TRIPLECTIC QUANTIZATION:
A GEOMETRICALLY COVARIANT DESCRIPTION OF THE
$Sp(2)$-SYMmetric Lagrangian Formalism

I. A. Batalin
I. E. Tamm Theory Division, P. N. Lebedev Physics Institute
Russian Academy of Sciences, 53 Leninski prosp., Moscow 117924, Russia

R. Marnelius
Institute of Theoretical Physics, Chalmers University, S-41296 Göteborg, Sweden

and

A. M. Semikhatov
I. E. Tamm Theory Division, P. N. Lebedev Physics Institute
Russian Academy of Sciences, 53 Leninski prosp., Moscow 117924, Russia

ABSTRACT
A geometric description is given for the $Sp(2)$ covariant version of the field-antifield quantization
of general constrained systems in the Lagrangian formalism. We develop differential geometry
on manifolds in which a basic set of coordinates (‘fields’) have two superpartners (‘antifields’).
The quantization on such a triplectic manifold requires introducing several specific differential-
geometric objects, whose properties we study. These objects are then used to impose a set of
generalized master-equations that ensure gauge-independence of the path integral. The theory
thus quantized is shown to extend to a level-1 theory formulated on a manifold that includes
antifields to the Lagrange multipliers. We also observe intriguing relations between triplectic and
ordinary symplectic geometry.
1 Introduction

The field-antifield formalism for quantization of general dynamical systems subjected to constraints has been developed in [1] and [2], and has found a variety of applications to quantization problems (for a review, see, e.g. [3]). For most applications, it was enough to work in the special coordinates which made the field-antifield associations explicit – the (anti)supersymmetric version of the Darboux coordinates.

More recently, in the application to the quantization of, probably, the most complicated field theory known, the string field theory, a covariant form of the field-antifield approach proved useful [4, 5]. Covariant here and henceforth refers to the manifold of fields of a given theory. The required generalization was constructed, in several steps, in refs. [7, 8, 9]; see also [6] and [10] for a review and more recent developments. Covariantization of the field–antifield formalism has led to realizing the existence of, and then solving, another interesting problem – that of constructing a hypergauge theory and of a multilevel generalization of the basic BV scheme. Both these problems were solved by appropriately generalizing the master equation. In order to ensure gauge invariance of the path integral for the partition function, an extra measure factor was identified and its transformation properties found. It was also found that the extra measure factor can be naturally included into the multilevel scheme.

On the other hand, the antibracket quantization in the Lagrangian formalism has got a powerful generalization to an $Sp(2)$-symmetric formalism [11]–[14]. (see also refs. [15]–[22]). The $Sp(2)$-symmetric formalism, however, although applicable to a very large class of gauge theories, has so far been limited to the description of the field space in ‘Darboux’ coordinates. It seems to us that the recent development in the quantization needs has, or will soon have, come to a point requiring the two approaches – quantization in arbitrary coordinates on the field space and the $Sp(2)$-symmetric scheme – to merge. Besides, this would look as a natural step from the point of view of the logical development of the field-antifield quantization scheme. This is why we decided to give in this paper an attempt for a construction of the $Sp(2)$-symmetric Lagrangian quantization in arbitrary coordinates (as already clear from the above, by quantization we mean constructing a finite-dimensional analogue of the path integral).

Introducing the $Sp(2)$-symmetric formalism, as it was implicit already in [11, 12], means departing from the well-established facts of (anti-)symplectic geometry. One introduces canonical triplets instead of canonical pairs. Such triplets contain two ‘antifields’ conjugate to one ‘field’, each one being conjugate with respect to one of the two antibracket structures (let us note in passing that when we talk about Darboux coordinates, these would be in such a triplectic version). These antibrackets are degenerate, and the essence of the antitriplectic (or, for short, simply triplectic) geometry consists in formulating two anti-Poisson structures – antibrackets – whose degeneracies are related to each other in a certain way.

Below, we will develop a differential-geometric setting for the antibrackets and other triplectic structures in arbitrary coordinates. A characteristic difference from the antisymplectic case [7–14] is that we have not only the antibrackets, but also a pair of distinguished vector fields, whose

---

1 The original motivation for it being the introduction of ghost-anti-ghost symmetry into the Hamiltonian BRST-anti-BRST-formalism [23, 24], see also refs. [25, 26].

2 We do not claim, however, having put the formalism on arbitrary manifolds, as the global aspects are left beyond the scope of the present paper. This subject is rather special also because the field-theoretic applications of the formalism mean that the manifolds become infinite-dimensional (see, however, [1]). Our analysis applies to a finite-dimensional model.
properties must be correlated with properties of the antibrackets and of the ‘odd Laplacians’. These are used to formulate the appropriately generalized master equations.

A special and remarkable feature is that the triplectic Lagrangian quantization also involves a relation to symplectic geometry. In fact, the geometric data required to define triplectic quantization on an arbitrary triplectic manifold \( M \) include a symplectic submanifold \( \mathcal{L}_1 \subset M \) (this comes together with a Lagrangian submanifold \( \mathcal{L}_0 \subset \mathcal{L}_1 \), which is the manifold of ‘classical’ fields of a given theory). The appearance of a symplectic structure, albeit on a submanifold in \( M \), accounts for the fact that the gauge-fixing amounts to specifying a bosonic function (see ref. [14]). The trick of the triplectic quantization is that a Poisson bracket, being an even (‘bosonic’) operation, cannot lead to an odd BRST-like transformation within the statistics assignments characteristic to the Lagrangian quantization. One therefore ‘encodes’ the information about the symplectic structure (the Poisson bracket) into certain fermionic vector fields \( V^a \), and then develops a version of the anticanonical formalism on a bigger space. The antibrackets on the extended space are, in a sense, a ‘square root’ of the Poisson bracket associated with the symplectic structure. This provides us by the way with a rather non-standard reformulation of the ordinary symplectic geometry. Alternatively, one can say that triplectic geometry naturally contains symplectic ‘subgeometry’.

Comparing the situation with what one has in the usual antisymplectic lagrangian quantization, one should keep in mind that there the antibracket is non-degenerate, and therefore its maximal isotropic subspaces determine Lagrangian submanifolds. In the triplectic case, however, (both) antibrackets allow a large isotropic subspace of functions that has nothing to do with defining a Lagrangian manifold. This degeneracy is precisely controlled by the Poisson bracket, and it is amazing that conditions that determine Lagrangian manifolds with respect to this Poisson structure can be recast into an antisymplectic form, in terms of the two brackets that are, in a certain sense, ‘ghosts’ of the original Poisson bracket.

After we have formulated the basics of triplectic differential geometry, we proceed to the quantization method. We will introduce two sets of master-equations (each one being two-component, as is always the case on a triplectic manifold). One of these is the ‘main’ master equation imposed on that part of the full action, \( W \), that contains the classical action of the theory to be quantized, the gauge generators, information about the (non)closedness of the gauge algebra, etc. The other master equation is imposed on a gauge-fixing part of the master action, which will be denoted by \( X \), that contains the gauge-fixing conditions and whose job in the path integral is ultimately to fix the gauge in such a way as to introduce no dependence on the specific gauge chosen. We will explain below in more detail what precisely is meant by gauge independence in the most general situation. This gauge-fixing procedure and the related quantities will often be referred to in this context as hypergauge-fixing in order to distinguish it from the traditional procedure which does not involve antifields. The approach that deals with two different master equations for two different quantities was first formulated in refs. [1], [3] for the usual antibracket quantization, while in this paper we develop this idea in the triplectic case, where it turns out that the two master equations, those for \( W \) and \( X \), are no longer two different copies of the same equation.

Our main concern will be the equations for the (hyper)gauge-fixing part of the master-action, \( X \), and gauge independence of the partition function. This can be studied via ‘fine-tuned’ changes of integration variables in the integral, which would guarantee that the integral itself does not change, even though an effective change of \( X \) (and, in particular, of the hypergauge conditions) has been induced. A novel point here is that the possibility (used extensively in the previous papers

\(^3\) Not anti-symplectic!
on the subject) to shift the Lagrange multipliers in the integral will now be encoded directly in the equations imposed on the master-action, and this action itself will be allowed, in principle, to have arbitrary dependence on the variables which, provided they enter at most linearly, become the Lagrange multipliers. This will give the most general, to our knowledge, formulation of gauge theories.

Recall that in ordinary gauge theories, gauge invariance is understood as actual independence of the path integral from gauge conditions or, more generally, from the gauge fermion. In the geometrically covariant formulation the gauge fermion is hidden into a set of hypergauge-fixing functions. These functions are arranged into a solution to the ‘X’-master-equation. Then the gauge independence takes the form of the independence from the natural arbitrariness inherent in this master-equation. This applies to the antisymplectic as well as the triplectic quantization. One may think (and in the antisymplectic case, prove) that this arbitrariness in the solution of the master equation effectively results in the freedom of choosing the gauge fermion encoded in the hypergauge-fixing functions.

The new equations imposed on the gauge-fixing part of the master-action can be neatly summarized in (and actually derived from) a higher-level master equation, the one that is defined on an extended space that includes antifields to the ‘generalized Lagrange multipliers’ λ. The idea is borrowed from refs. [7, 8], but a novel feature is that, like the rest of the theory, the sector of the antifields to λ has a triplectic structure. However, we do not give in this paper the details of quantization of the resulting level-1 theory (in the nomenclature where the original theory is level 0). Introducing the extended sector does anyway lead to drastic simplifications, both technically and conceptually, in studying the generalized master equations. In particular, this enables us to represent the automorphisms of these equations in a closed form. The knowledge of these automorphisms is necessary when studying admissible deformations of the gauge-fixing part of the master-action.

The paper is organized as follows: In section 2 we introduce the main geometrical tools to be used in the antitriplectic quantization. In section 3 the ordinary antisymplectic quantization in its most general formulation is reviewed. This formulation serves as a guide to our triplectic treatment in the following sections. In section 4 we consider the corresponding path integral in the triplectic case and derive the equations which must be satisfied by the master-action and the hypergauge-fixing action in order for the path integral to be ‘BRST’-invariant. In section 5 we discuss gauge invariance and show that the partition function is independent of the choice of a solution for the hypergauge-fixing action, and in section 6 we discuss a particular ansatz that gives rise to singular gauges. Finally we give some concluding remarks in section 7. In two appendices we demonstrate how to extract a Poisson bracket from the antibrackets and derive some useful properties of variations of the introduced geometric quantities.

2 Differential geometry of triplectic quantization:
Main definitions

In the following, the commutator [ , ] denotes the graded commutator \([A, B] = AB - (-1)^{\varepsilon(A)\varepsilon(B)} BA\).

2.1 Reminder on the canonical triplectic coordinates

In the standard canonical approach, one introduces a momentum \(p_A\) to each coordinate \(x^A\). In the anti-hamiltonian formalism of refs. [1, 2], the ‘momentum’ has statistics opposite to that of the
’coordinate’, and is called antifield \( x_A^* \). In the \( Sp(2) \)-symmetric case, there are two antifields \( x_{Aa}^* \), \( a = 1, 2 \), associated to every coordinate \([11, 12, 13]\). The triplets \( (x^A, x_{Aa}^*) \) can be considered as coordinates on a (neighbourhood on) a manifold \( M \) whose dimension is therefore a multiple of three, and which can be called ‘antitriplectic’. \( M \) is endowed with two antibrackets:

\[
(F, G)^a = F \frac{\partial}{\partial x^A} \frac{\partial}{\partial x_{Aa}^*} G - (-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)} (F \leftrightarrow G), \quad a = 1, 2. \tag{2.1}
\]

The symmetry properties of each of the brackets thus introduced, are those of the usual antibracket:

\[
(F, G)^a = -(G, F)^a (-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)} \tag{2.2}
\]

The next – and probably the most fundamental – object of the antisymplectic formalism is the ‘odd Laplacian’ \( \Delta \), whose properties have been widely discussed in the literature. In the present, triplectic, case we have two such operators. In the ‘Darboux’ coordinates \( (x^A, x_{Aa}^*) \) they are

\[
\Delta^a = (-1)^{\varepsilon_{Aa}} \frac{\partial}{\partial x^A} \frac{\partial}{\partial x_{Aa}^*}, \quad a = 1, 2. \tag{2.3}
\]

Each of these operators is nilpotent. In fact, they satisfy:

\[
\Delta^{\{a \Delta^b \}} = 0 \quad \text{(2.4)}
\]

where the curly brackets indicate symmetrization in the indices \( a \) and \( b \).

Now we are going to demonstrate how these definitions, given so far in the ‘Darboux’ coordinates, generalize to arbitrary coordinates. We will also introduce more objects, which will be used in our quantization procedure.

### 2.2 General coordinates

Let \( \Gamma^A, \varepsilon(\Gamma^A) \equiv \varepsilon_A, \ A = 1, \ldots, 3M \) be local coordinates on an antitriplectic manifold \( M \) and \( \varepsilon_A \in \{0, 1\} \) their Grassmann parities. Let us introduce (odd) tensors \( E^{ABa}, \ a = 1, 2 \), with the properties

\[
\varepsilon(E^{ABa}) = \varepsilon_A + \varepsilon_B + 1 \quad E^{ABa} = - E^{BAb} (-1)^{(\varepsilon_A+1)(\varepsilon_B+1)} \tag{2.5}
\]

We will also need a volume form on \( M \), which will be represented by a scalar density \( \rho \). Having such a density \( \rho \), one can define divergence of a vector field \( U = U^A \partial_A \) by

\[
\text{div} \ U = \rho^{-1} \partial_A (\rho U^A) (-1)^{\varepsilon_A} \tag{2.6}
\]

where \( \partial_A = \partial/\partial \Gamma^A \). Further, with both \( \rho \) and \( E \) at our disposal, we can define two fermionic operators that generalize eqs. (2.3):

\[
\Delta^a = \frac{1}{2} (-1)^{\varepsilon_A} \rho^{-1} \partial_A \rho E^{ABa} \partial_B \tag{2.7}
\]

We require them to satisfy

\[
\Delta^{\{a \Delta^b \}} = 0 \iff [\Delta^a, \Delta^b] = 0. \tag{2.8}
\]
It then follows that $E^{ABa}$ must satisfy the relations
\[ E^{AD}(\partial_D E^{BC})|b\rangle (-1)^{\langle\varepsilon_A+1\rangle}\langle\varepsilon_C+1\rangle + \text{cycle}(A\ B\ C) = 0 \] (2.9)
and
\[ (-1)^{\langle\varepsilon_A\rangle} \rho^{-1} \partial_A \left( \rho E^{AB}|a\rangle \partial_B (-1)^{\langle\varepsilon_C\rangle} \rho^{-1} \partial_C (\rho E^{CD})|b\rangle \right) = 0 \] (2.10)

Next, we introduce two antibrackets $(\ ,\ )^a$, $a = 1, 2$, by the formula
\[ \Delta^a(FG) = (\Delta^a F) G + F \Delta^a G (-1)^{\langle\varepsilon(F)\rangle} + (F, G)^a (-1)^{\langle\varepsilon(F)\rangle} \] (2.11)
which shows that $\Delta^a$ fails to differentiate the product of functions on $\mathcal{M}$, and it is the corresponding antibracket that measures the deviation of $\Delta^a$ from being a derivation. It follows that
\[ (F, G)^a = F \partial_A E^{ABa} \partial_B G \] (2.12)
Obviously,
\[ \varepsilon((F, G)^a) = \varepsilon(F) + \varepsilon(G) + 1 \] (2.13)
and the symmetry property (2.2) is fulfilled.

It now follows from (2.12) that the antibrackets differentiate the algebra of functions under multiplication:
\[ (F, GH)^a = (F, G)^a H + G(F, H)^a (-1)^{\langle\varepsilon(F)\rangle+1}\langle\varepsilon(G)\rangle} \] (2.14)
Applying eqs. (2.12)-(2.10) as $\Delta^a \Delta^b (FG) \equiv 0$ and $\Delta^a \Delta^b (FGH) \equiv 0$ we arrive at
\[ \Delta^a(F,G)^b = (\Delta^a F, G)^b + (F, \Delta^a G)^b (-1)^{\langle\varepsilon(F)\rangle+1} \] (2.15)
and
\[ ((F, G)^{a, H})^b (-1)^{\langle\varepsilon(F)\rangle+1}\langle\varepsilon(H)\rangle+1} + \text{cycle}(F, G, H) = 0 \] (2.16)
respectively. As before, the curly bracket indicates symmetrization in the indices $a$ and $b$. Equation (2.10) is a version of the Jacobi identity satisfied by the two antibrackets.$^4$

2.3 Antitriplectic quantities for quantization

From the results of [14] we expect the following properties to be necessary for $Sp(2)$-symmetric quantization of a general gauge theory: First we need an even number of bosonic as well as fermionic fields in the theory, which in turn requires us to consider an antitriplectic manifold $\mathcal{M}$ of dimension $6N^5$. We set therefore $\mathcal{M} = 2N$ from now on. Next, we need odd (fermionic) vector fields $V^a$, $a = 1, 2$ on $\mathcal{M}$. These are required to satisfy
\[ \left[ \Delta^a, \ V^b \right] \equiv \Delta^a V^b + V^b \Delta^a = 0 \] (2.17)

$^4$ Its Poisson counterpart, which says that two brackets are compatible, is in the basis of constructing bi-hamiltonian integrable systems [18, 19] and has been studied from various points of view, see, e.g., [20]. In the present situation, the compatibility condition for the antibrackets follows from eq. (2.8) imposed on the $\Delta$-operators which play a special role in the anticanonical case.

$^5$ Which is actually related to the fact, discussed below and in Appendix A, that there is a hidden symplectic structure in the triplectic quantization.
Applying this identity to a product of two functions \(FG\) and making use of (2.11), we see that each \(V^a\) differentiates both antibrackets:

\[
V^a(F, G)^b = (V^a F, G)^b + (F, V^a G)^b (-1)^{\varepsilon(F)+1}
\]  

(2.18)

Introducing the components as

\[
V^a \equiv (-1)^{\varepsilon A} V^{Aa} \partial_A, \quad \varepsilon(V^{Aa}) = \varepsilon_A + 1
\]  

(2.19)

we rewrite the conditions implied by (2.17) as the following equations (the first of which has the usual Lie-derivative form):

\[
(-1)^{\varepsilon_C} V^{Ca} \partial_C E^{ABb} = (-1)^{\varepsilon_A} V^{Aa} \partial_C E^{CBb} - (-1)^{(\varepsilon_A+1)(\varepsilon_B+1)}(-1)^{\varepsilon_B} V^{Ba} \partial_C E^{CAb}
\]  

(2.20)

\[
(-1)^{\varepsilon_A} \rho^{-1} \partial_A \left( \rho E^{AAb} \partial_B V^{Cb} \right) + (-1)^{\varepsilon_A+\varepsilon_B} V^{Ba} \partial_B \left( \rho^{-1} \partial_A (\rho E^{ACb}) \right) (-1)^{\varepsilon_C} = 0.
\]  

(2.21)

Furthermore, we need the \(V^a\) to satisfy:

\[
[V^a, V^b] \equiv V^a (V^b) = 0
\]  

(2.22)

which is simply

\[
(-1)^{\varepsilon_A} \Delta^{A(a} \partial_A V^{B|b)} = 0.
\]  

(2.23)

In addition to the above, we will choose \(V^a\) to be divergence-free:

\[
\text{div} V^a = 0
\]  

(2.24)

(in the sense of eq. (2.6)) in order to simplify the formulae. This condition is not of the same status as the previous ones however, in that it can be removed at the expense of altering several formulæ (see Remark 3 on page 22).

It will be useful to introduce the operator

\[
K = \epsilon_{ab} V^a V^b
\]  

(2.25)

that satisfies the projection property,

\[
V^a K = KV^a = 0
\]  

(2.26)

due to (2.22). In particular,

\[
K^2 = 0.
\]  

(2.27)

Note that the properties (2.8), (2.11) and (2.15) of \(\Delta^a\) are inherited by the operators

\[
\Delta^a_\pm = \Delta^a \pm \frac{i}{\hbar} V^a
\]  

(2.28)

due to the properties (2.17), (2.18) and (2.22) of \(V^a\).

The presence of the \(V^a\) vector fields will be noticeable in many places of our formalism, and their origin can be traced to the fact that, in triplectic geometry, the generalized BRST-type transformations, besides being ‘duplicated’, acquire a ‘transport’ term in addition to the antibracket term: we will thus widely use transformations of the general form

\[
\delta \Gamma^A = (\Gamma^A, \mathcal{H})^a \mu_a - \text{const} V^{Aa} \mu_a
\]  

(2.29)
where $\mu$ are odd (fermionic) parameters or functions. The actual value of the constant is related to the normalization of $V^a$. Note that all the defining relations for $V^a$ are homogeneous in $V^a$. As we will see, fixing the normalization of $V^a$ has to do with boundary conditions for generalized master equations and with the symplectic ‘subgeometry’ that can be extracted from the above axioms. In particular, although the normalization of $V^a$ is conventional, they cannot be scaled to zero as that would mean degeneration of the Poisson structure.

In [14], an $Sp(2)$-symmetric quantization was performed in ‘Darboux’ coordinates

$$\Gamma^A = \{\Phi^\alpha, \Phi^*_{aa}, \overline{\Phi}_\alpha, \pi^{\alpha a}\},$$

with $\Phi^\alpha$ being the original fields. In [14], $\Phi^*_{aa}$, $\overline{\Phi}_\alpha$ were considered to be antifields while $\pi^{\alpha a}$ were considered to be auxiliary field variables. This is in distinction to the present formulation where $\Phi^\alpha$, $\Phi^*_{aa}$ are considered to be field variables and $\Phi^*_{aa}$, $\pi^{\alpha a} \equiv \epsilon_{ab}\pi^{ab}$ antifields. In our conventions the $\Delta^a$ (2.7) and the antibrackets (2.12) have the following form in terms of the Darboux coordinates

$$\Delta^a = (-1)^{\varepsilon_\alpha} \frac{\partial}{\partial \Phi^\alpha} \frac{\partial}{\partial \Phi^*_{aa}} + (-1)^{\varepsilon_\alpha} \frac{\partial}{\partial \Phi^\alpha} \frac{\partial}{\partial \pi^{\alpha a}}$$

$$(F, G)^a = \left( F \frac{\partial}{\partial \Phi^\alpha} \frac{\partial}{\partial \Phi^*_{aa}} G + F \frac{\partial}{\partial \Phi^\alpha} \frac{\partial}{\partial \pi^{\alpha a}} G - (-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)}(F \leftrightarrow G) \right)$$

in agreement with the expressions (2.3) and (2.1). The vector fields $V^a$ (2.19) are here given by

$$V^a = \frac{1}{2} \epsilon^{ab} \left( \Phi^*_{ab} \frac{\partial}{\partial \Phi^\alpha} - (-1)^{\varepsilon_\alpha} \pi^{\alpha b} \frac{\partial}{\partial \Phi^\alpha} \right)$$

which is easily seen to satisfy (2.17).

In the present paper, however, we do not resort to the Darboux coordinates, nor even to their existence (which is not known to us as a theorem on the canonical form of two appropriately degenerate antibrackets and the corresponding $V^a$ vector fields).

### 2.4 A Poisson structure from triplectic geometry

We are going to demonstrate the relation of the triplectic geometry to symplectic (sub)geometry. As a motivation, note that the structure of the field space in the Darboux coordinates (2.30) suggests that it may be related to a canonical phase space structure. Indeed, for $F$ and $G$ being two arbitrary functions of $\Phi, \overline{\Phi}$ only, we find

$$(F, V^a G)^b = \frac{1}{2} \epsilon^{ab} \left( F \frac{\partial}{\partial \Phi^\alpha} \frac{\partial}{\partial \Phi^*_{aa}} G - (-1)^{\varepsilon(F)\varepsilon(G)}(F \leftrightarrow G) \right).$$

It is remarkable that a Poisson structure can in fact be extracted, under a few additional assumptions, from the general axioms introduced in sections 2.2 and 2.3. The additional assumptions are a geometrical counterpart of the boundary conditions that will be considered in section 4. They specify, among other things, non-degeneracy properties of the tensors $E^{ABa}$ and impose certain ‘boundary conditions’ on the structure of triplectic manifold $M$. 

7
Namely, in the triplectic manifold $\mathcal{M}$, with
\[ \dim \mathcal{M} = (4N - 2n)2N + 2n, \] (2.35)
there should exist a submanifold $\mathcal{L}_1 \subset \mathcal{M}$ of dimension
\[ \dim \mathcal{L}_1 = (2n2N - 2n) \] (2.36)
satisfying the four requirements listed and discussed below:

(i) non-degeneracy property: $E^{ABa}$ should define a non-degenerate pairing between the cotangent bundle to $\mathcal{L}_1$ and conormal bundle of $\mathcal{L}_1$ in the following sense: for any $\omega \in N\mathcal{L}_1 \subset T^*(\mathcal{M})$ being a 1-form that annihilates all the vectors tangent to $\mathcal{L}_1$ (i.e., $\omega(v) = 0$ for $v \in T\mathcal{L}_1$), we should have
\[ \omega_A E^{ABa} \omega_B = 0 \text{ for } a = 1, 2 \text{ and } \forall \omega \in T^*\mathcal{L}_1 \Rightarrow \omega_A = 0; \] (2.37)

(ii) all functions on $\mathcal{L}_1$ are annihilated by both operators $\Delta^a$:
\[ \Delta^a F = 0, \quad a = 1, 2, \quad F \in \mathcal{F}(\mathcal{L}_1). \] (2.38)

Since functions on a submanifold are closed under multiplication, we can apply $\Delta^a$ to a product of two such functions and use (2.11) to arrive at
\[ (F, G)^a = 0, \quad a = 1, 2, \quad F, G \in \mathcal{F}(\mathcal{L}_1) \] (2.39)

(iii) In addition, $\mathcal{L}_1$ should be such that
\[ (F, V^{(a}G^{b)}) = 0, \quad F, G \in \mathcal{F}(\mathcal{L}_1) \] (2.40)
(or equivalently $\Delta^a(FV^b)G = 0$). When this is satisfied, $(F, V^{a}G)^b$ is $\epsilon^{ab}$ times what will become the Poisson bracket. We thus define
\[ \{F, G\} \equiv \epsilon_{ba}(F, V^{a}G)^b = \epsilon_{ab}\Delta^a(FV^b)G(-1)^{\epsilon(F)} \] (2.41)
(the last equality follows from (2.11), (2.17) and (2.38)). This has then the symmetry properties of a Poisson bracket, $\{F, G\} = -\epsilon(F)\{G, F\}$ and satisfies the Leibnitz rule (see Appendix A). Before discussing the fourth requirement that would lead eventually to the Jacobi identity, consider in more detail the properties we have imposed so far.\footnote{Note that a related construction of extending a given Poisson bracket to a (single!) antibracket on a bigger manifold has been carried out in [30, 31] for a particular realization of the vector field $V$ in the ‘Darboux’ coordinates. In the present paper, conversely, we start with a (pair of) general $V^a$ vector fields and impose conditions on $V^a$ as well as the two antibrackets that would guarantee the existence of a Poisson subgeometry.}

Let us specify locally the submanifold $\mathcal{L}_1$ by a set of $4N$ equations $\varphi^\mu = 0$ and choose local coordinates $x^i$ on $\mathcal{L}_1$. Then $\Gamma^A = (x^i, \varphi^\mu)$ will be local coordinates on $\mathcal{M}$. From (2.39) it follows that, in such an adapted coordinate system, we should have
\[ E^{ij} = 0, \quad a = 1, 2, \quad i, j = 1, \ldots, 2N, \] (2.42)
while the non-degeneracy condition (2.37) takes the form
\[ \omega_\mu E^{ij} = 0 \Rightarrow \omega_\mu = 0. \] (2.43)
As to (2.40), we write \( V^a = (-1)\xi_i V^{i\alpha} \partial_i + (-1)^{\xi_\mu} V^{\mu a} \partial_\mu \) and then (2.40) becomes
\[
E^{\mu a} \partial_\mu V^{jb} + (a \leftrightarrow b) = 0.
\]
(2.44)

Let us see how these conditions behave under infinitesimal coordinate transformations
\[
\delta \Gamma^A = T^A (\Gamma) = (T^i, T^\mu) \quad \text{with} \quad \partial_\mu T^i = 0
\]
(2.45)
that consist of an arbitrary change of coordinates on \( L_1 \) (described by \( T^i(x) \)) and an arbitrary transformations of the remaining coordinates given by \( T^\mu(x, \varphi) \). First, the condition \( E^{ij a} = 0 \) varies according to (B.8), resulting in
\[
0 = -T^i \partial_C E^{Cja} - E^{iCa} \partial_C T^j = -T^i \partial_\mu E^{\mu ja} - E^{\mu a} \partial_\mu T^j
\]
(2.46)
which is satisfied by virtue of \( \partial_\mu T^i = 0 \). Next, evaluating the variation of (2.44) due to (B.7) and (B.11), we see that it vanishes again by virtue of \( \partial_\mu T^i = 0 \).

In the coordinates introduced above, the candidate Poisson structure on \( L_1 \) is determined by the tensor
\[
\omega^{ij} = \frac{1}{2} \epsilon_{ab} (-1)^{\xi_i} E^{ij b} \partial_\mu V^{j a}.
\]
(2.47)
Requiring it to be non-degenerate would impose (in combination with (2.37)) the appropriate rank conditions on the matrix \( \partial_\mu V^{ja} \).

We have finally to ensure that the operation \( \{ F, G \} \) leaves us in the same class of functions, namely functions on \( L_1 \). For any two such functions \( F \) and \( G \), consider \( \Delta^a \{ F, V^b G \} = 0 \), where we can use (2.15). This vanishes due to (2.17), hence \( \Delta^a \{ F, G \} = 0 \). However, this does not automatically imply \( \{ H, \{ F, G \} \} = 0 \) for \( H \in F(L_1) \). This latter condition is tantamount to
\[
\epsilon_{ab} E^{kuc} \partial_\nu (E^{ijb} \partial_\mu V^{ja}) = 0.
\]
(2.48)
Rewriting it (making use of (2.44)) as
\[
E^{kuc} \partial_\nu (E^{ijb} \partial_\mu V^{ja}) + (b \leftrightarrow c) = 0,
\]
(2.49)
or, in terms of three arbitrary functions on \( L_1 \),
\[
\{ F, (G, V^a H)^{(b)} c \} = 0,
\]
(2.50)
we can show that the Jacobi identity follows for the Poisson bracket, see Appendix A. Now, in view of the condition (2.37), eq. (2.48) states simply that \( \partial_\mu \omega^{ij} = 0 \), which obviously guarantees that the Poisson bracket of functions on \( L_1 \) does not acquire any dependence on the coordinates transversal to \( L_1 \). It is possible, however, to impose this only as a
(iv) ‘boundary condition’
\[
\partial_\mu \omega^{ij} \bigg|_{L_1} \equiv \frac{1}{2} \epsilon_{ab} (-1)^{\xi_i} \partial_\mu (E^{ij b} \partial_\mu V^{ja}) \bigg|_{L_1} = 0
\]
(2.51)
and define the Poisson bracket accordingly, by explicitly restricting to \( L_1 \):
\[
\{ F, G \} = F \partial_i \omega^{ij} G \bigg|_{L_1}.
\]
(2.52)
Eq. (2.51) will then be sufficient to guarantee the Jacobi identity. Thus the fourth requirement takes the form of a ‘boundary condition’, stating that \( \epsilon_{ab} E^{ijb} \partial_\mu V^{ja} \) has to be constant along the
directions transversal to \( L_1 \) only on the first infinitesimal neighbourhood of \( L_1 \). In fact, the relations (2.42), (2.43) (and the non-degeneracy conditions (2.37)), too, can be relaxed to those holding only on infinitesimal neighbourhoods of \( L_1 \).

This concludes those requirements on the triplectic quantities that are related to a fixed submanifold \( L_1 \subset \mathcal{M} \). In the quantization of gauge theories, this manifold serves also to impose boundary conditions on the master actions, as will be shown below.

3 General antisymplectic Lagrangian quantization

Before we tackle the construction of general hypergauge-fixed actions in the \( Sp(2) \)-symmetric case using the general triplectic formulation given in the previous section, we shall describe the corresponding construction in the antisymplectic case. In fact, even the ordinary BV theory can be given a more general and streamlined formulation. The antisymplectic construction is presented in this section in such a way as to make it easier to compare with the triplectic case. The presentation given here of the antisymplectic/antibracket quantization method is also (to our knowledge) the most general formulation. The reader who is interested only in the triplectic case may proceed directly to section 4.

3.1 The path integral and master equations

In the general antisymplectic theory one first defines an invariant quantum master action \( W(\Gamma, \bar{h}) \) on an antisymplectic manifold \( \mathcal{M} \) by the condition that it must satisfy the quantum master equation

\[
\Delta \exp \left( \frac{i}{\hbar} W \right) = 0 \iff \frac{1}{2}(W, W) = i\hbar \Delta W \tag{3.1}
\]

where \( \Delta \) and \( (\ , \) ) are the usual antisymplectic differential and the antibracket respectively (see ref. [10]; these structures can be thought of as being given by (2.7) and (2.12) with the index \( a \) dropped; however, in contrast to the triplectic case, the antibracket must be non-degenerate). \( W(\Gamma; \bar{h}) \) is assumed to be expandable in \( \bar{h} \). One then imposes the boundary condition

\[
W(\cdot; 0) \big|_{L_0} = S(\cdot) \tag{3.2}
\]

where \( L_0 \) is a Lagrangian submanifold in \( \mathcal{M} \) on which the action \( S \) is defined. This manifold must be specified as a part of the definition of the theory. Further, to formulate the remaining part of the boundary conditions on \( W \), let \( W^{(1)} \) be the restriction of \( W(\cdot; 0) \) to the first infinitesimal neighbourhood of \( L_0 \). It is then required that

\[
(S, W^{(1)}) = 0. \tag{3.3}
\]

The rest of the dependence of \( W \) on the fields off the ‘classical’ manifold \( L_0 \) is determined by the master equation, whose solutions ‘propagate’ from the boundary conditions (3.2) and (3.3).

We set \( \dim L_0 = (n|N-n) \) in the following.

Next, we introduce the gauge-fixing master-action \( X \) and consider the following ansatz for the partition function path integral

\[
Z = \int \exp \left\{ \frac{i}{\hbar} [W + X] \right\} \rho(\Gamma)[d\Gamma][d\lambda]. \tag{3.4}
\]
$X(\Gamma, \lambda; \hbar)$ is assumed to be expandable both in $\hbar$ and in the parametric variables $\lambda^\alpha$. The new variables $\lambda^\alpha$, $\alpha = 1, \ldots, N$, become Lagrange multipliers when the action is restricted to depend on them linearly. There must be $n$ bosons and $N - n$ fermions among the $\lambda^\alpha$ if dim $\mathcal{L}_0 = (n|N - n)$.

Now we come to imposing an equation on $X$; this will be a generalization of the master-equation of the form (3.1). The generalization is due to the fact that $X$, unlike $W$, is allowed to depend on $\lambda^\alpha$. The sought equation can be arrived at by requiring that the integral (3.4) be invariant under anticanonical transformations given by

$$\delta \Gamma^A = (\Gamma^A, -W + X) \mu$$

where $\mu$ is a fermionic constant. The form of (3.5) is such that the resulting conditions on $X$ will not involve $W$ when one makes use of the fact that $W$ satisfies the quantum master equation (3.1). In order to obtain as general conditions on $X$ as possible we allow, along with (3.5), a variation of $\lambda$ of the form (the factor $-2$ is chosen for convenience)

$$\delta \lambda^\alpha = -2R^\alpha \mu$$

where $R^\alpha(\Gamma, \lambda; \hbar)$ is an apriory arbitrary function expandable in both $\lambda^\alpha$ and $\hbar$, with $\varepsilon(R^\alpha) = \varepsilon_\alpha + 1$ where $\varepsilon_\alpha \equiv \varepsilon(\lambda^\alpha)$. Our claim now is that invariance of the integral (3.4) under the transformations (3.5) and (3.6) will follow once we impose the equation

$$i\hbar \alpha \Delta X - hX_{\partial \alpha} R^\alpha + i\hbar R^\alpha X_{\partial \alpha} = 0$$

or equivalently

$$\left(\Delta - \frac{i}{\hbar} (-1)^{\varepsilon_\alpha} R^\alpha \partial_\alpha - \frac{i}{\hbar} (-1)^{\varepsilon_\alpha} \partial_\alpha R^\alpha\right) \exp \left(\frac{i}{\hbar} X\right) = 0$$

where $\partial_\alpha \equiv \partial/\partial \lambda^\alpha$. This is a 'weak' form of the quantum master equation, where by weak we mean that the equation contains terms other than those with the $\Delta$ operator and the antibracket. The name actually refers to the fact that, in the most common case when $X$ depends on $\lambda^\alpha$ linearly via $G_\alpha \lambda^\alpha$, the integral is concentrated on the hypergauge locus $G_\alpha = 0$ and the term $X_{\partial \alpha} R^\alpha$ in (3.7) is in fact proportional to the $G_\alpha$.

Eq. (3.7) is an equation for both $X$ and $R^\alpha$. It is solved by Taylor expanding in $\hbar$ and $\lambda^\alpha$. If we first expand $X$ and $R^\alpha$ in $\hbar$, i.e.

$$X = X_0 + i\hbar X_1 + (i\hbar)^2 X_2 + \ldots, \quad R^\alpha = R^\alpha_0 + i\hbar R^\alpha_1 + (i\hbar)^2 R^\alpha_2 + \ldots$$

then eq. (3.7) leads to

$$\frac{1}{2}(X_0, X_0) = X_0_{\partial \alpha} R^\alpha_0$$

$$\left(\Delta X_0, X_1\right) - X_0 R^\alpha_{\partial \alpha} = X_0_{\partial \alpha} R^\alpha_1 + X_1_{\partial \alpha} R^\alpha_0 - R^\alpha_0_{\partial \alpha}$$

etc. An expansion in $\lambda^\alpha$ leads then to a further proliferation of equations since each power of $\lambda^\alpha$ must agree.

Note that by applying to the LHS of (3.7) the operator

$$\Delta + \frac{i}{\hbar} (X, \cdot) - \frac{i}{\hbar} R^\alpha (-1)^{\varepsilon_\alpha} \partial/\partial \lambda^\alpha$$

we arrive at a consistency condition

$$\frac{\partial}{\partial \lambda^\alpha} \left((i\hbar R^\alpha + R^\beta (-1)^{\varepsilon_\beta} \partial R^\alpha/\partial \lambda^\beta - \left(X, R^\alpha\right)) \epsilon^\lambda X\right) = 0$$

(3.13)
It should be realized that this is only a member in the chain of ‘higher consistency relations’. We
will now introduce a more powerful formalism that collects the different terms in the ‘weak’ master
equation as well as all the higher consistency relations into a ‘strong’ master equation which is
obtained by introducing antifields to $\lambda^\alpha$, as it will be done in subsection 3.2.

So far we have shown that the integral (3.4) is invariant under the transformations (3.5) and
(3.6) provided $X$ satisfies (3.7) for appropriately chosen $R^\alpha$ in (3.6). This is a global invariance since
$\mu$ was a constant in (3.5) and (3.6). The trick used in the BV scheme in order to prove independence
from the choice of the gauge condition consists in performing a field-dependent transformation of
the integration variables such that would result in an arbitrary infinitesimal shift of the gauge
conditions within the allowed class. Consider therefore the same transformations as above but now
with $\mu$ replaced by an arbitrary function, $\mu \rightarrow \mu(\Gamma, \lambda; \hbar)$. Note that we allow $\mu$ to have an a priori
arbitrary dependence on $\lambda$. The resulting transformation of the integral gives rise to a nonvanishing
Jacobian $1 + \delta J$:

$$\delta J = (-W + X, \mu) - 2 R^\alpha (\partial_\alpha \mu)(-1)^{\bar{\epsilon}\alpha}$$  (3.14)

At this point we need to perform an additional canonical transformation, which is obviously non-
trivial only for non-constant $\mu$:

$$\delta_1 \Gamma^A = \frac{\hbar}{i}(\Gamma^A, \mu)$$  (3.15)

This has an interesting effect: it cancels the $(-W, \mu)$ term in (3.14) while doubling the contribution
$(X, \mu)$. The additional transformation also contributes a Jacobian

$$\delta_1 J = 2 \frac{\hbar}{i} \Delta \mu$$  (3.16)

The result of putting all terms together amounts to the following effective change of the action in
(3.4):

$$\delta X = (X, f) - i\hbar \Delta f - (-1)^{\bar{\epsilon}\alpha} R^\alpha \partial_\alpha f$$  (3.17)

where we have defined

$$f \equiv 2 \frac{\hbar}{i} \mu$$  (3.18)

Therefore, by a transformation of dummy variables in the integral (3.4) we have rewritten it as a
similar integral with a deformed gauge-fixing master-action $X$.

There are two basic (related) requirements to be satisfied by the new, deformed $X$. First, it
must satisfy the (weak) master equation. One can check explicitly that eq. (3.7) is indeed invariant
under (3.17) and a simultaneous infinitesimal transformation of $R^\alpha$ of the form

$$\delta R^\alpha = (R^\alpha, f) - (-1)^{\bar{\epsilon}\alpha} F^{\alpha\beta} \partial_\beta f$$  (3.19)

where $F^{\alpha\beta}$ appears in the solution (3.24) of the condition (3.13). We will call a deformation of $X$
that goes through the master equation consistent. Thus (3.17) is a consistent variation of $X$, in the
sense that it preserves the equation imposed on $X$.

The second requirement on the variation $\delta X$ is that it describe the maximal arbitrariness allowed
by the equation and the boundary conditions. This arbitrariness is then the manifestation of gauge
freedom inherent in our formalism.

There exists a general way to prove the desired properties of the above variation using the
$\lambda^*$-extended formalism.

---

7 From this moment on, the $W$-part of the master action is no longer involved in the derivation. Such a decoupling
of $W$ and $X$ is the result of the opposite signs chosen in (3.4) and (3.14). In other words, these two equations involve
the different operators $\Delta_\pm$ (2.28).
3.2 $\lambda^*$-extended sector and strong master equation

Considerable simplifications will follow when we identify the different terms in eq. (3.8) as coming from a ‘strong’ master equation (i.e. an equation of the type of (3.1)) on a bigger manifold. In order to recast the weak master equation (3.8) into a strong one on an extended antisymplectic manifold we introduce the anticanonical pairs $(\lambda^\alpha, \lambda^*\alpha)$ where $\lambda^*\alpha (\varepsilon(\lambda^\alpha) = \varepsilon_\alpha + 1)$ are antifields to the variables $\lambda^\alpha$. On this extended manifold $\tilde{M}$ we may define an odd operator

$$\Delta_{\text{ext}} \equiv \Delta + (-1)^{\varepsilon_\alpha} \frac{\partial}{\partial \lambda^\alpha} \frac{\partial}{\partial \lambda^*\alpha}$$

(3.20)

and the corresponding antibracket

$$(F,G)_{\text{ext}} \equiv (F,G) + F \frac{\partial}{\partial \lambda^\alpha} G - F \frac{\partial}{\partial \lambda^*\alpha} G.$$  (3.21)

We define now the extended quantum master equation by

$$\Delta_{\text{ext}} \exp \left( \frac{i}{\hbar} \mathcal{X} \right) = 0.$$  (3.22)

The weak master-equation (3.8) for $X$ will now follow from (3.22) once we expand $\mathcal{X}$ in the antifields:

$$\mathcal{X} \equiv X - \lambda^*\alpha R^\alpha + \frac{1}{2} \lambda^*\alpha \lambda^*\beta F^{\alpha\beta} + \mathcal{O}(\lambda^*)^3$$

(3.23)

(where $F^{\beta\alpha} = (-1)^{(\varepsilon_\alpha+1)(\varepsilon_\beta+1)} F^{\alpha\beta}$). Further, one can observe that in the first order in $\lambda^*$ eq. (3.22) gives

$$i\hbar \Delta R^\alpha + (-1)^{\varepsilon_\alpha} \partial^\alpha R^\alpha - (X, R^\alpha) = (-1)^{\varepsilon_\alpha} (i\hbar \partial^\beta F^{\alpha\beta} - \partial^\beta X F^{\alpha\beta}),$$

(3.24)

which, in fact, implies the consistency condition (3.13). As is always the case with master equations formulated with the appropriate boundary conditions at all antifields set to zero, higher orders in the antifields generate higher compatibility conditions. However, the use of the $\lambda^*$-extended formalism is not limited to putting together the compatibility conditions. It is very useful in understanding gauge independence of the integral (3.4).

In order to ‘legitimatize’ the $\lambda^*$-extended formalism, consider the path integral (3.4) rewritten as

$$Z = \int \exp \left\{ \frac{i}{\hbar} \left[ W + \mathcal{X} \right] \right\} \delta(\lambda^*) \rho(\Gamma)[d\Gamma][d\lambda][d\lambda^*]$$

$$= \int \exp \left\{ \frac{i}{\hbar} \left[ W + \mathcal{X} + \lambda^*\alpha \eta^\alpha \right] \right\} \rho(\Gamma)[d\Gamma][d\lambda][d\lambda^*][d\eta]$$

(3.25)

where we have used that $\mathcal{X} \big|_{\lambda^* = 0} = X$. Now, the point of rewriting (3.4) in this form was that $\mathcal{X}$ satisfies a strong master equation. At the same time, we have represented the partition function as an integral over the extended manifold $\tilde{M}$, which bears the antibracket (3.20). However, (3.25) is obviously written in a very special hypergauge, namely the one that explicitly kills the $\lambda^*$-dependence.

To make contact with the general gauge theory on $\tilde{M}$ based on a solution to the strong master equation we have to show that it is possible to choose arbitrary hypergauges in the integral (3.25). As before, we do that infinitesimally by performing a non-constant shift of the integration variables and the accompanying transformation of the form of (3.15), but now on the extended manifold $\tilde{M}$,
using \((\cdot, \cdot)_{\text{ext}}\) etc. The crucial observation is that defining \(\tilde{W} = W + \lambda_\alpha^* \eta^\alpha\) we would obtain a solution of the master equation
\[
\frac{1}{i}(\tilde{W}, \tilde{W})_{\text{ext}} = i\hbar \Delta_{\text{ext}} \tilde{W}
\] (3.26)
Hence we can repeat on \(\tilde{\mathcal{M}}\) the steps (3.14)–(3.16) starting with \(\delta \tilde{\Gamma} = (\tilde{\Gamma}, -\tilde{W} + \mathcal{X})_{\text{ext}} \mu\) (and \(R^\alpha \to 0\)). We then arrive at a variation of \(\mathcal{X}\) which, as will be seen shortly, describes the arbitrariness in \(\mathcal{X}\) corresponding to the gauge freedom. We thus conclude that an arbitrary admissible gauge choice is allowed in the integral (3.25), thereby providing the bridge between the formalism of the previous subsection and the \(\lambda^*\)-extended formalism.

The advantage of introducing the \(\lambda^*\)-extended formalism is that, as follows from [32, 33], the automorphisms of the infinite algebra generated by the ‘strong’ master equation (3.22) can be described in a closed form as
\[
\exp\left(\frac{i}{\hbar} \mathcal{X}'\right) = \left(\exp\left[\Delta_{\text{ext}}, \tilde{\Psi}\right]\right) \exp\left(\frac{i}{\hbar} \mathcal{X}\right)
\] (3.27)
where \(\tilde{\Psi}\) is a fermionic function or operator and \(\Delta_{\text{ext}}\) acts to the right (so that, e. g., \([\Delta_{\text{ext}}, \tilde{\Psi}] (F) = \Delta_{\text{ext}}(\tilde{\Psi} F) + \tilde{\Psi} \Delta_{\text{ext}} F\)). Consider the case when \(\tilde{\Psi}\) is an infinitesimal function \(\tilde{f}\). Then,
\[
\delta \mathcal{X} \equiv \mathcal{X}' - \mathcal{X} = (\mathcal{X}, \tilde{f})_{\text{ext}} - i\hbar \Delta_{\text{ext}} \tilde{f}
\] (3.28)
If we assume \(\tilde{f}\) to be independent of \(\lambda_\alpha^*\) we find \((\tilde{f} = f)\)
\[
\delta X = \delta \mathcal{X}|_{\lambda^*_\alpha = 0} = (X, f) - (-1)^{\varepsilon_\alpha} R^\alpha \partial_\alpha f - i\hbar \Delta f
\] (3.29)
which is equal to (3.17). This shows that the variation \(\delta X\) (3.17) is consistent since it agrees with the formula for infinitesimal automorphisms. We also have
\[
\delta R^\alpha = - \left(\frac{\partial}{\partial \lambda^*_\alpha} \delta \mathcal{X}\right)|_{\lambda^*_\alpha = 0} = (R^\alpha, f) - (-1)^{\varepsilon_\beta} F^{\beta \alpha} \partial_\beta f
\] (3.30)
which reproduces eq. (3.19) that has been obtained as a part of the condition guaranteeing compatibility of the ‘induced’ variation (3.17) with the weak master-equation (3.7).

The most general consistent variation is obtained from the automorphism formula (3.27) when the infinitesimal function \(\tilde{f}\) has an arbitrary dependence on \(\lambda_\alpha^*\). Inserting
\[
\tilde{f} \equiv f - \lambda_\alpha^* Q^\alpha + O((\lambda^*)^2)
\] (3.31)
into (3.28) one arrives at
\[
\delta X = \delta \mathcal{X}|_{\lambda^*_\alpha = 0} = (X, f) - i\hbar \Delta f - (-1)^{\varepsilon_\alpha} R^\alpha \partial_\alpha f - X^\alpha \partial_\alpha Q^\alpha + i\hbar (-1)^{\varepsilon_\alpha} \partial_\alpha Q^\alpha
\] (3.32)
instead of (3.29). This can be obtained as an effective change of \(X\) induced by the above manipulations in the path integral (3.4) if one also adds the variation of \(\lambda^\alpha\)
\[
\delta \lambda^\alpha = Q^\alpha, \quad Q^\alpha \equiv Q^\alpha(\Gamma, \lambda; \hbar)
\] (3.33)
Notice that \(\delta R^\alpha\) in this case has a much more general form than (3.30).

We conclude that the partition function (3.4) is locally independent of the gauge-fixing action \(X\) provided equation (3.7) for the master-action \(X\) is satisfied. By a transformation of variables in
the integral we have induced a deformation of the gauge-fixing action \( X \) that preserves the equation imposed on \( X \).

**Remarks**

1. Note that the mapping (3.27) amounts to an anticanonical transformation. An important consequence is that it can therefore be applied to any function, not necessarily the master-action.

2. It is possible to ‘abelianize’ the hypergauge conditions \( G_\alpha = 0 \), expressing them in terms of a gauge fermion and a ‘rotation’ matrix \([10]\). In that case, one can trace how the independence of the solution of the master equation from the corresponding automorphism transformations reformulates as the arbitrariness in choosing the gauge fermion and the rotation matrix. This gives the precise relation between the arbitrariness of solutions to the master equation and the gauge freedom.

**3.3 On the structure of hypergauge-fixing actions**

Let us stress that \( X \) can in general be an arbitrary polynomial in \( \lambda^\alpha \). This corresponds to the possibility of introducing non-singular gauges such as e.g. the \( \alpha \)-gauges in Yang–Mills. On the other hand, singular hypergauges are those that explicitly involve the product of delta-functions in the integrand. Singular hypergauges result from taking \( X \) to be linear in \( \lambda^\alpha \), \( X = G_\alpha \lambda^\alpha \). Hence \( \lambda^\alpha \) are Lagrange multipliers for the hypergauges \( G_\alpha \), and integrating over \( \lambda^\alpha \) in (3.29) results in concentrating the integral on the locus \( G_\alpha = 0 \). For such \( X \)’s, there is an important grading by the so called Planck number \( \text{Pl} \). This is defined as \([1]-[10]\)

\[
\text{Pl}(FG) = \text{Pl}(F) + \text{Pl}(G), \quad \text{Pl}(\Gamma^A) = 0, \quad \text{Pl}(h) = \text{Pl}(\lambda^\alpha) = -\text{Pl}(\lambda^{\alpha\ast}) = 1.
\]

(3.34)

The master-actions \( X \) that lead to a singular hypergauge are characterized by \( \text{Pl}(X) = 1 \). Then all the exponents in (3.27) have Planck number zero, while in (3.29), e.g. \( R^\alpha \) has Planck number 2. A solution to the set of equations derived by expanding (3.7) is given by \( \[9\] \) (in the original notations, which may now appear somewhat non-systematic, with \( G_\alpha \lambda^\alpha = \text{our } X_0, H = \text{our } X_1 \))

\[
X = G_\alpha \lambda^\alpha + \text{i} h H \quad (3.35)
\]

\[
R^\alpha = \frac{1}{2} U^\alpha_{\beta\gamma} \lambda^\gamma \lambda^\beta (-1)^{\epsilon_{\beta\gamma}} - \text{i} h V^\alpha_{\beta} \lambda^\beta - (\text{i} h)^2 \tilde{G}^\alpha \quad (3.36)
\]

where the functions \( G_\alpha, H, U^\alpha_{\beta\gamma}, V^\alpha_{\beta} \) and \( \tilde{G}^\alpha \) satisfy

\[
(G_\alpha, G_\beta) = G_\gamma U^\gamma_{\alpha\beta} \quad (3.37)
\]

\[
(H, G_\alpha) = \Delta G_\alpha - U^\beta_{\beta\alpha} (-1)^{\epsilon_{\beta\alpha}} - G_\beta V^\beta_{\alpha} \quad (3.38)
\]

\[
\Delta H - \frac{1}{2} (H, H) + V^\alpha_{\alpha} = G_\alpha \tilde{G}^\alpha \quad (3.39)
\]

Next, \( H \) is what has been considered in the previous papers on the subject as an additional measure factor in the integral. Eq. (3.37) is the classical starting point for this solution and corresponds to (3.10). It says that the gauge functions \( G_\alpha \) are in involution.

**4 Antitriplectic Lagrangian quantization**

We now return to the main subject of this paper, the triplectic quantization in the geometrical setting developed in section 2. As in the antisymplectic case considered in the previous section, we
first introduce an invariant action $W(\Gamma; \hbar)$, which is now defined on a $6N$-dimensional antitriplectic manifold $M$ and is required to satisfy the generalized master equations with a ‘transport’ term due to the $V^a$:

$$\left( \Delta^a + \frac{i}{\hbar} V^a \right) \exp \left\{ \frac{i}{\hbar} W \right\} = 0 \quad (4.1)$$

or, equivalently,

$$\frac{1}{2} (W,W)^a + V^a W = i\hbar \Delta^a W \quad (4.2)$$

where $\Delta^a, V^a$ and the antibrackets are as defined in section 2 in arbitrary coordinates. Eq. (4.2) is solved by an expansion in $\hbar$ with the appropriate boundary conditions.

Recall that the triplectic quantization of a given theory requires introducing not only antifields, but also a ‘second set of fields’. Therefore, even with all the antifields set to zero, one has to single out a submanifold $L_0$ of the original fields of the theory that is being quantized. Most naively, one could think of the procedure of adding the new fields as duplicating the original variables $\phi^\alpha \in L_0$, that is introducing, instead of the action $S(\phi)$, a function of two sets of variables, $S(\phi^\alpha - \bar{\phi}^\alpha)$, possessing the obvious ‘gauge’ symmetry. However, the actual doubling of the degrees of freedom is not so naive, and the procedure consists not in adding another copy of the fields $\phi^\alpha$, but rather in going over to the cotangent bundle $T^*L_0$ to the manifold $L_0$. The new ‘fields’ are therefore certain $\bar{\phi}^\alpha$, where the index position points to their ‘momentum’ nature.

As to the boundary conditions on the master-action, one could therefore declare that, after projecting out all the antifields, one should be left with a cotangent bundle whose ‘coordinates’ (as opposed to the ‘momenta’) are the original variables of the theory. However, an obvious generalization, which we are going to make, is achieved by replacing the cotangent bundle by an arbitrary symplectic manifold and its Lagrangian submanifold. Then, the boundary conditions are imposed on $W$ in two steps.

To formulate the boundary conditions, one first restricts to a fixed symplectic manifold $L_1 \subset M$ with a chosen Lagrangian submanifold $L_0 \subset L_1$. Then, one imposes

$$W(\cdot; 0) \big|_{L_0} = S(\cdot) \quad (4.3)$$

Note that the dimensions are given by the formulae (2.36) and (2.35) and

$$\dim L_0 = (n|N - n) \quad (4.4)$$

The submanifolds $L_0 \subset L_1 \subset M$ are fixed and make up a part of the definition of the theory. The ‘classical’ gauge generators are encoded in the boundary conditions involving $W^{(1)}$, which is formulated as the appropriate symplectic analogue of (3.3).

Now we propose the following ansatz for the partition function path integral in the triplectic quantization:

$$Z = \int \exp \left\{ \frac{i}{\hbar} \left[ W + X \right] \right\} \rho(\Gamma)[d\Gamma][d\lambda] \quad (4.5)$$

where $X(\Gamma, \lambda; \hbar)$ is a gauge-fixing action. As before, $\lambda^\alpha, \alpha = 1, \ldots, N$, are parametric variables that generalize Lagrange multipliers for the hypergauge functions. We require that (4.7) be invariant under a generalized canonical transformation accompanied by a shift of the $\lambda^\alpha$:

$$\delta \Gamma^A = (\Gamma^A, -W + X)^a \mu_a - 2V^a \delta \mu_a ,$$

$$\delta \lambda^\alpha = -2\delta \mu^\alpha \mu_a \quad (4.6)$$

\*Note that the Grassmann parity is not reversed in this construction. We will actually use only the Poisson structure, but in order that the integrals be well-defined eventually, this must be non-degenerate.
where $\mu_a$ are two fermionic constants. This allows us to arrive at an equation we will then postulate for $X$. The form of (4.6) is such that the resulting conditions on $X$ will not involve $W$ when one makes use of the fact that $W$ satisfies the generalized quantum master equation (4.2). We claim that the integral (4.5) will be invariant under the transformations (4.6) provided

$$\frac{1}{2}(X, X)^a - i\hbar \Delta^a X - V^a X - X\partial_a R^{aa} + i\hbar R^{aa} \partial_a = 0$$

or equivalently

$$\left(\Delta^a - \frac{i}{\hbar} V^a - \frac{i}{\hbar}(-1)^{\varepsilon_a} R^{aa} \partial_a + (-1)^{\varepsilon_a} \partial_a R^{aa}\right) \exp\left(\frac{i}{\hbar} X\right) = 0$$

where as before $\partial_a \equiv \partial/\partial \lambda^a$ and $\varepsilon_a \equiv \varepsilon(\lambda^a)$. This is a ‘weak’ analogue of the generalized quantum master equation (4.2), with an opposite sign for the $V^a$-term. As in the antisymplectic case, solutions to (4.7) should be looked for in the form of (formal) power series $X(\Gamma, \lambda; \hbar)$ and $R^{aa}(\Gamma, \lambda; \hbar)$ in $\hbar$:

$$X = X_0 + i\hbar X_1 + (i\hbar)^2 X_2 + \ldots,$$

$$R^{aa} = R^{aa}_0 + i\hbar R^{aa}_1 + (i\hbar)^2 R^{aa}_2 + \ldots$$

In the lowest order, in particular, we find

$$\frac{1}{2}(X_0, X_0)^a - V^a X_0 - X_0 \partial_a R^{aa}_0 = 0$$

As before, coefficients for all powers of $\lambda^a$ must agree.

By applying to the LHS of eq. (4.7) the operator

$$\Delta^b - \frac{i}{\hbar} V^b + \frac{i}{\hbar} (X, \cdot)^b - \frac{i}{\hbar}(-1)^{\varepsilon_a} R^{ab} \partial_a$$

and symmetrizing in $a$ and $b$ we arrive at the consistency condition

$$\frac{\partial}{\partial \lambda^a} \left\{ \left(i\hbar(\Delta^a R^{ab}) + V^a R^{ab} - (X, R^{aa})^b + (-1)^{\varepsilon_{\beta}} R^{\beta(a} \partial_{\beta} R^{ab)}\right) e^{\frac{i}{\hbar} X} \right\} = 0$$

which is solved in the extended formalism introduced below, see (5.16).

5 Independence from the choice of hypergauge fixing and gauge invariance in the antitriplectic case

In this section we shall show that the path integral (1.3) is independent of local variations of hypergauge-fixing action $X(\Gamma, \lambda; \hbar)$ for a certain class of variations. This actually amounts to the statement of gauge invariance in the sense of independence from the ordinary gauge-fixing function. In the geometrically covariant version, this function is encoded into a set of hypergauge-fixing functions which satisfy master equations; thus gauge invariance takes the form of an adequate freedom in choosing hypergauge-fixing functions. The fundamental fact is that this freedom is determined just by the (master) equation satisfied by the gauge-fixing master-action $X$. Therefore gauge invariance is reformulated in terms of automorphisms of solutions to the master equation.

We start with a procedure that is a generalization of the previous approach in the antisymplectic case given in section 3, i.e. we shall look for a transformation of integration variables that can induce
a variation of $X$. Let us replace the fermionic constants $\mu_a$ in (4.6) by arbitrary functions of $\Gamma^A, \lambda^\alpha$ and $\hbar$ subjected to the condition that they are expandable in powers of both $\lambda^\alpha$ and $\hbar$. The integrand (4.5) is then no longer invariant. In fact, the transformations (4.6) give now rise to the following extra Jacobian $1 + \delta J$ in (4.5):

$$\delta J = (-W + X, \mu_a)^a - 2V^a\mu_a - 2R^{a\alpha}\partial_\alpha\mu_a(-1)^{\varepsilon_a}$$  (5.1)

(The second term on the RHS is the action of vector fields on the respective functions; we also remind the reader that $\partial_\alpha = \partial/\partial \lambda^\alpha$). The contribution $(-W, \mu_a)^a$ may be compensated for by the additional transformation

$$\delta_1 \Gamma^A = \hbar i(\Gamma^A, \mu_a)^a$$  (5.2)

which in turn yields a new Jacobian

$$\delta_1 J = 2\Delta^a\mu_a$$  (5.3)

and at the same time doubles the term $(X, \mu_a)^a$. In this way we finally obtain the following effective change of the integrand of (4.5):

$$\delta [\text{exponent of (4.5)}] = \frac{i}{\hbar}\delta X$$  (5.4)

where

$$\delta X = (X, f_a)^a - V^a f_a - i\hbar \Delta^a f_a - (-1)^{\varepsilon_a} R^{a\alpha}\partial_\alpha f_a$$  (5.5)

with

$$f_a \equiv \frac{2\hbar}{i} \mu_a.$$  (5.6)

We may also add a further variation of $\lambda^\alpha$ of the form (3.33) which results in the following effective change of $X$:

$$\delta X = (X, f_a)^a - V^a f_a - (-1)^{\varepsilon_a} R^{a\alpha}\partial_\alpha f_a - X\partial_\alpha Q^\alpha - i\hbar \Delta^a f_a + i\hbar(-1)^{\varepsilon_a} \partial_\alpha Q^\alpha.$$  (5.7)

Now, as in the antisymplectic case, we have to ensure that the master equation imposed on $X$ will be preserved after the deformation. However, not every $\delta X$ given by (5.7) would preserve the triplectic master equations (recall that these include the algebra of gauge conditions as well as other relations). The point is that the RHS of (5.7) involves two fermionic functions $f_a, a = 1, 2$ whereas, as we will see shortly, a natural arbitrariness in solutions to the master equation, as well as the gauge freedom, is described by one bosonic function. We can nevertheless show that there do exist certain variations of the form (5.7) that preserve the master equation. This will follow from the extended formalism, which is constructed as follows.

We introduce a linear space $\Lambda$ spanned by $(\lambda^\alpha, \lambda^{*\alpha}, \eta^{a\alpha})$, with $\epsilon(\lambda^\alpha) = \epsilon(\lambda^{*\alpha}) = \epsilon_a, \epsilon(\eta^{a\alpha}) = \epsilon(\eta^{\alpha a}) = \epsilon_a + 1$, and define an extended triplectic manifold $\tilde{\mathcal{M}} = \mathcal{M} \times \Lambda$. On $\tilde{\mathcal{M}}$, we introduce the operators

$$\Delta^a_{\text{ext}} = \Delta^a + (-1)^{\varepsilon_a} \frac{\partial}{\partial \lambda^\alpha} \frac{\partial}{\partial \lambda^{*\alpha}} + (-1)^{\varepsilon_a+1} \epsilon^{ab} \frac{\partial}{\partial \lambda^\alpha} \frac{\partial}{\partial \eta^{a\beta}}$$  (5.8)

and the corresponding antibrackets

$$(F, G)^a_{\text{ext}} = (F, G)^a + \left( F \frac{\partial}{\partial \lambda^\alpha} \frac{\partial}{\partial \lambda^{*\alpha}} G + e^{ab} F \frac{\partial}{\partial \eta^{a\beta}} \frac{\partial}{\partial \lambda^\alpha} G - (-1)^{(\epsilon(F)+1)(\epsilon(G)+1)} (F \leftrightarrow G) \right).$$  (5.9)
Another triplectic quantity are the vector fields $V^a$, which extend to $\tilde{\mathcal{M}}$ as

$$V^a = V^a - \frac{1}{2} \left( \epsilon^{ab} \lambda_{ab} \frac{\partial}{\partial \lambda_a} - (-1)^{\xi_1} \eta^{aa} \frac{\partial}{\partial \xi_1} \right)$$

and satisfy the necessary conditions $[\mathcal{L}_{\mathcal{Q}}$, $\Delta^b_{\text{ext}}] = 0$ etc. in accordance with the conditions given in section 3. (Notice that equations (5.8)–(5.10) follow from the Darboux expressions (2.31)–(2.33).)

Now we can introduce an extended quantum master equation

$$\left( \Delta^a_{\text{ext}} - i \frac{\hbar}{\hbar} \mathcal{L}^a \right) \exp \left( i \frac{\hbar}{\hbar} \mathcal{X} \right) = 0$$

or equivalently,

$$\frac{1}{2} (\mathcal{X}, \mathcal{X})_{\text{ext}}^a - \mathcal{L}^a \mathcal{X} - i \hbar \Delta^a_{\text{ext}} \mathcal{X} = 0.$$  

The ‘weak’ master equation (4.8) follows from (5.12) once we take $\mathcal{X}$ to be

$$\mathcal{X} (\Gamma, \lambda, \lambda^*, \overline{\lambda}, \eta) = \tilde{\mathcal{X}} (\Gamma, \lambda, \lambda^*, \overline{\lambda}) + \frac{1}{2} \lambda^* \eta$$

and further expand $\tilde{\mathcal{X}}$ as

$$\tilde{\mathcal{X}} (\Gamma, \lambda, \lambda^*, \overline{\lambda}) = X (\Gamma, \lambda) - \lambda^* \lambda X (\Gamma, \lambda) - \bar{X} \lambda^* \lambda X (\Gamma, \lambda) + \text{higher orders in } \lambda^*, \overline{\lambda}$$

(That the dependence of $\mathcal{X}$ on $\eta$ is linear can also be understood by reducing from the level-1 theory). Further, specifying the second-order terms in (5.14) as

$$\frac{1}{2} \lambda^* \lambda X (\Gamma, \lambda), \frac{1}{2} \lambda^* \lambda X (\Gamma, \lambda) = \bar{X} \lambda^* \lambda X (\Gamma, \lambda) + \text{higher orders in } \lambda^*, \overline{\lambda}$$

(with $F^{\beta a; \alpha a} = (-1)^{(\xi_1 + 1)(\xi_2 + 1)} F^{\alpha a; \beta a}$), we find from the first order in $\lambda^*$ in eq. (5.12):

$$i \hbar \Delta^a R^{ab} + V^a R^{ab} + (\lambda^* \lambda X + X) \epsilon^{ab} \partial_\beta R^{ab} - (X, R^{ab}) = (1) \epsilon^a (i \hbar \partial_\beta F^{\alpha a; \beta a} - \partial_\beta X F^{\beta a; \alpha a})$$

(5.16)

This solves consistency condition (4.12) upon symmetrization, while $\overline{R}^a$ represents the antisymmetrized part. From the first order in $\lambda^*$ in eq. (5.12) we get further relations constraining $\overline{R}^a$:

$$i \hbar \Delta^a \overline{R}^a + V^a \overline{R}^a + (\lambda^* \lambda X + X) \epsilon^{ab} \partial_\beta \overline{R}^a - (X, \overline{R}^a) = (1) \epsilon^a (i \hbar \partial_\beta \overline{R}^a - \partial_\beta X \overline{R}^a)$$

(5.17)

Let us show now how the integral (4.13) can be reformulated on the extended triplectic manifold using the extended master action $\mathcal{X}$. We have

$$Z = \int \exp \left\{ i \frac{\hbar}{\hbar} \mathcal{W} + \tilde{\mathcal{X}} \right\} \delta (\lambda^*) \delta (\overline{\lambda}) \rho (\Gamma) [d\Gamma] [d\lambda] [d\lambda^*] [d\overline{\lambda}]$$

$$= \int \exp \left\{ i \frac{\hbar}{\hbar} \mathcal{W} + \lambda^* \bar{\lambda} + \bar{\lambda} \lambda \eta \right\} \rho (\Gamma) [d\Gamma] [d\lambda] [d\lambda^*] [d\overline{\lambda}] [d\eta]$$

(5.18)

(5.18)

$$(\xi = \lambda^{(1)})$$. The integral can then be reformulated on the extended triplectic manifold using the extended master action $\mathcal{X}$. We have

$$Z = \int \exp \left\{ i \frac{\hbar}{\hbar} \mathcal{W} + \tilde{\mathcal{X}} \right\} \delta (\lambda^*) \delta (\overline{\lambda}) \rho (\Gamma) [d\Gamma] [d\lambda] [d\lambda^*] [d\overline{\lambda}]$$

(5.19)
Therefore, as in section 3, we can perform an arbitrary gauge variation in the integral (5.18), thus establishing the validity of the path integral on $\hat{M}$ in a general hypergauge. We also have at our disposal now all the power of the automorphism formula for solutions of the strong master equations. This is described as follows.

Consider eq. (5.11). We have a mapping $\mathcal{X} \mapsto \mathcal{X}'$ on the solutions given by \[ \exp \left( \frac{i}{\hbar} \mathcal{X}' \right) = \left( \exp \frac{\hbar}{i} \left[ \Delta^a_{\text{ext}} - \frac{i}{\hbar} \mathcal{V}^a, [\Delta^b_{\text{ext}} - \frac{i}{\hbar} \mathcal{V}^b, \tilde{\Phi}] \right] \right) \exp \left( \frac{i}{\hbar} \mathcal{X} \right) \] (5.20) where $\tilde{\Phi}$ is an arbitrary bosonic function or operator. When $\tilde{\Phi}$ is an infinitesimal function $\tilde{\varphi}$, we have

$$
\delta \mathcal{X} \equiv \mathcal{X}' - \mathcal{X} = \mathcal{K}\tilde{\varphi} + \epsilon_{ab} \left\{ (\mathcal{X}, (\mathcal{X}, \tilde{\varphi})^b)_{\text{ext}} - \mathcal{V}^a(\mathcal{X}, \tilde{\varphi})^b_{\text{ext}} - (\mathcal{X}, \mathcal{V}^b\tilde{\varphi})^a \right\} + i\hbar \left[ \mathcal{V}^a \Delta^b_{\text{ext}} + \Delta^a_{\text{ext}}\mathcal{V}^b \right] $$

$$
- \hbar^2 \Delta^a_{\text{ext}} \Delta^b_{\text{ext}} \tilde{\varphi} \right\} \right) \right. \quad (5.21)
$$

(where $\mathcal{K} = \epsilon_{ab} \mathcal{V}^a \mathcal{V}^b$). This expression rewrites as an $\mathcal{X}$-dependent transformation

$$
\delta \mathcal{X} = (\mathcal{X}, \tilde{f}_a)_{\text{ext}} - \mathcal{V}^a \tilde{f}_a - i\hbar \Delta^a_{\text{ext}} \tilde{f}_a \right\} \right) \right. \quad (5.22)
$$

where

$$
\tilde{f}_a = \epsilon_{ab} \left\{ (\mathcal{X}, (\mathcal{X}, \tilde{\varphi})^b)_{\text{ext}} - \mathcal{V}^b \tilde{\varphi} - i\hbar \Delta^b_{\text{ext}} \tilde{\varphi} \right\} \right. \quad (5.23)
$$

The form (5.13) of $\mathcal{X}$ depending on $\eta$ only via $\frac{1}{2} \lambda^a_{\alpha} \eta^{\alpha a}$ will be preserved once $\tilde{\varphi}$ is taken to be a function of $(\Gamma, \lambda, \lambda^*, \overline{\lambda})$. Then, the variation of the master action $X$ follows as $\delta X = \delta \mathcal{X} \right|_{\lambda^*, \overline{\lambda} = 0}$ and turns out to have the form of a variation (5.7) obtained above. From (5.14) and (5.21), we also get the variation of $R^{\alpha a}$ as

$$
\delta R^{\alpha a} = - \left( \frac{\partial}{\partial \lambda^a_{\alpha}} \delta \mathcal{X} \right) \right|_{\lambda^*, \overline{\lambda} = 0} \quad (5.24)
$$

Thus the automorphisms of the strong master-equation in the extended sector reduce to the original phase space as a particular realization (certain $X$-dependent transformations) of the variations (5.7) that were obtained by changing variables in the path integral. The partition function (5.18) is therefore independent of local changes of the gauge-fixing master action $X$ within the class of those $X$’s which satisfy the master equation; in other words, the subclass of the variations (5.7) consisting of the above $X$-dependent transformations do preserve the algebra of the constraints imposed by the master-action $X$.

To assert that the above transformations of $X$ describe gauge invariance of our formalism, one needs the statement that the automorphism formula (5.21) describes the maximum freedom allowed by (generic) solutions to the master equation. As we have noted, in the antisymplectic case a similar statement can be proven using ‘abelianization’, whose triplectic analogue has yet to be elaborated. It seems very probable, however, that the above automorphism formula describes correctly the maximal arbitrariness of a solution to the master equation, thereby allowing one to arrive at an arbitrary gauge (in the ordinary sense) by choosing different solutions to the master equation.

**Remarks.**

1. In similarity with the antisymplectic case, we can find a coordinate transformation that induces the above variation when applied to $X$; the power of the ‘coordinate representation’ of this variation
is that it can then be applied to any function on $\mathcal{M}$. It turns out that the desired infinitesimal diffeomorphism on $\mathcal{M}$ consists of a ‘hamiltonian’ piece (in the triplectic sense, i.e. generated by the antibrackets) and a term involving $V^a$. The ‘hamiltonian’, or canonical, transformations are called so by analogy with the (anti)symplectic formalism; they are considered systematically in Appendix B.

While a naive generalization of the canonical transformation to the triplectic case would be given by $F \mapsto (h_1, F)^1 + (h_2, F)^2$ which involves two ‘hamiltonians’ $h_a$, the intrinsic triplectic transformations are those for which the (fermionic) functions $h_a$ are expressed through one arbitrary bosonic function $\varphi$ and a solution $X$ of the master equation. One thus gets $X$-dependent transformations, which are a characteristic feature of the triplectic geometry.

To describe how the transformation generated by $\varphi$ acts on an arbitrary function $F$, one should start with the extended formalism. Consider the following infinitesimal mapping on functions on the extended manifold $\tilde{\mathcal{M}}$:

$$\tilde{F} \mapsto \tilde{F} + \epsilon_{ab}(\tilde{F}, (\mathcal{X}, \tilde{\varphi})_a^{\text{ext}})^a + 2\epsilon_{ab}(\tilde{F}, V^a\tilde{\varphi})^b_{a_{\text{ext}}} - \epsilon_{ab}(V^a\tilde{F}, \tilde{\varphi})^b_{a_{\text{ext}}} \quad (5.25)$$

where different $\tilde{\varphi}$'s label infinitesimal automorphisms and $\mathcal{X}$ is a fixed solution of the master equation. When applied to $\tilde{F} = \mathcal{X}$, this transformation reproduces the classical ($\bar{\hbar}$-independent) part of eq. (5.21) apart from the $K\tilde{\varphi}$ term. Now, by expansion in $\lambda^*, \tilde{\varphi}$, one arrives at a ‘weak’ form of the corresponding transformation. Its ‘main’ part will still be given by antibrackets and $V^a$ fields. To avoid overloading our formulae with the details that bear no principal importance, we will write out only this ‘main’ part of the coordinate transformation. It is obtained by taking the fields in (5.25) to depend on $(\Gamma, \lambda)$ only. Thus, consider an infinitesimal variation of functions $F$ on $\mathcal{M}$:

$$F \mapsto F + \epsilon_{ab}(F, (X, \varphi)^b_a + 2\epsilon_{ab}(F, V^a\varphi)^b_a - \epsilon_{ab}(V^aF, \varphi)^b_a \quad (5.26)$$

When applied to $X$, this results in the following transformation

$$\delta X = (X, f)^a_a - V^a f_a - i\hbar \Delta^a f_a , \quad (5.27)$$

with $f_a$ in turn depending on $X$ via

$$f_a = \epsilon_{ab} \left\{ (X, \varphi)^b_a - V^b \varphi - i\hbar \Delta^b \varphi \right\} . \quad (5.28)$$

The first two terms in the transformation (5.26) are given by a hamiltonian vector field $\mathcal{H}_{h_a(\varphi)}$ (see (3.3)) with the hamiltonians

$$h_a(\varphi) = -\epsilon_{ab}(X, \varphi)^b_a + 2\epsilon_{ab}V^b \varphi , \quad (5.29)$$

while the third term in (5.26) is a ‘transport’ term. Thus (5.26) rewrites as

$$\delta F = \mathcal{H}_{h_a(\varphi)} F - \epsilon_{ab}(V^aF, \varphi)^b . \quad (5.30)$$

This becomes purely ‘hamiltonian’ if $V^a F = 0$, an important example of which we will meet in section 6.

2. Some consequences of the automorphism formula, such as the invariance of the strong master equation under variations given by (5.22), (5.23), can also be verified independently: Consider varying the extended ‘strong’ master-equation

$$(\mathcal{X}, \delta \mathcal{X})_a^\text{ext} - V^a \delta \mathcal{X} = i\hbar \Delta^a \delta \mathcal{X} \quad (5.31)$$
where $\delta X$ is given by (5.22), (5.23). In order to demonstrate that this variation indeed satisfies (5.31) one needs the following identities (we suppress the index ‘ext’) [14]:

$$
\epsilon_{bc}(X, (X, X, \bar{\varphi})^c)^a = \epsilon_{bc}\left\{ (X, (\bar{\varphi}, (X, X)^b)^c)^a + (\bar{\varphi}, (X, (X, X)^b)^c)^a + 2((X, X)^b, (X, \bar{\varphi})^c)^a \right\}
$$

(5.32)

$$
\epsilon_{bc}\left\{ \Delta^a(X, (X, \bar{\varphi})^c)^b + (\bar{\varphi}, (X, \Delta^c X)^b)^a + (\Delta^c X, (X, \bar{\varphi})^b)^a - (X, (\Delta^c X, \bar{\varphi})^b)^a + (X, X, \Delta^c \bar{\varphi})^b + (X, \Delta^b(X, \bar{\varphi})^c)^a - (\Delta^b X, (X, \bar{\varphi})^c)^a \right\} =
$$

(5.33)

and

$$
\epsilon_{bc}\left\{ \Delta^a \Delta^b(X, \bar{\varphi})^c - \Delta^a(\Delta^c X, \bar{\varphi})^b + 2(\Delta^c X, \Delta^b \bar{\varphi})^a + (\bar{\varphi}, \Delta^a \Delta^c X)^a + (\Delta^a X, \Delta^b \bar{\varphi})^b + (\Delta^a \Delta^c \bar{\varphi})^a \right\} = 0
$$

(5.34)

They can be derived using (2.11) and the identities $\Delta^a \Delta^b \Delta^c(\Lambda^d \bar{\varphi}) = 0$, $\Delta^a \Delta^b \Delta^c(\Lambda^2 \bar{\varphi}) = 0$ and $\Delta^a \Delta^b \Delta^c(X \bar{\varphi}) = 0$ respectively. One may notice that these identities are even valid when $\Delta^a_{\text{ext}}$ is replaced by $\Delta^a_{\text{ext}} + \alpha V^a$ for any constant $\alpha$ (cf. (2.23)).

3. We have remarked in section 2 that the condition $\text{div } V^a = 0$ can be removed at the expense of several new terms appearing in some of our formulae. Namely, the master equation (5.12) will change to

$$
\frac{1}{2}(X, X)^a_{\text{ext}} - V^a X - i\hbar \Delta^a_{\text{ext}} X + \frac{1}{2}i\hbar \text{div } V^a = 0
$$

(5.35)

(recall that $\text{div } V^a = \rho^{-1} \partial_A (V^A)(-1)^{\varepsilon^A}$), which suggests introducing a differential operator

$$
\hat{V}^a = V^a + \frac{1}{2} \text{div } V^a
$$

(5.36)

(as $\Lambda$ is a linear space, it does not contribute to the density $\rho$; divergence of the components of $V$ along $\Lambda$ is zero). We still have to ensure $[\hat{V}^a, \hat{V}^b] = 0 \iff \hat{V}^{(a} \hat{V}^{b)} = 0$ and $[\Delta^a_{\text{ext}}, \hat{V}^b] = 0$. This reduces to requirements on the $V^A$ components, and results in the following modifications in the formulae in section 2. First, in addition to (2.23) we are going to have

$$
(-1)^{\varepsilon^A} V^A \partial_A \left( \rho^{-1} \partial_B (\rho V^B)(-1)^{\varepsilon^B} \right) = 0
$$

(5.37)

Eq. (2.20) does not change, while (2.21) acquires on the LHS an additional term

$$
E^{ABa} \partial_A \left( \rho^{-1} \partial_B (\rho V^B)(-1)^{\varepsilon^B} \right).
$$

(5.38)

Finally, new equations are

$$
\Delta^a \left( \rho^{-1} \partial_B (\rho V^B)(-1)^{\varepsilon^B} \right) = 0
$$

(5.39)

Note also that the master equation (5.2) changes accordingly, into

$$
(\Delta^a + i \frac{\hbar}{\hbar} \hat{V}^a) \exp \left\{ i \hbar \hat{W}^a \right\} = 0
$$

(5.40)

4. Note finally that the Poisson bracket extends to $\mathcal{L}_1 \times \Lambda_1$, where $\Lambda_1$ is spanned by $(\lambda^\alpha, \lambda_\alpha)$. 

22
6 On the structure of hypergauge-fixing actions

A useful illustration of the general scheme is provided by taking $X_0$, the classical part of the gauge-fixing master-action $X$, to depend on $\lambda^\alpha$ at most linearly, as

$$X_0 = G_\alpha \lambda^\alpha + Z$$

(6.1)

where $Z$ is independent of $\lambda^\alpha$. Then the functions $G_\alpha$ should eventually specify a Lagrangian submanifold and thereby fix a gauge, since integrating over $\lambda^\alpha$ introduces delta functions $\prod \delta(G_\alpha)$ into the integral (4.5). This makes the form (6.1) important in applications.

It follows that for master actions of this class one can define a ‘weakened’ analogue of the Planck number introduced in the antisymplectic case in section 3.3. In the triplectic case, however, we will have a filtration instead of a grading. First, we mimic (3.34):

$$\text{Pl}(FG) = \text{Pl}(F) + \text{Pl}(G),$$

$$\text{Pl}(\Gamma^A) = 0,$$

$$\text{Pl}(\bar{\eta}^\alpha) = -\text{Pl}(\eta^\alpha) = 1,$$

$$\text{Pl}(\lambda^\alpha) = -\text{Pl}(\eta^\alpha) = -2.$$  

(6.2)

However, by assigning a Planck number $p$ to a function we will now mean that the function can contain all terms whose individual Planck numbers are not bigger than $p$. Then, the ansatz (6.1) is characterized by Planck number 1. Such $X$’s provide a natural triplectic analogue of singular gauges (i.e. those resulting in an explicit insertion of delta-functions into the integrand in (4.5)).

For the classical part of the master equation, we find from eq. (4.10)

$$\frac{1}{2}(X_0, X_0) - V^a X_0 = G_\alpha R_0^{\alpha a}$$  

(6.3)

with structure functions $R_0^{\alpha a}$ (see (4.9)) that must now have the form

$$R_0^{\alpha a} = \frac{1}{2} U_{[\alpha}^{\alpha a} \lambda^\beta \lambda^{\gamma]} (\pm 1)^{\delta \beta} + U_\beta^{\alpha a} \lambda^\beta + \frac{1}{2} U^{\alpha a}$$

(6.4)

for some $\lambda^\alpha$-independent $U_{\beta}^{\alpha a}$, $U_{\beta}^{\alpha a}$, and $U^{\alpha a}$. Expanding in $\lambda^\alpha$, one readily extracts the involution relations of the hypergauge functions $G_\alpha$

$$(G_\alpha, G_\beta)^a = G_\alpha U^{\gamma a}_{\alpha \beta}$$

(6.5)

with structure functions $U^{\gamma a}_{\alpha \beta}$, together with

$$(Z, G_\alpha)^a - V^a G_\alpha = G_\beta U^{\beta a}_{\alpha} ,$$

(6.6)

and

$$(Z, Z)^a = G_\gamma U^{\gamma a}.$$  

(6.7)

The involution relation (6.5) is a straightforward generalization of what we had in the antisymplectic case. However, there are no counterparts to eqs. (6.6)–(6.7). This is not yet the final form of the equation: below, we will propose a further specialization of the form of $Z$.

Going over to higher-order terms in the expansion of $X$ in powers of $\hbar$ beyond the zeroth order given by (6.1), we extend the ansatz (6.1), which is a first-order polynomial in $\lambda^\alpha$, to an $X$ of Planck
This means in particular that $X_2$ etc. $= 0$, while $X_1$ is $\lambda$-independent. We then arrive at the following equations for the ‘measure’ $H \equiv X_1$:

$$
(H, G_\alpha)^a = \Delta^a G_\alpha - U^a_\beta \lambda (1)^\beta + G_\beta P^a_\alpha,
$$
$$
(H, Z)^a - V^a H = \Delta^a Z - U^a_\alpha + G_\alpha P^{aa},
$$
$$
\Delta^a H - \frac{1}{\hbar}(H, H)^a = P^{aa} - G_\alpha R^a_{2a}
$$

(6.8)

which corresponds to $R^a_{1a}$ (see (4.3) and (5.4)) having the structure

$$
R^a_{1a} = P^{aa}_\lambda \lambda^b + P^{aa},
$$

(6.9)

with $R^a_{2a}$ being $\lambda$-independent. The two terms in the $\hbar$-expansion can be called the ‘classical’ master-action and the triplectic measure.

Once we have chosen an $X_0$ having Planck number 1, we must ensure that the variation of $X$ does not increase the Planck number. To this end, we should extract from (5.21) terms of (individual) Planck number 2 (denoted below as $|\hbar|^2$), and require their vanishing. We replace $V^a X$ in (5.21) using (5.12); then we observe that $\tilde{\varphi}$ must have Planck number zero in order to avoid terms with yet higher Planck numbers. Then the condition for the vanishing of (strict-) Planck-number-2 terms becomes

$$
\epsilon_{ab} \left\{ (X', X', \tilde{\varphi})^b_{\text{ext}} \epsilon^{\alpha}_{\text{ext}} \left|^{(2)} \right. - \frac{1}{2} \left( (X', X')^a, \tilde{\varphi} \right)^b_{\text{ext}} \left|^{(2)} \right. + i\hbar (\Delta_{\text{ext}}^a X', \tilde{\varphi})^b_{\text{ext}} \left|^{(2)} \right. - \bar{\hbar} \Delta_{\text{ext}} \Delta_{\text{ext}} \tilde{\varphi} \left|^{(2)} \right. \right\} = 0
$$

(6.10)

Expanding this in $\lambda \lambda^{*} \lambda$ and $\hbar$ gives a set of relations guaranteeing preservation of the ansatz (6.1) under variations. In the classical part, the corresponding condition on $\varphi$ amounts to demanding the cancellation of $\lambda$-squared terms. This leaves us with the following constraint on (the classical part of) $\varphi = \tilde{\varphi}|_{\lambda^{*}=0,\lambda=0}$ and $G_\alpha, G_\beta$ (which all are functions on $M$):

$$
\epsilon_{ab} (G_\alpha, (G_\beta, \varphi)^b)^a + \epsilon_{ab} ((G_\alpha, \varphi)^b, G_\beta)^a (-1)^{\varepsilon_\beta + 1} = \epsilon_{ab} ((G_\alpha, G_\beta)^a, \varphi)^b
$$

(6.11)

which happens to coincide with the Jacobi identities – for the particular entries involved – in the antisymmetric sector (in $ab$). This is rather natural, since the fact that the $\lambda$s retain their role of Lagrange multipliers (setting the integral onto the locus $G_\alpha = 0$) means, by consistency, that the antibracket algebra of the gauge functions $G_\alpha$ must be preserved. For general transformations this is impossible just because of the lack of the Jacobi identity for two non-coincident antibrackets; only the compatibility condition (2.10) symmetrized in $ab$ exists in general.

We will now further specialize the ansatz for $X_0$ to

$$
X_0 = G_\alpha \lambda^a + KY
$$

(6.12)

where $Y$ is a function on $M$ and $K$ was introduced in (2.25). The $KY$ form of the $\lambda$-independent term does give a solution of the master equation in the Darboux coordinates [14] and, due to its invariant form, it can be taken over to the general case. Then, however, we have to make sure that this ansatz is stable under the ‘gauge’ variations that follow from (5.22), (5.23). We discuss this issue in Appendix B, where we also consider, more generally, form variations of the triplectic-geometric

---

9It appears rather non-trivial that a solution exists for the triplectic measure that has no $\hbar^2$-terms and no $\lambda$ in the $\hbar$ order. In the antisymplectic case, a similar solution is given by a subtle differential-geometric construction [14, 3, 23], which would be interesting to extend to the triplectic case.
quantities. The upshot is that the variation of the \( \lambda \)-independent part of \( X_0 \) can be represented as a combined effect of a coordinate variation of \( K \) and a variation of \( Y \). Recall that the automorphism transformations can be implemented by a coordinate transformation (Remark 1 on page 20); then \( K \) and \( Y \) should be brought to the new coordinate system by the appropriate transformation. On top of that, there is a form-variation of \( Y \), equal simply to the infinitesimal function \( \phi \).

The use of the \( KY \) term can be illustrated in the Darboux coordinates; in ref. [14], \( X_0 \) was given by (6.12) satisfying (6.3) with \( R^{\alpha a}_0 = 0 \). Using in this case the Poisson bracket (2.34) and differentiating (6.6) with respect to \( \pi^\alpha_a \) and \( \Phi^\alpha \), we arrive at

\[
\frac{\partial G_\beta}{\partial \Phi^\gamma} = -2 \left\{ G_\beta, \frac{\partial Y}{\partial \Phi^\gamma} \right\} (-1)^{\bar{\varepsilon}_\beta \varepsilon_\gamma}, \quad \frac{\partial G_\alpha}{\partial \Phi^\gamma} = -2 \left\{ G_\alpha, \frac{\partial Y}{\partial \Phi^\gamma} \right\} (-1)^{\bar{\varepsilon}_\alpha \varepsilon_\gamma}
\]

which constrains the gauge functions in the \( \Phi^\alpha, \Phi^\beta \)-sector and is solved by

\[
Y = \Phi^\alpha \Phi^\beta - 2F(\Phi) \tag{6.14}
\]
\[
G_\alpha = \Phi^\alpha - F(\Phi) \frac{\partial}{\partial \Phi^\alpha} \tag{6.15}
\]

where \( F(\Phi) \) is a bosonic function of \( \Phi^\alpha \) (‘the gauge boson’). This case illustrates clearly that the \( G_\alpha \) must not be annihilated by \( V^\alpha \). The gauge freedom contained in \( N \) functions \( G_\alpha \) has eventually reduced to one bosonic function, which is needed for gauge-fixing.

7 Concluding remarks

In this paper we have developed the basics of antitriplectic differential geometry. Based on this, we have suggested a quantization of general hypergauge theories in an \( Sp(2) \)-symmetric way in arbitrary coordinates on the field space. This generalizes what one has in the antisymplectic case, which is also reviewed and generalized here. We have also seen that the most complete framework for the hypergauge algebra generating relations is a triplectic extension of the multilevel-type \[3, 4\] antibracket formalism. It should be realized that the formalism constructed above is in fact developed in a context much more general than the original motivation, which was the \( Sp(2) \)-symmetric quantization and, in particular, the ghost-anti ghost symmetry.

We have also given a reformulation of the usual antibracket scheme based on the ‘weak’ master-equations, which resulted in allowing an a priori arbitrary dependence of the gauge-fixing master-action on the ‘Lagrange multipliers’ \( \lambda^\alpha \) (which do become true Lagrange multipliers to the hypergauge conditions as soon as the action is taken to depend on them linearly). This new possibility of introducing an arbitrary \( \lambda \)-dependence is common to both the antisymplectic and triplectic cases, although in the triplectic case the essence of the problem is stressed by the fact that \( \lambda \)-independent terms exist in the master-action \( X \) along with the \( \lambda \)-dependent ones already at the classical level. In the standard antisymplectic quantization, on the other hand, \( X_0 = X|_{h=0} \) is homogeneous in \( \lambda^\alpha \), and only the ‘quantum’ part (the measure \( H \)) is not proportional to \( \lambda^\alpha \). The difference between antisymplectic and triplectic cases is reflected in the character of the Planck number conservation: in the triplectic case, only a filtration by the Planck number is respected by the equations.

We have demonstrated that the partition function is independent of the deformations of gauge-fixing master-action \( X \) within a certain class allowed by the algebra of hypergauge conditions. The independence from the choice of a solution of the master equation is what replaces gauge invariance
in the general case. The point is that in the general hypergauge theories the gauge-fixing functions are hidden in the master-action $X$. This action is ‘dual’ to the master action $W$ whose expansion in antifields (once these are identified) starts with the classical action that is gauge-independent in the simplest sense of the word. On the other hand, the $X$ action is built starting from gauge-fixing functions (and hence is tautologically gauge-dependent). It is quite remarkable that both these actions satisfy the appropriate master equations. The freedom in choosing $X$ is then precisely the independence from the gauge fixing. The correct balance of degrees of freedom is maintained when one considers the maximal arbitrariness in solutions to the $X$-master-equation (such that is sufficient to establish gauge independence in the usual sense).

We have been able to ‘lift’ the formalism based on the weak master equation to the extended triplectic manifold obtained by introducing triplectic partners to the Lagrange multipliers $\lambda^\alpha$. This has allowed us to work with the strong master equations and, in particular, to use the automorphism formula. Moreover, strong master equations analogous to eq. (5.11) can be taken over to a tower of extended sectors in the antitriplectic counterpart to the multilevel field-antifield formalism [7]–[10]. It will be shown elsewhere how the procedure of reformulating the path integral on the extended triplectic manifold fits into the general multilevel scheme obtained by gradually extending the phase space by new variables $\lambda^{(n)}_{\alpha a}$, $\lambda^{(n)}_{(n+1)\alpha}$ and $\lambda^{(n)}_{(n+1)\alpha}$. Several interesting questions have to do with the vector fields $V^\alpha$ that we have used to formulate the master equations. One may e.g. ask if a ‘reduction’ (at least formal) is possible to the anticanonical case by simply removing the superscripts on $\Delta_\pm$ (2.28), $V$ and the antibrackets and dropping the terms with the $\epsilon$-symbols. This would result in the master equation of the form \[ \frac{1}{2}(X, X) - V X = i\hbar \Delta X, \] with $V$ being an odd, nilpotent, operator that commutes with $\Delta$ and differentiates the antibracket. Such an equation was introduced recently in [27] in a particular setting of string field theory, in the form of the boundary operator (differential) $\partial$. It seems very interesting to investigate the possibility of reformulating the general Lagrangian quantization in the case when there is room for such a $\partial$ to appear (e.g. when the antibracket degenerates).

There are several other interesting questions. Recall that an action with quadratic dependence on the ‘Lagrange multipliers’ $\lambda$ is related to non-singular gauges. In the traditional gauge theories, there exists a mechanism (a canonical transformation [29]) allowing one to derive these non-singular gauges from singular ones (imposed by delta-functions in the integral). It would be very interesting to understand higher orders in $\lambda^\alpha$ in the triplectic case along these lines. Another direction is a possible triplectic generalization of the algebraic scheme [1, 34, 35, 36] underlying antisymplectic Lagrangian quantization.

Acknowledgements

IAB and AMS are thankful to Lars Brink for warm hospitality at the Institute of Theoretical Physics, Chalmers University. The work of IAB and AMS is partially supported by Human Capital and Mobility program of the European Community under the Project INTAS-93-2058. AMS wishes to thank G. Ferretti and I. Tyutin for useful comments. IAB is also grateful to I. Tyutin for stimulating discussions at early stages of this work. IAB was supported in part also by Board of Trustees of ICASS and NATO Linkage Grant #931717.
A Poisson bracket from the antibrackets

We are going to show that, if one defines a \( \{ \cdot, \cdot \} \) by
\[
(F, V^a G)^b = \frac{1}{2} \varepsilon^{abc} \{F, G\},
\]
this would be a Poisson bracket on functions such that \( (F, G)^a = 0 \) and \( (F, V^a G)^b = 0 \). This will mean that one can consistently relate the LHS of (2.50) to the fixed Poisson bracket on the symplectic submanifold \( \mathcal{L}_1 \) introduced in section 2.4.

As mentioned in subsection 2.4, it follows from (2.39) that
\[
(F, V^a G)^b = -(-1)^{\varepsilon(F)\varepsilon(G)}(G, V^a F)^b \quad \text{for} \quad F, G \in \mathcal{F}(\mathcal{L}_1),
\]
which gives the correct symmetry properties. Next, let us show that the derivation property holds:
\[
\{F, GH\} = -\varepsilon_{ab}(F, V^a(GH))^b
\]
\[
= -\varepsilon_{ab}(F, (V^a G) H)^b - \varepsilon_{ab}(-1)^{\varepsilon(G)}(F, G V^a H)^b
\]
\[
= -\varepsilon_{ab}(F, V^a G)^b H - \varepsilon_{ab}(-1)^{\varepsilon(G)+\varepsilon(F)+\varepsilon(H)+1} G (F, V^a H)^b
\]
\[
= \{F, G\} H + (-1)^{\varepsilon(F)\varepsilon(G)} G \{F, H\}.
\]

Now, we claim that the Jacobi identity for the Poisson bracket follows from the condition
\[
(F, (G, V^c H)^{(b)}d) = 0 \quad F, G, H \in \mathcal{F}(\mathcal{L}_1)
\]
which we have arrived at in (2.50). Applying \( V^a \) to (2.4), we get
\[
(F, V^a (G, V^c H)^{(b)}d)
\]
\[
= (-1)^{\varepsilon(F)} V^a (F, (G, V^c H)^{(b)}d)
\]
\[
= (-1)^{\varepsilon(F)} ((V^a F, G)^{(b)}d, V^c H)^d + (-1)^{\varepsilon(F)\varepsilon(G)} (G, (V^a F, V^c H)^{(b)}d) \quad \text{by (2.16)}
\]
\[
= (-1)^{\varepsilon(F)\varepsilon(G)} (G, (V^a F, V^c H)^{(b)}d)
\]
\[
+ (-1)^{\varepsilon(F)\varepsilon(G)} (G, V^a (F, V^c H)^{(b)}d)
\]
\[
= -\frac{1}{2} (-1)^{\varepsilon(F)\varepsilon(G)+\varepsilon(F)+\varepsilon(F)+\varepsilon(G)} (H, V^c \{G, F\})^{(d, b)a} \quad \text{by (A.4)}
\]
\[
-\frac{1}{2} (-1)^{\varepsilon(F)\varepsilon(F)} (G, V^a \{F, H\})^{(d, b)c} \quad \text{by (A.4)}
\]
\[
-\frac{1}{2} (-1)^{\varepsilon(F)\varepsilon(F)} \varepsilon^{ac} (G, (F, KH)^{(b)}d) \quad \text{by (2.25)}
\]
\[
= -\frac{1}{4} (-1)^{\varepsilon(F)\varepsilon(G)+\varepsilon(F)+\varepsilon(G)+\varepsilon(G)} \{H, \{G, F\}\} \varepsilon^{(d, b)a}
\]
\[
-\frac{1}{4} (-1)^{\varepsilon(F)\varepsilon(G)} \{G, \{F, H\}\} \varepsilon^{(d, b)c}
\]
\[
-\frac{1}{2} (-1)^{\varepsilon(F)\varepsilon(F)+\varepsilon(F)} \varepsilon^{ac} (G, (F, KH)^{(b)}d)
\]

The LHS here, on the other hand, equals
\[
-\frac{1}{4} \{F, \{G, H\}\} \varepsilon^{(d, b)c}
\]

Now the terms that involve \( \varepsilon \)-symbols and symmetrization in \( bd \) are also symmetric in \( ac \), while the last term in (A.5) is antisymmetric in \( ac \) (and therefore has to vanish). The equality among the \( ac \)-symmetric terms is then equivalent to the Jacobi identity for the Poisson bracket.
B Variations in the triplectic geometry

In contrast to their (anti)symplectic counterparts, the triplectic antibrackets do vary under the ‘canonical’ transformations (which is largely due to the fact that the ‘canonical’ transformations are not truly canonical). Consider an even vector field $T ≡ T^A \partial_A$ on our manifold $\mathcal{M}$ (antisymplectic or triplectic). It will be convenient to define the Hamiltonian mapping from functions to vector fields. In the antisymplectic case this is standard,

$$\mathcal{H} : \mathcal{F}(\mathcal{M}) \rightarrow \text{Vect}(\mathcal{M}) : H \mapsto \mathcal{H}_H$$

such that

$$\mathcal{H}_H F = (H, F)$$

whence, writing $\mathcal{H}_H = \mathcal{H}_H^A \partial_A$, we find

$$\mathcal{H}_H^A = (-1)^{\varepsilon_A \varepsilon(H) + \varepsilon(H) + \varepsilon_A E^{AB} \partial_B H}$$

where $E^{AB}$ determines the antibracket by a formula directly analogous to (2.12). In the triplectic case, we have two such vector fields,

$$\mathcal{H}^a_H F = (H, F)^a,$$

$$\mathcal{H}^{aA}_H = (-1)^{\varepsilon_A \varepsilon(H) + \varepsilon(H) + \varepsilon_A E^{ABa} \partial_B H}$$

For an odd $H$, $\mathcal{H}^a_H$ are even, and it follows that $\Delta^a H = -\frac{1}{2} \text{div} \mathcal{H}^a_H$. For $H$ even, $\mathcal{H}_H$ is odd, and we either define divergence as $\text{div} V = \rho^{-1} \partial_A (\rho V^A)$ or introduce the components (as we did in the text) by writing $V = (-1)^{\varepsilon_A V^A} \partial_A$; then, $\Delta^a H = \frac{1}{2} \text{div} \mathcal{H}^a_H$, and thus in any case

$$\Delta^a H = \frac{1}{2} (-1)^{\varepsilon(H)} \text{div} \mathcal{H}^a_H.$$  

Now, the variation of $\Delta^a$ under an infinitesimal change of variables $\Gamma^A \mapsto \Gamma^A + T^A$ is given by the Lie derivative

$$\mathcal{L}_T \Delta^a = T \circ \Delta^a - \Delta^a \circ T$$

and by the same formula with the superscripts removed in the antisymplectic case. Using (2.11) to find the corresponding variation of the antibracket, we see that $\mathcal{L}_T (\cdot, \cdot)^a$ evaluates on two functions $F, G$ as

$$T(F, G)^a - (TF, G)^a - (F, TG)^a$$

which is nothing but

$$\mathcal{L}_T E^{ABa} = T^C \partial_C E^{ABa} - T^A \partial_C E^{CBa} - E^{ACa} \partial_C T^B.$$  

Consider next the important case when the vector field $T$ is itself hamiltonian. In the antisymplectic situation this means simply that $T = \mathcal{H}_h$ for a (fermionic) function $h$. Then, $T$ differentiates the antibracket by virtue of the Jacobi identity and therefore eq. (B.7) (with the upper indices dropped) vanishes. In the triplectic case the situation is more interesting. First, a hamiltonian vector field $T = \mathcal{H}_h$ is replaced by a sum $T = \mathcal{H}_{h_1} + \mathcal{H}_{h_2}$. In this case, the formula (B.6) rewrites as

$$(\mathcal{L}_T \Delta^a)(F) = (h_b, \Delta^a F)^b - \Delta^a (h_b, F)^b$$  

...
Now we take here $h$ as in (A.1). We find that the Lie derivative along $H$ of $F$ as

$$L_T V^a = [T, V^a].$$

Then, for a ‘hamiltonian’ vector field $T = \mathcal{H}^a_h$ we find, using (2.18),

$$(L_T V^a)(F) = -(V^a h_c, F)^c.$$  \hspace{1cm} (B.12)

The formulae (3.11) and (3.10) have an interesting consequence for the Poisson bracket defined as in (A.1). We find that the Lie derivative along $\mathcal{H}_c^a$ of $(\cdot, V^a \cdot)^b$ evaluates on a pair of functions $F, G$ on $\mathcal{L}_1$ as

$$\mathcal{H}_c^a(F, V^a G)^b - (\mathcal{H}_c^a F, V^b G)^a - (F, V^b \mathcal{H}_c^a G)^a.$$  \hspace{1cm} (B.13)

Now we take here $h_c = \epsilon_{cd} V^d f$ with $f$ being a function on $\mathcal{L}_1$, so that $(F, f)^a = 0$, $(F, V^a f)^b = 1/2 \epsilon^{ab} \{F, f\}$ and similarly for $G$, which allows us to show that eq. (B.13) becomes

$$\frac{1}{2} \epsilon^{ab} \left( -\{\{F, G\}, f\} + \{\{F, f\}, G\} + \{F, \{G, f\}\} \right)$$  \hspace{1cm} (B.14)

which is the LHS of Jacobi identity for the Poisson bracket and therefore vanishes. The effect of the chosen hamiltonian vector field is therefore that of a true hamiltonian vector field on a symplectic manifold. This suggests one of the possible definitions of hamiltonian vector fields in the triplectic geometry as $\epsilon_{ab} \mathcal{H}^a_{V^b \varphi}$, which acts on functions as

$$F \mapsto \epsilon_{ab} (V^b \varphi, F)^a$$  \hspace{1cm} (B.15)

and is thus an extension to $\mathcal{M}$ of the vector fields that are hamiltonian in the Poisson structure on $\mathcal{L}_1$.

Consider next the form (6.12) of the classical part of gauge-fixing master-action. We are going to apply the above relations to derive the transformation properties of the KY term from (6.12). From (3.11), the variation of the operator $K$ (2.27) generated by the ‘hamiltonians’ $h_a$ is given by

$$(L_T K)(F) = -2 \epsilon_{ab} (V^a h_c, V^b F)^c - (K h_c, F)^c.$$  \hspace{1cm} (B.16)

Now, once a master-action $X$ is chosen, there is a set of hamiltonians $h_a$ depending on one arbitrary function as in (5.29). The transformation generated by such $h_a$ is related by (5.30) to the automorphism transformations of solutions to the master equation. More precisely, since KY is the $\lambda$-independent part of the master-action $X$, we pick out $\lambda$-independent terms in (5.30). Thus we arrive at the hamiltonians

$$h_c = -\epsilon_{cd} (KY, \varphi)^d + 2 \epsilon_{cd} V^d \varphi$$  \hspace{1cm} (B.17)

Remarkably, for $F = KY$, the last term in (5.30) drops out due to (2.26), and we are left with a purely ‘hamiltonian’ piece!

Now we can evaluate (3.16) with the above $h_a$: for $\delta K \equiv L_T K$, we find

$$(\delta K)(F) = -\epsilon_{ab} ((KY, \varphi)^b, KF)^a - 2 \epsilon_{ab} K (\varphi, V^b F)^a + 2 \epsilon_{ab} V^b (\varphi, KF)^a + \epsilon_{ab} K ((KY, \varphi)^b, F)^a.$$  \hspace{1cm} (B.18)
where we have used (2.18) and the projector property (2.26) to pull some of the $V^a$ out of the antibrackets.

Returning now to (6.12), we extract from the automorphism transformations from section 5 the variation of $KY$. As in (5.27), (5.28), we will consider for simplicity only the ‘main’ part of the variation, given by the antibracket and $V^a$. Thus, the automorphism formula tells us that the $\lambda$-independent terms in (6.12) vary as

$$\delta(KY) = -\epsilon_{ab}((KY, \varphi)^b_Y, KY)^a - 2\epsilon_{ab}V^a(\varphi, KY)^b + K\varphi$$

(B.19)

Since the automorphism can be implemented by a coordinate transformation, and as long as the coordinate transformations do not leave $K$ invariant, we have to transform $K$ to the new coordinate system. Subtracting from the last expression the effect on $K$ of the coordinate transformation (B.18) with $F = Y$, we find

$$\delta(KY) - (\delta K)(Y) = K\varphi + 2\epsilon_{ab}K(\varphi, V^bY)^a - \epsilon_{ab}K((KY, \varphi)^b, Y)^a$$

(B.20)

which is of the desired form $K(\ldots)$. Moreover, the variation of $Y$ that follows from this formula can be written as

$$\delta Y = \varphi + \epsilon_{ab}(Y, (KY, \varphi)^b)^a + 2\epsilon_{ab}(Y, V^a\varphi)^b$$

(B.21)

where the two last terms are, again, the effect of the hamiltonian transformation generated by (5.29), that is, the $\lambda$-independent part of the coordinate transformation (5.29) (we have used the projector property (2.26)). The remaining term $\varphi$ is a form variation of $Y$. This establishes the ‘covariance’ properties and hence consistency of the ansatz (6.12) for the master-action $X$ that leads to singular (delta-function) gauges.
References

[1] I.A. Batalin and G.A. Vilkovisky, Phys. Lett. B102 (1981) 27.
[2] I.A. Batalin and G.A. Vilkovisky, Phys. Rev. D28 (1983) 2567.
[3] M. Henneaux, Nucl. Phys. B (Proc. Suppl.) 18A (1990) 47.
[4] H. Hata and B. Zwiebach, Ann. Phys. 229 (1994) 177.
[5] A. Sen and B. Zwiebach, Phys. Lett. B320 (1994) 29.
[6] A.S. Schwarz, Commun. Math. Phys. 155 (1993) 249.
[7] I.A. Batalin and I.V. Tyutin, Int. J. Mod. Phys. A8 (1993) 2333.
[8] I.A. Batalin and I.V. Tyutin, Mod. Phys. Lett. A8 (1993) 3673.
[9] I.A. Batalin and I.V. Tyutin, Mod. Phys. Lett. A9 (1994) 1707.
[10] I.A. Batalin and I.V. Tyutin, Generalized field-antifield formalism, to be published in the Memorial Volume to Prof. F.A. Berezin.
[11] I.A. Batalin, P.M. Lavrov, and I.V. Tyutin, J. Math. Phys. 31 (1990) 1487.
[12] I.A. Batalin, P.M. Lavrov, and I.V. Tyutin, J. Math. Phys. 32 (1990) 532.
[13] I.A. Batalin, P.M. Lavrov, and I.V. Tyutin, J. Math. Phys. 32 (1990) 2513.
[14] I.A. Batalin and R. Marnelius, Completely anticanonical form of Sp(2)-symmetric Lagrangian quantization, ITP-Göteborg report 94-30, hep-th/950230.
[15] M. Henneaux, Phys. Lett. B282, (1992) 372.
[16] G. Barnich, R. Constantinescu and P. Grégoire, Phys. Lett. B293 (1992) 353.
[17] P. Grégoire and M. Henneaux, J. Phys. A26 (1993) 6073.
[18] F. Magri, J. Math. Phys. 19 (1978) 1156.
[19] I.M. Gelfand and I.Ya. Dorfman, Funk. Analiz i Prilozh., 14 (1980) 77.
[20] A.G. Reyman and M.A. Semenov–Tian-Shansky, Phys. Lett. A130 (1988) 456.
[21] P.H. Damgaard and F. De Jonghe, Phys. Lett. B305 (1993) 59.
[22] L. Tătaru, A cohomological approach to the Batalin–Lavrov–Tyutin covariant quantization of irreducible hypergauge theory, preprint TUW 94-18.
[23] I.A. Batalin, P.M. Lavrov, and I.V. Tyutin, J. Math. Phys. 31 (1990) 6.
[24] I.A. Batalin, P.M. Lavrov, and I.V. Tyutin, J. Math. Phys. 31 (1990) 2708.
[25] P. Grégoire and M. Henneaux, Phys. Lett. B277 (1993) 372.
[26] P. Grégoire and M. Henneaux, Commun. Math. Phys. 157 (1993) 279.
[27] A. Sen and B. Zwiebach, Background independent algebraic structures in closed string field theory, MIT-CTP-2346 (August 1994).
[28] O.M. Khudaverdyan, private communication.
[29] I.A. Batalin and R.E. Kallosh, Nucl. Phys. B222 (1983) 139.
[30] O.M. Khudaverdyan and A.P. Nersessian, On the geometry of the Batalin-Vilkovisky formalism, UGVA-DPT 1993/03-807.
[31] S. Aoyama, Quantization of the topological sigma model and the master equation of the BV formalism, KUL-TF-93/39.
[32] I.A. Batalin and G.A. Vilkovisky, Nucl. Phys. B234 (1984) 106.
[33] B.L. Voronov and I.V. Tyutin, Teor. Mat. Fiz. 50 (1982) 333.
[34] M. Penkava and A. Schwarz, On some algebraic structures arising in string theory.
[35] B.H. Lian and G.J. Zuckerman, Commun. Math. Phys. 154 (1993) 613.
[36] E. Getzler, Commun. Math. Phys. 159 (1994) 265.