Theoretical analysis of Fresnel reflection and transmission in the presence of gain media

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Abstract

When a monochromatic electromagnetic plane-wave arrives at the flat interface between its transparent host (i.e., the incidence medium) and an amplifying (or gainy) second medium, the incident beam splits into a reflected wave and a transmitted wave. In general, there is a sign ambiguity in connection with the \( k \)-vector of the transmitted beam, which requires at the outset that one decide whether the transmitted beam should grow or decay as it recedes from the interface. The question has been posed and addressed most prominently in the context of incidence at large angles from a dielectric medium of high refractive index onto a gain medium of lower refractive index. Here, the relevant sign of the transmitted \( k \)-vector determines whether the evanescent-like waves within the gain medium exponentially grow or decay away from the interface. We examine this and related problems in a more general setting, where the incident beam is taken to be a finite-duration wavepacket whose footprint in the interfacial plane has a finite width. Cases of reflection from and transmission through a gainy slab of finite-thickness as well as those associated with a semi-infinite gain medium will be considered. The broadness of the spatiotemporal spectrum of our incident wavepacket demands that we develop a general strategy for deciding the signs of all the \( k \)-vectors that enter the gain medium. Such a strategy emerges from a consideration of the causality constraint that is naturally imposed on both the reflected and transmitted wavepackets.

1 Introduction

The well-known Fresnel reflection and transmission formulas describe the behavior of a monochromatic plane-wave of frequency \( \omega \), incident on a flat interface between two homogenous media [1]. The geometry is shown in Fig. 1a, in which media 1 and 2 are both semi-infinite, and we call this the “single-surface” problem. We will also consider the “finite-slab” problem shown in Fig. 1b, in which a finite thickness medium 2 is sandwiched between semi-infinite media 1 and 3. To calculate the response of the system by the usual Fresnel method, we assume the existence of two counter-propagating plane-waves in each medium (one for each wave-vector \( k \) allowed by Maxwell’s equations) except for the final transmission medium, for which we handpick one of the two \( k \)-vectors on the basis of some “commonsense” argument such as “energy should flow to the right” or “the field amplitude must decay as \( z \to \infty \).” Then, we enforce the boundary conditions at each interface to unambiguously determine the amplitude of each plane-wave. Next, to determine the response of the system to a more realistic stimulus, such as a beam or a pulse, we decompose the stimulus into plane-waves by Fourier transformation, apply the Fresnel response in the Fourier domain, and then recompose the resulting waves by an inverse Fourier transformation. With these two steps, (i) determining the response to a single \((k, \omega)\) plane-wave, and (ii) treating the incident stimulus as a superposition of such plane-waves, the problem is solved in generality, or so the conventional wisdom goes [1–3].

However, if the transmission medium has gain (as opposed to being lossy or lossless), both steps described above become problematic. First of all, in the single-surface problem, there is no longer a commonsense argument that the field amplitude must decay as \( z \to \infty \), leading to
an ambiguity in the choice of the transmitted wave-vector, and resulting in uncertainty about the Fresnel response. Secondly, and more surreptitiously, the usual Fourier transformation methods break down, because the gain medium can cause the system response function to be non-analytic in the upper-half of the complex frequency plane, $\omega$. By considering only the real $\omega$-axis, as is usually done, and not accounting for the imaginary part of $\omega$, these calculations can result in reflected and transmitted pulses that violate causality (see reference [4] for the behavior that results when the problem is treated in this naïve, but common, way).

The ambiguity in the direction of the transmitted wave-vector has been discussed most often in the context of total internal reflection (TIR) at the interface between a transparent dielectric and a gain medium [4–6]. TIR is a special case of the single-surface problem, where the real-valued $\varepsilon_1$ is greater than the real-valued $\varepsilon_2$, and a monochromatic plane-wave of frequency $\omega$ arrives at the incidence angle $\theta_{\text{inc}}$ that exceeds the critical TIR angle $\theta_{\text{TIR}} = \sin^{-1} \left( \sqrt{\varepsilon_2 / \varepsilon_1} \right)$. In this case, given the dispersion relation that yields the $z$-component of the transmitted $k$-vector in terms of $k_x$, $\omega$, and the speed of light in free space, $c$, as $k_{z2} = \pm (\omega/c) \sqrt{\varepsilon_2 - (ck_x/\omega)^2}$, the quantity under the radical becomes negative, resulting in an evanescent wave that could either decay or grow (exponentially) as it recedes from the interface.

When the transmission medium is lossless, the commonsense argument compels us to choose the exponentially decaying evanescent wave [1–3]. If the transmission medium happens to be weakly absorptive, its complex dielectric constant $\varepsilon_2' + i\varepsilon_2''$ brings about a small reduction in the Fresnel reflection coefficient at the interface by directing a fraction of the incident energy into medium 2, which this medium subsequently absorbs. The general characteristics of the quasi-evanescent wave, however, will not change drastically compared to the transparent case, provided that the imaginary part $\varepsilon_2''$ of $\varepsilon_2$ is reasonably small. Specifically, the transmitted wave continues to exponentially decay away from the interface, albeit with a small (non-zero) component of its Poynting vector directing the electromagnetic (EM) energy along the propagation direction, which we have taken to be the $z$-axis. In contrast, if the transmission medium has gain, which occurs when $\varepsilon_2''$ assumes a negative value at the frequency $\omega_{\text{inc}}$, the incident (monochromatic) plane-wave, there is no simple commonsense argument to decide whether the EM wave within medium 2 should grow or decay exponentially along the $z$-axis [4–6] (this question of TIR from a gain medium is not merely an academic curiosity, but one that has important practical applications and consequences in the context of fiber-optical lasers and amplifiers in which the core is passive and the cladding constitutes the gain medium [7]).

In this paper, we explain how to solve the Fresnel problem without having to handpick a particular wave-vector $k$ in the transmission medium. The key is to analyze the system’s response not just at a single frequency, but in the entire complex frequency plane, $\omega = \omega' + i\omega''$, and impose the requirement that the system obey causality. In this way, the correct choice for the direction of the wave-vector at every value of $\omega$ naturally emerges, eliminating the ambiguity that previously required the choice of one solution over another. The causality constraint thus replaces the requirement for “commonsense,” and the method works just as well for gainy media as it does for transparent and lossy media.

By choosing an incident wavepacket that has a finite duration as well as a finite footprint at the interface between media 1 and 2, we broaden the scope of the investigation to
include a continuum of temporal frequencies \( \omega \) as well as spatial frequencies \( k_x \) within the spectral profile of the incident wave. As it turns out, answering the narrow question of how to solve the Fresnel problem for a single incident plane-wave, \( (k_x, \omega) \), requires a comprehensive solution for the entire spatiotemporal spectrum of the incident wavepacket. This is true not only for the single-surface problem depicted in Fig. 1a, but also for the finite-slab problem of Fig. 1b. In the special case of TIR from a semi-infinite gain medium, the correct \( k_x \) will be seen to carry the plus sign for some values of \( (k_x, \omega) \) and minus sign for the others. Thus, the solution to this problem is not as simple as a prescription for which sign to choose below, and which sign above, the critical angle \( \theta_{\text{TIR}} \). Instead, the problem leads us to consider the entire complex \( k_x \)-plane and, along the way, it redefines our notion of what it means to “cross the critical angle.” The problem also illuminates important details about spatiotemporal spectral composition: the plane-waves that constitute a wavepacket must be carefully chosen to avoid violating causality.

The connection between a system’s frequency response and causality is taught at the undergraduate level, typically in the context of the Kramers–Kronig relations [2]. In that case, the system under consideration (e.g., an individual dipole) is driven by a time-dependent stimulus, the response to which unfolds as a function of time only. When the system’s stimulus and response happen to be functions of both space and time, as in the Fresnel problem, there are subtleties that require careful consideration. In what follows, we address these issues in the general framework of linear response theory before proceeding to apply the concepts to the Fresnel problem.

The organization of the paper is as follows. In the next section, we examine the connection between causality and the complex-plane representation of spatiotemporal frequencies \( (k, \omega) \) in a linear system. An examination of the analyticity of the system’s transfer function in the entire complex plane reveals the circumstances under which the superposition integrals involving the \( k_x \) variable are best carried out in the complex \( k_x \)-plane along a contour that deviates from the real \( k_x \)-axis. Then, in Sect. 3, we formulate the Fresnel problem in the presence of gainy media as one such linear system to which the aforementioned complex-plane techniques apply. In the sections that follow, we dissect the Fresnel problem’s transfer function and proceed to examine it one piece at a time.

Section 4 is devoted to a description of the Lorentz oscillators that underlie the frequency-dependent dielectric functions \( \varepsilon(\omega) \) of the material media—be they the nearly transparent incident and transmittance media 1 and 3, or the amplifying gain medium 2. In Sect. 5, we describe the Fresnel reflection and transmission coefficients for the two systems under consideration, namely, the system depicted in Fig. 1a involving a semi-infinite gain medium, and the system of Fig. 1b, where a finite-thickness gainy slab is sandwiched between two nearly transparent dielectric media. The geometric configuration of these systems is such that the \( k_x \)-vectors in each medium will have a component \( k_x \) along the \( x \)-axis and a component \( k_z \) along the \( z \)-axis. Given an incident plane-wave with \( (k, \omega) = (k_x \hat{x} + k_z \hat{z}, \omega) \), Maxwell’s boundary conditions obligate the excited plane-waves in media 1, 2, and 3 to have the same \( k_x \) value as the incident wave; this shared value of \( k_x \) will be treated as a complex entity and written as \( k_x' + ik'' \). The corresponding \( k_z \) in each medium is subsequently determined from the dispersion relation, \( k^2_x + k^2_z = (\omega/c)^2 \varepsilon(\omega) \), where \( c \) is the speed of light in vacuum. The dependence of \( k_z \) on \( \varepsilon(\omega) \) indicates that, for each pair of incident \( k_x \) and \( \omega \) values, there will be a \( k_z \) in medium 1, \( k_z \) in medium 2, and \( k_z \) in medium 3. Given that the dispersion relation specifies each \( k_x \) as the square root of a complex entity, there will be a sign ambiguity for each \( k_z \) that we will eventually resolve by a proper choice of the corresponding branch-cuts [9–11].

The branch-points and branch-cuts associated with the \( k_z \) components of the various \( k \)-vectors play a pivotal role in determining the shape of the aforementioned integration contour in the \( k_z \)-plane, as explained in Sect. 6. Also important in deciding the shape of the integration contour in the finite-slab problem are the \( k_z \)-plane trajectories of the singularities (e.g., poles) of the Fresnel reflection and transmission coefficients; this connection will be elucidated in Sect. 7.

Finally, we put the proposed method to use with numerical simulations aimed at computing the reflected and transmitted waves for a finite-duration, finite-width incident wavepacket. A simple model for the incident wavepacket is introduced in Sect. 8. (Although, for pedagogical reasons, we use this model of the incident packet in our numerical simulations, we are fully aware of its shortcomings as a realistic model. A more nuanced approach to constructing realistic incident wavepackets is outlined in Appendix A.) Our numerical simulation results are presented in Sect. 9, first for a semi-infinite gain medium, and then for a 5.0 micron-thick gainy slab. These simulations are primarily intended to demonstrate the viability of the proposed method of calculation using currently available computational resources. They also reveal the profound differences between the Fresnel reflection and transmission in the presence of gain media versus those involving only passive (i.e., transparent and/or lossy) media. The paper closes with a brief summary of the results and a few concluding remarks in Sect. 10.
According to Cauchy’s theorem of complex analysis, the integral from \(- \Omega\) to \(\Omega\) of an arbitrary function \(\phi(\omega)\) on the real-axis \(\omega'\) of the \(\omega\)-plane equals the integral of \(\phi(\omega)\) over the depicted semi-circular contour of radius \(\Omega\), provided that \(\phi(\omega)\) is analytic within the enclosed region between the semi-circle and the straight-line segment from \(- \Omega\) to \(\Omega\). The preceding statement remains valid in the limit when \(\Omega \to \infty\). Thus, if the domain of analyticity of \(\phi(\omega)\) is the entire upper-half \(\omega\)-plane and, furthermore, if \(\phi(\omega)\) approaches zero sufficiently rapidly when \(|\omega| \to \infty\) in the upper-half plane so that the integral over the semi-circle goes to zero, then Cauchy’s theorem guarantees the vanishing of the integral along the real axis, that is, 

\[
\int_{\infty}^{\infty} \phi(\omega') d\omega' = 0
\]

### 2 Spatiotemporal frequencies in the complex plane

To appreciate a well-known consequence of the causality constraint, consider a linear, shift-invariant system whose output \(g(t)\) is related to the input \(f(t)\) via a convolution with the system’s impulse-response \(h(t)\); that is, \(g(t) = f(t) * h(t)\). Denoting the frequency by the (real-valued) variable \(\omega'\), and the Fourier transforms of our time-dependent functions by \(\tilde{f}(\omega')\), \(\tilde{g}(\omega')\), and \(\tilde{h}(\omega')\), the response of the system to \(f(t)\) can be written as follows [8]:

\[
g(t) = (2\pi)^{-1} \int_{-\infty}^{\infty} \tilde{f}(\omega') \tilde{h}(\omega') e^{-i\omega't} d\omega'. \tag{1}
\]

The above integral, taken from \(-\infty\) to \(\infty\) along the real axis \(\omega'\) of the complex \(\omega\)-plane, may be regarded as an integral over the lower leg of a contour in the \(\omega\)-plane that is closed via a large semicircle in the upper half-plane, as shown in Fig. 2. If the input \(f(t)\) happens to be zero for \(t < 0\), then causality dictates that the response \(g(t)\) must similarly vanish for \(t < 0\). By Cauchy’s residue theorem [9–11], this implies that the integrand in Eq. (1) should be an analytic function of \(\omega\) in the upper half of the \(\omega\)-plane. More specifically, given that \(e^{-i\omega't}\) and \(\tilde{f}(\omega)\) are already well-behaved analytic functions—for any reasonable choice of the input \(f(t)\)—the implication is that the transfer function \(\tilde{h}(\omega)\) should also be analytic in the upper half-plane.

Next, consider a two-dimensional (2D) linear, shift-invariant system whose input \(f(x,t)\) is a function of both a spatial coordinate \(x\) and the time coordinate \(t\). The Fourier expansion of this function may be written as a 2D integral over the real variables \(k_x\) and \(\omega'\), as follows:

\[
f(x,t) = (2\pi)^{-2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(k_x,\omega') e^{ik_x x} e^{-i\omega't} dk_x d\omega'. \tag{2}
\]

Denoting the system’s transfer function in the Fourier domain by \(\tilde{h}(k_x,\omega')\), which is also a function of spatial and temporal frequencies \(k_x\) and \(\omega'\), the output \(g(x,t)\) of the system will be

\[
g(x,t) = (2\pi)^{-2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(k_x,\omega') \tilde{h}(k_x,\omega') e^{ik_x x} d\omega' dk_x. \tag{3}
\]

Once again, for causality to hold, the transfer function \(\tilde{h}(k_x,\omega)\) should be analytic in the upper-half \(\omega\)-plane. But now that \(\tilde{h}(k_x,\omega)\) is a multivariate function, there are subtleties in the above statement that require careful dissection, because the behavior of \(\tilde{h}(k_x,\omega)\) in the \(\omega\)-plane depends on the value of \(k_x\) at which we evaluate the transfer function. Writing \(k_x = k_x' + i k_x''\) and \(\omega = \omega' + i \omega''\), we must examine \(\tilde{h}(k_x,\omega)\) as a function of its four real variables \(k_x', k_x'', \omega', \) and \(\omega''\). In this four-dimensional space, suppose we conduct a search for the set of all points \(P_x = (k_x', k_x'', \omega', \omega'')\) where \(\tilde{h}(k_x,\omega)\) is non-analytic—say, due to the existence of singularities such as poles and/or branch-cuts. If any such points of non-analyticity happen to satisfy \(k_x'' = 0\) and \(\omega'' \geq 0\) (i.e., at least one \(P_x\) lands on the \(k_x''\)-axis of integration in Eq. (3) and in the upper-half of the \(\omega\)-plane), we may be tempted to declare that \(\tilde{h}(k_x,\omega)\) is not analytic in the upper-half \(\omega\)-plane and that, therefore, the causality of the system cannot be assured; see Fig. 3.

That conclusion, however, would be premature; we can rescue causality, because we have the freedom to switch to an alternative integration path in the \(k_x\)-plane—to be justified shortly. By moving the integration path away from the real \(k_x\)-axis and onto a contour that bypasses all \(k_x\)-plane singularities \(k_x'(k_x'')\) associated with \(\omega'' \geq 0\), one can restore analyticity to the upper-half \(\omega\)-plane, thereby affirming the causality of the system; a typical deformed integration path in the \(k_x\)-plane is shown in Fig. 3.

In summary, the recipe for treating such problems is:

1. Find all points \(P_x = (k_x', k_x'', \omega', \omega'')\) where \(\omega'' \geq 0\), and the resultant \(\tilde{h}(k_x,\omega)\) is non-analytic (say, due to the existence of a singularity such as a pole or a branch-cut).
2. Identify the projections \((k_x', k_x'')\) into the \(k_x\)-plane of all the singular points \(P_x\) with \(\omega'' \geq 0\).
3. Choose an integration contour in the \(k_x\)-plane that bypasses all such points of non-analyticity.
There exists a continuum of such singularities, of which five points along the trajectory have been highlighted. The singularities whose projections lie in the upper half of the $\omega$-plane are rendered moot by choosing an integration contour in the $k_x$-plane that bypasses those singularities. In this way, the transfer function remains analytic in the upper-half $\omega$-plane, which is needed to ensure the overall causality of the response of the system.

We have presented the above method in a way that suggests the same $k_x$-plane integration contour $C$ must be used for each and every value of $\omega'$ in the $\omega$-integral. As helpful as this may be as a way of thinking about the procedure for didactic purposes, in practice, it is not an actual restriction. When evaluating the inverse Fourier integral numerically, one may pick different $k_x$-plane integration contours for each and every value of $\omega'$, if that happens to be more convenient.

Finally, it must be pointed out that, in the aforementioned step iii, one may not be able to choose an integration path that bypasses all the singular points in the $k_x$-plane associated with the upper-half $\omega$-plane. When this happens, the method fails and the system is said to exhibit an absolute instability—as opposed to a convective instability [12–15].

This section has provided a bird’s eye view of the fundamental mathematical procedure used in the remainder of the paper. In what follows, we apply this method to the Fresnel problem to compute the reflected and transmitted wavepackets in the presence of a gain medium. Many important details that have been mentioned briefly here will be brought up again and expanded upon as our solution to the Fresnel problem is elaborated.
3 Statement of the problem

Having outlined a possible approach to ensuring the causality of a system whose physical evolution unfolds in space as well as in time, we now cast the Fresnel problem in this framework. The two systems of interest in this paper are schematically shown in Fig. 1. The incident wavepacket is uniform along the y-axis, thus rendering our analysis independent of the y coordinate. The fixed observation point \((x_0, z_0)\) may be located in medium 1 (where \(z_0 \leq 0\)), or in medium 2 (where \(z_0 \geq 0\) for the single-surface problem, and \(0 \leq z_0 \leq d\) for the finite-slab problem), or, in the case of the finite slab, in medium 3 (where \(z_0 \geq d\)). All three media are linear, isotropic, homogeneous, and non-magnetic, specified by their dielectric permittivity \(\varepsilon(\omega)\) and magnetic permeability \(\mu(\omega) = 1\). Thus, the optical properties of the \(j\)th medium are encapsulated in its dielectric function \(\varepsilon_j(\omega)\), which is related to the refractive index via the standard identity \(n_j(\omega) = \sqrt{\varepsilon_j(\omega)}\) [1]. The goal is to compute the reflected and transmitted \(E\)-field at the observation point as a function of time, namely, \(E(x_0, z_0, t)\).

In connection with the discussion in Sec.2, the transfer functions here are \(\rho(k_x, \omega') e^{-i k_x z_0}\) and \(\tau(k_x, \omega') e^{i k_x z_0}\). In these equations, \(k_{1z}\) and \(k_{2z}\) are the \(z\)-components of the \(k\)-vectors within the incidence and transmittance media, which—as aside from a sign ambiguity which we set out to resolve—are determined by the dispersion relation in Eq. (6) [1–3]. The Fresnel reflection and transmission coefficients \(\rho\) and \(\tau\) appearing in Eqs. (7) and (8) are themselves functions of \(k_{1z}\) and \(k_{2z}\), which are, in turn, related to \(k_x\) and \(\omega\); these relations will be discussed in some detail in the following sections. The guiding principle is to understand the behavior of the inverse Fourier integrands of Eqs. (7) and (8) in the complex \(k_x\) and \(\omega\) planes, and to choose a \(k_x\)-plane contour \(C\) that renders the integrands analytic in the upper half \(\omega\)-plane. As it turns out, the single-surface problem depicted in Fig. 1a provides an instructive example for how to deal with the various \(k\)-branch-cuts as the cause of non-analyticity of the integrand, whereas the finite-slab problem of Fig. 1b exemplifies the method of dealing with other types of singularity (e.g., poles) of the corresponding integrands.

\[
E^{\text{ref}}_y(x_0, z_0, t) = (2\pi)^{-2} \int_{\omega'=-\infty}^{\infty} d\omega' e^{-i \omega' t} \int_C \tilde{E}^{\text{inc}}_y(k_x, \omega') \rho(k_x, \omega') e^{i (k_x x_0 - k_{1z} z_0)} dk_x,
\]

\[
E^{\text{trans}}_y(x_0, z_0, t) = (2\pi)^{-2} \int_{\omega'=-\infty}^{\infty} d\omega' e^{-i \omega' t} \int_C \tilde{E}^{\text{inc}}_y(k_x, \omega') \tau(k_x, \omega') e^{i (k_x x_0 + k_{2z} z_0)} dk_x.
\]
4 Lorentz oscillators

In general, each of the three media, modelled as a collection of $K$ Lorentz oscillators, has its own dielectric permittivity $\varepsilon(\omega)$, as follows [1–3]:

$$\varepsilon(\omega) = 1 + \sum_{k=1}^{K} f_k \frac{\omega_p^2_k}{(\omega^2 - \omega_p^2 - i\gamma_k \omega)}. \tag{9}$$

The above equation contains the standard plasma frequency $\omega_p$, resonance frequency $\omega_0$, damping coefficient $\gamma$, and oscillator strength $f$ for each oscillator. If the resonance line-widths are sufficiently narrow (i.e., small $\gamma$) and the resonance frequencies $\omega_{r1}, \omega_{r2}, \ldots, \omega_{rK}$ are sufficiently far apart, the various oscillators act more or less independently of each other. Each oscillator then dominates a range of frequencies centered at $\omega_{r_k}$. Each medium has its own set of $4K$ parameters $(f_k, \omega_{p_k}, \omega_{r_k}, \gamma_k)$, with $k$ ranging from 1 to $K$. In the case of the passive media (1 and 3), all oscillator strengths $f_k$ are +1, whereas the active medium (i.e., gain medium 2) has at least one oscillator whose strength $f_k$ equals −1.\footnote{For non-magnetic media, where the relative permeability $\mu(\omega)$ is 1, the refractive index $n(\omega)$, the dielectric susceptibility $\chi(\omega)$, and the relative permittivity $\varepsilon(\omega)$ are related via the standard identity $n(\omega) \sqrt{\mu(\omega)} = 1 + \chi(\omega)$.}

The general aspects of the problem can be studied for single-oscillator media (i.e., $K = 1$). Multi-oscillator media are not expected to introduce conceptual or mathematical difficulties—at least in cases where the various resonance frequencies of each medium are sufficiently far apart from one another—beyond those already encountered in the case of single-oscillator media. Therefore, we will use the dielectric functions $\varepsilon_1(\omega)$ and $\varepsilon_2(\omega)$ of media 1 and 2 given by the single-oscillator Lorentz model [2, 3], as follows:

$$\varepsilon_1(\omega) = 1 + \frac{\omega_p^2_1}{\omega_{r1}^2 - \omega^2 - i\gamma_1 \omega}; \quad \varepsilon_2(\omega) = 1 + \frac{\omega_p^2_2}{\omega_{r2}^2 - \omega^2 - i\gamma_2 \omega}. \tag{10}$$

Note that the passive medium 1 is distinguished from the active (gain) medium 2 by the plus sign versus the minus sign appearing immediately after 1 on the right-hand side [16]. The dielectric function $\varepsilon_3(\omega)$ of medium 3 is similar to $\varepsilon_1(\omega)$, albeit with its own parameter set $(\omega_{p3}, \omega_{r3}, \gamma_3)$.

Although the Lorentz oscillators obey the Kramers–Kronig relations, thereby guaranteeing the causal response of the individual dipoles of each medium, the technique described in Sect. 2 is still needed to ensure that the system as a whole complies with the causality constraint.

5 Reflection and transmission coefficients

For an $s$-polarized incident wavepacket (i.e., one whose $E$-field is aligned with the $y$-axis, also known as a transverse electric or TE wave), the Fresnel reflection and transmission coefficients in the single-surface problem of Fig. 1a are [1–3]

$$\rho(k_z, \omega) = (k_{1z} - k_{2z}) / (k_{1z} + k_{2z}), \tag{11}$$

$$\tau(k_z, \omega) = 2k_{1z} / (k_{1z} + k_{2z}). \tag{12}$$

In the finite-slab problem of Fig. 1b, the reflection coefficient at the slab’s front facet ($z = 0$) and transmission coefficient at its rear facet ($z = d$) are given by [1–3]

$$\rho(k_z, \omega) = \frac{\rho_{1z} + \rho_{23} \exp(2ik_{2z}d)}{1 - \nu}, \tag{13}$$

$$\tau(k_z, \omega) = \frac{(1 + \rho_{1z})(1 + \rho_{23}) \exp(ik_{2z}d)}{1 - \nu}. \tag{14}$$

The transmission coefficient yielding the $E$-field inside the gain layer (medium 2) at $z = z_0$ is

$$\tau(k_z, \omega) = \frac{(1 + \rho_{1z}) \exp(ik_{2z}z_0) + \rho_{23} \exp[ik_{2z}(2d - z_0)]}{1 - \nu}. \tag{15}$$

In Eqs. (13)–(15), the roundtrip coefficient $\nu$, itself a function of $k_z$ and $\omega$, is given by

$$\nu = \rho_{21} \rho_{23} \exp(2ik_{2z}d). \tag{16}$$

For an $s$-polarized incident wave, the reflection coefficients $\rho_{1z}$, $\rho_{21}$, and $\rho_{23}$, are known to be [1–3]

$$\rho_{1z} = -\rho_{21} = (k_{1z} - k_{2z}) / (k_{1z} + k_{2z}), \tag{17}$$

$$\rho_{23} = (k_{2z} - k_{3z}) / (k_{2z} + k_{3z}). \tag{18}$$

The $z$-components of the $k$-vectors within media 1, 2, and 3, denoted by $k_{1z}$, $k_{2z}$, and $k_{2z}$ and given by Eq. (6), will be described in the following section. For now, it is important to recognize that the various $k_z$ are functions of both $\omega$ and $k_z$, where $\omega$ and $k_z$ are generally complex-valued. Each $k_z$, being the square root of a complex entity, requires a choice of plus or minus sign, with only one of the two signs being acceptable at any given point $(k_z, \omega)$.\footnote{The only exception to this rule is the sign of $k_{2z}$ in the case of the finite-slab of Fig. 1b, where Eqs. (13)–(18) yield the same values for $\rho(k_z, \omega)$ and $\tau(k_z, \omega)$ irrespective of the chosen sign for $k_{2z}$. Consequently, the choice of branch-cuts for $k_{2z}$ in the finite-slab problem is of no significance. In all other cases, one must carefully choose the branch-cuts for $k_{1z}$, $k_{2z}$, and $k_{3z}$, to ensure that each acquires its correct sign.} One must
ensure that $k_z(-k_z^*, -\omega^*) = -k_z^*(k_z, \omega)$, to guarantee the Hermitian symmetry relations $\rho(-k_z^*, -\omega^*) = \rho^*(k_z, \omega)$ and $\tau(-k_z^*, -\omega^*) = \tau^*(k_z, \omega)$, which are essential if the reflected and transmitted fields at all observation points are to be real-valued.

In connection with the reflected $E$-field computed in accordance with Eq. (7), the Fresnel reflection coefficient $\rho$ is given by Eq. (11) in the case of a semi-infinite gain medium, and by Eq. (13) in the case of a finite-thickness slab. Similarly, the transmitted $E$-field inside the semi-infinite gain medium of Fig. 1a is obtained from Eq. (8) using the Fresnel transmission coefficient $\tau$ given by Eq. (12). In the case of the finite-thickness slab of Fig. 1b, the transmitted $E$-field inside medium 3 is obtained using the Fresnel coefficient $\tau$ of Eq. (14), provided that $k_{3z}(z_0 - d)$ is substituted for $k_{z2}z_0$ within the exponential factor in Eq. (8). As for the $E$-field inside the slab itself, one must use Eq. (8) in conjunction with $\tau(k_z, \omega')$ of Eq. (15) and, given that Eq. (15) already incorporates the relevant propagation phase-factor within the slab, the term $\pm 2iz_0$ should be removed from the exponential factor in Eq. (8).

Inside the complex $\omega$-plane, the region of interest will be the upper half-plane, although certain symmetries allow us to focus our attention exclusively on the first quadrant ($Q_1$), where $\omega = \omega' + i\omega''$ has $\omega' \geq 0$ and $\omega'' \geq 0$. For all points in the second quadrant ($Q_2$), we have $\epsilon(-\omega'') = \epsilon^*(\omega)$, a direct consequence of the Lorentz oscillator model of Eq. (9). For these points, there is no need to keep track of the points in $Q_2$ of the $\omega$-plane (the symmetry between $Q_1$ and $Q_2$ of the $\omega$-plane also implies that the contribution to the $E$-field at $(x_0, z_0)$ by frequencies at that reside on the positive imaginary axis $\omega''$ must be real-valued). The bulk of our computational effort revolves around identifying and then eliminating the singularities of the inverse Fourier integrands of Eqs. (7) and (8) from $Q_1$ of the $\omega$-plane. As pointed out in Sect. 2, removing all these singularities is necessary to ensure that causality is satisfied. It is worth emphasizing here that the final step in the calculation of $E_y(x_0, z_0, t)$ is an inverse Fourier transformation over the real $\omega'$-axis, as seen in Eqs. (7) and (8). Considering that the contribution to the inverse Fourier integral of the negative half of this axis equals the conjugate of that from the positive half, one can simplify the calculation by taking the real part of the integral computed only for $\omega' = 0$ to $\omega'$.

Having identified all the terms of the integrands of Eqs. (7) and (8), we now examine these integrands for their regions of non-analyticity due to the existence of branch-cuts, and then poles.

\begin{displaryfigure}
\centering
\includegraphics[width=\textwidth]{fig4}
\caption{Trajectories of the zeros of $k_{1x}$, $k_{2x}$, and $k_{3x}$ in the complex $k_x$-plane, when $\omega'$ is fixed at zero value while $\omega''$ rises from 0 to $+\infty$. In compliance with the requirement to bypass all the singular points, the chosen integration contour $C$ avoids crossing the branch-point trajectories for all values of $\omega' \geq 0$.}
\end{displaryfigure}

### 6 Branch-points and branch-cuts

As discussed in Sect. 2, it is necessary to identify all the non-analytic points (i.e., poles and branch-cuts) of the integrands in Eqs. (7) and (8) that satisfy $\omega'' \geq 0$. Focusing on the branch-points in the present section, we note that the wave-vector components $k_{1x}$, $k_{2x}$, and $k_{3x}$ appear throughout the inverse Fourier integrands, both explicitly (in the exponential terms) and implicitly (in the Fresnel coefficients). Since the function $k_{1x}(k_z, \omega)$ involves a square root, a branch-cut is needed to uniquely evaluate this square root. To examine the branch-points and branch-cuts in the $k_x$-plane, we decompose $k_{1x}$ into the following product of two square roots:

$$
k_{1x} = \sqrt{(\omega/\epsilon')^2 \epsilon_z(\omega) - k_z^2} = -i[k_z - \omega \epsilon_z(\omega)/\epsilon]^{\frac{3}{2}}[k_z + \omega \epsilon_z(\omega)/\epsilon]^{\frac{1}{2}}.
\tag{19}
$$

Recall that, for a given $\omega' \geq 0$, $n_1(\omega')$ and $n_3(\omega')$ will each have a value in $Q_1$ and a symmetrically located value in $Q_3$ of the complex plane—a simple consequence of the fact that $n_1$ and $n_3$ are the square roots of $\epsilon_1$ and $\epsilon_3$, both of which are $Q_1$ complex numbers. In the case of $n_2(\omega')$, if the dominant oscillator in the vicinity of $\omega'$ happens to be gainy (i.e., $f_k < 0$), then the values of $n_2(\omega')$ will be in $Q_2$ and $Q_3$; otherwise, $n_2(\omega')$ will be in $Q_1$ and $Q_3$.

We start by choosing a fixed point on the $\omega$-plane integration path with $\omega' \geq 0$ and $\omega'' = 0$. Each of the two radicals on the right-hand side of Eq. (19) will then have a branch-point [9, 10] within the $k_x$-plane at $k_z = \pm \omega n_1(\omega')/c$. For $k_{1x}$ and $k_{3x}$, these branch-points always end up in $Q_1$ and $Q_3$ of the $k_x$-plane. For $k_{2x}$, the branch-points will be in $Q_2$ and $Q_4$ of the $k_x$-plane if $\omega'$ happens to be near the resonance frequency of a gainy oscillator; otherwise, they will land in $Q_1$ and $Q_3$. Having found the location of the branch-points
for \( \omega'' = 0 \), we now trace the trajectories of these branch-points as \( \omega'' \) rises from 0 to \(+\infty\), the reason being the need to identify all possible non-analytic points in the upper-half \( \omega \)-plane—not just those that fall on the \( \omega \)-plane integration path. The \( k_x \)-plane trajectories of the branch-points of \( k_{1z}, k_{2z}, \) and \( k_3 \), for a fixed value of \( \omega' > 0 \) are shown schematically in Fig. 4 (and computed precisely for a specific example in Sect. 9). The branch-point trajectories in \( Q_1 \) and \( Q_4 \) move upward, whereas the corresponding trajectories in \( Q_2 \) and \( Q_3 \) move downward (due to the inherent odd symmetry).

The branch-cuts should be drawn as straight vertical lines, originating from each branch-point and extending to infinity—although, to reduce clutter, some may find it convenient to designate the branch-point trajectories themselves as branch-cuts [11]. Note that the branch-point trajectories for \( k_{1z} \) and \( k_{2z} \) never cross the \( k'_x \)-axis, whereas the trajectories for \( k_3 \), that start in \( Q_2 \) and \( Q_4 \) (i.e., those corresponding to gainy oscillators) do cross the \( k'_x \)-axis. These branch-point trajectories for \( k_{2z} \) necessitate a deformation of the integration contour away from the \( k'_x \)-axis in the \( k_x \)-plane (i.e., inverse Fourier transformation with respect to \( k_x \)), lest the integration path crosses a branch-cut, which would result in a discontinuity of \( k_{2z} \) that heralds a violation of causality by rendering the integrand in the upper-half \( \omega \)-plane non-analytic.

One such deformed contour, though by no means a unique choice, appears in Fig. 4. Note that one could choose a different \( k_x \)-contour for each \( \omega' \) along the \( \omega \)-contour. Alternatively, one could draw the branch-point trajectories of Fig. 4 for all values of \( \omega' \geq 0 \), then choose a single \( k_x \)-contour that bypasses all of them at once. We choose the latter method in our numerical simulations, though, in the end, either method would give the same result. By avoiding the branch-point trajectories in the \( k'_x \)-plane, our chosen integration path ensures the analyticity of the wave-vector components \( k_c(k_x, \omega) \) in all three media.

Lastly, we must specify the range of the phase angles \( \phi \) at the branch-cuts associated with the individual terms whose square roots appear on the right-hand side of Eq. (19). The necessity for all three \( k_z \) s to approach \( \omega/c \) when \( \omega'' \to +\infty \) can be appreciated by examining the exponential factors in the integrands of Eqs. (7) and (8) in conjunction with the semi-circular integration path in the upper-half \( \omega \)-plane of Fig. 2, on which the integrands are required to vanish for \( t < |\tau_0|/c \).

Interestingly, the branch-cuts for \( k_{2z} \) play an important role in determining the integration contour \( C \) only in the single-surface problem depicted in Fig. 1a. For a gainy slab of finite thickness \( d \), such as that of Fig. 1b, the Fresnel reflection and transmission coefficients of the slab are insensitive to the choice of sign for \( k_{2z} \) (i.e., the coefficients remain the same under the transformation \( k_{2z} \to -k_{2z} \)); consequently, the branch-cuts of \( k_{2z} \) play no role whatsoever in determining the \( k_x \)-plane integration contour \( C \) for the finite-slab problem. In this case, the choice of integration contour is dictated by other singularities of the Fresnel reflection and transmission coefficients whose \( \omega \)-values happen to be in \( Q_1 \) of the \( \omega \)-plane. These singularities are discussed in the next section.

**Fig. 5** Branch-cuts for \( k_{1z} \) and \( k_{2z} \) in the \( k_x \)-plane corresponding to a fixed, positive, real-valued frequency \( \omega \). The branch-cuts are chosen to ensure that they do not cross the deformed integration contour \( C \). The branch-cuts are chosen to bypass all branch-cuts, \( k_{2z} \) varies continuously with \( k_x \) and \( \omega \), becoming an analytic function in the upper-half \( \omega \)-plane.
7 Singularities of the Fresnel reflection and transmission coefficients

In the case of a gainy slab of finite-thickness \( d \), when the roundtrip coefficient \( v \) defined in Eq. (16), itself a function of \( k_x \) and \( \omega \), equals 1.0, the Fresnel reflection and transmission coefficients \( (\rho \) and \( \tau \) diverge to infinity, causing the inverse Fourier integrands in Eqs. (7) and (8) to exhibit a singularity at the corresponding point \((k_x, \omega)\). Considering that \( v \) is a function of \( k_x^2 \), the \( k_x \)-values of the singularities always appear in symmetric pairs \( (\pm k_x, \omega) \). In general, a large number of such singularities exist, and one must conduct an exhaustive numerical search to identify all such singular points \((\pm k_x, \omega)\) for all \( Q_1 \) points of the \( \omega \)-plane.

In the single-surface problem of Fig. 1a, the singularities of \( \rho(k_x, \omega) \) and \( \tau(k_x, \omega) \) will be the zeros of \( k_1 + k_2 \); see Eqs. (11) and (12). If both media 1 and 2 happen to have only a single Lorentz oscillator, it can be shown that \( \rho \) and \( \tau \) will have no singularities in the upper-half \( \omega \)-plane. In the case of multi-oscillator media, however, a numerical search must be conducted to identify the singularities \((\pm k_x, \omega)\) associated with all points in \( Q_1 \) of the \( \omega \)-plane. The \( k_x \)-plane integration contour \( C \) must then bypass the \( \pm k_x \) singular points residing in \( Q_2 \) and \( Q_4 \) of the \( k_x \)-plane, in addition to avoiding all the branch-point trajectories of \( k \), as described previously.

Due to symmetries that allow the \( \omega \)-integral to be evaluated along the positive semi-axis \( \omega' > 0 \), we only need to find the singularities that fall within \( Q_1 \) of the \( \omega \)-plane. Most of the \( \pm k_x \) pairs associated with singular points \( \omega \) in \( Q_1 \) of the \( \omega \)-plane turn up in \( Q_1 \) and \( Q_3 \) of the \( k_x \)-plane. However, for a weakly amplifying medium, a limited number of such singular \( \pm k_x \) pairs appear in \( Q_2 \) and \( Q_4 \) of the \( k_x \)-plane, not too far away from the real \( k_x' \)-axis. By picking a deformed contour \( C \) in the \( k_x \)-plane in such a way as to avoid all such singularities—say, by staying above all singular points in \( Q_2 \) and below all singular points in \( Q_4 \), in the same manner as we bypassed the branch-point trajectories in Fig. 4—one can guarantee the absence of singularities in \( Q_1 \) of the \( \omega \)-plane. In this way, upon evaluating the inverse Fourier integrals over the \( k_x \) variable in Eqs. (7) and (8) along a properly deformed contour \( C \), the resulting function of \( \omega \) ends up being analytic throughout the entire \( Q_1 \) of the \( \omega \)-plane (the analyticity of this function in \( Q_2 \) automatically follows from its mirror symmetry with respect to the imaginary \( \omega'' \)-axis).

The fact that, for any given point in the upper-half \( \omega \)-plane, the corresponding singularities in the \( k_x \)-plane appear as \( \pm k_x \) pairs, accounts for the odd symmetry of the contour \( C \) with respect to the \( k_x'' \)-axis, as depicted in Figs. 4 and 5. Of course, the contour need not obey the same odd symmetry as the singularities, but is only required to properly bypass these non-analytic points.

We have previously pointed out that the symmetry between \( Q_1 \) and \( Q_2 \) of the \( \omega \)-plane obviates the need for evaluating the \( k_x \)-plane integral for a \( Q_2 \) frequency—once the integral for the corresponding \( Q_1 \) frequency has been evaluated. Nevertheless, if one felt inclined to evaluate the \( k_x \)-plane integral for a \( Q_2 \) frequency, then we must emphasize that, in going from a \( Q_1 \) point \( \omega \) to the corresponding \( Q_2 \) point \( -\omega' \), it is imperative to also switch \( k_x \) and \( k_x' \) to \( -k_x \) and \( -k_x' \), respectively. Thus, one must remember to flip the integration contour \( C \) around the imaginary \( k_x'' \)-axis. In the special case when the chosen \( \omega \) happens to be on the positive \( \omega'' \)-axis, the integration path taken in the \( k_x \)-plane could be either the contour \( C \) or its flipped version around the \( k_x'' \)-axis. As a matter of fact, on this dividing line between \( Q_1 \) and \( Q_2 \) of the \( \omega \)-plane, it is also allowed to directly integrate along the real \( k_x' \)-axis—with the accompanying benefit that it is now easy to demonstrate that the integral along the \( k_x' \)-axis is real-valued.

8 Incident wavepacket

In the preceding sections, for purposes of explaining the methodology, it sufficed to state that the incident wavepacket has a definite starting point in time and a finite footprint along the \( x \)-axis at the interface between media 1 and 2. For the numerical calculations that follow, we now specify a

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Fig. 6 Finite-width functions \( f(x/W) \) and \( g(t/T) \) whose product forms the envelope of the incident wavepacket
The time \( t \), angular frequency \( \omega \), distance \( x \), and wavenumber \( k_z \) are normalized by \( t_0 \), \( \omega_0 \), \( x_0 \), and \( k_0 \):

\[
t = t_0^* \quad \omega = \omega_0 \tilde{\omega} (\omega_0 = 1/t_0) \quad x = x_0^* \quad k_z = k_0 \tilde{k}_z (k_0 = 1/x_0)
\]

Table 2: Material parameters for the incidence medium (1), gain medium (2), and transmittance medium (3)

| Medium | \( \omega_p \) | \( \omega_g \) | \( \gamma_1 \) | \( d \) |
|--------|----------------|----------------|---------------|-------|
| Incidence (1) | \( 1.5 \times 10^{15} \) | \( 4.5 \times 10^{15} \) | \( 3.0 \times 10^{14} \) | |
| Gain (2) | \( 3.3 \times 10^{14} \) | \( 3.0 \times 10^{15} \) | \( 3.0 \times 10^{14} \) | \( 5 \mu m \) |
| Transmittance (3) | \( 1.2 \times 10^{15} \) | \( 4.8 \times 10^{15} \) | \( 3.0 \times 10^{14} \) | \( 5 \) |

When the gain medium is not semi-infinite, its thickness is specified as \( d \). Also shown are normalized parameter values.

\[
E^{(inc)}(x, t) = f(x/W)g(t/T) \cos (k_{xc}x - \omega_0 t)\hat{y}.
\] (20)

Here, \( f(x) \) and \( g(t) \), depicted schematically in Fig. 6, are finite-width functions of the spatial coordinate \( x \) and the time \( t \), respectively. The center frequency of the packet is \( \omega_z \), and its central wavenumber is denoted by \( k_{zc} \). Assuming the refractive index \( n_z(\omega) = \sqrt{\epsilon_z(\omega)} \) of the incidence medium is real-valued at \( \omega = \omega_z \), the incident packet's central ray is tilted away from the \( z \)-axis at an angle \( \theta_z = \sin^{-1} \left[ c k_{zc} / \omega_z n_z(\omega_z) \right] \). Needless to say, the finite duration of the packet causes a spreading of the frequency content of the incident beam around \( \omega_z \), while its finite footprint along the \( x \)-axis broadens the range of incidence angles around \( \theta_z \), so that, in principle, every \( \omega \) and every \( k_z \) makes a contribution to the incident packet. A judicious choice of the envelope functions \( f(x/W) \) and \( g(t/T) \), however, ensures that the spatiotemporal spectrum of the incident packet remains more or less confined to the vicinity of \( (k_{xc}, \omega_z) \).

For the numerical simulations reported in the next section, we have chosen \( f(x) \) to be a repeated convolution of the rectangular function \( \text{rect}(x) \) with itself, where \( \text{rect}(x) \) is defined as 1 when \( |x| \leq 1/2 \) and 0 otherwise. The Fourier transform of this function, \( \int_{-\infty}^{\infty} \text{rect}(x) e^{-ik_{xc}x} \, dx \), is \( \text{sinc}(k_{xc}/2\pi) \), where, by definition, \( \text{sinc}(x) = \sin(\pi x) / (\pi x) \). When \( \text{rect}(x) \) is repeatedly convolved with itself \( (n \text{ times}), \) the resulting function, \( \text{rect}(x)^n \text{rect}(x) \) \(* \text{...} \* \text{rect}(x) \), will be a fairly smooth function, having width \( n \) and Fourier transform \( \text{sinc}^n(k_{xc}/2\pi) \) [8]. Thus, the Fourier transform of our envelope function \( f(x/W) \) in Eq. (20) is given by \( \tilde{f}(k_z) = W \text{sinc}^n(\pi Wk_z/2\pi) \). Similarly, we have chosen the other envelope function, \( g(t/T) \), so that its Fourier transform is \( \tilde{g}(\omega) = T \text{sinc}^m(\pi T\omega/2\pi) \). Here, \( m \) and \( n \) are arbitrary (albeit small) positive integers. The 2D Fourier transform of the incident packet is thus found to be

\[
\tilde{E}^{(inc)}(k_z, \omega) = \frac{1}{2} \tilde{f}(k_z - k_{zc}) \tilde{g}(\omega - \omega_z) + \frac{1}{2} \tilde{f}(k_z + k_{zc}) \tilde{g}(\omega + \omega_z).
\] (21)

The simple wavepacket whose spatiotemporal spectral profile is given by Eq. (21) may be criticized on the grounds that it cannot be realistically created at the interface between the incidence medium (1) and the gain medium (2), the reason being that certain frequencies are bound to be strongly absorbed within the incidence medium before arriving at the interface, and also because evanescent waves having \( k_z > n_z(\omega)\omega/c \) do not survive propagation distances that are needed to reach the interface. While we concur with this critique of the model of the incident packet defined by Eqs. (20) and (21), we nevertheless contend that several important aspects of the problem of reflection from (and transmission through) gain media can be fruitfully examined by adopting such a less-than-ideal model. In Appendix A, we propose a more realistic approach to constructing the incident packet that does not suffer from the shortcomings of the model described by Eqs. (20) and (21), albeit at a significant cost in terms of the complexity of the additional numerical computations that would accompany this more realistic construction.

9 Numerical computations

To streamline the numerics and also to abbreviate references to numerical values of the various parameters, we use the normalization scheme summarized in Table 1. Consequently, our units of the time \( t \) are \( 3.33 \ldots \) femtoseconds (fs), frequencies \( \omega \) will be specified in multiples of \( 3 \times 10^{14} \) radians per second (rad/s), spatial coordinates \( x \) and \( z \) will be in microns \((\mu m)\), and the wavenumbers \( k_z \) and \( k_{zc} \) will be given in units of \( 10^6 \) inverse meters \((\text{m}^{-1})\). Note that this normalization scheme leaves products such as \( \omega t \), \( k_x \), and \( k_z \)
The pulse duration is $mT$, the footprint of the beam is $nW$, its center frequency is $\omega_c$, and its central $k_x$ value is $k_{xc}$. Also shown are the normalized values of these parameters.

| $\omega_c = 2.5 \times 10^{15}$ | $T = 1.0 \times 10^{-14}$ | $m = 4$ | $k_{xc} = 8.0 \times 10^6$ | $W = 5.0 \times 10^{-6}$ | $n = 5$ |
|-------------------------------|--------------------------|-------|---------------------|---------------------|-------|
| $\bar{\omega}_c = 8.3333$   | $\bar{T} = 3.0$          | $\bar{k}_{xc} = 8.0$ | $\bar{W} = 5.0$     |                     |       |

The pulse duration is $mT$, the footprint of the beam is $nW$, its center frequency is $\omega_c$, and its central $k_x$ value is $k_{xc}$. Also shown are the normalized values of these parameters.
pattern of the pole distribution and, most importantly, the spread of the poles below the real axis and down into $Q_4$, do not change when we repeat the calculation using different number of points and different rectangular regions.

The blue dots in Fig. 8a are the poles of the Fresnel reflection (or transmission) coefficient for a large number of points $\omega$ on the positive real frequency axis. b Trajectories of a few randomly selected poles in the $k_z$-plane when the corresponding frequency $\omega$ moves up from $\omega'$ to $\omega' + i\omega''$; the purple-to-red color coding indicates increasing values of $\omega''$ from 0 to 1. c Magnified view of a small section of the $k_z$-plane, showing the region $8.3 \leq k'_z \leq 12$ and $-1 \leq k''_z \leq 1$

![Integration contour (green) in $Q_4$ of the $k_z$-plane. A flipped copy of the same contour in $Q_2$ is also needed to complete the integration path.](a)

![Branch-point trajectories of $a$ $k_1$ and $b$ $k_2$, corresponding to a large number of complex frequencies $\omega = \omega' + i\omega''$, with $0 \leq \omega' \leq 40$ and $\omega''$ rising from 0 to 1 (purple to red). Also shown (in green) is the integration contour in $Q_4$ of the $k_z$-plane, which is used in our numerical evaluation of the inverse Fourier integrals over $k_z$. For the parameter values used here, the branch-point trajectories appear to be vertical lines at the scale of the plot, but zooming in would reveal their slight curviness.](b)
with an asymptotic approximation to the pole equation \( \nu(k_x, \omega) = 1 \), valid far from the real axis \( k' \), that the poles corresponding to complex frequencies in \( Q_1 \) of the \( \omega \)-plane are bounded from below in \( Q_2 \) of the \( k_x \)-plane—i.e., there exists a value of \( k'' \) below which no poles can be found. Therefore, we have confidence that our numerical method has found all of the pole trajectories, and that our chosen contour \( C \) avoids crossing any of them.

We have previously mentioned that the branch-cuts of \( k_{1z} \) and \( k_{2z} \) do not cross the \( k' \)-axis and, therefore, would not by themselves require deformation of the standard contour along the \( k' \)-axis. Figure 9a illustrates this point for \( k_{1z} \), showing that, for \( \omega' \geq 0 \), the branch-points for the passive semi-transparent medium 1 always land in \( Q_1 \) (and, by symmetry, in \( Q_3 \)) of the \( k_x \)-plane. The situation is different, however, for the gain medium 2, since, for \( \omega' \geq 0 \), some of the \( k_{2z} \) branch-points originate in \( Q_3 \) (and, by symmetry, also in \( Q_1 \)) of the \( k_x \)-plane; see Fig. 9b. Now, for the finite-slab problem of Fig. 1b, the branch-cuts of \( k_{2z} \) are inconsequential, since Eqs. (13)–(18) are indifferent to a sign-change of \( k_{2z} \). It is only for the single-surface problem of Fig. 1a that the \( k_{2z} \) branch-cuts need to be considered when constructing the integration contour \( C \) in the \( k_x \)-plane. Figure 9b shows an integration contour (green) that is properly chosen to stay below all \( k_{2z} \) branch-cuts in \( Q_3 \) of the \( k_x \)-plane. (Not shown in this figure is the other half of the same contour \( C \) that is symmetrically located in \( Q_1 \) of the \( k_x \)-plane.) The general rule is that the \( k_x \)-plane integration contour \( C \) must avoid crossing all branch-point trajectories as well as all the pole trajectories (if any) corresponding to frequencies \( \omega \) in \( Q_1 \) of the \( \omega \)-plane; this is what Figs. 8 and 9 aim to convey. Exceptions arise in the case of finite-slabs, where \( k_{2z} \) branch-points become inconsequential, and in the single-surface problem, where \( k_{3z} \) is non-existent.

As a reminder of the role played by the branch-points, branch-cuts, and the contour \( C \) in our numerical simulations, Fig. 10a shows that the location of a point \( k_z \) on \( C \) determines the two complex numbers on the right-hand side of Eq. (19) that are needed to identify unique values for \( k_{1z} \), \( k_{2z} \), and \( k_{3z} \). The indicated ranges for the angle \( \varphi \) associated with each branch-cut guarantee that all three \( k \) s approach \( \omega/c \) in the limit when \( \omega'' \to +\infty \), which is essential for the satisfaction of the causality constraint. Note that the chosen branch-cuts do not cross the contour \( C \), which is also necessary to ensure the analyticity of the integrands in Eqs. (7) and (8) by making the angles \( \varphi \) vary smoothly as the point \( k_{2z} \) travels along \( C \).

As an example, consider Fig. 10b, which shows the trajectory of \( k_{2z} = k'_{2z} + ik''_{2z} \) at the fixed frequency \( \bar{\omega} = 10 \), when \( k_{2z} \) moves along \( C \) from \( \bar{k}'' = 30 \) to 0. It is seen that, for \( |\bar{k}'| < 8.76 \), the imaginary part \( k''_{2z} \) of \( k_{2z} \) is negative, indicating that the corresponding plane-wave inside the semi-infinite gain medium 2 exponentially grows along the \( z \)-axis. Outside this interval, where \( |\bar{k}'| > 8.76 \), the plane-wave inside the gain medium decays away from the interfacial plane at \( z = 0 \). It is thus seen that the problem of TIR at the interface between a (nearly) transparent incidence medium 1 and a (weakly amplifying) semi-infinite gain medium 2 is not amenable to an elementary solution; rather, it is essential to consider the entire spatio-temporal spectrum of the incident packet to determine which plane-wave constituents of the incident beam participate in forward amplification upon entering the gain medium, and which ones support the backward propagating (and amplified) reflected beam.
configuration of the simulated system is depicted in Fig. 11. The incident packet is a 40fs linearly polarized (TE) light pulse with a 25 μm footprint, arriving at the interface between the passive medium 1 and the semi-infinite gain medium 2 at an oblique angle of ~ 63°. The incident light pulse is confined to the (normalized) time interval [\(t_{\text{min}}, t_{\text{max}}]\] = [-6, 6].

In Fig. 12, we show the reflected wavepackets at several locations within the interfacial plane (i.e., at \(z = 0\)); also shown for comparison are the incident packets at \(x_0 = 0\) and ± 5μm. The reflected E-field amplitude profiles are seen to be broadened (due to dispersion as well as diffraction), and also proportionately delayed with an increasing distance from the incident beam’s center at \((x, y, z) = (0, 0, 0)\). Although the reflected waves in the vicinity of the incident beam are relatively weak, they gain strength with an increasing distance \(|x_0|\) away from the center. The tilt of the incident beam is such that its spatial frequency content is heavily biased in favor of positive \(k_x\) values. Nevertheless, plane-waves having negative \(k_x\) values are also present within the spectral profile of the incident wavepacket; these backward-propagating waves (along the \(x\)-axis) are responsible for the growth of the reflected wave amplitudes along the negative \(x\)-axis. Causality is seen to be satisfied since the reflected pulses everywhere emerge only after a proper delay following the onset of the incident pulse at \(t = -6\).

Figure 13 shows the reflected packets at a few locations within the \(xy\)-planes at \(z = -5\) μm and \(z = -10\) μm, which are several wavelengths away from the interfacial plane at \(z = 0\). Recalling that the central incident ray arrives at the rather large oblique angle of \(\theta_c \approx 63^\circ\), we note that the reflected central ray is merely \(27^\circ\) or so away from the interfacial plane. Thus, for example, the reflected packets at \((x_0, z_0) = (20, -5\mu m)\) and \((20, -10\mu m)\) could just be the delayed (and possibly attenuated) versions of the reflected pulses that leave the interface at \((x_0, z_0) = (10\mu m, 0)\) and \((0, 0)\), respectively; see Fig. 12. Considering that the packet arriving at \((20, -5\mu m)\) has nearly twice the amplitude of that at \((20, -10\mu m)\), it is likely that the reduced amplitude at the latter location, being only one-fifth of that at \((0, 0)\), is caused by diffraction or by interference with other parts of the reflected light that return from the interfacial plane.

The transmitted waves at several points inside the gain medium are shown in Fig. 14. Note that the transmitted packets, in addition to being broadened and properly delayed, are also substantially amplified. The arrival time of the transmitted packet at \((x_0, z_0) = (0, 30\mu m)\) is consistent with the expected minimum delay of \(\Delta t = 30\). Similarly, within the \(xy\)-plane at \(z = 40\mu m\), the pulses arrive a short while after the expected minimum delay of \(\Delta t = 40\).

Our next set of numerical results pertains to reflected and transmitted packets for a 5 μm-thick gain medium (2) sandwiched between the incidence medium (1) and the transmittance medium (3). The geometrical configuration of the
Fig. 12 Plots of the incident (red) and reflected (blue) wavepackets at various locations in the interfacial plane (i.e., at $z = 0$) between the nearly transparent medium 1 and the weakly amplifying, semi-infinite medium 2. On the right-hand side, the reflected $E$-field packet is seen to rapidly grow with the positive distance $x_0$ away from the incident beam’s footprint. As for negative values of $x_0$ appearing on the left-hand side, the $E$-field amplitude is small at first, but it also begins to rise, albeit slowly, with distance from the incident packet’s center at $(x, y, z) = (0, 0, 0)$. The packets are seen to be broadened and also proportionately delayed while gaining strength as the observation point recedes from the incident beam’s footprint.
system is shown in Fig. 15. The incident packet is, once again, a 40fs linearly polarized (TE) light pulse with a 25μm footprint, which arrives at the interface between the passive medium 1 and the 5μm-thick gainy slab 2 at an oblique angle of ~ 63°. The light passing through the slab emerges into the nearly transparent passive medium 3.

Figure 16 shows the reflected wavepackets at several locations within the xy-plane at z = 0 (i.e., at the interface between the nearly transparent medium 1 and the 5μm-thick amplifying medium 2); also shown, for comparison with the reflected waves, are the incident packets at x₀ = 20, 30, 35μm. The packets are seen to be broadened (due to dispersion as well as diffraction) and also proportionately delayed with distance away from the incident beam’s footprint.

Fig. 13 Reflected wavepackets within the xy-planes located at z = −5μm (left column) and z = −10μm (right column). From top to bottom, the distance along the x-axis from the center of the incident beam at (x, y, z) = (0, 0, 0) is x₀ = 20, 30, 35μm. The packets are seen to be broadened (due to dispersion as well as diffraction) and also proportionately delayed with distance away from the incident beam’s footprint.
negative $k_x$ values) generates a backward-propagating wave along the negative $x$-axis, which gains strength as it moves away from the incident packet’s footprint. The exponential growth of this backward wave that travels along the negative $x$-axis, as well as that of the forward-propagating wave along the positive $x$-axis, are manifestations of the so-called “convective instability” associated with such weakly amplifying slabs [11–14].

Figure 17 shows several transmitted packets emerging from the rear facet of the gainy slab at $z = 5\mu m$. In general,
these packets are broadened, properly delayed with distance away from the incident beam’s footprint, and amplified in consequence of single or multiple passages through the slab. Occasionally, one can distinguish the light that directly reaches the rear facet from that which requires an extra bounce inside the slab.

The computed results reported here were obtained using 1000 points along the integration contour in the \(k_z\)-plane \((-40 \leq k_z \leq 40\); the number of points on the positive real \(\omega'\)-axis \((0 \leq \omega' \leq 40\) was 2000 for the semi-infinite gain medium and 4000 for the 5\(\mu\)m-thick slab. Numerical accuracy was achieved using 100 working digits, with 15 digits as the precision goal.

In general, more samples along the \(\omega''\)-axis and also along the integration contour \(C\) in the \(k_z\)-plane are needed for larger values of \(x_0\) and \(z_0\). Similarly, more computational resources must be summoned for larger gain coefficients and/or thicker slabs, which cause the \(k_z\)-plane integration contour \(C\) to move further away from the \(k_z''\)-axis. Given sufficient computing power, one could also contemplate more realistic representations of both passive and active media by introducing additional Lorentz oscillators into the models of the dielectric functions used for media 1, 2, and 3 in accordance with Eq. (9).

10 Concluding remarks

This paper has described a systematic approach to computing the Fresnel reflection and transmission of a finite-duration, finite-spatial-footprint wavepacket arriving within a nearly transparent incidence medium at two kinds of interfaces. In the first case, the interface is with a semi-infinite gain medium. In the second case, the interface is with a gain medium of finite-thickness, which is followed by another nearly transparent semi-infinite medium. The crucial step in both cases is to deform the integration path away from the real \(k''_z\)-axis in the complex \(k_z\)-plane in such a way as to avoid crossing the relevant branch-cuts and also to eliminate all the poles and singularities from the upper-half of the complex \(\omega\)-plane. In conjunction with a proper specification of the branch-cuts, this choice of the integration contour in the \(k_z\)-plane not only ensures the causality of the reflected and transmitted wavepackets, but also leaves no ambiguity regarding the correct signs of the square root expressions that define the \(k_z\) components of the various \(k\)-vectors.

In our numerical simulations, we used a single-oscillator Lorentz model to represent the dielectric functions \(\varepsilon(\omega)\) of the incidence, transmittance, and gain media. However, there should be no limitations in principle on the number of such oscillators that could be used to represent each medium. More realistic simulations, especially those involving incidence and transmittance media with reasonably large refractive indices over a broad range of frequencies, would require more than one Lorentz oscillator.

We also avoided the complications arising from material nonlinearities, including gain saturation [16]. It is well known that the field amplitudes inside a gain medium cannot grow indefinitely, and that the degree of population inversion—itself determined by the pump power and by the inherent properties of the material medium—limits the range over which the complex refractive index \(n_2(\omega) = n_2' + in_2''\) could be considered to be independent of the internal \(\mathcal{E}\)-field amplitudes. Our linear treatment of the gain medium (i.e., treatment in accordance with the so-called “small-signal model”) is thus reliable only up until the point in time when gain saturation begins to show its inexorable effects.

Although we chose in this paper to deform the integration contour in the \(k_z\)-plane while keeping the inverse Fourier integral over the real \(\omega'\)-axis of the \(\omega\)-plane, we could instead have invoked similar lines of reasoning (and more or less the same procedural steps) to deform the \(\omega\)-plane integration contour while keeping the inverse Fourier integral over the real \(k_z''\)-axis of the \(k_z\)-plane. The arguments based on causality, the analyticity of the inverse Fourier integrands, and the role of the branch-points and branch-cuts of \(k_{zz}, k_{zy},\) and \(k_{zy}\) would then have led to an alternative, albeit equally valid, formulation; for a more detailed discussion, see Appendix B. In this alternative approach, the deformed integration contour would be entirely in the upper half of the \(\omega''\)-plane and, if desired, it could be constructed to exhibit even symmetry with respect to the \(\omega''\)-axis (in contrast to the odd symmetry of the \(k_z''\)-plane contour around the \(k_z''\)-axis).

Fig. 15 A 40fL wavepacket, linearly polarized along the \(y\)-axis and having a 25\(\mu\)m footprint along the \(x\)-axis, arrives at the interface between a passive, nearly transparent medium 1 and a weakly amplifying medium 2. The light passing through the 5\(\mu\)m-thick gainy slab emerges into another nearly transparent, semi-infinite, passive medium 3. The central ray of the incident packet makes an angle \(\theta_x = 63^\circ\) with the \(z\)-axis. Plots of the complex refractive indices \(n_1, n_2,\) and \(n_3\) as functions of the temporal frequency \(\omega\) appear in Fig. 7.
Fig. 16 Plots of the incident (red) and reflected (blue) wavepackets at several locations in the interfacial plane (i.e., the \( xy \)-plane at \( z = 0 \)) between the nearly transparent medium 1 and the 5\( \mu \)m-thick weakly amplifying medium 2. On the right-hand side, the reflected \( E \)-field amplitude is seen to rapidly increase with the positive distance \( x_0 \) away from the center of the incident packet. As for the negative values of \( x_0 \) appearing on the left-hand side, the \( E \)-field amplitude is fairly small at first, but it also rises, albeit slowly, with the increasing distance from the incident packet’s center at \((x, y, z) = (0, 0, 0)\). The packets are broadened and proportionately delayed as the observation point recedes from the incident wave’s footprint.
Finally, let us emphasize the substantial freedom that is available to us in choosing the $k_x$-plane integration contour $C$, so long as the contour does not cross the relevant pole trajectories and/or the branch-cuts. In general, of course, the constraints imposed on the contour $C$ for the semi-infinite gain medium of Fig. 1a differ from those for the finite-thickness gainy slab of Fig. 1b. As such, the contours chosen for these two cases need not be identical, although, in our
Appendix A

Constructing a realistic incident wavepacket at the front facet of the gain medium

In Sect. 8, we described a simple model for the compact wavepacket that arrives at the interface between media 1 and 2. A more realistic description of the incident packet requires that the incidence medium 1 be modeled as a prism whose slanted facet makes an angle $\theta$ with the $x$-axis, as depicted in Fig. 18. The finite-width, finite-duration wavepacket now arrives at the slanted facet, which coincides with the $x'y'-z'$-plane of the tilted $x'y'z'$ coordinate system. The incident $E$-field amplitude in the $x'y'$-plane at $z' = 0$ is given by

$$E^{\text{inc}}(x', t) = f(x'/W)g(t/T)\cos(\omega t)\hat{y}. \quad (A1)$$

Assuming the Fourier transforms of the spatial and temporal profiles of the wavepacket are given by $\hat{f}(k_x') = W \sin^m(Wk_x'/2\pi)$ and $\hat{g}(\omega) = T \sin^n(T\omega/2\pi)$, where $m$ and $n$ are small positive integers, the Fourier transform of the incident $E$-field is readily found to be

$$E^{\text{inc}}_y(k_x', \omega) = \frac{1}{2} \hat{f}(k_x') \big[ \hat{g}(\omega - \omega_x) + \hat{g}(\omega + \omega_x) \big]. \quad (A2)$$

At the entrance facet of the prism, where $z' = 0$, we have $k_x' = (\omega/c)\sqrt{1 - (ck'_z/\omega)^2}$ in free space and

$$k_x' = (\omega/c)\sqrt{\varepsilon_1(\omega) - (ck'_z/\omega)^2}$$

in medium 1. In the case of $y$-polarized incident light, the Fresnel transmission coefficient at the prism’s slanted facet is given by

$$\tau' (k_x', \omega) = 2k_x'_{\text{inc}} / (k_x'_{\text{inc}} + k_x'_{\text{ref}}). \quad (A3)$$

Given that $k_x'$ is real and that $\omega$ is real and positive, one may simplify the computation by choosing the sign of the square roots so that $k_x'_{\text{inc}}$ and $k_x'_{\text{ref}}$ are in $Q_1$ of the complex plane; in other words, there is no need to resort to properly constructed branch-cuts (i.e., straight vertical lines in $Q_1$ and $Q_3$ of the $k_x$-plane). With reference to Fig. 18 and noting that, at the interface between media 1 and 2, $x' = x \cos \theta$ and $z' = \zeta_0 + x \sin \theta$, we now invoke Eqs. (A2) and (A3) to express the incident $E$-field arriving at the interfacial $xy$-plane located at $z = 0$ as the following inverse Fourier transform integral:

$$\begin{align*}
\tilde{E}_y^{\text{inc}}(x, t) &= (2\pi)^{-1} \int_{\omega = 0}^{\infty} \int_{k_x'_{\text{inc}}}^{\infty} \int_{k_x'_{\text{ref}}}^{-\infty} \tau' (k_x', \omega) \tilde{E}_y^{\text{inc}}(k_x', \omega) e^{ik_x'x \cos \theta + ik_x'_{\text{inc}}(\zeta_0 + x \sin \theta)} dk_x'. \\
&= (2\pi)^{-1} \int_{\omega = 0}^{\infty} \int_{k_x'_{\text{inc}}}^{\infty} \int_{k_x'_{\text{ref}}}^{-\infty} \tau' (k_x', \omega) \tilde{E}_y^{\text{inc}}(k_x', \omega) e^{ik_x'x \cos \theta + ik_x'_{\text{ref}}(\zeta_0 + x \sin \theta)} dk_x'.
\end{align*} \quad (A4)$$
In this equation, the inner integral over \( k'_x \) must be evaluated as a function of \( x \) for each real and positive value of \( \omega \). In what follows, this function will be referred to as \( F_y^{(inc)}(x, \omega) \), that is,

\[
F_y^{(inc)}(x, \omega) = (2\pi)^{-1} \int_{k'_x = -\infty}^{\infty} \tau'(k'_x, \omega) \tilde{E}_y^{(inc)}(k'_x, \omega) e^{ik'_x x + i\omega t} dx.
\] (A5)

Note that \( \zeta_0 \), the distance between the origins of the \( xyz \) and \( x'y'z' \) coordinates must be chosen such that \( \zeta_0 + x \sin \theta \) is non-negative for all values of \( x \) in the interval \([x_{\min}, x_{\max}]\); that is, \( \zeta_0 \geq |x_{\min}| \sin \theta \). Also, the integral in Eq. (A5) is evaluated over the real \( k'_x \) axis. Having computed \( F_y^{(inc)}(x, \omega) \) for all real and positive frequencies \( \omega \), we proceed to find its Fourier transform over \( x \), namely,

\[
\tilde{E}_y^{(inc)}(k_x, \omega) = \int_{-\infty}^{\infty} F_y^{(inc)}(x, \omega) e^{-ik_x x} dx.
\] (A6)

This is the function that now substitutes for \( \tilde{E}_y(k_x, \omega) \) in Eqs. (7) and (8).

**Appendix B**

**Deforming the integration path in the complex \( \omega \)-plane**

A glance at Figs. 8 and 9 reveals that the \( k_z \)-plane trajectories of poles and branch-points of a given system generally move upward when the temporal frequency \( \omega \), starting at a positive real value \( \omega' \), acquires a positive imaginary part \( \omega'' \) by moving up, parallel to the imaginary axis of the \( \omega \)-plane. Not shown in Figs. 8 and 9 are the simultaneous happenings on the left half of the \( k_z \)-plane, where, due to the inherent odd symmetry, all the pole and branch-point trajectories move downward. It is thus seen that the real \( k'_z \)-axis can be cleared of all the singularities of the inverse Fourier integrands of Eqs. (7) and (8) if the \( \omega \)-plane integration contour is sufficiently moved away from the real \( \omega' \)-axis and into the upper-half of the \( \omega \)-plane.

As a simple example, note that the Fourier transformation of the incident packet at \( z = 0 \) can be done on any straight-line \( \omega = \omega' + i\Omega_0 \) that is parallel to the \( \omega' \)-axis, provided that \( \Omega_0 \geq 0 \) (the vanishing of the incident packet for \( t < 0 \) guarantees the existence of its Fourier transform for any value of \( \omega'' \geq 0 \)). In this way, one can proceed to solve Maxwell’s equations for individual plane-waves in media 1, 2, and 3, match the boundary conditions at \( z = 0 \) and \( z = d \), and obtain the usual Fresnel reflection and transmission coefficients, \( \rho(k'_z, \omega) \) and \( \tau(k'_z, \omega) \), with the tacit assumption that \( \omega \) is an arbitrary point on the straight line parallel to and above the \( \omega' \)-axis [12].

At this point in the analysis, the existence of branch-points for \( k_{z_2} \) (i.e., \( k_z \) in medium 2) and/or poles associated with \( \rho \) and \( \tau \) in the upper-half \( \omega \)-plane imposes a lower bound on \( \Omega_0 \) that ensures the satisfaction of the all-important causality requirement [13–15] (causality decrees that the reflected and transmitted waves cannot reach the point \((x, y, z)\) prior to \( t = |z|/c \)). Causality also fixes the signs of \( k_{z_1}, k_{z_2}, \) and \( k_3 \) so that, referring to Eq. (6), there will be no ambiguity as to which one of the \( \pm \) signs should be picked at any given point \((k'_z, \omega' + i\Omega_0)\) in the Fourier domain. In this way, the reflected and transmitted EM fields at the observation point \((x_0, z_0, t)\) can be computed via a 2D inverse Fourier transformation, first over the \( k'_z \)-axis, and then along the straight-line \( \omega = \omega' + i\Omega_0 \) in the \( \omega \)-plane.

As we have argued in this paper, under certain circumstances, the Fourier transforms can be rearranged in such a way that the transform in the \( \omega \)-plane returns to the real \( \omega' \)-axis at the expense of carrying out the Fourier integral in the \( k_z \)-plane over a properly deformed contour—as opposed to over the real \( k'_z \)-axis. In accordance with the arguments advanced in Sect. 2, the deformed integration contour in the \( k_z \)-plane is chosen such that the associated singularities will disappear from the upper-half \( \omega \)-plane, thereby clearing the way for the integration path \( \omega = \omega' + i\Omega_0 \) to return to the \( \omega' \)-axis. The mathematical basis for these assertions, of course, continues to be the Cauchy–Goursat theorem of complex analysis [9–11]. It is worth emphasizing once again that the choice of the integration path, be it the straight-line \( \omega = \omega' + i\Omega_0 \) in the \( \omega \)-plane or a properly deformed contour in the \( k_z \)-plane, is dictated by the analyticity of the functions involved and by the requirement of causality. These constraints also automatically fix the signs of \( k_{z_1}, k_{z_2}, \) and \( k_3 \), without resort to any kind of “commonsense” physical argument.

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**Declarations**

**Conflict of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.

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