A CRITERION FOR COHOMOLOGICAL DIMENSION

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Abstract. We give a criterion for the cohomological dimension of a field, involving norm maps on Milnor $K$-theory; this criterion was originally formulated by Kato. The theorem we prove is a generalization of a theorem in Serre’s book on Galois cohomology.

Let $G$ be a profinite group. $G$ is said to have cohomological dimension at most $n$ if for every discrete torsion $G$-module $A$, $H^q(G, A) = 0$ for all $q > n$, see §I.3 of Serre [23]; the cohomological dimension of $G$ will be denoted $cd(G)$.

For a field $k$, one sets the cohomological dimension of $k$ to be the cohomological dimension of the absolute Galois group $G_k$. There is a classical criterion for fields of cohomological dimension of most one:

Theorem (Serre [23, §II.3]). Let $k$ be a perfect field. Then the cohomological dimension of $k$ is at most 1 if and only if for every finite extension $K/k$ and every finite Galois extension $L/K$, the norm map:

$$N : L^\times \to K^\times$$

is surjective.

The purpose of this note is to use motivic complexes to obtain an analogous criterion for fields of cohomological dimension at most $n$.

1. The main result

For a field $K$, let $K_n(K)$ denote the $n$th Milnor $K$-group of $K$. It is defined as:

$$K_n(K) = K^\times \otimes \ldots \otimes K^\times / I,$$

where $I$ is the ideal generated by $a_1 \otimes \ldots \otimes a_n$ such that $a_i + a_j = 1$ for some $i \neq j$.

Let $L/K$ be a finite Galois extension. Although $K_n(L)^{\text{Gal}(L/K)}$ is not in general isomorphic to $K_n(K)$ for $n \geq 2$, one can still define a norm map:

$$N : K_n(L)^{\text{Gal}(L/K)} \to K_n(K),$$

see Bass and Tate [5]. It was shown by Kato [10, §1] that these norm maps satisfy certain functorial properties which characterize them uniquely, see §IX.3 of Fesenko-Vostokov [8] for an overview. One can use motivic complexes to construct the norm map simply using the Galois action, see Remark B.2.

For the convenience of the reader, the properties of motivic complexes used in the course of the proof of the main result are included in Appendix A. We will also

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need some facts about Tate cohomology with coefficients in a complex; these can be found in Appendix B.

We have the following result:

**Theorem 1.1.** Let $k$ be a field; if the characteristic of $k$ is $p > 0$, assume furthermore that $[k : k^p] \leq p^{n-1}$. Then the cohomological dimension of $k$ is at most $n$ if and only if for every finite extension $K/k$ and every finite Galois extension $L/K$, the norm map:

$$N : K_n(L) \rightarrow K_n(K)$$

is surjective.

This criterion for $k$ was originally formulated by Kato [10]; in §3.3 Kato calls such a field $k$ a $\mathfrak{B}_n$-field.

**Proof.** Let $k$ be a field such that for all finite extensions $K/k$ and all finite Galois extensions $L/K$, the norm map:

$$N : K_n(L) \rightarrow K_n(K)$$

is surjective.

Let $\mathbb{Z}_B(n, L)$ denote the weight $n$ motivic complex for $L$ over the Zariski site, see Remark A.3. Our assumption implies that for all intermediate fields $K \subseteq E \subseteq L$:

$$\hat{H}^0(\text{Gal}(L/E), \mathbb{Z}_B(n, L)) = 0,$$

where $\hat{H}^q(-, -)$ denotes Tate cohomology, see Remark B.3.

By Hilbert’s Theorem 90, we have that:

$$\hat{H}^1(\text{Gal}(L/E), \mathbb{Z}_B(2, L)) = 0,$$

for any intermediate field $E$, see Remark A.2. Thus, by Corollary B.5 we have that:

$$\hat{H}^q(\text{Gal}(L/K), \mathbb{Z}_B(n, L)) = 0$$

for all $q \in \mathbb{Z}$.

Taking a limit over finite Galois extensions $L/K$, we see that:

$$H^q(G_K, \mathbb{Z}_\text{ét}(n, K)) = 0$$

for all $q > 0$ and all finite extensions $K/k$.

Now, let $p$ be a prime different from the characteristic of $k$. We have an exact triangle in the derived category of $G_K$-modules:

$$\mathbb{Z}_\text{ét}(n, K) \rightarrow \mathbb{Z}_\text{ét}(n, K) \rightarrow \mu_p^\otimes[n] \rightarrow \mathbb{Z}_\text{ét}(n, K)[1].$$

Thus, if $K/k$ is any finite extension, we have that:

$$H^q(G_K, \mu_p^\otimes) = 0$$

for all $q > n$.

Now, let $H$ be a $p$-Sylow subgroup of $G_k$. Let $S$ be the field corresponding to $H$. The extension $S/k$ is a limit over all finite Galois extensions $K/k$ contained in $S$. Taking a limit over these fields, we see that:

$$H^{n+1}(H, \mu_p^\otimes) = 0.$$

But $H$ acts trivially on $\mu_p$, because $S$ contains the $p^{th}$ roots of unity. Thus, we can identify the $n^{th}$ Tate twist $\mu_p^\otimes$ with $\mathbb{Z}/p\mathbb{Z}$, obtaining:

$$H^{n+1}(H, \mathbb{Z}/p\mathbb{Z}) = 0.$$
But by Proposition 21 in §I.4.1 of Serre [23], this implies that $cd_p(H) \leq n$, which implies that $cd_p(G_k) \leq n$.

Finally, if $p$ is the characteristic of $k$, then $cd_p(G_k) \leq 1$, see Proposition 3 in §II.2.2 of Serre [23]. Thus, the cohomological dimension of $k$ is at most $n$.

Next, suppose that $cd(G_k) \leq n$. If $K/k$ is any finite extension, then $G_K$ is isomorphic to a subgroup of finite index in $G_k$, so we have that

$$cd(G_K) \leq cd(G_k) \leq n,$$

see Proposition 22 in §I.4.1 of Serre [23].

For every prime $p$ not equal to the characteristic of $k$, $H^2(G_K, \mathbb{Z}_\text{ét}(n, K))(p) = 0$, by Lemma 1.1 of Saito [21]. This can be easily deduced using the exact triangle:

$$\mathbb{Z}_\text{ét}(n, K) \to \mathbb{Z}_\text{ét}(n, K) \to v(n)K \to \mathbb{Z}_\text{ét}(n, K)[1].$$

If the characteristic of $k$ is equal to $p > 0$, then there is an exact triangle in the derived category of $G_K$-modules:

$$\mathbb{Z}_\text{ét}(n, K) \to \mathbb{Z}_\text{ét}(n, K) \to v(n)K \to \mathbb{Z}_\text{ét}(n, K)[1],$$

where $v(n)K$ denotes the additive subgroup of the deRham complex $\Omega^\bullet_K$ generated by logarithmic differential $n$-forms, see Milne [17]. But, we have that

$$[K : K^p] \leq [k : k^p] \leq p^{n-1},$$

which implies that the dimension of $\Omega^1_K$ is at most $n - 1$. So, for completely elementary reasons, we have that $\Omega^1_K = 0$, and:

$$H^2(G_K, \mathbb{Z}_\text{ét}(n, K))(p) = 0$$

for $p$ equal to the characteristic of $k$.

Thus, for all finite extension $K/k$, we have that:

$$H^2(G_K, \mathbb{Z}_\text{ét}(n, K)) = 0.$$

Now, suppose that $L/K$ is a finite Galois extension. We have an inflation-restriction exact sequence:

$$0 \to H^2(\text{Gal}(L/K), \mathbb{Z}_B(n, L)) \to H^2(G_K, \mathbb{Z}_\text{ét}(n, K)) \to H^2(G_L, \mathbb{Z}_\text{ét}(n, L)),$$

see Remark A.4. This implies that:

$$H^2(\text{Gal}(L/K), \mathbb{Z}_B(n, L)) = 0$$

for all finite Galois extensions $L/K$ with $K/k$ finite. Again, we can use Hilbert’s Theorem 90 and Corollary [15] to see that:

$$\hat{H}^q(\text{Gal}(L/K), \mathbb{Z}_B(n, L)) = 0$$

for all $q \in \mathbb{Z}$.

In particular, setting $q = 0$ gives the surjectivity of the norm map:

$$N : K_2(L) \to K_2(K).$$

□

**Remark 1.2.** By setting $n = 1$, one recovers Serre’s proof in [23] of the Theorem stated in the introduction.
Remark 1.3. The condition that \([k : kp] \leq p^{n-1}\) if \(k\) is of characteristic \(p > 0\) is
only used to imply the triviality of \(H^2(G_K, \mathbb{Z}_{\text{et}}(n, K))(p)\). One could replace this
condition with the weaker assumption that:

\[ H^2(G_K, \mathbb{Z}_{\text{et}}(n, K))(p) = 0 \]

for all finite extensions \(K/k\), as in Proposition 5 in §II.3.1 of Serre [23].

Remark 1.4. Using the notation of Kato introduced in [10] and [11], one can
rephrase the main result: if \(k\) is any field, then \(k\) is a \(\mathcal{B}_n\)-field if and only if the
dimension of \(k\) is at most \(n\).

2. Applications of the main result

Application 2.1. Suppose that \(k\) is a global field, that is, a number field or the
function field of a curve over a finite field. Then it is well-known, see Serre [23],
that the cohomological dimension of \(k\) is at most 2, except in the case that \(k\) is a
number field with a real place. Furthermore, if \(k\) is a function field, then we have
that \([k : kp] = p\). Thus, for function fields and totally imaginary number fields, we
recover a theorem of Bak [4] on the surjectivity of the norm map on \(K_2\).

In the case that \(k\) is a number field with at least one real embedding, then we
have that \(cd_p(G_k) \leq 2\) for all \(p \neq 2\). We also recover the theorem of Bak loc. cit.
that for number fields with a real place, the norm map is surjective on \(K_2\) except
on the 2-primary part. This can be explicitly seen using the description of the
Milnor ring of a global field due to Bass-Tate [5], and the fact that any local field
is a \(\mathcal{B}_2\)-field.

Remark 2.2. In the course of the proof of the main result, we saw that \(k\) is a
\(\mathcal{B}_n\)-field if and only if for all finite extensions \(K/k\), a certain motivic cohomology
group vanishes; that is, that:

\[ H^2(G_K, \mathbb{Z}_{\text{et}}(n, K)) = 0. \]

For \(n = 1\), this motivic cohomology group is the Brauer group of \(K\). One should
think of these cohomology groups for \(n > 1\) as the “higher weight” Brauer groups
of \(K\).

One can rephrase the main result using this language: if \(k\) is a field such that
for all finite extensions \(K/k\), the weight \(n\) Brauer group of \(K\) is trivial, then the
cohomological dimension of \(k\) is at most \(n\). The converse is true, after adding the
additional assumption that either \([k : kp] \leq p^{n-1}\), or \(H^2(G_K, \mathbb{Z}_{\text{et}}(n, K))(p) = 0\).

Application 2.3. Let \(X = \text{Spec } R\), where \(R\) be a number ring with field of fractions
\(K\), a totally imaginary number field. As noted previously, \(K\) is a \(\mathcal{B}_2\)-field.
Let \(\nu\) be a finite place of \(K\), and let \(K_\nu\) denote the completion of \(K\) at \(\nu\). Then \(K_\nu\)
is a complete discrete valuation field with residue field \(k_\nu\), a finite field. Each \(k_\nu\) is
a \(\mathcal{B}_1\)-field.

So, \(X\) is a scheme whose generic point is a \(\mathcal{B}_2\)-field, and whose codimension one
points are \(\mathcal{B}_1\)-fields. We have a long exact sequence of cohomology groups coming
from localization theory:

\[ \ldots \to H^q(X, \mathbb{Z}_{\text{et}}(n, X)) \to H^q(G_K, \mathbb{Z}_{\text{et}}(n, K)) \to \bigoplus_{\nu} H^q(G_{k_\nu}, \mathbb{Z}_{\text{et}}(n-1, k_\nu)) \to \ldots. \]
The main result implies that the motivic cohomology groups $H^q(X,\mathbb{Z}_{\text{et}}(n,X))$ are trivial for $n > 1$ and $q > 1$. One has the same result when $X$ is a regular curve over a finite field.

**Application 2.4.** In [10], Kato proves that if $F$ is a complete discrete valuation field with residue field $k$, then $F$ is a $\mathfrak{B}_{n+1}$-field if and only if $k$ is a $\mathfrak{B}_n$-field. One can use the main result to give a simple proof of this fact, using Kato’s result [10] that the dimension of $F$ is equal to the dimension of $k$ plus one.

**Application 2.5.** Classically, $\mathfrak{B}_1$-fields play an important role in local and global class field theory. In the local theory, one uses the fact that this is a property of a finite fields, and of the maximal unramified extension of a local field.

In the global theory, one can use this property to understand the parallel significance of the cyclotomic extensions of a number field, and extensions of the base field of a function field. The maximal such extension is both cases is a $\mathbb{Z}$-extension, and the resulting field will be a $\mathfrak{B}_1$-field.

**Application 2.6.** As one might expect, $\mathfrak{B}_n$-fields for $n > 1$ make an appearance in the cohomological approach to higher local class field theory, as in Kato [10], Koya [12], Spiess [25] and Saito [21]. This concerns the study of the Galois group of $n$-local fields, that is, a complete discrete valuation field with residue field an $(n-1)$-local field. If $k$ is a $k$-local field, then $k$ is itself a $\mathfrak{B}_{n+1}$-field; the maximal unramified extension of $k$ in which all corresponding residue field extensions are unramified is the “smallest” field over $k$ which is a $\mathfrak{B}_n$-field.

One expects that $\mathfrak{B}_2$-fields will play an important role in a cohomological approach to global class field theory for arithmetic surfaces, if such an approach exists. One can see a manifestation of this phenomenon in the semi-global result of [2].

**Application 2.7.** It may be possible to relate Kato’s $\mathfrak{B}_n$ property to Serre’s $C_n$ property [23]. Recall that a field $k$ is called a $C_n$-field if every homogeneous polynomial of degree $d$ in more than $d^n$ variables has a nontrivial zero.

Kato [10 §3] showed that every $C_2$-field is a $\mathfrak{B}_2$ using the reduced norm. The converse is false, because it is well-known that field $\mathbb{Q}_2$ of 2-adic numbers is not a $C_2$-field, although its cohomological dimension is 2, which implies that it is a $\mathfrak{B}_2$-field.

Kato conjectured in [10] that every $C_n$-field is a $\mathfrak{B}_n$-field; using the main result, we see that this is equivalent to Serre’s conjecture in [23] that the cohomological dimension of a $C_n$-field is at most $n$.

**Appendix A. Motivic complexes**

Let $X$ be a scheme. In [15], Lichtenbaum predicted the existence of certain objects in the derived category of sheaves on the étale site over $X$, which have come to be known as motivic complexes. First, we present a list of properties (or “Axioms”) that motivic complexes are expected to satisfy. These properties have been modified slightly: we work with complexes that are acyclic in *positive* degrees. The reader will find that the grading used here is shifted down by $n$ from the standard choice *loc. cit.*

1. If $n > 0$, $\mathbb{Z}_{\text{et}}(n,X)$ is acyclic outside the interval $[-n+1,0]$. 
(2) Let $\alpha_\ast$ be the functor that assigns to every étale sheaf on $X$ the associated Zariski sheaf. Then $R^1\alpha_\ast Z_{\text{ét}}(n, X) = 0$.

(3) Let $l$ be a positive integer, prime to all residue field characteristics of $X$. Then there is an exact triangle:

$$Z_{\text{ét}}(n, X) \to Z_{\text{ét}}(n, X) \to \mu_l \otimes n \to Z_{\text{ét}}(n, X)[1],$$

where $\mu_l \otimes n$ denotes the $n^{th}$ Tate twist of the $l^{th}$ roots of unity $\mu_l$ on the étale site over $X$.

If $p$ is a residue field characteristic of $X$, then there is an exact triangle in the derived category of $G_K$-modules:

$$Z_{\text{ét}}(n, X) \to Z_{\text{ét}}(n, X) \to v(n) \to Z_{\text{ét}}(n, X)[1],$$

where $v(n)$ denotes the additive subsheaf of the deRham complex $\Omega^\bullet X$ generated by logarithmic differential $n$-forms, see Milne [17].

(4) There are product maps:

$$Z_{\text{ét}}(n, X) \otimes L Z_{\text{ét}}(m, X) \to Z_{\text{ét}}(n + m, X).$$

(5) The sheaves $H^0(Z_{\text{ét}}(n, X))$ are isomorphic to $\text{Gr}^n K_{n-1}$, where $K_\ast$ is the sheaf of algebraic $K$-groups on the étale site over $X$, and $\text{Gr}_\gamma$ is the gradation corresponding to Soulé’s $\gamma$-filtration [24].

(6) If $F$ is a field, then $H^0(G_F, Z_{\text{ét}}(n, F))$ is canonically isomorphic to $K^M_n(F)$.

**Remark A.1.** For small weights, the motivic complex is concentrated in degree zero. $Z_{\text{ét}}(0, X)$ is the constant sheaf $\mathbb{Z}$; $Z_{\text{ét}}(1, X)$ is the multiplicative group $\mathbb{G}_m$.

**Remark A.2.** If $K$ is a field, then the second axiom for $X = \text{Spec } K$ is an analogue of Noether’s generalization of Hilbert’s Theorem 90. It is equivalent to stipulating that:

$$H^1(G_K, Z_{\text{ét}}(n, K)) = 0.$$

**Remark A.3.** If $K$ is a field, and $L/K$ is a Galois extension, then there is a natural way to produce a complex of $\text{Gal}(L/K)$-modules from a complex of $G_K$-modules. It is given by:

$$C^\bullet \mapsto \tau_{\leq 0} R\Gamma(G_L, C^\bullet).$$

In the case that $C^\bullet$ is concentrated in degree zero, we recover the classical operation of taking $G_L$-invariants.

For the weight $n$ motivic complex over a field $K$, it is expected that the corresponding complex of $\text{Gal}(L/K)$-modules will have much in common with the étale complex. We denote

$$Z_B(n, L) = \tau_{\leq 0} R\Gamma(G_L, Z_{\text{ét}}(n, K)),$$

after Beilinson, who first envisioned motivic complexes for the Zariski site.

There is also a set of axioms for Beilinson’s complex $Z_B(n, X)$ of sheaves on the Zariski site of $X$:

- (1) If $n > 0$, then $Z_B(n, X)$ is acyclic outside of degrees $[-n + 1, 0]$.
- (2) If $l$ is invertible on $X$, then

$$Z_B(n, X) \otimes^L \mathbb{Z}/l\mathbb{Z} = \tau_{\leq 0} R\alpha_\ast \mathbb{Z}/l\mathbb{Z}(n)[n],$$

where $\mathbb{Z}/l\mathbb{Z}(n)$ denotes the $n^{th}$ Tate twist of $\mu_l$. 


(3) $Gr^n_k \gamma^j_k \simeq H^{n-j}(X, \mathbb{Z}_B(n, X))$ up to torsion, and probably up to "small factorials". Here $K_\alpha$ denotes the sheaf of $K$-groups on the Zariski site of $X$.

(4) If $X$ is smooth, then $H^0(X, \mathbb{Z}_B(n, X))$ is $K_n(X)$. In general, $\mathbb{Z}_B(n, X)$ should be given by $\tau \leq 0 R\alpha^* \mathbb{Z}_{\text{ét}}(n, X)$. In fact, Beilinson’s axioms follow directly from a strengthened version of the axioms for the étale site, see Lichtenbaum [15].

Remark A.4. Let $K$ be a field, and $L/K$ a finite Galois extension. Then the cohomology groups for the Zariski motivic complex $\mathbb{Z}_B(n, L)$ and the étale motivic complex $\mathbb{Z}_{\text{ét}}(n, K)$ are related through an inflation-restriction exact sequence:

$$0 \to H^2(\text{Gal}(L/K), \mathbb{Z}_B(n, K)) \to H^2(G_K, \mathbb{Z}_{\text{ét}}(n, K)) \to H^2(G_L, \mathbb{Z}_{\text{ét}}(n, L)).$$

This is a simple consequence of Hilbert’s Theorem 90 for $\mathbb{Z}_{\text{ét}}(n, K)$ and the fact that there is a quasi-isomorphism:

$$\mathbb{Z}_B(n, L) = \tau \leq 0 R\Gamma(G_L, \mathbb{Z}_{\text{ét}}(n, K)).$$

Remark A.5. In 1987, Lichtenbaum [16] constructed a candidate for the weight two motivic complex, using techniques from $K$-theory. He showed it satisfies most of the Axioms, and all the Axioms if $X = \text{Spec } K$.

Bloch was the first to construct a candidate for the weight $n$ motivic complex, using higher Chow groups, see [7]. Using another approach, Voevodsky, Suslin and Friedlander [27] have constructed a triangulated category that plays the role of the derived category of mixed motives. In the process, they have shown [26] that Bloch’s higher Chow group complex satisfies all of Lichtenbaum’s axioms, except perhaps the first. This is axiom is equivalent to the Beilinson-Soulé Vanishing Conjecture [24]. It is known that Soulé’s $\gamma$-filtration has finite length for $X = \text{Spec } K$, that is, that Bloch’s complex is bounded below. For the purpose of our result on cohomological dimension, this is sufficient.

Appendix B. Tate cohomology

When $G$ is a finite group, we use the standard two-sided resolution $P_\bullet$ of $\mathbb{Z}$ as a $G$-module to define Tate cohomology groups, see Neukirch-Schmidt-Wingberg [19]. The following definition was made by Koya [12]:

Definition B.1. Let $G$ be a finite group, $A^\bullet$ a bounded complex of $G$-modules, and $P_\bullet$ the standard two-sided resolution of $\mathbb{Z}$ as a $G$-module. The Tate cohomology group $\tilde{H}^q(G, A^\bullet)$ for $q \in \mathbb{Z}$ is defined to be the $q^{th}$ total homology of the double complex:

$$\text{Hom}_G(P_\bullet, A^\bullet).$$

Remark B.2. In this setting, one can introduce norm maps on hyper-cohomology groups. Let $N = \sum_{\sigma \in G} \sigma$ be the norm element of the group ring $\mathbb{Z}[G]$. If $A^\bullet$ is a complex of $G$-modules, then multiplication by $N$ yields a complex $NA^\bullet$ with the property that

$$\mathcal{H}^q(NA^\bullet) = N\mathcal{H}^q(A^\bullet).$$

The latter group can be considered as a subgroup of $H^0(G, \mathcal{H}^q(G, A^\bullet))$, which maps via an edge morphism to $H^q(G, A^\bullet)$, using the standard spectral sequence to
compute hyper-cohomology, see Weibel [28]. The norm map on hyper-cohomology groups is defined to be the composite:

\[ N : \mathcal{H}^q(A^\bullet) \to H^q(G, A^\bullet). \]

In the case where \( G = \text{Gal}(L/K) \) and \( A^\bullet \) is \( \mathbb{Z}_B(n, L) \), the weight \( n \) motivic complex on the Zariski site, we can recover the norm maps on Milnor \( K \)-theory defined by Bass-Tate [5] by setting \( q = 0 \). These definitions necessarily agree, because the definition of the norm through hyper-cohomology shares the same functoriality properties as in Kato [10, §1], which characterize the norm map uniquely.

**Remark B.3.** For the purposes of Tate cohomology, the assumption that \( A^\bullet \) is a bounded complex, that is, that \( A^\bullet \) has only finitely many non-zero entries, is necessary for the convergence of the spectral sequence associated to the double complex. If \( A^\bullet \) is a complex that is zero in degree larger than \( n \), then

\[ \hat{H}^q(G, A^\bullet) \simeq \mathcal{H}^q(\text{R}\Gamma(G, A^\bullet)) = H^q(G, A^\bullet) \]

for all \( q > n \). Furthermore, we have that

\[ \hat{H}^n(G, A^\bullet) = \mathcal{H}^n(\text{R}\Gamma(G, A^\bullet))/N\mathcal{H}^n(A^\bullet), \]

see Theorem 1.2 of Koya [12].

We have the following fundamental observation:

**Proposition B.4** (Dimension-shifting for complexes). Let \( G \) be a finite group and \( A^\bullet \) a bounded complex of \( G \)-modules. Then there exists a \( G \)-module \( A' \) such that for any subgroup \( H \) of \( G \),

\[ \hat{H}^q(H, A^\bullet) \simeq \hat{H}^q(H, A') \]

for all \( q \in \mathbb{Z} \).

See [1] for a proof.

**Corollary B.5.** [13, Prop. 4] Let \( G \) be a finite group, and \( A^\bullet \) a bounded complex of \( G \)-modules. Suppose that for every \( p \)-Sylow subgroup \( H \) of \( G \), there is an integer \( i_H \) such that

\[ \hat{H}^{i_H}(H, A^\bullet) = \hat{H}^{i_H+1}(H, A^\bullet) = 0. \]

Then \( \hat{H}^q(G, A^\bullet) = 0 \) for all \( q \in \mathbb{Z} \).

This follows immediately from the Proposition, and the corresponding classical fact for the Tate cohomology of a module, see Artin-Tate [3].

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