Rational-Valued, Small-Prime-Based Qubit-Qutrit and Rebit-Retrit Rank-4/Rank-6 Conjectured Hilbert-Schmidt Separability Probability Ratios

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Abstract

We implement a procedure—based on the Wishart-Laguerre distribution—recently outlined by both K. Życzkowski and the group of A. Khvedelidze, I. Rogojin and V. Abgaryan, for the generation of random (complex or real) $N \times N$ density matrices of rank $k \leq N$ with respect to Hilbert-Schmidt (HS) measure. In the complex case, one commences with a Ginibre matrix (of normal variates) $A$ of dimensions $k \times k + 2(N - k)$, while for a real scenario, one employs a Ginibre matrix $B$ of dimensions $k \times k + 1 + 2(N - k)$. Then, the $k \times k$ product $AA^\dagger$ or $BB^T$ is diagonalized—padded with zeros to size $N \times N$—and rotated by a random unitary or orthogonal matrix to obtain a random density matrix with respect to HS measure. Implementing the indicated procedure for rank-4 rebit-retrit states, for 800 million Ginibre-matrix realizations, 6,192,047 were found separable, for a sample probability of $.00774006$—suggestive of an exact value of $\frac{387}{5000} = \frac{3^2 \cdot 43}{2^5 \cdot 5^2} = .0774$. A prior conjecture for the HS separability probability of rebit-retrit systems of full rank is $\frac{860}{6561} = \frac{2^2 \cdot 5 \cdot 43}{3^8} \approx 0.1310775$ (while the two-rebit counterpart has been proven to be $\frac{29}{64} = \frac{2^3}{2^6}$, and the two-qubit one, very strongly indicated to be $\frac{8}{33} = \frac{2^3}{3^3 \cdot 11}$). Subject to these two conjectures, the ratio of the rank-4 to rank-6 probabilities would be $\frac{59049}{1000000} = \frac{3^{10}}{2^8 \cdot 5^6} \approx 0.059049$, with the common factor 43 cancelling. As to the intermediate rank-5 probability, application of a 2006 theorem of Szarek, Bengtsson and Życzkowski informs us that it must be one-half the rank-6 probability—its conjectured to be $\frac{27}{1000} = \frac{3^3}{2^3 \cdot 5^3}$, while for rank 3 or less, the associated probabilities must be 0 by a 2009 result of Ruskai and Werner. We are led to re-examine a 2005 qubit-qutrit analysis of ours, in these regards, and now find evidence for a $\frac{70}{2673} = \frac{2 \cdot 5 \cdot 7}{3^3 \cdot 11} \approx 0.0261878$ rank-4 to rank-6 probability ratio.

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I. INTRODUCTION

Pursuit of the problem “of quantum separability or inseparability from a measurement theoretical point of view” posed in 1998 by Žyczkowski, Horodecki, Sanpera and Lewenstein [1] has generated a considerable literature [2–15].

Of particular interest is the finding—motivated by the results reported in [5]—that the Hilbert-Schmidt PPT (positive-partial-transpose) probability of the generic class of $N \times N$ density matrices of rank $N - 1$ is one-half the probability of the density matrices of full rank ($N$) [7]. (For $N = 4, 6$, PPT-probability is equivalent to separability probability.) However, the interesting line of geometric reasoning (Archimedes’ formula,...) applied in [7] does not seem extendable to density matrices of rank $k = N - n$, for $n > 1$, so further investigative approaches seem necessary in such regards.

In [13], questions of this (reduced rank) nature were posed. However, the Hilbert-Schmidt separability probability of 0.1652 reported for the rank-three two-qubit states seemed inconsistent with the indicated analysis of Szarek, Bengtsson and Žyzckowski [7], since the evidence (both numerical and analytical)—though yet short of a formalized proof—is highly compelling that the Hilbert-Schmidt separability probability of full-rank (4) two-qubit states is $\frac{8}{33} \approx 0.242424$ [10, 11, 14].

We now implement a procedure—based on the Wishart-Laguerre distribution [16, 17]—recently outlined in email communications by both K. Žyczkowski and the group of A. Khvedelidze, I. Rogojin and V. Abgaryan for the generation of random (complex or real) $N \times N$ density matrices of rank $k$ with respect to Hilbert-Schmidt measure. In the complex case, one commences with a Ginibre matrix (of normal variates) $A$ of dimensions $k \times k + 2(N - k)$, while for a real scenario, one employs a Ginibre matrix $B$ of dimensions $k \times k + 1 + 2(N - k)$. Then, the $k \times k$ product $AA^\dagger$ or $BB^T$ is diagonalized, padded with zeros to size $N \times N$, and then rotated by a random unitary or orthogonal matrix to obtain, as desired, a random density matrix with respect to Hilbert-Schmidt measure.

In [15], conjectures of Hilbert-Schmidt separability probabilities of $\frac{860}{6561} = \frac{2^2 \cdot 5 \cdot 43}{3^8} \approx 0.1310775$ and $\frac{27}{1000} = \frac{3^3}{2^2 \cdot 5^3} \approx 0.027$ were advanced—based on 1,850,000,000 and 2,415,000,000 iterations—for generic rebit-retrit and qubit-qutrit systems, respectively. (In [13] Tab. 1] a qubit-qutrit probability estimate of 0.0270 was reported.) Additionally, in the 2005 study [5], the rank-4 qubit-qutrit Hilbert-Schmidt separability probability was reported to be close
FIG. 1: Estimates of rank-4 rebit-retrit Hilbert-Schmidt PPT/separability probability and conjectured value of \( \frac{387}{50000} = \frac{3^2 \cdot 43}{2^5 \cdot 5^3} = 0.00774 \).

\((\frac{1}{33.9982})\) to \(\frac{1}{54}\) as large as the full-rank probability, presently conjectured to be \(\frac{27}{1000}\). (The rank-4 two-rebit HS separability probability has been proven by Lovas and Andai to equal \(\frac{29}{64} = \frac{29}{2^6}\).)

II. ANALYSES

A. Rebit-retrit analysis

Implementing the indicated procedure for rank-4 rebit-retrit states, for 800 million Ginibre-matrix realizations, 6,192,047 were found separable for a sample probability of .00774006–suggestive of an exact value of \(\frac{387}{50000} = \frac{3^2 \cdot 43}{2^5 \cdot 5^3} = 0.00774\) (Fig. 1). Subject to such a conjecture and the indicated \(\frac{860}{6561} = \frac{2^5 \cdot 5 \cdot 43}{3^8}\) full-rank one, the ratio of the rank-4 to rank-6 probabilities would be \(\frac{59049}{10000000} = \frac{3^{10}}{2^{26} \cdot 5^6} = (\frac{243}{1000})^2 \approx 0.059049\), with the common factor 43 interestingly cancelling. For ranks of three and less, the 2009 theorem of Ruskai and Werner \([18]\) informs us that the associated separability probabilities are zero.

B. Qubit-qutrit analysis

However, this new–to us, intriguing–rebit-retrit conjecture of \(\frac{59049}{10000000} = \frac{3^{10}}{2^{26} \cdot 5^6}\), seemed somewhat different (perhaps more “elegant”) in nature than–at this point in time–its
FIG. 2: Estimates of rank-4 qubit-qutrit Hilbert-Schmidt PPT/separability probability and conjectured value of \( \frac{7}{9900} = \frac{7}{2^2 \cdot 3^2 \cdot 5^2 \cdot 11} = 0.000707071 \).

apparent qubit-qutrit counterpart, which would involve dividing \( \frac{27}{1000} \) by \( \frac{1}{34} \), yielding \( \frac{27}{34000} = \frac{3^3}{2^4 \cdot 5^3 \cdot 17} \approx 0.000794118 \). Since the “34” stemmed from an estimate of 33.9982 reported in our “long ago” 2005 study \([5]\)–relying upon quasi-Monte Carlo (Tezuka-Faure) numerical integration–we decided to re-examine it employing the new, detailed-above Wishart-Laguerre-based methodology of K. Žyczkowski and the group of A. Khvedelidze, I. Rogojin and V. Abgaryan.

Then, employing for hundred million \( 4 \times 8 \) complex-entry Ginibre matrices, we obtained an estimate of 0.000707020, of similar magnitude, but still markedly different from the indicated 0.000794118 (Fig. 2). (In the 2005 study, contrastingly, an Euler-angle parameterization of unitary matrices was employed. But it is not now quite clear there, in what manner the parameterization was adopted to the rank-4 analysis.) This result is suggestive of an exact value of \( \frac{7}{9900} = \frac{7}{2^2 \cdot 3^2 \cdot 5^2 \cdot 11} = 0.000707071 \). Subject to this conjecture and the indicated \( \frac{27}{1000} \) full-rank one, the ratio of the rank-4 to rank-6 probabilities would be \( \frac{70}{2673} = \frac{2 \cdot 5 \cdot 7}{3^5 \cdot 11} \approx 0.0261878 \).

C. Qubit-ququart analysis

In \([15]\), a Hilbert-Schmidt PPT-probability conjecture of \( \frac{16}{12375} = \frac{2^4}{3^2 \cdot 5^2 \cdot 11} \approx 0.0012929 \) was advanced for \( 2 \times 4 \) qubit-ququart systems. In a further analysis of ours, for rank-6 such systems, based on 149 million Ginibre-matrix realizations, we obtained a PPT-probability
estimate of 0.0000546242. Though we plan to extend this analysis, a tentative conjecture for
this last value is $\frac{169}{3093750} = \frac{13^2}{2^2 \cdot 3^2 \cdot 5^6 \cdot 11} \approx 0.0000546263$ with the rank-6/rank-8 ratio, then, being
$\frac{169}{4000} = \frac{13^2}{2^3 \cdot 3} \approx 0.04225$.

III. CONCLUDING REMARKS

In the course of the research reported above, we have, in particular, sought rational-valued Hilbert-Schmidt PPT/separability rebit-retrit and qubit-qutrit probability formulas. Certainly, we have no demonstration that this must, in fact, be the case. But in light of the proven nature of the two-rebit probability $\left(\frac{29}{64}\right)$ [12], and the strong evidence for the two-qubit $\left(\frac{8}{33}\right)$, two-quat[neronic]bit $\left(\frac{26}{323}\right)$,... counterparts [14], this seems a direction worth pursuing–especially in light of the elegant nature of the formulas so far found (not to mention also the “half-theorem” of Szarek, Bengtsson and Žyczkowski [7]). Also, in terms of the Hilbert-Schmidt measure, the two-qubit separability probability is equally divided between those states for which $|\rho| > |\rho^{PT}|$ and those for which $|\rho^{PT}| > |\rho|$ [19].

Needless to say, it would seem, the intrinsic high-dimensionality (twenty and thirty-five) of the problems under consideration above, and related ones, presents formidable challenges to exact, symbolic analyses, as contrasted with the numerical approach adopted here.

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