Space-charge effects in low-energy flat-beam transforms

Scott B Moroch 1*, Timothy W Koeth 1 and Bruce E Carlsten 2

1 University of Maryland, College Park; 4418 Stadium Dr, College Park, MD 20740, United States of America
2 Los Alamos National Laboratory, Los Alamos, New Mexico 87545, United States of America
* Author to whom any correspondence should be addressed.

E-mail: smoroch@umd.edu, koeth@umd.edu and bcarlsten@lanl.gov

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Abstract

Flat-beam transforms (FBTs) provide a technique for controlling the emittance partitioning between the beam’s two transverse dimensions. To date, nearly all FBT studies have been in regimes where the beam’s own space-charge effects can be ignored, such as in applications with high-brightness electron linacs where the transform occurs at high, relativistic, energies. Additionally, FBTs may provide a revolutionary path to high-power generation at high frequencies in vacuum electron devices where the beam emittance is currently becoming a limiting factor, which is the motivation for this paper. Electron beams in vacuum electron devices operate both at a much lower energy and a much higher current than in accelerators and the beam’s space-charge forces can no longer be ignored. Here we analyze the effects of space charge in FBTs and show there are both linear and nonlinear forces and effects. The linear effects can be compensated by retuning the FBT and by adding additional quadrupole elements. The nonlinear effects lead to an ultimate dilution of the lower recovered emittance and may lead to an eventual power limitation for high-frequency traveling-wave tubes and other vacuum electron devices.

1. Introduction

Introduced about two decades ago, a flat-beam transform (FBT) [1] is a linear transformation that allows arbitrary partitioning of the beam emittances between the two transverse planes, while preserving the product of the emittances (and thus the beam’s overall four-dimensional transverse phase-space volume), where the emittance is an important measure of beam quality and is defined in an rms sense as

\[
\varepsilon_{x,\text{norm}} = \gamma \beta \sqrt{\langle x'^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}
\]

for the horizontal dimension and in a similar manner for the vertical dimension [2]. Here the brackets indicate ensemble averages, the primes refer to axial derivatives \( \left( x' = \frac{dx}{dz} \right) \) and \( \gamma \) and \( \beta \) are the beam’s relativistic mass factor and velocity normalized to the speed of light \( c \), respectively. The beam’s initial emittance can be caused by, for example, the transverse thermal spread of the electron emission at the cathode or by the electron flow around a cathode shadow grid and can increase during beam transport due to nonlinear forces including nonlinear space-charge forces. The emittance from a thermionic gun is typically about 1 \( \mu \)m for beams in the regime we are interested in (nominally 0.25 to 2.5 A at 20 kV, or 5–50 kW of electron beam power), e.g., a cathode using a long-life scandate cathode operating at 2 A cm\(^{-2}\) [3] and at 1200 K with an edge radius of 4 mm would produce a 1-A electron beam with a thermal rms emittance of 0.9 \( \mu \)m. We typically call the beam’s initial thermal emittance plus any growth due to nonlinear fields in the gun diode region the beam’s intrinsic emittance to differentiate it from emittance growths in later stages.

The motivation for this work is to see if an FBT can be applied to future high-power-high-frequency vacuum electron devices (VEDs) such as planar traveling-wave tubes (TWTs) [4, 5] to help with beam-transport stability limitations. There is increasing interest to make high power (nominally kW to 10s of kW peak power) at high frequencies (nominally Ka-band to W-band, or roughly 30 GHz to 100 GHz), often with large bandwidths (10% or more), with compact vacuum electron devices [6–9] for radar, remote sensing, communications, and other
applications. Reference [10] contains an excellent summary of high-frequency applications requiring higher power than currently available that are driving the development of new higher-frequency, higher-power sources and amplifiers.

A common approach to making higher power VEDs at high frequencies is to employ a sheet electron beam [11–14]. (It is important here to point out that, in this paper, a ‘sheet’ beam means a high-aspect ratio elliptical beam. A true rectangular sheet beam is actually harder to efficiently generate than a high-aspect ratio elliptical beam plus it introduces significant transport issues.) While sheet beam devices have already been demonstrated with strong uniform solenoidal magnetic fields [13–17], periodic focusing [18–20] can lead to a more compact system without beam curling [21] and can be more desirable for many applications, especially those requiring the highest possible efficiency and/or compact sizes.

However, for a sheet beam, the beam emittance in the narrow plane has to be sufficiently small to make the transport stable. For the following discussion, we assume the planar beam is narrow in the horizontal, or x, dimension. Assuming a periodic axial magnetic field of the form $B = B_p \sin(2\pi z / \lambda)$ is used to confine the beam to a small horizontal rms size $\sigma_x$, the peak magnetic field needs to be [18]

$$B_p = \left( \frac{cm}{e} \right) \frac{2}{\sigma_y} \left( \frac{1}{\beta \gamma I_A} \frac{1}{\sigma_x} + \frac{e^2}{\sigma_x^2} \right)$$

where $m$ and $e$ are the electronic mass and charge, respectively, and $\sigma_y$ is the vertical rms size. As the frequency in a VED increases, the beam size $\sigma_x$ in the narrow dimension must get smaller to maintain good interaction between the beam and the RF, scaling inversely with the frequency, and the second term depending on the emittance dominates. The peak magnetic field cannot be made arbitrarily high to keep the beam confined at higher and higher frequencies, instead periodic focusing becomes unstable if the magnetic field strength is above an absolute limit of [22]

$$B_{p, limit} = 2 \sqrt{1.3 \frac{mc^2 \gamma}{e\lambda}}$$

Ideally the peak focusing field is about a factor of 2 less than this limit [23]. As there are limitations to how short the period $\lambda$ can be due to physical magnet sizes, equations (2) and (3) tell us that the emittance in the beam’s narrow dimension must decrease as the RF frequency is increased to keep the beam transport stable. Since the beam emittance in the vertical dimension has no such limitation, these arguments motivate the desire to use an FBT in the sheet-beam forming section of a high-frequency planar VED to move excess emittance from the horizontal dimension to the vertical dimension.

An FBT works by applying an axial magnetic field at the location of the cathode of an initially round beam which introduces an initial correlation between the two transverse dimensions. The configuration of a FBT is shown in figure 1. Three skew quadrupole magnets (meaning the quadrupoles are rotated 45° from their usual, or ‘normal’, orientation) remove the correlations, leading to a beam with one final emittance lower than the beam’s intrinsic emittance (i.e., the transverse emittance that would occur in the absence of the FBT) and the other final emittance larger than the beam’s intrinsic emittance. Significantly, the product of the final emittances is equal to the square of the intrinsic emittance, so the overall four-dimensional volume in transverse phase space is conserved. FBTs have been demonstrated [24, 25] and analyzed [26, 27] and simple approaches to determining the skew quadrupole strengths have been published [28, 29]. To date, space-charge effects in FBTs have been ignored with the exception of [27], where the space-charge effects at low energy, well before the FBT

![Figure 1. Flat-beam transform configuration. A solenoid generates an axial magnetic field at the location of the cathode. Once the beam leaves the solenoidal field it encounters three skew quadrupole which remove the x-y correlations. Image from [28].](image-url)
skew–quadrupole section at high energy, were studied. In this paper, alternatively, we consider the nature of the space-charge effects within the skew quadrupole section itself where the beam can be highly asymmetric.

An important concept needed for understanding the beam physics behind an FBT is the beam’s eigen-emittances. Eigen-emittances are conserved under symplectic linear transformations [26], like those found in ideal accelerator beam lines. In the absence of coupling between the transverse dimensions, the real transverse rms emittances (as defined by equation (1)) are equal to the beam’s eigen-emittances. If coupling is present, the real transverse rms emittances are larger than the eigen-emittances. Importantly, linear forces coupling the two transverse dimensions (as in the FBT skew–quadrupole section) can change the transverse rms emittances while the eigen-emittances remain unchanged. Reference [30] reviews the eigen-emittance theory and its applications to accelerators in great detail.

Specifically in an FBT, the axial magnetic field at the location of the electron-beam cathode causes the two transverse eigen-emittances to separate, one being larger than the beam’s intrinsic emittance and the other being smaller. Using \( \mathbf{L} \) to represent a normalized angular momentum [26],

\[
\mathbf{L} = \frac{eB_{\text{cath}}}{2mc} \sigma_{\text{cath}}^2
\]

where \( \sigma_{\text{cath}} \) is the beam’s transverse \((x \text{ or } y)\) rms size at the cathode and \( B_{\text{cath}} \) is the axial magnetic field at the location of the cathode, the two eigen-emittances are (for the case where the angular momentum contribution is much larger than the intrinsic emittance, \( \mathbf{L} \gg \varepsilon_0 \) [26])

\[
\varepsilon_{\text{eig},-} = \frac{\varepsilon_0^2}{2|\mathbf{L}|}
\]

\[
\varepsilon_{\text{eig},+} = 2|\mathbf{L}|.
\]

The actual, kinematic, beam rms emittances (as defined in equation (1)) due to the axial field on the cathode are

\[
\varepsilon_{\text{trans}} = \sqrt{L^2 + \varepsilon_0^2}
\]

once the beam reaches a field-free region, which are much larger than the beam’s actual intrinsic emittances \( \varepsilon_0 \).

The function of the FBT skew quadrupoles is to remove all the beam’s transverse \( x-y \) correlations (such that \( \langle xy \rangle, \langle xy' \rangle, \langle x'y \rangle, \langle x'y' \rangle \) are made to vanish after the skew–quadrupole section) and the \( x \) and \( y \) emittances evolve into the eigen-emittances in equations (5) and (6). This is exactly what we want for a high-power, high-frequency VED as then the \( x \) emittance can be made small at the expense of the \( y \)-emittance and we can achieve stable beam transport at higher frequencies.

The key assumptions in the simple FBT analysis are that the beam is round, has uniform density, and is at a waist (i.e., where the correlations \( \langle xx' \rangle \) and \( \langle yy' \rangle \) vanish) at the location of the first skew quadrupole and which is outside the magnetic field region. The beam is also assumed to have been round with uniform density at the location of the cathode. To date, FBTs have only been demonstrated on RF photoinjectors, with a typical final emittance ratio \( \varepsilon_{\text{eig},+}/\varepsilon_{\text{eig},-} \) of about 100. Space-charge effects in FBTs have been ignored with the exception of [27], where the space-charge effects were considered as the beam was accelerated in a photoinjector and before the skew–quadrupole transform section. To emphasize the difference in the space-charge in the transform region between a nominal 20-keV, 1-A beam for a VED and a 20-MeV, 20-A beam from a photoinjector, such as used in [27], the 20-kV beam has up to 6 orders of magnitude higher beam perveance than the 20-MeV beam, where the beam perveance is defined as

\[
P = \frac{2I}{I_4 \gamma^3 \beta^3}
\]

is a measure of the relative importance of space-charge effects, where \( I \) represents the beam current and \( I_4 \) is about 17 kA. Thus, the space-charge effects in an FBT for a VED need to be re-examined.

Since an FBT is a linear transform, it is reasonable to expect that the linear parts of the beam’s space-charge forces (including those that lead to additional \( x-y \) correlations) do not increase the eigen-emittances and their effect on the final beam emittances can be removed by retuning the FBT skew quadrupoles, while the nonlinear parts of the space-charge forces lead to increases in the eigen-emittances and also in the final beam emittances. Unfortunately, we see the form of these nonlinear forces will tend to increase the lower eigen-emittance more than the upper eigen-emittance and will lead to an ultimate limitation on using an FBT for a high-power, high-frequency VED.

The three major contributions of this paper are: (1) identification of the form of the linear space-charge effects in an FBT; (2) verification that the linear space-charge effects in an FBT can be completely removed with a combination of retuning the skew quadrupole strengths and by rotating the middle skew quadrupole; and (3) evaluation of the residual emittance dilution due to the nonlinear space-charge effect from the nonuniformity of
the beam density profile as the beam becomes squeezed in one dimension within the skew quadrupole section. Nominal FBT design equations (for zero space-charge forces) are presented in section 2, along with zero-current simulations verifying the FBT concept. The effect of linear space-charge forces is considered in section 3. In section 4 we evaluate the residual emittance dilution from the nonlinear component of the space-charge force for a 20-keV, 250-mA beam and compare it to a simple model. This study shows that, even with space-charge effects that lead to an ultimate limitation, FBTs can be used to significantly lower the emittance in one plane for low-voltage beams with currents relevant for planar high-frequency vacuum electron devices.

2. FBT design and zero-current simulations

In this section, we introduce the analytical solution for the quadrupole strengths in a FBT, neglecting space-charge forces, and present our nominal FBT design parameters and values. We proceed to model the beam line in A Space Charge Tracking Algorithm (ASTRA) to verify the ability to recover the eigen-emittances (i.e. eliminating the x-y correlations in the beam). This was conducted as an initial verification of the simulation tool and was included to provide baseline results for comparison with the linear and nonlinear space charge effects included in later sections.

2.1. FBT design equations

A solution for the skew quadrupole strengths can be found in [28] and [29], where the beam is at a waist at the location of the first skew quadrupole. Using

$$ A = \frac{L}{\beta_0 \gamma_0 \sigma_z^2} $$  \hspace{1cm} (9)



to represent the normalized angular momentum of the beam at the location of the first skew quadrupole,

$$ C_i = \frac{e}{\gamma_0 \beta_0 mc} \int_{-\infty}^{\infty} B_{i,j}'dz $$  \hspace{1cm} (10)


to represent the fields in the skew quadrupoles (indexed $i = 1, 2, 3$), and $D_{1-2}$ and $D_{2-3}$ to represent the distances between the centers of the first and second quadrupole and the second and third quadrupole, respectively, the skew quadrupole field strengths for the FBT are given by [28]:

$$ C_{2,0} = \frac{D_{2-3}}{D_{1-2} + D_{2-3}} - 2C_{2,0}A = \frac{1}{D_{1-2}D_{2-3}} = 0 $$  \hspace{1cm} (11)

$$ C_{1,0} = -\frac{1}{2} \left( \frac{1}{C_{2,0}D_{1-2}D_{2-3}} + \frac{C_{2,0}D_{2-3}}{D_{1-2} + D_{2-3}} \right) $$  \hspace{1cm} (12)

$$ C_{3,0} = -\frac{C_{2,0}D_{1-2}}{D_{1-2} + D_{2-3} - C_{2,0}D_{1-2}D_{2-3}} $$  \hspace{1cm} (13)

In equations (11)–(13), the ‘0’ subscripts represent solutions for zero current, and where we are using the normalization that a positive field amplitude means the focusing is perpendicular to the x-y diagonal looking upstream along the beam. Note that there are two solutions for $C_{2,0}$ which then also lead to two different solutions for $C_{1,0}$ and $C_{3,0}$. However, one solution leads to much smaller quadrupole field strengths which is preferable. With $A > 0$ and using the solution with the positive sign in front of the square root in the quadratic solution led to $C_{2} > 0$ and the smaller eigen-emittance ends up in the horizontal dimension. Table 1 describes the nominal FBT design conditions we are considering in this paper (an intrinsic emittance of 1 $\mu$m and beam radius of 1 mm at the cathode and at the location of the first skew quadrupole, and a 500 G axial magnetic field at the cathode).

We can additionally predict the nature of the nonlinear space-charge forces on the beam’s final emittances by considering the form of the nonlinear space-charge forces. Since the beam has low energy and a high-aspect ratio when it travels through the skew-quadrupole section, we expect the main effect to be an increase in the beam’s nonlinear rms divergence in the narrow dimension, and a negligible effect in the wide dimension.

By explicitly solving for the eigen-emittances as we increase one of the initial beam emittances [31], we find that if the emittance is increased by an increase in the beam’s rms divergence (keeping the rms beam size the same), the increase in the eigen-emittances is predominantly in the lower eigen-emittance. Conversely, if the emittance is increased by an increase in the rms beam size (keeping the beam’s rms divergence the same), the increase in the eigen-emittances is predominantly in the upper eigen-emittance. While there are additional beam dynamics within the skew-quadrupole section (e.g., the beam transverse correlations are being modified by the skew quadrupoles), this suggests the nonlinear space-charge force will largely degrade the lower
eigen-emittance and which will lead to an ultimate limitation in the usefulness of an FBT for a low-energy beam, which we will confirm in section 4.

2.2. Zero-current simulations

We use A Space Charge Tracking Algorithm (ASTRA) \([32]\) to model the electron beam. ASTRA is fully nonlinear, with an advanced multipole space-charge routine fully capable of accurately modeling the nonlinear space-charge forces from an elliptical beam at any orientation. We verified the simple FBT theory with ASTRA for the zero-current case with the nominal FBT parameters shown in table 1. The actual skew-quadrupole strengths in the ASTRA simulations had some minor differences due to simulation constraints. First, we needed to use a physical solenoid representation to provide the axial magnetic field at the location of the beam cathode. The solenoid’s length led to the beam diverging before the first skew quadrupole which had to be placed sufficiently downstream to be in a field-free region. As a result, we added a second, focusing, solenoid centered at \(z = 2.5 \text{ cm}\), as seen in figure 2, to bring the beam to a waist at the location of the first skew quadrupole, starting at \(z = 5 \text{ cm}\). The solenoidal field on axis \((x = y = 0)\) was modeled in POISSON SUPERFISH \([33]\) and imported into ASTRA which computed the off-axis field components to third order. Since the waist radius at the first skew quadrupole is slightly different from 1 mm, the skew-quadrupole strengths needed to recover the eigen-emittances are slightly different than those in table 1. (To quantify this effect, a 1% increase in the beam waist radius at the location of the first skew quadrupole reduces the strength of the first skew quadrupole by 1.6%, increases the strength of the second by 2.0%, and increases the strength of the third by 2.1%) The ASTRA solution quantitatively agreed with these variations from the nominal beam size.

The ASTRA results of modeling the zero current FBT are shown in figure 3, demonstrating the recovery of the predicted eigen-emittance values. Note the eigen-emittance variations where there is appreciable axial magnetic field which are due to our eigen-emittance algorithm using the beam’s kinetic momentum as opposed

![Figure 2. Plot of the axial magnetic field profile before the FBT region in the ASTRA simulation.](image)

| Table 1. Nominal FBT parameters. |
|----------------------------------|
| Beam energy | 20 keV |
| Beam current | 250 mA |
| Radius at cathode | 1 mm |
| Radius at first skew quad | 1 mm |
| Magnetic field at cathode | 500 G |
| Intrinsic emittance | 1.00 \(\mu\text{m}\) |
| Angular momentum term \(L\) | 3.66 \(\mu\text{m}\) |
| Effective quadrupole lengths | 0.5 cm |
| Quadrupole spacing | 5 cm |
| \(B_{Q1}\) | \(-5.18 \text{T m}^{-1}\) |
| \(B_{Q2}\) | \(0.365 \text{T m}^{-1}\) |
| \(B_{Q3}\) | \(-0.186 \text{T m}^{-1}\) |
| Lower eigen-emittance | 0.133 \(\mu\text{m}\) |
| Upper eigen-emittance | 7.46 \(\mu\text{m}\) |
| Emittance ratio | 53.8 |
to its canonical momentum. Thus, changes in the axial magnetic field lead to changes in the \(xy\) and \(yx\) correlations which show up as changes in the eigen-emittances. The dips occur where the axial magnetic field, \(B_z\), becomes equal to \(B_{cath}\left(\frac{z}{z_{cath}}\right)^2\) and the beam’s angular momentum vanishes (and so do the correlations). The eigen-emittances also jump up at the start of the skew quadrupoles, as they are calculated using a finite slice of the beam with non-zero longitudinal spread at an instant in time. The eigen-emittances subsequently reduce after the full slice has exited the quadrupole.

3. Effects of linear space-charge forces

Since linear forces, even ones coupling \(x\) and \(y\), don’t change the eigen-emittances, the addition of linear space-charge forces shouldn’t affect the final emittances after the FBT section. We expect we can recover the eigen-emittances by retuning the FBT quadrupoles. Here we show that is in fact true, and that only the second FBT quadrupole needs to be retuned—specifically its strength needs to be adjusted and it needs to be rotated so it is no longer perfectly skew. The rotation allows it to generate a normal quadrupole field in addition to the skew quadrupole field.

3.1. Linear space-charge theory

Next, we consider the space-charge forces of an ideal, uniformly filled, elliptical beam with some tilt, as shown in figure 4. The following expressions are intended to be representative of the beam’s linear space-charge forces within the FBT section.
A uniform filled ellipse has space charge forces,
\begin{align}
F_{x,sc} &= \frac{eI}{4\pi \varepsilon_0 \beta \gamma^2 c \sigma_x (\sigma_x + \sigma_y)} \hat{x} \\
F_{y,sc} &= \frac{eI}{4\pi \varepsilon_0 \beta \gamma^2 c \sigma_y (\sigma_x + \sigma_y)} \hat{y}
\end{align}
(14)

in the rotated coordinate frame \((\tilde{x}, \tilde{y})\) where the beam is an upright ellipse \([34]\) (figure 4) with \(\sigma_x\) and \(\sigma_y\) representing the rms horizontal and vertical beam sizes in that frame and \(\varepsilon_0\) is the permittivity of free space. The rotation angle \(\theta\) between the \((x, y)\) and \((\tilde{x}, \tilde{y})\) frames, in terms of the second moments of the \((x, y)\) coordinate system, is given by
\begin{equation}
\theta = \frac{1}{2} \tan^{-1} \left( \frac{2xy}{\sigma_x^2 - \sigma_y^2} \right)
\end{equation}
(16)

The space-charge forces in the \((x, y)\) coordinate system are then
\begin{align}
F_{x,sc} &= xk \left( \cos^2 \theta \frac{\sigma_x}{\sigma_y} + \sin^2 \theta \frac{\sigma_y}{\sigma_x} \right) + yk \sin (2\theta) \left( \frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2} \right) \\
F_{y,sc} &= yk \left( \cos^2 \theta \frac{\sigma_y}{\sigma_x} + \sin^2 \theta \frac{\sigma_x}{\sigma_y} \right) + xk \sin (2\theta) \left( \frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2} \right)
\end{align}
(17)

where \(k = \frac{eI}{4\pi \varepsilon_0 \beta \gamma^2 c (\sigma_x + \sigma_y)}\). The first terms in parentheses are non-coupling and the second terms couple the transverse dimensions. Note the coefficients of the first, non-coupling, term are a mixture of a symmetric focusing (or defocusing) force and an asymmetric, or normal, quadrupole force (i.e., focusing in one plane and defocusing in the other) and do not lead to a change in the transverse rms emittances. The second terms, coupling \(x\) and \(y\), lead to change in the rms emittances. Importantly, these terms are exactly of the form of the coupling forces from a skew quadrupole, and, as such, we should be able to compensate them by a change in the skew quadrupole strengths.

A more detailed analysis of these forces \([31]\), show the following two effects:

1. For the symmetric focusing (defocusing) space-charge force there is always an FBT solution with no modification of the skew quadrupoles positions or rotation angles needed; in other words, the symmetric part of the non-coupling space-charge force can be simply tuned out with the existing skew quadrupoles by changing their strengths a bit, and, in particular, only the middle skew quadrupole needs to be retuned.

2. For the case of an asymmetric quadrupole space-charge impulse, there may not be an FBT solution by simply retuning the skew quadrupole strengths term, and a normal quadrupole field component may be required to be added at the location of the middle skew quadrupole.

Thus, we anticipate that the second skew quadrupole may need to be rotated in addition to being retuned. Importantly, (1) above also suggests we can add symmetric focusing as needed within the FBT to ensure good beam transport without affecting the ability to recover the eigen-emittances, which could be accomplished either with another solenoid or multiple quadrupoles.

An additional small space-charge-induced quadrupole field may be introduced by the image charges in the beam-pipe wall \([35]\). This additional quadrupole force from the image charge has magnitude < 1% of the quadrupole component of the space-charge force shown in equation (18) for an elliptical beam with a very high aspect ratio that extends ½ of the way to the beam-pipe wall. The magnitude of this force grows to about 4% of that of the quadrupole component of the space-charge force if the elliptical beam extends ¾ of the way to the beam-pipe wall. Of course this effect can be compensated by a relatively small adjustment on the rotation and strength of the second skew quadrupole. The sextupole and higher order fields would be more problematic, but they drop off quickly (the magnitude of the sextupole and octupole forces are 0.02% and 0.001% of the space-charge forces, respectively, for the case the elliptical beam reaches ½ the way to the beam-pipe wall and 0.6% and 0.1%, respectively, for the case the elliptical beam reaches ¾ of the way to the beam-pipe wall).

### 3.2. Simulations with linear space charge forces

Here we use the code PUSHER \([28]\), which only includes linear forces on the particles (it uses the Lawson linear space-charge approximation for an elliptical beam \([34]\), in the form of equations (14) and (15)), to study the effects of the linear component of the beam’s space-charge forces. PUSHER is a simple time-stepping, particle-pushing code following an infinitely thin axial slice of the beam. In figure 5, we plot the achieved lower emittance...
along with the lower eigen-emittance for a 250-mA beam at the end of the third skew quadrupole, as a function of the amount of a normal quadrupole field added to the second skew quadrupole field, normalized to the skew quadrupole field magnitude. In this plot, all the skew quadrupole strengths are varied for each point in this plot to achieve the lowest possible final horizontal emittance, while keeping the ratio of the normal quadrupole field to the skew quadrupole field at the location of the second skew quadrupole fixed. The difference between the red and blue curves allows us to see that the linear components of the space-charge force are in fact removed with proper retuning of the skew quadrupoles and an effective rotation of the second skew quadrupole. It is worth pointing out that these simulations included all components of the space-charge forces as shown in equations (17) and (18): a symmetric non-coupling force, an asymmetric non-coupling force, and the symmetric coupling term. The coupling space-charge force and the symmetric part of the non-coupling space-charge force are both compensated by a change in the second skew quadrupole strength, while the asymmetric part of the non-coupling space-charge force can only be compensated by the addition of a normal quadrupole field. It is also important to note that, for this level of space charge, the second skew quadrupole needs to be nearly rotated a full 45° to eliminate the effects of the space charge (i.e., it ends up being essentially a slightly rotated normal quadrupole).

4. Effect of non-linear space charge

In this section, we consider the effects of non-linear space charge forces, with a specific focus on the skew-quadrupole region where the beam is highly asymmetric. We see the non-linear space-charge forces do indeed degrade the lower eigen-emittance and recovered lower rms emittance as we predicted. However, a significant reduction in the lower rms emittance is still possible with a FBT (a factor of four), for the parameters from table 1.

4.1. Non-linear space-charge simulations

We used ASTRA to include the nonlinear space-charge effects in the beam transformer. ASTRA includes a full 3D space charge algorithm, in which the bunch is modeled as a distribution of macro-charges [36]. The bunch is broken into Cartesian cells of constant charge density and Poisson’s equation is solved in the rest frame of the bunch, using a Fast-Fourier Transform method. We used a long 0.1-ns bunch to suppress longitudinal effects, consisting of $10^8$ macro particles in ASTRA. To emulate a DC beam (as present in VEDs) the emittances and transverse charge density profiles were calculated only for a central slice (1%) of the bunch. Therefore, despite the presence of longitudinal space charge effects in the overall simulated bunch, they are suppressed in the central slice. In the cathode region, ASTRA’s cylindrically symmetric space-charge solver is employed, and the bunch is split into radial and longitudinal cells. In the FBT section, the beam becomes highly asymmetric and requires ASTRA’s 3D Cartesian-based algorithm, which uses free-space boundary conditions [37]. The bunch is split into a $32 \times 32 \times 32$ Cartesian grid, corresponding to approximately 3 macro-charges per cell for computing the space-charge fields. This resolution is sufficient to see the effects of the nonlinear space-charge forces.
As described in the previous section, the beam in an FBT will become rotated with respect to the normal coordinate axis as in figure 4. Given that ASTRA employs a non-rotated Cartesian grid, we performed several test simulations to verify that the computed space-charge fields of a beam in the rotated frame agreed with the fields from a beam aligned to ASTRA’s axis. More specifically, we compared the growth in $\sigma_x$, $\sigma_y$, $\varepsilon_x$ and $\varepsilon_y$ for rotated and non-rotated, asymmetric beams with a Gaussian charge density profile sent through a drift in ASTRA. With a $32 \times 32 \times 32$ Cartesian grid, we found excellent agreement between the simulations, thus verifying ASTRA’s ability to handle the space-charge forces in an FBT.

A python-based wrapper was employed to vary the skew quadrupole strengths and rotation angles to search for the best recovery of the eigen-emittances. Contrary to the previous simulations, the non-linear forces lead to strong coupling between the quadrupole strengths and the beam eigen-emittances. However, as before, recovery of the lower eigen-emittance was possible at the correct second skew quadrupole strength and rotation, as shown in figure 6 where we superimposed a quadrupole field (equivalent to rotation of the skew quadrupole). Once a solution for the quadrupole parameters was found, the simulation was run again with a $64 \times 64 \times 64$ Cartesian grid and $5 \times 10^6$ macro-particles. This higher-resolution simulation showed that any errors introduced by the lower resolution ($32 \times 32 \times 32$) grid utilized during the optimization procedure were negligible. The optimized solution for the 250-mA beam current as a function of axial position is shown in figure 7. The emittance growth is dominated in the skew-quadrupole region, with the lower (0.25 $\mu$m) and upper (8.62 $\mu$m) emittances growing by 60% and 13%, respectively, compared to the zero-current case. A direct comparison of the evolution of the lower eigen-emittance in the zero-current case and the 250-mA case is shown in figure 8.

It is worth noting three significant artifacts observed in the lower eigen-emittance of figures 7 and 8. As seen in the zero-current simulations (figure 3), the eigen-emittances change in the solenoid fields in the cathode region, as they are calculated based on kinetic, not canonical momentum. As observed in the zero current case,
the eigen-emittances increase in the quadrupoles as the calculation is taken over a finite beam slice with non-zero longitudinal spread. However, dissimilar to the zero-current case, the eigen-emittances undergo explicit jumps after quadrupoles 2 and 3, at $z = 10.5$ cm and $z = 15.5$ cm respectively. The jump following the second skew-quadrupole is due to its $+15^\circ$ rotation with respect to the standard skew orientation, to introduce the necessary quadrupole field for compensating the linear space-charge effects. While there are no actual nonlinear forces introduced by a rotated ideal (i.e., hard-edged) quadrupole, a rotated quadrupole using the model in ASTRA introduces nonlinear cross terms in the fringe fields that lead to an abrupt increase in the lower eigen-emittance. This nonlinear correlation necessitates the rotation of the third quadrupole in the opposite direction ($-8^\circ$ in this optimized solution), leading to a corresponding drop in the eigen-emittance. The final lower eigen-emittance is the same as if we were able to cleanly superimpose the skew quadrupole and a normal quadrupole field as in the theory.

Most importantly, there is a gradual increase in the lower eigen-emittance between the second and third quadrupoles. This increase is attributed to the characteristic behavior of an FBT, in which the beam is squeezed in the narrow plane, resulting in a Gaussian charge density profile, as predicted by [38]. To exemplify this, the transverse beam profile in the rotated-frame of the beam, mid-way between the second and third quadrupoles is plotted in figure 9(a), where we are calling the narrow dimension $x$. It is evident that the charge density is non-uniform in the narrow-dimension and much more uniform in the wider-dimension.

Despite these non-ideal effects, for these parameters (a $20$ keV, $250$ mA, $1$ $\mu$m emittance beam), we were able to reduce the emittance of a planar beam in its narrow dimension by a factor of 4, which, from equation (2) indicates a possible decrease in the narrow size of the beam by a factor of 2.5 (or, essentially, an increase in a VED frequency by a factor of 2.5 with the same beam power). This is a significant result.

4.2. Estimate of the eigen-emittance growth from nonlinear space charge

As described, we attribute the growth in the lower eigen-emittance to the nonlinear space-charge forces, which result from the beam’s nonuniform density in its narrow dimension, as it gets squeezed in the FBT. To show this squeezing, the rms beam sizes in the rotated frame of the beam (as in figure 4), are plotted in figure 9(d). To estimate this emittance growth, we proceed by formulating an analytical expression for the emittance growth of a beam with nonuniform, gaussian charge density in the narrow-dimension ($x$) and uniform density in $y$ and $z$-dimensions. Motivated by the discussion at the end of section 2.1, we can check if the numerically calculated growth in the lower eigen-emittance is consistent with a characteristic amount of change in the beam’s divergence in its narrow dimension.

Using the same approach as in [27], we find that the change in the lower eigen-emittance (to be added in quadrature with the beam’s zero-current lower eigen-emittance value) due to nonlinear space-charge forces to be [39],

$$\Delta \varepsilon_{nl} = \frac{\pi - 3}{6} \frac{1}{\beta^2 \gamma^2} \frac{z_0 \sigma_z}{\Delta y} \frac{I}{I_A}$$

(19)

where $z_0$ is a characteristic length of the FBT and $\Delta y$ is the vertical height of the beam in the larger dimension, about four times $\sigma_y$ since the beam is largely uniform in that dimension. Looking at figure 9(d), we can estimate this growth over a length, $z_0$, of about 10 cm with the beam size of $\Delta y \approx 18$ mm and $\sigma_z \approx 1$ mm (using the beam sizes at $z = 12$ cm). For a 250-mA beam current, this emittance growth is about 0.16 $\mu$m,
corresponding to a final lower eigen-emittance of 0.21 $\mu$m when added in quadrature with the predicted eigen-emittance of 0.133 $\mu$m for the zero-current case. This is reasonably consistent with the actual simulated lower eigen-emittance of 0.25 $\mu$m (figure 6).

We can use this result to estimate at what beam current this FBT design now longer improves the narrow dimension beam emittance. Scaling from an emittance growth of about 0.16 $\mu$m for 250 mA and an intrinsic emittance of 1 $\mu$m, we find the induced emittance growth from the non-linear space charge will approach the intrinsic emittance for a beam current of about 1.5 A, indicating that would be the practical upper current limit. It is important to note that this value may be increased higher for an optimized FBT design (e.g., if the beam narrow dimension can be kept larger on average with focusing elements which we have shown would not affect our ability to achieve the same lower eigen-emittance).

**5. Conclusions**

In this paper, we have extended the existing FBT theory to include space-charge effects. Our motivation for this was to determine if FBTs could be applied to future low-voltage vacuum electron devices that require high-brightness sheet beams. We first analyzed the effects of the linear space-charge components which prevent a standard FBT from completely recovering the beam eigen-emittances. We uncovered, analytically, the ability to remove the emittance growth introduced by these linear space-charge forces in an FBT by re-tuning the skew-quadrupole strengths and introducing a normal, superimposed quadrupole field. Simulations, conducted with a particle-pushing code (PUSHER), supported these results, and demonstrated the ability to fully recover the beam eigen-emittances. To study the effects of nonlinear space-charge, we modeled the beamline in ASTRA, which includes a full 3D space-charge algorithm. These high-fidelity simulations showed that we were able to reduce the emittance of a planar beam in its narrow dimension by a factor of 4, which could translate to a VED frequency increase of a factor of 2.5 while maintaining the same power. This represents a potentially significant breakthrough over current limitations in reaching high power at high frequency.

These simulations also indicated that, in the skew-quadrupole region, the charge-density profile in the narrow-plane of the beam became Gaussian which led to nonlinear space-charge forces and a growth of the lower eigen-emittance of about $\sim$60% for our nominal parameters. We conclude that this emittance growth, caused by the squeezing of the beam as it rotates, represents a fundamental limiting factor of FBTs and will need...
to be considered when approaching higher power and higher frequency sheet-beam traveling wave tubes. It may be possible to mitigate this nonlinear effect by minimizing the squeezing by adding additional quadrupole focusing elements. While an FBT may not be a ‘silver bullet’ allowing arbitrary frequency increase of future VEDs by itself, these results indicate that it may be a useful part of a high-power high-frequency VED design strategy.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

ORCID iDs

Scott B Moroch @ https://orcid.org/0000-0002-3339-5083
Timothy W Koeth @ https://orcid.org/0000-0002-0082-0514
Bruce E Carlsten @ https://orcid.org/0000-0001-5619-907X

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