Condensate of a charged boson fluid at non-integer dimensions

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Abstract

Condensate of a charged boson fluid at non-integer dimensions between 2 and 3 is studied. Interaction between particles is assumed to be Coulombic, and Bogoliubov approximation is applied for a weak coupling regime. The condensate and the superfluid fraction at finite temperatures and at non-integer dimensions are calculated. The theoretical results are compared with the superfluid densities of superconducting films, which show a universal splitting behavior. The qualitative similarity between the charged boson fluid and the superconducting films gives a strong clue for the origin of the universality.

Key words: charged boson fluid, condensate fraction, low-dimensional system
I. INTRODUCTION

Charged boson fluid (CBF) has been attracting attention as an interesting model of many physical systems at low temperatures. Several theoretical and computational studies have been carried out on the 2D and the 3D CBF at zero and finite temperatures [1-4]. Also, a general expression for the superfluid density of dilute Bose gas for \( D > 2 \) was given by Fisher and Hohenberg [5].

It has been, recently, noted that porous media and films can be studied using fractal dimensionality between 2 < \( D < 3 \) [6]. This study also revealed that several prominent features of porous media originates from dimensionality and detailed nature of mutual interaction plays a much less prominent role.

In this paper, we study the condensate and the superfluid density of CBF at finite temperatures in the dimensionality between 2 < \( D < 3 \). Then, the result will be compared with the experimental data from superconducting thin films. It will be shown that the nature of gap states and detailed property of superconducting mechanism plays a much less prominent role than that of the geometrical factors.

We will consider a model of point-like spinless charged bosons embedded in a uniform neutralizing background and coupled by a Coulomb type interaction. The Bogoliubov approximation will be applied for the model Hamiltonian in weakly interacting regime to calculate the condensate. The condensate fraction will be calculated as a function of temperature and dimensionality between 2 and 3. Then, the result is utilized to calculate the superfluid fraction with the help of the \( D \)-dimensional Fisher and Hohenberg formula [5].

Finally, the theoretical results on the charged bosons in fractal dimensions between 2 and 3 will be compared to experimental data on superfluid density of superconducting films and the physical implications will be discussed.
II. BOGOLIUBOV APPROACH TO A CHARGED BOSON FLUID

We begin our theoretical scheme from the well-established Bogoliubov approach for the CBF on a uniform neutralizing background and generalize to non-integer dimensions. The model Hamiltonian is given by

$$H = \int d\mathbf{r} \psi^\dagger(\mathbf{r}) \left( -\frac{\nabla^2}{2m} - \mu \right) \psi(\mathbf{r}) + \frac{1}{2} \int \int d\mathbf{r} d\mathbf{r}' \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') V(|\mathbf{r} - \mathbf{r}'|) \psi(\mathbf{r}') \psi(\mathbf{r}),$$

where $V(|\mathbf{r} - \mathbf{r}'|) = Q^2/|\mathbf{r} - \mathbf{r}'|$, $m$ is the effective mass, and $Q$ is the effective charge of the CBF. $\psi(\mathbf{r})$ is the boson field operator and $\mu$ is the chemical potential. $\hbar = c = 1$ is assumed for convenience.

D-dimensional coupling strength is measured by the dimensionless parameter $r_s$ defined by $A r_s^D = (n a_B^D)^{-1}$. A large $r_s$ means a long range of interaction and a small density of particles. $A$ is the D-dimensional volume parameter given by $A = 2\pi^{D/2} D^{-1} \Gamma(D/2)^{-1}$ and $a_B$ is the effective Bohr radius of the system defined by $a_B = 1/mQ^2$.

We first apply the Bogoliubov transformation at $T=0$. The field operator is written as

$$\psi(\mathbf{r}, t) = \psi_0 + \tilde{\psi}(\mathbf{r}, t),$$

where $\psi_0$ is a macroscopic order parameter representing the condensate. $\tilde{\psi}$ describes the particles promoted out of the condensate and can be expressed as a linear transformation

$$\tilde{\psi}(\mathbf{r}, t) = \sum_k \left[ u_k(\mathbf{r}, t) a_k + v_k^\dagger(\mathbf{r}, t) a_k^\dagger \right],$$

where $a_k(a_k^\dagger)$ is the bosonic quasi-particle operator. Also, we have $\mu = 0$, and $n_0 = \psi_0^2$ for the uniform condensate.

A straightforward calculation produces the following results in D-dimensions

$$u_k^2 = \frac{1}{2} \left[ \epsilon_k^{-1} \left( n_0 U_D(k) + \frac{k^2}{2m} \right) + 1 \right],$$

$$v_k^2 = \frac{1}{2} \left[ \epsilon_k^{-1} \left( n_0 U_D(k) + \frac{k^2}{2m} \right) - 1 \right],$$

where
\[
\epsilon_k = \left\{ \frac{n_0 k^2 U_D(k)}{m} + \left( \frac{k^2}{2m} \right)^2 \right\}^{1/2}.
\] (6)

Here, \( \epsilon_k \) is the energy spectrum of the Bogoliubov transformation. Note that \( u_k^2 - v_k^2 = 1 \). \( U_D(k) \) is the Fourier transformation of the \( D \)-dimensional coupling \( V(r) \) and given by
\[ U_D(k) = D A Q^2 / k^{D-1}. \]

Note that \( U_3(k) = 4\pi Q^2 / k^2 \), and \( U_2(k) = 2\pi Q^2 / k \) as expected.

### III. Condensate Fraction at Finite Temperatures and Non-integer Dimensions

The extension of the theory to finite temperatures is effected by the Bose distribution function,
\[ f_k = 1 / \{ \exp(\beta \epsilon_k) - 1 \}. \]
The condensate density at nonzero temperature in \( D \)-dimensions is given by
\[
n_0 = n - \langle \psi^\dagger(r) \psi'(r) \rangle
= n - \sum_{k \neq 0} \left[ v_k^2 + f_k (u_k^2 + v_k^2) \right].
\] (7)

Substituting Eqs. (4), (5), and (6) into Eq. (7), we obtain the condensate fraction in \( D \)-dimensions as follows
\[
1 - \frac{n_0}{n} = \frac{1}{n} \sum_{k \neq 0} \left[ v_k^2 + f_k (u_k^2 + v_k^2) \right]
= \frac{1}{2n} \int_0^\infty dk \frac{DAk^{D-1}}{(2\pi)^D} \left[ \frac{n_0 U_D(k) + k^2}{m} \left( 1 + \frac{2}{e^{\beta \epsilon_k} - 1} \right) - 1 \right].
\] (8)

Eq. (8) can be converted into a compact form after some mathematical steps,
\[
\left( 1 - \frac{n_0}{n} \right) \left( \frac{n_0}{n} \right)^{-D/(D+1)} = F(D) I(T, D).
\] (9)

\( F(D) \) is the coefficient which is defined by
\[
F(D) = \frac{3}{D+1} \frac{2^{(-D^2+3D+2)/(D+1)} r_s^{D/(D+1)}}{D(D+2)/(D+1)} \frac{\Gamma^2(D/2)}{\Gamma^2(D/2)}. \] (10)

\( I(D, T) \) is the integral which is defined by
\[ I(D, T) = \int_0^\infty dx \, x^{-(D-2)(D-3)/D(D+1)} \left[ \frac{p(x)}{q(x)} \left( 1 + \frac{2}{e^{C(D,T)g(x)}q(x) - 1} \right) - 2x^{6/D} \right], \quad (11) \]

where

\[ p(x) = 1 + 2x^{12/D}, \quad (12) \]

\[ q(x) = \sqrt{1 + x^{12/D}}, \quad (13) \]

\[ C(D, T) = \left( \frac{Dn_0}{n} \right)^{2/(D+1)} \frac{2^{(3-D)/(D+1)}}{r_s^{2D/(D+1)}} \frac{Q^2}{a_Bk_BT}, \quad (14) \]

and

\[ g(x) = x^{6(3-D)/D(D+1)}. \quad (15) \]

The temperature dependence is enclosed in the function \( C(D, T) \). The integral \( I(D, T) \) is performed numerically.

The condensate fraction in D-dimensions is plotted as a function of temperature in FIG. 1. The unit of the temperature is \( Q^2/a_Bk_B \). The three coupling strengths were chosen for the calculation: (a) \( r_s = 0.8 \) for a weak coupling and (b) \( r_s = 1.2 \) for an intermediate coupling, and (c) \( r_s = 2.0 \) for a strong coupling, respectively. We see that the condensate fraction depends strongly on the interaction strength, \( r_s \). We also observe that for all the different values of the coupling strength, the condensate fractions have significant nonzero values and are relatively flat near \( T \sim 0K \) region.

\[ IV. \text{SUPERFLUID FRACTION AT FINITE TEMPERATURES AND NON-INTEGER DIMENSIONS} \]

In order to compare the theory with available experimental data on superconducting films, it is necessary to obtain the superfluid fraction which is directly related to the penetration depth of superconducting films. The superfluid density of dilute Bose gas in \( D \)-dimensions has been studied by Fisher and Hohenberg \( \text{s} \). It is given by a Landau quasiparticle formula based on the \( D \)-dimensional Bogoliubov energy spectrum \( \epsilon_k \) in Eq. \( \text{s} \) as
\[
\frac{n_s}{n} = 1 - \frac{\beta}{2nm} \sum_{k \neq 0} k^2 \frac{e^{\beta \epsilon_k}}{(e^{\epsilon_k} - 1)^2}.
\] (16)

Substitution of Eq. (6) into the Eq. (16) gives an explicit expression for the D-dimensional superfluid fraction,

\[
\frac{n_s}{n} = 1 - H(D, T) \int_0^\infty dx x^{D+1} \frac{e^{B(D, T)\sqrt{x^{1-D}+x^4}}}{\left\{e^{B(D, T)\sqrt{x^{1-D}+x^4}} - 1\right\}^2}.
\] (17)

\(B(D, T)\) and \(H(D, T)\) are defined by the following expressions:

\[
B(D, T) = 2^{(3-D)/(D+1)} \left(\frac{Dn_0r_s}{n}\right)^{2/(D+1)} \frac{Q^2}{a_Bk_BT},
\] (18)

and

\[
H(D, T) = \frac{A^2D^{(2D+1)/(D+1)}}{(2\pi)^D} \left(\frac{4n_0r_s}{n}\right)^{D/(D+1)} B(D, T).
\] (19)

The condensate fraction \(n_0/n\) of Eq. (9) is substituted into Eqs. (18) and (19) for the calculation of the superfluid fraction \(n_s/n\).

The \(D\)-dimensional superfluid fraction is obtained from Eq. (17), and plotted as a function of temperature for three different values of the coupling strength in FIG. 2. The same \(r_s\) values as in FIG. 1 were chosen for the comparison.

We find the basic structure between the condensate fraction in FIG. 1 and the superfluid fraction in FIG. 2 is similar for any coupling strength, \(r_s\). Therefore, it is clear that condensate density gives a strong and useful hint for the superfluid density for the CBF system.

V. SUPERFLUID DENSITY OF SUPERCONDUCTING FILMS

Measurements on the temperature dependent superfluid density of superconducting films \(La_{2-x}Sr_xCuO_4\) and \(Mo_7Ge_{23}\) drew considerable interests, in connection with the puzzling splitting behaviors shown in FIG. 3. In FIG. 3 we note that \(1/\lambda^2\) is proportional to the superfluid density, and inverse of the sheet inductance, \(1/L(T)\), is also proportional to the areal superfluid density, \(n_s d\).
We observe that the two figures in FIG. 3 exhibit the same generic splitting behavior for samples with various thickness. Considering differences in the detailed physical properties between LSCO and MoGe films, such qualitative similarity is quite surprising and, thus, it strongly implies that the generic behavior stems from some hidden common properties.

There have been several theoretical efforts to explain the general features of the superfluid density of superconducting films. It has been shown that the simple BCS and the d-wave symmetry just lead into a quadratic dependence of \(1/\lambda^2(T)\) \[9,10,13\] and, thus, can not explain the generic splitting behavior. Effect of thermal fluctuations is shown to explain the relative flat behavior of curves at low temperatures \[13\]. But it could not also explain the above mentioned behavior.

Since the superconducting mechanisms of LSCO is believed to be basically different from that of MoGe, any theory which explains the behavior should not be based on the microscopic details of the superconducting mechanisms of the two materials. Instead, it should mainly reflect the common geometric nature of superconducting films.

It has been shown that condensation of bound pairs can be used to explain the superconductivity with proper scaling \[14,15\]. In the previous section, we have shown that the superfluid density, which originates from the charged boson condensation in fractal dimensions, \(2 < D < 3\), exactly duplicates the observed experimental data from the superconducting films. Therefore, we believe that the theoretical results in FIG. 2 are the natural explanation of the experimental observations in FIG. 3.

We note also that the general features of the superfluid density curves of the superconducting films are qualitatively similar to those of liquid helium-4 in films and porous media \[16–19\].

VI. CONCLUSIONS

In this paper, we have studied the condensate of charged boson fluid at finite temperatures in non-integer dimensions between 2 and 3. The condensate and superfluid fraction
are obtained as functions of temperature and dimensions at various values of the coupling strength.

We have shown that the generic splitting behavior of the superfluid density universal for superconducting films may originate from the geometric nature of the films and detailed nature of the superconducting mechanism plays a much less prominent role than generally believed.

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REFERENCES

[1] Cherny A Y and Shanenko A A 1998 *Phys. Lett. A* **250** 170

[2] Moroni S, Conti S and Tosi M P 1996 *Phys. Rev. B* **53** 9688

[3] Alexandrov A S, Beere W H and Kabanov V V 1996 *Phys. Rev. B* **54** 15363

[4] Strepparola E, Minguzzi A and Tosi M P 2001 *Phys. Rev. B* **63** 104509

[5] Fisher D S and Hohenberg P C 1988 *Phys. Rev. B* **37** 4936

[6] Kim S-H, Kim C K and Nahm K 1999 *J. of Phys.-CM* **11** 10269

[7] Magro W R and Ceperley D M 1974 *Phys. Rev. Lett.* **73** 826

[8] Andreone A, Cassinese A, Di Chiara A and Vaglio R 1994 *Phys. Rev. B* **49** 6392

[9] Paget K M, Boyce B R and Lemberger T R 1999 *Phys. Rev. B* **59** 6549

[10] Paget K M, Guha S, Cieplak M Z and Trofimov I E 1999 *Phys. Rev. B* **59** 641

[11] Karpinska K, Cieplak, Guha S and Maliowski 2000 *Phys. Rev. Lett.* **84** 155

[12] Turneaure S J, Lemberger T R and Graybeal J M 2000 *Phys. Rev. Lett.* **84** 987

[13] Alexandrov A S and Gills R T 1999 *Physica C* **325** 35

[14] Lemberger T R, Pesetski A A, and Turneaure S J 2000 *Phys. Rev. B* **61** 1483

[15] Schafroth M R 1954 *Phys. Rev.* **96** 1149

[16] Brewer D F 1970 *J. Low Temp. Phys.* **3** 205

[17] Bretz M 1973 *Phys. Rev. Lett.* **31** 1447

[18] Finotello D, Gills K A, Wong A and Chan M H W 1988 *Phys. Rev. Lett.* **61** 1954

[19] Steele L M and Finotello D 1992 *J. Low Temp. Phys.* **89** 645
FIGURES

FIG. 1. The condensate fraction of the charged bosons in non-integer dimensions. From top to bottom $D = 3.0, D = 2.8, D = 2.6, D = 2.4$ (a) When $r_s = 0.8$, (b) when $r_s = 1.2$, and (c) when $r_s = 2.0$. The unit of the temperature is $Q^2/a_B k_B$.

FIG. 2. The superfluid fraction of the charged bosons in non-integer dimensions. From top to bottom $D = 3.0, D = 2.8, D = 2.6, D = 2.4$ (a) When $r_s = 0.8$, (b) when $r_s = 1.2$, and (c) when $r_s = 2.0$. The unit of the temperature is $Q^2/a_B k_B$.

FIG. 3. (a) Superfluid fraction of $La_{2-x}Sr_xCuO_4$ films measured by K. M. Paget et al.[9] The transition temperatures are between $20K - 30.5K$, and film thickness are between $5000\AA - 900\AA$. The film thickness decreases from top to bottom. (b) Superfluid fraction of the $Mo_{77}Ge_{23}$ superconducting films measured by S. J. Turneaure et al.[12] The transition temperatures are between $2.999K - 7.050K$, and film thickness are between $500\AA - 20\AA$ from top to bottom.
$r_s=0.8$ (a)
$n_0/n \quad (c)$

$T = 2.0$ (c)
\( r_s = 0.8 \) (a)
Temperature

\( r_s = 1.2 \) (b)
$r_s = 2.0$ (c)
