GLOBAL TOPOLOGIES OF REEB SPACES OF STABLE FOLD MAPS WITH NON-TRIVIAL TOP HOMOLOGY GROUPS

NAOKI KITAZAWA

Abstract. The Reeb space of a continuous map is the space of all (elements representing) connected components of preimages endowed with the quotient topology induced from the natural equivalence relation on the domain. These objects are strong tools in (differential) topological theory of Morse functions, fold maps, which are their higher dimensional variants, and so on: they are in general polyhedra whose dimensions are same as those of the targets. In suitable cases Reeb spaces inherit topological information such as homology groups, cohomology rings, and so on, of the manifolds.

This presents the following problem: what are global topologies of Reeb spaces of these smooth maps of suitable classes like? The present paper presents families of stable fold maps having Reeb spaces with non-trivial top homology groups with their (co)homology groups (and rings). Related studies on the global topologies from the viewpoints of the singularity theory of differentiable maps and differential topology have been presented by various researchers including the author. The author previously constructed families of fold maps with Reeb spaces with non-trivial top homology groups and with good topological properties. This paper presents new families, especially, generalized situations of some known situations.

1. Introduction.

For a continuous map \( c : X \rightarrow Y \), we can define the equivalence relation \( \sim_c \) on \( X \) so that \( x_1 \sim_c x_2 \) if and only if they are in a same connected component of a preimage. We can define the Reeb space of \( c \) as follows.

Definition 1. In the situation above, the quotient space \( W_c := X/\sim_c \) is the Reeb space of \( c \).

\( q_f : X \rightarrow W_c \) denotes the quotient map and \( \bar{c} \) denotes the map uniquely obtained so that \( c = \bar{c} \circ q_c \).

Reeb spaces are strong tools in (differential) topological theory of Morse functions, fold maps, which are their higher dimensional variants, and so on: they are in general polyhedra whose dimensions are same as those of the targets. In suitable cases Reeb spaces inherit topological information such as homology groups, cohomology rings, and so on, of the manifolds. They play important roles in applied mathematics or applications of mathematical science such as data analysis and visualizations. [53] is a related article.

Hereafter, \( p \in X \) is said to be a singular point if for a differentiable map \( c : X \rightarrow Y \) the rank of the differential \( dc_p \) there is smaller than both the dimensions of

---

Key words and phrases. Reeb spaces. Morse functions and fold maps. (Co)homology. Polyhedra.

2020 Mathematics Subject Classification: Primary 57R45. Secondary 57R19.
the domain and the target. The singular set of a differentiable map \( c \) is the set of all singular points, denoted by \( S(c) \). The singular value set of \( c \) means the image of the singular set. A singular value of \( c \) is a point in the singular value set. The regular value set of \( c \) means the complementary set of the singular value set. A regular value of \( c \) is a point in the regular value set.

For Morse functions and smooth functions which are not so wild, the Reeb spaces are graphs where the vertex sets are the sets of all (elements representing) connected components of preimages containing at least one singular point: see [50] for example. Reeb spaces are in such cases Reeb graphs.

We present simplest Morse functions and fold maps and their Reeb spaces. Fold maps are reviewed in the next section.

Example 1. For Morse functions on closed manifolds with exactly two singular points, if the dimensions of the manifolds are greater than 1, then the Reeb spaces are graphs with exactly two vertices and one edge.

Canonical projections of unit spheres are simplest examples of fold maps and special generic maps: we review the class of special generic maps as an important subclass in the next section. For such a map, the singular set is an equator. The images are unit discs.

These maps are higher dimensional versions of Morse functions on unit spheres obtained by considering the natural heights: these functions are simplest ones in the functions before.

If the dimensions of the manifolds are greater than the dimensions of the targets, then the Reeb spaces are unit discs of the targets.

For a copy of the 2-dimensional torus embedded naturally in the 3-dimensional Euclidean space, the Morse function obtained by the natural height has exactly four singular points and the Reeb graph is as follows.

1. The vertex set consists of exactly four vertices. For exactly two vertices, the degrees are 1 and for the remaining vertices, the degrees are 3.
2. The edge set consists of exactly four edges.
3. The \( j \)-th integral homology group of the Reeb space is isomorphic to \( \mathbb{Z} \) for \( j = 0, 1 \).

We present two problems on Reeb spaces.

Problem 1. For a given graph, can we construct a good smooth function of a given suitable class?

[54] is a pioneering study. [40], [43], [50], and so on, are important works and there exist other closely related works. [24], [26] and [27] are related works by the author.

The following problem is related to the present study more.

Problem 2. Construct good Morse functions, fold maps, and more general good smooth maps and investigate the topologies of their Reeb spaces?

Related to this, [37] concentrates on Reeb spaces of so-called stable maps into the plane on closed smooth manifolds whose dimensions are greater than 2. It is shown that for 2-dimensional polyhedra whose homology groups are isomorphic to that of a disc under the situation that the coefficient is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \) and having suitable local properties can be Reeb spaces of stable maps of suitable
classes. Stable maps are explained in the next section as ones forming an important subclass of smooth maps.

The author have studied explicit topologies of Reeb spaces of explicit fold maps via construction of the maps first in [17] and later in [23], [25], [28], [30], [31], and so on: for closely related studies on manifolds admitting such maps see [29] and [32] for example. Such studies are important and difficult and are also different from systematic theory of knowing existence of such maps in [2], [3], and so on. Their homology groups and cohomology rings are various. Moreover, for example, top-homology groups are not trivial and free. As another important remark, these maps and Reeb spaces are obtained via very explicit and simple operations changing maps and manifolds locally starting from very fundamental fold maps such as canonical projections of unit spheres.

The present paper concentrates on the following problem.

Main Problem. Construct explicit (families of) good Morse functions, fold maps, and more general smooth maps satisfying suitable differential topological properties and presenting Reeb spaces which do not collapse to lower dimensional polyhedra. Investigate their topologies.

Main Theorems are as follows, leaving expositions on several undefined notions and notation later.

The following is of a new type.

Main Theorem 1 (Theorem 3.). Let \( l \geq 0 \) and \( m > n \geq 1 \) be integers and \( \{X_j\}_{j=1}^l \) be a family of finitely many \( n \)-dimensional closed, connected and stably parallelizable manifolds Let \( \{(F_{j,1}, F_{j,2})\}_{j=1}^l \) be a family of pairs of \((m-n)\)-dimensional closed, connected and smooth manifolds for each of which a disjoint union of the two manifolds bounds a compact, connected and smooth manifold \( F_j \) obtained by attaching handles containing at most \( 1 \) 1-handle to \((F_{j,1} \sqcup F_{j,2}) \times (\{0\} \subset [0,1])\). Let \( X \) be an \( n \)-dimensional connected and compact manifold we can smoothly immerse into \( \mathbb{R}^n \). Then we have a simple fold map \( f \) on an \( m \)-dimensional closed and connected manifold \( M \) into \( \mathbb{R}^n \) satisfying the following two.

1. The Reeb space \( W_f \) is a branched manifold of the class \( A \) simple homotopy equivalent to a space obtained by a finite iteration of taking a bouquet starting from \( l + 1 \) polyhedra where the \( j \)-th polyhedron collapses to \( X_j \) for \( 1 \leq j \leq l \) and is \( X \) for \( j = l + 1 \).
2. There exist connected components of the preimage of a regular value diffeomorphic to \( F_{j,1} \) and \( F_{j,2} \) for each \( j \).

The following two generalize situations of ones of existing results, presented as Theorems 1 and 2, abstractly.

Main Theorem 2 (Theorem 4.). Let \( l \geq 0 \) and \( m > n \geq 1 \) be integers. Let \( \{(F_{j,1}, F_{j,2})\}_{j=1}^l \) be a family of pairs of \((m-n)\)-dimensional closed, connected and smooth manifolds for each of which a disjoint union of the two manifolds bounds a compact, connected and smooth manifold \( F_j \) obtained by attaching handles containing at most \( 1 \) 1-handle to \((F_{j,1} \sqcup F_{j,2}) \times (\{0\} \subset [0,1])\). Let \( X \) be an \( n \)-dimensional connected and compact manifold we can smoothly immerse into \( \mathbb{R}^n \). Let \( \{Y_j \subset X\}_{j=1}^l \) be a family of mutually disjoint \( n \)-dimensional compact, connected and smooth submanifolds of \( X \). Then we have a simple fold map \( f \) on an \( m \)-dimensional closed and connected manifold \( M \) into \( \mathbb{R}^n \) such that the Reeb space
$W_f$ is a branched manifold of the class $A$ obtained by attaching $l$ manifolds each of which is diffeomorphic to a double $DY_j$ of $Y_j$ identifying $Y_j$ in $X$ and $Y_j \subset DY_j$ canonically and that there exist connected components of the preimage of a regular value diffeomorphic to $F_{j,1}$ and $F_{j,2}$ for each $j$.

**Main Theorem 3** (Theorem 5.). In Theorem 4, assume that $l > 0$, that $X$ is obtained by attaching $h_p$ $p$-handles for $1 \leq p \leq n-1$ starting from $0$-handles and that the following four hold.

1. $H_1(X;\mathbb{Z}) \cong \mathbb{Z}^{h_1-1(l-1)}$.
2. $H_p(X;\mathbb{Z}) \cong \mathbb{Z}^{h_p}$ for $2 \leq p \leq n-1$.
3. $F_j$ is obtained by attaching $h_{j,p}$ $p$-handles for $1 \leq p \leq n-1$ starting from a 0-handle.
4. $X - \bigcup_{j=1}^{l} \text{Int} Y_j$ is obtained by attaching $h_p - \Sigma_{j=1}^{l} h_{j,p}$ $p$-handles to the product of a manifold diffeomorphic to the boundary and $\{0\} \subset [0,1]$ for all $1 \leq p \leq n-1$.

Then, we have the following facts on homology groups and cohomology groups and rings.

1. $H_p(W_f;\mathbb{Z})$ is isomorphic to the direct sum of $H_p(X;\mathbb{Z}) \oplus \mathbb{Z}^{\Sigma_{j=1}^{l} h_{j,n-p}}$ for $1 \leq p \leq n-1$ and $\mathbb{Z}^l$ for $p = n$.
2. $H^p(W_f;\mathbb{Z})$ is isomorphic to $A_p \oplus \oplus_{j=1}^{l} (H^p(Y_j;\mathbb{Z})) \oplus \oplus_{j=1}^{l} (H^p(DY_j - \text{Int} Y_j;\mathbb{Z}))$ where $A_p$ is isomorphic to $\mathbb{Z}^{h_{1}-\Sigma_{j=1}^{l} h_{j,1}-(l-1)}$ for $p = 1$ and $\mathbb{Z}^{h_{p}-\Sigma_{j=1}^{l} h_{j,p}}$ for $2 \leq p \leq n-1$. $H^n(W_f;\mathbb{Z})$ is isomorphic to $\mathbb{Z}^l$.
3. $H^*(X;\mathbb{Z})$ is regarded as a subalgebra of $H^*(W_f;\mathbb{Z})$ and the cohomology group is identified with $A_p \oplus \oplus_{j=1}^{l} (H^p(Y_j;\mathbb{Z})) \oplus \oplus_{j=1}^{l} (H^p(DY_j - \text{Int} Y_j;\mathbb{Z}))$ before for each degree $1 \leq p \leq n-1$. $H^*(\bigcup DY_j;\mathbb{Z})$ is regarded as a subalgebra of $H^*(W_f;\mathbb{Z})$ and the cohomology group is identified with $\{0\} \oplus \oplus_{j=1}^{l} (H^p(Y_j;\mathbb{Z})) \oplus \oplus_{j=1}^{l} (H^p(DY_j - \text{Int} Y_j;\mathbb{Z}))$ for each degree $1 \leq p \leq n-1$ and $\oplus_{j=1}^{l} (H^n(DY_j;\mathbb{Z})) \cong \mathbb{Z}^l$ for degree $p = n$.
4. For $1 \leq p_1, p_2 \leq n-1$, the product of an element of degree $p_1$, identified with an element of $A_{p_1} \oplus \oplus_{j=1}^{l} (H^{p_1}(Y_j;\mathbb{Z})) \oplus \oplus_{j=1}^{l} (H^{p_1}(DY_j - \text{Int} Y_j;\mathbb{Z}))$ and an element of degree $p_2$, identified with an element of $\{0\} \oplus \oplus_{j=1}^{l} (H^{p_2}(DY_j - \text{Int} Y_j;\mathbb{Z})) \subset A_{p_1} \oplus \oplus_{j=1}^{l} (H^{p_2}(Y_j;\mathbb{Z})) \oplus \oplus_{j=1}^{l} (H^{p_2}(DY_j - \text{Int} Y_j;\mathbb{Z}))$, vanishes in $H^*(W_f;\mathbb{Z})$.

In the next section, we review the definition and fundamental properties of fold maps and their Reeb spaces. We also review special generic maps as an important subclass and their Reeb spaces, which are compact and smoothly immersed manifolds in the Euclidean spaces. The class of simple fold maps extends the class in a natural way. The Reeb spaces of such maps are so-called branched manifolds. We also refer to stable maps as an important subclass of smooth maps as a kind of appendices. The third section is devoted to Main Theorems with supporting existing studies.

Hereafter, manifolds, maps between manifolds, (boundary) connected sums, and so on, are considered in the smooth category (or of the class $C^\infty$), unless otherwise stated. However, we also discuss in the PL, or as an equivalent category, the piecewise smooth category. For example, we consider PL bundles and smooth bundles as bundles whose fibers are smooth or PL manifolds or general polyhedra.
2. (Stable) fold maps and their Reeb spaces.

Definition 2. Let \( m \geq n \geq 1 \) be integers. Let \( M \) be an \( m \)-dimensional closed manifold and \( N \) be an \( n \)-dimensional manifold with no boundary. A smooth map \( f : M \to N \) is said to be a fold map if at each singular point \( p \) \( f \) is represented as
\[
(x_1, \ldots, x_m) \mapsto \left(x_1, \ldots, x_{n-1}, \sum_{j=n}^{m-i(p)} x_j^2 - \sum_{j=m-i(p)+1}^{m} x_j^2\right)
\]
via suitable coordinates and a suitable integer \( 0 \leq i(p) \leq \frac{m-n+1}{2} \).

For systematic studies and advanced expositions on fold maps, see [4] and [47] for example. Morse functions are for specific cases or \( n = 1 \) with \( N = \mathbb{R} \). [41] and [42] present fundamental information on Morse functions and as a related notion, \( j \)-handle and an handle attachment in [42] and so on.

Proposition 1. In Definition 2, for a fold map \( f \), \( i(p) \) is unique: we call the integer the index of \( p \). The restrictions to the singular set \( S(f) \) and the set of all singular points of a fixed index, which are proven to be \((n-1)\)-dimensional closed and smooth submanifolds with no boundaries, are immersions.

Definition 3. A fold map is said to be special generic if the indices of singular points are always 0.

A smooth manifold is well-known to be regarded as an object in the PL category or a polyhedron canonically. Hereafter, for a smooth manifold, a polyhedron means this. For the following, refer to [48], [49] and [55] (or fact 1).

Proposition 2. Let \( m > n \geq 1 \) be integers. Let \( M \) be an \( m \)-dimensional closed manifold and \( N \) be an \( n \)-dimensional manifold with no boundary. For a fold map \( f \), the Reeb space is an \( n \)-dimensional polyhedron uniquely induced from the manifold \( N \) and \( q_f \) and \( \bar{f} \) are piecewise smooth maps. The Reeb space of a special generic map \( f \) is a compact manifold smoothly immersed into \( \mathbb{R}^n \) via \( \bar{f} \).

Definition 4. A fold map \( f \) is said to be simple if the restriction of \( q_f \) to \( S(f) \) is injective.

See [47] and [52] for example for simple fold maps. In the following proposition, the Reeb space is not a manifold in general: it may be a so-called branched manifold. We define a branched manifold in Definition 8 as a polyhedron locally PL homeomorphic to a Reeb space in Propositions 5 and 6. 2-dimensional polyhedra of this kind are studied in various articles. [57] is a pioneering work on so-called 2-dimensional simple polyhedra or so-called shadows. [11], [39], [45], and so on, are closely related to this. These polyhedra form a class wider than that of the 2-dimensional branched manifolds here. [44] and [46] are on multibranched surfaces and regarded as slightly generalized versions of these 2-dimensional branched manifolds.

Proposition 3. A special generic map is simple. Let \( m > n \geq 1 \) be integers. Let \( M \) be an \( m \)-dimensional closed manifold and \( N \) be an \( n \)-dimensional manifold with no boundary. For a simple fold map \( f : M \to N \), \( W_f \) is a branched manifold.

Last we add an exposition on so-called (topologically) stable maps as a kind of appendices. See [4] for example for systematic expositions.
Definition 5. A smooth map $f$ between manifolds $M$ and $N$ with no boundaries is said to be a **stable** map if there exists an open neighborhood in the space of all smooth maps from $M$ into $N$ endowed with the so-called **Whitney $C^\infty$ topology**, and for any map $f'$ there, there exists a pair $(\Phi_{f'}, \phi_{f'})$ of diffeomorphisms satisfying $\phi_{f'} \circ f' = f \circ \Phi_{f'}$. If we replace the diffeomorphisms by homeomorphisms, then $f$ is said to be **topologically stable**.

The following fact is a very general fact. We omit the definition of a **Thom map** and precise expositions.

**Fact 1** ([55].) **For a fold map, (topologically) stable map, and more generally, a so-called Thom map, the Reeb space is a polyhedron.**

For smooth functions of suitable classes, stable maps into the plane on closed manifolds whose dimensions are greater than 2, this has been explicitly shown in [37] and so on.

**Proposition 4.** A fold map $f$ in Definition 2 is stable if and only if for each value $p \in f(S(f))$ the following conditions hold.

1. The preimage $f^{-1}(p)$ consists of exactly $l \geq 1$ points and is denoted by $\{p_j\}_{j=1}^l$.
2. The dimension of the intersection of all images of the differential $d_{f p_j}$ is $n - l$.

Definition 6. An immersion of a smooth manifold with no boundary into another smooth manifold with no boundary is said to have **normal crossings only**, if this satisfies the two conditions the restriction $f|_{S(f)}$ in Proposition 4 satisfies (where we change these two in a suitable way to apply here).

Example 2. A Morse function such that the preimage of a singular value always has exactly one singular point is stable. On the other hand, a Morse function is stable if and only if this condition holds. It is a well-known fundamental fact that such Morse functions exist densely on any (closed) manifold. A fold map such that the restriction to the singular set is an embedding is stable.

In fact, a fold map can be deformed into a stable one by a slight perturbation or we can find a stable fold map close to the original map under the Whitney $C^\infty$ topology. Moreover, we concentrate on simple fold maps in the present study and for such maps, this notion does not affect on our arguments essentially. The following is an important proposition in knowing homology groups and cohomology rings of the manifolds from Reeb spaces in specific cases.

Hereafter, a **standard sphere** is a smooth manifold which is diffeomorphic to a unit sphere $S^k$: $S^k$ denotes the $k$-dimensional unit sphere. For the $k$-dimensional Euclidean space $\mathbb{R}^k$ and $p \in \mathbb{R}^k$, $||p||$ denotes the distance between $p$ and the origin 0 where the Euclidean space is endowed with the standard Euclidean metric.

A **linear bundle** is a smooth bundle whose fiber is diffeomorphic to a unit disc $D^k := \{x \mid ||x|| \leq 1\} \subset \mathbb{R}^k$ or a unit sphere $S^k := \{x \mid ||x|| = 1\} \subset \mathbb{R}^{k+1}$ and whose structure group acts on the fiber linearly in a canonical way.

**Proposition 5.** Let $m > n \geq 1$ be integers. Let $M$ be an $m$-dimensional closed manifold and $N$ be an $n$-dimensional manifold with no boundary. Let $f$ be a simple fold map. Assume also the following three.
Then we have the following five.

1. Indices of singular points of \( f \) are always 0 or 1.
2. Preimages of regular values are disjoint unions of copies of standard spheres.
3. \( M \) is orientable if \( m - n > 1 \).

Then we have the following three.

1. For the image \( q_f(C) \) of each connected component \( C \) of the singular set consisting of singular points of index 0, there exists a regular neighborhood PL homeomorphic to \( q_f(C) \times [0,1] \) where \( q_f(C) \) is identified canonically with \( q_f(C) \times \{0\} \). Furthermore, the composition of the restriction of \( q_f \) to the preimage of the regular neighborhood with the canonical projection to \( q_f(C) \) gives a linear bundle whose fiber is diffeomorphic to \( D^{m-n+1} \).
2. For the image \( q_f(C) \) of each connected component \( C \) of the singular set consisting of singular points of index 1, there exists a regular neighborhood PL homeomorphic to the total space of a PL bundle over \( q_f(C) \) whose fiber is PL homeomorphic to \( K := \{(r \cos t, r \sin t) \mid 0 \leq r \leq 1, t = \frac{a}{2} \pi, a = 0, 1, 2 \} \subset \mathbb{R}^2 \). Furthermore the following three must hold.
   a. The structure group is isomorphic to the symmetric group \( S_2 \) of degree 2 preserving the distance \( |p| \) between each point \( p \in K \) and the origin 0 and fixing \( \{(r,0) \mid 0 \leq r \leq 1\} \).
   b. \( q_f(C) \) is identified canonically with \( q_f(C) \times \{0\} \) in the bundle.
   c. Furthermore, the composition of the restriction of \( q_f \) to the preimage of the regular neighborhood with the canonical projection to \( q_f(C) \) gives a smooth bundle whose fiber is diffeomorphic to a manifold obtained by removing three disjointly and smoothly embedded copies of \( D^{m-n+1} \) in a copy of \( S^{m-n+1} \).
3. \( q_f \) induces the isomorphisms \( \lambda_j : H_j(M;A) \to H_j(W_f;A), q_f^* : H^j(W_f;A) \to H^j(M;A) \) and \( \lambda_j : \pi_j(M) \to \pi_j(W_f) \) for \( 0 \leq j \leq m - n \). If \( f \) is special generic map, then these isomorphisms work for \( j = m - n \).
4. \( q_f^* : H^*(W_f;A) \to H^*(M;A) \) preserves cup products for any pair such that the sum of the degrees is smaller than \( m - n \) and we can replace ”smaller than” by ”smaller than or equal to” for a special generic map \( f \).
5. Cup products of \( M \) for any pair such that the degree of each class is smaller than \( m - n \) and that the sum of the degrees is greater than \( n \) always vanish. For a special generic map \( f \), we can replace ”smaller than” by ”smaller than or equal to” and ”greater than” by ”greater than or equal to”.

Definition 7. A stable fold map \( f \) satisfying the assumption of Proposition 5 is said to be standard-spherical.

This class, extending the class of special generic maps naturally, has been studied in various scenes. See [12], [13], [16], [20], [21], [22], [29], [32], [33], [34], [51] and so on. We also present a general proposition as Proposition 6, regarded as a proposition explaining Proposition 3 more precisely.

Proposition 6. Let \( m > n \geq 1 \) be integers. Let \( M \) be an \( m \)-dimensional closed manifold and \( N \) be an \( n \)-dimensional manifold with no boundary. Let \( f \) be a simple fold map. Then we have the following three.

1. For the image \( q_f(C) \) of each connected component \( C \) of the singular set consisting of singular points of index 0, there exists a small regular neighborhood PL homeomorphic to \( q_f(C) \times [0,1] \) where \( q_f(C) \) is identified canonically with \( q_f(C) \times \{0\} \). Furthermore, the composition of the restriction of
Let \( q_f \) to the preimage of the regular neighborhood with the canonical projection to \( q_f(C) \) gives a linear bundle whose fiber is diffeomorphic to \( D^{m-n+1} \).

(2) For the image \( q_f(C) \) of each connected component \( C \) of the singular set consisting of singular points of index \( j > 0 \), there exists a small regular neighborhood PL homeomorphic to the total space of a PL bundle over \( q_f(C) \) such that fibers satisfy either of the following two. Furthermore, the composition of the restriction of \( q_f \) to the preimage of the regular neighborhood with the canonical projection to \( q_f(C) \) gives a smooth bundle.

(a) The fiber is PL homeomorphic to \( K := \{(r \cos t, r \sin t) \mid 0 \leq r \leq 1, t = \frac{2a}{3} \pi, a = 0, 1, 2 \} \subset \mathbb{R}^2 \). Furthermore the following two must hold.
   (i) The structure group is isomorphic to the symmetric group \( S_2 \) of degree 2 preserving the distance \(||p||\) between each point \( p \in K \) and the origin 0 and fixing \( \{(r, 0) \mid 0 \leq r \leq 1 \} \).
   (ii) \( q_f(C) \) is identified canonically with \( q_f(C) \times \{0\} \) in the bundle.

(b) The fiber is PL homeomorphic to \( I := [-1, 1] \subset \mathbb{R} \). Furthermore the following two must hold.
   (i) The structure group is trivial.
   (ii) \( q_f(C) \) is identified canonically with \( q_f(C) \times \{0\} \) in the bundle.

(3) The composition of the restriction of \( q_f \) to the preimage of the regular neighborhood with the canonical projection to \( q_f(C) \) gives a smooth bundle.

3. **Main Theorems and existing supporting studies.**

Let \( n \) be a positive integer. Let \( \{G_j\}_{j=0}^n \) be a sequence of finitely generated commutative groups of length \( n \) such that \( G_0 \) is trivial, that \( G_{n-1} \) is free, that \( G_n \) is free and non-trivial, and that a suitable additional condition (A) holds.

Motivated by the importance and the difficulty of construction of explicit fold maps and obtaining information of the resulting manifolds, the author has obtained several results of the following types via *bubbling operations*, which are surgery operations to manifolds and maps. The property (B) denotes a suitable property on the structure of the (resulting) fold maps.

**Theorem 1.** Let \( f_0 : M_0 \to \mathbb{R}^n \) be a stable fold map on an \( m \)-dimensional closed and connected manifold \( M_0 \) into \( \mathbb{R}^n \) satisfying \( m \geq n \geq 1 \).

Then by a finite iteration of bubbling operations starting from \( f_0 \), we have a stable fold map \( f \) satisfying the property (B) and \( H^j(W_f; \mathbb{Z}) \cong H^j(W_{f_0}; \mathbb{Z}) \oplus G_j \) for \( 0 \leq j \leq n \).

Especially, if \( f_0 \) is standard-spherical, then we can construct a standard-spherical map \( f \) in this way.

**Theorem 2.** Let \( f_0 : M_0 \to \mathbb{R}^n \) be a stable fold map on an \( m \)-dimensional closed and connected manifold \( M_0 \) into \( \mathbb{R}^n \) satisfying \( m \geq n \geq 1 \).

Then by finite iterations of bubbling operations, we have a family \( \{f_\lambda : M_\lambda \to \mathbb{R}^n \} \) of infinitely many stable fold maps satisfying the property (B) and \( H^j(W_{f_\lambda}; \mathbb{Z}) \cong H^j(W_{f_0}; \mathbb{Z}) \oplus G_j \) for \( 0 \leq j \leq n \) and for distinct \( \lambda_1 \) and \( \lambda_2 \), the integral cohomology rings of the resulting Reeb spaces are mutually non-isomorphic.

As in Theorem 1, if the given map \( f_0 \) is standard-spherical, then we can construct these fold maps as standard-spherical maps.

We review a *bubbling operation* shortly.
First, in a connected component of the regular value set, we choose a suitable closed, and connected smooth submanifold with no boundary or a polyhedron represented as one obtained by a finite iteration of taking a bouquet of two such submanifolds starting from finitely many such submanifolds. We remove the interior of a connected component of the preimage of a small regular neighborhood of this. We attach a new smooth map so that the singular value set is the disjoint union of the original singular value set and a connected submanifold diffeomorphic and parallel to the boundary made after the removal. It is motivated by several operations for construction of new stable (fold) maps in [35], [36] and [37]. Especially, a bubbling surgery, first defined and systematically presented in [36], has motivated the author to study in this way. A bubbling surgery is a case where the submanifold is a point.

This is extended for cases where connected submanifolds or submanifolds in subpolyhedra are images of manifolds via immersions having normal-crossings only and cases where the submanifolds are standard spheres are studied in [30] and [31]. In the present paper, we do not concentrate on such cases.

For studies related to this, see [17], [23], [25], [28], [30], [31], and so on. For ones concentrating more on manifolds admitting the maps, see [29], [32], and so on. [12]–[16] are on round fold maps: they are, shortly, stable fold maps such that the singular values are embedded concentric spheres and canonical projections of unit spheres are simplest round fold maps. This class forms an important subclass of fold maps, introduced first by the author. In [36] and so on, Kobayashi has studied this class without defining the class. This class has also motivated the author to introduce the bubbling operations, extending construction of some round fold maps.

We present results or regarded as ones on global topologies of Reeb spaces of stable fold maps of suitable classes.

[40] and so on say, that a graph with no loops is regarded as the Reeb space of a smooth function of a suitable class on a closed surface. Posing additional conditions on the graphs, the graphs can be regarded as the Reeb graphs of Morse functions satisfying suitable conditions on closed surfaces and manifolds. Furthermore, if graphs may have loops, then by replacing the targets with circles, similar facts hold.

We explain higher dimensional cases. There exist several known related facts for 2-dimensional cases. We introduce several notions. Respecting Proposition 6, we can define a branched manifold of the class $A(B)$. It may be a notion to be defined in a more general form. However, in the present paper, we define the notion as follows.

**Definition 8.** A $k$-dimensional compact polyhedron $X$ satisfying the following two is said to be a branched manifold of the class $A$.

1. There exists a family $\{Y_j\}$ of finitely many compact and connected PL submanifolds with no boundaries.
2. $X - \bigsqcup Y_j$ is an $k$-dimensional manifold with no boundary.
3. For each $Y_j$, either of the following two holds.
   a. There exists a small regular neighborhood PL homeomorphic to $Y_j \times [0, 1]$ where $Y_j$ is identified canonically with $Y_j \times \{0\}$. 

There exists a small regular neighborhood PL homeomorphic to the total space of a PL bundle over $Y_j$ such that the fiber is PL homeomorphic to $K := \{(r \cos t, r \sin t) \mid 0 \leq r \leq 1, t = \frac{2a\pi}{3}, a = 0, 1, 2\} \subset \mathbb{R}^2$.

Furthermore the following two must hold.

(i) The structure group is isomorphic to the symmetric group $S_2$ of degree 2 preserving the distance $|p|$ between each point $p \in K$ and the origin 0 and fixing $\{(r, 0) \mid 0 \leq r \leq 1\}$.

(ii) $Y_j$ is identified canonically with $Y_j \times \{0\}$ in the bundle.

If we replace the phrase “the symmetric group $S_2$ of degree 2 preserving the distance $|p|$ of each point $p \in K$ and the origin 0 and fixing $\{(r, 0) \mid 0 \leq r \leq 1\}$” by “the symmetric group $S_3$ of degree 3 preserving the distance $|p|$ between each point $p \in K$ and the origin 0”, then $X$ is said to be a branched manifold of the class $B$.

A (2-dimensional) simple polyhedron with no vertices is a branched manifold of the class $B$. The following is essentially equivalent to Proposition 5.2 in [37].

**Fact 2** (Essentially shown in [37]). Let $m > 2$ be an integer. If a 2-dimensional branched manifold $X$ of the class $A$ satisfies $H_j(X; \mathbb{Z}/2\mathbb{Z}) \cong H_j(D^m; \mathbb{Z}/2\mathbb{Z})$ for any $j$, then it is PL homeomorphic to the Reeb space of a standard-spherical map on an $m$-dimensional closed and connected manifold into the plane.

The following is closely related and to some extent equivalent to Fact 2 and a result proven by a different argument.

**Fact 3** ([45]). If a 2-dimensional branched manifold $X$ of the class $B$ is simply-connected and satisfies $H_j(X; \mathbb{Z}) \cong H_j(D^2; \mathbb{Z})$ for any $j$, then it collapses to a point.

More precisely, he uses graphs with some labels which are fundamental and strong tools in representing the polyhedra, introduced in [39]. He also shows that this is of the class $A$ in proving this.

Note also that Naoe shows this to prove important results in low-dimensional differential topology there: he has shown that 3-dimensional closed manifolds represented by such shadows or admitting simple stable fold maps whose Reeb spaces are isomorphic to such polyhedra are 3-dimensional standard spheres and that 4-dimensional compact manifolds represented by such shadows are diffeomorphic to the 4-dimensional unit disc. For shadows and such manifolds represented by shadows, see [57] and see also [11], for example. Note also that Martelli’s studies such as [39] are closely related to these studies.

From this, we can see that such a polyhedron is PL homeomorphic to the Reeb space of a standard-spherical map on an $m$-dimensional closed and connected manifold into the plane where $m \geq 3$. We show a sketch of a proof of this fact. In Definition 8, we construct a smooth map on the preimage of the small regular neighborhood of $Y_j$ as a smooth family of a Morse function over $Y_j$ ($Y_j \times \{0\}$) satisfying either of the following two respecting the topology of the regular neighborhood of $Y_j$.

1. A product map of a Morse function on a copy of $D^{m-1}$ which is obtained by considering a natural height and represented as the form $(x_1, \ldots, x_{m-1}) \rightarrow \sum_{j=1}^{m-1} x_j^2$ for suitable coordinates and the identity map on $Y_j \times \{0\}$ for $Y_j$ such that the regular neighborhood is PL homeomorphic to $Y_j \times [0, 1]$. 

(2) A smooth family of a Morse function on a manifold obtained by removing the interiors of 3 disjointly and smoothly embedded copies of an \((m-1)\)-dimensional unit disc from an \((m-1)\)-dimensional standard sphere on \(Y_j \times \{0\}\) satisfying the following two for \(Y_j\) such that the regular neighborhood is PL homeomorphic to the total space of a bundle over \(Y_j\) whose fiber is PL homeomorphic to \(K\).

(a) The Morse function has exactly one singular point in the interior, the image is a closed interval and the preimage of the minimal value and that of the maximal value are exactly one connected component of the boundary and exactly two connected components of the boundary, respectively.

(b) The Morse function is invariant under a suitable smooth action by \(\mathbb{Z}/2\mathbb{Z}\): the family is trivial if the bundle over \(Y_j \times \{0\}\) whose fiber is PL homeomorphic to \(K\) is product and not trivial if the bundle over \(Y_j \times \{0\}\) whose fiber is PL homeomorphic to \(K\) is not trivial.

Last, on the preimage of the complementary set of the disjoint union of the interiors of the small regular neighborhoods for all \(Y_j\)'s, we construct the projection of a trivial bundle over the complementary set whose fiber is an \((m-2)\)-dimensional standard sphere. A kind of induction respecting collapsing for simplicial complexes of the branched manifolds presents a desired map.

Main Theorems give new explicit construction of (simple) stable fold maps and their Reeb spaces, which are branched manifolds of the class A.

Theorem 3 is of a new type. Theorems 4 and 5 are regarded as results generalizing situations of ones of existing results presented as Theorems 1–2 abstractly.

Before presenting them, we explain fundamental notions.

Definition 9. For a module \(A\) over a commutative ring \(R\) having a unique identity element \(1\) different from the zero element \(0\)

(1) If \(a \in A\) is not represented as \(a = ra'\) for any pair of an element \(r \in R\) which is not a unit and \(a' \in A\), then \(a\) is said to be a unit element.

(2) For an arbitrary unit element \(a \in A\), we can uniquely define a homomorphism \(a^*\) into \(R\) satisfying the following two and it is said to be the dual of \(a\).

\[
\begin{align*}
(a) & \quad a^*(a) = 1. \\
(b) & \quad a^*(b) = 0 \text{ for any submodule } B \text{ giving an internal direct sum decomposition } <a> \oplus B \text{ of } A \text{ and any } b \in B \text{ where } <a> \text{ is the submodule generated by } \{a\} \quad (a).
\end{align*}
\]

In the present discussions, duals of homology classes (for which we can define the duals) are important. They are regarded as cohomology classes in canonical ways.

Definition 10. For a smooth or PL manifold \(X\), a homology class \(c\) is said to be represented by a closed and smooth or PL submanifold \(Y\) with no boundary and with an orientation if \(c\) is realized as the value of the homomorphism induced by the inclusion at the so-called fundamental class: the fundamental class of an oriented closed and smooth or PL manifold is the generator of the top homology group respecting the orientation and note that we do not need the orientation if the coefficient is isomorphic to \(\mathbb{Z}/2\mathbb{Z}\).
A stably parallelizable manifold is a smooth manifold such that the Whitney sum of the tangent bundle and a 1-dimensional trivial real bundle over the manifold is a trivial real vector bundle.

**Theorem 3.** Let \( l \geq 0 \) and \( m > n \geq 1 \) be integers and \( \{X_j\}_{j=1}^l \) be a family of finitely many \( n \)-dimensional closed, connected and stably parallelizable manifolds. Let \( \{(F_{j,1}, F_{j,2})\}_{j=1}^l \) be a family of pairs of \((m-n)\)-dimensional closed, connected and smooth manifolds for each of which a disjoint union of the two manifolds bounds a compact, connected and smooth manifold \( F_j \) obtained by attaching handles containing at most 1 1-handle to \((F_{j,1} \sqcup F_{j,2}) \times ([1, 1])\). Let \( X \) be an \( n \)-dimensional connected and compact manifold we can smoothly immerse into \( \mathbb{R}^n \). Then we have a simple fold map \( f \) on an \( m \)-dimensional closed and connected manifold \( M \) into \( \mathbb{R}^n \) satisfying the following two.

1. The Reeb space \( W_f \) is a branched manifold of the class \( A \) simple homotopy equivalent to a space obtained by a finite iteration of taking a bouquet starting from \( l+1 \) polyhedra where the \( j \)-th polyhedron collapses to \( X_j \) for \( 1 \leq j \leq l \) and is \( X \) for \( j = l+1 \).
2. There exist connected components of the preimage of a regular value diffeomorphic to \( F_{j,1} \) and \( F_{j,2} \) for each \( j \).

**Proof.** Each \( X_i \) admits a stable fold map into \( \mathbb{R}^n \) by virtue of the theory of [2] and as a result it is special generic such that the singular set is a closed submanifold of dimension \( n-1 \) with no boundary whose normal bundle is trivial and whose complementary set is not connected and consists of exactly two connected components: see also [59] for example as an explicit study on [2], considering the theory for cases of 3-dimensional closed and orientable manifolds. For the singular set of each special generic map, take a small closed tubular neighborhood \( NS(X_i) \), regarded as a trivial linear bundle over the singular set whose fiber is diffeomorphic to \( D^1 \). We construct a Morse function on \( F_i \) satisfying the following two.

1. The preimage of the minimal value is the boundary \( F_{i,1} \sqcup F_{i,2} \) and contains no singular points.
2. At distinct singular points, the singular values are distinct.
3. The Reeb space is PL homeomorphic to \( K \) in Definition 8 and so on.

We have a product map of the Morse function and the identity map of the singular value set of the special generic map before. The Reeb space is PL homeomorphic to the product of \( K \) and the singular set of the special generic map.

We have trivial smooth bundles over two disjoint compact and connected submanifolds of \( X_i - \text{IntNS}(X_i) \) restoring the original manifold \( X_i - \text{IntNS}(X_i) \) by the disjoint union whose fibers are diffeomorphic to \( F_{i,1} \) and \( F_{i,2} \), respectively. Note that the \( n \)-dimensional compact manifolds are smoothly immersed into \( \mathbb{R}^n \) respecting the special generic map on \( X_i \). We can glue these maps to obtain a simple fold map such that the Reeb space collapses to \( X_i \) and that there exist connected components of the preimage of a regular value diffeomorphic to \( F_{i,1} \) and \( F_{i,2} \). This technique generalizes a method in the proof of Theorem 1 and so on in [15].

To obtain a desired map, we take a connected sum of \( l \) maps obtained in this way and a special generic map on an \( m \)-dimensional closed and connected manifold whose Reeb space is diffeomorphic to \( X \), smoothly immersed into \( \mathbb{R}^n \). Note that such a special generic map is constructed by virtue of a fundamental argument in [48] or we can construct such a map easily. We do not review a rigorous definition.
of a connected sum of two or more simple (stable) fold maps or general stable (fold) maps. However, we can understand the definition easily in a natural way. This notion is fundamental in various scenes. In [48], connected sums of special generic maps are fundamental and important, for example.

Theorem 4. Let \( l \geq 0 \) and \( m > n \geq 1 \) be integers. Let \( \{(F_{j,1}, F_{j,2})\}_{j=1}^{l} \) be a family of pairs of \((m-n)\)-dimensional closed, connected and smooth manifolds for each of which a disjoint union of the two manifolds bounds a compact, connected and smooth manifold \( F_{j} \) obtained by attaching handles containing at most \( 1 \)-handle to \((F_{j,1} \sqcup F_{j,2}) \times ([0, 1]) \). Let \( X \) be an \( n \)-dimensional connected and compact manifold we can smoothly immerse into \( \mathbb{R}^{n} \). Let \( \{Y_{j} \subset X\}_{j=1}^{l} \) be a family of mutually disjoint \( n \)-dimensional compact, connected and smooth submanifolds of \( X \). Then we have a simple fold map \( f \) on an \( m \)-dimensional closed and connected manifold \( M \) into \( \mathbb{R}^{n} \) such that the Reeb space \( W_{f} \) is a branched manifold of the class \( A \) obtained by attaching \( l \) manifolds each of which is diffeomorphic to a double \( DY_{j} \) of \( Y_{j} \) identifying \( Y_{j} \) in \( X \) and \( Y_{j} \subset DY_{j} \) canonically and that there exist connected components of the preimage of a regular value diffeomorphic to \( F_{j,1} \) and \( F_{j,2} \) for each \( j \).

Proof. First we construct a special generic map whose Reeb space is diffeomorphic to \( X \), smoothly immersed into \( \mathbb{R}^{n} \), as in Theorem 3. We can do this so that the restriction to the preimage of the complementary space of the interior of a small collar neighborhood of \( \partial X \) in the Reeb space is the projection of a trivial bundle over the space such that the fiber is diffeomorphic to \( S^{m-n} \) by virtue of the theory. We take \( Y_{j} \) and a small closed tubular neighborhood of \( \partial Y_{j} \) in \( X \). We remove the interior of the preimage of the closed tubular neighborhood for each \( j \) and we attach a new map. We construct this new map as a product map of a Morse function satisfying the following four and the identity map on a manifold diffeomorphic to \( \partial Y_{j} \).

1. The preimage of the minimal value is the disjoint union of two connected components of the boundary and diffeomorphic to \( F_{j,1} \sqcup F_{j,2} \) and contains no singular points.
2. The preimage of the maximal value is a connected component of the boundary and a standard sphere and contains no singular points.
3. At distinct singular points, the singular values are distinct.
4. The Reeb space is PL homeomorphic to \( K \) in Definition 8 and so on.

We have a collar neighborhood of \( \partial Y_{j} \subset Y_{j} \) in this situation naturally. Over the complementary space of the interior of the collar neighborhood, we construct a trivial smooth bundle whose fiber is diffeomorphic to \( F_{i,1} \sqcup F_{i,2} \) instead. By gluing these maps and composing the resulting map with the original immersion of \( X \) into \( \mathbb{R}^{n} \), we have a desired simple fold map.

Remark 1. We can obtain infinitely many stably parallelizable closed manifolds even if we restrict the class of the manifolds. For example, we can find such manifolds in the class of \( 4 \)-\( 7 \)-dimensional closed and simply-connected manifolds, studied or systematically explained in [1], [5], [38], [56], [58], [60], [61], and so on. Some well-known such explicit manifolds are given. In addition, for example, by considering smooth immersions or embeddings into suitable Euclidean spaces whose codimensions are greater than 2, we can consider normal bundles and the subbundles obtained by restricting the fibers to the unit spheres. The total spaces are...
such closed manifolds. Such facts imply that we can find various homology groups, cohomology rings, and so on, of the resulting Reeb spaces.

Example 3. Related to Remark 1, every 4-dimensional closed, connected and orientable manifold can be smoothly embedded into $\mathbb{R}^7$. We can have 6-dimensional closed, connected and stably parallelizable manifolds in the presented way for these manifolds. We can apply Theorems 3 and 4: for example by taking $X_j$ as such a 6-dimensional manifold, $X$ as a unit disc, and $n = 6$ in Theorem 3. Some 4-dimensional closed, connected and orientable manifolds present Reeb spaces such that the square of some cohomology class may not be divisible by 2. Such cases have not been studied in existing related studies (by the author).

Remark 2. Related to Main Theorems, [7], [8], [18], [19], and so on, present results stating that existence of a connected component of the preimage of a regular value which is not null-cobordant or which is not null-cobordant in a wider sense different from original well-known senses makes the top homology group of the Reeb space non-trivial for some coefficient.

Theorem 5. In Theorem 4, assume that $l > 0$, that $X$ is obtained by attaching $h_p$-handles for $1 \leq p \leq n - 1$ starting from $l$ 0-handles and that the following four hold.

1. $H_1(X; \mathbb{Z}) \cong \mathbb{Z}^{h_1-(l-1)}$.
2. $H_p(X; \mathbb{Z}) \cong \mathbb{Z}^{h_p}$ for $2 \leq p \leq n - 1$.
3. $F_j$ is obtained by attaching $h_{j,p}$ p-handles for $1 \leq p \leq n - 1$ starting from a 0-handle.
4. $X - \bigcup_{j=1}^l \text{Int} Y_j$ is obtained by attaching $h_p - \Sigma_{j=1}^l h_{j,p}$ p-handles to the product of a manifold diffeomorphic to the boundary and $\{0\} \subset [0,1]$ for all $1 \leq p \leq n - 1$.

Then, we have the following facts on homology groups and cohomology groups and rings.

1. $H_p(W_f; \mathbb{Z})$ is isomorphic to the direct sum of $H_p(X; \mathbb{Z}) \oplus \mathbb{Z}^{\Sigma_{j=1}^l h_{j,n-p}}$ for $1 \leq p \leq n - 1$ and $\mathbb{Z}^l$ for $p = n$.
2. $H^p(W_f; \mathbb{Z})$ is isomorphic to $A_p \oplus \mathbb{Z}^{\Sigma_{j=1}^l (H^p(Y_j; \mathbb{Z}) \oplus \mathbb{Z})} \oplus \mathbb{Z}^{\Sigma_{j=1}^l (H^p(DY_j - \text{Int} Y_j; \mathbb{Z}))}$ where $A_p$ is isomorphic to $\mathbb{Z}^{h_1 - \Sigma_{j=1}^l h_{j,1} - (l-1)}$ for $p = 1$ and $\mathbb{Z}^{h_p - \Sigma_{j=1}^l h_{j,p}}$ for $2 \leq p \leq n - 1$. $H^n(W_f; \mathbb{Z})$ is isomorphic to $\mathbb{Z}^l$.
3. $H^*(X; \mathbb{Z})$ is regarded as a subalgebra of $H^*(W_f; \mathbb{Z})$ and the cohomology group is identified with $A_p \oplus \mathbb{Z}^{\Sigma_{j=1}^l (H^p(Y_j; \mathbb{Z}) \oplus \mathbb{Z})} \oplus \mathbb{Z}^{\Sigma_{j=1}^l (H^p(DY_j - \text{Int} Y_j; \mathbb{Z}))}$ before for each degree $1 \leq p \leq n - 1$. $H^*(\bigcup_{j=1}^l DY_j; \mathbb{Z})$ is regarded as a subalgebra of $H^*(W_f; \mathbb{Z})$ and the cohomology group is identified with $\{0\} \oplus \mathbb{Z}^{\Sigma_{j=1}^l (H^p(Y_j; \mathbb{Z}) \oplus \mathbb{Z})} \oplus \mathbb{Z}^{\Sigma_{j=1}^l (H^p(DY_j - \text{Int} Y_j; \mathbb{Z}))}$ for each degree $1 \leq p \leq n - 1$ and $\mathbb{Z}^l$ for degree $p = n$.

Proof. We consider a Mayer-Vietoris sequence

$\rightarrow H_p(\bigcup_{j=1}^l Y_j; \mathbb{Z}) \rightarrow H_p(X; \mathbb{Z}) \oplus H_p(\bigcup_{j=1}^l DY_j; \mathbb{Z}) \rightarrow H_p(W_f; \mathbb{Z}) \rightarrow$

and the homomorphism from $H_p(\bigcup_{j=1}^l Y_j; \mathbb{Z})$ into $H_p(X; \mathbb{Z}) \oplus H_p(\bigcup_{j=1}^l DY_j; \mathbb{Z})$ is a monomorphism. Furthermore, each summand is induced by the natural inclusion and a monomorphism. Assumptions on handles and integral homology groups,
which are free, imply that in the argument we cannot cancel pairs of handles except 
\( l - 1 \) pairs of 0-handles and 1-handles. \( DY_j \) is represented as a double of \( Y_j \).

\[ DY_j \rightarrow \text{Int} Y_j \] is, by considering the duals to original handles for \( Y_j \), regarded as a
manifold obtained by attaching \( h_{j,p} \) \((n - p)\)-handles to the product of a manifold
diffeomorphic to the boundary and \( \{0\} \subset [0, 1] \) for \( 1 \leq p \leq n - 1 \). These arguments
on the topological structures of the manifolds complete the proof of all facts. \( \square \)

Example 4. [12], [13], [16], and so on, present explicit examples implicitly or explicit
ly for Theorem 5 where \( X \) and \( Y_j \) are \( n \)-dimensional standard discs. We can
know that manifolds represented as connected sums of total spa
ces of smooth bundles
over the \( n \)-dimensional standard sphere whose fibers are \((m - n)\)-dimensional
standard spheres admit such maps with \( m > n \geq 1 \). [17], [23], and so on, present
explicitly or implicitly more general examples.

4. Acknowledgment with additional remarks related to applications
of our studies on geometry of manifolds to machine-learnings and
related topics and data availability.

The author is a member of and supported by JSPS KAKENHI Grant Num-
ber JP17H06128 ”Innovative research of geometric topology and singularities of
differentiable mappings” (Principal Investigator: Osamu Saeki).

This is also closely related to a joint research project at Institute of Mathemat-
ics for Industry, Kyushu University (20200027), ”Geometric and constructive studies
of higher dimensional manifolds and applications to higher dimensional data”, prin-
cipal investigator of which is the author. The author would like to thank people
supporting the research project. This is a project on applications of mathemat-
ical (especially, geometric) theory on higher dimensional differentiable manifolds
developed through the studies of the author to data analysis, visualizations, and
so on. This is a kind of new project of applying the singularity theory of different-
able maps and (differential) topology to machine-learnings and related problems
such as multi-optimization problems, genetic algorithms, evolutionary com-
putations, and so on. [6], [9], [10], [53], and so on, are closely related studies and the
author has been interested in this field. For example, Naoki Hamada is an expert
of multi-optimization problems together with related topics and kindly proposed
a strategy of studying multi-functions for the problems via topological theory of
their Reeb spaces. More precisely, as one interesting application, he proposed an
idea that complexity of the topology of the Reeb space can measure how difficult
and complicated a multi-optimization problem is. This together with backgrounds
on geometry has motivated the author to study global topologies of Reeb spaces of
smooth maps of important classes further.

The author declares that all data supporting the present study are in the present
paper.

References

[1] D. Barden, Simply Connected Five-Manifolds, Ann. of Math. (3) 82 (1965), 365–385.
[2] Y. Eliashberg, On singularities of folding type, Math. USSR Izv. 4 (1970), 1119–1134.
[3] Y. Eliashberg, Surgery of singularities of smooth mappings, Math. USSR Izv. 6 (1972), 1302–
1326.
[4] M. Golubitsky and V. Guillemin, Stable mappings and their singularities, Graduate Texts in
Mathematics (14), Springer-Verlag (1974).
[5] R. E. Gompf and A. I. Stipsicz, 4-manifolds and Kirby calculus, Graduate Studies in Mathematics, Vol. 20, American mathematical Society, 1999.
[6] N. Hamada, K. Hayano, S. Ichiki, Y. Kabata and H. Teramoto, Topology of Pareto sets of strongly convex problems, SIAM Journal on Optimization, 30, no. 3, pp. 2659–2686 arXiv:1904.03615.
[7] J. T. Hiratuka and O. Saeki, Triangulating Stein factorizations of generic maps and Euler Characteristic formulas, RIMS Kokyuroku Bessatsu B38 (2013), 61–89.
[8] J. T. Hiratuka and O. Saeki, Connected components of regular fibers of differentiable maps, in "Topics on Real and Complex Singularities", Proceedings of the 4th Japanese-Australian Workshop (JARCS4), Kobe 2011, World Scientific, 2014, 61–73.
[9] N. Hamada and S. Ichiki, Simpliciality of strongly convex problems, to appear in Journal of the Mathematics Society of Japan, arXiv:1912:09328.
[10] N. Hamada and S. Ichiki, Characterisation of the equality of weak efficiency and efficiency on convex free disposal hulls, arXiv:1910:02867.
[11] M. Ishikawa and Y. Koda, Stable maps and branched shadows of 3-manifolds, Mathematische Annalen 367 (2017), no. 3, 1819–1863, arXiv:1403.0596.
[12] N. Kitazawa, On round fold maps (in Japanese), RIMS Kokyuroku Bessatsu B38 (2013), 45–59.
[13] N. Kitazawa, On manifolds admitting fold maps with singular value sets of concentric spheres, Doctoral Dissertation, Tokyo Institute of Technology (2014).
[14] N. Kitazawa, Fold maps with singular value sets of concentric spheres, Hokkaido Mathematical Journal Vol.43, No.3 (2014), 327–359.
[15] N. Kitazawa, Constructions of round fold maps on smooth bundles, Tokyo J. of Math. Volume 37, Number 2, 385–403, arxiv:1305.1708.
[16] N. Kitazawa, Round fold maps and the topologies and the differentiable structures of manifolds admitting explicit ones, submitted to a refereed journal, arXiv:1304.0618 (the title has changed).
[17] N. Kitazawa, Constructing fold maps by surgery operations and homological information of their Reeb spaces, submitted to a refereed journal, arxiv:1508.05630 (the title has been changed).
[18] N. Kitazawa, Smooth maps compatible with simplicial structures and preimages, arxiv:1802.06381.
[19] N. Kitazawa, Additional structures on preimages of regular values of smooth maps induced from the manifolds and cycles of the Reeb spaces, arXiv:1805.02243.
[20] N. Kitazawa, Lifts of spherical Morse functions, submitted to a refereed journal, arxiv:1805.05852.
[21] N. Kitazawa, Generalizations of Reeb spaces of special generic maps and applications to a problem of lifts of smooth maps, arxiv:1805.07783.
[22] N. Kitazawa, A new explicit way of obtaining special generic maps into the 3-dimensional Euclidean space, arxiv:1806.04581.
[23] N. Kitazawa, Notes on fold maps obtained by surgery operations and algebraic information of their Reeb spaces, arxiv:1811.04080.
[24] N. Kitazawa, On Reeb graphs induced from smooth functions on 3-dimensional closed orientable manifolds with finitely many singular values, submitted to a refereed journal, arxiv:1902.08841.
[25] N. Kitazawa, Explicit remarks on the torsion subgroups of homology groups of Reeb spaces of explicit fold maps, submitted to a refereed journal, arxiv:1906.00943.
[26] N. Kitazawa, On Reeb graphs induced from smooth functions on closed or open manifolds, arxiv:1908.04540.
[27] N. Kitazawa, Maps on manifolds onto graphs locally regarded as a quotient map onto a Reeb space and a new construction problem, arxiv:1909.10315.
[28] N. Kitazawa, New observations on cohomology rings of Reeb spaces of explicit fold maps and manifolds admitting these maps, arxiv:1911.09164.
[29] N. Kitazawa, Notes on explicit smooth maps on 7-dimensional manifolds into the 4-dimensional Euclidean space, arxiv:1911.11274.
[30] N. Kitazawa, Surgery operations to fold maps to construct fold maps whose singular value sets may have crossings, arxiv:2003.04147.
[31] N. Kitazawa, Surgery operations to fold maps to increase connected components of singular sets by two, arxiv:2004.03583.
[32] N. Kitazawa, Explicit fold maps on 7-dimensional closed and simply-connected manifolds of new classes, arxiv:2005.05281.
[33] N. Kitazawa, Special generic maps and fold maps and information on triple Massey products of higher dimensional differentiable manifolds, submitted to a refereed journal, arxiv:2006.08960v7.
[34] N. Kitazawa, Closed manifolds admitting no special generic maps whose codimensions are negative and their cohomology rings, submitted to a refereed journal, arxiv:2008.04226v4.
[35] M. Kobayashi, Stable mappings with trivial monodromies and application to inactive log-transformations, RIMS Kokyuroku. 815 (1992), 47–53.
[36] M. Kobayashi, Bubbling surgery on a smooth map, preprint.
[37] M. Kobayashi and O. Saeki, Simplifying stable mappings into the plane from a global viewpoint, Trans. Amer. Math. Soc. 348 (1996), 2607–2636.
[38] M. Kreck, On the classification of 1-connected 7-manifolds with torsion free second homology, to appear in the Journal of Topology, arxiv:1805.02391.
[39] B. Martelli, Four-manifolds with shadow-complexity zero, Int. Math. Res. Not. 2011 (2011), 1268–1351.
[40] Y. Masumoto and O. Saeki, A smooth function on a manifold with given Reeb graph, Kyushu J. Math. 65 (2011), 75–84.
[41] J. Milnor, Morse Theory, Annals of Mathematics Studies AM-51, Princeton University Press; 1st Edition (1963.5.1).
[42] J. Milnor, Lectures on the h-cobordism theorem, Math. Notes, Princeton Univ. Press, Princeton, N.J. 1965.
[43] L. P. Michalak, Realization of a graph as the Reeb graph of a Morse function on a manifold, to appear in Topol. Methods Nonlinear Anal., Advance publication (2018), 14pp, arxiv:1805.06727.
[44] M. E. Munoz and M. Ozawa, The maximum and minimum genus of a multibranched surface, Topology and its Appl. (2020) 107502, arXiv:2005.06765.
[45] H. Naoe, Shadows of 4-manifolds with complexity zero and polyhedral collapsing, Proc. Amer. Math. Soc. 145 (2017), 4561–4572.
[46] M. Ozawa, Multibranched surfaces in 3-manifolds, J. Math. Sci. 255 (2021), 193–208, arXiv:2005.07409.
[47] O. Saeki, Notes on the topology of folds, J. Math. Soc. Japan Volume 44, Number 3 (1992), 551–566.
[48] O. Saeki, Topology of special generic maps of manifolds into Euclidean spaces, Topology Appl. 49 (1993), 265–293.
[49] O. Saeki, Topology of special generic maps into $\mathbb{R}^3$, Workshop on Real and Complex Singularities (Sao Carlos, 1992), Mat. Contemp. 5 (1993), 161–186.
[50] O. Saeki, Reeb spaces of smooth functions on manifolds, to appear in International Mathematics Research Notices, arXiv:2006.01689.
[51] O. Saeki and K. Suzuoka, Generic smooth maps with sphere fibers J. Math. Soc. Japan Volume 57, Number 3 (2005), 881–902.
[52] K. Sakuma, On the topology of simple fold maps, Tokyo J. of Math. Volume 17, Number 1 (1994), 21–32.
[53] D. Sakurai, O. Saeki, H. Carr, H Wu, T. Yamamoto, D. Duke and S. Takahashi, Interactive Visualization for Singular Fibers of Functions $f : \mathbb{R}^3 \to \mathbb{R}^2$, IEEE Transactions on Visualization and Computer Graphics ( Volume: 22, Issue: 1, Jan. 31 2016), 945–954.
[54] V. Sharko, About Kronrod-Reeb graph of a function on a manifold, Methods of Functional Analysis and Topology 12 (2006), 389–396.
[55] M. Shiota, Thom’s conjecture on triangulations of maps, Topology 39 (2000), 383–399.
[56] S. Smale, On the structure of 5-manifolds, Ann. of Math. (2) 75 (1962), 38–46.
[57] V. G. Turaev, Shadow links and face models of statistical mechanics, J. Differential Geom. 36 (1992), 35–74.
[58] C. T. C. Wall, Classification problems in differential topology. V. On certain 6-manifolds, Invent. Math. 1 (1966), 355–374.
[59] M. Yamamoto, *Construction of fold map of Lens space $L(p,1)$ where singular set is a torus* (Singularity theory of differential maps and its applications), RIMS Kôkyûroku 2049 (2017), 45–56.

[60] A. Zhurbr, *Closed simply connected six-dimensional manifolds: proof of classification theorems*, Algebra i Analiz 12 (2000), no. 4, 126–230.

[61] A. Zhurbr (mainly edited by A. Zhurbr), *6-manifolds: 1-connected*, http://www.map.mpim-bonn.mpg.de/6-manifolds:1-connected.

Institute of Mathematics for Industry, Kyushu University, 744 Motooka, Nishi-ku Fukuoka 819-0395, Japan, TEL (Office): +81-92-802-4402, FAX (Office): +81-92-802-4405,

*Email address: n-kitazawa@imi.kyushu-u.ac.jp*

*Webpage: https://naokikitazawa.github.io/NaokiKitazawa.html*