A new variational approach and its application to heavy quarkonia

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By combining the variational principle with Heisenberg uncertainty principle in an effective Hamiltonian for heavy flavored mesons, we introduce a framework to estimate masses and radii of these states from an analytical constraint. In a novel manner, a model for quark velocity and a model for quark momentum width are introduced. These kinematical model parameters are obtained as analytical functions of inter quark separation in heavy quarkonia. The values of such quark parameters are then used in the calculation of S-wave annihilation decay rates of $c\bar{c}$ and $b\bar{b}$. To test the accuracy of our technique we first calculate the spin averaged masses, scalar radii and annihilation decay rates of charmonium and bottomonium without and with relativistic corrections by solving the Schrödinger wave equation with the appropriate parametrization of the Song-Lin potential. The Schrödinger wave equation is solved numerically with the matrix Numerov method and we observe a good agreement with the experimental measurements and other theoretical calculations and extract strong running coupling constant for $c\bar{c}$ and $b\bar{b}$ systems. In non-relativistic settings, heavy meson spectra have been obtained and extended to rather higher excited states within our framework by using bare masses of $c$ and $b$ quarks which we have extracted from analysis of experimental data.

Keywords: Non-relativistic potential model; charmonium and bottomonium; variational principle.

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1. Introduction

The study of heavy quarkonia in quantum chromodynamics (QCD) is a field theoretical non-perturbative problem which has promoted the development of several techniques appropriate to address this issue. Under certain considerations, it can be simplified to a non-relativistic (NR) quantum mechanical problem by exploring the interaction strength between quarks inside hadrons in the static limit of the interaction potential (SLIP) between quarks. Lattice QCD simulations \cite{1,2} suggest for this potential a Coulombic term at short distances plus a linearly rising part at large inter quark separations, which to-date has motivated a large number of phenomenological models \cite{3–19} that capture the quantum chromodynamic traits of strong interactions. Whilst the SLIP picture for light-quarks must be taken with a grain of salt, for the case of heavy mesons, $c\bar{c}$ and $b\bar{b}$, this picture is accurate enough to describe the mass spectra of heavy quarkonia. Assuming spherical symmetry, the SLIP can be cast in the general form $V(r) = -Ar^{-\alpha} + Br^{\beta} + V_0$, where $r$ is the inter quark distance, $A$, $B$ and $V_0$ are constants and $\alpha$ and $\beta$ are free parameters assessed either from Lattice QCD, see for example \cite{20}, or fixed by fitting the known experimental masses of hadronic states. The choice $\alpha = \beta = 1$ and $V_0 = 0$ corresponds to the Cornell potential. The choice $V_0 = 0$, $\alpha = 2 = \beta/2$ corresponds to the anharmonic potential \cite{21} and other choices give rise to commonly used potentials.

The relevant Schrödinger wave equation (SWE) with any of these SLIPs often has to be solved numerically. To that end, a vast number of numerical strategies have been implemented in literature among which, to count a few, we find the shooting method \cite{22}, the Asymptotic Iteration Method (AIM) \cite{23} and many others such as different forms of Runge-Kutta methods. Most numerical strategies implemented for solving the SWE are as precise as going up to $O(d^5)$ of grid spacing $d$. The Matrix Numerov Method (MNM) (see, for instance \cite{24}) which in the context of meson physics has been put forward by the Qena group \cite{25} and extended by our group \cite{26}, departs from a discretization of the kinetic term in the SWE in such a manner that the problem of solving the radial SWE is cast in the form of a matrix eigenvalue problem. In this form, its accuracy is of $O(d^6)$. On the other hand, many proposals of quark model (QM) potentials have used quark masses far higher than those reported in the Particle Data Booklet in various years \cite{3–19}. The quark model community has exploited the freedom in choosing quark masses (the so called constituent quark masses) and quark interaction energy (QIE) parameters in $m_{q_1} + m_{q_2} + \Sigma = M_{\text{meson}}$. This hides the factual and physical values of quark masses $m_{q_1}$, $m_{q_2}$ and those of the quark interaction energy $\Sigma$.

In our present work we study the spectrum of heavy $c\bar{c}$, $b\bar{b}$ states. For that purpose, we use bare quark masses in SWE extracted from experimental evidence got by various Collab-
orations. Regarding the SLIP, we are interested in the choice of parameters $\alpha = \beta = 1/2$ and $V_0 = 0$, known in literature as the Song-Lin (SL) potential [27]. We select this effective potential beyond the paradigmatic Cornell potential because there are a number of appealing features [28, 29] that the SL potential displays better than the Cornell model. Let us recall that the SL potential has been theoretically calculated by coupling the non-Abelian character of QCD to a heavy dilaton field [30]. From phenomenological perspective [27], the first term in this potential $b/\sqrt{r}$ is inspired from leptonic widths of vector mesons $\rho, \omega, \phi, J/\psi$ and $Y$ measured in experiments [31] and perturbative QCD at short distances, while the term $(-a/\sqrt{r})$ has been motivated by considering a chromo-electric flux tube [27] between a quark and antiquark $\bar{Q}$ in a quarkonium system where perturbative QCD is hard to manage, and the possible colour-screening effects on strong force which are considered of considerable importance [33–36]. In fact, the colour-screening in our parametrization of SL is far more simply added than in [33–36] using Cornell potential. The SL potential has been considered in [23,37,38] and applied only to low lying meson states with no or insignificant application to decay rates and other quarkonia observables. We calculate spin averaged masses of $c\bar{c}$, $b\bar{b}$ spectroscopic states, which are relevant for the prediction of hyperfine splittings between $^1L_L$ and $^3L_L$ states defined as $(^3L_L) - L_L$, where $^3L_L$ is spin averaged mass of the triplet $^3L_{L-1}, ^3L_{L+1}, ^3L_{L+3}$. By considering the spin averaged wave functions at contact, we readily obtain the leptonic, diphoton, digluon, trilphon, trilgluon and one photon plus 2 gluons decay rates of $S$-wave states by demanding agreement of 1S state for each one of these decays with the experiment. We do this to find the value of the QCD coupling constant ($\alpha_s$) and then use it for decay modes of 2S to 5S states. A comparison with known values of these decay widths is presented, rendering our full dynamical picture of the NR framework in fair agreement with the dynamics of these heavy meson states. We also include momentum width and velocity corrections to these decays and get their excellent agreement with experiment. We obtain wavefunctions, masses, radii, quark velocities, quark momentum widths, annihilation decay rates and strong running coupling constant for charmonium and bottomonium states in the following manner: we fit the parameters of the potential by demanding exactness of the theoretical mass of only one 1S spin averaged state of $c\bar{c}$ with experimental value and then calculate all remaining spin averaged masses for heavy mesons with the parameters $a = 0.7011 \text{ GeV}^{3/2}$, $b = 0.8912 \text{ GeV}^{1/2}$ and this parametrization of SL potential makes the SWE a robust dynamical equation of heavy quarkonia even beyond the original expectations of the model [27] and extends the achievements of other similar calculations [23, 37, 38]. We compare our findings against experimental results and other theoretical calculations and systematically extend our results up to 8S, 7P, 7D, 7F, 7G. This extension to higher states is allowed and reliable because suitable colour-screening effects are already present in confining term of our SL potential parametrization and this colour screening of the confining term in inter quark potential is seen to produce higher orbital and angular momentum excited states efficiently [33].

After preparing the firm ground from MNM we obtain the charmonium and bottomonium spin averaged masses and radii, in order to circumvent the struggle for numerical accuracy in the numerical solution, we introduce a variational approach combining the Heisenberg uncertainty principle and the variational principle in the Hamiltonian of the SWE. Such an approach, which we refer to simply as the variational principle (VP), imposes an algebraic transcendental relation between the masses and the radii of these states and is partially based in our previous work in [26]. The results from our VP are less uncertain than those from MNM. In the remaining part of this article we present the details as follows: Section 2 introduces MNM to solve the SWE. Section 3 presents our model of the quark velocities, and the decay widths for different decay modes of spin averaged $S$-states of charmonia, and bottomonia. In Sec. 4 we develop a strategy of combining the variation in mesonic Hamiltonian with Heisenberg UP through our momentum width-model to establish analytical relation between spin averaged masses and radii of heavy meson states. Conclusion and further possible developments are discussed in Sec. 5. In the Appendix A to this paper we have calculated spin averaged masses by including hyperfine, spin-orbit, and tensor interactions to SL potential and compare these against the spin averaged masses in Tables I and II obtained without these spin effects while in Appendix B we have estimated the bare quark masses. The color screening is explored in Appendix C.

2. Numerov method

The non-relativistic kinetic energy operator for a two quark system is

$$T = m_1 + m_2 + \frac{|\vec{p}_1|^2}{2m_1} + \frac{|\vec{p}_2|^2}{2m_2},$$

where $m_{1,2}$ and $\vec{p}_{1,2}$ are the corresponding quark masses and momenta. Then the Hamiltonian of this system is

$$\mathcal{H} = \sum_{i=1}^{2} \left( m_i + \frac{|\vec{p}_i|^2}{2m_i} \right) + V(\vec{r}_1, \vec{r}_2).$$

If these two quarks are moving in a spherically symmetric potential that depends only on their mutual separation $r = |\vec{r}_1 - \vec{r}_2|$, namely $V(\vec{r}_1, \vec{r}_2) = V(r)$, the problem can be decoupled into the center of mass motion of the two quark system and the relative motion of the quarks about their center of mass. Factorizing the former, the SWE for the relative motion reduces to the one-body eigenvalue problem

$$\left( m_1 + m_2 + \frac{|\vec{p}|^2}{2\mu} + V(r) \right) \Psi(\vec{r}) = \Delta \Psi(\vec{r}),$$

where $\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass of the system, $\Delta$ is the mass of the meson state and $\vec{p}$ is the momentum of quark about the center of mass of the meson.
Working in spherical coordinates and separating the angular part in Eq. (3) according to \( \Psi(\vec{r}) = R_{nl}(r)Y_{ml}(\theta, \phi) \), where \( Y_{ml}(\theta, \phi) \) are the spherical harmonic functions normalised to unity (with \( n, l \) and \( m \) the principal, orbital angular momentum and magnetic quantum numbers), the masses of heavy meson states are obtained by solving the radial one-body, one dimensional-like SWE \( \psi(r) = rR_{nl} \),

\[
(m_1 + m_2 + \Sigma (m_1, m_2, l, a, b; r)) \psi(r) = \Delta \psi(r),
\]

where we define

\[
\Sigma (m_1, m_2, l, a, b; r) = -\frac{1}{2\mu} \frac{d^2}{dr^2} + \frac{\ell(\ell + 1)}{2\mu r^2} + V(r),
\]

as an operator for QIE. In what follows, we obtain the charmonium and bottomonium spectra by solving the SWE with the SL potential [27]

\[
V(r) = -\frac{b}{\sqrt{r}} + a\sqrt{r},
\]

having \( a \) and \( b \) as free parameters. We fix these parameters to the lightest \( c\bar{c} \) mass, and then derive the rest of the heavy quarkonium full spectrum through the Matrix Numerov Eq. (10). Explicitly, these values are

\[
a = 0.7011 \text{ GeV}^2, \quad b = 0.8912 \text{ GeV}^2,
\]

whereas the bare quark masses are taken to be

\[
m_c = 1.2 \text{ GeV}, \quad m_b = 4.668 \text{ GeV}.
\]

In this form we parametrize charmonium mass spectrum as

\[
\Delta_{2S+1L_J}^2 = 2.4 + \Sigma
\]

while for bottomonia we take

\[
\Delta_{2S+1L_J}^2 = 9.334 + \Sigma.
\]

Our parametrization in Eq. (7) is unique and the same for \( c\bar{c} \) and \( b\bar{b} \) mesons. The masses in Eq. (8) are not arbitrary, see Appendix B for this point.

In order to solve the corresponding SWE in Eq. (4) we discretize its radial coordinate \( r \) into \( N \) equidistant points \( r_i \) separated a distance \( d \) within a preselected characteristic length \( r_{max} \) suitable for the description of heavy meson spectra and represent the kinetic operator in Eq. (4) in terms of the two trigonal matrices

\[
A_{N,N} = \frac{I_{-1} - 2I_0 + I_1}{d^2},
\]

\[
B_{N,N} = \frac{I_{-1} + 10I_0 + I_1}{12},
\]

such that Eq. (4) is expressed in the form as in Refs. [24–26]

\[
-\frac{1}{2\mu} A_{N,N} B^{-1}_{N,N} \psi_i + \left[ V_N(r_i) + \frac{\ell(i + 1)}{2\mu r_i^2} + m_1 + m_2 \right] \psi_i = \Delta \psi_i,
\]
diagonal (PD), in the diagonal one step below PD, and in the
diagonal one step above PD which are filled with entries as
1. Concretely, we use $r_{\text{max}} = 4 \text{ fm}$ and fix the number of
grid points with $N = 350$. These two numbers are obtained
from the stability of the inverse of the masses of $S$, $P$ and $D$
states against variation of the number of points and the size
of the domain of integration. For the sake of illustration, we
show the variation of the inverse of the masses for $S$-states
for charmonium and bottomonium with the number of grid
points in Fig. 1 and with the interval of integration in Fig. 2.

2.1. Spectra

Numerical solution of Eq. (10) gives wavefunctions corre-
sponding to spin averaged masses of charmonium and bot-
tomonium. These are our well behaved spin averaged or-
thonormal wavefunctions plotted for $S$, $P$ and $D$ states in
Figs. 3 and 4. The spin averaged masses are shown in Tables I
and II and compared against the experimental values [39] and

![Image](Figure 3. Wave functions for first five $S$-states (upper panel), $P$-
states (mid panel) and $D$-states (lower panel) of $c\bar{c}$ mesons in the SL
potential.)

![Image](Figure 4. Wave functions for first five $S$-states (upper panel), $P$-
states (mid panel) and $D$-states (lower panel) of $b\bar{b}$ meson in the SL
potential.)

other theoretical findings. It is important to note that finding
the correct interaction energy between quarks (QIE) is a key
feature for the success or failure of a quark model and is a
real essence of understanding the matter in visible universe.
The authors in Ref. [23] and Ref. [27] used charm quark mass
1.8 GeV which is $600 \text{ MeV}$ more than our bare charm quark
mass. It means that these authors estimated interaction en-
ergy $600 \text{ MeV}$ smaller in charmonium system than ours. As
for the bottom quark is concerned, these authors estimate the
interaction energy to be $1066 \text{ MeV}$ smaller when compared to
ours. The correct input mass to SWE produces correct predic-
tions and bare quark masses available so far are not less than
being the correct masses of charm in Eq. (B.1) and bottom in
Eq. (B.2). So our calculated interaction energies for charmo-
nia and bottomonia must be correct in our bare quark model
(BQM). A quick analysis of QIE in Tables I and II shows
interaction energy in $nL$ spin averaged states for bottomonia
is smaller than that for charmonia, and QIE for $(n+1)L$ state
is larger than for $nL$. As a complementary note, in the Appendix A we have calculated these mass spectra by including spin-spin, spin-orbit, and tensor interactions between quark $Q$ and antiquark $\bar{Q}$ as part of the dynamical Eq. (10) instead of using the leading order perturbation method. These are the so called quarkonia mass spectra with spin effects. We have compared these spectra with those in Tables I and II obtained without incorporating the spin effects in the form of mass-plots Fig. 7.

As for the size of the bound states, the root-mean-square radii $r_{\text{rms}}$ can be directly obtained from the numerical solution of the SWE as,

### Table I. Masses [GeV] for $c\bar{c}$ spin-averaged $S$, $P$ and $D$ States. We calculated spin averaged masses using $\left( \sum_J (2J+1)\Delta_J / \sum_J (2J+1) \right)$ from Ref. [62] for Cornell potential model in Refs. [27,49] for purpose of comparison with our predicted $5P$ and $4D$ spin averaged states. In the last column, N stands for “from Numerov Method”, E for “from Experimental mass of the meson” and VP is for “from our variational principle” (Sec. 4).

| $nL$ | Exp. Masses [33] | $m_c = 1.8$ GeV [22] | $m_c = 1.8$ GeV [26] | $m_c = 1.2$ GeV [Our work] | Our QIE [MeV] | N/E/VP |
|------|------------------|----------------------|----------------------|-----------------------------|--------------|--------|
| 1S   | 3.067            | 3.104                | 3.097                | 3.066                       | 666/         | 660/   | 662   |
| 2S   | 3.649            | 3.703                | 3.673                | 3.764                       | 1364/        | 1250/  | 1211  |
| 3S   | 4.040            | 4.090                | 4.017                | 4.208                       | 1808/        | 1640/  | 1624  |
| 4S   | 4.415            | 4.375                | 4.276                | 4.545                       | 2145/        | 2015/  | 1937  |
| 5S   | —                | 4.692                | 4.487                | 4.824                       | 2424/        | —      | —     |
| 1P   | 3.525            | 3.572                | 3.524                | 3.566                       | 1166/        | 1125/  | 1042  |
| 2P   | —                | 3.986                | 3.907                | 4.055                       | 1655/        | —      | —     |
| 3P   | —                | 4.280                | 4.186                | 4.418                       | 2018/        | —      | —     |
| 4P   | —                | 4.580                | 4.410                | 4.714                       | 2314/        | —      | —     |
| 5P   | —                | 5.034 [27]           | —                    | 4.969                       | 2569/        | —      | —     |
| 1D   | 3.769            | 3.806                | 3.791                | 3.902                       | 1502/        | 1369/  | 1388  |
| 2D   | 4.159            | 4.185                | 4.090                | 4.292                       | 1892/        | 1759/  | 1713  |
| 3D   | —                | 4.474                | 4.328                | 4.605                       | 2205/        | —      | —     |
| 4D   | —                | 4.898 [49]           | —                    | 4.871                       | 2471/        | —      | 2231  |
| 5D   | —                | —                    | —                    | 5.107                       | 2707/        | —      | 2444  |

### Table II. Masses [Gev] for $b\bar{b}$ spin-averaged $S$, $P$, $D$ States. We calculated spin averaged masses using $\left( \sum_J (2J+1)\Delta_J / \sum_J (2J+1) \right)$ from Ref. [62] for Cornell potential model in Ref. [48] for purpose of comparison with our predicted $5P$, $4D$ and $5D$ spin averaged states. In the last column, N stands for “from Numerov Method”, E for “from Experimental mass of the meson” and VP is for “from our variational principle” (Sec. 4).

| $nL$ | Exp. Masses [33] | $m_b = 5.2$ GeV [22] | $m_b = 5.2$ GeV [26] | $m_b = 4.668$ GeV [Our work] | Our QIE [MeV] | N/E/VP |
|------|------------------|----------------------|----------------------|-----------------------------|--------------|--------|
| 1S   | 9.444            | 9.473                | 9.460                | 9.444                       | 110/         | 108/   | 122   |
| 2S   | 10.023           | 10.024               | 10.034               | 10.098                      | 764/         | 687/   | 651   |
| 3S   | 10.355           | 10.327               | 10.356               | 10.482                      | 1148/        | 1019/  | 1017  |
| 4S   | 10.579           | 10.593               | 10.589               | 10.766                      | 1432/        | 1243/  | 1283  |
| 5S   | 10.865           | 10.788               | 10.776               | 10.998                      | 1664/        | 1529/  | 1498  |
| 1P   | 9.900            | 9.912                | 9.902                | 9.930                       | 596/         | 564/   | 496   |
| 2P   | 10.260           | 10.275               | 10.261               | 10.358                      | 1024/        | 924/   | 890   |
| 3P   | —                | 10.580               | 10.512               | 10.665                      | 1331/        | —      | 1178  |
| 4P   | —                | 10.703               | 10.711               | 10.911                      | 1577/        | —      | 1408  |
| 5P   | —                | 11.013 [48]          | —                    | 11.119                      | 1785/        | —      | 1602  |
| 1D   | 10.161           | 10.156               | 10.162               | 10.234                      | 900/         | 825/   | 810   |
| 2D   | —                | 10.434               | 10.433               | 10.565                      | 1231/        | —      | 1093  |
| 3D   | —                | 10.625               | 10.643               | 10.825                      | 1491/        | —      | 1328  |
| 4D   | —                | 10.934 [48]          | —                    | 11.043                      | 1709/        | —      | 1529  |
| 5D   | —                | 11.143 [48]          | —                    | 11.232                      | 1898/        | —      | 1703  |
Where the symbols $n$ and $\ell$ stand for the principal and orbital angular momentum quantum number of the meson, respectively. Furthermore, the heavy quarkonia states exhibit momentum width $\beta$ as function of the quarkonium size determined by their quantum numbers in our proposed functional form:

$$\beta = \frac{\delta}{\rho_{\text{rms}}}.$$  \hfill (12)

Radii from Eq. (11) and momentum width from Eq. (12) for different states are reported in Table III. A variation of momentum width with the size of the $S$, $P$ and $D$ states of charmonia and bottomonia are depicted on right panels in Figs. 5 and 6, along with the analytical expressions for momentum width as functions of instantaneous quark separation $r$. From Table III we observe that bottomonia are more compact objects than corresponding charmonia. Furthermore the charmonia sizes are roughly twice as large as those of the similar states of the bottomonia while momentum width for $c$-quark mesons is approximately half that of the $b$ quark.

### 3. Other quarkonia observables

From the discussion of the previous section, we obtain a few parameters that allow us to characterize the dynamics of these heavy quark systems.

3.1. A model for quark velocities

Quark velocity is a vital ingredient in taking a decision whether to describe quark dynamics in QCD perturbatively or non-perturbatively, and in the determination of hadron masses using string quark model Ref. [46]. The root-mean-square velocity of bottom quark in the $b\bar{b}$ ground state is about $0.3c$ and that of the charm quark in the lowest energy state of $c\bar{c}$ is $0.5c$ (where $c$ is the speed of light in vacuum; see, for instance, Ref. [47]). Moreover, and speaking naively, in the study of various types of decays of a hadron the distribution of angular momentum-momentum of the partons within it among possible decay products is a necessary item both in experimental measurements (for example the $J^{PC}$ of a meson) and in theoretical calculations of decays (as we do in Subsec. 3.2.1 in the form of momentum width and velocity corrections). Incidentally, how the momentum is distributed among constituents is equally important for inelastic scattering of leptons from composite particles as in deep inelastic scattering experiments. This concept is of prime importance for explanation of the effect observed by the European Muon Collaboration (EMC) [48], the EMC effect, where in Deep Inelastic Scattering (DIS) experiments the scattering cross-section of leptons from quarks in nucleons of heavy atomic nuclei is smaller than their scattering cross-section from quarks in nucleons of light atomic nuclei. In addition, this momentum distribution is fundamentally dependent on parton velocity, see for example Eqs. (69), (70) and (106) in Ref. [46] for a flux tube quark model. For these reasons, we introduce a model for the quark velocity which is applicable not only for the ground states of hadrons, but also for excited states. Another reason for promulgating this definition is of

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**Table III.** Radii $r_{\text{rms}}^{cc}$ of $c\bar{c}$ and $r_{\text{rms}}^{bb}$ of $b\bar{b}$ states in [fm] and corresponding momentum width $\beta^{cc}$ and $\beta^{bb}$ in [GeV] in spin averaged $S$, $P$, and $D$ quarkonia states.

| $n\ell$ | $r_{\text{rms}}^{cc}$ ($\text{SWE}$) $/ r_{\text{rms}}^{cc}$ ($\text{VP}$) $/ \text{Ref.}$ [63] | $\beta^{cc} / \beta_{\text{SWE}}^{cc} / \beta_{\text{VP}}^{cc}$ | $r_{\text{rms}}^{bb}$ ($\text{SWE}$) $/ r_{\text{rms}}^{bb}$ ($\text{VP}$) $/ \text{Ref.}$ [64] | $\beta^{bb} / \beta_{\text{SWE}}^{bb} / \beta_{\text{VP}}^{bb}$ |
|--------|---------------------------------|-------------------------------|---------------------------------|-------------------------------|
| 1S     | 0.439/ 0.3533/ 0.41             | 0.682/ 0.693                 | 0.225/ 0.179 / 0.233           | 1.335/ 1.369                 |
| 2S     | 0.915/ 0.922/ 0.91             | 0.765/ 0.760                 | 0.488/ 0.501 / 0.545           | 1.434/ 1.397                 |
| 3S     | 1.352/ 1.363/ 1.38             | 0.813/ 0.810                 | 0.737/ 0.755 / 0.805           | 1.493/ 1.457                 |
| 4S     | 1.762/ 1.773/ 1.87             | 0.851/ 0.846                 | 0.972/ 0.991 / 1.030           | 1.544/ 1.514                 |
| 5S     | 2.151/ 2.161/ 2.39             | 0.882/ 0.879                 | 1.200/ 1.216 / 1.232           | 1.588/ 1.562                 |
| 1P     | 0.697/ 0.775/ 0.71             | 0.717/ 0.645                 | 0.370/ 0.416 / 0.435           | 1.350/ 1.202                 |
| 2P     | 1.155/ 1.195/ 1.19             | 0.779/ 0.753                 | 0.628/ 0.658 / 0.711           | 1.432/ 1.368                 |
| 3P     | 1.577/ 1.601/ 1.67             | 0.824/ 0.812                 | 0.869/ 0.893 / 0.945           | 1.496/ 1.456                 |
| 4P     | 1.976/ 1.992/ –               | 0.860/ 0.853                 | 1.097/ 1.118 / 1.154           | 1.549/ 1.521                 |
| 5P     | 2.343/ 2.367/ –               | 0.891/ 0.887                 | 1.317/ 1.336 / 1.346           | 1.594/ 1.572                 |
| 1D     | 0.936/ 1.097/ 0.96             | 0.748/ 0.638                 | 0.507/ 0.601 / 0.593           | 1.379/ 1.165                 |
| 2D     | 1.380/ 1.472/ 1.44            | 0.797/ 0.747                 | 0.760/ 0.818 / –              | 1.448/ 1.345                 |
| 3D     | 1.791/ 1.850/ 1.94            | 0.838/ 0.811                 | 0.994/ 1.036 / –              | 1.508/ 1.448                 |
| 4D     | 2.178/ 2.220/ –               | 0.871/ 0.856                 | 1.218/ 1.251 / –              | 1.559/ 1.560                 |
| 5D     | 2.497/ 2.581/ –               | 0.900/ 0.891                 | 1.434/ 1.460 / –              | 1.604/ 1.575                 |

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Figure 5. The velocities and momenta of charm quark within $c\bar{c}$ versus radii of $c\bar{c}$ obtained from SWE and our VP.

direct use for SWE and for the variational approach which we describe below, in Sec. 4. The quark velocities in states other than the ground state of quarkonia have never been reported as far as we know. We propose two ways for calculating the quark velocity, a first parametrization is motivated from simple dimensional analysis and the idea that two particles are revolving around their common center of mass as if no QCD glue or some other medium is present between and around the two particles. The explicit parametrization is

$$v_F = \frac{\beta}{m_Q}, \quad (13)$$

where $\beta$ is the quark momentum found from Eq. (12), $m_Q$ is the bare quark mass. A second parametrization is based on our model that whatever is present in the meson in addition to quarks half of the meson mass should come from one quark and its dressing. In other words, the quark has to drag along it the field it is moving in. So we assume its inertia as half of the meson mass and thus we introduce

$$v_S = 2 \frac{\beta}{\Delta}, \quad (14)$$

where $\Delta$ is the $Q\bar{Q}$ meson mass. This definition can be extended to the case of baryons, composed entirely of heavy quarks, the effective quark inertia would be one third the mass of that baryon and we shall get a factor of three in Eq. (14) in place of a factor of two. In our view, Eq. (14) should be more realistic because, a priori, it takes into account whatever is present in the meson along with the quarks and Eq. (13) should be a special case of our model given by Eq. (14) in the limit where QIE is negligible as compared to sum of the bare quark masses. Thus, we adhere to Eq. (14) and report quark

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Figure 6. The velocities and momenta of bottom quark within $b\bar{b}$ versus radii of $b\bar{b}$ obtained from SWE and our VP.

velocities in $S$, $P$ and $D$ spin averaged states of heavy quarkonia in Tables IV and V, which are in fair agreement with the overall assessment reported in Ref. [47]. These quark velocities would be of essential importance for determining inclusive decay rates by using Eqs. (27) and (33) of Ref. [49] and the references therein, for all spectroscopic states of heavy quarkonia instead of the $B_c(1S\text{-wave})$ only.

We plot quark velocities versus radii of mesons in their different quantum states in Figs. 5 and 6. We observe that for $c\bar{c}$ and $b\bar{b}$, the velocities of charm and bottom quarks in $S$, $P$ and $D$ states follow a universal rule,

$$v_S(r) = \frac{\sigma}{\sqrt{r}} + \varepsilon \sqrt{r} + v_0^S,$$

while the momentum widths behave as

$$\beta(r) = \rho \sqrt{r} + \beta_0,$$

where $\sigma(\text{fm})$, $\varepsilon(\text{fm}^{-1/2})$, $v_0^S$, $\rho(\text{GeV} \cdot \text{fm}^{-1/2})$ and $\beta_0(\text{GeV})$ depend upon the principal- and orbital angular momentum-quantum numbers of the quarkonium state. An interesting conclusion that we reach at by comparing Tables IV and V is that velocities of charm and bottom quarks predicted by Eqs. (13) and (14) do not differ significantly for low lying mesons but this difference becomes conspicuous for higher quarkonia states. In fact, the difference in predictions for bottom quark in Table V is almost negligible. The latter means that as per our velocity model, the heavier quarks are surrounded by a little amount of fields contributing to the

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the mass of meson via QIE and vice versa, but as the heavier quarks get apart as in D-wave bottomonia, predictions start differing. This observation is also supported by comparing the QIE's of \((nL)_{c\bar{c}}\) with \((nL)_{b\bar{b}}\) in Tables I and II. It is not without interest when we compare our quark-velocity Tables IV and V with quark-velocity Table I of Ref. [46]. It has also been measured in experiments (see Ref. [50]) that first generation quarks in heavier atoms have velocities 10 to 20% smaller than their velocities in lighter atoms. This trend of EMC effect Ref. [48, 50] is predicted by our velocity model Eq. (14) as we look horizontally from left to right the direction of increasing mass of mesonic QQ atoms in Table IV for the second generation charm quark, and from left to right in Table V for the third generation bottom quark, and this is a practical success of our model of quark velocities. We state these findings as follows: quark velocity decreases due to an increase in its interaction energy with other quark, and this interaction energy is larger as the interquark separation grows bigger. Due to increase in both of the interaction energy and interquark separation, the mass of the bound state of quarks rises (compare for example 1S\(_{c\bar{c}}\) with 1P\(_{c\bar{c}}\) with 1D\(_{c\bar{c}}\)). Hence, quarks move at slow velocities in heavier bound states and vice versa.

### 3.2. Annihilation decay rates

To further strengthen our use of bare quark masses in the non-relativistic potential model and the claimed improved accuracy of the Matrix Numerov method, we have opted to calculate the various decay rates of spin averaged S-wave charmonia and bottomonia. Contrary to the general practice of proposing Gaussian, or hydrogenic functions with one or two free parameters, we have calculated the wave functions for the spin averaged quarkonia as solutions of SWE. The leptonic \(e^+e^-\), diphoton \((\gamma\gamma)\), digluon \((gg)\), trigluon \((ggg)\), and 1 photon-2 gluon decays are of pivotal importance in identifying and producing the resonances. These decays are also helpful for establishing the conventional mesons and other multi-quark structures Refs. [51, 52]. The decay rates of S-wave quarkonia are [53–55]

\[
\Gamma(nS \rightarrow e^-e^+) = \frac{|R_{ns}(0)|^2}{\Delta_{ns}^2} 4 \alpha_s^2 \left(1 - \frac{16 \alpha_s}{3\pi}\right),
\]

\[
\Gamma(nS \rightarrow g) = \frac{|R_{ns}(0)|^2}{m_Q^2} \frac{10(\pi^2 - 9)}{81\pi} \alpha_s^3 \times \left(1 - \frac{4.9\alpha_s}{\pi}\right),
\]

\[
\Gamma(nS \rightarrow \gamma\gamma) = \frac{|R_{ns}(0)|^2}{m_Q^2} \frac{4(\pi^2 - 9)}{3\pi} e^6 \alpha_s^3 \times \left(1 - \frac{12.6\alpha_s}{\pi}\right),
\]

---

**Table IV.** Velocities from SWE and our VP (Sec. 4) in natural units, of the charm quark in orbitally and radially excited states of charmonium.

| States (n) | S | P | D |
|-----------|---|---|---|
|           | \(v_F\) | \(v_{S,SWE/V_{S,V,P}}\) | \(v_F\) | \(v_{S,SWE/V_{S,V,P}}\) | \(v_F\) | \(v_{S,SWE/V_{S,V,P}}\) |
| 1         | 0.59 | 0.445/0.453 | 0.60 | 0.402/0.375 | 0.62 | 0.383/0.337 |
| 2         | 0.64 | 0.406/0.420 | 0.65 | 0.384/0.388 | 0.66 | 0.371/0.363 |
| 3         | 0.68 | 0.386/0.401 | 0.69 | 0.373/0.385 | 0.70 | 0.364/0.369 |
| 4         | 0.71 | 0.374/0.390 | 0.72 | 0.365/0.380 | 0.73 | 0.357/0.370 |
| 5         | 0.74 | 0.366/0.383 | 0.74 | 0.358/0.376 | 0.75 | 0.353/0.368 |

**Table V.** Velocities from SWE and our VP (Sec. 4) in natural units, of the bottom quark in orbitally and radially excited states of bottomonium.

| States (n) | S | P | D |
|-----------|---|---|---|
|           | \(v_F\) | \(v_{S,SWE/V_{S,V,P}}\) | \(v_F\) | \(v_{S,SWE/V_{S,V,P}}\) | \(v_F\) | \(v_{S,SWE/V_{S,V,P}}\) |
| 1         | 0.286 | 0.283/0.289 | 0.289 | 0.272/0.244 | 0.295 | 0.270/0.229 |
| 2         | 0.307 | 0.284/0.280 | 0.306 | 0.276/0.268 | 0.310 | 0.274/0.258 |
| 3         | 0.319 | 0.285/0.282 | 0.321 | 0.281/0.277 | 0.323 | 0.279/0.271 |
| 4         | 0.331 | 0.287/0.285 | 0.332 | 0.284/0.283 | 0.334 | 0.282/0.280 |
| 5         | 0.340 | 0.289/0.288 | 0.342 | 0.287/0.287 | 0.344 | 0.286/0.285 |
\[\Gamma(nS \rightarrow \gamma gg) = \frac{|R_{ns}(0)|^2}{m_Q^2} \frac{8(\pi^2 - 9)}{9\pi} \alpha_s^2 e_Q^2 \times \left(1 - \frac{6.7\alpha_s}{\pi}\right), \quad (20)\]

\[\Gamma(nS \rightarrow gg) = \frac{|R_{ns}(0)|^2}{m_Q^2} \frac{2\alpha_s^2}{3} \left(1 + \frac{4.8\alpha_s}{\pi}\right), \quad (21)\]

\[\Gamma(nS \rightarrow \gamma\gamma) = \frac{|R_{ns}(0)|^2}{m_Q^2} \frac{3\alpha_s^4}{4} \left(1 - \frac{3.4\alpha_s}{\pi}\right), \quad (22)\]

where \(R_{ns}(0)\) is the meson wave functions at contact, which we take from our numerical solutions of Eq. (10). \(\Delta_{ns}\) the mass corresponding to the meson state, \(m_Q\) the bare quark mass, \(e_Q\) the quark electric charge in elementary units \(e\), \(\alpha = e^2/(4\pi)\) the fine structure constant and \(\alpha_s\) the strong coupling constant. In each of the above expressions for decay rates, the last factor in parentheses comes from the one-loop radiative QCD corrections. It is important to mention that value of \(\alpha_s\) is not available in the two free parameters of SL potential, see Ref. [23, 27] and Eq. (7). Thus, to move forward, we fit the experimentally known one value of the decay rate of each channel from Eqs. (17) to (22) in the 1S state and extract the value of \(\alpha_s\) for that decay rate. Then we produce all remaining values for 2S, 3S, and so on states for each channel. The values extracted for \(\alpha_s\) pertaining to decay modes in Eqs. (17) to (22) are displayed in Table VI for charmonia and bottomonia. The over all average value of \(\alpha_s\) for \(QQ\) comes out to be 0.3515 \pm 0.1761.

A comprehensive review about the strong running coupling constant \(\alpha_s\) within the effective potential approach is given in section 4.3 of the Ref. [56]. Our values of the strong coupling constant are in agreement with the findings of that work.

Annihilation decay rates are reported in Tables VII to XVIII along with a comparison against experimental results and other theoretical calculations where available.

### 3.2.1. The Velocity and Momentum Width Corrections In Annihilation decay rates

Most NR quark model calculations usually fail to produce decay rates agreeable with experiment. The root of this failure lies in the absence of proper account of velocities and momentum widths of the two valence quarks going to annihilate each other in a meson. We have found the momentum correction factor

\[\Lambda_{pc} = \sqrt{1 - \left(\frac{\beta c}{m_c}\right)^2}, \quad (23)\]

in charmonia annihilation decays except for the decays to three gluons, where this factor is,

\[\Lambda_{p\gamma\gamma} = \sqrt{1 - \left(\frac{\beta c}{2m_c}\right)^2}. \quad (24)\]

### Table VI. The strong running coupling \(\alpha_s\)

| Quarkonia(nS) | \(\Gamma^e^e\) | \(\Gamma^{gg}\) | \(\Gamma^{\gamma\gamma}\) | \(\Gamma^{ggg}\) | \(\Gamma^g\) | \(\Gamma^{\gamma\gamma}\) | \(\alpha_s^{e^e}\) |
|---------------|---------------|---------------|---------------|---------------|---------------|---------------|----------------|
| \(c\bar{c}\)   | 0.0147        | 0.1834        | 0.1948        | 0.2244        | 0.2780        | 0.5435        | 0.2872 \pm 0.1328 |
| \(b\bar{b}\)  | 0.4268        | 0.2091        | 0.2475        | 0.0943        | 0.2968        | 0.7771        | 0.4058 \pm 0.1946 |

### Table VII. Leptonic decay widths [keV] of spin averaged S-wave \(c\bar{c}\) states.

| State | \(\Gamma(nS)/\Gamma^{e^e}(nS)\) | Exp. [38] | [39] | [40] | \((\Gamma(nS > e^+e^-)/\Gamma(1S > e^+e^-))\) our/Exp. [26] |
|-------|-----------------|-----------|------|------|--------------------------------------------------|
| 1S    | 5.55            | 5.55      | 5.63 | 3.112 | 1.00/1.00                                        |
| 2S    | 2.18            | 2.33      | 2.19 | 2.197 | 0.39/(0.45 \pm 0.08)                             |
| 3S    | 1.34/0.91       | 0.86      | 1.20 | 1.701 | 0.16/(0.16 \pm 0.04)                             |
| 4S    | 0.97/0.63       | 0.58      | 0.63 | -     | 0.11/(0.11 \pm 0.04)                             |
| 5S    | 0.76/0.47       | -         | 0.24 | -     | 0.08/-                                          |

### Table VIII. Three-gluon decay widths [keV] of spin averaged S-wave \(c\bar{c}\) states.

| State | \(\Gamma(nS)/\Gamma^{gg}(nS)\) | Exp. [38] | [41] | [42] | \((\Gamma(nS > ggg)/\Gamma(1S > ggg))\) our/Exp. |
|-------|-----------------|-----------|------|------|--------------------------------------------------|
| 1S    | 59.45           | 59.55     | 269.06 | 52.8 \pm 5 | 1.00/1.00                                       |
| 2S    | 35.26/30.48     | 31.38     | 112.03 | 23 \pm 2.6 | 0.51/0.53                                       |
| 3S    | 27.14/23.52     | -         | 94.57  | -     | 0.40/-                                          |
| 4S    | 22.85/19.85     | -         | 88.44  | -     | 0.33/-                                          |
| 5S    | 20.12/17.38     | -         | 85.30  | -     | 0.29/-                                          |
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TABLE IX. Three-photon decay widths [eV] of spin averaged $S$-wave $c\bar{c}$ states.

| State | Our $\Gamma(nS)/\Gamma^{c\bar{c}}(nS)$ | Exp. [38] | [41] | $(\Gamma(nS \rightarrow \gamma\gamma\gamma)/\Gamma(1S \rightarrow \gamma\gamma\gamma))$ our/Exp. |
|-------|-------------------------------------|-----------|------|-------------------------------------------------|
| 1S    | 1.08                               | 1.08±0.032| 3.95 | 1.00/1.00                                       |
| 2S    | 0.64/0.45                          | -         | 1.64 | 0.42/--                                         |
| 3S    | 0.49/0.33                          | -         | 1.39 | 0.31/--                                         |
| 4S    | 0.42/0.27                          | -         | 1.30 | 0.25/--                                         |
| 5S    | 0.37/0.23                          | -         | 1.25 | 0.21/--                                         |

TABLE X. Two-photon decay widths [keV] of spin averaged $S$-wave $c\bar{c}$ states.

| State | Our $\Gamma(nS)/\Gamma^{c\bar{c}}(nS)$ | Exp. [38] | [40] | [41] | $(\Gamma(nS \rightarrow \gamma\gamma)/\Gamma(1S \rightarrow \gamma\gamma))$ our/Exp. |
|-------|-------------------------------------|-----------|------|------|-------------------------------------------------|
| 1S    | 5.10                               | 5.1±0.4   | 6.96 | 6.62 | 1.00/1.00                                       |
| 2S    | 3.02/2.12                          | 2.15±0.6  | 10.45| 2.88 | 0.42/0.42                                       |
| 3S    | 2.33/1.58                          | -         | 1.03 | 2.44 | 0.31/--                                         |
| 4S    | 1.96/1.28                          | -         | -    | 2.30 | 0.25/--                                         |
| 5S    | 1.73/1.09                          | -         | -    | 2.21 | 0.21/--                                         |

TABLE XI. Two-gluon decay widths [MeV] of spin averaged $S$-wave $c\bar{c}$ states.

| State | Our $\Gamma(nS)/\Gamma^{c\bar{c}}(nS)$ | Exp. [38] | [44] | [45] | $(\Gamma(nS \rightarrow gg)/\Gamma(1S \rightarrow gg))$ our/Exp. |
|-------|-------------------------------------|-----------|------|------|-------------------------------------------------|
| 1S    | 28.64                               | 28.6±2.2  | 13.07| 15.70| 1.00/1.00                                       |
| 2S    | 16.99/10.05                         | 14±7      | 9.53 | 8.10 | 0.35/(0.49±0.28)                                |
| 3S    | 13.08/7.51                          | -         | 4.41 | -    | 0.26/--                                         |
| 4S    | 11.01/6.16                          | -         | -    | -    | 0.22/--                                         |
| 5S    | 9.71/5.22                           | -         | -    | -    | 0.18/--                                         |

TABLE XII. (One-photon,two-gluon) decay widths [keV] of spin averaged $S$-wave $c\bar{c}$ states.

| State | Our $\Gamma(nS)/\Gamma^{c\bar{c}}(nS)$ | Exp. [38] | [41] | $(\Gamma(nS \rightarrow \gamma gg)/\Gamma(1S \rightarrow \gamma gg))$ our/Exp. |
|-------|-------------------------------------|-----------|------|-------------------------------------------------|
| 1S    | 8.18                                | 8.18±0.25 | 9.00 | 1.00/1.00                                       |
| 2S    | 4.85/2.87                           | 2.93±0.16 | 3.75 | 0.35/0.36                                       |
| 3S    | 3.73/2.14                           | -         | 3.16 | 0.26/--                                         |
| 4S    | 3.14/1.76                           | -         | 2.96 | 0.22/--                                         |
| 5S    | 2.77/1.49                           | -         | 2.85 | 0.18/--                                         |

TABLE XIII. Leptonic decay widths [keV] of spin averaged $S$-wave $b\bar{b}$ states.

| State | Our $\Gamma(nS)$ | Exp. [38] | [46] | [47] | $(\Gamma(nS \rightarrow e^+e^-)/\Gamma(1S \rightarrow e^+e^-))$ |
|-------|-----------------|-----------|------|------|-------------------------------------------------|
| 1S    | 1.336           | 1.34±0.018| 0.998| 1.60 | 1.00/1.00                                       |
| 2S    | 0.610           | 0.612±0.011| 0.439| 0.64 | 0.46/0.46                                       |
| 3S    | 0.412           | 0.443±0.008| 0.341| 0.44 | 0.31/0.33                                       |
| 4S    | 0.318           | 0.322±0.041*| 0.298| 0.35 | 0.24/0.24                                       |
| 5S    | 0.262           | 0.310±0.070| 0.265| 0.29 | 0.20/0.23                                       |

* This decay width is not correct in Table (XVII) of the Ref. [59]
TABLE XIV. Three-gluon decay widths [keV] of spin averaged S-wave $b\bar{b}$ states.

| State | Our $\Gamma(nS)/\Gamma^{c}(nS)$ | Exp. [38] | [59] | [58] | $(\Gamma(nS > ggg)/\Gamma(1S > ggg))$ our/Exp. |
|-------|--------------------------------|-----------|------|------|-----------------------------------------------|
| 1S    | 44.14                          | 44.13±1.09| 50.8 | 47.6 | 1.00/1.00                                     |
| 2S    | 23.05/17.44                    | 18.8±1.59 | 28.4 | 26.3 | 0.40/0.43±0.04                               |
| 3S    | 16.78/12.36                    | 7.25±0.85 | 21   | 19.8 | 0.28/0.16                                     |
| 4S    | 13.67/9.82                     | –         | 16.7 | 15.1 | 0.22/–                                        |
| 5S    | 11.77/8.26                     | –         | 14.2 | 13.1 | 0.18/–                                        |

TABLE XV. Three-photon decay widths [keV] of spin averaged S-wave $b\bar{b}$ states.

| State | Our $\Gamma(nS)/\Gamma^{c}(nS)$ | Exp. [38] | [59] | [58] | $(\Gamma(nS > \gamma\gamma\gamma)/\Gamma(1S > \gamma\gamma\gamma))$ our/Exp. |
|-------|--------------------------------|-----------|------|------|-----------------------------------------------|
| 1S    | $1.95 \times 10^{-5}$/1.54×10^{-5} | –         | 1.94×10^{-5} | 1.7×10^{-5} | 1.00/1.00                                     |
| 2S    | $1.02 \times 10^{-5}$/7.72×10^{-6} | –         | 1.09×10^{-5} | 9.8×10^{-6} | 0.50/–                                        |
| 3S    | $7.43 \times 10^{-6}$/5.48×10^{-6} | –         | 8.04×10^{-6} | 7.6×10^{-6} | 0.36/–                                        |
| 4S    | $6.05 \times 10^{-6}$/4.35×10^{-6} | –         | 6.36×10^{-6} | 6.0×10^{-6} | 0.28/–                                        |
| 5S    | $5.21 \times 10^{-6}$/3.66×10^{-6} | –         | 5.43×10^{-6} | –         | 0.24/–                                        |

TABLE XVI. Two photon decay widths [keV] of spin averaged S-wave $b\bar{b}$ states.

| State | Our $\Gamma(nS)/\Gamma^{c}(nS)$ | Exp. [38] | [59] | [58] | $(\Gamma(nS > \gamma\gamma)/\Gamma(1S > \gamma\gamma))$ our/Exp. |
|-------|--------------------------------|-----------|------|------|-----------------------------------------------|
| 1S    | 1.05/0.83                      | –         | 1.05 | 0.94 | 1.00/1.00                                     |
| 2S    | 0.55/0.42                      | –         | 0.489 | 0.41 | 0.51/–                                        |
| 3S    | 0.40/0.29                      | –         | 0.323 | 0.29 | 0.35/–                                        |
| 4S    | 0.32/0.23                      | –         | 0.237 | 0.20 | 0.28/–                                        |
| 5S    | 0.28/0.20                      | –         | –     | –    | 0.24/–                                        |

TABLE XVII. Two gluon decay widths [keV] of spin averaged S-wave $b\bar{b}$ states.

| State | Our $\Gamma(nS)/\Gamma^{c}(nS)$ | Exp. [38] | [59] | [58] | $(\Gamma(nS > gg)/\Gamma(1S > gg))$ our/Exp. |
|-------|--------------------------------|-----------|------|------|-----------------------------------------------|
| 1S    | 17.90/14.08                    | –         | 17.9 | 16.6 | 1.00/1.00                                     |
| 2S    | 9.35/7.07                      | –         | 8.33 | 7.2  | 0.50/–                                        |
| 3S    | 6.80/5.01                      | –         | 5.51 | 4.9  | 0.36/–                                        |
| 4S    | 5.54/3.98                      | –         | 4.03 | 3.4  | 0.28/–                                        |
| 5S    | 4.77/3.35                      | –         | –    | –    | 0.24/–                                        |

TABLE XVIII. One photon-Two gluon decay widths [keV] of spin averaged S-wave $b\bar{b}$ states.

| State | Our $\Gamma(nS)/\Gamma^{c}(nS)$ | Exp. [38] | [59] | [58] | $(\Gamma(nS > \gamma gg)/\Gamma(1S > \gamma gg))$ our/Exp. |
|-------|--------------------------------|-----------|------|------|-----------------------------------------------|
| 1S    | 1.19                           | 1.19±0.33 | 1.32 | 1.2  | 1.00/1.00                                     |
| 2S    | 0.62                           | 0.612±0.011 | 0.739 | 0.68 | 0.52/0.51                                     |
| 3S    | 0.45/0.23                      | 0.20±0.04 | 0.547 | 0.52 | 0.19/0.17                                     |
| 4S    | 0.37/0.18                      | –         | 0.433 | 0.40 | 0.15/–                                        |
| 5S    | 0.32/0.15                      | –         | 0.370 | –    | 0.13/–                                        |

The momentum width correction factor in all bottomonia annihilation decays is

$$\Lambda_{ps} = \sqrt{1 - \left(\frac{2\beta b}{m_b}\right)^2}. \quad (25)$$

The velocity correction factor for all charmonia annihilation decays is

$$\Lambda_{vc} = \sqrt{1 - v_{Sc}^2}. \quad (26)$$

and this velocity factor in the case of bottomonia annihilation

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decay rates happens to be
\[ \Lambda_{v_b} = \sqrt{1 - v^2_{SB}}, \]  
Except for annihilation to \( \gamma gg \) for which the velocity correction factor is
\[ \Lambda_{v_{\gamma gg}} = \sqrt{1 - 2v_{SB}}, \]  
where \( v_{SC} \) and \( v_{SB} \) are, respectively, the velocities of \( c \) and \( b \) quarks from our velocity model in Eq. (14). For the case of quarkonia annihilations, the overall correction factor is \( \Lambda_R \), to be multiplied with expressions in Eqs. (17) to (22) while \( Q \) is either \( c \) or \( b \) quark. The corrected decay rates denoted by \( \Gamma^\ast \) are reported in Tables VII to XVIII.

The inclusion of momentum width and velocity corrections give decay rates in agreement with experiment and this is another validation about the correctness of our velocity model, Eq. (14), and the momentum width model, Eq. (38). The ratio \( \Gamma(\eta S)/\Gamma(1S) \) for \( n > 1 \) is seen to remain always less than 0.55 and more than 0.05. Equations (23) to (28) impose very tight constraints on the allowed momentum widths and velocities of heavy quarks while they are in a quarkonium. This is so because any change in the exponents or in the multiplication factor of velocity or momentum spoils the consistency of theoretical decay rates with experiment.

4. A Variational Approach

A subtle combination of the Heisenberg uncertainty Principle and the variation of mesonic Hamiltonian can be used to estimate the masses of charmonium and bottomonium as follows. The radial Eq. (4) can be written as,
\[ \left(m_1 + m_2 - \frac{1}{2\mu} \frac{d^2}{dr^2} r + \frac{\ell(\ell + 1)}{2\mu r^2} + V(r) \right) \times R_{n\ell}(r) = \Delta R_{n\ell}(r). \]  
This means radial Hamiltonian becomes (because radial component of momentum operator is \( p_r = \ell[1/r][\partial/\partial r] r \)),
\[ H_R = m_1 + m_2 + \frac{p^2_r}{2\mu} + \frac{\ell(\ell + 1)}{2\mu r^2} + V(r). \]  
We have modeled the quark momentum of this equation in a peculiar way which is described next.

We define symmetric point as the one for which all three cartesian components of any 3-vector are equal in magnitude. This means in cartesian space,
\[ r = \sqrt{3} \bar{x}, \]  
\[ p_r = \beta = \sqrt{3} \bar{p}_x. \]  
This definition turns Eq. (30) into the following effective Hamiltonian:
\[ H_R = m_1 + m_2 + \frac{3}{2\mu} \beta^2 + \frac{\ell(\ell + 1)}{6\mu \bar{x}^2} + V(\sqrt{3} \bar{x}). \]  
Here we invoke the Heisenberg uncertainty Principle,
\[ \Delta x \cdot \Delta \bar{p}_x \geq \frac{1}{2}. \]  
as follows: A little cerebration reveals that the equality sign in Eq. (34) would hold for ideal systems (for which experimental as well as theoretical uncertainties are frozen at the minimum possible values of \( \Delta x \) and \( \Delta \bar{p}_x \)), while the inequality sign in Eq. (34) would come when the system experiences all sorts of “dissipative and interactive” effects. From Tables II and III, we observe that even in the lowest 1S state the QIE is not zero, which means \( QQ \) system always has medium in which quarks move. Therefore, the equality sign in Eq. (34) cannot hold. With these key ideas on board, we parametrize the uncertainty relation in the form (that will become clear below)
\[ \Delta x \cdot \Delta \bar{p}_x = \frac{\delta}{3}. \]  
where \( \delta \) is real, positive number more than one and we define it as,
\[ \delta = \left( \sqrt{2(n - 1) + \ell + 3} \right) \delta_{n,(1-\ell)} + \left( 2(n - 1) + \ell + \frac{3}{2} \right) (1 - \delta_{n,(1-\ell)}) , \]  
where \( \delta_{n,(1-\ell)} \) is the Kronecker delta. This definition is in accordance with the momentum width of known botommonium and charmonium states (see, for instance, Refs. [25, 26, 67]) and is convenient for the discussion below, as we shortly explain. By keeping in mind the quantum indeterminacy in position and momentum of quark in a meson, let us define the notation in which \( \Delta x \) is \( \bar{x} \)-the mean of the position coordinate of quark relative to center of mass of the meson, and similarly \( \Delta \bar{p}_x \) by the mean quark momentum width \( \bar{\beta}_x \) about center of mass. With these definitions, whatever physical quantity \( A \) we calculate in our VP corresponds to the average \( \langle A \rangle \) of many measurements of \( A \) and in essence the same as expectation value of a Hermitian operator \( \bar{A} \) from some linear vector space acting on a Hilbert space of states the meson could be in.

Thus, from Eq. (35),
\[ \bar{\beta}_x = \frac{\delta}{3 \bar{x}}. \]  
The use of Eq. (31) and Eq. (32) gives,
\[ \bar{\beta} = \frac{\delta}{\bar{r}}. \]  
and, in our notation, Eq. (33) becomes
\[ \bar{H}_R = m_1 + m_2 + \frac{3}{2\mu} \bar{\beta}^2 + \frac{\ell(\ell + 1)}{6\mu \bar{x}^2} + V(\sqrt{3} \bar{x}). \]  
Now we use this expression for computing meson radii and masses in the following way. Equation (37) makes Eq. (39) altogether a function of one variable \( \bar{x} \) only,
\[ \bar{H}_R(\bar{x}) = (m_1 + m_2) + \frac{\delta^2 + \ell(\ell + 1)}{6\mu} \frac{1}{\bar{x}^2} + V(\sqrt{3} \bar{x}). \]
This is the Hamiltonian of our meson system \(Q\bar{Q}\) that depends on \(\bar{x}\), which we regard as a variational parameter. Thus, minimizing \(\bar{H}_R\) with respect to \(\bar{x}\), we reach to the constraint
\[
-\frac{\delta^2 + \ell(\ell + 1)}{3\mu} + \frac{\partial V(\sqrt{3\bar{x}})}{\partial \bar{x}} = 0. \tag{41}
\]
Its solution gives \(\bar{x} = x_{\text{min}}\), and when this value is substituted in Eq. (40) then we identify \(\bar{H}_R(x_{\text{min}})\) as the mass of the meson denoted by \(\Delta_{VP}\) written as,
\[
\Delta_{VP} = (m_1 + m_2) + \frac{\delta^2 + \ell(\ell + 1)}{6\mu} + \frac{1}{x_{\text{min}}} + V(\sqrt{3x_{\text{min}}}). \tag{42}
\]
Let us apply this general procedure to a concrete example in which SLIP is the Song-Lin potential,
\[
V(\bar{r}) = -\frac{b}{\sqrt[3]{r}} + a\sqrt{r}. \tag{43}
\]
Upon inserting this potential into Eq. (40), we reach at the transcendental relation
\[
\left(\frac{3}{2}\mu ay^5 + \frac{3}{2}\mu by^3 - 2(\delta^2 + \ell^2 + \ell)\right) = 0, \tag{44}
\]
where \(y = \sqrt{\bar{x}}\). Equation (44) has only one real root for which \(\bar{H}_R(\bar{x}_{\text{min}})\) is minimum and which we denote as \(y_{\text{min}}\). All other four roots are complex and hence discarded being extraneous. Thus solution of quintic equation gives,
\[
\bar{x} = x_{\text{min}} = y_{\text{min}}^2. \tag{45}
\]
Equation (45) used in Eq. (31) with our SL parametrization Eqs. (7) and (8) give the radii of \(c\bar{c}\) and \(b\bar{b}\) mesons reported in Table III along with root mean square radii obtained from SWE using Eq. (11) and compared with Refs. [44, 45]. The agreement of our VP with SWE for quarkonia radii is excellent.

Substitution of the root Eq. (45) in Eq. (42) produces the meson mass,
\[
\Delta_{VP} = (m_1 + m_2) + \frac{\delta^2 + \ell(\ell + 1)}{6\mu} + \frac{1}{x_{\text{min}}^2} - \frac{b}{\sqrt[3]{x_{\text{min}}}} + a\sqrt{3x_{\text{min}}}. \tag{46}
\]

The ground state mass formula is
\[
\Delta_{VP(1S)} = (m_1 + m_2) + \left(\frac{1}{2\mu} \frac{1}{(2x_{\text{min}})^2} - \frac{b}{\sqrt[3]{3x_{\text{min}}}} + a\sqrt{3x_{\text{min}}}\right), \tag{47}
\]
and the higher states have the mass formula,
\[
\Delta_{VP(nL)} = (m_1 + m_2) + \left(\frac{\delta^2 + \ell(\ell + 1)}{6\mu} - \frac{1}{(2x_{\text{min}})^2}\right) - \frac{b}{\sqrt[3]{3x_{\text{min}}}} + a\sqrt{3x_{\text{min}}}. \tag{48}
\]

The masses of \(c\bar{c}\) and \(b\bar{b}\) mesons from Eqs. (47) and (48) are reported in Table XXIII and compared with experimental values as well as those from SWE. Again the agreement of masses from our VP with experiment and SWE is remarkable. It is further emphasized that mass values from our VP are closer to experimental findings than those predicted by SWE through MNM. It implies our VP is more accurate than MNM. We also compare our VP masses of \(S_\pm\), \(D_\pm\), \(F_\pm\) and \(G_\pm\)-waves with sophisticated calculations of Refs. [33, 68–70] in Tables XIX to XXII. The SWE is found not accurate enough in predicting higher quarkonia masses, but Tables XIX to XXIII show our VP works better not only for low lying states but it is very much suitable for spectroscopy of higher quarkonia states as well.

There are some more add ons from our VP. We use Eq. (45) in Eq. (31) and then result in Eq. (37) to get \(\beta_{c\bar{c}}\) and \(\beta_{b\bar{b}}\), which are reported in Table III. We use Eqs. (38), (47) and (48) in Eq. (14) to get \(v_{S,V,P}\) for different states of heavy quarkonia reported in Table IV. The quantity within parentheses in Eqs. (47) and (48) is the QIE which we report in Tables I and II. So every thing done by SWE numerically is more accurately and efficiently done analytically by our quantum mechanical VP.
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5. Conclusion

In this article, we have explored the mass spectra of charmonia and bottomonia in a non-relativistic framework invoking the Song-Lin potential as the effective interaction that binds heavy quarks in these meson systems with a single set of parameters. By numerically solving the resulting SWE through the Numerov strategy, we were able to calculate spin-averaged masses of $S$, $P$, $D$, $F$ and $G$ states. For the sake of illustration, we have not included in the discussion the techniques and ideas of spin or tensor interactions. These findings are straightforwardly extended in the Appendix A.

We have calculated $S$-wave annihilation decay rates without and with momentum width and velocity corrections by using wave functions at contact obtained from MNM. The agreement of theory with experiments after applying the corrections indicate that expressions for these decay rates should be revisited by properly incorporating the quark velocities and momentum widths so that the factor $|\psi(r = 0)|^2$ of wave function at contact gets replaced with $|\psi(r = 0, v, \beta)|^2$. This factor may be calculated by identifying the string fragmentation in the Lund Model [71, 72] with decay of a meson and using our VP based momentum Eq. (16) as transverse momentum distribution in the Lund Area Law. The linear rise in scalar part of the Cornell potential is inappropriate for calculation of higher excited states of mesons unless other contributions like the readjustments of the constituent quark mass, the free parameters in SLIPS and the complicated relativistic corrections in heavy quarkonia are added. The remedy to this issue has been sought in the color screening of SLIPS (see Appendix C).

The quark kinetic energies found from Table III using $\mu v^2/2$ and $\beta^2/(2\mu)$ where $\mu$ is the reduced mass of two bare quarks, are in good agreement with each other. This establishes the validity of our models of quark-velocities and mo-

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**Table XXI.** Masses (in [GeV]) of $F$-waves heavy quarkonia from our VP compared with those from Screened Cornell potential (SP) Refs. [60].

| Quarkonia          | $1F$          | $2F$          | $3F$          | $4F$          | $5F$          | $6F$          | $7F$          |
|--------------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| $c\bar{c}_{VP/D.Ebert}$ | 4.056/4.071  | 4.318/4.406  | 4.556/-       | 4.770/-       | 4.964/-       | 5.141/-       | 5.305/-       |
| $b\bar{b}_{VP/SP}$   | 10.380/10.366 | 10.603/10.609 | 10.802/10.812 | 10.979/10.988 | 11.138/-      | 11.281/-      | 11.4135/-     |

**Table XXII.** Masses (in [GeV]) of $G$-wave heavy quarkonia from our VP compared with those from Screened Cornell potential (SP) Refs. [59] and D. Ebert [61].

| Quarkonia          | $1G$          | $2G$          | $3G$          | $4G$          | $5G$          | $6G$          | $7G$          |
|--------------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| $c\bar{c}_{VP/D.Ebert}$ | 4.2783/4.345 | 4.5008/-      | 4.7095/-      | 4.9023/-      | 5.0804/-      | 5.2457/-      | 5.4000/-      |
| $b\bar{b}_{VP/SP}$   | 10.570/10.534 | 10.756/10.747 | 10.929/10.929 | 11.087/-      | 11.232/-      | 11.366/-      | 11.490/-      |

**Table XXIII.** Masses [GeV] for $c\bar{c}$ and $b\bar{b}$ spin-averaged $S$, $P$ and $D$ states from our VP compared with those from SWE and from the experiments.

| $nL$ | Exp. [33] | (Numerov)$_{c\bar{c}}$ | Our VP | Exp. [33] | (Numerov)$_{b\bar{b}}$ | Our VP |
|------|-----------|------------------------|--------|-----------|------------------------|--------|
| 1S   | 3.067     | 3.066                  | 3.062  | 9.444     | 9.444                  | 9.458  |
| 2S   | 3.649     | 3.764                  | 3.611  | 10.023    | 10.098                 | 9.987  |
| 3S   | 4.040     | 4.208                  | 4.024  | 10.355    | 10.482                 | 10.353 |
| 4S   | 4.415     | 4.545                  | 4.337  | 10.597    | 10.766                 | 10.619 |
| 5S   | 4.487     | 4.824                  | 4.595  | 10.865    | 10.998                 | 10.834 |
| 1P   | 3.525     | 3.566                  | 3.442  | 9.900     | 9.930                  | 9.832  |
| 2P   | 3.907     | 4.055                  | 3.879  | 10.260    | 10.358                 | 10.226 |
| 3P   | 4.186     | 4.418                  | 4.213  | 10.512 [57] | 10.665                | 10.514 |
| 4P   | 4.409     | 4.714                  | 4.486  | 10.711 [57] | 10.911                | 10.744 |
| 5P   | 4.807     | 4.969                  | 4.720  | 10.014    | 11.119                 | 10.938 |
| 1D   | 3.769     | 3.902                  | 3.788  | 10.161    | 10.234                 | 10.146 |
| 2D   | 4.159     | 4.292                  | 4.113  | 10.432 [57] | 10.565                | 10.429 |
| 3D   | 4.328     | 4.605                  | 4.391  | 10.643 [57] | 10.825                | 10.664 |
| 4D   | 4.520     | 4.871                  | 4.631  | 11.011    | 11.043                 | 10.865 |
| 5D   | 4.885     | 5.107                  | 4.844  | 11.389    | 11.232                 | 11.039 |
Appendix

A. Spin effects

In order to test the role of spin-effects in the spin averaged masses of heavy quarkonium systems $Q\bar{Q}$, we choose the spin-orbit, and tensor interactions by using the expressions $10(a)$, $10(b)$ and $(11)$ in Ref. [27] and the spin-spin interaction as the Gaussian-smereaded interaction in Eq. (4). After properly evaluating the angular momentum factors through the rules of non-relativistic quantum mechanics, these interactions for the SL potential can be easily shown to be,

$$H(r)_{LS} = \frac{1}{2m^2r} \left( \frac{3b}{2r} - \frac{a}{2\sqrt{r}} \right) \lambda,$$

$$H(r)_{T} = \frac{11b}{24m^2r^2} \times \left( s(s+1)l(l+1) - \frac{3}{4} (\lambda + \lambda^2) \right),$$

$$H(r)_{SS} = \frac{32\pi\alpha_s}{9m^2} \left( \frac{\sigma}{\sqrt{3}} \right)^3 \times e^{-\sigma^2r^2} \left( \frac{s(s+1)}{2} - \frac{3}{4} \right),$$

where $s$ is total spin of the meson which is either 0 or 1, $l$ is orbital angular momentum of the quark, $m$ is the bare quark mass, $\sigma$ is the spread in Gaussian, $\alpha_s$ is the strong running coupling constant, $r$ is the interquark separation, $a$ and $b$ are SL potential parameters, and

$$\lambda = j(j+1) - l(l+1) - s(s+1),$$

with $j$ the total angular momentum of the meson from $|l-s|$ to $(l+s)$. By inserting $H(r)_{LS}$, $H(r)_{T}$ and $H(r)_{SS}$ in the Matrix Numerov radial SWE, Eq. (10), we have the masses of singlets $|1L_j\rangle$ and triplets $|3L_j\rangle$ which are then spin averaged as [40],

$$\Delta(nL) = \frac{\Sigma_j (2j+1) \Delta_j}{\Sigma_j (2j+1)}.$$  

The numerical values of spin averaged masses from (A.5) which include spin effects, and from Tables I and II, which do not include spin effects, have been plotted together in Fig. A.1. Our conclusion is: fine, tensor and hyperfine
interactions between quark and antiquark do not contribute in the spin averaged masses of charmonia and bottomonia except in the \( P \)-wave charmonia, where these effects appear mildly in \( 1P, 2P, \) and \( 3P \) states within the scheme we are using to incorporate such corrections. Let it be mentioned that the strength of the spin contribution might change depending upon it is considered to start with in an effective potential or as a perturbation [74].

**B. Extraction of bare masses of Charm and Bottom quarks**

The general practice in quark model calculations using SWE is to use the quark mass which fits to the experimentally measured masses of the mesons and then calculate the unknown meson masses. This is the so called constituent quark mass. We do not follow this approach in our current paper. Instead, we compute here the bare mass of quark using experimental results and then use it to fit with only one lowest lying experimental spin averaged mass of \( \eta_c(1S) \) and \( J/\psi(1S) \). Our only free parameters are the parameters in a given SLIP. Now, the PDG averages for quark masses include quark mass values obtained from various theoretical calculations, lattice QCD as well as analyses of the Data from different experiments done at high-end laboratories. We use only the experimental results. For the charming case, we take into account the experiments in Table XXIV. Our average for the bare mass of the charm quark is

\[
m_c = 1.227 \pm 0.052, \quad (B.1)
\]

and using Table XXV for the beautiful case, our average for the bare mass of bottom quark is

\[
m_b = 4.35 \pm 0.389. \quad (B.2)
\]

The uncertainty in Eqs. \((B.1)\) and \((B.2)\) have been found by using the experimental values from second column of Table XXIV and XXV in

\[
\delta m_Q^2 = \frac{1}{N} \sum_{k=1}^{N} (m_Q - m_Q^k)^2, \quad (B.3)
\]

where \((m_Q)^\text{av}\) is the root-mean-square mass of heavy quark \( Q \). The upper (lower) bound on bare charm mass is 1.279 (1.175) and for the bottom quark these bounds are 4.739 (3.961). It is important to note the values in Eq. \((8)\) obtained from Eqs. \((B.1)\) and \((B.2)\) not only lie within these upper (lower) bounds, their difference

\[
m_b - m_c = 3.47, \quad (B.4)
\]

is our same yield as from the experimental determinations reported in Table XXVI.

**TABLE XXIV.** Experiments leading to bare mass of charm quark.

| Sr. No | \((m_c)_{\text{exp.}}\) | Experiments | Reference |
|--------|------------------------|-------------|-----------|
| 1      | 1.290 \(\pm\) 0.077    | DESY-HERA-H1, DESY-HERA-ZEUS | [75] |
| 2      | 1.26 \(\pm\) 0.041     | DESY-HERA-H1, DESY-HERA-ZEUS | [76] |
| 3      | 1.196 \(\pm\) 0.050    | SLAC-PEP2-BABAR | [77] |
| 4      | 1.159 \(\pm\) 0.075    | CERN-WA-096 | [78] |

**TABLE XXV.** Experiments leading to bare mass of bottom quark.

| Sr. No | \((m_b)_{\text{exp.}}\) | Experiments | Reference |
|--------|------------------------|-------------|-----------|
| 1      | 4.049 \(\pm\) 0.138 \(\pm\) 0.118 | DESY-HERA-H1, DESY-HERA-ZEUS | [75] |
| 2      | 4.186 \(\pm\) 0.044 \(\pm\) 0.015 | SLAC-PEP2-BABAR | [77] |
| 3      | 4.243 \(\pm\) 0.049     | KEK-BF-BELLE | [79] |
| 4      | 4.07 \(\pm\) 0.17      | DESY-HERA-ZEUS | [80] |
| 5      | 5.26 \(\pm\) 1.2       | CERN-LEP-DELPHI | [81] |
| 6      | 4.19 \(\pm\) 0.40      | CERN-LEP-DELPHI | [82] |
| 7      | 4.33 \(\pm\) 0.06 \(\pm\) 0.10 | CESR-CLEO | [83] |

**TABLE XXVI.** Experimental measurements of mass difference between charm and bottom quarks.

| Sr. No | \((m_b - m_c)_{\text{exp.}}\) | Experiments | Reference |
|--------|-----------------------------|-------------|-----------|
| 1      | 3.472 \(\pm\) 0.032        | SLAC-PEP2-BABAR | [77] |
| 2      | 3.42 \(\pm\) 0.06           | CERN-LEP-DELPHI | [84] |
| 3      | 3.44 \(\pm\) 0.03           | SLAC-PEP2-BABAR | [85] |
C. The color screening

We compare the color screening effects from SL potential with other screened potentials shown in Fig. C.1. The screened potentials (SPs) [27, 33–36, 86] are seen to saturate to a scale of less than or equal to 2 GeV. This feature is inappropriate for correct reproduction of higher excited states of heavy quarkonia with bare quark masses because the kinetic part $\beta^2/(2\mu)$ of QIE in Tables I and II is less than 1 GeV (0.67 GeV for $c$ quark, and 0.53 vGeV for $b$ quark). With the unique parametrization of SL which we have achieved, we find it more appropriate for higher meson states as is reflected in Tables XIX to XXII without introducing any contribution by hand. In addition, our parametrization makes SL match with other screened potentials in the low lying meson states having spatial extensions (i.e. interquark separations) less than 1.0 fm. Another point which we want to stress is that we have included screening effects in the interaction potential without introducing any exponential damping factor $e^{-\mu r}$ as has been done, for example, in Refs. [27, 33–36, 86]. This reduces the mathematical living cost to a great extent in calculations with SWE, Bethe-Salpeter equation, Dirac equation, and other quantum field theories.

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