CYCLIC COVERINGS OF VIRTUAL LINK DIAGRAMS

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Abstract. A virtual link diagram is called mod \( m \) almost classical if it admits an Alexander numbering valued in integers modulo \( m \), and a virtual link is called mod \( m \) almost classical if it has a mod \( m \) almost classical diagram as a representative. In this paper, we introduce a method of constructing a mod \( m \) almost classical virtual link diagram from a given virtual link diagram, which we call an \( m \)-fold cyclic covering diagram. The main result is that \( m \)-fold cyclic covering diagrams obtained from two equivalent virtual link diagrams are equivalent. Thus we have a well-defined map from the set of virtual links to the set of mod \( m \) almost classical virtual links. Some applications are also given.

1. Introduction

Virtual links, introduced by L. H. Kauffman [12], correspond to abstract links [9] and stable equivalence classes of links in thickened surfaces [2, 9]. A virtual link diagram is called almost classical if it admits an Alexander numbering (cf. [16]), and it is called mod \( m \) almost classical if it admits an Alexander numbering in \( \mathbb{Z}_m \) (cf. [11]). A virtual link is called almost classical (resp. mod \( m \) almost classical) if it has an almost classical (resp. mod \( m \) almost classical) virtual link diagram as a representative. Every classical link diagram is almost classical, and every almost classical virtual link diagram is mod \( m \) almost classical. A virtual link diagram is checkerboard colorable if and only if it is mod 2 almost classical. It is known that Jones polynomials of mod 2 almost classical virtual links have a property that Jones polynomials of classical links have ([5, 6]). Alexander polynomials for mod \( m \) almost classical virtual links can be defined in a similar way to those for almost classical link diagrams [1].

In this paper, we introduce the notion of an oriented cut point and a cut system for a virtual link diagram, which is an extension of (unoriented) cut points introduced by H. Dye in [3, 4]. For any pair \((D, P)\) of a virtual link diagram \( D \) and a cut system \( P \), we construct a virtual link diagram \( \varphi_m(D, P) \) which is mod \( m \) almost classical. We call it an \( m \)-fold cyclic covering (virtual link) diagram of \((D, P)\).

It turns out that the strong equivalence class of \( \varphi_m(D, P) \) does not depend on \( P \), namely, for any cut systems \( P \) and \( P' \) of the same virtual link diagram \( D \), \( \varphi_m(D, P) \) and \( \varphi_m(D, P') \) are strongly equivalent (Lemma 5). Our main theorem (Theorem 6) states that if virtual link diagrams \( D \) and \( D' \) are equivalent, then \( \varphi_m(D, P) \) and \( \varphi_m(D', P') \) are

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equivalent. Thus, we obtain a well-defined map from the set of virtual links to the set of mod \( m \) almost classical virtual links.

As an application, we demonstrate how Theorem 6 is used to show that two virtual link diagrams are not equivalent. Theorem 6 implies Theorem 10 that if \( \varphi_m(D, P) \) is not equivalent to a disjoint union of \( m \) copies of \( D \) itself then \( D \) is never equivalent to a mod \( m \) virtual link diagram, i.e., the virtual link represented by \( D \) is not mod \( m \) almost classical.

This paper is organized as follows: In Section 2 we recall virtual link diagrams and Alexander numberings, and introduce the notions of an oriented cut point and a cut system. In Section 3 we give a method of construction of \( \varphi_m(D, P) \). It is shown that \( \varphi_m(D, P) \) is a mod \( m \) almost classical virtual link diagram. In Section 4, main results, Lemma 5 and Theorem 6, are introduced and proved. In Section 5 we give an alternative method of constructing cyclic covering virtual link diagrams. In Section 6 we show some applications.

2. Alexander numberings and cut systems

In this section we recall virtual link diagrams and Alexander numberings, and introduce the notions of an oriented cut point and a cut system, which are used for our construction of cyclic covering diagrams.

A \textit{virtual link diagram} is a generically immersed, closed and oriented 1-manifold in \( \mathbb{R}^2 \) with information of positive, negative or virtual crossing, on each double point. Here a \textit{virtual crossing} means an encircled double point without over-under information \cite{12}. \textit{Generalized Reidemeister moves} are the local moves depicted in Figure 1. The 3 moves on the top are \textit{(classical) Reidemeister moves} and the 4 moves on the bottom are so-called \textit{virtual Reidemeister moves}. Two virtual link diagrams \( D \) and \( D' \) are said to be equivalent (resp. strongly equivalent) if they are related by a finite sequence of generalized Reidemeister moves (resp. virtual Reidemeister moves) and isotopies of \( \mathbb{R}^2 \). A \textit{virtual link} (resp. a \textit{pre-virtual link}) is an equivalence class (resp. a strong equivalence class) of virtual link diagrams.

A \textit{virtual path} of a virtual link diagram \( D \) means a path (possibly a loop) on \( D \) on which there are no classical crossings. A virtual link diagram \( D' \) is said to be obtained from \( D \) by a \textit{detour move} if \( D' \) is obtained by replacing a virtual path of \( D \) with a path which is a virtual path of \( D' \). Two diagrams \( D \) and \( D' \) are strongly equivalent if and only if they are related by a finite sequence of detour moves and isotopies of \( \mathbb{R}^2 \) (cf. \cite{9, 12}).

Let \( D \) be a virtual link diagram. A \textit{semi-arc} of \( D \) is a virtual path which is an immersed arc between two classical crossings of \( D \) or an immersed loop. Let \( m \) be a non-negative integer. An \textit{Alexander numbering} (resp. a \textit{mod} \( m \) \textit{Alexander numbering}) of \( D \) is an assignment of a number of \( \mathbb{Z} \) (resp. \( \mathbb{Z}_m \)) to each semi-arc of \( D \) such that the numbers of 4 semi-arcs around each classical crossing are as shown in Figure 2 for some \( i \in \mathbb{Z} \) (resp. \( i \in \mathbb{Z}_m \)).
Reidemeister moves

Virtual Reidemeister moves

Figure 1. Generalized Reidemeister moves

Figure 2. Alexander numbering

Note that the numbers assigned to semi-arcs around a virtual crossing is depicted as in Figure 3.

Figure 3. Alexander numbering around a virtual crossing

An example of an Alexander numbering is depicted in Figure 4. A classical link diagram always admits an Alexander numbering.

Figure 4. An Alexander numbering of a classical link diagram

Not every virtual link diagram admits an Alexander numbering. The virtual link diagram depicted in Figure 5 (i) does not admit an Alexander numbering, and the virtual link diagram in Figure 5 (ii) does.

Figure 5 shows an example of a mod 3 Alexander numbering, which is not an Alexander numbering.

A virtual link diagram is almost classical (resp. mod m almost classical) if it admits an Alexander numbering (resp. a mod m Alexander numbering). A virtual link L is almost classical (resp. mod m almost classical) if there is an almost classical (resp. mod m almost classical) virtual link diagram of L.
Figure 5. Virtual link diagrams which does/does not admit an Alexander numbering

Figure 6. An mod 3 Alexander numbering of a virtual link diagram

H. Boden, R. Gaudreau, E. Harper, A. Nicas, L. White [1] studied mod $m$ almost classical virtual links. By definition, any almost classical virtual link diagram is mod $m$ almost classical. A virtual link diagram is checkerboard colorable if and only if it is mod 2 almost classical. It is shown in [1] that for a mod $m$ almost classical virtual knot $K$, if $D$ is a minimal virtual knot diagram of $K$, then $D$ is mod $m$ almost classical.

H. Dye introduced the notion of a cut point [3], which is an ‘unoriented’ cut point in our sense. The author [6] generalized the Kauffman-Murasugi-Thistlethwaite theorem ([11, 14, 15]) on the span of the Jones polynomial of a classical link to checkerboard colorable and proper virtual links. Using cut points, H. Dye [4] further extended this result to virtual link diagrams that are not checkerboard colorable.

Using (unoriented) cut points, the author constructed in [7, 8] a map from the set of virtual links to the set of checkerboard colorable virtual links, i.e., the set of mod 2 almost classical virtual links. In this paper, we generalize this to the mod $m$ case.

An oriented cut point or simply a cut point is a point on an arc at which a local orientation of the arc is given. In this paper we denote it by a small triangle on the arc as in Figure 7. Whenever cut points on a virtual link diagram are discussed, we assume that they are on semi-arcs of the diagram avoiding crossings. An oriented cut point is called coherent (resp. incoherent) if the local orientation indicated by the cut point is coherent (resp. incoherent) to the orientation of the virtual link diagram.

Figure 7. An oriented cut point on an arc

Let $D$ be a virtual link diagram and $P$ a set of oriented cut points of $D$. We say that $P$ is a cut system if $D$ admits an Alexander numbering such that at each oriented cut point,
the number increases by one in the direction of the oriented cut point (Figure 8). Such an Alexander numbering is called an Alexander numbering of a virtual link diagram with a cut system. See Figure 9 for examples.

![Figure 8. Alexander numbering of a virtual link diagram with a cut system](image)

![Figure 9. Alexander numberings of virtual link diagrams with cut systems](image)

For a virtual link diagram $D$ with a cut system $P$, let $\text{Arc}(D, P)$ be the set of arcs (or loops) obtained from semi-arcs of $D$ by cutting along $P$. (If there is a semi-arc of $D$ which is a loop and has no cut points of $P$, then $\text{Arc}(D, P)$ has the loop as an element.) An Alexander numbering of $D$ with $P$ is regarded as a map from $\text{Arc}(D, P)$ to $\mathbb{Z}$. For a semi-arc $a$ of $D$ not being a loop, we denote by $a^-$ (resp. $a^+$) the arc of $\text{Arc}(D, P)$ which contains the starting point (resp. the terminal point) of $a$.

**Lemma 1.** Let $f : \text{Arc}(D, P) \to \mathbb{Z}$ be an Alexander numbering of a virtual link diagram $D$ with a cut system $P$.

1. For any semi-arc $a$ of $D$ not being a loop, $f(a^+) - f(a^-)$ is the number of coherent cut points minus the number of incoherent cut points of $P$ appearing on $a$.
2. For any semi-arc $a$ of $D$ being loop, the number of coherent cut points minus the number of incoherent cut points of $P$ appearing on $a$ is 0.

**Proof.** It is obvious, since when we move along $a$ from $a^-$ to $a^+$, the numbers assigned by $f$ changes by $+1$ (resp. $-1$) at each coherent (resp. incoherent) cut point.

A canonical cut system of a virtual link diagram is a cut system which is obtained by introducing two oriented cut points as in Figure 10 around each classical crossing. It is really a cut system and an Alexander numbering looks as in Figure 10 around each virtual crossing.

The local transformations of oriented cut points depicted in Figure 11 are called oriented cut point moves. For a virtual link diagram with a cut system, the result by an oriented
cut point move is also a cut system of the same virtual link diagram. Note that the move III′ depicted in Figure 11 is obtained from the move III modulo the moves II.

Figure 11. Oriented cut moves

Theorem 2. Two cut systems of the same virtual link diagram are related by a sequence of oriented cut point moves.

Proof. Let $P$ and $P'$ be cut systems of a virtual link diagram $D$. Let $f$ (resp. $f'$) be an Alexander numbering of $D$ with cut system $P$ (resp. $P'$). Applying a finite number of oriented cut point moves III to $P$, we obtain a cut system $P''$ and an Alexander numbering $f''$ such that the numberings of 4 edges around each classical crossing are as same as those of $f'$. By Lemma 1, we see that for any semi-arc $a$ of $D$, the number of coherent cut points minus the number of incoherent cut points of $P''$ appearing on $a$ is equal to that of $P'$. Thus, by using oriented cut point moves I and II, $P''$ can be transformed to $P'$. □

Corollary 3. Let $D$ be a virtual link diagram and let $P$ be a cut system of $D$. The number of coherent cut points of $P$ equals that of incoherent cut points of $P$.

Proof. The canonical cut system for $D$ has the property that the number of coherent cut points equals that of incoherent cut points. Since each oriented cut point move preserves this property, by Theorem 2 we see that any cut system has the property. □

3. CYCLIC COVERINGS OF VIRTUAL LINK DIAGRAMS

In this section, we introduce a method of constructing a mod $m$ almost classical virtual link diagram $\varphi_m(D, P)$, which is determined up to strong equivalence, from a virtual link diagram $D$ with a cut system $P$.

We denote by a pair $(D, P)$ a virtual link diagram $D$ with a cut system $P$. Moving $(D, P)$ slightly by an isotopy of $\mathbb{R}^2$, we assume that each cut point $p$ of $P$ is on a horizontal line $\ell(p)$ in $\mathbb{R}^2$ such that $\ell(p)$ intersects $D$ transversely avoiding all crossings of $D$ and $p$ is a unique cut point of $P$ on $\ell(p)$. Let $(D^0, P^0), (D^1, P^1), \ldots, (D^{m-1}, P^{m-1})$ be $m$ parallel copies of $(D, P)$ with $(D^0, P^0) = (D, P)$ obtained from $(D, P)$ by sliding along the $x$-axis such that they appear from left to right in this order. For each cut point $p \in P$, we denote
by $p^k$ the copy of $p$ in $P^k$ for $k \in \{0, \ldots, m-1\}$. See Figure 12 for an example. (The Alexander numberings in the figure are used later.)

![Diagram](image)

**Figure 12.** 3 copies of a virtual link diagram with cut system

For each $p \in P$, let $N(\ell(p))$ be a regular neighborhood of the horizontal line $\ell(p)$ in $\mathbb{R}^2$. In Figure 13, $N(\ell(p))$ is the part between two dotted lines parallel to $\ell(p)$. The diagram $\bigcup_{k=0}^{m-1} D^k$ looks locally near $N(\ell(p))$ as in the upper part of Figure 13. Replace it as in the lower part of the figure for every $p \in P$, where the dotted arc drawn in the very bottom of the figure means a virtual path and we may put it anywhere as long as it contains only virtual crossings. The virtual link diagram obtained this way is denoted by $\varphi_m(D, P)$ and is called an $m$-fold cyclic covering (virtual link) diagram of $(D, P)$.

In the early stage of this construction, we modified $(D, P)$ by an isotopy of $\mathbb{R}^2$. When we modify $(D, P)$ differently, the diagram $\varphi_m(D, P)$ may change. However, it is preserved up to strong equivalence. Although this fact can be seen by observing how the diagram $\varphi_m(D, P)$ changes by a modification of $(D, P)$, we will show it in a more general situation as Theorem 7 in Section 5.

![Diagram](image)

**Figure 13.** Construction of $m$ cyclic covering

For example, for $(D, P)$ depicted in Figure 12 (i), a 3-fold cyclic covering virtual link diagram $\varphi_3(D, P)$ is shown in Figure 14.
Proposition 4. For a virtual link diagram $D$ with a cut system $P$, an $m$-fold cyclic covering virtual link diagram $\varphi_m(D, P)$ is mod $m$ almost classical.

Proof. Let $f$ be an Alexander numbering of $(D, P)$. For each $k \in \{0, \ldots, m - 1\}$, let $f^k$ denote the Alexander numbering of $(D^k, P^k)$ obtained from $f$ by shifting $k$. As shown in Figure 15, the Alexander numberings $f_0, \ldots, f_{m-1}$ induce a mod $m$ Alexander numbering of $\varphi_m(D, P)$. For example, see Figures 12 and 14. □

Figure 15. Alexander numbering of a cyclic covering virtual link diagram

4. The main theorem

In Section 3 we introduced an $m$-fold cyclic covering diagram $\varphi_m(D, P)$ for a virtual link diagram $D$ with a cut system $P$. In this section, we first show that $\varphi_m(D, P)$, up to strong equivalence, does not depend on $P$ (Lemma 5). Hence we may denote it by $\varphi_m(D)$. Our main theorem is that if $D$ and $D'$ are equivalent then $\varphi_m(D, P)$ and $\varphi_m(D', P')$ are equivalent (Theorem 6). This implies that we have a map $\varphi_m$ from the set of virtual links to the set of mod $m$ almost classical virtual links.

Lemma 5. Let $D$ be a virtual link diagram, and let $P_1$ and $P_2$ be cut systems of $D$. Then $\varphi_m(D, P_1)$ and $\varphi_m(D, P_2)$ are strongly equivalent.
Proof. Suppose that $P_1$ and $P_2$ are as in the left part of Figure 16. Then $\varphi_m(D, P_1)$ and $\varphi_m(D, P_2)$ are as in the right part of the figure, which are related by detour moves. The other cases of oriented cut moves are shown by a similar argument. □

![Figure 16. Results by an oriented cut point move](image)

The following is our main theorem. It implies that we have a map $\varphi_m$ from the set of virtual links to the set of mod $m$ almost classical virtual links.

**Theorem 6.** Let $(D, P)$ and $(D', P')$ be virtual link diagrams with cut systems. If $D$ and $D'$ are equivalent, then $\varphi_m(D, P)$ and $\varphi_m(D', P')$ are equivalent.

**Proof.** By Lemma 5 it is sufficient to consider the case that $P$ and $P'$ are canonical cut systems.

If $D'$ is related to $D$ by one of Reidemeister moves, then $\varphi_m(D, P)$ and $\varphi_m(D', P')$ are related by $m$ Reidemeister moves, which are copies of the original Reidemeister moves.

Suppose that $D'$ is related to $D$ by a virtual Reidemeister move I (resp. II) as in Figure 17 (i) (resp. (ii)). Let $P_*$ be the cut system obtained from $P$ by cut point moves I and II as in the figure. By Lemma 5 $\varphi_m(D, P)$ and $\varphi_m(D, P_*)$ are equivalent. On the other hand $\varphi_m(D', P')$ and $\varphi_m(D, P_*)$ are related by $m$ virtual Reidemeister moves I (resp. II). Thus $\varphi_m(D, P)$ and $\varphi_m(D', P')$ are equivalent.

Suppose that $D'$ is related to $D$ by a virtual Reidemeister move III as in Figure 17 (iii). Let $P_*$ (resp. $P'_*$) be the cut system obtained from $P$ (resp. $P'$) by cut point moves as in the figure. By Lemma 5 $\varphi_m(D, P)$ (resp. $\varphi_m(D', P')$) and $\varphi_m(D, P_*)$ (resp. $\varphi_m(D', P'_*)$)
are equivalent. On the other hand, \( \varphi_m(D, P) \) and \( \varphi_m(D', P') \) are related by \( m \) virtual Reidemeister moves III. Thus \( \varphi_m(D, P) \) and \( \varphi_m(D', P') \) are equivalent.

Suppose that \( D' \) is related to \( D \) by a virtual Reidemeister move IV as in Figure 17 (iv). Let \( P_k \) (resp. \( P'_k \)) be the cut system obtained from \( P \) (resp. \( P' \)) by cut point moves as in the figure. By Lemme 5, \( \varphi_m(D, P) \) (resp. \( \varphi_m(D', P'') \)) and \( \varphi_m(D, P_k) \) (resp. \( \varphi_m(D', P'_k) \)) are equivalent. On the other hand, \( \varphi_m(D, P_k) \) and \( \varphi_m(D', P'_k) \) are equivalent by \( m \) virtual Reidemeister moves IV. Thus \( \varphi_m(D, P) \) and \( \varphi_m(D', P') \) are equivalent. The other cases where the orientations of virtual link diagrams are different are shown by a similar argument.

\[ \begin{align*}
(D, P) & \quad (D, P_k) \\
(D, P') & \quad (D', P') \\
(D_k, P_k) & \quad (D_{k'}, P_{k'})
\end{align*} \]

Virtual Reidemeister move I

Virtual Reidemeister move II

\[ \begin{align*}
(D, P) & \quad (D, P_k) \\
(D, P') & \quad (D', P') \\
(D_k, P_k) & \quad (D_{k'}, P_{k'})
\end{align*} \]

Virtual Reidemeister move III

Virtual Reidemeister move IV

**Figure 17.** Diagrams related by a virtual Reidemeister move

5. **AN ALTERNATIVE CONSTRUCTION OF CYCLIC COVERING VIRTUAL LINK DIAGRAMS**

In this section, we introduce two methods of constructing cyclic covering virtual link diagrams. The first one is a more general method, denoted by \( \varphi_m^0(D, P) \), including the method introduced in Section 3 as a special case. The second one is a method which is also a special case of the first one. The reader who does not need it might skip this section.

In the construction of \( \varphi_m(D, P) \) introduced in Section 3, we first modified \( (D, P) \) so that each horizontal line \( \ell(p) \) through \( p \in P \) intersects \( D \) transversely avoiding the crossings of \( D \) and the other cut points of \( P \), and then we considered \( m \) parallel copies of \( (D, P) \). However, we may define \( \varphi_m(D, P) \) without this procedure.

Let \( (D, P) \) be a virtual link diagram with a cut system. Let \( (D^k, P^k), k = 0, \ldots, m - 1 \), be virtual link diagrams with cut systems such that each \( (D^k, P^k) \) is a copy of \( (D, P) \) and that the intersection of \( D^k \) and \( D^{k'} \) for \( k \neq k' \) is empty or consists of virtual crossings. (Furthermore, we may weaken the assumption that \( (D^k, P^k) \) is a copy of \( (D, P) \) so that \( (D^k, P^k) \) is isotopic to \( (D, P) \) by an isotopy of \( \mathbb{R}^2 \) or even that \( (D^k, P^k) \) is strongly equivalent to \( (D, P) \).) For each \( p \in P \), let \( N(p) \) be a regular neighborhood of \( p \) in \( D \),
which is a small arc on $D$ containing $p$. Let $p_-$ and $p_+$ be the endpoints of $N(p)$ such that the orientation of the virtual link diagram restricted to $N(p)$ is from $p_-$ to $p_+$. For each $k \in \{0, \ldots, m-1\} = \mathbb{Z}_m$, let $p^k$, $N(p^k)$, $p_-^k$ and $p_+^k$ be the corresponding copy of $p$, $N(p)$, $p_-$ and $p_+$ in $D^k$. Remove $N(p^k)$ for all $p \in P$ and $k \in \{0, \ldots, m-1\}$ from the diagram $\bigcup_{k=0}^{m-1} D^k$ and, for each $p \in P$ and $k \in \{0, \ldots, m-1\}$, connect the endpoint $p_-^k$ to $p_+^{k-\epsilon(p)}$ by any virtual path. Here $\epsilon(p)$ is $+1$ (resp. $-1$) if $p$ is coherent (resp. incoherent). We denote by $\varphi^0_m(D, P)$ a virtual link diagram obtained this way.

Consider an Alexander numbering $f$ of $(D, P)$ and let $f^k$ be the Alexander numbering of $(D^k, P^k)$ obtained from $f$ by shifting by $k$. Then $f^0, \ldots, f^{m-1}$ induce a mod $m$ Alexander numbering of $\varphi^0_m(D, P)$. Thus $\varphi^0_m(D, P)$ is mod $m$ almost classical.

The method of construction of $\varphi_m(D, P)$ introduced in Section 3 is a special case of the construction of $\varphi^0_m(D, P)$.

**Theorem 7.** For a virtual link diagram $D$ with a cut point $P$, a diagram $\varphi^0_m(D, P)$ is unique up to strong equivalence.

**Proof.** Let $D_1$ and $D_2$ be virtual link diagrams obtained from the same $(D, P)$ by the construction for $\varphi^0_m(D, P)$ introduced above. By definition of $\varphi^0_m(D, P)$, every classical crossing of $D_1$ (or $D_2$) can be labelled uniquely with $c^k$ for a classical crossing $c$ of $D$ and $k \in \{0, \ldots, m-1\}$. Thus there is a natural bijection between the classical crossings of $D_1$ and those of $D_2$. By an ambient isotopy of $\mathbb{R}^2$, we may assume that $D_1$ and $D_2$ coincide in a regular neighborhood of every classical crossing. Let $E$ denote the closure of the complement of the regular neighborhoods of all classical crossings of $D_1$ (or of $D_2$) in $\mathbb{R}^2$. The intersection $D_1 \cap E$ (or $D_2 \cap E$) consists of virtual paths which are properly immersed arcs or immersed loops in $E$.

Let $A(D_1)$ (resp. $A(D_2)$) be the set of properly immersed arcs of $D_1 \cap E$ (resp. $D_2 \cap E$), and let $L(D_1)$ (resp. $L(D_2)$) be the set of immersed loops of $D_1 \cap E$ (resp. $D_2 \cap E$).

Let $a_1 \in A(D_1)$ and $a_2 \in A(D_2)$ be virtual paths starting with the same point $a_1(0) = a_2(0)$ in $\partial(D_1 \cap E) = \partial(D_2 \cap E)$, and let $a_1(1)$ and $a_2(1)$ be their terminal points in $\partial(D_1 \cap E) = \partial(D_2 \cap E)$. We assert that $a_1(1) = a_2(1)$. This is seen as follows. Let $E(D)$ be the complement of the regular neighborhoods of all classical crossings of $D$ in $\mathbb{R}^2$. The intersection $D \cap E(D)$ consists of virtual paths which are properly immersed arcs or immersed loops in $E(D)$. Let $a(0)$ be a point of $\partial(D \cap E(D))$ corresponding to $a(0)$ and let $a$ be the virtual path of $D \cap E(D)$ starting at $a(0)$. Let $a(1)$ be the terminal point of $a$ in $\partial(D \cap E(D))$. Then $a(1) = a_2(0) = a(0)^k$, $a_1(1) = a(1)^{k'}$ and $a_2(1) = a(1)^{k''}$ for some $k, k', k'' \in \{0, \ldots, m-1\} = \mathbb{Z}_m$. Note that $k' - k$ is the sum of $-\epsilon(p)$ for all cut points $p \in P$ appearing on $a$, and so is $k'' - k$. Thus, we see that $k' = k''$ and $a_1(1) = a_2(1)$. Therefore, there is a bijection between $A(D_1)$ and $A(D_2)$ such that corresponding arcs $a_1 \in A(D_1)$ and $a_2 \in A(D_2)$ have the same starting point and the same terminal point.
Every loop of \( L(D_1) \) (or \( L(D_2) \)) can be labelled as \( \ell^k \) for a virtual path being an immersed loop \( \ell \) of \( D \) and \( k \in \{0, \ldots, m-1\} \). Thus there is a bijection between \( L(D_1) \) and \( L(D_2) \).

By detour moves, replace virtual paths which are elements of \( A(D_1) \) and \( L(D_1) \) with the corresponding elements of \( A(D_2) \) and \( L(D_2) \), and we can obtain \( D_2 \) from \( D_1 \). This implies that \( D_1 \) and \( D_2 \) are strongly equivalent.

We introduce another method of construction of cyclic covering virtual link diagrams, which is a special case of the method above. Let \((D, P)\) be a virtual link diagram with a cut system. Put \( m \) copies of \((D, P)\) in \( \mathbb{R}^2 \), say \((D_0, P_0), \ldots, (D_{m-1}, P_{m-1})\), such that all corresponding semi-arcs are in parallel as in Figure 18 and all crossings between \( D^k \) and \( D^{k'} \) for \( k \neq k' \) are virtual crossings. Here semi-arcs of \( D^{k+1} \) appears on the right of \( D^k \) with respect to the orientation of \( D \) as in Figure 18. See Figure 19 (i) and (ii) for an example with \( m = 3 \).

![Figure 18. Parallel virtual link diagrams](image)

![Figure 19. A 3-fold cyclic covering virtual link diagram](image)

For a cut point \( p \in P \), let \( p^k \) denote the corresponding cut point of \( P^k \). Remove regular neighborhoods of all \( p^k \) for \( p \in P \) and \( k \in \{0, \ldots, m-1\} = \mathbb{Z}_m \) from \( \bigcup_{k=0}^{m-1} D^k \), and connect the endpoints by virtual paths as in Figure 20 (iii) (resp. (iv)) if the cut point is coherent (resp. incoherent) as in Figure 20 (i) (resp. (ii)).

Then we obtain a virtual link diagram. Let us denote it by \( \varphi_m(D, P) \). See Figure 19 (iii) for an example. This concrete construction is also a special case of the general construction.
We call them cyclic covering (virtual link) diagrams.

From this construction we see the following.

**Corollary 8.** Let \((D, P)\) be a virtual knot diagram with cut system. Then \(\varphi_m(D, P)\) is an \(m\)-component virtual link diagram.

**Proof.** Consider \(\varphi_m(D, P)\). Since the number of coherent cut points of \(P\) equals that of incoherent cut points of \(P\) (Corollary 3), the number of twists as in Figure 20 (iii) appearing in \(\varphi_m(D, P)\) equals that of the opposite twists as in Figure 20 (iv). Thus \(\varphi_m(D, P)\) is an \(m\)-component virtual link diagram, and so is \(\varphi_m(D, P)\).

\(\square\)

### 6. Applications

First, we demonstrate how Theorem 6 is used to show that two virtual link diagrams are not equivalent.

Let \((D, P)\) and \((D', P')\) be virtual link diagrams with cut points depicted in Figure 21 (i) and (ii). Then \(\varphi_3(D, P)\) and \(\varphi_3(D', P')\) are as in the figure.

**Figure 20.** Replacement of neighborhoods of cut points

\(\varphi_0(D, P)\). By Theorem 7, \(\varphi_m(D, P)\), \(\varphi_1(D, P)\) and \(\varphi_0(D, P)\) are all strongly equivalent.

**Figure 21.** Example of mod 3 cyclic covering virtual link diagram
It is easily seen that $\varphi_3(D, P)$ and $\varphi_3(D', P')$ are not equivalent, since any pair of components of $\varphi_3(D, P)$ have linking number 0 and any pair of components of $\varphi_3(D', P')$ have linking number 1. By Theorem 6, we conclude that $D$ and $D'$ are not equivalent.

Theorem 6 implies Theorem 10 below, which can be used to show that some virtual link diagrams are never equivalent to mod $m$ almost classical virtual link diagrams.

**Lemma 9.** Let $D$ be a mod $m$ almost classical virtual link diagram. For any cut system $P$ of $D$, $\varphi_m(D, P)$ is strongly equivalent to a virtual link diagram which is a disjoint union of $m$ copies of $D$.

**Proof.** There is a cut system $P_0$ of $D$ such that for each semi-arc of $D$, there are no cut points on it or there are $m$ coherent (or incoherent) cut points on it. Each semi-arc of $D$ with $m$ coherent (or incoherent) cut points yields $m$ copies of such semi-arcs in the $m$ parallel copies of $D$, and $m$ virtual paths in $\varphi_m(D, P_0)$ as in Figure 22. These $m$ virtual paths in $\varphi_m(D, P_0)$ can be replaced with $m$ straight virtual paths by detour moves, and we obtain a disjoint union of $m$ copies of $D$. This implies that $\varphi_m(D, P_0)$ is strongly equivalent to the disjoint union of $m$ copies of $D$. Thus $\varphi_m(D, P_0)$ is strongly equivalent to a disjoint union of $m$ copies of $D$. By Lemma 9 (or Theorem 7), we see that $\varphi_m(D, P)$ is strongly equivalent to a disjoint union of $m$ copies of $D$. □

Figure 22. mod $m$ almost classical virtual link and its oriented cut points

**Theorem 10.** If $\varphi_m(D, P)$ is not equivalent to a disjoint union of $m$ copies of $D$, then $D$ is never equivalent to a mod $m$ almost classical virtual link diagram.

**Proof.** Suppose that $D$ is equivalent to a mod $m$ almost classical virtual link diagram $D'$. By Lemma 9 $\varphi_m(D', P')$ is equivalent to a disjoint union of $m$ copies of $D'$. By Theorem 6 $\varphi_m(D, P)$ and $\varphi_m(D', P')$ are equivalent. Thus, $\varphi_m(D, P)$ is equivalent to a disjoint union of $m$ copies of $D'$, and hence equivalent to a disjoint union of $m$ copies of $D$. This contradicts the hypothesis. □

Let $D'$ be the virtual link diagram depicted in Figure 21. For the cut system $P'$ in the figure, $\varphi_3(D', P')$ is not equivalent to a disjoint union of $D$, since a pair of its components have linking number 1. By Theorem 10 we can conclude that $D'$ is never equivalent to a mod 3 almost classical virtual link diagram.
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