The Entropy of Lagrange–Finsler Spaces and Ricci Flows

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Abstract

We formulate a statistical analogy of regular Lagrange mechanics and Finsler geometry derived from Grisha Perelman’s functionals and generalized for nonholonomic Ricci flows. Explicit constructions are elaborated when nonholonomically constrained flows of Riemann metrics result in Finsler like configurations, and inversely, when geometric mechanics is modelled on Riemann spaces with a preferred nonholonomic frame structure.

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1 Introduction

The Ricci flow theory became a very powerful method in understanding the geometry and topology of Riemannian manifolds [1 2 3 4] (see also reviews [5 6 7] on Hamilton–Perelman theory of Ricci flows). There
were proposed a number of important innovations in modern physics and mechanics.

Any regular Lagrange mechanics and analogous gravity theory can be naturally geometrized on nonholonomic Riemann manifolds as models of Lagrange, or Finsler, spaces [8, 9], see Refs. [10, 11, 12, 13] for details and applications to modern physics. One of the major goals of geometric mechanics is the study of symmetry of physical systems and its consequences. In this sense, the ideas and formalism elaborated in the Ricci flow theory provide new alternatives for definition of ‘optimal’ geometric configurations and physical interactions.

A Riemannian geometry is defined completely on a manifold provided with a symmetric metric tensor and (uniquely defined to be metric compatible and torsionless) Levi–Civita connection structures. Contrary, the Lagrange and Finsler geometries and their generalizations are constructed from three fundamental and (in general) independent geometric objects: the nonlinear connection, metric and linear connection. Such models were developed when the main geometric structures are derived canonically from a fundamental effective, or explicit, Lagrange (Finsler) function and have an alternative realization as a Riemann geometry with a preferred nonholonomic frame structure. Following such ideas, in Ref. [8], we proved that Ricci flows of Riemannian metrics subjected to nonholonomic constraints may result in effective Finsler like geometries and that any Lagrange–Finsler configuration can be ‘extracted’ from the corresponding nonholonomic deformations of frame structures. An important result is that the G. Perelman’s functional approach [2, 3, 4] to Ricci flows can be redefined for a large class of canonical metric compatible nonlinear and linear connections. For regular Lagrange systems, this allows us not only to derive the evolution equations and establish certain optimal geometric and topological configurations but also to construct canonical statistical and thermodynamical models related to effective mechanical, gravitational or gauge interactions.

The aim of this paper is to analyze the possible applications of the theory of Ricci flows to geometric mechanics and related thermodynamical models. We shall follow the methods elaborated in Sections 1-5 of Ref. [2] generalizing this approach to certain classes of Lagrange and Finsler metrics and connections (see a recent review on the geometry of nonholonomic manifolds and locally anisotropic spaces in Ref. [14, 13]). It should be emphasized that such constructions present not only a geometric extension from the canonical Riemannian spaces to more sophisticate geometries with local anisotropy

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1 for simplicity, in this work we shall consider only smooth and orientable manifolds
but launch a new research program [8, 9, 16, 17] on Ricci flows of geometric and physical objects subjected to nonholonomic constraints.

The paper is organized as follows: In section 2, we outline the main results on metric compatible models of Lagrange and Finsler geometry on nonholonomic manifolds. The G. Perelman’s functional approach to Ricci flow theory is generalized for Lagrange and Finsler spaces in section 3. We derive the evolution equations for Lagrange–Ricci systems in section 4. A statistical approach to Lagrange–Finsler spaces and Ricci flows is proposed in section 5. Finally, we discuss the results in the section 6. Some relevant formulae are presented in the Appendix.

2 Lagrange Mechanics and N–anholonomic Manifolds

Let us consider a manifold $V$, $\dim V = n + m, n \geq 2, m \geq 1$. Local coordinates on $V$ are labelled in the form $u = (x, y)$, or $u^\alpha = (x^i, y^a)$, where indices $i, j, ... = 1, 2, ..., n$ are horizontal (h) ones and $a, b, ... = 1, 2, ..., m$ are vertical (v) ones. We follow our convention to use "boldface" symbols for nonholonomic spaces and geometric objects on such spaces [12, 8, 13, 14]. The typical examples are those when $V = TM$ is a tangent bundle, $V = E$ is a vector bundle on $M$, or $V$ is a (semi–) Riemann manifold, with prescribed local (nonintegrable) fibred structure.

In this work, a nonholonomic manifold $V$ is considered to be provided with a nonitegrable (nonhlonomic) distribution defining a nonlinear connection (N–connection). This is equivalent to a Whitney sum of conventional h– and v–subspaces, $hV$ and $vV$,

$$TV = hV \oplus vV,$$

where $TV$ is the tangent bundle. Such manifolds are called, in brief, N–anholonomic (in literature, one uses two equivalent terms, nonholonomic and anholonomic). Locally, a N–connection is defined by its coefficients, $N = \{N^a_i\}$, stated with respect to a local coordinate basis, $N = N^a_i(u)dx^i \otimes \partial/\partial y^a$. We can consider the class of linear connections when $N^a_i(u) = \Gamma^a_{ib}(x)y^b$ as a particular case.

N–connections are naturally considered in Finsler and Lagrange geometry [10, 11, 12, 13]. They are related to (semi) spray configurations

$$\frac{dy^a}{ds} + 2G^a(x, y) = 0,$$

where $G^a(x, y)$ is the spray form of the Finsler function.
of a curve \( x^i(\varsigma) \) with parameter \( 0 \leq \varsigma \leq \varsigma_0 \), when \( y^i = dx^i/d\varsigma \) [spray configurations are obtained for integrable equations]. For a regular Lagrangian \( L(x, y) = L(x^i, y^a) \) modelled on \( V \), when the Lagrange metric (equivalently, Hessian) 
\[
L_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j} 
\]
is not degenerate, i.e. \( \det |g_{ij}| \neq 0 \), one finds the fundamental result (proof is a straightforward computation):

**Theorem 2.1** For \( 4G^j = L_{ij} \left( \frac{\partial^2 L}{\partial y^j \partial x^i} y^k - \frac{\partial L}{\partial x^i} \right) \), with \( L_{ij} \) inverse to \( L_{ij} \), the "nonlinear" geodesic equations \((2)\) are equivalent to the Euler–Lagrange equations 
\[
\frac{d}{d\varsigma} \left( \frac{\partial L}{\partial y^i} \right) - \frac{\partial L}{\partial x^i} = 0.
\]

Originally, the Lagrange geometry was elaborated on the tangent bundle \( TM \) of a manifold \( M \), for a regular Lagrangian \( L(x, y) \) following the methods of Finsler geometry \([10, 11]\) (Finsler configurations can be obtained in a particular case when \( L(x, y) = F^2(x, y) \) for a homogeneous fundamental function \( F(x, \lambda y) = \lambda F(x, y), \lambda \in \mathbb{R} \)). Lagrange and Finsler geometries can be also modelled on N–anholonomic manifolds \([12, 13]\) provided, for instance, with canonical N–connection structure

\[
N^a_i = \frac{\partial G^a}{\partial y^i}.
\]

**Proposition 2.1** A N–connection defines certain classes of nonholonomic preferred frames and coframes,

\[
e^a_\alpha = \left[ e_i = \frac{\partial}{\partial x^i} - N^a_i(u) \frac{\partial}{\partial y^a}, e_b = \frac{\partial}{\partial y^b} \right] \quad (5)
e^a = e^i dx^i, e^a = dy^a + N^a_i(x, y) dx^i. \quad (6)
\]

**Proof.** One computes the nontrivial nonholonomy coefficients \( W_{ib}^a = \partial N^a_i/\partial y^b \) and \( W_{ij}^a = \Omega_{ji}^a = e_i N^a_j - e_j N^a_i \) (where \( \Omega_{ij}^a \) are the coefficients of the N–connection curvature) for

\[
[e_\alpha, e_\beta] = e_\alpha e_\beta - e_\beta e_\alpha = W_{\alpha\beta}^\gamma e_\gamma. \quad (7)
\]

□

One holds:
Claim 2.1. Any regular Lagrange mechanics $L(x,y) = L(x^i,y^a)$ modelled on $V$, $\dim V = 2n$, defines a canonical metric structure

$$L_g = L_{ij}(x,y) [e^i \otimes e^j + e^i \otimes e^j].$$

Proof. For $V = TM$, the metric (8) is just the Sasaki lift of (3) on total space $[10]$. In abstract form, such canonical constructions can be performed similarly for any N–anholonomic manifold $V$. This approach to geometric mechanics follows from the fact that the (semi) spray configurations are related to the N–connection structure and defined both by the Lagrangian fundamental function and the Euler–Lagrange equations, see Theorem 2.1. □

Definition 2.1. A distinguished connection (d–connection) $D$ on $V$ is a linear connection preserving under parallel transports the Whitney sum (1).

In order to perform computations with d–connections we can use N–adapted differential forms like $\Gamma_{\alpha\beta} = \Gamma_{\alpha\beta\gamma} e^\gamma$ with the coefficients defined with respect to (6) and (5) and parametrized $\Gamma_{\alpha\beta} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc})$. The torsion of a d–connection is computed

$$T^\alpha = De^\alpha = de^\alpha + \Gamma^\alpha_{\beta\gamma} e^\gamma.$$

Locally, it is characterized by (N–adapted) d–torsion coefficients

$$T^i_{jk} = L^i_{jk} - L^i_{kj}, \quad T^i_{ja} = -T^i_{aj} = C^i_{ja}, \quad T^i_{ji} = \Omega^i_{ji},$$

$$T^a_{bi} = -T^a_{ib} = \frac{\partial N^a}{\partial y^b} - L^a_{bi}, \quad T^a_{bc} = C^a_{bc} - C^a_{cb}.$$ (10)

Theorem 2.2. There is a unique canonical d–connection $\hat{D} = \{\hat{\Gamma}^\alpha_{\beta\gamma} = (\hat{L}^i_{jk}, \hat{L}^a_{bk}, \hat{C}^i_{jc}, \hat{C}^a_{bc})\}$ which is metric compatible with the Lagrange canonical metric structure, $D (L_g) = 0$, and satisfies the conditions $\hat{T}^i_{jk} = \hat{T}^a_{bc} = 0$.

Proof. It follows from explicit formulas (8) and (1) and (A.1). □

A geometric model of Lagrange mechanics can be elaborated in terms of Riemannian geometry on $V$, as a noholonomic Riemann space, if we chose the Levi–Civita connection $\nabla = \{\Gamma^\gamma_{\alpha\beta}\}$ defined uniquely by the Lagrange metric $L_g$ but such constructions are not adapted to the N–connection splitting (1) induced by the (semi) spray Lagrange configuration. In an equivalent form, such constructions can be adapted to the N–connection structure.
if the canonical distinguished connection $\hat{\mathbf{D}} = \{\hat{\Gamma}^\gamma_{\alpha\beta}\}$ is considered. In this case, the geometric space is of Riemann–Cartan type, with nontrivial torsion induced by the N–connection coefficients under nonholonomic deformations of the frame structure.

**Conclusion 2.1** Any regular Lagrange mechanics (Finsler geometry) can be modelled in two equivalent canonical forms as a nonholonomic Riemann space or as a N–anholonomic Riemann–Cartan space with the fundamental geometric objects (metric and connection structures) defined by the fundamental Lagrange (Finsler) function.

Inverse statements when (semi) Riemannian metrics are modelled by certain effective Lagrange structures and corresponding Ricci flows also hold true but in such cases one has to work with models of generalized Lagrange geometry, see [8, 10, 11].

**Remark 2.1** One considers different types of d–connection structures in Finsler geometry. For instance, there is an approach based on the so–called Chern connection [15] which is not metric compatible and considered less suitable for applications to standard models in modern physics, see discussion in [13, 8].

For convenience, in Appendix, we outline the main formulas for the connections $\nabla$ and $\hat{\mathbf{D}}$ and their torsions, curvature and Ricci tensors.

### 3 The Perelman’s Functionals on Lagrange and Finsler Spaces

The Ricci flow equation was originally introduced by R. Hamilton [1] as an evolution equation

$$\frac{\partial g_{\alpha\beta}(\chi)}{\partial \chi} = -2 \cdot R_{\alpha\beta}(\chi)$$  \hspace{1cm} (11)

for a set of Riemannian metrics $g_{\alpha\beta}(\chi)$ and corresponding Ricci tensors $R_{\alpha\beta}(\chi)$ parametrized by a real parameter $\chi$. The Ricci flow theory is a
branch of mathematics elaborated in connection to rigorous study of topological and geometric properties of such equations and possible applications in modern physics.

In the previous section, see also related details in our works \[8, 13\], we proved that the Lagrange–Finsler geometries can be modelled as constrained structures on N–anholonomic Riemannian spaces. We concluded there that the Ricci flows of regular Lagrange systems (Finsler metrics) can be described by usual Riemann gradient flows but subjected to certain classes of nonholonomic constraints. It should be also noted that, inversely, it is possible to extract from respective nonholonomic Riemannian configurations the Lagrange or Finsler ones. Working with the canonical d–connection \(\hat{\mathbf{D}}\), we get a Ricci tensor \(\Lambda_6\) which, in general, is not symmetric but the metric \(g\) is symmetric. In such cases, we are not able to derive the equation \((11)\) on nonholonomic spaces in a self–consistent heuristic form following the analogy of R. Hamilton’s equations and the Einstein’s equations. We emphasize that one can be considered flows of nonholonomic Einstein spaces, when \(\hat{\mathbf{R}}_{\alpha\beta}\) is symmetric (we investigated such solutions in Refs. \[9, 16, 17\]), but more general classes of solution of the Ricci equations with nonholonomic constraints would result in nonsymmetric metrics, see discussions in Ref. \[8\].

The Grisha Perelman’s fundamental idea was to prove that the Ricci flow is not only a gradient flow but also can be defined as a dynamical system on the spaces of Riemannian metrics by introducing two Lyapunov type functionals. In this section, we show how the constructions can be generalized for N–anholonomic manifolds if we chose the connection \(\hat{\mathbf{D}}\).

The Perelman’s functionals were introduced for Ricci flows of Riemannian metrics. For the Levi–Civita connection defined by the Lagrange metric \(\mathbf{g}\), are written in the form

\[\mathcal{F}(\mathbf{L}, f) = \int_{\mathbf{V}} \left( \mathcal{L}_{\mathbf{g}} + |\nabla f|^2 \right) e^{-f} \, dV,\]

and

\[\mathcal{W}(\mathbf{L}, f, \tau) = \int_{\mathbf{V}} \left[ \tau \left( \mathcal{L}_{\mathbf{g}} + |\nabla f|^2 \right) + f - 2n \right] \mu \, dV,\]

where \(dV\) is the volume form of \(\mathbf{L}_{\mathbf{g}}\), integration is taken over compact \(\mathbf{V}\), function \(f\) is introduced in order to have the possibility to consider gradient flows with different measures, see details in \[2\], and \(\mathcal{L}_{\mathbf{g}}\) is the scalar curvature computed for \(\nabla\). For \(\tau > 0\), we have \(\int_{\mathbf{V}} \mu dV = 1\) when \(\mu = (4\pi\tau)^{-n} e^{-f}\).

\[3\] In our works \[13, 13\], we use left "up" and "low" indices as labels for some geometric/
The functional approach can be redefined for N–anholonomic manifolds:

Claim 3.1 For Lagrange spaces, the Perelman’s functionals for the canonical d–connection \( \hat{D} \) are defined

\[
\mathcal{F}(L, \hat{f}) = \int \left( R + S + |\hat{D}\hat{f}|^2 \right) e^{-\hat{f}} \, dV, \tag{13}
\]

\[
\mathcal{W}(L, \hat{f}, \tau) = \int \left[ \hat{\tau} \left( R + S + |^hD\hat{f}| + |^vD\hat{f}| \right)^2 + \hat{f} - 2n \right] \hat{\mu} \, dV, \tag{14}
\]

where \( dV \) is the volume form of \( ^Lg \), \( R \) and \( S \) are respectively the \( h \)- and \( v \)-components of the curvature scalar of \( \hat{D} \), see (A.7), for \( \hat{D} = (D_v, D_h) \), or \( \hat{D} = (hD, vD), |\hat{D}\hat{f}|^2 = |hD\hat{f}|^2 + |vD\hat{f}|^2 \), and \( \hat{f} \) satisfies \( \int V \hat{\mu} dV = 1 \) for \( \hat{\mu} = (4\pi\tau)^{-n} e^{-\hat{f}} \) and \( \tau > 0 \).

Proof. We can redefine equivalently the formulas (12) for some \( \hat{f} \) and \( f \) (which can be a non–explicit relation) when

\[
\tau(R + |\nabla f|^2) e^{-f} = \left( R + S + |^hD\hat{f}|^2 + |^vD\hat{f}|^2 \right) e^{-\hat{f}} + \Phi
\]

and re–scale the parameter \( \tau \to \hat{\tau} \) to have

\[
\int V \hat{\tau} (R + |\nabla f|^2 + f - 2n) \mu = \int V \hat{\tau} (R + S + |^hD\hat{f}| + |^vD\hat{f}|)^2 + \hat{f} - 2n |\hat{\mu} + \Phi_1
\]

for some \( \Phi \) and \( \Phi_1 \) for which \( \int V \Phi dV = 0 \) and \( \int V \Phi_1 dV = 0 \).

Elaborating a N–adapted variational calculus, we shall consider both variations in the so–called h– and v–subspaces as defined by the decompositions (1). We write, for simplicity, \( g_{ij} = ^Lg_{ij} \) and consider the \( h \)-variation \( ^h\delta g_{ij} = v_{ij} \), the \( v \)-variation \( ^v\delta g_{ab} = v_{ab} \), for a fixed N–connection structure in (8), and \( ^h\delta \hat{f} = ^h f, ^v\delta \hat{f} = ^v f \).

Lemma 3.1 The first N–adapted variations of (13) are given by

\[
\delta \mathcal{F}(v_{ij}, v_{ab}, ^h f, ^v f) = \int V \left[ -v_{ij}(R_{ij} + D_iD_j\hat{f}) + \left( 2 ^h f - ^h f \right) \left( 2 ^h \Delta \hat{f} - |^h D \hat{f}| \right) + R \right]
\]

\[
+ [-v_{ab}(R_{ab} + D_aD_b\hat{f}) + \left( 2 ^v f - ^v f \right) \left( 2 ^v \Delta \hat{f} - |^v D \hat{f}| \right) + S] \right] e^{-\hat{f}} dV
\]

physical objects, for instance, in order to emphasize that such values are induced by a Lagrangian, ot defined by the Levi-Civita connection.
where $\tilde{\Delta} = h\Delta + v\Delta$, $h\Delta = D_iD^i$, $v\Delta = D^aD_a$, $h_v = g^{ij}v_{ij}$, $v_v = g^{ab}v_{ab}$.

**Proof.** It is a N–adapted calculus similar to that for Perelman’s Lemma in [2]. We omit details given, for instance, in the proof from [5], see there Lemma 1.5.2, but we note that if such computations are performed on a N–anholonomic manifold, the canonical $d$–connection results in formulas (A.6), for the Ricci curvature and (A.7), for the scalar curvature of $\tilde{D}$. It should be emphasized that because we consider that variations of a symmetric metric, $h\delta g_{ij} = v_{ij}$ and $v\delta g_{ab} = v_{ab}$, are considered independently on $h$– and $v$–subspaces and supposed to be also symmetric, we get in (15) only the symmetric coefficients $R_{ij}$ and $R_{ab}$ but not $R_{ai}$ and $R_{ia}$. Admitting nonsymmetric variations of metrics, we would obtain certain terms in $\delta\tilde{F}(v_{ij}, v_{ab}, h_f, v_f)$ defined by the nonsymmetric components of the Ricci tensor for $\tilde{D}$. In this work, we try to keep our constructions on Riemannian spaces, even they are provided with N–anholonomic distributions, and avoid to consider the so–called Lagrange–Eisenhart, or Finsler–Eisenhart, geometry analyzed, for instance in Chapter 8 of monograph [10] (for nonholonomic Ricci flows, we discuss the problem in [3]). □

4 Evolution Equations for Lagrange Systems

The normalized (holonomic) Ricci flows, see details in Refs. [2] [5] [6] [7], with respect to the coordinate base $\partial_\alpha = \partial/\partial u^\alpha$, are described by the equations

$$
\frac{\partial}{\partial \chi} g_{\alpha\beta} = -2\tilde{R}_{\alpha\beta} + \frac{2r}{5} g_{\alpha\beta}, \tag{16}
$$

where the normalizing factor $r = \int RdV/dV$ is introduced in order to preserve the volume $V$.\footnote{We underline the indices with respect to the coordinate bases in order to distinguish them from those defined with respect to the ’N–elongated’ local bases [3] and [5].} We note that here we use the Ricci tensor $\tilde{R}_{\alpha\beta}$ and scalar curvature $\tilde{R} = g^{\alpha\beta}\tilde{R}_{\alpha\beta}$ computed for the connection $\nabla$. The coefficients $g_{\alpha\beta}$ are those for a family of metrics $Lg(\chi), \ Lg(\chi), \ Lg(\chi), \ Lg(\chi)$, rewritten with respect to the coordinate basis, $Lg(\chi) = g_{\alpha\beta}(\chi)du^\alpha \otimes du^\beta$, where

$$
g_{\alpha\beta}(\chi) = \begin{bmatrix}
 g_{ij}(\chi) = g_{ij} + N^a_i N^b_j g_{ab} & g_{ib}(\chi) = N^e_i g_{be} \\
 g_{ij}(\chi) = N^e_j g_{be} & g_{ab}(\chi) = N^e_a g_{be}
\end{bmatrix}, \tag{17}
$$

for $g_{\alpha\beta}(\chi) = g_{\alpha\beta}(\chi)$, when $g_{ij}(\chi) = Lg_{ij}(\chi, u), g_{ab}(\chi) = Lg_{ab}(\chi, u)$ and $N^a_i(\chi) = N^a_i(\chi, u)$ defined from a set of Lagrangians $L(\chi, u)$, respectively.
by formulas (3) and (4).

With respect to the N–adapted frames (5) and (6), when

\[ e_\alpha(\chi) = e_{a\alpha}(\chi) \partial_a \quad \text{and} \quad e^\alpha(\chi) = e^{\alpha}_{a\alpha}(\chi) du^a, \]

the frame transforms are respectively parametrized in the form

\[
e_{a\alpha}(\chi) = \begin{bmatrix}
  e^i_i = \delta^i_i & e^i_a = N^b_i(\chi) \delta^a_b \\
  e^i_a = 0 & e^a_a = \delta^a_a
\end{bmatrix}, \tag{18}
\]

\[
e^{\alpha}_{a\alpha}(\chi) = \begin{bmatrix}
  e^i_i = \delta^i_i & e^b_i = -N^b_k(\chi) \delta^i_k \\
  e^i_a = 0 & e^a_a = \delta^a_a
\end{bmatrix},
\]

where \( \delta^i_i \) is the Kronecker symbol, the Ricci flow equations (16) are

\[
\frac{\partial g_{ij}}{\partial \chi} = 2\left[N^a_i N^b_j (\hat{R}_{ab} - \lambda g_{ab}) - \hat{R}_{ij} + \lambda g_{ij}\right] - g_{cd} \frac{\partial}{\partial \chi} (N^c_i N^d_j), \tag{19}
\]

\[
\frac{\partial g_{ab}}{\partial \chi} = -2R_{ab} + 2\lambda g_{ab}, \tag{20}
\]

\[
\frac{\partial}{\partial \chi} (N^{c}_{j} g_{ae}) = -2 \hat{R}_{ia} + 2\lambda N^{c}_{j} g_{ae}, \tag{21}
\]

where \( \lambda = r/5 \) and the metric coefficients are defined by the ansatz (17).

If \( \nabla \rightarrow \hat{\nabla} \), we have to change \( R_{\alpha\beta} \rightarrow \hat{R}_{\alpha\beta} \) in (19)–(21). The N–adapted evolution equations for Ricci flows of symmetric metrics, with respect to local coordinate frames, are written

\[
\frac{\partial g_{ij}}{\partial \chi} = 2\left[N^a_i N^b_j (\hat{\hat{R}}_{ab} - \lambda g_{ab}) - \hat{\hat{R}}_{ij} + \lambda g_{ij}\right] - g_{cd} \frac{\partial}{\partial \chi} (N^c_i N^d_j), \tag{22}
\]

\[
\frac{\partial g_{ab}}{\partial \chi} = -2(\hat{\hat{R}}_{ab} - \lambda g_{ab}), \tag{23}
\]

\[
\hat{\hat{R}}_{ia} = 0 \quad \text{and} \quad \hat{\hat{R}}_{ai} = 0, \tag{24}
\]

where the Ricci coefficients \( \hat{\hat{R}}_{ij} \) and \( \hat{\hat{R}}_{ab} \) are computed with respect to coordinate coframes, being frame transforms (18) of the corresponding formulas (A.6) defined with respect to N–adapted frames. The equations (24) constrain the nonholonomic Ricci flows to result in symmetric metrics.

The aim of this section is to prove that equations of type (22) and (23) can be derived from the Perelman’s N–adapted functionals (13) and (14) (for simplicity, we shall not consider the normalized term and put \( \lambda = 0 \)).
Definition 4.1 A metric \( Lg \) generated by a regular Lagrangian \( L \) evolving by the (nonholonomic) Ricci flow is called a (nonholonomic) breather if for some \( \chi_1 < \chi_2 \) and \( \alpha > 0 \) the metrics \( \alpha Lg(\chi_1) \) and \( \alpha Lg(\chi_2) \) differ only by a diffeomorphism (in the \( N \)-anholonomic case, preserving the Whitney sum \( \{v\} \)). The cases \( \alpha =, <, > \) define correspondingly the steady, shrinking and expanding breathers (for \( N \)-anholonomic manifolds, one can be the situation when, for instance, the \( h \)-component of metric is steady but the \( v \)-component is shrinking).

Clearly, the breather properties depend on the type of connections are used for definition of Ricci flows.

Following a \( N \)-adapted variational calculus for \( \tilde{F}(L, \tilde{f}) \), see Lemma 3.1, with Laplacian \( \tilde{\Delta} \) and \( h \)– and \( v \)–components of the Ricci tensor, \( \tilde{R}_{ij} \) and \( \tilde{S}_{ij} \), defined by \( \tilde{D} \) and considering parameter \( \tau(\chi), \partial \tau/\partial \chi = -1 \), we prove

Theorem 4.1 The Ricci flows of regular Lagrange mechanical systems are characterized by evolution equations

\[
\frac{\partial g_{ij}}{\partial \chi} = -2\tilde{R}_{ij}, \quad \frac{\partial g_{ab}}{\partial \chi} = -2\tilde{R}_{ab}, \quad \frac{\partial \tilde{f}}{\partial \chi} = -\tilde{\Delta} \tilde{f} + \left| \tilde{D} \tilde{f} \right|^2 - R - S
\]

and the property that, for constant \( \int_V e^{-\tilde{f}} dV \),

\[
\frac{\partial}{\partial \chi} \tilde{F}(Lg(\chi), \tilde{f}(\chi)) = 2 \int_V \left[ \left| \tilde{R}_{ij} + D_i D_j \tilde{f} \right|^2 + \left| \tilde{R}_{ab} + D_a D_b \tilde{f} \right|^2 \right] e^{-\tilde{f}} dV.
\]

Proof. For Riemannian spaces, a proof was proposed by G. Perelman [2] (details of the proof are given for the connection \( \nabla \) in the Proposition 1.5.3 of [5], they can be similarly reproduced for the canonical \( d \)-connection \( \tilde{D} \)). For \( N \)-anholonomic spaces, we changed the status of such statements to a Theorem because for nonholonomic configurations there are not alternative ways of definition Ricci flow equations in \( N \)-adapted form following two different, heuristic and functional, approaches. The functional variant became the unique possibility for a rigorous proof containing \( N \)-adapted calculus. Finally, we note that for the Levi–Civita connection the functional \( F \) is nondecreasing in time and the monotonicity is strict unless we are on a steady gradient soliton (see, for instance, Ref. [5] for details on solitonic solutions and Ricci flows). This property sure depend on the type of connection is used and how solitons are defined. We shall not use it in this works and omit such considerations.□
The priority of the N–adapted calculus for the canonical d–connection \( \hat{D} = (hD, vD) \) is that from formal point of view we work as in the case with the connection \( \nabla \) but have to dub the results for the h– and v–components and redefine them with respect N–adapted bases. This analogy holds true for all (generalized) and Lagrange and Finsler metrics because \( \hat{D} \) is metric compatible and uniquely defined by the coefficients of \( Lg \), similarly to \( \nabla \).

It should be noted that even a closed formal analogy of formulas exist, the evolution equations, their solutions, and related geometrical and fundamental objects are different because \( \hat{D} \neq \nabla \). Following this property, we can formulate (the reader may check that its statements and proofs consist a N–adapted modification of Proposition 1.5.8 in [5] containing the details of the original result from [2]):

**Theorem 4.2** If a regular Lagrange (Finsler) metric \( Lg(\chi) \) and functions \( \hat{f}(\chi) \) and \( \hat{\tau}(\chi) \) evolve for \( \frac{\partial}{\partial \chi} \) = \(-1 \) and constant \( \int (4\pi\hat{\tau})^{-n}e^{-\hat{f}}dV \), as solutions of the system

\[
\frac{\partial g_{ij}}{\partial \chi} = -2\hat{R}_{ij}, \quad \frac{\partial g_{ab}}{\partial \chi} = -2\hat{R}_{ab}, \quad \frac{\partial \hat{f}}{\partial \chi} = -\Delta \hat{f} + \left| \hat{D}\hat{f} \right|^2 - R - S + \frac{n}{\tau},
\]

one holds the equality

\[
\frac{\partial}{\partial \chi} \hat{W}(Lg(\chi), \hat{f}(\chi), \hat{\tau}(\chi)) = 2\int _{\nabla} \hat{\tau}[|\hat{R}_{ij} + D_iD_j\hat{f} - \frac{1}{2\tau}g_{ij}|^2 +
\left|\hat{R}_{ab} + D_aD_b\hat{f} - \frac{1}{2\tau}g_{ab}|^2](4\pi\hat{\tau})^{-n}e^{-\hat{f}}dV.
\]

For the Levi–Civita connection \( \nabla \), the functional \( \hat{W}(Lg(\chi), f(\chi), \tau(\chi)) \) is nondecreasing in time and the monotonicity is strict unless we are on a shrinking gradient soliton. Similar properties can be formulated in N–adapted form, but it is not obvious if some of them hold true for \( \nabla \) they will be preserved for \( \nabla \to \hat{D} \).

The Lagrange–Ricci flows are are characterized by the evolutions of preferred N–adapted frames [18] (see proof in [8]):

**Corollary 4.1** The evolution, for all time \( \tau \in [0, \tau_0] \), of preferred frames on a Lagrange space \( e_\alpha(\tau) = e_\alpha^\beta(\tau, u)\partial_\alpha \) is defined by the coefficients

\[
e_\alpha^\beta(\tau, u) = \left[
\begin{array}{ccc}
e_i^L(\tau, u) & N_i^b(\tau, u) \ e_b^a(\tau, u) \\
0 & e_a^a(\tau, u)
\end{array}
\right],
\]
with \( L_{g_{ij}}(\tau) = e_i(\tau, u) e_j(\tau, u) \eta_{ij} \), where \( \eta_{ij} = \text{diag}[\pm 1, \ldots, \pm 1] \) states the signature of \( L_{g_{[0]}}(u) \), is given by equations

\[
\frac{\partial}{\partial \tau} e_\alpha^\alpha = L_{g^{\alpha\beta}} R_{\beta\gamma} e_\gamma^\alpha, \quad \text{for the Levi-Civita connection ;}
\]

\[
\frac{\partial}{\partial \tau} e_\alpha^\alpha = L_{g^{\alpha\beta}} \hat{R}_{\beta\gamma} e_\gamma^\alpha, \quad \text{for the canonical d–connection .}
\]

It should be emphasized that it would be a problem to prove directly the results of this section for Ricci flows of Finsler spaces with metric non-compatible d–connections like in Ref. [15]. Nevertheless, our proofs can be generalized also for nonmetric Lagrange–Finsler configurations if the nonmetricity is completely defined by the coefficients of the d–metric and N–connection structures. In such cases, we can prove the theorems and consequences as for metric compatible cases (for the Levi–Civita connection and/or Cartan d–connection) and then to distort the formulas in unique forms using corresponding deformation tensors.

5 Statistical Analogy for Lagrange–Finsler Spaces and Ricci Flows

Grisha Perelman showed that the functional \( \mathcal{W} \) is in a sense analogous to minus entropy [2]. We show that this property holds true for nonholonomic Ricci flows which provides a statistical model for regular Lagrange (Finsler) systems.

The partition function \( Z = \int \exp(-\beta E) d\omega(E) \) for the canonical ensemble at temperature \( \beta^{-1} \) is defined by the measure taken to be the density of states \( \omega(E) \). The thermodynamical values are computed in the form: the average energy, \( < E > = -\partial \log Z / \partial \beta \), the entropy \( S = \beta < E > + \log Z \) and the fluctuation \( \sigma = (E - < E >)^2 = \partial^2 \log Z / \partial \beta^2 \).

Let us suppose that a set of regular mechanical systems with Lagrangians \( L(\tau, x, y) \) is described by respective metrics \( L_{g}(\tau) \) and N–connection \( N^a_i(\tau) \) and related canonical linear connections \( \nabla(\tau) \) and \( \hat{D}(\tau) \) subjected to the conditions of Theorem 4.2. One holds

**Theorem 5.1** Any family of regular Lagrange (Finsler) geometries satisfying the evolution equations for the canonical d–connection is characterized
by thermodynamic values

\[
< \hat{E} > = -\tau^2 \int_V \left( R + S + \left| \nabla^\tau \hat{f} \right|^2 + \left| v \nabla^\tau \hat{f} \right|^2 - \frac{n}{\tau} \right) \hat{\mu} \, dV,
\]

\[
\hat{S} = -\int_V \left[ \tau \left( R + S + \left| \nabla^\tau \hat{f} \right|^2 + \left| v \nabla^\tau \hat{f} \right|^2 \right) + \hat{f} - 2n \right] \hat{\mu} \, dV,
\]

\[
\hat{\sigma} = 2 \tau^4 \int_V \left[ |\hat{R}_{ij} + D_i D_j \hat{f} - \frac{1}{2\tau} g_{ij}|^2 + |\hat{R}_{ab} + D_a D_b \hat{f} - \frac{1}{2\tau} g_{ab}|^2 \right] \hat{\mu} \, dV.
\]

**Proof.** It follows from a straightforward computation for \( \hat{Z} = \exp\{ \int_V [-\hat{f} + n] \hat{\mu} dV \} \). We note that similar values \( < \hat{E} >, \hat{S}, \text{ and } \hat{\sigma} \) can be computed for the Levi–Civita connection \( \nabla \) also defined for the metric \( \hat{g} \), see functionals (12). □

This results in

**Corollary 5.1** A \( N \)-anholonomic Lagrange (Finsler) model defined by the canonical \( d \)-connection \( \hat{D} \) is thermodynamically more (less, equivalent) convenient than a similar one defined by the Levi–Civita connection \( \nabla \) if \( \hat{S} < S \) \( (\hat{S} > S) \).

Following this Corollary, we conclude that such models are positively equivalent for integrable \( N \)-anholonomic structures with vanishing distortion tensor (see formulas (A.2) and (A.3)). For such holonomic structures, the anholonomy coefficients \( W_\gamma^{\alpha\beta} (7) \) are zero and we can work only with the Levi–Civita connection. There are necessary explicit computations of the thermodynamical values for different classes of exact solutions of non-holonomic Ricci flow equations [9, 16, 17] or of the Einstein equations with nonholonomic/ noncommutative variables [13] in order to conclude which configurations are thermodynamically more convenient for \( N \)-anholonomic or (pseudo) Riemannian configurations. In certain cases, some constrained (Finsler like, or more general) configurations may be more optimal than the Levi–Civita ones.

Finally, we would like to mention that there were elaborated alternative approaches to geometric and non–equilibrium thermodynamics, locally anisotropic kinetics and kinetic processes elaborated in terms of Riemannian and Finsler like objects on phase and thermodynamic spaces, see reviews of results and bibliography in Ref. [13]. Those models are not tailor-made for Ricci flows of geometric...
objects and seem not to be related to the statistical thermodynamics of metrics and connections which can be derived from (an) holonomic Perelman’s functionals. In a more general context, the ”Ricci flow thermodynamics” seem to be related to ”non-extensive” Tsallis statistics which is valid for non–equilibrium cases and is considered to be ”more fundamental” than the equilibrium Boltzman–Gibbs statistics, see Ref. [24] and references therein.

6 Conclusion and Discussion

In this paper, we have introduced an extension of Prelman’s functional approach to Ricci flows [2] in order to derive in canonical form the evolution equations for Lagrange and Finsler geometries and formulate a statistical analogy of regular mechanical systems. This scheme is of practical applicability to the problem of the definition of the most optimal geometric and topological configurations in geometric mechanics and analogous models of field interactions. In this context, we elaborate a new direction to geometrization of Lagrange systems following the theory of nonholonomic Ricci flows and generalized Riemann–Cartan and Lagrange–Finsler spaces equipped with compatible metric, nonlinear connection and linear connection structures [9, 16, 17, 13].

Since the initial works on Ricci flows [1, 5, 6, 7], the problem of definition of evolution equations of fundamental geometric objects was treated in a heuristic form following certain analogy with the original 'proof' of the Einstein equations when a symmetric Ricci tensor was set to be proportional to a 'simple' and physically grounded combination of coordinate/parametric derivatives of metric coefficients. In our works [13], we proved that Finsler like geometries can be modelled by preferred nonholonomic frame structures even as exact solutions in the Einstein and string gravity and has analogous interpretations in terms of geometric objects on generalized Lagrange spaces and nonholonomic manifolds [10, 11, 13]. Then, it was shown that flows of Lagrange–Finsler geometries can be extracted from flows of Riemannian metrics by imposing certain classes of nonholonomic constraints and deformations of the frame and linear connection structures [8].

In order to derive the first results on Lagrange–Ricci, or Finsler–Ricci flows, in a form more familiar to researches skilled in geometric analysis and Riemannian geometry, we worked in the bulk with the Levi–Civita connection for Lagrange, or Finsler, metrics and then sketched how the results can be redefined in terms of the canonical connections for 'locally isotropic' geometries. The advantage of the Perelman’s approach to the Ricci flow theory
is that it can easily be reformulated for a covariant calculus adapted to the nonlinear connection structure which is of crucial importance in generalized Riemann–Finsler geometry. For such geometries, the functional methods became a strong tool both for rigorous proofs of the nonholonomic evolution equations and formulating new alternative statistical models for regular Lagrange systems.

The two approaches are complementary in the following sense: the functional scheme gives more rigorous results when the type of geometric structures are prescribed and the holonomic or nonholonomic Ricci flows and the related statistical/thermodynamical models are constructed in the same class of geometries, whereas the heuristic ideas and formulas are best adapted for flow transitions from one type of geometries to another ones (for instance, from Finsler configurations to Riemannian ones, and inversely).

The next challenge in our program on nonholonomic Ricci flows and applications is to formulate a functional formalism for general nonholonomic manifolds in a form, when various type of nonholonomic Clifford, algebroid, noncommutative, solitonic ... structures can be extracted from flows of 'Riemannian' geometrical objects by imposing the corresponding classes of nonholonomic constraints and deformations of geometric objects. We discuss such results and provide a more detailed list of references on Ricci flows and applications to modern classical and quantum physics in our recent work [9, 16, 17, 25, 26, 27] (see references therein).

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Appendix

One exists a minimal extension of the Levi–Civita connection $\nabla$ to a canonical d–connection $\hat{D}$ which is defined only the coefficients of Lagrange metric $L g$ and canonical nonlinear connection $N^a_i$ which is also metric compatible, with $\hat{T}^i_{jk} = 0$ and $\hat{T}^a_{bc} = 0$, but $\hat{T}^i_{ja}, \hat{T}^a_{ji}$ and $\hat{T}^a_{bi}$ are not zero, see (10). The coefficient $\hat{\Gamma}^\gamma_{\alpha\beta} = (\hat{L}^i_{jk}, \hat{L}^a_{bk}, \hat{C}^i_{jc}, \hat{C}^a_{bc})$ of this connection, with
where the explicit components of the distortion tensor $\bar{\xi}_{ab}$ parametrize the coefficients in the form

$$L^a_{bc} = e_b N^a_c + \frac{1}{2} g^{ac} \left( e_k g_{bc} - g_{dc} e_b N^d_k - g_{db} e_c N^d_k \right),$$

$$C^i_{jc} = \frac{1}{2} g^{ik} e_c jk, \quad \bar{C}^a_{bc} = \frac{1}{2} g^{ad} (e_c g_{bd} + e_c g_{cd} - e_d g_{bc}),$$

where, for simplicity, we write $g_{jr}$ and $g_{bd}$ without label "L" we used for Hessian $L g_{ij}$. The Levi–Civita linear connection $\nabla = \{ \Gamma^\alpha_{\beta\gamma} \}$, uniquely defined by the conditions $\nabla g = 0$, is not adapted to the distribution $(1)$. Let us parametrize the coefficients in the form

$$\Gamma^\alpha_{\beta\gamma} = \left( \begin{array}{c} L^i_{jk}, \ L^a_{jk}, \ L^i_{bk}, \ L^a_{bk}, \ C^i_{jb}, \ C^a_{jb}, \ C^i_{bc}, \ C^a_{bc} \end{array} \right),$$

$$\nabla e_k (e_j) = L^i_{jk} e_i + L^a_{jk} e_a, \quad \nabla e_k (e_b) = L^i_{bk} e_i + L^a_{bk} e_a, \quad \nabla e_c (e_j) = C^i_{jc} e_i + C^a_{jc} e_a, \quad \nabla e_c (e_b) = C^i_{bc} e_i + C^a_{bc} e_a.$$

It is convenient to express

$$\Gamma^\alpha_{\beta\gamma} = \hat{\Gamma}^\alpha_{\beta\gamma} + Z^\alpha_{\beta\gamma}$$

where the explicit components of the distortion tensor $Z^\alpha_{\beta\gamma}$ are computed

$$Z^i_{bk} = \frac{1}{2} \Omega^c_{jk} g_{bc} g^{ij} - q_{jk}^{ih} C^j_{hb}, \quad Z^a_{jk} = - \frac{1}{2} q_{cb} e_j e_c = 0, \quad Z^a_{bc} = 0,$$

$$Z^i_{ab} = - \frac{g^{ij}}{2} \left( g_{cb} \Xi^c_{ij} + g_{ca} \Xi^c_{ij} \right), \quad Z^a_{jk} = - C^i_{jk} g_{ik} g^{ab} - \frac{1}{2} \Omega^a_{jk},$$

for $q_{ij}^{jk} = \frac{1}{2} (\delta_j^i \delta^k_h - g_{jk} g^{ih}), \quad q_{cd}^{ab} = \frac{1}{2} (\delta_c^a \delta^d_b \pm g_{cd} g^{ab}), \quad \Xi^c_{ij} = \left[ L^c_{ij} - e_a (N^c_j) \right].$

If $V = TM$, for certain models of Lagrange and/or Finsler geometry, we can identify $\hat{L}^i_{jk}$ to $\hat{L}^i_{bk}$ and $\hat{C}^i_{jc}$ to $\hat{C}^a_{bc}$ and consider the canonical d–connection as a couple $\hat{\Gamma}^\alpha_{\beta\gamma} = \left( \hat{L}^i_{jk}, \hat{C}^i_{jk} \right)$.

By a straightforward d–form calculus, we can find the N–adapted components of the curvature of a d–connection $D$,

$$R^\alpha_{\beta\gamma\delta} \equiv \nabla^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma} \wedge \Gamma^\alpha_{\delta\gamma} = R^\alpha_{\beta\gamma\delta} e^{\gamma} \wedge e^{\delta};$$

$$\nabla^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma} \wedge \Gamma^\alpha_{\delta\gamma} = R^\alpha_{\beta\gamma\delta} e^{\gamma} \wedge e^{\delta};$$

$$\nabla^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma} \wedge \Gamma^\alpha_{\delta\gamma} = R^\alpha_{\beta\gamma\delta} e^{\gamma} \wedge e^{\delta};$$

$$\nabla^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma} \wedge \Gamma^\alpha_{\delta\gamma} = R^\alpha_{\beta\gamma\delta} e^{\gamma} \wedge e^{\delta};$$

$$\nabla^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma} \wedge \Gamma^\alpha_{\delta\gamma} = R^\alpha_{\beta\gamma\delta} e^{\gamma} \wedge e^{\delta};$$

$$\nabla^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma} \wedge \Gamma^\alpha_{\delta\gamma} = R^\alpha_{\beta\gamma\delta} e^{\gamma} \wedge e^{\delta};$$

$$\nabla^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma} \wedge \Gamma^\alpha_{\delta\gamma} = R^\alpha_{\beta\gamma\delta} e^{\gamma} \wedge e^{\delta};$$

$$\nabla^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma} \wedge \Gamma^\alpha_{\delta\gamma} = R^\alpha_{\beta\gamma\delta} e^{\gamma} \wedge e^{\delta};$$

$$\nabla^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma} \wedge \Gamma^\alpha_{\delta\gamma} = R^\alpha_{\beta\gamma\delta} e^{\gamma} \wedge e^{\delta};$$

$$\nabla^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma} \wedge \Gamma^\alpha_{\delta\gamma} = R^\alpha_{\beta\gamma\delta} e^{\gamma} \wedge e^{\delta};$$

$$\nabla^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma} \wedge \Gamma^\alpha_{\delta\gamma} = R^\alpha_{\beta\gamma\delta} e^{\gamma} \wedge e^{\delta};$$

$$\nabla^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma} \wedge \Gamma^\alpha_{\delta\gamma} = R^\alpha_{\beta\gamma\delta} e^{\gamma} \wedge e^{\delta};$$
\[ \begin{align*}
R^i_{\, hjk} &= e_k L^j_{\, hj} - e_j L^i_{\, hk} + L^m_{\, hj} L^i_{\, mk} - L^m_{\, hk} L^i_{\, mj} - C^i_{\, ha} \Omega^a_{\, kj}, \\
R^a_{\, bjk} &= e_k L^b_{\, aj} - e_j L^a_{\, bk} + L^c_{\, bj} L^a_{\, ck} - L^c_{\, bk} L^a_{\, cj} - C^a_{\, bc} \Omega^c_{\, kj}, \\
R^i_{\, jka} &= e_a L^j_{\, ik} - D_k C^i_{\, ja} + C^i_{\, jb} T^b_{\, ka}, \\
R^i_{\, bka} &= e_a L^i_{\, bk} - D_k C^a_{\, ba} + C^i_{\, bc} T^c_{\, ka}, \\
R^i_{\, jbc} &= e_c C^j_{\, ib} - e_b C^i_{\, jc} + C^h_{\, jk} C^i_{\, hc} - C^h_{\, jk} C^i_{\, hb}, \\
R^a_{\, ced} &= e_d C^a_{\, be} - e_c C^a_{\, bd} + C^e_{\, bc} C^a_{\, ed} - C^e_{\, bd} C^a_{\, ec}. 
\end{align*} \]  

(A.5)

Contracting respectively the components of (A.5), one proves that the Ricci tensor \( R_{\alpha\beta} \bowtie R^\tau_{\alpha\beta\tau} \) is characterized by d–tensors,

\[ R_{ij} \bowtie R^k_{\, ijk}, \quad R_{ia} \bowtie -R^k_{\, ika}, \quad R_{ai} \bowtie R^b_{\, aib}, \quad R_{ab} \bowtie R^c_{\, abc}. \]  

(A.6)

It should be noted that this tensor is not symmetric for arbitrary d–connections \( D \). The scalar curvature of a d–connection is

\[ \hat{R} \bowtie g^{\alpha\beta} R_{\alpha\beta} = R + S, \quad R = g^{ij} R_{ij}, \quad S = g^{ab} R_{ab}, \]  

(A.7)

defined by a sum the h– and v–components of (A.6) and d–metric (8).

The Einstein tensor is defined and computed in standard form

\[ G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \ hat{R}. \]  

(A.8)

It should be noted that, in general, this Einstein tensor is different from that defined for the Levi–Civita connection but for the canonical d–connection and metric defined by a Lagrange model both such tensors are derived from the same Lagrangian and metric structure. Finally, we note that formulas (A.4) – (A.8) are defined in the same form for different classes of linear connection. For the canonical d–connection and the Levi–Civita connection, we label such formulas with respective 'hats' and left 'vertical lines', for instance, \( \hat{R}^2_{\gamma\delta} \) and \( R^2_{\gamma\delta}, \hat{R}_{\alpha\beta} \) and \( R_{\alpha\beta}, \ldots \)

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