Phase structure and confinement properties of noncompact gauge theories
II. $Z(N)$ Wilson loop and effective noncompact model

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Abstract

An approach to studying lattice gauge models in the weak coupling region is proposed. Conceptually, it is based on the crucial role of the original $Z(N)$ symmetry and the invariant gauge group measure. As an example, we calculate an effective model from the compact Wilson formulation of the $SU(2)$ gauge theory in $d = 3D3$ dimensions at zero temperature. Confining properties and phase structure of the effective model are studied in details.

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1 Introduction

In the previous article [1] we presented a brief review of noncompact lattice models and considered several examples aiming to clarify some aspects of the confinement problem in the noncompact formulation of lattice theories. Our conclusion was that one may construct a noncompact model which confines in the same way as compact model does if we take properly into account the contribution of the invariant measure and the centre subgroup of the original $SU(N)$ symmetry of the compact model. We are going to demonstrate here how to calculate a noncompact model from the Wilson compact formulation including both these conditions. It seems to be worthwhile to start our discussion with a brief summary of earlier investigations [1].

We calculated the $SU(2)$ partition function of the noncompact model at finite temperature approximating the initial action by its chromoelectric part (time-like plaquettes) and compared it with known expression for the $SU(2)$ partition function of the compact Wilson model in the same approach. We have found out that these two models coincide at high temperatures only and the noncompact version does not exhibit the confining features. The same result was obtained by avoiding an expansion of Polyakov loops around the unit matrix and preserving the compact integration over $A_0$ gauge potential. We get convinced by this example that compact Wilson models and noncompact lattice Yang-Mills theories might belong to different universality classes.

As the next step, we followed a proposition of Ref. [2] and considered the sine-Gordon model with a $Z(2)$ invariant potential. This potential had to reproduce a contribution of the invariant group measure for $A_0$ gauge potential in the noncompact formulation. However, the model occurred to break the global $Z(2)$ symmetry at all values of the coupling constants in $d > 2$ dimensions [3]. This fact seems to be in contradiction with the main idea of [2] since the invariant measure was introduced to preserve the $Z(2)$ symmetry of the vacuum. The behaviour of some important correlation functions in the sine-Gordon model resembles rather that of $U(1)$ Villain lattice model investigated in [3] and the sine-Gordon model is an effective model of the lattice abelian theory with the Villain action.

We concluded then that in order to reproduce the specific features of the $SU(2)$ Wilson theory we have to preserve not only the global centre symmetry but the local centre symmetry as well. In the absence of the local $Z(2)$ symmetry the global one can be spontaneously broken what is the case for the sine-Gordon model whereas there should be no such a breakdown in zero temperature $SU(2)$ model. Actually, the invariance of the theory with respect to global $Z(2)$ arises only from the potential term in the sine-Gordon model whereas the kinetic term is invariant under global $U(1)$ transformations. Meanwhile, compact Wilson action itself is invariant both under $Z(2)$ global and under $Z(2)$ local transformations. It is a very essential point in the $Z(N)$ mechanism of confinement that $SU(N)$ gauge group is broken up to its local $Z(N)$ subgroup [4, 5]. Hence, accepting the idea of Ref. [2] that one should simulate the invariant measure in a form of a local $Z(N)$ invariant potential we believe that an effective noncompact action should be itself invariant under $Z(N)$ local transformations.
but not only under global ones. To do so, it is not sufficient to introduce invariant measure into the effective action. A modification of the model [2] which respects local $Z(N)$ symmetry will be presented in this article. The second important observation which can be extracted from these discussions is the conclusion that we are not allowed to neglect dynamics of space gauge potentials. At least, we have to reproduce the contribution of $Z(N)$ configurations contained in $U_n(x)$ matrices.

Another hint supporting these ideas comes from the discussion of a version of the $XY$ model which can be an effective model of $SU(2)$ gauge theory at finite temperature [1, 6] in the strong coupling region. The corresponding effective action [see Eq.(56) of [1]] has only a global $Z(2)$ invariance which can be spontaneously broken at high temperature. However, the corresponding effective action at zero temperature has also local $Z(2)$ symmetry (see next section for details) and this is, according to Elitzur’s theorem, the reason of the absence of the phase transition at zero temperature in non-abelian gauge theories. If we set space gauge potentials to be equal to zero (considering their dynamics as being inessential for the confinement, as has been proposed in [2]) we get either the compact $XY$ model or the noncompact sine-Gordon model. In both cases the spontaneous breaking of $Z(2)$ symmetry takes place (actually, the resulting theory is invariant under global $U(1)$ transformations and just this symmetry is completely broken as it happens in zero temperature $U(1)$ lattice model where deconfinement of electrons takes place). To avoid this breakdown it is necessary to perform a summation over $Z(2)$ configurations contained in space gauge fields. In such a way we shall come to the theory with a local $Z(2)$ symmetry. This is what will be done in the next section. The corresponding noncompact theory could then have a form of the sine-Gordon model with local $Z(2)$. Doing so, we can construct a noncompact weak coupling limit of the compact model which exhibits confinement as compact theory does. We showed (this was shortly discussed in [7], too) how to define a noncompact model with necessary confining properties, but we did not give a definition of the correlation functions of Polyakov loops and Wilson loops. We will fill this gap in the present paper.

Thus, our basic observation is if we are willing to construct a noncompact theory confining in the same way as compact one, we have to 1) perform the summation over $Z(N)$ variables in the compact model and then to take a limit $g \to 0$ expanding gauge field matrices around unit matrix; 2) include invariant measure in a form of a local $Z(2)$ invariant potential $= 1$. We need to be very careful using the expansion of gauge field matrices around unit ones because we can miss relevant minima of an effective action which we get after summation over $Z(N)$ configurations. We have demonstrated this point in the previous article on a simple example of the noncompact model with compact integration over $A_0$ gauge potential. The expansion around all minima is required to come to the reliable model in the weak coupling region.

The outline of the paper is as follows.

In section 2 we discuss in detail a general strategy of our investigation of the Wilson lattice theory in the weak coupling region. We derive a representation for the partition function of the 3-dimensional $SU(2)$ gauge theory performing the summation over $Z(2)$ configurations. In section 3 we define the $XY$ model with a local $Z(2)$ symmetry and
study some of its properties. Here, we also present our effective noncompact model in the weak coupling region. The Wilson loops are defined in our model in section 4. We calculate $Z(2)$ Wilson loop in some simplest cases and show the finiteness of the string tension in the limit of vanishing bare coupling constant (continuum limit). We terminate with section 5 as a Summary discussing essential points of our approach, confinement mechanism in gauge theories as well as some still unsolved problems.

2 General representation for the partition function

We are going to accomplish the above programme taking as an example $SU(2)$ lattice gauge theory (LGT) in $d = 3D3$ dimensions. The corresponding partition function has the form \[ Z = 3D \int D\mu(U) \exp \left[ \lambda \sum_p \Omega_{\partial p}(U) \right], \] where $\Omega$ is a character of the fundamental representation of $SU(2)$ gauge group, $D\mu(U)$ is the invariant integration measure and $\lambda = 3D^2 g^2$. We are interested in a reliable expansion for the gauge matrices in the weak coupling region. It is obvious, that configurations dominating the partition function (1) in this limit are those for which $\Omega_{\partial p}(U)$ is close to unity. It does not follow, however, that a single gauge matrix $U_{\mu}(x)$ is close to unity. The naive expansion of $U_{\mu}(x)$ around unit matrix leads to the usual perturbative expansion implying the loss of confinement. Nevertheless, we may use this expansion if we choose a proper gauge which makes all matrices $U_{\mu}(x)$ as close to unity as possible. This so-called minimal Landau gauge was studied in details in [9, 10].

An effective action calculated in the thermodynamical limit differs from the Faddeev-Popov action by inclusion of a new term which preserves, however, renormalizability and asymptotic freedom. The corresponding weak coupling expansion differs from the Faddeev-Popov perturbation theory by terms which are finite in every order and are small only at high energies [9]. Under some plausible assumptions an area law for the Wilson loop in the newly developed theory has been proved [10]. Though we are fully aware of the success of this approach in the study of the weak coupling region of LGT we would like to develop a method which, on our view, could clarify in some aspects the role of $Z(N)$ symmetry and $Z(N)$ excitations in confinement and could be useful in an investigation of a mechanism of this phenomenon.

We rewrite (1) using a representation

\[ \Omega_{\partial p}(U) = 3DZ_{\partial p} \frac{1}{2} \Tr \bar{U}_{\partial p}, \] where $\bar{U}_{\partial p} \in SU(2)/Z(2) \sim SO(3)$ and $Z_{\partial p}$ is a product of $Z(2)$ elements along the minimal plaquette. For the invariant measure we have

\[ D\mu(U) = 3D^\frac{1}{2} \sum_{z=3D\pm1} D\mu(\bar{U}), \]
where \( D_\mu(U) \) is an invariant measure on \( SO(3) \) group. Let us recall, that invariant measure on \( SU(N)/Z(N) \) group coincides with \( SU(N) \) measure up to the restriction
\[
-\frac{\pi}{N} \leq \text{arg}[\text{Tr}U] \leq \frac{\pi}{N}.
\]
(4)

Setting \( \bar{U}_\mu(x) = 3DI \) everywhere, we get a \( Z(2) \) gauge model which confines static charges in the strong coupling region. \( Z(N) \) gauge model can be viewed as a classical solution of the \( SU(N) \) Wilson model
\[
U_\mu(x) = 3DV_x Z_\mu(x) V^+_{x+\mu},
\]
(5)
where \( V_x \) is an arbitrary \( SU(N) \) matrix \([11]\). The space-time structure of this solution is a two-dimensional closed surface and in the continuum limit it corresponds to singular gauge transformations. It should be emphasized that this solution, representing \( Z(N) \) excitations, is one of the few exact topological solutions of the Wilson model which could, in many cases, provide the same minimum of the Wilson action as the trivial one \( U_\mu(x) = 3DV_x V^+_{x+\mu} \). In our case this minimum is achieved when the product of \( Z(N) \) elements along minimal plaquettes equals 1. However, the \( Z(N) \) gauge model undergoes a phase transition to the deconfinement phase in the weak coupling region.

It was proposed in \([11]\) to calculate quantum corrections for \( Z(N) \) excitations coming from the following expansion for \( SO(3) \) gauge matrices
\[
\bar{U}_{\partial p} \approx 1 - F^2_{\mu\nu},
\]
(6)
where \( F_{\mu\nu}(A) \) is the strength tensor, aiming to avoid the phase transition from the \( SU(N) \) LGT (under assumption, of course, that \( Z(N) \) configurations play a central role in a formation of the confining forces). As a result, we have \( Z(N) \) gauge spin system connected to noncompact Yang-Mills potentials. It has been found, however, that the phase transition still exists, although the critical coupling moves to a smaller value. Thus, the string tension is vanishing in the weak coupling region. We have to conclude that the expansion (6) is not suitable in this region because no phase transition has been found in \( SU(2) \) and \( SU(3) \) gauge theories (it is possible that a roughening phase transition can happen since it is of an infinite order and does not affect the string tension) and leads to the false vacuum in the weak coupling region. Indeed, when the product of \( Z(2) \) elements along a plaquette equals 1 then the expansion (6) provides the global minimum of the action (if we do not fix a gauge). However, this expansion does not provide the same minimum when \( Z_{\partial p} = 3D - 1 \). We think that this is the main reason why \( Z(N) \) global symmetry occurred to be broken when \( g \to 0 \) (we shall specify mentioned symmetry below).

Our approach in investigation of the role of \( Z(2) \) configurations in the region of small coupling consists in the following. We expand the space gauge matrices around the unit matrix similarly to \([11]\)
\[
\bar{U}_n(x) = 3D1 + i\text{ag}A_n(x).
\]
(7)
However, unlike \([11]\) this expansion can be made more rigorous if we fix a gauge which makes all the matrices \( \bar{U}_n(x) \) as close to unity as possible. Since \( Z(2) \) transformation
under consideration acts on $U_0$ gauge matrices, we choose not to expand these matrices taking them in a diagonal form but not fixing a static gauge (this is, of course, an approximation which leads, however, to the same permanent features discussed below as an exact treatment \[14\]). This procedure results in a kind of the XY model for the $A_0$ gauge potential\[3\].

Thus, our starting point is the partition function

$$Z = 3D \sum_z \int \prod_{x,n} DA_n(x) \prod_x D\bar{U}_0(x) \exp \left[ \frac{\lambda}{2} \sum_{\bar{p}} Z(\partial \bar{p}) \text{Tr} \left( 1 - \frac{a^4 g^2}{2} F_{nm}^2(x) \right) + S_d + S_{g.f.} \right. $$

$$\left. + \frac{\lambda}{2} \sum_{p_0} Z(\partial p_0) \text{Tr} \bar{U}_0(x) (1 + iagA_n(x)) \bar{U}_0^+(x + n)(1 - iagA_n(x + m)) \right], \quad (8)$$

where we have omitted an irrelevant constant and introduced the following notations. $F_{nm}^2(x)$ is the Yang-Mills strength tensor, $S_{g.f.}$ is the gauge fixing term and $S_d$ is the Faddeev-Popov determinant (here and further Latin (Greek) indices mean two(three)-dimensional vectors, bars mean two-dimensional quantities). $\bar{p}$ ($p_0$) is the space(time)-like plaquette. The gauge fixing term could be taken, for example, as in \[9\] where we have to substitute $SU(2)/Z(2)$ matrices instead of $SU(2)$ ones. $\int \prod_{x,n} DA_n(x)$ is an integral over gauge potentials calculated over all noncompact region, while $D\bar{U}_0(x)$ means the invariant measure on $SO(3)$ group. We suppose, however, that the dynamics of space gauge potentials is not important in achieving an area law for the time-space Wilson loop even in the continuum limit\[4\]. In this approach $S_{g.f.}$ and $S_d$ might be excluded from the consideration as we have fixed the gauge only for $SO(3)$ part of the gauge group and thereby all these terms do not depend on $Z(2)$ variables. In this approach we have the following expression for the partition function:

$$Z = 3D \sum_{z_n(x)} \int \prod_x D\bar{U}_0(x) \exp \left[ \lambda \sum_{\bar{p}} Z(\partial \bar{p}) + \lambda \sum_{p_0} Z(\partial p_0) \cos(\phi_t(\bar{x}) - \phi_t(\bar{x} + n)) \right], \quad (9)$$

where we have denoted $\phi_t(\bar{x}) = 3D agA_0(x)$. Let us discuss briefly what shall we get if we do not set $A_n(x) = 3D0$? We can follow \[\prod\] and consider the quadratic fluctuations around $Z(2)$ space configurations fixing the Feynman gauge, i.e.

$$S_{g.f.} = 3D - \frac{1}{2} \sum_{x,n} [\Delta_n A_n^c(x)]^2, \quad (10)$$

where $\Delta_n$ is the difference operator on the lattice. This results in the gaussian path integral over space gauge potentials $A_n(x)$ calculating which we come to a bosonic determinant in the background $Z(2)$ and $A_0(x)$ gauge fields. Since in what follows we shall study \[\prod\] in the approach $A_n(x) = 3D0$, we adduce here only qualitative arguments why such integration cannot change the confinement picture which follows from the partition function \[\prod\]. The main effects of the discussed integration appear to

\[3\]In an exact procedure we would get a kind of $O(2) \ast O(2)$ model for $A_0$.

\[4\]The only difference in this respect from \[\prod\] is that we keep centre elements of space-gauge matrices
be the following. First of them was pointed out in [11]. Expanding the determinant in powers of the fluctuations $Z(\partial \bar{p}) - 1$, one finds that the first term generates the Wilson action leading to the substitution

$$\lambda \rightarrow \lambda - \frac{N_c^2 - 1}{4}$$

(11)
in (8). Obviously, it is not difficult to take into account this contribution. The second power in $Z(\partial \bar{p}) - 1$, as has been shown in [11], leads to an increasing of the disorder of the system and, as such, can only enhance the string tension and lower the critical coupling. It is very complicated to estimate reliably the contribution of higher order corrections. Another effect coming from time-like plaquettes is the generation of a new interaction between $Z(2)$ excitations and $A_0(x)$ gauge potential of the kind

$$\sum_{p_0} Z(\partial p_0) \cos(\phi_t(\bar{x}) + \phi_t(\bar{x} + n)).$$

(12)

Being added to the action in (3) this term reduces the original symmetry of the action to $Z(2)$ symmetry. Let us explain this in more detail. The original action (3) as well as the action in (3) is invariant under global $Z(2)$ transformations (diagonal gauge for $U_0$) $\phi(x) \rightarrow \phi(x) \pm \pi$ whereas it could seem that the action in (3) is invariant under global $U(1)$ transformations $\phi(x) \rightarrow \phi(x) \pm const.$ The integration over space gauge potentials recovers the original symmetry of the action. It does not follow, however, that the theory defined in (3) is invariant under more general $U(1)$ transformations since the group measure in the integrand of (3) is invariant only under $Z(2)$. In this paper we neglect the contributions of the type (12) because as we have explained the symmetries of both actions are actually the same. Moreover, it has been discussed in [1] that the terms in the lattice action like $\cos(\phi(x) + \phi(x + n))$ correspond in the continuum limit to the contribution of the local potentials up to the order $O(g)$ which we do not take into account here. Concerning the contribution of the local potential we assume that all such corrections are incorporated in the invariant measure which we simulate as a local $Z(2)$ invariant potential of the sine-Gordon type.

Let us turn now to the expression (5) for the partition function. We are going to perform the summation over $Z(2)$ variables. One possible way is to rewrite this sum as a sum over all closed surfaces in $d$-dimensional space. To do that, we present (5) as

$$Z = 3D \sum_{z_\mu(x)} \int \prod_x \bar{D}U_0(x) \prod_{\bar{p}} \left[ \cosh \lambda + \sinh \lambda Z(\partial \bar{p}) \right]$$

$$\prod_{p_0} \left[ \cosh(\lambda \cos \Delta \phi) + \sinh(\lambda \cos \Delta \phi)Z(\partial p_0) \right].$$

(13)

The notation $\Delta \phi = 3D\phi_t(\bar{x}) - \phi_t(\bar{x} + n)$ has been used. Summing over $z_\mu(x)$ we notice that the only plaquette configurations giving a non-zero contribution to $Z$ are those which form closed surfaces (every link must enter even number of times in the product over all plaquettes). As a result we have a sum over all possible products of all possible surfaces on the lattice. The surfaces which enter the same product have no plaquettes
in common. Let Ω be an arbitrary closed 2-surface constructed from | Ω | plaquettes, \( N_p \) (\( N_{p_0} \)) - a number of space(time)-like plaquettes and \( N_l \) - a number of links. \( N_p \) is a common number of plaquettes on the lattice. We put down the result in the form, recovering all constants

\[
Z = 3De^{-\lambda N_p} (\cosh \lambda)^{N_p} 2^{N_l} \sum_{\Omega} (\tanh \lambda)^{|\Omega_p|} \sum_{\sigma = 3D \pm 1} (\prod_{p_0 \in \Omega} \sigma) Z_{\Omega}(\sigma). \tag{14}
\]

We have introduced here new time-like plaquette’s variable \( \sigma \) for further convenience. Since the action \( \sum_{p_0} \cos \Delta \phi \) is diagonal in time indices, we can consider \( \sigma(p_0) = 3D \sigma_n(t) \) as link variable and then define

\[
Z_{\Omega}(\sigma) = 3D \prod_{t=3D1}^{N_l} Z_0^0(\sigma(t)), \tag{15}
\]

\[
Z_0^0(\sigma_n(t)) = 3D 2^{-N_p} \int \prod_x D\tilde{U}_0(x) \exp[\lambda \sum_{p_0} \sigma \cos \Delta \phi]. \tag{16}
\]

The summation over \( \sigma \) can be easily performed to produce

\[
\sum_{\sigma = 3D \pm 1} \prod_{p_0 \in \Omega} \sigma Z_0^0(\sigma) = 3D \int \prod_x D\tilde{U}_0(x) \prod_{p_0} \cosh(\lambda \cos \Delta \phi) \prod_{p_0 \in \Omega} \tanh(\lambda \cos \Delta \phi). \tag{17}
\]

In what follows we use both representation (14)-(16) and (14),(17) for the partition function. It is clear from the described procedure that this method can be directly applied for the full action in (8). Two additional terms, supported on 2-closed surfaces, would appear in this case: the Yang-Mills term \( F_{nm}^2(x) \) and a term describing interactions between noncompact Yang-Mills potentials and the compact \( \bar{U}_0 \) gauge field. The form of the latter can be found in [1] (terms \( S_1 \) and \( S_2 \) from the section 4).

Completing this section we want to discuss general phase structure of the model (14) and a representation for the invariant measure \( D\tilde{U}_0 \) on \( SO(3) \) gauge group. In absence of the interaction term in time-like plaquettes, i.e. \( \cos \Delta \phi = 3D1 \), we get the three-dimensional \( Z(2) \) gauge model. On the dual lattice it is equivalent to the \( Z(2) \) Ising model which has a phase of spontaneously broken \( Z(2) \) global symmetry. The corresponding phase transition on the original lattice corresponds to the deconfinement phase transition. In the weak coupling region the Wilson loop obeys the perimeter law behaviour. The term \( \cos \Delta \phi \) might drastically change this picture. On the dual lattice the leading configurations contributing to the partition function in the limit \( \lambda \to \infty \) are those when all dual spins are strongly disordered (ordered, deconfinement phase on the original lattice). At small \( \lambda \) the main contribution comes from the ordered configurations of the dual spins (this is the confinement phase on the original lattice). The term \( \cos \Delta \phi \) generates contributions which negative plaquettes \( Z_{\partial p} = 3D - 1 \) turn into positive ones, thus disordering the \( Z(2) \) system on the original lattice even at large \( \lambda \). Our hope is that this competition between the ordering mechanism of the \( Z(2) \) system and disordering mechanism of the \( XY \) system will shift the critical point to infinity, i.e. \( g \to 0 \). A roughening phase transition of the infinite order is still possible at a finite value of \( \lambda \). We shall turn to this principal point again in section 4.
We will see a hint on another phase transition if we neglect the space-like plaquettes in (3). Summing over $z_\mu(x)$ in this approach one has in the thermodynamical limit on the periodic lattice

$$Z = 3D2^{N_l} \int \prod_x D\bar{U}_0(x) \prod_{p_0} \cosh(\lambda \cos \Delta \phi).$$

This model resembles the $XY$ model on the square lattice which undergoes a phase transition related to the condensation of vortices. However, in addition to the usual $U(1)$ global symmetry, the model (18) possesses the local $Z(2)$ symmetry (see discussion in the next section). This property immediately means that any $Z(2)$ noninvariant observable or correlation function is identically equal to zero. $Z(2)$ invariant correlation function displays qualitatively different behaviour in the limits $\lambda \rightarrow 0$ and $\lambda \rightarrow \infty$ when $D\bar{U}_0(x) = 3Dd\phi(x)$ like in the standard $XY$ model. As will be discussed further, such correlation functions are related to the adjoint sources introduced in the original model (1). As well known, however, adjoint sources are screened in pure $SU(N)$ gauge theories at large distances [12] in the strong coupling limit and this behaviour is expected to be valid in the whole range of the coupling constant. We conclude from this that the $SO(3)$ part of the invariant measure could play an important role in the phase structure of our effective model. The invariant measure reduces $U(1)$ global symmetry to the $Z(2)$ global symmetry. The latter could be spontaneously broken in the $XY$ model. In the presence of the local $Z(2)$ it is impossible in force of Elitzur’s theorem. Thus, we should expect that adjoint sources will not show critical behaviour similar to the fundamental ones if we take properly into account the invariant measure. Usually, it is accepted that in the naive limit and in the perturbative theory we have to replace the invariant measure with the flat measure. This is not so obvious in the nonperturbative treatment of the weak coupling limit, and the opposite opinion has been advocated in [2]. We accept the following general form for the invariant measure contribution casted by Ref. [4]:

$$\int \prod_x D\bar{U}_0(x) = 3D \int_{-\pi/2}^{+\pi/2} \prod_x d\phi_x e^{\mu \sum_x V(\phi_x)},$$

where $V(\phi_x)$ is a local, $Z(2)$ invariant potential. Integration region for angle variables $\phi$ is restricted because of Eq. (4). The usual $SO(3)$ measure can be recovered if we set $\mu = 3D1$ and $V(\phi_x) = 3D \ln(1 - \cos^2 \phi_x)$.

This general phase structure described above is, in its main features, valid for $d = 3D4$ theory as well. The difference is that effective $d = 3D2$ model for $3 - d$ theory (18) demonstrates a phase transition in the absence of the invariant measure term ($\mu = 3D0$) but there should not be a spontaneous breaking of $U(1)$ global symmetry (this is the Kosterlitz-Thouless phase transition). The corresponding model in $3 - d$ theory could exhibit a spontaneous breaking of global continuous symmetry if we measure it with the corresponding source as described in this section. In the presence of the potential term (14) the phase transition disappears from the effective theory (18) in both dimensions. The model defined in (16)-(18) with (19) we call the $XY$ model with the local $Z(2)$ symmetry and use the notation $XYL$ to distinguish it from the usual $XY$ model. Its study is the subject of the next section.
3 The $XYL$ model and the noncompact model in the weak coupling region

The $d$-dimensional $XYL$ model can formally be viewed as a combined model of $XY$ models in the same dimension with ferromagnetic and antiferromagnetic couplings. Averaging is performed as a sum (difference) of the weights $e^{\lambda S_l}$ of these models, $S$ is a density of the action, $l$ is a link, if $l$ does not belong to $\Omega$ (if $l \in \Omega$). Our particular case is slightly different, however, since the plaquettes in adjacent time slices are not interacting, so that the only geometry of 2-surfaces is important. It is clear from (14)-(17) that we can define the corresponding partition function for 2-d theory as

$$Z_{\Omega} = 3D \prod_{t=3D1}^{N_t} \int d\phi_x e^{\mu \sum_x V(\phi_x)} \prod_{t \in L_t} \cosh(\lambda \cos \Delta \phi) \prod_{l \in L_t} \sinh(\lambda \cos \Delta \phi).$$  (20)

In our case, $L_t \in \Omega$ is a set of closed loops on a time slice $t = 3Dconst$ whose geometry is connected with the closed surfaces in $d = 3D3$ dimensions and

$$\sum_{t=3D1}^{N_t} L_t = 3D |\Omega_{p0}|,$$  (21)

where $|\Omega_{p0}|$ is a number of time-like plaquettes on a surface $\Omega$. Thus, as the first step in study of these systems we could digress from this connection and investigate a pure 2−d theory considering $L_t$ as arbitrary closed loops. To do that we rewrite (20) as

$$Z_{\Omega} = 3D \prod_{t=3D1}^{N_t} 2^{-N_t} \sum_{\sigma_l=3D1 \pm 1 \in L_t} \sigma_l Z_{XYL}(\sigma),$$  (22)

$$Z_{XYL}(\sigma) = 3D \int_0^{\pi} \prod_x d\phi_x \exp[\mu \sum_x V(\phi_x) + \lambda \sum_{x,n} \sigma_n(x) \cos(\phi_x - \phi_{x+n})].$$  (23)

We omitted all bars here considering that $x = 3D(x_1, x_2)$ to the end of this section, $n = 3D1, 2$. The symmetry properties of the model can be easily deduced from the last two equations. At $\mu = 3D0$ the model is invariant under global transformations

$$\phi_x \to \phi_x + const,$$  (24)

and under local discrete $Z(2)$ transformations

$$\phi_x \to \phi_x + \pi n_x, \quad \sigma_k(x) \to -\sigma_k(x), \quad k = 3D \pm n,$$  (25)

where $n_x$ is an arbitrary integer. The last symmetry one sees directly from (20): $\cosh \lambda \cos \Delta \phi$ is identically invariant under (23) whereas $\sinh \lambda \cos \Delta \phi$ changes a sign if $n_x$ is an odd number. Because the site variable $\phi_x$ must enter a closed loop = even number of times, the product of $\sinh \lambda \cos \Delta \phi$ is left invariant under (25). This is, of course, a consequences of the definition (17) where the variable $\phi$ is rather a link.
variable. Since every link enters two or four times in a closed surface, the partition
function in (14) is also invariant under the corresponding local transformations. This
property means that an expectation value of any $Z(2)$ noninvariant quantity equals
zero as a consequences of Elitzur’s theorem. However, the correlation functions which
are invariant may show a critical behaviour like in the $XY$ model showing the phase
transition of the Kosterlitz-Thouless type. Following the definition (19), the potential
$V(\phi_x)$ breaks the global symmetry (24). Hence, the full theory is invariant only under
the discrete transformations (25). We expect in this case that the only phase is available
in the whole range of couplings: correlation functions are left finite at large distances
that corresponds to the perimeter law for the adjoint sources.

In a similar manner we can define the sine-Gordon model with the local discrete
symmetry. Choosing the potential (19) in the sine-Gordon form one gets

$$Z_{SG} = 3D \int_{-\infty}^{\infty} d\phi_x \sum_{k_n(x)} \exp[\mu \sum_x \cos 2\phi_x - \lambda \sum_{x,n} (\phi_x - \phi_{x+n} + \pi k_n(x))^2].$$

The local symmetry $\phi_x \rightarrow \phi_x + \pi n_x$ presented here is similar to the discrete symmetry (24). There is a spontaneous breaking of the global $Z(2)$ symmetry in the standard sine-
Gordon model. As in the previous case we expect that the only phase with unbroken
local $Z(2)$ symmetry can be found in the model (26) and all the properties of the model
are very close to those of $XY_L$ model in the region of weak coupling.

A dual representation for $XY_L$ model one obtains similarly to $XY$ model. Summing
over $\sigma_n(x)$ we have on the dual lattice, keeping the notations of the original lattice for the indices

$$\sum_{\sigma_l=3D\pm 1} \prod_{l \in L_t} \sigma_l Z_{XY_L}(\sigma) = 3D \sum_{s_x=3D-s_{x+n}-j_{n}(x)}^{3D-\infty} \sum_{j_{n}(x)=3D-\infty}^{3D-\infty} \prod_{l \in L_t} I_2(s_x-s_{x+n-j_{n}(x)})(\lambda) \prod_{l \in L_t} I_2(s_x-s_{x+n-j_{n}(x)}+1)(\lambda) \prod_{p} B_{2j_p}(\mu),$$

$I_k$ is the modified Bessel function. The plaquette variable $j_p$ is defined as $j_p = 3Dj_n(x) + j_m(x+n) - j_{n}(x+m) - j_m(x)$. The coefficients $B_m$ are the following Fourier components

$$e^{\mu V(\phi)} = 3D \sum_{m} B_{2m}(\mu) e^{2im\phi}. $$

Only even terms are present here because of the property

$$V(\phi + \pi n) = 3DV(\phi).$$

$L$ in (27) is a set of links dual to links $L_t$ forming a closed loop on the original lattice.

To display the expected phase structure two types of correlation functions can be
defined and evaluated in the $XY_L$ model. The first one is the spin-spin correlation
function

$$\Gamma_{m,k} = 3D < e^{im\phi_0} e^{-ik\phi_R} > .$$
Let us discuss its behaviour when $R \to \infty$ at asymptotically small and large values of $\lambda$. We omit all calculations as they follow step by step similar calculations in $XY$ model. First of all it is clear that $\Gamma_{m,k} = 3D0$ if $m$ and/or $k$ is an odd number in force of the symmetry (25). One sees this from the calculation of the dual representation (27) or from the mean-field approach on the original lattice even before the summation over closed loops. The behaviour of the correlation functions $\Gamma_{2m,2k}$ is very close to spin-spin correlations in $XY$ model. In the next section we shall argue that these correlators carry a memory of the adjoint Wilson loop of the original model (1). Therefore, we expect that closed loops $L_t$ do not influence essentially its behaviour since they have appeared after summation over $Z(2)$ variables and the adjoint sources do not feel these configurations. Neglecting all $L_t$ one finds at $\mu = 3D0$ the following asymptotics ($m = 3Dk$)

$$\Gamma_{2m} \approx \exp[-R \ln \frac{(2m)!}{\lambda^{2m}}], \quad \lambda \to 0, \quad (31)$$

$$\Gamma_{2m} \approx \exp[-\frac{2m^2}{\lambda} \ln R], \quad \lambda \to \infty. \quad (32)$$

These asymptotics exhibit the Kosterlitz-Thouless phase transition: the expectation value of the spin $< e^{2im\phi_x} >= 3D0$ and the global symmetry (24) is unbroken at all $\lambda$. The potential $V(\phi)$ destroys the global symmetry: one finds that $< e^{2im\phi_x} >$ differs from zero and the correlation function $\Gamma_{2m}$ is finite in the limit $R \to \infty$ at all values of $\lambda$. This may correspond to the perimeter law behaviour of the adjoint loops as we have discussed before.

The second type of correlation functions is defined as follows

$$\Gamma_{\sigma_c} = 3D < \prod_{l \in C} \sigma_l >, \quad (33)$$

where $C$ is a closed loop and

$$\Gamma_{W}^{\sigma_c} = 3D < e^{i\phi_0} \prod_{l \in L} \sigma_l e^{-i\phi_R} >=, \quad (34)$$

where $L$ is a path between points 0 and $R$. Both correlation functions are invariant under local transformations (25). The correlation function (33) defines a time-like plaquette dependence of the expectation value of $Z(2)$ Wilson loop (see next section for detail) while (34) defines a similar dependence of $U(1)$ Wilson loop which could also be determined in the model (1). Unlike the spin-spin correlation functions, both $\Gamma_{\sigma_c}$ and $\Gamma_{W}^{\sigma_c}$ decrease exponentially when $C \to \infty$ and $R \to \infty$, correspondingly, at all values of $\lambda$. One finds this from the simple estimates on the original lattice when $\lambda$ tends zero and on the dual lattice in the opposite case. This is a hint on the area law for $Z(2)$ Wilson loop at all couplings and we shall verify this in the next section. All this is valid only in the case of absence of the loops $L_t$. However, the closed loops $L_t$ might be of great importance here, especially in the region of small gauge coupling $\lambda \to \infty$ and this influence must be studied separately.
We are ready now to present an effective noncompact model for \( A_0 \) gauge potential in the weak coupling limit. Our result is the noncompact sine-Gordon model defined in (26) with modification that includes the summation over 2-surfaces. The following formulae and the finite result are valid for the theory in \( d \)-dimensions. The usual way of the calculation in the region \( \lambda \to \infty \) is to expand \( \cos \Delta \phi \approx 1 - \frac{1}{2}(\Delta \phi)^2 \). However, this implies a loss of the local symmetry (25). To recover the symmetry we rewrite the expression (20), omitting an irrelevant here product over \( t \), as

\[
Z_\Omega = 3D \int \prod x \, d\phi_x e^{\mu \sum_x V(\phi_x)} \sum_{r_l=3D0,1} \exp[\lambda \cos(\Delta \phi_x + \pi r_l)] \prod_{l \in L_t} (-1)^{r_l}. \tag{35}
\]

Up to infinite constant we may sum in (35) over \( r_l \) from \(-\infty \) to \(+\infty \). This infinite constant will be cancelled from all expectation values. After this we can expand \( \cos(\Delta \phi_x + \pi r_l) \) around unity. This leads us to

\[
Z_\Omega = 3D \int_{-\infty}^{\infty} \prod x \, d\phi_x e^{\mu \sum_x V(\phi_x)} \sum_{r_l=3D-\infty}^{\infty} \exp[-\frac{\lambda}{2} \sum_l (\Delta \phi_x + \pi r_l)^2] \prod_{l \in L_t} (-1)^{r_l}. \tag{36}
\]

This procedure can be made more rigorous in the following way. Integrating out \( \phi_x \)-variables we get

\[
Z_\Omega = 3D \sum_{k_l} \sum_{m_x} \prod_{l \in L_t} I_{2k_l}(\lambda) \prod_{l \in L_t} I_{2k_l+1}(\lambda) \prod_x B_{2m_x}(\mu) \delta(m_x + \sum_{n=3D-d}^d k_n(x)). \tag{37}
\]

\( B_m \) was defined in (28), the symbol \( \delta \) means the Kronecker delta function and \( k_n(x) = 3D - k_n(x) \). Using the Poisson summation formula to calculate the sum over \( k_l \) we may change the Kronecker delta into the Dirac delta function since \( k_n(x) \) becomes a continuous variable. This again generates an infinite constant which is cancelled from locally invariant expectation values. On the other hand this preserves the local periodicity of the original action. Taking asymptotic of the Bessel function at large \( \lambda \)

\[
I_k(y) \approx \frac{1}{\sqrt{2\pi y}} e^y \exp[-\frac{k^2}{2y}], \tag{38}
\]

we can integrate over \( k_n(x) \) and sum over \( m_x \). This yields the formula (36) justifying the simple calculations presented above. Substituting (36) into (14) and remembering that in our case the closed loops \( L_t \) in the plane \( t = 3D\text{const} \) are connected to the 2-surfaces we get the final expression for the noncompact model

\[
Z = 3De^{-\lambda N_p (\cosh \lambda)^{N_p}} 2^N \sum_{\Omega} (\tanh \lambda)^{\vert \Omega_p \vert}
\prod_{t=3D1}^{N_t} \int_{-\infty}^{\infty} \prod x \, d\phi_x e^{\mu \sum_x V(\phi_x)} \sum_{r_{p_0}=3D-\infty}^{\infty} \exp[-\frac{\lambda}{2} \sum_{p_0} (\Delta \phi_x + \pi r_{p_0})^2] \prod_{p_0 \in \Omega} (-1)^{r_{p_0}}. \tag{39}
\]

Here, \( r_{p_0} \) belongs to a time-like plaquette and the effective action is diagonal in \( t \), so that we consider \( \phi_x \) as the site variable. (39) is a noncompact analogy of the compact
model (9) in the weak coupling region. The model is clearly different both from the naive noncompact model and from the model proposed in [2] to include the local $Z(2)$ symmetry and the sum over 2-surfaces as the result of the summation over all $Z(2)$ configurations of the original Wilson mode= 1.

4 Wilson loops in the effective models and the area law

The usual way to find out the confining properties of the pure gauge theory is to study the behaviour of the fundamental Wilson loop which gives a potential between the static quark-antiquark pair. Our approach allows us to restrict the calculations to the $Z(2)$ Wilson loop

$$W_z(C) = 3D \prod_{l \in C} z_l,$$

where $C$ is a closed rectangular loop in the $t - x$ plane. The expectation value of the loop is calculated with the partition function (9) and can be written down by help of the representation (13) in the form

$$\langle W_z(C) \rangle = 3DZ^{-1} \sum_{z_{\mu}(x)} \int D\bar{U}_0(x) \prod_{\bar{p}} [\cosh \lambda + \sinh \lambda Z(\bar{p})] \prod_{p_0} [\cosh(\lambda \cos \Delta \phi) + \sinh(\lambda \cos \Delta \phi) Z(\partial p_0)] \prod_{l \in C} z_l.$$  

Let $W$ be the $SU(2)$ Wilson loop $W = 3D_4 \prod_{l \in C} U_l$. From obvious inequality $\langle W \rangle \leq \langle W_z \rangle$ it follows that if $Z(2)$ Wilson loop obeys the area law behaviour, $SU(2)$ Wilson loop will show the same feature in the theory defined by (9) with the measure (19). Presumably, even stronger statement can be formulated if we remember the result of [13], namely

$$\langle W \rangle_{SU(N)} \leq \langle W_z \rangle_{Z(N)},$$

which tells us that $SU(N)$ Wilson loop will obey the area law in the Wilson $SU(N)$ gauge model if $Z(N)$ Wilson loop in $Z(N)$ gauge model obeys area law at the coupling constant $g^2/N$. As we have discussed many times, the main effect of $\cos \Delta \phi$ in (9) is to disorder the system at large $\lambda$. Thus, we expect that the following inequality has to be fulfilled

$$\langle W \rangle_{SU(2)} \leq \langle W_z \rangle,$$

where $\langle W_z \rangle$ is calculated in the statistical ensemble defined by the partition function (9).

First of all we need to calculate representations for $W_z$ both in the compact model (14) and in the noncompact model (39). Summing over $z_{\mu}(x)$ in (11) one obtains

$$\langle W_z(C) \rangle = 3DZ^{-1}(\cosh \lambda)^{N_p} 2^{N_l} \sum_{S(\partial C)} \sum_{\Omega/p \in S(\partial C)} (\tanh \lambda)^{|S_p|} (\tanh \lambda)^{|\Omega_p|}.$$
\[
\sum_{\sigma_{p_0}=3D \pm 1} \int_{-\pi/2}^{\pi/2} \prod_x d\phi_x \exp[\mu \sum_x V(\phi_x) + \lambda \sum_{p_0} \sigma_{p_0} \cos(\Delta \phi)] \left( \prod_{p_0 \in S(\partial C)} \sigma_{p_0} \right) \left( \prod_{p_0 \in \Omega} \sigma_{p_0} \right). \tag{43}
\]

Interpretation of this formula is obvious: the expectation value of the \(Z(2)\) Wilson loop is expressed through the sum over all possible surfaces \(S\) on the lattice whose boundary is the loop \(C\); \(|S_p|\) is a number of space-like plaquettes on the surface \(S\). Every term in the sum over \(S\) includes the sum over all 2-surfaces \(\Omega\) which have no common plaquettes with given surface \(S\). Repeating all the steps from the end of the previous section we come to the expression for the Wilson loop at large \(\lambda\) of the form (up to a constant)

\[
< W_z(C) > = 3DZ^{-1}(\cosh \lambda)^{N_\mu/2} \sum_{S(\partial C)} (\tanh \lambda)^{|S_p|} \sum_{\Omega/p \in S(\partial C)} (\tanh \lambda)^{|\Omega_p|} \sum_{r_{p_0}=3D-\infty}^\infty \int \prod_x d\phi_x \exp[\mu \sum_x V(\phi_x) - \lambda/2 \sum_{p_0} (\Delta \phi + \pi r_{p_0})^2] \left( \prod_{p_0 \in S(\partial C)} (-1)^{r_{p_0}} \right) \left( \prod_{p_0 \in \Omega} (-1)^{r_{p_0}} \right). \tag{44}
\]

The treatment of the adjoint Wilson loop \(W_{ad}(C)\) in the present approach is much simpler since it does not include \(Z(2)\) variables. We may put down the following equation for the adjoint loop making use of the diagonality of the effective action in time indices and the fact that the global symmetry \(\phi \to -\phi\) is unbroken in our model

\[
< W_{ad}(C) > = 3D < \prod_t e^{2i(\phi_0(t)-\phi_R(t))} > \tag{45}
\]

(we recall that in our approach the space gauge potentials \(A_n = 3D0\)).

To understand the behaviour of expectation values in Eqs. (43)-(45) we consider first the approximation where only time-like plaquettes are taken into account. We have from Eq. (43) in this case

\[
< W_z(C) >_0 = 3DZ^{-1} \sum_{\sigma_{p_0}=3D \pm 1} \left( \prod_{p_0 \in S_{min}(\partial C)} \sigma_{p_0} \right) \int \prod_x d\phi_x \exp[\mu \sum_x V(\phi_x) + \lambda \sum_{p_0} \sigma_{p_0} \cos(\Delta \phi)], \tag{46}
\]

where \(S_{min}\) is the minimal surface enclosed by the loop \(C\) and lying in the \((t-x)\)-plane (only such a surface survives in this limit since to go out of the \((t-x)\)-plane at least four space-like plaquettes must be present). Thus, we come to the expression for the fundamental Wilson loop of the form (summing up over \(\sigma\))

\[
< W_z(C) >_0 = 3D < \prod_{p_0 \in S_{min}} \tanh(\lambda \cos \Delta \phi) >, \tag{47}
\]

which allows us to make a simple estimate of the module of the expectation value

\[
<| W_z(C) | >_0 \leq (\tanh \lambda)^{S_{min}}. \tag{48}
\]
Similar appraisals performed for noncompact formulation (44) lead to the result

\[ < W_z(C) >_0 = 3D \prod_{p_0 \in S_{\min}} \frac{\theta_4(\gamma, i\lambda \pi \Delta \phi/2)}{\theta_3(\gamma, i\lambda \pi \Delta \phi/2)} >. \]  

(49)

Here, \( \theta_i \) is the Jacobi theta-function and we have denoted \( \gamma = 3D \exp(-\frac{\lambda}{2} \pi^2) \). From here, we find the following bound in the region \( \gamma \to 0 \) (\( g^2 \sim 0 \))

\[ < | W_z(C) | >_0 \leq e^{-4\gamma S_{\min}}. \]  

(50)

The adjoint Wilson loop (45) in the present approximation reduces to the form (30) displaying, hence, the expected perimeter law behaviour (see (31), (32) and the discussion around them).

One can say, of course, that, in fact, the time-like plaquette approximation is close to the strong coupling regime where only \( S_{\min} \) survives in the thermodynamical limit. As such, this is true and the bound (48) cannot be trusted in the weak coupling region. However, we consider that the bound (50) is a great achievement of the whole approach. We would like to recall at this point that naive noncompact lattice theory with time-like plaquettes only does not show area law behaviour at any couplings [1]. Thus, the periodicity of the compact theory, which we have tried to reproduce in our noncompact effective model, can be of great importance for confinement at any couplings (actually, the noncompact theory (39) is valid only at small couplings with respect to the original model [1] but, as such, can be formally studied at all couplings). We find it surprisingly good that the bound (50) is in agreement with the expected asymptotic freedom behaviour of the string tension that is

\[ \alpha = 3D4\gamma = 3D4 \exp\left[-\frac{\pi^2}{g^2}\right]. \]  

(51)

Eq.(51) leads to the usual perturbative relation

\[ g^2 = 3D - \frac{\pi^2}{2 \log(a\Lambda)}, \]  

(52)

if we impose the condition \( \frac{da}{da} = 3D0 \), where \( a \) is the lattice spacing.

Calculations in the full theory with space-like plaquettes included demand a very complicated treatment of the summation over 2-surfaces and will be considered in a separate publication [14]. Let us now discuss qualitatively what we expect in this case and why we think the theory (9) could be confining in the weak coupling region. We turn first to the compact theory and to the corresponding representation for the Wilson loop (43).

The expectation value of the Wilson loop in ordinary \( Z(N) \) gauge theory can be written down using (43) as

\[ < W_z(C) >= 3DZ^{-1}(\cosh \lambda)^{N_r 2^N_i} \sum_{S(\partial C)} (\tanh \lambda)^{|S|} \sum_{\Omega/\mu \in S(\partial C)} (\tanh \lambda)^{|\Omega|}. \]  

(53)
In the strong coupling region \((\tanh \lambda \sim 0)\) only surfaces of small sizes contribute to the partition function and we are allowed to neglect the restriction in the numerator of the last formula which forbids the summation over surfaces \(\Omega\) if they have common plaquettes with given surface \(S(\partial C)\). Hence, in this region we approximately have

\[
< W_z(C) > \sim \sum_{S(\partial C)} (\tanh \lambda)^{|S|} L(S),
\]

where \(L(S)\) is a number of the surfaces spanning the loop \(C\). Below the critical point this sum is known to be convergent and we have in the thermodynamical limit the area law

\[
< W_z(C) > \sim (\tanh \lambda)^{S_{\text{min}}}.\]

As \(\lambda\) grows, however, \(L(S)\) becomes growing faster than \((\tanh \lambda)^S\) goes to zero. This leads to the phase transition and to the perimeter law for the Wilson loop.

Two factors may, hopefully, change this behaviour. The first of them is already contained in the pure \(Z(N)\) system. When we move towards the critical point, Eq.(54) ceases to be a proper equation for the expectation value even below \(\lambda_{\text{cr}}\). The reason is that at the same time the number of 2-closed surfaces contributing to \(< W_z(C) >\) is much increased, too, and we may not neglect the restriction forbidding the summation over \(S(\partial C)\) having common plaquettes with closed surfaces. This strongly reduces effective \(L(S)\) (or, equivalently, lowers the effective coupling) but not sufficiently to preserve the system of the transition to the deconfinement phase. The second, most essential factor, appears from the integration over \(SU(2)/Z(2)\) subgroup on the time-like plaquettes. To see that we consider the following analog of Eq.(54) which we expect to approximately hold for \(< W_z(C) >\) in the whole range of couplings for Eq.(43)

\[
< W_z(C) > \approx \sum_S (\tanh \lambda)^{|S|} L_{\text{eff}}(S) < \prod_{p_0 \in S} \sigma_{p_0} > .
\]

Here, \(S = 3DS(\partial C), L_{\text{eff}}(S)\) is the effective number of surfaces containing \(| S |\) plaquettes reduced by the restriction stressed just above (the first factor). \(< \cdots >\) is calculated in the statistical ensemble defined by the partition function (44) either with the ordinary \(SU(2)\) measure or with the measure (13). On every time slice \(< \prod_{p_0 \in S} \sigma_{p_0} >\) forms a set of closed loops where this expectation value is reduced to the correlation function \(\Gamma_{\sigma_c} (33)\) in XYL model. This correlation function is expected to decrease exponentially at all values of coupling constant, which implies the area law behaviour for the expectation value in (55= ). We introduce now the effective coupling for time-like plaquettes as

\[
< \prod_{p_0 \in S} \sigma_{p_0} > = 3D[\gamma_{\text{eff}}(\lambda)]^{S_0},
\]

where \(S_0\) is a number of time-like plaquettes on the given surface \(S(\partial C)\). The simplest estimates show that \(\gamma_{\text{eff}}\) decreases much faster than the corresponding coupling of the pure \(Z(N)\) gauge model where \(\tanh \lambda \to 1\) when the gauge coupling goes to zero. We
suppose now that $L_{\text{eff}}(S)$ including a number of time-like plaquettes $S_0 > S_{\text{min}}$ is suppressed by the effective coupling $\gamma^{S_0}$. Then, we have

$$< W_z(C) > \approx (\gamma_{\text{eff}})^{S_{\text{min}}} \sum_S (\tanh \lambda)^{|S_p|} L_{\text{eff}}(S) \delta(S_0 - S_{\text{min}}).$$

This formula produces the expected area law in the thermodynamical limit and when $S_{\text{min}} \to \infty$ for all couplings. Yet, the critical point may still exist since the number of the surfaces with fixed $S_0$ but different configurations of space-like plaquettes can be larger than $(\tanh \lambda)^{|S_p|}$. The point is that in this critical point the string tension does not turn into zero as could be seen from the last equation unlike $Z(N)$ gauge model. Hence, we may interpret the critical point (if it really exists) as a point of a crossover transition (or a roughening transition) which has been observed in Monte-Carlo simulations many times and which does not affect the string tension.

Extending now these observations to the noncompact representation (44) we observe that the only difference is in $\gamma_{\text{eff}}$ which becomes in this case

$$\gamma_{\text{eff}}^{S_0} = 3D < \prod_{p_0 \in S} (-1)^{r_{p_0}} >.$$

Now, $< ... >$ is calculated with the partition function (39). We have the same qualitative feature of the decreasing of an effective coupling as in the previous case. Hence, all discussed arguments can be applied for the noncompact representation. As a result, we are convinced that both the model (3) and the noncompact model (39) could be confining models with the same mechanism of confinement. These arguments have a general character, therefore we believe that discussed mechanism could be a common mechanism for $SU(N)$ both in $d = 3D3$ and in $d = 3D4$ dimensions which preserves the system from the deconfining transition. For the real proof, however, one has to prove that representation (55) is a reliable one in the weak coupling region and to find a bound on $L_{\text{eff}}(S)$ at large $S$.

5 Summary

In this article we have assumed that the compact Wilson theory and noncompact lattice Yang-Mills theory belong to different universality classes and the latter in its naive form is unable to confine. We have demonstrated that if we take into account $Z(N)$ local invariance of the Wilson compact model and find weak coupling limit of the model after summation over $Z(N)$ variables the resulting model differs from the naive lattice Yang-Mills theory and possesses the necessary confining properties. Our real calculations were done for the $SU(2)$ Wilson model in three dimensions but it is clear from the whole procedure that the generalization to $SU(3)$ gauge group and to higher dimensions is straightforward.

Let us turn once more to our starting model (3) and summarize the facts justifying its investigation. The lines of arguments, leading from (4) to (4), indicate that the partition function (6) should describe the main contribution to the original partition
function in the gauge where all $SU(2)/Z(2)$ space-gauge matrices are chosen to be as close to unity as possible. Truly, we have not given a rigorous proof of that but hopefully it can be done using a method developed in [9]. Qualitative discussion of what might happen if we have taken into account corrections resulting from integration over noncompact Yang-Mills space-gauge potentials shows that these corrections cannot essentially spoil or even change the effective action in [9] (we speak of corrections to $Z(2)$ configurations; certainly, there will be a lot of contributions not directly related to $Z(2)$). Also, the inequality (42) is to be emphasized at this point. Had it been proved rigorously it would alone justify the investigation of the model [8] because our hope is to prove an area law for the space-time Wilson loop in this model for weak coupling. Having (42) in hand, one can claim the same is valid for full $SU(2)$ theory.

Summarizing our results we would like to mention two points. The first of them are local discrete transformations (25). It was essential to preserve this symmetry in the noncompact theory. Let us suppose that we neglect the symmetry, for example, expanding $\cos \Delta \phi$ simply as $1 - (\Delta \phi)^2/2$. It is easy to verify then that we would get almost no effect on the coupling $\gamma_{eff}$. Why it is so can be understood if we turn to the expression (9) for the partition function. Substituting this expansion into time-like plaquettes we come to the Yoneya expansion which we already discussed. In our terms it would lower somehow the effective coupling $\gamma_{eff}$ but obviously not sufficient to shift the critical point to zero. As shown by our estimations including the invariant measure cannot save the situation. We would, thus, expect that the similar symmetry must be present in a full noncompact version of the Wilson gauge theory with the space gauge-potential included.

The second point concerns again the invariant measure. In the end of section 2 and in section 3 we have discussed the role of the measure in providing the adjoint Wilson loop with the perimeter law for large loops at any couplings. Now, what can we say about the contribution of the measure into the fundamental ($Z(2)$ in our case) Wilson loop? The bounds (48), (49) do not depend on whether the measure is present in the partition function or not. Thus, in the strong coupling region the very fact of confinement does not depend on the invariant measure. That is clear because confining forces in this region have origin in the very $Z(N)$ system and the $SU(2)/Z(2)$ part of the action is not important here for $Z(2)$ Wilson loop. However, the situation could be quite different in the weak coupling region or if we consider $SU(2)$ Wilson loop. This is because the term $\mu V(\phi)$ affects significantly the expectation value in (53) or in (58) at large $\lambda$ lowering the effective constant $\gamma_{eff}$. The question may arise at this point, namely how to calculate the coupling $\mu$ at the invariant potential. As far as our calculations are concerned, we can work with the ordinary $SU(2)$ measure since all our Gaussian integrals are well defined in this case, too. It could be possible then to define $\mu$ in a renormalizable theory. Let us imagine that we have managed to calculate an expectation value of the Wilson loop and to find the string tension to be a function of $g^2$ and $\mu$. Then we can calculate the lines of constant physics which will give us the condition on the coupling constants as a function of the lattice spacing. Thus, $\mu = 3D\mu(a)$ we will find under assumption that the expectation value of the Wilson
loop in the theory with the term $\mu V$ is on the same line of constant physics as the Wilson loop in the theory with the usual $SU(2)$ measure. In this way the invariant term $\mu V$ could be included into the renormalizable noncompact Yang-Mills theory.

Concluding the summary we want to make some remarks on the intriguing question what mechanism of confinement do our approach and calculations support? It may seem, at least at the first sight, that this approach is more in the spirit of the mechanism of confinement caused by the $Z(N)$ vortex condensate proposed in [4]. It is plausible that it is the case but in order to show this the condensate (existence of which is a sufficient condition for confinement) defined in [4] has to be calculated in the weak coupling region. Since this problem is much easier to solve in our approach, we will return to this point in the next publication. At the present stage, however, we do not think that our approach is in contradiction with the monopole mechanism of confinement, e.g. the one discussed in the abelian projection method. Indeed, our effective 2-dimensional (the same can be said of $3-d$ theory) $XYL$ model includes vortex configurations (not that ones mentioned just above) playing an important role in the phase structure discussed in great details in sections 2 and 3. Truly, these properties concerned mainly the behaviour of the adjoint Wilson loop but it is possible that these configurations become relevant to the fundamental loop at the weak coupling. These vortices might be reminiscent of monopole configurations, contained in the $SU(2)/Z(2)$ sector of the original three (or four) dimensional model. Defining $U(1)$ Wilson loop in the model (9) by help of (34) one investigates this possibility within presented approach.

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References

[1] O.A. Borisenko, V.K. Petrov, G.M. Zinovjev and J. Boháčik, Phase structure and confinement properties of noncompact gauge theories I, [hep-lat/9508001], 1995.

[2] K. Johnson, L. Lelouch, J. Polonyi, Nucl.Phys. B367 (1991) 67= 5.

[3] M. Gópfert, G. Mack, Com.Math.Phys. 81 (1981) 97; 82 (1982) 545.

[4] G. Mack, V. Petkova, Ann.Phys. 125 (1980) 117.

[5] G. Mack, Phys.Lett. B78 (1978) 263.

[6] O.A. Borisenko, V.K. Petrov, G.M. Zinovjev, Theor.Mat.Fiz. 80 (1989) 381.

[7] O.A. Borisenko, V.K. Petrov, G.M. Zinovjev, Nucl.Phys. B (Proc. Suppl.) 42 (1995) 466.
[8] K. Wilson, Phys.Rev. D10 (1974) 2445.
[9] D. Zwanziger, Nucl.Phys. B378 (1992) 525.
[10] D. Zwanziger, Nucl.Phys. B412 (1994) 657.
[11] T. Yoneya, Nucl.Phys. B144 (1978) 195.
[12] K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440.
[13] E. Seiler, Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics, Springer-Verlag Berlin Heidelberg New-York, 1982.
[14] Surfaces in a modified $Z(N)$ model and the string tension in the weak coupling region, in preparation.