Field strength for graded Yang-Mills theory

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The field strength is defined for \( \text{osp}(2/1; \mathbb{C}) \) non-degenerate graded Lie algebra. We show that a pair of Grassman-odd scalar fields find their place as a constituent part of the graded gauge potential on the equal footing with an ordinary (Grassman-even) one-form taking values in the proper Lie subalgebra, \( \text{su}(2) \), of the graded Lie algebra. Some possibilities of constructing a meaningful variational principle are discussed.

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I. INTRODUCTION

As well-known the number of gauge bosons of a theory is given by the number of generators of the corresponding gauge Lie algebra. The requirement of being a Lie algebra stems from the reality of the action functional which follows from the properties of the Fermat principle in optics and the Feynman path integral. In particular, this leads to physically admissible evolution of a system in optics and the Feynman path integral. In particular, the requirement of being a Lie algebra stems from the reality of the action functional which follows from the properties of the Fermat principle in optics and the Feynman path integral. In particular, this leads to physically admissible evolution of a system in optics and the Feynman path integral. In particular, this leads to physically admissible evolution of a system with such an action functional. Nevertheless, no one is restricted from looking for not necessarily of Lie-type algebras as the gauge algebras provided they are physically meaningful.

II. GRADING IN GAUGE ALGEBRA

Such an algebra is advocated in the current contribution. This is a graded extension of \( \text{su}(2) \) gauge algebra by a pair of odd generators, \( \tau_A \), which anticommute with one another and commute with the three even generators, \( T_a \), of \( \text{su}(2) \). We use the square brackets to denote both commutation and anticommutation operations of the generators with understanding of their proper usage. The defining relations have the form, \([1, 2, 3] \):

\[
[T_a, T_b] = \varepsilon_{abc} T_c, \quad [T_a, \tau_A] = \frac{1}{2} (\sigma_a)_A B \tau_B, \quad [\tau_A, \tau_B] = \frac{1}{2} (\sigma^a)_{AB} T_a. \tag{1}
\]

Lowercase Roman indices run from 1 to 3; uppercase Roman indices run over 1, 2; \((\sigma_a)_A B = (\sigma_a)_A C \varepsilon_{CDB} \varepsilon_{abc} = (\varepsilon_{123} = 1) \) and \(\varepsilon_{AB} (\varepsilon_{12} = 1) = \) the Levi-Civita totally antisymmetric symbols in three and two dimensions; the Pauli matrices \((\sigma_a)_A B \) and \(\varepsilon_{AB} \) are given by

\[
(\sigma_a)_A B = (\sigma^a)_A B = \left( \begin{array}{ccc} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right), \quad \tau_A B = \left( \begin{array}{ccc} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right). \tag{2}
\]

The non-degenerate super Killing form, \( B(T_a, T_b) \), is defined by

\[
B(T_a, T_b) = \frac{2}{3} \text{str}(T_a T_b) = \left( \begin{array}{ccc} \delta_{ab} & 0 \\ 0 & 0 \end{array} \right)_A B, \tag{3}
\]

where the supertrace operation is adopted from \([4]\) and the Greek indices from the beginning of the alphabet run over the whole set of the graded Lie algebra generators. It turns out that all of the generators are grade star Hermitian: on the even ones being just Hermitian in an ordinary sense while the odd generators obey more complicated relations (cf. \([4, 5]\)). We assign a degree, deg\( T_A \), 0 to the even and 1 to the odd generators.

III. GAUGE POTENTIAL

Given an \( \text{su}(N) \) Lie algebra one defines a gauge potential, which takes values in the algebra, by introducing \((N^2 - 1) \times n\) one-forms \(A_{\mu}^a(x)dx^\mu\), \(n\) being the dimension of space-time, and transvection them with the algebra generators \(T_a\). Note that from the present standpoint we have a composite object of the degree \((0, 1)\), the first position shows that generator \(T_a\) is, by definition, an even element of the Lie algebra and the second position exhibits that \(A_{\mu}^a(x)dx^\mu\) is a one-form in the algebra of exterior differential forms on space-time. This has a suggestive generalization to the case when one also has the degree 1 odd part of a graded Lie algebra: (s)he needs to construct the homogeneous compliment of the expression above, namely, the element of degree \((1, 0)\). It has the form \(\tau_A \Phi^A(x)\) and must be added to the element of degree \((0, 1)\) to form the complete graded gauge potential

\[
A(x) = T_a A_{\mu}^a(x)dx^\mu + \tau_A \Phi^A(x) \tag{4}
\]

homogeneous = even \(\otimes\) odd + odd \(\otimes\) even

Here \(\Phi^A(x)\) are zero-forms on space-time. Thus one obtains a proper element \(A(x) = T_a A^a(x) [A^a(x) \equiv (A^a_{\mu} dx^\mu, \Phi^A)] \) of the total degree one in the direct product of two graded algebras (cf., \([6, \text{Eq. (2.45)}]\) and \([7, \text{p. 629}]\)).

IV. GRADED FIELD STRENGTH

The graded field strength, \(F(x)\), is defined by means of taking the exterior derivative of the graded gauge potential, \(A(x)\), and adding its wedge product with itself, \(A(x) \wedge A(x)\) multiplied by the interaction constant:

\[
F(x) = dA(x) + (ig/2)A(x) \wedge A(x). \tag{5}
\]

Now we shall clarify the meaning of both operations. The first term on the right-hand side of (4) is given by

\[
dA(x) = \frac{1}{2!} T_a [\partial_{\mu} A_{\nu}^a(x) - \partial_{\nu} A_{\mu}^a(x)] dx^\mu \wedge dx^\nu + \tau_A d\Phi^A(x). \tag{6}
\]

The second term on the right-hand side of (4) needs more explanations. First, as in the case with Yang-Mills theory, one defines the wedge product for algebra valued forms \(A(x)\). Second, we have to deal with the odd part of graded Lie algebra and zero-forms involved in the graded
The very possibility of defining the graded potential and field strength opens up a number of further questions. Firstly, a problem of graded gauge invariance arises. We hope that existing in the literature (see, e.g. (3)) introduction of Grassman-odd transformation parameters for odd generators of the graded Lie algebra could provide appropriate solution to this problem. Secondly, as mentioned at the beginning of the current contribution, one is interested in a definition of a real-valued Lagrangian density. This would lead to physically acceptable Euler-Lagrange equations. In particular, we are going to consider an invariant with respect to the graded Lie algebra automorphisms expression

$$S = \int \text{str}(\mathcal{F} \wedge * \mathcal{F})$$

as the relevant action functional and explore the corresponding equations of motion.

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**V. DISCUSSION AND OUTLOOK**

The very possibility of defining the graded potential and field strength opens up a number of further questions. Firstly, a problem of graded gauge invariance arises. We hope that existing in the literature (see, e.g. (3)) introduction of Grassman-odd transformation parameters for odd generators of the graded Lie algebra could provide appropriate solution to this problem. Secondly, as mentioned at the beginning of the current contribution, one is interested in a definition of a real-valued Lagrangian density. This would lead to physically acceptable Euler-Lagrange equations. In particular, we are going to consider an invariant with respect to the graded Lie algebra automorphisms expression

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Nапряженность поля для градуированной теории Янга-Миллса

Определяется градуированной напряженностью поля для $osp(2/1; C)$ низкоэнергетической калибровочной алгебры. Показано, что пара четных гамильтоновских полей может являться составной частью градуированного 4-потенциала калибровочного поля нарушение с обя- чайной (гамильтоново четной) одн-формой принимающей значения в максимальной собственной $su(2)$-подалгебре Ли градуированной алгебры $osp(2/1; C)$. Обсуждаются некоторые возможности построения на этой основе физически приемлемого вариационного принципа.

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