Inhomogeneous universes in observational coordinates

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Abstract

Isotropic inhomogeneous dust universes are analysed via observational coordinates based on the past light cones of the observer’s galactic worldline. The field equations are reduced to a single first–order ODE in observational variables on the past light cone, completing the observational integration scheme. This leads naturally to an explicit exact solution which is locally nearly homogeneous (i.e. FRW), but at larger redshift develops inhomogeneity. New observational characterisations of homogeneity (FRW universes) are also given.
1. Introduction

The high degree of isotropy of the cosmic microwave background radiation is usually taken as evidence that the late (matter–dominated) universe is isotropic about our worldline. When combined with the Copernican principle that we do not occupy a privileged position, this leads to isotropy about all worldlines, and thus to a Friedmann–Robertson–Walker (FRW) universe. (The proof of this background–radiation argument is the Ehlers–Geren–Sachs theorem [1], whose perturbed version is proved in [2].)

If we suspend the Copernican assumption in favour of a directly observational approach, as set out in detail in [3], then it turns out that isotropy of the background radiation is insufficient to force isotropy onto the spacetime geometry. However, it has been shown [4] that isotropy of galactic number counts and area distances (equivalently luminosity distances), together with the vanishing of cosmological proper motions and image distortion, does force isotropy of the universe about our worldline. The universe is then a Lemaître–Tolman–Bondi (LTB) model [5], including the special case of homogeneous FRW models.

Current galactic observations are nowhere near the accuracy and completeness of the background radiation observations, but they are not inconsistent with isotropy. However, observations of the galactic distribution do not imply homogeneity [6],[7]; indeed, they sometimes appear to indicate that inhomogeneities may not be smoothed out on larger and larger averaging scales. Therefore, motivated by observations, and leaving aside the Copernican principle, we assume isotropic galactic observations which are not necessarily (i.e. without further observational evidence) homogeneous. Observational characterisations of homogeneity are then needed in order to test the basis on which the standard FRW cosmologies rests. In [3] it was shown that an LTB universe is homogeneous if and only if the observed galactic area distances and number counts take precisely the FRW form (as functions of redshift). In Section 4, we derive a further characterisation of homogeneity within the observational framework, and show that for the parabolic case, the characterisation of [3] may be weakened: the single (area distance, number count) relation must take the Einstein–de Sitter form.

The 3 + 1 analysis of LTB spacetimes [3], based on spatial hypersurfaces \{t = \text{const.}\}, is effective for finding and analysing exact solutions, but is not directly suited to observational cosmology. Since observations lead to data not on \{t = \text{const.}\}, but on the past light cone of the observer, a 2 + 2 approach is directly adapted to the analysis.
of LTB universes from the viewpoint of observational cosmology. Furthermore, it is not possible simply to transform the 3 + 1 LTB solutions to observational coordinates, since such a transformation requires solution of the null geodesic equation, which is in general only possible numerically. In observational coordinates, the null geodesics are known directly, but the field equations are considerably more complicated, and the analogue of the 3 + 1 exact solutions is not known. In Section 2, we briefly review the observational coordinate approach.

This paper complements and corrects [8], where it is indicated how in principle the field equations can be integrated using the redshifts, number counts and area distances as data. An important technical mistake was made in [8] (see Section 4), which severely over-restricts the solutions obtained by that procedure. Here, in Section 3, we give an explicit reduction of the field equations in 2 + 2 observational coordinates to a single first-order ordinary differential equation (ODE) on the past light cone. The variables in the ODE are the area distance and a number count function. This corrects the integration procedure of [8], and is parallel to the 3 + 1 case of an explicit general solution.

An application of this reduction is the construction in Section 5 of a class of explicit exact solutions representing an observationally-based inhomogeneous generalisation of flat FRW universes. The solutions approach the Einstein–de Sitter solution near the observer, so that they are locally homogeneous (for small redshift). At greater redshifts, the solutions deviate appreciably from the Einstein–de Sitter solution: the matter distribution is locally homogeneous, but develops inhomogeneity at larger redshifts.

2. Metric and observations in observational coordinates

In 3 + 1 coordinates based on spatial hypersurfaces \( \{ t = \text{const.} \} \), the metric is

\[
 ds^2 = -dt^2 + \left[ \frac{\partial R(t, r)}{\partial r} \right]^2 \frac{dr^2}{1 - kf(r)^2} + R(t, r)^2 d\Omega^2 \tag{1}
\]

where \( k = 0, 1, -1 \) corresponds, respectively, to parabolic, elliptic and hyperbolic intrinsic geometry of the \( \{ t = \text{const.} \} \) hypersurfaces, and \( d\Omega^2 \) is the metric of the unit 2-sphere. The coordinates are comoving with the dust four-velocity \( u^\mu = \delta^\mu_t \). If \( r \) is chosen as a proper area coordinate, then the FRW case is given by

\[
 R(t, r) = ra(t) , \quad f(r) = r
\]
In the general case, the solution $R(t,r)$ of Einstein’s equations may be given explicitly \[^5\]. However these coordinates and solution are not adapted to observations, which are made not on a \(\{t = \text{const.}\}\) surface, but rather on our past light cone. (See \[^3\],\[^6\],\[^9\] for further discussion of the crucial implications of this point.) Light emitted from a source at radial distance $r_E$ at cosmic time $t_E$ propagates along the null geodesic \[ \sqrt{1 - k f^2} dt = -(\partial R/\partial r) dr \] before being observed at $r = 0$ at time $t_O > t_E$. The explicit analytic form $t = t(r)$ of the null geodesic cannot in general be determined. If we use observational coordinates based on the past light cones of the observer’s worldline, then the observations are given in the most simple and direct form. The price paid for this is that the field equations can no longer be solved explicitly. However, as we shall show, they may be reduced to a single first–order ODE.

In observational coordinates $x^\mu = (w, y, \theta, \phi)$, the metric takes the 2 + 2 form \[^8\]

\[ ds^2 = -A(w,y)^2 dw^2 + 2A(w,y)B(w,y) dwdy + C(w,y)^2 d\Omega^2 \] (2)

where \(\{w = \text{const.}\}\) are the past light cones along \(\{y = 0\}\), and $y$ is a comoving radial distance parameter down the light rays (null geodesics) \(\{w = \text{const.}, (\theta, \phi) = \text{const.}\}\).

Regularity on \(\{y = 0\}\) imposes on the metric the central limiting conditions \[^3\]: as $y \rightarrow 0$

\[
\begin{align*}
A(w, y) &= A(w,0) + O(y) & A(w,0) \neq 0 \\
B(w, y) &= B(w,0) + O(y) & B(w,0) \neq 0 \\
C(w, y) &= B(w,0)y + O(y^2)
\end{align*}
\] (3)

The nontrivial coordinate freedom is a scaling of $w$ and $y$:

\[
\begin{align*}
w &\rightarrow \tilde{w} = \tilde{w}(w) , & y &\rightarrow \tilde{y} = \tilde{y}(y) & \left(\frac{d\tilde{w}}{dw} \neq 0 \neq \frac{d\tilde{y}}{dy}\right)
\end{align*}
\] (4)

The first corresponds to a freedom to choose $w$ as any time parameter (e.g. proper time) along \(\{y = 0\}\). The second corresponds to a freedom to choose $y$ as any null distance parameter (e.g. affine parameter, redshift) on an initial light cone \(\{w = w_0\}\), after which $y$ is dragged along by the matter flow. These scalings imply a scaling of the metric functions $A, B$:

\[
\begin{align*}
A &\rightarrow \tilde{A} = \frac{dw}{d\tilde{w}} A , & B &\rightarrow \tilde{B} = \frac{dy}{d\tilde{y}} B
\end{align*}
\]

The metric function $C$ is the observer area distance (or angular diameter distance) on each past light cone, since $dS_E = C^2 d\Omega_E$, where $dS_E$ is the cross sectional area of
the source and $d\Omega_E$ the solid angle it subtends at the central observer \{y = 0\}. If $d_A$ denotes the area distance that is measured here–and–now, then

$$d_A(y) = C(w_0, y)$$

(5)

where $y$ may be chosen as the redshift, so that observations determine $C$ on \{w = w_0\}. By contrast, in the 3 + 1 approach, $d_A(r) = R(t(r), r)$, where $R$ is known from the explicit exact solution, but $r$ is not known and neither is the explicit form $t(r)$ of the past light cone of here–and–now. The luminosity distance is

$$d_L = (1 + z)^2 d_A$$

(6)

where the redshift $z$ of the source relative to the central observer is given by

$$1 + z = \frac{A(w_0, 0)}{A(w_0, y)}$$

(7)

The metric function $B$ determines the deviation of $y$ from the affine parameter $\nu$

$$B = \frac{1}{A} \frac{dv}{dy}$$

(8)

Two four–vectors are defined intrinsically by the matter and light rays: the dust velocity and photon wave vectors

$$w^\mu = A^{-1} \delta^\mu_w, \quad k^\mu = (AB)^{-1} \delta^\mu_y$$

(9)

The number of galactic sources counted by the central observer out to a distance $y$ is [see 8], which corrects the error in 3, equation (19)]

$$N(y) = 4\pi \int_0^y n(w_0, \bar{y}) B(w_0, \bar{y}) C(w_0, \bar{y})^2 d\bar{y}$$

(10)

on using (7) and (8). In (10), selection effects are ignored, the freedom (1) has been used to set $A(w_0, 0) = 1$, and $n$ is the number density of sources, so that the matter density is

$$\rho = 4\pi mn$$

where $4\pi m$ is the average galactic mass.
The rate of expansion of the dust is $3H = u^{\mu}u_{\mu}$, so that, by (2),
\[ H = \frac{1}{3A} \left( \frac{\dot{B}}{B} + \frac{2\dot{C}}{C} \right) \]  
(11)
where a dot denotes $\partial/\partial w$. For the central observer, relative to whom the recession of other worldlines is isotropic, $H$ is precisely the Hubble rate. In the homogeneous FRW case, $H$ is constant at each instant of time $t = t_0$. But in the general inhomogeneous case, $H$ varies with radial distance from $r = 0$ on $\{t = t_0\}$. By (3), the central behaviour of $H$ is
\[ H(w, y) = \left[ \frac{1}{A(w, 0)} \frac{\dot{B}(w, 0)}{B(w, 0)} \right] + O(y) \]
At any given instant $w = w_0$ along $\{y = 0\}$, the first term is just the Hubble constant $H_0 = H(w_0, 0)$ measured by the central observer:
\[ H_0 = \frac{1}{A_0} \frac{\dot{B}_0}{B_0} \]  
(12)
The covariant four–velocity of the dust is given by the gradient of proper time along the matter worldlines: $u_{\mu} = -t_{, \mu}$. It is also given by (3) and (2) as
\[ u_{\mu} = g_{\mu\nu}u^\nu = -Aw_{,\mu} + By_{,\mu} \]
Comparing these two forms, we get
\[ dt = Adw - Bdy \iff A = \frac{\partial t}{\partial w}, \quad B = -\frac{\partial t}{\partial y} \]  
(13)
which shows that the surfaces of simultaneity for the observer are given in observational coordinates by $Adw = Bdy$. The integrability condition of (13) is
\[ A' + \dot{B} = 0 \]  
(14)
where a prime denotes $\partial/\partial y$. This is precisely the momentum conservation equation (see 3).

Using (3), (5), (7), (12) and (14), it follows that, exactly as in the FRW case, the Hubble constant is determined by the initial slope of the $(dA, z)$ relation, or equivalently of the $(dL, z)$ relation:
\[ H_0^{-1} = \left( \frac{ddA}{dz} \right)_0 = \left( \frac{ddL}{dz} \right)_0 \]  
(15)
Since the acceleration and vorticity of the dust are zero, the only non-trivial kinematic quantities are the expansion (11) and the shear $\sigma$. Using (9), (11) and (14) in the general expressions given in [4], we find

$$\sigma = \frac{2}{\sqrt{3A}} \left| \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right|$$  \hspace{0.5cm} (16)

Thus the FRW case is given by $\dot{B}/B = \dot{C}/C$.

3. Reduction of the field equations

The conservation equations are (14) and the energy conservation equation $u^\mu \nabla_\mu (\rho) + 3H\rho = 0$, which gives, by (10) and (11)

$$\rho = \frac{mN'}{BC^2}$$  \hspace{0.5cm} (17)

The central behaviour of the density is $\rho(w, y) = \rho(w, 0) + O(y)$, and then (17) and (3) imply

$$\rho(w, y) = \left[ \frac{\rho_0 B_0^3}{B(w, 0)^3} \right] + O(y)$$

$$N(y) = \left[ \frac{\rho_0 B_0^3}{3m} \right] y^3 + O(y^4)$$  \hspace{0.5cm} (18)

In observational coordinates, the field equations lead to the following first integrals (see [8]):

$$\frac{\dot{C}}{A} + \frac{C'}{B} = F \equiv \frac{N'}{N}$$  \hspace{0.5cm} (19)

$$\frac{\dot{C}}{C} \frac{C'}{C} + A \frac{C'^2}{2B C^2} - AB \frac{2C^2}{2C^2} = -\frac{mN_*}{2C^3}$$  \hspace{0.5cm} (20)

where $N_*(y)$ is an arbitrary function, whose central behaviour is the same as that of the number counts:

$$N_*(y) = \left[ \frac{\rho_0 B_0^3}{3m} \right] y^3 + O(y^4)$$  \hspace{0.5cm} (21)

by (19), (3) and (18).

In [8], an integration scheme for the field equations, with observational data on the past light cone $\{w = w_0\}$, is presented, intending to give a formal exact solution.
Due to the over-restrictive nature of the condition $B = A$, it fails to do so. Here we give a somewhat different approach, which parallels the explicit exact solution in $3 + 1$ coordinates, and which corrects [8]. In terms of the intrinsically defined variables [see (9)]

$$u \equiv u^\mu \nabla_\mu(C) = \frac{\dot{C}}{A}, \quad v \equiv (u^\nu k_\nu)^{-1} k^\mu \nabla_\mu(C) = \frac{C'}{B} \quad (22)$$

(19) and (20) become

$$u + v = F \quad (23)$$
$$v^2 + 2uv = 1 + \frac{mN_*}{C}$$

Solving and eliminating $u, v$, we obtain

$$A = \frac{\dot{C}}{[F^2 - 1 + mN_*/C]^{1/2}} \quad (23)$$
$$B = \frac{C'}{F - [F^2 - 1 + mN_*/C]^{1/2}} \quad (24)$$

Thus the problem is reduced to determining $C$. Integrating (23) along the matter flow $y = \text{const}$, and using (13), we get

$$t + T(y) = \int \frac{dC}{[F^2 - 1 + mN_*/C]^{1/2}} \quad (25)$$

where $T$ is arbitrary. This is the LTB exact solution [5], provided we identify

$$F^2 = 1 - kf^2, \quad k = 0, \pm 1 \quad (26)$$

Of course, we could have started from this known solution, but then we would have to evaluate some of the field equations in observational coordinates anyway in order to determine $A$ and $B$. Furthermore, the observational coordinates allow us to incorporate observations directly into the field equations.

By (26), $N_*$ is a modified number count variable:

$$N_*(y) = \int_0^y \sqrt{1 - kf(\tilde{y})^2 N'(\tilde{y})} d\tilde{y} \quad (27)$$

so that $k = 0 \Rightarrow N_* = N$. 

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Now the cosmic time $t$ in (25) is not known explicitly in observational coordinates, so we must transform (25) into an equation involving only observational variables. This entails losing the explicit exact 3 + 1 general solution, but the balancing advantage, absent in the 3+1 case, is that the light rays are known explicitly, and the fundamental equation involves only observational variables.

We evaluate the integral in (25), differentiate down the light rays, and use (13), (24) and (27), to get:

\[
\frac{C'}{1 - \sqrt{\frac{mN}{C}}'} + \frac{2}{3} \left( \sqrt{\frac{C^3}{mN}} \right)' - T' = 0, \quad \text{for } k = 0
\]  

\[
\frac{C'}{\sqrt{1 - kf^2 - \frac{mN}{C}} - kf^2} = T' - \frac{1}{2} \begin{cases} 
    \frac{[mN_*(\Gamma - \sin \Gamma)/f^3]'}{\Gamma \equiv 2 \arcsin(f\sqrt{C/mN_*})} & \text{where } k = 1 \\
    \frac{[mN_*(\sinh \Gamma - \Gamma)/f^3]'}{\Gamma \equiv 2 \arcsinh(f\sqrt{C/mN_*})} & \text{where } k = -1
\end{cases}
\]

For each $k$, this is the fundamental equation, representing the observational reduction of the field equations, analogous to the 3 + 1 case. It is a first-order ODE for the area distance $C$ as a function of $y$ on each light cone. Though the arbitrary function $T'(y)$ generally leads to an infinite family of solutions, on our past light cone \(\{w = w_0\}\) this fundamental equation relates $T'(y)$ to observationally determined quantities. The observational data on \(\{w = w_0\}\) may be taken as $d_A(z)$ and $N(z)$. In summary, we can determine the arbitrary functions $f(y)$, $N_*(y)$ and $T'(y)$ from the data in the following way, very similar to what is suggested in [8]:

(a) Set $B = A$ on our past light cone. This amounts to a choice of $y$ on \(\{w = w_0\}\), and is valid since we do not specify $B = A$ for all $w$. (If we do not use this freedom, then it seems very difficult, if not impossible, to use all the available data on our past light cone in determining the solution.)

(b) Then integrate the Ehlers (or null Raychaudhuri) equation on our past light cone, as outlined in [8] [equations (31) to (33)]. This enables us to change our data from functions of $z$ (the observed form) to functions of $y$ on our past light cone.

(c) We can also determine $\dot{C}$ on our past light cone, as outlined in [8] (Section 5 iii). $C'$ is essentially given on our past light cone by the data. Thus, since we know both $A$ and $B$ on our past light cone, we can determine $f(y)$ from (19).
(d) Knowing \( f(y) \), we can use (27) to determine \( N_*(y) \) since we know \( N(z) \) from data and thus \( N(y) \) from (b). Thus we have determined the arbitrary functions \( f(y) \) and \( N_*(y) \) by using the data we presume we have available on our past light cone.

(e) Finally, as indicated above, we can use (28), (29) on \( \{ w = w_0 \} \) to find \( T'(y) \).

4. Characterisation of homogeneity

Observational characterisations of homogeneity (as opposed to unverifiable philosophical assumptions such as the Copernican principle [3]), are complicated by the fact that even the homogeneous FRW universes *appear* inhomogeneous - because observations are made down the light cone, which is a hypersurface of inhomogeneity [3],[6],[9]. Homogeneity of an FRW universe is implied indirectly by the specific forms of the observed quantities. Any deviation from these forms indicates real inhomogeneity (i.e. on spatial hypersurfaces). A precise observational characterisation of homogeneity within the LTB class of universes has been given [3],[8]:

*isotropic dust universes are homogeneous if and only if the (area distance, redshift) and (number count, redshift) relations measured by the central observer take exactly the FRW form.*

This characterisation may be slightly weakened in the parabolic \( (k = 0) \) case. In the fundamental equation (28), we may transform the independent variable to \( N \):

\[
\left[ 1 - \sqrt{mN/C} \right]^{-1} \frac{\partial C}{\partial N} + \frac{\partial}{\partial N} \left( \frac{2}{3} \sqrt{C^3/mN} \right) - \frac{\partial T}{\partial N} = 0
\]  

(30)

It is well known [3] that \( T = \text{const.} \) reduces the metric to FRW. The first order ODE (30) then shows that \( d_A(N) = C(w_0, N) \) takes the FRW form (see section 5 for the explicit form) if and only if \( T = \text{const.} \). Thus, for \( k = 0 \), one does not need the (area distance, redshift) and (number count, redshift) relations separately to take the FRW form, but only the single (area distance, number count) relation:

*a parabolic LTB universe is Einstein–de Sitter if and only if the (area distance, number count) relation measured by the central observer takes exactly the Einstein–de Sitter form.*

This result is consistent with [3], but it follows more directly via our approach. Note that the result does not extend to \( k \neq 0 \), essentially since in that case 2 arbitrary
functions \((T\) and \(f\)) need to be fixed by 2 independent observational relations.

We emphasize the ‘global’ nature of these characterisations, i.e. the fact that \(d_A(z)\) and \(N(z)\) must be FRW for all \(z\). Local observational parameters derived from \(d_A\) and \(N\) are insufficient to characterise homogeneity. For example, the Hubble constant, derived from \(d_A\) as in (13), cannot in itself distinguish between inhomogeneous and homogeneous universes. The dependence of \(H_0\) on the spatial geometry (i.e. on \(k\)) and on the density \(\rho_0\) at the observer, follows from (12), (23), (24) and (26) as

\[
H_0 = \sqrt{\frac{\rho_0}{3} - k \left[ \frac{f'(0)}{B_0} \right]^2}
\]

(31)

The FRW case has \(f'(0) = 1\) and \(B_0 = a_0\) (see (32) and (33) below), and for a given \(\rho_0\) and \(k\), the FRW \(H_0\) is observationally indistinguishable from the general LTB \(H_0\).

A particular consequence of (31) arises when an LTB under–dense region is matched to an FRW region. The low value for \(\rho_0\) implies a low value for \(H_0\) if \(k = 0, 1\). Thus cosmological models with a locally high Hubble constant arising from an under–dense inhomogeneous local region within a global FRW universe (see for example [10]), must have \(k = -1\) in the local LTB region. The conditions for smooth matching at the boundary (see [11]) then require \(k = -1\) in the FRW region, i.e. an open FRW universe.

We now return to the problem of characterising homogeneity, and derive an alternative condition (for all \(k\)). As pointed out in [12], the assertion in [8] that the coordinate freedom (4) could be used to set \(B(w, y) = A(w, y)\) (32) is erroneous, and in fact (32) implies restrictions on the spacetime geometry. This choice made in [8] automatically over–restricts the solutions obtained by that procedure to be homogeneous (FRW) solutions, as we show below. As long as \(y\) is specified to be comoving, one can set \(B = A\) only on one past light cone, i.e. \(B(w_0, y) = A(w_0, y)\), and not generally. Even though the \(ww\) and \(wy\) components of the Lie derivative of the metric \(g_{\mu\nu}\) along \(u^\mu\) are equal, the time derivatives of \(A\) and \(B\) are not given by the Lie derivatives. And there is no further freedom to specify them arbitrarily - as that freedom has already been used up by specifying \(y\) to be comoving. If, however, \(y\) is not chosen comoving, then one could set \(B(w, y) = A(w, y)\) without loss of generality.

This correction provides another characterisation of FRW universes within the LTB class:
an LTB universe is FRW if and only if there exists a choice of radial comoving coordinate such that \( g_{wy} = g_{ww} \) in observational coordinates.

The proof proceeds simply and directly from the formalism set up in sections 2 and 3. The assumption (32) allows us to integrate (14) and (19), to get

\[
A(w, y) = a(\eta) \, , \, \quad C(w, y) = a(\eta) \int_0^y \sqrt{1-kf(\tilde{y})^2} d\tilde{y} \, , \, \quad \eta \equiv w - y
\]  

(33)

where we have used the central condition (3) for \( C \). Then it follows immediately from (33) that the shear (16) vanishes, so that the spacetime is FRW.

For completeness, we present the FRW metric and observations in observational coordinates. By (33) we can write the metric (2) as

\[
ds^2 = a(\eta)^2 \left[ -d\eta^2 + dy^2 + \left( \int_0^y \sqrt{1-kf(\tilde{y})^2} d\tilde{y} \right)^2 d\Omega^2 \right]
\]

showing that \( \eta \) is conformal time. It follows that \( y \) is conformal proper radial distance \( \chi \), so that

\[
\int_0^\chi \sqrt{1-kf(\tilde{y})^2} d\tilde{y} = \frac{1}{\sqrt{k}} \sinh \sqrt{k} \chi
\]

(34)

Now (32), (33) and (34) reduce the field equation (see [8])

\[
\frac{\ddot{C}}{C} = \frac{\dot{A}}{AC} - \frac{A^2}{2C^2} + \frac{A \dot{C}}{BC} C' + \frac{A^2}{2B^2} C'^2
\]

to

\[
2a \frac{d^2a}{d\eta^2} - \left( \frac{da}{d\eta} \right)^2 - ka^2 = 0
\]

which has solutions

\[
a(w - \chi) = \alpha \begin{cases} 
(w - \chi)^2 & k = 0 \\
1 - \cos(w - \chi) & k = 1 \\
\cosh(w - \chi) - 1 & k = -1
\end{cases}
\]

(35)

where \( \alpha \) is constant and the big bang occurs at \( w = \chi \).

The derivation of the exact observational formulas in FRW universes is immediate and transparent in observational coordinates. If \( t_0 \) is the age of the universe (so that
$t_0 = w_0$, since $\chi = 0$ for the observer), then the constant $\alpha$ follows from (12) and (35) as

$$\alpha = \frac{1}{H_0} \begin{cases} 2t_0^{-3} \\ \sin t_0 (\cos t_0 - 1)^{-2} \\ \sinh t_0 (\cosh t_0 - 1)^{-2} \end{cases}$$

while the density parameter $\Omega_0 = \rho_0/3 H_0^2$ follows from (17) and (18):

$$\Omega_0 = \begin{cases} 1 \\ 2/(\cos t_0 + 1) \\ 2/(\cosh t_0 + 1) \end{cases}$$

Then (35), (36) and (37) lead directly to all relations between the redshift [see (7)], observer area distance [see (5), (33)], luminosity distance [see (6)] and number counts [see (10)]. For example, the number count/ luminosity distance relation is

$$N(d_L) = \frac{3}{4mH_0} \left[ 1 + \frac{1}{\Omega_0(\Omega_0 - 1)} - \frac{2}{3} \frac{\Omega_0(\Omega_0 - 1)^{-3/2}[2\chi(d_L) - \sin 2\chi(d_L)]}{\Omega_0(1 - \Omega_0)^{-3/2}[\sinh 2\chi(d_L) - 2\chi(d_L)]]} \right]$$

where, for $k = \pm 1$:

$$\chi(d_L) = \begin{cases} \arctan \left( \frac{2\sqrt{\Omega_0 - 1} + H_0 d_L}{1 + H_0 d_L} \right) + \\ - \arccos \left( \frac{\Omega_0 H_0 d_L}{\sqrt{\Omega_0^2 + (\Omega_0 - 1)(8H_0 d_L + 4)}} \right)^{-1/2} \right) \right) \\ \ln \left[ \Omega_0 H_0 d_L - 2\sqrt{\Omega_0 - 1 + 2H_0 d_L} \right] + \\ - \ln \left[ H_0 d_L (2 - \Omega_0) - 2(1 + H_0 d_L) \sqrt{\Omega_0} \right] \right) \right) \right) \right) \right)$$

5. Observational generalisation of the Einstein–de Sitter universe

The LTB solution (25) allows one to write down any number of explicit exact inhomogeneous generalisations of FRW solutions. One simply chooses $T(y)$. However, if we want an observationally based generalisation, then the LTB solution is not appropriate, since $t$ is not an observational variable, and deriving the observational variables requires integration of the null geodesics. In the observational coordinates, the key equation (28) or (29) is far more complicated, but it has the advantage that it is already in observational variables. Thus any explicit solution will be a readily interpreted observational generalisation of an FRW universe.
We have found one such solution in the case \( k = 0 \), i.e. an observational inhomogeneous generalisation of an Einstein–de Sitter universe. The equation (30) can be simplified by defining the observational variable

\[
D = \sqrt{\frac{C}{mN}}
\]  

(40)

Then (30) becomes

\[
6N \frac{\partial D}{\partial N} + 2D + 1 + \left[ 3 \frac{\partial T}{m \partial N} \right] \left( 1 - \frac{D}{D^3} \right) = 0
\]

(41)

By (40), (23), (24) and (17), we get

\[
A(w, N) = 2mND^2\dot{D}
\]

\[
B(w, N) = mMD^2(D + 2ND_N)/(D - 1)
\]

\[
C(w, N) = mND^2
\]

\[
\rho(w, N) = (D - 1)/[m^2N^2D^6(D + 2ND_N)]
\]  

(42)

Now it is clear that the ODE (41) is separable for \( \partial T/\partial N = \frac{1}{3}m\beta, \beta = \text{const} \). If \( \beta = 0 \), then \( T = \text{const.} \), which is the FRW case. In the observational variables, the Einstein–de Sitter solution following from (41) and (42) is

\[
A = \frac{1}{7}m\dot{P}(P - N^{1/3})^2, \quad B = \frac{1}{12}mMN^{-2/3}(P - N^{1/3})^2
\]

\[
C = \frac{1}{7}mN^{1/3}(P - N^{1/3})^2, \quad \rho = 192m^{-2}(P - N^{1/3})^{-6}
\]

(43)

where \( P(w) \) is an arbitrary function, subject to \( P(w_0) = 2(mH_0)^{-1/3} \), which ensures that \( \rho_0 = 3H_0^2 \). The (area distance, number count) relation is \( d_A(N) = C(w_0, N) \). Note that by (33), (43) and (17), the observational variable \( D \) on \( \{w = w_0\} \) determines the variation in number counts in terms of the variation in density:

\[
\frac{N'}{N} = \frac{D\rho'}{\rho}
\]  

(44)

The inhomogeneous case \( T = \frac{1}{3}m\beta N, \beta \neq 0 \), leads to the solution

\[
N = Q(w) \exp \left[ -\int \sqrt{C/mN} \frac{6D^3dD}{2D^4 + D^3 - \beta D + \beta} \right]
\]  

(45)
where \( Q \) is an arbitrary function. For small number counts [equivalently, by (3), (21) and (27)], the solution (15) on \( \{ w = w_0 \} \) has the form

\[
d_A(N) = (mH_0^{-2})^{1/3}N^{1/3} - (m^2H_0^{-1})N^{2/3} + \frac{1}{3}mN + \frac{1}{3}\beta(m^4H_0)N^{4/3} - \frac{1}{6}\beta(m^5H_0^2)N^{5/3} + O(N^2)
\]  

(46)

where \( Q(w_0) \) has been chosen so that the leading term in (46) agrees with that in (13). The first three terms in (46) are exactly the Einstein–de Sitter form given by (13). Thus by the characterisation proved in section 4, the new solution approaches the Einstein–de Sitter solution for small number counts and then deviates from it for larger number counts. Choosing \( y = z \) on \( \{ w = w_0 \} \), it follows from (17) and (18) that

\[
N^{1/3} = B_0 \left( \frac{\rho_0}{3m} \right)^{1/3} z + O(z^2)
\]

so that small number counts are equivalent to small redshift. This solution represents a universe that is locally homogeneous, but for larger redshift develops real inhomogeneity. If the Copernican principle is not invoked, and if large–scale observations of the galactic distribution are taken at face value, then such a model of the late universe is not automatically ruled out.

For completeness, we give the explicit form of the integral in (15), which depends on the roots \( \alpha_i \) of the quartic in the integrand (13):

(a) \( \beta = \beta_\pm \equiv 13 \pm \frac{15}{2}\sqrt{3} \Rightarrow \alpha_1 = \alpha_2 \equiv r, \alpha_3 = \bar{\alpha}_4 \equiv \lambda \neq \bar{\lambda} \neq r : \)

\[
N = Q(w) \left( 1 - \frac{1}{r} \sqrt{C/mN} \right)^q \left( 1 - \frac{1}{\lambda} \sqrt{C/mN} \right)^{\xi^2} \exp \left[ p \left( \sqrt{C/mN} - r \right)^{-1} \right]
\]  

(47)

where

\[
q = \frac{3r^2[2r(\lambda + \bar{\lambda}) - 3\lambda\bar{\lambda} - r^2]}{(\lambda - r)^2(\lambda - r)^2}
\]

\[
p = \frac{3r^3[-r(\lambda + \bar{\lambda}) + \lambda\bar{\lambda} + r^2]}{(\lambda - r)^2(\lambda - r)^2}
\]

\[
\xi = \frac{-3\lambda^3}{(\lambda - r)^2(\lambda - \lambda)^2}
\]
(b) $0 \neq \beta \neq \beta_{\pm} \Rightarrow$ no $\alpha_i$ equal:

$$N = Q(w) \left( 1 - \frac{1}{\alpha_1} \sqrt{\frac{C}{mN}} \right)^{\beta_1} \left( 1 - \frac{1}{\alpha_2} \sqrt{\frac{C}{mN}} \right)^{\beta_2} \times$$

$$\times \left( 1 - \frac{1}{\alpha_3} \sqrt{\frac{C}{mN}} \right)^{\beta_3} \left( 1 - \frac{1}{\alpha_4} \sqrt{\frac{C}{mN}} \right)^{\beta_4}$$

(48)

where

$$\beta_i = \frac{-3\alpha_i^3}{(\alpha_i - \alpha_j)(\alpha_i - \alpha_k)(\alpha_i - \alpha_l)}, \quad j < k < l$$

and $i$ is not equal to $j, k$ or $l$.

Further calculations show that the qualitative behaviour of the solutions (47) and (48) falls into two types, according to whether or not the past light cone reconverges.

6. Concluding remarks

Motivated by the direct observational evidence for isotropy, and by the need to identify indirect observational evidence for homogeneity, we have completed an exact observational analysis of isotropic dust cosmologies, i.e. LTB spacetimes. Such an analysis, based on the past light cone (which is the hypersurface on which observational data is given), was begun in [3] and continued in [8]. An important error in [8] has been corrected here, allowing us to properly complete the integration scheme, as described in Section 3. The main result is the reduction of the field equations to a single ODE on the past light cone [equations (28), (29)], which then allows us to show in detail how the observations, i.e. (area distance, redshift) and (number count, redshift) relations, provide the data needed on the past light cone to integrate the field equations and determine the metric and matter distribution.

The observational ODE (28) leads to a simplified characterisation of Einstein–de Sitter universes via the single (area distance, number count) relation (Section 4). Furthermore, we give in Section 4 a new observational characterisation of homogeneity, i.e. the condition that $B = A$ in observational coordinates. Finally, in Section 5 we
used (28) to construct an exact observational generalisation of the Einstein–de Sitter universe, given by (45), (46), (47) and (48). This spacetime is locally homogeneous but inhomogeneous on large scales. We are not proposing it as a realistic model of the universe, which is more generally believed to be locally inhomogeneous but homogeneous on large enough scales, but it is not ruled out by current galactic observations, and the method of deriving it could be extended to construct models with different behaviour of the inhomogeneity.

References

[1] Ehlers J, Geren P and Sachs R K 1968 *J. Math. Phys.* **9**, 1344

[2] Stoeger W R, Maartens R and Ellis G F R 1995 *Astrophys. J.* **443**, 1

[3] Ellis G F R, Nel S D, Maartens R, Stoeger W R and Whitman A P 1985 *Phys. Rep.* **124**, 315

[4] Maartens R and Matravers D R 1994 *Class. Quantum Grav.* **11**, 2693

[5] Bonnor W B 1974 *Mon. Not. R. Astr. Soc.* **167**, 55

[6] Ellis G F R 1979 *Gen. Relativ. Grav.* **11**, 281

[7] Peebles P J E 1993 *Principles of Physical Cosmology* (Princeton: Princeton University Press)

[8] Stoeger W R, Ellis G F R, and Nel S D 1992 *Class. Quantum Grav.* **9**, 509

[9] Ribeiro M B 1995 *Astrophys. J.* **441**, 477

[10] Moffat J W and Tatarski D C 1995 *Astrophys. J.* **453**, 17

[11] Bonnor W B and Chamorro A 1990 *Astrophys. J.* **361**, 21

[12] Stoeger W R, Ellis G F R and Xu C 1994 *Phys. Rev. D* **49**, 1845

[13] Stephani H 1982 *General Relativity* (Cambridge: Cambridge University Press)

[14] Gradshteyn I S and Ryzhik I M 1965 *Table of Integrals, Series and Products* (New York: Academic)