ENUMERATION OF $AGL(m, \mathbb{F}_{p^e})$-INVARIANT EXTENDED CYCLIC CODES

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Abstract. Let $p$ be a prime and let $r$, $e$, $m$ be positive integers such that $r|e$ and $e|m$. The enumeration of linear codes of length $p^m$ over $\mathbb{F}_{p^r}$ which are invariant under the affine linear group $AGL(m, \mathbb{F}_{p^e})$ is equivalent to the enumeration of certain ideals in a partially ordered set $(\mathcal{U}, \prec)$ where $\mathcal{U} = \{0, 1, \cdots, \frac{m}{e}(p-1)^e\}$ and $\prec$ is defined by an $e$-dimensional simplicial cone. When $e = 2$, the enumeration problem was solved in an earlier paper. In the present paper, we consider the cases $e = 3$. We describe methods for enumerating all $AGL(m, \mathbb{F}_{p^3})$-invariant linear codes of length $p^m$ over $\mathbb{F}_{p^r}$.

1. Introduction

Extended cyclic codes which are invariant under a certain affine linear group were first studied by Kasami, Lin and Peterson [9] and by Delsarte [7]. These codes were further investigated by Charpin [4], [5], by Berger [1], Berger and Charpin [2], [3] in the context of permutation groups, and by Charpin and Levy-Dit-Vehel [6] in conjunction with self-duality. Extended cyclicity follows from affine invariance except when the code is the full ambient space; see later in the introduction. Affine-invariant codes are interesting because of the large automorphism groups they possess. Examples of affine-invariant codes include the $q$-ary Reed-Muller codes which are precisely $AGL(m, \mathbb{F}_q)$-invariant codes of length $q^m$ over $\mathbb{F}_q$.

The interest of affine-invariant codes is not limited to coding theory. As we will see below, such codes are precisely submodule of a certain module over the group algebra $\mathbb{K}[AGL(n, \mathbb{F})]$ where $\mathbb{F}$ and $\mathbb{K}$ are two finite fields of the same characteristic. Therefore, affine-invariant codes provide concrete examples of modular representations of the affine linear group $AGL(n, \mathbb{F})$.

The present paper and its predecessor [8] deal with the enumeration of affine-invariant codes. Delsarte’s characterization of affine-invariant extended cyclic codes in terms of defining sets [7] is the foundation of our work. The starting point of our approach is a reformulation (Theorem 1.1) of Delsarte’s characterization; the reformulation changes the enumeration problem from an algebraic one to a combinatorial and geometric one.

A comprehensive introduction to affine-invariant extended cyclic codes can be found in [2]. A detailed introduction to our approach was given in [8]. Thus in the present introduction, we only give the essential facts to be used in the paper.

Let $p$ be a prime and $r$, $m$, $e$ positive integers such that $e|m$. Identify $\mathbb{F}_{p^m}$ with $\mathbb{F}_{p^e}$. Then the affine linear group $AGL(m, \mathbb{F}_{p^e})$ acts on $\mathbb{F}_{p^m}$ hence also acts on

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the group algebra \( F_p^e[(F_{p^e}, +)] \). Put

\[ G_{m,e} = AGL(\frac{m}{e}, F_p^e). \]

Then \( F_p^e[(F_{p^e}, +)] \) is an \( F_p^e[G_{m,e}] \)-module. Define

\[ \mathcal{M} = \{ \sum_{g \in F_{p^e}} a_g X^g \in F_p^e[(F_{p^e}, +)] : \sum_{g \in F_{p^e}} a_g = 0 \}. \]

\( F_p^e[G_{m,e}] \)-submodules of \( F_p^e[(F_{p^e}, +)] \) are \( G_{m,e} \)-invariant codes over \( F_p^e; F_p^e[G_{m,e}] \)-submodules of \( \mathcal{M} \) are \( G_{m,e} \)-invariant extended cyclic codes over \( F_p^e \). In fact, every proper \( F_p^e[G_{m,e}] \)-submodules of \( F_p^e[(F_{p^e}, +)] \) must be contained in \( \mathcal{M} \) \( [8] \).

As pointed out in \([8]\), in order to determine \( F_p^e[G_{m,e}] \)-submodules of \( \mathcal{M} \) for all \( r \), it suffices to determine those with \( r|e \). Thus we always assume \( r|e \).

Let

\[ P = \begin{bmatrix}
  p^0 & p^{e-1} & \cdots & p^1 \\
  p^1 & p^0 & \cdots & p^2 \\
  \vdots & \vdots & \ddots & \vdots \\
  p^{e-1} & p^{e-2} & \cdots & p^0
\end{bmatrix}. \]

For \( u, v \in \mathbb{R}^e \), we say \( u \prec v \) if \((u - v)P\) has all the coordinates \( \leq 0 \). Let \( \Delta \subset \mathbb{R}^e \) be the set of all linear combinations of the rows of

\[ (1 - p^e)P^{-1} = \begin{bmatrix}
  1 & 0 & 0 & \cdots & 0 & -p \\
  -p & 1 & 0 & \cdots & 0 & 0 \\
  0 & -p & 1 & \cdots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & 1 & 0 \\
  0 & 0 & 0 & \cdots & -p & 1
\end{bmatrix} \]

with nonnegative coefficients. Namely, \( \Delta \) is the \( e \)-dimensional simplicial cone spanned by the rows of \((1 - p^e)P^{-1}\). It is clear that \( u \prec v \) if and only if \( u \in v + \Delta \).

The relation \( \prec \) is a partial order in \( \mathbb{R}^e \).

Let

\[ A = \begin{bmatrix}
  0 & & & & 1 \\
  1 & 0 & & & \\
  & 1 & 0 & & \\
  & & \ddots & & \\
  & & & 0 & 1 \\
  & & & & 1
\end{bmatrix}_{e \times e} \]

be the circulant permutation matrix. Since \( AP = PA \), the matrix \( A \) preserves the partial order \( \prec \), i.e., \( u \prec v \) if and only if \( uA \prec vA \).

For any subset \( \Omega \subset \mathbb{R}^e \), \((\Omega, \prec)\) is a partially ordered set. An ideal of \((\Omega, \prec)\) is a subset \( I \subset \Omega \) such that for each \( u \in I \) and \( v \in \Omega \), \( v \prec u \) implies \( v \in I \).

Let

\[ \mathcal{U} = \left\{ 0, 1, \cdots, \frac{m}{e} (p - 1) \right\}^e. \]

For each \( s \in \{0, 1, \cdots, p^m - 1\} \), write

\[ s = s_0 p^0 + \cdots + s_{m-1} p^{m-1}, \quad 0 \leq s_i \leq p - 1, \]
and define
\[ \sigma(s) = \left[ \sum_{i \equiv 0 \pmod{e}} s_i, \sum_{i \equiv 1 \pmod{e}} s_i, \ldots, \sum_{i \equiv e-1 \pmod{e}} s_i \right] \in U. \]

The following is a reformulation of Delsarte's characterization of affine-invariant extended cyclic codes [7]:

**Theorem 1.1.** (8) There is a one-to-one correspondence between the \( \mathbb{F}_p[G_{m,e}] \)-submodules of \( \mathbb{F}_p^r[\mathbb{F}_{p^m}, +] \) and the \( A^r \)-invariant ideals of \( (U, \prec) \). If \( I \) is an \( A^r \)-invariant ideal of \( (U, \prec) \), the corresponding \( \mathbb{F}_p[G_{m,e}] \)-submodules of \( \mathbb{F}_p^r[\mathbb{F}_{p^m}, +] \) is

\[
M(I) := \left\{ \sum_{g \in \mathbb{F}_{p^m}} a g X^g \in \mathbb{F}_p^r[\mathbb{F}_{p^m}, +] : \sum_{g \in \mathbb{F}_{p^m}} a g g^s = 0 \right\}
\]

for all \( s \in \{0, 1, \ldots, p^m - 1\} \) with \( \sigma(s) \in I \).

In (1.1), \( 0^0 \) is defined as 1. Moreover, \( M(I) \subset M \) if and only if \( I \neq \emptyset \).

**Note.** When \( e = m \), i.e., when \( U = \{0, 1, \ldots, p - 1\}^e \), the partial order \( \prec \) in \( U \) is the cartesian product of linear orders. Namely, \( (x_1, \ldots, x_e) \prec (y_1, \ldots, y_e) \) in \( U \) if and only if \( x_i \leq y_i \) for all \( 1 \leq i \leq e \). However, this is not the case when \( 1 < e < m \).

**Example 1.2.** Let \( p = 3 \), \( m = 6 \), \( e = 3 \), \( r = 1 \) and \( I = \{ (0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 1, 1), (2, 0, 0), (0, 2, 0), (0, 0, 2), (3, 0, 0), (0, 3, 0), (0, 0, 3) \} \).

![Figure 1. The A-invariant ideal I](image-url)
It is easy to see that $I$ is an $A$-invariant ideal of $(\mathcal{U}, \prec)$. We have
\[
\sigma^{-1}(I) = \{0, 1, 2, 3, 4, 6, 9, 10, 12, 13, 18, 27, 28, 29, 30, 36, 39, 54, 55, 81, 82, 84, 87, 90, 91, 108, 117, 162, 165, 243, 244, 246, 247, 252, 261, 270, 273, 324, 325, 351, 486, 495\}.
\]
The $F_3[G_{0,3}]$-submodule of $M$ corresponding to $I$ is
\[
M(I) = \left\{ \sum_{g \in F_3^e} a_gX^g \in F_3[(F_3^e, +)] : \sum_{g \in F_3^e} a_gg^s = 0 \text{ for all } s \in \sigma^{-1}(I) \right\}.
\]
Therefore, the essential problem is how to enumerate the $A^r$-invariant ideals of $(\mathcal{U}, \prec)$. When $e = 1$, the problem is trivial. When $e = 2$, the problem has been solved in [8]. The present paper deals with the case $e = 3$. We will describe methods for enumerating all $A^r$-invariant ideals of $(\mathcal{U}, \prec)$ for $e = 3$.

2. Description of the Approach

For simplicity, an ideal of $(\Omega, \prec)$, where $\Omega \subset \mathbb{R}^r$, is called an ideal of $\Omega$.

**Lemma 2.1.** (i) Let $\Omega \subset \Gamma \subset \mathbb{R}^r$ such that $\Omega$ and $\Gamma$ are $A^r$-invariant. If $I$ is an $A^r$-invariant ideal of $\Omega$, then there is an $A^r$-invariant ideal $J$ of $\Gamma$ such that $I \cap \Gamma = J$.

(ii) Let $\Omega \subset \mathbb{R}^r$ and $\Gamma \subset \mathbb{R}^r$. Let $I$ be an ideal of $\Omega$ and $J$ an ideal of $\Gamma$ such that $I \cap \Gamma = J \cap \Omega$. Then $I \cup J$ is an ideal of $\Omega \cup \Gamma$ if and only if
\[
(I + \Delta) \cap \Gamma \subset J \quad \text{and} \quad (J + \Delta) \cap \Omega \subset I.
\]

**Proof.** (i) Let $J = (I + \Delta) \cap \Gamma$. Then $J$ is an $A^r$-invariant ideal of $\Gamma$. Since $I$ is an ideal of $\Omega$, we have $J \cap \Omega = (I + \Delta) \cap \Omega = I$.

(ii) ($\Rightarrow$) Since $(I + \Delta) \cap (\Omega \cup \Gamma)$ is the ideal of $\Omega \cup \Gamma$ generated by $I$, i.e., the smallest ideal of $\Omega \cup \Gamma$ containing $I$, and since $I \cup J$ is an ideal of $\Omega \cup \Gamma$, we have $(I + \Delta) \cap (\Omega \cup \Gamma) \subset I \cup J$. Hence
\[
(I + \Delta) \cap \Gamma = (I + \Delta) \cap (\Omega \cup \Gamma) \cap \Gamma
\subset (I \cup J) \cap \Gamma
= (I \cap \Gamma) \cup J
= (J \cap \Omega) \cup J
= J.
\]

In the same way, $(J + \Delta) \cap \Omega \subset I$.

($\Leftarrow$) We have
\[
(I + \Delta) \cap (\Omega \cup \Gamma) = [(I + \Delta) \cap \Omega] \cup [(I + \Delta) \cap \Gamma] \subset I \cup J
\]
since $(I + \Delta) \cap \Omega = I$ and, by (2.1), $(I + \Delta) \cap \Gamma \subset J$. In the same way, $(J + \Delta) \cap (\Omega \cup \Gamma) \subset I \cup J$. Therefore,
\[
[(I \cup J) + \Delta] \cap (\Omega \cup \Gamma) \subset I \cup J,
\]
which makes $I \cup J$ an ideal of $\Omega \cup \Gamma$. \hfill \Box

In general, all $A^r$-invariant ideals of $\mathcal{U}$ can be constructed using the following inductive strategy. Partition $\mathcal{U}$ into $A^r$-invariant subsets $\mathcal{U}_1, \ldots, \mathcal{U}_k$. Let $1 \leq i \leq k$ and assume that for each $j$ with $j < i$, an $A^r$-invariant ideal $I_j$ of $\mathcal{U}_j$ has been
constructed such that $\bigcup_{j<i} I_j$ is an ideal of $\bigcup_{j<i} \mathcal{U}_j$. Construct an $A^r$-invariant ideal $I_i$ of $\mathcal{U}_i$ such that for all $j < i$,

\[(I_i + \Delta) \cap \mathcal{U}_j \subset I_j \quad \text{and} \quad (I_j + \Delta) \cap \mathcal{U}_i \subset I_i.\]

Then by Lemma 2.1 (ii), $\bigcup_{j \leq i} I_j$ is an $A^r$-invariant ideal of $\bigcup_{j \leq i} \mathcal{U}_j$. Eventually, $I = \bigcup_{j \leq k} \mathcal{U}_j$ is an $A^r$-invariant ideal of $\mathcal{U}$ with $I \cap \mathcal{U}_i = I_i$ for all $1 \leq i \leq k$. We shall call an ideal $I_i$ of $\mathcal{U}_i$ satisfying (2.2) compatible with $I_j$ ($j < i$).

**Remarks.** (i) Constructing an $A^r$-invariant ideal $I$ in $\mathcal{U}$ is an $e$-dimensional geometric problem. By partitioning $\mathcal{U}$ suitably, constructing an $A^r$-invariant ideal $I_i$ in $\mathcal{U}_i$ becomes an $(e-1)$-dimensional geometric problem.

(ii) Since for each $A^r$-invariant ideal $I$ of $\mathcal{U}$, $I \cap \mathcal{U}_i$ ($1 \leq i \leq k$) is $A^r$-invariant ideal of $\mathcal{U}_i$, the above strategy does enumerate all $A^r$-invariant ideals of $\mathcal{U}$.

(iii) The existence of an $A^r$-invariant ideal $I_i$ of $\mathcal{U}_i$ compatible with $I_j$ ($j < i$) is guaranteed by Lemma 2.1. Hence the inductive construction can always be completed.

To turn the above strategy into an enumeration algorithm, what we essentially need are effective methods for enumerating all $A^r$-invariant ideals $I_i$ which are compatible with an existing sequence of $A^r$-invariant ideals $I_j$ ($j < i$). The main purpose of this paper is to provide such effective methods in the case $e = 3$.

Form now on, we assume $e = 3$. Put

$$n = \frac{m}{3}(p-1).$$

Since $r|e$, there are two possibilities for $r$: $r = 1$ or 3. When $r = 3$, we partition $\mathcal{U}$ as

\[(2.3) \quad \mathcal{U} = \bigcup_{i=0}^{n} \mathcal{U}_i \]

where

$$\mathcal{U}_i = \{(x, y, z) \in \mathcal{U} : z = i\}.$$  

When $r = 1$, we partition $\mathcal{U}$ as

\[(2.4) \quad \mathcal{U} = \bigcup_{i=0}^{n} \mathcal{V}_i \]

where

$$\mathcal{V}_i = \{(x, y, z) \in \mathcal{U} : x \leq i, y \leq i, z \leq i, \text{ and at least one of } x, y, z \text{ is } i\}.$$  

Section 4 deals with the case $r = 3$. We describe two methods for enumerating compatible ideals $I_i$ of $\mathcal{U}_i$. The method of forward slicing enumerates all ideals $I_i$ of $\mathcal{U}_i$ which are compatible with ideals $I_j$ of $\mathcal{U}_j$ where $0 \leq j < i$; the method of backward slicing enumerates all ideals $I_i$ of $\mathcal{U}_i$ which are compatible with ideals $I_j$ of $\mathcal{U}_j$ where $i < j \leq n$. Section 5 deals with the case $r = 1$. We describe a method for enumerating all $A$-invariant ideals $I_i$ of $\mathcal{V}_i$ compatible with $A$-invariant ideals $I_j$ of $\mathcal{V}_j$ where $0 \leq j < i$. In preparation for these attempts, in the next section, we first take a close look of the cross section of an ideal in $\mathcal{U}$ on a plane parallel to a coordinate plane. We also introduce the notion of walk in the next section.
3. Cross Sections and Walks

Let $c \in \mathbb{R}$. Observe that $\Delta \cap (\mathbb{R}^2 \times \{c\})$ consists of points $(x, y, c) \in \mathbb{R}^3$ satisfying

$$
\begin{cases}
  x + py + p^2 c \leq 0, \\
p^2 x + y + pc \leq 0, \\
p x + p^2 y + c \leq 0,
\end{cases}
$$

i.e.,

$$
\begin{cases}
  x + py \leq \min\{-p^2 c, -\frac{1}{p} c\}, \\
p^2 x + y \leq -pc.
\end{cases}
$$

The solution set of (3.1) is depicted in Figure 2 when $c \geq 0$ and in Figure 3 when $c < 0$.

Figure 2. The cross section of $\Delta$ on the plane $z = c$, $c \geq 0$

Figure 3. The cross section of $\Delta$ on the plane $z = c$, $c < 0$
Let
\[ D = \{(x, y) \in \mathbb{R}^2 : x + py \leq 0, \ p^2x + y \leq 0\} \]  
(Figure 4). We can write
\[ \Delta \cap (\mathbb{R}^2 \times \{c\}) = \begin{cases} 
(D - c(0, p)) \times \{c\}, & \text{if } c \geq 0, \\
(D - c(\frac{1}{p}, 0)) \times \{c\}, & \text{if } c < 0.
\end{cases} \]

Given \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) in \(\mathbb{R}^3\), \((x_1, y_1, z_1) \prec (x_2, y_2, z_2)\) if and only if \((x_1, y_1, z_1) - (x_2, y_2, z_2) \in \Delta \cap (\mathbb{R}^2 \times \{z_1 - z_2\})\). By (3.2), this happens if and only if
\[ (x_1, y_1) \in (x_2, y_2) + \begin{cases} 
(D - (z_1 - z_2)(0, p)), & \text{if } z_1 \geq z_2, \\
(D - (z_1 - z_2)(\frac{1}{p}, 0)), & \text{if } z_1 < z_2.
\end{cases} \]

Thus
\[ [(x_2, y_2, z_2) + \Delta] \cap (\mathbb{R}^2 \times \{z_1\}) = \begin{cases} 
[(x_2, y_2) + D - (z_1 - z_2)(0, p)] \times \{z_1\}, & \text{if } z_1 \geq z_2, \\
[(x_2, y_2) + D - (z_1 - z_2)(\frac{1}{p}, 0)] \times \{z_1\}, & \text{if } z_1 < z_2.
\end{cases} \]

By symmetry, we also see that \((x_1, y_1, z_1) \prec (x_2, y_2, z_2)\) if and only if
\[ (y_1, z_1) \in (y_2, z_2) + \begin{cases} 
(D - (x_1 - x_2)(0, p)), & \text{if } x_1 \geq x_2, \\
(D - (x_1 - x_2)(\frac{1}{p}, 0)), & \text{if } x_1 < x_2,
\end{cases} \]

which is equivalent to
\[ (z_1, x_1) \in (z_2, x_2) + \begin{cases} 
(D - (y_1 - y_2)(0, p)), & \text{if } y_1 \geq y_2, \\
(D - (y_1 - y_2)(\frac{1}{p}, 0)), & \text{if } y_1 < y_2.
\end{cases} \]

Lemma 3.1. Let \(c\) be an integer written in the form \(c = ap + b\) where \(a, b \in \mathbb{Z}, 0 \leq b \leq p - 1\). Then
\[ \left[c(\frac{1}{p}, 0) + D\right] \cap \mathbb{Z}^2 = \left\{(a, 0), (a + 1, -p^2 + pb)\right\} + D \cap \mathbb{Z}^2. \]
Proof. Note that
\[
\left[ \left( \frac{1}{p}, 0 \right) + D \right] \setminus \left\{ (a, 0), (a + 1, -p^2 + pb) \right\} + D
\]
is the indicated region in Figure 5. Obvious, this region does not contain any points in \( \mathbb{Z}^2 \). □

\[\text{Figure 5. Proof of Lemma 3.1}\]

The restriction of \( \prec \) on the \( xy \)-plane, still denoted by \( \prec \), is defined by the 2-dimensional cone \( D \): \((x_1, y_1) \prec (x_2, y_2)\) if and only if \((x_1, y_1) \prec (x_2, y_2) + D\). It is clear that for \( I \subset \Omega \subset \mathbb{R}^2 \) and \( c \in \mathbb{R} \), \( I \times \{ c \} \) is an ideal of \( \Omega \times \{ c \} \) if and only if \( I \) is an ideal of \( \Omega \).

For integers \( a \leq b \), let
\[
[a, b] = \{ x \in \mathbb{Z} : a \leq x \leq b \}.
\]
Following the approach in [8], we can characterize ideals of a rectangle in \( \mathbb{Z}^2 \) by their boundaries. Such boundaries are called walks.

**Definition 3.2.** Let \( a \leq b \) and \( c \leq d \) be integers. A walk in \([a, b] \times [c, d]\) is a sequence
\[
(x_0, y_0), (x_1, y_1), \ldots, (x_k, y_k)
\]
in \([a, b] \times [c, d]\) satisfying the following conditions.

(i) \( x_0 = a \) or \( y_0 = d \); \( x_k = b \) or \( y_k = c \).

(ii) For each \( 0 < i \leq k \), either \((x_i, y_i) = (x_{i-1} + h, y_{i-1})\) for some \( 1 \leq h \leq p \) or \((x_i, y_i) = (x_{i-1}, y_{i-1} - v)\) for some \( 1 \leq v \leq p^2 \). In the first case, \((x_{i-1}, y_{i-1}), (x_i, y_i)\) is called a horizontal step of length \( h \); in the second case, \((x_{i-1}, y_{i-1}), (x_i, y_i)\) is called a vertical step of length \( v \).

(iii) The steps in the sequence \((3.7)\) alternate between horizontal and vertical.
(iv) If \( a \leq x_0 < b \) and \( y_0 = d \), the first step is vertical; if \( x_k = b \) and \( c \leq y_k < d \), the last step is horizontal.

(v) If the first step is horizontal of length \( h \), then \( 1 \leq h \leq p - 1 \); if the last step is vertical of length \( v \), then \( 1 \leq v \leq p^2 - 1 \).

Let \( U = [a, b] \times [c, d] \). For each walk \( W = ((x_0, y_0), \ldots, (x_k, y_k)) \) in \( U \), denote by \( \iota_U(W) \) the lower left part of \( U \) bounded by \( W \) (see Figure 6), i.e.,

\[
\iota_U(W) = \{ (x, y) \in U : x \leq x_i \text{ and } y \leq y_i \text{ for some } 0 \leq i \leq k \}.
\]

Figure 6. A walk \( W \) in \([0, 10] \times [0, 10] \) and its corresponding ideal \( \iota(W) \), \( p = 2 \)

We denote the empty walk in \( U \) by \( \emptyset \) and define \( \iota_U(\emptyset) = \emptyset \). Then

\[
W \longmapsto \iota_U(W)
\]

is a bijection from the set \( \mathcal{W}_U \) of all walks in \( U \) to the set \( \mathcal{I}_U \) of all ideals of \( U \). In fact, the conditions in Definition 3.2 are necessary and sufficient to ensure that for every \( u \in \iota_U(W) \), \((u + D) \cap U \subseteq \iota_U(W) \). The inverse map \( \iota_U^{-1} : \mathcal{I}_U \to \mathcal{W}_U \) is denoted by \( \omega_U \). When \( U \) is clear from the context, \( \iota_U \) and \( \omega_U \) are simply written as \( \iota \) and \( \omega \). We call a walk \( W \) the boundary of the ideal \( \iota(W) \) and \( \iota(W) \) the ideal bounded by \( W \). We remind the reader that the boundary here is unrelated to the border in [2].

For two walks \( W_1, W_2 \in \mathcal{W}_U \), we say that \( W_1 \leq W_2 \) if \( \iota(W_1) \subset \iota(W_2) \), which simply means that \( W_1 \) is below and to the left of \( W_2 \). The partially ordered set \((\mathcal{I}_U, \subset)\) is a lattice where “\( \land \)” is “\( \cap \)” and “\( \lor \)” is “\( \cup \)”. Consequently, \((\mathcal{W}_U, \leq)\) is also a lattice with

\[
W_1 \land W_2 = \omega(\iota(W_1) \cap \iota(W_2))
\]

and

\[
W_1 \lor W_2 = \omega(\iota(W_1) \cup \iota(W_2)).
\]

We introduce some operations on walks. Let \( U_i = [a_i, b_i] \times [c_i, d_i] \) \((i = 1, 2)\), where \( a_i \leq b_i \) and \( c_i \leq d_i \) are integers, and assume \( U_1 \supset U_2 \). Let \( W \) be a walk in
Let $I = \iota_{U_1}(W)$. The restriction of $W$ in $U_2$, denoted by $W|_{U_2}$, is defined to be $\omega_{U_2}(I \cap U_2)$. If $U_2 \subset \iota_{U_1}(W)$, $W|_{U_2}$ is the point $(b_2, d_2)$; otherwise, $W|_{U_2}$ is the walk in $U_2$ consisting of steps and partial steps of $W$. (See Figure 7.)

![Figure 7. The restriction of a walk](image)

For $h, v \in \mathbb{Z}$, the shift of $W$ by $h$ horizontal units and $v$ vertical units is a walk in $[a_1 + h, b_1 + h] \times [c_1 + v, d_1 + v]$ and is denoted by $W + (h, v)$.

Let $Z$ be a walk in $U_2$ and let $J = \iota_{U_2}(Z)$. A walk $W$ in $U_1$ is called an extension of $Z$ if $W|_{U_2} = Z$. Let $\overline{Z}_{U_1}$ and $\underline{Z}_{U_1}$ be the highest and lowest (the largest and lowest with respect to $\leq$) extensions of $Z$ in $U_1$ respectively. Then $\overline{Z}_{U_1} = \omega_{U_1}(K)$ where $K$ is the largest ideal of $U_1$ such that $K \cap U_2 = J$ and $\underline{Z}_{U_1} = \omega_{U_1}(L)$ where $L$ is the smallest ideal of $U_1$ such that $L \cap U_2 = J$. In fact, $\overline{Z}_{U_1}$ is the boundary of $(J + D) \cap U_1$, $\underline{Z}_{U_1}$ can be obtained from $Z$ easily: If $Z$ is the point $(b_2, d_2)$ (i.e., $J = U_2$), $\overline{Z}_{U_1}$ is the point $(b_1, d_1)$. If $Z$ is not the point $(b_2, d_2)$ and $Z \neq \emptyset$, we extend $Z$ to the lower right with steps alternating between horizontal ones of largest possible lengths and vertical ones of length 1, and to the upper left with steps alternating between vertical ones of largest possible lengths and horizontal ones of length 1. If $Z = \emptyset$ and $(a_1, b_1) \neq (a_2, b_2)$, we start from the point $(\max\{a_2 - 1, a_1\}, \max\{b_2 - 1, b_1\})$ and extend to the lower right and to the upper left as described above. (See Figure 8.) If $Z = \emptyset$ and $(a_1, b_1) = (a_2, b_2)$, then $\overline{Z}_{U_1} = \emptyset$. $\underline{Z}_{U_1}$ is obtained in a similar way. (See Figure 9.)
We list some obvious properties of restrictions and extensions. Let $U_i$, $i = 1, 2, 3$, be rectangles in $\mathbb{Z}^2$ such that $U_1 \supset U_2 \supset U_3$. Let $W$ be a walk in $U_1$ and $Z$ a walk in $U_3$. We have

$$\left( W_{|U_2} \right)_{|U_3} = W_{|U_3},$$

$$\left( Z_{|U_2} \right)_{U_1} = Z_{U_1},$$

$$\left( Z_{|U_2} \right)_{U_1} = Z_{U_1},$$

$$\left( Z_{|U_2} \right)_{U_3} = \left( Z_{|U_2} \right)_{U_3} = Z.$$

4. Enumerating Ideals of $U$

In this section we assume $r = 3$. Since $A^3$ is the identity matrix, $A^3$-invariant ideals of $U$ are simply ideals of $U$. Recall that $n = \frac{m}{3}(p - 1)$. Put

$$U = [0, n]^2,$$

and partition $U$ as

$$U = \bigcup_{i=0}^{n}(U \times \{i\}).$$

A sequence of ideals $J_0, \ldots, J_{i-1}$ (or $J_{i+1}, \ldots, J_n$) of $U$ is called forward (respectively, backward) consistent if $\bigcup_{j=0}^{i-1}(J_j \times \{j\})$ is an ideal of $U \times [0, i-1]$ (respectively, $\bigcup_{j=i+1}^{n}(J_j \times \{j\})$ is an ideal of $U \times [i+1, n]$). An ideal $J_i$ of $U$ is said to be consistent
with $J_0, \ldots, J_{i-1}$ (or $J_{i+1}, \ldots, J_n$) if $J_0, \ldots, J_{i-1}, J_i$ (respectively, $J_i, J_{i+1}, \ldots, J_n$) is forward (backward) consistent.

**Note.** In the terminology of Section 2, the statement that $J_i$ is consistent with $J_0, \ldots, J_{i-1}$ means that $J_i \times \{i\}$ is compatible with $J_j \times \{j\}$, $0 \leq j < i$, with respect to the partition $\mathcal{U} = \bigcup_{j=0}^n (U \times \{j\})$. The meaning of the statement that $J_i$ is consistent with $J_{i+1}, \ldots, J_n$ is similar.

Given a forward consistent sequence of ideals $J_0, \ldots, J_{i-1}$ (or a backward consistent sequence $J_{i+1}, \ldots, J_n$), our goal in this section is to enumerate all ideals $J_i$ of $U$ which are consistent with $J_0, \ldots, J_{i-1}$ (or $J_{i+1}, \ldots, J_n$). When $n < p$, the problem is trivial: In this case, the partial order $\prec$ in $\mathcal{U}$ is the cartesian product of linear orders, hence $J_i$ is consistent with $J_0, \ldots, J_{i-1}$ (or $J_{i+1}, \ldots, J_n$) if and only if $J_i \subset J_{i-1}$ (or $J_i \supset J_{i+1}$). When $n \geq p$, the problem is more complex. The main result of this section is the determination of two walks $X_i$ and $Y_i$ in $U$, which can be computed from the boundaries of $J_0, \ldots, J_{i-1}$ (respectively, the boundaries of $J_{i+1}, \ldots, J_n$), such that $J_i$ is consistent with $J_0, \ldots, J_{i-1}$ (or $J_{i+1}, \ldots, J_n$) if and only if $X_i \leq \sigma(J_i) \leq Y_i$.

**Lemma 4.1.** Let $i$ be an integer with $0 \leq i \leq n$ and let $J_i$ be an ideal of $U$.

(i) Let $J_0, \ldots, J_{i-1}$ be a forward consistent sequence of ideals of $U$. Then $J_i$ is consistent with $J_0, \ldots, J_{i-1}$ if and only if

\[
J_i + D - (i-j)(0,p) \cap U \subset J_i, \quad 0 \leq j < i
\]

and

\[
J_i + D - (a,0) \cap U \subset J_{i-ap-b},
\]

\[
J_i + D - (a+1,-p^2+pb) \cap U \subset J_{i-ap-b}
\]

for all $a, b \in \mathbb{Z}$ with $a \geq 0$, $0 \leq b \leq p-1$, $ap + b \leq i$.

(ii) Let $J_{i+1}, \ldots, J_n$ be a backward consistent sequence of ideals of $U$. Then $J_i$ is consistent with $J_{i+1}, \ldots, J_n$ if and only if

\[
J_i + D - (j-i)(0,p) \cap U \subset J_j, \quad i < j \leq n
\]

and

\[
J_{i+ap+b} + D - (a,0) \cap U \subset J_i,
\]

\[
J_{i+ap+b} + D - (a+1,-p^2+pb) \cap U \subset J_i
\]

for all $a, b \in \mathbb{Z}$ with $a \geq 0$, $0 \leq b \leq p-1$, $i + ap + b \leq n$.

**Proof.** (i) By Lemma 2.1, $J_i$ is consistent with $J_0, \ldots, J_{i-1}$ if and only if for every $0 \leq j < i$,

\[
(J_j \times \{j\} + \Delta) \cap (U \times \{i\}) \subset J_i \times \{i\}
\]

and

\[
(J_i \times \{i\} + \Delta) \cap (U \times \{j\}) \subset J_j \times \{j\}.
\]

However, by (3.4), we see that (4.7) is equivalent to (4.1) and (4.8) is equivalent to

\[
J_i + D - (i-j)(0,p) \cap U \subset J_j.
\]

By Lemma 3.1, (4.9) is equivalent to (4.2) and (4.3).
The proof of (ii) is essentially the same.

**Lemma 4.2.** Let $J$ and $K$ be ideals of $U$ with boundaries $W$ and $Z$ respectively. Let $a \geq 0$ and $b \geq 0$ be integers and let $\overline{K}$ be the largest ideal of $[0, a + n] \times [-b, n]$ such that $\overline{K} \cap U = K$. Then the following conditions are equivalent.

(i) \[ J + D + (a, -b) \cap U \subset K. \]

(ii) \[ J + (a, -b) \subset \overline{K} \cap ((a, -b) + U). \]

(iii) \[ J + D + (a, -b) \cap ([0, a + n] \times [-b, n]) \subset \overline{K}. \]

(iv) \[ \left( \left( W + (a, -b) \right)_{[0, a + n] \times [-b, n]} \right) \left| U \right. \leq Z. \]

(v) \[ W + (a, -b) \leq \left( \left( Z_{[0, a + n] \times [-b, n]} \right) \left| [a, a + n] \times [-b, -b + n] \right. \right. \]

(vi) \[ \left( \left( W + (a, -b) \right)_{[0, a + n] \times [-b, n]} \right) \left| U \right. \leq \left( \left( Z_{[0, a + n] \times [-b, n]} \right) \left| [a, a + n] \times [-b, -b + n] \right. \right. \]

**Proof.** Condition (iv) is a restatement of (i) in terms of boundaries. In fact, \[ \left( \left( W + (a, -b) \right)_{[0, a + n] \times [-b, n]} \right) \left| U \right. \] is the boundary of \[ J + D + (a, -b) \cap U. \] In the same way, (ii) \iff (v) and (iii) \iff (vi). Condition (vi) follows from (iv) through the operation \( ( )_{[0, a + n] \times [-b, n]} \); condition (iv) follows from (vi) through the operation \( ( ) \left| U \right. \). Similarly, (v) \iff (vi) through operations \( ( )_{[0, a + n] \times [-b, n]} \) and \( ( )_{[a, a + n] \times [-b, -b + n]} \). □

![Figure 10. Illustration of Lemma 4.2](image-url)
Lemma 4.3. Let $J, K, L$ be ideals of $U$ and let $b, c$ be positive integers. If

\[(4.16) \quad [J + D + (0, -b)] \cap U \subset K\]

and

\[(4.17) \quad [K + D + (0, -c)] \cap U \subset L,\]

then

\[(4.18) \quad [J + D + (0, -b - c)] \cap U \subset L\]

Proof. Let $\bar{K}$ be the largest ideal of $[0, n] \times [-b - c, -c + n]$ such that

\[(4.19) \quad \bar{K} \cap ([0, n] \times [-c, -c + n]) = K + (0, -c).\]

Then by (4.16) and Lemma 4.2

\[(4.20) \quad J + (0, -b - c) \subset \bar{K} \cap ([0, n] \times [-b - c, -b - c + n]).\]

Let $\bar{L}$ be the largest ideal of $[0, n] \times [-b - c, -c + n]$ such that $\bar{L} \cap U = L$. Put $\hat{L} = \bar{L} \cap ([0, n] \times [-c, n])$. Clearly, $\hat{L}$ is the largest ideal of $[0, n] \times [-c, n]$ such that $\hat{L} \cap U = L$. Thus by (4.17) and Lemma 4.2

\[(4.21) \quad K + (0, -c) \subset \hat{L} \cap ([0, n] \times [-c, -c + n]) = \bar{L} \cap ([0, n] \times [-c, -c + n]).\]

Let $\hat{L}$ be the largest ideal of $[0, n] \times [-b - c, -c + n]$ such that

\[(4.22) \quad \hat{L} \cap ([0, n] \times [-c, -c + n]) = \bar{L} \cap ([0, n] \times [-c, -c + n]).\]

We claim that

\[(4.23) \quad \hat{L} = \bar{L} \cap ([0, n] \times [-b - c, -c + n]).\]

In fact, $\omega(\hat{L})$ is the highest extension of $\omega(\bar{L} \cap ([0, n] \times [-c, n])); \omega(\bar{L})$ is the highest extension of $\omega(\bar{L} \cap ([0, n] \times [-c, -c + n]))$. Since both extensions follow the same rules (described in the last paragraph of Section 3), the new steps (in $[0, n] \times [-b - c, -c]$) in both extensions are identical. Therefore (4.23) is proved.

Note that $\bar{K}$ is an ideal of $[0, n] \times [-b - c, -c + n]$ and that by (4.19) and (4.21),

\[\bar{K} \cap ([0, n] \times [-c, -c + n]) \subset \hat{L} \cap ([0, n] \times [-c, -c + n]).\]

By the maximality of $\hat{L}$, we have $\bar{K} \subset \hat{L}$. However, (4.23) implies that $\hat{L} \subset \bar{L}$. Thus we have $\bar{K} \subset \bar{L}$. Hence by (4.20), we have

\[J + (0, -b - c) \subset \bar{L} \cap ([0, n] \times [-b - c, -b - c + n]),\]

which, by Lemma 4.2 implies (4.18). \qed
Lemma 4.4. Let $J, K, L$ be ideals of $U$ and let $a, b, c, d$ be nonnegative integers. Assume that

\begin{equation}
J + D + (a, -b) \cap U \subset K
\end{equation}

and

\begin{equation}
K + D + (c, -d) \cap U \subset L.
\end{equation}

Furthermore, assume that $K \neq \emptyset$, $K \neq U$ and that $\omega(K)$ is not a single horizontal step. (Note that when $n \geq p$, $\omega(K)$ is never a single horizontal step.) Then we have

\begin{equation}
J + D + (a + c, -b - d) \cap U \subset L.
\end{equation}

Proof. Let $\bar{L}$ be the largest ideal of $[0, a + c + n] \times [-b - d, n]$ such that $\bar{L} \cap U = L$ and let $\bar{K}$ be the largest ideal of $[c, a + c + n] \times [-b - d, -d + n]$ such that

\[
\bar{K} \cap ([c, c + n] \times [-d, -d + n]) = K + (c, -d).
\]

By (4.24) and Lemma 4.2,

\begin{equation}
J + (a + c, -b - d) \subset \bar{K} \cap ([a + c, a + c + n] \times [-b - d, -b - d + n]).
\end{equation}

Put $\bar{L} = \bar{L} \cap ([0, c + n] \times [-d, n])$. Clearly, $\bar{L}$ is the largest ideal of $[0, c + n] \times [-d, n]$ such that

\begin{equation}
\bar{L} \cap U = L.
\end{equation}

By (4.25) and Lemma 4.2,

\begin{equation}
K + (c, -d) \subset \bar{L} \cap ([c, c + n] \times [-d, -d + n]).
\end{equation}

Let $\hat{K}$ be the smallest ideal of $[0, c + n] \times [-d, n]$ such that

\[
\hat{K} \cap ([c, c + n] \times [-d, -d + n]) = K + (c, -d).
\]
By (4.29) and the minimality of \(\hat{K}\), we have

\[
\hat{K} \subset \tilde{L}.
\]

The walk \(\omega(\hat{K})\) is an extension of \(\omega(K + (c, -d))\) to the upper left; the walk \(\omega(\tilde{K})\) is an extension of \(\omega(K + (c, -d))\) to the lower right. (See Figure 12.) Since \(K + (c, -d) \neq \emptyset\), \(K + (c, -d) \neq [c, c + n] \times [-d, -d + n]\), and since \(\omega(K + (c, -d))\) is not a single horizontal step, the union (in the obvious sense) of the walks \(\omega(\hat{K})\) and \(\omega(\tilde{K})\) is a walk in \([0, a + c + n] \times [-b - d, -b - d + n]\). Denote this walk by \(W\). Note that

\[
\iota(W) \cap U = \hat{K} \cap U \quad \text{(by (4.30))}
\]

Thus by the maximality of \(\tilde{L}\), we have \(\iota(W) \subset \tilde{L}\). Hence

\[
J + (a + c, -b - d)
\]

\[
\subset \hat{K} \cap ((a + c, a + c + n] \times [-b - d, -b - d + n]) \quad \text{(by (4.27))}
\]

\[
= \iota(W) \cap ((a + c, a + c + n] \times [-b - d, -b - d + n])
\]

\[
\subset \tilde{L} \cap ((a + c, a + c + n] \times [-b - d, -b - d + n])
\]

By Lemma 4.2, (4.26) follows. \(\square\)

**Remark.** If \(K = \emptyset\) or \(K = U\), or \(\omega(K)\) is a single horizontal step, the conclusion in Lemma 4.4 may not be true. Counterexamples are given in Figures 13–15.
Figure 13. A counterexample of Lemma 4.4: $K = \emptyset$

Figure 14. A counterexample of Lemma 4.4: $K = U$
Theorem 4.5. Let $i$ be an integer with $0 \leq i \leq n$. Let $J_i, \ldots, J_n$ be a backward consistent sequence of ideals of $U$ and let $J_i$ be an ideal of $U$. Then $J_i$ is consistent with $J_{i+1}, \ldots, J_n$ if and only if the following conditions are satisfied.

(i) $J_i \supset J_{i+1}$.
(ii) $[J_i + D - (0, p)] \cap U \subset J_{i+1}$.
(iii) $[J_{i+p} + D + (1, 0)] \cap U \subset J_i$. (If $i + p > n$, this condition is null.)
(iv) Let $\alpha_i$ be the largest integer such that $1 \leq \alpha_i \leq p - 1$, $i + \alpha_i \leq n$ and $J_{i+\alpha_i} \neq \emptyset$. Then $[J_{i+\alpha_i} + D + (1, -p^2 + p\alpha_i)] \cap U \subset J_i$. (If such an $\alpha_i$ does not exist, this condition is null.)

Proof. First note that the theorem holds when $n < p$. In fact, in this case, since the partial order $\prec$ in $U$ is the cartesian product of linear orders, $J_i$ is consistent with $J_{i+1}, \ldots, J_n$ if and only if (i) is satisfied. Meanwhile, as one can easily see, (ii) is automatically satisfied; (iii) is null; (iv) is either automatically satisfied or is null. Therefore we assume $n \geq p$.

We show that (4.4) – (4.6) in Lemma 4.1 together are equivalent to conditions (i) – (iv) in Theorem 4.5.

(⇒) Condition (i) follows from (4.5) with $a = 0$ and $b = 1$ since $[J_i+1 + D] \cap U = J_{i+1}$. Condition (ii) is a special case of (4.4). Conditions (iii) and (iv) are special cases of (4.5) and (4.6).

(⇐) To prove (4.4), let $i < j \leq n$. By (ii) and the fact that $J_{i+1}, \ldots, J_n$ is backward consistent, we have

$$\begin{cases}
[J_i + D - (0, p)] \cap U \subset J_{i+1}, \\
[J_{i+1} + D - (j - i - 1)(0, p)] \cap U \subset J_j,
\end{cases}$$

Thus by Lemma 4.3

$$[J_i + D - (j - i)(0, p)] \cap U \subset J_j.$$
$J_{i+1}, \ldots, J_n$ is backward consistent, we have
\begin{equation}
\begin{aligned}
&[J_{i+p} + D + (1, 0)] \cap U \subset J_i, \\
&[J_{i+ap} + D + (a - 1, 0)] \cap U \subset J_{i+p}.
\end{aligned}
\end{equation}
We claim that
\begin{equation}
[J_{i+ap} + D + (a, 0)] \cap U \subset J_i.
\end{equation}
In fact, if $J_{i+p} \neq \emptyset$ and $J_{i+p} \neq U$, then (4.32) follows from (4.31) and Lemma 4.4. If $J_{i+p} = \emptyset$, then by (i), $J_{i+ap} = \emptyset$ since $J_{i+1}, \ldots, J_n$ is backward consistent. Thus (4.32) holds. If $J_{i+p} = U$, by (i), we have $J_i = U$ and (4.32) also holds. Since $J_{i+ap+b} \subset J_{i+ap}$, we have
\begin{equation}
[J_{i+ap+b} + D + (a, 0)] \cap U \subset [J_{i+ap} + D + (a, 0)] \cap U \subset J_i.
\end{equation}

Finally, we prove (4.4). We may assume $b \geq 1$, since if $b = 0$, we have
\begin{equation}
\begin{aligned}
&[J_{i+ap} + D + (a, 0)] \cap U \\
&\subset [J_{i+ap} + D + (a, 0)] \cap U.
\end{aligned}
\end{equation}
Since $(1, -p^2) \in D$ and (by 4.32).

In (iv), if $\alpha_i$ does not exist or $\alpha_i < b$, then $J_{i+b} = \emptyset$. Hence $J_{i+ap+b} = \emptyset$ and we are done. So assume that $\alpha_i \geq b$. By (iv) and the fact that $J_{i+1}, \ldots, J_n$ is backward consistent, we have
\begin{equation}
\begin{aligned}
&[J_{i+ap} + D + (1, -p^2 + p\alpha_i)] \cap U \subset J_i, \\
&[J_{i+ap} + D + (0, -p(\alpha_i - b))] \cap U \subset J_{i+\alpha_i}, \\
&[J_{i+ap+b} + D + (a, 0)] \cap U \subset J_{i+b}.
\end{aligned}
\end{equation}
If neither of $J_{i+\alpha_i}$ and $J_{i+b}$ is $\emptyset$ or $U$, by (4.31) and Lemma 4.4, we have
\begin{equation}
[J_{i+ap+b} + D + (a+1, -p^2 + pb)] \cap U \subset J_i,
\end{equation}
which is (4.6). If one of $J_{i+\alpha_i}$ and $J_{i+b}$ is $\emptyset$ or $U$, then $J_{i+b} = \emptyset$ or $J_{i+b} = U$ or $J_{i+\alpha_i} = U$ since $J_{i+1} \neq \emptyset$. Thus $J_{i+ap+b} = \emptyset$ or $J_i = U$ and (4.6) also holds.

**Theorem 4.6.** Let $i$ be an integer with $0 \leq i \leq n$. Let $J_0, \ldots, J_{i-1}$ be a forward consistent sequence of ideals of $U$ and let $J_i$ be an ideal of $U$. Then $J_i$ is consistent with $J_0, \ldots, J_{i-1}$ if and only if the following conditions are satisfied.
\begin{enumerate}
\item $J_i \subset J_{i-1}$.
\item $J_i \supset [J_{i-1} + D - (0, p)] \cap U$.
\item $[J_i + D + (1, 0)] \cap U \subset J_{i-p}$. (If $i - p < 0$, this condition is null.)
\item Let $\beta_i$ be the largest integer such that $1 \leq \beta_i \leq p - 1$, $i - \beta_i \geq 0$ and $J_{i-\beta_i} \neq U$. Then $[J_i + D + (1, -p^2 + p\beta_i)] \cap U \subset J_{i-\beta_i}$. (If such a $\beta_i$ does not exist, this condition is null.)
\end{enumerate}

**Proof.** By the same reason in the proof of Theorem 4.5, we may assume $n \geq p$.

We show that (4.1) - (4.3) in Lemma 4.4 together are equivalent to conditions (i) - (iv) in Theorem 4.6. Since the proof is essentially the same as the proof of Theorem 4.5, we only show that (i) - (iv) of Theorem 4.6 imply (4.3).

Let $a, b$ be integers such that $a \geq 0$, $0 \leq b \leq p - 1$ and $i - ap - b \geq 0$. By an argument similar to (4.33), we may assume $b \geq 1$. In (iv), if $\beta_i$ does not exist or if
\( \beta_i < b \), then \( J_{i-b} = U \). Hence \( J_{i-ap-b} = U \) and (4.3) is obvious. So we may assume that \( \beta_i \geq b \). By (iv) and the fact that \( J_0, \ldots, J_{i-1} \) is forward consistent, we have
\[
\begin{align*}
(J_i + D + (1, -p^2 + p\beta_i)) \cap U &\subset J_{i-\beta_i}, \\
(J_{i-\beta_i} + D + (0, -p(\beta_i - b))) \cap U &\subset J_{i-b}, \\
(J_{i-b} + D + (a, 0)) \cap U &\subset J_{i-ap-b}.
\end{align*}
\]
(4.35)

If neither of \( J_{i-\beta_i} \) and \( J_{i-\beta_i} \) is \( \emptyset \) or \( U \); (4.3) follows from (4.35) and Lemma 4.4. If one of \( J_{i-\beta_i} \) and \( J_{i-\beta_i} \) is \( \emptyset \) or \( U \), then \( J_{i-b} = \emptyset \) or \( J_{i-b} = U \) or \( J_{i-\beta_i} = \emptyset \) since \( J_{i-\beta_i} \neq U \). Thus \( J_{i-ap-b} = U \) or \( J_i = \emptyset \); in either case, (4.3) holds.

Note. In (4.36), if \( i + p > n \), the walk after the first \( \lor \) is not defined; if \( \alpha_i \) does not exist, the walk after the second \( \lor \) is not defined. Our convention, here and later, is that any undefined walk in a \( \lor \) or \( \land \) operation is ignored.

Proof. The corollary is a restatement of Theorem 4.5 in terms of boundaries. In fact, conditions (i), (iii) and (iv) of Theorem 4.5 are equivalent to
\[
\begin{align*}
W_i \geq W_{i+1},
W_i \geq (W_{i+p} + (1, 0))_{[0, n+1] \times [0, n]} |U|,
W_i \geq (W_{i+p} + (1, -p^2 + p\alpha_i))_{[0, n+1] \times [-p^2 + p\alpha_i, n]} |U|,
\end{align*}
\]
where \( \alpha_i \) is defined in Theorem 4.5 (iv), and
(4.37)
\[
Y_i = (W_{i+1})_{[0, n] \times [-p, n]} |(0, n] \times [-p, -p+n] + (0, p).
\]
Then \( J_i \) is consistent with \( J_{i+1}, \ldots, J_n \) if and only if
(4.38)
\[
X_i \leq W_i \leq Y_i.
\]

Corollary 4.7. (Backward slicing) Let \( i \) be an integer with \( 0 \leq i \leq n \). Let \( J_{i+1}, \ldots, J_n \) be a backward consistent sequence of ideals of \( U \) and let \( J_i \) be an ideal of \( U \). Put \( W_j = \omega(J_j), i \leq j \leq n \). Let
(4.39)
\[
X_i = W_{i+1} \lor (W_{i+p} + (1, 0))_{[0, n+1] \times [0, n]} |U|,
\]
where \( \alpha_i \) is defined in Theorem 4.5 (iv), and
(4.40)
\[
Y_i = (W_{i+1})_{[0, n] \times [-p, n]} |(0, n] \times [-p, -p+n] + (0, p).
\]
By Lemma 4.2, condition (ii) of Theorem 4.5 is equivalent to
\[
W_i \leq (W_{i+1})_{[0, n] \times [-p, n]} |(0, n] \times [-p, -p+n] + (0, p).
\]

Corollary 4.8. (Forward slicing) Let \( i \) be an integer with \( 0 \leq i \leq n \). Let \( J_0, \ldots, J_{i-1} \) be a forward consistent sequence of ideals of \( U \) and let \( J_i \) be an ideal of \( U \). Put \( W_j = \omega(J_j), 0 \leq j \leq i \). Let
(4.39)
\[
X_i' = (W_{i-1} - (0, p))_{[0, n] \times [-p, n]} |U|
\]
and
(4.40)
\[
Y_i' = W_{i-1} \land (W_{i-p})_{[0, n+1] \times [0, n]} |(1, 0)] \land (W_{i-\beta_i})_{[0, n+1] \times [-p^2 + p\beta_i, n]} |(1, n+1] \times [-p^2 + p\beta_i, -p^2 + p\beta_i + n] - (1, -p^2 + p\beta_i),
\]

where $\beta_i$ is defined in Theorem 4.6 (iv). Then $J_i$ is consistent with $J_0, \ldots, J_{i-1}$ if and only if

\[
X'_i \leq W_i \leq Y'_i.
\]

**Proof.** The corollary is a restatement of Theorem 4.6 in terms of boundaries. By Lemma 4.2, conditions (i), (iii) and (iv) of Theorem 4.6 are equivalent to

\[
\begin{cases}
W_i \leq W_{i-1}, \\
W_i \leq (W_i-p)_{[0,n+1] \times [0,n]} \mid_{[1,n+1] \times [0,n]} - (1,0), \\
W_i \leq (W_i-\beta_i)_{[0,n+1] \times [-p^2+p\beta_i,n]} \mid_{[1,n+1] \times [-p^2+p\beta_i,-p^2+p\beta_i+n]} - (1,-p^2+p\beta_i).
\end{cases}
\]

Condition (ii) of Theorem 4.6 is equivalent to

\[
W_i \geq (W_{i-1}-(0,p))_{[0,n] \times [-p,n]} \mid_U.
\]

\[\square\]

**Example 4.9.** (Backward slicing) Let $p = 3$ and $m = 12$ ($n = \frac{m}{3}(p-1) = 8$). A backward consistent sequence of ideals $J_8, J_7, \ldots, J_0$ is illustrated in Figure 16 through their boundary walks $W_8, W_7, \ldots, W_0$. When choosing walk $W_i$, we first determine the lower bound $X_i$ and the upper bound $Y_i$ defined in Corollary 4.7. Figure 17 shows how $Y_1$ is determined and Figure 18 shows the procedure to find $X_1$. The ideal $I = \bigcup_{j=0}^8 (J_j \times \{j\})$ of $\mathcal{U}$ is depicted in Figure 19.
Figure 16. An example of backward slicing
Figure 17. Determination of $Y_1$

Figure 18. Determination of $X_1$
Example 4.10. (Forward slicing) Let $p = 3$ and $m = 9$ ($n = \frac{mp}{3}(p - 1) = 6$). A sequence of walks $W_0, W_1, \ldots, W_6$ satisfying (4.41) is given in Figure 20. The resulting ideal $I = \bigcup_{j=0}^{6}(\iota(W_j) \times \{j\})$ of $\mathcal{U}$ is illustrated in Figure 21.
Figure 20. An example of forward slicing

Figure 21. The ideal $I = \bigcup_{j=0}^{6} (\iota(W_j) \times \{j\})$
5. Enumerating $A$-Invariant Ideals of $U$

In this section, we consider the case $r = 1$. Therefore, we are interested in ideals of $U$ which are invariant (symmetric) under the action of $A$. The problem here is more difficult than the one in Section 4.

In order to enumerate the $A$-invariant ideals of $U$, we partition $U$ as

$$U = \bigcup_{i=0}^{n} V_i$$

where

$$V_i = \{(x, y, z) \in U : x \leq i, y \leq i, z \leq i \text{ and at least one of } x, y, z \text{ is } i\}.$$ 

For any subset $X \subset \mathbb{R}^3$, we denote its image under $A$, i.e., $\{xA : x \in X\}$, by $X^A$.

Put

$$V_i = [0, i]^2 \times \{i\}.$$

Then

$$V_i = V_i \cup V_i^A \cup V_i^{A^2}. $$

Let $I$ be an $A$-invariant ideal of $V_i$. Write

$$I \cap V_i = J \times \{i\}.$$ 

Then $J$ is an ideal of $[0, i]^2$ such that

$$I = (J \times \{i\}) \cup (J \times \{i\})^A \cup (J \times \{i\})^{A^2}$$

and

(5.1) \quad \{x : (x, i) \in J\} = \{y : (i, y) \in J\}.

On the other hand, if $J$ is any subset of $[0, i]^2$ satisfying (5.1), then the $A$-invariant subset $I = (J \times \{i\}) \cup (J \times \{i\})^A \cup (J \times \{i\})^{A^2} \subset V_i$ has the property that $I \cap V_i = J \times \{i\}$.

Let $J_j$ $(0 \leq j \leq i)$ be an ideal of $[0, j]^2$ such that

(5.2) \quad \{x : (x, j) \in J_j\} = \{y : (j, y) \in J_j\}.

We call the sequence $J_0, \ldots, J_{i-1}$ consistent if

$$\bigcup_{j=0}^{i-1} \left( (J_j \times \{j\}) \cup (J_j \times \{j\})^A \cup (J_j \times \{j\})^{A^2} \right)$$

is an $A$-invariant ideal of $[0, i-1]^3$. The ideal $J_i$ of $[0, i]^2$ is said to be consistent with $J_0, \ldots, J_{i-1}$ if the sequence $J_0, \ldots, J_{i-1}, J_i$ is consistent. Note that the meaning of consistency here is different from that of Section 4.

**Note.** In the terminology of Section 2, the statement that $J_i$ is consistent with $J_0, \ldots, J_{i-1}$ means that $\bigcup_{j=0}^{i} (J_j \times \{i\})^A$ is compatible with $\bigcup_{j=0}^{i} (J_j \times \{j\})^A$, $0 \leq j < i$, with respect to the partition $U = \bigcup_{j=0}^{i} V_j$.

Given a consistent sequence of ideals $J_0, \ldots, J_{i-1}$ and an ideal $J_i$ of $[0, i]^2$. Our goal in this section, roughly speaking, is to determined two walks $\Phi_i$ and $\Psi_i$ in $[0, i]^2$ such that $J_i$ is consistent with $J_0, \ldots, J_{i-1}$ if and only if $\Phi_i \leq \omega(J_i) \leq \Psi_i$. 
Lemma 5.1. Let $0 \leq i \leq n$. Let $J_j$ ($0 \leq j \leq i$) be an ideal of $[0,j]^2$ such that $J_0, \ldots, J_{i-1}$ is a consistent sequence. Write

$$
\bigcup_{j=0}^{i-1} \left[ (J_j \times \{j\}) \cup (J_j \times \{j\})^A \cup (J_j \times \{j\})^{A^2} \right] = \bigcup_{j=0}^{i-1} (J_{i,j} \times \{j\}),
$$

where $J_{i,j}$ ($0 \leq j \leq i-1$) is a ideal of $[0, i-1]^2$, and write

$$
\omega(J_i) = ((x_0, y_0), \ldots, (x_k, y_k)).
$$

Then $J_i$ is consistent with $J_0, \ldots, J_{i-1}$ if and only if the following conditions are satisfied:

1. $(x_0, y_0) = (y_k, x_k)$ if $y_0 = i$.
2. $(J_i \times \{j\} + \Delta) \cap (0, i-1]^2 \times \{j\}) \subset J_{i,j} \times \{j\}$ for all $0 \leq j < i$.
3. $(J_i \times \{j\} + \Delta) \cap V_i \subset J_i \times \{i\}$ for all $0 \leq j < i$.
4. $(J_i \times \{i\} + \Delta) \cap V_i^A \subset (J_i \times \{i\})^A$.
5. $(J_i \times \{i\} + \Delta) \cap V_i^{A^2} \subset (J_i \times \{i\})^{A^2}$.

Proof. Let $I = (J_i \times \{i\}) \cup (J_i \times \{i\})^A \cup (J_i \times \{i\})^{A^2}$ and denote by $I'$ the ideal of $[0,i]^3$ in $(5.3)$. *(⇒)* Equation $(5.4)$ follows from $(5.2)$. Since $J_0, \ldots, J_{i-1}, J_i$ is a consistent sequence of ideals, $I \cup I'$ is an ideal of $[0,i]^3$. By Lemma 2.1 (ii), we have

- $(I \cap V_i + \Delta) \cap [0, i-1]^3 \subset I'$,
- $(I' + \Delta) \cap V_i \subset I \cap V_i$,
- $(I \cap V_i + \Delta) \cap V_i^A \subset I \cap V_i^A$,
- $(I \cap V_i + \Delta) \cap V_i^{A^2} \subset I \cap V_i^{A^2}$.

These inclusions are equivalent to $(5.5) - (5.8)$ respectively.

*(⇐)* First, from $(5.3)$, we have

$$
\{ x : (x, i) \in J_i \} = \{ y : (i, y) \in J_i \}.
$$

Thus (cf. the statement after (5.1)),

$$
I \cap V_i = J_i \times \{i\}.
$$

From $(5.9)$, $(5.7)$, and $(5.8)$, we have

$$
(I \cap V_i + \Delta) \cap V_i^{A_k} \subset I \cap V_i^{A_k}, \quad k = 0, 1, 2.
$$

Hence

$$
(I \cap V_i + \Delta) \cap V_i \subset I.
$$

Since $I$ is $A$-invariant, we have

$$
(I + \Delta) \cap V_i \subset I,
$$

which means that $I$ is an ideal of $V_i$. 
From (5.9), (5.5), and (5.6), we have
\[
\begin{cases}
(I \cap V_i + \Delta) \cap [0, i-1] \subset I', \\
(I' + \Delta) \cap V_i \subset I \cap V_i.
\end{cases}
\]
Since both $I$ and $I'$ are $A$-invariant, we obtain
\[
\begin{cases}
(I \cap V_i^A + \Delta) \cap [0, i-1]^3 \subset I', \\
(I' + \Delta) \cap V_i^A \subset I \cap V_i^A,
\end{cases}
\quad k = 0, 1, 2.
\]
Therefore,
\[
(5.10) \quad \begin{cases}
(I + \Delta) \cap [0, i-1] \subset I', \\
(I' + \Delta) \cap V_i \subset I.
\end{cases}
\]
By (5.10) and Lemma 2.1 (ii), $I \cup I'$ is an ideal of $[0, i]^3$, i.e., $J_0, \ldots, J_{i-1}, J_i$ is consistent. □

**Lemma 5.2.** In Lemma 5.1, (5.4) – (5.6) imply (5.7).

**Proof.** First assume $i < p$. In this case, the partial order $\prec$ in $[0, i]^3$ is the cartesian product of linear orders and (5.7) follows from (5.4) trivially. (See Figure 22.)

![Figure 22](image-url)

Figure 22. $(J_i \times \{i\} + \Delta) \cap V_i^A \subset (J_i \times \{i\})^A$ when $i < p
So we assume that \( i \geq p \). Let \( I \) and \( I' \) be as in the proof of Lemma 5.1. Note that \( I \cap V_i = J_i \times \{ i \} \) by (5.4).

For each \( u = (x', i, z') \in (J_i \times \{ i \} + \Delta) \cap V_i^A \), we want to show that \( u \in (J_i \times \{ i \})^A \).

Note that there exists \( (x, y, i) \in J_i \) such that \( u \prec (x, y, i) \).

If \( y = i \), then \( (x, y, i) \in (J_i \times \{ i \}) \cap V_i^A = I \cap V_i \cap V_i^A \subset (I \cap V_i)^A = (J_i \times \{ i \})^A \).

Since \( (J_i \times \{ i \})^A \) is an ideal of \( V_i^A \), we have \( u \in (J_i \times \{ i \})^A \). Thus we assume \( y < i \).

If \( z' = i \), then \( (x', i, i) \prec (x, y, i) \) implies \( (x', i) \prec (x, y) \), hence \( (x', i, i) \in J_i \). By (5.4), \( (i, x') \in J_i \), hence \( u = (x', i, i) = (i, x', i)A \in (J_i \times \{ i \})^A \). Thus we assume \( z' < i \).

By (3.6), \( (x', i, z') \prec (x, y, i) \) if and only if
\[
(x', i, z') \prec (x, i, i) - (i - y)(p, 0, 0) = (x - (i - y)p, i, i).
\]

Thus we have
\[
(x', i, z') \prec (x', i, z') + (p, -1, 0)
\]
\[
\prec (x - (i - y)p, i, i) + (p, -1, 0)
\]
\[
= (x - (i - 1 - y)p, i, i)
\]
\[
= (x, y, i) + (i - 1 - y)(-p, 1, 0)
\]
\[
\prec (x, y, i),
\]
i.e.,
\[
(x', i, z') \prec (x' + p, i - 1, z') \prec (x, y, i).
\]

If \( (z', x') \prec (x, y) \), then \( (z', x') \in J_i \). Thus \( (x', i, z') = (x', x', i)A \in (J_i \times \{ i \})^A \).

Therefore, we assume \( (z', x') \neq (x, y) \).

We claim that
\[
x' < i - p.
\]

In fact, since \( (x', i, z') \prec (x - (i - y)p, i, i) \), we have \( (z', x') \prec (i, x - (i - y)p) \), i.e.,
\[
x' \leq x - (i - y)p + \frac{1}{p}(i - z').
\]

If \( z' > x \), (5.13) gives
\[
x' < x - (i - y)p + \frac{1}{p}(i - x)
\]
\[
= \frac{p - 1}{p} x + py - \frac{p^2 - 1}{p} i
\]
\[
\leq \frac{p - 1}{p} i + p(i - 1) - \frac{p^2 - 1}{p} i
\]
\[
= i - p.
\]

If \( z' \leq x \), since \( (z', x') \neq (x, y) \), we must have
\[
x' > y + \frac{1}{p}(x - z').
\]

Combining (5.13) and (5.14), we have
\[
x - (i - y)p + \frac{1}{p}(i - z') > y + \frac{1}{p}(x - z')
\]
which gives
\[(p - 1)y > \frac{1}{p} x - x + pi - \frac{1}{p} i = \frac{p^2 - 1}{p} i - \frac{1}{p} x,\]
i.e.,
\[y > \frac{p + 1}{p} i - \frac{1}{p} x \geq i,\]
which is a contradiction. Thus (5.12) is proved.

Now we have \((x', i - 1, y') \prec (x, y, i)\) and \((x' + p, i - 1, z') \in [0, i - 1]^3\). By (5.5), \((x' + p, i - 1, z') \in (J_i \times \{i\} + \Delta) \cap [0, i - 1]^3 \subseteq I'\). Thus we have
\[(x', i, z') \in (I' + \Delta) \cap V_i A \quad \text{(by (5.11))}\]
\[\subset (J_i \times \{i\}) A \quad \text{(by (5.9) and the A-symmetry of I').}\]

\[\square\]

**Lemma 5.3.** Assume that in Lemma 5.1, (5.4) – (5.7) are satisfied. Then (5.8) is equivalent to
\[(5.15) \max \{y : (i - 1, y) \in J_i\} \leq \max \{x : (x, i - p) \in J_i\} \quad \text{if } i \geq p.\]

**Proof.** First assume \(i < p\). Then (5.15) is satisfied without instance. Since in case, the partial order \(\prec\) in \([0, i]^3\) is the cartesian product of linear orders, (5.8) holds trivially. (See Figure 23.) So we assume that \(i \geq p\). Again, let \(I\) and \(I'\) be as in the proof of Lemma 5.1.

![Figure 23](image-url)
Proof of “(5.16) \Rightarrow (5.8)”. Let \( u = (i, y', z') \in (J_i \times \{i\} + \Delta) \cap V_i A^2 \). We want to show that \( u \in (J_i \times \{i\}) A^2 \).

Note that there exists \( (x, y) \in J_i \) such that \( u \prec (x, y, i) \). Also note that (5.4) implies that \( I \cap V_i = J_i \times \{i\} \).

If \( x = i \), then \( (x, y, i) \in (J_i \times \{i\}) \cap V_i A^2 = I \cap V_i \cap V_i A^2 \subset (I \cap V_i) A^2 = (J_i \times \{i\}) A^2 \). Since \( (J_i \times \{i\}) A^2 \) is an ideal of \( V_i A^2 \), we have \( u \in (J_i \times \{i\}) A^2 \).

Next, assume \( x < i - 1 \). By (3.5), \( (i, y', z') \prec (x, y, i) \) if and only if \( (i, y', z') < (i, y, i) - (i - x)(0, 0, p) = (i, y, i - p(i - x)) \).

Thus we have
\[
(i, y', z') \prec (i, y, i - p(i - x))
\]
\[
\prec (i, y, i - p(i - x)) + (i - x - 1)(-1, 0, p)
\]
\[
= (x + 1, y, i - p)
\]
\[
\prec (x, y, i),
\]

i.e.,
\[
(5.16) \quad u = (i, y', z') \prec (x + 1, y, i - p) \prec (x, y, i).
\]

If \( y = i \), then
\[
(x + 1, y, i - p) \in (J_i \times \{i\} + \Delta) \cap V_i A
\]
\[
\subset (J_i \times \{i\}) A \quad \text{(by (5.7)).}
\]

Thus
\[
u \in [(J_i \times \{i\}) A + \Delta] \cap V_i A^2
\]
\[
\subset [(J_i \times \{i\}) + \Delta] \cap V_i A^2
\]
\[
\subset (J_i \times \{i\}) A^2 \quad \text{(by (5.7) again).}
\]

If \( y < i \), then \( (x + 1, y, i - p) \in (J_i \times \{i\} + \Delta) \cap [0, i - 1]^2 \subset I' \). Hence we have
\[
u \in (I' + \Delta) \cap V_i A^2 \quad \text{(by (5.16))}
\]
\[
\subset (J_i \times \{i\}) A^2 \quad \text{(by (5.6) and the A-symmetry of I').}
\]

Finally, assume \( x = i - 1 \). By (3.5), we have
\[
((x, y, i) + \Delta) \cap V_i A^2 = \{i\} \times [(y, i - p) + D) \cap [0, i]^2].
\]

(See Figure 24.) However, by (5.14), \( y \leq \max\{x : (x, i - p) \in J_i\} \). Thus \( (y, i - p) \in J_i \). Therefore,
\[
u \in ((x, y, i) + \Delta) \cap V_i A^2
\]
\[
= \{i\} \times [(y, i - p) + D) \cap [0, i]^2]
\]
\[
\subset \{i\} \times J_i
\]
\[
= (J_i \times \{i\}) A^2.
\]
Figure 24. The cross section of \((i-1, y, i) + \Delta\) in \(V_i^2\)

Proof of “(5.8) \(\Rightarrow\) (5.15)”. We may assume that \(\{y : (i-1, y) \in J_i\} \neq \emptyset\). Let \(\bar{y} = \max\{y : (i-1, y) \in J_i\}\). Then \((i-1, \bar{y}) \in J_i\). Hence

\[
\{i\} \times \left[\left((\bar{y}, i-1, p) + D\right) \cap [0, i]^2\right] \\
= \left((i-1, \bar{y}, i) + \Delta\right) \cap V_i^2 \\
\subset (J_i \times \{i\} + \Delta) \cap V_i^2 \\
\subset (J_i \times \{i\})^2 \\
= \{i\} \times J_i.
\]

In particular, \((\bar{y}, i-p) \in J_i\). Therefore

\[
\bar{y} \leq \max\{x : (x, i-p) \in J_i\},
\]

which is (5.15). \(\square\)

Lemma 5.4. Let \(J\) be an ideal of \([0, i-1]^2\) and \(K\) an ideal of \([0, i]^2\). Let \(b \geq 0\) be an integer. Then

\[
(5.17) \quad [J + D + (0, -b)] \cap [0, i]^2 \subset K
\]

if and only if

\[
(5.18) \quad [J + D + (0, -b)] \cap [0, i-1]^2 \subset K \cap [0, i-1]^2.
\]

Proof. We only have to prove that (5.18) \(\Rightarrow\) (5.17). Let \((x, y) \in [J + D + (0, -b)] \cap [0, i]^2\), we want to show that \((x, y) \in K\).

If \((x, y) \in [0, i-1]^2\), we are done by (5.18). So assume \((x, y) \notin [0, i-1]^2\), i.e., \(x = i \text{ or } y = i\).

There exists \((x', y') \in J + (0, -b)\) such that \((x, y) \prec (x', y')\). If \(y' \geq 0\), then \((x', y') \in [0, i-1]^2\), hence \((x', y') \in [J + D + (0, -b)] \cap [0, i-1]^2 \subset K \cap [0, i-1]^2 \subset K\). Therefore \((x, y) \in K\).

If \(y' < 0\), since \((x, y) \prec (x', y')\), we must have \(x < x'\). By the assumption, \(y = i\). From Figure 25, we have

\[
(x, i) \prec (x' - (i-1 - y')p, i-1) \prec (x', y')
\]
and

\[ x' - (i - 1 - y')p \in [x, x'] \subset [0, i - 1]. \]

Hence \((x' - (i - 1 - y')p, i - 1) \in [J + D + (0, -b)] \cap [0, i - 1]^2 \subset K \cap [0, i - 1]^2 \subset K.\) Therefore, we also have \((x, y) \in K.\) \hfill \Box

![Diagram](image)

**Figure 25. Proof of Lemma 5.4**

**Lemma 5.5.** Let \(J\) and \(K\) be ideals of \([0, i - 1]^2\) where \(i \geq p\) and \(J \neq [0, i - 1]^2, J \neq 0\). Let \(b, c \geq 0\) be integers. Let \(J\) be the largest ideal of \([0, i]^2\) such that \(J \cap [0, i - 1]^2 = J\) and \(K\) the largest ideal of \([0, i]^2\) such that \(K \cap [0, i - 1]^2 = K\). If

\[(5.19) \quad [J + (b, -c) + D] \cap [0, i - 1]^2 \subset K,\]

then

\[(5.20) \quad [\hat{J} + (b, -c) + D] \cap [0, i]^2 \subset K.\]

**Proof.** Let \(\omega(J) = W\) and \(\omega(K) = Z\). Then \(\omega(\hat{K}) = Z_{[0, i]^2}\),

\[\omega(\hat{J} + (b, -c)) = (W + (b, -c))_{[b, b+i] \times [-c, -c+i]},\]

and

\[(5.21) \quad \omega([\hat{J} + (b, -c) + D] \cap [0, i]^2) = \omega(\hat{J} + (b, -c))_{[0, b+i] \times [-c, i]} \mid_{[0, i]^2} = Y \mid_{[0, i]^2},\]

where

\[Y = [(W + (b, -c))_{[b, b+i] \times [-c, -c+i]}]_{[0, b+i] \times [-c, i]}\].

Since \(J \neq [0, i - 1]^2\) and \(J \neq 0\), we have \(\hat{J} \neq [0, i]^2\) and \(\hat{J} \neq 0\). Thus \(\omega(\hat{J} + (b, -c))\) is neither \(0\) nor the single point \((b+i, -c+i)\). Since \(i \geq p\), \(\omega(\hat{J} + (b, -c))\) is not a single horizontal step. Therefore, the extension from \(\omega(\hat{J} + (b, -c))\) to \(Y\) requires the same additional steps as the extension from \(W + (b, -c)\) to \((W + (b, -c))_{[b, b+i-1] \times [-c, i]}\).

(See Figure 26.) Thus \(Y\) is the union (in the obvious sense) of

\[(5.22) \quad (W + (b, -c))_{[b, b+i-1] \times [-c, i]} \quad \text{and} \quad \omega(\hat{J} + (b, -c)).\]

By (5.19), we have

\[(5.23) \quad (W + (b, -c))_{[b, b+i-1] \times [-c, i-1]} \mid_{[0, i-1]^2} \leq Z.\]

By (5.22), \(Y\) is an extension of \((W + (b, -c))_{[b, b+i-1] \times [-c, i]}\), hence an extension of \((W + (b, -c))_{[b, b+i-1] \times [-c, i-1]} \mid_{[0, i-1]^2}\). Thus (5.23) gives \(Y \leq Z_{[0, b+i] \times [-c, i]}\). Taking restriction on \([0, i]^2\), we have

\[Y \mid_{[0, i]^2} \leq Z_{[0, i]^2}.\]
Using (5.21), we have
\[ \omega \left( \hat{J} + (b, -c) + D \right) \cap [0, i]^2 \leq \mathbb{Z}_{[0,i]^2} = \omega(K), \]
which proves (5.20). \qed

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure26}
\caption{Proof of Lemma 5.5}
\end{figure}

Lemma 5.6. In Lemma 5.1, let \( W_{i,j} = \omega(J_{i,j}) \) (0 \leq j < i) and \( W_i = \omega(J_i) \). Set
\begin{equation}
S_i = (W_{i,i-1} + (0, -p))_{[0,i] \times [-p,i]} \cap [0,i]^2
\end{equation}
and
\begin{equation}
T_i = (W_{i,i-1})_{[0,i]^2} \cap [(W_{i,i-p})_{[0,1+i] \times [0,i]} \cap [1,1+i] \times [0,i] -(1,0)] \cap
\left( (W_{i,i-\beta_i})_{[0,1+i] \times [-p^2+p\beta_i,i]} \cap [1,1+i] \times [-p^2+p\beta_i, -p^2+p\beta_i,i]- (1,-p^2+p\beta_i) \right),
\end{equation}
where \( \beta_i \) is the largest integer such that 1 \leq \beta \leq p - 1, i - \beta_i \geq 0 \) and \( J_{i,i-\beta_i} \neq [0,i-1]^2 \). (If \( \beta_i \) does not exist, the last walk at the right hand side of (5.25) is ignored.) Then (5.5) and (5.6) hold if and only if
\[ S_i \leq W_i \leq T_i. \]
\begin{proof}
We will show that (5.6) is equivalent to \( S_i \leq W_i \) and that (5.5) is equivalent to \( W_i \leq T_i \).
First we claim that (5.6) is equivalent to
\begin{equation}
[J_{i,j} + D - (i - j)(0,p)] \cap [0, i]^2 \subset J_i \quad \text{for all } 0 \leq j < i
\end{equation}
and that (5.5) is equivalent to
\begin{equation}
[J_i + D + (a,0)] \cap [0, i-1]^2 \subset J_{i,i-ap-b}
\end{equation}
and
\begin{equation}
[J_i + D + (a+1,-p^2+bp)] \cap [0, i-1]^2 \subset J_{i,i-ap-b},
\end{equation}

\end{proof}
where \(a, b \in \mathbb{Z}, a \geq 0, 0 \leq b \leq p - 1\) and \(ap + b \leq i\). The proof of these claims is the same as the proof of Lemma 4.1 (i).

Therefore, it suffices to establish the following relations:

\[
(5.26) \iff S_i \leq W_i; \\
(5.27) \text{ and } (5.28) \iff W_i \leq T_i.
\]

**Proof of “(5.26) \iff S_i \leq W_i”**. By Lemma 5.4, (5.26) is equivalent to

\[
(5.29) \quad [J_{i,j} + D - (i - j)(0, p)] \cap [0, i - 1]^2 \subset J_i \cap [0, i - 1]^2 \quad \text{for all } 0 \leq j < i.
\]

Since \(\bigcup_{j=0}^{i-1} (J_{i,j} \times \{j\})\) is an ideal of \([0, i - 1]^3\), we have (cf. 4.1)

\[
(5.30) \quad [J_{i,j} + D - (i - 1 - j)(0, p)] \cap [0, i - 1]^2 \subset J_{i,i-1} \quad \text{for all } 0 \leq j < i.
\]

Note that Lemma 4.3 remains true with \([0, i - 1]^2\) in place of \(U\). Thus by Lemma 4.3 and (5.30), we see that (5.29) holds for all \(0 \leq j < i\) if and only if it holds for \(j = i - 1\), i.e., if and only if

\[
(5.31) \quad [J_{i,i-1} + D - (0, p)] \cap [0, i - 1]^2 \subset J_i \cap [0, i - 1]^2.
\]

By Lemma 5.4 again, (5.31) is equivalent to

\[
(5.32) \quad [J_{i,i-1} + D - (0, p)] \cap [0, i]^2 \subset J_i.
\]

In terms of boundaries, (5.32) is equivalent to \(S_i \leq W_i\).

**Proof of “(5.27) and (5.28) \iff W_i \leq T_i”**. Let \(\hat{J}_{i,j}\) \((0 \leq j < i)\) be the largest ideal of \([0, i]^2\) such that \(\hat{J}_{i,j} \cap [0, i - 1]^2 = J_{i,j}\). We claim that \(W_i \leq T_i\) is equivalent to the following three conditions:

\[
(5.33) \quad J_i \subset \hat{J}_{i,i-1}.
\]

\[
(5.34) \quad [J_i + D + (1, 0)] \cap [0, i]^2 \subset \hat{J}_{i,i-1} \quad \text{if } i \geq p.
\]

\[
(5.35) \quad [J_i + D + (1, -p^2 + p\beta_i)] \cap [0, i]^2 \subset \hat{J}_{i,i-1} \quad \text{if } \beta_i \text{ does not exist.}
\]

In fact, (5.34) is equivalent to

\[
W_i \leq (W_{i,i-1})_{[0,i]^2}.
\]

By Lemma 4.2, (5.34) is equivalent to

\[
W_i + (1, 0) \leq (W_{i,i-1})_{[0,1+i] \times [0,i]} | [1,1+i] \times [0,i]
\]

\[
= (W_{i,i-1})_{[0,1+i] \times [0,i]} | [1,1+i] \times [0,i].
\]

and (5.35) is equivalent to

\[
W_i + (1, -p^2 + p\beta_i)
\]

\[
\leq (W_{i,i-1})_{[0,1+i] \times [-p^2+p\beta_i]} | [1,1+i] \times [-p^2+p\beta_i] \quad \text{if } \beta_i \text{ does not exist.}
\]

Thus (5.34) – (5.35) together are equivalent to \(W_i \leq T_i\).

Therefore, it remains to show that (5.27) and (5.28) \(\iff\) (5.33) – (5.35).
Proof of “(5.27) and (5.28) \Rightarrow (5.33) – (5.35)”. In (5.27), letting \(a = 0\) and \(b = 1\), we obtain
\[ J_i \cap [0, i - 1]^2 \subset J_{i,i-1}. \]
Hence \(J_i \subset J_{i,i-1}\). In a similar way, (5.33) follows from (5.27) with \(a = 1\), \(b = 0\); (5.35) follows from (5.28) with \(a = 0\), \(b = \beta_i\).

Proof of “(5.27) and (5.28) \Rightarrow (5.33) – (5.35)”. First assume \(i < p\). In this case, the partial order \(<\) in \([0, i]^3\) is the cartesian product of linear orders. Recall that (5.27) and (5.28) together are equivalent to (5.5) and note that (5.5) is equivalent to
\[ (J_i \times \{i\} + \Delta) \cap [0, i - 1]^3 \subset \bigcup_{j=0}^{i-1} (J_{i,j} \times \{j\}). \]
Thus it suffices to show that
\[ (5.36) \quad J_i \cap [0, i - 1]^2 \subset J_{i,j} \quad \text{for all } 0 \leq j < i. \]
Since \(\bigcup_{j=0}^{i-1} (J_{i,j} \times \{j\})\) is an ideal of \([0, i - 1]^3\), we have\(J_{i,j} \subset J_{i,j-1}\) for all \(0 \leq j < i\).
By (5.33), we also have \(J_i \cap [0, i - 1]^2 \subset J_{i,i-1}\). Hence (5.36) holds.

Now assume \(i \geq p\). Since \(\bigcup_{j=0}^{i-1} (J_{i,j} \times \{j\})\) is an ideal of \([0, i - 1]^3\), by Lemma 4.1 (i), we have
\[ \begin{align*}
&\left\{ [J_{i,j} + \Delta + (a, 0)] \cap [0, i - 1]^2 \subset J_{i,j-\alpha p - b} \right. \\
&\left. [J_{i,j} + \Delta + (a + 1, -p^2 + b)] \cap [0, i - 1]^2 \subset J_{i,j-\alpha p - b} \right. \\
\end{align*} \]
for \(a \geq 0\), \(0 \leq b \leq p - 1\) and \(\alpha p + b \leq j < i\). By Lemma 5.6 we have
\[ (5.37) \quad \begin{align*}
&\left\{ [\hat{J}_{i,j} + \Delta + (a, 0)] \cap [0, i]^2 \subset \hat{J}_{i,j-\alpha p - b} \right. \\
&\left. [\hat{J}_{i,j} + \Delta + (a + 1, -p^2 + b)] \cap [0, i]^2 \subset \hat{J}_{i,j-\alpha p - b} \right. \\
\end{align*} \]
for \(a \geq 0\), \(0 \leq b \leq p - 1\) and \(\alpha p + b \leq j < i\). Note that \(\beta_i\) is also the largest integer such that \(1 \leq \beta_i \leq p - 1\), \(i - \beta_i \geq 0\) and \(\hat{J}_{i,i-\beta_i} \neq [0, i]^2\). By (5.34), (5.35), (5.37) and the proof of Theorem 4.6 we have
\[ (5.38) \quad \begin{align*}
&\left\{ [J_i + \Delta + (a, 0)] \cap [0, i]^2 \subset \hat{J}_{i,i-\alpha p - b} \right. \\
&\left. [J_i + \Delta + (a + 1, -p^2 + b)] \cap [0, i]^2 \subset \hat{J}_{i,i-\alpha p - b} \right. \\
\end{align*} \]
for \(a \geq 0\), \(0 \leq b \leq p - 1\) and \(\alpha p + b \leq j < i\). Conditions (5.27) and (5.28) immediately follow form (5.38). \(\square\)

Remark. In Lemma 5.6 we always have
\[ S_i \leq T_i. \]
In fact, by Lemma 2.1 (i), there is at least one \(J_i\) satisfying all the conditions in Lemma 5.1. Thus there exists at least one walk \(W_i\) in \([0, i]^2\) such that \(S_i \leq W_i \leq T_i\).

Definition 5.7. Let \(0 \leq i \leq n\) and let \(J_i\) be an ideal of \([0, i]^2\). We call \(J_i\) of type I if \(J_i \cap (\{i - 1, i\} \times [0, i]) = \emptyset\);

type II if \(J_i \cap (\{i - 1, i\} \times [0, i]) \neq \emptyset\) but \(J_i \cap (\{i\} \times [0, i]) = \emptyset\);

type III if \(J_i \cap (\{i\} \times [0, i]) \neq \emptyset\).
Theorem 5.8. Let $1 \leq i \leq n$ and let $J_j (0 \leq j \leq i)$ be an ideal $[0, j]^2$. Assume that $J_0, \ldots, J_{i-1}$ is a consistent sequence of ideals and write
\[
\bigcup_{j=0}^{i-1} [(J_j \times \{j\}) \cup (J_j \times \{j\})^A \cup (J_j \times \{j\})^b] = \bigcup_{j=0}^{i-1} (J_{i,j} \times \{j\}),
\]
where $J_{i,j}$ is an ideal of $[0, i-1]^2$. Let $W_{i,j} = \omega(J_{i,j})$ ($0 \leq j < i$) and $W_i = \omega(J_i)$ and let $S_i$ and $T_i$ be as in Lemma 5.7.

(i) $J_i$ is of type I and consistent with $J_0, \ldots, J_{i-1}$ if and only if
\[
(0, i) \notin \iota(S_i), \quad (i - 1, 0) \notin \iota(S_i)
\]
and
\[
S_i \leq W_i \leq T_i',
\]
where
\[
T_i' = T_i \cap A_i \cap B_i,
\]
$A_i$ is the highest walk in $[0, i]^2$ starting from $(0, i - 1)$ and $B_i$ is the highest walk in $[0, i]^2$ ending at $(i - 2, 0)$.

(ii) $J_i$ is of type II and consistent with $J_0, \ldots, J_{i-1}$ if and only if
\[
(0, i) \notin \iota(S_i), \quad (i, 0) \notin \iota(S_i)
\]
and
\[
W_i = \begin{cases} W_i|_{[0, i-1] \times [0, i]} = (i - 1, v), (i - 1, 0) \\ \Gamma_i \leq W_i|_{[0, i-1] \times [0, i]} \leq \Lambda_i \end{cases}
\]
for some integer $v$ satisfying
\[
0 \leq v < \min\{p^2, \frac{p-1}{p}i + \frac{1}{p}\}
\]
\[
(i - 1, v) \in \iota(T_i), \quad (i - 1, v + i) \notin \iota(X_i) \\
(v, i - p) \in \iota(T_i) \quad \text{if} \quad i \geq p
\]
and for the walks $\Gamma_i$ and $\Lambda_i$ defined as follows.
\[
\Gamma_i = (S_i \cup E_{i,v})|_{[0, i-1] \times [0, i]} \cup C_{i,v},
\]
\[
\Lambda_i = (T_i \cap A_i)|_{[0, i-1] \times [0, i]} \cap D_{i,v},
\]
where $C_{i,v}$ is the lowest walk in $[0, i-1] \times [0, i]$ ending at $(i - 1, v)$, $D_{i,v}$ is the highest walk in $[0, i-1] \times [0, i]$ ending at $(i - 1, v)$, and
\[
E_{i,v} = \begin{cases} \text{the lowest walk in } [0, i]^2 \text{ passing through } (v, i - p), & \text{if } i \geq p, \\ 0, & \text{if } i < p. \end{cases}
\]

(iii) $J_i$ is of type III and consistent with $J_0, \ldots, J_{i-1}$ if and only if
\[
\Phi_i \leq W_i \leq \Psi_i
\]
for some integer $u$ satisfying
\[
0 \leq u \leq i
\]
\[
(i, u) \in \iota(T_i), \quad (u, i) \in \iota(T_i) \\
(i, u + 1) \notin \iota(S_i), \quad (u + 1, i) \notin \iota(S_i)
\]
and for the walks $\Phi_i$ and $\Psi_i$ defined as follows.
\[
\Phi_i = S_i \cup F_{i,u} \cup M_{i,u},
\]
\[
\Psi_i = T_i \cap G_{i,u} \cap N_{i,u}.
\]
where $F_{i,u}$ is the lowest walk in $[0, i]^2$ starting from $(u, i)$, $G_{i,u}$ is the highest walk in $[0, i]^2$ starting from $(u, i)$, $M_{i,u}$ is the lowest walk in $[0, i]^2$ ending at $(i, u)$, $N_{i,u}$ is the highest walk in $[0, i]^2$ ending at $(i, u)$.

Proof. Necessity. We first show the necessity in cases (i) – (iii). By Lemma 5.6, we have $S_i \leq W_i \leq T_i$.

(i) Since $J_i$ is of type I, $(i - 1, 0) \notin J_i$. By (5.4), $(0, i) \notin J_i$. Thus $(0, i) \notin \iota(S_i)$, $(i - 1, 0) \notin \iota(S_i)$ and $W_i \leq A_i \land B_i$. Hence $W_i \leq T_i$.

(ii) Since $J_i$ is of type II, we have

$$W_i|_{[i-1,i] \times [0,i]} = \left((i - 1, v), (i - 1, 0)\right)$$

for some $0 \leq v \leq i$. Since $(i, 0) \notin J_i$, by (5.3), $(0, i) \notin J_i$. Thus $(i - 1, v) \in J_i$ implies that $v < p^2$ and $v + (i - 1) \frac{1}{p} < i$, i.e.,

$$v < \min\{p^2, \frac{p - 1}{p} i + 1\}.$$  

Clearly, $(0, i) \notin \iota(S_i)$, $(i, 0) \notin \iota(S_i)$, $(i - 1, v) \in \iota(T_i)$ and $(i - 1, v + 1) \notin \iota(S_i)$. By (5.15), $(v, i - p) \in J_i \subset \iota(T_i)$ if $i \geq p$.

Since $W_i|_{[0,i-1] \times [0,i]}$ ends at $(i - 1, v)$, we have

(5.43)
$$C_{i,v} \leq W_i|_{[0,i-1] \times [0,i]} \leq D_{i,v}.$$  

Since $(0, i) \notin J_i$, we have $W \leq A_i$. In case $i \geq p$, Lemma 5.3 implies $(v, i - p) \in J_i$. Thus, whether $i \geq p$ or not, we always have $W \geq E_{i,v}$. It follows that

(5.44)
$$S_i \vee E_{i,v} \leq W_i \leq T_i \land A_i.$$  

Combining (5.43) and (5.44), we get

$$\Gamma_i \leq W_i|_{[0,i-1] \times [0,i]} \leq A_i.$$  

(iii) Assume that $W_i$ ends at $(i, u)$. By (5.4), $W_i$ starts with $(u, i)$. Thus $F_{i,u} \leq W_i \leq G_{i,u}$ and $M_{i,u} \leq W_i \leq N_{i,u}$. It follows that $\Phi_i \leq W_i \leq \Psi_i$. Condition (5.42) is obvious.

 Sufficiency. For the sufficiency in cases (i) – (iii), we only give the proof for case (iii). The proofs for cases (i) and (ii) are similar.

By Lemmas 5.1 and 5.2 it suffice to show that conditions (5.4) – (5.6) and (5.8) are satisfied. Since

$$F_{i,u} \vee M_{i,u} \leq W_i \leq G_{i,u} \land N_{i,u},$$  

$W_i$ must start from $(u, i)$ and end at $(i, u)$. Hence (5.4) holds. Since $S_i \leq W_i \leq T_i$, by Lemma 5.6 (5.5) and (5.6) follow. Let $v = \max\{y : (i - 1, y) \in J_i\}$. Then $v - u \leq p^2$. Thus $(v, i - p) < (u + p^2, i - p) < (u, i) \in J_i$. Hence $(v, i - p) \in J_i$ and consequently, (5.15) holds. By Lemma 5.3 (5.8) follows.

\[\square\]

**Lemma 5.9.** In case (i) of Theorem 5.8, condition (5.39) implies

(5.45)
$$S_i \leq T'_i.$$  

In case (ii), conditions (5.40) and (5.41) imply

(5.46)
$$\Gamma_i \leq \Lambda_i.$$  

In case (iii), condition (5.32) implies

(5.47)
$$\Phi_i \leq \Psi_i.$$
Remark. Lemma 5.9 assures the existence of $W_i$ in Theorem 5.8 provided condition (5.39) in case (i), or conditions (5.40) and (5.41) in case (ii), or condition (5.42) in case (iii) are satisfied.

Proof of Lemma 5.9. It is obvious that (5.39) implies (5.40) and that (5.42) implies (5.39). We only prove that (5.40) and (5.41) imply (5.46).

We show that each of the walks $S_i|_{[0,i-1] \times [0,6]}$, $E_i,v|_{[0,i-1] \times [0,6]}$ and $C_i,v$ is less than each of the walks $T_i|_{[0,i-1] \times [0,6]}$, $A_i|_{[0,i-1] \times [0,6]}$ and $D_i,v$. Most of these relations are obvious. The only ones that need proofs are

\begin{align}
(5.48) & \quad E_{i,v}|_{[0,i-1] \times [0,6]} \leq A_{i}|_{[0,i-1] \times [0,6]}, \\
(5.49) & \quad E_{i,v}|_{[0,i-1] \times [0,6]} \leq D_{i,v}, \\
(5.50) & \quad C_{i,v} \leq A_{i}|_{[0,i-1] \times [0,6]}.
\end{align}

To prove (5.48), we may assume $i \geq p$. It suffices to show that $(0,i) \notin \nu(E_{i,v})$, i.e., $i - p + \frac{1}{p}v < i$. (See Figure 27(a).) This is true since $v < p^2$.

To prove (5.49), we may again assume $i \geq p$. It suffices to show that $i - p \leq v + (i - 1 - v)p^2$, i.e., $v \leq i - \frac{p}{p+1}$. (See Figure 27(b).) This follows from the inequality $v < \frac{i}{p} + \frac{1}{p}$ in (5.41).

To prove (5.50), it suffices to have $v + \frac{1}{p}(i-1) < i$. (See Figure 27(c).) This is given by (5.41).}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{proofs.png}
\caption{Proofs of (5.48) – (5.50)}
\end{figure}

Example 5.10. Let $p = 3$ and $m = 9$ ($n = \frac{m}{3}(p - 1) = 6$). In this example, we exhibit a consistent sequence of ideals $J_0, \ldots, J_6$ using Theorem 5.8. Figure 28 gives the boundaries $W_i = \omega(J_i)$ ($0 \leq i \leq 6$) and the walks $S_i$ and $T_i$ which are needed for choosing $W_i$. The resulting $A$-invariant ideal of $[0,6]^3$,

\[ I = \bigcup_{i=0}^{6} \left[ (J_i \times \{j\}) \cup (J_i \times \{j\})^A \cup (J_i \times \{j\})^{A^2} \right], \]

is depicted in Figure 30. The cross sections of $I$ on the parallels of the $xy$-planes, i.e., $J_0, \ldots, J_6, J_6$ are given in Figure 29 in terms of their boundaries $W_{i,0}, \ldots, W_{i,5}, W_6$. The $A$-symmetry of $I$ is clearly visible in Figure 30. However, the fact that $I$ is an ideal in $(U, \prec)$ is not obvious from Figure 30.
\begin{figure}[h]
\centering
\begin{subfigure}[b]{0.24\textwidth}
\includegraphics[width=\textwidth]{W0}
\caption{$S_1 = \emptyset$
\hspace{1cm}$T_1 = (1,1)$}
\end{subfigure}
\begin{subfigure}[b]{0.24\textwidth}
\includegraphics[width=\textwidth]{W1}
\caption{$S_2 = \emptyset$
\hspace{1cm}$T_2 = (2,2)$}
\end{subfigure}
\begin{subfigure}[b]{0.24\textwidth}
\includegraphics[width=\textwidth]{W2}
\caption{$S_3 = \emptyset$
\hspace{1cm}$T_3 = (3,3)$}
\end{subfigure}
\begin{subfigure}[b]{0.24\textwidth}
\includegraphics[width=\textwidth]{W3}
\caption{$S_4$: \hspace{1cm}$T_4$:}
\end{subfigure}
\begin{subfigure}[b]{0.24\textwidth}
\includegraphics[width=\textwidth]{W4}
\caption{$S_5$: \hspace{1cm}$T_5$:}
\end{subfigure}
\begin{subfigure}[b]{0.24\textwidth}
\includegraphics[width=\textwidth]{W5}
\caption{$S_6 = \emptyset$
\hspace{1cm}$T_6$:}
\end{subfigure}
\begin{subfigure}[b]{0.24\textwidth}
\includegraphics[width=\textwidth]{W6}
\caption{}\end{subfigure}
\caption{Example 5.10: the walks $S_i$, $T_i$, and $W_i$}
\end{figure}
Figure 29. Example 5.10, boundaries of the cross sections of $I$

Figure 30. Example 5.10, the $A$-invariant ideal $I$
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