Classical nonlinear simulation of low-order modes of Lamb waves in plate

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Abstract. For the classical nonlinear research of low-order modes of Lamb waves, this paper firstly introduces the classical nonlinearity derived from the intrinsic nonlinear induced low-order Lamb waves (S0 and A0 modes). Theoretical and numerical calculations are studied in two aspects. The influence of nonlinear effects on the nonlinear effects of classical nonlinear low-order Lamb waves and the cumulative growth effect are analyzed by finite element simulation. The results show that the nonlinear effect produced by the superelastic material model is greater than the geometric nonlinearity, and the linear elastic material model does not produce nonlinear effects. In addition, as the third-order elasticity increases in the material, the amplitude of the second harmonic gradually increases. The second harmonic generated by the A0 mode with phase velocity mismatch is the S0 mode. It can be seen that the group velocity matching is not a necessary condition for generating the second harmonic. Since the phase velocity matching is not satisfied, there is no cumulative growth effect; The higher the S0 mode phase velocity matching degree, the more obvious the cumulative growth effect, and the second harmonic is the S0 mode.

1. Introduction
Plate-shaped metal structures are widely used in various major equipments, such as ship shells, train bodies, aircraft hatches, tank shells, and pressure vessels for nuclear power plant reactors. Compared with nonlinear body waves, Lamb waves have dispersion characteristics, and there is generally no significant or strong nonlinear effect. Breazeale, Thompson[1] and Hikata[2], etc., proposed the main reason for the generation of higher harmonics caused by nonlinearity and dislocation of elastic materials. These conclusions are important for the degradation of structural materials. Regarding the symmetry problem of nonlinear Lamb wave higher harmonics, the conclusions of scholars at home and abroad are still not uniform. The second harmonic displacement field deduced by Deng Mingxi[3-4] has symmetry properties and draws conclusions from this. Whether the fundamental wave is a symmetric mode or an antisymmetric mode, the second harmonic generated can only be a symmetric mode. Sri-vastava and Scala[5] pointed out that the anti-symmetric mode of a nonlinear Lamb wave can only be an odd-order harmonic; and the symmetric mode can be all higher-order harmonics. Lima and Hamilton[6] theoretically pointed out that a fundamental wave can only produce second harmonics of its kind, that
is, symmetric mode lamb waves can only produce symmetric second harmonics, antisymmetric mode blue Mbo can only produce anti-symmetric second harmonics. In the study of the cumulative growth effect of Lamb waves, Deng [7-8], Hamilton, and Lima use the mode expansion analysis method, respectively, the conditions that must be satisfied for the strong nonlinear effect are pointed out: the fundamental frequency Lamb wave and the second harmonic blue The phase velocity matching of the M-wave and the power flow from the fundamental frequency Lamb wave to the second harmonic Lamb wave mode are not zero. Muller et al. [9] pointed out that the group velocity of the Kisbol wave mode must be equal to the group velocity of the double-frequency Lamb wave mode, which is also a necessary condition for generating cumulative effects. Deng Mingxi [10] and Zhu [11] pointed out that the generation of the second harmonic of the Lamb wave is not limited by the group velocity matching condition through experiments and simulations.

At present, most of the researches focus on the Lamb wave study in which the fundamental wave is in the S1 and A1 modes or higher. Due to the multimode characteristics of the Lamb wave, no matter which method is used to excite the target Lamb mode, inevitably, other Lamb wave modes are generated, which results in a very complex time domain waveform of the received signal, and pattern recognition is very difficult, which increases the difficulty of signal processing. In practical engineering applications, it is easier and practical to stimulate a single low-frequency mode. In this paper, the causes of the classical nonlinear effects in materials, the symmetry of the second harmonic generated by low-order Lamb waves, and the accumulation of nonlinear effects are studied by theoretical and finite element simulations. The effectiveness of low-order Lamb waves on the degradation of material performance degradation.

2. Basic theory

2.1 Classical nonlinear theory analysis

The classical nonlinear acoustic theory mainly studies the influence of the intrinsic nonlinearity on the propagation characteristics of sound waves in the medium, and two basic nonlinear sources that cause nonlinear effects [12-13]. One is the geometric nonlinearity independent of the material properties, mainly with solids. The Lagrangian strain tensor of the medium motion is related to the Euler strain tensor; the other is the material nonlinearity inherent in the medium, which is mainly related to the expansion of stress and strain.

The nonlinear wave equation of the five-constant elasticity theory in solids [15-16]:

\[
\rho \frac{\partial^2 u}{\partial t^2} - \mu \frac{\partial^2 u}{\partial x^2} - \left( \lambda + \mu \right) \frac{\partial^2 u}{\partial x \partial y} = F_i
\]

Where is the density \( \rho \) of the material, \( u \) the displacement vector; Among them, the nonlinear drive term \( F_i \) is:

\[
F_i = \left( \frac{\mu A}{4} \right) \left( \frac{\partial^2 u}{\partial x^2} \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2} \frac{\partial u}{\partial y} + 2 \frac{\partial^2 u}{\partial x \partial y} \frac{\partial u}{\partial x} \right) + \left( \lambda + \mu \right) \left( \frac{\partial^2 u}{\partial x \partial y} \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y \partial y} \frac{\partial u}{\partial y} \right) + \left( \frac{A}{4} + B \right) \left( \frac{\partial^2 u}{\partial x^2} \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2} \frac{\partial u}{\partial y} \right) + \left( \lambda + B \right) \frac{\partial^2 u}{\partial x \partial y} \frac{\partial u}{\partial x} \frac{\partial u}{\partial y}
\]

It can be seen from the above equation that in the case of small amplitude, the nonlinear driving term is zero, and the wave equation becomes a linear wave equation. When ultrasonic waves are excited at a limited amplitude, the propagation of ultrasonic waves in the medium will have a nonlinear effect. Even if the third-order elastic constant in equation (2) is all zero, the driving term is not zero due to the existence of geometric nonlinearity, that is a nonlinear effect can also be produced.

2.2 Analysis of the cumulative effect of Lamb wave

According to the second-order perturbation theory, the second harmonic of the Lamb wave does not have an antisymmetric mode, so the second harmonic of the Lamb wave is a symmetric mode [17-19]. When the
fundamental frequency is equal to the phase velocity of the double-frequency Lamb wave, the second harmonic sound field is accumulated in the panel [20]:

\[
\begin{align*}
\tilde{u}_{x0}^{(2)} &= \sin \frac{\gamma}{\gamma} \sum_{n=1}^{\infty} u_{x0}^{(2,n)} \times \left[ (-1)^n \left( -\cos \theta_s + \frac{\Gamma_1(\xi)}{\Gamma_2(\xi)} \sin \theta_l \right) \frac{y}{d} \right] \\
&\quad + \sin \theta_s + \frac{\Gamma_1(\xi)}{\Gamma_2(\xi)} \cos \theta_l \frac{x}{d} \right] \sin \theta_\alpha \exp(i2K_{ls} \cdot r_\alpha + ir) \\
&\quad + \Gamma(\xi) \sin \frac{\gamma}{\gamma} \sum_{n=1}^{\infty} u_{s0}^{(2,n)} \times \left[ (-1)^n \left( -\cos \theta_s + \frac{\Gamma_1(\xi)}{\Gamma_2(\xi)} \sin \theta_l \right) \frac{y}{d} \right] \\
&\quad + \sin \theta_s + \frac{\Gamma_1(\xi)}{\Gamma_2(\xi)} \cos \theta_l \frac{x}{d} \right] \sin \theta_\alpha \exp(i2K_{ls} \cdot r_\alpha + ir)
\end{align*}
\]

Where \( u_{x0}^{(2)} \) is the displacement component of the second harmonic along the x-axis; \( u_{s0}^{(2)} \) is the displacement component of the second harmonic along the y-axis; L or S is the physical quantity related to the longitudinal or transverse wave; \( \theta_s, \theta_l \) is the partial wave vector sum \( L_nK \) and \( T_nK \) \((n =1, 2)\) the angle with the y axis; \( k \) is the component of \( K_{ls} \) and \( K_{s} \) along the z axis.

2.3 Nonlinear lamb wave numerical calculation
The position of the excitation source is \((x, y) = (0, d)\), when the excitation frequency is 100 kHz, according to equations (3) and (4), the cumulative second harmonic sound field of the fundamental S0 mode and the relationship between the amplitude of the displacement component and the propagation distance are obtained. Among them, the x component of the displacement is represented by \( u' \), and the y component is represented by \( v' \). The results are shown in Fig. 1 and Fig. 2. It can be seen from the figure that the second harmonic of the fundamental S0 mode is a symmetric mode. The in-plane displacement component and the off-plane displacement component of the second harmonic increase with the increase of the propagation distance to form a sinusoidal function distribution, and the amplitude of the in-plane displacement is greater than the off-plane displacement component.
3. Simulation Research on Low-order Modes of Nonlinear Lamb Waves

The low-order mode propagation problem of nonlinear Lamb waves is analyzed using COMSOL 5.2 finite element simulation. A two-dimensional simulation model of the nonlinear Lamb wave in the plate is established, as shown in Fig. 3, wherein the model has a length of 6000 mm and a thickness of 2 mm. The superelastic Murgangh material model was used and the material properties are shown in Table 1. In the simulation, the excitation signal is a 10-cycle single-audio signal modulated by Hanning window with a center frequency of 100 kHz.

![A0 mode excitation model](image)

![S0 mode excitation model](image)

Figure 3 finite element model

| Table 1 Material properties |
|-----------------------------|
| $\rho$ (kg·m$^{-3}$) | $\lambda$ (GPa) | $\mu$ (GPa) | $A$ (GPa) | $B$ (GPa) | $C$ (GPa) |
| 2700 | 50.33 | 25.9 | -416 | -131 | -150 |

When the second harmonic frequency is 200 kHz, the size and time step of the maximum mesh are 1 mm and 1E-7s, respectively, and the minimum grid size is set to 0.8 mm. It can be seen that there are two modes that can be excited. The in-plane displacement was extracted at 250 mm and the effects of the following three material models on the nonlinear Lamb wave were analysed.

3.1 Nonlinear Simulation of Fundamental S0 Mode

According to the dispersion curve of 2mm aluminum plate, when the excitation frequency is 100kHz, there are two low-order modes (S0 and A0) Lamb waves, which are used to excite the single symmetric S0 mode. The left end face applies a uniform load in the x direction as shown in Figure 3a. In order to better simulate the actual situation, the Lamb wave propagates in the aluminum plate, and the boundary condition of the excitation end is set to 1E+7 Mpa.

In order to illustrate the influence of material nonlinearity and geometric nonlinearity on the second harmonic amplitude, three constitutive models are used for analysis. The model parameters are shown in Table 2. The receiving node is located 500 mm from the left end face and the receiving direction is the in-plane displacement (x direction). Figure 4a shows the waveforms obtained under the three models. Obviously, there is no obvious difference between the waveforms of the three, and the three signals are Fourier transformed. The result is shown in Fig. 4b. It can be seen from the figure that the second harmonic amplitude generated by the superelastic model is larger than the geometric nonlinear model, while the linear elastic material does not produce nonlinear effects. It can be seen from equation (2) that as the third-order elastic constant increases, the magnitude of the driving term increases, and the resulting nonlinear effect increases. Figure 5 shows the detection signals and their spectra for the third-order elastic constants of multiples (1, 2, and 3). It can be seen from Fig. 5a that there is no significant difference between the waveforms of the three, and it can be seen from Fig. 5b that as the third-order elastic constant increases, the amplitude of the second harmonic increases monotonously.

| Table 2 Three constitutive models |
|----------------------------------|
| Constitutive model | Third-order elastic constant | strain |
| Line elasticity | $A = 0, B = 0, C = 0$ | $E^1$ |
The received time domain signal is filtered by a digital band pass filter to obtain a fundamental time domain signal and a second harmonic time domain signal. As shown in Fig. 6, it can be seen that the two wave packets are almost coincident. Since the phase velocity of the fundamental wave and the second harmonic are nearly matched, the phase velocities are 5419 m/s and 5412 mm/s, respectively, and another condition of the accumulation effect is also satisfied, that is, a non-zero power flow. In order to study the cumulative growth effect of the second harmonic of the S0 mode in the aluminum plate, Figure 7 shows the normalized spectrum of the in-plane displacement at different propagation distances. It can be seen from the partial enlargement that the second harmonic amplitude increases as the distance traveled increases. It is found that the nonlinear coefficient $\beta$ of the S0 mode Lamb wave normalization increases cumulatively with the propagation distance. This conclusion is completely consistent with the theoretical analysis, as shown in Fig. 8.
3.2 Nonlinear Simulation of Fundamental A0 Mode

In order to study the second harmonic symmetry generated by the classical nonlinearity of the asymmetric A0 mode Lamb wave in the plate, and the uniformity is applied in the y direction on the left end of the model. The load is used to excite the antisymmetric A0 mode, as shown in Figure 3b, and the other simulated parameters are the same as in the previous section. The in-plane displacements were extracted at 100, 300, and 500 mm, respectively, and the results are shown in Figure 9. It can be seen from the figure that as the propagation distance increases, the resulting new mode is slowly separated from the A0 mode. Applying a Hanning window to the new mode time domain signal in Figure 9 to modulate, and filtering the modulated time domain signal through a bandpass filter, as shown in Figure 10 is shown. Figure 11 shows the fundamental and second harmonic frequency domain signals. It can be seen that the frequency of the new mode generated is 200 kHz. In order to determine the newly generated waveform mode, the out-of-plane displacement of the range of 100 μs to 170 μs and 200 μs to 350 μs was extracted along the entire plate thickness at a propagation distance of 500 mm, and the results are shown in Figure 14. Since the off-plane displacement of the A0 mode is a symmetric distribution, the off-plane displacement of the S0 mode is an antisymmetric distribution. According to the result of Figure 12a, the wave packet at 100 μs to 170 μs in Figure 12 can be determined as the S0 mode, from Figure 12b. As a result, a wave packet of 200 μs to 350 μs can be obtained as the A0 mode.

It can be seen from the above results that the second harmonic S0 mode can be propagated independently of the fundamental A0 mode, and the phase velocities of the two are different, and there is an asynchronous interaction. For the amplitude of the second harmonic S0 mode. A "shooting" phenomenon will occur, so group velocity matching is not a necessary condition for second harmonic generation [17]. Figure 13 shows that the nonlinear coefficient $\beta$ normalized by different propagation distances accumulates with the propagation distance. It can be seen from the figure that the nonlinear coefficient of the symmetric mode S0 increases much more than the asymmetric mode A0. This shows that the second harmonic of the asymmetric A0 mode has low reliability in the nonlinear characterization of materials.
4. Conclusion
In this paper, based on the classical nonlinear study of Lamb wave low-order modes, the following conclusions are drawn.

1) Based on the classical nonlinear theory, the theoretical model of the nonlinear effect of the low-order Lamb wave S0 mode in the plate is established.

2) According to the nonlinear wave equation, and through the finite element calculation, the nonlinear effect produced by the superelastic material model is larger than the geometric nonlinearity, and the linear elastic material model has no nonlinear effect. In addition, the amplitude of the second harmonic is in a monotonically increasing relationship with the third-order elasticity in the material.

3) The numerical simulation results of the low-order Lamb wave in the plate show that the second harmonic generated by the phase velocity mismatched A0 mode is the S0 mode, which shows that the group velocity matching is not a necessary condition for generating the second harmonic. Since the phase velocity matching is not satisfied, there is no cumulative growth effect; the higher the S0 mode phase velocity matching degree, the more obvious the cumulative growth effect, and the second harmonic is the S0 mode.

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