Relativistic BCS-BEC Crossover at Zero Temperature

Lianyi He and Pengfei Zhuang

Physics Department, Tsinghua University, Beijing 100084, China

(Dated: April 27, 2018)

We investigate the BCS-BEC crossover at zero temperature in the frame of a relativistic model. The universality of the BCS-BEC crossover for non-relativistic systems breaks down in relativistic case and the crossover can be induced by changing the density. When the effective scattering length is much less than the fermion Compton wavelength, we recover the non-relativistic result if the gas is initially in non-relativistic state. At ultra-strong coupling where the scattering length is of the order of the Compton wavelength, a new BEC state appears. In this state the condensed bosons become nearly massless and anti-fermions are excited. The behavior of the Goldstone mode and the mixing between the amplitude and phase modes are significantly different in different condensed regions.

PACS numbers: 11.30.Qc, 12.38.Lg, 11.10.Wx, 25.75.Nq

I. INTRODUCTION

It was found many years ago that the Bardeen-Cooper-Shriffer (BCS) superfluidity/superconductivity in degenerate Fermi gas and the Bose-Einstein condensation (BEC) of composite molecules can be smoothly connected when Eagles and Leggett found that the ground state wave functions of BCS and BEC states are essentially the same[1, 2]. The BCS-BEC crossover phenomenon can be realized in high temperature superconductivity and atomic Fermi gas where the s-wave scattering length can be adjusted[3, 4, 5, 6].

Recently, the study on BCS-BEC crossover has been extended to relativistic Fermi systems with the Nozieres-Schmitt-Rink (NSR) theory[7, 8] and the Bose-Fermi model[9]. In the NSR theory above the critical temperature, a new feature of BCS-BEC crossover is found: There exist two kinds of BEC states[7], the non-relativistic BEC (NBEC) and the relativistic BEC (RBEC). In the RBEC state, the condensed bosons become nearly massless, anti-matter can be excited, and matter and anti-matter are nearly degenerate. A natural question is: Does the RBEC appear only in relativistic systems and never exist in non-relativistic systems? In this paper, we will argue that the RBEC can happen in any system if the attractive coupling is large enough, even though the system is initially in non-relativistic state. From text books[10], when the Fermi momentum of a system is much smaller than the fermion mass, we can treat the system non-relativistically. This statement is true only if the coupling is weak enough. For sufficiently strong coupling, the system will become relativistic even though the Fermi momentum is much smaller than the fermion mass.

We will extend in this paper the theory of BCS-BEC crossover in symmetry breaking state[3] at zero temperature to relativistic systems. The requirement for such an extension is that we should recover the non-relativistic theory in a proper limit. We will focus on two relativistic effects: the anti-fermion degrees of freedom and the non-trivial fermion mass. The first effect leads to the appearance of the RBEC state at sufficiently strong coupling, and the second effect breaks the universality of BCS-BEC crossover. We will also investigate the collective modes evolution in the BCS-BEC crossover.

II. RELATIVISTIC THEORY OF BCS-BEC CROSSOVER

The physical motivation why we need a relativistic theory of BCS-BEC crossover is mostly due to the study of QCD phase diagram, especially the dense quark matter which may exist in compact stars and can be created in heavy ion collisions. However, we will point out that the theory is also necessary for non-relativistic systems when the coupling is strong enough. To this end, let us first review the non-relativistic theory of BCS-BEC crossover in a dilute Fermi gas.

Leggett’s mean field theory[11] is successful to describe the non-relativistic BCS-BEC crossover at zero temperature. For a dilute Fermi gas with fixed density $n = k_f^3/(3\pi^2)$, where $k_f$ is the Fermi momentum, the BCS-BEC crossover can be found if one self-consistently solves the gap and number equations for the pairing gap $\Delta_0$ and the fermion chemical potential $\mu_n$,

\begin{equation}
\frac{k_f^3}{3\pi^2} = \int \frac{d^3k}{(2\pi)^3} \left(1 - \frac{\xi_k}{E_k}\right),
\end{equation}

where $E_k$ and $\xi_k$ are defined as $E_k = \sqrt{\xi_k^2 + \Delta_0^2}$ and $\xi_k = k^2/(2m) - \mu_n$, and $a_s$ is the s-wave scattering length. The fermion mass $m$ plays a trivial role here, since the BCS-BEC crossover depends only on the dimensionless parameter $\eta = 1/(k_f a_s)$, which is the so-called universality for non-relativistic systems. The BCS-BEC crossover can be characterized by the chemical potential $\mu_n$. It coincides with the Fermi energy $\epsilon_f = k_f^2/(2m)$ in the BCS limit $\eta \to -\infty$ but becomes negative in the BEC region. In the BEC limit $\eta \to +\infty$, one has $\mu_n \to -E_b/2$.
with \( E_b = 1/(ma^2) = 2\eta^2\epsilon_f \) being the molecular binding energy. Therefore, in the non-relativistic theory the chemical potential will tend to be negatively infinity in the BEC limit.

A problem arises if we look into the physics of the BEC limit from a relativistic point of view. In the relativistic description, the fermion dispersion becomes \( \xi_{\pm} = \sqrt{k^2 + m^2 \pm \mu_r} \), where + corresponds to fermion and anti-fermion degrees of freedom, and \( \mu_r \) is the chemical potential in relativistic theory. In the non-relativistic limit \( |k| \ll m \), if \( \mu_r \) is of the order of \( m \), we can neglect the anti-fermion degrees of freedom and recover the non-relativistic dispersion \( \xi_{\pm} \approx k^2/(2m) - (\mu_r - m) \). Therefore, not \( \mu_r \) itself but the quantity \( \mu_r - m \) plays the role of non-relativistic chemical potential \( \mu_n \). While the chemical potential \( \mu_n \) can be arbitrarily negative in non-relativistic theory, \( \mu_r \) is under some physical constraint in relativistic theory. Since the molecule binding energy can not be larger than two times the constituent mass, the absolute value of the non-relativistic chemical potential \( \mu_n = \mu_r - m = -E_b/2 \) at strong enough coupling can not exceed the fermion mass \( m \), and the relativistic chemical potential \( \mu_r \) should always be positive. If the number density \( n \) satisfies \( k_f \ll m \), the non-relativistic theory works well when the coupling is not very strong and the binding energy satisfies \( E_b \ll 2m \). However, if the coupling is strong enough to ensure \( E_b \sim 2m \), relativistic effects will appear. From \( E_b \approx 1/(ma^2) \), we can roughly estimate that the non-relativistic theory becomes unphysical for the s-wave scattering length \( a_s \sim 1/m \). This can be understood if we consider \( 1/m \) as the Compton wavelength \( \lambda_c \) of a particle.

What will then happen at \( \eta \to +\infty \) in an attractive Fermi gas? At sufficiently strong coupling with \( E_b \to 2m \) and \( \mu_r \to 0 \), the dispersions \( \xi_{\pm} \) and \( \xi_{\pm} \) for fermions and anti-fermions become nearly degenerate, and non-relativistic limit can not be reached even for systems with \( k_f \ll m \). This means that the anti-fermion pairs can be excited and become nearly degenerate with the fermion pairs, and the condensed bosons and anti-bosons become nearly massless. Without any model dependent calculation, we can confirm an important relativistic effect in BCS-BEC crossover: There should exist a relativistic BEC(RBEC) state which is smoothly connected to the non-relativistic BEC(NBEC) state. The RBEC is not a specific phenomenon for relativistic fermion systems, it should appear in any fermion system if the attractive coupling becomes strong enough, even though the initial non-interacting gas satisfies \( k_f \ll m \).

We now start to construct a general relativistic model, which we expect to recover the non-relativistic theory in a proper limit. We consider only fermion degrees of freedom in the original Lagrangian. The most general form of the Lagrangian density can be expressed as

\[
\mathcal{L} = \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi + \mathcal{L}_I, \tag{2}
\]

where \( \psi, \bar{\psi} \) denote the Dirac fields with mass \( m \), and \( \mathcal{L}_I \) describes the attractive interaction among the fermions. It is shown that the dominant interaction is in the scalar channel \( J^P = 0^+ \). For the sake of simplicity, we consider the zero range interaction and take the form

\[
\mathcal{L}_I = \frac{g}{4} (\bar{\psi} \gamma_\tau \sigma C \psi)^\dagger (\psi^\dagger \gamma^\tau \sigma C \psi), \tag{3}
\]

where \( g \) is the attractive coupling constant, and \( C = i\gamma^0\gamma_2 \) is the charge conjugation matrix. Generally, by increasing the attractive coupling, the crossover from condensation of spin-zero Cooper pairs at weak coupling to the Bose-Einstein condensation of bound bosons at strong coupling can be realized.

We start our calculation from the partition function in imaginary time formalism,

\[
Z = \int D\bar{\psi} D\psi e^{\int_0^\beta d\tau \int d^d x (\mathcal{L} + \mu \psi^\dagger \psi)}, \tag{4}
\]

where \( \beta = 1/T \), and \( \mu \) is the chemical potential corresponding to the net charge density \( \psi^\dagger \psi \) which is determined by the charge conservation. Similar to the method in the study of superconductivity, we introduce the Nambu-Gorkov spinors

\[
\Psi = \left( \begin{array}{c} \psi \\ \bar{\psi} \end{array} \right), \quad \bar{\Psi} = \left( \begin{array}{c} \bar{\psi} \\ \psi \end{array} \right) \tag{5}
\]

and the auxiliary pair field \( \Delta(x) = g\psi^\dagger(x)C\gamma_\tau \psi(x)/2 \), and then perform the Hubbard-Stratonovich transformation, the partition function can be written as

\[
Z = \int D\bar{\Psi} D\Psi D\Delta D\Delta^* e^{\int_0^\beta d\tau \int d^d x \left( \frac{1}{2} \bar{\Psi} \Sigma \psi - \frac{1}{2} \Delta^2 \right)} \tag{6}
\]

in terms of the inverse Nambu-Gorkov propagator

\[
G^{-1} = i\gamma^0 \partial_\mu - m + i\gamma_\tau \partial_\tau + \frac{i\gamma_\tau}{2} \Delta^\tau \sigma^-, \tag{7}
\]

where \( \sigma^\pm = (\sigma_1 \pm i\sigma_2)/2 \) are defined in Nambu-Gorkov space with \( \sigma_i (i = 1, 2, 3) \) being the Pauli matrices. Integrating out the fermion fields, we finally derive the partition function

\[
Z = \int D\Delta D\Delta^* \ e^{-S_{eff}} \tag{8}
\]

with the bosonic effective action

\[
S_{eff} = \int_0^\beta d\tau \int d^d x \left[ \frac{\Delta^2}{g} - \frac{1}{2\beta} \text{Tr} \ln[\beta G^{-1}] \right]. \tag{9}
\]

While in general case one should include the contribution from pair fluctuations, the Leggett mean field theory is already good to describe the BCS-BEC crossover at zero temperature. This can be seen from the fact that, the dominant contribution of fluctuations to the thermodynamic potential at low temperature is from the massless Goldstone modes and is proportional to \( T^4 \). At zero temperature it vanishes. While the mean field theory can describe well the BCS-BEC crossover, it can
not predict precise values of the universal constants in the unitary limit and the boson-boson scattering length in the BEC limit. Since our goal in this paper is to investigate the BCS-BEC crossover only, we will take the mean field approximation in the following.

In the mean field approximation, we consider the uniform and static saddle point $\Delta(x) = \Delta_0$ which satisfies the stationary condition $\delta S_{eff}[\Delta_0]/\delta \Delta_0 = 0$. The thermodynamic potential density $\Omega = S_{eff}[\Delta_0]/(\beta V)$ at the saddle point can be evaluated as

$$\Omega = \frac{\Delta_0^2}{g} - \int \frac{d^3k}{(2\pi)^3} \left(E_k^- + E_k^+ - \xi_k^- - \xi_k^+\right) \tag{10}$$

where we have defined the notations $E_k^\pm = \sqrt{(\xi_k^\pm)^2 + \Delta_0^2}$ and $\xi_k^\pm = \epsilon_k \pm \mu$ with $\epsilon_k = \sqrt{k^2 + m^2}$. Minimizing $\Omega$, we get the gap equation for the condensate $\Delta_0$,

$$\frac{1}{g} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left(\frac{1}{E_k^-} + \frac{1}{E_k^+}\right) \tag{11}$$

and the number equation for the fermion density $n = k_f^3/(3\pi^2)$,

$$\frac{k_f^3}{3\pi^2} = \int \frac{d^3k}{(2\pi)^3} \left[\left(1 - \frac{\xi_k^-}{E_k^-}\right) - \left(1 - \frac{\xi_k^+}{E_k^+}\right)\right]. \tag{12}$$

The first and second terms on the right hand side of Equations (11) and (12) correspond to fermion and anti-fermion degrees of freedom, respectively.

The model is non-renormalizable and a proper regularization is needed. We subtract the vacuum contribution $\partial \Omega/\partial \Delta_0|^T_{\eta = \Delta_0 = 0}$ from the gap equation, namely we replace the bare coupling $g$ by a renormalized coupling $U$,

$$-\frac{1}{U} = \frac{1}{g} - \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left(\frac{1}{\epsilon_k - m} + \frac{1}{\epsilon_k + m}\right). \tag{13}$$

To recover the non-relativistic theory correctly, the chemical potential in vacuum should be chosen as $\mu = m$ rather than $\mu = 0$. Such a subtraction is consistent with the formula derived from the two body scattering matrix in non-relativistic theory, the ultraviolet divergence can not be completely removed, and a momentum cutoff $\Lambda$ still exists in the theory. The relativistic Fermi energy $E_f$ in the theory can be define as $E_f = \sqrt{k_f^2 + m^2}$, which recovers the Fermi kinetic energy $E_f - m \simeq \epsilon_f = k_f^2/(2m)$ in non-relativistic limit.

III. RELATIVISTIC EFFECTS IN BCS-BEC CROSSOVER

In the non-relativistic theory, there are only two characteristic lengths $k_f^{-1}$ and $a_s$, and the BCS-BEC crossover shows the universality: After a proper scaling, all physical quantities depend only on the dimensionless coupling $\eta = 1/(k_f a_s)$. Especially, in the unitary limit $a_s \to \infty$, all scaled physical quantities become universal constants. Unlike the non-relativistic theory where the fermion mass $m$ plays a trivial role, in the relativistic theory a new length scale, namely the Compton wavelength $\lambda_c = m^{-1}$ appears. As a consequence, the BCS-BEC crossover should depend on not only the dimensionless coupling $\eta$, but also the quantity $\zeta = k_f/m = k_f \lambda_c$. Since the cutoff $\Lambda$ is needed, there is also a $\Lambda/m$ dependence. By scaling all energies by $\epsilon_f$ and momenta by $k_f$, the gap and number equations (11) and (12) become dimensionless,

$$-\frac{\pi}{2} \eta = \int_0^\infty x^2 dx \left[\frac{1}{E_x^-} - \frac{1}{\epsilon_x - 2\zeta^{-2}}\right. \left.\frac{1}{E_x^+} - \frac{1}{\epsilon_x + 2\zeta^{-2}}\right],$$

$$\frac{2}{3} = \int_0^\infty x^2 dx \left[\left(1 - \frac{\xi_x^-}{E_x^-}\right) - \left(1 - \frac{\xi_x^+}{E_x^+}\right)\right] \tag{14}$$

with the definitions $E_x^\pm = \sqrt{(\xi_x^\pm)^2 + (\Delta_0/\eta)^2}$, $\xi_x^\pm = \epsilon_x \pm \mu/\eta$, $\xi_x = 2\zeta^{-1}\sqrt{x^2 + \zeta^{-2}}$ and $\eta = 1/(k_f a_s) = (m/k_f)/(1/ma_s) \ll m/k_f = \zeta^{-1}$. Therefore, the complete condition for the non-relativistic limit can be expressed as $\zeta \ll 1$ and $\eta \ll 1$.

To confirm the above statement we solve the gap and number equations numerically. In Fig.1 we show the condensate $\Delta_0$ and non-relativistic chemical potential $\mu - m$ as functions of $\eta$ in the region $-1 < \eta < 1$ for several values of $\zeta < 1$. In this region the cutoff dependence is weak and can be neglected. For sufficiently small $\zeta$, we really recover the Leggett result. With increasing $\zeta$, however, the universality is broken and the deviation becomes more and more remarkable. On the other hand, when we increase the coupling $\eta$, especially for $\eta \to 1$, the difference between our calculation at any fixed $\zeta$ and the Leggett result becomes significant due to relativistic effects. This means that even in the case $\zeta \ll 1$ we can not recover the non-relativistic result when the cou-
pling \( \eta \) is of the order of \( \zeta^{-1} \). This can be seen clearly from the \( \eta \) dependence of \( \Delta_0 \) and \( \mu \) in a wider \( \eta \) region, shown in Fig.2. We found in our numerical calculation with the cutoff \( \Lambda/m = 10 \) a critical coupling \( \eta_c \approx 2\zeta^{-1} \) which is consistent with the above estimation. Beyond this point the chemical potential \( \mu \) approaches zero and the condensate \( \Delta_0 \) becomes of the order of the relativistic Fermi energy \( E_f \) or the fermion mass \( m \). In this region, the relativistic effect becomes significant, even though the initial non-interacting gas is in a non-relativistic state.

We can derive an analytical expression for the critical coupling \( \eta_c \) or \( U_c \) for the RBEC state. At the critical coupling, we can take \( \mu \approx 0 \) and \( \Delta_0 \ll m \) approximately, and the gap equation becomes

\[
\frac{1}{U_c} \approx \frac{m^2}{2\pi^2} \int_{0}^{\Lambda} \frac{dk}{\sqrt{k^2 + m^2}},
\]

from which we have

\[
U_c^{-1} = \frac{2}{\pi} U_0^{-1} f(\Lambda/m), \quad \eta_c = \frac{2}{\pi} \zeta^{-1} f(\Lambda/m),
\]

with the definitions \( f(x) = \ln(x + \sqrt{x^2 + 1}) \) and \( U_0 = 4\pi/m^2 = 4\pi\lambda_0^2 \). While in the non-relativistic region with \( \eta \ll \zeta^{-1} \) the solution is almost cutoff independent, in the RBEC region with \( \eta \sim \zeta^{-1} \) the solution becomes sensitive to the cutoff.

To see what happens in the region with \( \eta \zeta \sim 1 \), we discuss the fermion and anti-fermion momentum distributions \( n_-(k) \) and \( n_+(k) \),

\[
n_{\pm}(k) = \frac{1}{2} \left( 1 - \frac{\xi_k^{\pm}}{E_k^{\pm}} \right),
\]

In the non-relativistic BCS and BEC regions with \( \eta \zeta \ll 1 \), we have \( \Delta_0 \ll E_f \), and the anti-fermion degrees of freedom can be safely neglected, \( n_-(k) \approx 0 \). In the BCS limit with \( \Delta_0 \ll \epsilon_f \), \( n_-(k) \) deviates slightly from the standard Fermi distribution at the Fermi surface, especially we have \( n_-(0) \approx 1 \). In the deep BEC region, we have \( \Delta_0 \sim \eta \epsilon_f \) and \( |\mu - m| \sim \eta^2 \epsilon_f \) and in turn \( |\mu - m| \gg \Delta_0 \) and \( n_-(0) \ll 1 \), and \( n_-(k) \) becomes very smooth. However, in the RBEC region with \( \eta \zeta \sim 1 \), \( \mu \) approaches zero, and the anti-fermions become nearly degenerate with the fermions,

\[
n_-(k) \approx n_+(k) = \frac{1}{2} \left( 1 - \frac{\epsilon_k}{\sqrt{\epsilon_k^2 + \Delta_0^2}} \right).
\]

FIG. 1: The condensate \( \Delta_0 \) and non-relativistic chemical potential \( \mu - m \), scaled by the non-relativistic Fermi energy \( \epsilon_f \), as functions of \( \eta \) in the region \( -1 < \eta < 1 \) for several values of \( \zeta \). In the calculations we set \( \Lambda/m = 10 \).

FIG. 2: The condensate \( \Delta_0 \) and chemical potential \( \mu \), scaled by the relativistic Fermi energy \( E_f \), as functions of \( \eta \) in a wide region \( -1 < \eta < 20 \) for several values of \( \zeta \). In the calculations we set \( \Lambda/m = 10 \).
where \( n_\pm(0) \) can be large enough as long as \( \Delta_0 \) is of the order of \( m \).

In the non-relativistic BCS and BEC regions, the total net density \( n = n_- - n_+ \) is approximately \( n \simeq n_- = \int d^3k/(2\pi)^3 n_-(k) \), and the contribution from the anti-fermions can be neglected, \( n_+ = \int d^3k/(2\pi)^3 n_+(k) \simeq 0 \). However, when we approach to the RBEC region, the contributions from fermions and anti-fermions are almost equally important. In the RBEC region with \( \Delta_0 < m \) we can estimate

\[
n_- \simeq n_+ \simeq \frac{\Delta_0^2}{8\pi^2} \int_0^\Lambda dk \frac{k^2}{k^2 + m^2} = \frac{\Delta_0^2\Lambda}{8\pi^2} \left( 1 - \frac{m}{\Lambda} \arctan \frac{\Lambda}{m} \right). \tag{19}
\]

For \( m \ll \Lambda \), the second term in the bracket can be omitted, \( n_- \simeq n_+ \simeq \Delta_0^2\Lambda/(8\pi^2) \). The densities of fermions and anti-fermions are both much larger than \( n \), and their difference produces a conserved net density.

### IV. DENSITY EFFECT IN BCS-BEC CROSSOVER

In the non-relativistic BCS-BEC crossover, the coupling strength and number density are reflected in the theory in a compact way through the dimensionless quantity \( \eta = 1/(k_f a_s) \), and changing the density of the system can not induce a BCS-BEC crossover. However, if the universality is broken, there would exist an extra density dependence which may induce a BCS-BEC crossover. In non-relativistic Fermi gas, the breaking of the universality can be induced by a finite range interaction\[14]\).

In the relativistic theory, the universality is naturally broken by the \( \zeta = k_f/m \) dependence which leads to the extra density effect.

To study this phenomenon, we calculate the phase diagram in the \( U_0/U - k_f/m \) plane where \( U_0/U \) reflects the pure coupling constant effect and \( k_f/m \) reflects the pure density effect. The reason why we do not present the phase diagram in the \( \eta - \zeta \) plane is that both \( \eta = 1/(k_f a_s) \) and \( \zeta = k_f/m \) include the density effect. The phase diagram is shown in Fig.3. The BEC state is below the line \( \mu = m \) and the BCS-like state is above the line. We see clearly two ways to realize the BCS-BEC crossover, by changing the coupling constant at a fixed density and changing the density at a fixed coupling. Note that we only plot the line which separates the two regions with \( \mu > m \) and \( \mu < m \). Above and close to the line there should exist a crossover region, like the phase diagram in \[14]\).

The density induced BCS-BEC crossover can be realized in dense QCD such as QCD at finite isospin density\[15 16 17 18\] and two color QCD at finite baryon density. The new feature in QCD is that the quark mass \( m \) decreases with increasing density due to chiral symmetry restoration at finite density, which would lower the line in the phase diagram Fig.3.

![FIG. 3: The phase diagram in the \( U_0/U - k_f/m \) plane. The line is defined as \( \mu = m \).](image)

### V. COLLECTIVE MODE EVOLUTION

To investigate the Gaussian fluctuations above the saddle point \( \Delta_0 \), we write \( \Delta(x) = \Delta_0 + \phi(x) \) and expand the action to the second order in \( \phi \) to obtain

\[
S_{\text{Gauss}}[\phi] = S_{\text{crit}}[\Delta_0] + \frac{1}{2} \sum_Q \Phi^4(Q) M(Q) \Phi(Q), \tag{20}
\]

where \( M \) is defined as \( \Phi^4(Q) = (\phi^*(Q), \phi(-Q)) \), \( Q = (q, i\nu_m) \) with \( \nu_m = 2m\pi/\beta (m = 0, \pm 1, \pm 2, \ldots) \) is the four momentum of the collective mode, and \( \sum_Q = T \sum_m \int d^3q/(2\pi)^3 \) denotes integration over the three momentum \( q \) and summation over the frequency \( \nu_m \). The inverse propagator \( M \) of the collective mode is a \( 2 \times 2 \) matrix with the elements

\[
M_{11}(Q) = \frac{1}{g} + \frac{1}{2} \sum_K \text{Tr} \left( i\gamma E K + 1 \right)^2 \right), \\
M_{12}(Q) = \frac{1}{2} \sum_K \text{Tr} \left( i\gamma E K + 1 \right)^2 \right), \\
M_{21}(Q) = M_{12}(-Q), \\
M_{22}(Q) = M_{11}(-Q), \tag{21}
\]

where \( K = (k, i\omega_n) \) with \( \omega_n = (2n + 1)\pi/\beta (n = 0, 1, 2, \ldots) \) is the fermion four momentum, and \( G^{ij}(i, j = 1, 2) \) are the elements of the fermion propagator \( G = G[D_0] \) in the Nambu-Gorkov space given by

\[
G^{11}_0 = \frac{i\omega_n + k^−}{(i\omega_n)^2 - (E_k^2)^2} \Lambda_+ \gamma_0 + \frac{i\omega_n - k^+}{(i\omega_n)^2 - (E_k^2)^2} \Lambda_− \gamma_0, \\
G^{12}_0 = \frac{-i\Delta_0}{(i\omega_n)^2 - (E_k^2)^2} \Lambda_+ \gamma_0 + \frac{i\Delta_0}{(i\omega_n)^2 - (E_k^2)^2} \Lambda_− \gamma_0, \\
G^{21}_0 = G^{11}_0 (\mu \rightarrow -\mu), \\
G^{22}_0 = G^{12}_0 (\mu \rightarrow -\mu), \tag{22}
\]

with the energy projectors

\[
\Lambda_±(k) = \frac{1}{2} \left[ 1 \pm \frac{\gamma_0 (\vec{\eta} \cdot \vec{k} + m)}{\epsilon_k} \right]. \tag{23}
\]
At zero temperature, $M_{11}$ and $M_{12}$ can be evaluated as

\[
M_{11} = \frac{1}{g} + \int \frac{d^3k}{(2\pi)^3} \left[ \left( \frac{(v_{k}^{+})^2}{iv_{m} + E_{k}^+ - E_{k-q}^+} - \frac{(u_{k}^-)^2(u_{k+q}^-)^2}{iv_{m} - E_{k}^- + E_{k+q}^-} \right) T_+ \right. \\
+ \left. \left( \frac{(u_{k}^-)^2(v_{k}^+)^2}{iv_{m} + E_{k}^- - E_{k+q}^-} - \frac{(u_{k}^-)^2(v_{k+q}^-)^2}{iv_{m} + E_{k}^- + E_{k+q}^-} \right) T_+ \right. \\
+ \left. \left( \frac{(u_{k}^-)^2(v_{k}^+)^2}{iv_{m} - E_{k}^- - E_{k+q}^-} - \frac{(u_{k}^-)^2(v_{k+q}^-)^2}{iv_{m} - E_{k}^- + E_{k+q}^-} \right) T_- \right. \\
+ \left. \left( \frac{(u_{k}^-)^2(v_{k}^+)^2}{iv_{m} - E_{k}^- + E_{k+q}^-} - \frac{(u_{k}^-)^2(v_{k+q}^-)^2}{iv_{m} + E_{k}^- + E_{k+q}^-} \right) T_- \right],
\]

\[
M_{12} = \int \frac{d^3k}{(2\pi)^3} \left[ \left( \frac{(u_{k}^-)^2(v_{k}^+)^2}{iv_{m} + E_{k}^- + E_{k+q}^-} - \frac{(u_{k}^-)^2(v_{k-q}^-)^2}{iv_{m} + E_{k}^- - E_{k-q}^-} \right) T_+ \right. \\
+ \left. \left( \frac{(u_{k}^-)^2(v_{k}^+)^2}{iv_{m} + E_{k}^- - E_{k+q}^-} - \frac{(u_{k}^-)^2(v_{k-q}^-)^2}{iv_{m} + E_{k}^- + E_{k+q}^-} \right) T_+ \right. \\
+ \left. \left( \frac{(u_{k}^-)^2(v_{k}^+)^2}{iv_{m} - E_{k}^- + E_{k+q}^-} - \frac{(u_{k}^-)^2(v_{k-q}^-)^2}{iv_{m} - E_{k}^- - E_{k-q}^-} \right) T_- \right. \\
+ \left. \left( \frac{(u_{k}^-)^2(v_{k}^+)^2}{iv_{m} - E_{k}^- - E_{k+q}^-} - \frac{(u_{k}^-)^2(v_{k-q}^-)^2}{iv_{m} - E_{k}^- + E_{k+q}^-} \right) T_- \right],
\]

with $T_\pm = 1/2 \pm (k \cdot q + q^2)/2(\epsilon_k \epsilon_{k+q} + \epsilon_{k-q})$ and the coherent coefficients $(u_k^-)^2 = (1 + \xi_k^2/E_k^2)/2$ and $(v_k^+)^2 = (1 - \xi_k^2/E_k^2)/2$.

Taking the analytical continuation $iv_m \rightarrow \omega + i0^+$, the dispersion $\omega = \omega(q)$ of the collective mode is obtained by solving the equation

\[
\det M[q, \omega(q)] = 0.
\]

To make the result more physical, we decompose the complex fluctuation field $\phi(x)$ into its amplitude mode $\lambda(x)$ and phase mode $\theta(x)$, $\phi(x) = (\lambda(x) + i\theta(x))/\sqrt{2}$. Then the fluctuation part of the effective action takes the form

\[
(\lambda^* \theta^* \begin{pmatrix} M_{11}^+ + M_{12}^+ \\ -iM_{11}^- \\ IM_{11}^+ + M_{12}^+ \end{pmatrix} \begin{pmatrix} \lambda \\ \theta \end{pmatrix}),
\]

where $M_{ij}(q, \omega) = (M_{11}(q, \omega) \pm M_{11}(q, -\omega))/2$ are even and odd functions of $\omega$. From $M_{11}(q, 0) = 0$, the amplitude and phase modes decouple exactly at $\omega = 0$. Furthermore, $M_{11}(0,0) = M_{12}(0,0)$ ensures that the phase mode at $q = 0$ is gapless, i.e., the Goldstone mode.

We now determine the velocity of the Goldstone mode, $\omega(q) = c|q|$ for $|q| \ll \text{min}_q \{E_k^\pm\}$. For this purpose, we make a small $q$ and $\omega$ expansion of the effective action,

\[
M_{11}^+ + M_{12}^+ = A + C|q|^2 - D\omega^2 + \cdots,
\]

\[
M_{11}^+ - M_{12}^+ = Q|q|^2 - R\omega^2 + \cdots,
\]

\[
M_{11}^- = B\omega + \cdots.
\]

The Goldstone mode velocity now reads

\[
c^2 = \frac{Q}{B^2/A + R},
\]

and the corresponding eigenvector of $M$ is $(\lambda, \theta) = (-icq/B/A, 1)$, which is a pure phase mode at $q = 0$ but has an admixture of the amplitude mode controlled by $B$ at finite $q$. The explicit form of $A$, $B$, $R$ and $Q$ can be calculated as

\[
A = 4\Delta_0^2 R,
\]

\[
B = \frac{1}{4} \int \frac{d^3k}{(2\pi)^3} \left( \frac{\xi_k^+ - \xi_k^-}{E_k^3 - E_k^+} \right)^2,
\]

\[
R = \frac{1}{8} \int \frac{d^3k}{(2\pi)^3} \left( \frac{1}{E_k^+ + 1} + \frac{1}{E_k^- + 1} \right),
\]

\[
Q = \frac{1}{16} \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{E_k^3 - E_k^+} \left( \frac{\xi_k^+}{\epsilon_k} + \left( \frac{\Delta_0^2}{E_k} - \frac{\xi_k^-}{\epsilon_k} \right) \frac{k^2}{3\epsilon_k} \right) \right. \\
+ \left. \frac{1}{E_k^+} \left( \frac{\xi_k^+}{\epsilon_k} + \left( \frac{\Delta_0^2}{E_k} - \frac{\xi_k^-}{\epsilon_k} \right) \frac{k^2}{3\epsilon_k} \right) \right. \\
+ \left. 2 \left( \frac{1}{E_k^+} + \frac{1}{E_k^-} - 2 \frac{E_k^- - \xi_k^+}{E_k^+ E_k^-} \frac{\xi_k^+}{\epsilon_k} + \frac{\Delta_0^2}{E_k^2} \right) \right] \times \frac{1}{\epsilon_k} \left( 1 - \frac{k^2}{3\epsilon_k^3} \right).
\]

In the non-relativistic limit with $\zeta \ll 1$ and $\zeta \eta \ll 1$, we have $|\mu - m| \ll m$, all the terms that include anti-fermion energy can be neglected, the fermion dispersions $E_k^\pm$ and $\xi_k^\pm$ can be approximated by $E_k$ and $\xi_k$, and the function $T_\pm$ can be approximated by 1. In this limit the functions $M_{11}$ and $M_{12}$ are the same as the ones obtained in non-relativistic theory. In this case, we have

\[
B = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{\xi_k}{E_k^3},
\]

\[
R = \frac{1}{8} \int \frac{d^3k}{(2\pi)^3} \frac{1}{E_k^3},
\]

\[
Q = \frac{1}{16} \int \frac{d^3k}{(2\pi)^3} \frac{1}{E_k^3} \left( \frac{\xi_k}{m} + \frac{\Delta_0^2}{E_k^2} \right).
\]

In the BCS limit, all the integrated functions peak near the Fermi surface, we have $B = 0$ and $c^2 = Q/R$. Working out the integrals we recover the well known result $c = \zeta/\sqrt{3}$ for non-relativistic BCS superfluidity. In the BEC region, the Fermi surface does not exist and $B$ becomes non-zero. An explicit calculation shows that $c = \zeta/\sqrt{3\eta} \ll \zeta$. This result can be rewritten as $c^2 = 4\pi n_{B0}B/m_B^2$ where $n_B = 2m$, $a_B = 2a_s$, and $n_B = n/2$ are corresponding mass, scattering length and density of a dilute Bose gas, which recovers the result of the Bogoliubov theory of dilute Bose gas with short-range repulsive interaction. In the RBEC region with $\zeta \eta \sim 1$, we have $\mu \rightarrow 0$, and the terms include anti-fermion energy become nearly
and degenerate with the fermion terms and hence can not be neglected. Since the quantity $B$ vanishes at $\mu \to 0$, we have $c^2 = \lim_{\lambda \to \infty} Q/R$ in the RBEC region. Taking $\mu = 0$ which leads to

$$ R = \frac{1}{4} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{E_k^3}, $$

$$ Q = \frac{1}{8} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{E_k^3} \left( 3 - \frac{k^2}{\epsilon_k^2} + \frac{\Delta_0^2 k^2}{E_k^2 \epsilon_k^2} \right), \quad (31) $$

where $E_k = \sqrt{\epsilon_k^2 + \Delta_0^2}$ is now the degenerate dispersions in the limit $\mu \to 0$, we find $c = 1$ in this region. This is consistent with the Goldstone boson velocity calculated from the relativistic boson field theory of BEC[21].

In the ultra-relativistic BCS state with $k_f \gg m$ and $\Delta_0 \ll \mu \simeq E_f$, all the terms that include anti-fermion energy can be neglected again. In this case we have $B = 0$ and

$$ R = \frac{\mu^2}{16\pi^2} \int_0^\infty \frac{dk}{[(k - \mu)^2 + \Delta_0^2]^{3/2}} , $$

$$ Q = \frac{\mu^2}{32\pi^2} \int_0^\infty \frac{dk}{[(k - \mu)^2 + \Delta_0^2]^{5/2}} . \quad (32) $$

A simple algebra shows that $Q/R = 3$, and hence we obtain the well-known result $c = 1/\sqrt{3}$ for the ultra-relativistic BCS superfluidity.

The mixing of amplitude and phase modes is quite different in different regions. In the weak coupling BCS region, all the integrated functions peak near the Fermi surface and we have $B = 0$ due to the particle-hole symmetry, and hence the amplitude and phase modes decouple exactly. In the NBEC phase with $\zeta_\eta \ll 1$, while the anti-fermion term in $B$ can be neglected, we have $B \neq 0$ since the particle-hole symmetry is lost, and the mixing is strong. In the RBEC region, while both particle-hole and anti-particle–anti-hole symmetries are lost, they cancel each other and we have again $B = 0$. This can be seen from the fact that for $\mu \to 0$ the first and second terms in $B$ cancel to each other. Thus in the RBEC region, the amplitude and the phase modes decouple again. This can also be explained in the frame of the boson field theory of BEC. In non-relativistic theory, the off-diagonal elements of the inverse propagator are proportional to $i\mu_\omega[11,21]$, which induces a strong mixing between amplitude and phase modes. However, in relativistic theory, the off-diagonal elements are proportional to $\mu_\omega[11,21]$, which vanishes when BEC of massless bosons happens.

VI. SUMMARY

We have investigated the BCS-BEC crossover at zero temperature in a relativistic Fermi gas model. Unlike the non-relativistic theory, the fermion mass plays a nontrivial role and serves as a new length scale. As a consequence, the universality breaks down and there exists an extra number density effect on the BCS-BEC crossover. When the effective scattering length is much less than the fermion Compton wavelength and the Fermi momentum is much less than the fermion mass, we can recover the non-relativistic theory. At the ultra-strong coupling where the effective scattering length is of the order of the Compton wavelength, the RBEC state appears. In this state the condensed boson becomes nearly massless and anti-fermions are excited. The sound velocity of the Goldstone mode and the mixing between the amplitude and phase modes are quite different in different regions, and the results agree well with the boson field theory.

In this paper, we investigated only the relativistic BCS-BEC crossover at zero temperature where all pairs get condensed. At finite temperature, zero momentum condensed pairs can be thermally excited and one should go beyond the mean field theory to treat properly the non-condensed pairs[1]. The study on the relativistic BCS-BEC crossover in the symmetry breaking phase at finite temperature is in progress.

Acknowledgement: This work is supported by the grants NSFC10428510, 10435080 10575058 and SRFDP2004003103.

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