Two-Dimensional Coulomb Glass as a Model for Vortex Pinning in Superconducting Films

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A glass model of vortex pinning in highly disordered thin superconducting films in magnetic fields $B \ll H_{c2}$ at low temperatures is proposed. Strong collective pinning of a vortex system realized in disordered superconductors that are close to the quantum phase transition to the insulating phase, such as InO$_x$, NbN, TiN, MoGe, and nanogranular aluminum, is considered theoretically for the first time. Utilizing the replica trick developed for the spin glass theory, we demonstrate that such vortex system is in non-ergodic state of glass type with a large kinetic inductance per square. The distribution function of local pinning energies is calculated, and it is shown that it possesses a wide gap; i.e., the probability to find a weakly pinned vortex is extremely low.

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1. INTRODUCTION

In this work, we study strongly disordered superconducting films subject to magnetic field $B \ll H_{c2}$ at low temperatures. The main interest for this problem emerges from active experimental research in this area (see, e.g., review [1]; a more detailed discussion regarding some of experiments [2, 3] is present in the end of the paper). The main issue we need to study is the competition between strong pinning of each individual vortex by disorder and repulsion between vortices. Strong pinning corresponds to the energy variations of the order of vortex core energy itself when vortex is moved by distance of the order of the core size $\xi$.

Such strong pinning emerges because the order parameter itself is strongly fluctuating [4]. The regular vortex lattice in such situation does not appear, and even the short-range order is absent, but the vortex density is constant on average and is fixed by the external magnetic field. It is very important for such state to exist that the energy of shear deformations of the vortex lattice (in the model of weak collective pinning), or study pinning of isolated vortices neglecting the interaction between them; both approaches are inapplicable to the problem at hands. We also mention theory of strong pinning [11–13], where strong impurities were considered, and the interaction between vortices was considered by means of elasticity theory for the vortex lattice; it was possible due to low concentration of strong impurities. Our situation is different: the defects are strong and their concentration is high.

We develop a theory of vortex glass in a situation, which reminds the “Coulomb glass” state realized in the model of the Coulomb gap proposed by Efros and Shklovskii [14, 15], but in a situation when the interaction between particles (vortices in our case) is logarithmic repulsion $U(r) = U_0 \ln \frac{a}{r}$, instead of usual Coulomb one. Here, the constant is $U_0 = \frac{\Phi_0 d}{8\pi \lambda^2}$ for a thin superconducting film of thickness $d$, which is much smaller than the London penetration depth $\lambda$. Strictly speaking, on the largest distances $r \geq \lambda_{2D} = 2\lambda^2/d$ the interaction energy is no longer logarithmic, it decays as $\propto 1/r$; however, we will consider superconductors with a very high ratio $\lambda_{2D} \geq 100$ (which is easily realizable in thin films of strongly disordered superconductors), where finite value of $\lambda_{2D}$ does not play any role.
Phenomenological approach to the problem of vortices moving in the film, similar to one used in [14, 15], was developed in [16] (see also [17]). Here, we develop alternative approach based on the paper by Müller and Ioffe [18] (see also [19, 20]), where the problem of Coulomb gap was studied using spin glass theory methods, and a phase transition to the non-ergodic state with broken replica symmetry was predicted. However, unlike [18], we will not assume that the theory can be described by a purely local matrix model neglecting the spatial fluctuations of matrix fields describing the glass phase.

2. MODEL AND MEAN FIELD THEORY

We will use model assumption that vortices can occupy positions of a discrete regular lattice with lattice constant $a$. The configuration of vortices will be described by “occupation numbers” of each site $\langle n_i \rangle$. External magnetic field $B$ leads to a finite vortex density $\langle n_i \rangle \equiv K = B^2a^2/\Phi_0$. We neglect antivortices as well as vortices with charge $n_i > 1$. Finite concentration of vortices will be fixed by the chemical potential $\mu$. Finally, disorder in our model will be described by the random energy of a vortex core $e_v$, those correlation function is $\langle u_r \rangle = W^2\delta_{rr'}$. It leads to the Hamiltonian

$$H = \sum_{r,r'} \delta_{rr'} J_{rr'} + \sum_r (u_r - \mu) \delta_{rr},$$

where $\delta_{rr} \equiv n_r - K$ and $J_{rr'} = U_0 \ln \frac{L}{|r - r'|}$. Disorder strength is assumed to be large, $W \gg U_0$. In fact, in the superconductors we consider, $W \sim U_0$; in the conclusion, we will discuss why the model assumption $W \gg U_0$ will not affect our main results. We average the free energy over the disorder utilizing the replica trick, and perform a Hubbard–Stratonovich transformation of a non-local term introducing the auxiliary field $\varphi$ (it has the meaning of the dual variable to the superconducting phase). As a result, we arrive at the following expression for the partition function:

$$Z^n = \int \mathcal{D} \varphi \exp \left( -\frac{1}{2} \varphi(\beta J)^{-1} \varphi \right) \times \prod_r \text{Tr}_v \exp \left( \frac{1}{\beta} \sum_a (\beta^2 W^2 \delta_{nn'} + \frac{1}{2} \delta_{nn'} \mathcal{J}_{rr'} \varphi^a_\alpha \varphi^a_\alpha) \right),$$

where $\text{Tr}_v \equiv \sum_{\eta=0,1}$; Latin indices numerate replicas $a = 1, ..., n$ ($n \to 0$), matrix $\mathcal{J}^{ab} = 1$ describes the quenched disorder equivalent for all replicas, and the interaction $\mathcal{J}_{rr'} = \delta_{rr'}$ is diagonal in replica space. It is worth noting that in this expression, the “vortex” part of the action appears now purely local.

We characterize the glass state by means of diagonal in coordinate space (yet coordinate-dependent) matrix $\mathcal{G}_{rr}^{ab} = -\varphi^a_\alpha \varphi^b_\alpha$, which describes the correlations of slowly varying in space part of bilinear combination of fields. Glass transition corresponds to spontaneous replica symmetry breaking in such a matrix. The order parameter is introduced utilizing the identity (the integral over $\varphi$ is taken along the imaginary axis)

$$I = \int \mathcal{D} \varphi \mathcal{D} \varphi \exp \left( -\frac{1}{2} \text{Tr} \left( \hat{\mathcal{G}} \hat{\mathcal{G}} \right) - \frac{1}{2} \varphi \hat{\mathcal{D}} \varphi \right).$$

The fluctuations of the field $\varphi$ are described by the propagator with the screening length $l \sim a\sqrt{W}/U_0$. On the other hand, it is reasonable to assume that the fluctuations of the order parameter $\hat{\mathcal{G}}_{rr}$ will be correlated on much larger spatial scales in the glass phase and near the transition.

In order to deal with the interaction between vortex occupation numbers $n_i^e$ and $q_i^e$ field, we expand $\exp \left( \sum_a \varphi^a_\alpha \delta n^a_i \right)$ in the Taylor series and rewrite arbitrary term in the momentum representation:

$$e^{i \sum \varphi^a_\alpha \delta n^a_i} = \sum_{k=0}^\infty \sum_{q_{i_k}+q_{i_k} = 0}^{q_{i_k}} \frac{i^k}{k!} \delta n^a_{i_k} \ldots \delta n^a_{i_k} \varphi^a_{q_{i_k}} \ldots \varphi^a_{q_{i_k}}.$$ (4)

We wish to describe fluctuations of soft modes of the order parameter $\hat{\mathcal{G}}_{rr}$ with the wave vectors much smaller than typical wave vectors of $\varphi$ fields, that is $q_i \ll l^{-1}$. The main contribution to such fluctuations come from the terms in (4), where some pairs of wave vectors are anomalously close to each other $|q_i + q_j| \ll l^{-1}$; such “contractions” will then be replaced by $\mathcal{G}_{q_i+q_j}$ (meaning that the total wave vector is small). Therefore, in order to obtain the leading contribution to fluctuations of the slow modes, we need to consider all contractions of $\varphi$ fields in this expression. The terms with odd $k$ then describe the interaction between the slow and fast modes, and can be neglected in the leading order. This allows us to replace the interaction between the field $\varphi(r)$ and vortex degrees of freedom by the local interaction between vortices described by the term $\delta n_i^a \hat{\mathcal{G}}_{rr} \delta n^a_i/2$ in the exponent.

We finally perform the remaining Gaussian integration over $\varphi(r)$, and arrive at the following field theory describing fluctuations of slow modes of the matrix order parameter:

$$Z^n = \int \mathcal{D} \hat{\varphi} \mathcal{D} \hat{\varphi} \exp \left( -n S[\hat{\mathcal{G}}, \hat{\varphi}] \right),$$

$$n S[\hat{\mathcal{G}}, \hat{\varphi}] = \frac{1}{2} \text{Tr} \left( \hat{\mathcal{G}} \hat{\varphi} \right) + \frac{1}{2} \text{Tr} \ln (1 + \beta \hat{\varphi} \hat{\varphi}) + \beta n \sum_r F_r[\hat{\mathcal{G}}_{rr}].$$ (6)
where the local part of the free energy is given by the expression

\[
e^{-\beta n \mathcal{F}_1[\hat{g}]}
= \text{Tr}_r \exp \left( \frac{1}{2} \delta n (\hat{n}^2 W^2 \hat{g} + \hat{g}) \delta n + \beta \mu \sum_a \delta n_a^2 \right).
\] (7)

We begin the analysis of the action (6) by studying the spatially homogeneous saddle points:

\[
\frac{\delta S}{\delta \hat{g}} = \frac{1}{2} (\hat{g} + \hat{G}) = 0, \quad Q_{ab} = \langle \delta n_a \delta n_b \rangle_{\hat{g}},
\] (8)

where \( \hat{Q} \) is the density correlation function calculated in the local model (7).

The second saddle-point equation acquires the following form, in agreement with the definition of the \( \mathcal{G} \) matrix:

\[
\frac{\delta S}{\delta \hat{g}} = \frac{1}{2} (\hat{g} + \hat{G}) = 0, \quad \hat{G} = (\langle \beta \hat{J} \rangle^{-1} + \hat{g})^{-1}.
\] (9)

To illustrate the role of the \( \hat{G} \) matrix, let us introduce into a system a pair of infinitesimal vortices with charges \( q_{12} \ll 1 \) to the points \( r_{12} \) in the replicas \( a_{12} \). It corresponds to the following perturbation of the system Hamiltonian:

\[
V = \sum_r \left[ q_r J_{rr} \delta n^a_r + q_r J_{rr} \delta n^b_r \right] + q_r q_r J_{rr}.
\] (10)

Free energy response to such a perturbation determines interaction energy between two added vortices. After the Hubbard–Stratanovich transformation, one finds a correction to the action in the exponent in Eq. (3), equal to \( \langle \delta n^a_r \delta n^b_r \rangle_{\hat{g}} \) on the Gaussian integration over \( \phi \), one finds the following additional contribution to Eq. (6):

\[
\frac{\delta S}{\delta \hat{g}} = \frac{1}{2} \left( q_r^2 \langle G_{rr} \rangle^a + q_r^2 \langle G_{rr} \rangle^b \right) + q_r q_r \langle G_{rr} \rangle.
\] (11)

It means that the average value of \( \langle \hat{G} \rangle \) matrix can be identified with the effective interaction between two “infinitesimal” vortices:

\[
U_{\phi \phi}^{(\text{eff})}(r, r') = \left( \frac{\langle \delta n^a \delta n^b \rangle_{\hat{g}}}{\langle \delta n^a \delta n^b \rangle_{\hat{g}}} \right) = T \langle G_{rr} \rangle.
\] (12)

Finally, we write the equation for the chemical potential:

\[
\left\langle \frac{\partial S}{\partial (\beta \mu)} \right\rangle = \sum_a \left\langle \delta n_a^2 \right\rangle_{\hat{g}} = 0.
\] (13)

As \( W \) is assumed to be the largest parameter is the problem, the chemical potential in the leading order is determined by the “bare” density of states \( v(u) = \exp(-u^2/2W^2) / \sqrt{2\pi W} \) via the equation

\[
1 - 2K = \int v(u) du \tanh \frac{\beta (u - \mu)}{2} = \int v(u) d\text{sgn}(u - \mu),
\] (14)

which yields asymptotic expressions

\[
\mu = -W \sqrt{2\pi} \left( \frac{1}{2} - K \right), \quad |K - 1/2| \ll 1,
\] (15)

\[
2 \ln \left( \frac{1}{\sqrt{2\pi K}} \right)^{1/2}, \quad K \ll 1.
\]

3. HIGH-TEMPERATURE PHASE
AND GLASS TRANSITION

We begin from the high-temperature phase corresponding to the replica-symmetric solutions \( \mathcal{G}_{ab} = \mathcal{G}_0 \delta_{ab} + \mathcal{G}_1 \mathcal{J}_{ab} \) (and the same for \( \mathcal{Q} \)). Since the vortex variables \( \delta n_a \) are similar to the Ising spin variables, the following identity can be written:

\[
\delta n^2 = \delta n(1 - 2K) + K(1 - K) \quad \text{at} \quad K = 1/2
\]

As a result, screening appears in the propagator \( W \rightarrow \sqrt{W^2 + T^2} \mathcal{J}_1 \). Both effects are actually negligible because \( \mu - W \gg T, U_0 \).

We obtain the solutions

\[
\mathcal{G}_0 = W \int \frac{v(u) du}{(2 \cosh \frac{\beta (u - \mu)}{2})^2} \approx TV_0,
\] (16)

\[
\mathcal{G}_1 = K(1 - K) - \mathcal{G}_0.
\] (17)

Since \( W \) is larger, the density of states \( v(u) \) can actually be replaced with a constant:

\[
v_0 \equiv v(\mu) \approx \frac{1}{\sqrt{2\pi W}}, \quad |K - 1/2| \ll 1,
\] (18)

\[
\left( \frac{K}{2 \ln \left( \frac{1}{\sqrt{2\pi K}} \right)^{1/2}} \right), \quad K \ll 1.
\]

As a result, screening appears in the propagator

\[
G_{ab}(k) = G_0(k) \delta_{ab} + G_1(k) \mathcal{J}_{ab},
\]

\[
G_0(k) = \frac{2\pi \beta U_0}{k^2 + \Gamma^2}, \quad G_1(k) = -\mathcal{G}_0^2(k)/a^2
\] (19)

with \( l = 2a \sqrt{v_0 U_0} \approx a \sqrt{W/U_0} \). Finally, the order parameter is

\[
\mathcal{G}_0 = -\frac{\beta U_0}{2} \ln \frac{1}{v_0 U_0}, \quad \mathcal{G}_1 = \frac{\beta^2 U_0}{v_0} K(1 - K).
\] (20)
In order to study the stability of the replica-symmetric solution and deduce the freezing transition temperature, one needs to study the Hessian, i.e., the quadratic expansion of the action (6):

\[ nS^{(2)}[\delta \hat{G}, \delta \hat{\phi}] = \frac{1}{2} \text{Tr}(\delta \hat{G} \delta \hat{G}) - \frac{1}{4} \text{Tr}(\delta \hat{G} \delta \hat{G}) + \frac{1}{8} \sum_r Q_{(ab)(ab)} \delta \hat{G}_{ab} \delta \hat{G}_{ab}, \]

where we have introduced the correlation function

\[ Q_{(ab)(ab)} = \langle \delta n_{ab} \delta n_{ab} \delta n_{ab} \rangle_q. \]

Upon lowering the temperature, a singularity appears in the replicon mode. This mode corresponds to the linear subspace of matrices subject to the following two constraints: \( \delta \hat{G}_{aa} = 0 \) and \( \sum_a \delta \hat{G}_{ab} = 0 \). The action for the replicon fluctuations then reads:

\[ nS^{(2)} = \int \frac{d\mathbf{q}}{4\pi^2} \text{Tr}(\delta \hat{G}_q \delta \hat{G}_q) \left[ -Q_{22} \frac{1}{1 - \hat{B}_2(q)} \right] \left( \delta \hat{G}_q, \delta \hat{G}_q \right), \]

where symbol \( \text{Tr} \) corresponds to the trace with respect to replica space only, and the following notation was introduced:

\[ \hat{B}_2(q) = \int \frac{d\mathbf{u}}{2\cosh \left( \frac{\nu(u-m)}{2} \right)} = T\nu/6. \]

Quadratic expansion (24) corresponds to the ladder summation of diagram series for a four-point Green’s function of the \( \phi \) field. The action (24) yields the propagators

\[ \langle \langle \delta \hat{G}_{ab} \delta \hat{G}_{a'b'} \rangle \rangle_q = \frac{12\beta a^2}{\tau + q^2 l^2 / 6}, \]

\[ \langle \langle \delta \hat{G}_{ab} \delta \hat{G}_{a'b'} \rangle \rangle_q = \frac{\mathcal{G}_{ab} \mathcal{G}_{a'b'}}{\tau + q^2 l^2 / 6}, \]

where we have introduced the freezing temperature

\[ T_c \equiv U_0 / 12; \]

at this temperature the value \( \tau \equiv T/T_c - 1 \) changes sign, and the instability appears in the theory (24) leading to the spontaneous symmetry breaking.

The tensor \( \mathcal{G}_{ab} \) is the projector onto the replicon mode.

As it is shown in the Supplementary 1, the correlation function \( \langle \langle \delta \hat{G} \rangle \rangle \) has the following physical meaning: it describes the long-wavelength asymptotic of the mean square fluctuation of the polarizability:

\[ \langle \delta n_{ab} \delta n_{ab} \rangle = \lim_{n \to 0} \frac{1}{n(n-1)_{ab}} \left[ \langle \delta \hat{G}_{ab} \delta \hat{G}_{ab} \rangle \right]. \]

Finally, replica structure of the projector onto the replicon mode gives additional factor of \( \lim_{n \to 0} \frac{\mathcal{G}_{ab}}{n(n-1)_{ab}} = 3/2 \) to Eq. (27).

In the vicinity of the transition, when \( \tau \ll 1 \), the quadratic part of the action (24) can be approximately diagonalized by the transformation

\[ \left( \begin{array}{c} \Psi \\ \phi \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right) \left( \begin{array}{c} \delta \hat{G}_1 \\ \delta \hat{G}_2 \end{array} \right), \]

the mode \( \Psi \) appears to be soft, and the mode \( \phi \) is gapped and thus can be neglected. Expanding the functional with respect to \( \Psi \), we arrive at (the details of the calculation are given in the Supplementary 2) the following Ginzburg–Landau functional:

\[ nS[\Psi] = \nu \left( \frac{1}{24} \text{Tr}(\tau \dot{\Psi}^2 + (\nabla \dot{\Psi})^2 / 6) - \frac{1}{2160} \left[ 7 \text{Tr} \dot{\Psi}^4 \right] + 6 \sum_{abc} \Psi_{ab}^4 - \frac{1}{1024} \sum_{abc} \Psi_{ab}^4 \right). \]

Despite the large screening length \( l \gg a \) in our problem, all the coefficients in front of the non-linear terms are of the same order \( -\nu T_c \sim U_0 / W \). Consequently, the derived Ginzburg–Landau theory lacks a small parameter, and the Ginzburg region where the fluctuation effects are strong is of the width \( \mathcal{G}_i = O(1) \); thus, the mean field theory is inapplicable near the transition. The same conclusion applies to the three-dimensional counterpart of the same problem studied in [18]. Strong critical fluctuations prevent us from the study of the critical region itself; therefore, we switch to the low-temperature phase of the model, where fluctuation effects are suppressed by a small factor \( T/T_c \ll 1 \).

4. LOW-TEMPERATURE PHASE

IN THE 1-STEP REPLICA SYMMETRY BREAKING APPROXIMATION

The ratio between coefficients in front of two cubic terms in the action (31), \( c_1/c_2 = 6/7 < 1 \), which suggests that the full continuous replica symmetry breaking scheme due to Parisi [21] should be used; if the same ratio would be \( >1 \), then 1-step replica symmetry breaking scheme (1-RSB) [22] would be sufficient. In our problem the ratio \( c_1/c_2 \) is quite close to unity, thus we will try to apply the 1-RSB approximation and show \textit{a posteriori} that the obtained solution is a very good one numerically. 1-RSB scheme suggests the following form for the matrices:

\[ \mathcal{G}_{ab} = \mathcal{G}_{0ab} \mathcal{R}_{ab} + \mathcal{G}_{1ab} \mathcal{R}_{ab} + \mathcal{G}_{2ab} \mathcal{R}_{ab}, \]

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$$2_{ab} = 2_{a} \delta_{ab} + \frac{1}{m}(2_{1} - 2_{0}) R_{ab} + 2_{c} \beta_{ab},$$
(32)

The auxiliary matrix \(R_{ab} = \delta_{[a/m][b/m]}\) (here \(\ldots\) denotes the integer part) is a block-diagonal matrices with diagonal blocks of size \(m \times m\) being filled with unities, while off-diagonal blocks are filled with zeros. In the replica limit \(n \to 0\), the parameter \(m \in (0,1)\) becomes an additional variational parameter of our theory. The Green’s function \(G\), see Eq. (9), is parametrized in the same manner:

$$G_{ab}(k) = G_{0}(k) \delta_{ab} + \frac{1}{m} (G_{1}(k) - G_{0}(k)) R_{ab} + G_{2}(k) \beta_{ab},$$
(33)

with

$$G_{0}(k) = \frac{2 \pi \beta U_{0}}{k^{2} + l_{0}^{2}}, \quad G_{2}(k) = -2 \pi G_{1}^{2}(k)/a^{2},$$
(34)

where \(l_{0} = a(2 \pi \beta U_{0} / \beta_{0})^{-1/2}\) are two different screening lengths. The first group of saddle point equations, Eq. (9), reads:

$$\begin{align*}
\langle \beta_{0} \rangle &= -\beta U_{0} \ln(l_{0}/a) = \beta U_{0} \ln(\beta U_{0} \beta_{0})/2, \\
\langle \beta_{1} \rangle &= \beta U_{0} \ln(l_{0}/l_{1})/m = \beta U_{0} \ln(\beta_{1} / \beta_{0})/2m, \\
\langle \beta_{2} \rangle &= \pi \beta_{2}(\beta U_{0} l_{1}/a)^{2} = \beta U_{0} \beta_{2}/2\beta_{1}.
\end{align*}$$
(35)

The second group, Eq. (8), in the limit of \(W \gg U_{0}\), can be expressed in terms of the auxiliary function \(f_{a}(m, \beta_{0})\) (see Supplementary 3 for details):

$$\begin{align*}
\bar{\beta}_{0} &= -\frac{\nu_{0} T}{1 - m} \frac{\partial f_{a}}{\partial \beta_{1}}, \\
\bar{\beta}_{1} &= \nu_{0} T, \\
\bar{\beta}_{2} &= K(1 - K) + \left(\frac{1}{m} - 1\right) \beta_{0} - \frac{1}{m} \beta_{1},
\end{align*}$$
(36)

and the auxiliary function reads:

$$f_{a}(m, \beta_{0}) = \frac{2}{m} \int dz \left\{ \ln \Xi(z, m, \beta_{0}) - m \ln 2 \cosh \frac{z - m^{2} \beta_{0}}{2} \right\},$$
(37)

$$\Xi(z, m, \beta_{0}) = \int \frac{dy e^{-y^{2}/2\beta_{0}}}{\sqrt{2 \pi \beta_{0}}} \left[ 2 \cosh \frac{y - z}{2} \right]^{m}.$$  
(38)

Last equation of the group (36) is trivial consequence of the fact that diagonal elements are fixed via the relation \(\bar{\beta}_{aa} = K(1 - K)\); the second equation suggests that the screening length \(l_{1}\) coincides with the screening length in the replica-symmetric phase. Finally, to close the whole system of equations we need to add stationary equation for the 1-RSB parameter \(m\), which can be written in the form

$$-\frac{6 \nu_{0} T}{m^{2}} \left( 1 - \frac{1}{1 - m \partial / \partial \beta_{1}} \right) + \beta_{1} \frac{\partial f_{c}}{\partial \beta_{1}} - m \frac{\partial f_{c}}{\partial m} = 0.$$  
(39)

Among seven equations (35), (36), and (39), only equations for \((m, \beta_{1}, \bar{\beta}_{0})\) are nontrivial.

At low temperatures, the system of Eqs. (35), (36), and (39) has the solution (see Supplementary 3.1 for details)

$$m = 1.09(T / T_{c}), \quad \beta_{1} = 61.0(T_{c} / T)^{2},$$
(40)

$$\bar{\beta}_{0} = 1.43 \times 10^{-5} v_{0} T.$$  
(41)

5. PHYSICAL PROPERTIES

OF THE LOW-TEMPERATURE PHASE

As we have shown above (Eq. (12)), the \(G\) matrix describes the interaction energy for two probe vertices introduced to the system. It is known from the theory of spin glasses [21] that replica symmetry breaking physically corresponds to the breaking of the ergodicity and dependence of the system state on its history. In particular, two protocols are commonly considered: the Zero Field Cooling (ZFC), which corresponds to introducing the probe vertices after freezing into the glass state, and Field Cooling, which corresponds to introducing the vertices before freezing. In the replica technique it corresponds to the two response functions

$$U_{ZFC}^{(\text{eff})}(r_{1}, r_{2}) = \lim_{\nu_{0} \to 0} \left[ U_{ab}^{(\text{eff})}(r_{1}, r_{2}) - U_{ab}^{(\text{eff})}(r_{1}, r_{2}) \right]$$
(42)

$$= TG_{0}(r_{1} - r_{2}),$$
(43)

The extreme smallness of \(\bar{\beta}_{0}\) (Eq. (41)) and the relation (34) leads to very large value of a screening length for the ZFC-response in the glass phase \(l_{0} = 260l_{c}\) (\(l_{c}\) coincides with the screening length in the high-temperature phase). In any experimentally feasible situation, such value \(l_{0}\) can be considered as infinity. As a result, at low temperatures \(T \ll T_{c}\), logarithmic interaction between vortices is restored, and such phase is characterized by nonzero superfluid stiffness:

$$\rho_{ZFC}^{(s)} = \frac{T}{4 \pi} \lim_{\nu_{0} \to 0} k^{2} G_{0}(k) = \frac{U_{0}}{2 \pi},$$  
(44)

Another important physical quantity in the problem is the distribution function of the local potential of an individual vortex, \(P_{h}(u)\). Detailed calculation of this distribution function at low temperatures is described in Supplementary 2.2; here, we present the approximate result, which is valid at low temperatures:

$$P_{h} = \frac{u - \bar{u}}{T_{c}} = v_{0} \frac{1}{2} \text{erfc}(3.03 - 0.09\bar{h}).$$  
(45)
At low temperatures, the gap develops in the distribution function. The half-width of the gap is of the order of $-30T_c = 2.5U_0$, and the absolute value of the density of states exactly at the chemical potential is negligibly small $-10^{-5}$, albeit nonzero. This small value is the actual reason behind the small value of $\langle \Omega \rangle$ (see Eq. (41)), which can be expressed as follows

$$\langle \Omega \rangle = \int P(u)\,du \approx TP_0, \quad P_0 \equiv P(\mu). \quad (46)$$

The last equality takes into account the fact that the density of states is nearly constant at the scales $|u - \mu| \sim T$.

The low-temperature phase in the 1-RSB approximation is unstable: the replicon mode, which is responsible for additional replica symmetry breaking, has a negative eigenvalue. However, Eqs. (24)–(27) are still applicable to the replicon mode, with only difference being that the screening length $l$ should be replaced by $l_0$ and the density of states $v_0$ should be replaced by its renormalized value $P_0$. This is possible because the density of states, despite having a large gap of width $-30T_c$, can be considered almost constant at the scales $-T$ that we are interested in. In particular, at $T \ll T_c$ the value $\tau = T/T_c - 1 = -1$, and thus the mode $q = 0$ is indeed unstable. However, due to the value $l_0 = a/\sqrt{2\pi \beta U_0}$, having a large numerical factor $\approx 250$, the phase volume of unstable modes $q \leq 1/l_0$ appears to be extremely small. It leads to the natural assumption that approximations we have made here can be used to describe the system with a good precision.

The entropy in the 1-RSB approximation can be written as

$$S = v_0T \times \left[ f_c'(m, \Phi_1) + \frac{1}{2} \frac{\partial f_c}{\partial m} - \Phi_1 \frac{\partial f_c}{\partial \Phi_1} + \frac{\pi^2}{3} \right] - 3\beta T_c \langle \Omega \rangle. \quad (47)$$

At low temperatures, the behavior of the entropy is discussed in Supplementary 2.3; here we briefly state the results. At zero temperature, the entropy is negative, but its absolute value is extremely small (which also stems from the low density of states):

$$S(T = 0) = -3\beta T_c \langle \Omega \rangle \approx -4.29 \times 10^{-5}v_0T_c. \quad (48)$$

The freezing transition of the vortex glass can also be considered from another point of view, as a statistical mechanics problem of a particle in the logarithmically correlated random potential [23]. Indeed, albeit the bare disorder is short-range correlated, the effective random potential probed by a separate vortex has the form

$$u_{\text{eff}}(r) = u(r) + \sum_{\tau} J_{\tau} \delta n_{\tau}, \quad (49)$$

and its fluctuations on the scales $l_0 \gg r \gg l_1$ could be estimated (utilizing (34) and (40)) as follows:

$$\frac{\langle u_{\text{eff}}(r) - u_{\text{eff}}(0) \rangle^2}{m} = \frac{2(1 - m)T^2(G_0(r) - G_0(0))}{m} \approx 11T_c^2 \ln \frac{l_0}{a} \quad T \ll T_c.$$
Formally, the obtained 1-RSB solution is unstable, which signals that the theory utilizing the continuous Parisi scheme should be developed. However, the difference between such a full theory and the one presented in this work is expected to be extremely small, which is suggested by the value of the entropy per site \(-S_0 \approx 10^{-5}\). Furthermore, these discrepancies should in fact be described using the dynamical spin glass theory, since fluctuations at the scales of the order of \(l_0\) at \(T \ll T_c\) cannot occur in a thermodynamically equilibrium fashion. Finally, we wish to point that in general case \(K \neq \frac{1}{2}\), an additional contribution to the free energy in the low-temperature phase is expected, which can make the 1-RSB solution stable. The study of these issues is postponed for the future.

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