THE PSEUDOFOREST ANALOGUE FOR THE STRONG NINE DRAGON TREE CONJECTURE IS TRUE

LOGAN GROUT AND BENJAMIN MOORE

Abstract. We prove that for any positive integers $k$ and $d$, if a graph $G$ has maximum average degree at most $2k + \frac{2d}{d+k+1}$, then $G$ decomposes into $k+1$ pseudoforests $C_1, \ldots, C_{k+1}$ such that there is an $i$ such that for every connected component $C$ of $C_i$, we have that $e(C) \leq d$.

1. Introduction

Throughout this paper, all graphs are finite and may contain multiple edges, but have no loops. All undefined graph theory terminology can be found in [1]. For a graph $G$, $V(G)$ denotes the vertex set and $E(G)$ denotes the edge set. We will let $v(G) = |V(G)|$ and $e(G) = |E(G)|$. Given a graph $G$, an orientation of $G$ is obtained from $E(G)$ by taking each edge $xy$, and replacing $xy$ with exactly one of the arcs $(x, y)$ or $(y, x)$. For any vertex $v$, let $d(v)$ and $d^+(v)$ denote the degree and out-degree, respectively, of $v$. Let $\Delta(G)$ and $\Delta^+(G)$ denote the maximum degree and out-degree respectively. For any graph $G$, we will say a decomposition of $G$ is a set of edge disjoint subgraphs of $G$ such that the union of their edges sets is $E(G)$. We will say the maximum average degree of $G$ is

$$\text{mad}(G) = \max_{H \subseteq G} \frac{2e(H)}{v(H)}.$$ 

Recall, a pseudoforest is a graph where each connected component has at most one cycle. This paper will focus on decompositions into pseudoforests.

We motivate our main result as an analogue to the Strong Nine Dragon Tree Conjecture. The Strong Nine Dragon Tree Conjecture, proposed by Montassier, Ossona de Mendez, Raspaund, and Zhu in [6], is a statement about decomposing a graph into a minimal number of forests, such that one of the forests has additional structure. The statement is the following:

**Conjecture 1.1** (Strong Nine Dragon Tree Conjecture). Let $G$ be a graph where

$$\max_{H \subseteq G, v(H) > 1} \frac{e(H)}{v(H) - 1} \leq k + \frac{d}{d + k + 1},$$

then $G$ decomposes into $k + 1$ forests, where one of the forests has the property that every connected component has at most $d$ edges.

When $d = 1$, the conjecture was proven by Yang in [7] and with a slightly different proof by Jiang and Yang in [4]. When $k = 1$ and $d = 2$, the Strong Nine Dragon Tree Conjecture was proven by Kim, Kostochka, West, Wu, and Zhu [5].

Our main result is an proving the analogue of the Strong Nine Dragon Tree conjecture for pseudoforests (initially conjectured in [3]).
Theorem 1.2. If $\text{mad}(G) \leq 2k + \frac{3d}{d+k+1}$, then $G$ decomposes into $k + 1$ pseudoforests $T_1, \ldots, T_k, F$, such that each connected component of $F$ contains at most $d$ edges.

There has been previous work building towards Theorem 1.2. A weaker version of Theorem 1.2 was proven by Fan, Li, Song, and Yang [2], which implies Theorem 1.2 when $d = 1$.

Theorem 1.3 (2). If $\text{mad}(G) \leq 2k + \frac{3d}{d+k+1}$, then $G$ decomposes into $k + 1$ pseudoforests, where one of the pseudoforests has maximum degree at most $d$.

One can view Theorem 1.2 as a strengthening of Theorem 1.3. Interestingly, the proof of the Strong Nine Dragon Tree Conjecture when $k = 1$ and $d = 2$ in [5] implies Conjecture 1.2 when $k = 1$ and $d = 2$. The authors proved Theorem 1.2 when $d \leq 2k + 2$ in [3].

We split the proof up into three subsections. The proof will be by minimal counterexample, and the first subsection dedicated to describing how we will choose our counterexample. The second part describes a flipping operation which we will use to try and “improve” our counterexample. The final part is to apply the flipping operation plus a counting argument to show that either we could obtain a decomposition that we desire, or that there exists a subgraph with average degree larger than the bound in Theorem 1.2.

2. Decomposing sparse graphs into pseudoforests

2.1. Picking the minimal counterexample. Fix $k$ and $d$ as positive integers. We will assume $d \geq 2$, as the case where $d = 1$ is covered in [2]. Suppose $G$ is a vertex minimal counterexample to Theorem 1.2 for the fixed values of $k$ and $d$.

Our first step will be to obtain desirable orientations of $G$. For this, we use a lemma proved in [2] (the hypothesis is slightly different, however their proof gives us the desired result without any change). We omit the proof.

Lemma 2.1 (2). If $G$ is a vertex minimal counterexample to Theorem 1.2, then there exists an orientation of $G$ such that for all $v \in V(G)$, we have $k \leq d^+(v) \leq k + 1$.

Let $F$ be the set of orientations of $E(G)$ which satisfy Lemma 2.1. The first observation is that any orientation in $F$ gives rise to a natural pseudoforest decomposition.

Observation 2.3. Given a red-blue colouring of $G$, we can decompose our graph $G$ into $(k + 1)$-pseudoforests such that $k$ of the pseudoforests have all of their edges coloured blue, and the other pseudoforest has all of its edges coloured red.

Note that given an orientation in $F$, one can generate many different red-blue colourings. Recall that if a graph admits an orientation where each vertex has out-degree at most one, then the graph is a pseudoforest. Hence we have the following observation.

Definition 2.2. Let $\sigma \in F$. Given $\sigma$, a red-blue colouring of $G$ is a colouring of the edges obtained in the following way. For all $v \in V(G)$, if $d^+(v) = k + 1$, arbitrarily colour $k$ of the outgoing arcs blue and one of the arcs red. If $d^+(v) = k$, then colour all outgoing arcs blue.

Observation 2.3. Given a red-blue colouring of $G$, we can decompose our graph $G$ into $(k + 1)$-pseudoforests such that $k$ of the pseudoforests have all of their edges coloured blue, and the other pseudoforest has all of its edges coloured red.

Observe that one red-blue colouring can give rise to many different pseudoforest decompositions (when $k \geq 2$). Given a pseudoforest decomposition obtained from Observation 2.3 we will say a pseudoforest which has all arcs coloured blue is a blue pseudoforest, and the pseudoforest with all arcs coloured red is the red pseudoforest.
Definition 2.4. Let $f$ be a red-blue colouring of $G$, and let $C_1, \ldots, C_k, F$ be a pseudoforest decomposition obtained from $f$ by Observation 2.3. Then we say that $C_1, \ldots, C_k, F$ is a pseudoforest decomposition generated from $f$. We will always use the convention that $F$ is the red pseudoforest, and $C_i$ is a blue pseudoforest.

As $G$ is a counterexample, in every pseudoforest decomposition generated from a red-blue colouring, there is a connected component of the red pseudoforest which has more than $d$ edges.

Now we define a residue function which we will use to pick our minimal counterexample.

Definition 2.5. Let $f$ be a red-blue colouring and $C_1, \ldots, C_k, F$ be a pseudoforest decomposition generated by $f$. Let $T$ be the set of components $F$. We define the residue function, $\rho$, as:

$$\rho(F) = \sum_{K \in T} \max\{e(K) - d, 0\}.$$

Using a red-blue colouring, and the resulting pseudoforest decomposition, we define an induced subgraph of $G$ which we will focus our attention on.

Definition 2.6. Suppose that $f$ is a red-blue colouring of $G$, and suppose $D = (C_1, \ldots, C_k, F)$ is a pseudoforest decomposition generated from $f$. Let $R$ be an component of $F$ such that $e(R) > d$. We define the subgraph $H_{f,D,R}$ in the following manner. Let $S \subseteq V(G)$ where $v \in S$ if and only if there exists a path $P = v_1, \ldots, v_m$ such that $v_m = v$ where $v_1 \in V(R)$, and either $v_i v_{i+1}$ is an arc $(v_i, v_{i+1})$ coloured blue, or $v_i v_{i+1}$ is an arbitrarily directed arc coloured red. Then we let $H_{f,D,R}$ be the graph induced by $S$.

Given a graph $H_{f,D,R}$, we will say $R$ is the root. We will say the red components of $H_{f,D,R}$ are the components of $F$ contained in $H_{f,D,R}$, and given a vertex $x \in V(H_{f,D,R})$, we let $R^x$ denote the red component of $H_{f,D,R}$ containing $x$. As notation, given a subgraph $K$ of $G$, we will let $E_b(K)$ and $E_r(K)$ denote the set of edges of $K$ coloured blue and red, respectively. We let $e_b(K) = |E_b(K)|$ and $e_r(K) = |E_r(K)|$.

We make an important observation about the average degree of $H_{f,D,R}$ which allows us to focus on just the red edges.

Observation 2.7. If

$$\frac{e_r(H_{f,D,R})}{v(H_{f,D,R})} > \frac{d}{d + k + 1},$$

then

$$\frac{e(H_{f,D,R})}{v(H_{f,D,R})} > k + \frac{d}{d + k + 1}.$$

Proof. By definition, for every vertex $v \in V(H_{f,D,R})$, all blue outgoing arcs from $v$ are in $E(H_{f,D,R})$. Therefore,

$$\frac{e_b(H_{f,D,R})}{v(H_{f,D,R})} = k.$$

Hence since

$$\frac{e_r(H_{f,D,R})}{v(H_{f,D,R})} > \frac{d}{d + k + 1},$$

we have

$$\frac{e(H_{f,D,R})}{v(H_{f,D,R})} = \frac{e_r(H_{f,D,R})}{v(H_{f,D,R})} + \frac{e_b(H_{f,D,R})}{v(H_{f,D,R})} > k + \frac{d}{d + k + 1}.$$
With this observation, our goal will be to show that either we can obtain the desired decomposition, or apply the observation and contradict the maximum average degree bound. Towards this goal, we define the notion of a *troublesome component*.

**Definition 2.8.** Let $P$ be the red pseudoforest of some red-blue colouring. Let $K$ be a subgraph of $P$. We will say $K$ is *troublesome* if
\[
\frac{e_r(K)}{v(K)} < \frac{d}{d+k+1}.
\]

We now define the notion of a legal order of the components of the red pseudoforest in $H_{f,D,R}$.

**Definition 2.9.** We call an ordering $(R_1, \ldots, R_t)$ of the red components of $H_{f,D,R}$ legal if all red components are in the ordering, $R_1$ is the root component, and for all $j \in \{2, \ldots, t\}$ there exists an integer $i$ with $1 \leq i < j$ such that there is a blue arc $(u,v)$ such that $u \in V(R_i)$ and $v \in V(R_j)$.

Let $(R_1, \ldots, R_t)$ be a legal ordering. We will say that $R_i$ is a *parent* of $R_j$ if $i < j$ and there is a blue arc $(v_i, v_j)$ where $v_i \in R_i$ and $v_j \in R_j$. In this definition a red component may have many parents. To remedy this, if a red component has multiple parents, we arbitrarily pick one such red component and designate it as the only parent. If $R_i$ is the parent of $R_j$, then we say that $R_j$ is a *child* of $R_i$. We say a red component $R_i$ is an *ancestor* of $R_j$ if we can find a sequence of red components $R_{i_1}, \ldots, R_{i_m}$ such that $R_{i_1} = R_i$, $R_{i_m} = R_j$, and $R_{i_q}$ is the parent of $R_{i_{q+1}}$ for all $q \in \{1, \ldots, m-1\}$.

An important definition we need is the notion of vertices determining a legal order.

**Definition 2.10.** Given a legal order $(R_1, \ldots, R_t)$, we will say a vertex $v$ *determines the legal order* for $R_j$ if there is a blue arc $(u, v)$ such that $u \in R_i$ and $v \in R_j$ such that $i < j$.

Observe there may be many vertices which determine the legal order for a given red component. More importantly, for every component which is not the root, there exists a vertex which determines the legal order.

We also want to compare two different legal orders.

**Definition 2.11.** Let $(R_1, \ldots, R_t)$ and $(R'_1, \ldots, R'_t)$ be two legal orders. We will say $(R_1, \ldots, R_t)$ is *smaller* than $(R'_1, \ldots, R'_t)$ if the sequence $(e(R_1), \ldots, e(R_t))$ is smaller lexicographically than $(e(R'_1), \ldots, e(R'_t))$.

With this, we will pick our minimal counterexample in the following manner.

First, we pick our counterexample to be minimized with respect to the number of vertices. Then, we pick an orientation in $F$, a red-blue colouring $f$ of this orientation, a pseudoforest decomposition $D = (C_1, \ldots, C_k, F)$ generated by $f$, a root component $R$ of $F$, and lastly a legal order $(R_1, \ldots, R_t)$ of $H_{f,D,R}$ subject to the following conditions in the following order:

1. The number of cycles in $F$ is minimized,
2. subject to the above condition, we minimize the residue function $\rho$,
3. and subject to both of the above conditions, $(R_1, \ldots, R_t)$ is the smallest legal order.

From here on out, we will assume we are working with a counterexample picked in the manner described.
2.2. The flip operation. Let $f$ be the red-blue colouring of our counterexample, and let $C_1, \ldots, C_k$ be the blue pseudoforests, and $F$ the red pseudoforest. Let $(R_1, \ldots, R_t)$ be the legal ordering picked for our counterexample.

Definition 2.12. Let $f$ be a red-blue colouring. Let $(x, y)$ be an arc coloured blue, and suppose that $e = xv$ is an arbitrarily oriented red arc incident to $x$. To flip on $e$ and $(x, y)$ is to take a maximal directed red path $Q = v_1, v_2, \ldots, v_n, y$ where $(v_i, v_{i+1})$ is a red arc, $(v_n, y)$ is a red arc, reverse the direction of all arcs of $P$, change the colour of $(x, y)$ to red, reorient $(x, y)$ to $(y, x)$, change the colour of $e$ to blue, and (if necessary), reorient $(v, x)$ to $(x, v)$.

Observation 2.13. Suppose we flip on an edge $e = xv$ and $(x, y)$. Then the resulting orientation is in $F$.

Proof. If $xv$ is oriented $(x, v)$, then the flip operation changes the colour of $(x, y)$ to red and the colour of $e$ to blue, all internal vertices of $Q$ still have red out-degree at most one, $y$ has red out-degree one as we orient $(x, y)$ to $(y, x)$, and $v$ still has red out-degree at most one. By construction all the blue out-degrees stay the same. Hence this orientation is in $F$.

If $xv$ is oriented $(v, x)$, then the flip operation changes the colour of $(x, y)$ to red, and then reorients $(v, x)$ to $(x, v)$ and reverses the orientation on a maximal directed red path $Q$. By construction, the blue out-degrees of all vertices remain the same, and similarly, the red out-degrees stay the same. Hence this orientation is in $F$. □

To avoid repetitively mentioning it, we will implicitly make use of Observation 2.13.

2.3. The Key Lemmas. Now we are ready to prove the main lemmas. First we make a basic observation about red cycles.

Observation 2.14. Let $(x, y)$ be an edge which is coloured blue such that $R^x$ is distinct from $R^y$ and $R^y$ is a tree. Then $x$ does not lie in a cycle of $F$.

Proof. Suppose towards a contradiction that $x$ lies in a cycle of $F$. Let $e$ be an edge incident to $x$ which lies in the cycle coloured red. Now flip at $(x, y)$ and $e$. As $(x, y)$ was an arc between two red components, and $e$ was in the cycle coloured red, after performing the flip, we reduce the number of cycles in $F$ by one. However, this contradicts our choice of minimum counterexample. □

Now we show that if a red component plus any child of the red component must combine to have at least $d$ edges, or all children of the red component contain a cycle.
Lemma 2.15. Let $R^x$ and $R^y$ be red components such that $R^y$ is the child of $R^x$, $R^y$ does not contain a cycle, and $(x, y)$ is a blue arc from $x$ to $y$. Then $e(R^x) + e(R^y) \geq d$.

Proof. Suppose towards a contradiction that $e(R^x) + e(R^y) < d$. Hence $e(R^x) < d$. Thus $R^x$ is not the root component. Let $w$ be a vertex which determines the legal order for $R^x$. Observe that by Observation 2.13, $x$ does not belong to a cycle of $R^x$.

Case 1: $w \neq x$.
Let $e$ be the edge incident to $x$ in $R^x$ such that $e$ lies on the path from $x$ to $w$ in $R^x$. Then flip on $(x, y)$ and $e$. As $e(R^x) + e(R^y) < d$, all resulting red components have less than $d$ edges, and hence we do not increase the residue function. Furthermore, we claim we can find a smaller legal order. Let $R_i$ be the component in the legal order corresponding to $R^x$. Then consider the new legal order where the components $R_1, \ldots, R_{i-1}$ remain in the same position, we replace $R_i$ with $R^w$, and then complete the order arbitrarily. By how we picked $e$, $e(R^w)$ is strictly smaller than $e(R_i)$, and hence we have found a smaller legal order, a contradiction.

Case 2: $w = x$.

As $R^x$ is not the root component, let $R^{x_0}$ be an ancestor of $R^x$ such that $e(R^{x_0}) \geq 1$. Let $R^{x_0}, R^{x_{n-1}}, \ldots, R^{x_1}$ be a sequence of red components such that for $i \in \{1, \ldots, n-1\}$, $R^{x_i}$ is the child of $R^{x_{i+1}}$ and $R^x$ is the child of $R^{x_1}$. Up to relabelling the vertices, there is a path $P = x_n, \ldots, x_1, x, y$ such that $(x_{i+1}, x_i)$ is a arc coloured blue, and $(x_1, x)$ is a arc coloured blue. Let $e$ be a red edge incident to $x_n$. Now do the following. Colour $(x, y)$ red, and reverse the direction of all arcs in $P$. Colour $e$ blue, and orient $e$ away from $x_n$. By the same argument in Observation 2.13, the resulting orientation is in $F$. Furthermore as $e(R^x) + e(R^y) < d$, all resulting red components have less than $d$ edges, and hence $x$ did not increase. Finally, we can find a smaller legal order in this orientation, as we simply take the same legal order up to the component containing $x_n$, and then complete the remaining order arbitrarily. As the component containing $x_n$ has at least one less edge now, this order is a smaller legal order, a contradiction.

We will need a bound on the number of edges in a connected troublesome component.

Observation 2.16. Let $K$ be a connected troublesome component. Then

$$e(K) < \frac{d}{k+1}.$$ 

Proof. From the definition of troublesome component we have

$$\frac{e(K)}{v(K)} < \frac{d}{d + k + 1}.$$ 

As $K$ is connected, $K$ must be a tree, as if $K$ contains a cycle then $e(K) \geq v(K)$, and $\frac{d}{d + k + 1} < 1$, which implies $K$ is not troublesome. Since $K$ is a tree, we obtain the inequality

$$e(K)(d + k + 1) < d(e(K) + 1).$$

This inequality holds if and only if

$$e(K)(k + 1) < d,$$

which is equivalent to

$$e(K) < \frac{d}{k+1}.$$ 

□
Given a red component $K$, where $K$ has $K_1, \ldots, K_q$ as troublesome children, we will denote $K_C$ as the subgraph with vertex set $V(K_C) = V(K) \cup V(K_1) \cup \cdots \cup V(q)$, and contains all red edges from $K, K_1, \ldots, K_q$.

**Lemma 2.17.** Let $K$ be a red component such that $K$ is not troublesome. Let $K_1, \ldots, K_q$ be the troublesome children of $K$. Then

$$e(K_C) \geq \frac{d}{v(K_C)}.$$  

Furthermore, if $R^*$ is the root component, then the inequality is strict.

**Proof.** We split the proof into two cases, one where $q \leq k$, and the other where $q > k$.

**Case 1:** $q \leq k$.

By Lemma 2.15, we know for each $i \in \{1, \ldots, q\}$ we have that $e(K) + e(K_i) \geq d$. As $e(K_i) \geq 0$ for all $i \in \{1, \ldots, q\}$, it follows that $e(K_i) \geq \max\{0, d - e(K)\}$ for all $i \in \{1, \ldots, q\}$.

Expanding the definitions and some easy calculation gives us:

$$\frac{e(K_C)}{v(K_C)} = \frac{e(K) + \sum_{i=1}^{q} e(K_i)}{v(K) + \sum_{i=1}^{q} v(K_i)} \geq \frac{e(K) + \sum_{i=1}^{q} \max\{0, d - e(K)\}}{e(K) + 1 + q + \sum_{i=1}^{q} \max\{0, d - e(K)\}} \geq \frac{e(K) + \sum_{i=1}^{q} \max\{0, d - e(K)\}}{e(K) + 1 + k + \sum_{i=1}^{q} \max\{0, d - e(K)\}}.$$  

The first equality is simply applying the definition of $K_C$. The second inequality uses that $e(K_i) \geq \max\{0, d - e(K)\}$, and that as $K_i$ is troublesome, $K_i$ is a tree, so $v(K_i) = e(K_i) + 1$.

Finally, the last inequality is using that $q \leq k$.

Now we split this into two cases based on whether or not $\max\{0, d - e(K)\}$ is $0$ or $d - e(K)$.

**Subcase 1:** $\max\{0, d - e(K)\} = 0$.

If $\max\{0, d - e(K)\} = 0$, then $e(K) + e(K_i) \geq d$, we have that $e(K) \geq d$. Thus it follows that,

$$\frac{e(K) + \sum_{i=1}^{q} \max\{0, d - e(K)\}}{e(K) + 1 + k + \sum_{i=1}^{q} \max\{0, d - e(K)\}} \geq \frac{e(K)}{e(K) + k + 1} \geq \frac{d}{d + k + 1}.$$  

Observe that if $K$ is the root component, then $e(K) > d$, and thus the above inequality is strict in this case.

**Subcase 2:** $\max\{0, d - e(K)\} = d - e(K)$.

Calculating we obtain,

$$\frac{e(K) + \sum_{i=1}^{q} \max\{0, d - e(K)\}}{e(K) + 1 + k + \sum_{i=1}^{q} \max\{0, d - e(K)\}} \geq \frac{e(K) + q(d - e(K))}{e(K) + q(d - e(K)) + k + 1} \geq \frac{e(K) + d - e(K) + k + 1}{d - e(K) + e(K)} \geq \frac{d}{d + k + 1}.$$  

Again, if $K$ is the root component, then $e(K) > d$, and the above inequalities are strict. Thus the lemma follows when $q \leq k$. 

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Case 2: \( q > k \).

Note that each vertex has at most \( k \) blue outgoing edges. Hence in this case there are at least two distinct vertices in \( K \) which have blue outgoing edges to troublesome children.

Let \( S = \{x_1, \ldots, x_m\} \) be a maximal collection of vertices in \( K \) such that there exists vertices \( y_1, y_{a_1}, \ldots, y_{m}, y_{m,a_m} \) where \( (x_i, y_{i,j}) \) is a directed blue edge, \( R^{y_{i,j}} \) is a troublesome child of \( K \), and \( R^{y_{i,j}1} \neq R^{y_{i,j}2} \) if \( (i_1, j_1) \neq (i_2, j_2) \). As there are at least two vertices in \( K \) which have blue outgoing edges to troublesome children, \( m \geq 2 \). By Observation 2.11 none of \( x_1, \ldots, x_m \) lie in a red cycle of \( K \). Fix a vertex \( w \) such that \( w \) determines the legal order for \( K \) (in the event \( K \) is the root, we pick \( w \) arbitrarily). Note that \( S \setminus \{w\} \) is non-empty, as \( |S| \geq 2 \). We partition \( S \setminus \{w\} \) into a minimal number of sets \( S^1, \ldots, S^p \) such that for each \( S^i \), there is a vertex \( x \in S^i \) such that for every \( z \in S^i \), on any \((z, w)\)-path, \( x \) lies on this path.

For each vertex \( x_i \in S \setminus \{w\} \), let \( e_i \) be the edge incident to \( x_i \) such that \( e_i \) is in every \((x_i, w)\)-path. Note that \( e_i \neq e_j \) for \( i \neq j \). Let \( K^i \) denote the component of \( K \) which contains \( x_i \) after deleting \( e_i \). If flipping on \((x_i, y_{i,j})\) and \( e_i \) does not contradict our choice of counterexample, then \( e(K^i) + e(R^{y_{i,j}}) > d \), and the residue function increased, as otherwise we would be able to obtain a smaller legal order.

Observe that if \( x_i \in S^j \) and \( x_p \in S^2 \) where \( j \neq z \), then \( K^i \) is vertex disjoint from \( K^p \). Now we make a series of claims.

Claim 2.18. \( |S^i| = 1 \) for all \( i \in \{1, \ldots, p\} \).

Proof. Suppose towards a contradiction that \( |S^i| \geq 2 \). Let \( x_i \) be the vertex in \( S^i \) with largest distance from \( w \), and let \( x \) be the vertex in \( S^w \) such that all \((x, w)\)-paths contain \( x \).

For any vertex \( y_{i,z} \) where \( R^{y_{i,z}} \) is troublesome and \((x_i, y_{i,z})\) is a blue arc, we have that \( e(R^{y_{i,z}}) < \frac{dk}{k+1} \). Since \( e(K^i) + e(R^{y_{i,z}}) > d \) (as otherwise flipping would contradict our choice of counterexample), it follows that \( e(K^i) > \frac{dk}{k+1} \). Let \( e' \) be the edge incident to \( x \) which lies on every path from \( x \) to \( x_i \). Observe that flipping on \( e' \) and \((x, y)\) gives a component \( K' \) containing \( x_j \) such that \( e(K') < e(K) - \frac{dk}{k+1} + \frac{d}{k+1} \leq e(K) \). It follows that the residue function decreased if \( K \) had more than \( d \) edges, or we find a smaller legal order. In either case, we obtain a contradiction. \( \square \)

Claim 2.19. There is at least one troublesome child of \( K \), say \( R^w_1 \) such that \( w \) is the only vertex in \( K \) with a blue outgoing edge to \( R^w_1 \).

Proof. Suppose towards a contradiction that this is false, so for every troublesome child of \( K \), there is a vertex which is not \( w \) which has a blue outgoing arc to that troublesome component. Then \( p \geq 2 \), as otherwise we fall into case one.

Let \( e(B) \) denote the number of edges which are not in any \( K^i \) component for any \( i \in \{1, \ldots, q\} \), and \( v(B) \) the number of vertices not in any \( K^i \) component for any \( i \in \{1, \ldots, q\} \). Then we can partition the edge set of \( K^C \) into edges which lie in some \( K^i \) (observe these are disjoint by Claim 2.18), the edges which are in a troublesome component (disjoint by assumption), and any left over edges. Note that \( e(B) \geq p \), as \( K \) is connected. Thus, doing a calculation we obtain
The pseudoforest analogue for the Strong Nine Dragon Tree Conjecture is true.

\[
e(K_C) = \sum_{i=1}^{p} e(K_i) + \sum_{j=1}^{a_j} e(R^{y_{i,j}}) + e(B)\]
\[
v(K_C) = \sum_{i=1}^{p} v(K_i) + \sum_{j=1}^{a_j} v(R^{y_{i,j}}) + v(B)\]
\[
\ge \frac{dp + p}{(d + k + 1)p + 1} > \frac{d}{d + k + 1}.
\]

Hence if \( w \) does not have a troublesome child, the claim holds. \( \square \)

Claim 2.20. \( p = 1. \) That is, there is only one set \( S^1. \)

**Proof.** Suppose towards a contradiction that \( p \geq 2. \) Then \( K \) contains at least \( \frac{2dk}{k+1} + 2 \) edges. Observe that \( \frac{2dk}{k+1} + 2 > d \) if and only if \( (d - 1)k > -2k - 2, \) which always holds. We claim that flipping \( e_1 \) and any \( (x_1, y_{1,j}) \) will reduce the residue function. To see this, we consider two cases. Let \( K_w \) denote the component containing \( w \) after flipping.

If after flipping, both of the resulting components have more than \( d \) edges, then the residue function decreases by \( e(K) - d, \) and increases by \( e(K^1) + e(R^{y_{1,j}}) - d + e(K^w) - d. \) But
\[
e(K) - d < e(K^1) + e(R^{y_{1,j}}) + e(K^w) - 2d
\]
always holds as this is equivalent to
\[
d \geq e(R^{y_{1,j}})
\]
which holds as \( e(R^{y_{1,j}}) \leq \frac{d}{k+1}. \)

Therefore we can assume that \( e(R^w) \) has at most \( d \) edges.

In this case, the residue function only would not improve if \( e(R^w) \) had less than \( \frac{d}{k+1} \) edges. However, this is not true, as otherwise flipping on say, \( e_2 \) and any edge \( (x_2, y_{2,j}) \) would give a smaller legal order without increasing the residue function. Thus the claim follows. \( \square \)

Therefore we can assume that \( p = 1. \) Let \( e_w \) be the edge incident to \( w \) which lies on every path from \( w \) to \( x_1. \) Let \( (w, y) \) be a blue arc where \( R^y \) is a troublesome component of \( K. \)

Flipping on \( (w, y) \) and \( e_w \) must result in the component containing \( w \) having strictly more edges than \( K, \) otherwise we improve the legal order, or possibly reduce the residue function. Note that flipping on \( e_w \) and \( (w, y), \) the component containing \( w \) loses strictly more than \( \frac{dk}{k+1} \) edges, and gains strictly fewer than \( \frac{d}{k+1} \) edges. Hence the component containing \( w \) has strictly fewer edges, a contradiction. Hence the result follows. \( \square \)

Now we finish the proof. Let \( \mathcal{R} \) denote the set of red components which are not troublesome. By Lemma 2.15 it follows that,
\[
V(H) = \bigcup_{K \in \mathcal{R}} V(K_C).
\]
This follows since a troublesome component cannot have a troublesome child by Lemma 2.17. Therefore it follows that:

\[
E_r(H) = \bigcup_{K \in \mathcal{R}} E(K_C).
\]

Since \(K_C\) is not troublesome for any \(K\) by Lemma 2.15 and they partition \(H_{f,D,R}\), the maximum average degree of \(H_{f,D,R}\) must be at least \(2k + \frac{2d}{d+k+1}\). Furthermore, the root component \(R\) satisfies \(e(R) > d\), hence we get that \(H_{f,D,R}\) has maximum average degree strictly larger than \(2k + \frac{2d}{d+k+1}\), contradicting our assumption. Theorem 1.2 follows.

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(Logan Grout) Departement of Combinatorics and Optimization, University of Waterloo, Waterloo, ON, Canada
E-mail address: lcgrout@edu.uwaterloo.ca

(Benjamin Moore) Department of Combinatorics and Optimization, University of Waterloo, Waterloo, ON, Canada
E-mail address: brmoore@uwaterloo.ca