FOUR-DIMENSIONAL QUADRATIC FORMS OVER $\mathbb{C}(t)(X)$

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Abstract. For quadratic forms in 4 variables defined over the rational function field in one variable over $\mathbb{C}(t)$, the validity of the local-global principle for isotropy with respect to different sets of discrete valuations is examined.

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1. Introduction

Let $E$ be a field of characteristic different from 2 and let $E(X)$ denote the rational function field in one variable over $E$.

For $E = \mathbb{C}(t)$, the field of Laurent series in one variable over the complex numbers, the quadratic form

$$Y_1^2 + tY_2^2 + tY_3^2 + X(Y_1^2 + Y_2^2 + tY_4^2)$$

in the variables $Y_1, Y_2, Y_3, Y_4$ over $E(X)$ has no non-trivial zero, but it has a non-trivial zero over the completion of $E(X)$ with respect to any non-trivial valuation on $E(X)$ that is trivial on $E$. This is in contrast to the situation when $E$ is a finite field, by the Hasse-Minkowski Theorem (See [6, Chapter VI, Theorem 66.1]). Note that, in both cases, the field $E$ has a unique extension of each degree in a fixed algebraic closure.

By a $\mathbb{Z}$-valuation, we mean a valuation with value group $\mathbb{Z}$. A quadratic form is isotropic if it has a non-trivial zero, otherwise it is anisotropic. In all generality, an anisotropic quadratic form over $E(X)$ of dimension at most 3 remains anisotropic over the completion of $E(X)$ with respect to some $\mathbb{Z}$-valuation on $E(X)$ that is trivial on $E$; this follows for example from Milnor’s Exact Sequence [4, Theorem IX.3.1]. The case of 4-dimensional quadratic forms is the first case over $E(X)$ where the validity of such a local-global principle for isotropy depends on the base field $E$.

When $E$ is a nondyadic local field, using a result of Lichtenbaum [5], one obtains that a 4-dimensional anisotropic quadratic form over $E(X)$ remains anisotropic
over the completion of $E(X)$ with respect to some $\mathbb{Z}$-valuation on $E(X)$ that is trivial on $E$ (see [1, Remark 3.8]). This resembles the case where $E$ is a finite field.

In contrast to the situations where $E$ is a finite field or a local field, for $E = \mathbb{C}(t)$ the example of the quadratic form above shows that the local-global principle for isotropy of 4-dimensional quadratic forms over $E(X)$ fails with respect to $\mathbb{Z}$-valuations that are trivial on $E$. However, anisotropy of this quadratic form can be detected over the larger field $\mathbb{C}(X)((t))$, by using Springer’s Theorem (see [4, Proposition VI.1.9]).

Consider the more general situation where the field $E$ is complete with respect to any nondyadic $\mathbb{Z}$-valuation $v$. In this case, a local-global principle for isotropy was obtained in [1] using a geometric setup. Let $\mathcal{O}_v$ denote the valuation ring of $v$. By a model for $E(X)$ over $\mathcal{O}_v$ we mean a two-dimensional integral normal projective flat $\mathcal{O}_v$-scheme $\mathcal{X}$ whose function field is isomorphic to $E(X)$. Codimension-one points on a model of $E(X)$ over $\mathcal{O}_v$ correspond to certain $\mathbb{Z}$-valuations on $E(X)$.

For a model $\mathcal{X}$ of $E(X)$ over $\mathcal{O}_v$ let $\Omega_{\mathcal{X}}$ denote the set of $\mathbb{Z}$-valuations given by codimension-one points of $\mathcal{X}$. Consider the set $\hat{\Omega} = \bigcup_{x \in \mathcal{X}} \Omega_x$ where the union is taken over all models $\mathcal{X}$ of $E(X)$ over $\mathcal{O}_v$. It follows from [1, Theorem 3.1 and Remark 3.2] that an anisotropic quadratic form over $E(X)$ remains anisotropic over the completion of $E(X)$ with respect to some $\mathbb{Z}$-valuation in $\hat{\Omega}$. One may ask whether this remains true if one replaces $\hat{\Omega}$ by $\Omega_x$ for some well-chosen model $\mathcal{X}$ of $E(X)$ over $\mathcal{O}_v$.

The aim of this note is to show that this is not the case: if the residue field of $v$ is separably closed then, for any model $\mathcal{X}$ of $E(X)$ over $\mathcal{O}_v$, there exists an anisotropic 4-dimensional quadratic form over $E(X)$ which is isotropic over the completion of $E(X)$ with respect to any $w \in \Omega_{\mathcal{X}}$ (Corollary 2). Let $\pi \in \mathcal{O}_v$ be a uniformiser of $v$. For any model $\mathcal{X}$ of $E(X)$ over $\mathcal{O}_v$, the set $\{w(\pi) \mid w \in \Omega_{\mathcal{X}}\}$ is finite and hence it has an upper bound. However, for any positive integer $r$, the quadratic form

$$\varphi_r = (X^r - \pi)Y_1^2 + (X^{r+1} + \pi)Y_2^2 + \pi XY_3^2 + X(X^r + \pi)Y_4^2$$

is anisotropic over $E(X)$, but it is isotropic over the completion of $E(X)$ with respect to any $\mathbb{Z}$-valuation $w$ on $E(X)$ with $w(\pi) < r$ (Theorem). The construction of $\varphi_r$ is inspired by the example in [1, Remark 3.6] of an anisotropic 6-dimensional quadratic form over $\mathbb{Q}_p(X)$ where $p$ is an odd prime.

2. Results

We assume some familiarity with basic quadratic form theory over fields, for which we refer to [4]. We first fix some notation and recall some results.

By a quadratic form or simply a form we mean a regular quadratic form. Let $E$ always be a field of characteristic different from 2 and let $E^\times$ denote its multiplicative group. For $a_1, \ldots, a_n \in E^\times$ the diagonal form $a_1X_1^2 + \cdots + a_nX_n^2$ is denoted by $\langle a_1, \ldots, a_n \rangle$. 

Let \( v \) be a \( \mathbb{Z} \)-valuation on \( E \). We denote the corresponding valuation ring, its maximal ideal and its residue field respectively by \( \mathcal{O}_v, \mathfrak{m}_v \) and \( \kappa_v \). For an element \( a \in \mathcal{O}_v \), let \( \overline{a} \) denote the image of \( a + \mathfrak{m}_v \) under the residue map \( \mathcal{O}_v \to \kappa_v \). The completion of \( E \) with respect to \( v \) is denoted by \( E_v \). We say that \( v \) is henselian if it extends uniquely to every finite field extension of \( E \). Complete discretely valued fields are henselian (see [2, Theorem 1.3.1 and Theorem 4.1.3]). We recall a consequence of Hensel’s Lemma:

**Lemma.** Let \( v \) be a henselian \( \mathbb{Z} \)-valuation on \( E \) such that \( v(2) = 0 \). Then

(a) The form \( \langle u_1, u_2 \rangle \) over \( E \) is isotropic if and only if \( \frac{u_1}{u_2} \in -\kappa_v^{\times 2} \).

(b) If \( \kappa_v \) is separably closed, then every 3-dimensional form over \( E \) is isotropic.

**Proof:** Since \( \frac{u_1}{u_2} \in -\kappa_v^{\times 2} \) the polynomial equation \( t^2 + \frac{u_1}{u_2} \) has a solution in \( \kappa_v \) and since \( v(2) = 0 \) it follows by Hensel’s Lemma [2, Theorem 4.1.3(4)] that \( u_1u_2 = -E^2 \), whereby the quadratic form \( \langle u_1, u_2 \rangle \) over \( E \) is isotropic. Since \( \kappa_v \) is separably closed with \( v(2) = 0 \), we have that \( \pi \in -\kappa_v^{\times 2} \) for all \( u \in \mathcal{O}_v^{\times} \). Since every 3-dimensional quadratic form over \( E \) contains a 2-dimensional form isometric to \( \lambda(1, u) \) for some \( u \in \mathcal{O}_v^{\times} \) and \( \lambda \in E^{\times} \); (b) follows from (a).

The set of all \( \mathbb{Z} \)-valuations on \( E(X) \) is denoted by \( \Omega_{E(X)} \). For \( r \in \mathbb{N} \), we define

\[
\Omega_r = \{ w \in \Omega_{E(X)} \mid w(E^X) = i\mathbb{Z} \text{ for some } 0 \leq i \leq r \}.
\]

With this notation, \( \Omega_0 \) is the set of all \( E \)-trivial \( \mathbb{Z} \)-valuations on \( E(X) \). We recall that any monic irreducible polynomial \( p \in E[X] \) determines a unique \( \mathbb{Z} \)-valuation \( v_p \) on \( E(X) \) which is trivial on \( E \) and such that \( v_p(p) = 1 \). There is further a unique \( \mathbb{Z} \)-valuation \( v_\infty \) on \( E(X) \) such that \( v_\infty(f) = -\deg(f) \) for any \( f \in E[X] \setminus \{0\} \). Moreover, every \( \mathbb{Z} \)-valuation \( w \) on \( E(X) \) trivial on \( E \) is either equal to \( v_\infty \) or to \( v_p \) for some monic irreducible polynomial \( p \in E[X] \) (see [2, Theorem 2.1.4]), and in either of the two cases the residue field is a finite field extension of \( E \).

**Theorem.** Let \( v \) be a henselian \( \mathbb{Z} \)-valuation on \( E \) such that \( v(2) = 0 \). Assume that \( \kappa_v \) is separably closed. Let \( \pi \in E^{\times} \) be such that \( v(\pi) = 1 \) and let \( r \in \mathbb{N} \). Then the quadratic form

\[
\varphi_r = \langle X^r - \pi, X^{r+1} + \pi, \pi X, X(X^r + \pi) \rangle
\]

is isotropic over \( E(X)_w \) for every \( \mathbb{Z} \)-valuation \( w \in \Omega_{r-1} \) but anisotropic over \( E(X) \) for some \( w \in \Omega_r \).

**Proof:** Set \( F = E(X) \). We first show that \( \varphi_r \) is isotropic over \( F_w \) for all \( w \in \Omega_{r-1} \). Consider \( w \in \Omega_{r-1} \).

**Case 1:** \( w(\pi) = 0 = w(X) \). Then \( \kappa_w \) is a finite extension of \( E \). Since \( v \) is henselian, there is a unique extension \( v' \) of \( v \) to \( \kappa_w \), and \( v' \) again henselian. Furthermore, it follows by [2, Theorem 3.3.4] that \( v'(\kappa_w^{\times}) \) is isomorphic to \( \mathbb{Z} \) and \( \kappa_{v'} \) is separably closed. It follows by part (b) of the Lemma that every 3-dimensional quadratic form over \( \kappa_w \) is isotropic. We have that \( w = v_p \) for some monic irreducible
polynomial \( p \in E[X] \) such that \( p \neq X \). Note that, in this case at least three diagonal coefficients of \( \varphi_r \) are units in \( O_w \). It follows by Springer’s Theorem [4, Proposition VI.1.9] that \( \varphi_r \) is isotropic over \( F_w \).

**Case 2:** \( 0 \leq w(\pi) < r \) and \( 1 \leq w(X) \). Let \( u = (X^r\pi^{-1} - 1)(X^{r+1}\pi^{-1} + 1) \). Then \( w(u) = 0 \) and \( \pi' = -1 \in -\kappa_w^{-2} \). It follows by part (a) of the Lemma that the form \( \pi^{-1}\langle X^r - \pi, X^{r+1} + \pi \rangle \) is isotropic over \( F_w \). Thus \( \varphi_r \) is isotropic over \( F_w \).

**Case 3:** \( w(X) < 0 \leq w(\pi) < r \). Note that \( \kappa_w \) is either a finite extension of \( E \) or a rational function field over a finite extension of \( \kappa_v \); since \(-1 \in \kappa_w^{-2} \), we get in either case that \(-1 \in \kappa_w^{-2} \). Consider \( u = (1 + \pi X^{-r+1})(1 + \pi X^{-r}) \). We have that \( w(u) = 0 \) and \( \pi' = -1 \in \kappa_w^{-2} \). It follows by part (a) of the Lemma that the form \( X^{-r+1}\langle X^{r+1} + \pi, X(X^r + \pi) \rangle \) is isotropic over \( F_w \). Thus \( \varphi_r \) is isotropic over \( F_w \).

We have thus shown that \( \varphi_r \) is isotropic over \( F_w \) for every \( w \in \Omega_{r-1} \). Now we show that \( \varphi_r \) is anisotropic over \( F_w \) for some \( w \in \Omega_f \).

Let \( E' = E(s) \), where \( s = \sqrt[\pi]{\pi} \). Then \( v \) extends uniquely to a valuation on \( E' \) which we again denote by \( v \). Note that \( s^r = \pi \) in \( E' \) and hence \( v(\pi) = rv(s) \). Then \( v' = rv \) is a \( \mathbb{Z} \)-valuation on \( E' \).

Let \( L = E'(X) \) and let \( Y = \frac{\Delta}{s} \). Note that \( L = E'(Y) \). By [2, Corollary 2.2.2], there exists a unique extension of \( v' \) to \( L \) such that \( v(Y) = 0 \) and \( \overline{Y} \) is transcendental of \( \kappa_{v'} \); we further have that \( \kappa_w = \kappa_{v'}(\overline{Y}) \) and \( w(L^s) = v'(E'^s) = \mathbb{Z} \). Since \( w(Y) = 0 \), we have that \( w(X) = w(s) = 1 \). We get that

\[
\varphi_r = \langle s^r(Y^r - 1), s^r(sY + 1), s^{r+1}Y, s^{r+1}Y(Y^r + 1) \rangle
\]

Consider the forms \( \varphi_1 = \langle Y^r - 1, sY + 1 \rangle \) and \( \varphi_2 = \langle Y, Y(Y^r + 1) \rangle \).

Since \( \overline{Y} - 1, \overline{Y} + 1 \notin -\kappa_w^{-2} \), it follows by Springer’s Theorem [4, Proposition VI.1.9] that the quadratic form \( s^{-r}\varphi_r \) is anisotropic over \( L_w \). Hence \( \varphi_r \) is anisotropic over \( L_w \). We obtain that \( \varphi_r \) is anisotropic over \( F_{w|E} \). Note that, \( w(\pi) = w(s^r) = rw(s) = r \), thus \( w \in \Omega_r \).

We now provide a different perspective to the above theorem. For a subset \( \Omega \subseteq \Omega_{E(X)} \), we say that \( \Omega \) has the **finite support property** if for every \( f \in E(X)^x \) the set \( \{ w \in \Omega \mid w(f) \neq 0 \} \) is finite. It is well-known that \( \Omega_0 \) has the finite support property. When \( E \) carries a discrete valuation the set \( \Omega_{E(X)} \) does not have the finite support property. However, for any model \( \mathcal{X} \) of \( E(X) \) over \( \mathcal{O}_w \), the set \( \Omega_{E(X)} \) contains \( \Omega_0 \) and has the finite support property. We show the following:

**Corollary 1.** Let \( v \) be a henselian \( \mathbb{Z} \)-valuation on \( E \) with \( v(2) = 0 \). Assume that \( \kappa_v \) is separably closed. Let \( \Omega \subseteq \Omega_{E(X)} \) be a subset with the finite support property. Then there exists an anisotropic 4-dimensional quadratic form over \( E(X) \) which is isotropic over \( E(X)_w \) for every \( w \in \Omega \).

**Proof:** Let \( \pi \in E^x \) be such that \( v(\pi) = 1 \). Since \( \Omega \) has the finite support property, the set \( \{ w \in \Omega \mid w(\pi) \neq 0 \} \) is finite. Set \( r = 1 + \max\{w(\pi) \mid w \in \Omega \} \). Clearly
\( \Omega \subseteq \Omega_{r-1} \). Then the form \( \varphi_r \) in the Theorem is isotropic over \( E(X)_w \) for every \( w \in \Omega \), but anisotropic over \( E(X) \).

**Corollary 2.** Let \( v \) be a henselian \( \mathbb{Z} \)-valuation on \( E \) with \( v(2) = 0 \). Assume that \( \kappa_v \) is separably closed. Let \( \mathcal{X} \) be a regular model of \( E(X) \) over \( \mathcal{O}_v \). Then there exists an anisotropic 4-dimensional quadratic form over \( E(X) \) which is isotropic over \( E(X)_w \) for every \( w \in \Omega_{\mathcal{X}} \).

**Proof:** By [3, Chapter II, Lemma 6.1], for every element \( f \in E(X)^\times \) the set \( \{ w \in \Omega_{\mathcal{X}} \mid w(f) \neq 0 \} \) is finite, hence the statement follows by Corollary 1.

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