Smoothing with the Best Rectangle Window is Optimal for All Tapered Rectangle Windows

Kaan Gokcesu, Hakan Gokcesu

Abstract—We investigate the optimal selection of weight windows for the problem of weighted least squares. We show that weight windows should be symmetric around its center, which is also its peak. We consider the class of tapered rectangle window weights, which are nonincreasing away from the center. We show that the best rectangle window is optimal for such window definitions. We also extend our results to the least absolutes and more general case of arbitrary loss functions to find similar results.

I. INTRODUCTION

In many machine learning problems, smoothing of a data set is desirable to detect behavioral patterns, and to filter out noise and outliers [1]. Fundamentally, smoothing works by the assumption that closer data points in some ordering sense (e.g., time index for temporal data) should be more closely related to each other. Henceforth, unexpectedly high valued observed data points should be lower and low ones should be higher. Smoothing approaches, which are abundant in literature, can help in achieving a more flexible and robust analysis of datasets by constructing their more informative smoothed versions [2]. Because of the acquired more informative data, it has many applications in various fields from data analysis [3]–[7], signal processing [8]–[11], anomaly detection [12]–[15], and machine learning [16]–[23].

Moreover, it has applications in ECG analysis [35], specifically in QRS detection [36]. In mathematical terms, moving average is optimal in acquiring the true signal from a noisy observation, where the noise is Gaussian [37]. The moving average is optimal in acquiring the true signal from a noisy observation, where the noise is Gaussian [37]. This has application in many fields from edge detection [45] to mass spectrometry [46]; especially in image processing, because of its edge-preserving smoothing abilities [47].

All in all, given an observation sequence \( \{y_n\}_{n=1}^N \); smoothing techniques generally create the smoothed signal \( \{x_n\}_{n=1}^N \) by using a sliding window [48] of some length \( 2K + 1 \), i.e.,

\[
x_n = T(\{y_m\}_{m=n-K}^{n+K}),
\]

where the function \( T(\cdot) \) can be mean, median or some other function altogether. A more generalized definition is achieved by the incorporation of a window function \( \{w_k\}_{k=-K}^K \), where the smoothed signal is created as

\[
x_n = T(\{y_m, w_m\}_{m=n-K}^{n+K}),
\]

where mean, median, etc. becomes weighted mean, weighted median, etc. Henceforth, the design of such window functions are just as important.

Window functions [49] are utilized in spectral analysis [50] as well as antenna design [51] and beam-forming [52]. However, we mainly focus on their applications in the field of statistical analysis [53], for the smoothing of a dataset, where they define a weighting vector. In curve fitting [54] and in the field of Bayesian analysis [55], it is also referred as a kernel [56]. Kernel smoothing [57] is a statistical technique to create data estimates as a weighted average (where weights are defined by a kernel) of its neighboring observations. This weighting is such that it diminishes from its peak in all directions, i.e., closer data points are given higher weights. To this end, we tackle the problem of optimal kernel design, by considering the general class of tapered rectangle windows, where the peak is equally weighted with its immediate adjacent points and non-increasing further away.
II. Problem Description

Let us have the observed samples
\[ y = \{y_n\}_{n=1}^N. \]  
(3)

We smooth \( y \) and create our smoothed signal
\[ x = \{x_n\}_{n=1}^N. \]  
(4)

We start by focusing on the design of a moving average smoother. Hence, to create \( x \), we pass \( y \) through a weighted moving average (weighted mean) filter \( w = \{w_k\}_{k=-M}^M \) of window size \( 2M+1 \leq N \), where \( w \) is in a probability simplex. Hence, the weights are positive, i.e.,
\[ w_k \geq 0, \quad k \in \{-M, \ldots, M\}, \]  
(5)

and they sum to 1, i.e.,
\[ \sum_{k=-M}^M w_k = 1. \]  
(6)

Let \( N \) be odd, i.e., \( N = 2K + 1 \) for some natural number \( K \). Then, without loss of generality, we can define the weight vector \( w \) for a window length of \( N = 2K + 1 \) with some trailing zeros, i.e.,
\[ \tilde{w}_k = \begin{cases} w_k, & -M \leq k \leq M \\ 0, & \text{otherwise} \end{cases}, \]  
(7)

for \( k \in \{-K, \ldots, K\} \). Thus, without loss of generality, we can assume the window length is \( N \). The smoothed signal \( x \) is given by
\[ x_n = \sum_{k=-K}^K w_k y_{n+k}, \]  
(8)

where the mean filter is cyclic, i.e.,
\[ y_{n+k} = y_{(n+k-1) \mod N+1}. \]  
(9)

Note that the output \( x \) of this weighted moving average is a global minimizer of the following optimization problem (weighted cumulative cross error):
\[ \min_{x \in \mathbb{R}^N} \sum_{n=1}^N \sum_{k=-K}^K w_k (y_{n+k} - x_n)^2, \]  
(10)

since it is convex and the gradient is zero at its minimizer. Hence, given a weight vector \( w \), the solution of (10) is (8).

However, a question arises about which \( w \) is most suitable for smoothing purposes given an observation vector \( y \). To find the optimal \( w \), we minimize the objective function in (10) with respect to \( w \), i.e.,
\[ \arg \min_{w \in \mathbb{R}^N} \sum_{n=1}^N \sum_{k=-K}^K w_k (y_{n+k} - x_n)^2, \]  
(11)

where \( \mathbb{W} \) is a subset of the probability simplex of dimension \( N \), i.e., \( \mathbb{P}^N \). Later, we will specifically consider the set of tapered rectangle windows, where for every \( w \in \mathbb{W} \), we have
\[ w : \begin{cases} w_{i-1} = w_i = w_{i+1} \\ w_j \geq w_k, & i \leq j \leq k, \\ w_j \leq w_k, & j \leq k \leq i \end{cases}, \]  
(12)

where the window \( w \) diminishes from its peak \( w_i \).

III. Formulation as a Quadratic Programming

Let the objective function \( F(w, x) \) be defined as
\[ F(w, x) = \sum_{n=1}^N \sum_{k=-K}^K w_k (y_{n+k} - x_n)^2, \]  
(13)

where the optimization problem in (11) becomes
\[ \arg \min_{w \in \mathbb{W}} \min_{x \in \mathbb{R}^N} F(w, x). \]  
(14)

Let us define the objective with respect to \( w \), \( G(w) \) as
\[ G(w) = \min_{x \in \mathbb{R}^N} F(w, x), \]  
(15)

where the optimization problem becomes
\[ \arg \min_{w \in \mathbb{W}} G(w). \]  
(16)

We can write \( G(w) \) as follows:
\[ G(w) = \sum_{n=1}^N \sum_{k=-K}^K w_k \left( y_{n+k} - \sum_{m=-K}^K w_m y_{n+m} \right)^2. \]  
(17)

Definition 1. The autocorrelation of \( y \) is defined as follows:
\[ r_t = \sum_{n=1}^N y_n y_{n+t}, \]  
(18)

where \( y_n \) is cyclic as in (9).

Proposition 1. From Definition 1 and (9), we have the following properties:
- \( r_t \) is symmetric around 0, i.e., \( r_t = r_{-t} \).
- \( r_t \) is periodic with \( N \), i.e., \( r_t = r_{N+t} \).
- For any integer \( k \), we have \( \sum_{n=1}^N y_{n+k} y_{n+k+t} = r_t \).

Lemma 1. Using Proposition 1 we can reduce the optimization problem to the following:
\[ \arg \max_{w \in \mathbb{W}} w^T R w, \]  

where \( w \) is the weight vector and \( R \) is the autocorrelation matrix (i.e., \( R(i, j) = r_{j-i} \)).

Proof. Using Proposition 1 and after some algebra, we can alternatively write the objective function \( G(w) \) as
\[ G(w) = \sum_{k=-K}^K w_k r_0 - 2 \sum_{k=-K}^K w_k \sum_{m=-K}^K w_m r_{m-k} \]  
+ \[ \sum_{k=-K}^K w_k \sum_{m=-K}^K w_m r_{l-m}, \]  
(19)

\[ = r_0 - \sum_{k=-M}^M \sum_{m=-K}^K w_k w_m r_{m-k}, \]  
(20)

\[ = r_0 - w^T R w. \]  
(21)

We can change the objective in (16) to the following:
\[ \arg \max_{w \in \mathbb{W}} [r_0 - G(w)], \]  
(22)

which concludes the proof. \( \square \)
IV. SOLUTION IN A CONVEX POLYTOPE

We start by showing that the maximization problem in Lemma 1 is a convex problem.

Lemma 2. The autocorrelation matrix \( R \) is positive semidefinite, i.e.,
\[
u^T R v \geq 0,
\]
for any \( v \in \mathbb{R}^N \).

Proof. From Definition 1, we can write the matrix \( R \) as a sum of outer products
\[
R = \sum_{n=1}^{N} z_n z_n^T,
\]
where the vector \( z_n \) is equal to the vector \( y \) that is shifted by \( n \) in a cyclic manner. Hence,
\[
v^T R v = \sum_{n=1}^{N} v^T z_n z_n^T v = \sum_{n=1}^{N} \alpha_n^2,
\]
where \( \alpha_n \) is the inner product of \( v \) with \( z_n \). Since sum of squares are nonnegative, we conclude the proof.

Since \( R \) is positive semidefinite, the objective in Lemma 1 is the maximization of a convex function over the set \( W \). Let us consider the case that the set \( W \) is a convex polytope.

Definition 2. Let the subset \( W \) be a convex polytope defined by its \( I \) number of vertices \( v_i \) for \( i \in \{1, \ldots, I\} \). Let the vertex matrix \( V \) be such that the \( i^{th} \) column of \( V \) is \( v_i \).

When \( W \) is a convex polytope as defined in Definition 2, we have the following property.

Proposition 2. From Definition 2 we can write any point \( w \in W \) as a convex combination of the vertices \( \{v_i\}_{i=1}^{I} \), i.e.,
\[
w = \sum_{i=1}^{I} p_i v_i,
\]
where \( \{p_i\}_{i=1}^{I} \) is in the probability simplex \( \mathcal{P}^I \), i.e., \( p_i \geq 0 \) for \( i \in \{1, \ldots, I\} \) and \( \sum_{i=1}^{I} p_i = 1 \). Thus, we can write \( w \) as
\[
w = V p,
\]
where the \( i^{th} \) column of the matrix \( V \) is \( v_i \) and \( p = \{p_i\}_{i=1}^{I} \).

Lemma 3. When the set \( W \) is a convex polytope with \( I \) vertices defined by its vertex matrix \( V \) as in Definition 2 we can reduce the optimization problem to the following:
\[
\arg \max_{p \in \mathcal{P}^I} p^T C p,
\]
where \( C = V^T R V \) and the set \( \mathcal{P}^I \) is the probability simplex of dimension \( I \).

Proof. The proof comes from rewriting the optimization problem in Lemma 1 with Proposition 2.

For a convex maximization problem over a probability simplex, we have following fundamental result.

Lemma 4. A one-hot vector \( p^* = \{p_i^*\}_{i=1}^{I} \) (where \( p_i^* = 1 \) and \( p_j^* = 0 \) for \( i \neq j \), for some \( j \in \{1, \ldots, I\} \)) is a maximizer of a convex function \( H(p) \) over the probability simplex \( p \in \mathcal{P}^I \).

Proof. For any \( p \in \mathcal{P}^I \), we have
\[
p = \sum_{i=1}^{I} p_i q_i,
\]
where \( \{p_i\}_{i=1}^{I} = p \) and \( q_i \) is the one-hot vector whose \( i^{th} \) element is 1 and other elements are 0. Thus, from Jensen inequality, we have
\[
H(p) \leq \sum_{i=1}^{I} p_i H(q_i),
\]
which concludes the proof.

Using Lemma 4 in conjunction with Lemma 3 we get the following theorem.

Theorem 1. If the set \( W \) in the optimization problem of Lemma 1 is a convex polytope as in Definition 2 then one of its vertices \( v_i \) is a maximizer, i.e.,
\[
v_i^T R v_i = \max_{w \in W} w^T R w,
\]
for some \( i \in \{1, \ldots, I\} \).

Proof. When \( W \) is convex polytope as in Definition 2, the optimization problem of Lemma 1 can be written as in Lemma 3. We see that \( V^T R V \) is positive semidefinite, i.e.,
\[
c^T V^T R V c \geq 0,
\]
for any \( c \), since \( Vc \), itself, is a vector and \( R \) is positive semidefinite. Hence, this is a convex maximization problem over a probability simplex. From Lemma 4 we know that a one-hot vector for \( p \) is a maximizer. Since \( w = V p \), one of the vertices is a maximizer of the original problem, which concludes the proof.

We point out that a probability simplex itself is also a convex polytope. Hence, if the weights \( w \) can be any probability distribution of dimension \( N \), we have the following result.

Corollary 1. When \( W = \mathcal{P}^N \), a one-hot vector \( w^* \), i.e.,
\[
w_k^* = \begin{cases} 1, & k = i \\ 0, & k \neq i \end{cases},
\]
for some \( i \in \{-K, \ldots, K\} \), where \( \{w_k^*\}_{k=-K}^{K} = w^* \) is an optimizer.

Proof. The proof comes from Lemma 4.
V. DESIGNING THE WINDOW FUNCTION

For the design of the window function, i.e., the set of weights \( W \), we start by considering some properties.

**Lemma 5.** Let the vector \( w_n \) be a shifted by \( n \) (in a cyclic manner) version of the vector \( w \). Then, we have
\[
 w_n^T R w_n = w^T R w, 
\]
for any integer \( n \).

**Proof.** Cyclic shift of a vector \( w \) by 1 can be achieved by multiplying by a matrix \( S_1 \), whose rows are equal to the cyclic shift of the identity matrix by 1. We observe that
\[
 S_1^T R S_1 = R, 
\]
since \( R \) is a circulant matrix. Let \( S_n \) be the matrix that achieves a cyclic shift of \( n \), then we have
\[
 S_n = S_1^n, 
\]
where \( S_1^n \) is the successive multiplication of \( S_1 \) by \( n \) times. Hence,
\[
 S_n^T R S_n = R, 
\]
and
\[
 w_n^T R w_n = w^T R w, 
\]
which concludes the proof. \( \square \)

This lemma shows that any cyclic shift of the weight vector \( w \) will have the same objective value, which is intuitive since it would be equivalent to shifting the observation \( y \) itself. Thus, without loss generalization, we choose the solution which has its maximum at \( w_0 \), i.e., the weights have their peak at \( k = 0 \).

**Lemma 6.** Let the vector \( \tilde{w} = \{ \tilde{w}_k \}_{k=−K}^K \) be symmetric to the vector \( w = \{ w_k \}_{k=−K}^K \) around \( k = 0 \), i.e.,
\[
 \tilde{w}_k = w_{−k}, 
\]
for \( k \in \{−K, \ldots, K\} \). Then, we have
\[
 \tilde{w}^T R \tilde{w} = w^T R w. 
\]

**Proof.** We observe that
\[
 \tilde{w} = J w, 
\]
where \( J \) is an exchange matrix (or row-reversed identity matrix), whose anti diagonal is all 1 and 0 otherwise. Hence,
\[
 \tilde{w}^T R \tilde{w} = w^T J^T R J w. 
\]
Since \( R \) is a symmetric circulant matrix, we have
\[
 J^T R J = R, 
\]
which gives
\[
 \tilde{w}^T R \tilde{w} = w^T R w, 
\]
and concludes the proof. \( \square \)

Since this result shows that two weights that are symmetric around \( k = 0 \) to each other have the same loss, we consider the set of weights that are symmetric around \( k = 0 \).

For the set of symmetric window distributions, we define the following.

**Definition 3.** Let us define \( W_{\text{symm}} \) as the set weights \( w = \{ w_k \}_{k=−K}^K \) that satisfy
\[
 w_0 = (1 + \epsilon) w_1, \\
 w_k = w_{−k}, \quad \forall k, \\
 w_k = (1 + \epsilon) w_{K−k−1}, \quad k \geq 1, \\
 w_k = (1 + \epsilon) w_{K−k}, \quad k \leq −1, 
\]
for some \( \epsilon \geq 0 \).

Note that we have constrained \( w_0 \), because otherwise, the optimal solution would have been a one-hot vector. When \( \epsilon = 0 \), we have the set of tapered rectangle windows in \([12]\).

**Lemma 7.** The set \( W_{\text{symm}} \) defined in [Definition 3] is a convex polytope with \( K \) vertices \( \{ v_i \}_{i=1}^K \), where
\[
 v_i, k = \begin{cases} 
 1 + \epsilon, & k = 0 \\
 \frac{1}{2i + 1} + \epsilon, & 1 \leq |k| \leq i, \\
 0, & \text{otherwise}
\end{cases} 
\]
where
\[
 p_i = \begin{cases} 
 w_K (2K + 1 + \epsilon), & i = K \\
 (w_i - w_{i+1}) (2i + 1 + \epsilon), & 1 \leq i \leq K−1. 
\end{cases} 
\]

Let us define \( w_{K+1} = 0 \), we have
\[
 \sum_{i=1}^K p_i = \sum_{i=1}^K (w_i - w_{i+1}) (2i + 1 + \epsilon), 
\]
\[
 = \sum_{i=1}^K (1 + \epsilon) (w_i - w_{i+1}) + 2 \sum_{i=1}^K \sum_{j=i}^K (w_j - w_{j+1}), 
\]
\[
 = (1 + \epsilon) w_1 + 2 \sum_{i=1}^K w_i, 
\]
\[
 = \sum_{i=−K}^K w_k = 1, 
\]
since \( w_0 = (1 + \epsilon) w_1 \) and \( w_i = w_{−i} \) from [Definition 3]. Moreover, \( p_i \geq 0 \), which gives \( p = \{ p_i \}_{i=−K}^K \) is a probability vector. From [Proposition 2], \( W_{\text{symm}} \) is a convex polytope defined by its vertices \( \{ v_i \}_{i=1}^K \) for any \( \epsilon \geq 0 \), which concludes the proof. \( \square \)

**Corollary 2.** When \( \epsilon = 0 \), an unweighted moving average of some odd length \( i \geq 3 \) is an optimizer for the set \( W_{\text{symm}} \) in [Definition 3].

**Proof.** From [Lemma 7], we see that, when \( \epsilon = 0 \), \( W_{\text{symm}} \) is a convex polytope whose vertices consist of the moving average vectors with odd lengths of at least 3. From [Theorem 1], one of these vertices is an optimizer, which concludes the proof. \( \square \)
VI. EXTENSION TO MEDIAN FILTER

We observe that our derivations are not limited to the weighted mean filter and can also be applied to the weighted median filter as well. For a given observation vector \( y = \{ y_n \}_{n=1}^N \) and probability weights \( w = \{ w_k \}_{k=-K}^K \), let the output \( x = \{ x_n \}_{n=1}^N \) be created as follows:

\[
x_n = \text{wmedian} \left( \{ y_{n+k}, w_k \}_{k=-K}^K \right),
\]

where the function \( \text{wmedian}(\cdot) \) orders \( \{ y_{n+k}, w_k \}_{k=-K}^K \) according to \( \{ y_{n+k} \}_{k=-K}^K \). Let \( \{ y_{n+k}, w_k \}_{k=-K}^K \) be the ordered version of \( \{ y_{n+k}, w_k \}_{k=-K}^K \) and \( \{ \tilde{w}_k \}_{k=-K}^K \) be its corresponding re-indexed weights. Then, we have

\[
x_n = \tilde{y}_i, \quad \exists i : \sum_{k=-K}^{i-1} \tilde{w}_k \leq 0.5, \quad \sum_{k=i+1}^{K} \tilde{w}_k \leq 0.5.
\]

**Definition 4.** \( x = \{ x_n \}_{n=1}^N \) in (43) is a solution to the optimization problem

\[
\min_{x \in \mathbb{R}^N} F_A(w, x) \triangleq \min_{x \in \mathbb{R}^N} \sum_{n=1}^{N} \sum_{k=-K}^{K} w_k |y_{n+k} - x_n|.
\]

**Remark 1.** The objective function \( F_A(w, x) \) in Definition 4 is linear in its argument \( w \) for any fixed \( x \in \mathbb{R}^N \).

**Proposition 3.** The function \( G_A(w) \), which is the minimization of \( F_A(w, x) \) (in Definition 4) over \( x \), i.e.,

\[
G_A(w) \triangleq \min_{x \in \mathbb{R}^N} F_A(w, x)
\]

is concave in \( w \).

**Proof.** Since \( F_A(w, x) \) is linear, we have

\[
G_A(\lambda w_0 + (1 - \lambda) w_1) \\
= \min_{x \in \mathbb{R}^N} [\lambda F_A(w_0, x) + (1 - \lambda) F_A(w_1, x)],
\]

\[
\geq \min_{x \in \mathbb{R}^N} \lambda F_A(w_0, x) + \min_{x \in \mathbb{R}^N} (1 - \lambda) F_A(w_1, x),
\]

\[
\geq \lambda G_A(w_0) + (1 - \lambda) G_A(w_1),
\]

for any \( 0 \leq \lambda \leq 1 \), which concludes the proof.

**Remark 2.** We have the following concave minimization:

\[
\min_{w \in W} G_A(w).
\]

**Theorem 2.** If the set \( W \) in the optimization problem of Remark 2 is a convex polytope defined by its set of vertices \( \{ v_i \}_{i=1}^I \) as in Definition 2, then one of its vertices \( v_j \) is a minimizer, i.e.,

\[
G_A(v_j) = \min_{w \in W} G_A(w),
\]

for some \( j \in \{ 1, \ldots, I \} \).

**Proof.** The proof is similar as in Theorem 1 and comes from Lemma 4 since \( G_A(w) \) is concave from Proposition 3.

**Corollary 3.** For the set \( W_{\text{sym}} \) of Definition 3 when \( \epsilon = 0 \), an unweighted median filter of some odd length \( i \geq 3 \) is an optimizer for Remark 2

**Proof.** The proof is similar to the proof of Corollary 2.

VII. DISCUSSIONS AND CONCLUSION

Similar to the case with the absolute loss, we observe that any arbitrary loss function has the same result. Let \( \alpha \) be such that it is a minimization of the following objective function

\[
\min_{x \in \mathbb{R}^N} F_G(w, x),
\]

where

\[
F_G(w, x) \triangleq \sum_{n=1}^{N} \sum_{k=-K}^{K} w_k f(y_{n+k}, x_n),
\]

for some function \( f(\cdot, \cdot) \).

We observe that irrespective of the function \( f(\cdot, \cdot) \), \( F_G(w, x) \) is linear in \( w \). Thus, We have

\[
G_G(w) = \min_{x \in \mathbb{R}^N} F_G(w, x),
\]

which is also concave in \( w \). Similarly to the median filter setting, when \( W \) is a convex polytope, one of its vertices is an optimizer.

In general, to find an optimizer in the convex polytope \( W_{\text{symm}} \), all of the vertices need to be tried to find the one with the smaller loss \( G_G(w) \). For the median filtering problem, this has a polynomial in \( N \) complexity. However, for the weighted mean filtering problem, it is much more efficient.

**Remark 3.** For the quadratic optimization problem in Theorem 2 the optimal weights in \( W_{\text{symm}} \) can be found in linearity, i.e., \( O(N \log N) \) complexity.

**Proof.** From Wiener–Khinchin theorem \[27\], we can compute a row of the autocorrelation matrix \( R \) in \( O(N \log N) \) time with two Fast-Fourier Transforms (FFT) \[58\], since

\[
r = \text{FFT}(f_y \odot f_y^*),
\]

where \( r \) is the autocorrelation vector (a single row of \( R \)), the operation \( \odot \) is element-wise multiplication, the super-script star denotes the conjugate and

\[
f_y = \text{FFT}(y).
\]

After finding the autocorrelation, starting with the length 3 weight vector, the evaluation of the objective function in Lemma 1 can be recursively computed in \( O(1) \) time for every odd length weight vector. Hence, the evaluation process has linear time complexity, i.e., \( O(N) \) complexity and the total complexity is linearity, i.e., \( O(N \log N) \).

In conclusion, we have investigated the optimal selection of weight windows for the problem of weighted least squares. We have shown that weight windows should be symmetric around its center, which is also the peak of the window. Moreover, we have considered that the weights should be discounted with sample distances; and assumed that the weights are nonincreasing away from the center (as per the tapered rectangle window definition). Surprisingly, for such window definitions, an optimal weighting has been shown to be an unweighted window of some odd size at least 3. We have also extended our results to the least absolutes and the more general case of arbitrary loss functions, where we found similar results.
