LINEAR DIFFERENTIAL EQUATIONS FOR THE RESOLVENTS OF THE CLASSICAL MATRIX ENSEMBLES

ANAS A. RAHMAN AND PETER J. FORRESTER

ABSTRACT. The spectral density for random matrix \( \beta \) ensembles can be written in terms of the average of the absolute value of the characteristic polynomial raised to the power of \( \beta \), which for even \( \beta \) is a polynomial of degree \( \beta(N-1) \). In the cases of the classical Gaussian, Laguerre and Jacobi weights, we show that this polynomial, and moreover the spectral density itself, can be characterised as the solution of a linear differential equation of degree \( \beta + 1 \). This equation, and its companion for the resolvent, are given explicitly for \( \beta = 2 \) and 4 for all three classical cases, and also for \( \beta = 6 \) in the Gaussian case. Known dualities for the moments relating \( \beta \) to \( 4/\beta \) then imply corresponding differential equations in the case \( \beta = 1 \), and for the Gaussian ensemble, the case \( \beta = 2/3 \). We apply the differential equations to give a systematic derivation of recurrences satisfied by the spectral moments and by the coefficients of their \( 1/N \) expansions, along with first order differential equations for the coefficients of the \( 1/N \) expansions of the corresponding resolvents. We also present the form of the differential equations when scaled at the hard or soft edges.

1. Introduction

Two complementary statistical quantities associated with the eigenvalues of a random matrix ensemble are the averages of products of characteristic polynomials and the \( k \)-point correlation function. For so-called Wigner matrices (see e.g. [60]), for which the entries (up to a possible symmetry requirement) are independently distributed, the former is a lot simpler. To illustrate this, let \( X \) be any \( N \times N \) real random matrix with all entries independent and having zero mean and standard deviation unity. Then it is straightforward to verify that the real symmetric matrix \( G = \frac{1}{2}(X + X^T) \) has average characteristic polynomial

\[
\langle \det(\lambda I_N - G) \rangle = 2^{-N}H_N(\lambda),
\]

where \( H_N(\lambda) \) is the \( N^{th} \) Hermite polynomial. On the other hand, the lowest order \( k \)-point correlation function, i.e. the spectral density, exhibits no such simple formula in general.

If instead of Wigner matrices one considers invariant ensembles with probability density function (PDF) proportional to \( |\det X|^\alpha e^{-\sum_{l=1}^N \text{Tr} X^l} \), the averaged characteristic polynomials and the \( k \)-point correlation function are seen to be closely related. For such invariant matrix
ensembles the eigenvalue PDF has the form, to be referred to as $\text{ME}_{\beta,N}(w(x))$,

$$
\frac{1}{C_N} \prod_{i=1}^{N} w(x_i) \prod_{1 \leq j < k \leq N} |x_k - x_j|^{\beta},
$$

(1.2)

where $\beta = 1, 2$ or 4 corresponds to the diagonalising matrices being unitary matrices with real, complex or real quaternion entries respectively, and $w(x) = |x|^{\alpha_0}e^{-\sum_{l=1}^{N} a_l x^l}$. With $N$ replaced by $N + 1$ in (1.2), the one-point density $\rho_{(1),\beta,N+1}(x)$ is then given in terms of the absolute value of the $\beta^{th}$ moment of the characteristic polynomial according to

$$
\rho_{(1),\beta,N+1}(x) = \frac{(N + 1)C_N}{C_{N+1}} w(x) \left< \prod_{i=1}^{N} |x - x_i|^{\beta} \right>_{\text{GE}_{\beta,N}},
$$

(1.3)

where the average is over the PDF (1.2) (i.e. the eigenvalue PDF for $N$ eigenvalues).

Let us now specialise to one of the three classical weights$^1$

$$
w(x) = \begin{cases} 
e^{-x^2}, & \text{Gaussian} \\ x^a e^{-x} \chi_{x>0}, & \text{Laguerre} \\ x^a (1-x)^b \chi_{0<x<1}, & \text{Jacobi}, \end{cases}
$$

(1.4)

where $\chi_A = 1$ for $A$ true and $\chi_A = 0$ otherwise. In the Jacobi case the average in (1.3) is an example of a particular class of Selberg correlation integrals which have been shown to satisfy a first order matrix linear differential equation of order $N + 1$ [33], as well as a linear matrix recurrence relation in the parameter of the same order [31].

Being of order $N + 1$, the use of the linear differential equation in relation to analysing the density is, per se, feasible for only small values of $N$. However, this circumstance changes dramatically when one takes into consideration that for the classical ensembles the moments of the characteristic polynomial satisfy duality relations [23], [25], [9], [14], [27], [16]. The simplest of these is in the Gaussian case. With $w(x) = e^{-x^2}$ in (1.2) denoted $\text{GE}_{\beta,N}$, the duality reads [3]

$$
\left< \prod_{i=1}^{N} \left( x - \sqrt{\frac{2}{\beta}} x_i \right)^n \right>_{\text{GE}_{\beta,N}} = \left< \prod_{i=1}^{N} (x - ix_i)^N \right>_{\text{GE}_{4/\beta,n}}.
$$

(1.5)

This shows that for $\beta$ even, the average in (1.3) in the Gaussian case is equal to an average which satisfies a linear differential equation of order $\beta + 1$. Analogous dualities for the Laguerre and Jacobi cases [23], [25], [27] show that the same conclusion holds for those ensembles too.

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$^1$The terminology classical weight has its origin in the theory of orthogonal polynomials of a single variable. For a precise definition see e.g. [27] §5.4.1. These three weights correspond to eigenvalue densities with two soft edges, a soft and hard edge, and two hard edges, respectively.
The primary purpose of the present paper is to make explicit the order 3 for $\beta = 2$, and order 5 for $\beta = 4$, linear differential equations satisfied by the density $\rho_{\beta,N}(x)$ in the classical cases \((1.4)\). Another type of duality formula, relating spectral moments

$$ m_k = \int_I x^k \rho_{\beta,N}(x) \, dx, \quad I = \text{supp} \rho_{\beta,N}, \quad (1.6) $$
onumber

of the $\beta$ and $4/\beta$ ensembles \([19, 21, 32, 36, 64]\), then allows for the determination of a $5^{th}$ order differential equation satisfied by \((1.3)\) with $\beta = 1$.

For the ensemble corresponding to \((1.2)\) with $\beta = 2$ and the Gaussian weight (referred to as the GUE), the fact that the density satisfies a third order linear differential equation can be traced back to the work of Lawes and March \([48]\), where the setting is that of interpreting the GUE eigenvalue PDF as the squared ground state wave function of spinless non-interacting fermions in one dimension, in the presence of a harmonic confining potential. In the random matrix theory literature, this result has appeared in the work of Götze and Tikhomiroz \([38]\), making use of an earlier result of Haagerup and Thorbjørnsen \([39]\) characterising the two-sided Laplace transform of the density in terms of a hypergeometric function. Also contained in \([38]\) is a third order linear differential equation for the spectral density of the LUE (the Laguerre weight case of \((1.2)\) with $\beta = 2$); this equation can be found too in \([11]\). These characterisations were used to obtain optimal bounds for the rate of convergence to the limiting semi-circle law (for the GUE) and Marchenko-Pastur law (for the LUE). The earlier result of \([39]\) was used by Ledoux \([49]\) to deduce small deviation inequalities for the largest eigenvalue in the GUE. More recently Kopelevitch \([47]\) made use of the third order differential equation satisfied by the spectral density for the GUE to study the $1/N$ expansion of the average of a linear statistic.

In the Gaussian case with $\beta = 1$ or $\beta = 4$, corresponding to the well known GOE and GSE respectively, the fifth order homogeneous differential equations for the density were derived in \([64]\) using a method based on known evaluation of the density in terms of Hermite polynomials \([22]\). Very recently third order inhomogeneous differential equations have been derived for the same quantity, in which the inhomogeneous term involves the GUE density \([57]\). Our approach, using Selberg correlation integrals and duality formulas, is different, and moreover unifies all the classical cases. Furthermore, it opens the way for the future study of applications analogous to those in \([38], [49], [47], [57]\). An application in the present work will be to the characterisation of the moments of the spectral density through difference equations, supplementing results on this topic in \([41], [39], [49], [59], [12], [13], [11]\) as well as the related studies \([62], [59], [51], [53], [54], [10]\). Another will be to the characterisation of the soft and hard edge scaled densities for $\beta = 1, 2$ and 4 via differential equations, and at the soft edge for $\beta = 6$ and 2/3 as well.
In section 2, we derive homogeneous linear differential equations for the $\beta = 1, 2$ and $4$ Laguerre and Jacobi ensembles’ densities, and inhomogeneous analogues of these for the resolvents

$$W_{\beta,N}(x) = \int_I \frac{\rho_{(1),\beta,N}(\lambda)}{x - \lambda} \, d\lambda, \quad I = \text{supp} \rho_{(1),\beta,N}. \quad (1.7)$$

In a structural sense our approach is applicable in all three cases for all even $\beta$, with the moments duality relation then implying the analogous characterisation for the coupling $4/\beta$. However, at a technical level there is an increase in complexity as the weight changes from Gaussian, to Laguerre, to Jacobi, and an increase in complexity as $\beta$ is increased. Hence beyond the Jacobi weight with $\beta = 4$ and $1$, the next case in this ordering of complexity is the Gaussian weight with $\beta = 6$ and $\beta = 2/3$. In the final subsection of section 2, we work this case out in detail, giving the explicit forms of the 7th order linear differential equation specifying the densities.

For the weights considered in this paper, when the eigenvalue density is scaled to have finite support in the $N \to \infty$ regime, its resolvent admits a topological expansion [6]. That is, if $c_N$ is a scaling parameter such that to leading order $\rho_{(1),\beta,N}(c_N \lambda)$ has a finite support as a function of $\lambda$, then $W_{\beta,N}(c_N x)$ has an asymptotic $1/N$-expansion of the form

$$\frac{c_N}{N} W_{\beta,N}(c_N x) = \sum_{l=0}^{\infty} \frac{W_l^\beta(x)}{(N\sqrt{\kappa})^l}, \quad \kappa := \frac{\beta}{2}, \quad (1.8)$$

where the $W_l^\beta(x)$ are independent of $N$, but may depend on $\kappa$. In section 3, we apply the results of section 2 to derive recurrence relations for the spectral moments using the fact that (1.7) characterises the resolvent as a moment generating function. In the large-$N$ regime, we also derive recurrence relations for the $W_l^\beta(x)$ defined above, and the corresponding coefficients of the $N$-expansions of the spectral moments. Differential equations characterising the densities in the hard and soft edge limits are considered in section 4.

2. Differential Equations

It has been commented above that homogeneous differential equations for the densities of the Gaussian orthogonal, unitary, and symplectic ensembles are known from previous work. This is similarly true of the corresponding (inhomogeneous) differential equations for the corresponding resolvents. For future reference, we present the results here, using [64] as our source and thus choosing the Gaussian weight to be $e^{-Nx^2/(2g)}$ (the coupling constant $g$ determines the length scale – to leading order in $1/N$, the spectrum is supported on $(-2\sqrt{g}, 2\sqrt{g})$ – and we recall from (1.8) that $\kappa = \frac{\beta}{2}$).
Proposition 2.1. Define

\[
\mathcal{D}^{(G)}_{\beta,N} = \begin{cases} 
    \left( \frac{g}{N\sqrt{\kappa}} \right)^2 \frac{d^2}{dx^2} - y_{(G)}^2 \frac{d}{dx} + x, & \beta = 2 \\
    - \left( \frac{g}{N\sqrt{\kappa}} \right)^4 \frac{d^5}{dx^5} + 5 \left[ \frac{1}{2} y_{(G)}^2 - h \left( \frac{g}{N\sqrt{\kappa}} \right) \right] \left( \frac{g}{N\sqrt{\kappa}} \right)^2 \frac{d^3}{dx^3} - 3 \left( \frac{g}{N\sqrt{\kappa}} \right)^2 x \frac{d^2}{dx^2} \\
    - \left[ y_{(G)}^2 - 4h \left( \frac{g}{N\sqrt{\kappa}} \right)^2 \right] \frac{d}{dx} + \left[ y_{(G)}^2 - 2h \left( \frac{g}{N\sqrt{\kappa}} \right) \right] x, & \beta = 1, 4,
\end{cases}
\]

(2.1)

where \( h := \sqrt{\kappa} - 1/\sqrt{\kappa} \) and \( y_{(G)} = \sqrt{x^2 - 4g} \). Then, for \( \beta = 1, 2 \) and \( 4 \),

\[
\mathcal{D}^{(G)}_{\beta,N} \rho^{(G)}_{(1),\beta,N}(x) = 0
\]

(2.2)

and

\[
\mathcal{D}^{(G)}_{\beta,N} \frac{1}{N} W^{(G)}_{\beta,N}(x) = \begin{cases} 
    2, & \beta = 2 \\
    2y_{(G)}^2 - 10h \left( \frac{g}{N\sqrt{\kappa}} \right), & \beta = 1, 4.
\end{cases}
\]

(2.3)

Remark 2.2. (1) It can be observed that for \( \beta = 1 \) or \( 4 \),

\[
\mathcal{D}^{(G)}_{\beta,N} = - \left( \frac{g}{N\sqrt{\kappa}} \right)^2 \left[ \mathcal{D}^{(G)}_{2,\beta,N} + 2x \right] \frac{d^2}{dx^2} + \left[ y_{(G)}^2 - 5h \left( \frac{g}{N\sqrt{\kappa}} \right) \right] \mathcal{D}^{(G)}_{2,\beta,N}
\]

\[
+ \frac{1}{2} y_{(G)}^2 \left( \frac{g}{N\sqrt{\kappa}} \right)^2 \frac{d^3}{dx^3} - \left[ hy_{(G)}^2 - \left( \frac{g}{N\sqrt{\kappa}} \right) \right] \left( \frac{g}{N\sqrt{\kappa}} \right) \frac{d}{dx} + 3h \left( \frac{g}{N\sqrt{\kappa}} \right) x.
\]

(2.4)

This form of the differential operator should be compared with results derived recently in [57].

(2) The operator for \( \beta = 2 \) is even in \( N \). In the case of \( \beta = 1 \) and \( \beta = 4 \), it is invariant under the mapping \((N, \kappa) \mapsto (-N\kappa, 1/\kappa)\); this is consistent with a known duality [59] [64].

Proposition 2.1 will be used in section 4. The remainder of this section is devoted to deriving similar results for the Laguerre and Jacobi ensembles for \( \beta = 1, 2 \) and \( 4 \), and also extending our methods to the Gaussian ensemble with \( \beta = 6 \) and (through the aforementioned duality) \( 4/\beta = 6 \).

2.1. The Jacobi Ensemble Differential Equations. Our main object of interest here is

\[
I_{\beta,N}^{(j)}(x) := \left\langle \prod_{l=1}^{N} |x - x_l|^\beta \right\rangle_{\text{JE}_{\beta,N}(\pi, h)},
\]

(2.5)
where we have written $\text{JE}_{\beta,N}(a,b)$ for $\text{ME}_{\beta,N}(x^a(1-x)^b)$, the latter being defined above (1.2). The analogue of the duality (1.5) in the Jacobi case is [27, Ch.13]

$$
\frac{S_N(a,b,\kappa)}{S_N(a+n,b,\kappa)} \left\langle \prod_{l=1}^{N}(x_l - x)^n \right\rangle_{\text{JE}_{\beta,N}(a,b)} = \left\langle \prod_{l=1}^{n}(1-xx_l)^N \right\rangle_{\text{JE}_{\kappa,\beta,a}(a',b')} ,
$$

(2.6)

where $S_N$ is the Selberg integral [27, Ch.4]. In fact, both of these averages are known to equal [45]

$$
\text{}_2F_1^{(x)}(-N,(a+b+n+1)/\kappa+N-1,(a+n)/\kappa;(x)^n),
$$

(2.7)

where $\text{}_2F_1^{(x)}$ is the generalised multivariate hypergeometric function and $(x)^n$ is the $n$-tuple $(x,\ldots,x)$. Note that on the RHS of (2.6), the parameter $b'$ is in general less than $-1$, and so the integral must be understood in the sense of analytic continuation. The average (2.5) relates to the LHS of (2.6) in the case $n = \beta$ for even $\beta$, the latter detail being needed to remove the absolute value sign.

A simple change of variables shows that the average on the RHS of (2.6) is given by $(-x)^{nN} J_{n,0}^{(N)}(1/x)$, where

$$
J_{n,p}^{(N)}(x) := \left\langle \prod_{l=1}^{n}(x_l - x)^{N+\chi_l\leq p} \right\rangle_{\text{JE}_{\kappa,\beta,a}(a',b')} .
$$

(2.8)

Thus for $\beta$ an even positive integer,

$$
I_{\beta,N}^{(f)}(x) \propto (-x)^{\beta N} J_{\beta,0}^{(N)}(1/x) .
$$

(2.9)

The significance of the generalised average (2.8) is that it satisfies the differential-difference equation [23]

$$
(n-p)E_p J_{n,p+1}^{(N)}(x) = -(A_p x + B_p) J_{n,p}^{(N)}(x) + x(x-1) \frac{d}{dx} J_{n,p}^{(N)}(x) + D_p x(x-1) J_{n,p-1}^{(N)}(x),
$$

$$
A_p = (n-p) \left(a' + b' + \frac{2}{\kappa}(n-p-1) + 2N + 2 \right),
$$

$$
B_p = (p-n) \left(a' + N + 1 + \frac{1}{\kappa}(n-p-1) \right),
$$

$$
D_p = p \left(\frac{1}{\kappa}(n-p) + N + 1 \right),
$$

$$
E_p = a' + b' + \frac{1}{\kappa}(2n-p-2) + N + 2,
$$

(2.10)

later observed to be equivalent to a particular matrix differential equation [33].
Remark 2.3. The order \( p \) elementary symmetric polynomial on \( N \) variables is

\[
e_p(x_1, \ldots, x_N) = \sum_{1 \leq j_1 < j_2 < \cdots < j_p \leq N} x_{j_1} \cdots x_{j_p} \tag{2.11}
\]

so that \( e_p(x_1, \ldots, x_N) = 0 \) if \( p > N \). It is known \([27, \text{Ch.4}]\) that the above differential-difference equation \((2.10)\) is satisfied by a broader class of functions

\[
\tilde{f}^{(N)}_{n,p,q}(x) := \left( \prod_{l=1}^{n} |x_l - x|^N x_{x_1, \ldots, x_{N-q} < x} e_p(x - x_1, \ldots, x - x_N) \right)_{E_{\beta\beta}(\sigma', \sigma')}, \quad 0 \leq q \leq N, \tag{2.12}
\]

which, in the case \( q = N \), are proportional to \( f^{(N)}_{n,p}(x) \) due to the symmetry of the integrand in \((2.8)\).

Our present goal is to derive a scalar differential equation for \( f^{(N)}_{\beta,0}(x) \). The differential-difference equation \((2.10)\) is sufficient for this, since the values \( p = 0, 1, \ldots, \beta \) result in a closed system of equations. While this method is in principal applicable for all \( \beta \in 2\mathbb{N} \), the resulting equation will be of order \( \beta + 1 \), so we address only \( \beta = 2 \) and 4 in this paper.

Lemma 2.4. With \( p = 0 \) and \( n = \beta = 2 \), the function \( f^{(N)}_{2,0}(x) \) satisfies the differential equation

\[
0 = x^2(x - 1)^2 \frac{d^3}{dx^3} f^{(N)}_{2,0}(x) - [3C_2(x) - 2(1 - 2x)] x(x - 1) \frac{d^2}{dx^2} f^{(N)}_{2,0}(x)
+ \left[ ((a + N)(1 + 4N) + 4 + N) x(x - 1) + (2C_2(x) - 3(1 - 2x)) C_2(x) \right] \frac{d}{dx} f^{(N)}_{2,0}(x)
- 2N(a + N) [2C_2(x) - 3(1 - 2x)] f^{(N)}_{2,0}(x), \tag{2.13}
\]

where \( C_2(x) = (a + 2N)(x - 1) - b - 2x \).

Proof. Setting \( n = 2 \) and taking \( p = 0, 1, 2 \) in \((2.10)\), we obtain the matrix differential equation

\[
\frac{d}{dx} \begin{bmatrix}
f^{(N)}_{2,0}(x) \\
f^{(N)}_{2,1}(x) \\
f^{(N)}_{2,2}(x)
\end{bmatrix} = \begin{bmatrix}
A_0 x^2 + B_0 & 2E_0 & 0 \\
x(x-1) & A_1 + B_1 & E_1 \\
0 & x(x-1) & E_2
\end{bmatrix} \begin{bmatrix}
f^{(N)}_{2,0}(x) \\
f^{(N)}_{2,1}(x) \\
f^{(N)}_{2,2}(x)
\end{bmatrix}. \tag{2.14}
\]

The second row gives an expression for \( f^{(N)}_{2,0}(x) \) which transforms the third row into a differential equation involving only \( f^{(N)}_{2,0}(x) \) and \( f^{(N)}_{2,1}(x) \). This equation further transforms into an equation for just \( f^{(N)}_{2,0}(x) \) upon substitution of the expression for \( f^{(N)}_{2,1}(x) \) drawn from
where \( \tilde{C}_p(x) \) is similarly defined as Lemma \( k \) turn gives us differential equations for equation corresponding to the fifth row yields a fifth order differential equation for \( J_{2,0}(x) \),

\[
0 = x^2(x - 1)^2 \frac{d^3}{dx^3} J_{2,0}^{(N)}(x) - \left[ \tilde{C}_{2,0}(x) + \tilde{C}_{2,1}(x) + 1 - 2x \right] x(x - 1) \frac{d^2}{dx^2} J_{2,0}^{(N)}(x) \\
+ \left[ (D_2 E_1 + 2D_1 E_0 - A_1 - 2A_0 + 2)x(x - 1) + \tilde{C}_{2,0}(x) \tilde{C}_{2,1}(x) \right] \frac{d}{dx} J_{2,0}^{(N)}(x) \\
\left[ A_0 \tilde{C}_{2,1}(x) + (A_1 - D_2 E_1)(A_0 x + B_0) - 2D_1 E_0(1 - 2x) \right] J_{2,0}^{(N)}(x),
\]

where \( \tilde{C}_{2,p}(x) = (A_p - 2)x + B_p + 1 \) with \( n = \beta = 2 \). Substituting the appropriate values for the constants \( A_p, B_p, D_p \) and \( E_p \) gives the claimed result. \( \square \)

**Lemma 2.5.** With \( p = 0 \) and \( n = \beta = 4 \), the function \( J_{4,0}^{(N)}(x) \) satisfies the differential equation with polynomial coefficients

\[
0 = 4x^4(x - 1)^4 \frac{d^5}{dx^5} J_{4,0}^{(N)}(x) \quad 20 [(a + 4N)(x - 1) - b - 2x]^2 x^3(x - 1)^3 \frac{d^4}{dx^4} J_{4,0}^{(N)}(x)
\]

\[ + \left[ 5(a + 4N)^2 - 5a(a - 2) - 12 \right] x^3(x - 1)^3 \frac{d^3}{dx^3} J_{4,0}^{(N)}(x) + \cdots \]

where the (lengthier) specific forms of the coefficients of the lower order derivatives have been suppressed.

**Proof.** Like the preceding proof, setting \( n = 4 \) in (2.10) yields the matrix differential equation

\[
\frac{d}{dx} \begin{bmatrix} J_{4,0}^{(N)}(x) \\ J_{4,1}^{(N)}(x) \\ J_{4,2}^{(N)}(x) \\ J_{4,3}^{(N)}(x) \\ J_{4,4}^{(N)}(x) \end{bmatrix} = \begin{bmatrix} A_0 x + B_0 \\ x(x - 1) \\ 0 \\ 0 \\ 0 \\ \frac{2E_2}{x(x - 1)} \end{bmatrix} \begin{bmatrix} 4E_2 \\ x(x - 1) \\ 2E_2 \\ x(x - 1) \\ x(x - 1) \end{bmatrix} \begin{bmatrix} 4 x^2 \\ x(x - 1) \\ A_0 x + B_0 \\ x(x - 1) \\ x(x - 1) \end{bmatrix} \frac{d^3}{dx^3} J_{4,0}^{(N)}(x)
\]

\[ = \begin{bmatrix} A_0 x + B_0 \\ x(x - 1) \\ 0 \\ 0 \\ 0 \\ \frac{2E_2}{x(x - 1)} \end{bmatrix} \begin{bmatrix} 4E_2 \\ x(x - 1) \\ 2E_2 \\ x(x - 1) \\ x(x - 1) \end{bmatrix} \begin{bmatrix} 4 x^2 \\ x(x - 1) \\ A_0 x + B_0 \\ x(x - 1) \\ x(x - 1) \end{bmatrix} \frac{d^3}{dx^3} J_{4,0}^{(N)}(x)
\]

For \( 1 \leq p \leq 4 \), the \( p \)th row gives an expression for \( J_{4,p}^{(N)}(x) \) in terms of \( \frac{d}{dx} J_{4,p-1}^{(N)}(x) \) and \( J_{4,k}^{(N)}(x) \) for \( k < p \). Substituting these expressions (in the order of decreasing \( p \)) into the differential equation corresponding to the fifth row yields a fifth order differential equation for \( J_{4,0}^{(N)}(x) \) similar to that of lemma \( 2.4 \).

\( \square \)

Now we may easily obtain differential equations for \( I_{\beta,N}^{(f)}(x) \) for \( \beta = 2 \) and \( 4 \), which in turn give us differential equations for \( p_{(1),\beta,N}^{(f)}(x) \) for the same \( \beta \) values.
Proposition 2.6. Define
\[ D_{2,N}^{(j)} = x^3(1-x)^3 \frac{d^3}{dx^3} + 4(1-2x)x^2(1-x)^2 \frac{d^2}{dx^2} + [(a+b+2N)^2 - 14] x^2(1-x)^2 \frac{d}{dx} \]
\[ - [a^2(1-x) + b^2x - 2] x(1-x) \frac{d}{dx} + \frac{1}{2} [(a+b+2N)^2 - 4] (1-2x)x(1-x) \]
\[ + \frac{3}{2} [a^2 - b^2] x(1-x) - a^2(1-x) + b^2x. \]  

Then,
\[ D_{2,N}^{(j)} \rho_{(1),2,N}^{(j)}(x) = 0 \] \tag{2.19}
and
\[ D_{2,N}^{(j)} \frac{1}{N} W_{2,N}^{(j)}(x) = (a+b+N)(a(1-x) + bx). \] \tag{2.20}

Proof. We change variables \( x \mapsto 1/x \) in (2.13) to obtain a differential equation for \( f_{2,0}^{(N)}(1/x) \), using the fact that \( \frac{d}{d(1/x)} = -x^2 \frac{d}{dx} \). Since this differential equation is homogeneous, we can ignore constants of proportionality and substitute in
\[ f_{2,0}^{(N)}(1/x) = x^{-a-2N}(1-x)^{-b} \rho_{(1),2,N+1}^{(j)}(x) \]
according to (1.3) and (2.9). Repeatedly applying the product rule and then replacing \( N+1 \) by \( N \) gives (2.19). Applying the Stieltjes transform to this equation term by term (see appendix A) and substituting in the values of the spectral moments \( m_1^{(j)} \) and \( m_2^{(j)} \) with \( \beta = 2 \) \[27\] yields (2.20). \( \square \)

This differential equation is a generalisation of that given in \[27\] §4 for the \( \beta = 2 \) Legendre ensemble, which is the Jacobi ensemble with \( a = b = 0 \). It lowers the order by one relative to the fourth order differential equation for \( \rho_{(1),2,N}^{(j)}(x) \) given in the recent work \[11\] Prop. 6.7.

Remark 2.7. It has been observed in \[34\] §3.3 that the third order differential equation for the spectral density in the Gaussian and Laguerre cases for \( \beta = 2 \) are equivalent to particular \( \sigma \) Painlevé equations implicit from the characterisation of gap probabilities in those cases (see e.g. \[27\] Ch. 8). An analogous result holds true for the differential equation (2.19). Thus, with \( v_1 = v_3 = N + (a+b)/2, v_2 = (a+b)/2, v_4 = (b-a)/2, \) in studying the gap for the interval \((0,s)\) one encounters the nonlinear equation \[27\] eq. (8.76)
\[ (t(1-t)f''')^2 - 4t(1-t)(f'')^3 + 4(1-2t)(f')^2f + 4f'f^2 - 4f^2v_1^2 \]
\[ + (f')^2 \left( 4tv_1^2(1-t) - (v_2 - v_4)^2 - 4tv_2v_4 \right) + 4f' \left( -v_1^2 + 2tv_1^2 + v_2v_4 \right) = 0. \]
subject to the boundary condition \( f(t) \sim -\xi t P_{1,2,N}^{(j)}(t) \). Substituting this boundary condition for \( f \) and equating terms of order \( \xi^2 \) shows that \( u(t) := t P_{1,2,N}^{(j)}(t) \) satisfies the second order nonlinear differential equation

\[
(t(1-t)u''(t))^2 - 4(u(t))^2v_1^2 + (u'(t))^2(4tv_1^2(1-t) - (v_2 - v_4)^2 - 4tv_2v_4) + 4u(t)u'(t)(-v_1^2 + 2tv_1^2 + v_2v_4) = 0. \tag{2.21}
\]

Differentiating this, and simplifying, gives a third order linear differential equation that agrees with \( \text{(2.19)} \) in the \( t \to 0^+ \) regime. Indeed, if we instead take \( u(t) := t(1-t) P_{1,2,N}^{(j)}(t) \), differentiating \( \text{(2.21)} \) and simplifying yields \( \text{(2.19)} \) exactly.

**Proposition 2.8.** Recalling that \( \kappa = \beta/2 \), let

\[
a_\beta := \frac{a}{\kappa - 1}, \quad b_\beta := \frac{b}{\kappa - 1}, \quad N_\beta := (\kappa - 1)N
\]

so that \((a_4, b_4, N_4) = (a, b, N)\) and \((a_1, b_1, N_1) = (-2a, -2b, -N/2)\). For \( \beta = 1 \) or 4, define

\[
D_{\beta,N}^{(j)} = 4x^5(1-x)^5 \frac{d^5}{dx^5} + 40(1-2x)x^4(1-x)^4 \frac{d^4}{dx^4} + [5e_2 - 493] x^4(1-x)^4 \frac{d^3}{dx^3} - [5f_+(x; \bar{a}, \bar{b}) - 88] x^3(1-x)^3 \frac{d^3}{dx^3} + [19e_2 - 539] (1-2x)x^3(1-x)^3 \frac{d^2}{dx^2} + 41 \left[ \bar{a} - \bar{b} \right] x^3(1-x)^3 \frac{d^2}{dx^2} + 16(1-2x)x^2(1-x)^2 \frac{d^2}{dx^2} - 22f_-(x; \bar{a}, \bar{b})x^2(1-x)^2 \frac{d^2}{dx^2} + \left[ \bar{c}^2 - 64e_2 + 719 \right] x^3(1-x)^3 \frac{d}{dx} - \left[ (e_2 - 45) (\bar{a} + \bar{b} - 6) + (\bar{a} - \bar{b})^2 + (\bar{c}^2 - 37) (\bar{a} - \bar{b}) (1-2x) - 248 \right] x^2(1-x)^2 \frac{d}{dx} + \left[ f_+(x; \bar{a}^2, \bar{b}^2) - 14f_+(x; \bar{a}, \bar{b}) - 16 \right] x(1-x) \frac{d}{dx} + \left[ e_2^2 - 9 \right] (1-2x)x^2(1-x)^2 + \left[ 5(e_2 - 9)(\bar{a} - \bar{b}) \right] x^2(1-x)^2 - \left[ \frac{1}{2} \right] \left[ 4e_2 - 36 + \frac{3}{2}(\bar{a} - \bar{b})^2 \right] (1-2x)x(1-x) - \left[ \frac{1}{2} \right] \left[ (3e_2 - 35)f_-(x; \bar{a}, \bar{b}) + \frac{3}{2}(\bar{a}^2 - \bar{b}^2) + 4(\bar{a} - \bar{b}) \right] (1-x) + f_-(x; \bar{a}^2, \bar{b}^2) \tag{2.22}
\]

where

\[
\bar{a} = a_\beta(a_\beta - 2), \quad \bar{b} = b_\beta(b_\beta - 2), \quad \bar{c} = a_\beta + b_\beta + 4N_\beta - 1,
\]

\[
f_{\pm}(x; \bar{a}, \bar{b}) = \bar{a}(1-x) \pm \bar{b}x. \tag{2.23}
\]

Then, for \( \beta = 1 \) and 4,

\[
D_{\beta,N}^{(j)} P_{1,2,N}^{(j)}(x) = 0 \tag{2.24}
\]
and

\[
D_{\beta,N}^{(1)} \frac{1}{N} W_{\beta,N}^{(1)}(x) = (\bar{\epsilon} - 2N\beta)x(1 - x) \left( (a_\beta + b_{\bar{\beta}})(a_\beta + b_{\bar{\beta}} - 2)(a_\beta(1 - x) + b_{\bar{\beta}}x) + 4N_{\beta}(\bar{\epsilon} - 2N\beta)(2a_\beta(1 - x) + 2b_{\bar{\beta}}x - 1) - a_\beta b_{\bar{\beta}}(a_{\bar{\beta}} + b_{\bar{\beta}} - 6) - 4(a_\beta + b_{\bar{\beta}} - 1) \right) - (\bar{\epsilon} - 2N\beta)(a_\beta(a_\beta - 2)^2(1 - x)^2 + b_{\bar{\beta}}(b_{\bar{\beta}} - 2)^2x^2). \quad (2.25)
\]

**Proof.** The proof is done in four steps. First, to see that equation (2.24) holds for \( \beta = 4 \), one undertakes the same steps as in the proof of Proposition 2.6. That is, change variables \( x \mapsto 1/x \) in (2.16), substitute in

\[
f_{A,0}^{(N)}(1/x) = x^{-a - 4N/(1 - x)^{-b} \rho_{(1),A,N+1}(x)},
\]

and then replace \( N + 1 \) by \( N \). For the second step, apply the Stieltjes transform according to appendix A and substitute in the values of the spectral moments \( m_1^{(j)} \) to \( m_4^{(j)} \) to obtain (2.25) for \( \beta = 4 \). The third step proves (2.25) for \( \beta = 1 \) by employing the known duality \[21, 32, 36\]

\[
W_{4,N}^{(j)}(x; a, b) = W_{1,-N/2}^{(j)}(-2x; -2a, -2b),
\]

which leads us to formulate (2.25) in terms of the \((a_\beta, b_{\bar{\beta}}, N\beta)\) variables. Since this step essentially redefines constants, applying the inverse Stieltjes transform to this result returns (2.24) with the new constants.

In the way that (2.22) is presented, all of its dependencies on \( a, b, \) and \( N \) are captured by \( \bar{a}, \bar{b} \) and \( \bar{\epsilon} \). Moreover, it can be seen that this operator is invariant under the symmetry \((x, a, b) \leftrightarrow (1 - x, b, a)\), which is a property of \( \rho_{(1),\beta,N}^{(j)}(x) \). Equations (2.19) and (2.24) have been checked for \( N = 1, 2 \) and to hold in the large \( N \) limit, while (2.20) and (2.25) have been checked to be consistent with expressions for the topological expansion coefficients \( W_{\beta}^{(j),0}(x), \ldots, W_{\beta}^{(j),4}(x) \) generated via the method presented in [32]. It should be noted that while the spectral moments of the Jacobi \( \beta \) ensemble’s eigenvalue density are complicated rational functions of \( a, b, \) and \( N \), the RHS of (2.20) and (2.25) are relatively simple polynomials.

### 2.2. The Laguerre Ensemble Differential Equations

The eigenvalue density \( \rho_{(1),\beta,N}^{(L)}(x) \) of the Laguerre \( \beta \) ensemble can be obtained from that of the Jacobi \( \beta \) ensemble via a limiting procedure: From (1.3), upon straightforward scaling we see

\[
\rho_{(1),\beta,N+1}^{(L)}(x) = \lim_{b \to \infty} b^{(N+3)N_{\bar{\beta}}+(N+2)a_{\bar{\beta}}+N+1} \rho_{(1),\beta,N+1}^{(j)} \left( \frac{x}{b} \right).
\quad (2.26)
\]
This fact allows one to transform differential equations satisfied by $\rho_{(1),\beta,N}^{(I)}(x)$ into analogues satisfied by $\rho_{(1),\beta,N}^{(L)}(x)$, which will be presented in a moment.

As an aside, suppose that one would like to obtain differential equations for $\rho_{(1),\beta,N}^{(L)}(x)$ without prior knowledge of the analogous differential equations for $\rho_{(1),\beta,N}^{(I)}(x)$, i.e. if the results of §2.1 were not available, or if one were interested in ensembles with $\beta \notin \{1, 2, 4\}$. Then, it is actually more efficient to circumvent computation of the differential equations for $\rho_{(1),\beta,N}^{(I)}(x)$ and use the aforementioned limiting procedure indirectly. That is, let

$$L_{\beta,p}^{(N)}(x) = \lim_{b \to \infty} \left( -\frac{x}{b} \right)^{\beta N + p} L_{\beta,p}^{(N)}(b/x), \quad \tilde{L}_{\beta,p}^{(N)}(x) = x^a e^{-x} L_{\beta,p}^{(N)}(x)$$

so that by (1.3), $\rho_{(1),\beta,N+1}^{(I)}(x)$ is proportional to $\tilde{L}_{\beta,0}^{(N)}(x)$ when $\beta$ is a positive even integer. Equation (2.10) gives a differential-difference equation for $(-\frac{x}{b})^{\beta N + p} L_{\beta,p}^{(N)}(b/x)$ and taking the limit $b \to \infty$ thus gives a differential-difference equation for $L_{\beta,p}^{(N)}(x)$. Substituting $L_{\beta,p}^{(N)}(x) = x^{-\beta} e^{x} L_{\beta,p}^{(N)}(x)$ then gives a differential-difference equation for $\tilde{L}_{\beta,p}^{(N)}(x)$. These equations with $p = 0, 1, \ldots, \beta$ are equivalent to matrix differential equations not unlike (2.14) and (2.17). However, having taken the limit $b \to \infty$, we ensure that these new matrix differential equations are simpler, and have the added benefit of simplifying down to scalar differential equations for $\rho_{(1),\beta,N+1}^{(L)}(x)$ rather than for auxiliary functions. One may then apply the Stieltjes transform to obtain differential equations for the resolvents, and use appropriate $\beta \leftrightarrow 4/\beta$ moment dualities [19] [32] [56] to obtain mirror differential equations like those seen in Proposition 2.8.

Since we are interested in $\rho_{(1),\beta,N}^{(L)}(x)$ with $\beta \in \{1, 2, 4\}$ and we have differential equations for $\rho_{(1),\beta,N}^{(I)}(x)$ for these $\beta$ values, we use the more direct approach to give the following proposition.

**Proposition 2.9.** Retaining the definitions of $a_\beta$, $N_\beta$, and $\tilde{a}$ from Proposition 2.8, define

$$D_{2,N}^{(L)} = x^3 \frac{d^3}{dx^3} + 4x^2 \frac{d^2}{dx^2} - [x^2 - 2(a + 2N)x + a^2 - 2] x \frac{d}{dx} + [(a + 2N)x - a^2]$$

(2.27)
Then, for $\beta = 1, 2$ and $4,$
\[
D^{(L)}_{\beta,N} = 4x^5 \frac{d^5}{dx^5} + 40x^4 \frac{d^4}{dx^4} - 10 \frac{(x-x)}{x+1} \frac{x}{(x-x)} + 5a - 88 \right) x^3 \frac{d^3}{dx^3}
- \left[ 16 \frac{(x-x)}{x+1} + 22a - 16 \right] x^2 \frac{d^2}{dx^2}
+ \frac{1}{(x-x)} \left[ \frac{x}{x+1} + 2 \frac{(x-x)}{x+1} + (2 \frac{(x-x)}{x+1} + \tilde{a} - 2) \right] x^3 \frac{d}{dx}
- \left[ 4(\tilde{a} - 3) \frac{(x-x)}{x+1} - \tilde{a}^2 + 14\tilde{a} + 16 \right] x \frac{d}{dx} - \left( \frac{(x-x)}{x+1} \right)^3
+ \left( 2 \frac{(x-x)}{x+1} + \tilde{a} \right) \left( \frac{x}{x+1} \right)^2 - (3\tilde{a} + 4) \left( \frac{(x-x)}{x+1} + \tilde{a} \right)^2
\]
(2.28)

Then, for $\beta = 1, 2$ and $4,$
\[
D^{(L)}_{\beta,N} \rho^{(L)}_{1,\beta,N}(x) = 0
\]
(2.29)

and
\[
D^{(L)}_{\beta,N} \frac{1}{N} \rho^{(L)}_{\beta,N}(x) = \begin{cases}
(x + a), & \beta = 2 \\
\frac{4}{x+1} \left[ \frac{x}{x+1} + (2a - 1) \right] N \frac{1}{x+1} \left( \frac{x}{x+1} \right)^3 \\
+ \frac{1}{x+1} \left[ (a + 2) \left( \frac{x}{x+1} \right)^2 + (a + 4) \frac{x}{x+1} - a(a + 2) \right], & \beta = 1, 4.
\end{cases}
\]
(2.30)

Proof. To obtain (2.29) from (2.19) and (2.24), change variables $x \mapsto x/b,$ multiply both sides by $b^{N+2}(N-a+1)a+N$ according to (2.26), and then take the limit $b \to \infty.$ This is equivalent to changing variables $x \mapsto x/b,$ extracting terms of leading order in $b,$ then rewriting $\rho^{(L)}_{(1),\beta,N}$ as $\rho^{(L)}_{(1),\beta,N}.$

To obtain (2.30), apply the same prescription to (2.20) and (2.25). Alternatively, one may apply the Stieltjes transform to (2.29). This is considerably easier than in the Jacobi case (see appendix A), since
\[
\int_0^\infty \frac{x^n}{x-s} \frac{d^n}{dx^n} \rho^{(L)}_{(1),\beta,N}(x) dx = s^n \frac{d^n}{ds^n} \rho^{(L)}_{\beta,N}(s)
\]
can be computed through repeated integration by parts using the fact that the boundary terms vanish at all stages: For $a > -1$ real and $m > n$ non-negative integers, $\frac{x^m}{x-s} \frac{d^n}{dx^n} \rho^{(L)}_{(1),\beta,N}(x)$ has a factor of $x^{a+m-n}$ which dominates at $x = 0$ and a factor of $e^{-x}$ which dominates as $x \to \infty.$

Equation (2.29) has been checked for $N = 1$ and 2. From inspection, it seems that the natural variable of (2.29) and (2.30) is $x/(\kappa - 1),$ which is the limit $\lim_{b \to \infty} b \rho^{(L)}_{\beta} \left( \frac{x}{b} \right).$ This is in
keeping with the duality \cite{19,32}

\[ W_{\beta,N}^{(L)}(x; a) = W_{4/\beta,-\kappa N}^{(L)}(-x/\kappa; -a/\kappa). \]

Strictly speaking, changing variables to \( y = x/(\kappa - 1) \) is not natural and would be counterproductive since the corresponding weight \( [(\kappa - 1)y]^d e^{(1-\kappa)y} \) vanishes at \( \kappa = 1 \) and has different support depending on whether \( \kappa < 1 \) or \( \kappa > 1 \).

2.3. **Additional Gaussian Ensemble Differential Equations.** The previous subsection contains a discussion on how one would obtain differential equations for the densities and resolvents of Laguerre ensembles with even \( \beta \) without knowledge of differential equations for the corresponding Jacobi \( \beta \) ensembles’ densities and resolvents. We now elucidate those ideas by explicitly applying them to the Gaussian ensembles with \( \beta = 2/3 \). Indeed, since we haven’t investigated these \( \beta \) values in the Jacobi case, we cannot immediately apply the direct limiting approach used in the proof of Proposition \( 2.9 \) and it is in fact more efficient to instead use the method presented below.

Like in the Jacobi and Laguerre cases, our initial focus is the average

\[ I_{\beta,N}^{(G)}(x) := \left\langle \prod_{l=1}^{N} |x - x_l|^\beta \right\rangle_{GE_{\beta,N}}. \]  \hspace{1cm} (2.31)

Replacing \( x \) by \( \sqrt{\frac{2}{\beta}} x \) in duality (1.5) and then factoring \( (-i)^{nN} \) from the RHS shows that for even \( \beta \), \( I_{\beta,N}^{(G)}(x) \) is proportional to \( G_{\beta,0}^{(N)}(x) \) where

\[ G_{n,p}^{(N)}(x) := (-i)^{nN+p} \left\langle \prod_{l=1}^{n} (x_l + i\sqrt{\frac{2}{\beta}} x)^{N+X_l \leq p} \right\rangle_{GE_{\beta,n}}, \quad 0 \leq p \leq n. \]  \hspace{1cm} (2.32)

Setting \( a' = b' = L \) and changing variables \( x_l \rightarrow \frac{1}{2} \left( 1 + \frac{x_l}{L} \right) \) in \( I_{n,p}^{(N)}(x) \) (2.8), we see that

\[ G_{n,p}^{(N)}(x) = \lim_{L \to \infty} 4^{nL}(-2i\sqrt{L})^{nN+p}(2\sqrt{L})^{(n-1)/\kappa+n} \left\langle \prod_{l=1}^{n} \left( 1 - i\sqrt{\frac{2}{\beta L}} x \right) \right\rangle \bigg|_{a'=b'=L}. \]  \hspace{1cm} (2.33)

Thus, equation (2.10) simplifies to a differential-difference equation for \( G_{n,p}^{(N)}(x) \),

\[ (n-p)G_{n,p+1}^{(N)}(x) = \frac{(n-p)}{\sqrt{\kappa}} x G_{n,p}^{(N)}(x) - \frac{\sqrt{\kappa}}{2} \frac{d}{dx} G_{n,p}^{(N)}(x) + \frac{p}{\kappa}\left( \frac{1}{\kappa}(n-p) + N + 1 \right) G_{n,p-1}^{(N)}(x) \]  \hspace{1cm} (2.34)

(cf. \cite{35} eq. (5.5)). Taking the \( L \to \infty \) limit early has already yielded a simpler equation than (2.10). However, we can take this one step further by defining

\[ G_{n,p}^{(N)}(x) = e^{-x^2} G_{n,p}^{(N)}(x) \]  \hspace{1cm} (2.35)
so that \( \rho_{(1),\beta,N+1}^{(G)}(x) \) is proportional to \( \tilde{C}_{\beta,0}^{(N)}(x) \). It is then easy to obtain

\[
\frac{d}{dx} \tilde{G}_{n,p}^{(N)}(x) = \frac{p}{\sqrt{x}} \left( \frac{1}{\kappa} (n - p) + N + 1 \right) \tilde{G}_{n,p-1}^{(N)}(x) \\
+ 2 \left( \frac{1}{\kappa} (n - p) - 1 \right) x \tilde{G}_{n,p}^{(N)}(x) - \frac{2}{\sqrt{x}} (n - p) \tilde{G}_{n,p+1}^{(N)}(x). \quad (2.36)
\]

With \( n = \beta \in 2\mathbb{N} \) and \( p = 0, 1, \ldots, n \), this is equivalent to a matrix differential equation which is moreover equivalent to a scalar differential equation for \( \rho_{(1),\beta,N+1}^{(G)}(x) \).

**Proposition 2.10.** For \( \beta = 2/3 \) or \( 6 \), define

\[
\mathcal{D}_{\beta,N}^{(G)} = 81(\kappa - 1)^{7/2} \frac{d^7}{dx^7} + 1008 \left( 3N_\beta - \frac{2x^2}{\kappa - 1} + 2 \right) (\kappa - 1)^{5/2} \frac{d^5}{dx^5} + 2016x(\kappa - 1)^{3/2} \frac{d^4}{dx^4}
\]

\[
+ 64 \left( 21N_\beta - \frac{14x^2}{\kappa - 1} + 5 \right) \left( 21N_\beta - \frac{14x^2}{\kappa - 1} + 23 \right) (\kappa - 1)^{3/2} \frac{d^3}{dx^3}
\]

\[
+ 9984 \left( 3N_\beta - \frac{2x^2}{\kappa - 1} + 2 \right) x(\kappa - 1)^{1/2} \frac{d^2}{dx^2} + 256 \left[ 54N_\beta \left( 4N_\beta^2 + 8N_\beta + 3 \right)
\]

\[
- (432N_\beta^2 + 576N_\beta + 57) \frac{x^2}{\kappa - 1} + 96(3N_\beta + 2) \frac{x^4}{(\kappa - 1)^2} - \frac{64x^6}{(\kappa - 1)^3} - 20 \right] (\kappa - 1)^{1/2} \frac{d}{dx}
\]

\[
+ 256 \left( 144N_\beta^2 + 192N_\beta - 64(3N_\beta + 2) \frac{x^2}{\kappa - 1} + \frac{64x^4}{(\kappa - 1)^2} + 25 \right) x(\kappa - 1)^{-1/2} \quad (2.37)
\]

where we retain the definition \( N_\beta = (\kappa - 1)N \). Then, for these same \( \beta \) values,

\[
\mathcal{D}_{\beta,N}^{(G)} \rho_{(1),\beta,N}^{(G)}(x) = 0 \quad (2.38)
\]

and

\[
\mathcal{D}_{\beta,N}^{(G)} \frac{1}{N} W_{\beta,N}^{(G)}(x) = \frac{2^{11}}{\sqrt{\kappa - 1}} \left( \frac{4x^2}{\kappa - 1} - 6N_\beta - 7 \right)^2 - 3 \frac{2^{12}}{\sqrt{\kappa - 1}}. \quad (2.39)
\]
Proof. Take equation (2.36) with \( n = \beta = 6 \) and \( N \) replaced by \( N - 1 \) to obtain the matrix differential equation

\[
\frac{d}{dx} \begin{bmatrix}
G_{6,0}^{(N-1)}(x) \\
G_{6,1}^{(N-1)}(x) \\
G_{6,2}^{(N-1)}(x) \\
G_{6,3}^{(N-1)}(x) \\
G_{6,4}^{(N-1)}(x) \\
G_{6,5}^{(N-1)}(x) \\
G_{6,6}^{(N-1)}(x)
\end{bmatrix} = \begin{bmatrix}
2x & -4\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\
\frac{3N+5}{3\sqrt{3}} & \frac{4x}{\sqrt{3}} & -\frac{10}{\sqrt{3}} & 0 & 0 & 0 & 0 \\
0 & \frac{6N+8}{3\sqrt{3}} & \frac{2x}{\sqrt{3}} & -\frac{8}{\sqrt{3}} & 0 & 0 & 0 \\
0 & 0 & \sqrt{3}(N + 1) & 0 & -2\sqrt{3} & 0 & 0 \\
0 & 0 & 0 & \frac{12N+8}{3\sqrt{3}} & -\frac{2x}{\sqrt{3}} & -\frac{4}{\sqrt{3}} & 0 \\
0 & 0 & 0 & 0 & \frac{15N+5}{3\sqrt{3}} & -\frac{4x}{\sqrt{3}} & -\frac{2}{\sqrt{3}} \\
0 & 0 & 0 & 0 & 0 & 2\sqrt{3N} & -2x
\end{bmatrix} \begin{bmatrix}
G_{6,0}^{(N-1)}(x) \\
G_{6,1}^{(N-1)}(x) \\
G_{6,2}^{(N-1)}(x) \\
G_{6,3}^{(N-1)}(x) \\
G_{6,4}^{(N-1)}(x) \\
G_{6,5}^{(N-1)}(x) \\
G_{6,6}^{(N-1)}(x)
\end{bmatrix}
\] (2.40)

Like in the proofs of Lemmas 2.4 and 2.5 for \( 1 \leq p \leq 6 \), the \( p \th \) row of the above matrix differential equation gives an expression for \( \tilde{G}_p^{(N-1)}(x) \) in terms of \( \frac{d}{dx} \tilde{G}_{p-1}^{(N-1)}(x) \) and \( \tilde{G}_{k}^{(N-1)}(x) \) for \( k < p \). Substituting these expressions into the equation corresponding to the last row in the order of decreasing \( p \) then yields a \( 7 \th \) order differential equation satisfied by \( \tilde{G}_6^{(N-1)}(x) \). Since \( \rho_{(1),6,N}(x) \) is proportional to \( \tilde{G}_{6,0}^{(N-1)}(x) \), this equation is equivalent to (2.38) for \( \beta = 6 \). Taking the Stieltjes transform of this result and substituting in the spectral moments \( m_2^{(G)} \) and \( m_4^{(G)} \) from (64) then yields (2.39) for \( \beta = 6 \). Employing the duality

\[
W_{\beta,N}^{(G)}(x) = \frac{i}{\kappa \sqrt{\kappa}} W_{4/\beta,-xN}^{(G)}(ix/\sqrt{\kappa})
\] (2.41)

as is consistent with results of (179) (64) then shows that (2.39) also holds for \( \beta = 2/3 \). Finally, taking the inverse Stieltjes transform of this result shows that (2.38) holds for \( \beta = 2/3 \) as well.

Equation (2.38) has been checked for \( N = 1 \) and 2. Similar to the Laguerre case, it seems like \( x/\sqrt{\kappa - 1} \) is the natural variable in Proposition 2.10. This is evidently due to the duality (2.41) used in the proof of this proposition. Like in the Laguerre case, there is presently no benefit in changing variables in \( x/\sqrt{\kappa - 1} \).

It has been mentioned that equation (2.36) leads to a matrix differential equation which is equivalent to a scalar differential equation for \( \rho_{(1),6,N+1}^{(G)}(x) \) when \( \beta \) is even. For \( \beta = 2 \) and 4, these differential equations are respectively,

\[
\frac{d}{dx} \begin{bmatrix}
G_{2,0}^{(N-1)}(x) \\
G_{2,1}^{(N-1)}(x) \\
G_{2,2}^{(N-1)}(x)
\end{bmatrix} = \begin{bmatrix}
2x & -4 & 0 \\
N + 1 & 0 & -2 \\
0 & 2N & -2x
\end{bmatrix} \begin{bmatrix}
G_{2,0}^{(N-1)}(x) \\
G_{2,1}^{(N-1)}(x) \\
G_{2,2}^{(N-1)}(x)
\end{bmatrix},
\] (2.42)
The corresponding scalar differential equations for the density agree with those of Proposition 2.1 after scaling $x \mapsto \sqrt{\frac{N}{2\kappa}} x$. So too does the differential equation for $p^{(G)}_{(1),1,N}(x)$ obtained by applying duality (2.41).

3. Recurrence Relations

In this section, we present the first of our two promised applications of the differential equations derived in §2. Namely, these differential equations yield recursions for the spectral moments upon interpreting the resolvents as moment generating functions. The moments are polynomials in $N$ in the Gaussian and Laguerre cases, and rational functions in $N$ in the Jacobi case. Thus, after appropriate scaling, the spectral moments of the eigenvalue densities have $1/N$ expansions whose coefficients satisfy recursions presented in §3.2 which are in turn obtained from those in §3.1. Recursions satisfied by these moment-expansion coefficients have been studied in special cases, with particular attention paid to their topological or combinatorial interpretations. For instance, the GUE moment coefficients satisfy the Harer-Zagier recursion [41], and it was subsequently found that the Gaussian and Laguerre moments all have interpretations in terms of ribbon graphs for each of $\beta = 1, 2$ and 4 [55, 17, 8].

In §3.3, we use the results of the previous section to obtain differential equations for the coefficients of the topological expansions of the scaled resolvents. These differential equations yield a recursive process for computing the resolvents up to any desired order in $1/N$. We remark that topological recursion accomplishes the same task, although less efficiently, for general $\beta > 0$; see [22], [32] and references therein. Hence our work isolates extra structures for particular $\beta$. For now, checking that these differential equations are satisfied by the resolvent coefficients computed according to [32] will reaffirm the results of sections 2 and 3.
3.1. Recursions for the Spectral Moments. As mentioned in the introduction, use of the geometric series in (1.7) shows that

\[ W_{\beta,N}(x) = \sum_{k=0}^{\infty} \frac{m_k}{x^k+1}, \quad x \notin \text{supp } \rho_{(1),\beta,N} \]  \hspace{1cm} (3.1)

where \( m_k \) is the \( k \)th spectral moment as defined in equation (1.6). Substituting this into the differential equations of section 2 and then equating terms of equal order in \( x \) gives relations between the moments. The equations obtained from terms of negative order in \( x \) give the upcoming recursions, while the terms of order 1 and positive order in \( x \) give the first few moments required to run the recursions (the latter are also available in earlier literature; see e.g. [42] and references therein). This method applies to all the cases considered, so the following propositions will be presented without formal verification.

It should be noted that the moment recursions pertaining to the Laguerre ensembles can be obtained from those for the Jacobi ensembles by substituting in \( m_{k}^{(J)} = b^{-k} m_{k}^{(L)} \) and then taking the limit \( b \to \infty \), since (2.26) implies that as \( b \to \infty \), the terms of leading order in \( b \) of \( W_{\beta,N}^{(L)}(x) \) are proportional to those of \( W_{\beta,N}^{(J)}(bx) \).

**Proposition 3.1.** For \( \beta = 2 \) and \( k \geq 3 \), the moments of the Jacobi ensemble satisfy the third order linear recurrence

\[ m_{k}^{(J)} = \frac{1}{d_{2,0}^{(J)}} \sum_{l=1}^{3} d_{2,l}^{(J)} m_{k-l}^{(J)} \]  \hspace{1cm} (3.2)

where

\[ d_{2,0}^{(J)} = k \left[ (k-1)^2 - (a+b+2N)^2 \right], \]
\[ d_{2,1}^{(J)} = 3k^3 - 11k^2 - k \left[ 2(a+b+2N)^2 + a^2 - b^2 - 14 \right] + 3(a+b)(a+2N) + 6(N^2 - 1), \]
\[ d_{2,2}^{(J)} = (2k-3) \left[ 2N(a+b+N) + ab \right] - (k-2) \left[ 3k^2 - 10k - 3a^2 + 9 \right], \]
\[ d_{2,3}^{(J)} = (k-3) \left[ (k-2)^2 - a^2 \right]. \]  \hspace{1cm} (3.3)

The initial terms \( m_{0}^{(J)}, m_{1}^{(J)}, m_{2}^{(J)} \) are given in [52].

We remark that a third order linear recurrence for the difference of moments \( m_{2k} - m_{2k-2} \) is known from the earlier work of Ledoux [49]. Moreover, the second order recurrence on the LUE moments,

\[ (k+1)m_{k}^{(L)} = (2k-1)(a+2N)m_{k-1}^{(L)} + (k-2) \left[ (k-1)^2 - a^2 \right] m_{k-2}^{(L)}, \quad \beta = 2, k \geq 2, \]  \hspace{1cm} (3.4)

derived as a limiting case of (3.2) according to the prescription noted in the paragraph above Proposition 3.1 agrees with the recurrence obtained in [49] for \( \{ m_{k}^{(L)} \} \). It has been shown in
that this recurrence also holds for all \(k \in \mathbb{Z}\), assuming \(a\) is large enough for the negative moments to converge.

**Proposition 3.2.** For \(\beta = 1\) and \(4\), and \(k \geq 5\), the moments of the Jacobi ensemble satisfy the fifth order linear recurrence

\[
m_k^{(l)} = \frac{1}{d_{40}^{(l)}} \sum_{l=1}^{5} d_{4l}^{(l)} m_{k-l}^{(l)}
\]

where

\[
d_{40}^{(l)} = k(\tilde{e}^2 - (k - 2)^2)(\tilde{e}^2 - (2k - 1)^2),
\]

\[
d_{41}^{(l)} = \frac{1}{2}(\tilde{e}^2 - 9)^2 (5 - 6k) + \frac{1}{2} (\tilde{a} - \tilde{b}) \left( (\tilde{e}^2 - 9)(5 - 4k) + 2k(5(k - 1)(k - 5) + 4k) \right) + (\tilde{e}^2 - 9)k [5(4k - 3)(k - 3) + 2k] - 4k^2(k - 5) [5(k - 2)(k - 1) - 2],
\]

\[
d_{42}^{(l)} = \tilde{e}^4(3k - 5) + \tilde{e}^2 \left[ \frac{1}{4} (\tilde{a} + \tilde{b})(2k - 5) + 5(\tilde{a} - \tilde{b})(k - 2) - 30k^3 + 171k^2 - 339k + 230 \right] - \frac{1}{2} (\tilde{a} + \tilde{b}) \left[ 5k^3 - 44k^2 + 129k - 125 \right] - \frac{1}{2} (\tilde{a} - \tilde{b}) \left[ 35k^3 - 246k^2 + 581k - 460 \right] + \frac{1}{2} (2k - 5)(\tilde{a} - \tilde{b})^2 + 40k^4(k - 11) + 1966k^3 - 4443k^2 + 5056k - 2305,
\]

\[
d_{43}^{(l)} = \frac{1}{2} \tilde{e}^4(5 - 2k) + \tilde{e}^2 \left[ \frac{1}{4} (\tilde{a} + \tilde{b})(25 - 8k) + \frac{1}{4} (\tilde{a} - \tilde{b})(45 - 16k) + 20k^3 - 155k^2 + 401k - 345 \right] + \frac{5}{4} (\tilde{a} + \tilde{b}) \left[ 6k^3 - 62k^2 + 216k - 253 \right] - \frac{1}{4} (\tilde{a} - \tilde{b}) \left[ 90k^3 - 806k^2 + 2436k - 2485 \right] + \frac{1}{4} (\tilde{a}^2 - \tilde{b}^2)(15 - 4k) + \frac{1}{4} (\tilde{a} - \tilde{b})^2(25 - 8k) - 4k^3(10k^2 - 140k + 789) + 8923k^2 - 12600k + \frac{14125}{2},
\]

\[
d_{44}^{(l)} = (k - 4) \left[ k^3 + k^2 - 18k - (\tilde{e}^2 - \tilde{b} - 4k^2 + 29k - 51)(5k^2 - 29k + 40) \right] + \frac{1}{2} \tilde{a}^2(6k - 25) + \frac{1}{2} \tilde{a} \left[ (4k - 15)(\tilde{e}^2 - \tilde{b} - 10k^2 + 76k - 147) - 2(k - 5) \right],
\]

\[
d_{45}^{(l)} = (k - 5) \left[ 4(k - 5)(k - 4) - \tilde{a} \right] \left[ \tilde{a} - (k - 4)(k - 2) \right],
\]

and we retain the definitions of \(\tilde{a}, \tilde{b}\), and \(\tilde{e}\) given in Proposition 2.8. The required initial terms \(m_0^{(l)}\) to \(m_4^{(l)}\) can be computed through MOPS [20] or via the methods presented in [52, 32].

Applying the aforementioned limiting procedure to this result then yields the analogous recurrences in the Laguerre case.
Proposition 3.3. For $\beta = 1$ and 4, and $k \geq 4$, the moments of the Laguerre ensemble satisfy the fourth order linear recurrence

\[ m_k^{(L)} = \frac{1}{k + 1} \sum_{l=1}^{4} d_{4,l}^{(L)} (k - 1)^l m_{k-l}^{(L)} \]  

(3.7)

where

\[ d_{4,1}^{(L)} = (4k - 1) (a_\beta + 4N_\beta), \]
\[ d_{4,2}^{(L)} = (k - 1) (5k^2 - 11k + 4) - (2k - 3) \left[ \tilde{a} + 2 (a_\beta + 4N_\beta)^2 \right], \]
\[ d_{4,3}^{(L)} = (a_\beta + 4N_\beta) \left[ (4k - 11) \tilde{a} - 10k^3 + 68k^2 - 146k + 96 \right], \]
\[ d_{4,4}^{(L)} = (k - 4) \left[ 4(k - 4)(k - 3) - \tilde{a} \right] \left[ \tilde{a} - (k - 3)(k - 1) \right], \]

(3.8)

and we retain the definitions of $a_\beta, \tilde{a},$ and $N_\beta$ given in Proposition 2.8. The required initial terms $m_0^{(L)}$ to $m_3^{(L)}$ are given in [52], [32].

This recursion is a homogeneous version of equation (43) of [13], in which the inhomogeneous terms depend on the moments of the LUE. Another simple observation is that for each of $\beta = 1, 2$ and 4, the recursions for the spectral moments of the Laguerre ensembles contain one less term than the corresponding recursions for the Jacobi ensembles, and these terms are drastically simpler, as well. This is of course due to the loss of parameter $b$.

We remark that our method applied to Proposition 2.1 yields moment recursions for the GOE, GUE, and GSE which agree with those given in [41], [50], [64], and moreover agree with the appropriate limit of the corresponding Laguerre or Jacobi recursions as derived above. Our method allows us to go further and derive analogous recursions for the $\beta = 2/3$ and $\beta = 6$ Gaussian ensembles’ spectral moments.

Proposition 3.4. For $\beta = 2/3$ and 6, and $k \geq 12$, the moments of the Gaussian ensemble satisfy the sixth order linear recurrence

\[ m_k^{(G)} = \frac{1}{4(k + 2)} \sum_{l=1}^{6} d_{6,l}^{(G)} \left( \frac{k - 1}{4} \right)^l m_{k-2l}^{(G)} \]  

(3.9)
where
\[
\begin{align*}
    d_{6,1}^{(G)} &= 8(3k - 1)(3N_\beta + 2), \\
    d_{6,2}^{(G)} &= 48(8 - 3k)N_\beta(3N_\beta + 4) + 49k^3 - 216k^2 + 92k + 320, \\
    d_{6,3}^{(G)} &= 4(k - 5)(3N_\beta + 2) \left[ 24N_\beta(3N_\beta + 4) - 49k^2 + 304k - 442 \right], \\
    d_{6,4}^{(G)} &= 2(k - 5)^3 \left[ 294N_\beta(3N_\beta + 4) - 63(k - 6) - 274 \right], \\
    d_{6,5}^{(G)} &= 252(k - 5)^5(3N_\beta + 2), \\
    d_{6,6}^{(G)} &= 81(k - 5)^7, 
\end{align*}
\]
and we retain the definition $N_\beta = (\kappa - 1)N$. Here, $(x)_n = x(x-1) \cdots (x-n+1)$ is the falling Pochhammer symbol. The required initial terms $m_0^{(G)}, m_2^{(G)}, \ldots, m_{10}^{(G)}$ are given in [64].

This recurrence is different to those given earlier in this subsection, in that it runs over every second moment. This is because the odd moments vanish, since the eigenvalue density $\rho^{(G)}(x)$ of the Gaussian ensemble is an even function. Indeed, equation (3.9) holds trivially for $k$ odd. The methods employed in [50] manifest a coupling between the moments of the GUE and GOE, which remains mysterious from our viewpoint, as does the coupled recurrence between the LUE and LOE moments given in [13, eq. (43)]: as made clear in [13], both can be traced back to a structural formula for the $\beta = 1$ density in the classical cases, in terms of the $\beta = 2$ density plus what can be regarded as a rank 1 correction; see [2]. It is however hoped that such a coupling exists between the moments of the $\beta = 6$ Gaussian ensemble and the moments of the GOE and/or GUE.

### 3.2. Harer-Zagier Type 1-Point Recursions

It is known from Jack polynomial theory [19] [21] [22] that the spectral moments for the Gaussian, Laguerre, and Jacobi ensembles have the following structures
\[
\begin{align*}
    m_{2k}^{(G)} &= \sum_{l=0}^{k} M_{k,l}^{(G)} N^{k-l+1}, \\
    m_k^{(L)} &= \sum_{l=0}^{k} M_{k,l}^{(L)} N^{k-l+1}, \\
    m_k^{(J)} &= \sum_{l=0}^{\infty} M_{k,l}^{(J)} N^{k-l},
\end{align*}
\]
respectively. Recall that the odd moments of the Gaussian ensemble are zero, and that in general the moments of the Jacobi ensemble are rational in $N$. Here, the expansion coefficients $M_{k,l}^{(\cdot)}$ have no dependence on $N$. Substituting these expressions into the recurrences
given in the previous subsection and then equating terms of equal order in $N$ yield so-called 1-point recursions \([9]\).

Applying this procedure to the GUE retrieves the celebrated Harer-Zagier recursion \([41]\), while the GOE analogue has been studied in \([46, 50]\) and references therein. The motivation for the former came from the interpretation of $M_{k,l}^{(GUE)}$ as the number of ways of gluing the edges of a $2k$-gon to form a compact orientable surface of genus $l/2$. Similarly, $4^k M_{k,l}^{(GOE)}$ counts the number of gluings of a $2k$-gon that form a compact locally-orientable surface of Euler characteristic $2 - l$. It is rather straightforward to obtain the GSE analogue through the $\beta \leftrightarrow 4/\beta$ duality. Proposition \([3, 4]\) allows an extension of the GUE, GOE, GSE 1-point recursions to the $\beta = 2/3$ and $\beta = 6$ Gaussian ensembles. Missing are the initial conditions. For this and the other recurrences presented below, these can be computed according to the strategy outlined in the first paragraph of section \([3.1]\).

**Proposition 3.5.** Expand the moments of the Gaussian ensemble according to \([3.11]\). Then, for $\beta = 2/3$ and $6$, and $k \geq 6$,

$$
(k + 1)M_{k,l}^{(G)} = \sum_{i=1}^{6} \sum_{j=0}^{i} \frac{(\kappa - 1)^{2i-j}}{2^i} f_{i,j} M_{k-i,l-j}^{(G)}
$$

(3.14)

where

$$
f_{1,0} = 3(6k - 1), \quad f_{1,1} = 2(6k - 1), \quad f_{2,0} = 36(4 - 3k), \quad f_{2,1} = 48(4 - 3k),
$$

$$
f_{2,2} = 49k^3 - 108k^2 + 23k + 40, \quad f_{3,0} = 108(2k - 5), \quad f_{3,1} = 216(2k - 5),
$$

$$
f_{3,2} = 3(5 - 2k)(98k^2 - 304k + 189), \quad f_{3,3} = 2(5 - 2k)(98k^2 - 304k + 221),
$$

$$
f_{4,2} = \frac{441}{2}(2k - 5)_3, \quad f_{4,3} = 294(2k - 5)_3, \quad f_{4,4} = \frac{1}{2}(2k - 5)_3(126k(3 - k) - 137),
$$

$$
f_{5,4} = \frac{189}{8}(2k - 5)_5, \quad f_{5,5} = 63(2k - 5)_5, \quad f_{6,6} = \frac{81}{8}(2k - 5)_7
$$

and all other $f_{i,j}$ are zero. We also set $M_{k,l}^{(G)} = 0$ if $l < 0$ or $l > k$.

When \([3.14]\) is used to compute $M_{k,0}^{(G)}$, it reduces to

$$
16(k + 1)M_{k,0}^{(G)} = 12(6k - 1)(\kappa - 1)^2 M_{k-1,0}^{(G)} - 36(3k - 4)(\kappa - 1)^4 M_{k-2,0}^{(G)}
$$

$$
+ 27(2k - 5)(\kappa - 1)^6 M_{k-3,0}^{(G)} \quad \kappa = 1/3 \text{ or } 3. \quad (3.16)
$$

This is in keeping with the limiting scaled density of the Gaussian ensembles equaling the Wigner semi-circle law specified by the density $\frac{1}{\pi} \sqrt{2 - x^2}$ supported on $|x| < \sqrt{2}$: up to a scale factor the Catalan numbers are the even moments.

To expand the Laguerre and Jacobi spectral moments in $N$, we need to decide on how the parameters $a$ and $b$ scale with $N$, which has been inconsequential up until now. To make
our results suitable for most known (see e.g. [53]) or potential applications, we first consider the LUE moments with \( a = \alpha_1 N + \delta_1 \) where the \( \alpha_1 \) and \( \delta_1 \) are parameters of order unity, though our method easily accommodates for more general \( N \)-expansions of \( a \). Substituting this together with the expansion (3.12) into (3.4) gives the sought recurrence.

**Proposition 3.6.** For \( \beta = 2 \), and \( k \geq 2 \),

\[
(k + 1)M_{k,l}^{(L)} = (2k - 1)((2 + \alpha_1)M_{k-1,l}^{(L)} + \delta_1 M_{k-1,l-1}^{(L)})
- \alpha_1^2(k - 2)M_{k-2,l}^{(L)} + (k - 2)\left((k - 1)^2 - \delta_1^2\right)M_{k-2,l-2}^{(L)}
\tag{3.17}
\]

where we set \( M_{k,l}^{(L)} = 0 \) if \( l < 0 \) or \( l > k \). When this is used to compute \( M_{k,0}^{(L)} \) in the case \( a = \delta_1 = O(1) \) and thus \( \alpha_1 = 0 \), one retrieves the familiar Catalan recursion

\[
M_{k,0}^{(L)} = \frac{2(2k - 1)}{k + 1}M_{k-1,0}^{(L)}.
\]

This is consistent with the fact that for \( a = O(1) \), the limiting scaled spectral density for the Laguerre \( \beta \) ensemble is given by the particular Marchenko-Pastur density \( \frac{1}{2\pi} \sqrt{4/x - 1} \) [60], which has for its \( k \)th moment the \( k \)th Catalan number.

When \( \delta_1 = 0 \), equation (3.17) reduces to a recursion over even \( l \), and we may interpret \( l/2 \) as genus following Di Francesco [17]. In this interpretation, \( M_{k,l}^{(L)} \) counts the same gluings as those in the Harer-Zagier recursion, except that the base \( 2k \)-gon has vertices alternately coloured black and white and the gluings respect this bicolouring. Moreover, the coloured vertices are weighted according to the dimensions of the underlying complex random matrices, which relate to the \( a \) parameter in our notation. When we additionally have \( \alpha_1 = 0 \) so that \( a = 0 \), the recurrence (3.17) is equivalent to that of theorem 4.1 of [58] on the aforementioned gluings with the coloured vertices no longer weighted.

When we take \( \delta_1 \neq 0 \), the recurrence (3.17) is now over all integer \( l \geq 0 \), to which the interpretation in terms of gluings of \( 2k \)-gons does not immediately extend. On the other hand, most of the literature [5, 7, 12, 13, 62, 59, 51, 53, 54] relating the Laguerre and Jacobi ensembles to the problems of quantum cavities and quantum transport either does not specify the scaling behaviour of the \( a \) and \( b \) parameters, fixes \( a \) and \( b \) as precise values, or takes \( a, b \propto N \). In light of this, we proceed with \( a = \alpha_1 N \) and \( b = \alpha_2 N \) with \( \alpha_i = O(1) \) for the sake of brevity and clarity. The derivation is readily transferable to more general cases should such a need arise in the future. To complete this subsection, we present the \( M_{k,l}^{(j)} \) recurrences for the LOE, LSE, and JUE. We do not present the JOE and JSE \( M_{k,l}^{(j)} \) recurrences
due to their (relatively speaking) unwieldy form, but note that they can also be derived from the results of §3.1 in the same way as the other recurrences in this subsection.

**Proposition 3.7.** For \( \beta = 1 \) and \( 4, a = a_1N \) with \( a_1 = O(1), \) and \( k \geq 4, \)

\[
(k + 1) M^{(L)}_{k,l} = \sum_{i=1}^{4} \sum_{j=0}^{\infty} (k - 1)^j g_{i,j} M^{(L)}_{k-i,l-j}
\]

(3.18)

where

\[
g_{1,0} = (4k - 1) \left( a_1 + 4(k - 1) \right) (k - 1), \quad g_{2,0} = (3 - 2k)(a_1^2 + 2(a_1 + 4(k - 1))^2(k - 1)^2),
\]

\[
g_{2,1} = 2(2k - 3)a_1, \quad g_{2,2} = (k - 1)(5k^2 - 11k + 4),
\]

\[
g_{3,0} = (4k - 11)(a_1 + 4(k - 1))(2k - 1) (k - 1), \quad g_{3,1} = 2(11 - 4k)(a_1 + 4(k - 1))a_1(k - 1),
\]

\[
g_{3,2} = 2(3 - k) (5k^2 - 19k + 16) (a_1 + 4(k - 1))(k - 1), \quad g_{4,0} = (4 - k)a_1^4,
\]

\[
g_{4,1} = 4(k - 4)a_1^3, \quad g_{4,2} = (k - 4)(5k^2 - 32k + 47)a_1^2,
\]

\[
g_{4,3} = 2(4 - k)(k - 3)(5k - 17)a_1, \quad g_{4,4} = 4(1 - k)[(k - 4)(k - 3)]^2,
\]

(3.19)

and all other \( g_{i,j} \) are zero. We also set \( M^{(L)}_{k,l} = 0 \) if \( l < 0 \) or \( l > k. \)

**Proposition 3.8.** For \( \beta = 2, a = a_1N \) and \( b = a_2N, \) with \( a_i = O(1), \) and \( k \geq 3, \)

\[
k(a_1 + a_2 + 2)^2 M^{(I)}_{k,l} = k(k - 1)^2 M^{(I)}_{k,l-2} + \sum_{i=1}^{3} \left( h_{i,0} M^{(I)}_{k-i,l} + h_{i,1} M^{(I)}_{k-i,l-2} \right)
\]

(3.20)

where

\[
h_{1,0} = 2(4k - 3)(a_1 + a_2 + 1) + (3a_1(k - 1) + a_2k)(a_1 + a_2), \quad h_{1,1} = (1 - k)(3k^2 - 8k + 6),
\]

\[
h_{2,0} = 3a_1^2(2k) + (3 - 2k)((a_1 + 2)(a_2 + 2) - 2), \quad h_{2,1} = (k - 2)(3k^2 - 10k + 9),
\]

\[
h_{3,0} = a_1^2(k - 3), \quad h_{3,1} = (3 - k)(k - 2)^2,
\]

(3.21)

and we set \( M^{(I)}_{k,l} = 0 \) if \( l < 0. \)

**Proof.** Since the derivation of this recurrence is slightly different to that described earlier, we supply the details. Begin by substituting expansion (3.13) into (3.2) and then multiplying both sides by \( d_{2,0}^{(I)} \) from (3.3). Then, substitute \( a = a_1N \) and \( b = a_2N \) and equate terms of order \( 3 - l \) in \( N. \) Note that with our choice of \( a \) and \( b, \) the coefficients \( d_{2,0}^{(I)}, \ldots, d_{2,3}^{(I)} \) all contain a term of order \( 2 \) and a term of order \( 1 \) in \( N, \) and no other terms. The \( O(N^2) \) terms of \( d_{2,l}^{(I)} \) correspond to the coefficients of \( M_{k-i,l}^{(I)} \) in (3.20) while the terms of order \( 1 \) correspond to the coefficients of \( M_{k-i,l-2}^{(I)}. \)
There are two immediate observations relating to Proposition 3.8. Firstly, this recursion runs over even \( l \), similar to Proposition 3.6 with \( a = a_1N \). Indeed, when the first few moments \( m_0^{(j)}, m_1^{(j)}, m_2^{(j)} \) are expanded as series in \( 1/N \), they do not contain terms of even powers in \( N \), so recurrence (3.20) is trivially satisfied when \( l \) is odd. Secondly, unlike the analogous recurrence relations for the GUE and LUE, \( M_{k,l}^{(j)} \) depends via this recurrence on \( M_{k,l-2}^{(j)} \) instead of just \( M_{i,l}^{(j)} \) with \( i < k \). This means that computing the order \( N^l \) term of \( m_k^{(j)} \) through this recurrence requires one to find all other terms in \( m_l^{(j)} \) that are higher order in \( N \).

As yet it is not known whether the JUE spectral moments count some type of surface similar to those counted by the GUE and LUE spectral moments. If such an interpretation is possible for the \( M_{k,l}^{(j)} \), the fact that recurrence (3.20) runs over even \( l \) suggests that \( l/2 \) might again play the role of genus, and the second observation above might give a clue as to what sets the JUE apart from the GUE and LUE. Since setting \( a = 0 \) in Proposition 3.6 yields a simpler recurrence that retains an interpretation in terms of gluings of \( 2k \)-gons, we set \( a = b = 0 \) in Proposition 3.8 for comparison:

Proposition 3.9. For \( \beta = 2 \), \( a = b = 0 \), and \( k \geq 3 \),

\[
4kM_{k,l}^{(j)} = k(k - 1)^2M_{k,l-2}^{(j)} + 2(4k - 3)M_{k-1,l}^{(j)} + (1 - k)(3k^2 - 8k + 6)M_{k-1,l-2}^{(j)}
+ 2(3 - 2k)M_{k-2,l}^{(j)} + (k - 2)(3k^2 - 10k + 9)M_{k-2,l-2}^{(j)} + (3 - k)(k - 2)^2M_{k-3,l-2}^{(j)},
\] (3.22)

where we set \( M_{k,l}^{(j)} = 0 \) if \( l < 0 \).

3.3. Differential Equations for the Coefficients of the Topological Expansion. A feature of the moment expansions (3.11) and (3.12) for the Gaussian and Laguerre ensembles is that for large \( N \) they are proportional to \( N^{k+1} \), whereas the moment expansion (3.13) is proportional to \( N \). For the corresponding resolvents to admit a \( 1/N \) expansion of the form given in (1.8), a change of scale and of normalisation is required, so that all moments are to leading order unity. For this, the rescaled spectral density will be normalised to integrate to unity and will have compact support in the \( N \to \infty \) regime.

Following [64] and [32], this can be achieved by introducing the so-called smoothed densities as

\[
\tilde{\rho}_{(1),\beta,N}^{(G)}(x) = \sqrt{\frac{\kappa}{NN}} \rho_{(1),\beta,N}^{(G)}(\sqrt{NN}x),
\] (3.23)

\[
\tilde{\rho}_{(1),\beta,N}^{(L)}(x) = \kappa \rho_{(1),\beta,N}^{(L)}(N\kappa x),
\] (3.24)

\[
\tilde{\rho}_{(1),\beta,N}^{(J)}(x) = \frac{1}{N} \rho_{(1),\beta,N}^{(J)}(x),
\] (3.25)
and their corresponding scaled resolvents as

\[
\tilde{W}_{\beta,N}^{(G)}(x) = \sqrt{N\kappa} W_{\beta,N}^{(G)}(\sqrt{N\kappa}x), \tag{3.26}
\]

\[
\tilde{W}_{\beta,N}^{(L)}(x) = N\kappa W_{\beta,N}^{(L)}(N\kappa x); \tag{3.27}
\]

since \(\tilde{\rho}_{(1),\beta,N}(x)\) has the same support as \(\rho_{(1),\beta,N}(x)\), there is no need to scale \(W_{\beta,N}^{(j)}(x)\). In the \(N \to \infty\) limit the smoothed densities approach the Wigner semi-circle law in the Gaussian case, the Marchenko-Pastur law in the Laguerre case, and a functional form first deduced by Wachter in the Jacobi case. The exact functional forms of the latter two are dependent on the proportionality in \(N\) of the \(a\) and \(b\) parameters, and are given explicitly in [32].

Compared to the resolvents \(W_{\beta,N}(x)\) of the eigenvalue densities, the scaled resolvents are generating functions for the moments of the smoothed densities above, which converge for large \(x\), and can be expanded in \(1/N\) according to:

\[
\tilde{W}_{\beta,N}^{(G)}(x) = 2N \sum_{l=0}^{\infty} \frac{W_{\beta}^{(G),j}(l)(x)}{(2N\sqrt{\kappa})^l}, \tag{3.28}
\]

\[
\tilde{W}_{\beta,N}^{(L)}(x) = N \sum_{l=0}^{\infty} \frac{W_{\beta}^{(L),j}(l)(x)}{(N\sqrt{\kappa})^l}, \tag{3.29}
\]

\[
W_{\beta,N}^{(j)}(x) = N \sum_{l=0}^{\infty} \frac{W_{\beta}^{(j),l}(x)}{(N\kappa)^l}, \tag{3.30}
\]

where we again follow [64, 32] for consistency. Appropriately scaling the differential equations of section 2 yields differential equations for the scaled resolvents which in turn give first order differential equations for the expansion coefficients \(W_{\beta}^{(j)}(x)\), after substituting in the expansions above and then equating terms of equal order in \(N\). Compared to the topological recursion, these differential equations provide a tractable recursive process for computing the \(W_{\beta}^{(j)}(x)\), which is simpler in that there is no need for introducing multi-point correlators, but more restrictive in that they only apply for the \(\beta\)-values considered in this paper. Presently, we intend to use the upcoming differential equations to check earlier results of this section for consistency both internally and with [64, 32]. For this reason, we continue to treat the case \(a = \alpha_1 N\) and \(b = \alpha_2 N\), with \(\alpha_i = O(1)\).

We begin with the Gaussian ensemble with \(\beta \in \{2/3, 1, 4, 6\}\) and refer to [40] for the \(\beta = 2\) case.

**Proposition 3.10.** We reuse the notation of Proposition 2.1 with \(g = 1/2\) so that \(h = \sqrt{\kappa} - 1/\sqrt{\kappa}\) and \(y_{(G)} = \sqrt{x^2 - 2}\). Then for \(\beta = 1\) and \(4\), the expansion coefficients of \(\tilde{W}_{\beta,N}^{(G)}(x)\) (3.28) satisfy the
differential equations

\begin{equation}
y^2 (G) \frac{d}{dx} W^{(G),0}_\beta (x) - x W^{(G),0}_\beta (x) = -1,
\end{equation}

\begin{equation}
y^2 (G) \frac{d}{dx} W^{(G),1}_\beta (x) - x W^{(G),1}_\beta (x) = 4h \frac{d}{dx} W^{(G),0}_\beta (x) - \frac{h}{y^2 (G)} \left[ 2x W^{(G),0}_\beta (x) + 5 \right],
\end{equation}

and for \( l \geq 2 \), the general differential equation,

\begin{equation}
y^2 (G) \frac{d}{dx} W^{(G),l}_\beta (x) - x W^{(G),l}_\beta (x) = 4h l \frac{d}{dx} W^{(G),l-1}_\beta (x) - \frac{2hx}{y^2 (G)} W^{(G),l-1}_\beta (x)

+ \frac{1}{y^2 (G)} \left[ \frac{5y^2 (G)}{2} \frac{d^3}{dx^3} - 3x \frac{d^2}{dx^2} + \frac{d}{dx} \right] W^{(G),l-2}_\beta (x)

- \frac{5h}{y^2 (G)} \frac{d^3}{dx^3} W^{(G),l-3}_\beta (x) + \frac{1}{y^2 (G)} \frac{d^5}{dx^5} W^{(G),l-4}_\beta (x),
\end{equation}

where we set \( W^{(G),k}_\beta := 0 \) for \( k < 0 \).

**Proposition 3.11.** Retain the choice of \( g = 1/2 \) from Proposition 3.10. Then for \( \beta = 2/3 \) and 6, the expansion coefficients of \( \hat{W}^{(G)}_{\beta,N}(x) \) satisfy the differential equations

\begin{equation}
y^2 (G) \frac{d}{dx} W^{(G),0}_\beta (x) - x W^{(G),0}_\beta (x) = -1,
\end{equation}

\begin{equation}
y^2 (G) \frac{d}{dx} W^{(G),1}_\beta (x) - x W^{(G),1}_\beta (x) = 6h \frac{d}{dx} W^{(G),0}_\beta (x) - \frac{h}{y^2 (G)} \left[ 4x W^{(G),0}_\beta (x) - 7 \right],
\end{equation}

\begin{equation}
y^2 (G) \frac{d}{dx} W^{(G),2}_\beta (x) - x W^{(G),2}_\beta (x) = 6h \frac{d}{dx} W^{(G),1}_\beta (x) - \frac{4hx}{y^2 (G)} W^{(G),1}_\beta (x) - \frac{43}{3y^4 (G)}

+ \frac{1}{12y^4 (G)} \left[ 49y^4 (G) \frac{d^3}{dx^3} - 78y^2 (G) \frac{d^2}{dx^2} + 3(72 - 19x^2) \frac{d}{dx} + 25x \right] W^{(G),0}_\beta (x).
\end{equation}
and for \( l \geq 3 \), the general differential equation,

\[
\frac{d^2}{dx^2} W^{(G),l}_{\beta}(x) - x W^{(G),l}_{\beta}(x) = 6h \frac{d}{dx} W^{(G),l-1}_{\beta}(x) - \frac{4hx}{y^{(G)}_{\beta}} W^{(G),l-1}_{\beta}(x)
\]

\[
+ \frac{1}{12y^{(G)}_{\beta}} \left[ 49y^{(G)}_{\beta} \frac{d^3}{dx^3} - 78xy^{(G)}_{\beta} \frac{d^2}{dx^2} + 3(72-19x^2) \frac{d}{dx} + 25x \right] W^{(G),l-2}_{\beta}(x)
\]

\[
+ \frac{h}{3y^{(G)}_{\beta}} \left[ 49y^{(G)}_{\beta} \frac{d^3}{dx^3} + 39x \frac{d^2}{dx^2} - 10 \frac{d}{dx} \right] W^{(G),l-3}_{\beta} + \frac{7h}{y^{(G)}_{\beta}} \frac{d^5}{dx^5} W^{(G),l-5}_{\beta}
\]

\[
+ \frac{1}{18y^{(G)}_{\beta}} \left[ 63y^{(G)}_{\beta} \frac{d^5}{dx^5} + 63x \frac{d^4}{dx^4} + 230 \frac{d^3}{dx^3} \right] W^{(G),l-4}_{\beta} + \frac{3h}{4y^{(G)}_{\beta}} \frac{d^7}{dx^7} W^{(G),l-6}_{\beta}, \tag{3.37}
\]

where we set \( W^{(G),k}_{\beta} := 0 \) for \( k < 0 \).

This proposition has been checked against \([64]\) up to \( l = 6 \), and thus also serves as a check for differential equation \([2.39]\) up to order 6 in \( 1/N \). Interpreting \( W^{(G),l}_{\beta}(x) \) as a generating function for \( M_{l,N}^{(1)} \), we confirm \([3.5]\) up to \( l = 6 \) as well. Similar checks for consistency have been carried out for Propositions \([3.12]\) to \([3.15]\) below.

Moving on to the Laguerre ensembles, we have the following three propositions.

**Proposition 3.12.** The expansion coefficients of the LUE scaled density's resolvent \( W^{(L)}_{2,N}(x) \) satisfy the differential equation

\[
[4x - (\alpha - x)^2] x \frac{d}{dx} W^{(L),0}_{2}(x) + [(\alpha + 2)x - \alpha^2] W^{(L),0}_{2}(x) = x + \alpha, \tag{3.38}
\]

and for \( l \geq 2 \), the general differential equation,

\[
[(\alpha - x)^2 - 4x] x \frac{d}{dx} W^{(L),l}_{2}(x) - [(\alpha + 2)x - \alpha^2] W^{(L),l}_{2}(x)
\]

\[
= \left[ x^3 \frac{d^3}{dx^3} + 4x^2 \frac{d^2}{dx^2} + 2x \frac{d}{dx} \right] W^{(L),l-2}_{2}(x). \tag{3.39}
\]

In the \( a = \alpha_1 N \) setting considered here, \( W^{(L),k}_{2} = 0 \) when \( k \) is odd. Thus, differential-difference equation \([3.39]\) holds vacuously for odd \( l \), and should otherwise be interpreted as a recursion over even \( l \). This is in keeping with the discussion following Proposition \([3.6]\).

**Proposition 3.13.** Let \( y^{(L),1}_1 = \sqrt{(2\alpha_1 - x)^2 - 4x} \). The expansion coefficients of the LOE scaled density's resolvent \( W^{(L)}_{1,N}(x) \) satisfy the differential equations

\[
- xy^{(L),1}_1 \frac{d}{dx} W^{(L),0}_{1}(x) + 2 [(\alpha + 1)x - 2\alpha^2] W^{(L),0}_{1}(x) = x + 2\alpha_1, \tag{3.40}
\]
\[ xy_{(L),1}^4 \frac{d}{dx} W_{1}^{(L),1}(x) - 2y_{(L),1}^2 \left[(\alpha_1 + 1)x - 2\alpha_1^2\right] W_{1}^{(L),1}(x) \]
\[ = 8h\alpha_1 y_{(L),1}^2 \frac{d}{dx} W_{1}^{(L),0}(x) + 4h\alpha_1 \left[x^2 - 6(\alpha_1 + 1)x + 8\alpha_1^2\right] W_{1}^{(L),0}(x) \]
\[ + 2h \left[x(x-1) + 4\alpha_1 x + 8\alpha_1^2\right], \quad (3.41) \]

\[ xy_{(L),1}^4 \frac{d}{dx} W_{1}^{(L),2}(x) - 2y_{(L),1}^2 \left[(\alpha_1 + 1)x - 2\alpha_1^2\right] W_{1}^{(L),2}(x) \]
\[ = 8h\alpha_1 y_{(L),1}^2 \frac{d}{dx} W_{1}^{(L),1}(x) + 4h\alpha_1 \left[x^2 - 6(\alpha_1 + 1)x + 8\alpha_1^2\right] W_{1}^{(L),1}(x) \]
\[ + \frac{5}{2} x^3 y_{(L),1}^2 \frac{d^3}{dx^3} W_{1}^{(L),0}(x) + x^2 \left[8x^2 - 38(\alpha_1 + 1)x + 44\alpha_1^2\right] \frac{d^2}{dx^2} W_{1}^{(L),0}(x) \]
\[ + 2x \left[x^2 - 6(\alpha_1 + 1)x + 10\alpha_1^2\right] \frac{d}{dx} W_{1}^{(L),0}(x) \]
\[ + \left[4(\alpha_1 + 1)x - 8\alpha_1^2\right] W_{1}^{(L),0}(x) - 2(2\alpha_1 + x) \quad (3.42) \]

and for \( l \geq 3 \), the general differential equation,

\[ xy_{(L),1}^4 \frac{d}{dx} W_{1}^{(L),l}(x) - 2y_{(L),1}^2 \left[(\alpha_1 + 1)x - 2\alpha_1^2\right] W_{1}^{(L),l}(x) \]
\[ = 8h\alpha_1 y_{(L),1}^2 \frac{d}{dx} W_{1}^{(L),l-1}(x) + 4h\alpha_1 \left[x^2 - 6(\alpha_1 + 1)x + 8\alpha_1^2\right] W_{1}^{(L),l-1}(x) \]
\[ + \frac{5}{2} x^3 y_{(L),1}^2 \frac{d^3}{dx^3} W_{1}^{(L),l-2}(x) + x^2 \left[8x^2 - 38(\alpha_1 + 1)x + 44\alpha_1^2\right] \frac{d^2}{dx^2} W_{1}^{(L),l-2}(x) \]
\[ + 2x \left[x^2 - 6(\alpha_1 + 1)x + 10\alpha_1^2\right] \frac{d}{dx} W_{1}^{(L),l-2}(x) + \left[4(\alpha_1 + 1)x - 8\alpha_1^2\right] W_{1}^{(L),l-2}(x) \]
\[ + \left[-2h\alpha_1 x \left[5x^2 \frac{d^3}{dx^3} + 22x \frac{d^2}{dx^2} + 14 \frac{d}{dx}\right] W_{1}^{(L),l-3}(x) \right] \]
\[ - \left[x^5 \frac{d^5}{dx^5} + 10x^4 \frac{d^4}{dx^4} + 22x^3 \frac{d^3}{dx^3} + 4x^2 \frac{d^2}{dx^2} - 4x \frac{d}{dx}\right] W_{1}^{(L),l-4}(x), \quad (3.43) \]

where we set \( W_{1}^{(L),-1} := 0 \).

**Proposition 3.14.** The expansion coefficients of the LSE scaled density’s resolvent \( W_{4,N}^{(L)}(x) \) satisfy the differential equations presented in Proposition 3.13 upon replacing \( \alpha_1 \) with \( \alpha_1 / 4 \).

Finally, the analogous proposition for the JUE is as follows.
**Proposition 3.15.** The expansion coefficients of the JUE scaled density’s resolvent $W_{2,N}^{(j)}(x)$ satisfy the differential equation

$$
\left[(\alpha_1 + \alpha_2 + 2)^2 x^2 - 2(\alpha_1 + 2)(\alpha_1 + \alpha_2)x + a_1^2\right] x(x-1) \frac{d}{dx} W_{2,N}^{(j),0}(x)
+ [a_1^2(x-1)^3 + \alpha_1(\alpha_2 + 2)x(1-x)(1-2x) + (\alpha_2 + 2)^2 x^3 - 2(\alpha_2 + 1)(3x^2 + x)] W_{2,N}^{(j),0}(x)
= (\alpha_1 + \alpha_2 + 1)(\alpha_1(1-x) + \alpha_2 x),
$$

and for $l \geq 1$,

$$
\left[(\alpha_1 + \alpha_2 + 2)^2 x^2 - 2(\alpha_1 + 2)(\alpha_1 + \alpha_2)x + a_1^2\right] x(x-1) \frac{d}{dx} W_{2,N}^{(j),l}(x)
+ [a_1^2(x-1)^3 + \alpha_1(\alpha_2 + 2)x(1-x)(1-2x) + (\alpha_2 + 2)^2 x^3 - 2(\alpha_2 + 1)(3x^2 + x)] W_{2,N}^{(j),l}(x)
= x^4(x-1)^3 \frac{d^3}{dx^3} W_{2,N}^{(j),l-2}(x) - 4x^2(x-1)^2(1-2x) \frac{d^2}{dx^2} W_{2,N}^{(j),l-2}(x)
+ 2x(x-1)(7x(x-1) + 1) \frac{d}{dx} W_{2,N}^{(j),l-2}(x) - 2x(x-1)(1-2x) W_{2,N}^{(j),l-2}(x),
$$

where we set $W_{2,N}^{(j),-1}(x) = 0$.

Similar to Proposition 3.12 in the $a = \alpha_1 N$, $b = \alpha_2 N$ setting, $W_{2,N}^{(j),k}$ vanishes for odd $k$, and the differential equations of Proposition 3.15 constitute a recursion over even $l$. As in §3.2, we do not present the analogous differential equations for the JOE and JSE due to their cumbersome structure.

4. Differential Equations for the Soft and Hard Edge Scaled Densities

In this section, we present the second of our promised applications of the differential equations in section 2. Namely, we derive differential equations satisfied by the eigenvalue densities when they have been centred on either the largest or smallest eigenvalue and they have been scaled so that the mean spacing between this eigenvalue and its neighbour is order unity. Upon recentring and scaling in this manner, the eigenvalue densities considered fall into one of two universal classes: The limiting smoothed density exhibits either a square root profile, which we call a soft edge, or it exhibits an inverse square root singularity, which we call a hard edge (see e.g. [28, 63]). In particular, the Gaussian ensemble eigenvalue density exhibits only soft edges in accordance with the Wigner semicircle law. Likewise, the Laguerre ensemble eigenvalue density also has a soft edge at the largest eigenvalue. Both edges of the Jacobi density and the left edge of the Laguerre density behave either as hard edges when the corresponding parameter is of order unity -- $a$ for the regime of the smallest eigenvalue and $b$ for the largest eigenvalue -- or as soft edges if said parameter is of order
Theorem 4.1. Define the soft edge limiting forms of the differential operators introduced in section \ref{sec:definition} as

\[
D^{(soft)}_{\beta,\infty} = \begin{cases} 
\frac{d^3}{dx^3} - 4x \frac{d}{dx} + 2, & \beta = 2 \\
\frac{d^3}{dx^3} - 10\kappa x \frac{d^3}{dx^3} + 6\kappa x \frac{d^2}{dx^2} + 16\kappa^2 x^2 \frac{d}{dx} - 8\kappa^2 x, & \beta = 1, 4, \\
3 \frac{d^2}{dx^2} - 56\kappa x \frac{d^2}{dx^2} + 28\kappa \frac{d}{dx} + 764\kappa^2 x^2 \frac{d}{dx^2} - 208\kappa^2 x \frac{d}{dx} - 4\kappa^2 (64\kappa x^3 - 17) \frac{d}{dx} + 128\kappa^3 x^2, & \beta = 2/3, 6, \\
\end{cases}
\] (4.1)

where we recall that $\kappa = \beta/2$. Then for $\beta \in \{2/3, 1, 2, 4, 6\}$, at leading order, the Gaussian, Laguerre and Jacobi eigenvalue densities satisfy the following differential equation in the soft edge limit:

\[
D^{(soft)}_{\beta,\infty} \rho^{(soft)}_{(1),\beta,\infty}(x) = 0.
\] (4.2)

Proof. Make the change of variables $x \mapsto \sqrt{k} \left( \sqrt{2N} + \frac{x}{\sqrt{2N^{1/6}}} \right)$ in (2.2) and (2.38). Then multiply through by $N^{-1/2}$ for $\beta = 2$, $N^{-5/6}$ for $\beta = 1$ and 4, or $N^{-7/6}$ for $\beta = 2/3$ and 6. Equating terms of order one then yields (4.2) above, while all other terms vanish in the $N \to \infty$ limit. \hfill \square

For $\beta = 1, 2, 4$, the above proof can be replicated by instead considering the differential equations (2.19), (2.24), and (2.29) for the Jacobi and Laguerre eigenvalue densities due to universality. Explicitly, differential equation (4.2) is satisfied by the leading order term of the following scaled densities in the large $N$ limit \cite{34, 43, 28, 4}:

- In the regime of the largest eigenvalue,

\[
\rho^{(G)}_{(1),\beta,N} \left( \sqrt{k} \left( \sqrt{2N} + \delta_G + \frac{x}{\sqrt{2N^{1/6}}} \right) \right),
\]

where $\delta_G = o(N^{-1/6})$ is an arbitrary parameter (the regime of the smallest eigenvalue can be treated by exploiting the symmetry $x \mapsto -x$);

- In the regime of the largest eigenvalue with $a = O(1)$,

\[
\rho^{(L)}_{(1),\beta,N} \left( \kappa \left( 4N + \delta_L^{(a=O(1))} + 2(2N)^{1/3} x \right) \right),
\]

where $\delta_L^{(\cdot)} = o(N^{1/3})$ is henceforth an arbitrary parameter;
• In the regime of the largest eigenvalue with \( a = \alpha_1 N \) and \( \alpha_1 = O(1) \),
\[
\rho^{(L)}_{(1),\beta,N} \left( \kappa \left( q^2 N + \delta^{(l,a=O(N))}_L - q_+ \left( \frac{q-N}{q_+ + 1} \right)^{1/3} x \right) \right)
\]
where \( q_\pm = \sqrt{1 + \frac{\alpha_1}{\kappa}} \pm 1 \) are the endpoints of the support of the limiting smoothed density \( \rho^{(L)}_{(1),\beta,N} \) as defined by (3.23) (note that \( q_+ \to 2 \) as \( \alpha_1 \to 0 \), so we have consistency with the scaling given above for the \( a = O(1) \) regime);

• In the regime of the smallest eigenvalue with \( a = \alpha_1 N \) and \( \alpha_1 = O(1) \),
\[
\rho^{(L)}_{(1),\beta,N} \left( \kappa \left( q^2 N + \delta^{(s,a=O(N))}_L - q_- \left( \frac{q-N}{q_- + 1} \right)^{1/3} x \right) \right);
\]

• In the regime of either the largest or smallest eigenvalue, with either \( a = \alpha_1 N \) and \( \alpha_1, b = O(1) \), or \( a = \alpha_1 N, b = \alpha_2 N \) and \( \alpha_1, \alpha_2 = O(1) \), \( \rho^{(f)}_{(1),\beta,N} \) scaled according to (42), (44).

Remark 4.2. For even \( \beta \), \( \rho^{(soft)}_{(1),\beta,\infty}(x) \) has an explicit representation as a \( \beta \)-dimensional integral due to [15], while [30] provides alternate forms for \( \beta = 1, 2 \), and 4. To date, there is no explicit functional form for \( \rho^{(soft)}_{(1),2,3,\infty}(x) \). On the other hand, [18] shows for all \( \beta > 0 \) that at leading order as \( x \to \infty \),
\[
\rho^{(soft)}_{(1),\beta,\infty}(x) \propto e^{-\frac{4\beta^3}{3}}
\]
which is an extension of the even-\( \beta \) result of [29],
\[
\rho^{(soft)}_{(1),\beta,\infty}(x) \sim \frac{\Gamma(1 + \kappa)}{8\kappa^\kappa} e^{-\frac{4\beta^3}{3}} x^{\frac{3\kappa}{2}}.
\]  (4.3)

This result is consistent with the differential equations of Theorem 4.1. So too is the result [15],
\[
\rho^{(soft)}_{(1),\beta,\infty}(x) \sim \frac{\sqrt{|x|}}{\pi}, \quad \beta \in 2\mathbb{N}.
\]  (4.4)

Likewise, Theorem 4.4 below is consistent with the result [25],
\[
\rho^{(hard)}_{(1),\beta,\infty}(x) \sim \frac{1}{2\pi \sqrt{x}}, \quad \beta \in 2\mathbb{N}.
\]  (4.5)

Note that the asymptotic forms (4.3), (4.4), and (4.5) respectively capture the facts that moving past the soft edge results in exponential decay, moving from the soft edge into the bulk results in a square root profile, and moving from the hard edge into the bulk shows a square root singularity.
Remark 4.3. In the above, we characterise the soft edge limiting forms of the eigenvalue densities of interest. However, our methods can be pushed further to investigate the leading order corrections, in which there is much interest (see e.g. [34], [61]). For example, letting $\mu(x)$ denote the first scaled density from the list above multiplied by $N^{-7/6}$, we know that appropriately changing variables in (2.38) yields a differential equation – call it (†) – satisfied by $\mu(x)$ with $\beta$ set to either 2/3 or 6, and that (†) reduces to (4.2) upon taking the limit $N \to \infty$. Due to linearity of (2.38), it is an equally simple observation that replacing $\mu(x)$ by $N^r \left( \mu(x) - \rho^{(soft)}_{(1),\beta,\infty}(x) \right)$ in (†) produces a differential equation which, in general, vanishes as $N \to \infty$ for $r < 1/3$, and reduces to a non-trivial differential equation (††) for $r = 1/3$. This shows that $\mu(x)$ is of the form

$$
\mu(x) = \rho^{(soft)}_{(1),\beta,\infty}(x) + \frac{1}{N^{1/3}} \mu_c(x) + \cdots
$$

where the leading order correction $\mu_c(x)$ is some function which satisfies (††). It turns out that in this particular case, there is an optimal scaling such that the leading order correction is $O(N^{-2/3})$ instead of $O(N^{-1/3})$. Indeed, with $\delta_G = (1 - 1/\kappa) / (2\sqrt{2N})$ [34], replacing $\mu(x)$ by $N^r \left( \mu(x) - \rho^{(soft)}_{(1),\beta,\infty}(x) \right)$ in (†) produces a differential equation which is non-trivial in the $N \to \infty$ limit when $r = 2/3$. Likewise, a similar phenomenon can be observed for the Laguerre ensemble in the regime of the largest eigenvalue when we set

$$
\delta_L^{(a=O(1))} = 2a/\kappa, \quad \delta_L^{(a=O(N))} = \left( 1 - \frac{1}{\kappa} \right) \frac{a_1/\kappa}{\sqrt{a_1/\kappa} + 1}.
$$

Though we’ve yet to pin down a precise procedure for using the differential equations of section 2 to determine the optimal scaling, we can use them to characterise the leading order correction once such a scaling is known.

With regards to the following theorem, it should be noted that investigations into the optimal scaling have been undertaken at the hard edge by [35] and references therein; using their prescribed scaling, we confirm via the following theorem that the leading order correction at the hard edge for the Laguerre ensemble is $O(N^{-2})$.

**Theorem 4.4.** Define the hard edge limiting forms of the differential operators introduced in section 2 as

$$
D_{\beta,\infty}^{(hard)} = \begin{cases} 
 x^3 \frac{d^3}{dx^3} + 4x^2 \frac{d^2}{dx^2} + \left[ x - a^2 + 2 \right] x \frac{d}{dx} + \frac{1}{2} x - a^2, & \beta = 2 \\
 4x^5 \frac{d^5}{dx^5} + 40x^4 \frac{d^4}{dx^4} + \left[ 10kx - 5\tilde{a} + 88 \right] x^3 \frac{d^3}{dx^3} + \left[ 38kx - 22\tilde{a} + 16 \right] x^2 \frac{d^2}{dx^2} \\
 + \left[ (2kx - \tilde{a})^2 + 12kx - 14\tilde{a} - 16 \right] x \frac{d}{dx} + (2kx - \tilde{a})(kx - \tilde{a}) - 4kx, & \beta = 1, 4,
\end{cases}
$$

(4.7)
where we retain the definition of $\tilde{a}$ given in Proposition 2.8. Then for $\beta \in \{1, 2, 4\}$, at leading order, the Laguerre and Jacobi eigenvalue densities satisfy the following differential equation in the hard edge limit:

$$
D_{\beta,\infty}^{(\text{hard})} \rho_{(1),\beta,\infty}^{(\text{hard})}(x) = 0. \quad (4.8)
$$

Proof. Since the Gaussian ensemble eigenvalue densities do not exhibit hard edges, we turn to the Laguerre ensemble as it is the next-simplest to work with. Thus, we begin by changing variables $x \mapsto \kappa x / (4N)$ in (2.29). Equating terms of order one yields the differential equation (4.8) above, and we note that all other terms are $O(1/N)$. □

Similar to the Theorem 4.1, one can check that the differential equations (2.19) and (2.24) for the Jacobi ensemble eigenvalue densities can be scaled to show that the large $N$ limits of the following scaled densities satisfy (4.8) at leading order in $N$:

- In the regime of the smallest eigenvalue with $a, b = O(1)$
  $$
  \rho_{(1),\beta,N}^{(J)} \left( \frac{x}{4N^2} \right);
  $$

- In the regime of the smallest eigenvalue with $b = a_2 N$ and $a, a_2 = O(1)$
  $$
  \rho_{(1),\beta,N}^{(J)} \left( \frac{x}{4(1+a_2/\kappa)N^2} \right);
  $$

- Before scaling, we may exploit the symmetry $(x, a, b) \leftrightarrow (1-x, b, a)$ for the Jacobi ensemble to easily characterise the hard edges of the Jacobi ensemble at the largest eigenvalue.

As a final comment, we note that it is known that at leading order in $x$, the soft and hard edge limiting densities have forms which have $\beta$ as a continuous parameter. This is in stark contrast to our characterisation, as our differential equations are valid for specific $\beta$ and are of order $\beta + 1$. It is still an open question as to whether it is possible to $\beta$-deform the differential equations in this paper so that they may characterise general-$\beta$ ensembles.

Appendix A.

To take the Stieltjes transforms of equations (2.19) and (2.23), we need to compute terms of the form

$$
\int_0^1 \frac{x^p (1-x)^q}{s-x} \frac{d^n}{dx^n} \rho_{(1),\beta,N}^{(J)}(x) \, dx
$$

for $\beta = 2$ or $4$, $0 \leq n \leq 5$, $n \leq q \leq n + 2$, and $q \leq p \leq q + 1$ all integers. To this end, we define

$$
\mathcal{I}_\beta(s; p, q, n, k) := \int_0^1 \frac{x^p (1-x)^q}{(s-x)^k} \frac{d^n}{dx^n} \rho_{(1),\beta,N}^{(J)}(x) \, dx \quad (A.1)
$$
for integers $0 \leq n \leq q \leq p$ and $k \geq 0$. Then, integration by parts gives the identity
\[
\mathcal{I}_\beta(s; p, q, n, k) = (p + q)\mathcal{I}_\beta(s; p, q - 1, n - 1, k) - p\mathcal{I}_\beta(s; p - 1, q - 1, n - 1, k)
- k\mathcal{I}_\beta(s; p, q - 1, n - 1, k + 1).
\] (A.2)

Applying this identity $n$ times allows us to reduce $\mathcal{I}_\beta(s; p, q, n, k)$ to an expression involving terms of the form $\mathcal{I}_\beta(s; p, q, 0, k)$. Then, considering $(s - x)^{-k-1} = (-1)^k/k! \partial_x^k(s - x)^{-1}$ for $k \geq 0$ gives us
\[
\mathcal{I}_\beta(s; p, q, 0, k + 1) = \frac{(-1)^k}{k!} \frac{d^k}{ds^k} \mathcal{I}_\beta(s; p, q, 0, 1), \quad k \geq 0,
\] (A.3)

which allows us to further reduce to an expression involving terms of the form $\mathcal{I}_\beta(s; p, q, 0, 1)$. Finally, factorisation of $x^p - s^p$ for positive integer $p$ yields
\[
\mathcal{I}_\beta(s; p, q, 0, 1) = s^p \mathcal{I}_\beta(s; 0, q, 0, 1) - \sum_{l=0}^{p-1} s^{p-l-1} \mathcal{I}_\beta(s; 1, q, 0, 0), \quad p \geq 1
\] (A.4)

and likewise
\[
\mathcal{I}_\beta(s; 0, q, 0, 1) = \sum_{m=0}^{q} \left( \binom{q}{m} \right) (-1)^m \left[ s^m W_{\beta, N}^{(l)}(s) - \sum_{l=0}^{m-1} s^{m-l-1} m_l^{(l)} \right], \quad q \geq 1,
\] (A.5)

where $m_l^{(l)}$ is the $l$th moment of $\rho_{(1)\beta,N}^{(l)}(x)$. These last two equations thus reduce our expression to one involving powers of $s$, derivatives of $W_{\beta, N}^{(l)}(s)$, and moments of $\rho_{(1)\beta,N}^{(l)}(x)$.

To begin, we list
\[
\mathcal{I}_\beta(s; 1, 1, 1, 1) = s(1-s) \frac{d}{ds} W_{\beta, N}^{(l)}(s) - N
\]
\[
\mathcal{I}_\beta(s; 2, 2, 1) = s^2(1-s)^2 \frac{d^2}{ds^2} W_{\beta, N}^{(l)}(s) - 2N(s-2) - 6m_1^{(l)}
\]
\[
\mathcal{I}_\beta(s; 3, 3, 1) = s^3(1-s)^3 \frac{d^3}{ds^3} W_{\beta, N}^{(l)}(s) - 6N(s^2 - 3s + 3) - 24m_1^{(l)}(s-3) - 60m_2^{(l)}
\]
\[
\mathcal{I}_\beta(s; 4, 4, 1) = s^4(1-s)^4 \frac{d^4}{ds^4} W_{\beta, N}^{(l)}(s) - 24N(s^3 - 4s^2 + 6s - 4)
- 120m_1^{(l)}(s^2 - 4s + 6) - 360m_2^{(l)}(s-4) - 840m_3^{(l)}
\]
\[
\mathcal{I}_\beta(s; 5, 5, 1) = s^5(1-s)^5 \frac{d^5}{ds^5} W_{\beta, N}^{(l)}(s) - 120N(s^4 - 5s^3 + 10s^2 - 10s + 5)
- 720m_1^{(l)}(s^3 - 5s^2 + 10s - 10) - 2520m_2^{(l)}(s^2 - 5s + 10)
- 6720m_3^{(l)}(s-5) - 15120m_4^{(l)}
\]
All of the necessary $I_\beta(s; p, q, n, k)$ can be obtained from these through variants of identity (A.4) and integration by parts. For example,

$$I_\beta(s; n + 1, n, n, 1) = sI_\beta(s; n, n, n, 1) - I_\beta(s; n, n, n, 0) = sI_\beta(s; n, n, n, 1) + (-1)^n + 1 \int_0^1 \frac{d^n}{dx^n} (x^n (1 - x)^n)p_{(1, \beta,N)}^{(f)}(x) \, dx,$$

which only requires knowledge of the moments of $p_{(1, \beta,N)}^{(f)}(x)$, in addition to a term from the above list. It should be noted that the Stieltjes transform takes $x^p (1 - x)^q \frac{d^n}{dx^n} p_{(1, \beta,N)}^{(f)}(x)$ to $s^p (1 - s)^q \frac{d^n}{ds^n} W_{(1, \beta,N)}^{(f)}(s)$ plus terms that do not involve $W_{(1, \beta,N)}^{(f)}(s)$. It follows that the differential equations for the resolvent will be the same as those for the density, with additional inhomogeneous terms.

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School of Mathematics and Statistics, ARC Centre of Excellence for Mathematical and Statistical Frontiers, University of Melbourne, Victoria 3010, Australia

E-mail address: anas.rahman@live.com.au

School of Mathematics and Statistics, ARC Centre of Excellence for Mathematical and Statistical Frontiers, University of Melbourne, Victoria 3010, Australia

E-mail address: pjforr@unimelb.edu.au