Estimating parameterized entanglement measure

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Abstract
The parameterized entanglement monotone, the $q$-concurrence, is also a reasonable parameterized entanglement measure. By exploring the properties of the $q$-concurrence with respect to the positive partial transposition and realignment of density matrices, we present tight lower bounds of the $q$-concurrence for arbitrary $q \geq 2$. Detailed examples are given to show that the bounds are better than the previous ones.

Keywords Quantum entanglement · Entanglement measure · Concurrence

1 Introduction

Entanglement is one of the most remarkable phenomena in quantum mechanics [1, 2]. In recent years, great efforts have been made toward the understanding of the role played by the entanglement in quantum information theory [3]. It has been the most important resources in quantum information processing and communication such as quantum dense coding [4], quantum metrology [5, 6], quantum teleportation [7, 8], quantum secret sharing [9] and quantum cryptography [10]. These applications have strongly motivated the study on detection and quantification of entanglement in an operational way.
Detecting the entanglement of generic mixed states is still a hard problem. The positive partial transpose (PPT) criterion [11] says that for any separable bipartite state $\rho_{AB}$, the partial transposed matrix $\rho^\Gamma \geq 0$ ($\Gamma$ represents the partial transposition with respect to the subsystem $B$ in the following) is semi-positive. The PPT criterion is a necessary and sufficient condition of separability only for pure states and $2 \otimes 2$- and $2 \otimes 3$-dimensional mixed states [11, 12]. While the realignment criterion [13–16] says that the realigned matrix $R(\rho)$ of any separable $\rho_{AB}$ satisfies $\|R(\rho)\|_1 \leq 1$, where $\|X\|_1$ denotes the trace norm defined by $\|X\|_1 = \text{Tr} \sqrt{XX^\dagger}$. These separability criteria have been widely used for entanglement detection both theoretically and experimentally in quantum information processing [17].

The quantification of quantum entanglement for a given quantum state is also a difficult undertaking due to the intricate interplay between classical and quantum correlations [18, 19]. It has been proposed that a reasonable measure of entanglement should fulfill certain conditions [18–20]. Some interesting entanglement measures have been provided for bipartite systems such as concurrence [21–23], entanglement of formation [24–26], negativity [27], robustness of entanglement [28] and Rényi-$\alpha$ entropy of entanglement [29]. There are also some entanglement monotones [30] such as the convex-roof extension of negativity [31], Tsallis-$q$ entropy of entanglement [32], as well as the entanglement monotones induced by fidelity [33].

Nevertheless, most proposed entanglement measures or monotones involve extremizations which are difficult to handle analytically. Usually analytical results are only available for two-qubit states [25] or some special higher-dimensional mixed states [34–36] for certain special measures [31, 37]. Therefore, efforts have been made toward the estimation of entanglement measures for general mixed states [38]. The analytical lower bound for the concurrence has been derived in [23] based on the PPT and realignment criteria. In [39], the authors sharpened this bound by relating the concurrence to the local uncertainty relations and the correlation matrix criterion. By using the PPT and realignment criteria, a lower bound for the genuine tripartite entanglement concurrence was obtained in [40].

In particular, the concurrence plays a vital role in entanglement distributions such as entanglement swapping and remote preparation of bipartite entangled states [41]. For any pure state $|\Psi\rangle_{AB}$, the concurrence is given by $C(|\Psi\rangle_{AB}) = \sqrt{2(1 - \text{Tr} \rho_A^2)}$ with $\rho_A = \text{Tr}_B |\Psi\rangle \langle \Psi| [21]$. In fact, the concurrence is related to the specific Tsallis-$q$ entropy $T_q(\rho_A)$ with $q = 2$, $C(|\Psi\rangle_{AB}) = \sqrt{2T_2(\rho_A)} [42, 43]$. Noteworthily, the Tsallis-$q$ entropy provides a generalization of the traditional Boltzmann-Gibbs statistical mechanics and enables one to find a consistent treatment of dynamics in many nonextensive physical systems such as with long-range interactions, long-time memories and multifractal structures [44]. The Tsallis-$q$ entropy also provides many intriguing applications in the realms of quantum information theory [45–48]. Therefore, it is of great significance to provide an entanglement measure from the Tsallis entropy with $q > 2$. Recently, the authors in [49] has presented such parameterized entanglement monotone for $q > 2$.

In this paper, we present analytical tighter lower bounds for the parameterized entanglement monotone $q$-concurrence given in [49], which is also a well-defined entanglement measure. The rest of this paper is organized as follows. In Sect. 2, we
recall some necessary conditions for bipartite entanglement measures, and the concept of $q$-concurrence. We derive tighter lower bounds of the $q$-concurrence for general mixed states, and consider some detailed examples in Sect. 3. We make a conclusion in Sect. 4.

### 2 Bound on $q$-concurrence

Let $\mathcal{D}$ denote the set of bipartite states in finite dimensional Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ associated with subsystems $A$ and $B$. A well-defined quantum entanglement measure $E$ must satisfy certain conditions $[18–20]$ as follows:

(i) $E(\rho) \geq 0$ for any state $\rho \in \mathcal{D}$, where the equality holds only for separable states.

(ii) $E$ is invariant under local unitary transformations, $E(\rho) = E \left( U_A \otimes U_B \rho U_A^\dagger \otimes U_B^\dagger \right)$ for any local unitaries $U_A$ and $U_B$.

(iii) $E$ does not increase on average under stochastic LOCC,

$$E(\rho) \geq \sum_i p_i E(\rho_i)$$

for any $\rho \in \mathcal{D}$, where $p_i = \text{Tr} A_i \rho A_i^\dagger$ is the probability of obtaining outcome $i$, and $\rho_i = A_i \rho A_i^\dagger / p_i$ with $A_i$ the Kraus operators with respect to the stochastic LOCC such that $\sum_i A_i^\dagger A_i = I$.

(iv) $E$ is convex,

$$E\left( \sum p_i \rho_i \right) \leq \sum p_i E(\rho_i).$$

(v) $E$ cannot increase under LOCC, $E(\rho) \geq E(\Lambda(\rho))$ for any LOCC map $\Lambda$.

The condition (ii) can be removed if the condition (v) holds. $E$ is said to be an entanglement monotone $[30]$ if the first four conditions hold. In $[50]$, it has been shown that a convex function $E$ satisfies condition (v) if and only if it satisfies condition (ii) and

$$E\left( \sum p_i |i\rangle\langle i|_M \otimes \rho_i \right) = \sum_i p_i E(\rho_i),$$

where $M = A', B'$ is a flag system and $\{|i\rangle\}$ are the local orthogonal basic vectors. In addition, (3) is just the flag additivity which is equivalent to the average monotonicity and the convexity, i.e., the conditions (iii) and (iv) $[51]$. In this case, any entanglement monotone defined in $[30]$ is an entanglement measure.
Any pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ can be expressed in the Schmidt form under suitable local bases $\{|i\rangle_A\}$ and $\{|i\rangle_B\}$ of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively,

$$|\psi\rangle_{AB} = \sum_{i=1}^{d} \sqrt{\lambda_i} |i\rangle_A \langle i|_B,$$

(4)

where $\lambda_i$'s are the squared Schmidt coefficients with $\sum_{i=1}^{d} \lambda_i = 1$, and $d = \min\{d_A, d_B\}$ with $d_A$ and $d_B$ the dimensions of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively [3]. The parameterized entanglement monotone $q$-concurrence $C_q(|\psi\rangle_{AB})$ of a state $|\psi\rangle_{AB}$ [49] is defined by

$$C_q(|\psi\rangle_{AB}) = 1 - \text{Tr} \rho_A^q$$

(5)

for any $q \geq 2$, where $\rho_A = \text{Tr}_B (|\psi\rangle \langle \psi|)$ is the reduced density operator of the subsystem $A$. It is direct to verify that $C_q(|\psi\rangle_{AB}) = 1 - \sum_{i=1}^{d} \lambda_i^q$ and $0 \leq C_q(|\psi\rangle) \leq 1 - d^{1-q}$, where the left equality holds if $|\psi\rangle$ is a product state, and the right equality holds for the maximally entangled state $|\psi\rangle = 1/\sqrt{d} \sum_{i} |ii\rangle$. The $q$-concurrence for general mixed states $\rho \in \mathcal{D}$ is given by convex-roof extension,

$$C_q(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C_q(|\psi_i\rangle),$$

(6)

where the infimum is taken over all possible pure state decompositions of $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$, with $\sum_i p_i = 1$ and $p_i \geq 0$.

It has been proved in [49] that the $q$-concurrence $C_q(\rho)$ is already an entanglement monotone. Therefore, it is also an entanglement measure.

It is hard to derive an analytic formula of the entanglement measure $q$-concurrence for general mixed states. In the following, we estimate the $q$-concurrence by deriving its lower bounds based on the PPT and realignment criteria. For a given bipartite state $\rho = \sum_{ijkl} \rho_{ijkl} |ij\rangle \langle kl|$ in computational bases, the partial transposed matrix $\rho^\Gamma$ with respect to the subsystem $B$ is given by $\rho^\Gamma = \sum_{ijkl} \rho_{ijkl} |kl\rangle \langle ij|$, and the realigned matrix is given by $\mathcal{R}(\rho) = \sum_{ijkl} \rho_{ijkl} |ik\rangle \langle jl|$. For a given pure state $|\psi\rangle$ defined in (4), it is straightforward to show that [23],

$$1 \leq \|\rho^\Gamma\|_1 = \|\mathcal{R}(\rho)\|_1 = \left( \sum_{i=1}^{d} \sqrt{\lambda_i} \right)^2 \leq d,$$

(7)

where $\rho = |\psi\rangle \langle \psi|$. In particular, for $q = 2$, the $q$-concurrence becomes $C_2(|\psi\rangle) = 1 - \sum_{i=1}^{d} \lambda_i^2 = 2 \sum_{i<j} \lambda_i \lambda_j$. One has then [23],

$$C_2(|\psi\rangle) \geq \frac{1}{d(d-1)} (\|\rho^\Gamma\|_1 - 1)^2$$

(8)

for any pure state $|\psi\rangle$ on $\mathcal{H}_A \otimes \mathcal{H}_B$. 

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Before deriving a tight lower bound of \( q \)-concurrence for \( q \geq 2 \), we first show the following conclusion. For a given pure state \(|\psi\rangle\) defined in (4), let us analyze the monotonicity of the function \( f(q) \) given by,

\[
f(q) = \frac{1 - \sum_{i=1}^{d} \lambda_i^q}{1 - d^{1-q}}
\]

for any \( q \geq 2 \). Set

\[
G_{dq} = \sum_{i=1}^{d} \lambda_i^q \ln \lambda_i \left( d^{1-q} - 1 \right) - \left( 1 - \sum_{i=1}^{d} \lambda_i^q \right) d^{1-q} \ln d.
\]

We have

\[
\frac{\partial f}{\partial q} = \frac{G_{dq}}{(1 - d^{1-q})^2}.
\]

We see that the sign of the first derivative of \( f(q) \) with respect to \( q \) depends on the sign of the function \( G_{dq} \), with constraints \( \sum_{i=1}^{d} \lambda_i = 1 \) and \( \lambda_i > 0 \) for \( i = 1, ..., d \). Consider the minimum of \( G_{dq} \) by using Lagrange multipliers [36] subject to the constraints \( \sum_{i=1}^{d} \lambda_i = 1 \) and \( \lambda_i > 0 \). There is only one stable point under the constraints, \( \lambda_i = 1/d, i = 1, \cdots, d \), for which we have \( G_{dq} = 0 \).

The second derivative of \( f(q) \) with respect to \( q \) at the stable point is given by

\[
\frac{\partial^2 G_{dq}}{\partial \lambda_i^2} \bigg|_{\lambda_i = \frac{1}{d}} = d^{2-q} \left[ q (q - 1) \ln d - (2q - 1) \left( 1 - d^{1-q} \right) \right].
\]

Therefore, we get

\[
\frac{\partial^2 G_{dq}}{\partial \lambda_i^2} \bigg|_{\lambda_i = \frac{1}{d}} \geq 0
\]

for \( q \geq s \equiv 2.4721 \), \( d = 2 \) and \( q \geq 2 \), \( d \geq 3 \). In these cases, the minimum extreme point is the minimum point and \( \partial f/\partial q \geq 0 \). From the above analysis, it is straightforward to have the following conclusion.

**Corollary 1** If \( q \geq h \), then \( C_q(\rho) \geq \frac{1 - d^{1-q}}{1 - d^{1-q}} C_h(\rho) \) for either \( h \geq s \), \( d = 2 \) or \( h \geq 2 \), \( d \geq 3 \).

We now derive the main result of this paper.

**Theorem 1** For a general bipartite state \( \rho \in \mathcal{D} \), we have

\[
C_q(\rho) \geq \frac{1 - d^{1-q}}{(d - 1)^2} \left( \max \left( \|\rho^T\|_1, \|R(\rho)\|_1 \right) - 1 \right)^2.
\]
for either $q \geq 2$ with $d \geq 3$ or $q \geq 3$ with $d = 2$, and

$$C_q (\rho) > \frac{1 - 2^{1-q}}{2 - 2^{2-s}} \left[ \max (\| \rho \|^1, \| R (\rho) \|_1) - 1 \right]^2$$

(15)

for $s \leq q < 3$ with $d = 2$.

**Proof** For a given pure state (4), from (13) and (11), we can obtain that $f(q)$ in (9) is an increasing function with respect to $q$ in the cases of $q \geq 2$ with $d \geq 3$, and $q \geq s$ with $d = 2$.

In the first case of $d \geq 3$, we have

$$C_q (| \psi \rangle) \geq \frac{1 - d^{1-q}}{1 - d^{-1}} C_2 (| \psi \rangle) \geq \frac{1 - d^{1-q}}{(d - 1)^2} (\| \sigma \|^1 - 1)^2$$

(16)

for $q \geq 2$, where $\sigma = | \psi \rangle \langle \psi |$, the first inequality is due to the monotone increasing of $f(q)$, the second inequality is due to (8).

For case of $d = 2$, similar to (16), we can obtain when $q \geq 3$,

$$C_q (| \psi \rangle) \geq \frac{1 - 2^{1-q}}{1 - 2^{2-s}} C_3 (| \psi \rangle) = (1 - 2^{1-q}) 2C_2 (| \psi \rangle) \geq (1 - 2^{1-q}) (\| \sigma \|^1 - 1)^2,$$

(17)

where the equality is due to $f(2) = f(3)$. When $s \leq q < 3$, we have

$$C_q (| \psi \rangle) \geq \frac{1 - 2^{1-q}}{1 - 2^{1-s}} C_s (| \psi \rangle) > \frac{1 - 2^{1-q}}{1 - 2^{1-s}} C_2 (| \psi \rangle) > \frac{1 - 2^{1-q}}{2 - 2^{2-s}} (\| \sigma \|^1 - 1)^2,$$

(18)

where the second inequality is from the monotonic increase in $C_q (| \psi \rangle)$ with respect to $q$.

Let $\rho = \sum_i p_i | \psi_i \rangle \langle \psi_i |$ be the optimal pure state decomposition of $C_q (\rho)$ for a given mixed state $\rho \in \mathcal{D}$. For the cases of $q \geq 2$ with $d \geq 3$ and $q \geq 3$ with $d = 2$, we have

$$C_q (\rho) = \sum_i p_i C_q (| \psi_i \rangle)$$
\[ \begin{align*}
&\geq \frac{1 - d^{1-q}}{(d-1)^2} \sum_i p_i \left( \| \rho_i^\Gamma \|_1 - 1 \right)^2 \\
&\geq \frac{1 - d^{1-q}}{(d-1)^2} \left( \sum_i p_i \| \rho_i^\Gamma \|_1 - 1 \right)^2 \\
&\geq \frac{1 - d^{1-q}}{(d-1)^2} \left( \| \rho_1^\Gamma \|_1 - 1 \right)^2.
\end{align*} \]

(19)

where \( \rho_i = |\psi_i\rangle\langle \psi_i| \). The first inequality is from (16) and (17), the second inequality is obtained from the convexity of the function \( f(x) = x^2 \), the last inequality is due to the convex property of the trace norm and \( \| \rho_1^\Gamma \|_1 \geq 1 \) in (7).

From (7), similar to (16), (17) and (19), we have that

\[ C_q(\rho) \geq \frac{1 - d^{1-q}}{(d-1)^2} \left( \| \rho_1^\Gamma \|_1 - 1 \right)^2 \]

(20)

in the cases of \( q \geq 2 \) with \( d \geq 3 \) and \( q \geq 3 \) with \( d = 2 \). Combining (19) and (20), we can obtain (14). Similarly, from (18) one can obtain (15).

Theorem 1 gives tight lower bounds of the entanglement measure \( C_q(\rho) \). In [49], a lower bound of \( C_q(\rho) \) has been derived,

\[ C_q(\rho) \geq \left[ \max \left\{ \| \rho_1^\Gamma \|_1^{q-1}, \| \rho_1^\Gamma \|_1^{q-1} \right\} - 1 \right]^2. \]

(21)

In the next section, we calculate the \( q \)-concurrence of isotropic states and show that our lower bound is tighter than the one given in (21).

### 3 \( q \)-Concurrence for isotropic states

The isotropic states \( \rho_F \) are the convex mixtures of the maximally entangled state and the maximally mixed state [13],

\[ \rho_F = \frac{1 - F}{d^2 - 1} \left( I - |\Psi^+\rangle\langle \Psi^+| \right) + F |\Psi^+\rangle\langle \Psi^+|, \]

(22)

where \( |\Psi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |ii\rangle \) is the maximally entangled pure state, \( I \) is the identity operator and \( F \) is the fidelity of \( \rho_F \) with respect to \( |\Psi^+\rangle \). \( F = \langle \Psi^+ \rho_F | \Psi^+ \rangle \), \( 0 \leq F \leq 1 \). \( \rho_F \) is separable for \( F \leq 1/d \) and invariant under the operation \( U \otimes U^* \) for any unitary transformation \( U \) [13]. The entanglement of formation [34], tangle and concurrence [36], and Rényi \( \alpha \)-entropy entanglement [29] for the isotropic states have studied. Furthermore, it has proven that \( \| \rho_F^\Gamma \|_1 = \| \rho_1^\Gamma \|_1 = dF \) for \( F > 1/d \) [15, 27].

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By straightforward calculation, the $q$-concurrence for $\rho_F$ is given by

$$C_q (\rho_F) = \text{co} (\xi (F, q, d)),$$

where $\text{co} (\xi (F, q, d))$ denotes the largest convex function that is upper bounded by the function $\xi (F, q, d)$, and

$$\xi (F, q, d) = 1 - \gamma^{2q} - (d - 1) \delta^{2q},\quad (24)$$

where $\gamma = \sqrt{F}/\sqrt{d} + \sqrt{(d-1)(1-F)}/\sqrt{d}$, $\delta = \sqrt{F}/\sqrt{d} - \sqrt{1-F}/\sqrt{d}$ [49].

To show the tightness of our lower bound (14), let us first consider the case of $q = 3$.

(i) $d = 2$. (24) becomes

$$\xi (F, 3, 2) = \frac{3}{4} (2F - 1)^2,$$

where $F \in (1/2, 1]$. As the second derivative of $\xi (F, 3, 2)$ with respect to $F$ is nonnegative, we have

$$C_3 (\rho_F) = \begin{cases} 
0, & F \leq 1/2, \\
\frac{3}{4} (2F - 1)^2, & 1/2 < F \leq 1,
\end{cases}\quad (26)$$

which is just our lower bound (14). Therefore, for $q = 3$ and $d = 2$, our lower bound (14) is just the exact value of the $q$-concurrence for any two-qubit isotropic state $\rho_F$, while (21) gives rise to

$$C_3 (\rho_F) \geq \frac{(2F + 1)^2}{12} (2F - 1)^2,$$

whose lower bound is less than the exact value of $q$-concurrence.

In fact, from (9), we have $f (2) = f (3)$ for $d = 2$. Hence, $C_2 (\rho_F) = 2/3 C_3 (\rho_F)$, which is consistent with the 2-concurrence of any two-qubit isotropic state $\rho_F$ [36]. This implies that our lower bound (14) is exact for both $C_2 (\rho_F)$ and $C_3 (\rho_F)$.

(ii) $d = 3$. From (24), we have

$$\xi (F, 3, 3) = 1 - \gamma^6 - 2\delta^6$$

for any $F \in (1/3, 1]$, where $\gamma = \sqrt{F}/\sqrt{3} + \sqrt{2 - 2F}/\sqrt{3}$ and $\delta = \sqrt{F}/\sqrt{3} - \sqrt{1-F}/\sqrt{6}$. As the first derivative of $\xi (F, 3, 3)$ with respect to $F$ is always positive, $\xi (F, 3, 3)$ is monotonically increasing in the regime where $\rho_F$ is entangled, see Fig. 1. Since the second derivative of $\eta (F, 3, 3)$ with respect to $F$ becomes negative when $F \geq 0.86$, $\eta (F, 3, 3)$ is no longer a convex function for $F \in [0.86, 1]$. As $C_3 (\rho_F)$
is the largest convex function that is upper bounded by (28), we connect the point $F = 0.86$ with the point $F = 1$ by a straight line. Therefore, we obtain, see Fig. 1,

$$C_3(\rho_F) = \begin{cases} 
0, & F \leq 1/3, \\
\xi (F, 3, 3), & 1/3 < F \leq 0.86, \\
1.777F - 0.888, & 0.86 < F \leq 1.
\end{cases}$$ (29)

From Theorem 2, we get that

$$C_3(\rho_F) \geq \frac{2}{9} (3F - 1)^2,$$ (30)

while the lower bound (21) gives rise to

$$C_3(\rho_F) \geq \frac{(3F + 1)^2}{72} (3F - 1)^2.$$ (31)

Our lower bound of (30) is tighter than (31), see Fig. 2.

For the case $q = 4$, we have when $d = 2$,

$$\xi (F, 4, 2) = \frac{8 - (2F - 1)^2}{8} (2F - 1)^2.$$ (32)
for any $F \in (1/2, 1]$. As the second derivative of $\xi (F, 4, 2)$ with respect to $F$ is nonnegative, we obtain

$$C_4 (\rho_F) = \begin{cases} 0, & F \leq 1/2, \\ \frac{8 - (2F - 1)^2}{8} (2F - 1)^2, & 1/2 < F \leq 1. \end{cases}$$  \quad (33)$$

From Theorem 2, we have

$$C_4 (\rho_F) \geq \frac{7}{8} (2F - 1)^2,$$  \quad (34)$$

while from (21), one gets

$$C_4 (\rho_F) \geq \frac{(4F^2 + 2F + 1)^2}{56} (2F - 1)^2. \quad (35)$$

Obviously, our lower bound (34) is tighter than (35), see Fig. 3.

4 Conclusion

We have derived tighter lower bounds of the $q$-concurrence for $q \geq 2$ with $d \geq 3$ and $q \geq s$ with $d = 2$. Moreover, we calculated the $q$-concurrence for isotropic states. In particular, we obtained the analytical formulae of the $q$-concurrence for
isotropic states with \( q = 3 \) and \( d = 2, 3 \), as well as with \( q = 4 \) and \( d = 2 \). It turned out that our lower bound is exact for \( q = 2, 3 \) and \( d = 2 \). These results may highlight further investigations on implications of quantum entanglement to quantum information processing.

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