Multi-dimensional Diffeomorphism and Current Algebras from Virasoro and Kac-Moody Currents

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November 6, 2018

Abstract

The recently constructed Fock representations of \( N \)-dimensional diffeomorphism and current algebras are reformulated in terms of one-dimensional currents, satisfying Virasoro and affine Kac-Moody algebras.

1 Introduction

In a recent paper [8], I constructed Fock representations of diffeomorphism and current algebras in \( N \)-dimensional spacetime. More precisely, I considered the DGRO (Diffeomorphism, Gauge, Reparametrization, Observer) algebra \( DGRO(N, g) \), where \( g \) is a finite-dimensional Lie algebra. The Fock representations were obtained by expanding functions, valued in suitable spaces, in a multi-dimensional Taylor series around the points of a one-dimensional trajectory, prior to normal ordering. In the resulting expressions, the Fock oscillators only appear in bilinear combinations. Therefore, it suffices to find currents that satisfy the same algebraic relations as these bilinears. In the present paper I write down the relevant expressions in terms of affine Kac-Moody and Virasoro currents, and verify that they indeed satisfy \( DGRO(N, g) \). This means that new representations of the full DGRO algebra can be constructed using representations of these Kac-Moody and Virasoro algebras. In particular, I give a Sugawara construction for the reparametrization subalgebra.

The construction works for every finite jet order \( p \) (the order at which the Taylor expansion is truncated), but the abelian charges, i.e. the parameters multiplying the cocycles, diverge when \( p \to \infty \); the worst parameters behave like \( p^{N+2} \). An important problem is to construct factor modules such that the limit \( p \to \infty \) exists, at least for \( N \) sufficiently small and for some choice of \( g \). A first step in this direction is taken in Section [6]. In the direct sum of realizations with jet order ranging from \( p - r \) to \( p \), the leading divergences can be made to cancel. Then the abelian charges are independent of \( p \) for \( N = r \) and they vanish for \( N < r \).
The higher-dimensional analogs of Virasoro and loop algebras have not attracted much attention compared to their one-dimensional siblings. One reason may be that Pressley and Segal, in their influential book *Loop groups* [12], noted that the higher-dimensional Kac-Moody-like cocycle follows by pull-back from the one-dimensional case. Equivalently, the extension of map$(N,g)$, which is the algebra of $g$-valued functions, is obtained by restriction of these functions to some privileged one-dimensional curve ("the observer’s trajectory"). However, I believe that their claim is somewhat misleading, because it does not imply that all modules are inherited from $\hat{g}$. The realizations constructed in the present paper depend on more data: not only do they involve functions on the preferred curve, but also derivatives of these functions up to some fixed finite order $p$. Since this includes transverse derivatives, it is a truly higher-dimensional effect. Moreover, these realizations are expressed in terms of Kac-Moody currents, but the relevant algebra is not $\hat{g}$, except as a special case.

For diffeomorphisms the difference between $N = 1$ and $N > 1$ dimensions is even more dramatic. The observer’s trajectory is not preserved by diffeomorphisms, and thus the higher-dimensional Virasoro extensions are not central; in fact, the current algebra cocycle does not commute with diffeomorphisms either. Note also that the classical $diff(N)$ modules (tensor densities) depend crucially on the dimension.

Related work can be found in [1, 2, 7, 8, 9, 11, 13, 14]. Cocycles of the diffeomorphism algebra were classified by Dzumadil’daev [3] and reviewed in [10].

2 Background

2.1 DGRO Algebra

Let $\xi = \xi^\mu(x)\partial_\mu$, $x \in \mathbb{R}^N$, $\partial_\mu = \partial/\partial x^\mu$, be a vector field, with commutator $[\xi, \eta] \equiv \xi^\mu\partial_\mu\eta^\nu - \eta^\mu\partial_\mu\xi^\nu$. Greek indices $\mu, \nu = 1, 2, .., N$ label the space-time coordinates and the summation convention is used on all kinds of indices.

The diffeomorphism algebra (algebra of vector fields, Witt algebra) $diff(N)$ is generated by Lie derivatives $L_\xi$. In particular, we refer to diffeomorphisms on the circle as reparametrizations. They form an additional $diff(1)$ algebra with generators $L_f$, where $f = f(t)d/dt$, $t \in S^1$, is a vector field on the circle.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with basis $J^a$ (hermitian if $\mathfrak{g}$ is compact and semisimple), structure constants $f^{ab}_c$, and Killing metric $\delta^{ab}$. The brackets are

$$[J^a, J^b] = i f^{ab}_c J^c. \quad (2.1)$$

Let $\delta^a \propto \text{tr } J^a$ be proportional to the linear Casimir operator, satisfying $f^{ab}_c \delta^c \equiv 0$. Clearly, $\delta^a = 0$ if $\mathfrak{g}$ is semisimple, but it may be non-zero on abelian factors. Let map$(N,\mathfrak{g})$ denote the gauge algebra of maps from $N$-dimensional spacetime to $\mathfrak{g}$. It is the algebra of $\mathfrak{g}$-valued functions $X = X_a(x)J^a$ with commutator
Here \( J \) denotes its generators by \( \mathcal{J}_X \). The action of \( \text{diff}(N) \) on \( \text{map}(N, g) \) is given by \( \xi X = \xi^\mu \partial_\mu X^\alpha J^\alpha \).

Finally, let the \( \text{Obs}(N) \) be the space of local functionals of the observer’s trajectory \( q^\mu(t) \), i.e. polynomial functions of \( q^\mu(t), \dot{q}^\mu(t), ... \), \( \partial^k q^\mu(t)/\partial t^k \), \( k \) finite, regarded as a commutative algebra. \( \text{Obs}(N) \) is a \( \text{diff}(N) \) module in a natural manner.

The DGRO algebra \( \text{DGRO}(N, g) \) is an abelian but non-central Lie algebra extension of \( \text{diff}(N) \ltimes \text{map}(N, g) \oplus \text{diff}(1) \) by \( \text{Obs}(N) \):

\[
0 \rightarrow \text{Obs}(N) \rightarrow \text{DGRO}(N, g) \rightarrow \text{diff}(N) \ltimes \text{map}(N, g) \oplus \text{diff}(1) \rightarrow 0.
\]

The extension depends on the eight parameters \( c_j, j = 1, ..., 8 \), to be called abelian charges. The brackets are given by

\[
\begin{align*}
[\mathcal{L}_\xi, \mathcal{L}_\eta] &= \mathcal{L}_{[\xi, \eta]} + \frac{1}{2\pi i} \int dt \left( c_1 \partial_\rho \partial_\nu \xi^\mu(q(t)) \partial_\nu \eta^\rho(q(t)) + 
+ c_2 \partial_\rho \partial_\nu \xi^\mu(q(t)) \partial_\nu \eta^\rho(q(t)) \right), \\
[\mathcal{L}_\xi, \mathcal{J}_X] &= \mathcal{J}_{[\xi, X]} - \frac{c_7}{2\pi i} s^a \int dt \left( \dot{q}^a(t) X_\alpha(q(t)) \partial_\alpha \partial_\mu \xi^\mu(q(t)) \right), \\
[\mathcal{J}_X, \mathcal{J}_Y] &= \mathcal{J}_{[X,Y]} - \frac{c_5}{2\pi i} \left( c_5 \delta^{ab} + c_8 \delta^a \delta^b \right) \int dt \left( \dot{q}^a(t) \partial_\nu X_\alpha(q(t)) Y_\nu(q(t)) \right), \\
[L_f, \mathcal{L}_\xi] &= \frac{c_3}{4\pi i} \int dt \left( \dot{f}(t) - i\dot{f}(t) \right) \partial_\alpha \xi^\alpha(q(t)), \\
[L_f, \mathcal{J}_X] &= \frac{c_6}{4\pi i} \delta^a \int dt \left( \dot{f}(t) - i\dot{f}(t) \right) X_\alpha(q(t)), \\
[L_f, L_g] &= L_{[f, g]} + \frac{c_4}{24\pi i} \int dt \left( \dot{f}(t) \dot{g}(t) - \dot{f}(t) \dot{g}(t) \right), \\
[\mathcal{L}_\xi, q^\mu(t)] &= \xi^\mu(q(t)), \\
[L_f, q^\nu(t)] &= -f(t) \dot{q}^\nu(t), \\
[\mathcal{J}_X, q^\mu(t)] &= [q^\mu(s), q^\nu(t)] = 0,
\end{align*}
\]

extended to all of \( \text{Obs}(N) \) by Leibniz’ rule and linearity.

### 2.2 Tensor fields and Fock modules

In \( \mathbf{8} \) Fock representations of \( \text{DGRO}(N, g) \) were constructed. We started from classical fields transforming as

\[
\begin{align*}
[\mathcal{L}_\xi, \phi(x, t)] &= -\xi^\mu(x) \partial_\mu \phi(x, t) - \partial_\nu \xi^\mu(x, t) T^\nu_\mu \phi(x, t), \\
[\mathcal{J}_X, \phi(x, t)] &= -X_\alpha(x) J^\nu \phi(x, t), \\
[L_f, \phi(x, t)] &= -f(t) \partial_\nu \phi(x, t) - \lambda \dot{f}(t) \phi(x, t) + iwf \phi(x, t),
\end{align*}
\]

Here \( J^\nu \) satisfies \( g(2.1) \), and \( T^\mu_\nu \) satisfies \( gl(N) \), with brackets

\[
[T^\mu_\nu, T^\rho_\sigma] = \delta^\rho_\sigma T^\mu_\nu - \delta^\mu_\nu T^\rho_\sigma,
\]

\( X^\alpha = \xi^\mu \partial_\mu X^\alpha J^\alpha \).
and \( \phi(x, t) \) takes values in some \( g \oplus gl(N) \) module. Let \( \mathbf{m} = (m_1, m_2, ..., m_N) \), all \( m_\mu \geq 0 \), be a multi-index of length \( |\mathbf{m}| = \sum_{\mu=1}^{N} m_\mu \). Denote by \( \mu \) a unit vector in the \( \mu^{th} \) direction, so that \( \mathbf{m} + \mu = (m_1, ..., m_\mu + 1, ..., m_N) \), and let

\[
\phi_{,\mathbf{m}}(t) = \partial_\mathbf{m} \phi(q(t), t) = \frac{\partial_{m_1} \cdots \partial_{m_N} \phi(q(t), t)}{m_1! \cdots m_N!}
\]

be the \( |\mathbf{m}|^{th} \) order derivative of \( \phi(x, t) \) on the observer’s trajectory \( q^\mu(t) \). Such objects transform as

\[
\begin{align*}
[\mathcal{L}_\xi, \phi_{,\mathbf{m}}(t)] &= \partial_\mathbf{m} ([\mathcal{L}_\xi, \phi(q(t), t)]) + [\mathcal{L}_\xi, \phi_{,\mathbf{m}}(t)] \partial_\mu \partial_\mathbf{m} \phi(q(t), t) \\
&= - \sum_n T^n_m (\xi(q(t))) \phi_n(t), \\
[\mathcal{J}_X, \phi_{,\mathbf{m}}(t)] &= \partial_\mathbf{m} ([\mathcal{J}_X, \phi(q(t), t)]) \\
&= - \sum_n J^n_m (X(q(t))) \phi_n(t), \\
[L_f, \phi_{,\mathbf{m}}(t)] &= - f(t) \dot{\phi}_{,\mathbf{m}}(t) - \lambda \dot{f}(t) \phi_{,\mathbf{m}}(t) + iw f \phi_{,\mathbf{m}}(t),
\end{align*}
\]

where

\[
\begin{align*}
T^n_m (\xi) &= \binom{n}{m} \partial_{n-m+\nu} \xi^\nu T^\nu_m \\
&\quad + \binom{n}{m-\mu} \partial_{n-m+\mu} \xi^\mu - \delta^n_m \xi^\mu, \\
J^n_m (X) &= \binom{n}{m} \partial_{n-m} X_a J^a_m.
\end{align*}
\]

Here \( m! = m_1! m_2! \cdots m_N! \) and

\[
\binom{m}{n} = \frac{m!}{n!(m-n)!} = \binom{m_1}{n_1} \binom{m_2}{n_2} \cdots \binom{m_N}{n_N}.
\]

Some other properties of multi-dimensional binomial coefficients are listed in appendix [A].

Here and henceforth we use the convention that a sum over a multi-index runs over all values of length at most \( p \). Since \( T^n_m (\xi) \) and \( J^n_m (X) \) vanish whenever \( |n| > |m| \), the sums over \( n \) in (2.6) are in fact further restricted.

Add dual coordinates (jet momenta) \( (p_\mu(t), \pi^\mathbf{m}(t)) \), which satisfy

\[
\begin{align*}
[p_\mu(s), q^\nu(t)] &= \delta^\nu_\mu \delta(s-t), \\
\{\pi^\mathbf{m}(s), \phi_{,\mathbf{n}}(t)\} &= - \{\phi_{,\mathbf{n}}(t), \pi^\mathbf{m}(s)\} = \delta^\mathbf{m}_{\mathbf{n}} \delta(s-t),
\end{align*}
\]

and all other brackets vanish. For definiteness, we take the fields to be fermionic, as indicated by the curly brackets. Then it follows immediately from (2.4) that
the following operators define a realization of $DGRO(N, g)$ in Fock space:

$$\mathcal{L}_\xi = \int dt \left\{ :\xi^\mu(q(t))p_\mu(t): + \sum_{m,n} :\pi^m(t)T^n_{\mu}(\xi(q(t)))\phi_n(t): \right\},$$

$$J_X = \int dt \sum_{m,n} :\pi^m(t)J^m_n(X(q(t)))\phi_n(t); ,$$

$$L_f = \int dt \left\{ -f(t) :\dot{q}^\mu(t)p_\mu(t): + \sum_{m} :\pi^m(t)(f(t)\dot{\phi}_m(t) + \lambda f(t)\phi_m(t) - iw\phi_{m}) : \right\},$$

where double dots ($: :$) denote normal ordering with respect to frequency.

### 3 Virasoro / Kac-Moody Realization

By means of (2.7), we can rewrite (2.10) as

$$\mathcal{L}_\xi = \int dt \left\{ :\xi^\mu(q(t))p_\mu(t): - \xi^\mu(q(t))P_\mu(t) + \sum_{m,n} \left( \begin{array}{c} m \\ n \end{array} \right) \partial_{m-n}\xi^\mu(q(t))E^m_{n+\mu}(t) \right\} + T_{d\xi},$$

$$T_{d\xi} = \int dt \sum_{m,n} \left( \begin{array}{c} m \\ n \end{array} \right) \partial_{m-n+\nu}\xi^\mu(q(t))T^m_{n\nu}(t),$$

$$J_X = \int dt \sum_{m,n} \left( \begin{array}{c} m \\ n \end{array} \right) \partial_{m-n}\phi_n(t)J^m_n(t),$$

$$L_f = \int dt f(t)L(t),$$

where

$$P_\mu(t) = \sum_{m} E^m_{m+\mu}(t),$$

$$L(t) = - :\dot{q}^\mu(t)p_\mu(t): + F(t),$$

and

$$E^m_{n}(t) = :\pi^m(t)\phi_n(t): ,$$

$$J^m_n(t) = :\pi^m(t)J^\mu_n(\phi_n(t): ,$$

$$T^m_{n\mu}(t) = :\pi^m(t)T^\mu_n\phi_n(t): ,$$

$$F(t) = \sum_{m} :\pi^m(t)\dot{\phi}_m(t): .$$
The currents \( E^m_n(t) \) satisfy the Kac-Moody algebra \( \hat{gl}(\binom{N+p}{N}) \) with brackets

\[
[E^m_n(s), E^p_r(t)] = (\delta_n^r E^m_n(s) - \delta_s^m E^r_n(s))\delta(s-t) + \frac{k^{00}}{2\pi i} \delta^m_s \delta^n_r \delta(s-t).
\] (3.4)

The notation \( k^{00} \) for the central charge will be explained shortly. In particular,

\[
[P_\mu(s), P_\nu(t)] = 0,
\]

\[
[P_\mu(s), E^m_n(t)] = (E^{m-\mu}_n(s) - E^m_{n+\mu}(s))\delta(s-t) + \frac{k^{00}}{2\pi i} \delta^m_n \delta(s-t),
\] (3.5)

However, the currents \( J^m_n(t) \) and \( T^m_n(t) \) do not satisfy such a simple algebra. They span a vector space which is isomorphic to \( \mathfrak{g} \oplus gl(\binom{N+p}{N}) \) and \( gl(N) \oplus gl(\binom{N+p}{N}) \), respectively, but these spaces are not preserved by the bracket. Indeed,

\[
[J^m_n, J^p_s] = \frac{1}{2} \delta^s_n i f^{ab}_{rc} J^m_n J^p_s + J^{m(ab)}_n J^{r(ab)}_s,
\]

\[ J^{m(ab)}_n = \pi^m (J^a J^b + J^b J^a) \phi_n \]

contains the symmetric expression \( J^a J^b + J^b J^a \), belonging to \( \mathcal{U}(\mathfrak{g}) \), the universal enveloping algebra of \( \mathfrak{g} \). \( \mathcal{U}(\mathfrak{g}) \) has basis \( I^A \), where \( A = (), (a), (a_1 a_2), \ldots, (a_1 a_2 \ldots a_n), \ldots \) consists of \( n \)-tuples of \( \mathfrak{g} \) indices. These tuples are completely symmetric; any anti-symmetry can always be expressed in terms of lower-order tuples by means of (2.1). A typical element in \( \mathcal{U}(\mathfrak{g}) \) thus has the form \( \phi^{(a_1 a_2 \ldots a_n)} = J^{(a_1} J^{a_2} \ldots J^{a_n)} \), where parentheses denote symmetrization. We identify \( \mathfrak{g} \) with the one-tuples in \( \mathcal{U}(\mathfrak{g}) \), \( J^a = I^{(a)} \), and denote the empty set by \( () = \emptyset \). By construction, \( \mathcal{U}(\mathfrak{g}) \) is an associative algebra, and we denote its structure constants by \( g^{AB}_C \):

\[
I^A I^B = g^{AB}_C I^C.
\] (3.7)

E.g., \( g^{(a)(b)}_{(c)} = -g^{(b)(a)}_{(c)} = \frac{1}{2} i f^{ab}_{rc} \). For brevity, we shall not parenthesize single \( \mathfrak{g} \) indices henceforth.

Define \( \mathcal{U}(\mathfrak{g}, N, p) \) as the Lie algebra with basis \( I^{mA}_n \), where \( A \) is a \( \mathcal{U}(\mathfrak{g}) \) index and \( m, n \) are multi-indices of length at most \( p \). The brackets read

\[
[I^{mA}_n, J^B_s] = \delta^s_n g^{AB}_C J^{mC}_n - \delta^m_s g^{BA}_C I^{mC}_n.
\] (3.8)

That this is a Lie algebra follows from the explicit realization \( I^{mA}_n = \pi^m I^A \phi_n \).

Note that \( \mathcal{U}(\mathfrak{g}, N, p) \) is a highly reducible Lie algebra. It contains \( \binom{N+p}{N} \) different subalgebras isomorphic to \( \mathfrak{g} \), generated by \( I^{m(a)}_n \) (no sum on \( m \)), as well as a \( gl(\binom{N+p}{N}) \) subalgebra generated by \( I^{m0}_n \).

\( \mathcal{U}(\mathfrak{g}) \) has the universal property that every \( \mathfrak{g} \) representation \( M \) is given by a homomorphism \( \mathcal{U}(\mathfrak{g}) \to M \). We can therefore reinterpret \( A, B \) as \( M \) indices,

\(^1\)To avoid unduly wide hats, I put the Kac-Moody hat only over the algebra’s given name.
view $I^A$ as a basis for $M$, and let $g^{AB}_C$ be the structure constants for the associative product in $M$. The brackets (5,5) still define a Lie algebra, which we denote by $\mathcal{U}_M(g, N, p)$. Contrary to $\mathcal{U}(g, N, p)$, this algebra is finite-dimensional provided that $M$ is a finite-dimensional representation of $g$.

The corresponding Kac-Moody algebras are $\hat{\mathcal{U}}(g, N, p)$ and $\hat{\mathcal{U}}_M(g, N, p)$. A basis is given by $I_n^m(t), t \in S^1$, and the brackets read

$$[I_n^m(s), I_n^m(t)] = \delta_n^s \delta^A_C I_s^m C I_n^m(s) \delta(s - t) - \delta_n^s g^{BA}_C I_n^C(s) \delta(s - t) + \frac{k^{AB}}{2\pi i} \delta_n^s \delta(s - t).$$

(3.9)

If $\mathcal{U}(g, N, p)$ were semi-simple, the central charge matrix $k^{AB}$ would be proportional to $\delta^{AB}$, but this needs not be true in the presence of a linear Casimir.

Of particular interest is the case that $g = gl(N)$, due to the appearence of $gl(N)$ representations in $diff(N)$ representations (tensor fields). This is obtained from the general case by substituting $J^a \mapsto T^\mu_a$, so a $gl(N)$ basis is labelled by duplets $\mu^a$. More generally, we need the direct sum $g \oplus gl(N)$, with basis $J^a$ and $T^\mu_a$. The $g \oplus gl(N)$ basis is thus labelled by $\mu^a$. Now reinterpret the $\hat{\mathcal{U}}(g)$ labels $A$ as $\hat{\mathcal{U}}(g \oplus gl(N))$ labels, so a typical element in $\hat{\mathcal{U}}(g \oplus gl(N), N, p)$ has the form $I_n^m = I_n^{m(\mu_1, \ldots, \mu_k)(a_1, \ldots, a_l)}$. It is labelled by multi-indices $m, n$, symmetric tuples of $g$ indices $(a_1, \ldots, a_l)$ and tuples of $gl(N)$ indices $(\mu_1, \ldots, \mu_k)$, symmetric under simultaneous interchange of pairs $(\mu_i, \nu_j) \leftrightarrow (\mu_j, \nu_i)$. The three first operators in (3.3) are embedded into the corresponding Kac-Moody algebra (5,5) as follows:

$$E_n^m = I_n^{m()} = I_n^m,$$

$$J_m^n = I_n^{m(a)} = I_{n(a)}^m,$$

$$T_{\nu}^{\mu} = I_n^{m(\nu)} = I_{n(\nu)}^m.$$  

(3.10)

To reduce writing, we introduce an abbreviated notation where a label $M = m^A_n$ (a latin capital from the middle of the alphabet) stands for both $gl(N)$ multi-indices and $\hat{\mathcal{U}}(g \oplus gl(N), N, p)$ indices. The bracket in $\hat{\mathcal{U}}(g \oplus gl(N), N, p)$ can now be written as

$$[I^M, I^N] = i f^{MN}_R I^R,$$

(3.11)

and the corresponding Kac-Moody algebra $\hat{\mathcal{U}}(g \oplus gl(N), N, p)$ reads

$$[I^M(s), I^N(t)] = i f^{MN}_R I^R(s) \delta(s - t) + \frac{k^{MN}}{2\pi i} \delta(s - t).$$

(3.12)

Skew-symmetry and the Jacobi identities imply the relations

$$k^{NM} = k^{MN},$$

$$f^{MN} S k^{SR} = f^{NR} S k^{SM} = f^{RM} S k^{SN}.$$  

(3.13)
The structure constants are then given by

\[ i_{m}{}^{A} {}_{n}{}^{B} {}_{s}{}^{C} = \delta_{n}{}^{m} \delta_{s}^{u} \delta_{u}^{g} g^{AB} C - \delta_{s}^{m} \delta_{u}^{n} g^{BA} C, \]

\[ k_{m}{}^{A} {}_{n}{}^{B} = \delta_{n}^{m} \delta_{s}^{k} k^{AB}. \]  

(3.14)

The conditions (3.13) take the form

\[ k^{AB} = k^{BA}, \]

\[ g^{AB} D k^{DC} = g^{BC} D k^{DA} = g^{CA} D k^{DB}. \]  

(3.15)

We are mainly interested in what restrictions these conditions impose on the first few elements of the central charge matrix. For the sake of this argument, we temporarily ignore the \( gl(N) \) factor and only consider \( g \). Specialization to \( A = (a), B = C = \emptyset \) gives \( k^{a \emptyset} = k^{\emptyset a} \), which is already clear. The case \( A = (a), B = (b), C = \emptyset \) is more interesting:

\[ k^{ab} = k^{ba} = g_{b}^{a} k_{\emptyset}^{\emptyset} + g_{a}^{b} k_{c}^{c} + g_{(cd)}^{a} k^{(cd)\emptyset}. \]  

(3.16)

The symmetric part gives a condition on \( k^{(ab)\emptyset} \):

\[ k^{(ab)\emptyset} = k^{ab} + k^{ba} - (g_{b}^{a} + g_{a}^{b}) k_{\emptyset}^{\emptyset}, \]  

(3.17)

which we do not need here. The skew part implies that \( f_{c}^{ab} k^{c\emptyset} = 0 \), i.e. that \( k^{c\emptyset} \propto \delta^{c} \), the linear Casimir. Finally, the case \( A = (a), B = (b), C = (c) \) gives

\[ g_{b}^{a} k_{c}^{Dc} = g_{c}^{b} k_{Dc}^{Da} = g_{ca}^{D} k^{Db}. \]  

(3.18)

The even part of this condition yields

\[ (g_{b}^{a} + g_{a}^{b}) k_{Dc}^{\emptyset} + k^{(ab)c} = (g_{b}^{c} + g_{c}^{b}) k^{\emptyset a} + k^{(bc)a}, \]  

(3.19)

which constrains \( k^{(ab)c} \). The odd part leads to the usual Kac-Moody condition \( f_{d}^{bc} k^{da} = 0 \). One solution is the usual one proportional to the quadratic Casimir: \( k^{ab} = k^{\emptyset a} \), but there is also another solution due to the presence of a linear Casimir: \( k^{ab} \propto \delta^{a} \delta^{b} \).

We do not need the full central charge matrix to determine the abelian charges of the DGRO algebra, but only the components generated from the operators (3.10). The conditions we just studied imply that these components must be of the form

\[ k^{\mu \nu} = k_{1} \delta^{\mu}_{\nu} \delta^{a}_{c} + k_{2} \delta^{\mu}_{\nu} \delta^{b}_{c}, \]

\[ k^{\mu \emptyset} = k_{3} \delta^{\mu}_{\nu}, \]

\[ k^{\emptyset a} = k_{4}, \]

\[ k^{ab} = k_{5} \delta^{ab} + k_{6} \delta^{a} \delta^{b}, \]

\[ k^{a \emptyset} = k_{7} \delta^{a}, \]

\[ k^{\mu a} = k_{7} \delta^{\mu}_{\nu} \delta^{a}. \]  

(3.20)
Equation (3.20) defines eight independent parameters \( k_1 - k_8 \), which characterize the Kac-Moody algebra \( \hat{U}(\mathfrak{g} \oplus \mathfrak{gl}(N), N, p) \), at least in part. Its central charge matrix may have additional independent components, but if so they arise for higher-order operators, which do not affect the abelian charges of the DGRO algebra.

The realization (3.1) also depends on a fourth operator \( F(t) \), which generates the Virasoro algebra \( Vir \) commuting with the observer’s trajectory:

\[
[F(s), F(t)] = (F(s) + F(t))\delta(s - t) + \frac{c(N, p)}{24\pi i} (\delta'(s - t) + \delta(s - t)).
\]

The generators of \( \hat{U}(\mathfrak{g} \oplus \mathfrak{gl}(N), N, p) \) transform as weight one primary fields:

\[
[F(s), I^M(t)] = I^M(s)\delta(s - t) + \frac{d^M}{4\pi i} (\ddot{\delta}(s - t) + i\dot{\delta}(s - t)).
\]

The \( L_{II} \) Jacobi identity leads to the condition \( f^{MN R} d_R = 0 \), which means that the parameters \( d^M \) are proportional to the linear Casimir. In particular,

\[
\begin{align*}
d^0 &= d_0, \\
d^\nu &= d_1^\nu, \\
d^a &= d_2^a,
\end{align*}
\]

which defines three additional parameters \( d_0 - d_2 \). In particular, for the operators in (3.10), (3.23) takes the form

\[
\begin{align*}
[F(s), E^m_n(t)] &= E^m_n(s)\delta(s - t) + \frac{d_0}{4\pi i} \delta^m_n(\ddot{\delta}(s - t) + i\dot{\delta}(s - t)), \\
[F(s), J^m_a(t)] &= J^m_a(s)\delta(s - t) + \frac{d_2}{4\pi i} \delta^a \delta^m_n(\ddot{\delta}(s - t) + i\dot{\delta}(s - t)), \\
[F(s), T^{m\nu}_{n\mu}(t)] &= T^{m\nu}_{n\mu}(s)\delta(s - t) + \frac{d_1}{4\pi i} \delta^\nu \delta^m_n(\ddot{\delta}(s - t) + i\dot{\delta}(s - t)).
\end{align*}
\]

We have thus rewritten the Fock realization in terms of currents (3.3), satisfying (3.12), (3.21) and (3.23). In this process, all explicit reference to the Fock operators \( \pi^m(t) \) and \( \phi^m_n(t) \) has vanished. The result is summarized in the following theorem.

**Theorem 3.1** Embed \( E^m_n(t) \), \( J^m_a(t) \), \( T^{m\nu}_{n\mu}(t) \) and \( F(t) \) into \( Vir \times \hat{U}(\mathfrak{g} \oplus \mathfrak{gl}(N), N, p) \), with brackets given by (3.12), (3.21) and (3.23), by means for (3.10). Then the generators (3.1) satisfy the DGRO algebra \( DGRO(N, \mathfrak{g}) \) (2.4). The abelian
charges are given by

\[
c_1 = 1 - k_1 \left( \frac{N+p}{N} \right) - k_4 \left( \frac{N+p+1}{N+2} \right),
\]
\[
c_2 = -k_2 \left( \frac{N+p}{N} \right) - 2k_3 \left( \frac{N+p}{N+1} \right) - k_4 \left( \frac{N+p}{N+2} \right),
\]
\[
c_3 = 1 + d_1 \left( \frac{N+p}{N} \right) + d_0 \left( \frac{N+p}{N+1} \right),
\]
\[
c_4 = 2N + c(N,p),
\]
\[
c_5 = k_5 \left( \frac{N+p}{N} \right),
\]
\[
c_6 = d_2 \left( \frac{N+p}{N} \right),
\]
\[
c_7 = k_7 \left( \frac{N+p}{N} \right) + k_6 \left( \frac{N+p}{N+1} \right),
\]
\[
c_8 = k_8 \left( \frac{N+p}{N} \right),
\]

where the parameters \( k_1 - k_8 \) are defined in (3.20) and \( d_0 - d_2 \) are defined in (3.24).

**Proof.** It is clear that we have some realization of \( \text{DGRO}(N,g) \), because the Fock operators only enter through normal-ordered bilinear combinations. Appendix B contains an independent verification of all representation conditions and a calculation of the abelian charges.  \( \Box \)

An even more general realization is obtained by the redefinition \( F(t) \mapsto F(t) + \Delta F(t) \), where

\[
\Delta F(t) = -\lambda (\dot{D}(t) + iD(t)),
\]

where \( \lambda \) is a parameter and \( D(t) = \sum_m F_m^+(t) \). Hence \( \Delta L_f = \lambda \int dt \left( \dot{f}(t) - if(t) \right) D(t) \). One checks that \( D(t) \) obeys

\[
[F(s), D(t)] = D(s)\delta(s-t) + \frac{d_0}{4\pi i} \left( \frac{N+p}{N} \right) \left( \delta(s-t) + i\delta(s-t) \right)
\]
\[
[D(s), T^M(t)] = \frac{k^{M\theta}}{2\pi i} \delta(s-t).
\]

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In particular,

\[
[D(s), D(t)] = \frac{k_4}{2\pi i} \left( \frac{N + p}{N} \right) \delta(s - t),
\]

\[
[D(s), P_\mu(t)] = 0,
\]

\[
[D(s), E_n^m(t)] = \frac{k_4}{2\pi i} \delta_n^m \delta(s - t),
\]

\[
[D(s), J^m_{n\sigma}(t)] = \frac{k_6}{2\pi_i} \delta^m_n \delta\delta(s - t),
\]

\[
[D(s), T^{mn}_{\mu\nu}(t)] = \frac{k_3}{2\pi i} \delta^m_n \delta(s - t).
\]  

This leads to a change of some abelian charges:

\[
\Delta c_3 = -2\lambda (k_3 \left( \frac{N + p}{N} \right) + k_4 \left( \frac{N + p}{N + 1} \right)),
\]

\[
\Delta c_4 = 12(\lambda d_0 - \lambda^2 k_4) \left( \frac{N + p}{N} \right),
\]

\[
\Delta c_6 = -2\lambda k_6 \left( \frac{N + p}{N} \right),
\]

which can be accommodated by a shift in the parameters \(d_0 - d_2\) and \(c(N, p)\):

\[
\Delta d_0 = -2\lambda k_4 \quad \Delta d_2 = -2\lambda k_6 \quad \Delta d_1 = -2\lambda k_3 \quad \Delta c(N, p) = 12(\lambda d_0 - \lambda^2 k_4) \left( \frac{N + p}{N} \right).
\]

\[ (3.30) \]

4 Comparison with previous work

The results in [8] are recovered by choosing the fermionic Fock realization (3.3) for the Kac-Moody and Virasoro generators. Let \( \varrho \) be a \( gl(N) \) representation and \( M \) a \( g \) representation. In [8] I defined numbers \( k_0(\varrho), k_1(\varrho), k_2(\varrho), y_M, z_M \) and \( w_M \) by the following relations:

\[
\text{tr } 1 = \dim \varrho \dim M,
\]

\[
\text{tr } T_\nu^\mu = k_0(\varrho) \dim M \delta_\nu^\mu,
\]

\[
\text{tr } T_\nu^\mu T_\tau^\rho = (k_1(\varrho) \delta_\rho^\nu \delta_\tau^\mu + k_2(\varrho) \delta_\mu^\nu \delta_\tau^\rho) \dim M,
\]

\[
\text{tr } J^a = z_M \dim \varrho \delta^a,
\]

\[
\text{tr } J^a J^b = (y_M \delta^{ab} + z_M \delta^a \delta^b) \dim \varrho,
\]

\[
\text{tr } J^a T_\nu^\mu = z_M k_0(\varrho) \delta_\mu^\nu \delta^a,
\]

where the trace is taken in the \( g \oplus gl(N) \) representation \( M \oplus \varrho \). The parameters \( k_1 - k_8, d_0 - d_2 \) and \( c(N, p) \), which are defined in \( (3.21), (3.24) \) and \( (3.21) \),
respectively, are for this realization

\[
\begin{align*}
  k_1 &= k_1(g) \dim M, \\
  k_2 &= k_2(g) \dim M, \\
  k_3 &= k_0(g) \dim M, \\
  k_4 &= \dim \varrho \dim M, \\
  k_5 &= y_M \dim \varrho, \\
  k_6 &= z_M \dim \varrho, \\
  k_7 &= z_M k_0(\varrho), \\
  k_8 &= w_M \dim \varrho, \\
  d_0 &= \dim \varrho \dim M, \\
  d_1 &= k_0(\varrho) \dim M, \\
  d_2 &= z_M \dim \varrho
\end{align*}
\]

(4.2)

If we substitute these expressions into (3.26), the abelian charges in Theorems 1 and 3 of [8] are recovered.

There are some apparent discrepancies. First, we could use a bosonic Fock representation for (3.3), giving all parameters the opposite sign. Second, the modification (3.27) leads to the shift (3.30) in some of the abelian charges. Third, in my previous paper I introduced a parameter \( w \) which only affects a trivial cocycle; here I have set \( w = \lambda \).

5 Sugawara construction

An immediate corollary of Theorem 3.1 is that if we are able to construct a representation of \( \text{Vir} \ltimes \tilde{\mathcal{U}}(\mathfrak{g} \oplus \mathfrak{gl}(N), N, p) \), commuting with the oscillators \( q^\mu(t) \) and \( p^\mu(t) \), we have automatically constructed a representation of \( \text{DGRO}(N) \). As an example we employ the Sugawara construction to express the reparametrization Virasoro algebra in terms of the \( \tilde{\mathcal{U}}(\mathfrak{g} \oplus \mathfrak{gl}(N), N, p) \) generators. The Sugawara construction is of course well known, and it is described in many places [4, 5].

What is somewhat unusual is that \( \text{tr} \ I^M(t) \neq 0 \), at least in some cases, such as \( M = m^0 \) (no sum on \( m \)). Therefore, we do not assume that the central charge matrix is proportional to the unit matrix. Set

\[
F(t) = \gamma_{MN} :I^M(t)I^N(t):,
\]

(5.1)

where the coefficients \( \gamma_{MN} = \gamma_{NM} \) satisfy

\[
\gamma_{MN}(2k^{NR}\delta^M_T + f^{NR}_S f^{SM}_T) = \delta^R_T.
\]

(5.2)

One finds that the operators (5.1) satisfy (3.21) – (3.23), where

\[
\begin{align*}
  c(N, p) &= 2\gamma_{MN} k^{MN}, \\
  d^R &= i\gamma_{MN} f^{NR}_S k^{SM} = i\gamma_{MN} f^{MN}_S k^{SR} = 0,
\end{align*}
\]

(5.3)
and we used (3.13) in the last step. In particular, if we assume that
\[ k_{MN} f^{NR} S^{SM} = Q \delta^T_V, \]
where \( k_{MN} \) is the inverse of \( k^{MN} \): 
\[ k^{MN} k_{NR} = k_{RN} k^{NM} = \delta^M_R, \]
then
\[ \gamma_{MN} = \frac{k_{MN}}{2 + Q}, \quad c(N, p) = \frac{2}{2 + Q} \delta^M_M. \]
Since \( I^M = I^M_n \),
\[ \delta^M_M = \sum_{m, n} \delta^m_n \delta^m_A = \left( \frac{N + p}{N} \right) \delta^A_A, \]
by Lemma A.5 below. Clearly, \( \delta^A_A = \dim U(g \oplus gl(N)) = \infty \), so the abelian charge \( c_4 \) is infinite, which is unacceptable. However, the Sugawara construction is well defined if we replace \( \hat{U}(g \oplus gl(N), N, p) \) with \( \hat{U}_M(g \oplus gl(N), N, p) \), because the underlying associative algebra is finite-dimensional whenever \( M \) is a finite-dimensional representation. Hence
\[ c(N, p) = \frac{2}{2 + Q} \dim M \left( \frac{N + p}{N} \right), \]
\[ d_0 = d_1 = d_2 = 0. \]

We could also consider a hybrid representation, \( F(t) = F_g(t) + F_{gl(N)}(t) \), where we construct \( F_g(t) \) by Sugawara for \( U(g, N, p) \) and use a Fock representation for \( F_{gl(N)}(t) \).

6 A different realization

It is possible to realize the Virasoro operators \( L_f \) in a different fashion. This is based upon the observation that the jet \( \phi_m(t) \) in (2.5) can equivalently be defined in terms of a Taylor expansion around the observer’s trajectory:
\[ \phi(x, t) = \sum_{|m| \leq p} \frac{1}{m!} \phi_m(t)(x - q(t))^m, \]
where \( (x - q(t))^m = (x^1 - q^1(t))^{m_1} \ldots (x^N - q^N(t))^{m_N} \). The space spanned by \( \phi(x, t), x \in \mathbb{R}^N, t \in S^1 \), contains an invariant subspace consisting of \( t \)-independent fields. The condition \( \frac{\partial}{\partial t} \phi(x, t) = 0 \) translates into
\[ \dot{\phi}_m(t) = \dot{q}^\mu(t) \phi_{m+\mu}(t), \]
for all \( m \) such that \( |m| \leq p - 1 \) (we can not have \( |m| = p \), because then \( \phi_{m+\mu}(t) \) is undefined). Now substitute this relation into the expression for \( F(t) \) in (3.3):
\[ F(t) = \sum_m : \pi^m(t) \dot{\phi}_m(t) : = \sum_m expression \]
\[ = \dot{q}^\mu(t) \sum_m E_{m+\mu}(t) = \dot{q}^\mu(t) P_\mu(t). \]
Theorem 6.1 Embed $E^m_n(t)$, $J^m_n(t)$ and $T^m_n(t)$ into $\hat{U}(\mathfrak{g} \oplus \mathfrak{gl}(N), N, p)$ by means of (3.10). Set
\begin{align*}
L(t) &= -\dot{q}^\mu(t)p_\mu(t) + F(t), \\
F(t) &= \dot{q}^\mu(t)P_\mu(t),
\end{align*}
where $P_\mu(t) = \sum_r E^r_{r+\mu}(t)$. Set $L'(t) = L(t) + \Delta L(t)$, $J'(X) = J(X) + \Delta J(X)$, where
\begin{align*}
\Delta L(t) &= -\frac{i c_3}{4\pi i} \int dt \partial_\mu \xi^\mu(q(t)), \\
\Delta J(X) &= -\frac{i c_6}{4\pi i} \int dt \delta_a X^a(q(t))
\end{align*}
and $L(t)$ and $J(X)$ are given in (3.1). Then $L'(t)$, $J'(X)$ and $L_f = \int dt f(t)L(t)$ satisfy the DGRO algebra $\text{DGRO}(N, \mathfrak{g})$. The abelian charges are the same as in Theorem 3.1, with the following exceptions:
\begin{align*}
c_3 &= 1 + 2k_3 \left( \frac{N + p}{N + 1} \right) + 2k_4 \left( \frac{N + p}{N + 2} \right), \\
c_4 &= 2N, \\
c_6 &= 2k_6 \left( \frac{N + p}{N + 1} \right).
\end{align*}

The proof is deferred to Appendix C.

It should be noted that $F(t)$ defined in Theorem 6.1 does not satisfy the relations (3.22) required by Theorem 3.1, so this is a genuinely new realization. Instead, (3.22) is replaced by
\begin{align*}
[F(s), q^\mu(t)] &= 0, \\
[F(s), p_\nu(t)] &= P_\nu(s)\delta(s - t).
\end{align*}

7 Finiteness conditions

In the previous sections, we constructed realizations of $\text{DGRO}(N, \mathfrak{g})$ in terms of the loop algebra $\hat{U}(\mathfrak{g} \oplus \mathfrak{gl}(N), N, p)$. If we substitute the realization (3.3) of this loop algebra, we obtain manifestly well defined $\text{DGRO}(N, \mathfrak{g})$ Fock modules for each finite $p$. It is interesting to consider the limit $p \to \infty$. Then the Taylor expansion (6.1) is essentially equivalent to a spacetime field $\phi(x, t)$, under some suitable analyticity assumptions. However, taken at face value, the prospects for taking this limit appear bleak. When $p$ is large, $(m + p)^n/n!$, so the abelian charges (3.26) diverge; the worst case is $c_1 \approx c_2 \approx p^{N+2}/(N + 2)!$, which diverge in all dimensions $N > -2$.

There is one way out of this problem. We can consider a more general realization by taking the direct sum of operators corresponding to different values of the jet order $p$. Set therefore
\begin{align*}
F(t) &= F^{(0)}(t) + F^{(1)}(t) + F^{(2)}(t) + \ldots + F^{(r)}(t), \\
I^M(t) &= I_{(0)}^M(t) + I_{(1)}^M(t) + I_{(2)}^M(t) + \ldots + I_{(r)}^M(t).
\end{align*}
where \( F(i)(t) \) and \( I^M_{(i)}(t) \) form a basis for \( \text{Vir} \oplus \hat{\mathcal{U}}(\mathfrak{g} \oplus gl(N), N, p - i) \), and operators corresponding to different value of the label \((i)\) commute. Thus \((i)\) corresponds to the jet order \(p - i\), and if we only keep one term (i.e. \( r = 0 \)), we recover the situation studied previously. The equations (3.21), (3.23) and (3.12) are replaced by

\[
[F(i)(s), F(j)(t)] = \delta_{ij} \left\{ (F(i)(s) + F(i)(t)) \delta(s - t) + \right.
\]
\[
\left. + \frac{c(i)(N, p - i)}{24 \pi i} (\sigma(s - t) + \dot{\delta}(s - t)) \right\}, \tag{7.2}
\]
\[
[F(i)(s), I^M_{(j)}(t)] = \delta_{ij} \left\{ I^M_{(j)}(s) \delta(s - t) + \frac{d^M_{(j)}}{4 \pi i} (\dot{\delta}(s - t) + i \delta(s - t)) \right\},
\]
\[
[I^M_{(i)}(s), I^M_{(j)}(t)] = \delta_{ij} \left\{ i f^{MN} R R^R_{(i)}(s) \delta(s - t) + \frac{k^{MN}_{(j)}}{2 \pi i} \dot{\delta}(s - t) \right\}.
\]

It is immediate that (3.24), with these more general expressions (7.1) for the Virasoro and Kac-Moody currents, still yields a realization of \( \text{DGRO}(N, \mathfrak{g}) \).

The abelian charges are given by sum of terms like those in Theorem 3.1, e.g.,

\[
c_5 = \sum_{i=0}^r k_{5}^{(i)} \binom{N + p - i}{N}
\]
\[
= k_{5}^{(0)} \binom{N + p}{N} + k_{5}^{(1)} \binom{N + p - 1}{N} + \ldots + k_{5}^{(r)} \binom{N + p - r}{N}. \tag{7.3}
\]

We now want to choose the parameters \( k_{5}^{(i)} \) such that the expression for \( c_5 \) has a finite \( p \to \infty \) limit when \( N \leq r \). Because of the following lemma, the right choice is \( k_{5}^{(i)} = (-1)^i \binom{i}{0} k_5 \), where \( k_5 = k_{5}^{(0)} \).

**Lemma 7.1**

\[
\sum_{i=0}^r (-1)^i \binom{r}{i} \binom{N + p - i}{N} = \binom{N + p - r}{N - r}.
\]

**Proof.** Denote the LHS by \( c_{r,p,N} \). Then we use the recurrence formula in Lemma A.3 to write

\[
c_{r,p,N} = \sum_{i=0}^r (-1)^i \left\{ \binom{r - 1}{i} + \binom{r - 1}{i - 1} \right\} \binom{N + p - i}{N} \tag{7.4}
\]
\[
= \sum_{i=0}^r (-1)^i \binom{r - 1}{i} \binom{N + p - i}{N} + \sum_{j=0}^{r-1} (-1)^{j+1} \binom{r - 1}{j} \binom{N + p - j - 1}{N}
\]
\[
= c_{r - 1,p,N} - c_{r - 1,p - 1,N}. \]

But the recurrence formula also implies that

\[
\binom{N + p - r}{N - r} = \binom{N + p - r + 1}{N - r + 1} - \binom{N + p - r}{N - r + 1}. \quad \square \tag{7.5}
\]

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The abelian charges $c_6$ and $c_8$ are given by analogous expressions. For $c_3$ and $c_7$, the situation is somewhat more complicated, because there are two terms. To organize calculations, we first introduce the following function:

$$G_{r,p,N}(\{k^{(i)}\}_{i=0}^r) = \sum_{i=0}^r k^{(i)} \binom{N + p - i}{N}.$$  \hspace{1cm} (7.6)

**Lemma 7.2** The function (7.6) has the following properties:

i. $G_{r,p,N}(\{k^{(i)}\}_{i=0}^r) = \binom{N + p - r}{N - r} k$, if $k^{(i)} = (-)^i \binom{r}{i} k$,

ii. $G_{r,p,N}(\{k^{(i)}\}_{i=0}^r) + G_{r-1,p,N}(\{\bar{k}^{(i)}\}_{i=0}^{r-1}) = G_{r,p,N}(\{k^{(i)} + \bar{k}^{(i-1)}\}_{i=0}^r)$, where $\bar{k}^{(-1)} \equiv 0$.

iii. $G_{r,p,N}(\{k^{(i)}\}_{i=0}^r) = G_{r-1,p,N-1}(\{\sum_{j=0}^i k^{(j)}\}_{i=0}^{r-1})$, provided that $\sum_{i=0}^r k^{(i)} = 0$.

**Proof.** Property i is equivalent to Lemma 7.1. Property ii follows from a trivial shift in the summation variable in the second term. To prove property iii, set $a^{(0)} = k^{(0)}$, $a^{(i)} = a^{(i-1)} + k^{(i)} = \sum_{j=0}^i k^{(j)}$. Then

$$G_{r,p,N}(\{k^{(i)}\}_{i=0}^r) = k^{(0)} \binom{N + p}{N} + k^{(1)} \binom{N + p - 1}{N} + ... + k^{(r)} \binom{N + p - r}{N}$$

$$= a^{(0)} \binom{N + p}{N} + (a^{(1)} - a^{(0)}) \binom{N + p - 1}{N} + ... + (a^{(r)} - a^{(r-1)}) \binom{N + p - r}{N}$$

$$= a^{(0)} \binom{N + p - 1}{N - 1} + a^{(1)} \binom{N + p - 2}{N - 1} + ... +$$

$$+ a^{(r-1)} \binom{N + p - r}{N - 1} + a^{(r)} \binom{N + p - r}{N}$$

$$= G_{r-1,p,N-1}(\{a^{(j)}\}_{j=0}^{r-1}) + a^{(r)} \binom{N + p - r}{N}.$$  

by repeated use of the recurrence formula. \hspace{1cm} □

Using the function $G_{r,p,N}$ defined in (7.6), we rewrite the expression for $c_7$...
\[ c_7 = \sum_{i=0}^{r} k_7^{(i)} \left( \frac{N + p - i}{N} \right) + \sum_{i=0}^{r} k_6^{(i)} \left( \frac{N + p - i}{N + 1} \right) \]
\[ = G_{r,p,N}(\{k_7^{(i)}\}_{i=0}^{r}) + G_{r,p-1,N+1}(\{k_6^{(i)}\}_{i=0}^{r}) \]
\[ = G_{r,p,N}(\{k_7^{(i)}\}_{i=0}^{r}) + G_{r-1,p-1,N}(\{i \sum_{j=0}^{k_6^{(j)}r-1}\}_{i=0}^{r-1}) \quad (7.7) \]
\[ = G_{r,p,N}(\{k_7^{(i)} + \sum_{j=0}^{i} k_6^{(j)}r\}_{i=0}^{r}) \]
\[ = k_7 \left( \frac{N + p - r}{N - r} \right), \]
which holds provided that \( \sum_{i=0}^{r} k_6^{(i)} = 0 \) and \( k_7^{(i)} + \sum_{j=0}^{i} k_6^{(j)} = (-)^i \binom{r}{i} k_7^r \). The abelian charge \( c_3 \) is treated analogously.

For \( c_3 \), we must apply Lemma 7.2 twice to the last term:
\[ 1 - c_3 = \sum_{i=0}^{r} k_1^{(i)} \left( \frac{N + p - i}{N} \right) + \sum_{i=0}^{r} k_4^{(i)} \left( \frac{N + p - i + 1}{N + 2} \right) \]
\[ = G_{r,p,N}(\{k_1^{(i)}\}_{i=0}^{r}) + G_{r,p-1,N+2}(\{k_4^{(i)}\}_{i=0}^{r}) \]
\[ = G_{r,p,N}(\{k_1^{(i)}\}_{i=0}^{r}) + G_{r-1,p-1,N+1}(\{i \sum_{j=0}^{k_4^{(j)}r-1}\}_{i=0}^{r-1}) \quad (7.8) \]
\[ = G_{r,p,N}(\{k_1^{(i)}\}_{i=0}^{r}) + G_{r-2,p-1,N}(\{i \sum_{j=0}^{i-1} \sum_{\ell=0}^{j} k_4^{(\ell)}r\}_{i=0}^{r-2}) \]
\[ = G_{r,p,N}(\{k_1^{(i)} + \sum_{j=0}^{i-1} \sum_{\ell=0}^{j} k_4^{(\ell)}r\}_{i=0}^{r}) \]
\[ = k_1 \left( \frac{N + p - r}{N - r} \right), \]
where the conditions are
\[ \sum_{i=0}^{r} k_4^{(i)} = 0, \]
\[ \sum_{i=0}^{r-1} \sum_{j=0}^{i} k_4^{(j)} = 0, \quad (7.9) \]
\[ k_1^{(i)} + \sum_{j=0}^{i-1} \sum_{\ell=0}^{j} k_4^{(\ell)} = (-)^i \binom{r}{i} k_1^r. \]
$c_2$ is computed by repeatedly using the properties in Lemma 7.2:

$$
-c_2 = \sum_{i=0}^{r} k_2^{(i)} \left( \frac{N + p - i}{N} \right) + \sum_{i=0}^{r} 2k_3^{(i)} \left( \frac{N + p - i}{N + 1} \right) + \sum_{i=0}^{r} k_4^{(i)} \left( \frac{N + p - i}{N + 2} \right)
$$

$$
= G_{r,p,N} \{ k_2^{(i)} \}_{i=0}^{r} + G_{r,p-1,N+1} \{ 2k_3^{(i)} \}_{i=0}^{r} + G_{r,p-2,N+2} \{ k_4^{(i)} \}_{i=0}^{r-1}
$$

$$
= G_{r,p,N} \{ k_2^{(i)} \}_{i=0}^{r} + G_{r,p-1,N+1} \{ 2k_3^{(i)} \}_{i=0}^{r} + G_{r-1,p-2,N+1} \{ k_4^{(i)} \}_{i=0}^{r-1}
$$

$$
= G_{r,p,N} \{ k_2^{(i)} \}_{i=0}^{r} + G_{r,p-1,N+1} \{ 2k_3^{(i)} + \tilde{k}_4^{(i-1)} \}_{i=0}^{r}
$$

$$
= k_1 \left( \frac{N + p - r}{N - r} \right),
$$

where $\tilde{k}_4^{(i)} = \sum_{j=0}^{i} k_4^{(j)}$, and the conditions are

$$
\tilde{k}_4^{(r)} = 0,
$$

$$
\sum_{i=0}^{r} (2k_3^{(i)} + \tilde{k}_4^{(i-1)}) = 0,
$$

$$
k_2^{(i)} + \sum_{j=0}^{i-1} (2k_3^{(j)} + \tilde{k}_4^{(j-1)}) = (-)^i \binom{r}{i} k_2.
$$

Finally, the Virasoro central charge $c(N,p)$, when given by either (4.2) or (5.7), has the form $c(N,p) = c\left(\frac{N+p}{N}\right)$ for some parameter $c$ independent of both $N$ and $p$. The abelian charge $c_4$ then has the form $c_4 = 2N + c\left(\frac{N+p}{N}\right)$. When we add several contributions $c^{(i)}$, it becomes

$$
c_4 = 2N + \left( \frac{N + p - r}{N - r} \right),
$$

provided that $c^{(i)} = (-)^i \binom{r}{i} c$.

The results in this section can be summarized in the following theorem:

**Theorem 7.1** Let the operators $F(t)$ and $I^M(t)$ be given as in (7.1), and let $E_n(t)$, $J_m(t)$, $T_{nm}(t)$ be as in (3.1). Then the generators (3.1) satisfy the
DGRO algebra DGRO($N, g$) [24]. The abelian charges are given by

\[
\begin{align*}
c_1 & = 1 - k_1 \left( \frac{N + p - r}{N - r} \right), \\
c_2 & = -k_2 \left( \frac{N + p - r}{N - r} \right), \\
c_3 & = 1 + d_1 \left( \frac{N + p - r}{N - r} \right), \\
c_4 & = 2N + c \left( \frac{N + p - r}{N - r} \right), \\
c_5 & = k_5 \left( \frac{N + p - r}{N - r} \right), \\
c_6 & = d_2 \left( \frac{N + p - r}{N - r} \right), \\
c_7 & = k_7 \left( \frac{N + p - r}{N - r} \right), \\
c_8 & = k_8 \left( \frac{N + p - r}{N - r} \right),
\end{align*}
\] (7.13)
where \( k_n^{(0)} = k_n, d_n^{(0)} = d_n \), provided that the following conditions hold:

\[
\begin{align*}
    k_1^{(i)} + \sum_{j=0}^{i+1} \sum_{\ell=0}^j k_4^{(\ell)} &= (-)^i \binom{r}{i} k_1, \\
    k_2^{(i)} + \sum_{j=0}^{i-1} (2k_3^{(j)} + \sum_{\ell=0}^{j-1} k_4^{(\ell)}) &= (-)^i \binom{r}{i} k_2, \\
    \sum_{i=0}^{r} (2k_3^{(i)} + \sum_{\ell=0}^{i-1} k_4^{(\ell)}) &= 0, \\
    \sum_{i=0}^{r} k_4^{(i)} &= 0, \\
    \sum_{i=0}^{r-1} \sum_{j=0}^{i} k_4^{(j)} &= 0, \\
    k_5^{(i)} &= (-)^i \binom{r}{i} k_5, \\
    \sum_{i=0}^{r} k_6^{(i)} &= 0, \\
    k_7^{(i)} + \sum_{j=0}^{i-1} k_6^{(j)} &= (-)^i \binom{r}{i} k_7, \\
    k_8^{(i)} &= (-)^i \binom{r}{i} k_8, \\
    \sum_{i=0}^{r} d_0^{(i)} &= 0, \\
    d_1^{(i)} + \sum_{j=0}^{i-1} d_0^{(j)} &= (-)^i \binom{r}{i} d_1, \\
    d_2^{(i)} &= (-)^i \binom{r}{i} d_2, \\
    c^{(i)} &= (-)^i \binom{r}{i} c.
\end{align*}
\]

(7.14)

All abelian charges vanish if \( N < r \) and diverge when \( p \to \infty \) if \( N > r \). When \( N = r \), the abelian charges are independent of \( p \) and in general non-zero.

8 Discussion

In this paper, I reformulated the DGRO(\( N, g \)) Fock modules from [8] as realizations in terms of Virasoro and affine Kac-Moody currents. This gave rise
two new types of realizations: the Sugawara construction in Section 5 and the modification in Section 6, and it became possible to formulate conditions for the existence of the $p \to \infty$ limit.

The modules obtained in the last section are quite unnatural. True, they admit that we let $p \to \infty$, but in the same time they are highly reducible, being direct sums for finite $p$. So we are in the strange situation that there exist several well-defined modules for finite $p$, but only their sum and not the individual summands survive in the limit. Another problem is that the central charges of the underlying affine algebras have alternating signs, by Theorem 7.1, so they can not all be represented unitarily.

There is an attractive resolution to these problems. If we can find an invariant nilpotent fermionic operator intertwining between the individual Fock modules in the direct sum, the associated cohomology groups will also be modules. The modules at jet order $p$ can be thought of as quantum fields, whereas lower-order modules describe antifields of various degree. In the unpublished paper [1], I began to investigate a cohomology theory closely related to the Koszul-Tate cohomology arising in the Batalin-Vilkovisky approach to gauge theory [6]. Note that the better known BRST cohomology does not work, for two reasons. First, BRST both imposes constraints and identifies points on gauge orbits, which means that the gauge algebra acts trivially on the cohomology. Second, BRST is ill defined in the presence of non-trivial cocycles, which is precisely the case of interest here.

The conditions in Theorem 7.1 impose severe restrictions on the possible field content and on the form of the Euler-Lagrange equations. Preliminary calculations indicate that if we assume that the Euler-Lagrange equations are of first order for fermions and second order for bosons, and we only have irreducible gauge symmetries of one order higher, then the dimension of space-time must be four. Moreover, it is also necessary to have gauge symmetries for fermions, pointing toward some kind of supersymmetry. These issues will be addressed in a forthcoming publication.

A Lemmas

Lemma A.1

\[
\binom{m}{n} \binom{n}{r} = \binom{m}{r} \binom{m-r}{n}.
\]

Proof. Both sides can be written as

\[
\frac{m!}{n!(m-n)!} \frac{n!}{r!(n-r)!} \frac{(m-r)!}{(m-r)!}. \quad \square
\]
Lemma A.2
\[
\sum_{m,n,r} \binom{m}{n} \binom{n}{r} \partial_{m-n}f \partial_{n-r}g = \sum_{m,r} \binom{m}{r} \partial_{m-r}(fg).
\]

Proof.
\[
LHS = \sum_{m,n,r} \binom{m}{n} \binom{m-r}{m-n} \partial_{m-n}f \partial_{n-r}g
= \sum_{m,r} \binom{m}{r} \sum_{s} \binom{m-r}{s} \partial_{s}f \partial_{m-r-s}g = RHS,
\]
by Lemma A.1 and Leibniz’ rule. □

Lemma A.3
\[
\binom{m}{n} + \binom{m}{n-\mu} - \binom{m+\mu}{n} = 0.
\]

In one dimension, this is the recurrence formula:
\[
\binom{m+1}{n} = \binom{m}{n} + \binom{m}{n-1}.
\]

Proof. The three terms only differ in the \(\mu^{th}\) components, so the other components contribute a constant factor. It thus suffices to prove the one-dimensional recurrence formula:
\[
\binom{m}{n} + \binom{m}{n-1} - \binom{m+1}{n} = \binom{m}{n}(1 + \frac{n}{m-n+1} - \frac{m+1}{m+1-n}) = 0. \quad □
\]

Lemma A.4
\[
\binom{m}{n} \binom{n+\mu}{s} = \binom{m}{s} \binom{m-s}{m-n} + \binom{m}{n} \binom{n}{s-\mu}.
\]

Proof. First use Lemma A.3 on \(\binom{n+\mu}{s}\), then apply Lemma A.1. □

Lemma A.5
\[
\sum_{|m|\leq p} 1 = \binom{N+p}{N}.
\]
**Proof.** The statement is proved by induction. Define

\[
A(N, p) \equiv \sum_{|m| \leq p} 1 = \sum_{m_1=0}^{p} \sum_{m_2 + \ldots + m_N \leq p - m_1} 1 = \sum_{m_1=0}^{p} A(N-1, p - m_1). \quad (A.1)
\]

Thus,

\[
A(N, p + 1) = \sum_{m=0}^{p+1} A(N-1, p + 1 - m) = \sum_{m=0}^{p} A(N-1, p - m) + A(N-1, p + 1), \quad (A.2)
\]

where we substituted \(n = m - 1\). \(A(N, p)\) must therefore obey the recursion relation

\[
A(N, p + 1) = A(N, p) + A(N-1, p + 1). \quad (A.3)
\]

The equation has the solution

\[
A(N, p) = \binom{N + p}{N} \quad (A.4)
\]

due to the recurrence relation in Lemma A.3. Moreover, the boundary case \(p = 0\) is clear, since

\[
A(N, 0) = \sum_{|m|=0} 1 = 1 = \binom{N}{N}. \quad \square
\]

**Lemma A.6**

\[
\sum_{\mu} \sum_{|m| \leq p-1} \left( \begin{array}{c}
m + \mu \\
m \end{array} \right) \phi^{\mu} = \left( \begin{array}{c}
N + p \\
N + 1 \end{array} \right) \phi^{\mu}. \quad (A.5)
\]

**Proof.** By symmetry, we only need to evaluate the expression for \(\mu = 1\). Set

\[
B(N, p) \equiv \sum_{|m| \leq p} \left( \begin{array}{c}
m + \hat{1} \\
m \end{array} \right) = \sum_{|m| \leq p} (m_1 + 1), \quad (A.5)
\]

where \(\hat{1}\) denotes a unit vector in the first direction. Clearly,

\[
B(N, p) = \sum_{m_1=0}^{p} \sum_{m_2 + \ldots + m_N \leq p - m_1} (m_1 + 1) = \sum_{m=0}^{p} (m + 1) A(N-1, p - m). \quad (A.6)
\]
This function satisfies

\[
B(N, p + 1) = \sum_{m=0}^{p+1} (m + 1)A(N - 1, p + 1 - m)
\]
\[
= \sum_{n=0}^{p} (n + 2)A(N - 1, p - n) + A(N - 1, p + 1)
\]
\[
= B(N, p) + A(N, p) + A(N - 1, p + 1)
\]
\[
= B(N, p) + \binom{N + p + 1}{N}.
\]

By Lemma A.3, this recursion relation has the solution

\[
B(N, p) = \binom{N + p + 1}{N + 1}.
\]

The boundary case is clear, because

\[
B(N, 0) = \sum_{|m|=0} (m + 1) = 1 = \binom{N + 1}{N + 1}.
\]

Thus,

\[
\sum_{|m|\leq p-1} \binom{m+\hat{1}}{m} = B(N, p-1)\phi_1^1 = \binom{N+p}{N+1} \phi_1^1,
\]

and the lemma is proven. □

Lemma A.7

\[
\sum_{\mu, \nu} \sum_{|m|\leq p-1} \binom{m + \mu}{m} \binom{m + \nu}{m} \phi^\mu_\mu + \sum_{\mu \neq \nu} \sum_{|r|\leq p-2} \binom{r + \mu + \nu}{r + \nu} \binom{r + \mu + \nu}{r + \mu} \phi^\mu_\mu
\]
\[
= \binom{N + p + 1}{N + 2} \phi^\mu_\mu + \binom{N + p}{N + 2} \phi^\mu_\mu
\]

\[
(A.9)
\]

Proof. We have to distinguish two cases: \(\mu \neq \nu\) and \(\mu = \nu\). Consider the first case, say \(\mu = 1, \nu = 2\), and define

\[
C(N, p) = \sum_{|m|\leq p} \binom{m+\hat{1}}{m} \binom{m+\hat{2}}{m} = \sum_{|m|\leq p} (m_1 + 1)(m_2 + 1)
\]
\[
= \sum_{m_1=0}^{p} \sum_{m_2=0}^{p-m_1} \sum_{m_3 + \ldots + m_N \leq p-m_1-m_2} (m_1 + 1)(m_2 + 1)
\]
\[
(A.10)
\]
\[
= \sum_{m_1=0}^{p} \sum_{m_2=0}^{p-m_1} (m_1 + 1)(m_2 + 1)A(N - 2, p - m_1 - m_2).
\]
The recursion relation becomes

\[
C(N, p + 1) = \sum_{m_1=0}^{p+1} \sum_{m_2=0}^{p+1-m_1} (m_1 + 1)(m_2 + 1)A(N - 2, p + 1 - m_1 - m_2)
\]

\[
= \sum_{n_1=0}^{p} \sum_{m_2=0}^{p-n_1} (n_1 + 2)(m_2 + 1)A(N - 2, p - n_1 - m_2)
\]

\[
+ \sum_{m_2=0}^{p} (m_2 + 1)A(N - 2, p + 1 - m_2)
\]

\[
= C(N, p) + \sum_{n_1=0}^{p} B(N - 1, p - n_1) + B(N - 1, p + 1),
\]

where we used (A.6) twice. We now need the sum

\[
\Delta(N, p) = \sum_{m=0}^{p} B(N - 1, p - m) = \sum_{m=0}^{p} \binom{N + p - m}{N}.
\]

(A.12)

Recursion relation:

\[
\Delta(N, p + 1) = \sum_{n=0}^{p+1} \binom{N + p + 1 - m}{N}
\]

\[
= \sum_{n=0}^{p} \binom{N + p - n}{N} + \binom{N + p + 1}{N}
\]

(A.13)

\[
= \Delta(N, p) + \binom{N + p + 1}{N},
\]

with the solution

\[
\Delta(N, p) = \binom{N + p + 1}{N + 1}.
\]

(A.14)

We now substitute (A.14) into (A.11):

\[
C(N, p + 1) = C(N, p) + \binom{N + p + 1}{N + 1} + \binom{N + p + 1}{N}
\]

\[
= C(N, p) + \binom{N + p + 2}{N + 1}
\]

(A.15)

\[
= C(N, p) + B(N, p + 1),
\]

and thus

\[
C(N, p) = \binom{N + p + 2}{N + 2}.
\]

(A.16)
We check that the boundary value \( C(N,0) = 1 \) is correct. The \( \mu = 1, \nu = 2 \), contribution to the first term in [A.9] is thus \( C(N, p - 1)\phi_{21}^{12} \). To obtain the contribution to the second term, we note that

\[
\sum_{|r| \leq p-2} (r_1 + 1)(r_2 + 1) = C(N, p - 2).
\]

Hence the total contribution to [A.9] is

\[
C(N, p - 1)\phi_{21}^{12} + C(N, p - 2)\phi_{12}^{12}.
\]

(A.17)

Now we consider the case \( \mu = \nu = 1 \), say. The relevant sum is

\[
D(N, p) = \sum_{|m| \leq p} (m + 1)^2 = \sum_{m=0}^{p} (m + 1)^2 A(N - 1, p - m).
\]

(A.18)

Recursion relation:

\[
D(N, p + 1) = \sum_{m=0}^{p+1} (m + 1)^2 A(N - 1, p + 1 - m)
\]

\[
= \sum_{n=0}^{p} ((n + 1)^2 + 2(n + 1) + 1)A(N - 1, p - n) + A(N - 1, p + 1)
\]

\[
= D(N, p) + 2B(N, p) + A(N, p) + A(N - 1, p + 1)
\]

(A.19)

where [A.7] was used in the last step. Equation (A.19) has the solution

\[
D(N, p) = C(N, p) + C(N, p - 1) = \binom{N + p + 2}{N+2}^2 + \binom{N + p + 1}{N+2},
\]

because substitution of this expression into (A.19) gives rise to two copies of the identity (A.13) (for \( p \) and \( p - 1 \), respectively). The total contribution to the LHS is thus

\[
(C(N, p - 1) + C(N, p - 2))\phi_{11}^{11} = C(N, p - 1)\phi_{11}^{11} + C(N, p - 2)\phi_{11}^{11},
\]

which is of the same form as [A.17]. □

Lemma A.8

\[
\sum_{\mu, \nu} \sum_{|m| \leq p-2} \binom{m + \mu + \nu}{m} \phi_{\mu\nu}^{\mu\nu} = \binom{N + p}{N+2} \phi_{\mu\nu}^{\mu\nu}.
\]

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Proof. First consider the case $\mu = \nu = 1$, say. The relevant sum is
\[
E(N, p) = 2 \sum_{|m| \leq p} \left( \frac{m + 1}{m} \right) (A.20)
\]
\[
= \sum_{m_1=0}^{p} (m_1 + 2)(m_1 + 1)A(N - 1, p - m_1).
\]
Recursion relation:
\[
E(N, p + 1) = \sum_{m=0}^{p+1} (m + 2)(m + 1)A(N - 1, p + 1 - m)
\]
\[
= \sum_{n=0}^{p} (n + 3)(n + 2)A(N - 1, p - n) + 2A(N - 1, p + 1)
\]
\[
= E(N, p) + 2 \sum_{n=0}^{p} (n + 2)A(N - 1, p - n) + 2A(N - 1, p + 1)
\]
\[
= E(N, p) + 2B(N, p + 1) = E(N, p) + 2 \left( \frac{N + p + 2}{N + 1} \right).
\]
Hence
\[
E(N, p) = 2 \left( \frac{N + p + 2}{N + 2} \right)
\]
and
\[
\sum_{|n| \leq p-2} \left( \frac{m + 1 + \hat{1}}{m} \right) \phi_{11}^{11} = \frac{1}{2} E(N, p - 2) \phi_{11}^{11} = \left( \frac{N + p}{N + 2} \right) \phi_{11}^{11}.
\]
Now let $\mu \neq \nu$, say $\mu = 1, \nu = 2$.
\[
\sum_{|n| \leq p-2} \left( \frac{m + 1 + \hat{2}}{m} \right) \phi_{12}^{12} = C(N, p - 2) \phi_{12}^{12} = \left( \frac{N + p}{N + 2} \right) \phi_{12}^{12}.
\]
The lemma now follows by summing over contributions of the form (A.22) and (A.23). □

Lemma A.9 There is an embedding $\text{diff}(N) \hookrightarrow \text{diff}(N) \rtimes \text{map}(N, \text{gl}(N))$, given by $L'_{\xi} = L_{\xi} + T_d\xi$.

Proof. It suffices to prove the statement for the realization $L_{\xi} = \xi^\mu \partial_\mu$, $T_d\xi = \partial_\sigma \xi^\mu T_{\mu}^\nu$, where $T_{\mu}^\nu$ provide a realization of $\text{gl}(N)$ (2.4):
\[
[\mathcal{L}_{\xi}, \mathcal{L}_{\eta}] = [\xi^\mu \partial_\mu + \partial_\rho \xi^\mu T_{\rho}^\nu, \eta^\sigma \partial_\nu + \partial_\sigma \eta^\sigma T_{\nu}^\sigma]
\]
\[
= \xi^\mu \partial_\mu \eta^\nu \partial_\nu + \xi^\mu \partial_\mu \partial_\sigma \eta^\nu T_{\nu}^\sigma + \partial_\rho \xi^\mu \partial_\rho \eta^\sigma T_{\sigma}^\nu - \xi \leftrightarrow \eta
\]
\[
= \xi^\mu \partial_\mu \eta^\nu \partial_\nu + \partial_\sigma (\xi^\mu \partial_\rho \eta^\nu) T_{\sigma}^\nu - \xi \leftrightarrow \eta
\]
\[
= [\xi, \eta]^\nu \partial_\nu + \partial_\sigma [\xi, \eta]^\nu T_{\sigma}^\nu = L'_{[\xi, \eta]}.
\]
□
Proof of Theorem 3.1

B.1 \(\mathcal{JJ}\) bracket

To reduce writing, we suppress all arguments in single and double integrals throughout all proofs. Thus we write \(X_a = X_a(q(s)), Y_b = Y_b(q(t)), J^m_n = J^m_n(s), J^r_s = J^r_s(t), \delta = \delta(s - t)\) and \(\dot{\delta} = \dot{\delta}(s - t) = -\dot{\delta}(t - s)\). Further, \[\int F = \int \int ds dt F(s, t)\] for any functional \(F\), etc. The representation condition for the \(\mathcal{JJ}\) bracket reads

\[
\{J_X, J_Y\} = \sum_{m, n, r, s} \left( \binom{m}{n} \binom{r}{s} \right) \int \partial_{m-n} X_a \partial_{r-s} Y_b \cdot J^m_n \]

\[
\times \left( \delta^r g^{ab} c J^m_n C \delta - \delta^m g^{ba} C J^r_s C \delta + \frac{k^{ab}}{2\pi i} \delta^m \delta^r \dot{\delta} \right) \]

\[
= \sum_{m, n, r, s} g^{ab} C \int \left( \binom{m}{n} \right) \left( \binom{r}{s} \right) \partial_{m-n} X_a \partial_{r-s} Y_b \cdot J^m_n \]

\[- \sum_{m, n, r} g^{ba} C \int \left( \binom{m}{n} \right) \left( \binom{r}{m} \right) \partial_{m-n} X_a \partial_{r-m} Y_b \cdot J^r_s C \]

\[- \frac{k^{ab}}{2\pi i} \sum_{|m|, |n| \leq p} \left( \binom{m}{n} \right) \left( \binom{n}{m} \right) \int \partial_{m-n} \dot{X}_a \partial_{n-m} Y_b. \]

We now use Lemma A.2 to rewrite the first two terms as

\[
\sum_{m, n} \left( \binom{m}{n} \right) \left( \delta^m g^{ab} C - \delta^m \delta^m \cdot \delta^a \delta^b \right) \int \partial_{m-n} (X_a Y_b) J^m_n C = \mathcal{J}[X,Y]. \quad \text{(B.2)}
\]

The third term contains the factor \(\left( \binom{m}{n} \right)\), which is zero unless \(m = n\) (if \(m_\rho > n_\rho\) for some \(\rho\), \(\binom{n}{m} = 0\), and if \(n_\rho > m_\rho\), \(\binom{m}{n} = 0\)). Therefore, this term becomes

\[
- \frac{k^{ab}}{2\pi i} \sum_{|m| \leq p} 1 \int \dot{X}_a Y_b = - \frac{k_5 \delta^{ab} + k_8 \delta^a \delta^b}{2\pi i} \left( \binom{N}{N} + \binom{p}{N} \right) \int \dot{X}_a Y_b, \quad \text{(B.3)}
\]

by Lemma A.5. This fixes \(c_5 = k_5 \left( ^{N+p}_{N} \right) \) and \(c_8 = k_8 \left( ^{N+p}_{N} \right) \).
B.2 $\mathcal{L}\mathcal{J}$ bracket

Consider first the case $T_{\mu\xi} = 0$. Moreover, we initially ignore normal ordering, to get the regular terms:

$$[\mathcal{L}_\xi, \mathcal{J}_X] = \int \int [\xi^\mu (p_\mu - P_\mu) + \sum_{m,n} \left( \frac{m}{n} \right) \partial_{m-n+s} \delta_{n+\mu}^m \partial_{s+\mu}^m X_{a_s} J_{a_r}^n]$$

$$= \int \sum_{r,s} \left( \frac{r}{s} \right) \left( \xi^\mu \partial_{s+\mu} X_{a_s} J_{a_r}^n + \xi^\mu \partial_{s+\mu} X_{a_s} (J_{a_r}^n - J_{s+\mu}^r) \right)$$

$$+ \sum_{m,n} \left( \frac{m}{n} \right) \partial_{m-n+s} \delta_{n+\mu}^m \partial_{s+\mu}^m X_{a_s} (J_{a_r}^n - J_{s+\mu}^r) \right) \right)$$

$$= \int \sum_{r,s} \left( \frac{r}{s} \right) \left( \xi^\mu \partial_{s+\mu} X_{a_s} J_{a_r}^n + \xi^\mu \partial_{s+\mu} X_{a_s} (J_{a_r}^n - J_{s+\mu}^r) \right)$$

$$+ \sum_{m,n,r} \left( \frac{m}{n} \right) \partial_{m-n+s} \delta_{n+\mu}^m \partial_{s+\mu}^m X_{a_s} (J_{a_r}^n - J_{s+\mu}^r) \right) \right)$$

The first term vanishes due to Lemma A.3. In the second, we use Lemma A.3 and obtain

$$- \sum_{m,n,r} \left( \frac{m}{n} \right) \partial_{m-n+s} \delta_{n+\mu}^m \partial_{s+\mu}^m X_{a_s} (J_{a_r}^n - J_{s+\mu}^r) \right) \right)$$

After we set $s = n + \mu$ in the first term, it cancels the last term and we are left with the middle term, which equals $\mathcal{J}_X$.

The extension becomes

$$\text{ext}([\mathcal{L}_\xi, \mathcal{J}_X]) = \int \int \frac{k^0}{2\pi i} \left\{ - \sum_{r,s} \left( \frac{r}{s} \right) \xi^\mu \partial_{s+\mu} X_{a_s} \right\}$$

$$+ \sum_{m,n,r} \left( \frac{m}{n} \right) \partial_{m-n+s} \delta_{n+\mu}^m \partial_{s+\mu}^m X_{a_s} \right\} \delta^r$$

$$= \frac{k^0}{2\pi i} \int \left\{ \sum_{|s|<p-1} \left( \frac{s + \mu}{s} \right) \xi^\mu \partial_{s+\mu} X_{a_s} - \sum_{s} \sum_{|n|<p-1} \left( \frac{n}{s} \right) \left( \frac{n + \mu}{s} \right) \partial_{s-n} \delta_{n+\mu} X_{a_s} \right\}.$$
The last term is zero unless \( n_\mu \leq s_\mu \), \( s_\mu \leq n_\mu + 1 \), and \( n_\nu = s_\nu \) for all \( \nu \neq \mu \). This leaves two cases:

1. \( s = n \): \[- \sum_{|n| \leq p-1} \binom{n + \mu}{n} \partial_\mu \xi X_a, \] (B.7)

2. \( s = n + \mu \): \[- \sum_{|n| \leq p-1} \binom{n + \mu}{n} \partial_\mu \xi^\nu X_a. \] (B.8)

The first term cancels the first term in (B.6), so the total extension reads

\[- \frac{k_{ab}^\alpha}{2\pi i} \int \sum_{\mu} \sum_{|n| \leq p-1} \binom{n + \mu}{n} \partial_\mu \xi X_a = - \frac{k_{ab}^\alpha}{2\pi i} \frac{(N + p - 1)}{(N + 1)} \int \partial_\mu \xi X_a, \] (B.8)

by Lemma A.6. Now replace \( L_\xi \mapsto \dot{L}_\xi + T d_\xi \) and use the result in subsection B.1. The regular terms vanish because \([d_\xi, X] = 0\), and the extension is given by (B.3):

\[- \frac{k_{\nu a}^\alpha}{2\pi i} \frac{(N + p)}{N} \int \partial_\nu \xi X_a. \] (B.9)

Summing (B.8) and (B.9), we see that \( c_7 = k_6 \frac{(N + p)}{(N + 1)} + k_7 \frac{(N + s)}{N} \).

### B.3 \( \mathcal{L} \mathcal{L} \) bracket

Again we initially set \( T d_\xi = 0 \):

\[ [\mathcal{L}_\xi, \mathcal{L}_\eta] = \int \left[ :\xi^\mu p_\mu - \xi^\mu P_\mu + \sum_{m,n} \binom{m}{n} \partial_{m-n} \xi^\mu E_{m+n}^\mu \right] \] \[ \times \left[ :\eta^\nu p_\nu - \eta^\nu P_\nu + \sum_{r,s} \binom{r}{s} \partial_{r-s} \eta^\nu E_{r+s}^\nu \right]. \] (B.10)

We first ignore normal ordering and extensions, and verify that the regular terms come out right:

\[- \frac{k_{\nu a}^\alpha}{2\pi i} \frac{(N + p)}{N} \int \partial_\nu \xi X_a + \frac{k_{\nu a}^\alpha}{2\pi i} \frac{(N + p)}{N} \int \partial_\nu \dot{\xi} X_a \] \[ = \int \xi^\mu \partial_\mu \eta^\nu (p_\nu - P_\nu) + \sum_{r,s} \binom{r}{s} \xi^\mu \partial_{r-s} \eta^\nu (E_{r+s}^\nu - E_{r+s+\mu}^s) \] \[ + \sum_{m,n,r,s} \binom{m}{n} \binom{r}{s} \partial_{m-n} \xi^\mu \partial_{r-s} \eta^\nu \delta_{n \mu}^r \] \[ = \int \xi^\mu \partial_\mu \eta^\nu (p_\nu - P_\nu) + \sum_{r,s} \binom{r}{s} \left( - \binom{r + \mu}{s} - \binom{r}{s - \mu} \right) \xi^\mu \partial_{r-s+\mu} \eta^\nu E_{r+s}^\nu \] \[ + \sum_{m,n,s} \binom{m}{n} \binom{n + \nu}{s} \partial_{m-n} \xi^\mu \partial_{n+\nu} \eta^\nu E_{r+s}^\nu - \xi \leftrightarrow \eta, \] (B.11)
where $\xi \leftrightarrow \eta$ stands for terms obtained by interchanging $\xi$ and $\eta$ everywhere. The first sum vanishes due to Lemma A.3, whereas we use Lemma A.4 to rewrite the last sum as

$$\sum_{m,n,s} \left( \begin{pmatrix} m \cr s \end{pmatrix} (m - s) \right) \left( \begin{pmatrix} m \cr n \end{pmatrix} (n - s) + \begin{pmatrix} n \cr s \end{pmatrix} (n - m) \right) \partial_{m-n} \delta_{s+\mu-s} E_{s+\nu}^m. \quad \text{(B.12)}$$

In the first term, we set $r = m - n$ and use Leibniz’ rule:

$$\sum_{m,r,s} \left( \begin{pmatrix} m \cr s \end{pmatrix} \right) \left( \begin{pmatrix} m - s \cr r \end{pmatrix} \right) \partial_r \xi \partial_{m-n-s-r} E_{s+\nu}^m = \sum_{m,s} \left( \begin{pmatrix} m \cr s \end{pmatrix} \right) \partial_{m-s} (\xi \partial_{s+\nu}) E_{s+\nu}^m.$$

In the second term, we set $r = s - m$, and get

$$\sum_{m,n,r} \left( \begin{pmatrix} m \cr n \end{pmatrix} \right) \left( \begin{pmatrix} n \cr r \end{pmatrix} \right) \partial_{m-n} \delta_{s+\nu} E_{r+\mu+\nu}^m = \sum_{m,r} \left( \begin{pmatrix} m \cr r \end{pmatrix} \right) \partial_{m-r} (\xi \partial_{\nu}) E_{r+\mu+\nu}^m.$$

This term is symmetric under the interchange $\xi \leftrightarrow \eta$, so it cancels one of the unwritten terms. Summing up, we find that the regular terms combine to

$$\int \xi \partial_\nu (p_\nu - P_\nu) + \sum \left( \begin{pmatrix} m \cr s \end{pmatrix} \right) \partial_{m-s} (\xi \partial_{\nu}) E_{s+\nu} - \xi \leftrightarrow \eta = \mathcal{L}_{[\xi,\eta]}.$$

We now turn to the extensions. The current-independent term was calculated in [8] (and, using a different formalism, by Rao and Moody [14]). The result is

$$\text{ext}(\int \xi \partial_\nu (p_\nu ; \int \eta \partial_\nu (p_\nu ;)) = \frac{1}{2\pi i} \int \partial_\nu \xi \partial_\nu \eta'.$$

The rest of the extension becomes

$$\frac{k^{00}}{2\pi i} \int \left\{ - \sum_{r,s} \left( \begin{pmatrix} r \cr s \end{pmatrix} \right) \xi \partial_{r-s} \partial_{s+\nu} + \sum \left( \begin{pmatrix} m \cr n \end{pmatrix} \right) \partial_{m-n} \xi \partial_{s+\nu} \right\} \delta_{s+\mu+\nu}$$

$$+ \sum_{m,n,r,s} \left( \begin{pmatrix} m \cr n \end{pmatrix} \right) \left( \begin{pmatrix} n \cr s \end{pmatrix} \right) \partial_{m-n} \delta_{r-s} \partial_{s+\nu} \delta_{s+\mu+\nu} \right\} \delta_{s+\mu+\nu} \quad \text{(B.14)}$$

$$= \frac{k^{00}}{2\pi i} \int \left\{ \sum_{|s| \leq p-2} \left( \begin{pmatrix} s + \mu + \nu \cr s \end{pmatrix} \right) \xi \partial_{s+\nu} \partial_{\mu+\nu} + \sum_{|n| \leq p-2} \left( \begin{pmatrix} n + \mu + \nu \cr n \end{pmatrix} \right) \partial_{\mu+\nu} \xi \partial_{\nu} \right\}$$

$$- \sum_{|n| \leq p-2} \left( \begin{pmatrix} s + \nu \cr n \end{pmatrix} \right) \left( \begin{pmatrix} n + \mu \cr s \end{pmatrix} \right) \partial_{s-n+\nu} \partial_{s-n+\mu} \partial_{s-n+\nu}.$$

In the last term, there are two possibilities. First assume that $\mu \neq \nu$. Then $\binom{s+\nu}{n}$ vanishes unless $n_\nu \leq s_\nu + 1$, $n_\mu \leq s_\mu$, and $\binom{n+\mu}{s}$ vanishes unless $s_\mu \leq n_\nu + 1$,
\( s_\nu \leq s_\nu \). In addition, both binomial coefficients vanish unless \( n_\rho = s_\rho \) for all \( \rho \neq \mu, \nu \). This leaves four non-zero cases:

1. \( n = s \):
   \[- \sum_{|n| \leq p-1} \binom{n + \nu}{n} \binom{n + \mu}{n} \partial_\nu \xi^\mu \partial_\mu \eta^\nu,\]

2. \( n = s + \nu \):
   \[- \sum_{|n|,|s| \leq p-2} \binom{s + \nu}{s + \nu} \binom{s + \mu + \nu}{s} \partial_\nu \xi^\nu \partial_{\nu + \mu} \eta^\nu,\]

3. \( s = n + \mu \):
   \[- \sum_{|n|,|s| \leq p-2} \binom{n + \mu + \nu}{n + \mu} \binom{n + \mu}{n + \mu} \partial_{\mu + \nu} \xi^\mu \partial_{\mu + \nu} \eta^\nu,\]

4. \( n + \mu = s + \nu \):
   \[- \sum_{|r| \leq p-2} \binom{r + \mu + \nu}{r + \nu} \binom{r + \mu + \nu}{r + \mu} \partial_\mu \xi^\mu \partial_\nu \eta^\nu,\]

where \( n = r + \nu, s = r + \mu \) in the last line. When \( \mu = \nu \), the conditions become \( n_\nu \leq s_\nu + 1 \) and \( s_\mu \leq n_\mu + 1 \), together with \( n_\rho = s_\rho \) for all \( \rho \neq \mu = \nu \). The expression is now non-zero only in the first three cases in the list above, whereas the fourth possibility is subsumed by the first one: if \( \mu = \nu \), \( n + \mu = s + \nu \) whenever \( n = s \). The second and third cases cancel the two first terms in (B.14), which leaves us with

\[
- \frac{k_4}{2\pi i} \int \left\{ \sum_{\mu,\nu} \sum_{|n| \leq p-1} \binom{n + \nu}{n} \binom{n + \mu}{n} \partial_\nu \xi^\mu \partial_\mu \eta^\nu + \right.
\]

\[
+ \sum_{\mu \neq \nu} \sum_{|r| \leq p-2} \binom{r + \mu + \nu}{r + \nu} \binom{r + \mu + \nu}{r + \mu} \partial_\mu \xi^\mu \partial_\nu \eta^\nu \left. \right\} \quad (B.16)
\]

\[
= - \frac{k_4}{2\pi i} \left( N + p + 1 \right) \partial_\nu \xi^\mu \partial_\mu \eta^\nu - \frac{k_4}{2\pi i} \left( N + 1 \right) \partial_\mu \xi^\mu \partial_\nu \eta^\nu,
\]

by Lemma A.7.

We now replace \( \mathcal{L}_\xi \mapsto \mathcal{L}_\xi + T_d\xi \). That the regular terms still yield a realization of \( diff(N) \) is clear by Lemma A.8, so there remains to calculate the extension. Evidently,

\[
\text{ext}([\mathcal{L}_\xi + T_d\xi, \mathcal{L}_\eta + T_d\eta]) = \text{ext}([\mathcal{L}_\xi, \mathcal{L}_\eta]) + \text{ext}([\mathcal{L}_\xi, T_d\eta]) - \text{ext}([\mathcal{L}_\eta, T_d\xi]) + \text{ext}([T_d\xi, T_d\eta]).
\]

The first term is given by (B.13) and (B.16). The second term follows from (B.8):

\[
\text{ext}([\mathcal{L}_\xi, T_d\eta]) = - \frac{k_3}{2\pi i} \left( N + p \right) \left( N + 1 \right) \int \partial_\mu \xi^\mu \partial_\nu \eta^\nu
\]

\[
= - \frac{k_3}{2\pi i} \left( N + p \right) \left( N + 1 \right) \int \partial_\mu \xi^\mu \partial_\nu \eta^\nu. \quad (B.17)
\]

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The third term gives an identical contribution, whereas the fourth contribution follows from \( \text{(B.3)} \):

\[
\begin{align*}
\text{ext}(T_{d\xi}, T_{d\eta}) &= -\frac{k_1 \delta^{\rho}_{\mu} \delta^{\sigma}_{\nu}}{2\pi i} \left( N + p \right) \int \partial_{\rho \xi} \partial_{\sigma \eta} \phi \partial_{\sigma \eta} \phi' \\
&= -\frac{k_1}{2\pi i} \left( N + p \right) \int \partial_{\rho \xi} \partial_{\rho \eta} \phi - \frac{k_2}{2\pi i} \left( N + p \right) \int \partial_{\rho \xi} \partial_{\rho \eta} \phi'.
\end{align*}
\]

The total extension is thus the sum of \( \text{(B.13)}, \text{(B.16)}, \text{(B.17)} \) (twice), and \( \text{(B.18)} \). This fixes the abelian charges \( c_1 \) and \( c_2 \) to the values given in Theorem 3.1.

\[ B.4 \quad \mathcal{L} \mathcal{J} \text{ bracket} \]

\[
[L_f, J_X] = \int \int [f \left( -\dot{\xi}^\mu \partial_\mu - F \right), \sum_{m,n} \left( m \atop n \right) \partial_{m-n} \partial_{m+n} X_a J_{m+n}^a]
\]

\[
= \sum_{m,n} \left( m \atop n \right) \int \int f \left\{ -\dot{\xi}^\mu \partial_{m-n+\mu} \partial_{m+n} \delta + \partial_{m-n} X_a (J_{m+n}^a \dot{\delta} + \frac{d_2}{4\pi i} \delta^a \partial_{m+n} \delta + \sum_{|m| < p} \left( m \atop n \right) \partial_{m-n} \partial_{n+\mu} \partial_{m+n+\mu} \delta) \right\} \]

\[
= \frac{d_2}{4\pi i} \delta^a \sum_{|m| < p} 1 \int (\ddot{f} - i \ddot{f}) X_a.
\]

where we used that \( \dot{q}^\mu \partial_{m-n+\mu} X_a = \partial_{m-n} X_a \). The value of the abelian charge \( c_6 = d_2 \frac{N + p}{N} \) now follows from Lemma A.3.

\[ B.5 \quad \mathcal{L} \mathcal{L} \text{ bracket} \]

Again we initially set \( T_{d\xi} = 0 \):

\[
[L_f, \mathcal{L}_\xi] = \int \int [f \left( -\dot{\xi}^\mu \partial_\mu - F \right), \xi^\mu \partial_\mu - \xi^\mu \partial_\mu - \sum_{m,n} \left( m \atop n \right) \partial_{m-n} \xi^\mu E_{m+n}^\mu] \]

\[
= \int \int f \left\{ -\dot{\xi}^\mu \partial_\mu \xi^\mu \partial_\mu - \xi^\mu \partial_\mu - \sum_{m,n} \left( m \atop n \right) \dot{\xi}^\mu \partial_{m-n} \partial_{m-n+\mu} E_{m+n+\mu} \right\}
\]

\[
+ \xi^\mu \partial_\mu \delta_\mu \dot{\delta} - \xi^\mu \partial_\mu \dot{\delta} + \sum_{m,n} \left( m \atop n \right) \partial_{m-n} \xi^\mu E_{m+n+\mu} \dot{\delta} \]

\[
+ \frac{1}{4\pi i} (\partial_\mu \xi^\mu + d_0 \sum_{m,n} \left( m \atop n \right) \partial_{m-n} \xi^\mu \delta_{m+n+\mu} \delta (\ddot{\delta} + i \dot{\delta}) \right\}
\]

\[
= 0 + \frac{1}{4\pi i} \left( 1 + d_0 \sum_{|n| \leq p-1} \left( n + \mu \atop n \right) \right) \int (\ddot{f} - i \ddot{f}) \partial_\mu \xi^\mu.
\]
In the last step, we used that $\dot{\varphi}_\mu \delta_\mu \xi^\alpha = \dot{\xi}^\alpha$ to eliminate all regular terms. The abelian charge $c_3$ has two contributions: one from the observer’s trajectory and one from $F(t)$; its value now follows from Lemma A.6.

The contribution from $T_d\xi$ follows from [B.19] by replacing $d_2\delta_\mu X_a \mapsto d_1\partial_\mu \xi^\alpha$:

$$[\mathcal{L}_\xi, T_d\xi] = \frac{d_1}{4\pi i} \left( \frac{N + p}{N} \right) \int (\dddot{f} - i\dot{f}) \partial_\mu \xi^\alpha.$$  (B.20)

### B.6 $LL$ bracket

Since $[F(s), \dot{q}^\mu(t)p_\mu(t)] = 0$, $L_f = \int (-f \dot{q}^\mu p_\mu + fF)$ consists of two commuting Virasoro generators. The first consists of $N$ bosonic fields, each contributing $+2$ to $c_4$, and the second contributes $c(N, p)$ by assumption (3.21). Hence $c_4 = 2N + c(N, p)$.

### C Proof of Theorem 6.1

It is straightforward to show that the redefinitions (6.5) only affects the trivial cocycle in the $L\mathcal{E}$ and $L\mathcal{J}$ brackets. Thus the unprimed operators satisfy the DGRO algebra (2.2), with the following modifications:

$$[L_f, \mathcal{L}_\xi] = \frac{c_3}{4\pi i} \int dt \, \dddot{f}(t) \partial_\mu \xi^\alpha(q(t)),$$

$$[L_f, \mathcal{J}_X] = \frac{c_6}{4\pi i} \delta^\alpha \int dt \, \dddot{f}(t) X_a(q(t)).$$  (C.1)

#### C.1 $L\mathcal{J}$ bracket

$$[L_f, \mathcal{J}_X] = \int \int \left[ (-\dot{q}^\mu p_\mu + \dot{q}^\mu P_\mu) \sum_{m,n} \left( \frac{\mathbf{m}}{\mathbf{n}} \right) \delta_{m-n} X_a J_m^a \right]$$

$$= \sum_{m,n} \left( \frac{\mathbf{m}}{\mathbf{n}} \right) \int \int f \left\{ -\dot{q}^\mu \delta_{m-n-\mu} X_a J_m^a - \frac{k_0}{2\pi i} \delta_{n+\mu} \dot{\xi}^\alpha \right\}$$

$$= \sum_{m,n} \left\{ -\left( \frac{\mathbf{m}}{\mathbf{n}} \right) + \left( \frac{\mathbf{m} + \mu}{\mathbf{n}} \right) - \left( \frac{\mathbf{m}}{\mathbf{n} - \mu} \right) \right\} \int \int \dot{f} \dot{q}^\mu \delta_{m-n-\mu} X_a J_m^a$$

$$= \frac{k_0}{2\pi i} \sum_{|n| \leq p-1} \left( \mathbf{n} + \mu \right) \int \dot{f} \dot{q}^\mu \partial_\mu X_a$$

$$= 0 + \frac{k_0}{2\pi i} \frac{N + p}{N + 1} \int \dot{f} \dot{X}_a,$$

by Lemmas A.3 and A.5. Hence $c_6 = 2k_0 \left( \frac{N + p}{N + 1} \right)$.  

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C.2 \( LL \) bracket

We first set \( T_{d\xi} = 0 \).

\[
[L_f, \mathcal{L}_\xi] = \int \int [f(-:q^\mu p_\nu + \dot{q}^\mu P_\nu:) :\xi^\mu p_\nu - \xi^\mu P_\mu + \sum_{m,n} \left( \frac{m}{n} \right) \partial_{m-n} \xi^\mu E^m_{n+\mu}] 
\]

The regular terms yield

\[
\int f \left\{ -\dot{\xi}^\mu p_\mu + \dot{\xi}^\mu P_\mu - \sum_{m,n} \left( \frac{m}{n} \right) \partial_{m-n} \dot{\xi}^\mu E^m_{n+\mu} + \sum_{m,n} \left( \frac{m}{n} \right) \dot{q}^\mu \partial_{m-n} \xi^\mu (E^m_{n+\mu} - E^m_{n+\mu+p}) \right\} + \int \int f \xi^\mu (p_\mu - P_\mu) \hat{\delta} = 0,
\]

by Lemma \[A.3\] and \( \dot{q}^\mu \partial_\rho \xi^\mu = \dot{\xi}^\mu \).

The extension has one contribution \( \frac{1}{4\pi i} \int (\dddot{f} - i\ddot{f}) \partial_\mu \xi^\mu \) from the observer’s trajectory. The rest of the extension becomes

\[
\int \int f \dot{q}^\mu \sum_{m,n} \left( \frac{m}{n} \right) \partial_{m-n} \dot{\xi}^\mu E^m_{n+\mu} \frac{k_{00}}{2\pi i} \delta^m_{n+\mu+p} \hat{\delta} = -k_4 \int \sum_{|n| < p-2} \left( \frac{n + \mu + \rho}{n} \right) \dot{q}^\mu \partial_\mu \xi^\mu (C.4)
\]

where we used Lemma \[A.8\] in the last step.

The contribution from \( T_{d\xi} \) follows from \[C.2\] by replacing \( k_0 \delta^a X_a \mapsto k_3 \partial_\mu \xi^\mu \):

\[
[\mathcal{L}_\xi, T_{d\xi}] = \frac{k_3}{2\pi i} \left( \frac{N + p}{N + 1} \right) \int \ddot{f} \partial_\mu \xi^\mu (C.5)
\]

Taking the three contributions together, we find that

\[
c_3 = 1 + 2k_4 \left( \frac{N + p}{N + 2} \right) + 2k_3 \left( \frac{N + p}{N + 1} \right).
\] (C.6)

C.3 \( LL \) bracket

We note that \( p_\mu(t) + P_\mu(t) \) satisfy the same Heisenberg algebra as \( p_\mu(t) \):

\[
[p_\mu(s) - P_\mu(s), q^\nu(t)] = \delta^\nu_\mu \delta(s-t),
\]

\[
[p_\mu(s) - P_\mu(s), p_\nu(t) - P_\nu(t)] = 0.
\] (C.7)

Therefore, \( L_f' = -\int f :q^\mu (p_\mu - P_\mu) : \) satisfies the same algebraic relations as \( L_f^0 = -\int f :\dot{q}^\mu p_\mu : \) does, i.e. a Virasoro algebra with central charge \( c_4 = 2N \).
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