A lower bound on Seshadri constants of hyperplane bundles on threefolds

Kungho Chan

Abstract We give the lower bound on Seshadri constants for the case of very ample line bundles on threefolds. We consider the situation when the Seshadri constant is strictly less than 2 and give a version of Bauer’s theorem (Math Ann 313(3):547–583, 1999, Theorem 2.1) for singular surfaces so we can prove the same result for smooth threefolds.

1 Introduction

The Seshadri constant at a given point on a smooth projective variety was introduced by Demailly [2] to study Fujita’s conjecture. It measures how positive a nef line bundle locally is near a given point. Since then Seshadri constants were recognized as interesting invariants of algebraic varieties on their own.

Definition 1.1 Suppose $X$ is a projective variety of dimension $n$ and $L$ is a nef line bundle over $X$. Let $x$ be a point on $X$ and

$$
\pi : \tilde{X} \to X
$$

the blowup of $X$ at $x$ with the exceptional divisor $E$. Then, we define the Seshadri constant $\epsilon(L, x)$ of $L$ at $x$ as

$$
\epsilon(L, x) := \sup \{ \alpha \geq 0 \mid \pi^* L - \alpha E \text{ is nef} \}.
$$

Or, equivalently, it can be defined by

$$
\epsilon(L, x) := \inf_{x \in C \subset X} \left\{ \frac{L.C}{\text{mult}_x C} \right\}
$$

where the infimum is taken over all integral curves $C \subset X$ passing through $x$ [3, 5.1.5].
For the case that $X \subset \mathbb{P}^N$ is a smooth integral projective variety and $L$ is the restriction $O_X(1)$ on $X$ of the hyperplane bundle of $\mathbb{P}^N$, it is easy to see that $\epsilon(O_X(1), x) \geq 1$ for every $x \in X$. Obviously, the equality holds if there is a line in $X$ passing through $x$. For the case of smooth surfaces, Bauer proved the following.

**Theorem 1.2** [1, Theorem 2.1] Let $X \subset \mathbb{P}^N$ be a smooth irreducible surface.

(a) $\epsilon(O_X(1), x) = 1$ if and only if $X$ contains a line passing through $x$.

(b) Suppose $X$ is of degree $d \geq 4$ and $x$ is a point on $X$. If $X$ contains no line passing through $x$, then

$$\epsilon(O_X(1), x) \geq \frac{d}{d-1}.$$  

(c) If $X$ is of degree $d \geq 4$ and $x \in X$ is a point such that the Seshadri constant $\epsilon(O_X(1), x)$ satisfies the inequalities $1 < \epsilon(O_X(1), x) < 2$, then it is of the form

$$\epsilon(O_X(1), x) = \frac{a}{b},$$

where $a, b$ are integers with $3 \leq a \leq d$ and $a/2 < b < a$.

(d) All rational numbers $a/b$ with $3 \leq a \leq d$ and $a/2 < b < a$ occur as local Seshadri constants of smooth irreducible surfaces in $\mathbb{P}^3$ of degree $d$.

Bauer’s approach was to consider the intersection of $X$ and the tangent plane $T_x X$ of $X$ at $x$. If the Seshadri constant $\epsilon(O_X(1), x) < 2$, then it must be computed by a component of the intersection and the multiplicity of the component at $x$ is bounded above by the degree $d$ of $X$. In this paper, we prove a similar result for any integral surfaces in $\mathbb{P}^3$.

**Proposition 1.3** Let $X$ be a projective integral surface of degree $d \geq 3$ in $\mathbb{P}^3$. If $x \in X$ is a point of multiplicity $m$ and $X$ contains no line passing through $x$, then

(a) $\epsilon(O_X(1), x) \geq \frac{d}{d-1}$. 

(b) If $\epsilon(O_X(1), x) < \frac{m+1}{m}$, then

$$\epsilon(O_X(1), x) = \frac{a}{b}$$

for some integers $a, b$ such that $3 \leq a \leq md$ and $\frac{ma}{m+1} < b \leq \frac{a(d-1)}{d}$.

By constructing a singular surface in $\mathbb{P}^3$, we also show that the lower bound $\frac{d}{d-1}$ here is optimal. Furthermore, we give the same lower bound for smooth threefolds in projective spaces.

**Theorem 1.4** Let $X$ be an irreducible smooth projective threefold of degree $d \geq 4$ in $\mathbb{P}^N$, $x$ a point in $X$ and $T_x X$ the tangent linear subspace of $\mathbb{P}^N$ to $X$ at $x$. If $X$ contains no line through $x$, then

(a) If $\dim(T_x X \cap X) = 0$, then

$$\epsilon(O_X(1), x) \geq 2.$$