Fluid approach to two-sided Markov-modulated Brownian motion

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Abstract: We extend to Markov-modulated Brownian motion (MMBM) the renewal approach which has been successfully applied to the analysis of Markov-modulated fluid models. It has recently been shown that MMBM may be expressed as the limit of a parameterized family of Markov-modulated fluid models. We prove that the weak convergence also holds for systems with two reflecting boundaries, one at zero and one at \( b > 0 \), and that the stationary distributions of the approximating fluid models converge to the stationary distribution of the two-sided reflected MMBM. Thus, we obtain a new representation for the stationary distribution, effectively separating the limiting behaviour of the process at the boundaries from its behaviour in the interior of \((0, b)\).

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1. Introduction

Since the beginning of the last century, Brownian motions have been an important class of stochastic processes, with applications in increasingly diverse areas such as biology, queueing theory, physics, environmental modeling, and mathematical finance. Naturally, the effectiveness of Brownian motions as modeling tools has lead to their many generalizations, one of which are the class of Markov-modulated Brownian motions (MMBMs), where the drift and variance are driven by an independent, continuous-time finite-state Markov chain. Thus, Markov-modulated Brownian motions are not only mathematically fascinating but also applicable for a wide variety of real-life applications.

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Traditionally, the stationary distribution of MMBMs has been analyzed mainly via the theory of generators of Markov processes in Rogers [12], partial differential equations in Karandikar and Kulkarni [8], the theory of martingales in Asmussen [1] and Asmussen and Kella [2], and generalized Jordan chains in D’Auria et al. [6]. Recently, there appeared a fifth approach, via an approximation by Markov-modulated fluid flows (MMFFs). First, Ramaswami [11] constructed a parameterized family of MMFFs that converge weakly to a standard Brownian motion. Then, Latouche and Nguyen [10] generalized this construction to approximate MMBMs, with and without a reflecting boundary at level zero; the authors showed that the stationary distributions of approximating fluid processes converge to the stationary distribution of the limiting reflected one-sided MMBM, assuming that the latter process is positive recurrent.

Here, we apply the fluid-based approximation approach to carry out the stationary analysis for reflected two-sided Markov-modulated Brownian motions, with boundaries at zero and at \( b > 0 \). This provides us with a new representation for the stationary distribution, obtained via a proof significantly different from the ones that rely on the theory of generators [12] or on time-reversal arguments [7]. The new representation indicates that the stationary density is a product of two terms, one of which is about the limiting behaviour at the interior \((0, b)\) and the other is about the limiting behaviour at the boundaries. This opens the way to the analysis of more complex models.

In Section 2, we formally define Markov-modulated Brownian motions and Markov-modulated fluid flows, and describe the fluid-based approximation in [10]. We show in Section 3 that the stationary distributions of the approximating processes converge to the stationary distribution of the MMBM, and we determine in Section 4 the closed-form expression for the limiting stationary distribution. In Section 5, we draw comparison between our representation and the ones derived in [12, 7].

2. Preliminaries

2.1. Markov-modulated models

A Markov-modulated Brownian motion \( Y = \{Y(t), \kappa(t) : t \geq 0\} \) is a continuous-time two-dimensional Markov-process, where the phase \( \kappa(t) \) is a Markov-chain on a finite state space \( \mathcal{M} = \{1, \ldots, m\} \), the level \( Y(t) \in (-\infty, \infty) \) is a Brownian motion with drift \( \mu_i \) and variance \( \sigma_i^2 \) whenever \( \kappa(t) = i \in \mathcal{M} \). We denote by \( D \) the drift matrix \( \text{diag}(\mu_1, \ldots, \mu_m) \), by \( V \) the variance matrix \( \text{diag}(\sigma_1^2, \ldots, \sigma_m^2) \), and by \( Q \) the generator of \( \kappa(t) \), which we assume to be irreducible. We also assume the following.

**Assumption 2.1.** The initial level \( Y(0) \) is zero, the initial phase \( \kappa(0) \) has the stationary distribution \( \alpha \) (that is, \( \alpha Q = 0, \alpha 1 = 1 \)), and \( \sigma_i^2 \neq 0 \) for all \( i \in \mathcal{M} \).

Markov-modulated Brownian motions are sometimes referred to as second-order fluid models; similarly, Markov-modulated fluid flows are also known as...
first-order fluid models. A Markov-modulated fluid flow \( L = \{L(t), \varphi(t) : t \geq 0\} \) is a continuous-time two-dimensional Markov process, where the phase \( \varphi(t) \) is a Markov chain on a finite state space \( S \), the level \( L(t) \in (-\infty, \infty) \) is independent of \( \varphi(t) \) and \( \frac{d}{dt}L(t) = c_i \) if \( \varphi(t) = i \in S \).

### 2.2. A fluid-based approximation

Given the Markov-modulated Brownian motion \( \mathcal{Y} = \{Y(t), \kappa(t)\} \) defined above, and given Assumption 2.1, we construct a parameterised family of fluid flows \( \{L_\lambda(t), \beta_\lambda(t), \varphi_\lambda(t) : t \geq 0\} \) as follows. The phase process here is a two-dimensional Markov chain \( \{\beta_\lambda(t), \varphi_\lambda(t)\} \) on state space \( S = \{(k,i) : k \in \{1,2\} \text{ and } i \in \mathcal{M}\} \), with generator

\[
T_\lambda = \begin{bmatrix}
Q - \lambda I & \lambda I \\
\lambda I & Q - \lambda I
\end{bmatrix},
\]

where the components of \( T_\lambda \) are indexed according to lexicographic ordering of \( \{1,2\} \times \mathcal{M} \), the parameter \( \lambda \) is positive, and \( I \) denotes the identity matrix of appropriate dimensions. Whenever ambiguity might arise, we write \( I_n \) to denote the \( n \times n \) identity matrix. The rate matrix \( C_\lambda = \text{diag}(c_{k,i})_{k \in \{1,2\}, i \in \mathcal{M}} \) for the level \( L_\lambda(t) \) is given by

\[
C_\lambda = \begin{bmatrix}
D + \sqrt{\lambda}\Theta & \lambda V \\
\lambda V & D - \sqrt{\lambda}\Theta
\end{bmatrix}, \quad \text{where } \Theta = \sqrt{V}.
\]

As \( Q \) is by assumption irreducible, so is \( T_\lambda \), and for sufficiently large values of \( \lambda \) the matrix \( C_\lambda \) is invertible.

**Assumption 2.2.** The initial level \( L_\lambda(0) \) is zero, \( \beta_\lambda(0) \) has the stationary distribution \( \gamma = (1/2, 1/2) \), and \( \varphi_\lambda(0) \) has the stationary distribution \( \alpha \).

Informally, we duplicate for the constructed fluid model the state space \( \mathcal{M} \) of the phase process \( \kappa(t) \), and keep track of each copy via \( \beta_\lambda(t) \in \{1,2\} \). The process \( \{\beta_\lambda(t), \kappa_\lambda(t)\} \) switches from a phase in a copy (say, \( (1,i) \)) to the corresponding phase in the other copy (which would be \( (2,i) \)) with rate \( \lambda \); the dynamic between phases in a copy is the same as that of the phase process \( \kappa(t) \), governed by \( Q \). As \( \lambda \) tends to infinity, \( (\beta_\lambda(t), \kappa_\lambda(t)) \) switches between two corresponding phases faster and faster, effectively scaling time. The matrix \( C_\lambda \) implies, that for the duplicated phases we modify the original drifts by an increment of \( \sqrt{\lambda}\Theta \) for one copy and by a decrement of the same quantity for the other copy. As \( \lambda \) tends to infinity, so does the difference between two drifts of corresponding phases, effectively scaling space. The combined scaling of space and of time is the underlying reason for the convergence of parameterised fluid flows to the Markov-modulated Brownian motion \( \{Y(t), \kappa(t)\} \). This weak convergence, proved in [10], is formally stated in the following theorem.

**Theorem 2.3** ([10]). Given Assumptions 2.1 and 2.2, the processes \( \{L_\lambda(t), \varphi_\lambda(t) : t \geq 0\} \) converge weakly to \( \{Y(t), \kappa(t) : t \geq 0\} \), as \( \lambda \to \infty \).
2.3. Reflected one-sided processes

We can construct a similar approximation by fluid flows for MMBMs with a reflecting boundary at level zero. Denote by \( \hat{Y} = \{ \hat{Y}(t), \kappa(t) : t \geq 0 \} \) the reflected one-sided MMBM associated with \( \{ Y(t), \kappa(t) : t \geq 0 \} \), where

\[
\hat{Y}(t) = Y(t) - \inf_{0 \leq v \leq t} Y(v),
\]

and by \( \hat{L}_\lambda = \{ \hat{L}_\lambda(t), \beta_\lambda(t), \varphi_\lambda(t) : t \geq 0 \} \) the resulting fluid flow if we introduce \( \{ L_\lambda(t), \beta_\lambda(t), \varphi_\lambda(t) \} \) a reflecting boundary at zero:

\[
\hat{L}_\lambda(t) = L_\lambda(t) - \inf_{0 \leq v \leq t} L_\lambda(v).
\]

By applying the one-sided reflection map to \( Y(t) \) and to \( L_\lambda(t) \), we know that the process \( \hat{Y}(t) \) exists uniquely and so does \( \hat{L}_\lambda(t) \). The following result immediately follows from Theorem 2.3.

**Corollary 2.4** ([10]). The processes \( \{ \hat{L}_\lambda(t), \varphi_\lambda(t) : t \geq 0 \} \) weakly converge, as \( \lambda \to \infty \), to the reflected one-sided MMBM \( \{ \hat{Y}(t), \kappa(t) : t \geq 0 \} \).

If the process \( \{ \hat{Y}(t), \kappa(t) \} \) is positive recurrent, that is, if \( \alpha D \mathbf{1} < 0 \), then the two limits, of \( \lambda \) and of \( t \), are interchangeable, and the limiting distribution of \( \{ \hat{L}_\lambda(t), \varphi_\lambda(t) \} \) converge, as \( \lambda \to \infty \), to the joint stationary distribution of \( \{ \hat{Y}(t), \kappa(t) \} \) [10, Theorem 3.6].

3. Reflected two-sided Markov-modulated Brownian motions

Here, we consider processes with not only a reflecting boundary at level zero but also one at level \( b \), for some finite \( b > 0 \). Let \( \tilde{Y} = \{ \tilde{Y}(t), \kappa(t) : t \geq 0 \} \) be the reflected two-sided MMBM associated with \( \{ Y(t), \kappa(t) : t \geq 0 \} \), where

\[
\tilde{Y}(t) = Y(t) + W(t) - M(t) \in [0, b],
\]

with \( W(t) \) and \( M(t) \) being the local times at level zero and level \( b > 0 \), respectively. More specifically, \( W(t) \) and \( M(t) \) are processes that satisfy the following conditions: \( W(t) \) and \( M(t) \) are nondecreasing with \( W(0) = M(0) = 0 \); \( \tilde{Y}(s) = 0 \) if \( W(s) < W(t) \) for all \( t > s \); and \( \tilde{Y}(s) = b \) if \( M(s) < M(t) \) of all \( t > s \). By applying the two-sided reflection map on \([0, b]\) (Kruk et al. [9]), we obtain existence and uniqueness for \( \tilde{Y}(t) \).

We denote by \( \tilde{L}_\lambda = \{ \tilde{L}_\lambda(t), \beta_\lambda(t), \varphi_\lambda(t) : t \geq 0 \} \) the finite-buffer fluid process associated with the unbounded process \( \{ L_\lambda(t), \beta_\lambda(t), \varphi_\lambda(t) \} \), where for \( \beta_\lambda(t) = k \in \{1, 2\} \) and \( \varphi_\lambda(t) = i \in \mathcal{M} \)

\[
\frac{d}{dt} \tilde{L}_\lambda(t) = c_{k,i} \quad \text{if } \tilde{L}_\lambda(t) \in (0, b),
\]

\[
= \max\{0, c_{k,i}\} \quad \text{if } \tilde{L}_\lambda(t) = 0,
\]

\[
= \min\{0, c_{k,i}\} \quad \text{if } \tilde{L}_\lambda(t) = b.
\]
In other words, in between the boundaries at 0 and at b the process \( \tilde{L}_\lambda(t) \) evolves the same way \( L_\lambda(t) \) does. Upon hitting level 0 (or level \( b \)), \( \tilde{L}_\lambda(t) \) remains there until the phase process \( \{ \beta_\lambda(t), \varphi_\lambda(t) \} \) switches to a phase with positive rate (or, respectively, negative rate). The process \( \tilde{L}_\lambda(t) \) can be obtained by applying the two-sided reflection map on \([0, b]\) to \( L_\lambda(t) \), and therefore exists uniquely. The following corollary follows immediately from Theorem 2.3.

**Corollary 3.1.** The processes \( \{ \tilde{L}_\lambda(t), \varphi_\lambda(t) : t \geq 0 \} \) weakly converge to the reflected two-sided MMBM \( \{ \tilde{Y}(t), \kappa(t) : t \geq 0 \} \), as \( \lambda \to \infty \).

**Assumption 3.2.** The mean drift \( \alpha D_1 \) of \( \{ Y(t), \kappa(t) \} \) is different from zero.

The mean drift of \( \{ \tilde{L}_\lambda(t), \varphi_\lambda(t) \} \) is \( \gamma \otimes \alpha C_\lambda 1 \) and it is straightforward to verify that \( \gamma \otimes \alpha C_\lambda 1 = \alpha D_1 \) independently of \( \lambda \).

In order to determine that the joint stationary distributions of approximating one-sided fluids \( \{ \tilde{L}_\lambda(t), \varphi_\lambda(t) \} \) converge to that of the one-sided MMBM \( \{ \tilde{Y}(t), \kappa(t) \} \), Latouche and Nguyen [10, Theorem 3.6] show that the limiting distribution is equivalent to the stationary distribution of \( \{ \tilde{Y}(t), \kappa(t) \} \) obtained in Asmussen [1]. Here, we follow a more direct approach to show convergence of stationary distributions of two-sided processes.

Denote by \( \tilde{F}_\lambda(x) \) the joint stationary distribution vector of \( \{ \tilde{L}_\lambda(t), \varphi_\lambda(t) \} \), with components

\[
[\tilde{F}_\lambda(x)]_i = \lim_{t \to \infty} P[\tilde{L}_\lambda(t) \leq x, \varphi_\lambda(t) = i] \quad \text{for } x \in [0, b] \text{ and } i \in M. \tag{1}
\]

and by \( \tilde{F}(x) \) its element-wise limit, where

\[
[\tilde{F}(x)]_i = \lim_{\lambda \to \infty} [\tilde{F}_\lambda(x)]_i \quad \text{for } x \in [0, b] \text{ and } i \in M. \tag{2}
\]

We prove in the next section that the limit \( \tilde{F}(x) \) defined in (2) exists. Here, to preserve the flow we assume its existence and show that this limit is indeed the stationary distribution \( \tilde{G}(x) \) of \( \{ \tilde{Y}(t), \kappa(t) \} \), where

\[
[\tilde{G}(x)]_i = \lim_{t \to \infty} P[\tilde{Y}(t) \leq x, \kappa(t) = i] \quad \text{for } x \in [0, b] \text{ and } i \in M.
\]

First, we extend Theorem 2.3 by modifying its assumptions that \( L_\lambda(0) = 0 \) and \( Y(0) = 0 \).

**Theorem 3.3.** Assume that \( (Y(0), \kappa(0)) \) has the distribution \( \tilde{F} \) and that \( (L_\lambda(0), \varphi_\lambda(0)) \) has the distribution \( \tilde{F}_\lambda \). The family of processes \( \{ L_\lambda(t), \varphi_\lambda(t) : t \geq 0 \} \) converge to the Markov-modulated Brownian motion \( \{ Y(t), \kappa(t) : t \geq 0 \} \), as \( \lambda \to \infty \).

**Proof.** First, we prove that the finite-dimensional distributions of \( \{ L_\lambda(t), \varphi_\lambda(t) \} \) converge to those of \( \{ Y(t), \kappa(t) \} \) via convergence of moment generating functions, that is, we show that

\[
\lim_{\lambda \to \infty} E[e^{s L_\lambda(t)} 1_{\{ \varphi_\lambda(t) = j \}} | \{ L_\lambda(0), \varphi_\lambda(0) \} = \tilde{F}_\lambda] = E[e^{s Y(t)} 1_{\{ \kappa(t) = j \}} | \{ Y(0), \kappa(0) \} = \tilde{F}] \tag{3}
\]
The marginal stationary distribution of the phase $\varphi_\lambda$ is $\alpha$, and we may write that
\[
E[e^{sL_\lambda(t)}1_{(\varphi_\lambda(t)=j)}|(L_\lambda(0),\varphi_\lambda(0))=d\bar{F}_\lambda]
=\sum_{i\in\mathcal{M}}\alpha_iE[e^{sL_\lambda(t)}1_{(\varphi_\lambda(t)=j)}|L_\lambda(0)=d\bar{F}_{\lambda|i},\varphi_\lambda(0)=i]
\]
where $\bar{F}_{\lambda|i}$ is the conditional stationary distribution of $L_\lambda$, given that the phase is $i$,
\[
=\sum_{i\in\mathcal{M}}\alpha_iE[e^{s(L_\lambda(t)+\bar{F}_{\lambda|i})}1_{(\varphi_\lambda(t)=j)}|L_\lambda(0)=0,\varphi_\lambda(0)=i]
\]
where $\bar{F}_{\lambda|i}$ is a random variable with distribution $\bar{F}_{\lambda|i}$,
\[
=\sum_{i\in\mathcal{M}}\alpha_iE[e^{s\bar{F}_{\lambda|i}}|E[e^{sL_\lambda(t)}1_{(\varphi_\lambda(t)=j)}|L_\lambda(0)=0,\varphi_\lambda(0)=i]]
=\alpha\Gamma_\lambda(s)e^{\Delta_\lambda(s)e_j}
\]
where $\Gamma_\lambda(s)$ is a diagonal matrix with $E[e^{s\bar{F}_{\lambda|i}}]$, $i\in\mathcal{M}$, on the diagonal, $e_j$ is an $m \times 1$ vector with zeros in all entries except the $j$th one, and $\Delta_\lambda(s)$ is the Laplace matrix exponent of $\{L_\lambda(t),\varphi_\lambda(t)\}$ which satisfies
\[
[e^{\Delta_\lambda(s)}]_{ij}=\mathbb{E}[e^{sL_\lambda(t)}1_{(\varphi_\lambda(t)=j)}|L_\lambda(0)=0,\varphi_\lambda(0)=i]
\]
for $i, j \in \mathcal{M}$. Similarly,
\[
E[e^{s\kappa(t)}1_{(\kappa(t)=j)}|(Y(0),\kappa(0))=d\bar{F}]=\alpha\Gamma(s)e^{\Delta_Y(s)e_j},
\]
where $\Gamma(s)$, by the definition of $\bar{F}$, is the limit of $\Gamma_\lambda(s)$ as $\lambda \to \infty$ and $\Delta_Y(s)$ is the Laplace matrix exponent of $\{Y(t),\kappa(t)\}$. In addition, Theorem 2.4 in [10] states that $\lim_{\lambda\to\infty}e^{\Delta_\lambda(s)e_j}=e^{\Delta_Y(s)e_j}$, and so (4, 5) imply (3).

Now, we show that the family $\{L_\lambda(t),\beta_\lambda(t),\varphi_\lambda(t)\}$ is still tight under the new initial condition. By Theorem 8.3 in Billingsley [3] and Whitt [13], it is sufficient to verify the following two conditions

(i) for each $\eta > 0$, there exists $a$ such that

$$\mathbb{P}[L_\lambda(0) > a] \leq \eta \quad \text{for sufficiently large } \lambda,$$

(ii) for each $\varepsilon, \eta > 0$, there exists $\delta \in (0, 1)$ and $\lambda_0$ such that

$$\frac{1}{\delta}\mathbb{P}\left[\sup_{t \leq s \leq t+\delta}|L_\lambda(s) - L_\lambda(t)| \geq \varepsilon\right] \leq \eta \quad \text{for all } \lambda \geq \lambda_0 \text{ and } t > 0.$$
Condition (ii) follows from the proof of Theorem 2.7 in [10], which show that the family \( \{L_\lambda(t), \beta_\lambda(t), \varphi_\lambda(t)\} \) is tight given that \( L_\lambda(0) = 0 \) and \( \varphi_\lambda(0) = a \). Condition (i) is immediately satisfied by setting \( a = b \), the upper reflecting boundary. \( \square \)

We are now ready to show that the two limits, of \( \lambda \) and of \( t \), are also interchangeable in the two-sided case. In other words,

**Theorem 3.4.** The limiting distribution of \( \{\tilde{L}_\lambda(t), \varphi_\lambda(t) : t \geq 0\} \) converges, as \( \lambda \to \infty \), to the stationary distribution of \( \{Y(t), \kappa(t) : t \geq 0\} \).

**Proof.** Let \( \{0 = t_0 \leq t_1 \leq t_2 \leq \cdots\} \) be a sequence of arbitrary time epochs. Assume that \( \{L_\lambda(0), \varphi_\lambda(0)\} \) has the distribution \( F_\lambda \). For \( x \in [0, b] \) and \( i \in M \)

\[
P[\tilde{L}_\lambda(t_k) \leq x, \varphi_\lambda(t_k) = i] = P[\tilde{L}_\lambda(0) \leq x, \varphi_\lambda(0) = i] \quad \text{for all } k \geq 0. \tag{7}
\]

On the other hand, Theorem 3.3 implies that for \( x \in [0, b] \) and \( i \in M \),

\[
P[\tilde{Y}(t_k) \leq x, \kappa(t_k) = i] = \lim_{\lambda \to \infty} P[\tilde{L}_\lambda(t_k) \leq x, \varphi_\lambda(t_k) = i]
= \lim_{\lambda \to \infty} P[\tilde{L}_\lambda(0) \leq x, \varphi_\lambda(0) = i]
= [F(x)]_i
\]

independently of \( t_k \). Thus, \( F \) is the stationary distribution of \( \{\tilde{Y}(t), \kappa(t)\} \). \( \square \)

### 4. Stationary distribution of two-sided MMBM

In light of Theorem 3.4, a key component for obtaining the stationary distribution of the two-sided MMBBM \( \{\tilde{Y}(t), \kappa(t)\} \) is the stationary distribution of the finite-buffer fluid process \( \{\tilde{L}_\lambda(t), \beta_\lambda(t), \varphi_\lambda(t)\} \).

For \( k = 1, 2 \) and \( i \in M \), let

\[
\pi_{k,i}^{(b)}(x) = \lim_{t \to \infty} \frac{d}{dx} P[\tilde{L}_\lambda(t) \leq x, \beta_\lambda(t) = k, \varphi_\lambda(t) = i], \quad \text{for } 0 < x < b,
\]

\[
p_{k,i}^{(0)} = \lim_{t \to \infty} \frac{d}{dx} P[\tilde{L}_\lambda(t) = 0, \beta_\lambda(t) = k, \varphi_\lambda(t) = i],
\]

\[
p_{k,i}^{(b)} = \lim_{t \to \infty} \frac{d}{dx} P[\tilde{L}_\lambda(t) = b, \beta_\lambda(t) = k, \varphi_\lambda(t) = i]
\]

be the stationary density function and probability masses at two boundaries of \( \{\tilde{L}_\lambda(t), \beta_\lambda(t), \varphi_\lambda(t)\} \), respectively. Also, define the stationary density vector

\[
\pi^{(b)}(x) = (\pi_{k,i})_{k \in \{1, 2\}, i \in M} = \begin{bmatrix} \pi_{+}^{(b)}(x) & \pi_{-}^{(b)}(x) \end{bmatrix},
\]

and the stationary probability mass vectors

\[
p^{(0)} = (p_{k,i}^{(0)})_{k \in \{1, 2\}, i \in M} \quad \text{and} \quad p^{(b)} = (p_{k,i}^{(b)})_{k \in \{1, 2\}, i \in M}.
\]
By their physical interpretations, \( p^{(0)} = (0, p^{(0)}_{-}) \) and \( p^{(b)} = (p^{(b)}_{+}, 0) \). Da Silva Soares and Latouche [5, Theorems 4.4 and 5.1] give a representation for the stationary density and probability masses at boundaries of a finite-buffer fluid model, given that the rates of the fluid level are restricted to ±1. We extend their results to the case with general rates, for which we require some notation and definitions.

Let us partition the generator matrix \( T_{\lambda} \) and the rate matrix \( C_{\lambda} \) according to phases with positive and negative rates as follows

\[
T_{\lambda} = \begin{bmatrix} T_{++} & T_{+-} \\ T_{-+} & T_{--} \end{bmatrix} \quad \text{and} \quad C_{\lambda} = \begin{bmatrix} C_{+} & C_{-} \end{bmatrix},
\]

For notational convenience when dealing with expansion of infinite series later, we write \( \lambda = 1/\varepsilon^2 \). Next, define the matrices

\[
U_{\varepsilon} = |C_{-}|^{-1}T_{--} + |C_{-}|^{-1}T_{-+}\Psi_{\varepsilon},
U_{\varepsilon}^{*} = C_{+}^{-1}T_{++} + C_{+}^{-1}T_{-+}\Psi_{\varepsilon}^{*},
K_{\varepsilon} = C_{+}^{-1}T_{++} + \Psi_{\varepsilon}|C_{-}|^{-1}T_{-+},
K_{\varepsilon}^{*} = |C_{-}|^{-1}T_{--} + \Psi_{\varepsilon}^{*}C_{+}^{-1}T_{--},
\]

where \( \Psi_{\varepsilon} \) is the minimal nonnegative solution to the Riccati equation

\[
C_{+}^{-1}T_{-+} + C_{+}^{-1}T_{++}\Psi_{\varepsilon} + \Psi_{\varepsilon}|C_{-}|^{-1}T_{-+} + \Psi_{\varepsilon}|C_{-}|^{-1}T_{++} = 0,
\]

and \( \Psi_{\varepsilon}^{*} \) is the minimal nonnegative solution to the Riccati equation

\[
|C_{-}|^{-1}T_{++} + |C_{-}|^{-1}T_{-+}\Psi_{\varepsilon}^{*} + \Psi_{\varepsilon}^{*}C_{+}^{-1}T_{++} + \Psi_{\varepsilon}^{*}C_{+}^{-1}T_{-+}\Psi_{\varepsilon}^{*} = 0.
\]

It is well-known that \( \Psi_{\varepsilon} \) and \( \Psi_{\varepsilon}^{*} \) have probabilistic interpretations: \( \Psi_{\varepsilon} \) records return probabilities from above to initial level in the boundary-free fluid process \( \{L_{\lambda}(t), \varphi_{\lambda}(t)\} \), and \( \Psi_{\varepsilon}^{*} \) records return probabilities from below to initial level.

Lemma 4.1 ([10]).

\[
\Psi_{\varepsilon} = I + \varepsilon \Theta_{1} + O(\varepsilon^2), \quad (8)
\Psi_{\varepsilon}^{*} = I + \varepsilon \Theta_{1}^{*} + O(\varepsilon^2), \quad (9)
\]

where \( \Theta^{-1}_{1} \) and \( -\Theta^{-1}_{1} \) are solutions to the matrix quadratic equation

\[
\frac{1}{2} VX^{2} + DX + Q = 0,
\]

such that

(i) if \( \alpha D 1 > 0 \), \( \Theta_{1}^{-1} \) has one eigenvalue equal to zero and \( m-1 \) eigenvalues with strictly negative real part, it is the unique such solution; \( -\Theta_{1}^{-1} \) has \( m \) eigenvalues with strictly positive real parts, it, too, is the unique such solution,
(ii) if $\alpha D_1 < 0$, $\Theta^{-1}\Psi_1$ has $m$ eigenvalues with strictly negative part, and $-\Theta^{-1}\Psi_1^*$ has $m-1$ eigenvalues with strictly positive real parts and one eigenvalue equal to zero; both $\Psi^{-1}\Psi_1$ and $-\Theta^{-1}\Psi_1^*$ are still unique such solutions.

**Remark 4.2.** Lemma 3.4 in [10] gives the eigenvalue characterisations of $\Theta^{-1}\Psi_1$ and $-\Theta^{-1}\Psi_1^*$ in the case when the mean drift $\alpha D_1$ is negative. We employ analogous reasoning to extend the results to the case $\alpha D_1 > 0$.

**Remark 4.3.** Let $\tau_{x}^{\pm} = \inf\{t > 0 : \pm X(t) > x\}$ be the first passage times to the corresponding levels $x$ and $-x$ of the unbounded process $Y(t)$. Under the assumption that $\sigma_i > 0$ for all $i \in \mathcal{M}$, it is easy to confirm that $\Theta^{-1}\Psi_1$ and $-\Theta^{-1}\Psi_1^*$ are the same as, respectively, the generators $\Lambda_-$ and $\Lambda_+^{**}$ of the time-changed processes $\kappa(\tau_{x}^{-})$ and $\kappa(\tau_{x}^{+})$ in Ivanovs [7], and $\Theta^{-1}\Psi_1^*$ is the same as the matrix $U(t)$ in Breuer [4].

Now we are ready to express the stationary density for the general case.

**Theorem 4.4.** The stationary density vector $\pi^{(b)}(x)$, for $0 < x < b$, of the finite-buffer fluid process $\{\bar{L}_\lambda(t), \beta_\lambda(t), \varphi_\lambda(t)\}$ is given by

$$\pi^{(b)}(x) = y \begin{bmatrix} e^{K_x}x & 0 \\ 0 & e^{K_x(b-x)} \end{bmatrix} \begin{bmatrix} C_+^{-1} & \Psi e^{-1} \\ \Psi e^{-1} & |C_+^{-1}| \end{bmatrix}$$

(10)

where

$$y = [y_+ y_-] = \begin{bmatrix} p^{(b)}_+ & p^{(b)}_0 \end{bmatrix} \begin{bmatrix} 0 & T_{+-} \\ T_{-+} & 0 \end{bmatrix} N^{-1}$$

(11)

and

$$N = \begin{bmatrix} I & e^{K_x}b \Psi e^{-1} \\ e^{K_x}b \Psi e^{-1} & I \end{bmatrix}.$$ 

The boundary probability masses $p^{(b)}_+, p^{(b)}_0$ satisfy the system of equations

$$\begin{bmatrix} p^{(b)}_+ & p^{(b)}_0 \end{bmatrix} W_\varepsilon = 0,$$

(12)

$$\begin{bmatrix} p^{(b)}_+ & p^{(b)}_0 \end{bmatrix} 1 + \int_0^b [\pi^{(b)}_+(x) \pi^{(b)}_-(x)] 1 dx = 1,$$

(13)

with

$$W_\varepsilon = \begin{bmatrix} T_{++} & 0 \\ 0 & T_{-+} \end{bmatrix} + \begin{bmatrix} 0 & T_{+-} \\ T_{-+} & 0 \end{bmatrix} G^{(b)},$$

(14)

where the matrix $G^{(b)}$ defined as

$$G^{(b)} = \begin{bmatrix} \Lambda^{(b)}_+ & \Psi^{(b)}_+ \\ \Psi^{(b)}_- & \Lambda^{(b)}_- \end{bmatrix}$$

(15)

is the solution of the system

$$\begin{bmatrix} \Lambda^{(b)}_+ & \Psi^{(b)}_+ \\ \Psi^{(b)}_- & \Lambda^{(b)}_- \end{bmatrix} \begin{bmatrix} I & \Psi e^{U_+}b \\ \Psi e^{U_-}b & I \end{bmatrix} = \begin{bmatrix} e^{U_+} & \Psi \\ \Psi e^{U_-}b & I \end{bmatrix}.$$ 

(16)
Proof. This is shown by adapting the proof of [5, Theorem 4.4] to the general case where the fluid rates may be different from 1 or \(-1\).

We give below another expression for the stationary distribution, which will be more convenient in the sequel.

**Corollary 4.5.** The stationary density vector \(\pi_{\varepsilon}(b)(x)\), for \(0 < x < b\), and the probability masses \(p_{+}(b)\) and \(p_{-}(0)\) may also be written as

\[
\pi_{\varepsilon}(b)(x) = c \left[ \nu_+ \quad \nu_- \right] \begin{bmatrix} e^{K_+x} & 0 \\ 0 & e^{K_- (b-x)} \end{bmatrix} \begin{bmatrix} C_+^{-1} & \Psi_{\varepsilon} |C_-|^{-1} \\ \Psi_{\varepsilon}^* C_+^{-1} & |C_-|^{-1} \end{bmatrix} N^{-1} \tag{17}
\]

and

\[
\begin{bmatrix} p_{+}(b) \\ p_{-}(0) \end{bmatrix} = c \left[ \nu_+ \quad \nu_- \right] \begin{bmatrix} -T_+^{-1} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -T_-^{-1} \end{bmatrix}, \tag{18}
\]

where the vector \(\nu = [\nu_+ \quad \nu_-]\) is the stationary probability vector of the matrix

\[
H = G(b) \begin{bmatrix} -T_+^{-1} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -T_-^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ T_+ \\ 0 \end{bmatrix}. \tag{19}
\]

and the scalar \(c\) is the normalizing constant defined by

\[
\begin{bmatrix} p_{+}(b) \\ p_{-}(0) \end{bmatrix} 1 + \int_0^b \begin{bmatrix} \pi_{+}(b)(x) \\ \pi_{-}(b)(x) \end{bmatrix} 1 \mathrm{d}x = 1. \tag{20}
\]

Proof. The proof is in two steps. Firstly, we show that the right-hand side of (18) is a solution of the system (12). Indeed,

\[
\begin{aligned}
\nu G(b) \begin{bmatrix} -T_+^{-1} \\ 0 \end{bmatrix} & W = -\nu G(b) + \nu G(b) \begin{bmatrix} -T_+^{-1} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ T_+ \end{bmatrix} \tag{21}
\end{aligned}
\]

by definition of \(\nu\). This proves (18), where \(c\) is some scaling constant.

Secondly, the vector \(y\) defined in (11) may be written as

\[
\begin{aligned}
y & = c \left[ \nu_+ \quad \nu_- \right] G(b) \begin{bmatrix} -T_+^{-1} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ T_+ \end{bmatrix} N^{-1} \\
& = c \nu H N^{-1} \\
& = c \nu N^{-1},
\end{aligned}
\]

which proves (17).

To prove Theorem 4.11 below, we analyse in a succession of lemmas the behaviour of the factors in (17) as functions of \(\varepsilon\).
Lemma 4.6. The matrices $K_\varepsilon$, $K_\varepsilon^*$, $U_\varepsilon$, $U_\varepsilon^*$ and the inverse of $N$ are such that

$$K_\varepsilon = K_0 + O(\varepsilon) \quad \text{with} \quad K_0 = \Psi_1 \Theta^{-1} + 2V^{-1}D,$$

$$K_\varepsilon^* = K_0^* + O(\varepsilon) \quad \text{with} \quad K_0^* = \Psi_1^* \Theta^{-1} - 2V^{-1}D,$$

$$U_\varepsilon = \Theta^{-1} \Psi_1 + \varepsilon(\Theta^{-1}Q + V^{-1}D\Psi_1) + O(\varepsilon^2),$$

$$U_\varepsilon^* = \Theta^{-1} \Psi_1^* + \varepsilon(\Theta^{-1}Q - V^{-1}D\Psi_1) + O(\varepsilon^2)$$

and

$$N^{-1} = \left[ \begin{array}{cc} I & e^{K_0 b}^{-1} \\ e^{K_0 b} & I \end{array} \right] + O(\varepsilon).$$

Proof. The expressions for $K_\varepsilon$ and $K_\varepsilon^*$ are from [10, Lemma 3.6] and a similar proof gives the expressions for $U_\varepsilon$ and $U_\varepsilon^*$. Equation (25) follows from (8, 9, 21, 22). \hfill \Box

The matrices $\Lambda^{(b)}_{++}, \Lambda^{(b)}_{-+}, \Lambda^{(b)}_{+-}, \Lambda^{(b)}_{-\cdot}$ all have probabilistic interpretations. We write $P_{\tau(x,k,i)\tau(y,k,j)}$ as shorthand for $P_{\tau(x,k,i)\tau(y,k,j)} = k, \varphi(x) = i$ for $k \in \{1, 2\}$ and $i \in \mathcal{M}$, and let $\tau_x = \inf\{t > 0 : L(t) = x\}$ be the hitting time to level $x$, for $x \in [0,b]$. Then, for $i, j \in \mathcal{M}$

$$\Lambda^{(b)}_{++}[i,j] = P_{\tau(z,x,1,1)}[\tau_1 < \infty, \tau_1 \tau_0 < \tau_2, \beta(x) = 1, \varphi(x) = j],$$

$$\Lambda^{(b)}_{-+}[i,j] = P_{\tau(z,x,1,2)}[\tau_1 < \infty, \tau_1 \tau_0 < \tau_2, \beta(x) = 2, \varphi(x) = j],$$

$$\Lambda^{(b)}_{+-}[i,j] = P_{\tau(z,x,2,1)}[\tau_1 < \infty, \tau_1 \tau_0 < \tau_2, \beta(x) = 2, \varphi(x) = j],$$

$$\Lambda^{(b)}_{-\cdot}[i,j] = P_{\tau(z,x,2,2)}[\tau_1 < \infty, \tau_1 \tau_0 < \tau_2, \beta(x) = 1, \varphi(x) = j].$$

Lemma 4.7. The matrix $G^{(b)}$ is given by

$$G^{(b)} = J + \varepsilon G^{(b)}_1 + O(\varepsilon^2)$$

where

$$J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad G^{(b)}_1 = \begin{bmatrix} L_1 & P_1 \\ \bar{P}_1 & \bar{L}_1 \end{bmatrix}$$

$$L_1 = (\Psi_1 - P_1)e^{-\Theta^{-1}\Psi_1 b},$$

$$\bar{L}_1 = (\Psi_1^* - \bar{P}_1)e^{-\Theta^{-1}\Psi_1^* b},$$

$$P_1 = (\Psi_1 e^{\Theta^{-1}\Psi_1 b} - e^{\Theta^{-1}\Psi_1 b})^{-1},$$

$$\bar{P}_1 = (\Psi_1^* e^{\Theta^{-1}\Psi_1^* b} - e^{\Theta^{-1}\Psi_1^* b})^{-1}.$$ 

Proof. By Lemma 4.1 and (23, 24),

$$\Psi_1 e^{U_\varepsilon b} = e^{U_\varepsilon b} + \varepsilon \Psi_1 e^{U_\varepsilon b} + O(\varepsilon^2) = e^{\Theta^{-1}\Psi_1 b} + \varepsilon \Upsilon_1 + O(\varepsilon^2),$$

where $\varepsilon \Upsilon_1 \to 0$ as $\varepsilon \to 0$, and

$$\Psi_1^* e^{U_\varepsilon^* b} = e^{U_\varepsilon^* b} + \varepsilon \Psi_1^* e^{U_\varepsilon^* b} + O(\varepsilon^2) = e^{\Theta^{-1}\Psi_1^* b} + \varepsilon \Upsilon_1^* + O(\varepsilon^2),$$
where \( \Upsilon^*_1 \to 0 \) as \( \varepsilon \to 0 \). Then, we find from the system (16), that the matrices \( \Lambda^{(b)}_{++}, \Lambda^{(b)}_{+-}, \Psi^{(b)}_+ \), and \( \bar{\Psi}^{(b)}_+ \) can be written as

\[
\Lambda^{(b)}_{++} = L_0 + \varepsilon L_1 + O(\varepsilon^2), \quad \Lambda^{(b)}_{+-} = \bar{L}_0 + \varepsilon \bar{L}_1 + O(\varepsilon^2),
\]

\[
\Psi^{(b)}_+ = P_0 + \varepsilon P_1 + O(\varepsilon^2), \quad \bar{\Psi}^{(b)}_+ = \bar{P}_0 + \varepsilon \bar{P}_1 + O(\varepsilon^2).
\]

This leads to a new system of equations, the first of which is

\[
L_0 + \varepsilon L_1 + O(\varepsilon^2) + \{P_0 + \varepsilon P_1 + O(\varepsilon^2)\}\{e^{\Theta^{-1}\Psi^*_1b} + \varepsilon \Upsilon^*_1 + O(\varepsilon^2)\}
\]

\[
= e^{U^*_1b} - \varepsilon \Psi + O(\varepsilon^2)\}
\]

\[
\Psi^*_b = \Psi^*_1 e^{U^*_1b} + \varepsilon \Upsilon^*_1 + O(\varepsilon^2),
\]

the second, similarly, is

\[
\{\bar{P}_0 + \varepsilon \bar{P}_1 + O(\varepsilon^2)\}\{e^{\Theta^{-1}\Psi^*_1b} + \varepsilon \Upsilon^*_1 + O(\varepsilon^2)\} + \bar{L}_0 + \varepsilon \bar{L}_1 + O(\varepsilon^2)
\]

\[
= e^{\Theta^{-1} \Psi^*_1b} - \varepsilon \Psi^*_1 e^{U^*_1b} + \varepsilon \Upsilon^*_1 + O(\varepsilon^2),
\]

and the third and fourth are

\[
\{L_0 + \varepsilon L_1 + O(\varepsilon^2)\}\{e^{\Theta^{-1}\Psi^*_1b} + \varepsilon \Upsilon^*_1 + O(\varepsilon^2)\} + P_0 + \varepsilon P_1 + O(\varepsilon^2)
\]

\[
= I + \varepsilon \Psi + O(\varepsilon^2),
\]

\[
\bar{P}_0 + \varepsilon \bar{P}_1 + O(\varepsilon^2) + \{\bar{L}_0 + \varepsilon \bar{L}_1 + O(\varepsilon^2)\}\{e^{\Theta^{-1}\Psi^*_1b} + \varepsilon \Upsilon^*_1 + O(\varepsilon^2)\}
\]

\[
= I + \varepsilon \Psi^*_1 + O(\varepsilon^2).
\]

We match coefficients for \( \varepsilon^0 \) in both sides of (31)–(34) to obtain

\[
L_0 + P_0 e^{\Theta^{-1}\Psi^*_1b} = e^{\Theta^{-1}\Psi^*_1b},
\]

\[
\bar{P}_0 e^{\Theta^{-1}\Psi^*_1b} + \bar{L}_0 = e^{\Theta^{-1}\Psi^*_1b},
\]

\[
L_0 e^{\Theta^{-1}\Psi^*_1b} + P_0 = I,
\]

\[
\bar{P}_0 + \bar{L}_0 e^{\Theta^{-1}\Psi^*_1b} = I.
\]

Equations (35) and (37) imply that \( L_0\{I - e^{\Theta^{-1}\Psi^*_1b}e^{\Theta^{-1}\Psi^*_1b}\} = 0 \). By the proof of [10, Lemma 3.4], the matrices \( \Theta^{-1}\Psi^*_1b \) and \( \Theta^{-1}\Psi^*_1b \) are generators, one of which is for a transient Markov chain. Thus, \( e^{\Theta^{-1}\Psi^*_1b}e^{\Theta^{-1}\Psi^*_1b} \) is sub-stochastic and \( \{I - e^{\Theta^{-1}\Psi^*_1b}e^{\Theta^{-1}\Psi^*_1b}\} \) is invertible, which give \( L_0 = 0 \) and \( P_0 = I \). Similarly, (38) and (36) imply \( \bar{L}_0 = 0 \) and \( \bar{P}_0 = I \).

Next, we equate coefficients for \( \varepsilon \) in both sides of (31)–(34) to obtain

\[
L_1 + P_1 e^{\Theta^{-1}\Psi^*_1b} = -\Psi^*_1 e^{\Theta^{-1}\Psi^*_1b},
\]

\[
\bar{P}_1 e^{\Theta^{-1}\Psi^*_1b} + \bar{L}_1 = -\Psi^*_1 e^{\Theta^{-1}\Psi^*_1b},
\]

\[
L_1 e^{\Theta^{-1}\Psi^*_1b} + P_1 = \Psi^*_1,
\]

\[
\bar{P}_1 + \bar{L}_1 e^{\Theta^{-1}\Psi^*_1b} = \Psi^*_1.
\]
As $e^{\Theta^{-1}\Psi x}$ and $e^{\Theta^{-1}\Psi t}$ are invertible, constraints (41) and (42) prove (27) and (28), respectively, whereas (29) follow from (27) and constraint (39), and (30) follow from (28) and constraint (40).

By Lemma 4.7, $\lim_{\varepsilon \to 0} G^{(b)} = J$, which makes sense probabilistically. As $\varepsilon \to 0$, $G^{(b)}$ converges to the matrix of hitting and returning probabilities in the limiting Markov-modulated Brownian motion $\{Y(t), \kappa(t)\}$. Thus, for example, $\lim_{\varepsilon \to 0} \Lambda^{(b)}_{1+} = 0$ implies that the conditional probabilities of $Y(t)$ hitting the upper boundary $b$ before returning to the initial level $0$ are zero.

**Lemma 4.8.** The vector $\nu$ is such that $\nu = \nu_0 + O(\varepsilon)$, where $\nu_0$ is the unique probability vector, solution of the system $\nu_0 G^{(b)}_1 = 0$, $\nu_0 1 = 1$.

**Proof.** We readily observe from the definition of the matrix $T$ that

$$H = G^{(b)} \begin{bmatrix} 0 & I + \varepsilon^2 Q + O(\varepsilon^4) \\ I + \varepsilon^2 Q + O(\varepsilon^4) & 0 \end{bmatrix} = I + \varepsilon G^{(b)}_1 J + O(\varepsilon^2)$$

so that $\nu$ is of the form $\nu = \nu_0 + \varepsilon \nu_1 + O(\varepsilon^2)$. If we equate the coefficients of equal power of $\varepsilon$ on both sides of $\nu = \nu H$, $\nu 1 = 1$, we find that

$$\nu_0 = \nu_0, \quad \nu_1 = \nu_1 + \nu_0 G^{(b)}_1 J, \quad \nu_0 1 = 1,$$

or

$$\nu_0 G^{(b)}_1 J = 0, \quad \nu_0 1 = 1.$$  

Since $J$ is nonsingular and $J^2 = I$, we may rewrite the system above as

$$\nu_0 G^{(b)}_1 = \nu_0 J G^{(b)}_1 = 0, \quad \nu_0 1 = \nu_0 J 1 = 1. \quad (43)$$

Now, the matrix

$$J G^{(b)}_1 = \begin{bmatrix} \tilde{P}_1 & \tilde{L}_1 \\ L_1 & P_1 \end{bmatrix}$$

is an irreducible generator, as we show below, and this entails that the system $x J G^{(b)}_1 = 0$, $x 1 = 1$ has a unique solution, so that the lemma will be proved.

Since $G^{(b)}$ is stochastic, we conclude from (26) that $L_1$ and $\tilde{L}_1$ are both nonnegative, as well as all the off-diagonal elements of $P_1$ and of $\tilde{P}_1$. As $G^{(b)} 1 = 1$, $G^{(b)}_1 1 = 0$ and the diagonal elements of $P_1$ and of $\tilde{P}_1$ must be less than, or equal to zero. Finally, as $G^{(b)}$ is irreducible, all diagonal elements of $P_1$ and $\tilde{P}_1$ must be strictly negative.

**Lemma 4.9.** The last factor in (17) is

$$\begin{bmatrix} C_-^{-1} & \Psi \xi C_-^{-1} \\ \Psi \xi C_+^{-1} & |C_-|^{-1} \end{bmatrix} = \varepsilon \begin{bmatrix} \Theta^{-1} & \Theta^{-1} \\ \Theta^{-1} & \Theta^{-1} \end{bmatrix} + O(\varepsilon^2). \quad (44)$$

**Proof.** By the definition of $C$, we have

$$C_+^{-1} = (1/\varepsilon \Theta + D)^{-1} = \varepsilon \Theta^{-1} + O(\varepsilon^2)$$

and similarly $C_-^{-1} = -\varepsilon \Theta^{-1} + O(\varepsilon^2)$. To conclude the proof, we use (8, 9). 

□
Lemma 4.10. The normalizing constant $c$ in (17) is of the form
\[ c = \varepsilon^{-1}c_{-1} + c_0 + O(\varepsilon). \] (45)

Proof. The exact expression of $c_{-1}$ is not as important as the form of the right-hand side in (45) and we shall omit the details in the argument below. By (20),
\[ e^{-1} = \nu G^{(b)} \left[ \begin{array}{cc} -T_{++}^{-1} & 0 \\ 0 & -T_{--}^{-1} \end{array} \right] \nu_0 \]
\[ + \nu N^{-1} \int_0^b \left[ \begin{array}{cc} e^{K_{+,x}} & 0 \\ 0 & e^{K_{-,x}(b-x)} \end{array} \right] dx \left[ \begin{array}{cc} C_{++}^{-1} & \Psi_x C_{+-}^{-1} \\ \Psi_x^* C_{-+}^{-1} & |C_-|^{-1} \end{array} \right] \nu_0. \] (46)
By Lemmas 4.8, 4.9 and 4.6, the first term in (46) is $O(\varepsilon^2)$ and the second is $O(\varepsilon)$, thus $c^{-1}$ is $O(\varepsilon)$ and this justifies (45).

We may now bring together all our partial results.

Theorem 4.11. The stationary density of the two-sided Markov-modulated Brownian motion $\{(\tilde{Y}_t(t), \kappa(t))\}$ is given by
\[ \lim_{\varepsilon \to 0} \pi^{(b)}_x(x)(1_2 \otimes I_m) = c^* \nu_0 \left[ \begin{array}{cc} I & e^{K_{0b}} \\ e^{K_{0b}} & I \end{array} \right]^{-1} \left[ \begin{array}{cc} e^{K_{0x} \Theta^{-1}} & \Psi_x C_+^{-1} \\ \Psi_x^* C_-^{-1} & |C_-|^{-1} \end{array} \right] \nu_0, \] (47)
for $x \in (0,b)$, where $\nu_0$ is the unique probability vector that is solution of the system $\nu_0 G^{(b)}_1 = 0$, $\nu_0 1 = 1$, and $c^*$ is a normalizing constant.

The probability masses of $\{(\tilde{Y}_t(t), \kappa(t))\}$ at the two boundaries are zero.

Proof. This is a direct consequence of Lemmas 4.7 to 4.10.

Remark 4.12. Theorem 4.11 shows that the stationary density is made up of two components: the factor $\nu_0$ is about the limiting behaviour of $\{(\tilde{Y}_t(t), \kappa(t))\}$ at the boundaries, the matrix product is about its limiting behaviour in the interior $(0,b)$. This factorization implies that to modify the boundary behaviour of $\{(\tilde{Y}_t(t), \kappa(t))\}$ would affect the vector $\nu_0$ only.

5. Comparison with existing literature

Section 3.2 of Ivanovs [7] shows that, under assumption of all variances being positive, both [7] and [12] obtained the same stationary density of the two-sided Markov-modulated Brownian motion $\tilde{Y}(t)$ conditioned on the phase $\kappa(t)$:
\[ [f(x)]^T = \lim_{t \to \infty} \frac{d}{dx} P[\tilde{Y}(t) \leq x|\kappa(t)], \]
\[ = -\{e^{\tilde{\Pi}_+ + e^{(b-x)\tilde{\Pi}_+} \tilde{\Pi}_- e^{b \tilde{\Pi}_+}}(I - e^{b \tilde{\Pi}_+} e^{b \tilde{\Pi}_+})^{-1} \}, \] (48)
where $\tilde{\Pi}_+$ and $\tilde{\Pi}_-$ are respectively the generators of first passage times to level $x$ and level $-x$ in $\{\tilde{Y}(t), \pi(t) : t \geq 0\}$, the time-reversed version of the unbounded
MMBM \{Y(t), \kappa(t)\}. Each is a solution to one of the two matrix quadratic equations
\[
\frac{1}{2} V X^2 + DX + \Delta \frac{1}{\alpha} Q^\top \Delta = 0.
\] (49)

The proof of Theorem 3.7 in \[10\] gives a relationship between \(K_0\) and \(\Omega\):\[
\Omega^\top + \Delta_1/\alpha = \Delta \Theta K_0 \Theta^{-1} \Delta_1/\alpha.
\]
and, similarly, we also have \(\Omega_{-}^\top + \Delta_1/\alpha = \Delta \Theta K_0^* \Theta^{-1} \Delta_1/\alpha\). Thus, the conditional stationary density (48) can be rewritten as
\[
f(x) \Delta_\alpha = -\alpha \Theta(I - e^{bK_0} e^{bK_0^{*}})^{-1} (K_0 e^{xK_0} + e^{bK_0} K_0^* e^{(b-x)K_0^*}) \Theta^{-1} \Delta_1/\alpha,
\] (50)
and thus the joint stationary density for \{\tilde{Y}(t), \kappa(t)\} is given by
\[
f(x) \Delta_\alpha = -\alpha \Theta(I - e^{bK_0} e^{bK_0^{*}})^{-1} (K_0 e^{xK_0} + e^{bK_0} K_0^* e^{(b-x)K_0^*}) \Theta^{-1} \Delta_1/\alpha,
\]
which coincides with our (47) if
\[
-\alpha \Theta \begin{bmatrix} I - e^{bK_0} e^{bK_0^*} \end{bmatrix}^{-1} \begin{bmatrix} K_0 \\ e^{bK_0} K_0^* \end{bmatrix} = e^* \nu_0 \begin{bmatrix} I \\ e^{bK_0} \end{bmatrix}^{-1}.
\]
This is shown through tedious algebraic manipulations.

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