THE EXPLICIT EVALUATIONS FORMULA FOR RAMANUJAN’S SINGULAR MODULI AND RAMANUJAN- SELBERG CONTINUED FRACTION

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Abstract. At scattered places of his notebooks, Ramanujan recorded over 30 values of singular moduli \( \alpha_n \). All those results were proved by Berndt et. al by employing Weber-Ramanujan’s class invariants. In this paper, we initiate to derive the explicit evaluations formula for \( \alpha_{9n} \) and \( \alpha_{n/9} \) by involving class invariant. For this purpose, we establish several new \( P – Q \) mixed modular equations involving theta-functions. Further application of these modular equations, we derive a new formula to explicit evaluation of Ramanujan- Selberg continued fraction.

1. Introduction

Ramanujan’s general theta-function \( \Pi \) is defined by

\[
f(a, b) = \sum_{n=-\infty}^{n=\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.
\]

By using Jacobi’s fundamental factorization formula, the above theta function takes the form as follows:

\[
f(a, b) = (-a; ab)_\infty \cdot (-b; ab)_\infty \cdot (ab; ab)_\infty.
\]

The following definitions of theta functions \( \varphi, \psi \) and \( f \) with \( |q| < 1 \) are classical:

\[
\varphi(q) = (q, q) = \sum_{n=-\infty}^{\infty} q^n = (-q; q^2)_\infty (q^2; q^2)_\infty, \quad (1.1)
\]

\[
\psi(q) = (q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty}{(q^2; q^2)_\infty}, \quad (1.2)
\]

\[
f(-q) = f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_\infty, \quad (1.3)
\]

where, \( (a; q)_\infty = \prod_{n=0}^{\infty} (1 – aq^n) \).

If \( q = e^{2\pi i \tau} \), then, \( f(-q) = q^{-1/24} \eta(\tau) \), where, \( \eta(\tau) \) is classical Dedekind eta-function.

The ordinary or Gaussian hypergeometric function is defined by

\[
_{2}F_{1} \left( \begin{array}{c} a, b \\ c \end{array} ; z \right) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n, \quad |z| < 1
\]

where,

\[
(a)_n = \begin{cases} a(a+1)(a+2) \cdots (a+n-1), & n = 1, 2, 3, \ldots \nonumber \\
1, & n = 0. \nonumber 
\end{cases}
\]

Now, we shall recall the definition of modular equation from \( \Pi \). The complete elliptic integral of the first kind \( K(k) \) of modulus \( k \) is defined by

\[
K(k) = \int_{0}^{\pi/2} \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}} = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{(n!)^2} k^{2n} = \frac{\pi}{2} \frac{\varphi^2}{\varphi^2} \left( e^{-\pi \frac{\varphi^2}{\varphi^2}} \right), \quad (0 < k < 1)
\]

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and let $K' = K(k')$, where $k' = \sqrt{1 - k^2}$ is represented as the complementary modulus of $k$. Let $K, K', L,$ and $L'$ denote the complete elliptic integrals of the first kind associated with the moduli $k, k', l,$ and $l'$ respectively. In case, the equality

$$n \frac{K'}{K} = \frac{L'}{L} \quad \text{(1.5)}$$

holds for a positive integer $n$, then a modular equation of degree $n$ is the relation between the moduli $k$, and $l$, which is implied by equation (1.5). Ramanujan defined his modular equation involving $\alpha$, and $\beta$, where, $\alpha = k^2$ and $\beta = l^2$. Then we say $\beta$ is of degree $n$ over $\alpha$ if

$$\frac{2F_1(1/2, 1/2; 1; 1 - \beta)}{2F_1(1/2, 1/2; 1; \beta)} = \frac{n \cdot 2F_1(1/2, 1/2; 1; 1 - \alpha)}{2F_1(1/2, 1/2; 1; \alpha)}$$

Ramanujan recorded various degrees of modular equations in his note books. For instance, the modular equation of degree 3 (see [11, Entry 5(ii), (ix), Ch.19, p.230]) as following

$$(\alpha \beta)^{1/4} + ((1 - \alpha)(1 - \beta))^{1/4} = 1,$$  \hspace{1cm} (1.6)

$$(\alpha(1 - \beta))^{1/2} + (\beta(1 - \alpha))^{1/2} = 2(\alpha \beta(1 - \alpha)(1 - \beta))^{1/8}. \quad \text{(1.7)}$$

Also, Ramanujan defined mixed modular equation or modular equation of composite degrees, along with four distinct moduli. Ramanujan recorded 23 $P$-$Q$ modular equations in terms of their theta function in his notebooks [6]. All those proved by Berndt et al. by employing the theory of theta functions and modular forms.

If, as usually quoted in the theory of elliptic functions, $k = k(q)$ denotes the modulus, then, the singular moduli $k_n$ is defined by $k_n = (e^{-\pi \sqrt{n}})$, where $n$ is a positive integer. In terms of Ramanujan, set $\alpha = k^2$ and $\alpha_n = k_n^2$, he hypothesized the values of over 30 singular moduli in his notebooks. Also, he asserted the value of $k_{10}^2$ which he wrote in his second letter to Hardy [5, p. xxix]), which was earlier proved by Wastson [8] by using a remarkable formula. The formula found can also be in Ramanujan’s first notebook [6] Vol. 1, p. 320]. The same formula can also be used to evaluate various values of $\alpha_n$, for even values of $n$. On page 82 of his first notebook, Ramanujan stated three additional theorems for calculating $\alpha_n$, for even values of $n$. Particularly, he offered formulae for $\alpha_{4p}$, $\alpha_{6p}$, and $\alpha_{16p}$. Moreover, he recorded several values of $\alpha_n$ for odd values of $n$ in his first and second notebook. All these results were proved by Berndt et al. by employing Ramanujan’s class invariants $G_n$ and $g_n$. The Ramanujan’s class invariants [2] p.183, (1.3) defined by $q = e^{-\pi \sqrt{n}}$

$$G_n = 2^{-1/4}q^{-1/24}\chi(q) = \frac{f(q)}{2^{1/4}q^{1/24}f(-q^2)}; \quad g_n = 2^{-1/4}q^{-1/24}\chi(-q) = \frac{f(-q)}{2^{1/4}q^{1/24}f(-q^2)}, \quad \text{(1.8)}$$

where, $n$ is a positive rational number and $\chi(q) = (q; q^2)_\infty$. Ramanujan evaluated a total of 116 class invariants [2] p.189-204]. These class invariants were proved by various authors using techniques such as modular equations, Kronecker limit formula, and empirical process (established by Watson) [2] Chapter 34].

By Entry 12(v), (vi)[11, p. 124], and our conclusion (1.8) is that

$$G_n = (4\alpha_n(1 - \alpha_n))^{-1/24} \quad ; \quad g_n = (4\alpha_n(1 - \alpha_n)^{-2})^{-1/24}. \quad \text{(1.9)}$$

It follows the concept that if we know the explicit values of $G_n$ and $g_n$, then the corresponding values of $\alpha_n$ can be obtained by solving a quadratic equations. However, the expression one obtain are generally unattractive, and so, better algorithm can be sought for evaluations of $\alpha_n$.

Let, for $|q| < 1$,\n
$$N(q) : = 1 + \frac{q}{1} + \frac{q^2 + q^4}{1 + \frac{q^2}{1 + \frac{q^4}{1 + \frac{q^6}{1 + \frac{q^8}{\cdots}}}}} \quad \text{(1.10)}$$

In his notebooks [6] p. 290], Ramanujan recorded that

$$N(q) = \frac{(-q; q^2)_\infty}{(-q^2; q^4)_\infty}. \quad \text{(1.11)}$$
This formula was at first proved by Selberg [7] and the alternative proof was given by Ramanathan [4]. From [10] and [11], we obtain that

\[ S_1(q) := \frac{q^{1/8} N(q)}{1 + \frac{1}{1 + \frac{q + q^2}{1 + \frac{q^2 + q^4}{1 + \cdots}}}, \]

which is called the Ramanujan-Selberg continued fraction. Closely related to \( S_1(q) \) is the continued fraction \( S_2(q) \) (see [10] (1.8), (1.9)), which can be defined by

\[ S_2(q) := \frac{q^{1/8}}{1 + \frac{-q}{1 + \frac{-q + q^2}{1 + \frac{-q^2 + q^4}{1 + \cdots}}}} = \frac{q^{1/8}(q^2;q^2)_{\infty}}{(q;q^2)_{\infty}}, \quad |q| < 1 \]

This formula was at first proved by Selberg [7] and the alternative proof was given by Ramanathan [4].

Zhang [10] recorded a general formula to find the explicit evaluations of (1.12) by using Ramanujan’s singular moduli as follows:

\[ S_1 \left( e^{-\pi \sqrt{n}} \right) = \frac{\delta_{n/2}^{1/8}}{\sqrt{2}}. \]

Moreover, he established a general formula to calculate \( S_2(q) \) in terms of singular moduli.

The present paper is organized as follows. In Section 2, we state that a few lemmas which are essentials to prove our main results. We establish several new \( P-Q \) mixed modular equations involving theta-functions, which are presented in Section 3. As application of those modular equations, we derive some new general formulae involving Weber-Ramanujan’s class invariants for explicit evaluations of \( \alpha_{9n}, \alpha_{n/9}, S_1(q), \) and \( S_2(q) \), which are discussed in Section 4. Finally, calculate some explicit values of singular moduli and Ramanujan-Selberg continued fraction in Section 5.

2. Preliminaries

We list a few identities which are useful in establishing our main results.

Lemma 2.1. [1] Entry 12 (i), (ii), (iv) Ch.17, p.124 | We have

\[ f(q) = \sqrt{2} q^{1/6} \alpha (1 - \alpha)/q^{1/24}, \]

(2.1)

\[ f(-q) = \sqrt{2} q^{-1/6} (1 - \alpha)/q^{1/24}, \]

(2.2)

\[ f(-q^4) = \sqrt{2} q^{-2/3} (1 - \alpha)/q^{4/24}. \]

(2.3)

Lemma 2.2. [1] p.39, Entry 24 (iii),(iv) | We have

\[ \frac{f(q)}{f(q^2)} = \frac{f(-q^2) f(-q^4)}{f(-q) f(-q^4)}. \]

(2.4)

Lemma 2.3. [3] Theorem 3.1(3.2) | Let

\[ K = (256 \alpha \beta \gamma (1 - \alpha)(1 - \beta)(1 - \gamma)(1 - \delta))^{1/24} ; \quad R = \left( \frac{\gamma \delta (1 - \gamma)(1 - \delta)}{\alpha \beta (1 - \alpha)(1 - \beta)} \right)^{1/24}. \]

If \( \alpha, \beta, \gamma, \delta \) is of degree 1,3,3,9, then,

\[ 8 \left( K^3 + \frac{1}{K^3} \right) \left( R^3 + \frac{1}{R^3} + 1 \right) = \left( R^0 + \frac{1}{R^0} \right) + 10 \left( R^6 + \frac{1}{R^6} \right) + 19 \left( R^3 + \frac{1}{R^3} \right) + 36. \]

(2.5)

3. New Mixed Modular Equations

In this section, we discuss about the establishment of a few novel mixed modular equations using theory of theta functions.
We observe that the first factors of the aforemention equation vanish for that specific value. Dividing them by \( P \) does not vanish for that specific value. Thus, we obtain that

\[
(PQ + \frac{16}{PQ}) \left( \frac{P}{Q} + \frac{Q}{P} + 1 \right) = \left( \frac{P}{Q} + \frac{Q}{P} \right)^3 - 2 \left( \frac{P}{Q} + \frac{Q}{P} \right)^2 - 8 \left( \frac{P}{Q} + \frac{Q}{P} \right) - 8.
\]

**Proof.** Transcribing \( P \) and \( Q \) using (2.2) and (2.3), then simplifying, we arrive at

\[
\frac{P}{2} = \left( \frac{1 - \alpha}{\alpha \beta} \right)^{1/8}; \quad \frac{Q}{2} = \left( \frac{1 - \gamma}{\gamma \delta} \right)^{1/8}.
\]

Employing (3.1) in (1.3), we have

\[
K = \left( \frac{8PQ}{(P^2 + 4)(Q^2 + 4)} \right)^{1/3}; \quad R = \left( \frac{P^2Q + 4Q}{PQ^2 + 4P} \right)^{1/3},
\]

where, \( K \) and \( R \) are defined as in Lemma 2.3. Now, on applying (3.2) in (2.5), we obtain that

\[
A(P, Q)B(P, Q) = 0,
\]

where,

\[
A(P, Q) = P^6Q^6 - 8P^5Q^5 - 64P^4Q^4 - 80P^3Q^3 - 256P^2Q^2 - 64P^3Q^3 - 768P^2Q^2 - 1024PQ^4 + 2048PQ - 4096,
\]

\[
B(P, Q) = Q^6 - P^3Q^3 - 2PQ^5 - P^4Q^4 - 5P^2Q^2 - 5P^3Q^3 - 12PQ^4 - 16PQ^3 - 16P^3Q + P^6.
\]

We observe that the first factor of (3.3) does not vanish for \( q \to 0 \). Nevertheless, the second factor vanish for that specific value. Dividing them by \( P^3Q^3 \) and rearranging the terms. Hence we complete the proof.

**Theorem 3.2.** If \( P = \frac{f(-q)f(-q^3)}{f(-q^4)f(-q^{12})} \) and \( Q = \frac{q^{1/4}f(-q)f(-q^{12})}{f(-q^4)f(-q^{12})} \), then

\[
\left( \frac{1}{PQ} + \frac{PQ}{2} \right)^2 = \left( \frac{P}{Q} + \frac{Q}{P} \right)^2 \left( \frac{1}{PQ} + \frac{PQ}{2} + 1 \right) + 4.
\]

**Proof.** Transcribing \( P \) and \( Q \) using (2.2) and (2.3), then by simplifying, we arrive at

\[
P = \left( \frac{\beta(1 - \alpha)}{\alpha(1 - \beta)} \right)^{1/8}; \quad Q = \left( \frac{\delta(1 - \gamma)}{\gamma(1 - \delta)} \right)^{1/8}.
\]

Employing (3.5) in (1.4), we have

\[
(\alpha \beta(1 - \alpha)(1 - \beta))^{1/8} = \frac{2P^2}{P^4 + 1}; \quad (\gamma \delta(1 - \gamma)(1 - \delta))^{1/8} = \frac{2Q^2}{Q^4 + 1}.
\]

Now applying (3.6) in (2.3), we obtain the following result

\[
(P^2Q^6 - P^5Q^5 + PQ^5 + 2P^4Q^4 + Q^4 + 4P^3Q^3 + P^6Q^2 + 2P^2Q^2 + P^5Q - PQ + P^4) \times (P^2Q^6 + P^5Q^5 - PQ^5 + 2P^4Q^4 + Q^4 + 4P^3Q^3 + P^6Q^2 + 2P^2Q^2 - P^5Q + PQ - P^4) \times (P^4Q^6 - P^5Q^5 + PQ^5 + P^6Q^4 + 2P^3Q^4 - 4P^2Q^4 + 2Q^2 + P^5Q - PQ - P^4) \times (P^4Q^6 + P^5Q^5 - PQ^5 + P^6Q^4 + 2P^2Q^2 + Q^2 + P^4Q^2 + 2Q^2 - P^5Q - PQ + P^4) = 0.
\]

We observe that the first factors of the aforemention equation vanish for \( q \to 0 \), whereas, the other factors do not vanish for that specific value. Thus, we obtain that

\[
P^2Q^6 - P^5Q^5 + PQ^5 + 2P^4Q^4 + Q^4 + 4P^3Q^3 + P^6Q^2 + 2P^2Q^2 + P^5Q - PQ + P^4 = 0.
\]

Dividing the equation by \( P^3Q^3 \) and rearranging the terms, we arrive at the desired result.
Theorem 3.3. If \( P = \frac{f(-q)}{q^{1/3}f(-q^4)} \) and \( Q = \frac{f(-q^9)}{q^{9/8}f(-q^{36})} \), then,

\[
\left( P^4Q^4 + \frac{256}{P^4Q^4} \right) \left( \frac{P}{Q} + \frac{Q}{P} + 1 \right) = \left( \frac{P}{Q} + \frac{Q}{P} \right)^6 - 8 \left( \frac{P}{Q} + \frac{Q}{P} \right)^5 + 4 \left( \frac{P}{Q} + \frac{Q}{P} \right)^4 + 64 \left( \frac{P}{Q} + \frac{Q}{P} \right)^3
\]

\[
-16 \left( \frac{P}{Q} + \frac{Q}{P} \right)^2 - 160 \left( \frac{P}{Q} + \frac{Q}{P} \right) - 96.
\]

(3.7)

Proof. By Theorem 3.1, can be written in the following form,

\[
\left( \frac{f(-q)f(-q^9)f^2(-q^{12})}{q^4f(-q^4)f(-q^{36})f^2(-q^{12})} \right) + 16 \left( \frac{qf(-q)f(-q^9)f^2(-q^{12})}{f(-q)f(-q^9)f^2(-q^3)} \right) = \frac{h^3 - 2h^2 - 8h - 8}{h + 1},
\]

where, \( h = \frac{P}{Q} + Q/P \). Solving the above equation, we arrive at

\[
\frac{f(-q)f(-q^9)f^2(-q^3)}{q^4f(-q^4)f(-q^{36})f^2(-q^{12})} = \frac{h^3 - 2h^2 - 8h - 8 + vh}{2(h + 1)},
\]

where \( v = \pm \sqrt{h^2 - 4h^4 - 12h^2 + 16h + 32} \). Similarly, by Theorem 3.2, we obtain that

\[
\left( \frac{f(-q)f(-q^9)f^2(-q^{12})}{q^4f(-q^4)f(-q^{36})f^2(-q^3)} \right)^2 = \frac{h^2 - 2h - 6 + v}{2(h + 1)}.
\]

Now multiplying (3.8) and (3.9), then employing the value of \( v^2 \), we deduce that

\[
4h^8 - 24h^7 + v(4h^6 - 16h^5 - 44h^4 + 72h^3 + 256h^2 + 224h + 64) - 44h^6 + 256h^5 + 464h^4 + (-8P^4Q^4 - 512)h^3 + (-24P^4Q^4 - 1728)h^2 + (-24P^4Q^4 - 1408)h - 8P^4Q^4 - 384 = 0
\]

Isolating the terms containing \( v \) on one side of the above equation and squaring both sides, we arrive at

\[
(h + 1)^5(P^4Q^4h^6 - 8P^4Q^4h^5 + 4P^4Q^4h^4 + 64P^4Q^4h^3 - 16P^4Q^4h^2 - P^8Q^8h - 160P^4Q^4h - 256h - 8P^4Q^4 - 96P^4Q^4 - 256) = 0
\]

We observe that the second factors of above equation vanish for \( q \to 0 \) and the first factor does not vanish for that specific value. Dividing the aforementioned equation by \( P^4Q^4 \) and rearranging the terms. Hence we complete the proof. \( \square \)

Theorem 3.4. If \( P = \frac{f(-q)f(-q^9)}{q^{5/4}f(-q^4)f(-q^{36})} \) and \( Q = \frac{f(-q^3)f(-q^{12})}{q^{3/4}f(-q^{12})f(-q^{36})} \), then

\[
\left( \frac{P}{Q} + \frac{Q}{P} \right)^3 = Q^2 + \frac{16}{Q^2}.
\]

(3.10)

Proof. Theorem 3.2 can be written in the form

\[
u^2 - (u + 1)(P/Q + Q/P)^2 - 4 = 0,
\]

(3.11)

where, \( u = \frac{qf(-q)f(-q^9)}{f(-q^4)f(-q^{36})} + \frac{f(-q^3)f(-q^{12})}{f(-q^{12})f(-q^{36})} \).

Solving (3.11) for \( u \) and choosing the appropriate root, then employing in (3.10) we obtain, after a straightforward lengthy calculation that

\[
(Q^6 - P^3Q^5 + 3P^2Q^4 + 3P^4Q^2 - 16P^3Q + P^6)(Q^6 + P^3Q^5 + 3P^2Q^4 + 3P^4Q^2 + 16P^3Q + P^6)
\]

\[
\times (PQ^6 - 16PQ^5 + 3P^3Q^4 + 3P^5Q^2 - P^3Q + P^7)(PQ^6 + 16P^5Q^5 + 3P^3Q^4 + 3P^5Q^2 + P^3Q + P^7)
\]

We observe that the first factors of above equation vanish for \( q \to 0 \) and other factors does not vanish for that specific value. Dividing by \( P^3Q^2 \) and rearranging the terms. Hence we complete the proof. \( \square \)
4. General Formulae and the Explicit Evaluations

In this section, we establish the formulae involving Weber-Ramanujan’s class invariants for explicit evaluations of $\alpha_n$, $\alpha_{n/9}$, $S_1(q)$, and $S_2(q)$ by modular modular equations to those derived in pervious section.

**Theorem 4.1.** If $g_n$ is defined as in (1.8) respectively, then

\[
\alpha_{9n} = \left( \sqrt{g_n^{24} + 1} - g_n^{12} \right)^2 \left( \sqrt{g_n^8 + 1} - g_n^4 \right)^4 \times \left( \frac{g_n^8 + 1 + \sqrt{g_n^{16} - g_n^8 + 1}}{2} - \frac{\sqrt{g_n^{16} - g_n^8 + 1}}{2} \right)^8, \tag{4.1}
\]

\[
\alpha_{n/9} = \left( \sqrt{g_n^{24} + 1} - g_n^{12} \right)^2 \left( \sqrt{g_n^8 + 1} - g_n^4 \right)^4 \times \left( \frac{g_n^8 + 1 + \sqrt{g_n^{16} - g_n^8 + 1}}{2} + \frac{\sqrt{g_n^{16} - g_n^8 + 1}}{2} \right)^8. \tag{4.2}
\]

**Proof.** Combining (2.1) and (2.3) with $q = e^{-\pi \sqrt{n}}$, then simplifying, we obtain that

\[
\alpha_n = \left( \frac{f(q)}{2^{1/2} q^{1/4} f(-q^4)} \right)^{-8}. \tag{4.3}
\]

Employing (2.4) in Theorem 3.2.2 [9, p.21] along with replacing $q$ by $q^3$, we obtain that

\[
\left( \frac{f(-q^3)}{q^{3/4} f(-q^{12})} \right)^4 + 16 \left( \frac{q^{3/4} f(-q^{12})}{f(-q^3)} \right)^4 = \left( \frac{f(q^3)}{q^{1/4} f(-q^4)} \right)^{12}. \tag{4.4}
\]

Replacing $q$ by $-q$ in (3.10), then applying (4.4), and (4.3) with $q = e^{-\pi \sqrt{n/9}}$, we deduce that

\[
\left( \frac{\alpha_n^2}{\alpha_n \alpha_{n/9}} \right)^{1/8} - \left( \frac{\alpha_n \alpha_{n/9}}{\alpha_n^2} \right)^{1/8} = 2g_n^4.
\]

On solving the above equation and choosing the appropriate root, then we arrive at

\[
\alpha_n \alpha_{n/9} = \alpha_n^2 \left( \sqrt{g_n^8 + 1} - g_n^4 \right)^8. \tag{4.5}
\]

We observed that some representation for $\alpha_n$ in terms of $g_n$. This is given by [21 p.289, Eq.(9.27)]

\[
\frac{1}{\sqrt{\alpha_n}} - \sqrt{\alpha_n} = 2g_n^{12}. \tag{4.6}
\]

Employing (4.6) in (4.5), we conclude that

\[
\alpha_n \alpha_{n/9} = \left( \sqrt{g_n^{24} + 1} - g_n^{12} \right)^4 \left( \sqrt{g_n^8 + 1} - g_n^4 \right)^8. \tag{4.7}
\]

By Theorem 3.2 we obtain that

\[
l^2 - 2^{3/2} \left( \frac{1}{\sqrt{\alpha_n}} - \sqrt{\alpha_n} \right)^{2/3} = 0, \tag{4.8}
\]

where, $l = (\alpha_{9n}/\alpha_{n/9})^{1/8} + (\alpha_{n/9}/\alpha_{9n})^{1/8}$. Now, employing (4.6) in (4.8), then solving for $l$ and choosing positive real root, we deduce that

\[
\left( \frac{\alpha_n}{\alpha_{n/9}} \right)^{1/8} + \left( \frac{\alpha_{n/9}}{\alpha_n} \right)^{1/8} = 2 \left( g_n^8 + \sqrt{g_n^{16} - g_n^8 + 1} \right). \tag{4.9}
\]
On solving the above equation and choosing the appropriate root, we obtain that
\[ \frac{\alpha_{36}}{\alpha_{n/9}} = \left( \frac{g_n^8 + 1 + \sqrt{g_n^{16} - g_n^8 + 1}}{2} - \frac{g_n^8 - 1 + \sqrt{g_n^{16} - g_n^8 + 1}}{2} \right)^{16}. \] (4.10)

By combining (4.7) and (4.10), this completes the proof. □

**Corollary 4.1.** If \( S_1(q) \) and \( g_n \) are defined as in (1.12), and (1.8) respectively, then
\[ S_1 \left( e^{-\pi \sqrt{n/3}} \right) = \frac{1}{\sqrt{2}} \left( \sqrt{g_n^4 + 1 - g_n^{12}} \right)^{1/4} \left( \sqrt{g_n^4 + 1 - g_4^4} \right)^{1/2} \]
\[ \times \left( \sqrt{g_n^8 + 1 + \sqrt{g_n^{16} - g_n^8 + 1}} - \sqrt{g_n^8 - 1 + \sqrt{g_n^{16} - g_n^8 + 1}} \right), \] (4.11)
\[ S_1 \left( e^{-\pi \sqrt{n/3}} \right) = \frac{1}{\sqrt{2}} \left( \sqrt{g_n^4 + 1 - g_n^{12}} \right)^{1/4} \left( \sqrt{g_n^4 + 1 - g_4^4} \right)^{1/2} \]
\[ \times \left( \sqrt{g_n^8 + 1 + \sqrt{g_n^{16} - g_n^8 + 1}} + \sqrt{g_n^8 - 1 + \sqrt{g_n^{16} - g_n^8 + 1}} \right). \] (4.12)

**Proof.** Employing pervious theorem in (1.11), it is not difficult to deduce to our corollary. □

**Theorem 4.2.** If \( S_2(q) \) and \( G_n \) are defined as in (1.13), and (1.8) respectively, then
\[ S_2 \left( e^{-\pi \sqrt{n/3}} \right) = \frac{1}{\sqrt{2}} \left( G_n^{12} - \sqrt{G_n^{24} - 1} \right)^{1/4} \left( G_n^4 - \sqrt{G_n^{8} - 1} \right)^{1/2} \]
\[ \times \left( \sqrt{G_n^8 + 1 + \sqrt{G_n^{16} + G_n^8 + 1}} - \sqrt{G_n^8 - 1 + \sqrt{G_n^{16} + G_n^8 + 1}} \right), \] (4.13)
\[ S_2 \left( e^{-\pi \sqrt{n/3}} \right) = \frac{1}{\sqrt{2}} \left( G_n^{12} - \sqrt{G_n^{24} - 1} \right)^{1/4} \left( G_n^4 - \sqrt{G_n^{8} - 1} \right)^{1/2} \]
\[ \times \left( \sqrt{G_n^8 + 1 + \sqrt{G_n^{16} + G_n^8 + 1}} + \sqrt{G_n^8 - 1 + \sqrt{G_n^{16} + G_n^8 + 1}} \right). \] (4.14)

**Proof.** The proof of our theorem can be obtained by Theorem 4.2 and Theorem 3.4. Since the proof is analogous to Theorem 4.1 and so, we omit the details. □

## 5. Explicit evaluations

After obtaining class invariants \( G_n \), and \( g_n \), then, Theorem 4.1 Corollary 4.1 and Theorem 4.2 can be utilized to calculate several explicit values of Ramanujan’s singular moduli and Ramanujan-Selberg continued fraction. We conclude the present work with following the computations.

**Theorem 5.1.** We have
\[ \alpha_{36} = \left( \sqrt{2} - 1 \right)^4 \left( \sqrt{3} - \sqrt{2} \right)^4 \left( \sqrt{\frac{\sqrt{3} + 3}{2}} - \frac{\sqrt{\sqrt{3} + 1}}{2} \right)^8, \]
\[ \alpha_{4/9} = \left( \sqrt{2} - 1 \right)^4 \left( \sqrt{3} - \sqrt{2} \right)^4 \left( \sqrt{\frac{\sqrt{3} + 3}{2}} + \frac{\sqrt{\sqrt{3} + 1}}{2} \right)^8. \]
Proof. Letting \( n = 4 \), \( g_4 = 2^{1/8} \) [9, Theorem 4.1.2 (i)] and employing this value in (4.1), and (4.2), we evaluate that
\[
\alpha_{36} = (3 - 2\sqrt{2})^2 \left(\sqrt{3} - \sqrt{2}\right)^4 \left(\frac{\sqrt{3} + 3}{2} - \frac{\sqrt{3} + 1}{2}\right), \tag{5.1}
\]
\[
\alpha_{4/9} = (3 - 2\sqrt{2})^2 \left(\sqrt{3} - \sqrt{2}\right)^4 \left(\frac{\sqrt{3} + 3}{2} + \frac{\sqrt{3} + 1}{2}\right). \tag{5.2}
\]
By [2, p.284, Eq.(9.5)], we have
\[
3 - 2\sqrt{2} = \left(\sqrt{2} - 1\right)^2. \tag{5.3}
\]
Employing (5.3) in (5.1), and (5.2), we arrive at desired results. \( \square \)

**Theorem 5.2.** We have
\[
\alpha_{72} = \left(\frac{\sqrt{2} + 2}{2} - \frac{\sqrt{2}}{2}\right)^{16} \left(\frac{3\sqrt{2} + 6}{2} - \frac{3\sqrt{2} + 4}{2}\right), \tag{5.4}
\]
\[
\alpha_{8/9} = \left(\frac{\sqrt{2} + 2}{2} - \frac{\sqrt{2}}{2}\right)^{16} \left(\frac{3\sqrt{2} + 6}{2} + \frac{3\sqrt{2} + 4}{2}\right). \tag{5.5}
\]
Proof. Letting \( n = 8 \), \( g_8 = 2^{1/8} \left(\sqrt{2} + 1\right)^{1/8} \) [9, Theorem 4.1.2 (ii)]. It follows that
\[
\sqrt{g_8^{34} + 1} = 5 + 4\sqrt{2} ; \quad \sqrt{g_8^{8} + 1} = \sqrt{2} + 1 ; \quad \sqrt{g_8^{16} - g_8^{8} + 1} = 3 + \sqrt{2}. \tag{5.6}
\]
Applying (5.6) in (1.1), we deduce that
\[
\alpha_{72} = \left(5 + 4\sqrt{2} - \sqrt{56 + 40\sqrt{2}}\right)^2 \left(\sqrt{2} + 1 - \sqrt{2 + 2\sqrt{2}}\right)^4 \left(\frac{3\sqrt{2} + 6}{2} - \frac{3\sqrt{2} + 4}{2}\right). \tag{5.7}
\]
Now we apply Lemma 9.10 [2] p. 292] with \( r = 5 + 4\sqrt{2} \). Then \( t = (\sqrt{2} + 1)/2 \) and so
\[
5 + 4\sqrt{2} - \sqrt{56 + 40\sqrt{2}} = \left(\frac{\sqrt{2} + 2}{2} - \frac{\sqrt{2}}{2}\right)^4. \tag{5.8}
\]
Further,
\[
\sqrt{2} + 1 - \sqrt{2 + 2\sqrt{2}} = \left(\frac{\sqrt{2} + 2}{2} - \frac{\sqrt{2}}{2}\right)^2. \tag{5.9}
\]
From (5.7), (5.8), and (5.9), we deduce (5.4). Similarly we arrive at (5.5). \( \square \)
Theorem 5.3. We have

\[
S_1(e^{-6\pi}) = \frac{1}{\sqrt{2}} \left( \sqrt{2}-1 \right)^{1/2} \left( \sqrt{3} - \sqrt{2} \right)^{1/2} \left( \sqrt{\frac{\sqrt{3}+3}{2}} - \sqrt{\frac{\sqrt{3}+1}{2}} \right),
\]

\[
S_1(e^{-2\pi/3}) = \frac{1}{\sqrt{2}} \left( \sqrt{2}-1 \right)^{1/2} \left( \sqrt{3} - \sqrt{2} \right)^{1/2} \left( \sqrt{\frac{\sqrt{3}+3}{2}} + \sqrt{\frac{\sqrt{3}+1}{2}} \right),
\]

\[
S_1(e^{-\pi\sqrt{2}}) = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{\sqrt{2}+2}{2}} - \sqrt{\frac{\sqrt{2}}{2}} \right)^2 \left( \sqrt{\frac{3\sqrt{2}+6}{2}} - \sqrt{\frac{3\sqrt{2}+4}{2}} \right),
\]

\[
S_1(e^{-\pi\sqrt{3}}) = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{\sqrt{2}+2}{2}} - \sqrt{\frac{\sqrt{2}}{2}} \right)^2 \left( \sqrt{\frac{3\sqrt{2}+6}{2}} + \sqrt{\frac{3\sqrt{2}+4}{2}} \right).
\]

Proof. The proof of theorem can be obtained by (4.11), and (4.12). Since the proof is analogous to previous theorems, and so, we omit the details. \(\square\)

Theorem 5.4. We have

\[
S_2(e^{-3\pi\sqrt{2}}) = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{\sqrt{5}+3}{4}} - \sqrt{\frac{\sqrt{5}-1}{4}} \right)^2 \left( \sqrt{\frac{3\sqrt{5}+7}{4}} - \sqrt{\frac{3\sqrt{5}+3}{4}} \right),
\]

\[
S_2(e^{-\pi\sqrt{5}}) = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{\sqrt{5}+3}{4}} - \sqrt{\frac{\sqrt{5}-1}{4}} \right)^2 \left( \sqrt{\frac{3\sqrt{5}+7}{4}} + \sqrt{\frac{3\sqrt{5}+3}{4}} \right),
\]

\[
S_2(e^{-\pi\sqrt{7}}) = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{3-\sqrt{7}}{\sqrt{2}}} \right)^{1/2} \left( \sqrt{\frac{\sqrt{21}+5}{2}} - \sqrt{\frac{\sqrt{21}+3}{2}} \right),
\]

\[
S_2(e^{-\pi\sqrt{3}}) = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{3-\sqrt{7}}{\sqrt{2}}} \right)^{1/2} \left( \sqrt{\frac{\sqrt{21}+5}{2}} + \sqrt{\frac{\sqrt{21}+3}{2}} \right).
\]

Proof. Employing the class invariant \(G_5\), and \(G_7\) (see [2, p. 189]) in (1.13), and (1.14), we obtain all the above values. Since the proof is analogous to Theorem 5.3 and Theorem 5.4, and so, we omit the details. \(\square\)

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