Heuristic derivation of Casimir effect in minimal length theories

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We propose a heuristic derivation of Casimir effect in the context of minimal length theories based on a Generalized Uncertainty Principle (GUP). By considering a GUP with only a quadratic term in the momentum, we compute corrections to the standard formula of Casimir energy for the parallel-plate geometry, the sphere and the cylindrical shell. For the first configuration, we show that our result is consistent with the one obtained via more rigorous calculations in Quantum Field Theory. Experimental developments are finally discussed.

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\section{I. INTRODUCTION}

Quantum vacuum is not empty. Seen up close, it is crowded with all sort of virtual particles continuously popping in and out of existence. One of the most outstanding manifestations of vacuum fluctuations is the Casimir effect \cite{1,2}, which has recently aroused great interest in a large class of domains, ranging from quantum computing \cite{3} to biology \cite{4}. As well known, the Casimir effect originates from alterations of the zero-point energy induced by boundary conditions. The ensuing attractive force is obtained by differentiating the vacuum energy with respect to the separation between the boundaries. In passing, we mention that an alternative derivation was proposed in Ref. \cite{5}, where the Casimir effect was addressed by considering relativistic van der Waals forces between metallic plates.

Besides its intrinsic interest in the standard Quantum Field Theory (QFT), the Casimir effect provides a useful test bench for physics beyond the Standard Model \cite{6} and gravity theories \cite{7,8,9}. In Refs. \cite{6}, for instance, it was analyzed in connection with the unitary inequivalence between mass and flavor Fock spaces for mixed fields. Similarly, in Refs. \cite{8} and \cite{9} the computation of the Casimir energy density and pressure was exploited to fix some constraints on the characteristic free parameters appearing in the Standard Model Extension and in extended theories of gravity, respectively. Recently, extensive studies were also carried out in the context of the Generalized Uncertainty Principle (GUP) \cite{10,11,12,13,14}, where non-trivial corrections were shown to arise due to the existence of a minimal length at Planck scale. The present contribution fits in the last of the above lines of research, since it aims to investigate the connection between the Casimir effect and models which inherently embed this fundamental scale.

The concept of minimal length naturally emerges in quantum gravity theories in the form of an effective minimal uncertainty in position $\Delta x_{\text{min}} > 0$. Several different theoretical arguments (many of them in form of Gedanken experiments) show the impossibility to measure arbitrarily short distances, due to the very existence of gravity. This naturally leads to a modification of the Heisenberg position-momentum uncertainty principle (HUP). In a one-dimensional setting, studies of string theory, loop quantum gravity, deformed special relativity and black hole physics \cite{15,16,17,18,19,20,21,22} have converged on the idea that a proper generalization of the HUP would be

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left[ 1 + \beta \left( \frac{\Delta p c}{E_p} \right)^2 \right],$$

where $E_p$ is the Planck energy and we are retaining only the leading-order correction in the dimensionless parameter $\beta > 0$. Of course, in the limit $\beta \to 0$, the HUP of ordinary quantum mechanics is recovered, as it should be. Let us also remark that the deformation parameter $\beta$ is not fixed by the theory: in principle, it can be either constrained via experiments \cite{23} or estimated by computational techniques in different contexts \cite{24}, which yield $\beta \sim O(1)$ (for a recent

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over view on the various attempts to fix $\beta$, see Ref. [25]. However, given the high-energy scale at which modifications of the HUP should become relevant, the natural arenas for testing GUP effects are undoubtedly Hawking [25] and Unruh [27] radiations.

Note that, for mirror-symmetric states (with $\langle \hat{p} \rangle = 0$), Eq. (1) can be equivalently rephrased in terms of the generalized commutation relation

$$[\hat{x}, \hat{p}] = i\hbar \left[1 + \beta \left(\frac{\hat{p} c}{E_p}\right)^2 \right],$$

since $\Delta x \Delta p \geq (1/2) |\langle [\hat{x}, \hat{p}] \rangle|$. Vice-versa, the above relation implies the inequality (1) for any state. Moreover, in $n$ spatial dimensions, the commutator (2) can be cast in different forms, among which the most common is

$$[\hat{x}_i, \hat{p}_j] = \left[f(\hat{p}^2) \delta_{ij} + g(\hat{p}^2) \hat{p}_i \hat{p}_j \right], \quad i, j = 1, \ldots, n$$

with $f(\hat{p}^2) = 1 + \beta \left(\frac{\hat{p} c}{E_p}\right)^2$ and $g(\hat{p}^2) = 0$ (note that in $n$-dimensions, the functions $f(\hat{p}^2)$ and $g(\hat{p}^2)$ can be chosen in different ways [12][13]). In any case, they are not completely arbitrary, being related via the requirement of translational and/or rotational symmetry of the commutator [28].

Working in the outlined scenario, in the present paper we compute GUP-corrections to the Casimir energy for three different geometries: the parallel-plate configuration, the spherical and cylindrical shells. For the first case, we follow a field theoretical treatment first [10–13] and then a heuristic derivation. The two approaches are found to be consistent as concerns the dependence of the corrective term on the inverse fifth power of the distance between the plates. On the other hand, to the best of our knowledge, this is the first time that the Casimir effect for spherical and cylindrical geometries is addressed in the context of GUP. Thereby, we can only compare our results with the ones existing in literature in the limit of vanishing $\beta$.

The remainder of the work is organized as follows: in Section II we briefly review the standard calculation of the Casimir vacuum energy for the parallel-plate geometry. The obtained result is then extended to the context of GUP by quantizing the field in the formalism of maximally localized states. Motivated by the utility of heuristic procedures which help to develop physical intuition, in Section III we consider a similar derivation of the Casimir effect both from HUP and GUP based on simple quantum and thermodynamic arguments. In this regard, we also clarify the approach of Ref. [29], where the Casimir effect is deduced from the HUP by naively introducing an effective radius $r_e$. The above reasoning is then applied to the cases of a spherical and cylindrical shells in Sec. [IV]. Finally, conclusions and perspectives are given in Section [V].

II. CASIMIR EFFECT FOR PARALLEL PLATES: QFT APPROACH

In the framework of canonical QFT, the Casimir effect can be derived via different approaches and in a wide range of contexts. By referring to the original treatment by Casimir, in what follows we sketch the main steps leading to the relation between the zero-point energy $\Delta E$ and the distance $d$ between the plates.

A. Casimir effect from Heisenberg uncertainty principle

We consider the simplest three-dimensional geometry of two parallel plates separated by a distance $d$ along the $x$-axis. Let $L$ be the side of the plates (with $L \gg d$) and $S = L^2$ their surface area. The Casimir effect arises from the vacuum fluctuations of any quantum field in the presence of such boundary conditions on field modes. Consider, for example, the electromagnetic field $A(t, x)$ in the Coulomb gauge $\nabla \cdot A = 0$,

$$\hat{A}(t, x) = \sum_{\lambda=1,2} \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{(2\pi)^4}{\omega_p}} \left[ e_{p,\lambda} \hat{a}_{p,\lambda} \psi_p(t, x) + \text{h.c.} \right],$$

where $e_{p,\lambda}$ are the polarization vectors satisfying the relation $e_{p,\lambda} e_{p,\lambda'} = \delta_{\lambda\lambda'}$ and $\psi_p(t, x)$ are the plane waves (i.e. the standard position representation of momentum eigenstates) of frequency $\omega_p = cp/\hbar$. The ladder operators $\hat{a}_{p,\lambda}$ in Eq. (4) obey the canonical commutation relations.

Now, the vacuum energy responsible for the attractive force between the plates can be obtained by subtracting the infinite vacuum energy of the electromagnetic field in free space from the corresponding infinite energy between the perfectly conducting boundaries. Mathematically speaking, we have

$$\Delta E(d) \equiv E(d) - E_0 = \langle 0 | \hat{H}(d) - \hat{H} | 0 \rangle,$$
where the Hamiltonian is \( \hat{H} = \frac{1}{8\pi} \int d^3x \left[ (\partial_0 \hat{A})^2 - \hat{A} \cdot \nabla^2 \hat{A} \right] \). By using this relation, one can show that

\[
\Delta E(d) = cS \int \frac{d^3p_{\perp}}{(2\pi\hbar)^2} \left[ \frac{p_{\perp}}{2} + \sum_{n=1}^{\infty} \sqrt{|p_{\perp}|^2 + \frac{n^2\pi^2\hbar^2}{d^2}} - \int_0^{\infty} dn \sqrt{|p_{\perp}|^2 + \frac{n^2\pi^2\hbar^2}{d^2}} \right],
\]

where \( p_{\perp} = (p_x, p_z) \) is the transverse momentum and we have exploited the fact that the conditions of vanishing field on the plates only allow for a discrete set of values of the momentum along the \( x \)-axis, i.e. \( p_x = \frac{n\pi}{d} \), with \( n \) integer. Note that, in the above calculations, we have neglected surface corrections. The integral in Eq. (6) is divergent for large values of the momentum. A possible trick to remove this infinity is to introduce an ultraviolet momentum cutoff \( p_{\text{max}} \sim \hbar/d \) and remove the regularization only at the end of calculations\(^1\). By following this procedure and applying the asymptotic Euler–MacLaurin summation formula, we obtain the well-known expression for the energy shift

\[
\Delta E(d) = -\frac{\pi^2}{120} \frac{\hbar c S}{d^3}.
\]

For later convenience, we also write down the formula for the Casimir energy in one spatial dimension

\[
\Delta E(d) = -\frac{\pi}{12} \frac{\hbar c}{d}.
\]

B. Casimir effect in minimal length QED

Let us now investigate the Casimir effect in the presence of the generalized commutator \[^3\]. Note that, in this case, a minimization of the generalized uncertainty relation with respect to \( \Delta p_i \) gives the following nonzero minimal length (\( \Delta x_i \)\( _{\text{min}} = \sqrt{\beta} \ell_p \)), where \( \ell_p = \hbar c / E_p \) denotes the Planck length.

A quantum theoretical framework which implements the appearance of a nonzero minimal uncertainty in position has been described in Ref. \[^{17}\]. Unlike the standard quantum mechanics, in this context we do not have localized functions in the \( x \)-space, so we have to introduce the so-called quasi-position representation. This consists in projecting the state of the system onto the set of maximally localized states, which, by definition, are characterized by the minimal position uncertainty (\( \Delta x \)\( _{\text{min}} \)). In the momentum representation, the general (i.e. time-dependent) maximally localized state around the average position \( \mathbf{x} \) takes the form

\[
\tilde{\psi}_p(t, \mathbf{x}) = \frac{1}{(\sqrt{2\pi\hbar})^3} e^{-i[\tilde{\omega}_p t - \mathbf{p} \cdot \mathbf{x} / \hbar]},
\]

where

\[
\tilde{\omega}_p = \frac{E_p}{\hbar \sqrt{\beta}} \arctan \left( \frac{c \sqrt{\beta}}{E_p} \right), \quad \tilde{p}_i = \left[ \frac{E_p}{c \sqrt{\beta}} \arctan \left( \frac{c \sqrt{\beta}}{E_p} \right) \right] \mathbf{p}_i.
\]

Note that, for \( \beta \to 0 \), the quasi-position representation reduces to the standard plane-wave formalism, since \( \tilde{\omega}_p \to \omega_p \), \( \tilde{p}_i \to \mathbf{p}_i \) and (\( \Delta x_i \)\( _{\text{min}} = 0 \).

In terms of the maximally localized states, the electromagnetic field reads

\[
\hat{A}(t, \mathbf{x}) = \sum_{\lambda = 1, 2} \int \frac{d^3p}{(2\pi)^3} \left[ \frac{(2\pi)^4 \hbar c^2 \sqrt{\beta}}{E_p \arctan \left( \frac{c \sqrt{\beta}}{E_p} \right)} \right] \left[ \epsilon_{\beta, \lambda} \hat{a}_{\mathbf{p}, \lambda} \tilde{\psi}_p(t, \mathbf{x}) + \text{h.c.} \right],
\]

where the factor \( (1 + c^2 p^2 \beta / E_p^2)^{-1} \) arises from the modified completeness relation for the momentum eigenstates \( | \mathbf{p} \rangle \).

In order to derive GUP-corrections to the Casimir effect, let us replace the field expansion \[^{11}\] in the Hamiltonian. The calculation of the energy shift \[^{6}\] proceeds as usual \[^{13}\]. The remarkable difference, however, is that in this case we do not need to introduce any restriction on the momentum scale. Indeed, from Eq. (10), it follows the natural cutoff \( \tilde{p}_{\text{max}} = \pi E_p / (2c \sqrt{\beta}) \), leading to \( n_{\text{max}} = E_p d / (2\hbar c \sqrt{\beta}) \). Accordingly, the Casimir energy takes the form \[^{13}\]

\[
\Delta E(d) = -\frac{\pi^2}{720} \frac{\hbar c S}{d^3} \left[ 1 + \frac{2\pi^2 \beta}{3} \left( \frac{\hbar c}{E_p d} \right)^2 \right],
\]

\[^1\] Note that other common regularization techniques are the zeta function regularization and the point splitting \[^{30}\].
to be compared with the standard QED expression \[7\]. Note that the GUP term is attractive, since it contributes to increase the modulus of the energy.

In Fig. 1, the Casimir energy per unit surface area has been plotted as a function of the distance between the plates for three values of the deformation parameter \(\beta\). For distances large enough, the different curves overlap, being the effects of the minimal length negligible.

### III. CASIMIR EFFECT FOR PARALLEL PLATES: HEURISTIC APPROACH

Although the heuristic arguments we are going to discuss do not give the exact expression for the Casimir force, they allow us to better understand the origin of this effect, as well as the nature of GUP-corrections to the standard formula \[7\]. Thus, following the guidelines of the previous Section, we provide a heuristic computation of Casimir energy by using the Heisenberg Uncertainty Principle first and then working in the framework of minimal length theories based on the Generalized Uncertainty Principle \[1\]. Comparison with the corresponding QFT results is finally discussed.

#### A. Casimir effect from Heisenberg uncertainty principle

In Ref. \[29\], the Casimir effect is derived from the idea that the contribution to the vacuum energy at a point \(P_0\) of a plate is affected by the presence of the other boundary. Specifically, the author considers virtual photons produced by vacuum fluctuations somewhere in the space and arriving at \(P_0\). In order to compute the total Casimir energy \(\Delta E\), one has to take into account all the points on the surface \(S\) of the plate. Therefore, from the HUP

\[
\Delta x \Delta E \simeq \frac{\hbar c}{2},
\]

\((p = E/c \text{ for photons})\), the total contribution to the energy fluctuation \(\Delta E\) is given by those photons in a volume \(S\Delta x\) around the plate, where \(\Delta x\) is the position uncertainty of the single particle. Note that, if we had only one plate, \(\Delta x\) would be infinite, since photons may be created in any point of the space. However, this is no longer true in the presence of both the boundaries. In that case, indeed, virtual particles originating from behind the second plate cannot reach \(P_0\). Thus, the additional plate acts as a sort of shield.

The above situation can be depicted as follows: consider a sphere of radius \(R\) centered at the point \(P_0\) and enclosing both the plates (see Fig. 2). In the single-plate configuration, the effective volume \(S\Delta x\) corresponding to the entire

![Fig. 1: Casimir energy per unit surface versus the distance between the plates for different values of \(\beta\) (quantities are in Planck units). Note that the green (dash-dotted) line starts from the minimal distance \(d = \sqrt{\beta \ell_p} \simeq 3 \ell_p\).](image-url)
FIG. 2: Heuristic derivation of the zero-point energy for two parallel plates at distance $d$. The sphere of radius $R$ represents the whole space. The only photons which are allowed to impact on $P_0$ are those in the shadowed volume.

Space can be thought of as the total volume of the sphere $V_T = \frac{4}{3}\pi R^3$, with $R \to \infty$. Clearly, such a volume will be reduced by including the second plate, which will prevent particles that pop out in $P$ from impacting on $P_0$ (with reference to Fig. 2, the effective volume is represented by the shadowed region). As a result, we can write $S\Delta x = V_T - V_C$, where $V_C$ is the volume shielded by the second plate. In the case of infinite boundaries, or even better when $L/(2d) \to \infty$, one can show that $V_C = \frac{2}{3}\pi R^3$, yielding [29]

$$S\Delta x \approx \frac{2}{3}\pi R^3. \quad (14)$$

In the above treatment, no length scale has been considered, hence the volume $S\Delta x$ diverges as the radius $R$ increases. To cure such a pathological behavior, in Ref. [29] the author introduces a cutoff $r_e$ representing the effective distance beyond which photons have a negligible probability to reach the plate. In this way, Eq. (14) can be rewritten as

$$S\Delta x \approx \frac{2}{3}\pi r_e^3, \quad (15)$$

which has indeed a finite value. Combining this relation with the HUP, we then obtain

$$|\Delta E(r_e)| = \frac{3}{4} \frac{Sh c}{\pi r_e^3}, \quad (16)$$

which implies

$$r_e \approx d, \quad (17)$$

from comparison with the exact expression (7) for the Casimir energy. Strictly speaking, we would have

$$r_e = \frac{\sqrt{540}}{\pi} d \approx 2.6 d. \quad (18)$$

Although the above derivation is straightforward and very intuitive, the discussion on the physical origin of the length cutoff $r_e$ appears to be rather obscure in some points. Therefore, in order to clarify the meaning of Eq. (17), let us focus on the computation of the Casimir effect in a simplified one-dimensional system: similar reasonings can be readily extended to three dimensions.

From the Heisenberg uncertainty relation, it is well-known that large energy fluctuations live for very short time and, thus, hard virtual photons of energy $\Delta E$ can only travel short distances of order $hc/\Delta E$. As a consequence, the further these particles are created from a plate, the more negligible their contribution to the energy around that plate will be. Let us apply these considerations to the apparatus in Fig. 3. It is easy to see that virtual photons popping
FIG. 3: Setup for the heuristic derivation of the Casimir effect: two infinite parallel plates (bold lines) at distance $d$. The effective radius beyond which the creation of virtual photons does not give a significant contribution to the Casimir energy is denoted by $r_e$.

out in the strip of width $d$ on the right side of the right plate do not contribute to the Casimir effect, since their pressure is balanced by those photons originating between the plates. By contrast, photons coming from a distance greater than $d$ in the right region do not experience any compensation, because their symmetric “partners” on the left side are screened by the first plate. The overall result is a net force acting on the right plate from right to left. Of course, this argument can be symmetrically applied to the left boundary and provides a qualitative explanation for the origin of the attractive Casimir force.

Now, consider a point at a distance $x_0 > d$ from one of the plates, as in Fig. 3. Virtual photons originate from quantum fluctuations in a small region around that point. Such a region, however, cannot be smaller than the (reduced) Compton length of the electron, $\lambda_C = \hbar/(m_e c)$, otherwise the energy amplitude of the fluctuation would exceed the threshold $E \simeq m_e c^2$ for the production of electron-positron pairs. Besides, photons produced at $x_0$ can impact on the plate (and therefore contribute to the Casimir force) only if their energy $E$ is such that $0 < E < E_0$, where $E_0 = \hbar c/x_0$. Particles of higher energy $E > E_0$, indeed, would recombine before reaching the plate, since the distance they travel is $x = \hbar c/E < \hbar c/E_0 = x_0$.

We can now assume that photons coming from $x_0$ originate from fluctuations of energy $E$ with a probability given by a Boltzmann-like factor $f(E) = e^{-E/(m_e c^2)}$. Thus, the total linear energy density (i.e. the energy per unit length) arriving on the plate will be

$$|\Delta \varepsilon(E_0)| = \int_0^{E_0} \frac{dE}{\lambda_C^2 m_e c^2} f(E) = \frac{1}{\hbar c} \int_0^{E_0} dE E e^{-E/(m_e c^2)},$$

(19)

where, since we are dealing with the electromagnetic field, we have introduced the natural threshold of the electron mass/energy $m_e c^2$. In terms of the distance $x_0$, the above integral becomes

$$|\Delta \varepsilon(x_0)| = \hbar c \int_{x_0}^{\infty} \frac{dx}{x^3} e^{-\hbar c/(m_e c x)}.$$  

(20)

Finally, in order to get the contribution to the Casimir energy from all the photons which impact on the plate, we integrate over all the points $x_0$ such that $d < x_0 < \infty$, obtaining

$$|\Delta E(d)| = \int_d^{\infty} dx_0 \Delta \varepsilon(x_0).$$

(21)
The integrals in Eqs. (20) and (21) can be easily evaluated by observing that, for \( x \) large enough, the Boltzmann factor \( e^{-\hbar/(m_e x)} \) becomes approximately of order unity. This yields

\[
|\Delta \varepsilon(x_0)| \approx \frac{1}{2} \frac{\hbar c}{x_0^2},
\]

and, hence

\[
|\Delta E(d)| \approx \frac{1}{2} \frac{\hbar c}{d},
\]

which is in good agreement with the QFT prediction (8). Similarly, one can show that the generalization to three dimensions leads to a result consistent with Eq. (7).

The physical relevance of the above discussion becomes clearer if we observe that probability distributions like those in Eq. (19) or (20) allow us to naturally interpret the effective radius \( r_e \) in Eq. (15) as the distance from the plate below which the vast majority of photons contribute to the large part of the Casimir energy. More rigorously, we can define \( r_e > d \) as the distance within which photons carrying the fraction \( 0 < \gamma < 1 \) of the total Casimir energy are created. In other terms, we can write

\[
\frac{\hbar c}{2} \int_d^{d_e} dx \frac{dx}{x^2} = \gamma \Delta E(d),
\]

from which

\[
r_e = \frac{d}{1 - \gamma}.
\]

Thus, setting \( r_e \approx 2.6 d \) (as in Eq. (18)) amounts to consider a fraction \( \gamma \approx 0.62 \) of the total energy responsible for the Casimir effect.

The above picture is quite rough, since it relies on the adoption of a Boltzmann-like distribution for the energy of quantum vacuum fluctuations. As a result, it underestimates the fraction of photons produced within the distance \( r_e \) from the plate. Considerable improvements can be achieved by employing more realistic functions \( f(E) \) in Eq. (19). For further details on this topic, see Refs. [31] and therein.

### B. Casimir effect from Generalized uncertainty principle

Let us now extend the above arguments to the context of the GUP. As in the previous Subsection, we shall focus for simplicity on the one-dimensional case. The generalization to three dimensions proceeds in a very similar fashion.

We start from the modified uncertainty relation (1), here recast in the form

\[
\Delta x \Delta E \simeq \frac{\hbar c}{2} \left[ 1 + \beta \left( \frac{\Delta E}{E_p} \right)^2 \right].
\]

By solving with respect to \( \Delta E \), we obtain

\[
\Delta E = \frac{\Delta x E_p^2}{\hbar c \beta} \left[ 1 \pm \sqrt{1 - \beta \left( \frac{\hbar c}{\Delta x E_p} \right)^2} \right],
\]

where the only solution to be considered is the one with negative sign, as it reduces to the standard result for vanishing \( \beta \) (conversely, the solution with positive sign has no evident physical meaning). After expanding to the first order in \( \beta \), it follows that

\[
\Delta E = \frac{\hbar c}{2 \Delta x} \left[ 1 + \frac{\beta}{4} \left( \frac{\hbar c}{\Delta x E_p} \right)^2 \right].
\]

If we now neglect those photons coming from distances greater than the effective radius \( r_e \), it is natural to assume the uncertainty position \( \Delta x \) of the single photon to be of the order of \( r_e \) and, thus, of \( d \), according to Eq. (18). Then, by replacing \( \Delta x \approx 2.6 d \) into Eq. (28), the contribution to the Casimir energy at a given point reads

\[
|\Delta E(d)| \approx 0.2 \frac{\hbar c}{d} \left[ 1 + 0.04 \beta \left( \frac{\hbar c}{E_p d} \right)^2 \right],
\]

from which

\[
r_e = \frac{d}{1 - \gamma}.
\]
FIG. 4: Heuristic derivation of the zero-point energy for a spherical shell of radius $a$. The outer sphere of radius $R$ represents the whole space. The only photons which are allowed to impact on $P_0$ are those in the shadowed volume.

which indeed agrees with Eq. (8) in the limit $\beta \to 0$.

The above considerations can be now generalized to three-dimensions by taking into account the contribution to the zero-point energy at any point of the plates of area $S$. In doing so, straightforward calculations lead to

$$|\Delta E(d)| \simeq 0.03 \frac{\hbar c S}{d^3} \left[ 1 + 0.04 \beta \left( \frac{\hbar c}{E_p d} \right)^2 \right],$$

which is to be compared with Eq. (12). In spite of our skinny assumptions, one can see that the obtained expression agrees with the field theoretical result as concerns the dependence of the GUP correction on the inverse fifth power of the distance between the plates. We also notice that the exact numerical coefficient can be recovered by including a proper factor which accounts for the extension of the GUP (26) to a higher-dimensional system.

IV. CASIMIR EFFECT FOR SPHERICAL AND CYLINDRICAL SHELLS: HUP VS GUP APPROACHES

In this Section, we apply our heuristic approach to configurations other than parallel plates. Specifically, we compute the Casimir energy for a spherical and cylindrical shells. It is nevertheless worth mentioning that the following analysis may serve as a basis for the study of more sophisticated geometries too.

A. Casimir effect for a spherical shell

By following the same reasoning as for parallel plates, let us consider a spherical shell of (finite) radius $a$ enclosed in a larger sphere of (infinite) radius $R$ representing the whole space (see Fig. 4). By taking a point $P_0$ on the surface of the shell, one can easily understand that the only photons which are allowed to impact on $P_0$ are those originating from vacuum fluctuations in the shadowed volume. In light of this, we can write the effective volume $S\Delta x$ as the sum of the volume of the upper (external) hemisphere and the one of the shell, i.e.

$$S\Delta x = \frac{2}{3} \pi R^3 + \frac{4}{3} \pi a^3,$$

where $S = 4\pi a^2$ is the surface area of the shell. For the internal consistency of our formalism, we require $0 < a \leq R/\sqrt{2}$. By drawing a comparison with the parallel-plate system, the $a \to 0$ limit corresponds to the case in which the
two plates are stuck together. In this case, the spherical shell degenerates into a single point, since \( P_0 \) merges with its antipode. Consequently, the effective volume \( S_\Delta x \) will be twice the volume of the hemisphere of radius \( R \to \infty \), i.e. it will cover all the space\(^2\). On the other hand, for \( a \leq R/\sqrt{2} \to \infty \), the point \( P_0 \) is not affected by the presence of the walls of the shell. This amounts to the case where the two parallel plates are infinitely far apart from each other. Again, the effective volume will be equal to the whole space, i.e. \( S_\Delta x = 4\pi R^3/3 \to \infty \) from Eq. (31).

Now, by using Eq. (31), the position uncertainty \( \Delta x \) reads
\[
\Delta x = \frac{R^3 + 2a^3}{6a^2},
\]
that still diverges as \( R \) increases. As for the parallel plates, however, we can reasonably neglect photons coming from distances greater than the effective radius \( R \sim r_e \). By combining Eq. (32) with the HUP, it follows that
\[
|\Delta E(a, r_e)| = \frac{3\hbar c a^2}{r_e^3 + 2a^3}.
\]
If we now assume \( r_e \) to be of the order of the size of the system as in Eq. (18), i.e. \( r_e \approx 2.6(2a) \), we finally obtain
\[
|\Delta E(a)| = 0.02 \frac{\hbar c}{a},
\]
that matches with the QFT result of Refs. \( 32, 33 \), up to a factor 1/2 (as discussed after Eq. (25), one may improve the agreement by refining the considerations on the photon energy distribution as in Eq. (19)). Note that we cannot infer any kind of information on the sign of \( \Delta E \) and, thus, on the nature of the Casimir force (whether it is attractive or repulsive), since our heuristic calculations only allow to derive the absolute value of the energy shift. In this regard, we emphasize that the issue of the sign of the zero-point force for a spherical shell is quite controversial. In Refs. \( 32, 33 \), for instance, it is argued that a conducting sphere would tend to be expanded due to effects of vacuum fluctuations. On the other hand, if one roughly approximates the sphere as two parallel plates of area \( S = \pi a^2 \) and separation \( d \approx a \) and considers the standard expression \( \{7\} \) for the energy, the opposite sign for the Casimir force would be obtained\(^3\). A similar result is claimed to be valid for two slightly separated hemispheres (that is, a spherical shell sliced with a very subtle knife) and, more generally, for any symmetric configuration (see Kenneth-Klick’s no-go theorem \( \{35\} \)).

Let us now investigate to what extent the zero-point Casimir energy \( \{34\} \) gets modified by the Generalized Uncertainty Principle \( \{26\} \). To this aim, by replacing Eq. (32) into (28), we get
\[
|\Delta E(a, r_e)| = \frac{3\hbar c a^2}{r_e^3 + 2a^3} \left\{ 1 + \beta \left[ \frac{3\hbar c a^2}{E_p(r_e^3 + 2a^3)} \right]^2 \right\},
\]
where we have implicitly made use of the length cutoff \( R \sim r_e \). By setting \( r_e \approx 2.6(2a) \) as before, we find
\[
|\Delta E(a)| = 0.02 \frac{\hbar c}{a} \left[ 1 + 0.0004\beta \left( \frac{\hbar c}{E_p a} \right)^2 \right].
\]
Hence, on the basis of purely heuristic arguments, we obtain a GUP term scaling as the inverse cube of the radius of the spherical shell. As expected, the greater the sphere, the smaller the correction to the standard (HUP) result. Unlike the parallel-plate configuration, however, a full-fledged field theoretical calculation has not yet been carried out, thus preventing us from making any comparison.

### B. Casimir effect for a cylindrical shell

We now compute the zero-point energy for a cylindrical shell of radius \( a \) and height \( H \). As depicted in Fig. 5 we assume \( H > a \); however, similar considerations hold true for any size of the system.

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\(^2\) In order to compute the effective volume relative to \( P_0 \) for \( a \to 0 \), we must also take into account the symmetric contribution of those photons which impact on the antipode of \( P_0 \).

\(^3\) Note that the idea to describe a sphere as two (circular) parallel plates was originally proposed by Casimir to calculate the fine-structure constant \( \alpha \).
FIG. 5: Heuristic derivation of the zero-point energy for a cylindrical shell of radius $a$ and height $H > a$. The sphere of radius $R$ represents the whole space. As for the spherical configuration, the only photons which can reach $P_0$ are those in the shadowed volume.

Let us consider a point $P_0$ on the lateral surface of the cylinder. In this case, we can write the effective volume as

$$ S\Delta x = \frac{2}{3} \pi r_e^3 + \pi a^2 H , $$

where now $S = 2\pi a H$ is the lateral surface of the cylinder and we have already implemented the cutoff $R \sim r_e$ on the radius of the surrounding sphere. Again, as $a$ and $H$ increase, the cylindrical shell tends to cover the whole space, while for vanishing $a$ and $H$, it collapses in a single point. In both cases, the effective volume will be equal to the entire space (see the discussion after Eq. (31)).

By inverting Eq. (37) with respect to the uncertainty position of photons, we get

$$ \Delta x = \frac{2r_e^3 + 3a^2 H}{6aH} , $$

which leads to the following expression of the standard (HUP) energy uncertainty:

$$ |\Delta E(a, H, r_e)| = \frac{3\hbar c a H}{2r_e^3 + 3a^2 H} . $$

Let us now observe that, since we are neglecting vacuum fluctuations from distances greater than $r_e$, it is reasonable to assume $H \simeq 2r_e$ (in other terms, the only photons which can impact on $P_0$ moving from the inside to the outside of the cylinder are those in the volume $V \simeq \pi a^2 (2r_e)$). By setting $r_e \simeq 2.6 (2a)$ as before, we then obtain

$$ |\Delta E(a)| = \frac{|\Delta E(a)|}{H} = 0.01 \frac{\hbar c}{a^2} , $$

where we have computed the energy per unit length in order to compare our expression with the QFT result of Ref. [36]. Note that the two outcomes are in good agreement with each other.

As usual, corrections induced by the GUP can be estimated by inserting Eq. (38) into Eq. (28). Straightforward calculations yield

$$ |\Delta \varepsilon(a, r_e)| = \frac{3\hbar c a}{2r_e^3 + 6a^2 r_e} \left\{ 1 + \beta \left[ \frac{6\hbar c a r_e}{E_p (2r_e^3 + 6a^2 r_e)} \right]^2 \right\} , $$

which, for $r_e \simeq 2.6 (2a)$, becomes

$$ |\Delta E(a)| = 0.01 \frac{\hbar c}{a^2} \left[ 1 + 0.01 \beta \left( \frac{\hbar c}{E_p a} \right)^2 \right] . $$

As for the sphere, a QFT treatment of Casimir effect for a cylindrical shell is missing so far.
V. DISCUSSION AND CONCLUSIONS

We have computed the corrections to the Casimir energy in the framework of minimal length theories based on a generalized uncertainty principle with only a quadratic term in the momentum. Calculations have been carried out for three different systems: the parallel plates, the spherical and cylindrical shells. For the first geometry, the result derived via heuristic arguments has been compared with the more rigorous field theoretical expression, showing in both cases a dependence of the GUP-correction on the inverse fifth power of the distance between the plates. On the other hand, the absence of a QFT treatment of Casimir effect with GUP for non-planar configurations does not allow any comparison for the sphere and the cylinder. Nevertheless, such calculations, along with a possible extension of our formalism to arbitrary $D$-dimensional systems (see for example Ref. [33]) will be investigated in more detail in future works.

Finally, some remarks are in order here. First, we point out that, even though direct observations of GUP effects on the Casimir force are extremely challenging, current experiments [37] might enable us to fix an upper bound on the minimal length $\Delta x_{\text{min}} = \sqrt{3}\ell_p$, and, thus, on the parameter $\beta$. Furthermore, we emphasize that a similar analysis of GUP-induced corrections has been proposed in Ref. [27] in the framework of the Unruh vacuum radiation for accelerated observers. In that case, the existence of a nonzero minimal length manifests itself in the form of (in principle) non-thermal corrections to the Unruh spectrum, which however can be reinterpreted as a shift of the usual Unruh temperature for small deformations of the commutator. In passing, we mention that deviations of the Unruh effect from the well-known behavior (and, more generally, non-inertial corrections on standard predictions of QFT) have been recently pointed out also in other contexts [38–40]. In light of these considerations, it is clear that the study of all these unconventional aspects of fundamental quantum phenomena represents a fertile but still largely uncharted field of research, since it allows us to test QFT in Planck-scale regime at both theoretical and experimental levels. More work is inevitably required along these directions.

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