Abstract

In this paper, we explore the characteristic polynomials of degree sum adjacency matrix \( DS_A(G) \) of a simple undirected graph \( G \). We state a relation between the structure of a graph and the coefficients of its \( DS_A \) polynomial. A walk generating function is expressed in terms of \( DS_A \) polynomial. Then, we obtain the degree sum adjacency polynomial for some standard graphs, derived graphs and for graph operations.

Keywords: degree sum adjacency matrix, degree sum adjacency polynomial, number of walks

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1. Introduction

Spectral graph theory focuses on the study of the eigenvalues and its relation to the structural properties of a graph. Thus, for a given graph many matrices were defined in this field which records the information about the vertices and the edges of a graph. To state a few, the most explored and widely studied matrices are the adjacency matrix, the laplacian matrix, the signless laplacian matrix, and many more.

In chemistry, many matrices are defined with respect to the distance, incidence and other factors. This motivated many researchers to explore different matrices [11, 13, 14] and study their properties and energy [1, 10]. Zagreb index defined as the sum of the degrees of adjacent vertices
have been studied intensively [4, 5, 6, 7, 15], which relates to the degree sum adjacency (DS_A) matrix. This motivated us to explore the DS_A polynomial for a graph and its operations. In this paper, we consider the degree sum adjacency matrix defined by Zaferani [14] and we discuss relation between the structure of a graph and the coefficients of DS_A polynomial. Then we determine the generating function for the number of walks of each length with respect to the degree sum adjacency matrix. Later we study the DS_A polynomial of complementary graphs, some regular graphs, derived graphs and graph operations in terms of its adjacency polynomial. The proof techniques of the results in this paper are analogous to the results in [3].

Let $G$ be a simple graph with $n$ vertices and $m$ edges. The adjacency matrix of a graph $G$ is defined as $A(G) = [a_{ij}]$, where $a_{ij} = 1$, if $v_i$ is adjacent to $v_j$ and $a_{ij} = 0$ otherwise. The adjacency eigenvalues are denoted as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and they satisfy all the basic relations [3]. The adjacency polynomial of a graph $G$ is denoted by,

$$
\phi(G : \lambda) = \det(\lambda I - A) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n.
$$

(1)

The degree sum adjacency polynomial of a graph $G$ is defined as

$$
P_{DS_A(G)}(\beta) = \det(\beta I - DS_A(G)) = \beta^n + a_1\beta^{n-1} + a_2\beta^{n-2} + \cdots + a_n.
$$

(2)

As $DS_A(G)$ is a real symmetric matrix, its eigenvalues must be real and can be arranged as $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$.

**Lemma 1.1.** [2] The eigenvalues of matrix $xI + yJ$ of order $n \times n$ are $x + ny$ with multiplicity one and $x$ with multiplicity $n - 1$. 

2. Characteristic polynomial of degree sum adjacency matrix

In this section, first we obtain the explicit values of some coefficients of polynomial as defined in Eq. (2). Then obtain the relation between the $DS_A$ characteristic polynomial of a graph and that of its complement.

Some propositions relating the coefficients $a_i$ of $P_{DS_A(G)}(\beta)$ to structural properties of $G$:

A degree sum adjacency matrix of any simple graph $G$ is,

$$DS_A(G) = \begin{pmatrix} 0 & d_{s12} & \cdots & d_{s1n} \\ d_{s21} & 0 & \cdots & d_{s2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{sn1} & d_{sn2} & \cdots & 0 \end{pmatrix}. \quad (3)$$

Then the coefficients of $DS_A$ polynomial of $G$ can be expressed using Sach’s theorem as follows.

Let $G$ be a graph having $n$ vertices and $i$ be any positive number. Then Sach’s graphs $S_i$ are the subgraphs of $G$ with $i$ vertices having disjoint union of $K_2$ and/or $C_n$. Let the number of components of $s \in S$ and number of cycles of $s \in S$ be $P(s)$ and $c(s)$ respectively. Then the coefficient $a_i$ of $\beta^{n-i}$ in Eq. (2) is given by

$$a_i = \sum_{s \in S_i} (-1)^{P(s)} (\text{square of degree sum along the edge}) \cdot 2^{c(s)} \left(\text{product of degrees along the edges of } c(s)\right).$$

Here we state first few coefficients of $DS_A$ polynomial.

- $a_0 = 1$
- $a_1 = 0$
- $a_2 = -\sum_{j<k} d_{sjk}^2$
- $a_3 = -2(\text{multiplying sum of the degrees along the edges of the triangle})$
- $a_4 = \left(\sum_{i<j,k<l} d_{sij}^2 \cdot d_{skl}^2 \text{ where } d_{sij} \text{ and } d_{skl} \text{ are the matching edges}\right) - 2(\text{multiplying sum of degrees along the edges of } C_4)$
- $a_5 = 2(\text{multiplying sum of the degrees along the edges of the triangle and disjoint edge}) - 2(\text{multiplying sum of degrees along the edges of } C_5)$
- $a_6 = -\left(\sum_{i<j,k<l, m<n} d_{sij}^2 \cdot d_{skl}^2 \cdot d_{smn}^2 \text{ where } d_{sij}, d_{skl} \text{ and } d_{smn} \text{ are the matching edges}\right) + 2(\text{multiplying sum of degrees along the edges of } C_4 \text{ and an disjoint edge})$

+ $4(\text{multiplying sum of degrees along the edges of two disjoint triangles}) - 2(\text{multiplying sum of degrees along the edges of } C_6) \right).$

Relation between $DS_A$ polynomial of a graph and its complement:

A walk of length $k$ in a graph is any sequence of vertices $v_1, v_2, \ldots, v_{k+1}$ (not necessarily different) such that there is an edge from $v_i$ to $v_{i+1}$, for each $i = 1, 2, \ldots, k$. To obtain the $DS_A$ polynomial of a complement graph we first find the generating function to get the number of walks of length $k$ in $G$ with respect to its $DS_A$ matrix.

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Theorem 2.1. Let $\overline{G}$ be the complement of $G$ and let $H_{DS_A(G)}(t) = \sum_{k=0}^{\infty} N_k t^k$ be the function that generates the number $N_k$ of walks of length $k$ in $G$, $(k = 0, 1, 2, \ldots)$ with respect to its $DS_A$ matrix. Then

$$H_{DS_A(G)}(t) = \frac{1}{2rt} \left\{ \left( \frac{r}{n-r-1} \right)^n (-1)^n P_{DS_A(\overline{G})} \left[ - \frac{(1 + 2rt)}{t} \left( \frac{n - r - 1}{r} \right) \right] \right\}.$$  \hspace{1cm} (4)

Proof. The proof of this theorem is analogous to the proof obtained for adjacency matrix of a graph $G$ [3]. Let $\sum (A)$ denote the sum of all entries of matrix $A$.

$$\begin{align*}
|B + xJ| &= |B| + x \sum (adj B) \\
adj B &= B^{-1}|B| \\
\sum_{k=0}^{\infty} a^k t^k &= 1 + at + a^2 t^2 + \cdots = \frac{1}{1-at}.
\end{align*}$$  \hspace{1cm} (5)

$$N_k = \sum_{i,j} ds_{ij}^k = \sum (DS_A)^k,$$  \hspace{1cm} (8)

where $B$ is any $n$ ordered non singular matrix, $J$ is a square matrix whose all entries are equal to one, $x$ is any arbitrary number and $N_k$ is the number of all walks of length $k$ in $G$ with respect to the $DS_A$ matrix.

Let $H_{DS_A(G)}(t) = \sum_{k=0}^{\infty} N_k t^k$ denote the generating function that gives the number of walks $N_k$ each of length $k$ in $G$. Using Eq. (8), Eq. (7) and Eq. (6) we get.

$$\sum_{k=0}^{\infty} N_k t^k = \sum_{k=0}^{\infty} \sum (DS_A)^k t^k = \sum_{k=0}^{\infty} (DS_A)^k t^k = \sum (I - DS_A t)^{-1} = \sum adj(I - DS_A t) |I - DS_A t|.$$  \hspace{1cm} (9)

From Eq. (5) we have

$$\sum adj B = \frac{1}{x} \{|B + xJ| - |B|\}.$$

Substituting $B = I - DS_A t$, Eq. (9) reduces to

$$\sum_{k=0}^{\infty} N_k t^k = \frac{1}{x} \left\{ \frac{|I - DS_A t + xJ| - |I - DS_A t|}{|I - DS_A t|} \right\}.$$  \hspace{1cm} (10)

Substituting $x = 2rt$ in the Eq.(10) we get
\[ \sum_{k=0}^{\infty} N_k t^k = \frac{1}{2rt} \left\{ \left| I - DS_A t + 2rtJ \right| - \left| I - DS_A t \right| \right\}. \] (11)

But

\[ DS_A = 2(n - r - 1) (J - I - A) \]
\[ DS_A = 2(n - r - 1) \left( J - I - \frac{DS_A}{2r} \right) \]
\[ 2r(\overline{DS_A}) = 2(n - r - 1) (-2rI + 2rJ - DS_A). \]

Multiplying both sides by \( t \) we get,

\[ -DS_A t + 2rtJ = \frac{rt(\overline{DS_A})}{n - r - 1} + 2rtI. \]

Using the above result in Eq.(11) we get

\[ \sum_{k=0}^{\infty} N_k t^k = \frac{1}{2rt} \left\{ \left| I + \left( \frac{rDS_A}{n - r - 1} + 2rI \right) t \right| \right\} - 1 \]
\[ = \frac{1}{2rt} \left\{ \left| \left( \frac{1 + 2rt}{t} \right) I + \frac{rDS_A}{n - r - 1} \right| \right\} - 1 \]
\[ = \frac{(-1)^n \left( \frac{r}{n - r - 1} \right)^n P_{DS_A(\overline{G})} \left( \frac{1 + 2rt}{t} \right) \left( \frac{n - r - 1}{r} \right) - 1}{P_{DS_A(G)} \left( \frac{1}{t} \right)}. \]

Hence we get the required generating function. \( \square \)

**Theorem 2.2.** If \( G \) is a regular graph with degree \( r \) and \( n \) vertices, then \( DS_A \) polynomial of the complement \( \overline{G} \) is

\[ P_{DS_A(\overline{G})}(\beta) = (-1)^n \left( \frac{n - r - 1}{r} \right)^n \left[ \frac{\beta - 2(n - r - 1)^2}{\beta + (n - r - 1)(2r + 2)} \right] P_{DS_A(G)} \left( \frac{-r\beta - 2r(n - r - 1)}{n - r - 1} \right). \] (12)

**Proof.** Since \( G \) is a \( r \) regular graph, a walk can begin at any one vertex of \( G \) and may continue in \( r \) ways. Therefore, number of walks of length \( k \) in \( G \) is \( N_k = nr^k \).
Thus, for $DS_A(G)$ we have $\frac{N_k}{(2r)^k} = nr^k$.

Hence for the generating function $H_G(t)$ we have,

$$H_{DS_A(G)}(t) = \sum_{k=0}^{\infty} N_k t^k = \sum_{k=0}^{\infty} (DS_A)^k t^k$$

$$= \sum_{k=0}^{\infty} n.r^k.(2r)^k t^k = \sum_{k=0}^{\infty} n(2r^2 t)^k$$

$$= \frac{n}{1 - 2r^2 t}. \quad (13)$$

Using Eq.(4) we get

$$\frac{1}{2rt} \left\{ \left(\frac{r}{n-r-1}\right)^n (-1)^n P_{DS_A(\overline{G})} \left[ -\left(\frac{1+2rt}{t}\right) \left(\frac{n-r-1}{r}\right) \right] \right\} = \frac{n}{1 - 2r^2 t}. \quad (14)$$

Substituting $-\left(\frac{1+2rt}{t}\right) \left(\frac{n-r-1}{r}\right) = \beta$ in Eq.(14) we get

$$\left\{ \left(\frac{r}{n-r-1}\right)^n P_{DS_A(\overline{G})}(\beta) \right\} = \frac{-n}{1 + \frac{2(n-r-1)}{\beta + 2(n-r-1)}}$$

$$= \frac{-2(n-r-1)}{\beta + (n-r-1)(2r+2)}$$

$$\frac{(-1)^n \left(\frac{r}{n-r-1}\right)^n P_{DS_A(\overline{G})}(\beta)}{P_{DS_A(G)} \left[ -r\beta - 2r(n-r-1) \right]} - 1 = \frac{-2n(n-r-1)}{\beta + (n-r-1)(2r+2) + 1}$$

Simplifying we get the required $DS_A$ polynomial for $\overline{G}$ in terms of $DS_A$ polynomial of $G$. \qed

3. $DS_A$ polynomials and spectra of some regular graphs

**Theorem 3.1.** [14] The degree sum adjacency polynomial of a complete graph $K_n$ with $n$ vertices is

$$P_{DS_A(K_n)}(\beta) = [\beta + 2(n-1)]^{n-1} \left[ \beta - 2(n-1)^2 \right]. \quad (15)$$

This result can also be obtained by using lemma (1.1).
Then multiplying, we get the required result.

\[ P_{DS_A(K_2)}(\beta) = (\beta^2 - 4)^k. \] (16)

**Proof.** As each component of a 1-regular graph is isomorphic to \( K_2 \), by substituting \( n = 2 \) in Eq. (15) we obtain

\[ P_{DS_A(K_2)}(\beta) = (\beta - 2)(\beta + 2) = (\beta^2 - 4)^k. \]

\( \square \)

A cocktail-party graph is a complementary graph of 1-regular graph.

**Corollary 3.1.** The \( DS_A \)-polynomial of the cocktail-party graph with \( 2k \) vertices is

\[ P_{DS_A(CP(k))}(\beta) = \beta^k[\beta - 2(2k - 2)][\beta + 4(2k - 2)]^{k-1}. \] (17)

**Proof.** Let \( G \) be a 1-regular graph, then \( P_{DS_A(\overline{G})} = P_{DS_A(CP(k))} \). To obtain \( DS_A \) polynomial for cocktail-party graph, substitute \( n = 2k \) and \( r = 1 \) in Eq. (12)

\[
P_{DS_A(CP(k))}(\beta) = (-1)^{2k}(2k - 2)^{2k} \left[ \frac{\beta - 2(2k - 2)^2}{\beta + 2(2k - 2)4} \right] P_{DS_A(G)} \left[ -\frac{\beta - 2(2k - 2)}{2k - 2} \right]
\]

\[
= (2k - 2)^{2k} \left[ \frac{\beta - 2(2k - 2)^2}{\beta + 2(2k - 2)4} \right] \left\{ \left[ -\frac{\beta - 2(2k - 2)}{2k - 2} \right]^2 - 4 \right\}^k
\]

\[
= \beta^k[\beta - 2(2k - 2)^2][\beta + 4(2k - 2)]^{k-1}.
\]

\( \square \)

**Theorem 3.3.** If \( C_n \) is a cycle with \( n \) vertices, then eigenvalues of degree sum matrix of \( C_n \) are

\[ \beta_k = 8 \cos \left( \frac{2\pi k}{n} \right) \quad k = 0, 1, \ldots, n - 1. \] (18)

**Proof.** The eigenvalues of \( A(C_n) \) are \( \lambda_k = 2 \cos \frac{2\pi k}{n} \) where \( k = 0, 1, \ldots, n - 1 \). As \( DS_A(G) = 4A(G) \), the eigenvalues of \( DS_A(C_n) \) are \( \beta_k = 8 \cos \frac{2\pi k}{n} \) where \( k = 0, 1, \ldots, n - 1 \).

\( \square \)

A crown graph \( S_n^0 \) is obtained from the complete bipartite graph \( K_{n,n} \) by deleting the perfect matching edges.

**Theorem 3.4.** The \( DS_A \)-polynomial of a \( 2n \)-vertex crown graph \( S_n^0 \) is

\[ P_{DS_A(S_n^0)}(\beta) = [\beta^2 - 4(n - 1)^2]^{n-1} [\beta^2 - 4(n - 1)^4] \] (19)

**Proof.** The \( DS_A \)-matrix of crown graph will be of the form \( \begin{bmatrix} X & Y \\ Y & X \end{bmatrix} \). The \( DS_A \) matrix can be reduced to the form \( (X - Y)(X + Y) \), where \( X \) is a matrix of all zeros and \( Y \) is a matrix with all non diagonal entries as \( 2(n - 1) \) and the diagonal entries as zero. The matrix \( Y \) is of the form \( 2(n - 1)J - 2(n - 1)I \). Separately evaluating \( (X - Y) \) and \( (X + Y) \) by applying lemma (1.1) and then multiplying, we get the required result.

\( \square \)
4. **DS\textsubscript{A} polynomial of some graph operations**

*Line Graph* \(L(G)\) of a graph \(G\) is the graph which has one-to-one correspondence between the vertex set and the set of edges of the graph \(G\), with two vertices of \(L(G)\) being adjacent iff the corresponding edges are adjacent in \(G\) [8].

**Theorem 4.1.** If \(G\) is a \(r\) regular graph having \(n\) vertices and \(m = \frac{1}{2}nr\) edges and \(L(G)\) is a line graph, then \(DS\textsubscript{A}\) polynomial of \(L(G)\) in terms of \(DS\textsubscript{A}\) polynomial of \(G\) is

\[
P_{DS\textsubscript{A}(L(G))}(\beta) = (\beta + 8r - 8)^{m-n} \left(\frac{2r-2}{r} \right)^n P_{DS\textsubscript{A}}(G) \left[\frac{r}{2r-2} (\beta - 4r^2 + 12r - 8)\right]. \tag{20}
\]

*Proof.* Let \(A\) be an adjacency matrix of graph \(G\), \(B\) be an adjacency matrix of graph \(L(G)\) and \(R\) be the incidence matrix of \(G\) with \(D\) as the degree matrix. Then for \(G\), we have

\[
RR^T = A + D = \frac{DS\textsubscript{A}(G)}{2r} + D \quad \text{and} \quad R^TR = B + 2I = \frac{DS\textsubscript{A}(L(G))}{4r - 4} + 2I
\]

Taking \(r' = 4r - 4\) we have,

\[
\begin{align*}
\beta^m P_{RR^T}(\beta) &= \beta^n R^TR(\beta) \\
\beta^{m-n} |\beta I - RR^T| &= |\beta I - R^TR| \\
\beta^{m-n} \left|\beta - \frac{DS\textsubscript{A}(G)}{2r} - D\right| &= \left|\beta - \frac{DS\textsubscript{A}(L(G))}{r'} - 2I\right| \\
\beta^{m-n} (\beta - r) I - \frac{DS\textsubscript{A}(G)}{2r} &= \left(\beta - 2\right) I - \frac{DS\textsubscript{A}(L(G))}{r'} \\
\frac{\beta^{m-n}}{(2r)^n} \cdot (r')^m P_{DS\textsubscript{A}(G)}[2r(\beta - r)] &= P_{DS\textsubscript{A}(L(G))}[r'(\beta - 2)]
\end{align*}
\]

substituting \(r'(\beta - 2) = \beta\) and \(r' = 4r - 4\) we get the required result as shown in Eq. (20).

*Subdivision graph* \(s(G)\) of a simple graph \(G\) is the graph which is obtained by adding (inserting) a new vertex onto every edge of \(G\) [8].

**Theorem 4.2.** If \(G\) is a regular graph of degree \(r\) with \(n\) vertices and \(m = \frac{nr}{2}\) edges and \(s(G)\) is a subdivision graph, then \(DS\textsubscript{A}\) polynomial \(P_{DS\textsubscript{A}(s(G))}\) of \(s(G)\) in terms of its adjacency polynomial \(\phi(G)\) is,

\[
P_{DS\textsubscript{A}(s(G))}(\beta) = \beta^{m-n}(r + 2)^{2n}\phi \left( G : \frac{\beta^2}{(r + 2)^2} - r \right). \tag{21}
\]

*Proof.* For a \(r\)-regular graph \(G\) having \(n\) vertices, its degree sum adjacency matrix of subdivision graph \(s(G)\) of graph \(G\) is \(DS\textsubscript{A}(s(G))\). As vertex set of \(s(G)\) is partitioned into two sets, one with
$n$ vertices of degree $r$ and the other with $m$ vertices of degree 2, the characteristic polynomial of $DE(s(G))$ is obtained as follows.

$$P_{DS_A(s(G))} (\beta) = \begin{vmatrix} \beta I_n & -(r + 2)R^T \\ -(r + 2)R & \beta I_n \end{vmatrix} = \beta^{m-n} \left[ \beta^2 I_n - RR^T(r + 2)^2 \right] = \beta^{m-n} \left[ \beta^2 I_n - (r + 2)^2(A + r I_n) \right] = \beta^{m-n}(r + 2)^2 \left( \frac{\beta^2}{(r + 2)^2} - r \right) I_n - A = \beta^{m-n}(r + 2)^2 \phi \left( G : \left( \frac{\beta^2}{(r + 2)^2} - r \right) \right).$$

\[ \Box \]

_Semi total point graph_ $T_1(G)$ is a graph which is derived from graph $G$ by inserting (adding) a new vertex into every edge of $G$ and each new inserted vertex is then joined to the end points of the corresponding edge [3].

**Theorem 4.3**. The $DS_A$ polynomial $P_{DS_A(T_1(G))}$ of semi total point graph $T_1(G)$ of a $n$ ordered $r$-regular graph $G$ in terms of its adjacency polynomial $\phi(G)$ is

$$P_{DS_A(T_1(G))} (\beta) = \beta^{m-n} \left[ 4r\beta + (2r + 2)^2 \right] \phi \left( G : \left[ \frac{\beta^2 - r(2r + 2)^2}{4r\beta + (2r + 2)^2} \right] \right). \tag{22}$$

**Proof.** Let $G$ be a $r$-regular graph with $n$ vertices, where $m = nr/2$ new vertices are added to construct a $T_1(G)$ graph. Then the $DS_A$ polynomial of $T_1(G)$ is $DS_A(T_1(G)) = \det(\beta I - DS_A(T_1(G)))$.

$$P_{T_1(G)} (\beta) = \begin{vmatrix} \beta I_m & -(2r + 2)R^T \\ -(2r + 2)R & \beta I_n - 4r A \end{vmatrix} = \beta^m \left[ \beta I_n - 4r A - \frac{(2r + 2)^2 RR^T I_m}{\beta} \right] = \beta^{m-n} \left[ \beta^2 I_n - 4r A \beta - (2r + 2)^2(A + r I) \right] = \beta^{m-n} \left[ \left( \frac{\beta^2 - r(2r + 2)^2}{4r\beta + (2r + 2)^2} \right) I_n - \left[ \frac{4r\beta + (2r + 2)^2}{4r\beta + (2r + 2)^2} \right] A \right] = \beta^{m-n} \left[ 4r\beta + (2r + 2)^2 \right] \frac{(\beta^2 - r(2r + 2)^2) I_n}{4r\beta + (2r + 2)^2} = \beta^{m-n} \left[ 4r\beta + (2r + 2)^2 \right] \phi \left( G : \left( \frac{\beta^2 - r(2r + 2)^2}{4r\beta + (2r + 2)^2} \right) \right).$$

\[ \Box \]

_Semi total line graph_ $T_2(G)$ of a graph $G$, is the graph with vertex set $V(T_2(G)) = V(G) \cup E(G)$ in which two vertices are adjacent if they are on adjacent edges of $G$ or one is a vertex of $G$ and the other is an edge of $G$, incident to it [3].
Theorem 4.4. Let $G$ be a $r$-regular graph having $n$ vertices and $m$ edges and let $T_2(G)$ be a semi total line graph of $G$. Then the $DS_A$ polynomial $P_{DS_A(T_2(G))}$ of semi total line graph $T_2(G)$ of a graph $G$ in terms of its adjacency polynomial of line graph $\phi(L(G))$ is

$$P_{DS_A(T_2(G))}(\beta) = \beta^{n-m}(4r\beta + 9r^2)^m \phi \left( L(G) : \begin{bmatrix} \beta^2 - 18r^2 \\ 4r\beta + 9r^2 \end{bmatrix} \right).$$ (23)

Proof. For a $r$-regular graph $G$, the $DS_A$ polynomial of $T_2(G)$ is

$$P_{DS_A(T_2(G))}(\beta) = \begin{vmatrix} \beta I_n & 3rR \\ 3rR^T & \beta I_m - 4rB \end{vmatrix}$$

$$= \beta^{n-m} \left( (\beta I_m - 4rB)\beta - 9r^2 R^T R \right)$$

$$= \beta^{n-m} \left( (\beta I_m - 4rB)\beta - 9r^2 (B + 2I) \right)$$

$$= \beta^{n-m} (\beta^2 - 18r^2) I_m - (4r\beta + 9r^2) B$$

$$= \beta^{n-m} (4r\beta + 9r^2)^m \phi \left( L(G) : \begin{bmatrix} \beta^2 - 18r^2 \\ 4r\beta + 9r^2 \end{bmatrix} \right).$$

Thorn graph $G^{+k}$ is a graph which is obtained from graph $G$ by attaching $k$ pendent vertices to every edge of $G$. If $G$ is a graph with $n$ vertices and $m$ edges, then $G^{+k}$ has $n + nk$ vertices and $m + nk$ edges.

Theorem 4.5. The $DS_A$ polynomial $P_{DS_A(G^{+k})}$ of Thorn graph $G^{+k}$ of a $n$ ordered $r$-regular graph $G$ in terms of its adjacency polynomial $\phi(G)$ is

$$P_{DS_A(G^{+k})}(\beta) = \beta^n [2(r + k)]^n \phi \left( G : \begin{bmatrix} \beta \\ 2(r + k) \end{bmatrix} - \frac{k(r + k + 1)^2}{2(r + k)\beta} \right).$$ (24)

Proof. The $DS_A$ polynomial of Thorn graph can be written as,

$$P_{DS_A(G^{+k})}(\beta) = \begin{vmatrix} \beta I_n - 2(r + k)A & -(r + k + 1)J & -(r + k + 1)J & \cdots & -(r + k + 1)J \\ -(r + k + 1)J' & \beta I_k & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -(r + k + 1)J' & 0 & \beta I_k & \cdots & 0 \\ -2(r + k + 1)J' & 0 & 0 & \cdots & \beta I_k \end{vmatrix}$$

where $A$ is the adjacency matrix of $G$, $I$ is the unit matrix and $J$ is a block matrix of order $(n, k)$. For $\beta \neq 0$, multiply the rows (consisting of block matrices) numbered 2, 3, $\ldots$, $k+1$ by $\frac{1}{\beta}(r+k+1)$
and add the resulting rows to the first row. This reduces the determinant as follows.

\[
P_{G+k}(\beta) = \begin{vmatrix}
[\beta I_n - 2(r + k)A - \frac{k(r + k + 1)^2}{\beta}] & 0 & \cdots & 0 \\
-(r + k + 1)J' & \beta I_k & \cdots & 0 \\
-(r + k + 1)J' & 0 & \beta I_k & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-(r + k + 1)J' & 0 & 0 & \cdots & \beta I_k
\end{vmatrix}
\]

\[
= \beta^{nk} \left[ \left( \beta - \frac{k(r + k + 1)^2}{\beta} \right) I_n - 2(r + k)A \right]
\]

\[
= \beta^{nk}[2(r + k)]^n \phi \left( G : \left[ \frac{\beta}{2(r + k)} - \frac{k(r + k + 1)^2}{2(r + k)\beta} \right] \right)
\]

Hence the result. \(\square\)

**Total graph** \(T(G)\) of \(G\) is a graph with vertex set \(V(T(G)) = V(G) \cup E(G)\), with two vertices of \(T(G)\) being adjacent if and only if the corresponding elements of \(G\) are adjacent or incident [8].

**Theorem 4.6.** If \(G\) is a regular graph of degree \(r\) having \(n\) vertices and \(m\) edges, then the Total graph \(T(G)\) has \((m - n)\) \(DS_A\) eigenvalues equal to \(-8r\) and the other \(2n\) eigenvalues are given by,

\[
\frac{1}{2} \left( 4r^2 - 8r + 8r\lambda_i \right) \pm 4r\sqrt{r^2 + 4 + 4\lambda_i}
\]

where \(\lambda_i (i = 1, 2, \ldots, n)\) being the adjacency eigenvalues of \(G\).

**Proof.** Let \(G\) be a \(r\) regular graph with \(n\) vertices and \(m\) edges. As \(DS_A(T(G))\) can be expressed in terms of its adjacency matrix \(A\), adjacency matrix of line graph \(B\) and the incidence matrix \(R\) of a graph \(G\), we get

\[
DS_A(T(G)) = \begin{pmatrix}
4rA & 4rR \\
4rR^T & 4rB
\end{pmatrix}
\]

Its \(DS_A\) polynomial can be expressed as

\[
P_{DS_A(T(G))}(\beta) = \begin{vmatrix}
\beta I - 4rA & -4rR \\
-4rR^T & \beta I - 4rB
\end{vmatrix}
\]

As \(A + D = RR^T\) and \(B + 2I = R^TR\)

\(-4rA = 4r^2I - 4rRR^T\) and \(-4rB = 8rI - 4rR^TR\)

\[
P_{DS_A(T(G))}(\beta) = \begin{vmatrix}
\beta I + 4r^2I - 4rRR^T & -4rR \\
-4rR^T & \beta I + 8rI - 4rR^TR
\end{vmatrix}
\]

Applying series of elementary transformation,

- Second row = second row - \(R^T\) first row
• First row = First row + \( \frac{4rR}{\beta + 8r} \) second row

the determinant can be expressed as follows.

\[
P_{DS_A(T(G))}(\beta) = \begin{vmatrix}
(\beta + 4r^2)I - 4rRR^T & -4rR \\
-4rR^T - (\beta + 4r^2)IR^T - 4rR^TRR^T & (\beta + 8r)I
\end{vmatrix}
\]

\[
= \begin{vmatrix}
\beta I - 4rA & -4rR \\
-(\beta + 4r^2 + 4r)R^T + 4rR^TRR^T & (\beta + 8r)I
\end{vmatrix}
\]

\[
= (\beta + 8r)I_m \left\{ (\beta I - 4rA) + \left[ -(\beta + 4r^2 + 4r) + 4rRR^T \right] \frac{4rR}{\beta + 8r} \right\}
\]

\[
= (\beta I + 8r)^{m-n} \left\{ (\beta I - 4rA)(\beta + 8r) + \left[ -((\beta + 4r^2 + 4r) + 4rRR^T)(4rR) \right] \right\}
\]

\[
= (\beta I + 8r)^{m-n} \left\{ (\beta I - 4rA)(\beta + 8r) + [4rA - (\beta + 4r)I](4rA + 4r^2I) \right\}
\]

\[
= (\beta I + 8r)^{m-n} \left[ 16r^2A^2 + (16r^3 - 48r^2 - 8r\beta)A + (\beta^2 + 8r\beta - 4r^2\beta - 16r^3) \right]
\]

\[
\times \prod_{i=1}^{n} \left\{ \beta^2 - \beta(4r^2 - 8r + 8r\lambda_i) + [16r^2\lambda_i^2 + \lambda_i(16r^3 - 48r^2) - 16r^3] \right\}
\]

where \( \lambda_i \) \( (i = 1, 2, \ldots, n) \) are the eigenvalues of \( A \). Thus we have proved that there are exactly \( (m - n) \) \( DS_A \) eigenvalues of \( T(G) \) equal to \( \beta = -8r \).

Using \( b^2 - 4ac \), we find that the roots of the polynomial \( a\beta^2 + b\beta + c \) where \( a = 1, b = -(4r^2 - 8r + 8r\lambda_i) \) and \( c = 16r^2\lambda_i^2 + \lambda_i(16r^3 - 48r^2) - 16r^3 \).

On solving we get \( 2n \) eigenvalues of \( T(G) \) as

\[
\frac{1}{2} \left\{ (4r^2 - 8r + 8r\lambda_i) \pm 4r\sqrt{r^2 + 4 + 4\lambda_i} \right\}.
\]

The join \( G_1 \nabla G_2 \) of (disjoint) graphs \( G_1 \) and \( G_2 \) is the graph that is obtained from \( G_1 \cup G_2 \), by joining every vertex of \( G_1 \) to all vertices of \( G_2 \).

Theorem 4.7. Let \( G_1 \) and \( G_2 \) be two regular graphs with regularity \( r_1 \) and \( r_2 \) and with orders \( n_1 \) and \( n_2 \) respectively. Then the \( DS_A \)-polynomial of \( G_1 \nabla G_2 \) is given by the relation,

\[
P_{DS_A(G_1\nabla G_2)}(\beta) = \frac{P_{G_1}(r_1\beta)}{[\beta - 2r_1(r_1 + n_2)]}[\beta - 2r_1(r_1 + n_2)]
\]

\[
\times \frac{P_{G_2}(r_2\beta)}{[\beta - 2r_2(r_2 + n_1)][\beta - 2r_2(r_2 + n_1)]}[[\beta - 2r_1(r_1 + n_2)]
\]

\[
[\beta - 2r_2(r_2 + n_1)] - n_1n_2x^2 \}.
\]

where \( x = n_1 + n_2 + r_1 + r_2 \).
Proof. The $DS_A$-polynomial of $G_1 \nabla G_2$ is obtained as

$$P_{DS_A(G_1 \nabla G_2)}(\beta) = \det(\beta I - DS_A(G_1 \nabla G_2))$$

$$= \begin{vmatrix} \beta I_{n_1} - \left(\frac{r_1 + n_2}{r_1}\right) DS_A(G_1) & -xJ_{n_1 \times n_2} \\ -xJ_{n_2 \times n_1} & \beta I_{n_2} - \left(\frac{r_2 + n_1}{r_2}\right) DS_A(G_2) \end{vmatrix}$$

where $x = n_1 + n_2 + r_1 + r_2$ and $J$ is a matrix whose all entries are equal to unity. The above determinant can be written as,

$$\begin{vmatrix} \beta & -ds_{12} & \ldots & -ds_{1n_1} & -x & -x & \ldots & -x \\ -ds_{21} & \beta & \ldots & -ds_{2n_1} & -x & -x & \ldots & -x \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -ds_{n_11} & -ds_{n_12} & \ldots & \beta & -x & -x & \ldots & -x \\ -x & -x & \ldots & -x & -ds'_{12} & \beta & \ldots & -ds'_{2n_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -x & -x & \ldots & -x & -ds'_{n_21} & -ds'_{n_22} & \ldots & \beta \end{vmatrix}.$$  \hfill (26)

Where $ds_{ij}$ is the $ij^{th}$ entry $DS_A$ matrix of $G_1$ and $ds'_{ij}$ is the $ij^{th}$ entry $DS_A$ matrix of $G_2$. In $G_1$ each vertex is adjacent to all vertices of $G_2$, so its new vertex degree is $r_1 + n_2$ and as there are $r_1$ vertices adjacent to a vertex $v_i$ in $G_1$, therefore

$$\sum_{j=1}^{n_1} ds_{ij} = 2r_1(r_1 + n_2) \quad \text{for} \quad i = 1, 2, \ldots, n_1.$$ \hfill (27)

Similarly for $G_2$

$$\sum_{j=1}^{n_2} ds'_{ij} = 2r_2(r_2 + n_1) \quad \text{for} \quad i = 1, 2, \ldots, n_2.$$ \hfill (28)

We carry out a series of elementary transformations so that the determinant remains unchanged. Subtracting $(n_1 + 1)^{th}$ row from the rows $(n_1 + 2), (n_1 + 3), \ldots, (n_1 + n_2)$ of determinant (26), we get

$$\begin{vmatrix} \beta & -ds_{12} & \ldots & -ds_{1n_1} & -x & -x & \ldots & -x \\ -ds_{21} & \beta & \ldots & -ds_{2n_1} & -x & -x & \ldots & -x \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -ds_{n_11} & -ds_{n_12} & \ldots & \beta & -x & -x & \ldots & -x \\ -x & -x & \ldots & -x & -ds'_{12} & \beta & \ldots & -ds'_{2n_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -x & -x & \ldots & -x & -ds'_{n_21} & -ds'_{n_22} & \ldots & \beta \end{vmatrix}.$$
Add the columns \((n_1 + 2), (n_1 + 3), \ldots, (n_1 + n_2)\) to the \((n_1 + 1)^{th}\) column, using Eq. (28), and also taking into consideration \(ds'_{ij} = ds'_{ji}\) we arrive at the following determinant,

\[
\begin{vmatrix}
\beta & -ds_{12} & \cdots & -ds_{1n_1} & -n_2 & -x & \cdots & -x \\
-ds_{21} & \beta & \cdots & -ds_{2n_1} & -n_2x & -x & \cdots & -x \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
-ds_{n_11} & -ds_{n_12} & \cdots & \beta & -n_2x & -x & \cdots & -x \\
-x & -x & \cdots & -x & \beta - 2r_2(r_2 + n_1) & -ds'_{12} & \cdots & -ds'_{2n_2} + ds'_{1n_2} \\
0 & 0 & \cdots & 0 & 0 & \beta + ds'_{12} & \cdots & \beta + ds'_{1n_2} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 0 & -ds'_{2n_2} + ds'_{12} & \cdots & \beta + ds'_{1n_2}
\end{vmatrix}.
\]

On simplifying, the determinant reduces to

\[
|X| = \begin{vmatrix}
\beta & -ds_{12} & \cdots & -ds_{1n_1} & -n_2x \\
-ds_{21} & \beta & \cdots & -ds_{2n_1} & -n_2x \\
\vdots & \vdots & & \vdots & \vdots \\
-ds_{n_11} & -ds_{n_12} & \cdots & \beta & -n_2x \\
-x & -x & \cdots & -x & \beta - 2r_2(r_2 + n_1)
\end{vmatrix}.
\]

where

\[
|X| = \begin{vmatrix}
\beta + ds'_{12} & -ds'_{23} + ds'_{13} & \cdots & -ds'_{2n_2} + ds'_{1n_2} \\
-ds'_{32} + ds'_{12} & \beta + ds'_{13} & \cdots & -ds'_{3n_2} + ds'_{1n_2} \\
\vdots & \vdots & & \vdots \\
-ds'_{n_22} + ds'_{12} & -ds'_{n_33} + ds'_{13} & \cdots & \beta + ds'_{1n_2}
\end{vmatrix}.
\]

Subtracting first rows of determinant (29) from all other rows, we get

\[
|X| = \begin{vmatrix}
\beta & -ds_{12} & \cdots & -ds_{1n_1} & -n_2x \\
-ds_{21} & \beta + ds_{12} & \cdots & -ds_{2n_1} + ds_{1n_1} & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
-ds_{n_11} & -ds_{n_12} + ds_{12} & \cdots & \beta + ds_{1n_1} & 0 \\
-1 & -1 & \cdots & -1 & \beta - 2r_2(r_2 + n_1)
\end{vmatrix}.
\]

Adding columns 2, 3, \ldots, \(n_1\) to the first column and using Eq. (27) we get

\[
|X| = \begin{vmatrix}
\beta - 2r_1(r_1 + n_2) & -ds_{12} & \cdots & -ds_{1n_1} & -n_2x \\
0 & \beta + ds_{12} & \cdots & -ds_{2n_1} + ds_{1n_1} & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
0 & -ds_{n_12} + ds_{12} & \cdots & \beta + ds_{1n_1} & 0 \\
-n_1x & -x & \cdots & -x & \beta - 2r_2(r_2 + n_1)
\end{vmatrix}.
\]
Expanding the determinant along its first column we get

\[
\{[\beta - 2r_1(r_1 + n_2)]\Delta_1 - (-1)^{n_1}n_1\Delta_2\} |X|.
\]

(32)

Where

\[
\Delta_1 = \begin{vmatrix}
\beta + ds_{12} & -ds_{23} + ds_{13} & \cdots & -ds_{2n_1} + ds_{1n_1} & 0 \\
-ds_{32} + ds_{12} & \beta + ds_{13} & \cdots & -ds_{3n_1} + ds_{1n_1} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-ds_{n_12} + ds_{12} & -ds_{n_13} + ds_{13} & \cdots & \beta + ds_{1n_1} & 0 \\
-x & -x & \cdots & -x & \beta - 2r_2(r_2 + n_1)
\end{vmatrix}
\]

and

\[
\Delta_2 = \begin{vmatrix}
-ds_{12} & -ds_{13} & \cdots & -ds_{1n_1} & -n_2x \\
\beta + ds_{12} & -ds_{23} + ds_{13} & \cdots & -ds_{2n_1} + ds_{1n_1} & 0 \\
-ds_{32} + ds_{12} & \beta + ds_{13} & \cdots & -ds_{3n_1} + ds_{1n_1} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-ds_{n_12} + ds_{12} & -ds_{n_13} + ds_{13} & \cdots & \beta + ds_{1n_1} & 0
\end{vmatrix}
\]

The expression in (32) can be rewritten as

\[
\{[\beta - 2r_1(r_1 + n_2)][\beta - 2r_2(r_2 + n_1)]|Y| - n_1n_2|Y|\} |X|
\]

\[
= |X||Y|\{[\beta - 2r_1(r_1 + n_2)][\beta - 2r_2(r_2 + n_1)] - n_1n_2\}
\]

(33)

where

\[
|Y| = \begin{vmatrix}
\beta + ds_{12} & -ds_{23} + ds_{13} & \cdots & -ds_{2n_1} + ds_{1n_1} \\
-ds_{32} + ds_{12} & \beta + ds_{13} & \cdots & -ds_{3n_1} + ds_{1n_1} \\
\vdots & \vdots & \ddots & \vdots \\
-ds_{n_12} + ds_{12} & -ds_{n_13} + ds_{13} & \cdots & \beta + ds_{1n_1}
\end{vmatrix}
\]

The above determinant can be written as

\[
|Y| = \frac{1}{[\beta - 2r_1(r_1 + n_2)]} \times
\]

\[
\begin{vmatrix}
\beta - 2r_1(r_1 + n_2) & -ds_{12} & -ds_{13} & \cdots & -ds_{1n_1} \\
0 & \beta + ds_{12} & -ds_{23} + ds_{13} & \cdots & -ds_{2n_1} + ds_{1n_1} \\
0 & -ds_{32} + ds_{12} & \mu + ds_{13} & \cdots & -ds_{3n_1} + ds_{1n_1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & -ds_{n_12} + ds_{12} & -ds_{n_13} + ds_{13} & \cdots & \beta + ds_{1n_1}
\end{vmatrix}
\]

Using Eq. (27), the sum of the \(i\)-th row in the above determinant is \(\beta + ds_{i1}\) for \(i = 2, 3, \ldots, n_1\). Therefore, by subtracting the columns 2, 3, \ldots, \(n_1\) of above determinant from the first column, we obtain
\[ |Y| = \frac{1}{[\beta - 2r_1(n_1 + n_2)]} \times \]

\[
\begin{vmatrix}
\beta & -ds_{12} & -ds_{13} & \cdots & -ds_{1n_1} \\
-\beta - ds_{21} & \beta + ds_{12} & -ds_{23} + ds_{13} & \cdots & -ds_{2n_1} + ds_{1n_1} \\
-\beta - ds_{31} & -ds_{32} + ds_{12} & \beta + ds_{13} & \cdots & -ds_{3n_1} + ds_{1n_1} \\
\vdots & & & \ddots & \\
-\beta - ds_{n_11} & -ds_{n_12} + ds_{12} & -ds_{n_13} + ds_{13} & \cdots & \beta + ds_{1n_1} \\
\end{vmatrix}
\]

Adding first row to all other rows of the determinant, we get

\[ |Y| = \frac{1}{[\beta - 2r_1(n_1 + n_2)]} \times \]

\[
\frac{1}{[\beta - 2r_1(n_1 + n_2)]} P_{G_1}(\beta).
\]

Similarly, we can show that from Eq. (30) we get

\[ |X| = \frac{1}{[\beta - 2r_2(n_2 + n_1)]} P_{G_2}(\beta). \quad (35) \]

Substituting Eq. (34) and Eq. (35) into Eq. (33) results to Eq. (25). \hfill \Box

Let \( G \) be a graph with \( n_1 \) vertices and let \( H \) be a graph with \( n_2 \) vertices. Then the corona \( G \circ H \) is the graph with \( n_1 + n_1n_2 \) vertices, which is obtained by taking graph \( G \) and \( n \) copies of graph \( H \) and by joining \( i^{th} \) vertex of \( G \) to each vertex in the \( i \)-copy of \( H \) \((i = 1, \cdots, n_1)\).

**Theorem 4.8.** Let \( G \) and \( H \) be regular graphs with \( n_1 \) and \( n_2 \) vertices respectively. Then the \( DS_A \) polynomial \( P_{DS_A(G \circ H)} \) of the corona \( G \circ H \) in terms of its adjacency polynomials \( \phi(G) \) and \( \phi(H) \) is

\[
P_{DS_A(G \circ H)}(\beta) = 2^{n_1n_2 + n_1}(r_1 + n_2)^{n_1}(r_2 + 1)^{n_1n_2} \left\{ \phi \left( H : \left[ \frac{\beta}{2(r_2 + 1)} \right] \right) \right\}^{n_1}
\]

\[
\phi \left( G : \left[ \frac{\beta}{2(r_1 + n_2)} - \frac{m(r_1 + r_2 + n_2 + 1)^2}{2(r_1 + n_2)(\beta - 2(r_2 + 1))} \right] \right).
\]

(36)
Proof. Let $A$ be adjacency matrix of the $r_1$ regular graph $G$ with $n_1$ vertices and let $B$ be the adjacency matrix of the $r_2$ regular graph $H$ with $n_2$ vertices. Its $DS_A$ polynomial can be obtained as follows,

$$P_{DS_A(G \circ H)} = \begin{vmatrix} 
\beta I - 2(r_1 + n_2)A & -(r_1 + r_2 + 1 + n_2)J & \ldots & -(r_1 + r_2 + 1 + n_2)J \\
-(r_1 + r_2 + 1 + n_2)JT & \beta I - 2(r_2 + 1)B & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-(r_1 + r_2 + 1 + n_2)JT & 0 & \ldots & \beta I - 2(r_2 + 1)B 
\end{vmatrix}.$$

Multiply the rows (consisting of block matrices) numbered $2, 3, \ldots, n_1$ by $\frac{r_1 + r_2 + 1 + n_2}{\beta I - 2(r_2 + 1)}$, then the sum of rows of the block matrices to the respective row of the first block matrix. This reduces the determinant to

$$P_{DS_A(G \circ H)}(\beta) = \left| \beta I - \frac{m(r_1 + r_2 + 1 + n_2)^2}{\beta I - 2(r_2 + 1)} - 2(r_1 + n_2)A \right| I - 2(r_1 + n_2)A$$

On simplifying the determinant, we get

$$P_{DS_A(G \circ H)}(\beta) = |\beta I - 2(r_2 + 1)B|^{n_1} \left( \beta - \frac{m(r_1 + r_2 + 1 + n_2)^2}{\beta I - 2(r_2 + 1)} \right) I - 2(r_1 + n_2)A$$

Theorem 4.9. The $DS_A$-polynomial of cartesian product of complete graphs $K_2$ and $K_n$, $K_2 \square K_n$ is

$$P_{DS_A(K_2 \square K_n)}(\beta) = \left( \frac{\beta - 2n^2}{\beta + 2n^2} \right) \left[ (\beta + 2n)^2 - 4n^2 \right]^{n-1} \left[ (\beta + 2n)^2 - 4n^2(n - 1)^2 \right]$$

Proof. As complement of $2n$-vertex crown graph $S_n^0$ is the cartesian product of $K_2$ and $K_n$, $K_2 \square K_n$. Applying the result of Theorem (3.4) in Eq. (12) of Theorem(2.2) we get the required result. \qed
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