Balancing cyclic R-ary Gray codes

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Abstract
New cyclic n-digit Gray codes are constructed over \{0, 1, \ldots, R - 1\} for all \(R, n \geq 2\). These codes have the property that the distribution of the digit changes (transition counts) is close to uniform: For every \(n \geq 2\), the number of occurrences of each digit in the n-digit codes is within \(R - 1\) of the average \(R^n/n\), and for the 2-digit codes every transition count is either \(\lfloor R^2/2 \rfloor\) or \(\lceil R^2/2 \rceil\).

1 Introduction
For \(R, n \geq 2\), an \textbf{n-digit R-ary Gray code} is a sequence in which each n-digit string with digits from the set \{0, 1, \ldots, R - 1\} occurs once and any two consecutive strings differ in only one digit and the difference equals \(\pm 1\); that is, the Lee distance equals 1, where the \textbf{Lee distance} between two n-strings is defined to be

\[
d_L(a_1 \ldots a_n , b_1 \ldots b_n) = \sum_{j=1}^{n} \min\{|a_j - b_j|, R - |a_j - b_j|\}.
\]

When the Lee distance between the last and first strings also equals 1, the code is called \textbf{cyclic}. For instance,

\[
10, 11, 01, 00 \quad \text{and} \quad 20, 21, 22, 12, 02, 01, 11, 10, 00
\]

(1)

are cyclic 2-digit R-ary Gray codes, with \(R = 2\) and \(R = 3\), respectively.

It is helpful to consider a cyclic Gray code as a Hamiltonian cycle in the \textbf{R-ary n-cube}, the graph whose vertices are all n-strings with digits from the set \{0, 1, \ldots, R - 1\}
in which two vertices are adjacent if they differ in only one digit by $\pm 1$. Our usual representation of this graph will have the vertices arranged in an $R^{n-1} \times R$ array, where the first $n$ digits of a vertex $n$-string are given by its row label and the last digit by its column label. Figure 1 has the Hamiltonian cycles for the cyclic Gray codes in (1).

Figure 1: cyclic 2-digit R-ary Gray codes for $R = 2, 3, 4$.

The transition sequence of a cyclic Gray code records the successive digit changes in the code, beginning with the change from the first to the second string. For example, the transition sequences for the codes in (1) are:

$$2, 1, 2, 1 \quad \text{and} \quad 2, 2, 1, 1, 2, 1, 2, 1, 1.$$ 

For each $1 \leq j \leq n$, the transition count $TC(j)$ of the digit $j$ is defined to be the number of times $j$ occurs as a transition digit in the code. One measure of uniformity is the distribution of the transition counts. When all transition counts are equal (this common value must necessarily equal $R^n/n$ and $n$ must divide $R^n$), the code is called totally balanced or uniform. The binary and quaternary codes in Figure 1 are totally balanced. In [7, 10] totally balanced cyclic $n$-digit binary codes are constructed for every $n$ which is a power of 2.

When $R^n$ is not divisible by $n$, totally balanced cyclic codes cannot exist and it is reasonable to ask for codes in which every transition count is either $\lfloor R^n/n \rfloor$ or $\lceil R^n/n \rceil$, a phenomenon which we will call well-balanced. When $R$ is even, the cyclicness forces every transition count to be even and so (unlike the cyclic ternary code given above in (1)) for even $R$ every well-balanced cyclic $R$-ary Gray code must be totally balanced and so cannot exist for odd $n$. In [2], the authors asked if a well-balanced (non-cyclic) $n$-digit binary Gray code exists for every $n \geq 2$. Although this continues to be an open question, in [9, Chapter 3] I. N. Suparta has obtained further evidence supporting the conjecture. In Theorem 2 we construct well-balanced 2-digit R-ary Gray codes for every $R \geq 2$. The codes depend on the even/odd parity of $R$, and are produced by a natural inductive process in which well-balanced cyclic $(R+2)$-ary Gray codes are constructed from the $R$-ary ones.

A cyclic $n$-digit binary Gray code satisfying

$$|TC(i) - TC(j)| \leq 2 \quad \text{for every pair } i, j,$$

is referred to as balanced. In [2], G. S. Bhat and C. Savage constructed balanced cyclic $n$-digit Gray codes for every $n \geq 2$ when they completed some details which remained
from the construction of J. P. Robinson and M. Cohn [7]. In a comment added to [2], F. Ruskey noted that T. Bakos [1, pp. 28–37] had much earlier constructed balanced cyclic binary codes; more recently I. N. Suparta [9, Section 3.2] has developed an elegant variant of these constructions. In each, n-digit cyclic binary codes are constructed inductively, using four (n-2)-digit binary codes.

Non-binary codes have been considered, for example, in [4, 3, 5, 6, 8, 9], but the question of constructing fairly uniform cyclic R-ary Gray codes for a general radix \( R > 2 \) has received far less attention than the binary case. In addition to the well-balanced cyclic 2-digit codes we construct in the next section for a general radix, in Theorem 4 and Theorem 7, we show how the simple inductive construction given in Section 3 can be used to obtain fairly uniform cyclic n-digit R-ary Gray codes for all \( R, n \geq 3 \).

2 Well-balanced cyclic 2-digit R-ary Codes

Theorem 1. Let \( R \geq 2 \). If there exists a cyclic 2-digit \( R \)-ary Gray code with transition counts \( \lfloor R^2/2 \rfloor + a \) and \( \lceil R^2/2 \rceil + b \) for some constants \( a, b \), then there exists a cyclic 2-digit \( (R+2) \)-ary Gray code with transition counts \((R+2)^2/2\) + a and \((R+2)^2/2\) + b.

Proof. The given cyclic \( R \)-ary code can be normalized so that it begins with \( R-1 \ 0 \) and ends with the string \( 0 \ \ 0 \). (This can be done because at some point there must be consecutive terms of the form \( c \ d \); \( c \pm 1 \ d \). The required form is then obtained by translating and reflecting if necessary so that the last term is \( 0 \ \ 0 \) and the first term is \( R-1 \ 0 \). Since the transition counts remain the same under these operations, we may assume the original cyclic code has this form.) Therefore, the code corresponds to a Hamiltonian path from \( R-1 \ 0 \) to \( 0 \ 0 \) in the \( R \)-ary square to which the edge from \( 0 \ 0 \) to \( R+1 \ 0 \) can be added to form a Hamiltonian cycle.

Labeling the rows of the \( (R+2) \)-ary square as \( R+1 \ \ R \ \ldots \ 1 \ \ 0 \) and the columns as \( 0 \ \ 1 \ \ldots \ 0 \ \ R+1 \), we first construct a path from the vertex \( R+1 \ 0 \) to \( R-1 \ 0 \) by beginning with the vertex \( R+1 \ 0 \), proceeding horizontally through all vertices of the form \( R+1 \ i \) to \( R+1 \ R+1 \) and then vertically down to \( 0 \ R+1 \). From there we travel to \( 0 \ R \) and continue by proceeding vertically upward to \( R \ R \), traveling horizontally over to \( R \ 0 \) and then ending with the edge to \( R-1 \ 0 \). This initial path covers all vertices in which at least one component is either \( R \) or \( R+1 \). Its transition sequence is: \( R+1 \) twos, followed by \( R+1 \) ones, one 2, and then \( R \) ones, \( R \) twos and one 1, for a total of \( 2R + 2 \) copies of each of 1 and 2. Attaching the given Hamiltonian path from \( R-1 \ 0 \) to \( 0 \ 0 \) to the end of this initial segment results in a Hamiltonian path in the \( (R+2) \)-ary square, and the addition of the edge from \( 0 \ 0 \) to \( R+1 \ 0 \) results in a Hamiltonian cycle. (The quaternary code in Figure 1 was obtained in this way from the binary code, and the diagram in Figure 2 illustrates the general construction.)

Since

\[ \lfloor R^2/2 \rfloor + 2R + 2 = \lfloor (R+2)^2/2 \rfloor \quad \text{and} \quad \lceil R^2/2 \rceil + 2R + 2 = \lceil (R+2)^2/2 \rceil, \]

the code produced is a cyclic 2-digit \( (R+2) \)-ary Gray code with the required transition counts. \( \square \)
Theorem 2. For any $R \geq 2$ there exists a cyclic 2-digit $R$-ary Gray code which is well-balanced, and so for even $R$ the code is totally balanced.

Proof. Consider the statement: For every $n \geq 1$ there exist well-balanced cyclic 2-digit $R$-ary Gray codes for $R = 2n$ and $R = 2n + 1$. The codes given in (1) satisfy this statement for $n = 1$. Application of Theorem 1 with $a = b = 0$ therefore yields the result by induction on $n$. \qed

3 Constructing $(n+1)$-digit codes from $n$-digit codes

One simple way to construct a Hamiltonian path on an $L \times R$ rectangular grid is to traverse successive vertices across rows; that is,

$$1 1, 1 2, \ldots, 1 R; 2 R, \ldots, 2 2, 2 1 \text{ etc.},$$

as illustrated in Figure 3. Notice that the terminal vertex is either $L 1$ or $L R$, according to whether $L$ is even or odd.

Our construction is based on a modification of this simple idea, and will yield cyclic codes whose transition counts are distributed fairly uniformly. The construction involves partitioning the row indices of an $R$-ary $(n+1)$-cube into blocks. Specifically, the rows are first indexed by a fixed cyclic $n$-digit $R$-ary Gray code, $a_1, a_2, \ldots, a_{R^n}$, and then are partitioned into $L$ nonempty blocks of consecutive elements, say

$$B_1 := a_1, \ldots, a_{i_1}, B_2 := a_{i_1+1}, \ldots, a_{i_2}, \ldots, B_L := a_{i_{L-1}+1}, \ldots, a_{R^n}.$$
The digit changes from $a_{i_k}$ to $a_{i_{k+1}}$ will be called the connecting digits of the partition, and the number of times any digit occurs as a connecting digit will be referred to as its connecting multiplicity. This idea of using partitions is at least implicit in earlier constructions. For notational convenience, we assume the code is in a standard form in which the coordinates of the $n$-strings have been permuted if necessary so that 1 is the $L$-th connecting digit, the transition digit from $a_R^1$ to $a_1^1$.

We use this partition to construct a Hamiltonian path in the R-ary $(n+1)$-cube in the following way: First of all, the Hamiltonian path will “respect” the partition $B_1 \cup \ldots \cup B_L$; that is, for every column index $k$ and every block $B_j$, the path must traverse the vertices $a_{i_j+1}^k, a_{i_j+2}^k, \ldots, a_{i_j+k}^k$ consecutively either in that direction or in the reverse direction. This allows us to imagine the $(n+1)$-cube as an $L \times R$ array in which the rows are indexed by the blocks $B_1, \ldots, B_L$. If $R$ is odd, the construction given in (3) yields a Hamiltonian path on the R-ary n-cube (refer to Figure 4). When $R$ is odd and $L$ is even, the addition of the edge from $a_R^1$ to $a_1^0$ results in a Hamiltonian cycle.

When $R$ is even, the path can be adjusted to get a Hamiltonian cycle on the R-ary n-cube in the following way: Within the given R-ary n-cube (with rows labeled by the cyclic Gray code partitioned by $B_1 \cup \ldots \cup B_L$ and columns labeled 0 1 \ldots R-1), consider the $R^n \times (R-1)$ grid consisting of the vertices whose last digit is nonzero. Since $R - 1$ is odd, the Hamiltonian path given in (3) can be constructed on the blocks of this grid. Its initial vertex is $a_1^1$ and the terminal vertex is either $a_{R^n}^R-1$ or $a_{R^n}^1$, depending on whether $L$ is odd or even. In either event, the terminal vertex is adjacent to $a_{R^n}^0$. A Hamiltonian cycle on the R-ary $(n+1)$-cube is obtained by appending this edge to the initial path, following through all vertices with zero second digit in the order:

$$a_{R^n}^0, a_{R^n-1}^0, \ldots, a_1^0,$$

and then ending with the edge to $a_1^1$. This is pictured for $R = 4$ in Figure 5. Regardless of whether $R$ is even or odd, the construction gives a cyclic $(n+1)$-digit R-ary Gray code. In what follows this will be referred to as the code induced by the partition $B_1 \cup \ldots B_L$.

**Theorem 3.** Let $a_1, a_2, \ldots, a_{R^n}$ be any cyclic $n$-digit R-ary Gray code with transition counts $T_1, \ldots, T_n$ (where two digit positions have been transposed if necessary so that the digit change from $a_{R^n}$ to $a_1$ is in the first digit). Let $B_1 \cup \ldots B_L$ be any partition of the
code, and $k_j$ be the connecting multiplicity of the digit $j$. When $R$ is odd and $L$ is even, the $j$-th transition count of the induced cyclic $(n+1)$-digit Gray code is
\[
TC(j) = \begin{cases} 
RT_j - (R - 1)k_j & \text{if } j \leq n \\
(R - 1)L & \text{if } j = n + 1 
\end{cases}.
\] (4)

When $R$ is even the transition counts of the induced cyclic $(n+1)$-digit Gray code are
\[
TC(j) = \begin{cases} 
RT_j - (R - 2)k_j - 2 & \text{if } j = 1 \\
RT_j - (R - 2)k_j & \text{if } 1 < j \leq n \\
(R - 2)L + 2 & \text{if } j = n + 1 
\end{cases}.
\] (5)

**Proof.** First we consider the contribution from the initial Hamiltonian path. Setting $N$ equal to either $R$ or $R - 1$, whichever is odd, the Hamiltonian path on the $R^n \times N$ grid has $(N-1)L$ horizontal lines, and so this initial part of the process forming the Hamiltonian cycle accumulates $(N-1)L$ changes in the digit $n+1$. Every edge within the partition-blocks is traversed in every column, the edge corresponding to the connecting digit from $a_R^n$ to $a_1$ never occurs, and the edge corresponding to every other connecting digit occurs exactly once. Therefore, for every $1 < j \leq n$ the digit $j$ changes
\[
N(T_j - k_j) + k_j = NT_j - (N - 1)k_j
\]
times in this initial segment, and there is one less change for $j=1$.

When $R$ is odd, the cycle is completed by adding one edge which corresponds to a change in the first digit. Since $N = R$ holds in this case, (4) is obtained.

For even $R$, two horizontal edges are added to complete the cycle, giving
\[
TC(n + 1) = (N - 1)L + 2 = (R - 2)L + 2.
\]

As far as additional vertical edges, every edge in the original code except for the one corresponding to transitioning from $a_{R^n}$ to $a_1$ is added once. Since the transition from $a_{R^n}$ to $a_1$ is a change in the first digit, this gives
\[
TC(1) = (R - 1)T_1 - (R - 2)k_1 - 1 + T_1 - 1 = RT_1 - (R - 2)k_1 - 2,
\]
and for all \(2 \leq j \leq n\),

\[ TC(j) = (R - 1) T_j - (R - 2) k_j + T_j = R T_j - (R - 2) k_j. \]

This proves (5). \(\square\)

We end this section by constructing the cyclic 3-digit ternary code induced by a partition of the 2-digit ternary code in (1). Consider any 4-block partition in which 2 occurs only once as a connecting digit. (For instance, \(B_1 := \{20, 21\}, B_2 := \{22\}, B_3 := \{12, 02, 01\}, B_4 := \{11, 10, 00\}\) is such a partition, since the connecting digits are 2,1,1,1.) From (4), the induced code has

\[ TC(1) = 10; \quad TC(2) = 9; \quad TC(3) = 8. \]

Although the code is not totally balanced, \(|TC(j) - R^n/n| \leq 1\) does hold for all \(j\).

4 Balancing cyclic R-ary Gray Codes for odd R

**Theorem 4.** If \(R \geq 3\) is an odd integer then for every \(n \geq 2\) there exists a cyclic \(n\)-digit \(R\)-ary Gray code whose transition counts \(TC(1), \ldots, TC(n)\) satisfy

\[ \left| TC(j) - \frac{R^n}{n} \right| < R - 1 \quad \text{for all} \quad 1 \leq j \leq n. \] (6)

Before proving this result, we apply it to the case when \(n\) divides \(R^n\).

**Corollary 5.** Let \(R \geq 3\) be an odd integer. If \(n \geq 2\) divides \(R^n\), there exists a cyclic \(n\)-digit \(R\)-ary Gray code whose transition counts \(TC(1), \ldots, TC(n)\) satisfy

\[ \left| TC(j) - \frac{R^n}{n} \right| \leq R - 2 \quad \text{for all} \quad 1 \leq j \leq n. \] (7)

**Proof of Corollary 5.** By hypothesis, \(TC(j) - \frac{R^n}{n}\) is an integer, and so the strict inequality in (6) implies (7). \(\square\)

The following technical lemma is used in the proof of Theorem 4.

**Lemma 6.** Let \(R \geq 3\) be an odd integer, and \(n \geq 3\). If \(T\) is a positive integer such that

\[ \left| T - \frac{R^n - 1}{n - 1} \right| \leq \begin{cases} 1/2 & \text{for } n = 3 \\ 1 & \text{for } n = 4 \text{ and } R = 3 \\ R - 1 & \text{otherwise} \end{cases} \] (8)

then there exists an integer \(k, 1 \leq k < T\), such that

\[ 0 \leq RT - k(R - 1) - \frac{R^n}{n} < R - 1. \] (9)
Proof. Because the length of the interval

\[ RT - \frac{R^n}{n} - (R-1) < x \leq RT - \frac{R^n}{n} \]

equals \( R - 1 \), it contains a (unique) integer multiple of \( R - 1 \), which we will call \( k(R-1) \). Therefore,

\[ RT - \frac{R^n}{n} - (R-1) < k(R-1) \leq RT - \frac{R^n}{n}. \]  

(10)

Rearrangement of this gives

\[ 0 \leq RT - k(R-1) - \frac{R^n}{n} < R - 1, \]

which is (9), and the proof is completed by showing \( 1 \leq k < T \).

To prove \( k < T \) from (10) it suffices to show \( RT - R^n/n < (R-1)T \), which can be re-written as \( T - R^n/n < 0 \). Also, since \( k \geq 1 \) would follow from \( k(R-1) > 0 \), by (10) it is sufficient to prove

\[ T < \frac{R^n}{n} \quad \text{and} \quad RT - \frac{R^n}{n} - (R-1) > 0. \]  

(11)

It can be checked that each of the three alternatives in (8) implies

\[ T \leq R^{n-1}/(n-1) + (R-1) < R^n/(n-1) + (R-1), \]

and so for all \( n \geq 3 \),

\[ \frac{R^n}{n} - T > \frac{R^n}{n} - \frac{R^n}{n-1} - (R-1) \]

\[ = \frac{R^n - n}{n(n-1)} - (R-1). \]  

(12)

Defining the sequence \( \{a_n\}_{n \geq 3} \) by \( a_n := R^n/n(n-1) \), we see

\[ \frac{a_n}{a_{n-1}} = R \frac{n-2}{n} = R(1-2/n) \geq R/3 \geq 1 \]

and so \( a_n \geq a_3 = R^3/6 \) for all \( n \geq 3 \). Continuing from (12), we have

\[ \frac{R^n}{n} - T \geq \frac{R^3}{6} - R + 1 = \frac{R(R^2 - 6)}{6} + 1 > 0 \]

for \( R \geq 3 \). This proves the first inequality in (11).

For the second inequality: when \( n = 3 \),

\[ RT - \frac{R^3}{3} - (R-1) \geq \frac{R(R^2 - 9)}{6} + 1 > 0; \]

when \( R = 3 \) and \( n = 4 \):

\[ RT - \frac{R^4}{4} - (R-1) \geq 22 - \frac{81}{4} > 0; \]

for the other cases,

\[ RT - \frac{R^n}{n} - (R-1) \geq a_n - R^2 + 1 \geq \begin{cases} 
R^2(R^3 - 20)/20 + 1 & \text{if } n \geq 5, \ R \geq 3 \\
R^2(R^2 - 12)/12 + 1 & \text{if } n = 4, \ R \geq 5 
\end{cases} \]

is always positive, completing the proof of the second inequality. \( \square \)
**Proof of Theorem 4.** Let \( R \geq 3 \) be odd. For the duration of this proof, the term Balanced 2-digit Gray code will be used to refer to cyclic codes satisfying the conclusion of Theorem 2, while for \( R = 3, n = 3 \), it will refer to the code constructed at the end of the last section. For all other \( n \geq 3 \) and odd \( R \geq 3 \), an \( n \)-digit Gray code will be called Balanced if it is cyclic and its transition counts satisfy (6). It can be checked that under this definition every transition count of a Balanced \((n-1)\)-digit code satisfies (8).

Since Theorem 2 yields a 2-digit Balanced code for all \( R \geq 2 \), it suffices to prove that every Balanced \( N \)-digit code has a partition for which the induced \((N+1)\)-digit code is Balanced. Because of the form of the transition counts for induced codes (refer to (4)), we will use Lemma 6 to find a good partition—one whose induced code is Balanced.

For \( N \geq 2 \), let \( T_1, \ldots, T_N \) be the transition counts of a Balanced \( N \)-digit code. Then the lemma implies the existence of integers \( k_1, \ldots, k_N \) such that for all \( j = 1, \ldots, N \),

\[
1 \leq k_j < T_j \quad \text{and} \quad 0 \leq S_j < R - 1,
\]

where \( S_j := RT_j - k_j (R - 1) - \frac{R^{N+1}}{N+1} \). At this point, the coordinates of the original Gray code should be permuted so that

\[
S_1 \geq S_2 \geq \ldots \geq S_N \geq 0.
\]

Since each \( k_j \) satisfies \( k_j < T_j \), it is possible to build a partition of the \( N \)-digit code in which the connecting multiplicities are \( k_1, \ldots, k_N \). This partition will induce a code, provided the sum of the connecting multiplicities is even. By (4), each of the first \( N \) transition counts in any such code would have the form \( S_j + \frac{R^{N+1}}{N+1} \), and so by (13) satisfies (6). Although the induced code might fail to be Balanced because its last transition count is too small, we will prove there is a slight modification of the partition for which the induced code is Balanced.

Deciding how many connecting digits to add is equivalent to deciding the total (even) number of blocks in the final partition. Since \( \sum_{j=1}^{N} T_j \) equals \( R^n \), the number

\[
\frac{R^{N+1}}{N+1} - (R - 1) \sum_{j=1}^{N} k_j = \frac{R^{N+1}}{N+1} - \frac{N}{N+1} R^{N+1} - (R - 1) \sum_{j=1}^{N} k_j
\]

\[
= R \sum_{j=1}^{N} T_j - \sum_{j=1}^{N} \left( (R - 1)k_j + \frac{R^{N+1}}{N+1} \right) = \sum_{j=1}^{N} S_j
\]

lies in the interval \([0, N(R - 1)]\), by (13). This implies the existence of an integer \( M \), \( 0 \leq M < N \), such that

\[
0 < \frac{R^{N+1}}{N+1} - (R - 1) \left( \sum_{j=1}^{N} k_j + M \right) < R - 1,
\]

where the left-hand inequality is strict because equality would yield the inconsistency of the odd integer \( \frac{R^{N+1}}{N+1} \) being divisible by the even number \( R - 1 \). Setting \( 2L \) to be the even integer of the two consecutive integers

\[
\sum_{j=1}^{N} k_j + M \quad \text{or} \quad \sum_{j=1}^{N} k_j + M + 1,
\]
we obtain an even integer $2L$ such that
\[
\sum_{j=1}^{N} k_j \leq \sum_{j=1}^{N} k_j + M \leq 2L \leq \sum_{j=1}^{N} k_j + M + 1 \leq \sum_{j=1}^{N} k_j + N
\]

and by (15),
\[
-(R - 1) < \frac{R^{N+1}}{N+1} - (R - 1)2L < R - 1. \tag{17}
\]

We proceed by proving it’s possible to modify the original partition to obtain a partition with $2L$ blocks that induces a Balanced code. Since $1 \leq k_j < T_j$ for all $1 \leq j \leq N$, each digit $j$ occurs at least $k_j + 1$ times as a transition digit of the original Balanced $N$-digit code, and so it is possible to refine the partition by choosing each digit at least one more time as a connecting digit. Also, by definition of $2L$ in (16), $D := 2L - \sum_{j=1}^{N} k_j$ (the number of connecting digits which must be added) equals either $M$ or $M + 1$, where $0 \leq M < N$, and so $0 \leq D \leq N$. Each integer
\[
K_j := \begin{cases} k_j + 1 & \text{if } j \leq D \text{ (and } D \neq 0) \\ k_j & \text{if } j > D \text{ (and } D \neq N) \end{cases}
\]
satisfies $1 \leq K_j \leq k_j + 1 \leq T_j$, and it is therefore possible to select $K_j$ occurrences of $j$ as a transition digit in the original code. For any such choice of transition digits, any partition which has those digits as its connecting digits would have $2L$ blocks. Also, by (4) the transition counts $TC(1), \ldots, TC(N + 1)$ of the $(N+1)$-digit code induced by such a partition are
\[
TC(j) = \begin{cases} RT_j - (R - 1)(k_j + 1) & \text{if } j \leq D \\ RT_j - (R - 1)k_j & \text{if } D < j \leq N \\ (R - 1)2L & \text{if } j = N + 1 \end{cases} \tag{18}
\]

Inequalities (13) and (17) imply the code is Balanced unless there exists at least one $j \leq D$ such that
\[
TC(j) - R^{N+1}/(N + 1) = -(R - 1);
\]
that is, $S_j = 0$ for some $j \leq D$. But if this were to happen, then by (14), $0 = S_N = \ldots = S_D$, and
\[
0 = \sum_{j=1}^{N+1} TC(j) - R^{N+1} = \sum_{j=1}^{N+1} \left( TC(j) - \frac{R^{N+1}}{N+1} \right) = \sum_{j=1}^{N} S_j - D(R - 1) + \left( TC(N + 1) - \frac{R^{N+1}}{N+1} \right)
\]
\[
= \sum_{j=1}^{D-1} S_j - D(R - 1) + \left( 2L(R - 1) - \frac{R^{N+1}}{N+1} \right) < (D - 1)(R - 1) - D(R - 1) + (R - 1) = 0,
\]
a contradiction. Therefore, $S_j > 0$ holds for all $j \leq D$, and we have constructed a partition which induces a Balanced code. \hfill ∎
5 Balancing cyclic R-ary Gray Codes for even R

For even R, the construction from Section 3 can be used to get codes which satisfy the following theorem.

**Theorem 7.** If \( R \geq 4 \) is an even integer then for every \( n \geq 2 \) there exists a cyclic \( n \)-digit \( R \)-ary Gray code whose transition counts \( TC(1), \ldots, TC(n) \) satisfy

\[
\left| TC(j) - \frac{R^n}{n} \right| < R - 2 \quad \text{for all } 1 \leq j \leq n. \tag{19}
\]

Before proving this result, we apply it to the case when \( n \) divides \( R^n \).

**Corollary 8.** Let \( R \geq 4 \) be an even integer. If \( n \geq 2 \) divides \( R^n \), there exists a cyclic \( n \)-digit \( R \)-ary Gray code whose transition counts satisfy

\[
\left| TC(j) - \frac{R^n}{n} \right| \leq R - 3 \quad \text{for all } 1 \leq j \leq n. \tag{20}
\]

**Proof of Corollary 8.** By hypothesis, \( TC(j) - \frac{R^n}{n} \) is an integer, and so the strict inequality in (19) implies (20). \(\square\)

As in the last section, for fixed \( R \geq 4 \), the construction in our proof of Theorem 7 proceeds by induction on the number of digits, beginning with the 2-digit codes in Theorem 1. The following technical result is used in the proof.

**Lemma 9.** Let \( R \geq 4 \) be an even integer, and \( n \geq 3 \). If \( T \) is an integer such that

\[
\begin{cases} 
T - \frac{R^{n-1}}{n-1} = 0 & \text{if } n = 3 \\
T - \frac{R^{n-1}}{n-1} < R - 2 & \text{if } n \geq 4
\end{cases}
\]

then there exists an integer \( k, 1 \leq k < T \), such that

\[
0 \leq RT - k(R - 2) - \frac{R^n}{n} < R - 2. \tag{21}
\]

**Proof.** The length of the interval in (21) is \( R - 2 \), and so there exists an integer multiple of \( R - 2 \), say \( k(R - 2) \), which satisfies the following inequality

\[
RT - \frac{R^n}{n} - (R - 2) < k(R - 2) \leq RT - \frac{R^n}{n}.
\]

It therefore remains to prove that \( 1 \leq k < T \), which is ensured by proving

\[
RT - \frac{R^n}{n} - (R - 2) > 0 \quad \text{and} \quad 0 < \frac{R^n}{n} - 2T. \tag{22}
\]

When \( n = 3 \),

\[
RT - \frac{R^3}{3} - (R - 2) = \frac{R(R^2 - 6)}{6} + 2 > 0
\]

and

\[
\frac{R^3}{3} - 2T = \frac{R^2(R - 3)}{3} > 0,
\]

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proving (22) for \( n = 3 \).

When \( n \geq 4 \), \( T > \frac{R^{n-1}}{n-1} - (R - 2) \), and so the first of the inequalities in (22) would follow from
\[
a_n - (R + 1)(R - 2) > 0,
\]
where \( a_n = \frac{R^n}{n(n-1)} \) is the increasing sequence used earlier in Lemma 6. Therefore,
\[
a_n - (R + 1)(R - 2) \geq a_4 - (R + 1)(R - 2) = \frac{R^2(R^2 - 12) + 12(R + 2)}{12} > 0,
\]
Since \( T < \frac{R^{n-1}}{n-1} + R - 2 \), it suffices to prove \( b_n > 2(R - 2) \), where \( b_n := \frac{R^n}{n} - \frac{2R^{n-1}}{n-1} \).

The quotient of two consecutive terms of this sequence is
\[
\frac{b_{n+1}}{b_n} = R \left( 1 - \frac{2}{n+1} \right) \frac{nR - 2(n + 1)}{(n-1)(R-2)n} \geq R \frac{1}{2} \frac{1}{2} \geq 1,
\]
and so \( b_n \geq b_4 \). Since
\[
b_4 - 2(R - 2) = \frac{R(3R^2(R - 4) + 4(R^2 - 6))}{12} + 4 > 0
\]
the second inequality in (22) also holds for \( n \geq 4 \). \qed

\textbf{Proof of Theorem 7.} Let \( R \geq 4 \) be even. For this proof, we will use the term Balanced 2-digit code to refer to the codes satisfying the conclusion of Theorem 2, and so the codes will be totally balanced; that is, every transition count equals \( R^2/2 \). For \( n \geq 3 \), a cyclic \( n \)-digit Gray code will be called Balanced when its transition counts satisfy (19). It suffices to prove every Balanced \( N \)-digit code has a partition for which the induced \( (N+1) \)-digit code is Balanced.

For \( N \geq 2 \), let \( T_1, \ldots, T_N \) be the transition counts of a Balanced \( N \)-digit code. Then the lemma implies the existence of integers \( k_1, \ldots, k_N \) such that for all \( j = 1, \ldots, N \),
\[
1 \leq k_j < T_j \quad \text{and} \quad 0 \leq RT_j - k_j(R - 2) - \frac{R^{N+1}}{N+1} < R - 2.
\]
Since each \( 1 \leq k_j < T_j \), the \( N \)-digit code can be partitioned in such a way that each digit \( j \) occurs \( k_j \) times as a connecting digit. There are \( \sum_{j=1}^{N} k_j \) blocks in the partition, and the transition counts \( TC(1), \ldots, TC(N + 1) \) for the induced code are given in (5). By construction, the first \( N \) transition counts satisfy
\[
-2 \leq TC(1) - \frac{R^{N+1}}{N+1} \quad \text{and} \quad 0 \leq TC(1) - \frac{R^{N+1}}{N+1} < R - 2 \quad \text{for} \quad 2 \leq j \leq N, 0 \leq TC(j) - \frac{R^{N+1}}{N+1} < R - 2. \quad (23)
\]
As in the proof of Theorem 4, for this choice of partition the last transition count \( TC(N + 1) \) might be too small, and the proof is completed by demonstrating a modification of this partition for which the induced code is Balanced.
First we’ll determine the number of partition-blocks for which the last transition count lies in the correct interval. Let $M(R - 2)$ be the least integer multiple of $R - 2$ which is greater than or equal to

$$\frac{R^{N+1}}{N + 1} - TC(N + 1) = R^{N+1} - TC(N + 1) - \frac{R^{N+1}}{N + 1} = \sum_{j=1}^{N} \left( TC(j) - \frac{R^{N+1}}{N + 1} \right).$$

By (23) this number lies in the interval $(-2, N(R - 2) - 2)$, and so $0 \leq M \leq N$ holds, and by definition $M(R - 2)$ satisfies

$$-(R - 2) < \frac{R^{N+1}}{N + 1} - TC(N + 1) - M(R - 2) \leq 0.$$  

If $M = 0$ our partition doesn’t need to be modified, and so we may assume $1 \leq M \leq N$. It suffices to prove at least $M$ of the first transition counts $TC(1), \ldots, TC(N)$ satisfy

$$0 < TC(j) - \frac{R^{N+1}}{N + 1} < R - 2,$$  

because then the partition can be refined by choosing one more of each of these $M$ digits as a connecting digit. If (25) were to hold for at most $M - 1$ digits, then

$$0 = \sum_{j=1}^{N+1} TC(j) - R^{N+1} = TC(N + 1) - \frac{R^{N+1}}{N + 1} + \sum_{j=1}^{N} (TC(j) - \frac{R^{N+1}}{N + 1}) \leq TC(N + 1) - \frac{R^{N+1}}{N + 1} + (M - 1)(R - 2),$$

contrary to the left-hand inequality in (24). Therefore, at least $M$ digits satisfy (25). Let $J_1, \ldots, J_M$ be any $M$ of these digits.

Since each digit $j \leq N$ occurs less than $T_j$ times as a connecting digit, we can modify the partition by choosing one more connecting digit for each of the digits $J_1, \ldots, J_M$ (where the partition is left unchanged when $M = 0$.) The first $N$ transition counts of the new $(N+1)$-digit induced code are $TC^*(1), \ldots, TC^*(N)$ satisfying:

$$TC^*(j) = \begin{cases} 
TC(j) - (R - 2) & \text{if } j = J_i \text{ for some } i \\
TC(j) & \text{otherwise}
\end{cases}.$$  

From (23) and the fact that the stronger inequality (25) holds for $j = J_1, \ldots, J_M$,

$$-(R - 2) < TC^*(j) - \frac{R^{N+1}}{N + 1} < R - 2,$$  

and the last transition count is

$$TC^*(N + 1) = (R - 2) (L + M) + 2 = TC(N + 1) + M(R - 2),$$

which by (24) also satisfies $|TC^*(N + 1) - \frac{R^{N+1}}{N + 1}| < R - 2$. This proves the induced $(N+1)$-digit code is Balanced. □
We end with an application to binary codes.

**Corollary 10.** For each \( n \geq 2 \) there exists a cyclic \( 2n\)-digit binary Gray code whose transition counts \( TC(1), \ldots, TC(2n) \) are such that every average of the form \( \frac{TC(2i-1) + TC(2i)}{2} \) equals either \( \lfloor 2^{2n}/(2n) \rfloor \) or \( \lceil 2^{2n}/(2n) \rceil \).

**Proof.** Consider the following mapping of elements of \( Z_4 \) to 2-digit binary strings

\[
0 \mapsto 00 ; \quad 1 \mapsto 01 ; \quad 2 \mapsto 11 ; \quad 3 \mapsto 10 .
\]

This mapping extends to a mapping of quaternary \( n \)-strings to binary \((2n)\)-strings under which Lee distance is preserved. Therefore, any cyclic \( n \)-digit quaternary Gray code is mapped to a cyclic \( 2n \)-digit binary Gray code, and the transition counts \( TC(1), \ldots, TC(2n) \) of the binary code have the property that \( TC(2i-1) + TC(2i) \) equals the transition count of the digit \( i \) in the 4-ary code for every \( i = 1, \ldots, n \). Using any \( n \)-digit 4-ary Gray code guaranteed by Theorem 7, we have

\[
\left| \frac{TC(2i-1) + TC(2i)}{2} - \frac{2^{2n}}{2n} \right| = \frac{1}{2} \left| TC(2i-1) + TC(2i) - \frac{4^n}{n} \right| < 1 ,
\]

from (19). Since every transition count of a cyclic binary Gray code is even, each such average is an integer and the last inequality implies it must equal either \( \lfloor 2^{2n}/(2n) \rfloor \) or \( \lceil 2^{2n}/(2n) \rceil \), as claimed. \( \square \)

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