SHARP LOWER BOUND FOR THE FIRST EIGENVALUE OF THE WEIGHTED \( p \)-LAPLACIAN: THE POSITIVE CURVATURE CASE

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Abstract. Combined with our previous work [10], we prove sharp lower bound estimates for the first nonzero eigenvalue of the weighted \( p \)-Laplacian with \( 1 < p < \infty \) on a compact Bakry-Émery manifold \((M^n, g, f)\), without boundary or with a convex boundary and Neumann boundary condition, satisfying \( \text{Ric} + \nabla^2 f \geq \kappa g \) for some \( \kappa \in \mathbb{R} \).

1. Introduction

The determination of lower bounds for the first nonzero eigenvalue for elliptic operators is an important issue in both mathematics and physics, since this constant determines the convergence rate of numerical schemes in numerical analysis, describes the energy of a particle in the ground state in quantum mechanics, and determines the decay rate of certain heat flows in thermodynamics. Given its physical and mathematical significance, the sharp lower bound of the first nonzero eigenvalue of the Laplace-Beltrami operator and the weighted Laplacian on a compact Riemannian manifold or Bakry-Émery manifold (without boundary or with convex boundary and Neumann boundary condition) has been established via the efforts of many mathematicians including [8, 9, 17, 7, 5] around the year 2000. Simple alternative proofs via the estimates of modulus of continuity were given in [3, 4].

In the last two decades, the attention has been focused on nonlinear elliptic operators, especially the \( p \)-Laplacian \( \Delta_p \) and the weighted \( p \)-Laplacian \( \Delta_{p,f} \), defined for \( 1 < p < \infty \) by

\[
\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)
\]

and

\[
\Delta_{p,f} u := e^f \text{div}(e^{-f}|\nabla u|^{p-2}\nabla u),
\]

respectively. Sharp lower bounds for the first nonzero eigenvalue of \( \Delta_p \) on compact Riemannian manifolds satisfying \( \text{Ric} \geq (n-1)\kappa g \) have been obtained in [12] for \( \kappa > 0 \), [15] for \( \kappa = 0 \), and [13] for \( \kappa < 0 \), (see also [10] Theorem 1.3] for a unified statement). The goal of this paper is to prove sharp lower bound for the first nonzero eigenvalue of \( \Delta_{p,f} \) on compact Bakry-Émery manifolds, thus completing our previous work in [10].

A triple \((M^n, g, f)\) consisting of an \( n \)-dimensional Riemannian manifold \((M^n, g)\) and a function \( f \in C^\infty(M) \) is called a Bakry-Émery manifold if it satisfies

\[
\text{Ric} + \nabla^2 f \geq \kappa g
\]
Theorem 1.1. Let \(1 < p \leq 2\) and \(\kappa \in \mathbb{R}\). Here \(\text{Ric}\) denotes the Ricci curvature of \((M, g)\) and \(\nabla^2 f\) denotes the Hessian of the function \(f\). By taking \(f\) to be a constant function, we see that Bakry-Émery manifolds include all Riemannian manifolds with a lower bound on the Ricci curvature. The tensor \(\text{Ric} + \nabla^2 f\) is called the Bakry-Émery Ricci tensor and it shares many important properties as the Ricci curvature, see [11] and [16].

The first nonzero eigenvalue of \(\Delta_{p,f}\), denoted by \(\lambda_{p,f}\), is defined by

\[
\lambda_{p,f} = \inf \left\{ \frac{\int_M |\nabla u|^p e^{-f} d\mu}{\int_M |u|^p e^{-f} d\mu} : u \in W^{1,p}(M, e^{-f} d\mu) \setminus \{0\}, \int_M |u|^{p-2} u e^{-f} d\mu = 0 \right\}.
\]

It was shown in [14] that this infimum is achieved by an eigenfunction \(u \in C^{1,\alpha}(M)\) satisfying the Euler-Lagrange equation

\[
\Delta_{p,f} u = -\lambda_{p,f} |u|^{p-2} u.
\]

In case \(\partial M \neq \emptyset\), the Neumann boundary condition \(\frac{\partial u}{\partial \nu} = 0\) on \(\partial M\) is always imposed, where \(\nu\) is the outward unit normal vector along \(\partial M\). In our recent work [10], we obtained sharp lower bound estimate for \(\lambda_{p,f}\) in terms of \(p, \kappa,\) and the diameter of \(M\), provided that either \(1 < p \leq 2\) or \(\kappa \leq 0\). In this paper, we take care of the remaining case \(p > 2\) and \(\kappa > 0\).

Combined with our results in [10], we prove that

**Theorem 1.1.** Let \((M^n, g, f)\) be a compact Bakry-Émery manifold (possibly with \(C^2\) convex boundary) with diameter \(D\) and \(\text{Ric} + \nabla^2 f \geq \kappa g\) for \(\kappa \in \mathbb{R}\). For \(1 < p < \infty\), let \(\lambda_{p,f}\) be the first nonzero eigenvalue of the weighted \(p\)-Laplacian \(\Delta_{p,f}\) (with Neumann boundary conditions if \(\partial M \neq \emptyset\)). Then we have

\[
\lambda_{p,f} \geq \mu_p(\kappa, D),
\]

where \(\mu_p(\kappa, D)\) is the first nonzero Neumann eigenvalue of the one-dimensional eigenvalue problem

\[
(p-1)|\varphi'|^{p-2}\varphi'' - \kappa t |\varphi'|^{p-2}\varphi' = -\mu_p(\kappa, D)|\varphi|^{p-2}\varphi
\]

on \([-D/2, D/2]\).

When \(\kappa = 0\), the ODE (1.2) can be solved explicitly (see [6] or [15]) and we have

\[
\mu_p(0, D) = (p-1)^2 \left( \frac{\pi_p}{D} \right)^p, \quad \text{where} \quad \pi_p = \frac{2\pi}{p \sin(\pi/p)}.
\]

Therefore, we see that Theorem 1.1 reduces to the sharp lower bound of first nonzero eigenvalue of the Laplacian by Zhong and Yang [17] if \(p = 2\) and \(f \equiv 0\), and to the sharp lower bound of first nonzero eigenvalue of the \(p\)-Laplacian by Valtorta [15] if \(f \equiv 0\). Moreover, when \(\kappa = 0\), the equality in (1.1) is achieved when \(M\) is one-dimensional circle (when \(M\) has no boundary) or the line segment \([-D/2, D/2]\) (when \(M\) has boundary).

For general \(\kappa \in \mathbb{R}\), the \(p = 2\) case of Theorem 1.1 was due to Bakry and Qian [5] (see also Andrews and Ni [2] for a simple alternative proof and the demonstration of the sharpness). We refer the reader to [10] and the references therein for more historical developments and other related results.

In [10], we used the method of modulus of continuity estimates to give a simple proof of Theorem 1.1 for \(1 < p \leq 2\) and \(\kappa \in \mathbb{R}\). Recent years have witnessed the great success of this approach and sharp eigenvalues estimates were obtained in [2] [3] [4]. However, it seems difficult to handle the \(p > 2\) case with this method, as pointed out in Andrews’ survey [1] Section 8]. The proof presented below uses the classical gradient estimates method, which
indeed works for all $1 < p < \infty$ and $\kappa \in \mathbb{R}$. This method has been used successfully by various authors to estimate eigenvalues, including $[8, 9, 17, 7, 5, 15, 13, 10]$. In order to handle the $\kappa > 0$ case, we have to overcome two difficulties in this paper. The sharp gradient comparison theorem, which is the most important ingredient, was only proved for $\kappa \leq 0$ in $[10]$. In this paper, we use the Bochner formula for the weighted $p$-Laplacian (see Proposition 2.1) to prove the sharp gradient comparison theorem (see Theorem 3.1 below). This approach indeed works regardless of the sign of $\kappa$, and for all $1 < p < \infty$.

The paper is organized as follows. In section 2, we derive the Bochner formula for $\Delta_{p,f}$. In section 3, we use the Bochner formula for $\Delta_{p,f}$ to prove the sharp gradient comparison theorem. In section 4, we study the qualitative behavior of solutions to the ODE (1.2) and construct one-dimensional models for the purpose of applying the gradient comparison theorem effectively. The proof of Theorem 1.1 is given in section 5. Examples are constructed in section 6 to demonstrate the sharpness of Theorem 1.1.

2. Bochner Formula for the Weighted $p$-Laplacian

The goal of this section is to establish the Bochner formula for $\Delta_{p,f}$ in Proposition 2.1 which will play a key role in proving the sharp gradient comparison theorem in section 3. This is a slight extension of $[15$, Section 3], where the author proves the Bochner formula for the $p$-Laplacian.

Let $H_u$ denote the Hessian of $u$ and set

$$A_u = \frac{H_u(\nabla u, \nabla u)}{|\nabla u|^2}$$

whenever $|\nabla u| \neq 0$. The linearization of $\Delta_{p,f}$ near a function $u$ is given by

$$P_{u,f}(\eta) := \frac{d}{dt} \bigg|_{t=0} \Delta_{p,f}(u + t\eta)$$

$$= |\nabla u|^{p-2} \Delta_f \eta + (p-2)|\nabla u|^{p-4} H_u(\nabla u, \nabla u) + (p-2)\Delta_{p,f} u \frac{\langle \nabla u, \nabla \eta \rangle}{|\nabla u|^2}$$

$$+ 2(p-2)|\nabla u|^{p-4} H_u \left( \nabla u, \nabla \eta - \frac{\nabla u}{|\nabla u|} \left( \frac{\nabla u}{|\nabla u|} , \nabla \eta \right) \right)$$

wherever $|\nabla u| \neq 0$. It is easy to see that this operator is strictly elliptic where the gradient of $u$ does not vanish. Let $P_{u,f}^{II}(\eta)$ be defined by

$$P_{u,f}^{II}(\eta) := |\nabla u|^{p-2} \Delta_f \eta + (p-2)|\nabla u|^{p-4} H_u(\nabla u, \nabla u).$$

**Proposition 2.1** (Bochner formula for $\Delta_{p,f}$). Let $x \in M$ and $U \subset M$ be an open set containing $x$. For any function $u \in C^3(U)$ with $\nabla u \neq 0$ on $U$, we have at $x$,

$$\frac{1}{p} P_{u,f}^{II}(|\nabla u|^p) = |\nabla u|^{p-2} \langle \nabla \Delta_{p,f} u, \nabla u \rangle - (p-2)|\nabla u|^{p-2} A_u \Delta_{p,f} u$$

$$+ |\nabla u|^{2(p-2)} \left( |H_u|^2 + p(p-2) A_u^2 + \text{Ric}_f(\nabla u, \nabla u) \right),$$
where \( \text{Ric}_f = \text{Ric} + \nabla^2 f \).

**Proof.** Let \( P^I_u(\eta) := |\nabla u|^{p-2} \Delta \eta + (p - 2) |\nabla u|^{p-4} H_\eta(\nabla u, \nabla u) \) be the second order part of \( P_{u,f}(\eta) \). The Bochner formula for the \( p \)-Laplacian, shown in [15, Proposition 3.1], states that

\[
\frac{1}{p} P^I_u(|\nabla u|^p) = |\nabla u|^{p-2} \langle \nabla \Delta_p u, \nabla u \rangle - (p - 2) |\nabla u|^{p-2} A_u \Delta_p u
+ |\nabla u|^{2(p-2)} (|H_u|^2 + p(p - 2) A_u^2 + \text{Ric}(\nabla u, \nabla u)).
\]

We then compute, using the identities \( \Delta_p u = \Delta_{p,f} u + |\nabla u|^{p-2} \langle \nabla u, \nabla f \rangle \) and \( \text{Ric} = \text{Ric}_f - \nabla^2 f \), that

\[
\frac{1}{p} P^I_{u,f}(|\nabla u|^p) = |\nabla u|^{p-2} \langle \nabla \Delta_{p,f} u, \nabla u \rangle - (p - 2) |\nabla u|^{p-2} A_u \Delta_{p,f} u
+ |\nabla u|^{2(p-2)} (|H_u|^2 + p(p - 2) A_u^2 + \text{Ric}_f(\nabla u, \nabla u))
+ |\nabla u|^{2(p-2)} \langle \nabla f(\nabla u, \nabla u) \rangle
\]

\[
= |\nabla u|^{p-2} \langle \nabla \Delta_{p,f} u, \nabla u \rangle - (p - 2) |\nabla u|^{p-2} A_u \Delta_{p,f} u
+ |\nabla u|^{2(p-2)} (|H_u|^2 + p(p - 2) A_u^2 + \text{Ric}_f(\nabla u, \nabla u))
+ \frac{1}{p} |\nabla u|^{p-2} \langle \nabla \nabla u, \nabla f \rangle.
\]

The desired formula then follows from

\[
\frac{1}{p} P^I_{u,f}(|\nabla u|^p) = \frac{1}{p} P^I_u(|\nabla u|^p) - \frac{1}{p} |\nabla u|^{p-2} \langle \nabla f, \nabla \nabla u \rangle.
\]

\( \square \)

3. **Sharp Gradient Comparison Theorem**

In this section, we use the Bochner formula for \( \Delta_{p,f} \) to derive the sharp gradient comparison theorem. The proof works for all \( 1 < p < \infty \) and \( \kappa \in \mathbb{R} \).

**Theorem 3.1.** Let \((M^n, g, f)\) be a compact Bakry-Émery manifold (possibly with \( C^2 \) convex boundary) satisfying \( \text{Ric} + \nabla^2 f \geq \kappa g \) for some \( \kappa \in \mathbb{R} \). Let \( u \) be a solution of

\[
\Delta_{p,f} u = -\lambda |u|^{p-2} u,
\]

normalized so that \(-1 = \min\{u\} < 0 < \max\{u\} \leq 1\) (in case \( \partial M \neq \emptyset \), we assume that \( u \) satisfies Neumann boundary condition). Suppose \( w : [a, b] \rightarrow \mathbb{R} \) is a solution of the one-dimensional equation:

\[
(p - 1)|w'|^{p-2} w'' - \kappa t |w'|^{p-2} w' = -\lambda |w|^{p-2} w
\]

which is strictly increasing on \([a, b]\), and such that the range of \( u \) is contained in \([w(a), w(b)]\). Then we have for all \( x \in M \),

\[
|\nabla u(x)| \leq w' \left( w^{-1}(u(x)) \right).
\]
Proof. **Case 1:** $\partial M = \emptyset$.
We may assume that $[\min \{u\}, \max \{u\}] \subset (w(a), w(b))$, because otherwise we can replacing $u$ by $\alpha u$ and then letting $\alpha \to 1^-$.
Let $w := w_\varepsilon(t)$ be the solution of the initial value problem
\begin{equation}
\begin{aligned}
|w'|^p - 2w'' - (\kappa - \varepsilon t)|w'|^p - 2w' + \lambda|w|^p - 2w = 0,
\end{aligned}
\tag{3.3}
\end{equation}
which is strictly increasing on $[a, b_{\varepsilon}]$ with $\lim_{t \to 0} b_{\varepsilon} = b$.
For $\varepsilon > 0$ sufficiently small, we still have that $[\min \{u\}, \max \{u\}] \subset (w(a), w(b_{\varepsilon}))$.

Let $c_0 \geq 1$ be the number defined by
\[ c_0 = \inf \{c \geq 1 : Z_c(x) = |\nabla u(x)|^p - (cw')^p|_{(cw) - 1(u(x))} \leq 0 \text{ on } M \}. \]
Clearly $c_0$ is finite since $Z_c(x)$ approaches negative infinity uniformly as $c$ approaches infinity.
By continuity, there exists $x_0 \in M$ such that
\[ 0 = Z_{c_0}(x_0) = \max_{x \in M} Z_{c_0}(x). \]

For simplicity of notations, we use $t = (c_0 w)^{-1}(u(x))$ as an intermediate variable and write $\varphi(t) = (c_0 w)(t)$. Thus the function $Z_{c_0}(x) = |\nabla u|^p - (\varphi')^p$ attains zero maximum at $x_0$.

Note that $|\nabla u(x_0)| \neq 0$ since $\varphi$ has positive derivative.

Hereafter, we assume that $u \in C^3(U)$ for some open neighborhood $U$ of $x_0$. This is certainly the case if $u(x_0) \neq 0$ or if $p \geq 2$, as pointed out in Remark 3.2. In case $1 < p < 2$ and $u(x_0) = 0$, then $u$ has only $C^{2,\alpha}$ regularity near $x_0$. However, this regularity issue is not an obstacle to the argument, as we will see in Remark 3.3. Here and below, all the derivatives of $\varphi$ are evaluated at $t_0 = (c_0 w)^{-1}(u(x_0))$.

The first derivative test implies that $\nabla Z_{c_0}(x_0) = 0$, which produces the following identity at $x_0$,
\[ p|\nabla u|^p - 2H_u \nabla u = \frac{p}{p - 1} \mathcal{L}_p \varphi \nabla u, \]
where $\mathcal{L}_p \varphi := (p - 1)(\varphi'^p - 2\varphi''$ is the one-dimensional $p$-Laplacian. In particular, we have
\[ (p - 1)|\nabla u|^p - 2A_u = \mathcal{L}_p \varphi. \tag{3.4} \]

Next we calculate and estimate the second derivatives. Using Proposition 2.1, we obtain that at $x_0$,
\begin{align*}
\frac{1}{p} P_{u,f}^I(\nabla u)^p & = |\nabla u|^{p-2} \langle \Delta_{p,f} u, \nabla u \rangle - (p - 2)|\nabla u|^{p-2} A_u \Delta_{p,f} u \\
& + |\nabla u|^{2(p-2)} \left( |H_u|^2 + p(p - 2) A_u^2 + \text{Ric}_f(\nabla u, \nabla u) \right) \\
& \geq -\lambda |\nabla u|^{p-2} \langle |u|^{p-2} u, \nabla u \rangle - (p - 2)\lambda |u|^{p-2} |\nabla u|^p - 2A_u \\
& + |\nabla u|^{2(p-2)} \left( (p - 1)^2 A_u^2 + \kappa |\nabla u|^2 \right) \\
& = -(p - 1)\lambda u^{p-2} |\nabla u|^p - \frac{p - 2}{p - 1} \lambda |u|^{p-2} u \mathcal{L}_p \varphi \\
& + (\mathcal{L}_p \varphi)^2 + \kappa |\nabla u|^{2p-2}, \tag{3.5}
\end{align*}
where we have used $|H_u|^2 \geq A_u^2$ and $\text{Ric}_f(\nabla u, \nabla u) \geq \kappa |\nabla u|^2$ in the inequality, and \textbf{3.4} in getting the last equality.
On the other hand, direct calculation shows
\[
\frac{1}{p} \nabla (\varphi')^p = \frac{1}{p-1} \mathcal{L}_p \varphi \nabla u,
\]
\[
\frac{1}{p} \Delta f(\varphi')^p = \frac{1}{p-1} \mathcal{L}_p \varphi \Delta f u + \frac{1}{p-1} \frac{d}{dt} (\mathcal{L}_p \varphi) \frac{1}{\varphi'} |\nabla u|^2,
\]
\[
\frac{1}{p} H(\varphi')^p(\nabla u, \nabla u) = \frac{1}{p-1} \mathcal{L}_p \varphi A u |\nabla u|^2 + \frac{1}{p-1} \frac{d}{dt} (\mathcal{L}_p \varphi) \frac{1}{\varphi'} |\nabla u|^4.
\]
Putting the above identities together, we obtain that
\[
\frac{1}{p} P^{II}_{u,f}(\varphi')^p = \frac{1}{p} |\nabla u|^{p-2} \Delta f(\varphi')^p + \frac{1}{p} (p-2) |\nabla u|^{p-4} H(\varphi')^p(\nabla u, \nabla u)
\]
\[
= \frac{\mathcal{L}_p \varphi}{p-1} |\nabla u|^{p-2} (\Delta f u + A u) + |\nabla u|^p \frac{d}{dt} (\mathcal{L}_p \varphi) \frac{1}{\varphi'}
\]
\[
= -\frac{\lambda |u|^{p-2} u}{p-1} \mathcal{L}_p \varphi + |\nabla u|^p \frac{d}{dt} (\mathcal{L}_p \varphi) \frac{1}{\varphi'},
\]
where we have used $|\nabla u|^{p-2} (\Delta f u + A u) = \Delta p, f u = -\lambda |u|^{p-2} u$ in the last equality.

The second derivative test implies $P^{II}_{u,f}(Z_c) \leq 0$ at $x_0$, thus we have by (3.5) and (3.6) that
\[
0 \geq \frac{1}{p} P^{II}_{u,f}(Z_c) = \frac{1}{p} P^{II}_{u,f}(|\nabla u|^p) - \frac{1}{p} P^{II}_{u,f}(\varphi')^p
\]
\[
\geq -(p-1)\lambda |\nabla u|^{p-2} |\nabla u|^p - \frac{p-2}{p-1} \lambda |u|^{p-2} u \mathcal{L}_p \varphi
\]
\[
+ (\mathcal{L}_p \varphi)^2 + \kappa |\nabla u|^{2p-2} + \frac{\lambda |u|^{p-2} u}{p-1} \mathcal{L}_p \varphi - |\nabla u|^p \frac{d}{dt} (\mathcal{L}_p \varphi) \frac{1}{\varphi'}.
\]
Substituting $|\nabla u(x_0)|^p = (\varphi'(t_0))^p$ and $u(x_0) = \varphi(t_0)$ into (3.7) gives that at $t_0$,
\[
0 \geq -(p-1)\lambda \varphi^{p-2}(\varphi')^p + \frac{(p-2)}{p-1} \lambda |\varphi|^{p-2} \varphi \mathcal{L}_p \varphi
\]
\[
+ (\mathcal{L}_p \varphi)^2 + \kappa (\varphi')^{2p-2} + \frac{\lambda |\varphi|^{p-2} \varphi}{p-1} \mathcal{L}_p \varphi - (\varphi')^{p-1} \frac{d}{dt} (\mathcal{L}_p \varphi)
\]
\[
\geq \mathcal{L}_p \varphi \left( \mathcal{L}_p \varphi - (\kappa - \varepsilon) t(\varphi')^{p-1} + \lambda |\varphi|^{p-2} \varphi + \varepsilon (\varphi')^{2p-2} \right.
\]
\[
- (\varphi')^{p-1} \left( \frac{d}{dt} (\mathcal{L}_p \varphi) + (p-1) \lambda \varphi^{p-2} \varphi' - (\kappa - \varepsilon) t \varphi \mathcal{L}_p \varphi - (\kappa - \varepsilon)(\varphi')^{p-1} \right)
\]
Since $\varphi$ satisfies the ODE
\[
\mathcal{L}_p \varphi - (\kappa - \varepsilon) t(\varphi')^{p-1} + \lambda |\varphi|^{p-2} \varphi = 0,
\]

it follows that $\mathcal{L}_p \varphi$ satisfies
\[
\frac{d}{dt} (\mathcal{L}_p \varphi) - (\kappa - \varepsilon)(\varphi')^{p-1} - (\kappa - \varepsilon) t \mathcal{L}_p \varphi + (p-1) \lambda \varphi^{p-2} \varphi' = 0.
\]
We then easily get from (3.8) that
\[
0 \geq \varepsilon (\varphi')^{2p-2} > 0,
\]
which is clearly a contradiction. The desired gradient estimates follows immediately since the solutions $w_\varepsilon$ converges in $C^1$ to the solution $w$ as $\varepsilon \to 0$.

**Case 2:** $\partial M \neq \emptyset$ and $u$ satisfies the Neumann boundary condition.

The proof for Case 1 remains valid as long as $x_0$ is in the interior of $M$. If $x_0 \in \partial M$, we
follow the argument in [13] Lemma 18] to show that \( \nabla Z_{c_0}(x_0) = 0 \). Once this is established, it then follows that \( P_{IJ}(Z_{c_0}(x_0)) \leq 0 \) and the rest of proof proceeds as in Case 1. Thus it suffices to show the following claim.

**Claim:** The equation \( \nabla Z_{c_0}(x_0) = 0 \) remains valid even if \( x_0 \in \partial M \).

**Proof of Claim.** Let \( \nu \) be the outward unit normal vector field of \( \partial M \). Since \( Z_{c_0}(x) \) attains its maximum at \( x_0 \in \partial M \), we know that all tangential derivatives of \( Z_{c_0}(x) \) vanish at \( x_0 \) and

\[
0 \leq \langle \nabla Z_{c_0}, \nu \rangle(x_0) = p |\nabla u|^{p-2} H_u(\nabla u, \nu) - p(\varphi')^{p-2} \varphi'' \langle \nabla u, \nu \rangle = p |\nabla u|^{p-2} H_u(\nabla u, \nu) - p |\nabla u|^{p-2} II(\nabla u, \nabla u) \leq 0.
\]

Here the last step is because of the convexity of \( \partial M \). Therefore we have \( \nabla Z_{c_0}(x_0) = 0 \) and the claim is proved. \( \square \)

The proof of Theorem 3.1 is complete now. \( \square \)

**Remark 3.2.** The eigenfunction \( u \) is in general not smooth. We have \( u \in C^{1,\alpha}(M) \cap W^{1,p}(M) \), and elliptic theory ensures that \( u \) is smooth where \( \nabla u \neq 0 \) and \( u \neq 0 \). If \( \nabla u \neq 0 \) and \( u(x) = 0 \), then \( u \in C^{2,\alpha}(U) \) if \( p > 2 \) and \( u \in C^{2,\alpha}(U) \) if \( 1 < p < 2 \), where \( U \) is a small neighborhood of \( x \). We refer the reader to [11] for these results.

**Remark 3.3.** If \( 1 < p < 2 \) and \( u(x_0) = 0 \), we only know that \( u \) is \( C^{2,\alpha} \) near \( x_0 \) and \( Z \) is \( C^{1,\alpha} \) near \( x_0 \). Thus the \( P_{IJ}(Z) \) may not be defined since there are two diverging terms in it. As we can see in equation (3.1), these term are

\[
-(p-1)\lambda |u|^{p-2} |\nabla u|^p \text{ and } -|\nabla u|^p \frac{1}{p} \frac{d}{dt} (\mathcal{L}_p \varphi).
\]

Since \( \nabla u(x_0) \neq 0 \), there exists an open set \( U \) containing \( x_0 \) such that \( U \setminus \{u = 0\} \) is open and dense in \( U \). On this set, we see that these two terms exactly cancel each other, and all the other term in \( P_{IJ}(Z_{c_0}) \) are well-defined and continuous on \( U \). Thus the formula \( P_{IJ}(Z_{c_0}) \leq 0 \) is valid even in this low regularity setting.

4. **One-dimensional Models for \( \kappa > 0 \)**

In this section, we study the qualitative behavior of the one-dimensional equation

\[
|w'|^{p-2}w'' - \kappa t |w'|^{p-2}w' + \lambda |w|^{p-2}w = 0.
\]  

(4.1)

for \( \kappa > 0 \). The case \( \kappa \leq 0 \) was treated in [10].

We recall some basic definitions and properties of \( p \)-trigonometric functions and refer the reader to [12] Chapter 1] for more details. For \( 1 < p < \infty \), let \( \pi_p \) be the positive number defined by

\[
\pi_p = \int_{-1}^{1} \frac{ds}{(1 - s^p)^{1/p}} = \frac{2\pi}{p \sin(\pi/p)}.
\]

The \( p \)-sine function \( \sin_p : \mathbb{R} \to [-1, 1] \) is defined implicitly on \([-\pi_p/2, 3\pi_p/2] \) by

\[
\begin{align*}
t &= \int_{0}^{\sin_p(t)} \frac{ds}{(1 - s^p)^{1/p}} & \text{if } t \in [-\pi_p/2, \pi_p/2], \\
\sin_p(t) &= \sin_p(\pi_p - t) & \text{if } t \in [\pi_p/2, 3\pi_p/2],
\end{align*}
\]
and is periodic on $\mathbb{R}$ with period $2\pi p$. It’s easy to see that for $p \neq 2$ this function is smooth around noncritical points, but only $C^{1,\alpha}(\mathbb{R})$ with $\alpha = \min\{p-1, (p-1)^{-1}\}$. By defining
\[
\cos_p(t) = \frac{d}{dt} \sin_p(t) \quad \text{and} \quad \tan_p(t) = \frac{\sin_p(t)}{\cos_p(t)},
\]
we then have the following generalized trigonometric identities:
\[
|\sin_p(t)|^p + |\cos_p(t)|^p = 1, \quad \frac{d}{dt} \tan_p(t) = \frac{1}{|\cos_p(t)|^p} = 1 + |\tan_p(t)|^p, \quad \frac{d}{dt} \arctan_p(t) = \frac{1}{1 + |t|^p}.
\]
Let $\alpha = \left(\frac{\lambda}{p-1}\right)^{1/p}$. We introduce the $p$-polar coordinates $r$ and $\theta$ defined by
\[
\alpha w = r \sin_p(\theta), \quad w' = r \cos_p(\theta),
\]
or equivalently,
\[
r = \left(\left(w'\right)^p + \alpha^p w^p\right)^{\frac{1}{p}}, \quad \theta = \arctan_p \left(\frac{\alpha w}{w'}\right). \tag{4.3}
\]
If $w$ is a solution of (4.1) with $w(a) = -1$ and $w'(a) = 0$, then direct calculation shows that $\theta$ and $r$ satisfy
\[
\begin{align*}
\theta'(a) &= \alpha - \frac{\kappa t}{p-1} \cos_p^{-1}(\theta) \sin_p(\theta), \\
\theta(a) &= -\frac{\pi}{2p}, \quad (\text{mod } \pi_p); \\
\frac{d\theta}{dt} \log r &= \frac{\kappa t}{p-1} \cos_p(\theta), \\
r(a) &= \alpha. \tag{4.4}
\end{align*}
\]
Since both $\sin_p(t)$ and $\cos_p^{-1}(t)$ are Lipschitz functions with Lipschitz constant 1, we can apply Cauchy’s theorem to obtain existence, uniqueness, and continuous dependence on the parameters. Indeed, we have the following proposition.

**Proposition 4.1.** For any $a \in \mathbb{R}$, there exists a unique solution $w$ to (4.1) with $w(a) = -1$ and $w'(a) = 0$, defined on $\mathbb{R}$ with $w, (w')^{p-2}w' \in C^1(\mathbb{R})$. Moreover, the solution depends continuously on the parameters in the sense of local uniform convergence of $w$ and $w'$ in $\mathbb{R}$.

Let $\lambda_0$ be the first nonzero eigenvalue of the eigenvalue problem (4.1) on the real line, i.e.,
\[
\lambda_0 = \inf \left\{ \frac{\int_{\mathbb{R}} |\varphi'|^p e^{-\frac{\kappa t}{2}} ds}{\int_{\mathbb{R}} |\varphi|^p e^{-\frac{\kappa t}{2}} ds} : \varphi \in W^{1,p}(\mathbb{R}, e^{-\frac{\kappa t}{2}} ds) \setminus \{0\}, \quad \int_{\mathbb{R}} |\varphi|^{p-2} \varphi e^{-\frac{\kappa t}{2}} ds = 0 \right\}. \tag{4.6}
\]
Let’s first discuss the special case $p = 2$. In this case, equation (4.1) (when normalizes so that $\kappa = 1$) is the so-called Hermite’s differential equation and solutions with polynomial growth are given by Hermitian polynomials. In particular, we have $\lambda_0 = \kappa$ and the corresponding eigenfunction is $\kappa t$. On the other hand, it is known that the first nonzero eigenvalue of the $f$-Laplacian is bounded from below by $\kappa$ if $\text{Ric + V}^2 f \geq \kappa g$, see for example [10, Theorem 1.6]. Thus when $p = 2$, we have $\lambda_{p,f} \geq \lambda_0$ (strict inequality when $M$ is compact) and this inequality plays an important role in the construction of one-dimensional models in [5].
Our first step here is to get the non-sharp bound \( \lambda_{p,f} > \lambda_0 \) for all \( 1 < p < \infty \) and \( \kappa > 0 \). It turns out that this is a consequence of the sharp gradient estimates in Theorem 3.1 and the compactness of \( M \).

**Proposition 4.2.** Let \((M^n, g, f)\) be a compact Bakry-Émery manifold (possibly with \(C^2\) convex boundary) satisfying \( \text{Ric} + \nabla^2 f \geq \kappa g \) for some \( \kappa > 0 \). Let \( \lambda_{p,f} \) be the first nonzero eigenvalue of \( \Delta_{p,f} \) (with Neumann boundary condition if \( \partial M \neq \emptyset \)). Then

\[
\lambda_{p,f} > \lambda_0,
\]

where \( \lambda_0 \) is defined in (4.6).

We first prove an elementary lemma.

**Lemma 4.1.** For any \( \lambda \leq \lambda_0 \), the ODE

\[
(p-1)|w'|^{p-2}w'' - \kappa t |w'|^{p-2}w' + \lambda |w'|^{p-2}w = 0
\]

(4.7)

admits an odd solution \( w : \mathbb{R} \to \mathbb{R} \) satisfying \( w'(t) > 0 \) for all \( t \in \mathbb{R} \).

**Proof of Lemma 4.1.** By Proposition 4.1, there exist a solution of (4.7) satisfying \( w(0) = 0 \) and \( w'(0) = 1 \). The oddness of \( w \) follows from uniqueness of solutions. The condition \( w'(t) > 0 \) for all \( t \in \mathbb{R} \) is a consequence of \( \lambda \leq \lambda_0 \). Otherwise, we get a solution of (4.7) on some interval \([-a, a]\) with \( w'(-a) = w'(a) = 0 \) and \( w'(t) > 0 \) on \((-a, a)\), implying that \( \lambda \) is the first nonzero eigenvalue of (4.7) on \([-a, a]\), which would contradict \( \lambda \leq \lambda_0 \).

**Proof of Proposition 4.2.** We argue by contradiction. Suppose \( \lambda \leq \lambda_0 \), then by Lemma 4.1 there exists an odd solution \( v : \mathbb{R} \to \mathbb{R} \) of the ODE (4.7) satisfying \( v'(t) > 0 \) for all \( t \in \mathbb{R} \). Since \( v(t) \) is strictly increasing, we have that either \( v(t) \) approaches infinity as \( t \to \infty \) or \( \lim_{t \to \infty} v(t) = A \in (0, \infty) \).

Let \( u \) be an eigenfunction associated to the eigenvalue \( \lambda_{p,f} \) normalized so that \( -1 = \min \{u\} < 0 < \max \{u\} \leq 1 \). Consider two points \( x_0 \) and \( y_0 \) such that \( u(x_0) = \min_{x \in M} u(x) \), and \( u(y_0) = \max_{x \in M} u(x) \). Let \( L = d(x_0, y_0) \) and \( \gamma : [0, L] \to M \) be a unit speed geodesic joining \( x_0 \) and \( y_0 \). Define \( h(t) = u(\gamma(t)) \) and choose \( I \subset [0, L] \) such that \( h' > 0 \) on \( I \) and \( h^{-1} \) is well-defined in a subset of full measure of \([-1, \max \{u\}]\).

For any number \( c \) such that the range of \( cv \) contains the interval \([-1, 1]\), we can apply Theorem 3.1 with \( w = cv \) on the interval \([-w^{-1}(-1), w^{-1}(\max \{u\})]\) to conclude that

\[
D \geq \int_0^L dt \geq \int_{-1}^{\max \{u\}} \frac{dy}{h'(h^{-1}(y))} \geq \int_{-1}^{\max \{u\}} \frac{dy}{w'(w^{-1}(y))} = \int_{w^{-1}(\max \{u\})}^{w^{-1}(-1)} dt = w^{-1}(\max \{u\}) - w^{-1}(-1) \geq -w^{-1}(-1) = v^{-1}\left(\frac{1}{c}\right).
\]

If \( \lim_{t \to \infty} v(t) = \infty \), we see that the right hand side goes to infinity by letting \( c \to 0 \). If \( \lim_{t \to \infty} v(t) = A \in (0, \infty) \), the right hand side goes to infinity by letting \( c \) decrease to \( A^{-1} \). Either way, this is a contradiction to the finiteness of the diameter of \( M \). □

For the purpose of getting sharp eigenvalue estimates, we need to show that for any eigenfunction \( u \) of \( \Delta_{p,f} \) with eigenvalue \( \lambda > \lambda_0 \), there exist an interval \([a, b]\) and a solution \( w \) of (4.1) such that \( w \) is strictly increasing on \([a, b]\) with \( w(a) = \min \{u\} \) and \( w(b) = \max \{u\} \). As we shall see, this can be achieved varying the initial data.
The rest of this section is a slight modification of [10] Section 4. Fix $\lambda > \lambda_0$ and $\kappa > 0$. Let $w_a$ be the solution of the following initial value problem (IVP)

$$\begin{aligned}
|w'|^{p-2}w'' - \kappa t|w'|^{p-2}w' + \lambda |w|^{p-2}w &= 0, \\
w(a) = -1, w'(a) = 0,
\end{aligned}$$

(4.8)

with $a \in \mathbb{R}$.

**Proposition 4.3.** Fix $\lambda > \lambda_0$ and $\kappa > 0$. There exists a unique $\bar{a} > 0$ such that the solution $w_{-\bar{a}}$ to the IVP (4.8) is odd. In particular, $w_{-\bar{a}}$ restricted to $[-\bar{a}, \bar{a}]$ has nonnegative derivative and has maximum value equal to one.

**Proof.** Consider the initial value problem

$$\begin{aligned}
\dot{\theta} &= \alpha - \frac{\kappa t}{p-1} \cos^{p-1} \theta \sin p \theta, \\
\theta(0) &= 0.
\end{aligned}$$

(4.9)

The uniqueness of solutions implies that $\theta(t)$ is an odd function. The fact $\lambda > \lambda_0$ implies there exists $\bar{a} > 0$ such that $\theta(\bar{a}) = \pi p/2$. It’s easily seen that the corresponding solution $r(t)$ to (4.5) is even, regardless of its initial value. The proposition follows by translating obtained information on $\theta$ and $r$ back to $w$. □

For the solution $w = w_a$ of (4.8), we define

$$\begin{aligned}
b(a) &= \inf \{ b > a : w'(b) = 0 \}, \\
m(a) &= w_a(b(a)), \\
\delta(a) &= b(a) - a.
\end{aligned}$$

In other words, $b(a)$ is the first value $b > a$ such that $w'(b) = 0$ with the convention that $b(a) = \infty$ if such a value does not exist. Thus $w$ is strictly increasing on the interval $[a, b(a)]$ and $m(a)$ is the maximum of $w$ on $[a, b(a)]$. The function $\delta(a)$ measures the length of the interval where $w(a)$ increases from $-1$ to $m(a)$.

We are concerned with the range of the function $m(a)$ as $a$ varies. Clearly $m(a) > 0$ since the eigenfunction $w$ changes sign. Also, Proposition 3.3 implies $m(-\bar{a}) = 1$. We shall show in the next proposition that $m(a)$ goes to zero as $a$ goes to $-\infty$. It then follows from the Intermediate Value Theorem that the range of $m(a)$ covers $(0, 1]$ when $a$ varies in $(-\infty, -\bar{a}]$.

**Proposition 4.4.**

$$\lim_{a \to -\infty} m(a) = 0.$$

**Proof.** The argument is dual to the proof of [10] Proposition 4.3, and we omit it here. □

**Proposition 4.5.** We have $\delta(a) \geq \delta(-\bar{a})$ for all $a \leq -\bar{a}$ with strict inequality if $a \neq -\bar{a}$.

**Proof.** The argument is dual to the proof of [10] Proposition 4.4, and we omit it here. □

At last, we study $\delta := \delta(-\bar{a}) = 2\bar{a}$ as a function of $\lambda$, having fixed $p$ and $\kappa$. It’s easy to that $\delta$ is a strictly decreasing function and so invertible. Thus we can define its inverse $\lambda(\delta)$, which is a continuous and decreasing function. Moreover, it can be characterized in the following equivalent way.
Proposition 4.6. For fixed $\kappa > 0, 1 < p < \infty$, we have that given $\delta > 0$, $\lambda$ is the first nonzero Neumann eigenvalue of the one-dimensional problem
\[
(p - 1)|w'|^{p - 2}w' - \kappa t|w'|^{p - 2}w' = -\lambda|w|^{p - 2}w
\]
on $[-\delta/2, \delta/2]$.

5. Proof of Theorem 1.1

After all the preparations in the previous two sections, we finally prove Theorem 1.1 in this section.

Proof of Theorem 1.1. It suffices to prove the case $\kappa > 0$, as the case $\kappa \leq 0$ was proved in [10]. Let $u$ be an eigenfunction of $\Delta_{p,f}$ associated to the eigenvalue $\lambda$. In view of
\[
\int_M |u|^{p - 2}ue^{-f} = 0,
\]
we can normalize $u$ so that $\min\{u\} = -1$ and $\max\{u\} \in (0, 1]$.

By Proposition 4.3 and 4.4 there exists an interval $[a, b]$ and a solution $w$ of (4.8) such that $w$ is strictly increasing on $[a, b]$ with $w(a) = -1 = \min\{u\}$ and $w(b) = \max\{u\} \in (0, 1]$. Moreover, we have $\lambda = \lambda_p([a, b])$, the first nonzero Neumann eigenvalue of the eigenvalue problem $|w'|^{p - 2}w'' - \kappa t|w'|^{p - 2}w' + \lambda|w|^{p - 2}w = 0$ on $[a, b]$.

Let $x$ and $y$ be such that $u(x) = \min_M u$ and $u(y) = \max_M u$. Consider a unit speed minimizing geodesic $\gamma : [0, d(x, y)] \rightarrow M$ joining $x$ and $y$. Let $h(t) = u(\gamma(t))$ and choose $I \subset [0, d(x, y)]$ such that $h' > 0$ on $I$ and $h^{-1}$ is well-defined in a subset of full measure of $[-1, u_{\max}]$. Then we get, by change of variables and the sharp gradient estimate proved in Theorem 3.1 that
\[
D \geq \int_0^{w_{\max}} \frac{dy}{h'(h^{-1}(y))} \geq \int_{-1}^{w_{\max}} \frac{dy}{w'(w^{-1}(y))} = \int_a^{b(a)} dt = \delta(a) \geq \delta(\tilde{a}),
\]
where the last inequality is proved in Proposition 4.5. This and Proposition 4.6 yield immediately to the desired estimate. \hfill \Box

6. Sharpness of the Lower Bound

In this section, we show that the lower bound (1.1) given in Theorem 1.1 is sharp for $n \geq 3$ for any $\kappa$ or $n \geq 2$ for $\kappa \leq 0$. More precisely, for each $\varepsilon > 0$, we construct a Bakry-Émery manifold $(M, g, f)$ with diameter $D$ and satisfying $\mu_{p,f} < \mu_p(\kappa, D) + \varepsilon$.

For manifolds with boundary, the construction is rather simple. Take a cylinder $rS^{n-1} \times [-D/2, D/2]$ for $r$ sufficiently small, with quadratic potential $f = \frac{s^2}{2}$. It is easy to see this is a Bakry-Émery manifold with diameter $D(1 + o(r))$ and $\text{Ric} + \nabla^2 f \geq \kappa g$. Let $w$ be an eigenfunction associated to the eigenvalue $\mu_p(\kappa, D)$ of the one-dimensional eigenvalue problem 1.2 on $[-D/2, D/2]$. Substituting the test function $\psi(z, s) = w(s)$ into the Rayleigh
quotient yields
\[
\lambda_{p,f} \leq \frac{\int_M |\nabla \psi|^p e^{-f} d\mu_g}{\int_M |\psi|^p e^{-f} d\mu_g} = \frac{\int_{-D/2}^{D/2} |w'|^p e^{-f} ds}{\int_{-D/2}^{D/2} |w|^p e^{-f} ds} = \mu_p(\kappa, D)
\]

It follows that \(\lambda_{p,f} \to \mu_p(\kappa, D)\) as \(r \to 0\), proving the sharpness of the estimate (1.1) in Theorem 1.1.

To demonstrate the sharpness of (1.1) in the smaller class of manifolds without boundary, we need a more involved construction. The idea is to attach spherical caps to the ends of the above examples. The Bakry-Émery manifolds are constructed exactly the same as in [4]. The Riemannian manifold \((M, f)\), which is approximately a thin cylinder with hemispherical caps attached at each end, is constructed as follows. Let \(\gamma\) be the curve in \(\mathbb{R}^2\) with curvature \(k\) given as function of arc length for suitably small \(r > 0\) and \(\delta > 0\) small compared to \(r\), by

\[
k(s) = \begin{cases} \frac{1}{r} & s \in [0, \frac{\pi r}{2} - \delta], \\ \varphi \left(\frac{s - \frac{\pi r}{2}}{\delta}\right) \frac{1}{r} & s \in \left[\frac{\pi r}{2} - \delta, \frac{\pi r}{2} + \delta\right], \\ 0 & s \in \left[\frac{\pi r}{2} + \delta, D\right], \end{cases} \tag{6.1}
\]

and extended to be even under reflection in both \(s = 0\) and \(s = D/2\). Here \(\varphi\) is a smooth nonincreasing function with \(\varphi(s) = 1\) for \(s \leq -1\), \(\varphi(s) = 0\) for \(s \geq 1\), and satisfying \(\varphi(s) + \varphi(-s) = 1\). Geometrically, this corresponds to a pair of line segments parallel to the \(x\) axis, capped by semicircles of radius \(r\) and smoothed at the joins. Let \((x(s), y(s))\) be the corresponding embedding and we choose the point corresponding to \(s = 0\) to have \(y(0) = 0\) and \(y'(0) = 1\). Let \((M^n, g)\) be the hypersurface of rotation in \(\mathbb{R}^{n+1}\) given by \(\{(x(s), y(s), z) : s \in \mathbb{R}, z \in S^{n-1}\}\).

The function \(f\) on \(M\) is a function of \(s\) only, given by \(f'(0) = 0\) (the value of \(f(0)\) is immaterial) and

\[
f''(s) = \begin{cases} \kappa (1 - \frac{D}{\frac{\pi r}{2}}), & s \in [0, \frac{\pi r}{2} - \delta], \\ \kappa \varphi \left(\frac{s - \frac{\pi r}{2}}{\delta}\right) \left(1 - \frac{D}{\frac{\pi r}{2}}\right) + \kappa \left(1 - \varphi \left(\frac{s - \frac{\pi r}{2}}{\delta}\right)\right) & s \in \left[\frac{\pi r}{2} - \delta, \frac{\pi r}{2} + \delta\right], \\ \kappa & s \in \left[\frac{\pi r}{2} + \delta, D\right], \end{cases} \tag{6.2}
\]

This implies \(f'(D/2) = 0\). We also extend \(f\) to be even under the reflection in \(s = 0\) and \(s = D/2\).

With the above choices, let’s compute the Bakry-Émery Ricci tensor of \((M, g, f)\). The eigenvalues of the second fundamental form are \(k(s)\) (in the \(s\) direction) and \(\sqrt{1 - \varphi(s)^2}\) in the orthogonal directions. Therefore the Gauss-Codazzi equations imply that the Ricci tensor has eigenvalues \((n - 1)k(s)\sqrt{1 - (\varphi(s)^2)}\) in the \(s\) direction, and \(k(s)\sqrt{1 - (\varphi(s)^2)} + (n - 1)\frac{1 - \varphi(s)^2}{y^2}\) in the orthogonal directions. The eigenvalues of the Hessian of \(f\) can be calculated as follows: The curves of fixed \(z\) in \(M\) are geodesics parametrized by \(s\), so the Hessian in this direction is just \(f''\) as given in (6.2). Since \(f\) depends only on \(s\) we also have that \(\nabla^2 f(e_i, e_i) = 0\) for \(e_i\) tangent to \(S^{n-1}\) and \(\nabla^2 f(e_i, e_j) = \frac{D}{y^2} f'' \delta_{ij}\).
The identities $y(s) = \int_0^s \cos(\theta(\tau)) d\tau$ and $y'(s) = \cos(\theta(s))$, where $\theta(s) = \int_0^s k(\tau) d\tau$ applied to (6.1) yields that as $\delta \to 0$,

$$y(s) = \begin{cases} r \sin(s/r), & s \in [0, \frac{\pi r}{2} - \delta], \\ r(1 + o(\delta)), & s \in [\frac{\pi r}{2} - \delta, D], \end{cases}$$

and

$$y'(s) = \begin{cases} \cos(s/r), & s \in [0, \frac{\pi r}{2} - \delta], \\ o(\delta), & s \in [\frac{\pi r}{2} - \delta, \frac{\pi r}{2} - \delta], \\ 0, & s \in [\frac{\pi r}{2} + \delta, D]. \end{cases}$$

Straightforward calculations then give the following expressions for the Bakry-Émery Ricci tensor $\mathrm{Ric}_f = \mathrm{Ric} + \nabla^2 f$:

$$\mathrm{Ric}_f(\partial_s, \partial_s) = \begin{cases} \kappa + \frac{n-1}{r^2} - \frac{n_D}{r^2}, & s \in [0, \frac{\pi r}{2} - \delta], \\ \kappa + \varphi \left(2 - \frac{n}{r^2}\right) \left(\frac{n-1}{r^2} \left(1 + o(\delta)\right) - \frac{n_D}{r^2}\right), & s \in [\frac{\pi r}{2} - \delta, \frac{\pi r}{2} - \delta], \\ \kappa, & s \in [\frac{\pi r}{2} + \delta, D], \end{cases}$$

and

$$\mathrm{Ric}_f(e, e) = \begin{cases} \frac{n-1}{r^2} + \frac{n_D}{r \tan(s/r)}, & s \in [0, \frac{\pi r}{2} - \delta], \\ \frac{n-2}{r^2} + o(\delta), & s \in [\frac{\pi r}{2} - \delta, \frac{\pi r}{2} - \delta], \\ \frac{n-2}{r^2} (1 + o(\delta)), & s \in [\frac{\pi r}{2} + \delta, D], \end{cases}$$

while $\mathrm{Ric}_f(\partial_s, e) = 0$ for any unit vector $e$ tangent to $S^{n-1}$. Clearly, we have $\mathrm{Ric}_f \geq \kappa g$ for $r$ and $\delta$ small enough for any $\kappa \in \mathbb{R}$ if $n \geq 3$ and for $\kappa \leq 0$ if $n = 2$. Moreover, the diameter of $M$ is $D(1 + o(\delta))$.

Finally, we show that for the Bakry-Émery manifold $(M, g, f)$ constructed above, the first nonzero eigenvalue $\lambda_{p,f}$ of $\Delta_{p,f}$ can be made arbitrarily close to $\mu_p(\kappa, D)$ by choosing $r$ and $\delta$ sufficiently small. Let $\psi$ be defined by

$$\psi(z, s) = \begin{cases} w(s - D/2), & s \in [\frac{\pi r}{2} + \delta, D - (\frac{\pi r}{2} + \delta)], \\ w(D/2 - \frac{\pi r}{2} - \delta), & s \in [0, \frac{\pi r}{2} + \delta] \cup [D - \frac{\pi r}{2} - \delta, D] \end{cases},$$

where $w$ is the solution of

$$(p - 1)|w'|^{p-2}w'' - \kappa s|w'|^{p-2}w' + \mu_p(\kappa, D - \pi r - 2\delta)|w|^{p-2}w = 0$$

with $w(0) = 0$, $w'(0) = 1$ and $w'(D/2 - \frac{\pi r}{2} - \delta) = 0$. Plugging the test function $\psi$ into the Rayleigh quotient gives

$$\lambda_{p,f} \leq \frac{\int_M |\nabla \psi|^p e^{-f} d\mu_g}{\int_M |\psi|^p e^{-f} d\mu_g} \leq \frac{\mu_p(\kappa, D - \pi r - \delta) \int_{|s - D/2| \leq D/2 - \frac{\pi r}{2} - \delta/2} |w|^p e^{-f} d\mu_g}{\int_{|s - D/2| \leq D/2} |w|^p e^{-f} d\mu_g} \leq \mu_p(\kappa, D - \pi r - \delta).$$

It follows that $\lambda_{p,f} \to \mu_p(\kappa, D)$ as $r$ and $\delta$ approach zero, thus proving the sharpness of the estimate (1.1) in Theorem 1.1.
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