The virtual element method 
on polygonal pixel–based tessellations*

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Abstract
We analyze and validate the virtual element method combined with a boundary correction similar to the one in [1, 2], to solve problems on two dimensional domains with curved boundaries approximated by polygonal domains. We focus on the case of approximating domains obtained as the union of squared elements out of a uniform structured mesh, such as the one that naturally arises when the domain is issued from an image. We show, both theoretically and numerically, that resorting to polygonal elements allows the assumptions required for stability to be satisfied for any polynomial order. This allows us to fully exploit the potential of higher order methods. Efficiency is ensured by a novel static condensation strategy acting on the edges of the decomposition.

Keywords: Virtual element method, polygonal approximating domain, smooth boundary, curved boundary

1. Introduction
The simplest (and cheapest!) meshes that can be used to approximate a complex domain are the ones whose elements coincide with elements of a sufficiently

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fine squared/cubic uniform structured grid. This holds particularly true when the
domain is retrieved from the results of some imaging procedure, as it often happens,
for instance, in medical applications. In such a framework (an approximation of) the
physical domain is already given as the union of pixels/voxels that a segmentation
procedure tags as belonging to the physical object of interest. These can be seen
as elements of a very fine structured quadrangular/hexahedral mesh. The polygo-
nal/polyhedral domain obtained as their union can then be used as an approximation
of the physical domain, to be used in the numerical solution of a PDE, modeling some
physical behavior.

It is a well known fact that approximating the solution of a problem in a physical
domain by simply solving, by a finite element method, a problem in an approximated
polygonal domain, with a boundary condition somehow “copied” from the physical
boundary data, yields, for methods of higher order, a suboptimal result. For ho-
mogeneous Dirichlet boundary value problems, the fact itself of approximating the
physical domain (with curved boundary) introduces an error that, even in the best
of cases, can be of the order $\delta^{3/2}$, $\delta$ being the distance between physical and approxi-
mate boundary [3]. In and of itself this leads to suboptimality whenever the order $k$
of the method is greater than or equal to two. In the framework we are considering
the situation is worse, and the method turns out to be suboptimal also for $k = 1,$
with a convergence of order only $h^{1/2}$, $h$ denoting the meshsize, as observed, both
theoretically and numerically, in [4]. Nevertheless, such an approach is currently
used by many practitioners. Resorting to a so called microFEM approach ([5]), they
use the mesh whose elements coincide with the voxels in a microCT scan, in order
to simulate some physical phenomenon taking place in an underlying (unknown) do-
main. Of course, given the extreme fineness of the mesh, the results obtained by such
an approach turn out to be sufficiently accurate. The cost is however much higher,
when compared to the cost of the finite element method of the same accuracy on
polygonal physical domains.

Different options exist to counter the sources of error related to the approximation
of the domain, and thus obtain a more efficient method, provided, of course, we
can rely on information on the actual physical domain. By what means, and how
accurately, such information can be retrieved from available imaging data is a crucial
question that is, however, out of the scope of this paper (we refer to [6] and the
references therein for an up to date survey on edge detection methods that can
be used to this aim). Once this information is available, one option is to work on a
possibly coarser mesh and either use a fictitious domain approach, as in the finite cell
method [7], or state the problem on the actual domain by “cutting” the elements that
cross the boundary, as done in the cutFEM method [8]. Remark that preprocessing
the image by changing its resolution is an easy way to obtain a coarser versions of the domain approximation. These approaches can also be combined with different discretization methodologies such as isogeometric analysis [9].

A different approach consists in resorting, while working on the approximate domain, to techniques specifically designed to take into account the fact that its boundary does not coincide with the actual curve/surface where boundary conditions should be prescribed. The first example of this strategy was introduced already in the early ’70s in the seminal paper by Bramble, Dupont and Thomée [10]. There, for convex domains in 2D, a Taylor extrapolation along the direction normal to the boundary of the approximating polygonal domain was leveraged, within a Nitsche approach, to weakly impose the correct boundary conditions. Introduced in the late 2010s, and already tested in different application fields (see, e.g., [11] [12] [13] [14]), the shifted boundary method (SBM, see [15] [16]) overcomes many of the limitations of the original Bramble, Dupont and Thomée approach (BDT), by a careful choice of the extrapolation direction and a clever design of the stabilization term for the underlying Nitsche method. Initially limited to order up to two, an high order version of the SBM, allowing, in principle, to attain any arbitrary order $k$, has been recently introduced and analyzed in [1]. In [17] a variant of the BDT method is proposed, based on the Lagrange multiplier method for Dirichlet boundary conditions [18]. In the same paper, the relation with Nitsche’s method with boundary correction is also discussed, in the spirit of [19]. Nitsche’s method with boundary correction is also studied in the framework of the cutFEM method in [20], where the analysis is carried out for an a priori arbitrary choice of the extrapolation direction. Such direction is, however, in practice, chosen to be, as in SBM, normal to the boundary of the physical domain. The polynomial extension finite element method ([21]) avoids the problem of choosing an extrapolation direction by replacing Taylor extrapolation with an averaged Taylor polynomial extrapolation to also attain arbitrary order under suitable conditions. Similar ideas can be also leveraged in the discontinuous Galerkin framework ([22]).

Unfortunately, as the order $k$ of the method gets higher, the approximate domains that we are considering eventually fail, when discretized by finite elements, to satisfy one of the assumptions required for the stability of such methods. Indeed, the analysis of boundary correction methods requires that the ratio between the distance from the approximate to the true boundary and the diameter of the elements is lower than a constant that decreases to zero as $k$ increases. A way to overcome this limitation is to replace the fine mesh with a sufficiently coarser polygonal/polyhedral mesh, obtained by agglomeration. This allows to make the diameter of the elements larger, while keeping the distance of the two boundaries fixed. On the new polyg-
onal/polyhedral mesh, one can then combine a boundary correction strategy with one of the discretization methods capable of handling polytopal meshes. In this direction, in [23] the authors combine a polynomial extension method with a weak Galerkin finite element method. In [24, 2] a boundary correction method similar to the one in [20] is combined, both in 2 and 3D, with virtual elements (VEM, [25, 26, 27]). Thanks to its flexibility, its robustness (in particular with respect to the shape of the elements), and its potential for high accuracy (the discretization can be designed to be of arbitrarily high order, [28]), virtual elements have, since their introduction in the early 2010’s, rapidly gathered the attention of the scientific community. This resulted in extensions to deal with different type of equations ([29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43]), different formulations ([44, 45, 46, 47, 48]) and numerous applications ([49, 50, 51, 42, 52, 53]).

The aim of this paper is to propose and validate, initially in two dimensions, the use of the combination of virtual elements and boundary correction methods such as BDT or SBM, as a way to obtain an efficient solver in the context presented previously. To this aim we will need to adapt the theoretical results obtained in [2] to the present framework. Indeed, tessellations whose elements are obtained as the union of squares of a structured uniform mesh, do not satisfy the standard shape regularity assumptions under which virtual element methods are usually analyzed. We will also need to show how to efficiently handle elements with a large number of small edges, deriving from the agglomeration process.

The paper is organized as follows. In Section 2 we introduce a unified notation for a class of boundary correction methods that includes, among others, the BDT method and some of its variants, as well as the high order SBM. In Section 3 we recall definition and properties of the plain virtual element method, and we present the virtual element method for curved domains [2] obtained by combining the VEM with the previously introduced boundary correction methods. In Section 4 we present a static condensation procedure allowing to keep the resulting linear system small, also in the presence of elements with a large number of small edges. In Section 5 we present the numerical results attesting to the validity of our proposal and comparing the performance attainable with virtual elements with those obtained by the same boundary correction approach in the finite element framework. The proofs of several theoretical bounds and estimates, needed to extend known virtual element results to the present framework, are presented in three appendixes.
2. Boundary correction methods in the finite element framework

For the sake of simplicity, we focus on a simple model problem, namely on the Poisson equation

\[ -\Delta u = f, \text{ in } \Omega, \quad u = g \text{ on } \Gamma = \partial \Omega, \]  

where \( f \in L^2(\Omega) \), \( g \in H^{1/2}(\partial \Omega) \), and where \( \Omega \subseteq \mathbb{R}^2 \) is a domain with a curved boundary \( \Gamma = \partial \Omega \), assumed, for the sake of convenience, to be of class \( C^\infty \). We consider here a context where the domain is not directly available but is obtained as the result of an imaging process. We will then only have an approximation of \( \Omega \), which we will denote by \( \Omega_h \), naturally decomposed as the union of (tiny) squared elements (the pixels) of size \( h \times h \), with edges parallel to the axes. We will not address here the segmentation problem that needs to be solved in order to single the approximated domain \( \Omega_h \) out of the image. It is natural, in this context, to assume that for all \( x \in \partial \Omega_h \), the distance \( d(x, \partial \Omega) \) of \( x \) to \( \partial \Omega \) verifies \( d(x, \partial \Omega) \leq h \) (this is another way to say that \( \Omega_h \) is an approximation to \( \Omega \)). This happens for instance if \( \Omega_h \) is constructed by retaining all the squares contained in \( \Omega \).

The simplest method to solve (1) (and one of the most used by practitioners) is the finite element method on the quadrilateral mesh \( T_h \), whose elements are the pixels, in which \( \Omega_h \) is naturally split. The approximation space is chosen as the standard \( Q_k \) finite element space

\[ V_h = \{ v \in C^0(\Omega_h) : v|_K \in Q_k \quad \forall K \in T_h \}, \]

\( Q_k \) denoting the space of polynomials \( p(x, y) \) of degree at most \( k \) both in \( x \) and in \( y \). Using Nitsche’s method (see [34]) to impose the boundary conditions, the approximate solution is defined as the element \( u_h \in V_h \) such that for all \( v \in V_h \) it holds

\[
\int_{\Omega_h} \nabla u_h \cdot \nabla v - \int_{\partial \Omega_h} \partial_{\nu_h} u_h v - \int_{\partial \Omega_h} u_h \left( \partial_{\nu_h} v - \gamma h^{-1} v \right) = \int_{\Omega_h} f v - \int_{\partial \Omega_h} g^*(\partial_{\nu_h} v - \gamma h^{-1} v),
\]

with \( g^* \) suitably defined, where \( \nu_h \) is the outer unit normal to \( \partial \Omega_h \), \( \gamma > 0 \) is a suitable, sufficiently large, scalar constant, and where \( \partial_{\nu_h} v \in L^2(\partial \Omega_h) \) denotes the \( L^2(\partial \Omega_h) \) function coinciding with \( \nabla u_h \cdot \nu_h \) on each boundary edge of the mesh \( T_h \). It is however known that, in our framework, such a method is suboptimal already for \( k = 1 \) (see [4]). Several strategies have been proposed in order to retrieve the optimal
order of approximation for the finite element solution on approximating polygonal domains, relying on the idea of suitably correcting the Nitsche’s formulation (2) so that the boundary condition is somehow imposed on the original boundary, while still maintaining all computations on the approximating domain.

We consider here a general formulation that encompasses a number of such strategies. We let $\mathcal{E}^\partial$ denote the set of edges of $\mathcal{T}_h$ lying on $\partial \Omega_h$, and, for $x \in \partial \Omega_h$, we let $\sigma(x)$ denote an outward unit vector. Assuming that $\Omega_h \subseteq \Omega$, for $x \in \partial \Omega_h$ we let $\delta(x) > 0$ denote the distance to $\partial \Omega$ along the direction $\sigma$, that is, the smallest nonnegative scalar such that $x + \delta(x)\sigma(x) \in \partial \Omega$.

see Figure 3. Letting $\partial^j u = (\partial^j ) u$ denote the $L^2(\partial \Omega_h)$ function coinciding, on each edge, with the $j$-th partial derivative of $u$ in the $\sigma$ direction, and letting, for $x \in \partial \Omega_h$, $g^*(x)$ be defined as

$$g^*(x) = g(x + \delta(x)\sigma(x)),$$

we look for $u_h \in V_h$ such that for all $v \in V_h$ it holds that

$$\int_{\Omega_h} \nabla u_h \cdot \nabla v - \int_{\partial \Omega_h} \partial_{\nu h} u_h v - \sum_{e \in \mathcal{E}^\partial} \int_e (u_h + \mathcal{C}[u_h]) \left( \partial_{\nu e} v - \gamma h^{-1}(v + \mathcal{C}[v]) \right) = \int_{\Omega_h} fv - \sum_{e \in \mathcal{E}^\partial} \int_e g^* \left( \partial_{\nu e} v - \gamma h^{-1}(v + \mathcal{C}[v]) \right),$$

(3)

where the “correction” term for the trial and test functions are defined as

$$\mathcal{C}[w] = \sum_{j=1}^{k} \frac{\delta^j}{j!} \partial_{\sigma}^j w, \quad \mathcal{C}^*[w] = \sum_{j=1}^{\hat{k}} \frac{\delta^j}{j!} \partial_{\sigma}^j w.$$

(4)

Different choices have been proposed in the literature for the extrapolation direction $\sigma$ and for the parameter $\hat{k}$ involved in the definition of $\mathcal{C}^*[\cdot]$ in (4), resulting in different correction methods. The choice $\hat{k} = 0$ (which is to be interpreted as $\mathcal{C}^*[v] = 0$) and $\sigma = \nu_h$ yields the method originally proposed in the seminal work [10] by Bramble, Dupont and Thomée. Choosing $\hat{k} = 1$ and $\sigma(x) = \nabla d(x, \partial \Omega) = \nu^*$, (that is $\sigma(x) = \nu(x + \delta(x)\sigma(x))$, $\nu$ denoting the outer unit normal to $\partial \Omega$) yields the high order shifted boundary method (SBM) proposed in [1]. The choice $\hat{k} = 0$ and $\sigma$ a priori arbitrary is analyzed in [8] and in [54, 24], where it is respectively exploited in the context of the cut finite element method, and in the virtual element framework. In both cases $\sigma$ is in practice also chosen as the gradient of the distance function to the boundary, so that $\delta(x)$ is as small as possible.
Figure 1: Three possible elements of the tessellation $\mathcal{T}_H$. For the sake of the exposition, boundary edges (with vertices marked in red) are never agglomerated to form larger edges, even when this is possible. Agglomeration of interior edges (vertices in blue) into larger edges fits instead in our exposition.

Figure 2: Three examples of the auxiliary triangulation $\mathcal{T}_K$. As one can see in the leftmost example, the presence of two adjacent boundary edges with very different length results in a badly shaped triangle. Adding few nodes, as in the central example, may improve the shape regularity of the triangulation. As the $\mathcal{T}_K$ is allowed to have a number of elements as large as needed, the presence of a large number of very small edges does not, in itself, result in badly shaped triangulation (see the rightmost example).

Figure 3: An approximate domains $\Omega_h$ falling in our framework. The theoretical framework does not in principle require the extrapolation direction $\sigma$ to coincide with either the normal $\nu_h$ to the approximate boundary or the normal $\nu^*$ to the physical boundary, though the latter is generally the best choice.
For all these choices it is possible to prove that, under the condition that \( \delta_h = \max_{x \in \partial \Omega_h} |\delta(x)| < \tau h \), with \( \tau > 0 \) a constant depending on the order \( k \), and provided \( \gamma \) is large enough, the method is stable and converges with optimal order. The (small) constant \( \tau \) decreases to 0 as \( k \) increases. If we construct the domain and choose the extrapolation direction \( \sigma \), in such a way that \( \delta_h = o(h) \), for all these choices and for all order \( k \) there exists a \( h_0 \) such that, provided \( \gamma \) is large enough, for all \( h < h_0 \) the method is stable and converges with optimal order.

Unfortunately for the class of polygons that we are considering in our framework it is not possible to choose \( \sigma \) in such a way that condition \( \delta_h = o(h) \) is satisfied. Indeed, in general, for approximating domains issued from imaging, which are the union of equal square elements (pixels), we have that \( \delta_h \simeq h \), even for the best choice of \( \sigma \). Consequently, there exist a \( \bar{k} \) such that for \( k > \bar{k} \) our tessellations will not satisfy the condition \( \delta_h \leq \tau h \), and, as \( k \) increases all the considered methods will eventually lose stability.

Remark 2.1. Other boundary correction strategies can be found in the literature that we could include in the unified formulation \((3)\), provided we allow more general forms for the correction terms \( \mathcal{C} \) and \( \mathcal{C} \), than the ones in \((4)\). We recall the polynomial extension method \((21)\), see also \((22)\).

3. The Virtual Element Discretization

The main idea behind the method we are proposing is to discretize the polygonal approximate domain \( \Omega_h \) with a polygonal tessellation \( T_H \), with meshsize \( H \), whose elements are obtained as union of quadrilateral elements of the fine tessellation \( T_h \). On the tessellation \( T_H \) we can then use the virtual element Nitsche’s method with boundary correction proposed in \((24)\) \((2)\). By taking particular care in the implementation, this will result in a method with a much more favorable cost/performance ratio than the one obtained by plain finite elements as used in the microFEM approach, without the need for modifying the bulk bilinear form near the boundary. To this aim, we start by reviewing the definition of the method we will be employing, as proposed in \((2)\).

\(^1\)Unless the specific value of the constant \( C \) is explicitly needed, throughout the paper we will write \( A \lesssim B \) (resp. \( A \gtrsim B \)) to indicate that the quantity \( A \) is bounded from above (resp. from below) by a constant \( C \) times the quantity \( B \), with \( C \) possibly depending on \( \Omega \) as well as on the parameters \( \alpha_0 \) and \( N_0 \) appearing in the shape regularity assumption \((3.1)\), but otherwise independent of the shape and size of the elements of the tessellations. The notation \( A \simeq B \) will stand for \( A \lesssim B \lesssim A \).
3.1. The tessellation

We assume that the tessellation $\mathcal{T}_H$ of $\Omega_h$ into polygons, obtained by agglomeration of the square elements of $\mathcal{T}_h$, satisfies the following Assumption.

**Assumption 3.1.** All elements $K \in \mathcal{T}_H$ are simply connected union of squares of the cartesian mesh $\mathcal{T}_h$ of meshsize $h$. Moreover, letting for $K \in \mathcal{T}_H$

$$H_K = \max_{(x,y),(x',y') \in K} \max\{|x - x'|, |y - y'|\},$$

we have $H_K \approx H$, with $H = \max_{K \in \mathcal{T}_h} H_K$, and there exist constants $\alpha_0 < 1$ and $N_0 \geq 2$ such that:

(i) each element $K \in \mathcal{T}_H$ verifies $S(x_K, \alpha_0 H_K) \subset K \subset S(x_K, H_K)$, $x_K = (x_K, y_K)$ denoting the geometrical center of $K$ and $S(x_K, d)$ the square of center $x_K$ and side $d$;

(ii) for all $x \in (x_K - H_K/2, x_K + H_K/2)$ (resp. $y \in (y_K - H_K/2, y_K + H_K/2)$) there exist at most $N_0$ values $y \in \mathbb{R}$ (resp. $x \in \mathbb{R}$) such that $(x, y) \in \partial K_{\text{hor}}$ (resp. $(x, y) \in \partial K_{\text{ver}}$), $\partial K_{\text{hor}}$ and $\partial K_{\text{ver}}$ respectively denoting the union of horizontal and vertical edges of $K$.

In order to prove the virtual element approximation estimate, we will also need to make the following additional assumption, where the shape regularity constant of a triangulation is intended as the maximum over all triangles $T$ of the ratio $h_T/\rho_T$, $h_T$ and $\rho_T$ respectively denoting the diameter of the circumscribed and inscribed circle.

**Assumption 3.2.** There exists a constant $\alpha_1$ such that for each element $K \in \mathcal{T}_H$ there exists a conforming triangulation $\mathcal{T}_K$ of $K$ with shape regularity constant at most $\alpha_1$, whose set of boundary edges coincides with the set of edges of $K$.

In the above assumption we do not require the number of elements of $\mathcal{T}_K$ to be uniformly bounded. We underline that the triangulation $\mathcal{T}_K$ only plays a role in the theoretical analysis of the method, and its actual construction is not needed for the design and implementation of the method. Remark that the virtual element method allows to handle polygons with aligned edges. Then, whenever a geometrical straight edge of the element $K$ is the union of two or more edges of the fine mesh $\mathcal{T}_h$, the set $E^K$ of computational edges of $K$ can be defined in different ways, depending on which vertices of $\mathcal{T}_h$ lying within the geometrical edge are retained as vertices of $\mathcal{T}_H$. 
Remark 3.3. In the present framework, it is always possible to build a triangulation $\widetilde{T}_K$ of $K$, whose set of boundary edges coincides with $E^K$. In general, however, we can only guarantee that for $T$ in $\widetilde{T}_K$, $h_T/\rho_T \lesssim H/h$. Remark that the presence of very small edges is not problematic in itself, but, as illustrated in figure 3, the bad situations are rather the ones where very small edges are adjacent to large edges.

Assumption 3.1 is not sufficient to imply the validity of the standard shape regularity assumption on which the analysis of the virtual element method relies. In particular, it does not imply star shapedness of the element with respect to all points in a ball of radius of order $H$. We can however show that it is sufficient to obtain a number of bounds which are usually proven under more restrictive assumptions. In particular, the following bounds, which we prove in Appendix A, hold with constants only depending on $\alpha_0, \alpha_1$ and $N_0$.

**Lemma 3.4.** Under Assumption 3.1, for all $\varphi \in H^1(K)$ it holds that
$$\|\varphi\|_{0,\partial K} \lesssim H^{-1/2}\|\varphi\|_{0,K} + H^{1/2}|\varphi|_{1,K}. \quad (5)$$
Additionally, for all $p \in P_k$ we have that
$$\|p\|_{0,\partial K} \lesssim H^{-1/2}\|p\|_{0,K}. \quad (6)$$
Moreover, provided Assumption 3.2 also holds, for all $v \in H^r(K)$, $r \geq 1$, we have that
$$\sum_{e \in E^K} |v|^2_{r-1/2,e} \lesssim |v|^2_{r,K}, \quad (7)$$
where $E^K$ denotes the set of edges of the polygon $K$.

For the sake of notational simplicity we also make the minor assumption that the set of boundary edges of $T_H$ coincides with $E^\partial$, which, we recall, is the set of boundary edges of the fine mesh $T_h$, that is, all boundary nodes of $T_h$ are also boundary nodes of $T_H$ (see Figure 1). Remark however that all the result presented here carry over to the case where we allow also the boundary edges of $T_H$ to be agglomerations of boundary edges of $T_h$.

### 3.2. The virtual element space

We will consider the standard order $k \geq 2$ enhanced virtual element discretization space [25], whose definition we briefly recall. For each polygon $K \in T_H$ we let the space $B_k(\partial K)$ be defined as
$$B_k(\partial K) = \{ v \in C^0(\partial K) : v|_e \in P_k \ \forall e \in E^K \},$$
where, we recall, $\mathcal{E}^K$ denotes the set of edges of the polygon $K$. A local space $\tilde{V}^{K,k}$ is defined as
\[ \tilde{V}^{K,k} = \{ v \in H^1(K) : v|_{\partial K} \in \mathbb{B}_k(\partial K), \Delta v \in \mathbb{P}_k \}, \]
and we introduce the operator $\Pi_{\nabla}^{\nabla,k} : H^1(K) \to \mathbb{P}_k$ defined as
\[ \int_K \nabla \Pi_{\nabla}^{\nabla,k} v \cdot \nabla q = \int_K \nabla v \cdot \nabla q, \quad \forall q \in \mathbb{P}_k, \quad \int_K \Pi_{\nabla}^{\nabla,k} v = \int_K v. \]
The local VE space is then defined ([27]) as
\[ V^{K,k} = \{ v \in \tilde{V}^{K,k} : \int_K v q = \int_K \Pi_{\nabla}^{\nabla,k} v q, \quad \forall q \in \mathbb{P}_k \cap \mathbb{P}_{k-2} \}, \]
where $\mathbb{P}_k \cap \mathbb{P}_{k-2}$ denotes the $L^2(K)$ orthogonal complement of $\mathbb{P}_{k-2}$ in $\mathbb{P}_k$. The global discrete VE space $V_H$ is finally obtained by glueing the local spaces continuously:
\[ V_H = \{ v \in H^1(\Omega) : v|_K \in V^{K,k} \forall K \in \mathcal{T}_H \}. \]
A function in $V_H$ is uniquely determined by the following degrees of freedom
- its values at the vertices of the tessellation;
- for each edge $e$, its values at the $k - 1$ internal points of the $k + 1$-points Gauss-Lobatto quadrature rule on $e$;
- for each element $K$, its moments in $K$ up to order $k - 2$.

The following lemma, usually proven under stronger shape regularity assumptions, also holds under Assumption 3.1, as we show in Appendix B.

**Lemma 3.5.** For any given function $u \in H^2(\Omega)$ we can define a unique function $u_I \in V_H$ such that, if $u \in H^s(K)$ with $2 \leq s \leq k + 1$ we have
\[ |u - u_I|_{1,K} \lesssim H^{s-1}_K |u|_{s,K}. \]
The elliptic projector $\Pi_{\nabla}^{\nabla,k}: V^{K,k} \rightarrow P_k$ is computable \cite{23}, while the bilinear form $a : V_H \times V_H \rightarrow \mathbb{R}$ and its local counterpart $a^K : V^{K,k} \times V^{K,k} \rightarrow \mathbb{R}$

$$a(\varphi, \psi) = \int_{\Omega} \nabla \varphi \cdot \nabla \psi, \quad a^K(\varphi, \psi) = \int_K \nabla \varphi \cdot \nabla \psi,$$

are not. In the definition of the virtual element discretization, the bilinear form $a$ is replaced by a computable approximate bilinear form $a_H : V_H \times V_H \rightarrow \mathbb{R}$

$$a_H(\varphi, \psi) = \sum_K a^K_H(\varphi, \psi),$$

where the elemental approximate bilinear form $a^K_H : V^{K,k} \times V^{K,k} \rightarrow \mathbb{R}$ is defined as

$$a^K_H(\varphi, \psi) = a^K(\Pi_{\nabla}^{\nabla,k}(\varphi), \Pi_{\nabla}^{\nabla,k}(\psi)) + \beta S^K(\varphi - \Pi_{\nabla}^{\nabla,k}(\varphi), \psi - \Pi_{\nabla}^{\nabla,k}(\psi)),$$

$\beta > 0$ being a constant parameter to be chosen later, and the stabilizing bilinear form $S^K_a$ being any computable symmetric bilinear form satisfying

$$c_* a^K(\varphi, \varphi) \leq S^K_a(\varphi, \varphi) \leq C_* a^K(\varphi, \varphi), \quad \forall \varphi \in V^{K,k} \text{ with } \Pi \nabla \varphi = 0, \quad (11)$$

with $c_*$ and $C_*$ two positive constants independent of $K$. Different choices for the bilinear form $S^K_a$ are available in the literature (see \cite{25}), several of which reduce, when expressed in terms of the degrees of freedom, to a suitably scaled euclidean scalar product \cite{28}.

Letting $H^1(T_H)$ and $P_k(T_H)$ respectively denote the spaces of discontinuous piecewise $H^1$ functions and of discontinuous piecewise polynomials of order up to $k$, defined on the tessellation $T_H$:

$$H^1(T_H) = \{v \in L^2(\Omega_h) : v|_K \in H^1(K) \text{ for all } K \in T_H\},$$

$$P_k(T_H) = \{v \in L^2(\Omega_h) : v|_K \in P_k(K) \text{ for all } K \in T_H\},$$

it will be convenient in the following to introduce the global projector $\Pi_{\nabla} : H^1(T_H) \rightarrow P_k(T_H)$ defined by $\Pi_{\nabla}(v)|_K = \Pi_{\nabla,k}(v|_K)$ for all $K \in T_H$. Moreover, we let $\Pi^0 : L^2(\Omega_h) \rightarrow P_k(T_H)$ denote the $L^2(\Omega_h)$ orthogonal projection onto the space of discontinuous piecewise polynomials of degree at most $k$.

### 3.3. Nitsche’s method with boundary correction for VEM

In order to discretize Problem (1) we substantially follow \cite{2}, with some minor differences that will allow us to make the method more efficient (see Remark 4.1).
More precisely, we assume that $\Omega_h \subseteq \Omega$ and we choose, on $\partial \Omega_h$, an outward direction $\sigma$, not necessarily normal to $\partial \Omega$ or $\partial \Omega_h$, which we assume to be constant on each edge $e$ (see Figure 3). For $u \in L^2(\Omega_h)$ with $u \in C^m(\hat{K})$ for all $K \in T_H$, we let $\partial^m_{\sigma} u$ denote the $L^2(\partial \Omega_h)$ function coinciding, on each boundary edge of $T_H$, with the $m$-th derivative of $u$ in the direction $\sigma$. We recall that for $x \in \partial \Omega_h$, $\delta(x) > 0$ denotes the smallest non negative scalar such that

$$x + \delta(x)\sigma(x) \in \partial \Omega.$$

We look for $u_h \in V_H$ such that for all $v \in V_H$ it holds that

$$a_H(u_h, v) - \int_{\partial \Omega_h} \partial_{\nu} \nabla(u_h) v$$

$$- \int_{\partial \Omega_h} \left( \Pi \nabla(u_h) + \mathcal{C} \left[ \Pi \nabla(u_h) \right] \right) \left( \partial_{\nu_h} \Pi \nabla(v) - \gamma H^{-1}(\Pi \nabla(v) + \hat{\mathcal{C}} \left[ \Pi \nabla(v) \right]) \right)$$

$$= \int_{\Omega_h} f \Pi \nabla(v) - \int_{\partial \Omega_h} g^* \left( \partial_{\nu_h} \Pi \nabla(v) - \gamma H^{-1}(\Pi \nabla(v) + \hat{\mathcal{C}} \left[ \Pi \nabla(v) \right]) \right) \tag{12}$$

with $g^*(x) = g(x + \delta(x)\sigma)$, and with $\mathcal{C} [w]$ and $\hat{\mathcal{C}} [w]$ defined in (4).

The analysis of equation (12) relies on the assumption that the discrete boundary $\partial \Omega_h$ is sufficiently close to the true boundary $\partial \Omega$. More precisely, in order for (12) to be well posed and yield an optimal error estimate, we need to assume that for some constant $\tau \in (0, 1)$, sufficiently small, we have that

$$\max_{K \in T_H} \max_{x \in \partial K \cap \partial \Omega_h} \frac{\delta(x)}{H} \leq \tau. \tag{13}$$

When $\Omega_h$ is constructed as the union of all elements in the fine mesh $T_h$ which are included in $\Omega$, such assumption reduces to

$$\hat{\tau} := \frac{h}{H} \leq C \tau,$$

the constant $C$ being the constant such that $\delta/H_K \leq C h/H$. If such an assumption is satisfied, existence and uniqueness of the solution of (12) can be proven by an identical argument to the one in [24, 2], which also yields an error estimate (see [2]) in the norm $\| \cdot \|_{\Omega_h}$, defined as

$$\| \varphi \|_{\Omega_h}^2 = | \Pi \nabla \varphi |_{1,T_H}^2 + | \varphi - \Pi \nabla \varphi |_{1,T_H}^2 + H^{-1} | \Pi \nabla \varphi + \hat{\mathcal{C}} \left[ \Pi \nabla \varphi \right] |_{0,\partial \Omega_h}^2,$$

where we set $| \varphi |_{1,T_H} = \sum_{K \in T_H} | \varphi |_{1,K}^2$. More precisely, we have the following theorem.
**Theorem 3.6.** There exists $\beta_0 > 0$ and $\gamma_0 > 0$ such that, provided $\beta > \beta_0$, for all $\gamma > \gamma_0$, the following holds: there exists a constant $\tau_\gamma$, depending on $\gamma$, such that if (13) holds for $\tau < \tau_\gamma$, Problem (12) is well posed, and, if $u \in H^{k+1}(\Omega) \cap W^{m,\infty}(\Omega)$, $m \leq k + 1$ we have the following error bound:

$$
\|u - u_h\|_{1,\Omega_h} \lesssim H^k|u|_{k+1,\Omega} + H^{-1/2}h^m|u|_{m,\infty,\Omega}.
$$

Observe that as $\Pi^\nabla + \mathcal{H} \circ \Pi^\nabla$ preserves the constants, $\| \cdot \|_{1,\Omega_h}$ is indeed a norm on $H^1(\Omega_h)$. Theorem 3.6 has been proven in [24] for the case $\hat{k} = 0$ with a slightly different formulation of equation (12). The proof for the new, more general, formulation is fundamentally the same, but, for the sake of completeness we report it in Appendix C.

**Remark 3.7.** We point out that, as $\gamma_0$ is independent of the parameter $c_*$ appearing in the condition (11) on the stabilization bilinear form $S^K_a$, the parameter $\gamma$ in (12) can be chosen independently of the choice of $S^K_a$.

**Remark 3.8.** By assuming $\Omega$ to be of class $C^\infty$, and allowing $\sigma$ to be an arbitrary direction, we only displace the problems related with the choice of the extrapolation direction (particularly in the vicinity of the singular points of $\Omega$). Theorem 3.6 is valid for any choice of $\sigma$, but of course, if $\sigma$ is badly chosen, the conditions on $\delta$ required by such theorem for stability will be more difficult to satisfy. The actual choice of $\sigma$ will be guided both by the result of Theorem 3.6 (in particular, by the condition $\tau < \tau_\gamma$ that suggests that $\sigma$ be chosen so that $\delta$ is as small as possible), and by practical considerations related to the numerical evaluation of the edge integrals in the bilinear form (3). These suggest to choose $\sigma$ so that $|\sigma|_e$ is smooth for all edges even in the presence of corners in the physical domain. In this last case, we refer to [56] for a strategy for the choice of the extrapolation direction.

**Remark 3.9.** We recall that other approaches exist to adapting the VEM to deal with curved boundaries in 2D. Both [57] and [58] propose versions of the VEM that handle elements with curved edges, which can be fitted to the actual physical boundary. Several extensions and applications are found in [59, 60, 61, 62]. As always, there are advantages and disadvantages in both approaches. In particular, the use of elements with curved edges also allows to deal with curved interfaces, which we do not presently treat. Conversely, the boundary correction approach has the main advantage that, in a geometry adaptive procedure where the knowledge of the actual physical boundary (implicitly defined by the image) is iteratively improved, modifications of the boundary only imply the modification of the correction term, with no need of recomputing the bulk integrals in the boundary adjacent elements.
4. Static condensation of the “lazy” degrees of freedom

While, depending on the ratio $H/h$, the number of degrees of freedom for the VE space $V_H$ can be much lower than the one for the finite element space $V_h$, it might still be quite high and, more importantly, the discretization \([12]\) leads to a linear system with a matrix that may have some large dense blocks. This happens whenever an element $K$ presents a large number of small edges. When $\hat{\tau}$ is large, this always happens in our framework, at least for boundary elements. While in 2D this does not yet pose a major problem, in order for our approach to be viable also in 3D, we need to tackle this issue. We observe that resorting to a classical static condensation approach \([63]\), or to other strategies aimed at reducing the number of interior degrees of freedom \([64, 65, 66, 67]\), while feasible, would not really yield a solution to this problem. Indeed, the number of interior degrees of freedom only depends on the polynomial degree but is independent of the refinement level of the underlying fine grid. On the other hand, the number of degrees of freedom lying on the boundary of the elements increases with both the polynomial degree $k$ and the refinement level $H/h$, and soon becomes dominant as $\hat{\tau}$ increases. We then need to somehow locally eliminate those degrees of freedom. In order to do so, we start with the observation that the bilinear form on the left hand side of \([12]\) can be split into two components: a consistency component that only sees the test and trial functions after the action of the operator $\Pi^\nabla$, and a stability component, only needed for ensuring the well posedness of the discrete equation. The consistency component has, in our framework, a large kernel, whose elements are only seen by the stability part of the bilinear form. In principle, we could then factor out such a kernel and restrict the discrete problem to a smaller space without losing the approximation properties. Unfortunately, the splitting of $V_H$ into $\text{ker}(\Pi^\nabla) \oplus V'_H$ (with $V'_H \subset V_H$ being a complement, not necessarily orthogonal, of $\text{ker}(\Pi^\nabla)$ into $V_H$) is, a priori, a non local operation that would require a singular value decomposition of a large matrix. Then, a full computation of such a splitting is not a viable option. We can however single out blocks of degrees of freedom on which such an operation can be performed locally. To do so, we start by introducing the macro edges of the tessellation: a macro edge $E$ is either a maximal connected component of $\partial K \cap \partial K'$, or a maximal connected component of $\partial K \cap \partial \Omega_h$, with $K, K' \in \mathcal{T}_h$ (see Figure 4). We observe that, in our framework, it generally happens that, at least macro edges on the boundary, but possibly also interior macro edges, are the union of a significant number of edges of the tessellation.
Let then $E$ be a macro edge, and consider the subset $V_E \subset V_H$ defined as

$$V_E = \{ v \in V_H : v = 0 \text{ on } \cup K \partial K \setminus E, \int_K vp = 0, \forall K \in \mathcal{T}_H, \ p \in \mathbb{P}_{k-2} \}.$$  

We easily see that, for a non zero $v \in V_E$, $v|_K \neq 0$ if and only if $E$ is a macro edge of $K$. For $v \in V_E$, $E$ macro edge of $K$, it is not difficult to write down a necessary and sufficient condition for $\Pi^\nabla_k v = 0$. Indeed, for $v \in V_E$ we have

$$\int_K \nabla v \cdot \nabla p = \int_E v \nabla p \cdot \nu_K$$

whence

$$v \in \ker \Pi^\nabla_k \iff \int_E v \nabla p \cdot \nu_K = 0 \text{ for all } p \in \mathbb{P}_k.$$ 

As $\nabla \mathbb{P}_k \subseteq (\mathbb{P}_{k-1})^2$, we also have that

$$\int_E v \bar{p} \cdot \nu_K = 0 \quad \text{for all } \bar{p} = (p_1, p_2) \in (\mathbb{P}_{k-1})^2 \implies v \in \ker \Pi^\nabla_k. \quad (14)$$

We can immediately see that the condition $(14)$ is completely local to the macro edge. Moreover we observe that if $E$ is a macro edge common to $K$ and $K'$, then, as, on $E$, $\nu_{K'} = -\nu_K$, we have that

$$v \in \ker \Pi^\nabla_{k'} \iff \int_E v \bar{p} \cdot \nu_{K'} = -\int_E v \bar{p} \cdot \nu_K = 0 \text{ for all } \bar{p} \in (\mathbb{P}_{k-1})^2,$$
that is the sufficient conditions for having $v|_K \in \ker \tilde{\Pi}_K^{\nu}$ and $v|_{K'} \in \ker \tilde{\Pi}_{K'}^{\nu}$ coincide. We can then split $V_E$ as

$$V_E = \tilde{V}_E \oplus \bar{V}_E \quad \text{with} \quad \tilde{V}_E = \{v \in V_E : \int_E v\tilde{p} \cdot \nu_E = 0, \ \forall \tilde{p} \in (\mathbb{P}_{k-1})^2\},$$

where on a macroedge $E = \partial K \cap \partial K'$ we choose a normal direction, setting either $\nu_E = \nu_K$ or $\nu_E = \nu_{K'}$, and where $\tilde{V}_E$ satisfies $\dim(\tilde{V}_E) + \dim(\bar{V}_E) = \dim(V_E)$. In turn, this results in a splitting of $V_H$ as

$$V_H = \hat{V}_H \oplus \tilde{V}_H, \quad \text{with} \quad \tilde{V}_H = \oplus_E \tilde{V}_E, \quad \hat{V}_H = V_I^T \oplus V_X^T \oplus_E \bar{V}_E,$$

where $V_I^T$ and $V_X^T$ are, respectively, the space of functions in $V_H$ identically vanishing on all edges of the tessellation $\mathcal{T}_H$ (that is the subspace of functions with vanishing vertex and edges degrees of freedom), and the space spanned by the basis functions corresponding to macro vertex (that is of vertexes of macro edges) degrees of freedom. We observe that for $\tilde{v} \in \hat{V}_H$ not only we have that $\tilde{\Pi}^\nu(\tilde{v}) = 0$ (this is by construction), but we also have that $\Pi^0(\tilde{v}) = 0$. In a way the functions in $\hat{V}_H$ do not (at least not directly) contribute to the consistency/accuracy of the method, and we then dub them (and the corresponding set of degrees of freedom) as “lazy”.

If we test equation (12) with $v = \tilde{v} \in \hat{V}_H$ we obtain the following identity

$$S(\hat{u}_h - \Pi^\nu(\hat{u}_h), \tilde{v}) + S(\bar{u}_h, \tilde{v}) = 0, \quad \text{where} \quad S(\varphi, \psi) = \sum_K S^K_a(\varphi, \psi), \quad (15)$$

from which we immediately obtain

$$\bar{u}_h = -\Pi_S(\hat{u}_h - \Pi^\nu(\hat{u}_h)),$$

where $\Pi_S$ is the projection onto $\bar{V}_H$, orthogonal with respect to the scalar product $S(\cdot, \cdot)$. We can then plug back this expression in equation (12) and set $v = \hat{v} \in \hat{V}_H$, obtaining the following reduced problem in $\hat{V}_H$

$$\int_{\mathcal{T}_H} \nabla \Pi^\nu(\hat{u}_h) \cdot \nabla \Pi^\nu(\hat{v}) - \int_{\partial \Omega_h} \partial \nu_h \Pi^\nu(\hat{u}_h) \hat{v}$$

$$- \int_{\partial \Omega_h} \left( \Pi^\nu(\hat{u}_h) + \mathcal{C} [\Pi^\nu(\hat{u}_h)] \right) \left( \partial \nu_h \Pi^\nu(\hat{v}) - \gamma H^{-1}(\Pi^\nu(\hat{v}) + \mathcal{C} [\Pi^\nu(\hat{v})]) \right)$$

$$+ s((1 - \Pi_S)(\hat{u}_h - \Pi^\nu(\hat{u}_h)), (1 - \Pi_S)(\hat{v} - \Pi^\nu(\hat{v})))$$

$$= \int_{\Omega_h} f \Pi^0(\hat{v}) - \sum_{e \in E^d} \int_e g^* \left( \partial \nu_e \Pi^\nu(\hat{v}) - \gamma H^{-1}(\Pi^\nu(\hat{v}) + \mathcal{C} [\Pi^\nu(\hat{v})]) \right), \quad (16)$$

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where, to make the term involving the stabilization symmetric, we exploited the fact that $1 - \Pi_S$ is the self adjoint projector with respect to the scalar product $S(\cdot, \cdot)$. Remark that, compared to the full formulation in $V_H$, the only term that is modified is the stabilization term. Moreover, if we choose $S$ in such a way that the corresponding matrix is diagonal, the computation of $\Pi_S$ turns out to be particularly cheap.

**Remark 4.1.** As already mentioned before, the method proposed in [2] is slightly different. More precisely, the discrete equation considered in such a paper is the following

$$a_H(u_h, v) - \int_{\partial V} \hat{c}_v \Pi \nabla (\hat{u}_h)v$$

$$= \int_{\partial V} (u_h + C [\Pi \nabla (u_h)]) (\hat{c}_v \Pi \nabla (v) - \gamma H^{-1}v)$$

$$= \int_{\partial V} f \Pi 0 (v) - \int_{\partial V} g^* (\hat{c}_v \Pi \nabla (v) - \gamma H^{-1}v).$$

With respect to the above formulation, besides considering a more general design of the Nitsche stabilization term, the equation (12) replaces $u_h$ and $v$ on $\partial V_h$ with $\Pi \nabla (u_h)$ and $\Pi \nabla (v)$. Without this modification, the static condensation of the "lazy" degrees of freedom would require solving an equation of the form

$$S(\tilde{u}_h, \tilde{v}) + \gamma H^{-1} \int_{\partial V_h} \tilde{u}_h \tilde{v}$$

$$= \int_{\partial V_h} \hat{c}_v \Pi \nabla (\hat{u}_h)\tilde{v} - \gamma H^{-1} \int_{\partial V_h} (\hat{u}_h - C [\Pi \nabla (\hat{u}_h)]) \tilde{v} + \gamma H^{-1} \int_{\partial V_h} g^* \tilde{v}$$

instead of the simpler and much cheaper equation (15).

**Remark 4.2.** We point out that, while we focus here on the Poisson equation, the underlying ideas can be applied to a much larger class of differential operator, including general elliptic operators with non constant coefficients. Moreover, the idea carries over also to the case of different, possibly non diagonal, VEM stabilization strategies, such as the ones proposed in [55]. In such case, however, the computation of $\Pi_S$ will require solving a block diagonal linear system and will not be as cheap.

5. **Numerical results**

We devote this section to test the performance of the virtual element method with boundary correction for increasing values of $k$, and to compare its performance with
Extrapolation direction | Stabilization | Ref.
---|---|---
(A) SBM ($\sigma(x) = \nabla d(x, \partial\Omega)$) | SBM ($\hat{k} = 1$) | [1]
(B) SBM ($\sigma(x) = \nabla d(x, \partial\Omega)$) | BDT ($\hat{k} = 0$) | [20]
(C) p.w. constant SBM ($\sigma(x) = \nabla d(x_e, \partial\Omega)$) | BDT ($\hat{k} = 0$) | [2]

Table 1: The three boundary correction strategies considered for the numerical tests. Following the SBM recipe, for all three methods the extrapolation direction is given by the gradient of the distance to the boundary, which for the (C) case is evaluated at the midpoint $x_e$ of each edge.

the performance of analogous methods in the finite element framework. We consider different variants of the proposed method, as illustrated in Table 1. Remark that choosing $\sigma = \nu_h$, as proposed in the original version of the BDT method, is extremely disadvantageous in the present framework. Consequently, we do not consider such an option. For variants (A) and (B), the extrapolation direction will follow the SBM recipe, and will be chosen as the gradient of the distance to the boundary. For the case (C), we slightly modify the SBM recipe, and let $\sigma$ be defined to be constant on each edge of $\Omega_h$, equals to the gradient of the distance to the boundary evaluated at the midpoint of each edge.

We tested the proposed method on two different curved domains, with different characteristics, namely a disk and a non convex bean shaped domain with curved boundary, and a reentrant corner with interior angle equals to $3\pi/2$. We consider different meshes, obtained by agglomerations of squared elements from uniform structured meshes of meshsize $h$, for different values of the ratio $\bar{\tau} = h/H$, which, unless otherwise stated, is the same for all the elements of a given mesh. We underline that, as stated in Remark 3.3, the considered tessellations automatically satisfy Assumption 3.2 with $\alpha_1 \leq \bar{\tau}^{-1}$.

Letting $u_h$ denote the discrete solution obtained by the order $k$ VEM method proposed in the previous section, for all the tests we consider the relative error in the $H^1(\mathcal{T}_H)$ seminorm, as well as in the $L^2(\Omega_h)$ norm. For the VEM case, these are, as usual, approximated as

$$e_1^u := \frac{\|
abla u - \Pi_k^{0}(\nabla u_h)\|_{0,\Omega_h}}{|u|_{1,\Omega_h}}, \quad e_0^u := \frac{\|u - \Pi_k^0 u_h\|_{0,\Omega_h}}{\|u\|_{0,\Omega_h}}, \quad (17)$$

where $\Pi_k^0 : L^2(\Omega_h) \to \mathbb{P}_k(\mathcal{T}_H)$ is the $L^2$ orthogonal projection onto the space of discontinuous piecewise polynomials of order up to $k$. For both test cases, we consider values of $k$ between 1 and 6. To facilitate the interpretation of the results, which, for all tests, we plot in log-log scale, for each set of data we also plot a dotted line
which we fit, by linear regression, to the central part of the data set (we exclude one or two values, that we estimate less relevant, at each end). The slope of such a line is reported in the plots.

5.1. Test 1 – Disk domain

For the first test, the domain is the disk of center \((0.5, 0.5)\) and radius 0.5. We solve Problem \([1]\) with data chosen in such a way that the solution \(u\) of the problem is the Franke function \([68]\)

\[
\begin{align*}
\mathcal{F}_{\text{Frnk}}(x, y) & := \\
& = \frac{3}{4} e^{-\left((9x-2)^2+(9y-2)^2\right)/4} + \frac{3}{4} e^{-\left((9x+1)^2/49+(9y+1)/10\right)} \\
& \quad + \frac{1}{2} e^{-\left((9x-7)^2+(9y-3)^2\right)/4} + \frac{1}{5} e^{-\left((9x-4)^2+(9y-7)^2\right)}.
\end{align*}
\]

As for the boundary correction strategy, we test both the shifted boundary method (strategy (A)) and the Bramble, Dupont and Thomeé method with piecewise constant closest point extrapolation direction, as proposed in \([2]\) (strategy (C)). For the sake of comparison, we also present the results of the shifted boundary method and of the Bramble, Dupont, Thomeé method with closest point extrapolation direction (strategy (B), proposed in \([20]\)), in combination with order \(k\) rectangular finite elements on the structured uniform grid corresponding to the \(\hat{\tau} = 1\) case. For the first set of tests we set the stabilization parameter for the Nitsche’s method \(\gamma = 100\).

Figure 5: The solution to test problem 2 (top) and one of the meshes used in the tests. Remark that the approximate domain \(\Omega_h\) is included in \(\Omega\).

5.1.1. Effect of the refinement parameter \(\hat{\tau}\)

To demonstrate the effect of the choice of the refinement parameter \(\hat{\tau}\), we start by considering the case of no refinement \((\hat{\tau} = 1)\), and we test order \(k\) finite elements and virtual elements combined with different boundary correction versions, on standard
triangular and quadrilateral meshes. For all the versions of the method we choose
the parameter $\gamma$ according to the recipe proposed in [1, Section 10.1], where, for
quadrangular elements, the inverse inequality constant $C_{\text{inv}}$ is computed according
to [69]. We observe that, for $k \leq 3$, all versions of the boundary correction method
yield optimal convergence. On the other end of the polynomial order range, for
$k = 6$, all versions lose optimality, in a more or less pronounced way, as $H = h$
becomes small. In general, the (A) version (which for finite elements coincides with
the high order SBM proposed in [1]) appears to behave better than the (B) and
(C) versions, due, we believe, to the better property of the SBM stabilization term.
Also in general the $Q_k$ version of FEM behaves more poorly than the $P_k$ FEM and
the order $k$ VEM. We believe that this is due to the fact that the inverse inequality
constant for the space $Q_k$ of polynomials of degree less than or equal to $k$ in each
variable is strictly greater then the one for $P_k$.

Switching to polygonal elements, and focusing on VEM on quadrangular meshes,
the situation improves as we take smaller values of $\tau$. In such case, however, due
to the difficulty of evaluating the inverse inequality constant, we were unable to use
the recipe of [1, Section 10.1] to evaluate the best value for $\gamma$. In the following tests
we took $\gamma = 100$. In Figures 7 and 8 we present the results of our test on polygonal
meshes obtained by agglomeration of uniform square meshes, for $\tau = 0.5, 0.25$ and
0.125. As $\tau$ decreases, the instabilities progressively disappear and, for $\tau = .125$, the
behavior of the $H^1$ error for both strategies (A) and (C) appears optimal for all
$k$s in our range (we believe the increase of the error for $k = 6$ on the finest mesh to be
the result of round-off errors). We can however still see some slight oscillations in
the $L^2$ norm for strategy (C). We also tested lower values of $\tau$, namely 0.0625 and
0.03125, with similar results, that we do not report here.

5.1.2. Effect of the choice of the stabilization parameter $\gamma$

It is well known that the performance of Nitsche’s method is affected by the choice
of the stabilization parameter $\gamma$. The theoretical analysis of our method confirms
that, also in our case, this has to be chosen large enough, that is, larger than $\gamma_0$
that depends on $k$ (tracking the dependence on $k$ in the analysis of the method
suggests that $\gamma_0 \approx \gamma k^2$ for some positive $\gamma$). To assess how sensitive to the choice
of $\gamma$ the method actually is, we tested, for the case $\tau = 0.125$, three values of $\gamma$,
namely, $\gamma = 10, \gamma = 100$ and $\gamma = 150$. Figure 9 suggests that $\gamma = 10$ is too low
for $k > 2$, while $\gamma = 100$ is good up to $k = 5$, but too low for $k = 6$, for which
$\gamma = 150$ yields instead, optimal results for both strategies (A) and (C). We also
observe that increasing $\gamma$ does not seem to negatively affect the error, and then it
seems reasonable, in the absence of a more precise analysis on the dependence of such
Figure 6: Test case 1, test on standard triangular and quadrangular meshes: $H^1$ and $L^2$ error for $P_k$ (top) and $Q_k$ (middle) finite elements with boundary correction strategies (A) and (B), in comparison with VEM (bottom) with boundary correction strategies (A) and (C) on the same quadrangular meshes (bottom). The stabilization parameter for Nitsche’s method follows the recipe of [1, Section 10.1].

Remark 5.1. The virtual element method can be also combined with the penalty free shifted boundary method proposed in [70], and we believe that, provided the VEM stabilization constant $c_*$ in condition (11) is large enough, the analysis presented here carries over to such a formulation.

5.1.3. Robustness with respect to the parameter $\hat{\tau}$

We aim at demonstrating that, by applying the static condensation procedure proposed in Section 4, it is possible to lower the value of the parameter $\hat{\tau}$ without asymptotically increasing the number of degrees of freedom. In Figure 10 we plot, for $k = 1, \cdots, 6$ the number of active degrees of freedoms retained after the elimination of the lazy degrees of freedom by the approach presented in Section 4, for the meshes used for all the tests of Sections 5.1 and 5.2. Observe how, for $H$ sufficiently small and $\hat{\tau} \leq .25$ the curves are, for all values of $k$ essentially superposed, that is, the
Figure 7: Test case 1, $H^1$ convergence of the VEM method with boundary correction strategies (A) and (C) for $k = 1, \cdots, 6$, with different values of $\hat{\tau} = h/H$. For $\hat{\tau} = 1$ and $k \geq 5$ the method displays evident instabilities. For $\hat{\tau} = .5$ the method is only slightly suboptimal. For $\hat{\tau} \leq .25$ we observe an optimal behavior for all $k \leq 5$, and only some very mild oscillations for $k = 6$. The dotted lines are obtained by linear regression fitting to a subset of the data that excludes the coarsest as well as the two finest meshes.

number of degrees of freedom for a given value of $H$ does not increase as $\hat{\tau}$ goes to 0. For all $k$ it is therefore possible to choose such a parameter in such a way that the resulting method is stable, without increasing the size of the linear system to be solved.

Next, we study the overhead resulting from the static condensation procedure. In Table 2 we report the ratio between the computational cost needed for carrying out the elimination of the lazy degrees of freedom and the one needed for the assembly of the stiffness matrix. We see that for larger values of $H$ and/or $\hat{\tau}$, the overhead is negligible, while for smaller values of simultaneously $H$ and $\hat{\tau}$ the overhead becomes larger, though still acceptable. We recall that both assembly of the stiffness matrix and elimination of the lazy degrees of freedom are highly parallelizable, as they can be carried out independently of each element and macro edge respectively.

We then consider the condition number of the stiffness matrices resulting from our method. We test the matrices before and after the elimination of the lazy degrees of freedom, and we report, for the sake of comparison, the matrix relative to Nitsche’s method without boundary correction. As we can see in Figure 11, the condition number with and without elimination of the lazy degrees of freedom are practically superposed. The instability of the method for larger values of $k$ and $\hat{\tau}$ is confirmed,
while for smaller values of $\hat{\tau}$ the condition of the boundary corrected method is smaller than the one for the plain Nitsche’s method.

5.2. Test 2 – Bean domain

In this test we consider a non convex domain with a corner with interior angle equals to $3\pi/2$, obtained as the union of a quarter of a disk with two half disks:

\[
\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \quad \text{with} \quad \begin{cases} 
\Omega_1 = \{(x, y) : (x^2 + y^2) < 1, \ x < 0, y > 0\}, \\
\Omega_2 = \{(x, y) : (x + .5)^2 + y^2 < 0.25, x < 0\}, \\
\Omega_3 = \{(x, y) : x + (y - .5)^2 < 0.25, x < 0\}.
\end{cases}
\]

The load and the non homogeneous Dirichlet data are chosen so that the solution (see Figure 5), expressed in polar coordinates, is

\[
u(\rho, \vartheta) = \rho^{2/3} \sin(2\vartheta/3),
\]

a standard benchmark for corner singularities. We know that $u \in H^s(\Omega)$ for all $s < 3/2$, but $u \notin H^{3/2}(\Omega)$. As we are in the presence of an interior angle, we follow an $hp$ strategy by resorting to geometric meshes, progressively refined in the proximity of the singular point $(0, 0)$, while simultaneously increasing the order of the $H^1$ norm.
the method. In the case of a polygonal domain, this strategy is expected to yield an exponential convergence of the $H^1(\Omega_h)$ error satisfying the bound

$$\|u - u_h\|_{1,\Omega_h} \lesssim \exp(-\sqrt{N}),$$

where $N$ is the number of degrees of freedom [71]. Observing that the parameter $\tau_\gamma$ and $\gamma_0$ given by Theorem 3.6 depend on $k$ and that a rough analysis suggests that $\tau_\gamma \approx 1/k^2$ and $\gamma_0 \approx k^2$, as the order of the method increases, we decrease $h$ so that the first condition is satisfied, and possibly modify also the elements that are not refined, enlarging them through the union of squared elements of the finer squared mesh with meshsize $h$. We refer to this adjustment of the elements close to the boundary as $\delta$ refinement. Simultaneously, we adjust the stabilization parameter and set $\gamma = \bar{\gamma}k^2$.

Once again, we consider the virtual element method with both strategy (A) and (C) boundary correction, and we test different values of the parameter $\bar{\gamma}$. The results, presented in Figure 12, display a behaviour similar to the one obtained for polygonal domains, where no boundary correction is needed. This suggests that, in the framework of the virtual element method, that allows to adjust the distance of the approximate and true boundary by reducing the mesh size of the fine grid $T_h$, boundary correction approaches such as the shifted boundary method or the BDT method with closest point mapping are potentially applicable in the $hp$ framework.
Figure 10: Number of active degrees of freedom (independent of the chosen boundary correction strategy) for the meshes used for the tests on the disk domain, for different values of the order $k$ of the method and of the refinement parameter $\tau$.

Figure 11: Condition number of the matrices resulting from the VEM(\lambda) method, for different values of the parameter $\tilde{\tau}$ and of the order $k$ of the method, before (dash-dot line) and after (solid line) elimination of the lazy degrees of freedom. The condition number for the matrix relative to Nitsche’s method (dashed line) are plotted for comparison.
5.3. Application to elasticity

As it happens in the finite element framework, also in the virtual element framework boundary correction methods such as the ones considered in this paper can be extended to a wide variety of different problems. We test here the proposed approach on the following elasticity model problem:

$$-\text{div}(\sigma) = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial\Omega$$

(19)

with $\sigma: \Omega \to \mathbb{R}^{2 \times 2}$ is the stress tensor defined as

$$\sigma = 2\mu \varepsilon(u) + \lambda \text{div}(u) I,$$

where $\mu$ and $\lambda$ are the constant Lamé parameters. To discretize such a problem, we take each component of the vector valued displacement space in the usual non-enhanced virtual space for the Poisson problem [72]. The discrete problem then reads: find $u_h \in (V_h)^2$ such that for all $v \in (V_h)^2$ it holds that

$$2\mu \int_{\mathcal{T}_h} \Pi^0_{k-1} \varepsilon(u_h) : \Pi^0_{k-1} \varepsilon(v) + \lambda \int_{\mathcal{T}_h} \Pi^0_{k-1} \text{div}(u_h) \cdot \Pi^0_{k-1} \text{div}(v) + S_K(u_h, v)$$

$$- \int_{\partial\Omega_h} \left[ (2\mu \varepsilon(\Pi^\nabla u_h) + \lambda \text{div}(\Pi^\nabla u_h)) \nu_h \right] \cdot v$$

$$- \int_{\partial\Omega_h} \left[ (2\mu \varepsilon(\Pi^\nabla v) + \lambda \text{div}(\Pi^\nabla v)) \nu_h + \gamma H^{-1}(\Pi^\nabla v + \mathcal{E}[\Pi^\nabla v]) \right] \cdot (\Pi^\nabla u_h + \mathcal{E}[\Pi^\nabla u_h])$$

$$= \int_{\mathcal{T}_h} f_h \cdot v - \int_{\partial\Omega_h} \left( (2\mu \varepsilon(\Pi^\nabla v) + \lambda \text{div}(\Pi^\nabla v)) n + \gamma H^{-1}(\Pi^\nabla v + \mathcal{E}[\Pi^\nabla v]) \right) \cdot g^*.$$

We tested our method in the VEM(A) version on a unit circle, with Lamé coefficients $\mu = \lambda = 1$, and data taken in such a way that the continuous solution is $u = (u_1, u_2)$ with $u_1(x, y) = \sin(\pi x) \sin(\pi(x - y))$ and $u_2(x, y) = \cos(\pi(x + 1)) \sin(\pi x^2)$. The results reported in Figure 13 show a behaviour similar to the one observed for the Poisson equation.

6. Conclusions and perspectives

We evaluated the performance of a boundary corrected virtual element method, for the numerical solution of the Poisson equation on curved smooth domain approximated by a polygonal domain of the type that can be easily built out of images (i.e. domain obtained as the union of pixels). The use of polygonal elements obtained
Figure 12: Test case 2. Convergence of the VEM method with boundary correction strategies (A) and (C), for a singular solution, with a graded mesh and both $hp$ and $\delta$ refinement.

Figure 13: Convergence of the VEM(A) method on equation (19).

| $H$ | $\tilde{r} = 1/8$ | $\tilde{r} = 1/16$ |
|-----|-----------------|-----------------|
| $k = 1$ | 0.0113 0.0083 0.0171 0.0516 | 0.0060 0.0115 0.0271 0.0723 |
| $k = 2$ | 0.0173 0.0149 0.0564 0.1600 | 0.0099 0.0254 0.0811 0.2015 |
| $k = 3$ | 0.0204 0.0236 0.1028 0.2088 | 0.0129 0.0326 0.1180 0.2820 |
| $k = 4$ | 0.0204 0.0316 0.1460 0.2584 | 0.0152 0.0388 0.1683 0.3340 |
| $k = 5$ | 0.0229 0.0343 0.1895 0.1724 | 0.0197 0.0477 0.2005 0.2078 |
| $k = 6$ | 0.0257 0.0422 0.2275 0.1377 | 0.0219 0.0490 0.2208 0.1858 |

Table 2: Computational overhead resulting from the elimination of the “lazy” degrees of freedom: ratio between the time needed for the elimination, and the time needed for assembling the stiffness matrix for $\tilde{r} = 1/8$ and $\tilde{r} = 1/16$. 

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as agglomeration of pixels allows boundary correction methods such as the shifted boundary method to satisfy the assumptions guaranteeing stability and optimality of the error estimates for arbitrary values of the order of the method. Eliminating, by a cheap static condensation procedure, a large number of degrees of freedom that do not actively contribute to the consistency of the method, allows to retrieve a robust behaviour of the error as a function of the number of degrees of freedom, independently of the order of approximation, that can be arbitrarily high. The numerical results demonstrate the potential of the method. While the paper only deals with the two dimensional case, its stability and convergence properties carry over to three dimensions, as shown in the preliminary tests carried out in [2, Section 5.3], for \( k = 1, 2 \). This suggests that the method is well suited to be applied also in the three dimensional case, though in such a case, in the elimination of the “lazy” degrees of freedom extra care will be needed to deal with the ones lying on the wirebasket. The method is well suited to be eventually coupled with an adaptive reconstruction of the smooth continuous boundary from imaging data, which, together with the extension to three dimensions, will be the focus of a forthcoming paper.

Appendix A. Proof of Lemma 3.4

We let \((a, b), a < b,\) denote the interval of extrema \(a\) and \(b\). We start by observing that, under our assumptions, the elements \(K\) in our mesh are extension domains \([73]\). We can then embed them in a square of diameter \(\approx H\), which, for simplicity, we will assume to be the square \(\hat{S} = (0, H)^2\), and, for all \(v \in H^1(K)\) there exists \(\hat{v} \in H^1(\hat{S})\) with \(\|\hat{v}\|_{1,\hat{S}} \leq \|v\|_{1,K}\), the constant in the inequality independent of \(H\) and \(h\). Moreover, a Poincaré inequality holds in \(K\), of the form

\[ \inf_{\alpha \in \mathbb{R}} \|v - \alpha\|_{0,K} \leq H \|v\|_{1,K} \]

with constants independent of \(H\) and \(h\). For \(v \in C^1(\overline{K})\) we can write

\[ \|v\|_{2,0,\partial \Omega_h}^2 = \sum_{e \in \mathcal{E}_h^K} \|v\|_{0,e}^2 + \sum_{e \in \mathcal{E}_v^K} \|v\|_{0,e}^2, \]

where \(\mathcal{E}_h^K \subset \mathcal{E}_v^K\) and \(\mathcal{E}_v^K \subset \mathcal{E}_v^K\) are the sets of, respectively, horizontal and vertical edges of \(K\). Let us consider the contribution of vertical edges. We have, with \(e = \{x_e\} \otimes (y_e^0, y_e^1)\),

\[ |e| \leq (y_e^0 - y_e^1), \]

where \(y_e^0 \leq y_e^1\). The extension of the above result to horizontal edges is immediate.
\[
\sum_{e \in \mathcal{E}_e^{y_e^1}} \|v\|_{0,e}^2 = \sum_{e \in \mathcal{E}_e^{y_e^1}} \int_{y_e^0}^{y_e^1} |v(x_e, y)|^2 \, dy \lesssim \sum_{e \in \mathcal{E}_e^{y_e^1}} \int_{y_e^0}^{y_e^1} ||K||^{-1} \int_K v(\sigma, \tau) \, d\sigma \, d\tau \, dy \\
+ \sum_{e \in \mathcal{E}_e^{y_e^1}} \int_{y_e^0}^{y_e^1} |v(x_e, y) - |K||^{-1} \int_K v(\sigma, \tau) \, d\sigma \, d\tau |^2 \, dy = I + II.
\]

We can write
\[
I \lesssim \sum_{e \in \mathcal{E}_e^{y_e^1}} \int_{y_e^0}^{y_e^1} ||K||^{-1} \int_K (v(x_e, y) - v(\sigma, \tau)) \, d\sigma \, d\tau \, dy \lesssim \frac{|\partial K|}{|K|} \|v\|_{0,K}^2 \lesssim H^{-1} \|v\|_{0,K}^2.
\]

Now we have
\[
II = \sum_{e \in \mathcal{E}_e^{y_e^1}} \int_{y_e^0}^{y_e^1} ||K||^{-1} \int_K (\tilde{v}(x_e, y) - v(\sigma, \tau)) \, d\sigma \, d\tau \, dy \\
\lesssim \sum_{e \in \mathcal{E}_e^{y_e^1}} \int_{y_e^0}^{y_e^1} ||K||^{-1} \int_K (\tilde{v}(x_e, y) - \tilde{v}(\sigma, y)) \, d\sigma \, d\tau \, dy \\
+ \sum_{e \in \mathcal{E}_e^{y_e^1}} \int_{y_e^0}^{y_e^1} ||K||^{-1} \int_K (\tilde{v}(\sigma, y) - v(\sigma, \tau)) \, d\sigma \, d\tau |^2 \, dy = III + IV.
\]

We can bound III as
\[
III \lesssim \sum_{e \in \mathcal{E}_e^{y_e^1}} \int_{y_e^0}^{y_e^1} ||K||^{-1} \int_K |(v(x_e, y) - \tilde{v}(\sigma, y))|^2 \, d\sigma \, d\tau \, dy \\
= \sum_{e \in \mathcal{E}_e^{y_e^1}} \int_{y_e^0}^{y_e^1} ||K||^{-1} \int_K \left| \int_{x_e}^{x_e} \partial_x \tilde{v}(\xi, y) \, d\xi \right|^2 \, d\sigma \, d\tau \, dy \\
\lesssim \sum_{e \in \mathcal{E}_e^{y_e^1}} \int_{y_e^0}^{y_e^1} ||K||^{-1} \int_{x_e}^{x_e} |\partial_x \tilde{v}(\xi, y)|^2 \, d\xi \, d\sigma \, d\tau \, dy \\
\lesssim ||K||^{-1} \int_K \sum_{e \in \mathcal{E}_e^{y_e^1}} \int_{y_e^0}^{y_e^1} \left| \int_{0}^{H} \partial_x \tilde{v}(\xi, y) \, d\xi \right|^2 \, dy \, d\sigma \, d\tau \\
\lesssim H \sum_{e \in \mathcal{E}_e^{y_e^1}} \int_{y_e^0}^{y_e^1} \int_{0}^{H} \left| \partial_x \tilde{v}(\xi, y) \right|^2 \, d\xi \, dy \, d\tau \lesssim H \|\partial_x \tilde{v}\|_{0,\mathcal{S}}^2.
\]
while IV is bound as

\[
IV \lesssim \sum_{e \in \mathcal{E}^K_v} \int_{y_e^0}^{y_e^1} |K|^{-1} \int_K \left| (v(\sigma, y) - \hat{v}(\sigma, \tau)) \right|^2 d\sigma d\tau dy
\]

\[
= \sum_{e \in \mathcal{E}^K_v} \int_{y_e^0}^{y_e^1} |K|^{-1} \int_{\mathbb{T}_e} \left| \hat{\partial}_y \hat{v}(\sigma, \zeta) \right|^2 d\sigma d\tau dy
\]

\[
\lesssim \sum_{e \in \mathcal{E}^K_v} \int_{y_e^0}^{y_e^1} |K|^{-1} \int_{\mathbb{T}_e} \left| \hat{\partial}_y \hat{v}(\sigma, \zeta) \right|^2 d\sigma d\tau dy
\]

\[
\lesssim \sum_{e \in \mathcal{E}^K_v} \int_{y_e^0}^{y_e^1} H|K|^{-1} \int_0^H \int_0^H \left| \hat{\partial}_y \hat{v}(\sigma, \zeta) \right|^2 d\sigma d\tau dy \leq H \left\| \hat{\partial}_y \hat{v} \right\|_{0, \hat{\Sigma}}^2,
\]

finally yielding

\[
II \lesssim III + IV \lesssim H \left| \hat{v} \right|^2_{1, \hat{\Sigma}} \lesssim H \left| v \right|^2_{1, K}.
\]

The contribution of horizontal edges is bound by the same argument, allowing to conclude that (5) holds for \( v \) smooth. The result for a generic \( v \in H^1(K) \) is obtained by density. The bound (6) for polynomials is a direct consequence of the combination of the previous bound with the inverse inequality

\[
\|p\|_{1, S(a_1 H)} \lesssim H^{-1} \|p\|_{0, S(a_1 H)}.
\]

In order to prove (7), we rely on the triangulation \( \tilde{\mathcal{T}}_K \) provided by Assumption 3.2. For each edge \( e \in \mathcal{E}^K \) we let \( T_e \in \tilde{\mathcal{T}}_K \) denote the triangle having \( e \) as an edge. Thanks to the shape regularity of the triangulation we can write \( |v|_{r-1/2} \lesssim |v|_{r, T_e} \), the implicit constant in the inequality only depending on the constant \( \alpha_1 \).

Then

\[
\sum_{e \in \mathcal{E}^K} |v|_{r-1/2, e}^2 \lesssim \sum_{e \in \mathcal{E}^K} |v|_{r, T_e}^2 \lesssim \sum_{T \in \mathcal{T}_K} |v|_{r, T}^2 \lesssim |v|_{r, K}^2.
\]

**Appendix B. Proof of Lemma 3.5**

Given \( u \in H^s(K), 2 \leq s \leq k + 1 \), we aim at constructing a quasi interpolant \( u_I \in V^{K,k} \) such that we can prove an optimal estimate on \( u - u_I \), robust in \( h \) and \( H \). We start by recalling that for all \( v \in H^{1/2}(\partial K) \) it holds that

\[
\inf_{\varphi \in H^1(\partial K)} \| \varphi \|_{1, K} = |\mathcal{H}(v)|_{1, K},
\]

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where $\mathcal{H}(u)$ denotes the harmonic lifting of $u$. We can then consider, for $H^{1/2}(\partial K)$ the non standard norm

$$
\|v\|_{1/2,\partial K} = |\mathcal{H}(v)|_{1,K} + \|v\|_{0,\partial K}.
$$

Letting $\|\cdot\|_{-1/2,\partial K}$ be defined as

$$
\|\varphi\|_{-1/2,\partial K} = \sup_{v \in H^{1/2}(\partial K), v = 0} \frac{\int \varphi v}{\|v\|_{1/2,\partial K}},
$$

we easily see that, for $v \in H^1(K)$ with $\Delta v \in L^2(K)$, $\Delta v$ average free, it holds, uniformly in $h$ and $H$, that

$$
\|\nabla \cdot \nu_K\|_{-1/2,\partial K} \lesssim \|\Delta v\|_{(H^1(K)/\mathbb{R})'} + |v|_{1,K}. \tag{B.1}
$$

For $u \in H^s(K)$, $2 \leq s \leq k + 1$ we now let $\tilde{u}_I \in \tilde{V}^{K,k}$ be defined as

$$
\tilde{u}_I(x_i) = u(x_i) \text{ for all node } x_i \text{ of } T_H, \text{ and } \int_K \Delta(u - \tilde{u}_I)q = 0 \text{ for all } q \in \mathbb{P}_k(K),
$$

and we define $u_I \in V^{K,k}$ as

$$
u_I = \tilde{u}_I \text{ on } \partial K \quad \text{and} \quad \int_K (u_I - \tilde{u}_I)q = 0 \text{ for all } q \in \mathbb{P}_{k-2}(K).
$$

We claim that $u_I$ thus defined satisfies (10). To prove our claim we now let $W^K$ denotes the standard finite element space of continuous piecewise polynomial functions of degree at most $k$ defined on the auxiliary triangulation $\tilde{T}_K$ given by Assumption 3.2. We let $\tilde{w}_I$ denote the interpolant of $u$ in $W^K$. We have, by standard finite element approximation estimates

$$
|u - \tilde{w}_I|_{1,K} \lesssim H^{s-1}|u|_{s,K}. \tag{B.2}
$$

We then observe that we have

$$
\|u - \tilde{u}_I\|_{1/2,\partial K} = \|u - \tilde{w}_I\|_{1/2,\partial K} \lesssim |u - \tilde{w}_I|_{1,K} + \|u - \tilde{w}_I\|_{0,\partial K} \lesssim H^{s-1}|u|_{s,K}.
$$

Let us bound $|u - \tilde{u}_I|_{1,K}$. Integrating by parts, using (B.1) as well as an Aubin Nitsche type duality trick to bound the $(H^1(K)/\mathbb{R})'$ norm of $\Delta(u - \tilde{u}_I)$, plus some standard polynomial interpolation bound on $\partial K$, we can write
\[ |u - \tilde{u}_I|_{1,K}^2 = - \int_K (\Delta(u - \tilde{u}_I))(u - \tilde{u}_I) + \int_{\partial K} \nabla (u - \tilde{u}_I) \cdot \nu_K (u - \tilde{u}_I) \]
\[ \leq \|\Delta(u - \tilde{u}_I)\|_{(H^1(K)/\mathbb{R})'} |u - \tilde{u}_I|_{1,K} + \|\nabla (u - \tilde{u}_I) \cdot \nu_K\|_{-1/2,\partial K} \|u - \tilde{u}_I\|_{1/2,\partial K} \]
\[ \leq H^{s-1} |u|_{s,K} |u - \tilde{u}_I|_{1,K} + H^{2s-2} |u|_{s,K}^2, \]

yielding, for \( \varepsilon > 0 \) arbitrary and for some positive constants \( C, C' \) only depending on the shape regularity parameters and on \( s \),

\[ |\nabla (u - \tilde{u}_I)|_{1,K}^2 \leq \frac{C}{2\varepsilon} H^{2s-2} |u|_{s,K}^2 + \frac{C\varepsilon}{2} |u - \tilde{u}_I|_{1,K}^2. \]

Choosing \( \varepsilon = 1/C \) yields the bound

\[ |u - \tilde{u}_I|_{1,K} \leq H^{s-1} |u|_{s,K}. \]

We now need to bound \( |\tilde{u}_I - u_I|_{1,K} \). Letting \( \pi_\ell : L^2(K) \to \mathbb{P}_\ell(K) \) denote the orthogonal projection onto the space of polynomials of degree at most \( \ell \) on \( K \), integrating by parts and using the fact that both \( \Delta \tilde{u}_I \) and \( \Delta u_I \) are polynomials of degree at most \( k \) and that \( \tilde{u}_I - u_I \) is orthogonal to \( \mathbb{P}_{k-2}(K) \), we have

\[ |\tilde{u}_I - u_I|_{1,K}^2 = - \int_K \Delta(\tilde{u}_I - u_I) (\tilde{u}_I - u_I) \]
\[ = \int_K \left( \Delta(\tilde{u}_I - u_I) - \pi_{k-2}(\Delta(\tilde{u}_I - u_I)) \right) (\tilde{u}_I - \Pi_K^{\nabla k} \tilde{u}_I) \]

where we could replace \( u_I \) with \( \Pi_K^{\nabla k} \tilde{u}_I \) thanks to definition of \( u_I \). Indeed, for \( u_I \in V^{K,k} \) we have by definition that \( \int_K u_I q = \int_K \Pi_K^{\nabla k} \tilde{u}_I q \) for all polynomials \( q \in \mathbb{P}_k \cap \mathbb{P}_{k-2}^+ \), where \( \tilde{u}_I \in W^K \) is any element of \( W^K \) satisfying \( \int_K \tilde{u}_I p = \int_K u_I p \) for all \( p \in \mathbb{P}_{k-2} \) (see (8)). Then, adding and subtracting \( u \) at the second factor on the right hand side, we have

\[ |\tilde{u}_I - u_I|_{1,K} \leq \| (1 - \pi_{k-2}) \Delta(\tilde{u}_I - u_I) \|_{(H^1(K)/\mathbb{R})'} \| \Pi_K^{\nabla k} \tilde{u}_I - u_I \|_{1,K} + |u - \tilde{u}_I|_{1,K}. \]

Using an Aubin-Nitsche’s duality argument, a polynomial approximation bound and an inverse inequality for polynomials, we bound, for \( \ell = k, k - 2 \)

\[ \| (1 - \pi_\ell) \Delta(\tilde{u}_I - u_I) \|_{(H^1(K)/\mathbb{R})'} \leq H \| (1 - \pi_\ell) \Delta(\tilde{u}_I - u_I) \|_{0,K} \]
\[ \leq H \| \Delta(\tilde{u}_I - u_I) \|_{0,K} \leq |\tilde{u}_I - u_I|_{1,K}. \]
Moreover, adding and subtracting \( u \) and using a polynomial approximation bound, we have
\[
|\Pi_K^{\nabla,k} \tilde{u}_I - u|_{1,K} \leq |\Pi_K^{\nabla,k} \tilde{u}_I|_{1,K} - |u|_{1,K} + |u - \Pi_K^{\nabla,k} u|_{1,K} \lesssim H^{s-1} |u|_{s,K}.
\]
Combining the different bounds and using a triangular inequality yields the desired bound.

**Appendix C. Proof of Theorem 3.6**

From Lemma 3.4, we immediately obtain that, for all \( \varphi \in V_K^{k,k} \)
\[
\| \partial_{\nu} \Pi_K^{\nabla,k} (\varphi) \|_{0,\partial K} \leq \| \nabla \Pi_K^{\nabla,k} (\varphi) \|_{0,\partial K} \lesssim H^{-1/2} \| \nabla \Pi_K^{\nabla,k} (\varphi) \|_{0,K},
\]
as well as
\[
\| \partial_{\nu} \Pi_K^{\nabla,k} (\varphi) \|_{0,\partial K} \lesssim H^{1/2-j} \| \nabla \Pi_K^{\nabla,k} (\varphi) \|_{0,K}.
\]

Then, it is difficult to prove that the following bounds hold
\[
\| \mathcal{E} [\Pi^\nabla (\varphi)] \|_{0,\partial \Omega_h} + \| \mathcal{E} \mathcal{E} [\Pi^\nabla (\varphi)] \|_{0,\partial \Omega_h} \leq C_1 H^\frac{h}{H} \| \Pi^\nabla (\varphi) \|_{1,\tau_H}, \quad (C.1)
\]
\[
\| \partial_{\nu} \Pi_K^{\nabla,k} (\varphi) \|_{0,\partial \Omega_h} \leq C_2 H^{-1/2} \| \Pi^\nabla (\varphi) \|_{1,\tau_H}, \quad (C.2)
\]
\[
\| \varphi - \Pi^\nabla (\varphi) \|_{0,\partial \Omega_h} \leq C_3 H^{1/2} \| \varphi - \Pi^\nabla (\varphi) \|_{1,\tau_H} \quad (C.3)
\]
(we recall that the boundary correction operators \( \mathcal{E} [\cdot] \) and \( \mathcal{E}[\cdot] \) are defined in (4)).

Let now \( A_H \) be defined as
\[
A_H (\varphi, \psi) = a_H (\varphi, \psi) - \int_{\partial \Omega_h} \partial_{\nu} \Pi^\nabla (\varphi) \psi - \int_{\partial \Omega_h} \left( \Pi^\nabla (\varphi) + \mathcal{E} \mathcal{E} [\Pi^\nabla (\varphi)] \right) \left( \partial_{\nu} \Pi^\nabla (\psi) - \gamma H^{-1} \mathcal{E} [\Pi^\nabla (\psi)] \right), \quad (C.4)
\]
Continuity of the bilinear form \( A_H \) with respect to the norm \( \| \cdot \|_{\Omega_h} \) follows from the above bounds. Let us prove that, provided \( h/H < \tau \) with \( \tau \) small enough, the bilinear form \( A_H \) is also coercive. Letting
\[
\mathcal{E} [w] = w + \mathcal{E} [w] = \sum_{j=1}^{k} \frac{\delta_{\nu} j}{j!} \partial_{\nu}^j w, \quad \mathcal{E} [w] = \mathcal{E} [w] - \mathcal{E} [w] = \sum_{j=k+1}^{k} \frac{\delta_{\nu} j}{j!} \partial_{\nu}^j w,
\]


we can write, for \( \varepsilon > 0 \) arbitrary

\[
A_H(\varphi, \varphi) \geq |\Pi^\nabla(\varphi)|_{1,T_H}^2 + \beta c_*|\varphi - \Pi^\nabla(\varphi)|_{1,T_H}^2 + \gamma H^{-1}\|\hat{\mathcal{S}}[\Pi^\nabla(\varphi)]\|_{0,\partial\Omega_h}^2 \\
- \int_{\partial\Omega_h} \partial_{\nu_h} \Pi^\nabla(\varphi)(\varphi - \Pi^\nabla(\varphi)) - 2\int_{\partial\Omega_h} \partial_{\nu_h} \Pi^\nabla(\varphi) \hat{\mathcal{S}}[\Pi^\nabla(\varphi)] - \int_{\partial\Omega_h} \hat{\mathcal{S}}[\Pi^\nabla(\varphi)] \partial_{\nu_h} \Pi^\nabla(\varphi) \\
+ \int_{\partial\Omega_h} \partial_{\nu_h} \Pi^\nabla(\varphi) \hat{\mathcal{S}}[\Pi^\nabla(\varphi)] + \gamma H^{-1}\int_{\partial\Omega_h} \hat{\mathcal{S}}[\Pi^\nabla(\varphi)] \hat{\mathcal{S}}[\Pi^\nabla(\varphi)] \\
\geq \left( \frac{1}{2} - C_2^2\varepsilon - \frac{3}{2}C_1^2\tau^2 - (C_1^2 + C_2^2)\tau \right)|\Pi^\nabla(\varphi)|_{1,T_H}^2 + (\beta c_* - \frac{C_1^2}{4\varepsilon})|\varphi - \Pi^\nabla(\varphi)|_{1,T_H}^2 \\
+ \left( \frac{\gamma}{2} - 2C_2^2 \right) H^{-1}\|\hat{\mathcal{S}}[\Pi^\nabla(\varphi)]\|_{0,\partial\Omega_h}^2.
\] (C.5)

We now choose \( \varepsilon = 1/(4C_2^2) \) and let \( \beta_0 = C_2^2/(2C_1^2) \) and \( \gamma_0 = 4C_2^2 \), in such a way that for \( \beta > \beta_0 \) and \( \gamma > \gamma_0 \) we have \( \beta c_* - C_3^2/(4\varepsilon) > 0 \) and \( \gamma/2 - 2C_2^2 > 0 \). For \( \gamma > \gamma_0 \), let now \( \tau_0(\gamma) \) denote the only positive solution of the equation \( \frac{1}{2} - \frac{3}{2}C_1^2\tau^2 - (C_1^2 + C_2^2)\tau = 0 \). We easily see that for \( \tau < \tau_0(\gamma) \), the coefficients of the first terms on the right hand side of (C.5) is strictly positive and the bilinear form \( A_H \) is, therefore, coercive with respect to the norm \( \| \cdot \|_{\Omega_h} \). An unique discrete solution \( u_h \) does thus exist.

Let \( u_I \) denote the VEM interpolant given by Lemma 3.5 and \( u_\pi \in \mathbb{P}_k(T_H) \) the \( L^2(\Omega) \) projection of \( u \) onto the space of discontinuous piecewise polynomials, and set \( d_h = u_I - u_h \). As in [24], we obtain

\[
\|u_I - u_h\|_{\Omega_h}^2 \leq |E1| + |E2| + |E3| + |E4| + |E5| + |E6| + |E7|,
\] (C.6)

with

\[
E1 = a_h(u_I - u_\pi, d_h), \quad E2 = \sum_{K \in T_H} a^K(u_\pi - u, d_h),
\]

\[
E3 = \int_{\partial\Omega_h} \partial_{\nu_h}(u - \Pi^\nabla(u_I)) \hat{\mathcal{S}}[\Pi^\nabla(d_h)], \quad E4 = \int_{\partial\Omega_h} \partial_{\nu_h}(u - \Pi^\nabla(u_I)) \hat{\mathcal{S}}[\Pi^\nabla(d_h)]
\]

\[
E5 = \int_{\partial\Omega_h} \partial_{\nu_h}(u - \Pi^\nabla(u_I))(d_h - \Pi^\nabla(d_h)), \quad E6 = \int_{\Omega_h} (f - \Pi^0 f)d_h,
\]

\[
E7 = \gamma H^{-1}\int_{\partial\Omega_h} (g - \mathcal{S}[\Pi^\nabla(u)])(\gamma H^{-1}\hat{\mathcal{S}}[\Pi^\nabla(d_h)] - \partial_{\nu_h}\Pi^\nabla(d_h))
\]

We observe that we have

\[
\|\partial_{\nu_h}\Pi^\nabla(d_h) - \gamma H^{-1}\Pi^\nabla(d_h)\|_{0,\partial\Omega_h} \leq H^{-1/2}\|d_h\|_{\Omega_h},
\]

35
as well as
\[ \| d_h - \Pi \nabla (d_h) \|_{0, \partial \Omega_h} \leq H^{1/2} | d_h - \Pi \nabla (d_h) |_{1, \Omega_h} \leq H^{1/2} \| d_h \|_{\Omega_h}. \]

In view of these bound, of (C.1), and of the definition of the norm \( \| \cdot \|_{\Omega_h} \), all terms at the right hand side of (C.6) are bound as in [25] and [24], yielding
\[ \| u_I - u_h \|_{\Omega_h}^2 \leq \left( H^k | u |_{1, \Omega} + H^{-1/2} h^{k+1} \| u \|_{k+1, \infty, \Omega} \right) \| u_I - u_h \|_{\Omega_h}. \]

We obtain the desired bound by dividing both sides by \( \| u_I - u_h \|_{\Omega_h} \).

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