TWISTED VERTEX REPRESENTATIONS VIA SPIN GROUPS AND THE MCKAY CORRESPONDENCE

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Abstract. We establish a twisted analog of our recent work on vertex representations and the Mckay correspondence. For each finite group $G$ and a virtual character of $G$, we construct twisted vertex operators on the Fock space spanned by the super spin characters of the spin wreath products $oS_n$ and a double cover of the symmetric group $S_n$ for all $n$. When $G$ is a subgroup of $SL_2(\mathbb{C})$ with the Mckay virtual character, our construction gives a group theoretic realization of the basic representations of the twisted affine and twisted toroidal algebras. When $G$ is an arbitrary finite group and the virtual character is trivial, our vertex operator construction yields the spin character tables for $oS_n$.

1. Introduction

The connection among the direct sum of G rothendieck groups of the symmetric groups $S_n$ for all $n$ and the theory of symmetric functions $[F1, \text{Sym}]$ has a simple interpretation in terms of a Heisenberg algebra and vertex operators $[F1]$, see part one of $[F1]$. In the recent works $[W, F1, F2]$ we have realized a generalization of such a connection by substituting the symmetric group $S_n$ with the wreath product $_w S_n = \mathfrak{so}_n$ associated to an arbitrary finite group $G$. Moreover, we introduced a crucial modification of this connection that, in the case when $G$ is a finite subgroup of $SL_2(\mathbb{C})$, yields a group theoretic realization of the affine Lie algebra $\mathfrak{g}[FK, Sym]$ and of the toroidal Lie algebra $\mathfrak{g}[F2, MRRY]$, where $\mathfrak{g}$ is a complex simple Lie algebra of ADE type whose Dynkin diagram is related to $G$ via the Mckay correspondence $[FK, MRR]$. The main goal of the present work is to extend the above results to realize the twisted basic representation of an affine Lie algebra $\mathfrak{g}[1]$ and its toroidal counterpart by means of a spin cover $\mathfrak{e}_n$ of the wreath product $_w S_n$ associated to a subgroup $G$ of $SL_2(\mathbb{C})$. 

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The twisting of the basic representation of the affine Lie algebra under consideration is determined by the multiplication by \( \frac{1}{2} \) on the Cartan subalgebra \( h \) of \( g \), and can be viewed as an odd counterpart of the even (untwisted) case. This twisting was originally introduced as the first step towards the construction of the M onster module for the Monster group in [FLM 1,FLM 2]. As in the homogenous case one starts with a representation of the Heisenberg subalgebra \( \mathfrak{h}[\frac{1}{2}] \) and reconstructs the rest of the twisted affine Lie algebra \( \mathfrak{b}[\frac{1}{2}] \) using the twisted vertex operators.

The representation theory of the spin group \( S_n \) which is a double cover of the symmetric group \( S_n \) was initiated by I. Schur [S] (also see [Jo] for an exposition). Its connection with vertex operators was further studied in [J1]. These results will play an important role in our present work. The representation theory of \( E_n \) was also studied in [HH] from a Hopf algebra viewpoint.

In order to work effectively only with the spin representations of \( E_n \), i.e., those which do not factor through \( S_n \), we adopt the approach of [Jo] by introducing a superalgebra structure on the group algebra of \( E_n \) and consider its super modules. It turns out that the superstructure is preserved under the main operations such as induction and restriction. The direct sum of the Grothendieck groups of spin super modules of \( E_n \) carries a natural Hopf algebra and we remark that a Hopf algebra was constructed in [HH] on a different space. This allows us to realize the vertex operators acting in the twisted vertex representations constructed from the sum of the Grothendieck rings. Our group theoretic method naturally recovers the basic representations of twisted affine Lie algebras \( \mathfrak{b}[\frac{1}{2}] \) [FW,FLM 1,FLM 2]. As in [FW 1] we realize this by introducing a modified bilinear form associated to the McKay virtual character which is twice the trivial character minus the character of the two-dimensional natural representation of \( \mathfrak{sl}_2(\mathbb{C}) \).

Much of our construction is valid for an arbitrary finite group and we have introduced the modified bilinear form associated to an arbitrary virtual character of as well. In the special case when is the trivial character the twisted vertex operators generate an infinite dimensional generalized Clifford algebra, which recovers the twisted boson-fermion correspondence. We further obtain the super character tables of the spin group \( E_n \) for all \( n \), generalizing the results of [J1].

One may generalize the results of this paper to the quantum case as it was done in [FW 2] for the homogenous picture of quantum affine algebras [W]. Our results also suggest that various previous constructions associated to (quantum) vertex representations admit remarkable
interpretation via G rothendieck rings of certain nite groups which are
variations of wreath products, though every new step in this direction
is unpredictable and brings new surprises. It is a very interesting and
challenging problem to nd such a group theoretic realization.

The organization of the paper is as follows. In Sect. 2 we present
the representation theory and structures of the spin group $e_n$. In Sect. 3
we review superalgebras and super modules and de ne the Hopf algebra
of the super spin characters of $e_n$. In Sect. 4 we introduce the weighted
bilinear form s in the G rothendieck rings of super modules and construct
basic spin super modules. In Sect. 5 we de ne the twisted Heisenberg
algebras and their Fock spaces. In Sect. 6 we establish the isom ory
between the sum of G rothendieck rings of super modules of $e_n$ and the
Fock space of a twisted Heisenberg algebra. In Sect. 7 we construct
twisted vertex operators via the induction and restriction functors on
the G rothendieck rings. In Sect. 8 we obtain the twisted basic repre-
sentation of the a ne Lie algebra $g[1]$ and the corresponding toroidal
algebras. In Sect. 9 we derive the super spin character tables of $e_n$ for
all n from the twisted boson-ferm ion correspondence.

2. A double cover of the wreath product

2.1. The spin group $S_n$. In this subsection we discuss some of the
basic properties of the double covers of the symmetric group, which
were introduced by Schur in his seminal paper [3]. We will adopt the
modern account [5] of Schur's theory.

Let $S_n$ be the symmetric group of n letters, and we use the con-
vention of multiplying permutations from right to left (different from
[3, 5]). The spin group $S_n$ is the nite group generated by $z$ and
t; i = 1; n 1 subject to the relations:

\begin{align*}
(2.1) & \quad z^2 = 1; \quad t_i^2 = (t_i t_{i+1})^3 = z; \\
(2.2) & \quad t_i t_j = z t_j t_i; \quad i > j + 1; \\
(2.3) & \quad z t_i = t_i z;
\end{align*}

Let \( n \) be the hom orphism from \( S_n \) to \( S_n \) sending \( t_i \) to the trans-
position \((i; i+1)\) and \( z \) to 1. We see that \( S_n \) is a central extension
of \( S_n \) by the cyclic group \( Z_2 \):

\[ 1 \to Z_2 \to S_n \to S_n \to 1; \]

where the embedding sends the order 2 element in \( Z_2 \) to \( z \). Schur [3]
determined that \( H^2(S_n; \mathbb{C}) \cong Z_2 \) for \( n > 3 \). The group $S_n$ is one of
the two double covers of the symmetric group $S_n$ (n > 3). Our results
in this paper can be easily translated to the other double cover (cf. S\[12\]).

The group \( S_n \) has a parity given as follows. Let \( d \) be the homomorphism from the free group generated by \( f_1, z \) where \( f_1 = 1 \) to \( \mathbb{Z}_2 \) by \( d(t_i) = 1, i = 1; \quad n \) and \( d(z) = 0 \). It is easily seen that \( d \) preserves the relations \( \{ 1, \alpha^2 \} \). Thus \( d \) defines a homomorphism from \( S_n \) to \( \mathbb{Z}_2 \), which we still denote by \( d \). An element \( x \in S_n \) is called even (resp. odd) if \( d(x) = 0 \) (resp. \( d(x) = 1 \)). The parity in \( S_n \) given by \( d \) lifts the standard notion of even and odd permutations in the symmetric group \( S_n \).

The spin group \( S_n \) has a cycle presentation due to J.H. Conway and others (see [W]). Embed \( S_n \) into \( S_{n+1} \) by identifying their first \( n \) generators \( t_i, i = 1; \quad n \) and \( n+1 \) generator \( f \). For \( i = 1 \); \quad n \) we define \( x_i = t_i t_{i+1} \cdot t_{i+2} S_{n+1} \). For a sequence \( i; \quad m \) of distinct integers from \( f; 1; \quad n \) we can define cycles in \( S_{n+1} \) as follows.

\[
\begin{align*}
\text{(2.4) } & \quad [i_1, i_2, \ldots, i_m] = [1, z; x_{i_1} x_{i_2} \cdots x_{i_m}] ; \quad x < m ; \quad n, \\
\text{(2.5) } & \quad x_j [i_1, i_2, \ldots, i_m] = i z^{m-1} [i_1, i_2, \ldots, i_m] x_j (j \in (i, n)); \quad x_j^2 = 1;
\end{align*}
\]

It is known that \( (i_1, i_2, \ldots, i_m) \neq (i, i_2, \ldots, i_m) \) and \( n+1 (i_1) = (i, n+1) \).

We list some useful identities for the cycles.

\[
\begin{align*}
\text{(2.6) } & \quad [i_1, i_2, \ldots, i_m] = i z^{m-1} [i_1, i_2, \ldots, i_m]; \\
\text{(2.7) } & \quad [i_1, i_2, \ldots, i_m] = i z^{m-1} [i_1, i_2, \ldots, i_m]; \\
\text{(2.8) } & \quad [i_1, i_2, \ldots, i_m] \cdot [j_1, j_2, \ldots, j_k] = z^{m-1} k [i_1, i_2, \ldots, i_m]; \\
\text{(2.9) } & \quad [i_1, i_2, \ldots, i_m] \cdot [j_1, j_2, \ldots, j_k] \text{ are disjoint.}
\end{align*}
\]

Proposition 2.1. \( [j] \) Each element of \( S_n \) can be presented as

\[
z^p [i_1, i_2, \ldots, i_m] [j_1, j_2, \ldots, j_k] ;
\]

where \( f_1, \ldots, f_l \) is a partition of the set \( f; 1; \quad n \) and \( p = 0; 1 \). If

\[
z^0 c_0 c_2^0 = z^0 c_0 c_2^0 \quad \text{are two expressions of the same element in terms of cycles}\ c_0 \text{ and } c_2^0, \text{ then } l = 1 \text{ and there is a permutation } S_1 \text{ such that}
\]

\[
c_2^0 = c_0 z^{m-1}; \quad m \quad 1 \text{ (mod 2);}
\]

where \( j \) denotes the length of the cycle \( c_2 \). Moreover if \( [i_1, \ldots, i_m] \) is a cyclic permutation of \( f_1, \ldots, f_l \), then \( j = (i) \) for a cyclic permutation of \( f_1, \ldots, f_l \).
Let $\lambda$ be a partition and we identify with its Young diagram consisting of $l$ rows of $i_1$ squares respectively aligned to the left. A tableau $T$ of shape $\lambda$ is a numbering of the squares with integers $1; 2; \ldots; j$ each appearing exactly once. For each tableau $T$ with a numbering $a_{j1}; 1; a_{j2}; 2; \ldots; a_{21}; 2; a_{22}; 2$, we define the element $t$ of $S_n$ to be

$$t = [a_{11}, 1, 1]a_{21}, 2, 2]\ a_1; [a_{11}; a]$$

The permutation $Q_{i=1}^1 (a_{ii}; 1, 1)$ associated with $t$ will be denoted by $s(t)$. It follows from Proposition 2.11 that the general element in $S_n$ is of the form $\sigma(t)$. For a permutation $s \in S_n$, we also denote $t^s = Q_{i=1}^1 (s(a_{ii}); 1, 1)\ a$. The following can be checked by induction using (2.10) and (2.11).

Lemma 2.2. For any two elements $t; t'$ in $S_n$ associated to tableaux $T$ and $T'$ we have that

$$t t t^{-1} = z^{d(t')} d(t') t^{s(t')}$$

2.2. The spin wreath product $S_n$. In this subsection we introduce the main finite group $S_n$ in this work, and extend our discussion from $S_n$ to $E_n$.

Let $G$ be a finite group with $r + 1$ conjugacy classes. We denote by $\chi = f \in S_{r}$ the set of complex irreducible characters, where $\chi_0$ stands for the trivial character, and by $S_r$ the set of conjugacy classes. The character value ($\chi$) of $2$ at a conjugacy class $c_2$ yields the character table $f \in S_r$ of $c$.

Let $R(\lambda) = \bigoplus_{j=0}^\infty C_j \lambda$ be the space of complex valued class functions on $G$. For $c_2 \in S_r$, let $c$ be the order of the centralizer of an element in the class $c$, so the order of the class is then $j = c$. The usual bilinear form on $R(\lambda)$ is defined as follows:

$$hf; gi = \frac{1}{j!} X \ f(h)g(x) = \frac{1}{c_2} X c_2 f(c)g(c)$$

where $c_1$ denotes the conjugacy class $fx_1; 1 \ x_2 \ c$. Clearly $c = c_1$. We will often write $h; i$ for $h; i$ when no ambiguity may arise. It is well known that

$$h_1; j = i_1; j$$

$$X (c) (c_1) = c_{c_2} c_1; c_1 c_2$$

$$X (c) (c_1) = c_{c_2} c_1; c_1 c_2$$
Thus $R_Z(\cdot) = \mathbb{Z}_\pi Z_1$ endowed with this bilinear form becomes an integral lattice in $R(\cdot)$.

Given a positive integer $n$, let $\pi = n$ be the $n$-th direct product of $\pi$, and let $0$ be the trivial group. The spin group $S_n$ acts on $n$ through the action of the group $S_n$ by permuting the indices: $t(g;_n) = g(g_1;_n); g;_n)$; and $z(g;_n) = g(g_1;_n); g;_n)$. The wreath product $e_n = \omega S_n$ of with $S_n$ is dened to be the semidirect product

$$e_n = n \circ S_n = f(g; t); g = (g_1;_n) \cdot g;_n) \cdot t 2 S_n g$$

with the multiplication

$$(g; t)(h; s) = (gt(h); ts);$$

Note that $e_n$ reduces to $S_n$ when $n$ is trivial. Clearly $e_n$ is a central extension of $n$ by $Z_2$ and $\bar{f} j = 2n! j f$.

We define a a parity for $e_n$ by extending the parity of $S_n$. Let $d:e_n \to Z_2 = f0;1g$ be the homomorphism from $e_n$ to $Z_2$ given by

$$d(g; t) = 1; \quad d(gz) = 0;$$

Clearly the degree 0 subset $e^0_n$ is the wreath product $\omega A_n$, where $A_n$ is the alternating group, and the degree one part $e^1_n$ is the complementary subset.

Let $a$ be a section from $n$ to $e_n$ such that $a = 1$. An element $x 2 n$ is called split if $(x)$ is not conjugate to $z(\cdot x)$. Otherwise $x$ is said to be non-split. Clearly this definition does not depend on the choice of the section and two conjugate elements are simultaneously split or non-split. A conjugacy class of $n$ is called split if its elements are split. We will also say that an element $x 2 e_n$ is split (resp. non-split) if $(x)$ is split (resp. non-split). Clearly the class $C$ splits if and only if the preimage $C = (C)$ splits into two conjugacy classes in $e_n$.

Any representation of $e_n$ can be viewed as a representation of $e_n$. Such a representation of $e_n$ satisfies the property $(z) = \text{Id}$. A representation of $e_n$ is called spin if $(z) = \text{Id}$. It follows that the characters of spin representations vanish on non-split classes. In this paper we only consider spin representations.

We remark that spin representations are sometimes referred to as negative or projective representations in the literature.

2.3. Conjugacy classes of $e_n$. Let $= (1; 2; \pi)$ be a partition of the integer $j = j + \mu$, where $1 \leq j \leq 1$. The integer $l$ is called the length of the partition and is denoted by $l(\cdot)$. We will
identify the partition \((1; 2; \ldots; 1)\) with \((1; 2; \ldots; 0; \ldots; 0)\). We will also make use of another notation for partitions:

\[ \pi = (1^m 2^n) \]

where \(m\) is the number of parts in \(\pi\) equal to 1. The number \(n(\pi)\) is defined to be \(1^2, \ldots, 0\), where 0 is the dual partition associated to \(\pi\). We will use the dominance order on partitions. For two partitions \(\pi\) and \(\sigma\), we write \(\pi > \sigma\) if \(1^l \cdot 1 + 2 \cdot 2, \ldots, \) etc.

A partition \(\pi\) is called strict if its parts are distinct integers (excluding the trivial parts of zero), in which case all the multiplicities \(m_i\) are 1.

We will use partitions indexed by \(X\) and \(Y\). For a finite set \(X\) and \(Y\), we write \(X = (x)_{k \in X}\) a family of partitions indexed by \(X\), we write

\[ X \]

\[ k_1 \leq k_2 \leq \cdots \leq k_n \]

\[ j(x) \]

\[ X_{\times Y} \]

It is convenient to regard \(X\) = \((x)_{k \in X}\) as a partition-valued function on \(X\). We denote by \(P(X)\) the set of all partitions indexed by \(X\) and \(P_n(X)\) the set of all partitions in \(P(X)\) such that \(k_1 = n\). The total number of partitions, denoted by \(l(X) = \sum_{x \in X} l(x)\), in the partition-valued function \(X = (x)_{k \in X}\) is called the length of \(X\). The dominance order is extended to partition-valued functions as follows. We denote \(X > Y\) if \(x > y\) for each \(x \in X\). We say that \(X > Y\) and \(X \neq Y\) for each \(x \in X\).

For a partition-valued function \(X\) we define

\[ n(0) = X \]

\[ n(c) = X \]

\[ n(c) = \frac{1}{2} : \]

\[ c \]

\[ c \neq 1 \]

Let \(OP(X)\) be the set of partition-valued functions \((x)_{k \in X}\) in \(P(X)\) such that all parts of the partitions \((x)\) are odd integers, and let \(SP(X)\) be the set of partition-valued functions \(X \in P\) such that each partition \((x)\) is strict. When \(X\) consists of a single element, we will omit \(X\) and simply write \(P\) for \(P(X)\), thus the notation \(OP\) or \(SP\) will be used similarly.

Lemma 2.3. \(\|P_n(X)\| = \|SP_n(X)\|\).

Proof. The generating function of the cardinalities of strict partition-valued functions is

\[ \frac{Y^n}{1 - q^n} = \frac{1}{(1 - q^n)^n} \]

which is the generating function of \(SP_n(X)\). \(\square\)
We also define a parity on partitions. For each partition \( \lambda \) we define \( d(\lambda) = \sum_{j \in \lambda} j \). For a partition-valued function \( \{ (x)_k \} \), we define \( d(\lambda) = \sum_{ij \in \lambda} j (x)_j = k \). It is clear that the conjugacy class of type \( \lambda \) in \( S_n \) is even if and only if \( d(\lambda) \) is even. We define the parity of the partition-valued function \( \{ (x)_k \} \) to be the parity of \( d(\lambda) \). We define

\[
P_n^0(\lambda) = f \ 2 \ P_n(\lambda) j \ d(\lambda) \quad 0 \ (\text{mod} \ 2) g;
\]

\[
P_n^1(\lambda) = f \ 2 \ P_n(\lambda) j \ d(\lambda) \quad 1 \ (\text{mod} \ 2) g;
\]

and define \( SP_n^i(\lambda) = P_n^i(\lambda) \setminus SP_n(\lambda) \) for \( i = 0, 1 \).

We now recall the description of conjugacy classes of \( E_n \) [11]. Let \( x = (g; ) \) be an element in a conjugacy class of \( E_n \), where \( g = (g_1; g_2; \ldots; g_n); g \) for each cycle \( y = (i_1 i_2 \ldots k) \) in the permutation, the element \( g_y = g_{i_1} g_{i_2} \ldots g_k \) is called the cycle-product of \( x \) corresponding to the cycle \( y \). For each cycle \( c_2 \) and \( i \), let \( m_i(c) \) be the number of \( i \)-cycles in the permutation such that the cycle products \( g_y \) lie in the conjugacy class \( c \). Then \( c = (c_1; c_2; c_3; \ldots) \) defines a partition-valued function on \( E_n \). It is known that the partition-valued function \( (c)_k \) is in one-to-one correspondence with the conjugacy class of \( x = (g; ) \) in \( E_n \) and is called the type of the conjugacy class. We will also say that an element has conjugacy type \( c \) if this element is contained in the conjugacy class.

Let \( 1^j \) be the representation of \( E_n \) given by \( x \) \( (1^j)^x \). A representation \( 1^j \) of \( E_n \) is called a double spin representation if

\[
1^j = 1^j.
\]

If \( 0 = (1^j)^x \), then \( 0 \) and \( 0 \) are called associate spin representations of \( E_n \).

The following result was proved in [11] for a double cover of any finite group.

**Proposition 2.4.** The number of split conjugacy classes of \( E_n \) is equal to the number of irreducible spin representations of \( E_n \).

2.4. Split conjugacy classes of \( E_n \). We define an order of conjugacy classes of \( E_n \) as \( c^0 = \phi g; c^j \). For each partition-valued function \( \{ (c)_j \} \), we let \( t_{(c)} \) be the element of \( E_n \) associated to a tableau \( T \) of shape \( (c) \) using the numbers \( j, i_1, j, (c)_j+1, j, i_j, (c)_j \) and we define the element \( t \) to be

\[
t = t_{(c)} t_{(c)} t_{(c)}
\]
which depends on the sequence $T$ of the tableaux $T_{i2}$, $i2T$.

Theorem 2.5. Let $\lambda = (c)_{2^k}$ be the type of a conjugacy class $C$ in $S_n$. Then the preimage $n^{-1}(C)$ splits into two conjugacy classes in $S_n$ if and only if

(1) when the class $C$ is even and all the $(c)_{2^k}$ are partitions with odd integer parts, i.e., $2OP_n(\lambda)$;

(2) when the class $C$ is odd and all the $(c)_{2^k}$ are strict partitions, i.e., $2SP_n(\lambda)$.

Proof. (1) Let $d(\lambda)$ be even and let each partition $(c)_{2^k}$ have odd integer parts. Assume the contrary that $(g; t)$ and $z(g; t)$ are conjugate in $n^{-1}(C)$, where $t$ is associated to a sequence of tableaux (see Eqn. (2.16)). Then for some $0; t \in S_n$

$$
(0; t)(g; t)(0; t)^{-1} = \lambda \quad s(\lambda)(g) \quad s(\lambda)s(\lambda; t; t)^{-1}
$$

(2.17)

where we have used the fact that $s(\lambda)s(\lambda) = s(\lambda)s(\lambda)$. It follows from Lemma 2.2 that $zt = z^{d(\lambda)}t^{\lambda} = t^{\lambda}$, since $d(\lambda) = 0 \pmod{2}$. Let $t = c_1c_2 \cdots$ and $t^{\lambda} = c_1c_2 \cdots$ be their cycle representations. Then $c_1 = z^{m_1}; c_2 \cdots$ and $m_1 = j_2^j + 1 \pmod{2}$ for some $2 \leq j$ by Proposition 2.1. Since each cycle length $j_2$ is odd, all the cycles commute with each other. Substituting $c_1 = z^{m_1}; c_2 \cdots$ back and rearranging the cycles, we have

$$
1 = z^{1+d(\lambda)} = z^{1+d(\lambda)} = z;
$$

which is a contradiction.

Now suppose that for some $c2$ there is an even cycle in $(c)$ of the class $C$ of type $\lambda$. That is, there is an element $(g; t)2^{-1}(C)$ such that $t = [i_1 \cdots i_k]$. Consider the element $j_2^j t e_n$, where $t = [i_1 \cdots i_k]$, and $h = (h_1; \cdots ; n)$ with $h_j = 1$ for $j \in i_n$ and $h_i = g_i; s; t = 1; \cdots ; 2k$. We claim that

$$
(0; t)(g; t)(0; t)^{-1} = \lambda h s(\lambda)(g)s(\lambda; t; t)^{-1} = (g; z t);
$$

which is shown by two steps. First we consider the $j$th component of $hs(\lambda)(g)s(\lambda; t; t)^{-1}$ in $S_n$. It equals $1$ when $j \notin i_n$, and it
equals \( g_u g_u \), \( g_u^{-1} = g_u \) for \( j = i_u \). Secondly we have
\[
t^{t^{-1}} = z i_{i_u} [i_{2x} i_{i_u}] i = z t
\]
by using Eqn. (2.17) and \( d( ) \equiv 0 \pmod{2} \) again. Thus \((g; t)\) is conjugate to \(( g; t)\). Therefore all partitions \((c) \) must be from \( \text{OP}( ) \) if \( ^1(c) \) splits.

(2) Let \( d( ) \) be odd. Assume all partitions \((c) \) are strict partitions. If on the contrary \((g; t)\) is conjugate to \(( g; t)\), then using \( d( ) = 1 \) we have as in (2.17) that \( zt = z^{d(1)} t^{s(1)} \) for the permutation \( s( ) \) \( 2 S_n \) associated to some \( 2 P_n( ) \). Let \( t = c_0 c_1 \) and \( t^{s(1)} = c_0 c_1^{0} 0^{1} c \) be their cycle representations. Then \( c_k = z^{h_k} i_{j}^{0} ( ) \) for some \( 2 S_i \).

Since \( s( ) \) cyclically permutes the indices in each cycle of \( s( ) \) we have \( d( ) = d( ) \). On the other hand, note that each cycle \( q \) corresponds to one part in \((c) \) for some \( c 2 \) and any conjugation of \( c \) still corresponds to a part in the same \((c) \). When we plug the equations \( c_k = z^{h_k} i_{j}^{0} ( ) \) back to \( zt = z^{d(1)} t^{s(1)} \) we see that \( \) is actually the identity since \((c) \) is strict. Therefore \( z^{h_k} i_{j}^{0} ( ) = z^{s(1)} \). Then \( d( ) = 1 \) \( d( ) + 1 \pmod{2} \), which is a contradiction. Hence \( n = 1(c) \) splits.

Now suppose \( n = 1(c) \) splits. If there are two identical parts in \((c) \) for some conjugacy class \( c 2 \), say \( t = [i_k][i_{j}^{1}] \) and \( h_a = 1 \) for \( a \in i_u ; j \), and
\[
\begin{align*}
x g_{i_j} g_{i_k} & = x g_{i_k} g_{i_j}, \\
\begin{cases}
\frac{x \times \frac{g}{j}}{g} = g_u g_{i_u} i_i : g
\end{cases}
\end{align*}
\]
Consider the element \((h; t)\) such that \( t = [i_k][i_{j}^{1}] \) and \( h_a = 1 \) for any \( i_u j \), and
\[
\begin{align*}
h_{i_k} & = g_{i_k} i_i \times (g_{i_k} i_{j}^{1}) \times g \quad s = 1; \\
h_{i_k} & = g_{i_k} i_i \times (g_{i_k} i_{i}^{1}) \times g \quad s = 1;
\end{align*}
\]
Clearly \( h_{i_k} = x \) and \( h_{i_k} = x^{-1} \) by Eqn. (2.19). Therefore we have the following equations for \( s = 1; 2; \) \( ;k \pmod{2} \)
\[
\begin{align*}
h_{i_k} & = g_{i_k} h_{i_k} g_{i_k}^{-1} \quad h_{i_k} = g_{i_k} h_{i_k} g_{i_k}^{-1}; \\
h_{i_k} & = g_{i_k} h_{i_k} g_{i_k}^{-1} \quad h_{i_k} = g_{i_k} h_{i_k} g_{i_k}^{-1};
\end{align*}
\]
which imply that \( h s( ) (g) s( ) (h^{-1}) = g \). Note also that \( s( ) s( ) = s( ) s( ) \) and \( d( ) = k \pmod{2} \). We see that the conjugation
\[
(h; t)(g; t) (h; t) \equiv (h s( ) (g) s( ) (h^{-1}); z^k t^{s(1)}) = (g; z^k t^{s(1)});
\]
where we used \( d(\cdot) = 1 \). Observe that by Eqn. \((2.23)\)
\[
t^\ast = \left[ j_k \right] \beta_j \]
Plugging this into Eqn. \((2.24)\) we obtain that \( (p_t) (g_t) \) \( (p_t) \) \( = z(g_t) \), and this contradiction says that each partition \( (c) \) must be strict.

Let \( C \) be a conjugacy class in \( n \) of type \( = ( (c)_2 \) \( 2 \) \( P_n(\cdot) \). We fix an order of the conjugacy classes of as before; \( c^0 \). Let \( T (c) \) be the special tableau such that the numbers \( (1 + j_0 j (c) + 1) \), \( P_j = k \) \( (c) \) \( j \) \( c \)
appear in the natural order from the left to right and up to bottom in the Young diagram of shape \( (c) \), and thus
\[
t (c) = \left[ (a + 1_1) ; \beta_j + (a_1 \left( 1 \right) \right]
\]
\[(2.21)]
where \( a_1 = \left( \begin{array} \end{array} \right) \) and \( (c) = ( (c)_1 ; \beta_j + (a_1 \left( 1 \right) \right) \). We define the special element \( t \) by
\[
t = t (c) t (c) t (c)
\]
For each split conjugacy class \( C \) in \( n \) of type \( , \) we define the conjugacy class \( D^+ \) in \( E_n \) to be the conjugacy class containing the element \( g_t \). We also define \( D = zD^+ \). Then \( n^{-1} (c) = D^+ \). Let \( (D^+) \) \( = \) \( 2 \) \( D^+ g \). We remark that \( (D^+) \) \( = z^0 (c)(D^-) \), where \( n(0) \) is defined in \((2.13)\) and \( \beta \) is the partition-valued function given by \( \beta (c) = ( (c)_1 \). \( G \)

Given a partition \( = (2^3 ; 2^2 \cdots) \), we define
\[
z = \sum_{i=1}^{m} i!
\]
\[(2.22)]
We note that \( z \) is the order of the centralizer of an element of cycle-type \( \in S_{j \cdot j} \).

For each partition-valued function \( = ( (c)_2 \) we define
\[
Z = \sum_{c_2} z (c)_1 \beta_j \]
which is the order of the centralizer of an element of conjugacy type \( = ( (c)_2 \) (see \( M_1^j \) ).

Proposition 2.6. The order of the centralizer of an element of conjugacy type \( \in E_n \) is given by
\[
z^c = 2Z; \text{ C is split}
\]
\[
z = Z; \text{ C is non-split.}
Proof. Let $C$ be a conjugacy class in $n$. If $^1(C)$ does not split, then $^1(C)$ is a conjugacy class in $e_n$, so its centralizer has the order $j_{n} j^1(C) j = j_{n} j^D j = Z$. Otherwise $^1(C) = D^+[D]$, and $j_{n} j^D j = 2Z$.

**Theorem 2.7.** [HH] (1) The number of conjugacy classes of $n$ and $^0_n$ are given by the following formulas:

- Split classes of the group $n$: $j = 2 j P^1_n(j) + j P^0_n(j)$.
- Split classes of the group $^0_n$: $j = j P^1_n(j) + 2 j P^0_n(j)$.

(2) The number of irreducible double spin representations is equal to the number of even strict partition-valued functions on $n$, and the number of pairs of irreducible associate spin representations is equal to the number of odd strict partition-valued functions on $n$.

Proof. The first statement in Part (1) is a corollary of Theorem 2.5 and Lemma 2.3. To see the second equation in (1), we observe that an irreducible spin representation $j$ decomposes as follows when restricting to the subgroup $e^0_n$: $j = j P^1_n(j) + j P^0_n(j)$; $j$ is a double spin, $j$ is an associate spin.

Moreover a pair of the associated spin representations, when restricted to $e^0_n$, become the same irreducible representation. Applying the counting formulas in (1) we obtain Part (2).

**Remark.** That the number of split conjugacy classes of $n$ contained in $^0_n$ is equal to $j P^1_n(j) + j P^0_n(j) = j P_n(j)$ by Theorem 2.7.

### 3. The Hopf algebra $R$ of super spin characters

#### 3.1. Superalgebras and super modules. We basically follow the exposition of [13] in this subsection. A complex superalgebra $A = A_0 \oplus A_1$ is a $Z_2$-graded complex vector space with a binary product $A \cdot A \rightarrow A$ such that $A_0 A_1 \rightarrow A_{1+1}$. A vector space $V = V_0 \oplus V_1$ is a super module for a superalgebra $A = A_0 \oplus A_1$ if $A_0 V_j \rightarrow V_{i+j}$. Elements of $V_1$ are called homogenous.

A linear map $f : M \rightarrow N$ between two $A$-super modules $M$ and $N$ is homogenous of degree $i$ if $f(M_j) = M_{i+j}$ and for any homogenous element $a \in A$ and any homogenous vector $m \in M$ we have

$$f(am) = (1)^{(a_1)} f(m)$$
Let
\[ \text{Hom}_A(\mathcal{M};\mathcal{N}) = \text{Hom}_A(\mathcal{M};\mathcal{N})_0 \oplus \text{Hom}_A(\mathcal{M};\mathcal{N})_1; \]
where \( \text{Hom}_A(\mathcal{M};\mathcal{N})_1 \) consists of \( A \)-super-homomorphisms of degree 1 from \( \mathcal{M} \) to \( \mathcal{N} \).

Let \( V = V_0 \oplus V_1 \) and \( W = W_0 \oplus W_1 \) be two supermodules. The tensor product \( V \otimes W \) is also a supermodule with \( (V \otimes W)_k = \sum_{k+1 \equiv m \pmod{2}} V_k \otimes W_m \). Submodules, irreducible or simple supermodules are defined similarly as usual. Two examples of complex simple superalgebras are given in order.

Let \( r; s \in 2 \mathbb{N} \). We define \( M(\mathcal{M};\mathcal{N}) \) to be the \( C \)-superalgebra of \( (r+s) \)-square matrices with the grading
\[ M(\mathcal{M};\mathcal{N})_0 = f \begin{array}{rr} A & 0 \\ 0 & D \end{array} A 2 M_{r+s}(C); D 2 M_{s+m}(C)g; \]
\[ M(\mathcal{M};\mathcal{N})_1 = f \begin{array}{rr} 0 & B \\ C & 0 \end{array} B 2 M_{r+s}(C); C 2 M_{s+m}(C)g; \]
and the operations are the underlying usual matrix addition and multiplication. As in the ungraded case, \( M(\mathcal{M};\mathcal{N}) \) can also be viewed as the superalgebra of \( \mathbb{Z}_2 \)-graded linear maps of \( \mathcal{M}^r \mathcal{N} = \mathcal{M}^r \mathcal{N}^c \) with the usual superpositions of maps. It is easily seen that \( M(\mathcal{M};\mathcal{N}) \) is a simple \( C \)-superalgebra and \( \mathcal{M}^r \mathcal{N} \) is a simple \( M(\mathcal{M};\mathcal{N}) \)-supermodule.

Another example is the \( C \)-superalgebra \( Q(\mathcal{N}) \) as a supervector space it is defined by
\[ Q(\mathcal{N})_0 = f \begin{array}{rr} A & 0 \\ 0 & A \end{array} A 2 M_{n+m}(C)g; \]
\[ Q(\mathcal{N})_1 = f \begin{array}{rr} 0 & B \\ B & 0 \end{array} B 2 M_{n+m}(C)g; \]
The superalgebra structure is given by the usual matrix multiplication. The space \( C^{n,h} \) is also a \( Q(\mathcal{N}) \)-supermodule under the usual matrix multiplication.

Wall [6] showed that these two simple superalgebras are the only two types of simple superalgebras over \( C \). In the sequel we will call the supermodule \( \mathcal{M}^r \mathcal{N} \) of type \( M \) if it is considered as a \( M(\mathcal{M};\mathcal{N}) \)-supermodule and \( C^{n,h} \) of type \( Q \) if it is considered as a \( Q(\mathcal{N}) \)-supermodule.

For any finite group \( G \) and a subgroup \( H \) of index 2, we set the parity of elements of \( H \) (resp. \( G \cap H \)) to be even (resp. odd). The corresponding group superalgebra of \( G \) is semisimple (see [23]). In the case of the spin wreath product \( e_n \) and the subgroup \( e_n^0 = n oA_n^0 \),
this parity agrees with the parity given by the homomorphism \( d \) (see (2.12)). As a superalgebra, \( C[\mathfrak{e}_n] \) is given by

\[
C[\mathfrak{e}_n] = f \sum a_g g \mathfrak{e}^0_n g;
\]

\[
C[\mathfrak{e}_n] = f \sum a_g g \mathfrak{e}^1_n g;
\]

and the product is the usual multiplication.

**Proposition 3.1.** There exists an isomorphism of \( C \)-superalgebras

\[
C[\mathfrak{e}_n]' = M (m^i_j) \otimes Q (n^i_j);
\]

Any finite dimensional \( C[\mathfrak{e}_n] \)-super module is isomorphic to a direct sum of simple super modules of type \( M \) and \( Q \).

By the definition of spin representations and Lemma 2.3 we know the number of irreducible spin super modules for \( \mathfrak{e}_n \).

**Proposition 3.2.** The number of irreducible spin super modules of \( \mathfrak{e}_n \) is equal to \( \mathfrak{sp}_n(\mathbb{R}) \), the number of strict partition-valued functions on \( \mathbb{R} \).

If \( V \) is an irreducible \( \mathfrak{e}_n \)-super module of type \( M \), then its underlying \( \mathfrak{e}_n \)-module is irreducible. If \( V \) is an irreducible \( \mathfrak{e}_n \)-super module of type \( Q \), then its underlying \( \mathfrak{e}_n \)-module decomposes into two irreducible \( \mathfrak{e}_n \)-modules \( U \) and \( U^0 \), where \( U^0 = U \) as a vector space and its action is given by \( a u = (1)^{d(a)} a u \) for any homogeneous element \( a \).

3.2. Induced super modules. Let \( G \) be a finite group with a central involution \( z \) and a parity epimorphism \( d: G \rightarrow \mathbb{Z}_2 \) such that \( d(z) = 0 \). Let \( H \) be a subgroup of \( G \) containing \( z \) such that the restriction of \( d \) on \( H \) is not identically zero. Such a pair \( (G; H) \) of finite groups will be called an admissible pair of finite groups.

The group algebras \( C[G] \) and \( C[H] \) become superalgebras with \( G^0 = \ker(d); H^0 = \ker(d_H) \) and \( G^1 = G; H^1 = H \).

Let \( W \) be a \( C[H] \)-super module. We define the induced super module \( \text{Ind}^G_H W \) for \( C[G] \) by

\[
\text{Ind}^G_H W = C[G] \otimes C[H] W
\]

with the action given by \( g(h w) = ghw \). Clearly \( \text{Ind}^G_H W \) is a spin super module if \( W \) is a spin super module.

The following lemma can be checked similarly as the ordinary case \( \mathfrak{g}[n] \).
Lem m a 3.3. Let \((G; H)\) be an adm issible pair of finite groups. Let \(V\) be a \(C[G]\)-supermodu le and \(W\) a \(C[H]\)-sub-supermodu le of \(V\). Then \(V\) is equal to \(\text{Ind}_H^G W\) if and only if

\[
V = \bigoplus_{s \in \text{Int } \text{Ind}_H^G W} sW;
\]

where \(sW\) denotes the subspace \(xW (x \in s)\) of the supermodu le \(V\).

Let \((G; K)\) be another adm issible pair of finite groups. Consider the double cosets \(HsK\) of \(H\) and \(K\) in \(G\). For \(s \in 2\text{Int }H\cap \text{Ind}_H^G W\) the set \(Hs = sHsK\) is a subgroup of \(K\). The following analog of Mac key’s theorem can be proved similarly as in the ordinary case using Lem m a 3.3.

Prop os i t i on 3.4. Let \((G; H)\) and \((G; K)\) be adm issible pairs of finite groups as above. Then we have

\[
\text{Res}_K \text{Ind}_H^G W = \bigoplus_{s \in 2\text{Int }H\cap \text{Ind}_H^G W} \text{Ind}_K^G \text{Res}_H^G W;
\]

as supermodules.

3.3. The space \(R\).

A spin class function on \(e_n\) is a class function map from \(e_n\) to \(C\) such that

\[
f(zx) = f(x);
\]

Thus spin class functions vanish on non-split conjugacy classes. A spin super class function on an \(e_n\) is a spin class function \(f\) on \(e_n\) such that \(f\) vanishes further on odd strict conjugacy classes. In other words, \(f\) corresponds to a complex functional on \(OP\) in view of Theorem 2.3.

Let \(R\) be the \(C\)-span of spin super class functions on \(e_n\). Let \(R\) be the \(C\)-span of class functions on \(e_n\). Let \(R^0\) be the subspace of the class functions \(f(x)\) such that \(f(zx) = f(x); x \in e_n\), and let \(R^1\) be the space of spin class functions. Then we have

\[
\begin{align*}
R(e_n) &= R^0(e_n), R^1(e_n); \\
R(e_n) &= R^1(e_n), R^0(e_n).
\end{align*}
\]

In this paper we will focus on the space \(R(e_n)\). We remark that \(R^1(e_n)\) can be identified as a vector space with the Grothendieck ring of spin representations of \(e_n\), and it is not difficult to recover \(R^1(e_n)\) from \(R(e_n)\) using Proposition 3.2.
The standard inner product $h_{j|i}$ on $R(\mathcal{E}_n)$ induces an inner product on $R(\mathcal{E}_n)$. For two spin super class functions $f; g \in R(\mathcal{E}_n)$, we define

$$h_{f; g} = h_{f; g_{e_0}}$$

(3.6)

$$= \frac{1}{\mathcal{P}} \sum_{x_1, x_2 \in \mathcal{E}_n} f(x)g(x^\dagger) = \frac{1}{2} h_{f; g_{e_0}};$$

where $h_{f; g_{e_0}}$ is the inner product of $f_{e_0}$ and $g_{e_0}$ in the space of class functions on the subgroup $\mathcal{E}_n^0$. Since every split conjugacy class of $\mathcal{E}_n$ have the form $fD^+g$, and $f(D^+) = f(D)$, we can rewrite the inner product by using Proposition 2.6.

$$h_{f; g} = \sum_{x \in \mathcal{P}} \frac{1}{2} f(x)g(x);$$

(3.7)

where $f(\_\_\_) = f(D^+)$ and $^\dagger$ is defined in Sect. 2.4. In the sequel we will take the value of a class function at $2 \mathcal{P}$ to be the value at the conjugacy class $D^+$.

Let $V = V_0 \oplus V_1$ be a $\mathcal{E}_n$-super module. We define the character of $V$ as the function $\chi : x \mapsto \text{tr}(x), x \in \mathcal{E}_n$. Clearly $\chi(\mathcal{E}_n^1) = 0$.

Proposition 3.5. The characters of irreducible spin $\mathcal{E}_n$-super modules form a $C$-basis of $R(\mathcal{E}_n)$. Let $f$ and $g$ be two irreducible characters of spin super modules, then

$$h_{f; g} \geq 1 \text{ if type } M$$

(3.8)

$$\geq 0 \text{ otherwise}$$

Conversely, if $h_{f; g} = 1$ for $f \in R(\mathcal{E}_n)$, then $f$ is an irreducible spin $\mathcal{E}_n$-super module of type $M$. If $h_{f; g} = 2$, then either $f$ is the character of an irreducible spin super module of type $Q$ or $f$ is a sum or difference of two irreducible characters of spin super modules of type $M$.

Proof. Let $c$ be the characteristic function on the conjugacy class $c$. Then $R(\mathcal{E}_n)$ is spanned by $c_r$, where $r$ ranges over the set of split even classes. Thus $\dim(R(\mathcal{E}_n)) = \mathcal{P}_n(\_\_\_\_\_\_)$. On the other hand we see that the characters of spin super modules are class functions in $R(\mathcal{E}_n)$ since the trace of any odd endomorphism is zero. Let $f$ and $g$ be the characters of two irreducible spin super modules of $\mathcal{E}_n$. It follows from Proposition 3.2 that the underlying
module of or is either irreducible module or the sum of two associated irreducible modules according to their types, which implies immediately the orthogonality relation (3.8). Therefore the matrix of the inner product is orthogonal on the set of super spin characters. Then by Lemma 2.3 and Proposition 3.2

$$\dim (R \langle e_n \rangle) \neq \dim (R \langle e_m \rangle)$$

Thus the two inequalities above become equality, and so the irreducible characters of spin $e_n$-supers modules form a $Z$-basis in $R \langle e_n \rangle$.

The last characterization of irreducible supers modules follows from the semi-simplicity of the superalgebra $C[\mathbb{Z}]$ and the usual orthogonality of ordinary irreducible characters. \[\square\]

3.4. Hopf algebra structure on $R$. We now define one of our main objects

$$R = \bigoplus_{n=0}^{M} R \langle e_n \rangle.$$ 

Let $e_n \sim e_m$ be the direct product of $e_n$ and $e_m$ with a twisted multiplication

$$(t; t^0) (s; s^0) = (tsz d(t^0)s^0; t^0 s^0),$$

where $s, t \in e_n$; $s^0, t^0 \in e_m$ are homogeneous. Note that $f_n \sim e_m j = f_n f_m j$ where $j$ is the spin product of $e_n$ and $e_m$ (see [HH]) by

$$e_n \sim e_m = e_n \sim e_m = f(l; 1); (z; z)g;$$

which can be embedded into the spin group $e_{n+m}$ canonically by letting

$$(g; t^0_i); l \equiv (g; t^0_i); l \equiv (g; t^0_{i+j})$$

where $i = 1; \ldots; n$, $j = 1; \ldots; m$. We will identify $e_n \sim e_m$ with its image in $e_{n+m}$ and regard it as a subgroup of $e_{n+m}$. Clearly

$$e_{n+m} \langle e_n \sim e_m \rangle$$

is the pull-back of $e_{n+m}$.

Remark 3.6. Partition $f_{1; 2} \ldots n + m$ into a disjoint union of subsets $I$ and $J$ with $j \in I$ and $j \in J$. Then $e_n \sim e_m$ can be embedded into $e_{n+m}$ using a similar map as (3.10) by mapping the generators of $S_n$ and $S_m$ to the generators of $S_{n+m}$ indexed by $I$ and $J$ respectively. One can check that all such embeddings of $e_n \sim e_m$ are conjugate subgroups in $e_{n+m}$.

The subgroup $e_n \sim e_m$ has a distinguished subgroup of index 2 consisting of even elements given by $d$. We define $R \langle e_n \sim e_m \rangle$ to be the
space of spin class functions on $e_n \wedge e_m$ that vanish on odd conjugacy classes of $e_n \wedge e_m$.

For two spin super modules $U$ and $V$ of $e_n$ and $e_m$ we define the super (outer)-tensor product $U \cdot V$ by

$$(t; s) \cdot (u \cdot v) = (d(t) \cdot d(s)) \cdot (tu \cdot sv);$$

where $s$ and $u$ are homogeneous elements. We see immediately that

$$(z; 0) \cdot (u \cdot v) = (u \cdot v);$$

$$(z; 1) \cdot (u \cdot v) = (u \cdot v).$$

This says that $U \cdot V$ is a spin $e_n \wedge e_m$-super module.

The following is a direct generalization of a result in [1] for trivial.

Proposition 3.7. Let $U$ and $V$ be simple super modules for $e_n$ and $e_m$ respectively. Then

1) If both $U$ and $V$ are of type $M$, then $U \cdot V$ is a simple $e_n \wedge e_m$-super module of type $M$.

2) If $U$ and $V$ are of different type, then $U \cdot V$ is a simple $e_n \wedge e_m$-super module of type $Q$.

3) If both $U$ and $V$ are of type $Q$, then $U \cdot V \cdot N \cdot N$ for some simple $e_n \wedge e_m$-super module $N$ of type $M$.

The super (outer)-tensor product defines an isometric isomorphism

$$R^{(e_n)} R^{(e_m)} \cdot_\mu \cdot_\nu R^{(e_n \wedge e_m)};$$

which is actually an isomorphism over $Q$ by Proposition 3.7.

The space $R$ carries a multiplication defined by the composition

$$m : R^{(e_n)} R^{(e_m)} \cdot_\mu \cdot_\nu R^{(e_n \wedge e_m)} \cdot_{Ind} \cdot_{Res} R^{(e_{n+m})};$$

and a composition defined by the composition

$$: R^{(e_n)} \cdot_{Res} \cdot_{Ind} R^{(e_{n+m} \wedge e_m)};$$

Here $Ind$ and $Res$ denote the induction (see (3.3)) and restriction functors respectively. The isomorphism $\cdot_\mu \cdot_\nu$ is equal to $0 \cdot_{Ind} \cdot_{Res} 1 \cdot_{Ind} \cdot_{Res}$ (see (3.11)).

Theorem 3.8. The above operations define a Hopf algebra structure for $R$. 

Proof. Using Remark 3.2 twice we observe that the following two embeddings give rise to two conjugate subgroups in $\mathcal{E}_{n+m+1}$ (see Remark 3.6):

$$(\mathcal{E}_n \cap \mathcal{E}_m)^{e_1} \subseteq \mathcal{E}_{n+m+1} - \mathcal{E}_m \cap \mathcal{E}_n^{e_1}$$

Using this and Lemma 3.3 we can easily check the associativity of the product.

For a simple super module $V$ we define

$$(3.14) \quad c(V) = \begin{cases} 0 & \text{if } V \text{ is type } \mathcal{M} \\ 1 & \text{if } V \text{ is type } \mathcal{Q} \end{cases}$$

Let $V_1, V_2$, and $V_3$ be simple super modules for $C[\mathcal{E}_n], C[\mathcal{E}_m]$, and $C[\mathcal{E}_l]$ respectively. It is easy to see that $c(V_1; V_2) = c(V_1)c(V_2)$ satisfies the cocycle condition

$$(3.15) \quad c(V_1; V_2) + c(V_1; V_2; V_3) = c(V_2; V_3) + c(V_1; V_2; V_3)$$

Therefore we can define $c(V_1; V_2; V_3)$ to be either of the above expressions.

Using the cocycle $c$ we prove the coassociativity as follows. Let $U$ be a $C[\mathcal{E}_n]$-super module and suppose that $\text{Res}^{e_n \cap e_m \cap e_k} U = \sum_{(m; l; k)} U_{\mathfrak{M}_m, \mathfrak{M}_l, \mathfrak{M}_k}$ as an irreducible decomposition. Then we have

$$(1 \quad 1 \quad 1 \quad 1) (U) = \sum_{(m; l; k)} \text{Res}^{e_n \cap e_m \cap e_k} U_{\mathfrak{M}_m, \mathfrak{M}_l, \mathfrak{M}_k}$$

where we used the cocycle condition in the third equation, and the notation $2^{(U_{\mathfrak{M}_m, \mathfrak{M}_l, \mathfrak{M}_k})}$ stands for the multiplicity-free summation of the irreducible components (c.f. Proposition 3.7 and definition of $\text{Ind}$).

Finally we look at the compatibility of multiplication and comultiplication. Fix $m$ and $n$, it follows from Proposition 3.7 that

$$(U \quad V) = \sum_{k+l=m+n} \text{Res}^{e_k \cap e_l \cap e_m \cap e_n} \text{Ind}^{e_m \cap e_n} U \quad V$$

where we used the cocycle condition in the third equation, and the notation $2^{(U_{\mathfrak{M}_m, \mathfrak{M}_l, \mathfrak{M}_k})}$ stands for the multiplicity-free summation of the irreducible components (c.f. Proposition 3.7 and definition of $\text{Ind}$).
where \( s \) runs through the double cosets \( e_m^ne_n^ne_m+n^e_k^ne_1 \). Notice that the double cosets are in one-to-one correspondence with the double cosets \( m^nn^m+n^k^ne_1 \). Again by the cocycle property of \( c \) and counting the double cosets we can check that the last summation is exactly \((U) (V)\).

Remark 3.9. Our Hopf algebra is different from that of \([HH]\) where a bigger space than our \( R \) was used.

The standard bilinear form in \( R \) is defined in terms of those on \( R(e_n) \) as follows:

\[
\lambda_u; v = \sum_n h_n; v_n; e_n; \quad u = \sum_n u_n \text{ and } v = \sum_n v_n \text{ with } u_n; v_n \in e_n.
\]

4. Basic spin representations of \( e_n \)

4.1. A weighted bilinear form on \( R(\cdot) \) and \( R(e_n) \). In \([FW1]\) we introduced the notion of weighted bilinear form on \( R(\cdot) \) and coherently combined several examples in this concept. We will also similarly define weighted bilinear forms on the space \( R(e_n) \).

Let \( c \) be a self-dual class function in \( R(\cdot) \), i.e., \( (c) = (c^1) \). Let \( \lambda(x) \) denote the product of two characters in \( R(\cdot) \), which is a orded by the tensor product. Let \( a_{ij} \in C \) be the (virtual) multiplicities of \( j \) in \( i \):

\[
\sum_j a_{ij}\delta_{ij} = \lambda(x) \quad \text{or} \quad \sum_{ij} a_{ij} \delta_{ij} = (\lambda(x))_0.
\]

We denote further by \( A \) the \((r+1) \times (r+1)\) matrix \( (a_{ij})_0 \). Then the weighted bilinear form \( h f; g i \) is defined by

\[
h f; g i = h \sum_{ij} a_{ij} \delta_{ij} f; gi \in R(\cdot);
\]

Alternatively it can be explicitly given by

\[
h f; gi = \frac{1}{X} \sum_{ij} a_{ij} \delta_{ij} \lambda(x) \quad \text{or} \quad \sum_{ij} a_{ij} \delta_{ij} \lambda(x)
\]

\[
(4.2)
\]

(4.3)
In particular, Eqn. (4.3) is equivalent to
\[(4.4)\]
\[h_{ij} = a_{ij};\]

The self-duality implies that $A$ is a symmetric matrix. Note that the
weighted bilinear form becomes the standard bilinear form when $\omega = 0$,
the trivial character.

Let $V$ be a spin super module for $\mathbb{C}_n$ and $W$ a module for $n$. As a
$\mathbb{Z}_2$-graded vector space $W = V \mathbb{V} \mathbb{W}$ and the action of
$\mathbb{C}_n$ is determined by
\[(4.5)\]
\[(g; z^p) (w, v) = (g; s(\ )) w (g^2z) v; \quad g 2^n; \quad 2 P (\ );\]

It is easy to check that the tensor product $V \mathbb{W}$ is a spin $\mathbb{C}_n$
super module. This construction defines a morphism:
\[(4.6)\]
\[R (n) \quad R (\mathbb{C}_n) \quad R (\mathbb{C}_n);\]

Let us recall the construction of character $n(\ )$ in $[N, [F, W, 1]]$. Let
be an irreducible character of a graded by the $\mathbb{Z}_m$ module $V$, the tensor
product $V \mathbb{W}$ is naturally a $\mathbb{Z}_m$ module by the direct product action of $n$
composed with permutation action of the symmetric group $S_n$. The
resulting character of $n$ is denoted by $n(\ )$. Furthermore we can
eextend $n$ from to $R (\ )$. The character value of $n(\ )$ at the class
$= (c)$ is given by
\[(4.7)\]
\[n(\ ) = \frac{1}{c_2} Y (c);\]

It is clear that the class function $n(\ )$ is self-dual as long as $h$ is.

We now introduce a weighted bilinear form on $R (\mathbb{C}_n)$ by letting
\[h_{f;g} = h_{n(\ )} f;g; 2 R (\mathbb{C}_n);\]

where we use the map (4.3). The self-duality of $n(\ )$ implies that the
bilinear form $h; i$ is symmetric.

Remark 4.1. When $n = 1$, this weighted bilinear form obviously reduces to the weighted bilinear form on $R (\ )$ defined in (4.2).
4.2. Basic spin representations. Let the Pauli spin matrices be

\[
\begin{align*}
0 & = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ; & 1 & = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \\
2 & = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; & 3 & = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} ;
\end{align*}
\]

Let \( C_{2k} \) be the Clifford algebra generated by \( e_i, e_j \), \( 2k \) with relations:

\[
(4.8) \quad f e_i e_j g = e_i e_j + e_j e_i = 2 \delta_{ij} ;
\]

Thus \( e_j^2 = 1 \). The Clifford algebra \( C_{2k} \) is endowed with a natural superalgebra structure by letting the parity of \( e_i \) to be odd for each \( i \).

When \( k = 1 \), one has that \( C_2 \) is \( M(1 | 1) = \text{End}(C^{13}) \) and the action of \( C_2 \) on \( C^{13} \) is given by the Pauli spin matrices:

\[
\begin{align*}
e_1 & \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ; & e_2 & \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \\
e_3 & \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; & e_4 & \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} ;
\end{align*}
\]

More generally we have \( C_{2k} = \text{End}(kC^{13}) \) \( \cong M(2^k | 2^k \times 1) \). The tensor product \( kC^{13} \) admits a canonical supermodule structure for the Clifford algebra \( C_{2k} \) under the action

\[
(4.9) \quad e_{2j} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ;
\]

\[
(4.10) \quad e_{2j} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ;
\]

The above formulas define explicitly the structure of a simple \( C_{2k} \)-superalgebra on \( kC^{13} \).

Let \( C_{2k+1} \) be the Clifford algebra generated by \( e_i, i = 1, \ldots, 2k+1 \) with similar relations like (4.8). We embed \( C_1 \) into \( \text{End}(C^{13}) \) by \( 1 \equiv \text{Id}, e_1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Then

\[
C_{2k+1} \rightarrow C_{2k} \quad C_1 \rightarrow \text{End}(kC^{13})
\]

gives a \( C_{2k+1} \)-superalgebra structure on \( kC^{13} \). The explicit action is given by the same formulae (4.9-10), except that \( j = 1, \ldots, k \) and in (4.10) observe that \( C_{2k+1} \cong Q(2^k) \).

It is well-known (see e.g. [J]) that there exists an embedding of \( \mathcal{S}_n \) into the multiplicative Clifford group of units in \( C_{2n} \). Therefore \( kC^{13} \) can be regarded as a \( \mathcal{S}_n \)-superalgebra, which is called the basic spin supermodule for \( \mathcal{S}_n \). More explicitly we have

Proposition 4.2. \( \mathcal{S}_n, J \) The basic spin supermodule for \( \mathcal{S}_n \) is \( kC^{13} \) with the action

\[
(4.11) t_j \equiv \begin{pmatrix} j+1 \\ 2j \end{pmatrix} e_j ; \quad j = 1, \ldots, n.
\]
Here we take $e_0 = 0$. Its character $n$ is given by

$$n(\pi) = \begin{cases} 2^{l(1) - 2} & \text{if } 2 \text{ OP}_n, n \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

(4.12)

In particular $n(1) = 2^{d_0 + e}$. Here $d$ denotes the largest integer $a$.

Proposition 4.3. Let $n \geq 1$ be an odd integer. The basic spin super module $S^{(n)}_{+}$ is an irreducible $S_n$-module under the action (4.11). Its character $n$ is given by the second equation of (4.12). In particular $n(1) = 2^{n - 1 - 2}$.

2) Let $n \geq 1$ be an even integer. The basic spin super module is a reducible $S_n$-module under the action (4.11) and decomposes into two irreducible $S_n$-modules whose characters $n$ are given by

$$n(\pi) = \begin{cases} 2^{l(1) - 2} & \text{if } 2 \text{ OP}_n, \\ 0 & \text{otherwise.} \end{cases}$$

(4.13)

In particular, $n(1) = 2^{n - 2 - 2}$.

4.3. The spin character $n(\pi)$ of $e_n$. Let $V$ be a module ordered by the character $2 \text{ R}(\pi)$, and let $U$ be a spin super module (resp. module) of $S_n$ with the character $\pi$. The tensor product $V^\otimes U$ has a canonical spin super module (resp. module) structure for $e_n$ as follows (compare (4.3)). For any $g = (g_1; \ldots; g_n) \in G_n$ let $(g; z^2 t)$ be an element in $e_n$. The super module (resp. module) structure is defined by

$$(g; z^2 t) : (v_1 \otimes \ldots \otimes v_n; u) = g_1 v_{s(g_1)}(1) \otimes \ldots \otimes g_n v_{s(g_n)}(t u);$$

We denote by $n(\pi)$ the character of the constructed spin super module (resp. module).

Recall that the conjugacy class $D^+$ contains an element $(g; t)$ (see (2.3)).

Proposition 4.4. Let $\pi$ be the character of a spin $S_n$-super module. Then the character values of $n(\pi)$ at the conjugacy classes $D$ (2 $O P_n(\pi)$) are given by

$$n(\pi) \in D = \begin{cases} \chi(t) \chi^{(c)}(c) & \text{if } (c) \in (c) \in (c) \in (c) \in (c) \in (c). \end{cases}$$

(4.14)
Proof. Consider \((g; z^p t) 2 \in \mathfrak{e}_n\); where \(g = (g_1; \ldots; g_n) 2 \in \mathfrak{e}_n\) and \(t\) is an \(n\)-cycle, say \(t = [12 \ldots n]\). Denote by \(e_1; \ldots; e_n\) a basis of \(V\), and we write \(g e_j = a_{ij}(g) e_i, a_{ij}(g) 2 C\): It follows that
\[
(g; z^p t) (e_{j_1} \cdots e_{j_k}) \quad = \quad g_1 (e_{j_1}) \quad g (e_{j_1}) \cdots \quad g (e_{j_k}) \quad z^p t (u);
\]

Thus we obtain
\[
n(\ ) (z^p t) = \quad \text{trace} a (g_1) a (g_2) \cdots a (g_{n-1}) \quad (z^p t) = \quad \text{trace} a (g_1, g_2) \cdots a (g_{n-1}) \quad (z^p t) = \quad (c) \quad (z^p t);
\]

where we notice that \(g_1, g_2, \ldots, g_{n-1}\) lies in \(C\).

Given \(x^y 2 \mathfrak{e}_n\), where \(x 2 \mathfrak{e}_n\) and \(y 2 \mathfrak{e}_n\), we clearly have
\[
n(\ ) (x^y) = n(\ ) (x) n(\ ) (y).
\]

Thus it follows that for the conjugacy class \(D^+ 2 \mathfrak{e}_n\) of type , we have
\[
n(\ ) (D) = \quad (c) \quad (c)^{11} (c);
\]

where \(jj = n\).

Since the sign character is trivial at even classes, we can extend naturally \(n(\ )\) to a map from \(R(\ )\) to \(R(\mathfrak{e}_n)\) (compare with \([\mathbb{W}, \mathbb{F}, \mathbb{W}^{-1}])\).

For two \(-\)characters and we denote
\[
X^n_{\mathfrak{e}_n}(\ ) = \quad \text{Ind}_{\mathfrak{e}_n}^{\mathfrak{e}_n} [n \quad m(\ ) \quad m(\ )];
\]

When \(n\) is even, the character \(n(\ )\) of the basic spin super module (see Sect. 4.2) decomposes into the sum of irreducible characters \(n(\ )\) of \(\mathfrak{e}_n\)-modules. For each \(c 2\), we denote the special partition-valued function \(c^{(n)} 2 P(\ )\) such that
\[
c^{(n)} (c) = (n); \quad c^{(n)} (c) = ; \quad \text{for } c 6 c;
\]

The following corollary is an immediate consequence of Propositions 4.4 and 4.5.

Corollary 4.5. (1) The character value of \(n(\ )\) at the conjugacy class \(D^+\) of type is
\[
n(\ ) (D) = \quad 2^d (\pi) - 2^0 \quad c_2 \quad (c)^{11} (c) \quad 2 \quad \text{O} \quad P_n(\ ) \quad 0\quad \text{otherwise};
\]

where \(\pi\) is 0 or 1 depending on whether \(n\) is even or odd.
(2) Let $n$ be an even positive integer. The character values of $v_{n}(\cdot)$ at the conjugacy class $D^{+}$ of type $\alpha$ are

$$v_{n}(\cdot)(\cdot) = \begin{cases} \frac{P}{2} & \text{for } n \equiv 0 \mod 2 \text{ and } \alpha \equiv 0 \mod 2 \text{ or } \alpha \equiv 1 \mod 2, \\ 0 & \text{otherwise} \end{cases}$$

(4.18)

4.4. Two specializations. Let $d_i = i(c^{\theta})$ be the dimension of the irreducible representation of $\alpha$ by the character $\chi_i$. Let $A$ be the matrix of the bilinear form $h$ on $R(\cdot)$ with respect to the basis $i$. Observe that the vector

$$v_i = \{ i(c^{0}); i(c^{1}); \ldots; i(c^{r}) \}; \quad (i = 0; \ldots; r)$$

is an eigenvector of the matrix $A$ with eigenvalue $i(c^{\theta})$.

Two special choices of the weight function will be our prototype examples. The first choice is that $i = 0$, the trivial character.

Let $a_0$ be the character of the 2-dimensional representation of $G$ given by the embedding of $\alpha$ in $SL_2(C)$. Let

$$a_0 = 2_0.$$ 

Then the weighted bilinear form $h; i$ on $R$ becomes a symmetric semidefinite matrix. The radical of this bilinear form is one-dimensional and spanned by the character of the regular representation of $\alpha$

$$\alpha = \{ d_0 \}.$$ 

The following is the well-known list of finite subgroups of $SL_2(C)$: the cyclic, binary dihedral, tetrahedral, octahedral and icosahedral groups.

McKay observed that they are in one-to-one correspondence to simple-laced Dynkin diagrams of affine types $[M^C_i]$: $a_{ij} = 2$ for all $i$ if $\theta = 2Z$ and $i \neq j$ then $a_{ij} = 0$ or $1$. If $\theta = 2Z$ then $a_{ij} = 2$.

5. Twisted Heisenberg algebras and $\mathfrak{h}_{n}$

5.1. Twisted Heisenberg algebra $\mathfrak{h}_n; [1]$. Let $\mathfrak{h}_n; [1]$ be the infinite-dimensional Heisenberg algebra over $C$, associated with a finite group and a self-dual class function $2R(\cdot)$, with generators $a_{m}(\cdot); m \geq 2Z + 1$; $2$ and a central element $C$ subject to the relations:

$$[a_m(\cdot); a_n(0)] = \frac{m}{2} a_{m+n}h; \quad 0; iC; \quad m; n \geq 2Z + 1; \quad 0; 2.$$ 

We extend $a_m(\cdot)$ to all \begin{pmatrix} r \end{pmatrix}$$s_i h; \quad 2R(\cdot) (s_i C)$ by linearity:

$$a_m(\cdot) = \sum s_i a_m(\cdot).$$
The Heisenberg algebra may contain a large center because the bilinear form $\mathbf{h}$; may be degenerate. The center of $\mathbf{h}$, $\{ 1 \}$ is spanned by $C$ together with $a_m ( )$; $m \geq 2$; $2 \mathbb{Z} + 1$; $2 \mathbb{R}^0$, the radical of the bilinear form $\mathbf{h}$; in $\mathbb{R} ( )$.

Form $2 \mathbb{Z} + 1; c 2$ we introduce another basis for $\mathbf{h}$, $\{ 1 \}$:

$$a_m ( c) = \frac{m}{2} c \frac{1}{m} (c) a_m ( ) ;$$

(5.2)

The orthogonality of the irreducible characters of (2.11) implies that:

$$a_m ( ) = \frac{m}{2} c \frac{1}{m} (c) a_m ( ) ;$$

Proposition 5.1. The commutation relations among the new basis for $\mathbf{h}$, are given by

$$[ a_m ( c) a_n ( ) ] = \frac{m}{2} m ; n c \frac{1}{c} (c) C ; c ; c 2 ;$$

where $m ; n 2 \mathbb{Z} + 1$.

Proof. The proof is similar to the untwisted case [FW].

5.2. Action of $\mathbf{h}$, $\{ 1 \}$ on $S$ and $\mathbf{S}$, Denote by $S$, the symmetric algebra generated by $a_n ( )$; $n 2 \mathbb{Z} + 1$; $2$. There is a natural degree operator on $S$

$$\text{deg} (a_n ( )) = n ; n 2 \mathbb{Z} + 1 ;$$

which makes $S$ into a $\mathbb{Z}_+$-graded space.

We define an action of $\mathbf{h}$, $\{ 1 \}$ on $S$, as follows: $a_n ( )$; $n > 0$ acts as a multiplication operator on $S$ and $C$ as the identity operator; $a_n ( )$; $n > 0$ acts as a derivation of the symmetric algebra

$$a_n ( ) a = n_1 (1) a_n (2) \cdots a_n (k)$$

$$= \text{X}^k$$

$$i=1$$

$$n_1 \mathbf{h} ; i a_n (1) a_n (2) \cdots a_n (l) \cdots a_n (k) ;$$

Here $n_i > 0$; $i 2 \mathbb{R} ( )$ for $i = 1; \cdots ; k$, and $a_n (1)$ means the very term is deleted. In other words, the operator $a_n ( )$; $n > 0$; $2 \mathbb{R}^0$ acts as $0$, and $a_n ( )$; $n > 0$; $2 \mathbb{R} ( )$ acts as certain non-zero derivative operator. Note that $S$ is not an irreducible representation over $\mathbf{h}$, in general since the bilinear form $\mathbf{h}$; $i$ may be degenerate.

Denote by $S^0$ the ideal in the symmetric algebra $S$, generated by $a_n ( )$; $n 2 \mathbb{N} ; 2 \mathbb{R}^0$. Denote by $S$ the quotient $S = S^0$. It follows from the definition that $S^0$ is a subrepresentation of $S$ over the
Heisenberg algebra $\mathfrak{h}$; $[1]$. In particular, this induces a Heisenberg algebra action on $S$ which is irreducible. The unit 1 in the symmetric algebra $S$ is the highest weight vector. We will also denote by 1 its image in the quotient $S$. 

5.3. The bilinear form on $S$. The space $S$ admits a bilinear form $h; i^0$ determined by

\[(5.3) \quad h1; i^0 1 = 1; \quad a_n( ) = a_{n}( ); \quad n \neq 2Z + 1;\]

Here $a_n( )$ denotes the adjoint of $a_n( )$.

For any partition $\lambda = (1; 2; \cdots) 2 \text{ OP}$ and $\lambda \neq 2Z$, we define

\[a( ) = a_1( )a_2( )\cdots\]

For $\lambda = (c) \neq 2 \text{ OP}(\lambda)$, we define

\[a = a_{( )}\]

It is clear that $a; 2 \text{ OP}(\lambda)$ form a basis for $S$.

Similarly we define

\[a(c) = a_1(c)a_2(c)\cdots\]

for any partition $\lambda = (1; 2; \cdots) 2 \text{ OP}$ and $\lambda \neq 2 \text{ OP}$. For any $\lambda = (c) \neq 2 \text{ OP}(\lambda)$, we further define

\[a_0 = a_{(c)}\]

The elements $a_0; 2 \text{ OP}(\lambda)$ provide a new $C$-basis for $S$.

Recall that $2 \text{ OP}(\lambda)$ is given by assigning to $c_2$ the partition $(c)$, which is the composition of $\lambda$ with the involution on $c$ given by $c^{1/2}c^{1/2}$. It follows from Proposition 5.3.3 that

\[(5.4) \quad h a_0; a_0 - 1 = \frac{Z}{2^{\lambda(c)}} Y (c)\lambda(c); \quad \lambda \neq 2 \text{ OP}(\lambda); \]

Remark 5.2. $S^0$ can be characterized as the radical of the bilinear form $h; i^0$ in $S$. Thus the bilinear form $h; i^0$ induces a bilinear form on $S$ which will be denoted by the same notation.
6. Isometry between $R$ and $S$

6.1. The characteristic map $ch$. We define a $C$-linear map $ch : R \to S$ by letting

$$
(6.1) \quad ch(f) = \frac{X}{2^{|f|/2}} f(a^0); 
$$

where $f(\cdot) = f(D^+)$. The map $ch$ is called the characteristic map (compare with \[S,Jo\] for trivial).

Fix $n \in 2\mathbb{Z} + 1$ in this paragraph. Denote by $D_n(c)^+; c \geq 2$ the conjugacy class in $\mathfrak{e}_n$ of elements $(x; t_a) 2_n$ such that $x$ is an $n$-cycle and the cycle product of $(x; t_a)$ is $c$. Then set $D_n(c) = zD_n(c)^+$. Thus $D_n(c)$ are the associated split conjugacy classes of type $c^n$ (see (4.15)). Denote by $n(c)$ the super class function on $\mathfrak{e}_n$ which takes value $\frac{\bar{\alpha}}{2}$ on elements in the conjugacy classes $D_n(c)$, and 0 elsewhere. For $f = f^{(c)} \in \mathcal{O}_2(\mathfrak{g}) 2 \mathcal{O}_n(\mathfrak{g})$, $f^{(c)}(x) = \frac{1}{2^2} f^{(c)}(x) z^{(c)}$ is the class function of $\mathfrak{e}_n$ which takes value $\frac{2^{|f|/2}}{n^2}$ on the conjugacy classes $n(c)$ and 0 elsewhere. Given $2 \mathbb{R}(\cdot)$, we denote by $n(\cdot)$ the class function on $\mathfrak{e}_n$ which takes value $\frac{\bar{\alpha}}{2}$ on $D_n(c); c \geq 2$, and 0 elsewhere.

The following lemma is not difficult to verify.

Lemma 6.1. The map $ch$ sends $a^0$ to $a^0$. In particular, it sends $n(c)$ to $n(c)$ in $S$ and $n(\cdot)$ to $n(\cdot)$ for $n \in 2\mathbb{Z} + 1$.

In Sect. 3.2, we will see that the space $S$ has another distinguished basis consisting of generalized Schur $Q$-functions, which give rise to some integral basis in $R$ ($\mathfrak{e}_n$).

6.2. The image of $n(\cdot)$ under $ch$. Recall that we have defined a map from $R(\cdot)$ to $R(\mathfrak{e}_n)$ (Subsection 4.3).  

Proposition 6.2. For any $2 \mathbb{R}(\cdot)$, we have

$$
(6.2) \quad \sum_{n=0}^{\infty} \frac{X}{2^{n-2}} ch(n(\cdot)) z^n = \exp \left( \sum_{n=0}^{\infty} \frac{2^{|n|/2}}{n!} a(n(\cdot)) z^n \right); 
$$

where $\Pi$ is 0 or 1 according to $n$ is even or odd.
Proof. The character value of \( n(\ ) \) is given in Corollary 4.5, and we have
\[
2^{n-2} \text{ch}(n(\ )) Z^n = \sum_{c^2} 2^{1(l(\ ) a^n_0 (c) z^{l(\ )})) x \text{ch}(n(\ )) Z^n \text{ch}(n(\ )) Z^n
\]
\[
= \exp \left( \frac{2}{n} \right) a_n() Z^n
\]
Let \( ; \) be the characters of two representations of \( R \). It follows from (4.15) that
\[
2^{n-2} \text{ch}(n(\ )) Z^n = \sum_{c^2} 2^{1(l(\ ) a^n_0 (c) z^{l(\ )})) x \text{ch}(n(\ )) Z^n \text{ch}(n(\ )) Z^n
\]
\[
= \exp \left( \frac{2}{n} \right) a_n() Z^n
\]
Therefore the proposition holds for \( R_Z(\)\), and so for any element \( 2 R_Z(\)\).

Corollary 6.3. The formula (4.17) holds for any \( 2 R(\)\). In particular \( n(\)\) is self-dual if \( Z(\)\) is self-dual.

Component-wise, we obtain
\[
\text{ch}(n(\ )) = 2^{n-2} \sum_{c^2} 2^{1(l(\ ) a^n_0 (c) z^{l(\ )})) x \text{ch}(n(\ )) Z^n \text{ch}(n(\ )) Z^n
\]
where the sum runs through all the partitions of \( n \) into odd integers.

6.3. Isom etry between \( R \) and \( S \). It is well known that there exists a natural Hopf algebra structure on the symmetric algebra \( S \) with the usual multiplication and the comultiplication characterized by
\[
(a_n(\ )) = a_n(\ ) + 1 + 1 a_n(\ ) Z^n + 1;
\]
Recalling the Hopf algebra structure on $R$ defined in Sect. 3.4, we can easily verify the following proposition as in the untwisted case.

**Proposition 6.4.** The characteristic map $\text{ch}: R \to S$ is an isomorphism of Hopf algebras.

**Proof.** By counting dimensions of homogeneous degree subspaces it is easy to see that $\text{ch}$ is an isomorphism of vector spaces. The algebra isomorphism follows simply from the Frobenius reciprocity. To check the coalgebra isomorphism we use Proposition 6.2 to pass from the generators $a_n(\cdot)$ to the character $n(\cdot)$. It is then a simple calculation to verify that $n(\cdot)$ is group-like under the comultiplication (3.13), and this shows that $\text{ch}$ is a Hopf algebra isomorphism by using (6.3). □

Recall that we have defined a bilinear form $h; i^0$ on $R$ and a bilinear form on $S$ denoted by $h; i^0$. The lemma below follows from our definition of $h; i^0$ and the comultiplication .

**Lemma 6.5.** The bilinear form $h; i^0$ on $S$ can be characterized by the following two properties:

1. $h a_n(\cdot); a_{m}(\cdot) \circ \circ = \frac{n}{2} n.m h ; \circ \circ$; $2$ $m$ $n$ $2$ $Z$ $+ 1$;
2. $h f g; i^0 = h f; g i^0$; where $f; g$ on $S$, and the bilinear form on the r.h.s of 2), which is defined on $S$, is induced from $h; i^0$ on $S$.

**Theorem 6.6.** The characteristic map $\text{ch}$ is an isometry from the space $(R; h; i)$ to $(S; h; i^0)$.

**Proof.** Let $f$ and $g$ be any two super class functions in $R$ ($\mathcal{C}_n$). By definition of $\text{ch}$ (6.1) it follows that

\[
\text{ch}(f); \text{ch}(g)i^0 = \frac{2(1(\cdot)) \circ (c) = 2}{Z Z_0} f(\cdot) g(\cdot) h a_0 - a_0 - i^0
\]

\[
= \frac{2(1(\cdot)) \circ (c) = 2}{Z Z_0} f(\cdot) g(\cdot) \frac{Z_0}{2(1(\cdot))} c_2 Y (c)^{(c)} (c)
\]

\[
= \frac{1}{Z} f(\cdot) g(\cdot) c^2 (c)^{(c)}\]

\[
= \text{hf}; gi;
\]

where we have used the inner product identity (5.4). □
Remark 6.7. We can also prove it by showing that the characteristic map preserves the inner product of basis elements $2 \mathbb{R}$ and that of $a = \chi(1) = 2S$ as in \([\text{FLM 2}]\).

From now on we will identify the inner product $h; i$ on $\mathbb{R}$ with the inner product $h; i^0$ on $S$. As a special case, the standard Hermitian form on $\mathbb{R}$ is compatible via the characteristic map $\chi$ with the Hermitian form characterized by \((5.3)\) on $S$.

7. Vertex operators and $R$

7.1. A central extension of $\mathbb{R}_Z(2) = 2\mathbb{R}_Z()$. From now on we assume that $Z$ is a self-adjoint virtual character of $\mathbb{R}$, and thus $\mathbb{R}_Z()$ is an integral lattice under the symmetric bilinear form $h; i$.

Let $2\mathbb{R}_Z()$ be the sublattice of $\mathbb{R}_Z()$ consisting of elements $2$ of $\mathbb{R}_Z()$. The quotient $\mathbb{R}_{F_2()} = \mathbb{R}_Z() / 2\mathbb{R}_Z()$ has an induced abelian group structure and it can also be viewed as an $(r+1)$-dimensional vector space over $F_2 = \mathbb{Z}/2$. We will denote by $\overline{\cdot}$ the natural image of in $\mathbb{R}_{F_2()}$. Define $c_1$ to be the alternating form $\mathbb{R}_{F_2(\overline{\cdot})}$, $\mathbb{R}_{F_2()}!$, $\mathbb{R}_{F_2()}!$, $\mathbb{R}_{F_2(\overline{\cdot})}$, $\mathbb{R}_{F_2()}!$, $\mathbb{R}_{F_2(\overline{\cdot})}$.

The elements of $\mathbb{R}_{F_2(\overline{\cdot})}$ can be presented as $e^{- \overline{\cdot}}$, where $2 \mathbb{R}_Z()$, which implies that $\dim (\mathbb{R}_{F_2(\overline{\cdot})}) = 2^{r+2}$. We note that $e^{-2} \mathbb{R}_{F_2(\overline{\cdot})}$ satisfies $e^{-2} = 1$.

Let $\mathbb{R}$ be a subgroup of $\mathbb{R}_Z()$ which is maximal such that the alternating form $c_1$ vanishes on $2\mathbb{R}_Z()$. A variant of the following lemma was given in \([\text{FLM 2}]\).

Lemma 7.1. There are $2^{(r+1)}$ irreducible $\mathbb{R}_{F_2(\overline{\cdot})}$-module structures on the space $C[\mathbb{R}_Z() = ]$ such that $12 \mathbb{R}_{F_2(\overline{\cdot})}$ acts faithfully and

$$e^{- \overline{\cdot}} = e^{- \overline{\cdot}} (1^{r+1})$$

as operators on $C[\mathbb{R}_Z() = ]$. The dimension of $C[\mathbb{R}_Z() = ]$ is equal to $2^{r+2}$.

We will denote the elements of $C[\mathbb{R}_Z() = ]$ by $e^{|\cdot|}$, where $|\cdot| = + 2 \mathbb{R}_Z()$.

Clearly

$$e^{|\cdot|} = 1; e^{|\cdot|} = e^{|\cdot|}.$$
For $2 \mathbb{R}_Z(\cdot)$ we write the action of $\mathbb{R}^Z_{F_2}(\cdot)$ on $C \mathbb{R}_Z(\cdot) = [\cdot]$ as

$$e^\pm = (\ ; \ ) e^\pm.$$  

Then one can check that $\mathbb{R}^Z_{F_2}(\cdot)$ is a well-defined cocycle map from $\mathbb{R}_Z(\cdot)$ to $\mathcal{M}_1$. One also has $(\ ; \ ) = (\ ; \ )$.

7.2. Twisted Vertex Operators $X(\cdot; z)$. Fix an irreducible $\mathbb{R}^Z_{F_z}(\cdot)$-module structure on $C \mathbb{R}_Z(\cdot) = [\cdot]$ described in Eqn. (7.3).

We extend the actions of $e^\pm$ to the space of tensor product

$$F = R \mathbb{C} \mathbb{R}_Z(\cdot) = [\cdot];$$

by letting them act on the $R$ part trivially.

Introduce the operators $H_n(\cdot); 2 \mathbb{R}_Z(\cdot); n > 0$ as the following compositions of maps:

$$H_n(\cdot) : R (e_m) \mathbb{Z}^{\mathbb{Z}^{2n-2}} [1] \otimes_R (e_n \wedge e_m)^{\text{Ind}} \otimes_R (e_{n+m})$$

$$H_n(\cdot) : R (e_m) \mathbb{R}^{n+1} \otimes_R (e_n \wedge e_m) \mathbb{R}^{n} [1]; R (e_{n+m});$$

Define

$$X_n(\cdot) z^n = X_n(\cdot) z^n,$$

$$H_n(\cdot) z^n = H_n(\cdot) z^n.$$

We now define the twisted vertex operators $X_n(\cdot); 2 \mathbb{Z}, 2 \mathbb{R}$ by the following generating functions:

$$(7.4) \quad X^+(\cdot; z) = X(\cdot; z) X_n(\cdot) z^n = X_n(\cdot) z^n.$$  

We also denote

$$X(\cdot; z) = X(\cdot; z) = X_n(\cdot) z^n = X_n(\cdot) z^n.$$

The operators $X_n(\cdot)$ are well-defined operators acting on the space $F$. We extend the bilinear form $\langle \cdot, \cdot \rangle$ on $R$ to $F$ by letting

$$\langle h f(\cdot), g(\cdot) \rangle = \langle h, g \rangle \langle f, g \rangle 2 \mathbb{R}_Z(\cdot); 2 \mathbb{R}_Z(\cdot);$$

We extend the $Z_+$-gradation from $R$ to $F$ by letting

$$\deg a_n(\cdot) = n; \quad \deg e = 0.$$
Similarly we extend the bilinear form $h; i$ to the space

$$V = S \cdot C [R_Z()] = \{ \}$$

and extend the $Z_+$-gradation on $S$ to a $Z_+$-gradation on $V$.

The characteristic map $\chi$ will be extended to an isometry from $F$ to $V$ by fixing the subspace $C [R_Z()] = \{ \}$. We will denote this map again by $\chi$.

7.3. Twisted Heisenberg algebra and $R$. We define $a_n( ); n 2 2Z_+ + 1$ to be a map from $R$ to itself by the following composition

$$R (\mathcal{E}_m) \rightarrow \chi (\mathcal{E}_n) \rightarrow R (\mathcal{E}_m) \rightarrow R (\mathcal{E}_m + \chi_i R (\mathcal{E}_m + \chi_i R (\mathcal{E}_{m+n}));$$

We also define $a_n( ); n 2 2Z_+ + 1$ to be a map from $R$ to itself as the composite

$$R (\mathcal{E}_m) \rightarrow \chi (\mathcal{E}_n) \rightarrow R (\mathcal{E}_m) \rightarrow R (\mathcal{E}_m) \rightarrow R (\mathcal{E}_m + \chi_i R (\mathcal{E}_m + \chi_i R (\mathcal{E}_{m+n}));$$

We denote by $R^0$ the radical of the bilinear form $h; i$ in $R$ and denote by $\overline{R}$ the quotient $R = R^0$, which inherits the bilinear form $h; i$ from $R$.

Theorem 7.2. $R$ is a representation of the twisted Heisenberg algebra $\overline{R} ; [1]$ by letting $a_n( )$ (n 2 2Z_+ + 1) act as $a_n( )$ and $C$ as 1. $R^0$ is a subrepresentation of $R$ over $\overline{R} ; [1]$ and the quotient $\overline{R}$ is irreducible. The characteristic map $\chi$ is an isomorphism of $R$ (resp. $R^0, \overline{R}$) and $S$ (resp. $S^0, \overline{S}$) as representations over $\overline{R} ; [1]$.

7.4. The characteristic map of twisted vertex operators. We extend the characteristic map $\chi$ to a linear map $\chi: \text{End}(R) \rightarrow \text{End}(S)$ by

$$(7.5) \quad \chi (f) \chi (v) = \chi (f \circ v); f \in \text{End}(R); v \in R ;$$

The relation between the vertex operators defined in (7.2) and the Heisenberg algebra $\overline{R} ;$ is revealed in the following theorem.

Theorem 7.3. For any $2 R( )$, we have

$$\chi H_+ ( ; z) = \exp \frac{X}{2} a_n( ) z^n; n 1, 2 \text{odd}$$

and

$$\chi H ( ; z) = \exp \frac{X}{2} a_n( ) z^n; n 1, 2 \text{odd}$$
Proof. Observe that the operator $H_+ ( ; z) \adjoint$ is the adjoint operator of $H ( ; z^-)$ with respect to the bilinear form $h \cdot i$. Then the theorem follows from Lemma 6.1 and Proposition 6.2 by invoking the characteristic map.

As a consequence we have

$$\text{ch} \, X ( ; z) = \exp \frac{2 a_n( ) z^n}{n \; \text{odd}} \exp \frac{2 a_n( ) w^n}{n \; \text{odd}}.$$ 

Thus the characteristic map identifies the twisted vertex operators $X ( ; z)$ defined via finite groups $\mathcal{E}_n$ with the usual twisted vertex operators of $\mathcal{FLM}_1, \mathcal{FLM}_2$.

8. Vertex representations and the McKay correspondence

8.1. Product of two vertex operators. The normal ordered product $X ( ; z) X ( ; w) ; ; 2 \mathbb{R} ( )$ of two vertex operators is defined as follows:

$$:X ( ; z) X ( ; w) = H_+ ( ; z) H_+ ( ; w) H ( ; z) H ( ; w) :.$$ 

In the following theorem and later the expression $\frac{z^w}{z + w} h \cdot i$ represents the power series expansion in the variable $\frac{w}{z}$.

Theorem 8.1. For $; ; 2 \mathbb{R} ( )$ one has the following operator product expansion identity for twisted vertex operators.

$$X ( ; z) X ( ; w) = ( ; ) :X ( ; z) X ( ; w) : \frac{z^w}{z + w} h \cdot i.$$ 

Proof. It suffices to compute that

$$\text{ch} \, (H ( ; z) H_+ ( ; w)) = \exp \frac{2 a_n( ) z^n}{n \; \text{odd}} \exp \frac{2 a_n( ) w^n}{n \; \text{odd}} X ( ; z) = \text{ch} \, (H_+ ( ; w) H ( ; z)) \exp \frac{2 z^w}{n \; \text{odd}} \frac{h}{w}$$

$$= \text{ch} \, (H_+ ( ; w) H ( ; z)) \frac{z^w}{z + w} h \cdot i.$$ 

□
The following proposition is easy to check.

Proposition 8.2. Given \( 2 \mathbb{R}(\cdot); 2 \mathbb{R}_z(\cdot) \) and \( n \in \mathbb{Z} + 1 \), we have

\[
[a_n(\cdot); X(\cdot; z)] = h; \quad iX(\cdot; z)z^n.
\]

8.2. Twisted affine Lie algebra \( \mathfrak{g}[1] \) and twisted toroidal Lie algebra \( \hat{\mathfrak{g}}[1] \). Let \( g \) be a rank \( r \) complex simple Lie algebra of ADE type, and let \( \Delta \) be the root system generated by a set of simple roots \( \alpha_1, \ldots, \alpha_r \). Let \( \Delta_{\text{max}} \) be the highest root. The Lie algebra is generated by the Chevalley generators \( e_i, e_i^\vee; h_i = h_i^\vee \). We normalize the invariant bilinear form on \( g \) by \( (\alpha_{\text{max}}; \alpha_{\text{max}}) = 2 \).

Let \( ! \) be an automorphism of \( g \) of order \( k \) and let \( ! = \exp(2i\pi k) \). The automorphism \( ! \) induces a \( \mathbb{Z} = k \mathbb{Z} \)-gradation for \( g \):

\[
M_{\mathfrak{g}} = g_i; \quad g_i = fg_2 g_j (g) = !^i gg;
\]

and

\[
[g_i; g_j] = g_{i+j};
\]

The twisted affine Lie algebra \( \mathfrak{g}[1] \) is the graded vector space

\[
(8.1) \quad \mathfrak{g}[1] = \bigoplus_{i=0}^{M} g_i \quad t^{k} \quad t^{k} \quad C
\]

with the commuting relations

\[
(8.2) \quad [a(n); b(m)] = [a; b](n + m) + \frac{n}{k} a_m b_C;
\]

\[
(8.3) \quad [C; a(n)] = 0; \quad a; b 2 g; n; m 2 \mathbb{Z} ;
\]

where we used the notation

\[
a(n) = a t^n; \quad a 2 g; n 2 \mathbb{Z} ;
\]

When \( a(n) = 0 \) is the identity, \( \mathfrak{g}[1] \) becomes the (untwisted) affine Lie algebra \( \mathfrak{g} \). Let \( A = (a_{ij})_0 \) be the affine Cartan matrix associated to \( \mathfrak{g} \). The submatrix \( (a_{ij})_1 \) is the Cartan matrix of \( g \). The linear map

\[
e_i^\vee ! e_i^\vee \quad h_i^\vee ! h_i^\vee
\]

de nes an involution of the Lie algebra \( g \). We will denote the associated twisted affine Lie algebra by \( \hat{\mathfrak{g}}[1] \). Let \( k = \frac{1}{2}(e + e^\vee) \) and \( p = \ldots \)
\[ \begin{pmatrix} e & e \\ e & e \end{pmatrix}, \] where \( 2 \). It is easily seen that
\[
\begin{align*}
g_0 &= M C_k; \\
g_1 &= M C p M \
\end{align*}
\]

The basic twisted representation \( V \) of \( \mathfrak{b}[1] \) is the irreducible highest weight representation generated by a highest weight vector which is annihilated by \( a(n); n 2 \mathbb{Z} \); \( a(2) \) and \( C \) acts on \( V \) as the identity operator.

We now introduce the complex twisted toroidal Lie algebra \( \mathfrak{b}[1] \) (associated to \( g \)) with the following presentation: the generators are
\[
C; h_1(m); x_n(\_i); m 2 \mathbb{Z} + 1; n 2 \mathbb{Z}; i = 0; \ldots; r;
\]
and the relations are given by: \( C \) is central, and
\[
\begin{align*}
x_n(\_i) &= \begin{pmatrix} 1 \\ f_x(n(\_i)) \end{pmatrix}; \\
h_1(m); h_1(m^0) &= \frac{m}{2} a_{ij}; m \in \mathbb{C}; \\
h_1(n); x_m(\_j) &= a_{ij} x_{n+m}(\_j); \\
[x_n(\_i); x_n^0(\_i)] &= 8fh_1(n + n^0) + n n; n \in \mathbb{C} g;
\end{align*}
\]
\[
\begin{align*}
X^{a_{ij}} &= \sum_{s=0}^{a_{ij}} a_{ij} \left[ x_{n+s}(\_i); x_n^0 a_{ij} s(\_j) \right] = 0; & \text{if } a_{ij} = 0 \\
(1^f) X^{a_{ij}} &= \sum_{s=0}^{a_{ij}} a_{ij} \left[ x_{n+s}(\_i); x_n^0 a_{ij} s(\_j) \right] = 0; & \text{if } a_{ij} < 0
\end{align*}
\]
where \( n; n^0 2 \mathbb{Z}, m; m^0 2 \mathbb{Z} + 1, i; j = 0; 1; \ldots; r \), and \( h_1(2n) = 0 \) for \( n 2 \mathbb{Z} \). The twisted toroidal algebra is the \( q! 1 \) limit of the twisted quantum current algebra \( \mathfrak{g}[2] \) (see a slightly different form in [8.11]).

Set \( k_1(2n) = \frac{1}{2} x_{2n}(\_i) \) and \( p_1(2n + 1) = \frac{1}{2} x_{2n+1}(\_i) \). One can check that the relations given in (8.4) are consequences of the twisted algebra \( \mathfrak{g}[1] \) for the case of \( = 1 \) deduced in [8.2] (cf. the proof of Theorem 8.3 later).

8.3. Realization of twisted vertex representations. Let be a finite subgroup of \( SL_2(\mathbb{C}) \) and the virtual character to be twice the trivial character minus the character of the two-dimensional defining representation of \( SL_2(\mathbb{C}) \). It is known [5.3] that the matrix \( A = (a_{ij})_{0 \leq i \leq r} \) in Sect. 5.2 is the Cartan matrix for the corresponding algebra \( \mathfrak{g} \).

The following theorem provides a finite group realization of the vertex representation of the twisted toroidal Lie algebra \( \mathfrak{g}[1] \) on \( F \).
Theorem 8.3. A vertex representation of the twisted toroidal Lie algebra $b[1]$ is defined on the space $F$ by letting
\[
x_n (i) \equiv X_n (i); \quad x_n (i) \equiv (i; i)X_n (i);
\]
\[
h_1 (m) \equiv a_m (i); \quad C \equiv 1;
\]
where $n \in \mathbb{Z}; m \in 2 \mathbb{Z} + 1; 0 \leq r.$

Proof. All the commutation relations without binomial coefficients are easy consequences of Proposition 8.2 and Theorem 8.1 by the usual vertex operator calculus in the twisted picture (see [FLM, JL]). The corresponding relations with binomial coefficients in $V$ are equivalent to
\[
(z + w)^{a_{ij}}[X (i; z); X (j; w)] = 0; \quad a_{ij} > 0;
\]
\[
(z - w)^{a_{ij}}[X (i; z); X (j; w)] = 0; \quad a_{ij} < 0:
\]
This is again proved by using Theorem 8.1 with the same method as in the quantum vertex operators [JL].

Recall that $P = \prod_{i=0}^{r} d_i$ generates the one-dimensional radical $R^0$ of the bilinear form $h; i$ in $R_z (\cdot)$, where $d_i$ is the degree of the irreducible character $i$ of $\mathbb{C}$. The lattice $R_z (\cdot)$ in this case can be identified with the root lattice for the corresponding a ne Lie algebra. The quotient lattice $R_z (\cdot)/R^0$ inherits a positive definite integral bilinear form. Denote by $\overline{\cdot}$ the set of non-trivial irreducible characters of $\mathbb{C}$.

Let $R_{\mathbb{C}} (\overline{\cdot})$ be the sublattice of $R_z (\cdot)$ generated by $\overline{\cdot}$. Denote by $\overline{\text{Sym}} (\overline{\cdot})$ the symmetric algebra generated by $a_n (i); n \in 2 \mathbb{Z} + 1; i = 1; \ldots; r$. Equipped with the bilinear form $h; i$, $\text{Sym} (\overline{\cdot})$ is isometric to $\overline{S}$ which is in turn isometric to $\overline{R}$ as well. The irreducible $\overline{\text{K}_2} (\overline{\cdot})$-module $\overline{C} \overline{R}_z (\overline{\cdot})$ induces an irreducible $\overline{\text{K}_2} (\overline{\cdot})$-module structure on $\overline{C} \overline{R}_z (\overline{\cdot})$ given by the restriction of the alternating form $c_1$. We let $\mathcal{T}_0$ denote the rank of the restriction of $c_1$, then the statement of Lemma 7.1 also holds for the sublattice $R_{\mathbb{C}} (\overline{\cdot})$ and $\overline{\text{K}_2} (\overline{\cdot})$. In this case if the determinant in the Cartan matrix is an odd integer, then $\mathcal{T}_0 = 0$ and the space $\overline{C} \overline{R}_z (\overline{\cdot})$ is trivial.

We define
\[
\overline{V} = \overline{S} \otimes \overline{C} \overline{R}_z (\overline{\cdot}) = \overline{\text{Sym}} (\overline{\cdot}) \otimes \overline{C} \overline{R}_z (\overline{\cdot}) = \overline{C} \overline{R}_z (\overline{\cdot}) = \overline{\mathbb{C}};
\]
\[
\overline{F} = \overline{R} \otimes \overline{C} \overline{R}_z (\overline{\cdot}) = \overline{\mathbb{C}};
\]
Obviously, when restricted to $\overline{V}$, is an isometric isomorphism onto $\overline{V}$.

The space $F$, associated to the lattice $R_Z$, is isomorphic to the tensor product of the space $\overline{F}$ associated to $\overline{R}_Z$ and the space associated to the rank 1 lattice $Z$ equipped with the zero bilinear form.

The identity for a product of vertex operators $X(i;z)$ associated to $2$ (cf. Theorem 8.3) implies that $\overline{V}$ provides a realization of the vertex representation of $\mathfrak{g}[-1]$ on $\overline{V}$ (cf. [FLM]). The following theorem establishes a direct link from the finite group $2\text{SL}_2(\mathbb{C})$ to the affine Lie algebra $\mathfrak{g}[-1]$. This gives a twisted version of the new form of the McKay correspondence given in [FJW].

Theorem 8.4. The operators $X_n(\cdot);\quad 2^{-1}a_n(i);i=1;2;\ldots;n\quad 2\quad Z$ define an irreducible representation of the affine Lie algebra $\mathfrak{g}[-1]$ on $F$, isomorphic to the twisted basic representation.

9. Vertex operators and irreducible characters of $\mathfrak{e}_n$

In this section we specialize to be the trivial character $\delta_0$ of $\mathfrak{e}_n$. We will describe how to obtain the character table for the spin supermodules of $\mathfrak{e}_n$ from our vertex operator approach, generalizing [J1].

9.1. Algebra of vertex operators for $\mathfrak{e}_n$. In this case the weighted bilinear form reduces to the standard one $h; i$ and $R_Z$, $F$ is isomorphic to the lattice $Z^{r+1}$ with the standard integral bilinear form. Recall that $h_i; j = \delta_{ij}$. For simplicity we will only consider the vertex representations on the space $R^\infty$. In the following result the bracket $f; g$ denotes the anti-commutator.

Theorem 9.1. The operators $X^+_n(i); X^-_n(i) \in 2Z; 0 i r$ generate a generalized Clifford algebra:

\[
\begin{align*}
[K^+_n(i); X^-_n(j)] &= 0; \quad i \neq j \\
[K^-_n(i); X^+_n(j)] &= 0; \quad i \neq j \\
fX^+_n(i); X^-_n(j)g &= 2(1)_{n_i}; r_i;
\end{align*}
\]

(9.1)

\[
\begin{align*}
fX^-_n(i); X^+_n(j)g &= 2(1)_{n_i}; r_i; \\
K^+_n(i); X^-_n(j)g &= 0; \quad i \neq j \\
fX^-_n(i); X^-_n(j)g &= 2 \quad n_i; r_i;
\end{align*}
\]

Proof. It follows from the standard vertex operator calculus (cf. [J1]) by using Theorem [8.3].
Therefore we see that $\mathbb{R}$ is isomorphic to the tensor product of $r + 1$ copies of the space $\mathbb{R}$, the sum of Grothendieck groups of spin characters of $\mathbb{E}_n$-supernodes.

Remark 9.2. Let $A = (1_{ij})_{(r+1)^2}$, the matrix of the alternating form $c_1$ over $F_2$, then $A^2 = 0$. Here $T = 0$ if $r$ is even and 1 if $r$ is odd. Consequently it follows from Lemma 7.1 there are exactly $2^{r+1}$ irreducible $R_{F_2}(\cdot)$-modules on the $2^{2^{r+1}}$-dimensional space $C[R_2(\cdot)]=\cdot$. One of the (at most) two irreducible module structures is given by the cocycle $(i; j) = 1$, for $i < j$, and $(i; j) = 1$, for $i > j$. Then the vertex operators $X_n(\cdot)$ generate the twisted Clifford algebra on the space $F$ defined by $fX_n(\cdot)X_n^0(\cdot)g = 2\cdot1^i_{ij}n; \cdot\cdot\cdot$ and $fX_n^+(\cdot)X_n^-(\cdot)g = 2\cdot1_{ij}n; \cdot\cdot\cdot$.

9.2. Super spin character tables of $\mathbb{E}_n$ and vertex operators. We now use the twisted vertex operators to construct all irreducible characters of spin supernodes of $\mathbb{E}_n$ for all $n$.

Let $R_n(\mathbb{E}_n)$ be the lattice generated by the characters of spin irreducible $\mathbb{E}_n$-supernodes. Then $R_n(\mathbb{E}_n) = \mathbb{C} \cdot R(\mathbb{E}_n)$.

First we construct a special orthonormal basis in $R$ and then use them to give irreducible characters of spin $\mathbb{E}_n$-supernodes. The vertex operator $X_n(\cdot)$ is defined as in (7.2) except that we drop $e$. The following is easily seen (cf. (7.2)).

Lemma 9.3. For $n \geq 0$, we have

$$X_n(\cdot)1 = n; 0; \cdot\cdot\cdot$$

For a $m$-tuple index $= (1; \cdot\cdot\cdot; m) \in 2^m$ we denote

$$X(\cdot) = X_{1}(\cdot) \cdot\cdot\cdot n(X)1; \cdot\cdot\cdot$$

$$X(\cdot) = (X_{1}(\cdot)1) \cdot\cdot\cdot n(X)1; \cdot\cdot\cdot$$

We also define the raising operator $R_{ij}$ by

$$R_{ij}(1; \cdot\cdot\cdot m) = (1; \cdot\cdot\cdot i+1; \cdot\cdot\cdot j; \cdot\cdot\cdot m);$$

Then we define the action of the raising operator $R_{ij}$ on $X(\cdot)$ or $x(\cdot)$ by $X_{R_{ij}}(\cdot)$ or $x_{R_{ij}}(\cdot)$.

Given $2 \cdot P(\cdot)$, we define

$$x = \frac{Y}{2} X(\cdot);$$

Similarly we define $x = \frac{Q}{2} x(\cdot);$. 


Theorem 9.4. The vectors $X$ for $n = (1)$ live in an orthogonal basis in the vector space $R$ with $hX; i = 2^{l(i)}$.

Moreover, we have that

\begin{align}
X &= Y Y \frac{1}{1+R_{ij}} X ( ) \\
&= x + c; x;
\end{align}

where $c; 2 \in Z$, and $R_{ij}$ is the raising operator.

**Proof.** The generalized Clifford algebra structure (9.1) implies that the nonzero elements

\[ X_{n_l}( ) \quad n_l \downarrow \downarrow \downarrow \]

of distinct indices generate a spanning set in the space $R$. To see that they satisfy the raising operator expansion we compute that

\[ X ( ; z_l) X_l( ; z; z) = :X ( ; z) X_l( ; z; z) : \]

Using the result in [J1] it follows that this is exactly the generating function of the raising operator expansion at the case is trivial under the isomorphism $c$. In other words, equation (9.2) is true when is a characteristic partition-valued function.

Next the orthogonality follows from the generalized Clifford algebra commutation relations in Theorem 9.1. The orthogonality relations show that the raising operator is not aected by the character $c$, hence the general case follows by multiplying the raising operator formula for each $c$.

The corresponding basis in $S$ are the classical symmetric functions called Schur’s $Q$-function [J1]. For any strict partition $\lambda = (1; \ldots; 1)$, the Schur’s $Q$-function $Q$ is determined by $[\lambda']$

\[ Q (y_1; \ldots; y_n) = 2^{l_1} \frac{Y^{(1)}}{Y^{(1)}} \frac{Y^{(2)}}{Y^{(2)}} \cdots \frac{Y^{(l)}}{Y^{(l)}} \]

where $S_n$ acts on $y_1; \ldots; y_n$ and we allow $j = 0$ for $j = l+1; \ldots; n$.

It is known (see for example [M, J1]) that $Q$, $2 \in S_n$ form a basis in the subring of symmetric functions generated by the power sums $p_1; p_2; \ldots$

We can think of $a_n ( ); n > 0; 2$ as the $n$-th power sum in a sequence of variables $y = (y_1)_1$. In this way we identify the space
S with a distinguished subspace of symmetric functions generated by odd degree power sums indexed by \( y \). In particular given a strict partition we denote by \( Q( ) \) the Schur's Q-function associated to \( y \). We also denote by \( Q( ) \) the corresponding element in \( S \) by the identification of \( S \) and \( R \). For \( 2 \ P( ) \), we denote

\[
Q = Q( ) 2 S:
\]

(9.5)

For \( 2 \ SP( ) \) we define

\[
\overline{Q} = 2 \ (l( ) d( ))^{-\frac{1}{2}} \overline{Q};
\]

(9.6)

where \( d( ) \) is the parity of \( (\overline{Q}) \) (see (9.7)). Similarly we define \( \overline{Q} = 2 \ (l( ) d( ))^{-\frac{1}{2}} \overline{Q}( ) \), where \( \overline{Q}( ) = Q( ) \). In particular we have

\[
\text{ch}( ) = \overline{Q} = 2 \overline{Q}( ) = 2 \overline{Q}( ) ;
\]

(9.7)

The following result immediately follows from Theorem 9.4 combined with the characteristic map ch and (9.6).

Proposition 9.5. For strict partition-valued function \( 2 \ SP( ) \) we have

\[
\overline{Q} \overline{Q} i = 0 \quad \text{if } \epsilon ;
\]

(9.7)

\[
\overline{Q} \overline{Q} i = 1 \quad \text{if } \epsilon \text{ is even,}
\]

\[
\overline{Q} \overline{Q} i = 2 \quad \text{if } \epsilon \text{ is odd.}
\]

The following result generalizes a similar result of [1.8] for trivial.

Lemma 9.6. Under the characteristic map ch the symmetric functions \( \overline{Q} \) correspond to a character in \( R_\epsilon(\epsilon_n) \) and \( \overline{Q} \) correspond to a virtual character in \( R_\epsilon(\epsilon_n) \).

Proof. Observe that the tensor product of two irreducible supermodules of type \( Q \) is a sum of two irreducible supermodule of type \( M \) (see Proposition 3.7). Computing its inner product we know that for positive odd integers \( m \) and \( n \) the character \( \chi_n( ) \) is twice of some irreducible character. Then by induction we see that \( \chi_1( ) \) is a character in \( R_\epsilon(\epsilon_n) \). From Theorem 9.4 we see that \( \overline{Q} \) is a \( Z \)-linear combination of \( \overline{Q} \) with \( \text{ch}_1( ) \), hence \( \overline{Q} \) is a virtual character of \( \epsilon_n \). \( \square \)

For \( 2 \ SP_n( ) \), we define \( e = e( )^\wedge \wedge e( )^\wedge \). For a partition and an irreducible character of \( e \), we define the spin character \( ( ) \) of \( e \) to be \( ( ) \) (see Corollary 4.5).
Theorem 9.7. For each strict partition-valued function \( 2 \mathbb{S}P_n(1) \), the vector \( Q \) corresponds, under the characteristic map \( \text{ch} \), to the irreducible character of the spin \( E_n \)-supermodule given by a \( Z \)-linear combination of

\[
\text{Ind}^E_n_{(\varepsilon)}(\cdot) = (\cdot); \tag{9.8}
\]

where \( b \) and the rest sum and is = with multiplicity one. The parity of \( n \) is equal to \( d(\cdot) = n \) \( l(\cdot) \) (mod 2). Its character at the conjugacy class of type \( 2 \mathbb{O}P_n(1) \) is equal to the matrix coefficient

\[
2^{d(\cdot)} 1^{l(\cdot)} \text{deg}(\cdot) = 2_X; \tag{9.9}
\]

Moreover the degree of the character is equal to

\[
2^{b+n} \text{deg}(\cdot) = 2_X; \tag{9.10}
\]

where \( b \) denotes the smallest integer \( a \).

\textbf{Proof.} Suppose we know that \( Q \) corresponds to the character of an irreducible \( E_n \)-supermodule. By (5.4) and definition of the characteristic map we see immediately that the linear combination (9.2) is given by the vertex operator structure and the matrix coefficient (5.5) give the character table of all irreducible supermodules.

First we observe that the number of irreducible spin supermodules of type \( M \) is equal to the number of even strict partition-valued functions, which are realized by the vectors \( 2^{d(\cdot)} 1^{l(\cdot)} X(\cdot) \) \( 2 \mathbb{S}P^0(\cdot) \) up to signs. As for the vectors \( 2^{d(\cdot)} 1^{l(\cdot)} X(\cdot) \) \( 2 \mathbb{S}P^1(\cdot) \), it follows from Theorem 9.4 and Proposition 3.3 that each of such vectors corresponds to a virtual irreducible character in \( R_x(E_n) \) of type \( Q \), since the case of sum or difference of two irreducible characters of type \( M \) is ruled out by the orthogonality. To show that they correspond to actually irreducible characters it is sufficient to show that the value of

\[
\text{ch} X(\cdot) = 2^{d(\cdot)} 1^{l(\cdot)} X(\cdot)
\]

at the conjugacy class of the identity element of \( E_n \) is positive.

Let \( c^2 \) be the class consisting of the identity in \( \mathcal{E} \). The type of the identity element in \( E_n \) is the partition-valued function such that \( p(c^2) = (1^n) \) and \( (c) = 0 \) for \( c \neq c^2 \). Recall from (5.2) that \( a_n(c^2) = 2 \text{deg}(\cdot) a_n(\cdot) \). By comparing weights and using orthogonality
(Theorem 5.4) we have
\[ hX \cdot a^0_\i \cdot a^\i_\j \cdot a^{n^i} \cdot hX \]
\[ = h \cdot X \cdot () \cdot () \cdot (\deg a) \cdot a \cdot () \cdot ^n_\i \]
\[ = n! \cdot (\deg a) \cdot (j)! \cdot Y \]
\[ = n! \cdot (\deg a) \cdot (j)! \cdot hX \cdot () \cdot a^\i_\j \cdot () \cdot i:
\]

By the result of (6.51) in [1] we have
\[ hX \cdot () \cdot a^\i_\j \cdot () \cdot i = \frac{j! \cdot Y}{i! \cdot \i(\ ) \cdot j!(\ ) \cdot i < j} + j(\ )
\]

which implies the formula (9.10), thus the theorem is proved. □

The irreducible spin \( \mathfrak{S}_n \)-supermodules can be described easily as follows. For each irreducible character \( \mathbf{2} \) let \( U \) be the irreducible module a ordering. For each strict partition let \( V \) be the corresponding irreducible spin supermodule of \( \mathfrak{S}_n \). Using the construction of Sect. 9.3, we see that \( U \cdot V \) is a spin \( \mathfrak{S}_n \)-supermodule.

**Proposition 9.8.** For each strict partition-valued function \( i = (\ ) \) \( \mathbf{2} \) \( \mathbf{S}P_n(\ ) \), with \( m \) of the partitions \( (\ ) \) being odd, the super tensor product
\[ Y \cdot U^l(\ ) \cdot V^l(\ ) \]
decomposes completely into \( 2^{m-2n} \) copies of an irreducible spin \( \mathfrak{S} \)-supermodule. Denote this irreducible module by \( W \). Then the induced supermodule \( \text{Ind}^n_\mathfrak{S} W \) is the irreducible spin \( \mathfrak{S}_n \)-supermodule corresponding to \( (\ ) \), and it is of type M or Q according to \( d(\ ) = n \cdot l(\ ) \) is even or odd.

**Proof.** Let \( V \) be the irreducible spin \( \mathfrak{S}_n \)-supermodule corresponding to \( (\ ) \). It follows from Theorem 5.4 that \( V \) is an irreducible component of \( \text{Ind}^n_\mathfrak{S} W \).

Note that the supermodule \( U^l(\ ) \cdot V^l(\ ) \) is irreducible and is of type M (or type Q) according to \( d(\ ) = j(\ ) \cdot l(\ ) \) even (or odd). Let \( (\ ) \), \( (\ ) \) be odd partitions, and let \( (\ ) \), \( (\ ) \) be even partitions. Then the parity of \( m \) equals the parity of \( m \), i.e., \( d(\ ) = n \cdot l(\ ) \equiv m \pmod{2} \).

It follows from Proposition 9.7 that \( U^Q \cdot V^l(\ ) \) decomposes completely into \( 2^{m-2n} \) copies of the irreducible supermodule \( W \).
of $e$, and $W$ is of type $M$ if $m$ is even and of type $Q$ otherwise. The degree of $\text{Ind}_e^n W$ equals to $\frac{f_{n+1}}{f_j} \deg (W)$, and we have

$$
\deg (W) = 2^{dm+2e} \prod_{i<j} \deg (j)i!(1) \deg (V) \prod_{i<k} i!
$$

$$
= 2^{dm+2e} \prod_{i<j} \deg (j)i!(1) \deg (V) \prod_{i<k} i!
$$

where we have used the degree formula (9.17) for the special case of $S_n$. The exponents of 2 in the first factor and the product sum up to

$$
\sum_{X \in \text{bd}(V)} 2^{dm+2e} = \sum_{s=0}^n \frac{d((\lambda_s)) + 1}{2} + \sum_{s=m}^{n-1} \frac{d((\lambda_s))}{2}
$$

$$
= \frac{n}{2} \cdot \frac{1}{2} + \frac{m}{2} = \frac{n}{2} \cdot \frac{1}{2}
$$

where $\bar{m} = 0$ or 1 according to $m$ is even or odd. Thus the degree of $\text{Ind}_e^n W$ is exactly the one given by Eqn. (9.17). Hence $\text{Ind}_e^n W$ is the irreducible spin $e_n$-supermodule $V$ corresponding to $\lambda$. \hfill \Box

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