Perturbative Gravity from QCD Amplitudes

Z. Bern
Department of Physics,
University of California at Los Angeles,
Los Angeles, CA 90095-1547

and

A. K. Grant
Department of Physics,
Harvard University,
Cambridge, MA 02138

Abstract
We demonstrate that QCD gluon amplitudes can be used to construct a Lagrangian for gravity. This procedure makes use of perturbative 'squaring' relations between gravity and gauge theory that follow from string theory. We explicitly carry out the construction for up to five-point interactions and discuss a set of field variables in the Einstein-Hilbert Lagrangian for interpreting the Lagrangian obtained from QCD. A spin-off from our analysis is that it can be used to provide simpler tree-level gravity Feynman rules than for conventional gauges.

* Research supported by the US Department of Energy under grant DE-FG03-91ER40662.
† Research supported by the National Science Foundation grant PHY-9802709.
1 Introduction

Although both gravity and gauge theories contain a local symmetry, they have rather disparate physical properties. QCD, for example, is a confining theory while gravity is not. Similarly, in four dimensions QCD is renormalizable while field theories of gravity are non-renormalizable. Nonetheless, within the context of the perturbative expansions some remarkable ‘squaring’ relations exist between the tree-level $S$-matrix elements of gravity and gauge theories in both string and field theories \[1,2,3,4,5\]. These squaring relations imply that gauge symmetry is more closely related to the diffeomorphism invariance of gravity than one might suspect based on the respective Lagrangians. Recently, there has also been a resurgence of interest in perturbative gravity both in anti-de Sitter spaces \[6\] and in phenomenological applications of large compact dimensions \[7\], further motivating an investigation of these relations.

In this letter we will show that one can construct a low energy Lagrangian for gravity directly from QCD $S$-matrix elements and then discuss how this Lagrangian relates to the usual Einstein-Hilbert Lagrangian. The starting point of our investigation is the Kawai, Lewellen and Tye (KLT) relations expressing closed string tree amplitudes in terms of open string amplitudes \[1\]. These relations arise from the property that the integrand of a closed string amplitude is composed of left- and right-mover open string components (see e.g. ref. \[8\]). In the low energy limit where string theory reduces to field theory, the KLT relations express gravity tree amplitudes in terms of a sum of products of ‘left’ gauge theory amplitudes and ‘right’ gauge theory amplitudes.

The KLT relations were first exploited by Berends, Giele and Kuijf to obtain an infinite sequence of maximally helicity violating pure gravity tree amplitudes \[2\] using known gauge theory results. By unitarity, tree-level relations necessarily imply that loop amplitudes in the two theories are also related. Indeed, the method of constructing $S$-matrix elements via their analytic properties (see refs. \[9\]) provides a means for exploiting this to obtain loop-level (super) gravity amplitudes. Using these ideas, the divergence structure of $N = 8$ supergravity has been investigated with the result that it appears to be less divergent than previously thought \[3\]. Two infinite sequences of maximally helicity violating one-loop amplitudes in gravity theories have also been constructed \[5\]. These may be viewed as results in an effective field theory of gravity which necessarily must match the low energy limit of a more fundamental theory, such as string or $M$ theory. The ability to use the squaring relation between gravity and gauge theory to perform non-trivial gravity computations suggests that one can develop a deeper understanding of perturbative gravity by exploring this relationship.

From the point of view of the field theory Lagrangians, the KLT relations are rather mysterious: the Einstein-Hilbert Lagrangian does not factorize in any obvious way in terms of the Yang-Mills Lagrangian. It is not even completely clear what the notion of ‘left’ and ‘right’ parts of the theory mean given that the graviton is a symmetric tensor. Another obvious difficulty is that the Yang-Mills Lagrangian has only three- and four-point interactions while the Einstein-Hilbert Lagrangian has an infinite set of interactions.

Nevertheless, the gravity $S$-matrix does have the property that it is composed of products of two gauge theory amplitudes and that a graviton, very roughly speaking, is composed of ‘left’ and ‘right’ gauge fields, i.e., $h_{\mu\nu} \sim \tilde{A}_{\mu}A_{\nu}$. When one has a property of the $S$-matrix, a natural idea is to organize
the underlying formalism so that it reflects that property. As a well known example, the general superiority of Feynman diagrams as compared to time ordered perturbation theory follows from their preservation of manifest Lorentz symmetry. Here we wish to rearrange the gravity Lagrangian so it captures properties associated with the factorization of the S-matrix into sums of products of gauge theory amplitudes.

The analysis of this letter will consist of two parts. In the first part we will systematically construct a gravity Lagrangian from QCD gluon amplitudes using the KLT relations. We explicitly carry out the procedure through five graviton interactions; in principle, the procedure can be carried out to arbitrarily high orders, although as a practical matter the complexity of the computations increases rapidly with the number of gravitons. Nevertheless, the procedure shows that a Lagrangian for gravity can be obtained using QCD amplitudes. The gravity Lagrangian obtained in this way has the properties that (a) by construction it produces the correct tree-level amplitudes and (b) graviton Lorentz indices associated with the ‘left’ gauge theory do not contract with ones associated with the ‘right’ gauge theory. A Lagrangian with the property that the associated Feynman diagrams factorize into left and right parts has been presented previously by Siegel [10]. A spin-off from our analysis is that it can be used to provide a simpler set of gravity Feynman rules than the conventional de Donder gauge rules. Moreover, in our construction, the gravity three vertex is given directly as sums of squares of gauge theory vertices.

In the second part, we will find a set of field variables which allows us to interpret the gravity Lagrangian obtained from QCD in terms of the conventional Einstein-Hilbert Lagrangian. Non-linear field redefinitions and gauge fixings have been used previously by van de Ven [12] to simplify the computation of two-loop divergences in gravity. Here we wish to consider such field redefinitions and gauge fixings in order to find a particular set of field variables which helps clarify the connection to QCD. This construction indicates that there is a closer correspondence between ordinary gauge invariance and the diffeomorphism invariance of Einstein gravity than one might have suspected.

2 Review of the KLT relations and applications

A basic property of integrands for closed string amplitudes is that they factorize into open string integrands (except for the momentum zero mode). At tree-level, KLT [1] used this property to find simple relationships expressing closed string amplitudes in terms of products of open string amplitudes. In the infinite string tension limit where string theory reduces to an effective field theory, for the four- and five-point amplitudes these relations are,

\[
\mathcal{M}^\text{tree}_4(1, 2, 3, 4) = -i\left(\frac{\kappa}{2}\right)^2 s_{12} A^\text{tree}_4(1, 2, 3, 4) A^\text{tree}_4(1, 2, 4, 3),
\]

\[
\mathcal{M}^\text{tree}_5(1, 2, 3, 4, 5) = i\left(\frac{\kappa}{2}\right)^3 \left[ s_{12}s_{34} A^\text{tree}_5(1, 2, 3, 4, 5) A^\text{tree}_5(2, 1, 4, 3, 5) + s_{13}s_{24} A^\text{tree}_5(1, 3, 2, 4, 5) A^\text{tree}_5(3, 1, 4, 2, 5) \right],
\]

where, \(\kappa^2 = 32\pi G_N\), \(s_{ij} = (k_i + k_j)^2\), the \(A_n\) are color-ordered gauge theory partial amplitudes [13] and the \(\mathcal{M}_n\) are gravity amplitudes. Color does not appear in these relations because the gauge theory partial amplitudes are independent of color. The arguments of the amplitudes label the
external legs. These relations hold for any particle states appearing in any closed string theory since they follow from the basic factorization of the string integrands into ‘left’ and ‘right’ sectors. In particular, they hold in pure Einstein gravity, gravity coupled to a dilaton, anti-symmetric tensor or gauge field, and in supergravity theories. Moreover, these KLT relations generalize to an arbitrary number of external legs. (Explicit expressions may be found in Appendix A of ref. \[5\].) A basic property of the KLT formulas is that any Lorentz indices associated with the ‘left’ gauge theory amplitude do not contract with the Lorentz indices of the ‘right’ gauge theory amplitude.

The KLT relations give tree-level gravity amplitudes directly from known gauge theory amplitudes, which are generally far easier to compute. This was first applied by Berends, Giele and Kuijf \[2\] to obtain an infinite sequence of maximally helicity violating $n$-point Einstein gravity tree amplitudes using known QCD gluon amplitudes. Beyond tree-level, one may exploit the KLT relations using the cutting method developed for performing QCD amplitude calculations of phenomenological interest \[9\]. With this method one can reconstruct complete loop amplitudes (in a given dimensional regularization scheme) from their $D$-dimensional unitarity cuts. The one-loop amplitudes are specified in terms of tree amplitudes which satisfy the KLT relations; one can then obtain higher loop amplitudes by iterating the procedure. In this way, one can obtain loop-level (super) gravity $S$-matrix elements without reference to a Lagrangian or Feynman rules \[2, 4, 5\]. The efficiency of the computational method follows from the fact that one is recycling previously performed gauge theory calculations to obtain new results in gravity theories.

\[
\sum_{\text{gauge states}} A_{\text{tree}}^4(1, 2, \ell_2, -\ell_1) \times A_{\text{tree}}^4(3, 4, \ell_1, -\ell_2),
\]

summed over all states in the gravity theory that can cross the cut. From the KLT relations \(\mathcal{M}\) we may re-express this product in terms of gauge theory amplitudes,

\[
\sum_{\text{gauge states}} s_{12} A_{\text{tree}}^4(1, 2, \ell_2, -\ell_1) \times A_{\text{tree}}^4(3, 4, \ell_1, -\ell_2)
\times \sum_{\text{gauge states}} s_{12} A_{\text{tree}}^4(1, 2, \ell_1, -\ell_2) \times A_{\text{tree}}^4(3, 4, -\ell_2, \ell_1).
\]

The sum over the gravity theory states necessarily factorize into a sum over the states of two gauge theories because the underlying string theory has this property. This is a generic property holding for all cuts. In this way, one can re-express cut gravity amplitudes in terms of gauge theory ones. This has led to the construction of a number of non-trivial gravity loop amplitudes \[3, 4, 5\].

Figure 1: One can recycle tree amplitudes into loop amplitudes via $D$-dimensional unitarity cuts.

For example, consider the unitarity cuts of a one-loop four-point amplitude in any gravity theory coming from the low energy limit of a string theory. As depicted in fig. 1 a gravity tree amplitude appears on each side of the cut. Thus, on the cut one must evaluate the product,
3 Construction of a gravity Lagrangian from QCD

We now discuss a procedure for constructing an off-shell low energy Lagrangian for pure gravity starting from QCD gluon amplitudes. We explicitly carry this out for up to five graviton interactions. By construction, the Lagrangian that we obtain is equivalent to the Einstein-Hilbert Lagrangian in that it produces identical tree-level $S$-matrix elements.

In field theory one usually takes a given Lagrangian and constructs Feynman rules that can then be used to obtain the $S$-matrix elements. Here we wish to reverse this process, since from the KLT relations we have the gravity $S$-matrix in terms of QCD amplitudes, but wish instead to obtain a Lagrangian that preserves the ‘left’–‘right’ factorization of Lorentz indices.

To obtain the $n$-graviton term in the Lagrangian we start with the tree-level $n$-graviton amplitudes, given in terms of the QCD amplitudes, and subtract all diagrams containing up to $(n - 1)$-graviton interaction vertices. By iterating this procedure we can systematically build a gravity Lagrangian. However, to jump-start this process we first need a propagator and three vertex.

Consider the graviton propagator. The standard de Donder gauge propagator,

$$ P_{\mu\nu;\alpha\beta}(k) = \langle h_{\mu\alpha} h_{\nu\beta} \rangle_0 = \frac{1}{2} \left[ \eta_{\mu\nu} \eta_{\alpha\beta} + \eta_{\mu\beta} \eta_{\nu\alpha} - \frac{2}{D - 2} \eta_{\mu\alpha} \eta_{\nu\beta} \right] \frac{i}{k^2 + i\epsilon}, \tag{4} $$

is unacceptable since it contracts ‘left’ and ‘right’ indices; for example, the last term contracts $\mu$ with $\alpha$. Moreover, it contains explicit dependence on the dimension $D$, which must somehow cancel from the tree-level $S$-matrix elements since there is no such dependence in the KLT relations or in the gauge theory amplitudes. (We have chosen $\eta_{\mu
u}$ to have signature $(+,-,-,-)$.) A better propagator is

$$ P_{\mu\nu;\alpha\beta}(k) = i \frac{2}{\eta_{\mu\nu} \eta_{\alpha\beta} + \eta_{\mu\beta} \eta_{\nu\alpha}} \frac{i}{k^2 + i\epsilon}. \tag{5} $$

However, even this propagator is unacceptable because of the second term, which contracts a left index with a right one. To prevent such contractions, we wish to use instead the propagator,

$$ P_{\mu\nu;\alpha\beta}(k) = \eta_{\mu\nu} \eta_{\alpha\beta} \frac{i}{k^2 + i\epsilon}. \tag{6} $$

In a sense, for each graviton $h_{\mu\alpha}$ we assign the index $\mu$ to be a ‘left’ index and $\alpha$ to be a ‘right’ index. Of course, since $h_{\mu\alpha}$ is symmetric it does not matter which index is assigned to the left and which to the right, but once the choice is made we demand that left indices never contract with right ones. At first sight the gravity propagator (6) might seem rather peculiar since it appears to violate the fundamental property of the graviton being a symmetric tensor field. Nevertheless, it is not difficult to show that one may use the propagator (6) instead of (5) with no effect on the $n$-graviton $S$-matrix elements, provided that all vertices satisfy a rigid left–right interchange symmetry. By a rigid symmetry we mean that each $n$-point vertex be symmetric under a simultaneous interchange of all left and right Lorentz indices,

$$ V_{n}^{\mu_{1}\mu_{2}\cdots \mu_{n};\alpha_{1}\alpha_{2}\cdots \alpha_{n}}(k_{1}, k_{2}, \ldots, k_{n}) = V_{n}^{\alpha_{1}\alpha_{2}\cdots \alpha_{n};\mu_{1}\mu_{2}\cdots \mu_{n}}(k_{1}, k_{2}, \ldots, k_{n}), \tag{7} $$

where the $\mu_{i}$ are the left indices associated with each graviton and the $\alpha_{i}$ are the right indices.
With this requirement, one can show that there is no need to symmetrize the graviton propagator in the left and right indices. Consider, for example, the diagram in fig. 2. The external legs of the diagrams do not need to be explicitly symmetrized since the external graviton polarization tensors automatically symmetrize the legs. The interchange of the $\mu$ and $\alpha$ indices can be undone by performing a rigid interchange of left and right indices on the left-most vertex. Although the rigid interchange will also flip the indices on the external legs, this can be undone since the indices of each external leg are contracted with a symmetric tensor polarization. The same type of argument works for general diagrams. Thus, the propagator (6) can be used instead of the propagator (5) since the symmetrization of indices is automatically taken care of by the rigidly symmetrized vertices and by the symmetric polarization tensors on the external legs.

![Diagram](https://via.placeholder.com/150)

**Figure 2:** A sample diagram for showing that we may use propagator (6) instead of (5).

Given the propagator (6) we must also choose a three vertex. Any three vertex which agrees on shell with the three-graviton vertex is suitable for our purposes. (Kinematic restrictions prevent a massless particle from decaying into two others, but in terms of formal polarization tensors the vertex does not vanish; below we shall also obtain the same vertex from the Einstein-Hilbert action.) From string theory (see e.g. ref. [8]), the on-shell three-graviton vertex can be expressed in terms of products of gauge theory vertices. This motivates the choice of the three graviton vertex,

$$iG_3^{\mu\alpha,\nu\beta,\rho\gamma}(k, p, q) = \frac{i}{2} \left[ V_{\text{GN}}^{\alpha\beta\gamma}(k, p, q) + V_{\text{GN}}^{\mu\mu\rho}(p, k, q) \times V_{\text{GN}}^{\beta\alpha\gamma}(p, k, q) \right],$$

where

$$V_{\text{GN}}^{\mu\nu}(1, 2, 3) = i\sqrt{2}(k_1^\rho \eta^{\mu\nu} + k_2^\rho \eta^{\nu\rho} + k_3^\rho \eta^{\mu\rho}),$$

is the color ordered Gervais-Neveu [15] Yang-Mills three vertex, from which the color factor has been stripped. Our main reasons for using the vertex (8) is its simplicity and the fact that it makes the relationship to gauge theory manifest. By construction, this vertex is symmetric under the rigid interchange of left and right indices.

Armed with the propagator (6) and the three vertex (8) we may then use the KLT relations to find a gravity Lagrangian using QCD gluon amplitudes as input. At the first step of the process one defines a four-vertex by subtracting from the four-point $S$-matrix obtained via the KLT relations all diagrams containing a kinematic pole, as shown in fig. 3. This four vertex automatically has the property that left Lorentz indices do not contract with right ones, since the gravity $S$-matrix and diagrams containing the three-point vertex (8) have this property. Moreover, the vertex defined in this way also has the rigid symmetry under an interchange of left and right indices.

We may then convert this four-vertex to an $h^4$ term in the Lagrangian by inverting the usual procedure of obtaining Feynman vertices from Lagrangians. Of course, there is some ambiguity in this process since one can always add terms which vanish on shell. As long as these terms do not mix left and right Lorentz indices and satisfy a rigid left-right symmetry, which can always be imposed...
Figure 3: One can obtain a four vertex with the left-right factorization property by subtracting the diagrams containing the kinematic poles from the S-matrix.

by hand, they are acceptable terms; differences in the four-vertex then induce differences in the deduced higher-point vertices.

Once the four-graviton terms in the Lagrangian have been chosen we can then continue the process to obtain a five vertex by subtracting from the five-graviton amplitude all diagrams containing three- and four-point vertices. In principle, one can continue in this way to an arbitrary number of external legs, allowing one to build a Lagrangian for gravity order by order in perturbation theory using QCD gluon amplitudes. By construction this Lagrangian has the property that it generates all tree-level S-matrix elements and that it never contracts left indices with right ones.

Starting from QCD gluon tree amplitudes, we have carried out the construction for up to five points yielding the local gravity Lagrangian

\[ L = \sum_i L_i, \]

where

\[ L_2 = -\frac{1}{2} h_{\mu\nu} \partial^2 h_{\mu\nu}, \]

\[ L_3 = \kappa \left[ \frac{1}{2} h_{\mu\rho} h_{\sigma\mu} h_{\rho\sigma} + h_{\nu\mu} h_{\rho\mu} h_{\rho\sigma} \right], \]

\[ L_4 = -\kappa^2 \left[ \frac{1}{32} h_{\mu\nu,\lambda} h_{\mu\nu,\lambda} h_{\rho\sigma} + \frac{1}{2} h_{\mu\nu,\lambda} h_{\mu\rho} h_{\nu\sigma} h_{\sigma\lambda} + \frac{1}{8} h_{\mu\nu,\lambda} h_{\mu\rho} h_{\nu\sigma} h_{\sigma\lambda} - \frac{1}{4} h_{\mu\nu} h_{\mu\lambda} h_{\rho\sigma} h_{\rho\sigma,\lambda} 
- \frac{1}{4} h_{\mu\lambda} h_{\rho\sigma} h_{\rho\sigma,\nu} + \frac{1}{16} h_{\mu\nu} h_{\lambda\rho} h_{\lambda\rho,\sigma} + \frac{1}{24} h_{\mu\nu} h_{\lambda\rho} h_{\mu\rho,\sigma} \right], \]

\[ L_5 = \kappa^3 \left[ \frac{3}{64} h_{\mu\nu,\lambda} h_{\mu\nu,\lambda} h_{\rho\sigma,\lambda} h_{\rho\sigma,\lambda} - \frac{1}{12} h_{\mu\nu,\lambda} h_{\rho\sigma,\lambda} h_{\rho\sigma,\lambda} h_{\rho\sigma,\lambda} - \frac{1}{48} h_{\mu\nu,\lambda} h_{\mu\nu,\lambda} h_{\rho\sigma,\lambda} h_{\rho\sigma,\lambda} 
- \frac{1}{12} h_{\mu\nu,\lambda} h_{\rho\sigma,\lambda} h_{\rho\sigma,\lambda} h_{\rho\sigma,\lambda} - \frac{1}{32} h_{\mu\nu} h_{\mu\rho} h_{\nu\sigma} h_{\sigma\lambda} h_{\rho\sigma,\lambda} - \frac{1}{12} h_{\mu\nu} h_{\mu\rho} h_{\nu\sigma} h_{\rho\sigma,\lambda} h_{\rho\sigma,\lambda} 
- \frac{1}{6} h_{\mu\nu} h_{\mu\rho} h_{\nu\sigma} h_{\lambda\sigma} h_{\lambda\sigma} h_{\lambda\sigma} h_{\lambda\sigma} - \frac{1}{6} h_{\mu\nu} h_{\mu\rho} h_{\nu\sigma} h_{\lambda\sigma} h_{\lambda\sigma} h_{\lambda\sigma} h_{\lambda\sigma} 
+ \frac{1}{12} h_{\mu\nu} h_{\mu\rho} h_{\nu\sigma} h_{\lambda\sigma} h_{\lambda\sigma} h_{\lambda\sigma} h_{\lambda\sigma} + \frac{1}{8} h_{\mu\nu} h_{\mu\rho} h_{\nu\sigma} h_{\lambda\sigma} h_{\lambda\sigma} h_{\lambda\sigma} + \frac{1}{24} h_{\mu\nu} h_{\mu\rho} h_{\nu\sigma} h_{\lambda\sigma} h_{\lambda\sigma} h_{\lambda\sigma} \right], \]

where \( h_{\mu\nu,\lambda} = \partial_\lambda h_{\mu\nu} \). In Minkowski space, one of any two contracted indices should be taken to be a raised index using \( \eta^{\mu\nu} \), but we have suppressed this. In principle it might have been necessary to introduce auxiliary fields for locality to hold; indeed, as discussed below, when comparing this Lagrangian to the Einstein-Hilbert gravity Lagrangian it is useful to introduce an auxiliary dilaton.

Although the Lagrangian (10) is not unique since the terms can be modified by adding or subtracting contributions that vanish on shell and appropriately modifying the higher-point contributions, it is a relatively simple one. More importantly, as we shall see below, it allows for a relatively straightforward match to the conventional Einstein-Hilbert Lagrangian. By adjusting the four-point terms in the Lagrangian we have found a solution for \( L_5 \) that contains only six terms, but then the connection to the Einstein-Hilbert Lagrangian is a bit more complicated.

For compactness we have removed the rigid symmetrization between left and right indices; how-
ever, if one uses the propagator (3), each vertex must be rigidly symmetrized between left and right indices, e.g.,
\[ L_3 \rightarrow \frac{\kappa}{2} \left[ h_{\mu\nu} h_{\rho\sigma,\mu\nu} h_{\rho\sigma} + h_{\nu\mu} h_{\rho\sigma,\nu\mu} h_{\rho\sigma} + h_{\mu\nu} h_{\mu\rho,\sigma} h_{\sigma\rho,\nu} \right]. \] (11)
Although we do not explicitly give the generated Feynman rules here, it is quite straightforward to obtain these. The advantages of using Feynman rules generated by this Lagrangian instead of the conventional de Donder gauge rule are clear: besides the fact that one can use the propagator (3) instead of the more complicated de Donder gauge propagator (4) the three-, four- and five-point vertices are quite a bit simpler than the corresponding de Donder gauge vertices. Once the interaction terms have been rigidly symmetrized, when deriving the Feynman rules we can in a sense treat \( h_{\mu\nu} \) as a tensor field with no special index symmetry. Note that the Feynman diagrams generated by this Lagrangian do not contain explicit factors of \( D \). In the conventional de Donder gauge such factors do appear, but somehow cancel from the S-matrix elements.

The relative simplicity of these Feynman rules is related to the preservation of the S-matrix property that left and right Lorentz indices should not contract with each other. Although the KLT relations might appear to suggest that field variables exist where gravity can be reformulated as a polynomial theory with no more than four-point interactions as in the gauge theory case, we have been unsuccessful in finding such a Lagrangian.

The Lagrangian (10) does contain general coordinate invariance although it is not manifest since the symmetry has been gauge fixed. To make this explicit, we now relate the Lagrangian (10) to the usual Einstein-Hilbert Lagrangian which contract left with right graviton indices, i.e,
\[ h_{\mu\nu}, \quad h_{\mu\nu} h_{\nu\lambda} h_{\lambda\mu}, \quad \ldots, \quad \text{Tr}[h^{2m+1}], \] (12)
where \( \text{Tr}[h^n] \equiv h_{\mu_1\mu_2} h_{\mu_2\mu_3} \cdots h_{\mu_n\mu_1} \). Because of the way that these types of terms are tangled in the Einstein-Hilbert Lagrangian, it is not obvious how one can accomplish this. Nevertheless, the existence of the Lagrangian (10) implies that there must be some rearrangement with the desired property.

In ref. [14] the dilaton was introduced to allow for a field redefinition which removes the graviton trace from the quadratic terms in the Lagrangian. The appearance of the dilaton as an auxiliary field to help rearrange the Lagrangian is motivated by string theory which requires the presence of such a field. Following the discussion of ref. [14], we then consider the Lagrangian for gravity coupled to a dilaton,
\[ L_{\text{EH}}^{\text{EH}} = \frac{2}{\kappa^2} \sqrt{-g} R + \sqrt{-g} \partial^\mu \phi \partial_\mu \phi. \] (13)
(Our conventions are those of Weinberg [16], except that we have flipped the signature of the metric, \( g_{\mu\nu} \rightarrow -g_{\mu\nu} \); this then induces an overall sign flip in the Lagrangian.) In de Donder gauge, for example, taking \( g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \), the quadratic part of the Lagrangian is
\[ L_2 = -\frac{1}{2} h_{\mu\nu} \partial^2 h_{\mu\nu} + \frac{1}{4} h_{\mu\nu} \partial^2 h_{\nu\mu} - \phi \partial^2 \phi. \] (14)
The term involving $h_{\mu\nu}$ can be eliminated with the simultaneous field redefinitions \[14\],

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \eta_{\mu\nu}\sqrt{\frac{2}{D-2}} \phi, \quad \phi \rightarrow \frac{1}{2} h_{\mu\mu} + \sqrt{\frac{D-2}{2}} \phi,$$

so that the Lagrangian reduces to

$$L_2 \rightarrow -\frac{1}{2} h_{\mu\nu} \partial^2 h_{\mu\nu} + \phi \partial^2 \phi.$$ \[16\]

(This field redefinition is a bit different than the one in ref. \[14\] because we have chosen to normalize the dilaton kinetic term differently so as to slightly simplify the induced interaction terms discussed below.)

For the case of purely graviton external states the dilaton will not contribute to the tree-level $S$-matrix because of the selection rule that $\phi$ must be created or annihilated in pairs. Of course, if the dilaton, or any matter fields appearing in the theory, are taken as external states they can propagate in intermediate states. In this case, one would need to specify the precise matter content of the theory before proceeding with the analysis. Nevertheless, the fundamental factorization of the $S$-matrix implied by the KLT relations would still hold, since it follows from the underlying string theory.

Here we wish to generalize the field redefinitions \[13\] to all orders in $\kappa$ so as to remove all terms of the form \[12\] from the Lagrangian. Our generalization for the case of no gauge fixing is

$$g_{\mu\nu} = e^\kappa h_{\mu\nu} \equiv e^{\sqrt{\frac{2}{D-2}} \kappa \phi} (\eta_{\mu\nu} + \kappa h_{\mu\nu} + \frac{\kappa^2}{2} h_{\mu\rho} h_{\rho\nu} + \cdots),$$ \[17\]

where $h_{\mu\nu}$ is the graviton field, followed by the change of variables

$$\phi \rightarrow -\sqrt{\frac{2}{D-2}} (\phi + \frac{1}{2} h_{\mu\mu}).$$ \[18\]

We have verified through $O(h^6)$ that this choice eliminates all terms \[12\] which mix left and right Lorentz indices, even before fixing the gauge. As yet we have not performed any gauge fixing so the action is generally coordinate invariant, even if the choice of field variables obscures this.

One might be concerned that the field redefinition \[17\] would alter the gravity $S$-matrix. However, the $S$-matrix is guaranteed to be invariant under non-linear field redefinitions or under linear ones that do not alter the coupling to the external traceless polarization tensors. Indeed, our explicit calculations respect this property, as required.

An important remaining question is how one can choose a gauge fixing so that the terms in the Einstein-Hilbert Lagrangian resemble the terms of the Lagrangian \[14\] deduced from the QCD amplitudes. We have found a solution, which is to replace the field redefinition \[18\] with a non-linear generalization,

$$\phi \rightarrow -\sqrt{\frac{2}{D-2}} \left[ (\phi + \frac{1}{2} \text{Tr} h) + \kappa \left( \frac{1}{4} \phi^2 - \frac{1}{8} \text{Tr} (h^2) \right) + \kappa^2 \left( \frac{1}{12} \phi^3 - \frac{1}{8} \phi \text{Tr} (h^2) \right) + \cdots \right].$$ \[19\]
Then we add a gauge fixing term to the Lagrangian, \((F_\mu)^2\) following the standard procedure, where

\[
F_\mu = \left(h_{\mu\nu} + \phi_{,\mu}\right) + \kappa \left(-\frac{1}{4}\text{Tr}(h^2)_{,\mu} - \frac{1}{2}h_{\mu\nu} - h_{\mu\nu}\phi_{,\nu}\right) + \kappa^2 \left(\frac{1}{16}\text{Tr}(h^2)_{,\mu} + \frac{1}{8}\text{Tr}(h^2)_{,\mu}h_{\mu\nu} - \frac{1}{12}h_{\mu\nu}h_{\lambda\rho}\phi_{,\lambda}\phi_{,\rho} + \frac{1}{6}h_{\mu\nu,\rho}h_{\lambda\nu}h_{\lambda\rho} + \frac{1}{24}h_{\mu\nu}h_{\lambda\rho,\phi_{,\rho}} - \frac{1}{8}(\phi\text{Tr}(h^2))_{,\mu}\right) + \cdots.
\]

\((20)\)

With these choices we then obtain the desired form of the Lagrangian through \(O(h^4)\), which we express in terms of the Lagrangian (10) derived from QCD amplitudes,

\[
L_{2}^{\text{EH}} = L_2 + \phi\partial^2\phi,
\]

\[
L_{3}^{\text{EH}} = L_3 - \frac{\kappa}{2}h_{\mu\nu,\phi}h_{\mu\nu,\phi},
\]

\[
L_{4}^{\text{EH}} = L_4 + \kappa^2 \left[\frac{1}{32}h_{\mu\nu,\phi}h_{\mu\nu,\phi}h_{\rho\sigma} + \frac{1}{2}\phi h_{\mu\nu}h_{\lambda\sigma,\mu}h_{\lambda\sigma,\nu} - \frac{1}{4}\phi_{,\mu}h_{\mu\nu}h_{\lambda\sigma}h_{\lambda\sigma} + \frac{1}{2}\phi_{,\mu}h_{\mu\nu}h_{\lambda\sigma,\nu}h_{\lambda\sigma}
- \frac{1}{2}\phi_{,\mu}h_{\mu\nu}h_{\lambda\sigma,\rho}h_{\lambda\sigma} - \phi h_{\mu\nu}h_{\lambda\sigma,\mu}h_{\lambda\sigma,\nu} - \frac{1}{8}\phi_{,\mu}h_{\mu\nu,\rho}h_{\lambda\sigma}h_{\lambda\sigma} + \frac{1}{8}\phi^2 h_{\mu\nu,\phi}h_{\mu\nu,\phi}\right].
\]

\((21)\)

For the case of pure graviton external states we may integrate out \(\phi\) from the path integral. Equivalently, we may substitute the equation of motion for \(\phi\),

\[
\phi = \frac{\kappa}{4}\frac{1}{\partial^2}(h_{\mu\nu,\phi}h_{\mu\nu,\phi}) + \cdots
\]

\((22)\)

into the Lagrangian. For the two- and three-graviton terms in the Lagrangian, this gives exactly the terms in the Lagrangian (10). For the four-point we obtain exactly \(L_4\) plus a non-local piece that vanishes for on-shell gravitons,

\[
\Delta L_4 = \frac{\kappa^2}{16}h_{\mu\nu,\phi}h_{\mu\nu,\phi}h_{\rho\sigma,\lambda\lambda} - \frac{1}{16}h_{\mu\nu,\phi}h_{\mu\nu,\phi}h_{\rho\sigma,\lambda\lambda}h_{\rho\sigma,\lambda\lambda} - \frac{1}{32}h_{\mu\nu,\phi}h_{\mu\nu,\phi}h_{\rho\sigma,\lambda\lambda}h_{\rho\sigma,\lambda\lambda}.
\]

\((23)\)

Since the four-point Lagrangians differ a bit off-shell, at the five-point level, terms that are non-zero on-shell need to be added to \(L_5\),

\[
\Delta L_5 \bigg|_{\text{on shell}} = \kappa^3 \left[\frac{1}{64}h_{\mu\nu,\mu}h_{\rho\sigma,\phi_{,\rho}}h_{\rho\sigma,\phi_{,\sigma}}h_{\lambda\lambda}h_{\lambda\lambda} - \frac{1}{16}h_{\mu\nu,\phi}h_{\mu\nu,\phi}h_{\rho\sigma,\lambda\lambda}h_{\rho\sigma,\lambda\lambda} - \frac{1}{32}h_{\mu\nu,\phi}h_{\mu\nu,\phi}h_{\rho\sigma,\lambda\lambda}h_{\rho\sigma,\lambda\lambda} \right].
\]

\((24)\)

In a sense we may take \(L^{\text{EH}}\) as the more fundamental one since it comes directly from the Einstein-Hilbert Lagrangian, while the Lagrangian in eq. (10) provides the guidance needed to obtain it. This shows the connection of the Lagrangian obtained from QCD (10) and the Einstein-Hilbert Lagrangian for a very particular set of field variables and and gauge fixing. Although the two Lagrangians differ somewhat even after integrating out the auxiliary dilaton, we have explicitly shown that for up to five gravitons they generate the same on-shell scattering amplitudes.

Presumably, similar results can be obtained without introducing the dilaton; nevertheless, we found it useful for clarifying the required reorganization of the Einstein-Hilbert Lagrangian. Other reorganizations based on introducing other fields such as an anti-symmetric tensor, which is also motivated by string theory, are also possible [10].

At loop level there are also ghost contributions that can be obtained via the usual methods. (The Jacobian generated by the field redefinition is trivial, at least in perturbation theory, since it
is a point transformation.) Additionally the auxiliary dilaton would propagate in the loops, which would then need to be subtracted (see e.g., ref. \[14\]). At loop level the unitarity method advocated in refs. \[9, 3, 4, 5\] is, however, an efficient way to obtain the $S$-matrix without the need for Feynman rules.

4 Sample applications

As one simple application, one may obtain the soft factor for gravity amplitudes directly from the soft behavior of QCD amplitudes. Gravity tree amplitudes have the well known behavior \[17\],

$$M_{\text{tree}}^{n}(1, 2, \ldots, n+1) \xrightarrow{k_n \to 0} \frac{k}{2} S^{\text{gravity}}(n^+) \times M_{\text{tree}}^{n-1}(1, 2, \ldots, n-1),$$

as the momentum of graviton $n$, which we have taken to carry positive helicity, becomes soft. Using the three-graviton vertex \[8\] which is expressed in terms of the QCD three-gluon vertex, the gravity soft factor can then be expressed in terms of the QCD soft factor:

$$S^{\text{gravity}}(n^+) = \sum_{i=1}^{n-1} s_{ni} S^{\text{QCD}}(q_i, n^+, i) \times S^{\text{QCD}}(q_r, n^+, i),$$

where $S^{\text{QCD}}(q, n^+, i) = \langle q i \rangle / \langle q n \rangle \langle n i \rangle$ is the eikonal factor for a positive helicity soft gluon in QCD. The $\langle ij \rangle = \langle i^- j^+ \rangle$ are spinor inner products and $|i^\pm \rangle$ are massless Weyl spinors of momentum $k_i$, labeled with the sign of the helicity (see e.g., \[13\]). Although not manifest, the soft factor \[26\] is independent of the choices of null helicity ‘reference momenta’ $q_l$ and $q_r$. By choosing $q_l = k_1$ and $q_r = k_{n-1}$ we recover the form of the soft graviton factor for $k_n \to 0$ used in, for example, refs. \[2, 4, 5\].

As a less trivial example, the explicit factorization of the left and right indices of the interaction vertices would imply that the all-plus helicity graviton current satisfying eqs. (B.10) and (B.11) of ref. \[5\] do actually follow from Einstein gravity. In ref. \[5\] the hitch in obtaining these recursion relations from Einstein gravity was the different choices of helicity reference momenta on the left and on the right. (Although, not derived directly from Einstein gravity these recursion relations were useful for defining ‘half-soft’ functions which serve as building blocks for one-loop gravity amplitudes.) In this case, $\varepsilon_+^{\mu} \varepsilon_+^{\nu} \neq 0$, which prevented a derivation of the recursion relations from Einstein gravity. With the factorization of left and right Lorentz indices, such contractions simply do not occur.

5 Discussion

Since the tree-level $S$-matrix is generated by solving perturbatively the classical equations of motion, one might suspect that it is possible to relate more general solutions of the classical equations of motion for gravity to gauge theory solutions. The property that the $S$-matrix can be expressed in terms of ‘left’ and ‘right’ sectors is generic in string theory. For this reason, it should be possible to extend the discussion in this letter to theories containing, for example, the anti-symmetric tensor
or to supergravity theories. It also seems reasonable that gravity theories in curved spaces can be reformulated to express tree-level amplitudes in terms of gauge theory ones. Moreover, the intimate connection of perturbative gravity amplitudes with gauge theory ones suggests a closer connection of diffeomorphism invariance with non-abelian gauge symmetry than one might have suspected. We feel that these issues deserve further attention.

We thank S. Cherkis, L. Dixon and J. Schwarz for helpful discussions.

References

[1] H. Kawai, D.C. Lewellen and S.-H.H. Tye, Nucl. Phys. B269:1 (1986).
[2] F.A. Berends, W.T. Giele and H. Kuijf, Phys. Lett. B 211:91 (1988).
[3] Z. Bern, L. Dixon, D.C. Dunbar, M. Perelstein and J.S. Rozowsky, Nucl. Phys. B530:401 (1998) [hep-th/9802162].
[4] Z. Bern, L. Dixon, M. Perelstein, J.S. Rozowsky, Phys. Lett. B444:273 (1998) [hep/9809160].
[5] Z. Bern, L. Dixon, M. Perelstein, J.S. Rozowsky, preprint [hep-th/9811140].
[6] S. Gubser and I. Klebanov, Phys. Lett. B413:41 (1997) [hep-th/9708005]; J. Maldacena, Adv. Theor. Math. Phys. 2:231 (1998) [hep-th/9711200]; S. Gubser, I. Klebanov and A. Polyakov, Phys. Lett. B428:105 (1998) [hep-th/9802103]; [hep-th/9802116]; E. Witten, Adv. Theor. Math. Phys. 2:253 (1998) [hep-th/9802150].
[7] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B429:263 (1998) [hep-ph/9803315]; E.A. Mirabelli, M. Perelstein, M.E. Peskin, preprint [hep-ph/9811337]; T. Han, J.D. Lykken, R.-J. Zhang, preprint [hep-ph/9811350]; J.L. Hewett, preprint [hep-ph/9811350].
[8] M.B. Green, J.H. Schwarz and E. Witten, Superstring Theory (Cambridge University Press, 1987).
[9] Z. Bern, L. Dixon, D.C. Dunbar and D.A. Kosower, Nucl. Phys. B425:217 (1994) [hep-ph/9403227]; Nucl. Phys. B435:59 (1995) [hep-ph/9409265]; Z. Bern and A.G. Morgan, Nucl. Phys. B467:479 (1996) [hep-ph/9511336]; Z. Bern, L. Dixon and D.A. Kosower, Ann. Rev. Nucl. Part. Sci. 46:109 (1996) [hep-ph/9602228]; Nucl. Phys. B513:3 (1998) [hep-ph/9708239].
[10] W. Siegel, Phys. Rev. D47:5453 (1993) [hep-th/9302030]; Phys. Rev. D48:2826 (1993) [hep-th/9305073]; in Proceedings of Strings 1993, eds. M.B. Halpern, A. Sevrin and G. Rivlis (World Scientific, Singapore, 1994) [hep-th/9308133].
[11] B.S. DeWitt, Phys. Rev. 162:1239 (1967);  
M. Veltman, in Les Houches 1975, Proceedings, Methods In Field Theory, eds. R. Balian and  
J. Zinn-Justin (North-Holland, Amsterdam, 1976);  
S. Sannan, Phys. Rev. D34:1749 (1986).  

[12] A.E.M. van de Ven, Nucl. Phys. B378:309 (1992).  

[13] M. Mangano and S.J. Parke, Phys. Rep. 200:301 (1991);  
L. Dixon, in Proceedings of Theoretical Advanced Study Institute in Elementary Particle Physics  
(TASI 95), ed. D.E. Soper [hep-ph/9601353].  

[14] Z. Bern, D.C. Dunbar and T. Shimada, Phys. Lett. B312:277 (1993) [hep-th/9307001];  
D.C. Dunbar and P.S. Norridge, Nucl. Phys. B433:181 (1995) [hep-th/9408014].  

[15] J.L. Gervais and A. Neveu, Nucl. Phys. B46:381 (1972).  

[16] S. Weinberg, Gravitation and Cosmology (John Wiley and Sons, 1972).  

[17] S. Weinberg, Phys. Lett. 9:357 (1964); Phys. Rev. 140:B516 (1965).