Twist Field as Three String Interaction Vertex in Light Cone String Field Theory

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Abstract

It has been suggested that matrix string theory and light-cone string field theory are closely related. In this paper, we investigate the relation between the twist field, which represents string interactions in matrix string theory, and the three-string interaction vertex in light-cone string field theory carefully. We find that the three-string interaction vertex can reproduce some of the most important OPEs satisfied by the twist field.

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1 Introduction

Retrospecting recent progress in understanding various interesting effects in string theory, we are led to the desire of constructing a complete off-shell formulation of string theory. As will be explained below, at present there are two formulations for light-cone quantization of type IIB closed superstring theory. However, neither of them is considered to be complete.

One of the formulations is the light-cone superstring field theory (LCSFT) \[1, 2\] constructed from supersymmetry algebra. The starting point is the Green-Schwarz action for free strings. We can construct the Hamiltonian and the supercharges satisfying the supersymmetry algebra out of it. The interaction terms are added to these charges by requiring that the total charges satisfy the supersymmetry algebra perturbatively. The first order interaction term is given as

\[
H_{1}^{123} = Z_{ij}^{i}v^{ij}(|V|)_{123},
\]

\[
Q_{1}^{123} = \bar{Z}_{i}^{i}v^{i\bar{a}|\Lambda}|V|_{123},
\]

\[
\tilde{Q}_{1}^{123} = Z_{i}^{i}\bar{s}^{i\bar{a}|\Lambda}|V|_{123}.
\]

Here \(|V|_{123}\) is the three-string interaction vertex constructed by the overlapping condition and \(Z_{ij}^{i}\) (\(\bar{Z}_{i}^{i}\)) is the holomorphic (anti-holomorphic) part of the bosonic momentum at the interaction point, whose divergence is regularized as follows:

\[
\left( P^{i} + \frac{1}{2\pi\alpha}X^{\mu}|(\sigma)|_{123} \sim \frac{1}{\sqrt{\sigma - \sigma_{I}}}Z^{i}|V|_{123},
\]

with \(\alpha = p^{+}\) and \(\sigma_{I}\) being the interaction point. \(\Lambda\) is the regularization of the fermionic momentum at the interaction point and \(v^{ij}(\Lambda)\), \(s^{i\bar{a}}(\Lambda)\) and \(\bar{s}^{i\bar{a}}(\Lambda)\) are known but intricate functions of \(\Lambda\). The program of constructing the interaction terms is successful at the first order, though it is too complicated to proceed to higher orders.

The other formulation is matrix string theory (MST) \[3, 4\], which stems from the Matrix formulation of light-cone quantization of M-theory \[5\] and takes the form of (1+1)-dimensional super Yang-Mills theory. To relate MST to the perturbative string, we first note that the Yang-Mills coupling \(g_{YM}\) is related to the string coupling \(g_{s}\) and the string length \(\sqrt{\alpha'}\) by \(g_{YM}^{-1} = g_{s}\sqrt{\alpha'}\). Hence, the free string limit corresponds to the IR limit and the first order interaction term to the least irrelevant operator. From the requirement of the dimension counting and the locality of the interaction, we expect that the first order interaction term is written as dimension three operator constructed essentially out of the twist field. The interaction term of MST is proposed to be \[4\]

\[
H_{1} = \sum_{m,n} \int d\sigma \left( \tau^{i}\Sigma^{i}\bar{r}^{j}\Sigma_{j}^{j} \right)_{m,n},
\]

where \(\tau^{i}\) is the excited twist field defined as

\[
\partial X^{i}(z, \sigma) \sim \frac{1}{\sqrt{z}}\tau^{i}(0),
\]
with $\sigma(z, \bar{z})$ being the $\mathbb{Z}_2$ twist field and $\Sigma^i(z)$ being the spin field for the Green-Schwarz fermions. The indices $m$ and $n$ of the twist fields denote the string bits where the “exchange” interaction takes place. These indices have to be summed over in calculating the string amplitude.

The expression (5) in MST seems somewhat formal compared with that in LCSFT (1), though it is more promising to go beyond the first order in MST than in LCSFT [6]. Hopefully we can obtain some information in LCSFT from MST. For this purpose, we would like to relate LCSFT to MST carefully. We can easily find a close analogy between (1) and (5) and between (4) and (6), if we regard $\sigma(z, \bar{z})$ as $|V\rangle_{123}$ and $\tau^i(z, \bar{z})$ as $Z^i|V\rangle_{123}$. Following this analogy between LCSFT and MST, two supercharges of MST were written down explicitly in [7]. These arguments of supercharges are consistent with the pioneering but primitive argument in [4] and with the relation between LCSFT and MST proposed in [6].

In this paper, we would like to proceed further to investigate the relation between the twist field $\sigma(z, \bar{z})$ and the three-string interaction vertex $|V\rangle_{123}$ scrupulously. In particular, in addition to the defining OPE of the excited twist field (6), we would like to realize the OPE of two twist fields [8, 9] (for each dimension)

$$\sigma(z, \bar{z}) \cdot \sigma(0) \sim \frac{1}{|z|^{1/4} (\ln |z|)^{1/2}},$$

in terms of the three-string interaction vertex $|V\rangle_{123}$. To realize the OPE (7), we identify the interaction point $\sigma_I$ of $|V\rangle_{123}$ with the insertion point $z$ of the twist field $\sigma(z, \bar{z})$. We multiply two string interaction vertices with a short intermediate time $T$ to see whether the effective interaction vertex reproduces the reflector (which corresponds to the identity operator in CFT) with the suitable singularity.

A natural question arises here. In LCSFT the three-string interaction vertex $|V\rangle_{123}$ is always accompanied by the level-matching projection

$$P_r = \oint \frac{d\theta}{2\pi} e^{i\theta(L_0^{(r)} - \bar{L}_0^{(r)})},$$

on each string $r = 1, 2, 3$. We would like to see which one corresponds to the twist field; the interaction vertex with projections, $P_1P_2P_3|V\rangle_{123}$, or the vertex without them, $|V\rangle_{123}$. Our answer to this question is as follows. To calculate the amplitude in LCSFT, we need to integrate over the intermediate string length ($\alpha = p^+$) and perform the level-matching projection at each string. The level-matching projection is equivalent to integrating over the diagrams by shifting the interaction point by an angle. These two integrations are combined into a simple summation over string bits $m$ and $n$ in (5). Since the twist field by itself is the expression before the summations, the corresponding interaction vertex in LCSFT should not contain any summations. Hence, the interaction vertex corresponding to the twist field is the one $|V\rangle_{123}$ without the level-matching projection and the intermediate string length integration.

There are two ways to realize (7) because the interaction vertices can be connected in two different ways. One of them is the four-point tree diagram connecting the long string of two
interaction vertices (fig. 1) and the other is the two-point 1-loop diagram connecting two short strings (fig. 2). We shall evaluate these two diagrams in the next section to see that both of the results are proportional to the reflector with the same singularity as that in (7).

Note that it is desirable to perform all the computations of the above diagrams in the superstring theory, if we want to relate LCSFT to MST. However, here we shall utilize the bosonic string theory for simplicity [10]. It should not be too difficult to generalize our computation to the supersymmetric case. Also note that the separation of the two interaction points in the above diagrams is in the worldsheet time direction, while we separate two insertion operators is given in appendix B.

The content of this paper is as follows. In the next section, we shall first recapitulate some necessary ingredients of LCSFT. The first subsection is devoted to the computation of the tree diagram and in the second and third subsections we compute the 1-loop diagram. Finally, we conclude with some further directions. A short review of Neumann coefficients is given in appendix A. A somewhat related result about free field realization of boundary changing operators is given in appendix B.

2 LCSFT computation

In this section, we would like to evaluate the two diagrams mentioned in the introduction to see the correspondence between the twist field $\sigma(z, \bar{z})$ and the interaction vertex $|V\rangle_{123}$. For this purpose, let us briefly review the closed LCSFT here. Three-string interaction vertex for LCSFT with $\alpha = p^+$ fixed is given as

$$|V(1_{\alpha_1}, 2_{\alpha_2}, 3_{\alpha_3})\rangle = [\mu(\alpha_1, \alpha_2, \alpha_3)]^2 \int \delta(1, 2, 3) e^{E(1, 2, 3) + E(1, 2, 3)|p_1\rangle_1 |p_2\rangle_2 |p_3\rangle_3},$$

where $\alpha_1 + \alpha_2 + \alpha_3 = 0$ and

$$\mu(\alpha_1, \alpha_2, \alpha_3) = \exp \left(-\tau_0 \sum_{r=1}^{3} \frac{1}{\alpha_r}\right), \quad \tau_0 = \sum_{r=1}^{3} \alpha_r \log |\alpha_r|,$$

$$\int \delta(1, 2, 3) = \int \frac{d^d p_1}{(2\pi)^{d-2}} \frac{d^d p_2}{(2\pi)^{d-2}} \frac{d^d p_3}{(2\pi)^{d-2}} (2\pi)^{d-2} \delta^{d-2}(p_1 + p_2 + p_3),$$

$$E(1, 2, 3) = \frac{1}{2} \sum_{r=1}^{3} \sum_{n, m \geq 1} N^r_{m, n} a^{(r)\dagger}_{m} a^{(r)\dagger}_{n} + \sum_{n \geq 1} N^{a}_{n} a^{(a)\dagger}_{n} \mathbb{P}^{a}_{123} - \frac{\tau_0}{2\alpha_1 \alpha_2 \alpha_3} \mathbb{P}^{2}_{123},$$

with $\mathbb{P}^{a}_{123} = \alpha_1 p^a_1 + \alpha_2 p^a_2 + \alpha_3 p^a_3$ ($i = 1, \cdots, d - 2$). We define the normalized left-moving oscillators $a^i_n = \alpha^i_n / \sqrt{n}$, $\bar{a}^i_n = \alpha^{-i\dagger}_n / \sqrt{n}$ for $n \geq 1$ satisfying $[a^i_n, a^{(j)\dagger}_m] = \delta_{n, m} \delta^{i\dagger j}$ and $a^i_n |p\rangle = 0$. $\bar{a}^i_n$ is the right-moving cousin and $E(1, 2, 3)$ is defined by replacing $a^i_n$ by $\bar{a}^i_n$ in $E(1, 2, 3)$.

For later convenience, let us note that $E(1, 2, 3)$ can be recast into the following form

$$E(1, 2, 3) = \frac{1}{2} \mathbf{a}^{(3)\dagger T} N^{3, 3} \mathbf{a}^{(3)} + \mathbf{a}^{(3)\dagger T} N^{3, 12} \mathbf{a}^{(12)} + \frac{1}{2} \mathbf{a}^{(12)\dagger T} N^{12, 12} \mathbf{a}^{(12)}$$
if we adopt the matrix notation for the indices of infinite oscillation modes
\[(a^{(r)})_m = a_m^{(r)}, \quad (a^{(r)\dagger})_m = a_m^{(r)\dagger}, \quad (N^r)_m = N_m^r, \quad (N^{r,s})_{mn} = N_{mn}^{r,s}, \tag{14}\]
and define
\[a^{(12)\dagger} = \begin{pmatrix} a^{(1)\dagger} \\ a^{(2)\dagger} \end{pmatrix}, \quad N^{12} = \begin{pmatrix} N^1 \\ N^2 \end{pmatrix}, \]
\[N^{3,12} = \begin{pmatrix} N^{3,1} & N^{3,2} \end{pmatrix}, \quad N^{12,3} = \begin{pmatrix} N^{1,3} \\ N^{2,3} \end{pmatrix}, \quad N^{12,12} = \begin{pmatrix} N^{1,1} & N^{1,2} \\ N^{2,1} & N^{2,2} \end{pmatrix}. \tag{15}\]

In terms of the matrix notation, the Neumann coefficients satisfy among others:
\[N^{3,12}N^{12,3} + N^{3,3}N^{12,3} = 1, \quad N^{3,12}N^{12,12} + N^{3,3}N^{12,12} = 0, \quad N^{12,12}N^{12,12} + N^{12,3}N^{12,12} = 1, \]
\[N^{3,12}N^{12} + N^{3,3}N^{12} = -N^3, \quad N^{12,12}N^{12} + N^{12,3}N^3 = -N^{12}, \]
\[N^{3T}(N^{12,3})^{-1}N^{12} = -\frac{\tau_0}{\alpha_1\alpha_2\alpha_3}, \tag{16}\]
which play important roles in our later computation. More details about the Neumann coefficients can be found in appendix A.

The reflector for closed LCSFT is given as
\[\langle R(1, 2) \rangle = \int \delta(1, 2) \langle p_1 | p_2 \rangle e^{-\langle a^{(1)\dagger} a^{(2)} + \bar{a}^{(1)\dagger} \bar{a}^{(2)} \rangle}, \tag{17}\]
with
\[\int \delta(1, 2) = \int \frac{dd-2p_1}{(2\pi)^{d-2}} 2^{d-2} \frac{dd-2p_2}{(2\pi)^{d-2}} \delta^{d-2}(p_1 + p_2), \tag{18}\]
and \(\langle p | p' \rangle = (2\pi)^{d-2} \delta^{d-2}(p - p'), \langle p | a_1^{\dagger} \rangle = 0\) and \(\langle p | a_1^{\dagger} \rangle = 0\).

We shall utilize the following formula in our later calculation.
\[\langle 0 | \exp\left(\frac{1}{2} a^T M a + a^T k\right) \exp\left(\frac{1}{2} a^{(1)\dagger} N a + a^{(1)\dagger} l\right) | 0 \rangle\]
\[= [\det(1 - MN)]^{-1/2} \exp\left(l^T \frac{1}{1 - MN} k + \frac{1}{2} k^T N \frac{1}{1 - MN} k + \frac{1}{2} l^T \frac{1}{1 - MN} M l\right). \tag{19}\]

\(^1\)One may wonder why \(N^{12,3}\) is invertible because it does not look like a square matrix. To clarify this point, let us regularize the size of the infinitely-dimensional Neumann matrices by truncating it at a finite level. Due to the expression of the mode expansion \([63]\), the UV cutoff of the worldsheet \(\Delta \sigma_r\) of string \(r\) is related to the truncation level \(L_r\) by \(L_r \Delta \sigma_r / |\alpha_r| \sim 1\). If we fix the UV cutoff of the worldsheet to a constant and choose string 3 to be the long string, \(|\alpha_3| = |\alpha_1| + |\alpha_2|\), the truncation levels of three strings have to be related by \(L_3 = L_1 + L_2\). In this sense \(N^{12,3}\) is a square matrix.
2.1 Tree diagram

We now turn to the evaluation of the four-point tree diagram with two incoming strings 1 (with length \( \alpha_1(>0) \)) and 2 (with length \( \alpha_2(>0) \)), joining and splitting again into two outgoing strings 4 and 5 of the same length as 1 and 2, respectively. (See fig. 1.) Note that since only the two twist fields which exchange the same string bits enjoy the OPE (7), we have to choose \( \alpha_4 = -\alpha_1 \) and \( \alpha_5 = -\alpha_2 \) so that the exchange interactions take place at the same string coordinate. The effective four-string interaction vertex of this diagram is given as \( (\alpha' = 2) \)

\[
|A(1, 2, 4, 5)\rangle = \langle R(3, 6)|e^{-\frac{T}{\alpha_3}(L_0^{(3)} + \bar{L}_0^{(3)})}|V(1_{\alpha_1}, 2_{\alpha_2}, 3_{\alpha_3})\rangle|V(4_{-\alpha_1}, 5_{-\alpha_2}, 6_{-\alpha_3})\rangle, \tag{20}
\]

with \( L_0 = p^ip^j/2 + n_{n\geq 1} n_n^i a_n^j - 1 \) and \( \bar{L}_0 = p^ip^j/2 + \sum_{n\geq 1} n_n^i \bar{a}_n^j - 1 \). Note that under the simultaneous change of signs \( \alpha_1 \rightarrow -\alpha_1, \alpha_2 \rightarrow -\alpha_2, \alpha_3 \rightarrow -\alpha_3 \), \( N_{r, s} \) is even while \( N^r \) is odd. Hence the Neumann coefficients in \( |V(4_{-\alpha_1}, 5_{-\alpha_2}, 6_{-\alpha_3})\rangle \) can be written in terms of those of \( |V(1_{\alpha_1}, 2_{\alpha_2}, 3_{\alpha_3})\rangle \). From the OPE (7) we expect that \( |A(1, 2, 4, 5)\rangle \) is proportional to the reflector \( |R(1, 4)\rangle|R(2, 5)\rangle \) with the coefficient divergent as \( (T^{-1/4}(\ln T)^{-1/2})^{24} = (T(\ln T)^2)^{-6} \) if we take the limit \( T \rightarrow +0 \). Here we identify the coordinate \( z \) of the twist field \( \sigma(z, \bar{z}) \) with the intermediate propagation time \( T \) up to a numerical factor when we are interested only in the short distance behavior of two operators. Though generally \( z \) may be related to \( T \) in a complicated way, the relation is approximately linear at a short distance.

![Figure 1: Four-string tree diagram in the \( \rho \)-plane.](image)

In order to perform our calculation simply, let us first rewrite the proper time expression of the propagator \( e^{-\frac{T}{\alpha_3}(L_0^{(3)} + \bar{L}_0^{(3)})} \) into \( e^{-\frac{T}{\alpha_3}(L_0^{(3)} + L_0^{(3)} + L_0^{(6)} + L_0^{(6)})} \). By applying the formula (19) with

\[
\begin{align*}
\mathbf{a} &= \mathbf{a}^{(36)}, & M &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, & N &= \begin{pmatrix} N_{T/2}^{3, 3} & 0 \\ 0 & N_{T/2}^{3, 3} \end{pmatrix}, \\
\mathbf{k} &= \mathbf{0}, & l &= \begin{pmatrix} N_{T/2}^{3, 12} & 0 \\ 0 & N_{T/2}^{3, 12} \end{pmatrix} \mathbf{a}^{(45)\dagger} + \begin{pmatrix} N_{T/2}^{3, 12} \mathbf{1}_{123} \\ -N_{T/2}^{3, 12} \mathbf{1}_{456} \end{pmatrix}, \tag{21}
\end{align*}
\]
with $N_{3,3}^T = e^{-\frac{\pi}{\nu_1} C} N_{3,3} e^{-\frac{\pi}{\nu_1} C}$, $N_{3,12}^T = e^{-\frac{\pi}{\nu_1} C} N_{3,12}$ and $N_{3}^T = e^{-\frac{\pi}{\nu_1} C} N_{3}$, we can compute $|A(1, 2, 4, 5)|$ without difficulty:

$$|A(1, 2, 4, 5)| = A_T \int \delta(1, 2, 4, 5) e^{F_T(1, 2, 4, 5)} |p_1| |p_2| |p_4| |p_5|_5,$$

with

$$A_T = \left|\mu(\alpha_1, \alpha_2, \alpha_3)\right|^2 \det \frac{d-2}{\pi} \left(1 - N_{T/2}^{3,3} N_{T/2}^{3,3}\right)^2,$$

$$\int \delta(1, 2, 4, 5) = \int \frac{d^{d-2}p_1}{(2\pi)^{d-2}} \frac{d^{d-2}p_2}{(2\pi)^{d-2}} \frac{d^{d-2}p_4}{(2\pi)^{d-2}} \frac{d^{d-2}p_5}{(2\pi)^{d-2}} (2\pi)^{d-2} \delta^{d-2}(p_1 + p_2 + p_4 + p_5).$$

The exponent $F_T(1, 2, 4, 5)$ takes a complicated expression. However, in the limit $T \to +0$ we can evaluate it formally with the use of (16) and $\mathbb{P}_{123} - \mathbb{P}_{456} = \alpha_3(p_1 + p_4)$ which holds because of the momentum conservation $(2\pi)^{d-2} \delta^{d-2}(p_3 + p_5)$. The formal result of it is as follows.

$$\lim_{T \to +0} F_T(1, 2, 4, 5) = -\left(\alpha^{(12)}|^T a^{(45)}|^T + \bar{a}^{(12)}|^T a^{(45)}|^T\right)$$

$$- (p_1 + p_4) \alpha_3 N_{3T}^{3T} (N_{12,3}^{3T})^{-1} \left(\alpha^{(12)}|^T + \bar{a}^{(12)}|^T + \bar{a}^{(45)}|^T\right)$$

$$- (p_1 + p_4)^2 \alpha_3^2 N_{3T}^{3T} (1 - (N_{3,3}^{3T})^2)^{-1} N_3^T.$$

Note that the first term gives the nonzero modes of the reflector. This is the first sign that our expectation works. Also the last term will give the zero mode part of the reflector $\delta^{d-2}(p_1 + p_4)$ if the quantity $b_0 = \alpha_3^2 N_{3T}^{3T} (1 - (N_{3,3}^{3T})^2)^{-1} N_3^T$ is divergent. Actually this seems to be true from numerical analysis. We can fit as $b_0 \simeq (1 \sim 3) \log L + (\text{constant})$, where $L$ is the size of Neumann matrices which we used in our numerical computation.

To regularize it properly, let us retrieve the intermediate time $T$ in our calculation. We find that, instead of the divergent quantity $b_0$, we have $b_T = \alpha_3^2 N_{3T}^{3T} (1 - (N_{3,3}^{3T})^2)^{-1} N_3^T$ which, according to (C.18), (C.20) and (C.21) in [11], is identified to be

$$b_T = \alpha_3^2 N_{3T}^{3T} (1 - (N_{3,3}^{3T})^2)^{-1} N_3^T = - \log(1 - Z_5).$$

Here we have mapped the worldsheet in the “light-cone” type $\rho$-plane into the whole complex $\rho$-plane by the Mandelstam map

$$\rho(z) = \alpha_1 (\log(z - Z_1) - \log(z - Z_4)) + \alpha_2 (\log(z - Z_2) - \log(z - Z_5)),$$

and fixed the gauge by choosing $Z_1 = \infty, Z_2 = 1, Z_4 = 0$ and $0 < Z_5 < 1$. Note that, without the insertion of the level-matching projections, the moduli parameter, $Z_5$, runs only along the real axis. To see the behavior of (26) in the limit $T \to +0$, all we have to do is to relate $T$ to $Z_5$ as in Chapter 11 of [12]. For this purpose, we note that $T$ can be regarded as the difference of two stationary points $z_{\pm}$ in the $\rho$-plane:

$$T = \rho(z_+) - \rho(z_-), \quad \frac{d\rho}{dz}\bigg|_{z = z_\pm} = 0.$$
Since the limit $T \to +0$ corresponds to the $t$-channel limit, $Z_5$ should approach $Z_2 = 1$ in this limit. After an explicit calculation, we find

$$\frac{T}{|\alpha_3|} \sim 4 \sqrt{\frac{\alpha_1 \alpha_2}{\alpha_3}} \sqrt{1 - Z_5},$$  \hspace{1cm} (29)$$

and hence $b_T \sim 2 \log(|\alpha_3|/T)$. This is consistent with the numerical result if we identify the regularization parameter by $|\alpha_3|/T \sim L$. Therefore, the contribution of the exponential of the last term in (25) is

$$e^{-b_T(p_1+p_4)^2} \sim \left[ \frac{\pi}{2 \log(|\alpha_3|/T)} \right]^{d-2} \delta^{d-2}(p_1 + p_4).$$ \hspace{1cm} (30)$$

This implies that

$$(2\pi)^{d-2} \delta^{d-2}(p_1 + p_2 + p_4 + p_5)e^{-b_T(p_1+p_4)^2} \alpha \sim (2\pi)^{d-2} \delta^{d-2}(p_1 + p_4)(2\pi)^{d-2} \delta^{d-2}(p_2 + p_5),$$ \hspace{1cm} (31)$$

which gives the zero mode part of the reflector.

The determinant factor $A_T$ in (22) is already evaluated as ($d = 26$)

$$A_T \sim 2^{10} \left| \frac{\alpha_1 \alpha_2}{\alpha_3} \right|^2 \left[ \frac{T}{|\alpha_3|} \right]^{-6},$$ \hspace{1cm} (32)$$
in \[13,11]\). (See also appendix B of \[14\].) Combining all the contributions, we find that in the limit $T \to +0$,

$$|A(1, 2, 4, 5)\rangle \sim 2^{-26} \pi^{-12} \left[ \frac{T}{|\alpha_{123}|} \left( \log \frac{T}{|\alpha_{123}|} \right)^2 \right]^{-6} |R(1, 4)|R(2, 5)\rangle,$$ \hspace{1cm} (33)$$

with $\alpha_{123} = (\alpha_1 \alpha_2 \alpha_3)^{1/3}$. This is consistent with our expectation.

### 2.2 1-loop diagram

In the previous subsection, we have computed one realization of the OPE (7). Here we would like to proceed to the other realization via the 1-loop diagram: the incoming string 6 splits into two short strings and join again into the outgoing string 3. (See fig. [2]) For this purpose, let us calculate ($\alpha_1, \alpha_2 > 0$)

$$|B(3, 6)\rangle = \langle R(2, 5)\langle R(1, 4)|e^{-\frac{T}{\alpha_1}(L_0^{(1)} + L_0^{(4)}) - \frac{T}{\alpha_2}(L_0^{(2)} + L_0^{(5)})}|V(1_{\alpha_1}, 2_{\alpha_2}, 3_{\alpha_3})\rangle|V(4_{-\alpha_1}, 5_{-\alpha_2}, 6_{-\alpha_3})\rangle.$$ \hspace{1cm} (34)$$

The calculation is parallel to the previous case of the tree diagram. Using (19) we obtain

$$|B(3, 6)\rangle = B_T \int \frac{d^{d-2}p_1}{(2\pi)^{d-2}} \int \delta(3, 6) e^{F_T(3, 6, p_1)}|p_3\rangle_3|p_6\rangle_6.$$ \hspace{1cm} (35)$$
with

\[ B_T = \left[ \mu(\alpha_1, \alpha_2, \alpha_3) \right]^2 \det^{-\frac{d-2}{2}} \left( 1 - N_{12,12}^T N_{12,12}^T \right), \quad (36) \]

and

\[ N_{12,12}^T = \text{diag}(e^{-\frac{2}{\alpha_1}C}, e^{-\frac{2}{\alpha_2}C}) N_{12,12} \text{diag}(e^{-\frac{2}{\alpha_1}C}, e^{-\frac{2}{\alpha_2}C}). \]

Again, the exponent \( F_T(3, 6, p_1) \) can be evaluated in the limit \( T \to +0 \) using various formulas of Neumann coefficients (16) as

\[ \lim_{T \to +0} F_T(3, 6, p_1) = -(a^{(3)\dag}a^{(6)\dag} + \bar{a}^{(3)\dag}\bar{a}^{(6)\dag}). \quad (37) \]

Namely, (35) is proportional to \( |R(3, 6)| \) including the zero mode sector. However, the integration of the loop momentum \( p_1 \) gives a divergent constant \( \delta^{d-2}(0) \) for \( T = 0 \). Therefore, we need to regularize \( |B(3, 6)| \) by the intermediate time \( T \) again:

\[ F_T(3, 6, p_1) = F_T(3, 6, p_1)_{\text{osc}} + c_T \left[ p_1 - \frac{\alpha_1}{\alpha_3} p_3 \right]^2 + C_T^T (a^{(3)\dag} - a^{(6)\dag} + \bar{a}^{(3)\dag} - \bar{a}^{(6)\dag}) \left[ p_1 - \frac{\alpha_1}{\alpha_3} p_3 \right] + \left[ \frac{\alpha_3^2}{\alpha_1 \alpha_2} \right] T, \quad (38) \]

with \( F_T(3, 6, p_1)_{\text{osc}} \) being the oscillator bilinear part of \( F_T(3, 6, p_1) \) and \( c_T \) and \( C_T \) being

\[ c_T = 2\alpha_3^2 \left( \frac{T}{2} - \frac{\tau_0}{\alpha_1 \alpha_2 \alpha_3} \right) + N_{12}^T (1 - N_{12,12}^T N_{12,12}^T)^{-1} (N_{12}^T + N_{12,12}^T N_{12}^T), \quad (39) \]

\[ C_T = \alpha_3 \left[ N_{12}^2 + N_{3,12}^2 (1 - N_{12,12}^T N_{12,12}^T)^{1/2} (N_{12}^T + N_{12,12}^T N_{12}^T) \right], \quad (40) \]

where we have used \( \mathbb{P}_{123} = \mathbb{P}_{456} = \alpha_3 p_1 - \alpha_1 p_3 \) and defined \( N_{12}^T = \text{diag}(e^{-\frac{2}{\alpha_1}C}, e^{-\frac{2}{\alpha_2}C}) N_{12}^T \). Note that our previous result (37) is equivalent to the following statement.

\[ \lim_{T \to +0} F_T(3, 6, p_1)_{\text{osc}} = -(a^{(3)\dag}a^{(6)\dag} + \bar{a}^{(3)\dag}\bar{a}^{(6)\dag}) \quad \text{with} \quad \lim_{T \to +0} c_T = 0, \quad \lim_{T \to +0} C_T = 0. \quad (41) \]

After we perform the loop momentum \( p_1 \) integration, the result \( |B(3, 6)| \) for \( T \neq 0 \) becomes

\[ |B(3, 6)| = B_T(4\pi c_T)^{-\frac{d-2}{2}} \int \delta(3, 6) e^{-\frac{1}{4\pi c_T} (C_T^T (a^{(3)\dag} - a^{(6)\dag} + a^{(3)\dag} - a^{(6)\dag})^2 + (\frac{\alpha_3^2}{\alpha_1 \alpha_2} - \frac{\alpha_3^2}{\alpha_1 ^2 \alpha_2}) T} \]
\[
\times e^{F_T(3,6,p_1)|_{\text{loss}}|P_3|_3|P_6|_6}.
\]

If we can further prove that \((n, m \geq 1)\)
\[
\lim_{T \to +0} \frac{(C_T)_m(C_T)_n}{c_T} = 0,
\]
we have
\[
|B(3, 6)\rangle \sim K_T|R(3, 6)\rangle,
\]
with \(K_T = B_T(4\pi c_T)^{-\frac{d-2}{2}}\) for \(T \to +0\). This assumption \([13]\) seems to be true from our numerical analysis, though it is still desirable to prove it algebraically. The numerical analysis strongly suggests that our result is proportional to the reflector \(|R(3, 6)\rangle\). In the next subsection, we would like to turn to the evaluation of the leading order of \(K_T\) for \(d = 26\), to see the singular behavior of the OPE \((7)\).

### 2.3 Evaluation of \(K_T\)

It is difficult to calculate \(K_T\) in \([14]\) directly using the Neumann coefficients. For the evaluation of \(K_T\) let us contract \(|B(3, 6)\rangle\) with two tachyon states. Since the full propagator including the light-cone directions is given as
\[
\Delta_r = \frac{1}{-2p^+_r p^-_r + L^{(r)}_0 + \bar{L}^{(r)}_0} = \int_0^\infty \frac{dT}{\alpha_r} e^{-\frac{T}{\alpha_r}(-2p^+_r p^-_r + L^{(r)}_0 + \bar{L}^{(r)}_0)},
\]
the total amplitude (without applying the level-matching projections \(\mathcal{P}_1\mathcal{P}_2\) on string 1 and 2) is given as
\[
S_{36} = 3\langle -k_3|_6\langle -k_6|R^{\text{LC}}(2, 5)\rangle|R^{\text{LC}}(1, 4)\rangle\Delta_1 \Delta_2 |V^{\text{LC}}(1, 2, 3)\rangle|V^{\text{LC}}(4, 5, 6)\rangle
= (2\pi)^d \delta^d(k_3 + k_6) \int d\alpha_1 \int_0^\infty dT_1 \int_0^\infty dT_2 \delta(T_2 - T_1) e^{2T_1 k^-_3} \frac{4\pi \alpha_1 \alpha_2}{4\pi \alpha_1 \alpha_2} K_T,
\]
where \(T = T_1 = T_2\) and the light-cone directions are included in the tachyon state \(|k]\), the reflector \(|R^{\text{LC}}\rangle\) and the interaction vertex \(|V^{\text{LC}}\rangle\). According to \([15]\) this light-cone expression can be calculated in the \(\alpha = p^+\) HIKKO string field theory:
\[
S_{36} = 3\langle -k_3|_6\langle -k_6|R^{\alpha=p^+}\rangle(2, 5)\rangle|R^{\alpha=p^+}(1, 4)\rangle \frac{b^{(1)}_0\overline{b}^{(1)}_0}{L^{\text{tot}(1)}_0 + \overline{L}^{\text{tot}(1)}_0} \frac{b^{(2)}_0\overline{b}^{(2)}_0}{L^{\text{tot}(2)}_0 + \overline{L}^{\text{tot}(2)}_0}
\times |V^{\alpha=p^+}(1, 2, 3)\rangle|V^{\alpha=p^+}(4, 5, 6)\rangle.
\]
Here the reflector \(|R^{\alpha=p^+}\rangle\) and the interaction vertex \(|V^{\alpha=p^+}\rangle\) are those of the \(\alpha = p^+\) HIKKO string field theory and \(L^{\text{tot}}_0\) and \(\overline{L}^{\text{tot}}_0\) are the total Virasoro operators including the light-cone directions of the matter part and the ghost part.
We can calculate it with CFT by mapping the light-cone \( \rho \)-plane into the torus \( u \)-plane, where the incoming string 6 and the outgoing string 3 in the \( \rho \)-plane are mapped to \( U_6 \) and \( U_3 \) in the \( u \)-plane respectively. (See fig. 3.) The Mandelstam map \( \rho(u) \) [16], which corresponds to fig. 3 is given as

\[
\rho(u) = |\alpha_3| \log \frac{\vartheta_1(u - U_6|\tau)}{\vartheta_1(u - U_3|\tau)} - 2\pi i \alpha_1 u . \tag{48}
\]

Here \( \tau \) and \( U_6 - U_3 \) are pure imaginary because we do not insert the level-matching projections \( P_1 P_2 \) in (47). The moduli parameter \( \tau \) of the torus is related to \( T \) in fig. 2 by finding the stationary points of the Mandelstam map as in (28),

\[
T = \rho(u_-) - \rho(u_+) , \quad \left. \frac{d\rho}{du} \right|_{u=\pm} = 0 . \tag{49}
\]

In the degenerating limit \( T \to +0 \), we have [17]:

\[
e^{-i\pi \tau} \sim \frac{T}{8|\alpha_3| \sin(\pi \alpha_1/|\alpha_3|)} . \tag{50}
\]

![Figure 3: The Mandelstam map from the \( \rho \)-plane to the \( u \)-plane. The left figure is the light-cone \( \rho \)-plane corresponding to fig. 2 while the right one represents the \( u \)-plane of the torus with periods 1 and \( \tau \). They are related by \( \rho(U_6) = -\infty, \rho(U_3) = \infty, \rho(u_+) = \rho_+ \) and \( \rho(u_-) = \rho_- \).]

Using the Mandelstam map (48) and the proper time representation of \( 1/(L_0^{\text{tot}(r)} + \bar{L}_0^{\text{tot}(r)}) \), (47) can be put into the CFT expression on the torus \( u \)-plane:

\[
S_{36} = \int_0^\infty dT_1 \int_0^\infty dT_2 \left\langle (\alpha_1 \alpha_2)^{-1}(\alpha_1 \alpha_2)^2 b_{T_1} b_{T_2} b_{T_2} V(k_3; U_3, \bar{U}_3) V(k_6; U_6, \bar{U}_6) C \right\rangle_\tau . \tag{51}
\]

Note that we have to put the measure \( (\alpha_1 \alpha_2)^{-1} \) inside the CFT correlator because the Mandelstam map (48) depends on \( \alpha_r \). Here \( T_1 \) and \( T_2 \) are the worldsheet proper time of the propagating strings 1 and 2 while \( b_{T_i} \) and \( V(k; u, \bar{u}) \) are defined as

\[
b_{T_i} = \int_{C_i} \frac{du \, d\bar{u}}{2\pi i d\rho} b(u) , \quad \bar{b}_{T_i} = \int_{\bar{C}_i} \frac{d\bar{u} \, d\bar{u}}{2\pi i d\rho} b(\bar{u}) . \tag{52}
\]
with the integration contours $C_i$ and $C_j$ shown in fig. [3]. Note that in the $\alpha = p^+$ HIKKO string field theory, the propagator $b_0\bar{b}_0/(L_0^{tot} + L_0^{tot})$ is originally defined on the light-cone $\rho$-plane, but the vertex operator $V(k; u, \bar{u})$ comes from that constructed on the unit disk. Therefore we have to take the conformal factor into account. The factor $(\alpha_1 \alpha_2)^2$ is from mapping four of $b_0$ in the $\rho$-plane of each string into the total $\rho$-plane, while the factor $C$ is the conformal factor for the tachyon vertices mapped from the local unit disk $w_\tau$ of each string into the torus $u$-plane:

$$ C = \left. \frac{du}{dw_3} \right|_{u = U_3} \left. \frac{du}{dw_6} \right|_{u = U_6} |k_3^2|^{-2}, $$

(54)

where

$$ \left. \frac{du}{dw_3} \right|_{u = U_3} \left. \frac{du}{dw_6} \right|_{u = U_6} = - \left( \frac{\vartheta_1(U_6 - U_3|\tau)}{\vartheta_1'(0|\tau)} \right)^2 e^{2\pi i \alpha_3 (U_6 - U_3 - \frac{\tau}{\alpha_3})}, $$

(55)

with $\vartheta_1'(\nu|\tau) = \partial_\nu \vartheta_1(\nu|\tau)$ and

$$ \rho = \begin{cases} 
\alpha_3 \log w_3 + T/2 & \text{Re} \, \rho > T/2 \\
-\alpha_3 \log w_6 - T/2 & \text{Re} \, \rho < -T/2 
\end{cases}. $$

(56)

Now all we have to do is to evaluate each sector of (51). This was done explicitly in [18] for the open string case. The ghost sector and the nonzero mode contribution of the matter sector are exactly the square of the open string case. The only difference comes from the zero mode contribution of the matter sector and still it can be evaluated similarly to the open string case. The contribution from the ghost part is (See (6.5) in [18].)

$$ \langle b_{T_1} \bar{b}_{T_1} b_{T_2} \bar{b}_{T_2} c(U_3) \bar{c}(U_3) c(U_6) \bar{c}(U_6) \rangle_\tau = \left| \frac{R}{2\pi} G \right|^2, $$

(57)

where $R$ and $G$ are defined as

$$ R^{-1} = |\alpha_3| (g'_1(u_+ - U_6|\tau) - g'_1(u_+ - U_3|\tau)), \quad G = \frac{2\pi i}{|\alpha_3|} \eta(\tau)^2, $$

(58)

with $g'_1(\nu|\tau) = \partial_\nu [\log \vartheta_1(\nu|\tau)]$ and $\eta(\tau) = e^{\frac{\pi i}{\alpha_3}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})$. The contribution from the matter part is

$$ \langle e^{ik_3 X(U_3, U_3)} : e^{ik_6 X(U_6, U_6)} : \alpha_1 \alpha_2 C \rangle_\tau = (2\pi \text{Im} \tau) \int d\alpha_1 \delta(T_2 - T_1) $$

$$ \times \delta^d(k_3 + k_6)(2\text{Im} \tau)^{-\frac{d}{2}} e^{-\frac{\pi k_3^2}{\text{Im} \tau} (U_3 - U_6 - (U_3 - U_6))^2} |\eta(\tau)|^{-2d} \left| \frac{\vartheta_1(U_3 - U_6|\tau)}{\vartheta_1'(0|\tau)} \right|^{-2k_3^2} \alpha_1 \alpha_2 C. $$

(59)
The derivation of the first line from the zero mode sector of the matter part is technical. See [18] for more details. Note that \( \delta(T_2 - T_1) = \delta(\rho(u + \tau) - \rho(u)) \) implies that the correlator gives a nonzero result only when

\[
U_6 - U_3 = \frac{\alpha_1}{|\alpha_3|} \tau,
\]

is satisfied.

Combining the ghost, matter contribution and the conformal factor, we find \( (d = 26) \)

\[
\left\langle \alpha_1 \alpha_2 b_{T1} \bar{b}_{T1} b_{T2} \bar{b}_{T2} V(k_3; U_3, \bar{U}_3) V(k_6; U_6, \bar{U}_6) C \right\rangle_{\tau}
\]

\[
\sim (2\pi)^d \delta^d(k_3 + k_6) \int d\alpha_1 \delta(T_2 - T_1) 2^{-28} \pi^{-13} \frac{\alpha_1 \alpha_2}{\alpha_3^4} \left[ \frac{\log T}{|\alpha_3|} \right]^2 \left[ \frac{\log \frac{T}{|\alpha_3|}}{\alpha_3} \right]^{-6},
\]

for \( T = T_1 = T_2 \to +0 \). Therefore we find \( |B(3, 6)| \) is proportional to the reflector with the singular coefficient as expected from (7):

\[
|B(3, 6)| \sim 2^{-26} \pi^{-12} \left[ \frac{T}{|\alpha_{123}|} \left( \log \frac{T}{|\alpha_{123}|} \right) \right]^2 \left[ \frac{T}{|\alpha_{123}|} \right]^{-6} |R(3, 6)|,
\]

if we compare the result of LCSFT (40) and that of the \( \alpha = p^+ \) HIKKO string field theory (61).

3 Discussion

In this paper, we investigated the correspondence between the twist field \( \sigma(z, \bar{z}) \) and the three-string interaction vertex \( |V\rangle_{123} \) in LCSFT. We evaluated two diagrams corresponding to the OPE (7) and found that both of them, (33) and (62), showed the same behavior including the log factor as expected from the calculation of the twist field.

We would like to list several further directions.

- Due to some technical difficulties, our computation of the Neumann matrices is not completely satisfactory. First of all, we only perform the numerical analysis for (43) instead of proving it algebraically. Secondly, to evaluate \( K_T \) we have to detour to the CFT techniques and the \( \alpha = p^+ \) HIKKO string field theory. We hope we will have more direct computation tools in the future.

- One of our original motivations comes from construction of LCSFT. After relating the twist field \( \sigma(z, \bar{z}) \) with the three-string vertex \( |V\rangle_{123} \) carefully in this paper, we would like to see how matrix string theory can help in the construction of LCSFT. As explained in the introduction, it is difficult to proceed to construction of higher order contact terms in LCSFT. We would like to see whether we can construct higher order terms explicitly with the help of MST. Our realization of the twist field via the three-string interaction vertex has been considered in the bosonic string theory in this paper. The first step should be to generalize our computation to the superstring case.
In the fermionic sector, it is a popular fact that the spin fields can be realized by the fundamental free bosons. It is interesting to see whether the free field realization has anything to do with our realization via the three-string interaction vertex in LCSFT. Though there is no simple free field realization for the twist field, we can construct one for its open string cousin, the boundary changing operator. Since the boundary changing operator changes the boundary conditions between the Neumann type and the Dirichlet type, or in other words, changes the signs of the anti-holomorphic part, it can be regarded as the twist field in the open string sector. We present the free field realization of the boundary changing operator in appendix B by applying a result of [21]. Hopefully, we can relate the free field realization with our current realization via the three-string vertex in the future.

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A Neumann coefficients

In this appendix, we would like to briefly review the Neumann coefficients. The Neumann coefficient matrix are constructed from the overlapping condition

$$\left( P^{(1)}(\sigma)\Theta(-\pi\alpha_1 < \sigma < \pi\alpha_1) + P^{(2)}(\sigma - \pi\alpha_1)\Theta(\pi\alpha_1 < \sigma) + P^{(2)}(\sigma + \pi\alpha_1)\Theta(\sigma < -\pi\alpha_1)
+ P^{(3)}(\pi(\alpha_1 + \alpha_2) - \sigma)\Theta(0 < \sigma) + P^{(3)}(-\pi(\alpha_1 + \alpha_2) - \sigma)\Theta(\sigma < 0) \right)|V\rangle_{123} = 0,$$

of the momentum function of each string

$$P^{(r)}(\sigma) = \frac{1}{2\pi|\alpha_r|} \left[ p^{(r)c} \cos \frac{n\sigma}{|\alpha_r|} + p^{(r)s} \sin \frac{n\sigma}{|\alpha_r|} \right],$$

which states that in the string interaction process the momentum is conserved along the string worldsheet. $\Theta$(inequality) denotes the step function, which takes value 1 if the inequality holds and otherwise 0.
Our next task is to rewrite the overlapping condition in terms of the mode expansion. If we normalize the momentum \( \pi \) so that the trigonometric functions have the unit norm:

\[
\pi^{(i)c} = \frac{1}{2\pi|\alpha_i|} \left( \frac{p^{(i)}}{\sqrt{C p^{(i)c}}} \right), \quad \pi^{(i)s} = \frac{1}{2\pi|\alpha_i|} \sqrt{C p^{(i)s}},
\]

the overlapping condition can be recast into the following form

\[
\pi^{(3)c} = (U_1 U_2) \left( \begin{pmatrix} \frac{-\alpha_1/\alpha_3}{\sqrt{C/2}} B \end{pmatrix} = \begin{pmatrix} 0^T \\ \frac{-\alpha_1/\alpha_3}{\sqrt{C/2}} A^{(1)} \end{pmatrix} \right), \quad \pi^{(3)s} = (V_1 V_2) \left( \begin{pmatrix} \frac{-\alpha_1/\alpha_3}{\sqrt{C/2}} \end{pmatrix} \right),
\]

where \( U_1, U_2, V_1, V_2 \) are defined as

\[
U_1 = \begin{pmatrix} \sqrt{-\frac{\alpha_1}{\alpha_3}} \sqrt{\frac{C}{2}} B & -\frac{\alpha_1}{\alpha_3} \sqrt{C A^{(1)}} \frac{1}{\sqrt{C}} \end{pmatrix}, \quad U_2 = \begin{pmatrix} -\frac{\alpha_2}{\alpha_3} \sqrt{\frac{C}{2}} B & \frac{\alpha_2}{\alpha_3} \sqrt{C A^{(2)}} \frac{1}{\sqrt{C}} \end{pmatrix},
\]

\[
V_1 = \sqrt{-\frac{\alpha_3}{\alpha_1}} C A^{(1)} \sqrt{C}, \quad V_2 = \sqrt{-\frac{\alpha_3}{\alpha_2}} C A^{(2)} \sqrt{C},
\]

with \( A^{(1)}, A^{(2)}, B \) and \( C \) being

\[
(A^{(1)})_{mn} = \sqrt{\frac{n}{m}} \frac{(-1)^m}{\pi \alpha_1} \int_0^{\pi \alpha_1} 2 \cos \frac{n \sigma}{\alpha_1} \cos \frac{m \sigma}{\alpha_3} d\sigma = \sqrt{\frac{m}{n}} \frac{(-1)^m}{\pi \alpha_3} \int_0^{\pi \alpha_3} 2 \sin \frac{n \sigma}{\alpha_1} \sin \frac{m \sigma}{\alpha_3} d\sigma, \quad (68)
\]

\[
(A^{(2)})_{mn} = \sqrt{\frac{n}{m}} \frac{(-1)^m}{\pi \alpha_2} \int_0^{\pi (\alpha_1 + \alpha_2)} 2 \cos \frac{n (\sigma - \pi \alpha_1)}{\alpha_2} \cos \frac{m \sigma}{\alpha_3} d\sigma = \sqrt{\frac{m}{n}} \frac{(-1)^m}{\pi \alpha_3} \int_0^{\pi (\alpha_2 + \alpha_3)} 2 \sin \frac{n (\sigma - \pi \alpha_1)}{\alpha_2} \sin \frac{m \sigma}{\alpha_3} d\sigma, \quad (69)
\]

\[
(B)_m = \frac{2(-1)^{m+1}}{\sqrt{m \pi \alpha_1 \alpha_2}} \int_0^{\pi \alpha_1} \cos \frac{m \sigma}{\alpha_3} d\sigma = \frac{2(-1)^m}{\sqrt{m \pi \alpha_1 \alpha_2}} \int_{\pi \alpha_1}^{\pi (\alpha_1 + \alpha_2)} \cos \frac{m \sigma}{\alpha_3} d\sigma, \quad (70)
\]

\[
(C)_{mn} = m \delta_{mn}. \quad (71)
\]

Since the overlapping condition relates the incoming string momentum with the outgoing string momentum, it does not drop any information. Therefore, the transformation matrices \( (U_1, U_2) \) and \( (V_1, V_2) \) should be unitary \[19\]. By requiring the unitarity of these matrices, we have \( (r, s = 1, 2) \)

\[
-\frac{\alpha_r}{\alpha_3} A^{(r)T} C A^{(s)} = \delta_{rs} C, \quad A^{(r)T} C B = 0, \quad \frac{1}{2} \alpha_1 \alpha_2 B^T C B = 1, \quad -\frac{\alpha_3}{\alpha_r} A^{(r)T} \frac{1}{C} A^{(s)} = \delta_{rs} \frac{1}{C},
\]

\[
\sum_{t=1}^3 \alpha_t A^{(t)} \frac{1}{C} A^{(T)t} = \frac{1}{2} \alpha_1 \alpha_2 \alpha_3 B B^T, \quad \sum_{r=1}^3 \frac{1}{\alpha_r} A^{(r)} C A^{(r)T} = 0, \quad (72)
\]
if we define \((A^{(3)})_{mn} = \delta_{mn}\) in addition. In fact, these identities are proved in [1].

As in [1], the three-string interaction vertex can be constructed by matching the momentum eigenstates. After the Gaussian integration, we find the result is given by (9) with the Neumann coefficient matrices given as

\[
N^r_s = \delta^r_s - 2A^{(r)T} \Gamma^{-1} A^{(s)}, \quad N^r = -A^{(r)T} \Gamma^{-1} B, \quad \Gamma = 1 + A^{(1)} A^{(1)T} + A^{(2)} A^{(2)T}, \quad (73)
\]

With this formal expression of the Neumann coefficients, we can prove all the formulas we need in this paper (16), as well as [20]

\[
\sum_{t=1}^3 N^{r,t} N^{t,s} = \delta_{r,s}, \quad \sum_{t=1}^3 N^{r,t} N^{t} = -N^r, \quad \sum_{t=1}^3 N^{rT} N^{t} = \frac{2\tau_0}{\alpha_1 \alpha_2 \alpha_3}. \quad (74)
\]

### B Free field realization of boundary changing operators

In this appendix we would like to present a free field realization of the boundary changing operators. In [22] a certain class of boundary deformations

\[
S_{\text{int}} = -\frac{1}{2} \int d\theta \left( g \exp \frac{iX(\theta)}{\sqrt{2}} + \bar{g} \exp -\frac{iX(\theta)}{\sqrt{2}} \right), \quad (75)
\]

was solved exactly by the boundary state

\[
\langle B \rangle = \langle N \rangle \exp \left( -i\pi (g_r J_0^+ + \bar{g}_r J_0^-) \right), \quad (76)
\]

when the target space is compactified at the self-dual radius. Here \(g_r\) is the renormalized coupling constant, which equals to 0 when the boundary interaction satisfies the Neumann boundary condition, \(g = 0\), and to 1/2 when it satisfies the Dirichlet boundary condition, \(g = \infty\). In [21] the relation between the bare coupling constant \(g\) and the renormalized one \(g_r\) was worked out explicitly and the coupling constant \(g\) is further generalized into an external source \(g(\theta)\) depending on the boundary coordinate \(\theta\). The result is

\[
\langle B \rangle = \langle N \rangle \exp \left[ \int d\theta \left( \frac{1}{2} g_r(g(\theta), \bar{g}(\theta)) e^{i\sqrt{2}X_L(\theta)} + \frac{1}{2} \bar{g}_r(g(\theta), \bar{g}(\theta)) e^{-i\sqrt{2}X_L(\theta)} \right) \right], \quad (77)
\]

\[
g_r(g(\theta), \bar{g}(\theta)) = \frac{2}{\pi |g(\theta)|} \frac{g(\theta)}{|g(\theta)|} \arctan \left[ \tanh \left( \frac{\pi}{2} |g(\theta)| \right) \right]. \quad (78)
\]

If we choose \(g(\theta)\) to be

\[
g(\theta) = g \Theta(\phi_1 < \theta < \phi_2), \quad (79)
\]

and take the limit \(g \to \infty\) finally, we can realize a situation with one part of the boundary satisfying the Neumann boundary condition while the other satisfying the Dirichlet boundary condition. This boundary is the same as that with two boundary changing operators inserted.
Hence we can consider the realization of the boundary changing operators using the result of [21]. With this choice of \(g(\theta)\) we have

\[
g_e(g(\theta), \bar{g}(\theta)) = \frac{1}{2} \Theta(\phi_1 < \theta < \phi_2).
\]

In terms of the mode expansion \(J^a(z) = \sum_{m=-\infty}^{\infty} J^a_m / z^{m+1}\), our result is expressed by

\[
\langle B | = \langle N | \exp \sum_{n=-\infty}^{\infty} \frac{(w_{1}^{-n} - w_{2}^{-n})}{2n} J^1_n,
\]

with \(w_i = e^{-i\phi_i}\). Here we have used the Fourier expansion of the step function,

\[
\Theta(\phi_1 < \theta < \phi_2) = \frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{i \sin(\theta - \phi_2) - e^{-i\phi_1}}{4n}.
\]

Note that we have used the bookkeeping notation for the zero mode \(n = 0\). More correctly, the zero mode should be spelled out as \((\phi_2 - \phi_1)/4\).

Therefore, a naive candidate for the boundary changing operators is

\[
\sigma_\pm(w) \simeq \exp \pm \left( \sum_{m \neq 0} \frac{w^{-m}}{2m} J^1_m - \log w J^1_0 \right).
\]

The sign \(\pm\) is chosen depending whether the boundary operator changes the Neumann type into the Dirichlet type or vice versa. In the expression of (83) we are not careful about the zero mode and the normal ordering. This ambiguity can be fixed by requiring that the boundary changing operators are the primary fields with dimension \((1/16, 0)\). Since we have

\[
\sum_{m \neq 0} \frac{w^{-m}}{2m} J^1_m - \log w J^1_0 - \frac{i}{\sqrt{2}} \chi_L(w) = -i X_L(w),
\]

if we replace the direction 1 by the direction 3 in the exponent of (83), our boundary changing operators should be written as

\[
\sigma_\pm(w) = e^{-i\pi J^3_0/2} : \exp \pm \left( -\frac{i X_L(w)}{2\sqrt{2}} \right) : e^{i\pi J^3_0/2},
\]

which are \((1/16, 0)\) primary fields.

It is difficult to proceed to simplify the expression (85) of the boundary changing operators but we can check the OPEs

\[
\sigma_+(z) \cdot \sigma_-(0) \sim \frac{1}{z^{1/8}}, \quad \partial X_L(z) \cdot \sigma_\pm(0) \sim \frac{1}{\sqrt{z}} \tau_\pm(0).
\]

The latter one is shown by transforming the operator \(\partial X_L(z)\) by the SU(2) rotation, instead of rotating the boundary changing operators. Since under the \(J^0_0\) rotation \(J^3(z) = i \partial X_L(z)/\sqrt{2}\) is transformed into \(J^1(z) =: \cos \sqrt{2} X_L(z) :\), the latter one of (86) is easily found by noting

\[
: e^{i\sqrt{2} X_L(z)} : \cdot : e^{-i \pi J^3_0}/z \cdot X_L(0) : \sim \frac{1}{\sqrt{z}} : e^{i \pi J^3_0}/z \cdot X_L(0) :.
\]

Here we have used \(X_L(z) \cdot X_L(0) \sim -\ln z\).
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