LIMITS, STANDARD COMPLEXES AND fr-CODES

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ABSTRACT. For a strongly connected category $C$ with pair-wise coproducts, we introduce a cosimplicial object, which serves as a sort of resolution for computing higher derived functors of $\lim : \text{Ab}^C \to \text{Ab}$. Applications involve Künneth theorem for higher limits and $\lim$-finiteness of fr-codes. A dictionary for the fr-codes with words of length $\leq 3$ is given.

1. INTRODUCTION

Let $G$ be a group. By $\text{Pres}(G)$ we denote the category of presentations of $G$ with objects being free groups $F$ together with epimorphisms to $G$. Morphisms are group homomorphisms over $G$. For a functor $F : \text{Pres}(G) \to \text{Ab}$ from the category $\text{Pres}(G)$ to the category of abelian groups, one can consider the (higher) limits $\lim^i F$, $i \geq 0$, over the category of presentations. The limits $\lim^i F$ are studied in the series of papers [5], [6], [11], [12].

Let $\text{Ring}$ be the category of rings. The group ring functor $\mathbb{Z}[-] : \text{Pres}(G) \to \text{Ring}$, $(F \to G) \mapsto \mathbb{Z}[F]$ has two functorial ideals $f$ and $r$ defined as

$$f(F \to G) = \ker\{\mathbb{Z}[F] \to \mathbb{Z}\}, \quad r(F \to G) = \ker\{\mathbb{Z}[F] \to \mathbb{Z}[G]\}$$

For different products of ideals $f$ and $r$, their sums and intersections, like

$$\text{fr} + rf, \quad r^2 \cap f^3$$

one can consider their higher limits. It turns out, such limits, which depend functorially on $G$, cover a rich collection of various functors on the category of groups, including certain homological functors, derived functors etc.

(Finite) sums of monomials formed from letters $f$ and $r$ we call fr-sentences or fr-codes. By translation we mean a description of the functors $\lim^i (\text{fr-code})$, $i \geq 1$, fr-codes viewed as functors $\text{Pres}(G) \to \text{Ab}$. For the moment we do not have a unified method of translation of a given code and, in every new case, in order to translate a code, we find specific tricks. At the end of the paper we present a dictionary of all nontrivial translations of codes with monomials of length $\leq 3$. In order to illustrate the diversity of functors which

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appear in this way, we give the following examples:

\[
\lim^1 (rff + frr) = \text{Tor}(H_2(G), G_{ab}), \\
\lim^1 (rr + ffr + rff) = H_2(G, G_{ab}), \\
\lim^1 (rr + ffr) = H_3(G), \\
\lim^2 (rr + ffr) = g \otimes \mathbb{Z}[G] g.
\]

Here \(H_i(G)\) is the \(i\)th integral homology of \(G\), \(g\) the augmentation ideal in \(\mathbb{Z}[G]\), \(G_{ab}\) the abelianization of \(G\).

Since the category \(\text{Pres}(G)\) is strongly connected, the \(\lim^1(\text{fr-code})\) has a natural interpretation as the maximal constant subfunctor of \(f/(\text{fr-code})\) (see [5], [6]). For example,

\[
\lim^1 rr + fff = \lim^0 \frac{f}{rr + fff} = \frac{(rr + fff) \cap (fr + fff)}{rr + fff} = \text{Tor}(G_{ab}, G_{ab}), \\
\lim^1 rf + fr = \lim^0 \frac{f}{rf + fr} = \frac{ff}{rf + fr} = g \otimes \mathbb{Z}[G] g.
\]

The point of this theory (which we also call metaphorically as \(\text{fr-language}\)), is that the formal manipulations with codes in two letters may induce deep and unexpected transformations of functors. Simple transformations of \(\text{fr}\)-codes, like changing the symbol \(r\) by \(f\) in a certain place, adding a monomial to the \(\text{fr}\)-code etc, induce natural transformations of (higher) limits determined by these \(\text{fr}\)-codes. For example, the transformation of the \(\text{fr}\)-codes

\[
rr + ffr \leadsto rr + ffr + rff
\]

induces the natural transformation of functors

\[
H_3(G) = \lim^1 (rr + ffr) \leadsto H_2(G, G_{ab}) = \lim^1 (rr + ffr + frr).
\]

Here the map \(H_3(G) \to H_2(G, G_{ab})\) is constructed as \(H_3(G) = H_2(G, g) \to H_2(G, g/g^2) = H_2(G, G_{ab})\), where the last map is induced by the natural projection \(g \twoheadrightarrow g/g^2 = G_{ab}\).

This paper has two main parts. The first part is more abstract, we prove that any (finite) \(\text{fr}\)-code has only finite number of non-zero higher limits (see Theorem 4.4). In order to prove this statement, we develop a general theory of standard complexes constructed for elements of categories with pairwise coproducts (such as our category \(\text{Pres}(G)\)). More precisely, for any object \(c\) of a category with pairwise coproducts we introduce a cosimplicial object \(\mathcal{G}(c)\), such that, for any functor \(\mathcal{F}\) from our category to abelian groups, the (higher) limits \(\lim^i \mathcal{F}\) are naturally isomorphic to the cohomotopy groups \(\pi^i \mathcal{F} \mathcal{G}(c)\) (Theorem 2.12). It follows from Theorem 4.4 that, given a \(\text{fr}\)-code, the number of its non-zero higher limits is finite. In the second part we present concrete translations. We form a dictionary of the various \(\text{fr}\)-codes using spectral sequences, Grünberg resolution, Künneth-type formulas and collections of tricks. Observe that, not all \(\text{fr}\)-codes can be translated using homological algebra only, in some cases (like the case \(rr + ffr + ffr + rff\)), nontrivial statements from the theory of groups and group rings are useful.
2. The standard complex

**Definition 2.1.** A category $\mathcal{C}$ is called *strongly-connected* if for any two objects $c, c' \in \mathcal{C}$

$$\text{Hom}_\mathcal{C}(c, c') \neq \emptyset.$$  

Moreover, if for any $c, c' \in \mathcal{C}$ there exists coproduct $c \sqcup c'$, we say that $\mathcal{C}$ is a *category with pairwise coproducts* (i.e. with finite non-empty coproducts).

**Definition 2.2.** Let $\mathcal{F} : \mathcal{C} \to \text{Ab}$ be a functor. The *Higher limits* $\lim^i \mathcal{F}$ of $\mathcal{F}$ are the right derived functors of the limit functor:

$$\lim^i \mathcal{F} = \mathbb{R}^i \lim \mathcal{F}, \quad \lim : \text{Ab}^\mathcal{C} \to \text{Ab}.$$  

We will assume that in the functor category there are enough injective objects, so higher limits of any functor exists, provided $\mathcal{C}$ is small. In a general case, as in section (4), where $\mathcal{C} = \text{Pres}(G)$, the existence of higher limits for functors of interest can be established, using Grothendieck-Tarsky theory, as in [6].

For a cochain complex of functors the relation between higher limits of its terms and limits of its cohomology is given by the following spectral sequence.

**Proposition 2.3** ([5], (2.5), (2.6)). Let $\mathcal{F}^\bullet$ be a bounded below cochain complex of functors with lim-acyclic cohomology. Then there exists a convergent spectral sequence

$$E^{p,q}_1 = \lim^q \mathcal{F}^p \Rightarrow \lim H^{p+q}(\mathcal{F}^\bullet)$$

with the differential on the first page induced by the differential of $\mathcal{F}^\bullet$.  

**Remark.** For a functor $\mathcal{F}$ consider its subfunctor of the invariants $\text{inv} \mathcal{F} : \mathcal{C} \to \text{Ab}$:

$$\text{inv} \mathcal{F}(c) = \{ x \in \mathcal{F}(c) | \forall c' \in \mathcal{C}, \varphi, \psi : c \to c', \mathcal{F}(\varphi)(x) = \mathcal{F}(\psi)(x) \}.$$  

In strongly connected categories this functor is constant and its value is equal to $\lim \mathcal{F}$, see (4.1) in [6]. Moreover, it is known [11] that the limit of a functor from a strongly connected category with pair-wise coproducts is equal to the equalizer

$$\lim \mathcal{F} \cong \text{eq}(\mathcal{F}(c) \rightrightarrows \mathcal{F}(c \sqcup c))$$

for any $c \in \mathcal{C}$. In particular, this equalizer does not depend on $c$.

To generalize the relation between limits and invariants to the level of derived functors we introduce the following notion:

**Definition 2.4.** For $c \in \mathcal{C}$ consider the following cosimplicial object $\mathfrak{B} : \Delta \to \mathcal{C}$, which we will call the *standard complex* associated with $c$:

$$\mathfrak{B}(c)^n = \bigcup_{j=0}^n c,$$

$$\mathfrak{B}(c)([n] \overset{f}{\rightarrow} [m]) = \bigsqcup_{j=0}^n c^{(i_f(0), \ldots, i_f(n))} \rightarrow \bigsqcup_{k=0}^m c$$
here $i_j : c \to \bigsqcup_{k=0}^m c$, $0 \leq j \leq m$ are canonical inclusions and notation $(g_0, \ldots, g_n)$, $g_k : c \to c'$ stands for the unique map $c^{\downarrow n+1} \to c'$ such that $(g_0, \ldots, g_n) \circ i_j = g_j$.

By definition, cofaces and codegeneracies of $\mathfrak{B}(c)$

$$d^i : c^{\downarrow n+1} \to c^{\downarrow n+2}, \quad s^i : c^{\downarrow n+1} \to c^{\downarrow n}$$

are given by

$$(4) \quad d^i = (i_0, \ldots, \hat{i}_j, \ldots, i_{n+1}), \quad s^i = (i_0, \ldots, i_j, \ldots, i_n).$$

This complex is very similar to the so-called canonical resolution, associated with the monad $(c\sqcup(-), \nabla, i_2)$, here $\nabla : c\sqcup c\sqcup(-) \xrightarrow{(i_1, i_2)\circ \text{id}} c\sqcup(-)$, see (17) (8.6.8). This similarity will become an identification, if there is an initial object 0 in $\mathcal{C}$. In this case though all higher limits of the functor $\mathcal{F} : \mathcal{C} \to \text{Ab}$ are trivial, provided $\mathcal{F}(0) = 0$. Alternatively, since $(\mathcal{C}, \sqcup)$ can be considered as a strong monoidal category (without unit) and every object is a monoid with respect to this structure, for any $c$ the standard resolution $\mathfrak{B}(c)$ can be considered as a unique monoidal functor $\Delta \to \mathcal{C}$ which sends $[0]$ to $c$, as in (7.5) of [9].

Now we will study some homotopical properties of the standard complex $\mathfrak{B}(c)$.

**Definition 2.5** ([10], (2.1)). Let $f, g : X \to Y$ be two morphisms between cosimplicial objects $X$ and $Y$. A cosimplicial homotopy between $f$ and $g$ is a collection of maps $k^i : X^{n+1} \to Y^n$, $0 \leq i \leq n$, satisfying the following identities:

$$(5) \quad k^0 d^0 = g, \quad k^i d^{i+1} = f$$

$$(6) \quad k^i d^i = \begin{cases} d^i k^{i-1}, & i < j \\ k^{i-1} d^i, & i = j > 0 \\ d^{i-1} k^i, & i > j + 1 \end{cases}$$

$$(7) \quad k^i s^i = \begin{cases} s^i k^{i+1}, & i \leq j \\ s^{i-1} k^i, & i > j \end{cases}$$

We will use the following definition of the Moore complex and the alternate sum complex for the abelian case, which are dualizations of the standard definitions, as in [3]:

**Definition 2.6.** Let $A$ be a cosimplicial object in an abelian category $\mathcal{C}$

- The *Moore complex* $QA$ of $A$ is a cochain complex

$$(QA)^n = \text{coker}\left\{ \bigoplus_{i=1}^n A^{n-1} \xrightarrow{d^i} A^n \right\}$$

- The *alternate sum complex* $CA$ of $A$ is a cochain complex

$$CA^n = A^n, \quad d = \sum_{i=0}^{n+1} (-1)^i d^i$$
Both constructions are functorial, with $Q : \mathcal{C}^\Delta \to \text{Ch}_{<0}(\mathcal{C})$ being an exact functor, and as in the simplicial case, these two complexes are chain homotopic to each other. Since a cosimplicial homotopy $\{k^i\}_{i=0}^\infty$ between $f$ and $g$ induces a chain homotopy

$$k = \sum_{i=0}^n (-1)^i k^i$$

between $Cf$ and $Cg$, $Qf$ and $Qg$ are also homotopic.

The Moore complex $QA$ also has a convenient iterative description in terms of the $d'\text{ecalage}$ of $A$, which is a cosimplicial object $\text{Dec} A$ with the following structure:

$$(\text{Dec} A)^n = A^{n+1},
\begin{array}{l}
d^i_{\text{Dec} A} = d^i_{A}, \\
s^j_{\text{Dec} A} = s^j_{A}
\end{array}$$

**Proposition 2.7.** The following formula holds:

$$(QA)^n = \text{coker} \{(QA)^{n-1} \xrightarrow{d^1} (Q\text{Dec} A)^{n-1}\}.$$

**Proof.** Consider the following diagram:

The diagonal arrow here represents a map to a “total” cokernel of the square (the cokernel of the natural map from the push-out to the right-bottom corner), which is equal to a “sequential” cokernel, represented by the rightmost vertical arrows.

Turns out, on a strongly connected $\mathcal{C}$ the standard complex construction is constant up to homotopy:

**Theorem 2.8.** Let $\mathcal{C}$ be a category with pair-wise coproducts. Then for any two maps $f, g : c \to c'$ the induced morphisms $\mathfrak{A}(c) \to \mathfrak{A}(c')$ are homotopic.

**Proof.** Consider the following collection of maps $\{k^i : \mathfrak{A}(c)^{n+1} \to \mathfrak{A}(c')^n\}_{i=0}^\infty$.

$$k^i = (i_0 f, \ldots, i_i f, i_i g, \ldots, i_n g) = s^i \alpha^i, \quad \alpha^i := f \sqcup \cdots \sqcup f \sqcup g \sqcup \cdots \sqcup g$$

First we consider how $\alpha^j$ commutes with cofaces and codegeneracies. For fixed $i < j$:

$$\alpha^j d^i = (i_0 f, \ldots, i_j f, i_{j+1} g, \ldots, i_n g)(i_0, \ldots, \hat{i}_i, \ldots, i_n) = (i_0 f, \ldots, \hat{i}_i f, \ldots, i_j f, i_{j+1} g, \ldots, i_n g)$$

$$= (i_0, \ldots, \hat{i}_i, \ldots, i_n g)(i_0 f, \ldots, i_{j-1} f, i_j g, \ldots, i_n g) = d^i \alpha^{j-1}$$

(continued on the next page)
For $i > j + 1$:

$$\alpha^j d^i = (i_0 f, \ldots, i_j f, i_{j+1} g, \ldots, i_n g) = d^i \alpha^j$$

For codegeneracies if $i \leq j$:

$$\alpha^j s^i = (i_0 f, \ldots, i_j f, i_{j+1} g, \ldots, i_n g) = s^i \alpha^{j+1}$$

And similarly for $i > j$:

$$\alpha^j s^i = (i_0 f, \ldots, i_j f, i_{j+1} g, \ldots, i_i g, \ldots, i_n g) = s^i \alpha^j$$

Returning to $k^i$ and using the cosimplicial identities:

$$k^i d^i = s^i \alpha^j d^i = \begin{cases} s^i d^i \alpha^{j-1}, & i < j \\ s^i d^i \alpha^j, & i > j + 1 \end{cases} = \begin{cases} d^i s^{j-1} \alpha^{j-1}, & i < j \\ d^i s^j \alpha^j, & i > j + 1 \end{cases} = \begin{cases} d^i k^{j-1}, & i < j \\ d^i k^j, & i > j + 1 \end{cases}$$

$$k^i s^i = s^j \alpha^j s^i = \begin{cases} s^i s^j \alpha^{j+1}, & i \leq j \\ s^i s^j \alpha^j, & i < j \end{cases} = \begin{cases} s^i k^{j+1}, & i \leq j \\ s^i k^j, & i < j \end{cases}$$

Finally we consider relations for $k^j d^j$ and the boundaries of the homotopy $k^0 d^0, k^n d^{n+1}$:

$$k^j d^j = (i_0 f, \ldots, i_j f, i_{j+1} g, \ldots, i_n g) = (i_0 f, \ldots, i_j f, i_{j+1} g, \ldots, i_n g) = k^j d^i$$

$$k^0 d^0 = (i_0 f, i_0 g, \ldots, i_n g) = (i_0 g, \ldots, i_n g) = g \sqcup \cdots \sqcup g = \mathcal{B}(g)^n$$

$$k^n d^{n+1} = (i_0 f, \ldots, i_n f, i_n g) = (i_0 f, \ldots, i_n f) = f \sqcup \cdots \sqcup f = \mathcal{B}(f)^n$$

This shows that $\{k^i\}_{i=0}^\infty$ defined above is indeed a cosimplicial homotopy between $\mathcal{B}(f)$ and $\mathcal{B}(g)$.

\[ \square \]

**Corollary 2.9.** Let $\mathcal{F} : \mathcal{C} \to \text{Ab}$ be a functor on a strongly connected $\mathcal{C}$ with pair-wise coproducts. Then the cohomology groups

$$\pi^n \mathcal{F} \mathcal{B}(c) := H^n C \mathcal{F} \mathcal{B}(c)$$

are independent of $c \in \mathcal{C}$. 

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Remark. If a category $C$ is not strongly connected, $\mathcal{B}$ can be quite far from being homotopically constant, as the following example shows (see [1], [16]). Let $k$ be a field and $\mathcal{C} = k-Alg$ be a category of commutative $k$-algebras and a coproduct is given by a tensor product over $k$. Let $F = U : k-Alg \to k-Mod$ be a forgetful functor, then for $A \in k-Alg$ the (coaugmented) alternate sum complex

$$U \mathcal{B}(A) : k \to A \to A \otimes_k A \to A \otimes_k A \otimes_k A \to \ldots$$

$$d : (a_1 \otimes \ldots \otimes a_k) \mapsto (a_1 \otimes \ldots a_{i-1} \otimes 1 \otimes a_i \otimes \ldots \otimes a_k)$$

is called the Amitsur complex and its cohomology broadly depends on $A$. For example, for $A = k$, $U \mathcal{B}(k) = k$ and the complex is contractible. But for $A$ being a finite dimensional extension of $k$ it can be shown (see [1]) that $H^2(U \mathcal{B}(A))$ is the Brauer group of the corresponding extension.

Let $F : \mathcal{C} \to \text{Ab}$ be a functor. Below we will study cocycles and (co)homotopy groups of the cosimplicial object $F \mathcal{B}(c)$.

**Lemma 2.10.** Cofaces \(^{(4)}\) induce isomorphisms on higher limits of $F$:

$$\lim^m F(\sqcup^{n+1} c) \xrightarrow{F(d')^*} \lim^m F(\sqcup^{n+2} c)$$

**Proof.** First two cofaces $i_1, i_2 : c \to c \sqcup c$ in $\mathcal{B}(c)$ are inducing isomorphisms

$$\lim^n F(c) \xrightarrow{F(i_k)^*} \lim^n F(c \sqcup c)$$

by (3.6) in [5]. Modifying the proof of this lemma, one can see that the similar fact holds for all canonical inclusions $i_k : c \to \sqcup^n c$. This can be seen by considering a functor $\Phi_n : c \mapsto \sqcup^n c$ together with a natural transformation $i_k : id \to \Phi_n$ such that for any $c' \in \mathcal{C}$ the comma category $(\Phi_n \downarrow c')$ is contractible. Now consider the diagrams

$$k < i + 1 : \quad k \geq i + 1 :$$

After applying $F$ and $\lim^n$ diagonal arrows become isomorphisms, hence a horizontal arrow, which is a map, induced by coface, is an isomorphism too. \(\square\)

Cocycles $Z^n F \mathcal{B}(c)$ of the standard complex serve as a natural generalization of the functor of invariants $\mathcal{B}$:

**Lemma 2.11.** For $c \in \mathcal{C}$ the following formula holds:

$$Z^n F \mathcal{B}(c) = \{ x \in F(\sqcup^{n+1} c) | \forall c', \varphi_0, \ldots, \varphi_{n+1} : c \to c' \sum_{j=0}^{n+1} (-1)^j F((\varphi_0, \ldots, \hat{\varphi_j}, \ldots, \varphi_{n+1}))(x) = 0 \}$$
Proof. By definition,
\[ Z^n F \mathcal{B}(c) = \{ x \in F(\sqcup^{n+1}) \mid \sum_{j=0}^{n+1} (-1)^j F(d_j)(x) = 0 \} \]

Let’s denote the right hand side of (9) by \( \text{inv}^n F(c) \). The inclusion \( \text{inv}^n F(c) \subset Z^n F \mathcal{B}(c) \) is obvious. Now for any collection of maps \( \varphi_0, \ldots, \varphi_{n+1} : c \to c' \) there is a unique morphism \( \Phi = (\varphi_0, \ldots, \hat{\varphi_j}, \ldots, \varphi_{n+1}) \) such that \( \varphi_j = \Phi i_j \) and moreover \( (\varphi_0, \ldots, \hat{\varphi_j}, \ldots, \varphi_{n+1}) = \Phi \circ (i_0, \ldots, i_j, \ldots, i_{n+1}) \). Hence for \( x \in Z^n F \mathcal{B}(c) \):

\[
\sum_{j=0}^{n+1} (-1)^j F((\varphi_0, \ldots, \hat{\varphi_j}, \ldots, \varphi_{n+1}))(x) = \\
\sum_{j=0}^{n+1} (-1)^j F(\Phi) \circ F((i_0, \ldots, \hat{i_j}, \ldots, i_{n+1}))(x) = \\
F(\Phi)(\sum_{j=0}^{n+1} (-1)^j F((i_0, \ldots, \hat{i_j}, \ldots, i_{n+1}))(x)) = F(\Phi)(0) = 0
\]

and \( x \in \text{inv}^n F(c) \) \( \square \)

The gap between the higher invariants \( \text{inv}^n \) and the higher limits of the functor \( F \) is given by the coboundaries of \( F \mathcal{B}(c) \) as the following theorem shows and hence the standard complex (2.4) can be used as a sort of resolution for computing \( \lim^n F \):

**Theorem 2.12.** For strongly connected category \( \mathcal{C} \) with pair-wise coproducts and a functor \( F : \mathcal{C} \to \text{Ab} \) for any \( c \in \mathcal{C} \)

(10) \( \lim^n F = \pi^n F \mathcal{B}(c) \)

Proof. By (2.9) the (co)homotopy groups of \( F \mathcal{B}(c) \) are independent of \( c \), in particular, a cochain complex \( C F \mathcal{B}(-) \) is bounded below, has \( \lim \)-acyclic cohomology and there is a spectral sequence (2.3):

\[ E_1^{p,q} = \lim^q F \mathcal{B}(\sqcup^p c) = \lim^q F \mathcal{B}(c) \Rightarrow \lim \pi^{p+q} F \mathcal{B}(c) = \pi^{p+q} F \mathcal{B}(c) \]

The first page differential in this spectral sequence (which is acting horizontally) is a morphism, induced on \( \lim^q \) by the differential of the alternate sum complex: \( \sum_j (-1)^j F(d_j) \). Each summand in this differential is an isomorphism by (2.10) and hence the first and
second page of the spectral sequence look like this:

\[
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
\lim^2 F(c) & 0 & \lim^2 F(c \sqcup c) \\
\lim^1 F(c) & 0 & \lim^1 F(c \sqcup c) \\
\lim F(c) & 0 & \lim F(c \sqcup c) \\
\end{array}
\]

\[
\begin{array}{ccc}
\lim^2 F(c) & 0 & 0 \\
\lim^1 F(c) & 0 & 0 \\
\lim F(c) & 0 & 0 \\
\end{array}
\]

Assertion follows. \(\square\)

**Definition 2.13.** We say that the functor \(F : \mathcal{C} \to \text{Ab}\) has the degree \(\deg F \leq n\) if \(Q(F \mathfrak{B}(c))^k = 0\) for all \(k > n\) for some \(c \in \mathcal{C}\).

This definition of the degree is a generalization (see [14]) of the usual notion of the degree of a polynomial functor between abelian categories [2]. We will sketch the (dual version of) main ideas from [14].

For a category \(\mathcal{C}\) let \(\mathcal{C}_{(1)}\) be a category of splitting monomorphisms of the form \(c \to c \sqcup c'\), iteratively \(\mathcal{C}_{(k)} = (\mathcal{C}_{(k-1)})_{(1)}\). Given a functor \(F : \mathcal{C} \to \text{Ab}\) its coderivative is defined as

\[
\mathcal{F}_{(1)}(c \to c \sqcup c') = \text{coker} \left\{ F(c) \to F(c \sqcup c') \right\}
\]

Similarly the higher orders coderivatives of \(F\) are defined. Then the dual version of Proposition 1.7 of [14] holds:

**Proposition 2.14.** Let \(F\) be a functor such that \(F_{(k)} = 0\) for some \(k\). Then \(\deg F \leq k - 1\).

**Proof.** For a cosimplical object \(X\) define \(k\)-cubes \(c_k(X)\) iteratively as

\[
c_0(X) = X_0, \quad c_{k+1}(X) = c_k(X) \xrightarrow{d} c_k(\text{Dec} X)
\]

Then for \(X = F \mathfrak{B}(c)\), \(\mathcal{F}_{(k-1)}(c_{k-1}(X)) = \text{coker} \left\{ \mathcal{F}_{(k-1)}(c_{k-1}(X)) \xrightarrow{d} \mathcal{F}_{(k-1)}(c_{k-1}(\text{Dec} X)) \right\}\) and from (2.7) and the induction we get that

\[
(QX)^k = \mathcal{F}_{(k)}(c_k(X))
\]

\(\square\)

The degree functor \(\deg\) behaves in a predictable way with a tensor product of functors:

**Theorem 2.15.** Let \(F\) and \(G\) be functors of degrees \(\leq n\) and \(\leq m\) respectively. Then their tensor product \(F \otimes G\) has degree \(\leq n + m - 1\).

**Proof.** For a given split monomorphism \(f : c \to c'\) in \(\mathcal{C}\) the map \((F \otimes G)(f)\) divides into composition of two split monomorphisms

\[
(F(c) \otimes G(c)) \xrightarrow{\text{id} \otimes G(f)} (F(c) \otimes G(c')) \xrightarrow{F(f) \otimes \text{id}} (F(c') \otimes G(c'))
\]
hence coderivative of the tensor product splits as
\[(F \otimes G)_{(1)}(f) = F(c) \otimes G_{(1)}(f) \oplus F_{(1)}(f) \otimes G(c')\]
By iterating this formula we get
\[(F \otimes G)_k = \bigoplus_{i+j=n} s^i F_{(j)} \otimes t^j G_{(i)}\]
Result now follows from this formula and Proposition 2.14.

3. KÜNNETH THEOREM

We can use the fact that \(\lim^n F\) can be expressed as cohomology groups of a well-understood complex to determine the higher limits of a tensor product of functors, using a Künneth-type spectral sequence as in (6.8) of [15]. For later use in (4) we will expand our universe of functors and describe the Künneth formula in this generalized setting.

As in [5], let \(\text{Mod}_R\) denote the category of pairs \((R, M)\), where \(R\) is a ring and \(M\) is a right \(R\)-module. Morphisms are pairs \((f, \varphi) : (R, M) \to (S, N)\) consisting of ring homomorphism \(f : R \to S\) and \(R\)-linear map \(\varphi : M \to N\), where \(R\) acting on \(N\) through \(f\). There is a natural projection \(\text{Mod}_r \to \text{Ring}\). Similarly, \(\text{Mod}_l\) will denote the category of left modules over arbitrary rings.

**Definition 3.1** ([5], (3.2)). Let \(O : \mathcal{C} \to \text{Ring}\) be a \(\text{Ring}\)-valued functor. Then the right \(O\)-module \(F : \mathcal{C} \to \text{Mod}\) is a functor, such that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Mod}_r & \xrightarrow{F} & \text{Ring} \\
\text{C} & \xrightarrow{O} & \text{Ring}
\end{array}
\]

Definition of the left \(O\)-module is completely symmetric.

Note that the higher limits \(\lim^\bullet F\) have a structure of a graded module over graded ring \(\lim^\bullet O\). For the right \(O\)-module \(F\) and the left \(O\)-module \(G\) their tensor product over \(O\) is defined as a functor:

\[F \otimes_O G : \mathcal{C} \to \text{Ab}, \ F \otimes_O G(c) = F(c) \otimes_{O(c)} G(c)\]

**Theorem 3.2.** Let \(O : \mathcal{C} \to \text{Ring}\) be a functor such that \(\lim^\bullet O\) is of finite global dimension. For a right \(O\)-module \(M\) and left \(O\)-module \(N\) of finite degree, such that \(N(c)\) is a flat \(O(c)\)-module for all \(c \in \mathcal{C}\) there is a second quadrant spectral sequence

\[E_2^{p,q} = \text{Tor}_{p}^{\lim^\bullet O}(\lim^\bullet M, \lim^\bullet N)_q \Rightarrow \lim^\bullet M \otimes_O N\]

**Proof.** The proof is a direct combination of the cosimplicial version of Theorem 6 of [15] and (2.12). Fix \(c \in \mathcal{C}\) and consider the projective resolution \(P_\bullet\) of \(M \mathfrak{B}(c)\) over \(O \mathfrak{B}(c)\) such that \(\pi^\bullet P_i\) are free \(\pi^\bullet O \mathfrak{B}(c)\) modules for all \(i\). The resolution \(P_\bullet\) can be constructed in a
way that $\pi^* P_\bullet$ is a free resolution of $\lim^\bullet M$ over $\lim^\bullet \mathcal{O}$ and hence there is an isomorphism of graded abelian groups

$$\pi^* P_\bullet \otimes_{\lim^\bullet \mathcal{O}} \lim^\bullet N \cong \pi^*(P_\bullet \otimes_{\mathcal{O}(c)} N \mathcal{O}(c))$$

Applying the Moore chain complex functor $Q$ horizontally to the cosimplicial chain complex $D = P_\bullet \otimes_{\mathcal{O}(c)} N \mathcal{O}(c)$ and switching to the homological notation, we obtain a second quadrant double complex. Further argument is standard. Consider two spectral sequences, associated with $D$:

- $E^{2}_{p,q} = H^h_q H^p D = H^p Q(T^\mathcal{O}(c)(M \mathcal{O}(c), N \mathcal{O}(c)))$ Provided $N \mathcal{O}(c)$ is free as $\mathcal{O}(c)$-module, only the bottom line is nontrivial on the second page and the spectral sequence converges to $\lim^\bullet M \otimes_{\mathcal{O}} N$

- $E^{2}_{p,q} = H^v_q H^p D = H^q(\pi^* P_\bullet \otimes_{\lim^\bullet \mathcal{O}} \lim^\bullet N) = (\mathcal{T}\text{or}^\lim^\bullet \mathcal{O}(\lim^\bullet M, \lim^\bullet N))_q$

Since $\mathcal{T}\text{or}$-functors vanish above the certain line this spectral sequence converges to the same limit, as the first one.

4. fr-CODES

We denote by $\text{Pres}$ the category whose objects are all presentations $c : F \to G$ and morphisms are commutative squares

$$
\begin{array}{ccc}
F & \xrightarrow{\tilde{\varphi}} & F' \\
\downarrow c & & \downarrow c' \\
G & \xrightarrow{\varphi} & G'
\end{array}
$$

$(\varphi, \tilde{\varphi}) : c \to c'$.

For each group $G$ the category $\text{Pres}(G)$ is a subcategory of $\text{Pres}$. Then for a functor

$$\mathcal{F} : \text{Pres} \to \text{Ab}$$

and any $i \geq 0$ we have a map

$$G \mapsto \lim^i \mathcal{F}_{\text{Pres}(G)}.$$

Here $c : F \to G$ can be considered as an object of $\text{Pres}(G)$ and we can take $\mathcal{O}(c)$ that we will denote by $\mathcal{O}_G(c)$ in order to emphasize that we take it in the category $\text{Pres}(G)$ but not in the whole category $\text{Pres}$. By Theorem 2.12 we have an isomorphism

$$\lim^i \mathcal{F}_{\text{Pres}(G)} \cong H^i \mathcal{F} \mathcal{O}_G(c).$$

Moreover any morphism (11) in the category $\text{Pres}$ gives a morphism of cosimplicial objects

$$\mathcal{O}_{\varphi, \tilde{\varphi}} : \mathcal{O}_G(c) \to \mathcal{O}_{G'}(c').$$

Then the morphism $(\varphi, \tilde{\varphi})$ induces a homomorphism

$$
\begin{array}{ccc}
\lim^i \mathcal{F}_{\text{Pres}(G)} & \to & \lim^i \mathcal{F}_{\text{Pres}(G')} \\
\text{Pres}(G) & & \text{Pres}(G')
\end{array}
$$
Lemma 4.1. The homomorphism $\varphi$ depends only on $\varphi$ and does not depend on the choice of presentations and $\tilde{\varphi}$. Moreover, these homomorphisms define a functor

$$\lim^i \mathcal{F} : \text{Gr} \to \text{Ab}. $$

Proof. Assume that we have two presentations $c_i : F_i \to G$, $i = 1, 2$ for $G$, two presentations $c_i' : F'_i \to G'$ for $G'$. Assume also that we have two morphisms $(\varphi, \tilde{\varphi}) : c_i \to c_i'$ in Pres. Consider the presentations $c_1 \ast c_2 : F_1 \ast F_2 \to G$ and $c_1' \ast c_2' : F_1' \ast F_2' \to G'$, the morphism $(\varphi, \tilde{\varphi}_1 \ast \tilde{\varphi}_2) : c_1 \ast c_2 \to c_1' \ast c_2'$ and the commutative diagram

$$
\begin{array}{ccc}
\mathbb{A}_G(c_1) & \xrightarrow{a_{\varphi, \tilde{\varphi}_1}} & \mathbb{A}_{G'}(c_1') \\
\downarrow & & \downarrow \\
\mathbb{A}_G(c_1 \ast c_2) & \xrightarrow{a_{\varphi, \tilde{\varphi}_1 \ast \tilde{\varphi}_2}} & \mathbb{A}_{G'}(c_1' \ast c_2') \\
\downarrow & & \downarrow \\
\mathbb{A}_G(c_2) & \xrightarrow{a_{\varphi, \tilde{\varphi}_2}} & \mathbb{A}_{G'}(c_2')
\end{array}
$$

By Theorem 2.8 the vertical arrows induce isomorphisms on $H^i \mathcal{A}(-)$. The assertion follows. \qed

The group ring functor $\mathbb{Z}[-] : \text{Pres} \to \text{Ring}$, $(F \to G) \mapsto \mathbb{Z}[F]$ has two functorial ideals $(\mathbb{Z}[F]$-modules in sense of Definition 3.1) $f$ and $r$ defined as

$$f(F \to G) = \ker \{ \mathbb{Z}[F] \to \mathbb{Z} \}, \ r(F \to G) = \ker \{ \mathbb{Z}[F] \to \mathbb{Z}[G] \}$$

Definition 4.2. The $\mathbb{Z}[F]$-module $c : \text{Pres} \to \text{Ab}$ is called an fr-code, if it is a functorial ideal of $\mathbb{Z}[F]$, formed by products of the ideals $f$ and $r$, their sums and intersections.

Usually we consider $c$ as a functor from $\text{Pres}(G) \to \text{Ab}$ for a fixed $G$, limits always are taken over $\text{Pres}(G)$. We need it to be defined on the category $\text{Pres}$ only for the functors

$$\lim^i c : \text{Gr} \to \text{Ab}$$

to be well-defined. (see Lemma 4.1).

The notion of degree (2.13) seems to be a reasonable invariant of fr-code for the estimation of its $\lim^*$-dimension, since the Moore chain complex functor $Q$ is exact and the property of a functor being a degree $\leq k$ is closed under extensions. But already $f$ itself has an infinite degree, although it is $\mathbb{Z}[F]$-additive (i.e. of degree one with respect to $\mathbb{Z}[F]$), as shown in [6]. But since all fr-codes are subfunctors of $f$, this difficulty can be overcome by introducing the following notion:

Definition 4.3. An f-degree of an fr-code $c$ is a degree of the quotient $f/c$.

Since $f$ has trivial limits, it is straightforward that if $\deg^f c \leq n$ then $\lim^i c = 0$ for $i > n + 1$. 
Theorem 4.4. Every (finite) \(fr\)-code \(c\) has a finite \(f\)-degree and hence only a finite number of the non-zero higher limits.

Proof. Let \(n\) be a minimal power of \(r\) such that \(r^n \subset c\), then we have an epimorphism \(f/r^n \twoheadrightarrow f/c\) which induces a surjection on the level of cochain complexes:

\[
\frac{Q^f_{r^n}}{Q^f_c} \twoheadrightarrow \frac{Q^f_c}{Q^f_c}
\]

and hence it is sufficient to prove finiteness of \(f/r^n\). The sequence of the short exact sequences

\[
(13) \quad r^n/r^{n+1} \hookrightarrow f/r^{n+1} \twoheadrightarrow f/r^n
\]

starts with a constant functor \(f/r = g = \ker \{Z[G] \to \mathbb{Z}\}\) and the problem is reduced to the functors \(r^n/r^{n+1} = (r/r^2)^{\otimes Z[F]}\) (see Lemma 5.1). Covering this tensor product by the tensor product over \(Z\) and applying Theorem 2.15, only the case \(n = 1\) need to be shown.

Note that \(r/r^2\) is a free \(Z[F]\)-module with basis formed by elements \(r-1, r \in R, \) see [4], hence a natural map \(R_{ab} \to r/r^2, r \mapsto r-1\) factors through \(Z[F] \otimes R_{ab} \to r/r^2\) and this map is an isomorphism.

Finally, the functor \(R_{ab} = r/fr\) has a finite degree, since it is embedded in \(f/fr = f \otimes_{Z[F]} Z[G]\) which is an additive functor. Indeed (see also [17]):

\[
f(F \ast F') \otimes_{Z[F \ast F']} Z[G] = (f(F) \otimes_{Z[F]} Z[F \ast F'] \oplus f(F') \otimes_{Z[F']} Z[F \ast F']) \otimes_{Z[F \ast F']} Z[G]
\]

\[
= f(F) \otimes_{Z[F]} Z[G] \oplus f(F') \otimes_{Z[F']} Z[G]
\]

which concludes the proof. □

5. Dictionary

In this section, we give a dictionary for all codes written on \(fr\)-language which consist of words with length \(\leq 3\). If one can not find a code in our table, this means that either it has trivial translation, i.e. all \(\lim^{i} = 0\), or has the same translation as its mirror image, which is in our dictionary. For example, the codes \(rf + ffr\) and \(fr + ffr\) have the same translations. As mention in Introduction, by translation we mean a description of the functors \(\lim^{i}(fr - code), i \geq 1\), \(fr\)-codes viewed as functors from the category of free group presentations to the category of abelian groups.

We will omit the translation of simple codes given in [6], like \(rr + fff\), or \(rr + ffr\), \(rrf + ffr\).

In construction of the dictionary, we will use the following statements.

Lemma 5.1 (Lemma 5.9 in [6]). Let \(a' \subset a, \ b' \subset b\) be ideals of \(Z[F]\) and \(\text{Tor}(Z[F]/a, Z[F]/b) = 0\), then there is a natural isomorphism

\[
\frac{a}{a'} \otimes_{Z[F]} \frac{b}{b'} = \frac{ab}{ab' + a'b}.
\]
Lemma 5.2. For any functor $\mathcal{F}(F, R)$ and a non-constant functor $\mathcal{H}(F)$, which depends only on $F$, $\lim^i \mathcal{F} \otimes \mathcal{H} = 0$, $i \geq 0$.

Similarly one can show (see [4]) that, for a fr-code with all words started with f, all limits are zero.

Lemma 5.3 (Lemma 6 in [11]). Let $\mathcal{F}$ be a constant functor. Then any subfunctor $\mathcal{G} \hookrightarrow \mathcal{F}$ and any epimorphic image $\mathcal{F} \twoheadrightarrow \mathcal{H}$ are constant functors.

We will also use the spectral sequence 2.3, especially applied to the 4-term complexes. For convenience, let us reformulate the statement about convergence of the spectral sequence 2.3 in a more explicit form. Let $\mathcal{F}^*$ be a complex of functors $\text{Pres}(G) \rightarrow \text{Ab}$

$$\cdots \rightarrow \mathcal{F}^n \rightarrow \mathcal{F}^n + 1 \rightarrow \cdots$$

Assume that $\mathcal{F}^*$ is bounded below (i.e. $\mathcal{F}^n = 0$ for $n << 0$) and that $H^n(\mathcal{F}^*)$ is constant for any $n$. Then there exists a converging spectral sequence $E_i$ with differentials

$$d^r : E_i^{i,j} \rightarrow E_{i+r,j-r+1}$$

such that

$$E_1^{i,j} = \lim^i \mathcal{F}^i \Rightarrow H^{i+j}(\mathcal{F}^*).$$

Now we proceed to the computations.

**rfr+frf:** Tensoring the short exact sequence $\frac{r}{fr} \hookrightarrow \frac{f}{fr} \twoheadrightarrow g$ by $- \otimes \frac{f}{fr+rf}$ and taking the group homology $H_i(G, -)$, we get the long exact sequence

$$(14) \quad H_1\left(G, \frac{f}{fr} \otimes \frac{f}{fr+rf}\right) \rightarrow H_2\left(G, \frac{f}{fr} \otimes \frac{f}{fr+rf}\right) \rightarrow H_2\left(G, \frac{f}{fr} \otimes \frac{f}{fr+rf}\right) \rightarrow$$

$$\rightarrow \frac{r}{fr} \otimes \frac{f}{fr} \rightarrow \frac{f}{fr} \otimes \frac{f}{fr} \rightarrow g \otimes \frac{f}{fr}.$$

Here we used the property that, for any $G$-module $M$, there is a natural isomorphism $H_1(G, g \otimes M) = H_2(G, M)$. Since $f/\text{fr}$ is a free $\mathbb{Z}[G]$-module, $\frac{f}{fr} \otimes \frac{f}{fr+rf}$ is weak projective and hence $H_i(G, \frac{f}{fr} \otimes \frac{f}{fr+rf}) = 0$, $i \geq 1$. By Lemma 5.1

$$\frac{r}{fr} \otimes \frac{f}{fr+rf} = \frac{rf}{rfr+rf}, \quad \frac{f}{fr} \otimes \frac{f}{fr+rf} = \frac{ff}{frf+rf}.$$

And $\frac{ff}{frf+rf}$ has trivial limits by Lemma 5.2. From a spectral sequence of Proposition 2.3 applied to a four-term exact sequence (14), it can be seen that

$$\lim \frac{rf}{rfr+rf} = \lim^1 (rfr+frf) = \lim H_2\left(G, \frac{f}{fr} \otimes \frac{f}{fr+rf}\right)$$

and there is a short exact sequence

$$\lim^1 H_2\left(G, \frac{f}{fr} \otimes \frac{f}{fr+rf}\right) \hookrightarrow \lim^2 (rfr+frf) \twoheadrightarrow g^2 \otimes \mathbb{Z}[G] g.$$
Here $g \otimes_{\mathbb{Z}[G]} g^2 = \lim(g \otimes_{\mathbb{Z}[G]} \frac{f}{fr + rf}) = \lim^1(rf + ffr)$. To determine the $\lim$ and $\lim^1$ of $H_2(\frac{f}{fr + rf})$, consider the short exact sequence

$$\frac{ff}{fr + rf} \hookrightarrow \frac{f}{fr + rf} \rightarrow \frac{ff}{fr}$$

and the associated homology long exact sequence

$$H_3(G) \otimes F_{ab} \rightarrow H_2\left(G, g \otimes_{\mathbb{Z}[G]} g\right) \rightarrow H_2\left(G, \frac{f}{fr + rf}\right) \rightarrow H_2(G) \otimes F_{ab} \rightarrow H_1\left(G, g \otimes_{\mathbb{Z}[G]} g\right)$$

Any map from $H_n(G) \otimes F_{ab}$ to a constant functor (which depends only on $G$) factors through $H_n(G) \otimes G_{ab}$. This follows from elementary properties of colimits (see [7]), namely from $\text{colim}(H_n(G) \otimes F_{ab}) = H_n(G) \otimes G_{ab}$. Therefore, after truncating (15) and applying Proposition 2.3 together with Lemma 5.3 to it, we obtain

$$\lim^1(rf + frf) = \text{coker}\{H_3(G) \otimes G_{ab} \rightarrow H_2\left(G, g \otimes_{\mathbb{Z}[G]} g\right)\}.$$  

and

$$\lim^1 H_2\left(G, \frac{f}{fr + rf}\right) = \text{im}\{H_2(G) \otimes G_{ab} \rightarrow H_1\left(G, g \otimes_{\mathbb{Z}[G]} g\right)\}.$$  

Hence, there is a short exact sequence

$$\text{im}\{H_2(G) \otimes G_{ab} \rightarrow H_1\left(G, g \otimes_{\mathbb{Z}[G]} g\right)\} \hookrightarrow \lim^2(rf + frf) \rightarrow g^2 \otimes_{\mathbb{Z}[G]} g.$$  

**rr+frf+ff:** Consider the Grünberg resolution which consists of free $\mathbb{Z}[G]$-modules:

$$\cdots \rightarrow \frac{fr}{fr} \rightarrow \frac{r}{fr} \rightarrow \frac{f}{fr}$$

Tensoring it with $G_{ab} = \frac{f}{fr + rf}$ over $\mathbb{Z}[G]$, we obtain the complex

$$\cdots \rightarrow \frac{fr}{fr} \otimes_{\mathbb{Z}[G]} \frac{f}{fr + rf} \rightarrow \frac{r}{fr} \otimes_{\mathbb{Z}[G]} \frac{f}{fr + rf} \rightarrow \frac{f}{fr} \otimes_{\mathbb{Z}[G]} \frac{f}{fr + rf}$$

which can be written, by Lemma 5.1 as

$$\cdots \rightarrow \frac{frf}{fr + frf} \rightarrow \frac{rf}{fr + frf} \rightarrow \frac{ff}{fr + frf}.$$  

Hence, there is a natural isomorphism

$$H_2(G, G_{ab}) = \frac{rf \cap (fr + fff)}{fr + frf + fff}.$$  

For two ideals $I, J \subset f$ there is a short exact sequence:

$$\frac{f}{I \cap J} \hookrightarrow \frac{f}{I} \oplus \frac{f}{J} \rightarrow \frac{f}{I + J}$$
And hence we get the following 4-term exact sequence

\[ H_2(G, G_{ab}) \hookrightarrow \frac{f}{rr + rff + frf} \to \frac{f}{fr} \oplus \frac{f}{fr + fff} \to G_{ab} \otimes G_{ab}. \]

From the associated spectral sequence we obtain the identifications

\[ \lim^1(rr + frf + rff) = H_2(G, G_{ab}), \]
\[ \lim^2(rr + frf + rff) = G_{ab} \otimes G_{ab}, \]
\[ \lim^i(rr + frf + rff) = 0, \quad i \geq 3. \]

Now observe that, the statement written in Introduction, that the transformation of the \(fr\)-codes

\[ rr + frf \simeq rr + frf + rff \]

induces the natural transformation of functors

\[ H_3(G) = \lim^1(rr + frf) \simeq H_2(G, G_{ab}) = \lim^1(rr + frf + frr) \]

follows immediately from the identifications

\[ H_3(G) = \frac{rf \cap fr}{rr + frf} \to \frac{rf \cap (fr + fff)}{rr + frf + rff}. \]

**rrf+rfr+frr:** Taking the tensor product over \(\mathbb{Z}[G]\) of the Gr\"unberg resolution with \(H_2(G) = \frac{r\cap fr}{fr + rf}\) and \(\frac{r}{fr + rf}\) respectively, we obtain

\[ H_2(G, H_2(G)) = \frac{r(r \cap ff) \cap (ffr + frf)}{fr(r \cap ff) + rrf + frf}, \]
\[ H_2 \left( G, \frac{r}{fr + rf} \right) = \frac{rr \cap (ffr + frf)}{rrf + rfr + frr}. \]

Since \(\lim^1rr = \lim^1(ffr + frf) = 0\),

\[ \lim^1(rrf + rff + frr) = \lim H_2 \left( G, \frac{r}{fr + rf} \right). \]

The natural map \(H_2(G, H_2(G)) \to H_2 \left( G, \frac{r}{fr + rf} \right)\) is injective. Indeed, the above terms can be decomposed as

\[ 0 \to H_2(G) \otimes H_2(G) \to H_2(G, H_2(G)) \to \text{Tor}(G_{ab}, H_2(G)) \to 0 \]

and

\[ 0 \to H_2(G) \otimes \frac{r}{fr + rf} \to H_2 \left( G, \frac{r}{fr + rf} \right) \to \text{Tor} \left( G_{ab}, \frac{r}{fr + rf} \right) \to 0. \]

Using the fact that \(\frac{r}{fr + rf} = \text{coker} \{ H_2(G) \hookrightarrow \frac{r}{fr + rf} \}\) is torsion-free (since it is a subgroup of \(f/ff = F_{ab}\), we see that the natural map \(H_2(G) \otimes H_2(G) \to H_2(G) \otimes \frac{r}{fr + rf}\) is injective and \(\text{Tor}(G_{ab}, H_2(G)) \to \text{Tor} \left( G_{ab}, \frac{r}{fr + rf} \right)\) is an isomorphism. The natural map
$H_1(G, H_2(G)) \to H_1 \left( G, \frac{r}{\text{fr} + \text{rf}} \right)$ is also injective, by the same reason. Hence, we have the following short exact sequence

\[ 0 \to H_2(G, H_2(G)) \to H_2 \left( G, \frac{r}{\text{fr} + \text{rf}} \right) \to H_2(G) \otimes \frac{r}{\text{rf} \cap \text{ff}} \to 0. \]  

It follows that

\[ \lim^1 H_2(G, \frac{r}{\text{fr} + \text{rf}}) = \lim^1 (H_2(G) \otimes \frac{r}{\text{rf} \cap \text{ff}}) \]

To compute the latter one, we use Künneth theorem 3.2 which in this case degenerates to

\[ \lim^1 (H_2(G) \otimes \frac{r}{\text{rf} \cap \text{ff}}) = H_2(G) \otimes \lim^1 \frac{r}{\text{rf} \cap \text{ff}} = H_2(G) \otimes G_{ab} \]

Applying Proposition 2.3 to the 4-term exact sequence

\[ 0 \to \text{Tor}(G_{ab}, H_2(G)) \to H_2(G) \otimes \frac{r}{\text{rf} \cap \text{ff}} \to H_2(G) \otimes \frac{f}{\text{ff}} \to H_2(G) \otimes G_{ab} \to 0, \]

we obtain the following description

\[ \lim H_2(G) \otimes \frac{r}{\text{rf} \cap \text{ff}} = \text{Tor}(G_{ab}, H_2(G)), \quad \lim^1 H_2(G) \otimes \frac{r}{\text{rf} \cap \text{ff}} = H_2(G) \otimes G_{ab}. \]

The isomorphism (16) and the exact sequence (17) now imply that there exists the following natural short exact sequence

\[ 0 \to H_2(G, H_2(G)) \to \lim^1 (\text{rrf} + \text{rfr} + \text{frr}) \to \text{Tor}(G_{ab}, H_2(G)) \to 0 \]

In order to understand $\lim^2 (\text{rrf} + \text{rfr} + \text{frr})$, consider the spectral sequence applied to the 4-term sequence

\[ 0 \to H_2 \left( G, \frac{r}{\text{rf} + \text{fr}} \right) \to \frac{\text{rr}}{\text{rrf} + \text{rfr} + \text{frr}} \to \frac{\text{fr}}{\text{frf} + \text{frr}} \to \frac{\text{fr}}{\text{rr} + \text{frf} + \text{frr}} \to 0. \]

Putting the values of $\lim^i (\text{rr} + \text{frf} + \text{frr})$, $\lim H_2 \left( G, \frac{r}{\text{rf} + \text{fr}} \right)$ into the cells of the spectral sequence and noting that $\lim^i \text{rr} = g \otimes g$, $\lim^i \text{rr} = 0$, $i \neq 2$, we obtain the following diagram which gives a description of $\lim^2 (\text{rrf} + \text{rfr} + \text{frr})$ as a functor glued from three pieces

\[
\begin{array}{c}
H_2(G) \otimes G_{ab} \\
\downarrow \\
\lim^1 \frac{\text{rr}}{\text{rrf} + \text{rfr} + \text{frr}} \longrightarrow \lim^2 (\text{rrf} + \text{rfr} + \text{frr}) \longrightarrow \ker \{ g \otimes g \to G_{ab} \otimes G_{ab} \} \\
\downarrow \\
H_2(G, G_{ab})
\end{array}
\]
First observe that,
\[
\lim_i (\text{ffr} + \text{rff} + \text{ffff}) = 0, \ i \geq 0.
\]
This follows from the isomorphism
\[
G_{ab} \otimes F_{ab} \otimes G_{ab} = \frac{f}{r + \text{ff}} \otimes \frac{f}{r + \text{ff}} \otimes \frac{f}{r + \text{ff}} = \frac{f}{r + \text{ff}} \otimes \frac{f}{r + \text{ff}} \otimes \frac{f}{r + \text{ff}} = \frac{\text{ffff}}{f + r + \text{rff} + \text{ffr}}.
\]
Consider the following exact sequence
\[
\frac{\text{ff}}{r \cap \text{ff}} \otimes \frac{\text{gg}}{G} \frac{\text{ff}}{r \cap \text{ff}} \rightarrow \frac{\text{ff}}{\text{ffr} + \text{rff}} \rightarrow \frac{\text{ffff}}{f + \text{rff} + \text{ffr} + \text{ffff}} \rightarrow 0.
\]
The left hand term is \( g^2 \otimes \frac{\text{gg}}{G} \). Since an epimorphic image of a constant functor is a constant functor, (18) implies that
\[
\lim_1 (\text{ffr} + \text{rff}) = \frac{g^2 \otimes \frac{\text{gg}}{G} g^2}{\sim}, \quad \lim_i (\text{ffr} + \text{rff}) = 0, \ i \geq 2,
\]
where \( \frac{g^2 \otimes \frac{\text{gg}}{G} g^2}{\sim} \) is the image of the left hand map in the last exact sequence, i.e.
\[
\frac{g^2 \otimes \frac{\text{gg}}{G} g^2}{\sim} = \frac{\text{ffff}}{(\text{ffr} + \text{rff}) \cap \text{ffff}}.
\]
Define one more quotient of \( g^2 \otimes \frac{\text{gg}}{G} g^2 \) as follows:\footnote{Observe that, there exists a natural exact sequence}
\[
\frac{g^2 \otimes \frac{\text{gg}}{G} g^2}{\sim} := \frac{(r + \text{ff})^2}{\text{rr} + \text{ffr} + \text{rff}}.
\]
There is a natural epimorphism \( \frac{g^2 \otimes \frac{\text{gg}}{G} g^2}{\sim} \rightarrow \frac{g^2 \otimes \frac{\text{gg}}{G} g^2}{\approx} \). The short exact sequence
\[
0 \rightarrow \frac{g^2 \otimes \frac{\text{gg}}{G} g^2}{\approx} \rightarrow \frac{\text{ff}}{\text{r} + \text{ffr} + \text{rff}} \rightarrow \frac{\text{ff}}{(r + \text{ff})^2} \rightarrow 0
\]
implies that
\[
\lim_2 (\text{rr} + \text{ffr} + \text{rff}) = \lim_2 ((r + \text{ff})^2)
\]
and there is an exact sequence
\[
0 \rightarrow \frac{g^2 \otimes \frac{\text{gg}}{G} g^2}{\approx} \rightarrow \lim_1 (\text{rr} + \text{ffr} + \text{rff}) \rightarrow \lim_1 ((r + \text{ff})^2) \rightarrow 0.
\]
Now consider the short exact sequence

\[
0 \to \frac{f}{r + ff} \otimes \frac{ff}{fr + fff} \to \frac{f}{r + ff} \otimes \frac{ff + r}{fr + fff} \to \frac{f}{r + ff} \otimes \frac{ff + r}{ff} \to 0
\]

The left hand term has zero limits, since it is isomorphic to \(G_{ab} \otimes F_{ab} \otimes G_{ab}\). Since the diagonal action of \(F\) on the middle and the right head terms are trivial, they are isomorphic to \(f(f + ff)\) and \(f(ff + r)\) respectively. Hence,

\[
\lim_1((r + ff)^2) = \lim_1(rr + fff) = \text{Tor}(G_{ab}, G_{ab}), \\
\lim_2((r + ff)^2) = \lim_2(rr + fff) = G_{ab} \otimes G_{ab}. 
\]

We obtain the needed description

\[
\lim_2(rr + ffr + rff) = G_{ab} \otimes G_{ab}
\]

and the short exact sequence

\[
0 \to \frac{g^2 \otimes [G]}{g^2} \approx \lim_1(rr + ffr + rff) \to \text{Tor}(G_{ab}, G_{ab}) \to 0.
\]

**frr+rfr:** There is an isomorphism

\[
\frac{ff}{fr + rf} \otimes_{[F]} r = \frac{ffr}{frr + rfr}.
\]

This is a particular case of the functor \(A \otimes_{[F]} r\), where \(A\) is a constant, in this case \(A = g \otimes_{[G]} g\). Since \(\lim^* [F] = \mathbb{Z}\) is of finite global dimension and \(r\) is a free \([F]\) -module, K"unneth theorem 3.2 can be applied to \(A \otimes_{[F]} r\), and it degenerates to a series of usual K"unneth short exact sequences

\[
\bigoplus_{i+j=n} \lim^i A \otimes \lim^j r \hookrightarrow \lim^n A \otimes_{[F]} r \to \bigoplus_{i+j=n+1} \text{Tor} \lim^i A, \lim^j r
\]

which computes the only non-trivial higher limit as

\[
\lim^1 A \otimes_{[F]} r = A \otimes g.
\]

In this way, we obtain the description

\[
\lim_2(frr + rfr) = (g \otimes_{[G]} g) \otimes g, \quad \lim^i(frr + rfr) = 0, \ i \neq 2.
\]

In the same way we have

\[
\lim_2(rr + ffr) = G_{ab} \otimes g, \quad \lim^i(rr + ffr) = 0, \ i \neq 2.
\]

**rfr+frr:** There is an isomorphism

\[
\frac{r}{fr + rf} \otimes \frac{r + ff}{ff} = \frac{rr + rff}{rff + frr}.
\]
We have the following descriptions of the limits of above terms
\[
\lim \frac{r}{fr + rf} = H_2(G), \quad \lim \frac{r}{fr + rf} = G_{ab}, \quad \lim i \frac{r}{fr + rf} = 0, \quad i \geq 2
\]
and
\[
\lim \frac{r + ff}{ff} = G_{ab}, \quad \lim i \frac{r + ff}{ff} = 0, \quad i \neq 1.
\]
As noted before, the abelian group \( \frac{r + ff}{ff} = \frac{r}{rf + ff} \) is torsion-free, hence the K"unneth formula implies the following
\[
\lim \frac{rr + rff}{rff + frr} = \text{Tor}(H_2(G), G_{ab}),
\]
\[
H_2(G) \otimes G_{ab} \hookrightarrow \lim \frac{rr + rff}{rff + frr} \twoheadrightarrow \text{Tor}(G_{ab}, G_{ab}),
\]
\[
\lim^2 \frac{rr + rff}{rff + frr} = G_{ab} \otimes G_{ab},
\]
\[
\lim^i \frac{rr + rff}{rff + frr} = 0, \quad i \geq 3.
\]
Comparing this description with the values of \( \lim^i (rr + rff) = \lim^i (rr + fff) \), we obtain the following:
\[
\lim^1 (rr + fff) = \text{Tor}(H_2(G), G_{ab}),
\]
\[
F \hookrightarrow \lim^2 (rr + fff) \twoheadrightarrow \ker \{ g \otimes G_{ab} \twoheadrightarrow G_{ab} \otimes G_{ab} \},
\]
\[
H_2(G) \otimes G_{ab} \hookrightarrow F \twoheadrightarrow \text{Tor}(G_{ab}, G_{ab}).
\]

**ffr+frf+rff+rr:** Consider the short exact sequence
\[
\begin{array}{c}
\text{ffr} + \text{frf} + \text{rff} + \text{rr} \cap \text{fff} \\
\text{ffr} + \text{frf} + \text{rff} + \text{rr} \cap \text{fff} \\
\text{ffr} + \text{frf} + \text{rff} + \text{rr} \cap \text{fff} \\
\text{ffr} + \text{frf} + \text{rff} + \text{rr} \cap \text{fff} \\
\text{ff} + \text{rr} + \text{fff}.
\end{array}
\]
The left hand term is a natural quotient of \( g \otimes \mathbb{Z}[G] \otimes g \otimes \mathbb{Z}[G] \), hence,
\[
\lim^2 (ffr + frf + rff + rr) = \lim^2 (rr + fff) = G_{ab} \otimes G_{ab},
\]
\[
\lim^2 (ffr + frf + rff + rr) = 0, \quad i \geq 3.
\]
Next observe that,
\[
rr \cap fff \subset ffr + frf + rff.
\]
This follows from the identification of the intersection of augmentation ideals:
\[
\Delta^2(R) \cap fff = \Delta^3(R) + \Delta(R) \Delta(R \cap [F, F]) + \Delta([R, R] \cap [[F, F], F])
\]
and the identity \( R' \cap \gamma_3(F) = \)
\[\text{In the free group ring } \mathbb{Z}[F], \quad rr = \Delta^2(R) + rrf.\]
\[ [R \cap F', R] \] see [8]. Hence, the left hand term in the above short exact sequence is \( g \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G] \) and we have a short exact sequence
\[
g \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G] \hookrightarrow \lim^1(\text{ffr } + \text{frf } + \text{rff } + \text{rr}) \twoheadrightarrow \text{Tor}(G_{ab}, G_{ab}).
\]
We collect the results in the following table. By \( F'' \oplus G \) we mean an extension of the form \( F \hookrightarrow * \rightarrow G \).

\[ * \]

\[ \text{3A simple proof of this identity is the following. Observe that, } \Lambda^2(R/(R \cap [F, F])) = \frac{[R, R]}{[R, R]/[[F, F], F]}, \text{ where } \Lambda^2 \text{ is the exterior square, and } \Lambda^2(F_{ab}) = [F, F]/[[F, F], F]. \text{ Now the needed identity follows from the inclusion } \Lambda^2(R/(R \cap [F, F])) \hookrightarrow \Lambda^2(F_{ab}), \text{ which is induced by the inclusion } R/R \cap [F, F] \hookrightarrow F_{ab}. \]
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