The Generalized Power Graph of a Finite Group

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Abstract. Graphs associated with groups and other algebraic structures have been actively investigated, since they have valuable applications in data mining. For a finite group G, let $\Gamma_G$ be the graph with the non-identity elements of G as the vertex set, and two vertices are adjacent if they respectively lie in two conjugate proper subgroups of G. $\Gamma_G$ is called the generalized power graph with respect to G. This paper explores how the graph theoretical properties of $\Gamma_G$ can affect on the group theoretical properties of G.

1. Introduction
Graphs associated with groups and other algebraic structures have been actively investigated, since they have valuable applications (cf. [1]) and are related to automata theory (cf. [2, 3]), such as, Cayley graphs as classifiers for data mining (cf. [4]). Also, study of algebraic structures by graphs associated with them gives rise to many recent and interesting results in the literature. This field is relatively new, and over the years different types of graphs derived from a group have been defined, such as the prime graph [5], the non-commuting graph [6], the power graph, and of course Cayley graphs, which have a long history.

The non-commuting graph was first considered by Neumann [6] and this graph are studied by many authors. Recently, Erfanian et al. [7] introduced the relative n-th non-commuting graph of a finite group, which is a generalized non-commuting graph. The directed power graph of a semigroup was defined by Kelarev and Quinn [8]. Chakrabarty et al. [9] introduced the (undirected) power graph of a finite group.

Let G be a finite group with identity element $e$. The power graph $P_G$ of G is a simple graph whose vertex set is G and two vertices x and y in G are adjacent if and only if $x=y^m$ or $y=x^m$ for some positive integer m. Moghaddamfar et al. [10] discussed the subgraph of $P_G$ induced by $G \setminus \{e\}$, where $e$ is the identity element of G. Now consider the graph $\Gamma_G$ as follows: Take $G \setminus \{e\}$ as the vertex set and join two vertices if they respectively lie in two conjugate proper subgroups of G. For example, Fig. 1 is the generalized power graph of the quaternion group $Q_8$ with 8 elements and Fig. 2 is the generalized power graph of the symmetric group $S_3$ on 3 letters.

![Fig. 1 Q_8](image1)

![Fig. 2 S_3](image2)

By the definition, if G is non-cyclic then the subgraph of $P_G$ induced by $G \setminus \{e\}$ is a subgraph of $\Gamma_G$. If G is a cyclic group with order n, then $\Gamma_G$ has at least $\Phi(n)$ isolate vertices which are all generators of...
G (In fact, if n=2p for some prime p, then the number of the isolate vertices of $\Gamma_G$ is $\Phi(n)+1$; otherwise it is $\Phi(n)$), and denoted by the set S, here $\Phi$ is the Euler totient function. In the case, the subgraph induced by $G \setminus \{e\} \setminus S$ is also a subgraph of $\Gamma_G$. So call $\Gamma_G$ the generalized power graph of G. Let x, y $\in G \setminus \{e\}$. If $\langle x, y \rangle \neq G$, then x is adjacent to y in $\Gamma_G$. In general, x ~ y does not imply that $\langle x, y \rangle$ is a proper subgroup of G (an example is showed in Fig. 2, $\langle 1,2 \rangle \sim (1,3)$ but $\langle 1,2 \rangle \sim (1,3) = S_3$). However, if G is restricted on Dedekind group (G is called a Dedekind group if its every subgroup is normal, for example, every abelian group is Dedekind), then x and y are joined by an edge of $\Gamma_G$ if and only if there exists a proper subgroup H of G such that x, y $\in H$.

This paper explores how the graph theoretical properties of $\Gamma_G$ can affect on the group theoretical properties of G. Section 2 discusses the connectivity of generalized power graphs. In Section 3, all groups whose generalized power graphs are empty, bipartite, planar and complete are obtained.

2. The connectivity of $\Gamma_G$ The exponent of group G is defined as the least common multiple of the orders of all elements of G, is denoted by exp(G). Denote by $\pi(n)$ the set of the prime divisors of a positive integer n. For convenience, write $\pi(G)$ and $\pi(x)$ instead of $\pi(|G|)$ and $\pi(|x|)$, respectively. In this section, for the groups G with $|\pi(G)|=1$ or 2, some sufficient and necessary conditions for which $\Gamma_G$ is connected are given. It also shows that if $|\pi(G)|=3$ then $\Gamma_G$ is connected.

Theorem 2.1 Let G be a p-group. Then $\Gamma_G$ is connected if and only if G is non-cyclic and G is not isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$.

Proof. First let $\Gamma_G$ be connected. It is clear that G is not cyclic. Assume, to the contrary, that G is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Then exp(G)=p. Thus, for some non-identity element g of G, <g> is a maximal subgroup of G and hence $\Gamma_G$ is constructed by some complete subgraphs isomorphic to $K_p$. Now let a, b $\in G$, and let $\Gamma_{<a>$} and $\Gamma_{<b>$} be two distinct complete subgraphs of $\Gamma_G$ induced by the subgroups <a> and <b>, respectively. If there is an edge a ~ b between $\Gamma_{<a>$} and $\Gamma_{<b>$}, where 1 $\leq i, j \leq p$, then $\langle a', b' \rangle$ is a proper subgroup of G, which is impossible since $\langle a' \rangle = <a>$ and $\langle b' \rangle = <b>$. It follows that $\Gamma_G=\mathbb{Z}_p \times \mathbb{Z}_p$ is isomorphic to $(p+1)K_p$. Certainly, $\Gamma_G$ is not connected, a contradiction.

Conversely, suppose that G is non-cyclic and is not isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Since H is self-conjugate for every proper subgroup H of G, the subgraph of $\Gamma_G$ induced by H\{e\} is complete. If the Frattini subgroup of G, is non-trivial, then $\Gamma_G$ is connected; if not, G is an elementary abelian p-group. Therefore G is isomorphic to $(\mathbb{Z}_p)^n$, where $n \geq 3$. In the case, it is easy to see that $\langle a, b \rangle$ is a proper subgroup of G for all a, b $\in G$. It means that $\Gamma_G$ is complete.

Theorem 2.2 Let p and q be two distinct prime integers, G a group with $\pi(G)=[p, q]$. Then $\Gamma_G$ is connected if and only if G is non-cyclic and has a proper subgroup S such that $\pi(S)=[p, q]$.

Proof. Suppose that $\Gamma_G$ is connected. It is straightforward that G is non-cyclic. Since $\pi(G)=[p, q]$ there exist two elements x and y of G\{e\} such that $|x|=p$ and $|y|=q$. In view of the connectivity of $\Gamma_G$, there is a path: $x \sim v_1 \sim v_2 \sim \cdots \sim v_n \sim y$. Since x is adjacent to $v_1$, there exist two conjugate proper subgroups S and K in G such that $x \in S$ and $y \in K$. If $\pi(S)=[p, q]$, then this completes the proof. Thus, now assume that S is a p-group and next, consider the edge $v_1 \sim v_2$. Similarly, there is a proper subgroup S; of G, such that $v_i \in S$, and $\pi(S)=[p, q]$, where $1 \leq i \leq n$. This procedure shall prove the result.

Now suppose that G is non-cyclic and has a proper subgroup S such that $\pi(S)=\pi(G)$. Assume that x and y are two distinct vertices of $\Gamma_G$. Since $\pi(G)=[p, q]$, $\pi(x)$ has three probabilities: [p, q], [p] and [q], and so is $\pi(y)$. Without loss of generality, consider four cases.

Case 1. $\pi(x)=[p, q]$ and $\pi(y)=[p]$. Clearly, G has two Sylow p-subgroups $P_1$ and $P_2$ (here $P_1$ and $P_2$ could be equal) such that $x \in P_1$ and $y \in P_2$: particularly, $P_1$ and $P_2$ are conjugate in G. Thus x and y are joined by an edge of $\Gamma_G$.

Case 2. $\pi(x)=[p, q]$ and $\pi(y)=[q]$. Choose a p-subgroup $P_0$ of $\langle x \rangle$ and let v be a non-identity element of $P_0$. Case 1 shows that y ~ v. Note that $\langle x \rangle$ is a proper subgroup of G, $x \sim v$. So $x \sim v \sim y$, as desired.

Case 3. $\pi(x)=[p]$ and $\pi(y)=[q]$. Assume that $x_1$ and $y_1$ are two p-element and q-element of S respectively. Therefore $x_1 \sim y_1$. Case 1 means that $x_1 \sim x$ and $y_1 \sim y$. Thus there exists a path $x \sim x_1 \sim y_1 \sim y$, as required.
Case 4. \( \pi(x) = \{p, q\} \) and \( \pi(y) = \{p, q\} \). Clearly, there are p-element \( x_1 \) and q-element \( y_1 \), such that \( x_1 \sim x \) and \( y_1 \sim y \). Now Case 3 ends the proof.

Let \( P_i \) be the Sylow \( p_i \)-subgroup of \( G \), where \( p_i \) is prime for every \( 1 \leq i \leq s \). \( \{P_1, P_2, \ldots, P_s\} \) is called a Sylow basis of \( G \) provided that \( P_i \not\subseteq P_j \) for all \( 1 \leq i, j \leq s \). A well-known theorem of P. Hall shows that \( G \) is solvable if and only if \( G \) possesses a Sylow basis.

**Theorem 2.3** Let \( G \) be a non-cyclic group such that \(|\pi(G)| = 3\). Then \( \Gamma_G \) is connected.

**Proof.** First assume that \( G \) is solvable. Let \( \pi(G) = \{p_1, p_2, p_3\} \) and let \( \{P_1, P_2, P_3\} \) be a Sylow basis of \( G \). Now choose \( x \) in \( G \setminus \{e\} \), if \( \pi(x) = \pi(G) \), then an argument similar to the one used in Theorem 2.2 ends the proof. Therefore, now assume that \( \pi(x) \) is a proper subset of \( \pi(G) \), and let \( y \) be any vertex of \( \Gamma_G \). If \( \pi(x) \cap \pi(y) \neq \emptyset \), then there exists at least a prime number \( p \in \pi(G) \), such that \( P_i \) and \( P_j \) are two \( p \)-subgroups of \( \langle x \rangle \) and \( \langle y \rangle \) respectively. Let \( u \) be a non-identity element belonging to \( P_i \) and let \( v \) be a non-identity element belonging to \( P_j \). Case 1 of Theorem 2.2 indicates that \( u \) and \( v \) is adjacent. So \( x \sim u \sim v \sim y \) is a path in \( \Gamma_G \), as required. Now suppose that \( \pi(x) \cap \pi(y) = \emptyset \). Then pick \( u, v \in G \setminus \{e\} \) such that \( u \in P_x \leq P_xP_y^h \) and \( v \in P_y \leq P_yP_x^h \), where \( P_x \) is a \( p \)-subgroup of \( \langle x \rangle \), \( P_y \) is a \( p \)-subgroup of \( \langle y \rangle \), \( P_x \) and \( P_y \) are Sylow \( p \)-subgroup and Sylow \( p \)-subgroup of \( G \) respectively, \( g, h \in G \) and \( 1 \leq i, j \leq 3 \). Set now \( w \in P_y \} \{e\} \) and \( z \in P_x \{e\} \). Thus, as assuming that \( \{P_1, P_2, P_3\} \) is a Sylow basis, \( w \sim x \). It means that \( x \sim u \sim w \sim z \sim v \sim y \) is a path. It also follows that \( \Gamma_G \) is connected.

Now suppose that \( G \) is non-solvable. Corollary 4 of [11] tells us that, if \( G \) is a non-solvable group with \(|\pi(G)| = 3\), then one of the following groups is involved in \( G \): \( L_2(4) \), \( L_2(7) \), \( L_2(8) \), \( L_2(17) \), \( L_3(3) \). By GAP, it follows that the generalized power graphs of these groups above are connected. The proof is now complete.

3. Groups characterized by their generalized power graphs

In the section, all groups whose generalized power graphs are empty, bipartite, planar and complete are obtained.

**Theorem 3.1** Let \( G \) be a group. Then \( \Gamma_G \) is empty if and only if \( G \) is isomorphic to one of the groups: \( Z_2 \times Z_2, Z_4, Z_p \) for some prime \( p \).

**Proof.** It is easy to check that each of the generalized power graphs of \( Z_2 \times Z_2, Z_4 \) and \( Z_p \) is empty, where \( p \) is a prime. Now \( \Gamma_G \) is empty. Let \( x \) be an isolate vertex of \( \Gamma_G \). If \( \langle x \rangle = G \) then \( G \) is cyclic. Thus, suppose now that \( \langle x \rangle \neq G \). Since both \( x \) and \( x^{-1} \) belong to \( \langle x \rangle \), \( |x| = 2 \). If there exists an element \( g \) in \( G \) such that \( \langle x \rangle \neq \langle x^g \rangle \), then \( \langle x \rangle \neq \langle x^{x^{-1}} \rangle \). Therefore, \( x \) is adjacent to \( x^{-1} \), a contradiction. It means that \( \langle x \rangle \) is normal in \( G \). On the other hand, clearly, \( \langle x \rangle \) is a maximal subgroup of \( G \). Thus, assume now that \( |G| = 2^p \), for some odd prime \( p \) and some integer \( n \). If \( G \) is a 2-group then \( |G| = 4 \). Otherwise, \( |G| = 2^p \) and so \( G \) is isomorphic to \( Z_2 \times Z_p \), which is a cyclic group. It means that \( G \) is isomorphic to \( Z_2 \times Z_2 \) or a cyclic group. If \( G \) is a cyclic group with order composite number, then \( n = 4 \). Now it follows that \( G \) is one of the groups: \( Z_2 \times Z_2, Z_4, Z_p \), where \( p \) is a prime.

**Theorem 3.2** Let \( G \) be a group. Then \( \Gamma_G \) is bipartite if and only if \( G \) is isomorphic to one of the groups: \( Z_2 \times Z_2, Z_4, Z_6, Z_8, Z_9 \times Z_9, Z_{10} \), \( Z_2 \times Z_2, D_8, Q_8, Z_8, Z_4 \times Z_2, Z_9 \) for some prime \( p \).

**Proof.** The sufficiency is obvious. Now assume that \( \Gamma_G \) is a bipartite graph. It is well known that a graph is bipartite if and only if the graph contains no odd cycles. Thus, \( G \) has no proper subgraphs of order greater than or equal 4. If \(|\pi(G)| = 1\) then \( |G| = 2^2 \), \( 2^3 \) or \( Z_p \), where \( p \) is prime. It is easy to verify that \( G \) is one group of the groups: \( Z_2 \times Z_2, Z_4, Z_6 \times Z_6, Z_8, Z_9 \times Z_9 \).

If \(|\pi(G)| = 2\) then the order of \( G \) equal to 6. Since \( \Gamma_3 \) contains a triangle induced by \( \{1, 2\}, \{1, 3\}, \{2, 3\}\), which is non-bipartite. So \( G \) is isomorphic to \( Z_{12} \), as desired.

**Theorem 3.3** Let \( G \) be a group. Then \( \Gamma_G \) is planar if and only if \( G \) is isomorphic to one of the groups: \( Z_2 \times Z_2, Z_6, S_3, Z_9, Z_2 \times Z_2, Z_{15}, Z_{10}, Z_{25}, Z_2 \times Z_2, D_8, Q_8, Z_8, Z_4 \times Z_2, Z_9 \) for some prime \( p \).

**Proof.** First assume that \( \Gamma_G \) is planar. If there exists a proper subgroup \( H \) of \( G \) of order greater than or equal 6, then the subgraph of \( \Gamma_G \) induced by \( H \{e\} \) is isomorphic to the complete graph \( K_6 \{e\} \), which is a contradiction since \( K_6 \) is non-planar. It follows that the order of any proper subgroup of \( G \) must be less than or equal to 5. It follows that \( \pi(G) \) is a subset of \( \{2, 3, 5\} \) by Sylow theorems. Now consider three cases.
Case 1. $|\pi(G)|=3$. In this case, it is easy to see that $|G|=2^3\times 3^5$ or $2^3\times 3^5$. If $|G|=2^3\times 3^5$, then $G$ has at least a normal Sylow subgroup $P$. By Burnside’s $p^aq^b$-theorem, it follows that $G/P$ is solvable and so is $G$. Since solvable group possesses Hall $\pi$-subgroups for all sets $\pi$ of primes of $\pi(G)$, $G$ has at least a subgroup of order 15, a contradiction.

Now suppose that $|G|=2^3\times 3^5$. If $G$ is simple, then $G$ is isomorphic to $A_5$, the alternating group on 5 letters. It is well known that $\langle(3,4,5), (1,2)(4,5)\rangle$ is a subgroup of $A_5$ of order 6. Hence, It follows that $\Gamma_G$ is non-planar. Thus now assume that $G$ is not simple. Let $H$ and $P$ be a non-trivial normal subgroup and a Sylow 5-subgroup of $G$, respectively. Considering the subgroup $HP$, it is clear that $|HP|=5|H|$ if $H\cap P=1$. If $HP$ is a proper subgroup of $G$, then $|HP|>5$, a contradiction; otherwise $G=HP$ and so $|H|=12$, a contradiction again. It means that $H\cap P=P$. If $P$ is a proper subgroup of $H$, then, a contradiction as $|P|=5$. This implies $H=P$, that is, $P$ is normal in $G$. It follows that $G$ is solvable. Similarly as above, also a contradiction.

Case 2. $|\pi(G)|=2$. Obviously, $|G|=2^3\times 3^5, 2^3\times 3^5, 2^2\times 3^3$ or $2^2\times 3^5$. It is easy to verify that $\Gamma_{DS_3}$, $\Gamma_{D_{10}}$, $\Gamma_{S_3}$ and $\Gamma_{Z_{15}}$ are planar. Also, it is easy to see that $\Gamma_{D_{12}}$ is non-planar since the subgraph induced by the set of all elements of order 2 is $K_5$.

Let $|G|=2^3\times 5$. Then, in view of Sylow theorem, the Sylow 5-subgroup $P$ is normal in $G$. Let $H$ be a subgroup of $G$ of order 2. One can see from $(|H|, |P|)=1$ that $|HP|=10$, a contradiction.

Suppose that $|G|=2^3\times 5$. It is clear that $G$ is isomorphic to one of the groups: $Z_{12}, Z_2\times Z_6, D_{12}, Q_{12}, A_4$.

If $G$ is not isomorphic to $A_4$, then there exists at least a subgroup of $G$ of order 6, which is impossible. Considering $A_4$, a fact is that $A_4$ has eight Sylow 3-subgroups, and they are conjugate each other in $G$. It is easy to see that $K_5$ is a subgraph of $\Gamma_{A_4}$, a contradiction.

Case 3. $\pi(G)|=1$. It is straightforward that in this case, $G$ is isomorphic to one of $Z_2, Z_2\times Z_2, Z_3, Z_3\times Z_5, Z_2\times Z_8, D_6, Q_8, Z_2\times Z_2\times Z_2, Z_2\times Z_2, Z_2\times Z_2, Z_2, Z_2, Z_2$, where $p$ is a prime. By checking, it is easy to see that the generalized power graph of $Z_2\times Z_2\times Z_2$ is isomorphic to $K_8$, which is non-planar, others are planar.

The proof of the converse is clear. The proof is now complete.

Denote by $d(G)$ the minimum possible size of all generating sets for $G$. Clearly, if $d(G)\geq 3$ for any group $G$, then $\Gamma_G$ is complete.

**Corollary 3.4** Let $G$ be a Dedekind group. Then $\Gamma_G$ is complete if and only if $d(G)\geq 3$.

By the fundamental theorem for finitely generated abelian groups, if $G$ is isomorphic to $\langle r_1\rangle \times \langle r_2\rangle \times \cdots \times \langle r_s\rangle$, where $|r_i|$ is a divisor of $|r_j|$ for all $i=1, 2, \ldots, s-1$, then $r_i$ is uniquely determined by $G$, and $d(G)\geq r_i$. Now, in view of Corollary 3.4, the following result is obtained.

**Corollary 3.5** Let $G$ be an abelian group. Then $\Gamma_G$ is complete if and only if $G$ is isomorphic to $\langle r_1\rangle \times \langle r_2\rangle \times \cdots \times \langle r_s\rangle$, where $|r_i|$ divides $|r_{i+1}|$ for all $i=1, 2, \ldots, s-1, s$ and $s\geq 3$.

Clearly, if a group $G$ is isomorphic to a group $H$ then $\Gamma_G$ is isomorphic to $\Gamma_H$. However, the converse is not true in general. For example, it is easy to see that $Q_8$ and $D_8$ are not isomorphic, but their generalized power graphs are isomorphic. Moreover, Theorem 3.1 also presents an example, the generalized power graph of $Z_8$ is isomorphic to the generalized power graph of $Z_2\times Z_2$; however, $Z_6$ and $Z_2\times Z_2$ are clearly non-isomorphic. Now, using generalized power graphs, characterize all cyclic groups except for the group with order 4.

**Theorem 3.6** Let $G$ be a group and $n\neq 4$. Then $\Gamma_G$ is isomorphic to the generalized power graph of $Z_n$ if and only if $G$ is isomorphic to $Z_n$.

**Proof.** The sufficiency follows trivially. Now suppose that $G$ is isomorphic to the generalized power graph of $Z_n$. Then there exists at least a vertex $x$ of $\Gamma_G$ such that $\deg(x)=0$. It is clear that $|G|=n$. By the proof of Theorem 3.1, it follows that $G$ is isomorphic to $Z_2\times Z_2$ or a cyclic group. Since $n\neq 4$, $G$ is cyclic, as desired.

Finally, an applications of Theorem 3.6 is given.

**Theorem 3.7** Let $G$ be a group such that $\Gamma_G$ contains an end-vertex, and $R$ a group. Then $\Gamma_G$ is isomorphic to $\Gamma_R$ if and only if $G$ is isomorphic to $R$. 

Proof. The proof of the sufficiency is straightforward. In order to prove the necessity, assuming that \( x \) is an end-vertex of \( \Gamma_G \) and \( y \) is the one vertex of \( \Gamma_G \) such that \( x \sim y \) belongs to \( E(\Gamma_G) \). So there are two conjugate subgroups \( H \) and \( K \) such that \( x \in H \) and \( y \in K \).

It is easy to see that \(|x|=3 \) or \( 2 \) by deg(\( x \))=1. If \(|x|=2 \) then \(|H|=2 \). Hence \(|K|=2 \) and so the order of \( y \) is also \( 2 \). Thus, suppose now that \(|G|=2^r \), where \( (2,r)=1 \). Since \( <x> \) is a maximal subgroup of \( G \), \( n=1 \). Since deg(\( x \))=1, \( <x> \) and \( <y> \) are two Sylow \( 2 \)-subgroups of \( G \). While \( n_2=1 \) (mod 2), where \( n_2 \) is the number of all Sylow \( 2 \)-subgroups of \( G \). Now the contradiction implies that \(|x|=3 \).

It is clear that \( <x> \) is a normal subgroup of \( G \). Now it follows that \(|G|=9 \) or \( 3q \), where \( q \) is a prime number such that \( q \neq 3 \). If \(|G|=9 \), then \( G=Z_9 \) or \( Z_3 \times Z_3 \). Now let \(|G|=3q \). If \( q=2 \) then \( G=S_3 \) or \( Z_6 \). Otherwise, it follows from Sylow theorems that \( G \) is isomorphic to \( Z_3 \times Z_q \). It follows that \( G \) is one of the groups: \( S_3, Z_3 \times Z_3, Z_3p \) for some prime number \( p \).

If \( G=Z_{3p} \), then Theorem 3.6 completes the proof. If \( G=S_3 \), then \( R=Z_6 \) or \( S_3 \). Since the generalized power graph of \( S_3 \) has no isolate vertices, \( R=S_3 \), as desired. Similarly, if \( G \) is isomorphic to \( Z_3 \times Z_3 \), then \( R \) is also isomorphic to \( Z_1 \times Z_3 \), as desired.

4. Conclusion

In this paper, the generalized power graph of a finite group is introduced. Based on the results and discussions presented above, the conclusions are obtained as below:

1. Let \( G \) be a \( p \)-group. Then, \( \Gamma_G \) is connected if and only if \( G \) is non-cyclic and \( G \) is not isomorphic to \( Z_p \times Z_p \).

2. Let \( p \) and \( q \) be two distinct prime integers, and let \( G \) be a group with \( \pi(G)={p, q} \). Then, \( \Gamma_G \) is connected if and only if \( G \) is non-cyclic and has a proper subgroup \( S \) such that \( \pi(S)=\pi(G) \).

3. Let \( G \) be a non-cyclic group such that \(|\pi(G)|=3 \). Then \( \Gamma_G \) is connected.

4. All finite groups whose generalized power graphs are empty, bipartite, planar and complete were classified.

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References

[1] Abawajy J, Kelarev A V, et al. Rees semigroups of digraphs for classification of data[J]. Semigroup Forum, 2016, 92: 121-134.

[2] Kelarev A V. Graph Algebras and Automata[M]. New York: Marcel Dekker, 2003.

[3] Kelarev A V. Labelled Cayley graphs and minimal automata[J]. Australasian Journal of Combinatorics, 2004, 30: 95-101.

[4] Kelarev A V, Ryan J, et al. Cayley graphs as classifiers for data mining: The influence of asymmetries[J]. Discrete Mathematics, 2009, 309: 5360-5369.

[5] Williams J S. Prime graph components of finite groups[J]. Journal of Algebra, 1981, 69: 487-513.

[6] Neumann B H. A problem of Paul Erdős on groups[J]. Journal of the Australian Mathematical Society Series A, 1976, 21: 467-472.

[7] Erfanian A, Tolue B. Relative n-th non-commuting graphs of finite groups[J]. Bulletin of the Iranian Mathematical Society, 2013, 39: 663-674.

[8] Kelarev A V, Quinn S J. A combinatorial property and power graphs of groups[J]. Contribution to General Algebra, 2000, 12: 229-235.

[9] Chakrabarty I, Ghosh S, et al. Undirected power graphs of semigroups[J]. Semigroup Forum, 2009, 78: 410-426.

[10] Moghaddamfar A R, Rahbariyan S, et al. Certain properties of the power graph associated with a finite group[J]. Journal of Algebra and Its Applications, 2014, 13: 1450040, 18pp.

[11] Thompson J G. Non-solvable finite groups all of whose local subgroups are solvable[J]. Bulletin
of the American Mathematical Society, 1968, 74: 383-437.