ON THE HILBERT COEFFICIENTS, DEPTH OF ASSOCIATED GRADED RINGS AND REDUCTION NUMBERS

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Abstract. Let \((R, m)\) be a \(d\)-dimensional Cohen-Macaulay local ring, \(I\) an \(m\)-primary ideal of \(R\) and \(J = (x_1, \ldots, x_d)\) a minimal reduction of \(I\). We show that if \(J_{d-1} = (x_1, \ldots, x_{d-1})\) and \(\sum_{n=1}^{\infty} \lambda(I^{n+1} \cap J_{d-1})/(JI^n \cap J_{d-1}) = i\) where \(i = 0, 1\), then depth \(G(I) \geq d - i - 1\). Moreover, we prove that if \(e_2(I) = \sum_{n=2}^{\infty} (n-1)\lambda(I^n/JI^{n-1}) - 2;\) or if \(I\) is integrally closed and \(e_2(I) = \sum_{n=2}^{\infty} (n-1)\lambda(I^n/JI^{n-1}) - i\) where \(i = 3, 4\), then \(e_1(I) = \sum_{n=1}^{\infty} \lambda(I^n/JI^{n-1}) - 1\). In addition, we show that \(r(I)\) is independent. Furthermore, we study the independence of \(r(I)\) with some other conditions.

1. INTRODUCTION

Throughout the paper we will assume that \((R, m)\) is a \(d\)-dimensional Cohen-Macaulay local ring having an infinite residue field and \(I\) an \(m\)-primary ideal of \(R\). An ideal \(J \subseteq I\) is called a reduction of \(I\) if \(I^{r+1} = JI^r\) for some nonnegative integer \(r\) (see [20]). The least such \(r\) is called the reduction number of \(I\) with respect to \(J\) and denoted by \(r_J(I)\). A reduction \(J\) is called a minimal reduction if it does not properly contain a reduction of \(I\), under our assumption it is generated by a regular sequence. The reduction number of \(I\) is defined as \(r(I) = \min\{r_J(I) : J\) is a minimal reduction of \(I\}\). The reduction number \(r(I)\) is said to be independent if \(r(I) = r_J(I)\) for all minimal reduction \(J\) of \(I\). Sally in [26] raised the following question: If \((R, m)\) is a \(d\)-dimensional Cohen-Macaulay local ring having an infinite residue field, then is \(r(m)\) independent? A natural extension of this question is to replace \(r(m)\) with \(r(I)\). Let \(G(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}\) be the associated graded ring of \(I\). Huckaba in [10] and Trung in [29] independently proved that if depth \(G(I) \geq d - 1\), then \(r(I)\) is independent (see also [18], [8], [9] and [28]). Moreover, Wu in [34] with some conditions proved that if depth \(G(I) \geq d - 2\), then \(r(I)\) is independent. However if \(d \geq 2\) and depth \(G(I) \leq d - 2\), then \(r(I)\) is not independent in general. Counterexamples have been obtained in [10], [18] and [17].

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The Hilbert function of $I$ is given by $H_I(n) = \lambda(R/I^n)$ (where $\lambda$ denotes length) with the convention that $I^n = R$ for $n \leq 0$. The Hilbert polynomial of $I$ is written in the form

$$P_I(n) = \sum_{i=0}^{d} (-1)^i e_i(I) \binom{n + d - i - 1}{d - i},$$

where $e_0(I), e_1(I), \ldots, e_d(I)$ are uniquely determined by $I$ and called the Hilbert coefficients of $I$. It is well known that $H_I(n) = P_I(n)$ for all $n \geq 0$.

Valabrega and Valla in [30] obtained that $G(I)$ is Cohen-Macaulay if and only if there exists a minimal reduction $J$ of $I$ such that $I^n \cap J = I^{n-1}J$ for all $n$. Later on, Guerrieri in [4] asked that if $J$ is a minimal reduction of $I$ such that $\sum_{n \geq 1} \lambda(I^n \cap J/I^{n-1}J) = t$, then is depth $G(I) = d - t$? The case $t = 0$ is simply a restatement of the Valabrega-Vall theorem, whereas case $t = 1$ was proved in [4]. Some partial answers in the case $t = 2, 3$ were also proved in [5] and [7]. Huckaba and Marley in [13] and Vaz Pinto in [31] independently showed that $e_1(I) \leq \sum_{n \geq 1} \lambda(I^n/I^{n-1}J)$ and equality holds if and only if depth $G(I) \geq d - 1$ for some minimal reduction $J$ of $I$. Another closely related conjecture was raised by Wang in [32] while attempting to prove Guerrieri’s conjecture. Namely, he asked whether the difference $\sum_{n \geq 1} \lambda(I^n/I^{n-1}J) - e_1(I) = s \geq 0$, implies depth $G(I) \geq d - s - 1$. Wang first showed that an affirmative answer to his conjecture implies the validity of Guerrieri’s conjecture. Then in [32] he settled the case $s = 1$ (see also [22]). Unfortunately, both conjectures fail in general as shown in [33].

Corso, Polini and Rossi in [2] established a general upper bound on $e_2(I)$, which is reminiscent of the bound on $e_1(I)$ due to Huckaba and Marley in [13] and Vaz Pinto in [31]. Namely, it holds that $e_2(I) \leq \sum_{n \geq 2} (n - 1)\lambda(I^n/I^{n-1}J)$ for any minimal reduction reduction $J$ of $I$. Furthermore, the upper bound is attained if and only if depth $G(I) \geq d - 1$. In addition, if $e_2(I) \geq \sum_{n \geq 2} (n - 1)\lambda(I^n/I^{n-1}J) - 2$ or if $I$ is integrally closed and $e_2(I) \geq \sum_{n \geq 2} (n - 1)\lambda(I^n/I^{n-1}J) - 4$, then depth $G(I) \geq d - 2$.

In this paper we prove the following results.

**Theorem 1.1.** Let $J = (x_1, \ldots, x_d)$ be a minimal reduction of $I$.

1. If one of the following conditions holds:
   
   (i) $e_2(I) = \sum_{n=2}^{\infty} (n - 1)\lambda(I^n/JI^{n-1}) - 2$;
   
   (ii) $I$ is integrally closed and $e_2(I) = \sum_{n=2}^{\infty} (n - 1)\lambda(I^n/JI^{n-1}) - i$, where $i = 3, 4$.

   Then $e_1(I) = \sum_{n=1}^{\infty} \lambda(I^n/JI^{n-1}) - 1$. Moreover, we give a counterexample such that the converse in general is not true.
(2) If $J_{d-1} = (x_1, ..., x_{d-1})$ and $\sum_{n=1}^{\infty} \lambda(J^{n+1} \cap J_{d-1})/(J^n \cap J_{d-1}) = i$ where $i=0,1$, then depth $G(I) \geq d-i-1$.

Theorem 1.2. With our assumption of the above theorem, we show that $r(I)$ is independent. Also, we study the independence of $r(I)$ with some other conditions.

For any unexplained notation or terminology, we refer the reader to [1] and [25].

2. Preliminary

In this section we recall some known results which is studied in [14]. An element $x \in I \setminus I^2$ is said to be superficial for $I$ if there is an integer $c$ such that $(I^{n+1} : x) \cap I^c = I^n$ for all $n \geq c$. If grade $I \geq 1$ and $x$ is a superficial element, then $x$ is a regular element of $R$ and by Artin-Rees Theorem $I^{n+1} : x = I^n$ for all $n$ sufficiently large. If $R/m$ is infinite, then a superficial element for $I$ always exists. A sequence $x_1, ..., x_s$ is called a superficial sequence for $I$ if $x_1$ is superficial for $I$ and $x_i$ is superficial for $I/(x_1, ..., x_{i-1})$ for $2 \leq i \leq s$. If $J$ is a minimal reduction of $I$, then there is a superficial sequence $x_1, ..., x_d$ in $I$ such that $J = (x_1, ..., x_d)$. For any element $x \in I$ we let $x^*$ denote the image of $x$ in $I/I^2$. We note that if $x^*$ is a regular element of $G(I)$, then $x$ is a regular element of $R$ and $G(I/(x)) \cong G(I)/(x^*)$.

A set of ideals $F = \{I_n\}_{n \in \mathbb{N}}$ where $I_0 = R$ and $I_1 = I$, of $R$ is called a Hilbert filtration if we have (i) $I_{n+1} \subseteq I_n$ for all $n \geq 0$, (ii) $I_n I_m \subseteq I_{n+m}$ for all $n, m \geq 0$, and (iii) there is a $k \geq 0$ such that $I^n \subseteq I_n \subseteq I^{n-k}$ for all $n \geq 0$. Let $F$ be a Hilbert filtration and $x = x_1, ..., x_l \in I_1$ a regular sequence on $R$ and a superficial sequence for $F$. Huckaba and Marley in [14] constructed the Koszul complex $C(x, \mathcal{F}, n)$ which has the following form

$$0 \rightarrow R/I_{n+l} \rightarrow (R/I_{n+l+1})^l \rightarrow (R/I_{n+l+2})^l \rightarrow ... \rightarrow R/I_n \rightarrow 0.$$

Let $C(n) = C(x_1, x_2, ..., x_l, \mathcal{F}, n)$ and $C'(n) = C(x_1, x_2, ..., x_{l-1}, \mathcal{F}, n)$. For any $n$ there is an exact sequence of complexes

$$0 \rightarrow C'(n) \rightarrow C(n) \rightarrow C'(n-1)[-1] \rightarrow 0.$$

Thus, we have the corresponding long exact sequence on homology:

$$\cdots \rightarrow H_i(C'(n)) \rightarrow H_i(C(n)) \rightarrow H_{i-1}(C'(n-1)) \rightarrow x_i \rightarrow H_{i-1}(C'(n)) \rightarrow \cdots (\ast)$$

Since $F$ is a Hilbert filtration, $H_i(C(x, \mathcal{F}, n))$ has finite length for all $i$ and $n$. For $i \geq 1$, consider

$$h_i(x, \mathcal{F}) := \sum_{n=1}^{\infty} \lambda(H_i(C(x, \mathcal{F}, n)))$$
Lemma 3.1. \( \lambda \) and also by using [11, Lemma 2.7] we have

By Lemma 3.1, we have

Therefore, for each \( n \)

and

\[
k_i(\mathcal{x}, \mathcal{F}) := \sum_{n=2}^{\infty} (n-1)\lambda(H_i(C, (\mathcal{x}, \mathcal{F}, n))).
\]

These integers are well-defined by [14, Lemma 3.6]. Although, \( h_i(\mathcal{x}, \mathcal{F}) = k_i(\mathcal{x}, \mathcal{F}) = 0 \) for all \( i \geq l \). For \( \mathcal{x}' = x_1, ..., x_{l-1} \) we define

\[ h_i'(\mathcal{x}', \mathcal{F}) := \sum_{n=1}^{\infty} \lambda(H_i(C, (\mathcal{x}', \mathcal{F}, n))). \]

3. The results

Lemma 3.2. \( \lambda \) Let \( \mathcal{F} \) be a Hilbert filtration and \( \mathcal{x} = x_1, ..., x_l \) a regular sequence on \( R \) and a superficial sequence for \( \mathcal{F} \). Then for each \( i \geq 1 \)

\[
\sum_{j \geq i} (-1)^{j-i} h_j(\mathcal{x}, \mathcal{F}) \geq 0.
\]

Moreover, equality occurs if and only if \( \text{grade}(\mathcal{x}^*) \geq l - i + 1 \).

Proof. Fix \( i \geq 1 \) and for each \( n \) let \( B_n \) be the kernel of the map \( H_i(C(n)) \rightarrow H_{i-1}(C'(n-1)) \) given in (*)). Then, for each \( n \), we have the exact sequence

\[
0 \rightarrow H_i(C(n)) \rightarrow ... \rightarrow H_i(C'(n-1)) \rightarrow H_i(C'(n)) \rightarrow B_n \rightarrow 0.
\]

Therefore, for each \( n \), we have

\[
\lambda(B_n) = \sum_{j=i+1}^{l} (-1)^{j-i-1} \lambda(H_j(C(n))) + \sum_{j=i}^{l} (-1)^{j-i} \Delta[\lambda(H_j(C'(n)))]
\]

and also by using [11, Lemma 2.7] we have

\[
\sum_{n=2}^{\infty} (n-1)\Delta[\lambda(H_j(C'(n)))] = -\sum_{n=1}^{\infty} \lambda(H_j(C'(n))) = -h_j'(\mathcal{x}', \mathcal{F}).
\]

Thus we see that

\[
\sum_{n=2}^{\infty} (n-1)\lambda(B_n) = \sum_{j=i+1}^{l} (-1)^{j-i-1} k_j(\mathcal{x}, \mathcal{F}) - \sum_{j=i}^{l} (-1)^{j-i} h_j'(\mathcal{x}', \mathcal{F}). (**)\]

By Lemma 3.1, we have \( \sum_{j=i}^{l} (-1)^{j-i} h_j'(\mathcal{x}', \mathcal{F}) \geq 0 \) and so \( \sum_{j=i+1}^{l} (-1)^{j-i-1} k_j(\mathcal{x}, \mathcal{F}) \geq 0 \) for each \( i \geq 1 \).

By [14, Proposition 3.3], if \( \text{grade}(\mathcal{x}^*) \geq l - i + 1 \) then \( H_j(C(n)) = 0 \) for all \( n \) and
$j \geq i$. Thus, $k_j(x, F) = 0$ for $j \geq i$.

Conversely, suppose for $i \geq 1$

$$\sum_{j=i+1}^{l} (-1)^{j-i-1} k_j(x, F) = 0.$$

Then by ($**$) we have $\sum_{n=2}^{\infty} (n-1)\lambda(B_n) = 0$ and so by Lemma 3.1 we obtain $\text{grade}(x^*) \geq l - i + 1$.

\[ \square \]

Remark 3.3. By [14, §4.] we have

$$\Delta^d[P_F(n) - H_F(n)] = \lambda(I_n/JI_{n-1}) - \sum_{i=2}^{d} (-1)^i \lambda(H_i(C,(n)))$$

and

$$e_i(F) = \sum_{n=i}^{\infty} (n-1) \Delta^d[P_F(n) - H_F(n)].$$

Therefore we can obtain the following

$$e_1(F) = \sum_{n=1}^{\infty} \lambda(I_n/JI_{n-1}) - \sum_{i=2}^{d} (-1)^i h_i(x, F)$$

and

$$e_2(F) = \sum_{n=2}^{\infty} (n-1)\lambda(I_n/JI_{n-1}) - \sum_{i=2}^{d} (-1)^i k_i(x, F).$$

Proposition 3.4. (compare with [2, Theorem 3.1]) Let $F$ be a Hilbert filtration and $J = (x_1, \ldots, x_d)$ be a minimal reduction of $F$. Then

$$e_2(F) \leq \sum_{n=2}^{\infty} (n-1)\lambda(I_n/JI_{n-1})$$

with equality if and only if depth $G(F) \geq d - 1$.

Proof. By Remark 3.3 and Lemma 3.2 we have the following

$$e_2(F) = \sum_{n=2}^{\infty} (n-1)\lambda(I_n/JI_{n-1}) - \sum_{i=2}^{d} (-1)^{i-2} k_i(J, F),$$

and

$$\sum_{i=2}^{d} (-1)^{i-2} k_i(J, F) \geq 0.$$

Thus

$$e_2(F) \leq \sum_{n=2}^{\infty} (n-1)\lambda(I_n/JI_{n-1}).$$

Also the equality follows by Lemma 3.2. \[ \square \]
Remark 3.5. By [14, Lemma 3.2] and (*) we can obtain the following exact sequence

\[ 0 \rightarrow (I_{n-1} : (x))/I_{n-1} \rightarrow (I_{n-1} : (x'))/I_{n-1} \xrightarrow{x_1} (I_{n-1} : (x'))/I_{n-1} \rightarrow \ldots \]

\[ \rightarrow (I_{n-1} \cap (x'))/(x'I_{n-2} \xrightarrow{x_1} (I_n \cap (x'))/(x'I_{n-2}) \rightarrow (I_n \cap (x'))/x'I_{n-1} \rightarrow \]

\[ R/(I_{n-1}, (x')) \xrightarrow{x_1} R/(I_n, (x')) \rightarrow R/(I_n, (x')) \rightarrow 0. \]

If \( A_n \) is the kernel of the map \( R/(I_{n-1}, (x')) \xrightarrow{x_1} R/(I_n, (x')) \), then \( A_n = \left( (I_n, (x')) : x_i \right)/(I_{n-1}, (x')) \).

If \( B_n \) is the kernel of the map \( (I_{n-1} \cap (x'))/(x'I_{n-1} \rightarrow R/(I_{n-1}, (x')) \) or the cokernel of the map \( (I_{n-1} \cap (x'))/(x'I_{n-1} \xrightarrow{x_1} (I_n \cap (x'))/(x'I_{n-2}) \), then \( B_n = (I_n \cap (x'))/((x'I_{n-1} + x_I(I_{n-1} \cap (x')))) \). Thus if \( \text{grade}(x^*) \geq l - 1 \), then \( \left( (I_n, (x')) : x_i \right)/(I_{n-1}, (x')) \cong (I_n \cap (x'))/(x'I_{n-1} \cdot \cdot \cdot) \)...

Proposition 3.6. Let \( d \geq 2 \) and \( J \) be a minimal reduction of \( I \) such that \( \sum_{i=2}^{d} (-1)^{i-2}h_i = 1 \).

Then \( \text{depth} G(I) \geq d - 2 \).

Proof. If \( \sum_{i=2}^{d} (-1)^{i-2}h_i = 1 \), then by Remark 3.3, \( \sum_{n=0}^{\infty} \lambda(I^{n+1}/JI^n) - e_1(I) = 1 \) and so by [32] Theorem 3.1 we have \( \text{depth} G(I) \geq d - 2 \). \( \square \)

Proposition 3.7. Let \( J \) be a minimal reduction of \( I \) such that

\[ \sum_{n=1}^{\infty} \lambda((I^{n+1} \cap J_{d-1})/(JI^n \cap J_{d-1})) = 0. \]

Then \( \text{depth} G(I) \geq d - 1 \).

Proof. By using induction on \( n \), we prove that \( I^{n+1} \cap J_{d-1} = J_{d-1}I^n \) for every \( n \geq 0 \).

For \( n = 0 \), there is nothing to prove. Assume that \( n \geq 1 \) and \( I^n \cap J_{d-1} = J_{d-1}I^{n-1} \).

From the following equalities:

\[ I^{n+1} \cap J_{d-1} = J_{d-1}I^n \]

\[ = (J_{d-1}I^n + x_{d}I^n) \cap J_{d-1} \]

\[ = J_{d-1}I^n + (x_{d}I^n \cap J_{d-1}) \]

\[ = J_{d-1}I^n + x_{d}(I^n \cap J_{d-1}) \]

\[ = J_{d-1}I^n + x_{d}(J_{d-1}I^{n-1}) = J_{d-1}I^n. \]
and using Valabrega and Valla’s theorem the result follows.

\[ \square \]

**Proposition 3.8.** Let \( J \) be a minimal reduction of \( I \) such that
\[
\sum_{n=1}^{\infty} \lambda((I^{n+1} \cap J_{d-1})/(JI^n \cap J_{d-1})) = 1.
\]
Then \( \text{depth}(G(I)) \geq d - 2 \).

**Proof.** Let \( B_{n+1} \) be the kernel of the map \( H_1(C,(n+1)) \to H_0(C',(n)) \) given in (\(*\)). Consider the following exact sequence
\[
0 \to H_d(C,(n+1)) \to \ldots \to H_1(C,(n)) \xrightarrow{\partial} H_1(C',(n+1)) \to B_{n+1} \to 0.
\]
Therefore by [14, Lemma 3.2] we have the following exact sequence
\[
\ldots \to I^n \cap J_{d-1}/J_{d-1}I^{n-1} \xrightarrow{\partial} I^{n+1} \cap J_{d-1}/J_{d-1}I^n \to B_{n+1} \to 0
\]
and \( B_{n+1} = I^{n+1} \cap J_{d-1}/JI^n \cap J_{d-1} \). Thus
\[
\sum_{n=1}^{\infty} \lambda(B_{n+1}) = \sum_{n=1}^{\infty} \lambda((I^{n+1} \cap J_{d-1})/(JI^n \cap J_{d-1})) = \sum_{i=1}^{d} (-1)^{i-2} h_i = 1
\]
and so by Proposition 3.6 we have \( \text{depth}(G(I)) \geq d - 2 \). \( \square \)

**Remark 3.9.** Let \( B_n \) be the kernel of the map \( H_1(C,(n)) \to H_0(C',(n-1)) \) given in (\(*\)). Then for each \( n \) we have the exact sequence
\[
0 \to H_d(C,(n)) \to \ldots \to H_1(C',(n-1)) \xrightarrow{\partial} H_1(C',(n)) \to B_n \to 0.
\]
Therefore, for each \( n \) we have
\[
\lambda(B_n) = \sum_{i=2}^{d} (-1)^{i-2} \lambda(H_i(C,(n))) + \sum_{i=1}^{d} (-1)^{i-1} \Delta[\lambda(H_i(C',(n)))]
\]
and by using [11] Lemma 2.7
\[
\sum_{n=2}^{\infty} (n-1) \Delta[\lambda(H_i(C',(n)))] = -\sum_{n=1}^{\infty} \lambda(H_i(C',(n))) = -h'_i
\]
and \( \sum_{n=1}^{\infty} \Delta[\lambda(H_i(C',(n)))] = 0 \).

Since \( \sum_{n=2}^{\infty} (n-1) \lambda(B_n) = \sum_{i=2}^{d} (-1)^{i-2} k_i - \sum_{i=1}^{d} (-1)^{i-1} h'_i \) and \( \sum_{n=1}^{\infty} \lambda(B_n) = \sum_{i=2}^{d} (-1)^{i-2} h_i \),

by Lemma 3.1, \( \sum_{i=1}^{d} (-1)^{i-1} h'_i = 0 \) if and only if \( \text{depth}(G(I)) \geq d - 1 \) if and only if \( \sum_{n=2}^{\infty} (n-1) \lambda(B_n) = 0 \). Thus \( \sum_{i=2}^{d} (-1)^{i-2} k_i = 1 \) cannot be happen. If \( \sum_{i=2}^{d} (-1)^{i-2} k_i = 2 \), then we have \( \sum_{i=1}^{d} (-1)^{i-1} h'_i = 1 \) and \( \sum_{n=2}^{\infty} (n-1) \lambda(B_n) = 1 \). In this case we obtain \( \lambda(B_2) = 1 \) and \( \lambda(B_n) = 0 \) for any \( n \neq 2 \). Therefore \( \sum_{n=1}^{\infty} \lambda(B_n) = \sum_{i=2}^{d} (-1)^{i-2} h_i = 1 \) and by Proposition 3.6 depth \( G(I) \geq d - 2 \).

If \( I \) is integrally closed, then \( I^2 \cap J = JJ \) and by [14] Lemma 3.2 we have \( H_1(C,(3)) = 0 \) and \( B_2 = 0 \).
If $\sum_{i=2}^{d} (-1)^{i-2}k_i = 2$, then $\lambda(B_2) = 1$ and this is contradiction. Thus, this case can not be happen.

If $\sum_{i=2}^{d} (-1)^{i-2}k_i = 3$, then we have $\sum_{n=2}^{\infty} (n-1)\lambda(B_n) = 1$ or 2. The case $\sum_{n=2}^{\infty} (n-1)\lambda(B_n) = 1$ can not be happen because $\lambda(B_2) = 0$. If $\sum_{n=2}^{\infty} (n-1)\lambda(B_n) = 2$, then $\lambda(B_3) = 1$ and $\lambda(B_n) = 0$ for all $n \neq 3$, so $\sum_{i=2}^{d} (-1)^{i-2}h_i = 1$ and depth $G(I) \geq d-2$.

If $\sum_{n=2}^{\infty} (n-1)\lambda(B_n) = 3$, then $\lambda(B_4) = 1$ and $\lambda(B_n) = 0$ for any $n \neq 4$. Thus $\sum_{i=2}^{d} (-1)^{i-2}h_i = 1$ and depth $G(I) \geq d-2$.

In the following result we compare [2, Theorem 3.1 and 3.3] with [32, Theorem 3.1].

**Theorem 3.10.** Let $J$ be a minimal reduction of $I$. If one of the following conditions holds:

1. $e_2(I) = \sum_{n=2}^{\infty} (n-1)\lambda(I^n/JI^{n-1}) - 2$;
2. $I$ is integrally closed and $e_2(I) = \sum_{n=2}^{\infty} (n-1)\lambda(I^n/JI^{n-1}) - i$, where $i = 3, 4$.

Then $e_1(I) = \sum_{n=1}^{\infty} \lambda(I^n/JI^{n-1}) - 1$.

**Proof.** (1) If $e_2(I) = \sum_{n=2}^{\infty} (n-1)\lambda(I^n/JI^{n-1}) - 2$, then $\sum_{i=2}^{d} (-1)^{i-2}k_i = 2$ and by Remark 3.9 $\sum_{n=1}^{\infty} \lambda(B_n) = \sum_{i=2}^{d} (-1)^{i-2}h_i = 1$. Therefore by Remark 3.3 we have $e_1(I) = \sum_{n=1}^{\infty} \lambda(I^n/JI^{n-1}) - 1$.

(2) If $I$ is integrally closed and $e_2(I) = \sum_{n=2}^{\infty} (n-1)\lambda(I^n/JI^{n-1}) - i$ where $i = 3, 4$ then $\sum_{i=2}^{d} (-1)^{i-2}k_i = 3$ or 4. Therefore by Remark 3.9 $\sum_{n=1}^{\infty} \lambda(B_n) = \sum_{i=2}^{d} (-1)^{i-2}h_i = 1$ and so $e_1(I) = \sum_{n=1}^{\infty} \lambda(I^n/JI^{n-1}) - 1$.

The following example show that the converse of Theorem 3.10 in general is not true.
Example 3.11. Let $R = k[x, y]_{(x, y)}$, where $k$ is a field and $I = (x^6, y^6, x^5y + x^2y^4)$. Then by using Macaulay 2 [3] we can obtain the following Hilbert polynomial

$$P_I(n) = 36 \left( \frac{n+1}{2} \right) - 15 \left( \frac{n}{1} \right) + 11$$

and $e_1(I) = \sum_{n=1}^{\infty} \lambda(I^n/JI^{n-1}) - 1$ but $e_2(I) = \sum_{n=2}^{\infty} (n-1)\lambda(I^n/JI^{n-1}) - 3$ and $I$ is not integrally closed.

Let $A = \bigoplus_{m \geq 0} A_m$ be a Noetherian graded ring where $A_0$ is an Artinian local ring, $A$ is generated by $A_1$ over $A_0$ and $A_+ = \bigoplus_{m > 0} A_m$. Let $H^i_{A_+}(A)$ denote the $i$-th local cohomology module of $A$ with respect to the graded ideal $A_+$ and set $a_i(A) = \max\{m| H^i_{A_+}(A)|_m \neq 0\}$ with the convention $a_i(A) = -\infty$, if $H^i_{A_+}(A) = 0$. The Castelnuovo-Mumford regularity is defined by $\text{reg}(A) := \max\{a_i(A) + i | i \geq 0\}$. In the following theorem, for simply, we use $a_i$ instead of $a_i(A(G))$.

**Theorem 3.12.** Let $J$ denote a minimal reduction of $I$. Suppose that one of the following conditions holds:

1. $e_2(I) = \sum_{n=2}^{\infty} (n-1)\lambda(I^n/JI^{n-1}) - 2$

2. $I$ is integrally closed and $e_2(I) = \sum_{n=2}^{\infty} (n-1)\lambda(I^n/JI^{n-1}) - i$, where $i = 3, 4$.

Then $r(I)$ is independent.

**Proof.** (1) By Remark 3.9, $\sum_{i=2}^{d} (-1)^{i-2}k_i = 2$ and depth $G(I) \geq d-2$. If depth $G(I) \geq d-1$, then by [17] Theorem 2] $r_J(I) = \text{reg}(G(I))$ and so the result in this case follows. Now we assume that depth $G(I) = d-2$ and $\sum_{i=2}^{d} (-1)^{i-2}k_i = 2$. By Remark 3.9, $\lambda(B_2) = 1$ and $\lambda(B_n) = 0$ for any $n \neq 2$. Therefore we have $\lambda(I^2 \cap Jd_{-1}/Jd_{-1}I) = 1$, $\lambda(I^n \cap Jd_{-1}/Jd_{-1}I^{n-1}) = 0$ for any $n \neq 2$, $\lambda((I^2 + Jd_{-2}) : xd_{-1}/I) = 1$ and $\lambda((I^n + Jd_{-2}) : xd_{-1}/I^{n-1} + Jd_{-2}) = 0$ for any $n \neq 2$. If depth $G(I) = d-2$, then by applying [15] Theorem 2.1] there are two cases:

(i) If $a_{d-1} \leq a_d$, then $r_J(I) = a_d + d = \text{reg}(G(I))$.

(ii) If $a_d < a_{d-1}$, then $r_J(I) = a_{d-1} + d - 1 = \text{reg}(G(I))$ and $a_{d-1} = \max\{n| (I^{n+d-1} + Jd_{-2}) : xd_{-1} \neq I^{n+d-2} + Jd_{-2}\}$. Therefore by the above process $a_{d-1} = 3 - d$ and so for any reduction $J$ of $I$, $r_J(I) \leq 2$. Hence $r(I)$ is independent.

(2) Let $i = 3$. Then by Remark 3.9, $\lambda(B_3) = 1$ and $\lambda(B_n) = 0$ for any $n \neq 3$. Thus

$$\lambda(I^3 \cap Jd_{-1}/Jd_{-1}I^2) = 1 = \lambda((I^3 + Jd_{-2}) : xd_{-1}/I^2)$$

and

$$\lambda(I^n \cap Jd_{-1}/Jd_{-1}I^{n-1}) = 0 = \lambda((I^n + Jd_{-2}) : xd_{-1}/I^{n-1} + Jd_{-2})$$
for any \( n \neq 3 \). If depth \( G(I) = d - 2 \), then by \([13]\) Theorem 2.1 there is two cases:

(i) If \( a_{d-1} \leq a_d \), then \( r_J(I) = a_d + d = \text{reg}(G(I)) \).

(ii) If \( a_d < a_{d-1} \), then \( r_J(I) \leq a_{d-1} + d - 1 = \text{reg}(G(I)) \) and \( a_{d-1} = \max\{n|I^{n+d-1} + J_{d-2} : x_{d-1} \neq I^{n+d-2} + J_{d-2}\} \). Therefore by the above process \( a_{d-1} = 4 - d \) and so for any reduction \( J \) of \( I \), \( r_J(I) \leq 3 \). Since \( I \) is integrally closed, then \( r(I) \) is independent.

Let \( i = 4 \). Then by Remark 3.9, \( \sum_{i=2}^{d} (-1)^{i-2}k_i = 4 \), \( \lambda(B_4) = 1 \) and \( \lambda(B_n) = 0 \) for any \( n \neq 4 \). Therefore we have \( \lambda(I^4 \cap J_{d-1}/J_{d-1}I^3) = 1 = \lambda((I^4 + J_{d-2}) : x_{d-1}/I^3) \), \( \lambda(I^n \cap J_{d-1}/J_{d-1}I^{n-1}) = 0 = \lambda((I^n + J_{d-2}) : x_{d-1}/I^{n-1} + J_{d-2}) \) for any \( n \neq 4 \).

If depth \( G(I) = d - 2 \), then by applying \([13]\) Theorem 2.1 there is two cases:

(i) If \( a_{d-1} \leq a_d \), then \( r_J(I) = a_d + d = \text{reg}(G(I)) \).

(ii) If \( a_d < a_{d-1} \), then \( r_J(I) \leq a_{d-1} + d - 1 = \text{reg}(G(I)) \) and \( a_{d-1} = \max\{n|I^{n+d-1} + J_{d-2} : x_{d-1} \neq I^{n+d-2} + J_{d-2}\} \). Therefore by the above process \( a_{d-1} = 5 - d \) and so for any reduction \( J \) of \( I \), \( r_J(I) \leq 4 \). If \( r_J(I) \leq 2 \) for some \( J \), then \( G(I) \) is Cohen-Macaulay and this is a contradiction with depth \( G(I) = d - 2 \). If \( r_J(I) = 3 \) for some \( J \), then \( I^4 = JI^3 \) and so \( \lambda(I^4 \cap J_{d-1}/J_{d-1}I^3) = 0 \) and this is a contradiction. Hence for any minimal reduction \( J \) of \( I \) we have \( r_J(I) = 4 \) and so \( r(I) \) is independent.

Let \( a \) be an ideal of grade at least 1 in a Noetherian ring \( R \). The Ratliff-Rush closure of \( a \) is defined as the ideal \( \bar{a} = \bigcup_{n \geq 1}(a^{n+1} : a^n) \). It is a refinement of the integral closure of \( a \) and \( \bar{a} = a \) if \( a \) is integrally closed (see \([24]\)).

**Proposition 3.13.** (compare with \([2]\) Theorem 3.3(b))] Let \( d = 3 \) and \( J \) be a minimal reduction of \( I \). If \( I = I \) and \( e_2(I) = \sum_{n=2}^{\infty} (n-1)\lambda(I^n/JI^{n-1}) - 3 \) then depth \( G(I) \geq 1 \).

**Proof.** If \( e_2(I) = \sum_{n=2}^{\infty} (n-1)\lambda(I^n/JI^{n-1}) - 3 \), then \( \sum_{i=2}^{d} (-1)^{i-2}k_i = 3 \). Hence by Remark 3.9, \( \sum_{i=1}^{d} (-1)^{i-1}h'_i = 1 \) and \( \sum_{n=2}^{\infty} (n-1)\lambda(B_n) = 2 \). Therefore we have the following cases:

1) \( \lambda(B_3) = 1 \) and \( \lambda(B_n) = 0 \) for any \( n \neq 3 \).

2) \( \lambda(B_2) = 2 \) and \( \lambda(B_n) = 0 \) for any \( n \neq 2 \).

For the first case, by Proposition 3.6. depth \( G(I) \geq 1 \) and the result follows. Let consider the second case. Since \( I = I \), we can obtain that \( I^2 : x = I \) for all
superficial element. Since \( \lambda(B_2) = 2 \), it follows \( \lambda(I^2 \cap J_2/J_2I) = 2 \). It therefore follows \( \lambda(I^2 \cap J_1/J_1I) = 0 \). Hence \( \sum_{i=1}^{d} (-1)^{i-1} h_i' \geq 2 \) and this is a contradiction.

Northcott in [19] proved that \( e_1(I) \geq e_0(I) - \lambda(R/I) \) and after that Huneke in [13] showed that \( e_1(I) = e_0(I) - \lambda(R/I) \) if and only if \( I^2 = JI \). When this is the case, \( G(I) \) is Cohen-Macaulay. Sally in [27] proved that if \( d \geq 2 \), \( e_1(I) - e_0(I) + \lambda(R/I) = 1 \) and \( e_2(I) \neq 0 \), then depth \( G(I) \geq d - 1 \) (see also [13] Corollary 4.15 and [6] Proposition 3.1). Also Itoh in [15] with this conditions proved that if \( I \) is integrally closed, then \( G(I) \) is Cohen-Macaulay. In the following example we show that the condition integrally closed ideal \( I \) for Cohen-Macaulayness of \( G(I) \) is essential.

The following example appear in [23].

**Example 3.14.** Let \( R = k[x, y, z]_{(x, y, z)} \), where \( k \) is a field, and \( I = (x^2 - y^2, y^2 - z^2, xy, yz, xz) \). Then, by Macaulay 2, we have \( e_0(I) = 8 \), \( e_1(I) = 4 \), \( e_2(I) = 0 \) and depth \( G(I) = 0 \).

**Lemma 3.15.** Let \( J \) be a minimal reduction of \( I \). If \( e_1(I) - e_0(I) + \lambda(R/I) = 2 \) and \( I = I \), then depth \( G(I) \geq d - 1 \).

**Proof.** By using the Sally machine and the good behaviour of \( e_1(I) \) modulo superficial elements, we may reduce the statement to dimension two. If \( e_1(I) - e_0(I) + \lambda(R/I) = 2 \), by [24] Corollary 1.5 we have \( r_J(I) \leq 3 \) for any minimal reduction \( J \) of \( I \). If there exist a minimal reduction \( J \) of \( I \) such that \( r_J(I) = 2 \), then by [16] Lemma 2.1 \( G(I) \) is Cohen-Macaulay. If for any minimal reduction \( J \) of \( I \), \( r_J(I) = 3 \) then by [24] Remark 1.8 \( r_J(I) \leq e_1(I) - e_0(I) + \lambda(R/I) + 2 - \lambda(I^2/JI) \).

Hence \( r(I) = 3 \leq 4 - \lambda(I^2/JI) \) and \( \lambda(I^2/JI) \leq 1 \) and so by [24] Corollary 1.7 depth \( G(I) \geq 1 \).

**Proposition 3.16.** Let \( J \) be a minimal reduction of \( I \). If \( e_1(I) - e_0(I) + \lambda(R/I) \leq 3 \) and \( I = I \), then \( r(I) \) is independent.

**Proof.** By Lemma 3.15 and the above explanation, we can assume \( e_1(I) - e_0(I) + \lambda(R/I) = 3 \) and also by [24] Corollary 4.7 and [10] Lemma 1.1 we may assume that \( d = 2 \). If depth \( G(I) \geq 1 \), then by [18] Corollary 2.2 we have \( r_J(I) = \text{reg}(G(I)) \) and so \( r(I) \) is independent. Now we may assume that depth \( G(I) = 0 \). Since \( e_1(I) - e_0(I) + \lambda(R/I) = 3 \), by [24] Corollary 1.5 \( r_J(I) \leq 4 \) for any minimal reduction \( J \) of \( I \). If for some \( J \), \( r_J(I) = 2 \) then by [16] Lemma 2.1 \( G(I) \) is Cohen-Macaulay and this is a contradiction with depth \( G(I) = 0 \). If for some \( J \), \( r_J(I) = 4 \) then by [24] Remark 1.8 \( r_J(I) \leq e_1(I) - e_0(I) + \lambda(R/I) + 2 - \lambda(I^2/JI) \).

Therefore \( \lambda(I^2/JI) \leq 1 \) and so depth \( G(I) \geq 1 \) and this is also a contradiction.
Hence we can assume that for any minimal reduction $J$ of $I$, $r_J(I) = 3$ and so $r(I)$ is independent.

**Corollary 3.17.** Let $J$ be a minimal reduction of $I$. If $e_1(I) - e_0(I) + \lambda(R/I) \leq r_J(I) - 1$, $I = \bar{I}$ and depth $G(I) \geq d - 2$, then $r(I)$ is independent.

**Proof.** By using the Sally machine and the good behaviour of $e_1(I)$ modulo superficial elements, we may reduce the statement to dimension two. If $e_1(I) - e_0(I) + \lambda(R/I) \leq r_J(I) - 1$, then by [23, Remark 1.8] $r_J(I) \leq e_1(I) - e_0(I) + \lambda(R/I) + 2 - \lambda(I^2/JI)$. Hence $\lambda(I^2/JI) \leq 1$ and so depth $G(I) \geq d - 1$. Therefore the result follows by [17, Theorem 2].

**Example 3.18.** Let $R = k[x, y]_{(x, y)}$ where $k$ is a field, and $I = (x^6, y^6, x^5y, x^3y^3, x^2y^4, xy^5)$. Then by Macaulay 2 we have $e_0(I) = 36$, $e_1(I) = 15$ and $\lambda(R/I) = 22$. Hence $e_1(I) - e_0(I) + \lambda(R/I) = 1$ but depth $G(I) = 0$ and $r_J(I) = 2 = \text{reg}(G(I))$ for all minimal reduction $J$ of $I$.

**Example 3.19.** Let $R = k[x, y]_{(x, y)}$ where $k$ is a field, and $I = (x^6, y^6, x^5y, x^3y^3, x^2y^4)$. Then by Macaulay 2 we have $e_0(I) = 36$, $e_1(I) = 15$ and $\lambda(R/I) = 23$. Hence $e_1(I) - e_0(I) + \lambda(R/I) = 2$ but depth $G(I) = 0$ and $r_J(I) = 2 = \text{reg}(G(I))$ for all minimal reduction $J$ of $I$.

**Example 3.20.** Let $R = k[x, y]_{(x, y)}$ where $k$ is a field, and $I = (x^6, y^6, x^5y, x^3y^3, xy^5)$. Then by Macaulay 2 we have $e_0(I) = 36$, $e_1(I) = 15$ and $\lambda(R/I) = 23$. Hence $e_1(I) - e_0(I) + \lambda(R/I) = 3$ but depth $G(I) = 0$ and $r_J(I) = 2 = \text{reg}(G(I))$ for all minimal reduction $J$ of $I$.

**Example 3.21.** Let $R = k[x, y]_{(x, y)}$ where $k$ is a field, and $I = (x^6, y^6, x^5y, x^3y^3, xy^5)$. Then by Macaulay 2 we have $e_0(I) = 36$, $e_1(I) = 15$ and $\lambda(R/I) = 24$. Hence $e_1(I) - e_0(I) + \lambda(R/I) = 3$ but for two minimal reduction $J_1 = (x^6, x^5y + y^6)$ and $J_2 = (x^6, y^6)$ we have $r_{J_1}(I) = 2$ and $r_{J_2}(I) = 3$ and depth $G(I) = 0$ because $I$ is not integrally closed.

The following example due to Huckaba and Huneke [12].

**Example 3.22.** Let $R = k[x, y, z]_{(x, y, z)}$ where $k$ is a field of characteristic $\neq 3$. Let $\mathfrak{a} = (x^4, x(y^3 + z^3), y(y^3 + z^3), z(y^3 + z^3))$ and set $I = \mathfrak{a} + m^3$ where $m$ is the maximal ideal of $R$. The ideal $I$ is an integral closer $m$-primary ideal whose associated graded ring $gr_I(R)$ has depth 2. We checked that $e_0(I) = 76$, $e_1(I) = 48$ and $\lambda(R/I) = 31$ so $e_1(I) - e_0(I) + \lambda(R/I) = 3$ and $r(I)$ is independent.

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