The higher derivative regularization and quantum corrections in $\mathcal{N} = 2$ supersymmetric theories

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Abstract

We construct a new version of the higher covariant derivative regularization for a general $\mathcal{N} = 2$ supersymmetric gauge theory formulated in terms of $\mathcal{N} = 1$ superfields. This regularization preserves both supersymmetries of the classical action, namely, the invariance under the manifest $\mathcal{N} = 1$ supersymmetry and under the second hidden on-shell supersymmetry. The regularizing $\mathcal{N} = 2$ supersymmetric higher derivative term is found in the explicit form in terms of $\mathcal{N} = 1$ superfields. Thus, $\mathcal{N} = 2$ supersymmetry is broken only by the gauge fixing procedure. Then we analyze the exact NSVZ $\beta$-function and prove that in the considered model its higher loop structure is determined by the anomalous dimension of the chiral superfield $\Phi$ in the adjoint representation which is the $\mathcal{N} = 2$ superpartner of the gauge superfield $V$. Using the background field method we find that this anomalous dimension is related with the anomalous dimension of the hypermultiplet and vanishes if the effective action is invariant under $\mathcal{N} = 2$ background supersymmetry. As a consequence, in this case the higher loop contributions to $\beta$-function also vanish. The one-loop renormalization structure in the considered regularization is also studied by the explicit calculations of the one-loop renormalization constants.

Keywords: supersymmetry, higher covariant derivative regularization, renormalization, $\beta$-function, supergraphs.

1 Introduction

Supersymmetric theories possess remarkable properties on the quantum level. These properties are provided by non-renormalization theorems according to which supersymmetric theories have a much better ultraviolet behavior than the non-supersymmetric ones. The most famous example is, certainly, the $D = 4$, $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM) theory, which is finite conformal invariant quantum field theory model [1, 2, 3, 4]. Perturbative quantum corrections in $D = 4$, $\mathcal{N} = 2$ SYM theories are finite starting from the two-loop approximation [5, 6, 7]. $D = 4$, $\mathcal{N} = 1$ supersymmetric theories have less number of independent renormalization constants in comparison with the non-supersymmetric ones and the superpotential has
no quantum corrections \[7\]. Moreover, it is possible to find an expression for a \(\beta\)-function of \(\mathcal{N} = 1\) supersymmetric theories, which is exact in all orders \[8, 9, 10, 11, 12, 13\]. (For theories containing chiral matter superfields this expression relates the \(\beta\)-function with the anomalous dimension of the matter superfields.) This expression for the \(\beta\)-function is called the exact Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) \(\beta\)-function.

The most elegant approach to the non-renormalization theorems is obtained in the framework of the superfield formulation of supersymmetric theories. In this case, the proof of the non-renormalization theorems is based on three points: (i) superspace structure of superpropagators, containing the delta-functions of anticommuting variables that allow to shrink the loops into dots in \(\theta\) space, (ii) the superfield background field method for supersymmetric gauge theories, and (iii) an assumption about existence of a regularization manifestly preserving the supersymmetry (see e.g. \[14, 15, 16\]). First two points are realized in the explicit form. As to the last one, it is not very clear from the beginning, how to construct a regularization which preserves the supersymmetry and which will be convenient for practical computations of supergraphs (see e.g. \[17\]). Therefore, a part of the proofs of the non-renormalization theorems based on the above assumption needs an additional justification.

It is known that the usually used dimensional regularization \[18, 19, 20, 21\], explicitly breaks the supersymmetry (see e.g. \[22\]), because numbers of bosonic and fermionic degrees of freedom differently depend on the space-time dimension. Most calculations of quantum corrections in supersymmetric theories are done using the regularization by the dimensional reduction \[23\], which is a special modification of the dimensional regularization, and the DR-scheme. Using DR-scheme the finiteness of the \(\mathcal{N} = 4\) SYM theory was verified by explicit calculations in one- \[24\], two- \[25, 26\], three- \[27\] \[28, 29\], and four-loop \[30\] approximations. Vanishing of two- and three-loop contributions to the \(\beta\)-function of the \(\mathcal{N} = 2\) SYM theory was explicitly demonstrated in \[31\]. The \(\beta\)-function for a general \(\mathcal{N} = 1\) SYM theory was calculated in one- \[31\], two- \[33\], three- \[31, 32, 34\], and four-loop \[35\] approximations. (The result agrees with the exact NSVZ \(\beta\)-function only in the one- and two-loop approximations, where a \(\beta\)-function is scheme-independent. In the higher orders the NSVZ \(\beta\)-function can be obtained only after a specially constructed finite renormalization \[32\].)

However, it is known that the dimensional reduction is not consistent from the mathematical point of view \[36\]. Due to this inconsistency any \(\mathcal{N}\) supersymmetry can be broken by quantum corrections in higher loops \[37, 38\]. This means that the non-renormalization theorems are not completely justified in framework of dimensional reduction and the problem of their justification is in general open.

Other methods can be also used \[39, 40\] for calculations of quantum corrections. In principle, it is possible even to use non-invariant regularizations, if a subtraction scheme is tuned in such a way that the Slavnov–Taylor identities are valid for the renormalized effective action \[41, 42, 43, 44, 45\]. However, for practical purposes it is much better to use an invariant regularization. Moreover, the existence of a regularization which preserves supersymmetries of a theory is a key step for proving the non-renormalization theorems. In order to construct an invariant regularization it is convenient to formulate a theory in terms of \(\mathcal{N} = 1\) superfields, because in this case \(\mathcal{N} = 1\) supersymmetry is a manifest symmetry. A mathematically consistent invariant regularization, which does not break \(\mathcal{N} = 1\) supersymmetry, is the higher covariant derivative regularization \[46, 47\]. In the supersymmetric case it can be formulated in terms of \(\mathcal{N} = 1\) superfields \[48, 49\] and, therefore, does not break \(\mathcal{N} = 1\) supersymmetry.

In addition to the supersymmetric regularization, manifestly supersymmetric quantization of a theory also requires supersymmetric gauge fixing procedure. In the case of \(\mathcal{N} = 1\) supersymmetric theories structure of divergences can be studied either using the component fields in the Wess–Zumino gauge, or using the superfield formulation. In the latter case the gauge can be fixed without breaking \(\mathcal{N} = 1\) supersymmetry \[50\] and the quantum corrections are calculated...
in a manifestly $\mathcal{N} = 1$ supersymmetric way that makes this procedure very convenient. Application of the higher covariant derivative regularization (complemented by a supersymmetric gauge condition) to calculation of quantum corrections for $\mathcal{N} = 1$ supersymmetric theories allows to explain naturally the origin of the exact NSVZ $\beta$-function. Loop integrals for the $\beta$-function appear to be integrals of total derivatives \cite{51, 52, 53} and even integrals of double total derivatives \cite{54, 55, 56, 57, 58}. A qualitative explanation of this fact can be given by analyzing Feynman rules \cite{59} using a method proposed in \cite{60}. Because the integrands in integrals which determine a $\beta$-function are total derivatives, at least one of the loop integrals can be calculated analytically. This gives the NSVZ relation between the $\beta$-function and the anomalous dimension which are defined in terms of the bare coupling constant \cite{61}. For the $\mathcal{N} = 1$ supersymmetric electrodynamics this was proved in all orders \cite{58}. As a consequence, in the Abelian case the NSVZ $\beta$-function was obtained exactly in all orders of the perturbation theory for the renormalization group functions defined in terms of the bare coupling constant \cite{61}. If the renormalization group functions are defined in terms of the renormalized coupling constant, the NSVZ $\beta$-function is obtained in a special subtraction scheme, which can be naturally constructed if the theory is regularized by higher covariant derivatives \cite{61, 62}. This (NSVZ) scheme is obtained by imposing the boundary conditions \cite{69} on the renormalization constants. However, so far there is no proof that in non-Abelian theories the exact NSVZ $\beta$-function is obtained with the higher covariant derivative regularization in all orders. Nevertheless, arguments based on anomalies \cite{11} and explicit calculations in the lowest loops \cite{52, 53, 55, 56, 57} allow to suggest this. Therefore, using the invariant regularization for $\mathcal{N} = 1$ supersymmetric theories it is possible to make general conclusions concerning the structure of divergences.

Existence of an invariant regularization is also needed for proving that the $\beta$-function of $\mathcal{N} = 2$ SYM theories vanishes beyond the one-loop approximation \cite{5, 4, 6}. The higher covariant derivative regularization is formulated in terms of $\mathcal{N} = 1$ superfields. Certainly, $\mathcal{N} = 2$ SYM theories can be written in terms of $\mathcal{N} = 1$ superfields (see e.g. \cite{14}). However, in this case only $\mathcal{N} = 1$ supersymmetry is manifest, the second supersymmetry being hidden and on-shell. Versions of the higher covariant derivative regularization so far used for explicit calculations preserve only $\mathcal{N} = 1$ supersymmetry. (A version of the higher derivative regularization for $\mathcal{N} = 2$ supersymmetric theories was constructed in \cite{63}, but the higher derivative term, which is invariant under both supersymmetries, was not presented.) Therefore, the $\mathcal{N} = 1$ higher derivative regularization being applied to $\mathcal{N}$ extended supersymmetric theories can only state that $\mathcal{N} = 1$ supersymmetry is preserved. It does not guarantee that total $\mathcal{N} = 2$ supersymmetry is not broken by quantum corrections. Therefore, the effective action is invariant only under the manifest supersymmetry, and it is not clear, whether it is invariant under the second (hidden) supersymmetry.

In this paper, using the formulation of $\mathcal{N} = 2$ SYM theories in terms of $\mathcal{N} = 1$ superfields, we construct a manifestly $\mathcal{N} = 1$ supersymmetric higher covariant derivative regularization, which is also invariant under the hidden supersymmetry. This regularization guarantees that all loop quantum corrections are automatically manifestly $\mathcal{N} = 1$ supersymmetric, and invariance under the hidden supersymmetry can be broken only by the gauge fixing procedure. (This situation is similar to the use of the Wess-Zumino gauge in $\mathcal{N} = 1$ supersymmetric theories: supersymmetry is also broken only by the gauge fixing procedure.) We find that if the effective action is invariant under the background gauge transformations and background $\mathcal{N} = 2$ supersymmetry, then all anomalous dimensions of the chiral superfields vanish. Staring from the exact NSVZ $\beta$-function we prove that beyond one loop the divergences are completely determined by the anomalous dimension $\gamma_{\Phi}$ of the chiral superfield $\Phi$ (which forms the $\mathcal{N} = 2$ vector supermultiplet together with the gauge superfield $V$). As a result, vanishing the $\beta$-function beyond the one-loop approximation depends on whether the this anomalous dimension vanishes or not.

This paper is organized as follows: In Sect. 2 we construct the higher derivative regulariza-
tion for $\mathcal{N} = 2$ supersymmetric theories which is manifestly $\mathcal{N} = 1$ supersymmetric and also possesses the additional hidden supersymmetry. A higher derivative term and the Pauli–Villars determinants proposed in this section are invariant under both supersymmetries. (A derivation of the $\mathcal{N} = 2$ supersymmetric higher derivative term by the Noether method is described in Appendix A.) However, the second supersymmetry is broken by the gauge fixing procedure. The renormalization of the considered model is discussed in Sect. 3. In Sect. 4 we prove that if the effective action is invariant under the background $\mathcal{N} = 2$ supersymmetry, then the anomalous dimension $\gamma_{\phi}$ vanishes. In Sect. 5 starting from the exact NSVZ $\beta$-function we derive a relation between $\beta$-function and the function $\gamma_{\phi}$. According to this relation, the NSVZ $\beta$-function gets no corrections beyond one-loop if $\gamma_{\phi}$ vanishes. One-loop calculation of quantum corrections with the constructed regularization is presented in Sect. 6. The results are briefly discussed in the Conclusion.

2 The higher covariant derivative regularization for $\mathcal{N} = 2$ supersymmetric theories.

In this paper we consider the $\mathcal{N} = 2$ SYM theory with matter. It is convenient to describe this theory in terms of $\mathcal{N} = 1$ superfields [15, 14, 16]. Using this notation the action can be written as

$$S = \frac{1}{2e_0^2} \text{tr} \left( \text{Re} \int d^4x \, d^2\theta \, W^a W_a + \int d^4x \, d^4\theta \, \Phi^+ e^{2V} \Phi e^{-2V} \right) + \frac{1}{4} \int d^4x \, d^4\theta \left( \phi^+ e^{2V} \phi + \bar{\phi}^+ e^{-2V} \bar{\phi} \right) + \left( \frac{i}{\sqrt{2}} \int d^4x \, d^2\theta \, \bar{\phi}^t \Phi \phi + \frac{1}{2} m_0 \int d^4x \, d^2\theta \, \bar{\phi}^t \phi + \text{c.c.} \right),$$

(1)

where $e_0$ is a bare coupling constant, and the real superfield $V$ contains the gauge field $A_\mu$ as a component. A superfield strength of the gauge superfield $V$ is defined by

$$W_a = \frac{1}{8} D^2 (e^{-2V} D_a e^{2V}).$$

(2)

(In our notation indices of right spinors are denoted by the Latin letters, and indices of left spinors are denoted by Latin letters with dots. Vector indices are denoted by Greek letters.) The chiral superfield $\Phi$ belongs to the adjoint representation of the gauge group. Together with the superfield $V$ it forms the multiplet of the $\mathcal{N} = 2$ SYM theory. The chiral superfields $\phi$ and $\tilde{\phi}$ form an $\mathcal{N} = 2$ hypermultiplet. The superfield $\phi$ lies in a representation $R_0$, which can be, in general, reducible. The superfield $\tilde{\phi}$ lies in the conjugated representation $\overline{R_0}$. For simplicity the action is written for a theory with a single coupling constant (i.e. the gauge group is simple) and a single mass $m_0$ (this corresponds to the irreducible representation $R_0$). The results described below can be easily generalized to more complicated cases.

The theory [11] is invariant under the supersymmetric gauge transformations

$$e^{2V} \to e^{-A^+} e^{2V} e^{-A}, \quad W_a \to e^A W_a e^{-A}, \quad \Phi \to e^A \Phi e^{-A},$$

$$\phi \to e^A \phi; \quad \tilde{\phi} \to e^{-A^t} \tilde{\phi},$$

(3)

where the parameter $A$ is an arbitrary chiral scalar superfield which takes values in the Lie algebra of the gauge group. (In the second string $A$ should be certainly presented as $\text{dim} \, R_0 \times \text{dim} \, R_0$ matrix.) Also the theory [11] is invariant under two supersymmetries. The first supersymmetry
is a manifest symmetry, because the action is written in terms of the $\mathcal{N} = 1$ superfields. The transformations of this supersymmetry can be also written in terms of the $\mathcal{N} = 1$ superfields \([14]\). For this purpose it is convenient to present the exponent of the gauge superfield as

$$ e^{2V} = e^{\Omega^+} e^\Omega $$

and define the right and left spinor gauge covariant derivatives

$$ \nabla_a = e^{-\Omega^+} D_a e^{\Omega^+}; \quad \nabla_a = e^{\Omega^+} D_a e^{-\Omega}, $$

respectively. Then the transformations of the manifest supersymmetry in terms of the $\mathcal{N} = 1$ superfields can be written as

$$ \delta e^\Omega = -8i D^a \xi e^{\Omega^+} W_a; \quad \delta e^{\Omega^+} = 8i D^a \xi W_a e^{\Omega^+}; $$

$$ \delta \Phi = i \bar{D}^2 \left[ e^{-2V} D^a (e^{2V} \Phi e^{-2V}) e^{2V} D_a \xi \right]; $$

$$ \delta \phi = i \bar{D}^2 \left[ e^{-2V} D^a \xi D_a (e^{2V} \phi) + \frac{1}{2} D^2 \xi \phi \right]; $$

$$ \delta \bar{\phi} = i \bar{D}^2 \left[ e^{2V} D^a \xi D_a (e^{-2V} \bar{\phi}) + \frac{1}{2} D^2 \xi \bar{\phi} \right], $$

where $\xi$ is a real scalar superfield which does not depend on the space-time coordinates. This superfield is a parameter of the transformations of the manifest supersymmetry. The action \([1]\) is also invariant under the transformations of the second on-shell supersymmetry. In terms of the $\mathcal{N} = 1$ superfields these transformations have the form

$$ \delta e^\Omega = i \eta^* e^{\Omega^+} \Phi; \quad \delta e^{\Omega^+} = -i \eta^+ e^{\Omega^+}; \quad \delta \Phi = -\frac{i}{2} W^a D_a \eta; $$

$$ \delta \phi = -\frac{1}{4\sqrt{2}} \left( \bar{D}^2 (\eta^* e^{-2V} \phi^*) - 4m_\Omega \eta \phi \right); \quad \delta \bar{\phi} = \frac{1}{4\sqrt{2}} \left( \bar{D}^2 (\eta^* e^{2V} \bar{\phi}^*) - 4m_\Omega \eta \bar{\phi} \right), $$

where $\eta$ is a chiral superfield independent of the space-time coordinates.

In order to regularize the theory \([1]\) we add to its action a term $S_\Lambda$ with higher covariant derivatives. Certainly, this term is not uniquely defined, because a number of derivatives can be arbitrary. In this paper we construct the simplest variant of this term, which is proportional to $\Lambda^{-2}$, where $\Lambda$ is a dimensionful parameter with the dimension of a mass. In Appendix \([1]\) using the Noether method we construct an expression for the action $S_\Lambda$ invariant under $\mathcal{N} = 2$ supersymmetry. It can be written in the following form:

$$ S_\Lambda = -\frac{1}{16c_\delta} \Lambda^{2} \int d^4 x \left\{ \frac{1}{2} \text{Re} \int d^2 \theta (e^{\Omega} W^a e^{-\Omega}) \bar{\nabla}^2 \nabla^2 (e^{\Omega} W_a e^{-\Omega}) 
+ \int d^4 \theta \left( \frac{1}{2} (e^{-\Omega^+} \Phi^+ e^{\Omega^+}) \bar{\nabla}^2 \nabla^2 (e^{\Omega} \Phi e^{-\Omega}) + 4(e^{\Omega} W^a e^{-\Omega}) \left[ \nabla_a (e^{\Omega} \Phi e^{-\Omega}), (e^{-\Omega^+} \Phi^+ e^{\Omega^+}) \right] 
+ 4(e^{-\Omega^+} W^a e^{\Omega^+}) \left[ (e^{\Omega} \Phi e^{-\Omega}), \nabla_a (e^{-\Omega^+} \Phi^+ e^{\Omega^+}) \right] - 8 \left[ (e^{\Omega} \Phi e^{-\Omega}), (e^{-\Omega^+} \Phi^+ e^{\Omega^+}) \right]^2 \right\}. $$

It is easy to see that after adding this term the divergences remain only in one-loop supergraphs, which is a typical feature of the higher derivative regularization \([64]\). Therefore, it is necessary
to regularize the remaining one-loop divergences. Usually for this purpose the Pauli–Villars determinants should be inserted into the generating functional \[65\]. However, in the considered case it is necessary to do this very carefully, because this procedure should not break \(\mathcal{N} = 2\) supersymmetry. It is necessary to introduce two different sets of the Pauli–Villars fields: the first set cancels one-loop divergences originated by the \(\mathcal{N} = 2\) gauge supermultiplet and the second one cancels one-loop divergences originated by the hypermultiplet. Taking into account absence of quadratic divergences for the \(\mathcal{N} = 2\) SYM theory, in order to cancel one-loop divergences of the gauge supermultiplet (and ghosts), it is sufficient to use a single Pauli–Villars determinant

\[
\text{Det}(PV, M)^{-1} = \int D\varphi \, D\tilde{\varphi} \exp(iS_\varphi),
\]

where the action for the Pauli–Villars fields \(\varphi\) and \(\tilde{\varphi}\) (in the adjoint representation of the gauge group) is given by

\[
S_\varphi = \frac{1}{2\epsilon_0} \text{tr} \int d^4x \, d^4\theta \left( \varphi^+ e^{2V} \varphi e^{-2V} + \tilde{\varphi}^+ e^{2V} \tilde{\varphi} e^{-2V} \right) + \text{c.c.},
\]

\[
+ \frac{1}{\epsilon_0} \text{tr} \left( \int d^4x \, d^2\theta \left( i\sqrt{2} \tilde{\varphi} [\Phi, \varphi] + M_0 \varphi \tilde{\varphi} \right) + \text{c.c.} \right).
\]

We choose the mass \(M_0\) of these Pauli–Villars fields proportional to the dimensionful parameter \(\Lambda\) in the higher derivative term:

\[
M_0 = a_0 \Lambda,
\]

where the finite constant \(a_0\) does not depend on the bare coupling constant. Introducing the Pauli–Villars fields \(\varphi\) and \(\tilde{\varphi}\) is motivated by the analogy with the \(\mathcal{N} = 4\) SYM theory. Really, in the \(\mathcal{N} = 4\) SYM theory 3 chiral superfields in the adjoint representation of the gauge group compensate divergences from the gauge supermultiplet and ghosts. This allows to guess that two Pauli–Villars fields \(\varphi\) and \(\tilde{\varphi}\) compensate at least a one-loop divergence originated by the gauge supermultiplet and one chiral superfield \(\Phi\) in the adjoint representation (including one-loop divergences of the ghost loop). This statement is verified by the explicit calculation made in Sect. \[6\]. Moreover, the action \[10\] is evidently invariant under both supersymmetries, because it coincides with the action of the massive \(\mathcal{N} = 2\) hypermultiplet in the adjoint representation of the gauge group.

Also we insert in the generating functional the Pauli–Villars determinants which cancel one-loop divergences originated by the hypermultiplet:

\[
\prod_{I=1}^{n} \text{Det}(PV, M_I)^{c_I},
\]

where the coefficients \(c_I\) satisfy the conditions

\[
\sum_{I=1}^{n} c_I = 1; \quad \sum_{I=1}^{n} c_I M_I^2 = 0.
\]

Again, it is convenient to present the Pauli–Villars determinants in the form

\[
\text{Det}(PV, M_I)^{-1} = \int D\phi_I \, D\tilde{\phi}_I \exp(iS_I),
\]

where \(\phi_I\) lies in the same representation \(R_0\) as the fields \(\phi\),

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\[ S_I = \frac{1}{4} \int d^4 x \, \bar{\theta} \left( \phi^+_I e^{2V} \phi_I + \tilde{\phi}^+_I e^{-2V} \tilde{\phi}_I \right) + \left( \int d^4 x \, d^2 \theta \left( \frac{i}{\sqrt{2}} \bar{\phi}^I \Phi \phi_I + \frac{1}{2} M_I \bar{\phi}^I \phi_I \right) + \text{c.c.} \right), \tag{15} \]

and the masses are proportional to the parameter \( \Lambda \):

\[ M_I = a_I \Lambda, \tag{16} \]

\( a_I \) being independent of \( e_0 \). Both Pauli–Villars actions are invariant under the transformations of \( N = 2 \) supersymmetry, because they coincide with the actions for the massive \( N = 2 \) hypermultiplets. Therefore, the regularization procedure is also invariant under both supersymmetries.

The next step is gauge fixing. We will do this in the framework of the \( N = 1 \) superfield background field method [14], which allows to get a manifestly \( N = 1 \) supersymmetric effective action preserving the classical gauge invariance. The gauge superfield \( V \) is split into the background field and the quantum field by making the substitution

\[ e^\Omega \rightarrow e^{\Omega_T} = e^\Omega e^\Omega; \quad e^{\Omega^+} \rightarrow e^{\Omega_T^+} = e^{\Omega^+} e^{\Omega^+}. \tag{17} \]

Then the background superfield \( V \) and the quantum superfield \( V \) are defined by

\[ e^{2V} \equiv e^{\Omega^+} e^\Omega; \quad e^{2V} \equiv e^{\Omega^+} e^\Omega. \tag{18} \]

The background-quantum splitting for the chiral superfield \( \Phi \) is trivial:

\[ \Phi \rightarrow \Phi_T = \Phi + \Phi, \tag{19} \]

where in the right hand side \( \Phi \) is the background superfield and \( \Phi \) is the quantum superfield. The gauge fixing term used in this paper does not include the superfields \( \Phi \) and \( \Phi \). As a consequence, the effective action depends only on the sum \( \Phi + \Phi \). This can be easily verified by making the linear substitution \( \Phi \rightarrow \Phi_T \) in the generating functional. Due to the same reason we do not use the background field method for the other chiral matter superfields.

It is convenient to fix a gauge without breaking the background gauge invariance

\[ e^\Omega \rightarrow e^{iK} e^\Omega e^{-A}; \quad e^{\Omega^+} \rightarrow e^{-A^+} e^{\Omega^+} e^{-iK}; \quad e^\Omega \rightarrow e^{\Omega^+} e^{-iK}; \quad e^{\Omega^+} \rightarrow e^{iK} e^{\Omega^+}; \]

\[ V \rightarrow e^{iK} V e^{-iK}; \quad \Phi \rightarrow e^A \Phi e^{-A}; \quad e^A \Phi e^{-A}; \quad \phi \rightarrow e^A \phi; \quad \tilde{\phi} \rightarrow e^{-A^t} \tilde{\phi}, \tag{20} \]

where \( K \) is an arbitrary real scalar superfield and \( A \) is an arbitrary chiral superfield. For this purpose we use the background covariant derivatives which are defined by

\[ \nabla_a = e^{-\Omega^+} D_a e^{\Omega^+}; \quad \bar{\nabla}_a = e^\Omega D_a e^{-\Omega}. \tag{21} \]

Note that the theory is also invariant under the quantum gauge transformations

\[ e^\Omega \rightarrow e^\Omega e^{-A}; \quad e^{\Omega^+} \rightarrow e^{-A^+} e^{\Omega^+}; \quad e^\Omega \rightarrow e^\Omega; \quad e^{\Omega^+} \rightarrow e^{\Omega^+}; \quad e^\Omega \Phi e^{-\Omega} \rightarrow e^A (e^\Omega \Phi e^{-\Omega}) e^{-A}; \quad e^\Omega \Phi e^{-\Omega} \rightarrow e^A (e^\Omega \Phi e^{-\Omega}) e^{-A}; \quad e^{\Omega^+} \Phi e^{-\Omega} \rightarrow e^A (e^{\Omega^+} \Phi e^{-\Omega}) e^{-A}; \quad e^{\Omega^+} \Phi e^{-\Omega} \rightarrow e^A (e^{\Omega^+} \Phi e^{-\Omega}) e^{-A}; \]

\[ e^\Omega \phi \rightarrow e^A (e^\Omega \phi); \quad e^{-\Omega^+} \tilde{\phi} \rightarrow e^{-A^t} (e^{-\Omega^+} \tilde{\phi}), \tag{22} \]

where \( A \) and \( A^+ \) are arbitrary background-(anti)chiral superfields.
\[ \nabla_\alpha A = 0; \quad \nabla_\alpha A^+ = 0. \] (23)

The generating functional can be formally written as

\[ Z = \int D\mu \text{Det}(PV, M_0)^{-1} \prod_{l=1}^{n} \text{Det}(PV, M_l)^{c_l} \exp \left( iS + iS_A + iS_{\text{sources}} \right), \] (24)

where the action \( S + S_A \) and the Pauli–Villars determinants are invariant under \( N = 2 \) supersymmetry by construction. Then according to standard procedure \[64\] we insert into the generating functional

\[ 1 = \Delta[V] \cdot \int DA DA^+ \delta(\nabla^2 V(A) - f)\delta(\nabla^2 V(A) - f^+), \] (25)

where \( f \) and \( f^+ \) are background-(anti)chiral superfields which satisfy the conditions

\[ \bar{\nabla}_\alpha f = 0; \quad \nabla_\alpha f^+ = 0. \] (26)

The quantum gauge superfield transformed under the infinitesimal quantum gauge transformations is denoted by

\[ V(A) = \frac{1}{2} \ln \left( e^{-A^+} e^{2V} e^{-A} \right) \approx V + \left( \frac{V}{1 - e^{2V}} \right)_{\text{Adj}} A^+ - \left( \frac{V}{1 - e^{-2V}} \right)_{\text{Adj}} A. \] (27)

The generating functional obtained after this insertion of 1 is defined by

\[ Z[j, f] = \int DF DF^+ \exp \left( -\frac{i}{16e_0^2} \text{tr} \int d^4 x d^4 \theta f^+ \left( 1 - \frac{\nabla^2 \nabla^2}{16A^2} \right) f \right) \times \exp \left( -\frac{i}{16e_0^2} \text{tr} \int d^4 x d^4 \theta f^+ \left( 1 - \frac{\nabla^2 \nabla^2}{16A^2} \right) C \right). \] (28)

The anticommuting background (anti)chiral Nielsen–Kallosh ghosts \( C \) and \( C^+ \) (in the adjoint representation of the gauge group) can be expressed in terms of the (anti)chiral superfields \( C \) and \( C^+ \) as

\[ C = e^{\Omega} C e^{-\Omega}; \quad C^+ = e^{-\Omega^+} C^+ e^{\Omega^+}. \] (29)

The integration over \( C \) and \( C^+ \) cancels the determinant appearing after the integration over the fields \( f \) and \( f^+ \), which are defined using equations similar to (29). It is convenient to present the corresponding contribution in the form

\[ \int DC DC^+ \exp (iS_C), \] (30)

where

\[ S_C = -\frac{1}{16e_0^2} \text{tr} \int d^4 x d^4 \theta C^+ \left( 1 - \frac{\nabla^2 \nabla^2}{16A^2} \right) C \] (31)

is the action for the Nielsen–Kallosh ghosts. (It is assumed that the fields \( C \) and \( C^+ \) are expressed in terms of \( C \) and \( C^+ \) using Eq. (29).)

Substituting the explicit expression for the functional \( Z[j, f] \) and taking the integrals over \( f \) and \( f^+ \) we obtain that the gauge fixing term
is effectively added to the classical action (1). As usually, $\Delta [V]$ is presented as an integral over the Faddeev–Popov ghost fields and gives the ghost action:

$$\Delta [V] = \int D\bar{c} Dc D\bar{c}^+ Dc^+ \exp (iS_{\text{ghost}}) \quad (33)$$

with

$$(f_0 + f_1 V + f_2 V^2 + \ldots) \quad (35)$$

The ghost $c$ and the antighost $\bar{c}$ are background-chiral; the ghost $c^+$ and the antighost $\bar{c}^+$ are background-antichiral. The ghost fields can be expressed in terms of the (anti)chiral fields $c, c^+, \bar{c},$ and $\bar{c}^+$ using equations similar to (29).

Thus, the generating functional can be written as

$$Z = \int D\mu \operatorname{Det}(PV, M_0)^{-1} \prod_{I=1}^{n} \operatorname{Det}(PV, M_I)^{\epsilon_I} \exp \left( iS + iS_{\Lambda} + S_{\text{gf}} + S_{\text{ghost}} + iS_{\text{C}} + iS_{\text{sources}} \right), \quad (36)$$

where $d\mu$ denotes the integration measure, $S$ is the original action of a $\mathcal{N} = 2$ supersymmetric theory (which can also contain hypermultiplet superfields), $S_{\Lambda}$ is the regularizing action constructed in this paper, and $S_{\text{sources}}$ is the action for the sources. The higher derivative term $S_{\Lambda}$ and the Pauli–Villars determinants are invariant under the transformations of both supersymmetries. However, the gauge fixing term and the ghost action are invariant only under the transformations of the manifest supersymmetry. Therefore, $\mathcal{N} = 2$ supersymmetry is broken only by the gauge fixing procedure.

### 3 Renormalization

The results of the previous section show that all ingredients for proving the $\mathcal{N} = 1$ non-renormalization theorem take place in the considered theory. The theory is renormalizable and renormalization preserves the manifest $\mathcal{N} = 1$ supersymmetry. Therefore, the divergences can be absorbed into the redefinitions of the coupling constant, fields, and masses:

$$\frac{1}{c_0^2} = \frac{Z_3}{e^2}; \quad m_0 = Z_m m; \quad \Phi_T = \sqrt{Z_\Phi} \Phi_{TR}; \quad V = Z_V V_R;$$

$$\phi = \sqrt{Z_{\phi}} \phi_R; \quad \tilde{\phi} = \sqrt{Z_{\tilde{\phi}}} \tilde{\phi}_R; \quad c = Z_c c_R; \quad \bar{c} = Z_{\bar{c}} \bar{c}_R. \quad (37)$$

Here we took into account that (as we already mentioned above) the effective action depends only on $\Phi_T = \Phi + \Phi$, and, therefore, it is not necessary to introduce two different renormalization constants for the background and quantum parts of this field.
According to the standard prescription, the renormalization constants $Z$ should be constructed so that to cancel the divergences, appearing in loop integrals. $\Omega_R$ can be defined by the equation

$$e^{2V_R} = e^{\Omega_R^+} e^{\Omega_R}. \quad (38)$$

It is important that in Eqs. (37) and (38) $V$ denotes the quantum gauge superfield. The background gauge superfield $V$ is not renormalized due to the unbroken background gauge invariance (20). Therefore, after the renormalization the total gauge superfield is renormalized as

$$e^{2V_T} \equiv e^{\Omega_T^+} e^{\Omega_T} \equiv e^{\Omega^+} e^{2Z_V V_R} e^{\Omega}. \quad (39)$$

In our notation the renormalized fields in the adjoint representation are presented in the form

$$V = V_R = e(V_R) At^A; \quad V_R = e(V_R) At^A; \quad \Phi_R = e(\Phi_R) At^A; \quad \Phi_R = e(\Phi_R) At^A, \quad (40)$$

where $e$ is the renormalized coupling constant, and $t^A$ denotes the generators of the fundamental representation, which are normalized by the condition

$$\text{tr}(t^A t^B) = \frac{1}{2} \delta^{AB}. \quad (41)$$

The similar equations for the bare superfields have the form

$$V = e_0 V At^A; \quad V = e_0 V At^A; \quad \Phi = e_0 (\Phi) At^A; \quad \Phi = e_0 (\Phi) At^A, \quad (42)$$

where $e_0$ is the bare coupling constant. Therefore, in components

$$V_A = \sqrt{Z_3} (V_R)_A; \quad V_A = Z_V \sqrt{Z_3} (V_R)_A;$$
$$\Phi_A = \sqrt{Z_3} Z_3 (\Phi_R)_A \equiv Z_A^B (\Phi_R)_B; \quad \Phi_A = \sqrt{Z_3} Z_3 (\Phi_R)_A = Z_A^B (\Phi_R)_B, \quad (43)$$

where we have introduced the notation

$$Z_A^B \equiv \sqrt{Z_3} Z_3 \delta_A^B. \quad (44)$$

After substitution (37)

$$S = \frac{Z_3}{2e^2} \text{tr} \int d^4 x \left( \text{Re} \int d^2 \theta W^a W_a + Z_3 \int d^4 \theta \Phi_R^+ e^{2V_T} \Phi_R e^{-2V_T} \bigg) \right.$$
$$+ \frac{Z_3}{4} \int d^4 x d^4 \theta \left( \phi^+_R e^{2V_T} \phi_R + \bar{\phi}^+_R e^{-2V_T} \bar{\phi}_R \right) + Z_3 \left( \frac{Z_3^{1/2}}{\sqrt{2}} \int d^4 x d^2 \theta \bar{\phi}_R \Phi_T R \Phi_R \right.$$}
$$+ \frac{1}{2} Z_3 m \int d^4 x d^2 \theta \bar{\phi}_R \Phi_R + c.c. \bigg), \quad (45)$$

where $W_a$ is constructed from the total gauge superfield $V_T$ given by Eq. (39). The quantum gauge superfield is present in the action only in the combination $e^{2V} = e^{\Omega^+} e^{\Omega}$. This quantum gauge superfield should be substituted by

$$V = Z_V V_R. \quad (46)$$
The ghost Lagrangian is renormalized as

\[ S_{\text{ghost}} = \frac{Z_3}{e^2} \text{tr} \int d^4x d^4\theta (\bar{c}_R + \bar{c}_R^\dagger) \left[ \left( \frac{V}{1 - e^{2V}} \right)_{\text{Adj}} c_R^+, \left( \frac{-V}{e^{-2V}} \right)_{\text{Adj}} c_R \right], \tag{47} \]

where \( V \) should be also substituted by \( Z_V V_R \) according to Eq. (46).

Using the \( \mathcal{N} = 1 \) nonrenormalization theorem [7] according to which the superpotential

\[ \frac{i}{\sqrt{2}} \int d^4x d^2\theta \Phi \phi + \frac{1}{2} m_0 \int d^4x d^2\theta \bar{\phi}^\dagger \phi \tag{48} \]

is not renormalized, from Eq. (45) we immediately obtain

\[ Z_{\Phi}^{1/2} = Z_m = Z_{\bar{\phi}}^{-1}. \tag{49} \]

Certainly, finite renormalizations are possible, but in this paper we assume that for finite terms the renormalization constants are chosen equal to 1.

## 4 Counterterms and the second \( \mathcal{N} = 1 \) supersymmetry

Naively, it is possible to suggest that due to existence of the second supersymmetry two first terms in Eq. (1) are renormalized in the same way. As a consequence, the second supersymmetry would lead to some restrictions to the renormalization constants. However, this question is rather subtle. Really, the regularized action proposed in this paper is invariant under both supersymmetries of the \( \mathcal{N} = 2 \) supersymmetric theory, but the gauge fixing term and the ghost actions are invariant only under the manifest supersymmetry. It is known [68] that any symmetry of the classical action corresponds to a symmetry of the renormalized action and the renormalized effective action. Nevertheless, we can not state that in the case under consideration the invariance corresponding to the second supersymmetry automatically fixes the renormalization constant \( Z_{\Phi} \). However, here we try to understand what is needed for fixing the renormalization constant \( Z_{\Phi} \).

Using the background field method we rewrite the transformations (7) making the background–quantum splitting for the gauge superfield and the chiral superfield \( \Phi \):

\[
\delta(e^\Omega e^{\bar{\Omega}}) = i\eta^* e^\Omega \Phi + \Phi; \quad \delta(e^{\Omega^+} e^{\bar{\Omega}^+}) = -i\eta(\Phi^+ + \Phi^+) e^{\Omega^+} e^{\bar{\Omega}^+};
\]

\[
\delta(\Phi + \Phi^+) = -\frac{i}{2} \left( W^a + \frac{1}{8} e^{-\Omega} \bar{\nabla}^2 (e^{-2V} \bar{\nabla} e^{2V}) e^\Omega \right) D_a \eta;
\]

\[
\delta \phi = \frac{1}{4\sqrt{2}} \left( D^2 (\eta^* e^{-2V} \phi^+) - 4m_0 \eta \phi \right); \quad \delta \bar{\phi} = \frac{1}{4\sqrt{2}} \left( D^2 (\eta^* e^{2V} \phi^+) - 4m_0 \eta \bar{\phi} \right),
\]

where \( V_T \) is given by Eq. (39) and

\[ W_a \equiv \frac{1}{8} D^2 (e^{-2V} D_a e^{2V}). \tag{51} \]

These transformations can be obtained if we set

\[
\delta e^\Omega = i\eta^* e^\Omega \Phi; \quad \delta e^{\Omega^+} = -i\eta \Phi^+ e^{\Omega^+}; \quad \delta \Phi = -\frac{i}{2} W^a D_a \eta
\]

for the background superfields and
\[
\delta e^\Omega = i\eta^* e^\Omega \Phi e^{-\Omega}; \quad \delta e^{\Omega^+} = -i\eta e^{-\Omega^+} \Phi e^{\Omega^+};
\]
\[
\delta \Phi = -\frac{i}{16} e^{-\Omega} \nabla^2 (e^{-2V} \nabla^a e^{2V}) e^\Omega D_a \eta;
\]
\[
\delta \phi = -\frac{1}{4\sqrt{2}} \left( D^2 (\eta^* e^{-2V} \tilde{\phi}^*) - 4m_0 \eta \tilde{\phi} \right); \quad \delta \tilde{\phi} = \frac{1}{4\sqrt{2}} \left( D^2 (\eta^* e^{2V} \phi^*) - 4m_0 \eta \phi \right)
\]
for the quantum superfields.

We will show that if the effective action is invariant under the hidden supersymmetry transformations (52), then
\[
Z_\Phi = 1.
\]
Actually, this is a manifestation of the general statement that a symmetry can impose restrictions on the renormalization constants. In this sense Eq. (54) is similar, for example, to the equation \(Z_3 = 1\) which follows from the symmetry under transformations of the conformal group, see e.g. [69]. Note that the regularization proposed in this paper is important, because it ensures the invariance of the regularized action under the BRST and \(N = 2\) supersymmetry transformations.

Now let us prove Eq. (54) assuming the invariance of the effective action under the transformations (52). If the sources for the hypermultiplet and quantum fields are set to 0, then the invariance of the effective action under the transformations (52) can be expressed by the equation
\[
0 = \text{tr} \int d^8 x \left\{ \frac{\delta \Gamma}{\delta V^A} \delta V^A + \frac{\delta \Gamma}{\delta \Phi} \cdot D^2 \left(-\frac{i}{2} W^a D_a \eta \right) + \frac{\delta \Gamma}{\delta \Phi^A} \cdot \bar{D}^2 \left(\frac{i}{2} \bar{W}^a \bar{D}_a \eta^* \right) \right\},
\]
where \(\delta \eta V\) is obtained from the equation
\[
\delta \eta (e^{2V}) = i\eta^* e^{2V} \Phi - i\eta \Phi^* e^{2V},
\]
which follows from Eq. (52). As a consequence,
\[
\delta \eta V = \frac{i}{2} \eta^* \Phi - \frac{i}{2} \eta \Phi^* + O(V).
\]
Let us differentiate Eq. (55) with respect to \(V^B\) and \(\Phi^A\), where the subscripts denote points in the superspace. Then, after setting all (background) fields to 0 we obtain
\[
\int d^8 x \eta_x D^2 \delta^8 \frac{\delta^2 \Gamma}{\delta V^A \delta V^B} = - \int d^8 x D^a \eta_x (D_a) x \delta^8 \frac{\delta^2 \Gamma}{\delta \Phi^A \delta \Phi^B},
\]
Because the effective action satisfies this relation, it is possible to choose such a subtraction scheme in which this relation is also valid for the renormalized action \(S_R\):
\[
\int d^8 x \eta_x D^2 \delta^8 \frac{\delta^2 S_R}{\delta V^A \delta V^B} = - \int d^8 x D^a \eta_x (D_a) x \delta^8 \frac{\delta^2 S_R}{\delta \Phi^A \delta \Phi^B},
\]
where \(S_R\) is constructed from the classical action by substituting the bare fields by the renormalized ones. In particular, a part of the renormalized action \(S_R\) corresponding to the two-point functions of the background fields has the form
\[ \frac{1}{4} \int \frac{d^4 p}{(2\pi)^4} d^4 \theta \left( - V^A_{\tilde{R}}(\theta, -p) \partial^2 \Pi_{1/2}^V V^A_{\tilde{R}}(\theta, p) + \Phi^+_A(\theta, -p) \Phi^+_A(\theta, p) \right) \]

\[ = \frac{1}{4} \int \frac{d^4 p}{(2\pi)^4} d^4 \theta \left( - \frac{1}{Z_3} V^A(\theta, -p) \partial^2 \Pi_{1/2} V^A(\theta, p) + (Z^{-2}) A^B \Phi^+_A(\theta, -p) \Phi_B(\theta, p) \right), \]

where \( \partial^2 \Pi_{1/2} \equiv -D^a D^a D_a/8 \) is the supersymmetric projection operator and \( (Z^{-1}) A^B \) denotes a matrix inverse to \( Z^A_A \). Substituting this expression into Eq. \((64)\) we obtain

\[ \frac{1}{2Z_3} \int d^8 x \eta_x \partial^2 \Pi_{1/2} \delta_{xy} \delta_{zm} \delta_{AB} = \frac{(Z^{-2}) A^B}{16} \int d^8 x \partial^2 \Pi_{1/2} \delta_{xy} \partial^2 \Pi \delta_{zm} \delta_{AB}. \]

Integrating \( D^2 \) by parts and taking into account that the supersymmetry transformation parameter \( \eta \) is a space-time constant, after some simple transformations involving the algebra of the covariant derivatives the integral in the right hand side of this equation can be rewritten as

\[ \int d^8 x \partial^2 \Pi_{1/2} \delta_{xy} \partial^2 \Pi \delta_{zm} \delta_{AB} = 8 \int d^8 x \eta_x \delta_{xy} \delta_{zm} \partial^2 \Pi \delta_{AB}. \]

Therefore, the renormalization constants \( Z_3 \) and \( Z^A_A \) are related by the equation

\[ Z^A_A = \sqrt{2} \delta_{AB}. \]

Comparing this result with Eq. \((59)\), we see that the superfield \( \Phi \) is not renormalized, \( Z_\Phi = 1 \), and we prove Eq. \((61)\). As a result we get the criterion whether the effective action is invariant under the background hidden supersymmetry. This criterion is given by Eq. \((62)\). It means, in particular, that under this condition the renormalization group function \( \gamma_{\Phi} = 0 \).

## 5 Finiteness of \( N = 2 \) supersymmetric theories beyond the one-loop approximation and the NSVZ \( \beta \)-function

Due to \( N = 1 \) supersymmetry a \( \beta \)-function of SYM theories is related with the anomalous dimension of the matter superfields by the NSVZ relation \([8, 9, 10, 11, 12, 13]\). For our purposes it is convenient to write it in the following form \([32]\):

\[ \beta(\alpha_0) = -\frac{\alpha_0^2 \left( 3C_2 - T(R) + C(R) \gamma^j_j(\alpha_0)/r \right)}{2\pi (1 - C_2 \alpha_0/2\pi)}, \]

where

\[ \alpha_0 = \frac{e_0^2}{4\pi} \]

is a bare coupling constant, \( \gamma^j_j(\alpha_0) \) is the anomalous dimension of the matter superfields, and the following notation is used:

\[ \text{tr} (T^A T^B) \equiv T(R) \delta^{AB}; \quad (T^A)_k^j \equiv C(R) \delta^j_j; \quad f^{ACD} f^{BCD} \equiv C_2 \delta^{AB}; \quad r = \delta_{AA}. \]

The renormalization group functions in Eq. \((54)\) are expressed in terms of the bare coupling constant \( \alpha_0 \). These functions are defined according to the following prescription:
\[ \beta(\alpha_0, \Lambda/\mu) \equiv \frac{\alpha_0(\alpha_0, \Lambda/\mu)}{d \ln \Lambda} \bigg|_{\alpha_0=\text{const}}; \quad \gamma_i^j(\alpha_0, \Lambda/\mu) \equiv -\frac{d \ln Z_i^j(\alpha_0, \Lambda/\mu)}{d \ln \Lambda} \bigg|_{\alpha_0=\text{const}}, \]  

(67)

where \( \alpha \) denotes the renormalized coupling constant \( \alpha = e^2/4\pi \). The matter is that these are the functions for which the NSVZ relation is obtained at least in the Abelian case if the theory is regularized by higher derivatives \[58\]. Usually the renormalization group functions are defined by a different way, in terms of the renormalized coupling constant:

\[ \tilde{\beta}(\alpha_0, \Lambda/\mu) \equiv \frac{\alpha_0(\alpha_0, \Lambda/\mu)}{d \ln \mu} \bigg|_{\alpha_0=\text{const}}; \]
\[ \tilde{\gamma}_i^j(\alpha_0, \Lambda/\mu) \equiv \frac{d \ln Z_i^j(\alpha_0, \Lambda/\mu)}{d \ln \mu} \bigg|_{\alpha_0=\text{const}}. \]  

(68)

It is well-known that the \( \beta \)-function and the anomalous dimension defined according to this prescription are scheme-dependent. However \[61 \ 62\], if the boundary conditions

\[ Z_3(\alpha, x_0) = 1; \quad Z_i^j(\alpha, x_0) = 1 \]  

(69)

are imposed on the renormalization constants in an arbitrary (but fixed) point \( x_0 = \ln \Lambda/\mu_0 \), then the renormalization group functions \( \tilde{\beta} \) and \( \tilde{\gamma}_i^j \) coincide with the renormalization group functions \( \beta \) and \( \gamma_i^j \):

\[ \tilde{\beta}(\alpha_0) = \beta(\alpha); \quad \tilde{\gamma}_i^j(\alpha_0) = \gamma_i^j(\alpha). \]  

(70)

This implies that the boundary conditions \( \tilde{\beta} \) at least in the Abelian case give the NSVZ scheme in all orders of the perturbation theory if the theory is regularized by higher derivatives. This statement was verified by the explicit three-loop calculations in Refs. \[61 \ 62\].

All features of the higher covariant derivative regularization, in particular, factorization of integrals into integrals of (double) total derivatives \[51 \ 54 \ 58\], which gives the NSVZ relation for the renormalization group functions \( \beta \) in the Abelian case, in the lowest loops also take place in the non-Abelian case \[52 \ 55 \ 56 \ 57\]. Although the all-loop derivation of the NSVZ relation is not so far completed for SYM theories, it seems reasonable to suggest that in the non-Abelian case the NSVZ relation is also obtained for the renormalization group functions defined in terms of the bare coupling constant with the higher covariant derivative regularization. In this section we prove that under this assumption the finiteness of \( \mathcal{N} = 2 \) SYM theories beyond the one-loop approximation can be very easily derived from the NSVZ relation if the effective action is invariant under the transformations \[52\]. Thus, the regularization proposed in this paper possibly allows not only to justify the non-renormalization theorems, but also to derive one of them in the easiest way.

The main observation is that \( \mathcal{N} = 2 \) SYM theories can be considered as a special case of \( \mathcal{N} = 1 \) supersymmetric Yang–Mills theories, the representation \( R \) for the matter superfield being reducible and equal to the direct sum

\[ R = \text{Adj} + R_0 + \overline{R}_0. \]  

(71)

\[ ^1 \text{For a fixed regularization the renormalization group functions are scheme-independent, see e.g. } \ 61. \]
\[ ^2 \text{Although the first equation in Eq. } \ 69 \text{ looks similar to the condition } Z_3 = 1, \text{ which can be imposed for obtaining the conformal symmetry limit of a theory } \ 69, \text{ there is a very important difference: in Eq. } \ 69 \text{ } Z_3 = 1 \text{ only in a single (but arbitrary) point } x_0, \text{ while the conformal symmetry limit is obtained if } Z_3 = 1 \text{ for arbitrary values of } x. \]
Here $\text{Adj}$ denotes the adjoint representation of the gauge group corresponding to the superfield $\Phi$. (This superfield together with the gauge superfield $V$ forms the $\mathcal{N} = 2$ gauge supermultiplet.) The fields in the representations $R_0$ and $\mathbf{T}_0$ ($\phi$ and $\tilde{\phi}$, respectively) form the hypermultiplet.

Let us find the constants $C(R)^i_j$ and $T(R)$ for the reducible representation \( (71) \). For this purpose we note that the generators of the considered representation can be written in the form

$$T^A(R) = \begin{pmatrix} T^A(\text{Adj}) & 0 & 0 \\ 0 & T^A(R_0) & 0 \\ 0 & 0 & -(T^A(R_0))^t \end{pmatrix}. \tag{72}$$

It is easy to see that for the adjoint representation $T(\text{Adj}) = C_2$. Therefore,

$$T(R) = C_2 + 2T(R_0). \tag{73}$$

Also we obtain

$$C(R)^i_j = \begin{pmatrix} C_2 \cdot \delta^B_A & 0 & 0 \\ 0 & C(R_0) & 0 \\ 0 & 0 & C(R_0) \end{pmatrix}. \tag{74}$$

We will prove that if the supersymmetric higher covariant derivative regularization is used, the anomalous dimension of the superfield $\Phi_A$ (defined in terms of the bare coupling constant) is related with a $\beta$-function. This anomalous dimension is calculated according to the following prescription:

$$\gamma(\alpha_0)_A^B \equiv -2 \cdot \left. \frac{d \ln Z^B_A}{d \ln \Lambda} \right|_{\alpha = \text{const}}, \tag{75}$$

where $\Phi_A = Z^B_A(\Phi_R)B$ and the limit $m_0 \to 0$ is assumed. Then using Eq. (44) we obtain

$$\gamma(\alpha_0)_A^B = -2 \cdot \frac{d \ln Z^B_A}{d \ln \Lambda} = -\frac{d \ln (Z_3 Z_\phi)}{d \ln \Lambda} \delta_A^B = \left( \frac{d \ln \alpha_0/\alpha}{d \ln \Lambda} - \frac{d \ln Z_\phi}{d \ln \Lambda} \right) \delta_A^B = \left( \frac{\beta(\alpha_0)}{\alpha_0} + \gamma_\phi(\alpha_0) \right) \delta_A^B, \tag{76}$$

where

$$\gamma_\phi(\alpha_0, \Lambda/\mu) \equiv \left. -\frac{d \ln Z_\phi(\alpha, \Lambda/\mu)}{d \ln \Lambda} \right|_{\alpha = \text{const}} \tag{77}$$

is the anomalous dimension of the superfield $\Phi$ defined in terms of the bare coupling constant.

The anomalous dimension of the hypermultiplet can be expressed through $\gamma_\phi$ using Eqs. (19):

$$(\gamma_\phi)^i_j \equiv -\frac{d \ln Z_\phi}{d \ln \Lambda} \cdot \delta^i_j = \frac{1}{2} \frac{d \ln Z_\phi}{d \ln \Lambda} \cdot \delta^i_j = -\frac{1}{2} \gamma_\phi(\alpha_0) \cdot \delta^i_j. \tag{78}$$

Therefore, the anomalous dimension can be written as

$$\gamma^i_j(\alpha_0) = \begin{pmatrix} (\beta(\alpha_0)/\alpha_0 + \gamma_\phi(\alpha_0)) \cdot \delta^B_A & 0 & 0 \\ 0 & -\gamma_\phi(\alpha_0)/2 \cdot \delta^i_j & 0 \\ 0 & 0 & -\gamma_\phi(\alpha_0)/2 \cdot \delta^i_j \end{pmatrix}. \tag{79}$$

Substituting the expressions for $T(R)$, $C(R)^i_j$, and $\gamma^i_j$ in Eq. (17) we obtain

---

\(^3\text{For the superfields } \Phi_A^Z \text{ in Eq. (77) corresponds to } (Z^2)_A^B. \text{ As a consequence, we obtain the factor 2 in Eq. (75).} \)
\[
\beta(\alpha_0) = -\frac{\alpha_0^2}{2\pi}\left(2C_2 - 2T(R_0) + C_2\left(\frac{\beta(\alpha_0)}{\alpha_0} + \gamma_\Phi(\alpha_0)\right) - T(R_0)\gamma_\Phi(\alpha_0)\right)\frac{1}{2\pi(1 - C_2\alpha_0/2\pi)}. \tag{80}
\]

Solving this equation for \(\beta(\alpha_0)\), after some simple transformations we find that a \(\beta\)-function of the considered theory is

\[
\beta(\alpha_0) = -\frac{\alpha_0^2}{\pi}\left(C_2 - T(R_0)\right)\left(1 + \frac{1}{2}\gamma_\Phi(\alpha_0)\right). \tag{81}
\]

Thus, we see that in the theory under consideration the higher loop structure of the NSVZ \(\beta\)-function is determined by the function \(\gamma_\Phi(\alpha_0)\). If \(\gamma_\Phi = 0\), the expression (81) contains only \(\alpha_0^2\). Therefore, the NSVZ \(\beta\)-function for an arbitrary renormalizable \(\mathcal{N} = 2\) supersymmetric Yang-Mills theory does not vanish only in the one-loop approximation and coincides with conventional one-loop \(\beta\)-function:

\[
\beta(\alpha_0) = -\frac{\alpha_0^2}{\pi}\left(C_2 - T(R_0)\right). \tag{82}
\]

As the result, we can conclude that the \(\mathcal{N} = 2\) non-renormalization theorem is equivalent to the statement \(\gamma_\Phi = 0\).

The equality \(\gamma_\Phi = 0\) follows from the invariance of the renormalized action under the background transformations of the hidden supersymmetry (see Section 4). It is evident that this invariance takes place if both a regularization and a gauge fixing procedure are invariant under the complete \(\mathcal{N} = 2\) supersymmetry. However, the considered gauge fixing term is invariant only under the manifest supersymmetry. Nevertheless, we can present here some indirect arguments in favor of this equality. It is known that the \(\beta\)-function is gauge independent if the minimal substraction scheme is used for renormalization (see e.g. [70]). Therefore, if there exists a completely \(\mathcal{N} = 2\) supersymmetric gauge, then the regularized effective action will be invariant under the same amount of supersymmetries as the classical action and according to (54) one gets \(\gamma_\Phi = 0\) and the \(\beta\)-function vanishes beyond one-loop. Since the \(\beta\)-function is gauge invariant, the same result will be valid in any gauge, in particular in the gauge used in this paper. But the completely \(\mathcal{N} = 2\) invariant gauge does actually exist [4]. However, the \(\mathcal{N} = 2\) supersymmetric gauge used for derivation of Eq. (54) was formulated in terms of \(\mathcal{N} = 2\) superfields while the proposed regularization is formulated in terms of \(\mathcal{N} = 1\) superfields and it is unclear whether these gauge and regularization are consistent one with another.

Thus, if we accept that \(\gamma_\Phi = 0\), then up to a possibility of making finite renormalizations we obtain the following values of the renormalization constants (exactly in all orders):

\[
Z_3 = 1 + \frac{\alpha}{\pi}\left(C_2 - T(R_0)\right)\ln\frac{\Lambda}{\mu}; \quad Z_\Phi = 1; \quad Z_\phi = 1; \quad Z_m = 1, \tag{83}
\]

where \(\mu\) is a renormalization parameter. Values of \(Z_V\), \(Z_c\) and \(Z_\xi\) are not so far defined. In the next section we describe the one-loop calculation, which allows to find values of these renormalization constants in the one-loop approximation.

6 One-loop renormalization with the higher covariant derivative regularization

In this section we calculate one-loop divergences using the version of the higher covariant derivative regularization constructed in this paper.
In the one-loop approximation the point Green function of the hypermultiplet, $G_{ij}$, defined by the equation

\[ \Gamma^{(2)}(\phi) = \frac{1}{4} \int \frac{d^4 p}{(2\pi)^4} d^4 \theta \left( \phi^* (\theta, -p)^i \phi(\theta, p)_j + \tilde{\phi}^* (\theta, -p)^i \tilde{\phi}(\theta, p)_j \right) G_{ij}(\alpha_0, \Lambda_0/p, m_0/\Lambda), \]  

(84)

can be obtained by calculating two diagrams presented in Fig. 1. It is easy to see that these diagrams cancel each other:

\[
G_{ij}(\alpha_0, \Lambda_0/p, m_0/\Lambda) = \delta_{ji} - C(R)j^i \int \frac{d^4 q}{(2\pi)^4} \frac{2e_0^2}{q^2(1 + q^2/\Lambda^2)(q + p)^2 + m_0^2} 
+ C(R)j^i \int \frac{d^4 q}{(2\pi)^4} \frac{2e_0^2}{q^2(1 + q^2/\Lambda^2)(q + p)^2 + m_0^2} + O(e_0^4) = \delta_{ij} + O(e_0^4) \]  

(85)

This result is in a complete agreement with Eq. (83). (For finite terms we always choose the renormalization constants equal to 1.) Using exactly the same arguments we prove that the Pauli–Villars fields $\phi$, $\tilde{\phi}$, $\phi^I$, and $\tilde{\phi}^I$ are not renormalized in the one-loop approximation.

The two-point Green function of the superfield $\Phi$ in the one-loop approximation is determined by diagrams presented in Fig. 2. Calculating these diagrams we obtain the function $G$ defined by the equation

\[ \Gamma^{(2)}(\Phi) = \frac{1}{2e_0^2} \text{tr} \int \frac{d^4 p}{(2\pi)^4} d^4 \theta \Phi^+(\theta, -p) \Phi(\theta, p) G(\alpha_0, \Lambda_0/p, m_0/\Lambda), \]  

(86)

where $\Phi = e_0 \Phi_A t^A$ and, for simplicity, we assume that the gauge group is simple. (The background superfield $\Phi$ is omitted, because the effective action depends only on $\Phi + \Phi$. ) We are interested in the divergent part of the function $G$. Taking into account that in the one-loop approximation logarithmically divergent terms are proportional to $\ln \Lambda$, it can be found by differentiating the result for the function $\ln G$ (in the one-loop approximation this is equivalent to differentiating the function $G$) with respect to $\ln \Lambda$ in the limit of the vanishing external momentum.
Here the first term (proportional to $T(R_0)$) is a contribution of the hypermultiplet and the corresponding Pauli–Villars fields (the second diagram in Fig. 2). The second term consists of the contributions of the first diagram in Fig. 2 and the Pauli–Villars fields $\varphi$ and $\tilde{\varphi}$ (the second diagram in Fig. 2). The third and the fourth terms correspond to the third and the fourth diagrams in Fig. 2, respectively, and cancel each other. Thus, we see that this expression is finite and can be easily calculated. In order to do this we note that Eq. (87) can be rewritten as an integral over a double total derivative:

\[
\frac{d \ln G}{d \ln \Lambda} \Bigg|_{p \to 0} = -\frac{e_0^2}{4\pi^2} \left( T(R_0) - C_2 \right) + O(e_0^4). \tag{88}
\]

Taking into account that the Pauli–Villars masses $M_I$ and $M_0$ are proportional to the parameter $\Lambda$, we easily obtain (setting $m_0 = 0$)

\[
\gamma(\alpha_0) = \frac{d \ln G}{d \ln \Lambda} \Bigg|_{p \to 0; m_0 = 0} = \frac{e_0^2}{4\pi^2} (T(R_0) - C_2) + O(e_0^4) = \frac{\alpha_0}{\pi} (T(R_0) - C_2) + O(\alpha_0^2). \tag{89}
\]

As a consequence,

\[
G = 1 + \frac{\alpha_0}{\pi} (T(R_0) - C_2) \ln \Lambda + \text{finite terms} + O(\alpha_0^2). \tag{90}
\]

Evidently, the Nielsen–Kallosh ghosts are not renormalized, because they interact only with the background gauge superfield. The one-loop renormalization of the Faddeev–Popov ghosts can be found by calculating a diagram presented in Fig. 3.

Figure 3: This diagram gives a two-point Green function of the Faddeev–Popov ghost superfields in the one-loop approximation.

It is easy to see that contributions of the various ghosts fields cancel each other and this diagram is convergent and gives the vanishing contribution. Therefore, in the one-loop approximation it is possible to choose $Z_c = 1$.\footnote{The considered Green function is also finite in the infrared limit if $p \neq 0$. In Eq. (87) it is possible to take the limit $p \to 0$ due to the derivative with respect to $\ln \Lambda$.}
Renormalization of the coupling constant can be investigated by calculating the two-point Green function of the background gauge superfield \( V \) in the one-loop approximation. Due to the Slavnov–Taylor identity [66, 67] this Green function is transversal:

\[
\Gamma^{(2)}_V = -\frac{1}{8\pi} \text{tr} \int \frac{d^4 p}{(2\pi)^4} d^4 \theta \: V(\theta, -p) \frac{\partial^2 \Pi_{1/2} V(\theta, p)}{\partial^2 \Pi_{1/2} V(\theta, p)} d^{-1}(\alpha_0, \Lambda/p, m_0/\Lambda).
\] (91)

In the one-loop approximation the function \( d^{-1} \) can be obtained by calculating the diagrams presented in Fig. 4. The result has the following form:

\[
\frac{d}{d \ln \Lambda} (d^{-1} - \alpha_0^{-1}) \bigg|_{p \to 0} = 4\pi \cdot \frac{d}{d \ln \Lambda} \int \frac{d^4 k}{(2\pi)^4} \left( C_2 \left( \frac{1}{k^4} + \frac{2}{\Lambda^4(1 + k^2/\Lambda^2)^2} \right) 
- C_2 \left( \frac{1}{k^4} - \frac{2}{(k^2 + m_0^2)^2} \right) + 2T(R_0) \left( \frac{1}{(k^2 + m_0^2)^2} - \sum_{I=1}^n c_I \frac{1}{(k^2 + M_I^2)^2} \right) 
- 2C_2 \left( \frac{1}{k^4} - \frac{1}{(k^2 + M_0^2)^2} \right) \right) + O(\epsilon_0^2).
\] (92)

The diagrams containing an internal loop of the quantum gauge superfield \( V \) (the first column in Fig. 4) give a vanishing contribution in the limit \( p \to 0 \). The diagrams with an internal loop of \( \Phi \) (the second column in Fig. 4) give the first term in Eq. (92). This term is exactly canceled by a contribution of the diagrams containing a loop of the Nielsen–Kallosh ghosts \( C \), which is given by the second term in Eq. (92). The third term in Eq. (92) corresponds to the contribution of the hypermultiplet \( \phi, \tilde{\phi} \) and its Pauli–Villars fields \( \phi_I, \tilde{\phi}_I \) (the fourth column in Fig. 4). The last term in Eq. (92) consists of the Faddeev–Popov ghosts \( (c, \tilde{c}) \) contribution and the contribution of the Pauli–Villars fields \( \varphi, \tilde{\varphi} \).

Taking into account that

\[
\frac{d}{d \ln \Lambda} (d^{-1} - \alpha_0^{-1}) \bigg|_{p \to 0; m_0=0} = -\frac{d}{d \ln \Lambda} (\alpha_0^{-1}) \bigg|_{m_0=0} = \frac{\beta(\alpha_0)}{\alpha_0^2},
\] (93)

we obtain that a \( \beta \)-function of the considered theory is given by integrals of double total derivatives:

\[
\frac{\beta(\alpha_0)}{\alpha_0^2} = -4\pi \int \frac{d^4 k}{(2\pi)^4} \frac{d}{d \ln \Lambda} \frac{\partial}{\partial k^\mu} \frac{\partial}{\partial k^\nu} \left( \frac{T(R_0)}{2k^2} \left( \ln(k^2 + m_0^2) - \sum_{I=1}^n c_I \ln(k^2 + M_I^2) \right) \right)
\] (92)
\[-\frac{C_2}{2k^2} \left( \ln k^2 - \ln (k^2 + M_0^2) \right) \right|_{m_0=0} + O(\alpha_0) = \frac{1}{\pi} \left( T(R_0) - C_2 \right) + O(\alpha_0). \]  

(94)

In the considered approximation this result agrees with the exact expression \([51]\). Comparing it with Eq. (89) we verify Eq. (79) in the considered (one-loop) approximation.

Due to the Slavnov–Taylor identity the two-point Green function of the quantum gauge superfield is also transversal:

\[ \Gamma^{(2)}_V - S^{(2)}_{\text{gf}} = -\frac{1}{8\pi} \text{tr} \int \frac{d^4p}{(2\pi)^4} d^4\theta V(\theta, -p) \partial^2 \Pi_{1/2} V(\theta, p) d^{-1}_q(\alpha_0, \Lambda/p, m_0/\Lambda). \]  

(95)

The function \(d^{-1}_q\) can be also found by calculating the diagrams presented in Fig. 4. The only difference is that the Nielsen–Kallosh ghosts \(C\) do not contribute to the renormalization of the quantum gauge superfield. The result has the following form:

\[
\begin{align*}
\frac{d}{d\ln \Lambda} \left( d^{-1}_q - \alpha_0^{-1} \right) \bigg|_{p \to 0} &= 4\pi \cdot \frac{d}{d\ln \Lambda} \int \frac{d^4k}{(2\pi)^4} \left( C_2 \left( \frac{1}{k^4} + \frac{2}{\Lambda^4(1 + k^2/\Lambda^2)^2} \right) \
-C_2 \left( \frac{3}{k^4} + \frac{2}{\Lambda^4(1 + k^2/\Lambda^2)^2} \right) \right) \left( \frac{1}{(k^2 + m_0^2)^2} \right) \sum_{I=1}^{n} c_I \left( \frac{1}{(k^2 + M_I^2)^2} \right) \right) + O(\epsilon_0^2). \\
&+ \frac{2C_2}{(k^2 + M_0^2)^2} \right) + O(\epsilon_0^2). \\
\end{align*}
\]

(96)

The expression in the right hand side of this equation is finite at finite values of \(\Lambda\) and coincides with the corresponding expression in Eq. (82). The contributions of the superfield \(\Phi\), the hypermultiplet (with the corresponding Pauli–Villars fields), and the Pauli–Villars fields \(\varphi\) and \(\bar{\varphi}\) are calculated exactly as earlier. However, contributions of the quantum gauge superfield and ghosts are different, if the external lines correspond to the quantum gauge superfield \(V\). As we have already mentioned above, the Nielsen–Kallosh ghosts do not contribute to the renormalization of the quantum gauge superfield, because their action depends only on the background gauge superfield. The Faddeev–Popov ghosts give only noninvariant terms proportional to \(\text{tr} V^2\), which exactly cancel similar terms coming from the diagrams with a loop of the quantum gauge superfield. The diagrams with a loop of the quantum gauge superfield also give invariant contribution, which is given by the second term in Eq. (90).

It is also expedient to compare Eqs. (88), (92), and (96). For this purpose we write the one-loop divergences of the considered two point functions in the following form (taking into account that the effective action depends on the superfield \(\Phi_T = \Phi + \bar{\Phi}\)):

\[
\frac{1}{2\epsilon_0^2} \text{tr} \int d^4x d^4\theta \left( - V \partial^2 \Pi_{1/2} V - V \partial^2 \Pi_{1/2} V + \Phi_T^+ \Phi_T \right) \ln \Lambda \left\{ - \epsilon_0^2 \int \frac{d^4k}{(2\pi)^4} \frac{d}{d\ln \Lambda} \frac{\partial}{\partial k^\mu} \frac{\partial}{\partial k^\mu} \right. \\
\left. \times \left( \frac{T(R_0)}{2k^2} \left( \ln(k^2 + m_0^2) - \sum_{I=1}^{n} c_I \ln(k^2 + M_I^2) \right) - \frac{C_2}{2k^2} \left( \ln k^2 - \ln(k^2 + M_0^2) \right) \right) \right\} + O(\epsilon_0^2). \\
\]

(97)

From this equation we see that the regularization constructed in this paper in the considered approximation allows to obtain the manifestly \(\mathcal{N} = 2\) supersymmetric effective action, although the gauge fixing procedure is not \(\mathcal{N} = 2\) supersymmetric. From Eq. (97) we also conclude that the superfield \(\Phi\) is not renormalized, \(Z_\Phi = 1\), because all divergences are absorbed into the coupling constant renormalization. This completely agrees with Eq. (51). Certainly, it is possible
to make a finite renormalization of the superfield $\Phi$. However, such a finite renormalization destroys $\mathcal{N} = 2$ supersymmetry and we will not make it. Moreover, we see that the quantum field $V$ is not renormalized in the one-loop approximation, so that it is possible to choose $Z_V = 1$.

Thus, we have verified that the proposed regularization does regularize the one-loop divergences and gives the correct values of the renormalization group functions in the one-loop approximation. In particular, we confirm Eq. (83) by the explicit calculation in the one-loop approximation and also obtain

$$Z_V = 1; \quad Z_cZ_{\bar{c}} = 1. \quad (98)$$

### 7 Conclusion

We have proposed a new version of the higher covariant derivative regularization for general $\mathcal{N} = 2$ SYM theories formulated in terms of $\mathcal{N} = 1$ superfields. At the classical level such theories are manifestly invariant under $\mathcal{N} = 1$ supersymmetry by construction, but these theories are also invariant under additional hidden on-shell supersymmetry.

For calculation of quantum corrections it is convenient to define the effective action in the framework of the background field method and fix a gauge without breaking the background gauge invariance. In order to regularize the theory by higher covariant derivatives, we constructed the gauge invariant higher derivative functional which is invariant under the same amount of supersymmetries as the classical action. Adding this functional to the classical action we regularize all divergences beyond the one-loop approximation in the gauge invariant way. The remaining one-loop divergences are regularized by inserting appropriate Pauli–Villars determinants into the generating functional. We show that these determinants preserve all supersymmetries of the classical action by construction. As a result, the hidden supersymmetry is broken only by the gauge fixing procedure. In this paper we have found that if the effective action is invariant under the background transformation of the hidden supersymmetry, the renormalization of the coupling constant is related with the renormalization of the superfields $\Phi_A$ (or, equivalently, the superfield $\Phi = e_0\Phi_A t^A$ is unrenormalized, $Z_\Phi = 1$). The exact NSVZ $\beta$-function is naturally obtained with help of the higher derivative regularization. Thus, it is possible to use the relation (81), which follows from the exact NSVZ $\beta$-function. This, in turn, implies that the higher loop structure of exact NSVZ $\beta$-function is determined by the anomalous dimension $\gamma_\Phi(a_0)$. If the function $\gamma_\Phi$ vanishes, the NSVZ $\beta$-function is reduced to a purely one-loop expression. Therefore, the equality $\gamma_\Phi = 0$ discussed above can be considered as the exact criterium of finiteness of $\mathcal{N} = 2$ SYM theories with matter beyond the one-loop approximation. We want to emphasize once more, that all previous proofs of the $\mathcal{N} = 2$ non-renormalization theorem were based on the assumption of existence of a regularization preserving all symmetries of the classical action in an arbitrary loop. However, all known regularizations do not satisfy this assumption. In this paper we actually presented such a regularization and showed how it works.

Also, we would like to point out that a completely off-shell $\mathcal{N} = 2$ supersymmetric regularization can in principle be developed within the harmonic superfield approach to $\mathcal{N} = 2$ supersymmetric theories [74]. This approach allows to formulate $\mathcal{N} = 2$ SYM theories in terms of off-shell $\mathcal{N} = 2$ superfields. Moreover, the background field formalism and off-shell $\mathcal{N} = 2$ supersymmetric gauge fixing procedure are developed in the harmonic superfield approach [72, 73]. Therefore, for constructing a manifestly $\mathcal{N} = 2$ supersymmetric regularization it is necessary to construct an appropriate gauge invariant higher derivative functional in terms of harmonic superfields. We plan to study this problem in a forthcoming work.
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Appendix

A Higher derivative term invariant under $\mathcal{N} = 2$ supersymmetry

In order to construct the action $S_\Lambda$, given by Eq. (8), it is convenient to use the Noether method \[15, 16\] writing the supersymmetry transformations in terms of $\mathcal{N} = 1$ superfields \[14\]. As a starting point we consider the action $S_0 = -\frac{1}{32\epsilon_0^2\Lambda^2} \text{tr} \int d^4x \left\{ \text{Re} \int d^2\theta (e^{\Omega} W^a e^{-\Omega}) \nabla^2 \nabla^2 (e^{\Omega} W_a e^{-\Omega}) \right. \left. + \int d^4\theta (e^{-\Omega^+} \Phi^+ e^{\Omega^+}) \nabla^2 \nabla^2 (e^{\Omega} \Phi e^{-\Omega}) \right\}$, (99)

where $\Lambda$ is a regularization parameter. (Its dimension is equal to the dimension of a mass.) In order to construct an action invariant under the transformations (7) by the Noether method, at the first step we calculate the variation of the action $S_0$. The result is given by the following (non-vanishing) expression:

$$
\delta S_0 = -\frac{i}{32\epsilon_0^2\Lambda^2} \text{tr} \int d^4x d^4\theta \left\{ -4\eta e^{\Omega} W^a e^{-\Omega} \left[ e^{-\Omega^+} \Phi^+ e^{\Omega^+}, \nabla^2 (e^{\Omega} W_a e^{-\Omega}) \right] 
-4\eta^* \nabla^2 (e^{-\Omega^+} W_a e^{\Omega^+}) \left[ e^{-\Omega^+} \Phi^+ e^{\Omega^+}, e^{\Omega} \Phi e^{-\Omega} \right] + \eta^* e^{-\Omega^+} \Phi^+ e^{\Omega^+} \left[ e^{\Omega} \Phi e^{-\Omega}, \nabla^2 \nabla^2 (e^{\Omega} \Phi e^{-\Omega}) \right] 
-\eta e^{-\Omega^+} \Phi^+ e^{\Omega^+} \nabla^2 \left[ e^{\Omega} \Phi e^{-\Omega}, \nabla^2 (e^{\Omega} \Phi e^{-\Omega}) \right] + \eta e^{-\Omega^+} \Phi^+ e^{\Omega^+} \nabla^2 \left[ e^{\Omega} \Phi e^{-\Omega}, \nabla^2 (e^{\Omega} \Phi e^{-\Omega}) \right] 
-\eta e^{-\Omega^+} \Phi^+ e^{\Omega^+} \nabla^2 \left[ e^{\Omega} \Phi e^{-\Omega}, \nabla^2 (e^{\Omega} \Phi e^{-\Omega}) \right] \right\}.
$$

(100)

These terms can be canceled by adding

$$
S_1 = -\frac{1}{4\epsilon_0^2\Lambda^2} \text{tr} \int d^4x d^4\theta \left( e^{\Omega} W^a e^{-\Omega} \left[ \nabla_a (e^{\Omega} \Phi e^{-\Omega}), (e^{-\Omega^+} \Phi^+ e^{\Omega^+}) \right] 
+ (e^{-\Omega^+} W^a e^{\Omega^+}) \left[ (e^{\Omega} \Phi e^{-\Omega}), \nabla_a (e^{-\Omega^+} \Phi^+ e^{\Omega^+}) \right] \right)
$$

(101)

to the action $S_0$. The sum $S_0 + S_1$ is also not invariant under the transformations (7):
\[ \delta(S_0 + S_1) = -\frac{i}{2\epsilon^2 \Lambda^2} \text{tr} \int d^4x d^4\theta \left( D_\alpha \eta \left[ e^{\Omega W^\alpha} e^{-\Omega}, e^{-\Omega^+} \Phi^+ e^{\Omega^+} \right] \ight. \\
+ D_\alpha \eta^+ \left[ e^{-\Omega^+} W^\alpha e^{\Omega^+}, e^{\Omega} \Phi e^{-\Omega} \right] \left[ (e^{\Omega} \Phi e^{-\Omega}), (e^{-\Omega^+} \Phi^+ e^{\Omega^+}) \right]. \] (102)

These terms can be canceled by adding the term

\[ S_2 = \frac{1}{2\epsilon^2 \Lambda^2} \text{tr} \int d^4x d^4\theta \left[ (e^{\Omega} \Phi e^{-\Omega}), (e^{-\Omega^+} \Phi^+ e^{\Omega^+}) \right]^2 \] (103)

to the action. Then the sum

\[ S_\Lambda = S_0 + S_1 + S_2 \] (104)

is invariant under the transformations (7).

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