BEYOND THE KÄHLER CONE

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To Friedrich Hirzebruch

Abstract. The moduli space of nonlinear $\sigma$-models on a Calabi–Yau manifold contains a complexification of the Kähler cone of the manifold. We describe a physically natural analytic continuation process which links the complexified Kähler cones of birationally equivalent Calabi–Yau manifolds. The enlarged moduli space includes a complexification of Kawamata’s “movable cone”. We formulate a natural conjecture about the action of the birational automorphism group on this cone.

Many mathematicians were taken by surprise during 1984–85 when we found physicists knocking on our doors, asking whether we knew anything about Riemannian 6-manifolds with a metric whose holonomy lies in $SU(3)$. Fortunately, Yau had solved the Calabi conjecture nearly 10 years earlier, so we were able to provide some answers: any smooth complex projective threefold with trivial canonical bundle admits a metric of this type. Also fortunately, these manifolds—now called Calabi–Yau threefolds—had been studied in some detail by algebraic geometers, in part due to the distinguished rôle they play in the classification theory of algebraic varieties.

During the following year, the questions became more and more specific, focusing primarily on the physicists’ desire for examples $X$ whose Euler number

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$e(X)$ satisfies $e(X) = \pm 6$.\textsuperscript{1} A lot of work was done on this question at the Max-Planck-Institut in Bonn during 1985–86, and much of it is described by Hirzebruch in the notes of his very nice lectures [24]. (Other papers from the same period which explain the rôle of Calabi–Yau manifolds in physics can be found in [53]; cf. also [18].) One of my purposes in this paper is to update the story, and explain what has been happening recently in the study of Calabi–Yau manifolds from the point of view of physics.

The focus of the investigations by string theorists has shifted from a search for specific Calabi–Yau threefolds to a study of the general properties of the physical theories built from such manifolds. The subject was re-invigorated a few years ago by the discovery of the surprising phenomenon known as mirror symmetry. I will review this, with a focus on recent applications of mirror symmetry to the study of the moduli spaces of the physical theories. It has recently been discovered [51, 1] that there is a kind of analytic continuation which links the physical theories associated to different birational models of a single Calabi–Yau manifold. The primary purpose of this paper is to explain this analytic continuation in a very general setting, for arbitrary Calabi–Yau threefolds. This work is an outgrowth of my collaboration with Paul Aspinwall and Brian Greene [1, 2], whose contribution I would like to acknowledge at the outset.

1. Moduli spaces of $\sigma$-models

A Calabi–Yau manifold is a compact connected orientable manifold $X$ of dimension $2n$ which admits Riemannian metrics whose (global) holonomy lies in $SU(n)$. Physicists have constructed some two-dimensional quantum field theories associated to these manifolds which are known as nonlinear $\sigma$-models. Quantum field theories remain problematic for mathematicians, since they have not been shown to make sense as rigorous mathematical theories except in very limited cases (cf. [19]). However, certain aspects of the quantum field theories built from Calabi–Yau manifolds can be studied purely mathematically, without making reference to the underlying physical theory. This is the strategy we shall adopt here.

The two-dimensional quantum field theories in question are constructed out of the space of maps from (variable) surfaces $\Sigma$ to a (fixed) Calabi–Yau manifold.

\textsuperscript{1}In retrospect, perhaps someone should have wondered why the physicists didn’t know whether they wanted $e(X) = 6$ or $e(X) = -6$. This ambiguity was one of the early hints of the phenomenon of “mirror symmetry”, which I will discuss later.
Each such theory is based on a “Lagrangian” functional on the space of such maps, and the standard Lagrangian used (cf. [10]) depends on the choice of a pair \((g_{ij}, B)\), where \(g_{ij}\) is a Riemannian metric on \(X\) with holonomy contained in \(\text{SU}(n)\), and \(B\) is the de Rham cohomology class of a real closed 2-form on \(X\). We consider two such pairs to be equivalent, written \((g_{ij}, B) \sim (g'_{ij}, B')\), if there is a diffeomorphism \(\varphi : X \to X\) such that \(\varphi^* (g_{ij}) = g'_{ij}\) and \(\varphi^* (B) - B' \in H^2_{\text{DR}}(X, \mathbb{Z})\), where \(H^2_{\text{DR}}(X, \mathbb{Z})\) denotes the image of the integral cohomology in the de Rham cohomology. This definition of equivalence arises as follows. First, the appearance of \(B\) in the Lagrangian (when applied to the map \(f : \Sigma \to X\)) takes the form \(\int_\Sigma f^* (B)\), so only the de Rham class of \(B\) matters.\(^3\) Second, the appearance of this Lagrangian in physically measurable quantities always involves an exponentiation in which this term becomes \(\exp(2\pi i \int_\Sigma f^* (B))\). Thus, shifting \(B\) by an integral class will not affect the physical theory.\(^4\)

As in [36], we regard the set of equivalence classes of pairs

\[ M_\sigma := \{(g_{ij}, B)\}/\sim \]

as a first approximation to a moduli space for these theories, which we call the one-loop semiclassical nonlinear \(\sigma\)-model moduli space, or just the nonlinear \(\sigma\)-model moduli space for short. This space may differ from the actual moduli space in several ways. First, the physical theory may fail to converge for some values of \((g_{ij}, B)\). This statement has no mathematical content in the absence of an adequate mathematical definition of quantum field theories; however, certain parts of the physical theory which can be formulated in purely mathematical terms (such as the three-point functions described below) should be expected to converge, and their failure to converge at a certain place is evidence that the physical theory is badly behaved there.

\(^2\)More precisely, the metrics which are needed in the physical theory are perturbations of these metrics with restricted holonomy. However the perturbed metrics, like the original metrics with restricted holonomy, are expected to be uniquely determined by the cohomology class of the associated Kähler form, once a complex structure has been specified on \(X\).

\(^3\)Note that this term—and indeed the entire Lagrangian—is also invariant under a simultaneous change of the sign of \(B\) and the orientation of \(\Sigma\). (I am grateful to Paul Aspinwall and Jacques Distler for discussions on this point.) We can safely ignore this here, though, since we have not specified an orientation of \(\Sigma\).

\(^4\)A bit more generally, one should also include a contribution to \(\exp(2\pi i \int_\Sigma f^* (B))\) coming from torsion in \(H_2(X, \mathbb{Z})\), as in [45, 5]; we will suppress that contribution in this paper.
Second, the family of quantum field theories may admit an analytic continuation (regarding them purely as two-dimensional quantum field theories) which no longer has a nonlinear $\sigma$-model interpretation. In fact, our two-dimensional quantum field theories are of a type called “superconformal”, and we are interested in deformations which preserve only this “superconformal” property and not the more restrictive “$\sigma$-model” property. Varying $g_{ij}$ and the class of $B$ gives a locally complete family of such deformations, but globally there may be deformations which do not preserve the $\sigma$-model structure.

Third, two superconformal field theories may be isomorphic by an isomorphism which does not preserve the $\sigma$-model structure. So we may have to enlarge the set of identifications among pairs which we are making.

In sum, we may need to shrink our moduli space a bit to ensure convergence, we may need to enlarge it (in other directions) to get a complete family, and we may need to mod out by further discrete identifications. In spite of these limitations, we can still obtain significant information about the structure of the moduli space by studying the more primitive “one-loop semiclassical nonlinear $\sigma$-model” version we have formulated above.

We immediately need a slight refinement of the nonlinear $\sigma$-model moduli space, which we call the $N=2$ moduli space. This is defined to be

$$\mathcal{M}_{N=2} := \{(g_{ij}, B, t)\}/\sim$$

where $t$ denotes a complex structure on $X$ with respect to which $g_{ij}$ is Kähler. (That such complex structures exist is a consequence of the holonomy being contained in $U(n)$.) By the Bogomolov–Tian–Todorov theorem [7, 43, 44], deformations of complex structure on $X$ are unobstructed and the Kodaira–Spencer map of each versal deformation is an isomorphism. By way of notation, we let $X_t$ denote the complex manifold “$X$ equipped with the structure $t$”, which we sometimes treat as if it were a fiber of a universal family $\mathcal{X} \to \mathcal{M}_{\text{complex}}$ over the moduli space of complex structures modulo diffeomorphism. (Such families do not in general exist, unless the moduli problem represented by $\mathcal{M}_{\text{complex}}$ has been formulated to include level structures and polarizations, so this use of $X_t$ is strictly speaking an abuse of notation.) There is a natural diagram

$$\begin{array}{ccc}
\mathcal{M}_\sigma & \leftarrow & \mathcal{M}_{N=2} \\
\downarrow & & \\
\mathcal{M}_{\text{complex}} & & 
\end{array}$$
of “forgetful” maps which relates these moduli spaces.

The fibers of the map $\mathcal{M}_{N=2} \to \mathcal{M}_\sigma$ are determined by the precise nature of the holonomy group. We henceforth restrict our attention to the case $h^{2,0}(X_t) = 0$, in which the map $\mathcal{M}_{N=2} \to \mathcal{M}_\sigma$ is known to be finite (cf. [4]). We still need to understand the fibers of the map $\mathcal{M}_{N=2} \to \mathcal{M}_{\text{complex}}$. To this end, let us fix a complex structure $t$. Then each $g_{ij}$ corresponding to a point in the fiber over $t$ determines a Kähler form $\omega := (\sqrt{-1}/2) \sum g_{ij} \bar{\beta} d\zeta^\alpha \wedge d\bar{\zeta}^\beta$. This is a closed, nondegenerate, real 2-form which can be regarded as specifying a symplectic structure on $X$. The de Rham classes of all possible $\omega$’s (called Kähler classes) form an open convex cone $\mathcal{K}_t \subset H^2_{\text{DR}}(X, \mathbb{R})$, the Kähler cone of $X_t$. There is also a closely related cone $(\mathcal{K}_t)_+$, the nef cone, defined by

$$(\mathcal{K}_t)_+ := \text{Hull} (\mathcal{K}_t \cap H^2_{\text{DR}}(X, \mathbb{Q})) .$$

(This cone includes the rationally defined subsets of the boundary of $\mathcal{K}_t$, while omitting any irrational parts of the boundary.)

By the theorems of Calabi [9] and of Yau [52], each class $J$ in $\mathcal{K}_t$ uniquely determines a metric $g_{ij}$ on $X_t$ with holonomy in $\text{SU}(n)$ whose associated Kähler form $\omega$ lies in the class $J$. The fibers of the map $\mathcal{M}_{N=2} \to \mathcal{M}_{\text{complex}}$ can thus be written in the form $\Gamma_t \mathcal{D}_t$, where $\mathcal{D}_t = H^2_{\text{DR}}(X_t, \mathbb{R}) + i \mathcal{K}_t$, and $\Gamma_t = H^2_{\text{DR}}(X, \mathbb{Z}) \rtimes \text{Aut}(X_t)$. For if we are given $(g_{ij}, B)$ associated to a particular complex structure $t$, the corresponding complexified Kähler class $B + i J$ naturally takes values in $H^2_{\text{DR}}(X_t, \mathbb{Z}) \mathcal{D}_t$. But we must also mod out by $\text{Aut}(X_t)$, to take care of diffeomorphisms which preserve the complex structure $t$.

The Kähler cone has now appeared. Soon, we will need to go “beyond” it.

2. THREE-POINT FUNCTIONS

The physical theory also determines two trilinear maps called three-point functions. The first of these is known as the B-model three-point function, since it can be calculated using Witten’s “B-model” [50]—a close relative of the original $\sigma$-model. This “three-point function” can be regarded as a trilinear map among certain bundles on the complex moduli space $\mathcal{M}_{\text{complex}}$, defined by using a universal family $\pi : \mathcal{X} \to \mathcal{M}_{\text{complex}}$. The arguments of the three-point function are (local) sections of certain bundles: we take $\alpha \in \Gamma(\mathcal{F}^{n-p+1}/\mathcal{F}^{n-p+2})$, $\beta \in \Gamma(R^1 \pi_* T_{\mathcal{X}/\mathcal{M}})$, $\gamma \in \Gamma(\mathcal{F}^p/\mathcal{F}^{p+1})$, where the $\mathcal{F}^p$ are the Hodge bundles for
the family \( \pi \), and \( T_{X/\mathcal{M}} \) is the relative holomorphic tangent bundle of the family. The B-model three-point function is then defined to be:

\[
\langle \alpha, \beta, \gamma \rangle_{\text{B-model}} := \int_{X_t} \alpha \wedge \nabla_{\beta} \gamma \in \Gamma(O_{\mathcal{M}}),
\]

where \( \nabla \) is the Gauss–Manin connection and \( \nabla_{\beta} \) is the directional derivative determined from \( \beta \) via the Kodaira–Spencer isomorphism.

The second three-point function is called the \textit{A-model three-point function}. (We will only define this in the case \( h^{2,0}(X_t) = 0 \).) The definition involves a bit of a technical digression. It will be formulated in terms of certain invariants, called the \textit{Gromov–Witten invariants}, which measure the rational curves on \( X_t \). Verifying mathematically that these invariants exist and have the properties expected by the physicists is an area of intense study, with much recent progress \([25, 26, 39, 40, 32, 30, 29]\). We will give a heuristic description of these invariants, following Witten’s original discussion \([48, 49]\), and use them as if they had all of the expected properties (including in particular the “multiple cover formula” of \([3]\), which we build into our definitions).

For each class \( \eta \in H_2(X, \mathbb{Z}) \), consider the moduli space of maps

\[
\mathcal{M}^t_{\eta} := \{ \text{generically injective holomorphic maps } f : \mathbb{P}^1 \to X_t \text{ with } [f(\mathbb{P}^1)] = \eta \}.
\]

A naïve dimension estimate suggests that \( \dim(\mathcal{M}^t_{\eta}) = \dim(X) \), and in fact by a theorem of McDuff \([34]\), this is true provided that one deforms the complex structure \( t \) to a nearby (non-integrable) almost-complex structure on \( X \). In order to formulate our heuristic description, we pass to such a nearby deformation.

For each point \( P \in \mathbb{P}^1 \), there is an evaluation map \( e_P : \mathcal{M}^t_{\eta} \to X_t \) given by \( e_P(f) = f(P) \). Then the Gromov–Witten invariants should be (heuristically) defined as

\[
G^t_{\eta}(A, B, C) := e^*_0(A) \cup e^*_1(B) \cup e^*_\infty(C)|_{[\mathcal{M}^t_{\eta}]},
\]

for \( A \in H^{p-1, p-1}(X_t) \), \( B \in H^{1,1}(X_t) \), and \( C \in H^{n-p, n-p}(X_t) \). (Note: if \( p = 1 \) or \( p = n \), then the intersection in the moduli space cannot be made transverse and \( G^t_{\eta}(A, B, C) = 0 \).) The difficulty with this attempted definition is that \( \mathcal{M}^t_{\eta} \) is not compact, so quite a bit of care must be used in trying to evaluate \((2.1)\). Much of the recent work on these invariants has been based on Gromov’s compactification
BEYOND THE KÄHLER CONE

[22] of $\mathcal{M}_\eta'$, but other compactifications have also been used. It is not yet clear that all proposed definitions agree.

We will assume that Gromov–Witten invariants can be defined somehow, and use them to define the A-model three-point functions in the following way. Assume for simplicity that $H^2(X,\mathbb{Z})$ and $H_2(X,\mathbb{Z})$ have no torsion. Let $C \subset K_t$ be an open cone of the form $\mathbb{R}_{>0}e^1 + \cdots + \mathbb{R}_{>0}e^r$ for some basis $e^1,\ldots,e^r$ of $L := H^2(X,\mathbb{Z})$, and let $e_1,\ldots,e_r$ be the dual basis of $H^2(X,\mathbb{Z})$. We express elements of $L \cap (L^\mathbb{R} + iC)$ in the form $\sum a_j e_j$, and the coefficients $a_j$ must satisfy $\text{Im}(a_j) > 0$. The class of this element in $L \setminus (L^\mathbb{R} + iC)$ can then be described by the quantities $q_j := \exp(2\pi i a_j)$, which provide coordinates on $L \setminus (L^\mathbb{R} + iC)$. Note that those coordinates are subject to the constraint $0 < |q_j| < 1$. In fact, the space $L \setminus (L^\mathbb{R} + iC)$ is isomorphic to $(\Delta^*)^r$, where $\Delta^*$ is the punctured disk. It admits a natural partial compactification $(\Delta^*)^r \subset \Delta^r$ with a distinguished boundary point 0.

The three-point function associated to this region $L \setminus (L^\mathbb{R} + iC) \subset \Gamma_t \setminus D_t$ is given by:

\begin{equation}
\langle A, B, C \rangle_{\text{A-model}} := A \cdot B \cdot C + \sum_{0 \neq \eta \in H_2(X,\mathbb{Z})} \frac{q^n}{1 - q^n} G'_{\eta}(A, B, C),
\end{equation}

where $q^n$ denotes $\prod (q_j)^{\eta_j}$ when $\eta = \sum \eta_j e_j$. All nonzero terms in this series have $\eta_j > 0$. However, no convergence properties of the series are known, so at present, the three-point function must be considered to take values in the formal power series ring $\mathbb{C}[[q_1,\ldots,q_n]]$, the completion of the coordinate ring of $\Delta^r$ at the distinguished boundary point 0. If we could learn something about the radius of convergence of this function, we would gain some information about the domain within $\mathcal{M}_\sigma$ in which the quantum field theory converges.

Notice that in the case of dimension 3 (i.e., $n = 3$), the three-point function involves the Gromov–Witten invariants only when all three of $A$, $B$, and $C$ come from $H^{1,1}(X_t)$ (i.e., when $p = 2$). In this case, each Gromov–Witten invariant simply counts the number of rational curves in the corresponding homology class (with appropriate signs), multiplied by the degrees of the class with respect to the given divisors. In other words,
As written, this formula only makes sense for generic almost-complex structures (for which \( M'_\eta / \text{PSL}(2, \mathbb{C}) \) is a finite set); even for them, care must be taken when calculating the “number” of points of the set, as some of them should be counted with a minus sign. (See [13, §8] for a discussion of this issue.)

In spite of an apparent dependence on the complex structure \( t \), in fact the Gromov–Witten invariants are independent of \( t \).\(^5\) It follows that the A-model three-point function depends on the “complexified Kähler” parameters \( q_j \) (which are local coordinates on \( \Gamma_t \backslash \mathcal{D}_t \)) but not on the complex structure parameter \( t \).

The cure for the apparent dependence on the choice of cone \( C \)—called a framing in [36]—is more difficult. The most ideal circumstances are represented by varieties which satisfy the

**Cone Conjecture** [36]. There is a rational polyhedral cone

\[ \Pi \subset H^2(X, \mathbb{R}) \]

the union of whose translates \( \gamma(\Pi) \) by automorphisms \( \gamma \in \text{Aut}(X_t) \) covers the nef cone \( (K_t)_+ \).

(This conjecture was originally made to ensure that the space \( \Gamma_t \backslash \mathcal{D}_t \) could be partially compactified using a construction of Looijenga [33] related to the Satake–Baily–Borel compactification.) The polyhedron \( \Pi \) in the cone conjecture can in fact be chosen to be the closure of a fundamental domain for the \( \text{Aut}(X_t) \)-action, and such a \( \Pi \) can be subdivided into cones \( C_j \) which are generated by bases of \( L \); these cones \( C_j \) can then be used in the definition of three-point functions as above.

The cone conjecture clearly holds whenever the nef cone is itself rational polyhedral (a not uncommon occurrence), and it has been verified in at least one nontrivial example [20].

As an alternative to the cone conjecture, it is possible to interpret the symbols \( q^0 \) as belonging to the group ring \( \mathbb{C}[H_2(X, \mathbb{Z})] \), and to interpret the three-point functions (2.2) as taking values in a certain formal completion of that ring.\(^6\) This provides another way to give an intrinsic meaning to Eq. (2.2), independent of choices of bases and cones.

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\(^5\)This is simply because the definition as we have formulated it uses a generic nearby almost-complex structure. It would be desirable to have definitions of these invariants purely within algebraic geometry—for such definitions, the independence from \( t \) will be more difficult to verify.

\(^6\)I am grateful to A. Givental for this remark.
3. Predictions from mirror symmetry

Mirror symmetry [16, 31, 14, 21] predicts that Calabi–Yau manifolds often come in mirror pairs $(X, Y)$, related by the existence an isomorphism between nonlinear $\sigma$-models on $X$ and $Y$ which permutes the data in a certain specified way.\(^7\) In particular, the roles of the moduli spaces $\mathcal{M}_{\text{complex}}$ and $\Gamma_t \backslash \mathcal{D}_t$ should be exchanged when passing from $X$ to $Y$, and the three-point functions of A-model and B-model type should be reversed.

This prediction has one very puzzling aspect: whenever the cone conjecture holds, $\Gamma_t \backslash \mathcal{D}_t$ is a bounded domain, covered by a finite number of “punctured polydisks” $L \backslash (L_R + i\mathcal{C}) \cong (\Delta^*)^7$. On the other hand, for each choice of polarization, the open subset $\mathcal{M}^{\text{pol}}_{\text{complex}} \subset \mathcal{M}_{\text{complex}}$ of polarized complex structures has the structure of a quasi-projective variety (by a theorem of Viehweg [46]). These properties would appear at first sight to be incompatible with a mirror symmetry isomorphism.

There are two potential resolutions to this puzzle. It may be that, after shrinking $\mathcal{D}_t$ to the natural domain of definition of the physical theory $\mathcal{D}$, there is a much larger symmetry group $\Gamma$ which acts on $\mathcal{D}$ (representing identifications between conformal field theories which are not visible as identifications between nonlinear $\sigma$-models), in such a way that $\Gamma \backslash \mathcal{D}$ is quasi-projective. Alternatively, it may be that the physical theory can be analytically continued beyond $\mathcal{D}_t$, and that when one attaches $\Gamma_t \backslash \mathcal{D}_t$ to other parameter spaces for other types of conformal field theory, one obtains a quasi-projective variety.\(^8\)

Analytically continuing outside $\mathcal{D}_t$ would take us “beyond the Kähler cone”. This is what in fact happens in the physical theory, as has been demonstrated recently in [51, 1]. We explain this in the next section.

4. Flops and $\sigma$-models

4.1. The boundary of the Kähler cone.

For the remainder of this paper, we will specialize to the case of algebraic threefolds, where the techniques of Mori theory (cf. [15]) are available for study-

\(^7\)The predicted relationship between the Euler numbers of these manifolds is $e(Y) = (-1)^{\dim X} e(X)$, which leads to the Euler number sign ambiguity (such as the $e(X) = \pm 6$ issue mentioned above) present in the early searches for specific Calabi–Yau threefolds.

\(^8\)One can also imagine combinations of these two scenarios, in which an enlargement of $\mathcal{D}_t$ admits an action by a larger group.
ing the Kähler cone. A detailed analysis of the cone for Calabi–Yau threefolds has been carried out by Kawamata [27] and Wilson [47]. The boundary of the closure of $\mathcal{K}_t$ has certain rational walls of codimension 1: each is the intersection of the nef cone $(\mathcal{K}_t)_+$ with a hyperplane $\Gamma^\perp$, where $\Gamma \subset \mathcal{X}_t$ is a curve which is log-extremal in the sense of Mori theory. Any linear system whose numerical class lies at an interior point of a rational wall $\Gamma^\perp$ determines a “contraction mapping” which contracts to points all effective irreducible curves $\Gamma'$ which are numerically equivalent to $\lambda \Gamma$ for some real number $\lambda$. The contraction mappings associated to rational walls come in several varieties:

(1) flopping contractions contract a finite number of curves to points,

(2) divisorial contractions contract divisors to subvarieties of lower dimension, and

(3) Mori fibrations have images of lower dimension.

(We use similar names for the rational walls, calling them flopping walls, divisorial walls, and Mori-fibration walls.) We will primarily focus on flopping walls, but will briefly return to the other two types at the end of the paper.

4.2. Flops.

Our initial concern is to understand the behavior of the Kähler metric $g_{ij}$ as we allow the complexified Kähler class $B + iJ$ to approach a flopping wall. The simplest kind of flop is centered on a curve $\Gamma \subset \mathcal{X}_t$ with normal bundle $O(-1) \oplus O(-1)$. The behavior of the metric itself is not known;\(^9\) we will settle for a (local) analysis of the behavior of the associated symplectic structure.

To analyze the symplectic structure, we first give a description of this flop in terms of variable symplectic reductions of a fixed $\mathbb{C}^*$-action, similar in spirit to [23] and [42]. We begin with $\mathbb{C}^4$ with coordinates $(w, x, y, z)$, and consider the action of $\mathbb{C}^*$ on $\mathbb{C}^4$ given by

$$(w, x, y, z) \mapsto (sw, sx, s^{-1}y, s^{-1}z)$$

for $s \in \mathbb{C}^*$. Fix a symplectic form

$$\omega := \frac{\sqrt{-1}}{2} (dw \wedge d\bar{w} + dx \wedge d\bar{x} + dy \wedge d\bar{y} + dz \wedge d\bar{z})$$

\(^9\)However, Candelas and de la Ossa [11] have given a very interesting local analysis of metrics in this situation.
on $\mathbb{C}^4$. There is then a moment map $\mu : \mathbb{C}^4 \to \mathbb{R}$ for the action of $\mathbb{C}^*$, given by

$$\mu(w, x, y, z) := \frac{1}{2} (|w|^2 + |x|^2 - |y|^2 - |z|^2).$$

The fibers of the moment map are invariant under the maximal compact subgroup $S^1 \subset \mathbb{C}^*$. One can then form the symplectic reductions $\mu^{-1}(r)/S^1$, for various values $r$ in the image of the moment map. The symplectic form $\omega$ induces a symplectic form $\omega_r$ on the set of smooth points of the reduced space $\mu^{-1}(r)/S^1$.

This $\mathbb{C}^*$-action can also be studied directly in algebraic geometry. By a variant of the Kirwan–Ness theorem [28, 38], for each value of $r$ there is an algebraic set $\Sigma_r \subset \mathbb{C}^4$ such that the symplectic reduction $\mu^{-1}(r)/S^1$ is isomorphic to the geometric invariant theory quotient $(\mathbb{C}^4 - \Sigma_r) // \mathbb{C}^*$. The set $\Sigma_r$ can be characterized as the union of all $\mathbb{C}^*$-orbits whose closures are disjoint from $\mu^{-1}(r)$. (The usual version of the Kirwan–Ness theorem applies to group actions on projective varieties; for the quasi-projective version used here, see [28, p. 115].)

When $r = 0$, $\Sigma_r$ is empty and the geometric invariant theory quotient $\mathbb{C}^4 // \mathbb{C}^*$ is simply the spectrum of the ring of invariants. Computing that ring is not hard: it is generated by the polynomials

$$A := wy, \quad B := wz, \quad C := xy, \quad D := xz$$

subject to the relation

$$AD - BC = 0;$$

it follows that

$$(4.1) \quad \mathbb{C}^4 // \mathbb{C}^* = \text{Spec } \mathbb{C}[A, B, C, D]/(AD - BC).$$

If $r < 0$, then $\Sigma_r$ is the set $\{y = z = 0\}$. It is not difficult to check that the quotient coincides with the blowup of $\text{Spec } \mathbb{C}[A, B, C, D]/(AD - BC)$ along the ideal $A = B = 0$. If $r > 0$, then $\Sigma_r$ is the set $\{w = x = 0\}$. Similarly, the quotient this time is the blowup of (4.1) along the ideal $A = C = 0$.

Moving from $r > 0$ to $r < 0$ is thus geometrically described as the familiar process of blowing down the original curve $\Gamma \subset X_t$ to a point, and then blowing back up in a different way to obtain a new curve $\hat{\Gamma}$ in a birationally equivalent space $\hat{X}_t$. (At $r = 0$, the natural geometric model is the singular one $\overline{X}_t$, in
which $\Gamma$ has been contracted to a point $P \in \overline{\mathcal{X}}_t$.) Since the proper transform map gives a natural identification between $H^2(\mathcal{X}_t)$ and $H^2(\tilde{\mathcal{X}}_t)$, we can regard the Kähler cones $\mathcal{K}_t$ and $\tilde{\mathcal{K}}_t$ as lying in the same space. When this is done, the homology classes of $\Gamma$ and $\tilde{\Gamma}$ are related by $[\tilde{\Gamma}] = -[\Gamma]$. Moreover, by the Duistermaat–Heckman theorem [17], the cohomology class of $\omega_r$ moves along a piecewise linear path in the cohomology group (cf. [23]), with $[\omega_r] \in \mathcal{K}_t$ for $r > 0$ and $[\omega_r] \in \tilde{\mathcal{K}}_t$ for $r < 0$. We thus see that the nef cones $(\mathcal{K}_t)_+$ and $(\tilde{\mathcal{K}}_t)_+$ meet along a common rational wall—itself naturally associated to the singular space $\overline{\mathcal{X}}_t$—which is a flopping wall for both.

4.3. The effect on three-point functions.

Having found that the Kähler cones of $\mathcal{X}_t$ and $\tilde{\mathcal{X}}_t$ can be fit together in a natural way, we now examine the A-model three-point functions in these two cases. We assume that $\mathcal{X}_t$ is sufficiently generic so that all effective curves in the numerical equivalence class $\Gamma$ have normal bundle $O(-1) \oplus O(-1)$, and that each is disjoint from all other rational curves on $\mathcal{X}_t$. It then follows that the birational correspondence between $\mathcal{X}_t$ and $\tilde{\mathcal{X}}_t$ does not disturb any curves other than those numerically equivalent to $\Gamma$ and $\tilde{\Gamma}$. Let $n_\Gamma = n_{\tilde{\Gamma}}$ be the number of such curves. We can write the A-model three-point functions of $\mathcal{X}_t$ in the form

$$\langle A, B, C \rangle^{\mathcal{X}_t}_{\text{A-model}} = A \cdot B \cdot C + \frac{q^{[\Gamma]}}{1 - q^{[\Gamma]}} (A \cdot \Gamma)(B \cdot \Gamma)(C \cdot \Gamma) n_\Gamma$$

$$+ \sum_{\eta \in H_2(X, \mathbb{Z}) \atop \eta \neq \lambda \Gamma} \frac{q^n}{1 - q^n} G_\eta'(A, B, C),$$

and there is a similar expression for the three-point function of $\tilde{\mathcal{X}}_t$, in which only the first two terms are different. Note that $D_t \subset \{ q : |q^{[\Gamma]}| < 1 \}$ so that the first two terms in the sum (4.2) are well-defined throughout $D_t$. These first two terms can clearly be analytically continued to the region of $q$’s satisfying $|q^{[\Gamma]}| > 1$, where they can be compared with the first terms of the corresponding three-point functions for $\tilde{\mathcal{X}}_t$.

\(^{10}\)We are blurring the distinction between the form $\omega_r$ (constructed using the local analysis near $\Gamma$) and global properties of its cohomology class $[\omega_r]$. This does not affect our statements, however, thanks to the “equivalence between algebraic and $\sigma$-model coordinates” discussed in [2].
Lemma [51, 2]. For all $q \in (\mathcal{D}_t)_+ \cup (\mathcal{D}_t')_+ \text{ with } q^{[\Gamma]} \neq 1$ we have

$$A \cdot B \cdot C + \frac{q^{[\Gamma]}}{1 - q^{[\Gamma]}} (A \cdot \Gamma)(B \cdot \Gamma)(C \cdot \Gamma) n_{\Gamma}$$

$$= \hat{A} \cdot \hat{B} \cdot \hat{C} + \frac{q^{[\hat{\Gamma}]}}{1 - q^{[\hat{\Gamma}]}} (\hat{A} \cdot \hat{\Gamma})(\hat{B} \cdot \hat{\Gamma})(\hat{C} \cdot \hat{\Gamma}) n_{\hat{\Gamma}},$$

where $\hat{A}$, $\hat{B}$, and $\hat{C}$ are the proper transforms of $A$, $B$, and $C$.

In other words, the change in the topological term is precisely compensated for by the change in the $q^{[\Gamma]}$ term.

Proof. First consider the case—illustrated in Fig. 1—in which $A$ and $B$ meet $\Gamma$ transversally (at, say, $a$ and $b$ points, respectively), and $\hat{C}$ meets $\hat{\Gamma}$ transversally (at, say, $c$ points), so that $C$ contains $\Gamma$ with multiplicity $c$. (In Fig. 1 we illustrate the configuration of divisors in the case $a = b = c = 1$.) $A$ and $B$ have no intersection points along $\Gamma$, but both $\hat{A}$ and $\hat{B}$ contain $\hat{\Gamma}$, and they meet $\hat{C}$. The total number of intersection points of $\hat{A}$, $\hat{B}$ and $\hat{C}$ (counted with multiplicity) which lie in $\hat{\Gamma}$ is thus $abc$. 
Since a similar thing happens for each curve in the numerical equivalence class, we see that

\[ \hat{A} \cdot \hat{B} \cdot \hat{C} - A \cdot B \cdot C = abc n_\Gamma = -(A \cdot \Gamma)(B \cdot \Gamma)(C \cdot \Gamma) n_\Gamma \]

(using \( A \cdot \Gamma = a, B \cdot \Gamma = b, C \cdot \Gamma = -c \)). On the other hand, since \( [\hat{\Gamma}] = -[\Gamma] \) and \( n_\Gamma = n_\Gamma \), we can compute:

\[
\frac{q^{[\Gamma]}}{1 - q^{[\Gamma]}} (A \cdot \Gamma)(B \cdot \Gamma)(C \cdot \Gamma) n_\Gamma - \frac{q^{[\Gamma]}}{1 - q^{[\Gamma]}} (\hat{A} \cdot \hat{\Gamma})(\hat{B} \cdot \hat{\Gamma})(\hat{C} \cdot \hat{\Gamma}) n_\hat{\Gamma} \\
= \frac{q^{[\Gamma]}}{1 - q^{[\Gamma]}} (A \cdot \Gamma)(B \cdot \Gamma)(C \cdot \Gamma) n_\Gamma + \frac{q^{-[\Gamma]}}{1 - q^{-[\Gamma]}} (A \cdot \Gamma)(B \cdot \Gamma)(C \cdot \Gamma) n_\Gamma \\
= (A \cdot \Gamma)(B \cdot \Gamma)(C \cdot \Gamma) n_\Gamma
\]

Equating (4.3) and (4.4) proves the formula in this case.

To prove the formula in general, note it is linear in \( A, B, \) and \( C \), and so it suffices to prove the formula when \( A, B, \) and \( C \) are very ample divisors. In particular we may assume from the outset that \( A \) and \( B \) meet \( \Gamma \) transversally, and that \( C \cdot \Gamma > 0 \). Then \((-\hat{\Gamma}) \cdot \hat{\Gamma} > 0\) as well, and we can find ample divisors whose difference is \(-\hat{C}\). Passing to a multiple, we can in fact assume that there are very ample divisors \( \hat{H} \) and \( \hat{H}' \) which meet \( \hat{\Gamma} \) transversally, with \(-\hat{C} = \frac{1}{N}(\hat{H} - \hat{H}')\). Applying the formula for \((A, B, H)\) and for \((A, B, H')\) (which satisfy the hypotheses of the special case), we deduce it for \((A, B, C)\).

The conclusion to draw from this lemma is that if the other terms in the three-point function converge in some domain \( D \) contained within \((D_t)_+ \cup (\hat{D}_t)_+\), then the entire three-point function converges in \( D \), and it is the same function for \( X_t \) as for \( \hat{X}_t \).

5. Moduli spaces of birational \( \sigma \)-models

We would like to construct a moduli space which incorporates this phenomenon of analytic continuation between \( \sigma \)-models on birationally equivalent Calabi–Yau threefolds. There are several technical difficulties in doing so, and the discussion we give here can only be regarded as a very preliminary attempt. Ideally,
the moduli problem would be formulated in terms of birational metrics (or birational symplectic forms and birational complex structures), modulo birational diffeomorphism. Since we don’t quite understand how to do that, we will take a somewhat less natural approach.

The first step is to fix a complex structure $t$ and construct an enlargement of the space of complexified Kähler classes. Our construction is based on a result of Kawamata [27], who showed that the set of Kähler cones of all birational models of $X_t$, when transported to $H^2(X_t, \mathbb{R})$ via proper transforms by all possible birational maps, gives a chamber structure to the convex hull of their union. (Taking the convex hull simply adds the walls between adjacent chambers to the union of the chambers.) The resulting cone $\text{Mov}(X_t)$, called the movable cone by Kawamata, can also be characterized as the interior of the closely related cone

$$\text{Mov}(X_t)_+ := \text{Hull} \left( \overline{\text{Mov}(X_t)} \cap H^2_{\text{DR}}(X, \mathbb{Q}) \right),$$

which is the convex hull of the set of “movable” divisor classes—those whose associated linear system has base locus of codimension at least two. The cone $\text{Mov}(X_t)_+$ can also be described as being the union of the proper transforms of the nef cones of all birational models of $X_t$.

We use the movable cone to define a birational Kähler moduli space of $X_t$ in the form $\Gamma_t^{\text{birat}} \backslash D_t^{\text{birat}}$, where

$$D_t^{\text{birat}} = H^2_{\text{DR}}(X_t, \mathbb{R}) + i \text{Mov}(X_t),$$

consists of all complex second cohomology classes whose imaginary part lies in the movable cone, and where

$$\Gamma_t^{\text{birat}} = H^2_{\text{DR}}(X, \mathbb{Z}) \times \text{Bir}(X_t),$$

includes the entire group $\text{Bir}(X_t)$ of birational automorphisms of $X_t$. (Note that the action of this group will permute the various chambers in Kawamata’s chamber structure, that is, the various Kähler cones of birational models of $X_t$.) As in the case of the $\sigma$-model moduli space itself, we should regard this space as only a first approximation to the true moduli space of the physical theories.

In order for this space to be well-behaved, we need some control over the action of the birational automorphism group. As a natural generalization of both the cone conjecture and a conjecture of Batyrev [6], we make the following
Birational Cone Conjecture. There is a rational polyhedral cone
\[ \mathcal{P} \subset H^2(X, \mathbb{R}) \]
the union of whose translates \( \gamma(\mathcal{P}) \) by birational automorphisms \( \gamma \in \text{Bir}(\mathcal{X}_t) \) covers the cone \( \text{Mov}(\mathcal{X}_t)_+ \).

This is known to hold in at least one non-trivial example—Calabi–Yau threefolds which are fiber products of generic rational elliptic surfaces with section (as studied by Schoen [41]). The finiteness of the action of the birational automorphism group on the set of birational models was checked by Namikawa [37], and the cone conjecture was checked by Grassi and the author [20]. Combining the two shows that the birational cone conjecture holds in this case.

The lemma from section 4 shows that the A-model three-point functions will generally have poles somewhere within \( \Gamma_{t}^{\text{birat}} \setminus \mathcal{D}_{t}^{\text{birat}} \). Points at which poles occur should be removed from \( \Gamma_{t}^{\text{birat}} \setminus \mathcal{D}_{t}^{\text{birat}} \) if we hope to describe an actual moduli space for the physical theories, which would only be expected to contain points represented by smooth Calabi–Yau manifolds. On the other hand, to understand the natural limit points of this moduli space we should leave \( \Gamma_{t}^{\text{birat}} \setminus \mathcal{D}_{t}^{\text{birat}} \) intact and try to construct a compactification (or at least a partial compactification) of it. This can be done using Looijenga’s semi-toric construction [33] whenever the birational cone conjecture holds for \( \mathcal{X}_t \).

The entire moduli space of birational \( \sigma \)-models should somehow be constructed as the union of the birational Kähler moduli spaces:
\[ \mathcal{M}_{N=2}^{\text{birat}} := \bigcup_{t \in \mathcal{M}_{\text{complex}}} \Gamma_{t}^{\text{birat}} \setminus \mathcal{D}_{t}^{\text{birat}}. \]

Unfortunately, at present, we do not know how either \( \text{Mov}(\mathcal{X}_t) \) or \( \text{Bir}(\mathcal{X}_t) \) vary with parameters, so it is difficult to topologize this space \( \mathcal{M}_{N=2}^{\text{birat}} \) or discuss its properties in any detail.

6. And beyond

We have seen that it is natural to go beyond the Kähler cone of \( \mathcal{X}_t \) to the movable cone, which describes physical theories based on all birational models of \( \mathcal{X}_t \). Can we go even further?
In the case of K3 surfaces, the elementary transformations play an important rôle in understanding the structure of the period map [8, 35]. These have a natural analogue for Calabi–Yau threefolds [47, 12], which lead a bit beyond the movable cone. Any divisorial contraction which contracts a divisor $E$ to a curve $C$ of genus $g \geq 1$ has its associated divisorial wall of the form $\Gamma^\perp$ with $\Gamma$ a generic fiber of the induced map $E \to C$. Since $E \cdot \Gamma = -2$, there is an associated reflection in cohomology

$$H \mapsto H + (H \cdot \Gamma)E,$$

which can be used to reflect the movable cone of $X_t$ through the wall $\Gamma^\perp$. As shown in [12, §9], the A-model three-point functions have a natural analytic continuation into this reflected cone, compatible with the reflection mapping, and the physical theory on the other side of the wall is isomorphic to the $\sigma$-model theory on some birational model of $X_t$. We can thus extend our moduli space to the “reflected movable cone”, which includes the images of $\Mov(X_t)$ under all such reflections (on all birational models of $X_t$).

What about other walls? For those Calabi–Yau threefolds which can be realized as hypersurfaces in toric varieties, it is possible to study the full analytic continuation\footnote{More precisely, what is studied is the full analytic continuation of that part of the Kähler moduli space which comes from the ambient toric variety.} of the Kähler moduli space [1, 2] by combining a special formulation of the physical theories on those spaces due to Witten [51] with an analysis based on mirror symmetry. When this is done, it is found that there are indeed analytic continuations beyond other kinds of walls—in fact, the full moduli space exhibits a vast chamber structure which includes many regions that meet the original Kähler and movable cones only at the origin. The currently available descriptions of the physical theories from these other regions depend on the special formulation of [51], and appear to be linked to the ambient toric variety. An abstract description of these theories—or at least of their associated cones—will be needed before any attempt can be made to formulate a completely general account of going beyond the “reflected movable cone” for arbitrary Calabi–Yau threefolds.

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