Mathematical Physics

Positive gravitational energy in arbitrary dimensions

Énergie gravitationnelle positive en dimension quelconque

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Abstract

We present a streamlined, complete proof, valid in arbitrary space dimension \( n \), and using only spinors on the oriented Riemannian space \((M^n; g)\), of the positive energy theorem in General Relativity.

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Résumé

On démontre un théorème d’énergie gravitationnelle positive en dimension quelconque utilisant seulement des spineurs liés au groupe \( \text{Spin}(n) \) sur une section d’espace Riemannien \((M^n, g)\).

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Un espace-temps Einsteinien est une variété Lorentzienne \((M^{n+1}, g)\) qui satisfait les équations d’Einstein

\[
R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = T_{\alpha\beta};
\]

on supposera qu’il satisfait la condition d’énergie dominante, \( u_\alpha T^\alpha\beta \) temporel pour tout vecteur temporel \( u \). Sur chaque section spatiale \( M^n \) la métrique induite \( g \) et la courbure extrinsèque \( K \) satisfont les contraintes qui s’écrivent dans un repère orthonormé d’axes \( e_i \) tangents et \( e_0 \) orthogonal à \( M^n \)

\[
R_{0j} \equiv \partial_j K^h_i - D_h K^k_j = T_{0j},
\]

\[
S_{00} \equiv \frac{1}{2} \{ R - |K|^2 + (\text{tr} K)^2 \} = T_{00}.
\]

On suppose que \( M^n \) est l’union d’un compact \( W \) et d’un nombre fini \( N \) d’ensembles \( \Omega_i \), appelés bouts (ends), difféomorphes au complément d’une boule de \( R^n \). On utilise une partition lisse \( f_i, f_K \) de l’unité sur \( M^n \) de supports contenus dans un \( \Omega_i \) ou un ouvert \( W_K \) difféomorphe à une boule de \( R^n \), l’union (finie) des \( W_K \) recouvrant \( W \). Tous ces ouverts sont munis de coordonnées locales \( x^i \) et de la métrique euclidienne \( \epsilon \equiv \eta_{ij} dx^i dx^j \). Un tenseur \( u \) sur \( M^n \) est une somme de tenseurs \( u_i = f_i u \), \( u_K = f_K u \). On utilise la métrique euclidienne pour définir les normes de ces tenseurs. Un espace de Banach \( C^p \) ou de Hilbert \( H_{p,s} \) d’un tenseur \( u \) sur \( M^n \) est défini à l’aide du \( sup \) ou de la somme des normes des tenseurs.

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Les choix différents de partition de l’unité donnent des normes équivalentes. La variété Riemannienne \((M^n, g)\) est dite asymptotiquement euclidienne (A.E.) si
\[
h := g - g \in H_{s,\delta} \cap C_{n-2}^1, \quad s > \frac{n}{2} + 1, \quad \frac{n}{2} - 2 > \delta > -\frac{n}{2},
\]
où \(g\) est une métrique lisse identique dans chaque \(\Omega_l\) à la métrique euclidienne \(e\).

La définition des masses gravitationnelles \(m_l\) et des moments \(p_l\), dits ADM, d’un espace-temps muni d’une section A.E. \((M^n, g)\) et de \(K \in H_{s-1,\delta+1}\) provient de la formulation Hamiltonienne des équations d’Einstein, on a dans \(\Omega_l\)
\[
m_l := \lim_{r \to \infty} \frac{1}{2} \int_{S_r^{n-1}} \left( \frac{\partial g_{ij}}{\partial x^l} - \frac{\partial g_{jl}}{\partial x^l} \right) n_i \mu_e, \quad r := \left\{ \sum (\alpha')^2 \right\}^{\frac{1}{2}},
\]
\[
p_l^h := \lim_{r \to \infty} \int_{S_r^{n-1}} p^{ih} n_i \mu_e, \quad p^{ih} := (K^{ih} - g^{ih} \text{tr} K).
\]

On reprend l’idée spinorielle de Witten pour démontrer, mais en utilisant seulement un spineur sur \(M^n\) lié à l’algèbre de Clifford \(Cl(n)\), la positivité de la masse d’un espace-temps Einsteinien quand \(R > 0\), donc sous la condition d’énergie dominante, quand \(M^n\) est une hypersurface maximale. Une formulation simple liée au moment \(P\), qui ne fait intervenir que la même sorte de spineurs sur \(M^n\), permet de montrer que \(m_l \geq |p_l|\) donc \(m \geq |p|\) sans condition sur \(R\) ni autre hypothèse sur les sources. Les démonstrations reposent sur un théorème d’existence pour la solution d’une équation de Dirac complétée, elliptique, sur une variété asymptotiquement euclidienne.

1. Introduction

The most elegant and convincing proof of the positive energy theorem is by using spinors, as did Witten\(^1\) in the case \(n = 3\) inspired by heuristic works of Deser and Grisaru originating from supergravity. The aim of this Note is to present a streamlined, complete proof, valid in arbitrary space dimension \(n\), and using only spinors on the oriented Riemannian space \((M^n; g)\), without invoking spacetime spinors.

We first give the notations and the definitions we use.

2. Definitions

2.1. Asymptotically Euclidean space

\(M^n\) is a smooth manifold union of a compact set \(W\) and a finite number of sets \(\Omega_l\), diffeomorphic to the complement of a ball in \(R^n\). One covers \(W\) by a finite number of open sets \(W_k\) each diffeomorphic to a ball in \(R^n\). We denote by \(\alpha^i\) local coordinates for a domain \(\Omega_l\) or \(W_k\). We set \(r := |\sum (\alpha')^2|^{\frac{1}{2}}\) and take \(\alpha_0 > 0\) such that \(\Omega_l := \{ r > \alpha_0 \}\). \(\Omega_l \cap W_k = \emptyset\) if \(r < 2\alpha_0\). We consider a preparation of \(M^n\), i.e. a smooth partition of unity, \(f_l, f_k, f_k\) with support in \(W_k\), \(f_l\) support in \(\Omega_l\) and \(f_l = 1\) for \(r > 2\alpha_0\). The Riemannian metric \(g\) is continuous and uniformly bounded above and below in each \(\Omega_l\), \(W_k\) by constant positive definite quadratic forms. A tensor field \(u\) on \(M^n\) is written as \(u = \sum u_l + \sum u_k\) with \(u_l := f_l u\), \(u_k := f_k u\). Norms on spaces of tensor fields are defined through their components in the \(\Omega_l\), \(W_k\), each endowed with the Euclidean metric \(e := \eta_{ij} dx^i dx^j = \sum (dx^i)^2\), with pointwise norm \(|.|\) and volume element \(\mu_e\). We use the Banach and Hilbert spaces \(C^k_p\) and \(H_{s,\delta}\) with norms
\[
\|u\|_{C^k_p} = \sup_{l,k} \left\{ \sup_{\Omega_l} \left| D^k u_l \right|, \sup_{W_k} \left| D^k u_k \right| \right\}, \quad D^k := \frac{\partial^k}{\partial x^1 \ldots \partial x^k},
\]
\[
\|u\|_{H_{s,\delta}}^2 = \sum_{l=1}^{\infty} \int_{\Omega_l} \int_{\sum_{k=\delta+1}^{s} |D^k u_l|^2 \mu_e + \sum_{k=1}^{\infty} \int_{W_k} \sum_{0 \leq k \leq s} |D^k u_k|^2 \mu_e.}
\]

Different preparations of \(M^n\) give equivalent norms. A Riemannian manifold \((M^n, g)\) is called asymptotically Euclidean (A.E.) if
\[
h_l := f_l (g - e) \in H_{s,\delta} \cap C_{n-2}^1, \quad f_k g \in H_s, \quad s > \frac{n}{2} + 1, \quad \frac{n}{2} - 2 > \delta > -\frac{n}{2}.
\]

It can be proved (using the fact that \(H_{s,\delta}\) is an algebra if \(s > \frac{n}{2}, \delta > -\frac{n}{2}\)) that an A.E. \((M^n, g)\) admits in each end \(\Omega_l\) an orthonormal coframe

\(^1\) For references prior to 1983 one can consult my survey on positive energy theorems for les Houches 1983 school reproduced in Y. Choquet-Bruhat [1].
\[ \theta^j := a^j_t \, dx^t, \quad a^j_t = \delta^j_t + \frac{1}{2} \chi^j_t, \quad \chi^j_t \in H_{s, \delta} \cap C^1_{n-2}. \]  
\[ (9) \]

In the following, components in the coordinates \( x^t \) are underlined. In \( \Omega_1 \) it holds that

\[ g^{ij} = \sum_h d^j_h d^i_h = \eta^{ij} + h^{ij}, \quad \eta^{ij} := \delta^i_j, \]
\[ h^{ij} = \frac{1}{2} (\lambda^j_i + \lambda^i_j) + \frac{1}{4} \sum_h \lambda^j_i \lambda^i_j, \quad \lambda^j_i \lambda^i_j \in H_{s, 2\delta + \frac{3}{2}} \cap C^1_{2n-4}. \]  
\[ (10) \]

The rotation coefficients \( c^i_h \) of the coframe \( \theta^i \) are, with \( (b^i_j) \) the matrix inverse of \( (a^i_j) \) and \( \partial_i \) the Pfaff derivative with respect to \( \theta^i \), \( d\theta^i = \frac{1}{2} c^i_h \theta^i \wedge \theta^j \).

\[ c^i_h = b^k_i \partial_k \lambda^i_h - b^i_k \partial_k \lambda^i_h = \frac{1}{2} (\partial_i \lambda^i_h - \partial_j \lambda^j_h) + \chi^i_h, \quad \chi^i_h \in H_{s, 1, 2, 1, +1/2}. \]  
\[ (11) \]

We choose the coframe such that

\[ \partial_i (\lambda^i_j - \lambda^j_i) \in H_{s, 1, 2, 1, +1/2}, \]

the components \( \omega_{i, jh} \) of the Riemannian connection \( \omega \) in the coframe \( \theta^i \) are then computed to be

\[ \omega_{i, jh} = \frac{1}{2} \partial_i h_{ij} - \frac{1}{2} \partial_j h_{ih} + \zeta_{i, hj}, \quad \zeta_{i, hj} \in H_{s, 1, 2, 1, +1/2}. \]  
\[ (12) \]

2.2. Global mass \( m \) and linear momentum \( \mu \)

We say that \( (M^n, g, K) \) is A.E. if \( (M^n, g) \) is A.E. and \( K \in H_{s, 1, 4} \cap C_{n-1}^0 \). The mass \( m \) and linear momentum \( \mu \) associated to an end \( \Omega \) define a spacetime vector \( E \) with components

\[ E^0 := m := \lim_{r \to \infty} \frac{1}{2} \int_{S^{n-1}} (\partial_i h_{ij} - \partial_j h_{ih}) n_i \mu_{\tilde{g}}, \quad E^h := p^h := \lim_{r \to \infty} \int_{S^{n-1}} p^h n_i \mu_{\tilde{g}}. \]
\[ (13) \]

The uniform bound of \( n_i \) and the equivalence of \( \mu_{\tilde{g}} \) with \( r^{-n-1} \tilde{g}_{n-1} \) show that the limits exist.

We always assume that the constraints \((1)\) and \((2)\) are satisfied on \( M^n \) and that \( T \) obeys the dominant energy condition.

3. Spinor fields and Dirac operator

The gamma matrices associated with an orthonormal coframe \( \theta^i \) of \( g \) at \( x \in M^n \) are linear endomorphisms of a complex vector space \( S \) of dimension \( p := 2^n/2 \) which satisfy the identities

\[ \gamma_i \gamma_j + \gamma_j \gamma_i = 2 n_{ij} I_p, \quad i, j = 1, \ldots, n, \quad I_p \text{ identity matrix}. \]
\[ (14) \]

The \( \gamma_i \) are chosen hermitian, i.e. \( \gamma_i = \tilde{\gamma}_i \), as is possible for an \( O(n) \) group.

The spinor group \( \text{Spin}(n) \), double covering of \( \text{SO}(n) \), can be realized by the group of invertible linear maps \( \Lambda \) of \( S \) which satisfy, with \( O := (O_i^j) \) an \( n \times n \) orthogonal matrix

\[ \Lambda^{-1} = O_i^j \gamma^j \quad \text{and} \quad \det \Lambda = 1. \]
\[ (15) \]

In a subset \( \Omega_1 \) or \( W_K \) with a given field \( \rho_0 \) of orthonormal frames a spinor field \( \psi \) is represented by a mapping \( (x^t) \mapsto \psi(x^t) \in S \). Under an \( O \in \text{SO}(n) \) change of frame, \( \rho = O \rho_0 \) the spinor \( \psi \) becomes represented by \( \psi' = \Lambda \psi \), where some choice has been made for the correspondence between \( \Lambda \) and \( O \). This can be made consistently on \( M^n \) if it admits a spin structure; that is, a homomorphism of a \( \text{Spin}(n) \) principal bundle \( P_{\text{Spin}(n)} \) onto the principal bundle of oriented orthonormal frames. It is a topological property of \( M^n \), the vanishing of its second Stiefel–Whitney class, always true for an orientable \( M^3 \). A spinor field on \( M^n \) is then a section of a vector bundle \( P_{\text{Spin}(n)} \) associated with \( P_{\text{Spin}(n)} \), with base \( M^n \) and typical fiber \( S \). To a space of spinors corresponds a space of cospinors, replacing \( S \) by the adjoint (complex dual) vector space \( S \) and the change of representation by \( \phi' = \phi \Lambda^{-1} \). Using dual frames \( e_A \) of \( S \) and \( \theta^A \) of \( S \) we have \( \psi = \psi^A e_A, \quad \phi = \phi^A \phi_A, \quad A = 1, \ldots, p \), we denote the duality relation by

\[ \phi \psi = (\phi, \psi) := \phi_A \psi^A, \quad \text{a frame independent scalar}. \]

By \( (15) \) \( \tilde{\psi} \) represented by \( \tilde{\psi}_A := (\psi^A)^* \) is a cospinor if \( \psi \) is a spinor, \( |\psi|^2 \equiv \tilde{\psi} \psi \) is positive definite.
A spin connection $\sigma$ on $(M^n, g)$ is deduced from an $O(n)$ connection $\omega$ by the isomorphism between the Lie algebras of $O(n)$ and $\text{Spin}(n)$ obtained by differentiation of (15), it is represented in each domain of the preparation by

$$\sigma_i \equiv \frac{1}{4} \gamma^h \gamma^k \omega_{i,hk}, \quad i = 1, \ldots, n. \quad (16)$$

The covariant derivative of a spinor $\psi$, resp. cospinor $\phi$, is a covariant vector spinor, resp. cospinor, with components in the frames $\theta^i \otimes e_A$, resp. $\theta^i \otimes \theta^A$;

$$(D_1 \psi)^A \equiv \partial_i \psi^A + (\sigma_i) \psi^A, \quad (D_1 \phi)_A \equiv \partial_i \phi_A - (\phi \sigma_i)_A.$$

The hermiticity of $\gamma_i$ and (14) show that $\tilde{\sigma}_i = -\sigma_i$, hence $D_1 \tilde{\psi} \equiv D_1 \psi$.

The Riemannian connection together with the spin connection define a first order derivation operator mapping tensor–spinor–cospinor fields into tensor–spinor–cospinor fields with one more covariant index. The gamma matrices are the components of a vector–spinor–cospinor which have covariant derivative zero.

The spin curvature $\rho$ is a 2-tensor–spinor–cospinor, image by the mapping of Lie algebras of the curvature tensor of $g$.

The Ricci identity for spinors reads

$$D_i D_j \psi - D_j D_i \psi \equiv \rho_{ij} \psi \quad \text{with} \quad \rho_{ij} := \frac{1}{4} R_{ij,hk} \gamma^h \gamma^k. \quad (17)$$

The Dirac operator on sections of the vector bundle $\Psi(n)$ reads locally

$$D \psi \equiv \gamma^i D_i \psi \equiv \gamma^i (\partial_i + \frac{1}{4} \omega_{i,hk} \gamma^h \gamma^k \psi), \quad \text{hence} \quad D \psi \equiv D_1 \tilde{\psi} \gamma^i. \quad (18)$$

The algebraic Bianchi identity together with (14) and (17) lead to the formula (see, for instance, A. Lichnerowicz [3]):

$$D^2 \psi \equiv \eta^{ij} D_i D_j \psi + \frac{1}{2} \gamma^i \gamma^j \rho_{ij} \psi \equiv \eta^{ij} D_i D_j \psi - \frac{1}{4} R \psi. \quad (19)$$

The Dirac operator is a first order linear operator with principal symbol $(\eta^{ij} \xi_i \xi_j)^2 \psi$, hence elliptic. Weighted Sobolev spaces for spinor fields on a prepared $M^n$ are defined as for tensor fields after setting $\psi = \sum (f_i \psi + f_K \psi)$ and using representations $\psi$. A known theorem (Y. Choquet-Bruhat and D. Christodoulou [2]) gives:

**Theorem 1.** On an A.E. $(M^n, g)$ the Dirac operator is a Fredholm operator from spinors in $H_{s,\delta}$ to spinors in $H_{s-1,\delta+1}$, it is an isomorphism if injective. The same is true of $D^2 \psi + f \psi$ if $f$ is a bounded linear map from spinors in $H_{s,\delta}$ to spinors in $H_{s-1,\delta+1}$.

### 4. Gravitational mass

We prove for arbitrary $n > 2$ the fundamental fact used by Witten for $n = 3$.

**Theorem 2.** Let $(M^n, g)$ be A.E. The mass $m$ of an end $\Omega_1$ is equal to

$$\lim_{r \to \infty} \int \mathcal{U}_0 n_i \mu_{\tilde{g}} \equiv \frac{m}{2}, \quad \mathcal{U}_0 := \text{Re} \{ \tilde{\psi}_0 (\eta^{ij} - \gamma^i \gamma^j) \sigma_j \psi_0 \}, \quad (20)$$

with $S^{n-1}_r$ the submanifold of the end $\Omega_1$ with equation $\{ (x^i)^2 \}^{\frac{1}{2}} = r$, $n_i$ its unit normal, $\mu_{\tilde{g}}$ the volume element induced by $g$ and $\psi_0$ a spinor constant in $\Omega_1$ (i.e. $\frac{\partial \psi_0}{\partial n_i} = 0$) and $|\psi_0| = 1$.

**Proof.** We first remark that, using $\gamma^i = \gamma^i$ and $\tilde{\sigma}_i = -\sigma_i$, one finds

$$\text{Re} (\tilde{\psi}_0 \eta^{ij} \sigma_j \psi_0) \equiv \frac{1}{2} \tilde{\psi}_0 \eta^{ij} (\sigma_j + \tilde{\sigma}_j) \psi_0 \equiv 0.$$ 

The definition (16) of $\sigma_j$ then implies

$$\mathcal{U}_0 = \frac{1}{2} \sum_{j,h,k} \tilde{\psi}_0 \omega_{j,hk} (\gamma^i \gamma^h \gamma^k - \gamma^j \gamma^k \gamma^j) \psi_0.$$

The property (12) of $\omega$ on A.E. $(M^n, g)$ shows that $r^{n-1} (\omega_{j,hk} - \frac{1}{2} \partial_k h_{jk} + \frac{1}{2} \partial_k h_{jk})$ tends uniformly to zero as $r$ tends to infinity. The uniform bound of $n_i$ and the equivalence of $\mu_{\tilde{g}}$ with $r^{n-1} \mu_{\tilde{g}}$ show that the limit in (20) exists. Calculations using (14), (12) and the symmetry of $\tilde{\omega}_k h_{ij}$ in $h$ and $j$ lead to

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*2 Similar to those done by P. Chrusciel in his Krakow lectures on Energy in General Relativity.*
\[
\lim_{r \to \infty} \int_{S_{r}^{n-1}} \mathcal{U}_{0}n_{i} \mu_{\tilde{g}} = \frac{1}{2} \lim_{r \to \infty} \int_{S_{r}^{n-1}} \omega_{j,ij} |\psi_{0}|^{2} n_{i} \mu_{\tilde{g}} = \frac{1}{2} m \text{ if } |\psi_{0}|^{2} = 1. \quad \Box
\]

To study the positivity of the mass one defines a vector \(\mathcal{U}^{i}\) on \(M^{n}\) such that the integrals on \(S_{r}^{n-1}\) of \(\mathcal{U}^{i}\) and \(\mathcal{U}_{0}^{i}\) have the same limit when \(r \to \infty\). The Stokes formula applied to the integral of the divergence of \(\mathcal{U}^{i}\) will give information on this limit. We set

\[
\mathcal{U}^{i} := \text{Re} \left( \tilde{\psi} \left( \eta^{i} D_{j} \psi - \gamma^{i} \gamma^{j} D_{j} \psi \right) \right).
\]

**Lemma 3.** On an A.E. manifold \((M^{n}, g)\) it holds that

1. \(D_{i} \mathcal{U}^{i} \geq 0\) if \(R \geq 0\) and \(D \psi = 0\).
2. If \(\psi = \psi_{0} + \psi_{1}\) with \(\partial_{i} \psi_{0} = 0\) in \(\Omega_{1}\) and \(\psi_{1} \in H_{s,\delta}\) then in \(\Omega_{l}\)

\[
\lim_{r \to \infty} \int_{S_{r}^{n-1}} \mathcal{U}_{0}^{i} n_{i} \mu_{\tilde{g}} = \lim_{r \to \infty} \int_{S_{r}^{n-1}} \mathcal{U}^{i} n_{i} \mu_{\tilde{g}}. \tag{22}
\]

**Proof.** By elementary computation, using \(D_{i} \tilde{\psi} = \tilde{D}_{i} \psi\) and the identity (19) one finds

\[
D_{i} \mathcal{U}^{i} = |D \psi|^{2} - |D \tilde{\psi}|^{2} + \frac{1}{4} R |\psi|^{2}. \tag{23}
\]

Therefore \(D_{i} \mathcal{U}^{i} \geq 0\) if \(R \geq 0\) and \(\psi\) satisfies the equation \(D \psi = 0\).

2. To study the limit of the integral on \(S_{r}^{n-1}\) of \(\mathcal{U}^{i}\) when \(\psi = \psi_{0} + \psi_{1}\) we write

\[
\mathcal{U}^{i} = \mathcal{U}_{0}^{i} + \frac{1}{2} \text{Re} \left( \tilde{\psi}_{0} \left[ \gamma^{j}, \gamma^{i} \right] D_{j} \psi_{1} + \tilde{\psi}_{1} \left[ \gamma^{j}, \gamma^{i} \right] D_{j} \psi \right). \tag{24}
\]

Hence

\[
D_{i} \mathcal{U}^{i} = D_{i} \mathcal{U}_{0}^{i} + D_{i} \mathcal{V}^{i}. \tag{25}
\]

Embedding and multiplication properties of Sobolev spaces give

\[
\tilde{\psi}_{1} \left[ \gamma^{j}, \gamma^{i} \right] D_{j} \psi \in H_{s,\delta} \times \left\{ C_{0}^{1} \cap H_{s-1,\delta+1} \right\} \subset H_{s-1,2\delta+1+\frac{n}{2}} \subset C_{0}^{\beta}.
\]

\[
\beta < 2\delta + 1 + \frac{n}{2} + \frac{n}{2} < 2n - 3.
\]

To estimate the other term one remarks that

\[
D_{j} \left( \tilde{\psi}_{0} \left[ \gamma^{j}, \gamma^{i} \right] D_{j} \psi_{1} \right) = D_{i} D_{j} \left( \tilde{\psi}_{0} \left[ \gamma^{j}, \gamma^{i} \right] \psi_{1} \right) - D_{i} \left( D_{j} \tilde{\psi}_{0} \left[ \gamma^{j}, \gamma^{i} \right] \psi_{1} \right). \tag{26}
\]

the first parenthesis is an antisymmetric 2-tensor hence its double divergence \(D_{i} D_{j}\) is identically zero. The second parenthesis is

\[
D_{j} \tilde{\psi}_{0} \left[ \gamma^{j}, \gamma^{i} \right] \psi_{1} \in C_{n-1}^{0} \times H_{s,\delta} \subset C_{0}^{\beta}.
\]

The Stokes formula implies, with \(M^{n}_{r} := M^{n} - \{ \Omega_{1} \cap \sum(x_{i})^{2} \geq r^{2} \}\)

\[
\int_{M^{n}_{r}} D_{i} \mathcal{U}^{i} \mu_{\tilde{g}} = \int_{S_{r}^{n-1}} \mathcal{U}^{i} n_{i} \mu_{\tilde{g}} = \int_{S_{r}^{n-1}} \left( \mathcal{U}_{0}^{i} + \mathcal{V}^{i} \right) n_{i} \mu_{\tilde{g}}. \tag{27}
\]

The fall off properties found for \(\mathcal{V}^{i}\) complete the proof. \(\Box\)

**Lemma 4.** If \((M^{n}, g)\) is A.E. \(R \geq 0\) and \(\psi_{0}\) is a smooth spinor constant in \(\Omega_{1}\) and zero in the other ends there exists on \(M^{n}\) a spinor \(\psi = \psi_{0} + \psi_{1}\), such that \(D \psi = 0\), \(\psi_{1} \in H_{s,\delta}\).

**Proof.** The hypotheses made on \(\psi_{0}\) show that \(D \psi_{0} \in H_{s-1,\delta+1}\). Theorem 1 implies the existence of \(\psi_{1}\). \(\Box\)

The lemmas imply that \(m \geq 0\) if \(R \geq 0\), that is if \((M^{n}, g)\) is a maximal submanifold of \((M^{n+1}, g)\); equivalently, if the pointwise gravitational momentum \(P\) on \(M^{n}\) has a vanishing trace. We will now lift this restriction, proving moreover that \(m \geq |p|\).
5. Positive energy

We define a real vector \( \mathcal{P} \) on an A.E. \((M^n, g, K)\) by \(^3\)

\[
\mathcal{P}^i := \frac{1}{2} \bar{\psi} \gamma^h P_{ih} \psi = \frac{1}{2} \bar{\psi} (\gamma^h K_{ih} - \gamma^i \gamma^j \gamma^h K_{jh}) \psi, \quad P_{ih} = K_{ih} - \delta_{ih} \text{tr} K.
\]

If \( \psi_0 \) is as before a smooth spinor constant in one end of \( M^n \) and zero in the other ends and \( \psi \) is a spinor on \( M^n \) such that \( \psi - \psi_0 \in C^0_n \), \( \beta > 0 \), then

\[
\lim_{r \to \infty} \int_{S_{r}^{n-1}} \mathcal{P}^i n_i \mu = \frac{1}{2} \bar{\psi}_0 \gamma^h p^h \psi_0, \quad p^h := \lim_{r \to \infty} \int_{S_{r}^{n-1}} P_{ih} n_i \mu.
\]

It is elementary to check using the properties of the \( \gamma \)'s that \( \gamma^h p^h \) is a hermitian matrix with eigenvalues \( \pm |p| \). If we choose for \( \psi_0 \) an eigenvector of the eigenvalue \( -|p| \) we then have

\[
\bar{\psi}_0 \gamma^h p^h \psi_0 = -|\psi_0|^2 |p|.
\]

To estimate the limit (28), we use again the Stokes formula, with

\[
D_i \mathcal{P}^i = \frac{1}{2} D_i (\bar{\psi} \gamma^h P_{ih} \psi) = \frac{1}{2} D_i (\bar{\psi} \gamma^h \psi) P_{ih} + \frac{1}{2} \bar{\psi} \gamma^h \psi D_i P_{ih}.
\]

The momentum constraint (2) gives

\[
D_i P_{ih} = -T^b_0.
\]

On the other hand, the identity (23) together with the Hamiltonian constraint implies that

\[
D_i \mathcal{U}^i \equiv |D \psi|^2 - |D \psi|^2 + \left( \frac{1}{2} T_{00} + \frac{1}{4} |K|^2 - \frac{1}{4} \text{tr} K^2 \right) |\psi|^2.
\]

We introduce the notations \(^4\)

\[
\nabla_i \psi := D_i \psi + \frac{1}{2} \gamma^h K_{ih} \psi, \quad \nabla_i \psi := \gamma^i \nabla_i \equiv \left( D + \frac{1}{2} \text{tr} K \right) \psi.
\]

Elementary computation using \( D_i \gamma^h = 0, D_j \gamma^h = 0 \) gives

\[
|\nabla \psi|^2 := \eta^i j \nabla_i \psi \nabla_j \psi \equiv |D \psi|^2 + \frac{1}{2} D_i (\bar{\psi} \gamma^h \psi) K_{ih} + \frac{1}{4} |K|^2 |\psi|^2.
\]

The identity (31) can therefore be written after simplification

\[
D_i \mathcal{U}^i \equiv |\nabla \psi|^2 - |D \psi|^2 + \left( \frac{1}{2} T_{00} - \frac{1}{4} \text{tr} K^2 \right) |\psi|^2 - \frac{1}{2} D_i (\bar{\psi} \gamma^h \psi) K_{ih}.
\]

We deduce from the definition

\[
|\nabla \psi|^2 \equiv |D \psi|^2 + \frac{1}{2} D_i (\bar{\psi} \gamma^h \psi) \text{tr} K + \frac{1}{4} |\text{tr} K|^2 |\psi|^2
\]

which gives

\[
D_i \mathcal{U}^i \equiv |\nabla \psi|^2 - |\nabla \psi|^2 + \frac{1}{2} T_{00} |\psi|^2 - \frac{1}{2} D_i (\bar{\psi} \gamma^h \psi) P_{ih}.
\]

**Lemma 5.** If \((M^n, g, K)\) is A.E. then it holds that

\[
D_i (\mathcal{U}^i + \mathcal{P}^i) \geq 0
\]

if the dominant energy condition holds and \( \psi \) satisfies the equation \( \nabla \psi = 0 \).

\(^3\) Remark that we do not introduce a matrix \( \gamma_0 \).

\(^4\) Note that \( \nabla \) is a linear operator mapping space spinor into space spinors, not the trace on \( M^n \) of the covariant derivative of a spacetime spinor.
Proof. The identities (30) and (35) lead to
\[ D_i (U_i + P_i) \equiv |\nabla \psi|^2 - |\nabla \psi|^2 + T, \quad T := \frac{1}{2} (T_{00} |\psi|^2 - \tilde{\psi} \gamma^h \psi T_{0h}), \] (37)
with \( T \geq 0 \) under the dominant energy condition, because
\[ |\tilde{\psi} \gamma^h \psi T_{0h}| \equiv |\psi|^2 \left( \eta^{ih} T_{0i} T_{0h} \right)^{\frac{1}{2}} \leq T_{00} |\psi|^2 \]
with \( T \geq 0 \) under the dominant energy condition, because
\[ |\tilde{\psi} \gamma^h \psi T_{0h}| \equiv |\psi|^2 \left( \eta^{ih} T_{0i} T_{0h} \right)^{\frac{1}{2}} \leq T_{00} |\psi|^2. \]
Lemma 6. If \((M^n, g, K)\) is A.E., then the equation \( \nabla \psi = 0 \) has a solution \( \psi = \psi_0 + \psi_1, \psi_0 \) smooth, constant in \( \Omega_1 \) and zero in the other ends, and \( \psi_1 \in H_{s,\delta} \).

Proof. The operator \( \nabla \psi \) has the same principal part as \( \mathcal{D} \), therefore is also elliptic. It maps \( H_{s,\delta} \) into \( H_{s-1,\delta+1} \). The equation \( \nabla \psi_1 = -\nabla \psi_0 \in H_{s-1,\delta+1} \) has one and only one solution if \( \nabla \psi \) is injective on \( H_{s,\delta} \). To show injectivity\(^5\) we remark that the identity (37) was established without restriction on \( \psi \), starting from the definitions of \( U_i \) and \( P_i \). We make \( \psi = \psi_1 \) in (37) and integrate it on \( M^n \), the fall off of \( \psi_1 \) implies that the divergence gives no contribution, the equation \( |\nabla \psi_1|^2 = 0 \) implies therefore that on \( M^n \), if \( T \geq 0 \)
\[ |\nabla \psi_1|^2 = 0, \quad \text{i.e.} \quad D_i \psi_1 + \frac{1}{2} \gamma^h K_{ih} \psi_1 = 0, \quad \text{with} \quad \gamma^h K_{ih} \in H_{s-1,\delta+1}. \]
The Poincaré inequality (see, for instance [1, Appendix 3, Sobolev spaces, p. 541]) in weighted Hilbert spaces leads to \( \psi_1 = 0 \) if \( \psi_1 \in H_{s,\delta} \), \( s > \frac{n}{2} + 1 \) and \(-2 + \frac{n}{2} > \delta > -\frac{n}{2} \).\(^\Box\)

The lemmas, after choice of \( \psi_0 \) satisfying (29), prove the following theorem:

Theorem 7. If an Einsteinian spacetime satisfies the dominant energy condition, the energy momentum vector \( E^0 = m, E^i = p^i \) of each end of an A.E. slice \((M^n, g, K)\) satisfies the inequality
\[ m \geq |p|. \]

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References

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[3] A. Lichnerowicz, Champs spinoriels et propagateurs en relativité générale, Bull. Soc. Math. France 92 (1964) 11–100.

\(^5\) See a similar proof in Chrusciel Krakow lecture notes.