Numerical Semigroups and Codes

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Abstract

A numerical semigroup is a subset of \( \mathbb{N} \) containing 0, closed under addition and with finite complement in \( \mathbb{N} \). An important example of numerical semigroup is given by the Weierstrass semigroup at one point of a curve. In the theory of algebraic geometry codes, Weierstrass semigroups are crucial for defining bounds on the minimum distance as well as for defining improvements on the dimension of codes. We present these applications and some theoretical problems related to classification, characterization and counting of numerical semigroups.

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Introduction

Numerical semigroups are probably one of the most simple mathematical objects. However they are involved in very hard (and some very old) problems. They can also be found in several applied fields such as error-correcting codes, cryptography, or combinatorial structures for privacy applications.

In the present chapter we present numerical semigroups with some of the related classical problems and we explore their importance in the field of algebraic-geometry codes.

The material is divided into two parts. In the first part we give a brief introduction to Weierstrass semigroups as the paradigmatic example of numerical semigroups, we present some classical problems related to general numerical semigroups, we deal with some problems on classification and characterization of numerical semigroups which have an application to coding theory, and we finally present a conjecture on counting numerical semigroups by their genus.

In the second part we present one-point algebraic-geometry codes and we focus on the applications that numerical semigroups have for defining bounds on the minimum distance as well as for defining improvements on the dimension of these codes. Based on the decoding algorithm for one-point codes one can deduce sufficient conditions for decoding, and from these conditions one can define minimal sets of parity checks (and so codes with improved correction capability) either for correcting any kind of error or at least for guaranteeing the correction of the so-called generic errors. The decoding conditions are
related to the associated Weierstrass semigroups and so the improvements can be defined in terms of semigroups.

1 Numerical semigroups

1.1 Paradigmatic example: Weierstrass semigroups on algebraic curves

1.1.1 Algebraic curves

Consider a field \( K \) and a bivariate polynomial \( f(x, y) \in K[x, y] \). If \( \bar{K} \) is the algebraic closure of \( K \), the (plane) affine curve associated to \( f \) is the set of points in \( \bar{K}^2 \) at which \( f \) vanishes. Now given a homogeneous polynomial \( F(X, Y, Z) \in K[X, Y, Z] \) the (plane) projective curve associated to \( F \) is the set of points in \( \mathbb{P}^2(\bar{K}) \) at which \( F \) vanishes. We use the notation \( \mathcal{X}_F \) to denote it.

From the affine curve defined by the polynomial \( f(x, y) \) of degree \( d \) we can obtain a projective curve defined by the homogenization of \( f \), that is, \( \bar{f}^*(X, Y, Z) = Z^d f(\frac{X}{Z}, \frac{Y}{Z}) \). Conversely, a projective curve defined by a homogeneous polynomial \( F(X, Y, Z) \) defines three affine curves with dehomogenized polynomials \( F(x, y, 1), F(1, u, v), F(w, 1, z) \). The points \((a, b) \in \bar{K}^2 \) of the affine curve defined by \( f(x, y) \) correspond to the points \((a : b : 1) \in \mathbb{P}^2(\bar{K}) \) of \( \mathcal{X}_{\bar{f}} \). Conversely, the points \((X : Y : Z) \) with \( Z \neq 0 \) (resp. \( X \neq 0, Y \neq 0 \)) of a projective curve \( \mathcal{X}_F \) correspond to the points of the affine curve defined by \( F(x, y, 1) \) (resp. \( F(1, u, v), F(w, 1, z) \)) and so they are called affine points of \( F(x, y, 1) \). The points with \( Z = 0 \) are said to be at infinity.

In the case \( K = \mathbb{F}_q \), any point of \( \mathcal{X}_F \) is in \( \mathbb{P}^2(\mathbb{F}_q^m) \) for some \( m \). If \( L/K \) is a field extension we define the \( L \)-rational points of \( \mathcal{X} \) as the points in the set \( \mathcal{X}_F(L) = \mathcal{X}_F \cap L^2 \).

We will assume that \( F \) is irreducible in any field extension of \( K \) (i.e. absolutely irreducible). Otherwise the curve is a proper union of two curves.

If two polynomials in \( K(X, Y, Z) \) differ by a multiple of \( F \), when evaluating them at a point of \( \mathcal{X}_F \) we obtain the same value. Thus it makes sense to consider \( K(X, Y, Z)/F \). Since \( F \) is irreducible, \( K(X, Y, Z)/F \) is an integral domain and we can construct its field of fractions \( Q_F \). For evaluating one such fraction at a projective point we want the result not to depend on the representative of the projective point. Hence, we require the numerator and the denominator to have one representative each, which is a homogeneous polynomial and both having the same degree. The function field of \( \mathcal{X}_F \), denoted \( K(\mathcal{X}_F) \), is the set of elements of \( Q_F \) admitting one such representation. Its elements are the rational functions of \( \mathcal{X}_F \). We say that a rational function \( f \in K(\mathcal{X}_F) \) is regular in a point \( P \) if there exists a representation of it as a fraction \( \frac{G(X, Y, Z)}{H(X, Y, Z)} \) with \( H(P) \neq 0 \). In this case we define \( f(P) = \frac{G(P)}{H(P)} \). The ring of all rational functions regular in \( P \) is denoted \( O_P \). Again it is an integral domain and this time its field of fractions is \( K(\mathcal{X}_F) \).
Let $P \in \mathcal{X}_F$ be a point. If all the partial derivatives $F_X, F_Y, F_Z$ vanish at $P$ then $P$ is said to be a singular point. Otherwise it is said to be a simple point. Curves without singular points are called non-singular, regular or smooth curves.

From now on we will assume that $F$ is absolutely irreducible and that $\mathcal{X}_F$ is smooth.

The genus of a smooth plane curve $\mathcal{X}_F$ may be defined as

$$g = \frac{(\deg(F) - 1)(\deg(F) - 2)}{2}.$$ 

For general curves the genus is defined using differentials on a curve which is out of the purposes of this survey.

### 1.1.2 Weierstrass semigroup

**Theorem 1.** Consider a point $P$ in the projective curve $\mathcal{X}_F$. There exists $t \in \mathcal{O}_P$ such that for any non-zero $f \in K(\mathcal{X}_F)$ there exists a unique integer $v_P(f)$ with

$$f = t^{v_P(f)}u$$

for some $u \in \mathcal{O}_P$ with $u(P) \neq 0$. The value $v_P(f)$ depends only on $\mathcal{X}_F, P$.

If $G(X, Y, Z)$ and $H(X, Y, Z)$ are two homogeneous polynomials of degree 1 such that $G(P) = 0$, $H(P) \neq 0$, and $G$ is not a constant multiple of $F_X(P)X + F_Y(P)Y + F_Z(P)Z$, then we can take $t$ to be the class in $\mathcal{O}_P$ of $G(X, Y, Z)$.

An element such as $t$ is called a local parameter. If there is no confusion we will write $\frac{G(X, Y, Z)}{H(X, Y, Z)}$ for its class in $\mathcal{O}_P$. The value $v_P(f)$ is called the valuation of $f$ at $P$. The point $P$ is said to be a zero of multiplicity $m$ if $v_P(f) = m > 0$ and a pole of multiplicity $-m$ if $v_P(f) = m < 0$. The valuation satisfies that $v_P(f) \geq 0$ if and only if $f \in \mathcal{O}_P$ and that in this case $v_P(f) > 0$ if and only if $f(P) = 0$.

**Lemma 2.**

1. $v_P(f) = \infty$ if and only if $f = 0$
2. $v_P(\lambda f) = v_P(f)$ for all non-zero $\lambda \in K$
3. $v_P(fg) = v_P(f) + v_P(g)$
4. $v_P(f + g) \geq \min\{v_P(f), v_P(g)\}$ and equality holds if $v_P(f) \neq v_P(g)$
5. If $v_P(f) = v_P(g) \geq 0$ then there exists $\lambda \in K$ such that $v_P(f - \lambda g) > v_P(f)$.

Let $L(mP)$ be the set of rational functions having only poles at $P$ and with pole order at most $m$. It is a $K$-vector space and so we can define $l(mP) = \dim_K(L(mP))$. One can prove that $l(mP)$ is either $l((m-1)P)$ or $l((m-1)P) + 1$. There exists a rational function $f \in K(\mathcal{X}_F)$ having only one pole at $P$ with $v_P(f) = -m$ if and only if $l(mP) = l((m-1)P) + 1$.

Let $A = \bigcup_{m \geq 0} L(mP)$, that is, $A$ is the ring of rational functions having poles only at $P$. Define $\Lambda = \{-v_P(f) : f \in A \setminus \{0\}\}$. It is obvious that $\Lambda \subseteq \mathbb{N}_0$, where $\mathbb{N}_0$ denotes the set of all non-negative integers.
Lemma 3. The set $\Lambda \subseteq \mathbb{N}_0$ satisfies

1. $0 \in \Lambda$
2. $m + m' \in \Lambda$ whenever $m, m' \in \Lambda$
3. $\mathbb{N}_0 \setminus \Lambda$ has a finite number of elements

Proof. 1. Constant functions $f = a$ have no poles and satisfy $v_P(a) = 0$ for all $P \in X_F$. Hence, $0 \in \Lambda$.

2. If $m, m' \in \Lambda$ then there exist $f, g \in A$ with $v_P(f) = -m$, $v_P(g) = -m'$. Now, by Lemma 2, $v_P(fg) = -(m + m')$ and so $m + m' \in \Lambda$.

3. The well-known Riemann-Roch theorem implies that $l(mP) = m + 1 - g$ if $m \geq 2g - 1$. On one hand this means that $m \in \Lambda$ for all $m \geq 2g$, and on the other hand, this means that $l(mP) = l((m - 1)P)$ only for $g$ different values of $m$. So, the number of elements in $\mathbb{N}_0$ which are not in $\Lambda$ is equal to the genus.

The three properties of a subset of $\mathbb{N}_0$ in the previous lemma will constitute the definition of a numerical semigroup. The particular numerical semigroup of the lemma is called the Weierstrass semigroup at $P$ and the elements in $\mathbb{N}_0 \setminus \Lambda$ are called the Weierstrass gaps.

1.1.3 Examples

Example 4 (Hermitian curve). Let $q$ be a prime power. The Hermitian curve $\mathcal{H}_q$ over $\mathbb{F}_{q^2}$ is defined by the affine equation $x^{q+1} = y^q + y$ and homogeneous equation $X^{q+1} - Y^q Z - Y Z^q = 0$. It is easy to see that its partial derivatives are $F_X = X^q$, $F_Y = -Z^q$, $F_Z = -Y^q$ and so there is no projective point at which $\mathcal{H}_q$ is singular. The point $P_{\infty} = (0 : 1 : 0)$ is the unique point of $\mathcal{H}_q$ at infinity.

We have $F_X(P_{\infty})X + F_Y(P_{\infty})Y + F_Z(P_{\infty})Z = -Z$ and so $t = \frac{X}{Z}$ is a local parameter at $P_{\infty}$. The rational functions $\frac{1}{Z}$ and $\frac{X}{Z}$ are regular everywhere except at $P_{\infty}$. So, they belong to $\bigcup_{m \geq 0} L(m P_{\infty})$. One can derive from the homogeneous equation of the curve that $t^{q+1} = \left(\frac{X}{Z}\right)^q + \frac{X}{Z}$. So, $v_{P_{\infty}}\left(\left(\frac{X}{Z}\right)^q + \frac{X}{Z}\right) = q + 1$. By Lemma 2 one can deduce that $v_{P_{\infty}}\left(\frac{x}{Z}\right) = q + 1$ and so $v_{P_{\infty}}\left(\frac{X}{Z}\right) = -(q + 1)$. On the other hand, since $\left(\frac{X}{Z}\right)^{q+1} = \left(\frac{X}{Z}\right)^q + \frac{X}{Z}$, we have $(q + 1) v_{P_{\infty}}\left(\frac{X}{Z}\right) = -q(q + 1)$. So, $v_{P_{\infty}}\left(\frac{X}{Z}\right) = -q$.

We have seen that $q, q + 1 \in \Lambda$. In this case $\Lambda$ contains what we will call later the semigroup generated by $q, q + 1$ whose complement in $\mathbb{N}_0$ has $\frac{q(q - 1)}{2}$ elements. Since we know that the complement of $\Lambda$ in $\mathbb{N}_0$ also has $g$ elements, this means that both semigroups are the same.

For further details on the Hermitian curve see [66, 35].
Example 5 (Klein quartic). The Klein quartic over \( \mathbb{F}_q \) is defined by the affine equation \( x^3y + y^3 + x = 0 \). We shall see that if \( \gcd(q, 7) = 1 \) then \( \mathcal{K} \) is smooth. Its defining homogeneous polynomial is \( F = X^3Y + Y^3Z + Z^3X \) and its partial derivatives are \( F_X = 3X^2Y + Z^3, F_Y = 3Y^2Z + X^3, F_Z = 3Z^2X + Y^3 \). If the characteristic of \( \mathbb{F}_q \) is 3 then \( F_X = F_Y = F_Z = 0 \) implies \( X^3 = Y^3 = Z^3 = 0 \) and so \( X = Y = Z = 0 \). Hence there is no projective point \( P = (X : Y : Z) \) at which \( \mathcal{K} \) is singular. Otherwise, if the characteristic of \( \mathbb{F}_q \) is different than 3 then \( F_X = F_Y = F_Z = 0 \) implies \( X^3Y = -3Y^3Z \) and \( Z^3X = -3X^3Y = 9Y^3Z \). Now the equation of the curve translates to \( -3Y^3Z + Y^3Z + 9Y^3Z = 7Y^3Z = 0 \). By hypothesis \( \gcd(q, 7) = 1 \) and so either \( Y = 0 \) or \( Z = 0 \). In the first case, \( F_Y = 0 \) implies \( X = 0 \) and \( F_X = 0 \) implies \( Y = 0 \), a contradiction, and in the second case, \( F_Y = 0 \) implies \( X = 0 \) and \( F_X = 0 \) implies \( Y = 0 \), another contradiction.

Let \( P_0 = (0 : 0 : 1) \). One can easily check that \( P_0 \in \mathcal{K} \). We have \( F_X(P_0)X + F_Y(P_0)Y + F_Z(P_0)Z = X \) and so \( t = \frac{X}{Y} \) is a local parameter at \( P_0 \). From the equation of the curve we get \( \left( \frac{X}{Y} \right)^3 + \frac{Z}{X} + \left( \frac{Z}{Y} \right)^3 = 0 \). So, at least one of the next equalities holds

\[ 3v_{P_0}(\frac{X}{Y}) = v_{P_0}(\frac{Z}{X}) \]
\[ 3v_{P_0}(\frac{X}{Y}) = 3v_{P_0}(\frac{Z}{X}) + v_{P_0}(\frac{Z}{Y}) \]
\[ v_{P_0}(\frac{X}{Y}) = 3v_{P_0}(\frac{Z}{X}) + v_{P_0}(\frac{Z}{Y}) \]

Since \( t = \frac{X}{Y} \) is a local parameter at \( P_0 \), \( v_{P_0}(\frac{X}{Y}) = -1 \). Now, since \( v_{P_0}(\frac{X}{Y}) \) is an integer, only the third equality is possible, which leads to the conclusion that \( v_{P_0}(\frac{Z}{X}) = 2 \). Similarly, \( \left( \frac{X}{Y} \right)^3 + \left( \frac{Y}{Z} \right)^3 + \frac{Z}{X} = 0 \) gives that at least one of the next equalities holds

\[ 3v_{P_0}(\frac{X}{Y}) + v_{P_0}(\frac{Z}{X}) = 3v_{P_0}(\frac{Z}{Y}) \]
\[ 3v_{P_0}(\frac{X}{Y}) + v_{P_0}(\frac{Z}{X}) = v_{P_0}(\frac{Z}{Y}) \]
\[ 3v_{P_0}(\frac{Z}{X}) = v_{P_0}(\frac{Z}{Y}) \]

Again only the third equality is possible and this leads to \( v_{P_0}(\frac{Z}{X}) = 3 \).

Now we consider the rational functions \( f_{ij} = \frac{Y^i Z^j}{X} \). We have already seen that \( v_{P_0}(f_{ij}) = -2i - 3j \) and we want to see under which conditions \( f_{ij} \in \cup_{m \geq 0} L(mP_0) \).

This is equivalent to see when it has no poles rather than \( P_0 \). The poles of \( f_{ij} \) may only be at points with \( X = 0 \) and so only at \( P_0 \) and \( P_1 = (0 : 1 : 0) \). Using the symmetries of the curve we get \( v_{P_1}(\frac{X}{Y}) = -1, v_{P_1}(\frac{Y}{Z}) = 2 \). So, \( v_{P_1}(f_{ij}) = -i + 2j \). Then \( f_{ij} \in \cup_{m \geq 0} L(mP_0) \) if and only if \( -i + 2j \geq 0 \). We get that \( \Lambda \) contains \( \{2i + 3j : i, j \geq 0, 2j \geq i\} = \{0, 3, 5, 6, 7, 8, \ldots \} \). This has 3 gaps which is exactly the genus of \( \mathcal{K} \). So,

\[ \Lambda = \{0, 3, 5, 6, 7, 8, 9, 10, \ldots \} \]

It is left as an exercise to prove that all this can be generalized to the curve \( \mathcal{K}_m \) with defining polynomial \( F = X^m Y + Y^m Z + Z^m X \), provided that \( \gcd(1, m^2 - m + 1) = 1 \). In this case \( v_{P_0}(f_{ij}) = -(m - 1)i - mj \) and \( f_{ij} \in \cup_{m \geq 0} L(mP_0) \) if and only if
\[ -i + (m-1)j \geq 0. \text{ Since } (m-1)i + mj = (m-1)i' + mj' \text{ for some } (i',j') \neq (i,j) \]
if and only if \( i \geq m \) or \( j \geq m - 1 \) we deduce that
\[
\{ -v_0(f_{ij}) : f_{ij} \in \cup_{m \geq 0} L(mP_0) \} = \{(m-1)i + mj : (i,j) \neq (1,0), (2,0), \ldots, (m-1,0)\}.
\]
This set has exactly \( \frac{m(m-1)}{2} \) gaps which is the genus of \( K_m \). So it is exactly the Weierstrass semigroup at \( P_0 \). For further details on the Klein quartic we refer the reader to [56, 35].

1.2 Basic notions and problems

A numerical semigroup is a subset \( \Lambda \) of \( \mathbb{N}_0 \) containing 0, closed under summation and with finite complement in \( \mathbb{N}_0 \). A general reference on numerical semigroups is [59].

1.2.1 Genus, conductor, gaps, non-gaps, enumeration

For a numerical semigroup \( \Lambda \) define the genus of \( \Lambda \) as the number \( g = \#(\mathbb{N}_0 \setminus \Lambda) \) and the conductor of \( \Lambda \) as the unique integer \( c \in \Lambda \) such that \( c - 1 \notin \Lambda \) and \( c + \mathbb{N}_0 \subseteq \Lambda \). The elements in \( \Lambda \) are called the non-gaps of \( \Lambda \) while the elements in \( \mathbb{N}_0 \setminus \Lambda \) are called the gaps of \( \Lambda \). The enumeration of \( \Lambda \) is the unique increasing bijective map \( \lambda : \mathbb{N}_0 \to \Lambda \). We will use \( \lambda_i \) for \( \lambda(i) \).

Lemma 6. Let \( \Lambda \) be a numerical semigroup with conductor \( c \), genus \( g \), and enumeration \( \lambda \). The following are equivalent.

(i) \( \lambda_i \geq c \)
(ii) \( i \geq c - g \)
(iii) \( \lambda_i = g + i \)

Proof. First of all notice that if \( g(i) \) is the number of gaps smaller than \( \lambda_i \), then \( \lambda_i = g(i) + i \). To see that (i) and (iii) are equivalent notice that \( \lambda_i \geq c \iff g(i) = g \iff g(i) + i = g + i \iff \lambda_i = g + i \). Now, from this equivalence we deduce that \( c = \lambda_{c-g} \). Since \( \lambda \) is increasing we deduce that \( \lambda_i \geq c \iff \lambda_i = \lambda_{c-g} \) if and only if \( i \geq c - g \).

1.2.2 Generators, Apéry set

The generators of a numerical semigroup are those non-gaps which can not be obtained as a sum of two smaller non-gaps. If \( a_1, \ldots, a_l \) are the generators of a semigroup \( \Lambda \) then \( \Lambda = \{n_1a_1 + \cdots + n_la_l : n_1, \ldots, n_l \in \mathbb{N}_0\} \) and so \( a_1, \ldots, a_l \) are necessarily coprime. If \( a_1, \ldots, a_l \) are coprime, we call \( \{n_1a_1 + \cdots + n_la_l : n_1, \ldots, n_l \in \mathbb{N}_0\} \) the semigroup generated by \( a_1, \ldots, a_l \) and denote it by \( \langle a_1, \ldots, a_l \rangle \).

The non-gap \( \lambda_1 \) is always a generator. If for each integer \( i \) from 0 to \( \lambda_1 - 1 \) we consider \( w_i \) to be the smallest non-gap in \( \Lambda \) that is congruent to \( i \) modulo \( \lambda_1 \), then each non-gap of \( \Lambda \) can be expressed as \( w_i + k\lambda_1 \) for some \( i \in \mathbb{N}_0 \).
\{0, \ldots, \lambda_1 - 1\} \text{ and some } k \in \mathbb{N}_0. \text{ So, the generators different from } \lambda_1 \text{ must be in } \{w_1, \ldots, w_{\lambda_1 - 1}\} \text{ and this implies that there is always a finite number of generators. The set } \{w_0, w_1, \ldots, w_{\lambda_1 - 1}\} \text{ is called the Apéry set of } \Lambda \text{ and denoted } Ap(\Lambda). \text{ It is easy to check that it equals } \{l \in \Lambda : l - \lambda_1 \notin \Lambda\}. \text{ References related to the Apéry set are } [1, 25, 60, 62, 43].

\subsection*{1.2.3 Frobenius' coin exchange problem}
Frobenius suggested the problem to determine the largest monetary amount that can not be obtained using only coins of specified denominations. A lot of information on the Frobenius' problem can be found in Ramírez Alfonsín's book [57].

If the different denominations are coprime then the set of amounts that can be obtained form a numerical semigroup and the question is equivalent to determining the largest gap. This is why the largest gap of a numerical semigroup is called the Frobenius number of the numerical semigroup.

If the number of denominations is two and the values of the coins are \(a, b\) with \(a, b\) coprime, then Sylvester's formula [69] gives that the Frobenius number is

\[ab - a - b.\]

However, when the number of denominations is larger, there is no closed polynomial form as can be derived from the next result due to Curtis [18].

\textbf{Theorem 7.} There is no finite set of polynomials \(\{f_1, \ldots, f_n\}\) such that for each choice of \(a, b, c \in \mathbb{N}\), there is some \(i\) such that the Frobenius number of \(a, b, c\) is \(f_i(a, b, c)\).

\subsection*{1.2.4 Hurwitz question}
It is usually attributed to Hurwitz the problem of determining whether there exist non-Weierstrass numerical semigroups, to which Buchweitz gave a positive answer, and the problem of characterizing Weierstrass semigroups. For these questions we refer the reader to [21, 38, 41] and all the citations therein.

A related problem is bounding the number of rational points of a curve using Weierstrass semigroups. One can find some bounds in [68, 42, 30].

\subsection*{1.2.5 Wilf conjecture}
The Wilf conjecture ([23, 19]) states that the number \(e\) of generators of a numerical semigroup of genus \(g\) and conductor \(c\) satisfies

\[e \geq \frac{c}{c - g}.\]

It is easy to check it when the numerical semigroup is symmetric, that is, when \(c = 2g\). In [19] the inequality is proved for many other cases. In [8] it was proved by brute approach that any numerical semigroup of genus at most 50 also satisfies the conjecture.
1.3 Classification

1.3.1 Symmetric and pseudo-symmetric numerical semigroups

**Definition 8.** A numerical semigroup $\Lambda$ with genus $g$ and conductor $c$ is said to be **symmetric** if $c = 2g$.

Symmetric numerical semigroups have been widely studied. For instance in [39, 35, 16, 12].

**Example 9.** Semigroups generated by two integers are the semigroups of the form

$$\Lambda = \{ma + nb : a, b \in \mathbb{N}_0\}$$

for some integers $a$ and $b$. For $\Lambda$ having finite complement in $\mathbb{N}_0$ it is necessary that $a$ and $b$ are coprime integers. Semigroups generated by two coprime integers are symmetric [39, 35].

Geil introduces in [29] the norm-trace curve over $\mathbb{F}_{q^r}$ defined by the affine equation

$$x^{(q^r-1)/(q-1)} = y^{q^r-1} + y^{q^r-2} + \cdots + y$$

where $q$ is a prime power. It has a single rational point at infinity and the Weierstrass semigroup at the rational point at infinity is generated by the two coprime integers $(q^r-1)/(q-1)$ and $q^r-1$. So, it is an example of a symmetric numerical semigroup.

Properties on semigroups generated by two coprime integers can be found in [39]. For instance, the semigroup generated by $a$ and $b$, has conductor equal to $(a-1)(b-1)$, and any element $l \in \Lambda$ can be written uniquely as $l = ma + nb$ with $m, n$ integers such that $0 \leq m < b$.

From the results in [39, Section 3.2] one can get, for any numerical semigroup $\Lambda$ generated by two integers, the equation of a curve having a point whose Weierstrass semigroup is $\Lambda$.

Let us state now a lemma related to symmetric numerical semigroups.

**Lemma 10.** A numerical semigroup $\Lambda$ with conductor $c$ is symmetric if and only if for any non-negative integer $i$, if $i$ is a gap, then $c - 1 - i$ is a non-gap.

The proof can be found in [39, Remark 4.2] and [35, Proposition 5.7]. It follows by counting the number of gaps and non-gaps smaller than the conductor and the fact that if $i$ is a non-gap then $c - 1 - i$ must be a gap because otherwise $c - 1$ would also be a non-gap.

**Definition 11.** A numerical semigroup $\Lambda$ with genus $g$ and conductor $c$ is said to be **pseudo-symmetric** if $c = 2g - 1$.

Notice that a symmetric numerical semigroup can not be pseudo-symmetric. Next lemma as well as its proof is analogous to Lemma 10.

**Lemma 12.** A numerical semigroup $\Lambda$ with odd conductor $c$ is pseudo-symmetric if and only if for any non-negative integer $i$ different from $(c-1)/2$, if $i$ is a gap, then $c - 1 - i$ is a non-gap.
Example 13. The Weierstrass semigroup at $P_0$ of the Klein quartic of Example 5 is $\Lambda = \{0, 3\} \cup \{i \in \mathbb{N}_0 : i \geq 5\}$. In this case $c = 5$ and the only gaps different from $(c - 1)/2$ are $l = 1$ and $l = 4$. In both cases we have $c - 1 - l \in \Lambda$. This proves that $\Lambda$ is pseudo-symmetric.

In [58] the authors prove that the set of irreducible semigroups, that is, the semigroups that can not be expressed as a proper intersection of two numerical semigroups, is the union of the set of symmetric semigroups and the set of pseudo-symmetric semigroups.

1.3.2 Arf numerical semigroups

Definition 14. A numerical semigroup $\Lambda$ with enumeration $\lambda$ is called an Arf numerical semigroup if $\lambda_i + \lambda_j - \lambda_k \in \Lambda$ for every $i, j, k \in \mathbb{N}_0$ with $i \geq j \geq k$ [17].

For further work on Arf numerical semigroups and generalizations we refer the reader to [2, 61, 13, 46]. For results on Arf semigroups related to coding theory, see [4, 17].

Example 15. It is easy to check that the Weierstrass semigroup in Example 5 is Arf.

Let us state now two results on Arf numerical semigroups that will be used later.

Lemma 16. Suppose $\Lambda$ is Arf. If $i, i + j \in \Lambda$ for some $i, j \in \mathbb{N}_0$, then $i + kj \in \Lambda$ for all $k \in \mathbb{N}_0$. Consequently, if $\Lambda$ is Arf and $i, i + 1 \in \Lambda$, then $i \geq c$.

Proof. Let us prove this by induction on $k$. It is obvious for $k = 0$ and $k = 1$. If $k > 0$ and $i, i + j, i + kj \in \Lambda$ then $(i + j) + (i + kj) - i = i + (k + 1)j \in \Lambda$. \qed

Consequently, Arf semigroups are sparse semigroups [46], that is, there are no two non-gaps in a row smaller than the conductor.

Let us give the definition of inductive numerical semigroups. They are an example of Arf numerical semigroups.

Definition 17. A sequence $(H_n)$ of numerical semigroups is called inductive if there exist sequences $(a_n)$ and $(b_n)$ of positive integers such that $H_1 = \mathbb{N}_0$ and for $n > 1$, $H_n = a_n H_{n-1} \cup \{m \in \mathbb{N}_0 : m \geq a_n b_{n-1}\}$. A numerical semigroup is called inductive if it is a member of an inductive sequence [55, Definition 2.13].

One can see that inductive numerical semigroups are Arf [17].

Example 18. Pellikaan, Stichtenoth and Torres proved in [54] that the numerical semigroups for the codes over $\mathbb{F}_{q^2}$ associated to the second tower of Garcia-Stichtenoth attaining the Drinfeld-Vlăduţ bound [27] are given recursively by $\Lambda_1 = \mathbb{N}_0$ and, for $m > 0$,

$$\Lambda_m = q \cdot \Lambda_{m-1} \cup \{i \in \mathbb{N}_0 : i \geq q^m - q^{(m+1)/2}\}.$$

They are examples of inductive numerical semigroups and hence, examples of Arf numerical semigroups.
Example 19. Hyperelliptic numerical semigroups. These are the numerical semigroups generated by 2 and an odd integer. They are of the form

$$\Lambda = \{0, 2, 4, \ldots, 2k - 2, 2k, 2k + 1, 2k + 2, 2k + 3, \ldots \}$$

for some positive integer $k$.

The next lemma is proved in [17].

Lemma 20. The only Arf symmetric semigroups are hyperelliptic semigroups.

In order to show which are the only Arf pseudo-symmetric semigroups we need the Apéry set that was previously defined.

Lemma 21. Let $\Lambda$ be a pseudo-symmetric numerical semigroup. For any $l \in Ap(\Lambda)$ different from $\lambda_1 + (c - 1)/2, \lambda_1 + c - 1 - l \in Ap(\Lambda)$.

Proof. Let us prove first that $\lambda_1 + c - 1 - l \in \Lambda$. Since $l \in Ap(\Lambda), l - \lambda_1 \not\in \Lambda$ and it is different from $(c - 1)/2$ by hypothesis. Thus $\lambda_1 + c - 1 - l = c - 1 - (l - \lambda_1) \in \Lambda$ because $\Lambda$ is pseudo-symmetric.

Now, $\lambda_1 + c - 1 - l - \lambda_1 = c - 1 - l \not\in \Lambda$ because otherwise $c - 1 \in \Lambda$. So $\lambda_1 + c - 1 - l$ must belong to $Ap(\Lambda)$.

Lemma 22. The only Arf pseudo-symmetric semigroups are $\{0, 3, 4, 5, 6, \ldots \}$ and $\{0, 3, 5, 6, 7, \ldots \}$ (corresponding to the Klein quartic).

Proof. Let $\Lambda$ be an Arf pseudo-symmetric numerical semigroup. Let us show first that $Ap(\Lambda) = \{0, \lambda_1 + (c - 1)/2, \lambda_1 + c - 1\}$. The inclusion $\supseteq$ is obvious. In order to prove the opposite inclusion suppose $l \in Ap(\Lambda), l \not\in \{0, \lambda_1 + (c - 1)/2, \lambda_1 + c - 1\}$. By Lemma 21 $\lambda_1 + c - 1 - l \in \Lambda$ and since $l \not\in \lambda_1 + c - 1, l = \lambda_1 + c - 1 - l \not\in \lambda_1$. On the other hand, if $l \not= 0$ then $l \not\in \lambda_1$. Now, by the Arf condition, $\lambda_1 + c - 1 - l + l = c - 1 \in \Lambda$, which is a contradiction.

Now, if $\# Ap(\Lambda) = 1$ then $\lambda_1 = 1$ and $\Lambda = N_0$. But $N_0$ is not pseudo-symmetric.

If $\# Ap(\Lambda) = 2$ then $\lambda_1 = 2$. But then $\Lambda$ must be hyperelliptic and so $\Lambda$ is not pseudo-symmetric.

So $\# Ap(\Lambda)$ must be 3. This implies that $\lambda_1 = 3$ and that 1 and 2 are gaps. If $1 = (c - 1)/2$ then $c = 3$ and this gives $\Lambda = \{0, 3, 4, 5, 6, \ldots \}$. Else if $2 = (c - 1)/2$ then $c = 5$ and this gives $\Lambda = \{0, 3, 5, 6, 7, \ldots \}$. Finally, if $1 \neq (c - 1)/2$ and $2 \neq (c - 1)/2$, since $\Lambda$ is pseudo-symmetric, $c - 2, c - 3 \in \Lambda$. But this contradicts Lemma 16.

The next two lemmas are two characterizations of Arf numerical semigroups. The first one is proved in [17] Proposition 1.

Lemma 23. The numerical semigroup $\Lambda$ with enumeration $\lambda$ is Arf if and only if for every two positive integers $i, j$ with $i \geq j$, $2\lambda_i - \lambda_j \in \Lambda$.

Lemma 24. The numerical semigroup $\Lambda$ is Arf if and only if for any $l \in \Lambda$, the set $S(l) = \{l' - l : l' \in \Lambda, l' \geq l\}$ is a numerical semigroup.
Proof. Suppose \( \Lambda \) is Arf. Then \( 0 \in S(l) \) and if \( m_1 = l' - l, m_2 = l'' - l \) with \( l', l'' \in \Lambda \) and \( l', l'' \geq l, l' = l'' = l - l \), then \( m_1 + m_2 = l' + l'' - l - l \). Since \( \Lambda \) is Arf, \( l' + l'' - l \in \Lambda \) and it is larger than or equal to \( l \). Thus, \( m_1 + m_2 \in S(l) \). The finiteness of the complement of \( S(l) \) is a consequence of the finiteness of the complement of \( \Lambda \).

On the other hand, if \( \Lambda \) is such that \( S(l) \) is a numerical semigroup for any \( l \in \Lambda \) then, if \( \lambda_i \geq \lambda_j \geq \lambda_k \) are in \( \Lambda \), we will have \( \lambda_i - \lambda_k \in S(\lambda_k), \lambda_j - \lambda_k \in S(\lambda_k) \) and therefore \( \lambda_i + \lambda_j - \lambda_k \in \Lambda \).

1.3.3 Numerical semigroups generated by an interval

A numerical semigroup \( \Lambda \) is generated by an interval \( \{i, i+1, \ldots, j\} \) with \( i, j \in \mathbb{N}_0, i \leq j \) if

\[
\Lambda = \{n_i i + n_{i+1} (i + 1) + \cdots + n_j j : n_i, n_{i+1}, \ldots, n_j \in \mathbb{N}_0\}.
\]

A study of semigroups generated by intervals was carried out by García-Sánchez and Rosales in [28].

Example 25. The Weierstrass semigroup at the rational point at infinity of the Hermitian curve (Example 4) is generated by \( q \) and \( q + 1 \). So, it is an example of numerical semigroup generated by an interval.

Lemma 26. The semigroup \( \Lambda_{\{i, \ldots, j\}} \) generated by the interval \( \{i, i+1, \ldots, j\} \) satisfies

\[
\Lambda_{\{i, \ldots, j\}} = \bigcup_{k \geq 0} \{ki, ki + 1, ki + 2, \ldots, kj\}.
\]

This lemma is a reformulation of [28] Lemma 1]. In the same reference we can find the next result [28] Theorem 6].

Lemma 27. \( \Lambda_{\{i, \ldots, j\}} \) is symmetric if and only if \( i \equiv 2 \mod j - i \).

Lemma 28. The only numerical semigroups which are generated by an interval and Arf, are the semigroups which are equal to \( \{0\} \cup \{i \in \mathbb{N}_0 : i \geq c\} \) for some non-negative integer \( c \).

Proof. It is a consequence of Lemma 16 and Lemma 26.

Lemma 29. The unique numerical semigroup which is pseudo-symmetric and generated by an interval is \( \{0, 3, 4, 5, 6, \ldots\} \).

Proof. By Lemma 26 for the non-trivial semigroup \( \Lambda_{\{i, \ldots, j\}} \) generated by the interval \( \{i, \ldots, j\} \), the intervals of gaps between \( \lambda_0 \) and the conductor satisfy that the length of each interval is equal to the length of the previous interval minus \( j - i \). On the other hand, the intervals of non-gaps between 1 and \( c - 1 \) satisfy that the length of each interval is equal to the length of the previous interval plus \( j - i \).

Now, by Lemma 12 \( (c - 1)/2 \) must be the first gap or the last gap of an interval of gaps. Suppose that it is the first gap of an interval of \( n \) gaps. If it is
equal to 1 then \( c = 3 \) and \( \Lambda = \{0, 3, 4, 5, \ldots\} \). Otherwise \((c-1)/2 > \lambda_1\). Then, if \( \Lambda \) is pseudo-symmetric, the previous interval of non-gaps has length \( n-1 \).

Since \( \Lambda \) is generated by an interval, the first interval of non-gaps after \((c-1)/2\) must have length \( n-1 + j - i \) and since \( \Lambda \) is pseudo-symmetric the interval of gaps before \((c-1)/2\) must have the same length. But since \( \Lambda \) is generated by an interval, the interval of gaps previous to \((c-1)/2\) must have length \( n + j - i \). This is a contradiction. The same argument proves that \((c-1)/2\) can not be the last gap of an interval of gaps. So, the only possibility for a pseudo-symmetric semigroup generated by an interval is when \((c-1)/2 = 1\), that is, when \( \Lambda = \{0, 3, 4, 5, 6, \ldots\} \).

1.3.4 Acute numerical semigroups

**Definition 30.** We say that a numerical semigroup is **ordinary** if it is equal to

\[ \{0\} \cup \{i \in \mathbb{N}_0 : i \geq c\} \]

for some non-negative integer \( c \).

Almost all rational points on a curve of genus \( g \) over an algebraically closed field have Weierstrass semigroup of the form \( \{0\} \cup \{i \in \mathbb{N}_0 : i \geq g + 1\} \). Such points are said to be **ordinary**. This is why we call these numerical semigroups ordinary \([32, 21, 67]\). Caution must be taken when the characteristic of the ground field is \( p > 0 \), since there exist curves with infinitely many non-ordinary points \([68]\).

Notice that \( \mathbb{N}_0 \) is an ordinary numerical semigroup. It will be called the **trivial** numerical semigroup.

**Definition 31.** Let \( \Lambda \) be a numerical semigroup different from \( \mathbb{N}_0 \) with enumeration \( \lambda \), genus \( g \) and conductor \( c \). The element \( \lambda_{\lambda^{-1}(c)-1} \) will be called the **dominant** of the semigroup and will be denoted \( d \). For each \( i \in \mathbb{N}_0 \) let \( g(i) \) be the number of gaps which are smaller than \( \lambda_i \). In particular, \( g(\lambda^{-1}(c)) = g \) and \( g(\lambda^{-1}(d)) = g' < g \). If \( i \) is the smallest integer for which \( g(i) = g' \) then \( \lambda_i \) is called the **subconductor** of \( \Lambda \) and denoted \( c' \).

**Remark 32.** Notice that if \( c' > 0 \), then \( c' - 1 \notin \Lambda \). Otherwise we would have \( g(\lambda^{-1}(c'-1)) = g(\lambda^{-1}(c')) \) and \( c' - 1 < c' \). Notice also that all integers between \( c' \) and \( d \) are in \( \Lambda \) because otherwise \( g(\lambda^{-1}(c')) < g' \).

**Remark 33.** For a numerical semigroup \( \Lambda \) different from \( \mathbb{N}_0 \) the following are equivalent:

(i) \( \Lambda \) is ordinary,

(ii) the dominant of \( \Lambda \) is 0,

(iii) the subconductor of \( \Lambda \) is 0.

Indeed, (i)\(\Leftrightarrow\) (ii) and (ii)\(\Rightarrow\) (iii) are obvious. Now, suppose (iii) is satisfied. If the dominant is larger than or equal to 1 it means that 1 is in \( \Lambda \) and so \( \Lambda = \mathbb{N}_0 \) a contradiction.
Definition 34. If $\Lambda$ is a non-ordinary numerical semigroup with enumeration $\lambda$ and with subconductor $\lambda_i$ then the element $\lambda_i - 1$ will be called the subdominant and denoted $d'$.

It is well defined because of Remark 33.

Definition 35. A numerical semigroup $\Lambda$ is said to be acute if $\Lambda$ is ordinary or if $\Lambda$ is non-ordinary and its conductor $c$, its subconductor $c'$, its dominant $d$ and its subdominant $d'$ satisfy $c - d \leq c' - d'$.

Roughly speaking, a numerical semigroup is acute if the last interval of gaps before the conductor is smaller than the previous interval of gaps.

Example 36. For the Hermitian curve over $\mathbb{F}_{16}$ the Weierstrass semigroup at the unique point at infinity is
\[
\{0, 4, 5, 8, 9, 10\} \cup \{i \in \mathbb{N}_0 : i \geq 12\}.
\]
In this case $c = 12$, $d = 10$, $c' = 8$ and $d' = 5$ and it is easy to check that it is an acute numerical semigroup.

Example 37. For the Weierstrass semigroup at the rational point $P_0$ of the Klein quartic in Example 5 we have $c = 5$, $d = c' = 3$ and $d' = 0$. So, it is an example of a non-ordinary acute numerical semigroup.

Lemma 38. Let $\Lambda$ be a numerical semigroup.

1. If $\Lambda$ is symmetric then it is acute.
2. If $\Lambda$ is pseudo-symmetric then it is acute.
3. If $\Lambda$ is Arf then it is acute.
4. If $\Lambda$ is generated by an interval then it is acute.

Proof. If $\Lambda$ is ordinary then it is obvious. Let us suppose that $\Lambda$ is a non-ordinary semigroup with genus $g$, conductor $c$, subconductor $c'$, dominant $d$ and subdominant $d'$.

1. Suppose that $\Lambda$ is symmetric. We know by Lemma 10 that a numerical semigroup $\Lambda$ is symmetric if and only if for any non-negative integer $i$, if $i$ is a gap, then $c - 1 - i \in \Lambda$. If moreover it is not ordinary, then 1 is a gap. So, $c - 2 \in \Lambda$ and it is precisely the dominant. Hence, $c - d = 2$. Since $c' - 1$ is a gap, $c' - d' \geq 2 = c - d$ and so $\Lambda$ is acute.

2. Suppose that $\Lambda$ is pseudo-symmetric. If $1 = (c - 1)/2$ then $c = 3$ and $\Lambda = \{0, 3, 4, 5, 6, \ldots\}$ which is ordinary. Else if $1 \neq (c - 1)/2$ then the proof is equivalent to the one for symmetric semigroups.

3. Suppose $\Lambda$ is Arf. Since $d \geq c' > d'$, then $d + c' - d'$ is in $\Lambda$ and it is strictly larger than the dominant $d$. Hence it is larger than or equal to $c$. So, $d + c' - d' \geq c$ and $\Lambda$ is acute.
4. Suppose that $\Lambda$ is generated by the interval $\{i, i + 1, \ldots, j\}$. Then, by Lemma 26, there exists $k$ such that $c = ki, c' = (k - 1)i, d = (k - 1)j$ and $d' = (k - 2)j$. So, $c - d = k(i - j) + j$ while $c' - d' = k(i - j) - i + 2j$. Hence, $\Lambda$ is acute.

In Figure 1 we summarize all the relations we have proved between acute semigroups, symmetric and pseudo-symmetric semigroups, Arf semigroups and semigroups generated by an interval.

**Remark 39.** There exist numerical semigroups which are not acute. For instance,

$$\Lambda = \{0, 6, 8, 9\} \cup \{i \in \mathbb{N}_0 : i \geq 12\}.$$

In this case, $c = 12, d = 9, c' = 8$ and $d' = 6$.

On the other hand there exist numerical semigroups which are acute and which are not symmetric, pseudo-symmetric, Arf or interval-generated. For example,

$$\Lambda = \{0, 10, 11\} \cup \{i \in \mathbb{N}_0 : i \geq 15\}.$$

In this case, $c = 15, d = 11, c' = 10$ and $d' = 0$. 
1.4 Characterization

1.4.1 Homomorphisms of semigroups

Homomorphisms of numerical semigroups, that is, maps $f$ between numerical semigroups such that $f(a + b) = f(a) + f(b)$, are exactly the scale maps $f(a) = ka$ for all $a$, for some constant $k \geq 0$. Indeed, if $f$ is a homomorphism then $\frac{f(a)}{a}$ is constant since $f(ab) = a \cdot f(b) = b \cdot f(a)$. Furthermore, the unique surjective homomorphism is the identity. Indeed, for a semigroup $\Lambda$, the set $k\Lambda$ is a numerical semigroup only if $k = 1$.

1.4.2 The $\oplus$ operation, the $\nu$ sequence, and the $\tau$ sequence

Next we define three important objects describing the addition behavior of a numerical semigroup.

**Definition 40.** The operation $\oplus : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ associated to the numerical semigroup $\Lambda$ (with enumeration $\lambda$) is defined as

$$i \oplus j = \lambda^{-1}(\lambda_i + \lambda_j)$$

for any $i, j \in \mathbb{N}_0$. Equivalently,

$$\lambda_i + \lambda_j = \lambda_{i \oplus j}.$$

The subindex referring to the semigroup may be omitted if the semigroup is clear by the context. The operation $\oplus$ is obviously commutative, associative, and has 0 as identity element. However there is in general no inverse element. Also, the operation $\oplus$ is compatible with the natural order of $\mathbb{N}_0$. That is, if $a < b$ then $a \oplus c < b \oplus c$ and $c \oplus a < c \oplus b$ for any $c \in \mathbb{N}_0$.

**Example 41.** For the numerical semigroup $\Lambda = \{0, 4, 5, 8, 9, 10, 12, 13, 14, \ldots\}$ the first values of $\oplus$ are given in the next table:

| $\oplus$ | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  | 15  | 16  | 17  | 18  | 19  | 20  | 21  | 22  | 23  | 24  | 25  | 26  | 27  | 28  | 29  | 30  | 31  |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|        | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  | 15  | 16  | 17  | 18  | 19  | 20  | 21  | 22  | 23  | 24  | 25  | 26  | 27  | 28  | 29  | 30  | 31  |

|        | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  | 15  | 16  | 17  | 18  | 19  | 20  | 21  | 22  | 23  | 24  | 25  | 26  | 27  | 28  | 29  | 30  | 31  |

**Definition 42.** The partial ordering $\preceq_{\Lambda}$ on $\mathbb{N}_0$ associated to the numerical semigroup $\Lambda$ is defined as follows:

$$i \preceq_{\Lambda} j \text{ if and only if } \lambda_j - \lambda_i \in \Lambda.$$
Equivalently

\[ i \preceq_{\Lambda} j \text{ if and only if there exists } k \text{ with } i \oplus_{\Lambda} k = j. \]

As before, if \( \Lambda \) is clear by the context then the subindex may be omitted.

**Definition 43.** Given a numerical semigroup \( \Lambda \), the set \( N_i \) is defined by

\[ N_i = \{ j \in \mathbb{N}_0 : j \preceq_{\Lambda} i \} = \{ j \in \mathbb{N}_0 : \lambda_i - \lambda_j \in \Lambda \}. \]

The sequence \( \nu_i \) is defined by \( \nu_i = \#N_i \).

**Example 44.** The \( \nu \) sequence of the trivial semigroup \( \Lambda = \mathbb{N}_0 \) is \( 1, 2, 3, 4, \ldots \).

Next lemma states the relationship between \( \oplus \) and the \( \nu \) sequence of a numerical semigroup. Its proof is obvious.

**Lemma 45.** Let \( \Lambda \) be a numerical semigroup and \( \nu \) the corresponding \( \nu \) sequence. For all \( i \in \mathbb{N}_0 \),

\[ \nu_i = \#\{ (j, k) \in \mathbb{N}_0^2 : j \oplus k = i \}. \]

**Example 46.** If we are given the table in Example 41 we can deduce \( \nu_7 \) by counting the number of occurrences of 7 inside the table. This is exactly 6. Thus, \( \nu_7 = 6 \). Repeating the same process for all \( \nu_i \) with \( i < 7 \) we get \( \nu_0 = 1, \nu_1 = 2, \nu_2 = 2, \nu_3 = 3, \nu_4 = 4, \nu_5 = 3, \nu_6 = 4 \).

This lemma implies that any finite set in the sequence \( \nu \) can be determined by a finite set of \( \oplus \) values. Indeed, to compute \( \nu_i \) it is enough to know \( \{ j \oplus k : 0 \leq j, k \leq i \} \).

**Definition 47.** Given a numerical semigroup \( \Lambda \) define its \( \tau \) sequence by

\[ \tau_i = \text{max}\{ j \in \mathbb{N}_0 : \text{exists } k \text{ with } j \leq k \leq i \text{ and } j \oplus_{\Lambda} k = i \}. \]

Notice that \( \tau_i \) is the largest element \( j \) in \( N_i \) with \( \lambda_j \leq \lambda_i/2 \). In particular, if \( \lambda_i/2 \in \Lambda \) then \( \tau_i = \lambda^{-1}(\lambda_i/2) \). Notice also that \( \tau_i \) is 0 if and only if \( \lambda_i \) is either 0 or a generator of \( \Lambda \).

**Example 48.** In Table 1 we show the \( \nu \) sequence and the \( \tau \) sequence of the numerical semigroups generated by 4, 5 and generated by 6, 7, 8, 17.

One difference between the \( \tau \) sequence and the \( \nu \) sequence is that, while in the \( \nu \) sequence not all non-negative integers need to appear, in the \( \tau \) sequence all of them appear. Notice for instance that 7 does not appear in the \( \nu \) sequence of the numerical semigroup generated by 4 and 5 nor the numerical semigroup generated by 6, 7, 8, 17. The reason for which any non-negative integer \( j \) appears in the \( \tau \) sequence is that if \( \lambda_i = 2\lambda_j \) then \( \tau_i = j \). Furthermore, the smallest \( i \) for which \( \tau_i = j \) corresponds to \( \lambda_i = 2\lambda_j \).
Table 1: $\nu$ and $\tau$ sequences of the numerical semigroups generated by 4, 5 and by 6, 7, 8, 17.

| $i$ | $\lambda_i$ | $\nu_i$ | $\tau_i$ |
|-----|--------------|----------|----------|
| 0   | 0            | 0        | 0        |
| 1   | 4            | 2        | 0        |
| 2   | 5            | 2        | 0        |
| 3   | 8            | 3        | 1        |
| 4   | 9            | 4        | 1        |
| 5   | 10           | 3        | 2        |
| 6   | 12           | 4        | 1        |
| 7   | 13           | 6        | 2        |
| 8   | 14           | 6        | 2        |
| 9   | 15           | 4        | 2        |
| 10  | 16           | 5        | 3        |
| 11  | 17           | 8        | 3        |
| 12  | 18           | 9        | 4        |
| 13  | 19           | 8        | 4        |
| 14  | 20           | 9        | 5        |
| 15  | 21           | 10       | 4        |
| 16  | 22           | 12       | 5        |
| 17  | 23           | 12       | 5        |
| 18  | 24           | 13       | 6        |
| 19  | 25           | 14       | 6        |
| 20  | 26           | 15       | 7        |
| 21  | 27           | 16       | 7        |
| 22  | 28           | 17       | 8        |
| 23  | 29           | 18       | 8        |
|     | :            | :        | :        |
| 0   | 0            | 0        | 0        |
| 1   | 6            | 2        | 0        |
| 2   | 7            | 2        | 0        |
| 3   | 8            | 2        | 0        |
| 4   | 12           | 3        | 1        |
| 5   | 13           | 4        | 1        |
| 6   | 14           | 5        | 2        |
| 7   | 15           | 4        | 2        |
| 8   | 16           | 3        | 3        |
| 9   | 17           | 2        | 0        |
| 10  | 18           | 4        | 1        |
| 11  | 19           | 6        | 2        |
| 12  | 20           | 8        | 3        |
| 13  | 21           | 8        | 3        |
| 14  | 22           | 8        | 3        |
| 15  | 23           | 8        | 3        |
| 16  | 24           | 9        | 4        |
| 17  | 25           | 10       | 4        |
| 18  | 26           | 11       | 5        |
| 19  | 27           | 12       | 5        |
| 20  | 28           | 13       | 6        |
| 21  | 29           | 14       | 6        |
| 22  | 30           | 15       | 7        |
| 23  | 31           | 16       | 7        |
|     | :            | :        | :        |

(a) $< 4, 5 >$  
(b) $< 6, 7, 8, 17 >$
1.4.3 Characterization of a numerical semigroup by \(\oplus\)
The next result was proved in [6, 7].

**Lemma 49.** The \(\oplus\) operation uniquely determines a semigroup.

*Proof.* Suppose that two semigroups \(\Lambda = \{\lambda_0 < \lambda_1 < \ldots\}\) and \(\Lambda' = \{\lambda'_0 < \lambda'_1 < \ldots\}\) have the same associated operation \(\oplus\). Define the map \(f(\lambda_i) = \lambda'_i\). It is obviously surjective and it is a homomorphism since \(f(\lambda_i + \lambda_j) = f(\lambda_i \oplus \lambda_j) = \lambda'_i + \lambda'_j = f(\lambda_i) + f(\lambda_j)\). So, \(\Lambda = \Lambda'\). \(\square\)

Conversely to Lemma 49 we next prove that any finite set of \(\oplus\) values is shared by an infinite number of semigroups. This was proved in [7].

**Lemma 50.** Let \(a, b\) be positive integers. Let \(\Lambda\) be a numerical semigroup with enumeration \(\lambda\) and let \(d\) be an integer with \(d \geq 2\). Define the numerical semigroup \(\Lambda' = d\Lambda \cup \{i \in \mathbb{N} : i \geq d\lambda_{a\oplus b}\}\). Then \(i \oplus_{\Lambda'} j = i \oplus_{\Lambda} j\) for all \(i \leq a\) and all \(j \leq b\), and \(\Lambda' \neq \Lambda\).

*Proof.* It is obvious that \(\Lambda' \neq \Lambda\). Let \(\lambda'\) be the enumeration of \(\Lambda'\). For all \(k \leq a \oplus \Lambda\) \(b, \lambda'_k = d\lambda_k\). In particular, if \(i \leq a\) and \(j \leq b\) then \(\lambda'_i = d\lambda_i\) and \(\lambda'_j = d\lambda_j\). Hence, \(\lambda'_{i\oplus\Lambda'j} = \lambda'_i + \lambda'_j = d\lambda_i + d\lambda_j = d\lambda_{i\oplus\Lambda}j = \lambda'_{i\oplus\Lambda}j\). This implies \(i \oplus_{\Lambda'} j = i \oplus_{\Lambda} j\). \(\square\)

By varying \(d\) in Lemma 50 we can see that although the values \((i \oplus j)_{0 \leq i, j}\) of a numerical semigroup uniquely determine it, any subset \((i \oplus j)_{0 \leq i \leq a, 0 \leq j \leq b}\) is exactly the corresponding subset of infinitely many numerical semigroups.

1.4.4 Characterization of a numerical semigroup by \(\nu\)

We will use the following result on the values \(\nu_i\). It can be found in [39, Theorem 3.8].

**Lemma 51.** Let \(\Lambda\) be a numerical semigroup with genus \(g\), conductor \(c\) and enumeration \(\lambda\). Let \(g(i)\) be the number of gaps smaller than \(\lambda_i\) and let

\[
D(i) = \{l \in \mathbb{N}_0 \setminus \Lambda : \lambda_i - l \in \mathbb{N}_0 \setminus \Lambda\}.
\]

Then, for all \(i \in \mathbb{N}_0\),

\[
\nu_i = i - g(i) + \#D(i) + 1.
\]

In particular, for all \(i \geq 2c - g - 1\) (or equivalently, for all \(i\) such that \(\lambda_i \geq 2c - 1\)), \(\nu_i = i - g + 1\).

*Proof.* The number of gaps smaller than \(\lambda_i\) is \(g(i)\) but it is also \(\#D(i) + \#\{l \in \mathbb{N}_0 \setminus \Lambda : \lambda_i - l \in \Lambda\}\). So,

\[
g(i) = \#D(i) + \#\{l \in \mathbb{N}_0 \setminus \Lambda : \lambda_i - l \in \Lambda\}.
\]

(1)

On the other hand, the number of non-gaps which are at most \(\lambda_i\) is \(i + 1\) but it is also \(\nu_i + \#\{l \in \Lambda : \lambda_i - l \in \mathbb{N}_0 \setminus \Lambda\}\) = \(\nu_i + \#\{l \in \mathbb{N}_0 \setminus \Lambda : \lambda_i - l \in \Lambda\}\). So,

\[
i + 1 = \nu_i + \#\{l \in \mathbb{N}_0 \setminus \Lambda : \lambda_i - l \in \Lambda\}.
\]

(2)
From equalities 1 and 2 we get
\[ g(i) - \#D(i) = i + 1 - \nu_i \]
which leads to the desired result.

The next lemma shows that if a numerical semigroup is non-trivial then there exists at least one value \( k \) such that \( \nu_k = \nu_{k+1} \).

**Lemma 52.** Suppose \( \Lambda \neq \mathbb{N}_0 \) and suppose that \( c \) and \( g \) are the conductor and the genus of \( \Lambda \). Let \( k = 2c - g - 2 \). Then \( \nu_k = \nu_{k+1} \).

**Proof.** Since \( \Lambda \neq \mathbb{N}_0 \), \( c \geq 2 \) and so \( 2c - 2 \geq c \). This implies \( k = \lambda^{-1}(2c - 2) \) and \( g(k) = g \). By Lemma 51, \( \nu_k = k - g + \#D(k) + 1 \). But \( D(k) = \{c - 1\} \). So, \( \nu_k = k - g + 2 \). On the other hand, \( g(k+1) = g \) and \( D(k+1) = \emptyset \). By Lemma 51 again, \( \nu_{k+1} = k - g + 2 = \nu_k \). \( \square \)

**Lemma 53.** The trivial semigroup is the unique numerical semigroup with \( \nu \) sequence equal to \( 1, 2, 3, 4, 5, \ldots \).

**Proof.** As a consequence of Lemma 52, for any other numerical semigroup there is a value in the \( \nu \) sequence that appears at least three times. \( \square \)

The next result was proved in [5, 6].

**Theorem 54.** The \( \nu \) sequence of a numerical semigroup determines it.

**Proof.** If \( \Lambda = \mathbb{N}_0 \) then, by Lemma 53, its \( \nu \) sequence is unique.

Suppose that \( \Lambda \neq \mathbb{N}_0 \). Then we can determine the genus and the conductor from the \( \nu \) sequence. Indeed, let \( k = 2c - g - 2 \). In the following we will show how to determine \( k \) without the knowledge of \( c \) and \( g \). By Lemma 52 it holds that \( \nu_k = \nu_{k+1} = k - g + 2 \) and by Lemma 51, \( \nu_i = i - g + 1 \) for all \( i > k \), which means that \( \nu_{i+1} = \nu_i + 1 \) for all \( i > k \). So,
\[ k = \max\{i : \nu_i = \nu_{i+1}\} \]

We can now determine the genus as
\[ g = k + 2 - \nu_k \]
and the conductor as
\[ c = \frac{k + g + 2}{2} \]

At this point we know that \( \{0\} \subseteq \Lambda \) and \( \{i \in \mathbb{N}_0 : i \geq c\} \subseteq \Lambda \) and, furthermore, \( \{1, c - 1\} \subseteq \mathbb{N}_0 \setminus \Lambda \). It remains to determine for all \( i \in \{2, \ldots, c - 2\} \) whether \( i \in \Lambda \). Let us assume \( i \in \{2, \ldots, c - 2\} \).

On one hand, \( c - 1 + i - g > c - g \) and so \( \lambda_{c-1+i-g} > c \). This means that \( g(c - 1 + i - g) = g \) and hence
\[ \nu_{c-1+i-g} = c - 1 + i - g - g + \#D(c - 1 + i - g) + 1 \]
On the other hand, if we define $\bar{D}(i)$ to be

$$\bar{D}(i) = \{ l \in \mathbb{N}_0 \setminus \Lambda : c - 1 + i - l \in \mathbb{N}_0 \setminus \Lambda, i < l < c - 1 \}$$

then

$$D(c - 1 + i - g) = \begin{cases} 
\bar{D}(i) \cup \{ c - 1, i \} & \text{if } i \in \mathbb{N}_0 \setminus \Lambda 
\bar{D}(i) & \text{otherwise.}
\end{cases} \quad (4)$$

So, from (3) and (4),

$$i \text{ is a non-gap } \iff \nu_{c - 1 + i - g} = c + i - 2g + \# \bar{D}(i).$$

This gives an inductive procedure to decide whether $i$ belongs to $\Lambda$ decreasingly from $i = c - 2$ to $i = 2$.

**Remark 55.** From the proof of Theorem 54 we see that a semigroup can be determined by $k = \max\{ i : \nu_i = \nu_{i+1} \}$ and the values $\nu_i$ for $i \in \{ c - g + 1, \ldots, 2c - g - 3 \}$.

**Remark 56.** Lemma 49 is a consequence of Lemma 45 and Theorem 54.

Conversely to Theorem 54 we next prove that any finite set of $\nu$ values is shared by an infinite number of semigroups. This was proved in [7]. Thus, the construction just given to determine a numerical semigroup from its $\nu$ sequence can only be performed if we know the behavior of the infinitely many values in the $\nu$ sequence.

**Lemma 57.** Let $k$ be a positive integer. Let $\Lambda$ be a numerical semigroup with enumeration $\lambda$ and let $d$ be an integer with $d \geq 2$. Define the semigroup $\Lambda' = d\Lambda \cup \{ i \in \mathbb{N} : i \geq d\lambda_k \}$ and let $\nu^\Lambda$ and $\nu'^\Lambda$ be the $\nu$ sequence corresponding to $\Lambda$ and $\Lambda'$ respectively. Then $\nu'^\Lambda_i = \nu^\Lambda_i$ for all $i \leq k$ and $\Lambda' \neq \Lambda$.

**Proof.** It is obvious that $\Lambda' \neq \Lambda$. Let $\lambda'$ be the enumeration of $\Lambda'$. For all $i \leq k$, $\lambda'_i = d\lambda_i$. In particular, if $j \leq i \leq k$, then $\lambda'_i - \lambda'_j = d(\lambda_i - \lambda_j) \in \Lambda' \iff \lambda_i - \lambda_j \in \Lambda$. Hence, $\nu'^\Lambda_i = \nu^\Lambda_i$.

As a consequence of Lemma 57, although the sequence $\nu$ of a numerical semigroup uniquely determines it, any subset $(\nu_i)_{0 \leq i < k}$ is exactly the set of the first $k$ values of the $\nu$ sequence of infinitely many semigroups. In fact, by varying $d$ among the positive integers, we get an infinite set of semigroups, all of them sharing the first $k$ values in the $\nu$ sequence.

It would be interesting to find which sequences of positive integers correspond to the sequence $\nu$ of a numerical semigroup. By now, only some necessary conditions can be stated, for instance,

- $\nu_0 = 1$,
- $\nu_1 = 2$,
- $\nu_i \leq i + 1$ for all $i \in \mathbb{N}_0$,
- there exists $k$ such that $\nu_{i+1} = \nu_i + 1$ for all $i \geq k$,
1.4.5 Characterization of a numerical semigroup by $\tau$

In this section we show that a numerical semigroup is determined by its $\tau$ sequence. This was proved in [10].

Lemma 58. Let $\Lambda$ be a numerical semigroup with enumeration $\lambda$, conductor $c > 2$, genus $g$, and dominant $d$. Then

1. $\tau_{(2c-g-2)+2i} = \tau_{(2c-g-2)+2i+1} = (c-g-1) + i$ for all $i \geq 0$

2. At least one of the following statements holds
   \begin{itemize}
   \item $\tau_{(2c-g-2)-1} = c - g - 1$
   \item $\tau_{(2c-g-2)-2} = c - g - 1$
   \end{itemize}

Proof. 1. If $i \geq 1$ then $\lambda_{(2c-g-2)+2i} = 2c - 2 + 2i$ and $\lambda_{(2c-g-2)+2i}/2 = c - 1 + i \in \Lambda$. So $\tau_{(2c-g-2)+2i} = \lambda^{-1}(c - 1 + i) = c - 1 + i - g$. On the other hand, $\lambda_{(2c-g-2)+2i+1} = 2c - 2 + 2i + 1 = (c - 1 + i) + (c - 1 + i + 1)$ and so $\tau_{(2c-g-2)+2i+1} = \lambda^{-1}(c - 1 + i) = c - 1 + i - g$.

If $i = 0$ then $\lambda_{(2c-g-2)+2i} = \lambda_{2c-g-2}$ and since $c > 2$ this is equal to $2c - 2$. Now $\lambda_{2c-g-2}/2 = c - 1$ and the largest non-gap which is at most $c - 1$ is $d$. On the other hand, $\lambda_{2c-g-2} - d = 2c - 2 - d \geq c$ because $c \geq d + 2$. Consequently $\lambda_{2c-g-2}-d \in \Lambda$ and $\tau_{2c-g-2} = \lambda^{-1}(d) = c - g - 1$. Similarly, the largest non-gap which is at most $\lambda_{2c-g-2}/2$ is $d$ and $\lambda_{2c-g-2} - d = 2c - 1 - d \in \Lambda$. So, $\tau_{2c-g-2} = c - g - 1$.

2. If $c = 3$ then $g = 2$ and $\lambda_{(2c-g-2)-2} = \lambda_0$ and $\tau_{(2c-g-2)-2} = 0 = c - g - 1$. Assume $c \geq 4$. If $d = c - 2$ then $\lambda_{(2c-g-2)-2} = 2c - 4 = 2d$, so $\tau_{(2c-g-2)-2} = \lambda^{-1}(d) = c - g - 1$. If $d = c - 3$ then $\lambda_{(2c-g-2)-1} = 2c - 3 = d + c$, so $\tau_{(2c-g-2)-1} = \lambda^{-1}(d) = c - g - 1$. Suppose now $d < c - 4$. In this case $\lambda_{(2c-g-2)-2}/2 = c - 2$, which is between $d$ and $c$, and $\lambda_{(2c-g-2)-2} - d = 2c - 4 - d \geq c$. So $\lambda_{(2c-g-2)-2} - d \in \Lambda$. This makes $\tau_{(2c-g-2)-2} = \lambda^{-1}(d) = c - g - 1$. \hfill $\Box$

Lemma 59. The trivial semigroup is the unique numerical semigroup with $\tau$ sequence equal to 0, 0, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, . . .

Proof. It is enough to check that for any other numerical semigroup there is a value in the $\tau$ sequence that appears at least three times. If $c = 2$ then $\tau_0 = \tau_1 = \tau_2 = 0$. If $c > 2$, by Lemma $58\tau_{2c-g-2} = \tau_{2c-g-1}$ and they are equal to at least one of $\tau_{2c-g-3}$ and $\tau_{2c-g-4}$. \hfill $\Box$

Theorem 60. The $\tau$ sequence of a numerical semigroup determines it.

Proof. Let $k$ be the minimum integer such that $\tau_{k+2i} = \tau_{k+2i+1}$ and $\tau_{k+2i+2} = \tau_{k+2i+1} + 1$ for all $i \in \mathbb{N}_0$. If $k = 0$, by Lemma $59\Lambda = \mathbb{N}_0$. Assume $k > 0$. 22
By Lemma 58 if \( c > 2, k = 2c - g - 2 \) and \( \tau_k = c - g - 1 \). So,

\[
\begin{align*}
\begin{cases}
c &= k - \tau_k + 1 \\
g &= k - 2\tau_k
\end{cases}
\]

This result can be extended to the case \( c = 2 \) since in this case \( c = 2, g = 1, k = 1 \) and \( \tau_k = 0 \).

This determines \( \lambda_i = i + g \) for all \( i \geq c - g \). Now we can determine \( \lambda_{c-g-1}, \lambda_{c-g-2}, \) and so on using that the smallest \( j \) for which \( \tau_j = i \) corresponds to \( \lambda_j = 2\lambda_i \). That is, \( \lambda_i = \frac{1}{2} \min \{ \lambda_j : \tau_j = i \} \).

We have just seen that any numerical semigroup is uniquely determined by its \( \tau \) sequence. The next lemma shows that no finite subset of \( \tau \) can determine the numerical semigroup. This result is analogous to \([7, \text{Proposition 2.2.}]\). In this case it refers to the \( \nu \) sequence instead of the \( \tau \) sequence.

**Lemma 61.** Let \( r \) be a positive integer. Let \( \Lambda \) be a numerical semigroup with enumeration \( \lambda \) and let \( m \) be an integer with \( m \geq 2 \). Define the semigroup \( \Lambda' = m\Lambda \cup \{ i \in \mathbb{N}_0 : i \geq m\lambda_i \} \) and let \( \tau^\Lambda \) and \( \tau^\Lambda' \) be the \( \tau \) sequence corresponding to \( \Lambda \) and \( \Lambda' \) respectively. Then \( \tau_i^{\Lambda'} = \tau_i^\Lambda \) for all \( i \leq r \) and \( \Lambda' \neq \Lambda \).

**Proof.** It is obvious that \( \Lambda' \neq \Lambda \). Let \( \lambda' \) be the enumeration of \( \Lambda' \). For all \( i \leq r \), \( \lambda_i' = m\lambda_i \). In particular, if \( j \leq i \leq r \), then it exists \( k \) with \( j \leq k \leq i \) and \( \lambda_j + \lambda_k = \lambda_i \) if and only if it exists \( k \) with \( j \leq k \leq i \) and \( \lambda_j' + \lambda_k' = \lambda_i' \). Hence, by the definition of the \( \tau \) sequence, \( \tau_i^{\Lambda'} = \tau_i^\Lambda \).

As a consequence of Lemma 61, although the sequence \( \tau \) of a numerical semigroup uniquely determines it, any subset \( \{ \tau_i \}_{0 \leq i \leq r-1} \) is exactly the set of the first \( r \) values of the \( \tau \) sequence of infinitely many semigroups. In fact, by varying \( m \) among the positive integers, we get an infinite set of semigroups, all of them sharing the first \( r \) values in the \( \tau \) sequence.

### 1.5 Counting

We are interested on the number \( n_g \) of numerical semigroups of genus \( g \). It is obvious that \( n_0 = 1 \) since \( \mathbb{N}_0 \) is the unique numerical semigroup of genus 0. On the other hand, if \( 1 \) is in a numerical semigroup, then any non-negative integer must belong also to the numerical semigroup, because any non-negative integer is a finite sum of 1’s. Thus, the unique numerical semigroup with genus 1 is \( \{0\} \cup \{ i \in \mathbb{N}_0 : i \geq 2 \} \) and \( n_1 = 1 \). In \([10]\) all terms of the sequence \( n_g \) are computed up to genus 37 and the terms of genus up to 50 are computed in \([8]\). Recently we computed \( n_{51} = 164253200784 \) and \( n_{52} = 266815155103 \). It is conjectured in \([8]\) that the sequence given by the numbers \( n_g \) of numerical semigroups of genus \( g \) asymptotically behaves like the Fibonacci numbers and so it increases by a portion of the golden ratio. More precisely, it is conjectured that (1) \( n_g \geq n_{g-1} + n_{g-2} \), (2) \( \lim_{g \to \infty} \frac{n_{g-1} + n_{g-2}}{n_g} = 1 \), (3) \( \lim_{g \to \infty} \frac{n_g}{n_{g-1}} = \phi \), where \( \phi = \frac{1 + \sqrt{5}}{2} \) is the golden ratio. Notice that (2) and (3) are equivalent.
By now, only some bounds are known for $n_g$, which become very poor when $g$ approaches infinity \[9, 20\]. Other contributions related to this sequence can be found in \[40, 41, 65, 44, 12, 11, 74, 37, 3\].

In Table 2 there are the results obtained for all numerical semigroups with genus up to 52. For each genus we wrote the number of numerical semigroups of the given genus, the Fibonacci-like-estimated value given by the sum of the number of semigroups of the two previous genus, the value of the quotient $\frac{n_{g-1} + n_{g-2}}{n_g}$, and the value of the quotient $\frac{n_g}{n_{g-1}}$. In Figure 2 and Figure 3 we depicted the behavior of these quotients. From these graphics one can predict that $\frac{n_{g-1} + n_{g-2}}{n_g}$ approaches 1 as $g$ approaches infinity whereas $\frac{n_g}{n_{g-1}}$ approaches the golden ratio as $g$ approaches infinity.

The number $n_g$ is usually studied by means of the tree rooted at the semi-
| $g$ | $n_g$ | $n_{g-1} + n_{g-2}$ | $\frac{n_{g-1} + n_{g-2}}{n_{g-1}}$ | $\frac{n_{g-2}}{n_{g-1}}$ |
|-----|-------|---------------------|----------------------------------|---------------------|
| 0   | 1     | 1                   | 1                                | 1                   |
| 1   | 1     | 2                   | 2                                | 2                   |
| 2   | 2     | 3                   | 0.75                             | 2                   |
| 3   | 4     | 6                   | 0.857143                         | 1.75                |
| 4   | 7     | 11                  | 0.916667                         | 1.71429             |
| 5   | 12    | 19                  | 0.826087                         | 1.91667             |
| 6   | 23    | 35                  | 0.897436                         | 1.69565             |
| 7   | 39    | 62                  | 0.925373                         | 1.71795             |
| 8   | 118   | 106                 | 0.898305                         | 1.76119             |
| 9   | 204   | 185                 | 0.906863                         | 1.72881             |
| 10  | 343   | 322                 | 0.938776                         | 1.68137             |
| 11  | 592   | 547                 | 0.923986                         | 1.72595             |
| 12  | 1001  | 935                 | 0.934066                         | 1.69088             |
| 13  | 1693  | 1593                | 0.940933                         | 1.69131             |
| 14  | 2857  | 2694                | 0.942947                         | 1.68754             |
| 15  | 4806  | 4550                | 0.946733                         | 1.68218             |
| 16  | 8045  | 7663                | 0.952517                         | 1.67395             |
| 17  | 13467 | 12851               | 0.954259                         | 1.67396             |
| 18  | 22464 | 21512               | 0.957621                         | 1.66808             |
| 19  | 37396 | 35931               | 0.960825                         | 1.66471             |
| 20  | 62194 | 59860               | 0.962472                         | 1.66312             |
| 21  | 103246| 99590               | 0.964589                         | 1.66006             |
| 22  | 170963| 165440              | 0.967695                         | 1.65588             |
| 23  | 282828| 274209              | 0.969526                         | 1.65432             |
| 24  | 467224| 453791              | 0.971249                         | 1.65197             |
| 25  | 770832| 750052              | 0.973042                         | 1.64981             |
| 26  | 1270267| 1238056            | 0.974642                         | 1.64792             |
| 27  | 2091030| 2041099            | 0.976121                         | 1.64613             |
| 28  | 3437839| 3361297            | 0.977735                         | 1.64409             |
| 29  | 5646773| 5528869            | 0.979120                         | 1.64254             |
| 30  | 9266788| 9084612            | 0.980341                         | 1.64108             |
| 31  | 15195070| 14913561          | 0.981474                         | 1.63973             |
| 32  | 24896206| 24461858          | 0.982354                         | 1.63844             |
| 33  | 40761087| 40091276          | 0.983567                         | 1.63724             |
| 34  | 66687201| 65657293          | 0.984556                         | 1.63605             |
| 35  | 109032500| 107448288        | 0.985470                         | 1.63498             |
| 36  | 178158289| 175719701       | 0.986312                         | 1.63399             |
| 37  | 290939807| 287190789       | 0.987114                         | 1.63304             |
| 38  | 474851445| 46998096         | 0.987884                         | 1.63213             |
| 39  | 774614284| 765791252       | 0.988610                         | 1.63128             |
| 40  | 1262992840| 124945279       | 0.989290                         | 1.63048             |
| 41  | 2058356522| 2037607124     | 0.989919                         | 1.62975             |
| 42  | 3353191846| 3321349362     | 0.990504                         | 1.62906             |
| 43  | 5460401576| 5411548368     | 0.991053                         | 1.62842             |
| 44  | 8888486816| 8813593422     | 0.991574                         | 1.62781             |
| 45  | 1446363648| 14348888392    | 0.992067                         | 1.62723             |
| 46  | 23527845502| 23552120464   | 0.992531                         | 1.62669             |
| 47  | 38260496374| 37991479150   | 0.992969                         | 1.62618             |
| 48  | 62200036752| 61788341876   | 0.993381                         | 1.62570             |
| 49  | 101090300128| 100460533126  | 0.993770                         | 1.62525             |
| 50  | 164253200784| 16329036880  | 0.994138                         | 1.62482             |
| 51  | 266815155103| 26534350912  | 0.994484                         | 1.62441             |

Table 2: Computational results on the number of numerical semigroups up to genus 52.
Figure 4: Recursive construction of numerical semigroups of genus $g$ from numerical semigroups of genus $g - 1$. Generators larger than the conductor are written in bold face.

group $\mathbb{N}_0$ and for which the children of a semigroup are the semigroups obtained by taking out one by one its generators larger than or equal to its conductor [8, 9, 11, 20]. This tree was previously used in [64, 63]. It is illustrated in Figure 4. It contains all semigroups exactly once and the semigroups at depth $g$ have genus $g$. So, $n_g$ is the number of nodes of the tree at depth $g$. Some alternatives for counting semigroups of a given genus without using this tree have been considered in [12, 8, 74].

2 Numerical semigroups and codes

2.1 One-point codes and their decoding

2.1.1 One-point codes

Linear codes A linear code $C$ of length $n$ over the alphabet $\mathbb{F}_q$ is a vector subspace of $\mathbb{F}_q^n$. Its elements are called code words. The dimension $k$ of the code is the dimension of $C$ as a subspace of $\mathbb{F}_q^n$. The dual code of $C$ is $C^\perp = \{v \in \mathbb{F}_q^n : v \cdot c = 0 \text{ for all } c \in C\}$. It is a linear code with the same length as $C$ and with dimension $n - k$.

The Hamming distance between two vectors of the same length is the number of positions in which they do not agree. The weight of a vector is the number of its non-zero components or, equivalently, its Hamming distance to the zero vector. The minimum distance $d$ of a linear code $C$ is the minimum Hamming distance between two code words in $C$. Equivalently, it is the minimum weight of all code words in $C$. The correction capability of a code is the maximum num-
ber of errors that can be added to any code word, with the code word being still uniquely identifiable. The correction capability of a linear code with minimum distance \( d \) is \( \left\lfloor \frac{d-1}{2} \right\rfloor \).

**One-point codes**  Let \( P \) be a rational point of the algebraic smooth curve \( X \) defined over \( \mathbb{F}_q \) with Weierstrass semigroup \( \Lambda \). Suppose that the enumeration of \( \Lambda \) is \( \lambda \). Recall that \( A = \bigcup_{m \geq 0} L(mP) \) is the ring of rational functions having poles only at \( P \). We will say that the order of \( f \in A \setminus \{0\} \) is \( s \) if \( v_P(f) = -\lambda_s \). The order of 0 is considered to be either \(-1\) or \(-\infty\). In the present work we will consider the order of 0 to be \(-1\) although both would be fine. We denote the order of \( f \) by \( \rho(f) \).

One can find an infinite basis \( z_0, z_1, \ldots, z_i, \ldots \) of \( A \) such that \( v_P(z_i) = -\lambda_i \) or, equivalently, \( \rho(z_i) = i \). Consider a set of rational points \( P_1, \ldots, P_n \) different from \( P \) and the map \( \varphi : A \to \mathbb{F}_q^n \) defined by \( \varphi(f) = (f(P_1), \ldots, f(P_n)) \). To each finite subset \( W \subseteq \mathbb{N}_0 \) we associate the one-point code

\[
C_W = \langle \varphi(z_i) : i \in W >^1 = \langle (z_i(P_1), \ldots, z_i(P_n)) : i \in W >^1. 
\]

We say that \( W \) is the set of parity checks of \( C_W \). The one-point codes for which \( W = \{0, 1, \ldots, m\} \) are called classical one-point codes. In this case we write \( C_m \) for \( C_W \).

2.1.2 Decoding one-point codes

This section presents a sketch of a decoding algorithm for \( C_W \). The aim is to justify the conditions guaranteeing correction of errors. Suppose that a code word \( c \in C_W \) is sent and that an error \( e \) is added to it so that the received word is \( u = c + e \). We will use \( t \) for the number of non-zero positions in \( e \).

**Definition 62.** A polynomial \( f \) is an error-locator of an error vector \( e \) if and only if \( f(P_i) = 0 \) whenever \( e_i \neq 0 \). The footprint of \( e \) is the set

\[
\Delta_e = \mathbb{N}_0 \setminus \{\rho(f) : f \text{ is an error-locator}\}. 
\]

It is well known that \( \#\Delta_e = t \) and that \( \Delta_e \) is \( \preceq \)-closed. That is, if \( i \preceq j \) and \( j \in \Delta_e \) then \( i \in \Delta_e \). If for each minimal element in \( \mathbb{N}_0 \setminus \Delta_e \) with respect to \( \preceq \) we can find an error-locator with that order then localization of errors is guaranteed.

**Definition 63.** Define the syndrome of orders \( i, j \) as

\[
s_{ij} = \sum_{l=1}^{n} z_i(P_l)z_j(P_l)e_l. 
\]

The syndrome matrix \( S^{r'} \) is the \((r + 1) \times (r' + 1)\) matrix with \( S_{ij}^{r'} = s_{ij} \) for \( 0 \leq i \leq r \) and \( 0 \leq j \leq r' \).
The matrix $S^{r’r}$ is the matrix $S^{r’r}$, transposed. By Lemma 2 if $i \oplus j = k$ then there exist $a_0, \ldots, a_{k}$ such that $z_i z_j = a_k z_k + \cdots + a_0 z_0$. Define $s_k = \sum_{i=1}^{n} z_k(P_i) e_i$. Then

$$s_{ij} = a_k s_k + \cdots + a_0 s_0.$$  \hfill (5)

The syndromes depend on $e$ which is initially unknown. So, in general, $s_{ij}$ and $s_k$ are unknown. For a polynomial $f = z_r + a_{r-1} z_{r-1} + \cdots + a_0 z_0$, being an error locator means that $(a_0, \ldots, a_r)S^{r’r} = 0$ for all $r’ > 0$. Conversely, there exists $M$ such that if $(a_0, \ldots, a_r)S^{r’r} = 0$ for all $r’$ with $r \oplus r’ \leq M$ then $f$ is an error locator.

Hence, we look for pairs $(r, r’)$ with $r \oplus r’$ large enough such that $(x_0, \ldots, x_{r-1}, 1)S^{r’r} = 0$ has a non-zero solution (and indeed we look for this solution). Notice that if $(x_0, \ldots, x_{r-1}, 1)S^{r’r} = 0$ has a non-zero solution then so does $(x_0, \ldots, x_{r-1}, 1)S^{r’’r}$ for all $r’’ < r’$.

The first difficulty is that only a few syndromes are known. This is overcome by using a majority voting procedure.

We proceed iteratively, considering the non-gaps of $\Lambda$ by increasing order. Suppose that all syndromes $s_{ij}$ are known for $i \oplus j < k$ and we want to compute the syndromes $s_{ij}$ with $i \oplus j = k$. By equation 5, this is equivalent to finding $s_k$. If $k \in W$ then the computation can be done by just using the definition of $C_W$: $s_k = \sum_{i=1}^{n} z_k(P_i)e_i = \sum_{i=1}^{n} z_k(P_i) u_{i} - \sum_{i=1}^{n} z_k(P_{i}) e_i = \sum_{i=1}^{n} z_k(P_{i}) u_{i}$. Otherwise we establish a voting procedure to determine $s_k$.

In the voting procedure the voters are the elements $i \leq k$ for which $(x_0, \ldots, x_{i-1}, 1)S^{i,k\oplus i+1} = 0$ and $(y_0, \ldots, y_{k\oplus i-1}, 1)S^{k\oplus i,i-1} = 0$ have non-zero solutions. We consider the value

$$\hat{s}_{i,k\oplus i} = (s_{i,(k\oplus i)}, \ldots, s_{i,(k\oplus i)}) \cdot (x_0, \ldots, x_{i-1}) = (s_{i,0}, \ldots, s_{i,(k\oplus i)-1}) \cdot (y_0, \ldots, y_{(k\oplus i)-1}).$$

as a candidate for $s_{i,k\oplus i}$. Notice that if $s_{i,k\oplus i} \neq \hat{s}_{i,k\oplus i}$ then $(x_0, \ldots, x_{i-1}, 1)S^{i,k\oplus i} = 0$ and $(y_0, \ldots, y_{k\oplus i-1}, 1)S^{k\oplus i,i-1} = 0$. Otherwise, if $s_{i,k\oplus i} = \hat{s}_{i,k\oplus i}$ then there exist no error-locators of order $i$ and no error-locators of order $k \oplus i$. Since $\hat{s}_{i,k\oplus i}$ is a candidate for $s_{i,k\oplus i}$, the associated candidate $\hat{s}_k$ for $s_k$ will be derived from the equation $\hat{s}_{i,k\oplus i} = a_k \hat{s}_k + a_{k-1} \hat{s}_{k-1} + \cdots + a_0 s_0$, where $a_0, \ldots, a_k$ are such that $z_i \hat{s}_{k\oplus i} = a_k \hat{s}_k + \cdots + a_0 s_0$. That is, $\hat{s}_k = \hat{s}_{i,k\oplus i} - a_{k-1} \hat{s}_{k-1} - \cdots - a_0 s_0.$

**Lemma 64.** If $i \in N_k$ and $i, k \oplus i \notin \Delta_e$ then $i$ is a voter and its vote coincides with $s_k$. 

---

1. An explanation for this can be found in [9][2][4].

2. If $(x_0, \ldots, x_{i-1}, 1)S^{i,j-1} = 0$ and $(y_0, \ldots, y_{j-1}, 1)S^{j,i-1} = 0$ then $(s_0, \ldots, s_{i-1}) \cdot (x_0, \ldots, x_{i-1}) = (s_0, \ldots, s_{j-1}) \cdot (y_0, \ldots, y_{j-1})$. Indeed, $(x_0, \ldots, x_{i-1}, 1)S^{i,j-1} = 0$ implies that $(s_0, \ldots, s_{i-1}) = -(x_0, \ldots, x_{i-1})(S^{i-1,j-1})^{-1}$ and similarly $(y_0, \ldots, y_{j-1}, 1)S^{j,i-1} = 0$ implies that $(s_0, \ldots, s_{i-1}) = -(y_0, \ldots, y_{j-1})(S^{j-1,i-1})^{-1}$.

Now, $(s_0, \ldots, s_{i-1})\cdot (x_0, \ldots, x_{i-1}) = -(y_0, \ldots, y_{j-1})S^{i-1,j-1} \cdot (x_0, \ldots, x_{i-1}) = \ldots = -(y_0, \ldots, y_{j-1})S^{i-1,j-1} \cdot (x_0, \ldots, x_{i-1}) = -(x_0, \ldots, x_{i-1})S^{i-1,j-1} \cdot (y_0, \ldots, y_{j-1}) = (x_0, \ldots, x_{i-1})S^{i-1,j-1} \cdot (y_0, \ldots, y_{j-1})$.
• If a voter \( i \) votes for a wrong candidate for \( s_k \) then \( i, k \odot i \in \Delta_e \).

• If \( \nu_k > 2#(N_k \cap \Delta_e) \) then a majority of voters vote for the right value \( s_k \).

Proof. The first two items are deduced from what has been said before. Consider the sets 
\[ A = \{ i \in N_k : i, k \odot i \in \Delta_e \}, \]
\[ B = \{ i \in N_k : i \in \Delta_e, k \odot i \notin \Delta_e \}, \]
\[ C = \{ i \in N_k : i \notin \Delta_e, k \odot i \in \Delta_e \}, \]
\[ D = \{ i \in N_k : i, k \odot i \notin \Delta_e \}. \]

By the previous items, the wrong votes are at most \( \#A \) while the right votes are at least \( \#D \).

Obviously, \( \nu_k = \#A + \#B + \#C + \#D \), \( \#(N_k \cap \Delta_e) = \#A + \#B = \#A + \#C \).
So, the difference between the right and the wrong votes is at least \( \#D - \#A = \nu_k - 2#A - \#B - \#C = \nu_k - 2#(N_k \cap \Delta_e) > 0 \).

The conclusion of this section is the next theorem.

**Theorem 65.** If \( \nu_i > 2#(N_i \cap \Delta_e) \) for all \( i \notin W \) then \( e \) is correctable by \( C_W \).

### 2.2 The \( \nu \) sequence, classical codes, and Feng-Rao improved codes

From the equality \( \#\Delta_e = t \) we deduce the next lemma.

**Lemma 66.** If the number \( t \) of errors in \( e \) satisfies \( t \leq \lfloor \frac{\nu_i - 1}{2} \rfloor \), then \( \nu_i > 2#(N_i \cap \Delta_e) \).

#### 2.2.1 The \( \nu \) sequence and the minimum distance of classical codes

Theorem 65 and Lemma 66 can be used in order to get an estimate of the minimum distance of a one-point code. The next definition arises from [23, 35, 39].

**Definition 67.** The order bound on the minimum distance of the classical code \( C_W \), with \( W = \{0, \ldots, m\} \) is

\[ d_{ORD}(C_m) = \min\{\nu_i : i > m\} \]

The order bound is also referred to as the Feng-Rao bound. The order bound is proved to be a lower bound on the minimum distance for classical codes [23, 35, 39].

**Lemma 68.** \( d(C_m) \geq d_{ORD}(C_m) \).

From Lemma 51 we deduce that \( \nu_{i+1} \leq \nu_{i+2} \) and so \( d_{ORD}(C_i) = \nu_{i+1} \) for all \( i \geq 2c - g - 2 \).

A refined version of the order bound is

\[ d_{ORD}^{P_1, \ldots, P_n}(C_m) = \min\{\nu_i : i > m, C_i \neq C_{i+1}\} \]

While \( d_{ORD} \) only depends on the Weierstrass semigroup, \( d_{ORD}^{P_1, \ldots, P_n} \) depends also on the points \( P_1, \ldots, P_n \). Since our point of view is that of numerical semigroups we will concentrate on \( d_{ORD} \).

Generalized Hamming weights are a generalization of the minimum distance of a code with many applications to coding theory but also to other fields.
such as cryptography. For the generalized Hamming weights of one-point codes there is a generalization of the order bound based also on the associated Weierstrass semigroups. We will not discuss this topic here but the reader interested in it can see \[34, 22].

2.2.2 On the order bound on the minimum distance

In this section we will find a formula for the smallest \( m \) for which \( d_{\text{ORD}}(C_i) = \nu_{i+1} \) for all \( i \geq m \), for the case of acute semigroups. At the end we will use Munuera-Torres and Oneto-Tamone’s results to generalize this formula.

**Remark 69.** Let \( \Lambda \) be a non-ordinary numerical semigroup with conductor \( c \), subconductor \( c' \) and dominant \( d \). Then, \( c' + d \geq c \). Indeed, \( c' + d \in \Lambda \) and by Remark 33 it is strictly larger than \( d \). So, it must be larger than or equal to \( c \).

**Theorem 70.** Let \( \Lambda \) be a non-ordinary acute numerical semigroup with enumeration \( \lambda \), conductor \( c \), subconductor \( c' \) and dominant \( d \). Let

\[
 m = \min\{\lambda^{-1}(c + c' - 2), \lambda^{-1}(2d)\}. \tag{6}
\]

Then,

1. \( \nu_m > \nu_{m+1} \)
2. \( \nu_i \leq \nu_{i+1} \) for all \( i > m \).

**Proof.** Following the notations in Lemma \[51\] for \( i \geq \lambda^{-1}(c) \), \( g(i) = g. \) Thus, for \( i \geq \lambda^{-1}(c) \) we have

\[
 \nu_i \leq \nu_{i+1} \text{ if and only if } \#D(i+1) \geq \#D(i) - 1. \tag{7}
\]

Let \( l = c - d - 1 \). Notice that \( l \) is the number of gaps between the conductor and the dominant. Since \( \Lambda \) is acute, the \( l \) integers before \( c' \) are also gaps. Let us call \( k = \lambda^{-1}(c' + d) \). For all \( 1 \leq i \leq l \), both \( (c' - i) \) and \( (d + i) \) are in \( D(k) \) because they are gaps and

\[
(c' - i) + (d + i) = c' + d.
\]

Moreover, there are no more gaps in \( D(k) \) because, if \( j \leq c' - l - 1 \) then \( c' + d - j \geq d + l + 1 = c \) and so \( c' + d - j \in \Lambda \). Therefore,

\[
 D(k) = \{c' - i : 1 \leq i \leq l\} \cup \{d + i : 1 \leq i \leq l\}.
\]

Now suppose that \( j \geq k \). By Remark \[69\] \( \lambda_k \geq c \) and so \( \lambda_j = \lambda_k + j - k = c' + d + j - k \). Then,

\[
 D(j) = A(j) \cup B(j),
\]
where

\[
A(j) = \begin{cases}
\{c' - i : 1 \leq i \leq l - j + k\} \\
\cup \{d + i : j - k + 1 \leq i \leq l\} \\
\emptyset
\end{cases}
\]

and

\[
B(j) = \begin{cases}
\emptyset \\
\{d + i : 1 \leq i \leq \lambda_j - 2d - 1\} \\
\{d + i : \lambda_j - d - c + 1 \leq i \leq l\} \\
\emptyset
\end{cases}
\]

Notice that \(A(j) \cap B(j) = \emptyset\) and hence

\[
\#D(j) = \#A(j) + \#B(j).
\]

We have

\[
\#A(j) = \begin{cases}
2l - j + k & \text{if } \lambda_k \leq \lambda_j \leq c + c' - 2, \\
0 & \text{otherwise.}
\end{cases}
\]

and

\[
\#B(j) = \begin{cases}
0 & \text{if } \lambda_k \leq \lambda_j \leq 2d + 1, \\
\lambda_j - 2d - 1 & \text{if } 2d + 2 \leq \lambda_j \leq c + d, \\
2c - 1 - \lambda_j & \text{if } c + d \leq \lambda_j \leq 2c - 2, \\
0 & \text{if } \lambda_j \geq 2c - 1.
\end{cases}
\]

So,

\[
\begin{aligned}
\#A(j + 1) &= \begin{cases}
\#A(j) - 2 & \text{if } \lambda_k \leq \lambda_j \leq c + c' - 2, \\
\#A(j) & \text{otherwise.}
\end{cases} \\
\#B(j + 1) &= \begin{cases}
\#B(j) & \text{if } \lambda_k \leq \lambda_j \leq 2d \\
\#B(j) + 1 & \text{if } 2d + 1 \leq \lambda_j \leq c + d - 1 \\
\#B(j) - 1 & \text{if } c + d \leq \lambda_j \leq 2c - 2 \\
\#B(j) & \text{if } \lambda_j \geq 2c - 1
\end{cases}
\end{aligned}
\]

Notice that \(c + c' - 2 < c + d\). Thus, for \(\lambda_j \geq c + d\),

\[
\#D(j + 1) = \begin{cases}
\#D(j) - 1 & \text{if } c + d \leq \lambda_j \leq 2c - 2, \\
\#D(j) & \text{if } \lambda_j \geq 2c - 1.
\end{cases}
\]

Hence, by (7), \(\nu_i \leq \nu_{i+1}\) for all \(i \geq \lambda^{-1}(c + d)\) because \(\lambda^{-1}(c + d) \geq \lambda^{-1}(c)\).

Now, let us analyze what happens if \(\lambda_j < c + d\).

If \(c + c' - 2 \leq 2d\) then

\[
\begin{aligned}
\#D(j + 1) &= \begin{cases}
\#D(j) - 2 & \text{if } \lambda_k \leq \lambda_j \leq c + c' - 2, \\
\#D(j) & \text{if } c + c' - 1 \leq \lambda_j \leq 2d, \\
\#D(j) + 1 & \text{if } 2d + 1 \leq \lambda_j \leq c + d - 1.
\end{cases}
\end{aligned}
\]

and if \(2d + 1 \leq c + c' - 2\) then

\[
\begin{aligned}
\#D(j + 1) &= \begin{cases}
\#D(j) - 2 & \text{if } \lambda_k \leq \lambda_j \leq 2d, \\
\#D(j) - 1 & \text{if } 2d + 1 \leq \lambda_j \leq c + c' - 2, \\
\#D(j) + 1 & \text{if } c + c' - 1 \leq \lambda_j \leq c + d - 1.
\end{cases}
\end{aligned}
\]

So, by (7) and since both \(c + c' - 2\) and \(2d\) are larger than or equal to \(c\), the result follows. \(\square\)
Corollary 71. Let $\Lambda$ be a non-ordinary acute numerical semigroup with enumeration $\lambda$, conductor $c$ and subconductor $c'$. Let

$$m = \min \{ \lambda^{-1}(c + c' - 2), \lambda^{-1}(2d) \}.$$ 

Then, $m$ is the smallest integer for which

$$d_{ORD}(C_i) = \nu_{i+1}$$

for all $i \geq m$.

Example 72. Recall the Weierstrass semigroup at the point $P_0$ on the Klein quartic that we presented in Example 5. Its conductor is 5, its dominant is 3 and its subconductor is 3. In Table 3 we have, for each integer from 0 to $\lambda - 1(2c - 2)$, the values $\lambda_i$, $\nu_i$ and $d_{ORD}(C_i)$.

For this example, $\lambda^{-1}(c + c' - 2) = \lambda^{-1}(2d) = 3$ and so, $m = \min \{ \lambda^{-1}(c + c' - 2), \lambda^{-1}(2d) \} = 3$. We can check that, as stated in Theorem 70, $\nu_3 > \nu_4$ and $\nu_i \leq \nu_{i+1}$ for all $i > 3$. Moreover, as stated in Corollary 71, $d_{ORD}(C_i) = \nu_{i+1}$ for all $i \geq 3$ while $d_{ORD}(C_2) \neq \nu_3$.

Lemma 73. Let $\Lambda$ be a non-ordinary numerical semigroup with conductor $c$, subconductor $c'$ and dominant $d$.

1. If $\Lambda$ is symmetric then $\min \{ c + c' - 2, 2d \} = c + c' - 2 = 2c - 2 - \lambda_1$,

2. If $\Lambda$ is pseudo-symmetric then $\min \{ c + c' - 2, 2d \} = c + c' - 2$,

3. If $\Lambda$ is Arf then $\min \{ c + c' - 2, 2d \} = 2d$,

4. If $\Lambda$ is generated by an interval then $\min \{ c + c' - 2, 2d \} = c + c' - 2$.

Proof.

1. We already saw in the proof of Lemma 38 that if $\Lambda$ is symmetric then $d = c - 2$. So, $c + c' - 2 = d + c' \leq 2d$ because $c' \leq d$. Moreover, by Lemma 10 any non-negative integer $i$ is a gap if and only if $c - 1 - i \in \Lambda$. This implies that $c' - 1 = c - 1 - \lambda_1$ and so $c' = c - \lambda_1$. Therefore, $c + c' - 2 = 2c - 2 - \lambda_1$. 

| $i$ | $\lambda_i$ | $\nu_i$ | $d_{ORD}(C_i)$ |
|-----|-------------|---------|---------------|
| 0   | 0           | 1       | 2             |
| 1   | 3           | 2       | 2             |
| 2   | 5           | 2       | 2             |
| 3   | 6           | 3       | 2             |
| 4   | 7           | 2       | 4             |
| 5   | 8           | 4       | 4             |
2. If $\Lambda$ is pseudo-symmetric and non-ordinary then $d = c - 2$ because $1$ is a gap different from $(c - 1)/2$. So, $c + c' - 2 = d + c' \leq 2d$.

3. If $\Lambda$ is Arf then $c' = d$. Indeed, if $c' < d$ then $d - 1 \in \Lambda$ and, by Lemma 16 $d - 1 \geq c$, a contradiction. Since $d \leq c - 2$, we have $2d \leq c + c' - 2$.

4. Suppose $\Lambda$ is generated by the interval $\{i, i + 1, \ldots, j\}$. By Lemma 26 there exists $k$ such that $c = ki$ and $d = (k - 1)j$. We have that $c - d \leq j - i$, because otherwise $(k + 1)i - kj = c - d - (j - i) > 1$, and hence $kj + 1$ would be a gap greater than $c$. On the other hand $d - c' \geq j - i$, and hence $2d - (c + c' - 2) = d - c + d - c' + 2 \geq i - j + j - i + 2 = 2$.

Example 74. Consider the Hermitian curve over $\mathbb{F}_{16}$. Its numerical semigroup is generated by 4 and 5. So, this is a symmetric numerical semigroup because it is generated by two coprime integers, and it is also a semigroup generated by the interval $\{4, 5\}$.

In Table 4 we include, for each integer from 0 to 16, the values $\lambda_i, \nu_i$ and $d_{\text{ORD}}(C_i)$. Notice that in this case the conductor is 12, the dominant is 10 and the subconductor is 8. We do not give the values in the table for $i > \lambda^{-1}(2c - 1) - 1 = 16$ because $d_{\text{ORD}}(C_i) = \nu_{i+1}$ for all $i \geq \lambda^{-1}(2c - 1) - 1$. We can check that, as follows from Theorem 70 and Lemma 73 \( \lambda^{-1}(c + c' - 2) = 12 \) is the largest integer $m$ with $\nu_m > \nu_{m+1}$ and so the smallest integer for which $d_{\text{ORD}}(C_i) = \nu_{i+1}$ for all $i \geq m$. Notice also that, as pointed out in Lemma 73 $c + c' - 2 = 2c - 2 - \lambda_1$.

Furthermore, in this example there are 64 rational points on the curve different from $P_\infty$ and the map $\varphi$ evaluating the functions of $A$ at these 64 points satisfies that the words $\varphi(f_0), \ldots, \varphi(f_{57})$ are linearly independent whereas $\varphi(f_{58})$ is linearly dependent to the previous ones. So, $d_{\text{ORD}}^{P_1, \ldots, P_n}(C_i) = d_{\text{ORD}}(C_i)$ for all $i \leq 56$.

Example 75. Let us consider now the semigroup of the fifth code associated to the second tower of Garcia and Stichtenoth over $\mathbb{F}_4$. As noticed in Example 18 this is an Arf numerical semigroup. We set in Table 5 the values $\lambda_i, \nu_i$ and $d_{\text{ORD}}(C_i)$ for each integer from 0 to 25. In this case the conductor is 24, the dominant is 20 and the subconductor is 20. As before, we do not give the values for $i > \lambda^{-1}(2c - 1) - 1 = 25$. We can check that, as follows from Theorem 74 and Lemma 73 \( \lambda^{-1}(2d) = 19 \) is the largest integer $m$ with $\nu_m > \nu_{m+1}$ and so, the smallest integer for which $d_{\text{ORD}}(C_i) = \nu_{i+1}$ for all $i \geq m$.

Munuera and Torres in [45] and Oneto and Tamone in [47] proved that for any numerical semigroup $m \leq \min\{c + c' - 2 - g, 2d - g\}$. Notice that for acute semigroups this inequality is an equality.

Munuera and Torres in [45] introduced the definition of near-acute semigroups. They proved that the formula $m = \min\{c + c' - 2 - g, 2d - g\}$ not only applies for acute semigroups but also for near-acute semigroups. Next we give the definition of near-acute semigroups.

Definition 76. A numerical semigroup with conductor $c$, dominant $d$ and subdominant $d'$ is said to be a near-acute semigroup if either $c - d \leq d - d'$ or $2d - c + 1 \notin \Lambda$. 
Oneto and Tamone in [47] proved that \( m = \min\{c + c' - 2 - g, 2d - g\} \) if and only if \( c + c' - 2 \leq 2d \) or \( 2d - c + 1 \notin \Lambda \). Let us see next that these conditions in Oneto and Tamone’s result are equivalent to having a near-acute semigroup.

**Lemma 77.** For a numerical semigroup the following are equivalent

1. \( c - d \leq d - d' \) or \( 2d - c + 1 \notin \Lambda \),

2. \( c + c' - 2 \leq 2d \) or \( 2d - c + 1 \notin \Lambda \).

**Proof.** Let us see first that (1) implies (2). If \( 2d - c + 1 \notin \Lambda \) then it is obvious. Otherwise the condition \( c - d \leq d - d' \) is equivalent to \( d' \leq 2d - c \) which, together with \( 2d - c + 1 \in \Lambda \) implies \( c' \leq 2d - c + 1 \) by definition of \( c' \). This in turn implies that \( c + c' - 2 < c + c' - 1 \leq 2d \).

To see that (1) is a consequence of (2) notice that by definition, \( d' \leq c' - 2 \). Then, if \( c + c' - 2 \leq 2d \), we have \( d - d' \geq d - c' + 2 \geq c - d \).

From all these results one concludes the next theorem.

**Theorem 78.** 1. For any numerical semigroup \( m \leq \min\{c + c' - 2 - g, 2d - g\} \).

2. \( m = \min\{c + c' - 2 - g, 2d - g\} \) if and only if the corresponding numerical semigroup is near-acute.

In [48] Oneto and Tamone give further results on \( m \) and in [49] the same authors conjecture that for any numerical semigroup,

\[ \lambda_m \geq c + d - \lambda_1. \]
Table 5: Garcia-Stichtenoth tower

| $i$ | $\lambda_i$ | $\nu_i$ | $d_{ORD}(C_i)$ |
|-----|-------------|---------|-----------------|
| 0   | 0           | 1       | 2               |
| 1   | 16          | 2       | 2               |
| 2   | 20          | 2       | 2               |
| 3   | 24          | 2       | 2               |
| 4   | 25          | 2       | 2               |
| 5   | 26          | 2       | 2               |
| 6   | 27          | 2       | 2               |
| 7   | 28          | 2       | 2               |
| 8   | 29          | 2       | 2               |
| 9   | 30          | 2       | 2               |
| 10  | 31          | 2       | 2               |
| 11  | 32          | 3       | 2               |
| 12  | 33          | 2       | 2               |
| 13  | 34          | 2       | 2               |
| 14  | 35          | 2       | 2               |
| 15  | 36          | 4       | 2               |
| 16  | 37          | 2       | 2               |
| 17  | 38          | 2       | 2               |
| 18  | 39          | 2       | 4               |
| 19  | 40          | 5       | 4               |
| 20  | 41          | 4       | 4               |
| 21  | 42          | 4       | 4               |
| 22  | 43          | 4       | 6               |
| 23  | 44          | 6       | 6               |
| 24  | 45          | 6       | 6               |
| 25  | 46          | 6       | 6               |
2.2.3 The $\nu$ sequence and Feng-Rao improved codes

The one-point codes whose set $W$ of parity checks is selected so that the orders outside $W$ satisfy the hypothesis of Lemma 66 and $W$ is minimal with this property are called Feng-Rao improved codes. They were defined in [24, 35].

**Definition 79.** Given a rational point $P$ of an algebraic smooth curve $X_F$ defined over $\mathbb{F}_q$ with Weierstrass semigroup $\Lambda$ and sequence $\nu$ with associated basis $z_0, z_1, \ldots$ and given $n$ other different points $P_1, \ldots, P_n$ of $X_F$, the associated Feng-Rao improved code guaranteeing correction of $t$ errors is defined as

$$C_{\tilde{R}(t)} = \langle (z_i(P_1), \ldots, z_i(P_n)) : i \in \tilde{R}(t) \rangle^\perp,$$

where

$$\tilde{R}(t) = \{ i \in \mathbb{N}_0 : \nu_i < 2t + 1 \}.$$

2.2.4 On the improvement of the Feng-Rao improved codes

Feng-Rao improved codes will actually give an improvement with respect to classical codes only if $\nu_i$ is decreasing at some $i$. We next study this condition.

**Lemma 80.** If $\Lambda$ is an ordinary numerical semigroup with enumeration $\lambda$ then

$$\nu_i = \begin{cases} 
1 & \text{if } i = 0, \\
2 & \text{if } 1 \leq i \leq \lambda_1, \\
i - \lambda_1 + 2 & \text{if } i > \lambda_1.
\end{cases}$$

**Proof.** It is obvious that $\nu_0 = 1$ and that $\nu_1 = 2$ whenever $0 < \lambda_i < 2\lambda_1$. So, since $2\lambda_1 = \lambda_{i+1}$, we have that $\nu_i = 2$ for all $1 \leq i \leq \lambda_1$. Finally, if $\lambda_i \geq 2\lambda_1$ then all non-gaps up to $\lambda_i - \lambda_1$ are in $N_i$ as well as $\lambda_i$, and none of the remaining non-gaps are in $N_i$. Now, if the genus of $\Lambda$ is $g$, then $\nu_i = \lambda_i - \lambda_1 + 2 - g$ and $\lambda_i = i + g$. So, $\nu_i = i - \lambda_1 + 2$. \qed

As a consequence of Lemma 80 the $\nu$ sequence is non-decreasing if $\Lambda$ is an ordinary numerical semigroup. We will see in this section that ordinary numerical semigroups are in fact the only semigroups for which the $\nu$ sequence is non-decreasing.

**Lemma 81.** Suppose that for the semigroup $\Lambda$ the $\nu$ sequence is non-decreasing. Then $\Lambda$ is Arf.

**Proof.** Let $\lambda$ be the enumeration of $\Lambda$. Let us see by induction that, for any non-negative integer $i$,

(i) $N_{\lambda^{-1}(2\lambda_i)} = \{ j \in \mathbb{N}_0 : j \leq i \} \cup \{ \lambda^{-1}(2\lambda_i - \lambda_j) : 0 \leq j < i \}$, where $\cup$ means the union of disjoint sets.

(ii) $N_{\lambda^{-1}(\lambda_i + \lambda_{i+1})} = \{ j \in \mathbb{N}_0 : j \leq i \} \cup \{ \lambda^{-1}(\lambda_i + \lambda_{i+1} - \lambda_j) : 0 \leq j < i \}$. 

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Notice that if (i) is satisfied for all \(i\), then \(\{ j \in \mathbb{N}_0 : j \leq i \} \subseteq \Lambda_{\lambda-1}(2\lambda_i)\) for all \(i\), and hence by Lemma 80, \(\Lambda_{\lambda-1}\) is Arf.

It is obvious that both (i) and (ii) are satisfied for the case \(i = 0\).

Suppose \(i > 0\). By the induction hypothesis, \(\nu_{\lambda-1}(\lambda_{i-1}+\lambda_i) = 2i\). Now, since \((\nu_i)\) is not decreasing and \(2\lambda_i > \lambda_{i-1} + \lambda_i\), we have \(\nu_{\lambda-1}(2\lambda_i) \geq 2i\). On the other hand, if \(j, k \in \mathbb{N}_0\) are such that \(j \leq k\) and \(\lambda_j + \lambda_k = 2\lambda_i\), then \(\lambda_j \leq \lambda_i\) and \(\lambda_k \geq \lambda_i\). So, \(\Lambda(\Lambda_{\lambda-1}(2\lambda_i)) \subseteq \{\lambda_j : 0 \leq j \leq i\} \cup \{2\lambda_i - \lambda_j : 0 \leq j < i\}\) and hence \(\nu_{\lambda-1}(2\lambda_i) \geq 2i\) if and only if \(\Lambda_{\lambda-1}(2\lambda_i) = \{j \in \mathbb{N}_0 : j \leq i\} \cup \{\lambda_i - \lambda_j : 0 \leq j < i\}\). This proves (i).

Finally, (i) implies \(\nu_{\lambda-1}(2\lambda_i) = 2i + 1\) and (ii) follows by an analogous argumentation.

Theorem 82. The only numerical semigroups for which the \(\nu\) sequence is non-decreasing are ordinary numerical semigroups.

Proof. It is a consequence of Lemma 81, Lemma 86, Theorem 70 and Lemma 80.

Corollary 83. The only numerical semigroup for which the \(\nu\) sequence is strictly increasing is the trivial numerical semigroup.

Proof. It is a consequence of Theorem 82 and Lemma 80.

As a consequence of Theorem 82, we can show that the only numerical semigroups for which the associated classical codes are not improved by the Feng-Rao improved codes, at least for one value of \(t\), are ordinary semigroups.

Corollary 84. Given a numerical semigroup \(\Lambda\) define \(m(\delta) = \max\{i \in \mathbb{N}_0 : \nu_i < \delta\}\). There exists at least one value of \(\delta\) for which \(\{i \in \mathbb{N}_0 : \nu_i < \delta\} \subset \{i \in \mathbb{N}_0 : i \leq \nu(\delta)\}\) if and only if \(\Lambda\) is non-ordinary.

2.3 Generic errors and the \(\tau\) sequence

All the results in these sections are based on [14, 15, 10]. Correction of generic errors has already been considered in [53, 50, 36].

2.3.1 Generic errors

Definition 85. The points \(P_1, \ldots, P_t\) (where \(P_j \neq P\)) are generically distributed if no non-zero function generated by \(z_0, \ldots, z_{t-1}\) vanishes in all of them. In the context of one-point codes, generic errors are those errors whose non-zero positions correspond to generically distributed points. Equivalently, \(e\) is generic if and only if \(\Delta_e = \Delta_t := \{0, \ldots, t - 1\}\).

Generic errors of weight \(t\) can be a very large proportion of all possible errors of weight \(t\) [33]. Thus, by restricting the errors to be corrected to generic errors the decoding requirements become weaker and we are still able to correct almost all errors. In some of these references, generic errors are called independent errors.
**Example 86 (Generic sets of points in $\mathcal{H}_q$).** Recall that the Hermitian curve from Example 84. It is defined over $\mathbb{F}_q^2$ and its affine equation is $x^{q+1} = y^q + y$.

The unique point at infinity is $P_\infty = (0 : 1 : 0)$. If $b \in \mathbb{F}_q$ then $b^q + b = Tr(b) = 0$ and the unique affine point with $y = b$ is $(0, b)$. There are a total of $q$ such points. If $b \in \mathbb{F}_q^2 \setminus \mathbb{F}_q$ then $b^q + b = Tr(b) \in \mathbb{F}_q \setminus \{0\}$ and there are $q + 1$ solutions of $x^{q+1} = b^q + b$, so, there are $q + 1$ different affine points with $y = b$. There are a total of $(q^2 - q)(q + 1)$ such points. The total number of affine points is then $q + (q^2 - q)(q + 1) = q^3$.

If we distinguish the point $P_\infty$, we can take $z_0 = 1, z_1 = x, z_2 = y, z_3 = x^2, z_4 = xy, z_5 = y^2, \ldots$.

Non-generic sets of two points are pairs of points satisfying $x^{q+1} = y^q + y$ and simultaneously vanish at $f = z_1 + a z_0 = x + a$ for some $a \in \mathbb{F}_q^2$. The expression $x + a$ represents a line with $q$ points. There are $q^2$ such lines. There are a total of $q^2 \binom{q}{2}$ pairs of colinear points over lines of the form $x + a$ and so $q^2 \binom{q}{2}$ non-generic errors.

Consequently, the portion of non-generic errors of weight 2 is

$$\frac{q^2 \binom{q}{2}}{\binom{q^2}{2}} = \frac{1}{q^2 + q + 1}.$$ Note that the Hermitian curve from $\mathcal{H}_q$.

A set of three points is non-generic if the points satisfy $x^{q+1} = y^q + y$ and simultaneously vanish at $f = z_1 + a z_0 = x + a$ for some $a \in \mathbb{F}_q^2$ or at $f = z_2 + a z_1 + b z_0 = y + a x + b$ for some $a, b \in \mathbb{F}_q^2$.

The expression $x + a$ represents a line (which we call of type 1) with $q$ points. There are $q^2$ lines of type 1.

The line $y + a x + b$ is called of type 2 if $a^{q+1} = b^q + b$ and of type 3 otherwise. There are $q^3$ lines of type 2 and $q^4 - q^3$ lines of type 3.

Lines of type 2 have only one point. Indeed, a point on $\mathcal{H}_q$ and on the line $y + a x + b$ must satisfy $x^{q+1} = (-a x - b)^q + (-a x - b) = -(a x)^q - a x - a^{q+1}$. Notice that $(x + a^q)^{q+1} = (x + a)^q (x + a^q) = (x^q + a)(x + a^q) = x^{q+1} + x^q a^q + a x + a^{q+1}$.

So, $x = -a^q$ is the unique solution to $x^{q+1} = -(a x)^q - a x - a^{q+1}$ and so the unique point of $\mathcal{H}_q$ on the line $y + a x + b$ is $(-a^q, a^{q+1} - b)$.

Lines of type 3 have $q + 1$ points. This follows by a counting argument. On one hand, as seen before, a point on $\mathcal{H}_q$ and on the line $y + a x + b$ must satisfy $x^{q+1} = -(a x)^q - a x - b^q - b$. There are at most $q + 1$ different values of $x$ satisfying this equation and so at most $q + 1$ different points of $\mathcal{H}_q$ on the line $y + a x + b$. On the other hand there are a total of $\binom{q}{2}$ pairs of affine points. Each pair meets only in one line. The number of pairs sharing lines of type 1 is $q^2 \binom{q}{2}$, the number of pairs sharing lines of type 2 is 0 and the number of pairs sharing lines of type 3 is at most $q^3(q - 1)\binom{q - 1}{2}$, with equality only if all lines of type 3 have $q + 1$ points. Since $q^2 \binom{q}{2} + q^3(q - 1)\binom{q - 1}{2} = \binom{q^3}{2}$, we deduce that all the lines of type 3 must have $q + 1$ points.

In total there are $q^2 \binom{q}{2}$ sets of three points sharing a line of type 1 and $(q^4 - q^3)\binom{q + 1}{3}$ sets of three points sharing a line of type 3.

The portion of non-generic errors of weight 3 is then

$$\frac{q^2 \binom{q}{2} + q^3(q - 1)\binom{q + 1}{3}}{\binom{q^3}{2}} = \frac{1}{q^2 + q + 1}.$$
2.3.2 Conditions for correcting generic errors

In the next lemma we find conditions guaranteeing the majority voting step for generic errors. It is a reformulation of results that appeared in [51, 4, 10].

**Lemma 87.** Let \( \Sigma_t = \mathbb{N}_0 \setminus \Delta_t = t + \mathbb{N}_0 \). The following conditions are equivalent.

1. \( \nu_k > 2 \#(N_k \cap \Delta_t) \),
2. \( k \in \Sigma_t \oplus \Sigma_t \),
3. \( \tau_k \geq t \).

**Proof.** Let \( A = \{ i \in N_k : i, k \oplus i \in \Delta_t \} \), \( D = \{ i \in N_k : i, k \oplus i \in \Sigma_t \} \). By an argument analogous to that in the proof of Lemma 64, \( \nu_k > 2 \#(N_k \cap \Delta_t) \) is equivalent to \( \#D > \#A \). If this inequality is satisfied then \( \#D > 0 \) and so \( k \in \Sigma_t \oplus \Sigma_t \). On the other hand, \( \min \Sigma_t \oplus \Sigma_t = t \oplus t > (t-1) \oplus (t-1) = \max \Delta_t \oplus \Delta_t \).

So, \( \Sigma_t \oplus \Sigma_t \cap \Delta_t \oplus \Delta_t = \emptyset \) and, if \( k \in \Sigma_t \oplus \Sigma_t \) then \( k \notin \Delta_t \oplus \Delta_t \) and so \( \#A = 0 \) implying \( \#D > \#A \).

The equivalence of \( k \in \Sigma_t \oplus \Sigma_t \) and \( \tau_k \geq t \) is straightforward. \( \square \)

The one-point codes whose set \( W \) of parity checks is selected so that the orders outside \( W \) satisfy the hypothesis of Lemma 87 and \( W \) is minimal with this property are called improved codes correcting generic errors. They were defined in [4, 14].

**Definition 88.** Given a rational point \( P \) of an algebraic smooth curve \( X_F \) defined over \( \mathbb{F}_q \) with Weierstrass semigroup \( \Lambda \) and sequence \( \nu \) with associated basis \( z_0, z_1, \ldots \) and given \( n \) other different points \( P_1, \ldots, P_n \) of \( X_F \), the associated improved code guaranteeing correction of \( t \) generic errors is defined as

\[
C_{\tilde{R}^*}(t) = \langle z_i(P_1), \ldots, z_i(P_n) : i \in \tilde{R}^*(t) \rangle^\perp,
\]

where \( \tilde{R}^*(t) = \{ i \in \mathbb{N}_0 : \tau_i < t \} \).

2.3.3 Comparison of improved codes and classical codes correcting generic errors

Classical evaluation codes are those codes for which the set of parity checks corresponds to all the elements up to a given order. Thus, the classical evaluation code with maximum dimension correcting \( t \) generic errors is defined by the set of checks \( R^*(t) = \{ i \in \mathbb{N}_0 : i \leq m(t) \} \) where \( m(t) = \max \{ i \in \mathbb{N}_0 : \tau_i < t \} \). Then, by studying the monotonicity of the \( \tau \) sequence we can compare \( \tilde{R}^*(t) \) and \( R^*(t) \) and the associated codes.

It is easy to check that for the trivial numerical semigroup one has \( \tau_{2i} = \tau_{2i+1} = i \) for all \( i \in \mathbb{N}_0 \). That is, the \( \tau \) sequence is

\[
0, 0, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, \ldots
\]

The next lemma determines the \( \tau \) sequence of all non-trivial ordinary semigroups.
Lemma 89. The non-trivial ordinary numerical semigroup with conductor \( c \) has \( \tau \) sequence given by

\[
\tau_i = \begin{cases} 
0 & \text{if } i \leq c \\
\lfloor i - c + 1 \rfloor & \text{if } i > c
\end{cases}
\]

Proof. Suppose that the numerical semigroup has enumeration \( \lambda \). On one hand, \( \lambda_1, \ldots, \lambda_\ell \) are all generators and thus \( \tau_i = 0 \) for \( i \leq c \). For \( i > c \), \( \lambda_i = c + i - 1 \geq 2c \). So, if \( \lambda_i \) is even (which is equivalent to \( c + i \) being odd) then

\[
\tau_i = \lambda^{-1}(\frac{1}{2}) = \frac{i+c+1}{2} - c + 1 = \lfloor \frac{i+c+1}{2} \rfloor. 
\]

If \( \lambda_i \) is odd (which is equivalent to either both \( c \) and \( i \) being even or being odd) then

\[
\tau_i = \lambda^{-1}(\frac{1}{2}) = \frac{i+c+1}{2} - c + 1 = \frac{i+c+2}{2} - c + 1 = \frac{i+c}{2} = \lfloor \frac{i+c}{2} \rfloor. 
\]

Remark 90. The formula in Lemma 89 can be reformulated as \( \tau_j = 0 \) for all \( j \leq c \) and, for all \( i \geq 0 \), \( \tau_{c+2i+1} = \tau_{c+2i+2} = i + 1 \).

The next lemma gives, for non-ordinary semigroups, the smallest index \( m \) for which \( \tau \) is non-decreasing from \( \tau_m \) on. We will use the notation \( \lfloor a \rfloor_\Lambda \) to denote the semigroup floor of a non-negative integer \( a \), that is, the largest non-gap of \( \Lambda \) which is at most \( a \).

Lemma 91. Let \( \Lambda \) be a non-ordinary semigroup with dominant \( d \) and let \( m = \lambda^{-1}(2d) \), then

1. \( \tau_m = c - g - 1 > \tau_{m+1} \),
2. \( \tau_i < c - g - 1 \) for all \( i < m \),
3. \( \tau_i \leq \tau_{i+1} \) for all \( i > m \).

Proof. For statement 1 notice that both \( 2d \) and \( 2d + 1 \) belong to \( \Lambda \) because they must be larger than the conductor. Furthermore, \( \tau_{\lambda^{-1}(2d)} = \lambda^{-1}(d) = c - g - 1 \) while \( \tau_{\lambda^{-1}(2d+1)} = \tau_{\lambda^{-1}(2d)+1} < \lambda^{-1}(d) \) because \( d + 1 \notin \Lambda \).

Statement 2 follows from the fact that if \( \lambda_i < 2d \) then \( \tau_i < \lambda^{-1}(d) = c - g - 1 \).

For statement 3 suppose that \( i > m \). Notice that \( 2d \) is the largest non-gap that can be written as a sum of two non-gaps both of them smaller than the conductor \( c \). Then if \( j \leq k \leq i \) and \( \lambda_j + \lambda_k = \lambda_i \) it must be \( \lambda_k \geq c \) and so \( \tau_i = \lambda^{-1}([\lambda_i - c]_\Lambda) \). Since both \( \lambda^{-1} \) and \( [\cdot]_\Lambda \) are non-decreasing, so is \( \tau \), for \( i > m \).

Corollary 92. The only numerical semigroups for which the \( \tau \) sequence is non-decreasing are ordinary semigroups.

A direct consequence of Corollary 92 is that the classical code determined by \( R^*(t) \) is always worse than the improved code determined by \( \tilde{R}^*(t) \) at least for one value of \( t \) unless the corresponding numerical semigroup is ordinary. From Lemma 91 we can derive that \( \tilde{R}^*(t) \) and \( R^*(t) \) coincide from a certain point and we can find this point. We summarize the results of this section in the next Corollary.
Corollary 93.  
1. $\widetilde{R}^*(t) \subseteq R^*(t)$ for all $t \in \mathbb{N}_0$.

2. $\widetilde{R}^*(t) = R^*(t)$ for all $t \geq c - g$.

3. $\widetilde{R}^*(t) = R^*(t)$ for all $t \in \mathbb{N}_0$ if and only if the associated numerical semigroup is ordinary.

Proof. Statement 1 is a consequence of the definition of $R^*(t)$. Statement 2 is clear if the associated semigroup is ordinary. Otherwise it follows from the fact proved in Lemma 91 that the largest value of $\tau_i$ before it starts being non-decreasing is precisely $c - g - 1$ and that before that all values of $\tau_i$ are smaller than $c - g - 1$. Statement 3 is a consequence of Corollary 92.

2.3.4 Comparison of improved codes correcting generic errors and Feng–Rao improved codes

In next theorem we compare $\tau_i$ with $\lfloor \frac{\nu_i - 1}{2} \rfloor$ and this will give a new characterization of Arf semigroups. Recall that $t \leq \lfloor \frac{\nu_i - 1}{2} \rfloor$ guarantees the computation of syndromes of order $i$ when performing majority voting (Lemma 64).

Theorem 94. Let $\Lambda$ be a numerical semigroup with conductor $c$, genus $g$, and associated sequences $\tau$ and $\nu$. Then

1. $\tau_i \geq \lfloor \frac{\nu_i - 1}{2} \rfloor$ for all $i \in \mathbb{N}_0$.

2. $\tau_i = \lfloor \frac{\nu_i - 1}{2} \rfloor$ for all $i \geq 2c - g - 1$.

3. $\tau_i = \lfloor \frac{\nu_i - 1}{2} \rfloor$ for all $i \in \mathbb{N}_0$ if and only if $\Lambda$ is Arf.

Proof. Let $\lambda$ be the enumeration of $\Lambda$.

1. Suppose that the elements in $N_i$ are ordered $N_{i,0} < N_{i,1} < N_{i,2} < \cdots < N_{i,\nu_i - 1}$. On one hand $\tau_i = N_{i,\lfloor \frac{\nu_i - 1}{2} \rfloor}$. On the other hand $N_{i,j} \geq j$ and this finishes the proof of the first statement.

2. The result is obvious for the trivial semigroup. Thus we can assume that $c \geq g + 1$. Notice that $\tau_i = N_{i,\lfloor \frac{\nu_i - 1}{2} \rfloor} = \lfloor \frac{\nu_i - 1}{2} \rfloor$ if and only if all integers less than or equal to $\lfloor \frac{\nu_i - 1}{2} \rfloor$ belong to $N_i$. Now let us prove that if $i \geq 2c - g - 1$ then all integers less than or equal to $\lfloor \frac{\nu_i - 1}{2} \rfloor$ belong to $N_i$. Indeed, if $j \leq \lfloor \frac{\nu_i - 1}{2} \rfloor$ then $\lambda_j \leq \lambda_i/2$ and $\lambda_i - \lambda_j \geq \lambda_i - \lambda_i/2 = \lambda_i/2 \geq c - 1/2$. Since $\lambda_i - \lambda_j \in \mathbb{N}_0$ this means that $\lambda_i - \lambda_j \geq c$ and so $\lambda_i - \lambda_j \in \Lambda$.

3. Suppose that $\Lambda$ is Arf. We want to show that for any non-negative integer $i$, all non-negative integers less than or equal to $\lfloor \frac{\nu_i - 1}{2} \rfloor$ belong to $N_i$. By definition of $\tau_i$ there exists $k$ with $\tau_i \leq k \leq i$ and $\lambda_i + \lambda_k = \lambda_i$. Now, if $j$ is a non-negative integer with $j \leq \lfloor \frac{\nu_i - 1}{2} \rfloor$, by statement 1 it also satisfies $j < \tau_i$. Then $\lambda_i - \lambda_j = \lambda_i + \lambda_k - \lambda_j \in \Lambda$ by the Arf property, and so $j \in N_i$. 

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On the other hand, suppose that \( \tau_i = \left\lfloor \frac{\nu_i - 1}{2} \right\rfloor \) for all non-negative integer \( i \). This means that all integers less than or equal to \( \tau_r \) belong to \( N_r \) for any non-negative integer \( r \). If \( i \geq j \geq k \) then \( \tau_{\lambda^{-1}(\lambda_i + \lambda_j)} \geq j \geq k \) and by hypothesis \( k \in N_{\lambda^{-1}(\lambda_i + \lambda_j)} \), which means that \( \lambda_i + \lambda_j - \lambda_k \in \Lambda \). This implies that \( \Lambda \) is Arf.

\[
\text{Statement 1) of Lemma } 58 \text{ for the case when } i > 0 \text{ is a direct consequence of Theorem 94 and Lemma 51.}
\]

Finally, Theorem 94 together with Lemma 58 has the next corollary. Different versions of this result appeared in 

\[
\begin{align*}
1. \quad \tilde{R}(t) & \subseteq \tilde{R}^*(t) \quad \text{for all } t \in \mathbb{N}_0. \\
2. \quad \tilde{R}^*(t) = \tilde{R}(t) \quad \text{for all } t \geq c - g. \\
3. \quad \tilde{R}^*(t) = \tilde{R}(t) \quad \text{for all } t \in \mathbb{N}_0 \text{ if and only if the associated numerical semigroup is Arf.}
\end{align*}
\]

\[
\text{Proof. Statement 1.) and 3.) follow immediately from Theorem 94 and the fact that } \tilde{R}(t) = \{ i \in \mathbb{N}_0 : \left\lfloor \frac{\nu_i - 1}{2} \right\rfloor < t \} \text{ and } \tilde{R}^*(t) = \{ i \in \mathbb{N}_0 : \tau_i < t \}. \text{ For statement 2.) we can use that for } i \geq 2c - g - 1, \tau_i = \left\lfloor \frac{\nu_i - 1}{2} \right\rfloor \text{ (Theorem 24) and that for } i \geq 2c - g - 1, \tau_i \geq c - g - 1 \text{ (Lemma 58), being } c - g - 1 \text{ the largest value of } \tau_j \text{ before it starts being non-decreasing (Lemma 31).}
\]

\[
\text{Further reading}
\]

We tried to cite the specific bibliography related to each section within the text. Next we mention some more general references: The book [59] has many results on numerical semigroups, including some of the problems presented in the first section of this chapter but also many others. The book [57] is also devoted to numerical semigroups from the perspective of the Frobenius’ coin exchange problem. Algebraic geometry codes have been widely explained in different books such as [72, 67, 56]. For one-point codes and also their relation with Weierstrass semigroups an important reference is the chapter [35].

\[
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\]

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