ARTICLE TYPE

Local Metric with Parameterized Evolution

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We present a canonical Hamiltonian formulation of GR in which \( \tau \), the parameter of system evolution, is external to spacetime, playing a role similar to what we call time in nonrelativistic mechanics. This approach, known as Stueckelberg-Horwitz-Piron (SHP) theory, inherits the full computational power of classical analytical mechanics while maintaining manifest covariance throughout and eliminating possible conflict with general diffeomorphism invariance. In particular, SHP simplifies the initial value problem with potential applications in highly dynamical interactions, such as black hole collisions. By allowing the energy-momentum tensor and metric to depend explicitly on \( \tau \), we may describe particle motion in geodesic form with respect to a dynamically evolving background metric. As a toy model, we consider a \( \tau \)-dependent mass \( \mathcal{M}(\tau) \), first as a perturbation in the Newtonian approximation and then for a Schwarzschild-like metric. As expected, the extended Einstein equations imply a non-zero energy-momentum tensor, proportional to \( \mathcal{M}(\tau)/\tau \), representing a flow of mass and energy into spacetime that corresponds to the changing source mass. In \( \tau \)-equilibrium, this system becomes a generalized Schwarzschild solution for which the extended Ricci tensor and energy-momentum tensor vanish.

KEYWORDS:
general relativity, evolution theories, initial value problem

1 | INTRODUCTION

In general relativity, matter tells spacetime how to curve and spacetime curvature tells matter how to move. More precisely, a mass distribution given by \( T^{\alpha\beta}(x) \) induces a local metric \( g_{\alpha\beta}(x) \) through the Einstein equations, and the metric enters the equations of motion for particles. One may, in principle, construct the \( T^{\alpha\beta}(x) \) associated with particles \( x^\mu(\tau) \), find the induced metric and obtain equations of motion for matter in the resulting gravitational field. But calculations of this type may be intractable — even for linearized field equations. As Fock observed Fock (1937), it may be impractical to replace \( \tau \) by \( t \) as parameter of particle motion, in order to obtain \( T^{\alpha\beta} \) as a function of \( x^\mu \) alone. Later, Stueckelberg Stueckelberg (1941a, 1941b) argued that to fully characterize all possible spacetime trajectories, the 8D phase space \( x^\mu(\tau), \dot{x}^\mu(\tau) \) must be unconstrained, and \( \tau \) cannot be identified with the proper time of the motion, which may not be well-defined. Horwitz and Piron Horwitz & Piron (1973) developed this idea into a manifestly covariant canonical mechanics for the many-body problem Horwitz (2015); Horwitz & Arshansky (2018), in which gauge fields may depend explicitly on \( \tau \) so that the Stueckelberg-Horwitz-Piron (SHP) theory remains well-posed Saad, Horwitz, & Arshansky (1989).

The classical and quantum mechanics of particles in a spacetime with local metric \( g_{\mu\nu}(x) \) has been studied extensively by Horwitz Horwitz (2019) and will not be discussed at length here. Our goal is to find a consistent prescription for extending general relativity to accommodate an energy-momentum tensor \( T^{\alpha\beta}(x, \tau) \) and a metric \( g_{\alpha\beta}(x, \tau) \) satisfying \( \tau \)-dependent Einstein equations. As a guide we recall the construction of the SHP particle action as \( S_{\text{Maxwell}} \rightarrow S_{\text{SHP}} \), where

\[
S_{\text{Maxwell}} = \int d\tau \frac{1}{2} \mathcal{M}(\tau) \dot{x}^\mu \dot{x}_\mu + \frac{\mathcal{E}}{c} \dot{x}^\mu A_\mu(x^\lambda) \tag{1}
\]
for $\lambda, \mu, \nu = 0, 1, 2, 3,$ and
\[
S_{\text{SHP}} = \int d\tau \left( \frac{1}{2} M \ddot{x}^\mu \dot{x}_\mu + \frac{e}{c} \dot{x}^\mu a_\mu (x^5, \tau) + \frac{e}{c} c_5 a_5 (x^5, \tau) \right)
\]
\[
= \int d\tau \frac{1}{2} M \ddot{x}^\mu \dot{x}_\mu + \frac{e}{c} \dot{x}^\mu a_\mu (x^5, \tau) (2)
\]

where $a, \beta, \gamma = 0, 1, 2, 3, 5,$ and in analogy to $x^0 = ct,$ we write $x^5 = c_5 \tau.$ Variation with respect to $x^\mu$ leads to the Lorentz force \cite{Land & Horwitz(1991)} and expressing a wide freedom in assigning coordinate freedoms to the 4D manifold, before applying a dynamical principle.

\section{Canonical Mechanics}

\subsection{An approach to a $\tau$-dependent metric}

Standard general relativity considers the invariant interval
\[
\delta x^2 = g_{\mu \nu} \delta x^\mu \delta x^\nu = (x_2 - x_1)^2
\]

between two neighboring spacetime points, viewed as an instantaneous displacement. This invariance is a geometrical statement expressing a wide freedom in assigning coordinates to the manifold. To transform geometry into dynamics, a spacetime path maps an arbitrary parameter $\zeta$ to a continuous sequence of events $x^\mu(\zeta)$ such that any two points on the path have timelike separation, and so the proper time can be taken as the parameter. Consistent with the notion of a 4D block universe, the path consists of instantaneous displacements, observed as "motion" through changes in $x^0(\zeta)$ with $\zeta.$ Parameterizing by proper time $\tau,$ the line element
\[
\delta x^2 = g_{\mu \nu} \delta x^\mu \delta x^\nu = g_{\mu \nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \delta \tau^2 = g_{\mu \nu} \dot{x}^\mu \dot{x}^\nu \delta \tau^2
\]

suggests a dynamical description of the path by the action
\[
S = \int dx = \int d\tau \sqrt{-g_{\mu \nu} \dot{x}^\mu \dot{x}^\nu}
\]

and leading to geodesic equations of motion as an expression of the equivalence principle. The geodesic equations can also be derived from the action
\[
S = \int d\tau \frac{1}{2} g_{\mu \nu} \dot{x}^\mu \dot{x}^\nu
\]

which removes the constraint $\dot{x}^2 = -c^2$ associated with (7).

In SHP a physical event $x^\mu(\tau)$ theory occurs at time $\tau$ and chronologically precedes events occurring at subsequent times. The physical picture describes the evolution of one 4D block universe defined at time $\tau$ to another, infinitesimally close, at $\tau + d\tau.$ Evolution slows to zero in equilibrium. The $\tau$-dependence of $g_{\mu \nu}(x, \tau)$ reminds us that while geometric relations on spacetime are defined within a given block universe, the dynamics are defined in the transition from one 4D block manifold to another. We therefore consider the interval
\[
dx^2 = \tilde{x}^\alpha (\tau + \delta \tau) - x^\alpha (\tau)
\]

between an event $x^\alpha$ occurring at time $\tau$ and an event $\tilde{x}^\alpha$ occurring at a different spacetime location at a subsequent time $\tau + \delta \tau.$ With $\delta x^5 = c_5 \delta \tau$ we may write
\[
dx^2 = g_{\alpha \beta}(x, \tau) \delta x^\alpha \delta x^\beta
\]

which contains both the geometrical distance $\delta x^\alpha$ between two neighboring points in one manifold, and the dynamical distance $\delta x^5$ between events occurring at two sequential times. This line element suggests the free particle Lagrangian
\[
L = \frac{1}{2} M g_{\alpha \beta}(x, \tau) \dot{x}^\alpha \dot{x}^\beta
\]

containing both geometry and dynamics. In a formal and restricted sense, this system is defined on 5D coordinates $(x^\mu, x^5),$ as was done for the fields in (2) to (4). Nevertheless, we maintain the distinction between geometry and evolution. The parameter $x^5$ does not become a dynamical degree of freedom, and no five dimensional symmetries are ascribed to this 5D structure. While the terms $c_5 g_{55}(x, \tau) \dot{x}^\mu$ and $c_5^2 g_{55}(x, \tau)$ appear as interactions, we view them as features of an extended equivalence principle, in which $g_{\mu \nu}$ expresses geodesic motion within the curved geometry of a 4D block universe at any fixed $\tau,$ while $g_{55}$ express geodesic motion along the evolving structure of one 4D block universe to another.

\subsection{General 5D spacetime}

If (1) were a 5D free particle Lagrangian for $x^\alpha(\xi)$ then the Euler-Lagrange equations would provide geodesic equations
\[
0 = D\gamma_{\alpha \beta} - \gamma_{\alpha \gamma} \Gamma_{\gamma \beta}^{\alpha \gamma}
\]

where $D/D\xi$ is the absolute derivative (in the notation of \cite{Weinberg(1972)}) and
\[
\Gamma_{\alpha \beta}^{\gamma} = \gamma_{\beta \delta} \Gamma_{\alpha \beta}^{\gamma} = \frac{1}{2} \gamma_{\beta \delta} \left( \partial_\alpha g_{\delta \beta} + \partial_\beta g_{\delta \alpha} - \partial_\delta g_{\alpha \beta} \right)
\]

is the standard Christoffel symbol in 5D.

\footnote{Horwitz and Gershon \cite{Gershon & Horwitz(2009)} have shown that a covariant action with scalar potential is equivalent to a free action with local conformal metric. In their work, metric component $g_{5\alpha}$ and related components of the connection similarly appear.}
Writing the canonical momentum
\[ p_a = \frac{\partial L}{\partial \dot{x}^a} = M g_{a\beta} \dot{x}^\beta \rightarrow \dot{x}^a = \frac{1}{M} g^{a\beta} p_\beta \] (14)
the scalar Hamiltonian, representing particle mass,
\[ K = \dot{x}^a p_a - L = \frac{1}{2M} g^{a\beta} p_a p_\beta = L \] (15)
is conserved, as seen directly through
\[ \frac{d}{d\tau} \left( \frac{1}{2} M g_{a\beta} \dot{x}^a \dot{x}^\beta \right) = M g_{a\beta} \ddot{x}^\beta \frac{D \dot{x}^a}{D\tau} = 0 \] (16)
where we used metric compatibility \( D g_{a\beta} / D\xi = 0 \). This result may also be seen from the canonical equations of motion
\[ \dot{x}^a = \frac{d x^a}{d\xi} = \frac{\partial K}{\partial p_a} \quad \dot{p}_a = \frac{d p_a}{d\xi} = -\frac{\partial K}{\partial x^a} \] (17)
and the Poisson bracket
\[ \{F, G\} = \frac{\partial F}{\partial x^a} \frac{\partial G}{\partial p_a} - \frac{\partial F}{\partial p_a} \frac{\partial G}{\partial x^a} \] (18)
so that
\[ \frac{d K}{d\xi} = \{K, K\} + \frac{\partial K}{\partial z} = -\frac{1}{2M} g^{a\beta} p_a p_\beta \frac{\partial g_{a\beta}}{\partial z} = 0 \] (19)
using the \( \xi \)-independence of the metric.

2.3 Breaking the 5D symmetry to 4D+1

We break the 5D symmetry by putting \( x^5(\tau) = c_5 \tau \) in (11), which becomes
\[ L = \frac{1}{2} M g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{1}{2} M c^2_5 \dot{x}^5 + \frac{1}{2} M c^2_5 g_{55} . \] (20)
Comparing (12) with the equations of motion
\[ 0 = \ddot{x}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta + 2 c_5 \Gamma^\mu_{5\alpha} \dot{x}^\alpha + c^2_5 \Gamma^\mu_{55} \] (21)
\[ 0 = \frac{D\dot{x}^5}{D\tau} = \frac{d\dot{x}^5}{d\tau} = d c_5 \quad \frac{d\dot{x}^5}{d\tau} \] (22)
the prescription for symmetry breaking connection is
\[ \Gamma^\mu_{5\alpha} = \frac{1}{2} \delta^{\mu\nu} \left( \delta_{5\alpha} + \dot{a}_\alpha g_{55} - \dot{a}_5 g_{55} \right) \quad \Gamma^5_{a\beta} \equiv 0 \] (23)
We notice in particular the violation of the 5D symmetry
\[ \Gamma_{5\mu\nu} = -\Gamma_{5\mu\nu} + \partial_\nu g_{5\alpha} \] (24)
which no longer applies because the label 5 indicates a scalar quantity, and is not a tensor index.

Because \( x^5 \) is not a dynamical quantity and has no conjugate momentum, the Hamiltonian can be written
\[ K = \frac{1}{2} M g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - \frac{1}{2} M c_5^2 g_{55} \] (25)
which is not conserved for a \( \tau \)-dependent metric, as seen from
\[ \frac{d K}{d\tau} = \frac{\partial K}{\partial \tau} = -\frac{1}{2} M \dot{x}^\mu \dot{x}^\nu \frac{\partial g_{\mu\nu}}{\partial \tau} - \frac{1}{2} M c^2_5 \frac{\partial g_{55}}{\partial \tau} . \] (26)
Expression (26) can also be found by taking the total \( \tau \)-derivative of (25) and inserting the equations of motion (21).

2.4 Einstein equations

We define \( n(x, \tau) \) to be the number of events per spacetime volume, so that the total number of events is
\[ N(\tau) = \int d\Omega \ n(x, \tau) \] (27)
where \( d\Omega \) is a scalar spacetime volume element, and
\[ j^a(x, \tau) = \rho(x, \tau) \dot{x}^a(\tau) = M n(x, \tau) \dot{x}^a(\tau) \] (28)
is the 5-component event current. The continuity equation in flat space
\[ \partial_a j^a = \partial_\mu j^\mu + \partial_5 j^5 = \partial_\mu j^\mu + \frac{\partial \rho}{\partial \tau} = 0 \] (29)
leads to conservation of total event number through
\[ \frac{d}{d\tau} \int d\Omega \ n(x, \tau) + \frac{1}{M} \int d\Omega \partial_\mu j^\mu = \frac{dN}{d\tau} = 0 . \] (30)
With a local metric (29) is generalized (in the notation of Wald (1984)) to
\[ \nabla_a j^a = \nabla_a X^a = \partial_a j^a + j^\alpha \Gamma^a_{\alpha a} = 0 \] (31)
and since \( j^5 \) is a scalar on physical grounds, prescription (23) requires \( \Gamma^5_{a\beta} \neq 0 \), so that the continuity equation becomes
\[ \nabla_a j^a = \frac{\partial \rho}{\partial \tau} + \nabla_\mu j^\mu = 0 . \] (32)

Generalizing the 4D stress-energy-momentum tensor to 5D, we write the mass-energy-momentum tensor (Land (2019)) as
\[ T^{a\beta} = M n \dot{x}^a \dot{x}^\beta = \rho \dot{x}^a \dot{x}^\beta \] (33)
where in addition to the standard 4D components \( T^{\mu\nu} \), we have the current density \( T^{5\beta} = \dot{x}^a \dot{x}^\beta \). Conservation of mass-energy-momentum \( \nabla_\beta T^{a\beta} = 0 \) follows from the geodesic equations \( \dot{x}^\alpha \nabla_\beta \dot{x}^\alpha = D \dot{x}^\alpha / D\tau = 0 \) and continuity \( \nabla_\mu j^\mu = 0 \), when the equations of motion (21) and (22) are evaluated under the prescription (23).

The Einstein equations are similarly extended to
\[ G_{a\beta} = R_{a\beta} - \frac{1}{2} R g_{a\beta} = \frac{8\pi G}{c^4} T_{a\beta} \] (34)
where the Ricci tensor \( R_{a\beta} \) and scalar \( R \) are found by contracting indices of the curvature tensor \( R^a_{\nu\mu\beta} \) found from the Christoffel symbols. Conservation of \( T_{a\beta} \) depends on the prescription (23), and so in order to insure
\[ \nabla_\beta G^{a\beta} = 0 \] (35)
we must similarly suppress \( \Gamma^5_{a\beta} \) when constructing the Ricci tensor. Writing \( R_{a\beta} = R^\alpha_{a\beta\nu} = R^\nu_{a\beta\alpha} = R^\nu_{a\beta5} \) we proceed by separating out terms containing the 5 index, leading to
\[ R_{5\nu} = \left( R_{\nu5} \right)^{4D} \] (36)
\[ R_{\mu5} = 1/c_5 \partial_\mu \Gamma^\lambda_{\lambda5} - \partial_\lambda \Gamma^\lambda_{\mu5} + \Gamma^\lambda_{\sigma5} \Gamma^\sigma_{\mu5} - \Gamma^\lambda_{\sigma5} \Gamma^\sigma_{\mu5} \] (37)
\[ R_{55} = 1/c_5 \partial_5 \Gamma^\lambda_{55} - \partial_\lambda \Gamma^\lambda_{55} + \Gamma^\lambda_{55} \Gamma^5_{\lambda5} - \Gamma^\lambda_{55} \Gamma^5_{\lambda5} \] (38)
with new 5-terms from \( g_{5\mu} \) and the \( \tau \)-dependence of \( g_{\mu\nu} \).

### 3.1 Newtonian Approximation

At low energy the velocity can be approximated as

\[
\dot{x} = \left( c \frac{dt}{dr}, \frac{dx}{dr}, c_5 \right) \approx \left( c \frac{dt}{dr}, 1, 0, c_5 \right)
\]

so the equations of motion reduce to

\[
0 \approx \ddot{x} + c_5^2 \dddot{\Gamma}_{00} \ddot{c} + 2c_5 c_3 \dddot{\Gamma}_{00} i + c_5^2 \dddot{\Gamma}_{55} i
\]

Taking \( \partial_0 g_{\mu\nu} = 0 \) and \( g_{0k} = 0 \) in the weak field approximation

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}
\]

we have the new Christoffel symbols

\[
\Gamma_{50}^\mu = \frac{1}{2c_5} g^{\mu0} \partial_0 h_{00} \quad \Gamma_{55}^\mu = - \frac{1}{2} g^{\mu\nu} \partial_\nu h_{55}
\]

The equations of motion

\[
\ddot{x} = \frac{1}{2} c^2 \mathbf{g}^{\mu\nu} \partial_\nu h_{00} - c \left( g^{\mu0} \partial_\mu h_{00} \right) i + \frac{1}{2} c^2 s^\mu \partial_\mu h_{55}
\]

split into

\[
\ddot{i} = \left( \partial_\tau h_{00} \right) i \quad \ddot{x} = \frac{1}{2} c^2 \mathbf{\nabla} h_{00} + \frac{1}{2} c^2 \mathbf{\nabla} h_{55}
\]

and writing

\[
\frac{d^2 x}{dt^2} = \frac{d}{dt} \left( \frac{dt}{dr} \frac{dx}{dt} \right) = i^2 \frac{d^2 x}{dt^2} + \left( \partial_\tau h_{00} \right) \frac{dx}{dt}
\]

we obtain

\[
\frac{d^2 x}{dt^2} = \frac{1}{2} c^2 \mathbf{\nabla} h_{00} + \frac{1}{2} c^2 \mathbf{\nabla} h_{55} - \left( \partial_\tau h_{00} \right) \frac{dx}{dt}
\]

The unperturbed motion \( \left( \partial_\tau h_{\mu\nu} = 0 \right) \) and \( \nabla h_{55} = 0 \) recovers Newtonian gravitation

\[
\frac{d^2 x}{dt^2} = -\frac{Gm}{r^2} \quad r = -\nabla \phi \quad \phi = -\frac{Gm}{r}
\]

in the usual manner, with

\[
g_{00} = -1 - \frac{2GM}{rc^2}
\]

As a perturbation at the Newtonian level, we treat the mass \( M = M(\tau) \) as \( \tau \)-dependent. As was shown in [Land (2019)], under certain circumstances, an accelerated charged particle in SHP electrodynamics may radiate in such a way that its energy and momentum vary independently, leading to a change in the particle’s mass. This mass is transferred to the electromagnetic field, producing mass radiation terms in the mass-energy-momentum tensor. Now the \( \tau \) equation

\[
\ddot{i} = \left( \partial_\tau h_{00} \right) i = \frac{2GM}{rc^2} i
\]

can be solved as

\[
i = \exp \left( \frac{2GM(\tau)}{rc^2} \right)
\]

where \( \Delta M(\tau) = M(\tau) - M(\tau_0) \) is the change in mass from its initial value. Neglecting \( (\Delta M)^2 \) for small changes in mass and putting \( h_{55} = 0 \), the equations for Newtonian gravity in plane polar coordinates (with \( \theta = \theta/\ell dr/dt \)) become

\[
\frac{d^2 r}{dt^2} - r \dot{\theta}^2 + \frac{2GMdr}{rc^2} \frac{dr}{dt} = -\frac{GM}{r^2}
\]

\[
\frac{1}{r} \frac{d}{dt} \left( r^2 \dot{\theta} \right) + \frac{2GMd\theta}{c^2} = 0
\]

after separating the radial and angular parts. We see that the nonrelativistic angular momentum \( M r^2 \dot{\theta} \) is not generally conserved in this limit, because the transition to \( t \) as evolution parameter in (45) introduced a non-radial velocity-dependent force component. Neglecting deviation from radial descent (taking \( \dot{\theta}^2 \approx 0 \)), the radial equation reduces to

\[
\frac{d^2 r}{dt^2} + \frac{2GMd}{c^2} \frac{d}{dt} \ln r = -\frac{GM}{r^2}
\]

describing an object accelerating in a Newtonian gravitational field with an additional dissipative term.

### 4.1 Extended Schwarzschild Solution

We extend the spherically symmetric metric by permitting \( g_{00} = B \) and \( g_{11} = A \) (in the notion of Weinberg [1972]) to be \( r \)-dependent (but still \( t \)-independent) and introducing a fifth metric component \( g_{55} = Q \). Using the spherical symmetry to put \( \theta = \pi/2 \), the Lagrangian becomes

\[
L = \frac{1}{2} \left[ -c^2 B(r, \tau) \dot{i}^2 + A(r, \tau) \dot{r}^2 + r^2 \dot{\phi}^2 + c^2 s Q(r, \tau) \right]
\]

from which we find new non-zero Christoffel symbols

\[
\Gamma^0_{50} = \frac{1}{2c_5} \quad \Gamma^1_{55} = \frac{1}{2} \frac{\partial Q}{A} \quad \Gamma^1_{15} = \frac{1}{2c_5} \frac{\partial A}{A}
\]

and new non-zero Ricci tensor components

\[
R_{05} = -\frac{1}{rc^2} \frac{\partial_\tau B}{\partial r} \quad R_{15} = \frac{1}{2c_5} \left[ \frac{\partial_\tau \partial_\tau B}{B} + 2 \frac{\partial_\tau B}{rB} \right]
\]

\[
R_{55} = \frac{1}{2} \left( \frac{c^2 Q}{B} - \frac{\left( \partial_\tau Q \right) \left( \partial_\tau B \right)}{B^2} \right) + \frac{1}{2c_5} \left( \frac{\partial_\tau B}{B} \right)^2 + \frac{\partial_\tau Q}{rB}
\]

suggesting a highly dynamical system. Taking \( M = M(\tau) \) to generalize the Schwarzschild solution

\[
B(r, \tau) = A^{-1}(r, \tau) = 1 - \frac{2GM(\tau)}{rc^2}
\]

we still have \( R_{\mu\nu} = (R_{\mu\nu})^{4D} = 0 \), but \( R_{05}, R_{15}, \) and \( R_{55} \) will contain non-vanishing terms proportional to \( dM/dr \). Setting the terms in \( R_{55} \) containing \( Q \) to zero we find

\[
Q = \int r \frac{B}{r^2} dr = \frac{1}{r} \left( \frac{M(\tau)G}{c^2 r} \right) = \frac{1}{2r} (3 - B)
\]
leading to the Einstein tensor \cite{54} with components

\[ G_{\mu\nu} = -\gamma_r^2 \left( 1 - \frac{MG}{c^2 r} \right)^{-1} g_{\mu\nu} \quad G_{55} = \frac{1}{r^2} \gamma_r^2 \]
\[ G_{50} = \frac{2\gamma_r}{r} \left( 1 - \frac{2MG}{c^2 r} \right)^{-1} \quad G_{51} = -\frac{\gamma_r}{r} \left( 1 - \frac{2MG}{c^2 r} \right)^{-1} \]

(59)

where \( \gamma_r = \frac{G}{c^2} \frac{dM}{d\tau} \left( 1 - \frac{2MG}{c^2 r} \right)^{-1} \).

The equations of motion found from Lagrangian \cite{54} admit solutions

\[ i = \frac{1}{B} = \left( 1 - \frac{2GM(\tau)}{rc^2} \right)^{-1} \quad \dot{\phi} = \frac{J}{r^2} \]

(60)

for constant \( J \) and the radial equation

\[ \frac{d}{d\tau} \left( -c^2 \frac{1}{B} + A\dot{\varphi}^2 + \frac{J^2}{r^2} - c^2 Q \right) = -\frac{1}{2} \dot{r}^2 \partial_r A - \frac{1}{2} c^2 \partial_r Q \]

(61)

We recognize the LHS as the Hamiltonian

\[ K = \frac{1}{2} \dot{x}^\mu \dot{x}^\mu - \frac{1}{2} c^2 s_{55} \]

(62)

in this parameterization, which as we saw in \cite{26} is not conserved when the metric is \( r \)-dependent. In equilibrium, when \( dM/d\tau = 0 \), we recover

\[ R = 0 \quad G_{a\beta} = 0 \quad j_a = 0 \quad \frac{dK}{d\tau} = 0 \]

(63)

describing a massless system (away from the origin), and the radial equation admits a first integral solution.

\section{5 DISCUSSION}

Interactions in SHP electrodynamics form an integrable system in which event evolution generates an instantaneous current defined over spacetime at \( \tau \), and in turn, these currents induce \( r \)-dependent fields that act on other events at \( \tau \). We expect that in a similar way, a fully-developed SHP formulation of general relativity will describe how the instantaneous distribution of mass and energy at \( \tau \) expressed through \( T_{a\beta}(x, \tau) \) induces the local metric \( g_{a\beta}(x, \tau) \), which in turn, determines geodesic equations of motion for any particular event at \( x^\mu(\tau) \).

Through the simplified example of a \( r \)-dependent mass point, we found that the effect of the mass shift on the metric leads, by way of the Einstein equations, to mass and energy radiation into the radial and time directions of spacetime. We also saw that this metric induces non-conservation of mass in a particle in extended geodesic motion. In this sense, as in SHP electrodynamics, matter may exchange mass through the medium of local spacetime. We therefore expect that SHP general relativity will be a useful computational tool in relativistic dynamics.

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