leq \| G_\lambda \|_{L^{l+\theta,\theta-1}}.
\]
Proof. We will be short here as most of the steps will be same as in Lemma 2.2.
\[
\|\chi(t)e^{i|\nabla|v_0}\|_{W^{s,a}_x} = \lambda^{s-a}\|\chi(t)e^{i|\nabla|v_0}\|_{L^p_x L^q_t} + \lambda^{s-a}\|\langle \tau - |\xi|\rangle^{\beta} F((\lambda + |\partial_t|^a Q_{\leq \lambda^2} \chi(t)e^{i|\nabla|v_0})\|_{L^p_x L^q_t}
\]
\[
+ \lambda^{\beta-1}\|\langle \tau - |\xi|\rangle F(Q_{\leq \lambda^2} \chi(t)e^{i|\nabla|v_0})\|_{L^p_x L^q_t}
\]

The first term above can be controlled using properties (2.6) and (2.3) respectively. For the second term, we have using (2.9)
\[
\lambda^{s-a}\|\langle \tau - |\xi|\rangle^{\beta} F((\lambda + |\partial_t|^a Q_{\leq \lambda^2} \chi(t)e^{i|\nabla|v_0})\|_{L^p_x L^q_t}
\]
\[
= \lambda^{s-a}\|\langle \tau - |\xi|\rangle^{\beta} F((\lambda + |\tau|^\beta) P(t)^{\langle \lambda \rangle \chi(t)v_0})\|_{L^p_x L^q_t}
\]
\[
\leq \max \left\{ \lambda^{s-a}\|\langle \tau - |\xi|\rangle^{\beta+2\theta} F((\lambda + |\tau|^\beta) P(t)^{\langle \lambda \rangle \chi(t)}}\|_{L^p_x L^q_t}, \lambda^{\beta}\|\langle \tau - |\xi|\rangle^{\beta} F((\lambda + |\tau|^\beta) P(t)^{\langle \lambda \rangle \chi(t)}}\|_{L^p_x L^q_t} \right\}
\]
\[
\lesssim \lambda^{s-a}\|P(t)^{\langle \lambda \rangle \chi(t)v_0})\|_{H_t^p L_x^q} \lesssim \lambda^s \|v_0\|_{L^2_x}
\]

Similarly, for the last term, we have from the choice of the parameter \(\beta\)
\[
\lambda^{\beta-1}\|\langle \tau - |\xi|\rangle F(Q_{\leq \lambda^2} \chi(t)e^{i|\nabla|v_0})\|_{L^p_x L^q_t} = \lambda^{\beta-1}\|\langle \tau - |\xi|\rangle F(e^{i|\nabla|v_0})P(t)^{\langle \lambda \rangle \chi(t)}}\|_{L^p_x L^q_t} \lesssim \lambda^s \|v_0\|_{L^2_x}
\]

We consider the \(W^\lambda\) norm for the Duhamel integral now. The non-linearity \(G_\lambda\) in the \(L^p_t L_x^q\) term is decomposed into high and low modulation and is treated exactly as in Lemma 2.3. The low modulation norm of the Duhamel integral is written as
\[
\lesssim \lambda^{s-a}\|\langle \tau - |\xi|\rangle^{\beta} F((\lambda + |\partial_t|^a Q_{\leq \lambda^2} \chi(t)\int_0^t e^{i(t-s)|\nabla|Q_{\leq \lambda^2} G_\lambda(s)ds))\|_{L^p_x L^q_t}
\]
\[
+ \lambda^{s-a}\|\langle \tau - |\xi|\rangle^{\beta} F((\lambda + |\partial_t|^a Q_{\leq \lambda^2} \chi(t)\int_0^t e^{i(t-s)|\nabla|Q_{\leq \lambda^2} G_\lambda(s)ds))\|_{L^p_x L^q_t} \equiv (1.1)+(1.2)
\]

(1.1) is controlled using (2.4). For (1.2), we consider two cases:

(i) \(a > 0\): Using (2.4) and the choice of the parameters \(a\) and \(\beta\), we have
\[
(1.2) \lesssim \lambda^{s-a}\|\langle \tau - |\xi|\rangle^{\beta} F(Q_{\lambda^2} \chi(t)\int_0^t e^{i(t-s)|\nabla|Q_{\lambda^2} G_\lambda(s)ds))\|_{L^p_x L^q_t}
\]
\[
\lesssim \lambda^{s+a+2\theta}\|\langle \tau - |\xi|\rangle^{-1} F(Q_{\lambda^2} G_\lambda)\|_{L^p_x L^q_t} \lesssim \lambda^{s+a+2\theta} \|F(Q_{\lambda^2} G_\lambda)\|_{L^p_x L^q_t} \lesssim \lambda^{\beta-1}\|F(Q_{\lambda^2} G_\lambda)\|_{L^p_x L^q_t}.
\]

(ii) \(a = 0\): Using (2.9) and Sobolev embedding, (1.2) can be bounded by
\[
\lambda^{\beta}\|P(t)^{\langle \lambda \rangle \chi(t)}\|_{H_t^p L_x^q}, \quad \|v_0\|_{L^2_x} = \frac{2}{3-2\theta},
\]

which is written in equivalent norm as
\[
\lambda^s \left( \left\|P(t)^{\langle \lambda \rangle \chi(t)}\|_{L^p_t L_x^q} \right\|_{L^p_t L_x^q} + \left\|P(t)^{\langle \lambda \rangle \chi(t)}\|_{L^p_t L_x^q} \right\|_{L^p_t L_x^q} \right) \equiv (1.21)+(1.22)
\]
\[
(1.21) \lesssim \lambda^s \|\chi(t)\|_{L^p_t L_x^q} \left\|P(t)^{\langle \lambda \rangle \chi(t)}\|_{L^p_t L_x^q} \right\|_{L^p_t L_x^q} \lesssim \lambda^{s-2}\|Q_{\lambda^2} G_\lambda\|_{L^p_t L_x^q}.
\]
\[
(1.22) \lesssim \lambda^s \left\|P(t)^{\langle \lambda \rangle \chi(t)}\right\|_{L^p_t L_x^q} \left\|P(t)^{\langle \lambda \rangle \chi(t)}\right\|_{L^p_t L_x^q} \equiv (1.21)+(1.22)
\]

(1.21) can be treated like (1.21) while for (1.22), we decompose the time cutoff to get
\[
(1.22) \lesssim \lambda^s \left\|P(t)^{\langle \lambda \rangle \chi(t)}\right\|_{L^p_t L_x^q} \lesssim \lambda^{s-2}\|Q_{\lambda^2} G_\lambda\|_{L^p_t L_x^q}
\]

The first term in the above sum vanishes and for the second, we use Bernstein’s and Hölder’s inequality to obtain the bound
\[
\lambda^{s-1}\|\langle \tau - |\xi|\rangle F(Q_{\lambda^2} \chi(t)\int_0^t e^{i(t-s)|\nabla|Q_{\lambda^2} G_\lambda(s)ds))\|_{L^p_x L^q_t} \lesssim \lambda^{s-1}\|F(Q_{\lambda^2} G_\lambda)\|_{L^p_x L^q_t}.
\]

The high modulation Duhamel integral is again decomposed with the first term controlled using (2.4) i.e
For the other term, we have
\[
\lambda^{\beta - 1} \left\| \langle \tau - |\xi| \rangle F(Q_{\tilde{x}} \chi(t) \int_0^t e^{i(t-s)|\nabla|} Q_{\tilde{x}} G_\lambda(s) ds) \left\|_{L^2_{t,\tilde{x}}}
\right.
\]
\[
= \lambda^{\beta - 1} \left\| \left[ \frac{p(t)}{\tilde{x}} \chi(t) \int_0^t p(s) e^{-ist|\nabla|} G_\lambda(s) ds \right]_{H^1 L^2_t} \right.
\]
\[
\lesssim \lambda^{\beta - 1} \left\| \left[ \frac{p(t)}{\tilde{x}} \chi(t) \int_0^t p(s) e^{-ist|\nabla|} G_\lambda(s) ds \right]_{L^2_{t,\tilde{x}}} + \lambda^{\beta - 1} \left\| \left[ \frac{p(t)}{\tilde{x}} \chi(t) \int_0^t p(s) e^{-ist|\nabla|} G_\lambda(s) ds \right]_{L^2_{t,\tilde{x}}} \right.
\]
\[
=: (\text{II}) + (\text{III}).
\]
Then, using (2.4), the choice of \( \beta \) and \( |m|_W \gtrsim 1 \).
\[
(\text{II}) \lesssim \lambda^{\beta - 1} \left\| \chi(t) \int_0^t p(s) e^{-ist|\nabla|} G_\lambda(s) ds \right\|_{L^2_{t,\tilde{x}}} \lesssim \lambda^{\beta - 1} \left\| \langle \tau - |\xi| \rangle F(Q_{\tilde{x}} \chi(t) G_\lambda) \right\|_{L^2_{t,\tilde{x}}}
\]
\[
\lesssim \lambda^{1 - a} \left\| \langle \tau - |\xi| \rangle^a F((\lambda + |\partial_\theta|)^a K_{\tilde{x}} G_\lambda) \right\|_{L^2_{t,\tilde{x}}}.
\]
\[
(\text{III}) \lesssim \lambda^{\beta - 1} \left\| \left[ \frac{p(t)}{\tilde{x}} \chi(t) \int_0^t p(s) e^{-ist|\nabla|} G_\lambda(s) ds \right]_{L^2_{t,\tilde{x}}} + \lambda^{\beta - 1} \left\| \left[ \frac{p(t)}{\tilde{x}} \chi(t) \int_0^t p(s) e^{-ist|\nabla|} G_\lambda(s) ds \right]_{L^2_{t,\tilde{x}}} \right.
\]
\[
The first term of (III) is treated like (II) and the second decomposed as follows
\[
\lesssim \lambda^{\beta - 1} \left\| \left[ \frac{p(t)}{\tilde{x}} \chi(t) \int_0^t p(s) e^{-ist|\nabla|} G_\lambda(s) ds \right]_{L^2_{t,\tilde{x}}} \right.
\]
\[
\lesssim \lambda^{\beta - 1} \left\| \left[ \frac{p(t)}{\tilde{x}} \chi(t) \int_0^t p(s) e^{-ist|\nabla|} G_\lambda(s) ds \right]_{L^2_{t,\tilde{x}}} + \lambda^{\beta - 1} \left\| \left[ \frac{p(t)}{\tilde{x}} \chi(t) \int_0^t p(s) e^{-ist|\nabla|} G_\lambda(s) ds \right]_{L^2_{t,\tilde{x}}} \right.
\]
\[
where the choice of the parameter \( \beta \) and \( |m|_W \gtrsim 1 \) ensure that the last inequality holds.
\]
2.6. Multilinear estimates. We recall some already known estimates in this section.

Lemma 2.4. ([3, Proposition 4.4], [2, Corollary 3.6]) Let \( d \in \{2, 3\} \) and \( f, g_1, g_2 \in L^2(\mathbb{R}^{d+1}) \) be such that
\[
\|f\|_{L^2} = \|g_1\|_{L^2} = \|g_2\|_{L^2} = 1.
\]
For \( k = 1, 2 \) let
\[
\text{supp}(f) \subset \left\{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d : \frac{\lambda}{2} \leq |\xi| \leq 2\lambda \right\} \cap W^\pm_L,
\]
\[
\text{supp}(g_k) \subset \left\{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d : \frac{\lambda_k}{2} \leq |\xi| \leq 2\lambda_k \right\} \cap S_L,
\]
where the frequencies \( \lambda, \lambda_1, \lambda_2 \) and the modulations \( L, L_1, L_2 \) satisfy
\[
1 \ll \lambda \lesssim \lambda_1 \sim \lambda_2, \quad L, L_1, L_2 \lesssim \lambda_2^2.
\]
Then, for
\[
I(f, g_1, g_2) = \int f(\xi_1 - \xi_2) g_1(\xi_1) g_2(\xi_2) d\xi_1 d\xi_2,
\]
\[
\xi_1 = (\tau_1, \xi_1), i = 1, 2,
\]
the following estimate holds
\[
\|I(f, g_1, g_2)\| \lesssim \frac{LL_1L_2^2}{\lambda_1^7} \log \lambda_1.
\]

Remark 2.5. For \( d = 2 \), there is no log term in the RHS of (2.11) but this does not affect the following analysis.

Lemma 2.6. (Bilinear Strichartz estimates) ([3, Proposition 4.3], [2, Proposition 3.3]) Let \( \hat{u}, \hat{v} \) denote the space-time Fourier transforms of \( u, v \) respectively for \( u, v \in L^2(\mathbb{R}^{d+1}), \ i = 1, 2 \).

(i) Let \( d \in \{2, 3\} \). Let \( u_k \) be dyadically Fourier-localised such that
\[
\text{supp}(\hat{u}_k) \subset \left\{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d : \frac{\lambda_k}{2} \leq |\xi| \leq 2\lambda_k \right\} \cap S_L,
\]
for \( L, \lambda_k \gtrsim 1 \). Then the following estimate holds
\[
\|u_k u_k\|_{L^2_{t,x}} \lesssim \frac{\lambda_k^d_1 L_1 L_2^2}{\lambda_1^d_1} \left\| u_k \right\|_{L^2_{t,x}} \left\| u_k \right\|_{L^2_{t,x}}.
\]

(ii) Let \( d \leq 3 \). Let \( u, v \) be dyadically Fourier-localised such that
\[
\text{supp}(\hat{u}) \subset \left\{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d : \frac{\lambda_1}{2} \leq |\xi| \leq 2\lambda_1 \right\} \cap S_L,
\]
\[
\text{supp}(\hat{v}) \subset \left\{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d : \frac{\lambda_2}{2} \leq |\xi| \leq 2\lambda_2 \right\} \cap S_L,
\]
for \( L, \lambda_k \gtrsim 1 \). Then the following estimate holds
\[
\|u v\|_{L^2_{t,x}} \lesssim \min\{\lambda_1, 2\} \frac{\lambda_2^{d-1}}{\lambda_1^{d-1}} (L_1 L_2)^2 \left\| u \right\|_{L^2_{t,x}} \left\| v \right\|_{L^2_{t,x}}.
\]
Remark 2.7. (i) If the frequencies $\lambda_1$ and $\lambda_2$ are such that $\lambda_1 \ll \lambda_2$ or $\lambda_2 \ll \lambda_1$, then the estimate (2.12) holds for $d = 1$ as well.
(ii) The estimates (2.12) and (2.13) remain valid if we replace the functions on the LHS by their complex conjugates.

Lemma 2.8. (Product estimate)[8, Lemma 2.7] Let $a \in \mathbb{R}$, $1 \leq p, q, r, s, t \leq \infty$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and $\frac{1}{s} = \frac{1}{t} + \frac{1}{r}$.

Then, for all $\mu > 0$,

$$
\|\langle \mu + |\partial_t|\rangle^a \langle \mu + |\partial_x|\rangle^b v \rangle \|_{L_t^r L_x^s} \lesssim \mu^{-|a|} \|\langle \mu + |\partial_t|\rangle^a v \rangle \|_{L_t^r L_x^s} \|\langle \mu + |\partial_x|\rangle^b u \rangle \|_{L_t^r L_x^s}.
$$

With all the required tools at our disposal, we head to prove the multilinear estimates.

3. Multilinear estimates for Schrödinger non-linearity

Theorem 3.1. Let $d \leq 3$ and $s, t$ in the range (1.3). There exist $a, b, s, t, l, \beta, \theta \in \mathbb{R}$ such that the estimate

$$
\|u R e(v)\|_{N_{s, l, \beta, \theta}^+} \lesssim \|u\|_{S_{s, t, \beta, \theta}^+} \|v\|_{W_{s, t, \beta, \theta}}^2
$$

(3.1)

holds.

Proof. We choose the parameters $a, b, s', l, \beta, \theta'$ as in (2.2) and begin by noting the following characterisation of the $N_{s, l, \beta, \theta}^+$-norm for the non-linearity in the case $0 \leq a < \frac{1}{4}$:

$$
\|F_{\lambda} \|_{N_{s, l, \beta, \theta}^+} \approx \lambda^s \|\tau + |\xi|^2 \beta^{-1} F(C_{\lambda < \lambda^2} F_{\lambda})\|_{L_2^s L_2^x}.
$$

(3.2)

For $\frac{1}{4} \leq a < \frac{1}{2}$ using Bernstein's inequality and Sobolev embedding, we have

$$
\lambda^{a+2\theta-3} \|P_{\lambda < \lambda^2} P_{\lambda} \|_{L_2^s L_2^x} \lesssim \lambda^{a+2\theta-2-a} \|P_{\lambda < \lambda^2} F_{\lambda} \|_{L_2^s L_2^x}
$$

(3.3)

$$
\lesssim \lambda^{a+2\theta-2-2a} \|\lambda + |\partial_t|\|_{\lambda < \lambda^2} F_{\lambda} \|_{L_2^s L_2^x}.
$$

(3.4)

$$
\lesssim \lambda^{a+2\theta-2-2a+b} \|\lambda + |\partial_t|\|_{\lambda < \lambda^2} C_{\lambda > \lambda^0} \|P_{\lambda} (uv)\|_{L_2^s L_2^x}.
$$

(3.5)

In the case $\frac{1}{2} \leq a < \frac{1}{2}$, we additionally need to prove the following:

$$
\left( \sum_{\lambda_0 \in 2^\mathbb{N}} \lambda_0^{2s' - 2a + b} \|P_{\lambda < \lambda^2} P_{\lambda} (uv)\|_{L_2^s L_2^x} \right)^\frac{1}{2} \lesssim \|u\|_{L_\infty H^s_x} \|v\|_{L_\infty H^s_x}.
$$

(3.6)

We now proceed to prove (3.4) and (3.5). We decompose the non-linearity $uv$ as

$$
uv = \sum_{\lambda_0 \in 2^\mathbb{N}} P_{\lambda_0} (uv)
$$

Further, we distinguish the high-low, low-high and the balanced interactions as follows:

$$
P_{\lambda_0} (uv) = \sum_{\lambda_1 \in 2^\mathbb{N}} P_{\lambda_0} (u_{\lambda_1} v) = \sum_{\lambda_1 \in \lambda_0 2^\mathbb{N}} P_{\lambda_0} (u_{\lambda_1} v) + \sum_{\lambda_0 < \lambda_1 \in 2^\mathbb{N}} P_{\lambda_0} (u_{\lambda_1} v) + \sum_{\lambda_0 \sim \lambda_1 \in 2^\mathbb{N}} P_{\lambda_0} (u_{\lambda_1} v).
$$

We state that high (low) modulation for the spatially localised Schrödinger solution $u_{|\xi|}$ means $\langle \tau + |\xi|^2 \rangle \gg (\ll) |\xi|^2$, while for the wave $v_{|\xi|}$, it means $\langle \tau - |\xi|^2 \rangle \gg (\ll) |\xi|^2$. We abbreviate high modulation by $H$ and low modulation by $L$. The subscripts 1 and 2 are appended with $\tau$ and $\xi$ to distinguish the temporal frequencies and the spatial frequencies of the Schrödinger and wave solutions. The output temporal frequencies are given by $\tau_0 = \tau_1 + \tau_2$. For the Schrödinger solution, low modulation occurs when the temporal frequencies are of size $\sim |\xi|^2 (\tau = -|\xi|^2)$. For the wave solution, temporal frequencies of size $\sim |\xi|^2 (\tau = |\xi|^2)$ lead to a free wave solution. Other than that, the sizes of the modulation and the temporal frequencies go hand in hand for a frequency localised wave solution.

All the possible interactions are treated individually. In cases with low output modulation, the required bilinear estimates are reduced to trilinear estimates by duality using the property (vii) from section 2.4. To distinguish
the dual term, we use \(-s\) and \(-l\) instead of \(s\) and \(l\), respectively, in the subscripts of the norms for the Schrödinger and wave components. Each subcase can be summed up depending on the size of the interacting frequencies. The constraints required to obtain the estimate for the full norms are listed at the end of each subcase, and we do not exclude points with logarithmic losses since our result does not cover the boundaries.

**Remark** 3.2. In the following, we treat the case of high spatial (and temporal) frequencies \(|\xi|, |\tau_1| \gg 1, i = 0, 1, 2\). In most of the cases, the estimates for the low frequencies (all the frequencies are of size \(\sim 1\) or the size of the lowest frequency in the interaction is \(\sim 1\)) follow using the same arguments without having to decompose the space-time Fourier supports of the interacting solutions. Hence, we do not mention it explicitly for each case. However, in some cases the arguments need to be modified for the low frequency cases. We shall do it wherever required.

**Case I. Low to high interaction** \((\lambda_1 \ll \lambda_0)\)

We decompose \(u_{\lambda_1}\) and \(v_{\lambda_0}\) as follows:

\[
u_{\lambda_1} = C_{< \lambda_1^2} u_{\lambda_1} + C_{> \lambda_1^2} u_{\lambda_1}, \quad \nu_{\lambda_0} = C_{< \lambda_0^2} v_{\lambda_0} + C_{> \lambda_0^2} v_{\lambda_0}.
\]

The following interactions can be distinguished on the basis of the size of the modulation:

1. \(H \times H \rightarrow H\)

We require to prove (3.5). Using the size of the modulation, the product estimate and Bernstein’s inequality respectively, we have

\[
\lambda_1^{\theta - 2a + b} \| (\tau + |\xi|^2)^{1 - \theta} F((\lambda_1 + |\partial_1|)a C_{< \lambda_1^2} (C_{> \lambda_1^2} u_{\lambda_1}, Q_{> \lambda_0^2} v_{\lambda_0}) \|_{L^2_{t, \xi}} \\
\lesssim \lambda_1^{\theta - 2a + b + 2\theta - 2} \lambda_1^{\frac{a}{2}} \| (\lambda_1 + |\partial_1|)a C_{< \lambda_1^2} u_{\lambda_1} \|_{L^2_{t, \xi}} \langle \lambda_1 + |\partial_1| \rangle^a \| Q_{> \lambda_0^2} v_{\lambda_0} \|_{L^2_{t, \xi}} \\
\lesssim \lambda_1^{\theta - 2a + b + 2\theta - 3} \lambda_1^{\frac{a}{2}} \| u_{\lambda_1} \|_{S_{-1, \xi}^{1, \theta}} \| v_{\lambda_0} \|_{W^1_{1, \theta}}.
\]

The last inequality above follows from (2.5). Since \(s' + 2\theta = s + 2\theta\), we can sum up the subcase in \(\lambda_1 \ll \lambda_0\) to obtain (3.5) provided \(s - \beta \leq 3 - 2\theta - b\) and \(\beta \geq -3 + \frac{d}{2} + 2\theta + a + b\).

2. \(H \times H \rightarrow L\)

We consider two cases for the temporal frequencies:

\(a.\ |\tau_1| \lesssim \lambda_1^2, |\tau_2| \sim \lambda_1^2\)

We prove (3.4) using duality. We consider the expression I defined in (2.10) and use Cauchy-Schwarz inequality as follows:

\[
|I(\mathcal{F}(Q_{< \lambda_1^2} v_{\lambda_0}, \mathcal{F}(C_{< \lambda_1^2} w_{\lambda_0}), \mathcal{F}(C_{\lambda_1^2 \leq \lambda_1^2} u_{\lambda_1})) | \lesssim \| Q_{< \lambda_1^2} v_{\lambda_0} \|_{L^2_{t, \xi}} \| C_{\lambda_1^2 \leq \lambda_1^2} u_{\lambda_1} \|_{W_{1, \theta}}.
\]

On decomposing the space-time Fourier supports of \(C_{\lambda_1^2 \leq \lambda_1^2} u_{\lambda_1}, \mathcal{C}_{< \lambda_1^2} w_{\lambda_0}\), into pieces \(L_1\) and \(L_0\), respectively and applying the bilinear estimate (2.12), we get

\[
(3.7) \lesssim \| Q_{< \lambda_1^2} v_{\lambda_0} \|_{L^2_{t, \xi}} \sum_{L_0 \ll \lambda_0^2, L_1 \lesssim \lambda_1^2} (L_0 L_1)^{\frac{1}{2}} \lambda_1^{\frac{1}{2}} \lambda_0^{-\frac{1}{2}} \| C_{L_1^1} u_{\lambda_1} \|_{L^2_{t, \xi}} \| C_{L_0} w_{\lambda_0} \|_{L^2_{t, \xi}} \\
\lesssim \lambda_1^{-\beta - \frac{d}{2} + s + 2\theta} \lambda_1^{-\frac{1}{2}} \lambda_1^{\frac{1}{2}} \lambda_1^{-\frac{1}{2}} \| u_{\lambda_1} \|_{S_{-1, \xi}^{1, \theta}} \| v_{\lambda_0} \|_{W_{1, \theta}^{1, \theta}} \| w_{\lambda_0} \|_{S_{2, \theta}^{-1, 1, -s}}.
\]

We require \(s - \beta \leq \frac{d}{2} - 2\theta\) and \(\beta \geq -3 + 2\theta + \frac{d}{2} + a\) to sum the above up and obtain the estimate for the full norms. In case \(\lambda_1 \sim \lambda_0 \sim 1\), we have

\[
|I(\mathcal{F}(Q_{< \lambda_1^2} v_{\lambda_0}), \mathcal{F}(C_{< \lambda_1^2} w_{\lambda_0}), \mathcal{F}(C_{\lambda_1^2 \leq \lambda_1^2} u_{\lambda_1})) | \lesssim \| Q_{< \lambda_1^2} v_{\lambda_0} \|_{L^2_{t, \xi}} \| C_{\lambda_1^2 \leq \lambda_1^2} u_{\lambda_1} \|_{L^2_{t, \xi}} \| C_{< \lambda_1^2} w_{\lambda_0} \|_{L^2_{t, \xi}} \\
\lesssim \| u_{\lambda_1} \|_{S_{-1, \xi}^{1, \theta}} \| v_{\lambda_0} \|_{W_{1, \theta}^{1, \theta}} \| w_{\lambda_0} \|_{S_{2, \theta}^{-1, 1, -s}}.
\]
We require

|I(F(Q_{\lambda^0}v_0), F(C_{\lambda^0}w_0), F(C_{\lambda^0}u_1))|
\lesssim \lambda_0^2 \|C_{\lambda^0}u_1\|_{L_t^\infty L_x^2} \|C_{\lambda^0}w_0\|_{L_t^\infty L_x^2} \\
\lesssim \lambda_0^{\frac{3}{2} - s + 2a} \lambda_0^{-2a - \beta + 2(\theta' - \theta) + s} \lambda_1^{\frac{1}{2} - 2a} \|C_{\lambda^0}u_1\|_{L_t^\infty L_x^2} \\
\times C_{\lambda^0}w_0 \|C_{\lambda^0}u_1\|_{L_t^\infty L_x^2} \\
\lesssim \lambda_0^{\frac{3}{2} - s + 2a} \lambda_0^{-2a - \beta + 2(\theta' - \theta) + s} \|u_1\|_{S^{s,1,0}_x W^{1,s,0}_t} \|w_0\|_{S^{-s,1,0}_x}.

Note that the penultimate inequality follows from Bernstein’s inequality for $\frac{1}{p} = \theta - \frac{1}{2}$ while the ultimate comes from the embedding relation (2.7). For summability, we require $s - \beta \leq 2 + (\theta' - \theta) + 2a$ and $\beta \geq -2 + \frac{d}{2}$.

3. $H \times L \to H$

Using the size of the modulation, the product estimate and Bernstein’s inequality, we obtain

\[
\lambda_0^{s' - 2a + b} \|C_{\lambda^0}u_1\|_{L_t^\infty L_x^2} \|C_{\lambda^0}w_0\|_{L_t^\infty L_x^2} \\
\lesssim \lambda_0^{s' - 2a + b + 2(\theta' - \theta) + s} \|C_{\lambda^0}u_1\|_{L_t^\infty L_x^2} \|C_{\lambda^0}w_0\|_{L_t^\infty L_x^2} \\
\lesssim \lambda_0^{s' - 2a + b + 2(\theta' - \theta) + s} \|C_{\lambda^0}u_1\|_{L_t^\infty L_x^2} \|C_{\lambda^0}w_0\|_{L_t^\infty L_x^2} \\
\times \lambda_1^{\frac{1}{2} - s + a - 2\theta} \|u_1\|_{S^{s,1,0}_x W^{1,s,0}_t} \|w_0\|_{S^{-s,1,0}_x}.
\]

Provided $s - l \leq 2 - 2\theta + a - b$ and $l \geq -2 + \frac{d}{2} + b$, we can sum the above to obtain (3.5).

4. $H \times L \to L$

From the relation $|\tau_0 + |\xi_0|^2| = |\tau_1 + |\xi_1|^2| + |\xi_2| + |\xi_0|^2| \ll \lambda_0^2$, we conclude that $|\tau_1| \sim \lambda_0^2$. Using Hölder’s inequality, we have

\[
|I(F(Q_{\lambda^0}v_0), F(C_{\lambda^0}w_0), F(C_{\lambda^0}u_1))| \lesssim \|C_{\lambda^0}u_1\|_{L_t^\infty L_x^2} \|C_{\lambda^0}w_0\|_{L_t^\infty L_x^2}.
\]

The spatial frequency support of $C_{\lambda^0}u_1$ is localised to frequencies of size $\sim \lambda_0$. Using orthogonality, we can reduce the estimate to the case when the spatial supports of $Q_{\lambda^0}v_0$ and $C_{\lambda^0}w_0$ are also localised to frequencies of size $\sim \lambda_1$. Noting this, decomposing the space-time Fourier supports of $Q_{\lambda^0}v_0$ and $C_{\lambda^0}w_0$ into pieces $L_2$ and $L_0$ respectively and applying the bilinear estimate (2.13), we have

\[
|I(F(Q_{\lambda^0}v_0), F(C_{\lambda^0}w_0), F(C_{\lambda^0}u_1))| \lesssim \|C_{\lambda^0}u_1\|_{L_t^\infty L_x^2} \sum_{L_0, L_2 = \lambda_0^2} \|Q_{L_2}v_0\|_{L_t^\infty L_x^2} \|C_{L_0}w_0\|_{L_t^\infty L_x^2} \\
\lesssim \lambda_0^{-1 - \frac{3}{2} + 2(\theta' - \theta) - 2a} \lambda_1^{\frac{1}{2} - s + 2a} \lambda_1^{\frac{1}{2} - 2a} \|C_{\lambda^0}u_1\|_{L_t^\infty L_x^2} \\
\times \lambda_0^{s' - 2a + b + 2(\theta' - \theta) + s} \|u_1\|_{S^{s,1,0}_x W^{1,s,0}_t} \|w_0\|_{S^{-s,1,0}_x}.
\]

We require $s - l \leq \frac{3}{2} - 2(\theta' - \theta) + 2a$ and $l \geq -2 + \frac{d}{2}$ for the summability of the above estimate.
5. $L \times H \to H$

Using the size of the modulation, the product estimate and Bernstein’s inequality, we have

\[
\lambda_0^{s'} - 2a + b \| (\tau + |\xi|^2)^{\theta - 1} F((\lambda_0 + |\partial_t|)^a C_{\xi,\lambda}^\gamma u_{\lambda_0} Q_{\lambda_0} v_{\lambda_0}) \|_{L^2_x} \leq \lambda_0^{s' - 3a + b + 2d - 2} \lambda_1^{\frac{d}{a}} \left( \lambda_0 + |\partial_t| \right)^a C_{\xi,\lambda}^\gamma u_{\lambda_0} \|_{L^2_x} \| (\lambda_0 + |\partial_t|)^a Q_{\lambda_0} v_{\lambda_0} \|_{L^2_x}
\]

Both the cases above can be summed up provided $s - \delta \lesssim 3 - 2\theta - b$ and $\beta \geq -3 + \frac{d}{2} + 2\theta + a + b$.

6. $L \times H \to L$

If $|\tau_2| \gg \lambda_0^2$, the output will have a high modulation. Hence it suffices to consider $|\tau_2| \sim \lambda_0^2$. Then, we have

\[
|I(F(Q_{\lambda_0} v_{\lambda_0}), F(C_{\xi,\lambda}^\gamma w_{\lambda_0}))| \lesssim \| Q_{\lambda_0} v_{\lambda_0} \|_{L^2_x} \sum_{L_0 < \lambda_0^2, L_0 \ll \lambda_1^2} \left( \frac{L_0}{L_1} \right) \frac{\lambda_0^{\beta - s}}{\lambda_1^{d - s}} \| C_{L_1} u_{\lambda_1} \|_{L^2_x} \| C_{L_0} w_{\lambda_0} \|_{L^2_x} \leq \lambda_0^{\beta - s} \| (\tau + |\xi|^2)^{1 - \theta} F(C_{\xi,\lambda}^\gamma u_{\lambda_1}) \|_{L^2_x} \lambda_0^{\beta - 1} \| (\tau - |\xi|) F(Q_{\lambda_0} v_{\lambda_0}) \|_{L^2_x} \times \lambda_0^{-s} \| (\tau + |\xi|^2)^{1 - \theta} F(C_{\xi,\lambda}^\gamma w_{\lambda_0}) \|_{L^2_x}
\]

We decompose the space-time Fourier supports of $C_{\xi,\lambda}^\gamma u_{\lambda_1}$ and $\overline{C_{\xi,\lambda}^\gamma w_{\lambda_0}}$ into pieces $L_1$ and $L_0$ respectively, and apply the bilinear estimate (2.12) to obtain

\[
(3.9) \lesssim \| Q_{\lambda_0} v_{\lambda_0} \|_{L^2_x} \sum_{L_0 < \lambda_0^2, L_0 \ll \lambda_1^2} \left( \frac{L_0}{L_1} \right) \frac{\lambda_0^{\beta - s}}{\lambda_1^{d - s}} \| C_{L_1} u_{\lambda_1} \|_{L^2_x} \| C_{L_0} w_{\lambda_0} \|_{L^2_x}
\]

We require $s - \delta \leq \frac{d}{2} - 2\theta$ and $\beta \geq -3 + \frac{d}{2} + 2\theta$ to sum the above estimate and obtain (3.4).

For $\lambda_1 \sim \lambda_0 \sim 1$, $d = 1$, we have

\[
|I(F(Q_{\lambda_0} v_{\lambda_0}), F(C_{\xi,\lambda}^\gamma w_{\lambda_0}), F(C_{\xi,\lambda}^\gamma u_{\lambda_1}))| \lesssim \| Q_{\lambda_0} v_{\lambda_0} \|_{L^2_x} \| C_{\xi,\lambda}^\gamma u_{\lambda_1} \|_{L^2_x} \| C_{\xi,\lambda}^\gamma w_{\lambda_0} \|_{L^2_x} \leq \| v_{\lambda_0} \|_{W^{1,\infty}_x} \| u_{\lambda_1} \|_{S^{1,\delta}_x} \| w_{\lambda_0} \|_{S^{-1,1-\delta}}.
\]

7. $L \times L \to H$

For $d = 3$ : Using the size of the modulation, the product estimate and the endpoint Strichartz space $L^2_x L^6_t$, we have

\[
\lambda_0^{s' - 2a + b} \| (\tau + |\xi|^2)^{\theta - 1} F((\lambda_0 + |\partial_t|)^a C_{\xi,\lambda}^\gamma u_{\lambda_0} Q_{\lambda_0} v_{\lambda_0}) \|_{L^2_x} \leq \lambda_0^{s' - 3a + b + 2d - 2} \lambda_1^\frac{d}{a} \| (\lambda_0 + |\partial_t|)^a C_{\xi,\lambda}^\gamma u_{\lambda_1} \|_{L^2_x} \| (\lambda_0 + |\partial_t|)^a Q_{\lambda_0} v_{\lambda_0} \|_{L^2_x} \leq \lambda_0^{s' - a + b + 2d - 2} \lambda_1^\frac{d}{a} \| u_{\lambda_1} \|_{S^{1,\delta}_x} \| v_{\lambda_0} \|_{W^{1,\infty}_x}, \lambda_0 + \lambda_1^\delta \sim \lambda_0^\delta.
\]

The constraints $s - l \leq 2 - 2\theta + a - b$ and $l \geq -\frac{d}{2} + 2\theta + b$ are required to sum the above cases.

For $d = 2$ : We consider two subcases for the size of the wave modulation:

a. $(\tau - |\xi|) \ll \lambda_0$: For $\lambda_0 \ll |\tau_2| \ll \lambda_0^2$, the wave has a modulation of size $\gg \lambda_0$, so it suffices to consider
\[ |\tau_2| \sim \lambda_0. \] Using the size of the output modulation and the bilinear estimate (2.13), we have
\[
\lambda_0^{\sigma - 2a + b} \left\| \{ \tau + |\xi|^2 \}^{\theta - 1} F(\lambda_0 + |\partial_t|) \right\| \lesssim \lambda_0^{\lambda - 2a + b + 2b' - 2} (\lambda_0 + \lambda_0^2)^\theta \sum_{L_1 \leq \lambda_0^2} (L_1 L_2)^{\frac{1}{\lambda_0^2}} \| C_{L_1} u_{L_1} \|_{L_{p_1}^{\infty} L_{q_1}^\infty} \| Q_{L_2} P_{\infty} Q \|_{L_{p_2}^{\infty} L_{q_2}^\infty} \|_{L_{p_3}^{\infty} L_{q_3}^\infty}.
\]
In the case \( 1 \sim \lambda_1 \ll \lambda_0 \), we have
\[
\lambda_0^{\sigma - 2a + b} \left\| \{ \tau + |\xi|^2 \}^{\theta - 1} F(\lambda_0 + |\partial_t|) \right\| \lesssim \lambda_0^{\lambda - 2a + b + 2b' - 2} (\lambda_0 + \lambda_0^2)^\theta \sum_{L_1 \leq \lambda_0^2} (L_1 L_2)^{\frac{1}{\lambda_0^2}} \| C_{L_1} u_{L_1} \|_{L_{p_1}^{\infty} L_{q_1}^\infty} \| Q_{L_2} P_{\infty} Q \|_{L_{p_2}^{\infty} L_{q_2}^\infty} \|_{L_{p_3}^{\infty} L_{q_3}^\infty}.
\]

b. \( \lambda_0 \ll \langle \tau - |\xi| \rangle \ll \lambda_0^2 \) : We use the size of the modulation, the product estimate and Bernstein’s inequality to obtain
\[
\lambda_0^{\sigma - 2a + b} \left\| \{ \tau + |\xi|^2 \}^{\theta - 1} F(\lambda_0 + |\partial_t|) \right\| \lesssim \lambda_0^{\lambda - 2a + b + 2b' - 2} (\lambda_0 + \lambda_0^2)^\theta \sum_{L_1 \leq \lambda_0^2} (L_1 L_2)^{\frac{1}{\lambda_0^2}} \| C_{L_1} u_{L_1} \|_{L_{p_1}^{\infty} L_{q_1}^\infty} \| Q_{L_2} P_{\infty} Q \|_{L_{p_2}^{\infty} L_{q_2}^\infty} \|_{L_{p_3}^{\infty} L_{q_3}^\infty}.
\]
Cases a and b can be summed up provided \( s - l \leq 2 - \theta + a - b \) and \( l \geq -1 + \theta + b \).

\( d = 1 \) : As for \( d = 3 \), we have
\[
\lambda_0^{\sigma - 2a + b} \left\| \{ \tau + |\xi|^2 \}^{\theta - 1} F(\lambda_0 + |\partial_t|) \right\| \lesssim \lambda_0^{\lambda - 2a + b + 2b' - 2} (\lambda_0 + \lambda_0^2)^\theta \sum_{L_1 \leq \lambda_0^2} (L_1 L_2)^{\frac{1}{\lambda_0^2}} \| C_{L_1} u_{L_1} \|_{L_{p_1}^{\infty} L_{q_1}^\infty} \| Q_{L_2} P_{\infty} Q \|_{L_{p_2}^{\infty} L_{q_2}^\infty} \|_{L_{p_3}^{\infty} L_{q_3}^\infty}.
\]
Provided \( s - l \leq 2 - 2\theta + a - b \) and \( l \geq -\frac{3}{2} + 2\theta + b \), we can sum the above estimates to obtain (3.5).

From the conclusions made at the end of each subcase, we see that following conditions on the parameters are required to ensure the validity of the estimates (3.4) and (3.5) for \( d \leq 3 \):

- \( l \geq -\frac{3}{2} + 2\theta + b \)
- \( \beta \geq -\frac{3}{2} + 2\theta + a + b \)
- \( s - l \leq 2 - 2\theta + a - b \)
- \( s - \beta \leq \min \left\{ 3 - 2\theta - b, \frac{5}{2} - 2\theta \right\} \).

\textbf{Case II. High to high interaction (} \( \lambda_1 \sim \lambda_0 \))

For \( \mu \ll \lambda_1 \), we decompose \( u_{\lambda_1} \) and \( v_{\mu} \) as follows:
\[
u_{\lambda_1} = C_{\infty} u_{\lambda_1} + C_{\pi} v_{\lambda_1}, \quad v_{\mu} = Q_{\pi} v_{\mu} + Q_{\infty} v_{\mu}.
\]
The following interactions can be distinguished on the basis of the size of the modulation:

1. \( H \times H \rightarrow H \)

We consider two cases for the temporal frequencies:

\textbf{a.} \( |\tau_1| \ll \lambda_0^2 \) or \( |\tau_2| \ll \lambda_0^2 \)

Since at least one of the temporal frequencies \( \tau_2 \) or \( \tau_1 \) has size \( \ll \lambda_0^2 \), we can apply Bernstein’s inequality in the time variable. We use the size of the modulation, the product estimate and Bernstein’s inequality in space.
and time variables to obtain

\[
\lambda_0^{s' - 2a + b} \| (\tau + |\xi|^2)^{\theta - 1} F((\lambda_0 + |\partial_t|)^a C_{\lambda_0^2} u_{\lambda_1} Q_{\mu^2} v_\mu) \|_{L^2_{\xi, \tau}} \\
\lesssim \lambda_0^{s' - 3a + b + 2\theta - 1} \mu^\frac{2}{\theta} \| (\lambda_1 + |\partial_t|)^a C_{\lambda_1^2} u_{\lambda_1} \|_{L^2_{\xi, \tau}} \| (\lambda_0 + |\partial_t|)^a Q_{\mu^2} v_\mu \|_{L^2_{\xi, \tau}} \\
\lesssim \lambda_0^{a + b - 1 - \beta} \mu^{\frac{2}{\theta} + 1 - \beta} \sup_{|\tau_2| > \mu^2} \frac{(\lambda_0 + |\tau_2|)^a}{(\lambda_1 + |\partial_t|)^a} \lambda_1^{s' - 2a} \| (\tau + |\xi|^2)^{\theta'} F((\lambda_1 + |\partial_t|)^a C_{\lambda_1^2} u_{\lambda_1}) \|_{L^2_{\xi, \tau}} \\
\times \mu^{-\beta - 1} \| (\tau - |\xi|) F(Q_{\mu^2} v_\mu) \|_{L^2_{\xi, \tau}} \\
\lesssim \begin{cases} \\
\lambda_0^{a + b - 1 - \beta} |u_{\lambda_1}|_{S_{\lambda_1^{s', i, \theta} - i}^i v_\mu}_{W_{\mu, i, \theta}} \|_{\tau_2 \sim \lambda_0} \\
\lambda_0^{a + b - 2 - \theta + 2 \beta - 1} |u_{\lambda_1}|_{S_{\lambda_1^{s', i, \theta} - i}^i v_\mu}_{W_{\mu, i, \theta}}, \lambda_0 \lesssim |\tau_2|.
\end{cases}
\]

Since \(b < 1\) and \(-a + b < 1\), we require \(\beta \geq -2 + \frac{d}{2} + a\) to sum the above estimates.

**b.** \(|\tau_1|, |\tau_2| \gg \lambda_0^2\)

We employ the same steps as in the previous case but cannot use Bernstein’s inequality wrt time. Instead, we make use of the high modulation of the wave.

\[
\lambda_0^{s' - 2a + b} \| (\tau + |\xi|^2)^{\theta - 1} F((\lambda_0 + |\partial_t|)^a C_{\lambda_0^2} u_{\lambda_1} Q_{\lambda_0^2} v_\mu) \|_{L^2_{\xi, \tau}} \\
\lesssim \lambda_0^{s' - 3a + b + 2\theta' - 1} \mu^\frac{2}{\theta} \| (\lambda_1 + |\partial_t|)^a C_{\lambda_1^2} u_{\lambda_1} \|_{L^2_{\xi, \tau}} \| (\lambda_0 + |\partial_t|)^a Q_{\lambda_0^2} v_\mu \|_{L^2_{\xi, \tau}} \\
\lesssim \lambda_0^{a + b + 2\theta' - 2} \mu^{\frac{2}{\theta} - \beta + 1 - \beta} \sup_{|\tau_2| > \lambda_0^2} \frac{(\lambda_0 + |\tau_2|)^a}{(\lambda_1 + |\partial_t|)^a} \lambda_1^{s' - 2a} \| (\lambda_1 + |\partial_t|)^a C_{\lambda_1^2} u_{\lambda_1} \|_{L^2_{\xi, \tau}} \| (\tau - |\xi|) F(Q_{\lambda_0^2} v_\mu) \|_{L^2_{\xi, \tau}} \\
\lesssim \lambda_0^{a + b + 2\theta' - 4} \mu^{\frac{2}{\theta} - \beta - 1} \| u_{\lambda_1} \|_{S_{\lambda_1^{s', i} - i}^i v_\mu}_{W_{\mu, i, \theta}}.
\]

We can sum the above up provided \(\beta \geq -3 + \frac{d}{2} + 2\theta + a + b\).

**2.** \(H \times H \to L\)

We consider the expression \(I\) and apply Hölder’s and Bernstein’s inequality to obtain

\[
|I(F(Q_{\mu^2} v_\mu), F(C_{\lambda_0^2} u_{\lambda_1}), F(C_{\lambda_1^2} u_{\lambda_1}))| \\
\lesssim \mu^\frac{2}{\theta} \| C_{\lambda_0^2} u_{\lambda_1} \|_{L^2_{\xi, \tau}} \| Q_{\mu^2} v_\mu \|_{L^2_{\xi, \tau}} \| C_{\lambda_1^2} u_{\lambda_1} \|_{L^2_{\xi, \tau}} \| Q_{\mu^2} v_\mu \|_{L^2_{\xi, \tau}} \\
\lesssim \mu^\frac{2}{\theta} \| C_{\lambda_0^2} u_{\lambda_1} \|_{L^2_{\xi, \tau}} \| C_{\lambda_1^2} u_{\lambda_1} \|_{L^2_{\xi, \tau}} \| (\tau + |\xi|^2)^{\theta} F((\lambda_1 + |\partial_t|)^a C_{\lambda_1^2} u_{\lambda_1}) \|_{L^2_{\xi, \tau}} \mu^{-\beta - 1} \| (\tau - |\xi|) F(Q_{\mu^2} v_\mu) \|_{L^2_{\xi, \tau}} \\
\times \lambda_0^a \| C_{\lambda_0^2} u_{\lambda_1} \|_{L^2_{\xi, \tau}} \| C_{\lambda_1^2} u_{\lambda_1} \|_{L^2_{\xi, \tau}} \\
\lesssim \lambda_1^{-a} \mu^{\frac{2}{\theta} - 1 - \beta} \| u_{\lambda_1} \|_{S_{\lambda_1^{s', i, \theta} - i}^i v_\mu}_{W_{\mu, i, \theta}} \| v_{\lambda_1} \|_{S_{\lambda_0^{s', i, \theta} - i}^i v_\mu}_{W_{\mu, i, \theta}}
\]

To obtain the last second inequality, we use Bernstein’s inequality for \(\frac{1}{\mu} = \theta - \frac{1}{\theta}\) and the last inequality follows from (2.7).

We require \(\beta \geq -2 + \frac{d}{2} + a\) to sum this estimate and obtain (3.4).

**3.** \(H \times L \to H\)

Using the size of the modulation weight, the product estimate and Bernstein’s inequality, we get

\[
\lambda_0^{s' - 2a + b} \| (\tau + |\xi|^2)^{\theta - 1} F((\lambda_0 + |\partial_t|)^a C_{\lambda_0^2} u_{\lambda_1} Q_{\mu^2} v_\mu) \|_{L^2_{\xi, \tau}} \\
\lesssim \lambda_0^{s' - 3a + b + 2\theta' - 1} \mu^\frac{2}{\theta} \| (\lambda_1 + |\partial_t|)^a C_{\lambda_1^2} u_{\lambda_1} \|_{L^2_{\xi, \tau}} \| (\lambda_0 + |\partial_t|)^a Q_{\mu^2} v_\mu \|_{L^2_{\xi, \tau}} \\
\lesssim \lambda_0^{a + b - 2} \mu^{\frac{2}{\theta} - l - a} \sup_{|\tau_2| > \mu^2} \frac{(\lambda_0 + |\tau_2|)^a}{(\lambda_1 + |\partial_t|)^a} \lambda_1^{s' - 2a} \| (\tau + |\xi|^2)^{\theta} F((\lambda_1 + |\partial_t|)^a C_{\lambda_1^2} u_{\lambda_1}) \|_{L^2_{\xi, \tau}} \\
\times \| (\tau - |\xi|) F(Q_{\mu^2} v_\mu) \|_{L^2_{\xi, \tau}} \\
\lesssim \begin{cases} \\
\lambda_0^{a + b - 2} \mu^{\frac{2}{\theta} - l} \| u_{\lambda_1} \|_{S_{\lambda_1^{s', i, \theta} - i}^i v_\mu}_{W_{\mu, i, \theta}}, \| v_{\lambda_1} \|_{S_{\lambda_0^{s', i, \theta} - i}^i v_\mu}_{W_{\mu, i, \theta}}, \lambda_0 \lesssim |\tau_2| \\
\lambda_0^{a - b - 2} \mu^{\frac{2}{\theta} - l + c} \| u_{\lambda_1} \|_{S_{\lambda_1^{s', i, \theta} - i}^i v_\mu}_{W_{\mu, i, \theta}} + \| v_{\lambda_1} \|_{S_{\lambda_0^{s', i, \theta} - i}^i v_\mu}_{W_{\mu, i, \theta}}.
\end{cases}
\]

The constraint \(l \geq -2 + \frac{d}{2} + b\) is required to sum the above up.
4. $L \times H \to H$

Using the size of the modulation, the product estimate and Bernstein’s inequality, we have

$$
\lambda_0^{\delta_2+b} \|(\tau + |\xi|^2)^{\theta - 1} F((\lambda_0 + |\partial_t|)^{\alpha} C_{\leq L_0^\beta} Q_{\leq L_0^\beta} v_\mu)\|_{L^2_{\tau,\xi}} \\
\lesssim \lambda_0^{\delta_2+b+2\theta-\beta} \mu_0^{\beta} \|(\lambda_1 + |\partial_t|)^{\alpha} C_{\leq L_0^\beta} Q_{\leq L_0^\beta} v_\mu\|_{L^2_{\tau,\xi}} \\
\lesssim \lambda_0^{\delta_2+b+2\theta-\beta} \mu_0^{\beta} \sup_{|\tau| \geq \mu_0} \langle \lambda_1 + |\partial_t| \rangle^{\alpha} C_{\leq L_0^\beta} Q_{\leq L_0^\beta} v_\mu\|_{L^2_{\tau,\xi}} \\
\lesssim \left\{ \begin{array}{ll}
\lambda_0^{\delta_2+b+2\theta-\beta} \mu_0^{\beta} \|u_{\lambda_1}\|_{S^{1,\theta}} \|v_\mu\|_{W^{1,\theta},\theta}, & \|\tau_2\| \lesssim \lambda_0 \\
\lambda_0^{\delta_2+b+2\theta-\beta} \mu_0^{\beta} \|u_{\lambda_1}\|_{S^{1,\theta}} \|v_\mu\|_{W^{1,\theta},\theta}, & \lambda_0 \lesssim |\tau_2|.
\end{array} \right.
$$

With $b \leq 2 - 2\theta$ and $\beta \geq -3 + \frac{d}{2} + 2\theta + a + b$, we can sum this estimate to obtain (3.5).

5. $L \times H \to L$

Since $u_{\lambda_1}$ has temporal frequencies of size $\sim \lambda_1^2$, we observe that the temporal frequencies $\tau_2$ are such that $\mu_0^2 \lesssim |\tau_2| \lesssim \lambda_0^2$.

$d = 2, 3$: We have the standard decomposition:

$$
|I(F(Q_{\mu}^{\leq \lambda_0^2} v_\mu), F(C_{\leq L_0^\beta} w_{\lambda_0}), F(C_{\leq L_0^\beta} u_{\lambda_1}))| \lesssim \left| \int \sum_{L_1, L_2 \leq \lambda_0^2} \sum_{L_0 \leq \lambda_0^2} \sum_{L_1 \leq \lambda_1^2} F(C_{L_1} u_{\lambda_1}) \right| \\
\lesssim \sum_{L_1, L_2 \leq \lambda_0^2} \lambda_1^{\frac{d}{2}} \log \lambda_1 (L_0 L_1 L_2)^\frac{d}{2} \|Q_{L_2} v_\mu\|_{L^2_{\tau,\xi}} \|Q_{L_1} w_{\lambda_0}\|_{L^2_{\tau,\xi}} \|Q_{L_0} w_{\lambda_0}\|_{L^2_{\tau,\xi}} \\
\lesssim \mu^{-\beta} \lambda_0^{2\theta - \frac{d}{2}} \log \lambda_1 \log |\tau + |\xi|^2| \|F(C_{\leq \lambda_1^2} u_{\lambda_1})\|_{L^2_{\tau,\xi}} \times \mu^\beta \|Q_{\mu}^{\leq \lambda_0^2} v_\mu\|_{L^2_{\tau,\xi}} \lambda_0^{-\delta} \|Q_{\mu}^{\leq \lambda_0^2} v_\mu\|_{L^2_{\tau,\xi}} \\
\lesssim \mu^{-\beta} \lambda_0^{2\theta - \frac{d}{2}} \log \lambda_1 \mu \lambda_0^{-\delta} \|Q_{\mu}^{\leq \lambda_0^2} v_\mu\|_{L^2_{\tau,\xi}} \|Q_{L_1} w_{\lambda_0}\|_{L^2_{\tau,\xi}} \|Q_{L_0} w_{\lambda_0}\|_{L^2_{\tau,\xi}} \\
\lesssim \lambda_0^{\delta_2+2\theta} \log \mu \lambda_0^{-\beta} \|Q_{\mu}^{\leq \lambda_0^2} v_\mu\|_{L^2_{\tau,\xi}} \|Q_{L_1} w_{\lambda_0}\|_{L^2_{\tau,\xi}} \|Q_{L_0} w_{\lambda_0}\|_{L^2_{\tau,\xi}} \\
\lesssim \lambda_0^{\delta_2+2\theta} \log \mu \lambda_0^{-\beta} \|Q_{\mu}^{\leq \lambda_0^2} v_\mu\|_{L^2_{\tau,\xi}} \|Q_{L_1} w_{\lambda_0}\|_{L^2_{\tau,\xi}} \|Q_{L_0} w_{\lambda_0}\|_{L^2_{\tau,\xi}} \\
\lesssim \lambda_0^{\delta_2+2\theta} \log \mu \lambda_0^{-\beta} \|u_{\lambda_1}\|_{S^{1,\theta}} \|v_\mu\|_{W^{1,\theta},\theta} \|w_{\lambda_0}\|_{S^{1,\theta}} \\
\lesssim \lambda_0^{\delta_2+2\theta} \log \mu \lambda_0^{-\beta} \|u_{\lambda_1}\|_{S^{1,\theta}} \|v_\mu\|_{W^{1,\theta},\theta} \|w_{\lambda_0}\|_{S^{1,\theta}}.
$$

The constraint $\beta > -\frac{d}{2} + 2\theta$ enables us to sum the above estimate for $d \leq 3$.

For $d \geq 1$ in $L \leq 3$, we have

$$
|I(F(Q_{\mu}^{\leq \lambda_0^2} v_\mu), F(C_{\leq L_0^\beta} w_{\lambda_0}), F(C_{\leq L_0^\beta} u_{\lambda_1}))| \lesssim \left| \int \sum_{L_1, L_2 \leq \lambda_0^2} \sum_{L_0 \leq \lambda_0^2} \sum_{L_1 \leq \lambda_1^2} F(C_{L_1} u_{\lambda_1}) \right| \\
\lesssim \mu^{-\beta} \lambda_0^{2\theta - \frac{d}{2}} \log \lambda_1 \log |\tau + |\xi|^2| \|F(C_{\leq \lambda_1^2} u_{\lambda_1})\|_{L^2_{\tau,\xi}} \times \mu^\beta \|Q_{\mu}^{\leq \lambda_0^2} v_\mu\|_{L^2_{\tau,\xi}} \|Q_{L_1} w_{\lambda_0}\|_{L^2_{\tau,\xi}} \|Q_{L_0} w_{\lambda_0}\|_{L^2_{\tau,\xi}} \\
\lesssim \mu^{-\beta} \lambda_0^{2\theta - \frac{d}{2}} \log \lambda_1 \mu \lambda_0^{-\delta} \|Q_{\mu}^{\leq \lambda_0^2} v_\mu\|_{L^2_{\tau,\xi}} \|Q_{L_1} w_{\lambda_0}\|_{L^2_{\tau,\xi}} \|Q_{L_0} w_{\lambda_0}\|_{L^2_{\tau,\xi}} \\
\lesssim \mu^{-\beta} \lambda_0^{2\theta - \frac{d}{2}} \log \mu \lambda_0^{-\delta} \|Q_{\mu}^{\leq \lambda_0^2} v_\mu\|_{L^2_{\tau,\xi}} \|Q_{L_1} w_{\lambda_0}\|_{L^2_{\tau,\xi}} \|Q_{L_0} w_{\lambda_0}\|_{L^2_{\tau,\xi}} \\
\lesssim \mu^{-\beta} \lambda_0^{2\theta - \frac{d}{2}} \log \mu \lambda_0^{-\delta} \|u_{\lambda_1}\|_{S^{1,\theta}} \|v_\mu\|_{W^{1,\theta},\theta} \|w_{\lambda_0}\|_{S^{1,\theta}}.
$$

The constraint $\beta > -\frac{d}{2} + 2\theta$ enables us to sum the above estimate for $d \leq 3$.

6. $L \times L \to L$

$d = 2, 3$: We consider the expression $I$, decompose the space-time Fourier supports of $C_{\leq \lambda_1^2} u_{\lambda_1}, Q_{\leq \lambda_0^2} v_\mu$ and

$$
\text{Remark 3.3.} \text{ In case the wave temporal frequencies are of size } \sim \lambda_0^2, \text{ a simpler argument applies.}
$$
If the size of the modulation, Bernstein’s inequality and the product estimate, we get

\[
|I(F(Q_{\leq \lambda_0^2} v_\mu), F(C_{\leq \lambda_0^2} w_{\lambda_0}), F(C_{\leq \lambda_0^2} u_{\lambda_1}))| \lesssim \sum_{L_1, L_2, L_0 \leq \lambda_0^2} |I(F(Q_{L_2} u_\mu), F(C_{L_0} w_{\lambda_0}), F(C_{L_1} u_{\lambda_1}))|.
\]

(3.12)

From lemma 2.4, we have

\[
(3.12) \lesssim \sum_{L_1, L_2, L_0 \leq \lambda_0^2} \lambda_1^{\frac{1}{2}} \log \lambda_1 (L_0 L_1 L_2)^{\frac{1}{2}} \|C_{L_1} u_{\lambda_1}\|_{L^2_{t,x}} \|Q_{L_2} v_\mu\|_{L^2_{t,x}} \|C_{L_0} w_{\lambda_0}\|_{L^2_{t,x}}
\]

\[
\lesssim \mu^{-\frac{1}{2}} \lambda_0^{-\frac{1}{2}} \log \lambda_1 (\tau + |\xi|^2)^{\theta} F(C_{\leq \lambda_0^2} u_{\lambda_1}) \|_{L^2_{t,x}}
\]

\[
\times \mu^{-\alpha} \|\tau - |\xi|^2\| F(Q_{\leq \lambda_0^2} v_\mu) \|_{L^2_{t,x}} \lambda_0^s \|\tau + |\xi|^2\|^{1-\theta} F(C_{\leq \lambda_0^2} w_{\lambda_0}) \|_{L^2_{t,x}}
\]

\[
\lesssim \mu^{-\frac{1}{2}} \lambda_0^{-\frac{1}{2}} \log \lambda_1 \|u_{\lambda_1}\|_{S^{1,s}_{t,x}} \|v_\mu\|_{W^{1,s}_{t,x}} \|w_{\lambda_0}\|_{S^{-1,1-\theta}}.
\]

\(d = 1\): Using Cauchy Schwarz inequality, we have

\[
|I(F(Q_{\leq \lambda_0^2} v_\mu), F(C_{\leq \lambda_0^2} w_{\lambda_0}), F(C_{\leq \lambda_0^2} u_{\lambda_1}))| \lesssim \|C_{\leq \lambda_0^2} u_{\lambda_1}\|_{L^2_{t,x}} \|Q_{\leq \lambda_0^2} v_\mu\|_{L^2_{t,x}} \|C_{\leq \lambda_0^2} w_{\lambda_0}\|_{L^2_{t,x}}.
\]

(3.13)

We decompose the space-time Fourier support of \(Q_{\leq \lambda_0^2} v_\mu\) and \(C_{\leq \lambda_0^2} w_{\lambda_0}\) into pieces of size \(L_2\) and \(L_0\) respectively and apply the bilinear estimate (2.13) for wave-Schrödinger interaction:

\[
(3.13) \lesssim \|C_{\leq \lambda_0^2} u_{\lambda_1}\|_{L^2_{t,x}} \sum_{L_0, L_2 \leq \lambda_0^2} (L_0 L_2)^{\frac{1}{2}} \lambda_0^{\frac{1}{2}} \|Q_{L_2} v_\mu\|_{L^2_{t,x}} \|C_{L_0} w_{\lambda_0}\|_{L^2_{t,x}}
\]

\[
\lesssim \lambda_0^{\frac{1}{2} + 2\theta} \mu^{-\frac{1}{2}} \lambda_1^{\frac{1}{2}} \|\tau + |\xi|^2\|^{\theta} F(C_{\leq \lambda_0^2} u_{\lambda_1}) \|_{L^2_{t,x}} \mu^{-\alpha} \|\tau - |\xi|^2\|^{1-\theta} F(C_{\leq \lambda_0^2} w_{\lambda_0}) \|_{L^2_{t,x}}
\]

\[
\times \lambda_0^s \|\tau + |\xi|^2\|^{1-\theta} F(C_{\leq \lambda_0^2} w_{\lambda_0}) \|_{L^2_{t,x}}
\]

\[
\lesssim \lambda_0^{\frac{1}{2} + 2\theta} \mu^{-\frac{1}{2}} \|u_{\lambda_1}\|_{S^{1,s}_{t,x}} \|v_\mu\|_{W^{1,s}_{t,x}} \|w_{\lambda_0}\|_{S^{-1,1-\theta}}.
\]

Provided \(l > -\frac{1}{2} + 2\theta\), we can sum the estimate for \(d \leq 3\) to obtain (3.4). For \(\mu \sim \lambda_0 \sim 1\), the estimate (3.4) holds with trivial modification.

We conclude that we require

\[
* \quad l > \max \left\{ -\frac{1}{2} + b, -\frac{3}{2} + 2\theta \right\} \quad * \quad \beta > \frac{3}{2} + 2\theta + a + b
\]

(3.14)

to obtain the estimates (3.4) and (3.5) when \(\lambda_0 \sim 1\) for \(d \leq 3\).

**Case III. Low to high interaction**\((\lambda_0 \ll \lambda_1)\)

We decompose \(u_{\lambda_1}\) and \(v_{\lambda_1}\) as follows:

\[
u_{\lambda_1} = C_{\leq \lambda_0^2} v_{\lambda_1} + C_{\geq \lambda_0^2} u_{\lambda_1}, \quad v_{\lambda_1} = Q_{\leq \lambda_0^2} v_{\lambda_1} + Q_{\geq \lambda_0^2} v_{\lambda_1}
\]

1. \(H \times H \rightarrow H\)

Using the size of the modulation, Bernstein’s inequality and the product estimate, we get

\[
\lambda_0^{\frac{\epsilon}{2} - 2a + b} \|\tau + |\xi|^2\|^{\theta - 1} |F((\lambda_0 + |\partial_\xi|)^{a} C_{\geq \lambda_0^2} (C_{\geq \lambda_0^2} (C_{\geq \lambda_0^2} v_{\lambda_1}))| \|_{L^2_{t,x}}
\]

\[
\lesssim \lambda_0^{\frac{\epsilon}{2} - 2a + 1} \lambda_1^{\alpha} \lambda_0^{s + a - \beta + 1} \sup_{|\tau| \geq \lambda_0^2} \frac{(\lambda_1 + |\tau_2|)^{\frac{\alpha}{2} \lambda_1^{\epsilon - 2a} (\lambda_1 + |\tau_2|)^{\lambda_1^{\epsilon - 2a} (\lambda_1 + |\tau_2|)}}{\lambda_1^{\epsilon - 2a} (\lambda_1 + |\tau_2|)^{\lambda_1^{\epsilon - 2a} (\lambda_1 + |\tau_2|)} \|v_{\lambda_1}\|_{L^2_{t,x}}
\]

\[
\times \lambda_1^{\beta \epsilon - 1} \|\tau - |\xi|^2\|^{\theta - 1} F(Q_{\geq \lambda_0^2} v_{\lambda_1}) \|_{L^2_{t,x}}
\]

\[
\lesssim \lambda_0^{\frac{\epsilon}{2} - 2a + 1} \lambda_1^{s - \beta + 3a - 1} \|u_{\lambda_1}\|_{S^{1,s}_{t,x}} \|v_{\lambda_1}\|_{W^{1,s}_{t,x}}
\]

If \(s + \beta \geq -1 + 3a\) and \(\beta \geq -3 + \frac{\epsilon}{2} + 2\theta + a + b\), we can obtain (3.5) by summing up the above estimate.

2. \(H \times H \rightarrow L\)

Since \(|\tau_2| \geq \lambda_1^2\), from the relation \(|\tau_0 + |\xi_0|^2| = |\tau_1 + \tau_2 + |\xi_0|^2| \leq \lambda_0^2\), it follows that \(|\tau_1| \geq \lambda_1^2\). We consider the
expression $I$ and use Hölder’s and Bernstein’s inequality as follows:

$$|I(F(Q_{\leq \lambda^2} u_1), F(C_{\leq \lambda^2} w_0), F(C_{\geq \lambda^2} u_1))|$$

$$\lesssim \lambda_0^{\frac{d}{2}} \|C_{\geq \lambda^2} u_1\|_{L^2_{t,x}} \|Q_{\geq \lambda^2} u_1\|_{L^2_{t,x}} \|C_{\leq \lambda^2} w_0\|_{L^\infty_t L^2_x}$$

$$\lesssim \lambda_0^{\frac{d}{2}+s+\theta-1-\lambda_1^{s-\theta}} \|C_{\leq \lambda^2} u_1\|_{L^2_{t,x}} \|Q_{\geq \lambda^2} u_1\|_{L^2_{t,x}} \|C_{\leq \lambda^2} w_0\|_{L^\infty_t L^2_x}$$

$$\lesssim \lambda_0^{\frac{d}{2}+s+\theta-1-\lambda_1^{s-\theta}} \|C_{\leq \lambda^2} u_1\|_{L^2_{t,x}} \|Q_{\geq \lambda^2} u_1\|_{L^2_{t,x}} \|C_{\leq \lambda^2} w_0\|_{L^\infty_t L^2_x}$$

$$\lesssim \lambda_0^{\frac{d}{2}+s+\theta-1-\lambda_1^{s-\theta}} \|C_{\leq \lambda^2} u_1\|_{L^2_{t,x}} \|Q_{\geq \lambda^2} u_1\|_{L^2_{t,x}} \|C_{\leq \lambda^2} w_0\|_{L^\infty_t L^2_x}$$

where the last inequality follows from the embedding (2.7) applied for $\frac{d}{2} = \theta - \frac{d}{2}$.

Since $s + \theta - 1 > 0$, we require $\beta \geq -2 + \frac{d}{2}$ in order to obtain (3.4).

3. $H \times L \to H$

Using the size of the modulation, the product estimate and Bernstein’s inequality, we have

$$\lambda_0^{s-2a+\frac{d}{2}} \|C_{\leq \lambda^2} u_1\|_{L^2_{t,x}} \|C_{\geq \lambda^2} u_1\|_{L^2_{t,x}} \|Q_{\leq \lambda^2} v_1\|_{L^2_{t,x}} \|Q_{\geq \lambda^2} v_1\|_{L^\infty_t L^2_x}$$

$$\lesssim \lambda_0^{s-2a+\frac{d}{2}} \|C_{\leq \lambda^2} u_1\|_{L^2_{t,x}} \|C_{\geq \lambda^2} u_1\|_{L^2_{t,x}} \|Q_{\leq \lambda^2} v_1\|_{L^\infty_t L^2_x}$$

The estimate can be summed up provided $s + l \geq -2\theta + 2a$ and $l \geq -2 + \frac{d}{2} + b$.

4. $H \times L \to L$

Since low modulation for the wave $v_\lambda$, implies $|\tau_2| < \lambda_1^2$, we observe from the relation

$$|\tau_0 + |\xi_0|^2| = |\tau_1 + \tau_2| + |\xi_1| + |\xi_2| = \lambda_1^2,$$

that the temporal frequencies $|\tau_1| \ll \lambda_1^2$. Using this information, we consider the following subcases for the size of the temporal frequencies $\tau_1$ and $\tau_2$ and make conclusions in the last two columns:

| $|\tau_1|$ | $|\tau_2|$ | $|\tau_0|$ | Conclusion |
|----------------|----------------|----------------|----------------|
| $\leq \lambda_1$ | $\leq \lambda_1$ | $\leq \lambda_1$ | $\lambda_0^a \leq \lambda_1$ |
| $\leq \lambda_1$ | $\leq \lambda_1$ | $\leq \lambda_1$ | $\lambda_0^a \leq \lambda_1$ |
| $\leq \lambda_1$ | $\leq \lambda_1$ | $\leq \lambda_1$ | $|\tau_0| \sim \lambda_1, \lambda_1 \ll \lambda_0^a$ |
| $\leq \lambda_1$ | $\leq \lambda_1$ | $\leq \lambda_1$ | $|\tau_0| \sim \lambda_1, \lambda_1 \ll \lambda_0^a$ |
| $\leq \lambda_1$ | $\leq \lambda_1$ | $\leq \lambda_1$ | $|\tau_0| \sim \lambda_1, \lambda_1 \ll \lambda_0^a$ |
| $\leq \lambda_1$ | $\leq \lambda_1$ | $\leq \lambda_1$ | $|\tau_0| \sim \lambda_1, \lambda_1 \ll \lambda_0^a$ |

Table 1. $H \times L \to L$

In all the above cases, we can bound the weight $(\lambda_1 + |\tau_1|)^{-a}(\lambda_1 + |\tau_2|)^{-a}$ by $\lambda_1^{-a} \lambda_0^{-2a}$. We proceed to prove (3.4) by duality and consider the expression $I$. Using Cauchy-Schwarz inequality, we get

$$|I(F(Q_{\leq \lambda^2} u_1), F(C_{\leq \lambda^2} w_0), F(C_{\geq \lambda^2} u_1))| \lesssim \|C_{\leq \lambda^2} u_1\|_{L^2_{t,x}} \|Q_{\leq \lambda^2} v_1\|_{L^2_{t,x}} \|C_{\geq \lambda^2} w_0\|_{L^\infty_t L^2_x}.$$  

(3.15)

In order to apply the bilinear estimate (2.13), we decompose the space-time Fourier supports of the two terms in RHS of the above display into pieces of size $L_2$ and $L_0$, respectively and obtain:

$$(3.15) \lesssim \|C_{\leq \lambda^2} u_1\|_{L^2_{t,x}} \sum_{L_0 < \lambda^2 \leq L^2_0} \frac{\lambda_0^{d/2}}{\lambda_0^d} (L_0 L_2)^{1/2} \|Q_{\leq \lambda^2} v_1\|_{L^2_{t,x}} \|C_{\geq \lambda^2} w_0\|_{L^\infty_t L^2_x}$$

$$\lesssim \lambda_0^{d/2+2\theta-2a} \lambda_1^{-s-2\theta-2a} \lambda_1^{s-\theta} \|C_{\leq \lambda^2} u_1\|_{L^2_{t,x}} \|Q_{\leq \lambda^2} v_1\|_{L^2_{t,x}} \|C_{\geq \lambda^2} w_0\|_{L^\infty_t L^2_x}$$

$$\lesssim \lambda_0^{d/2+2\theta-2a} \lambda_1^{-s-2\theta-2a} \lambda_1^{s-\theta} \|C_{\leq \lambda^2} u_1\|_{L^2_{t,x}} \|Q_{\leq \lambda^2} v_1\|_{L^2_{t,x}} \|C_{\geq \lambda^2} w_0\|_{L^\infty_t L^2_x}$$

$$\lesssim \lambda_0^{d/2+2\theta-2a} \lambda_1^{-s-2\theta-2a} \lambda_1^{s-\theta} \|C_{\leq \lambda^2} u_1\|_{L^2_{t,x}} \|Q_{\leq \lambda^2} v_1\|_{L^2_{t,x}} \|C_{\geq \lambda^2} w_0\|_{L^\infty_t L^2_x}.$$
5. $L \times H \to L$

We observe that the temporal frequencies $|\tau_2| \sim \lambda_1^2$. Using Hölder’s inequality, we have

$$|I(F(Q_{-\lambda_1^2}v_{\lambda_1}), F(C_{e_{\lambda_1^2}^2 w_{\lambda_0}}), F(C_{e_{\lambda_1^2}^2 u_{\lambda_1}}))| \leq \|Q_{-\lambda_1^2}v_{\lambda_1}\|_{L_{T_x}^2} \|C_{e_{\lambda_1^2}^2 L_{\lambda_1}^2}u_{\lambda_1}\|_{C_{e_{\lambda_1^2}^2 L_{\lambda_0}^2} w_{\lambda_0}} \|L_{T_x}^2\|_{L_{T_x}^2}$$

(3.16)

We apply the bilinear estimate (2.12) to the above by decomposing the space-time Fourier supports of $C_{e_{\lambda_1^2}^2 u_{\lambda_1}}$ and $C_{e_{\lambda_1^2}^2 w_{\lambda_0}}$ into pieces of size $L_1$ and $L_0$ respectively:

$$\left(3.16\right) \lesssim \|Q_{-\lambda_1^2}v_{\lambda_1}\|_{L_{T_x}^2} \sum_{L_0 \ll \lambda_1^2} (L_0 L_1) \frac{\lambda_1^{-1}}{\lambda_1^2} \|C_{L_1 u_{\lambda_1}}\|_{L_{T_x}^2} \|C_{L_0 w_{\lambda_0}}\|_{L_{T_x}^2}$$

$$\lesssim \lambda_0^{\frac{s}{2} + \gamma + 2\theta} \frac{\lambda_1^{-1}}{\lambda_1^2} \|\langle \tau + |\xi|^2 \rangle^\gamma F(C_{e_{\lambda_1^2}^2 u_{\lambda_1}})\|_{L_{T_x}^2} \lambda_1^{-\beta} \|\langle \tau - |\xi|^2 \rangle^\gamma F(Q_{-\lambda_1^2}v_{\lambda_1})\|_{L_{T_x}^2}$$

$$\times \lambda_0^{s} \|\langle \tau + |\xi|^2 \rangle F(C_{e_{\lambda_1^2}^2 w_{\lambda_0}})\|_{L_{T_x}^2}$$

$$\lesssim \lambda_0^{\frac{s}{2} + \gamma + 2\theta} \frac{\lambda_1^{-1}}{\lambda_1^2} \|u_{\lambda_1}\|_{L_{T_x}^2} \|w_{\lambda_0}\|_{L_{T_x}^2} \|L_{T_x}^2v_{\lambda_1}\|_{L_{T_x}^2} \|w_{\lambda_0}\|_{L_{T_x}^2} \|\langle \tau + |\xi|^2 \rangle^\gamma F(C_{e_{\lambda_1^2}^2 u_{\lambda_1}})\|_{L_{T_x}^2}$$

One can sum the above estimate to obtain (3.4) provided $s + \beta \geq -\frac{3}{2}$ and $\beta \geq -3 + \frac{d}{2} + 2\theta$.

For $d = 1$, in the case $\lambda_0 \sim \lambda_1 \sim 1$, we have

$$|I(F(Q_{-\lambda_1^2}v_{\lambda_1}), F(C_{e_{\lambda_1^2}^2 w_{\lambda_0}}), F(C_{e_{\lambda_1^2}^2 u_{\lambda_1}}))| \leq \|Q_{-\lambda_1^2}v_{\lambda_1}\|_{L_{T_x}^2} \|C_{e_{\lambda_1^2}^2 L_{\lambda_1}^2}u_{\lambda_1}\|_{L_{T_x}^2} \|C_{e_{\lambda_1^2}^2 L_{\lambda_0}^2} w_{\lambda_0}\|_{L_{T_x}^2}$$

$$\lesssim \|u_{\lambda_1}\|_{L_{T_x}^2} \|w_{\lambda_0}\|_{L_{T_x}^2} \|\langle \tau + |\xi|^2 \rangle^\gamma F(C_{e_{\lambda_1^2}^2 u_{\lambda_1}})\|_{L_{T_x}^2}$$

6. $L \times L \to H$

From the constraint $\tau_0 = \tau_1 + \tau_2$, we note that the output temporal frequencies are of size $\sim \lambda_1^2$. This implies that the output has a modulation of size $\sim \lambda_1^2$. We use the size of the modulation and the temporal frequencies and employ the bilinear estimate for wave-Schrödinger interaction. By orthogonality arguments, we reduce the estimates to the case when the spatial Fourier supports of $C_{e_{\lambda_1^2}^2 u_{\lambda_1}}$ and $Q_{-\lambda_1^2}v_{\lambda_1}$ are of size $\sim \lambda_0$.

$$\lambda_0^{s - 2a + b} \|\langle \tau + |\xi|^2 \rangle^\gamma F(C_{-\lambda_1^2 u_{\lambda_1}})\|_{L_{T_x}^2}$$

$$\lesssim \lambda_0^{s - 2a + b} \lambda_1^2 \|\langle \tau + |\xi|^2 \rangle^\gamma F(C_{e_{\lambda_1^2}^2 u_{\lambda_1}})\|_{L_{T_x}^2}$$

$$\lesssim \lambda_0^{s - 2a + b} \lambda_1^{2\theta - 2a + 2\theta} \|C_{e_{\lambda_1^2}^2 u_{\lambda_1}}\|_{L_{T_x}^2} \|Q_{e_{\lambda_1^2}^2 v_{\lambda_1}}\|_{L_{T_x}^2}$$

$$\lesssim \lambda_0^{s - 2a + b} \lambda_1^{2\theta - 2} \|u_{\lambda_1}\|_{L_{T_x}^2} \|w_{\lambda_0}\|_{L_{T_x}^2} \|\langle \tau + |\xi|^2 \rangle^\gamma F(C_{e_{\lambda_1^2}^2 u_{\lambda_1}})\|_{L_{T_x}^2}$$

To sum this estimate, we require $s + l \geq -\frac{3}{2} + 2\theta + 2a + b$ and $l \geq -3 + \frac{d}{2} + 2\theta + b$.

7. $L \times H \to H$

With Bernstein’s inequality and the product estimate, we get

$$\lambda_0^{s - 2a + b} \|\langle \tau + |\xi|^2 \rangle^\gamma F(C_{-\lambda_1^2 u_{\lambda_1}})\|_{L_{T_x}^2}$$

$$\lesssim \lambda_0^{s - 2a + b + 2\theta - 2a + 2\theta} \lambda_1^2 \|\langle \tau + |\xi|^2 \rangle^\gamma F(C_{e_{\lambda_1^2}^2 u_{\lambda_1}})\|_{L_{T_x}^2}$$

$$\lesssim \lambda_0^{s - 2a + b + 2\theta - 2a + 2\theta} \lambda_1^{-a + \beta + 1} \sup_{|\tau_2| \geq \lambda_1^2} (\lambda_1 + |\tau_2|) \lambda_1^a \|\langle \tau + |\xi|^2 \rangle^\gamma F(C_{e_{\lambda_1^2}^2 u_{\lambda_1}})\|_{L_{T_x}^2}$$

$$\lesssim \lambda_0^{s - 2a + b + 2\theta - 2a + 2\theta} \lambda_1^{-s - 3a - 1} \|u_{\lambda_1}\|_{L_{T_x}^2} \|w_{\lambda_0}\|_{L_{T_x}^2}$$

If $s + \beta \geq -1 + 3a$ and $\beta \geq -3 + \frac{d}{2} + 2\theta + a + b$, we can obtain (3.5) by summing up the above estimate.

Hence, for $\lambda_0 \ll \lambda_1$, we conclude that the following constraints are required for summability in dimension $d \leq 3$:

- $l \geq -\frac{3}{2} + 2\theta + b$
- $\beta \geq -\frac{3}{2} + 2\theta + a + b$
- $s + l \geq \max\left\{-2\theta + 2a, -\frac{5}{2} + 2\theta' + 2a\right\}$
- $s + \beta \geq -1 + 3a$. (3.17)
For \( \frac{1}{2} < a < 1 \), the estimate (3.6) holds provided \( l > -\frac{3}{2} + 2\theta, s + l \geq 0 \) and \( s - l \leq 3 - 2\theta \). It can be proved by imitating the proof of the product estimate \( \|fg\|_{H^{l-s}+\omega} \lesssim \|f\|_{H^l}\|g\|_{H^s} \) for each case above.

From (3.10), (3.14) and (3.17), we conclude that

\[
\begin{align*}
\bullet & \quad l > \frac{3}{2} + 2\theta + b \\
\bullet & \quad \beta > -\frac{3}{2} + 2\theta + a + b \\
\bullet & \quad s - l \leq \min\{2 - 2\theta + a - b, 3 - 2\theta\} \\
\bullet & \quad s - \beta \leq \min\{3 - 2\theta - b, \frac{5}{2} - 2\theta\} \\
\bullet & \quad s + l \geq \max\{\frac{5}{2} + 2\theta' + 2a, -2\theta + 2a\} \\
\bullet & \quad s + \beta \geq -1 + 3a
\end{align*}
\]

are the requirements for the estimate (3.1) to hold.

\( \square \)

4. Multilinear estimates for wave non-linearity

**Theorem 4.1.** Let \( d \leq 3 \) and \( s, l \) satisfy (1.3). There exist \( a, b, s, l, \beta, \theta \in \mathbb{R} \) such that the estimate

\[
\|\nabla(\overline{\tau}\psi)\|_{R^{\tau,s-\theta,1}} \lesssim \|\varphi\|_{S^{1,s}}\|\psi\|_{S^{1,s}} \tag{4.1}
\]

holds.

**Proof.** We choose the parameters as in (2.2). Using the definition of the norms, it suffices to show the following:

\[
\begin{align*}
\left( \sum_{\mu \in 2^{\mathbb{N}}} \mu^{2l(1-a+1)}\|\tau - |\xi_l|\theta F((\mu + |\partial_l|)^{\mu} P_\mu(\tilde{\varphi}\psi))\|_{L^2_{x,t}}^{2} \right)^{\frac{1}{2}} & \lesssim \|\varphi\|_{S^{1,s}}\|\psi\|_{S^{1,s}} \\
\left( \sum_{\mu \in 2^{\mathbb{N}}} \mu^{2\beta}\|Q_{\mu} P_\mu(\tilde{\varphi}\psi))\|_{L^2_{x,t}}^{2} \right)^{\frac{1}{2}} & \lesssim \|\varphi\|_{S^{1,s}}\|\psi\|_{S^{1,s}} \\
\left( \sum_{\mu \in 2^{\mathbb{N}}} \mu^{2l(1-2\theta - 2)}\|P_\mu(\tilde{\varphi}\psi))\|_{L^2_{x,t}}^{2} \right)^{\frac{1}{2}} & \lesssim \|\varphi\|_{S^{1,s}}\|\psi\|_{S^{1,s}}
\end{align*}
\]

The definition of high and low modulation remains the same as in section 3, so do the abbreviations \( H \) and \( L \). We now append 1 or 2 as subscripts with \( \tau \) to denote the temporal frequencies of \( \varphi \) and \( \psi \). The same is done for the spatial frequencies.

Employing the frequency trichotomy, we decompose the non-linearity \( P_\mu(\tilde{\varphi}\psi) \) into high-low, balanced (high-high) and low-high interactions as follows:

\[
P_\mu(\tilde{\tau}\psi) = P_\mu(\tilde{\tau}\psi_{\leq \mu}) + \sum_{\mu \leq \lambda_1 \sim \lambda_2} P_\mu(\tilde{\tau}\psi_{\lambda_1}^\lambda, \psi_{\lambda_2}^\lambda) + P_\mu(\tilde{\tau}\psi_{\leq \mu}) = \sum_{\lambda \leq \mu} P_\mu(\tilde{\tau}\psi_\lambda) + \sum_{\mu \leq \lambda_1 \sim \lambda_2} P_\mu(\tilde{\tau}\psi_{\lambda_1}^\lambda, \psi_{\lambda_2}^\lambda) + \sum_{\lambda \leq \mu} P_\mu(\tilde{\tau}\psi_{\lambda}) \tag{4.5}
\]

It suffices to consider the first two interactions above.

**Remark 4.2.** As in section 3, we consider the cases where \( |\xi_i|, |\tau_i| \gg 1, i = 1, 2 \). For the very low frequency cases \( |\xi_i| \sim 1, i = 1, 2 \) or the lowest frequency is very small, most of the estimates follow analogously without having to decompose the space-time Fourier supports. The cases in which these arguments do not follow will be treated separately.

**Case I. High-low interaction \( (\mu \gg \lambda) \)**

We decompose the spatially localised waves \( \tilde{\tau}\mu \) and \( \psi_\lambda \) as follows:

\[
\tilde{\tau}\mu = C_{\lambda_1}^2 \varphi_{\mu} + C_{\lambda_2}^2 \varphi_{\mu}, \quad \psi_\lambda = C_{\lambda_1}^2 \varphi_\lambda + C_{\lambda_2}^2 \varphi_\lambda.
\]

The following cases can be identified from the size of the modulation.

1. \( H \times H \rightarrow H \)

Since \( \mu^2 \lesssim |\tau_0 - |\xi_0|| = |\tau_1 - \tau_2 - |\xi_0|| \), we conclude that at least one of the temporal frequencies \( \tau_1 \) or \( \tau_2 \) has size \( \sim \mu^2 \). WLOG, let \( \tau_1 \) be of size \( \sim \mu^2 \). Using Hölder’s and Bernstein’s inequality, we have

\[
\mu^2\|Q_{\mu}^2 (C_{\lambda_1}^2 \varphi_{\mu} C_{\lambda_2}^2 \varphi_{\mu})\|_{L^2_{x,t}} \lesssim \mu^2 \lambda^2 \|C_{\lambda_1}^2 \varphi_{\mu} \|_{L^2_{x,t}} \|C_{\lambda_2}^2 \varphi_{\mu} \|_{L^2_{x,t}} \|C_{\lambda_1}^2 \varphi_{\mu} \|_{L^2_{x,t}} \|C_{\lambda_2}^2 \varphi_{\mu} \|_{L^2_{x,t}}
\]

\[
\lesssim \mu^{2-s-2\theta-b} \lambda^{s-2a+b} \|\tau + |\xi_0|\theta F((\mu + |\partial_l|)^{\mu} C_{\lambda_1}^2 \varphi_{\mu} \|_{L^2_{x,t}} \times \lambda^{s-2a} \|\tau + |\xi_0|\theta F((\lambda + |\partial_l|)^{\lambda} C_{\lambda_2}^2 \varphi_{\mu} \|_{L^2_{x,t}}
\]

\[
\lesssim \mu^{2-s-2\theta-b} \lambda^{s-2a} \|\varphi\|_{S^{1,s}}^2 \|\psi\|_{S^{1,s}}.
\]

We require \( s - \beta \geq -2\theta - b \) and \( 2s - \beta \geq \frac{3}{2} - 2\theta + a - b \).
2. $H \times H \to L$

We consider the expression $I$ and employ Hölder’s and Bernstein’s inequality to obtain
\[
|I(F(Q_{\leq \mu^2} \eta_0), F(C_{\geq \lambda^2} \psi_\lambda), F(C_{\geq \mu^2} \varphi_\mu))| \\
\lesssim \lambda^\frac{d}{2} \|C_{\geq \mu^2} \varphi_\mu\|_{L^2_{t,x}} \|C_{\geq \lambda^2} \psi_\lambda\|_{L^2_{t,x}} \|Q_{\leq \mu^2} \eta_0\|_{L^p_x L^\infty_t L^2_x} \\
\lesssim \lambda^{\frac{d}{2} - s + a - 2b} \|\partial_t\|_{s + a + b - \mu^{s - 2a + b}} \|\tau + |\xi|^2\|^\theta F((\mu + |\partial_t|)^\mu C_{\geq \lambda^2} \psi_\lambda)\|_{L^2_{t,\xi}} \\
\lesssim \lambda^{\frac{d}{2} - s + a - 2b} \|\phi_\mu\|_{S^{\mu,1,\theta}_x} \|\psi_\lambda\|_{S^{\lambda,1,\theta}_x} \|\eta_0\|_{W^{2,1,-1,-\theta}_x}. 
\]

The last inequality follows from the embedding (2.7) for $\frac{1}{p} = \frac{1}{2} - \frac{1}{d}$. We require $s - l \geq a - b$, $2s - l \geq \frac{d}{2} - 1 + 2a - b$ to obtain (4.2).

3. $H \times L \to H$

Using the relation $\mu^2 \lesssim |\tau_0 - |\xi_0| = |\tau_1 - \tau_2 - \mu|$ and $|\tau_2| \sim \lambda^2$, we find that $|\tau_1| \gtrapprox \mu^2$. Employing Hölder’s and Bernstein’s inequality, we obtain
\[
\mu^2 \|F(Q_{\leq \mu^2} (P^{(l)}_{\mu^2,2} C_{\leq \lambda^2} \psi_\lambda C_{\leq \lambda^2} \psi_\lambda))\|_{L^2_{t,\xi}} \\
\lesssim \mu^\gamma \frac{\lambda^\frac{d}{2}}{\lambda^\frac{d}{2}} \|P^{(l)}_{\mu^2,2} C_{\leq \lambda^2} \psi_\lambda\|_{L^2_{t,\xi}} \|C_{\leq \lambda^2} \psi_\lambda\|_{L^2_{t,x}} \|Q_{\leq \mu^2} \eta_0\|_{L^2_{t,x}} \\
\lesssim \mu^{\frac{d}{2} - s + b - 2b} \mu^{\frac{d}{2} - s + 2a + b} \|\tau + |\xi|^2\|^\theta F((\mu + |\partial_t|)^\mu C_{\geq \lambda^2} \psi_\lambda)\|_{L^2_{t,\xi}} \\
\lesssim \mu^{\frac{d}{2} - s + b - 2b} \|\psi_\lambda\|_{S^{\lambda,1,\theta}_x} \|\eta_0\|_{W^{2,1,-1,-\theta}_x}. 
\]

For $s - b \gtrapprox -b - 2\theta$ and $2s - b \gtrapprox 2a - b$, we can sum this estimate and obtain (4.3).

4. $H \times L \to L$

From the relation $|\tau_0| = |\tau_1 - \tau_2| \ll \mu^2$ and $|\tau_2| \sim \lambda^2$, we conclude that the temporal frequencies $\tau_1$ are such that $|\tau_1| \ll \mu^2$. We use Cauchy-Schwarz inequality for the expression $I$ as follows
\[
|I(F(Q_{\leq \mu^2} \eta_0), F(C_{\leq \lambda^2} \psi_\lambda), F(C_{\leq \mu^2} \varphi_\mu))| \lesssim \|C_{\leq \mu^2} \varphi_\mu\|_{L^2_{t,x}} \|C_{\leq \lambda^2} \psi_\lambda\|_{L^2_{t,x}} \|Q_{\leq \mu^2} \eta_0\|_{L^2_{t,x}}. 
\]

To employ the bilinear estimate (2.13) for the last term in the above display, we decompose the space-time Fourier supports of $C_{\leq \mu^2} \varphi_\mu$ and $Q_{\leq \mu^2} \eta_0$ into pieces of size $L_2$ and $L_0$ respectively:
\[
(4.6) \lesssim \|C_{\leq \mu^2} \varphi_\mu\|_{L^2_{t,x}} \sum_{L_2 \ll \lambda^2} \sum_{L_0 \ll \mu^2} \lambda^\frac{s-1}{2} \|L_1 L_0\|^\frac{s-1}{2} \|C_{L_1} \psi_\lambda\|_{L^2_{t,x}} \|Q_{L_0} \eta_0\|_{L^2_{t,x}} \\
\lesssim \mu^{s + a - b + 1} \lambda^{s - 1} \mu^{s - 2a + b} \|\tau + |\xi|^2\|^\theta F((\mu + |\partial_t|)^\mu C_{\leq \lambda^2} \psi_\lambda)\|_{L^2_{t,\xi}} \\
\times \lambda^\theta \|\tau + |\xi|^2\|^\theta F((\mu + |\partial_t|)^\mu C_{\leq \lambda^2} \psi_\lambda)\|_{L^2_{t,\xi}} \mu^{l - 1} \|\tau - |\xi|^2\|^\theta F((\mu + |\partial_t|)^\mu C_{\leq \lambda^2} \psi_\lambda)\|_{L^2_{t,\xi}} \\
\lesssim \mu^{s + a - b + 1} \lambda^{s - 1} \|\varphi_\mu\|_{S^{\mu,1,\theta}_x} \|\psi_\lambda\|_{S^{\lambda,1,\theta}_x} \|\eta_0\|_{W^{2,1,-1,-\theta}_x}. 
\]

Provided $s - l \gtrapprox a - b$ and $2s - l \gtrapprox 2a - b$, we can sum this estimate.

5. $L \times H \to H$

We consider two cases for the temporal frequencies $\tau_0$:

a. $|\tau_2| \lesssim \mu^2$: We apply the bilinear estimate (2.12) directly by decomposing the space-time Fourier supports of $C_{\leq \mu^2} \varphi_\mu$ and $C_{\geq \lambda^2} \psi_\lambda$ into pieces of size $L_1$ and $L_2$ respectively.
\[
\mu^2 \|F(Q_{\leq \mu^2} (C_{\leq \mu^2} \varphi_\mu C_{\leq \lambda^2} \psi_\lambda))\|_{L^2_{t,\xi}} \\
\lesssim \mu^\beta \sum_{L_1 \ll \mu^2} \sum_{L_2 \ll \lambda^2} \lambda^\frac{d-1}{2} \|L_1 L_2\|^\frac{d-1}{2} \|C_{L_1} \varphi_\mu\|_{L^2_{t,x}} \|C_{L_2} \psi_\lambda\|_{L^2_{t,x}} \\
\lesssim \mu^{\frac{d}{2} - s + a - b} \lambda^{\frac{d}{2} - s + a - b} \|\tau + |\xi|^2\|^\theta F(C_{\leq \mu^2} \varphi_\mu)\|_{L^2_{t,\xi}} \lambda^{\frac{d}{2} - 2a + b} \|\tau + |\xi|^2\|^\theta F((\lambda + |\partial_t|)^\mu C_{\geq \lambda^2} \psi_\lambda)\|_{L^2_{t,\xi}} \\
\lesssim \mu^{\frac{d}{2} - s + a - b} \varphi_\mu\|_{S^{\mu,1,\theta}_x} \|\psi_\lambda\|_{S^{\lambda,1,\theta}_x}. 
\]

b. $|\tau_2| \ll \mu^2$: We consider $|\tau_2| \ll \mu^2$ case.
Provided \( s - \beta \geq -\frac{1}{2} \) and \( 2s - \beta \geq \frac{d}{2} - 1 + a - b \), we can sum this estimate to obtain (4.3). For very small spatial frequencies i.e. \( \lambda \sim \mu \sim 1 \), we have
\[
\mu^6 \| F(Q_{\mu^2}(C_{\mu^2}\varphi_\mu C_{\mu^2}\psi_\lambda)) \|_{L_{t,x}^2} \lesssim \| C_{\mu^2}\varphi_\mu \|_{L_{t,x}^\infty} \| C_{\lambda^2}\psi_\lambda \|_{L_{t,x}^2} \lesssim \| \varphi_\mu \|_{S_{\mu,1,\sigma}^{1,0}} \| \psi_\lambda \|_{S_{\lambda,1,\sigma}^{1,0}}. \tag{4.7}
\]

**b. \( |\tau_2| \gg \mu^2 \):** We make use of the high modulation of \( \psi_\lambda \) by using Hölder’s and Bernstein’s inequality:
\[
\mu^2 \| F(Q_{\mu^2}(C_{\mu^2}\varphi_\mu C_{\mu^2}\psi_\lambda)) \|_{L_{t,x}^2} \lesssim \mu^2 \lambda^\frac{d}{2} \| C_{\mu^2}\varphi_\mu \|_{L_{t,x}^2} \| C_{\mu^2}\psi_\lambda \|_{L_{t,x}^2}
\]
\[
\lesssim \mu^{2-2\theta-2a-s} \| \varphi_\mu \|_{L_{t,x}^\infty} \| \psi_\lambda \|_{L_{t,x}^2} \lesssim \mu^{2-2\theta-2a-s} \| \varphi_\mu \|_{S_{\mu,1,\sigma}^{1,0}} \| \psi_\lambda \|_{S_{\lambda,1,\sigma}^{1,0}}.
\]
We require \( s - \beta \geq -2\theta - 2a \) and \( 2s - \beta \geq \frac{d}{2} - 2\theta - b \) for summability.

**6. \( L \times H \to L \)**

From the relation \( |\tau_1| = |\tau_1 - \tau_2| \ll \mu^2 \) nd \( |\tau_1| \sim \mu^2 \), we conclude that the temporal frequencies \( |\tau_2| \sim \mu^2 \). Cauchy-Schwarz inequality then gives
\[
|I(F(Q_{\mu^2}\eta_\mu), F(C_{\mu^2}\psi_\lambda))| \lesssim \| C_{\mu^2}\psi_\lambda \|_{L_{t,x}^2} \| P_\mu(C_{\mu^2}\varphi_\mu Q_{\mu^2}\eta_\mu) \|_{L_{t,x}^2} \tag{4.8}
\]
In order to apply the bilinear estimate (2.13), we decompose the Fourier supports of the last two terms in the above display into pieces of size \( L_1 \) and \( L_0 \) respectively. Since the spatial support of \( \psi \) is localised to frequencies of size \( \sim \lambda \), the estimate reduces to the case where the spatial supports of \( \varphi \) and \( \eta \) are also localised to frequencies of size \( \sim \lambda \). Using this reduction, we have
\[
(4.8) \lesssim \| C_{\mu^2}\psi_\lambda \|_{L_{t,x}^2} \sum_{L_0, L_1 \ll \mu^2} \lambda^{\frac{d}{2}} (L_1 L_0)^{\frac{s}{2}} \| C_{L_0}\psi_\mu \|_{L_{t,x}^2} \| Q_{L_0}\eta_\mu \|_{L_{t,x}^2}
\]
\[
\lesssim \mu^{-\frac{1}{2} + 2(\theta' - \theta) - 2a - l - s - b} \| \varphi_\mu \|_{L_{t,x}^\infty} \| \psi_\lambda \|_{L_{t,x}^2} \sum_{l_1, l_2 \ll \mu} \lambda^{\frac{d}{2}} (L_1 L_2)^{\frac{s}{2}} \| C_{L_1}\psi_\mu \|_{L_{t,x}^2} \| C_{L_2}\psi_\lambda \|_{L_{t,x}^2}
\]
\[
\lesssim \mu^{-\frac{1}{2} + 2(\theta' - \theta) - 2a - l - s - b} \| \varphi_\mu \|_{S_{\mu,1,\sigma}^{1,0}} \| \psi_\lambda \|_{S_{\lambda,1,\sigma}^{1,0}} \| \eta_\mu \|_{W_{\mu,1,\sigma}^{1,0}}.
\]
Provided \( s - l \geq -\frac{1}{2} + 2(\theta' - \theta) - 2a \) and \( 2s - l \geq -\frac{1}{2} - 1 - b \), we can obtain (4.1) by summing the estimate.

**7. \( L \times L \to H \)**

We employ the bilinear Strichartz estimate (2.12) by dividing the space time Fourier support of \( C_{\mu^2}\varphi_\mu \) and \( C_{\lambda^2}\psi_\lambda \) into pieces of size \( L_1 \) and \( L_2 \) respectively:
\[
\mu^3 \| F(Q_{\mu^2}(C_{\mu^2}\varphi_\mu C_{\lambda^2}\psi_\lambda)) \|_{L_{t,x}^2} \lesssim \mu^3 \sum_{L_1, L_2 \ll \mu^2} \lambda^{\frac{d}{2}} (L_1 L_2)^{\frac{s}{2}} \| C_{L_1}\varphi_\mu \|_{L_{t,x}^2} \| C_{L_2}\psi_\lambda \|_{L_{t,x}^2}
\]
\[
\lesssim \mu^{\frac{1}{2} - s} \lambda^{\frac{d}{2}} \| \varphi_\mu \|_{L_{t,x}^\infty} \| \psi_\lambda \|_{L_{t,x}^2} \lesssim \mu^{\frac{1}{2} - s} \lambda^{\frac{d}{2}} \| \varphi_\mu \|_{S_{\mu,1,\sigma}^{1,0}} \| \psi_\lambda \|_{S_{\lambda,1,\sigma}^{1,0}}.
\]
The constraints \( 2s - \beta \geq \frac{d}{2} - 1 \) and \( s - \beta \geq -\frac{1}{2} \) are required for summability. The argument in (4.9) can be mimicked for the very small spatial frequency case. We conclude that the following are the requirements for the validity of the estimates in the case \( \lambda \lesssim \mu \) for \( d \leq 3 \):
\[
\bullet \ s - \beta \geq -\frac{1}{2} \quad \bullet \ s - l \geq \max \left\{ a - b, \frac{1}{2} + 2(\theta' - \theta) - 2a \right\} \quad \bullet \ 2s - l \geq \frac{1}{2} + 2a - b \quad \bullet \ 2s - \beta \geq \frac{1}{2} + a - b.
\]

**Case II. High-high interaction (\( \mu \lesssim \lambda_1 \sim \lambda_2 \))**

We decompose \( \varphi_{\lambda_1} \) and \( \psi_{\lambda_2} \) as follows:
\[
\varphi_{\lambda_1} = C_{\mu^2}\varphi_{\lambda_1} + C_{\lambda_2^2}\varphi_{\lambda_1}, \quad \psi_{\lambda_2} = C_{\mu^2}\psi_{\lambda_2} + C_{\lambda_2^2}\psi_{\lambda_2}
\]

The following cases can be distinguished on the basis of the size of the modulation:

1. $H \times H \to H$

We use Hölder’s and Bernstein’s inequality as follows:

\[ \mu^2 \| F(Q_{\beta_1}(C_{\beta_2} \varphi_{\lambda_1} C_{\beta_2} \psi_{\lambda_2}))) \|_{L^2_{\tau,\xi}} \]
\[ \leq \mu^{\beta+\frac{d}{2}} \| C_{\beta_1} \varphi_{\lambda_1} \|_{L^2_{\tau,\xi}} \| C_{\beta_2} \psi_{\lambda_2} \|_{L^\infty_{\tau,\xi}} \]
\[ \leq \mu^{\beta+\frac{d}{2}} \lambda_1^{-2a-2b} \lambda_2^{-2a+b} \| (\tau + |\xi|^2)^{\theta} F((\lambda_1 + |\partial_\xi|)^n C_{\beta_1} \varphi_{\lambda_1} \|_{L^2_{\tau,\xi}} \]
\[ \leq \mu^{\beta+\frac{d}{2}} \lambda_1^{-2a-2b} \| \varphi_{\lambda_1} \|_{S^{1,\theta}_{\lambda_1}} \| \psi_{\lambda_2} \|_{S^{1,\theta}_{\lambda_2}} \]

We can sum the estimate in spatial frequencies $\mu \ll \lambda_1 \sim \lambda_2$ provided $2s - \beta \geq \frac{d}{2} - 2\theta + a - b$, by noting that $\beta + \frac{d}{2} > 0$ for $d \geq 1$ and $\beta$ as in (2.2).

2. $H \times H \to L$

We use Hölder’s and Bernstein’s inequality to obtain

\[ |I(F(Q_{\beta_1}(\varphi_{\lambda_1}), F(C_{\beta_2} \varphi_{\lambda_2})), F(C_{\beta_1} \varphi_{\lambda_1}))| \]
\[ \leq \mu^{\beta+\frac{d}{2}} \| C_{\beta_1} \varphi_{\lambda_1} \|_{L^2_{\tau,\xi}} \| C_{\beta_2} \psi_{\lambda_2} \|_{L^\infty_{\tau,\xi}} \| Q_{\mu \eta_0} \|_{L^2_{\tau,\xi}} \]
\[ \leq \mu^{\beta+\frac{d}{2}} \lambda_1^{-2a-2b} \lambda_2^{-2a+b} \| (\tau + |\xi|^2)^{\theta} F((\lambda_1 + |\partial_\xi|)^n C_{\beta_1} \varphi_{\lambda_1} \|_{L^2_{\tau,\xi}} \mu^{-l-1} \| Q_{\mu \eta_0} \|_{L^2_{\tau,\xi}} \]
\[ \leq \mu^{\beta+\frac{d}{2}} \lambda_1^{-2a-2b} \| \varphi_{\lambda_1} \|_{S^{1,\theta}_{\lambda_1}} \| \psi_{\lambda_2} \|_{S^{1,\theta}_{\lambda_2}} \]

where the last inequality follows from the embedding (2.7).

By noting that $\frac{d}{2} + l + 2\theta > 0$ we conclude that we require $2s - l \geq \frac{d}{2} - 2\theta + a - b$ for summability.

3. $H \times L \to H$

From Hölder’s and Bernstein’s inequality, we get

\[ \mu^2 \| F(C_{\beta_1} \varphi_{\lambda_1}, C_{\beta_2} \psi_{\lambda_2} \|_{L^2_{\tau,\xi}} \]
\[ \leq \mu^{\beta+\frac{d}{2}} \| C_{\beta_1} \varphi_{\lambda_1} \|_{L^2_{\tau,\xi}} \| C_{\beta_2} \psi_{\lambda_2} \|_{L^\infty_{\tau,\xi}} \]
\[ \leq \mu^{\beta+\frac{d}{2}} \lambda_1^{-2a-2b} \lambda_2^{-2a+b} \| (\tau + |\xi|^2)^{\theta} F((\lambda_1 + |\partial_\xi|)^n C_{\beta_1} \varphi_{\lambda_1} \|_{L^2_{\tau,\xi}} \]
\[ \leq \mu^{\beta+\frac{d}{2}} \lambda_1^{-2a-2b} \| \varphi_{\lambda_1} \|_{S^{1,\theta}_{\lambda_1}} \| \psi_{\lambda_2} \|_{S^{1,\theta}_{\lambda_2}} \]

Provided $2s - \beta \geq \frac{d}{2} - 2\theta + a - b$, we can obtain (4.2).

4. $L \times H \to H$

This interaction can be handled case by case with the roles of $\varphi$ and $\psi$ reversed.

5. $L \times L \to H$

We treat $d = 1, 2, 3$ separately.

$d = 3$ : We use Hölder’s and Bernstein’s inequality and the endpoint Strichartz space $L^2_t L^6_x$.

\[ \mu^2 \| F(C_{\beta_1}(\varphi_{\lambda_1}, C_{\beta_2} \psi_{\lambda_2} \|_{L^2_{\tau,\xi}} \leq \mu^{\beta+\frac{d}{2}} \| C_{\beta_1} \varphi_{\lambda_1} \|_{L^2_{\tau,\xi}} \| C_{\beta_2} \psi_{\lambda_2} \|_{L^\infty_{\tau,\xi}} \]
\[ \leq \mu^{\beta+\frac{d}{2}} \lambda_1^{-2a} \| \varphi_{\lambda_1} \|_{S^{1,\theta}_{\lambda_1}} \| \psi_{\lambda_2} \|_{S^{1,\theta}_{\lambda_2}} \]

$d = 2$ : We employ the bilinear Strichartz estimate (2.12) by decomposing the space-time Fourier supports of $C_{\beta_1} \varphi_{\lambda_1}$ and $C_{\beta_2} \psi_{\lambda_2}$ into pieces of size $L_1$ and $L_2$, respectively:

\[ \mu^2 \| F(C_{\beta_1}(\varphi_{\lambda_1}, C_{\beta_2} \psi_{\lambda_2} \|_{L^2_{\tau,\xi}} \leq \mu^2 \sum_{L_1, L_2 \leq \lambda_1} (L_1 L_2) \frac{1}{2} \| C_{\beta_1} \varphi_{\lambda_1} \|_{L^2_{\tau,\xi}} \| C_{\beta_2} \psi_{\lambda_2} \|_{L^\infty_{\tau,\xi}} \]
\[ \leq \mu^2 \lambda_1^{-2a} \| \varphi_{\lambda_1} \|_{S^{1,\theta}_{\lambda_1}} \| \psi_{\lambda_2} \|_{S^{1,\theta}_{\lambda_2}} \]

$d = 1$ : An application of Hölder’s and Bernstein’s inequality gives

\[ \mu^2 \| F(C_{\beta_1}(\varphi_{\lambda_1}, C_{\beta_2} \psi_{\lambda_2} \|_{L^2_{\tau,\xi}} \leq \mu^{\beta+\frac{d}{2}} \| C_{\beta_1} \varphi_{\lambda_1} \|_{L^\infty_{\tau,\xi}} \| C_{\beta_2} \psi_{\lambda_2} \|_{L^2_{\tau,\xi}} \leq \mu^{\beta+\frac{d}{2}} \lambda_1^{-2a} \| \varphi_{\lambda_1} \|_{S^{1,\theta}_{\lambda_1}} \| \psi_{\lambda_2} \|_{S^{1,\theta}_{\lambda_2}} \]
For $d \leq 3$, we require $s \geq 0$, $2s - \beta \geq \frac{1}{2}$ to sum the above subcase.

6. $L \times L \to L$

$d = 2, 3$ : We decompose the Fourier supports of $C_{\ll \lambda_1^2} \varphi_{\lambda_1}, C_{\ll \lambda_2^2} \psi_{\lambda_2}$ and $Q_{\ll \mu^2} \eta_{\mu}$ into pieces of size $L_1, L_2$ and $L_0$, respectively, and then apply lemma 2.4.

\[
\begin{align*}
&\lvert I(F(Q_{\ll \mu^2} \eta_{\mu}), F(C_{\ll \lambda_1^2} \varphi_{\lambda_1}), F(C_{\ll \lambda_2^2} \varphi_{\lambda_2}))\rvert \\
&\lesssim \sum_{L_1, L_2 \ll \lambda_1^2 \lambda_2^2 \ll \mu^2} \lvert I(F(C_{L_1} \eta_{L_1}), F(C_{L_2} \psi_{L_2}), F(C_{L_1} \varphi_{L_1}))\rvert \\
&\lesssim \sum_{L_1, L_2 \ll \lambda_1^2 \lambda_2^2 \ll \mu^2} \frac{(L_0 L_1 L_2)^{\frac{3}{2}}}{(\lambda_1^2 + \lambda_2^2)^{\frac{1}{2}}} \log \lambda_1 \lVert C_{L_1} \varphi_{\lambda_1} \rVert_{L_t^2, \infty} \lVert C_{L_2} \psi_{\lambda_2} \rVert_{L_t^2, \infty} \lVert Q_{L_0} \eta_{\mu} \rVert_{L_t^2, \infty} \\
&\lesssim \lambda_1^{-\frac{1}{2} - 2s} \mu^{2\theta + 1} \log \lambda_1 \lambda_1^s \lVert (\tau + |\xi|^2)^{\theta} F(C_{\ll \lambda_1^2} \varphi_{\lambda_1}) \rVert_{L_t^2, \infty} \lambda_2^s \lVert (\tau + |\xi|^2)^{\theta} F(C_{\ll \lambda_2^2} \psi_{\lambda_2}) \rVert_{L_t^2, \infty} \\
&\quad \times \mu^{-1-1} \lvert (\tau - |\xi|) \rvert^{-1-\theta} F(Q_{\ll \mu^2} \eta_{\mu}) \lVert_{L_t^2, \infty} \\
&\lesssim \lambda_1^{-\frac{1}{2} - 2s} \mu^{2\theta + 1} \log \lVert \varphi_{\lambda_1} \rVert_{S^{1, \theta, s}} \lVert \psi_{\lambda_2} \rVert_{S^{1, \theta, s}} \lVert \eta_{\mu} \rVert_{W^{1, \theta + 1 - s}}. 
\end{align*}
\]

$d = 1$ : We apply Hölder’s inequality and bilinear Strichartz estimate (2.13) by decomposing the space-time Fourier supports of $C_{\ll \lambda_1^2} \psi_{\lambda_2}$ and $Q_{\ll \mu^2} \eta_{\mu}$ into pieces of size $L_2$ and $L_0$, respectively.

\[
\begin{align*}
&\lvert I(F(Q_{\ll \mu^2} \eta_{\mu}), F(C_{\ll \lambda_1^2} \varphi_{\lambda_1}), F(C_{\ll \lambda_1^2} \psi_{\lambda_2}))\rvert \\
&\lesssim \lVert C_{\ll \lambda_1^2} \varphi_{\lambda_1} \rVert_{L_t^2, \infty} \lVert C_{\ll \lambda_1^2} \psi_{\lambda_2} \rVert_{L_t^2, \infty} \lVert Q_{L_0} \eta_{\mu} \rVert_{L_t^2, \infty} \\
&\lesssim \lVert C_{\ll \lambda_1^2} \varphi_{\lambda_1} \rVert_{L_t^2, \infty} \frac{1}{L_0^{\frac{1}{2}} L_2^{\frac{1}{2}}} (L_0 L_2)^{\frac{1}{2}} \lVert C_{L_2} \psi_{\lambda_2} \rVert_{L_t^2, \infty} \lVert Q_{L_0} \eta_{\mu} \rVert_{L_t^2, \infty} \\
&\lesssim \mu^{1/2} \lambda_1^{1/2 - 2s} \lambda_1^{s} \lVert (\tau + |\xi|^2)^{\theta} F(C_{\ll \lambda_1^2} \varphi_{\lambda_1}) \rVert_{L_t^2, \infty} \lambda_2^s \lVert (\tau + |\xi|^2)^{\theta} F(C_{\ll \lambda_2^2} \psi_{\lambda_2}) \rVert_{L_t^2, \infty} \\
&\quad \times \mu^{-1-1} \lvert (\tau - |\xi|) \rvert^{-1-\theta} F(Q_{\ll \mu^2} \eta_{\mu}) \lVert_{L_t^2, \infty} \\
&\lesssim \mu^{1/2} \lambda_1^{1/2 - 2s} \lVert \varphi_{\lambda_1} \rVert_{S^{1, \theta, s}} \lVert \psi_{\lambda_2} \rVert_{S^{1, \theta, s}} \lVert \eta_{\mu} \rVert_{W^{1, \theta + 1 - s}}. 
\end{align*}
\]

One requires $2s - l > -\frac{1}{2} + 2\theta$ to sum the subcase for $d \leq 3$. To conclude,

\[
\begin{align*}
&\bullet \ 2s - l \geq \max \left\{ -\frac{1}{2} + 2\theta, \frac{3}{2} - 2\theta + 2a - b \right\} \quad \bullet \ 2s - \beta > \max \left\{ \frac{3}{2} - 2\theta + a - b, \frac{1}{2} \right\} 
\end{align*}
\]

are the requirements for summability in this case.

The estimate (4.4) holds provided $s - l \geq -2 + 2\theta$, $s \geq 0$, $2s - l \geq -\frac{1}{2} + 2\theta$, and follows by imitating the proof of the product estimate $\lVert f \g \rVert_{L_t^2, \infty} \lesssim \lVert f \rVert_{L_t^2, \infty} \lVert g \rVert_{L_t^2, \infty}$ for both the cases above.

From (4.10) and (4.11), we list the required constraints on the parameters for the estimate (4.1) to hold true for $d \leq 3$:

\[
\begin{align*}
&\bullet \ s - l \geq \max \left\{ -2 + \theta, a - b, -\frac{1}{2} + 2(\theta - \theta') - 2a \right\} \quad \bullet \ s - \beta \geq -\frac{1}{2} \\
&\bullet \ 2s - l \geq \max \left\{ -\frac{1}{2} + 2\theta + 2a - b, -\frac{1}{2} + 2\theta \right\} \quad \bullet \ 2s - \beta \geq \max \left\{ \frac{3}{2} - 2\theta + a - b, \frac{1}{2} \right\}.
\end{align*}
\]

\[\square\]

5. Proof of Theorem 1.1

Given the non-linear estimates proved in sections 3 and 4, we can achieve a small data local well posedness result, see [8, section 5] for a simplified small data local well-posedness argument. To achieve a large data result, we need to extract a small power of $T$ on the RHS of the non-linear estimates.

We start with proving a slightly weaker form of the energy inequalities but with a small power of $T$ on the RHS. Note that for the $X^{s, \theta}$ type part of the norms, the required factor comes from property (2.8). It remains to extract the factor for the $L_t^2 L_x^d$ part of the norm.

To that end, we note for $0 < T < 1$ and a smooth time cutoff $\chi_T$

\[
\lambda^4 \lVert \chi_T \mathcal{I}_S [C_{\ll \lambda_1^2} \rho_{\ll \lambda_2} F_{\lambda}] \rVert_{L_t^\infty L_x^d} \lesssim \lambda^4 \lVert \chi_T C_{\ll \lambda_2^2} \rho_{\ll \lambda_2} F_{\lambda} \rVert_{L_t^1 L_x^d} \lesssim \lambda^4 T \lVert C_{\ll \lambda_2^2} \rho_{\ll \lambda_2} F_{\lambda} \rVert_{L_t^\infty L_x^d}.
\]
Interpolating this with $\lambda^s \|\chi_T \mathcal{I}_S[C_{\lambda \chi^2} F_{\lambda} P_{\lambda}(x) F_{\lambda}]\|_{L^p_{-t} L^2_x} \lesssim \lambda^{s-2} \|P_{\lambda}(x) F_{\lambda}\|_{L^p_{-t} L^2_x}$, we obtain for some $0 < \delta_1 < -\frac{1}{2} + \theta \ll 1$,
\[ \lambda^s \|\chi_T \mathcal{I}_S[C_{\lambda \chi^2} F_{\lambda} P_{\lambda}(x) F_{\lambda}]\|_{L^p_{-t} L^2_x} \lesssim \lambda^{s-2+2\delta_1} T^{\delta_1} \|C_{\lambda \chi^2} F_{\lambda} P_{\lambda}(x) F_{\lambda}\|_{L^p_{-t} L^2_x} \lesssim T^{\delta_1} \|F_{\lambda}\|_{N_s L_0^1 S^{s,0}}. \tag{5.1} \]

Note that we considered only the low ($\ll \lambda^2$) temporal frequency localised non-linearity because for the other case, we can again use property (2.8) since $m_S(\tau) \geq 1$.

As stated, from property (2.8) in section 2.4, for $\delta_2 > 0$, $-\frac{1}{2} < \theta - 1 < \theta - 1 + \delta_2 < \frac{1}{2}$, we have only for the $X^{s,\theta}$ part of the $N$ norm
\[ \|F\|_{N_s L_0^1 S^{s,\theta-1}(T)} \lesssim T^{\delta_2} \|F\|_{N_s L_0^1 S^{s,\theta+\delta_2}(T)}. \tag{5.2} \]

From (5.1) and (5.2) we conclude
\[ \|\chi_T \mathcal{I}_S[F]\|_{S^{s,\theta}(T)} \lesssim T^\delta \|F\|_{N_s L_0^1 S^{s,\theta}(T)} \tag{5.3} \]
where $\delta = \min\{\delta_1, \delta_2\} > 0$ and $\hat{\theta} = \theta - 1 + \delta$.

Using the same arguments for the wave solution and wave non-linearity, we have
\[ \|\chi_T \mathcal{I}_W[G]\|_{W^{s,\delta}(T)} \lesssim T^{\delta} \|G\|_{R^{s,\delta}(T)}. \tag{5.4} \]

We now head to prove theorem 1.1.

**Proof.** We call $(u, v) \in S^{s,\theta}(T) \times W^{l,\theta}(T)$ a solution to the system (2.1) – (2.2) with initial data $(u_0, v_0) \in H^{s} \times H^l$ if
\[ u(t) = \chi(e^{it\Delta} u_0 + \chi_T \mathcal{I}_S[uRe(v)])(t) \]
\[ v(t) = \chi(e^{it\Delta} v_0 + \chi_T \mathcal{I}_W[|\nabla|u|^2])(t), \tag{5.5} \]
for all $0 < T < 1$.

To apply a contraction mapping argument, we write (5.5) as $\Gamma(u, v)(t) = (u, v)(t)$. We prove that $\Gamma$ is a contraction on the space $S^{s,\theta}(T) \times W^{l,\theta}(T)$ for a suitably chosen $T$. Define $R := \|u_0\|_{H^{s}_2} + \|v_0\|_{H^{l}_2}$. We shall drop the superscripts on the norms for notational convenience. Then, using lemma 2.2, lemma 2.3, (5.3), (5.4), (3.1), (4.1), we have
\[ \|\Gamma(u, v)\|_{S \times W(T)} = \|\chi(t) e^{it\Delta} u_0 + \chi_T \mathcal{I}_S[uRe(v)]\|_{S(T)} + \|\chi(t) e^{it\Delta} v_0 + \chi_T \mathcal{I}_W[|\nabla|u|^2]\|_{W(T)} \]
\[ \lesssim (\|u_0\|_{H^{s}_2} + T^\delta \|u_0\|_{S(T)} + \|v_0\|_{H^{l}_2} + T^\delta \|v_0\|_{S(T)}^2) \]
\[ \lesssim C(\|u_0\|_{H^{s}_2} + \|v_0\|_{H^{l}_2} + T^\delta (\|u, v\|_{S \times W(T)} + \|u, v\|_{S \times W(T)}^2)). \]

Hence, $\Gamma$ is well-defined on a ball of radius $2C^2R$ in the space $S^{s,\theta}(T) \times W^{l,\theta}(T)$ if $T$ is chosen such that $C_{T^\delta} R < 1$ for some constant $C_1$. In order to show that $\Gamma$ is a contraction, we consider
\[ \|\Gamma(u_1, v_1) - \Gamma(u_2, v_2)\|_{S \times W(T)} \]
\[ = \|\chi_T(t) \mathcal{I}_S[u_1 Re(v_1) - u_2 Re(v_2)]\|_{S(T)} + \|\chi_T(t) \mathcal{I}_W[|\nabla|(|u_1|^2 - |u_2|^2)|w]\|_{W(T)} \]
\[ \lesssim T^\delta (\|u_1 Re(v_1) - u_2 Re(v_1)\|_{N_s L_0^1 S_s(T)} + \|u_1^2 - u_2^2\|_{R^{l,s}(T)}) \]
\[ \lesssim T^\delta (\|u_1 - u_2\|_{S(T)} + \|u_2 - u_1\|_{S(T)} + \|v_1 - v_2\|_{W(T)} + \|v_2 - v_1\|_{W(T)}) \]
\[ \lesssim T^\delta (\|u_1\|_{S(T)} + \|v_1\|_{W(T)} + \|u_2\|_{S(T)} + \|v_2\|_{W(T)})(\|u_1 - u_2\|_{S(T)} + \|v_1 - v_2\|_{W(T)}) \]
\[ = CT^\delta (\|u_1, u_2\|_{S \times W(T)} + \|v_1, v_2\|_{S \times W(T)})(\|u_1 - u_2, v_1 - v_2\|_{S \times W(T)}), \]
which becomes a contraction if $T$ is chosen such that $C_{T} T^\delta R < 1$, for a constant $C_2$. This proves that $(u, v) \in S^{s,\theta}(T) \times W^{l,\theta}(T)$ is a fixed point of $\Gamma$ i.e. a solution to the system (2.1) – (2.2) with initial data $(u_0, v_0)$. Standard arguments then imply uniqueness of the solution and Lipschitz continuity of the data to solution map for the system. \qed

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Akansha Sanwal, Faculty of Mathematics, Bielefeld University, Bielefeld, 33615, Germany.
E-mail - asanwal@math.uni-bielefeld.de