PRODUCT TYPE POTENTIAL ON THE XY MODEL:
SELECTION OF MAXIMIZING PROBABILITY AND A LARGE
DEVIATION PRINCIPLE

J. MOHR
IME, UFRGS - PORTO ALEGRE, BRASIL

Abstract. Given an interval \([a, b]\) the associated XY model is the space \(\Omega = [a, b]^\mathbb{N}\) with an a priori probability \(\nu\) on the state space \([a, b]\). In most of the cases the normalized Lebesgue probability is the a priori probability. One can consider some natural metrics on \(\Omega\) in such way that \(\Omega\) is compact.

Using the a priori probability \(\nu\) and a Ruelle operator one can define entropy for an invariant probability for the shift \(\sigma\) acting on \(\Omega\).

Given a Lipschitz potential \(f : [a, b]^\mathbb{N} \to \mathbb{R}\) one can ask: among the invariant probabilities which one is the equilibrium probability \(\mu\) for the interaction described by \(f\)? As usual the equilibrium probability for \(f\) is the one maximizing pressure. The above question can be analyzed via the Ruelle operator technique.

We will present here the case of the product type potential on the XY model and in this setting we can show the explicit expression of the equilibrium probability.

We will also consider questions about Ergodic Optimization, maximizing probabilities, subactions and we will show selection of a maximizing probability, when temperature goes to zero.

Finally we show a large deviation principle when temperature goes to zero and we present an explicit expression for the deviation function.

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1. Introduction

Let \(\Omega = [0, 1]^\mathbb{N}\) be the symbolic space XY and the a priori probability \(da\) (Lebesgue).

We consider the metric in \(\Omega = [0, 1]^\mathbb{N}\) given by:

\[
d(x, y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n}
\]

where \(x = (x_1, x_2, \ldots)\) and \(y = (y_1, y_2, \ldots)\) are on \(\Omega\). Note that \(\Omega\) is compact by Tychonoff’s theorem.

We denote by \(C\) the space of continuous functions from \(\Omega \to \mathbb{R}\).
Given a continuous function $f : \Omega \to \mathbb{R}$ let $L_f : C \to C$ be the Ruelle operator that sends $\varphi \mapsto L_f(\varphi)$, which is defined for each $x \in \Omega$ by the following expression

$$L_f(\varphi)(x) = \int_0^1 e^{f(a,x_1,x_2, \ldots)} \varphi(a,x_1,x_2, \ldots) da.$$  

As usual, we define the dual of the Ruelle operator, denoted by $L_f^*$, on the space of Borel measures on $\Omega$ as the operator that sends a measure $\mu$ to the measure $L_f^*(\mu)$ defined, for each $\varphi \in C$, by

$$\int_\Omega \varphi dL_f^*(\mu) = \int_\Omega L_f(\varphi) d\mu.$$  

The general case, where $\Omega = M^N$, $M$ is a compact set and the a priori probability is not necessarily Lebesgue is studied in [6] and is called one-dimensional lattice system theory. If we suppose $M = [0,1]$ and the a priori probability is Lebesgue, this is so-called XY one-dimensional model. It is a classical problem in Physics to analyze the Statistical Mechanics of lattices when the spin are on $S^1$ (see [5]). It is shown in [6] and [2] that if $f$ is Lipschitz then there exists a strictly positive Lipschitz eigenfunction $h_f$ for $L_f$ associated to a positive eigenvalue $\lambda_f$ and also the existence of an eigenprobability for $L_f^*$. Moreover, the eigenvalue $\lambda_f$ is simple (which means the eigenfunction is unique up to a multiplicative constant).

We denote by $M_\sigma$ the set of invariant measures for the shift map, $\sigma : \Omega \to \Omega$, defined by $\sigma(x_1,x_2,x_3,\ldots) = (x_2,x_3,x_4,\ldots)$. In [6] was defined the entropy $h(\mu)$ of $\mu \in M_\sigma$ and was proved a variational principle: given a Lipschitz potential $f$ and $\lambda_f$ is the maximal eigenvalue of $L_f$ then

$$\log \lambda_f = \sup_{\mu \in M_\sigma} \left\{ h(\mu) + \int f(x) d\mu(x) \right\}. \quad (1)$$

Moreover the supremum is attained on the eigenprobability of the dual of the Ruelle operator.

These are theoretical questions on the Thermodynamic Formalism for the XY model which were already addressed on some recent papers. However, there is lack of interesting examples where the theory can be applied. Here we will present several results and explicit examples on the Thermodynamic Formalism of the XY model in order to fill this gap.

We consider a continuous potential $f : \Omega \to \mathbb{R}$ of the form

$$f(x) = f(x_1,x_2,x_3,\ldots) = \sum_{j=1}^{\infty} f_j(x_j)$$

where $f_j : [0,1] \to \mathbb{R}$ are fixed functions. We say that the function $f$ is of the product type. We will also suppose that $\sum_{j=1}^{\infty} f_j(x_j)$ is absolutely convergent, for all $x \in \Omega$.

We will assume in some examples that each function $f_j$, $j \in \mathbb{N}$, is a Lipschitz functions with Lipschitz constant smaller than $\frac{1}{2}$. In this case one can show that $f : \Omega \to \mathbb{R}$ is Lipschitz.

Functions of the product type are studied in [4] in the case $\Omega = M^N$ where $M$ is a finite or countable alphabet. In [4] was shown, among other things, explicit formulae for the leading eigenvalue, the eigenfunction and eigenmeasure of the Ruelle operator.
In section 2 we will exhibit the explicit expression of the maximal eigenvalue, of the positive eigenfunction of the Ruelle operator and of the eigenprobability of the dual of the Ruelle operator, when \( M = [0, 1] \). If \( f \) is Lipschitz we know, by [6], that the eigenprobability satisfies a variational principle, and hence this measure is the equilibrium probability for \( f \).

Let \( \beta = 1/T \) be the inverse of the temperature \( T \), if we consider the potential \( \beta f \) and we denote by \( \tilde{\mu}_\beta \) the eigenprobability of \( L^*_{\beta f} \), its well known that the limits (in the weak* topology) of \( \tilde{\mu}_\beta \), when \( \beta \to \infty \), are related with the following problem: given \( f : \Omega \to \mathbb{R} \) Lipschitz continuous, we want to find probabilities that maximize \( \int_{\Omega} f(x) d\mu(x) \) over \( \mathcal{M}_\sigma \). If we define

\[
m(f) = \max_{\mu \in \mathcal{M}_\sigma} \left\{ \int f \, d\mu \right\},
\]

any measure that attains the maximal value is called a maximizing measure for \( f \).

See [6] for general results in ergodic optimization theory, when \( M = [0, 1] \).

It is shown in [6]: if for some subsequence we have \( \tilde{\mu}_{\beta_n} \rightharpoonup \mu_\infty \), when \( n \to \infty \), then \( \mu_\infty \) is a maximizing measure.

One interesting question is: \( \tilde{\mu}_\beta \) converges to a maximizing measure, when \( \beta \to \infty \) ? In the affirmative case we say we have selection of this maximizing measure. The problem of selection and non selection of a maximizing measure was studied in several works, see [10] and [3] for examples of non selection in the case \( M \) is the unitary circle.

We will show in section 3 that we have selection of a maximizing measure in the case \( f \) is of the product type and \( f(a, a, a, \ldots) \) has one or two maximum points in \( [0, 1] \), also a large deviation principle is true for this convergence.

In [7] was shown a large deviation principle in the case \( M = [0, 1] \) and the maximizing probability is unique for a potential that depends only in two coordinates. In the present work we do not suppose the maximizing probability is unique and the potential can depends on all coordinates.

2. Explicit expressions for eigenfunction and eigenprobability of functions of product type

Let us consider a continuous potential of the product type \( f : \Omega \to \mathbb{R} \) defined by

\[
f(x) = f(x_1, x_2, x_3, \ldots) = \sum_{j=1}^{\infty} f_j(x_j),
\]

where \( f_j : [0, 1] \to \mathbb{R} \) are fixed functions and such that \( \sum_{j=1}^{\infty} f_j(x_j) \) is absolutely convergent, for all \( x \in \Omega \).

Sometimes is more convenient use the following notation: \( g_i(a) = e^{f_i(a)} \), then

\[
e^{f(x_1, x_2, x_3, \ldots)} = e^{\sum_{j=1}^{\infty} f_j(x_j)} = \prod_{j=1}^{\infty} g_j(x_j) := g(x_1, x_2, x_3, \ldots).
\]

In this way, we can write

\[
\mathcal{L}_f(\varphi)(x) = \int_0^1 g_1(a) \prod_{j=2}^{\infty} g_j(x_{j-1}) \varphi(a, x_1, x_2, \ldots) da.
\]

In this section we will show the explicit expressions for the maximal eigenvalue and for positive eigenfunction of \( \mathcal{L}_f \) and for the eigenprobability of \( \mathcal{L}^*_f \).
Proposition 1. Suppose $f$ is continuous of the product type and that 
\( a) \sum_{j=1}^{\infty} f_j(x_j) \) is absolutely convergent, 
\( b) \sum_j \sum_{i>j} f_i(x_j) < \infty, \) for all \( x = (x_1, x_2, x_3, \ldots) \in \Omega. \)

If we define \( h_f(x) = \prod_{j=1}^{\infty} h_j(x_j), \) where \( h_j(b) = \prod_{i>j} g_i(b) \) and \( \lambda = \int_0^{1} \prod_{j=1}^{\infty} g_j(b) \) \( db. \) Then \( \mathcal{L}_f(h_f) = \lambda_f h_f. \)

Proof:
First we will see that \( \lambda_f = \int_0^{1} \prod_{j=1}^{\infty} g_j(b) \) \( db < \infty, \) in fact, note that
\( \prod_{j=1}^{\infty} g_j(b) = e^{\sum_{j=1}^{\infty} f_j(b)} = e^{f(b,b,\ldots)}, \) as \( f \) is continuous and \( \sum_{j=1}^{\infty} f_j(b) \) is absolutely convergent we have \( \lambda < \infty. \)

To see that \( h_f \) is well defined, note that a classical result claims that for a sequence of positive number \( a_j, \) the product \( \prod_{j=1}^{\infty} a_j \) is well defined, if and only if, \( \sum_j \log(a_j) < \infty. \) As \( f \) satisfies \( b) \) we have that \( \sum_j \log h_j(x_j) = \sum_j \sum_{i>j} \log g_i(x_j) = \sum_j \sum_{i>j} f_i(x_j) < \infty \) this implies \( h_f(x) = \prod_{j=1}^{\infty} h_j(x_j) < \infty. \)

Now we will show that \( h_f \) is an eigenfunction to \( \mathcal{L}_f: \) as \( h_j(b) = \prod_{i>j} g_i(b), \) multiplying it by \( g_j(b) \) we obtain
\( g_j(b) h_j(b) = g_j(b) \prod_{i>j} g_i(b) = \prod_{i>j-1} g_i(b) = h_{j-1}(b). \)

In particular \( g_1(a)h_1(a) = \prod_{i=1}^{j} g_i(a) \) and \( g_j(x_{j-1}) h_j(x_{j-1}) = h_{j-1}(x_{j-1}). \) This implies
\[
\mathcal{L}_f(h_f)(x) = \int_0^{1} g_1(a) \prod_{j=2}^{\infty} g_j(x_{j-1}) h_1(a) \prod_{j=2}^{\infty} h_j(x_{j-1}) da =
\int_0^{1} g_1(a) h_1(a) \prod_{j=2}^{\infty} g_j(x_{j-1}) h_j(x_{j-1}) da =
\int_0^{1} \prod_{i=1}^{\infty} g_i(a) da \prod_{j=2}^{\infty} h_{j-1}(x_{j-1}) = \lambda_f h_f(x).
\]

Note that \( h_f(x) = \prod_{j=1}^{\infty} \prod_{i>j} g_i(x_j) = \prod_{j=1}^{\infty} \prod_{i>j} e^{f_i(x_i)} = e^{\sum_{j=1}^{\infty} \sum_{i>j} f_i(x_i)}. \)

Proposition 2. Assume that the functions \( f_j, \) \( j \in \mathbb{N} \) are Lipschitz with Lipschitz constant smaller than \( \frac{1}{2} \) and that for some \( \bar{x} \) we know that \( \sum_{j=1}^{\infty} \sum_{i>j} f_i(\bar{x}_j) < \infty. \) Then, the hypothesis of Proposition 1 are true.

Proof:
\[
\left| \sum_{j=1}^{\infty} \sum_{i>j} f_i(x_j) - \sum_{j=1}^{\infty} \sum_{i>j} f_i(\bar{x}_j) \right| \leq \sum_{j=1}^{\infty} \sum_{i=j+1}^{\infty} \frac{1}{2^i} = \sum_{j=1}^{\infty} \frac{1}{2^{j-1}} = 2.
\]
We say that a potential \( \tilde{f} \) is normalized if \( \mathcal{L}_f(1) = 1 \) or \( \int_0^1 e^{\tilde{f}(a,x_1,x_2,...)} \, da = 1 \). Given a potential \( f, h_f \) and \( \lambda_f \) the eigenfunction and the eigenvalue of the operator \( \mathcal{L}_f \), we define the normalized potential associated to \( f \), as usual by \( \tilde{f} = f + \log h_f - \log h_f \circ \sigma - \log \lambda_f \).

In the exponential scale \( e^{\tilde{f}} \) became

\[
\tilde{g}(x) = \frac{g(x)h_f(x)}{h_f \circ \sigma(x) \lambda_f} = \frac{g(x)}{\lambda_f} \prod_{i=1}^{\infty} \frac{h_i(x_i)}{h_i(x_{i+1})} = \frac{g(x)h_1(x_1)}{\lambda_f} \prod_{i=1}^{\infty} \frac{h_{i+1}(x_{i+1})}{h_i(x_{i+1})},
\]

now using equation (2) and the definition of \( g \) we get

\[
\tilde{g}(x) = \prod_{i=1}^{\infty} g_i(x_i) h_1(x_1) = g_1(x_1) h_1(x_1) = \frac{g_1(x_1) g_2(x_1) h_2(x_1)}{\lambda_f} = ... \]

\[
... = \prod_{i=1}^{\infty} g_i(x_i) \int \prod_{j=1}^{\infty} g_j(b) db = \frac{e^{\sum_{i=1}^{\infty} f_i(x_i)}}{\int e^{\sum_{j=1}^{\infty} f_j(b)} db} \int e^{\tilde{f}(x_1,x_1,...)} db
\]

This implies \( \tilde{g} \) (and \( \tilde{f} \)) depends only on the first coordinate of \( x \).

It is known from [6] that if \( f \) is Lipschitz continuous then there exists a unique eigenprobability \( \tilde{\mu}_f \) for \( \mathcal{L}_f^* \), and that the measure \( \mu_f = \frac{1}{\tilde{\mu}_f} \tilde{\mu}_f \) is an eigenmeasure for \( \mathcal{L}_f^* \), where \( h_f \) is the unique eigenfunction of \( \mathcal{L}_f \) associated to the maximal eigenvalue \( \lambda_f \). The next proposition exhibits the explicit form of these measures.

**Proposition 3.** Suppose \( f \) is continuous and satisfies the hypothesis of Proposition 4. Let \( \mu_f = \otimes_{n=1}^{\infty} \mu_n \) and \( \tilde{\mu}_f = \otimes_{n=1}^{\infty} \tilde{\mu}_0 \) be measures of the product type given by the following expressions

\[
d\mu_n(a) = \frac{\prod_{i=1}^{n} g_i(a) da}{\int \prod_{j=1}^{\infty} g_j(b) db}, \quad d\tilde{\mu}_n(a) = \frac{\prod_{i=1}^{n} g_i(a) da}{\int \prod_{j=1}^{\infty} g_j(b) db} = \tilde{g}(a) da.
\]

Then, we have \( \mathcal{L}_f^*(\mu_f) = \lambda_f \mu_f, \mathcal{L}_f^*(\tilde{\mu}_f) = \tilde{\mu}_f \) and \( \mu_f = \frac{1}{\tilde{\mu}_f} \tilde{\mu}_f \).

**Proof:**
Let \( \varphi : [0,1]^N \to \mathbb{R} \), then

\[
\int \varphi d\mathcal{L}_f^*(\mu_f) = \int \mathcal{L}_f(\varphi) d\mu_f = \]

\[
= \int_{[0,1]^N} \int_{[0,1]} g(a,x_1,x_2,x_3,...) \varphi(a,x_1,x_2,x_3,...) da d\mu_1(x_1) d\mu_2(x_2) d\mu_3(x_3)... = \]

\[
= \int_{[0,1]^N} \int_{[0,1]} \varphi(a,x_1,x_2,x_3,...) g_1(a) g_2(x_1) g_3(x_2) g_4(x_3)... da d\mu_1(x_1) d\mu_2(x_2) d\mu_3(x_3)... = \]

\[
= \int_{[0,1]^N} \int_{[0,1]} \varphi(a,x_1,x_2,x_3,...) g_1(a) da g_2(x_1) d\mu_1(x_1) g_3(x_2) d\mu_2(x_2) g_4(x_3) d\mu_3(x_3)... = \]

\[
= \lambda_f \int_{[0,1]^N} \int_{[0,1]} \varphi(a,x_1,x_2,x_3,...) g_1(a) da \frac{g_2(x_1)}{\lambda_f} g_3(x_2) d\mu_1(x_1) g_3(x_2) d\mu_2(x_2) g_4(x_3) d\mu_3(x_3)... \]

Note that \( g_{n+1}(x_n) d\mu_n(x_n) = \prod_{i=n+1}^{\infty} g_i(x_i) dx_i = d\mu_{n+1}(x_n) = d\mu_1(a) \), hence we get

\[
\int \varphi d\mathcal{L}_f^*(\mu_f) =
\]
and this implies that $\mathcal{L}_f^*(\mu_f) = \lambda_f \mu_f$.

Let $\varphi : [0, 1]^n \to \mathbb{R}$, then

$$
\int \varphi \, d\mathcal{L}_f^*(\tilde{\mu}_f) = \int \mathcal{L}_f(\varphi) \, d\tilde{\mu}_f = \int_{[0,1]^n} \hat{g}(a) \varphi(a, x_1, x_2, x_3, ...) \, da \, d\tilde{\mu}_0(x_1) \, d\tilde{\mu}_0(x_2) \, d\tilde{\mu}_0(x_3) = \int_{[0,1]^n} \varphi(a, x_1, x_2, x_3, ...) \, da \, d\tilde{\mu}_0(x_1) \, d\tilde{\mu}_0(x_2) \, d\tilde{\mu}_0(x_3) = \int_{[0,1]^n} \varphi(a, x_1, x_2, x_3, ...) \, da \, d\tilde{\mu}_0(x_1) \, d\tilde{\mu}_0(x_2) \, d\tilde{\mu}_0(x_3) = \int \varphi \, d\tilde{\mu}_f,
$$

and this implies that $\mathcal{L}_f^*(\tilde{\mu}_f) = \lambda_f \tilde{\mu}_f$.

Finally, let us show that $\mu_\varphi = \frac{1}{\lambda_f} \tilde{\mu}_f$ as $\hat{g}(x_j) = \prod_{i=1}^{\infty} \frac{g_i(x_j)}{\prod_{j=1}^{\infty} g_j(b_j) db}$ and $h_j(x_j) = \prod_{i=1}^{\infty} g_i(x_j)$, we have that $\frac{\tilde{\mu}_f(x_j)}{\tilde{\mu}_f(\varphi(x_j))} = \prod_{i=1}^{\infty} \frac{g_i(x_j)}{\prod_{j=1}^{\infty} g_j(b_j) db}$, therefore

$$
\frac{d\tilde{\mu}_f(x)}{h_f(x)} = \frac{\hat{g}(x_1)}{h_1(x_1)} dx_1 \otimes \frac{\hat{g}(x_2)}{h_2(x_2)} dx_2 \otimes \frac{\hat{g}(x_3)}{h_3(x_3)} dx_3 \otimes \ldots = d\mu_1(x_1) \otimes d\mu_2(x_2) \otimes d\mu_3(x_3) \otimes \ldots
$$

From [6] we get that, if $\tilde{f}$ is normalized and $\tilde{\mu}_f$ is such that $\mathcal{L}_f^*(\tilde{\mu}_f) = \tilde{\mu}_f$, then the entropy of $\tilde{\mu}_f$ is given by,

$$
h(\tilde{\mu}_f) = -\int_{[0,1]^n} \tilde{f}(x) \, d\tilde{\mu}_f(x) = \int \log \hat{g}(x) \, d\tilde{\mu}_f(x) = -\int_{[0,1]^n} \log \hat{g}(x) \, d\tilde{\mu}_f(x) = \int \log \prod_{i=1}^{\infty} g_i(x_1) - \log \prod_{i=1}^{\infty} g_i(b) \, db \, d\tilde{\mu}_f(x) = \int \log \prod_{i=1}^{\infty} g_i(x_1) - \log \prod_{i=1}^{\infty} g_i(b) \, db \, d\tilde{\mu}_f(x) = \int \log \lambda_f - \int_{[0,1]} \sum_{i=1}^{\infty} f_i(a) \hat{g}(a) \, da.
$$

This is an explicit expression for the entropy of this example.

Also we compute

$$
\int_{[0,1]^n} \log g \, d\mu_f = \int_{[0,1]^n} \log \prod_{i=1}^{\infty} g_i(x_1) \hat{g}(x_1) dx_1 \hat{g}(x_2) dx_2 \ldots \hat{g}(x_i) dx_i \ldots = \sum_{i=1}^{\infty} \int_{[0,1]} \log g_i(x_1) \hat{g}(x_1) dx_1 \hat{g}(x_2) dx_2 \ldots \hat{g}(x_i) dx_i \ldots = \sum_{i=1}^{\infty} \int_{[0,1]} \log g_i(x_1) \hat{g}(x_1) dx_1 = \sum_{i=1}^{\infty} \int_{[0,1]} \log g_i(a) \hat{g}(a) da = \sum_{i=1}^{\infty} \int_{[0,1]} \log g_i(a) \hat{g}(a) da.
$$
And this implies that $h(\hat{\mu}) = \log \lambda_f - \int_{[0,1]} \log g d\hat{\mu}$ or
\[
\log \lambda_f = h(\hat{\mu}) + \int_{[0,1]} \log g d\hat{\mu}.
\]

This shows that $\hat{\mu}$ satisfies a variational principle, as in [6], i.e., let $f$ be a Lipschitz continuous potential and $\lambda_f$ be the maximal eigenvalue of $L_f$, then
\[
\log \lambda_f = P(f) = \sup_{\mu \in M_\sigma} \left\{ h(\mu) + \int f(x) d\mu(x) \right\},
\]
where $M_\sigma$ denote the set of $\sigma$ invariant Borel probability measures over $\mathcal{B}$. And the supremum is attained on the measure $\hat{\mu}_f$.

3. Zero temperature, selection of the maximizing measure and large deviation principle

Now we will analyze the question of zero temperature, when $\beta \to \infty$, for this example. General results on Ergodic Optimization and selection when temperature goes to zero, for the case $\Omega = \{1,\ldots,d\}^\mathbb{N}$, can be found in [1].

For each $\beta > 0$ we consider the potential $\beta f(x) = \sum_{j=1}^\infty \beta f_j(x_j)$, so the eigenfunction of $L_{\beta f}$ is given by $h_\beta(x) = e^{\sum_{j=1}^\infty \beta f_j(x_j)} = e^{\beta \sum_{j=1}^\infty \sum_{i>j} f_i(x_i)}$. And, the equilibrium probability is given by $\tilde{\mu}_\beta = \otimes_{n=1}^\infty \tilde{\mu}_{0,\beta}$ where
\[
\tilde{\mu}_{0,\beta}(a) = \frac{e^{\beta \sum_{j=1}^\infty f_j(a)}}{\int e^{\beta \sum_{i=1}^\infty f_i(b)} db} da.
\]

As usual, we would like to investigate the limits of $\tilde{\mu}_\beta$ and $\frac{1}{\beta} \log h_\beta(x)$, when $\beta \to \infty$.

The limits of $\tilde{\mu}_\beta$ are related with the following problem: given $f : \mathcal{B} \to \mathbb{R}$ Lipschitz, we want to find probabilities that maximize the value
\[
\int f(x) d\mu(x).
\]

We define
\[
m(f) = \max_{\mu \in M_\sigma} \left\{ \int f d\mu \right\}.
\]

Any of the probability measures which attains the maximal value will be called a maximizing probability measure, which will be denoted generically by $\mu_\infty$.

We say that $u$ is a calibrated subaction if
\[
m(f) = \max_{a \in [0,1]} \{ f(ax) + u(ax) - u(x) \}.
\]

We know that by [6] that, if the potential $f$ is Lipschitz continuous, then

i) $\lim_{\beta \to \infty} \frac{1}{\beta} \log \lambda_\beta = m(f)$,

where $\lambda_\beta = \int e^{\beta \sum_{j=1}^\infty f_j(a)} da$.

ii) Any limit, in the uniform topology,
\[
u := \lim_{n \to \infty} \frac{1}{\beta_n} \log(h_{\beta_n f}),
\]
is a calibrated subaction for $f$. 

Note that  

$$
\frac{1}{\beta} \log h_\beta(x) = \sum_{j=1}^{\infty} \sum_{i \geq j} f_i(x_j).
$$

does not depends on $\beta$, hence by the previous result we have that $u(x) = \sum_{j=1}^{\infty} \sum_{i \geq j} f_i(x_j)$ is a calibrated subaction.

**Remark:** We can also show directly that $u$ is a calibrated subaction by the following argument:

$$
u(x) = \sum_{i=2}^{\infty} f_i(x_1) + \sum_{i=3}^{\infty} f_i(x_2) + \sum_{i=4}^{\infty} f_i(x_3) + \ldots, \quad \text{and} \quad u(ax) = \sum_{i=2}^{\infty} f_i(ax_1) + \sum_{i=3}^{\infty} f_i(ax_1) + \sum_{i=4}^{\infty} f_i(ax_2) + \ldots, \quad \text{hence}$$

$$u(ax) - u(x) = \sum_{i=2}^{\infty} f_i(ax_1) + \sum_{i=3}^{\infty} f_i(ax_1) - \sum_{i=2}^{\infty} f_i(ax_2) + \sum_{i=4}^{\infty} f_i(ax_2) + \ldots = \sum_{i=2}^{\infty} f_i(ax_1) - \sum_{i=2}^{\infty} f_i(x_1).$$

Therefore

$$f(ax) + u(ax) - u(x) = f_1(a) + \sum_{i=2}^{\infty} f_i(x_{i-1}) + \sum_{i=2}^{\infty} f_i(a) - \sum_{i=2}^{\infty} f_i(x_{i-1}) = \sum_{i=1}^{\infty} f_i(a).$$

And

$$\max_{a \in [0,1]} \{ f(ax) + u(ax) - u(x) \} = \max_{a \in [0,1]} \sum_{i=1}^{\infty} f_i(a).$$

This shows that $u(x) = \sum_{j=1}^{\infty} \sum_{i \geq j} f_i(x_j)$ is a calibrated subaction.

**Lemma 4.** Suppose $W_t : M \to \mathbb{R}$ converges uniformly to $W : M \to \mathbb{R}$, when $t \to +\infty$. Then

$$\lim_{t \to \infty} \frac{1}{t} \log \int_M e^{\mu W_t(a)} d\nu(a) = \max_{a \in M} W(a).$$

For the proof of this Lemma see [8].

We use this lemma to obtain

$$m(f) = \lim_{\beta \to \infty} \frac{1}{\beta} \log \int_0^1 e^{\beta \sum_{j=1}^{\infty} \sum_{i \geq j} f_i(a) da} = \max_{a \in [0,1]} \sum_{j=1}^{\infty} f_j(a).$$

**Lemma 5.** Suppose $l(\beta) = \int_a^\beta e^{\beta F(t)} dt$, where $\beta$ is real and positive, $F(t), F''(t)$ and $F'''(t)$ are real and continuous in $\alpha \leq t \leq \beta$. Let $t = a$ be the only point of maximum of $F(t)$ in $[a, \delta]$, with $\alpha < a < \delta$, thus the asymptotic approximation as $\beta \to \infty$ is

$$\int_{\alpha}^{\delta} e^{\beta F(t)} dt = \left( \frac{-2\pi}{\beta F''(a)} \right)^{\frac{1}{2}} e^{\beta F(a)} + e^{\beta F(a)} O(\beta^{-\frac{3}{2}}).$$

For the proof of this Lemma see [8].

We will use Lemma 5 to show that we have selection of the maximizing measure in the following cases:
Theorem 6. Let \( F(b) = f(b, b, ...) = \sum_{i=1}^{\infty} f_i(b) \) and suppose \( F(b), F'(b) \) and \( F''(b) \) are real and continuous in \( 0 < b < 1 \).

a) Suppose \( F \) has only one maximum in \( a_1 \in (0, 1) \) then \( \lim_{\beta \to \infty} \mu_{0, \beta}(a) = \delta_{a_1} \) and \( \lim_{\beta \to \infty} \hat{\mu}_\beta = \otimes_{n=1}^{\infty} \delta_{a_n} \).

b) Suppose \( F \) has two maximum points in \( (0, 1) \), say \( 0 < a_1 < a_2 < 1 \), then we have \( \lim_{\beta \to \infty} \mu_{0, \beta} = \mu_{0, \infty} = p_1 \delta_{a_1} + p_2 \delta_{a_2} \) and \( \lim_{\beta \to \infty} \hat{\mu}_\beta = \otimes_{n=1}^{\infty} p_1 \delta_{a_1} + p_2 \delta_{a_2} \).

where \( p_1 + p_2 = 1 \) and \( \frac{\mu_{a_1}}{\mu_{a_2}} = \sqrt{\frac{F'(a_2)}{F'(a_1)}} \).

Proof:

a) If \( F \) has only one maximum in \( a_1 \in (0, 1) \) then

\[
\int_0^1 e^{\beta F(b)} db = \left( \frac{-2\pi}{\beta F''(a_1)} \right)^{\frac{1}{2}} e^{\beta F(a_1)} + e^{\beta F(a_1)} O(\beta^{-\frac{3}{2}}).
\]

Note that \( \tilde{\mu}_{0, \beta}(da) = \int_{a_1}^{a_2} e^{\beta F(b)} db \). Therefore, for each \( a \in [0, 1] \) we have

\[
\int_0^{a_1} e^{\beta F(b)} db = \int_{a_2}^{a_1} e^{\beta F(b)} db = \frac{e^{\beta F(a_1)} - e^{\beta F(a_1) - F(a_1)}}{\beta F''(a_1)} + O(\beta^{-\frac{3}{2}}).
\]

We conclude that the above expression goes to 0 if \( a \neq a_1 \) and goes to \( \infty \) if \( a = a_1 \), when \( \beta \to \infty \). Hence, \( \lim_{\beta \to \infty} \mu_{0, \beta}(a) = \delta_{a_1} \) and \( \lim_{\beta \to \infty} \hat{\mu}_\beta = \otimes_{n=1}^{\infty} \delta_{a_n} \) and we have selection of the maximizing measure.

b) Now we consider the case where \( F \) has two maximum points in \( (0, 1) \), say \( 0 < a_1 < a_2 < 1 \) we can divide \([0, 1]\) in two intervals, each one containing only one maximum point and apply the lemma to obtain

\[
\int_0^1 e^{\beta F(b)} db = \left( \frac{-2\pi}{\beta F''(a_1)} \right)^{\frac{1}{2}} e^{\beta F(a_1)} + e^{\beta F(a_1)} O(\beta^{-\frac{3}{2}}) +
\]

\[
\frac{-2\pi}{\beta F''(a_2)} \frac{1}{2} e^{\beta F(a_2)} + e^{\beta F(a_2)} O(\beta^{-\frac{3}{2}}) =
\]

\[
e^{\beta F(a_1)} \left[ \left( \frac{-2\pi}{\beta F''(a_1)} \right)^{\frac{1}{2}} + \left( \frac{-2\pi}{\beta F''(a_2)} \right)^{\frac{1}{2}} + O(\beta^{-\frac{3}{2}}) \right],
\]

and

\[
\int_0^1 e^{\beta F(b)} db = \frac{e^{\beta F(a)} - e^{\beta F(a_1)}}{\beta F''(a_1)} + \frac{e^{\beta F(a) - F(a_1)}}{\beta F''(a_2)} + O(\beta^{-\frac{3}{2}}).
\]

Therefore, if \( a \neq a_1 \) and \( a \neq a_2 \) the density of \( \tilde{\mu}_{0, \beta} \) goes to 0.

Consider now \( \mu_{0, \beta}(a_1 - \varepsilon, a_1 + \varepsilon) = \int_{a_1 - \varepsilon}^{a_1 + \varepsilon} e^{\beta F(b)} db \), therefore if \( \varepsilon_1, \varepsilon_2 > 0 \) are such that \( a_2 \notin (a_1 - \varepsilon_1, a_1 + \varepsilon_1) \) and \( a_1 \notin (a_2 - \varepsilon_2, a_2 + \varepsilon_2) \), then

\[
\frac{\tilde{\mu}_{0, \beta}(a_1 - \varepsilon_1, a_1 + \varepsilon_1)}{\tilde{\mu}_{0, \beta}(a_2 - \varepsilon_2, a_2 + \varepsilon_2)} = \frac{\int_{a_1 - \varepsilon_1}^{a_1 + \varepsilon_1} e^{\beta F(a)} da}{\int_{a_2 - \varepsilon_2}^{a_2 + \varepsilon_2} e^{\beta F(a)} da} =
\]
We can also prove a large deviation principle and exhibit the deviation function:

**Proposition 7.** We denote \( u(x) = \sum_{j=1}^{\infty} \sum_{i>j} f_i(x_j) \) the calibrated subaction, where \( x = (x_1, x_2, \ldots, x_j, \ldots) \). Consider the function

\[
I(x) = \sum_{j \geq 1} u(\sigma^j(x)) - u(\sigma^{j-1}(x)) - f(\sigma^{j-1}(x)) + m,
\]

then

i) \( I(x) = \sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} -f_i(x_j) + m \right) = \sum_{j=1}^{\infty} \left( -f(x_j, x_j, x_j, \ldots) + m \right) \)

ii) For each cylinder \( D = A_1 \times \ldots \times A_n \), where \( A_i \) are intervals of \([0,1]\), the following limit exists

\[
\lim_{\beta \to \infty} \frac{1}{\beta} \log \tilde{\mu}_\beta(D) = - \inf_{x \in D} I(x).
\]

**Proof:** i)

\[
I(x) = -u(x) + \sum_{j=1}^{\infty} \left( -f(\sigma^{j-1}(x)) + m \right) =
\]

\[
= \sum_{j=1}^{\infty} \sum_{i>j} -f_i(x_j) + \sum_{j=1}^{\infty} \left( -f(\sigma^{j-1}(x)) + m \right) =
\]

\[
= \sum_{j=1}^{\infty} \sum_{i>j} -f_i(x_j) + \sum_{j=1}^{\infty} \left( \sum_{k=1}^{\infty} -f_k(x_{k+j-1}) + m \right) =
\]

\[
= \sum_{i=2}^{\infty} -f_i(x_1) + \sum_{i=3}^{\infty} -f_i(x_2) + \sum_{i=4}^{\infty} -f_i(x_3) + \ldots +
\]

\[
+ \sum_{k=1}^{\infty} -f_k(x_k) + m + \sum_{k=1}^{\infty} -f_k(x_{k+1}) + m + \sum_{k=1}^{\infty} -f_k(x_{k+2}) + m + \ldots =
\]

\[
= \sum_{i=2}^{\infty} -f_i(x_1) + \sum_{i=3}^{\infty} -f_i(x_2) + \sum_{i=4}^{\infty} -f_i(x_3) + \ldots -
\]

\[
- f_1(x_1) - f_2(x_2) - f_3(x_3) + \sum_{k=4}^{\infty} -f_k(x_k) + m -
\]

\[
- f_1(x_2) - f_2(x_3) + \sum_{k=3}^{\infty} -f_k(x_{k+1}) + m - f_1(x_3) + \sum_{k=2}^{\infty} -f_k(x_{k+2}) + m + \ldots =
\]
\[
\begin{align*}
&= \sum_{i=1}^{\infty} -f_i(x_1) + \sum_{i=1}^{\infty} -f_i(x_2) + \sum_{i=1}^{\infty} -f_i(x_3) + \ldots + \sum_{k=4}^{\infty} -f_k(x_k) + m + \\
&\quad + \sum_{k=3}^{\infty} -f_k(x_{k+1}) + m - \sum_{k=2}^{\infty} -f_k(x_{k+2}) + m + \ldots = \\
&= \sum_{j=1}^{\infty} \left( \sum_{i=1}^{j} -f_i(x_j) + m \right) = \sum_{j=1}^{\infty} \left( -f(x_j, x_j, \ldots) + m \right) = \sum_{j=1}^{\infty} \left( -F(x_j) + m \right).
\end{align*}
\]

ii) Let \( D = A_1 \times \ldots \times A_n \) be a cylinder of \( \mathbb{R}^n \), then

\[
\tilde{\mu}_\beta(D) = \tilde{\mu}_{0, \beta}(A_1)\tilde{\mu}_{0, \beta}(A_2)\ldots\tilde{\mu}_{0, \beta}(A_n) = \frac{\int_{A_1} e^{\beta F(a)} da}{\int_{[0,1]} e^{\beta F(b)} db} \frac{\int_{A_2} e^{\beta F(a)} da}{\int_{[0,1]} e^{\beta F(b)} db} \ldots \frac{\int_{A_n} e^{\beta F(a)} da}{\int_{[0,1]} e^{\beta F(b)} db}.
\]

and

\[
\log \tilde{\mu}_\beta(D) = \log \left( \int_{A_1} e^{\beta F(a)} da \right) + \log \left( \int_{A_2} e^{\beta F(a)} da \right) + \ldots + \\
+ \log \left( \int_{A_n} e^{\beta F(a)} da \right) - n \log \left( \int_{[0,1]} e^{\beta F(b)} db \right).
\]

Therefore

\[
\lim_{\beta \to \infty} \frac{1}{\beta} \log \tilde{\mu}_\beta(D) = \lim_{\beta \to \infty} \frac{1}{\beta} \log \left( \int_{A_1} e^{\beta F(a)} da \right) + \ldots + \lim_{\beta \to \infty} \frac{1}{\beta} \log \left( \int_{A_n} e^{\beta F(a)} da \right) - \\
- n \lim_{\beta \to \infty} \frac{1}{\beta} \log \left( \int_{[0,1]} e^{\beta F(b)} db \right) = \\
= \max_{x_1 \in A_1} F(x_1) + \ldots + \max_{x_n \in A_n} F(x_n) - n \max_{a \in [0,1]} F(a) = \\
= - \inf_{x_1 \in A_1, \ldots, x_n \in A_n} \sum_{j=1}^{n} \left( F(x_j) - \max_{a \in [0,1]} F(a) \right) = - \inf_{x \in D} I(x).
\]

Note that as \( m(f) = \max_{a \in [0,1]} \sum_{i=1}^{\infty} f_i(a) = \max_{a \in [0,1]} F(a) \), we have

\[
I(x) = \sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} -f_i(x_j) + \max_{a \in [0,1]} \sum_{i=1}^{\infty} f_i(a) \right) = \sum_{j=1}^{\infty} \left( -F(x_j) + \max_{a \in [0,1]} F(a) \right)
\]

This implies that \( I(x) \geq 0 \), and \( I(x_1, x_2, \ldots, x_j, \ldots) = 0 \), if and only if, \( x_j \in \text{argmax } F \), for all \( j \in \mathbb{N} \).

Note that \( I(x_1, \ldots, x_n, x_1, \ldots, x_n, \ldots) = \infty \), if there exists \( x_j \notin \text{argmax } F \), \( 1 \leq j \leq n \).

Note also that to have \( I(x) < \infty \) is necessary that \( F(x_j) \to m(f) \).

\section*{Example 1}

Let us define \( f(x) = \sum_{i=1}^{\infty} -(x_i)^{2i} \) and suppose that we take \([-\frac{1}{2}, \frac{1}{2}] \) instead of \([0,1]\). Then \( f_i(a) = -a^{2i} \) and note that \( \frac{df_i(a)}{da} = -2ia^{2i-1} \), if we define \( c_i := \sup_{a \in [-\frac{1}{2}, \frac{1}{2}]} \left| \frac{df_i(a)}{da} \right| = 2i2^{-2i+1} = i2^{-2i+2} \), hence \( c_i < 2^{-i} \) for each \( i \geq 5 \).

Note also that \( c_i \leq 4 \cdot 2^{-i} \) for \( i = 1, 2, 3 \) and \( 4 \).
Therefore, we get that the Lipschitz constant of \( f_i \) is smaller than \( 4 \cdot 2^{-i} \), for all \( i \).

Then, \( |f(x) - f(y)| \leq \sum_{i=1}^{\infty} |f_i(x_i) - f_i(y_i)| \leq \sum_{i=1}^{\infty} \frac{4|x_i - y_i|}{2^i} = 4d(x, y) \), i.e., \( f \) is Lipschitz with constant 4.

Also,

\[
F(a) = \sum_{i=1}^{\infty} f_i(a) = -\frac{1}{1 - a^2} + 1 = 1 + \frac{1}{a^2 - 1}.
\]

In this case \( m(f) = 0 \) and if \( x = (x_1, x_2, ..., x_j, ...) \) then we get

\[
I(x) = -\sum_{j=1}^{\infty} \left( 1 + \frac{1}{x_j - 1} \right).
\]

and \( u(x) = \sum_{j=1}^{\infty} \sum_{i>j} f_i(x_j) = \sum_{j=1}^{\infty} \sum_{i>j} - (x_j)^{2i} \).

**Example 2:** Suppose we take \([-1, 1]\) instead \([0, 1]\) and \( f_i(a) = a^i i^{-\gamma}, \gamma > 1 \), then

\[
F(a) = \sum_{i=1}^{\infty} f_i(a) = \sum_{i=1}^{\infty} \frac{a^i}{i^{\gamma}}.
\]

This is the polylogarithm function.

Each \( f_i \) is a Lipschitz function: in the same way as before we consider

\[
c_i := \sup_{a \in [-1, 1]} \left| \frac{df_i(a)}{da} \right| = \sup_{a \in [-1, 1]} \left| \frac{a^{i-1}}{i^{\gamma-1}} \right| = i^{1-\gamma}.
\]

In this example we have

\[
\log(h_f(x)) = \sum_{j=1}^{\infty} \sum_{i>j} \frac{(x_j)^i}{i^{\gamma}}.
\]

This function is not Lipschitz but satisfies the hypothesis of Proposition 1 when \( \gamma > 2 \). Indeed,

\[
\sum_{i>j} \frac{(x_j)^i}{i^{\gamma}} \leq \sum_{i>j} \frac{|x_j|^i}{i^{\gamma}} \leq \sum_{i=j+1}^{\infty} \frac{1}{i^{\gamma}} \leq \int_{j}^{\infty} x^{-\gamma} \, dx = \lim_{b \to \infty} \frac{x^{-\gamma+1}}{-\gamma+1} \bigg|_{b}^{1} = \frac{j^{-\gamma+1}}{-\gamma+1},
\]

if \( \gamma > 1 \).

Hence,

\[
\log(h_f(x)) \leq \sum_{j=1}^{\infty} \frac{j^{-\gamma+1}}{-\gamma+1} = \frac{1}{1-\gamma} + \sum_{j=2}^{\infty} \frac{j^{-\gamma+1}}{-\gamma+1} \leq \frac{1}{1-\gamma} + \int_{1}^{\infty} x^{-\gamma+1} \, dx = \frac{1}{1-\gamma} + \lim_{b \to \infty} \frac{x^{-\gamma+2}}{(-\gamma + 1)(-\gamma + 2)} \bigg|_{b}^{1} = \frac{1}{1-\gamma} + \frac{1}{(1-\gamma)(2-\gamma)} < \infty,
\]

if \( \gamma > 2 \).

Note that \( \max_{a \in [0,1]} F(a) \) occurs when \( a = 1 \), hence \( \max_{a \in [0,1]} F(a) = \zeta(\gamma) \).
Then,
\[ I(x) = \sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} -f_i(x_j) + \max_{a \in [0,1]} \sum_{i=1}^{\infty} f_i(a) \right) = \sum_{j=1}^{\infty} \left( -\sum_{i=1}^{\infty} \frac{x_i^j}{i^\gamma} + \zeta(\gamma) \right) \]
and
\[ u(x) = \sum_{j=1}^{\infty} \sum_{i>j} f_i(x_j) = \sum_{j=1}^{\infty} \sum_{i>j} \frac{x_i^j}{i^\gamma}. \]

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