THE DERIVATIVE OF GLOBAL SURFACE-HOLONOMY FOR A NON-ABELIAN GERBE

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Abstract. Starting with a non-abelian gerbe represented by a non-abelian differential cocycle, with values in a given crossed-module, this paper explicitly calculates a formula for the derivative of the associated surface holonomy of squares mapped into the base manifold; with spheres later considered as a special case. While the definitions in this paper used for gerbes, their connections, and the induced holonomy will initially be simplicial, translations into a cubical setting will be provided to aide in explicit coordinate-based calculations. While there are many previously published results on the properties of these non-abelian gerbes, including some calculations of the derivative over a single open set, this paper endeavors to take these local calculations and glue them together across multiple open sets in order to obtain a single expression for the change in surface holonomy with respect to a one-parameter family of squares.

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1. Introduction

In [TWZ], an equivariantly closed differential form is associated to an abelian gerbe with connection by considering the derivative of the induced 2-holonomy. Originating as a first step in generalizing their work, this paper focuses on differentiating (Theorem 4.1) the global 2-holonomy (Definition 3.1) for a non-abelian $G$-gerbe (Definition 2.13).

Following Schreiber and Waldorf ([SWI], [SWII], [SWIII], and [SWIV]), the local cocycle description for a non-abelian gerbe with connection on a smooth manifold is reviewed and adopted. This paper uses their local transport data for bigons implicitly in Section 2.4.2 but translates that data into local transport data for squares as the process of differentiation became more manageable and organized in a cubical setting.

The method for gluing this local data together to provide a global definition of $\text{Hol}$ (Definition 3.1) in a cubical setting was borrowed from Martins and Picken (specifically, Figure 3 in [MP2]). Their papers proved many properties for a global holonomy on the group-level which can be found in Section 3.3, where these properties are reviewed in addition to some other relevant observations. This paper adds to those references by providing for a global formula, comprised of local data, for the derivative of global surface holonomy.

In Section 4, the main theorem of this paper, Theorem 4.1, is stated and proven, which essentially reads

$$d(\text{Hol}) = \text{Hol} \cdot \int_{Sq} H \quad \text{(modulo terms on the boundary of } Sq),$$

where $\int_{Sq} H$ represents fiber-integration of the 3-curvature terms for the given non-abelian gerbe through the interior of the square. The proof of this theorem, given in Section 4.4, amounts to considering a 1-parameter family of squares, considering the associated arrangement of cubes from the local data, and organizing the terms in the derivative accordingly.

Finally, in Sections 5.1 and 5.2, some examples are offered where Theorem 4.1 has an even cleaner representation. In the case where the surfaces which are integrated over are spheres, $d(\text{Hol})$ has only a boundary term at the base point. In the case where the gerbe is abelian, the well known situation where there are no terms for $d(\text{Hol})$ on the boundary is reproduced.

The hope after this paper is to continue the work of finding a non-abelian analogue of the work done in [TWZ], via a subsequent paper which will use the result of Theorem 4.1 and the appropriate cohomology theory, to find some equivariantly closed element representing a non-abelian gerbe with connection. In a separate project, the goal is to extend this paper’s derivative for 2-dimensional holonomy (landing in crossed-modules) to the derivative of 3-dimensional holonomy (landing in the appropriate version of a 2-crossed module). Furthermore, some of the geometric comments regarding the “globalness” of this 2-holonomy are planned to be formalized in current joint work with Micah Miller, Thomas Tradler, and Mahmoud Zeinalian.

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2. Conventions, Notation, and Setup

In order to arrive at a definition for global 2-holonomy, Definition 3.1 it is necessary to introduce some preliminary definitions and conventions.

2.1. Diffeological Spaces. Since we will be working on the space of smooth maps, $M^{Sq} := \{ \Sigma : Sq \to M \}$, where $Sq := [0,1] \times [0,1]$ is the standard square and $M$ is a smooth manifold, it is convenient to use the language of diffeological spaces and plots as described in [BaHo] and [IZ] (see [C2] for an early reference on “plots”). For our purposes, $M^{Sq}$ is a diffeology whose plots are given by maps $P : U \to C^\infty(M, Sq)$ which are smooth in both variables, i.e. the map $(r, x) \mapsto P(r)(x)$ is a smooth map from the subset $U \subset \mathbb{R}^n$ to $M$.

2.1.1. Covering $M^{Sq}$ With Open Sets $\mathcal{N}$. It is possible that any square mapped into $M$ will not be entirely contained within one open set, $U_\alpha$, of a chosen cover, $\mathcal{U}$. Instead, the square, $Sq$, is subdivided into a grid whereby each sub-square, $Sq(p,q)$, is mapped entirely inside some open set $U_{i(p,q)} \in \mathcal{U}$, as was the key idea borrowed from [MP2] and is now explained below. Note that while a finite collection of open sets is usually indexed, $U_{i_1}, U_{i_2}, \ldots, U_{i_n}$, it is helpful in this context to use the indexing convention, $i(p,q)$, which will range over the grid: $U_{i(1,1)}, U_{i(1,2)}, \ldots, U_{i(n,m)}$.

**Definition 2.1.** For a fixed open cover, $\mathcal{U}$, of $M$, a choice of a grid $I = \{1, \ldots, n\} \times \{1, \ldots, m\}$, and a choice of open sets $\{U_{i(p,q)}\}_{(p,q) \in I}$, define

$$\mathcal{N}_I := \{ \Sigma \in M^{Sq} \mid \text{for each } (p, q) \in I, \Sigma(Sq_{(p,q)}) \subset U_{i(p,q)} \},$$

where $Sq_{(p,q)} := \left[ \frac{p-1}{n}, \frac{p}{n} \right] \times \left[ \frac{q-1}{m}, \frac{q}{m} \right] \subset [0,1] \times [0,1]$. When the indexing set, $I$, is understood, we will simply refer to the open set as $\mathcal{N}$.

It will be useful later in Section 3 to have the notion of a grid on $\Sigma \in \mathcal{N}_I$.

**Definition 2.2.** Given a square, $\Sigma \in \mathcal{N}_I$, define the grid on $\Sigma$ by the following set of data: For each $i = (p, q) \in I$ define the $i$-face by $\gamma_i := \Sigma|_{Sq_i}$; For each $i = (p, q), j = (p+1, q)$ define the $ij$-vertical edge, $\gamma_{ij}^v$, by

$$\gamma_{ij}^v := \Sigma_i|_{\left[ \frac{q+1}{n}, \frac{q+1}{n} \right] \times \left[ \frac{p}{n}, \frac{p}{n} \right]} = \Sigma_j|_{\left[ \frac{1}{n}, \frac{1}{n} \right] \times \left[ \frac{q-1}{m}, \frac{q-1}{m} \right]} = \Sigma|_{\left[ \frac{1}{n}, \frac{1}{n} \right] \times \left[ \frac{q-1}{m}, \frac{q-1}{m} \right]}.$$

For each $i = (p, q), j = (p, q+1)$ define the $ij$-horizontal edge, $\gamma_{ij}^h$, by

$$\gamma_{ij}^h := \Sigma_i|_{\left[ \frac{p+1}{n}, \frac{p+1}{n} \right] \times \left[ \frac{q+1}{n}, \frac{q+1}{n} \right]} = \Sigma_j|_{\left[ \frac{q-1}{m}, \frac{q-1}{m} \right] \times \left[ \frac{1}{n}, \frac{1}{n} \right]} = \Sigma|_{\left[ \frac{q-1}{m}, \frac{q-1}{m} \right] \times \left[ \frac{1}{n}, \frac{1}{n} \right]}.$$

Any such $ij$-edge has a source vertex and target vertex labeled by $x_{ij}^0$ and $x_{ij}^1$, respectively. For each $ijkl$ where $i = (p, q), j = (p, q+1), k = (p+1, q), l = (p+1, q+1)$ define the $ijkl$-vertex, $x_{ijkl}$, by

$$x_{ijkl} := \Sigma|_{\left[ \frac{1}{n}, \frac{1}{n} \right] \times \left[ \frac{1}{n}, \frac{1}{n} \right]}.$$

For each $i_N = (p, 1), i_S = (p, m), i_W = (1, q), i_E = (n, q)$, define the northern boundary edge, $\gamma_i^N$, the southern boundary edge, $\gamma_i^S$, the western boundary edge, $\gamma_i^W$, and the eastern boundary edge, $\gamma_i^E$, respectively by:

$$\gamma_i^N := \Sigma_i|_{\left[ \frac{1}{n}, \frac{1}{n} \right] \times \{0\}}, \quad \gamma_i^S := \Sigma_i|_{\left[ \frac{1}{n}, \frac{1}{n} \right] \times \{1\}}, \quad \gamma_i^W := \Sigma_i|_{\{0\} \times \left[ \frac{1}{n}, \frac{1}{n} \right]} = \Sigma_i|_{\{1\} \times \left[ \frac{1}{n}, \frac{1}{n} \right]}.$$

Every diffeology on $X$ induces a topology on $X$, called the $\mathcal{D}$-topology. Page 54 of [IZ] states the following characterization of the $\mathcal{D}$-topology:
Proposition 2.3. A subset \( A \) of a diffeological space, \( X \), is open for the \( D \)-topology if and only if for every plot, \( \rho : U \to X \), \( \rho^{-1}(A) \) is open in \( U \).

It is straightforward to check that the sets \( N \) are open in the diffeology \( M^{Sq} \) under its \( D \)-topology:

Proposition 2.4. Consider the diffeological space \( M^Y := \{ f : Y \to M \mid f \text{ is smooth } \} \), with plots \( \rho : U \to M^Y \) given by smooth maps \( \tilde{\rho} : U \times Y \to M \), where \( (r, y) \mapsto \rho(r)(y) \). If \( K \subset Y \) is compact and \( U \subset M \) is open, then

\[
N(K, U) := \{ f \in M^Y \mid f|_K \subset U \}
\]

is an open set in \( M^Y \).

Corollary 2.5. Each \( N_I \) is an open subset of \( M^{Sq} \).

Proposition 2.6. For a fixed open cover \( U \), the open sets \( N_I \), ranging over all choices from Definition 2.1, cover \( M^{Sq} \).

2.2. Crossed Module Conventions and Relations. An early reference for the understanding that crossed modules were helpful structures for dealing with 2-dimensional algebra is [BrSp]. In this section, following [GiPf] as a reference for notation and convention, the definition of crossed modules, and some related properties that will be essential later on, are now reviewed.

Definition 2.7. A crossed module of Lie Groups is a pair of Lie groups, \((H, G)\), with a smooth group homomorphism, \((H \xrightarrow{t} G)\), called the target, and an action \( \alpha : G \to \text{Aut}(H) \), written \( \alpha_g(h) \), so that \( t \) and \( \alpha \) are required to satisfy the following compatibility conditions:

\[
(2.1) \quad t(\alpha_g(h)) = gt(h)g^{-1} \\
(2.2) \quad \alpha_{t(h)}(h') = hh'h^{-1}
\]

Example 2.8. If \( H \) is a Lie group, then \( G := \text{Aut}(H) \) induces a crossed module \((H \xrightarrow{t} \text{Aut}(H))\) via the target \( t : H \to \text{Aut}(H) \) given by

\[
t(h)(h') := hh'h^{-1}
\]

and the action \( \alpha : G \to \text{Aut}(H) \) given by the identity automorphism.

Example 2.9. Define the crossed module \( BS^1 := (S^1 \xrightarrow{t} \{\ast\}) \) given by the trivial target and identity action \( \alpha \). This is the crossed module often used in the study of abelian gerbes.

Associated to such a crossed module is a crossed module of Lie algebras:

Definition 2.10. A crossed module of Lie algebras is a pair of Lie algebras, \((\mathfrak{h}, \mathfrak{g})\), with a Lie algebra map, \((\mathfrak{h} \xrightarrow{t} \mathfrak{g})\), called the target, and a map \( \alpha : \mathfrak{g} \to \text{Der}(\mathfrak{h}) \), written \( \alpha_A(B) \), so that the two maps \( t \) and \( \alpha \) must satisfy the following compatibility conditions:

\[
(2.3) \quad t(\alpha_X Y) = [X, t(Y)] \\
(2.4) \quad \alpha_{t(Y_1)}(Y_2) = [Y_1, Y_2]
\]

A useful proposition regarding the center of \( H \) is as follows:
Proposition 2.11. The kernel of the target map, \( t : H \to G \), is in the center, \( Z(H) \), of \( H \). Similarly, the kernel of the target map, \( t : \mathfrak{h} \to \mathfrak{g} \) is in the center, \( Z(\mathfrak{h}) \), of \( \mathfrak{h} \).

Proof. If \( t(h) = 1 \), then for any \( h' \in H \), \( h' = \alpha_1(h') = \alpha_{t(h)}(h') = hh'h^{-1} \), thus \( h \in Z(H) \). A similar proof can be applied to the statement for \( X \in \mathfrak{h} \) with \( t(X) = 0 \):

\[
0 = \alpha_0(Y) = \alpha_{t(X)}(Y) = [X, Y].
\]

An important and well-known lemma, [KN1, Proposition 1.4] in calculating the derivative of the \( \alpha \) map is provided now for later reference

Lemma 2.12. For the function \( \alpha : G \times H \to H \) defined above, given functions \( g(t) : \mathbb{R} \to G \) and \( h(t) : \mathbb{R} \to H \), we have

\[
(2.5) \quad \frac{\partial}{\partial t} \bigg|_{t=t_0} (\alpha_{g(t)}(h(t))) = (\alpha_{g(t_0)})_{*} \left( \frac{\partial}{\partial t} \bigg|_{t=t_0} h(t) \right) + (\alpha_{h(t_0)})_{*} \left( \frac{\partial}{\partial t} \bigg|_{t=t_0} g(t) \right)
\]

as an equality in \( T_{\alpha_{g(t_0)}(h(t_0))}H \), the tangent space of \( H \) at \( \alpha_{g(t_0)}(h(t_0)) \). We will sometimes write

\[
\alpha \left( \frac{\partial}{\partial t} \bigg|_{t=t_0} g(t) \right) h(t_0) := (\alpha_{h(t_0)})_{*} \left( \frac{\partial}{\partial t} \bigg|_{t=t_0} g(t) \right)
\]

and refer to these as “path terms”.

2.3. Local Differential Data for a Gerbe. In [NW] it is shown that the cocycle description of a non-abelian gerbe is equivalent to the other three common formulations: classifying maps, groupoid bundle gerbes, and principal 2-bundles. In [SWIII] they go on to provide their simplicial formulation for a connection on, and associated parallel transport for, a given non-abelian gerbe. Following [SWIII], and since it is assumed we are working on a good open cover, i.e. one where all open sets and their \( n \)-fold intersections are contractible, the following (normalized) local data for a gerbe are used:

Definition 2.13. Given a smooth manifold, \( M \), an open cover \( \mathcal{U} = \{U_i\} \) of \( M \), and a crossed module of Lie groups \( \mathcal{G} = (H \xrightarrow{t} G) \), a \( \mathcal{G} \)-gerbe with a connection on \( M \), subordinate to the cover \( \mathcal{U} \), is defined by the following (normalized) local cocycle data:

- On each open set, \( U_i \) a pair
  \[ (A_i \in \Omega^1(U_i, \mathcal{G}), B_i \in \Omega^2(U_i, \mathfrak{h})) \]

- On each intersection, \( U_{ij} := U_i \cap U_j \) a pair
  \[ (g_{ij} \in \Omega^0(U_{ij}, \mathcal{G}), a_{ij} \in \Omega^1(U_{ij}, \mathfrak{h})) \]

- On each triple intersection, \( U_{ijk} := U_i \cap U_j \cap U_k \) a function
  \[ f_{ijk} \in \Omega^0(U_{ijk}, H) \]

satisfying the relations.
(1) On each open set,
\[ R_i := dA_i + \frac{1}{2}[A_i \wedge A_i] = t(B_i) \]
\[ g_{ii} = 1 \]
\[ a_{ii} = 0. \]

(2) On each intersection,
\[ A_j = g_{ij} A_i g_{ij}^{-1} - dg_{ij} g_{ij}^{-1} - t(a_{ij}) \]
\[ B_j = \alpha g_{ij} (B_i) - \nabla_j (a_{ij}) - \frac{1}{2}[a_{ij} \wedge a_{ij}] \]
\[ f_{ii} = f_{ij} = 1. \]

(3) On each triple intersection,
\[ g_{ik} = t(f_{ijk}) g_{jk} g_{ij} \]
\[ f_{ijk} \cdot a_{ik} \cdot f_{ijk}^{-1} = (\alpha g_{ij}) \cdot (a_{ij}) + a_{jk} + ((\alpha f_{ijk}) \cdot (A_k)) \cdot f_{ijk}^{-1} + df_{ijk} \cdot f_{ijk}^{-1}. \]

(4) On each quadruple intersection,
\[ f_{ikl} \alpha g_{kl} (f_{ijk}) = f_{ij} f_{jkl}. \]

where \( \nabla_j (\omega) := d\omega + \alpha A_j (\omega), g \cdot A := (L_g)_*(A), \) and \( A \cdot g := (R_g)_*(A). \)

Girelli and Pfeiffer [GiPf], as well as Baez and Schreiber [BaSc], define the curvature 3-form of a gerbe \( H_i \in \Omega^3(U_i, \mathfrak{h}) \) by
\[ H_i := \nabla_i (B_i) := dB_i + \alpha A_i (B_i) \]
where \( B_i \in \Omega^2(U_i, \mathfrak{h}) \) and \( (A_i, B_i) \) define the 2-connection of our gerbe on \( U_i \). It is straight-forward to check the following properties of the curvature 3-form.

**Proposition 2.14.** The curvature 3-form, \( H_i \), of a gerbe, satisfies
\[ (1) \ t(H_i) = 0 \]
\[ (2) \ \nabla_i (H_i) = 0. \]
\[ (3) \ H_j = \alpha g_{ij} (H_i), \text{ on each intersection of open sets, } U_{ij}. \]

2.4. **Local 2-Holonomy.** In this section, the simplicial definition of gerbes will give rise to a cubical notion of 2-holonomy.

2.4.1. **Comments on the local geometry of 2-holonomy.** In the works of Schreiber and Waldorf [SWI, SWII, SWIII] as well as Martins and Picken [MP1, MP2], the higher-holonomy is constructed as a map of 2-groupoids from a certain path 2-groupoid (see [SWII section 2.1] or [MP2 section 2.3]) to the 2-groupoid described by a crossed module of Lie groups with on object (see [BrSp, Theorem A] for this second 2-groupoid). Having such a map requires certain coherence conditions: for example that the algebraic target for the 2-holonomy of a surface be related to the 1-holonomy of the geometric boundary for that surface; and that the holonomy of a composition of surfaces/paths corresponds to the algebraic product of the holonomies of these surfaces/paths.

In both of these sets of authors’ respective path 2-groupoids (Martins and Picken using a cubical special double groupoid a la Brown and Spencer versus Schreiber and Waldorf using a simplicial double groupoid), the objects are points in the base manifold, and the 1-morphisms are thin homotopy classes of paths. That is, the
morphisms are equivalence classes of paths where the relation is given by certain homotopies between two paths having at most rank 1 [SWII, Definition 1.1]. Since this paper formally uses the data from [SWII], their simplicial path 2-groupoid is technically being used. However, by considering a square mapped into a manifold as a bigon whose source is the \((t = 0, s = 0)\)-vertex of the square and whose target is the boundary of the square, we can apply the higher holonomy maps of [SWII] to our squares mapped into the manifold and proceed from there.

2.4.2. Local Transport Data for Squares. Fix a gerbe on \(M\) subordinate to the cover \(U\), as defined in Definition 2.13. Then, following a modified version of the transport data for bigons given in [SWII, Section 1.3], define smooth functions

\[
\text{hol}_i : U_i^{[0,1]} \to G, \quad \text{Hol}_i : U_i^{Sq} \to H, \quad \text{Hol}_{ij} : U_{ij}^{[0,1]} \to H, \quad \text{and} \quad \text{Hol}_{ijkl} : U_{ijkl} \to H.
\]

These local functions satisfy the differential equations below. Essential to the construction of these functions in [SWII] are the concepts that (a) each path, \(\gamma : [0, 1] \to U_i\), can be extended trivially to a path with the same image in \(M\), \(\tilde{\gamma}\), having domain \(R\), and (b) from such a path, \(\tilde{\gamma} : R \to U_i\), we can consider the one-parameter family of paths, \(\tilde{\gamma}_t\), where for each \(t \in R\), the path’s image is that of \(\gamma\big|_{[0,t]}\) while being re-parametrized to still satisfy the definition of being a morphism in the path groupoid of \(M\) [SWII Section 1.1]. Similarly, squares \(\Sigma \in U_i^{Sq}\) produce one-parameter families of squares \(\Sigma_s\) whose image is that of \(\Sigma_{[0,1] \times [0, s]}\) while being 2-morphisms in the path 2-groupoid of \(M\) [SWII Section 2.1].

(D1) For a one-parameter family of paths, \(\tilde{\gamma}_t\), induced by the path, \(\gamma(t)\):

\[
\frac{\partial}{\partial t} \text{hol}_i(\gamma_t)^{-1} = \text{hol}_i \cdot A_i \left( \frac{\partial}{\partial t} \right)
\]

(D2) For a one-parameter family of squares, \(\Sigma_s\), induced by the square, \(\Sigma(t, s)\):

\[
\frac{\partial}{\partial s} \text{Hol}_i(\Sigma_s) = \text{Hol}_i(\Sigma_s) \cdot \int_0^1 (a_{\text{hol}_i(\cdot, s)})^* (B_i) dt \left( \frac{\partial}{\partial s} \right)
\]

(D3) For a one-parameter family of paths, \(\tilde{\gamma}_t\), induced by the path, \(\gamma(t)\):

\[
\frac{\partial}{\partial t} \bigg|_{t=0} (\text{Hol}_{ij}(\gamma_t)) = \alpha_{g_{ij}^{-1}(\gamma(0))} \left( a_{ij} \bigg|_{\gamma(0)} \left( \frac{\partial}{\partial t} \bigg|_{t=0} \right) \right)
\]

The collection of local transport data for squares \(\text{Hol}_i\) and \(\text{Hol}_{ij}\) have the following targets:

- For \(\text{Hol}_i\):

\[
t(\text{Hol}_i) = \text{hol}_i(-, 0)^{-1} \text{hol}_i(t, -)^{-1} \text{hol}_i(-, s) \text{hol}_i(0, -)
\]
which will be pictured diagrammatically as:

\[ (2.6) \]

\[
\begin{array}{c}
\text{hol}_i(\gamma(-, 0))^{-1} \\
\text{hol}_i(\gamma(0, -)) \\
\text{hol}_i(\gamma(t, -))^{-1} \\
\text{hol}_i(\gamma(-, s))
\end{array}
\]

- For \( \text{Hol}_{ij} \):
  
  \[ t(\text{Hol}_{ij}) = \text{hol}_i^{-1} g_{ij}^{-1}(y) \text{hol}_j g_{ij}(x) \]

which will be pictured diagrammatically as:

\[ (2.7) \]

\[
\begin{array}{c}
\text{hol}_i(\gamma)^{-1} \\
\text{g}_{ij}(\gamma(0)) \\
\text{g}_{ij}(\gamma(1))^{-1} \\
\text{hol}_j(\gamma)
\end{array}
\]

- For \( \text{Hol}_{ijkl} := \alpha g_{ik}^{-1} \left( \alpha g_{kj}^{-1} (f_{jkl}) f_{ijkl}^{-1} \right) \):
  
  \[ t(\text{Hol}_{ijkl}) = g_{ik}^{-1} g_{kl}^{-1} g_{jl} g_{ij} \]

which will be pictured diagrammatically as:

\[ (2.8) \]

\[
\begin{array}{c}
g_{ik}^{-1} \\
\text{g}_{il} \\
\text{Hol}_{ijkl} \\
g_{jl}^{-1}
\end{array}
\]

2.4.3. Glueing-Paths. Some of the conventions and computations which are necessary to glue the local transport data for squares defined above are laid in this section. Much of the details in this section can be described via the following proposition which can be seen as a result of the works of Brown, Spencer, and Higgins in [BrSp][BrHi][Br].
Proposition 2.15 ([BrSp] Theorem A]). Let $\mathcal{DG}$ be the category whose objects are special double groupoids with special connection and whose arrows are the morphisms of double groupoids preserving the connection, and $\mathcal{C}$ be the category of crossed modules. If we consider the full sub-category of $\mathcal{DG}$ whose double groupoids have exactly one object labeled, $\mathcal{DG}^1$, then there is an equivalence of categories $\gamma: \mathcal{DG}^1 \rightarrow \mathcal{C}$.

As a medium of discussion, the interchange law for our “glueing paths” is proved. At the end of this section, the established conventions are used to prove Propositions 2.18 and 2.17.

If two (horizontally) adjacent squares are to glued, define horizontal multiplication by using the following procedure:

\[
\text{Hol}_1 \cdot \alpha (d^{-1}c^{-1}b)(\text{Hol}_2) = \text{Hol}_1 \cdot \alpha (d^{-1}c^{-1}b)(\text{Hol}_2)
\]

(2.9)

Similarly, for vertically adjacent squares, vertical multiplication is defined by using the following procedure:

\[
\text{Hol}_2 \cdot \alpha (a^{-1})(\text{Hol}_1) = \text{Hol}_2 \cdot \alpha (a^{-1})(\text{Hol}_1)
\]

(2.10)

Proposition 2.16. The procedures above for composing 2-squares satisfy the “interchange law”. Moreover, any grid of squares glues together to a single square providing a unique element in $H$ associated to the grid.

\[\text{This procedure is helpful to be conceptualized as a “zip/unzip” procedure, in order to easily follow some calculations.}\]
Proof. The proof is exhibited in diagram-form. When first composing horizontally and then composing vertically:

\[
\begin{array}{c}
\bullet\quad a \rightarrow \quad b \rightarrow \\
c \quad Hol_1 \quad d \quad Hol_2 \quad e \\
\downarrow \quad f \rightarrow \quad g \rightarrow \\
h \quad Hol_3 \quad i \quad Hol_4 \quad j \\
\downarrow \quad k \rightarrow \quad l \rightarrow \\
\end{array}
\]

\[
\begin{array}{c}
\bullet \quad \text{Hol}_1 \cdot \alpha(c^{-1} f^{-1} d)(\text{Hol}_2) \quad e \\
\downarrow \\
\end{array}
\]

\[
\begin{array}{c}
\bullet \quad \text{Hol}_3 \cdot \alpha(h^{-1} k^{-1} i)(\text{Hol}_4) \quad j \\
\downarrow \\
\end{array}
\]

\[
\begin{array}{c}
\bullet \quad \text{Hol}_1 \cdot \alpha(c^{-1} f^{-1} d)(\text{Hol}_2) \cdot \alpha(c^{-1})(\text{Hol}_3 \cdot \alpha(h^{-1} k^{-1} i)(\text{Hol}_4)) \quad je \\
\downarrow \\
\end{array}
\]

whereas if the squares are composed vertically and then horizontally:

\[
\begin{array}{c}
\bullet\quad a \rightarrow \quad b \rightarrow \\
c \quad Hol_1 \quad d \quad Hol_2 \quad e \\
\downarrow \quad f \rightarrow \quad g \rightarrow \\
h \quad Hol_3 \quad i \quad Hol_4 \quad j \\
\downarrow \quad k \rightarrow \quad l \rightarrow \\
\end{array}
\]

\[
\begin{array}{c}
\bullet \quad \text{Hol}_1 \cdot \alpha(c^{-1})(\text{Hol}_3) \quad \text{Hol}_2 \cdot \alpha(d^{-1})(\text{Hol}_4) \\
\downarrow \\
\end{array}
\]

\[
\begin{array}{c}
\bullet \quad \text{Hol}_1 \cdot \alpha(c^{-1})(\text{Hol}_3) \cdot \alpha((hc)^{-1} k^{-1} i)(\text{Hol}_2 \cdot \alpha(d^{-1})(\text{Hol}_4)) \quad je \\
\downarrow \\
\end{array}
\]

It is straightforward to check that these two expressions are equal via the crossed module relations and observing the boundaries/targets of the \(\text{Hol}_*\) terms. \(\square\)

The following two propositions follow from [SWII, Lemmas 2.19 and 2.20], and state how these \(\text{Hol}\) functions relate to one another.
Proposition 2.17 (Edge Cube). Given a square, $\Sigma$, there is an equation at each intersection $U_{ij}$

$$\Hol_{ij}(-,0) = \Hol_{ij}(0,-) \cdot \alpha_{g_{ij}(0,0)}^{-1} \Hol_{ij}(0,-)^{-1} g_{ij}(0,s) \Hol_{ij}(0,-) (\Hol_{ij}(\Sigma))$$

$$\cdot \alpha_{g_{ij}(0,0)}^{-1} \Hol_{ij}(0,-)^{-1} g_{ij}(0,s) (\Hol_{ij}(-,s)) \alpha_{g_{ij}(0,0)}^{-1} (\Hol_{ij}^{-1}(\Sigma))$$

expressed by the following cube-diagram (2.11)

\[
\begin{array}{c}
\text{hol}_i(-,0) \quad \text{hol}_j(-,0) \\
\text{hol}_i(0,-) \quad \text{hol}_j(0,s) \\
\text{hol}_i(-,s) \quad \text{hol}_j(0,-)
\end{array}
\]

Proposition 2.18 (Vertex Cube). For a path $\gamma$ in $U_{ijkl}$, with $x = \gamma(0)$ and $y = \gamma(1)$, the local data for squares of Section 2.4.4 satisfies

$$\Hol_{ijkl}(x) = \Hol_{ij}(x) \cdot \alpha_{g_{ijkl}(x)}^{-1} \Hol_{ij}(x) \cdot \alpha_{g_{ijkl}(y)}^{-1} \Hol_{ijkl}(y)$$

expressed by the following cube-diagram (2.12)
3. Glueing Together Local 2-Holonomy

In this section, a definition is provided for 2-holonomy of a square mapped into $M$, $\Sigma \in \mathcal{N} \subset M^{S^2}$ (Definition 2.1), which lands in multiple open sets $U_i \subset M$. Recall that in Section 2.1, following the notation in [TWZ], an open cover of $M^{S^2}$ is given by sets $N_I = \{ \Sigma \in M^{S^2} | (p, q) \in I, \Sigma(Sq(p,q)) \subset U_i(p,q) \}$.

While the notation and inspiration for this particular paper is attributed to [TWZ], the broader scope of the ideas in this section (in particular the non-abelian approach to glueing squares) can be traced back to [BrHi] while the details and diagrams of [MP2] (e.g., Fig 3) were not only inspirational to this paper, but serve as a reference for this cubical approach to “patching together local holonomies”.

3.1. Comments on the globalness of 2-Holonomy. Before providing a definition for a “global” 2-holonomy in the next section, it is important to address the extent to which it is global. It is helpful to first review what this means for the 1-holonomy given by a principal $G$-bundle with connection, given by local data subordinate to an open cover, $\{ U_i \}$. The first thing one obtains from this local data is a functor of groupoids, $\mathcal{P}(U_i) \xrightarrow{hol_i} BG$, from the path 1-groupoid referenced above to the groupoid with a single object and morphism space equal to the Lie group, $G$. The fact that this is a functor implies in particular, that the holonomy of a constant path is the identity, i.e. $hol_i(\text{constant path}) = id_G$ as well as that holonomy respects composition of paths, i.e. $hol_i(\gamma_1 \circ \gamma_2) = hol_i(\gamma_1) \cdot hol_i(\gamma_2)$. The space of paths on the base manifold, up to thin homotopy again, has open sets of the form $N_{\{i_1, i_2, \ldots, i_n\}}$, where similar to definition 2.1, the paths in this open set are decomposed into pieces which land in the open sets $U_{i_1}, \ldots, U_{i_n}$. By applying the $g_{ij}$ transition maps at the vertices of these decomposed paths, and multiplying the $hol_i$ maps to the pieces of path, we obtain maps, $hol_N : \mathcal{N} \rightarrow G$. Next, restricting to loops, we can consider the sub-groupoid of loops $\mathcal{L}(M)$ over the base manifold, $M$, with similar open sets, and see that these maps $hol_N$ differ between two open sets $\mathcal{N}$ and $\mathcal{N}'$ by conjugation of $g_{i_1i'_1}$ applied to the based point of the loop in question. Finally, we can therefore say there is a well-defined map of groupoids, $hol : \mathcal{L}(M) \rightarrow B$, where $B$ is a bundle of groupoids, with trivializations of the form $\mathcal{N} \times BG$. It is in such a context that one can say that the properties of 1-holonomy can provide a global map.

Note however that for 2-holonomy, we may not have a transitive relation when we try to build a bundle of 2-groupoids (since our $g_{ij}$ do not satisfy the cocycle condition). In future work based on a current joint project with Micah Miller, Thomas Tradler, and Mahmoud Zeinalian, we hope to formally establish the previous paragraph and then setup and prove the following claim:

The 2-holonomy maps, $Hol_{\mathcal{N}}$, described in definition 2.1, glue together to construct a map of 2-groupoids $Hol : \mathcal{S}_2(M) \rightarrow \hat{G}$ from the sphere 2-groupoid to a gerbe of 2-groupoids over $M$ with structure 2-groupoid given by the crossed module, $G$.

While the details required to carefully setup and prove all of the statements implied in this subsection are not all contained in this paper, the following subsection goes
3.2. Semi-Global 2-Holonomy. For the remainder of this section, fix a good open cover \( \mathcal{U} = \{ U_i \} \) of \( M \) and an open set \( \mathcal{N} \subset M^{\text{Sq}} \) as in Definition 2.1.

**Definition 3.1.** Given local transport data for squares, \( \{ \text{Hol}_i \}, \{ \text{Hol}_{ij} \}, \{ \text{Hol}_{ijkl} \} \), define \( \text{Hol}^N : \mathcal{N} \to H \) on a square \( \Sigma \in \mathcal{N} \subset M^{\text{Sq}} \) by assembling the local data on the grid (Definition 2.2) as shown:

\[
\begin{array}{ccccccc}
\text{Hol}_a & \text{Hol}_{ab}^{-1} & \text{Hol}_b & \text{Hol}_{bc}^{-1} & \text{Hol}_c & \cdots & \text{Hol}_d \\
\text{Hol}_e & \text{Hol}_{ef}^{-1} & \text{Hol}_f & \text{Hol}_{fg}^{-1} & \text{Hol}_g & \cdots & \text{Hol}_h \\
\text{Hol}_i & \text{Hol}_{ij}^{-1} & \text{Hol}_j & \text{Hol}_{jk}^{-1} & \text{Hol}_k & \cdots & \text{Hol}_l \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\text{Hol}_m & \text{Hol}_{mn}^{-1} & \text{Hol}_n & \text{Hol}_{np}^{-1} & \text{Hol}_o & \cdots & \text{Hol}_p \\
\end{array}
\]

(3.1)

Using the multiplication conventions (diagrams (2.9) and (2.10)) for squares, glue first vertically and then horizontally to obtain the following expression

\[
\text{Hol}^N := \text{Hol}_a \cdot \text{Hol}_{ab}^{-1} \cdot \text{Hol}_b \cdot \text{Hol}_{bc}^{-1} \cdot \text{Hol}_c \cdot \cdots \cdot \text{Hol}_d \cdot \text{Hol}_{ab}^{-1} \cdot \text{Hol}_{ef}^{-1} \cdot \text{Hol}_e \cdot \text{Hol}_{fg}^{-1} \cdot \text{Hol}_f \cdot \text{Hol}_{gh}^{-1} \cdot \text{Hol}_g \cdot \cdots \cdot \text{Hol}_p 
\]

(3.2)

where \( \text{Hol}_i \) is evaluated on the face \( \Sigma_i \), \( \text{Hol}_{ij} \) is evaluated on the edge \( \gamma_{ij} \), and \( \text{Hol}_{ijkl} \) is evaluated on the vertex \( x_{ijkl} \) as described in Definition 2.2. The overline decoration on each, \( \overline{\text{X}} \), is described in Convention 3.2 below. Note that each face, \( \Sigma_a \), and edge, \( \gamma_{ab} \), technically corresponds to (products of) subintervals of \( I \). However, note that reparametrization does not change the output since these functions are based on the thin-homotopy-invariant functions of [SWII].

**Convention 3.2.** In general, the overline decoration is shorthand for an \( \alpha \)-action, where \( \overline{X} = \alpha_{\overline{\text{X}}} \) (X) for the appropriate element, \( g \in G \), which we now explain. The rules for gluing horizontally and vertically (diagrams (2.9) and (2.10)) determine the element, \( g \), once the order of the squares is chosen. When the order is not explicitly stated, the convention of first gluing vertically and then horizontally is...
assumed. In turn, the action on $X$ is given by the following 1-holonomy: start at the upper left corner of the grid, then move down the far left edge of the grid until you reach the bottom, then move right until you are at the left side of the column of $X$, then move up until you end at the upper left corner of the $X$-square as seen below:

\[
\begin{array}{|c|c|c|}
\hline
& & \\
\hline
& & \\
\hline
\end{array}
\]

For example, in (2.9), we would write $\text{Hol}_2 := \alpha_{d+1}c^{-1}b(\text{Hol}_2)$, while in (2.10) we would write $\text{Hol}_2 := \alpha_{a-1}(\text{Hol}_2)$.

3.3. Properties of $\text{Hol}_N$ for Squares. This section will provide a summary of the properties one would expect $\text{Hol}_N$ to have. Note that the propositions here are mostly modifications to the works of Schreiber and Waldorf along with the works of Martins and Picken. In places where the proof might be original to the context of this paper, some additional comments are supplied.

Proposition 3.3. $\text{Hol}_N$ is invariant under thin homotopy.

Proof. See [SWII, Lemma 2.16].

Proposition 3.4. The target of $\text{Hol}_N$ is equal to the one-holonomy along the boundary of the grid. Explicitly, assuming the grid (3.1), is

\[
t(\text{Hol}_N) = \text{hol}_{a-1} \cdot g_{ab} \cdot \text{hol}_{b-1} \cdot \text{hol}_{c-1} \cdot g_{cd} \cdot \text{hol}_{d-1} \cdot g_{de} \cdot \text{hol}_{e-1} \cdot g_{ae} \cdot \text{hol}_{e-1} \cdot g_{ae} \cdot \text{hol}_{a}.
\]

Furthermore, $\text{Hol}_N$ respects composition of squares.

Proposition 3.5. $\text{Hol}_N$ is invariant under subdivision.

Proof. Subdivision induces a new grid, on which the added transition data will be shown to be the identity. Consider the following vignette of what subdivision might produce near an $ijkl$ vertex:

\[
\begin{array}{|c|c|c|}
\hline
\text{Hol}_i & \text{Hol}_{i-1} & \text{Hol}_k \\
\hline
\text{Hol}_j & \text{Hol}_{j-1} & \text{Hol}_l \\
\hline
\text{Hol}_{ij} & \text{Hol}_{ij-1} & \text{Hol}_{ik} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
\text{Hol}_i & \text{Hol}_{i-1} & \text{Hol}_k \\
\hline
\text{Hol}_j & \text{Hol}_{j-1} & \text{Hol}_l \\
\hline
\text{Hol}_{ij} & \text{Hol}_{ij-1} & \text{Hol}_{ik} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
\text{Hol}_i & \text{Hol}_{i-1} & \text{Hol}_k \\
\hline
\text{Hol}_j & \text{Hol}_{j-1} & \text{Hol}_l \\
\hline
\text{Hol}_{ij} & \text{Hol}_{ij-1} & \text{Hol}_{ik} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
\text{Hol}_i & \text{Hol}_{i-1} & \text{Hol}_k \\
\hline
\text{Hol}_j & \text{Hol}_{j-1} & \text{Hol}_l \\
\hline
\text{Hol}_{ij} & \text{Hol}_{ij-1} & \text{Hol}_{ik} \\
\hline
\end{array}
\]
where equality comes from $\text{Hol}_{ii}$, $\text{Hol}_{iii}$, and $\text{Hol}_{ijij}$ all being the identity in $H$. □

**Proposition 3.6.** $\text{Hol}^N$ transforms across open sets $\mathcal{N}_I$ and $\mathcal{N}_{I'}$ by $\alpha_{g_{ii'}}$ at the base point and by $\text{Hol}_{jj'}$ and $\text{Hol}_{jj'k'k''}$ along the boundary. In particular:

$$\text{Hol}^{\mathcal{N}_{I'}} = \alpha_{g_{ii'}}(\text{Hol}^{\mathcal{N}_I}) \cdot \prod_{\partial \Sigma} \text{Hol}_{jj'} \cdot \prod_{\partial \Sigma} \text{Hol}_{jkj'k''}$$

**Proof.** Let $\Sigma \in \mathcal{N}_I \cap \mathcal{N}_{I'}$. By Proposition 3.5 there exists a subdivision using open sets $\mathcal{N}_I$ and $\mathcal{N}_{I'}$ which both use a grid of size $n$ ($t$-direction) by $m$ ($s$-direction) such that $\text{Hol}^{\mathcal{N}_I_0} = \text{Hol}^{\mathcal{N}_I}$ and $\text{Hol}^{\mathcal{N}_{I'}_0} = \text{Hol}^{\mathcal{N}_{I'}}$.

Consider the grid of $\text{Hol}^{\mathcal{N}_{I'}}$ analogous to that from Definition 3.1 but replace each face $\text{Hol}_{ij}'$ with a cube (Proposition 2.17), replace each horizontal edge $i'j'$ with a cube (Proposition 2.18), and similarly each vertical edge $i'k'$ with a cube. Although it has not been laid out in a previous proposition, one can divide an $ijkli'j'k'l'$ cube into $i_1i_2i_3i_4$-tetrahedra to build a cube:

$$\text{(3.5)}$$

Replacing each function in the grid of $\text{Hol}^{\mathcal{N}_{I'}}$ with their associated cube from above, all of the interior transition data cancels, leaving only transition data at the boundary. Note that the $\alpha_{g_{ii'}}$ from the statement comes from changing the basepoint from $U_i$ to $U_{i'}$ in the upper left corner of the square. This proves the statement of the proposition. □

### 4. $d(\text{Hol})$ for Squares

The focus of this section is the proof of Theorem 4.1, the main result of this paper, which says that the deRham differential applied to global 2-holonomy amounts to replacing one $B_i$ in any summand with one $H_i$; in addition to some terms associated to the boundary of $\Sigma$.

#### 4.1. A warmup example for the Main Theorem.

Before stating the main theorem of this paper formally in the next section, a warmup to the notation and the main idea is provided here. Consider a good open cover, $\mathcal{U} := \{U_1, U_2, U_3, U_4\}$,
whose Čech nerve contains the cells given by the diagram illustrated below:

\[
\begin{array}{c}
U_1 \quad U_{13} \quad U_3 \\
\downarrow U_{12} \quad U_{34} \\
U_2 \quad U_{24} \quad U_4 \\
\end{array}
\]

(4.1)

Given a non-abelian gerbe with connection (Definition 2.13), \( \{g_{ij}\}, \{f_{ijk}\}, \{A_i\}, \{a_{ij}\}, \{B_i\} \), and a one-parameter family of squares, \( \Sigma_r \in \mathcal{N} \) (Definition 2.2), where for our open sets we have \( \mathcal{N} = \mathcal{N}_I \), so that \( I = \{1, 2\} \times \{1, 2\} \) with \( U_{(1,1)} := U_1, U_{(1,2)} := U_2, U_{(2,1)} := U_3, \) and \( U_{(2,2)} := U_4 \).

By definition 3.1, we have for each \( r \in \mathbb{R} \), an element, \( \text{Hol}_N(\Sigma_r) \in H \), using the appropriately modified version of that definition’s diagram below:

\[
\begin{array}{c}
\text{Hol}_1 \quad \text{Hol}_{13} \quad \text{Hol}_3 \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{Hol}_{12} \quad \text{Hol}_{1234} \quad \text{Hol}_{34} \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{Hol}_2 \quad \text{Hol}_{24} \quad \text{Hol}_4 \\
\end{array}
\]

(4.2)

The goal of Theorem 4.1 is to offer a formula for \( \frac{d}{dr} (\text{Hol}_N(\Sigma_r)) \mid_{r=0} \). Recall that the left action for our crossed module of Lie Groups, \( \alpha : G \times H \rightarrow H \) induces a map \( \alpha_h : G \rightarrow H \), for each choice of \( h \in H \). In turn, we obtain from the pushforward a map \( (\alpha_h)_* : T_gG \rightarrow T_{\alpha_g(h)}H \) and thus a map \( (\alpha_h)_* : g \rightarrow T_hH \). Similarly, \( \alpha \) induces for each fixed \( g \in G \), a map \( \alpha_g : H \rightarrow H \) whose pushforward gives a map \( (\alpha_g)_* : h \rightarrow h \). For a fixed \( h \in H \) we also have the pushforward of, for example, multipliciton on the left, \( (L_h)_* : T_{h}H \rightarrow T_{h,h}H \) and thus \( (L_h)_* : h \rightarrow T_hH \).

Finally, we are ready to examine the content of Theorem 4.1 in our example with four open sets. As stated below, the abbreviated version of the formula for \( d(\text{Hol}) \) is

\[
d(\text{Hol}) = -\langle \alpha_{\text{Hol}} \rangle_*(A_{i(1,1)}) + \text{Hol} \cdot \int_{S^q} H + \text{Hol} \cdot \left( \int_{\partial S^q} B + \sum a \right)
\]

The term, \(-\langle \alpha_{\text{Hol}} \rangle_*(A_{i(1,1)})\) is the pushforward \(-\langle \alpha_{\text{Hol}} \rangle_* \) applied to \( A_1 \left( \frac{d}{dr} \Sigma_r(0,0) \right)_{r=0} \). The term, \( \int_{S^q} H \) is a sum of four terms, one for each 3-form,
$H_i \in \Omega^3(U_i, \mathfrak{h})$:

$$\int_{S^2 \Omega} H = \int_{S^2 \Omega(1,1)} (\alpha_{\text{path}(1,1)})_* H_1 + \int_{S^2 \Omega(1,2)} (\alpha_{\text{path}(1,2)})_* H_2 + \int_{S^2 \Omega(2,2)} (\alpha_{\text{path}(2,2)})_* H_3 + \int_{S^2 \Omega(2,2)} (\alpha_{\text{path}(2,2)})_* H_4.$$  

To be precise, and using the north, south, east, east notation of Definition 2.2, we can use the $(2,2)$-square as an example to note that

$$\int_{S^2 \Omega(2,2)} (\alpha_{\text{path}(2,2)})_* H_4 = \left( \alpha_{\text{hol}^{-1}(\lambda^1)} \cdot g_{12}^{-1}(\lambda^2) \cdot \alpha_{\text{hol}^{-1}(\lambda^3)} \cdot g_{24}^{-1}(\lambda^4) \right)_* \left( \int_{S^2 \Omega} (\alpha_{\text{path}(1,1)})_* (H_4) \right),$$

where $\text{path}(-,-) : [0,1]^2 \to G$ is locally defined so that $\text{path}(s,t)$ is the 1-holonomy vertically “down” to level $s$ and then horizontally “right” over to time, $t$. Note that one can show the choice of path (as long as it is smooth) does not matter since $t(H_4) = 0$. Since $\int_{S^2 \Omega} (\alpha_{\text{path}})_* H_4$ is valued in $\mathfrak{h}$, then the pushforward of left-multiplication by $\text{Hol}$ means that the term, $\text{Hol} \cdot \int_{S^2 \Omega} H$, takes values in $T_{\text{Hol}} H$.

Now that the notation has been explored a bit further, the meaning of the boundary terms becomes clearer, of which there are twelve:

(4.3)

4.2. The Main Theorem. For the remainder of this section, a grid, $\mathcal{N}$ is once again fixed.
Theorem 4.1. For the function $Hol = Hol^N : N \to H \subset \text{Mat}$ defined on the open set $N \subset M^{2q}$, we can compute the derivative $d(Hol) \in \Omega^1(M^{2q}, \text{Mat})$ as follows:

\begin{equation}
(4.4) \quad d(Hol) = - (\alpha_{Hol})_* (A_i(1,1)) + Hol \cdot \int_{Sq} H + Hol \cdot \left( \int_{\partial Sq} B + \sum_{\partial Sq} a \right)
\end{equation}

where $i_{(1,1)}$ is the index of the upper left open set $U_{i_{(1,1)}}$, and

\begin{equation}
(4.5) \quad \int_{Sq} H := \sum_{k=1,\ldots,n} \int_{Sq(k,i)} (\alpha_{\text{path}_k})_* (H_{i_{(1,1)}}).
\end{equation}

While the terms in the parenthesis on the right in (4.4) are boundary terms:

\begin{equation}
(4.6) \quad \int_{\partial Sq} B := - \sum_{k=1,\ldots,n} \int_{\partial Sq(k,i)} (\alpha_{\text{path}_k})_* (B_{i_{(1,1)}}) - \sum_{l=1,\ldots,m} \int_{\partial Sq} (\alpha_{\text{path}_l})_* (B_{i_{(1,1)}})
\end{equation}

\begin{equation}
\sum_{\partial Sq} a := - \sum_{k=1,\ldots,n-1} (\alpha_{\text{path}_{(k+1,k+1)}})_* (a_{i_{(1,1)}})_{(k+1,k+1)}
\end{equation}

\begin{equation}
- \sum_{l=1,\ldots,m-1} (\alpha_{\text{path}_{(n,l+1)}})_* (a_{i_{(1,1)}})_{(n,l+1)}
\end{equation}

\begin{equation}
+ \sum_{k=1,\ldots,n-2} (\alpha_{\text{path}_{(k+1,m)}})_* (a_{i_{(1,1)}})_{(k+1,m)}
\end{equation}

\begin{equation}
+ \sum_{l=m,\ldots,2} (\alpha_{\text{path}_{(1,l-1)}})_* (a_{i_{(1,1)}})_{(1,l-1)}
\end{equation}

where the expressions $\alpha_{\text{path}_k}(X)$ use an element $\text{path}_k \in G$ which is given by the appropriate path through the grid.

First, a lemma which can be found, for example, in Theorem 2.30 of [MP2], and which is the local version of this paper’s Theorem 4.1 is recalled:

Lemma 4.2. For the local 2-holonomy function $Hol_i : U_{i}^{Sq} \to H$ as defined in Section 2.4.2 we have

\begin{equation}
(4.7) \quad d(Hol_i) = - (\alpha_{Hol_i})_* (A_i(0,0)) + Hol_i \cdot \int_{Sq} H + Hol_i \cdot \int_{\partial Sq} a
\end{equation}

4.3. The Edge and Vertex Relations. Here two technical lemmas are proved, which give relations amongst the edges and vertices for when $\frac{\partial}{\partial x}(Hol)$ is later computed. Recall the expression for 2-holonomy, Definition 3.1:

\begin{equation}
Hol^N = Hol_{ab} \cdot Hol_{ac} \cdot Hol_{ad} \cdot \ldots \cdot Hol_{m} \cdot Hol_{ab}^{-1} \cdot Hol_{ac}^{-1} \cdot \ldots \cdot Hol_{mn}^{-1}
\end{equation}

2Where Mat is some matrix algebra.
Applying the Leibniz rule, the derivative of global holonomy will start with the expression,

\[
\frac{\partial}{\partial r} \bigg|_{r=0} \text{Hol} := \frac{\partial}{\partial r} \bigg|_{r=0} \left( \cdots \cdot \text{Hol}_{ij} \cdot \text{Hol}_{ij}^{-1} \cdot \cdots \cdot \text{Hol}_{ij} \right) \\
= \cdots + \cdots \frac{\partial}{\partial r} \bigg|_{r=0} \text{Hol}_{ij} \cdot \text{Hol}_{ij}^{-1} \cdot \cdots \cdot \text{Hol}_{ij} \cdot \cdots \\
\vdots \\
= \cdots \cdot \text{Hol}_{ij} \cdot \text{Hol}_{ij}^{-1} \cdot \cdots \cdot \frac{\partial}{\partial r} \bigg|_{r=0} \cdots \\
+ \cdots \\
\]  

To state the first lemma, the following setup is used:

Assume for the moment that the open set \( \mathcal{N} \) only requires four open sets \( U_i, U_j, U_k, \) and \( U_l \), written in counterclockwise order from the upper left of \( \Sigma \). Now, suppose a one-parameter family of squares, \( \Sigma(\rho) \), is given with \( \Sigma(0) = \Sigma \). In particular, then, given the grid associated to \( \mathcal{N} \), there is a path, \( \rho \), through the vertex \( x_{ijkl} \). Momentarily writing \( \rho(0) = x \) and \( \rho(r) = y \), there is a vertex cube associated to \( \rho \), given by Proposition 2.13 which corresponds to the equation:

\[
\text{Hol}_{ijkl}(x) = \text{Hol}_{ij} \cdot \alpha_{g_{ij}^{-1}(x)g_{ij}(y)g_{ij}(g_{ij}(y))} \cdot \alpha_{g_{ij}^{-1}(x)} \cdot \alpha_{g_{ij}^{-1}(x)} \cdot \alpha_{g_{ij}^{-1}(x)g_{ij}(y)g_{ij}(g_{ij}(y))} \cdot \text{Hol}_{ijkl}(y).
\]

Thus \( \text{Hol}_{ijkl}(x) \) in \( \text{Hol} \) can be replaced, to obtain the following equality of local transport data:

\[
(4.9)
\]
The above diagram can be “glued” together in the following way:

\[
Hol^{ijkl} = Hol_i(\Sigma_i(0)) \cdot \cdots \cdot Hol^{-1}_{ik}(\gamma_{ik}) \cdot \left(\text{Hol}_{ijkl}(x)\right) \cdot Hol^{-1}_{jl}(\gamma_{jl}) \cdot \cdots \cdot Hol_l(\Sigma_l(0))
\]

\[
= Hol_i(\Sigma_i(0)) \cdot \cdots \cdot Hol^{-1}_{ik}(\gamma_{ik}) \cdot \\
\left(\text{Hol}_{ij}(\rho)Hol^{-1}_{ik}(\rho)\text{Hol}_{ijkl}(y)\text{Hol}_{jl}(\rho)\text{Hol}^{-1}_{kl}(\rho)\right) \cdot Hol^{-1}_{jl}(\gamma_{jl}) \cdot \cdots \cdot Hol_l(\Sigma_l(0))
\]

(4.11)

where the notation \(\hat{X}\) is a visual aid for the reader to note that the path-action of \(X\), which uses the \(r\) direction; i.e. \(\hat{X}\) suggests that the path had to move “up or down” in the \(r\)-direction due to a \(Hol(\rho)\) term being placed out of order in the gluing process. Note that throughout a computation, the \(\bullet\) and \(\hat{\bullet}\) notation is implicit and might change when terms are rearranged following the crossed module relations.

The rewrite in (4.10) demonstrates how the global holonomy is unchanged when the vertex term is replaced with the remaining 5 faces of the vertex cube from Proposition 2.18. Now, one can rearrange the terms in (4.11), using only the crossed module relation (2.2) as follows:

\[
Hol^{ijkl} = Hol_i(\Sigma_i(0)) \cdot Hol_{ij}(\gamma_{ij}) \cdot Hol_{ij}(\rho)Hol_j(\Sigma_j(0)) \cdot Hol^{-1}_{ik}(\gamma_{ik})
\]

\[
\cdot Hol^{-1}_{ik}(\rho)\text{Hol}_{ijkl}(y)\text{Hol}_{jl}(\rho)\text{Hol}^{-1}_{kl}(\rho) \cdot Hol^{-1}_{jl}(\gamma_{jl})
\]

\[
\cdot Hol_k(\Sigma_k(0)) \cdot Hol_{kl}(\gamma_{kl}) \cdot Hol_l(\Sigma_l(0)).
\]

(4.12)

Here, some faces of the vertex cube were moved to a more convenient location, according to the following order:

3 Convenient in the sense that comparing these non-commutative terms will be easier later on.
Differentiating $\text{Hol}(\bullet)$ yields the following lemma:

**Lemma 4.3** (Vertex Cube Equation, (VCE)). For a one-parameter family of squares, $\Sigma_r$, at each vertex $x_{ijkl}$, there is a vertex cube equation, denoted by (VCE), where certain terms in the arbitrarily-long product are isolated.

\[
\begin{align*}
\text{(4.13)} & \quad \ldots \cdot \text{Hol}^{-1}_{ik} \cdot d(\text{Hol}_{ijkl}) \left( \frac{\partial}{\partial r} \right) \cdot \text{Hol}^{-1}_{jl} \cdot \ldots \\approx \\
\text{(4.14)} & \quad = - \ldots \cdot \text{Hol}_{ij} \cdot \alpha_{g_{ij}(x_{ijkl})^{-1}} \left( a_{ij} \mid x_{ijkl} \left( \frac{\partial}{\partial r} \right) \right) \cdot \text{Hol}_{ij} \cdot \ldots \\approx \\
\text{(4.15)} & \quad - \sum_{\bullet_2} \ldots \cdot \left( \alpha_{\text{hol}_j} (-t(a_{ij})(\frac{\partial}{\partial r}))_{g_{ij}} \right) (\text{Hol}_{\bullet_2}) \cdot \ldots \\approx \\
\text{(4.16)} & \quad - \ldots \cdot \text{Hol}_{ij} \cdot \ldots \cdot \left( \alpha_{\text{hol}_j} (-t(a_{ij})(\frac{\partial}{\partial r}))_{g_{ij}} \right) (\text{Hol}_{ik}^{-1}) \cdot \text{Hol}_{ijkl} \cdot \ldots \\approx \\
\text{(4.17)} & \quad + \ldots \cdot \text{Hol}_{ik}^{-1} \cdot \alpha_{g_{ik}^{-1}(x_{ijkl})} \left( a_{ik} \mid x_{ijkl} \left( \frac{\partial}{\partial r} \right) \right) \cdot \text{Hol}_{ijkl} \cdot \ldots \\approx \\
\text{(4.18)} & \quad - \ldots \cdot \text{Hol}_{ik}^{-1} \cdot \alpha_{\text{hol}_j} (A_j(\frac{\partial}{\partial r}) + d_{g_{ij}}(\frac{\partial}{\partial r}))_{g_{ij}} \cdot g_{ij} (\text{Hol}_{ijkl}) \cdot \text{Hol}_{ij}^{-1} \cdot \ldots \\approx \\
\text{(4.19)} & \quad - \ldots \cdot \text{Hol}_{ijkl} \cdot \alpha_{g_{ik}^{-1}(x_{ijkl})} \left( a_{ik} \mid x_{ijkl} \left( \frac{\partial}{\partial r} \right) \right) \cdot \text{Hol}_{ijkl} \cdot \ldots \\approx \\
\text{(4.20)} & \quad - \sum_{\bullet_3} \ldots \cdot \left( \alpha_{\text{hol}_i} (-t(a_{ik})(\frac{\partial}{\partial r}))_{g_{ik} \text{hol}_k \ldots} \right) (\text{Hol}_{\bullet_3}) \cdot \ldots
\end{align*}
\]
\[ (4.21) \quad - \ldots \cdot \overline{Hol}^{-1}_{ji} \cdot \left( \alpha_{hol} \left( -t(a_{ij}) \left( \frac{\partial}{\partial r} \right) \right) g_{ik} \alpha_{hol} \right) \cdot \overline{Hol}_{kl} \cdot \ldots \]

\[ (4.22) \quad + \ldots \cdot \overline{Hol}_{k} \cdot \alpha_{g_{ik}}^{-1} \left( a_{kl} \bigg|_{x_{ijkl}} \left( \frac{\partial}{\partial r} \right) \right) \cdot \overline{Hol}_{kl} \cdot \ldots \]

where, from Definition 2.13:

\[ -t(a_{ij}) = A_{j} + d g_{ij} g_{ij}^{-1} - g_{ij} A_{i} g_{ij}^{-1} \]

and where Hol is any face in the grid appearing above Hol, and Hol is any face in the grid appearing above Hol.

Next, vertical and horizontal edges are considered using the edge cubes from Proposition 2.17. The Lemma below yielding the horizontal and vertical edge cube equations follows from a proof analogous to that of Lemma 4.3 where the following expressions and corresponding diagrams for horizontal edges are used:

For a horizontal edge, \((ij)\), the one parameter family of squares used in this section, \(\Sigma(r)\), is restricted to its associated one-parameter family of horizontal edges, \(\gamma^h_{ij}(t, r)\). The edge at height \(r\) will be written \(\gamma^h_{ij}(-, r)\). This family of edges also creates a face, \(\gamma_{ij}(-, -)\), and two vertical edges, \(\gamma_{ij}(0, -)\) and \(\gamma_{ij}(1, -)\). The horizontal portion of Proposition 2.17 yields

\[ (4.23) \quad Hol = Hol^{(ij)}_{h-edge} \]

\[ (4.24) \quad := \ldots \cdot \overline{Hol}^{-1}_{pi} \left( \gamma^v_{pi} \right) \cdot \overline{Hol}_{pqij}(x_{pqij}) \cdot \overline{Hol}^{-1}_{qj} \left( \gamma^v_{qj} \right) \cdot \ldots \]

\[ (4.25) \quad \ldots \cdot \overline{Hol}_{i}(\Sigma) \cdot \overline{Hol}_{ij}(\gamma^h_{ij}(0, -)) \cdot \overline{Hol}_{ij} \left( \gamma^h_{ij}(-, -) \right) \cdot \overline{Hol}_{ij} \left( \gamma^h_{ij}(-, r) \right) \]

\[ (4.26) \quad \cdot \overline{Hol}^{-1}_{ij} \left( \gamma^h_{ij}(-, -) \right) \cdot \overline{Hol}^{-1}_{ij} \left( \gamma^h_{ij}(1, -) \right) \cdot \overline{Hol}_{ij}(\Sigma) \cdot \ldots \]

\[ (4.27) \quad \ldots \cdot \overline{Hol}^{-1}_{ik} \left( \gamma^v_{ik} \right) \cdot \overline{Hol}_{ijkl}(x_{ijkl}) \cdot \overline{Hol}^{-1}_{jl} \left( \gamma^v_{jl} \right) \cdot \ldots \]
Similarly, there is a one parameter family of vertical edges, $\gamma_{ij}^v (-, -)$, and thus the following expressions and diagram for vertical edges:

\begin{align}
\hat{\text{Hol}}_p^{ij} &= \ldots \cdot \hat{\text{Hol}}_{pqij} (x_{pqij}) \cdot \hat{\text{Hol}}_{ij}^{-1} (\gamma_{ij}^v (1, -)) \cdot \hat{\text{Hol}}_{ij}^{-1} (\gamma_{ij}^v (-, r)) \cdot \hat{\text{Hol}}_{ij} (\gamma_{ij}^v (0, -)) \cdot \hat{\text{Hol}}_{ij} (\gamma_{ij}^v (-, -)) \cdot \hat{\text{Hol}}_{ijkl} (x_{ijkl}) \cdot \ldots \\
\hat{\text{Hol}}_{ij} &= \ldots \cdot \hat{\text{Hol}}_{ijkl} (x_{ijkl}) \cdot \hat{\text{Hol}}_{ij} (\gamma_{ij}^v (-, r)) \cdot \hat{\text{Hol}}_{ij} (\gamma_{ij}^v (-, -)) \cdot \hat{\text{Hol}}_{ij} (\gamma_{ij}^v (-, -)) \cdot \hat{\text{Hol}}_{ijkl} (x_{ijkl}) \cdot \ldots \\
\hat{\text{Hol}}_{ij}^{-1} &= \ldots \cdot \hat{\text{Hol}}_{ijkl} (x_{ijkl}) \cdot \hat{\text{Hol}}_{ij} (\gamma_{ij}^v (-, -)) \cdot \hat{\text{Hol}}_{ij} (\gamma_{ij}^v (-, -)) \cdot \hat{\text{Hol}}_{ij} (\gamma_{ij}^v (-, -)) \cdot \hat{\text{Hol}}_{ijkl} (x_{ijkl}) \cdot \ldots
\end{align}
Differentiating these equations gives the following lemma:

**Lemma 4.4** (Edge Cube Equations, (ECEh) and (ECEv)). For a one-parameter family of squares, $\Sigma_r$, the following horizontal edge cube equation at $\gamma_{ij}^h$, denoted by (ECEh), holds where certain terms in the arbitrarily-long product are isolated

$$
\ldots \cdot \text{Hol}_i \cdot d(\text{Hol}_{ij}) \left( \frac{\partial}{\partial r} \right) \cdot \text{Hol}_j \cdot \ldots
$$

$$
(4.31) = - \ldots \cdot \text{Hol}_i \cdot \alpha_{g_{ij}^h(\gamma_{ij}^h(0))^{-1}} \left( a_{ij} \Big|_{\gamma_{ij}^h(0)} \right) \left( \frac{\partial}{\partial r} \right) \cdot \text{Hol}_{ij} \cdot \ldots
$$

$$
(4.32) - \ldots \cdot \text{Hol}_i \cdot \int_{\gamma_{ij}^h} \alpha_{\ast}(B_j) \left( \frac{\partial}{\partial r} \right) \cdot \text{Hol}_{ij} \cdot \ldots
$$

$$
- \ldots \cdot \text{Hol}_i \cdot (\alpha \left( A_{ij} \Big|_{\gamma_{ij}^h(0)} \left( \frac{\partial}{\partial r} \right) + dg_{ij} \Big|_{\gamma_{ij}^h(0)} \left( \frac{\partial}{\partial r} \right) g_{ij} \right) g_{ij}) (\text{Hol}_{ij}) \cdot \text{Hol}_j \cdot \ldots
$$

$$
+ \ldots \cdot \text{Hol}_{ij} \cdot \int_{\gamma_{ij}^h} \alpha_{\ast}(B_j) \left( \frac{\partial}{\partial r} \right) \cdot \text{Hol}_j \cdot \ldots
$$

$$
+ \ldots \cdot \text{Hol}_{ij} \cdot \alpha_{g_{ij}^h(\gamma_{ij}^h(1))^{-1}} \left( a_{ij} \Big|_{\gamma_{ij}^h(1)} \right) \left( \frac{\partial}{\partial r} \right) \cdot \text{Hol}_{ij} \cdot \ldots
$$

Similarly, the following vertical edge cube equation at $\gamma_{ij}^v$, denoted by (ECEv), holds:

$$
(4.33) \ldots \cdot \text{Hol}_{pqij} \cdot d(\text{Hol}_{ij}^{-1}) \left( \frac{\partial}{\partial r} \right) \cdot \text{Hol}_{ikjl} \cdot \ldots
$$
\begin{align}
(4.34) & = + \ldots \Hol_{pi} \cdot \int_{\gamma_{ij}} \alpha_\gamma(B_i) \left( \frac{\partial}{\partial r} \right) \cdot \Hol_{ij} \ldots \\
& - \sum_{a} \ldots \left( \alpha \left( \text{dhol}_{\gamma_{ij}}(\frac{\partial}{\partial r}) \right)_{\text{hol}_{i} g_{pi}} \ldots \right) \left( \Hol_{\bullet_2} \right) \ldots \\\n& - \ldots \Hol_{ik} \ldots \left( \alpha \left( \text{dhol}_{\gamma_{ij}}(\frac{\partial}{\partial r}) \right)_{\text{hol}_{i} g_{pi}} \ldots \right) \left( \Hol_{\bullet_2} \right) \ldots \\
\end{align}

\begin{align}
(4.35) & = + \ldots \Hol_{piqj} \cdot \alpha_{g_{ij}(\gamma_{ij}(0))}^{-1} \left( a_{ij} \mid \gamma_{ij}(0) \left( \frac{\partial}{\partial r} \right) \right) \cdot \Hol_{ij} \ldots \\
(4.36) & = - \ldots \Hol_{piqj} \left( \alpha \left( A \mid \gamma_{ij}(1) \left( \frac{\partial}{\partial r} \right) + \text{dhol}_{\gamma_{ij}}(\frac{\partial}{\partial r}) \right)_{\text{hol}_{i} g_{pi}} \ldots \right) \left( \Hol_{\bullet_1} \right) \ldots \\
(4.37) & = - \ldots \Hol_{ij} \ldots \left( \alpha \left( A \mid \gamma_{ij}(1) \left( \frac{\partial}{\partial r} \right) + \text{dhol}_{\gamma_{ij}}(\frac{\partial}{\partial r}) \right)_{\text{hol}_{i} g_{pi}} \ldots \right) \left( \Hol_{\bullet_1} \right) \ldots \\
(4.38) & = - \ldots \Hol_{ij} \ldots \left( \alpha \left( A \mid \gamma_{ij}(1) \left( \frac{\partial}{\partial r} \right) + \text{dhol}_{\gamma_{ij}}(\frac{\partial}{\partial r}) \right)_{\text{hol}_{i} g_{pi}} \ldots \right) \left( \Hol_{\bullet_1} \right) \ldots \\
(4.39) & = - \ldots \Hol_{ij} \ldots \left( \alpha \left( A \mid \gamma_{ij}(1) \left( \frac{\partial}{\partial r} \right) + \text{dhol}_{\gamma_{ij}}(\frac{\partial}{\partial r}) \right)_{\text{hol}_{i} g_{pi}} \ldots \right) \left( \Hol_{\bullet_1} \right) \ldots \\
(4.40) & = - \ldots \Hol_{ij} \ldots \left( \alpha \left( A \mid \gamma_{ij}(1) \left( \frac{\partial}{\partial r} \right) + \text{dhol}_{\gamma_{ij}}(\frac{\partial}{\partial r}) \right)_{\text{hol}_{i} g_{pi}} \ldots \right) \left( \Hol_{\bullet_1} \right) \ldots \\
(4.41) & = - \ldots \Hol_{ij} \ldots \left( \alpha \left( A \mid \gamma_{ij}(1) \left( \frac{\partial}{\partial r} \right) + \text{dhol}_{\gamma_{ij}}(\frac{\partial}{\partial r}) \right)_{\text{hol}_{i} g_{pi}} \ldots \right) \left( \Hol_{\bullet_1} \right) \ldots \\
(4.42) & = - \ldots \Hol_{ij} \ldots \left( \alpha \left( A \mid \gamma_{ij}(1) \left( \frac{\partial}{\partial r} \right) + \text{dhol}_{\gamma_{ij}}(\frac{\partial}{\partial r}) \right)_{\text{hol}_{i} g_{pi}} \ldots \right) \left( \Hol_{\bullet_1} \right) \ldots \\
(4.43) & = - \ldots \Hol_{ij} \ldots \left( \alpha \left( A \mid \gamma_{ij}(1) \left( \frac{\partial}{\partial r} \right) + \text{dhol}_{\gamma_{ij}}(\frac{\partial}{\partial r}) \right)_{\text{hol}_{i} g_{pi}} \ldots \right) \left( \Hol_{\bullet_1} \right) \ldots \\
\end{align}

where the abbreviation \text{dhol} is given by

\begin{align}
\text{dhol}_{\gamma} \left( \frac{\partial}{\partial r} \right) = A \mid \gamma(1) \left( \frac{\partial}{\partial r} \right) + \text{dhol}_{\gamma} \left( \frac{\partial}{\partial r} \right) \hol^{-1} - \hol A \mid \gamma(0) \left( \frac{\partial}{\partial r} \right) \hol^{-1}
\end{align}

and where \Hol_{\bullet_2} is any square in the grid appearing above \Hol_{piqj}, and \Hol_{\bullet_3} is any square in the grid appearing above \Hol_{aij}.

4.4. **Proof of Theorem 4.1.** In this section, it is shown that \text{d(Hol)} does not have differential information at the interior edges or vertices. At the same time, the boundary terms are gathered into a convenient expression. It is important to note that \text{d(Hol)}, applied to \text{dhol}_{\gamma} \mid_{r=0}, can be written, using the Liebniz rule as in (4.8). This expression has the following types of terms

1. **3-curvature terms:** namely the terms in \text{d(Hol)} from Lemma 4.2 where one \text{B}_{i} is replaced with an \text{H}_{i}
2. **Boundary terms:** these are terms which occur on the boundary of \Sigma.
3. **Edge and Vertex terms:**
   - (a) The remaining 2 types of terms coming from \text{d(Hol)}: four edge terms, written as one integral around the four sides in Lemma 4.2 and a corner term where \text{A}_{i} is applied to the upper left corner of the square.
   - (b) \text{d(Hol}_{ij}) and \text{d(Hol}_{ijk}): These are the de Rham differentials applied to any (vertical or horizontal) edges or vertices, see Lemmas 4.3 and 4.4.
(c) **Path terms**: These are the terms which appear from differentiating the \( \alpha_{\text{path}} \) of any \( \mathcal{H}o\text{t}_{\bullet} = \alpha_{\text{path}}(\mathcal{H}o\text{t}_{\bullet}) \) term in \( d(\mathcal{H}o) \) (see Lemma 2.12).

4.4.1. **Summary of Cancelation.** For the moment, all of the terms which end up canceling are collected and compared. Begin by using Lemmas 4.3 and 4.4 to replace any \( \frac{\partial}{\partial r} \bigg|_{r=0} \mathcal{H}o\text{t}_{ij} \) or \( \frac{\partial}{\partial r} \bigg|_{r=0} \mathcal{H}o\text{t}_{ijkl} \) with the corresponding expression and then showing that the \( \mathcal{H}o\text{t}_{ij} \) (edge) terms cancel with each other. In other words, for each term \( \mathcal{H}o\text{t}_{\bullet} \) or \( \mathcal{H}o\text{t}_{\bullet} \), applying \( \frac{\partial}{\partial r} \) to such terms and using Lemma 2.12, yields terms of the form \( \frac{\partial}{\partial r} \bigg|_{r=0} \mathcal{H}o\text{t}_{\bullet} \).

The following table gives a summary of all instances where two such terms will appear in \( d(\mathcal{H}o) \) with opposite sign. Note that for the reader’s convenience the **Local Lemma** 4.2 is labelled as “(LL)”, the **Vertex Cube Equation** from Lemma 4.3 as “(VCE)”, and the vertical and horizontal edge cube equations from Lemma 4.4 as “(ECEv)” and “(ECEh)”, respectively.

| Label | Term | Found in |
|-------|------|---------|
|       | \( \ldots \mathcal{H}o\text{t}_{p} \cdot \mathcal{H}o\text{t}_{pi} \cdot \int_{\gamma_{pi}} \alpha_{i} \cdot \mathcal{H}o\text{t}_{i} \ldots \) | (LL) (ECEh) |
| (A1)  | \( \ldots \mathcal{H}o\text{t}_{p} \cdot \int_{\gamma_{pi}} \alpha_{i} \cdot \mathcal{H}o\text{t}_{i} \ldots \) | (LL) (ECEh) |
| (A2)  | \( \ldots \mathcal{H}o\text{t}_{p} \cdot \int_{\gamma_{pi}} \alpha_{i} \cdot \mathcal{H}o\text{t}_{i} \ldots \) | (LL) (ECEh) |
| (A3)  | \( \ldots \mathcal{H}o\text{t}_{p} \cdot \int_{\gamma_{pi}} \alpha_{i} \cdot \mathcal{H}o\text{t}_{i} \ldots \) | (LL) (ECEv) |
| (A4)  | \( \ldots \mathcal{H}o\text{t}_{p} \cdot \int_{\gamma_{pi}} \alpha_{i} \cdot \mathcal{H}o\text{t}_{i} \ldots \) | (LL) (ECEv) |
| (B1)  | \( \ldots \mathcal{H}o\text{t}_{p} \cdot \int_{\gamma_{pi}} \alpha_{i} \cdot \mathcal{H}o\text{t}_{i} \ldots \) | (VCE) (ECEv) |
| (B2)  | \( \ldots \mathcal{H}o\text{t}_{p} \cdot \int_{\gamma_{pi}} \alpha_{i} \cdot \mathcal{H}o\text{t}_{i} \ldots \) | (VCE) (ECEv) |
| (B3)  | \( \ldots \mathcal{H}o\text{t}_{p} \cdot \int_{\gamma_{pi}} \alpha_{i} \cdot \mathcal{H}o\text{t}_{i} \ldots \) | (VCE) (ECEv) |
| (B4)  | \( \ldots \mathcal{H}o\text{t}_{p} \cdot \int_{\gamma_{pi}} \alpha_{i} \cdot \mathcal{H}o\text{t}_{i} \ldots \) | (VCE) (ECEv) |

The other terms which appear from applying \( \frac{\partial}{\partial r} \bigg|_{r=0} \) to each term \( \mathcal{H}o\text{t}_{\bullet} \) or \( \mathcal{H}o\text{t}_{\bullet} \) are the **path terms** from Lemma 2.12

\[
\alpha(\frac{\partial}{\partial r} \bigg|_{r=0} \ldots)(\mathcal{H}o\text{t}_{\bullet}),
\]

and are labelled “path-\( d(\mathcal{H}o) \)”.

The two tables below summarize all of the instances where **path terms** show up an even number of times, with opposite signs. In the first table, the focus is on **path terms** resulting from \( d(\mathcal{H}o) \); i.e. they show up one time in differentiating the path approaching each term in the expression \( 3.2 \) for \( \mathcal{H}o\text{t}^{N} \). The second table deals with **path terms** which cancel amongst the other relations.
Note that in (C3) one can have $wx = ix$ and in (D4) one can have $ij = wx$. In (D2) one can have $ij = yz$ but then the cancellation is due to $(\text{path-d}(\text{Hol}))$ and $(\text{ECEh})$.

Note that in (E3) one can have $wx = iq$, in (E4) one can have $wxyz = wigg$, and in (F1) one can have $z = i$. The only terms that are left, for the interior of $\Sigma$, after all of this cancelation are the 3-curvature terms, $H_i$ integrated over the square; i.e. the term $\int H$ in Theorem 4.1.

4.1. Getting the Boundary Right. Modulo the boundary, it has thus far been shown that $d(\text{Hol}) \equiv \text{Hol} \cdot \int_{\Sigma} H$. In order to get the boundary terms to work out properly, it remains to check that all terms that accumulate at the boundary of $\Sigma$, due to not being able to cancel with a missing adjacent square, either cancel or are of the form $B_i$ or $a_{ij}$ as described in the statement of Theorem 4.1. In particular, consider again the general grid from Definition 3.1. Note that when $d(\text{Hol}_{ij})$ is

| Label | Term | Found in |
|-------|------|----------|
| (C1)  | $\ldots \alpha_{(\ldots, \text{hol}, \ldots)} (\text{Hol}_z) \cdot \ldots \ldots$ | $(\text{path-d}(\text{Hol}))$ (ECEv) |
| (C2)  | $\ldots \alpha_{(\ldots, \text{hol}, \ldots)} (\text{Hol}_{yz}) \cdot \ldots \ldots$ | $(\text{path-d}(\text{Hol}))$ (ECEv) |
| (C3)  | $\ldots \alpha_{(\ldots, \text{hol}, \ldots)} (\text{Hol}_{wx}) \cdot \ldots \ldots$ | $(\text{path-d}(\text{Hol}))$ (ECEv) |
| (C4)  | $\ldots \alpha_{(\ldots, \text{hol}, \ldots)} (\text{Hol}_{wxyz}) \cdot \ldots \ldots$ | $(\text{path-d}(\text{Hol}))$ (ECEv) |
| (D1)  | $\ldots \alpha_{(\ldots, d_{g,y})} (\text{Hol}_z) \cdot \ldots \ldots$ | $(\text{path-d}(\text{Hol}))$ (VCE) |
| (D2)  | $\ldots \alpha_{(\ldots, d_{g,y})} (\text{Hol}_{yz}) \cdot \ldots \ldots$ | $(\text{path-d}(\text{Hol}))$ (VCE) |
| (D3)  | $\ldots \alpha_{(\ldots, d_{g,y})} (\text{Hol}_{wx}) \cdot \ldots \ldots$ | $(\text{path-d}(\text{Hol}))$ (VCE) |
| (D4)  | $\ldots \alpha_{(\ldots, d_{g,y})} (\text{Hol}_{wxyz}) \cdot \ldots \ldots$ | $(\text{path-d}(\text{Hol}))$ (VCE) |

Note that in (E1) one can have $wx = iq$, in (E2) one can have $wxyz = wigg$, in (E3) one can have $wxyz = wigg$, and in (F1) one can have $z = i$. The only terms that are left, for the interior of $\Sigma$, after all of this cancelation are the 3-curvature terms, $H_i$ integrated over the square; i.e. the term $\int H$ in Theorem 4.1.

4.1. Getting the Boundary Right. Modulo the boundary, it has thus far been shown that $d(\text{Hol}) \equiv \text{Hol} \cdot \int_{\Sigma} H$. In order to get the boundary terms to work out properly, it remains to check that all terms that accumulate at the boundary of $\Sigma$, due to not being able to cancel with a missing adjacent square, either cancel or are of the form $B_i$ or $a_{ij}$ as described in the statement of Theorem 4.1. In particular, consider again the general grid from Definition 3.1. Note that when $d(\text{Hol}_{ij})$ is
replaced with its corresponding cube equation at the northern boundary of the square, \( \Sigma \), a \( Hol_{ij} \) is differentiated as it collapses to the northern boundary, placing an \( a_{ij} \) at that spot. In such cases, it will be useful to write any term

\[
(4.44) \quad Hol_a \ldots Hol_{m} \cdot a_{ab} \left( \frac{\partial}{\partial r} \right) \cdot Hol_{ab}^{-1} \ldots Hol_p
\]

as \( \overline{a}_{ab} \cdot Hol \), simply by changing the path-action, \( \overline{\bullet} \), for the \( a_{ij} \) term. The easier terms to deal with will be on the Northern and Eastern boundaries. By using the algebra of the crossed module, the equation (4.44) can be rewritten as desired. However, explaining this algebra is a lot easier by recalling

\[
hh' = \alpha_t(h)(h')h
\]

and observing the equality as a picture:

\[
(4.45)
\]

In a similar fashion, based on the ordering of the Edge Cube equations (Lemmas 4.3 and 4.4) and the Local Lemma 4.2, all of the \( B_i \) and \( a_{ij} \) terms appearing along the Northern and Eastern boundary can be factored outside of \( Hol \).

For the Western and Southern boundaries, there is one extra tool needed. Considering again an \( a_{ij} \) term, let us consider the term

\[
(4.46) \quad Hol_a \cdot Hol_{ae} \cdot Hol_e \cdot Hol_{ei} \cdot a_{ei} \left( \frac{\partial}{\partial r} \right) \cdot Hol_i \ldots Hol_p
\]

which will be useful to rewrite as \( Hol \cdot \overline{a}_{ei} \). The tool here is to realize there are leftover terms on the boundary which assemble precisely to \( -[a_{ei}, -] \). To see this, one last type of term coming from \( d(Hol) \) is finally used, which has not been previously incorporated: the derivative of the path-action terms along the Western and Southern boundaries coming from each \( \overline{\bullet} \) in the expression for \( Hol \). Note that these terms did not appear for the Northern and Eastern boundaries since the convention is to use the path approaching a term going along the Western boundary, then along the Southern boundary, and then up towards the desired location through the interior. In other words, (4.47) can be written as

\[
(4.47) \quad Hol_a \cdot Hol_{ae} \cdot Hol_e \cdot Hol_{ci} \cdot a_{ci} \left( \frac{\partial}{\partial r} \right) \cdot Hol_i \ldots Hol_p
\]

which will momentarily be written as

\[
(4.48) \quad = Hol_1 \cdot Hol_{ci} \cdot \alpha_{hol_a^{-1} \cdot hol_e^{-1} \cdot hol_{ci}^{-1} \cdot g_{ci}}^{-1} (Hol_i) \ldots Hol_p
\]

Using all of the various terms occurring at this Southern \( ci \) boundary-corner, they can be combined in a useful way, where the reference to which set of relations the term comes from is listed in place of an equation label:

(d(Hol)-path) \quad = Hol_1 \cdot Hol_{ci} \cdot \alpha_{\ldots} g_{ci}^{-1} (dg_{ci} g_{ci}^{-1}) (Hol_i \cdot Hol_{ci})
This calculation above demonstrates the appearance of the term,

$$\text{Hol} \cdot \left( \sum_{\partial S_q} a \right) ,$$

from equation (4.1).

To demonstrate the appearance of the term,

$$\text{Hol} \cdot \left( \int_{\partial S_q} B \right) ,$$

from equation (4.1), a similar technique is applied to the $B_i$ integrated along the Western and Southern boundaries using the vanishing fake curvature condition $t(B_i) = R_i$, which is now shown below.

In a similar manner to the above, first write (3.2) as

$$\text{Hol}_1 \cdot \text{Hol}_2 \cdot \alpha_{\gamma_i} \cdot (\text{Hol}_2),$$

After differentiating, side terms along the path, $\gamma_S$, are obtained which can be rewritten in the desirable fashion:

$$-\text{Hol}_1 \cdot \text{Hol}_2 \cdot \int_{\gamma_n} \alpha_{\gamma} \cdot (B_n) \cdot \text{Hol}_2$$

$$+ \text{Hol}_1 \cdot \text{Hol}_2 \cdot \alpha_{\gamma} \cdot (R_n) \cdot (\text{Hol}_2)$$

$$= -\text{Hol}_1 \cdot \text{Hol}_2 \cdot \int_{\gamma_n} \alpha_{\gamma} \cdot (B_n) \cdot \text{Hol}_2$$

$$+ \text{Hol}_1 \cdot \text{Hol}_2 \cdot \alpha_{\gamma} \cdot (B_n) \cdot (\text{Hol}_2)$$

$$= -\text{Hol}_1 \cdot \text{Hol}_2 \cdot \int_{\gamma_n} \alpha_{\gamma} \cdot (B_n) \cdot \text{Hol}_2$$

$$+ \text{Hol}_1 \cdot \text{Hol}_2 \cdot \alpha_{\gamma} \cdot (B_n) \cdot (\text{Hol}_2)$$

$$= -\text{Hol}_1 \cdot \text{Hol}_2 \cdot \int_{\gamma_n} \alpha_{\gamma} \cdot (B_n) \cdot \text{Hol}_2$$
\[ + \text{Hol}_1 \cdot \text{Hol}_n \cdot \left( \int_{\gamma_n^2} \alpha_*(B_n), \text{Hol}_2 \right) \]
\[ = \text{Hol} \cdot \int_{\gamma_n^2} \alpha_*(B_n) \]

5. SPECIAL CASES

The results of this paper are now briefly stated as three special cases:

1. The surface holonomy of spheres, \( M^{S^2} \to H \).
2. The surface holonomy, \( M^{Sq} \to H \), which uses a crossed module \( (H \leftarrow G, \alpha) \), whose \( \alpha \)-action is given by inner-automorphisms.
3. The surface holonomy for abelian gerbes.

5.1. Surface Holonomy of Spheres. First some useful propositions for \( \text{Hol}^N \) for the case when \( \Sigma \in M^{S^2} \) are recorded below. By way of applying Proposition 3.4 to the case where the eastern and western boundaries are identified with each other, the northern edge is collapsed to a point, and the southern edge is collapsed to a point, we obtain the following two corollaries just as one can find in [MP2] and [SWIII]:

**Corollary 5.1.** The 2-holonomy of a sphere takes its values in the center of the Lie group, \( H \).

**Corollary 5.2.** The transformation of 2-holonomy between open sets in \( M^{S^2} \) is given by
\[ \text{Hol}^{N'}(\Sigma) = \alpha_{g_{i',0}}(0,0)(\text{Hol}^N) \]
for \( \Sigma \in N_I \cap N_{I'} \subset M^{S^2} \).

Similarly, in the case of \( M^{S^2} \) Theorem 4.1 can be simplified to an original corollary as follows.

**Corollary 5.3.** The total derivative of 2-holonomy, \( d(\text{Hol}) \), can be written:
\[
(5.1) \quad d(\text{Hol}) = -(\alpha_{\text{Hol}})_*(A_i(1,1)) + \text{Hol} \cdot \int_{S^2} H
\]
where \( \text{Hol} = \text{Hol}^N : N \to H \subset \text{Mat} \) is defined on the open set \( N \subset M^{S^2} \).

Note that \( \int_{S^2} H \in \Omega^1(N, \mathfrak{h}) \) is a one-form on the open subset \( N \subset M^{S^2} \); where \( \mathfrak{h} \) is the Lie algebra of \( H \), as usual. For two open subsets \( N_I, N_{I'} \subset M^{S^2} \), the transformation of \( \int_{S^2} H \) in \( N_I \cap N_{I'} \subset M^{S^2} \) can be written as follows.

**Corollary 5.4.** The integral of the 3-curvature over a sphere transforms in the following way:
\[ \int_{S^2} H^N = \alpha_{g_{i',0}}(0,0)(\int_{S^2} H^{N'}) \]
where by \( H^{N'} \) we mean to use the local 3-curvature as defined by the local data on \( N_{I'} \).
5.1.1. \( \alpha : G \to \text{Inn}(H) \). In this section the following special case is assumed, which will allow for considerable simplifications.

**Setting 5.5.** Suppose that \( \alpha \) factors through the inner automorphisms of \( H \):

\[
\begin{array}{c}
G \\
\xrightarrow{\alpha} \\
\xrightarrow{\alpha} \\
\text{Inn}(H)
\end{array}
\]

\( G \xrightarrow{\alpha} \text{Aut}(H) \)

\( \alpha \to \text{Inn}(H) \)

Remark 5.6. Note that if \( Y \in Z(\mathfrak{h}) \) then in Setting 5.5, \( \alpha_g(Y) = Y \) for any \( g \in G \). Similarly, if \( h \in Z(H) \), then \( \alpha_X(h) = 0 \) for any \( X \in \mathfrak{g} \), and for \( Y \in Z(\mathfrak{h}) \), \( X \in \mathfrak{g} \) then \( \alpha_X(Y) = 0 \).

Recall from Proposition 5.1 that the 2-holonomy, a function on \( N \subset M^{S^2} \), has trivial target yielding \( \text{Hol}^N \in Z(H) \), where \( Z(H) \) is the center of \( H \). It was shown in Proposition 5.2 that \( \text{Hol} \) transforms between open subsets of \( M^{S^2} \) via \( \alpha_{gij} \) but since the domain is in \( M^{S^2} \) and the action factors through an inner automorphism, it follows that \( \text{Hol}^N \) functions agree on overlaps (also proven in [MP2]):

**Corollary 5.7.** The function \( \text{Hol} : M^{S^2} \to H \) given by \( \text{Hol} |_N := \text{Hol}^N \) is well-defined (globally).

**Corollary 5.8.** In the case of Setting 5.7, the local 3-curvature forms, \( H_i \), glue together to a global 3-form, \( H \in \Omega^3(M, \mathfrak{h}) \).

As a consequence of the above fact we obtain the final original result of this paper,

**Corollary 5.9.** The total derivative of 2-holonomy, \( d(\text{Hol}) = \text{Hol} \cdot \int_{S^2} H \in \Omega^1(M^{S^2}, \text{Mat}) \), is globally defined.

5.2. **Abelian Gerbes and Surface Holonomy.** For this section, assume that squares, \( \Sigma : [0,1]^2 \to M \), are closed surfaces in \( M \). Recall that in an abelian gerbe, the structure crossed module \( (H \to 1, t) \) is used, where \( H \) is an abelian group, and the trivial group, 1, takes the place of the group, \( G \). Note that in this case the \( \alpha \)-action is trivial: \( \alpha(h) = h \), and so in the notation of this paper, it follows that \( \text{Hol}_\bullet = \text{Hol}_\circ \). Such a gerbe, \( G \), has \( g_{ij} = 1, A_i = 0 \). Note, then that we can recover the analogous results found in [TWZ].

**Corollary 5.10.** The surface holonomy function is well defined on \( M \). In particular we have \( \text{Hol}^{N_i}_\circ = \text{Hol}^{N_i}_\bullet \).

**Corollary 5.11.** For the function \( \text{Hol} = : M \to H \subset \text{Mat} \) defined on any open set \( N \subset M^{S^2} \), we can compute its total derivative \( d(\text{Hol}) \in \Omega^1(M^{S^2}, \text{Mat}) \) as follows:

\[
d(\text{Hol}) = \text{Hol} \cdot \int_{S^2} H
\]

where \( H \) is the global 3-form of Corollary 5.8.

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