Gradient estimates and entropy formulae of porous medium and fast diffusion equations for the Witten Laplacian

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Abstract. We consider gradient estimates to positive solutions of porous medium equations and fast diffusion equations:

\[ u_t = \Delta \phi (u^p) \]

associated with the Witten Laplacian on Riemannian manifolds. Under the assumption that the \( m \)-dimensional Bakry-Emery Ricci curvature is bounded from below, we obtain gradient estimates which generalize the results in [20] and [13]. Moreover, inspired by X. -D. Li’s work in [19] we also study the entropy formulae introduced in [20] for porous medium equations and fast diffusion equations associated with the Witten Laplacian. We prove monotonicity theorems for such entropy formulae on compact Riemannian manifolds with non-negative \( m \)-dimensional Bakry-Emery Ricci curvature.

Keywords. porous medium equation, fast diffusion equation, entropy formulae, Witten Laplacian

Mathematics Subject Classification. Primary 35B45, Secondary 35K55

1 Introduction

Let \((M^n, g)\) be an \(n\)-dimensional complete Riemannian manifold. Li and Yau [16] studied positive solutions of the heat equation

\[ u_t = \Delta u \] (1.1)

and obtained the following gradient estimates:

**Theorem A (Li-Yau [16]).** Let \((M^n, g)\) be a complete Riemannian manifold with \(\text{Ric}(B_p(2R)) \geq -K\), \(K \geq 0\). Suppose that \(u\) is a positive solution of (1.1) on \(B_p(2R) \times [0, T]\). Then on \(B_p(R)\),

\[ \frac{\nabla u^2}{u^2} - \alpha \frac{u_t}{u} \leq \frac{C(n)\alpha^2}{R^2} \left( \frac{\alpha^2}{\alpha - 1} + \sqrt{KR} \right) + \frac{na^2 K}{2(\alpha - 1)} + \frac{na^2}{2t}, \] (1.2)

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where $\alpha > 1$ is a constant. Moreover, when $R \to \infty$, (1.2) yields the following estimate on complete noncompact Riemannian manifold $(M^n, g)$:

$$\frac{\|\nabla u\|^2}{u^2} - \alpha \frac{u_t}{u} \leq \frac{n\alpha^2 K}{2(\alpha - 1)} + \frac{n\alpha^2}{2t}. \quad (1.3)$$

Recently, J. F. Li and X. J. Xu [15] obtained new Li-Yau type gradient estimates for positive solutions of the heat equation (1.1) on Riemannian manifolds. For the related research and some improvements on Li-Yau type gradient estimates of the equation (1.1), see [2, 9, 12, 18, 27, 28] and the references therein. The equation

$$u_t = \Delta (u^p) \quad (1.4)$$

with $p > 1$ is called the porous medium equation, which is a nonlinear version of the classical heat equation. For various values of $p > 1$, it has arisen in different applications to model diffusive phenomena (see [1, 20, 30] and the references therein). The equation (1.4) with $p \in (0, 1)$ is called the fast diffusion equation, which appears in plasma physics and in geometric flows. However, there are marked differences between the porous medium equations and the fast diffusion equation, see [8, 29]. For gradient estimates of (1.4), see [1, 13, 30, 34].

In [20], Lu, Ni, Vázquez and Villani studied gradient estimates of (1.4) and proved the following results (see Theorem 3.3 in [20]):

**Theorem B (P. Lu, L. Ni, J. Vázquez, C. Villani [20]).** Let $(M^n, g)$ be a complete Riemannian manifold with $\text{Ric}(\mathbb{B}_p(2R)) \geq -K, K \geq 0$. Suppose that $u$ is a positive solution to (1.4) with $p > 1$. Let $v = \frac{p}{p-1}u^{p-1}$ and $M = (p-1)\max_{\mathbb{B}_p(2R) \times [0,T]} u$. Then for any $\alpha > 1$, on the ball $\mathbb{B}_p(R)$, we have

$$\frac{\|\nabla v\|^2}{v} - \alpha \frac{v_t}{v} \leq C(n)M\alpha^2 \left( \frac{\alpha^2}{\alpha - 1} \frac{ap^2}{p-1} + (1 + \sqrt{KR}) \right) + \frac{\alpha^2}{\alpha - 1} aMK + \frac{a\alpha^2}{t}, \quad (1.5)$$

where $a = \frac{n(p-1)}{n(p-1)+2}$. Moreover, when $R \to \infty$, (1.5) yields the following estimate on complete noncompact Riemannian manifold $(M^n, g)$:

$$\frac{\|\nabla v\|^2}{v} - \alpha \frac{v_t}{v} \leq \frac{\alpha^2}{\alpha - 1} aMK + \frac{a\alpha^2}{t}. \quad (1.6)$$

Now, we rewrite the inequality (1.6) as

$$\|\nabla v\|^2 - av_t \leq \frac{\alpha^2}{\alpha - 1} aMKv + \frac{a\alpha^2v}{t}. \quad (1.7)$$

Since $(p-1)v = pu^{p-1}$, we have $(p-1)v \to 1$ as $p \to 1$. Hence, $M \to 1$,

$$\frac{\|\nabla v\|^2}{u^2}, \quad v_t \to \frac{u_t}{u}, \quad av \to \frac{n}{2}.$$
as $p \to 1$. As a result, \((1.7)\) becomes the inequality \((1.3)\) in Theorem A of Li-Yau.

Therefore, for complete noncompact Riemannian manifold \((M^n, g)\), the estimate \((1.6)\) in
Theorem B of Lu, Ni, Vázquez and Villani reduces to the estimate \((1.3)\) in Theorem A of Li-Yau when $p \to 1$.

Let $\phi \in C^2(M^n)$. The Witten Laplacian associated with $\phi$ is defined by
$$\Delta_\phi = \Delta - \nabla \phi \cdot \nabla$$
which is symmetric with respect to the $L^2(M^n)$ inner product under the weighted measure
$$d\mu = e^{-\phi} dv,$$
that is,
$$\int_{M^n} u \Delta_\phi v d\mu = - \int_{M^n} \nabla u \nabla v d\mu = \int_{M^n} v \Delta_\phi u d\mu, \quad \forall \, u, v \in C^\infty_0(M^n).$$

The $m$-dimensional Bakry-Emery Ricci curvature associated with the Witten Laplacian is
given by
$$\text{Ric}_\phi^m = \text{Ric} + \nabla^2 \phi - \frac{1}{m-n} d\phi \otimes d\phi,$$
where $m > n$ and $m = n$ if and only if $\phi$ is a constant. Define
$$\text{Ric}_\phi = \text{Ric} + \nabla^2 \phi.$$

Then $\text{Ric}_\phi$ can be seen as the $\infty$-dimensional Bakry-Emery Ricci curvature. In this paper,
we study the following equation associated with the Witten Laplacian:
$$u_t = \Delta_\phi (u^p) \quad (1.8)$$
with $p > 0$ and $p \neq 1$. For $p > 1$ and $p \in (0, 1)$, we derive estimates of Lu, Ni, Vázquez and Villani and Davies’s type estimate. Moreover, for $p > 1$, we obtain Hamilton’s type estimate and estimates of J. F. Li and X. J. Xu. In particular, our results generalize the ones in \cite{13}.

First we consider gradient estimates of \((1.8)\) under the assumption that the $m$-dimensional
Bakry-Emery Ricci curvature is bounded from below, and obtain the following results:

**Theorem 1.1.** Let \((M^n, g)\) be a complete Riemannian manifold with $\text{Ric}_\phi^m(B_p(2R)) \geq -K$, $K \geq 0$. Suppose that $u$ is a positive solution to the porous medium equation \((1.8)\) with $p > 1$. Let $v = \frac{p}{p-1} u^{p-1}$ and $M = (p-1) \max_{B_p(2R) \times [0, T]} v$. Then for any $\alpha > 1$, on the ball $B_p(R)$, we have
$$\frac{\left\| \nabla v \right\|^2}{v} - \alpha \frac{v_t}{v} \leq \tilde{a} \alpha^2 M \frac{C(m)}{R^2} \left\{ \frac{\alpha^2}{\alpha - 1} - \frac{\tilde{a} p^2}{p-1} + \left( 1 + \sqrt{KR} \coth(\sqrt{KR}) \right) \right\} + \frac{\alpha^2}{(\alpha - 1)} \tilde{a} MK + \tilde{a} \alpha^2 \frac{t}{t}, \quad (1.9)$$
where $\tilde{a} = \frac{m(p-1)}{m(p-1)+2}$. Moreover, when $R \to \infty$, \((1.9)\) yields the following estimate on complete noncompact Riemannian manifold \((M^n, g)\):
$$\frac{\left\| \nabla v \right\|^2}{v} - \alpha \frac{v_t}{v} \leq \frac{\alpha^2}{\alpha - 1} \tilde{a} MK + \tilde{a} \alpha^2 \frac{t}{t}. \quad (1.10)$$
Theorem 1.2. Let \((M^n, g)\) be a complete Riemannian manifold with \(\text{Ric}^n_g(B_p(2R)) \geq -K, K \geq 0.\) Suppose that \(u\) is a positive solution to the fast diffusion equation (1.8) with \(p \in (1 - \frac{2}{m}, 1)\). Let \(v = \frac{1}{p-1} u^{p-1}\) and \(M = (1-p) \max_{B_p(2R) \times [0, T]} (-v)\). Then for any \(0 < \alpha < 1\), on the ball \(B_p(R)\), we have

\[-\frac{\nabla v^2}{v} + \frac{v_t}{v} \leq \left(\frac{2}{m} \alpha^2 M C(m) \left(\frac{2}{m} \alpha^2 M^2 - \frac{1}{2(1-\alpha)}(\alpha - 1) \frac{1}{1 - \alpha} \right) + \left(1 + \sqrt{K} \text{coth}(\sqrt{K})\right)\right)\]

\[+ \frac{(-\alpha)^2 M K}{\sqrt{1 - \alpha}(1 - \alpha - \tilde{a})} A(\epsilon_1, \epsilon_2) + \frac{(-\alpha)^2}{A(\epsilon_1, \epsilon_2) t},\]

where \(\tilde{a} = \frac{m(p-1)}{m(p-1)+2}\) and positive constants \(\epsilon_1, \epsilon_2 \in (0, 1)\) satisfying

\[A(\epsilon_1, \epsilon_2) := \left[1 - \tilde{a}(1 - \alpha)\right] - \frac{(1 + \epsilon_2)^2(1 - \tilde{a})^2}{(1 - \epsilon_1)(1 - \alpha - \tilde{a})} > 0.\]

When \(R \to \infty\) and \(\alpha \to 1\), (1.11) yields the following estimate on complete noncompact Riemannian manifold \((M^n, g)\) with \(\text{Ric}^n_g \geq 0:\)

\[-\frac{\nabla v^2}{v} + \frac{v_t}{v} \leq \tilde{a},\]

(1.12)

Remark 1.1. Clearly, our estimate (1.10) reduces to (1.6) of Lu, Ni, Vázquez and Villani (see [20]) by letting \(m = n\). Moreover, for \(p \in (0, 1)\), Theorem 4.1 in [20] of Lu, Ni, Vázquez and Villani can be obtained from our Theorem 1.2 by taking \(m = n\).

Theorem 1.3. Let \((M^n, g)\) be a complete Riemannian manifold with \(\text{Ric}^n_g(B_p(2R)) \geq -K, K \geq 0.\) Suppose that \(v\) is a positive solution to the porous medium equation (1.8) with \(p > 1\). Let \(v = \frac{1}{p-1} u^{p-1}\) and \(M = (p-1) \max_{B_p(2R) \times [0, T]} v\). Then for any \(\alpha > 1\), on the ball \(B_p(R)\), we have

\[-\frac{\nabla v^2}{v} + \frac{v_t}{v} \leq \tilde{a} \alpha^2 \left\{ \frac{\tilde{a}^2 \alpha M^2}{(p-1)^2(\alpha - 1)^2(\alpha - 1)} \frac{C(m)}{R} + \left[\frac{1}{t} + \frac{MK}{2(\alpha - 1)}\right] \right\}^{\frac{1}{2}}\]

\[+ M \left(\frac{C(m)}{R^2} \left(1 + \sqrt{K} \text{coth}(\sqrt{K})\right)\right)^{\frac{1}{2}}:\]

(1.13)

where \(\tilde{a} = \frac{m(p-1)}{m(p-1)+2}\). Moreover, when \(R \to \infty\), (1.13) yields the following estimate on complete noncompact Riemannian manifold:

\[-\frac{\nabla v^2}{v} + \frac{v_t}{v} \leq \alpha^2 \tilde{a} M K + \frac{\tilde{a} \alpha^2}{t}.\]

(1.14)

Theorem 1.4. Let \((M^n, g)\) be a complete Riemannian manifold with \(\text{Ric}^n_g(B_p(2R)) \geq -K, K \geq 0.\) Suppose that \(v\) is a positive solution to the fast diffusion equation (1.8) with \(p \in (1 - \frac{2}{m}, 1)\). Let \(v = \frac{1}{p-1} u^{p-1}\) and \(M = (1-p) \max_{B_p(2R) \times [0, T]} (-v)\). Then for any
0 < \alpha < 1, on the ball $B_p(R)$, we have

$$
- \frac{\left| \nabla v \right|^2}{v} + \alpha \frac{v_t}{v} \leq \left\{ C(\tilde{\alpha}, \alpha) \frac{p}{(1-p)^2} M^{\frac{2}{R}} C + \left[ \frac{\alpha^2}{2(1-\alpha)} + 2(1-\tilde{\alpha}) \right] MK + \frac{1-\alpha-\tilde{\alpha}}{t} \right\} + (1-p)(1-\alpha-\tilde{\alpha}) M \frac{C(m)}{R^2} \left( 1 + \sqrt{KR \coth(\sqrt{KR})} \right) \right\},
$$

where $\tilde{\alpha} = \frac{m(p-1)}{m(p-1)+2}$. When $R \to \infty$, (1.15) yields the following estimate on complete noncompact Riemannian manifold $(M^n, g)$:

$$
- \frac{\left| \nabla v \right|^2}{v} + \alpha \frac{v_t}{v} \leq \left( \frac{\alpha^2}{2(1-\alpha)} + 2(1-\tilde{\alpha}) \right) MK + \frac{1-\alpha-\tilde{\alpha}}{t}.
$$

**Remark 1.2.** Our Theorem 1.3 reduces to Theorem 1.1 of [13] by letting $m = n$ and the estimate (1.13) improves (1.10) on complete noncompact Riemannian manifolds. For complete noncompact Riemannian manifolds with $p \in (0, 1)$, Lu, Ni, Vázquez and Villani [20] proved (see Corollary 4.2 in [20]) the following results: If $\mathrm{Ric} \geq -K$ and $0 < \alpha < 1$, then for any $\varepsilon > 0$ satisfying $C(a, \alpha, \varepsilon) := 1 + (-a)(1-\alpha) - \frac{(1-\alpha)(1-\alpha)^2}{(1-\alpha)-a-(1-\alpha)\varepsilon} > 0$, our estimate (1.16) reduces to (1.17) of Lu, Ni, Vázquez and Villani when $m = n$ and $\alpha \to 1$. Moreover, (1.16) is independent of $\varepsilon$.

**Theorem 1.5.** Let $(M^n, g)$ be a complete Riemannian manifold with $\mathrm{Ric}^m_p(B_p(2R)) \geq -K$, $K \geq 0$. Suppose that $u$ is a positive solution to the porous medium equation (1.8) with $p > 1$. Let $v = \frac{p}{p-1} u^{p-1}$ and $M = (p-1) \max_{B_p(2R) \times [0, T]} v$. Then for any $\alpha > 1$, on the ball $B_p(R)$, we have

$$
\frac{\left| \nabla v \right|^2}{v} - \alpha(t) \frac{v_t}{v} \leq \tilde{\alpha} \alpha^2(t) M \frac{C(m)}{R^2} \left( \frac{p^2 \tilde{\alpha} \alpha^2(t)}{2(p-1)(\alpha(t)-1)} + 3 + \sqrt{KR \coth(\sqrt{KR})} \right) + \tilde{\alpha} \alpha^2(t) \frac{t}{T},
$$

where $\tilde{\alpha} = \frac{m(p-1)}{m(p-1)+2}$ and $\alpha(t) = e^{2MKt}$. Moreover, when $R \to \infty$, (1.19) yields the following estimate on complete noncompact Riemannian manifold:

$$
\frac{\left| \nabla v \right|^2}{v} - \alpha(t) \frac{v_t}{v} \leq \tilde{\alpha} \alpha^2(t) \frac{t}{T}.
$$

**Remark 1.3.** Our Theorem 1.5 becomes Theorem 1.2 in [13] as long as we let $m = n$.

**Theorem 1.6.** Let $(M^n, g)$ be a complete Riemannian manifold with $\mathrm{Ric}^m_p(B_p(2R)) \geq -K$, $K \geq 0$. Suppose that $u$ is a positive solution to the porous medium equation (1.8)
with $p > 1$. Let $v = \frac{p}{p-1}u^{p-1}$ and $M = (p-1)\max_{B_p(2R) \times [0,T]} v$. Then on the ball $B_p(R)$, we have
\[
\frac{\lvert \nabla v \rvert^2}{v} - \alpha(t) \frac{v_t}{v} - \varphi(t) \leq \tilde{\alpha} M \frac{C(m)}{R^2} \left\{ 1 + \sqrt{K} R \coth(\sqrt{K} R) + \frac{\tilde{\alpha} p^2}{(p-1) \tanh(MKt)} \right\}, \tag{1.21}
\]
where $\tilde{\alpha} = \frac{m(p-1)}{m(p-1)+2}$, $\alpha(t)$ and $\varphi(t)$ are given by
\[
\varphi(t) = \tilde{\alpha} MK \{ \coth(MKt) + 1 \}, \quad \alpha(t) = 1 + \frac{\cosh(MKt) \sinh(MKt) - MKt}{\sinh^2(MKt)} \tag{1.22}
\]
Moreover, when $R \to \infty$, $(1.21)$ yields the following estimate on complete noncompact Riemannian manifold:
\[
\frac{\lvert \nabla v \rvert^2}{v} - \alpha(t) \frac{v_t}{v} - \varphi(t) \leq 0. \tag{1.23}
\]

**Theorem 1.7.** Let $(M^n, g)$ be a complete Riemannian manifold with $\text{Ric}^n_0(B_p(2R)) \geq -K$, $K \geq 0$. Suppose that $u$ is a positive solution to the porous medium equation $(1.3)$ with $p > 1$. Let $v = \frac{p}{p-1}u^{p-1}$ and $M = (p-1)\max_{B_p(2R) \times [0,T]} v$. Then on the ball $B_p(R)$, we have
\[
\frac{\lvert \nabla v \rvert^2}{v} - \alpha(t) \frac{v_t}{v} - \varphi(t) \leq \tilde{\alpha}^2(t) M \frac{C(m)}{R^2} \left\{ 1 + \sqrt{K} R \coth(\sqrt{K} R) + \frac{\tilde{\alpha} p^2 \alpha^2(t)}{(p-1) \tanh(MKt)} \right\}, \tag{1.24}
\]
where $\tilde{\alpha} = \frac{m(p-1)}{m(p-1)+2}$, $\alpha(t)$ and $\varphi(t)$ are given by
\[
\varphi(t) = \frac{\tilde{\alpha}}{t} + \tilde{\alpha} MK + \frac{\tilde{\alpha}}{3} (MK)^2 t, \quad \alpha(t) = 1 + \frac{2}{3} MK t. \tag{1.25}
\]
Moreover, when $R \to \infty$, $(1.24)$ yields the following estimate on complete noncompact Riemannian manifold:
\[
\frac{\lvert \nabla v \rvert^2}{v} - \alpha(t) \frac{v_t}{v} - \varphi(t) \leq 0. \tag{1.26}
\]

**Remark 1.4.** Our Theorems 1.6 and 1.7 reduce to Theorems 1.3 and 1.4 in [13] by taking $m = n$, respectively. Moreover, when $t$ is small enough, $\alpha(t), \varphi(t)$ defined by $(1.22)$ and $(1.25)$ both satisfy $\alpha(t) \to 1$ and $\varphi(t) \leq 2\tilde{\alpha} MK + \frac{\tilde{\alpha}}{t}$. Hence, $(1.23)$ and $(1.26)$ show
\[
\frac{\lvert \nabla v \rvert^2}{v} - \alpha(t) \frac{v_t}{v} \leq 2\tilde{\alpha} MK + \frac{\tilde{\alpha}}{t}. \tag{1.27}
\]
Clearly, for $t$ small enough, $(1.27)$ is better than $(1.10)$. Therefore, $(1.23)$ and $(1.26)$ improve $(1.10)$ on complete noncompact Riemannian manifolds in this sense.

Denote by $R$ the scalar curvature of the metric $g$. In [24], Perelman introduced the $W$-entropy functional as follows:
\[
W(g, f, \tau) = \int_M [\tau(R + \lvert \nabla f \rvert^2) + f - n] \frac{e^{-f}}{(4\pi \tau)^{\frac{n}{2}}} dv, \tag{1.28}
\]
where \( \tau \) is a positive scale parameter and \( f \in C^\infty(M^n) \) satisfies
\[
\int_{M^n} \frac{e^{-f}}{(4\pi \tau)^2} \, dv = 1.
\]
By [24], we know that the \( W \)-entropy is monotone increasing under the Ricci flow, and its critical points are given by gradient shrinking solitons. In [21, 22], Ni considered the \( W \)-entropy for the linear heat equation
\[
u_\tau = \Delta u \tag{1.29}
\]
on complete Riemannian manifolds. More precisely, for the \( W \)-entropy associated with (1.29):
\[
W(g, f, \tau) = \int_{M^n} \tau |\nabla f|^2 + f - n \frac{e^{-f}}{(4\pi \tau)^2} \, dv, \tag{1.30}
\]
where \( u = \frac{e^{-f}}{(4\pi \tau)^2} \) is a positive solution to (1.29) and \( \int_{M^n} u \, dv = 1 \), Ni [21] proved
\[
\frac{d}{d\tau} W(g, f, \tau) = -2 \int_{M^n} \tau \left( |\nabla^2 f - \frac{g}{2\tau}|^2 + \text{Ric}(\nabla f, \nabla f) \right) u \, dv. \tag{1.31}
\]
In particular, if the Ricci curvature is non-negative, then \( W \)-entropy defined by (1.31) is monotone non-increasing on complete Riemannian manifolds. For the research of the monotonicity of \( W \)-entropy to other geometric heat flows on Riemannian manifolds, see [10, 14, 20–22]. In [19], Li studied the \( W_m \)-entropy associated with the Witten Laplacian to the linear heat equation
\[
u_\tau = \Delta_\phi u \tag{1.32}
\]
on complete Riemannian manifolds satisfying the \( \mu \)-bounded geometry condition. More precisely, for the \( W_m \)-entropy associated with (1.32):
\[
W_m(g, f, \tau) = \int_{M^n} \tau |\nabla f|^2 + f - m \frac{e^{-f}}{(4\pi \tau)^2} \, d\mu, \tag{1.33}
\]
where \( u = \frac{e^{-f}}{(4\pi \tau)^2} \) is a positive solution to (1.32), Li [19] proved that if there exist two constants \( m > n \) and \( K \geq 0 \) such that \( \text{Ric}_\phi^m \geq -K \), then
\[
\frac{d}{d\tau} W_m(g, f, \tau) = -2 \int_{M^n} \tau \left( |\nabla^2 f - \frac{g}{2\tau}|^2 + \text{Ric}_\phi^m(\nabla f, \nabla f) \right) u \, d\mu
\]
\[\quad - \frac{2}{m - n} \int_{M^n} \tau \left( \nabla_\phi \nabla f + \frac{m - n}{2\tau} \right)^2 u \, d\mu. \tag{1.34}\]
In particular, if the \( \text{Ric}_\phi^m \geq 0 \), then \( W_m(g, f, \tau) \) is non-increasing along the heat equation (1.32). For the study to the Witten Laplacian associated with the \( m \)-dimensional Bakry-Emery Ricci curvature on complete Riemannian manifolds, see [3–5, 11, 18, 23, 25, 26, 31–33]. Let \( u \) be a positive solution to (1.4), and let \( v = \frac{p}{p - 1} u^{p-1} \). In [20], Lu, Ni, Vázquez and Villani introduced the following:
\[
N_p(g, u, t) = -t^n \int_{M^n} uv \, dv
\]
and
\[ W_p(g,u,t) = \frac{d}{dt}[tN_p(g,u,t)] = t^{a+1} \int_{M^n} \left( p \frac{|\nabla v|^2}{v} - \frac{a + 1}{t} \right) uv \, dv, \] (1.35)
where \( a = \frac{n(p-1)}{m(p-1)+2} \). They proved that if \( M^n \) is compact, then
\[
\frac{d}{dt} W_p(g,u,t) = -2(p-1)t^{a+1} \int_{M^n} \left( \frac{p-1}{n(p-1)+2t} \right)^2 + \text{Ric}(\nabla v, \nabla v) \right) uv \, dv \\
- 2t^{a+1} \int_{M^n} (p-1)\Delta v + \frac{a}{t}^2 \, uv \, dv. \] (1.36)

In particular, if the Ricci curvature is non-negative, then the entropy defined in (1.35) is monotone non-increasing on compact Riemannian manifolds when \( p > 1 \). For \( p < 1 \), using the Cauchy-Schwarz inequality, they proved from (1.36) that
\[
\frac{d}{dt} W_p(g,u,t) \leq -2t^{a+1} \int_{M^n} \left[ \frac{n(p-1)+1}{n(p-1)} \right] (p-1)\Delta v + \frac{a}{t} \right)^2 \] (1.37)
and
\[
\frac{d}{dt} W_p(g,u,t) \leq -2t^{a+1} \int_{M^n} (p-1)\Delta v + \frac{a}{t} \right)^2 \] (1.38)
where \( v = \frac{n}{p-1}u^{p-1} \) and \( \tilde{a} = \frac{m(p-1)}{m(p-1)+2} \). Under the \( m \)-dimensional Bakry-Emery Ricci curvature is bounded from below, we prove the following:

**Theorem 1.8.** Let \((M^n, g)\) be a compact Riemannian manifold. If \( u \) is a positive solution to the porous medium equation (1.8) with \( p > 1 \), then
\[
\frac{d}{dt} N_{p,m}(g,u,t) = -t^{\tilde{a}} \int_{M^n} \left( (p-1)\Delta v + \frac{\tilde{a}}{t} \right) uv \, d\mu, \] (1.39)
where \( \tilde{a} = \frac{m(p-1)}{m(p-1)+2} \). Under the \( m \)-dimensional Bakry-Emery Ricci curvature is bounded from below, we prove the following:

**Theorem 1.8.** Let \((M^n, g)\) be a compact Riemannian manifold. If \( u \) is a positive solution to the porous medium equation (1.8) with \( p > 1 \), then
\[
\frac{d}{dt} N_{p,m}(g,u,t) = -t^{\tilde{a}} \int_{M^n} \left( (p-1)\Delta v + \frac{\tilde{a}}{t} \right) uv \, d\mu, \] (1.40)
where \( v = \frac{n}{p-1}u^{p-1} \) and \( \tilde{a} = \frac{m(p-1)}{m(p-1)+2} \). In particular, if \( \text{Ric}^m \geq 0 \), then \( \frac{d}{dt} N_{p,m}(g,u,t) \leq 0 \) and \( N_{p,m}(g,u,t) \) is monotone non-increasing in \( t \). Moreover,
\[
W_{p,m}(g,u,t) = t^{\tilde{a}+1} \int_{M^n} \left( p \frac{|\nabla v|^2}{v} - \frac{\tilde{a} + 1}{t} \right) uv \, d\mu \] (1.41)
and
\[
\frac{d}{dt} W_{p,m}(g,u,t) = -2(p-1)\tilde{t}^{\tilde{a}+1} \int_{M^n} \left\{ \left| \nabla^2 v + \frac{g}{m(p-1) + 2|t|} \right|^2 + \frac{1}{m-n} |\nabla \phi \nabla v - \frac{m-n}{|m(p-1) + 2|t|} + \text{Ric}_\phi(m) (\nabla v, \nabla v) \right\} uv \, d\mu + \frac{1}{m-n} \left| \nabla \phi \nabla v - \frac{m-n}{|m(p-1) + 2|t|} \right|^2 \right\}uv \, d\mu.
\]

In particular, if Ric$_\phi^m \geq 0$, then \( \frac{d}{dt} W_{p,m}(g,u,t) \leq 0 \) and \( W_{p,m}(g,u,t) \) is monotone non-increasing in \( t \).

**Theorem 1.9.** If \( u \) is a positive solution to the fast diffusion equation (1.8) with \( p \in (0,1) \), then
\[
\frac{d}{dt} N_{p,m}(g,u,t) = -\tilde{t}^{\tilde{a}} \int_{M^n} \left( (p-1)\Delta \phi v + \frac{\tilde{a}}{\tilde{t}} \right) uv \, d\mu,
\]
where \( v = \frac{p}{p-1} u^{p-1} \) and \( \tilde{a} = \frac{m(p-1)}{m(p-1) + 2} \). In particular, if Ric$_\phi^m \geq 0$ and \( p \in (1-\frac{2}{m}, 1) \), then \( \frac{d}{dt} N_{p,m}(g,u,t) \leq 0 \) and \( N_{p,m}(g,u,t) \) is monotone non-increasing in \( t \). Moreover,
\[
W_{p,m}(g,u,t) = \tilde{t}^{\tilde{a}+1} \int_{M^n} \left( \frac{|\nabla v|^2}{v} - \frac{\tilde{a} + 1}{\tilde{t}} \right) uv \, d\mu
\]
and for any positive constant \( \varepsilon \geq m - n \) and \( 1 - \frac{1}{n+\varepsilon} \leq p \leq 1 - \frac{m-n}{m\varepsilon} \),
\[
\frac{d}{dt} W_{p,m}(g,u,t) \leq 2\tilde{t}^{\tilde{a}+1} \int_{M^n} \left\{ (1-p)\text{Ric}_\phi^m (\nabla v, \nabla v) + \left( \frac{1 - n(1 - p)}{n(1-p)} - \frac{\varepsilon}{n} \right) |(p-1)\Delta \phi v + \frac{\tilde{a}}{\tilde{t}}|^2 \right\}uv \, d\mu + \left( \frac{m(1-p)}{n(m-n)} - \frac{1}{n\varepsilon} \right) \left| \nabla \phi \nabla v - \frac{m-n}{|m(p-1) + 2|t|} \right|^2 \right\}uv \, d\mu.
\]

In particular, if Ric$_\phi^m \geq 0$, then \( \frac{d}{dt} W_{p,m}(g,u,t) \leq 0 \) and \( W_{p,m}(g,u,t) \) is monotone non-increasing in \( t \).

**Remark 1.5.** In particular, if \( m = n \), then we have that \( \phi \) is a constant. Then (1.42) becomes (5.6) of Lu, Ni, Vázquez and Villani in [20]. By letting \( m = n \) and \( \varepsilon \to 0 \), (1.45) becomes (1.32), which is Corollary 5.10 in [20].

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## 2 Proofs of Theorem 1.1 and 1.2

Let \( v = \frac{p}{p-1} u^{p-1} \). By virtue of the equation (1.8), we have \( v_t = (p-1)v\Delta \phi v + |\nabla v|^2 \) which is equivalent to
\[
\frac{v_t}{v} = (p-1)\Delta \phi v + \frac{|\nabla v|^2}{v}. \tag{2.1}
\]
Lemma 2.1. As in [20], we introduce the following differential operator
\[ \mathcal{L} = \partial_t - (p-1)v \Delta \phi. \]

Let \( F = \frac{|\nabla v|^2}{v} - \alpha \frac{w}{v} - \varphi \), where \( \alpha = \alpha(t) \) and \( \varphi = \varphi(t) \) are functions depending on \( t \).

1. If \( p > 1 \), then
\[
\mathcal{L}(F) \leq -\frac{1}{\tilde{a}} [(p-1)\Delta \phi v]^2 - 2(p-1)Ric^m_\phi(\nabla v, \nabla v) + 2p\nabla v \nabla F \\
+ (1 - \alpha) \left( \frac{v_t}{v} \right)^2 - \alpha' \frac{v_t}{v} - \varphi',
\]

2. If \( p \in (0,1) \), then
\[
\mathcal{L}(F) \geq -\frac{1}{\tilde{a}} [(p-1)\Delta \phi v]^2 - 2(p-1)Ric^m_\phi(\nabla v, \nabla v) + 2p\nabla v \nabla F \\
+ (1 - \alpha) \left( \frac{v_t}{v} \right)^2 - \alpha' \frac{v_t}{v} - \varphi',
\]

where \( \tilde{a} = \frac{m(p-1)}{m(p-1)+2} \).

Proof. We only give the proof to the case that \( p > 1 \). The proof to \( p < 1 \) is similar, so we omit it here.

By a direct calculation, we have
\[
\mathcal{L} \left( \frac{f}{g} \right) = \frac{1}{g} \mathcal{L}(f) - \frac{f}{g^2} \mathcal{L}(g) + 2(p-1) v \nabla \left( \frac{f}{g} \right) \nabla \log g, \quad \forall \ f, g \in C^\infty(M).
\]

Using (2.1) we obtain
\[
\mathcal{L}(v_t) = (p-1) v_t \Delta \phi v + 2 \nabla v \nabla v_t.
\]

It is well known that for the \( m \)-dimensional Bakry-Emery Ricci curvature, we have the following Bochner formula (for the elementary proof, see [17],[18]):
\[
\frac{1}{2} \Delta \phi (|\nabla w|^2) = |\nabla^2 w|^2 + \nabla w \nabla \Delta \phi w + \text{Ric}^m_\phi(\nabla w, \nabla w) \\
\geq \frac{1}{n} |\Delta w|^2 + \nabla w \nabla \Delta \phi w + \text{Ric}^m_\phi(\nabla w, \nabla w) \\
\geq \frac{1}{m} |\Delta \phi w|^2 + \nabla w \nabla \Delta \phi w + \text{Ric}^m_\phi(\nabla w, \nabla w).
\]

It follows from \( p > 1 \) that
\[
\mathcal{L}(|\nabla v|^2) \leq 2 \nabla v \nabla v_t - 2(p-1)v \left( \frac{1}{m} |\Delta \phi v|^2 + \nabla v \nabla \Delta \phi v + \text{Ric}^m_\phi(\nabla v, \nabla v) \right) \\
= 2 \nabla v[(p-1)v \Delta \phi v + |\nabla v|^2] - 2(p-1)v \left( \frac{1}{m} |\Delta \phi v|^2 \\
+ \nabla v \nabla \Delta \phi v + \text{Ric}^m_\phi(\nabla v, \nabla v) \right) \\
= 2(p-1)|\nabla v|^2 \Delta \phi v + 2 \nabla v \nabla (|\nabla v|^2) - \frac{2(p-1)v}{m} (\Delta \phi v)^2 \\
- 2(p-1)v \text{Ric}^m_\phi(\nabla v, \nabla v).
Applying (2.5) and (2.6) into (2.4) yields

\[
\mathcal{L} \left( \frac{v_t}{v} \right) = (p-1) \frac{v_t}{v} \Delta \phi v + \frac{2}{v} \nabla v \nabla v_t - \frac{v_t |\nabla v|}{v}^2 + 2(p-1)v \nabla \left( \frac{v_t}{v} \right) \nabla (\log v),
\]

\[
\mathcal{L} \left( \frac{|\nabla v|^2}{v} \right) \leq 2(p-1) \frac{|\nabla v|^2}{v} \Delta \phi v + \frac{2}{v} \nabla v \nabla (|\nabla v|^2) - \frac{2(p-1)}{m} (\Delta \phi v)^2 - 2(p-1)\text{Ric}_\phi^m (\nabla v, \nabla v) - \frac{|\nabla v|^2}{v^2} + 2(p-1) v \nabla \left( \frac{|\nabla v|^2}{v} \right) \nabla (\log v)
\]

and hence

\[
\mathcal{L}(F) = \mathcal{L} \left( \frac{|\nabla v|^2}{v} \right) - \alpha \mathcal{L} \left( \frac{v_t}{v} \right) - \alpha \frac{v_t}{v} - \phi'
\]

\[
\leq 2(p-1) \frac{|\nabla v|^2}{v} \Delta \phi v + \frac{2}{v} \nabla v \nabla (|\nabla v|^2) - \frac{2(p-1)}{m} (\Delta \phi v)^2 - 2(p-1)\text{Ric}_\phi^m (\nabla v, \nabla v) - \frac{|\nabla v|^2}{v^2} + 2(p-1) v \nabla \left( \frac{|\nabla v|^2}{v} \right) \nabla (\log v)
\]

Noticing

\[
2(p-1)v \nabla \left( \frac{|\nabla v|^2}{v} \right) \nabla (\log v) - 2\alpha(p-1)v \nabla \left( \frac{v_t}{v} \right) \nabla (\log v) = 2(p-1) \nabla v \nabla F,
\]

\[
\frac{2}{v} \nabla v \nabla (|\nabla v|^2) - \frac{2}{v} \nabla v \nabla v_t = \frac{2}{v} \nabla v \nabla [(F + \phi)v] = 2(F + \phi) \frac{|\nabla v|^2}{v} + 2 \nabla v \nabla F,
\]

we have

\[
2(p-1)v \nabla \left( \frac{|\nabla v|^2}{v} \right) \nabla (\log v) - 2\alpha(p-1)v \nabla \left( \frac{v_t}{v} \right) \nabla (\log v) + \frac{2}{v} \nabla v \nabla (|\nabla v|^2) - \alpha \frac{2}{v} \nabla v \nabla v_t
\]

\[
= 2p \nabla \nabla F + 2(F + \phi) \frac{|\nabla v|^2}{v}
= 2p \nabla \nabla F + 2 \left( \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \right) \frac{|\nabla v|^2}{v}.
\]

On the other hand, using (2.4) again, we have

\[
2(p-1) \frac{|\nabla v|^2}{v} \Delta \phi v - \frac{|\nabla v|^4}{v^2} - \alpha(p-1) \frac{v_t}{v} \frac{|\nabla v|^2}{v} = 2 \frac{|\nabla v|^2}{v} \left( \frac{v_t}{v} - \frac{|\nabla v|^2}{v^2} \right) - \frac{|\nabla v|^4}{v^2} - \alpha \frac{v_t}{v} \frac{|\nabla v|^2}{v^2} + \alpha \frac{v_t}{v} \frac{|\nabla v|^2}{v} = 2(\alpha + 2) \frac{v_t}{v} \frac{|\nabla v|^2}{v^2} - 3 \frac{|\nabla v|^4}{v^2} - \alpha \left( \frac{v_t}{v} \right)^2.
\]
Let \( \psi \) be a cut-off function such that \( \xi(r) = 1 \) for \( r \leq 1, \xi(r) = 0 \) for \( r \geq 2, 0 \leq \xi(r) \leq 1, \) and
\[
0 \geq \xi'(r) \geq -c_1 \xi''(r),
\]
\[
0 \geq \xi''(r) \geq -c_2,
\]
for positive constants \( c_1 \) and \( c_2. \) Denote by \( \rho(x) = d(x, p) \) the distance between \( x \) and \( p \) in \( \mathbb{M}^n. \) Let
\[
\psi(x) = \xi \left( \frac{\rho(x)}{R} \right).
\]
Making use of an argument of Calabi \([8]\) (see also Cheng and Yau \([7]\)), we can assume without loss of generality that the function \( \psi \) is smooth in \( B_p(2R). \) Then, we have
\[
\frac{\nabla \psi}{\psi} \leq \frac{C}{R^2}.
\]
By the comparison theorem with respect to the Witten Laplacian (see p. 1324, \([18]\))
\[
\Delta_\phi \psi \geq \sqrt{(m-1)K} \coth \left( \frac{K}{m-1} \right),
\]
we have
\[
\Delta_\phi \psi = \xi'' \Delta_\phi \rho + \frac{\xi'' |\nabla \rho|^2}{R^2} \geq -\frac{C(m)}{R^2} \left( 1 + \sqrt{K}R \coth(\sqrt{K}R) \right). \tag{2.12}
\]
Define \( \tilde{F} = \frac{\nabla \psi}{\psi} + \alpha \frac{\nabla \rho}{\rho}, \) where \( \alpha > 1 \) is a constant. Under the assumption that \( \text{Ric}_\phi'' \geq -K, \) \([22]\) shows that
\[
\mathcal{L}(\tilde{F}) \leq -\frac{1}{\alpha} [(p-1)|\Delta_\phi \psi|^2 + 2(p-1)K |\nabla \psi|^2 + 2p \nabla v \nabla \tilde{F} \]
\[
\leq -\frac{1}{\alpha} [(p-1)|\Delta_\phi \psi|^2 + 2MK |\nabla \psi|^2 + 2p \nabla v \nabla \tilde{F}]. \tag{2.13}
\]
Define $G = t\psi \tilde{F}$. Next we will apply maximum principle to $G$ on $B_p(2R) \times [0, T]$. Assume $G$ achieves its maximum at the point $(x_0, s) \in B_p(2R) \times [0, T]$ and assume $G(x_0, s) > 0$ (otherwise the proof is trivial), which implies $s > 0$. Then at the point $(x_0, s)$, it holds that

$$\mathcal{L}(G) \geq 0, \quad \nabla \tilde{F} = -\frac{\tilde{F}}{\psi} \nabla \psi$$

and by use of (2.13), we have

$$0 \leq \mathcal{L}(G) = s\psi \mathcal{L}(\tilde{F}) - (p - 1)v \tilde{F} \Delta_{\phi} \psi - 2s(p - 1)v \nabla \tilde{F} \nabla \psi + \psi \tilde{F}$$

$$= s\psi \mathcal{L}(\tilde{F}) - (p - 1)v \frac{\Delta_{\phi} \psi}{\psi} G + 2(p - 1)v \frac{|\nabla \psi|^2}{\psi^2} G + \frac{G}{s}$$

$$\leq s\psi \left( -\frac{1}{a} [(p - 1) \Delta_{\phi} \psi]^2 + 2MK \frac{|\nabla \psi|^2}{\psi^2} + 2p \nabla \psi \nabla \tilde{F} \right)$$

$$- (p - 1)v \frac{\Delta_{\phi} \psi}{\psi} G + 2(p - 1)v \frac{|\nabla \psi|^2}{\psi^2} G + \frac{G}{s}$$

$$\leq -\frac{s\psi^2}{a} [(p - 1) \Delta_{\phi} \psi]^2 + 2s\psi MK \frac{|\nabla \psi|^2}{\psi^2} + 2 \frac{p}{(p - 1)^\frac{1}{2}} M^\frac{1}{2} G \frac{|\nabla \psi|}{\psi^{\frac{1}{2}}}$$

$$- (p - 1)v (\Delta_{\phi} \psi) G + 2(p - 1)v \frac{|\nabla \psi|^2}{\psi} G + \frac{G}{s}. \quad (2.14)$$

Applying

$$[(p - 1) \Delta_{\phi} \psi]^2 = \frac{1}{\alpha^2} \tilde{F}^2 + \frac{2(\alpha - 1)}{\alpha^2} \tilde{F} \frac{|\nabla \psi|^2}{\psi} + \left( \frac{\alpha - 1}{\alpha} \right)^2 \frac{|\nabla \psi|^4}{\psi^2}$$

into (2.14), we obtain

$$0 \leq -\frac{1}{\tilde{a} \alpha^2} G^2 - \frac{2(\alpha - 1)}{\alpha^2} G \frac{|\nabla \psi|^2}{\psi} - \frac{s\psi^2}{a} \left( \frac{\alpha - 1}{\alpha} \right)^2 \frac{|\nabla \psi|^4}{\psi^2}$$

$$+ 2s\psi^2 MK \frac{|\nabla \psi|^2}{\psi} + 2 \frac{p}{(p - 1)^\frac{1}{2}} M^\frac{1}{2} \psi^\frac{1}{2} G \frac{|\nabla \psi|}{\psi^{\frac{1}{2}}}$$

$$- (p - 1)v (\Delta_{\phi} \psi) G + 2(p - 1)v \frac{|\nabla \psi|^2}{\psi} G + \frac{G}{s}. \quad (2.15)$$

By virtue of the inequality $-Ax^2 + Bx \leq \frac{B^2}{4A}$, we have

$$-\frac{s\psi^2}{a} \left( \frac{\alpha - 1}{\alpha} \right)^2 \frac{|\nabla \psi|^4}{\psi^2} + 2s\psi^2 MK \frac{|\nabla \psi|^2}{\psi} \leq \frac{\tilde{a} \alpha^2 s^2 M^2 K^2}{(\alpha - 1)^2},$$

$$-\frac{2(\alpha - 1)}{\tilde{a} \alpha^2} G \frac{|\nabla \psi|^2}{\psi} + 2 \frac{p}{(p - 1)^\frac{1}{2}} M^\frac{1}{2} \psi^\frac{1}{2} G \frac{|\nabla \psi|}{\psi^{\frac{1}{2}}} \leq \frac{\tilde{a} \alpha^2 p^2 M}{2(p - 1)(\alpha - 1)} \frac{|\nabla \psi|^2}{\psi} G.$$
Hence, (2.15) yields
\[
0 \leq -\frac{1}{\tilde{a}s\alpha^2}G^2 + \frac{\tilde{a}\alpha^2 s\psi^2 M^2 K^2}{(\alpha - 1)^2} + \frac{\tilde{a}\alpha^2 p^2 M}{2(p - 1)(\alpha - 1)} \frac{|\nabla \psi|^2}{\psi} G \\
- (p - 1)v(L\psi)G + 2(p - 1)v(\nabla \psi)^2 \psi G + \frac{\psi G}{s}
\]
\[
\leq -\frac{1}{\tilde{a}s\alpha^2}G^2 + \left\{ \frac{\tilde{a}\alpha^2 p^2 M}{2(p - 1)(\alpha - 1)} \frac{C(m)}{R^2} + (p - 1)M \frac{C(m)}{R^2} \left( 1 + \sqrt{KR \coth(\sqrt{KR})} \right) + \frac{\psi}{s} \right\} G \\
+ \tilde{a}\alpha^2 s\psi^2 M^2 K^2 \\
\frac{(\alpha - 1)^2}{G}.
\]
Solving the quadratic inequality of \(G\) in (2.16) yields
\[
G \leq \tilde{a}s\alpha^2 \left\{ \frac{\tilde{a}\alpha^2 p^2 M}{2(p - 1)(\alpha - 1)} \frac{C(m)}{R^2} + M \frac{C(m)}{R^2} \left( 1 + \sqrt{KR \coth(\sqrt{KR})} \right) + \frac{\psi}{s} \right\}
\]
\[
+ \left[ \frac{\tilde{a}\alpha^2 p^2 M}{2(p - 1)(\alpha - 1)} \frac{C(m)}{R^2} + M \frac{C(m)}{R^2} \left( 1 + \sqrt{KR \coth(\sqrt{KR})} \right) + \frac{\psi}{s} \right]^2
\]
\[
+ \frac{4\psi^2 M^2 K^2}{(\alpha - 1)^2}
\]
\[
\leq \tilde{a}s\alpha^2 \left\{ \frac{\tilde{a}\alpha^2 p^2 M}{2(p - 1)(\alpha - 1)} \frac{C(m)}{R^2} + M \frac{C(m)}{R^2} \left( 1 + \sqrt{KR \coth(\sqrt{KR})} \right) + \frac{\psi}{s} + \psi MK \frac{(\alpha - 1)^2}{G} \right\}.
\]
Hence we have
\[
G(x, T) \leq G(x_0, s)
\]
\[
\leq \tilde{a}T\alpha^2 \frac{C(m)}{R^2} \left\{ \frac{\alpha^2}{(p - 1)(\alpha - 1)} \tilde{a}p^2 M + M \left( 1 + \sqrt{KR \coth(\sqrt{KR})} \right) \right\}
\]
\[
+ \frac{\alpha^2}{(\alpha - 1)} \tilde{a}MK + \tilde{a}\alpha^2.
\]
For all \(x \in B_p(R)\), from (2.17), it holds that
\[
F(x, T) \leq \tilde{a}\alpha^2 M \frac{C(m)}{R^2} \left\{ \frac{\alpha^2}{\alpha - 1} \tilde{a}p^2 M + \left( 1 + \sqrt{KR \coth(\sqrt{KR})} \right) \right\}
\]
\[
+ \frac{\alpha^2}{(\alpha - 1)} \tilde{a}MK + \tilde{a}\alpha^2 \frac{T}{R^2}.
\]
Since \(T\) is arbitrary, we complete the proof of Theorem 1.1.

**Proof of Theorem 1.2.** When \(p \in (0, 1)\) we have \(v < 0\) and from (2.3)
\[
\mathcal{L}(-\tilde{F}) \leq \frac{1}{\tilde{a}}[(p - 1)\Delta \phi v]^2 + 2(p - 1)\text{Ric}(\nabla v, \nabla v) + 2p \nabla v \nabla (-\tilde{F})
\]
\[
- (1 - \alpha) \left( \frac{v_i}{v} \right)^2
\]
\[
\leq \frac{1}{\tilde{a}}[(p - 1)\Delta \phi v]^2 + 2MK \frac{|\nabla v|^2}{-v} + 2p \nabla v \nabla (-\tilde{F})
\]
\[
- (1 - \alpha) \left( \frac{v_i}{v} \right)^2.
\]
Define \( G = \psi (-\hat{F}) \). Next we will apply maximum principle to \( G \) on \( B_p(2R) \times [0,T] \). Assume \( G \) achieves its maximum at the point \((x_0,s) \in B_p(2R) \times [0,T]\) and assume \( G(x_0,s) > 0 \) (otherwise the proof is trivial), which implies \( s > 0 \). Then at the point \((x_0,s)\), it holds that

\[
\mathcal{L}(G) \geq 0, \quad \nabla (-\hat{F}) = -\frac{-\hat{F}}{\psi} \nabla \psi
\]

and by use of (2.18), we have

\[
0 \leq \mathcal{L}(G) = s\psi \mathcal{L}(-\hat{F}) - (p-1)v \frac{\Delta \psi}{\psi} G + 2(p-1)v \frac{|\nabla \psi|^2}{\psi^2} G + \frac{G}{s} \\
\leq s\psi \left( \frac{1}{\hat{a}}(p-1)\Delta \psi v^2 + 2MK \frac{|\nabla \psi|^2}{\psi^2} + 2p\nabla \psi \nabla (-\hat{F}) \right) \\
- (p-1)v \frac{\Delta \psi}{\psi} G + 2(p-1)v \frac{|\nabla \psi|^2}{\psi^2} G + \frac{G}{s} - (\alpha - 1)s\psi \left( \frac{\psi}{\psi} \right)^2 \\
\leq s\psi \left( (p-1)\Delta \phi v^2 + 2s\varphi MK \frac{|\nabla \psi|^2}{\psi^2} + 2p \frac{\varphi (1-p)^{\frac{1}{2}}} {\varphi (1-p)^{\frac{1}{2}}} M\frac{|\nabla \psi|^2}{\psi^2} \right) \\
- (p-1)v \frac{\Delta \psi}{\psi} G + 2(p-1)v \frac{|\nabla \psi|^2}{\psi^2} G + \frac{G}{s} - (\alpha - 1)s\psi \left( \frac{\psi}{\psi} \right)^2.
\]

Applying

\[
((p-1)\Delta \psi v^2) = \frac{1}{\alpha^2} \hat{F}^2 + \frac{2(\alpha - 1)}{\alpha^2} \hat{F} \frac{|\nabla \psi|^2}{\psi^2} + \left( \frac{\alpha - 1}{\alpha} \right)^2 \frac{|\nabla \psi|^4}{\psi^4},
\]

\[
\left( \frac{\psi}{\psi} \right)^2 = \frac{1}{\alpha^2} \left( -\hat{F} + \frac{|\nabla \psi|^2}{\psi^2} \right)^2 = \frac{1}{\alpha^2} (-\hat{F})^2 + 2\frac{2}{\alpha^2} (-\hat{F}) \frac{|\nabla \psi|^2}{\psi^2} + \frac{1}{\alpha^2} \frac{|\nabla \psi|^4}{\psi^4}
\]

into (2.19), we obtain

\[
0 \leq \frac{1}{\hat{a} s \alpha^2} \left\{ [-1 - \hat{a} (1 - \alpha)]G^2 - 2(1 - \hat{a})(1 - \alpha)s\psi G \frac{|\nabla \psi|^2}{\psi^2} \\
+ s^2 \psi^2 (1 - \alpha) (1 - \hat{a}) \frac{|\nabla \psi|^4}{\psi^4} \right\} + 2s\varphi^2 MK \frac{|\nabla \psi|^2}{\psi^2} \\
+ 2 \frac{p}{(1-p)^{\frac{1}{2}}} M\frac{1}{\varphi} \frac{|\nabla \psi|}{\psi} \frac{|\nabla \psi|}{\psi - \varphi} - (p-1)v(\Delta \phi) G \\
+ 2(p-1)v \frac{|\nabla \psi|^2}{\psi} G + \frac{G}{s}.
\]

Next we take the similar method as in Theorem 4.1 of [20]. Since \( p \in (1 - \frac{2}{m}, 1) \), we have \( \hat{a} < 0 \). Thus, we have for any positive constants \( \varepsilon_1, \varepsilon_2 \),

\[
2s\varphi^2 MK \frac{|\nabla \psi|^2}{\psi^2} \leq -\varepsilon_1 s^2 \psi^2 (1 - \alpha) (1 - \alpha - \hat{a}) \frac{|\nabla \psi|^4}{\psi^4} - \frac{1}{\varepsilon_1} \frac{\hat{a} s \alpha^2 (p-1)^2 \psi^2 M^2 K^2}{(1 - \alpha)(1 - \alpha - \hat{a})},
\]

\[
2 \frac{p}{(1-p)^{\frac{1}{2}}} M\frac{1}{\varphi} \frac{|\nabla \psi|}{\psi} \frac{|\nabla \psi|}{\psi - \varphi} \leq -\varepsilon_2 \frac{2}{\hat{a} s \alpha^2} (1 - \hat{a}) (1 - \alpha) s\psi G \frac{|\nabla \psi|^2}{\psi^2} \\
- \frac{\hat{a} s \alpha^2 p^2 M}{2\varepsilon_2 (1 - \hat{a})(1 - \alpha)(1-p)} \frac{|\nabla \psi|^2}{\psi} G.
\]
Hence, we get from (2.20) that

\[
0 \leq -\frac{1}{\tilde{a}s\alpha^2} \left\{ -|1 - \tilde{a}(1 - \alpha)|G^2 + 2(1 + \varepsilon_2)(1 - \tilde{a})(1 - \alpha)s\psi G|\nabla\psi|^2 + (1 - \varepsilon_1)s^2\psi^2(1 - \alpha)(1 - \alpha - \tilde{a})|\nabla\psi|^4 \right\} - \frac{1}{\varepsilon_1}(1 - \alpha)(1 - \alpha - \tilde{a})
\]

\[
- \frac{\tilde{a}\alpha^2p^2M}{2\varepsilon_2(1 - \tilde{a})(1 - \alpha)(1 - p)} \left|\nabla\psi\right|^2 G - (p - 1)v(\nabla\phi)G
\]

\[
+ 2(p - 1)v \left|\nabla\psi\right|^2 G + \frac{\psi G}{s}.
\]

Taking \(\varepsilon_1, \varepsilon_2\) such that

\[
[1 - \tilde{a}(1 - \alpha)] - \frac{(1 + \varepsilon_2)^2(1 - \tilde{a})^2(1 - \alpha)}{(1 - \varepsilon_1)(1 - \alpha - \tilde{a})} := A(\varepsilon_1, \varepsilon_2) > 0,
\]

then (2.21) yields

\[
0 \leq -\frac{1}{(\tilde{a}s\alpha^2)A(\varepsilon_1, \varepsilon_2)}G^2 + \left\{ \frac{(-\tilde{a})\alpha^2p^2M}{2\varepsilon_2(1 - \tilde{a})(1 - \alpha)(1 - p)} \frac{C}{R^2} + M \frac{C(m)}{R^2} \left(1 + \sqrt{KR} \coth(\sqrt{KR})\right) + \frac{\psi}{s} \right\}G + \frac{(-\tilde{a})\alpha^2\psi^2M^2K^2}{\varepsilon_1(1 - \alpha)(1 - \alpha - \tilde{a})}.
\]

Solving the quadratic inequality of \(G\) in (2.23) yields

\[
G \leq \frac{(-\tilde{a})\alpha^2}{A(\varepsilon_1, \varepsilon_2)} \left\{ \frac{(-\tilde{a})\alpha^2p^2M}{2\varepsilon_2(1 - \tilde{a})(1 - \alpha)(1 - p)} \frac{C}{R^2} + M \frac{C(m)}{R^2} \left(1 + \sqrt{KR} \coth(\sqrt{KR})\right) + \frac{\psi}{s} \right\} + \frac{\sqrt{\varepsilon_1(1 - \alpha)(1 - \alpha - \tilde{a})} \sqrt{A(\varepsilon_1, \varepsilon_2)}}{\sqrt{\varepsilon_1(1 - \alpha)(1 - \alpha - \tilde{a})} \sqrt{A(\varepsilon_1, \varepsilon_2)}}.
\]

Hence we have

\[
G(x, T) \leq G(x_0, s)
\]

\[
\leq \frac{(-\tilde{a})T\alpha^2M C(m)}{A(\varepsilon_1, \varepsilon_2)} \left\{ \frac{(-\tilde{a})\alpha^2p^2}{2\varepsilon_2(1 - \tilde{a})(1 - \alpha)(1 - p)} + \left(1 + \sqrt{KR} \coth(\sqrt{KR})\right) \right\}
\]

\[
+ \frac{(-\tilde{a})T\alpha^2MK}{\sqrt{\varepsilon_1(1 - \alpha)(1 - \alpha - \tilde{a})} A(\varepsilon_1, \varepsilon_2)} \frac{(-\tilde{a})\alpha^2}{A(\varepsilon_1, \varepsilon_2)}.
\]

and for \(x \in B_p(R)\),

\[
-F(x, t) \leq \frac{(-\tilde{a})\alpha^2M C(m)}{A(\varepsilon_1, \varepsilon_2)} \left\{ \frac{(-\tilde{a})\alpha^2p^2}{2\varepsilon_2(1 - \tilde{a})(1 - \alpha)(1 - p)} + \left(1 + \sqrt{KR} \coth(\sqrt{KR})\right) \right\}
\]

\[
+ \frac{(-\tilde{a})\alpha^2MK}{\sqrt{\varepsilon_1(1 - \alpha)(1 - \alpha - \tilde{a})} A(\varepsilon_1, \varepsilon_2)} \frac{(-\tilde{a})\alpha^2}{A(\varepsilon_1, \varepsilon_2)t}.
\]
This completes the proof of Theorem 1.2.

3 Proofs of Theorem 1.3-1.7

Under the assumption that $\text{Ric}^m \geq -K$ and $p > 1$, (2.3) shows that

$$\mathcal{L}(F) \leq -\frac{1}{\alpha}[(p-1)\Delta_{\phi}v]^2 + 2(p-1)K|\nabla v|^2 + 2p\nabla v \nabla F$$

$$+ (1-\alpha)\left(\frac{v_{\psi}}{v}\right)^2 - \alpha'\frac{v_{\psi}}{v} - \varphi'.$$

(3.1)

Following the methods in [13], we can prove that Theorem 1.3, 1.5, 1.6, 1.7 hold respectively.

Next we are in a position to prove Theorem 1.4. Define $F = \frac{|\nabla v|^2}{v} - \alpha \frac{v_{\psi}}{v}$, where $0 < \alpha < 1$ is a constant. Then (2.3) shows that

$$\mathcal{L}(-F) \leq \frac{1}{\alpha}[(p-1)\Delta_{\phi}v]^2 + 2MK\frac{|\nabla v|^2}{-v} + 2p\nabla v \nabla (-F) - (1-\alpha)\left(\frac{v_{\psi}}{v}\right)^2$$

$$= \frac{1}{\alpha^2}\left(-F - (1-\alpha)\frac{|\nabla v|^2}{-v}\right)^2 + 2MK\frac{|\nabla v|^2}{-v} + 2p\nabla v \nabla (-F)$$

$$- \frac{1-\alpha}{\alpha^2}\left(-F - \frac{|\nabla v|^2}{-v}\right)^2.$$ (3.2)

Let $G = tv(-F)$. We apply maximum principle to $G$ on $B_p(2R) \times [0,T]$ and assume that $G$ achieves its maximum at the point $(x_0,s) \in B_p(2R) \times [0,T]$ with $G(x_0,s) > 0$ (otherwise the proof is trivial). Then at the point $(x_0,s)$, it holds that

$$\mathcal{L}(G) \geq 0, \quad \nabla(-F) = -\frac{-F}{\psi} \nabla \psi$$

and by use of (3.2), we have

$$0 \leq \mathcal{L}(G) = s\psi \mathcal{L}(-F) - (p-1)v \frac{\Delta_{\phi}v}{\psi} G + 2(p-1)v \frac{|\nabla v|^2}{\psi^2} G + \frac{G}{s}$$

$$\leq \frac{s\psi}{\alpha^2}\left(-F - (1-\alpha)\frac{|\nabla v|^2}{-v}\right)^2 + 2s\varphi MK\frac{|\nabla v|^2}{-v} + 2p \frac{p}{(1-p)^{\frac{1}{2}}} M^2G\frac{|\nabla v|}{(-v)^{\frac{1}{2}}}$$

$$- \frac{1-\alpha}{\alpha^2} s\psi\left(-F - \frac{|\nabla v|^2}{-v}\right)^2 - (p-1)v \frac{\Delta_{\phi}v}{\psi} G + 2(p-1)v \frac{|\nabla v|^2}{\psi^2} G + \frac{G}{s}.$$ (3.3)

Let $\frac{|\nabla v|^2}{v} = \mu(-F)$ at the point $(x_0,s)$. Then we have $\mu \geq 0$ and

$$0 \leq \frac{1}{\alpha^2} \frac{1}{s\psi}[1 - (1-\alpha)\mu]G^2 + \frac{2\mu}{sG} \frac{p}{(1-p)^{\frac{1}{2}}} M^2G^\frac{3}{2} \frac{|\nabla \psi|}{\psi}$$

$$- \frac{1-\alpha}{\alpha^2} \frac{1}{s\psi}(1-\mu)G^2 - (p-1)v \frac{\Delta_{\phi}v}{\psi} G + 2(p-1)v \frac{|\nabla v|^2}{\psi^2} G + \frac{G}{s}.$$
Multiplying the both sides of (3.3) by \( \frac{s\psi}{G} \) yields
\[
0 \leq \frac{1}{\tilde{A} A^2} [1 - (1 - \alpha)\mu]^2 G + 2\mu MK s\psi + 2\mu^2 s^2 \frac{p}{(1 - p)^2} M^2 \frac{1}{\psi^2} |\nabla \psi|^2 G + \frac{1 - \alpha}{\alpha^2} (1 - \mu)^2 G - (p - 1) s v \Delta_\phi \psi + 2(p - 1) s v \frac{|\nabla \psi|^2}{\psi} + \psi
\]
(3.4)
where
\[
\tilde{A} = \frac{1}{-\tilde{a}\alpha^2} [1 - (1 - \alpha)\mu]^2 + \frac{1 - \alpha}{\alpha^2} (1 - \mu)^2,
\]
\[
\tilde{B} = \mu^2 s^2 \frac{p}{(1 - p)^2} M^2 \frac{1}{\psi^2},
\]
\[
\tilde{C} = 2\mu MK s\psi + (1 - p) s(-v) \left( -\Delta_\phi \psi + 2 \frac{|\nabla \psi|^2}{\psi} \right) + \psi.
\]
It is easy to see that
\[
\frac{1}{\tilde{A}} = \frac{(-\tilde{a})\alpha^2}{[1 - (1 - \alpha)\mu]^2 + (-\tilde{a})(1 - \alpha)(1 - \mu)^2}
\]
\[
= \frac{(-\tilde{a})\alpha^2}{1 + (-\tilde{a})(1 - \alpha) - 2(1 - \alpha)(1 - \tilde{a})\mu + (1 - \alpha)(1 - \alpha - \tilde{a})\mu^2}
\]
\[
\leq 1 - \alpha - \tilde{a},
\]
\[
\frac{2\mu}{\tilde{A}} = \frac{2(-\tilde{a})\alpha^2 \mu}{1 + (-\tilde{a})(1 - \alpha) - 2(1 - \alpha)(1 - \tilde{a})\mu + (1 - \alpha)(1 - \alpha - \tilde{a})\mu^2}
\]
\[
\leq \frac{(-\tilde{a})\alpha^2}{\sqrt{[1 + (-\tilde{a})(1 - \alpha)](1 - \alpha - \tilde{a}) - (1 - \alpha)(1 - \tilde{a})}}
\]
\[
= \sqrt{\frac{1}{1 - \alpha} + (-\tilde{a})[1 - \alpha - \tilde{a}] + (1 - \tilde{a})
\]
\[
\leq \frac{\alpha^2}{2(1 - \alpha)} + 2(1 - \tilde{a}),
\]
where the last inequality used \( \sqrt{x+y} \leq \frac{1}{2}(x+y) \) and there exists a constant \( C(\tilde{a}, \alpha) \) such that \( \frac{\mu}{\tilde{A}} \leq C(\tilde{a}, \alpha) \). From the inequality \( \tilde{x}^2 - 2\tilde{B} x \leq \tilde{C} \), we have \( x \leq \frac{2\tilde{B}}{\tilde{A}} + \sqrt{\frac{\tilde{C}}{\tilde{A}}} \). Applying this inequality into (3.3) by letting \( x = G^2 \) gives
\[
G^2 \leq C(\tilde{a}, \alpha) \frac{s}{\sqrt{2}} \frac{p}{(1 - p)^2} M^2 \frac{1}{R^2} C + \left[ \frac{\alpha^2}{2(1 - \alpha)} + 2(1 - \tilde{a}) \right] MK s + 1 - \alpha - \tilde{a}
\]
+ \( (1 - p)(1 - \alpha - \tilde{a}) MS \frac{C(m)}{R^2} \left( 1 + \sqrt{KR} \coth(\sqrt{KR}) \right) \]
(3.7)
Hence, for \( x \in B_p(R) \), we have
\[
-\frac{|\nabla v|^2}{v} + \alpha \frac{v_t}{v} \leq \left\{ C(\tilde{a}, \alpha) \frac{p}{(1 - p)^2} M^2 \frac{1}{R^2} C + \left[ \frac{\alpha^2}{2(1 - \alpha)} + 2(1 - \tilde{a}) \right] MK + \frac{1 - \alpha - \tilde{a}}{t}
\]
+ \( (1 - p)(1 - \alpha - \tilde{a}) MS \frac{C(m)}{R^2} \left( 1 + \sqrt{KR} \coth(\sqrt{KR}) \right) \]
\[
\frac{1}{2}
\]
(3.8)
We complete the proof of Theorem 1.4.

4 Proofs of Theorem 1.8 and 1.9

Lemma 4.1. If $M^n$ is a compact Riemannian manifold and $u$ is a positive solution to (1.8) with $p \neq 0$, then

$$\frac{d}{dt} \int_{M^n} uv \, d\mu = (p - 1) \int_{M^n} (\Delta \phi v) uv \, d\mu = -p \int_{M^n} |\nabla v|^2 u \, d\mu. \quad (4.1)$$

Proof. From (2.1), we have $(uv)_t = vu_t + uv_t = v\Delta \phi (u^p) + (p - 1)uv \Delta \phi v + u|\nabla v|^2$. It follows from $\nabla (u^p) = u \nabla v$ that

$$\int_{M^n} [v\Delta \phi (u^p) + u|\nabla v|^2] \, d\mu = \int_{M^n} [-\nabla v \nabla (u^p) + u|\nabla v|^2] \, d\mu = 0.$$

Hence

$$\frac{d}{dt} \int_{M^n} uv \, d\mu = \int_{M^n} (uv)_t \, d\mu$$

$$= \int_{M^n} [v\Delta \phi (u^p) + (p - 1)uv \Delta \phi v + u|\nabla v|^2] \, d\mu$$

$$= (p - 1) \int_{M^n} (\Delta \phi v) uv \, d\mu$$

$$= p \int_{M^n} (\Delta \phi v) u^p \, d\mu$$

$$= - p \int_{M^n} \nabla v \nabla (u^p) \, d\mu$$

$$= - p \int_{M^n} |\nabla v|^2 u \, d\mu.$$

We complete the proof of Lemma 4.1. \qed

Lemma 4.2. If $M^n$ is a compact Riemannian manifold and $u$ is a positive solution to (1.8) with $p \neq 0$, then

$$\frac{d}{dt} \int_{M^n} (\Delta \phi v) uv \, d\mu = 2 \int_{M^n} [p - 1)(\Delta \phi v)^2 + |\nabla v|^2 + \text{Ric}_\phi(\nabla v, \nabla v)] uv \, d\mu. \quad (4.2)$$

Proof. Noticing

$$\frac{d}{dt} \int_{M^n} (\Delta \phi v) uv \, d\mu = \int_{M^n} [(\Delta \phi v)_t uv + (\Delta \phi v)(uv)_t] \, d\mu. \quad (4.3)$$
A direct calculation gives
\[
(\Delta_{\phi} v)_t = \Delta_{\phi} [(p - 1)v \Delta_{\phi} v + |\nabla v|^2] \\
= (p - 1)[(\Delta_{\phi} v)^2 + 2\nabla v \nabla \Delta_{\phi} v + v \Delta_{\phi}^2 v] + \Delta_{\phi} |\nabla v|^2 \\
= (p - 1)(\Delta_{\phi} v)^2 + 2p\nabla v \nabla \Delta_{\phi} v + (p - 1)v \Delta_{\phi}^2 v + 2[|\nabla v|^2 + \text{Ric}_{\phi}(\nabla v, \nabla v)].
\]

We derive from \((p - 1)\nabla (uv^2) = (2p - 1)uv \nabla v\) that
\[
\int_{M^n} \nabla v \nabla (\Delta_{\phi} v) uv d\mu \\
= \int_{M^n} \Delta_{\phi} v[\nabla (uv^2) + 2|\nabla v|^2 + \text{Ric}_{\phi}(\nabla v, \nabla v)] uv d\mu. \tag{4.4}
\]

On the other hand,
\[
\int_{M^n} \nabla v \nabla (\Delta_{\phi} v) uv d\mu \\
= \int_{M^n} [\nabla (uv^2) + 2|\nabla v|^2 + \text{Ric}_{\phi}(\nabla v, \nabla v)] uv d\mu. \tag{4.5}
\]

Inserting (4.4) and (4.5) into (4.3) concludes the proof of Lemma 4.2.

\[\square\]

**Proof of Theorem 1.8 and 1.9.** By Lemma 4.1, we have
\[
\frac{d}{dt} N_{p,m}(g, u, t) = -\tilde{a}t^{\tilde{a}-1} \int_{M^n} uv d\mu - (p - 1)t^{\tilde{a}} \int_{M^n} (\Delta_{\phi} v) uv d\mu \\
= -t^{\tilde{a}} \int_{M^n} \left( (p - 1)\Delta_{\phi} v + \frac{\tilde{a}}{t} \right) uv d\mu.
\]

We obtain (1.40) and (1.43). On the other hand, from the definition of \(W_{p,m}(g, u, t)\) in (1.39), we have
\[
W_{p,m}(g, u, t) = \frac{d}{dt}[tN_{p,m}(g, u, t)] \\
= N_{p,m}(g, u, t) + t \frac{d}{dt} N_{p,m}(g, u, t) \\
= t^{\tilde{a}+1} \int_{M^n} \left( p\frac{|\nabla v|^2}{v} - \frac{\tilde{a} + 1}{t} \right) uv d\mu,
\]
where the Lemma 4.1 was used in the last equality. Hence, we derive (1.41) and (1.44).

Noticing that the estimate (1.10) also holds for compact Riemannian manifolds. Taking $K = 0$ and then letting $\alpha \to 1$ in (1.10) yields

$$(p - 1)\Delta_\phi v + \frac{\tilde{a}}{t} = \frac{v_t}{v} - \frac{|\nabla v|^2}{v} + \frac{\tilde{a}}{t} \geq 0,$$

which concludes that if $\text{Ric}_{\phi}^m \geq 0$, then $\frac{d}{dt}N_{p,m}(g, u, t) \leq 0$ and $N_{p,m}(g, u, t)$ is a monotone non-increasing in $t$. When $p \in (1 - \frac{2}{m}, 1)$ and $\text{Ric}_{\phi}^m \geq 0$, we also get from (1.12) that

$$(p - 1)\Delta_\phi v + \frac{\tilde{a}}{t} = \frac{v_t}{v} - \frac{|\nabla v|^2}{v} + \frac{\tilde{a}}{t} \leq 0,$$

which shows that $\frac{d}{dt}N_{p,m}(g, u, t) \leq 0$ and $N_{p,m}(g, u, t)$ is also a monotone non-increasing in $t$.

Next we are in a position to prove (1.42). From (1.40), we have

$$\frac{d}{dt} \left[ t \frac{d}{dt} N_{p,m}(g, u, t) \right] = \frac{d}{dt} \left[ -t^{\tilde{a}+1} \int_{M^n} (p - 1)(\Delta_\phi v)uw \, d\mu - \tilde{a} t^{\tilde{a}} \int_{M^n} \nabla v, \nabla v \right]$$

$$= \frac{d}{dt} \left[ -t^{\tilde{a}+1} \int_{M^n} (p - 1)(\Delta_\phi v)uw \, d\mu + \tilde{a} N_{p,m}(g, u, t) \right]$$

$$= -2t^{\tilde{a}+1} \int_{M^n} \left[ (p - 1)^2(\Delta_\phi v)^2 + (p - 1)|\nabla^2 v|^2 + (p - 1)\text{Ric}_\phi(\nabla v, \nabla v) \right] uv \, d\mu$$

$$- (\tilde{a} + 1)t^{\tilde{a}} \int_{M^n} (p - 1)(\Delta_\phi v)uw \, d\mu - \tilde{a} t^{\tilde{a}} \int_{M^n} \left( (p - 1)\Delta_\phi v + \frac{\tilde{a}}{t} \right) uv \, d\mu,$$

where the last equality used the Lemma 4.2. Hence,

$$\frac{d}{dt} W_{p,m}(g, u, t)$$

$$= -2t^{\tilde{a}+1} \int_{M^n} \left[ (p - 1)^2(\Delta_\phi v)^2 + (p - 1)|\nabla^2 v|^2 + (p - 1)\text{Ric}_\phi(\nabla v, \nabla v) \right] uv \, d\mu$$

$$- (\tilde{a} + 1)t^{\tilde{a}} \int_{M^n} (p - 1)(\Delta_\phi v)uw \, d\mu - (\tilde{a} + 1)t^{\tilde{a}} \int_{M^n} \left( (p - 1)\Delta_\phi v + \frac{\tilde{a}}{t} \right) uv \, d\mu \quad (4.6)$$

$$= -2t^{\tilde{a}+1} \int_{M^n} \left[ (p - 1)^2(\Delta_\phi v)^2 + (p - 1)|\nabla^2 v|^2 + (p - 1)\text{Ric}_\phi(\nabla v, \nabla v) \right] uv \, d\mu$$

$$+ (p - 1) \frac{\tilde{a} + 1}{t} \Delta_\phi v + \frac{\tilde{a}^2 + \tilde{a}}{2t^2} \right] uv \, d\mu.$$

Noticing

$$(p - 1)^2(\Delta_\phi v)^2 + (p - 1) \frac{\tilde{a} + 1}{t} \Delta_\phi v + \frac{\tilde{a}^2 + \tilde{a}}{2t^2}$$

$$= (p - 1)^2 \Delta_\phi v + \frac{m(p - 1) + 2}{m(p - 1) + 2t^2} \Delta_\phi v + \frac{2(p - 1)}{m(p - 1) + 2t^2} \Delta_\phi v + \frac{(p - 1)m}{m(p - 1) + 2t^2}.$$
and hence
\[(p-1)^2(\Delta_v^2) + (p-1)\frac{\tilde{a} + 1}{t} \Delta_v + \frac{\tilde{a}^2 + \tilde{a}}{2t^2} + (p-1)|\nabla^2 v|^2 + \frac{p-1}{m-n}|\nabla \phi \nabla v|^2 \]
\[= \left[(p-1)\Delta_v + \frac{m(p-1)}{m(p-1) + 2|t|} \right]^2 \quad (4.7) \]
\[+ (p-1)\left|\nabla^2 v + \frac{g}{m(p-1) + 2|t|}\right|^2 + \frac{p-1}{m-n}\left|\nabla \phi \nabla v - \frac{m-n}{m(p-1) + 2|t|}\right|^2. \]

We complete the proof of (1.42) by putting (4.7) into (1.46).

When \( p \in (0,1) \), by the Cauchy-Schwarz inequality, we have
\[-(p-1)\left|\nabla^2 v + \frac{g}{m(p-1) + 2|t|}\right|^2 \geq - \frac{p-1}{n}\left|\Delta v + \frac{n}{m(p-1) + 2|t|}\right|^2 \]
\[= - \frac{1}{n(p-1)} \left[(p-1)\Delta_v + \frac{\tilde{a}}{t}\right]^2 - \frac{p-1}{n}\left|\nabla \phi \nabla v - \frac{m-n}{m(p-1) + 2|t|}\right|^2 \]
\[= \frac{1-n(1-p)}{n(1-p)} \left[(p-1)\Delta_v + \frac{\tilde{a}}{t}\right]^2 + \frac{m(1-p)}{n(m-n)} \left|\nabla \phi \nabla v - \frac{m-n}{m(p-1) + 2|t|}\right|^2 \]
\[\geq \left(\frac{1-n(1-p)}{n(1-p)} - \frac{\varepsilon}{n}\right) \left[(p-1)\Delta_v + \frac{\tilde{a}}{t}\right]^2 \]
\[+ \left(\frac{m(1-p)}{n(m-n)} - \frac{1}{n\varepsilon}\right) \left|\nabla \phi \nabla v - \frac{m-n}{m(p-1) + 2|t|}\right|^2, \quad (4.8) \]

where \( \varepsilon \geq m-n \) is a positive constant and satisfies \( 1 - \frac{1}{n+\varepsilon} \leq p \leq 1 - \frac{m-n}{m\varepsilon} \). Inserting (4.8) into (1.42) gives
\[\frac{d}{dt} W_{p,m}(g,u,t) \leq 2t^m + \int_{M^n} \left[ (1-p)\text{Ric}^m_{\phi}(\nabla v, \nabla v) \right. \]
\[\left. + \left(\frac{1-n(1-p)}{n(1-p)} - \frac{\varepsilon}{n}\right) \left[(p-1)\Delta_v + \frac{\tilde{a}}{t}\right]^2 \right. \]
\[\left. + \left(\frac{m(1-p)}{n(m-n)} - \frac{1}{n\varepsilon}\right) \left|\nabla \phi \nabla v - \frac{m-n}{m(p-1) + 2|t|}\right|^2 \right] \ dv \ du. \quad (4.9) \]

Therefore, we complete the proof of (1.45).
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