Small Count Privacy and Large Count Utility in Data Publishing

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ABSTRACT

While the introduction of differential privacy has been a major breakthrough in the study of privacy preserving data publication, some recent work has pointed out a number of cases where it is not possible to limit inference about individuals. The dilemma that is intrinsic in the problem is the simultaneous requirement of data utility in the published data. Differential privacy does not aim to protect information about an individual that can be uncovered even without the participation of the individual. However, this lack of coverage may violate the principle of individual privacy. Here we propose a solution by providing protection to sensitive information, by which we refer to the answers for aggregate queries with small counts. Previous works based on \(\ell\)-diversity can be seen as providing a special form of this kind of protection. Our method is developed with another goal which is to provide differential privacy guarantee, and for that we introduce a more refined form of differential privacy to deal with certain practical issues. Our empirical studies show that our method can preserve better utilities than a number of state-of-the-art methods although these methods do not provide the protections that we provide.

1. INTRODUCTION

The ultimate source of the problem with privacy preserving data publishing is that we must also consider the utility of the published data. The problem is intriguing to begin with because we have a pair of seemingly contradictory goals of utility and privacy. Whenever we are able to provide some useful information with the published data, there is the question of privacy breach because of that information.

The statistician Tore Dalenius advocated the following privacy goal in [8]: *Anything that can be learned about a respondent from the statistical database should be learnable without access to the database.* To aim for this goal, some previous works have considered the approach where prior and post beliefs about an individual are to be similar [18, 27, 4]. As discussed in [13], this privacy goal may be self-contradictory and impossible in the case of privacy preserving data publication. The goal of the published data is for a receiver to know something about the population, it is by definition that the receiver can discover something about an individual in the population, and the receiver could happen to be the adversary. Due to the seeming impossibility of the above goal, research in differential privacy moves away from protecting the information about a row in the data table that can be learned from other rows [5]. The argument is that such information is derivable without the participation of the corresponding individual in the dataset, and hence it is not under the control of the individual. However, although not under the individual’s control, such information could nevertheless be sensitive.

An important goal of our work here is to show that it is possible to protect sensitive information that can be acquired from the published dataset provided that the data publisher, with control over the dataset, can act on behalf of each individual. The principle of protecting individual privacy may dictate that the publisher either provides this kind of protection or does not publish the data. It is desirable that on top of ensuring that the participation of a user makes little difference to the results of data analysis, the publisher also guarantees protection for sensitive information that can be derived from the published dataset, with or without the data of the individual involved. Such a solution is our goal. While Dalenius’s original goal may be impossible, it is also an overkill. A “relaxed” goal suffices: *Anything “sensitive” that can be learned about a respondent from the statistical database should be learnable without access to the database.* The obvious question is what should be considered sensitive. We provide a plausible answer here.

Let us consider an example given in [14], where a dataset \(D’\) tells us that almost everyone involved in a dataset has one left foot and one right foot. We would agree that knowing with high certainty that a respondent is two footed from \(D’\) is not considered a problem since almost everyone is two footed. Note that even if an individual does not participate in the data collection, the deduction can still be made based on a simple assumption of an i.i.d. data generation. Differential privacy and all of the proposed privacy models so far do not exclude the possibilities of deriving information of such form. In fact, by definition of data utility, such a derivation should be supported. This example is not alarming since it involves a large population. However there will be cases where the information becomes sensitive and requires protection. Let us consider a medical data set. Suppose lung cancer is not a common disease. Also suppose there are only five females aged 70, with postal code 2980 and all of them have lung cancer, the linkage of the corresponding (gender, age, postal code) with lung cancer in this case is 100%; if we maintain high utility for accurately extracting such information or concepts, the privacy of the five females will be compromised. The reason why this is alarming is because accurate answers to
queries of small counts can disclose highly sensitive information. A problem with many existing techniques lies in non-discriminative utilities for all concepts. We propose to consider discriminative utilities which are based on the population sizes: queries involving large populations can be answered relatively accurately while queries with a very small population base should not. A similar idea is found in the literature of security for statistical databases [1, 20, 21, 29] (see Section 10 on related work).

Protecting queries of small counts is implicit in many previous works. For example, the principle of ℓ-diversity [25] essentially protects against accurate answers to queries about the sensitive values of individuals, which may become small count queries given that the adversary has knowledge about the non-sensitive values of an individual and therefore is capable of linkage attack [29, 28]. We shall show that our mechanism provides better protection when compared to ℓ-diversity approaches.

Our major contributions are summarized as follows. We point out the dilemma that utility is a source of privacy breach, so that on top of differential privacy we must also protect sensitive information that can be derived from the published data. We propose a mechanism for privacy preserving data publication which provides three lines of protection: (1) differential privacy to protect information that may be attained from the data of an individual tuple, (2) protection for concepts with small counts which can be derived from the entire published data set, and (3) a guarantee that the published data does not narrow down the set of possible sensitive values for each individual. We enforce a stronger ℓ'-diverse privacy guarantee by setting ℓ = 0. We support discriminative utilities so that concepts with large counts can be preserved. While ℓ-diversity methods are vulnerable to adversary knowledge that eliminates ℓ − 1 possible values, our method is resilient to such attacks. We have conducted experiments on a real dataset to show that our method provides better utilities for the large sum queries than several state of the art methods which do not have the above guarantee.

The rest of the paper is organized as follows. In Section 2, we revisit ℓ'-diverse privacy for non-interactive database sanitization. We point out issues about ℓ' and about known presence. Then we introduce our model of ℓ'-diverted zero-differential privacy. Section 3 describes a first attempt of a solution using an existing randomization method, we show that this method cannot guarantee zero-differential privacy. Section 4 describes our proposed mechanism A' which generates D'. Section 5 is about count estimation given D'. Section 6 shows that mechanism A' supports high utility for large counts and high inaccuracies for small counts. Section 7 is about multiple attribute aggregations. Section 8 is a discussion about auxiliary knowledge that may be possessed by the adversary. Section 9 reports on the empirical study. Related works are summarized in Section 10 and we conclude in Section 11.

2. ℓ'-DIVERTED PRIVACY

Our proposed method guarantees a desired form of differential privacy with the additional protection against the disclosure of sensitive information of small counts. In this section we shall introduce our definition of privacy guarantee based on differential privacy. First we examine some relevant definitions from previous works. The following is taken from [6].

**Definition 1 (A(D) and ℵ-Differential Privacy).** For a database D, let A be a database sanitization mechanism, we will say that A(D) induces a distribution over outputs. We say that mechanism A satisfies ℵ-differential privacy if for all neighboring databases D₁ and D₂ (i.e., D₁ and D₂ differ in at most one tuple), and for all sanitized outputs D₁, D₂,  

\[
\Pr[A(D_1) = \hat{D}] \leq e^\epsilon \Pr[A(D_2) = \hat{D}]
\]

The above definition says that for any two neighboring databases, the probabilities that A generates any particular dataset for publication are very similar. However, there are some practical problems with this definition.

2.1 The problem with ℵ

In ℵ-differential privacy, the parameter ℵ is public. The sanitized data is released to the public, and the public refers to a wide spectrum of users and applications. It is not at all clear how we may have the parameter ℵ decided once and for all. In [14], it is suggested that we tend to think of ℵ as, say, 0.01, 0.1, or in some cases, ln 2 or ln 3. Evidently the value can vary a lot. For example, for the above suggested values, e^ℵ ranges from 1.01, 1.105 to 2 and 3.

A second problem with the setting of ℵ is that it may compromise privacy. Suppose that for all pairs of neighboring databases D₁ and D₂, where D₂ contains t while D₁ does not, \( \Pr[A(D_1) = \hat{D}] \) is 1/3, while \( \Pr[A(D_2) = \hat{D}] \) is 1. If we set ℵ to ln 3, then ℵ-differential privacy is satisfied, but the existence of t can be estimated with 75% confidence.

The above concerns call for the elimination of ℵ. We can do so by setting ℵ to zero. This is in fact the best guarantee since it means that there is no difference between D₁ and D₂ in terms of the probability of generating D'. We shall refer to this guarantee as zero-differential privacy.

2.2 The issue of known presence

While the initial definition of differential privacy aims at hiding the presence or absence of an individual’s data record, it is often the case that the presence is already known. As discussed in [13], in such cases, rather than hiding the presence, we wish to hide certain values in an individual’s row. We shall refer to such values that need to be hidden as the sensitive values. The definition of differential privacy need to be adjusted accordingly. The phrase “differ in at most one tuple” in Definition 1 can be converted to “have a symmetric difference of at most 2”. This is so that in two datasets D₁ and D₂, if only the data for one individual is different, then we shall find two tuples in the symmetric difference of D₁ and D₂. The two tuples are tuples of the same individual in the two datasets, but the sensitive values differ. However, with this definition, the counts for sensitive values in D₁ or D₂ would deviate from the original data set D. For a neighboring database, we prefer to preserve as much as possible the characteristics in D. In the following subsection, we introduce a definition of differential privacy that addresses the above problems.

2.3 ℵ'-DIVERTED zero-differential privacy

Given a dataset (table) D which is a set of N tuples, the problem is how to generate sensitive values for the tuples in D to be published in the output dataset D'. We assume that there are two kinds of attributes in the dataset, the non-sensitive attributes (NSA) and a sensitive attribute (SA) S. Let the domain of S be domain(S) = \{s₁, ..., s_m\}. We do not perturb the non-sensitive values but we may alter the sensitive values in the tuples to ensure privacy. We first introduce our definition of neighboring databases which preserves the counts of sensitive values, and we minimize the moves by swapping the sensitive values of exactly one arbitrary pair of tuples with different sensitive values. In the following we use t.s to denote the value of the sensitive attribute of tuple t.

**Definition 2.** (Neighbor w.r.t. t). Suppose we have two databases D₁ and D₂ containing tuples for the same set of individuals, and D₁ and D₂ differ only at two pairs of tuples, t, t’ in D₁ and D₂.
and $\ell$, $\ell'$ in $D_2$. Tuples $t$ and $t'$ are for the same individual, and $\ell$ and $\ell'$ are for another individual, with $t.s \neq \ell.s$, $t.s = \ell'.s$, and $\ell.s = t'.s$. Then we say that $D_2$ is a neighboring database to $D_1$ with respect to $t$.

Our definition of neighbors bears some resemblance to the concept of Bounded Neighbors in [23], where the counts of tuples are preserved. As in [23], our objective is a good choice of neighbors of $D$ (the original dataset) which should be difficult to distinguish from each other. Our differential privacy model retains the essence in Definition 1 from [6].

**Definition 3.** ($\ell'$-diverted privacy). We say that a non-interactive database sanitization mechanism $A$ satisfies $\ell'$-diverted zero-differential privacy, or simply $\ell'$-diverted privacy, if for any given $D_1$, for any tuple $t$ in $D_1$, there exists $\ell' - 1$ neighboring databases $D_2$ with respect to $t$, such that for all sanitized outputs $\hat{D}$, $Pr[A(D_1) = \hat{D}] = Pr[A(D_2) = D]$.

The above definition says that any individual may take on any of $\ell'$ different sensitive values by swapping the sensitive information with other individuals in the dataset, and it makes no difference in the probability of generating any neighboring $D$. It seems that our definition depends on the parameter $\ell'$. However, not knowing which $\ell' - 1$ neighboring databases it should be in the definition, an adversary will not be able to narrow down the possibilities. Therefore, even in the case where the adversary knows all the information about all individuals except for 2 individuals, there is still no certainty in the values for the 2 individuals.

Our task is to find a mechanism that satisfies $\ell'$-diverted zero-differential privacy while at the same time supports discriminative query answering. We need to derive a different technique.

3. Randomization: An Initial Attempt

In the search for a technique to guarantee a tapering accuracy for the estimated values from large counts to small counts, the law of large numbers [23,24] naturally comes to mind. Random perturbation has been suggested in [30], the reason being that “If a query set is sufficient large, the law of large numbers causes the error in the query to be significantly less than the perturbations of individual records.” Indeed, we have seen the use of i.i.d. for the randomization of datasets with categorical attributes. In [4], an identity perturbation scheme for categorical sensitive values is proposed. This scheme keeps the original sensitive value in a tuple with a probability of $p$ and randomly picks any other value to substitute for the true value with a probability of $(1 - p)$, with equal probability for each such value. Theorem 1 in [4] states that their method can achieve good estimation for large dataset sizes. Therefore, it is fair to ask if this approach can solve our problem at hand. Unfortunately, as we shall show in the following, this method cannot guarantee zero-differential unless $p$ is equal to $1/M$ with a domain size of $M$, which renders the generated data a totally random dataset. Let us examine this approach in more details.

Suppose that the tuple $t$ of an individual has sensitive value $t.s$ in $D$. The set of sensitive values is given by $\{s_1, \ldots, s_m\}$. We generate a sanitized value for the individual by selecting $s_i$ with probability $p_i$, so that

$$p_i = \begin{cases} 
    p & \text{for } s_i = t.s; \\
    q & \text{for } s_i \neq t.s 
\end{cases}$$

where $\sum_i p_i = 1$.

Let us refer to the anonymization mechanism above by $A$.

Let $D'$ be a dataset published by $A$ which contains a tuple for individual $I$. Consider two datasets $D_1, D_2$ which differ only in the sensitive value for the single tuple for $I$.

We are interested in the probability $Pr[A(D_1) = D']$ that $D'$ is generated from $D_1$ by $A$, and $Pr[A(D_2) = D']$. In particular, we shall show that when $p = q$, $A$ is zero-differential.

Let the tuples in $D_1$ be $t_1, \ldots, t_N$. Let the tuples in $D_2$ be $t'_1, \ldots, t'_N$.

**Lemma 1.** For mechanism $A$, if $p = q$, then $A$ satisfies zero-differential privacy according to Definition 7 with neighboring databases having a symmetric difference of at most 2.

**Proof:** Since all the non-sensitive values are preserved and only the sensitive values may be altered by $A$, we consider the probability that each tuple in $D_1$ or $D_2$ may generate the corresponding sensitive value in $D'$. For $D_k$, $k \in \{1,2\}$, let $p_k(t_i, s_i)$ be the probability that $A$ will generate $s_i$ for tuple $t_i$.

Mechanism $A$ handles each tuple independently. Hence $Pr[A(D_k) = D']$ is a function of $p_k(t_i, s_i)$ for all $i, j, k \in \{1,2\}$.

$$Pr[A(D_k) = D'] = f(p_k(t'_1, s_1), p_k(t'_1, s_2), \ldots, p_k(t'_1, s_m), \ldots, p_k(t'_N, s_1), \ldots, p_k(t'_N, s_m))$$

Given a tuple $t$ with sensitive value $t.s$, the probability that a sensitive value $s_j$ will be generated in $D'$ for $t$ depends only on the value of $t.s$.

Without loss of generality, let $D_1$ and $D_2$ differ only in the sensitive value for $t_i$. We have $p_1(t'_i, s_i) = p_2(t'_i, s_i)$ for all $j$ and all $i \neq r$. Obviously if we set $p = q = \frac{1}{2m}$, then the probability to generate any value given any original sensitive value will be the same. Although $t'_r, s_i \neq t'_i, s_i$, we have $p_1(t'_i, s_i) = p_2(t'_i, s_i)$ for all $j$. Hence $Pr[A(D_1) = D'] = Pr[A(D_2) = D']$ and Mechanism $A$ satisfies zero-differential privacy. □

**Lemma 2.** For mechanism $A$, if $p \neq q$, then $A$ does not satisfy zero-differential privacy according to Definition 7 with neighboring databases having a symmetric difference of at most 2.

**Proof:** We prove by constructing a scenario where we are given datasets $D_1, D_2$ differing in only one tuple, and a sanitized table $D'$, and $Pr[A(D_1) = D'] \neq Pr[A(D_2) = D']$. Consider the case where all tuples are unique in terms of the non-sensitive attributes. For $1 \leq i \leq n$, let $p_1(t_i, s_i) = p$ if $t'_i, s_i = t_i, s_i$, and $p_1(t_i, s_i) = q$ if $t'_i, s_i \neq t_i, s_i$. We have $Pr[A(D_1) = D'] = \prod p_1(t_i)$. Similarly we define $p_2(t_i)$ for $1 \leq i \leq n$. $Pr[A(D_2) = D'] = \prod p_2(t_i)$.

Furthermore, let $t'_i, s_i = t_k, s_k$ and $t'_i, s_i \neq t_k, s_k$. Therefore, for $t_k$, $p_1(t_k, s_k) = p$ and $p_2(t_k, s_k) = q$, while $p_1(t_i, s_i) = p_2(t_i, s_i)$ for $i \neq k$.

$$Pr[A(D_1) = D'] = \frac{p}{q}$$

Since $p \neq q$, it follows that $Pr[A(D_1) = D'] \neq Pr[A(D_2) = D']$ and therefore $A$ does not satisfy zero-differential privacy. □

**Lemma 3.** For mechanism $A$, if $p \neq q$, then $A$ is not $\ell'$-diverted zero-differential according to Definition 3.

**Proof:** We prove by showing a scenario where given $D_1$, and a neighboring database $D_2$ with respect to a tuple $t$, and an anonymized table $D'$, $Pr[A(D_1) = D'] \neq Pr[A(D_2) = D']$. Let $D_1$ and $D_2$ agree on all tuples except for $t'_2$ and $t'_2$ in $D_1$ and corresponding tuples $t'_2$ and $t'_2$ in $D_2$. Let all tuples be unique in terms of the non-sensitive attributes.
Furthermore, let $D_1 A \rightarrow D'$ be very low. Here we introduce a simple mechanism called differential privacy without sacrificing too much utility. We make an initialization step, whereby the dataset $D$ is an initialization step, whereby the dataset $D$ is initialized with the values in such a way that in each partition, the sensitive value of a tuple $t$ is not zero differential. 

From the previous analysis, in order to make the probability $\Pr[A(D_1) = D']$ equal to $\Pr[A(D_2) = D']$, all values need to be selected with probability equal to $\frac{p}{q}$. This would be the same as random data and it would have great cost in the utility.

4. PROPOSED MECHANISM

From the previous section on mechanism $A$, we see that for generating a dataset $D'$ from a given dataset $D$, randomization with uniform probability can attain zero-differential privacy. However, if the probability is uniform over the entire domain, the utility will be very low. Here we introduce a simple mechanism called $A'$ which introduces uniform probability over a subset of the domain. We shall show that this mechanism satisfies $\ell'$-diverted zero-differential privacy without sacrificing too much utility. We make the same assumption as in previous works [25, 33] that the dataset is eligible, so that the highest frequency of any sensitive attribute value does not exceed $N/\ell'$. Furthermore we assume that $N$ is a multiple of $\ell'$ (it is easy to ensure this by deleting no more than $\ell' - 1$ tuples from the dataset).

4.1 Mechanism $A'$

Mechanism $A'$ generates a dataset $D'$ given the dataset $D$. We assume that there is a single sensitive attribute ($SA$) $S$ in $D$. We shall show that $A'$ satisfies $\ell'$-diverted zero-differential privacy. There are four main steps for $A'$:

1. First we assume that the tuples in $D$ have been randomly assigned unique tuple id's independent of their tuple contents. Include the tuple id as an attribute $id$ in $D$. The first step of $A'$ is an initialization step, whereby the dataset $D$ goes through a projection operation on $id$ and the $SA$ attribute $S$. Let the resulting table be $D_s$. That is, $D_s = \Pi_{id,S}(D)$. Note that the non-sensitive values have no influence on the generation of $D_s$.

2. The set of tuples in $D_s$ is partitioned into sets of size $\ell'$ each in such a way that in each partition, the sensitive value of each tuple is unique. In other words, let there be $r$ partitions, $P_1, \ldots, P_r$; in each partition $P_i$, there are $\ell'$ tuples, and $\ell'$ different sensitive values. We call each partition a decoy group. If tuple $t$ is in $P_i$, we say that the elements in $P_i$ are the decoys for $t$. We also refer to $P_i$ as $P(t)$. With a little abuse of terminology, we also refer to the set of records in $D$ with the same id's as the tuples in this decoy group as $P(t)$.

One can adopt some existing partitioning methods in the literature of $\ell$-diversity. We require that the method be deterministic. That is, given a $D_s$, there is a unique partitioning from this step.

3. For each given tuple $t$ in $D_s$, we determine the partition $P(t)$. Let the sensitive values in $P(t)$ be $\{s'_1, \ldots, s'_{r}\}$. For each of these decoy values, there is a certain probability that the value is selected for publication as the sensitive value for $t$. For a value not in $\{s'_1, \ldots, s'_r\}$, the probability of being published as the value for $t$ is zero. In the following we shall also refer to the set $\{s'_1, \ldots, s'_r\}$ as $\mathcal{A}(t)$.

4. Suppose that a tuple $t$ has sensitive value $s$ in $D$. Create tuple $t'$ and initialize it to $t$. Next we generate a value to replace the $S$ value in $t'$ by selecting $s_i$ with probability $p_i$, so that

\[
p_i = p_p, \quad p_i = q = (1-p) \frac{1}{\ell'} \quad \text{for } s_i \neq t_s, \quad s_i \in \mathcal{A}(t) \\
p_i = 0 \quad \text{for } s_i \notin \mathcal{A}(t)
\]

The pseudocode for mechanism $A'$ is given in Algorithm 1. At first glance, mechanism $A'$ looks similar to partitioning based methods for $\ell$-diversity [25]. However, for the second step in $A'$, we can adopt an existing partitioning algorithm such as the one in [34] which has been designed for bucketization. However, $A'$ differs from these previous approaches in important ways.

Firstly, the generation of dataset $D'$ is based on a probabilistic assignment of values to attributes in the tuples. There is a non-zero probability that an SA (sensitive attribute) value that exists in $D$ does not exist in $D'$. In known partitioning based methods, the SA values in $D'$ are honest recording of the values in $D$, although in some algorithms they may be placed in buckets separated from the remaining values.

Secondly, the partitioning information is not released by $A'$, in contrast to previous approaches, in which the anonymized groups or buckets are made known in the data publication. For $\ell$-diversity methods, since the partitioning is known, each tuple has a limited set of possible values. By withholding the partitioning information, plus the possibility that a value existing in $D$ may not exist in $D'$, there is essentially no limit to the possible values for $S$ except for the entire domain for any tuple in $D'$.

Algorithm 1 - Mechanism $A'$

**Require:** $D$ is a $N$-tuple dataset, with random tuple id's, sensitive attribute $S$, and set of non-sensitive attributes $N\ SA$

1: $Ds \leftarrow \Pi_{id,S}(D)$
2: partition $Ds$ into decoy groups of size $\ell'$ each so that each decoy group has $\ell'$ unique sensitive values.
3: for each partition $P$ do
4: for each tuple $t$ in $P$ do
5: let $\mathcal{A}(t) = \{s'_1, \ldots, s'_{r}\}$
6: create tuple $t'$ and set $t'.id = t.id$
7: if $t.s = s'_i$ then
8: set $t'.s = s'_i$ with probability $p$
9: set $t'.s$ to $s'_j \neq s'_i$ with probability $q$
10: $Ds' \leftarrow \Pi_{N\ SA,S}(\Pi_{id,N\ SA,D}) \timesid Ds'$
11: shuffle tuples in $D'$ and publish $D'$ and $\ell'$
12: [Note that no other information about the partitions is published]


4.2 $\ell'$-diverted zero-differential guarantee

For the privacy guarantee, we shall show that if $p = q$, then $A'$ satisfies $\ell'$-diverted zero-differential privacy, otherwise, it does not. First we need to state a fact about $A'$.

**FACT 1.** In mechanism $A'$, let $p = q = \frac{1}{\ell'}$, so that $p = q = \frac{1}{\ell'}$. When executing $A'$, for two tuples $t$ and $t'$ in the same partition $(P(t) = P(t'))$, and any sensitive value $s$, the probability that $t$ will be assigned $s$ by $A'$ is equal to that for $t'$.

The following theorem says that we should set $p = q = 1/\ell'$ in mechanism $A'$.

**THEOREM 1.** For mechanism $A'$, if $p = q = \frac{1}{\ell'}$, then $A'$ satisfies $\ell'$-diverted zero-differential privacy.

**Proof:** Let $D'$ be a published dataset. Given a dataset $D_1$ which may generate $D'$, and a tuple $t$ in $D_1$, we find $\ell' - 1$ neighboring databases $D_2$ as follows:

We execute $A'$ on top of $D_1$. In the first step, $D_1^1$ is generated from $D_1$. In the second step, $D_1^2$ is partitioned into sets of size $\ell'$. Let $P(t)$ be the partition (decoy group) formed by $A'$ for $t$ in $D_1$. Pick one element $\tilde{t}$ in $P(t)$ where $\tilde{t} \neq t$. Form $D_2$ by swapping the non-sensitive values of $t$ and $\tilde{t}$ in $D_1$. By definition $D_2$ is a neighboring database of $D_1$.

Let $D_2^1$ be the table generated from Step 1 of $A'$ on $D_2$. Since we have only swapped the non-sensitive values of $t$ and $\tilde{t}$, $D_1^1 = D_2^1$. The partitioning step of $A'$ is deterministic, meaning that the same partitioning will be obtained for $D_1$ and $D_2$. From the above, we know that $t$ and $\tilde{t}$ are in the same partition for $D_1$, i.e., $P(t) = P(\tilde{t})$. When we consider the generation of sensitive values for $t$ and $\tilde{t}$, since they are in the same partition $P(t)$, by Fact 1 they have the same probabilities for different outcomes. Since the $SA$ values for different tuples are generated independently, $Pr\{A'(D_1) = D'\} = Pr\{A'(D_2) = D'\}$.

There are $\ell' - 1$ possible $D_2$ given $D_1$, we have shown that $A'$ satisfies $\ell'$-diverted zero-differential privacy.

**THEOREM 2.** For mechanism $A'$, if $p \neq q$, then $A'$ does not satisfy $\ell'$-diverted zero-differential privacy.

**Proof:** We prove by giving an instance where $A'$ is not $\ell'$-diverted zero-differential. We say that a dataset $D$ is $A'$-consistent with $D'$ if there is a non-zero probability that $D'$ is generated from $D$ by $A'$. Consider $D_1$ and $D_2$, each being consistent with $D'$. Let the tuples in $D_1$ be $t_1', ..., t_N'$. Let the tuples in $D_2$ be $t_1, ..., t_N$. The two sets of tuples are for the same set of individuals. Furthermore, assume that $D_1$ and $D_2$ differ in only 2 tuples for a pair of individuals; let the pair of tuples in $D_1$ be $t_1', t_2'$, and that in $D_2$ be $t_1, t_2$. Assume that all tuples have unique non-sensitive values, and

\[ t_{1,i} = s, t_{2,i} = t'_{2,i}, \]
\[ t_{1,i} = t_{2,i}, t'_{2,i} = t'_{2,i}, \]
\[ t_{1,i} = t_{2,i}, t'_{2,i} = t'_{2,i}. \]

For $1 \leq i \leq N$, let $p_1(t_1) = p$ if $t_{1,i} = t'_{2,i}$, and $p_1(t_1) = q$ if $t_{1,i} = t'_{2,i}$. Similarly we define $p_2(t_1)$ for $1 \leq i \leq N$. Therefore, for $s$, $p_1(t_1) = p$ and $p_2(t_1) = q$, while $p_1(t_1) = p_2(t_1)$ for $i \notin \{a, b\}$.

\[ Pr\{A'(D_1) = D'\} = \prod_{i=1}^{N} p_1(t_i) = \frac{p^2}{q^2}. \]

Since $p \neq q$, it follows that $Pr\{A'(D_1) = D'\} \neq Pr\{A'(D_2) = D'\}$ and therefore $A$ is not zero differential.

The above theorems show that in order to enforce $\ell'$-diverted zero-differential privacy, we should set both $p$ and $q$ to $1/\ell'$. This will be the assumption in our remaining discussions about $A'$.

5. AGGREGATE ESTIMATION

In this section we examine how to answer counting queries for the sensitive attribute based on the published dataset $D'$.

Let $|D'| = N$, so that there are $N$ tuples in $D$. Consider a sensitive value $s$. Let the true frequency of $s$ in $D$ be $f_s$. By mechanism $A'$, there will be $f_s$ decoy groups which contain $s$ in the decoy value sets. Each tuple in these groups has a probability of $p = \frac{1}{\ell'}$ to be assigned $s$ in $D'$. The probability that it is assigned other values $\bar{s}$ is $1 - p$. There are $f'_s \ell'$ such tuples.

Let $N'_s$ denote the number of times that $s$ is published in $D'$. The random variable $N'_s$ has the binomial distribution with parameters $f_s, \ell'$ and $p$.

\[ P[N'_s = x] = \binom{f_s \ell'}{x} p^x (1 - p)^{f_s \ell' - x}. \]

The expected value is $f_s \ell' p$, and $\sigma^2 = f_s \ell' p(1 - p)$.

Since we set $p = q = 1/\ell'$, the expected count of $s$ in $D'$ is given by $e_s = p \ell' f_s = f_s$; we have

\[ e_s = f_s. \]

That is, to estimate the true count of an $SA$ value $s$, we simply take the count of $s$ in $D'$, $f'_s$.

**THEOREM 3.** The estimation of $f_s$ by $f'_s$ is a maximum likelihood estimation (MLE).

**Proof:** Let $L(D)$ be the likelihood of the observation $f'_s$ in $D'$, given the original dataset $D$. $L(D) = Pr\{f'_s|D\}$.

From Mechanism $A'$, given $f_s$ occurrences of $s$ in $D$, there will be exactly $\ell' f_s$ tuples that generates $s$ in $D'$ with a probability of $p$. The remaining tuples have zero probability of generating a $s$ value. The probability that $f'_s$ occurrences of $s$ is generated in $D'$ is given by

\[ L(D) = Pr\{f'_s|D\} = \left( \frac{f'_s}{f_s} \right)^{f'_s} \frac{(1 - p)^{f_s \ell' - f'_s}}{\ell'}. \]

where $p = 1/\ell'$.

This is a binomial distribution function which is maximized when $f'_s$ is at the mean value of $\ell' f_s p = f_s$.

To examine the utility of the dataset $D'$, we ask how likely $f'_s$ is close to $f_s$, and hence the estimation $e_s$ is close to the true count $f_s$? However, we also need to provide protection for small counts. In the next section we shall analyze these properties of the published dataset.

6. PRIVACY, UTILITY, AND THE SUM

As discussed in Section 1, the utility of the dataset must be bounded so that for certain facts, in particular, those that involve very few individuals, the published data should provide sufficient protection. Here we consider the relationship between the utility and the number of tuples $n$ that is related to a sensitive value. Is it possible to balance between disclosing useful information where $n$ is large and hence safe and not disclosing accurate information when $n$ is small and hence need protection? We explore these issues in the following.
6.1 Utility for large sums

To answer the question about the utility for large sums, we make use of the Chebychev’s inequality which gives a bound for the likelihood that an observed value deviates from its mean.

**Chebychev’s Theorem:** If $X$ is a random variable with mean $\mu$ and standard deviation $\sigma$, then for any positive $k$, $Pr(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$ and $Pr(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$.

Let $X_1, X_2, ..., X_n$ be a sequence of independent, identically distributed random variables, each with mean $\mu$ and variance $\sigma^2$. Define the new sequence of $\overline{X}_i$ values by

$$\overline{X}_i = \frac{1}{n} \sum_{i=1}^{n} X_i, \ n = 1, 2, 3, ...$$

From Chebychev’s inequality, $P \left[ |\overline{X}_i - \mu| \geq \epsilon \right] \leq \frac{\sigma^2}{\epsilon^2 n}$ where $\mu_{\overline{X}_i} = E[\overline{X}_i] = \mu$, $\sigma_{\overline{X}_i}^2 = E[(\overline{X}_i - \mu)^2] = \frac{\sigma^2}{n}$, and $k$ is any positive real number. Choose $k = \frac{\epsilon}{\sigma}$ for some $\epsilon > 0$, we get

$$Pr \left[ |\overline{X}_i - \mu| \geq \epsilon \right] \leq \frac{\sigma^2}{\epsilon^2 n} \tag{1}$$

The above reasoning has been used to prove the law of large numbers. Let us see how it can help us to derive the utility of our published data for large sums. If there are $f_s$ tuples with $s$ value, then $n = \ell f_s$ tuples in $D$ will have a probability of $p$ to be assigned $s$ in $D'$. The setting of value $s$ to the tuples in $D'$ corresponds to a sequence of $\ell f_s$ independent Bernoulli random variables, $X_1, ..., X_{\ell f_s}$, each with parameter $p$. Here $X_i = 1$ corresponds to the event that $s$ is chosen for the $i$-th tuple, while $X_i = 0$ corresponds to the case where $s$ is not chosen.

The mean value $\mu_{\overline{X}_i} = p$. Also, $\sigma_{\overline{X}_i}^2 = (1-p)/n$. From Inequality (1),

$$Pr \left[ |\overline{X}_i - \mu| \geq \epsilon \right] \leq \frac{p(1-p)}{\epsilon^2 n^2} = \frac{p(1-p)}{\epsilon^2 \ell^2 f_s^2}$$

From Section 2.3 we set $p = \frac{1}{\ell}$, hence

$$Pr \left[ |\overline{X}_i - \mu| \geq \epsilon \right] \leq \frac{1}{\ell \epsilon^2 f_s^2} \tag{2}$$

Figure 1 shows the relationship between the possible values of $T_E$ and $T_f$. The utility is better for small $T_E$, and the value of $T_E$ becomes very small when the count is increasing towards 900. Note that utility is the other side of privacy breach, it also means that for concepts with large counts, privacy protection is not guaranteed since the accuracy in the count will be high.

### Definition 4 (Thresholds $T_E$ and $T_f$)

Given an original dataset $D$ and an anonymized dataset $D'$. A value $s$ has a $(\epsilon, T_E, T_f)$ utility guarantee if for a frequency $f_s$ of $s$ above the frequency threshold of $T_f$ in $D$,

$$Pr \left[ |f_s' - f_s| \geq \epsilon f_s \right] \leq T_E \text{ for } f_s \geq T_f \tag{4}$$

The above definition says that a value $s$ has a $(\epsilon, T_E, T_f)$ guarantee if whenever the frequency $f_s$ of $s$ is above $T_f$ in $D$, then the probability of a relative error of more than $\epsilon$ is at most $T_E$.

**Lemma 4.** Mechanism $\mathcal{A}'$ provides a $(\epsilon, T_E, T_f)$ utility guarantee for each sensitive value, where

$$\frac{1}{\ell \epsilon^2 T_E} = T_f^2 \tag{5}$$

Hence given, $\epsilon$ and $T_E$, we can determine the smallest count which can provide the utility guarantee.

**Example 1.** Consider some possible values for the parameters. Suppose $T_E = 0.02$ and $\ell = 10$. If $\epsilon = 0.2$, then $T_f = 11$. If $\epsilon = 0.001$, or $\epsilon = 0.02$ then $T_f = 49$.

### 6.2 Privacy for small sums

Next we show how our mechanism can inherently provide protection for small counts. From Inequality (3), small values of $f_s$ will weaken the guarantee of utility. We can in fact give a probability for relative errors based on the following analysis.

The number of $s$ in $D'$ is the total number of successes in $f_s \ell'$ repeated independent Bernoulli trials with probability $p'/\ell$ of success on a given trial. It is the binomial random variable with parameters $n = \ell f_s$ and $p = \frac{1}{\ell}$. The probability that this number is $x$ is given by

$$\binom{n}{x} p^x q^{n-x} = \binom{\ell f_s}{x} \left( \frac{1}{\ell} \right)^x \left( 1 - \frac{1}{\ell} \right)^{\ell f_s - x} \tag{6}$$

**Example 2.** If $f_s = 5$, $\ell = 10$, for an $\epsilon = 0.3$ bound on the relative error, we are interested to know how likely $f_s'$ is close to 5 within a deviation of 1. The probability that $f_s'$ is between 4 to 6 is given by

$$\sum_{x=4}^{6} \binom{n}{x} p^x q^{n-x} = \sum_{x=4}^{6} \binom{50}{x} 0.1^x 0.9^{50-x} \approx 0.52$$

Table 1 shows the relationship between the possible values of $T_E$ and $T_f$. The utility is better for small $T_E$, and the value of $T_E$ becomes very small when the count is increasing towards 900.
Hence the probability that $f_s'$ deviates from $f_s$ by more than 0.3$f_s$ is about 0.52.

**Definition 5 (Privacy Guarantee).** We say that a sensitive value $s$ has a $(\varepsilon, \mathcal{F}_P)$ privacy guarantee if the probability that the estimated count of $s$, $f_s'$, has a relative error of more than $\varepsilon$ is at least $\mathcal{F}_P$.

In Example 2, the value $s$ has a (0.3, 0.48) privacy guarantee. A graph is plotted in Figure 2 for the expected error for small values of $f_s$. Here the summation in the above probability is taken from $f_s' = [0.7f_s]$ to $f_s' = [1.3f_s]$. We have plotted for different $f_s$ values the probability given by

$$1 - \frac{\sum_{x \in [0.7f_s]} \binom{n}{x} p^x q^{n-x}}{q^{n-x}}$$

This graph shows that the relative error in the count estimation is expected to be large for sensitive values with small counts.

**7. Multiple Attribute Predicates**

In this section we consider the counts for sets of values. For example, we may want to know the count of tuples with both lung cancer and smoking, or the count of tuples with gender = female, age = 60 and disease = allergy. The problem here is counting the occurrences of values of an attribute set. Firstly we shall consider counts for predicates involving a single sensitive attribute, then we extend our discussion to predicates involving multiple sensitive attributes.

**7.1 Predicates involving a single $S_A$**

Assume that we have a set of non-sensitive attributes $NSA$ and a single sensitive attribute $SA$, let us consider queries involving both $NSA$ and $SA$. We may divide such a query into two components: $P$ and $s$, where $P \in \text{domain}(NA)$ ($NA \subseteq NSA$), and $s \in \text{domain}(SA)$. For example $P = (\text{female}, 60)$ and $s = (\text{allergy})$. Note that the non-sensitive attributes are not distorted in the published dataset. This can be seen as a special case of generating a non-sensitive value for the individual $t$ by selecting $s_i$ with probability $p_i$, so that

$$p_i = \begin{cases} 1 & \text{for } s_i = t.s. \\ 0 & \text{for } s_i \neq t.s. \end{cases}$$

Suppose we are interested in the count of the co-occurrences of non-sensitive values $P$ and $SA$ $s$.

**Definition 6 (State $i$).** There are 4 conjunctive predicates concerning $P$ and $s$, namely, $p_0 = \overline{P} \land \overline{s}$, $p_1 = P \land \overline{s}$, $p_2 = P \land s$, and $p_3 = P \land s$. If a tuple satisfies $p_i$, we say that it is at state $i$.

The distributions of the predicates in $D$ and $D'$ are given by $cnt(p_i)$ and $cnt'(p_i)$, respectively. Here $cnt(p_i)(cnt'(p_i))$ is the number of tuples satisfying $p_i$ in $D$ ($D'$).

For simplicity we let $x_1 = cnt(p_i)$ and $y_1 = cnt'(p_i)$, hence the a priori distribution concerning the states in $D$ is given by $x = \{x_0, x_1, x_2, x_3\}$, and the distribution in $D'$ is given by $y = \{y_0, y_1, y_2, y_3\}$. Hence $y$ contains the observed frequencies.

**Definition 7 (Transition Matrix $M$).** The probability of transition for a tuple from an initial state $i$ in $D$ to a state $j$ in $D'$ is given by $a_{ij}$. The values $a_{ij}$ forms a transition matrix $M$.

The values of $a_{ij}$ are given in Figure 3.

| $y_0(P \overline{s})$ | $y_1(P \overline{s})$ | $y_2(P \overline{s})$ | $y_3(P \overline{s})$ |
|----------------------|----------------------|----------------------|----------------------|
| $a_{00} = a_{01} = a_{02} = 0$ | $a_{03} = 0$ | $1 - a_{01} \frac{1}{N} x_{\overline{s}}$ | $x_{\overline{s}}$ |
| $a_{10} = a_{11} = a_{12} = 0$ | $a_{13} = 0$ | $\frac{\ell - 1}{\ell}$ | $\frac{\ell}{\ell}$ |
| $a_{20} = a_{21} = 0$ | $a_{22} = 0$ | $1 - a_{23} \frac{1}{N} x_{s}$ | $x_{s}$ |
| $a_{30} = a_{31} = 0$ | $a_{32} = 0$ | $\frac{\ell - 1}{\ell}$ | $\frac{1}{\ell}$ |

Figure 3: State transition probabilities

Let $Pr(r_i|x)$ be the probability that a tuple has state $i$ in $D'$ given vector $x$ for the initial state distribution. The following can be derived.

$$Pr(r_0|x) = \frac{1}{N} \left( 1 - \frac{x_1 + x_3}{N} x_0 + \frac{\ell - 1}{\ell} x_1 \right)$$

$$Pr(r_1|x) = \frac{1}{N} \left( \frac{x_1 + x_3}{N} x_0 + \frac{\ell}{\ell} x_1 \right)$$

$$Pr(r_2|x) = \frac{1}{N} \left( 1 - \frac{x_1 + x_3}{N} x_2 + \frac{\ell - 1}{\ell} x_3 \right)$$

$$Pr(r_3|x) = \frac{1}{N} \left( \frac{x_1 + x_3}{N} x_2 + \frac{\ell}{\ell} x_3 \right)$$

The above equations are based on the mechanism generating $D'$ from $D$. Let us consider the last equation, the other equations are derived in a similar manner. For each true occurrence of $(P, s)$, there is a $\frac{1}{N}$ probability that it will generate such an occurrence in $D'$. If there are $x_3$ such tuples, then the expected number of generated instances will be $x_3.\ell$. The other occurrences of $(P, s)$ in $D'$ may be generated by the $x_2$ tuples satisfying $P$ but with $t.s \neq s$ ($P \overline{s}$). Each such tuple $t$ satisfies $P$ for the non sensitive values and it is possible that $s \in decoys(t)$. We are interested to know how likely $s \in decoys(t)$.

There are in total $\frac{N}{x_2}$ partitions. There can be at most one such tuple in each partition. Hence $f_s$ of the partitions contain $s$ in the decoy set, and if a tuple $t$ is in such a partition, then $s \in decoys(t)$. The probability of having $s$ in $decosys(t)$ for a tuple $t$ with $t.s \neq s$ is the probability that $t$ is in one of the $f_s$ partitions above. Since mechanism $A'$ does not consider the $NSA$ values in the randomization process, all such tuples $t$ have equal probability of being in any of the $f_s$ partitions, and the probability is given by $f_s/N.\ell = f_s.\ell$. Since $f_s = x_1 + x_3$, this probability is $\frac{x_1 + x_3}{N.\ell}$. 
The total expected occurrence of \((P, s)\) is given by
\[
\frac{x_i}{c'} + \left( \frac{x_1 + x_3}{N} \right) \frac{x_3}{c'}
\]
We can convert this into a conditional probability that a tuple in \(D'\) satisfies \((P, s)\) given \(x\), denoted by \(Pr(r_3|x)\). This gives Equation (9).

Rewriting Equations (9) to (12) with the transition probabilities in Figure 3 gives the following:
\[
Pr(r_3|x) = \sum_{j=0}^{3} a_{ji} \frac{x_j}{N}
\]
Equation (10) shows that \(a_{ji}\) is the probability of transition for a tuple from an initial state \(j\) in \(D\) to a state \(i\) in \(D'\).

We adopt the iterative Bayesian technique for the estimation of the counts of \(x_0, ..., x_3\). This method is similar to the technique in for reconstructing multiple column aggregates.

Let the original states of tuples \(t_1, ..., t_N\) in \(D\) be \(U_1, ..., U_N\), respectively. Let the states of the corresponding tuples in \(D'\) be \(V_1, ..., V_N\). From Bayes rule, we have
\[
Pr(U_k = i|V_k = j) = \frac{P(V_k = j|U_k = i)P(U_k = i)}{P(V_k = j)}
\]
Since \(Pr(U_k = i) = x_i/N\), and \(Pr(V_k = j|U_k = i) = a_{ij}\),
\[
Pr(U_k = i|V_k = j) = \frac{a_{ij} x_j}{\sum_{i=0}^{3} a_{ij} x_i}
\]
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\]
Since \(Pr(U_k = i) = x_i/N\), and \(Pr(V_k = j|U_k = i) = a_{ij}\),
\[
Pr(U_k = i|V_k = j) = \frac{a_{ij} x_j}{\sum_{i=0}^{3} a_{ij} x_i}
\]
7.2 Multiple sensitive attributes

So far we have considered that there is a single sensitive attribute in the given dataset. Suppose instead of a single sensitive attribute \((SA)\), there are multiple \(SA\)s, let the sensitive attributes be \(S_1, S_2, ..., S_w\). We can generalize the randomization process by treating each \(SA\) independently, building decoy sets for each \(S_i\).

For predicates involving \((P_1, s_1, s_2, ..., s_w)\), where \(P\) is a set of values for a set of non-sensitive attributes, \(s_i \in \text{domain}(S_i)\), there will be \(K = 2^{w+1}\) different possible states for each tuple. Let \((P(s_1, s_2, ..., s_w))\) stand for \((P \land s_1 \land s_2 \land ... \land s_w)\). For reconstruction of the count for \((P_1, s_1, s_2, ..., s_w)\), we form a transition matrix for all the \(K = 2^{w+1}\) possible states. It is easy to see that the case of a single \(SA\) in Section 7.1 is a special case where the transition matrix \(M\) is the tensor product of two matrices \(M_0\) and \(M_1\), where \(M_0\) is for the set of non-sensitive values and \(M_1\) is for \(S_1\), and they are defined as follows:

\[
M_0 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
\[
M_1 = \begin{pmatrix}
1 & \frac{t_x}{r_x} & \frac{t_x}{r_x} \\
\frac{t_x}{r_x} & 1 & \frac{t_x}{r_x} \\
\frac{t_x}{r_x} & \frac{t_x}{r_x} & 1
\end{pmatrix}
\]

In general, with sensitive attributes \(S_1, ..., S_w\), the transition matrix is given by \(M = M_0 \otimes M_1 \otimes ... \otimes M_w\).

Let the entries in matrix \(M\) be given by \(m_{ij}\). We initialize \(x^0 = y\) and iteratively update \(x\) by the following equation
\[
x_{i+1}^t = \sum_{j=0}^{K-1} \sum_{r=0}^{m_{ij}} \frac{r}{x_j}
\]

In Equation (13), \(x^t\) is the value of \(x\) at iteration \(t\). \(a_{ij}\) refer to the value of \(m_{ij}\) at iteration \(t\), meaning that the value of \(m_{ij}\) depends on setting the values of \(x = x^t\). We iterate until \(x^{t+1}\) does not differ much from \(x^t\). The value of \(x\) at this fixed point is taken as the estimated \(x\) values. In particular \(x_{K-1}\) is the estimated count of \((P, s_1, ..., s_w)\).

8. BELIEF ABOUT AN INDIVIDUAL

An adversary may be armed with auxiliary knowledge in the attack on the sensitive value of an individual. In general auxiliary knowledge allows an adversary to rule out possibilities and sharpen their belief about the sensitive value of an individual. For example, a linkage attack refers to an attack with the help of knowledge about another database which is linked to the published data. The other database could be a voter registration list, and it has been discovered that only the values of birthdate, sex and zip code are often sufficient to identify an individual [22, 23].

In the design of \(\ell\)-diversity [24], the set of tuples are divided into blocks and there should be \(\ell\) well represented sensitive values in each block. The adversary needs \(\ell - 1\) damaging pieces of auxiliary knowledge to eliminate \(\ell - 1\) possible sensitive values and uncover the private information of an individual. Our method is an improvement over the \(\ell\)-diversity model since the possible sensitive values in our case is the entire domain of the sensitive attribute, including values that do not appear in the dataset. Hence if the domain size is \(m\), the adversary would need \(m - 1\) pieces of auxiliary knowledge to rule out \(m - 1\) possible values, but in that case, the adversary knows a priori the exact value without examining \(D'\).

Another form of auxiliary knowledge is knowledge about the sanitization mechanism. Since many known approaches aim to minimize the distortion to the data, they suffer from minimality attack [31]. Our method does not involve any distortion minimization step and therefore minimality attack will not be applicable.
9. EMPIRICAL STUDY

We have implemented our mechanism $A'$ and compared with some existing techniques that are related in some way to our method.

For step 2 of mechanism $A'$, we need to partition tuples in $D_s$ into sets of size $\ell'$ each and each partition contains $\ell'$ different sensitive values. We have adopted the group creation step in the algorithm for Anatomy \[44\]. In this algorithm, all tuples of the given table are hashed into buckets by the sensitive values, so that each bucket contains tuples with the same $SA$ value. The group creation step consists of multiple iterations. In each iteration a partition (group) with $\ell'$ tuples is created. Each iteration has two sub-steps: (1) find the set $L$ with the $\ell'$ hash buckets that currently have the largest number of tuples. (2) From each bucket in $L$, randomly select a tuple to be included in the newly formed partition. Note that the random selection in step (2) can be made deterministic by picking the tuple with the smallest tuple id.

9.1 Experimental setup

The experiments evaluate both effectiveness and efficiency of mechanism $A'$ for $\ell'$-diverted privacy. We also compare our method with three other approaches, Anatomy for $\ell$-diversity, differential privacy by means of Laplacian perturbation, and global randomization (mechanism $A$). Our code is written in C++ and executed on a PC with CORE(TM) i3 3.10 GHz CPU and 4.0 GB RAM. The dataset is generated by randomly sampling 500k tuples from the CENSUS\[2\] dataset which contains the information for American adults. We further produce five datasets from the 500k dataset, with cardinalities ranging from 100k to 500k. The default cardinality is 100k. Occupation is chosen as the sensitive attribute, which involves 50 distinct values.

In the experiment we consider count queries, which have been used for utility studies for partition-based methods \[34\] and randomization-based methods \[27\]. A pool of 5000 count queries is generated according to the method described in Appendix 10.9 in \[7\]. Specifically, we generate random predicates on the non-sensitive attributes, each of which is combined with each of the values in the domain of the sensitive attribute to form a query. We count the tuples satisfying a condition of the form $A_1 = v_1 \land \ldots \land A_d = v_d \land SA = v_s$, where each $A_i$ is a distinct non-sensitive attribute, $SA$ is the sensitive attribute, and the $v_i$ and $v_s$ are values from the domains of $A_i$ and $SA$, respectively. The selectivity of a query is defined as the percentage of tuples that satisfy the conditions in the query. For each selectivity $s$ that is considered we report on the average relative error of the estimated count for all queries that pass the selectivity threshold $s$. In later analysis, we group queries according to their distinct selectivities.

Given queries in the pool, we calculate the average relative error between the actual count (from the original dataset) and estimated count (from the published dataset) as the metric for utility. As discussed earlier, we differentiate between small counts and large counts. Specifically, we vary the selectivity (denoted by $s$, which is the ratio of the actual count to the cardinality of dataset) from 0.5% up to 5% for large counts. For small counts, we require the actual count to be no more than 10 (selectivity less than 0.1%). We evaluate the influence of various $\ell'$ values, and also the cardinalities of dataset on the utility. To assess the efficiency, we record and show the running time of our data publishing algorithm.

9.2 Utility for large counts

First we shall examine the impact of varying $\ell'$, while we have

\[1\] Downloadable at http://www.ipums.org

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{Relative error}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{Relative error for small counts}
\end{figure}

separate plot for distinct selectivity. In particular, the average relative error is computed for $\ell'$ ranging from 2 to 10, as shown in Figure 4 where selectivity of large counts is concerned. For large selectivity (i.e., large counts) between 2% and 5%, the error is as low as 20%. The error is also bounded by 40% for other selectivities, which is acceptable. Another observation is a trend that, as $\ell'$ increases, the error for most selectivities first decreases but soon start to rise. This can be explained by the fact that more restricted privacy (larger $\ell'$) requirement may compromise the utility. For the special case where the query involves only the sensitive attribute, the relative errors of both small and large counts are shown in Figure 5. The results agree with our analysis in Section 6. The relative error is as well shown against the selectivity in Figure 7.

9.3 Error for small counts

We plot the error of queries with small counts separately in Figure 5 where the counts are smaller than 10. As one can observe, the error is sufficiently high to ensure privacy, consistent with our requirement that answer for small count should be inaccurate enough to prevent privacy leakage. The relative error also displays a positive linear correlation with $\ell'$. In other words, as $\ell'$ becomes bigger (higher privacy), privacy for small counts is also ensured at a higher level.

9.4 Comparison with other models

To our knowledge there is no known mechanism for $\ell'$-diverted privacy. We would like to compare the utilities of our method with other models although it is not a fair comparison since our method provides guarantees not supported by the other models. We have chosen to compare with Anatomy because we have used a similar partitioning mechanism, and Anatomy is an improvement over previous $\ell$-diversity methods since it does not distort the non-sensitive values. We compare with the distortion based differential privacy method since it has been the most vastly used technique in differ-
utility, we plot the relative error against the graph. To see how
with depend on
Figure 11. Obviously the relative error from our method does not
100k dataset is used, and the average relative error is shown for
and we choose the
vacy, no matter which
for smaller
of queries that can be submitted to
Figure 7: Relative error versus selectivity for SA querying
The results for the global randomization mechanism
A are shown in Figures 8 and 9, respectively. The overall
error for multiple-dimension aggregates involving two
in Figure 10. We set the value of \( p \) to \( 1/\ell' \) so that the probability to
retain the original sensitive value in each tuple is the same in both
methods. It can be seen that our method has much better utility for
all the selectivities in our experiment.

9.5 Multiple sensitive values
We also consider the utility in scenarios where a query involves
more than one sensitive value. To this end, we choose Age and
Occupation as the sensitive attributes. The two sensitive attributes
are randomized independently and then combined for data publica-
tion. To allow queries of large selectivities, we first generalize the
domain of Age into ten intervals; without this step, most of the re-
sulting counts are too small and the range of selectivities is limited.
The relative error for multiple-dimension aggregates involving two

![Figure 6: Relative error versus \( \ell' \) for SA querying](image)

![Figure 8: Utility for Anatomy](image)

![Figure 9: Mechanism \( B' \) for \( \ell' \)-diverted privacy](image)

![Figure 10: Comparison of our method (\( \ell' \)-diverted) with differ-
ential privacy and global randomization by mechanism \( A \)](image)
sensitive attributes is shown in Figure 12 where $\ell'$ ranges from 2 to 8. Although given the diminished selectivities (0.1% to 0.7% for this case), the overall accuracy can match that in single-sensitive-attribute scenario.

9.6 Computational overhead

The computational overhead mainly comes from the partitioning process. We have adopted the partitioning method of Anatomy. This algorithm can be implemented with a time complexity of $O(N(1 + \frac{1}{V}))$, where $N$ is the cardinality of the table, and $V$ is the number of distinct values of the sensitive attribute. We show the running time for the case of single sensitive attribute on the largest 500K dataset, varying $\ell'$ from 2 to 10. For all chosen $\ell'$ values, our algorithm can finish within 10 seconds for a 500K dataset, which is practical to be deployed in real applications.

We also consider the querying efficiency at the user side. To estimate the answer, a user will compute each component of the vector $y$, and do matrix multiplications to iteratively converge at the answer $x$. When each component of $y$ changes by no more than 1%, we terminate the iteration and measure the querying time and number of iterations. In our experiments, SQLite\[^2\] serves for querying $y$, and we consider the case with two sensitive attributes which involves the most number of components in $y$, implying the largest computational cost. The result shows that the Bayesian iterative process takes negligible time, while the major cost comes from the querying step. In particular, it takes less than 1 ms in average, and 10 ms in the worst case, for the iterative process to converge. The median and average of the number of iterations is 16 and 325, respectively. In total, the average measured time for a query is 1612 ms, which poses little computational burden on users.

10. RELATED WORK

Differential privacy has been a break-through in the study of privacy preserving information releases. $\epsilon$-differential privacy has been introduced for query answering and the common technique is based on distortion to the query answer by a random noise that is i.i.d. from a Laplace distribution and calibrated to the sensitivity of the querying \[15\] [12]. Laplace noise has been used in many related works on differential privacy including recent works on reducing relative error \[33\] and the publication of data cubes in \[10\]. Since the data release can be for different purposes, in some tasks, the addition of noise makes no sense. For example, a utilization function might map databases to strings, strategies, or trees. The problem of optimizing the output of such a function while preserving $\epsilon$-differential privacy is addressed in \[26\]. For database publication, \[6\] shows that given a large enough dataset, a synthetic database can be generated that is approximately correct for all concepts in a given concept class; the minimal data size depends on the quality of the approximation, the log of the size of the universe, the privacy parameter $\epsilon$ and the Vapnik-Chervonenkis dimension of the concept class. Further results can be found in \[17\]. In most previous works, the definition of error is an absolute error \[11\] [16] [6] \[17\]. The algorithm iReduct in \[33\] considers relative errors and injects noise to query results according to the values of the results. A recent work \[23\] points out that differential privacy may not guarantee privacy when deterministic statistics have been previously published. In contrast we consider a more basic possible privacy leak which is due to the fact that differential privacy does not aim to protect information that can be derived from the published data, deeming such a task impossible. All previous works on differential privacy consider $\epsilon$-differential privacy for non-zero $\epsilon$ values. None of the works in the above considers the guarantee of protection of small sums, which is a major objective in our mechanism.

In the literature of statistical databases, the protection of small counts has been well-studied in the topic of security in statistical databases \[1\]. A concept similar to ours is found in \[30\] where the aim is to ensure that the error in queries involving a large number of tuples will be significantly less than the perturbation of individual tuples. It has been pointed out in previous works \[20\] [21] that the security of a database is endangered by allowing answers to counting queries that involve small counts, i.e. the number of tuples involved in the query is small. In \[9\], random sampling has been used to ensure large errors for small query set sizes. However, these previous works are about the secure disclosure of statistics from a dataset and do not deal with the problem of sanitization of a dataset for publication, and they have not considered the guarantee of differential privacy. Discriminative privacy protection has been considered in some previous work in privacy preserving data publication such as \[35\] [37], however, such works are based on personalized privacy requirements. There have been studies that the utility of published dataset can lead to privacy breach \[22\] [32], however, they focus on partition-based methods for $\ell$-diversity and they have pointed out the problems while no solution is proposed.

Randomization technique has been used in previous works in privacy preservation. The usefulness of such a technique is shown in \[3\], where the published data is used to build a decision tree which achieves classification accuracy comparable to the accuracy of classifiers built with the original data. An effective reconstruction method for data perturbation is introduced in \[2\]. In \[3\], random perturbation is adopted for privacy preserving computation for multidimensional aggregates in data horizontally partitioned at multiple clients. Randomization of transaction datasets for the mining of association rules has been considered in \[19\].

\[^2\]See http://docs.python.org/library/sqlite3.html
11. CONCLUSION

We have introduced a new mechanism in the problem of privacy preserving data publication with the following properties. Firstly, it satisfies $\ell$'-diverted zero-differential privacy, which makes sure that the resulting data analysis will have no difference whether an individual keeps its true sensitive value or swap the true value with other individuals. Secondly, the randomization process makes use of the law of large numbers in ensuring that large counts, which are not as sensitive, can be estimated with high accuracies while the small counts will be hidden by relatively large errors. Our method is parameter free except for the value of $\ell'$, however, the choice of $\ell'$ has little effect on the privacy and as shown in our experiments, setting $\ell'$ to 5 or above will do well in terms of the utilities. Furthermore, the sensitive value of a tuple in the published data can be any value in the attribute domain, so the mechanism is resilient to auxiliary knowledge which eliminates possible values. Our empirical studies on a real dataset show superior utility performance compared to other state-of-the-art methods which do not have the above guarantees. For future work, we may consider how to handle skewed sensitive microdata \cite{84}. Another direction for future work is to consider mechanisms such as small domain randomization for further boosting the utilities for large counts \cite{7}. The consideration of sequential data releases is another open problem.

As a final remark, all existing privacy models inherently release information that can be derived from the published datasets, and the same is true with our approach. It is important to make known to the users what kind of information they should expect to be released or derivable. In our case, it will be relatively accurate answer to queries with large sums.

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