The length of a shortest geodesic loop

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Abstract

We give a lower bound for the length of a non-trivial geodesic loop on a simply-connected and compact manifold of even dimension with a non-reversible Finsler metric of positive flag curvature. Harris and Paternain use this estimate in their recent paper [HP] to give a geometric characterization of dynamically convex Finsler metrics on the 2-sphere.

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On compact and simply-connected Riemannian manifold with positive sectional curvature $0 < K \leq 1$ the length of a non-constant geodesic loop is bounded from below by $2\pi$. This result is due to Klingenberg [Kl] and is of importance in proofs of the classical sphere theorem.

For a compact manifold $M$ with non-reversible Finsler metric $F$ the author introduced in [R1] the reversibility $\lambda := \max\{F(-X); F(X) = 1\} \geq 1$. In this short note we show how one can use the results and methods from [R1] to obtain the following estimate for the length of a geodesic loop depending on the flag curvature and the reversibility.

Proposition 1 Let $M$ be a compact and simply-connected differentiable manifold of even dimension $n \geq 2$ equipped with a non-reversible Finsler metric $F$ and flag curvature $K$ satisfying $0 < K \leq 1$. Then the length $l$ of a shortest non-constant geodesic loop is bounded from below: $l \geq \pi \left( 1 + \lambda^{-1} \right)$. In addition the injectivity radius satisfies: $\text{inj} \geq \pi / \lambda$.

In [R1] Theorem 4] it is shown that with the same assumptions the length of a closed geodesic $c$ satisfies this estimate. Therefore Proposition 1 follows from Proposition 3 which we are going to prove in this note. Proposition 1 answers a question posed to the author by G. Paternain. Using results by Hofer, Wysocki and Zehnder [HWZ] and the statement of Proposition 1 Harris and Paternain obtain the following geometric characterization of dynamically convex Finsler metrics on the 2-sphere:
Proposition 2 (Harris-Paternain [HP, Theorem B, Section 6])

Let $F$ be a non-reversible Finsler metric on the 2-sphere with reversibility $\lambda$ and flag curvature
$$\left(1 - \frac{1}{1 + \lambda}\right)^2 < K \leq 1.$$ Then the Finsler metric is dynamically convex, in particular there are either two geometrically distinct closed geodesics or there are infinitely many geometrically distinct ones.

For existence results for closed geodesic of Finsler metrics on the 2-sphere we refer to the recent survey [Lo] by Long and to [R3]. On the $n$-sphere $S^n$ there is a 1-parameter family $F_\epsilon$, $\epsilon \in [0,1)$ of Finsler metrics (called Katok metrics) with the following properties: $F_0$ is the standard Riemannian metric, for every $\epsilon \in (0,1)$ the metric is a non-reversible Finsler metric of constant flag curvature $1$, the reversibility is $\lambda = (1 + \epsilon)/(1 - \epsilon)$ and the shortest geodesic loop is a closed geodesic of length $\pi(1 + \lambda - 1)$. This shows that the estimate given in Proposition [1] is sharp. In addition the number of closed geodesics for $n = 2$ is two if $\epsilon$ is irrational. It is an open problem whether there is a non-reversible Finsler metric on $S^2$ with a finite number $N > 2$ of geometrically distinct closed geodesics.

We use the following notations on a compact manifold $M$ with Finsler metric $F$ introduced in [R1]: For points $p, q \in M$ let $\theta(p,q)$ be the minimal length of a piecewise differentiable curve $c : [0,1] \to M$ joining $p = c(0)$ and $q = c(1)$. For a non-reversible Finsler metric we have in general $\theta(p,q) \neq \theta(q,p)$, i.e. $\theta$ defines in general a non-symmetric metric on $M$. Therefore $\theta(p,q)$ equals the length of a minimal geodesic $c : [0,1] \to M$ joining $p = c(0)$ and $q = c(1)$, i.e. a geodesic with $L(c) = \theta(c(0),c(1))$. Then $d : M \times M \to \mathbb{R}$; $d(p,q) = (\theta(p,q) + \theta(q,p))/2$ defines a symmetric metric on $M$. For a point $p \in M$ and an unit vector $v \in T_pM$; $F(v) = 1$ let $c_v : \mathbb{R} \to M$ be the geodesic with $p = c(0); v = c'(0)$ and define $t_v := \sup\{t > 0; \theta(c(0), c(t)) = t\}$. Then $c_v(t_v)$ is a cut point of the point $p$ and the cut locus $\text{Cut}(p)$ of the point $p$ is given by $\text{Cut}(p) = \{c_v(t_v); v \in T_pM, F(v) = 1\}$.

We consider the following invariants and their relations: The \textit{symmetrized injectivity radius} $d := \inf\{d(p,q); q \in \text{Cut}(p)\}$, the length $L$ of a shortest (nontrivial) closed geodesic and the length $l$ of a shortest (nontrivial) geodesic loop.

Lemma 1 Let $(M, F)$ be a compact Finsler manifold, then the symmetrized injectivity radius $d$ and the length $l$ resp. $L$ of a shortest non-trivial geodesic loop resp. closed geodesic satisfy: $2d \leq l \leq L$.

Proof. Let $c : [0,l] \to M$ be a shortest geodesic loop parametrized by arc length with $p = c(0) = c(l)$. Let $q = c(t), t \in (0,l)$ be the cut point, i.e.
$c|[0, t]$ is minimal. It follows that $l = L(c) \geq \theta(p, q) + \theta(q, p) = 2d(p, q) \geq 2d$. The inequality $L \geq l$ is obvious.

The next ingredient in the Proof of Proposition 3 is the following result:

**Lemma 2** [R1, Lemma 7] Let $(M, F)$ be a compact Finsler manifold with reversibility $\lambda$, symmetrized injectivity radius $d$ and flag curvature $K \leq 1$. If $2d < \pi(1 + \lambda^{-1})$ then the length $l$ of a shortest non-trivial geodesic loop satisfies: $l = 2d$.

With the help of these two Lemmata we prove the following

**Proposition 3** Let $(M, F)$ be a compact manifold with Finsler metric $F$ with reversibility $\lambda$ and flag curvature $K \leq 1$. If the symmetrized injectivity radius $d$ satisfies $2d < \pi(1 + \lambda^{-1})$ then every shortest geodesic loop is a closed geodesic, hence $L = l = 2d$.

**Proof.** Let $c : [0, l] \to M$ be a shortest geodesic loop parametrized by arc length with $c(0) = c(l) = p$. Let $q = c(t), t \in (0, l)$ be the cut point, i.e. $c|[0, t]$ is minimal. We assume that $l < \pi(1 + \lambda^{-1})$. By Lemma 1 we obtain $2d < \pi(1 + \lambda^{-1})$. Then we conclude from Lemma 2 that $l = L(c) = 2d$. Since

$$2d = l = L(c) \geq \theta(p, q) + \theta(q, p) = 2d(p, q) \quad (1)$$

and $q \in \text{Cut}(p)$ it follows from the definition of the symmetrized injectivity radius $d$ that equality holds in Inequality (1). Therefore $c|[t, l]$ is a minimal geodesic joining $q$ and $p$. For sufficiently small $\epsilon > 0$ with $p_\epsilon = c(\epsilon)$ there is $t_\epsilon \in (t, 1)$ such that $q_\epsilon = c(t_\epsilon) \in \text{Cut}(p_\epsilon)$, i.e. the geodesic $c|[\epsilon, t_\epsilon]$ is minimal. We conclude from the triangle inequality:

$$2d(p_\epsilon, q_\epsilon) = \theta(p_\epsilon, q_\epsilon) + \theta(q_\epsilon, p_\epsilon) \leq \theta(p_\epsilon, q_\epsilon) + \theta(q_\epsilon, p) + \theta(p, p_\epsilon) = \theta(p, q) + \theta(q, p) = L(c) = 2d(p, q).$$

From the definition of the symmetrized injectivity radius $d$ it follows that actually equality holds, i.e. the geodesic loop is a closed geodesic.

**Proof of Proposition 4.** We assume that $l < \pi(1 + \lambda^{-1})$ and conclude from Lemma 1 $2d = l < \pi(1 + \lambda^{-1})$. Then Proposition 3 implies that $L = l = 2d < \pi(1 + \lambda^{-1})$. But in [R1, Theorem 4] it is shown that under the assumptions of Proposition 4 the length $L$ of a shortest closed geodesic satisfies: $L \geq \pi(1 + \lambda^{-1})$. Therefore we obtain a contradiction, i.e. $l \geq \pi(1 + \lambda^{-1})$. □
Remark 1  Under the assumptions of Proposition 3 we have shown that for any point $p \in M$ with a cut point $q \in M$ satisfying $d(p, q) = d$ there is a shortest closed geodesic $c : [0, 2d] \to M$ parametrized by arc length passing through $p$ and $q$, i.e. $p = c(0) = c(2d); q = c(t); t \in (0, 2d)$. Hence the restrictions $c_1 = c| [0, t]$ and $c_3 = c| [t, 2d]$ are minimal geodesics. The cut point $q = c(t)$ is not a conjugate point since $t = \theta(p, q) < \pi$ and $K \leq 1$. This implies that there is another minimal geodesic $c_2 : [0, t] \to M$ joining $p$ and $q$. Therefore Proposition 3 excludes the second case discussed in [R1, Remark 1] resp. [R2, Lemma 9.7(b)].

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