Unique Continuation and Observability Estimates for 2-D Stokes Equations with the Navier Slip Boundary Condition

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Abstract

This paper presents a unique continuation estimate for 2-D Stokes equations with the Naiver slip boundary condition in a bounded and simply connected domain. Consequently, an observability estimate for this equation from a subset of positive measure in time follows from the aforementioned unique continuation estimate and the new strategy developed in [16]. Several applications of the above-mentioned observability estimate to control problems of the Stokes equations are given.

Keywords. Stokes equations, observability estimate, unique continuation estimate, bang-bang property

AMS Subject Classifications. 93B07, 35Q30, 35B60

1 Introduction and Main Results

Let \( T > 0 \) and \( \Omega \subset \mathbb{R}^2 \) be a bounded and simply connected domain with a \( C^3 \) boundary \( \partial \Omega \). Let \( \mathbf{n} \) be the unit exterior normal vector to \( \partial \Omega \) and \( \tau \) be the unit tangent vector to \( \partial \Omega \) such that \((\mathbf{n}, \tau)\) is positively oriented. Consider the following Stokes equations:

\[
\begin{aligned}
\mathbf{u}_t - \Delta \mathbf{u} - \nabla p &= 0 \quad \text{in } \Omega \times (0, T), \\
\nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega \times (0, T), \\
\text{rot } \mathbf{u} &= 0, \quad \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \partial \Omega \times (0, T).
\end{aligned}
\]

The boundary condition in (1.1) is a special Navier slip boundary condition. In general, the Navier slip boundary condition reads (see for instance [4])

\[
\begin{aligned}
\mathbf{u} \cdot \mathbf{n} &= 0, \\
\bar{\sigma} \mathbf{u} \cdot \tau + (1 - \bar{\sigma})n_i \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tau_j &= 0,
\end{aligned}
\]

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where $\sigma$ is a constant in $[0, 1)$. Here we used the Einstein’s summation convention.

For the unique continuation of Stokes equations, there have been many literatures. Here, we would like to mention the classical reference [6] (see also [7]), where some qualitative unique continuation results are provided. The controllability of Stokes equations or Navier-Stokes equations, with either the Dirichlet boundary condition or the Navier-slip boundary condition are investigated in a large number of references. For Stokes equations with Dirichlet boundary condition, we refer to [5], [8] and the references therein. In [9], Jean-Michel Coron studied the approximate controllability of the 2-D Navier-Stokes equations with the general Navier-slip boundary conditions (1.2).

However, to our best knowledge, very limited works are concerned with the quantitative unique continuation for Navier-Stokes equations. Here, we would like to mention the paper [11] where an upper bound was given for the size of the nodal set of the vorticity for a solution (but not the nodal set of a solution) of the 2-D periodic Navier-Stokes equations.

The main purpose of this paper is to present the quantitative unique continuation for Equations (1.1). It should be mentioned that the current study is greatly motivated by recent papers [15], [16] and [2], where some kinds of unique continuation estimates for heat (or parabolic) equations were established. To present the main result of this paper, we begin by introducing the following notations:

$$L^2_\sigma(\Omega) = \{ u \in L^2(\Omega; \mathbb{R}^2) : \nabla \cdot u = 0, \ u \cdot n|_{\partial \Omega} = 0 \};$$

$$H^1_\sigma(\Omega) = \{ u \in H^1(\Omega; \mathbb{R}^2) : \nabla \cdot u = 0, \ u \cdot n|_{\partial \Omega} = 0 \}.$$  

When $O$ is a subset in $\mathbb{R}^2$, we will write accordingly $L^2(O)$ and $H^1(O)$ for $L^2(O; \mathbb{R}^2)$ and $H^1(O; \mathbb{R}^2)$, if there is no chance to make any confusion. We will denote by $(\cdot, \cdot)$ the usual inner product in $L^2(\Omega)$ and by $\chi_E$ the characteristic function of the subset $E$. In this paper, $N(\cdot)$ stands for a positive constant depending on what are enclosed in the brackets. It maybe vary in different contexts.

The main results of this paper are included in the following theorem:

**Theorem 1.1.** Let $T > 0$ and $\Omega \subset \mathbb{R}^2$ be a bounded and simply connected domain with a $C^3$ boundary $\partial \Omega$. Let $\omega \subset \Omega$ be a nonempty open subset. Then, (i) There are $N = N(\Omega, \omega)$ and $\alpha = \alpha(\Omega, \omega)$, with $\alpha \in (0, 1)$, such that when $0 \leq t_1 < t_2 \leq T$ and $u_0 \in L^2_\sigma(\Omega)$, the solution $u$ to Equations (1.1), with the initial condition $u(0) = u_0$, verifies

$$\| u(t_2) \|_{L^2(\Omega)} \leq N e^{\frac{\alpha}{2 - \alpha}} \left( \int_0^t \| u(t_2) \|_{L^2(\omega)}^\alpha \right)^\frac{1}{\alpha} \| u(t_1) \|_{L^2(\Omega)}^{1 - \frac{\alpha}{2}}. \quad (1.3)$$

(ii) For each subset $E \subset (0, T)$ of positive measure, there exists $N = N(\Omega, \omega, E, T)$ such that when $u_0 \in L^2_\sigma(\Omega)$, the solution $u$ to Equations (1.1), with the initial condition $u(0) = u_0$, satisfies

$$\| u(T) \|_{L^2(\Omega)} \leq N \int_0^T \chi_E \| u(t) \|_{L^2(\omega)} dt. \quad (1.4)$$
Our strategy to prove the estimate (1.3) is as follows: (a) We transform Equations (1.1) into one of parabolic type via the method of stream function; (b) We apply the estimate established in [2] (see also [15] and [16] for the case where $\Omega$ is convex), together with a type of the Sobolev interpolation inequality and some properties of heat equations, to get a unique continuation estimate for the stream functions; (c) We pull the unique continuation estimate for stream functions back to the desired estimate (1.3).

Three remarks are in order:

- The estimate (1.3) is not a trivial consequence of the corresponding unique continuation estimate for heat equations built up in [2] (see also [15] and [16]).
- In Step (c), it will be used that $\Omega$ is simply connected.
- Based on the estimate (1.3), the observability estimate (1.4) follows from the new strategy developed in [16] at once. The estimate (1.4) leads to the null-controllability of Equations (1.1) with controls restricted over $\omega \times E$. Since $E$ is a measurable subset in time, such null-controllability for Stokes equations seems to be new.

The rest of this paper is organized as follows: Section 2 presents some preliminaries; Section 3 proves Theorem 1.1; Section 4 provides some applications of Theorem 1.1 in the control theory of Stokes equations and Section 5, i.e., Appendix contains the proof of some elementary results used in this study.

## 2 Some Preliminaries

This section is devoted to review some classical results on the decomposition of two-dimensional vector fields and to prove the well-posedness of Equations (1.1). A comprehensive discussion of the decomposition of 2-D vector fields can be found in [15, Appendix I, pp. 458-469] or [10, pp. 18-56].

For each $\psi \in H^1(\Omega)$ and each $u = (u_1, u_2) \in H^1(\Omega)$, define

$$\text{curl} \psi = (\partial_2 \psi, -\partial_1 \psi), \quad \text{rot} u = \partial_1 u_2 - \partial_2 u_1.$$ 

It can be easily verified that

$$\text{curl} \text{rot} u = -\Delta u, \quad \text{when } u \in H^1_\sigma(\Omega) \cap H^2(\Omega);$$

$$\text{curl} \psi \in L^2_\sigma(\Omega), \quad \text{rot} \text{curl} \psi = -\Delta \psi, \quad \text{when } \psi \in H^1_\sigma(\Omega) \cap H^2(\Omega). \quad (2.1)$$

The following lemma gives a kind of Green formula connected with operators rot and curl:

**Lemma 2.1.** For any $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$,

$$\int_\Omega (\text{curl} \text{rot} u) \cdot v \, dx = -\int_{\partial \Omega} (\text{rot} u) (v \cdot \tau) \, ds + \int_\Omega (\text{rot} u)(\text{rot} v) \, dx.$$
Proof. Set \( \phi = \text{rot } u \). From the standard Green formula,

\[
\int_{\Omega} (\text{curl } \phi) \cdot v \, dx = \int_{\Omega} (\partial_2 \phi v_1 - \partial_1 \phi v_2) \, dx
\]

\[
= \int_{\partial \Omega} \phi(v_1 n_2 - v_2 n_1) \, ds - \int_{\Omega} \phi(\partial_2 v_1 - \partial_1 v_2 n_1) \, dx
\]

\[
= -\int_{\partial \Omega} \phi(v \cdot \tau) \, ds + \int_{\Omega} \phi \text{rot } v \, dx,
\]

which leads to the desired equality.

The next lemma concerns with the well-posedness of the parabolic-type equation satisfied by stream functions. Such well-posedness for a similar equation as Equation (2.3) in the next lemma was built up in \([12, \text{Théorème 6.10, pp. 88}]\). For the sake of completion, we give its proof in Appendix.

**Lemma 2.2.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with a \( C^3 \) boundary \( \partial \Omega \). Then, for each \( \psi_0 \in H^1_0(\Omega) \), there exists a unique solution \( \psi \), with

\[
\psi \in C([0, T]; H^1_0(\Omega)) \cap C^1((0, T]; H^1_0(\Omega)) \cap C((0, T]; H^2(\Omega)),
\]

and \( \Delta \psi \in C((0, T]; H^1_0(\Omega)) \),

(2.2)

to the equation

\[
\begin{cases}
\Delta \psi_t - \Delta^2 \psi = 0 & \text{in } \Omega \times (0, T), \\
\Delta \psi = 0, \quad \psi = 0 & \text{on } \partial \Omega \times (0, T), \\
\psi(\cdot, 0) = \psi_0(\cdot) & \text{in } \Omega.
\end{cases}
\]

(2.3)

**Lemma 2.3.** Let \( \Omega \) be a bounded and simply connected domain with a \( C^2 \) boundary \( \partial \Omega \). Then, for each \( u \in L^2(\Omega) \), there exists a unique stream function \( \psi \in H^1_0(\Omega) \) such that \( \text{curl } \psi = u \).

The proof of lemma 2.3 is essentially contained in \([10, \text{Theorem 3.1, pp. 37-40}]\), where the curl equation \( \text{curl } \psi = u \) was studied for the case that \( \Omega \) is multi-connected. The corresponding result there reads: for each \( u \in L^2(\Omega) \), the curl equation has a solution in \( H^1(\Omega) \), which is a constant on each connected component of \( \partial \Omega \). These constants may be different in different connected components. Hence, it may happen that any solution of the curl equation is not in \( H^1(\Omega) \) in that case. For the sake of convenience, we provide a proof for Lemma 2.3 in Appendix of this paper.

**Proposition 2.4.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded and simply connected domain with a \( C^3 \) boundary \( \partial \Omega \). Then, for each \( u_0 \in L^2(\Omega), \) Equations (1.1), with the initial condition \( \psi(\cdot, 0) = u_0 \), has a unique solution

\[
u \in C([0, T]; L^2(\Omega)) \cap C^1((0, T]; L^2(\Omega)) \cap C((0, T]; H^2(\Omega) \cap H^1(\Omega)),
\]

(2.4)

for some \( p \in L^2_{\text{loc}}(0, T; L^2(\Omega)) \). Moreover, \( u \in L^2(0, T; H^1(\Omega)) \) and

\[
\max_{s \in [0, T]} \|u(s)\|_{L^2(\Omega)}^2 + \int_0^T \|u(t)\|_{H^2(\Omega)}^2 \, dt \leq N(\Omega) \|u_0\|_{L^2(\Omega)}^2.
\]

(2.5)
Proof. According to Lemma 2.2, there is a unique \( \psi_0 \in H^1_0(\Omega) \) such that \( \text{curl} \psi_0 = u_0 \). Then, by Lemma 2.2 Equation (2.2), with the aforementioned \( \psi_0 \), has a unique solution \( \psi \) verifying (2.2). We claim that the vector field \( u := \text{curl} \psi \) satisfies Equations (1.1), as well as (2.4). In fact, it can be checked readily that \( \nabla \cdot u = \nabla \cdot \text{curl} \psi = 0 \) and that

\[
\text{rot} u = \text{rot} \text{curl} \psi = -\Delta \psi = 0, \quad \text{on} \quad \partial \Omega \times (0, T);
\]

\[
u \cdot n = \text{curl} \psi \cdot n = \frac{\partial \psi}{\partial \tau} = 0, \quad \text{on} \quad \partial \Omega \times (0, T).
\]

Also, by Equation (2.3),

\[
\text{rot}(u_t - \Delta u) = \text{rot}(\text{curl} \psi_t - \Delta \text{curl} \psi) = 0.
\]

Because \( \Omega \) is simply connected, for a.e. \( t \in (0, T) \), there exists a unique function \( p(t) \in H^1(\Omega) \) up to a constant such that (see for instance [10, Theorem 2.9, pp. 31])

\[
u_t - \Delta \nu = \nabla p.
\]

From the Poincaré inequality, it follows that there exists \( p \in L^2_{loc}(0, T; L^2(\Omega)) \) provided that \( \int_{\Omega} p(t, x) \, dx = 0 \) for a.e. \( t \in (0, T) \).

To justify (2.5), we multiply the first equation of (1.1) by \( u \), and then integrate it from \( \epsilon > 0 \) (sufficiently small) to \( s \in (0, T] \). Now, Lemma 2.1 leads to

\[
\|u(s)\|_{L^2(\Omega)}^2 + 2 \int_\epsilon^s \|\text{rot} u(t)\|_{L^2(\Omega)}^2 \, dt = \|u(\epsilon)\|_{L^2(\Omega)}^2. \tag{2.6}
\]

Sending \( \epsilon \to 0 \) in (2.6), we see that

\[
\|u(s)\|_{L^2(\Omega)}^2 + 2 \int_0^s \|\text{rot} u(t)\|_{L^2(\Omega)}^2 \, dt = \|u_0\|_{L^2(\Omega)}^2.
\]

This, together with the simply connectedness of \( \Omega \) and the decomposition theorem (see, e.g., [10] Remark 3.5, pp. 45) or [18] Lemma 1.6, pp. 465), indicates the estimate (2.5), from which, the uniqueness follows at once. \( \square \)

## 3 Unique Continuation Estimates

This section is devoted to prove Theorem 1.1. We first establish an estimate for the gradient of the stream function, and then present the proof of estimate (1.3) and (1.4), where the simply connectedness of the domain is used. In what follows, we will denote by \( \psi \) a solution to the equation:

\[
\begin{cases}
\Delta \psi - \Delta^2 \psi = 0 & \text{in} \quad \Omega \times (0, T), \\
\Delta \psi = 0, \quad \psi = 0 & \text{on} \quad \partial \Omega \times (0, T), \\
\psi(0) \in H^1_0(\Omega). 
\end{cases} \tag{3.1}
\]

**Lemma 3.1.** For any \( 0 \leq s < t \leq T \), \( \|\nabla \psi(t)\|_{L^2(\Omega)} \leq \|\nabla \psi(s)\|_{L^2(\Omega)} \). Moreover, \( \nabla \psi(t) = 0 \) for all \( t \in [0, T] \) whenever \( \nabla \psi(T) = 0 \).

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Proof. First, set \( e(t) = \int_{\Omega} |\nabla \psi(t)|^2 \, dx, \ t \in [0, T]. \)

Since \( \Delta \psi = 0 \) and \( \psi = 0 \) on \( \partial \Omega \times (0, T) \), we have that for each \( t > 0 \),

\[
\dot{e}(t) = \int_{\Omega} 2\nabla \psi \cdot \nabla \psi_t \, dx = 2 \int_{\partial \Omega} \psi \frac{\partial \psi_t}{\partial n} \, ds - 2 \int_{\Omega} \psi \Delta \psi_t \, dx
\]

\[
= -2 \int_{\Omega} \psi \Delta(\Delta \psi) \, dx = -2 \int_{\partial \Omega} \psi \frac{\partial \Delta \psi}{\partial n} \, ds
\]

\[
= -2 \int_{\Omega} |\Delta \psi|^2 \, dx.
\]

Namely, it holds that

\[
\dot{e}(t) = -2 \int_{\Omega} |\Delta \psi|^2 \, dx. \tag{3.2}
\]

Next,

\[
\ddot{e}(t) = -4 \int_{\Omega} \Delta \psi \Delta \psi_t \, dx = -4 \int_{\Omega} \Delta \psi \Delta(\Delta \psi) \, dx
\]

\[
= -4 \int_{\partial \Omega} \Delta \psi \frac{\partial (\Delta \psi)}{\partial n} \, ds + 4 \int_{\Omega} \nabla \Delta \psi \cdot \nabla \psi \, dx = 4 \int_{\Omega} |\nabla (\Delta \psi)|^2 \, dx. \tag{3.3}
\]

Integrating by parts and the Cauchy-Schwartz inequality lead to

\[
\int_{\Omega} |\Delta \psi|^2 \, dx = -\int_{\Omega} \nabla (\Delta \psi) \cdot \nabla \psi \, dx \leq \left( \int_{\Omega} |\nabla \Delta \psi|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla \psi|^2 \, dx \right)^{\frac{1}{2}}.
\]

This, together with (3.2) and (3.3), shows that

\[
\ddot{e}(t)e(t) \geq (\dot{e}(t))^2, \ t \in (0, T). \tag{3.4}
\]

If \( e(t) = 0 \) for all \( 0 \leq t \leq T \), we are done. Otherwise there exists a closed interval \([t_1, t_2] \subset [0, T]\) on which

\[
e(t) > 0 \quad \text{for all} \quad t \in [t_1, t_2); \quad \text{and} \quad e(t_2) = 0. \tag{3.5}
\]

Now, write \( h(t) := \log e(t), \ t_1 \leq t < t_2 \). Then, it follows from (3.3) that

\[
h''(t) \geq 0 \quad \text{for all} \quad t \in (t_1, t_2),
\]

i.e., \( h(t) \) is convex on \((t_1, t_2)\). Thus,

\[
h \left( \frac{t_1 + t}{2} \right) \leq h(t_1) + \frac{h(t)}{2}, \ t \in (t_1, t_2).
\]

This implies that

\[
e \left( \frac{t_1 + t}{2} \right) \leq (e(t_1))^{\frac{1}{2}}(e(t))^{\frac{1}{2}}, \ t \in (t_1, t_2).
\]

sending \( t \to t_2 \) in the above identity, we get \( e \left( \frac{t_1 + t_2}{2} \right) = 0 \), which contradicts (3.5). \( \square \)
Lemma 3.2. Let $\omega$ be a nonempty open subset of $\Omega$. Then, there are constants $N = N(\Omega, \omega)$ and $\alpha = \alpha(\Omega, \omega) \in (0, 1)$, such that when $0 \leq t_1 < t_2 \leq T$,

$$
\|\nabla \psi(t_2)\|_{L^2(\Omega)} \leq \left( N e^{\frac{N}{12}t_1} \|\nabla \psi(t_2)\|_{L^2(\omega)} \right)^{\alpha} \|\nabla \psi(t_1)\|_{L^2(\Omega)}^{\frac{1-\alpha}{12}},
$$

(3.6)

for all solutions to Equation $\text{(3.7)}$. Consequently, $\psi \equiv 0$ when $\nabla \psi(T, x) = 0$ for a.e. $x \in \omega$.

Proof. It suffices to prove the estimate $\text{(3.6)}$ when $0 < t_1 < t_2 \leq T$. Indeed, if it is the case, then, by taking $t_1 = \frac{t}{2}$ and noting that $\|\nabla \psi(t_1)\|_{L^2(\Omega)} \leq \|\nabla \psi(0)\|_{L^2(\Omega)}$ (see Lemma 3.1), we reach the estimate $\text{(3.6)}$ for $t_1 = 0$.

Because $\omega$ is a non-empty open set, there exists a ball $B_r$, centered at a point $x_0 \in \omega$ and of radius $r > 0$, such that $B_r \subset \omega$. Since $\Delta \psi(\cdot)$ satisfies the heat equation with the zero Dirichlet boundary condition, it follows from $\text{[2, Theorem 6]}$ (see also $\text{[15, Proposition 2.1]}$ or $\text{[16, Proposition 2.2]}$) that there are constants $N = N(\Omega, B_r)$ and $\alpha = \alpha(\Omega, B_r) \in (0, 1)$, such that

$$
\|\Delta \psi(t_2)\|_{L^2(\Omega)} \leq \|\Delta \psi(t_3)\|_{L^2(\Omega)} \left( N e^{\frac{N}{12}t_2} \|\Delta \psi(t_2)\|_{L^2(B_r)} \right)^{1-\alpha},
$$

(3.7)

when $0 < t_1 < t_3 < t_2 \leq T$, and $\psi$ solves Equation $\text{(3.1)}$.

Because $\psi = 0$ on $\partial \Omega \times (0, T)$, we obtain from the regularity of elliptic equations that

$$
\|\nabla \psi(t_2)\|_{L^2(\Omega)} \leq N(\Omega)\|\Delta \psi(t_2)\|_{L^2(\Omega)}.
$$

(3.8)

From the estimate:

$$
\|\Delta \psi(t_2)\|_{L^2(B_r)}^2 \leq \|\nabla \psi(t_2)\|_{H^1(B_r)}^2
$$

and the Sobolev interpolation inequality (see, for instance, $\text{[11, Theorem 5.2, pp.135]}$ or $\text{[13, pp. 43-44]}$):

$$
\|\nabla \psi(t_2)\|_{H^1(B_r)} \leq N(B_r)\|\nabla \psi(t_2)\|_{H^2(B_r)}^{\frac{1}{2}}\|\Delta \psi(t_2)\|_{L^2(B_r)}^{\frac{1}{2}},
$$

it follows that

$$
\|\Delta \psi(t_2)\|_{L^2(B_r)}^2 \leq N(B_r)\|\nabla \psi(t_2)\|_{H^2(B_r)}\|\Delta \psi(t_2)\|_{L^2(B_r)}.
$$

(3.9)

On the other hand, since $\Delta \psi = 0$ on $\partial \Omega \times (0, T)$, integrating by parts and the Cauchy-Schwartz inequality lead to

$$
\|\Delta \psi(t_3)\|_{L^2(\Omega)}^2 \leq \|\nabla \Delta \psi(t_3)\|_{L^2(\Omega)}\|\nabla \psi(t_3)\|_{L^2(\Omega)}.
$$

(3.10)

Combining inequalities $\text{(3.7)}$—$\text{(3.10)}$, we deduce that

$$
\|\nabla \psi(t_2)\|_{L^2(\Omega)} \leq I_1^{\frac{N}{12}} I_2^{\frac{N}{12}} I_3^{\frac{N}{12}} I_4^{\frac{N}{12}},
$$

(3.11)

where

\[
\begin{align*}
I_1 &= \|\nabla \Delta \psi(t_3)\|_{L^2(\Omega)}, \\
I_2 &= \|\nabla \psi(t_3)\|_{L^2(\Omega)}, \\
I_3 &= \|\nabla \psi(t_2)\|_{H^2(B_r)}, \\
I_4 &= N e^{\frac{N}{12}t_2} \|\nabla \psi(t_2)\|_{L^2(B_r)}, \quad N = N(\Omega, B_r).
\end{align*}
\]
Next, write \( \{ \lambda_i \}_{i \geq 1} \), with \( 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \), for the eigenvalues of the Laplace operator \(-\Delta\) with zero Dirichlet boundary condition, and \( \{ e_i \}_{i \geq 1} \) for the corresponding set of \( L^2(\Omega) \)-normalized eigenfunctions. Since \( \Delta \psi \) satisfies the heat equation with zero Dirichlet boundary condition, it holds that
\[
\Delta \psi(t_3) = \sum_{i \geq 1} e^{-\lambda_i(t_3-t_1)}(\Delta \psi(t_1), e_i)e_i.
\]
Thus,
\[
I_1^2 = \sum_{i \geq 1}|(\Delta \psi(t_1), e_i)|^2 \lambda_i e^{-2\lambda_i(t_3-t_1)}
\leq \sup_{i \geq 1} \left( \lambda_i e^{-\lambda_i(t_3-t_1)} \right) \sum_{i \geq 1}|(\Delta \psi(t_1), e_i)|^2 e^{-\lambda_i(t_3-t_1)}
\leq \frac{1}{t_3-t_1} \left\| \Delta \psi \left( \frac{t_1 + t_3}{2} \right) \right\|_{L^2(\Omega)}^2. \tag{3.12}
\]
Because \( \psi \) verifies the heat equation with zero Dirichlet boundary condition, it stands that
\[
\psi \left( \frac{t_1 + t_3}{2} \right) = \sum_{i \geq 1} e^{-\lambda_i(t_3-t_1)/2}(\psi(t_1), e_i)e_i.
\]
Hence,
\[
\left\| \Delta \psi \left( \frac{t_1 + t_3}{2} \right) \right\|_{L^2(\Omega)}^2 = \sum_{i \geq 1} \lambda_i^2 e^{-\lambda_i(t_3-t_1)}|\psi(t_1), e_i|^2
\leq \sup_{i \geq 1} \left( \lambda_i e^{-\lambda_i(t_3-t_1)} \right) \sum_{i \geq 1} \lambda_i |\psi(t_1), e_i|^2 \tag{3.13}
\leq \frac{1}{t_3-t_1} \left\| \nabla \psi(t_1) \right\|_{L^2(\Omega)}^2.
\]
This, together with (3.12), leads to
\[
I_1^2 \leq \frac{1}{(t_3-t_1)^2} \left\| \nabla \psi(t_1) \right\|_{L^2(\Omega)}^2. \tag{3.14}
\]
Since \( \psi = 0 \) and \( \Delta \psi = 0 \) on \( \partial \Omega \times (0, T) \), applying the elliptic regularity and the Poincaré inequality, we obtain that
\[
\left\| \nabla \psi(t_2) \right\|_{H^2(B_r)} \leq \left\| \psi(t_2) \right\|_{H^3(\Omega)} \leq N(\Omega) \left\| \Delta \psi(t_2) \right\|_{H^1(\Omega)} \leq N(\Omega) \left\| \nabla \Delta \psi(t_2) \right\|_{L^2(\Omega)}.
\]
By similar arguments as those to derive (3.12)–(3.14), we can verify that
\[
\left\| \nabla \Delta \psi(t_2) \right\|_{L^2(\Omega)} \leq \frac{1}{(t_2-t_1)^2} \left\| \nabla \psi(t_1) \right\|_{L^2(\Omega)}^2.
\]
Now, the above two estimates yield that
\[
I_3^2 = \left\| \nabla \psi(t_2) \right\|_{H^2(B_r)}^2 \leq \frac{N(\Omega)}{(t_2-t_1)^2} \left\| \nabla \psi(t_1) \right\|_{L^2(\Omega)}^2.
\]
In what follows, we find that estimate (1.4) from (1.3).

The desired estimate (3.6) stands if we replace $\frac{\alpha}{2}$ by $\alpha$ in (3.15).

Finally, the unique continuation in the second part of this lemma is an immediate consequence of the estimate (3.6) and Lemma 3.1.

**Proof of Theorem 1.1.** Arbitrarily fix a $u_0 \in L^2(\Omega)$. Since $\Omega$ is simply connected, according to Lemma 2.3 there exists a unique stream function $\psi_0 \in H^1(\Omega)$ such that $\text{curl} \, \psi_0 = u_0$. Let $\psi$ be the solution to Equation (2.3) with the aforementioned $\psi_0$. By Proposition 2.3 $u := \text{curl} \, \psi$ is the unique solution of Equations (1.1) with the initial condition $u(0) = u_0$. From Lemma 3.2 it follows that $\psi$ holds the estimate (3.6). Since

$$\|\nabla \psi(t)\|_{L^2(\Omega)} = \|\text{curl} \, \psi(t)\|_{L^2(\Omega)} = \|u(t)\|_{L^2(\Omega)} \quad \text{for each } t \in [0, T],$$

where $\Omega$ is either $\Omega$ or $\omega$, and because $u_0$ was arbitrarily taken from $L^2(\Omega)$, the desired estimate (1.3) follows from (3.6) at once.

Consequently, if $\|u(t)\|_{L^2(\Omega)} = 0$ for some $t > 0$, then by the estimate (3.6) and Lemma 3.2 we find that $u \equiv 0$.

Finally, by making use of the new strategy in [10], we can derive the observability estimate (1.3) from (1.3).

**4 Applications**

Let $T > 0$ and $\Omega \subset \mathbb{R}^2$ be a bounded and simply connected domain with a $C^3$ boundary $\partial \Omega$. Consider the following controlled Stokes equations:

$$\begin{cases}
    u_t - \Delta u - \nabla p = f & \text{in } \Omega \times (0, T), \\
    \nabla \cdot u = 0 & \text{in } \Omega \times (0, T), \\
    \text{rot} \, u = 0, \quad u \cdot n = 0 & \text{on } \partial \Omega \times (0, T), \\
    u(t, 0) = u_0(\cdot) & \text{in } \Omega.
\end{cases} \quad (4.1)
$$

In what follows, $H^{-1}_0(\Omega)$ stands for the dual of $H^1_0(\Omega)$, and $\langle \cdot, \cdot \rangle$ the scalar product between $H^{-1}_0(\Omega)$ and $H^1_0(\Omega)$. We first define the weak solution to equation (4.1).
Definition 4.1. For each \( f \in L^2(0,T;L^2(\Omega)) \) and each \( u_0 \in L^2_0(\Omega) \), \( u \) is called a weak solution of Equations (4.1), if

\[
\begin{align*}
&\mathbf{u} \in C([0,T];L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_\sigma(\Omega)), \quad \mathbf{u}_t \in L^2(0,T;H^{-1}_\sigma(\Omega)) \quad \text{with} \quad \mathbf{u}(\cdot,0) = \mathbf{u}_0 \\
&\int_0^T \langle \mathbf{u}_t, \mathbf{v} \rangle \, dt + \int_0^T (\text{rot} \mathbf{u}, \text{rot} \mathbf{v}) \, dt = \int_0^T \langle \mathbf{f}, \mathbf{v} \rangle \, dt, \quad \forall \mathbf{v} \in L^2(0,T;H^1_\sigma(\Omega)). \quad (4.2)
\end{align*}
\]

Remark 4.2. It can be verified that the solution \( \mathbf{u} \) obtained in Proposition 4.1 is a weak solution of Equations (4.1) with \( f = 0 \).

The following proposition is concerned with the existence and uniqueness of the weak solution of Equations (4.1) and we leave its proof in Appendix of this paper.

Proposition 4.3. For each \( f \in L^2(0,T;L^2(\Omega)) \) and each \( u_0 \in L^2_0(\Omega) \), Equations (4.1) has a unique weak solution.

In what follows, we will denote by \( \mathbf{u}(\cdot;f) \) the weak solution to Equation (4.1) corresponding to the exterior force \( f \), when the initial datum \( u_0 \) is given.

### 4.1 Null Controllability of the Stokes Equations

In this subsection, we will show that Theorem 1.1 implies the null controllability of Stokes equations with control restricted over \( \omega \times E \), where \( \omega \subset \Omega \) is a nonempty open subset, and \( E \) is a subset of positive measure in \((0,T)\). Denoted by \( \mathbf{u}(\cdot;f \chi_\omega \chi_E) \) the unique weak solution to Equations (4.1) corresponding to the control \( f \) restricted on the subset \( \omega \times E \).

Corollary 4.4. For each \( u_0 \in L^2_0(\Omega) \), there exists a control \( f \in L^\infty(0,T;L^2(\Omega)) \), with

\[
\|f\|_{L^\infty(0,T;L^2(\Omega))} \leq N\|u_0\|_{L^2(\Omega)}, \quad N = N(\Omega,\omega,E,T), \quad (4.3)
\]

such that \( \mathbf{u}(T;f \chi_\omega \chi_E) = 0 \).

Before giving the proof of Corollary 4.4, we first state an interpolation lemma quoted from [18, pp. 260-261].

Lemma 4.5. Let \( V,H,V' \) be three Hilbert spaces, each space included in the following one: \( V \subset H \equiv H' \subset V' \), \( V' \) being the dual of \( V \). If a function \( u \in L^2(0,T;V) \) and its derivative \( u' \in L^2(0,T;V') \), then \( u \) is almost everywhere equal to a continuous function from \([0,T]\) into \( H \) and we have the following equality, which holds in the scalar distribution sense on \((0,T)\):

\[
\frac{d}{dt}\|u\|_H^2 = 2\langle u', u \rangle_{V',V}. \quad (4.4)
\]

Remark 4.6. The equality (4.4) is meaningful since \( \langle u'(t), u(t) \rangle_{V',V} \) is integrable on \((0,T)\). Using (4.4) for \( u + v \), we have

\[
\frac{d}{dt}(u,v)_H = \langle u', v \rangle_{V',V} + \langle v', u \rangle_{V',V}, \quad (4.5)
\]

for all \( u,v \in L^2(0,T;V) \) with \( u', v' \in L^2(0,T;V') \).
Proof of Corollary 4.4. We first introduce the following adjoint system of Equations (4.1):

\[
\begin{align*}
    v_t + \Delta v + \nabla q &= 0 & \text{in} \ Ω \times (0, T), \\
    \nabla \cdot v &= 0 & \text{in} \ Ω \times (0, T), \\
    \text{rot} v &= 0, \ v \cdot n &= 0 & \text{on} \ ∂Ω \times (0, T), \\
    v(\cdot, T) &= v_T(\cdot) & \text{in} \ Ω.
\end{align*}
\]

(4.6)

For each \(v_T \in L^2_ε(Ω)\), according to Proposition 2.4, Equations (4.6) has a unique solution \(v \in L^2(0, T; H^1_0(Ω))\).

By Theorem 1.1 there exists a positive constant \(N = N(Ω, ω, E, T)\) such that

\[
\|v(0)\|_{L^2(Ω)} \leq N \int_0^T \chi_E \|v(t)\|_{L^2(ω)} \, dt.
\]

(4.7)

Now, set

\[
X \triangleq \{v = v_1 \chi_E : \ v \text{ solves Equations (4.1)} \}.
\]

Let \(X\) be endowed with the norm of \(L^1(0, T; L^2(Ω))\). Clearly, it is a subspace of \(L^1(0, T; L^2(Ω))\). Next, we define a linear functional \(F : X \to \mathbb{R}\) by

\[
F(v) = -(v(0), u_0).
\]

Note that \(F\) is well defined. In fact, if \(v_1 = v_2\), then it follows from Theorem 1.1 that \(v_1 = v_2\). By (4.7), we have that

\[
|F(v)| \leq \|v(0)\|_{L^2(Ω)} \|u_0\|_{L^2(Ω)} \leq N\|u_0\|_{L^2(Ω)} \int_0^T \|v(t)\|_{L^2(ω)} \, dt.
\]

Hence, \(F\) is a bounded linear functional on \(X\). By the Hahn-Banach theorem, \(F\) can be extended to a bounded linear functional on \(L^1(0, T; L^2(Ω))\). Using the Riesz representation theorem, we get that there exists \(f \in L^∞(0, T; L^2(Ω))\) such that

\[
F(h) = \int_0^T \int_Ω f(x, t) \cdot h(x, t) \, dx \, dt
\]

and

\[
|F(h)| \leq N\|u_0\|_{L^2(Ω)} \|h\|_{L^1(0, T; L^2(Ω))}, \text{ for all } h \in L^1(0, T; L^2(Ω))
\]

In particular, for each \(v \in X\), we have that

\[
-(v(0), u_0) = \int_0^T \int_Ω f(x, t) \cdot h(x, t) \, dx \, dt = \int_0^T \int_Ω \langle f_χ_ω E, v \rangle \, dt.
\]

(4.8)

We next verify that \(u(T; f_χ_ω E) = 0\). To serve this purpose, we first use (4.9) to get

\[
\int_0^T \langle u_t, v \rangle \, dt + \int_0^T \langle f_1 E, v \rangle \, dt = \int_0^T \langle u_0, v \rangle + \langle v, u \rangle \, dt.
\]

(4.9)

Then, by (4.2), we obtain that

\[
\int_0^T \langle u_t, v \rangle + \langle v, u \rangle \, dt = \int_0^T \langle \text{rot} v, \text{rot} u \rangle \, dt + \int_0^T \langle f_χ_ω E, v \rangle \, dt
\]

\[
- \int_0^T \langle \text{rot} u, \text{rot} v \rangle \, dt = \int_0^T \langle f_χ_ω E, v \rangle \, dt.
\]
This, along with (4.8) and (4.9), leads to

\[ (u(T; f\chi_{\omega E}), v_T) = 0, \text{ for all } v_T \in L^2_\sigma(\Omega). \]

Hence \( u(T; f\chi_{\omega E}) = 0 \). This completes the proof.

4.2 The Bang-bang Property of the Time and Norm Optimal Control Problem

In the sequel, we make use of Corollary 4.4 to get the bang-bang property for the minimal norm and minimal time control problems for Stokes equations. We begin with introducing these problems. Let \( \omega \subset \Omega \) be a nonempty open subset, and let \( u_0 \in L^2_\sigma(\Omega) \setminus \{0\} \). For each \( T > 0 \), define the following control constraint set:

\[ \mathcal{F}_T = \{ f \in L^\infty(0,T; L^2(\Omega)) : u(T; f\chi_{\omega}) = 0 \}. \]

According to Corollary 4.4, the set \( \mathcal{F}_T \) is nonempty. Consider the minimal norm control problem:

\[ (NP)_T : \quad M_T = \min \{ \| f \|_{L^\infty(0,T; L^2(\Omega))} : f \in \mathcal{F}_T \}. \]

Since \( \mathcal{F}_T \) is not empty, it follows from the standard arguments (see, e.g., [16]) that Problem \( (NP)_T \) has solutions. A solution of this problem is called a minimal norm control.

Now, one can use the same methods as those in [16] to prove the following consequence of Corollary 4.4:

Corollary 4.7. Problem \( (NP)_T \) has the bang-bang property: any minimal norm control \( f \) satisfies that \( \| f(t)\chi_{\omega} \|_{L^2(\Omega)} = M_T \) for a.e. \( t \in (0,T) \). Consequently, this problem has a unique minimal norm control in \( L^\infty(0,T; L^2(\omega)) \).

Next, for each \( M > 0 \), we define the following control constraint set:

\[ \mathcal{U}_M = \{ g \in L^\infty(\mathbb{R}^+; L^2(\Omega)) : \| g(t) \|_{L^2(\Omega)} \leq M \text{ for a.e. } t \in \mathbb{R}^+ \}. \]

Consider the minimal time control problem:

\[ (TP)^M : \quad T_M = \min \{ t > 0 : u(t; g\chi_{\omega}) = 0 \}, \]

where \( u(\cdot; g\chi_{\omega}) \) is the solution to

\[ \begin{align*}
  u_t - \Delta u - \nabla p &= g\chi_{\omega} \quad \text{in } \Omega \times (0,\mathbb{R}^+), \\
  \nabla \cdot u &= 0 \quad \text{in } \Omega \times (0,\mathbb{R}^+), \\
  \text{rot } u &= 0, \quad u \cdot n = 0 \quad \text{on } \partial\Omega \times (0,\mathbb{R}^+), \\
  u(\cdot,0) &= u_0(\cdot) \quad \text{in } \Omega.
\end{align*} \]  

(4.10)

According to Corollary 4.4 and the energy decay property of solutions to homogeneous Stokes equations, we see that \( \mathcal{U}_M \) is nonempty. By the standard arguments (see, e.g., [17, Lemma 3.2]), Problem \( (TP)^M \) has solutions. A solution of this problem is called a minimal time control.

One can follow the similar way as that in [19], [14] or [2] to show the following consequence of Corollary 4.4.
Corollary 4.8. Problem \((TP)^M\) has the bang-bang property: any minimal time control \(f\) satisfies that 
\[
\|f(t)\chi_\omega\|_{L^2(\Omega)} = M \quad \text{for a.e. } t \in (0, T_M).
\]
Consequently, this problem has a unique minimal time control in \(L^\infty(0, T_M; L^2(\omega))\).

Proof. Let \(f^*\) be the minimal control and \(t^*\) be the minimal time for the problem \((TP)^M\). Suppose by contradiction that there were \(E \subset (0, t^*)\) of positive measure and \(\varepsilon > 0\) such that
\[
\|f^*(t)\chi_\omega\|_{L^2(\Omega)} \leq M - \varepsilon \quad \text{for a.e. } t \in E.
\]
It suffices to show that there are a positive number \(\delta\) with \(\delta < t^*\) and a control \(g_\delta \in U_M\) such that the following holds:
\[
u(t^* - \delta; \chi_\omega g_\delta) = 0. \tag{4.11}
\]
This means that \(t^*\) could not be the optimal time, which leads to a contradiction. Indeed, from Corollary 4.4, it follows that for each positive number \(\delta\) sufficiently small, there exists a control \(f_\delta\), with the estimate
\[
\|f_\delta(t)\|_{L^\infty(0, t^* - \delta; L^2(\Omega))} \leq \frac{\varepsilon}{2},
\]
such that
\[
z_\delta(t^* - \delta) = 0,
\]
where \(z_\delta(\cdot)\) is the weak solution to the following controlled equation:
\[
\begin{aligned}
\Delta z_\delta - \nabla p_\delta + f_\delta(t)\chi_\omega & = 0 \quad \text{in } \Omega \times (0, t^* - \delta), \\

\Delta z_\delta & = 0 \quad \text{in } \partial \Omega \times (0, t^* - \delta), \\
\end{aligned}
\]
Here \(E_\delta = \{t > 0 : t + \delta \in E\}\) (Clearly, \(|E_\delta| \geq |E| - \delta\). Set
\[
g_\delta(t) = \begin{cases} 
    f^*(t + \delta) + \chi_{E_\delta}(t)f_\delta(t), & t \in (0, t^* - \delta), \\
    0, & t \in [t^* - \delta, \infty).
\end{cases}
\]
Clearly, \(g_\delta \in U_M\). It is easy to check that if we choose \(g_\delta\) as the control in Equations (4.10), the equality (4.11) will be valid. The uniqueness of the optimal control follows directly from the bang-bang property and the parallelogram identity (see, e.g., [19]).

5 Appendix

Proof of Lemma 2.2. First, let \(\phi = \Delta \psi\). It’s clear that \(\phi\) solves the following heat equation:
\[
\begin{aligned}
\phi_t - \Delta \phi & = 0 \quad \text{in } \Omega \times (0, T), \\
\phi & = 0 \quad \text{on } \partial \Omega \times (0, T), \\
\phi(\cdot, 0) & = \Delta \psi_0(\cdot) \quad \text{in } \Omega.
\end{aligned}
\]
Since $\Delta \psi_0 \in H^{-1}(\Omega)$, by the classical results of heat equation, it has a unique solution
\[\phi \in C([0, T]; H^{-1}(\Omega)) \cap C^1((0, T]; H^1(\Omega)) \cap C((0, T]; H^1_0(\Omega)) \text{ with } \phi(0) = \Delta \psi_0.\]

Consequently, $\Delta \psi \in C((0, T]; H^1_0(\Omega))$. Next, for each $t_1, t_2 \in [0, T]$, we consider the following elliptic equation:
\[
\begin{cases}
\Delta (\psi(t_1) - \psi(t_2)) = \phi(t_1) - \phi(t_2) & \text{in } \Omega, \\
\psi(t_1) - \psi(t_2) = 0 & \text{on } \partial \Omega,
\end{cases}
\]

the elliptic regularity indicates that
\[
\|\psi(t_1) - \psi(t_2)\|_{H^1_0(\Omega)} \leq N(\Omega)\|\phi(t_1) - \phi(t_2)\|_{H^{-1}(\Omega)};
\]

which, together with $\phi \in C([0, T]; H^{-1}(\Omega))$, implies that $\psi \in C((0, T]; H^1_0(\Omega))$. Similarly, $\psi \in C^1((0, T]; H^1_0(\Omega)) \cap C((0, T]; H^3(\Omega))$. The uniqueness is obvious.  

**Proof of Lemma 2.23** For each $u \in L^2_0(\Omega)$, there exists a sequence of $\{u_n\}_{n \geq 1} \subset \{v \in C^\infty_0(\Omega; \mathbb{R}^2) : \nabla \cdot v = 0\}$ (see, e.g., [18, pp. 13–16]) such that
\[
u_n \to u \text{ in } L^2(\Omega). \quad (5.1)
\]

For each $n \geq 1$, let $\psi_n \in H^1_0(\Omega)$ be the unique solution to
\[
\begin{cases}
-\Delta \psi_n = \text{rot } u_n & \text{in } \Omega, \\
\psi_n = 0 & \text{on } \partial \Omega. 
\end{cases} \quad (5.2)
\]

From classical results on elliptic equations, it follows that
\[
\|\psi_n\|_{H^1_0(\Omega)} \leq N(\Omega)\|u_n\|_{L^2(\Omega)}.
\]

Because $\psi_n = 0$ on $\partial \Omega$, we have
\[
\text{curl } \psi_n \cdot \mathbf{n} = \frac{\partial \psi_n}{\partial \tau} = 0 \text{ over } \partial \Omega.
\]

This implies that $\text{curl } \psi_n \in L^2_0(\Omega)$. Observe that Equation (5.2) can also be written as $\text{rot}(\text{curl } \psi_n - u_n) = 0$. Since $\Omega$ is simply connected, we get that $\text{curl } \psi_n - u_n \in \ker(\text{rot}) \cap L^2_0(\Omega) = 0$ (see, for instance, [10, Remark 2.2, pp. 32]). In fact, $\ker(\text{rot}) \cap L^2_0(\Omega)$ is isomorphic to the first space of cohomology $H^1(\Omega; \mathbb{R})$, which is zero for a simply connected domain (we refer the readers to [18, Appendix 1, Remark 1.1, pp. 463] for a detailed argument for such topics). Hence, $\text{curl } \psi_n = u_n$.

Since
\[
\begin{cases}
-\Delta (\psi_n - \psi_m) = \text{rot}(u_n - u_m) & \text{in } \Omega, \\
\psi_n - \psi_m = 0 & \text{on } \partial \Omega,
\end{cases}
\]

we have that
\[
\|\psi_n - \psi_m\|_{H^1_0(\Omega)} \leq N(\Omega)\|u_n - u_m\|_{L^2(\Omega)}.
\]

This, along with (5.1), indicates that $\{\psi_n\}_{n \geq 1}$ is a Cauchy sequence in $H^1_0(\Omega)$ and hence converges to some $\psi \in H^1_0(\Omega)$. In conclusion,
\[
\begin{aligned}
\text{curl } \psi_n \to \text{curl } \psi & \text{ in } L^2(\Omega), \\
\text{curl } \psi_n \to u_n & \text{ in } L^2(\Omega).
\end{aligned}
\]

Therefore, $\text{curl } \psi = u$.  

\[\square\]
Proof of Proposition 4.3. We first define on $H^1_0(\Omega) \times H^1_0(\Omega)$ a bilinear functional for a.e. $t \in (0,T)$ by

$$a(t; u, v) = (\text{rot} u, \text{rot} v).$$

(5.3)

Since $\Omega$ is simply connected, we have that (see, e.g., [10, Remark 3.5, pp. 45])

$$\| \text{rot} v \|_{L^2(\Omega)}^2 \geq N(\Omega) \| v \|_{H^1(\Omega)}^2, \quad v \in H^1_0(\Omega).$$

It can be verified that

$$\left\{ \begin{array}{l}
|a(t; u, v)| \leq M \| u \|_{H^1(\Omega)} \| v \|_{H^1(\Omega)}, \forall \ t \in [0,T], \forall \ u, v \in H^1_0(\Omega).
\end{array} \right.$$

$$\left\{ \begin{array}{l}
a(t; v, v) \geq \Lambda \| v \|_{H^1(\Omega)}^2, \forall \ t \in [0,T], \forall \ v \in H^1_0(\Omega), \text{ where } \Lambda > 0.
\end{array} \right.$$

By the Lions theorem (see, e.g., [3, Chapitre X, Commentaires, pp. 218]), we obtain that there exists $u \in C([0,T]; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega))$, $u_t \in L^2(0, T; H^{-1}_0(\Omega))$ with $u(\cdot, 0) = u_0$, such that

$$\langle u_t, v \rangle = -\langle \text{rot} u(t), \text{rot} v \rangle + \langle f(t), v \rangle, \text{ for a.e. } t \in [0,T] \text{ and for all } v \in H^1_0(\Omega).$$

In particular, for any $v \in L^2(0, T; H^1_0(\Omega))$,

$$\langle u_t, v(t) \rangle = -\langle \text{rot} u(t), \text{rot} v(t) \rangle + \langle f(t), v(t) \rangle, \text{ for a.e. } t \in [0,T].$$

Since the right hand side of above equality is integrable on $(0, T)$, we immediately get the desired equality (4.2). By a standard energy method, it follows that

$$\max_{s \in [0,T]} \| u(s) \|_{L^2(\Omega)}^2 \leq \| u_0 \|_{L^2(\Omega)}^2 + N(\Omega) \| f \|_{L^2(0,T; L^2(\Omega))^2},$$

and this completes the proof. \hfill \Box

Remark 5.1. The bilinear functional defined by (5.3) induces a self-adjoint maximal monotone operator $A$ in $L^2_0(\Omega)$ with

$$D(A) = \{ u \in H^2(\Omega) \cap H^1_0(\Omega); \text{rot} u|_{\partial \Omega} = 0 \}$$

and $Au = -P \Delta u$ when $u \in D(A)$. Here $P$ is the Helmholtz projection operator.

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