1. Introduction

Let $\Gamma$ be an infinite, countable group acting by isometries on a metric space $(X, d)$, $\mu$ a probability measure on $\Gamma$ and $z_0 \in X$ a base point. A $(\mu, z_0)$-random walk on $X$, or random walk on $X$ for short, is the image under the orbital map $\gamma \mapsto \gamma \cdot z_0$ of the random walk on $\Gamma$ driven by the measure $\mu$. We denote with $(z_n)_{n \in \mathbb{N}} \in X^\mathbb{N}$ the sequence of the successive positions of the walk. We refer to Section 2.1 for basics on random walks.

We assume that the random variable $d(z_0, z_1)$ has a finite exponential moment. We refer to that property saying that ‘$\mu$ has a finite exponential moment’. In the sequel $\mathbb{P}$ denotes the law of the random walk and $\mathbb{E}$ the corresponding expectation.
The rate of escape of the random walk is defined as the limit

\[ l := \lim_{n \to \infty} \frac{\mathbb{E}(d(z_n, z_0))}{n} . \]

(The existence of the limit follows from sub-additivity.) It follows from Kingman’s sub-additive ergodic theorem that \( l \) is also the \( \mathbb{P} \) almost sure limit of the ratio \( d(z_n, z_0)/n \).

This article addresses the question of large deviations with respect to this last convergence: we are looking for estimates of the probability that the distance \( d(z_n, z_0)/n \) deviates from \( l \) by an error of order 1, either from below or from above. More precisely, we investigate the case where the space \( X \) is geodesic and hyperbolic and the measure \( \mu \) is admissible. A probability measure \( \mu \) on \( \Gamma \) is said to be admissible when its support generates a semi-group which contains two independent loxodromic elements; see Subsection 2.2. Note we do not assume that \( X \) is proper.

This setting has recently attracted a lot of attention as it encompasses several natural actions such as Gromov hyperbolic groups acting on their Cayley graphs, mapping class groups of surfaces acting on their curve complexes, relatively hyperbolic groups acting on their conned off spaces and many others. We refer to the introduction of [MT18a][Section 1.2] for more details and references on the topic.

In [MT18a] and [MT18b], the authors investigate the escape rate of random walks driven by non elementary measures. They show in particular that it is positive in this setting. Their approach focus on the boundary theory; they also manage to identify the Poisson boundary of the random walk with the Gromov boundary on the underlying hyperbolic space under the assumption that the action is WPD. In [MS] a different approach was proposed based on deviation inequalities (and thus without any reference to boundary theory). Under the assumption that the action is acylindrical, the authors manage to prove a central limit theorem for the rate of escape on the group itself.

Let

\[ l_{\max} := \sup_{a \in \mathbb{R}} \left\{ \limsup_{n \to \infty} \frac{-\ln \left( \mathbb{P}(d(z_n, z_0) \geq an) \right)}{n} < \infty \right\} . \]

Note that \( l_{\max} < \infty \) if \( \mu \) has finite support. Note also that \( l_{\max} \) may be infinite.

Our main theorem is the following

**Theorem 1.1** (main theorem). Let \( \Gamma \) be a countable group acting by isometries on a geodesic hyperbolic space \( X \), \( \mu \) an admissible probability measure on \( \Gamma \) with a finite exponential moment and \( z_0 \in X \). Then there is a convex function \( \psi : \mathbb{R} \to \mathbb{R} \cup \{\infty\} \) which vanishes at \( l \) only such that

\[
\begin{align*}
\text{(deviations from above) for any } a > l \text{ such that } a \neq l_{\max}, \\
\quad -\ln \left( \frac{\mathbb{P}(d(z_n, z_0) \geq an)}{n} \right) &\to \psi(a) ; \\
\text{(deviations from below) for any } a \leq l, \\
\quad -\ln \left( \frac{\mathbb{P}(d(z_n, z_0) \leq an)}{n} \right) &\to \psi(a) .
\end{align*}
\]

To the best of our knowledge large deviations had not been studied in the context of Theorem 1.1 so far. Even in the special case where \( \Gamma \) is hyperbolic, Theorem 1.1 seems new. The assumption that \( \mu \) has a finite exponential moment is sharp, see
Subsection 2.1.

Note however that there are large deviations principle for Lyapounov exponents associated to random matrices product. We refer to the introduction of Sert’s PhD thesis [Ser16] and the references therein for more details.

When Γ is hyperbolic and μ has a finite support, a possible alternative approach would be to exploit the spectral gap property of the image of the random walk on the boundary of the group. We refer to [Gou17, end of page 4].

For a surface group with the standard presentation and a driving measure with a finite exponential moment, large deviation estimates follow from the regeneration structure introduced in [HMM18].

As we were completing the present work, we were informed of a parallel work of C. Sert and A. Sisto. In their paper, these two authors compared the large deviations probabilities for the distance \(d(z_n, z_0)\) and the translation length \(\tau_n\). They proved that, when μ has a finite support, then both quantities behave the same: in the statement of Theorem 1.1, one may replace the distance \(d(z_n, z_0)\) by \(\tau_n\). Not only will the probability of a deviation for the translation length converge to some limiting rate function but the rate function is the same \(\Psi\) as in Theorem 1.1.

**Some remarks on \(\Psi\).** The large deviations function \(\Psi\) may be very degenerated. For example, let \(\Gamma := \mathbb{F}_2 := \langle A, B \rangle\) be the free group with two generators seen as acting on itself. We make it a metric tree \(X\) by considering the word distance associated to the generating system \(\{A, A^{-1}, B, B^{-1}\}\) and we mark \(z_0\) as the identity of \(\Gamma\). Let then \(\mu\) be the measure \(\mu(A) = \mu(B) = \frac{1}{2}\). In this example the space \(X\) is hyperbolic and geodesic. The probability measure \(\mu\) is supported by the set \(\{A, B\}\) and, as such, has a finite exponential moment and generates a non elementary semi-group \((A, B\text{ themselves are independent and loxodromic})\). In this case one has for all \(n \in \mathbb{N}\)

\[d(z_0, z_n) = n,\]

so that the function \(\Psi\) values 0 at 1 and \(\infty\) otherwise.

For finitely supported probability measure, the function \(\Psi\) will be infinite on a neighbourhood of \(\infty\) as well. However, it is easy to see that \(\Psi\) is finite on some interval \([0, l_{\text{max}}]\) with \(l_{\text{max}} > l\) whenever the semi-group generated by \(\mu\) contains the identity. Indeed, we may accelerate or decelerate the random walk (with an exponential cost) by just adjusting the frequency of ‘identity elements’ in the trajectories using an argument similar to the one used in the proof of [MS, Theorem 4.12].

**Overview of the article.** The article is mostly self-contained and proofs only use a combination of elementary geometric and probabilistic arguments. In particular, unlike [MT18a], we make no use of any boundary whatsoever.

In Section 2 we recall some basics on random walks and hyperbolic geometry.

A very first observation is that a general sub-additivity argument due to Hamana [Ham01], that we recall in Appendix A, gives an upper-bound on the probability of a deviation from above: for any \(\epsilon > 0\) one has

\[
(1.2) \quad \liminf_{n \to \infty} \frac{-\ln(P(d(z_n, z_0) - \ln n \geq cn))}{n} > 0.
\]
Inequality (1.2) holds for any group acting by isometries on any metric space without any restriction.

**Remark 1.3.** We observe that the exponential decay of the probability of a deviation from below cannot hold in the same generality as (1.2). In the examples below, we equip a group $\Gamma$ with any left-invariant metric. We choose $z_0 = \text{id}$ to be the identity element in $\Gamma$. We assume the rate of escape does not vanish for otherwise it makes no sense to compute deviations from below.

1. **When $\Gamma$ is an amenable group and $\mu$ is a symmetric probability measure whose finite support generates $\Gamma$, then Kesten’s theorem implies that the probability $\mathbb{P}(z_n = z_0)$ does not decay exponentially fast:**

$$-\frac{1}{n} \ln \mathbb{P}(z_n = z_0) \xrightarrow{n \to \infty} 0.$$  

Therefore deviations from below have a sub-exponential decay.

2. **It is also possible to give examples of random walks on non-amenable groups for which deviations from below have a sub-exponential decay. Indeed start with an amenable group $\tilde{\Gamma}$ and a finitely supported symmetric driving measure $\tilde{\mu}$ as in 1. Then let $\Gamma$ be the direct product of $\tilde{\Gamma}$ with your preferred non-amenable group, say the free group on two generators $\mathbb{F}_2 := \langle A, B \rangle$. Then $\Gamma$ is non-amenable. We endow $\Gamma$ with the metric $d$ it inherits from the metric we chose on $\tilde{\Gamma}$, say $\tilde{d}$, and the usual word metric on $\mathbb{F}_2$ that we denote with $d^{(2)}$. Let $\mu$ be the product measure of $\frac{1}{2} \delta_{\text{id}} + \frac{1}{2} \tilde{\mu}$ on $\tilde{\Gamma}$ with the lazy simple random walk driving measure $\frac{1}{2} \delta_{\text{id}} + \frac{1}{2} (\delta_A + \delta_{A^{-1}} + \delta_B + \delta_{B^{-1}})$ on $\mathbb{F}_2$. The two components of the random walk driven by $\mu$, say $(z_n)$, are then a random walk on $\Gamma$ driven by $\tilde{\mu}$ for the first component, say $(\tilde{z}_n)$ and a lazy simple symmetric random walk on $\mathbb{F}_2$ for the second component, say $(z_n^{(2)})$. The two random walks $(\tilde{z}_n)$ and $(z_n^{(2)})$ are independent. The rate of escape $l$ of the random walk $(z_n)$ is therefore the sum of the rate of escape of the random walk $(\tilde{z}_n)$ with respect to $\tilde{d}$, say $\tilde{l}$, and the rate of escape of the lazy simple random walk $(z_n^{(2)})$, say $l^{(2)}$.

For any real $a$ such that $l^{(2)} < a < l = l^{(2)} + \tilde{l}$, we have that

$$\mathbb{P}(d^{(2)}(\text{id}, z_n) \leq an) \geq \mathbb{P}(\tilde{z}_n = \text{id}) \mathbb{P}(d^{(2)}(\text{id}, z_n^{(2)}) \leq an).$$

As in example 1., the term $\mathbb{P}(\tilde{z}_n = \text{id})$ has a sub-exponential decay. Since $l^{(2)} < a$, the second term $\mathbb{P}(d^{(2)}(\text{id}, z_n^{(2)}) \leq an)$ tends to 1. Therefore $\mathbb{P}(d(\text{id}, z_n) \leq an)$ has a sub-exponential decay.

Let us come back to the setting of Theorem 1.1. We denote with $(y, z)_{x}$ the Gromov product of $y, z \in X$ seen from $x$:

$$(y, z)_x := \frac{1}{2} (d(y, x) + d(z, x) - d(y, z)).$$

Our main geometric tool is the existence of a Schottky set as defined in the next

**Definition 1.4** (Schottky set). Let $X$ be a metric space, $z_0 \in X$ and $S$ a finite subset of $\text{Isom}(X)$. We say that $S$ is a **Schottky set** if there is a constant $C > 0$ such that for any pair $y, z \in X$ we have

$$\frac{|\left\{ s \in S , \ (y, s \cdot z)_{z_0} \leq C \right\}|}{|S|} \geq \frac{2}{3}.$$ 

In Appendix B, we use a variation of the ping-pong lemma to prove that, when $X$ is hyperbolic and geodesic and if the probability measure $\mu$ is admissible then there exists $p \in \mathbb{N}$ such that the support of $\mu^{*p}$ contains a Schottky set.
We then deal separately with large deviations from above and from below.

As far as deviations from above are concerned, we already mentioned that the fact that a deviation from above has an exponentially small probability follows from Hamana’s argument. In Section 3, we explain how the existence of the limit \( \lim_{n \to \infty} - \frac{1}{n} \ln \mathbb{P}(d(z_n, z_0) \geq an) \) for all \( a > l \) follows from a sub-additivity argument.

In order to compare \( \mathbb{P}(d(z_{n+m}, z_0) \geq a(n + m)) \) with the product \( \mathbb{P}(d(z_n, z_0) \geq an) \mathbb{P}(d(z_m, z_0) \geq am) \), following [DPPS11], we use a Schottky set. We implement this approach using an insertion trick as in [HK02].

Let us now discuss deviations from below. It is immediate, again by sub-additivity, that the limit \( \lim_{n \to \infty} - \frac{1}{n} \ln \mathbb{P}(d(z_n, z_0) \leq an) \) exists for all \( a < l \) and defines a convex function; see Section 4. The hardest (and hopefully most interesting) part is to show that the limit is positive.

Our starting point is a clever way to decompose a trajectory of a random walk that was introduced by A. Asselah and B. Schapira [AS17] to study large deviations for the range of random walks on \( \mathbb{Z}^d \). Adapted to our context, it yields the following quite general criterion for deviations from below to be exponentially small.

**Proposition 1.5.** Let \( \Gamma \) be a countable group acting by isometries on a metric space \( X \) and \( \mu \) a probability measure on \( \Gamma \) with a finite exponential moment. If one has

\[
\liminf_{p \to \infty} \sup_{x \in X} \frac{\mathbb{E}((x, z_p)_{z_0})}{p} = 0,
\]

then there is a positive convex function \( \Psi : ]0, l[ \to ]0, +\infty[ \) such that for all \( a \in ]0, l[ \)

\[
- \frac{\ln \mathbb{P}(d(z_n, z_0) \leq an)}{n} \to \Psi(a).
\]

Proposition 1.5 is proved in Section 4. Note that, in Proposition 1.5, we do not need assume \( X \) is hyperbolic or geodesic.

As a corollary of the above Proposition, we have the following

**Corollary 1.8.** Let \( \Gamma \) be a finitely generated amenable group and \( \mu \) a symmetric finitely-supported probability measure on \( \Gamma \) whose support generates \( \Gamma \). Equip \( \Gamma \) with any left-invariant metric \( d \). Assume the rate of escape does not vanish. Then

\[
\inf_{p} \sup_{x \in X} \frac{\mathbb{E}((x, z_p)_{z_0})}{p} \neq 0.
\]

Corollary 1.8 follows from Proposition 1.5 and Kesten’s theorem. As in Remark 1.3, one also shows that there exist examples of random walks on non-amenable groups for which (1.6) fails.

It now remains to show that (1.6) holds in the setting of Theorem 1.1. This will be a consequence of more precise exponential bounds on the tail of the law of the Gromov product \( (z_n, x)_{z_0} \) stated in Proposition 1.13 below.

We start quantifying the rough idea that, given any point \( x \in X \), with high probability, the random walk tends to walk away from \( x \). The next Theorem 1.10 plays the central role in the proof of Proposition 1.13. It is proved in Sections 5 and 6.
Theorem 1.10 (walking away uniformly). Let $\Gamma$ be a countable group acting by isometries on a metric space $X$, $\mu$ a probability measure on $\Gamma$ with a finite exponential moment and $z_0 \in X$. If the semi-group generated by $\mu$ contains a Schottky set and has unbounded orbits, then there is $\epsilon, c_1, c_2 > 0$ such that for any $x \in X$ and all $n \in \mathbb{N}$ we have
\[
P(d(z_n, x) - d(z_0, x) \leq \epsilon n) \leq c_1 e^{-c_2 n}.
\]
Note that we do not require $X$ to be hyperbolic nor geodesic.

Theorem 1.10 was already proved in [MS, Theorem 9.1] when the action of $\Gamma$ on $X$ is acylindrical. The approach proposed here is different.

Theorem 1.10 in particular implies that the rate of escape does not vanish. More precisely, it implies the following linear progress with exponential tail property.

Definition 1.11 (Linear progress). Let $X$ be a metric space. We say that a random path $(z_n)$, with values in $X$, has linear progress with exponential tail if there is a constant $\epsilon > 0$ such that
\[
\liminf_{n \to \infty} \frac{-\ln \left( \mathbb{P}(d(z_n, z_0) \leq \epsilon n) \right)}{n} > 0.
\]
Note that the linear progress with exponential tail property is proved in [MT18a] under the extra assumption that $\mu$ has finite support.

The last two sections of the paper are devoted to deducing Proposition 1.13 from the walking away uniformly theorem. We shall rely on deviation inequalities. The next Proposition 1.12 is a variant of [MS, Theorem 11.1]. It is proved in Section 7.

Proposition 1.12 (exponential-tail deviation inequalities). Let $\Gamma$ be a countable group acting by isometries on a geodesic hyperbolic space $X$, $\mu$ an admissible probability measure on $\Gamma$ with a finite exponential moment and $z_0 \in X$. If the random walk has linear progress with exponential tail, there are $\epsilon, c_1, c_2 > 0$ such that for all $0 \leq i \leq n$ and all $R > 0$ one has
\[
P((z_n, z_i)_{z_0} \geq R) \leq c_1 e^{-c_2 R}.
\]
In Section 8, combining Proposition 1.12 and the walking away property from Theorem 1.10, we finally derive exponential bounds on the Gromov product $(z_n, x)_{z_0}$ as announced.

Proposition 1.13 (uniform punctual deviations). Let $\Gamma$ be a countable group acting by isometries on a geodesic hyperbolic space $X$, $\mu$ an admissible probability measure on $\Gamma$ with a finite exponential moment and $z_0 \in X$. Then there are constants $c_1, c_2 > 0$ such that for all $x \in X$, all $n \in \mathbb{N}$ and all $R > 0$ one has
\[
P((z_n, x)_{z_0} \geq R) \leq c_1 e^{-c_2 R}.
\]
Integrating with respect to $R$ the bound in Proposition 1.13, one easily checks condition (1.6). The proof of Theorem 1.1 is now complete.

We observe that, taking $n$ to $\infty$ in Proposition 1.13, we immediately derive bounds on the harmonic measure. We refer to Section 2 for all definitions regarding the next statement.

Corollary 1.14 (harmonic measure). Let $\Gamma$ be a countable group acting by isometries on a geodesic hyperbolic space $X$, $\mu$ an admissible probability measure on $\Gamma$...
with a finite exponential moment and \( z_0 \in X \). There exists \( D, C > 0 \) such that for any \( \zeta \in \partial X \) and any \( r > 0 \) the harmonic measure \( \nu \) on \( \partial X \) satisfies
\[
\nu(B(\zeta, r)) \leq C \cdot r^{D},
\]
where \( B(\zeta, r) \) stands for the ball (with respect to the Gromov metric) on \( \partial X \) centred at \( \zeta \) of radius \( r \).

Harmonic measures were studied in great details for some proper hyperbolic spaces (see for example [Kif90] [KL90] [BHM11] [BH]). In particular the Hausdorff dimension of \( \nu \) can then be computed and its multi-fractal spectrum described as in [Tan19]. If \( \Gamma \) is hyperbolic and \( \mu \) has a finite support, the inequality in Corollary 1.14 holds when \( D \) is replaced by the Hausdorff dimension [BHM11]. In our context of a more general action, an upper-bound on the harmonic measure of a ball as in Corollary 1.14 is proved in [Mah12] but only when \( \mu \) has a finite support.

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2. First definitions and preliminary remarks

2.1. Basics on random walks. As a general reference on the topic, we recommend [Woe00]. Let \( \Gamma \) be an infinite, countable group and \( \mu \) be a probability measure on \( \Gamma \). Let \( (\Omega, \mathcal{F}) \) be a probability space and \( (\omega_i)_{i \in \mathbb{N}} : \Omega \rightarrow \Gamma \) a sequence of I.I.D. random variables following the law \( \mu \). We call such a sequence the increments of the random walks. We then form the sequence of random variables
\[
\gamma_n := \omega_1 \cdot \omega_2 \cdot ... \cdot \omega_n .
\]
Let \( \Gamma \) act on a metric space \((X, d)\) with a marked point \( z_0 \in X \). The push-forward of the random walk with respect to the orbital map is defined by
\[
\Gamma \rightarrow X \quad \gamma \mapsto \gamma \cdot z_0 .
\]
We denote with
\[
z_n := \omega_1 \cdot \omega_2 \cdot ... \cdot \omega_n \cdot z_0
\]
the image of the sequence \((\gamma_n)\). We call \((z_n)\) the positions of the image random walk under the orbital map. We will often use the notation \( d_n := d(z_0, z_n) \) for short.

Remark 2.1. Note that the sequence of random variables \((z_n)_{n \in \mathbb{N}}\) may not have the Markov property, even though the random walk \((\gamma_n)_{n \in \mathbb{N}}\) is a Markov process.

The triangular inequality gives
\[
d(z_0, z_{m+n}) \leq d(z_0, z_n) + d(z_n, z_{n+m}) .
\]
If we suppose that \( \mu \) has a finite first moment (i.e. \( \mathbb{E}(d(z_0, z_1)) < \infty \)) we then get the inequality
\[
\mathbb{E}(d(z_0, z_{m+n})) \leq \mathbb{E}(d(z_0, z_n)) + \mathbb{E}(d(z_n, z_{n+m})) = \mathbb{E}(d(z_0, z_n)) + \mathbb{E}(d(z_0, z_n)) .
\]
For the last equality, we used the fact that \( d(z_n, z_{n+m}) \) and \( d(z_0, z_m) \) have the same law. The sequence \( (\mathbb{E}(d_n))_{n \in \mathbb{N}} \) is therefore sub-additive and Fekete’s lemma implies that the following limit exists

\[
l := \lim_{n \to \infty} \frac{\mathbb{E}(d_n)}{n} = \inf_{n \in \mathbb{N}} \frac{\mathbb{E}(d_n)}{n}.
\]

We call \( l \) the \textbf{rate of escape} of the image random walk. Note that Kingman’s sub-additive ergodic theorem [Kin68] (see also [Ste89]) implies that the sequence \( (\frac{d_n}{n})_{n \in \mathbb{N}} \) also \( \mathbb{P} \)-almost surely converges towards \( l \).

We observe that if one has a large deviations estimates as in Theorem 1.1, then the measure \( \mu \) has a finite exponential moment. Indeed the triangular inequality implies that, for any \( a, \alpha \in \mathbb{R} \), we have

\[
\mathbb{P}(d_1 \geq an) \mathbb{P}(d_{n-1} \leq an) \leq \mathbb{P}(d_n \geq (\alpha-a)n).
\]

In particular, for \( a > l \) and \( \alpha > 2a \), the definition of \( l \) imposes

\[
\mathbb{P}(d_{n-1} \leq an) \to 1
\]

whereas the large deviations estimates from above imply that

\[
n \mapsto \mathbb{P}(d_n \geq (\alpha-a)n)
\]

has an exponential decrease. Therefore, the sequence \( (\mathbb{P}(d_1 \geq an))_{n \in \mathbb{N}} \) must also decrease exponentially fast.

### 2.2. Basics on hyperbolicity

As general references on the topic one can recommend [Gro87], [KB02] and [V ¨05] for the non proper setting. For a general metric space \( (X, d) \) we define the \textbf{Gromov product} of \( x_1, x_2 \) with respect to \( x_0 \) as

\[
(x_1, x_2)_{x_0} = \frac{1}{2} (d(x_0, x_1) + d(x_0, x_2) - d(x_1, x_2)).
\]

**Definition 2.2.** A metric space \( (X, d) \) is said to be \textbf{Gromov-hyperbolic} if there is a constant \( \delta > 0 \) such that for any four points \( \{x_i\}_{0 \leq i \leq 3} \) we have

\[
(x_1, x_2)_{x_0} \geq \min\{(x_3, x_1)_{x_0}, (x_3, x_2)_{x_0}\} - \delta.
\]

In this article, we will mostly deal with geodesic spaces. Recall that a metric space \( (X, d) \) is geodesic if the distance between any two points \( x, y \) is given by the length of a rectifiable path whose endpoints are \( x \) and \( y \).

The following definition is to explain the terminology involved in the statement of Proposition 1.14. Let \( X \) be a hyperbolic metric space and \( x_0 \in X \) a base point.

**Definition 2.3.** We define the \textbf{Gromov boundary}, that we denote by \( \partial X \), as the set of all sequences \( (x_n)_{n \in \mathbb{N}} \in X^\mathbb{N} \) such that

\[
\lim_{n, m \to \infty} (x_n, x_m)_{x_0} = \infty
\]

mod out the equivalence relation \( (x_n) \sim (y_n) \) if \( (x_n, y_n)_{x_0} \xrightarrow{n \to \infty} \infty \). We denote by \( [(x_n)] \) the class of such a sequence.

One can easily verify that the construction of \( \partial X \) does not depend on the base point \( x_0 \).

Choose \( \zeta := [(x_\zeta^n)] \in \partial X \) and \( r > 0 \) and set

\[
B(\zeta, r) := \{ \zeta_2 := [(y_n)] \in \partial X \, , \, \lim \inf_{n \to \infty} e^{-(x_\zeta^n, y_n)_{x_0}} \leq r \}.
\]
We define a topology on $\partial X$ by choosing the above sets as a generating basis of the open sets. The resulting topological space $\partial X$ is metrizable. The sets $B(\zeta, r)$ are 'almost' balls of radius $r$. We refer to [V05, Section 5] for more details.

**Definition 2.4.** Let $\lambda, C > 0$ and $I$ a sub interval of $\mathbb{N}$. A $(\lambda, C)$-quasi-geodesic indexed by $I$ (simply called quasi-geodesic when not ambiguous) is a sequence $(x_n)_{n \in I}$ such that for any $n, m \in I$

$$\lambda^{-1}|n - m| - C \leq d(x_n, x_m) \leq \lambda|n - m| + C.$$ 

In other words, a quasi-geodesic is a quasi-embedding of $I$ into $X$.

One can easily verify that quasi-geodesics indexed by $\mathbb{N}$ define a unique point in $\partial X$. Recall the statement of the celebrated Morse’s lemma.

**Lemma 2.5 (Morse’s lemma).** For any $\lambda, C > 0$ there is a constant $L = L(\Lambda, C, \delta)$ such that any $(\lambda, C)$-quasi-geodesic having same endpoints are $L$-close from one another.

The following definitions are to explain the terminology 'non elementary'.

**Definition 2.6.** An isometry $\gamma$ of a hyperbolic space $X$ is called loxodromic if for a point $x \in X$ (equivalently any) the sequence $(\gamma^n \cdot x)_{n \in \mathbb{Z}}$ is a quasi-geodesic.

In particular, a loxodromic element defines two points in the Gromov boundary $\gamma^+ \text{ and } \gamma^-$ corresponding to the classes of the two quasi-geodesics defined by the future and the past. We say that two loxodromic elements $\gamma_1, \gamma_2$ are independent if the fours points $\gamma_{1}^+, \gamma_{1}^-, \gamma_{2}^+, \gamma_{2}^-$ are all disjoints.

**Definition 2.7.** A semi-group acting on $X$ by isometries is called non elementary if it contains two independent loxodromic elements.

Recall a probability measure $\mu$ on a group $\Gamma$ acting by isometries on a hyperbolic space $X$ is said to be admissible when its support generates a non elementary semi-group.

Non elementary groups have a lot of elements spreading apart points of $X$. The proof of the following lemma is a variation around the proof of the well known ping-pong lemma. As we could not find any ready-to-use reference in this generality, we inserted a proof in Appendix B.

**Proposition 2.8 (Existence of Schottky sets).** Let $\Gamma$ be group acting by isometries on a geodesic hyperbolic space $X$, $z_0 \in X$ and $\mu$ an admissible probability measure on $\Gamma$. Then there is $p \in \mathbb{N}$ such that $\text{supp}(\mu^p)$ contains a Schottky set.

### 3. Deviations from above

The goal of this section is to show that we have large deviations estimates from above. We recall here this first part of the statement of Theorem 1.1 for the reader’s convenience.

**Proposition 3.1.** Let $\Gamma$ be a countable group acting by isometries on a geodesic hyperbolic space $X$, $\mu$ an admissible probability measure on $\Gamma$ with a finite exponential moment and $z_0 \in X$. Then there is a positive convex function $\Psi : [1, \infty[ \to \mathbb{R}_+ \cup \{\infty\}$ such that for any $a \neq l_{\max}$

$$-\ln \mathbb{P}(d(z_n, z_0) \geq an) \xrightarrow{n \to \infty} \Psi(a).$$

The part concerning $\Psi > 0$ follows from Hamana’s argument taken from [Ham01]. Namely, we will show in Appendix A that
Let us define

\[ x - P(3.6) \]

We now substitute \( x \) by \( am + c \) and \( y \) by \( an + c \) in order to get that for all \( m, n, p \geq p \):

\[ (3.6) \quad \mathbb{P}(d_{m+n+p} \geq x + y - c) \geq c^{-1} \cdot \mathbb{P}(d_m \geq x) \mathbb{P}(d_n \geq y) \cdot \]

We now substitute \( x \) by \( am + c \) and \( y \) by \( an + c \) in order to get that for all \( m, n, p \geq p \):

\[ \mathbb{P}(d_{m+n+p} \geq a(m + n) + c) \geq c^{-1} \cdot \mathbb{P}(d_{m-p} \geq an + c) \mathbb{P}(d_{n-p} \geq an + c) . \]

Thus we see that the sequence \((- \ln \left(c^{-1} \mathbb{P}(d_{n-p} \geq an + c)\right))_{n \geq p}\) is sub-additive. Let us define

\[ \psi_n(a) := - \frac{\ln \left(c^{-1} \mathbb{P}(d_{n-p} \geq an + c)\right)}{n} . \]

Fekete’s lemma implies that, for all \( a \), \( (\psi_n(a))_{n \geq p} \) converges; we denote with \( \Psi(a) \) the limit.

We now show that \( \Psi \) is convex. Indeed, using Inequality (3.6) one gets that, for any \( a, b > 0 \) and for any \( n \geq p \) then

\[ \psi_{2n} \left( \frac{a + b}{2} \right) \leq \frac{1}{2} \left( \psi_n(a) + \psi_n(b) \right) , \]

which shows, letting \( n \to \infty \), that \( \Psi \) is convex. Note, in particular, that \( \Psi \) is continuous on the interval \([0, l_{\text{max}}]\). We now conclude showing that the sequence \((- \frac{1}{n} \ln \mathbb{P}(d_n \geq an))_{n \in \mathbb{N}}\) converges to \( \Psi(a) \) for \( a \neq l_{\text{max}} \).

We start with the observation that for any \( \epsilon > 0 \) we have for \( n \) large enough

\[ \mathbb{P}(d_{n-p} \geq a - \epsilon) \geq \mathbb{P}(d_{n-p} \geq a(n - p)) \geq \mathbb{P}(d_{n-p} \geq an + c) . \]

Therefore

\[ \Psi(a - \epsilon) \leq \lim \inf \frac{1}{n} \ln \mathbb{P}(d_n \geq an) \]

\[ \leq \lim \sup \frac{1}{n} \ln \mathbb{P}(d_n \geq an) \leq \Psi(a) . \]
The above inequality implies that if $a > l_{\max}$ then $\Psi(a) = \infty$. In particular, using again the above inequality, if $a > l_{\max}$ we get
\[
\liminf -\frac{1}{n} \ln \mathbb{P}(d_n \geq an) \geq \Psi \left( \frac{l_{\max} + a}{2} \right) = \infty = \Psi(a).
\]
We conclude showing that $-\frac{1}{n} \ln \mathbb{P}(d_n \geq an)$ converges to $\Psi(a)$ for $a \in \llbracket l, l_{\max} \rrbracket$. Since $\Psi$ is convex and finite on $\llbracket l, l_{\max} \rrbracket$ it is in particular continuous. Letting $\epsilon \to 0$ in (3.7) we get that the sequence $(-\frac{1}{n} \ln \mathbb{P}(d_n \geq an))_{n \in \mathbb{N}}$ converges to $\Psi(a)$ on $\llbracket l, l_{\max} \rrbracket$.

One is left to show that Proposition 3.4 holds. Our strategy is inspired by the replacement trick proposed in [HK01] and by the use of Proposition 2.8, inspired by [DPPS11].

**Proof of Proposition 3.4.** The very first remark, made in [HK01], is that a way to construct a path of $n + m + p$ steps is to start with two paths, one of $n$ steps, another one of $m$ steps, and a way to connect them using $p$ steps. Requiring something on the finitely many steps in the middle, corresponding to the $p$ steps, should have finite probability cost. In our case, we shall constrain our $p$-paths to have endpoints in a Schottky set $S$.

Recall that Proposition 2.8 implies the existence of a Schottky set $S$ in the support of $\mu^p$ for some $p$. We choose such a $p$ and let $C$ denote the constant from Definition 1.4. We also let
\[
\zeta := \inf_{s \in S} \mathbb{P}(\gamma_p = s),
\]
and $S_{\text{sup}} := \sup_{s \in S} d(z_0, s \cdot z_0)$.

By definition of the Gromov product, for all $m, n, p > 0$ we have
\[
d_{m+n+p} = d_n + d(z_n, z_{m+n+p}) - 2(d(z_0, z_{n+m+p})_{z_n}).
\]
On the event $\gamma_n^{-1} \cdot (\gamma_n + p) \in S$, by the triangular inequality we also have

$$d(z_n, z_{m+n+p}) \geq d(z_{n+p}, z_{m+n+p}) - S_{sup}.$$ 

Therefore, on the intersection of the events $\gamma_n^{-1} \cdot (\gamma_n + p) \in S$, $d_n \geq x$, $d(z_n, z_{m+n+p}) \geq y$ and $(z_0, z_{m+n+p})_{z_0} \leq C$ we have

$$d_{m+n+p} \geq x + y - 2C - S_{sup}.$$ 

Let us set $c_2 := 2C + S_{sup}$. We get that, for all $s \in S$,

$$P(d_{m+n+p} \geq x + y - c_2) \geq P(d_n \geq x, d(z_{n+p}, z_{n+p+m}) \geq y, \gamma_n^{-1} \cdot (\gamma_n + p) = s, (z_0, z_{m+n+p})_{z_0} \leq C).$$

Note when $\gamma_n^{-1} \cdot (\gamma_n + p) = s$, then

$$(z_0, z_{m+n+p})_{z_0} = (\gamma_n^{-1} \cdot z_0, s \cdot \gamma_n^{-1} \cdot z_{m+n+p})_{z_0}.$$ 

Therefore, we have for all $s \in S$

$$P(d_{m+n+p} \geq x + y - c_2) \geq P(d_n \geq x, d(z_{n+p}, z_{n+p+m}) \geq y, \gamma_n^{-1} \cdot (\gamma_n + p) = s, (z_0, s \cdot \gamma_n^{-1} \cdot z_{m+n+p})_{z_0} \leq C).$$

The event $\gamma_n^{-1} \cdot (\gamma_n + p) = s$ only depends on the increments $n+1 \ldots n+p$. Therefore it is independent of $d_n \geq x, d(z_{n+p}, z_{n+p+m}) \geq y$ and $(\gamma_n^{-1} \cdot z_0, s \cdot \gamma_n^{-1} \cdot z_{m+n+p})_{z_0} \leq C$. Thus, we have for all $s \in S$

$$P(d_{m+n+p} \geq x + y - c_2) \geq \zeta \cdot P(d_n \geq x, d(z_{n+p}, z_{n+p+m}) \geq y, (\gamma_n^{-1} \cdot z_0, s \cdot \gamma_n^{-1} \cdot z_{m+n+p})_{z_0} \leq C).$$

Let $A := \{d_n \geq x, d(z_{n+p}, z_{n+p+m}) \geq y, \mathbf{1} = (\gamma_n^{-1} \cdot z_0, s \cdot \gamma_n^{-1} \cdot z_{m+n+p})_{z_0} \leq C\}$. Averaging with respect to $S$ gives

$$P(d_{m+n+p} \geq x + y - c_2) \geq \frac{\zeta}{\#S} \sum_{s \in S} P(A \cap (\gamma_n^{-1} \cdot z_0, s \cdot \gamma_n^{-1} \cdot z_{m+n+p})_{z_0} \leq C).$$

Rewriting the right above probabilities as expectations and permuting them with the summation yields

$$\frac{1}{\#S} \sum_{s \in S} P(A \cap (\gamma_n^{-1} \cdot z_0, s \cdot \gamma_n^{-1} \cdot z_{m+n+p})_{z_0} \leq C) = P(A) \mathbb{E}(\mathbf{1}_A : \#S).$$

Because of Proposition 2.8, we always have the following uniform lower bound

$$\frac{1}{\#S} \sum_{s \in S} P(A \cap (\gamma_n^{-1} \cdot z_0, s \cdot \gamma_n^{-1} \cdot z_{m+n+p})_{z_0} \leq C) \geq 2.$$ 

Therefore,

$$P(d_{m+n+p} \geq x + y - c_2) \geq \frac{2\zeta}{3} P(A) = \frac{2\zeta}{3} \mathbb{E}(\mathbf{1}_A).$$

Recall that $A = \{d_n \geq x, d(z_{n+p}, z_{n+p+m}) \geq y\}$. The event $d_n \geq x$ (which depends only on the $1 \ldots n$ first increments) being independent of the event $d(z_{n+p}, z_{n+p+m}) \geq y$ (which depends only on the $n + p + 1 \ldots n + m + p$ increments), we have

$$P(d_{m+n+p} \geq x + y - c_2) \geq \frac{2\zeta}{3} \cdot P(d_n \geq x) \cdot P(d(z_{n+p}, z_{n+p+m}) \geq y) \geq \frac{2\zeta}{3} \cdot P(d_n \geq x) \cdot P(d_m \geq y),$$

since $d_m$ and $d(z_{n+p}, z_{n+p+m})$ follow the same law. We conclude setting

$$c := \max\left(c_2, \frac{3}{2\zeta}\right).$$
4. Deviations from below

This section is dedicated to investigate the deviations from below. The strategy of the proof of the following proposition is inspired from [AS17].

Proposition 4.1. Let $\Gamma$ be a countable group acting on a metric space $X$ and $\mu$ a probability measure with a finite exponential moment on $\Gamma$ such that

$$\liminf_{p \to \infty} \sup_{x \in X} \mathbb{P}\left(\frac{(x, z_p)_{z_0}}{p}\right) = 0.$$  

Then there is a positive convex function $\Psi : ]0, l[ \to \mathbb{R}_+ \cup \{\infty\}$ such that for all $a \in ]0, l[$

$$- \ln \mathbb{P}\left(d(z_n, z_0) \leq an\right) \xrightarrow{n \to \infty} \Psi(a).$$

We shall start proving that the limit defining the function $\Psi$ exists. This only requires sub-additivity. We will then prove the most difficult part of the proof, namely that $\Psi > 0$ under the assumption (4.2).

Proof that the limit exists. The proof does not require Assumption (4.2). By the triangular inequality and independence again we have

$$\mathbb{P}(d_{n+m} \leq a(n + m)) \geq \mathbb{P}(d_m \leq am) \mathbb{P}(d_n \leq an).$$

Therefore the sequence $(- \ln \mathbb{P}(d_n \leq an)_{n \in \mathbb{N}}$ is sub-additive. Let us define

$$\left(\Psi_n(a)\right)_{n \in \mathbb{N}} := \left(- \ln \left(\mathbb{P}(d_n \leq an)\right)\right)_{n \in \mathbb{N}}.$$

Fekete’s lemma then gives that the sequence $\left(\Psi_n(a)\right)_{n \in \mathbb{N}}$ converges; we denote the limit with $\Psi(a)$.

To show that $\Psi$ is convex, one has to show that for all pairs $(a, b) \in ]0, l[^2$ we have

$$\Psi\left(\frac{a + b}{2}\right) \leq \frac{\Psi(a) + \Psi(b)}{2}.$$  

Using again the triangular inequality we get

$$\mathbb{P}\left(d_{2n} \leq \frac{a + b}{2} \cdot 2n\right) \geq \mathbb{P}(d_n \leq an) \mathbb{P}(d_n \leq bn),$$

and then

$$\Psi_{2n}\left(\frac{a + b}{2}\right) \leq \frac{\Psi_n(a) + \Psi_n(b)}{2}.$$  

We conclude by letting $n$ tend to $\infty$.

Proof that $\Psi > 0$. We will now use Assumption (4.2).

Let us start noticing that Proposition 4.1 is invariant under acceleration: given $k \in \mathbb{N}$ a measure $\mu$ with a finite exponential moment satisfies the conclusion of Proposition 4.1 if and only if the measure $\mu^k$ satisfies it.

Given a trajectory, we chop it into pieces of size $p \in \mathbb{N}$ and write the distance between the base point $z_0$ and the endpoint $z_n$ (where $n = mp$ for some integer $m$)
as a summation of I.I.D. random variables and a defect term.

Recall that the Gromov product of two points \( x, y \in \Gamma \) seen from \( z_0 \) is defined as
\[
(x, y)_{z_0} := \frac{1}{2} \left( d(z_0, x) + d(z_0, y) - d(x, y) \right).
\]
In particular, we have for any \( m, p > 0 \)
\[
2(z_0, z_{mp})_{z(m-1)p} = d(z_0, z_{(m-1)p}) + d(z_{mp}, z_{(m-1)p}) - d(z_0, z_{mp}).
\]
Equivalently,
\[
d_{mp} = d_{(m-1)p} + d(z_{mp}, z_{(m-1)p}) - 2(z_0, z_{mp})_{z(m-1)p}.
\]
By an immediate induction we get
\[
d_{mp} = \sum_{1 \leq i \leq m} d(z_{ip}, z_{(i-1)p}) - 2 \sum_{1 \leq i \leq m} (z_0, z_{ip})_{z(i-1)p}.
\]
Since the Gromov product is non-negative, one has the following set inclusion
\[
\begin{align*}
\{d_{mp} \leq an\} & \subset \left\{ \sum_{1 \leq i \leq m} d(z_{ip}, z_{(i-1)p}) \leq a + \frac{l}{2n} \right\} \cup \left\{ \sum_{1 \leq i \leq m} (z_0, z_{ip})_{z(i-1)p} \geq \frac{l-a}{4n} \right\},
\end{align*}
\]
which implies that
\[
P(d_{mp} \leq an) \leq P \left( \sum_{1 \leq i \leq m} d(z_{ip}, z_{(i-1)p}) \leq a + \frac{l}{2n} \right) + P \left( \sum_{1 \leq i \leq m} 2(z_0, z_{ip})_{z(i-1)p} \geq \frac{l-a}{2n} \right).
\]

We shall see that there exists \( p \) such that both the above probabilities go exponentially fast to 0. The argument for the first one only uses classical large deviations estimates for I.I.D. random variables whereas the control of the second one will be handled using Assumption (4.2).

We start with the top probability appearing in (4.4). The random variables
\[
(d(z_{ip}, z_{(i-1)p}))_{i \in \mathbb{N}}
\]
are I.I.D. and follow the law of \( d_p \). Therefore large deviations estimates for I.I.D. random variables with a finite exponential moment imply that
\[
P \left( \sum_{1 \leq i \leq m} d(z_{ip}, z_{(i-1)p}) \leq \frac{a+l}{2n} \right)
\]
has an exponential decay as soon as \( \frac{\mathbb{E}(d_p)}{p} > \frac{a+l}{2} \).

On the other hand, we already know that \( \frac{\mathbb{E}(d_p)}{p} \) converges to \( l \) and \( l > \frac{a+l}{2} \). Thus we conclude that there exists \( p_0 \) such that for all \( p \geq p_0 \), we have
\[
\lim inf -\frac{1}{n} \ln P \left( \sum_{1 \leq i \leq m} d(z_{ip}, z_{(i-1)p}) \leq \frac{a+l}{2n} \right) > 0.
\]
We now deal with the second probability appearing in (4.4) using Assumption (4.2).
Let us set $\epsilon := \frac{l-a}{2}$ and let $\lambda > 0$. We start the inequality
\[
P \left( 2 \sum_{1 \leq i \leq m} (z_0, z_{i\lambda}) z_{(i-1)\lambda} \geq \frac{l-a}{2} n \right) \leq e^{-\lambda \epsilon n} \cdot \mathbb{E} \left( \exp \left( \lambda \sum_{1 \leq i \leq m} (z_0, z_{i\lambda}) z_{(i-1)\lambda} \right) \right).
\]

We introduce the random variables
\[
\Pi_m(\lambda, p) := \exp \left( \lambda \sum_{1 \leq i \leq m} (z_0, z_{i\lambda}) z_{(i-1)\lambda} \right),
\]
and note that
\[
\Pi_m(\lambda, p) = \Pi_{m-1}(\lambda, p) \cdot \exp \left( \lambda(z_0, z_{mp}) z_{(m-1)p} \right).
\]

Let us denote with $(\mathcal{F}_i)_{i \in \mathbb{N}}$ the filtration naturally associated to the random walk. We compute
\[
\mathbb{E}(\Pi_m(\lambda, p)) = \mathbb{E} \left( \mathbb{E}(\Pi_{m-1}(\lambda, p) \cdot \exp(\lambda(z_0, z_{mp}) z_{(m-1)p}) | \mathcal{F}_{(m-1)p}) \right)
\]
\[
= \mathbb{E} \left( \mathbb{E}(\Pi_{m-1}(\lambda, p) \cdot \exp(\lambda(\gamma^{-1}_{(m-1)p} z_0, \gamma^{-1}_{(m-1)p} z_{mp}) z_0) | \mathcal{F}_{(m-1)p}) \right)
\]
\[
= \mathbb{E} \left( \Pi_{m-1}(\lambda, p) \cdot \mathbb{E}(\exp(\lambda(\gamma^{-1}_{(m-1)p} z_0, \gamma^{-1}_{(m-1)p} z_{mp}) z_0) | \mathcal{F}_{(m-1)p}) \right).
\]
The last equality holds because $\Pi_{m-1}(\lambda, p)$ is measurable with respect to $\mathcal{F}_{(m-1)p}$. Moreover, since $\gamma^{-1}_{(m-1)p} z_{mp}$ is independent of $\mathcal{F}_{(m-1)p}$ and since $\gamma^{-1}_{(m-1)p} z_{mp}$ follows the same law than $z_p$, we have
\[
\mathbb{E}(\Pi_m(\lambda, p)) \leq \mathbb{E} \left( \Pi_{m-1}(\lambda, p) \cdot \sup_{x \in \mathcal{X}} \mathbb{E}(\exp(\lambda(x, \gamma^{-1}_{(m-1)p} z_{mp}) z_0)) \right)
\]
\[
\leq \mathbb{E} \left( \Pi_{m-1}(\lambda, p) \cdot \sup_{x \in \mathcal{X}} \mathbb{E}(\exp(\lambda(x, z_p)) \right).
\]
An immediate induction yields
\[
\mathbb{E}(\Pi_m(\lambda, p)) \leq \delta(p, \lambda)^m,
\]
where
\[
\delta(p, \lambda) := \sup_{x \in \mathcal{X}} \mathbb{E}(\exp(\lambda(x, z_p)) z_0).
\]

Therefore,
\[
\mathbb{P} \left( 2 \sum_{1 \leq i \leq m} (z_0, z_{i\lambda}) z_{(i-1)\lambda} \geq \frac{l-a}{2} n \right) \leq e^{-\lambda \epsilon n} \delta(p, \lambda)^m
\]
\[
\leq e^{-\lambda \epsilon n + n \ln(\delta(p, \lambda))}
\]
\[
\leq e^{n[\ln(\delta(p, \lambda)) - \lambda p]}.
\]

We shall prove, using Assumption (4.2), that for all $\epsilon' > 0$ there exist $p \geq p_0 \in \mathbb{N}$ and $\lambda > 0$ such that
\[
\ln(\delta(p, \lambda)) \leq \frac{\ln(\delta(p, \lambda))}{\lambda p} \leq \epsilon'.
\]

We choose $\epsilon' := \epsilon/2$ with $p$ and $\lambda$ such that (4.6) holds. Then
\[
\mathbb{P} \left( 2 \sum_{1 \leq i \leq m} (z_0, z_{i\lambda}) z_{(i-1)\lambda} \geq \frac{l-a}{2} n \right) \leq e^{-\lambda \epsilon n} \delta(p, \lambda)^m \leq e^{-n \frac{\epsilon'}{2}}.
\]
does indeed decrease exponentially fast to 0 as \( n \to \infty \).

It remains to prove Inequality (4.6).

Note first that for any \( x \in X \) we have
\[
E \left( \exp \left( \lambda (x, z_p)_{z_0} \right) \right) \leq 1 + \lambda E \left( (x, z_p)_{z_0} \right) + \lambda^2 E \left( (x, z_p)_{z_0}^2 \exp \left( \lambda (x, z_p)_{z_0} \right) \right),
\]

since \( e^x \leq 1 + x + x^2 e^x \).

Using the upper bound \( (x, z_p)_{z_0} \leq d(z_0, z_p) \), we get that
\[
E \left( \exp \left( \lambda (x, z_p)_{z_0} \right) \right) \leq 1 + \lambda E \left( (x, z_p)_{z_0} \right) + \lambda^2 E_{z_0} \left( d_{z_0}^2 e^{\lambda d_p} \right).
\]

Assumption (4.2) provides us with some \( p^1 \) such that, for all \( p \geq p^1 \), we have
\[
\sup_{x \in X} E \left( (x, z_p)_{z_0} \right) \leq \epsilon' p^2.
\]

We choose \( p \geq \max(p_0; p^1) \).

Then, taking the sup on \( x \in X \), we get
\[
\delta(p, \lambda) \leq 1 + \lambda \epsilon' p^2 + \lambda^2 E \left( d_{p}^2 e^{\lambda d_p} \right).
\]

We now chose \( \lambda = \lambda(p) \) small enough such that
\[
\lambda^2 E \left( d_{p}^2 e^{\lambda d_p} \right) \leq \frac{\epsilon' p^2}{2}.
\]

Then
\[
\delta(p, \lambda) \leq 1 + \lambda \epsilon' p,
\]

and therefore, since \( \ln(1 + x) \leq x \),
\[
\ln(\delta(p, \lambda)) \leq \lambda \epsilon' p.
\]

5. Walking away uniformly

\textbf{Definition 5.1.} A sequence of random variables \((Z_n)_{n \in \mathbb{N}}\) taking values in a metric space \( X \) is said to satisfy the \textbf{walking away uniformly property} if there are constants \( \epsilon, \alpha, C > 0 \) such that for all \( x \in X \) and for all \( n \in \mathbb{N} \)
\[
\mathbb{P} \left( d(Z_n, x) - d(z_0, x) \leq \epsilon n \right) \leq Ce^{-\alpha n}.
\]

Note that the above definition does not actually depend on the random variables \((Z_n)_{n \in \mathbb{N}}\) but only on their laws. We shall use this fact in the proof of the following theorem by exhibiting a special set of random variables which have the desired law.

\textbf{Theorem 5.2.} Let \( \Gamma \) be a countable group acting by isometries on a metric space \( X \) and \( \mu \) a probability measure on \( \text{Isom}(X) \) whose support generates a semi-group which contains a Schottky set and which has unbounded orbits. Then, (the law of) \((z_n)_{n \in \mathbb{N}}\) satisfies the walking away uniformly property.
5.1. Overview of the argument. The proof of the above theorem is quite intricate. Let us start by noticing that Theorem 5.2 is invariant under acceleration: given \( k \in \mathbb{N} \) a measure \( \mu \) with a finite exponential moment satisfies the conclusion of Theorem 5.2 if and only if the measure \( \mu^k \) satisfies it. Therefore, since we assumed that \( \mu \) is admissible and up to taking some power of \( \mu \), we can suppose that \( \text{supp}(\mu) \) contains a Schottky set as in Proposition 2.8.

We start by showing that the above theorem is also invariant under sampling. More precisely, we will sample the positions \((z_n)_{n \in \mathbb{N}}\) along the times when drawing increments in a given set \( S \). We shall then use this sampling with respect to a Schottky set.

To make it precise, we will first exhibit a special family of increments \((\omega_i)_{i \in \mathbb{N}}\) (following the law \( \mu \)) through the introduction of the following random variables.

Let \( S \subset \text{supp}(\mu) \) be any finite set and
\[
\zeta := \inf_{\gamma \in S} \mu(\gamma) > 0 .
\]

Let \((\eta_i)_{i \in \mathbb{N}}\) be independent random variables following the Bernoulli law of parameter \( \zeta \). Let also \((V_i)_{i \in \mathbb{N}}\) i.i.d. random variables independent of the \( \eta_i \)s taking value in \( \Gamma \) with (common) distribution
\[
P(V_i = \gamma) := \begin{cases} (1 - \zeta)^{-1} (\mu(\gamma) - \zeta^{-1}) & \text{if } \gamma \in S \\ (1 - \zeta)^{-1} \mu(\gamma) & \text{if } \gamma \notin S \end{cases}
\]

This distribution defines a probability measure on \( \Gamma \) since \( \mu(\gamma) - \zeta^{-1} \geq 0 \) by definition of \( \zeta \) and since its total mass is 1 by construction. Note also that the random variables \((d(V_i, z_0, z_0))_{i \in \mathbb{N}}\) have a finite exponential moments since the \((d(\omega_i, z_0, z_0))_{i \in \mathbb{N}}\) has one too (their laws are proportional on all but finitely many \( \gamma \in \Gamma \)).

Let us now introduce the last set of random variables that we will need. Let \((S_i)_{i \in \mathbb{N}}\) be i.i.d. random variables uniformly distributed on \( S \) independent of all the \( \eta_i \)s and of the \( V_i \)s:
\[
P(S_i = \gamma) := \begin{cases} (\sharp S)^{-1} & \text{if } \gamma \in S \\ 0 & \text{if } \gamma \notin S \end{cases}
\]

In total, we are left with three sets of random variables which are all independent from one another. Finally, note that the following defined random variables (also taking values in \( \Gamma \))
\[
\omega_i := \begin{cases} S_i & \text{if } \eta_i = 1 \\ V_i & \text{if } \eta_i = 0 \end{cases}
\]

follow the law of \( \mu \). Indeed, by construction of the \( V_i \)s the \( S_i \)s and the \( \eta_i \)s one has
\[
P(\omega_i = \gamma) = \begin{cases} P(V_i = \gamma) P(\eta_i = 0) + P(\eta_i = 1) P(S_i = \gamma) & \text{if } \gamma \in S \\ P(V_i = \gamma) P(\eta_i = 0) & \text{if } \gamma \notin S \end{cases}
\]

We endow our new probability space with the filtration \((\mathcal{F}_i)_{i \in \mathbb{N}}\) corresponding to events which can be expressed using the random variables defined above only with indices \( \leq i \).

Let \( p \in \mathbb{N} \) fixed. We now define the \((S, p)\)-sampling we will use through the following sequence of stopping times, defined inductively
\[
\tau(1) := \inf \{k \geq p \, , \, \eta_k = 1\}
\]
\[
\tau(i) := \inf \{k \geq p + \tau(i - 1) \, , \, \eta_k = 1\} \text{ if } i > 1 .
\]
The reason why we introduce an extra parameter $p$ will become clear later. Intuitively, we will use this parameter in order to guarantee that the average distance the random walk travels between positions at times $\tau(i)$ and $\tau(i+1)$ is large compared to the hyperbolic constant $\delta$ and the constant $C$ appearing in Proposition 2.8.

Note that the random variables $(\tau(i+1)-\tau(i))_{i>1}$ are i.i.d. following the law of $\tau(1)$ since the $(\eta_k)_{k \in \mathbb{N}}$ are i.i.d.

The sampling on $\Gamma$ is defined according to the previously defined stopping time. Namely, it is the random walk whose successive positions are

$$\gamma_{\tau(n)} = \omega_1 \cdot ... \cdot \omega_{\tau(n)}.$$ 

By construction (the $(\tau(i+1)-\tau(i))_{i>1}$ are i.i.d.) the random variable $\gamma_{\tau(n)}$ follows the law $\mu_{\tau_n}$, where

$$\mu_{\tau}(\gamma) := \mathbb{P}(\omega_1 \cdot ... \cdot \omega_{\tau(1)} = \gamma).$$

**Definition 5.3.** The corresponding image random walk on $X$, whose positions are $z_{\tau(n)} = \gamma_{\tau(n)} \cdot z_0$, is called the $(S,p)$-sampling of $(z_n)_{n \in \mathbb{N}}$.

The following lemma guarantees that one can prove Theorem 5.2 for sampled random walk instead of for the initial one.

**Proposition 5.4.** Let $\mu$ be a probability measure with a finite exponential moment on a group $\Gamma$ which acts on a metric space $X$. Let $S \subseteq \text{supp}(\mu)$, $p > 0$ and $z_0 \in X$.

The image random walk driven by $\mu$ satisfies the walking away uniformly property if and only if its $(S,p)$-sampling satisfies it too.

In order to keep this subsection as an overview, we postpone the proof of the above Proposition to Subsection 5.3. The proof relies on the following simpler lemma whose proof is also postponed.

**Lemma 5.5.** Let $\mu$ be a probability measure with a finite exponential moment on a group $\Gamma$ which acts on a metric space $X$, $z_0 \in X$ and $\tau(1)$ as above. Then, the random variables $d(z_{\tau(1)}, z_p)$, $d(z_{\tau(1)}, z_0)$ and $d(z_{\tau(1)-1}, z_p)$ have a finite exponential moment.

We will then prove that the $(S,p)$-sampled random walk satisfies the walking away uniformly property. In order to do so, we shall introduce a last type of random walks. Intuitively, a $(S,p)$-sampling can be thought as a process in two steps. First, we ignore the first $p$ increments and we do not draw from $S$ (corresponding to $\eta_k = 0$) for a random time which follows a geometric law. Secondly, we draw an element uniformly from the set $S$. We shall make this precise by showing that a $(S,p)$-sampled random walk can be seen as a random walk whose odd increments correspond to the first step described above and the even ones to the second step, as in the following definition.

Let $\mu_1$ be a probability measure on $\Gamma$, $(X_i)_{i \in \mathbb{N}+1}$ i.i.d. random variables following the law $\mu_1$ and $(Y_i)_{i \in \mathbb{N}}$ i.i.d. random variables uniformly distributed on the set $S$ and independent of the $X_i$'s.

**Definition 5.6.** Let $(X_i)_{i \in \mathbb{N}+1}$ and $(Y_i)_{i \in \mathbb{N}}$ as above. We call (the laws of the) the following sequence of random variables a $(\mu_1,S)$-random walk

$$z^\mu_{n,S} := \begin{cases} 
X_1 \cdot Y_2 \cdot X_3 \cdot ... \cdot X_n \cdot z_0 & \text{if } n \text{ is odd} \\
X_1 \cdot Y_2 \cdot X_3 \cdot ... \cdot Y_n \cdot z_0 & \text{if } n \text{ is even}
\end{cases}$$

The following lemma relates the position at time $n$ of a $(S,p)$-sampled random walk to the position at time $2n$ of an $(\mu_1,S)$-random walk.
Lemma 5.7. Let $\mu$ be a probability measure on a group $\Gamma$ which acts on a space $X$, $z_0 \in X$ and $(\tau(i))_{i \in \mathbb{N}}$ as above. The sequence of random variables $(z_{\tau(n)})_{n \in \mathbb{N}}$ follows the law of the sequence $(z_{\mu_1, S})_{n \in \mathbb{N}}$ with

$$\mu_1(\gamma) := P(\omega_1 \cdot \ldots \cdot \omega_p \cdot \ldots \cdot \omega_{\tau(1)-1} = \gamma) .$$

In particular a $(S,p)$-sampled random walk satisfies the walking away uniformly property if and only if its associated $(\mu_1, S)$-random walk satisfies it too.

**Proof.** We first set the random variables $X_i$s and $Y_i$s as

$$\begin{cases} Y_{2i} := \omega_{\tau(i)} \\ X_{2i+1} := \omega_{\tau(i)+1} \cdot \ldots \cdot \omega_{\tau(i+1)-1} . \end{cases}$$

Then, by definition, $z_{\tau(n)} = X_1 \cdot Y_2 \cdot X_3 \cdot \ldots \cdot X_{2n-1} \cdot Y_{2n} \cdot z_0$. It follows from the independence properties of the random variables $X_i$s, $Y_i$s and $\eta_i$s that the random variables $(X_{2i-1}, Y_{2i})_{i \geq 1}$ are I.I.D. Using the fact that, on the set $\tau(1) = k$, we have $Y_2 = S_k$ and $X_1 = V_1 \cdot \ldots \cdot V_{k-1}$, it is also easy to see that $X_1$ and $Y_2$ are independent.

The next step is to find a criterion on $\mu_1$ which guarantees that if $S$ is a Schottky set then the associated $(\mu_1, S)$-random walk satisfies the walking away uniformly property. Let then fix $S \subset \text{supp}(\mu)$ a Schottky set as in Proposition 2.8. The following Proposition is the key and its proof will occupy all Section 6.

**Proposition 5.8.** For any Schottky set $S$ there is a constant $M > 0$ such that the following holds. For any probability measure $\mu_1$ with a finite exponential moment and

$$\sum_{\gamma \in \Gamma} \mu_1(\gamma) \ d(z_0, \gamma \cdot z_0) > M$$

the $(\mu_1, S)$-random walk satisfies the walking away uniformly property.

Let us see how to deduce Theorem 5.2 with all the material introduced above. Recall that we fix $\mu$ a probability measure on $\Gamma$ and $S$ a Schottky set contained in the support of $\mu$. Proposition 5.4 implies that it is sufficient to prove the walking away uniformly property for the $(S,p)$-sampled random walk. Because of Lemma 5.7, we know that the $(S,p)$-sampled random walk is also a $(\mu_1, S)$-random walk with

$$\mu_1(\gamma) := P(\omega_1 \cdot \ldots \cdot \omega_p \cdot \ldots \cdot \omega_{\tau(1)-1} = \gamma) .$$

It remains to show that the resulting $(\mu_1, S)$-random walk satisfies the conditions of Proposition 5.8. Note that Lemma 5.5 already asserts that $\mu_1$ has a finite exponential moment. The following lemma ensures that we can choose $p$ such that the mean $\sum_{\gamma \in \Gamma} \mu_1(\gamma) \ d(z_0, \gamma \cdot z_0)$ exceeds $M$.

**Lemma 5.9.** Let $\Gamma$ be a countable group acting by isometries on a metric space $X$ and $\mu$ be a probability measure on $\text{Isom}(X)$ whose support generates a semi-group with unbounded orbits and assume that $\mu$ has a finite first moment. Then

$$\limsup_{p \to \infty} E(d(z_0, z_{\tau(1)-1})) = \infty .$$

The following subsections are devoted to the proofs of all the above lemmata, except Proposition 5.8 which will be proven in Section 6.
5.2. Proof of Lemma 5.5. The differences of any two of the three random variables appearing in Lemma 5.5 obviously have a finite exponential moment. It is therefore sufficient to prove Lemma 5.5 for one of them only; say \(d(z_p, z_{\tau(1)})\). The proof is a straightforward computation. It only uses that \(\mu\) has a finite exponential moment together with the fact that \(\tau(1) - p\) follows a geometric law of parameter \(1 - \zeta\). Given \(\lambda > 0\) we compute

\[
\mathbb{E}\left(e^{\lambda d(z_{\tau(1)}, z_p)}\right) \leq \mathbb{E}\left(\exp\left(\lambda \sum_{p \leq i \leq \tau(1)-1} d(z_i, z_{i+1})\right)\right) \\
\leq \sum_{k \in \mathbb{N}} \mathbb{E}\left(\exp\left(\lambda \sum_{p \leq i \leq k-1} d(z_i, z_{i+1})\right) \mid \tau(1) = k\right) \mathbb{P}(\tau(1) = k) \\
\leq \zeta \sum_{k \in \mathbb{N}} \mathbb{E}\left(\exp\left(\lambda \sum_{p \leq i \leq k-1} d(z_i, z_{i+1})\right) \mid \tau(1) = k\right) (1 - \zeta)^{k-p}.
\]

We shall now see that for all \(\epsilon > 0\) there is \(\lambda > 0\) such that for every \(k \in \mathbb{N}\)

\[
(5.10) \quad E_k := \mathbb{E}\left(\exp\left(\lambda \sum_{p \leq i \leq k-1} d(z_i, z_{i+1})\right) \mid \tau(1) = k\right) < (1 + \epsilon)^k.
\]

It concludes the proof since we can choose \(\epsilon\) such that \((1 + \epsilon)(1 - \zeta)^{-1} < 1\).

Let us check (5.10). The event \(\tau(1) = k\) is defined as \(\eta_0 = 0, \eta_{p+1} = 0, ..., \eta_k = 1\). Therefore, by construction of the \(X_i\)s, we have

\[
E_k = \mathbb{E}\left(\exp\left(\lambda \sum_{p \leq i \leq k-1} d(V_i \cdot z_0, z_0)\right) e^{\lambda d(z_0, S_k \cdot z_0)} \mid \tau(1) = k\right).
\]

But \(\tau(1)\) is a function of the \(\eta_i\)s only and therefore is independent of \(S_k\) and independent of the \((V_i)_{1 \leq i \leq k-1}\). It yields

\[
E_k = \mathbb{E}\left(\exp\left(\lambda \sum_{p \leq i \leq k-1} d(V_i \cdot z_0, z_0)\right) e^{\lambda d(z_0, S_k \cdot z_0)}\right) = \left(\mathbb{E}\left(e^{\lambda d(V_1 \cdot z_0, z_0)}\right)\right)^{k-1-p} \mathbb{E}\left(e^{\lambda d(z_0, S_k \cdot z_0)}\right),
\]

since the \(V_i\)s are I.I.D. and independent of \(S_k\). This concludes the proof since we already saw that \(d(V_i \cdot z_0, z_0)\) has a finite exponential moment and since \(S_k\) has finite support.

5.3. Proof of Proposition 5.4. We will prove that: if the random walk \((z_{\tau(n)})_{n \in \mathbb{N}}\) satisfies the walking away uniformly property then \((z_n)_{n \in \mathbb{N}}\) satisfies it too. This is the only implication we need in this paper. The proof of the other implication is very similar.

Let \(\epsilon, C\) and \(\alpha\) such that for any \(x \in X\) we have

\[
\mathbb{P}(d(z_{\tau(n)}, x) - d(z_0, x) \leq \epsilon n) \leq Ce^{-\alpha n}.
\]

We set \(\beta := \mathbb{E}(\tau)\). We will show that \((z_{\beta n})_{n \in \mathbb{N}}\) satisfies the walking away uniformly property, which implies the result using again the invariance under acceleration.
Rewriting \( d(z_0, z_{\beta n}) \) as \( d(z_0, z_{\beta n}) - d(z_0, z_{\tau(n)}) + d(z_0, z_{\tau(n)}) \), we have
\[
\left\{ \frac{d(z_{\beta n}, x) - d(z_0, x)}{2} \leq \epsilon n \right\} \subset \\
\left\{ \frac{d(z_{\tau(n), x} - d(z_0, x)}{\epsilon n} \} \cup \left\{ d(x, z_{\tau(n)}) - d(x, z_{\beta n}) \geq \frac{\epsilon n}{2} \right\} .
\]

And then:
\[
P \left( \frac{d(z_{\beta n}, x) - d(z_0, x)}{2} \leq \epsilon n \right) \leq \\
P \left( \frac{d(z_{\tau(n), x} - d(z_0, x)}{\epsilon n} \} + P \left( d(x, z_{\tau(n)}) - d(x, z_{\beta n}) \geq \frac{\epsilon n}{2} \right) .
\]

Since we assumed that \((z_{\tau(n)})_{n\in\mathbb{N}}\) satisfies the walking away uniformly property we already know that \(P \left( \frac{d(z_{\tau(n), x} - d(z_0, x)}{\epsilon n} \right)\) has an exponential decay to 0, uniformly in \(x\).

It remains then to show that
\[
P \left( d(x, z_{\tau(n)}) - d(x, z_{\beta n}) \right) \geq \frac{\epsilon n}{2}
\]
decreases exponentially fast in \(n\), uniformly in \(x\). We will actually show that for all \(a > 0\)
\[
P \left( d(x, z_{\tau(n)}) - d(x, z_{\beta n}) \right) \geq a n
\]
decreases exponentially fast, uniformly in \(x\). By the triangular inequality we have
\[
\sum_{i \in \{\tau(n), \beta n\}} d(z_i, z_{i+1}) \geq d(z_{\beta n}, z_{\tau(n)}) \geq d(x, z_{\tau(n)}) - d(x, z_{\beta n}) ,
\]
and therefore
\[
P \left( \sum_{i \in \{\tau(n), \beta n\}} d(z_i, z_{i+1}) \right) \geq a n
\]
Note that the left hand side does not depend on \(x\) anymore. Define \(Z_i := d(z_i, z_{i+1})\).

The desired result will follow once we prove that for all \(a > 0\)
\[
P \left( \sum_{i \in \{\tau(n), \beta n\}} Z_i \right) \geq a n
\]
decreases exponentially. The above summation is a summation of I.I.D. random variables over a random time interval. In order to control it, we shall first control the random time with a large deviations estimate for I.I.D. random variables and conclude by controlling the summation using again a large deviations estimate for I.I.D. random variables

Recall that, by construction of the sampling, one has
\[
\tau(n) = \sum_{i \leq i \leq n} (\tau(i+1) - \tau(i)) ,
\]
the \((\tau(i+1) - \tau(i))\),\(\beta\) being I.I.D. distributed as \(\tau(1)\) (in particular they have a finite exponential moment).

Let \(\alpha > 0\) such that
\[
\alpha \cdot E(Z_1) \leq \frac{a}{4} .
\]
We use the large deviations estimate for \(\tau(n)\) (which is a summation of I.I.D. random variables with a finite exponential moment): let \(c_1, c_2 > 0\) such that
\[
P(|\tau(n) - \beta n| \geq \alpha n) \leq c_1 e^{-c_2 n} .
\]
Recall that $\beta$ is the mean of $\tau$. Therefore,

$$\Pr\left( \sum_{i \in \{\tau(n), \beta n\}} Z_i \geq a n \right) \leq c_1 e^{-c_2 n} + \Pr\left( \left\{ \sum_{i \in \{\tau(n), \beta n\}} Z_i \geq a n \right\} \cap \{|\tau - \beta n| \leq a n\} \right).$$

Since the $Z_i$s are non-negative, one has

$$\Pr\left( \left\{ \sum_{i \in \{\tau(n), \beta n\}} Z_i \geq a n \right\} \cap \{|\tau - \beta n| \leq a n\} \right) \leq \Pr\left( \sum_{(\beta - \alpha)n \leq i \leq (\beta + \alpha)n} Z_i \geq a n \right).$$

We conclude rewriting the right member of the above inequality as

$$\Pr\left( \sum_{(\beta - \alpha)n \leq i \leq (\beta + \alpha)n} (Z_i - \mathbb{E}(Z_i)) \geq (a - 2\alpha \cdot \mathbb{E}(Z_i))n \right).$$

Recall that we chose $\alpha$ such that $a - 2\alpha \cdot \mathbb{E}(Z_i) \geq \frac{a}{2}$. The $Z_i$s are I.I.D. with a finite exponential moment. Hence they satisfy large deviations estimates and the above probability decreases exponentially fast.

5.4. Proof of Lemma 5.9. By Lemma 5.5 we know that $\mathbb{E}(d(z_p, z_{\tau(1)}))$ is finite; besides, by construction of $\tau$, it does not depend on $p$. Therefore, by the triangular inequality and linearity of the expectation, Lemma 5.9 will follow once we have proved that

$$\limsup_{p \to \infty} \mathbb{E}(d(z_0, z_p)) = \infty.$$  

We start noticing that, for any $R > 0$, the following stopping time

$$\tau_R := \inf\{k \in \mathbb{N} \mid d(z_0, z_k) \geq R\}$$

is almost surely finite. Indeed, there is at least one element $\gamma_0$ in the semi-group $\Gamma_n$ generated by $\text{supp}(\mu)$ such that $\gamma_0 \cdot B(z_0, R) \cap B(z_0, R) = \emptyset$: recall we assumed that $\Gamma_n$ has unbounded orbits. Therefore there exists $k_0$ such that $\Pr(\gamma_{k_0} = \gamma_0) > 0$. With probability one, there will be infinitely many times $k$ such that $\gamma_{-k}^{-1} \gamma_{k_0 + k} = \gamma_0$. This last property implies that almost any path eventually leaves the ball of radius $R$ around $z_0$.

We conclude the proof of Lemma 5.9 with the following

**Lemma 5.11.** Let $\mu$ be a probability measure on a group $\Gamma$ acting by isometries on a metric space $X$ and $z_0 \in X$. If for any $R > 0$ the time $\tau_R$ is almost surely finite, then

$$\limsup_{p \to \infty} \mathbb{E}(d(z_0, z_p)) = \infty.$$  

**Proof.** We have for any $n \in \mathbb{N}$ and any $R > 0$

$$\Pr(z_n \notin B(z_0, R)) \geq \Pr(\tau_{2R} \leq n \mid d(z_{r_2R}, z_n) \leq R) \geq \sum_{0 \leq k \leq n} \Pr(\tau_{2R} = k \mid d(z_k, z_n) \leq R) \geq \sum_{0 \leq k \leq n} \Pr(\tau_{2R} = k) \cdot \Pr(d(z_k, z_n) \leq R)$$

since the event $\tau_{2R} = k$, that only depends on the first $k$ increments of the walk and $d(z_k, z_n)$, that only depends on the later increments of the walk, are independent.
The random variable $d(z_k, z_n)$ follows the same law as $d(z_0, z_{n-k})$. Therefore

$$
\mathbb{P}(z_n \notin B(z_0, R)) \geq \sum_{0 \leq k \leq n} \mathbb{P}(\tau_{2R} = k) \cdot \mathbb{P}(d(z_0, z_{n-k}) \leq R)
$$

$$
\geq \mathbb{P}(\tau_{2R} \leq n) \inf_{0 \leq k \leq n} \mathbb{P}(d(z_0, z_k) \leq R)
$$

$$
\geq \mathbb{P}(\tau_{2R} \leq n)(1 - \sup_{0 \leq k \leq n} \mathbb{P}(z_k \notin B(z_0, R))).
$$

Let $a_n := \sup_{0 \leq k \leq n} \mathbb{P}(z_k \notin B(z_0, R)) \geq \mathbb{P}(z_n \notin B(z_0, R))$. We have shown that

$$
a_n \geq \mathbb{P}(\tau_{2R} \leq n)(1 - a_n),
$$

Recall we are assuming that $\tau_{2R}$ is almost surely finite. Therefore there exists $n$ such that $\mathbb{P}(\tau_{2R} \leq n) \geq 1/2$. For such an $n$, we get

$$
a_n \geq \frac{1}{2}.
$$

Therefore there exists $0 \leq k \leq n$ such that

$$
\mathbb{P}(z_k \notin B(z_0, R)) \geq \frac{1}{2},
$$

which implies in particular for the same $k$ that

$$
\mathbb{E}(d(z_0, z_k)) \geq \frac{R}{2},
$$

thus concluding the proof.

\[ \blacksquare \]

6. Proof of Proposition 5.8

Let us first give Proposition 5.8 a more precise statement.

**Proposition 6.1.** Let $S$ be a Schottky set and $\mu_1$ be a probability measure with a finite exponential moment such that

$$
\sum_{\gamma \in \Gamma} \mu_1(\gamma) \ d(z_0, \gamma \cdot z_0) > 6C + 6S_{\sup}
$$

where $C$ is as in Proposition 2.8 and $S_{\sup} := \sup_{s \in S} d(z_0, s \cdot z_0)$. Then the corresponding $(\mu_1, S)$-random walk has the walking away uniformly property.

**Proof.** In order not to burden the notations, we shall denote by $(Z_n)_{n \in \mathbb{N}}$ (instead of $(z_\mu^{\mu_1})_{n \in \mathbb{N}}$) the successive positions in $X$ of the $(\mu_1, S)$-random walk. To simplify a bit the exposition, let us first note that one can suppose the even increments of the walk to be $\mu_1$-increments and the odd ones to be Schottky increments. Indeed, since we assumed that $\mu_1$ has a finite exponential moment, the walking away uniformly property does not depend on the first increment of the walk. With the notation introduced in Part 5 to define the $(\mu_1, S)$ random walk, we have $Z_n = Y_n \cdot z_0$ with

$$
Y_n := \begin{cases} 
Y_1 \cdot X_2 \cdots X_n & \text{if } n \text{ is even} \\
Y_1 \cdot X_2 \cdots Y_n & \text{if } n \text{ is odd} 
\end{cases}.
$$

We start with the obvious equality

$$
\mathbb{P}(d(Z_{2n}, x) - d(z_0, x) \leq \epsilon n) = \mathbb{P} \left( \sum_{0 \leq i \leq n} d(Z_{2i+2}, x) - d(Z_{2i}, x) \leq \epsilon n \right).
$$

For any $x, y, z \in X$, we let

$$
B_x(z, y) := d(z, x) - d(y, x),
$$
so that
\[ \sum_{0 \leq i \leq n} d(Z_{2i+2}, x) - d(Z_{2i}, x) = \sum_{0 \leq i \leq n} B_2(Z_{2i+2}, Z_{2i}) := S_n(x). \]
Since \( d(Z_{2i+2}, z_0) - d(Z_{2i}, z_0) \leq d(Z_{2i+2}, Z_{2i}) \), the random variable \( B_2(Z_{2i+2}, Z_{2i}) \) has a finite exponential moment. Using Markov inequality for a small enough \( \lambda > 0 \), we get that
\[ \mathbb{P}(S_n(x) \leq e^n) \leq e^{\lambda n} \mathbb{E}(e^{-\lambda S_n(x)}). \]
We will be done once we prove that there exists \( \lambda > 0 \) and \( 0 < \delta < 1 \) (which may depend on \( \lambda \)) such that for all \( x \)
\[ (6.2) \mathbb{E}(e^{-\lambda S_n(x)}) \leq \delta^n. \]
Recall that we denoted by \((\mathcal{F}_n)_{n \in \mathbb{N}}\) the filtration of \( \Omega \) with respect to the increments of the walk. Conditioning the right member with respect to \( \mathcal{F}_{2n-2} \) gives
\[ \mathbb{E}(e^{-\lambda S_n(x)}) = \mathbb{E}(\mathbb{E}(e^{-\lambda S_{n-1}(x)} | \mathcal{F}_{2n-2})) = \mathbb{E}(e^{-\lambda S_{n-1}(x)} \cdot \exp(-\lambda B_{\mathcal{F}_{2n-2}^{-1}(z_0)}(2n-1, \mathcal{F}_{2n-2})) \mathcal{F}_{2n-2} \mathcal{F}_{2n-2}) \]
since \( S_{n-1}(x) \) is \( \mathcal{F}_{2n-2} \) measurable. Because \( \mathcal{F}_{2n-2}^{-1}(z_0) \) is \( \mathcal{F}_{2n-2} \) measurable and \( \mathcal{F}_{2n-2}^{-1}(z_0) \) is independent of \( \mathcal{F}_{2n-2} \), we have
\[ \mathbb{E}(\exp(-\lambda B_{\mathcal{F}_{2n-2}^{-1}(z_0)}(2n-1, \mathcal{F}_{2n-2}))) \leq \sup_{y \in X} \mathbb{E}(\exp(-\lambda B_{\mathcal{F}_{2n-2}^{-1}(z_0)}(2n-1, \mathcal{F}_{2n-2}))) \]
\[ = \sup_{y \in X} \mathbb{E}(\exp(-\lambda B_{\mathcal{F}_{2n-2}^{-1}(z_0)}(2n-1, \mathcal{F}_{2n-2}))) \]
We get by an immediate induction that
\[ \mathbb{E}(e^{-\lambda S_n(x)}) \leq \delta(\lambda)^n, \]
where
\[ \delta(\lambda) := \sup_{y \in X} \mathbb{E}(e^{-\lambda B_{\mathcal{F}_{2n-2}^{-1}(z_0)}}). \]
We end this proof by showing the
\[ \text{Lemma 6.3. There is } \lambda > 0 \text{ such that} \]
\[ \sup_{y \in X} \mathbb{E}(e^{-\lambda B_{\mathcal{F}_{2n-2}^{-1}(z_0)}}) < 1 \].

**Proof.** We denote by \( A^c \) the complement of a set \( A \). Given \( y \in X \), we use the decomposition
\[ B_y(Z_2, z_0) = B_y(Z_2, z_0) \mathbb{1}_A + B_y(Z_2, z_0) \mathbb{1}_{A^c}, \]
where \( A := \{(Z_2, y)_{z_0} \leq C\} \) and \( C \) is the constant given by Proposition 2.8.

Note that on \( A \), since the first increment of the walk is in \( S \), we have
\[ B_y(Z_2, z_0) \geq d(Z_2, z_0) - 2C \]
\[ \geq d(Z_2, z_0) - 2C - d(Z_2, z_0) \]
\[ \geq d(Z_2, z_0) - 2C - S_{\sup}, \]
(we recall that we set \( S_{\sup} := \sup_{s \in S} (d(z_0, s \cdot z_0)) \).
On $A^c$ we use the trivial lower bound
\[ B_0(Z_2, z_0) \geq -d(Z_2, z_0) \geq -d(Z_2, Z_1) - S_{\text{sup}}. \]

We thus obtain the inequality
\[
e^{-\lambda B_0(Z_2, z_0)} \leq e^{-\lambda(d(Z_2, Z_1) - 2C - S_{\text{sup}})} P_A + e^{-\lambda d(Z_2, Z_1) - S_{\text{sup}}} I_{A^c} = e^{\lambda S_{\text{sup}}} \left( e^{-\lambda d(Z_2, Z_1) - 2C} P_A + e^{\lambda d(Z_2, Z_1)} I_{A^c} \right)\]

Since the distances appearing in the exponentials do not depend on $Y_1$ but only on $X_2$, we have
\[
\mathbb{E} \left( e^{-\lambda B_y(Z_2, z_0)} \mid X_2 \right) \leq e^{\lambda S_{\text{sup}}} \left( e^{-\lambda d(Z_2, Z_1) - 2C} \mathbb{P}(A \mid X_2) + e^{\lambda d(Z_2, Z_1)} \mathbb{P}(A^c \mid X_2) \right)
\leq e^{\lambda S_{\text{sup}}} \left( e^{-\lambda d(Z_2, Z_1) - 2C} \alpha(X_2) + e^{\lambda d(Z_2, Z_1)} (1 - \alpha(X_2)) \right),
\]
where we set
\[
\alpha(X_2) := \mathbb{P}(A \mid X_2) = \mathbb{P}((Z_2, y)_z \leq C \mid X_2) = \frac{\sharp \{ s \in S, (s \cdot X_2 \cdot z_0, y)_z \leq C \}}{\sharp S},
\]
using the fact that the first increment is uniformly distributed on $S$. Because $S$ is a Schottky set, we readily get that
\[
(6.4) \quad \alpha(X_2) \geq \frac{2}{3},
\]
for all $y$.

Next, we use the lower bound on $\mathbb{E}(d(Z_2, Z_1))$ to argue that, in the upper-bound above, out of the two competing exponentials, the main contribution comes from the term $e^{-\lambda d(Z_2, Z_1) - C}$.

Recall the general upper bound, $e^x \leq 1 + x + x^2 e^{|x|}$ and set
\[ R(\lambda, X_2) := \lambda^2 \left( d(Z_2, Z_1) + 2C \right)^2 e^{\lambda d(Z_2, Z_1)}. \]

We then estimate
\[
\mathbb{E} \left( e^{-\lambda B_y(Z_2, z_0)} \mid X_2 \right) 
\leq e^{\lambda S_{\text{sup}}} \left( \alpha(X_2) - \lambda d(Z_2, Z_1) - 2C \right) \alpha(X_2) + 1 - \alpha(X_2) + \lambda d(Z_2, Z_1) \left( 1 - \alpha(X_2) \right) + e^{2\lambda C} R(\lambda, X_2) \right) 
\leq e^{\lambda S_{\text{sup}}} \left( \left( 1 - \lambda d(Z_2, Z_1) \left( 2 \alpha(X_2) - 1 \right) + 2\lambda C \alpha(X_2) + e^{2\lambda C} R(\lambda, X_2) \right) \right) 
\leq e^{\lambda S_{\text{sup}}} \left( \left( 1 - \frac{1}{3} \lambda d(Z_2, Z_1) + 2\lambda C + e^{2\lambda C} R(\lambda, X_2) \right) \right).
\]

We used the bound (6.4) and the fact that $\alpha(X_2) \leq 1$. Taking the expectation in this last inequality and using the lower bound on $\mathbb{E}(d(Z_2, Z_1))$, we get that
\[
\mathbb{E} \left( e^{-\lambda B_y(Z_2, z_0)} \right) \leq e^{\lambda S_{\text{sup}}} \left( 1 - 2\lambda S_{\text{sup}} + e^{2\lambda C} \mathbb{E}(R(\lambda, X_2)) \right).
\]

Choose $\lambda_0 > 0$ such that
\[ C_2 := \mathbb{E} \left( \left( d(Z_1, Z_2) + C \right)^2 e^{\lambda_0 d(Z_1, Z_2)} \right) < \infty. \]

Then, for all $\lambda \leq \lambda_0$,
\[
\mathbb{E}(R(\lambda, X_2)) \leq C_2 \lambda^2
\]
and we get that
\[ E\left(e^{-\lambda S_n(z_0,z_0)}\right) \leq e^{\lambda S_{\sup}} \left(1 - 2\lambda S_{\sup} + C_2 \lambda^2 e^{2\lambda C}\right). \]
The right hand side of this last inequality is clearly $< 1$ for some positive but small enough $\lambda$.

\[ \blacksquare \blacksquare \]

7. Deviation inequalities

We show that a random walk which satisfies linear progress with exponential tail also satisfies the following property.

**Definition 7.1.** [MS] Let $(z_n)_{n \in \mathbb{N}}$ be a random path in a metric space $X$. We say that $(z_n)_{n \in \mathbb{N}}$ satisfies the exponential-tail deviation inequality if there are constants $C_1, C_2 > 0$ such that for all $0 \leq i \leq n$ and all $R > 0$ one has
\[ \mathbb{P}(\{z_i \geq R\}) \leq C_1 e^{-C_2 R}. \]

We adapt the proof of [MS, Theorem 11.1] to prove the following

**Proposition 7.2.** Let $\Gamma$ be a countable group acting by isometries on a geodesic hyperbolic space $X$ and $\mu$ a probability measure on $\Gamma$ with a finite exponential moment. If the random walk has linear progress with exponential tail then it satisfies the exponential-tail deviation inequalities.

**Remark 7.3.** In the case where $\Gamma$ acts acylindrically on a geodesic hyperbolic space, the Proposition is already proved in [MS, Theorem 10.7].

**Proof.** Given a geodesic $v$, we denote by $\pi_v$ a choice of nearest point projection from $X$ to $v$. Given two points $x, y \in X$ we denote by $[x, y]$ the choice of any geodesic path joining $x$ to $y$. Given any $y \in X$ we define
\[ N_v(y) := \{ x \in X \mid d(\pi_v(y), \pi_v(x)) \geq d(x, \pi_v(x))\}. \]
Note that the above set actually depends on $\pi_v(y)$ only and, in particular, not on $d(y, v)$. We refer to [MT18a, Section 2] and [Mah10, Section 3] for more details about the nearest point retraction.

Let $(x_i)_{i \in I}$ be a discrete path whose endpoints lie on the geodesic $v$. Given $k \in I$ we define
\[ \begin{align*}
  k_1 & := \sup \{ j < k \mid [x_j, x_{j+1}] \cap N_v(x_k) \neq \emptyset \} \\
  k_2 & := \inf \{ j \geq k \mid [x_j, x_{j+1}] \cap N_v(x_k) \neq \emptyset \}.
\end{align*} \]
Note that $d(x_{k_i}, x_k) \geq d(x_k, \pi_v(x_k)) - 100\delta$ for $i = 1, 2$ (see [MS, Lemma 11.4, Claim 1] and Figure 2). The following lemma is the geometric key of the proof.

**Lemma 7.4.** [MS, Lemma 11.4] For any $\epsilon > 0$ there are constants $c_1, c_2 > 0$ such that if

1. $d(x_{k_1}, x_{k_2}) \geq \epsilon (k_2 - k_1)$;
2. $d(x_{k_1}, x_{k_{i+1}}) \leq d(x_{k_1}, x_{k_2})/100$;
3. $d(x_{k_2}, x_{k_{i+1}}) \leq d(x_{k_1}, x_{k_2})/100$,

then
\[ \sum_{i \in [k_1+1, k_2-1]} d(x_i, x_{i+1}) \geq c_2 e^{c_1(k_2 - k_1)}. \]

The statement above is a simplified version of [MS, Lemma 11.4]. The proof follows the same line and is illustrated through Figure 2.
We now use Lemma 7.4 with the successive positions of the random walk \((x_i = z_i)\). Recall that we want to show that there are constants \(C_1, C_2\) such that for any \(k, n > 0\) and any \(R > 0\) we have

\[
\mathbb{P}((z_n, z_0)_z > R) \leq C_1 e^{-C_2 R}.
\]

We fix \(k, n\) and \(R > 0\). For a path \((z_j)\) satisfying \((z_n, z_0)_z > R\), we define the times \(k_1, k_2\) as in Lemma 7.4.

We distinguish two cases, depending on whether or not \(k_2 - k_1\) is large with respect to \(R\).

The next lemma addresses the case of paths with a small value for \(k_2 - k_1\).

**Lemma 7.5.** There are constants \(c_3, c_4, C > 0\) such that

\[
\mathbb{P}((z_n, z_0)_z > R) \cap \{k_2 - k_1 \leq c_3 R\} \leq C e^{-c_4 R}.
\]

**Proof.** We will look at all the possible values of \(k_1, k_2\) and conclude using the union bound.

Since we assumed that \((z_n, z_0)_z > R\) and by construction of \(N_v(z_k)\), we have that \(d(z_{k_1}, z_k) \geq R - 100\delta\) [MS, Lemma 11.4, Claim 1]. Let \(0 \leq m < c_3 R\) for some \(c_3\) that we will fix later on. Choose \(\alpha < k\) and \(\beta > k\) such that \(\beta - \alpha = m\). We have

\[
\mathbb{P}((z_n, z_0)_z > R), k_2 = \beta, k_1 = \alpha) \leq \mathbb{P} (d(z_{\alpha}, z_k) \geq R - 100\delta).
\]
Using the triangular inequality we get
\[
\mathbb{P}(d(z_\alpha, z_k) \geq R - 100\delta) \leq \mathbb{P}\left( \sum_{\alpha \leq i \leq k - 1} d(z_i, z_{i+1}) \geq R - 100\delta \right) \\
\leq \mathbb{P}\left( \sum_{\alpha \leq i \leq \beta - 1} d(z_i, z_{i+1}) \geq R - 100\delta \right) \\
= \mathbb{P}\left( \sum_{0 \leq i \leq m - 1} d(z_i, z_{i+1}) \geq R - 100\delta \right).
\]
Taking the Laplace transform and using Markov’s inequality we get that, for all \( \lambda > 0 \),
\[
\mathbb{P}\left( \sum_{0 \leq i \leq m - 1} d(z_i, z_{i+1}) \geq R - 100\delta \right) \leq e^{\lambda(R - 100\delta)} \left( \mathbb{E}(e^{\lambda d(z_0, z_1)}) \right)^m \\
\leq C e^{-\lambda R} \left( \mathbb{E}(e^{\lambda d(z_0, z_1)}) \right)^{c_3 R}.
\]
From this last inequality, provided we choose \( \lambda \) such that \( \mathbb{E}(e^{\lambda d(z_0, z_1)}) < \infty \) and \( c_3 \) small enough, we deduce that
\[
\mathbb{P}\left( (z_n, z_0)_z \gg R , k_2 - k_1 = \beta , k_1 = \alpha \right) \leq C e^{-cR},
\]
for some constants \( C \) and \( c \). Summing over the possible choices of \( \beta \) and \( \alpha \), we get the Lemma.

The next lemma deals with the remaining case corresponding to \( k_2 - k_1 \geq c_1 R \) and concludes the proof of Proposition 7.2.

**Lemma 7.6.** For any \( c_1 > 0 \), there are constants \( c_5, c_6 \) such that we have
\[
\mathbb{P}\left( (z_n, z_0)_z \gg R , k_2 - k_1 = \beta \right) \leq c_5 e^{-c_6 R}.
\]

**Proof.** We shall prove that, for any \( m > 0 \), then
\[
\mathbb{P}\left( (z_n, z_0)_z \gg R , k_2 - k_1 = m \right) \leq c_5 e^{-c_6 m}.
\]
The Lemma follows by summing over all \( m \geq c_1 R \) (with slightly different values for \( c_5 \) and \( c_6 \)).

Let us then fix \( m > 0 \). In the same way as for the proof of Lemma 7.5 we first fix \( k_1 = \alpha \) and \( k_2 = \beta \) with \( \beta - \alpha = m \) and then use the union bound.

Recall that since we assumed that the walk has linear progress with exponential tail one has constants \( \epsilon, c_7, c_8 > 0 \) such that
\[
\mathbb{P}\left( d(z_\alpha, z_\beta) \leq \epsilon m \right) \leq c_7 e^{-c_8 m}.
\]
There are also constants \( c_9, c_{10} \) such that
\[
\mathbb{P}\left( d(z_\alpha, z_{\alpha+1}) \geq \frac{d(z_\alpha, z_\beta)}{100} \right) \leq \mathbb{P}\left( d(z_\alpha, z_{\alpha+1}) \geq \frac{\epsilon m}{100} \right) + c_7 e^{-c_8 m} \\
\leq c_9 e^{-c_{10} m}.
\]

since \( d(z_\alpha, z_{\alpha+1}) \) has a finite exponential moment. A similar bound applies to \( d(z_\beta, z_{\beta+1}) \).
Remark 8.2. If we further assume square of the quantity we want to bound deduce Proposition 8.1 from Proposition 1.12. Indeed, let us rewrite as follows the \( d \)

\[ \sum_{\alpha \leq i \leq \beta} d(z_i, z_{i+1}) \geq c_2 e^{\epsilon_1 m}. \]

The probability of the above event is (super)-exponentially small in \( m = \beta - \alpha \).

\[ \square \]

8. Hitting measure

The purpose of this section is to prove the following proposition.

**Proposition 8.1.** Let \( \Gamma \) be a countable group acting non elementarily and by isometries on a geodesic hyperbolic space \( X \) and \( \mu \) an admissible probability measure on \( \Gamma \). Then, there are constants \( C, \alpha > 0 \) such that for any \( p \in \mathbb{N} \) and any \( x \in X \), \( R > 0 \) we have

\[ \mathbb{P}((z_p, x)_{z_0} \geq R) \leq C e^{-\alpha R}. \]

The above proposition implies that Assumption (4.2) holds since it implies that for any \( x \in X \)

\[ E((z_p, x)_{z_0}) \leq \frac{C}{\alpha}. \]

**Remark 8.2.** If we further assume \( \mu \) is symmetric, then there is an easy way to deduce Proposition 8.1 from Proposition 1.12. Indeed, let us rewrite as follows the square of the quantity we want to bound

\[ \mathbb{P}_{z_0} ((z_m, x)_{z_0} \geq R)^2 = \mathbb{P}_{z_0} ((z_m, x)_{z_0} \geq R) \cdot \mathbb{P}_{z_0} ((z_m, x)_{z_0} \geq R) \]

\[ = \mathbb{P}_{z_0} ((z_m, x)_{z_0} \geq R, (\bar{x}_m, x)_{z_0} \geq R) \]

where \( \bar{x}_m \) is an independent copy of \( x_m \). The hyperbolicity of \( X \) implies that, for any four points \((x_1)_{0 \leq i \leq 3} \) such that \((x_1, x_2)_{z_0} \geq R \) and \((x_2, x_3)_{z_0} \geq R \) then

\[ (x_3, x_1)_{z_0} \geq \min(R, R) - \delta = R - \delta. \]

Therefore

\[ \mathbb{P}_{z_0} ((z_m, x)_{z_0} \geq R, (\bar{x}_m, x)_{z_0} \geq R) \leq \mathbb{P}_{z_0} ((z_m, \bar{x}_m)_{z_0} \geq R - \delta). \]

Because we assumed the measure \( \mu \) symmetric, the random variable \((z_m, \bar{x}_m)_{z_0}\) has the same law as \((z_{2m}, z_0)_{z_m}\). Therefore

\[ \mathbb{P}_{z_0} ((z_m, \bar{x}_m)_{z_0} \geq R - \delta) = \mathbb{P}_{z_0} ((z_{2m}, z_0)_{z_m} \geq R - \delta). \]

We conclude the proof using the exponential-tail deviation inequality from Proposition 1.12:

\[ \mathbb{P}_{z_0} ((z_{2m}, z_0)_{z_m} \geq R - \delta) \leq c_1 e^{-c_2 (R - \delta)}. \]

\[ \square \]

**Proof.** We will use the walking away property from Theorem 1.10, the linear progress property from Definition 1.11 and exponential-tail deviation inequality from Proposition 1.12.

The geometric key of the proof is

**Lemma 8.3.** Let \( z, x, q, y \in X \). There is \( R_0 = R_0(\delta) > 0 \) large enough such that for every \( R \geq R_0 \) if

- \((x, q)_{z} \geq R\);
- \( \frac{d_{\mathbb{R}}}{d_{\mathbb{R}}} \leq d(z, y) \leq R \);


\[ (z, q)_y \leq \frac{R}{5}, \text{ then} \]
\[ d(z, x) - d(y, x) \geq \frac{R}{2}. \]

The proof of the above lemma is quite simple and is illustrated through Figure 3.

![Figure 3](image_url)

**Figure 3.** The geodesics from \( z \) to \( q \) and \( x \) fellow-travel for at least \( R \): the Gromov product \((y, q)_z\) is then \( O(\delta) \)-close from \((x, y)_z\) since we assumed \( d(z, y) \leq R \). Therefore, \((z, x)_y \leq R/5 + O(\delta)\) using \((z, q)_y \leq R/5\). We conclude using \( 4R/5 \leq d(z, y) \).

Let us see how to use Lemma 8.3 (with \( z = z_0, x = x, q = z_p \) and with \( y = z_k \) for some \( k \)) to get Proposition 8.1.

Let \( A_R \) be the event \( A_R := \{ d(z_{\beta R}, z_0) \leq R \} \). The linear progress with exponential tail property implies there exists \( \beta, c_1 > 0 \) such that \( P(A_R) \leq c_1^{-1} e^{-c_1 R} \).

Using large deviations estimates for I.I.D. random variables we know that there is \( \alpha, c_2 > 0 \) such that for \( 0 \leq j \leq \alpha R \) we have
\[ P \left( \sum_{1 \leq i \leq j} d(z_i, z_{i+1}) \geq R/2 \right) \leq c_2^{-1} e^{-c_2 R}. \]

In particular, setting \( B_R := \{ \exists i \in [0, \alpha R], d(z_i, z_0) \geq R/2 \} \) and using the triangular inequality together with the union bound, it gives a constant \( c_3 > 0 \) such that \( P(B_R) \leq c_3^{-1} e^{-c_3 R} \).

Observe we may choose \( \alpha \) and \( \beta \) above such that \( \alpha \leq \beta \).

Using that \( \mu \) has a finite exponential moment and the union bound we get a constant \( c_4 > 0 \) such that, setting \( C_R := \{ \exists i \in [0, \beta R], d(z_i, z_{i+1}) \geq R/5 \} \), we have \( P(C_R) \leq c_4^{-1} e^{-c_4 R} \).

Finally, using the exponential-tail deviation inequality, the union bound and setting \( D_R := \{ \exists i \in [0, \min(\beta R, p)], (z_p, z_i)_z \geq R/5 \} \), we get a constant \( c_5 > 0 \) such that \( P(D_R) \leq c_5^{-1} e^{-c_5 R} \).
Note that none of the constants introduced above depends on $x$.

It now remains to prove that there is a constant $c_6 > 0$ such that for all $p \in \mathbb{N}$ we have
\[
P\left(\{(z_p, x)_{z_0} \geq R\} \cap A_R \cap B_R \cap C_R \cap D_R \right) \leq c_6^{-1} e^{-c_6 R}.
\]

Note that a path in $A_R \cap B_R \cap C_R$ is such that all the steps are of length at most $R/5$; the path remains in $B(z_0, R/2)$ for the first $\alpha R$ steps but is outside the ball $B(z_0, R)$ at time $\min(p, \beta R)$. Observe these conditions imply that $p \geq \alpha R$. We thus conclude that the event $A_R \cap B_R \cap C_R$ is contained in the set
\[
\{\exists i \in [\alpha R, \min(p, \beta R)] : 4R/5 \leq d(z_0, z_i) \leq R\}.
\]

Note also that any path in $D_R$ must satisfy that for any $0 \leq i \leq \min(p, \beta R)$ then
\[
(z_p, z_0)_{z_i} \leq R/5.
\]

Therefore, the event
\[
\{(z_p, x)_{z_0} \geq R\} \cap A_R \cap B_R \cap C_R \cap D_R
\]
is contained in the event
\[
\{(z_p, x)_{z_0} \geq R\} \cap \{\exists i \in [\alpha R, \min(p, \beta R)] : 4R/5 \leq d(z_0, z_i) \leq R \text{ and } (z_p, z_0)_{z_i} \leq R/5\}.
\]

Using the union bound,
\[
P\left(\{(z_p, x)_{z_0} \geq R\} \cap \{\exists i \in [\alpha R, \min(p, \beta R)] : 4R/5 \leq d(z_0, z_i) \leq R \text{ and } (z_p, z_0)_{z_i} \leq R/5\}\right)
\leq \sum_{\alpha R \leq i \leq \beta R} P\left((z_p, x)_{z_0} \geq R, 4R/5 \leq d(z_0, z_i) \leq R, (z_p, z_0)_{z_i} \leq R/5\right).
\]

To conclude the proof, we show that there is a constant $c_7$ independent of $p$ and $x$ such that for for all $\alpha R \leq i \leq \beta R$ we have
\[
P\left((z_p, x)_{z_0} \geq R, 4R/5 \leq d(z_0, z_i) \leq R, (z_p, z_0)_{z_i} \leq R/5\right) \leq c_7^{-1} e^{-c_7 R}.
\]

Using Lemma 8.3, we get that, for any $\alpha R \leq i \leq \beta R$, we have
\[
P\left((z_p, x)_{z_0} \geq R, 4R/5 \leq d(z_0, z_i) \leq R, (z_p, z_0)_{z_i} \leq R/5\right)
\leq P\left(d(z_0, x) - d(z_i, x) \geq \frac{R}{2}\right)
\leq P\left(d(z_0, x) - d(z_i, x) \geq 0\right).
\]

Using the walking away property, we get a constant $c > 0$ independent of $x$ such that for every $\alpha R \leq i \leq \beta R$
\[
P\left(d(z_0, x) - d(z_i, x) \geq 0\right) \leq c^{-1} e^{-ci}
\leq c^{-1} e^{-c\alpha R}.
\]
Appendix A. Hamana’s argument

We mainly repeat arguments from [Ham01] requiring only sub-additivity. Let \(X\) be metric space, \(\mu\) a probability measure on \(\text{Isom}(X)\) with a finite exponential moment and \(z_0 \in X\). Recall that we denoted by \(z_n\) the position in \(X\) at time \(n\) of the random walk driven by \(\mu\). By the triangular inequality we have for every \(n, m \in \mathbb{N}\).

\[
d(z_0, z_{n+m}) \leq d(z_0, z_n) + d(z_n, z_{n+m}).
\]

Recall that by sub-additivity and because \(d(z_0, z_m)\) follows the same law as \(d(z_n, z_{n+m})\), the following limit is well defined:

\[
l := \lim_{n \to \infty} \frac{\mathbb{E}(d_n)}{n},
\]

where we denoted \(d_n := d(z_0, z_n)\).

Moreover, since we assumed that \(\mu\) has a finite exponential moment and that the increments are i.i.d., there exists \(\lambda_0 > 0\) such that for all \(\lambda < \lambda_0\) one has

\[
\mathbb{E}(e^{\lambda d_{n+m}}) \leq \mathbb{E}(e^{\lambda d_n}) \cdot \mathbb{E}(e^{\lambda d_m}).
\]

We conclude using the following purely analytical lemma.

Lemma A.1. Let \((d_n)_{n \in \mathbb{N}}\) be a sequence of non negative real valued random variables such that

1. \(d_1\) has a finite exponential moment;
2. there is \(\lambda_0 > 0\) such that for any \(0 \leq \lambda < \lambda_0\) and for any \(m, n \in \mathbb{N}\) one has

\[
\mathbb{E}(e^{\lambda d_{m+n}}) \leq \mathbb{E}(e^{\lambda d_n}) \cdot \mathbb{E}(e^{\lambda d_m}).
\]

Then for any \(a > l := \lim_{n \to \infty} \left( \frac{\mathbb{E}(d_n)}{n} \right)\) one has

\[
\liminf_{n \to \infty} -\ln \left( \frac{\mathbb{P}(d_n \geq an)}{n} \right) > 0.
\]

The range of validity of the above proposition is much wider than for random walks. It could be used in the setting of a sub-additive defective adapted cocycle as defined in [MS] for example.

Proof. First observe that the condition \(\mathbb{E}(e^{\lambda d_{m+n}}) \leq \mathbb{E}(e^{\lambda d_n}) \cdot \mathbb{E}(e^{\lambda d_m})\) implies that \(\mathbb{E}(d_{n+m}) \leq \mathbb{E}(d_n) + \mathbb{E}(d_m)\) and therefore that the limit defining \(l\) does exist.

Let us introduce the notation

\[
\Lambda_n(\lambda) := \ln \mathbb{E}(e^{\lambda d_n}).
\]

Our two assumptions imply that there is \(\lambda_0 > 0\) such that for all \(n \in \mathbb{N}\) and all \(\lambda < \lambda_0\) we have

\[
\mathbb{E}(e^{\lambda d_n}) < \infty.
\]

Since \(\mathbb{E}(e^{\lambda d_{m+n}}) \leq \mathbb{E}(e^{\lambda d_n}) \cdot \mathbb{E}(e^{\lambda d_m})\), we have \(\Lambda_{n+m}(\lambda) \leq \Lambda_n(\lambda) + \Lambda_m(\lambda)\), which is to say that the sequence \((\Lambda_n(\lambda))_{n \in \mathbb{N}}\) is sub-additive. Fekete’s lemma implies that

\[
\frac{\Lambda_n(\lambda)}{n} \to \Lambda(\lambda) := \inf_{p \in \mathbb{N}} \left( \frac{\Lambda_p(\lambda)}{p} \right).
\]

Using Markov’s inequality we get that, for all \(n\) and \(\lambda > 0\),

\[
\mathbb{P}(d_n \geq an) = \mathbb{P}(e^{\lambda d_n} \geq e^{\lambda an}) \leq e^{-\lambda an} \mathbb{E}(e^{\lambda d_n}).
\]
Applying the logarithm and dividing by $\lambda n$ we get
\[
\frac{1}{\lambda} \ln \left( \frac{P(d_n \geq an)}{n} \right) \leq -a + \frac{\Lambda_n(\lambda)}{\lambda n}.
\]
Therefore, for all $\lambda > 0$,
\[
\limsup_{n \to \infty} \frac{1}{\lambda} \ln \left( \frac{P(d_n \geq an)}{n} \right) \leq -a + \frac{\Lambda(\lambda)}{\lambda}.
\]
It remains then to show that
\[
\limsup_{\lambda \to 0} \left( \frac{\Lambda(\lambda)}{\lambda} \right) \leq l.
\]
At the cost of slightly reducing the value of $\lambda_0$, one can suppose that $\mathbb{E}(d_n e^{\lambda_0 d_n}) < \infty$ for all $n > 0$. Because of the upper bound $e^x \leq 1 + x + x^2 e^x$, we have for all $\lambda < \lambda_0$.
\[
\mathbb{E}(e^{\lambda d_n}) \leq 1 + \lambda \mathbb{E}(d_n) + \lambda^2 \mathbb{E}(d_n) e^{\lambda d_n}) \leq 1 + \lambda \mathbb{E}(d_n) + \lambda^2 \mathbb{E}(d_n) e^{\lambda d_n}) \leq 1 + \lambda \mathbb{E}(d_n) + \lambda^2 C_n,
\]
where $C_n := \mathbb{E}(d_n^2 e^{\lambda_0 d_n})$.

Applying the logarithm, dividing both sides by $n$ gives and using the inequality $\ln(1 + x) \leq x$,
\[
\frac{1}{n} \ln \left( \mathbb{E}(e^{\lambda d_n}) \right) \leq \lambda \frac{\mathbb{E}(d_n)}{n} + C_n \lambda^2 \frac{d_n}{n}.
\]
Therefore, for all $n > 0$ and for all $\lambda > 0$
\[
\frac{\Lambda_n(\lambda)}{n} \leq \lambda \frac{\mathbb{E}(d_n)}{n} + C_n \lambda^2 \frac{d_n}{n}.
\]
In particular for all $\lambda < \lambda_0$ and all $n \in \mathbb{N}$
\[
\Lambda(\lambda) = \inf_{k \in \mathbb{N}} \left( \frac{\Lambda_k(\lambda)}{k} \right) \leq \lambda \frac{\mathbb{E}(d_n)}{n} + C_n \lambda^2 \frac{d_n}{n}.
\]
Letting $\lambda \to 0$ we deduce that for all $n \in \mathbb{N}$
\[
\limsup_{\lambda \to 0} \left( \frac{\Lambda(\lambda)}{\lambda} \right) \leq \frac{\mathbb{E}(d_n)}{n}.
\]
Finally taking $n$ to $\infty$ gives
\[
\limsup_{\lambda \to 0} \left( \frac{\Lambda(\lambda)}{\lambda} \right) \leq l.
\]

**APPENDIX B. EXISTENCE OF SCHOTTKY SETS**

We prove Proposition 2.8 that we recall here for the reader’s convenience.

**Proposition B.1** (Existence of Schottky sets). Let $\Gamma$ be a countable group acting by isometries on a geodesic hyperbolic space $X$, $z_0 \in X$ and $\mu$ an admissible probability measure on $\Gamma$. Then there is $p \in \mathbb{N}$ such that $\text{supp}(\mu^p) \text{ contains a Schottky set.}$

**Proof.** We first reduce the proof to a purely geometric statement. Since we assumed that $\text{supp}(\mu)$ generates a non elementary discrete semi-group there are two independent loxodromic elements $\gamma_1, \gamma_2 \in \Gamma$ and $p_1, p_2 \in \mathbb{N}$ such that
\[
\mu^{p_1}(\gamma_1) > 0
\]
for \( i \in \{1, 2\} \). In particular we have
\[
\begin{align*}
\mu^{(p_1, p_2)}(\gamma_1^{p_1}, \gamma_2^{p_2}) & > 0 \\
\mu^{(p_1, p_2)}(\gamma_2^{p_1}, \gamma_1^{p_2}) & > 0 .
\end{align*}
\]

Because \( \gamma_1^{p_2} \) (resp. \( \gamma_2^{p_1} \)) has the same fixed points than \( \gamma_1 \) (resp. \( \gamma_2 \)), the pair \((\gamma_1^{p_2}, \gamma_2^{p_1})\) is still a pair of two independent loxodromic isometries. Therefore, up to taking some powers of \( \mu \) one can suppose that \( \text{supp}(\mu) \) contains two independent loxodromic elements.

For any pair \( \gamma_1, \gamma_2 \in \Gamma \), let \( S_k(\gamma_1, \gamma_2) \subset \Gamma \) be the set of all elements of \( \Gamma \) which can be written as a product of exactly \( k \) elements in \( \{\gamma_1, \gamma_2\} \). Note that \( S_k(\gamma_1, \gamma_2) \) is contained in the support of \( \mu^{*k} \). Proposition B.1 is an immediate consequence of the following

**Proposition B.2.** Let \( \gamma_1, \gamma_2 \) two independent loxodromic isometries. Then there is \( k \in \mathbb{N} \) such that \( S_k(\gamma_1, \gamma_2) \) contains a Schottky set as in Definition 1.4.

**Proof.** For any points \( x, y \in X \) and any \( C > 0 \), we define
\[
\mathcal{O}_C(x, y) := \{ z \in X , (y, z)_x \geq d(x, y) - C \} ,
\]
which we call the \( C \)-shadow of \( y \) seen from \( x \). Note that one could have defined it equivalently as
\[
\mathcal{O}_C(x, y) := \{ z \in X , (x, z)_y \leq C \} .
\]

Which is to say, when \( X \) is geodesic and up to a factor \( \delta \), the set of all points \( z \) such that any geodesic from \( z \) to \( x \) passes through the ball \( B(y, R) \).

An easy consequence of Morse’s lemma is the following. We denote by \( A^c \) stands for the complementary set of the set \( A \).

**Lemma B.3.** For any \( \lambda, C > 0 \) there is a constant \( K > 0 \) such that for any \( (\lambda, C) \)-quasi-geodesic \( (x_n)_{n \in \mathbb{Z}} \) and any \( m \leq n \leq p \)
\[
(\mathcal{O}_K(x_m, x_n))^c \subset \mathcal{O}_K(x_p, x_n) .
\]

**Proof.** Let \( L \) be the constant given by Morse’s lemma for the quasi-geodesic \( (x_n)_{n \in \mathbb{N}} \).

Recall that \( X \) is hyperbolic if and only if for every \( k > 0 \) there is a constant \( C_k \) such that for any points \( x_1, ..., x_k \in X \) there is a metric finite tree \( T \) which is a \((1, C_k)\)-quasi-isometry from the finite set \( \{(x_1, ..., x_k), d\} \) to \( T \).

Given \( x \in X \), let \( T \) be an approximating tree of the fours points \( x, x_n, x_m, x_p \in X \). We denote by using the upper index \( T \) the corresponding point in \( T \) (for example \( x^T \) denotes the point corresponding to \( x \) in \( T \)). In the tree \( T \) we have
\[
(\mathcal{O}_L^T(x_m, x_n))^c \subset \mathcal{O}_L^T(x_p, x_n) ,
\]
since the unique geodesic from \( x_m^T \) to \( x_p^T \) must pass through the ball centered at \( x_n^T \) of radius \( L \).

In order to get the inclusion (B.4) we must take into account the constant \( C_4 \) coming with the approximating tree. A \((1, C_4)\)-quasi-isometry can change at most the Gromov product by an amount of \( 3C_4/2 \). Therefore, any choice \( K \geq 3C_4 + L \) will lead to the set inclusion (B.4).
Note that a tree comparison argument also shows that for any \( m \leq n \leq p \), any \((\lambda, C)\)-quasi geodesic \((x_n)_{n \in \mathbb{Z}}\) and any \( K > 0 \) we have
\[
O_K(x_n, x_m) \subset O_{K + 3C_4}(x_p, x_m)
\]
We will also need the next lemma to set the ping-pong table through.

**Lemma B.6.** Let \( x_0 \in X \) and \( \gamma \) be a loxodromic isometry of \( X \). Then, there exists \( K_2 > 0 \) such that for any \( n > 0 \)
\[
\gamma^{2n}(O_{K_2}(x_0, \gamma^{-n}x_0)) \subset O_{K_2}(x_0, \gamma^n \cdot x_0).
\]

Note that, since \( \gamma \) is an invertible isometry, we also have
\[
\gamma^{-2n}(O_{K_2}(x_0, \gamma^n x_0)) \subset O_{K_2}(x_0, \gamma^{-n} \cdot x_0).
\]

**Proof.** Since, by definition, the sequence \((\gamma^n \cdot x_0)_{n \in \mathbb{Z}}\) is a quasi-geodesic one deduces from Inclusion (B.5) that
\[
O_K(x_0, \gamma^{-n} \cdot x_0) \subset O_{K + 3C_4}(\gamma^n \cdot x_0, \gamma^{-n} \cdot x_0),
\]
where we chose \( K \) as in Lemma B.6. (\( K \) only depends on the coefficients \((\lambda, C)\) of the quasi-geodesic \((\gamma^n \cdot x_0)_{n \in \mathbb{Z}}\).) Taking the complementary sets, we get that
\[
(\gamma_{K + 3C_4}(x_0, \gamma^{-n} \cdot x_0))^c \subset (O_K(\gamma^n \cdot x_0, \gamma^{-n} \cdot x_0))^c.
\]
We set \( K_2 := K + 3C_4 \). Now we apply \( \gamma^{2n} \) to get
\[
\gamma^{2n}(O_{K_2}(x_0, \gamma^{-n}x_0))^c \subset \gamma^{2n}(O_K(\gamma^n \cdot x_0, \gamma^{-n} \cdot x_0))^c
\]
\[
\subset O_K(\gamma^{3n} \cdot x_0, \gamma^n \cdot x_0).
\]
We now use (B.4) with \( 3n \leq n \leq 0 \) to get that
\[
\gamma^{2n}(O_K(\gamma^{3n} \cdot x_0, \gamma^n \cdot x_0))^c \subset O_K(x_0, \gamma^n \cdot x_0)
\]
\[
\subset O_{K_2}(x_0, \gamma^n \cdot x_0),
\]
concluding.

Let us now set the ping-pong table by using the above lemmas. Let \( \gamma_1, \gamma_2 \) be any two independent loxodromic isometries. Because we assumed \( \gamma_1, \gamma_2 \) to be independent, the four half quasi-geodesics \((\gamma^n_{1})_{n \in \mathbb{N}}\) and \((\gamma^n_{2})_{n \in \mathbb{N}}\) have four different limit points in \( \partial X \). In particular for \( n \) large enough the four following sets (determined by both the value of \( i \in \{1, 2\} \) and the sign \( \pm \)) are disjoint
\[
O_{K_i}(x_0, \gamma^{n \pm}_i \cdot x_0).
\]
Here \( K_i \) correspond to the constants appearing in Lemma B.6 (They depend on \( \gamma_i \), the point \( x_0 \) and on the hyperbolicity constant \( \delta \)).

Lemma B.6 implies that some power of \( \gamma_1, \gamma_2 \) satisfies the celebrated ping pong lemma with respect to the above defined shadows. In particular, for any \( j > 0 \) there is \( k > 0 \) such that there are \( 2j \) independent loxodromic elements \( \gamma_i^{\pm} \in S_k \) (the symbol \( \pm \) standing for a choice of sign; \( \pm \in \{-, +\} \)).

Repeating the above argument to some power of all the \( \gamma_i \) (with a larger value for \( k \) if necessary) one can suppose moreover that there are \( 2j \) disjoints shadows
\[
(O_{K_i}(x_0, \gamma_i^{k} \cdot x_0))_{1 \leq i \leq k \pm \in \{-, +\}}
\]
such that
\[
\gamma_j^{2}(O_{K_j}^{-}(x_0, \gamma_j^{-1} \cdot x_0))^c \subset O_{K_j}^{+}(x_0, \gamma_j \cdot x_0);
\]
\[
\gamma_j^{-2}(O_{K_j}^{+}(x_0, \gamma_j \cdot x_0))^c \subset O_{K_j}^{-}(x_0, \gamma_j^{-1} \cdot x_0).
\]
Let us now prove Proposition B.1. We fix \( j = 6 \) and we choose \( k \) such that 
\[
S := \{ \gamma_1, \gamma_2, ..., \gamma_6 \} \subset S_k
\]
where the \( \gamma_i \)’s are as above. For the constant \( C \) we set 
\[
C := \max_{1 \leq i \leq 6} (d(x_0, \gamma_i \cdot x_0) + K_i + 2\delta),
\]
where the \( K_i \)’s are as in Lemma B.6.

Let now \( y, z \in X \). Note first that, wherever \( y \) might be, its image under the set \( S \) must have at least 5 points in 5 different shadows of the family \((\mathcal{O}_{K_i}(x_0, \gamma_i x_0))_{1 \leq i \leq 6}\), see Figure 4.

\[
\text{Figure 4. This picture is with } j = 3 \text{ in order to burden it. Here, } y \text{ was taken deep in one of the 'repulsive' shadows, represented by the half circle in orange. Its images under the set } S \text{ are in all the others 'positives' shadows, expect the one corresponding to the 'attracting shadow' opposing the 'repulsive' one where } y \text{ was taken (the other orange half circle in the picture).}
\]

Let \( y_1, y_2, ..., y_5 \) be five of these images. Note also that the condition 
\[
(z, y_i)_{x_0} < C
\]
implies \( z \) does not belong to any of the shadows containing one of the \( y_i \)’s. Therefore, since all the shadows are disjoint, the inequality 
\[
(z, y_i)_{x_0} > C
\]
can be satisfied for at most one of the \( y_i \)’s. Therefore 
\[
\sharp \{ s \in S \mid (z, s \cdot y)_{x_0} < C \} \geq 4,
\]
and then
\[ \mu \left( \{ s \in S : (z, s \cdot y)_x^z < C \} \right) \geq \frac{4}{6} = \frac{2}{3}, \]
concluding.

References

[AS17] Amine Asselah and Bruno Schapira. Moderate deviations for the range of a transient random walk: path concentration. *Ann. Sci. Éc. Norm. Supér. (4)*, 50(3):755–786, 2017.

[BH] Yves Benoist and Dominique Hulin. Harmonic measures on negatively curved manifolds. *Annales Institut Fourier (to appear).*

[BHM11] Sébastien Blachère, Peter Haïssinsky, and Pierre Mathieu. Harmonic measures versus quasiconformal measures for hyperbolic groups. *Ann. Sci. Éc. Norm. Supér. (4)*, 44(4):683–721, 2011.

[DPPS11] Françoise Dal’Bo, Marc Peigné, Jean-Claude Picaud, and Andrea Sambusetti. On the growth of quotients of Kleinian groups. *Ergodic Theory Dynam. Systems*, 31(3):835–851, 2011.

[Gou17] Sébastien Gouëzel. Analyticity of the entropy and the escape rate of random walks in hyperbolic groups. *Discrete Anal.*, pages Paper No. 7, 37, 2017.

[Gro87] M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

[Ham01] Yuji Hamana. Asymptotics of the moment generating function for the range of random walks. *J. Theoret. Probab.*, 14(1):189–197, 2001.

[HK01] Yuji Hamana and Harry Kesten. A large-deviation result for the range of random walk and for the Wiener sausage. *Probab. Theory Related Fields*, 120(2):183–208, 2001.

[HK02] Yuji Hamana and Harry Kesten. Large deviations for the range of an integer valued random walk. *Ann. Inst. H. Poincaré Probab. Statist.*, 38(1):17–58, 2002.

[HMM18] Peter Haïssinsky, Pierre Mathieu, and Sebastian Müller. Renewal theory for random walks on surface groups. *Ergodic Theory Dynam. Systems*, 38(1):155–179, 2018.

[KB02] Ilya Kapovich and Nadia Benakli. Boundaries of hyperbolic groups. In *Combinatorial and geometric group theory (New York, 2000/Hoboken, NJ, 2001)*, volume 296 of *Contemp. Math.*, pages 39–93. Amer. Math. Soc., Providence, RI, 2002.

[Kif90] Yuri Kifer. A lower bound for Hausdorff dimensions of harmonic measures on negatively curved manifolds. *Israel J. Math.*, 71(3):339–348, 1990.

[Kin68] J. F. C. Kingman. The ergodic theory of subadditive stochastic processes. *J. Roy. Statist. Soc. Ser. B*, 30:499–510, 1968.

[KL90] Yuri Kifer and François Ledrappier. Hausdorff dimension of harmonic measures on negatively curved manifolds. *Israel J. Math.*, 71(3):339–348, 1990.

[Mah10] Joseph Maher. Linear progress in the complex of curves. *Trans. Amer. Math. Soc.*, 362(6):2963–2991, 2010.

[Mah12] Joseph Maher. Exponential decay in the mapping class group. *J. Lond. Math. Soc. (2)*, 86(2):366–386, 2012.

[MS] Pierre Mathieu and Alessandro Sisto. Deviation inequalities for random walks. *accepted at Duke Math.*

[MT18a] Joseph Maher and Giulio Tiozzo. Random walks on weakly hyperbolic groups. *J. Reine Angew. Math.*, 742:187–239, 2018.

[MT18b] Joseph Maher and Giulio Tiozzo. Random walks, wpd actions, and the cremona group. 2018.

[Ser16] Cagri Sert. *Joint Spectrum and Large Deviation Principles for Random Products of Matrices*. Thèses, Université Paris-Saclay, December 2016.

[Ste89] J. Michael Steele. Kingman’s subadditive ergodic theorem. *Ann. Inst. H. Poincaré Probab. Statist.*, 25(1):93–98, 1989.

[Tan19] Ryokichi Tanaka. Dimension of harmonic measures in hyperbolic spaces. *Ergodic Theory Dynam. Systems*, 39(2):474–499, 2019.

[Vö5] Jussi Väisälä. Gromov hyperbolic spaces. *Expo. Math.*, 23(3):187–231, 2005.

[Woe00] Wolfgang Woess. *Random walks on infinite graphs and groups*, volume 138 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2000.