Chaos in Static Axisymmetric Spacetimes II : non-vacuum case

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Abstract

We examine the effect of local matter on the chaotic behavior of a relativistic test particle in non-vacuum static axisymmetric spacetimes. We find that the sign of the sectional curvature in the geodesic deviation equation defined by the Riemann curvature does not always become a good tool to judge the occurrence of chaos in the non-vacuum case. However, we show that the locally unstable region (LU region) defined by the Weyl curvature can provide information about chaos even in non-vacuum spacetime as well as in vacuum spacetime. Since the Weyl tensor affects only the shear part of the geodesic congruence, it works effectively to stretch some directions of geodesic congruence, which helps to cause the chaotic behavior of geodesics. Actually, the orbit moving around an unstable periodic orbit (UPO) becomes strongly chaotic if it passes through an LU region, which means that the LU region can be used as a good tool to know in which situation the chaos by homoclinic mixing occurs around a UPO.

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1 Introduction

General Relativity (GR) is the most plausible theory of gravitation to explain relativistic astrophysical phenomena in our universe. It has succeeded in explaining many observational results in cosmology and astronomy by using idealized models such as Friedmann Robertson-Walker (FRW) spacetime or Schwarzschild spacetime.

However these kinds of idealized models cannot cover all of the astrophysical phenomena which might actually happen in our universe. Chaos may be one of these phenomena because these idealizations usually force a strong symmetry on spacetime, which extinguishes the possibility of complicated behavior such as chaos. In a Hamiltonian system chaos can occur only in the non-integrable system where the number of integrals is less than the dimension of configuration space [1].

For example, a Bianchi-IX universe, one of the spacetimes which is generalized from a closed FRW universe by dropping isotropy, is known to show stochastic behavior [2]. However, there has been much discussion about the definition of chaos independent of the choice of time coordinate [3].

As for test particle motion in GR, the Kerr-Newman spacetime, which includes the Schwarzschild, Kerr and Reissner-Nordström spacetimes as special cases, is known to be integrable. This can be explained by the fact that it is a Petrov type D spacetime and in addition to the energy $E$, angular momentum $L$ and super-hamiltonian $H \equiv \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$, there is a fourth conservative quantity described by the Killing tensor [4]. Thus bound orbits in this spacetime are strictly constrained to a torus in the phase space, and none of them behaves chaotically. However, in realistic situations the spacetime around compact objects such as a black hole or a neutron star is not expected to have such a high degree of symmetry, because of the distortion of the gravitational source itself or of the effect of other gravitational sources such as gravitational waves, magnetic fields or other astronomical objects. There is a possibility that the deviations of these spacetimes from Petrov D extinguish the Killing tensor and cause strong chaos in the test particle motion around those compact objects [5].

In fact more generic spacetimes are usually non-integrable and some of them exhibit chaotic behavior in test particle motions [6]-[12]. In GR, an unstable periodic orbit (UPO) exists in axisymmetric and static spacetime in contrast with Newtonian mechanics. This is one of the main factors which causes chaos for geodesics in the spacetime. For example, chaos is caused by periodic perturbations around a UPO in an integrable static and axisymmetric
spacetime, such as the Schwarzschild spacetime [10][11]. Although Schwarzschild spacetime is static, spherically symmetric and integrable, the perturbations extinguish the time symmetry or axisymmetry. This type of chaos can be explained by the homoclinic tangle around a UPO [13]. Such chaos can also occur by non-periodic perturbations like adding a small gravitational source to the spacetime, because it brings about the asymmetry around a UPO [8][9]. On the other hand, it is not trivial to determine whether or not chaos occurs if there is no perturbation around a UPO. It seems necessary to find tools to judge where and with what strength chaos occurs in these spacetimes, in order to make a quantitative analysis of chaos.

In the previous paper (paper I [8]), we paid attention to the fact that in GR, free particle motion in a spacetime is described by a geodesic, so that local instability may be determined from the sign of the curvature tensor. We showed that in axisymmetric static vacuum spacetimes, geodesics become strongly chaotic after passing through an LU region defined by the distribution of eigenvalues of the Weyl tensor. Recently, Vieira and Letelier showed, with their careful analysis around UPO, that chaos predicted by the LU region also originates from a homoclinic tangle, so that the local instability criterion is not so strict as to predict the chaotic property for the orbit passing through an LU region [14]. This result suggests that in addition to the passage through an LU region, it seems necessary for the orbit to pass by the UPO to cause strong chaos. However, from a physical viewpoint, the origin of the chaos around a UPO is still unclear without information about the LU region. For example, why the torus suddenly begins to break around UPO in Fig.5 of [8] cannot be explained just by the homoclinic tangle. Since the LU region can be utilized as an effective tool for judging the occurrence of chaos even in such a case, it is still important to analyze the correlation between the LU region and chaos, and to give its physical meaning.

In the previous paper, we examined the eigenvalues of the Weyl tensor, because in vacuum spacetime the Weyl tensor coincides with the Riemann tensor which is used to determine the local instability in the geodesic deviation equation. In the non-vacuum case, however, the situation is not simple, because the Riemann tensor is no longer the same as the Weyl tensor. In this paper, to see the effect of matter, we first analyze the deviation of geodesics by dividing the Riemann tensor into the Weyl tensor, the Ricci tensor and the scalar curvature in section 2. This procedure makes it clear how the eigenvalues of the Riemann tensor relate to those of the Weyl tensor. In section 3, we will numerically show that the LU region defined by the Weyl tensor still remains a good tool to determine chaos even in spacetimes with a matter field. We will also show that even if the local matter effect is strong enough to make the sectional curvature $\mathcal{K}(u, n)$ negative everywhere in bound regions, chaos can
still occur around an LU region. This result supports the claim of Vieira and Letelier on the weakness of the local instability determined by the sign of $K(u, n)$. In section 4, in order to reveal the physical origin of such chaos, we will analyze this type of chaos from another viewpoint, the shear effect of the geodesic congruence, and show the role of the Weyl tensor on the eigenvalues of the shear matrix. We will show that the LU region has the character of possessing two independent stretch directions for geodesic congruence, which might explain the contribution of the LU region to the chaos of geodesics passing by a UPO. Finally, we will give our conclusions and some remarks in section 5.

2 Eigenvalues of the Riemann tensor

2.1 Derivation of the eigenvalues of the Riemann tensor

In this paper, we use the same notation as in paper I [8]. In GR, the motion of a free particle is described by a geodesic and its deviation $n^\mu$ is given by the equation

$$\frac{D^2 n^\mu}{D\tau^2} = -R^\mu_{\nu\rho\sigma} u^\nu n^\rho u^\sigma,$$

where $u^\mu$ is the 4-velocity and $n^\mu$ is orthogonal to $u^\mu$, i.e., $u^\mu u_\mu = -1$, and $u^\mu n_\mu = 0$. Equation (2.1) is also reduced by the sectional curvature $K(u, n)(\equiv -R(u, n, u, n)/\|n\|^2)$ of the plane spanned by $u$ and $n$ to,

$$\frac{d^2 \|n\|}{d\tau^2} = K(u, n)\|n\| + \frac{1}{2\|n\|} \left\|n \times \frac{Dn}{D\tau}\right\|^2,$$

(2.2)

where we use the notation $\|V\| \equiv (\|V\| V^\mu V^\mu)^{1/2}$ for a 4-vector $V^\mu$. From (2.2), if $K(u, n)$ is positive, a geodesic becomes locally unstable since the second term in the right hand side of (2.2) is always positive. As described in paper I, the sectional curvature takes critical values in the direction of the eigenvectors of the Riemann tensor. Here we will show how these eigenvalues are expressed in a static axisymmetric spacetime.

In general, the Riemann tensor $R_{\mu\nu\rho\sigma}$ can be decomposed by using the Weyl tensor $C_{\mu\nu\rho\sigma}$, the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar $R$ into the form

$$R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} + g_{[\rho}R_{\sigma]\nu} - g_{[\nu}R_{\sigma]\rho} - \frac{1}{3}Rg_{[\rho}g_{\sigma]\nu}.$$

(2.3)

In bivector formalism, the Riemann and Weyl tensors are regarded as $6 \times 6$ matrices which are decomposed as follows:

$$\mathcal{R} = \begin{pmatrix} \mathcal{E} & \mathcal{H} \\ -\mathcal{H}^T & \mathcal{F} \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} \mathcal{E} & \mathcal{H} \\ -\mathcal{H} & \mathcal{E} \end{pmatrix}, \quad (2.4)$$
where \( \mathcal{E} , \hat{\mathcal{E}} , \mathcal{H} , \hat{\mathcal{H}} , \mathcal{F} \) and \( \hat{\mathcal{F}} \) are \( 3 \times 3 \) matrices \(^{10}\). In the vacuum case the parts composed of the Ricci tensor and Ricci scalar vanish from Einstein’s equation, and the Riemann tensor coincides with the Weyl tensor. However, in the non-vacuum case, we do not have such an advantage. We have to deal with both \( \mathcal{R} \) and \( \mathcal{C} \), separately. If the spacetime is static, the matrices \( \mathcal{R} \) and \( \mathcal{C} \) are expressed as follows:

\[
\mathcal{R} = \begin{pmatrix} \hat{\mathcal{E}} & 0 \\ 0 & \hat{\mathcal{F}} \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} \mathcal{E} & 0 \\ 0 & \mathcal{E} \end{pmatrix}
\]  
(2.5)

In this case, we can always diagonalize the matrices \( \mathcal{R} \) and \( \mathcal{C} \) by using an appropriate orthonormal real tetrad basis \( \{ e_0 , e_1 , e_2 , e_3 \} \) since the matrices \( \mathcal{E} , \hat{\mathcal{E}} \) and \( \hat{\mathcal{F}} \) in \( \mathcal{R} \) and \( \mathcal{C} \) are all symmetric. We denote the six eigenvalues of \( \mathcal{R} \) as follows

\[
\kappa_{0i}^R = -R(e_0 , e_i , e_0 , e_i) \quad (i < j, \ i, j = 1 \sim 3)
\]
\[
\kappa_{jk}^R = R(e_j , e_k , e_j , e_k) = \kappa_{ji}^R \quad (j < k, \ j, k = 1 \sim 3).
\]  
(2.6)

Then, the sectional curvature \( \mathcal{K}(u, n) \) is expressed by \( \kappa_{\mu\nu}^R \) as follows.

\[
\mathcal{K}(u, n) \equiv -R(u , n , u , n) = -u^{(\alpha)}n^{(\beta)}u^{(\gamma)}n^{(\epsilon)}R(e_\alpha , e_\beta , e_\gamma , e_\epsilon)
\]
\[
= \sum_{\mu > \nu} S_{\mu\nu}(u^{(\mu)}n^{(\nu)} - u^{(\nu)}n^{(\mu)})^2 \kappa_{\mu\nu}^R,
\]  
(2.7)

where \( S_{\mu\nu} \) is 1 if \( \nu \) is equal to zero, and otherwise it is \(-1\). From (2.7), a positive value of \( \kappa_{0i}^R \) contributes to the local instability ( \( \mathcal{K}(u, n) > 0 \) ), while a positive value of \( \kappa_{ij}^R \) contributes to the local stability ( \( \mathcal{K}(u, n) < 0 \) ).

For static and axisymmetric spacetimes, the metric is written as

\[
ds^2 = -e^{2U}dt^2 + e^{-2U}[\rho^2d\phi^2 + e^{2k}(dp^2 + dz^2)],\]
\[
(2.8)
\]
where \( t \) and \( \phi \) are the coordinates related to two Killing vectors \( \partial/\partial t \) and \( \partial/\partial \phi \) and both \( U \) and \( k \) are functions depending only on \( \rho \) and \( z \).\(^{11}\) In these coordinates, \( e_0 \) and \( e_3 \) coincide with the normalized Killing vectors \( e_0 \) and \( e_3 \), respectively. Then \( \kappa_{03}^R \) and \( \kappa_{12}^R \) always become \( \mathcal{R}^3_3 \) and \( \mathcal{R}^6_6 \), respectively, and the other eigenvalues are

\[
\kappa_{01}^R = \frac{1}{2} \left[ (\mathcal{R}^2_2 + \mathcal{R}^1_1) + \sqrt{(\mathcal{R}^2_2 - \mathcal{R}^1_1)^2 + 4(\mathcal{R}^3_3)^2} \right],
\]
\[
\kappa_{02}^R = \frac{1}{2} \left[ (\mathcal{R}^2_2 + \mathcal{R}^1_1) - \sqrt{(\mathcal{R}^2_2 - \mathcal{R}^1_1)^2 + 4(\mathcal{R}^3_3)^2} \right].
\]  
(2.9)

where the matrix \( \mathcal{R}^A_B \) is the Riemann tensor \( R \) in bivector formalism and the components \( A \) and \( B \) of \( \mathcal{R}^A_B \) run \( 1 \sim 6 \) which denote the tetrad components \( (\hat{t}\hat{\rho}) \), \( (\hat{t}\hat{z}) \), \( (\hat{t}\hat{\phi}) \), \( (\hat{z}\hat{\phi}) \), \( (\hat{\phi}\hat{\rho}) \),\(^{12}\)

\(^{11}\) We use units of \( G = c = 1 \), but we explicitly write \( G \) or \( c \) when it may help our discussion.
and $(\hat{\rho} \hat{z})$, respectively. $\kappa_{23}^R$ and $\kappa_{31}^R$ can be derived by changing the indices 1 and 2 in (2.9) to 4 and 5, respectively. We can also get the eigenvalues of $C$ using the same tetrad basis as follows,

$$
\begin{align*}
\kappa_{0i}^c &= -C(e_{(0)}, e_{(i)}, e_{(0)}, e_{(i)}) \\
\kappa_{jk}^c &= C(e_{(j)}, e_{(k)}, e_{(j)}, e_{(k)}) = \kappa_{kj}^c, \quad (j < k, \ j, k = 1 \sim 3). \quad (2.10)
\end{align*}
$$

Here we have $\kappa_{0i}^c = \kappa_{jk}^c$. For the metric (2.8), these eigenvalues, $\kappa_{\mu\nu}^c$ are derived by changing the Riemann tensor $R$ in (2.9) into the Weyl tensor $C$ as

$$
\begin{align*}
\kappa_{01}^c &= \kappa_{23}^c = \frac{1}{2} \left[ C_1^1 + C_2^2 + \sqrt{(C_1^1 - C_2^2)^2 + 4(C_2^1)^2} \right], \\
\kappa_{02}^c &= \kappa_{31}^c = \frac{1}{2} \left[ C_1^1 + C_2^2 - \sqrt{(C_1^1 - C_2^2)^2 + 4(C_2^1)^2} \right]. \quad (2.11)
\end{align*}
$$

and $\kappa_{03}^c = \kappa_{12}^c = C_3^3$. In [8], we called a region where $\kappa_{03}^c < 0$, $\kappa_{02}^c > 0$ and $\kappa_{01}^c > 0$ a locally unstable (LU) region, since $\kappa_{01}^c > 0$ and $\kappa_{02}^c > 0$ means that the sectional curvatures spanned by $e_{(0)}$ and one of the two eigenvectors orthogonal to two Killing directions, $K(e_{(0)}, e_{(1)})$ and $K(e_{(0)}, e_{(2)})$ are positive. In fact, we showed in [8] that this region strongly affects the chaotic behavior of free particle motion.

In the non-vacuum case, however, the eigenvalues of the Riemann tensor are not the same as those of the Weyl tensor and no longer degenerate. We cannot divide the spacetime by the eigenvalues of Riemann tensor as simply as in the vacuum case, because we have six independent eigenvalues.

In order to judge the chaos from curvature information, we have to know the relationship among these eigenvalues and clarify how the effect of local matter changes the value of $\kappa_{\mu\nu}^c$ from that of $\kappa_{\mu\nu}^R$. By using the relationship between the Riemann tensor and the Weyl tensor (2.3) and Einstein’s equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$, we get the relationship between $\kappa_{\mu\nu}^c$ and $\kappa_{\mu\nu}^R$ (see Appendix A for the detail).

$$
\begin{align*}
\kappa_{0i}^R &= \kappa_{0i}^c - \frac{4\pi}{3} \{ \rho_0 - p_i + 2(p_j + p_k) \} \\
\kappa_{jk}^R &= \kappa_{jk}^c + \frac{4\pi}{3} \{ 5(\rho_0 - p_i) - 2(p_j + p_k) \},
\end{align*}
$$

(2.12)

where $\rho_0 \equiv T(e_{(0)}, e_{(0)})$ and $p_i \equiv T(e_{(i)}, e_{(i)})$.

From (2.7), both positive values of $\kappa_{0i}^R$ and negative values of $\kappa_{jk}^R$ contribute to make $K(u, n)$ positive. In general we cannot give any concrete inequality between $\kappa_{\mu\nu}^c$ and $\kappa_{\mu\nu}^R$ by specifying energy conditions.
However, we can find that the electromagnetic field always weakens the local instability determined by the Weyl curvature. That is, in this case $T$ vanishes, so that (A.3) in Appendix A becomes

$$
\kappa_{0i}^R = \kappa_{0i}^c - 4\pi\{\rho_0 - p_i\} \\
\kappa_{jk}^R = \kappa_{jk}^c + 4\pi\{\rho_0 - p_i\} \quad (i, j, k = 1 \sim 3) \tag{2.13}
$$

and $\rho_0 - p_i$ is always positive, because of the dominant energy condition [15]. Hence the eigenvalues $\kappa_{0i}^R$ are always smaller than $\kappa_{0i}^c$, while $\kappa_{ij}^R$ is always bigger than $\kappa_{ij}^c$.

If the matter effect is not strong enough to change the sign of $\kappa_{02}^c$ or $\kappa_{03}^c$ in the LU region, the region where both $\kappa_{02}^R$ and $\kappa_{03}^R$ are positive, which we call the RLU region, remains inside the LU region. Then we may use the RLU region for a criterion of chaos. However, if the RLU region does not exist inside the LU region, it seems to be difficult to explain chaos around the LU region by the positivity of $\mathcal{K}(u, n)$. In fact in the vacuum case, the LU region was characterized as the region where $\mathcal{K}(e_{(0)}, e_{(1)}) > 0$ and $\mathcal{K}(e_{(0)}, e_{(2)}) > 0$ in [8]. However it is nothing but the RLU region, if a matter field exists.

In the next section, we will numerically examine the relationship between the chaotic behavior of a geodesic and its passage through an LU or RLU region by using some analytically given spacetimes.

### 3 Numerical analysis

Here we examine the relationship between chaos and LU or RLU regions by numerical calculations for some spacetimes with matter fields. First, we reanalyze the Majumdar-Papapetrou (MP) spacetimes as examples of spacetimes where both LU and RLU regions appear. Then, we study the Ernst universe as an example of a spacetime where no RLU region exists inside the LU region. Finally, we treat the spacetime with a scalar field as an example of the spacetime with a matter field which does not satisfy any energy condition.

#### 3.1 Chaos around N-maximally charged black holes

First, we examine the MP solution. This describes the system of several extreme Reissner-Nordström (RN) black holes that are balanced by the attractive gravitational and the repulsive electric forces. In this spacetime, $N$ black holes can be located at random in a three dimensional hypersurface. The metric is [18],

$$
 ds^2 = -V^{-2}dt^2 + V^2(dx^2 + dy^2 + dz^2) \tag{3.14}
$$
\[ V = 1 + \sum_{i=1}^{N} \frac{M_i}{r_i}, \quad (3.15) \]

where \( M_i \) is the mass of the \( i \)th black hole and \( r_i \equiv [(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2]^{1/2} \) is the Euclidean coordinate distance from the location of the \( i \)th black hole, \((x_i, y_i, z_i)\), to a point \((x, y, z)\). If \( N = 1 \), it is just the extreme RN solution. Since the RN solution belongs to Petrov type D, two eigenvalues are degenerate everywhere and neither LU nor RLU regions appear. This is consistent with the fact that stationary axisymmetric Petrov D spacetimes are integrable and chaos does not occur there.

The case of \( N > 1 \) has been examined by several authors [6][12]. They indicated that these spacetimes cause chaotic type motions characterized by fractal boundaries of initial conditions for both timelike and null geodesics on the meridian plane.

Here we examine the case with the \( N \) black holes located on the \( z \) axis, since we are interested only in the axisymmetric case. In this case, the spacetime is described by using the two functions \( U \) and \( k \) in the metric (2.8) with

\[ U = -\ln(1 - U_C), \]
\[ k = 0, \quad (3.16) \]

where \( U_C \) corresponds to that of \( N \)-Curzon spacetime [20] and is

\[ U_C = -\frac{G}{\sum_{i=1}^{N} \frac{M_i}{r_i}}. \quad (3.17) \]

First, we study the case \( N = 2 \) with non-zero angular momentum of a test particle, \( L \). In this case, we can easily find an LU region as in 2-Curzon or 2-ZV spacetime [6], since each of two gravitational sources on the \( z \) axis attracts the geodesics toward itself (Fig. 1(a)). Then if the energy of test particle is appropriately large, the bound orbit can be overlapped with LU region and strong chaos occurs beyond some critical energy (Fig. 1(b)). Here we can also find a RLU region inside an LU region, since the effect of the electric field produced by the electric charge of each black hole is not so strong as to extinguish it.

Next, we consider the case of \( N = 3 \), with one of the black holes located at the origin and the others at the same distance on both sides of the \( z \)-axis. In this case, we can also see an RLU region inside an LU region. When the mass of the central black hole is much more than that of the others, the LU region does not overlap the bound region. We have not seen any chaotic behavior there, even if we increase the energy up to the value of the UPO. However, as the mass of the central black hole reduces to that of the others, the bound region are more and more overlapped with the LU region. Here we analyzed the case that the energy
of geodesic is a little higher than the energy of UPO, $E_{UPO}$, where the classically admitted region does not become a bound region, but has the tiny throat connecting it with a black hole. In this case, we can check whether or not the chaos determined by an LU region always has the origin of homoclinic mixing as Vieira and Letelier claimed, since the orbit passing by the throat falls into a black hole. In this case when the orbit passes through the LU region, the torus in Poincaré map is completely broken (Fig.2), although the orbit falls into black hole with a finite time interval. These results certainly suggest that passing through the LU region (and RLU region) contributes to the occurrence of chaos (though it must be defined with finite time interval) as for geodesic motion in a multi-black hole system, regardless of homoclinic tangle.

### 3.2 Chaos around the Ernst universe

Next, we reexamine the Ernst solution which includes a magnetic field. Axisymmetric stationary spacetimes with magnetic field $B_0$ along the $z$ axis can be derived from the corresponding vacuum spacetimes [19]. In particular, when we start with the metric form (2.8) for the static vacuum case, we find the function $U_B$ and $k_B$ of the spacetime with magnetic field to be:

$$
U_B = U \ln \left(1 + \frac{1}{4} B_0^2 \rho^2 e^{-2U}\right)
$$

$$
k_B = k + 2 \ln \left(1 + \frac{1}{4} B_0^2 \rho^2 e^{-2U}\right)
$$

(3.18)

where $B_0$ is the magnitude of the magnetic field along the $z$ axis.

Several years ago, Karas and Vokrouhlicky found that both neutral and charged particles behave chaotically in one of the Ernst solutions, i.e., around a Schwarzschild black hole immersed in a magnetic field [7]. These particles show chaotic behavior more and more strongly as the energy increases and approaches the critical value, $E_{UPO}$, beyond which the particles fall into the black hole. This suggests that the chaos occurs through the homoclinic mixing around a UPO. However the origin of such a homoclinic mixing is still unclear, at least in the respect that the metric has a reflection symmetry on the equatorial plane to retain a homoclinic orbit around the UPO. Here we examine whether it is possible to explain the chaos by passing through an LU region. We reanalyze the case which they examined, that is, the case that $U$ and $k$ correspond to the functions of Schwarzschild spacetime and $B_0 = 0.15 /M$. In Fig.8 we plot the LU region and bound region of a neutral test particle with $(E^2, L) = (29.12 \mu^2, 25.0 \mu M)$, where $\mu$ is the rest mass of a test particle. From this figure, we can see that the magnetic field $B_0$ plays a role in causing an LU region around
a single black hole, since no LU region appears for the case of $B_0 = 0$. Certainly, it is possible to explain the strong chaos shown in [7] by the passage through an LU region, since the orbit moves inside the bound region almost ergodically and passes through the LU region regardless of the initial conditions. On the other hand, we cannot see an RLU region anywhere inside an LU region, because the matter effect is strong enough to change the sign of $\kappa_{01}^R$ or $\kappa_{02}^R$ into negative through equation (2.13). This result makes it difficult to explain chaos by local instability determined by geodesic deviation equation, since the local instability in the $\rho$-$z$ plane is determined by $\kappa_{01}^R$ and $\kappa_{02}^R$, rather than $\kappa_{01}^C$ and $\kappa_{02}^C$.

So here, in order to see the effect of the absence of an RLU region, we consider the eigenvalues of the following matrices,

$$\hat{R}^\mu_\rho = R^\mu_{\nu\sigma\rho} u^\nu u^\sigma$$
$$\hat{C}^\mu_\rho = C^\mu_{\nu\sigma\rho} u^\nu u^\sigma.$$  (3.19)

The sectional curvature, $\mathcal{K}(u, n)$ for any pair of $(u, n)$ is described by the linear combination of the non-zero eigenvalues $\alpha_i$, $(i = 1, \ldots, 3)$ of matrix $\hat{R}$ as follows,

$$\mathcal{K}(u, n) = \sum_{i=1}^{3} \alpha_i (n^i)^2. $$  (3.20)

(See Appendix B.) In order to fix the 4-velocity $u$ in (3.19), we use the parameter $\Theta_*$ defined as,

$$u^{(1)} = v_* \cos \Theta_*$$
$$u^{(2)} = v_* \sin \Theta_*,$$  (3.21)

where $v_*$ is the meridian velocity which we defined in [8] as follows,

$$v_*^2(x, E, L) = \frac{(E^2 - V^2_{\text{eff}}(x, L))}{2\|\partial/\partial t\|^2},$$  (3.22)

where $x = (\rho, z)$ and $V^2_{\text{eff}}$ is the effective potential for the particle with the angular momentum $L$. The rest components, $u^{(0)}$ and $u^{(3)}$ are determined by $E$ and $L$, respectively.

We numerically examined the distribution of eigenvalues $\alpha_i$ in the bound region and found that all of the eigenvalues $\alpha_i$ become negative everywhere inside the bound region for any pair of $(v_*, \Theta_*)$ with $(E^2, L) = (29.12 \mu^2, 25.0 \mu M)$. This means that chaos cannot be explained by the curvature term in (2.1) or (2.2), since $\mathcal{K}(u, n)$ becomes negative for any pair of $(u, n)$ from (3.20). This result seems inconsistent with the result that strong chaos occurs for the same value of $E$ and $L$, which leads us to new analysis by a shear effect of the geodesic congruence on chaos rather than that of the geodesic deviation, as we will see in section 4.
3.3 The spacetime with scalar field

Finally, we examine a spacetime with a scalar field, as an example of a spacetime with a matter field which satisfies neither the strong nor dominant energy condition. In this case, an RLU region need not be included in the LU region, since the matter field may increase the local instability.

Here we make use of the solution in which \( N \)-scalar charged ZV-type singularities are located on the \( z \) axis \[21\]. We examine the case of two scalar charged singularities balanced against one another on the \( z \) axis, by the gravitational force and the repulsive force induced by the scalar charge. This balance condition is attained by choosing each scalar charge so as to satisfy the relationship \( M_i = \Sigma_i \) (\( i = 1, 2 \)), where \( M_i \) is the mass of the \( i \)th singularity and \( \Sigma_i \) is its scalar charge. This condition is similar to that of the MP solution, where extreme black holes are balanced together. In this case, the metric is described by setting \( k = 0 \) in the corresponding 2-ZV solution, where \( M_i = m_i \delta \) ((3.10) in \[8\]) and this spacetime becomes asymptotically flat. As we showed in section 2, the RLU region is not always included in the LU region in general, in contrast with the case with an electromagnetic field. In fact, the RLU region can even be seen outside the LU region (Fig.4). However, the local matter effect is not strong enough to separate the RLU region from the LU region completely, so that chaos occurs for a particle passing through both of these regions. Hence, in this case we cannot clearly determine through which region chaos occurs, that is, through the LU or the RLU region.

4 The shear effect of the LU region on geodesic congruence

In section 3.2, we showed that no RLU region appears inside an LU region and \( \mathcal{K}(u, n) \) becomes negative everywhere inside the bound region in spite of the occurrence of strong chaos. This failure of the \( \mathcal{K}(u, n) \) criterion may come from the fact that the positivity of \( \mathcal{K}(u, n) \) is a sufficient condition for the local instability of geodesic but may not be necessary condition, because of the second term in the right side of (2.2). This second term could induce an instability of \( \|n\| \) against the negative contribution of the \( \mathcal{K}(u, n) \) term. In this case, we could not explain chaos simply by a local instability determined from the sign of \( \mathcal{K}(u, n) \). On the other hand, the LU region still works well to judge the occurrence of chaos even in the Ernst case, since any chaotic bound trajectory passes through an LU region.

In order to see the effect of an LU region even in this case, we examine the deviation of nearby geodesics by expansion \( \theta \) and shear \( \sigma_{\mu\nu} \) of the geodesic congruence. It is well known
that \( \theta \) and \( \sigma_{\mu\nu} \) satisfy the following equations \[13\],

\[
\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} - R_{\mu\nu}u^\mu u^\nu, \tag{4.1}
\]

\[
\frac{D\sigma_{\mu\nu}}{D\tau} = -\frac{2}{3}\theta\sigma_{\mu\nu} - \sigma_{\mu\rho}\sigma^{\rho\nu} + \frac{1}{3}h_{\mu\rho}\sigma^{\rho\sigma} + \hat{C}_{\mu\nu} + \frac{1}{2}R_{\mu\nu}^{(TF)}, \tag{4.2}
\]

where \( R_{\mu\nu}^{(TF)} \) is the trace-free part of \( R_{\mu\nu} \) and defined by

\[
R_{\mu\nu}^{(TF)} = h_{\mu\rho}h_{\nu\sigma}R^{\rho\sigma} - \frac{1}{3}h_{\mu\rho}h_{\nu\sigma}R^{\rho\sigma}, \tag{4.3}
\]

where \( h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu \) is the metric of 3 space perpendicular to \( u_\mu \). Here we assume that the twist term \( \omega_{\mu\nu} \) vanishes and examine the time evolution of a geodesic congruence whose initial value of \( \theta, \sigma_{\mu\nu} = 0 \). From (4.1), if the Ricci tensor \( R_{\mu\nu} \) satisfies the condition

\[
R_{\mu\nu}u^\mu u^\nu \geq 0 \tag{4.4}
\]

for an arbitrary 4-velocity \( u \), \( \theta \) negatively diverges, which is followed by the creation of a conjugate point of the geodesic. From Einstein’s equation \( G_{\mu\nu} = 8\pi T_{\mu\nu} \), the condition (4.4) is attained, if and only if the energy-stress tensor \( T_{\mu\nu} \) satisfies the following strong energy condition (see \[15\]).

\[
T_{\mu\nu}u^\mu u^\nu \geq -\frac{1}{2}T. \tag{4.5}
\]

As long as the condition (4.5) is satisfied, geodesic congruence will converge everywhere. On the other hand, the above energy condition tells us nothing about the shear of the geodesic congruence. It is possible for the geodesic to be stretched exponentially by the shear term \( \sigma \) even in a spacetime with a matter field satisfying the strong energy condition (4.5). This stretching property of shear seems to be an important cause of chaos, as we see in the famous baker’s transformation \[17\]. Since the Weyl tensor directly affects the time evolution of shear in (4.2), it is expected to play an important role in chaos through the stretching of nearby geodesics in combination with their bending at the edge of the bound region.

Especially for spacetimes with an electromagnetic field, since the time derivative of expansion \( \theta \) is negative in (4.1), the volume of geodesic congruence always converges because of the strong energy condition (4.3). So shear may be one of the important factors that causes local instability for geodesics in spacetimes with an electromagnetic field. In the equation of shear (4.2), only the last two terms on the right side,

\[
\Omega_\nu^\mu = \hat{C}_\nu^\mu + \frac{1}{2}R_{\nu(\mu}^{\rho TF)} \tag{4.6}
\]

are independent of the initial values of \( n \) and \( Dn/D\tau \). Hence \( \Omega_\nu^\mu \) is important when examining the effect of shear on chaos, since chaotic behavior of a trajectory is generally independent.
of the initial choice of $n$ around a given geodesic. Since each part of (1.2) is trace-free, the sum of all eigenvalues of $\Omega^\mu_\nu$ vanishes. Hence one of the eigenvalues of $\Omega^\mu_\nu$ is always positive and diverges as the geodesic approaches the gravitational source singularity, since this stretch comes from the tidal force by the gravitational source.

For the asymptotically flat spacetime with a single gravitational source singularity, the largest eigenvalues of $\Omega^\mu_\nu$, $\zeta_1$, always becomes positive and decreases monotonically to zero as $r \to \infty$ (Fig.5). The direction of its corresponding eigenvector projected on the $\rho$-$z$ plane is almost parallel to that of $u$ (Fig.3). (Note that those eigendirections are orthogonal to $u$ in the 4-dimensional spacetime because of conditions I and II in Appendix B.) The rest of the eigenvalues always become negative and monotonically increase to zero as $r \to \infty$. These properties for the eigenvalues of $\Omega^\mu_\nu$ also hold for those of the Weyl tensor, since the tidal force in the direction of the gravitational source is characterized by the positive eigenvalue of the Weyl tensor. On the other hand, in multi-black holes spacetimes or Ernst spacetime, one of the negative eigenvalues of the Weyl tensor becomes positive inside the LU region. (In fact, the sign change of this eigenvalue defines the LU region.) Since $\hat{C}^\mu_\nu$ is defined by the Weyl tensor (8.19), one of the eigenvalues of $\hat{C}^\mu_\nu$ also becomes positive in the LU region. In the non-vacuum case, the local matter part $\frac{1}{2}R^\mu_\nu(\text{TF})$ should also be considered in $\Omega^\mu_\nu$. In the Ernst case, the local matter effect $\frac{1}{2}R^\mu_\nu(\text{TF})$ in (1.6) is not strong enough to change the sign of the two positive eigenvalues, $\zeta_1$ and $\zeta_2$ of $\Omega^\mu_\nu$ in the LU region back to negative (Fig.5). Moreover, the eigendirections of $\Omega^\mu_\nu$ are almost the same as those of $\hat{C}^\mu_\nu$ and the Weyl tensor.

In order to see the effect of its two positive eigenvalues on shear, we compare the eigendirections of $\Omega^\mu_\nu$ with those of $\sigma^\mu_\nu$. As we can see in Fig.3, these eigendirections are almost the same, which means that the two positive eigenvalues of the Weyl tensor have the effect of stretching the geodesic congruence in two independent directions through the shear effect. As we can see in Fig.3, the second eigendirection whose eigenvalue changes from negative to positive is not parallel to $u$ on the $\rho$-$z$ plane, in contrast with the first eigendirection of $\zeta_1$. It is suggested that the second positive eigenvalue of the Weyl tensor contributes to stretching the congruence in the direction independent of $u$ and causes chaos for bound orbits passing through the LU region.

To substantiate this speculation, we examine the magnitude of the positive eigenvalue $\zeta_2$ of $\Omega^\mu_\nu$ inside a bound region and compare it with the value of the Lyapunov exponent $\lambda$ of chaotic motion inside this region. From Fig.8(a), we can see that for a given value of $L$, the peak of $\zeta_2$ becomes higher and higher, as $E$ gets larger. This is consistent with the result in our previous paper [8] that the Lyapunov exponent $\lambda$, that is, the strength of chaos, is proportional to $v_*$, since the effect on shear becomes larger as $v_*$ increases. For concrete
values of $\lambda$, in the Ernst case with $(E^2, L) = (29.12 \mu^2, 25.0 \mu M)$, we find the numerical value of $\lambda \sim 3.15 \times 10^{-2}/M$, regardless of initial conditions, which is the same order of magnitude as the peak value of $\zeta_2$.

As for the $\Theta_*$ dependence of $\zeta_2$, its peak becomes larger and larger as the geodesic crosses the equatorial plane orthogonally, although the change is not so large as to change the order of the value (Fig. 9(b)). We also find the same results in the relationship between shear and the LU region for multi-black hole systems. Thus these results substantiate our claim that the shear effect in the eigendirection corresponding to $\zeta_2$ is correlated with the chaotic motion of the geodesic.

5 Concluding Remarks

In this paper, we have examined the LU region criterion for chaos in the non-vacuum case, by analyzing test particle motion in several axisymmetric static spacetimes as well as in the vacuum case.

In multi-black hole systems, the RLU region was always seen inside the LU region, while in a spacetime with 2 scalar-charged singularities, the RLU region was seen even outside the LU region. In both cases, we found that passing through an LU or RLU region gives us good information about the occurrence of chaos, although it is indeterminate which region is more fundamental for the chaos because of the slight separation of these regions.

In an Ernst universe, however, the RLU region cannot be seen, because one of the two positive eigenvalues of the Weyl tensor is made negative by the strong local matter effect. In this case the local instability of geodesics is not explained by the curvature term in (2.1) or (2.2), since all of the eigenvalues of $\hat{R}^\mu_\nu$ become negative, which means that $\mathcal{K}(u, n)$ becomes negative for any direction of $(u, n)$ at any point in bound region. Then we examined the character of the LU region to explain such a chaotic behavior by dividing the matrix $\hat{R}^\mu_\nu$ into the shear part, $\Omega^\mu_\nu$ and the expansion part $R_{\mu\nu}u^\mu u^\nu$. We find that the eigenvalues of $\Omega^\mu_\nu$ are strongly affected by those of the Weyl tensor, so that in addition to the positive eigenvalue determined by the tidal force of gravitation, the second eigenvalue of $\Omega^\mu_\nu$ also becomes positive around the LU region. It follows from this property that two eigenvalues of the shear matrix $\sigma^\mu_\nu$ also become positive in almost the same directions as the eigen directions of $\Omega^\mu_\nu$, as we numerically showed in sec.4. Thus the LU region is characterized as the region in which the geodesic congruence is stretched in the direction independent of $u$ in addition to the direction of $u$. We can speculate that this additional stretch helps to cause strong chaos for geodesics passing through the LU region, especially around UPO. In fact we
found that the Lyapunov exponent of those chaotic motions can be estimated by the peak of the additional second type eigenvalue. These results suggest that the Weyl tensor plays a much more important role in determining the chaos of geodesics than the sign of sectional curvature $\mathcal{K}(u, n)$.

Strictly speaking, the chaos for the orbit passing through the LU region cannot completely be determined by the shear effect. For example, we cannot explain, just by the LU region criterion, why the chaos occurs only for the orbit whose energy is almost near to $E_{UPO}$ and why non-chaotic orbit coexists with chaotic orbit even if both of them pass through LU region for the fixed value of $E$ and $L$ (see, for example, Fig.2 in [14]). As Vieira and Letelier showed in [14], the chaos also seems correlated with the homoclinic tangle around the UPO. It might be because the stretching property of the LU region helps homoclinic mixing and makes the occurrence of chaos much easier around UPO. Hence, by our criterion we can just give a good tool to find a chaotic orbit.

Another interesting point we wish to stress is as follows. As we showed in sec.3.1, such a chaotic behavior can also be seen for the orbit with the energy beyond $E_{UPO}$, which cannot be explained by the homoclinic tangle. In a strict sense, we cannot define the chaos in such an unbounded case, since the orbits remain within the bound region only in the finite time interval. However, this result suggests that the LU region or the eigenvalues of the Weyl tensor could have some correlation with the unbounded chaotic type phenomena such as chaotic scattering or a fractal basin [12][22]. Further analysis along this line will be left for the future work.

Here we restricted ourselves to the static case, where there is no rotation effect of space-time involved in the geodesic equation. However, it will be interesting to extend our analysis to rotating spacetimes not only for the free particle motion but also for spinning particles [23], because the spin-orbit or spin-spin interaction may provide new physical ingredient. All of these speculations lead us to future works which may reveal some important roles for chaos in realistic relativistic astronomical phenomena in our universe.

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Starting from the Einstein’s equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$, we rewrite Eq.(2.3) using the energy stress tensor $T_{\mu\nu}$ as follows
\begin{equation}
R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} + 4\pi \left\{ g_{\mu[j} T_{\sigma]\nu} - g_{\nu[j} T_{\sigma]\mu} \right\} + \frac{8\pi}{3} T g_{\mu[j} g_{\sigma]\nu}. \tag{A.1}
\end{equation}

Then, we obtain
\begin{equation}
R(u, n, u, n) = C(u, n, u, n) + 4\pi \{ -T(n, n) + ||n||^2 T(u, u) \} + \frac{8\pi}{3} T ||n||^2. \tag{A.2}
\end{equation}

With the definition $\mathcal{K}(u, n) \equiv -R(u, n, u, n)/||n||^2$ and $\mathcal{K}_c(u, n) \equiv -C(u, n, u, n)/||n||^2$, Eq.(A.2) is now
\begin{equation}
\mathcal{K}(u, n) = \mathcal{K}_c(u, n) + 4\pi \left\{ \frac{T(n, n)}{||n||^2} - T(u, u) \right\} - \frac{8\pi}{3} T, \tag{A.3}
\end{equation}

where $T(u, u)$ represents the energy density $\rho_0$ as measured by the observer whose 4-velocity is $u$ and $T(n, n)/||n||^2$ represents the principal pressure $p_n$ in the direction of $n$ [15]. Then (A.3) becomes
\begin{equation}
\mathcal{K}(u, n) = \mathcal{K}_c(u, n) - 4\pi \{ \rho_0 - p_n \} - \frac{8\pi}{3} T \tag{A.4}
\end{equation}

Note that it is not guaranteed that the space-space components of $T_{\mu\nu}$ are always diagonalized [?]. However, it is easily shown that as long as the Riemann tensor is diagonalized, its components are inevitably diagonalized by the principal pressure $p_i$ for the tetrad basis $\{e_0, e_1, e_2, e_3\}$. Hence by contracting two of the tetrad basis elements $\{e_0, e_1, e_2, e_3\}$ with Eq.(A.1), we find the relation between the eigenvalues of the Weyl tensor and those of Riemann tensor as follows:
\begin{align}
\kappa^R_{0i} &= \kappa^c_{0i} - 4\pi \{ \rho_0 - p_i \} - \frac{8\pi}{3} T \\
\kappa^R_{jk} &= \kappa^c_{jk} + 4\pi \{ \rho_0 - p_i \} - \frac{8\pi}{3} T \quad (i, j, k = 1 \sim 3), \tag{A.5}
\end{align}

where $\rho_0 \equiv T(e_0, e_0)$ and $p_i \equiv T(e_i, e_i)$. By substituting the connection $T = -\rho_0 + \sum_i p_i$, these components become (2.12).

We first pay attention to the following properties of matrices, $\hat{\mathcal{R}}$ and $\hat{\mathcal{C}}$ in (3.19).

I ) One of the eigenvalue of matrix $\hat{\mathcal{R}}$ ( or $\hat{\mathcal{C}}$) is trivial, i.e., the corresponding eigenvector is the 4-velocity $u$ and its eigenvalue is zero.
II ) Any two eigenvectors of matrix $\hat{\mathcal{R}}$ ( or $\hat{\mathcal{C}}$) are orthogonal each other. This is not so trivial, since matrix the $\hat{\mathcal{R}}$ is not symmetric on all indices.

II can be proven by taking the property of the Riemann tensor and the definition of $\hat{\mathcal{R}}$ into account.

From Property II, it is always possible to define an orthonormal triad basis, $E_{i}^{\mu}$ ($i = 1, \ldots, 3$), orthogonal to $u$ by using the normalized eigenvectors of $\hat{\mathcal{R}}$ and expand by them the normalized deviation vector $\hat{n}$ defined as $\hat{n} \equiv n/\|n\|$ like

$$\hat{n} = \sum_{i=1}^{3} n^{i} E_{i}^{i},$$

from the condition, $(u, n) = 0$.

From (B.6) and the definition (2.7), $\mathcal{K}(u, n)$ can be described as

$$\mathcal{K}(u, n) = - R(u, n, u, n)$$

$$= - \sum_{i=1}^{3} \sum_{j=1}^{3} n^{i} n^{j} R(u, E_{i}^{i}, u, E_{j}^{j})$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} n^{i} n^{j} (E_{i}^{i}, \hat{\mathcal{R}} E_{j}^{j}).$$

(B.7)

From the condition $\hat{\mathcal{R}} E_{i}^{i} = \alpha_{i} E_{i}^{i}$, (B.7) becomes

$$\mathcal{K}(u, n) = \sum_{i=1}^{3} \sum_{j=1}^{3} n^{i} n^{j} \alpha_{j} (E_{i}^{i}, E_{j}^{j}).$$

(B.8)

From the condition $(E_{i}^{i}, E_{j}^{j}) = \delta_{ij}$, we can get (3.20).
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Figure 1: (a) LU and RLU region (lightly shaded) of the 2-black hole spacetime with two equal masses, $M$, located at $\pm 2GM/c^2$ on the z axis (black dots) and the bound region (dotted) of a test particle with the angular momentum $L = 3.4 G\mu M/c$ and energy, $E^2 = 0.664 (\mu c^2)^2$ corresponding to $E_{UPO}$. (b) Poincaré map for the orbit in the bound region with the initial condition $p^\rho_0 = 0.0$, $z_0 = 0$ and $\rho_0 = 3.0 GM/c^2$.

Figure 2: (a),(b) classically admitted region in 3 black hole case with the mass ratio of central black hole to each of outside black hole (a)1:0.9, (b)1:0.95. The parameter for each region is $E^2 = (a)0.5712, (b)0.72$ and $L = (a)5.2, (b)6.0$, respectively. (c) Poincaré map for the orbit in the bound region of (a) with the initial condition of each orbit is $p^\rho_0 = 0.0, z_0 = 0$ and $\rho_0 = 1.5, 1.6, 1.7, 1.8, 1.9 GM/c^2$. Chaos cannot be seen in this case. (d) the orbit in the admitted region of (b) with the initial condition, $p^\rho_0 = 0.0, z_0 = 0$ and $\rho_0 = 1.5 GM/c^2$ and (e) it’s Poincaré map. The Poincaré map of the orbit passing through an LU region certainly becomes chaotic, before it falls into the central black hole through tiny throat.

Figure 3: LU and bound region in Ernst case with magnetic field $B_0 = 0.15 c^2/GM$ around Schwarzschild black hole. The parameter for bound region is $L = 25.0 G\mu M/c$ and energy, $E^2 = 29.116816 (\mu c^2)^2$. 


Figure 4: (a) LU and RLU region in the spacetime with scalar field, where two singularities with scalar charge $Q$ is fixed on $z$ axis at $z \pm 2.0$. The bound region (dotted) with $L = 6.9 \, G \mu M/c$ and $E^2 = 0.90913 \, (\mu c^2)^2$ corresponding to $E_{\text{UPO}}$ is also depicted. RLU region is almost overlapped with LU region, but not included in it. (b) Poincaré map for the orbit in the bound region with the initial condition $p^0 = 1.0, z_0 = 0$ and $\rho_0 = 2.8 \, GM/c^2$.

Figure 5: $\zeta_1$ and $\zeta_2$ of RN solution on equatorial plane. Here we used the parameter $\Theta_*=0$, $L = 2.92$ and $(E/\mu c^2)^2 = 0.8663$ corresponding to $E_{\text{UPO}}$. $\zeta_1$ decreases monotonically as the coordinate $\rho$ increases, while $\zeta_2$ increases monotonically and approaches to zero.

Figure 6: The distribution of eigenvector field on $\rho$-$z$ plane for the Ernst case, Fig.3 with $\Theta_*=0$. The eigenvector fields corresponding to (a) $\zeta_1$ and (b) $\zeta_2$ are depicted.

Figure 7: (a)$\zeta_1$ and $\zeta_2$ of $\hat{\mathcal{C}}_{\nu}^\mu$ for Ernst solution on equatorial plane. Here we used the parameter $\Theta_*=0$, $L = 25.0 \, G \mu M/c$ and $(E/\mu c^2)^2 = 29.116816$ corresponding to $E_{\text{UPO}}$. (b)$\zeta_1$ and $\zeta_2$ of $\Omega_{\nu}^\mu$ for the same condition as (a). Both of the cases show that $\zeta_1$ decreases monotonically, while $\zeta_2$ has the peak around LU region.

Figure 8: The direction of eigenvector on $\rho$-$z$ plane for matrix $\sigma_{\nu}^\mu$ and $\Omega_{\nu}^\mu$ along the orbit with the initial condition, $p^0 = 0.0, z_0 = 0$ and $\rho_0 = 4.0 \, GM/c^2$ in the case of Fig.3. (a) and (b) are the two eigendirections for two positive eigenvalues of $\sigma_{\nu}^\mu$ and $\Omega_{\nu}^\mu$, respectively. (c) and (d) are the two eigendirections for two negative eigenvalues of $\sigma_{\nu}^\mu$ and $\Omega_{\nu}^\mu$, respectively. The tendency of the changes of positive eigendirections of $\sigma_{\nu}^\mu$ on $\rho$-$z$ plane along the orbit is almost the same as those of $\Omega_{\nu}^\mu$. So is the tendency of the changes of negative eigendirections.

Figure 9: $E^2$ and $\Theta_*$ dependence of $\zeta_2$ of matrix $\hat{\mathcal{C}}$ on equatorial plane for the same situation as fig.3. (a) $E^2$ dependence of $\zeta_2$ for $\Theta_*=0$, $L = 25.0 \, G \mu M/c$ and $(E/\mu c^2)^2 =29.116816[(i)]$, 27.0[(ii)],26.0[(iii)]. We also added $\zeta_2$ of matrix $\hat{\mathcal{C}}$ in Schwarzschild spacetime for comparison, where we used the parameter $\Theta_*=0$, $L=3.55$ and $(E/\mu c^2)^2 =0.9025$ corresponding to $E_{\text{UPO}}$. (b)$\Theta_*$ dependence of $\zeta_2$ for $\Theta_*= 0[(i)], 0.0 [(ii)],\pi/6.0[(iii)]/\pi/3.0, L = 25.0 \, G \mu M/c$ and $(E/\mu c^2)^2 = 29.116816$. As the $\Theta_*$ increases and the direction become parallel to equatorial plane, the eigenvalue becomes smaller and smaller.