BERGE-GABAI KNOTS AND L-SPACE SATELLITE OPERATIONS

JENNIFER HOM, TYE LIDMAN, AND FARAMARZ VAFAEE

Abstract. Let $P(K)$ be a satellite knot where the pattern, $P$, is a Berge-Gabai knot (i.e., a knot in the solid torus with a non-trivial solid torus Dehn surgery), and the companion, $K$, is a non-trivial knot in $S^3$. We prove that $P(K)$ is an L-space knot if and only if $K$ is an L-space knot and $P$ is sufficiently positively twisted relative to the genus of $K$. This generalizes the result for cables due to Hedden [Hed09] and the first author [Hom11].

1. Introduction

In [OS04d], Oszváth and Szabó introduced Heegaard Floer theory, which produces a set of invariants of three- and four-dimensional manifolds. One example of such invariants is $\widehat{HF}(Y)$, which associates a graded abelian group to a closed 3-manifold $Y$. When $Y$ is a rational homology three-sphere, $\text{rk } \widehat{HF}(Y) \geq |H_1(Y;\mathbb{Z})|$ [OS04c]. If equality is achieved, then $Y$ is called an $L$-space. Examples include lens spaces, and more generally, all connected sums of manifolds with elliptic geometry [OS05]. L-spaces are of interest for various reasons. For instance, such manifolds do not admit co-orientable taut foliations [OS04a, Theorem 1.4].

A knot $K \subset S^3$ is called an $L$-space knot if it admits a positive L-space surgery. Any knot with a positive lens space surgery is then an L-space knot. In [Ber], Berge gave a conjecturally complete list of knots that admit lens space surgeries, which includes all torus knots [Mos71]. Therefore it is natural to look beyond Berge’s list for L-space knots. In [Vaf13], the third author classifies the twisted $(p,kp \pm 1)$-torus knots admitting L-space surgeries, some of which are known to live outside of Berge’s collection. Another related goal is to classify the satellite operations on knots that produce L-space knots. By combining work of Hedden [Hed09] and the first author [Hom11], the $(m,n)$-cable of a knot $K \subset S^3$ is an L-space knot if and only if $K$ is an L-space knot and $n/m \geq 2g(K) - 1$. (Here, $m$ denotes the longitudinal winding.) We generalize this result by introducing a new L-space satellite operation using Berge-Gabai knots [Gab90] as the pattern.

Definition 1.1. A knot $P \subset S^1 \times D^2$ is called a Berge-Gabai knot if it admits a non-trivial solid torus filling.\footnote{Berge-Gabai knots, in the literature, are defined to be 1-bridge braids in solid tori with non-trivial solid tori fillings. We relax that definition to include torus knots as a proper subfamily.}

To see that this satellite operation is a generalization of cabling, it should be noted that any torus knot with the obvious solid torus embedding is a Berge-Gabai knot [Sei33]. Note also that any Berge-Gabai knot $P$ which is isotopic to a positive braid, when considered as a knot in $S^3$, admits a positive lens space surgery; for if performing appropriate surgery on $P$ in one of the solid tori in the genus one Heegaard splitting of $S^3$ returns a solid torus, then the corresponding surgery on the knot in $S^3$ will result in a lens space. For positive braids, this surgery is positive by Lemma 2.1 and [Mos71, Proposition 3.2].

It is shown in [Gab89] that any Berge-Gabai knot must be either a torus knot or a 1-bridge braid in $S^1 \times D^2$. More precisely, every Berge-Gabai knot $P \subset V = S^1 \times D^2$ is necessarily of the following
Berge-Gabai knots are knots in $S^1 \times D^2$ with non-trivial solid tori fillings. Such knots are always the closure of the braid $(\sigma_b \sigma_{b-1} \ldots \sigma_1)(\sigma_{w-1} \sigma_{w-2} \ldots \sigma_1)^t$ where $0 \leq b \leq w - 2$, and $|t| \geq 1$. (a) An example of a braid in a solid cylinder $I \times D^2$ that closes to form a Berge-Gabai knot with $b = 2$, $t = 3$, and $w = 5$. (The fact that the picture depicted above represents a Berge-Gabai knot is verified in [Gab90, Example 3.8].) Recall that we write $t = t_0 + qw$, where here $t_0 = 3$ and $q = 0$. (b) An immersed annulus $A$ that can be arranged to be an embedded surface in $V = S^1 \times D^2$ joining $P$ to $T = \partial V$ by performing oriented cut and paste and adding a $2\pi t/w$ twist. Note that the embedded surface $A$ provides, in the exterior of $P$, a homology from $w\ell + tm$ in $T$ to $\Lambda$ in $J = \partial \text{nb}(P)$.

We only consider the case where this construction produces a knot, rather than a link. This construction forms a torus knot if $b = 0$ and a 1-bridge representation of $P$ in $V$ if $1 \leq b \leq w - 2$. We call $w$ the **winding number**, $b$ the **bridge width**, and $t$ the **twist number** of $P$. Note that the twist number can be written as $t = t_0 + qw$ for some integers $t_0$ and $q$ where $t_0$ can be chosen so that $1 \leq t_0 \leq w - 1$.\(^2\) See Figure 1(a). Also, note that if $b \neq 0$ then the possibility of $t_0 = w - 1$ is disallowed as otherwise we would obtain a link with at least two components [Gab90].

\(^2\)Our construction of Berge-Gabai knots, which enables us to define them up to isotopy of the knot in $S^1 \times D^2$, is slightly different than that of Gabai [Gab90]. In Gabai's original construction, he always took $q = 0$ and considered knots in the solid torus up to homeomorphism of $S^1 \times D^2$ taking one knot to the other.
Remark 1.2. Note that if $t < 0$, then the braid $\sigma = (\sigma_b \sigma_{b-1} \cdots \sigma_1)(\sigma_{w-1} \sigma_{w-2} \cdots \sigma_1)^t$ is isotopic to a negative braid:

\[
\sigma \sim (\sigma_b \sigma_{b-1} \cdots \sigma_1)(\sigma_{w-1} \sigma_{w-2} \cdots \sigma_1)^t \\
\sim (\sigma_{w-1} \sigma_{w-2} \cdots \sigma_{b+1})^{-1}(\sigma_{w-1} \sigma_{w-2} \cdots \sigma_1)^{t+1}.
\]

We are now ready to state the main result. Let $P(K)$ denote a satellite knot with pattern $P$ and companion $K$.

**Theorem 1.3.** Let $P$ be a Berge-Gabai knot with bridge width $b$, twist number $t$, and winding number $w$, and let $K$ be a non-trivial knot in $S^3$. Then the satellite $P(K)$ is an L-space knot if and only if $K$ is an L-space knot and

\[
\frac{b+tw}{w^2} \geq 2g(K) - 1.
\]

Note that when $b = 0$, we can take $w = m$ and $t = n$, and Theorem 1.3 reduces to the cabling result of [Hed09, Hom11]. A version of the “if” direction of Theorem 1.3 appears in [Mot14, Proposition 7.2].

The outline of the proof of Theorem 1.3 is as follows. By applying techniques developed in [Gab90, Gor83] to carefully explore the framing change of the solid torus surgered along $P$, we prove the “if” direction of the theorem. More precisely, surgery on $P(K)$ corresponds to first doing surgery on $P$ (namely removing a neighborhood of $P$ from $S^3 \times D^2$ and Dehn filling along the new toroidal boundary component) and, second, attaching this to the exterior of $K$. Therefore, if one chooses the filling on $P$ such that the result is a solid torus (using that $P$ is a Berge-Gabai knot), then the overarching surgery on $P(K)$ corresponds to attaching a solid torus to the exterior of $K$ (performing surgery on $K$). Moreover, note that by positively twisting $P$ by performing a positive Dehn twist on $S^1 \times D^2$ (i.e., increasing $q$), we can obtain an infinite family of Berge-Gabai knots. Fixing an L-space knot $K$, for sufficiently large $q$, the satellite $P(K)$ will admit a positive L-space surgery. Finally, the “only if” direction is proved by methods similar to those used in [Hom11].

In order to prove Theorem 1.3, we establish the following lemma, which may be of independent interest.

**Lemma 1.4.** Let $P \subset S^1 \times D^2$ be a negative braid and $K \subset S^3$ be an arbitrary knot. Then the satellite knot $P(K)$ is never an L-space knot.

We point out that Lemma 1.4 can be extended more generally to the case that $P$ is a homogeneous braid which is not isotopic to a positive braid [Sta78, Theorem 2]. The proof of Lemma 1.4 was inspired by the arguments in [BM14].

We have the following corollary concerning the Ozsváth-Szabó concordance invariant $\tau$ and the smooth 4-ball genus.

**Corollary 1.5.** Let $P \subset S^1 \times D^2$ be a Berge-Gabai knot and $K \subset S^3$ be an L-space knot. If

\[
\frac{b+tw}{w^2} \geq 2g(K) - 1,
\]

then

\[
\tau(P(K)) = \tau(P) + w\tau(K),
\]

and

\[
g_4(P(K)) = g_4(P) + wg_4(K),
\]

where $\tau(P)$, respectively $g_4(P)$, denotes $\tau$, respectively the 4-ball genus, of the knot obtained from the standard embedding of $S^1 \times D^2$ into $S^3$.

**Proof.** If $J$ is an L-space knot, then $\tau(J) = g_4(J) = g(J)$ by [Ni07, Corollary 1.3] and [OS05, Corollary 1.6]. Furthermore, by Lemma 2.6,

\[
g(P(K)) = g(P) + wg(K).
\]
By assumption, $K$ is an L-space knot. The result is clear if $K$ is trivial, so assume that $K$ is non-trivial. Since $P$ is a Berge-Gabai knot with a necessarily positive twist number, it follows that $P$ is isotopic to a positive braid. Therefore, by the discussion following Definition 1.1, $P$ has a positive lens space surgery, and thus is an L-space knot. Furthermore, by Theorem 1.3, we also have that $P(K)$ is an L-space knot, and the result follows. □

Theorem 1.3 allows one to construct new examples of L-spaces as follows. First, begin with any L-space knot and then satellite with a Berge-Gabai knot satisfying the conditions in Theorem 1.3. Sufficiently large positive surgery will then result in an L-space. Using this technique, we will construct L-spaces with any numbers of hyperbolic and Seifert fibered pieces in the JSJ decomposition.

**Theorem 1.6.** Let $r$ and $s$ be non-negative integers such that at least one is non-zero. Then there exist infinitely many irreducible L-spaces whose JSJ decompositions consist of exactly $r$ hyperbolic pieces and $s$ Seifert fibered pieces.

As discussed, an L-space cannot admit a co-orientable taut foliation. Therefore, Theorem 1.6 will yield irreducible rational homology spheres without co-orientable taut foliations whose JSJ decompositions consist of any numbers of hyperbolic and Seifert fibered pieces. We remark that all rational homology spheres with Sol geometry are L-spaces [BGW13].

It is also natural to ask in what sense Theorem 1.3 generalizes; in particular, given a satellite knot which is an L-space knot, what must hold for the pattern or the companion? We propose the following conjecture (see also [BM14, Question 22]).

**Conjecture 1.7.** If $P(K)$ is an L-space knot, then so are $K$ and $P$.

Similarly, we conjecture that the converse holds as well, contingent on the pattern being embedded “nicely” in the solid torus (e.g., as a strongly quasipositive braid closure) and sufficiently “positively twisted” (akin to the condition in Theorem 1.3). We will not attempt to make these notions precise in this paper.

As supporting evidence for Conjecture 1.7, we will study it from the viewpoint of left-orderability. Recall that a non-trivial group $G$ is left-orderable if there exists a left-invariant total order on $G$ (see Section 3 for a more detailed discussion). We recall the conjecture of Boyer, Gordon, and Watson relating Heegaard Floer homology to the left-orderability of three-manifold groups.

**Conjecture 1.8** (Boyer-Gordon-Watson [BGW13]). Let $Y$ be an irreducible rational homology sphere. Then $Y$ is an L-space if and only if $\pi_1(Y)$ is not left-orderable.

We point out that the computational strengths of Heegaard Floer homology and left-orderability tend to be fairly different. It is hopeful that if Conjecture 1.8 is true then the strengths of each theory could be combined to derive new topological consequences. We utilize this philosophy to establish Conjecture 1.7 under the assumption of Conjecture 1.8.

**Proposition 1.9.** Assuming Conjecture 1.8, if $P(K)$ is an L-space knot, then so are $P$ and $K$.

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2. The main result

In this section, we provide background on 1-bridge braids in solid tori and Dehn surgery on satellite knots. See [Ber91, Gab90, Gor83] for further details. Throughout the rest of the paper, we assume that $P$ is a Berge-Gabai knot in $V = S^1 \times D^2$ (i.e., $P$ admits a non-trivial solid torus surgery) unless otherwise stated. We also consider the standard embedding of $S^1 \times D^2$ into $S^3$ such that $S^1 \times \{\ast\}$ bounds an embedded disk in $S^3$. When it is clear from context, we will not distinguish between the Berge-Gabai knot $P \subset V$ and $P \subset S^3$.

2.1. Berge-Gabai knots. The primary goal of this subsection is to highlight the Dehn surgeries on $P \subset V$ that will return a solid torus. In what follows, we provide a setup similar to that of [Gab90].

An arbitrary knot $P$ in $V$ is called a 1-bridge braid if $P$ can be isotoped to be a braid in $V$ that lies in $S^1 \times \partial D^2$ except for one arc that is properly embedded in $V$, and $P$ is not a torus knot. Gabai [Gab89] showed that any knot in a solid torus with a non-trivial solid torus surgery must be either a torus knot or a 1-bridge braid in $S^1 \times D^2$, and Berge [Ber91] classified all 1-bridge braids in $S^1 \times D^2$ with non-trivial solid torus fillings. We denote the braid index of $P$ by $w$.

We will consider $\hat{V}$, the exterior of $P \subset V$. Let $T = \partial V$ and $J = \partial \text{nb}(P)$. We equip $T$ with the homological generators $(m, \ell)$ where $\ell$ is the longitude $S^1 \times \{\ast\}$ of $T$ and $m$ is $\{\ast\} \times \partial D^2$; therefore, $\ell$ becomes null-homologous after standardly embedding $V$ in $S^3$ and removing $\text{nb}(P)$. We equip $J$ with homological generators $(\mu, \Lambda)$ as follows. The generator $\mu$ is the meridian of $P$. Note that $m$ is homologous to $w\mu$ in $\hat{V}$. To define $\Lambda$, consider the immersed annulus $A$ connecting $J$ to $T$ with $b$ arcs of self-intersection in Figure 1(b). By doing oriented cut and paste to the arcs of self-intersection we can arrange $A$ to be an embedded surface in $\hat{V}$ joining $J$ to $T$. Define $\Lambda$ to be $A \cap J$. Orient $m$, $\ell$, $\mu$, and $\Lambda$ as in Figure 1(b). Note that $A \cap T = w\ell + tm$, and so $w\ell + tm$ is homologous to $\Lambda$ in $\hat{V}$.

Let $\lambda$ be the simple closed curve on $J$ that is homologous to $\Lambda - w\mu \in H_1(J; \mathbb{Z})$. Thus, we have the following equalities in $H_1(\hat{V}; \mathbb{Z})$:

$$[\lambda] = [\Lambda - w\mu] = [w\ell + tm - w\mu] = [w\ell],$$

where the last equality follows from the fact that $m$ is homologous to $w\mu$. In particular, $\lambda$ becomes null-homologous after standardly embedding $V$ in $S^3$ and removing $\text{nb}(P)$. Now the equation $[\lambda] = [\Lambda - w\mu]$ can be used to switch from $(\mu, \Lambda)$- to $(\mu, \lambda)$-coordinates, where $(\mu, \lambda)$ are the usual meridian-longitude coordinates on $P$ when $V$ is standardly embedded in $S^3$.

We recall that a 1-bridge braid in $S^1 \times D^2$ with winding number $w$, bridge width $b$, and twist number $t$ can be represented via the braid word $\sigma = (\sigma_b \sigma_{b-1} \ldots \sigma_1)(\sigma_{w-1} \sigma_{w-2} \ldots \sigma_1)^t$ where $|t| \geq 1$, and $1 \leq b \leq w - 2$. The following lemma is a consequence of [Gab90, Lemma 3.2]:

**Lemma 2.1.** Let $P$ be a 1-bridge braid in $V$ and $s$ a positive integer. If filling $\hat{V}$ along a curve $\alpha = d\mu + s\Lambda$ in $J$ yields $S^1 \times D^2$, then $s = 1, d \in \{b, b + 1\}$, and $\gcd(w, d) = 1$.

In $(\mu, \lambda)$-coordinates these possible exceptional surgeries are $\alpha = (tw + d)\mu + \lambda$ where $d \in \{b, b + 1\}$.

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[3] We have stated Lemma 2.1 so that the orientation of $(\mu, \lambda)$ agrees with the standard convention that $\mu \cdot \lambda = 1$. In Gabai’s paper [Gab90], $\mu$ is oriented opposite to that of Figure 1(b).
Note that when \( P \) is an \((m,n)\)-torus knot in \( V \), there are infinitely many surgeries on \( P \) that will return a solid torus, including \( mn + 1 = tw + b + 1 \); this follows, for instance, from the proof of [Mos71, Proposition 3.2].

Let \((P; n_1/n_2)\) denote the result of filling \( \hat{V} \) along the curve \( n_1\mu + n_2\lambda \). Lemma 2.1 shows that if \( P \) is a Berge-Gabai knot, then \((P; p_d)\) will be homeomorphic to \( S^1 \times D^2 \) for at least one of the coefficients \( p_d = tw + d, d \in \{b, b + 1\} \).

Note that adding a positive full-twist to all of the \( w \) strands of \( P \) results in a new knot \( P' \) where \( t \) changes into \( t + w \). Correspondingly, there exists a homeomorphism of the solid torus (doing a positive meridional twist), which takes \( P \) to \( P' \). Iterating this process \( q \) times, we get the following:

**Proposition 2.2.** Let \( P \) be a Berge-Gabai knot in \( S^1 \times D^2 \), standardly embedded in \( S^3 \), so that \((P;p)\) is homeomorphic to a solid torus. Let \( P' \) be the knot obtained from \( P \) by adding \( q \) positive Dehn twists. Then

\[
(P'; p + qw^2) \cong S^1 \times D^2.
\]

Hence if we have a Berge-Gabai knot \( P \) with twist number \( t \), adding \( q \) full twists to all \( w \) strands of \( P \) will produce a Berge-Gabai knot with twist number \( t + qw \).

### 2.2. Surgery on \( P(K) \)

Let \( P(K) \) be a satellite knot with pattern \( P \subset V \) and companion \( K \). Let \( f : V \to \text{nb}(K) \) be a homeomorphism that determines the zero framing of \( K \), i.e., \([f(S^1 \times \{ \ast \})] = 0 \in H_1(X; \mathbb{Z})\) where \( X = S^3 - \text{nb}(K) \). Thus \( P(K) = f(P) \).

Recall that \( m, \ell \in H_1(T; \mathbb{Z}) \) are the natural meridian and longitude coordinates of \( T = \partial V \), oriented such that \( m \cdot \ell = 1 \). Recall also that \( \hat{V} = V - \text{nb}(P) \). Note that \( H_1(\hat{V}) = \mathbb{Z}(\ell) \oplus \mathbb{Z}(\mu) \) where \( \mu \) is the class of the meridian of \( \text{nb}(P) \). When \( P \) is viewed as a knot in \( S^3 \), let \( \lambda \subset \partial \text{nb}(P) \) be the unique curve on \( \partial \text{nb}(P) \) which is null-homologous in \( S^3 - \text{nb}(P) \) (i.e., the zero framing of \( P \)). That is, if \( f \) is as above, then \( f(\lambda) \) is the zero framing of \( P(K) \). Thus, \( S^3_{p_1/p_2}(P(K)) \cong X \cup_f (P; p_1/p_2) \), where the notation means \( \partial X \) and \( \partial (P; p_1/p_2) \) are identified via the restriction of \( f \) to \( \partial (P; p_1/p_2) = \partial V \).

With the above notation:

**Lemma 2.3** ([Gor83, Lemma 3.3]). For relatively prime integers \( p_1, p_2 \), and \( P \subset V \) with winding number \( w \):

1. \( H_1((P; p_1/p_2); \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_{\gcd(w, p_1)} \).
2. If \( w \neq 0 \), the kernel of \( H_1(\partial(P; p_1/p_2); \mathbb{Z}) \to H_1((P; p_1/p_2); \mathbb{Z}) \) is the cyclic group generated by

\[
\frac{p_1}{\gcd(w, p_1)} m + \frac{p_2 w^2}{\gcd(w, p_1)} \ell.
\]

Note that Lemma 2.3 is valid regardless of whether or not \( P \) is a Berge-Gabai knot. However, when \( P \) is a Berge-Gabai knot, we can use Lemma 2.3 to relate surgeries on \( K \) and \( P(K) \) in the following sense.

**Corollary 2.4.** Let \( P \) be a Berge-Gabai knot in \( V \) with winding number \( w \) so that \( (P; p) \cong S^1 \times D^2 \). Then

\[
S^3_p(P(K)) \cong S^3_{p/w^2}(K).
\]

**Proof.** The result essentially follows from the fact that

\[
S^3_p(P(K)) \cong X \cup_f (P; p).
\]
By assumption, \((P; p)\) is homeomorphic to a solid torus. Therefore, in order to find the corresponding surgery coefficient on \(K\), one needs to determine the slope of the meridian of \(\partial(P; p)\) under the canonical identification with \(\partial V\), and where it is sent under \(f\).

Note that the slope of the meridian of \((P; p)\) is precisely the generator of

\[
\ker \left( H_1(\partial(P; p); \mathbb{Z}) \to H_1((P; p); \mathbb{Z}) \right).
\]

Using the identification of \(\partial V\) and \(\partial(P; p)\), we have that the slope of the meridian, in \((m, \ell)\)-coordinates, is given by \((p, w^2)\) by Lemma 2.3. Since \(f\) sends \(m\) (respectively \(\ell\)) to the meridian (respectively longitude) of \(K\), the result follows. \(\square\)

Combining Lemma 2.1 with Corollary 2.4, we deduce the following:

**Proposition 2.5.** Let \(P\) be a Berge-Gabai knot with bridge width \(b \neq 0\), winding number \(w\), and twist number \(t\), and let \(K\) be an arbitrary knot in \(S^3\). Then for at least one \(d \in \{b, b + 1\}\),

\[
S^3_{d+tw}(P(K)) \cong S^3_{d+tw}(K).
\]

Note that \(\gcd(d + tw, w^2) = 1\) (see Lemma 2.1). We end this subsection by stating the following lemma, which turns out to be useful during the course of proving Theorem 1.3. Let \(\Delta_K(T)\) denote the symmetrized Alexander polynomial of \(K\). Recall the behavior of the Alexander polynomial for satellites (see for instance [Lic97]):

\[
\Delta_{P(K)}(T) = \Delta_P(T)\Delta_K(T^w). \tag{2.2.1}
\]

**Lemma 2.6.** Let \(P(K)\) be a fibered satellite knot where \(P\) has winding number \(w\). Then

\[
g(P(K)) = g(P) + wg(K).
\]

Furthermore, if \(P\) is a Berge-Gabai knot as above with \(t > 0\), then

\[
g(P) = \frac{(t - 1)(w - 1) + b}{2}. \tag{2.2.2}
\]

**Proof.** Since \(P(K)\) is a fibered knot, we deduce that \(\deg \Delta_{P(K)}(T) = g(P(K))\). It also follows that \(K\) and \(P\) are both fibered [HMS08]. Combining these two facts with (2.2.1), we see that

\[
g(P(K)) = g(P) + wg(K).
\]

In order to calculate \(g(P)\), notice that \(P\) is a positive braid if \(t > 0\). Hence, the Seifert surface \(R\) obtained from Seifert’s algorithm is a minimal genus Seifert surface for \(P\) [Sta78]. Then

\[
\chi(R) = 1 - 2g(P) \Rightarrow w - b - t(w - 1) = 1 - 2g(P).
\]

\(\square\)

2.3. Input from Heegaard Floer theory. In this subsection we mainly use the notation of [Hom11]. Recall that an \(L\)-space \(Y\) is a rational homology sphere with the simplest possible Heegaard Floer homology, i.e., \(\text{rk} \, \widehat{HF}(Y) = |H_1(Y; \mathbb{Z})|\). We say that a knot \(K\) in \(S^3\) is an \(L\)-space knot if it admits a positive \(L\)-space surgery.

We let \(\tau(K)\) denote the integer-valued concordance invariant from [OS03]. Let \(\mathcal{P}\) denote the set of all knots \(K\) for which \(g(K) = \tau(K)\). (Recall from [Hed10] that for fibered knots, \(g(K) = \tau(K)\) is equivalent to being strongly quasipositive.) If \(K\) is an \(L\)-space knot, then \(K \in \mathcal{P}\). This follows from [OS05, Corollary 1.6] and the fact that \(L\)-space knots are fibered [Ni07, Corollary 1.3].
Let 

\[ s_K = \sum_{i \in \mathbb{Z}} \left( \text{rk } H_*(\widehat{A}^K_i) - 1 \right), \]

where \( \widehat{A}^K_i \) is the subquotient complex of \( CFK^\infty(K) \) defined in [OS08]. It is proved in [Hom11] that \( \text{rk } H_*(\widehat{A}^K_i) \) is always odd, and so \( s_K \) is always a non-negative even integer. For a pair of relatively prime non-zero integers \( m \) and \( n > 0 \), let

\[ t_{m/n}^K = 2 \max(0, n(2\nu(K) - 1) - m). \tag{2.3.1} \]

Observe that

\[ t_{m/n}^K = 0 \quad \text{if and only if} \quad m/n \geq 2\nu(K) - 1. \tag{2.3.2} \]

The term \( \nu(K) \) is another integer-valued invariant of \( K \), defined in [OS11, Definition 9.1], which is bounded below by \( \tau(K) \) and above by \( g(K) \). In particular, if \( K \in P \), then \( \nu(K) = g(K) \).

Let \( m \) and \( n \) be as above, and suppose that \( \nu(K) \geq \nu(\hat{K}) \) where \( \hat{K} \) denotes the mirror of \( K \). (This condition is automatically satisfied for \( K \in P \).) If \( \nu(K) > 0 \) or \( m > 0 \), then

\[ \text{rk } \widehat{H}(S^3_{m/n}(K)) = m + ns_K + t_{m/n}^K \tag{2.3.3} \]

by [OS11, Proposition 9.6].

By (2.3.3), when \( m > 0 \) we have that

\[ S^3_{m/n}(K) \text{ is an L-space if and only if } t_{m/n}^K = 0 \text{ and } s_K = 0. \tag{2.3.4} \]

By [OS04b, Theorem 4.4], the group \( H_*(\widehat{A}^K_i) \) is isomorphic to \( \widehat{H}(S^3_{n}(K), [i]) \) for \( N \gg 0 \) and \( |i| \leq N/2 \). Thus, we have that

\[ K \text{ is an L-space knot if and only if } s_K = 0. \tag{2.3.5} \]

In fact, if \( K \) is a non-trivial L-space knot, \( S^3_{m/n}(K) \) is an L-space if and only if \( m/n \geq 2g(K) - 1 \). This follows from (2.3.2), (2.3.3), and the fact that for \( K \) a non-trivial L-space knot, \( \nu(K) = g(K) > 0 \). (The original argument for the forward direction is given in [KMOS07].)

### 2.4. Proof of Theorem 1.3

This subsection is devoted to the proof of Theorem 1.3. We begin with the proof of Lemma 1.4. We do not review the concept of a quasipositive Seifert surface but instead refer the reader to [Hed10, Rud98].

**Proof of Lemma 1.4.** Suppose for contradiction that \( P(K) \) is an L-space knot. Recall that L-space knots are fibered [Ni07, OS05]. It is also a well-known fact that a minimal genus Seifert surface for a negative braid can be expressed as a plumbing of negative Hopf bands [Sta78, Theorem 2]. (See also [AO01, Theorem 1] for an explicit construction in the case of torus knots.) Since \( P(K) \) is fibered, this implies that \( K \) is fibered and \( P \) is fibered in the solid torus [HMS08], so the fiber for \( P(K) \) is constructed by patching the fiber for \( P \) in the solid torus to \( w \) copies of the fiber for \( K \). As a result, when \( P \) is a negative braid, the fiber surface for \( P(K) \) contains (at least) as many negative Hopf bands as the one for \( P \).

By the above description of the fiber surface, we can deplumb a negative Hopf band. This means we can decompose the fiber surface for \( P(K) \) as a Murasugi sum, where one of the summands is not a quasipositive surface. By [Rud98], if a Seifert surface is a Murasugi sum, it is quasipositive if and only if all of the summands are quasipositive. Thus, the fiber surface for \( P(K) \) is not a quasipositive surface. However, since \( P(K) \) is an L-space knot, it is strongly quasipositive [Hed10], which gives a contradiction. \( \square \)
We prove Theorem 1.3 only for the cases where \( b \neq 0 \) (consequently \( 1 \leq t_0 \leq w - 2 \)) and refer the reader to [Hed09, Hom11] for the case \( b = 0 \).

**Proof of Theorem 1.3.** (\( \Leftarrow \)) The proof of this direction follows from Proposition 2.5, which tells us that

\[
S^3_{d + tw}(P(K)) \cong S^3_{d + tw}(K).
\]

Since \( K \) is a non-trivial L-space knot and \( \frac{b + tw}{w^2} \geq 2g(K) - 1 > 0 \), it follows that \( S^3_{d + tw}(K) \) is an L-space. Here we are using that \( d \geq b \). Therefore, \( P(K) \) is an L-space knot.

(\( \Rightarrow \)) For the case that \( t < 0 \) (see Remark 1.2), we apply Lemma 1.4 to see that \( P(K) \) cannot be an L-space knot. Therefore, we can assume that \( t > 0 \) and \( P(K) \) is an L-space knot. For simplicity of notation, we set \( m = d + t_0 w + qw^2 \) where \( d \in \{b, b+1\} \) is such that \( (P; m) \cong S^1 \times D^2 \). Again from Proposition 2.5 we have

\[
\text{rk } \widehat{HF}(S^3_m(P(K))) = \text{rk } \widehat{HF}(S^3_{m/w^2}(K)). \tag{2.4.1}
\]

Since \( P(K) \) is an L-space knot, it follows that \( g(P(K)) = \tau(P(K)) \), and we see that

\[
t^m_{P(K)} = 2 \max(0, 2g(P(K)) - 1 - m). \tag{2.4.2}
\]

We first suppose that \( \nu(K) \geq \nu(K) \). Since \( m > 0 \), we may combine (2.3.3), (2.3.5), and (2.4.1) to obtain

\[
m + t^m_{P(K)} = m + w^2 s_K + t^m_{K/w^2},
\]

or equivalently

\[
t^m_{P(K)} = w^2 s_K + t^m_{K/w^2}. \tag{2.4.3}
\]

Note that by Lemma 2.6, (2.2.2), and (2.4.2), we have that

\[
t^m_{P(K)} = \max(0, 4wg(K) - 2w - 2t_0 - 2qw + 2b - 2d). \tag{2.4.4}
\]

**Claim.** The equality in (2.4.3) does not hold unless both sides are identically zero.

**Proof of the Claim.** If \( t^m_{P(K)} \neq 0 \) then we have two cases:

**Case 1.** Suppose \( t^m_{K/w^2} = 0 \). Using (2.4.4), we see (2.4.3) is equivalent to

\[
4wg(K) - 2w - 2t_0 - 2qw + 2b - 2d = w^2 s_K.
\]

It follows that \( w \) divides \( 2t_0 + 2d - 2b \). Since \( d - b \in \{0, 1\} \) and \( 1 \leq t_0 \leq w - 2 \), we conclude that \( w = 2t_0 + 2d - 2b \). Since

\[
4wg(K) - 2w - 2qw = w^2 s_K,
\]

then

\[
4g(K) - 3 - 2q = ws_K.
\]

The right side is an even number and the left side is odd which is a contradiction.

**Case 2.** Suppose \( t^m_{K/w^2} \neq 0 \). By expanding both sides of (2.4.3) and again using (2.4.4), we see that

\[
4wg(K) - 2w - 2t_0 - 2qw + 2b - 2d = w^2 s_K + 4w^2 \nu(K) - 2w^2 - 2d - 2t_0 w - 2qw^2.
\]

By rearranging terms, we get

\[
4wg(K) - 2w + 2(b - t_0) - 2qw + 2t_0 w = w^2 (4\nu(K) - 2 - 2q + s_K).
\]

Therefore \( w \) divides \( 2(b - t_0) \). Since \( b \) and \( t_0 \) are both bounded above by \( w - 2 \), we have either \( 2(b - t_0) = \pm w \) or \( b = t_0 \).
Recall that we described $P$ as a braid closure in Section 1. Viewing this braid as a mapping class of the disk with $w$ punctures, it is straightforward to verify that if $b = t_0$, the $(t_0 + 1)^{th}$ puncture is fixed. Therefore, in this case $P$ has at least two components, which contradicts $P$ being a knot. Thus, we must have $2(b - t_0) = \pm w$.

Substituting and dividing by $w$ gives:

$$4g(K) - 2 \pm 1 - 2q + 2t_0 = w(4\nu(K) - 2 - 2q + s_K).$$

As in Case 1, comparing the parities of each side gives a contradiction.

□

Having proved the claim, all the terms in (2.4.3) are identically zero. Since $s_K = 0$, (2.3.5) gives that $K$ is an L-space knot. Also, $t_P^m(0) = 0$ together with (2.4.4) implies

$$\frac{t_0 + qw + d - b}{w} \geq 2g(K) - 1. \quad (2.4.5)$$

Since $1 \leq t_0 \leq w - 2$ and $(d - b) \in \{0, 1\}$, we have that $0 \leq t_0 + d - b < w$. Note that $2g(K) - 1$ is an integer, so we deduce that (2.4.5) holds if and only if

$$q \geq 2g(K) - 1,$$

which implies that

$$\frac{b + t_0w + qw^2}{w^2} \geq 2g(K) - 1,$$

as desired.

Now suppose that $\nu(K) < \nu(\overline{K})$. We claim that in this case, $P(K)$ is not an L-space knot, which is a contradiction. Recall from [OS11, Equation (34)] that $\nu(K)$ is equal to either $\tau(K)$ or $\tau(K) + 1$, and from [OS03, Lemma 3.3] that $\tau(\overline{K}) = -\tau(K)$. Thus, when $\nu(K) < \nu(\overline{K})$, it follows that $\nu(\overline{K}) > 0$. By [OS04c, Proposition 2.5], the total rank of $\widehat{HF}(Y)$, for a closed three-manifold $Y$, is independent of the orientation of $Y$, i.e.,

$$\text{rk } \widehat{HF}(Y) = \text{rk } \widehat{HF}(-Y). \quad (2.4.6)$$

By combining (2.4.6), Proposition 2.5, and the fact that

$$S_{m/n}^3(K) \cong -S_{-m/n}^3(\overline{K}) \quad (2.4.7)$$

we deduce that

$$\text{rk } \widehat{HF}(S_{m/n}^3(P(K))) = \text{rk } \widehat{HF}(S_{-m/n}^3(\overline{K})). \quad (2.4.8)$$

By combining (2.3.3), (2.3.5), and (2.4.8), since $P(K)$ is an L-space knot, we have

$$m + t_P^m(0) = -m + w^2s_K + \overline{t^{m/w^2}}. \quad (2.4.9)$$

Using (2.3.1) and the fact that $\nu(\overline{K}) > 0$, we observe that $\overline{t^{m/w^2}} \neq 0$.

Claim. The equality in (2.4.9) never holds.

Proof of the Claim. We prove the claim by considering the following two cases:

Case 1. Suppose $t_P^m(0) \neq 0$. Using (2.4.4), by expanding both sides of (2.4.9) we get that

$$d + t_0w + qw^2 + 4wg(K) - 2w - 2t_0 - 2qw + 2b - 2d
= -d - t_0w - qw^2 + w^2s_K + 4w^2\nu(\overline{K}) - 2w^2 + 2d + 2t_0w + 2qw^2.$$
A similar reasoning as in Case 1 of the previous part of the proof shows that this equality gives a contradiction.

Case 2. Suppose \( t_{P(K)}^m = 0 \). Using (2.4.4), we see that (2.4.9) is equivalent to

\[
d + t_0 w + qw^2 = -d - t_0 w - qw^2 + w^2 \nu(K) + 4w^2 \nu(K) - 2w^2 + 2d + 2t_0 w + 2qw^2.
\]

This equation reduces to \( 2w^2 = w^2 \nu(K) + 4w^2 \nu(K) \). However, this equation has no solutions, since \( \nu(K) > 0 \) and \( \nu(K) \geq 0 \).

Having proved the claim, it follows that if \( \nu(K) < \nu(K) \), then \( P(K) \) could not have been an L-space knot. This completes the proof.

\[ \square \]

3. Proofs of Theorem 1.6 and Proposition 1.9

Before proving Theorem 1.6 and Proposition 1.9 we remind the reader of a standard fact about geometric structures and Dehn surgery which we will make use of repeatedly without reference (see [Hei74, Proposition 5] and [Thu80, Section 5]). Suppose that \( M \) is a compact, orientable, irreducible manifold with incompressible torus boundary (e.g., the exterior of a non-trivial knot in \( S^3 \)). Then all but finitely many Dehn fillings of \( M \) are irreducible and have the same numbers of hyperbolic and Seifert fibered pieces in their JSJ decompositions as \( M \).

3.1. JSJ decompositions and L-spaces.

**Proof of Theorem 1.6.** In order to construct the family of manifolds described in the statement of the theorem, we will first construct an L-space satellite knot \( K_{s,r} \) with \( s \) Seifert fibered pieces and \( r \) hyperbolic pieces in the JSJ decomposition. The knot \( K_{s,r} \) will be constructed by a sequence of satellite operations using cables and Berge-Gabai knots. As discussed, all but finitely many surgeries on \( K_{s,r} \) will then be irreducible rational homology spheres with the desired JSJ decomposition. Since all surgeries with slope at least \( 2g(K_{s,r}) - 1 \) will result in L-spaces (see Subsection 2.3), sufficiently large surgeries on \( K_{s,r} \) will produce the desired infinite family.

Recall that if \( P \) is a torus knot standardly embedded in the solid torus, then the exterior of \( P \) is Seifert fibered over the annulus with a single cone point. We first construct a knot \( K_0 \) as an \( s \)-fold iterated torus knot with appropriately chosen cabling parameters. More specifically, we construct \( K_0 \) as follows. If \( s \) is 0, we simply take \( K_0 \) to be the unknot. Otherwise, we begin with \( K_1 \), the positive \((m_1,n_1)\)-torus knot, for some \( m_1, n_1 \geq 2 \). Perform the \((m_2,n_2)\)-cable, choosing \( n_2/m_2 \geq 2g(K_1) - 1 \), to obtain the knot \( K_2 \). Inductively, we construct \( K_i \) to be the \((m_i,n_i)\)-cable of \( K_{i-1} \), where we choose \( n_i/m_i \geq 2g(K_{i-1}) - 1 \). The JSJ decomposition of the exterior of \( K_s \) now consists of \( s \) Seifert pieces. Further, by [Hed09], \( K_s \) is an L-space knot.

Let \( P_1 \) be a positively twisted hyperbolic Berge-Gabai knot satisfying \( \frac{b + t_0 w + gw^2}{w^2} \geq 2g(K_s) - 1 \). We can construct \( P_1 \) as follows. Begin with any hyperbolic Berge-Gabai knot (i.e., hyperbolic in \( S^1 \times D^2 \); see [Ber91, Theorem 3.2 and p.17] to obtain explicit examples). Now add sufficiently many positive twists until the desired inequality is satisfied (fix \( b, t_0, w \), and increase \( q \) to obtain \( P_1 \). As discussed in Section 2.1, adding positive twists preserves the property of being a Berge-Gabai knot; furthermore, this does not change the type of geometry on the knot exterior, and thus \( P_1 \) will still be hyperbolic. If \( s \neq 0 \), we define \( K_{s,1} \) as the satellite knot with companion \( K_s \) and pattern \( P_1 \). By Theorem 1.3, \( K_{s,1} \) is an L-space knot. If \( s = 0 \), take \( K_{s,1} \) to be any hyperbolic L-space knot, such as the \((-2,3,7)\)-pretzel knot [FS80]. We now repeat this process \( r \) times, i.e., to obtain \( K_{s,i} \), satellite \( K_{s,i-1} \) with pattern a hyperbolic Berge-Gabai knot satisfying

\[ \square \]
below, we remind the reader that we will be assuming Conjecture 1.8. As discussed above, this completes the proof. □

3.2. Left-orderability. Recall that a non-trivial group $G$ is left-orderable if there exists a left-invariant total order on $G$. Examples of left-orderable groups include $\mathbb{Z}$ and $\text{Homeo}_+(\mathbb{R})$, while any group with torsion (e.g., a finite group) is not left-orderable. It is natural to ask which three-manifold groups can be left-ordered. Such groups are well-suited for this study due to the following theorem.

**Theorem 3.1** (Boyer-Rolfsen-Wiest [BRW05]). Let $Y$ be a compact, connected, irreducible, $P^2$-irreducible three-manifold. If there exists a non-trivial homomorphism $f : \pi_1(Y) \to G$ where $G$ is left-orderable, then $\pi_1(Y)$ is left-orderable. In particular, if there exists a non-zero degree map from $Y$ to $Y'$, where $\pi_1(Y')$ is left-orderable, then $\pi_1(Y)$ is left-orderable.

Rather than define $P^2$-irreducible, we simply point out that if $Y$ is orientable, then irreducibility implies $P^2$-irreducibility. For compact, orientable, irreducible three-manifolds with $b_1 > 0$, it then follows that their fundamental groups are always left-orderable. However, there are more interesting phenomena for rational homology spheres; for example $+3/2$-surgery on the left-handed trefoil has left-orderable fundamental group, while $-3/2$-surgery has torsion-free, non-left-orderable fundamental group (this can be deduced for instance from [BRW05, Theorem 1.3]). Surprisingly, the left-orderability of the fundamental groups of three-manifolds is conjecturally characterized by Heegaard Floer homology. The following conjecture was made in [BGW13]:

**Conjecture 1.8** (Boyer-Gordon-Watson). Let $Y$ be an irreducible rational homology sphere. Then $Y$ is an L-space if and only if $\pi_1(Y)$ is not left-orderable.

There exists a large amount of support for this conjecture, as it is known to be true for manifolds with Seifert or Sol geometry, branched double covers of non-split alternating links, graph manifold integer homology spheres, and many other families of examples (see for instance [BB13, BGW13, Pet09]). We also remark that irreducibility is necessary, as $\Sigma(2,3,7)\#\Sigma(2,3,5)$ has non-left-orderable fundamental group, but is not an L-space.

In the proof of Proposition 1.9 below, we remind the reader that we will be assuming Conjecture 1.8.

**Proof of Proposition 1.9.** Suppose that $P(K)$ is an L-space knot. Then for all $\alpha \in \mathbb{Q}$ with $\alpha \geq 2g(P(K)) - 1$, we have $S_\alpha^3(P(K))$ is an L-space. For all but finitely many such $\alpha$, we have that $S_\alpha^3(P(K))$ is irreducible as well. Thus, by Conjecture 1.8, we have that $\pi_1(S_\alpha^3(P(K)))$ is not left-orderable for $\alpha \gg 2g(P(K)) - 1$.

We first study the pattern $P$. By [CW11, Proposition 13], for such $\alpha$, $\pi_1(S_\alpha^3(P))$ is not left-orderable. Furthermore, for all but finitely many $\alpha$, we have that $S_\alpha^3(P)$ is irreducible. Therefore, we appeal to Conjecture 1.8 to conclude that $P$ is an L-space knot.

We modify the argument of [CW11, Proposition 13] to study the companion $K$. Recall that $w$ represents the winding number of $P$ in the solid torus $V$. We also consider the basis $(m, \ell)$ for $H_1(\partial V; \mathbb{Z})$ as given in Section 2. We choose $n \in \mathbb{Z}$ such that $\gcd(w, n) = 1$ and $n \gg 2g(P(K)) - 1$. As discussed, we have $S_\alpha^3(P(K))$ is irreducible and $\pi_1(S_\alpha^3(P(K)))$ is not left-orderable. We consider the manifold $(P; n)$. We have that the kernel of $i_* : H_1(\partial(P; n); \mathbb{Z}) \to H_1((P; n); \mathbb{Z})$ is generated by $nm + w^2\ell$ by Lemma 2.3. Since $\gcd(w, n) = 1$ by assumption, we have that the element $nm + w^2\ell$ is represented by a simple closed curve on $\partial(P; n)$ which bounds in $(P; n)$. It then follows that there exists a degree one map $\phi : (P; n) \to S^1 \times D^2$, which restricts to a homeomorphism on the
boundary (see for instance [Ron95, Lemma 2.2]). Since \( nm + w^2 \ell \) bounds in \( (P; n) \), we must have that \( \phi(nm + w^2 \ell) \) is isotopic to \( \{\ast\} \times D^2 \).

By extending \( \phi \) to be the identity on the exterior of \( K \), one obtains a degree one map from \( S^3_{nm}(P(K)) \) to \( S^3_{n/w^2}(K) \). Since \( S^3_{nm}(P(K)) \) is irreducible and \( \pi_1(S^3_{nm}(P(K))) \) is not left-orderable, we have that \( \pi_1(S^3_{n/w^2}(K)) \) is not left-orderable by Theorem 3.1. Since \( w \) is fixed, by choosing sufficiently large \( n \) with \( \gcd(w, n) = 1 \), we can arrange that \( S^3_{n/w^2}(K) \) is irreducible as well. Again, by Conjecture 1.8, \( K \) is an L-space knot. \( \square \)

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Department of Mathematics, Columbia University, New York, NY 10027
E-mail address: hom@math.columbia.edu

Department of Mathematics, The University of Texas, Austin, TX 78712
E-mail address: tlid@math.utexas.edu

Department of Mathematics, Michigan State University, East Lansing, MI 48824
E-mail address: vafaee@msu.edu