Normal Coordinates in Kähler Manifolds and the Background Field Method

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Abstract

Riemann normal coordinates (RNC) are unsuitable for Kähler manifolds since they are not holomorphic. Instead, Kähler normal coordinates (KNC) can be defined as holomorphic coordinates. We prove that KNC transform as a holomorphic tangent vector under holomorphic coordinate transformations, and therefore that they are natural extensions of RNC to the case of Kähler manifolds. The KNC expansion provides a manifestly covariant background field method preserving the complex structure in supersymmetric nonlinear sigma models.

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1 Introduction

The equivalence principle asserts that general coordinate transformations on curved space-times do not alter any physics, so that one can consider the coordinates that make a given application the simplest. Riemann normal coordinates (RNC) represent one such set of coordinates for Riemann manifolds [1, 2, 3]. They are defined as coordinates along geodesic lines starting from a chosen point. Hence, any point in a patch of RNC has one-to-one correspondence with a tangent vector at the chosen point.

In most superstring theories, extra dimensions of the higher-dimensional space-time are compactified to a Calabi-Yau manifold [4], which is a Ricci-flat Kähler manifold. This can be described by conformally invariant supersymmetric nonlinear sigma models in two dimensions, whose target spaces are Kähler manifolds [5]. For perturbative (or non-perturbative) analyses, we need to expand the Lagrangian in terms of fluctuating fields around the background fields [6]. A generally covariant expansion that preserves the complex structure of the target space is most suitable in these analyses. RNC provide a generally covariant expansion, but they are not holomorphic, whereas Kähler normal coordinates (KNC) give us such an expansion [7]. KNC are defined as coordinates satisfying some gauge conditions on the derivatives of the metric without recourse to geodesics [8].

In this paper, we prove that KNC transform as a holomorphic tangent vector, and therefore that they are a natural extension of RNC to the case of Kähler manifolds. The KNC expansion of the Lagrangian is a manifestly covariant expansion under holomorphic coordinate transformations of the target space. The relation between RNC and KNC is also shown: we find that they differ by terms proportional to the curvature tensor and its covariant derivatives, and hence they coincide only in flat space. We also give the KNC expansion of tensor fields that can be applied to the KNC expansion of the Lagrangian with a potential term or a higher order derivative term.

This paper is organized as follows. In §2, after a short review of the RNC expansion in Riemann manifolds, we show that RNC in Kähler manifolds are not holomor-
In §3, after recalling the definition of KNC, we discuss their basic properties. We then state a theorem elucidating the geometrical interpretation of KNC. The KNC expansion of tensor fields is also given. In §4, we apply KNC to the background field method in the supersymmetric nonlinear sigma models on Kähler manifolds. Some applications of the KNC expansion (the Wilsonian renormalization group, low energy theorems of Nambu-Goldstone bosons in four dimensions, and the effective field theory on a domain wall solution) are discussed in §5. In Appendix A, we summarize the geometry of Kähler manifolds. The relation between RNC and KNC is discussed in detail in Appendix B. We give a proof of the theorem presented in §3 in Appendix C.

2 Riemann Normal Coordinates

First, to compare with Kähler manifolds, we recall some properties of RNC in Riemann manifolds, following Ref. [1]. Then, we discuss the RNC in Kähler manifolds. It is observed that RNC are not holomorphic coordinates in Kähler manifolds. Therefore they are not suitable in cases of Kähler manifolds.

2.1 Riemann Normal Coordinates in Riemann Manifolds

Let \( \{x^A\} \) be the coordinates of a Riemann manifold \( M \) \((A = 1, \cdots, \dim M)\). To define RNC, we choose an expansion point \( \varphi^A \) and consider a geodesic \( \lambda^A(t) \) starting from this point, with \( t \) being an affine parameter \((0 \leq t \leq 1)\). We consider the endpoint \( \lambda^A(1) \) as a general point \( \varphi^A + \pi^A \) in the manifold. The geodesic equation in a Riemann manifold can be written

\[
\ddot{\lambda}^A(t) + \Gamma^A_{BC}(\lambda)\dot{\lambda}^B(t)\dot{\lambda}^C(t) = 0,
\]

where the dot denotes differentiation with respect to \( t \), and \( \Gamma^A_{BC} \) is the connection. The geodesic may be expanded in powers of the affine parameter according to

\[
\lambda^A(t) = \sum_{N=0}^{\infty} \frac{1}{N!}\lambda^{A(N)}(0)t^N,
\]
where $\lambda^{A(N)}$ is the $N$-th derivative of the geodesic with the initial condition
\[
\dot{\lambda}^A(0) \equiv \xi^A,
\] (2.3)
where $\xi^A$ is a tangent vector at the point $\varphi^A$. Here, $\xi^A$ is actually tangent to the geodesic. Recursive use of the geodesic equation (2.1) gives the relations
\[
\lambda^{A(N)}(t) = -\Gamma^A_{B_1B_2\ldots B_N}(\lambda)\dot{\lambda}^{B_1}(t)\dot{\lambda}^{B_2}(t)\cdots\dot{\lambda}^{B_N}(t).
\] (2.4)
Here, the coefficients $\Gamma^A_{B_1B_2\ldots B_N}$ are defined by
\[
\Gamma^A_{B_1B_2\ldots B_N} = \nabla^B_1\nabla^B_2\cdots\nabla^B_N - \frac{2}{N!}\Gamma^A_{B_1B_2\ldots B_N} \partial^B_1 \xi^{B_2} \cdots \xi^{B_N}.
\] (2.5)
where $\nabla$ denotes the covariant derivative acting on only lower indices of the connection. For instance,
\[
\Gamma^A_{B_1B_2B_3} = \nabla^B_1 \Gamma^A_{B_2B_3} = \partial^B_1 \Gamma^A_{B_2B_3} - \Gamma^C_{B_1B_3} \Gamma^A_{CB_2} - \Gamma^C_{B_1B_2} \Gamma^A_{CB_3}.
\] (2.6)
We thus obtain the coefficients in (2.2) as
\[
\lambda^A(t) = \varphi^A + \xi^A t - \sum_{N=2}^\infty \frac{1}{N!} \Gamma^A_{B_1B_2\ldots B_N} |_{\varphi} \xi^{B_1} \xi^{B_2} \cdots \xi^{B_N} t^N,
\] (2.7)
where the index $\varphi$ indicates quantities evaluated at the initial expansion point $\varphi^A$. Since the endpoint of the geodesics is $\varphi^A + \pi^A = \lambda^A(1)$, we have
\[
\pi^A = \xi^A - \sum_{N=2}^\infty \frac{1}{N!} \Gamma^A_{B_1B_2\ldots B_N} |_{\varphi} \xi^{B_1} \xi^{B_2} \cdots \xi^{B_N}.
\] (2.8)
This can be regarded as a coordinate transformation, and the RNC are defined by inverting this equation to obtain $\xi^A$ as a function of $\pi$. Therefore there is one-to-one correspondence between a tangent vector in the tangent space at $\varphi$ and a point in a patch of RNC around $\varphi$.

Now, let us discuss the properties of RNC. Since any geodesic can be written as $\lambda^A(t) = \xi^A t$ in RNC\(^1\), the expansion of RNC themselves in term of the tangent vector $\xi^A$ gives the relations
\[
\bar{\Gamma}^A_{(B_1B_2\ldots B_N)} |_{\varphi} = 0,
\] (2.9)
\(^1\) The degrees of freedom in general coordinate transformations preserving $x^A = \varphi^A$, $\pi^A = x^A - \varphi^A = c\pi^A + \sum_{N=2}^\infty c^A_{B_1\ldots B_N} \pi^{B_1} \cdots \pi^{B_N}$ coincide with the number of coefficients $\Gamma^A_{B_1\ldots B_N}$ at (2.7) of each order. Hence, there exist coordinates in which any geodesic from the origin becomes a straight line. These coordinates are KNC.
where the bar indicates quantities in RNC and the parentheses indicate the symmetrization with respect to indices:

$$T_{(A_1 A_2 \cdots A_N)} = \frac{1}{N!} (T_{A_1 A_2 \cdots A_N} + T_{A_2 A_1 \cdots A_N} + \cdots).$$

These conditions for RNC are equivalent to

$$\partial (B_i \partial B_2 \cdots \partial B_{N-2} \Gamma^A_{B_{N-1} B_N}) |_{\phi} = 0 \quad (2.10).$$

In RNC, any tensor can be expanded in terms of a tangent vector $\xi^A$ easily, using the identity (2.9) or (2.10). For instance, the metric tensor $g_{AB}$ can be expanded as

$$g_{AB}(x) = g_{AB}|_{\phi} - \frac{1}{3} R_{ACBD}|_{\phi} \xi^C \xi^D - \frac{1}{3!} D E R_{ACBD}|_{\phi} \xi^C \xi^D \xi^E$$

$$- \frac{1}{5!} \left( 6 D C D D R_{AEBF} - \frac{16}{3} R_{CBD G} R_{EAF G} \right) |_{\phi} \xi^C \xi^D \xi^E \xi^F + O(\xi^5), \quad (2.11)$$

in which no bars appear, because once we obtain an expansion of a tensor in RNC, it can be regarded as an expansion in terms of a tangent vector. Hence it is a tensor equation and holds in any coordinate system.

The RNC expansion can be applied to the background field method of nonlinear sigma models [1, 2]. (For another derivation of RNC, see Ref. [3].)

### 2.2 Riemann Normal Coordinates in Kähler Manifolds

As in Riemann manifolds considered above, we now consider RNC in a Kähler manifold $M$. Let $\{ z^i, z^* i \}$ be coordinates in the Kähler manifold ($i = 1, \cdots, \dim_{\mathbb{C}} M$).

We consider a geodesic $\lambda^i(t)$ with affine parameter $t$ (0 ≤ $t$ ≤ 1), starting at a point $\lambda^i(0) = \varphi^i$ and ending at a point $\lambda^i(1) = \varphi^i + \pi^i$. The geodesic equation in a Kähler manifold is given by

$$\ddot{\lambda}^i(t) + \Gamma^i_{jk}(\lambda, \lambda^*) \dot{\lambda}^j(t) \dot{\lambda}^k(t) = 0, \quad (2.12)$$

and its complex conjugate, where the dot denotes differentiation with respect to $t$.

In the same way, we can obtain the expansion of $\lambda^i(t)$ in terms of the tangent vectors $\xi^i$ and $\xi^*_i$. The first few orders are given by

$$\lambda^i(t) = \varphi^i + \xi^i t - \frac{1}{2} \Gamma^i_{ji,j2} |_{\phi} \xi^j \xi^j t^2 - \frac{1}{3!} \Gamma^i_{ji,j2,j3} |_{\phi} \xi^j \xi^j \xi^j t^3$$

$$- \frac{1}{3!} R^i_{ji,j2,j3} |_{\phi} \xi^j \xi^j \xi^j \xi^j t^3 + O(t^4), \quad (2.13)$$
in which $\Gamma^i_{j_1j_2j_3}$ is defined by

$$
\Gamma^i_{j_1j_2j_3} \equiv \partial_{j_1} \Gamma^i_{j_2j_3} - \Gamma^i_{j_1j_2} \Gamma^i_{j_3} - \Gamma^i_{j_1j_3} \Gamma^i_{j_2j_3} \equiv \nabla_{j_1} \Gamma^i_{j_2j_3}.
$$

This is the restriction of (2.6) to holomorphic indices. The expansion (2.13) can be obtained from the expansion (2.7) in Riemann manifolds by identifying real coordinates with the holomorphic and anti-holomorphic coordinates as $\{x^A\} = \{z^i, z^{*i}\}$.

The endpoint $\varphi^i + \pi^i = \lambda^i(1)$ of the geodesic can be expressed by

$$
\pi^i = \xi^i - \frac{1}{2} \Gamma^i_{j_1j_2} |_{\varphi} \xi^{j_1} \xi^{j_2} - \frac{1}{3!} \Gamma^i_{j_1j_2j_3} |_{\varphi} \xi^{j_1} \xi^{j_2} \xi^{j_3} - \frac{1}{3!} R^i_{j_1k_1j_2} |_{\varphi} \xi^{j_1} \xi^{j_2} \xi^{k_1} + O(\xi^4).
$$

The RNC obtained by inverting this equation depend on both $\pi$ and $\pi^*$: $\xi^i = \xi^i(\pi, \pi^*)$. Hence, the coordinate transformation from the holomorphic coordinates $z^i$ to the RNC $\xi^i$ is not holomorphic. It is thus seen that *Riemann normal coordinates are generally not holomorphic*. Such non-holomorphic terms in the transformation (2.15) appear in conjunction with covariant tensors like the curvature tensor $R^i_{j_1k_1j_2}$. This is very different from the case of Riemann manifolds.

In summary, we have the following:

1. The transformation (2.13) can be directly obtained from the transformation (2.8) in Riemann manifolds with the identification of coordinates $\{x^A\} = \{z^i, z^{*i}\}$,

2. All non-holomorphic terms in (2.13) appear with coefficients of covariant tensors, and therefore they exist in general, unless the Kähler manifold is flat. (See comments in §3.1 and discussion in Appendix B.)

### 3 Kähler Normal Coordinates

As shown in the last section RNC are inappropriate for Kähler manifolds, since they are not holomorphic. KNC are normal coordinates that are holomorphic. In this section, after recalling the definition of KNC and giving some discussions in the first subsection, we present a theorem that clarifies the geometric properties of KNC in
the second subsection. A proof of this theorem is given in Appendix C. The KNC expansion of tensor fields is also given in the last subsection.

### 3.1 Definition of Kähler Normal Coordinates

Let $K(z, z^*)$ be the Kähler potential so that $g_{ij^*}(z, z^*) = K_{,i_{j^*}}(z, z^*)$, where the comma denotes partial differentiation with respect to the coordinates. Then, decompose the coordinate $z^i$ into an expansion point $\varphi^i$ and a deviation $\pi^i$ from it: $z^i = \varphi^i + \pi^i$. We define the KNC $\{\omega^i, \omega^{*i}\}$, whose origin coincides with the expansion point $z^i = \varphi^i$, as coordinates such that the quantities $\hat{K}_{,i_{j^*}i_{j^*}...i_N}(\omega, \omega^*)|_0 = \hat{g}_{i_{j^*}i_{j^*}...i_N}(\omega, \omega^*)|_0 = 0$, 

\[
\hat{K}_{,i_{j^*}i_{j^*}...i_N}(\omega, \omega^*)|_0 = \hat{g}_{i_{j^*}i_{j^*}...i_N}(\omega, \omega^*)|_0 = 0 ,
\]

where the hat indicates quantities in KNC, and the index “0” indicates a value evaluated at the origin of KNC, $\omega^i = 0$. These conditions are equivalent to

\[
\partial_{i_1} \cdots \partial_{i_{N-2}} \hat{K}_{,i_{N-1}i_N}(\omega, \omega^*)|_0 = 0 ,
\]

which are similar to the conditions (2.10) for RNC in Riemann manifolds, except for symmetrization with respect to indices. The given coordinates $z^i$ (or $\pi^i$) can be transformed to such KNC by the holomorphic coordinate transformation $[7]

\[
\omega^i = \pi^i + \sum_{N=2}^{\infty} \frac{1}{N!} [g^{ij^*} K_{,i_{j^*}i_{j^*}...i_N}(z, z^*)]_{\varphi^i} \pi^1 \cdots \pi^N ,
\]

where the index $\varphi$ indicates that the quantity in question is evaluated at the expansion point, $z^i = \varphi^i$, of the original coordinates.

In KNC, using (3.1), the Kähler potential can be expanded as

\[
\hat{K}(\omega, \omega^*) = \hat{K}|_0 + \hat{F}(\omega) + \hat{F}^*(\omega^*) + \hat{g}_{ij^*}|_0 \omega^i \omega^{*j} + \sum_{M,N \geq 2} \frac{1}{M!N!} \hat{K}_{,i_{i_i}i_{j^*}i_{j^*}...i_{j_N}}|_0 \omega^i \omega^{*j} \cdots \omega^{*j_N} ,
\]

where $\hat{F}(\omega)$ is a holomorphic function of $\omega$, so that it can be eliminated by a Kähler transformation. It has been shown that all coefficients of the expansion (3.4) are
covariant tensors \[\]. For instance, the fourth order coefficient is \(\hat{K}_{i_1 i_2 j_1 j_2} |_0 = \hat{R}_{i_1 i_2 j_1 j_2} |_0\). An explicit expression of the coefficients in terms of the curvature tensor and its covariant derivatives up to sixth order is given in Ref. [7] by

\[
\hat{K}(\omega, \omega^*) = \hat{K} |_0 + \hat{F}(\omega) + \hat{F}^*(\omega^*) + \hat{g}_{i j^*} |_0 \omega^i \omega^{j^*} + \frac{1}{4} \hat{R}_{i j^* k l^*} |_0 \omega^i \omega^{j^*} \omega^k \omega^{l^*} \\
+ \frac{1}{12} \hat{D}_m \hat{R}_{i j^* k l^*} |_0 \omega^m \omega^i \omega^k \omega^{j^*} \omega^{l^*} + \frac{1}{12} \hat{D}_m \hat{R}_{i j^* k l^*} |_0 \omega^i \omega^k \omega^j \omega^{l^*} \omega^m \omega^{n} \\
+ \frac{1}{24} \hat{D}_n \hat{D}_m \hat{R}_{i j^* k l^*} |_0 \omega^n \omega^m \omega^i \omega^k \omega^{j^*} \omega^{l^*} + \frac{1}{12} \hat{D}_n \hat{D}_m \hat{R}_{i j^* k l^*} |_0 \omega^i \omega^k \omega^j \omega^{l^*} \omega^m \omega^{n} \\
+ \frac{1}{36} \left( \hat{D}_n \hat{D}_m \hat{R}_{i j^* k l^*} + 3 \hat{g}^{* m r^*} \hat{R}_{o(j^* m l^* k) r^*} \right) |_0 \omega^m \omega^i \omega^k \omega^{j^*} \omega^{l^*} \omega^{n} + O(\omega^7), \tag{3.5}
\]

where \(O(\omega^n)\) denotes terms of the order \(n\) in \(\omega\) and \(\omega^*\). Here, the parentheses enclosing indices indicate symmetrization with respect to the holomorphic and anti-holomorphic indices, respectively, e.g. \(T_{(i_1 i_2 \ldots i_M j_1 j_2 \ldots j_N)} \equiv T_{(i_1 i_2 \ldots i_M j_1 j_2 \ldots j_N)} \equiv \frac{1}{N! M!} (T_{(i_1 i_2 \ldots i_M j_1 j_2 \ldots j_N)} + T_{(i_2 i_1 \ldots i_M j_1 j_2 \ldots j_N)} + T_{(i_1 i_2 \ldots i_M j_2 j_1 \ldots j_N)} + \cdots)\). \[2\]

From the expansion of the Kähler potential \(\hat{K}_{i j^*}(\omega, \omega^*)\) we can calculate the KNC expansion of the metric tensor through fourth order, obtaining

\[
\hat{g}_{i j^*}(\omega, \omega^*) = \hat{g}_{i j^*} |_0 + \hat{R}_{i j^* k l^*} |_0 \omega^k \omega^{l^*} + \frac{1}{2} \hat{D}_m \hat{R}_{i j^* k l^*} |_0 \omega^m \omega^k \omega^{l^*} + \frac{1}{2} \hat{D}_m \hat{R}_{i j^* k l^*} |_0 \omega^i \omega^k \omega^{j^*} \omega^{l^*} \\
+ \frac{1}{6} \hat{D}_n \hat{D}_m \hat{R}_{i j^* k l^*} |_0 \omega^n \omega^m \omega^i \omega^k \omega^{j^*} \omega^{l^*} + \frac{1}{6} \hat{D}_n \hat{D}_m \hat{R}_{i j^* k l^*} |_0 \omega^i \omega^k \omega^j \omega^{l^*} \omega^m \omega^{n} \\
+ \frac{1}{4} \left( \hat{D}_n \hat{D}_m \hat{R}_{i j^* k l^*} + 3 \hat{g}^{* m r^*} \hat{R}_{o(j^* m l^* k) r^*} \right) |_0 \omega^m \omega^i \omega^k \omega^{j^*} \omega^{l^*} \omega^{n} + O(\omega^5). \tag{3.6}
\]

Comparing this result with the metric expansion in RNC \((2.11)\), it is seen that the coefficients in both expansions are quite different. The relation between KNC and RNC expansions is discussed in Appendix \[3\].

\[2\] It should be noted that we use notation that differs from that of Ref. [7], in which we used parentheses to indicate cyclic permutation without any numerical factor.

\[3\] All tensors in this expansion are symmetric in (anti-)holomorphic indices. We do not need the symmetrization, except for the last term, due to the identities summarized in Appendix [A]. KNC are coordinates for which these identities become manifest.
Let us now consider the inverse transformation of (3.3) in order to compare with the transformation laws (2.8) and (2.15) for RNC in Riemann and Kähler manifolds, respectively. We can show that the inverse of the transformation (3.3) is given by

$$\pi^i = \omega^i - \sum_{N=2}^{\infty} \frac{1}{N!} \Gamma^i_{j_1 j_2 \cdots j_N} |_{\phi} \omega^{j_1} \omega^{j_2} \cdots \omega^{j_N}.$$  \hspace{1cm} (3.7)

Here, $\Gamma^i_{j_1 j_2 \cdots j_N}$ is defined by

$$\Gamma^i_{j_1 j_2 \cdots j_N} = \nabla_{j_1} \nabla_{j_2} \cdots \nabla_{j_N-2} \Gamma^i_{j_{N-1} j_N},$$ \hspace{1cm} (3.8)

in which $\nabla$ is the covariant derivative acting on the lower indices. Equation (3.7) can be understood as follows: if we take $\pi^i$ to be $\omega^i$, we obtain the conditions $\Gamma^i_{(j_1 j_2 \cdots j_N)} |_{\phi} = 0$, which are actually equivalent to the condition (3.1) or (3.2) for KNC.

We summarize this subsection as follows:

1. The transformation law (3.7) coincides with the restriction of (2.8) to holomorphic indices. In other words, only the differences between the transformation laws (2.15) for RNC and (3.7) for KNC in Kähler manifolds consist of non-holomorphic terms associated with covariant tensors. (See also the comments in §2.2 and Eq. (B.2) in Appendix B.)

2. Using a holomorphic coordinate transformation, any coordinates can be transformed into KNC, because the freedom expressed by (3.7) and (3.3) coincide. However, we cannot set $\Gamma^A_{(B_1 \cdots B_n)} |_{\phi} = 0$ with any holomorphic coordinate transformation. This is the reason that geodesics are not straight lines in KNC. (Compare this with footnote 1 concerning the case of RNC.)

Before closing this section, we give an example of KNC.

Example: A simple example of KNC is given by the standard coordinates in the Fubini-Study metric of $\mathbb{C}P^1$. Let $z$ be a holomorphic coordinate. Then the Kähler potential can be written as

$$K(z, z^*) = \log(1 + |z|^2).$$ \hspace{1cm} (3.9)
By the equation \( \partial_z \partial_{z^*} N K = \frac{(-1)^N N!}{(1 + |z|^2)^{N+1}} \), the condition (3.1) holds, and therefore \( z \) is a KNC. Geodesics in KNC, \( z(t) = \frac{\xi}{|\xi|} \tan(|\xi|t) = \xi t + \frac{1}{3} |\xi|^2 \xi t^3 + \cdots \), in which \( \xi \) is a tangent vector of the geodesic, are not linear in \( t \).

3.2 The Transformation Law of Kähler Normal Coordinates

RNC in a Riemann manifold are defined by a tangent vector at the origin. However, the geometric properties of KNC are unclear, since KNC are not defined by geodesics. The following theorem clarifies the geometric meaning of KNC.

**Theorem**

KNC transform like a holomorphic tangent vector at the origin of KNC, i.e.

\[
\omega^i \rightarrow \omega'^i = \frac{\partial z'^i}{\partial z^j} \omega^j, \tag{3.10}
\]

under holomorphic coordinate transformations preserving \( z^i = \varphi^i \) given by \( \pi^i \rightarrow \pi'^i = \pi'^i(\pi) = c^i_{j_1} \pi^{j_1} + c^i_{j_1 j_2} \pi^{j_1} \pi^{j_2} + \cdots \).

A proof of this theorem is given in Appendix \( \text{C} \).

This situation is quite different from that for RNC, because RNC transform like a tangent vector, but they are not holomorphic in Kähler manifolds. From this theorem, we find that there is one-to-one correspondence in the vicinity of the origin between a point represented by KNC and a holomorphic tangent vector at the origin. Therefore, KNC are a quite natural extension of RNC to the case of a Kähler manifold.

We can regard the expansion (3.5) as an expansion in terms of a holomorphic tangent vector. Hence (3.5) is a tensor equation and holds for any holomorphic coordinates \( z^i \) because of the transformation law (3.10). The expansion of the Kähler potential around \( z^i = \varphi^i \) is given by

\[
K(z, z^*) = K|_{\varphi} + F(\omega) + F^*(\omega^*) + g_{ij*} |_{\varphi} \omega^i \omega^*^j + \frac{1}{4} R_{ijkl*} |_{\varphi} \omega^i \omega^k \omega^*^j \omega^*^l + \cdots. \tag{3.11}
\]

Note that \( z^i = \varphi^i \) represents the same point in the manifold as \( \omega^i = 0 \) in KNC.
3.3 The Kähler Normal Coordinate Expansion of Tensor Fields

In this subsection we discuss the covariant expansion of a tensor field using KNC. Any tensor $T_{i_1...i_n}(z, z^*)$ can be expanded easily in KNC as

$$\hat{T}_{i_1...i_n}(\omega, \omega^*) = \sum_{M,N=0}^{\infty} \frac{1}{M!N!} \hat{T}_{i_1...i_n, k_1...k_Ml_1...l_N}^* |o^{k_1} \omega^k \ldots |o^{k_M} \omega^{*k_l} \ldots |o^{*l_N} , \quad (3.12)$$

where the hat indicates quantities in KNC. All coefficients are tensors in general holomorphic coordinates.

As an example, a vector with the holomorphic index $T_i(z, z^*)$ can be expanded as

$$\hat{T}_i(\omega, \omega^*) = \hat{T}_i|0 + \hat{T}_{i,j}|0 \omega^j + \hat{T}_{i,k^*}|0 \omega^{*k} + \frac{1}{2} \hat{T}_{i,j, j_2}|0 \omega^{j_1} \omega^{j_2} + \frac{1}{2} \hat{T}_{i,k_1^* k_2^*}|0 \omega^{*k_1} \omega^{*k_2} + \hat{T}_{i, k_1^* j_1}|0 \omega^{j_1} \omega^{k_1^*} + O(\omega^3) . \quad (3.13)$$

Using (3.2), each coefficient can be rewritten as a covariant tensor in KNC as

$$\hat{T}_{i,j}|0 = \hat{D}_j \hat{T}_i|0 , \quad \hat{T}_{i,k^*}|0 = \hat{D}_{k^*} \hat{T}_i|0 ,$$
$$\hat{T}_{i,j_1 j_2}|0 = \hat{D}_j \hat{D}_{j_2} \hat{T}_i|0 , \quad \hat{T}_{i,k_1^* k_2^*}|0 = \hat{D}_{k_1^*} \hat{D}_{k_2^*} \hat{T}_i|0 ,$$
$$\hat{T}_{i, j_1 k_1^*}|0 = \hat{D}_{j_1} \hat{D}_{k_1^*} \hat{T}_i|0 = \hat{D}_{j_1} \hat{D}_{k_1^*} \hat{T}_i|0 + \hat{R}_{j_1 j_1^*} \hat{T}_i|0 . \quad (3.14)$$

(Note that covariant expressions are not unique in general, as seen in the last equation.) Hence the expansion of the tensor $T_i$ in terms of general holomorphic coordinates can be obtained as

$$T_i(z, z^*) = T_i|\varphi + D_j T_i|\varphi \omega^j + D_{k^*} T_i|\varphi \omega^{*k} + \frac{1}{2} D_{j_1} D_{j_2} T_i|\varphi \omega^{j_1} \omega^{j_2} + \frac{1}{2} D_{k_1} D_{k_2} T_i|\varphi \omega^{*k_1} \omega^{*k_2} + D_{j_1} D_{k_1^*} T_i|\varphi \omega^{j_1} \omega^{k_1^*} + O(\omega^3) , \quad (3.15)$$

where no hats appear, because this is a tensor equation, as seen from theorem (3.10), and therefore, it is valid in any holomorphic coordinates (see 3.2). In the case of a holomorphic vector $T_i(z)$ [for instance $T_i(z) = \partial_i W(z)$], this expansion reduces to

$$T_i(z) = T_i|\varphi + D_j T_i|\varphi \omega^j + \frac{1}{2} D_{j_1} D_{j_2} T_i|\varphi \omega^{j_1} \omega^{j_2} + O(\omega^3) . \quad (3.16)$$
Actually, the expansion of any holomorphic tensor $T_{i_1 \cdots i_M}(z)$ can be carried out to all orders:

$$T_{i_1 \cdots i_M}(z) = \sum_{N=0}^{\infty} \frac{1}{N!} D_{j_1} \cdots D_{j_N} T_{i_1 \cdots i_M} |\varphi^{j_1} \cdots \varphi^{j_N}|.$$ (3.17)

In the same way, a rank two tensor $T_{ij^*}(z, z^*)$ can be expanded as

$$T_{ij^*}(z, z^*) = T_{ij^*} |\varphi + \frac{1}{2} D_{k_1} D_{k_2} T_{ij^*} |\varphi \omega^{k_1} \omega^{k_2} + \frac{1}{2} D_{l_1} D_{l_2} T_{ij^*} |\varphi \omega^{l_1} \omega^{l_2} + (D_{l_1} D_{k_1} T_{ij^*} + R_{i k_1 l_1} T_{m j^*}) |\varphi \omega^{k_1} \omega^{l_1} + O(\omega^3).$$ (3.18)

In the case of the metric tensor $g_{ij^*}$, this reduces to (3.6) of this order, because of the metric compatibility $D_k g_{ij^*} = D_{k^*} g_{ij^*} = 0$.

4 Kähler Normal Coordinates in the Background Field Method

In this section we apply the KNC expansion to the background field method in supersymmetric nonlinear sigma models. The target spaces of $D = 2, \mathcal{N} = 2$ (or $D = 4, \mathcal{N} = 1$) supersymmetric nonlinear sigma models must be Kähler manifolds [5]. The Lagrangian of supersymmetric nonlinear sigma models with scalar fields $A^i(x)$ and Weyl fermions $\psi^i(x)$ is given (after elimination of auxiliary fields) by (see Ref. [19])

$$\mathcal{L} = -g_{ij^*}(A, A^*) \partial_\mu A^i \partial^\mu A^{j^*} - i g_{ij^*}(A, A^*) \bar{\psi}^j A^i \bar{\psi}^i D_\mu \psi^i$$

$$+ \frac{1}{4} R_{i k^* l^*} (A, A^*) \psi^i \psi^k \bar{\psi}^j \bar{\psi}^l,$$ (4.1)

where the covariant applied to fermions is defined by $D_\mu \psi^i = \partial_\mu \psi^i + \partial_\mu A^l \Gamma^i_{l k} (A, A^*) \psi^k$.

Scalar fields are coordinates of a Kähler manifold. Under the holomorphic field redefinition of the scalar fields $A^i \rightarrow A'^i = A^i(A)$, the fermions and the quantity $\partial_\mu A^i(x)$ transform like holomorphic tangent vectors:

$$\psi^i(x) \rightarrow \psi'^i(x) = \frac{\partial A'^i}{\partial A^j} \psi^j(x),$$

$$\partial_\mu A^i(x) \rightarrow \partial_\mu A'^i(x) = \frac{\partial A'^i}{\partial A^j} \partial_\mu A^j(x).$$ (4.3)
By its definition, $D_\mu \psi^i$ also transforms like a holomorphic vector. Therefore, the Lagrangian (4.1) is invariant under holomorphic coordinate transformations of the target space.

Next, we consider the background field method applied to supersymmetric non-linear sigma models. A manifestly supersymmetric expansion of the Lagrangian using either RNC or KNC in terms of superfields is impossible. If we were to promote transformation (2.8) or (3.3) to a relation between superfields, the connection $\Gamma$ in its transformation law would depend on both the holomorphic and anti-holomorphic coordinates of the background, and therefore chirality would not be preserved.\footnote{We take the opportunity here to correct an error in Ref. \cite{1}. A manifestly supersymmetric expansion in KNC is impossible even around bosonic backgrounds. A superfield expansion in KNC is possible only in constant backgrounds. We would like to thank Thomas E. Clark for pointing this out.}

We present here the background field expansion for the Lagrangian (4.1) in component fields using KNC. To this end, we decompose the complex scalar fields $A^i(x)$ into background fields $\varphi^i(x)$ and fields $\pi^i(x)$ fluctuating around them:

$$A^i(x) = \varphi^i(x) + \pi^i(x). \quad (4.4)$$

To expand the Lagrangian in terms of the fluctuations, we would like to transform $\pi^i(x)$ into KNC fields $\hat{\pi}^i(x)$. To do this, we must consider the expansion of the kinetic term, because the definition of the KNC depends on the space-time coordinates through the background fields $\varphi^i(x)$ [see Eqs. (4.5) and (4.6), below]. This was actually recognized in the RNC expansion in Ref. \cite{1}. Here, we generalize the treatment given in in Ref. \cite{1} to the case of KNC.

Promoting (3.7) to a relation among fields, the KNC fields $\hat{\pi}^i(x)$ can be expanded in terms of tangent vector fields $\hat{\omega}^i(x)$ as

$$\hat{\pi}^i(x) = \hat{\omega}^i(x) - \frac{1}{2} \hat{\Gamma}^i_{k1k2} [\varphi \hat{\omega}^{k1}(x) \hat{\omega}^{k2}(x)] + O(\hat{\omega}^3), \quad (4.5)$$

where hats indicate quantities in KNC. When no space-time derivatives act on $\hat{\pi}^i$, the KNC fields $\hat{\pi}^i$ coincide with the tangent vector fields: $\hat{\pi}^i(x) = \hat{\omega}^i(x)$. However,\footnote{In the case of a Kähler manifold with isometry, an expansion in terms of superfields is given by Clark and Love in Ref. \cite{10} defining new holomorphic quantities. We do not know their relation with KNC.}
when the space-time derivative is applied, the connection $\tilde{\Gamma}^i_{jk}$ in (3.3) is also differentiated and remains non-zero:

$$\partial_\mu \tilde{\pi}^i(x) = \hat{D}_\mu \tilde{\pi}^i(x) - \frac{1}{2} \partial_\mu \varphi^{*j}(x) \hat{R}^i_{k_1 k_2} |\varphi^{k_1}(x)\varphi^{k_2}(x) + O(\omega^3).$$ (4.6)

We have defined the covariant derivative on a tangent vector $V^i$ at $\varphi^i$ by $D_\mu V^i \equiv \partial_\mu V^i + \partial_\mu \varphi^j \Gamma^i_{jk} |\varphi V^k$ and used the fact that it is simply $\hat{D}_\mu V^i = \partial_\mu V^i$ in KNC, due to (3.2). For general holomorphic coordinates of fluctuations $\pi^i(x)$, Eq. (4.6) becomes

$$\partial_\mu \pi^i(x) = D_\mu \omega^i(x) - \frac{1}{2} \partial_\mu \varphi^{*j}(x) R^i_{k_1 k_2} |\varphi^{k_1}(x)\omega^{k_2}(x) + O(\omega^3),$$ (4.7)

because of the transformation laws of (4.3) and (3.10). This is a tensor equation as seen from (3.10).

We have already given the KNC expansion of the metric in (3.6):

$$g_{ij^*}(\varphi + \pi, \varphi^{*} + \pi^{*}) = g_{ij^*} |\varphi + R_{ij^* kl^*} |\varphi^{k} \omega^{l^*} + O(\omega^3).$$ (4.8)

Using Eqs. (4.7) and (4.8), we obtain the expansion of the bosonic kinetic term of the Lagrangian to second order in the fluctuations as

$$-L_{\text{boson}} = g_{ij^*} |\varphi \varphi^{*j} \varphi^{ij} + g_{ij^*} |\varphi (D_\mu \omega^i \partial_\mu \varphi^{*j} + \partial_\mu \varphi^i D_\mu \omega^{*j}) + g_{ij^*} |\varphi D_\mu \omega^i D_\mu \omega^{*j} + R_{ij^* kl^*} |\varphi \left(\omega^k \omega^{l^*} \partial_\mu \varphi^{i} \partial_\mu \varphi^{*j} - \frac{1}{2} \omega^i \omega^k \partial_\mu \varphi^{*j} \partial_\mu \varphi^{ij} - \frac{1}{2} \omega^{*j} \omega^{l^*} \partial_\mu \varphi^{i} \partial_\mu \varphi^{ik}\right) + O(\omega^3).$$ (4.9)

Next, we give the expansion of the fermion kinetic term. The expansion of the connection in KNC can be obtained from (3.3) as

$$\Gamma^i_{lk}(\varphi + \pi, \varphi^{*} + \pi^{*}) = R^i_{lk^* k} |\varphi \omega^{k^* 1} + \frac{1}{2} D_{k_1} R^i_{lk^* k} |\varphi \omega^{k^* 1} \omega^{* k_2} + D_{j^* k} R^i_{lk^* k} |\varphi \omega^{j^* k^* 1} + O(\omega^3).$$ (4.10)

Then, the expansion of the fermion kinetic term to second order in $\omega$ can be obtained as

$$-L_{\text{fermion}} = ig_{ij^*} |\varphi \tilde{\psi}^j \tilde{\sigma}^i \varphi |\varphi \partial_\mu \varphi^j (\tilde{\psi}^j \tilde{\sigma}^i \psi^k) \omega^{* k_1} + iR_{ij^* kk^*} |\varphi \partial_\mu \varphi^j (\tilde{\psi}^j \tilde{\sigma}^i \psi^k) \omega^{* k_1} + iR_{jj^* kk^*} |\varphi (\tilde{\psi}^j \tilde{\sigma}^j \psi^k) D_\mu \omega^{j^* 1} \omega^{* k_1} + \frac{i}{2} D_{k_1} R_{ij^* kk^*} |\varphi \partial_\mu \varphi^j (\tilde{\psi}^j \tilde{\sigma}^i \psi^k) \omega^{* k_1} \omega^{* k_2} + \frac{i}{2} D_{j^* k} R_{ij^* kk^*} |\varphi \partial_\mu \varphi^j (\tilde{\psi}^j \tilde{\sigma}^i \psi^k) \omega^{j^* 1} \omega^{* k_2} + O(\omega^3).$$ (4.11)
Here we make the following comments:

1. The expansions of (4.9) and (4.11) in KNC coincide with those in RNC at this order, since the difference between coordinates first appears at third order, as seen in Eq. (B.1). To preserve holomorphic structures beyond this order, we must use the KNC expansion given in this section.

2. In a constant background, with $\partial_\mu \phi^i = 0$, the expansion of (4.9) and (4.11) reduces to the expansion given in Ref. [7].

Supersymmetric nonlinear sigma models possess the potential term [19]

$$L_{\text{potential}} = -g^{ij}(A, A^*)D_i W(A)D_j W(A^*)$$

$$- \frac{1}{2} D_i D_j W(A) \psi^i \psi^j - \frac{1}{2} D_i D_j W^*(A^*) \bar{\psi}^i \bar{\psi}^j,$$  \hspace{1cm} (4.12)

where $W(A)$ is a holomorphic function called a “superpotential”. Using (3.17) and the relation $g^{ij}\ast (A, A^*) = g^{ij}\ast (\partial_\mu \phi^i)$, the potential term can be expanded as

$$-L_{\text{potential}} = [g^{ij}\ast D_i W D_j W^*] \phi$$

$$+[g^{ij}\ast (D_{k_1} D_i W) D_j W^*] \phi \omega^{k_1} + [g^{ij}\ast D_i W(D_{l_1} D_j W^*)] \phi \omega^{l_1}$$

$$+ \frac{1}{2} [g^{ij}\ast (D_{k_1} D_{k_2} D_i W) D_j W^*] \phi \omega^{k_1} \omega^{k_2} + \frac{1}{2} [g^{ij}\ast D_i W(D_{l_1} D_{l_2} D_j W^*)] \phi \omega^{l_1} \omega^{l_2}$$

$$+[g^{ij}\ast (D_{k_1} D_i W) (D_{l_1} D_j W^*) + R^{ij}\ast_{k_1 l_1} D_i W D_j W^*] \phi \omega^{k_1} \omega^{l_1}$$

$$+ \frac{1}{2} (D_i D_j W) \phi + D_{k_1} D_i D_j W \phi \omega^{k_1} + \frac{1}{2} D_{k_1} D_{k_2} D_i D_j W \phi \omega^{k_1} \omega^{k_2} \psi^i \psi^j$$

$$+ \frac{1}{2} (D_i D_j W^*) \phi + D_{k_1} D_i D_j W^* \phi \omega^{k_1} + \frac{1}{2} D_{k_1} D_{k_2} D_i D_j W^* \phi \omega^{k_1} \omega^{k_2} \bar{\psi}^i \bar{\psi}^j$$

$$+ O(\omega^3). \hspace{1cm} (4.13)$$

5 Discussion

We discuss some applications of the KNC expansion in this section.

1. Nonlinear sigma models are renormalizable in two dimensions. Perturbative methods [6], however, cannot be used in the large coupling regime. On the contrary, the Wilsonian renormalization group (WRG) [11] can be applied in this region, and
it may lead new results. To derive the WRG equation, we need to expand the Lagrangian around the background field. Using the KNC expansion, we can derive a WRG equation that is generally covariant under the reparameterization of the background field. This would provide a better understanding of non-perturbative aspects of supersymmetric nonlinear sigma models, combined with non-perturbative analysis of Hermitian symmetric spaces using the large $N$ method \cite{12} and related models applied to Ricci-flat Kähler manifolds \cite{13}.

2. In four dimensions, nonlinear sigma models can be considered as effective field theories corresponding to theories at higher energy scales, such as supersymmetric QCD or the minimal supersymmetric standard model. When symmetry is spontaneously broken down, there appear massless (quasi-)Nambu-Goldstone bosons in addition to fermionic superpartners \cite{14}. Using the KNC expansion, low energy theorems of scattering amplitudes for these bosons are studied in Ref. \cite{15}. A manifestly supersymmetric four derivative term with a rank four tensor was recently reported in Ref. \cite{16}. Hence it appears possible to obtain a supersymmetric extension of the chiral perturbation theory by applying the KNC expansion of tensor fields given in §3.3 to higher rank tensors.

3. Some supersymmetric nonlinear sigma models with suitable potentials admit BPS domain wall solutions, which break a half of the original supersymmetry (see, e.g., Ref. \cite{17}). The effective field theory on a wall is very interesting in the brane world scenario. To obtain this, we need to expand the Lagrangian around the domain wall background, as was done in the case of linear models in Ref. \cite{18}. We believe that the KNC expansion [with the potential term (4.13)] will be found to be a very powerful tool to construct effective field theories on BPS domain walls in supersymmetric nonlinear sigma models.

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A Kähler Manifolds

In this appendix, we summarize the geometry of Kähler manifolds. Here, uppercase Roman letters are used for both holomorphic and anti-holomorphic indices: \( \{x^A\} = \{z^i, z^{*i}\} \). The complex structure \( J \) and the Hermitian metric \( g \) are given on Hermitian manifolds. The Kähler form \( \Omega \equiv ig_{ij}dz^i \wedge dz^{*j} \) is closed on Kähler manifolds: \( d\Omega = 0 \). From this condition, the metric can be written as

\[
g_{ij} = \frac{\partial^2 K(z, z^*)}{\partial z^i \partial z^{*j}} = K_{ij\ast} (z, z^*) ,
\]

using a real function \( K \) called the Kähler potential. There exists an ambiguity in the definition of \( K \) in the sense that given Kähler potential \( K, K' = K + f(z) + f^\ast(z^*) \), with arbitrary holomorphic function \( f \), is also a Kähler potential.

Components of the affine connection with mixed indices disappear as a result of the compatibility condition of the complex structure, \( DJ = 0 \). The non-zero components of the connection are

\[
\Gamma^k_{ij} = g^{kl\ast}g_{jl\ast,i} = g^{kl\ast}K_{ijl\ast} ,
\]

and their conjugates. The independent components of the curvature tensor are

\[
R_{ij\ast k\ast l\ast} = \partial_k \Gamma_{ij\ast l\ast} = \partial_k (g^{mi\ast}g_{mj\ast,l\ast})
\]

and their conjugates. The curvature tensor with lower indices

\[
R_{ij\ast k\ast l\ast} \equiv g_{im\ast}R_{mj\ast,k\ast l\ast} = K_{ij\ast k\ast l\ast} - g_{mn\ast}K_{mj\ast,l\ast}K_{n\ast ik}
\]

has some symmetries among its indices. In addition to the symmetries of the curvature tensor on Riemann manifolds,

\[
R_{ABCD} = -R_{ABDC} = -R_{BACD} = R_{CDAB} ,
\]
there exist the symmetries
\[ R_{ij^{*}kl^{*}} = R_{kj^{*}il^{*}} = R_{il^{*}kj^{*}}, \] (A.6)
as a result of the Kähler condition.

The Bianchi identity \( D_A R_{BCDE} + D_C R_{ABDE} + D_B R_{CABE} = 0 \) on Riemann manifolds reduces to
\[ D_m R_{ij^{*}kl^{*}} = D_i R_{mj^{*}kl^{*}} \] (A.7)
in the case of Kähler manifolds. Commutators of covariant derivatives of an arbitrary tensor \( T_{C_1 \cdots C_n} \) are given by
\[ \left[ D_A, D_B \right] T_{C_1 \cdots C_n} = \sum_{a=1}^{n} R_{ABC_a} D T_{C_1 \cdots C_{a-1} D C_{a+1} \cdots C_n}. \] (A.8)
The equations \( \left[ D_i, D_j \right] = \left[ D_i^{*}, D_j^{*} \right] = 0 \) hold as a result of the Kähler property. KNC are the coordinates such that these identities become manifest.

**B Relation Between Kähler and Riemann Normal Coordinates**

In §2.2, we considered geodesics in general holomorphic coordinates. Considering geodesics in the KNC \( \omega^i \) instead of the general coordinates \( \pi^i = z^i - \varphi^i \), we can obtain the relation between KNC and RNC. Their relation up to fourth order is obtained instead of (2.13) as
\[ \omega^i = \xi^i - \frac{1}{3!} \hat{R}^i_{j_1 j_2 j_3} |0\xi^{j_1} \xi^{j_2} \xi^{k_1} - \frac{2}{4!} \hat{D}_{j_1} \hat{R}^i_{j_2 j_3 k_1 j_3} |0\xi^{j_1} \xi^{j_2} \xi^{j_3} \xi^{k_2} \xi^{k_3} \]
\[- \frac{1}{4!} \hat{D}_{k_1} \hat{R}^i_{j_1 j_2 j_3} |0\xi^{j_1} \xi^{j_2} \xi^{k_1} \xi^{k_2} \xi^{k_3} + O(\xi^5), \] (B.1)
where the hat indicates quantities in KNC, and the condition (3.2) has been used. In general, all coefficients in the expansion of the transformation from RNC to KNC are covariant tensors \( T \) composed of the curvature and metric tensors and their covariant derivatives, as follows from (B.2):
\[ \omega^i = \xi^i - \sum_{M=2, N=1}^{\infty} \hat{T}^i_{j_1 \cdots j_M k_1 \cdots k_N} (\hat{D}, \hat{R}, \hat{g}) |0\xi^{j_1} \cdots \xi^{j_M} \xi^{k_1} \cdots \xi^{k_N}. \] (B.2)
Here we give some comments:
1. KNC and RNC coincide if and only if the Kähler manifold is flat.

2. In the case of Hermitian symmetric spaces, the equation $DR = 0$ holds. Therefore the tensors $T$ in (B.2) are composed of only the curvature and metric tensors.

Next we demonstrate the relation between KNC and RNC with regard to the expansion of the metric tensor in these coordinates. From the transformation law (B.1), the Jacobian can be calculated to give

$$
\frac{\partial \omega^i}{\partial \xi^l} = \delta_i^l - \frac{1}{3} \hat{R}^i_{j_1 k_1} |_{0} \xi^{j_1} \xi^{k_1} - \frac{1}{4} \hat{D}_{j_2} \hat{R}^i_{j_2 k_2} |_{0} \xi^{j_2} \xi^{k_2} + O(\xi^4),
$$

$$
\frac{\partial \omega^i}{\partial \xi^* l} = -\frac{1}{6} \hat{R}^i_{j_1 * j_2} |_{0} \xi^{j_1} \xi^{j_2} - \frac{1}{12} \hat{D}_{j_3} \hat{R}^i_{j_3 * j_3} |_{0} \xi^{j_1} \xi^{j_2} \xi^3
$$

where the bar indicates the tensors in RNC. Note that the tensors on the right-hand sides of these equations are tensors in KNC (or general holomorphic coordinates). There appear non-Hermitian components, $\bar{g}_{ij}$ and its conjugate, since the transformation (B.2) is not holomorphic. The components given in (B.4) coincide with the RNC expansion of the metric (2.11), identifying real coordinates as $\{x^A\} = \{z^i, z^{*i}\}$ and enforcing the Kähler condition on the curvature tensor as $R_{ijkl} = R_{ij*kl} = R_{ijkl*} = 0$ on the right-hand side of (2.11). We would like to emphasize again that the RNC expansion of the metric includes unwanted non-Hermitian terms.

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C A Proof of the Theorem

In this appendix, we show that KNC in a Kähler manifold can be interpreted as a holomorphic tangent vector at the origin, and therefore they are a natural extension of RNC to the case of a Kähler manifold. To this end, we consider the relation between a set of KNC defined by a set of general holomorphic coordinates \( z^i \) and \( z'^i = z'^i(z) \), which are transformed under a holomorphic coordinate transformation preserving the origin. (In this appendix, we take the expansion point to be the origin, \( \varphi^i = 0 \), for simplicity, but the entire treatment holds for general expansion points, replacing \( z^i \) by \( \pi^i = z^i - \varphi^i \).)

First, we need the transformation law of the “generalized connection” \( K_{j^i i_1 \cdots i_N}(z, z^*) \) in the definition of KNC, given by the following lemma.

**Lemma**

The transformation law of \( K_{j^i i_1 \cdots i_N}(z, z^*) \) under a holomorphic coordinate transformation \( z^i \rightarrow z'^i = z'^i(z) \) is given by

\[
K_{j^i i_1 \cdots i_N}(z, z^*) \rightarrow K_{j'^i i'_1 \cdots i'_N}(z', z'^*) = \sum_{n=1}^{N} \frac{1}{n!} K_{d^i k_1 \cdots k_n}(z, z^*) \frac{\partial}{\partial z^*_{j^i}} \left[ \frac{\partial^N(z^{k_1} \cdots z^{k_n})}{\partial z'^{k_1} \cdots \partial z'^{k_n}} \right], \tag{C.1}
\]

where [\( \cdot \cdot \cdot \)*] possesses the meaning that terms including \( z \) that are differentiated by no \( z' \) are omitted.

The term for \( n = N \) is a homogeneous (tensorial) term, but all of the other terms are non-homogeneous terms. The \( N = 2 \) case corresponds to the ordinary connection: \( K_{j^i i_1 i_2} = g^{j^i}_{k^i} \Gamma_{i_1 i_2}^k \).

**(Proof)** We use mathematical induction for the proof.

i) First, we consider the \( N = 1 \) case. In this case, Eq. (C.1) is

\[
K_{j^i i_1} \rightarrow K_{j'^i i'_1} = K_{d^i k_1} \frac{\partial}{\partial z'^{k_1}} \frac{\partial}{\partial z'^{i_1}}, \tag{C.2}
\]

This is obvious, because \( K_{j^i i_1} = g^{j^i}_{i_1} \).

ii) We assume that Eq. (C.1) holds for \( N \). Differentiation of Eq. (C.1) with respect to \( z'^{i_N+1} \) gives

\[
K_{j'^i i'_1 \cdots i'_{N+1}} = \sum_{n=1}^{N} \frac{1}{n!} K_{d^i k_1 \cdots k_{n+1}} \frac{\partial}{\partial z'^{k_{n+1}}} \left[ \frac{\partial^N(z^{k_1} \cdots z^{k_n})}{\partial z'^{k_1} \cdots \partial z'^{k_n}} \right],
\]

where [\( \cdot \cdot \cdot \)*] possesses the meaning that terms including \( z \) that are differentiated by no \( z' \) are omitted.

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\]

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\[
K_{j'^i i'_1 \cdots i'_{N+1}} = \sum_{n=1}^{N} \frac{1}{n!} K_{d^i k_1 \cdots k_{n+1}} \frac{\partial}{\partial z'^{k_{n+1}}} \left[ \frac{\partial^N(z^{k_1} \cdots z^{k_n})}{\partial z'^{k_1} \cdots \partial z'^{k_n}} \right],
\]
The first term can be rewritten as
\[
\sum_{n=2}^{N+1} \frac{1}{(n-1)!} K_{t^* k_1 \cdots k_n} \frac{\partial z^{*l}}{\partial z^{*j}} \frac{\partial z^{k_n}}{\partial z^{t_{N+1}}} \left[ \frac{\partial^N (z^{k_1} \ldots z^{k_{n-1}})}{\partial z^{t_1} \ldots \partial z^{t_N}} \right].
\] (C.4)

Therefore, we have
\[
K_{i^* j^* \cdots t^*_{N+1}} = K_{i^* k_1} \frac{\partial z^{*l}}{\partial z^{*j}} \frac{\partial z^{k_1}}{\partial z^{t_1}} \ldots \frac{\partial z^{k_{N-1}+1}}{\partial z^{t_{N+1}}}
+ \sum_{n=2}^{N} \frac{1}{n!} K_{i^* k_1 \cdots k_n} \frac{\partial z^{*l}}{\partial z^{*j}} \frac{\partial z^{k_n}}{\partial z^{t_{N+1}}}
\left\{ \sum_{N=2}^{N+1} \frac{1}{(n-1)!} K_{i^* k_1 \cdots k_n} \frac{\partial z^{*l}}{\partial z^{*j}} \frac{\partial z^{k_n}}{\partial z^{t_{N+1}}} \right\}
+ \frac{\partial}{\partial z^{t_{N+1}}} \left[ \frac{\partial^N (z^{k_1} \ldots z^{k_{N}})}{\partial z^{t_1} \ldots \partial z^{t_N}} \right].
\] (C.5)

With regard to the term in the curly brackets on the right-hand side, the relation
\[
\frac{\partial}{\partial z^{t_{N+1}}} \left[ \frac{\partial^N (z^{k_1} \ldots z^{k_{N}})}{\partial z^{t_1} \ldots \partial z^{t_N}} \right]
= \left[ \frac{\partial^{N+1} (z^{k_1} \ldots z^{k_{N}})}{\partial z^{t_1} \ldots \partial z^{t_{N+1}}} \right].
\] (C.6)

holds, where symmetrization of the first term on the left-hand side is implied. We thus obtain
\[
K_{i^* j^* \cdots t^*_{N+1}} = \sum_{n=1}^{N+1} \frac{1}{n!} K_{i^* k_1 \cdots k_n} \frac{\partial z^{*l}}{\partial z^{*j}} \frac{\partial z^{k_n}}{\partial z^{t_{N+1}}} \left[ \frac{\partial^{N+1} (z^{k_1} \ldots z^{k_{N}})}{\partial z^{t_1} \ldots \partial z^{t_{N+1}}} \right].
\] (C.7)

iii) From i) and ii) the lemma is proved. (Q.E.D.)

We call a holomorphic coordinate transformation that leaves the origin invariant (i.e. \( z^i = 0 \) implies \( z'^i = 0 \) and vice versa), a “holomorphic coordinate transformation preserving the origin”. We immediately obtain the following corollary from the lemma:

**Corollary**

Under holomorphic coordinate transformations preserving the origin, \( K_{i^* j^* \cdots t^*_{N}} \) transforms according to
\[
K_{i^* j^* \cdots t^*_{N}} \big|_0 \rightarrow K_{i^* j^* \cdots t^*_{N}} \big|_0 = \sum_{n=1}^{N} \frac{1}{n!} K_{i^* k_1 \cdots k_n} \big|_0 \left[ \frac{\partial z^{*l}}{\partial z^{*j}} \frac{\partial z^{k_n}}{\partial z^{t_{N+1}}} \right].
\]
\[ = \sum_{n=1}^{\infty} \frac{1}{n!} K_{i^*k_1 \ldots k_n} \left| 0 \left[ \frac{\partial z^{i^*} \partial^N (z^{k_1} \ldots z^{k_n})}{\partial z^{i_{i_1}} \ldots \partial z^{i_{i_N}}} \right]_0 \right) \]  

(C.8)

at the origin, where the subscripts "0" indicate that the values are evaluated at the origin: \( z^i = 0 \) or \( z'^i = 0 \). The second equality holds because the term \([\cdots]_0\) vanishes when \( n > N \).

We are now ready to prove the theorem, which reveals the geometric meaning of KNC.

**A proof of the theorem:** Using the definition (3.3) and the corollary (C.8), the left-hand side of Eq. (3.10) can be explicitly calculated as

\[
\omega^{i^*} = \sum_{n=1}^{\infty} \frac{1}{n!} (g^{i'^*j} K_{j'^*i'_1 \ldots i'_n})_0 z'^{i_{i_1}} \ldots z'^{i_{i_n}} \\
= \sum_{n=1}^{\infty} \frac{1}{n!} g^{i'^*j} \left( \sum_{m=1}^{\infty} \frac{1}{m!} K_{i'^*k_1 \ldots k_m} \left| 0 \left[ \frac{\partial z^{i'^*} \partial^m (z^{k_1} \ldots z^{k_m})}{\partial z^{i_{i_{i_1}}} \ldots \partial z^{i_{i_m}}} \right]_0 \right) \right) z'^{i_{i_1}} \ldots z'^{i_{i_n}} \\
= \frac{\partial z'^{i'^*}}{\partial z^k} \left| 0 \left( \sum_{n=1}^{\infty} \frac{1}{n!m!} (g^{kJ} K_{i'^*k_1 \ldots k_m})_0 \left[ \frac{\partial^m (z^{k_1} \ldots z^{k_m})}{\partial z^{i_{i_{i_1}}} \ldots \partial z^{i_{i_m}}} \right]_0 \right) \right) z'^{i_{i_1}} \ldots z'^{i_{i_n}} \\
= \frac{\partial z'^{i'^*}}{\partial z^k} \left| 0 \left( \sum_{m=1}^{\infty} \frac{1}{m!} (g^{kJ} K_{i'^*k_1 \ldots k_m})_0 z^{k_{k_1}} \ldots z^{k_{k_m}} \right) \right) = \frac{\partial z'^{i'^*}}{\partial z^k} \left| 0 \right) \omega^k \].

(C.9)

(Q.E.D.)

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