The NIM Inertial Mass Measurement Project

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Abstract

An inertial mass measurement project, which is expected to precisely measure the Planck constant, \( h \), for possible comparisons with known gravitational mass measurement projects, e.g., the watt balance and the Avogadro project, is being carried out at the National Institute of Metrology, China. The principle, apparatus, and experimental investigations of the inertial mass measurement are presented. The prototype of the experiment and the Planck constant with relative uncertainty of several parts in \( 10^4 \) have been achieved for principle testing.

II. Principle

Details of the principle and theoretical analysis for inertial mass determination based on quasi-elastic electrostatic oscillation method have been reported in [8], and a brief review is presented here. The differential equation for the oscillation system as shown in Fig. 1 is expressed as (1) with small oscillation amplitude around the balancing position.

(1)

\[
(\beta_1 + 2m)\frac{d^2\theta}{dt^2} + (\beta_2 - kU^2)\theta = 0.
\]

In (1), \( m = m_1 = m_2 \) is the weighing test mass; \( \theta \) is the pivot angle; \( U \) is the applied dc voltage between electrodes 1 and 2 (3); \( \beta_1 \) and \( \beta_2 \) are mechanical factors that defined as

(2)

\[
\beta_1 = \frac{J_0}{L^2} + 2m_0, \quad \beta_2 = \frac{M_0 g \delta}{L^2},
\]

where \( J_0 \) denotes the moment of inertia of the balance beam, \( m_0 \) the mass of the suspension (except for the test mass), \( L = L_1 = L_2 \) the beam length of the balance oscillator and \( M_0 \) the mass of the balance beam. Here we define \( z \) as the vertical
Fig. 1. Schematic diagram of the quasi-elastic electrostatic oscillation method. O is the central knife and G is the mass center of the balance beam (OG = δ). L₁ and L₂ are left and right beam lengths. m₁ and m₂ are weighing masses. The movable electrodes of the twin Kelvin capacitor system (A and B) move vertically in oscillation. Electrode 1 is the high potential terminal while electrodes 2 and 3 are grounded. d₀ is the distance between electrodes 1 and 2 at balancing.

upward displacement of electrode 2B with respect to its equilibrium position, and k, the second order capacitance coefficient along the vertical direction z, can be written as

$$k = \frac{\partial^2 (C_{12A} + C_{12B} + C_{13A} + C_{13B})}{\partial z^2},$$  

(3)

where $C_{12A}$, $C_{12B}$, $C_{13A}$, and $C_{13B}$ is the capacitance between electrodes 1 and 2, 1 and 3 for capacitor A and B respectively. The periodic solution for (1) is written as

$$T = 2\pi \sqrt{\frac{\beta_1 + 2m}{\beta_2 - kU^2}}.$$  

(4)

In order to obtain the SI-1990 electrical ratio $\gamma$ and the Planck constant $h$, the SI value for $\beta_1$ and the electrical value for $\beta_2$ should be known. Here the substitution method is applied to solve these two values. The first step is to make $U = 0$, and the oscillation periods $T_1$ and $T_2$ are measured with different test masses $m = m_1$ and $m = m_2$ respectively. Then $\beta_1$ is solved as

$$\beta_1 = \frac{2(m_1T_2^2 - m_2T_1^2)}{T_1^2 - T_2^2}.$$  

(5)

It can be seen from (5) that $\beta_1$ is determined in SI value. Similarly, we make the test mass $m$ unchanged, and the oscillation periods $T_3$ and $T_4$ are measured when the dc voltage is set as $U = U_1$ and $U = U_2$. The calculated $\beta_2$ in electrical unit is as

$$\beta_2 = \frac{k(U_1^2T_3^2 - U_2^2T_4^2)}{T_3^2 - T_4^2}.$$  

(6)

Then the SI-1990 electrical ratio $\gamma$ and the Planck constant $h$ are respectively determined as

$$\gamma = \frac{\frac{4\pi^2}{\beta_1 + 2m}}{\frac{\beta_2 - kU^2}{90}},$$  

(7)

$$h = \frac{4\gamma}{R_{K-90}K_{J-90}}.$$  

(8)

where $R_{K-90}$ and $K_{J-90}$ are conventional values for the von Klitzing constant and the Josephson constant. The measurement for the quasi-elastic electrostatic oscillation method is divided into two phases. One is to measure the capacitance coefficient $k$ by measuring $\Sigma C = C_{12A} + C_{12B} + C_{13A} + C_{13B}$ as a function of the vertical displacement $z$. The other phase is measuring oscillation periods $T_1$, $T_2$, $T_3$, and $T_4$ at different conditions.

It can be seen all measurement quantities: the displacement of electrodes 2, the capacitance of $C_{12}$ and $C_{13}$, the applied dc voltage, and the oscillation period, can be measured accurately. Besides, it is noticed that three approximations are applied in the approach: 1) $m_1 = m_2 = m$, 2) $L_1 = L_2 = L$, and 3) $\theta \rightarrow 0$. Approximations 1) and 2) are obtained by adjusting and exchanging two equal masses in left and right weighing pans. Approximation 3) can be corrected by linear extrapolations at small measurement intervals.
Fig. 2. The prototype of the inertial mass measurement project. ① balance beam (central knife); ② actuator; ③ velocity sensor; ④ test mass; ⑤ mass servo system; ⑥ laser beam of an interferometer; ⑦ Kelvin capacitor.

III. APPARATUS

A. Overview

A prototype of the inertial mass project has been built as shown in Fig. 2. A conventional beam balance with 0.5m beam length is employed as the mainstay for modifications. The mass center of the balance beam is adjusted to a low position below the central knife by taking off the aluminum block above the central knife and adding two adjustable copper blocks below the central knife. A magnetic velocity sensor is designed for compensating the energy consumption during the oscillation by passing a current into a linear actuator. The weighing pans are connected to side knives by flexible structures. A servo system is used to take test masses (1kg, 2kg) synchronously on and off weighing pans. Two Kelvin capacitors (copper, gold-coated) with the same geometry parameters are symmetrically assembled below the weighing pans, and the distance between movable electrode 2 and fixed electrodes 1 (3) is 10mm at balancing position. A laser beam of an interferometer is set in the left pan for measuring positions of movable electrodes. The whole apparatus is placed on an isolated platform with reduced ground vibrations. A glass chamber is used to cut off the air flow. The room temperature is controlled with ±0.5°C by air conditioning system.

B. Velocity sensor and actuator

A magnetic velocity sensor and an linear actuator are employed. The velocity sensor is a 20000-turn copper coil fixed at the left end of the balance beam. The magnetic circuit is shown in Fig.3 (a). Two opposite faced permanent magnets are set in the iron magnet yoke to supply a radial magnetic flux density in the air gap with approximate uniformity in the measurement range of ±2mm along the vertical direction. The output signal is the induced voltage when the coil is moving in the air gap. Note that the coil frame is made by glass material to avoid eddy currents during the oscillation.

The schematic diagram of the actuator is shown in Fig.3 (b). Two windings $W_1$ and $W_2$ are connected in subtractive series and excited by the same dc current. A permanent magnet is fixed in the middle of $W_1$ and $W_2$ at the right end of the balance beam to drive the beam into different positions by exciting different amplitude dc currents in windings. The linearity of the actuator is adjusted by changing the virtual distance $Z$ of two windings. The measured function between the vertical displacement $z$ of the movable electrode and the excitation current in windings when $Z = 40$mm is as shown in Fig.4. In the range of ±2mm, the actuator has a linearity of 0.5%, and the residual value performs as a cubic function with the excitation current.

C. Weighing sensitivity and beam length equality

The mass center modification of balance beam will reduce the weighing sensitivity for the beam balance in theory. However, the high resolution of capacitance measurement for the Kelvin capacitor makes up, or even improves the weighing sensitivity. The test result of $C_{12A}$ with weighing different small values masses is shown in Fig.5. The initial state for the balance was
weighing two 1kg test masses. When sheet masses (50mg, 100mg) were added on or taken off weighing pans, the capacitance $C_{12A}$ changed obviously with a sensitivity of 0.012pF/mg. Compared to the sensitivity of 1mg before mass center modification, the weighing sensitivity for the balance is now improved to several tens of $\mu g$.

In the approach, test masses and beam lengths should be equal, i.e., $m_1 = m_2 = m$ and $L_1 = L_2 = L$. The equivalence of test masses can be easily realized by precision definition. The beam length equality is adjusted by exchanging test masses in pans to make a torque balance, i.e., $m_1 g L_1 = m_2 g L_2$, $m_2 g L_1 = m_1 g L_2$. $E_2$ class masses are used for preliminary tests. The capacitance $C_{12A}$ with weighing different mass is shown in Fig.5. It can be seen when 1kg test masses are take off two pans ($m_1 = m_2 = 0$kg) or two 2kg masses are added ($m_1 = m_2 = 2$kg), the capacitance change of $C_{12A}$ is less than 0.01pF, therefore the beam length equality is about several parts in $10^7$.

D. Voltage source

The dc voltage source is designed as a negative feedback system based on a 160:1 resistance divider as shown in Fig.6. $U_S$ is a adjustable rippled dc voltage supply from a rectifier (up to 3000V). A power resistor $R_L$ (100$\Omega$, 100w) is to limit the charge current and capacitor $C$ (5000$\mu$F) is used as a smoothing filter. $U_{r1}$ and $U_{r2}$ are both 10V voltage references (Fluke 732B) and $R_s = 100k\Omega$ is a sampling resistor with temperature coefficient lower than 1ppm/$^\circ C$. By the feedback of the regulator, a
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Fig. 5. Test results of weighing sensitivity and beam length equality for the beam balance. The marked horizontal coordinate is the mass weighing in left pan while the vertical coordinate is the mass in right pan (unit, kg).

Fig. 6. Circuit of the designed voltage source. Several terminals are placed in $R_1$ to output different dc voltages (0V-700V-1000V-1200V-1400V-1600V).

100µA dc current is passing through the resistance divider $R_1 = 100k\Omega$ ($100k\Omega \times 1$) and $R_2 = 16M\Omega$ ($100k\Omega \times 160$). In order to obtain a stable resistance ratio, all elements in $R_1$ and $R_2$ are the same type resistors with similar temperature coefficient lower than 1ppm/°C. A typical experimental test in Fig.7 shows the stability of the designed voltage source is about 1ppm (peak-peak) in 8 hours. Note the test is operated in air and the quadratic drift is caused due the self heating of the resistance divider, which is considered to be improved by a better temperature control system (e.g., a oil tank) in future.

**IV. EXPERIMENT AND DISCUSSION**

**A. Capacitance coefficient measurement**

Two capacitances for each Kelvin capacitor, $C_{12}$ and $C_{13}$, are measured by a commercial transformer capacitance bridge AH2700A at different suspension positions. The positioning is currently controlled by an open-loop circuit as shown in Fig.8 (a). The 20bit DA outputs dc voltage between -5V and +5V. Resistor $r_1 = 100\Omega$ is made by twenty $500\Omega$ elements (2 series as one component, 10 components in parallel) to reduce the heating problem. The velocity sensor is shorted as a damper in capacitance measurement phase to reduce unwanted mechanical vibrations. The position of movable electrodes is measured by an interferometer.

To reduce the uncertainty from mechanical vibrations, the capacitance and the vertical position are synchronously measured. Besides, as the mathematical models of $C_{12}$ and $C_{13}$ are well known [13], a best fit according to the models is applied in the data analysis. A typical measurement result of function $\Sigma C(z)$ in range of $-500\mu m < z < 500\mu m$ is shown in Fig.9. Based on the symmetry of the twin Kelvin capacitor system, the function between $\Sigma C(z)$ and $z$ should follow the following equation

$$\Sigma C(z) = C_0 + \alpha_2 z^2 + \alpha_4 z^4 + \ldots,$$

(9)
where $C_0$ is the fixed component; $\alpha_2$ and $\alpha_4$ are the second-order and fourth order coefficients. As the oscillation equation is obtained with zero amplitude, the nonlinearity of the measurement must be corrected. Here a linear extrapolation between the calculated $k$ and the length of the fit interval $\Delta z$ (shown in Fig.9) is applied. And $k$ is determined as the value when $\Delta z = 0$ as

$$k = \frac{\partial^2 \sum C(z)}{\partial z^2} \mid_{\Delta z=0}. \quad (10)$$

The linear extrapolation result is shown in Fig.9. The $k$ value with zero oscillation amplitude is $7.11966 \mu F/mm^2$. The error bars (standard deviation) are about several parts in $10^4$ and the residual values of the linear fit are several parts in $10^5$.

**B. Periods measurement**

It is known that the damping of the system, which is mainly caused by air resistance and mechanical friction, will slow down the oscillation. In the periods measurement phase, a linear velocity feedback circuit is designed as shown in Fig.8(b) to compensate the energy loss in each period. The compensation can be expressed mathematically as the following equation

$$\left( \beta_1 + 2m \right) \frac{d^2 \theta}{dt^2} + \left( \xi - \frac{\varepsilon_1 \varepsilon_2}{r_2} \right) \frac{d \theta}{dt} + \left( \beta_2 - kU^2 \right) \theta = 0, \quad (11)$$

where $\xi$ is the natural damping ratio, $\varepsilon_1$ is the proportion of the velocity sensor as $u = \varepsilon_1 \frac{d \theta}{dt}$ ($u$ is the output of the velocity sensor) while $\varepsilon_2$ is the proportion of the actuator as $\tau_0 = \varepsilon_2 u/r_2$ ($\tau_0$ is the output moment of the actuator). It can be seen that the damping will be changed by choosing different resistance values of $r_2$.  

Fig. 7. A typical stability test of the voltage source. Each measurement contains 300 points. Note the quadratic drift repeats with several $\mu V$.

Fig. 8. Controlling circuits at different measurement phases. (a) capacitance coefficient measurement; (b) periods measurement.
Experimental damping behaviors of the oscillation system is shown in Fig. 10. It can be seen that the natural damping ratio $\xi$ is approximate constant when the oscillation amplitude $z_{pp} > 200 \mu m$. In the measurement, the resistor $r_2$ is selected with resistance value of 550$\Omega$ to keep the attenuation of the oscillation in a slow speed of 0.2dB/min. During the oscillation, both the oscillation period and the amplitude are simultaneously measured. The period is measured using a commercial frequency counter SR620 triggered by a rectified square waveform signal, which is converted from the velocity signal by a rectifier.

Note that for a under damping system (the damping coefficient $0 < \varsigma < 1$), the damping will introduce an error $e_d$ for the period measurement, expressed as

$$e_d = \frac{\omega - \omega_0}{\omega_0} = \sqrt{1 - \varsigma^2} - 1 \approx -\frac{\varsigma^2}{2},$$

where $\omega$ is the damped frequency and $\omega_0$ is the frequency without any damping. In the presented case when $r_2 = 550\Omega$, the damping coefficient $\varsigma$ is calculated as $6 \times 10^{-4}$ with a typical oscillation period of 10 seconds, and hence the error for measuring the oscillation period due to the damping effect is $-1.8 \times 10^{-7}$ according to (12). To achieve the measurement uncertainty of 2 parts in $10^8$ for the Planck constant determination, $e_d$ should be corrected with at least a 0.1 accuracy level.

Similar to the capacitance coefficient measurement, the nonlinear correction for periods measurement is also required. As restoring moments, either the mechanical component or the electrostatic component, are odd functions, thus the oscillation
period $T$ performs as an even function as

$$T = T_0(1 + \rho_2 z_0^2 + \rho_4 z_0^4 + ...),$$

(13)

where $T_0$ is the oscillation period with zero amplitude; $z_0$ is the oscillation amplitude; $\rho_2$ and $\rho_4$ are Taylor coefficients.

The measurement relations of periods with different test masses (0kg, 0.4kg, 1kg, 2kg) and different voltages (0V, 700V, 1000V) are shown in Fig.11 [14]. Note all the period values applied are with zero amplitude, which are calculated by linear extrapolations of $T$ and $z_0^2$. A typical nonlinearity measurement of $T$ and $z_0^2$ when $m = 1$kg is demonstrated in Fig.12. For each period measurement, the standard deviation is about several parts in $10^5$. It is concluded from Fig.12 that the electrostatic restoring moment performs a stronger nonlinearity than the mechanical component, which has been discussed in [15].
C. The Planck constant

Based on the measured oscillation period $T_0$ as a two-dimensional function of the weighing mass $m$ and the applied dc voltage $U$, both the SI value and 1990 conventional electrical value for both $\beta_1$ and $\beta_2$ can be calculated by a least-squares fit. Knowing the SI-1990 electrical ratio $\gamma$, the Planck value is obtained by (7). In the analysis, no significant systematic error is found on the principle demonstration for measuring the Planck constant with a relative uncertainty of several parts in $10^4$.

It is found by experiment that the main uncertainty (3 parts in $10^5$) of the measurement is caused by the mechanical deformation of the balance beam when different test masses are added on the mass pan. On the current stage, the balance beam is simply realized by modification of a conventional weighing balance, whose rigidity, however, is not strong enough to ensure the stability of $\beta_2$ with different masses. A further wheel realization of the balance beam, which is similar to the NIST-3 watt balance design [10] with optimized moment of inertia, can reduce the deformation effect by a factor of more than 20. In the meanwhile, an accurate correction model of this effect based on limited measurements are under development. We hope the related uncertainty component can be suppressed below $1 \times 10^{-7}$ by conjunction with mechanical optimizations and corrections.

V. Conclusion

A inertial mass measurement approach, the quasi-elastic electrostatic oscillation project, is introduced at NIM. The method avoids the difficult mass center measurement of a conventional inertial mass determination. The absolute and 1990 electrical units are related by a torque transformation on a beam balance oscillator using twin-Kelvin capacitor system. The principle, a prototype, and several experimental investigations for the inertial mass measurement are presented as the principle demonstration. A relative uncertainty of several parts in $10^4$ for the Planck constant measurement is obtained.

A new measurement system, including the wheel balance beam oscillator, a more precise capacitor manufacture, new sensors, a precision PID position control system, and a whole interferometer system, is now being considered to reduce the measurement uncertainty. A result comparison between the inertial mass measurement and the gravitational mass measurement of the Joule Balance experiment is as a first step expected in the future.

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REFERENCES

[1] N. Jones, "Tough science: five experiments as hard as finding the Higgs," Nature, vol.481, no.5, pp.14-17, Jan. 2012.
[2] M. Stock, "Watt balance experiments for the determination of the Planck constant and the redefinition of the kilogram," Metrologia, vol.50, no.1, pp.R1-R16, Dec. 2012.
[3] R. Steiner, "History and progress on accurate measurements of the Planck constant," Rep. Prog. Phys., vol.76, no.1, pp.016101-46, Dec. 2012.
[4] B. P. Kibble, "A measurement of the gyromagnetic ratio of the proton by the strong field method," Atomic Masses and Fundamental Constants 5ed, J H Sanders and A H Wapstra (New York: Plenum), pp.545-551.
[5] B. Andreas B et al, "2011 determination of the Avogadro constant by counting the atoms in a 28Si crystal," Phys. Rev. Lett., vol.106, no.3, pp.030801-04, Jan. 2011.
[6] I. M. Mills, P. J. Mohr, T. J. Quinn, B. N. Taylor and E. R. Williams, "Redefinition of the kilogram, ampere, kelvin and mole: a proposed approach to implementing CIPM recommendation 1 (CI-2005)," Metrologia, vol.43, no.3, pp.227-246, Jun. 2006.
[7] P. Umberto, S. Danilo, "Prototype of a pendulum for deriving the kilogram from electrical quantities," IEEE Trans. Instrum. Meas., vol.58, no.4, pp.930-935, Apr. 2009.
[8] S. Li, Z. Zhang, Q. He, et al, "A proposal for absolute determination of inertial mass by measuring oscillation periods based on the quasi-elastic electrostatic force," Metrologia, vol.50, no.1, pp.9-14, Dec. 2012.
[9] V. Bego, J. Butorac J and K. Poljancic, "Voltage balance for replacing the kilogram," IEEE Trans. Instrum. Meas., vol.44, no.2, pp.579-582, Apr. 1995.
[10] V. Bego, J. Butorac and D. Ilic, "Realization of the kilogram by measuring at 100 kV with the voltage balance ETF," IEEE Trans. Instrum. Meas., vol.48, no.2, pp.212-215, Apr. 1999.
[11] S. Li, B. Han, Z. Li, et al, "Precisely measuring the Planck constant by electromechanical balances," Measurement, vol.45, no.1, pp.1-13, Jan. 2012.
[12] T. J. Quinn, "News from the BIPM," Metrologia, vol.26, no.1, 69-74, Jan. 1989.
[13] W. C. Heerens, F. C. Vermeulen, "Capacitance of Kelvin guard-ring capacitors with modified edge geometry," Journal of Applied Physics, vol.46, no.6, pp.2486-2490, Jun. 1975.
[14] S. Li, Z. Zhang, Q. He, et al, "An Inertial Mass Measurement Prototype at NIM", Conference on Precision Electromagnetic Measurements (CPEM 2014) Rio de Janeiro, Brazil, Aug. 2014.
[15] S. Li, J. Lan, B. Han, H. Tan, Z. Li, "Nonlinearity and periodic solution of a standard-beam balance oscillation system", Chinese Physics B, vol.21, no.6, pp.064601-5, Jun. 2012.
[16] S. Schlammingen, D. Haddad, F. Seifert, et al, "Determination of the Planck constant using a watt balance with a superconducting magnet system at the National Institute of Standards and Technology", Metrologia, vol51, no2, pp.S15-S24, Mar. 2014.