Difference-Type-Exponential Estimators Based on Dual Auxiliary Information Under Simple Random Sampling

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Abstract: Auxiliary information plays a vital role at the selection and/or estimation stage to achieve the efficient estimates of the unknown population parameters. Dual use of auxiliary information, one the original and second the ranks of the auxiliary variable help to increase the efficiency of the estimators. In this article, we proposed and evaluated the performance of difference-type-exponential estimators based on dual auxiliary information for population mean under simple random sampling. Mathematical expressions for the bias and the mean squared error of the proposed estimators are obtained. Three real-life data sets and Monte Carlo simulation studies are carried out for illustration. The results of the empirical and the simulation studies, in terms of mean square errors and percentage relative efficiencies indicate that the proposed estimators perform better as compared to their counterparts.

Key words: Auxiliary variable; Mean square error; Percent absolute relative bias; Percentage relative efficiency; Ranked auxiliary variable.

1. Introduction

Over the last few decades, survey sampling has evolved into an extensive body of theory, methods and operations used daily all over the world. Survey sampling is broadly used in agriculture, business management, demography, economics, education, engineering, industry, medical sciences, political science, social sciences and many others. In sample surveys, there are many estimators of finite population mean under simple random sampling that rely on auxiliary information. It is fact that the appropriate use of auxiliary information in probability sampling results in considerable reduction in variance of the estimator of unknown population parameter(s). The existing estimation procedures are based only on the original form of the supplementary information provided by auxiliary variable(s).

Recently, Haq et al. [1] initiated an idea of utilizing the additional information of the auxiliary variable along with its original information to boost the efficiency of estimators. This additional information is in form of ranks of the auxiliary variable, called ranked auxiliary variable.

The motivation behind this article is to explore more efficient estimators by using the dual auxiliary information. We proposed two difference-type-exponential estimators based on the original and the ranked auxiliary information for the efficient estimation of population mean under simple random sampling scheme.

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Consider a sample of size \( n \) is drawn using simple random sampling without replacement (SRSWOR) scheme from a population of size \( N \), for \( i = 1, 2, 3, \ldots, N \). Let \( x_i, y_i \) and \( r_{x,i} \) denote the observations on the auxiliary variable, study variable and ranked auxiliary variable respectively for the \( i \)th unit of the population. Some useful measures are explained in Table 1. To obtain bias, mean square error (MSE) and minimum MSE of the proposed estimators, we define the following relative error terms and their expectations.

\[
\bar{\xi}_0 = \frac{y - \bar{Y}}{Y}, \quad \bar{\xi}_1 = \frac{r_x - \bar{R}_x}{\bar{R}_x} \quad \text{and} \quad \bar{\xi}_2 = \frac{x - \bar{X}}{\bar{X}}
\]

such that

\[
E(\bar{\xi}_0) = E(\bar{\xi}_1) = E(\bar{\xi}_2) = 0 \quad E(\bar{\xi}_0^2) = \psi C_y^2, \quad E(\bar{\xi}_1^2) = \psi C_r^2, \quad E(\bar{\xi}_2^2) = \psi C_x^2
\]

\[
E(\bar{\xi}_0 \bar{\xi}_1) = \psi \rho_{xy} C_y C_r, \quad E(\bar{\xi}_0 \bar{\xi}_2) = \psi \rho_{xy} C_y C_x, \quad E(\bar{\xi}_1 \bar{\xi}_2) = \psi \rho_{x,y} C_y C_r
\]

where

\[
\psi = (\frac{1}{n} - \frac{1}{N}), \quad \rho_{xy} = (S_y S_x)^{-1} S_{xy}, \quad \rho_{yr} = (S_y S_r)^{-1} S_{yr}, \quad \rho_{rx} = (S_x S_r)^{-1} S_{xr}
\]

\[
S_{xy} = (N - 1)^{-1} \sum_{i=1}^{N} (y_i - \bar{Y})(x_i - \bar{X}), \quad S_{yr} = (N - 1)^{-1} \sum_{i=1}^{N} (y_i - \bar{Y})(r_{x,i} - \bar{R}_x)
\]

\[
S_{xr} = (N - 1)^{-1} \sum_{i=1}^{N} (x_i - \bar{X})(r_{x,i} - \bar{R}_x)
\]

Some important expressions used in upcoming sections are defined below:

\[
R = \frac{\bar{Y}}{\bar{X}}, \quad \bar{R}_x = \frac{\bar{R}_x}{\bar{R}_x}, \quad \bar{R}_y = \frac{\bar{R}_y}{\bar{R}_y}, \quad \gamma = \frac{\alpha \bar{X}}{\alpha \bar{X} + \beta}
\]

\[
\kappa = \rho_{xy} \frac{C_y}{C_x}, \quad \phi_1 = \psi C_y^2, \quad \phi_2 = 1 + \psi C_y^2, \quad \phi_3 = \psi C_r^2
\]

\[
\phi_4 = \psi C_x^2 (\kappa - 1), \quad \phi_5 = \psi C_x^2 (\kappa - \frac{1}{2}), \quad \phi_6 = \psi \rho_{xy} C_y C_r
\]

\[
\phi_7 = \psi \rho_{x,y} C_x C_r, \quad \phi_8 = \psi C_y^2 (2\kappa - 1), \quad \phi_9 = \frac{\psi C_x^2}{2}
\]

2. Traditional and Existing Exponential-type Estimators

Several authors have used ratio, product and regression-type estimators to estimate population mean when both study and auxiliary variables are directly observable. For detail, see the following references: Kadilar and Cingi [2-3], Gupta and Shabbir [4], Grover and Kaur [5-6], Singh and Solanki [7], Haq and Shabbir [8], Shabbir et al. [9], Ekpenyong and Enang [10], Khan et al. [11], Solanki and Singh [12], Srisodaphol et al. [13], Singh and Pal [14], Singh et al. [15], Irfan et al. [16-17], Javed et al. [18] etc.

This section gives a brief introduction of traditional estimators i.e. unbiased, ratio, product and regression and well-known exponential-type estimators of population mean under simple random sampling.
2.1. Commonly used unbiased, ratio, product and regression estimators of the population mean $\bar{Y}$ are

$$\hat{Y} = \bar{y}$$  \hspace{1cm} (2.1)

$$\bar{Y}_R = y \left( \frac{X}{x} \right), \quad x \neq 0$$  \hspace{1cm} (2.2)

$$\bar{Y}_p = y \left( \frac{x}{X} \right)$$  \hspace{1cm} (2.3)

$$\bar{Y}_{REG} = \bar{y} + b (\bar{X} - \bar{x}) \quad \text{where} \quad b = \frac{\rho_{xy} S_y}{S_x} \text{is the slope coefficient.}$$  \hspace{1cm} (2.4)

The expressions for the bias of $\bar{Y}_R$ and $\bar{Y}_p$ are given by

$$\text{Bias}\left( \bar{Y}_R \right) \approx \psi \bar{Y} C_x \left( C_x - \rho_{xy} C_y \right)$$  \hspace{1cm} (2.5)

$$\text{Bias}\left( \bar{Y}_p \right) \approx \psi \bar{Y} C_x \left( C_x + \rho_{xy} C_y \right)$$  \hspace{1cm} (2.6)

2.2. Following ratio and product exponential type estimators are suggested by Bahl and Tuteja [19]

$$\bar{Y}_{BT,R} = \bar{y} \exp \left( \frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right)$$  \hspace{1cm} (2.7)

$$\bar{Y}_{BT,P} = \bar{y} \exp \left( \frac{\bar{x} - \bar{X}}{\bar{x} + \bar{X}} \right)$$  \hspace{1cm} (2.8)

Average of Equation (2.7) and Equation (2.8) can be written as

$$\bar{Y}_{BT,Avg} = \bar{y} \left[ \exp \left( \frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right) + \exp \left( \frac{\bar{x} - \bar{X}}{\bar{x} + \bar{X}} \right) \right]$$

2.3. Haq and Shabbir [8] proposed three improved exponential type estimators based on original auxiliary information, given by

$$\bar{Y}_{HS1} = \left[ \frac{\lambda_1}{2} \frac{\bar{y}}{y} \left( \frac{\bar{X}}{x} + \frac{x}{X} \right) + \lambda_2 \left( \bar{X} - \bar{x} \right) \right] \exp \left( \frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right)$$  \hspace{1cm} (2.9)

$$\bar{Y}_{HS2} = \left[ \lambda_1 \bar{Y}_{BT,Avg} + \lambda_3 \left( \bar{X} - \bar{x} \right) \right] \exp \left( \frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right)$$  \hspace{1cm} (2.10)

$$\bar{Y}_{HS3} = \left[ \frac{\lambda_1}{2} \bar{Y}_{BT,Avg} \left( \frac{\frac{\bar{X}}{x} + \frac{x}{X}}{\bar{X} + \bar{x}} \right) + \lambda_6 \left( \bar{X} - \bar{x} \right) \right] \exp \left( \frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right)$$  \hspace{1cm} (2.11)

Expressions for bias of $\bar{Y}_{HS1}, \bar{Y}_{HS2}$ and $\bar{Y}_{HS3}$ are given by

$$\text{Bias}\left( \bar{Y}_{HS1} \right) \approx \frac{1}{8} \left[ -8 \bar{Y} + \bar{Y} \left[ 8 + \psi C_x \left( 7 C_x - 4 \rho_{xy} C_y \right) \right] \lambda_1 + 4 \bar{X} \psi C_x^2 \lambda_2 \right]$$  \hspace{1cm} (2.12)
\begin{equation}
\text{Bias}(\bar{Y}_{HS2}) \approx \frac{1}{2} \left[ -2\bar{Y} + \bar{Y} \left\{ 2 + \psi C_x \left( C_x - \rho_{xy} C_y \right) \right\} \lambda_3 + \bar{X} \psi C_x^2 \lambda_4 \right] \tag{2.13}
\end{equation}

\begin{equation}
\text{Bias}(\bar{Y}_{HS3}) \approx \frac{1}{2} \left[ -2\bar{Y} + \bar{Y} \left\{ 2 + \psi C_x \left( 2C_x - \rho_{xy} C_y \right) \right\} \lambda_3 + \bar{X} \psi C_x^2 \lambda_6 \right] \tag{2.14}
\end{equation}

2.4. Ekpenyong and Enang [10] proposed following two efficient exponential ratio estimators as below

\[ \bar{Y}_{j1} = \lambda_7 \bar{Y} + \lambda_8 \left( \bar{X} - \bar{x} \right) \exp \left( \frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right) \tag{2.15} \]

\[ \bar{Y}_{j2} = \lambda_9 \bar{Y} + \lambda_{10} \left( \bar{X} - \bar{x} \right) \exp \left( \frac{2(\bar{X} - \bar{x})}{\bar{X} + \bar{x}} \right) \tag{2.16} \]

Bias of \( \bar{Y}_{j1} \) and \( \bar{Y}_{j2} \) are given as

\[ \text{Bias}(\bar{Y}_{j1}) \equiv \bar{Y} \left[ (\lambda_7 - 1) + \lambda_8 R\psi \frac{C_x^2}{2} \right] \tag{2.17} \]

\[ \text{Bias}(\bar{Y}_{j2}) \equiv \bar{Y} \left[ (\lambda_9 - 1) + \lambda_{10} R\psi C_x^2 \right] \tag{2.18} \]

2.5. Haq et al. [1] suggested an improved class of estimators following the lines of Shabbir and Gupta [20] and Grover and Kaur [5-6]. This class is based on the original and the ranked auxiliary information.

\[ \bar{Y}_{HA} = \left[ \lambda_{11} \bar{Y} + \lambda_{12} \left( \bar{X} - \bar{x} \right) + \lambda_{13} \left( \bar{R}_x - \bar{r}_x \right) \right] \exp \left( \frac{\alpha(\bar{X} - \bar{x})}{\alpha(X - x) + 2\beta} \right) \tag{2.19} \]

where \( \alpha \) and \( \beta \) may be any constant values or functions of the known parameters of the auxiliary variable.

Expression for bias of \( \bar{Y}_{HA} \) is stated as

\[ \text{Bias}(\bar{Y}_{HA}) \equiv \frac{1}{8} \left[ -8\bar{Y} + 4\psi C_x \left( \bar{X} C_x \lambda_{12} + \bar{R}_x C_x \lambda_{13}\rho_{xy} \right) + \bar{Y} \lambda_{11} \left\{ 8 + \psi\gamma C_x \left( 3\gamma C_x - 4C_y \rho_{xy} \right) \right\} \right] \tag{2.20} \]

Remark 2.1. As we know, \( \lambda_i, i = 1, 2, \ldots, 13 \) appearing in the above equations are the unknown weights determined such that the MSEs are minimized. So, the optimal values of \( \lambda_i \) are obtained by using the following condition:

\[ \frac{\partial \text{MSE}}{\partial \lambda_i} = 0; \quad i = \bar{Y}_{HS1}, \bar{Y}_{HS2}, \bar{Y}_{HS3}, \bar{Y}_{j1}, \bar{Y}_{j2}, \bar{Y}_{HA} \quad (i = 1, 2, 3, \ldots, 13). \]

On solving, we get the following:

\[ \lambda_1 = \frac{8 + 3\psi C_x^2}{8[1 + \psi C_x^2 + \psi C_y^2(1 - \rho_{xy})]}, \quad \lambda_2 = \frac{\bar{Y} \left[ 8C_x \rho_{xy} + C_x \left\{ -4 + \psi(3\rho_{xy} C_x - 4C_y \rho_{xy}) \right\} \right]}{8XC_x \left[ 1 + \psi C_x^2 + \psi C_y^2(1 - \rho_{xy}) \right]} \]
\[
\lambda_3 = \frac{4}{4 + \psi C_y^2 - 4\psi C_x^2(-1 + \rho_{yx}^2)}, \quad \lambda_4 = \frac{\bar{Y}}{2X} \left[ \frac{-8C_x + 8\psi C_y^2 \rho_{yx}}{C_x[4 + \psi C_x^2 - 4\psi C_y^2(-1 + \rho_{yx}^2)]} \right]
\]

\[
\lambda_5 = \frac{4 + 2\psi C_x^2}{4 + 5\psi C_x^2 - 4\psi C_y^2(-1 + \rho_{yx}^2)}, \quad \lambda_6 = \frac{\bar{Y}}{2X} \left[ \frac{8C_x \rho_{yx} + C_x \{-4 + \psi(4\rho_{yx} C_y C_x - 4\psi C_y^2(-1 + \rho_{yx}^2))\}}{2XC_x[4 + 5\psi C_x^2 - 4\psi C_y^2(-1 + \rho_{yx}^2)]} \right]
\]

\[
\lambda_7 = \frac{\phi_1 + \phi_4 \phi_6}{\phi_1 \phi_2 - \phi_3^2}, \quad \lambda_8 = R \left( \frac{\phi_3 + \phi_5 \phi_6}{\phi_1 \phi_2 - \phi_3^2} \right)
\]

\[
\lambda_9 = \frac{\phi_1 + \phi_4 \phi_6}{\phi_1 \phi_2 - \phi_4^2}, \quad \lambda_{10} = R \left( \frac{\phi_3 + \phi_5 \phi_6}{\phi_1 \phi_2 - \phi_4^2} \right)
\]

\[
\lambda_{11} = \frac{8 - \psi \gamma^2 C_x^2}{8 \left[ 1 + \psi C_x^2(1 - R_{yx}^2) \right]}\]

\[
\lambda_{12} = \frac{\bar{Y} \left[ \psi \gamma^2 C_y^2(-1 + \rho_{yx}^2) + (-8C_y + \psi \gamma^2 C_x^2)C_x \left( \rho_{yx} - \rho_{xy} \rho_{yx} \right) + 4\gamma C_y(-1 + \rho_{yx}^2)\{-1 + \psi \gamma^2 (1 - R_{yx}^2)\} \right]}{8XC_x(-1 + \rho_{yx}^2) \left[ 1 + \psi C_y^2 (1 - R_{yx}^2) \right]}
\]

\[
\lambda_{13} = \frac{\bar{Y} \left( 8 - \psi \gamma^2 C_x^2 \right)C_x \left( \rho_{yx} \rho_{xy} - \rho_{yx} \right)}{8R_{yx}C_s(-1 + \rho_{yx}^2) \left[ 1 + \psi C_y^2 (1 - R_{yx}^2) \right]}
\]

**Remark 2.2.** MSEs and minimum MSEs at the optimal values of \(\lambda_i, i = 1, 2, ..., 13\) of the estimators presented in Equation (2.1) to Equation (2.19) are given below.

\[
MSE\left( \hat{Y} \right) = V\left( \hat{Y} \right) = \psi \bar{Y}^2 C_y^2
\]  

\[
MSE\left( \bar{Y}_R \right) \equiv \psi \bar{Y}^2 \left[ C_y^2 + C_x^2 - 2 \rho_{yx} C_y C_x \right]
\]  

\[
MSE\left( \bar{Y}_P \right) \equiv \psi \bar{Y}^2 \left[ C_y^2 + C_x^2 + 2 \rho_{yx} C_y C_x \right]
\]  

\[
MSE\left( \bar{Y}_{REG} \right) \equiv \psi \bar{Y}^2 C_y^2 \left[ 1 - \rho_{yx}^2 \right]
\]  

\[
MSE\left( \bar{Y}_{BT,R} \right) \equiv \frac{\psi}{4} \bar{Y}^2 \left[ 4C_y^2 + C_x^2 - 4 \rho_{yx} C_y C_x \right]
\]  

\[
MSE\left( \bar{Y}_{BT,P} \right) \equiv \frac{\psi}{4} \bar{Y}^2 \left[ 4C_y^2 + C_x^2 + 4 \rho_{yx} C_y C_x \right]
\]  

\[
MSE_{\text{min}}\left( \bar{Y}_{HS1} \right) = \frac{\psi \bar{Y}^2 \left[ -25\psi C_y^4 + 16(-1 + \rho_{yx}^2)(-4 + \psi C_y^2)C_y^2 \right]}{64[1 + \psi C_y^2 + \psi C_y^2(1 - \rho_{yx}^2)]}
\]
\[
MSE_{\min}(\bar{Y}_{HS2}) = \frac{\psi \bar{Y}^2 \left[ -\psi C_x^2 + 4(-1 + \rho_{xy}^2)(-4 + \psi C_y^2)C_y^2 \right]}{4[4 + \psi C_x^2 - 4\psi C_y^2(-1 + \rho_{xy}^2)]} 
\] (2.28)

\[
MSE_{\min}(\bar{Y}_{HS3}) = \frac{\psi \bar{Y}^2 \left[ -9\psi C_x^4 + 4(-1 + \rho_{xy}^2)(-4 + \psi C_y^2)C_y^2 \right]}{4[4 + 5\psi C_x^2 - 4\psi C_y^2(-1 + \rho_{xy}^2)]} 
\] (2.29)

\[
MSE_{\min}(\bar{Y}_{II}) = \bar{Y}^2 \left( 1 - \frac{\varphi_1 + 2\varphi_2 + \varphi_3 + \varphi_4}{\varphi_2 - \varphi_4^2} \right) 
\] (2.30)

\[
MSE_{\min}(\bar{Y}_{II}) = \bar{Y}^2 \left( 1 - \frac{\varphi_1 + 2\varphi_2 + \varphi_3 + \varphi_4}{\varphi_2 - \varphi_4^2} \right) 
\] (2.31)

\[
MSE_{\min}(\bar{Y}_{II}) = \psi \bar{Y}^2 \left[ 64C_y^2(1 - R_{xy}^2) - \psi\gamma^4 C_x^4 - 16\psi\gamma^2 C_x^2C_y^2(1 - R_{xy}^2) \right] 
\] (2.32)

where
\[
R_{xy}^2 = \frac{\rho_{xy}^2 + \rho_{yx}^2 - 2\rho_{xy}\rho_{yx}\rho_{xr}}{1 - \rho_{xy}^2} 
\]

3. Proposed Estimators

In this section, we proposed two new difference-type-exponential estimators for population mean under SRSWOR. These estimators are based on the dual use of auxiliary information. 1) The auxiliary variable, uses the original/actual measurements of the auxiliary variable. 2) The ranked auxiliary variable, uses the ranks of the auxiliary variable. Mathematical properties such as bias, MSE and minimum MSE of the proposed estimators are derived up to first order of approximation. The bias of an estimator is the difference between the estimator's expected value and the true value of the parameter being estimated i.e. \( \text{Bias}(\hat{Y}) = E(\hat{Y} - \bar{Y}) \) and mean square error (MSE) can be defined as the divergence of the estimator values from the true parameter value i.e. \( MSE(\hat{Y}) = E((\hat{Y} - \bar{Y})^2) \).

3.1. First proposed estimator

\[
\bar{Y}_{P1} = \frac{\lambda_{14}}{2} \left( \frac{X}{\bar{x}} + \frac{\bar{x}}{X} \right) \bar{Y}_{BT, Avg} + \lambda_{15} \left( R_x - \bar{r} \right) + \lambda_{16} \left( \bar{X} - \bar{x} \right) \exp \left( \frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right) \] (3.1)

where \( \lambda_{14}, \lambda_{15} \text{ and } \lambda_{16} \) are the suitably chosen constants.

Using relationship (1.1) to rewrite the estimator in Equation (3.1), then subtract \( \bar{Y} \) from both sides and get the expression up to first order of approximation in this way

\[
\bar{Y}_{P1} - \bar{Y} = \left[ \lambda_{14} \bar{Y} + \frac{5\lambda_{14} \bar{Y} \bar{r}_x^2}{8} + \lambda_{14} \bar{Y} \bar{r}_x - \lambda_{15} \bar{r}_x \bar{r}_y - \lambda_{16} \bar{X} \bar{r}_y + \frac{\lambda_{16} \bar{X} \bar{r}_y^2}{2} \right] - \bar{Y} 
\] (3.2)
Applying expectation on both sides of Equation (3.2), we get the bias of proposed estimator as

\[
\text{Bias}\left(\bar{Y}_{p1}\right) \equiv (\lambda_{i4} - 1)\bar{Y} + \frac{\theta_i}{2} \left(\lambda_{i6} \bar{X} + \frac{5\lambda_{i4} \bar{Y}}{4}\right) \tag{3.3}
\]

Squaring both sides of Equation (3.2) up to first order of approximation, we have

\[
\left(\bar{Y}_{p1} - \bar{Y}\right)^2 \equiv \left[\bar{Y}^2 + \lambda_{i4} \bar{Y}^2 + \lambda_{i4}^2 \bar{Y}^2 \varepsilon_0^2 + \lambda_{i5} R_s \varepsilon_1^2 + \lambda_{i6}^2 \bar{X}^2 \varepsilon_2^2 + \frac{5}{4} \lambda_{i4} \bar{Y}^2 \varepsilon_2^2 + \lambda_{i4} \lambda_{i6} \bar{X} \bar{Y} \varepsilon_2^2 - 2\lambda_{i4} \bar{Y}^2 \varepsilon_2^2 - \frac{5}{4} \lambda_{i4} \bar{Y}^2 \varepsilon_2^2 - 2\lambda_{i4} \lambda_{i5} \bar{X} \bar{Y} \varepsilon_2^2 - 2\lambda_{i4} \lambda_{i6} \bar{X} \bar{Y} \varepsilon_2^2 - \lambda_{i6} \bar{X} \bar{Y} \varepsilon_2^2\right] \tag{3.4}
\]

The MSE of \(\bar{Y}_{p1}\) is obtained by taking the expectation on both sides of Equation (3.4)

\[
MSE\left(\bar{Y}_{p1}\right) \equiv \bar{Y}^2 \left[1 + \lambda_{i4}^2 \phi_2 + \frac{5}{4} \lambda_{i4} \phi_1 + \lambda_{i5}^2 \varepsilon_1^2 + \lambda_{i6}^2 \varepsilon_2^2 + 2\lambda_{i4} \phi_1 - \frac{5}{4} \lambda_{i4} \phi_1\right] \tag{3.5}
\]

Now, we have to choose the weights of \(\lambda_{i4}, \lambda_{i5}\) and \(\lambda_{i6}\) such that the resulting MSE of \(\bar{Y}_{p1}\) will be minimized. So, the optimal weights of \(\lambda_{i4}, \lambda_{i5}\) and \(\lambda_{i6}\) are selected with the help of following equations.

\[
\frac{\partial \text{MSE}\left(\bar{Y}_{p1}\right)}{\partial \lambda_{i4}} = (8\phi_2 + 10\phi_1) \lambda_{i4} - 8\lambda_{i5} \phi_6 - 4\lambda_{i6} \phi_8 - 8 - 5\phi_i
\]

\[
\frac{\partial \text{MSE}\left(\bar{Y}_{p1}\right)}{\partial \lambda_{i5}} = 2\lambda_{i5} \phi_3 - 2\lambda_{i4} \phi_6 + 2\lambda_{i6} \phi_7 - \lambda_{i4} \phi_8
\]

\[
\frac{\partial \text{MSE}\left(\bar{Y}_{p1}\right)}{\partial \lambda_{i6}} = 2\lambda_{i6} \phi_3 - 2\lambda_{i3} \phi_6 - \lambda_{i4} \phi_8 - \phi_i
\]

Setting \(\frac{\partial \text{MSE}\left(\bar{Y}_{p1}\right)}{\partial \lambda_{i}} = 0, i = 14, 15, 16\) and solving simultaneously, we get

\[
\lambda_{i4(\text{opt})} = \frac{E_1 E_2 - 2E_2 \phi_3 \phi_5}{E_2 E_4 - 2E_3^2}
\]

\[
\lambda_{i5(\text{opt})} = \frac{2\phi_6 (E_1 E_2 - 2E_2 \phi_3 \phi_5) + \phi_7 \left(E_1 E_3 - \phi_3 E_4\right)}{2 \phi_3 R_s \left(E_2 E_4 - 2E_3^2\right)}
\]

\[
\lambda_{i6(\text{opt})} = \frac{\phi_5 \phi_4 E_4 - E_1 E_3}{2 \phi \left(E_2 E_4 - 2E_3^2\right)}
\]

where

\[
E_1 = 8\phi_3 + 5\phi_3 \phi_3, \quad E_2 = \phi_3 \phi_5 - \phi_3^2
\]

\[
E_3 = 2\phi_6 \phi_3 - \phi_3 \phi_5, \quad E_4 = 8\phi_3 \phi_5 + 10\phi_3 \phi_5 - 8\phi_3^2
\]
Inserting optimal weights of $\lambda_{14}, \lambda_{15}$ and $\lambda_{16}$ in Equation (3.5), we get the minimum MSE of the proposed estimator as

$$
MSE_{\text{min}}(\bar{Y}_{p1}) = \frac{\bar{Y}^2}{4\varphi_3 F_1^2} \left[ 4\varphi_3 F_1^2 + \left( 4\varphi_2 \varphi_3 - 4\varphi_6^2 + 5\varphi_4 \varphi_3 \right) F_2^2 + \left( \varphi_1 \varphi_3 - \varphi_7^2 \right) F_3^2 - \left( 8 + 5\varphi_1 \right) \varphi_3 F_1 F_2 \right] ^2
$$

(3.6)

where

$$
F_1 = E_2 E_4 - 2 E_3^2, \quad F_2 = E_4 E_2 - 2 \varphi_3 E_3, \quad F_3 = \varphi_4 E_4 - E_4 E_3
$$

3.2. Second proposed estimator

$$
\bar{Y}_{p2} = \lambda_{17} \bar{Y} + \lambda_{18} \left( \bar{R}_x - \bar{r}_x \right) + \lambda_{19} \left( \bar{X} - \bar{x} \right) \exp \left( \frac{2 \left( \bar{X} - \bar{x} \right)}{\bar{X} + \bar{x}} \right)
$$

(3.7)

where $\lambda_{17}, \lambda_{18}$ and $\lambda_{19}$ are the suitably chosen constants.

Following the same procedure mentioned in section 3.1, we have the following expressions:

$$
\bar{Y}_{p2} - \bar{Y} = \left[ \lambda_{17} \bar{Y} + \lambda_{17} \bar{Y} \xi_0 - \lambda_{18} R_x \xi_1 - \lambda_{19} \bar{X} \xi_2 + \lambda_{19} \bar{X} \xi_2 - \bar{Y} \right]
$$

(3.8)

$$
\text{Bias} \left( \bar{Y}_{p2} \right) = \bar{Y} \left[ (\lambda_{17} - 1) + \lambda_{19} \varphi_1 \rho \right]
$$

(3.9)

$$
\left( \bar{Y}_{p2} - \bar{Y} \right)^2 = \bar{Y}^2 + \lambda_{17}^2 \bar{Y}^2 + \lambda_{18}^2 R_x \xi_1 + \lambda_{19}^2 \bar{X} \xi_2 + 2 \lambda_{17} \lambda_{19} \bar{Y} \bar{X} \xi_2 - 2 \lambda_{17} \bar{Y}^2 - 2 \lambda_{17} \bar{Y}^2
$$

(3.10)

$$
\text{MSE} \left( \bar{Y}_{p2} \right) = \bar{Y}^2 \left[ 1 + \lambda_{17}^2 \varphi_2 + \lambda_{18}^2 R_x \varphi_3 + \lambda_{19}^2 R_x \varphi_4 - 2 \lambda_{17} \right]
$$

(3.11)

The optimal weights of $\lambda_{17}, \lambda_{18}$ and $\lambda_{19}$ are obtained in this way

$$
\frac{\partial \text{MSE} \left( \bar{Y}_{p2} \right)}{\partial \lambda_{17}} = 2 \left( \lambda_{17} \varphi_2 - 1 - \lambda_{19} \varphi_3 - \lambda_{18} \varphi_4 \right)
$$

$$
\frac{\partial \text{MSE} \left( \bar{Y}_{p2} \right)}{\partial \lambda_{18}} = 2 \left( \lambda_{18} \varphi_3 - \lambda_{17} \rho \right)
$$

$$
\frac{\partial \text{MSE} \left( \bar{Y}_{p2} \right)}{\partial \lambda_{19}} = 2 \left( \lambda_{19} \varphi_4 - \lambda_{17} \rho \right)
$$

Setting $\frac{\partial \text{MSE} \left( \bar{Y}_{p2} \right)}{\partial \lambda_i} = 0, i = 17, 18, 19$ and solving simultaneously, we get
\[
\lambda_{17(\text{opt})} = \frac{\varphi_3 E_5 - (\varphi_2 E_7 + \varphi_4 E_8)}{\varphi_2 (E_5 E_8 - E_6^2)}
\]

\[
\lambda_{18(\text{opt})} = \frac{\varphi_6 E_5 - E_6 E_7}{R (E_5 E_8 - E_6^2)}
\]

\[
\lambda_{19(\text{opt})} = \frac{E_7 E_4 - \varphi_6 E_6}{R (E_5 E_8 - E_6^2)}
\]

where

\[
E_5 = \varphi_3 - \varphi_4^2, \quad E_6 = \varphi_2 E_7 - \varphi_6, \quad E_7 = \varphi_2 + \varphi_4, \quad E_8 = \varphi_2^2 - \varphi_6^2
\]

Inserting optimal weights of \(\lambda_1, \lambda_8\) and \(\lambda_9\) in Equation (3.11), we get the minimum MSE of the proposed estimator as below.

\[
\text{MSE}_{\text{min}}(\bar{Y}_{\text{opt}}) = \frac{\bar{Y}^2}{\varphi_2 F_4^2} \left[ \varphi_2 F_4^2 + \varphi_4 F_2 + \varphi_2 \varphi_3 F_6^2 + F_7^2 - 2(\varphi_1 F_4 - \varphi_2 F_6) \varphi_2 F_5 \right]
\]

where

\[
F_4 = E_5 E_8 - E_6^2, \quad F_5 = E_7 E_8 - \varphi_6 E_6
\]

\[
F_6 = \varphi_6 E_5 - E_6 E_7, \quad F_7 = \varphi_2 \varphi_3 E_5 - \varphi_2 \varphi_7 E_6 - \varphi_6 E_6 E_7 + \varphi_4 E_7 E_8
\]

**Remark 3.1.** It is important to mention that the parameters \(\rho_x, \rho_y, \rho_{xy}, C_y, C_x\) and \(C_r\) appearing in the expressions of optimal weights and the minimum MSEs are generally unknown. However, these parameters can be estimated quite accurately from the preliminary data or from the repeated surveys based on sampling over several occasions. The utilization of prior information on parameters at the estimation stage has been dealt by various authors including Singh and Singh [21] and Vishwakarma and Kumar [22].

### 4. Applications

In this section, three real-life data sets are used to evaluate the performance of proposed estimators as compared to the existing estimators in terms of percent absolute relative bias (PARB), mean squared error (MSE) and percentage relative efficiencies (PRE). For more details of these measures see Rao et al. [23], Silva and Skinner [24] and Nidhi et al. [25] etc. MSEs are calculated using the expressions defined in sections 2-3. PARB and PRE of an estimator can be computed through the following expressions:

\[
\text{PARB}(\star) = \left( \frac{\bar{Y} - \bar{\hat{Y}}}{\bar{Y}} \right) \times 100
\]

\[
\text{PRE} = \frac{\text{MSE}(\hat{\bar{Y}})}{\text{MSE}(\bar{Y})} \times 100
\]

where \(\star = \bar{Y}, \bar{\hat{Y}}_\text{REG}, \bar{\hat{Y}}_{\text{HS1}}, \bar{\hat{Y}}_{\text{HS2}}, \bar{\hat{Y}}_{\text{HS3}}, \bar{\hat{Y}}_{J1}, \bar{\hat{Y}}_{J2}, \bar{\hat{Y}}_{HA}, \bar{\hat{Y}}_{P1}, \bar{\hat{Y}}_{P2}\)
Population 1: (Source: Singh and Mangat [26], p. 369)

\[
N = 69, \quad n = 12, \quad \bar{Y} = 135.2608, \quad \bar{X} = 345.7536
\]
\[
C_y = 0.8422, \quad C_x = 0.8479, \quad C_r = 0.5747, \quad \beta_{2(x)} = 7.2159
\]
\[
\bar{R} = 34.9565, \quad \rho_{yx} = 0.9224, \quad \rho_{yr} = 0.7136, \quad \rho_{sx} = 0.8185
\]

Population 2: (Source: Ekpenyong and Enang [10])

\[
N = 923, \quad n = 180, \quad \bar{Y} = 436.4345, \quad \bar{X} = 11440.5
\]
\[
C_y = 1.7183, \quad C_x = 1.8645, \quad C_r = 0.577, \quad \beta_{2(x)} = 18.7208
\]
\[
\bar{R} = 461.9642, \quad \rho_{yx} = 0.9543, \quad \rho_{yr} = 0.6442, \quad \rho_{sx} = 0.6306
\]

Population 3: (Source: Kadilar and Cingi, [27])

\[
N = 854, \quad n = 290, \quad \bar{Y} = 2930.12, \quad \bar{X} = 37600.11
\]
\[
C_y = 5.8379, \quad C_x = 3.8509, \quad C_r = 0.1883, \quad \beta_{2(x)} = 312.0651
\]
\[
\bar{R} = 426.8747, \quad \rho_{yx} = 0.9165, \quad \rho_{yr} = 0.2585, \quad \rho_{sx} = 0.3458
\]

The PARB of all the estimators are shown in Table 2 and the MSEs and PREs are given in Table 3.

5. Simulation Study Based on Real Data Sets

Monte Carlo simulation study is carried out to check the potential of the proposed estimators over the competing estimators through R software.

A step by step approach for the simulation study is as below:
1) Select a SRSWOR of size \(n\) from the population of size \(N\).
2) Use sample data from step 1 to find PARB and MSE of all the estimators under study.
3) Repeat 50,000 times step 1 and step 2.
4) Obtain 50,000 values for PARB and MSE of all the estimators.
5) Average of 50,000 values obtained in step 4 are the PARB and MSE.
6) PARB and PREs (with respect to sample mean \(\hat{Y}\)) of all the estimators are calculated.

6. Important Findings

✓ From Table 2, the percentage absolute relative bias of the proposed estimators are least as compared to all the competing estimators for all three data sets.
✓ From Table 3, it is imperative to mention that the proposed estimators \(\bar{Y}_m\) and \(\bar{Y}_{r_2}\) have minimum MSEs and maximum PREs than all the traditional and existing estimators in all populations.
From Table 4, simulation study indicates that the percentage absolute relative bias (PARB) of the proposed estimators are lesser than the competing estimators for all three data sets used in our study.

From Table 5, simulation study again reveals that $\bar{Y}_{P1}$ and $\bar{Y}_{P2}$ have maximum gain in PREs than all other estimators under competition. This phenomenon is observed in all populations under study.

So, above findings confirmed that the proposed estimators outperform than the competitors under study.

7. Concluding Remarks

This manuscript considers the improved estimation of finite population mean under simple random sampling without replacement. Efficient utilization of auxiliary information can play vital role in this regard. So, the purpose is achieved by using the original information of the auxiliary variable and additionally the ranks of auxiliary variable. Some new difference-type-exponential estimators based on above idea of dual auxiliary information are proposed. Mathematical properties including bias, MSE and minimum MSE of the proposed estimators are derived up to first degree of approximation. To assess the potentiality of the proposed estimators over the competing estimators, real data analysis as well as simulation study is carried out. Three natural data sets are used for the empirical study. The outcome of this comparison indicates that the proposed estimators are more efficient and less biased than the traditional and other well-known existing estimators. Thus, the researchers are encouraged to use the proposed estimators for estimating the population mean under SRSWOR.

The present work could be extended to estimate the: 1) finite population mean under other sampling designs like stratified random sampling and two-phase sampling etc. 2) other unknown finite population parameters including median, variance and proportions etc. 3) population mean of a sensitive variable in the presence of non-sensitive auxiliary information.

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**Table Captions**

**Table 1** Some useful measures
**Table 2** Numerical comparison of percent absolute relative bias
**Table 3** MSE’s and PRE’s of the existing and proposed estimators
**Table 4** Percent absolute relative bias of estimators w.r.t \( \hat{Y} \) based on simulation study
**Table 5** PREs of estimators w.r.t \( \hat{Y} \) based on simulation study

| Measures                     | Study variable (y) | Auxiliary variable (x) | Ranked auxiliary variable (r_x) |
|------------------------------|-------------------|------------------------|---------------------------------|
| Sample Mean                  | \( \bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i \) | \( \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \) | \( \bar{r}_x = \frac{1}{n} \sum_{i=1}^{n} r_{x,i} \) |
| Population mean              | \( \bar{Y} = \frac{1}{N} \sum_{i=1}^{N} y_i \) | \( \bar{X} = \frac{1}{N} \sum_{i=1}^{N} x_i \) | \( \bar{R}_x = \frac{1}{N} \sum_{i=1}^{N} r_{x,i} \) |
| Population Variance          | \( S_y^2 = (N-1) \sum_{i=1}^{N} (y_i - \bar{Y})^2 \) | \( S_x^2 = (N-1) \sum_{i=1}^{N} (x_i - \bar{X})^2 \) | \( S_{rx}^2 = (N-1) \sum_{i=1}^{N} (r_{x,i} - \bar{R}_x)^2 \) |
| Population coefficient of variation | \( C_y^2 = \left( \frac{1}{\bar{Y}^2} \right) S_y^2 \) | \( C_x^2 = \left( \frac{1}{\bar{X}^2} \right) S_x^2 \) | \( C_{rx}^2 = \left( \frac{1}{\bar{R}_x^2} \right) S_{rx}^2 \) |
| Estimators | Population 1 | Population 2 | Population 3 |
|------------|--------------|--------------|--------------|
| \(\hat{Y}\) | --- | --- | --- |
| \(\overline{Y}_{\text{REG}}\) | --- | --- | --- |
| \(\overline{Y}_{\text{HS}1}\) | 0.5901 | 0.1062 | 1.1345 |
| \(\overline{Y}_{\text{HS}2}\) | 0.6905 | 0.1153 | 1.1993 |
| \(\overline{Y}_{\text{HS}3}\) | 0.5439 | 0.1017 | 1.1069 |
| \(\overline{Y}_{\text{JI}1}\) | 0.6399 | 0.1115 | 1.1195 |
| \(\overline{Y}_{\text{JI}2}\) | 0.4642 | 0.0961 | 0.9136 |
| \(\overline{Y}_{\text{HA} (1)} \alpha = 1, \beta = C_x\) | 0.6857 | 0.1129 | 1.1855 |
| \(\overline{Y}_{\text{HA} (2)} \alpha = 1, \beta = \beta_{2(x)}\) | 0.6863 | 0.1130 | 1.1857 |
| \(\overline{Y}_{\text{HA} (3)} \alpha = \beta_{2(x)}, \beta = C_x\) | 0.6857 | 0.1129 | 1.1855 |
| \(\overline{Y}_{\text{HA} (4)} \alpha = C_x, \beta = \beta_{2(x)}\) | 0.6865 | 0.1130 | 1.1856 |
| \(\overline{Y}_{\text{HA} (5)} \alpha = 1, \beta = \rho_{yx}\) | 0.6857 | 0.1130 | 1.1855 |
| \(\overline{Y}_{\text{HA} (6)} \alpha = C_x, \beta = \rho_{yx}\) | 0.6858 | 0.1129 | 1.1855 |
| \(\overline{Y}_{\text{HA} (7)} \alpha = \rho_{yx}, \beta = C_x\) | 0.6857 | 0.1129 | 1.1855 |
| \(\overline{Y}_{\text{HA} (8)} \alpha = \beta_{2(x)}, \beta = \rho_{yx}\) | 0.6857 | 0.1130 | 1.1855 |
| \(\overline{Y}_{\text{HA} (9)} \alpha = \rho_{yx}, \beta = \beta_{2(x)}\) | 0.6863 | 0.1129 | 1.1858 |
| \(\overline{Y}_{\text{HA} (10)} \alpha = 1, \beta = N \bar{X}\) | 0.6981 | 0.1138 | 1.1974 |
| \(\overline{Y}_{\text{P1}}\) | 0.3343 | 0.0845 | 0.9117 |
| \(\overline{Y}_{\text{P2}}\) | 0.3588 | 0.0938 | 0.8730 |
| Estimators | Population 1 | Population 2 | Population 3 |
|------------|--------------|--------------|--------------|
|            | MSE’s        | PRE’s        | MSE’s        | PRE’s        | MSE’s        | PRE’s        |
| $\hat{Y}$  | 893.344      | 100.000      | 2515.074     | 100.000      | 666353.000   | 100.000      |
| $\overline{Y}_{REG}$ | 133.267      | 670.342      | 224.611      | 1119.747     | 106635.000   | 624.891      |
| $\overline{Y}_{HS1}$ | 107.982      | 827.308      | 202.369      | 1242.816     | 97410.320    | 684.068      |
| $\overline{Y}_{HS2}$ | 126.334      | 707.129      | 219.747      | 1144.532     | 102974.400   | 647.105      |
| $\overline{Y}_{HS3}$ | 99.528       | 897.581      | 193.846      | 1297.460     | 95034.750    | 701.168      |
| $\overline{Y}_{J1}$ | 117.484      | 760.396      | 212.505      | 1183.536     | 96116.890    | 693.274      |
| $\overline{Y}_{J2}$ | 84.936       | 1051.785     | 183.133      | 1373.359     | 78444.970    | 849.453      |
| $\overline{H}^{(1)}_{HA}$ | 125.473      | 711.981      | 215.232      | 1168.541     | 101787.500   | 654.651      |
| $\overline{H}^{(2)}_{HA}$ | 125.577      | 711.391      | 215.239      | 1168.503     | 101806.400   | 654.529      |
| $\overline{H}^{(3)}_{HA}$ | 125.460      | 712.055      | 215.231      | 1168.546     | 101787.300   | 654.652      |
| $\overline{H}^{(4)}_{HA}$ | 125.597      | 711.278      | 215.235      | 1168.525     | 101792.300   | 654.620      |
| $\overline{H}^{(5)}_{HA}$ | 125.474      | 711.975      | 215.232      | 1168.541     | 101787.300   | 654.652      |
| $\overline{H}^{(6)}_{HA}$ | 125.477      | 711.958      | 215.232      | 1168.541     | 101787.300   | 654.652      |
| $\overline{H}^{(7)}_{HA}$ | 125.474      | 711.975      | 215.232      | 1168.541     | 101787.500   | 654.651      |
| $\overline{H}^{(8)}_{HA}$ | 125.460      | 712.055      | 215.231      | 1168.546     | 101787.300   | 654.652      |
| $\overline{H}^{(9)}_{HA}$ | 125.587      | 711.335      | 215.239      | 1168.503     | 101808.100   | 654.519      |
| $\overline{H}^{(10)}_{HA}$ | 127.734      | 699.378      | 216.793      | 1160.127     | 102806.400   | 648.162      |
| $\overline{P}_{P1}$ | 61.170       | 1460.428     | 161.068      | 1561.498     | 78281.210    | 851.229      |
| $\overline{P}_{P2}$ | 65.656       | 1360.643     | 178.828      | 1406.421     | 74957.120    | 888.979      |
| Table 4 | Estimators | Population 1 | Population 2 | Population 3 |
| --- | --- | --- | --- | --- |
| | | $n = 12$ | $n = 14$ | $n = 16$ | $n = 180$ | $n = 200$ | $n = 230$ | $n = 290$ | $n = 310$ | $n = 330$ |
| | $\hat{Y}$ | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| | $\bar{Y}_{REG}$ | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| | $\bar{Y}_{HS1}$ | 0.5140 | 0.4522 | 0.3982 | 0.0967 | 0.0873 | 0.0747 | 0.9971 | 0.9283 | 0.8595 |
| | $\bar{Y}_{HS2}$ | 0.6106 | 0.5190 | 0.4469 | 0.1056 | 0.0942 | 0.0795 | 1.0573 | 0.9771 | 0.8997 |
| | $\bar{Y}_{HS3}$ | 0.4688 | 0.4207 | 0.3753 | 0.0923 | 0.0839 | 0.0723 | 0.9703 | 0.9065 | 0.8415 |
| | $\bar{Y}_{JI1}$ | 0.5639 | 0.4870 | 0.4236 | 0.1019 | 0.0913 | 0.0775 | 0.9871 | 0.9196 | 0.8522 |
| | $\bar{Y}_{JI2}$ | 0.3864 | 0.3654 | 0.3356 | 0.0867 | 0.0796 | 0.0694 | 0.7930 | 0.7621 | 0.7223 |
| | $\bar{Y}_{HA}^{(1)}$ | 0.5430 | 0.4661 | 0.4055 | 0.1024 | 0.0913 | 0.0770 | 1.0094 | 0.9322 | 0.8654 |
| | $\bar{Y}_{HA}^{(2)}$ | 0.5387 | 0.4646 | 0.4051 | 0.1025 | 0.0906 | 0.0774 | 1.0246 | 0.9345 | 0.8645 |
| | $\bar{Y}_{HA}^{(3)}$ | 0.5361 | 0.4689 | 0.4108 | 0.1026 | 0.0910 | 0.0771 | 1.0217 | 0.9388 | 0.8603 |
| | $\bar{Y}_{HA}^{(4)}$ | 0.5305 | 0.4658 | 0.4042 | 0.1030 | 0.0914 | 0.0766 | 1.0184 | 0.9236 | 0.8598 |
| | $\bar{Y}_{HA}^{(5)}$ | 0.5353 | 0.4633 | 0.4062 | 0.1029 | 0.0912 | 0.0771 | 1.0113 | 0.9283 | 0.8589 |
| | $\bar{Y}_{HA}^{(6)}$ | 0.5400 | 0.4706 | 0.4057 | 0.1024 | 0.0914 | 0.0767 | 1.0237 | 0.9407 | 0.8521 |
| | $\bar{Y}_{HA}^{(7)}$ | 0.5311 | 0.4678 | 0.4061 | 0.1024 | 0.0904 | 0.0776 | 1.0153 | 0.9345 | 0.8636 |
| | $\bar{Y}_{HA}^{(8)}$ | 0.5373 | 0.4666 | 0.4083 | 0.1027 | 0.0915 | 0.0769 | 1.0172 | 0.9308 | 0.8786 |
| | $\bar{Y}_{HA}^{(9)}$ | 0.5368 | 0.4658 | 0.4054 | 0.1021 | 0.0911 | 0.0767 | 1.0168 | 0.9285 | 0.8590 |
| | $\bar{Y}_{HA}^{(10)}$ | 0.5462 | 0.4721 | 0.4071 | 0.1039 | 0.0910 | 0.0773 | 1.0277 | 0.9446 | 0.8663 |
| | $\bar{Y}_{PI1}$ | 0.2253 | 0.1965 | 0.1880 | 0.0744 | 0.0697 | 0.0619 | 0.7530 | 0.7232 | 0.6876 |
| | $\bar{Y}_{PI2}$ | 0.2347 | 0.1769 | 0.1574 | 0.0834 | 0.0767 | 0.0669 | 0.7210 | 0.6971 | 0.6664 |
| Estimators | Population 1 | | Population 2 | | Population 3 | |
|------------|--------------|--------------|--------------|--------------|--------------|
|            | $n = 12$     | $n = 14$     | $n = 16$     | $n = 180$    | $n = 200$    | $n = 230$    | $n = 290$    | $n = 310$    | $n = 330$    |
| $\hat{Y}$  | 100.000      | 100.000      | 100.000      | 100.000      | 100.000      | 100.000      | 100.000      | 100.000      | 100.000      |
| $\bar{Y}_{REG}$ | 766.586      | 749.207      | 729.755      | 1206.283     | 1193.833     | 1178.324     | 676.225      | 618.575      | 556.940      |
| $\bar{Y}_{HS1}$ | 987.347      | 913.926      | 858.046      | 1347.447     | 1313.015     | 1274.462     | 734.719      | 678.189      | 617.416      |
| $\bar{Y}_{HS2}$ | 812.755      | 785.375      | 758.909      | 1234.139     | 1217.667     | 1197.828     | 696.543      | 639.131      | 577.600      |
| $\bar{Y}_{HS3}$ | 1098.132     | 990.408      | 914.524      | 1411.218     | 1365.724     | 1315.974     | 753.621      | 697.281      | 637.084      |
| $\bar{Y}_{JI1}$ | 888.710      | 842.743      | 803.538      | 1279.251     | 1255.959     | 1228.712     | 740.498      | 683.856      | 622.881      |
| $\bar{Y}_{JI2}$ | 1382.002     | 1165.650     | 1032.976     | 1502.994     | 1473.015     | 1417.636     | 898.397      | 850.142      | 798.285      |
| $\bar{Y}_{(1)}^{HA}$ | 908.289      | 868.816      | 833.734      | 1272.931     | 1257.712     | 1236.909     | 722.773      | 663.921      | 600.093      |
| $\bar{Y}_{(2)}^{HA}$ | 907.008      | 869.154      | 832.204      | 1269.818     | 1254.615     | 1234.438     | 744.068      | 723.019      | 717.034      |
| $\bar{Y}_{(3)}^{HA}$ | 905.622      | 866.220      | 839.286      | 1261.567     | 1244.772     | 1233.626     | 736.168      | 722.732      | 714.374      |
| $\bar{Y}_{(4)}^{HA}$ | 905.313      | 849.367      | 834.382      | 1263.093     | 1260.664     | 1242.487     | 733.023      | 725.365      | 710.147      |
| $\bar{Y}_{(5)}^{HA}$ | 903.575      | 864.829      | 836.822      | 1262.015     | 1259.152     | 1240.036     | 732.665      | 716.237      | 707.388      |
| $\bar{Y}_{(6)}^{HA}$ | 911.161      | 868.626      | 832.772      | 1268.188     | 1250.123     | 1245.323     | 730.110      | 724.998      | 717.704      |
| $\bar{Y}_{(7)}^{HA}$ | 897.334      | 865.880      | 848.572      | 1262.963     | 1252.886     | 1244.779     | 726.805      | 717.746      | 713.321      |
| $\bar{Y}_{(8)}^{HA}$ | 904.338      | 862.441      | 840.217      | 1259.486     | 1253.556     | 1240.967     | 733.072      | 719.673      | 714.668      |
| $\bar{Y}_{(9)}^{HA}$ | 907.359      | 878.727      | 838.201      | 1281.329     | 1251.296     | 1234.161     | 729.270      | 724.919      | 708.541      |
| $\bar{Y}_{(10)}^{HA}$ | 894.141      | 853.038      | 839.442      | 1263.584     | 1247.958     | 1232.534     | 725.638      | 715.839      | 707.742      |
| $\bar{Y}_{PI}$   | 3258.479     | 1952.065     | 1524.815     | 1748.603     | 1642.411     | 1536.096     | 937.697      | 889.288      | 836.048      |
| $\bar{Y}_{F2}$   | 3055.020     | 1903.397     | 1464.315     | 1561.014     | 1493.770     | 1423.147     | 968.348      | 922.812      | 872.949      |
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Muhammad Irfan obtained his Ph.D. degree in Statistics at Zhejiang University, Hangzhou, China in 2018. He holds the position of Assistant Professor in the Department of Statistics, Government College University, Faisalabad Pakistan. He has more than 20 research publications in well reputed journals. His areas of interest are Sampling Theory, Probability Distributions and Time Series Analysis.

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