Splitting the source term for the Einstein equation to classical and quantum parts

T.S. Biró and P. Ván

Heavy Ion Research Group

MTA Wigner Research Centre for Physics, Budapest

(Dated: December 5, 2013)

We consider the special and general relativistic extensions of the action principle behind the Schrödinger equation distinguishing classical and quantum contributions from the wave function field. Postulating a particular quantum correction to the source term in the classical Einstein equation we identify the conformal content of the above action and obtain classical gravitation for massive particles, but with a cosmological term representing off-mass-shell contribution to the energy-momentum tensor. In this scenario the - on the Planck scale surprisingly small - cosmological constant stems from quantum binding with a Bohr radius $a$ as being $\Lambda = \frac{3}{a^2}$. This is the same relation as for the de Sitter cosmological horizon.

Contents

I. Introduction

II. Nonrelativistic Quantum Mechanics
   A. Schrödinger equation with Madelung variables
   B. Schrödinger equation from action principle

III. Special Relativistic Quantum Mechanics
   A. Klein-Gordon Lagrangian
   B. Action principle with Madelung variables
   C. Klein-Gordon Energy-Momentum tensor
   D. Bohm-Takabayashi Energy-Momentum Tensor

IV. General Relativistic Quantum Mechanics
   A. Modification of the Einstein equation
   B. Classical and quantum terms in the Einstein equation
   C. An order of magnitude estimate for the cosmological constant

Summary

Acknowledgement

References

I. INTRODUCTION

The enigma of the cosmological constant, in the modern view interpreted as dark energy, is mainly due to its surprisingly small, yet nonzero magnitude: should it have namely a quantum gravity origin (analogous to a symmetric broken phase with nonzero Higgs fields in the Standard Model), then the natural scale for a ground state (vacuum) energy density would be of $M_P^2$ order, with $M_P$ being the Planck mass. In the symmetric phase on the other hand it should be exactly zero. According to astronomical observations, however, the effect is about 120 decimal orders of magnitudes too small for the energy density (and still 30 orders too small in the linear energy scale) while it is definitely non zero [1–16].

In spite of this naturalness problem the dark energy is responsible for $68 - 72\%$ of the evolution of the Universe observed presently in the standard cosmological models based on Friedmann’s first calculation. A remaining $24 - 28\%$ of the effect is called dark matter, about which more ideas have been already discussed in the literature.

In this paper we present an interesting observation based on a conformal treatment of the Schrödinger equation: as if the quantum mechanical problem of obtaining wave functions could be splitted to a massive and a conformal part in line with a classical – quantum partition [20–23]. After considering in a relativistic setting (but without spin effects) the free Klein-Gordon action is inspected. Identifying the quantum part as belonging to a traceless relativistic energy-momentum tensor, we suggest to modify its Bohm-Takabayashi form [24–29] and connect the remaining classical part to Einstein’s gravity equation [30–32].
in form of a dust matter source of massive point particles moving on Bohm trajectories. In this scenario the quantum nature of the wave function revals itself in deviations from the classical on-mass-shell relation $P_{\mu}P^\mu = (mc)^2$, and our suggested natural coupling to gravity makes a simple conformal transformation of the full Einstein tensor expedient.

After this transformation the classical part (dust gravity) separates from quantum effects which among others include a cosmological term. This term represents negative pressure e.g. for stationary quantum bound states in a simple attractive $-1/r$ potential.

In this paper we first recall the Schrödinger equation with complex magnitude – phase variables, together with the underlying action principle. Then the Klein-Gordon quantum action is analyzed in the same way, aiming at the determination of the physically correct energy-momentum tensor. Here we emphasize the quantum-conformal (in the Bohm-like contribution traceless) construction possibility. Based on this we suggest a dilaton-like [33–38] modification of the Einstein equation (but without introducing an extra dilaton field), which is however without any effect for the classical part. Finally a naturally emerging conformal transformation identifies the classical Einstein tensor, the quantization (wave function normalization) volume, the conformal symmetry of the quantum part and a cosmological term proporitonal to the off-mass-shell part of wave functions. Based on the latter we estimate the Bohr radius of a bounded state and find it slightly above the Hubble radius of the universe, according to the last obervations.

II. NONRELATIVISTIC QUANTUM MECHANICS

A. Schrödinger equation with Madelung variables

Although the Madelung picture [39–42]of the Schrödinger equation has been criticized due to various reasons (see e.g. [43–45]), we would like to explore here another important aspect of the use of the magnitude and phase of the complex wave function field, $\varphi$. Being interested in a splitting of the fundamental quantum equation into a "classical" and a "rest" part namely, the representation

$$\varphi = Re^{\frac{i}{\hbar} \alpha}$$

is of genuine use. Here $\alpha$ plays the role of the classical action for the corresponding classical dynamics and the canonical classical momentum and energy are derived accordingly as

$$E = -\frac{\partial \alpha}{\partial t}, \quad P = \nabla \alpha.$$ 

The Schrödinger equation in its well-known form,

$$-\frac{\hbar^2}{2m} \nabla^2 \varphi + V(x)\varphi = i\hbar \frac{\partial \varphi}{\partial t},$$

then can be rewritten in terms of the classical momentum, energy and the quantum factor $R$ by observing the following derivatives:

$$\frac{\partial}{\partial t} \varphi = \left( \frac{1}{R} \frac{\partial R}{\partial t} - \frac{i}{\hbar} E \right) \varphi, \quad \nabla \varphi = \left( \frac{1}{R} \nabla R + \frac{i}{\hbar} P \right) \varphi.$$ 

The Laplacian becomes

$$\nabla^2 \varphi = \left[ \nabla \left( \frac{\nabla R}{R} + \frac{i}{\hbar} P \right) + \left( \frac{\nabla R}{R} + \frac{i}{\hbar} P \right)^2 \right] \varphi.$$ 

Now the Schrödinger equation [3] is separated into its real and imaginary parts as follows: The real part connects the classical energy and momentum according to the classical formula, $E = P^2/2m$, and reveals a quantum correction, called the Bohm potential [24–26, 46]:

$$E = V - \frac{\hbar^2}{2m} \left[ \nabla \left( \frac{\nabla R}{R} + \frac{i}{\hbar} P \right) \right]^2 - \frac{P^2}{\hbar^2}$$
The interpretation of this energy expression (6) as a sum of a classical energy and a quantum modification,
\[ E = \left( \frac{P^2}{2m} + V \right) - \frac{\hbar^2}{2m} \nabla^2 R, \] (7)
reveals a position-dependent quantum correction to the classical energy, \( E \). The imaginary part on the other hand leads to a first order time-evolution constraint equation,
\[ \frac{i\hbar}{R} \frac{\partial R}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial}{\partial t} \left[ \nabla P + \frac{2}{R} P \cdot \nabla R \right]. \] (8)
Simplifying this imaginary part leads to an analogue of the mass density continuity equation,
\[ m \left( \frac{\partial R^2}{\partial t} + \nabla \left( R^2 P \right) \right) = 0. \] (9)
Upon introducing the velocity field via \( P = m v \) and the local fluid density \( \rho = R^2 = |\varphi|^2 \), this relation was interpreted as a continuity equation for the mass current carried by a "Madelung fluid"
\[ \frac{\partial \rho}{\partial t} + \nabla (\rho v) = 0. \] (10)
Similarly equation (7) is an integral of the corresponding rotation free momentum balance of a special Korteweg fluid \[47, 48\] or an off mass shell relation \[49\]. We do not pursue further the fluid aspect \[50, 51\], but follow the classical – quantum separation hint by using the variables \( R \) and \( \alpha \). At the end we shall realize that exactly this splitting makes it possible to identify a conformal part in the quantum dynamics of the massive particles having a traceless contribution to the energy-momentum tensor.

**B. Schrödinger equation from action principle**

According to Schrödinger's original article about his equation the following Action Principle can be formulated: instead of fulfilling the classical Hamilton-Jacobi equation \[52\], it is violated so that its space-time integral weighted by \( |\varphi|^2 \) achieves a variational extremum. The use of this form of the weighting factor may be argued for by noting that only this leads to a linear Euler-Lagrange variational equation.

The Quantum Action Principle behind the Schrödinger equation is given by \[20–22\]
\[ S = \int \left( \frac{\partial S}{\partial t} + \frac{\left| \nabla S \right|^2}{2m} + V \right) |\varphi|^2 d^3 x dt. \] (11)
It has been interpreted via a "Boltzmannian" eikonal ansatz: \( S = \frac{\hbar}{i} \ln \varphi \). Using this ansatz leads to the following complex bilinear form of the quantum action:
\[ \mathcal{S} = \int \left[ \frac{\hbar}{i} \varphi^* \frac{\partial \varphi}{\partial t} + \frac{\hbar^2}{2m} \nabla \varphi^* \cdot \nabla \varphi + V \varphi^* \varphi \right] d^3 x dt \] (12)
Finally variation against \( \varphi^* \) delivers the well-known Schrödinger equation, linear in the complex wave function \( \varphi \):
\[ \frac{\delta \mathcal{S}}{\delta \varphi^*} = \frac{\hbar}{i} \frac{\partial \varphi}{\partial t} - \frac{\hbar^2}{2m} \nabla^2 \varphi + V \varphi = 0 \] (13)
It is straightforward to check by variation against \( \varphi \) followed by a complex conjugation that the eikonal coefficient, \( \hbar/i \), has to be pure imaginary.

Now we re-investigate this quantum action with magnitude-phase variables in order to see the effect of the quantum – classical splitting considered in the previous subsection. Indeed the action also splits into quantum and classical parts,
\[ \mathcal{S} = \int \left[ \frac{\hbar^2}{2m} (\nabla R)^2 + R^2 \left( \frac{\partial \alpha}{\partial t} + V + (\nabla \alpha)^2 \right) \right] d^3 x dt \] (14)
The characteristic Lagrangian structure contained in this Quantum Action Principle can be summarized as follows:

\[ \mathcal{L} = \hbar^2 \text{ (quantum kinetic) } + R^2 \text{ (classical Hamilton-Jacobi equation)} \]

Finally we make some remarks about the relation between the pure classical action, \( \alpha \) and the complex action variable in the eikonal form, \( S \) (more commonly used in derivations). In fact one realizes that \( S = \alpha - i\hbar \ln R \), i.e. the real part of \( S \) is the classical \( \alpha \). Certainly for \( R = 1 \) the classical dynamics is recovered. In the quantum propagation of massive objects, however, \( \alpha \) and its derivatives, \( E \) and \( P \), are not constants, their evolution couples to that of \( R(x, t) \) exactly via the Schrödinger equation. This fact typically reflects deviations from the classical momentum and energy, and - as we shall see - also from the on-mass-shell dispersion relation.

### III. SPECIAL RELATIVISTIC QUANTUM MECHANICS

#### A. Klein-Gordon Lagrangian

Disregarding the spin of the electron, the Schrödinger equation can be viewed as the non-relativistic approximation to the Klein-Gordon equation – in analogy to the non-relativistic approximation to the relativistic Hamilton-Jacobi equation based on the energy-momentum dispersion relation of the mass point \( m \). Although the Klein-Gordon equation does not describe the quantum energy levels of the H-atom precisely, for the study of a quantum – classical splitting it is more suitable due to its simplicity.

The quantum action is based on the Lagrange density

\[ \mathcal{L} = -\frac{1}{2} \partial_\mu \psi^* \partial^\mu \psi - \frac{1}{2} \left( \frac{mc}{\hbar} \right)^2 \psi^* \psi, \]  

(15)

containing a complex \( \psi(x, t) \) field. The action is a Lorentz-invariant integral,

\[ \Theta = \int \mathcal{L} \, d^4x, \]  

(16)

with \( dx^4 = (cdt, d\vec{r}) \). We use physical units in which \( [\mathcal{L}] = \text{energy density} / c = [mc/L^3] \) and the Lorentz form \( \text{diag}(1, -1, -1, -1) \).

In order to keep the relation to the non-relativistic wave function description on the one hand and to the classical relativistic mass point action (Maupertuis action) on the other hand, we include some further factors. The complex scalar field related to the wave function is written as

\[ \psi = \frac{\hbar}{\sqrt{mc}} R e^{i\alpha}. \]  

(17)

Here \( \alpha \) is the (real) classical action, as used in the previous section. The physical units of \( R \) can be obtained from the mass term in the Lagrange density:

\[ \left( \frac{mc}{\hbar} \right)^2 \psi^* \psi = mcR^2 \]  

(18)

is part of \( \mathcal{L} \), so it follows that \( R^2 \) is a number density. Comparing this with the Maupertuis action for a classical mass point:

\[ -\frac{1}{2} \int \left( \int mc^2 R^2 d^3x \right) dt = -\int mc^2 d\tau \]  

(19)

we conclude to a normalization \( \int R^2 d^3x = 2 \) in the comoving volume. Here we note that not having the factor \( 1/2 \) in the Lagrange density eq.(15) would have resulted in a normalization to 1. This does not make a fundamental difference as long as \( R^2 = |\varphi|^2 \) is normalized to a finite value.

We consider now the derivatives of the complex field, \( \psi \) in Lorentz-covariant notation. The first derivative of \( \psi \) is given by

\[ \partial_\mu \psi = \left( \frac{\partial_\mu R}{R} + \frac{i}{\hbar} \partial_\mu \alpha \right) \psi. \]  

(20)

The derivative of the classical action is again a classical four-momentum and a four-velocity field also can be introduced analog-
gous to the non-relativistic treatment:

\[ P_\mu = \partial_\mu \alpha, \quad u_\mu = P_\mu/(mc). \]  
(21)

B. Action principle with Madelung variables

The special relativistic quantum action of the free massive particle can again be split into a classical and a quantum part by using the magnitude-phase variables. As a functional of the fields \( R(x) \) and \( \alpha(x) \) it reads as

\[ \mathcal{S} = \frac{\hbar^2}{2mc} \int \left[ \partial_\mu R \partial^\mu R + \frac{R^2}{\hbar^2} \left( \partial_\mu \alpha \partial^\mu \alpha - (mc)^2 \right) \right] d^4x. \]  
(22)

Rewriting this expression, it is transformed into \( \hbar^2 \) times quantum kinetic plus \( R^2 \) times classical part:

\[ \mathcal{S} = \int \left[ \frac{\hbar^2}{2mc} \partial_\mu R \partial^\mu R + \frac{R^2}{2mc} \left( P_\mu P^\mu - (mc)^2 \right) \right] d^4x \]  
(23)

Now the classical part is the relativistic energy-momentum mass-shell expression, which is classically zero, but in the quantum mechanics in general it differs from zero, unless \( R \) is a constant.

As it is well-known the Klein-Gordon action possesses a \( U(1) \) phase symmetry of the \( \psi(x) \) field. The corresponding \( U(1) \) Noether current is given by

\[ -J_\mu = i \frac{\hbar}{2} \left( \psi \partial^\mu \psi^* - \psi^* \partial^\mu \psi \right) = \frac{1}{mc} R^2 P_\mu = R^2 u_\mu \]  
(24)

constituting a number density 4-current \( R^2 u^\mu = \rho u^\mu \) based on the fluid picture. It is interesting to realize that the variation of the quantum action \( \mathcal{S} \) with respect to the classical action (phase) \( \alpha \) results in the conservation of this current:

\[ \delta \mathcal{S} / \delta \alpha = -\partial_\mu \left( \frac{1}{mc} R^2 \partial^\mu \alpha \right) = \partial_\mu J_\mu = 0. \]  
(25)

For our present seek for the quantum - classical splitting of the content of quantum physics it is, however, more important to study the other Euler-Lagrange equation of motion, the one obtained by variation against \( R \). It delivers

\[ \delta \mathcal{S} / \delta R = -\frac{\hbar^2}{mc} \Box R + \frac{R}{mc} \left( P_\mu P^\mu - (mc)^2 \right) = 0. \]  
(26)

Here \( \partial_\mu \partial^\mu = \Box \). This equation constitutes an off-mass shell dispersion relation for the classical 4-momentum

\[ P_\mu P^\mu - (mc)^2 = \hbar^2 \frac{\Box R}{R}. \]  
(27)

Either one interprets this as quantum effects causing the free scalar field be off-mass shell even without any further interaction, or one speculates that perhaps the underlying space-time metric receives corrections if \( R(x) \) is not a constant. In the latter case we consider a metric view:

\[ g_{\mu\nu} u^\mu u^\nu = 1 + \left( \frac{\hbar}{mc} \right)^2 \frac{\Box R}{R}. \]  
(28)

This is a Compton wavelength scaled, locally Lorentzian spacetime metric. This finding paves the way towards the investigation of effects due to the quantum nature of the matter on the spacetime metric in the light of the Einstein equation (c.f. section IV).

C. Klein-Gordon Energy-Momentum tensor

In order to execute such a program one has to investigate the source term of gravity, i.e. the energy-momentum tensor more closely. So, before turning to the Einstein equation, we turn to the calculation of the Klein-Gordon energy-momentum tensor. First we review the textbook derivation [53], the one using \( \psi \) and \( \psi^* \). As a first step the canonically conjugated complex
“momentum” field is obtained,

\[ \Pi_\mu = \frac{\delta \mathcal{L}}{\delta \partial_\mu \psi} = \frac{1}{2} \partial_\mu \psi^*, \]  

(29)

and then according to the familiar Legendre-transformation-like definition the following energy-momentum tensor is presented:

\[ T_{\mu\nu} = \Pi_\mu \partial_\nu \psi + \Pi_\mu^* \partial_\nu \psi^* - g_{\mu\nu} \mathcal{L}. \]  

(30)

This can be rewritten in terms of \( R \) and \( \alpha \) as follows:

\[ T_{\mu\nu} = mcR^2 \left( u_\mu u_\nu - \frac{1}{2} g_{\mu\nu} (u_\alpha u^\alpha - 1) \right) + \frac{\hbar^2}{2mc} \left( \partial_\mu R \partial_\nu R - \frac{1}{2} g_{\mu\nu} \partial_\alpha R \partial^\alpha R \right). \]  

(31)

Here we note that the term proportional to \( (u_\alpha u^\alpha - 1) \) is also of quantum nature, in the order of \( \hbar^2 \). The only classical contribution to \( T_{\mu\nu} \) is therefore \( mcR^2 u_\mu u_\nu \), that of the dust consisting of point-particles with mass \( m \) moving on Bohm trajectories according to the velocity field \( u_\mu(x) \).

Replacing back the off-mass-shell relation (28) into this expression leads to:

\[ T_{\mu\nu} = mcR^2 u_\mu u_\nu + \frac{\hbar^2}{2mc} \left( \partial_\mu R \partial_\nu R - g_{\mu\nu} (\partial_\alpha R \partial^\alpha R + R \Box R) \right). \]  

(32)

Here the \( \mathcal{O}(\hbar^2) \) part is the quantum contribution, the rest is classical dust. There are, however, other derivations of the energy-momentum tensor with a formally different result \([54–56]\). In the next subsection we review this.

D. Bohm-Takabayashi Energy-Momentum Tensor

Although the Bohm-Takabayashi energy-momentum tensor \([28, 29]\) was originally derived in the Madelung fluid picture, its validity is independent of the fluid interpretation. To begin with one takes the derivative of the off-mass-shell equation (28) and multiplies it by \( R^2 / 2 \):

\[ \frac{R^2}{2} \partial_\mu \left[ u_\nu u^\nu - 1 - \frac{\hbar^2}{(mc)^2} \frac{\Box R}{R} \right] = 0. \]  

(33)

Introducing now the Compton wavelength \( L_C = \hbar / mc \) and expanding the derivative of \( u_\nu u^\nu \) we obtain

\[ R^2 u^\nu \partial_\mu u_\nu - \frac{1}{2} L_C^2 R^2 \partial_\mu \left( \frac{\Box R}{R} \right) = 0. \]  

(34)

One utilizes also the following identity (the Madelung fluid is irrotational)

\[ \partial_\mu u_\nu = \frac{1}{mc} \partial_\mu \partial_\nu \alpha = \frac{1}{mc} \partial_\nu \partial_\mu \alpha = \partial_\nu u_\mu. \]  

(35)

Therefore

\[ R^2 u^\nu \partial_\mu u_\nu = R^2 u^\nu \partial_\nu u_\mu = \partial_\nu (R^2 u^\nu u_\mu) - u_\mu \partial_\nu (R^2 u^\nu) \]  

(36)

and due to continuity (eq.9) the last term vanishes.

By these manipulations we obtain

\[ \partial_\nu \left( R^2 u^\nu u_\mu \right) = \frac{1}{2} L_C^2 R^2 \partial_\mu \left( \frac{\Box R}{R} \right). \]  

(37)

Further use of the Leibniz rule in this formula leads to

\[ R^2 \partial_\mu \left( \frac{\Box R}{R} \right) = R \Box \partial_\mu R - \partial_\mu R \Box R = \partial^\nu \left( R \partial_\nu \partial_\mu R - \partial_\nu R \partial_\mu R \right). \]  

(38)
This form already reveals a vanishing divergence of the Bohm-Takabayashi tensor

\[ T_{\mu\nu} = mcR^2 u_\mu u_\nu - \frac{\hbar^2}{2mc} \left( R\partial_\mu \partial_\nu R - \partial_\mu R \partial_\nu R \right). \]  \hfill (39)

Obviously this expression differs from the Klein-Gordon one \hfill (32) by

\[ \Delta_{\mu\nu} = T_{\mu\nu} - T_{\mu\nu} = \frac{\hbar^2}{2mc} \left( \partial_\mu R \partial_\nu R + R\partial_\mu \partial_\nu R - g_{\mu\nu} (\partial_\alpha R \partial^\alpha R + R \Box R) \right). \]  \hfill (40)

This difference does not spoil the energy-momentum conservation, because it has a vanishing divergence. We note that

\[ (g_{\mu\nu} \Box - \partial_\mu \partial_\nu) \frac{R^2}{2} = g_{\mu\nu} (\partial_\alpha R \partial^\alpha R + R \Box R) - \partial_\mu R \partial_\nu R - R \partial_\mu \partial_\nu R. \]  \hfill (41)

Using this identity one realizes that the difference between the familiar Klein-Gordon and the Bohm-Takabayashi tensor is,

\[ \Delta_{\mu\nu} = \frac{\hbar^2}{4mc} \left( \partial_\mu \partial_\nu - g_{\mu\nu} \Box \right) R^2, \]  \hfill (42)

has a vanishing divergence \hfill [57]. This difference can also be written as a divergence of a three-index tensor, \[ \Delta_{\mu\nu} = \partial^\alpha f_{\alpha\mu\nu} \] with

\[ f_{\alpha\mu\nu} = \frac{\hbar^2 R}{2mc} \left( g_{\alpha\mu} \partial_\nu R - g_{\mu\nu} \partial_\alpha R \right). \]  \hfill (43)

We note that in general it is allowed to add a term to the energy-momentum tensor with vanishing divergence, such a term does not change the conservation law. Energy, however, has a physical meaning. In fact, the continuous symmetries of the underlying action govern the correct expression. The proper energy-momentum tensor can be obtained by taking into account all continuous symmetries via their infinitesimal generators, according to a procedure described in \hfill [58, 59]. The difference \[ \Delta_{\mu\nu} \] is related to the realization of the conformal symmetry.

The full energy-momentum tensor can be obtained as the proper mixture of the above expressions. The general tensor contains a parameter \( \lambda \) multiplying \[ \Delta_{\mu\nu} \] and added to the Bohm-Takabayashi tensor \hfill (39):

\[ T_{\mu\nu} = mcR^2 u_\mu u_\nu + \frac{\hbar^2}{2mc} \left( R\partial_\mu \partial_\nu R - \partial_\mu R \partial_\nu R \right) + \lambda \Delta_{\mu\nu}. \]  \hfill (44)

The conformal part can be identified by inspecting the trace of the energy-momentum tensor,

\[ \Sigma^\mu = \left( 1 + L^2 \frac{1 - 3\lambda}{4} \right) (mcR^2) \]  \hfill (45)

For \( \lambda = 0 \) one arrives at the original Bohm-Takabayashi tensor. For \( \lambda = 1 \) the original Klein-Gordon case emerges. Finally, for \( \lambda = 1/3 \) only classical dust contributes to the trace and all terms proportional to \( L^2 \propto \hbar^2 \) – the quantum part of the energy-momentum tensor – are altogether traceless. This will be the basis of the classical - quantum splitting of the Einstein equation when considering classical gravity with quantum sources.

In order to prepare this study in the next section, we express the general energy-momentum tensor in scaling variables. Using \( R = e^\sigma / \sqrt{V} \), where \( V \) is constant, one easily gets

\[ \Sigma^\nu = \frac{me^2 \sigma}{V} e^\sigma u_\mu u_\nu + \frac{\hbar^2}{2meV} e^{2\sigma} \Xi^\nu \]  \hfill (46)

with

\[ \Xi^\nu = 2\lambda \partial^\mu \sigma \partial_\mu \sigma + (\lambda - 1) \partial^\mu \partial_\mu \sigma - \lambda \delta^\mu_\nu (2\partial_\mu \sigma \partial^\alpha \sigma + \Box \sigma) \]  \hfill (47)

This expression readily reminds us to a dilaton field \( \sigma(x) \) \hfill [33, 58].
IV. GENERAL RELATIVISTIC QUANTUM MECHANICS

A. Modification of the Einstein equation

In order to proceed by simplifying the above general energy-momentum tensor when coupled to gravity, we propose to consider the following modified Einstein equation

$$G_{\mu\nu} + \Lambda \delta_{\mu\nu} = \frac{8\pi G}{c^3} e^{-2\sigma} T_{\mu\nu}. \quad (48)$$

For $\sigma = 0$ (i.e. $R = 1/\sqrt{V}$, meaning a purely classical source) it coincides with the original Einstein equation (with a possible cosmological term).

For depicting gravity effects we shall use from now on the Schwarzschild length (half Schwarzschild radius), defined by $L_S = \frac{G m}{c^3}$. In this way two length scales appear in the equation describing the gravitational effects of the energy-momentum tensor of the quantum wave function:

$$G_{\mu\nu} + \Lambda \delta_{\mu\nu} = \frac{8\pi L_S}{V} u_{\mu} u_{\nu} + \frac{4\pi L_S^2 L_C^2}{V} \left(2\lambda \delta_{\mu\nu} - (\lambda - 1) \partial_{\mu} \delta_{\nu\nu} - \lambda \delta_{\mu\nu} (2\partial_{\alpha} \sigma \partial^\alpha \sigma + \Box \sigma)\right). \quad (49)$$

Since the structure of the source term is quite complex at this stage, we search for a corresponding representation of the Einstein tensor on the left hand side. A conformal transformation by $\Omega(x) = e^{s(x)}$ changes the metric tensor, the Riemann and Ricci tensor as well as the Einstein tensor, making from (an originally flat) spacetime (with metric $\eta_{\mu\nu}$) a curved one:

$$\bar{g}_{\mu\nu} = e^{2s} \eta_{\mu\nu}. \quad (50)$$

induces an additive change in the Christoffel symbol,

$$\delta \Gamma^\nu_{\mu\alpha} = \partial_{\mu} s \delta^\nu_{\alpha} + \partial_{\alpha} s \delta^\nu_{\mu} - \partial^\nu s \eta_{\mu\alpha}, \quad (51)$$

an additive change in the Ricci tensor

$$\delta R_{\mu\nu} = 2 (\partial_{\mu} s \partial_{\nu} s - \partial_{\nu} s \partial_{\mu} s) - (\Box s + 2 \partial_{\alpha} s \partial^\alpha s) \eta_{\mu\nu} \quad (52)$$

and the additive change

$$\delta \mathcal{R} = -6 e^{-2s} \left(\Box s + \partial_{\mu} s \partial^\mu s\right) \eta_{\mu\nu} \quad (53)$$

in the Ricci scalar. Finally we arrive at the following form of the Einstein tensor

$$G_{\mu\nu} = \bar{G}_{\mu\nu} + (2\Box s + \partial_{\alpha} s \partial^\alpha s + \Lambda) \delta^\mu_{\mu} - 2\partial^\mu \partial_{\nu} s + 2\partial^\mu s \partial_{\nu} s. \quad (54)$$

Noting this we are well prepared for a term-by-term identification of different contributions.

B. Classical and quantum terms in the Einstein equation

As a first step we regard a conformally transformed Einstein tensor. Equation (48) after a suitable conformal transformation becomes:

$$\left[\bar{G}_{\mu\nu} + (2\Box s + \partial_{\alpha} s \partial^\alpha s + \Lambda) \delta^\mu_{\mu} - 2\partial^\mu \partial_{\nu} s + 2\partial^\mu s \partial_{\nu} s\right] = \frac{8\pi G}{c^3} e^{-2\sigma} \bar{T}_{\mu\nu}. \quad (55)$$

We want now to connect $s$ and $\sigma$ in a way to reduce this equation to its possibly simplest form. As a source term we insert our general $\lambda$-tagged energy-momentum tensor as obtained above, cf. equation (46).
The term by term identification suggests that we have to satisfy the following equations separately:

\[ \overline{\mathcal{G}}_{\mu\nu} = \frac{8\pi L S}{V} u^\mu u_\nu, \]
\[ (2\Box s + \partial_\mu s \partial^\mu s + \Lambda) = -\lambda \frac{4\pi L S L_C^2}{V} (2\partial_\mu \sigma \partial^\mu \sigma + \Box \sigma), \]
\[ -2\partial^\mu \partial_\nu s = (\lambda - 1) \frac{4\pi L S L_C^2}{V} \partial^\mu \partial_\nu \sigma, \]
\[ 2\partial^\mu s \partial_\nu s = 2\lambda \frac{4\pi L S L_C^2}{V} \partial^\mu \sigma \partial_\nu \sigma. \]  

(56)

Obviously the choice \( s = \sigma \) simplifies the most! Based on this we arrive at a classical Einstein equation governed by the dust moving on Bohmian trajectories:

\[ \overline{\mathcal{G}}_{\mu\nu} = \frac{8\pi L S}{V} u^\mu u_\nu. \]  

(57)

As a further bonus it is compulsory to choose \( \lambda = 1/3 \) based on the two last lines in eq. (56), i.e. the quantum part is conformal. Moreover the quantization volume is expressed by the two underlying natural length scales,

\[ V = \frac{4\pi}{3} L S L_C. \]  

(58)

is the Planck volume scaled by \( M_P/m \).

Finally we have a cosmological term proportional to the off-mass-shell effect on the Bohm trajectories:

\[ \Lambda = -3 (\Box \sigma + \partial_\mu \sigma \partial^\mu \sigma) = -3 \frac{\Box R}{R}. \]  

(59)

This result delivers a key to a new thinking about the cosmological constant. In this scenario quantum effects, in particular an attractive interaction, may lower the effective \( P_\mu P^\mu \) for a particle below the classical mass shell value, acting this way as an effective cosmological constant (cf. eq. 27):

\[ \Lambda = \frac{3}{\hbar^2} \left( (mc)^2 - P_\mu P^\mu \right). \]  

(60)

Summarizing we have splitted the slightly modified Einstein equation with cosmological term, e.q. (48), in the form

\[ G_{\mu\nu}^{\text{flat}} + \Lambda g_{\mu\nu}^{\text{flat}} = \frac{8\pi G}{c^3} e^{-2\sigma} \left( T_{\mu\nu}^{\text{classical}} + T_{\mu\nu}^{\text{quantum}} \right) \]  

(61)

to a series of equivalent expressions (56), including a classical dust-like source for the deformed Einstein tensor,

\[ G_{\mu\nu}^{\text{curved}} = \frac{8\pi G}{c^3} T_{\mu\nu}^{\text{classical}}, \]  

(62)

and obtained a quantum interpretation for the cosmological term in flat metric \( \Lambda \). In fact we constructed a particular Jordan-Einstein frame change, which is optimally suited to the wave function.

C. An order of magnitude estimate for the cosmological constant

Based on this view we make a new order of magnitude estimate for the value of the cosmological constant. A plane-wave solution of the Klein-Gordon equation leads to zero. Therefore we consider an external vector field \( A^\mu \) in the Klein-Gordon equation and look for bound solutions. Then it is easy to see, that the only modification in the previous train of thought comprises a modification of the four-momentum \( \tilde{P}^\mu = \partial^\mu \alpha - A^\mu \). The form of equations (61) and (62) does not change, the cosmological term is still given by eq. (59). Assuming a particular reference frame and introducing a potential \( A^0(r) = V(r) = -b/r \), the solution of the Schrödinger equation (3) and the Klein-Gordon equation (25)-(27) becomes \( R = Ke^{-r/a} \), with \( a \) being the
corresponding “Bohr radius” [61, 70]. We hence obtain the following cosmological term (cf. eq. (59))

$$\Lambda = 3 \frac{\nabla^2 R}{R} = 3 \left( \frac{1}{a^2} - \frac{2}{ar} \right) = \frac{3}{a^2} + V(r)$$  \hspace{1cm} (63)

with \( a = 6/b \). Here the constant part must belong to the cosmological effect. The quantum energy part is a spatial constant, \( 3/a^2 \) which may be in the correct order of magnitude.

We seek for the proper “Bohr radius” value possibly explaining the observations. Recent data of the Hubble constant and omega \( \Omega_\Lambda = 0.692 \pm 0.010, H_\Lambda = 66.7 \pm 1.1 \text{ km s}^{-1} \text{Mpc}^{-1} \) [15] give \( \Lambda = (1.08 \pm 0.05) \times 10^{-52} \text{m}^{-2} \). From this we obtain [71]

$$a = \sqrt{\frac{3}{\Lambda}} = (1.67 \pm 0.04) \times 10^{36} m = (1.76 \pm 0.04) \times 10^{10} \text{ly}. \hspace{1cm} (64)$$

We note that the corresponding \( C_{\mu\nu}^{\text{curved}} \) displays a de Sitter radius with the same value of ‘\( a \’ \), a little larger the radius of the observed universe to date.

**Summary**

In summary we have explored the classical – quantum splitting of the Schrödinger equation by using the magnitude-phase representation of the complex wave function. By doing so not the Madelung fluid interpretation, but the partial conformal symmetry hidden in the relativistic Klein-Gordon Lagrangian, a simple relativistic generalization behind the Schrödinger Quantum Action, was in focus. Although the mass term breaks conformal invariance, in the limit of zero mass the rest of the theory should restore this. Accordingly the determination of the proper energy-momentum tensor has to take this symmetry into account.

Following the general mathematical recipe [57, 58, 63], we concluded that neither the naive expression - frequently found in textbooks - nor the Bohm-Takabayashi form of \( T_{\mu\nu} \) takes care of this symmetry. Surprisingly (or not), a conformal transformation of the Einstein tensor can be carried out which separates a classical fluid-like contribution of the free particle wave function to the classical gravity from quantum corrections in the energy-momentum tensor, \( T_{\mu\nu} \), by assuming a simple (in the 0 limit vanishing) modification of the Einstein equation.

Moreover this quantum – classical splitting of the source term of the Einstein equation functions only if the Bohm potential part of \( T_{\mu\nu} \) is traceless (\( \Lambda = 1/3 \)). Beyond this a cosmological term arises which was found to be proportional to the off-mass-shell measure of particles moving on Bohmian trajectories (\( \Lambda = -3 \Box R/R \)). As a small bonus the natural reference quantization volume belonging to the normalization of the magnitude of the wave function, \( (R = |\varphi|, \int R^2 d^3x = 2) \), becomes a Planck-scale based quantity \( (V = \frac{4\pi}{3} L_s L_c^2 = \frac{M_{\text{Pl}}}{\hbar} \frac{a^3}{T}) \).

Finally we investigated the wave-function induced cosmological term in case of quantum bound states in a simple, static \(-b/r\) potential and estimated that for the observed value such a Bohr–de Sitter radius (\( a = 6/b \) with \( \Lambda = 3/a^2 \)) may be responsible. Experimental data solidicate an estimate of this radius close to the size of the visible universe.

Recent approaches to quantum geometry recognize the connection to conformal transformation and Weyl geometry from several different points of view. Carroll [64, 65] summarizes different works in this respect. On the other hand Koch reviews and further elaborates some issues regarding reservations against some nonstandard quantum interpretations [66, 67] (see also [68, 69] related specifically to the mentioned work of Wallstrom [45]). Our treatment is based on energetic considerations trying to avoid the traps of interpretational issues.

In the view presented in this paper the scalar field is not an extra entity in the Universe, but it is caused by the total wave function of all bound matter filling it.

**Acknowledgement**

We thank Manfried Faber for detailed discussions. Antal Jakovác, András Patkós and Reinhard Alkofer contributed with inspiring remarks at the ACHT (Austrian-Croatian-Hungarian Triangle) Meeting in Retzhof, June 2013. This work was supported by the Hungarian National Research Fund OTKA (K81161, K104260).

---

[1] W. de Sitter. On Einstein’s theory of gravitation and its astronomical consequences. *Monthly Notices of the Royal Astronomical Society*, 76:699–728, 1916.
[46] P. R. Holland. *The Quantum Theory of Motion*. Cambridge University Press, Cambridge, 1993.

[47] P. Ván and T. Fülöp. Weakly nonlocal fluid mechanics - the Schrödinger equation. *Proceedings of the Royal Society, London A*, 462(2066):541–557, 2006, quant-ph/0304062.

[48] Ván P. Weakly nonlocal non-equilibrium thermodynamics - variational principles and Second Law. In Ewald Quak and Tarmo Soomere, editors, *Applied Wave Mathematics (Selected Topics in Solids, Fluids, and Mathematical Methods)*, chapter III, pages 153–186. Springer-Verlag, Berlin-Heidelberg, 2009, arXiv:0902.3261.

[49] T. Fülöp and S. D. Katz. A frame and gauge free formulation of quantum mechanics. 1998, quant-ph/9806067.

[50] B. Bistrovic, R. Jackiw, H. Li, V. P. Nair, and S.-Y. Pi. Non-Abelian fluid dynamics in Lagrangian formulation. *Physical Review D*, (67):025013(11), 2003, hep-th/0210143.

[51] R. Jackiw, V. P. Nair, S-Y. Pi, and A. P. Polychronakos. Perfect fluid theory and its extensions. *Journal of Physics A*, 37:R327–R432, 2004, arXiv:hep-ph/0407101.

[52] E. Schrödinger. Über eine bemerkenswerte Eigenschaft der Quantenbahnen eines einzelnen Elektrons. *Zeitschrift für Physik*, 12(1):13–23, 1923.

[53] C. Itzykson and J.B. Zuber. *Quantum field theory*. McGraw-Hill, New York, etc., 1980.

[54] D.H. Delphenic. A geometric origin for the Madelung potential. arXiv:gr-qc/0211065.

[55] D.H. Delphenic. A strain tensor that couples to the Madelung stress tensor. arXiv:1303.3582.

[56] R. Carroll. Remarks on geometry and the quantum potential. arXiv:math-ph/0701007.

[57] C.G. Callan, S. Coleman, and R. Jackiw. A new improved energy-momentum tensor. *Annals of Physics*, 59:42–73, 1970.

[58] M. Forger and H. Römer. Currents and the energy-momentum tensor in classical field theory. *Annals of Physics*, 309:306–389, 2004.

[59] J. M. Pons. Noether symmetries, energy-momentum tensors, and conformal invariance in classical field theory. *Journal of Mathematical Physics*, 52:012904, 2011.

[60] R. M. Wald. *General Relativity*. The University of Chicago Press, Chicago and London, 1984.

[61] A. Galindo and P. Pascual. *Quantum Mechanics I*. Springer Verlag, Berlin, etc., 1990.

[62] J. Beringer, et al., and (Particle Data Group). The review of particle physics. *Physical Review D*, 86(1):010001, 2012.

[63] M. J. Gotay and J. E. Marsden. Stress-energy-momentum tensors and the Belinfante-Rosenfeld formula. *Contemporary Mathematics*, 132:367–392, 1992.

[64] R. Carroll. *On the emergence theme of physics*. World Scientific, 2010.

[65] R. Carroll. Remarks on gravity and quantum geometry. arXiv:1007.4744 [math.ph].

[66] B. Koch. Quantizing geometry or geometrizing the quantum? In *QUANTUM THEORY: Reconsideration of Foundations* - 5., volume 1232 of *AIP Conference Proceedings*, pages 313–320. American Institute of Physics, 2010, arXiv:1004.2879v2 [hep-th].

[67] B. Koch. A geometrical dual to relativistic Bohmian mechanics - the multi particle case. arXiv:0901.4106.

[68] L. Smolin. Could quantum mechanics be an approximation to another theory? arXiv:quant-ph/0609109.

[69] I. Schmelzer. An answer to the Walstrom objection against Nelsonian statistics. arXiv:1101.5774v2 [quant-ph].

[70] Dong, Shi-Hai, *Wave Equations in Higher Dimensions*. Springer, 2011.

[71] Assuming that the previous potential is of gravitational origin results in $m/2 = 16.2 \pm 0.2$ kg value of reduced mass.