PERIODIC PERTURBATIONS OF A COMPOSITE WAVE OF TWO VISCOUS SHOCKS FOR 1-D FULL NAVIER-STOKES EQUATIONS

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Abstract. This paper is concerned with the asymptotic stability of a composite wave of two viscous shocks under spatially periodic perturbations for the 1-D full Navier-Stokes equations. It is proved that as time increases, the solution approaches the background composite wave with a shift for each shock, where the shifts can be uniquely determined if both the periodic perturbations and strengths of two shocks are small. The key of the proof is to construct a suitable ansatz such that the anti-derivative method works.

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1. INTRODUCTION

The one-dimensional full compressible Navier-Stokes equations in the Lagrangian coordinates reads

\[
\begin{aligned}
\partial_t v - \partial_x u &= 0, \\
\partial_t u + \partial_x p(v, \theta) &= \mu \partial_x \left( \frac{\partial_x u}{v} \right), \\
\partial_t E + \partial_x (p(v, \theta) u) &= \kappa \partial_x \left( \frac{\partial_x \theta}{v} \right) + \mu \partial_x \left( \frac{u^2}{v} \right),
\end{aligned}
\]

where \( v(x, t) > 0 \) is the specific volume, \( u(x, t) \in \mathbb{R} \) is the velocity, \( \theta(x, t) > 0 \) is the absolute temperature, the pressure \( p(v, \theta) \) satisfies

\[ p(v, \theta) = \frac{R \theta}{v}, \]

and the total energy is given by

\[ E = e + \frac{1}{2} u^2, \]

where the internal energy \( e \) is

\[ e = \frac{R}{\gamma - 1} \theta + \text{const.} \]

Qian Yuan is supported by the China Postdoctoral Science Foundation funded projects 2019M660831 and 2020TQ0345.

The research of Yuan Yuan is partially supported by the National Natural Science Foundation of China (Grants No. 11901208).
In this paper, we are concerned about a composite wave of the 1-viscous shock and 3-viscous shock under spatially periodic perturbations, i.e. we consider a Cauchy problem for (1.1) with the initial data
\[(v, u, E)(x, 0) = (v_0(x), u_0(x), E_0(x)), \quad x \in \mathbb{R}, \tag{1.5}\]
satisfying
\[(v_0, u_0, E_0)(x) \rightarrow \begin{cases} (\overline{v}_l, \overline{u}_l, \overline{E}_l) + (\phi_{0l}, \psi_{0l}, w_{0l})(x) & \text{as } x \rightarrow -\infty, \\ (\overline{v}_r, \overline{u}_r, \overline{E}_r) + (\phi_{0r}, \psi_{0r}, w_{0r})(x) & \text{as } x \rightarrow +\infty, \end{cases} \tag{1.6}\]
where \(\overline{v}_i > 0, \overline{u}_i, \overline{E}_i \geq 0\) are constants with \(\overline{E}_i = \frac{\gamma - 1}{\gamma} \left( \overline{E}_i \left( \frac{1}{2} \overline{u}_i^2 \right) \right) > 0\), and \((\phi_{0l}, \psi_{0l}, w_{0l})(x)\) are periodic functions with period \(\tau_i > 0\) for \(i = l, r\). The composite wave has an intermediate state \((\overline{v}_m, \overline{u}_m, \overline{E}_m)\) such that \((\overline{v}_i, \overline{u}_i, \overline{E}_i)\) connects \((\overline{v}_m, \overline{u}_m, \overline{E}_m)\) by a 1-shock with speed \(s_1 < 0\), and \((\overline{v}_m, \overline{u}_m, \overline{E}_m)\) connects \((\overline{v}_r, \overline{u}_r, \overline{E}_r)\) by a 3-shock with speed \(s_3 > 0\), where the constants satisfy the Rankine-Hugoniot conditions,
\[-s_i \left[ [v]_i \right] - [u]_i = 0, \\ -s_i \left[ [u]_i \right] + [p]_i = 0, \quad \text{for } i = 1, 3, \tag{1.7}\]
and the entropy conditions,
\[\lambda_1(\overline{v}_m, \overline{\theta}_m) < s_1 < \lambda_1(\overline{v}_r, \overline{\theta}_l), \tag{1.8}\]
\[\lambda_3(\overline{v}_r, \overline{\theta}_r) < s_3 < \lambda_3(\overline{v}_m, \overline{\theta}_m),\]
where we let \(\left[ \cdot \right]_1\) and \(\left[ \cdot \right]_3\) denote the jumps when crossing the 1-shock and 3-shock, respectively,
\[\left[ A \right]_1 = \overline{\lambda}_m - \overline{\lambda}_l \quad \text{and} \quad \left[ A \right]_3 = \overline{\lambda}_r - \overline{\lambda}_m, \tag{1.9}\]
and \(\lambda_1(v, \theta) = -\sqrt{2p(v, \theta)}\) and \(\lambda_3(v, \theta) = -\lambda_1(v, \theta)\) are the first and third eigenvalues of the system (1.1), respectively, when \(\mu = \kappa = 0\). Moreover, the periodic perturbations in (1.6) are assumed to be of zero average
\[\int_0^{\tau_i} (\phi_{0l}, \psi_{0l}, w_{0l})(x) dx = 0, \quad i = l, r. \tag{1.10}\]
It is noted that the zero average condition (1.10) is necessary for the stability of the background wave. Otherwise, by adding the constant averages of these periodic perturbations onto the constants \((\overline{v}_i, \overline{u}_i, \overline{E}_i)\) for \(i = l, r\), the new states may generate other kinds of Riemann solutions, which is not the topic of this paper.

Since the system (1.1), the R-H conditions (1.7), and the entropy conditions (1.8) are invariant under Galilean transform, without loss of generality, one can let \(u - \overline{u}_m\) substitute \(u\), and \(\overline{u}_l - \overline{u}_m, 0, \overline{\lambda}_r - \overline{\lambda}_m\) substitute \(\overline{u}_l, \overline{u}_m, \overline{\lambda}_r\), respectively, to assume that \(\overline{u}_m = 0. \tag{1.11}\)

For \(i = 1, 3\), the \(i\)-viscous shock wave \((v_i^s, u_i^s, E_i^s) (x - s_i t)\) is a traveling wave solution to (1.1), solving the corresponding ordinary system,
\[
\begin{cases}
-s_i \left( v_i^s \right)' - u_i^s = 0, \\
-s_i \left( u_i^s \right)' + (p \left( v_i^s, \theta_i^s \right))' = \mu \left( \frac{u_i^s}{v_i^s} \right)' \\
-s_i \left( E_i^s \right)' + (p \left( v_i^s, \theta_i^s \right) u_i^s)' = \kappa \left( \frac{u_i^s}{v_i^s} \right)' + \mu \left( \frac{u_i^s}{v_i^s} \right)' \tag{1.12}
\end{cases}
\]
with the end-behaviors
\[
\begin{align*}
(v^S_1, u^S_1, E^S_1)(-\infty) &= (\bar{v}_1, \bar{u}_1, \bar{E}_1), \\
(v^S_3, u^S_3, E^S_3)(-\infty) &= (\bar{v}_3, \bar{u}_3, \bar{E}_3), \\
(v^S_1, u^S_1, E^S_1)(+\infty) &= (\bar{v}_m, \bar{u}_m, \bar{E}_m), \\
(v^S_3, u^S_3, E^S_3)(+\infty) &= (\bar{v}_r, \bar{u}_r, \bar{E}_r),
\end{align*}
\]
where \( \theta^S_i = \frac{\gamma-1}{R} (E^S_i - \frac{1}{2} (u^S_i)^2) \).

There have been lots of literatures studying the stability of viscous shocks. The first result was proved by I’lin-Oleinik [6] for the one-dimensional scalar viscous convex conservation laws, by using the maximum principle for the anti-derivative variables. For the systems, Matsumura-Nishihara [10] and Goodman [1] independently applied the energy method to prove the stability of a single viscous shock provided that the initial perturbation has zero mass on \( \mathbb{R} \). And then by introducing diffusion waves propagating along other families of characteristics and establishing their point-wise estimates, Liu [8], Szepessy-Xin [11] and Liu-Zeng [9] removed the zero-mass condition of the perturbation around a single viscous shock. For the combination wave of two viscous shocks for the full Navier-Stokes equations, there is only one diffusion wave propagating along the linearly degenerate characteristic, Huang-Matsumura [2] successfully utilized the basic energy method to show the stability.

If the initial perturbation around a shock is a periodic function that is not integrable on \( \mathbb{R} \), Z.Xin and the authors showed the \( L^\infty(\mathbb{R}) \) stability of shocks and rarefaction waves for the convex conservation laws in both inviscid and viscous cases; see [12, 13, 14]. And then Huang-Yuan [3] extended the result to the case for a single shock with large amplitude for the isentropic Navier-Stokes equations, if the initial periodic perturbations satisfy a zero-mass condition. We also refer to [4, 3] for the planar rarefaction wave under multi-dimensional periodic perturbations.

In this paper, we prove the stability of the combination wave of two viscous shocks under periodic perturbations for the full Navier-Stokes equations, without the zero-mass condition. The key is to construct a suitable ansatz such that the anti-derivative method works. Different from the usual way in [10, 2, 9] that the ansatz is a composite wave with the a constant shift for each of two shock waves and probably a linear diffusion wave, in this paper we choose appropriate shift function (of time) for each conservative variable \( v, u \) and \( E \), and assign a constant difference between the locations of 1-shock and 3-shock to deal with the periodic perturbations at \( |x| = +\infty \); see similar arguments in [4]. The shift curves are derived from each equation of the Navier-Stokes system, to fulfill the constraints from zero masses (2.26). By assuming the difference between the locations of 1-shock and 3-shock to be a constant, and choosing three appropriate shift curves derived from each equation of the Navier-Stokes system, we finally find six constraints satisfied by six independent variables (see (2.30)), which can be uniquely determined by the implicit theorem if the periodic perturbations and the strength of shocks are both small enough. And then the ansatz can be well-defined in terms of the so determined constants and shift curves.

The rest of the paper is organized as follows. In Section 2, we will introduce some notations and some useful lemmas, then show the construction of the ansatz and the main result. In Section 3, we will introduce the anti-derivative variables of the perturbations and their error terms, and then we will give the reformulated problem and prove the a priori estimates, and thus finish the prove of the main result. The proves of two lemmas about the shift curves and the error terms of the ansatz are supplemented in the last section.
2. ANSATZ AND MAIN RESULTS

Denote
\[
\delta_1 = |v_t - \varpi_m| + |u_t| + |\theta_t - \vartheta_m|, \quad \delta_3 = |v_r - \varpi_m| + |u_r| + |\theta_r - \vartheta_m|, \quad \delta := \min\{\delta_1, \delta_3\}.
\]

(2.1)

Similar to [2], we also assume that
\[
0 < \delta_1 + \delta_3 \leq C\delta \quad \text{as} \quad \delta_1 + \delta_3 \to 0.
\]

(2.2)

It means that the approach of this paper cannot be applied to the case for one single shock where either \(\delta_1\) or \(\delta_3\) is zero.

**Lemma 2.1** ([7, 2]). Under the conditions (1.7) and (1.8), assume that \(\gamma \in (1, 2]\) and (2.2) holds with \(\delta > 0\) small enough. Then (1.12) admits a viscous shock wave \((v^i, u^i, E^i)(x - s, t)\) for \(i = 1, 3\). Moreover, there exist two constant \(c_0 > 0\) and \(C > 0\), independent of \(t\) or \(\delta\), such that
\[
\begin{align*}
\left| (v^i(x) - \varpi_m, u^i(x), \theta^i(x) - \vartheta_m) \right| &\leq C\delta_1 e^{-c_0\delta_1|x|} \quad \text{for} \quad x > 0, \\
\left| (v^i(x) - \varpi_m, u^i(x), \theta^i(x) - \vartheta_m) \right| &\leq C\delta_3 e^{-c_0\delta_3|x|} \quad \text{for} \quad x < 0, \\
\hat{c} \cdot u^i(x) &< 0, \quad \left| \hat{c} \cdot (v^i, u^i, \theta^i) \right| \leq C\delta_1 e^{-c_0\delta_1|x|} \quad \text{for} \quad x \in \mathbb{R}, \quad i = 1, 3.
\end{align*}
\]

It follows from (2.2) and Lemma 2.1 that
\[
\begin{align*}
|A^i(x - s t) - \mathcal{A}_m| \cdot |B^i (x - s t) - \mathcal{B}_m| \leq C\delta^2 e^{-c_1|t - s_3|x|}, \quad x \in \mathbb{R}, \quad t > 0,
\end{align*}
\]

(2.3)

where \(A\) and \(B\) represent either \(v, u, E\) or \(\theta\) and \(c_1 = c_0 \min\{|s_1|, s_3|\}\).

Then we give some properties of periodic solutions to (1.1) in the following lemma.

**Lemma 2.2.** Assume that \((v_0, u_0, E_0)(x) \in H^k(0, \pi)\) with \(k \geq 2\) is periodic with period \(\pi > 0\) and average \((\bar{v}, \bar{u}, \bar{E})\). Then there exists \(\varepsilon_0 > 0\) such that if
\[
\varepsilon = \left\| (v_0, u_0, E_0) - (\bar{v}, \bar{u}, \bar{E}) \right\|_{H^k(0, \pi)} \leq \varepsilon_0,
\]

(2.4)

then there exists a unique periodic solution
\[
(v, u, E)(x, t) \in C(0, +\infty; H^k(0, \pi))
\]

to (1.1) with the initial data \((v_0, u_0, E_0)\). Moreover, it holds that
\[
\left\| (v, u, E) - (\bar{v}, \bar{u}, \bar{E}) \right\|_{H^k(0, \pi)}(t) \leq C\varepsilon e^{-2\alpha t}, \quad t \geq 0,
\]

(2.5)

where the constants \(C > 0\) and \(\alpha > 0\) are independent of \(\varepsilon\) or \(t\).

Hence, for the initial data satisfying (1.6), we let \((v_i, u_i, E_i)(x, t)\) \((i = l, r)\) denote the unique periodic solutions to (1.1) with the periodic initial data
\[
(v_i, u_i, E_i)(x, 0) = (\bar{v}_i, \bar{u}_i, \bar{E}_i) + (\phi_0i, \psi_0i, w_0i)(x),
\]

(2.6)

and define the perturbations as
\[
(\phi, \psi, w)(x, t) := (v_i, u_i, E_i)(x, t) - (\bar{v}_i, \bar{u}_i, \bar{E}_i), \quad \zeta_i(x, t) := \theta_i(x, t) - \bar{\theta}_i,
\]

(2.7)

where the temperature \(\theta_i(x, t) = \frac{\gamma - 1}{\gamma} \left( E_i - \frac{1}{2} u_i^2 \right)(x, t)\) has the periodic initial data
\[
\zeta_0i(x) := \frac{\gamma - 1}{\gamma} \left( w_{0i} - \frac{1}{2} \psi_{0i}^2 - \bar{u}_i \psi_{0i} \right)(x).
\]

(2.8)
Then due to the conservative form of (1.1), one has that the periodic functions $(\phi_i, \psi_i, w_i)(x,t)$ for $i = l, r$ have zero averages for any $t \geq 0$.

For the viscous shocks $(v_i^S, u_i^S, E_i^S)(x-s_it)$ for $i = 1, 3$, let

$$
g_1(x) := \frac{v_1^S(x) - \bar{v}_l}{\|v\|_1}, \quad h_1(x) := \frac{E_1^S(x) - \bar{E}_l}{\|E\|_1},
$$

$$
g_3(x) := \frac{v_3^S(x) - \bar{v}_m}{\|v\|_3}, \quad h_3(x) := \frac{E_3^S(x) - \bar{E}_m}{\|E\|_3}.
$$

Then it follows from the R-H conditions (1.7) and the ordinary differential systems (1.12) that

$$
g_1(x) = \frac{u_1^S(x) - \bar{u}_l}{-\bar{u}_l} \quad \text{and} \quad g_3(x) = \frac{u_3^S(x)}{\bar{u}_r},
$$

where we used the assumption that $\bar{u}_m = 0$. Then it holds that $0 < g_1(x), h_i(x) < 1$ and $g_i'(x), h_i'(x) > 0$ for all $x \in \mathbb{R}$.

For convenience, we define two shift operators $\tau^1$ and $\tau^3$ as

$$
\tau_i^k(A)(x,t) := A(x - s_i(t - b(t))), \quad i = 1, 3,
$$

where $s_i$ is the speed of $i$-shock, and $A = A(x)$ and $b = b(t)$ are any measurable functions.

**Ansatz.** Now we are ready to construct the ansatz. Let $X(t), Y(t), Z(t)$ be three $C^1$ curves on $[0, +\infty)$ and $\sigma \in \mathbb{R}$ be a constant, all of which will be determined later.

Set

$$
v^t(x,t) := v_l(t)\left[1 - \tau_1^1(v_1)(x,t)\right] + v_r(t)\tau_1^{3,\sigma}(g_1)(x,t),
$$

$$
u^t(x,t) := u_l(t)\left[1 - \tau_1^3(h_1)(x,t)\right] + u_r(t)\tau_1^{3,\sigma}(h_3)(x,t),
$$

$$
E^t(x,t) := E_l(t)\left[1 - \tau_1^3(h_1)(x,t)\right] + E_r(t)\tau_1^{3,\sigma}(h_3)(x,t),
$$

and set

$$
\theta^t(x,t) := -\gamma + \frac{1}{R}\left[\|\theta^t\|_2 - \frac{1}{2}(\theta^t)^2\right](x,t).
$$

We aim to find appropriate curves $X(t), Y(t), Z(t)$ and number $\sigma$ to make the anti-derivative variable method available, even though the initial perturbations in (1.6) are not integrable on $\mathbb{R}$. By plugging the ansatz $(v^t, u^t, E^t)$ into (1.1) with direct calculations, one can obtain that

$$
\begin{align*}
\partial_t v^t - \partial_x u^t &= \partial_x F_{1,1} + f_{1,2} + X'f_{1,3}, \\
\partial_t u^t + \partial_x p(v^t, \theta^t) - \mu \partial_x \left(\frac{\partial_x^2 u^t}{v^t}\right) &= \partial_x F_{2,1} + f_{2,2} + Y'f_{2,3}, \\
\partial_t E^t + \partial_x \left(p(v^t, \theta^t) u^t\right) - \kappa \left(\frac{\partial_x \theta^t}{v^t}\right) &= \partial_x F_{3,1} + f_{3,2} + Z'f_{3,3},
\end{align*}
$$

where

$$
\begin{align*}
F_{1,1} &= u_l\left[\tau_1^3(g_1) - \tau_1^1(g_1)\right] - u_r\left[\tau_1^{3,\sigma}(g_3) - \tau_1^{3,\sigma}(g_3)\right], \\
f_{1,2} &= [s_l(v_l - \bar{v}_m) + u_l]\tau_1^1(g_1) - [s_3(v_r - \bar{v}_m) + u_r]\tau_1^{3,\sigma}(g_3), \\
f_{1,3} &= (v_l - \bar{v}_m)\tau_1^1(g_1) - (v_r - \bar{v}_m)\tau_1^{3,\sigma}(g_3),
\end{align*}
$$

(2.12)
\[
\begin{align}
F_{2,1} &= p(v, \theta) - p(v_1, \theta_1) \left[1 - \tau_1^3(g_1)\right] - p(v_r, \theta_r) \tau_3^3 + \sigma(g_3), \\
F_{2,2} &= \left\{\begin{array}{l}
[\alpha v - \alpha y (1 - \tau_3^3(g_1))] - \frac{\alpha v - \alpha y}{v_y} \tau_3^3 + \sigma(g_3), \\
[-s_1 u - p(v, \theta)](g_1) \tau_3^3, \\
[s_1 u - p(v_r, \theta_r)](g_3), \\
[u, \tau_3^3(g_1)] - \tau_3^3 + \sigma(g_3),
\end{array}\right.
\end{align}
\]
\[
(2.13)
\]
\[
F_{3,1} = p(v, \theta) u - p(v_1, \theta_1) u_1 \left[1 - \tau_1^3(h_1)\right] - p(v_r, \theta_r) u_r \tau_3^3 + \sigma(h_3), \\
F_{3,2} = \left\{\begin{array}{l}
[-s_1 (E_l - E_m) - p(v, \theta_1) u_1 + \frac{\alpha v - \alpha y}{v_y} \tau_3^3 + \sigma(h_3)], \\
[-s_3 (E_r - E_m) - p(v_r, \theta_r) u_r + \frac{\alpha v - \alpha y}{v_y} \tau_3^3 + \sigma(h_3)],
\end{array}\right.
\]
\[
(2.14)
\]
\[
F_{3,3} = (E_l - E_m) \tau_3^3 + \sigma(h_3) - (E_r - E_m) \tau_3^3 + \sigma(h_3).
\]
\[
(2.15)
\]
It is noted that since \((v, v^*, E^*, \theta^*)\) tends to \((v_1, v^*, E_1, \theta_1)\) and \((v_r, v^*, E_r, \theta_r)\) as \(x \to \pm \infty\), respectively, one can verify easily that \(F_{1, i}(x, t) (i = 1, 2, 3)\) vanishes as \(|x| \to \infty\) for each \(t \geq 0\). To make the system (2.11) as a conservative form, the curves \(X, Y\) and \(Z\) should satisfy
\[
\begin{align}
X'(t) &= -\frac{\int_R f_1(2, x, t) dx}{\int_R f_1(3, x, t) dx}, \\
Y'(t) &= -\frac{\int_R f_2(2, x, t) dx}{\int_R f_2(3, x, t) dx}, \\
Z'(t) &= -\frac{\int_R f_3(2, x, t) dx}{\int_R f_3(3, x, t) dx},
\end{align}
\]
where the denominators in (2.15) are away from zero if the periodic perturbations \((\phi_0, \psi_0, \zeta_0)\) \((i = l, r)\) are small enough (see Lemma 2.2). The curves \(X, Y\) and \(Z\) can be uniquely determined as long as the corresponding initial data \(X_0, Y_0\) and \(Z_0\) are given. And before locating these initial points, we first give the following lemma.

**Lemma 2.3.** Assume that the R-H conditions (1.7) hold and the periodic perturbations satisfy (1.10) and
\[
\varepsilon := \|\phi_0, \psi_0, w_0\|_{H^3(0, \pi_1)} + \|\phi_0, \psi_0, w_0\|_{H^3(0, \pi_r)} \leq \varepsilon_0,
\]
where \(\varepsilon_0 > 0\) is small enough. Then given any constant triple \((X_0, Y_0, Z_0)\), there exists a unique solution \((X, Y, Z) (t) \in C^1[0, +\infty)\) to the system (2.15) with the initial data \((X, Y, Z) (0) = (X_0, Y_0, Z_0)\). Moreover, the solution has the large time behaviors
\[
|(X', Y', Z') (t)| + |(X, Y, Z) (t) - (X_\infty, Y_\infty, Z_\infty)| \leq C\varepsilon e^{-2\alpha t}, \quad t \geq 0,
\]
where \(\alpha > 0\) is the constant given in Lemma 2.2, the constant \(C > 0\) is independent of \(\varepsilon, t\), and the corresponding final locations \(X_\infty, Y_\infty, Z_\infty\) can be computed (in terms of the initial data (1.6) and the periodic solutions (2.6)) as follows,
\[
\begin{align}
X_\infty &= X_0 + \frac{1}{\varpi_r - \varpi_l} \left\{ \int_{-\infty}^{0} [\phi_0 (x) g_1 (x - X_0) - \phi_0 (x) g_3 (x - X_0 - \sigma)] dx \\
&\quad - \int_{0}^{+\infty} [\phi_0 (x) (1 - g_1 (x - X_0)) - \phi_0 (x) (1 - g_3 (x - X_0 - \sigma))] dx \right\} \\
&\quad + \frac{1}{\varpi_l} \int_{0}^{\pi_l} \phi_0 (y) dy dx - \frac{1}{\varpi_r} \int_{0}^{\pi_r} \phi_0 (y) dy dx \\
&:= H_1 (X_0, \sigma),
\end{align}
\]
\[
(2.16)
\]
\[ \mathcal{Y}_x = \mathcal{Y}_0 + \frac{1}{\pi r - \pi l} \left\{ \int_{-\infty}^{0} [\psi_{0l}(x)g_1(x - \mathcal{Y}_0) - \psi_{0r}(x)g_3(x - \mathcal{Y}_0 - \sigma)] \, dx ight. \\
\left. \quad - \int_{0}^{+\infty} [\psi_{0l}(x)(1 - g_1(x - \mathcal{Y}_0)) - \psi_{0r}(x)(1 - g_3(x - \mathcal{Y}_0 - \sigma))] \, dx \\
\right. \\
\left. + \frac{1}{\pi l} \int_{0}^{\pi_l} \int_{0}^{x} \psi_{0l}(y) \, dy \, dx - \int_{0}^{+\infty} \frac{1}{\pi l} \int_{0}^{\pi_l} [p(v_l, \theta_l) - p(\overline{v}_l, \overline{\theta}_l)] \, dx \, dt \\
\right. \\
\left. \quad - \frac{1}{\pi r} \int_{0}^{\pi_r} \int_{0}^{x} \psi_{0r}(y) \, dy \, dx + \int_{0}^{+\infty} \frac{1}{\pi r} \int_{0}^{\pi_r} [p(v_r, \theta_r) - p(\overline{v}_r, \overline{\theta}_r)] \, dx \, dt \\
\right. \\
\left. + \frac{\mu}{\pi l} \int_{0}^{\pi_l} \log(\overline{v}_l + \phi_{0l}(x)) \, dx - \frac{\mu}{\pi r} \int_{0}^{\pi_r} \log(\overline{v}_r + \phi_{0r}(x)) \, dx \right\} \\
:= H_2(\mathcal{Y}_0, \sigma), \]

\[ \mathcal{Z}_x = \mathcal{Z}_0 + \frac{1}{E_r - E_l} \left\{ \int_{-\infty}^{0} [w_{0l}(x)h_1(x - \mathcal{Z}_0) - w_{0r}(x)h_3(x - \mathcal{Z}_0 - \sigma)] \, dx ight. \\
\left. \quad - \int_{0}^{+\infty} [w_{0l}(x)(1 - h_1(x - \mathcal{Z}_0)) - w_{0r}(x)(1 - h_3(x - \mathcal{Z}_0 - \sigma))] \, dx \\
\right. \\
\left. + \frac{1}{\pi l} \int_{0}^{\pi_l} \int_{0}^{x} w_{0l}(y) \, dy \, dx - \frac{1}{\pi r} \int_{0}^{\pi_r} \int_{0}^{x} w_{0r}(y) \, dy \, dx \\
\right. \\
\left. + \int_{0}^{+\infty} \frac{1}{\pi l} \int_{0}^{\pi_l} \left[ \frac{\partial_x \theta_l}{v_l} + \frac{u_l \partial_x u_l}{v_l} - \left( p(v_l, \theta_l)u_l - p(\overline{v}_l, \overline{\theta}_l)\overline{u}_l \right) \right] \, dx \, dt \\
\right. \\
\left. - \int_{0}^{+\infty} \frac{1}{\pi r} \int_{0}^{\pi_r} \left[ \frac{\partial_x \theta_r}{v_r} + \frac{u_r \partial_x u_r}{v_r} - \left( p(v_r, \theta_r)u_r - p(\overline{v}_r, \overline{\theta}_r)\overline{u}_r \right) \right] \, dx \, dt \right\}, \]

\[ := H_3(\mathcal{Z}_0, \sigma). \]

Note that due to Lemma 2.2, all the integrals in Lemma 2.3 are bounded and thus \( X_x, \mathcal{Y}_x \) and \( \mathcal{Z}_x \) are well-defined. Since the proof of Lemma 2.3 is similar to that in [13], we place it in the last section, Section 4.

Define the composite wave of 1- viscous shock and 3- viscous shock with the corresponding shifts \( b = b(t) \) and \( d = d(t) \) as

\[ u_{l,d}^S := \tau_b^1 (v_l^S) - v_m + \tau_d^3 (v_d^S), \]

\[ u_{l,d}^S := \tau_b^1 (u_l^S) + \tau_d^3 (u_d^S), \]

\[ E_{l,d}^S := \tau_b^1 (E_l^S) - E_m + \tau_d^3 (E_d^S), \]

\[ \theta_{l,d}^S := \frac{\gamma - 1}{R} \left[ E_{l,d}^S - \frac{1}{2} (u_{l,d}^S)^2 \right] \]

\[ = \tau_b^1 (\theta_l^S) - \theta_m + \tau_d^3 (\theta_d^S) - \frac{\gamma - 1}{R} \tau_b^1 (u_l^S) \tau_d^3 (u_d^S). \]

For convenience, we omit the lower indices of the variables above as \( v^S, u^S, E^S \) and \( \theta^S \) when \( b = d = 0 \). Moreover, we denote \( A \sim B \) when

\[ \| A - B \|_{L^\infty(R)} \leq C e^{-\alpha t} + C \delta^3 e^{-\epsilon_1 \delta t}, \quad t > 0, \]

(2.20)
holds, where \( C > 0 \) is independent of either \( \varepsilon, \delta \) or \( t \). And for later use, as in [2], we denote \( A \approx B \) when they satisfy the pointwise estimates

\[
|A - B| \leq C \left( \delta^2 + |\eta| \delta^2 \right) e^{-c\delta t - c|\eta|} + C \frac{|\eta|}{(1 + t)^2} e^{-c|\eta|} + C(\delta + |\eta|) e^{-c\delta t - c|\eta|} \tag{2.21}
\]

holds for some \( c > 0 \) and \( C > 0 \).

Then by Lemmas 2.2 and 2.3 and (2.3), the ansatz (2.9) satisfies that

\[
\begin{align*}
v^4(x, t) &\sim v_1^4(x - s_1 t - X_\infty) + v_3^4(x - s_3 t - X_\infty - \sigma) - \theta_m = v_{(X, X_\infty + \sigma)}(x, t), \\
u^4(x, t) &\sim u_1^4(x - s_1 t - Y_\infty) + u_3^4(x - s_3 t - Y_\infty - \sigma) = u_{(Y, Y_\infty + \sigma)}(x, t), \\
E^4(x, t) &\sim E_1^4(x - s_1 t - Z_\infty) + E_3^4(x - s_3 t - Z_\infty - \sigma) - \bar{E}_m = E_{(Z, Z_\infty + \sigma)}(x, t),
\end{align*}
\]

Constraints from coinciding limits. From [2], it is plausible to require \( X_\infty = Y_\infty = Z_\infty \), denoted by \( \xi \), otherwise, either

\[
(v_1^4(x - s_1 t - X_\infty), u_1^4(x - s_1 t - Y_\infty), E_1^4(x - s_1 t - Z_\infty))
\]

or

\[
(v_3^4(x - s_3 t - X_\infty - \sigma), u_3^4(x - s_3 t - Y_\infty - \sigma), E_3^4(x - s_3 t - Z_\infty - \sigma))
\]

is not a traveling wave solution to (1.1). Thus, from Lemma 2.3 we give the first constrain on the five free variables \( \xi, \sigma, X_0, Y_0 \) and \( Z_0 \) as

\[
\xi = H_1(X_0, \sigma) = H_2(Y_0, \sigma) = H_3(Z_0, \sigma). \tag{2.22}
\]

Under the condition (2.22), it then follows from Lemmas 2.1 and 2.3 that for all \( t \geq 0 \),

\[
\begin{align*}
v^4 &\sim v_{(X, X_\infty + \sigma)} \sim v_{(Y, Y_\infty + \sigma)} \sim v_{(Z, Z_\infty + \sigma)} \sim v_{(\xi, \xi + \sigma)}; \\
u^4 &\sim u_{(X, X_\infty + \sigma)} \sim u_{(Y, Y_\infty + \sigma)} \sim u_{(Z, Z_\infty + \sigma)} \sim u_{(\xi, \xi + \sigma)}; \\
E^4 &\sim E_{(X, X_\infty + \sigma)} \sim E_{(Y, Y_\infty + \sigma)} \sim E_{(Z, Z_\infty + \sigma)} \sim E_{(\xi, \xi + \sigma)}; \\
\theta^4 &\sim \theta_{(X, X_\infty + \sigma)} \sim \theta_{(Y, Y_\infty + \sigma)} \sim \theta_{(Z, Z_\infty + \sigma)} \sim \theta_{(\xi, \xi + \sigma)}.
\end{align*}
\tag{2.23}
\]

Similar to [8, 2], a diffusion wave propagating along the second family of characteristics \( r_2 = \left( 1, 0, \frac{p_m}{R} \right)^T \) should be considered in the case for general perturbations without the zero-mass condition. Set

\[
\begin{align*}
\tilde{v} :&= v^4 + \Theta, \quad \tilde{u} : = u^4 + a \partial_x \Theta, \quad \tilde{E} : = E^4 + \frac{\bar{P}_m}{\gamma - 1} \Theta, \\
\tilde{\Theta} :&= \frac{\gamma - 1}{R} \left( \tilde{E} - \frac{1}{2} \tilde{u}^2 \right), \\
\tilde{p} :&= \frac{\tilde{R} \tilde{\Theta}}{\tilde{v}} = p^4 - \frac{\Theta}{\tilde{v}} \left( p^4 - \bar{P}_m \right) - \frac{a(\gamma - 1)}{\tilde{v}} \left( \frac{a \left( \partial_x \Theta \right)^2}{2} + u^4 \partial_x \Theta \right), \tag{2.24}
\end{align*}
\]

where \( \Theta(x, t) = \frac{\eta}{\sqrt{4\pi(1 + t)}} e^{-\frac{x^2}{4\pi(1 + t)}} \) is a smooth diffusion wave for \( t \geq 0 \), satisfying

\[
\partial_t \Theta = a \partial_x^2 \Theta \quad \text{with} \quad a = \frac{(\gamma - 1)\kappa}{\gamma R \bar{P}_m} > 0, \quad \int \Theta(x, t) dx = \eta, \tag{2.25}
\]

where \( \eta \in \mathbb{R} \) is a constant to be determined.
Constraints from zero masses. From the equations (1.1), Lemma 3.1 and (2.25), one can get that the perturbations \( v - \tilde{v}, u - \tilde{u} \) and \( E - \tilde{E} \) carry zero masses for all \( t \geq 0 \), as long as the initial data do, i.e.
\[
\int_R (v - \tilde{v}) (x, 0) dx = 0, \quad \int_R (u - \tilde{u}) (x, 0) dx = 0, \quad \int_R (E - \tilde{E}) (x, 0) dx = 0 \quad (2.26)
\]
By direct calculations, the first identity in (2.26) yields that
\[
\eta = \int_R (v_0(x) - v^S(x, 0)) \, dx
= \int_R [v_0(x) - v_1^S (x - \xi_0) - v_3^S (x - \xi_0 - \sigma)] + \bar{v}_m
\quad - \phi_0(x)(1 - g_1(x - \xi_0)) - \phi_0(x)g_3(x - \xi_0 - \sigma) \, dx
= \int_R [v_1^S (x) - v_1^S (x - \xi_0)] \, dx + \int_R [v_3^S (x) - v_3^S (x - \xi_0 - \sigma)] \, dx
\quad + \int_{-\infty}^{0} (v_0 - v^S - \phi_0) (x) dx + \int_{0}^{\infty} (v_0 - v^S - \phi_0) (x) dx
\quad + \int_{-\infty}^{0} \phi_0(x)g_1 (x - \xi_0) - \phi_0(x)g_3 (x - \xi_0 - \sigma) \, dx
\quad - \int_{0}^{\infty} [\phi_0(x)(1 - g_1 (x - \xi_0)) - \phi_0(x) (1 - g_3 (x - \xi_0 - \sigma))] \, dx.
\]
By Lemma 2.3, one has that
\[
\eta = (\bar{v}_m - \bar{v}_l) \xi_0 + (\bar{v}_r - \bar{v}_m) (\xi_0 + \sigma) + \int_{-\infty}^{0} (v_0 - v^S - \phi_0) (x) dx
\quad + \int_{0}^{\infty} (v_0 - v^S - \phi_0) (x) dx + (\bar{v}_r - \bar{v}_l) (\xi - \xi_0)
\quad - \frac{1}{\pi_l} \int_{0}^{\pi_l} \int_{0}^{x} \phi_0(y) dy dx + \frac{1}{\pi_r} \int_{0}^{\pi_r} \int_{0}^{x} \phi_0(y) dy dx
= (\bar{v}_r - \bar{v}_l) \xi + (\bar{v}_r - \bar{v}_m) \sigma + C_1, \quad (2.27)
\]
where the constant \( C_1 \) denotes the sum of the remaining four integrals, which is independent of the six variables \( \xi_0, \theta_0, Z_0, \sigma, \xi \) or \( \eta \). By similar calculations, one can use Lemma 2.3 to get that
\[
0 = \int_R (u_0(x) - u^S(x, 0)) \, dx = (\bar{u}_r - \bar{u}_l) \xi + (\bar{u}_r - \bar{u}_m) \sigma + C_2, \quad (2.28)
\]
\[
\frac{\bar{p}_m}{\gamma - 1} \eta = \int_R (E_0(x) - E^S(x, 0)) \, dx = (\bar{E}_r - \bar{E}_l) \xi + (\bar{E}_r - \bar{E}_m) \sigma + C_3, \quad (2.29)
\]
where
\[
C_2 = \int_{-\infty}^{0} (u_0 - u^S - \psi_0 (x)) dx + \int_{0}^{\infty} (u_0 - u^S - \psi_0 (x)) dx
\quad - \frac{1}{\pi_r} \int_{0}^{\pi_r} \int_{0}^{x} \psi_0(y) dy dx + \int_{0}^{\infty} \frac{1}{\pi_l} \int_{0}^{\pi_l} [p(v_l, \theta_l) - p(\bar{v}_l, \bar{\theta}_l)] dx dt
\quad + \frac{1}{\pi_r} \int_{0}^{\pi_r} \int_{0}^{x} \psi_0(y) dy dx - \int_{0}^{\infty} \frac{1}{\pi_r} \int_{0}^{\pi_r} [p(v_r, \theta_r) - p(\bar{v}_r, \bar{\theta}_r)] dx dt
\]
\[
- \frac{\mu}{\pi_l} \int_{0}^{\pi_l} \log (\pi_l + \phi_{0}(x)) \, dx + \frac{\mu}{\pi_r} \int_{0}^{\pi_r} \log (\pi_r + \phi_{0r}(x)) \, dx,
\]
\[
C_3 = \int_{-\infty}^{0} (E_0 - E^S - w_{0l})(x) \, dx + \int_{0}^{+\infty} (E_0 - E^S - w_{0r})(x) \, dx
\]
\[
- \frac{1}{\pi_l} \int_{0}^{\pi_l} \int_{0}^{x} w_{0l}(y) \, dy \, dx + \frac{1}{\pi_r} \int_{0}^{\pi_r} \int_{0}^{x} w_{0r}(y) \, dy \, dx
\]
\[
- \int_{-\infty}^{+\infty} \frac{1}{\pi_l} \int_{0}^{\pi_l} \left[ \frac{\kappa}{v_l} \frac{\partial x}{v_l} + \mu \frac{u_l \partial x}{u_l} - \left( p(v_l, \theta_l) u_l - p(\pi_l, \theta_l) \pi_l \right) \right] (x, t) \, dx \, dt
\]
\[
+ \int_{0}^{+\infty} \frac{1}{\pi_r} \int_{0}^{\pi_r} \left[ \frac{\kappa}{v_r} \frac{\partial x}{v_r} + \mu \frac{u_r \partial x}{u_r} - \left( p(v_r, \theta_r) u_r - p(\pi_r, \theta_r) \pi_r \right) \right] (x, t) \, dx \, dt,
\]
both of which are independent of \(X_0, Y_0, Z_0, \sigma, \xi\) or \(\eta\).

Collecting (2.22) and (2.27) to (2.29), the six free variables \(X_0, Y_0, Z_0, \sigma, \xi\) and \(\eta\) should satisfy the following six identities,
\[
\begin{align*}
\xi - H_1(X_0, \sigma) &= 0, \\
\xi - H_2(Y_0, \sigma) &= 0, \\
\xi - H_3(Z_0, \sigma) &= 0, \\
(\pi_r - \pi_l) \xi + (\pi_r - \pi_m) \sigma - \eta + C_1 &= 0, \\
(\pi_r - \pi_l) \xi + (\pi_r - \pi_m) \sigma + C_2 &= 0, \\
(\pi_r - \pi_l) \xi + (\pi_r - \pi_m) \sigma - \frac{\rho_m}{\gamma - 1} \eta + C_3 &= 0.
\end{align*}
\]
(2.30)

The Jacobian of the system (2.30) is given by
\[
\det \begin{bmatrix}
\partial_{X_0} H_1 & 0 & 0 & \partial_{\sigma} H_1 & -1 & 0 \\
0 & \partial_{Y_0} H_2 & 0 & \partial_{\sigma} H_2 & -1 & 0 \\
0 & 0 & \partial_{Z_0} H_3 & \partial_{\sigma} H_3 & 0 & 0 \\
0 & 0 & 0 & \pi_r - \pi_m & \pi_r - \pi_l & -1 \\
0 & 0 & 0 & \pi_r - \pi_m & \pi_r - \pi_l & 0 \\
0 & 0 & 0 & \pi_r - \pi_m & \pi_r - \pi_l & \frac{\rho_m}{\gamma - 1}
\end{bmatrix}
\]
\[
= \partial_{X_0} H_1 \cdot \partial_{Y_0} H_2 \cdot \partial_{Z_0} H_3 \cdot \det \begin{bmatrix}
\left[ v \right]_3 & \left[ v \right]_1 + \left[ v \right]_3 & -1 \\
\left[ u \right]_3 & \left[ u \right]_1 + \left[ u \right]_3 & 0 \\
\left[ E \right]_3 & \left[ E \right]_1 + \left[ E \right]_3 & \frac{\rho_m}{\gamma - 1}
\end{bmatrix}
\]
(2.31)

Recall that \(r_2 = (1, 0, \frac{\rho_m}{\gamma - 1})^T\) is a 2-eigenvector. And for weak \(i\)-shock, the vector \((\left[ v \right]_i, \left[ u \right]_i, \left[ E \right]_i)^T\) is close to be parallel to the \(i\)-eigenvector \(r_i\) for \(i = 1, 3\). Thus, when the strengths of the waves \(\delta_1\) and \(\delta_3\) are both small enough, the determinant of the matrix in (2.31) is nonzero. On the other hand, it holds that
\[
\partial_{X_0} H_1 = 1 - \frac{1}{\pi_r - \pi_l} \int_{R} \phi_{0}(x) g_1(x - X_0) \, dx,
\]
\[
\partial_{Y_0} H_2 = 1 - \frac{1}{\pi_r - \pi_l} \int_{R} \psi_{0}(x) g_1(x - Y_0) \, dx,
\]
\[
\partial_{Z_0} H_3 = 1 - \frac{1}{\pi_r - \pi_l} \int_{R} \psi_{0}(x) g_1(x - Z_0) \, dx,
\]
all of which are away from zero, provided that the perturbations are small enough. Therefore, we have the following lemma.
Lemma 2.4. If the strengths of the shock waves, $\delta_1$ and $\delta_3$, and the amplitudes of the perturbations, $\sum_{i=l,r} \|(\phi_{0i}, \psi_{0i}, w_{0i})\|_{L^\infty(\mathbb{R})}$ are small enough, then the system (2.30) admits a unique solution $(X_0, Y_0, Z_0, \sigma, \xi, \eta) \in \mathbb{R}^6$.

Thus, by Lemma 2.4, the desired ansatz (2.24) is well defined. And for convenience, with the determined constant shifts $\xi$ and $\sigma$, we let

$$
(V_1, U_1, E_1, \Theta_1)(x, t) := (v_1^S, u_1^S, E_1^S, \theta_1^S)(x - s_1 t - \xi),
$$

$$
(V_3, U_3, E_3, \Theta_3)(x, t) := (v_3^S, u_3^S, E_3^S, \theta_3^S)(x - s_3 t - \xi - \sigma),
$$

(2.32)

and let $P_i := p(V_i, \Theta_i)$ for $i = 1, 3$.

Now we are ready to present the main result.

Theorem 2.5. Assume that the constants in (1.5) satisfy the R-H conditions (1.7) with the entropy conditions (1.8), and the periodic perturbations $(\phi_{0i}, \psi_{0i}, w_{0i}) \in H^2([0, \pi])$ satisfy (1.10) for $i = l, r$. Then there exist $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that if the strength of the shocks satisfies $\delta < \delta_0$ and (2.2), and the periodic perturbations satisfy

$$
\sum_{i=l,r} \|(\phi_{0i}, \psi_{0i}, w_{0i})\|_{H^2([0, \pi])} < \varepsilon_0.
$$

Then the problem (1.1) and (1.5) admits a unique solution satisfying

$$
(v - \bar{v}, u - \bar{u}, \theta - \bar{\theta}) \in C \left(0, +\infty; H^2(\mathbb{R})\right), \quad v - \bar{v} \in L^2 \left(0, +\infty; H^1(\mathbb{R})\right),
$$

$$
(u - \bar{u}, \theta - \bar{\theta}) \in L^2 \left(0, +\infty; H^2(\mathbb{R})\right),
$$

where $(\bar{v}, \bar{u}, \bar{\theta})$ is defined in (2.24) and the constants $X_0, Y_0, Z_0, \xi, \sigma$ and $\eta$ therein are uniquely determined by (2.30). Moreover, it holds that

$$
\|(v, u, \theta) - (V_1 + V_3 - \bar{v}_m, U_1 + U_3, \Theta_1 + \Theta_3 - \bar{\theta}_m)\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{as } t \to +\infty,
$$

where $(V_i, U_i, \Theta_i)$ is the shifted background $i$-viscous shock defined in (2.32) for $i = 1, 3$.

Remark 2.6. The ansatz $(\bar{v}, \bar{u}, \bar{\theta})$ is more complicated than that in [2]. Here we have to choose appropriate shift functions (of time) $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ for each variable $v, u$ and $E$, respectively, such that the anti-derivative method works (see (2.9) and (2.15)). While in [2], the ansatz is a composite wave in which each of two shock waves is shifted by a constant.

In the next section, we will first reformulate the problem and prove a priori estimates, and thus finish the proof of Theorem 2.5. And the detailed proof of Lemmas 2.3 and 3.1 will be shown in Section 4.

3. Reformulation of the Problem and A priori estimates

From (2.11) and (2.24), the ansatz $(\bar{v}, \bar{u}, \bar{E})$ satisfies that

$$
\begin{cases}
\partial_t \bar{v} - \partial_x \bar{u} = \partial_x F_1, \\
\partial_t \bar{u} + \partial_x \bar{p} = \mu \partial_x \left( \frac{\partial \bar{u}}{\partial x} \right) + \partial_x F_2 + \partial_x \bar{R}_1, \\
\partial_t \bar{E} + \partial_x (\bar{p} \bar{u}) = \kappa \partial_x \left( \frac{\partial \bar{E}}{\partial x} \right) + \mu \partial_x \left( \frac{\partial \bar{u}^2}{\partial x} \right) + \partial_x F_3 + \partial_x \bar{R}_2,
\end{cases}
$$

(3.1)
where $F_i$ ($i = 1, 2, 3$) denotes the anti-derivative variables of the source terms in (2.11), i.e.

\[
F_1(x, t) := F_{1,1}(x, t) + \int_{-\infty}^{x} f_{1,2}(y, t) dy + \mathcal{X}'(t) \int_{-\infty}^{x} f_{1,3}(y, t) dy,
\]

\[
F_2(x, t) := F_{2,1}(x, t) + \int_{-\infty}^{x} f_{2,2}(y, t) dy + \mathcal{Y}'(t) \int_{-\infty}^{x} f_{2,3}(y, t) dy,
\]

\[
F_3(x, t) := F_{3,1}(x, t) + \int_{-\infty}^{x} f_{3,2}(y, t) dy + \mathcal{Z}'(t) \int_{-\infty}^{x} f_{3,3}(y, t) dy,
\]

and the remainders satisfy that

\[
\begin{align*}
\hat{R}_1 &= a \hat{\vartheta}_t \Theta + \hat{p} - p^r - \mu \left( \frac{\partial_x \hat{u}}{v} - \frac{\hat{u}^r}{v^r} \right), \\
\hat{R}_2 &= \frac{\kappa \theta_m}{\gamma \theta_m} \hat{\vartheta}_x \Theta + \hat{\vartheta} u - p^r u^r - \kappa \left( \frac{\partial_x \hat{\vartheta} \theta}{v} - \frac{\partial_x \theta^r}{v^r} \right) - \mu \left( \frac{\hat{\vartheta} \partial_x \hat{u}}{v} - \frac{\hat{u}^r \partial_x u^r}{v^r} \right).
\end{align*}
\]  

(3.3)

Denote $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R})}$ and $\|\cdot\|_k = \|\cdot\|_{H^k(\mathbb{R})}$ for $k \geq 1$. And we have the following lemma.

**Lemma 3.1.** Under the assumptions of Theorem 2.5, the anti-derivative variables (3.2) exist and satisfy that

\[
\| (F_1, F_2, F_3) (\cdot, t) \|_2 \leq C \varepsilon e^{-\alpha t} + C \delta^{3/2} e^{-\alpha t},
\]

(3.4)

where $\alpha > 0$ is the constant in Lemma 2.2, and the constant $C > 0$ is independent of either $\varepsilon, \delta$ or $t$.

The proof is based on Lemma 2.2 and the construction of the ansatz, and we place it in Section 4 for brevity.

Introduce

\[
\begin{align*}
\hat{v} &:= V_1 + V_3 - \tau_m + \Theta, \\
\hat{u} &:= U_1 + U_3 + a \hat{\vartheta}_x \Theta, \\
\hat{E} &:= E_1 + E_3 - \overline{E}_m + \overline{\tau}_m \Theta, \\
\hat{\vartheta} &:= \gamma - \frac{1}{R} \left( \hat{E} - \frac{1}{2} \hat{u}^2 \right) \quad \text{and} \quad \hat{p} := \frac{R \hat{\vartheta}}{\overline{v}}.
\end{align*}
\]

(3.5)

We remark that (3.5) is exactly the ansatz constructed in [2], in which the initial perturbations around the shock waves are in the $H^1(\mathbb{R})$ space, i.e. the periodic perturbations $(\phi_{0,i}, \psi_{0,i}, w_{0,i})$ vanishes for $i = l, r$. Comparing (3.5) to the ansatz (2.24) of this paper, when $A$ represents either $v, u, E, \theta$ or $p$, direct calculations yield that

\[
A^r = \hat{A} + \mathcal{E} \quad \text{and} \quad \partial_x^k A^r = \partial_x^k \hat{A} + \mathcal{E}, \quad k = 1, 2,
\]

(3.6)

where and hereafter we use $\mathcal{E}$ to represent the sums of error terms when they satisfy the relation (2.20), i.e. $\mathcal{E} \sim 0$.

By Lemma 2.1 and (2.3), when $A$ and $B$ represent either $v, u, E, \theta$ or $p$, it holds that

\[
\begin{align*}
|A_i^S (x - s_1 t) - \overline{A}_m| \cdot |B_i^S (x - s_3 t) - \overline{B}_m| &\approx 0, \\
|A_i^S (x - s_1 t) - \overline{A}_m| \cdot |\Theta| &\approx 0, \quad i = 1, 3,
\end{align*}
\]

\[
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\]
and then
\[ w^i \partial_x \Theta = (\hat{u} + \mathcal{E}) \partial_x \Theta \approx 0, \quad \hat{\Theta} \approx \theta^i = \frac{\hat{v}_m}{v} \Theta. \] (3.7)

Since \( \hat{p}^i = \hat{p} + \mathcal{E} \approx P_1 + P_3 - \hat{p}_m + \mathcal{E} \) (see [2, (2.23)]), one can get from (2.24) and (3.7) that
\[ \hat{p} - \hat{p}^i = \frac{R\hat{\theta}}{\hat{v}} - \frac{R\theta^i}{v^2} \approx \frac{R(\theta^i + \frac{\hat{v}_m}{v} \Theta)}{v^2} - \frac{R\theta^i}{v^2} = \frac{\Theta}{\hat{v}} (\hat{p}^i - \hat{p}_m) \] (3.8)
\[ \Rightarrow \hat{p} - \hat{p}^i \approx \frac{\Theta}{\hat{v}} (P_1 - \hat{p}_m + P_3 - \hat{p}_m + \mathcal{E}) \approx 0. \]

By similar arguments, one can obtain the following lemma.

**Lemma 3.2.** Under the assumptions of Theorem 2.5, it holds that
\[ |\hat{R}_i|, |\partial_x \hat{R}_i| \approx 0, \quad i = 1, 2. \] (3.9)

Define the perturbations
\[ \phi := v - \hat{v}, \quad \psi := u - \hat{u}, \quad w := E - \hat{E} \quad \text{and} \quad \zeta := \theta - \hat{\theta}. \]

Following the idea of [2], due to the decreasing property of shock waves, we should consider the anti-derivatives of the perturbations. Let
\[ \Phi(x, t) := \int_{-\infty}^{x} \phi(y, t)dy, \quad \Psi(x, t) := \int_{-\infty}^{x} \psi(y, t)dy, \quad W(x, t) := \int_{-\infty}^{x} w(y, t)dy, \]
\[ \Xi = \frac{\gamma - 1}{R} (W - \hat{u} \Psi), \] (3.10)
which leads to
\[ v - \hat{v} = \partial_x \Phi, \quad u - \hat{u} = \partial_x \Psi, \quad \theta - \hat{\theta} = \partial_x \Xi - \frac{\gamma - 1}{R} \left( \frac{1}{2} (\partial_x \Psi)^2 - \partial_x \hat{u} \Psi \right). \] (3.11)

And then (1.1) and (3.1) give the reformulated problem
\[
\begin{align*}
\partial_t \Phi - \partial_x \Psi & = -F_1, \\
\partial_t \Psi - \left( \frac{\hat{v}}{v} - \frac{\mu \partial_x \hat{u}}{v} \right) \partial_x \Phi & + \frac{\gamma - 1}{v} \partial_x \hat{u} \Psi = \frac{\mu}{v} \partial_x \Psi + J_1 - F_2 - \hat{R}_1, \\
\frac{R}{\gamma - 1} \partial_t \Xi & + \left( \frac{\hat{v}}{v} - \frac{\mu \partial_x \hat{u}}{v} \right) \partial_x \Psi + \partial_x \hat{u} \Psi - \frac{\kappa (\gamma - 1)}{\gamma R} \partial_x (\partial_x \hat{u} \Psi) & + \frac{\mu}{v} \partial_x \Psi \partial_x \Phi \\
& = \frac{\gamma - 1}{2v} \partial_x \Xi + J_2 - F_3 + \hat{u} F_2 - \hat{R}_2 + \hat{u} \hat{R}_1.
\end{align*}
\] (3.12)

where \( J_1 \) and \( J_2 \) are the sums of higher order terms, given by
\[ J_1 = \frac{\gamma - 1}{2v} (\partial_x \Psi)^2 + \frac{\mu}{vv} \partial_x \hat{u} (\partial_x \Phi)^2 - \frac{\mu}{vv} \partial_x \Psi \partial_x \Phi - \left( p - \hat{p} + \frac{\hat{v}}{v} \partial_x \Phi - \frac{R}{\hat{v}} \hat{z} \right), \]
\[ J_2 = \left( \frac{\hat{v}}{v} - p \right) \partial_x \Psi + \mu \left( \frac{\partial_x u}{v} - \frac{\partial_x \hat{u}}{v} \right) \partial_x \Psi - \frac{\kappa (\gamma - 1)}{\gamma R} \partial_x \Psi \partial_x \Psi - \frac{\partial_x \Phi}{v} \partial_x \Psi \partial_x \Psi. \] (3.13)

Now we set the following a priori assumptions. For \( T > 0 \), let
\[ N(T) := \sup_{t \in [0, T]} \| (\Phi, \Psi, \Xi)(t) \|_2 \leq \nu_0, \] (3.14)
where the constant \( \nu_0 > 0 \) is small enough.
Proposition 3.3 (A priori estimates). Under the assumptions of Theorem 2.5, there exist positive constants \( \delta_0, \varepsilon_0 \) and \( \nu_0 \), which are independent of \( T \), such that if \( \delta + |\eta| \leq \delta_0, \varepsilon \leq \varepsilon_0 \) and \( N(T) \leq \nu_0 \), then

\[
\sup_{t \in (0, T)} \| \Phi, \Psi, \Xi \|_2^2 + \int_0^T \left( \| \partial_x \Phi \|_1^2 + \| \partial_x \Psi, \partial_x \Xi \|_2^2 \right) dt + \int_0^T \int_{\mathbb{R}} \left( |\partial_x U_1| + |\partial_x U_3| \right) (\Psi^2 + \Xi^2) \, dx \, dt \\
\leq C \left( \| \Phi_0, \Psi_0, W_0 \|_2^2 + \varepsilon_0 + \delta_0^{\frac{1}{2}} \right). \tag{3.15}
\]

Under the assumptions of Proposition 3.3, one can get that if \( \delta_0, \varepsilon_0 \) and \( \nu_0 \) are small enough, then

\[
\inf_{x,t} \theta \geq \frac{\delta_0}{2}, \quad \inf_{x,t} v \geq \frac{\nu_0}{4}, \quad \inf_{x,t} \theta \geq \frac{\theta_0}{2}, \quad \inf_{x,t} \theta \geq \frac{\theta_0}{4},
\tag{3.16}
\]

and the higher-order terms \( J_1 \) and \( J_2 \) in (3.13) satisfy

\[
J_1 = O(1) \left( |\partial_x \Phi|^2 + |\partial_x \Xi|^2 + |\partial_x \Psi|^2 + |\partial_x \Xi|^2 + |\partial_x \Psi|^2 \right),
\]

\[
J_2 = O(1) \left( |\partial_x \Phi|^2 + |\partial_x \Psi|^2 + |\partial_x \Xi|^2 + |\partial_x \Psi|^2 + |\partial_x \Xi|^2 \right). \tag{3.17}
\]

Lemma 3.4. Under the assumptions of Proposition 3.3, there exist small \( \delta_0 > 0, \varepsilon_0 > 0 \) and \( \nu_0 > 0 \) such that

\[
\sup_{t \in (0, T)} \| \Phi, \Psi, \Xi \|_2^2 + \int_0^T \| \partial_x \Psi, \partial_x \Xi \|_2^2 \, dt + \int_0^T \int_{\mathbb{R}} \left( |\partial_x U_1| + |\partial_x U_3| \right) (\Psi^2 + \Xi^2) \, dx \, dt \\
\leq C \| \Phi_0, \Psi_0, W_0 \|_2^2 + C(\nu_0 + \delta_0^{\frac{1}{2}}) \int_0^T \| \partial_x \Phi, \partial_x \Xi, \partial_x \Psi \|_2^2 \, dt + C(\varepsilon_0 + \delta_0^{\frac{1}{2}}).
\tag{3.18}
\]

Proof. When \( \delta_0, \varepsilon_0 \) are small enough, it holds that

\[
\inf_{x,t} \left( \tilde{p} - \frac{\mu \partial_x \tilde{u}}{v} \right) \geq c \inf_{x,t} \theta - C \delta_0 - C \varepsilon_0 \geq \frac{\theta_0}{4} > 0,
\]

\[
\inf_{x,t} \left( p^x - \frac{\mu \partial_x \tilde{u}}{v^2} \right) \geq \frac{\theta_0}{4} \quad \text{and} \quad \inf_{x,t} \left( \tilde{p} - \frac{\mu \partial_x \tilde{u}}{v} \right) \geq \frac{\theta_0}{4}
\tag{3.19}
\]

for some constants \( c > 0 \) and \( C > 0 \). Thus, one can define \( \tilde{L} := (\tilde{p} - \frac{\mu \partial_x \tilde{u}}{v})^{-1} \), \( L^x := (p^x - \frac{\mu \partial_x \tilde{u}}{v^2})^{-1} \) and \( \tilde{L} := (\tilde{p} - \frac{\mu \partial_x \tilde{u}}{v})^{-1} \).

Then (3.12)1 \( \Phi + (3.12)2 \cdot \tilde{v} \tilde{L} \Psi + (3.12)3 \cdot R \tilde{L}^2 \Xi \) gives that

\[
\partial_t N_1 + N_2 + N_3 + N_4 = \partial_x (\cdots) - F_1 \Phi + (J_1 - F_2 - \tilde{R}_1) \tilde{v} \tilde{L} \Psi \tag{3.20}
\]

\[
+ (J_2 - F_3 + \tilde{u} F_2 - \tilde{R}_2 + \tilde{u} \tilde{R}_1) R \tilde{L}^2 \Xi,
\]
where

\[ N_1 = \frac{1}{2} \left( \Phi^2 + \tilde{v} \tilde{L} \Psi^2 + \frac{R^2}{\gamma - 1} \tilde{L}^2 \Xi^2 \right), \]

\[ N_2 = \tilde{b} \Psi^2 + \mu \partial_x \tilde{L} \Psi \partial_x \Psi + \mu \tilde{L} \left( \partial_x \Psi \right)^2 \quad \text{with} \quad \tilde{b} = -\frac{1}{2} \partial_t \left( \tilde{v} \tilde{L} \right) + (\gamma - 1) \tilde{L} \partial_x \tilde{u}, \]

\[ N_3 = -\frac{R^2}{\gamma - 1} \tilde{L} \partial_t \tilde{L} \Xi^2 + \partial_x \left( \frac{\kappa R}{v} \tilde{L} \right) \Xi \partial_x \Xi + \frac{\kappa R}{v} \tilde{L}^2 (\partial_x \Xi)^2, \]

\[ N_4 = R \left( \tilde{L}^2 \partial_t \tilde{u} - \partial_x \tilde{L} \right) \Psi \Xi + \frac{(\gamma - 1) \kappa}{v} \partial_x \tilde{u} \tilde{L}^2 \Psi \partial_x \Xi + \kappa (\gamma - 1) \partial_x \tilde{u} \partial_x \left( \frac{\tilde{L}^2}{v} \right) \Psi \Xi + \frac{\kappa R}{v} \partial_x \tilde{L}^2 \partial_x \Psi \Xi, \]

and \( \partial_x (\cdots) \) denotes the terms vanishing after integration on \( \mathbb{R} \).

Combining (3.6), (3.8) and [2, (2.23) and (3.16)], one has that

\[ \partial_t \tilde{b} \approx \partial_t \| \tilde{L} \|^2 \approx \partial_t \left( P_1 + P_3 - \tilde{p}_m \right) + \mathcal{C}, \]

\[ \partial_t \left( \frac{\mu \partial_x \tilde{u}}{v^2} \right) \approx \partial_t \left( \frac{\mu \partial_x \tilde{u}}{v^2} \right) + \mathcal{C} \approx \partial_t \left( \frac{\mu \partial_x U_1}{V_1} + \frac{\mu \partial_x U_3}{V_3} \right) + \mathcal{C}, \]

\[ \partial_t \tilde{L} \approx \partial_t \| \tilde{L} \|^2 \approx \partial_t \tilde{L} + \mathcal{C} \approx \partial_t L_1 + \partial_t L_3 + \mathcal{C}, \]

\[ \partial_x \tilde{L} \approx \partial_x L_1 + \partial_x L_3 + \mathcal{C}, \]

where \( L_i := (P_i - \frac{\mu \partial_x U_i}{V_i})^{-1} = (b_i - s_i^2 V_i)^{-1} \) with \( b_i = \tilde{p}_m + s_i^2 \tau_m \) for \( i = 1, 3 \).

Now we estimate \( H_2, H_3, H_4, J_1 \tilde{v} \tilde{L} \Psi \) and \( J_2 R \tilde{L}^2 W \) in (3.20) one by one. Firstly, similar to the arguments of (3.22), one has that the coefficient \( \tilde{b} \) in \( H_2 \) satisfies that

\[ \tilde{b} \approx -\frac{1}{2} \partial_t \left( \tilde{v} \tilde{L} \right) + (\gamma - 1) \tilde{L} \partial_x \tilde{u}^2 \]

\[ = -\frac{1}{2} \partial_t \left( \tilde{v} \tilde{L} \right) + (\gamma - 1) \tilde{L} \partial_x \tilde{u} + \mathcal{C} \]

\[ \approx \sum_{i=1,3} \left[ -\frac{1}{2} \partial_t \left( L_i V_i \right) + (\gamma - 1) L_i \partial_x U_i \right] + \mathcal{C} \]

\[ = \sum_{i=1,3} -\frac{1}{2} \partial_x U_i L_i^2 \left( b_i - 2(\gamma - 1) L_i^{-1} \right) + \mathcal{C}, \]

\[ \geq c \sum_{i=1,3} \left| \partial_x U_i \right| \left( (3 - \gamma) \tilde{p}_m - C \delta_i \right) \| \Psi \|_{L^\infty(\mathbb{R})}, \]

and \( \tilde{L} \) satisfies that

\[ \tilde{L} \approx L_1 + L_3 + \mathcal{C} \geq \sum_{i=1,3} \left( \tilde{p}_m - C \delta_i \right)^{-1} - \| \Psi \|_{L^\infty(\mathbb{R})} \geq c - \| \Psi \|_{L^\infty(\mathbb{R})}, \]

for some constant \( c > 0 \). Then for small \( \delta_0 \) and \( \varepsilon_0 \), one has

\[ \mu \partial_x \tilde{L} \Psi \partial_x \Psi \approx \sum_{i=1,3} \mu s_i^2 L_i^2 \partial_x V_i \Psi \partial_x \Psi + \mathcal{C} \Psi \partial_x \Psi \]

\[ \geq -\sum_{i=1,3} \left( C s_i^2 L_i^3 \left( \partial_x V_i \right)^2 \Psi^2 + \frac{1}{2} \mu L_i \left( \partial_x \Psi \right)^2 \right) - \| \Psi \|_{L^\infty(\mathbb{R})} | \Psi | \left| \partial_x \Psi \right| \]
For brevity, we will deal with the error terms arising from the relation "\( \approx \)" later. Collecting the inequalities above, one has that

\[
N_2 \geq c(\varepsilon_0 + \delta) \sup_{t \in (0, T)} \| \Psi \|^2,
\]

which implies that

\[
\int_0^T \int_\mathbb{R} N_2 dx dt \geq c \int_0^T \left\| \left( \varepsilon_0 + \delta \frac{1}{2} \right) \sup_{t \in (0, T)} \| \Psi \|^2 \right\| dt,
\]

if \( \varepsilon \leq \varepsilon_0 \) and \( \delta \leq \delta_0 \) are small enough.

For \( N_3 \), by (3.22) and the fact that \( \partial_t L_i = s_i L_i^2 s_i \partial_x V_i = -s_i L_i^2 |\partial_x U_i|, \ i = 1, 3 \), one has that

\[
\frac{R}{\gamma - 1} L \partial_t \partial_x \Xi \geq C(\varepsilon_0 + \delta) \Xi^2 - \| \mathcal{E} \|_{L^\infty(\mathbb{R})} \Xi^2,
\]

which implies that

\[
\int_0^T \int_\mathbb{R} N_3 dx dt \geq c \int_0^T \left\| \left( \varepsilon_0 + \delta \right) \Xi^2 \right\| dt.
\]
Thus, it holds that
\[
|N_4| \leq C\delta_0^{\frac{1}{2}} (|\partial_x U_1| + |\partial_x U_3|) (\Psi^2 + \Xi^2) + C\delta_0^{\frac{1}{2}} (|\partial_x \Phi|^2 + |\partial_x \Xi|^2) + \|\tilde{E}\|_{L^\infty(\mathbb{R})} (\Psi^2 + \Xi^2 + |\partial_x \Phi|^2 + |\partial_x \Xi|^2),
\]
which implies that
\[
\int_0^T \int_{\mathbb{R}} |N_4| \, dx \, dt \leq C (\varepsilon_0 + \delta_0^{\frac{1}{2}}) \int_0^T \left( |\partial_x U_1| + |\partial_x U_3| \right)^{\frac{1}{2}} (\Psi, \Xi, \partial_x \Phi, \partial_x \Xi) \, dt + C (\varepsilon_0 + \delta_0^{\frac{1}{2}}) \sup_{t \in (0, T)} \|\Psi, \Xi\|^2.
\] (3.25)

For the right-hand side of (3.20), note that
\[
\left| \partial \tilde{L}_1 J_1 \Psi \right| \leq C \|\Psi\|_{L^\infty(\mathbb{R})} \left[ |\partial_x \Phi|^2 + |\partial_x \Psi|^2 + |\partial_x \Xi|^2 + |\partial_x^2 \Psi|^2 \right. \\
+ \left( |\partial_x U_1| + |\partial_x U_3| + \|\tilde{E}\|_{L^\infty(\mathbb{R})} \right) (\Psi^2), \tag{3.26}
\]
\[
\left| R \tilde{L}_2 J_2 \Xi \right| \leq C \|\Xi\|_{L^\infty(\mathbb{R})} \left[ |\partial_x \Phi|^2 + |\partial_x \Psi|^2 + |\partial_x \Xi|^2 + |\partial_x^2 \Psi|^2 \right. \\
+ \left( |\partial_x U_1| + |\partial_x U_3| + \|\tilde{E}\|_{L^\infty(\mathbb{R})} \right) (\Psi^2 + |\partial_x \zeta|^2). \tag{3.27}
\]

Thus, by using Cauchy inequality, one can prove that
\[
\int_0^T \int_{\mathbb{R}} \left| \text{RHS of (3.20)} \right| \, dx \, dt \\
\leq C \int_0^T \left| F_1, F_2 \right| \|\Phi, \Psi, \Xi\| \, dt + CN(T) \int_0^T \left( |\partial_x (\Phi, \Psi, \Xi)| \right)^2 \\
+ \left| \partial_x^2 \Psi, \partial_x \zeta \right|^2 + \left( |\partial_x U_1| + |\partial_x U_3| \right)^{\frac{1}{2}} (\Psi) \, dt + C (\varepsilon_0 + \delta_0^{\frac{1}{2}}) \sup_{t \in (0, T)} \|\Psi\|^2 \\
\leq C (\varepsilon_0 + \delta_0^{\frac{1}{2}}) + C (\varepsilon_0 + \delta_0^{\frac{1}{2}}) \sup_{t \in (0, T)} \|\Phi, \Psi, \Xi\|^2 + C\nu_0 \int_0^T \left( |\partial_x (\Phi, \Psi, \Xi)| \right)^2 \\
+ \left( |\partial_x U_1| + |\partial_x U_3| \right)^{\frac{1}{2}} (\Psi) \, dt + C\nu_0 \int_0^T \|\partial_x^2 \Psi, \partial_x \zeta \|^2 \, dt. \tag{3.28}
\]

At last, we deal with all the error terms arising from the relation “≈”, which is same as [2, Lemma 3.1]. The integral of them on \((0, t) \times \mathbb{R}\) can be bounded by
\[
\int_0^T \int_{\mathbb{R}} \left| \tilde{R} \right| (|\Phi| + |\Psi| + |\Xi| + |\partial_x \Phi| + |\partial_x \Psi| + |\partial_x \Xi|) \, dx \, dt \\
\leq C \int_0^T \left( (\delta^\frac{1}{2} + |\eta|) e^{-\varepsilon_0 t} + \frac{|\eta|}{(1 + t)^{\frac{1}{2}}} \right) \|\Phi, \Psi, \Xi, \partial_x \Phi, \partial_x \Psi, \partial_x \Xi\|^2 \, dx \, dt \tag{3.29}
\]
\[
\leq C \delta_0^{\frac{1}{2}} \left( \sup_{t \in (0, T)} \|\Phi, \Psi, \Xi\|^2 + \int_0^T \|\partial_x (\Phi, \partial_x \Psi, \partial_x \Xi)\|^2 \, dt \right),
\]
where \(\tilde{R} \approx 0\) represents the error terms which were eliminated in the previous estimates (3.23) to (3.28). Then (3.18) follows from (3.23) to (3.29).
Lemma 3.5. Under the assumptions of Proposition 3.3, there exist small \( \delta_0 > 0, \varepsilon_0 > 0 \) and \( v_0 > 0 \) such that

\[
\sup_{t \in (0, T)} \left( \| \Phi \|^2 + \| \Psi, \Xi \|^2 \right) + \int_0^T \left| \partial_x \Phi, \partial_x \Psi, \partial_x \Xi \right|^2 dt + \int_0^T \int_R \left( | \partial_x U_1 | + | \partial_x U_3 | \right) \left( \Psi^2 + \Xi^2 \right) dx dt \\
\leq C \left( \| \Phi_0 \|^2_1 + \| \Psi_0, W_0 \|^2 \right) + C \left( \nu_0 + \delta_0^2 \right) \int_0^T \left| \tilde{\partial}_x \zeta, \tilde{\partial}_x \Psi \right|^2 dt + C \left( \varepsilon_0 + \delta_0^2 \right).
\]

(3.30)

Proof. Substituting \( \tilde{\partial}_x \Psi \) by \( \tilde{\partial}_x \Phi + F_1 \) in (3.12) gives

\[
\tilde{\partial}_t \left( \frac{\mu}{2 \tilde{v}} (\tilde{\partial}_x \Phi)^2 \right) = \frac{\mu}{2 \tilde{v}} \left( \tilde{\partial}_x \Phi \right)^2 - \tilde{\partial}_x \Phi \tilde{\partial}_t \Phi + \frac{1}{\tilde{v} \lambda} \tilde{\partial}_x \Phi \nonumber
\]

(3.32)

And then by multiplying it by \( \tilde{\partial}_x \Phi \), one has

\[
\tilde{\partial}_t \left( \frac{\mu}{2 \tilde{v}} (\tilde{\partial}_x \Phi)^2 \right) - \frac{\mu}{2 \tilde{v}} \left( \tilde{\partial}_x \Phi \right)^2 - \tilde{\partial}_x \Phi \tilde{\partial}_t \Phi + \frac{1}{\tilde{v} \lambda} \tilde{\partial}_x \Phi = \left( \frac{\tilde{R}_1}{\tilde{v}} \tilde{\partial}_x \Xi + \frac{\gamma - 1}{\tilde{v}} \tilde{\partial}_x \tilde{u} \Phi - J_1 + \tilde{R}_1 - \frac{\mu}{\tilde{v}} \tilde{\partial}_x F_1 + F_2 \right) \tilde{\partial}_x \Phi.
\]

Since \( \tilde{\partial}_x \Phi \tilde{\partial}_t \Phi = \tilde{\partial}_x (\tilde{\partial}_x \Phi \Phi) - \tilde{\partial}_x (\tilde{\partial}_t \Phi \Phi) + (\tilde{\partial}_x \Phi)^2 - F_1 \tilde{\partial}_x \Phi \) and if \( \delta_0 > 0 \) and \( \varepsilon_0 > 0 \) are small enough,

\[
- \tilde{\partial}_t \left( \frac{\mu}{2 \tilde{v}} \right) + \frac{1}{2 \tilde{v} \lambda} = \tilde{\partial}_t \Phi - \frac{\mu}{\tilde{v} \lambda} \left( \tilde{\partial}_x \tilde{u} \Phi \right) \geq \frac{\tilde{\partial}_m}{4} - C \| \tilde{\partial}_x F_1 \|_{L^2(R)},
\]

then integrating (3.32) on \( \mathbb{R} \) yields that

\[
\int_R \left( \frac{\mu}{2 \tilde{v}} (\tilde{\partial}_x \Phi)^2 - \tilde{\partial}_x \Phi \Phi \right)(x, T) dx + \int_0^T \int_R \frac{1}{2 \tilde{v} \lambda} (\tilde{\partial}_x \Phi)^2 dx dt \\
\leq C \left( \| \Phi_0 \|^2_1 + \| \Psi_0 \|^2 \right) + C \int_0^T \| \tilde{\partial}_x \Phi, \tilde{\partial}_x \Xi \|^2 dt + C \delta_0 \int_0^T \int_R \left( | \tilde{\partial}_x U_1 | + | \tilde{\partial}_x U_3 | \right) \Psi^2 dx dt \\
+ C \int_0^T \| \Psi \|^2 dt + C \int_0^T \left( \| \tilde{R}_1 \|^2 + \| F_1 \|^2_1 + \| F_2 \|^2 \right) dt (3.33)
\]

From Lemmas 3.1 and 3.2 and (3.17), it holds that

\[
\int_0^T \| \mathcal{E} \|_{L^2(\mathbb{R})} \| \Psi \|^2 dt \leq C \left( \varepsilon_0 + \delta_0^2 \right) \sup_{t \in [0, T]} \| \Psi \|^2,
\]

\[
\int_0^T \left( \| \tilde{R}_1 \|^2 + \| F_1 \|^2_1 + \| F_2 \|^2 \right) dt \leq C \left( \varepsilon_0^2 + \delta_0 \right),
\]

\[
\int_0^T \| \tilde{\partial}_x F_1 \|_{L^2(\mathbb{R})} \| \tilde{\partial}_x \Phi \|^2 dt \leq C \left( \varepsilon_0 + \delta_0^2 \right) \int_0^T \| \tilde{\partial}_x \Phi \|^2 dt,
\]

\[
\| J_1 \|^2 \leq C \nu_0 \| \tilde{\partial}_x \Phi, \tilde{\partial}_x \Psi, \tilde{\partial}_x \Xi, \tilde{\partial}_x^2 \Psi \|^2 + C \delta_0 \int_R \left( | \tilde{\partial}_x U_1 | + | \tilde{\partial}_x U_3 | \right) \Psi^2 dx.
\]

Then (3.30) can follow from Lemma 3.4 and (3.33). \( \square \)
Lemma 3.6. Under the assumptions of Proposition 3.3, there exist small $\delta_0 > 0, \varepsilon_0 > 0$ and $\nu_0 > 0$ such that

\[
\sup_{t \in (0,T)} \|\phi, \psi, \zeta\|^2 + \int_0^T \|\partial_x \phi, \partial_x \zeta\|^2 dt \\
\quad \leq C \|\phi_0, \psi_0, \zeta_0\|^2 + C(\varepsilon_0 + \delta_0^{\frac{1}{3}}).
\]

\[
\sup_{t \in (0,T)} \|\Phi, \Psi, \Xi\|^2 + \int_0^T (\|\partial_x \Phi\|^2 + \|\partial_x \Psi\| \|\zeta\|^2) dt + \int_0^T \int_R (|\partial_x U_1| + |\partial_x U_3|)(\Psi^2 + \Xi^2) dx dt \\
\quad \leq C \left( \|\Phi_0, \Psi_0, W_0\|_1^2 + \varepsilon_0 + \delta_0^{\frac{1}{3}} \right).
\] (3.34)

Proof. Here we provided a different proof from that in [2][Lemma 3.3]. Subtracting (3.1)_2 from (1.1)_2 gives that

\[
\partial_t \psi - \mu \partial_x \left( \frac{\partial_x \psi}{v} \right) = -\partial_x \left( p - \tilde{p} + \mu \frac{\partial_x \bar{u}}{v} \phi + F_2 + \tilde{R}_1 \right). \tag{3.35}
\]

Multiplying $\psi$ on two sides and integrating the resulting equation on $\mathbb{R} \times (0, T)$ give that

\[
\|\psi\|^2(T) + \int_0^T \|\partial_x \psi\|^2 dt \leq C \|\psi_0\|^2 + C \int_0^T \left( \|p - \tilde{p}\| + \|\phi\| + \|F_2\| + \|\tilde{R}_1\| \right) \|\partial_x \psi\| dt \\
\quad \leq C \|\psi_0\|^2 + C \int_0^T \left( \|\zeta\| + \|\phi\| + \|F_2\|^2 + \|\tilde{R}_1\|^2 \right) \|\partial_x \psi\| dt,
\]

which implies that

\[
\|\psi\|^2(T) + \int_0^T \|\partial_x \psi\|^2 dt \leq C \|\psi_0\|^2 + C \int_0^T \left( \|\zeta\|^2 + \|\phi\|^2 + \|F_2\|^2 + \|\tilde{R}_1\|^2 \right) dt. \tag{3.36}
\]

From (3.11), it holds that

\[
\int_0^T \|\zeta\|^2 dt \leq C \int_0^T \left( \|\partial_x \Xi\|^2 + \|\partial_x \Psi\|^2 \|\partial_x \phi\|^2 + \|\partial_x \bar{u} \Phi\|^2 \right) dt \\
\quad \leq C \int_0^T \|\partial_x \Xi, \partial_x \Psi\|^2 dt + C \int_0^T \left( |\partial_x U_1| + |\partial_x U_3| \right) \|\frac{1}{2} \Phi\|^2 dt \\
\quad \quad + C(\varepsilon_0 + \delta_0^{\frac{1}{3}}) \sup_{t \in [0,T]} \|\Psi\|^2.
\]

Combining (3.36) and Lemma 3.5, if $\varepsilon_0, \nu_0$ and $\delta_0$ are small enough, one has that

\[
\|\psi\|^2(T) + \int_0^T \|\partial_x \psi\|^2 dt \\
\quad \leq C \|\Phi_0, \Psi_0\|_1^2 + \|W_0\|^2 + C(\nu_0 + \delta_0^{\frac{1}{3}}) \int_0^T \|\partial_x \zeta\|^2 dt + C(\varepsilon_0 + \delta_0^{\frac{1}{3}}). \tag{3.37}
\]

Similarly, subtracting (3.1)_3 from (1.1)_3 gives that

\[
\frac{R}{\gamma - 1} \partial_t \zeta - \kappa \partial_x \left( \frac{\partial_x \zeta}{v} \right) = -\frac{p - \tilde{p}}{\partial_x \psi} \partial_x \tilde{u} - \frac{p}{\partial_x \psi} \partial_x \psi - \partial_x \left( \kappa \frac{\partial_x \phi}{v} + F_3 + \tilde{R}_2 \right) \\
\quad - \frac{\mu (\partial_x \bar{u})^2}{v} \phi + \frac{\mu}{v} \partial_x \tilde{u} \partial_x \psi + \partial_x (F_2 + \tilde{R}_1). \tag{3.38}
\]
Then multiplying \( \zeta \) on two sides and integrating the resulting equation on \( \mathbb{R} \times (0, T) \), and then by similar arguments above including integration by parts, one can verify that

\[
\| \zeta \|^2(T) + \int_0^T \| \partial_x \zeta \|^2 dt \leq C \int_0^T \| \phi, \zeta, \partial_x \psi \|^2 + C \int_0^T \| F_2, F_3, \bar{R}_1, \bar{R}_2 \|^2 dt.
\]

Then combining (3.37) and Lemmas 3.1, 3.2 and 3.5 can finish the proof of (3.34).

**Proof of Proposition 3.3.** With Lemma 3.6, to prove (3.15), it suffices to control the terms

\[
\sup_{t \in (0, T)} \| \partial_x \phi, \partial_x \psi, \partial_x \zeta \|^2 + \int_0^T (\| \partial_x \phi \|^2 + \| \partial_x^2 \psi, \partial_x^2 \zeta \|^2) dt
\]

The proof is similar to that of Lemmas 3.5 and 3.6 when dealing with the resulting equations after taking the first derivative on (3.31), (3.35) and (3.38), respectively. So it is omitted here.

**Proof of Theorem 2.5.** With the a priori estimate (3.15), it is routine to obtain that the a priori assumption (3.14) can be fulfilled with small \( \| \Phi_0, \Psi_0, W_0 \|, \varepsilon_0 \) and \( \delta_0 \), the solution exists globally. And moreover,

\[
(v, u, \theta) \rightarrow (\tilde{v}, \tilde{u}, \tilde{\theta}) \rightarrow (V_1 + V_3 - \overline{v}_m, U_1 + U_3, \overline{\Theta}_1 + \overline{\Theta}_3 - \overline{\theta}_m) \text{ in } L^\infty(\mathbb{R}) \text{ as } t \rightarrow +\infty.
\]

Refer to [2] for details.

## 4. Proof of Lemmas 2.3 and 3.1

**Proof of Lemma 2.3.** When the perturbations are small enough, the existence, uniqueness and regularities of \( X, Y, Z \) can be easily obtained by the ODE (2.15). The large time behaviors can be proved by (1.7) and Lemma 2.2.

Now we calculate \( X, Y, Z \). For any fixed \( y \in [0, 1], t > 0 \) and integer \( N > 0 \), we define the domain

\[
\Omega_y^N := \{(\tau, x); 0 < \tau < t, \Gamma_1^N(\tau) < x < \Gamma_r^N(\tau)\},
\]

where \( \Gamma_1^N(\tau) := s_1 \tau + \mathcal{X}(\tau) + (-N + x) \pi_l \)

\( \Gamma_r^N(\tau) := s_3 \tau + \mathcal{X}(\tau) + (N + x) \pi_r \).

Integrating (2.11) over \( \Omega_y^N \) with respect to \( (x, t) \) and \( [0, 1] \) with respect to \( y \), integrating by parts and taking the limit \( N \rightarrow \infty \), yields that

\[
\lim_{N \rightarrow \infty} \int_0^1 \int_{\Omega_y^N} (\partial_t v^2 - \partial_x u^2) dx d\tau dy = 0,
\]

\[
\lim_{N \rightarrow \infty} \int_0^1 \left\{ \int_{\Gamma_1^N(0)}^{\Gamma_1^N(t)} v\varphi_1(x) dx + \int_0^t [v^2(s_3 + \mathcal{X'}) + u^2] (\Gamma_r^N(\tau), \tau) d\tau 
- \int_{\Gamma_1^N(t)}^{\Gamma_1^N(0)} v\varphi_1(x, t) dx - \int_0^t [v^2(s_1 + \mathcal{X'}) + u^2] (\Gamma_1^N(\tau), \tau) d\tau \right\} dy = 0.
\]
For the integrals on \( \tau = 0 \) and \( \tau = t \),
\[
\int_{\Gamma_1^N} v_1^S(x)dx - \int_{\Gamma_1^N(t)} v_1^N(t, x)dx
\]
\[= \int_{\Gamma_1^N} \left[ \phi_{w}(1 - \tau \chi_0(g_1)) + \phi_{w} \tau^3 \chi_0 + \sigma(g_3) + \tau^3 \chi_0(v_1^S) + \tau^3 \chi_0 + \sigma(v_3^S) - \overline{v}_m \right]dx
\]
\[+ \int_{\Gamma_1^N(t)} \left[ \tau^3 \chi_0(v_1^S) + \tau^3 \chi_0 + \sigma(v_3^S) - \overline{v}_m \right]dx + \mathcal{O}(e^{-\alpha t})
\]
\[= \int_{\Gamma_1^N} \left[ \phi_{w}(x)(1 - g_1(x - \chi_0)) + \phi_{w}(x)g_3(x - \chi_0 - \sigma) \right]dx
\]
\[- \overline{v}_m \left[ \Gamma_1^N(0) - \Gamma_1^N(0) \right] + \overline{v}_m \left[ \Gamma_1^N(t) - \Gamma_1^N(t) \right]
\[- \int_{(s_3 - s_1) + (N + y) \pi_x + \sigma}^{(s_3 - s_1) + (N + y) \pi_x + \sigma} v_3^S(x)dx + \int_{(s_3 - s_1) + (N + y) \pi_x - \sigma}^{(s_3 - s_1) + (N + y) \pi_x - \sigma} v_3^S(x)dx + \mathcal{O}(e^{-\alpha t})
\]
\[
\frac{\phi_1(x)dy}{N \rightarrow \infty} \lim_{N \rightarrow \infty} \int_{0}^{1} \left\{ \int_{\Gamma_1^N} \left[ \phi_{w}(x)(1 - g_1(x - \chi_0)) + \phi_{w}(x)g_3(x - \chi_0 - \sigma) \right]dx \right\} dy
\]
\[- \overline{v}_m (s_3 - s_1) t + \mathcal{O}(e^{-\alpha t})
\]
\[= - \int_{-\infty}^{+N} \left[ \phi_{w}(x) \tau (x - \chi_0) - \phi_{w}(x)g_3(x - \chi_0 - \sigma) \right]dx
\]
\[- \frac{1}{\pi_t} \int_{0}^{\pi t} \int_{0}^{x} \phi_0(y)dydx + \frac{1}{\pi_r} \int_{0}^{\pi r} \int_{0}^{x} \phi_0(y)dydx - \overline{v}_m (s_3 - s_1) t + \mathcal{O}(e^{-\alpha t})
\]
Here \( v_1^S(x) \rightarrow \overline{v}_m \) as \( x \rightarrow \infty \) and \( v_3^S(x) \rightarrow \overline{v}_m \) as \( x \rightarrow -\infty \). Since \( \phi_{w}, \phi_{w} \) have zero average, \( \int_{0}^{1} \chi_0 + (N + x) \pi_x \phi_0(y)dydx = \frac{1}{\pi_x} \phi_0(y)dydx. \)

For the integrals on \( \Gamma_1^N \) and \( \Gamma_1^N(t) \),
\[
\lim_{N \rightarrow \infty} \int_{0}^{t} \left\{ \int_{0}^{t} [v^x(s_3 + X') + u^x]^2(\Gamma_1^N(\tau), \tau)d\tau \right\} dy
\]
\[= \frac{1}{\pi_t} \int_{0}^{t} \int_{0}^{\pi t} \left[ v_r(s_3 + X') + u_r \right] (s_3 \tau + X(\tau) + N \pi_x + x, \tau)d\tau dx
\]
\[= \overline{v}_r(s_3 t + X - \chi_0) + \overline{u}_r t.
\]
Therefore, we have
\[
- \int_{-\infty}^{0} \left[ \phi_{w}(x) \tau (x - \chi_0) - \phi_{w}(x)g_3(x - \chi_0 - \sigma) \right]dx
\]
\[+ \int_{0}^{+\infty} \left[ \phi_{w}(x)(1 - g_1(x - \chi_0)) - \phi_{w}(x)(1 - g_3(x - \chi_0 - \sigma)) \right]dx
\]
\[- \frac{1}{\pi_t} \int_{0}^{\pi t} \int_{0}^{x} \phi_0(y)dydx + \frac{1}{\pi_r} \int_{0}^{\pi r} \int_{0}^{x} \phi_0(y)dydx - \overline{v}_r(s_3 - s_1)t + \mathcal{O}(e^{-\alpha t})
\]
\[+ \overline{v}_r(s_3 t + X - \chi_0) + \overline{u}_r t - \overline{v}_r(s_1 t + X - \chi_0) - \overline{u}_r t = 0.
\]
By the definition of the shock speed $s_3, s_1$, (2.16) is proved. The other two formulas, (2.17) and (2.18), are similar and thus their proves are omitted here. Refer to [13] for more details of such calculations.

**Proof of Lemma 3.1.** Here we give only the proof of $F_3$, since the proofs of the other two are similar and easier.

First, by the equation of $Z'$, (2.15), one has that

$$F_3(x, t) = F_{3,1}(x, t) + \int_{-\infty}^{x} f_{3,2}(y, t)dy + Z'(t) \int_{-\infty}^{x} f_{3,3}(y, t)dy, \quad (4.1)$$

$$= F_{3,1}(x, t) - \int_{x}^{+\infty} f_{3,2}(y, t)dy - Z'(t) \int_{x}^{+\infty} f_{3,3}(y, t)dy. \quad (4.2)$$

**Case 1.** For $x < s_3t$, we decompose $F_3(x, t)$ according to (4.1),

$$F_3 = p(v^3, \theta^3)u^3 - p(v_1, \theta_1)u_1 - k \left( \frac{\partial_x \theta^2}{v^4} - \frac{\partial_x \theta_1}{v_1} \right) - \mu \left( \frac{u^3 \partial_x u^3}{v^4} - \frac{u_1 \partial_x u_1}{v_1} \right)$$

$$+ p(\bar{v}, \bar{\theta})u \tau \ell_2(h_1) - p(\bar{v}, \bar{\theta})u \tau \ell_3(h_3)$$

$$- (s_1 [E]_1 + p(\bar{v}, \bar{\theta})u) \tau \ell_1(h_1) - (s_3 [E]_3 - p(\bar{v}, \bar{\theta})u) \tau \ell_3(h_3) + D_1,$$

where the remaining term $D_1$ is the sum of products of some well-decaying terms, that is

$$D_1 = \left[ p(v_1, \theta_1)u_1 - p(\bar{v}, \bar{\theta})u \right] - k \left( \frac{\partial_x \theta_1}{v_1} - \mu \frac{u_1 \partial_x u_1}{v_1} \right) \tau \ell_1(h_1)$$

$$- \left[ p(v, \theta)u_r - p(\bar{v}, \bar{\theta})u_r \right] - k \left( \frac{\partial_x \theta r}{v_r} - \mu \frac{u_r \partial_x u_r}{v_r} \right) \tau \ell_3(h_3)$$

$$+ \int_{-\infty}^{x} \left[ s_1 (E_1 - \bar{E})_1 - p(v_1, \theta_1)u_1 + p(\bar{v}, \bar{\theta})u \right]$$

$$+ \mu \frac{u_l \partial_x u_l}{v_1} + Z'(E_1 - \bar{E}) \tau \ell_1(h_1)$$

$$- \int_{-\infty}^{x} \left[ s_3 (E_r - \bar{E}) - p(v, \theta)u_r + p(\bar{v}, \bar{\theta})u_r \right]$$

$$+ \frac{u_r \partial_x u_r}{v_r} + Z'(E_r - \bar{E}) \tau \ell_3(h_3) dy,$$

where each square bracket $[\cdots]$ decays exponentially fast with respect to $t$ in the $W^{1, \infty}(\mathbb{R})$ norm, by using Lemma 2.2.

Then it holds that,

$$\sum_{k=0}^{2} \int_{-\infty}^{x} |\partial_x^k D_1(x, \tau)|^2 dx \leq C\varepsilon^2 e^{-4\alpha t} \sum_{k=0}^{1} \int_{-\infty}^{x} \left[ |\partial_x^k \tau \ell_2(h_1)|^2 + |\partial_x^k \tau \ell_3(h_3)|^2 \right] dy$$

$$\leq C\varepsilon^2 e^{-4\alpha t} \left[ C + (s_3 - s_1) t \right],$$

where $C > 0$ is independent of $\varepsilon$ or $t$. And

$$F_3 - D_1 = \left[ p(v^3, \theta^3)u^3 - p(v_1, \theta_1)u_1 - \left( p(v, \theta)u_r - p(\bar{v}, \bar{\theta})u_r \right) \right]$$

$$\left( \frac{u^3 \partial_x u^3}{v^4} - \frac{u_1 \partial_x u_1}{v_1} \right)$$

$$\left( \frac{u_r \partial_x u_r}{v_r} - \frac{u_1 \partial_x u_1}{v_1} \right).$$
\[-\kappa \left( \frac{\partial_x \theta^v}{v^2} - \frac{\partial_x \theta_l}{v_l} - \frac{\partial_x \theta^S_{(\xi, \xi + \sigma)}}{v^S_{(\xi, \xi + \sigma)}} \right) \]
\[-\mu \left( \frac{u^2 \partial_x u^v}{v^2} - \frac{u_l \partial_x u_l}{v_l} - \frac{u^S_{(\xi, \xi + \sigma)} \partial_x u^S_{(\xi, \xi + \sigma)}}{v^S_{(\xi, \xi + \sigma)}} \right) \]
\[+ \left[ p \left( v^S_{(\xi, \xi + \sigma)}, \theta^S_{(\xi, \xi + \sigma)} \right) u^S_{(\xi, \xi + \sigma)} - \tau_1^1 \left( p(v^1_1, \theta^1_1) u^1_1 \right) - \tau_1^3 (\theta^1_1 \theta^S_1) u^S_3 \right] \]
\[-\kappa \left( \frac{\partial_x \theta^S_{(\xi, \xi + \sigma)}}{v^S_{(\xi, \xi + \sigma)}} - \tau_1^1 \left( \frac{\partial_x \theta^S_1}{v^S_1} \right) - \tau_1^3 \left( \frac{\partial_x \theta^S_3}{v^S_3} \right) \right) \]
\[-\mu \left[ \frac{u^S_{(\xi, \xi + \sigma)} \partial_x u^S_{(\xi, \xi + \sigma)}}{v^S_{(\xi, \xi + \sigma)}} - \tau_1^1 \left( \frac{u^S_1 \partial_x u^S_1}{v^S_1} \right) - \tau_1^3 \left( \frac{u^S_3 \partial_x u^S_3}{v^S_3} \right) \right] \]
\[+ \left[ s_1 \left( \tau_2^1 - \tau_2^3 \right) (E^S_1) + s_3 \left( \tau_3^3 \tau_2^3 \right) (E^S_3) \right] \]
\[:= \sum_{i=1}^{7} I_i. \quad (4.3) \]

We calculate the terms one by one.

\[ I_1 = u^x \int_0^1 (\partial_x p) \left( (v^x - v_l, \theta^x) \right) d\rho(v^x - v_l) \]
\[ + u^x \int_0^1 (\partial_\theta p) \left( (v^x, \theta_l + \rho(\theta^x - \theta_l)) \right) d\rho(\theta^x - \theta_l) + \int_0^1 p(v^x, \theta_l) (u^x - u_l) \]
\[ - u^S_{(\xi, \xi + \sigma)} \int_0^1 (\partial_x p) \left( (v^S_{(\xi, \xi + \sigma)} - v_l), \theta^S_{(\xi, \xi + \sigma)} \right) d\rho(v^S_{(\xi, \xi + \sigma)} - v_l) \]
\[ - u^S_{(\xi, \xi + \sigma)} \int_0^1 (\partial_\theta p) \left( (v^S_{(\xi, \xi + \sigma)}, \theta^S_{(\xi, \xi + \sigma)} - \theta_l) \right) d\rho(\theta^S_{(\xi, \xi + \sigma)} - \theta_l) \]
\[ - p(v_l, \theta_l) (u^S_{(\xi, \xi + \sigma)} - u_l). \]

Note that

\[ v^x - v_l = (v^v - v_l) \tau^1_1 (g_1) + (v_r - v_l) \tau^3_3 (g_3) \]
\[ = (v^v - v_l) \tau^1_1 (g_1) + (v_r - v_l) \tau^3_3 (g_3) + I_{1,1}, \]
\[ = v^S_{(\xi, \xi + \sigma)} - v_l + I_{1,1}, \]

\[ \theta^x - \theta_l = (\theta^v - \theta_l) \tau^2_2 (h_1) + (\theta_r - \theta_l) \tau^3_3 (h_3) \]
\[ + \frac{\gamma - 1}{2R} \left[ u^2_r (1 - \tau^2_2 (h_1)) + u^2_r \tau^3_3 (h_3) - (u_l (1 - \tau^1_1 (g_1)) + u_r \tau^3_3 (g_3)) ^2 \right] \]
\[ = (\theta^v - \theta_l) \tau^1_1 (h_1) + (\theta_r - \theta_l) \tau^3_3 (h_3) \]
\[ + \frac{\gamma - 1}{2R} \left[ u^2_r \tau^1_1 (h_1 + g_1 (2 - g_1)) + u^2_r \tau^3_3 (h_3 - g_3) \right] \]
Similarly, get that where it holds that

\[
I = (J_1 - J'_1) \left( v^S_{\xi,\xi+\sigma} - \bar{v}_l \right) + J_1 I_{1,1} + (J_2 - J'_2) \left( \theta^S_{\xi,\xi+\sigma} - \bar{\theta}_l \right) + J_2 I_{1,2} + p(v_l, \theta_l) I_{1,3} + \left[ p(v_l, \theta_l) - p(\bar{v}_l, \bar{\theta}_l) \right] \left( u^S_{\xi,\xi+\sigma} - \bar{u}_l \right).
\]

Since \( J_1 \sim J'_1, J_2 \sim J'_2 \) and \( p(v_l, \theta_l) \sim p(\bar{v}_l, \bar{\theta}_l) \), one has that

\[
\sum_{k=0}^{2} \int_{-\infty}^{s_{lt}} |\delta^k_x I_1|^2 \, dx \leq C \varepsilon^2 e^{-2\alpha t}.
\]

For \( I_2 \), it holds that

\[
-\kappa^{-1} I_2 = \delta_x \left( \theta^S - \theta_l \right) \left( \frac{1}{v^2} - \frac{1}{v_l} \right) - \delta_x \left( \theta^S_{\xi,\xi+\sigma} - \bar{\theta}_l \right)
\]

\[
= \delta_x I_{1,2} \left( \frac{1}{v^2} - \frac{1}{v_l} \right) + \delta_x \theta_l \left( \frac{1}{v^2} - \frac{1}{v_l} \right) + \left( \frac{1}{v^2} - \frac{1}{v_{\xi,\xi+\sigma}} \right) \delta_x \left( \theta^S_{\xi,\xi+\sigma} - \bar{\theta}_l \right),
\]

where \( I_{1,2} \) is given by (4.5). Using Lemma 2.2 and the fact that \( v^2 \sim v^S_{\xi,\xi+\sigma} \), one can get that

\[
\sum_{k=0}^{2} \int_{-\infty}^{s_{lt}} |\delta^k_x I_2|^2 \, dx \leq C \varepsilon^2 e^{-2\alpha t}.
\]

Similarly, \( I_3 \) satisfies that

\[
-2 \mu^{-1} I_3 = \frac{1}{v^2} \delta_x \left[ \left( u^2 \right)^2 - u^2_l + \left( u^S_{\xi,\xi+\sigma} \right)^2 - \bar{u}^2_l \right] + \delta_x u^2 \left( \frac{1}{v^2} - \frac{1}{v_l} \right) + \delta_x \left( u^S_{\xi,\xi+\sigma} \right) \left( \frac{1}{v^2} - \frac{1}{v_{\xi,\xi+\sigma}} \right).
\]

It follows from (4.6) that

\[
(u^2 - u_l^2 - \left( u^S_{\xi,\xi+\sigma} \right)^2 + \bar{u}^2_l) = I_{1,3}(u^2 + u_l) + \left( u^S_{\xi,\xi+\sigma} - \bar{u}_l \right) \left( u^2 - u^S_{\xi,\xi+\sigma} + u_l - \bar{u}_l \right).
\]

Thus, it holds that

\[
\sum_{k=0}^{2} \int_{-\infty}^{s_{lt}} |\delta^k_x I_3|^2 \, dx \leq C \varepsilon^2 e^{-2\alpha t}.
\]

For \( I_4 \), it holds that

\[
I_4 = \left[ p \left( v^S_{\xi,\xi+\sigma}, \theta^S_{\xi,\xi+\sigma} \right) - \tau^1 \xi \left( p(v^1, \theta^1) \right) \right] \left[ \tau^1 \xi \left( u^S_{\xi,\xi+\sigma} \right) \right]
\]

\( I_{4,1} \).
\[ + \left[ p \left( v^S_{(\xi,\xi+\sigma)}, \theta^S_{(\xi,\xi+\sigma)} \right) - \tau^3_{\xi+\sigma} \left( p(v^S_3, \theta^S_3) \right) \right] \tau^3_{\xi+\sigma} (u^S_3), \]

where

\[ I_{4,1} = \int_0^1 (\partial_x \rho) \left( (1 - \rho) \tau^1_\xi (v^S_1) + \rho v^S_{(\xi,\xi+\sigma)}, \theta^S_{(\xi,\xi+\sigma)} \right) d\rho \left( v^S_{(\xi,\xi+\sigma)} - \tau^1_\xi (v^S_1) \right) \]

\[ + \int_0^1 (\partial_\theta \rho) \left( \tau^3_\xi (v^S_1), (1 - \rho) \tau^1_\xi (\theta^S_1) + \rho \theta^S_{(\xi,\xi+\sigma)} \right) d\rho \left( \theta^S_{(\xi,\xi+\sigma)} - \tau^1_\xi (\theta^S_1) \right) \]

\[ = \int_0^1 (\partial_x \rho) (\cdots) d\rho \left( \tau^3_{\xi+\sigma} (v^S_3) - \varpi_m \right) \]

\[ + \int_0^1 (\partial_\theta \rho) (\cdots) d\rho \left[ \tau^3_{\xi+\sigma} (\theta^S_3) - \vartheta_m - \frac{\gamma - 1}{R} \tau^1_\xi (u^S_1) \tau^3_{\xi+\sigma} (u^S_3) \right]. \]

Similarly,

\[ I_{4,2} = \int_0^1 (\partial_x \rho) (\cdots) d\rho \left( \tau^1_\xi (v^S_1) - \varpi_m \right) \]

\[ + \int_0^1 (\partial_\theta \rho) (\cdots) d\rho \left[ \tau^1_\xi (\theta^S_1) - \vartheta_m - \frac{\gamma - 1}{R} \tau^1_\xi (u^S_1) \tau^3_{\xi+\sigma} (u^S_3) \right]. \]

It follows from (2.3) that

\[ \sum_{k=0}^2 \int_{\mathbb{R}} \left| \partial_x^k I_4 \right|^2 \, dx \leq C\delta^4 \int_{\mathbb{R}} e^{-2\delta \xi - 2\delta |x|} \, dx \leq C\delta^4 e^{-2\delta \lambda}. \quad (4.10) \]

And \( I_5 \) and \( I_6 \) satisfy that

\[ -\kappa^{-1} I_5 = -\frac{1}{v^S_{(\xi,\xi+\sigma)}} \left[ \tau^1_\xi \left( \frac{\partial_x \theta^S_1}{v^S_1} \right) \left( \tau^3_{\xi+\sigma} (v^S_3) - \varpi_m \right) \right] \]

\[ + \tau^3_{\xi+\sigma} \left[ \frac{\partial_x \theta^S_3}{v^S_3} \right] \left( \tau^1_\xi (v^S_1) - \varpi_m \right) + \frac{\gamma - 1}{R} \partial_x \left( \tau^1_\xi (u^S_1) \tau^3_{\xi+\sigma} (u^S_3) \right) \right]; \]

\[ -2\mu^{-1} I_6 = -\frac{1}{v^S_{(\xi,\xi+\sigma)}} \left[ \tau^1_\xi \left( \frac{\partial_x (u^S_1)^2}{v^S_1} \right) \left( \tau^3_{\xi+\sigma} (v^S_3) - \varpi_m \right) \right] \]

\[ + \tau^3_{\xi+\sigma} \left[ \frac{\partial_x (u^S_3)^2}{v^S_3} \right] \left( \tau^1_\xi (v^S_1) - \varpi_m \right) - 2\partial_x \left( \tau^1_\xi (u^S_1) \tau^3_{\xi+\sigma} (u^S_3) \right) \right]. \]

Then similar to (4.10), one can get that

\[ \sum_{k=0}^2 \int_{\mathbb{R}} \left( \left| \partial_x^k I_5 \right|^2 + \left| \partial_x^k I_6 \right|^2 \right) \, dx \leq C\delta^3 e^{-2\delta \lambda}. \quad (4.11) \]

And by Lemma 2.3, it is easy to prove that

\[ \sum_{k=0}^2 \int_{\mathbb{R}} \left| \partial_x^k I_4 \right|^2 \, dx \leq C\varepsilon^2 e^{-4\alpha t}. \quad (4.12) \]
Hence, it holds that
\[
\sum_{k=0}^{2} \int_{-\infty}^{s_3 t} |e_k F_3|^2 \, dx \leq C \varepsilon^2 e^{-2\alpha t} + C \delta^3 e^{-2\alpha t}. \tag{4.13}
\]

Case 2. If \( x > s_3 t \), we decompose \( F_3 \) according to (4.2).

\[
F_3 = p(v^\ast, \theta^\ast) u^t - p(v_r, \theta_r) u_r - \kappa \left( \frac{c_x \theta^\ast}{v^\ast - c_x \theta_r}{v_r} \right) - \mu \left( \frac{u^\ast c_x u^t}{v^t} - \frac{u_r c_x u_r}{v_r} \right) - p(\bar{v}_l, \bar{\theta}_l) [1 - \tau^1 \varepsilon(h_1)] + p(\bar{v}_r, \bar{\theta}_r) [1 - \tau^2 \varepsilon(h_3)] + (s_1 [E]_1 + p(\bar{v}_l, \bar{\theta}_l) [1 - \tau^1 \varepsilon(h_1)] + (s_3 [E]_3 - p(\bar{v}_r, \bar{\theta}_r)) [1 - \tau^3 \varepsilon(h_3)] + D_2,
\]

\[
= p(v^\ast, \theta^\ast) u^t - p(v_r, \theta_r) - p(v^S_{(\xi, \xi+\sigma)}, \theta^S_{(\xi, \xi+\sigma)}) u^S_{(\xi, \xi+\sigma)} + p(\bar{v}_r, \bar{\theta}_r) [1 - \tau^1 \varepsilon(h_1)] + p(\bar{v}_r, \bar{\theta}_r) [1 - \tau^3 \varepsilon(h_3)] + D_2,
\]

where \( I_4 \) to \( I_7 \) are the terms defined in (4.3) and the remaining term

\[
D_2 = - \left[ p(v_l, \theta_l) u_l - p(\bar{v}_l, \bar{\theta}_l) \bar{u}_l - \kappa \frac{c_x \theta_l}{v_l} - \mu \frac{u_l c_x u_l}{v_l} \right] (1 - \tau^1 \varepsilon(h_1))
\]

\[
+ \left[ p(v_r, \theta_r) u_r - p(\bar{v}_r, \bar{\theta}_r) \bar{u}_r - \kappa \frac{c_x \theta_r}{v_r} - \mu \frac{u_r c_x u_r}{v_r} \right] (1 - \tau^3 \varepsilon(h_3))
\]

\[
- \int_{x}^{+\infty} \left[ s_1 (E_l - \bar{E}_l) - p(v_l, \theta_l) u_l + p(\bar{v}_l, \bar{\theta}_l) \bar{u}_l + \kappa \frac{c_x \theta_l}{v_l} \right]
\]

\[
+ \mu \frac{u_l c_x u_l}{v_l} + Z' (E_l - \bar{E}_m) \tau^1 \varepsilon(h'_1) \, dy
\]

\[
+ \int_{x}^{+\infty} \left[ s_3 (E_r - \bar{E}_r) - p(v_r, \theta_r) u_r + p(\bar{v}_r, \bar{\theta}_r) \bar{u}_r + \kappa \frac{c_x \theta_r}{v_r} \right]
\]

\[
+ \mu \frac{u_r c_x u_r}{v_r} + Z' (E_r - \bar{E}_m) \tau^3 \varepsilon(h'_3) \, dy
\]

where each square bracket \([ \cdots ]\) decays exponentially fast with respect to \( t \) in the \( W^{1,\infty}(\mathbb{R}) \) norm. Thus \( D_2 \) satisfies

\[
\sum_{k=0}^{2} \int_{x}^{+\infty} |e_k D_2(x, t)|^2 \, dx \leq C \varepsilon^2 e^{-4\alpha t} + \sum_{k=0}^{2} \int_{x}^{+\infty} \left( |e_k [1 - \tau^1 \varepsilon(h_1)]|^2 + |e_k [1 - \tau^3 \varepsilon(h_3)]|^2 \right) dx
\]

\[
\leq C \varepsilon^2 e^{-4\alpha t}, \quad t > 0,
\]
where \( C > 0 \) is independent of \( \varepsilon \) or \( t \). And similar to proof of \( I_1 \) to \( I_3 \) in Case 1, one can get that
\[
\sum_{k=0}^{2} \sum_{i=1}^{3} \int_{s_{s,t}^3}^{s_{s,t}^4} \left| \partial_x^k f_i \right|^2 \, dx \leq C \varepsilon^2 e^{-4\alpha t} \sum_{k=0}^{2} \int_{s_{s,t}^3}^{s_{s,t}^4} \left( \left| \partial_x^k [1 - \tau_{1,0}(h_1)] \right|^2 + \left| \partial_x^k [1 - \tau_{3,0}(h_3)] \right|^2 \right) \, dx
\]
\[
\leq C \varepsilon^2 e^{-4\alpha t},
\]
And thus,
\[
\sum_{k=0}^{2} \int_{s_{s,t}^3}^{s_{s,t}^4} \left| \partial_x^k F_3 \right|^2 \, dx \leq C \varepsilon^2 e^{-4\alpha t} + C \delta^3 e^{-2\epsilon t}. \tag{4.14}
\]
Collecting two cases above, one can get that
\[
\| F_3 \|_2^2 = \sum_{k=0}^{2} \left[ \int_{-\infty}^{s_{s,t}^3} \left| \partial_x^k F_3 \right|^2 \, dx + \int_{s_{s,t}^3}^{s_{s,t}^4} \left| \partial_x^k F_3 \right|^2 \, dx \right] \leq C \varepsilon^2 e^{-2\alpha t} + C \delta^3 e^{-2\epsilon t}.
\]

\[
\square
\]

5. Appendix

Proof of Lemma 2.2. For convenience, by using \( u - \overline{u} \) to substitute \( u \), one can assume that \( \overline{u} = 0 \).

Denote the perturbation terms
\[
\phi(x, t) = u(x, t) - \overline{u}, \quad \psi(x, t) = u(x, t) - \overline{u} = u(x, t),
\]
\[
w(x, t) = E(x, t) - \overline{E}, \quad \zeta(x, t) = \theta(x, t) - \overline{\theta},
\]
which satisfies
\[
\partial_t \phi - \partial_x \psi = 0,
\]
\[
\partial_t \psi + \partial_x \left( \frac{R \zeta}{v} \right) + \partial_x \left( \frac{R \theta}{v} \right) = \mu \partial_x \left( \frac{\partial_x \psi}{v} \right), \tag{5.1}
\]
\[
\frac{R}{\gamma - 1} \partial_t \zeta + p \partial_x \psi = \kappa \partial_x \left( \frac{\partial_x \zeta}{v} \right) + \frac{\mu}{v} (\partial_x \psi)^2.
\]
Assume that \( k \geq 2 \) and
\[
\delta = \sup_{t \in [0, T]} \| \phi, \psi, \zeta \|_{H^k(0, \pi)} (t) > 0 \tag{5.2}
\]
is small enough.

Multiplying \( -R \overline{\theta} \left( \frac{1}{\gamma} - \frac{1}{\gamma} \right) \) on (5.1)1, \( \psi \) on (5.1)2, and \( -\overline{\theta} \left( \frac{1}{\gamma} - \frac{1}{\gamma} \right) = \frac{\zeta}{\theta} \) on (5.1)3, respectively, and summing the results together yield that
\[
\partial_t \left[ \frac{1}{2} \psi^2 + R \overline{\theta} \Phi \left( \frac{\psi}{\overline{\theta}} \right) + \frac{R}{\gamma - 1} \overline{\theta} \Phi \left( \frac{\theta}{\overline{\theta}} \right) \right] + \frac{\mu}{v} (\partial_x \psi)^2 + \frac{\kappa}{v \theta} (\partial_x \zeta)^2
\]
\[
= \partial_x \left[ \frac{\mu}{v} \psi \partial_x \psi + \frac{\kappa}{v \theta} \zeta \partial_x \zeta - R \left( \frac{\theta}{v} - \frac{\overline{\theta}}{v} \right) \psi \right] + \frac{\kappa}{v \theta^2} \zeta (\partial_x \zeta)^2 + \frac{\mu}{v \theta} (\partial_x \psi)^2,
\]
where \( \Phi(s) = s - \ln s - 1 \). It follows from (5.2) that
\[
\frac{d}{dt} \int_{0}^{\pi} \left[ \frac{1}{2} \psi^2 + R \overline{\theta} \Phi \left( \frac{\psi}{\overline{\theta}} \right) + \frac{R}{\gamma - 1} \overline{\theta} \Phi \left( \frac{\theta}{\overline{\theta}} \right) \right] \, dx + 2c_1 \| \partial_x \psi, \partial_x \zeta \|^2 \leq 0. \tag{5.3}
\]
for some constant $c_1 > 0$, independent of $t$. By the conservative forms of (1.1), one has that
\[
\int_0^\pi (\phi, \psi, w) (x, t) \equiv 0, \quad t \geq 0.
\] (5.4)
Then the Poincaré inequality yields that
\[
\|\phi\| \leq a \|\partial_x \phi\|, \quad \|\psi\| \leq a \|\partial_x \psi\|, \quad \|w\| \leq a \|\partial_x w\|,
\] (5.5)
for some constant $a > 0$. This, together with $\zeta = \frac{\gamma - 1}{R} (w - \frac{1}{2} \psi^2)$, yields that
\[
\|\partial_x \zeta\|^2 \geq a^{-2} \|\zeta\|^2 - \frac{(\gamma - 1)^2}{4R^2a^2} \|\psi\|^2_{L^\infty} \|\zeta\|^2 - \frac{(\gamma - 1)^2}{R^2} \|\partial_x \psi\|^2
\geq a^{-2} \|\zeta\|^2 - \frac{(\gamma - 1)^2}{4R^2a^2} \|\psi\|^2 - \frac{(\gamma - 1)^2}{R^2} \|\partial_x \psi\|^2.
\] (5.6)
Thus, by (5.3), (5.5) and (5.6), if $\delta > 0$ is small enough, one has that
\[
\frac{d}{dt} \int_0^\pi \left[ \frac{1}{2} \psi^2 + R\overline{\Phi} \left( \frac{v}{\theta} \right) + \frac{R}{\gamma - 1} \overline{\Phi} \left( \frac{\theta}{\theta} \right) \right] dx + c_1 \|\partial_x \psi, \partial_x \zeta\|^2 + c_2 \|\psi, \zeta\|^2 \leq 0,
\] (5.7)
By using $(5.1)_1$, $(5.1)_2$ is equivalent to
\[
\partial_t \psi + \frac{R}{v} \partial_x \zeta = \partial_t \left( \frac{\mu}{v} \partial_x \phi \right) + \frac{R\theta}{v^2} \partial_x \phi.
\]
Then multiplying this equation by $\frac{\partial_x \phi}{v}$ and using $(5.1)_1$ again, one has that
\[
\frac{d}{dt} \int_0^\pi \left[ \frac{1}{2} \psi^2 + R\overline{\Phi} \left( \frac{v}{\theta} \right) + \frac{R}{\gamma - 1} \overline{\Phi} \left( \frac{\theta}{\theta} \right) \right] dx + c_3 \|\partial_x \phi\|^2 \leq C \|\partial_x \psi, \partial_x \zeta\|^2.
\] (5.8)
Collecting (5.7) and (5.8), and using (5.5), one has that
\[
\frac{d}{dt} \int_0^\pi \left\{ M_1 \left[ \frac{1}{2} \psi^2 + R\overline{\Phi} \left( \frac{v}{\theta} \right) + \frac{R}{\gamma - 1} \overline{\Phi} \left( \frac{\theta}{\theta} \right) \right] + \frac{\mu}{2v^2} (\partial_x \phi)^2 - \frac{\psi}{v} \partial_x \phi \right\} dx
+ c_4 \|\partial_x \phi, \partial_x \psi, \partial_x \zeta\|^2 + c_5 \|\phi, \psi, \zeta\|^2 \leq 0.
\] (5.9)
If $M_1 > 0$ is large enough, the terms in $\{ \cdots \}$ in (5.9) satisfy that
\[
c_5^{-1} \|\phi, \psi, \zeta, \partial_x \phi\|^2 \leq \int_0^\pi \{ \cdots \} dx \leq c_5 \|\phi, \psi, \zeta, \partial_x \phi\|^2,
\] (5.10)
which implies that
\[
\|\phi, \psi, \zeta, \partial_x \phi\|^2 (t) \leq \left( \|\phi_0\|^2 + \|\psi_0\|^2 \right) e^{-\frac{c_5 t}{2a}}.
\] (5.11)
Since the estimates for the higher order derivatives are standard and their exponential decay rates can be proved in the same way with aid of Poincaré inequality, thus we omit the details.
\[\square\]
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