THE SPECTRUM OF THE LERAY TRANSFORM FOR CONVEX REINHARDT DOMAINS IN $\mathbb{C}^2$

DAVID E. BARRETT AND LOREDANA LANZANI

Abstract. The Leray transform and related boundary operators are studied for a class of convex Reinhardt domains in $\mathbb{C}^2$. Our class is self-dual; it contains some domains with less than $C^2$-smooth boundary and also some domains with smooth boundary and degenerate Levi form. $L^2$-regularity is proved, and essential spectra are computed with respect to a family of boundary measures which includes surface measure. A duality principle is established providing explicit unitary equivalence between operators on domains in our class and operators on the corresponding polar domains. Many of these results are new even for the classical case of smoothly bounded strongly convex Reinhardt domains.

1. Introduction

The Leray transform $\mathcal{L}$ is a higher-dimensional analog of the classical Cauchy transform for planar domains. It belongs to a family of operators, the Cauchy-Fantappié transforms, projecting functions on the boundary onto the space of holomorphic boundary values. These operators play an essential role in higher-dimensional function theory, just as the original Cauchy transform does in the one-dimensional setting. (See for instance Kerzman and Stein [KST] and the monographs [HeLe], [Kra] and [Ran].)

Though the Cauchy-Fantappié construction is not canonical in general, the Leray transform is distinguished by the simple explicit construction of the corresponding kernel function and by the presence of a good transformation law under linear fractional transformations ([Bol2], Thm. 3). The construction of the Leray transform requires that the domain under study satisfy the geometric condition of “$\mathcal{C}$-linear convexity.”

Date: May 15, 2009.

2000 Mathematics Subject Classification. 32A26.

* Supported in part by the AWM and the NSF (grant No. DMS-0700815.)
In this paper we provide rather detailed information about the Leray transform on certain convex Reinhardt domains in $\mathbb{C}^2$. In particular, we learn that

(A) $L$ is $L^2$-bounded on some, but not all, smoothly bounded weakly convex domains;

(B) $L$ is $L^2$-bounded on some, but not all, strongly convex domains whose boundaries are less than $C^2$-smooth;

(C) it is important to give thought to the choice of boundary measure – in particular, measures involving (suitably-chosen) powers of the Levi form work as well as (or better than) surface measure;

(D) there is a duality rule relating the qualitative and quantitative behavior of $L$ on a domain $D$ to the corresponding behavior on the polar domain $D^*$ (defined in (7.1)). This provides a surprising linkage between the previous topics (A) and (B).

The Reinhardt designation means that $D$ is invariant under all rotations of the form
\begin{equation}
(z_1, z_2) \mapsto (e^{i\theta_1}z_1, e^{i\theta_2}z_2).
\end{equation}

Reinhardt domains occur naturally in various contexts in several complex variables (for instance, the domains of convergence of power series of holomorphic functions are Reinhardt domains) and are often a source of meaningful examples which serve as models for more general theories. One class of domains singled out in our work is the class $\tilde{R}$ consisting of bounded convex complete $C^1$-smooth Reinhardt domains in $\mathbb{C}^2$ that are $C^2$-smooth and strongly convex away from the axes $\{\zeta_1\zeta_2 = 0\}$. (See Proposition 7 for an alternate description of $\tilde{R}$.)

The class $\tilde{R}$ contains the subclass $\mathcal{P}$ consisting of weighted $L^p$-balls; that is,
\begin{equation}
\mathcal{P} = \{D_{p,a_1,a_2} : a_1 > 0, a_2 > 0, 1 < p < \infty\},
\end{equation}
where we have let
\begin{equation}
D_{p,a_1,a_2} = \{(z_1, z_2) \in \mathbb{C}^2 : a_1|z_1|^p + a_2|z_2|^p < 1\}.
\end{equation}

Finally, we let $\mathcal{R}$ denote the class of domains in $\tilde{R}$ that are well-modeled by a domain $D_{p_j,a_{1,j},a_{2,j}} \in \mathcal{P}$ near boundary points on each of the axes $\zeta_j = 0$, $j = 1, 2$. (See Definition 16 for the formal description.)

We have $\mathcal{P} \subsetneq \mathcal{R} \subsetneq \tilde{R}$.

The smoothness of a domain in $\mathcal{R}$ is determined by the size of the exponents $p_1, p_2$. On the one hand, if $1 < p_1 < 2$, an $\mathcal{R}$-domain will be strongly geometrically convex (in the sense of [Pol]) and $C^{1,p_1-1}$-smooth
near \( \{ \zeta_1 = 0 \} \); on the other hand, for \( p_1 \geq 2 \) the domain will be at least \( C^2 \)-smooth near \( \{ \zeta_1 = 0 \} \), but strong geometric convexity and strong Levi pseudoconvexity will fail if \( p_1 > 2 \). The size of \( p_2 \) similarly determines the qualitative behavior of the domain near \( \{ \zeta_2 = 0 \} \).

We will show in Proposition 8 below that for \( D \in \tilde{R} \) and \( \zeta \in bD \setminus \{ \zeta_1 \zeta_2 = 0 \} \) there is a unique \( D_{p(\zeta), a_1(\zeta), a_2(\zeta)} \in \mathcal{P} \) osculating \( D \) at \( \zeta \) in the sense that all data up through second order will match the re. If \( D \in R \), then setting \( p(\zeta) = p_1 \) when \( \zeta_1 = 0 \) and \( p(\zeta) = p_2 \) when \( \zeta_2 = 0 \) we get a continuous function \( p(\zeta) \) defined on all of \( bD \) (see Proposition 17).

For a \( C^2 \)-smooth convex domain \( D \) in \( \mathbb{C}^2 \) the Leray integral \( \mathbb{L} = \mathbb{L}_D \) is defined by letting

\[
\mathbb{L}f(w) = \int_{\zeta \in bD} f(\zeta) L(\zeta, w)
\]

for \( w \in D \), where

\[
L(\zeta, w) = \frac{1}{(2\pi i)^2} \frac{j^*(\partial \rho \wedge \overline{\partial} \rho)(\zeta)}{[\partial \rho(\zeta) \cdot (\zeta - w)]^2}
\]

is the Leray kernel defined for \( \zeta \in bD, w \in D \); here \( \rho \) is a defining function for \( bD \), \( j^* \) denotes the pullback of the inclusion \( j : bD \to \mathbb{C}^2 \) acting on three-forms, and \( \partial \rho(\zeta) \cdot (\zeta - w) \) denotes the action of the linear functional \( \partial \rho(\zeta) \) on the vector \( \zeta - w \), namely

\[
\partial \rho(\zeta) \cdot (\zeta - w) = \frac{\partial \rho}{\partial \zeta_1}(\zeta)(\zeta_1 - w_1) + \frac{\partial \rho}{\partial \zeta_2}(\zeta)(\zeta_2 - w_2).
\]

It follows from the convexity of \( D \) that \( \partial \rho(\zeta) \) is a so-called “generating form” for \( D \); if \( bD \) contains no line segments we have in particular that the expression in (1.6) is non-zero for each \( \zeta \in bD \) and for each \( w \in D \setminus \{ \zeta \} \) (see [Ran], §IV.3.1 and §IV.3.2).

The kernel \( L(\zeta, w) \) is independent of the choice of defining function \( \rho \) (see [Ran], §IV.3.2, also [Ler], [Nor], [Aiz]).

The function \( \mathbb{L}f \) will be holomorphic in \( D \) when the integral (1.4) converges, and \( \mathbb{L} \) reproduces a holomorphic function from its boundary values.

We should mention that the Leray integral is defined more generally for \( \mathbb{C} \)-linearly convex domains, that is, for domains whose complement is a union of complex hyperplanes. (These are also known as “linearly convex” domains.) But \( \mathbb{C} \)-linearly convex complete Reinhardt domains are automatically convex (see Example 2.2.4 in [APS]) so in the current work we focus only on convex Reinhardt domains.
When $D$ satisfies additional hypotheses (e.g. strong convexity) then $L$ extends to a singular integral operator on the boundary, also denoted by $L$ (see [KS1], page 207, and [LS]).

For domains $D \in \tilde{R}$ the theory outlined above does not apply directly, but we will show in particular that the reproducing property for holomorphic functions is still valid (see Corollary 24 and Proposition 32).

In order to consider bounds and adjoints for $L$ we will need to introduce measures on $bD$; specifically, we will consider measures $\mu$ that are invariant under the rotations (1.1) and are absolutely continuous with respect to surface measure. We will take particular interest in boundary measures that are continuous positive multiples of $|L(\zeta)|^{1-\eta}d\sigma(\zeta)$, where $q$ is a fixed real exponent, $d\sigma$ is surface measure and $|L|$ is the Euclidean norm

\begin{equation}
|L| = -\frac{j^*(\partial \rho \wedge \overline{\partial} \rho)}{|\nabla \rho|^2 d\sigma}
\end{equation}

of the Levi-form. (Here we interpret the three-form $j^*(\partial \rho \wedge \overline{\partial} \rho)$ as a measure on $bD$.) We will say that such a measure has order $q$ (see Definition 43 below).

We are ready now to state our main results.

**Theorem 1.** Suppose $D \in R$ and $\mu$ is a rotation-invariant boundary measure of order $q$ with $q$ satisfying the condition

\begin{equation}
|q| < \min_{j=1,2} \left| \frac{p_j}{p_j - 2} \right| = \min_{j=1,2} \left| \frac{1}{p_j - 1} - \frac{1}{p_j^*} \right|^{-1}.
\end{equation}

(Here $p_1$ and $p_2$ are as in the description of $R$ above, and $p_j^*$ denotes the conjugate exponent to $p_j$ — thus $1/p_j + 1/p_j^* = 1$.)

Then the Leray transform $L$ is bounded on $L^2(bD, \mu)$.

Moreover, the operator $L_{\mu}^* L$ admits an orthogonal basis of eigenfunctions, and the essential spectrum of $L_{\mu}^* L$ is equal to

\begin{equation}
\{0\} \cup \left\{ \frac{\sqrt{p(\zeta) p^*(\zeta)}}{2} : \zeta \in bD \right\} \cup \left\{ \lambda_{p_j, q, n} : j = 1, 2, n \geq 0 \right\}.
\end{equation}
here, $L^* \mu$ is the adjoint of $L$ in $L^2(bD, \mu)$, $p(\zeta)$ is the function discussed above (see Proposition 3 and Proposition 17), $p^*(\zeta)$ denotes the conjugate exponent to $p(\zeta)$ (thus $1/p(\zeta) + 1/p^*(\zeta) = 1$), and

\begin{equation}
\lambda_{p,q,n} = \frac{\Gamma \left( \frac{2n}{p} + 1 + q \left( \frac{1}{p} - \frac{1}{p^*} \right) \right) \Gamma \left( \frac{2n}{p^*} + 1 + q \left( \frac{1}{p^*} - \frac{1}{p} \right) \right) \Gamma^2(n + 1) \left( \frac{2}{p} \right)^{\frac{4n+1+q}{p} + \frac{5}{p^*} + 1 + q \left( \frac{1}{p} - \frac{1}{p^*} \right)} \left( \frac{2}{p^*} \right)^{\frac{4n+1+q}{p^*} + \frac{5}{p} + 1 + q \left( \frac{1}{p^*} - \frac{1}{p} \right)} \right) \right)
\end{equation}

(For the definition and basic properties of the essential norm and the essential spectrum, see Propositions 36 and 37 and adjacent material). Note that the interval $|q| \leq 1$ is always included in (1.8). In Corollary 18 below we will show that if $D \in \mathcal{R}$ is a smooth domain then $p_1 = p_2 = 2$ so that (1.8) holds for all values $q \in \mathbb{R}$. On the other hand, if at least one of the $p_j$ is different from 2 then (1.8) defines a proper subinterval of the real line.

Theorem 1 may be compared with previous work by Bonami and Lohoué [BL] and Hansson [Han] (which we specialize here to complex dimension $n = 2$), as follows. Given $1 < p_j < +\infty$ set

$$D = \{ (\zeta_1, \zeta_2) \in \mathbb{C}^2 : |\zeta_1|^{p_1} + |\zeta_2|^{p_2} < 1 \}.$$ 

Note that $D$ belongs to the class $\mathcal{R}$. Bonami and Lohoué study Cauchy-Fantappié transforms and related operators for $D$ as above when $p_j > 2$, $j = 1, 2$ and $\mu$ is a measure of order $q = 1$. Hansson proves that for $D$ as above the operator $L$ is bounded on $L^2(bD, \mu)$ when $p_j > 2$ are positive integers and $\mu$ is a measure of order $q = 0$. In either case $bD$ is $C^k$-smooth ($k \geq 2$) and weakly pseudoconvex (its Levi form is singular at boundary points that lie along the axes \{ $\zeta_1 \zeta_2 = 0$ \}). When $D \in \mathcal{R}$ is as above but $p_j < 2$ it follows that $D$ is strongly convex but non-smooth and the construction of the Cauchy-Fantappié kernels investigated by Bonami and Lohoué becomes problematic (see comments below after Corollary 5), whereas the Leray transform $L$ is still well defined and by Theorem 1 it is bounded in $L^2(bD, \mu)$ for all measures $\mu$ of order $q$ with $q$ ranging in the interval (1.8). In fact, more is true: in [7] we present a duality result providing an explicit unitary equivalence of the Leray transform for a domain $D \in \widetilde{\mathcal{R}}$ (resp. $D \in \mathcal{R}$) and the Leray transform for its polar domain $D^* \in \widetilde{\mathcal{R}}$ (resp. $D^* \in \mathcal{R}$). On the one hand, we see that the polar of a smooth, weakly pseudoconvex domain may be non-smooth and strongly convex; for example, the polar of the domain $D \in \mathcal{R}$ given above with $p_j > 2$ is

$$D^* = \{ (\zeta_1, \zeta_2) : |\zeta_1|^{p_1^*} + |\zeta_2|^{p_2^*} < 1 \} \in \mathcal{R}$$
where \( p_j^* < 2 \) is the conjugate exponent of \( p_j \); see Theorem 17 for the precise statement in the general case. On the other hand, combining this duality with (1.9) and (1.10) in Theorem 1 we see that, modulo a switch of measure (from \( \mu \) of order \( q \) to \( \tilde{\mu} \) of order \(-q\)), from the point of view of the spectral theory of the Leray transform any domain \( D \) in the class \( \mathcal{R} \) is qualitatively and quantitatively indistinguishable from its polar domain \( D^* \).

Lanzani and Stein show in [LS] that \( \mathbb{L} \) is \( L^2 \)-bounded with respect to surface measure when \( D \) is a bounded strongly (\( \mathcal{C} \)-linearly) convex domain in \( \mathbb{C}^n \) with \( C^{1,1} \)-smooth boundary. The examples discussed above show that neither strong convexity nor \( C^{1,1} \)-smoothness of the boundary is a necessary condition for \( L^2 \)-boundedness of \( \mathbb{L} \). On the other hand, in §6 we present examples showing that if we try to settle for weak convexity or \( C^{1,\alpha} \)-smoothness of the boundary with no further conditions then \( \mathbb{L} \) may fail to be \( L^2 \)-bounded with respect to any reasonable boundary measure.

Following Kerzman and Stein ([KS1], [KS2]) we will use the notation \( \mathbb{A}_\mu \) for the anti-self-adjoint operator \( \mathbb{L}^* - \mathbb{L} \).

**Theorem 2.** In the setting of Theorem 1, the operator \( \mathbb{A}_\mu \) admits an orthogonal basis of eigenfunctions, and the essential spectrum of \( \mathbb{A}_\mu \) is equal to

\[
\{0\} \cup \left\{ \pm i \sqrt{p(\zeta)p^*(\zeta)} - 1 : \zeta \in \partial D \right\} \\
\cup \left\{ \pm i \sqrt{\lambda_{p_j,q,n} - 1} : j = 1, 2, n \geq 0 \right\}.
\]

As mentioned above, Corollary 18 below will show that if \( D \in \mathcal{R} \) is a smooth domain, then \( p_1 = p_2 = 2 \) so that for all \( q \in \mathbb{R} \) we have \( \lambda_{p_j,q,n} = 1, j = 1, 2 \); thus the choice of \( q \) is no longer relevant in the description of our class of measures and we obtain the following results.

**Theorem 3.** Suppose \( D \subset \mathbb{C}^2 \) is a \( C^2 \)-smooth, strongly convex Reinhardt domain and let \( \mu \) be any rotation-invariant continuous positive multiple of surface measure.

Then \( \mathbb{L} \) is bounded on \( L^2(bD, \mu) \).

Moreover, the operator \( \mathbb{L}^* \mathbb{L} \) admits an orthogonal basis of eigenfunctions, and the essential spectrum of \( \mathbb{L}_\mu \mathbb{L} \) is equal to

\[
\{0\} \cup \left\{ \sqrt{\frac{p(\zeta)p^*(\zeta)}{2}} : \zeta \in \partial D \right\}
\]
or, equivalently,
\[ \{0\} \cup \left\{ \frac{p(\zeta)}{2\sqrt{p(\zeta)} - 1} : \zeta \in bD \right\}. \]

The essential norm of \( L \) is
\[ \max \left\{ \sqrt[4]{\frac{p(\zeta)p^*(\zeta)}{4}} : \zeta \in bD \right\}. \]

Theorem 4. In the setting of Theorem 3, the operator \( A_{\mu} \) admits an orthogonal basis of eigenfunctions, and the essential spectrum of \( A_{\mu} \) is equal to
\[ \{0\} \cup \left\{ \pm i \sqrt[4]{\frac{p(\zeta)p^*(\zeta)}{2}} - 1 : \zeta \in bD \right\} \]
or, equivalently,
\[ \{0\} \cup \left\{ \pm i \sqrt[4]{\frac{p(\zeta)}{2\sqrt{p(\zeta)} - 1}} - 1 : \zeta \in bD \right\}. \]

The essential norm of \( A_{\mu} \) is
\[ \max \left\{ \sqrt[4]{\frac{p(\zeta)p^*(\zeta)}{2}} - 1 : \zeta \in bD \right\}. \]

Combining Theorem 4 with Proposition 15 below we obtain the following.

Corollary 5. In the setting of Theorem 4, the operator \( A_{\mu} \) will be compact in \( L^2(bD, \mu) \) if and only if \( D \) is a domain of the form
\[ \{(z_1, z_2) : a_1|z_1|^2 + a_2|z_2|^2 < 1\}. \]

(See the end of §3 for related results.)

The results outlined above should be contrasted with work of Kerzman and Stein [KS1] on a related operator \( \mathbb{H} \) of Cauchy-Fantappié type due to Henkin [Hen] and Ramírez [Ram]. This operator is based on the (quadratic) Levi polynomial rather than the linear functions of \( w \) appearing in (1.6); it may be defined on any strongly pseudoconvex domain with \( C^\infty \)-smooth boundary. Kerzman and Stein show that the operator \( \mathbb{H} \) is a compact perturbation of the Szegő projection defined with respect to surface measure \( \sigma \) (see the end of §3); it follows that \( \mathbb{H}^* \mathbb{H} \) has essential spectrum \( \{0, 1\} \), and \( \mathbb{H}^* - \mathbb{H} \) has essential spectrum
{0}. Thus $H$ provides more direct access to the Szegő projection, while $L$ has a more informative spectral theory.

The plan of the paper proceeds as follows.

In §2 we provide more details about the classes of domains under study, relating properties of a Reinhardt domain $D$ to the geometry of the curve $\gamma_+ = bD \cap \mathbb{R}^2_+$. We prove the osculation results mentioned above; the corresponding exponent $p$ defines a continuous map from $\gamma_+$ to the interval $(1, \infty)$. We also introduce a special parameter $s$ on $\gamma_+$ which plays an important role throughout the rest of the paper, and we characterize the classes $\tilde{\mathcal{R}}$, $\mathcal{R}$ and $\mathcal{P}$ in terms of $p$ as a function of $s$.

In §3 we present the basic theory of the Leray transform for domains in the class $\tilde{\mathcal{R}}$, confirming in particular that the reproducing property for holomorphic functions still holds even when the domains are less than $C^2$-smooth. We introduce a special class of measures on $bD$, the admissible measures; in essence, a rotation-invariant measure $\mu$ on $bD$ is admissible if and only if $\mu$ is finite and $L$ maps $L^2(bD, \mu)$ to holomorphic functions on $D$. We also discuss norms of the Fourier pieces of $L^*\mu$ and $A\mu$ and explain their relation to properties of the overall operators.

§4 contains more information about boundary measures and geometry, confirming in particular that for $D \in \mathcal{R}$ a measure of order $q$ is admissible if and only if condition (1.8) holds.

In §5 we perform some asymptotic analysis of the norms of the Fourier pieces and use these results to prove Theorems 1 and 2.

§6 contains examples of domains for which the $L^2$-boundedness of the Leray transform fails (with respect to any admissible measure, in particular surface measure) due to lack of boundary regularity or lack of strong convexity away from the axes. It also contains an example of a domain in $\tilde{\mathcal{R}} \setminus \mathcal{R}$ with the property that surface measure is not admissible but measures of order $q$ are admissible when $|q| < 1$. In this case, $L$ is not bounded on $L^2(bD, \mu)$ for any rotation-invariant measure $\mu$.

In §7 we present the duality results mentioned earlier, and §8 contains a few concluding remarks.

Acknowledgment. We are grateful to M. Lacey for helpful discussions, and to and E. M. Stein for raising the questions that have led to Examples 1 and 2 in §6.
2. Geometric considerations

Let $D \subset \mathbb{C}^2$ be a Reinhardt domain. Set

\begin{align}
(2.1) \quad \gamma &= \gamma_D = bD \cap \mathbb{R}^2_{\geq 0} = bD \cap ([0, \infty) \times [0, \infty)) ; \\
(2.2) \quad \gamma_+ &= bD \cap \mathbb{R}^2_+ = (bD \setminus \{\zeta_1\zeta_2 = 0\}) \cap \mathbb{R}^2_{\geq 0}.
\end{align}

(Here we are viewing $\mathbb{R}^2$ as a submanifold of $\mathbb{C}^2$.)

Proposition 6. In this situation, if $D$ has $C^k$-smooth boundary ($k \geq 1$) then the following will hold.

\begin{align}
(2.3a) \quad i\mathbb{R}^2 &\subset T_\zeta bD \text{ for each } \zeta \in \gamma. \\
(2.3b) \quad bD &\text{ meets } \mathbb{R}^2 \text{ transversally.} \\
(2.3c) \quad \gamma &\text{ is a } C^k \text{-smooth 1-manifold.} \\
(2.3d) \quad \text{If } \zeta \in \gamma \text{ and } \zeta_1 = 0 \text{ then } T_\zeta bD = \mathbb{C} \times i\mathbb{R} \text{ and } T_\zeta \gamma = \mathbb{R} \times \{0\}. \\
(2.3e) \quad \text{If } \zeta \in \gamma \text{ and } \zeta_2 = 0 \text{ then } T_\zeta bD = i\mathbb{R} \times \mathbb{C} \text{ and } T_\zeta \gamma = \{0\} \times \mathbb{R}.
\end{align}

Proof. If $\zeta_1 \zeta_2 \neq 0$ then (2.3a) follows from the fact that

$$\{(e^{i\theta_1} \zeta_1, e^{i\theta_2} \zeta_2) : \theta_1, \theta_2 \in \mathbb{R}\} \subset bD.$$ 

The continuous dependence of $T_\zeta bD$ on $\zeta$ now forces (2.3a) to hold also when $\zeta$ lies on one of the axes.

It follows now that $T_\zeta bD + \mathbb{R}^2 = \mathbb{C}^2$ for all $\zeta \in \gamma$ which shows that (2.3b) holds, and the transverse intersection theorem now implies (2.3c).

Item (2.3d) follows from (2.3a) and the invariance of $T_\zeta bD$ under rotations in the $\zeta_1$ variable. The proof of (2.3e) is similar. \(\square\)

As in the introduction, we let $\tilde{\mathcal{R}}$ denote the space of bounded convex complete $C^1$-smooth Reinhardt domains in $\mathbb{C}^2$ that are $C^2$-smooth and strongly convex away from the axes $\{\zeta_1\zeta_2 = 0\}$. Then $\gamma$ will be a $C^1$-smooth curve meeting both axes, while $\gamma_+$ will be $C^2$-smooth with non-vanishing curvature. It follows easily that $\gamma$ will be the graph of a concave function, and in fact we easily verify the following.

Proposition 7. A Reinhardt domain $D$ belongs to $\tilde{\mathcal{R}}$ if and only if it may be described as

\begin{equation}
D = \{(z_1, z_2) : |z_2| < \phi(|z_1|), \ |z_1| < b_1\}
\end{equation}
where \( b_1 > 0 \) and \( \phi \) is a continuous function on \([0, b_1]\) satisfying

\[
\begin{align*}
(2.5a) & \quad \phi > 0 \text{ on } [0, b_1]; \\
(2.5b) & \quad \phi(b_1) = 0; \\
(2.5c) & \quad \phi' \text{ is continuous on } [0, b_1) \text{ and negative on } (0, b_1); \\
(2.5d) & \quad \phi'(0) = 0; \\
(2.5e) & \quad \phi'(t) \to -\infty \text{ as } t \to b_1; \\
(2.5f) & \quad \phi'' \text{ is continuous and negative on } (0, b_1).
\end{align*}
\]

Let \( R \) be the map \( bD \to \gamma, (\zeta_1, \zeta_2) \mapsto (|\zeta_1|, |\zeta_2|) \). Then any function \( f \) on \( \gamma \) induces a rotation-invariant function \( f \circ R \) on \( bD \), and every rotation-invariant function \( f \) on \( bD \) may be recovered from its values on \( \gamma \) by the formula

\[
(2.6) \quad f = f \circ R.
\]

We will use \((r_1, r_2)\) as coordinates on \( \mathbb{R}^2_{\geq 0} \); thus

\[
\gamma = \{(r_1, r_2) : 0 \leq r_1 \leq b_1, r_2 = \phi(r_1)\}.
\]

Extending these functions via (2.6) we also have \( r_j = |z_j| \) on \( bD \).

Away from the axes, domains in \( \tilde{\mathcal{R}} \) are modeled after the \( \mathcal{P} \)-domains described in (1.2) and (1.3) in the following sense.

**Proposition 8.** Suppose \( D \in \tilde{\mathcal{R}} \). Then, for every \( \zeta \in bD \) with \( \zeta_1 \zeta_2 \neq 0 \) there is a unique \( D_{p(\zeta), a_1(\zeta), a_2(\zeta)} \in \mathcal{P} \) osculating \( bD \) to second order at \( \zeta \).

**Proof.** We start by considering points \( \zeta = (r_1, r_2) \in \gamma_+ \). Noting that the curve \( bD_{p, a_1, a_2} \cap \mathbb{R}^2_+ \) is given by \( r_2 = \sqrt{\frac{1-a_1 r_1^p}{a_2}} \), we see that we need to determine \( p = p(\zeta) \), \( a_1 = a_1(\zeta) \) \( a_2 = a_2(\zeta) \) so that

\[
\phi(r_1) = \sqrt{\frac{1-a_1 r_1^p}{a_2}}
\]

\[
\phi'(r_1) = \frac{d}{dr_1} \sqrt{\frac{1-a_1 r_1^p}{a_2}} = -\frac{a_1 r_1^{p-1} \left( \frac{1-a_1 r_1^p}{a_2} \right) \frac{1}{2}}{a_2}
\]

and

\[
\phi''(r_1) = \frac{d^2}{dr_1^2} \sqrt{\frac{1-a_1 r_1^p}{a_2}} = \frac{a_1(1-p)r_1^{p-2} \left( \frac{1-a_1 r_1^p}{a_2} \right) \frac{1}{p}}{(1-a_1 r_1^p)^2}.
\]
Substituting \(1 - a_1 r_1^p = a_2 \phi^p(r_1)\) throughout the second and third equations and solving for \(a_1, a_2\) we obtain
\[
a_1(\zeta) = \frac{(1 - p) \phi'(r_1)^2}{r_1^p \phi(r_1) \phi''(r_1)} \quad (2.7)
\]
\[
a_2(\zeta) = \frac{(p - 1) \phi'(r_1)}{r_1 \phi^p(r_1) \phi''(r_1)} \quad (2.8)
\]
Plugging these values back into the first equation and solving for \(p\) we obtain
\[
p(\zeta) = 1 + \frac{r_1 \phi(r_1) \phi''(r_1)}{\phi'(r_1)(\phi(r_1) - r_1 \phi'(r_1))}. \quad (2.9)
\]
Using (2.5) it is easy to check that \(p(\zeta) > 1\) and that \(a_1(\zeta)\) and \(a_2(\zeta)\) are positive.

We finish by extending \(p, a_1\) and \(a_2\) to functions on \(bD \setminus \{\zeta_1 \zeta_2 = 0\}\) by setting \(p = p \circ R, a_1 = a_1 \circ R\) and \(a_2 = a_2 \circ R\) as in (2.6); rotation-invariance guarantees that the extended functions do what is required.

Let \(D\) be an \(\tilde{R}\)-domain. Much of what we do below is made simpler by the introduction of the following auxiliary parameter on \(\gamma_+\):
\[
s = \frac{-r_1 \phi'(r_1)}{\phi(r_1) - r_1 \phi'(r_1)} = \frac{-r_1^{-1} dr_2}{r_1^{-1} dr_1 - r_2^{-1} dr_2} = 1 - \frac{r_1^{-1} dr_1}{r_1^{-1} dr_1 - r_2^{-1} dr_2}. \quad (2.10)
\]
We note for later use that
\[
\frac{dr_2}{r_2} = -\frac{s}{1-s} \frac{dr_1}{r_1} \quad (2.11)
\]
and
\[
\frac{dr_2}{dr_1} = -\frac{s}{1-s} \frac{r_2}{r_1} \quad (2.12)
\]
Our assumptions (2.5) on \(\phi\) yield
\[
s > 0 \text{ on } \gamma_+ \quad (2.13a)
\]
\[
\lim_{\zeta \to (0, b_2)} s(\zeta) = 0 \quad (2.13b)
\]
\[
\lim_{\zeta \to (b_1, 0)} s(\zeta) = 1 \quad (2.13c)
\]
moreover, differentiating (2.10) with respect to \(r_1\) and using (2.9) we obtain
\[
\frac{ds}{dr_1} = \frac{sp}{r_1}. \quad (2.14)
\]
Thus $s$ is $C^1$-smooth on $\gamma_+$ and extends to a monotone continuous function (hence a homeomorphism) mapping $\gamma$ onto the interval $[0, 1]$.

Applying (2.10) to (2.14) we obtain the companion formula

$$\frac{d(1-s)}{dr_2} = \frac{(1-s)p}{r_2}.$$  

(2.15)

The functions $s$ and $p$ determine the coordinate functions $r_1, r_2$ (up to multiplicative constants) as follows:

$$r_1(\zeta) = b_1 \exp \left(- \int_{(b_1,0)}^{(1-s)} \frac{ds}{sp} \right)$$

$$r_2(\zeta) = b_2 \exp \left( \int_{0}^{s} \frac{d(1-s)}{(1-s)p} \right).$$

(The integrals are taken over arcs of $\gamma$.)

Let

$$\dot{p} = p \circ s^{-1} : (0, 1) \to (1, \infty)$$

i.e., $\dot{p}$ gives $p$ as a function of $s$. Then we have

$$r_1 = b_1 \exp \left(- \int_{s}^{1} \frac{dt}{t \dot{p}(t)} \right)$$

(2.17a)

$$r_2 = b_2 \exp \left(- \int_{0}^{s} \frac{dt}{(1-t) \dot{p}(t)} \right)$$

(2.17b)

on $\gamma_+$ and so

$$\gamma_+ = \left\{ \left( b_1 \exp \left(- \int_{s}^{1} \frac{dt}{t \dot{p}(t)} \right), b_2 \exp \left(- \int_{0}^{s} \frac{dt}{(1-t) \dot{p}(t)} \right) \right) : 0 < s < 1 \right\};$$

thus also

$$bD \setminus \{\zeta_1 \zeta_2 = 0\}$$

$$= \left\{ \left( b_1 \exp \left(- \int_{s}^{1} \frac{dt}{t \dot{p}(t)} \right) e^{i\theta_1}, b_2 \exp \left(- \int_{0}^{s} \frac{dt}{(1-t) \dot{p}(t)} \right) e^{i\theta_2} \right) : 0 < s < 1, \theta_1 \in [0, 2\pi), \theta_2 \in [0, 2\pi) \right\}.$$
THE SPECTRUM OF THE LERAY TRANSFORM

constants and \( \tilde{p} : (0, 1) \to (1, \infty) \) is a continuous function satisfying

\[
\int_0^1 \frac{ds}{s \tilde{p}(s)} = \infty \tag{2.20a}
\]

\[
\int_0^1 \frac{ds}{(1 - s) \tilde{p}(s)} = \infty \tag{2.20b}
\]

\[
\int_0^1 \frac{ds}{s \tilde{p}^*(s)} = \infty \tag{2.20c}
\]

\[
\int_0^1 \frac{ds}{(1 - s) \tilde{p}^*(s)} = \infty \tag{2.20d}
\]

Here, \( \tilde{p}^*(s) \) denotes the dual exponent of \( \tilde{p}(s) \) (that is \( 1/\tilde{p}^*(s) + 1/\tilde{p}(s) = 1 \).

**Proof.** Suppose that \( D \in \tilde{\mathcal{R}} \). Then condition (2.20a) follows from (2.13b) and (2.17a). Similarly, condition (2.20b) follows from (2.13c) and (2.17b). Next, we observe that (2.17a) yields

\[
\int_s^1 \frac{dt}{t \tilde{p}^*(t)} = -\log s + \log(r_1/b_1). \tag{2.21}
\]

Moreover, conditions (2.5d) and (2.12) imply

\[
\frac{s}{r_1} \to 0 \quad \text{as} \quad s \to 0, \tag{2.22}
\]

so that (2.20c) follows from (2.21) and (2.22). Identity (2.20d) follows by a parallel argument.

Suppose now that we are given positive constants \( b_1, b_2 \) together with a continuous function \( \tilde{p} \) satisfying (2.20a) through (2.20d). Then (2.18) describes an open arc \( \gamma_+ \) in \( \mathbb{R}^2_+ \), and conditions (2.20a) and (2.20b) imply that \( \gamma_+ \) extends to a closed arc \( \gamma \) in \( \mathbb{R}^2 \geq 0 \) with endpoints at \((0, b_2)\) and \((b_1, 0)\). The monotonicity of the resulting \( r_1 \) and \( r_2 \) as functions of \( s \) (see (2.17a), (2.17b)) shows that \( \gamma \) is the graph of a continuous decreasing function \( \phi \) on \([0, b_1]\) satisfying (2.5a) and (2.5b). Moreover, using (2.17a) and (2.17b) we find that

\[
\phi'(r_1) = \frac{dr_2/ds}{dr_1/ds} = -\frac{s}{1 - s} \frac{r_2}{r_1} \tag{2.23}
\]

is continuous and negative on \((0, b_1)\). Taking (2.21) into account, we find that condition (2.20c) implies (2.22), and using (2.23) we see that \( \phi'(r_1) \to 0 \) as \( r_1 \to 0 \); thus we have verified (2.5c) and (2.5d). A similar argument allows us to deduce (2.5e) from (2.20d). Finally, using

\[
\frac{ds}{dr_1} = \frac{1}{dr_1/ds} = \frac{s \tilde{p}(s)}{r_1}
\]
to differentiate (2.23) we find that
\[ \phi''(r_1) = -\frac{(\bar{p}(s) - 1)s}{(1-s)^2} \]
verifying (2.5f). We have shown that \( \phi \) satisfies the conditions of Proposition 7, thus we may use (2.4) to define the desired domain \( D \in \tilde{\mathcal{R}} \). \( \square \)

**Definition 10.** We refer to the domain \( D \) constructed at the end of the previous proof as the domain generated by \( \bar{p}, b_2, b_1 \).

**Remark 11.** The parameterizations (2.18) and (2.19) extend to parameterizations of all of \( \gamma \) and \( bD \), respectively, with \( s \) ranging over the closed interval \([0, 1]\).

**Lemma 12.** For \( D \in \tilde{\mathcal{R}} \), \( (w_1, w_2) \in \overline{D} \), \( (r_1, r_2) \in \gamma_+ \) with \( (|w_1|, |w_2|) \neq (r_1, r_2) \) we have
\[ \frac{s}{r_1}|w_1| + \frac{1-s}{r_2}|w_2| < 1. \]  
Proof. The strict convexity of \( \overline{D} \cap \mathbb{R}^2_+ \) implies that \( (|w_1|, |w_2|) \) lies below the tangent line \( x_2 = r_2 + \phi'(r_1)(x_1 - r_1) \) to \( \gamma_+ \) at \( (r_1, r_2) \), that is,
\[ |w_2| - r_2 < \phi'(r_1)(|w_1| - r_1). \]  
Combining this with \( \frac{s}{r_1} = \frac{-\phi'(r_1)}{r_2 - r_1 \phi'(r_1)} \) and \( \frac{1-s}{r_2} = \frac{1}{r_2 - r_1 \phi'(r_1)}, \) see (2.10), we obtain (2.24).
\[ \square \]

**Lemma 13.** For \( D \in \tilde{\mathcal{R}} \) we have
\[ \frac{s}{r_1} \leq \frac{1}{b_1} \text{ and } \frac{1-s}{r_2} \leq \frac{1}{b_2} \]
on \( \gamma_+ \).

Proof. This follows from (2.24) by setting \( w = (b_1, 0) \) and \( (0, b_2) \), respectively.
\[ \square \]

**Lemma 14.** For \( D \in \tilde{\mathcal{R}} \) the functions \( \frac{s}{r_1} \) and \( \frac{1-s}{r_2} \) extend to continuous functions on \( \gamma \), and the function \( \left( \frac{s}{r_1} \right)^2 + \left( \frac{1-s}{r_2} \right)^2 \) extends to a continuous positive function on \( \gamma \).

Proof. This is a consequence of the limits \( \lim_{\zeta \to (0, b_2)} \frac{s}{r_1} = 0, \lim_{\zeta \to (b_1, 0)} \frac{s}{r_1} = \frac{1}{b_1}, \lim_{\zeta \to (0, b_2)} \frac{1-s}{r_2} = \frac{1}{b_2}, \lim_{\zeta \to (b_1, 0)} \frac{1-s}{r_2} = 0. \) (See (2.12) to check the first and fourth limits.)
\[ \square \]

In the case of a \( \mathcal{P} \)-domain the \( s \)-parametrization of \( \gamma \) given in (2.18) takes the following especially simple form:
\[ (2.27) \gamma = \{(b_1 s^{1/p}, b_2 (1-s)^{1/p}) : 0 \leq s \leq 1\}. \]
The Spectrum of the Leray Transform

Proposition 15. Suppose \( D \in \tilde{\mathcal{R}} \). If the function \( p \) is constant then \( D \in \mathcal{P} \).

Proof. If \( p \) is constant then (2.18) matches (2.27). \( \square \)

For general \( D \in \tilde{\mathcal{R}} \) there will be no control on the behavior of \( p \) along \( \gamma_+ \) as we approach one of the endpoints, so we will also consider the following smaller class of domains.

Definition 16. Let \( \mathcal{R} \) denote the class of domains

\[
\{(z_1, z_2) : |z_1| < b_1, |z_2| < \phi(|z_1|)\}
\]

with \( b_1 \) a given positive constant and \( \phi \) a continuous decreasing concave function on \([0, b_1]\) which is \( C^2 \)-smooth on \((0, b_1)\) and satisfies

\[
\begin{align*}
(2.28a) & \quad \phi''(r_1) < 0 \quad \text{for } 0 < r_1 < b_1; \\
(2.28b) & \quad \phi(r_1) = b_2 - c_2 r_1^{p_2} + \epsilon_1(r_1) \quad \text{for } r_1 \text{ near } 0; \\
(2.28c) & \quad \phi(r_1) = \sqrt{\frac{b_1 - r_1 + \epsilon_2(\phi(r_1))}{c_1}} \quad \text{for } r_1 \text{ near } b_1
\end{align*}
\]

where \( b_j > 0, c_j > 0 \) and \( p_j > 1 \) are constants and \( \epsilon_j(r_j) \) are functions satisfying

\[
\begin{align*}
(2.29a) & \quad \epsilon_j \text{ is of class } C^1 \text{ for } r_1 \geq 0; \\
(2.29b) & \quad \epsilon_j \text{ is of class } C^2 \text{ for } r_1 > 0; \\
(2.29c) & \quad \epsilon_j(0) = 0; \\
(2.29d) & \quad \epsilon_j'(0) = 0; \\
(2.29e) & \quad \epsilon_j''(r_j) = o(r_j^{p_j-2})
\end{align*}
\]

for \( j = 1, 2 \).

The conditions (2.28) imply the conditions (2.3) and so \( \mathcal{R} \) is contained in \( \tilde{\mathcal{R}} \).

Condition (2.28c) is equivalent to the condition that \( \psi = \phi^{-1} \) satisfies

\[
\psi(r_2) = b_1 - c_1 r_2^{p_1} + \epsilon_2(r_2) \quad \text{for } r_2 \text{ near } 0.
\]

The class \( \mathcal{R} \) is invariant under permutation of the coordinates \( z_1, z_2 \); thus we will often transfer work on behavior near the axis \( z_1 = 0 \) to get corresponding results near \( z_2 = 0 \).

Note that the assumptions (2.29) imply that

\[
\begin{align*}
(2.31a) & \quad \epsilon_1'(r_1) = o(r_1^{p_1-1}) \\
(2.31b) & \quad \epsilon_1(r_1) = o(r_1^{p_1}).
\end{align*}
\]
As mentioned in the introduction, the class \( \mathcal{R} \) contains the \( \mathcal{P} \)-domains \([1,2]\). For a \( \mathcal{P} \)-domain, the constants in \((2.28a), (2.28b) \text{ and } (2.28c)\) are determined in terms of \(p\) and \(a_1, a_2\) by

\[
p_1 = p_2 = p, \quad b_2 = \frac{1}{\sqrt[p]{a_2}}, \quad b_1 = \frac{1}{\sqrt[p]{a_1}}, \quad c_2 = \frac{a_1}{p \sqrt[p]{a_2}}, \quad c_1 = \frac{a_2}{p \sqrt[p]{a_1}},
\]

the function \(\epsilon_1(|\zeta_1|)\) is the error term of the first-order expansion of \(\phi(|\zeta_1|) = \sqrt[p]{1 - a_1|\zeta_1|^p} = b_2 \sqrt[p]{1 - b_1^{-p}|\zeta_1|^p}\) in powers of \(|\zeta_1|^p\) about \(\zeta_1 = 0\), while \(\epsilon_2(|\zeta_2|)\) is similarly determined by

\[
\psi(|\zeta_2|) = \sqrt[p]{1 - a_2|\zeta_2|^p} = b_1 \sqrt[p]{1 - b_2^{-p}|\zeta_2|^p}.
\]

**Proposition 17.** Suppose \(D \in \mathcal{R}\). Then the functions \(p(\zeta), a_1(\zeta)\) and \(a_2(\zeta)\) described in Proposition 8 extend to continuous functions on all of \(bD\) with \(p(\zeta) = p_1\) when \(\zeta_1 = 0\) and \(p(\zeta) = p_2\) when \(\zeta_2 = 0\).

**Proof.** Using \(2.6\) as before it will suffice to show that the functions \(p, a_1\) and \(a_2\) extend continuously from \(\gamma_+\) to \(\gamma\) (with \(p\) taking the indicated boundary values).

Combining \(2.31a\) with \(2.28b\) and \(2.10\) we find that

\[
(2.33) \quad s = \frac{p_1c_2r_1^{p_1} - r_1 \epsilon_1'(r)}{b_2 + (p_1 - 1)c_2r_1^{p_1} + \epsilon_1(r_1) - r_1 \epsilon'(r_1)} = \frac{p_1c_2}{b_2} r_1^{p_1} + o(r_1^{p_1})
\]

and

\[
(2.34) \quad \frac{ds}{dr_1} = \frac{p_1^2c_2}{b_2} r_1^{p_1 - 1} + o(r_1^{p_1 - 1}).
\]

We note for future reference that \((2.33)\) may be rewritten in the form

\[
(2.35) \quad r_1 = s^{1/p_1} \left( \left( \frac{b_2}{p_1c_2} \right)^{1/p_1} + o(1) \right).
\]

From \(2.14\) we now obtain

\[
p(\zeta) = \frac{r_1}{s} \frac{ds}{dr_1} = p_1 + o(1)
\]
as \(\zeta \to (0, b_2)\).
Applying a similar analysis to (2.7) and (2.8) we find that
\[ a_1(\zeta) \to \frac{p_2 c_1}{b_2} \text{ and } a_2(\zeta) \to \frac{1}{b_2} \text{ as } \zeta \to (0, b_2). \]
Transferring these results to the other axis we have
\[ (2.36) \quad 1 - s = \frac{p_2 c_1}{b_1} r_2^{p_2} + o(r_2^{p_2}); \]
also \( p(\zeta) \to p_2, a_2(\zeta) \to \frac{p_2 c_1}{b_1} \) and \( a_1(\zeta) \to \frac{1}{b_1} \) as \( \zeta \to (b_1, 0) \). \( \square \)

**Corollary 18.** Suppose \( D \) is a \( C^2 \)-smooth strongly convex Reinhardt domain in \( \mathbb{C}^2 \). Then the function \( p \) defined by (2.9) extends to a continuous rotation-invariant function on \( bD \) satisfying \( p(\zeta) = 2 \) when \( \zeta_1 \zeta_2 = 0 \).

**Proof.** Such a domain satisfies Definition 16 with \( p_1 = p_2 = 2 \). \( \square \)

**Theorem 19.** A domain generated by \( \bar{\tilde{p}}, b_2 \) and \( b_1 \) as in (2.18) belongs to \( \mathcal{R} \) if and only if \( \bar{\tilde{p}} \) satisfies the conditions
\[ (2.37a) \quad \bar{\tilde{p}} \text{ extends to a continuous function } [0, 1] \to (1, \infty); \]
\[ (2.37b) \quad \int_0^1 \left( \frac{1}{\bar{\tilde{p}}(s)} - \frac{1}{\bar{\tilde{p}}(0)} \right) \frac{ds}{s} \quad \text{and} \quad \int_0^1 \left( \frac{1}{\bar{\tilde{p}}(s)} - \frac{1}{\bar{\tilde{p}}(1)} \right) \frac{ds}{1 - s} \]
converge as improper integrals.

(The condition (2.37a) means that \( \lim_{s \to 0^+} \int_s^1 \left( \frac{1}{\bar{\tilde{p}}(t)} - \frac{1}{\bar{\tilde{p}}(0)} \right) \frac{dt}{t} \) and \( \lim_{s \to 1^-} \int_0^s \left( \frac{1}{\bar{\tilde{p}}(t)} - \frac{1}{\bar{\tilde{p}}(1)} \right) \frac{dt}{1-t} \) exist and are finite.)

Note that (2.37a) implies (2.20a)-(2.20d).

**Proof.** Suppose our domain is in \( \mathcal{R} \). Then Proposition 17 shows that (2.37a) holds with \( \bar{\tilde{p}}(0) = p_1, \bar{\tilde{p}}(1) = p_2 \). Combining this with (2.17) we obtain
\[ \int_s^1 \left( \frac{1}{\bar{\tilde{p}}(t)} - \frac{1}{\bar{\tilde{p}}(0)} \right) \frac{dt}{t} = - \log \frac{r_1}{b_1} + \frac{1}{p_1} \log s \]
\[ = \frac{1}{p_1} \log \frac{s b_1^{p_1}}{r_1^{p_1}}. \]

Furthermore, (2.33) guarantees that the expression above converges to \( \frac{1}{p_1} \log \frac{p_1 c_2 b_1^{p_1}}{b_2} \). Then a similar argument establishes the other half of (2.37b).

Suppose now that the conditions (2.37) hold. Note that (2.37a) implies the conditions in (2.20), so \( D \in \mathcal{R} \). We need to specify constants \( b_2, p_1, p_2, c_1 \) and \( c_2 \) so that all the conditions of Definition 16 hold. We set \( p_1 = \bar{\tilde{p}}(0), p_2 = \bar{\tilde{p}}(1) \) and \( b_2 = \phi(0) \).
For any $c_2 > 0$ we find that $\epsilon_1$ defined from (2.28), that is,

\begin{equation}
\epsilon_1(r_1) = r_2 - b_2 + c_2 r_1^{p_1},
\end{equation}

satisfies conditions (2.29a) through (2.29d) for $j = 1$, and we are left to determine $c_2$ so that (2.29e) is satisfied. We set

\begin{equation}
c_2 = \frac{b_2}{p_1 b_1^{p_1}} \exp \left( p_1 \int_0^1 \left( \frac{1}{p(s)} - \frac{1}{p_1} \right) \frac{ds}{s} \right)
\end{equation}

Since (2.38) holds as before, we find that

\begin{equation}
s = \frac{p_1 c_2}{b_2} r_1^{p_1} + o(r_1^{p_1}).
\end{equation}

Differentiating (2.39) twice with the use of (2.12) and (2.14) we obtain that

\begin{align*}
\epsilon'_1(r_1) &= -s \frac{r_2}{1 - s r_1} + c_2 p_1 r_1^{p_1 - 1}, \\
\epsilon''_1(r_1) &= -(p - 1) \frac{s r_2}{(1 - s)^2 r_1^2} + (p_1 - 1) c_2 p_1 r_1^{p_1 - 2}.
\end{align*}

Combining these with (2.40) and $r_2 = b_2 + o(1)$, $p = p_1 + o(1)$ we find that (2.29e) holds for $j = 1$.

A similar argument takes care of $j = 2$. \qed

3. Construction and basic properties of the Leray transform for domains in the class $\tilde{\mathcal{R}}$

In this section we compute the Leray kernel for domains in the class $\tilde{\mathcal{R}}$ and check that the associated Leray transform $L$ reproduces holomorphic functions from their boundary values. We introduce the notion of admissible measure and provide formulae for various norms and spectra. (Unless explicitly stated, at this stage $L$ is not assumed to be $L^2$-bounded.)

We base our computations on the function

\begin{equation}
\rho(\zeta_1, \zeta_2) = |\zeta_2| - \phi(|\zeta_1|)
\end{equation}

where $\phi$ is as in (2.5). This function will fail to be differentiable at points where $\zeta_1 \zeta_2 = 0$; moreover,

\begin{equation}
|\nabla \rho(\zeta)| = \sqrt{1 + (\phi'(|\zeta_1|))^2},
\end{equation}

will not be bounded above where defined. So $\rho$ is a defining function for $bD \setminus \{\zeta_1 \zeta_2 = 0\}$, but not for $bD$.

For $w \in D$, (1.5) still defines a three-form on $bD \setminus \{\zeta_1 \zeta_2 = 0\}$ which is independent of the particular choice of defining function. When integrating expressions involving this form over $bD$ we simply ignore
the points where $\zeta_1 \zeta_2 = 0$. (The set of such points has measure zero with respect to all boundary measures considered below.)

The classical proof (see [Ran], §IV.3.2) of the reproducing property for holomorphic functions no longer applies, but we remedy this in Corollary 24 and Proposition 32 below.

**Lemma 20.** Let $D \in \tilde{\mathbb{R}}$. Then, representing $\zeta \in bD \setminus \{\zeta_1 \zeta_2 = 0\}$ by the coordinates $(s, \theta_1, \theta_2)$ as in (2.19), we have

\[
L(\zeta, w) = \frac{ds \wedge d\theta_1 \wedge d\theta_2}{4\pi^2 \left(1 - e^{-i\theta_1} \frac{s r_1}{s r_1} w_1 - e^{-i\theta_2} \frac{1 - s}{r_2} w_2\right)^2}.
\]

**Proof.** From (2.10), (2.9), (2.14) and (2.15) we obtain

\[
\phi'(r_1) = -\frac{s}{s r_1} - \frac{r_2}{s r_1}, \quad \phi''(r_1) = -\frac{(p-1)s}{(1-s)^2} \frac{r_2}{r_1}, \quad dr_1 = \frac{r_2}{p} ds, \quad dr_2 = -\frac{r_2}{p} \frac{ds}{1-s}.
\]

Using (1.5) to compute $L(\zeta, w)$ we first compute $\partial \rho \wedge \bar{\partial} \partial \rho$ with $\rho$ as in (3.1); then, setting $z_j = r_j e^{i\theta_j}$ and applying the above formulae we obtain

\[
j^* (\partial \rho \wedge \bar{\partial} \partial \rho) = -\frac{r_2^2}{4(1-s)^2} ds \wedge d\theta_1 \wedge d\theta_2.
\]

Turning our attention to the denominator we find that

\[
(\rho(\zeta) \bullet (\zeta - w))^2 = \left(\frac{1}{2} e^{-i\theta_1} \frac{s}{1-s} r_1 w_1 + \frac{1}{2} e^{-i\theta_2} \frac{1-s}{r_2} w_2\right)^2.
\]

Dividing and simplifying we obtain (3.3). □

From (2.21) we have $|e^{-i\theta_1} \frac{s}{r_1} w_1 + e^{-i\theta_2} \frac{1-s}{r_2} w_2| < 1$. Thus by the differentiated geometric series we have

\[
L(\zeta, w) = \frac{ds \wedge d\theta_1 \wedge d\theta_2}{4\pi^2} \sum_{j=0}^{\infty} (j + 1) \left(e^{-i\theta_1} \frac{s}{r_1} w_1 + e^{-i\theta_2} \frac{1-s}{r_2} w_2\right)^j.
\]

Using the binomial theorem we obtain the following result.

**Lemma 21.** The Leray kernel admits the expansion

\[
L(\zeta, w) = \frac{ds \wedge d\theta_1 \wedge d\theta_2}{4\pi^2} \sum_{n,m=0}^{\infty} \frac{(n + m + 1)!}{n! m!} \left(\frac{s}{r_1}\right)^n \left(\frac{1-s}{r_2}\right)^m \cdot w_1^n w_2^m e^{-i(n\theta_1 + m\theta_2)}
\]
converging uniformly (with exponential speed) for \( w \) in any compact subset of \( D \). (In fact, the convergence is uniform on compact subsets of \( \overline{D} \setminus \{\zeta\} \).

**Definition 22.** We say that a function \( f \) on \( bD \) is an \((n,m)\)-monomial if it takes the form

\[
f(\zeta) = g(s) e^{i(n\theta_1 + m\theta_2)}.
\]

**Corollary 23.** If \( f \) is an \((n,m)\)-monomial of the form (3.5) then for \( w \in D \) we have

\[
L f(w) = \begin{cases}
0 & \text{if } \min\{n,m\} < 0; \\
\frac{(n+m+1)!}{n! m!} \left( \int_0^1 g(s) \left( \frac{s}{r_1} \right)^n \left( \frac{1-s}{r_2} \right)^m ds \right) w_1^n w_2^m & \text{if } \min\{n,m\} \geq 0.
\end{cases}
\]

**Proof.** This follows from Lemma 21 (or Lemma 20 and a residue computation).

If \( f(\zeta) = \zeta_1^n \zeta_2^m \) then applying Corollary 23 with \( g(s) = r_1^n r_2^m \) and recalling that

\[
\int_0^1 s^n (1-s)^m ds = \frac{n! m!}{(n + m + 1)!}
\]

we find that \( L f(w) = w_1^n w_2^m \) for \( w \in D \). Taking sums we obtain the following.

**Corollary 24.** The operator \( L \) reproduces holomorphic polynomials from their restrictions to \( bD \).

Returning to Corollary 23 we see that when \( f \) is an \((n,m)\)-monomial \( g(s) e^{i(n\theta_1 + m\theta_2)} \) then \( L f \) extends continuously to \( \overline{D} \) with boundary values given (in the non-trivial cases) by

\[
L f(R_1 e^{i\theta_1}, R_2 e^{i\theta_2}) = \frac{(n + m + 1)!}{n! m!} \left( \int_0^1 g(s) \left( \frac{s}{r_1} \right)^n \left( \frac{1-s}{r_2} \right)^m ds \right) R_1^n R_2^m e^{i(n\theta_1 + m\theta_2)}.
\]

In particular, \( L \) maps \((n,m)\)-monomials to \((n,m)\)-monomials.

Let \( \mu \) be a rotation-invariant measure on \( bD \) described by

\[
d\mu = \frac{1}{4\pi^2} \omega(s) ds d\theta_1 d\theta_2,
\]

where \( \omega(s) \) is measurable and positive a.e.
Definition 25. Let \( L^2_{n,m}(bD, \mu) \) denote the space of \((n, m)\)-monomials that are in \( L^2(bD, \mu) \).

The spaces \( L^2_{n_1, m_1}(bD, \mu) \) and \( L^2_{n_2, m_2}(bD, \mu) \) are orthogonal subspaces of \( L^2(bD, \mu) \) unless \((n_1, m_1) = (n_2, m_2)\). Note also that if \( f_1(\zeta) = g_1(s)e^{i(n_1\theta_1 + m_1\theta_2)} \) and \( f_2(\zeta) = g_2(s)e^{i(n_2\theta_1 + m_2\theta_2)} \) are in \( L^2_{n,m}(bD, \mu) \) then the Hermitian inner product \( \langle f_1, f_2 \rangle \) of the monomials in \( L^2_{n,m}(bD, \mu) \) is just \[ \int_0^1 g_1(s)g_2(s)\omega(s)\,ds. \]

Proposition 26. When \( n, m \geq 0 \), the restriction \( \mathbb{L}_{n,m} \) of \( \mathbb{L} \) to \( L^2_{n,m}(bD, \mu) \) is a rank-one projection operator with \( L^2 \) operator norm given by

\[ \|\mathbb{L}_{n,m}\|^2_\mu = \left( \frac{(n + m + 1)!}{n! m!} \right)^2 \int_0^1 \left( \frac{s}{r_1} \right)^{2n} \left( \frac{1 - s}{r_2} \right)^{2m} \frac{1}{\omega(s)}\,ds \cdot \int_0^1 r_1^{2n} r_2^{2m} \omega(s)\,ds. \]

Proof. Set

\[ \kappa_{n,m} = \frac{(n + m + 1)!}{n! m!} \left( \frac{s}{r_1} \right)^n \left( \frac{1 - s}{r_2} \right)^m \frac{1}{\omega(s)} e^{i(n\theta_1 + m\theta_2)}, \]
\[ \tau_{n,m} = r_1^n r_2^m e^{i(n\theta_1 + m\theta_2)}. \]

Then from (3.8) and Corollary 23 and using the formula above for the inner product in \( L^2_{n,m}(bD, \mu) \) we have

\[ \mathbb{L}_{n,m}(f) = \langle f, \kappa_{n,m} \rangle \tau_{n,m} \]
and (3.7) yields

\[ \langle \tau_{n,m}, \kappa_{n,m} \rangle = 1 \]
so that

\[ \mathbb{L}_{n,m}^2 = \mathbb{L}_{n,m} \]
and

\[ \|\mathbb{L}_{n,m}\|^2_\mu = \|\kappa_{n,m}\|^2_\mu \|\tau_{n,m}\|^2_\mu. \]

Theorem 27. Let \( D \in \tilde{\mathbb{R}} \) and let \( \mu \) be a rotation-invariant measure on \( bD \) described by (3.9) with \( \omega(s) \) measurable and positive a.e. Then the following conditions are equivalent.

(3.16a) \[ \int_0^1 \frac{1}{\omega(s)}\,ds \] and \( \int_0^1 \omega(s)\,ds \) are finite.
(3.16b) The measure \( \mu \) is finite, and the functional \( f \mapsto (\mathbb{L}f)(0) \) is bounded on \( L^2(bD, \mu) \).

(3.16c) The measure \( \mu \) is finite, and for each \( w \in D \) the functional \( f \mapsto (\mathbb{L}f)(w) \) is bounded on \( L^2(bD, \mu) \).

(3.16d) \( \|L_{0,0}\|_\mu < \infty \).

(3.16e) For each \((n,m)\) we have \( \|L_{n,m}\|_\mu < \infty \).

Proof. The equivalence of items (3.16a) and (3.16d) is immediate from (3.10). Similarly, (3.10) together with Lemma 14 and the boundedness of \( r_1 \) and \( r_2 \) show in turn that items (3.16d) and (3.16e) are equivalent.

To see that items (3.16b) and (3.16a) are equivalent, note that

\[
\mu(bD) = \int_0^1 \omega(s) \, ds
\]

and that from (3.3) and (1.4) we have that

\[
\mathbb{L}f(0) = \int_{bD} \frac{1}{\omega(s)} f(\zeta) \, d\mu = \langle f, 1/\omega \rangle
\]

is a linear functional on \( L^2(bD, \mu) \) with norm \( \sqrt{\int_0^1 \frac{1}{\omega(s)} \, ds} \).

Finally, to see that items (3.16b) and (3.16c) are equivalent, fix \( w \in D \) and note that Lemmas 12 and 14 allow us to conclude that

\[
\sup_{\zeta = (r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \in bD} \left| e^{-i\theta_1} \frac{s}{r_1} w_1 + e^{-i\theta_2} \frac{1-s}{r_2} w_2 \right| \leq 1.
\]

Consulting (3.3) we see that

\[
\frac{1}{2} \leq \inf_{\zeta \in bD} \left| \frac{L(\zeta, w)}{L(\zeta, 0)} \right| \leq \sup_{\zeta \in bD} \left| \frac{L(\zeta, w)}{L(\zeta, 0)} \right| < \infty,
\]

from which the desired result follows immediately.

Definition 28. We will call a measure of the form (3.9) admissible if it satisfies the equivalent conditions of Theorem 27.

Remark 29. It is easy to see that any rotation-invariant measure \( \mu \) on \( bD \) satisfying condition (3.16b) in Theorem 27 must in fact be of the form (3.9) with \( \omega(s) \) measurable and positive a.e.

Assume now that \( \mu \) is admissible. Because \( D \) is Reinhardt, any \( f \in L^2(bD, \mu) \) may be written uniquely as a sum \( f = \sum_{n,m \in \mathbb{Z}} f_{n,m} \) converging in \( L^2(bD, \mu) \) where each \( f_{n,m} \) is an \((n,m)\)-monomial (3.5) and

\[
\|f\|^2_\mu = \sum_{n,m \in \mathbb{Z}} \|f_{n,m}\|_\mu^2.
\]
If \( L \) is to define a bounded operator on \( L^2(bD, \mu) \) it must be given by
\[
(3.17) \quad Lf = \sum_{n,m \geq 0} L_{n,m} f_{n,m}
\]
and thus
\[
\|Lf\|_\mu^2 = \sum_{n,m \geq 0} \|L_{n,m} f_{n,m}\|_\mu^2.
\]
From this we easily obtain the following.

**Theorem 30.** \( L \) defines a bounded operator on \( L^2(bD, \mu) \) if and only if the quantities \( \|L_{n,m}\|_\mu \) given in (3.10) are uniformly bounded for \( n, m \geq 0 \); moreover,
\[
\|L\|_\mu = \sup \{\|L_{n,m}\|_\mu : n, m \geq 0\}.
\]
Condition (3.16e) in Theorem 27 shows that when \( \mu \) is admissible then the boundary values of holomorphic polynomials lie in \( L^2(bD, \mu) \).

This observation motivates the following.

**Definition 31.** The Hardy space \( H^2(bD, \mu) \) is the closure in \( L^2(bD, \mu) \) of the boundary values of holomorphic polynomials.

From Corollaries 23 and 24 and Proposition 26 we obtain the following.

**Proposition 32.** If \( L \) defines a bounded operator on \( L^2(bD, \mu) \) then \( L \) is a projection operator from \( L^2(bD, \mu) \) onto \( H^2(bD, \mu) \).

When \( L \) defines a bounded operator on \( L^2(bD, \mu) \) then \( L \) admits an adjoint \( L^*_\mu \).

From (3.17) we have \( L^*_\mu = \sum_{n,m \geq 0} (L_{n,m})^*_\mu \). From (3.12) we see that
\[
(L_{n,m})^*_\mu \text{ maps } L^2_{n,m}(bD, \mu) \text{ to } L^2_{n,m}(bD, \mu) \text{ via the formula}
\]
\[
(3.18) \quad (L_{n,m})^*_\mu(f) = \langle f, \tau_{n,m} \rangle \kappa_{n,m}.
\]
Of course, the norms of \( (L_{n,m})^*_\mu \) and \( L^* \) match those of \( L_{n,m} \) and \( L \).

**Proposition 33.** The self-adjoint operator \( (L_{n,m})^*_\mu L_{n,m} \) has rank one with spectrum given by
\[
\{0, \|L_{n,m}\|_\mu^2\}.
\]

**Proof.** From (3.12), (3.18) and (3.13) we have
\[
(L_{n,m})^*_\mu L_{n,m} f = \|\tau_{n,m}\|_\mu^2 \langle f, \kappa_{n,m} \rangle \kappa_{n,m}
\]
which is \( \|\tau_{n,m}\|_\mu^2 \|\kappa_{n,m}\|_\mu^2 = \|L_{n,m}\|_\mu^2 \) times the orthogonal projection onto the line through \( \kappa_{n,m} \).
Remark 34. It is clear that \((\mathbb{L}_{n,m})^*_\mu \mathbb{L}_{n,m}\) admits an orthonormal basis of eigenfunctions.

Corollary 35. If \(\mu\) is admissible, then \((\mathbb{L}_{n,m})^*_\mu \mathbb{L}_{n,m}\) is unitarily equivalent to \(\mathbb{L}_{n,m} (\mathbb{L}_{n,m})^*_\mu\) in \(L^2(bD,\mu)\). Moreover, if \(\mathbb{L}\) is bounded in \(L^2(bD,\mu)\), then \((\mathbb{L})^*_\mu \mathbb{L}\) is unitarily equivalent to \(\mathbb{L} (\mathbb{L})^*_\mu\).

Turning the attention to essential norms and spectra, we recall that the essential norm of an operator \(T\) on a Hilbert space \(H\) is the distance (in the operator norm) of \(T\) from the space of compact operators \(\mathcal{K}(H)\) (see [Pel], page 25) while the essential spectrum of a bounded operator \(T \in \mathcal{L}(H)\) is the spectrum of the projection of \(T\) on the Calkin algebra \(\mathcal{L}(H)/\mathcal{K}(H)\) (see [HR], page 32), and we have

**Proposition 36 ([HR], Proposition 2.2.2).** For a self-adjoint or anti-self-adjoint operator admitting an orthonormal basis of eigenfunctions, the essential spectrum consists of limits of sequences of eigenvalues together with isolated eigenvalues of infinite multiplicity.

In general, the essential spectrum includes the continuous spectrum, which is absent in our work but does appear in analysis of the Kerzman-Stein operator for many non-smooth planar domains (see [Bo1]).

**Proposition 37** (see [CM], §3.2; [ÖT], §2). The essential norm of an operator \(T\) is the square root of the largest value in the essential spectrum of \(T^*T\).

Using Propositions 33, 36 and 37 we obtain the following.

**Theorem 38.** The essential norm of \(\mathbb{L}\) on \(L^2(bD,\mu)\) is given by

\[
\limsup \{ \|\mathbb{L}_{n,m}\|_\mu : n, m \geq 0 \},
\]

where \(\limsup q_{n,m}\) is defined by

\[
\inf \left\{ \sup \{ q_{n,m} \in (\mathbb{N} \times \mathbb{N}) \setminus F \} : F^{\text{finite}} \subset \mathbb{N} \times \mathbb{N} \} \right\}.
\]

The essential spectrum of \(\mathbb{L}^*_\mu \mathbb{L}\) consists of 0 together with all values of

\[
\lim_{j \to \infty} \|\mathbb{L}_{n_j,m_j}\|_\mu^2
\]

taken along sequences \((n_j,m_j)\) with \(\max\{n_j,m_j\} \to \infty\) as \(j \to \infty\) along which the above limit exists.

As in the introduction we set

\[
A_\mu = \mathbb{L}^*_\mu - \mathbb{L}
\]
and
\[(A_{n,m})_{\mu} = (\mathbb{L}_{n,m})_{\mu}^* - \mathbb{L}_{n,m}.\]

These operators are anti-self-adjoint.

**Proposition 39.** \((A_{n,m})_{\mu}\) is a rank-two operator with norm given by
\[\| (A_{n,m})_{\mu} \|_{\mu}^2 = \| \mathbb{L}_{n,m} \|_{\mu}^2 - 1.\]

The spectrum of \((A_{n,m})_{\mu}\) is the set
\[\{0, \pm i \sqrt{\| \mathbb{L}_{n,m} \|_{\mu}^2 - 1}\}.\]

**Proof.** Using the notation from (3.11) and the identities (3.12), (3.13) (but dropping subscripts), set
\[\lambda = \tau - \langle \tau, \kappa \rangle \kappa = \tau - \frac{\kappa}{\| \kappa \|_{\mu}^2} \kappa.\]

Then \(\lambda \perp \kappa\) and \(\| \lambda \|_{\mu}^2 = \| \tau \|_{\mu}^2 - \frac{\langle \kappa, \tau \rangle^2}{\| \kappa \|_{\mu}^2} = \| \tau \|_{\mu}^2 - \frac{1}{\| \kappa \|_{\mu}^2}.\)

Using (3.12) and (3.18) we have
\[\langle f, \tau \rangle \kappa - \langle f, \kappa \rangle \tau = \langle f, \lambda \rangle \kappa - \langle f, \kappa \rangle \lambda\]
(3.19)
so
\[\| (A_{n,m})_{\mu} (f) \|_{\mu}^2 = \| f, \lambda \|_{\mu}^2 \| \kappa \|_{\mu}^2 + \| f, \kappa \|_{\mu}^2 \| \lambda \|_{\mu}^2.\]

If \(\lambda = 0\) then it follows \(A_{n,m} = 0\) and also \(\tau = \kappa/\| \kappa \|_{\mu}^2\), so (3.15) shows \(\| \mathbb{L} \|_{\mu} = 1\), which proves the desired result.

If on the other hand \(\lambda \neq 0\) then we may write (3.20) as
\[\| (A_{n,m})_{\mu} (f) \|_{\mu}^2 = \left( \left\langle f, \frac{\lambda}{\| \kappa \|_{\mu}^2} \right\rangle \right)^2 + \left\langle f, \frac{\kappa}{\| \kappa \|_{\mu}^2} \right\rangle \| \kappa \|_{\mu}^2 \| \lambda \|_{\mu}^2.\]

By Bessel’s inequality this is less than or equal to
\[\| f \|_{\mu}^2 \| \kappa \|_{\mu}^2 \| \lambda \|_{\mu}^2 = \| f \|_{\mu}^2 \left( \| \kappa \|_{\mu}^2 \| \tau \|_{\mu}^2 - 1 \right)\]
with equality holding if and only if \(f\) is in \(\text{Span}\{\kappa, \tau\}\). Thus
\[\| (A_{n,m})_{\mu} \|_{\mu}^2 = \| \kappa \|_{\mu}^2 \| \tau \|_{\mu}^2 - 1.\]

The eigenvalues of \(A_{n,m}\) on \(\text{Span}\{\kappa, \tau\}\) are \(\pm i \sqrt{\| \kappa \|_{\mu}^2 \| \tau \|_{\mu}^2 - 1}\), and \(A_{n,m}\) vanishes on \(\text{Span}\{\kappa, \tau\}^\perp\). Thus the spectrum of \(A_{n,m}\) is
\[\{0, \pm i \sqrt{\| \kappa \|_{\mu}^2 \| \tau \|_{\mu}^2 - 1}\}.\]

Invoking (3.15), the proof is complete. \(\square\)
Remark 40. It is clear that \((A_{n,m})_\mu\) admits an orthonormal basis of eigenfunctions.

Assembling the pieces as in Theorems 30 and 38 we have the following.

Theorem 41. The norm of \(A_\mu\) acting on \(L^2(bD,\mu)\) is

\[
\sup \left\{ \sqrt{\| L_{n,m} \|_\mu^2 - 1} : n, m \geq 0 \right\}.
\]

The essential norm of \(A_\mu\) is given by

\[
\limsup \left\{ \sqrt{\| L_{n,m} \|_\mu^2 - 1} : n, m \geq 0 \right\}.
\]

In particular, \(A_\mu\) is compact on \(L^2(bD,\mu)\) if and only if

\[
\limsup \left\{ \sqrt{\| L_{n,m} \|_\mu^2 - 1} : n, m \geq 0 \right\} = 0.
\]

The spectrum of \(A_\mu\) is the closure of

\[
\{0\} \cup \left\{ \pm i \sqrt{\| L_{n,m} \|_\mu^2 - 1} : n, m \geq 0 \right\}.
\]

The essential spectrum of \(A_\mu\) consists of 0 together with all values of

\[
\lim_{j \to \infty} \pm i \sqrt{\| L_{n_j,m_j} \|_\mu^2 - 1}
\]

taken along sequences \((n_j, m_j)\) with \(\max\{n_j, m_j\} \to \infty\) as \(j \to \infty\) along which the above limit exists.

For a geometric interpretation of these results, let \(\theta_{n,m} \in [0, \frac{\pi}{2})\) be the angle between \(\kappa_{n,m}\) and \(\tau_{n,m}\) in \(L^2(bD,\mu)\). (Thus \(\langle \kappa_{n,m}, \tau_{n,m} \rangle = \cos \theta_{n,m} \cdot \|\kappa_{n,m}\| \cdot \|\tau_{n,m}\|\). From (3.13), (3.15) and (3.21) we find that

\[
\| L_{n,m} \|_\mu = \sec \theta_{n,m} \quad \text{and} \quad \| (A_{n,m})_\mu \| = \tan \theta_{n,m}.
\]

Returning to Proposition 32, note that \(L\) will be the orthogonal projection from \(L^2(bD,\mu)\) to \(H^2(bD,\mu)\) (the Szegő projection for \(\mu\)) if and only if \(L^*_\mu = L\); this is in turn equivalent to any one of the following conditions:

\[
\begin{align*}
& \bullet A_\mu = 0; \\
& \bullet (A_{n,m})_\mu = 0; \\
& \bullet \| L_{n,m} \|_\mu = 1; \\
& \bullet \theta_{n,m} = 0; \\
& \bullet \| L \|_\mu = 1.
\end{align*}
\]

Examining (3.11) we see that this will happen if and only if

\[
\omega(s) = c_{n,m} \left( \frac{s}{r_1^2} \right)^n \left( \frac{1-s}{r_2^2} \right)^m
\]
for each \((n, m)\) with \(c_{n,m}\) a positive constant. Selecting \((n, m) = (0, 0), (1, 0)\) and \((0, 1)\) in turn we see that this can happen if and only if 
\[
\frac{s}{r_1} - \frac{1-s}{r_2}\quad \text{and} \quad \omega(s)\ 
\]
are constant. Applying (2.14), (2.15) and Proposition 15 we obtain the following.

**Proposition 42.** Let \(D \in \tilde{R}\) and let \(\mu\) be an admissible measure on \(bD\). Then \(L\) will be the Szegő projection for \(\mu\) if and only if \(D\) is of the form 
\[
\{(z_1, z_2) : a_1|z_1|^2 + a_2|z_2|^2 < 1\}\quad \text{and} \quad \omega(s)\ 
\]
is constant.

Bolt has shown that the Leray transform for a strongly \((C\text{-linearly)}\) convex bounded domain in \(\mathbb{C}^n\) with \(C^3\)-smooth boundary will coincide with the Szegő projection (for a suitably-chosen measure) if and only if the domain is a complex-affine image of the unit ball \([\text{Bo}2, \text{Bo}3]\).

4. **More on boundary measures and geometry**

From formulas (3.8) and (3.10) we see that our theory becomes simplest with the use of the measure 
\[
d\mu_0 = \frac{1}{4\pi^2} \, ds \, d\theta_1 \, d\theta_2
\]
on \(bD\). When \(D\) is smooth and strongly convex this measure will be comparable to surface measure \(d\sigma\) but in general this will not be so. Indeed, from (3.2), (2.14) and (2.15) we find that 
\[
d\sigma = r_1 r_2 \left(1 + (\phi'(r_1))^2\right)^{1/2} \, dr_1 \, d\theta_1 \, d\theta_2
\]
\[
= r_1 r_2 \sqrt{dr_1^2 + dr_2^2} \, d\theta_1 \, d\theta_2
\]
\[
= \frac{r_1^2 r_2^2}{ps(1-s)} \sqrt{\left(\frac{s}{r_1}\right)^2 + \left(\frac{1-s}{r_2}\right)^2} \, ds \, d\theta_1 \, d\theta_2,
\]
where \(p\) is as in (2.9). From (1.7) and (3.4) we deduce 
\[
|L| \, d\sigma = \frac{1}{4 \left(\left(\frac{s}{r_1}\right)^2 + \left(\frac{1-s}{r_2}\right)^2\right)} \, ds \, d\theta_1 \, d\theta_2.
\]

Lemma 14 now shows that \(d\mu_0\) is comparable to \(|L| \, d\sigma\). In particular we see that \(d\mu_0\) will not be comparable to \(d\sigma\) unless \(|L|\) is bounded above and below. For \(D \in \mathcal{P}\), for example, it is easy to check using (2.27) that this happens if and only if \(p = 2\).

Formula (4.1) motivates the following

**Definition 43.** We will say that a rotation-invariant measure on the boundary of a domain \(D \in \tilde{R}\) has order \(q\) if it is a continuous positive multiple of \(|L|^{1-q} \, d\sigma\).
Thus surface measure $d\sigma$ has order $q = 1$, and the special measure $d\mu_0$ has order $q = 0$.

From (4.1) and (4.2) we find

$$|\mathcal{L}| = \frac{ps(1-s)}{4r_1^2r_2^2\left(\left(\frac{s}{r_1}\right)^2 + \left(\frac{1-s}{r_2}\right)^2\right)^{3/2}}.$$  

It follows that $\mu$ has order $q$ if and only if $\mu$ is expressed as

$$\varphi(s)\left(\frac{r_1^2r_2^2}{ps(1-s)}\right)^q ds\,d\theta_1\,d\theta_2,$$

where $\varphi$ is assumed to be positive and continuous on all of $bD$. In particular, the Fefferman measure $d\mu_{bD}^{\text{Fe}}$ has order $q = 2/3$, where

$$d\mu_{bD}^{\text{Fe}} \overset{\text{def}}{=} |\mathcal{L}|^{1/3} d\sigma = \left(\frac{r_1^2r_2^2}{2ps(1-s)}\right)^{2/3} ds\,d\theta_1\,d\theta_2$$

(see p. 259 of [Fef]; also [Bar1]). This measure may be defined on general smooth pseudoconvex domains in $\mathbb{C}^2$ and plays a distinguished role in complex analysis due to the fact that it transforms by the rule

$$F^* (d\mu_{bD_2}^{\text{Fe}}) = |\det F'|^{2/3} d\mu_{bD_1}^{\text{Fe}}$$

under a biholomorphic mapping $F$ mapping $bD_1$ to $bD_2$. (Modified versions of this construction work also in higher dimensions.)

Note for comparison that integrals of the form $\int_{bD} |\mathcal{L}|^{-1} d\sigma$ (corresponding to $q = 2$) appear in work on spectral asymptotics of the $\overline{\partial}$-Neumann problem by Metivier [Met].

If $D \in \mathcal{R}$ then combining Proposition 17 with (2.33), (2.36) and (4.3) we find that a measure of order $q$ is given by the following expression

$$\varphi(s)\left(\frac{r_1^2r_2^2}{s^{1/3}r_2^{-1}(1-s)^{r_2^{-1}}}\right)^q ds\,d\theta_1\,d\theta_2$$

$$= \varphi(s) s^q\left(\frac{1}{r_1} - \frac{1}{r_1}\right)^q (1-s)^q\left(\frac{1}{r_2} - \frac{1}{r_2}\right)^q ds\,d\theta_1\,d\theta_2,$$

where $\varphi$ is positive and continuous on $bD$. Recalling Definition 28 we easily obtain the following result.

**Proposition 44.** If $D \in \mathcal{R}$ then a rotation-invariant measure of order $q$ is admissible if and only if (1.8) holds.

Applying this to values of $q$ just discussed we see that $q = 2/3$ (indeed, any $q \in [0, 1]$) will always work, while $q = 2$ works if and only if both of the $p_j$ lie in the interval $(4/3, 4)$. 
For later reference, we close this section with a quick look at the differential geometry of \( bD \). A computation based on the parameterization (2.19) shows that the principal curvatures of \( bD \) are given by

\[
\begin{align*}
\kappa_1 &= \frac{s}{r_1^2 \sqrt{\left(\frac{s}{r_1} \right)^2 + \left(\frac{1-s}{r_2} \right)^2}} \\
\kappa_2 &= \frac{1-s}{r_2^2 \sqrt{\left(\frac{s}{r_1} \right)^2 + \left(\frac{1-s}{r_2} \right)^2}} \\
\kappa_3 &= (p-1) \frac{s(1-s)}{r_1^2 r_2^2} \left(\left(\frac{s}{r_1} \right)^2 + \left(\frac{1-s}{r_2} \right)^2\right)^{3/2}.
\end{align*}
\]

For a point in \( \gamma_+ \) the corresponding principal directions are given by \((i,0), (0,i)\) and the tangent to \( \gamma_+ \). The principal directions at other points are found by rotation.

As a check, note that for the unit sphere we have \( p = 2, r_1 = \sqrt{s}, r_2 = \sqrt{1-s} \) and thus \( \kappa_1 = \kappa_2 = \kappa_3 = 1 \).

5. Asymptotics in \( \mathcal{R} \)

In this section we perform asymptotic analysis of the norms of \( \mathbb{L}_{n,m} \) and use these results to prove Theorems 1 and 2.

**Theorem 45.** Suppose that \( D \in \mathcal{R} \) and that \( \mu \) is an admissible measure on \( bD \) of order \( q \) (as in Definition 43).

Let \((n_j, m_j)\) be a sequence in \( \mathbb{N} \times \mathbb{N} \) with \( \max\{n_j, m_j\} \to \infty \) as \( j \to \infty \).

(a) If \( \min\{n_j, m_j\} \to \infty \) and \( \frac{n_j}{m_j} \to u \in [0, \infty] \) then

\[
\|\mathbb{L}_{n_j, m_j}\|_\mu^2 \to \frac{\sqrt{\bar{p}(\frac{u}{1+u}) \bar{p}^\ast(\frac{u}{1+u})}}{2},
\]

where \( \bar{p} \) was defined in (2.16).

(b) If \( n_j \) is independent of \( j \) then

\[
\|\mathbb{L}_{n_j, m_j}\|_\mu^2 \to \frac{\Gamma \left( \frac{2n_0}{p_1} + 1 - q \left( \frac{1}{p_1} - \frac{1}{p_2} \right) \right) \Gamma \left( \frac{2n_0}{p_1} + 1 - q \left( \frac{1}{p_1} - \frac{1}{p_2} \right) \right)}{\Gamma^2(n_0+1) \left( \frac{2}{p_1} \right)^{2n_0+1+q \left( \frac{1}{p_1} - \frac{1}{p_2} \right)} \left( \frac{2}{p_2} \right)^{2n_0+1+q \left( \frac{1}{p_1} - \frac{1}{p_2} \right)}},
\]

where \( n_0 \) is the common value of the \( n_j \) and \( p_1 \) is as in Definition 16.
(c) If \( m_j \) is independent of \( j \) then
\[
\|L_{n_j, m_j}\|_\mu^2 \to \frac{\Gamma\left( \frac{2m_0}{p_2} + 1 - q\left( \frac{1}{p_2} - \frac{1}{p_2^*} \right) \right)}{\Gamma^2(m_0 + 1)\left( \frac{2}{p_2^{*}} \right)^{\frac{2m_0}{p_2} + 1 - q\left( \frac{1}{p_2} - \frac{1}{p_2} \right)}}
\]

where \( m_0 \) is the common value of the \( m_j \) and \( p_2 \) is as in Definition 16.

Proof of Theorems 1 and 2, assuming Theorem 45.

Suppose that
\[
(*) \lim \|L_{n_j, m_j}\|_\mu \text{ exists (possibly equal to } +\infty \text{) along some sequence } (n_j, m_j) \text{ with } \max\{n_j, m_j\} \to \infty \text{ as } j \to \infty.
\]

We consider the sequence of quotients: \( \{n_j/m_j\} \subset [0, +\infty] \) and distinguish the two cases: \( \min\{n_j, m_j\} \to \infty \); \( \min\{n_j, m_j\} \not\to \infty \). In either case, passing to a subsequence we may arrange for one of the following three conditions to hold:

\begin{enumerate}
  \item[(5.1a)] \( \min\{n_j, m_j\} \to \infty \) and \( \frac{n_j}{m_j} \) approaches some value \( u \in [0, \infty] \), or
  \item[(5.1b)] \( n_j \) is independent of \( j \), or
  \item[(5.1c)] \( m_j \) is independent of \( j \).
\end{enumerate}

Then Theorem 45 provides the limiting value of \( \|L_{n_j, m_j}\|_\mu \) in each of these three cases.

In particular, since Theorem 45 shows that none of these limiting values can be infinite, we conclude that the set \( \{\|L_{n,m}\|_\mu : n, m \geq 0\} \) is bounded. Then Theorem 30 shows that \( \mathbb{L} \) is bounded on \( L^2(bD, \mu) \).

The orthonormal basis of eigenfunctions for \( \mathbb{L}_\mu^* \mathbb{L} \) is obtained by combining eigenfunction bases for each \( L^2_{n,m}(bD, \mu) \).

Finally, we use Theorem 38 and the above description of limiting values of \( \|L_{n_j, m_j}\|_\mu \) to verify the description of the essential spectrum of \( \mathbb{L}_\mu^* \mathbb{L} \).

This completes the proof of Theorem 1. The proof of Theorem 2 proceeds in similar fashion using Theorem 11.

Proof of Theorem 45, part (a). This is a variation on Laplace’s method for asymptotic expansion of integrals. (See for example Chapter 3 of [Mil].)

From (3.10) and (3.7) we have
\[
(5.2) \quad \|L_{n,m}\|_\mu^2 = \frac{I_{n,m,1} \cdot I_{n,m,1}}{I_{n,m,0}^2},
\]
where

\[ I_{n,m,k} = \int_0^1 g_{1,k}^n g_{2,k}^m h_k \, ds, \]
\[ g_{1,k} = r_1^{2k} s^{1-k}, \]
\[ g_{2,k} = r_2^{2k} (1-s)^{1-k}, \]
\[ h_k = \omega^k. \]

Using (2.14) and (2.15) we have

\[ \frac{d}{ds} \left( \log g_{1,k}^n g_{2,k}^m \right) = \left( \frac{2k}{\tilde{p}(s)} + 1 - k \right) \left( \frac{n - (n + m)s}{s(1-s)} \right) \]
\[ = \left( \frac{2k}{\tilde{p}(s)} + 1 - k \right) \left( \frac{n - m}{s - 1-s} \right). \]  

(5.3)

Noting that

\[ \frac{2k}{\tilde{p}(s)} + 1 - k = \begin{cases} 
2/\tilde{p}^*(s) & \text{for } k = -1 \\
1 & \text{for } k = 0 \\
2/\tilde{p}(s) & \text{for } k = 1 
\end{cases} \]

and recalling (2.37a) we see that

\[ C_k \overset{\text{def}}{=} \inf_{0 \leq s \leq 1} \left( \frac{2k}{\tilde{p}(s)} + 1 - k \right) > 0. \]  

(5.4)

(5.5)

It follows easily that \( \log g_{1,k}^n g_{2,k}^m \) takes its maximum value at

\[ s_{n,m} \overset{\text{def}}{=} \frac{n}{n + m}. \]

(We’ll assume for the remainder of this proof that \( n, m > 0 \) and thus \( 0 < s_{n,m} < 1 \)) Integrating (5.3) from \( s_{n,m} \) to \( s \) and applying (5.5) we find that in fact

\[ \log \left( \frac{g_{1,k}^n(s) g_{2,k}^m(s)}{g_{1,k}^n(s_{n,m}) g_{2,k}^m(s_{n,m})} \right) \leq C_k \log \left( \frac{s^n (1-s)^m}{s_{n,m}^n (1-s_{n,m})^m} \right). \]  

(5.6)

We set

\[ A_{n,m,k} = \sqrt{\frac{2nm}{\left( \frac{2k}{\tilde{p}(s_{n,m})} + 1 - k \right) (n + m)^3}}. \]  

(5.7)
(The reader interested in tracing the motivation for the computations to come may wish to note that $A_{n,m,k} = \sqrt{-2/ (\log g_{1,k}^n g_{2,k}^m)''(s_{n,m})}$, though $(\log g_{1,k}^n g_{2,k}^m)''(s)$ may not exist for other values of $s$.)

Using (5.4) and (5.5) it is easy to check that

\begin{equation}
A_{n,m,k} \geq \sqrt{\frac{nm}{(n+m)^2}} = s_{n,m} \sqrt{\frac{m}{n(n+m)}} = (1 - s_{n,m}) \sqrt{\frac{n}{m(n+m)}},
\end{equation}

\begin{equation}
A_{n,m,k} \leq \frac{\sqrt{2}s_{n,m}}{\sqrt[C_k]{n}},
\end{equation}

\begin{equation}
A_{n,m,k} \leq \frac{\sqrt{2}(1 - s_{n,m})}{\sqrt[C_k]{m}},
\end{equation}

\begin{equation}
A_{n,m,k}^2 \leq \frac{2}{C_k s_{n,m}} (1 - s_{n,m}),
\end{equation}

\begin{equation}
A_{n,m,k}^2 \leq \frac{2}{C_k s_{n,m}} (1 - s_{n,m})^2,
\end{equation}

where $C_k$ is as in (5.5). In particular we also have

\begin{equation}
A_{n,m,k} = o(s_{n,m}) \text{ and } A_{n,m,k} = o(1 - s_{n,m}) \text{ as } \min\{n,m\} \to \infty.
\end{equation}

We define functions $g_{n,m,k}$ and $h_{n,m,k}$ on $\mathbb{R}$ by setting

\begin{equation}
g_{n,m,k}(t) = \frac{g_{1,k}^n(s_{n,m} + tA_{n,m,k})g_{2,k}^n(s_{n,m} + tA_{n,m,k})}{g_{1,k}^n(s_{n,m})g_{2,k}^n(s_{n,m})},
\end{equation}

\begin{equation}
h_{n,m,k}(t) = \frac{h_k(s_{n,m} + tA_{n,m,k})}{h_k(s_{n,m})}
\end{equation}

for $t \in J_{n,m,k} \overset{\text{def}}{=} \left(-\frac{s_{n,m}}{A_{n,m,k}}; 1 - \frac{s_{n,m}}{A_{n,m,k}}\right)$, with $g_{n,m,k}(t) = h_{n,m,k}(t) = 0$ otherwise.

Note that $g_{n,m,k}(t)$ assumes a maximum value of 1 at $t = 0$. Note also that from (5.6) and (5.4) and the monotonicity properties of $g_{n,m,0}$ we have

\begin{equation}
g_{n,m,k}(t) \leq g_{n,m,0} \left(\frac{tA_{n,m,k}}{A_{n,m,0}}\right) \leq g_{n,m,0} \left(\frac{t}{\sqrt{2}}\right)
\end{equation}

for $k = \pm 1$. 


We claim that

\[(5.17) \quad g_{n,m,k}(t) \leq e^{C_k(1-2^{-|k|/2}|t|)} \text{ for all } t \text{ when } \min\{n, m\} \geq 2, k = -1, 0, 1.\]

In view of (5.16) it will suffice to prove

\[(5.18) \quad g_{n,m,0}(t) \leq e^{1-|t|} \text{ for all } t \text{ when } \min\{n, m\} \geq 2.\]

This is trivial for \(t \notin J_{n,m,0}\). For \(|t| < 1\) it follows from \(g_{n,m,k}(t) \leq 1\). For \(t \in J_{n,m,0}, |t| > 1\), we use

\[g_{n,m,0}(t) = \left(1 + t \sqrt{\frac{2m}{n(n+m)}}\right)^n \left(1 - t \sqrt{\frac{2n}{m(n+m)}}\right)^m\]

to verify that \((\log g_{n,m,0})''(t) < 0\) so that \(\log g_{n,m,0}\) is concave on \(J_{n,m,k}\). In particular, for \(t \in J_{n,m,0}, t > 1\) we have

\[
\log g_{n,m,0}(t) \leq \log g_{n,m,0}(1) + (\log g_{n,m,0})'(1) \cdot (t - 1) \\
\leq 0 + (\log g_{n,m,0})'(1) \cdot (t - 1) \\
= -\frac{2}{1 + \sqrt{\frac{2m}{n(n+m)}}} \cdot (t - 1) \\
\leq -\frac{2}{1 + \sqrt{\frac{2m}{n(n+m)}}} \cdot (t - 1) \\
\leq -(t - 1)
\]

showing that \((5.18)\) holds. A symmetric argument shows that \((5.18)\) holds also in the remaining case: \(t \in J_{n,m,0}, t < -1\).

Next we consider the pointwise behavior of \(g_{n,m,k}(t)\) as \(\min\{n, m\} \to \infty\). Note that (5.13) guarantees that each fixed \(t\) lies in \(J_{n,m,k}\) when \(\min\{n, m\}\) is large enough. Using (5.3) we have

\[
(\log g_{n,m,k})'(t) = -\left(\frac{2k}{\tilde{p}(s_{n,m} + A_{n,m,k}t)} + 1 - k\right) \\
\cdot \frac{(n + m)A_{n,m,k}^2}{(s_{n,m} + A_{n,m,k}t)(1 - s_{n,m} - A_{n,m,k}t)}
\]
and so

\[
\log g_{n,m,k}(t) = -(n + m)A_{n,m,k}^2 \int_0^t \left( \frac{2k}{\tilde{p}(s_{n,m} + A_{n,m,k}\tau)} + 1 - k \right) \cdot \frac{\tau}{(s_{n,m} + A_{n,m,k}\tau)(1 - s_{n,m} - A_{n,m,k}\tau)} d\tau.
\]

Letting \(\min\{n, m\} \to \infty\) we find with the use of (5.8), (5.10) that the above integral is asymptotic to

\[
\int_0^t \left( \frac{2k}{\tilde{p}(s_{n,m})} + 1 - k \right) \cdot \frac{\tau}{s_{n,m}(1 - s_{n,m})} d\tau
\]

hence

\[
\lim g_{n,m,k}(t) = \lim \exp \left( -(n + m)A_{n,m,k}^2 \left( \frac{2k}{\tilde{p}(s_{n,m})} + 1 - k \right) \cdot \frac{t^2}{2s_{n,m}(1 - s_{n,m})} \right)
\]

(5.19)

\[= e^{-t^2}. \]

Turning now to \(h_{n,m,k}\), see (5.15), we first use (4.4) and (1.8) to verify that \(h_k\) takes the form

(5.20)

\[h_k(s) = \phi(s)s^{B_1}(1 - s)^{B_2}\]

with \(\phi\) positive and continuous on \([0, 1]\) and \(B_1, B_2 \in (-1, 1)\). Then we see

(5.21)

\[h_{n,m,k}(t) \leq C \quad \text{when} \quad -\frac{s_{n,m}}{2A_{n,m,k}} \leq t \leq \frac{1 - s_{n,m}}{2A_{n,m,k}}\]

and using (5.15) and (5.13) we obtain

(5.22) \(h_{n,m,k}(t) \to 1\) as \(\min\{n, m\} \to \infty\), uniformly on bounded sets.

We claim that also

(5.23) \(\int_{-\infty}^{\infty} |h_{n,m,k}(t) - 1| e^{C_k(1-2^{-|k|/2}|t|)} dt \to 0\) as \(\min\{n, m\} \to \infty\).

To see this, decompose the integral into five pieces

\[
\int_{-\infty}^{-\frac{s_{n,m}}{2A_{n,m,k}}} + \int_{-\frac{s_{n,m}}{2A_{n,m,k}}}^{-\frac{1-s_{n,m}}{2A_{n,m,k}}} + \int_{-\frac{1-s_{n,m}}{2A_{n,m,k}}}^{\frac{1-s_{n,m}}{2A_{n,m,k}}} + \int_{\frac{1-s_{n,m}}{2A_{n,m,k}}}^{\frac{s_{n,m}}{2A_{n,m,k}}} + \int_{\frac{s_{n,m}}{2A_{n,m,k}}}^{\infty}.
\]
The first and fifth terms involve intervals on which $h_{n,m,k}$ vanishes, so they reduce to 
$$
\int_{-\infty}^{s_{n,m}} e^{C_k(1+2^{-|k|/2}t)} \, dt 
+ \int_{s_{n,m}}^{\infty} e^{C_k(1-2^{-|k|/2}t)} \, dt,
$$
respectively, and thus tend to zero by (5.13). Using the dominated convergence theorem with the support of (5.21) and (5.22) we see that the third term tends to zero. Setting

$$
v = s_{n,m} + tA_{n,m,k}$$

we find that the second term may be written as

$$
\frac{s_{n,m}}{A_{n,m,k}(1-s_{n,m})} e^{-C_k s_{n,m} \frac{2^{-|k|/2}A_{n,m,k}}{A_{n,m,k}}} \leq \frac{2}{C_k} \left( \frac{s_{n,m}}{A_{n,m,k}} \right)^3 e^{-C_k s_{n,m} \frac{2^{-|k|/2}A_{n,m,k}}{A_{n,m,k}}} \to 0.
$$

(The inequality stems from (5.11).) A similar argument takes care of the fourth term.

We are now ready to compute that

$$
I_{n,m,k} \frac{h_k(s_{n,m}) g^n_{1,k}(s_{n,m}) g^m_{2,k}(s_{n,m})}{A_{n,m,k} h_k(s_{n,m}) g^n_{1,k}(s_{n,m}) g^m_{2,k}(s_{n,m})} = \frac{\int_0^1 g^n_{1,k}(s) g^m_{2,k}(s) h_k(s) \, ds}{A_{n,m,k} h_k(s_{n,m}) g^n_{1,k}(s_{n,m}) g^m_{2,k}(s_{n,m})}
$$

$$
= \int_{-\infty}^{\infty} h_{n,m,k}(t) g_{n,m,k}(t) \, dt
$$

$$
= \int_{-\infty}^{\infty} (h_{n,m,k}(t) - 1) g_{n,m,k}(t) \, dt
$$

$$
+ \int_{-\infty}^{\infty} g_{n,m,k}(t) \, dt
$$

$$
\to 0 + \int_{-\infty}^{\infty} e^{-t^2} \, dt = \sqrt{\pi}
$$

as $\min\{n, m\} \to \infty$, where we have used (5.11) and (5.23) to find the limit of the first term and (5.17), (5.19) and dominated convergence to find the limit of the second term.

Combining these results and simplifying we have

$$
\frac{2}{\sqrt{\hat{p}(s_{n,m}) \hat{p}^*(s_{n,m})}} \left\| I_{n,m} \right\|^2_{L^2} = \frac{A^2_{n,m,0}}{A_{n,m,-1} A_{n,m,1}} \cdot \frac{I_{n,m,-1} \cdot I_{n,m,1}}{I^2_{n,m,0}} \to 1
$$

as $\min\{n, m\} \to \infty$, which implies part (a) of Theorem 45.

□

Lemma 46. Suppose that
• $H$ is a continuous function on $[0, 1]$;
• $H(0) \neq 0$;
• $g$ is a non-negative $C^1$ function on $[0, 1]$ with a strict maximum at 0;
• $g'(0) < 0$;
• $\sigma > -1$.

Then

\[ \int_0^1 g^m(s) s^\sigma H(s) \, ds \sim H(0) \left( \frac{-g'(0)}{g(0)} \right)^{-\sigma-1} \Gamma(\sigma + 1) m^{-\sigma-1} g^m(0) \]

as $m \to \infty$.

Proof. This is a minor variation of Watson’s lemma. (See for example Chapter 2 of [Mil].)

First note that the hypotheses on $g$ imply that $\frac{\log(g(s)/g(0))}{s}$ extends to a negative continuous function on $[0, 1]$ and so $g(s) \leq g(0) \exp(-\epsilon s)$ for some $\epsilon > 0$; thus for some $C > 0$ we have

\[ \left( \frac{g(s/m)}{g(0)} \right)^m s^\sigma H(s/m) \leq C \exp(-\epsilon s) s^\sigma \]

on $[0, m]$. Also note that

\[ m (\log g(s/m) - \log g(0)) \to s (\log g)'(0) = \frac{g'(0)}{g(0)} s \]

as $m \to \infty$.

Using the change of variables formula and the dominated convergence theorem with the support of (5.27) and (5.28) we have

\[
m^{\sigma+1} g^{-m}(0) \int_0^1 g^m(s) s^\sigma H(s) \, ds \\
= \int_0^m \left( \frac{g(s/m)}{g(0)} \right)^m s^\sigma H(s/m) \, ds \\
\to \int_0^\infty \exp \left( \frac{g'(0)}{g(0)} s \right) s^\sigma H(0) \, ds \\
= H(0) \left( -\frac{g(0)}{g'(0)} \right)^{\sigma+1} \int_0^\infty \exp(-s) s^\sigma \, ds \\
= H(0) \left( -\frac{g(0)}{g'(0)} \right)^{\sigma+1} \Gamma(\sigma + 1)
\]

which is equivalent to (5.26). \qed
Proof of Theorem 45, part (b). We focus on the same three integrals as in the proof of part (a).

Let $n_0$ be the common value of the $n_j$. To apply Lemma 46, we use (2.35) and (4.4) to match the integrals to the left-hand side of (5.26) and we use (2.17b) to evaluate $d\log r_2$ and $d\log((1-s)/r_2)$ at $s = 0$. The resulting approximations read as follows:

1. $\int_0^1 s^{n_0}(1-s)^m ds \sim \Gamma(n_0 + 1)m^{-n_0-1}$;
2. $\int_0^1 r_1^{2n_0}r_2^{2m_j}\omega(s) ds \sim \frac{h(0)}{b_2} \frac{2n_0}{c_2^{p_1}} \Gamma\left(\frac{2n_0}{p_1} + 1 + q\left(\frac{1}{p_1} - \frac{1}{p_1^*}\right)\right)$;
3. $\int_0^1 \left(\frac{s}{r_1}\right)^{2n_0} \left(\frac{1-s}{r_2}\right)^{2m_j} \frac{1}{\omega(s)} ds \sim \frac{2n_0}{p_1^{p_1^*}} \Gamma\left(\frac{2n_0}{p_1} + 1 + q\left(\frac{1}{p_1} - \frac{1}{p_1^*}\right)\right)$.

Plugging these results into (3.10) and (3.7) we find that (5.29)

\[
\|L_{n_0,m_j}\|_2^2 \rightarrow \frac{\Gamma\left(\frac{2n_0}{p_1} + 1 + q\left(\frac{1}{p_1} - \frac{1}{p_1^*}\right)\right)}{\Gamma^2(n_0 + 1) \left(\frac{2n_0}{p_1} + 1 + q\left(\frac{1}{p_1} - \frac{1}{p_1^*}\right)\right)}
\]

as claimed.

Proof of Theorem 45, part (a). This is parallel to the proof of part (b).

6. Examples

Example 1. Let $\tilde{p}$ be a smooth map from $[0, 1]$ to $[1, \infty)$ satisfying

1. $\tilde{p}\left(\frac{1}{2}\right) = 1$;
2. $\tilde{p}(s) > 1$ for $s \neq \frac{1}{2}$;
3. $\tilde{p}(s) \equiv 2$ for $s$ near 0 and for $s$ near 1.

Let $D$ be the domain generated by $\tilde{p}, 1, 1$ as in (2.19) and Definition 10. We claim that $D$ has the following properties:
(6.1a) $D$ is a convex Reinhardt domain with $C^\infty$-smooth boundary;

(6.1b) $D \notin \mathbb{R}$;

(6.1c) $D$ is strictly convex (i.e., $bD$ contains no line segments);

(6.1d) the principal curvatures $\kappa_1$ and $\kappa_2$ are strictly positive, but $\kappa_3$ vanishes precisely on the torus corresponding to $s = 1/2$;

(6.1e) the conditions of Theorem 27 are still equivalent and thus can still be used to define the notion of an admissible measure as in Definition 28;

(6.1f) measures of order $q$ are admissible for all $q \in \mathbb{R}$;

(6.1g) $L_D$ fails to be bounded on $L^2(bD, \mu)$ when $\mu$ is any admissible measure given by $\frac{1}{4\pi^2} \omega(s) \, ds \, d\theta_1 \, d\theta_2$ with $\omega(s)$ positive and continuous for $s \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$.

The first four items follow easily from material in the beginning of §2 along with (4.5) (see in particular Theorem 9 to check item (6.1b)). Item (6.1e) follows from the continued validity of the conclusions of Lemmas 12 and 14. Item (6.1f) follows from (4.3). (In fact, the value of $q$ is irrelevant here.)

To verify item (6.1g), consider a sequence $(n_j, m_j)$ in $\mathbb{N} \times \mathbb{N}$ with $\frac{n_j}{m_j} \to u \in (0, 1) \cup (1, \infty)$. Referring to the proof of Theorem 45, part (a) and in particular to (5.24) we find that

$$
\lim \inf \frac{I_{n_j,m_j,k}}{A_{n_j,m_j,k} h_k(s_{n_j,m_j}) g_{1,k}(s_{n_j,m_j}) g_{2,k}(s_{n_j,m_j})} \geq \int_{-1}^{1} e^{-t^2} \, dt
$$

for $k = -1, 1$. Combining as in (5.25) we find that

$$
\lim \inf \|L_{n_j,m_j}\|_\mu^2 \geq \frac{\sqrt{\bar{\rho} \left( \frac{u}{1+u} \right) \bar{\rho}^* \left( \frac{u}{1+u} \right)}}{5}.
$$

Since the right-hand side above approaches infinity as $u \to 1$ we see from Theorem 30 that $L$ fails to be bounded on $L^2(bD, \mu)$.

**Example 2.** Pick $0 < \nu < 1$ and let $\bar{\rho}$ be a continuous map from $[0, 1]$ to $[2, \infty]$ satisfying

- $\bar{\rho}(s) = (s - \frac{1}{2})^{-\nu}$ for $s$ near $\frac{1}{2}$;
- $\bar{\rho}(s)$ is finite and smooth for $s \neq \frac{1}{2}$;
- $\bar{\rho}(s) \equiv 2$ for $s$ near $0$ and for $s$ near $1$.

Let $D$ be the domain generated by $\bar{\rho}, 1, 1$ as in (2.19) and Definition 10. We claim that $D$ has the following properties:

(6.2a) $D$ is a convex Reinhardt domain;

(6.2b) $bD$ is of class $C^{1,1/\nu}$ (but not better);
(6.2c) $bD$ is of class $C^\infty$ away from the torus corresponding to $s = \frac{1}{2}$;
(6.2d) $D \notin \tilde{\mathcal{R}}$;
(6.2e) the principal curvatures $\kappa_1$ and $\kappa_2$ have positive lower and upper bounds, while $\kappa_3$ has a positive lower bound but tends to infinity as we approach the torus corresponding to $s = \frac{1}{2}$;
(6.2f) the conditions of Theorem 27 are still equivalent and thus can still be used to define the notion of an admissible measure as in Definition 28;
(6.2g) measures of order $q$ are admissible if and only if $|q| < 1/\nu$;
(6.2h) $L_D$ fails to be bounded on $L^2(bD, \mu)$ when $\mu$ is any admissible measure given by $\frac{1}{4\pi^2} \omega(s) \, ds \, d\theta_1 \, d\theta_2$ with $\omega(s)$ positive and continuous for $s \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$.

Item (6.2c) is clear from the construction. Item (6.2d) may be verified with the use of the series expansion

$$\phi(r_1) = \phi(r_1^*) - (r_1 - r_1^*) - Q(r_1 - r_1^*)^{1 + \frac{1}{\nu + 1}} + \ldots$$

where $r_1^*$ is the value of $r_1$ corresponding to $s = \frac{1}{2}$ and $Q$ is a positive constant.

The other items are verified as in Example 1.

We note that $D$ is strongly convex in the sense of [Pol].

**Example 3.** Let

$$\tilde{p}(s) = \frac{\log(10/s)}{\log(10/s) - 1/2}, \quad b_2 = 1, \quad b_1 = \sqrt{\log 10}$$

and let $D$ be the domain generated by $\tilde{p}, b_1, 1$ as in (2.19) and Definition 10. We claim that $D$ has the following properties:

(6.3a) $D$ is a convex Reinhardt domain;
(6.3b) $bD$ is of class $C^1$ but not in any stronger Hölder class;
(6.3c) $D \in \tilde{\mathcal{R}} \setminus \mathcal{R}$;
(6.3d) the conditions of Theorem 27 are still equivalent and thus can still be used to define the notion of an admissible measure as in Definition 28;
(6.3e) surface measure is not admissible;
(6.3f) measures of order $q$ are admissible for $|q| < 1$;
(6.3g) $L_D$ fails to be bounded on $L^2(bD, \mu)$ when $\mu$ is any admissible measure given by $\frac{1}{4\pi^2} \omega(s) \, ds \, d\theta_1 \, d\theta_2$ with $\omega(s)$ positive and continuous for $s \in (0, 1)$. 
Note that
\[ \dot{p^*}(s) = 2 \log(10/s), \]
\[ \frac{1}{sp(s)} = \frac{d}{ds} \left( \log s + \frac{1}{2} \log(\log(10/s)) \right), \]
\[ \frac{1}{sp^*(s)} = \frac{d}{ds} \left( -\frac{1}{2} \log(\log(10/s)) \right). \]

It is easy to check now that conditions (2.20a) through (2.20d) hold but (2.37a) fails, showing that (6.3c) holds.

Away from the \( \zeta_2 \)-axis \( D \) behaves like a domain in \( \mathbb{R} \).

To understand the behavior near the \( \zeta_2 \)-axis we note that
\[ r_1 = s \sqrt{\log(10/s)} \]
(by (2.17a)), while \( r_2 \to 1 \) as \( s \to 0 \). Item (6.3b) can now be deduced from (2.12). Using (4.3) we see that a measure of order \( q \) takes the form (3.9) with \( \omega(s) \) a positive continuous multiple (near \( s = 0 \)) of
\[ (s \log(10/s))^q; \]
it follows easily that such a measure is admissible if and only if \( |q| < 1 \), establishing (6.3e) and (6.3f).

The proof of (6.3g) goes along the same lines as the proof of (6.1g), but this time we let \( u \) approach 0.

The other items are verified as in the previous examples.

7. Duality

Given a bounded convex Reinhardt domain \( D \subset \mathbb{C}^2 \), the polar of \( D \) is the bounded convex Reinhardt domain
\[ (7.1) \quad D^* \overset{\text{def}}{=} \{ z \in \mathbb{C}^2 : \text{Re} \langle z, \zeta \rangle < 1 \text{ for all } \zeta \in D \}, \]
where \( \langle \cdot, \cdot \rangle \) denotes the standard Hermitian inner product on \( \mathbb{C}^2 \).

For \( \zeta \in bD \) there is \( z \in bD^* \) satisfying \( \text{Re} \langle z, \zeta \rangle = 1 \); the rotational symmetries of \( D \) in fact imply that
\[ (7.2) \quad \langle z, \zeta \rangle = 1. \]

Now assume \( bD \) is \( C^1 \)-smooth; then (7.2) uniquely determines \( z \in bD^* \) (the tangent space to \( bD \) at \( \zeta \) is given by \( \text{Re} \langle z, \zeta \rangle = 1 \)).

Assume further that \( D \) is strictly convex (i.e., \( bD \) contains no line segments). Then the map \( T : bD \to bD^* \) defined by \( T(\zeta) = z \) is injective (since, for \( z \) and \( \zeta \) as in (7.2), we have \( \{ \eta \in \overline{D} : \text{Re} \langle z, \eta \rangle = 1 \} = \{ \zeta \} \)). Compactness arguments show that \( T \) is a homeomorphism. It is easy to check that \( T \) restricts to a homeomorphism from \( \gamma = \gamma_D \)
to $\gamma^* \overset{\text{def}}{=} \gamma^*$ (see (2.1)); moreover, the restriction of $T$ to $\gamma$ determines the whole map $T$ via the formula

$$T : (r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \mapsto (T_1(r_1, r_2)e^{i\theta_1}, T_2(r_1, r_2)e^{i\theta_2}).$$

As before, we set $\gamma_+ = \gamma \cap \mathbb{R}^2_+$ and $\gamma^*_+ = \gamma^* \cap \mathbb{R}^2_+$.

**Theorem 47.** Suppose $D \in \tilde{\mathcal{R}}$. Then the following will hold.

(a) $D^* \in \tilde{\mathcal{R}}$.
(b) If $D \in \mathcal{R}$ then $D^* \in \mathcal{R}$.
(c) The mapping $T \big|_{\gamma^*_+} : \gamma^*_+ \to \gamma^*_+ \text{ is a } C^1 \text{-smooth diffeomorphism.}$
(d) The following relations hold along $\gamma^*_+$:

\begin{align*}
(7.3) & \quad r_1 \cdot (r_1 \circ T) + r_2 \cdot (r_2 \circ T) = 1 \\
(7.4) & \quad (r_1 \circ T) dr_1 + (r_2 \circ T) dr_2 = 0 \\
(7.5) & \quad r_1 d(r_1 \circ T) + r_2 d(r_2 \circ T) = 0 \\
(7.6) & \quad r_1 \cdot (r_1 \circ T) = s \\
(7.7) & \quad r_2 \cdot (r_2 \circ T) = 1 - s \\
(7.8) & \quad s \circ T = s \\
(7.9) & \quad \frac{1}{p} + \frac{1}{p \circ T} = 1.
\end{align*}

(e) If $D$ is generated by $\tilde{p}, b_2, b_1$ then $D^*$ is generated by $\tilde{p}^*, b_2^{-1}, b_1^{-1}$.
(f) If $d\mu = \omega(s) ds d\theta_1 d\theta_2$ describes an admissible measure on $bD$ then $d\tilde{\mu} = \frac{1}{\omega(s)} ds d\theta_1 d\theta_2$ describes an admissible measure on $bD^*$.
(g) The operator

$$U_\mu : f \mapsto (f \circ T) \cdot \omega^{-1}$$

defines an isometry between $L^2(bD^*, \tilde{\mu})$ and $L^2(bD, \mu)$.
(h) If $\mathbb{L}_D$ is bounded on $L^2(bD, \mu)$ then $\mathbb{L}_{D^*}$ is bounded on $L^2(bD^*, \tilde{\mu})$ and the isometry $U_\mu$ intertwines $\mathbb{L}$ with its adjoints in the following sense:

$$\mathbb{L}_{D^*\mu} \circ U_\mu = U_\mu \circ \mathbb{L}_{D^*}$$
$$\mathbb{L}_D \circ U_\mu = U_\mu \circ \mathbb{L}_{D^*\mu}.$$

The norm of $\mathbb{L}_{D^*}$ on $L^2(bD^*, \tilde{\mu})$ equals the norm of $\mathbb{L}_D$ on $L^2(bD, \mu)$, and the spectral data for $\mathbb{L}_{D^*\mu}$ and $\mathbb{A}_{\tilde{\mu}}$ on $bD^*$ match the spectral data for $\mathbb{L}_{D^*\mu}$ and $\mathbb{A}_\mu$ on $bD$, respectively.

**Proof.** The relation (7.3) follows from (7.2).

Holding $z$ fixed in (7.2) and differentiating with respect to $\zeta \in \gamma_+$ we obtain (7.3).
Solving (7.2) and (7.4) for \( r_1 \circ T \) and \( r_2 \circ T \) and recalling (2.10) we obtain

\[
\begin{align*}
    r_1 \circ T &= \frac{dr_2}{r_1 dr_2 - r_2 dr_1} = \frac{s}{r_1} \\
    r_2 \circ T &= \frac{-dr_1}{r_1 dr_2 - r_2 dr_1} = \frac{1 - s}{r_1}
\end{align*}
\]

establishing (7.6) and (7.7). It follows that \( T \) is a \( C^1 \)-smooth map from \( \gamma^+ \) to \( \gamma^*+ \); thus we may also hold \( \zeta \) fixed in (7.2) and differentiate with respect to \( z \) to obtain (7.5). Define \( s \) on \( \gamma^*+ \) by using the middle third of (2.10); applying (7.5), (7.6) and (7.7) to (2.10) we verify (7.8).

To verify (7.9) we note that from (2.14), (7.6) and (7.8) we have

\[
\begin{align*}
    \frac{1}{p} + \frac{1}{p o T} &= \frac{d \log r_1}{d \log s} + \frac{d \log(r_1 \circ T)}{d \log(s \circ T)} \\
    &= \frac{d \log r_1}{d \log s} + \frac{d \log s - d \log r_1}{d \log s} \\
    &= 1.
\end{align*}
\]

From (7.8) and (7.9) we obtain \( \tilde{p}_{D^*} = \tilde{p}_D \). Parts (a) and (b) of the current theorem follow now from Theorems 9 and 19; using the limits from the proof of Lemma 14 to sort out the \( b_j \)s we also obtain (e).

Reversing our reasoning we see that \( T^{-1} \) is also \( C^1 \)-smooth on \( \gamma^*+ \). Item (f) is an immediate consequence of Definition 28, item (3.16a) of Theorem 27 and the relation (7.8).

A direct computation shows that the operator \( U_\mu \) defined in item (g) is norm-preserving.

The intertwining relations in item (h) are verified by checking each Fourier piece using (3.11), (3.12) and (3.18). The remaining claims in (h) follow from the isometric nature of \( U_\mu \) and general principles. □

Remark 48. Aspects of the duality presented here are treated for smooth strongly \( \mathbb{C} \)-linearly convex domains in arbitrary dimension without the Reinhardt assumption in [Bar2].

8. Closing remarks

(A) The following result highlights the special role played by the measure \( \mu_0 \) given by \( d\mu_0 = \frac{1}{4\pi} ds d\theta_1 d\theta_2 \).

Proposition 49. Let \( D \in \widetilde{\mathcal{R}} \) and suppose that \( L \) is bounded on \( L^2(bD, \mu) \) for some admissible measure \( \mu \) on \( bD \). Then the following conditions are equivalent.
(8.1a) The restriction of the operator $L^*_\mu L$ to the orthogonal complement of its kernel admits an orthogonal basis of eigenfunctions that consists of $(n,m)$-monomials and is closed under multiplication.

(8.1b) The measure $\mu$ is a constant multiple of $\mu_0$.

Proof. Suppose that (8.1b) holds. Then referring to identity (3.11) and Proposition 33 we see that the functions
\[
\left\{ \left( \frac{s}{r_1} \right)^n \left( \frac{1-s}{r_2} \right)^m e^{i(n\theta_1+m\theta_2)} : n, m \geq 0 \right\}
\]
provide the desired basis.

Suppose now that (8.1a) holds. Referring again to (3.11) and Proposition 33 we see that our basis must contain constant multiples of the eigenfunctions $\frac{1}{\omega(s)}$ and $\frac{1}{r_1 \omega(s)} e^{i\theta_1}$, and that furthermore the product of these two eigenfunctions must be a constant multiple of the second eigenfunction. It follows that $\omega(s)$ is constant, as claimed. □

(B) The methods we have employed here rely significantly on the circular symmetry of complete Reinhardt domains. We plan to examine in a future paper the question of which of our results generalize to non-Reinhardt $\mathbb{C}$-linearly convex domains. Of course, it will also be interesting to see what happens in higher dimension.

References

[Aiz] L. A. Aizenberg, Integral representations of functions which are holomorphic in convex regions of $\mathbb{C}^n$ space, Soviet Math. (Dokl. Akad, Nauk) 151 (1963), 1149–1152.

[APS] M. Andersson, M. Passare and R. Sigurdsson, Complex convexity and analytic functionals, Progress in Mathematics 225, Birkhäuser Verlag, 2004.

[Bar1] D. Barrett, A floating body approach to Fefferman’s hypersurface measure, Math. Scand., 98 (2006), 69-80.

[Bar2] D. Barrett, Holomorphic projection and duality for domains in complex projective space, \texttt{arXiv:0810.0858} preprint.

[Bol1] M. Bolt, Spectrum of the Kerzman-Stein operator for model domains, Integral Equations Operator Theory 50 (2004), 305–315.

[Bol2] M. Bolt, A geometric characterization: complex ellipsoids and the Bochner-Martinelli kernel, Illinois J. Math. 49 (2005), 811–826.

[Bol3] M. Bolt, The Möbius geometry of hypersurfaces, Michigan Math. J. 56 (2008), 603-622.

[BL] A. Bonami and N. Lohoué, Projecteurs de Bergman et Szegő pour une classe de domaines faiblement pseudo-convexes et estimations $L^p$ Compositio Math. 46 (1982), no. 2, 159–226.
[CM] C. Cowen and B. MacCluer, Composition Operators on Spaces of Analytic Functionals, CRC Press, 1995.

[Fef] C. Fefferman, Parabolic invariant theory in complex analysis, Adv. in Math. 31 (1979), 131–262.

[Han] T. Hansson, On Hardy spaces in complex ellipsoids, Ann. Inst. Fourier (Grenoble) 49 (1999), 1477–1501.

[Hen] G. M. Henkin, Integral representation of functions which are holomorphic in strictly pseudoconvex regions, and some applications, (Russian) Mat. Sb. (N.S.) 78 (1969), 611–632.

[HeLe] G. Henkin and J. Leiterer, Theory of functions on complex manifolds, Birkhäuser, 1984.

[HR] N. Higson and N. Roe, Analytic K-Homology, Oxford Math. Monographs (2000).

[KS1] N. Kerzman and E. M. Stein, The Szegő kernel in terms of Cauchy-Fantappié kernels, Duke Math. J. 45 (1978), 197–224.

[KS2] N. Kerzman and E. M. Stein, The Cauchy kernel, the Szegő kernel, and the Riemann mapping function, Math. Ann. 236 (1978), 85–93.

[Kra] S. Krantz, Function theory of several complex variables (2nd ed.), Wadsworth & Brooks/Cole, 1992.

[LL] E. Lieb and M. Loss, Analysis (2nd ed.), American Mathematical Society, 2001.

[LS] L. Lanzani and E. M. Stein, Cauchy-type integrals on non-smooth domains in C^n, in preparation.

[Ler] J. Leray, Le calcul différentiel et intégral sur une variété analytique complexe (Problème de Cauchy, III, Bull. Soc. Math. France 82 (1959), 6–180.

[Met] G. Metivier, Spectral asymptotics for the H∞-Neumann problem, Duke Math. J. 48 (1981), 779–806.

[Mil] P. Miller, Applied Asymptotic Analysis, American Mathematical Society, 2006.

[Nor] F. Norguet, Répresentations intégrales des fonctions de plusieurs variables complexes, C. R. Acad. Sci. Paris 250 (1960), 1780–1782.

[ÖT] H. Özbay and A. Tannenbaum, A skew Toeplitz approach to the H∞ optimal control of multivariable distributed systems, SIAM J. Control Optim. 28 (1990), 653–670.

[Pei] V. Peller, Hankel operators and their applications, Springer Monographs in Math. (2003).

[Pol] E. S. Polovinkin, Strongly convex analysis, (Russian) Mat. Sb. 187 (1996), 103–130; translation in Sb. Math. 187 (1996), 259–286.

[Ram] E. Ramírez de Arellano, Ein Divisionsproblem und Randintegraldarstellungen in der komplexen Analysis, Math. Ann. 184 (1969/1970), 172–187.

[Ran] R. M. Range, Holomorphic functions and integral representations in several complex variables, Springer-Verlag 1986.

[RS] M. Reed and B. Simon, Methods of modern mathematical physics, Vol I: Functional analysis, revised and enlarged edition, Academic Press, 1980.

DEPT. OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109-1043 USA
