Non-Perturbative Quark-Antiquark Production From a Constant Chromo-Electric Field via the Schwinger Mechanism

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(Dated: October 2, 2018)

Abstract

We obtain an exact result for the non-perturbative quark (antiquark) production rate and its $p_T$ distribution from a constant SU(3) chromo-electric field $E^a$ with arbitrary color index $a$ by directly evaluating the path integral. Unlike the WKB tunneling result, which depends only on one gauge invariant quantity $|E|$, the strength of the chromo-electric field, we find that the exact result for the $p_T$ distribution for quark (antiquark) production rate depends on two independent Casimir (gauge) invariants, $E^a E^a$ and $[d_{abc} E^a E^b E^c]^2$.

PACS numbers: PACS: 11.15.-q, 11.15.Me, 12.38.Cy, 11.15.Tk
Non-perturbative quark anti-quark pair production from a constant chromo-electric field is widely employed to study hadronization at low $p_T$ in high energy $e^+e^-$ and pp collisions [1]. In these approaches the color flux-tube energy density or the string tension is related to the constant chromo-electric field strength $|E|$. In high energy heavy-ion collisions at the RHIC and the LHC [2] a classical chromo field might be formed just after two nuclei pass through each other [3, 4]. In order to study the production of a quark-gluon plasma from a classical chromo field it is necessary to know how quarks and gluons are formed from the latter. In a recent paper [5] we have derived a formula for the rate for non-perturbative gluon pair production and its $p_T$ distribution from a constant SU(3) chromo-electric field with arbitrary color index $a$ via vacuum polarization. In this paper we will extend our study to quark-antiquark pair production.

We will not employ Schwinger’s proper time method [6] in our calculation. Although this method is widely used to obtain the total fermion production rate $dN/d^4x$ [6, 7] (for a review see [8]), this method cannot be used to obtain the $p_T$ distribution of the rate $dN/d^4xd^2p_T$. For this purpose the WKB tunneling method [9, 10] has been used in the past to approximate the $p_T$ distribution for the quark (antiquark) production rate. Although the WKB tunneling result for the $p_T$ distribution is correct in QED it is not necessarily true in QCD because of the presence of the non-trivial color generators in the fundamental and adjoint representations of SU(3). For this reason we will directly evaluate the path integral in this paper and obtain an exact result for the $p_T$ distribution of the non-perturbative quark (antiquark) production rate from a constant chromo-electric field with arbitrary color in the gauge group SU(3). We find that unlike the WKB tunneling result, which depends on one gauge invariant quantity $|E|$, the strength of the chromo-electric field, [9, 10] the exact result for the $p_T$ distribution for the quark (antiquark) production rate depends on two independent gauge invariants, $E^aE^a$ and $[d_{abc}E^aE^bE^c]^2$.

We obtain the following formula for the number of non-perturbative quarks (antiquarks) produced per unit time, per unit volume and per unit transverse momentum from a given constant chromo-electric field $E^a$

\[
\frac{dN_{q,\bar{q}}}{dt d^3x d^2p_T} = -\frac{1}{4\pi^3} \sum_{j=1}^{3} |g\lambda_j| \ln[1 - e^{-\frac{\pi(p_T^2 + m^2)}{|g\lambda_j|}}],
\]

(1)

where $m$ is the mass of the quark. This result is gauge invariant because it depends on the
following gauge invariant eigenvalues

\[\lambda_1 = \sqrt{\frac{C_1}{3}} \cos \theta,\]
\[\lambda_2 = \sqrt{\frac{C_1}{3}} \cos (\frac{2\pi}{3} - \theta),\]
\[\lambda_3 = \sqrt{\frac{C_1}{3}} \cos (\frac{2\pi}{3} + \theta),\]

(2)

where \(\theta\) is given by

\[\cos^2 3\theta = \frac{3C_2}{C_1}.\]

(3)

These eigenvalues only depend on two independent Casimir invariants for SU(3)

\[C_1 = E^a E^a, \quad C_2 = [d_{abc}E^aE^bE^c]^2,\]

(4)

where \(a, b, c = 1, \ldots, 8\). Note that \(0 \leq \cos^2 3\theta \leq 1\) because \(C_3 \geq 3C_2\) and both \(C_1\) and \(C_2\) are positive. The integration over \(p_T\) in eq. (1) reproduces Schwinger’s result for total production rate \(dN/d^4x\) [7].

The exact result in eq. (1) can be contrasted with the following formula obtained by the WKB tunneling method [9]

\[\frac{dN_{q,\bar{q}}}{dtd^3xd^2p_T} = \frac{-|gE|}{4\pi^3} \ln[1 - e^{-\frac{\pi(p_T^2 + m^2)}{|gE|}}].\]

(5)

In our result in eq. (1) the symmetric tensor \(d_{abc}\) appears. Hence the WKB tunneling method does not reproduce the correct result for the \(p_T\) distribution of the quark (antiquark) production rate from a constant chromo-electric field \(E^a\). We now present a derivation of eq. (1).

The Lagrangian density for a quark in a non-abelian background field \(A^a_\mu\) is given by

\[\mathcal{L} = \bar{\psi}^i \left[(\delta_{ij} \hat{p} - gT^a_{ij}A^a) - m\delta_{ij}\right] \psi^j = \frac{\bar{\psi}^i}{M_{ij}[A]} \psi^j,\]

(6)

where \(\hat{p}_\mu\) is the momentum operator and \(T^a_{ij}\) is the generator in the fundamental representation of gauge group SU(3) with \(a=1,2,\ldots,8\) and \(i, j = 1, 2, 3\). The vacuum to vacuum transition amplitude in the presence of the non-abelian background field \(A^a_\mu\) is given by

\[<0|0> = \frac{\int [d\bar{\psi}] [d\psi] e^{i\int d^4x \bar{\psi}^j M_{jk}[A] \psi^k} \int [d\bar{\psi}] [d\psi] e^{i\int d^4x \bar{\psi}^j M_{jk}[0] \psi^k} = \frac{\text{Det}[M[A]]}{\text{Det}[M[0]]} = e^{i\bar{S}(1)}.\]

(7)
The one loop effective action becomes

\[ S_{q\bar{q}}^{(1)} = -i \text{ Tr } \ln[(\delta_{ij} \hat{p} - gT_{ij}^a A^a) - m\delta_{ij}] + i\text{ Tr } \ln[\delta_{ij} \hat{p} - m\delta_{ij}] . \] (8)

The trace \( \text{Tr} \) contains an integration over \( d^4x \), a sum over color and Lorentz indices and a trace over Dirac matrices. Since the trace is invariant under transposition we also get

\[ S_{q\bar{q}}^{(1)} = -i \text{ Tr } \ln[(\delta_{ij} \hat{p} - gT_{ij}^a A^a) + m\delta_{ij}] + i\text{ Tr } \ln[\delta_{ij} \hat{p} + m\delta_{ij}] . \] (9)

Adding both the above equations we get

\[ 2S_{q\bar{q}}^{(1)} = -i \text{ Tr } \ln[(\delta_{ij} \hat{p} - gT_{ij}^a A^a)^2 - m^2\delta_{ij}] + i\text{ Tr } \ln[\delta_{ij} \hat{p}^2 - m^2\delta_{ij}] , \] (10)

which can be written as

\[ 2S_{q\bar{q}}^{(1)} = -i \text{ Tr } \ln[(\delta_{ij} \hat{p} - gT_{ij}^a A^a)^2 + \frac{g}{2}\sigma_{\mu\nu}T_{ij}^a F^{a\mu\nu} - m^2\delta_{ij}] + i\text{ Tr } \ln[\delta_{ij} \hat{p}^2 - m^2\delta_{ij}] . \] (11)

Since it is convenient to work with the trace of the exponential we replace the logarithm by

\[ \ln \frac{a}{b} = \int_0^\infty \frac{ds}{s} [e^{is(b+ie)} - e^{is(a+ie)}] . \] (12)

We assume that the constant electric field is along the \( z \)-axis (the beam direction) and we choose the gauge \( A_0^a = 0 \) so that \( A_3^a = -E^a \hat{x}_0^a \). The color indices \( (a=1,\ldots,8) \) are arbitrary. Since \( \Lambda_{ij} = T_{ij}^a E^a \) has three eigenvalues we write after diagonalization

\[ (\Lambda_d)_{ij} = (\lambda_1, \lambda_2, \lambda_3) . \] (13)

The trace over the Dirac matrices \( (\text{tr}_D) \) give

\[ \text{tr}_D[e^{is\frac{g}{2}\sigma_{\mu\nu}T_{ij}^a F^{a\mu\nu}}] = 4 \cosh(sgT_{ij}^a E^a ) . \] (14)

To reduce this problem to the motion of one harmonic oscillator, we make a similarity transformation [5, 11] (we also make a similarity transformation in the group space) and obtain

\[ \text{tr}_D e^{is[\delta_{ij} \hat{p} - gT_{ij}^a A^a]^2 + \frac{g}{2}\sigma_{\mu\nu}T_{ij}^a F^{a\mu\nu} - m^2\delta_{ij}] = 4 \cosh(sg(\Lambda_d)_{il})[e^{ip_0p_0/g(\Lambda_d)} e^{is(\hat{p}\hat{p} - \hat{p}\hat{r} - m^2 - g^2(\Lambda_d^2)\hat{w}^2)} e^{-ip_0p_0/g(\Lambda_d)} ]_{lj} , \] (15)
where \( p_T = \sqrt{p_T^2 + p_2^2} \) is the transverse momentum of the quark or antiquark (transverse to the electric field direction). Hence from eqs. (11) and (12) we find

\[
2S^{(1)}_{q\bar{q}} = i \int_0^\infty \frac{ds}{s} \sum_{j=1}^3 \text{tr} \left[ e^{ip_3 p_0 / g \lambda_j} e^{i s (\hat{p}^2 - \hat{p}_T^2 - m^2 - g^2 \lambda_j^2 s^2 + i \epsilon)} e^{-ip_3 p_0 / g \lambda_j} \right. \\
\left. \times \left[ 4 \cosh s g \lambda_j - 4 e^{i s (\hat{p}^2 - \hat{p}_T^2 - m^2 + i \epsilon)} \right] \right]. \quad (16)
\]

The trace \( \text{tr} \) denotes an integral over a complete set of \( x \) eigenstates. We add complete sets of \( p_j \) eigenstates, and obtain

\[
2S^{(1)}_{q\bar{q}} = i \int_0^\infty \frac{ds}{s} \sum_{j=1}^3 \frac{1}{4 \pi^3} \int d^4x \int d^2p_T e^{-is(p_T^2 + m^2)} - s \epsilon \left[ |g \lambda_j| \frac{\cosh s g \lambda_j}{\sinh s |g \lambda_j|} - \frac{1}{s} \right]. \quad (17)
\]

The \( s \)-integral at fixed \( p_T \) is convergent at \( s \to 0 \), but the integration over \( p_T \) yields an extra factor \( 1/s \) so now it seems divergent. However charge renormalization cures this ultraviolet problem by subtracting also the term linear in \( s \) in the expansion of \( \cosh s g \lambda_j / \sinh s |g \lambda_j| \). The integral is well behaved as \( s \to \infty \). To perform the \( s \) contour integration we use the well-known expansion

\[
\frac{1}{\sinh x} = \frac{1}{x} + 2x \sum_{n=1}^\infty \frac{(-1)^n}{x^2 + n^2 \pi^2},
\]

and then we formally replace \( s \) by \( -is \) (as first advocated by Schwinger in QED). The integral is now real, except for half-circles around the poles at \( s |g \lambda_j| = -in\pi \) for \( n=1,2,3... \). The \( 1/x \) term in (18) cancels against the \( 1/s \) term in (17). This yields the probability for quark (antiquark) production per unit time and per unit volume

\[
W_{q\bar{q}} = 2\text{Im}\mathcal{L}^{(1)} = \frac{1}{4 \pi^3} \int d^2p_T \sum_{n=1}^\infty \sum_{j=1}^3 \frac{|g \lambda_j|}{n} e^{-n \pi (p_T^2 + m^2) / s |g \lambda_j|}. \quad (19)
\]

Now all that is left is to determine the eigenvalues \( \lambda_j \) \((j = 1,2,3)\) of the matrix \( \Lambda_{ij} = T^a_{ij} E^a \) in the fundamental representation of the gauge group SU(3). Evaluating the traces of \( \Lambda_{ij} \), \( \Lambda^2_{ij} \) and the determinant of \( \Lambda_{ij} \) we find

\[
\lambda_1 + \lambda_2 + \lambda_3 = 0, \quad (20)
\]

\[
\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \frac{1}{2} E^a E^a, \quad (21)
\]

and

\[
\lambda_1 \lambda_2 \lambda_3 = \frac{1}{12} [d_{abc} E^a E^b E^c], \quad (22)
\]
the solution of which is given by eq. \(2\).

In this letter we have obtained an exact result for the rate for non-perturbative quark (antiquark) production and its \(p_T\) distribution in a constant chromo-electric field \(E^a\) with arbitrary color index \(a\) via vacuum polarization. We have used the background field method of QCD with the gauge group SU(3). The \(p_T\) distribution for quark (antiquark) production can be applied at the RHIC and the LHC colliders. We find that, unlike the WKB tunneling method, the \(p_T\) distribution of the quark or antiquark production rate depends on two independent Casimir (gauge) invariants, \(E^aE^a\) and \([d_{abc}E^aE^bE^c]^2\).

**Acknowledgments**

I thank Peter van Nieuwenhuizen for discussions and useful suggestions during the completion of this work and Jack Smith for a careful reading of the manuscript. This work was supported in part by the National Science Foundation, grants PHY-0071027, PHY-0098527, PHY-0354776 and PHY-0345822.

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