Exact diagonalisation of 1-d interacting spinless Fermions

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Abstract
A new method of constructing an infinite set of exact eigenfunctions of 1–d interacting spinless Fermionic systems is derived. Creation and annihilation operators for the interacting system are found and thereby the many–body Hamiltonian is diagonalised. The formalism is applied to several examples. One instance is the theory of Jack polynomials. For the Calogero-Moser-Sutherland Hamiltonian a direct proof of the correctness of the asymptotic Bethe Ansatz is given.

1 Introduction
The study of one–dimensional integrable models of interacting particles has a long history going back to Bethe [Bet31]. In physics there has been a renewed interest in one–dimensional integrable systems in the last years in the study of cold atom gases and Bose–Einstein condensates [FRZ04, CZ04a, CZ04b].

The standard way of constructing eigenfunctions is Bethe’s Ansatz. The crucial condition for Bethe’s Ansatz to be successful is that the two–body scattering matrix $S_{ab}(k_j - k_k)$ of a particle of type $a$ with momentum $k_j$ with a particle of type $b$ with momentum $k_k$ fulfills the Yang–Baxter (star triangle) equation, which serves as the starting point for the algebraic Bethe Ansatz.

Imposing periodic boundary conditions for the $N$ particle wave function, from Bethe’s Ansatz one (or a set) of Fredholm integral equations for the density of states is obtained. They are referred to as Bethe Ansatz equations.

Applied either to 1–d quantum mechanical, or to 2–d classical lattice theories, Bethe’s Ansatz has been extraordinarily successful. For one reason, because many of the most important lattice models, as for instance the Hubbard model, Heisenberg model, Ising model etc. only nearest neighbor interaction is assumed. Applied to continuous models Bethe’s Ansatz is in its simplest form

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constrained to particles with δ–interaction [LL63, Yan67, Gau66], being the only strictly local interaction.

The problems, connected with non–local interactions in continuum models, were partly overcome by the asymptotic Bethe Ansatz (ABA). It was introduced by Sutherland [Sut71a, Sut71b] in order to obtain thermodynamical quantities for Calogero–Moser–Sutherland (CMS) models.

The basis assumption of ABA is that the Bethe Ansatz equations still hold for non–local interactions as long as the \( N \)–body \( S \)–matrix factorizes into a product of 2–body \( S \)–matrices. Using the ABA hypothesis it is therefore possible to obtain thermodynamical quantities without detailed knowledge of the wave function in the interacting region. The correctness of this assumption has been proven for the trigonometric CMS model, where a complete set of eigenfunctions can be constructed with Jack polynomials [Mac95, For92, For93, For95]. In that specific model even some thermodynamical correlation functions could be calculated [Ha95].

Although the correctness of ABA has been proven also in some other cases [Kaw92], the different treatment of local and non–local interactions within the Bethe Ansatz is unsatisfactory. Moreover ABA yields no clue of how to construct eigenfunctions of systems with non–local interaction. Therefore in this work I wish to put forward an approach, alternative to Bethe’s Ansatz which treats local and non–local interactions on the same basis.

The exact \( N \)–body wave function is not constructed via an Ansatz and by adjusting parameters but by the successive application of a creation operator onto the vacuum ground state. Thereby an integral representation for an arbitrary \( N \) particle eigenfunction is obtained. The explicit construction of this creation operator and its corresponding annihilation operator for certain interaction potentials is the main result of this work.

The basic idea roots on the following observation: a class of multidimensional integration formulae – some of them are well known for a long time – allow for a natural interpretation as Fermionic creation operators of a one–dimensional many–body Hamiltonian. These integrals share the common property that the integration domain \( I^{(N)} \) of a set \( x' \) of \( N \) integration variables \( x'_n \), \( 1 \leq n \leq N \), is defined by the interlacing condition

\[
x_1 > x'_1 > x_2 > \ldots > x_N > x'_N > x_{N + 1}.
\]

The integral itself is therefore in general a function of a set \( x \) of \( N + 1 \) arguments \( x_i \). Due to condition (1) this function vanishes, whenever two arguments are equal. Thus by construction it obtains the nature of a Fermionic wave function, when the \( N + 1 \) arguments are interpreted as particle positions. One instance is the following version of the Dixon–Anderson integral [Dix05, And91]

\[
\begin{align*}
\frac{\Gamma^{N + 1}(\lambda + 1)}{\Gamma((N + 1)(\lambda + 1))} &= \int_{I^{(N)}}^{(N)} \prod_{i=1}^{N} dx'_i \mu_\lambda(x, x') \\
\mu_\lambda(x, x') &= \frac{\Delta_N(x')}{\Delta_{N+1}^{2\lambda}(x)} \prod_{i=1}^{N} \prod_{j=1}^{N} (x_i - x'_j)^\lambda \prod_{j=1}^{N} \prod_{i=1}^{N+1} (x'_j - x_i)^\lambda.
\end{align*}
\]
It might be considered a representation of a power of Vandermonde’s determinant
\[
\Delta_{N+1}(x) \equiv \prod_{n<m} (x_n - x_m)
\] (4)
as an integral over \(\Delta_N(x')\), if both sides are multiplied with \(\Delta^\lambda_{N+1}(x)\). In Ref. [GK02] group integrals of the form
\[
\phi^{(\lambda)}_N(k, x) = \int_{U \in U(N)} d\mu(U) \exp(-itru^{-1}xUk),
\] (5)
were considered. Here \(U(N)\) is a classical Lie group manifold and \(x, k\) are diagonal \(N \times N\) matrices, and \(d\mu(U)\) is the Haar measure of the group. The parameter \(\lambda\) depends on the group under consideration.\(^1\) It was found that \(\phi^{(\lambda)}_N(k, x)\) can be constructed recursively in \(N\) as
\[
\phi^{(\lambda)}_{N+1}(k^+, x) \propto \int_{I^{(N)}} \prod_{i=1}^N dx'_i \mu_\lambda(x, x') e^{i \kappa_{N+1}(tr x - tr x')} \phi^{(\lambda)}_N(k, x'),
\] (6)
where \(\kappa_{N+1}\) is the additional element of the new \((N+1) \times (N+1)\) matrix \(k^+\) on the left hand side. Okounkov and Olshanski [OO97] found an integration formula for symmetric Jack polynomials \(J_\lambda^N(n, x)\) in \(N\) variables:\(^2\)
\[
J_{N+1}^\lambda(n, x) \propto \int_{I^{(N)}} \prod_{i=1}^N dx'_i \mu_\lambda(x, x') J_N^\lambda(n, x'),
\] (7)
which relates Jack polynomials with \(N+1\) arguments to Jack polynomials with \(N\) arguments to the same partition \(n = \{n_1, \ldots, n_N\}, n_i \in \mathbb{N}\).

Eq. (2), Eq. (6) and Eq. (7) share the same structure: An integral of a function \(f\) of \(N\) variables multiplied with an integration kernel \(\mu_\lambda(x, x')\) reproduces the same function \(f\) with the number of variables increased by one. I will embed these instances in a general framework and show that the proper generalization of \(\mu_\lambda(x, x')\) has a most natural interpretation as the coordinate representation of a particle creation operator \(a_k^\dagger\).

Whereas some of the integral formulae arising from the creation operator \(a_k^\dagger\) are already known, the construction of the annihilation operator is completely new. The resulting integral representation of the annihilation operator yields interesting new integral identities, which might be useful in applications.

This paper is the first in a series, dealing with the construction of creation and annihilation operators in 1–d integrable systems. It focuses on spinless Fermions. Bosons and spin 1/2 Fermions will be addressed in separate publications.

The paper is organized as follows. The first section supplies a precise description of the problem, whose solution will be presented in Theorem [1].

\(^1\) \(\lambda\) is related to the parameter \(\beta\) of [GK02] by \(\lambda = \beta/2 - 1\).

\(^2\) \(\lambda\) is related to the parameter \(\alpha\) of MacDonalds book by \(\lambda = 1/\alpha - 1\).
discussion of Theorem 1 follows. In Sec. 3 the results of Theorem 1 are illustrated in several applications. Proofs will be given in Sec. 4.

2 Statement of the result

We consider a Hamiltonian, describing one dimensional non–relativistic spinless interacting particles with mass $1/2$

$$H = \int_{\Omega} \psi^\dagger(x) \left( -\frac{d^2}{dx^2} \right) \psi(x) dx + \int_{\Omega} dx dx' \psi^\dagger(x) \psi^\dagger(x') V(x-x') \psi(x') \psi(x).$$ (8)

The integration domain $\Omega$ is the real axis or a compact interval. For the first choice the spectrum of $H$ will be continuous. This has the implication that the eigenvalue problem for $H$ might have no solutions in the Hilbert space $L^2$ of square integrable $C^1$–functions in $\mathbb{R}$. In this case we look for solutions in an enhanced (rigged) Hilbert space, which allows for eigenstates of $H$ which are not normalisable. In other words, we consider the Gelfand triplet $\mathcal{D} \subset L^2 \subset \mathcal{D}'$, where $\mathcal{D}$ is the space of test functions and $\mathcal{D}'$ is the space of distributions, dual to $\mathcal{D}$. In particular $\mathcal{D}'$ includes eigenfunctions, which behave in the asymptotic limit as plane waves (scattering solutions of the Schrödinger equation). For this behavior in the asymptotic limit of the wave function we introduce the term scattering boundary condition (SBC). We consider

$$\Omega = \begin{cases} \mathbb{R}, & \text{for scattering boundary conditions (SBC)}, \\ [0,L], & \text{for periodic boundary conditions (PBC)}. \end{cases}$$ (9)

If we denote the one–particle Hilbert space by $\mathcal{L}(\Omega)$, we have

$$\mathcal{L}(\Omega) = \begin{cases} \mathcal{D}', & \text{for scattering boundary conditions (SBC)}, \\ L^2, & \text{for periodic boundary conditions (PBC)}. \end{cases}$$ (10)

In Eq. 8, $\psi(x)$ and $\psi^\dagger(x)$ are Fermionic creation (annihilation) operators obeying the anticommutation relation

$$\{\psi(x),\psi^\dagger(x')\} = \psi(x)\psi^\dagger(x') + \psi^\dagger(x')\psi(x) = \delta(x-x').$$ (11)

For a fixed number of particles $N$, the Hamiltonian can be written in first quantization in coordinate representation as

$$H_N(x) = -\sum_{n=1}^N \frac{\partial^2}{\partial x_n^2} + \sum_{i\neq j} V(x_i-x_j),$$ (12)

where $x$ denotes the set of particle positions $x = \{x_1,\ldots,x_N \}$. The Hamiltonian is the direct sum $H = \bigoplus_{N=0}^\infty H_N$ acting on the Fock space $\mathcal{H} = \bigcup_{N=0}^\infty \mathcal{H}_N$, where

$$\mathcal{H}_N = \{\psi(x) \in \mathcal{L}(\Omega^N) | \psi(x) \text{ completely antisymmetric} \}.$$ (13)
and $L(\Omega^N) = L^N(\Omega)$. On $\mathcal{H}_N$ a scalar product is defined by the $N$–fold integral

$$\langle \psi_N | \phi_N \rangle = \int_{\Omega^N} d^N[x] \psi_N^*(x) \phi_N(x) , \quad (14)$$

where the infinitesimal volume element $d^N[x] = \prod_{n=1}^{N} dx_n$ was introduced.

Our goal is to map the Hamiltonian as given in Eq. (8) onto the quadratic form

$$H = \sum_k \epsilon(k) a_k a_k^\dagger , \quad (15)$$

where $a_k$ ($a_k^\dagger$) annihilates (creates) a particle with quasimomentum $k$. The Hamiltonian (15) acts on the complete Fock space and is block diagonal in the basis of eigenstates of the particle number operator

$$N = \sum_k a_k^\dagger a_k . \quad (16)$$

In contrast, the creation and annihilation operators define a map which does not conserve particle number

$$a_k : \mathcal{H}_N \rightarrow \mathcal{H}_{N-1} \quad , \quad a_k |\psi_N\rangle \in \mathcal{H}_{N-1} ,$$
$$a_k^\dagger : \mathcal{H}_N \rightarrow \mathcal{H}_{N+1} \quad , \quad a_k^\dagger |\psi_N\rangle \in \mathcal{H}_{N+1} . \quad (17)$$

This mapping reads in configuration space

$$\psi_{N+1}(x) = \int_{\Omega^N} d^N[x'] a_k^\dagger(x, x') \psi_N(x') ,$$
$$\psi_{N-1}(x) = \int_{\Omega^N} d^N[x'] a_k(x, x') \psi_N(x') . \quad (18)$$

This means, in configuration space the creation operator is an integral operator whose kernel is a complex valued function of two sets of coordinates $x = \{x_1, \ldots, x_{N+1}\}$ and $x' = \{x'_1, \ldots, x'_N\}$. We call $a_k^\dagger(x, x')$ \textit{creation function}. The annihilation operator is an integral operator whose kernel $a_k(x, x')$ is a complex valued function of $x = \{x_1, \ldots, x_{N-1}\}$ and $x' = \{x'_1, \ldots, x'_N\}$. We call $a_k(x, x')$ \textit{annihilation function}.

The creation and annihilation operators are defined by the basis independent commutator relation

$$[H, a_k] = -\epsilon(k) a_k$$
$$[H, a_k^\dagger] = \epsilon(k) a_k^\dagger . \quad (19)$$
They read in coordinate representation

\[ 0 = \left[ H_{N-1}(x) + \epsilon(k) \right] \int_{\Omega^N} d^N[x'] a_k(x, x') \psi_N(x') \]

\[ - \int_{\Omega^N} d^N[x'] a_k(x, x') H_N(x') \psi_N(x') , \]

\[ 0 = \left[ H_{N+1}(x) - \epsilon(k) \right] \int_{\Omega^N} d^N[x'] a^\dagger_k(x, x') \psi_N(x') - \int_{\Omega^N} d^N[x'] a^\dagger_k(x, x') H_N(x') \psi_N(x') . \]  

(20)

This translates into a set of partial differential equations for the creation functions \( a^\dagger_k(x, x') \) and the annihilation functions \( a_k(x, x') \)

\[ [H_{N-1}(x) - H_N(x')] a_k(x, x') = -\epsilon(k) a_k(x, x') , \]

\[ [H_{N+1}(x) - H_N(x')] a^\dagger_k(x, x') = \epsilon(k) a^\dagger_k(x, x'). \]  

(21)

With a set of operators fulfilling the commutator relations (19) simultaneous eigenfunctions of the Hamiltonians Eq. (15) and Eq. (12) to the eigenvalue \( E = \sum_{n=1}^N \epsilon(k_n) \) can be constructed by

\[ |\psi_N(k)\rangle = \prod_{n=1}^N a^\dagger_{k_n} |0\rangle . \]  

(22)

Since two operators which have the same eigenfunctions to the same eigenvalue are equal, the conditions specified in Eqs. (19) to (21) are sufficient to prove that the Hamiltonians in the both forms as given in Eq. (8) and in Eq. (15) are equal up to a basis rotation.

Restricting ourselves to Fermions, we require the wave function \( \langle x|\psi_N(k)\rangle \in \mathcal{H}_N \) to be completely antisymmetric in two sets of arguments \( k = \{ k_1, \ldots, k_N \} \) and \( x = \{ x_1, \ldots, x_N \} \), thus the quasiparticle creation and annihilation operators have to obey the Fermionic anticommutation rules

\[ \{ a_k, a^\dagger_{k'} \} = \begin{cases} \delta(k - k') , & \text{SBC} , \\ \delta_{k,k'} , & \text{PBC} . \end{cases} \]  

(23)

We introduce the following notation: \( \psi_n(k, x) \) denotes always a function with \( n \in \mathbb{N} \) arguments \( \{ k_1, \ldots, k_n \} \) and with \( n \) arguments \( \{ x_1, \ldots, x_n \} \).

We now state the main result of this section as a theorem. It states a sufficient condition on the interaction potential \( V(x) \) for the existence of the above defined creation and annihilation operators and gives an explicit construction of the eigenstates.

**Theorem 1 (Spinless Fermions)** 1. Let \( f(x) \) be an antisymmetric function satisfying the condition

\[ f(x)f(y) + f(x)f(z) + f(y)f(z) = \text{const.} , \]

\[ x + y + z = 0 \]  

(24)
and $F(x) = \int x \, dx' f(x')$, such that $F(x) = F(-x)$, then annihilation functions $a_k(x, x')$ and creation functions $a_k^\dagger(x, x')$ satisfying the conditions specified in Eq. (21) are given by

$$a_k(x, x') = \exp \left[ -\sum_{n<m}^{N-1} F(x_n - x_m) + \sum_{n,m}^N F(x_n - x_m') - \sum_{n<m}^N F(x_n' - x_m') + i k \left( \sum_{n=1}^{N-1} x_n - \sum_{m=1}^N x_m' \right) \right],$$

$$a_k^\dagger(x, x') = \exp \left[ -\sum_{n<m}^{N+1} F(x_n - x_m) + \sum_{n,m}^N F(x_n - x_m') - \sum_{n<m}^N F(x_n' - x_m') + i k \left( \sum_{n=1}^{N+1} x_n - \sum_{m=1}^N x_m' \right) \right]. \tag{25}$$

2. The interaction potential in the Hamiltonian (12) is related to the function $f$ by

$$V(x) = f^2(x) - f'(x) + \text{const.} \tag{26}$$

3. With the complex functions $a_k(x, x')$ and $a_k^\dagger(x, x')$ defined in Eq. (25), eigenfunctions $\psi_{N\pm1}(k, x)$ to the $N \pm 1$ particle Hamiltonian Eq. (13) can be constructed recursively as

$$\psi_{N+1}(k, x) = (x|a_{kN+1}^\dagger|\psi_N(k))$$

$$= \frac{C_N(k)}{\sqrt{N+1}} \int_{i_{\text{in}}}^N d^N[x'] a_{kN+1}^\dagger(x, x') \psi_N(k, x'), \tag{27}$$

$$\psi_{N-1}(k, x) = (x|a_{kN}|\psi_N(k))$$

$$= R \sqrt{N} C_{N-1}(k) \int_{j_{\text{out}}}^N d^N[x'] a_{kN}(x, x') \psi_N(k, x'), \tag{28}$$

where $R = (2\pi)^{-1}$ for SBC and $R = L^{-1}$ for PBC. The integration domains $i_{\text{in}}^N$ and $j_{\text{out}}^N$ are defined by

$$\int_{i_{\text{in}}}^N d^N[x'] \, \ldots = \int_{x_2}^{x_1} dx_1' \int_{x_3}^{x_2} dx_2' \ldots \int_{x_{N+1}}^{x_N} dx_N' \, \ldots \tag{29}$$

$$\int_{j_{\text{out}}}^N d^N[x'] \, \ldots = \int_{x_2}^{x_1} dx_1' \int_{x_3}^{x_2} dx_2' \ldots \int_{-\infty}^{x_{N-1}} dx_N' \, \ldots, \text{(SBC)}$$

$$\int_{j_{\text{out}}}^N d^N[x'] \, \ldots = \int_{x_2}^{x_1} dx_1' \int_{x_3}^{x_2} dx_2' \ldots \int_{0}^{x_{N-1}} dx_N' \, \ldots, \text{(PBC)}$$

The normalization constant $C_N(k)$ is coordinate independent. For the most important potentials the explicit value will be given below (see Proposition 2).
4. The dispersion relation of the quasimomenta is quadratic:

\[ \epsilon(k) = k^2 . \]  

(30)

The functions (27) and (28) are eigenfunctions to the \( N \pm 1 \) particle Hamiltonians (12) and to the center of mass momentum operator

\[ P_{N \pm 1}(x) = -i \sum_{n=1}^{N \pm 1} \frac{\partial}{\partial x_n} , \]  

(31)

such that

\[ H_{N \pm 1}(x) \psi_{N \pm 1}(k, x) = \left( \sum_{n=1}^{N \pm 1} k_n^2 \right) \psi_{N \pm 1}(k, x) , \]  

(32)

\[ P_{N \pm 1}(x) \psi_{N \pm 1}(k, x) = \left( \sum_{n=1}^{N \pm 1} k_n \right) \psi_{N \pm 1}(k, x) . \]  

(33)

5. With respect to the scalar product Eq. (14) the orthogonality relation

\[ \langle \psi_N(k') | \psi_N(k) \rangle \propto \left\{ \begin{array}{ll} \det[2\pi \delta(k'_i - k_j)]_{1 \leq i,j \leq N} , & \text{SBC} \\
\det [L \delta_{k'_i,k_j} ]_{1 \leq i,j \leq N} , & \text{PBC}. \end{array} \right. \]  

(34)

holds.

The complete proof of Theorem 1 will be given in Sec. 4. Here we discuss some of its consequences.

**N-particle wave function** As a corollary to Theorem 1 every \( N \)-particle wave function can be written as a \( N(N-1)/2 \)-fold integral as follows

\[ \psi_{N+1}(k, x) = \prod_{n=1}^{N} \left( \frac{C_n(k)}{\sqrt{n + 1}} \int_{a_n} d^n[x^{(n)}] a_{k_n}^{\dagger} (x^{(n+1)}, x^{(n)}) \right) . \]  

(35)

The coordinate sets \( x \) and \( x^{(N+1)} \) are identified. Eq. (35) is obtained by iterating Eq. (27).

**Condition on the potential** The functional equation (24) is a special case of the functional equation

\[ f(x)f(y) + f(x)f(z) + f(y)f(z) = W(x) + W(y) + W(z) \]
\[ x + y + z = 0 , \]  

(36)

which was found by Sutherland [Sut71a, Sut75] to be the necessary condition for a product wave function

\[ \psi_N(0, x) = \prod_{i<j}^{N} \exp[F(x_i - x_j)] \]  

(37)
to be an eigenfunction to eigenvalue zero of an $N$–body Hamiltonian with two–body interaction only. The general solution of (36) was found by Calogero [Cal75b, Cal75a]. It is given by
\begin{equation}
  f(x) = \lambda \zeta(x|\omega,\omega') + \beta x ,
\end{equation}
where $\zeta(x|\omega,\omega')$ is the Weierstrass $\zeta$–function [AS72] and $\omega, \omega'$ are the two periods of the corresponding Weierstrass elliptic function $P$. The right hand side of (36) is determined by the Frobenius–Stickelberger equation for the Weierstrass $\zeta$–function [FS80]
\begin{equation}
  (\zeta(x) + \zeta(y) + \zeta(z))^2 = -\zeta'(x) - \zeta'(y) - \zeta'(z)
\end{equation}
to
\begin{equation}
  W(x) = -\frac{1}{2} (\lambda f'(x) + f^2(x)) - \frac{\lambda \beta}{2} .
\end{equation}
The condition $3W(x) = \text{const.}$ therefore requires
\begin{equation}
  -\frac{3}{2} (\lambda f'(x) + f^2(x) - \lambda \beta) = \text{const.}
\end{equation}
as well. Eq. (41) can be integrated. The most general simultaneous solution to Eq. (41) and Eq. (24) is, with const. $= -z^2 \lambda^2$
\begin{equation}
  f(x) = z \lambda \coth(z(x + x_0)) ,
\end{equation}
for arbitrary complex parameters $x_0$, $z$ and $\lambda$. Requirement of a real potential restricts the values of $z = a + ib$ to be either real $z = a$ or purely imaginary $z = ib$ and $\lambda$ to be real. These solutions are obtained from Eq. (38) by setting $\beta \rightarrow a^2 \lambda/3$ and $\omega' \rightarrow i \pi/2a$, $\omega \rightarrow \infty$ or $\omega \rightarrow \pi/2b$, $\omega' \rightarrow \infty$. A series of potentials can be derived by taking various limits. The most important cases are listed in Table 1. They are the interactions of the trigonometric (I), rational (II) and hyperbolic (III) Calogero–Sutherland–Moser (CMS) type. Remarkably also the sign–function respectively the $\delta$–distribution are obtained from the shifted hyperbolic CMS Hamiltonian in the limit $a \rightarrow \infty$, $\lambda \rightarrow 0$ with $a \lambda = c$ finite.

**Structure of creation and annihilation functions** The creation (annihilation) functions as defined in Eq. (25) are strictly speaking also functions of the particle number $N$. We suppressed this obvious $N$ dependence in order to unburden notation. However, to describe the relation between $a_k$ and $a_k^\dagger$ it is useful to indicate the $N$ dependence by the symbols $a_k^{(N)}(x,x')$ and by $a_k^{\dagger(N)}(x,x')$. Then we have
\begin{equation}
  a_k^{(N+1)}(x',x) = a_k^{\dagger(N)}(x,x') .
\end{equation}
The general structure of the creation (annihilation) function $a_k^\dagger(x, x')$ ($a_k(x, x')$) factorizes into a $k$ independent part $a_k^\dagger(x, x')$ ($a_k(x, x')$) and a $k$ dependent part. The $k$ dependent part is for all types of potentials a product of plane waves of the primed and the unprimed coordinates. They may be considered arbitrary eigenfunctions of the non–interaction $V(x) = 0$ many–body Hamiltonian. On the other hand, as mentioned above, the basic ingredient of the $k$–independent part is the product function $\psi_N(0, x) = \prod_{i<j} \exp[F(x_i - x_j)]$, which is an eigenfunction of the interacting system to eigenvalue zero. Using this general structure one might extend Theorem 1 to systems, where translation invariance is broken by an external potential, but whose ground state wave function can yet be written in the product form of Eq. (37).

**Boundaries of the integral operator** Since we wish to create Fermionic eigenstates we cannot take $\Omega^N$ as integration domain in Eqs. (27) and (28). Because the integrand is by construction completely antisymmetric in the $N$ integration variables $x'$ this would yield the integral unavoidably zero. By the same argument the integral becomes zero, whenever two integration variables have the same integration domain. The integration domains $I^{(N)}_{\text{in}}$ and $I^{(N)}_{\text{out}}$ as chosen in Eq. (29) guarantee that the wave functions $\psi_{N \pm 1}(k, x)$ are completely antisymmetric. By the definition of $I^{(N)}_{\text{in}}$ and $I^{(N)}_{\text{out}}$ no ordering of particles is introduced. The newly created wave functions are defined in any of the $(N \pm 1)!$ disjunct regions of $\Omega^N \pm 1$.

It is non–trivial to see that by the introduction of the domains $I^{(N)}_{\text{in}}$ and $I^{(N)}_{\text{out}}$ the commutator relations in coordinate representation (24) are not affected. Since $H_{N \pm}(x)$ acts also on the boundaries it has to be shown explicitly that no additional boundary terms occur. For periodic boundary conditions this yields a restrictive condition onto the form of the creation (annihilation) functions.

**Proposition 1 (Periodicity)** For PBC the creation (annihilation) function $a_k^\dagger(x)$ ($a_k(x)$) itself must be periodic with period $L$.

### Table 1: List of interaction potentials for which creation and annihilation operators can be constructed. They are based on the addition theorem for the cot function: $\cot(x) \cot(y) = \cot(x+y)\cot(x) + \cot(y) + 1$.

| $b \to 0$ (I): | $f(x)$ | $V(x)$ | $F(x)$ | const. |
|--------------|--------|--------|--------|--------|
| $\lambda b \cot(bx)$ | $b^2 \lambda (\lambda + 1) \sin^2(bx)$ | $\lambda \ln |\sin(bx)|$ | $b^2 \lambda^2$ |
| $b \to ia$ (I): | $\lambda^2 \frac{\alpha \cot(\alpha x)}{\sin^2(\alpha x)}$ | $\lambda \ln |\sinh(\alpha x)|$ | $-a^2 \lambda^2$ |
| $x \to x + i\pi/2$ (III): | $a \lambda \coth(ax)$ | $-a^2 \lambda (\lambda + 1) \sinh^2(ax)$ | $\lambda \ln |\cosh(ax)|$ | $-a^2 \lambda^2$ |
| $\lambda \to 0$ (IV): | $\epsilon \text{sgn}(x)$ | $-2c \theta(x)$ | $c|x|$ | $-c^2$ |
The proof of this statement is given in App. Proposition 1 essentially fixes the boundary condition for all potentials in Table 1. Potential (I) is periodic with period $\pi/b$. Therefore the asymptotic regime is never reached and only periodic boundary conditions $L = \pi/b$ are allowed for this potential. By Prop. 1, for potentials of type (II) to (IV) only scattering boundary conditions (SBC) are allowed. Only for particles with $\delta$–interaction both boundary conditions can be imposed.

Analyticity at the Boundaries For the integrals (27) and (28) to exist the creation and annihilation functions can have at most an integrable singularity in their integration domain. For potentials (I) to (III) this yields a restriction for the coupling parameter $\lambda$ to the range $\lambda \in [-1, \infty)$. This restriction is consistent with the well known fact that for CMS–Hamiltonians the minimal value of the coupling constant $g = \lambda(\lambda + 1)$ is $g = -1/4$ [OP83]. For these potentials the behavior of $\psi_N(k, x)$ when two particles come close to each other can also be extracted directly from the creation and annihilation functions. The wave function vanishes with a typical power

$$\lim_{x_i \to x_{i+1}} \psi_N(k, x) \propto (x_i - x_{i+1})^{\lambda+1} + O((x_i - x_{i+1})^{\lambda+2}) .$$  \hspace{1cm} (44)

This shows again that $\lambda > -1$ must be imposed.

Normalisation For potentials (II)–(V) with SBC the wave function is not normalisable. The constant $C_N(k)$ can be evaluated by requiring that $\psi_N(x, k)$ becomes the free particle solution in the asymptotic regime, i.e., in the regime where all distances $x_i - x_j$, $1 \leq i, j \leq N$ are large. For potential (I) the wave–function is normalisable and the calculation is more involved. The result for all cases can be summarized as follows.

Proposition 2 Depending on the potentials listed in Table 1 the $k$–dependent normalisation constant $C_N(k)$ is given by

$$C_N(k) = \begin{cases} \left( \frac{L}{2\pi i} \right)^{-N} \prod_{i=1}^{N} \frac{(2i)^\lambda \text{sgn}(k_i - k_{N+1})}{B(\frac{L}{2\pi i}|k_i - k_{N+1}|, \lambda + 1)} & \text{for (I)} , \\ \Gamma^N(\lambda + 1) \prod_{i=1}^{N} (k_i - k_{N+1})^\lambda & \text{for (II)} , \\ \prod_{i=1}^{N} (ik_{N+1} - ik_i) & \text{for (III)–(V)} , \end{cases}$$  \hspace{1cm} (45)

where $B(x, y)$ is Euler’s beta–function. The proof of Prop. 2 is given in Sec. 4.2.

3 Continuing potential (II) periodically leads to potential (I).
3 Applications

The $N(N - 1)/2$ integrals in the representation Eq. (35) of the exact $N$ particle state can in some cases be evaluated exactly. However, many important properties of the wave function can actually be extracted from Eq. (35) without solving the integral. In the following we discuss free Fermions as well as the interaction potentials (I), (II) and (V) of Tab. 1.

3.1 Free particles

Since it is instructive to see, how Theorem 1 works in the simplest case, we illustrate it first for free Fermions. A general $N + 1$ particle state is given as the Slater determinant

$$\langle x | \psi_{N+1}(k) \rangle = \frac{1}{\sqrt{(N+1)!}} \text{det} \begin{bmatrix} e^{ik_n x_m} \end{bmatrix}_{1 \leq n, m \leq N+1} \equiv \psi_{N+1}^{(0)}(k, x). \quad (46)$$

This result can be derived from Theorem 1 by induction as follows: Assume the $N$ particle states have been constructed, then we find the $N+1$ particle states by

$$\langle x | \psi_{N+1}(k) \rangle = \langle x | a_{k_{N+1}}^\dagger | \psi_N(k) \rangle = \frac{C_N(k)}{\sqrt{(N+1)!}} \int_{J^{(N)}_{\text{in}}} d^N[x'] a_{k_{N+1}}^\dagger(x, x') \text{det} \begin{bmatrix} e^{ik_n x_m} \end{bmatrix}_{1 \leq n, m \leq N}$$

$$\times \exp \left( ik_{N+1} \sum_{n=1}^{N+1} x_n - ik_{N+1} \sum_{m=1}^{N} x'_m \right). \quad (47)$$

With the integration domain $J^{(N)}_{\text{in}}$ given by Eq. (29) the integral can be performed yielding again a determinant

$$\langle x | \psi_{N+1}(k) \rangle = \frac{1}{\sqrt{(N+1)!}} \frac{C_N(k)}{\prod_{n=1}^{N+1} (ik_{k} - ik_{N+1})} \text{det} \begin{bmatrix} e^{ik_n x_m} \end{bmatrix}_{1 \leq n, m \leq N+1}, \quad (48)$$

which is the desired result (compare with Prop. 2). The action of $a_k$ on an $N$ particle state is given by

$$\langle x | \psi_{N-1}(k) \rangle = \langle x | a_{k} | \psi_N(k) \rangle = \frac{RC_{N-1}(k)}{\sqrt{(N-1)!}} \int_{J^{(N)}_{\text{out}}} d^N[x'] a_{k}(x, x') \text{det} \begin{bmatrix} e^{ik_n x_m} \end{bmatrix}_{1 \leq n, m \leq N}$$

$$\times \exp \left( ik \sum_{n=1}^{N-1} x_n - ik \sum_{m=1}^{N} x'_m \right). \quad (49)$$
If we choose SBC we have to equip the quasimomentum \( k \) with a positive imaginary increment \( i \epsilon \) for the \( x'_1 \) integration and with a negative increment for the \( x'_N \) integration. Then the integrals are convergent and yield again a determinant

\[
\langle x | \psi_{N-1}(k) \rangle = \frac{1}{2\pi} \frac{C_{N-1}(k)}{\sqrt{(N-1)!}} \begin{vmatrix}
2\pi \delta(k - k_1) & e^{ik_1 x_1} & \cdots & e^{ik_1 x_{N-1}} \\
\frac{ik_1 - ik}{ik_1 - ik} & \cdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
2\pi \delta(k - k_N) & e^{ik_N x_1} & \cdots & e^{ik_N x_{N-1}} \\
\end{vmatrix}.
\]

(50)

From Eq. (50) it follows

\[
ak | \psi_N(k) \rangle = 0, \text{ if } k \neq k_i, i = 1 \ldots N
\]

\[
\langle x | ak | \psi_N(k) \rangle = \frac{C_{N-1}(k)}{\prod_{i=1}^{N} (ik_i - ik)} \frac{1}{\sqrt{(N-1)!}} \det \left[ e^{ik_n x_m} \right]_{1 \leq n \leq N-1, 1 \leq m \leq N}, \text{ for } k = k_j.
\]

(51)

which is the desired result (compare with Prop. 2 with \( j = N \)).

For PBC we require \( \psi_N^{(0)}(k, x) \) to be a periodic function with period \( L \). Therefore \( k = \frac{2\pi}{L} n \), where \( n \) is a set of \( N \) integers \( n_i \in \mathbb{Z} \). For the action of the creation operator nothing changes as compared to SBC. For the action of the annihilation operator we find that the \( \delta \)-distribution is substituted by a Kronecker–delta

\[
\delta(k_i - k_j) \rightarrow \frac{L}{2\pi} \delta_{k_i, k_j}.
\]

(52)

This completes the construction for free spinless Fermions.

### 3.2 Particles with \( \delta \)-interaction

For spinless Fermions the \( \delta \)-interaction is invisible and the wave function becomes identical with the wave function of free Fermions as given in Eq. (46). This result is quickly derived using Theorem 1. Using the free solution \( \psi_N^{(0)}(x, k) \) for \( \psi_N(x, k) \) in Eq. (27), we obtain

\[
\langle x | \psi_{N+1}(k) \rangle = \langle x | a_k^\dagger | \psi_N(k) \rangle
\]

\[
= \frac{C_N(k)}{\sqrt{(N+1)!}} \int_{\mathcal{I}_N^{(N)}} d[x'] a_k^\dagger(x, x') \det \left[ e^{ik_n x_m'} \right]_{1 \leq n,m \leq N}
\]

(53)
For the creation function $a^\dagger_{kN+1}(x,x')$ we use Eq. (25) and extract $F(x) = c|x|$ from Tab. 1

$$a^\dagger_{kN+1}(x,x') = \exp \left( ik_{N+1} \sum_{n=1}^{N+1} x_n - ik \sum_{m=1}^{N} x'_m \right)$$

$$\exp \left( -c \sum_{n>m} |x_n - x_m| + c \sum_{n,m} |x_n - x'_m| - c \sum_{n>m} |x'_n - x'_m| \right).$$

It is readily seen that the exponent in the second line of Eq. (54) drops out completely for an arbitrary order of the particles. Eq. (53) reduces to the free particle expression Eq. (47). The corresponding result is obtained for the action of the annihilation operators.

### 3.3 Trigonometric Calogero–Moser–Sutherland system

For the interaction potential (I) the model is called trigonometric CMS–model. As was pointed out in Sec. 2 the Hamiltonian has to be considered with PBC. First we recall some facts of the model.

The eigenfunctions of the trigonometric CMS–models can be written as

$$\psi_N(n,z) = \psi_N(0,z) \prod_{i=1}^{N} z^K_{i} J^{(\lambda)}_{N,K}(n,z),$$

$$\psi_N(0,z) = \Delta_N^{\lambda+1}(z) \prod_{i=1}^{N} z_i^{-\lambda+2(N-1)},$$

where the arguments $z_i$ are related to the particle positions $x_i$ by the relation

$$z_i = \exp \left( 2\pi i x_i / L \right).$$

Here $\psi_N(0,z)$ is the ground state wave function of $H_N$ and $J^{(\lambda)}_{N,K}(n,z)$ is a symmetric polynomial in $N$ variables $z_i$ labeled by a partition $n = (n_1, \ldots, n_N)$ of integers $n_1 \leq n_2 \leq \ldots n_N$. These polynomials are called Jack polynomials and were extensively studied [Mac95, Sta89]. The parameter $\lambda$ is related to the parameter $\alpha$ of McDonald's book [Mac95] by $\lambda = 1/\alpha - 1$. The center of mass momentum $K$ is a real parameter. Jack polynomials are defined as eigenfunctions of the operator $H'_{N,K}$, which is obtained by adjunction of $H_N$ with the ground state wave function $\psi_N(0,z)$ times the Galilean boost $\prod_{i=1}^{N} z_i^K

$$H'_{N,K} = \left( \psi_N(0,z) \prod_{i=1}^{N} z_i^K \right)^{-1} \left( H_N - E^{(N,K)}_0 \right) \left( \psi_N(0,z) \prod_{i=1}^{N} z_i^K \right)$$

$$= \left( \frac{2\pi}{L} \right)^2 \sum_{i=1}^{N} \left( z_i \frac{\partial}{\partial z_i} \right)^2 + [(\lambda + 1)(N-1) + 2K] \sum_{i} z_i \frac{\partial}{\partial z_i}$$

$$+ 2(\lambda + 1) \sum_{i<j} z_i z_j \left( \frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right).$$
The ground state energy $E_0^{(N,K)}$ is given by

$$E_0^{(N,K)} = \frac{1}{12} (N + 1)N(N - 1)(\lambda + 1)^2 + NK(\lambda + 1)(N - 1).$$ \hspace{1cm} (58)

For $K = 0$ and $\lambda = 0$ this is identical to the ground state energy of free spinless Fermions. The wave function $\psi_N(n,z)$ in Eq. (55) vanishes with the power $\lambda + 1$, when two particles come close to each other, but it is not antisymmetric under interchange of two particles. Rather it obtains a phase $\pi(\lambda + 1)$ under the action of the permutation operator $P$

$$P_{nm}\psi(x_1, \ldots, x_n, \ldots, x_m, \ldots, x_N) = \psi(x_1, \ldots, x_m, \ldots, x_n, \ldots, x_N), \forall n, m.$$

(59)

For $\lambda$ even $\psi_N(n,z)$ is Fermionic, for $\lambda$ odd it is Bosonic. For arbitrary real $\lambda$ it is a wave function with anyonic statistics \[Ha95\]. A Fermionic wave function can always be obtained from Eq. (55) by the substitution

$$\Delta_{N-1}^{\lambda+1}(z) \prod_{i=1}^N z^{-\frac{\lambda+1}{2}(N-1)} \rightarrow |\Delta_N(z)|\Delta_N(z) \prod_{i=1}^N z^{-(N-1)/2}.$$

(60)

Jack polynomials depend strictly speaking also on the center of mass momentum $K$. In the following we fix the Galilean boost to $K \equiv -\frac{\lambda+1}{2}(N-1)$ and suppress the $K$ dependence of the Jack polynomial by setting

$$J_{N-\Delta_{N-1}^{\lambda+1}(N-1)}^{(\lambda)}(n,z) \equiv J_N^{(\lambda)}(n,z).$$

(61)

We now prove three statements

**Proposition 3** The asymptotic Bethe Ansatz equation \[Sut71\]

$$k_i = \frac{2\pi}{L} \left( I_i + \frac{\lambda}{2} \sum_{j=1}^{N+1} \text{sgn}(k_i - k_j) \right), \hspace{1cm} I_i \in \mathbb{Z}$$

(62)

for the quasimomenta $k_i$ are equivalent to the periodicity condition Prop. 2 of the creation (annihilation) operator $a_k^\dagger (a_k)$.

**Proof** In the region

$$x_1 > x_2 > \ldots > x_{N+1}$$

(63)

the creation function $a_k^\dagger(x,x')$ can be written as

$$a_k^\dagger(x,x') = \left( \frac{\prod_{i<j}^N \sin(\pi(x'_i - x'_j)/L)}{\prod_{i<j}^{N+1} \sin(\pi(x_i - x_j)/L)} \right)^\lambda \prod_{i=1}^N \prod_{j=1}^N \sin(\pi(x'_j - x_i)/L) \prod_{j=1}^{N+1} \prod_{i=j+1}^N \sin(\pi(x'_j - x_i)/L) \exp \left( ik_{N+1} \sum_{j=1}^{N+1} x_j - ik_{N+1} \sum_{j=1}^{N} x'_j \right).$$

(64)
After introducing complex variables
\[ z_i = \exp \left( \frac{2\pi i x_i}{L} \right), \quad z'_i = \exp \left( \frac{2\pi i x'_i}{L} \right) \] (65)

Eq. (27) becomes an integral representation for \( \psi_{N+1}(k, z) \)
\[ \psi_{N+1}(k, z) = \frac{C_N(k)}{\sqrt{N+1}} \left( \frac{L}{2\pi i} \right)^N \int_{\mathbb{I}_N^{(N)}} d^N[z'] 
\left( \prod_{i=1}^{N+1} \prod_{j=1}^N (z_i - z'_j) \prod_{j=1}^{N+1} (z'_j - z_i) \right)^{\lambda} 
\left( 2i \right)^{-\lambda N} \prod_{i=1}^{N+1} z_i - \lambda - 1 \frac{L k_{N+1} + 1}{2\pi} \psi_N(k, z') \) (66)

with \( d^N[z'] = \prod_{i=1}^{N+1} dz'_i \). Using the form of Eq. (55) for \( \psi_N(k, z') \) as well as for \( \psi_{N+1}(k, z) \) on both sides of Eq. (66)
\[ J^{(\lambda)}_{N+1}(n, z) = \frac{C_N(k)(2i)^{-\lambda N}}{\sqrt{N+1}} \left( \frac{L}{2\pi i} \right)^N \prod_{i=1}^{N+1} z_i - \lambda - 1 \frac{L k_{N+1} + 1}{2\pi} \mu_{\lambda}(z, z') \] (67)

is obtained. We recall that \( \mu_{\lambda}(z, z') \) was defined in Eq. (3) in the introductory section. Here \( n \) is the same partition of length \( N \) on both sides. Now periodicity
of \( a^\dagger_{k_{N+1}}(x, z') \) requires
\[ n_{N+1} = \frac{L}{2\pi} k_{N+1} + N(\lambda + 1), \quad \text{with} \ n_{N+1} \in \mathbb{Z} \] (68)

This equation can be iterated
\[ k_i = \frac{2\pi}{L} (n_i - (\lambda + 1)(i - 1)) \] (69)

where we assume that all \( n_i \in \mathbb{N} \). We can restrict ourselves to positive integers, since any negative integers can be absorbed by an appropriate boost. Subtracting the center of mass momentum \( K \) from every \( k_i \) yields
\[ k_i = \frac{2\pi}{L} \left( n_i + \frac{\lambda + 1}{2}(N + 1 - 2i) \right), \quad 1 \leq i \leq N + 1 \] (70)

It is easy to verify that the \( k_i \) in Eq. (70) are solutions of the Bethe equation (62). The integers \( I_i \) are related to \( n_i \) by \( I_i = n_i + (N + 1 - 2i)/2 \). This completes the proof of Prop. 3.

**Proposition 4** The action of the creation operator \( a^\dagger \) is equivalent to the integral representation for Jack polynomials, found by Olshaniski and Okounkov in [OO97].
Eq. (67) is written as

\[
\prod_{i=1}^{N} z_i \quad J_N(n, z) = J_N^{(\lambda)}(\{n_1 + 1, n_2 + 1, \ldots \}, z) \equiv J_N^{(\lambda)}(n + 1, z). \tag{71}
\]

Eq. (72) is written as

\[
J_{N+1}^{(\lambda)}(n - n_{N+1}, z) = \frac{C_N(k)(2i)^{-\lambda N} \left( \frac{L}{2\pi i} \right)^N}{\sqrt{N+1}} \int_{i_{in}} d^N z' |\mu_{\lambda}(z, z') J_N^{(\lambda)}(n - n_{N+1}, z')|. \tag{72}
\]

This is exactly the result by Okounkov and Olshanski ([OO97], Proposition 6) for a Jack polynomial with partition \( n - n_{N+1} \), or equivalently for a Jack polynomial \( J_{N,K-n_{N+1}} \) (boosted by \(-n_{N+1}\)) with partition \( n \), if we adjust the normalisation constants

\[
C_N(k)(2i)^{-\lambda N} \left( \frac{L}{2\pi i} \right)^N = \prod_{i=1}^{N} [B(n_i - n_{N+1} + (N+1-i)(\lambda+1), \lambda+1)]. \tag{73}
\]

Here \( B(x, y) \) is Eulers beta–function. The quasimoments \( k_i \) on the l. h. s. are related to the integers \( n_i \) on the r. h. s. by Eq. (69). This completes the proof of Prop. 4.

**Proposition 5** Let \( J_N^{(\lambda)}(n, z) \) be a Jack polynomial to partition \( n = \{n_1, \ldots, n_N\} \) and let \( n \) be a positive integer. Define

\[
\nu_{n,\lambda}(z, z') = \frac{\Delta_N(z')}{\Delta_{N+1}(z)} \prod_{i=1}^{N} z^{-2\lambda-1} z_i^{-N-1} \prod_{i=1}^{N} \prod_{j=i+1}^{N} (z_i - z_j)^\lambda \prod_{j=i}^{N} (z_j' - z_i)^\lambda. \tag{74}
\]

Then the following integral representation for Jack polynomials holds

\[
\oint d z_1 \oint d z_2 \ldots \oint d z_N \nu_{n,\lambda}(z, z') J_N^{(\lambda)}(n, z') = B_N \delta_{n,n_m} J_{N-1}^{(\lambda)}(n - n, z). \tag{75}
\]

For \( \lambda \in \mathbb{N} \), \( n_m = n_i + (\lambda + 1)(N - i) \), \( 1 \leq i \leq N \). For \( \lambda \notin \mathbb{N} \), \( m = N \). The contour integral is over the unit circle. The normalisation constant is

\[
B_N = 2\pi i \prod_{i=1}^{N} [B(n_i - n_m + (m - i)(\lambda + 1), \lambda + 1)]. \tag{76}
\]

**Proof** The action of the annihilation operator \( a_k \) is defined by Eq. (28). It is given in coordinate free notation by

\[
a_k |\psi_N(k)\rangle = \sum_{i} (-1)^{i+1} \delta_{k,k_i} |\psi_{N-1}(k_{m_i})\rangle, \tag{77}
\]

17
where in the state $|\psi_{N-1}(k_{x_i})\rangle$ a particle with quasimomentum $k_i$ has been deleted. After the variable transformation (65) using Eq. (55) yields

$$J^{(\lambda)}_{N-1}(n, z) = \frac{\sqrt{N}}{L} C_{N-1}(k) (2i)^{-\lambda(N-1)} \left( \frac{L}{2\pi i} \right)^N \int_{\mathcal{I}^{(N)}_{\text{out}}} d^N[z'] \nu_{n,\lambda}(z, z') J^{(\lambda)}_N(n, z') ,$$

(78)

where we have set

$$n = \frac{L}{2\pi} k + (\lambda + 1)(N - 1) .$$

(79)

The relation of the set of integers $n$ to the set of quasimomenta $k$ is given by Eq. (69). Periodicity of $a_k(x, x')$ requires $n \in \mathbb{N}$. Therefore the Kronecker–delta for the quasimomenta transforms to

$$\delta_{k, k_i} \rightarrow \delta_{n, n_i + (\lambda+1)(N-i)} .$$

(80)

For $i = N$ this is just $\delta_{n, n_N}$. For $i \neq N$ it can only be non–zero for $\lambda \in \mathbb{N}$. This is one assertion of Prop. 5. Due to the antisymmetry of the integrand in Eq. (28), the lower bounds in $\mathcal{I}^{(N)}_{\text{out}}$ can be extended for all integration variables $x_i'$ to zero, respectively for all $z_i'$ in Eq. (78) to one, without changing the integral. This yields Eq. (75) and completes the proof of Prop. 5.

### 3.4 Rational Calogero–Moser–Sutherland system

For the type (II)–interaction potential the model is called rational CMS–model. Recursion formula (27) was derived in a slightly different form in [GK02]. There, instead of $\psi_N(k, x)$,

$$\phi_N(k, x) = \prod_{n<m} \text{sgn}(k_n - k_m) \text{sgn}(x_n - x_m) \frac{\psi_N(k, x)}{|\Delta_N(x)\Delta_N(k)|^{\lambda+1}}$$

(81)

was considered. $\phi_N(k, x)$ is completely symmetric in $x$ and in $k$, as well as under interchange of the two sets $x$ and $k$. For the special values $\lambda = -1/2, 0, 1$ it is the group integral (6) over $U(N)$, where $U(N)$ is the unitary group over the real field ($\lambda = -1/2$), the complex field ($\lambda = 0$) or the quaternion field ($\lambda = 1$). The parameter $\beta$ of Ref. [GK02] is related to the coupling constant $\lambda$ by $\lambda = \beta/2 - 1$ (see footnote 1). The measure function

$$\mu_{\lambda}(x, x') = |\Delta_{N+1}(x)|^{-\lambda-1} a_0^\dagger(x, x') |\Delta_N(x')|^{\lambda+1}$$

(82)

has the geometrical interpretation as the invariant Haar measure over the coset

$$\frac{\hat{U}(N)}{U(N-1)}, \quad \hat{U}(N) = \frac{U(N)}{U(1) \otimes \ldots U(1)} ,$$

(83)
in a special parametrization, called Gelfand–Zetlin coordinates [GT50, GK02].
$G_1 \otimes G_2$ denotes the direct product group. Up to now there exist explicit results for wave functions of the interacting rational CMS–model only for $\lambda = 1$ for small particle number up to $N = 4$ and for three particles for arbitrary $\lambda$ [BH03].

4 Proofs
In this section we prove Theorem 1 and Proposition 2

4.1 Proof of Theorem 1
We prove the five points of Theorem 1

1. In order to prove part 1 of Theorem 1 we show that the annihilation function $a_k(x, x')$ and the creation function $a_k^\dagger(x, x')$ defined in (25) are solutions of the differential equations (21). We first focus on the creation function $a_k^\dagger(x, x')$. We make for $a_k^\dagger(x, x')$ the Ansatz

$$a_k^\dagger(x, x') = \exp \left[ - \sum_{n<m}^{N+1} F(x_n - x_m) + \sum_{n,m}^N F(x'_n - x'_m) - \sum_{n<m}^N F(x'_n - x'_m) + ik \left( \sum_{n=1}^{N+1} x_n - \sum_{m=1}^{N} x'_m \right) \right],$$  

(84)

where $F(x)$ is an arbitrary even function. Acting with $\sum_{n=1}^{N+1} \frac{\partial^2}{\partial x_n^2}$ and with $\sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2}$ on $a_k^\dagger(x, x')$ yields

$$\sum_{n=1}^{N+1} \frac{\partial^2}{\partial x_n^2} a_k^\dagger(x, x') = \left[ \sum_{n \neq m} \left( [f(x_n - x_m)]^2 - f'(x_n - x_m) \right) + \sum_{l \neq m \neq n}^N f(x_n - x_m) f(x_n - x_l) - 2 \sum_{n \neq m}^N f(x_n - x_m) f(x_n - x'_l) + \sum_{l \neq m}^{N+1} \sum_{n}^N f(x_n - x'_m) f(x_n - x'_l) - (N+1)k^2 + \sum_{n}^{N+1} \sum_{m}^N [f^2(x_n - x'_m) + f'(x_n - x'_m) + ik f(x_n - x'_m)] \right] a_k^\dagger(x, x'),$$  

(85)
and, by the same token

\[
\sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2} a_k^\dagger(x,x') = \left[ \sum_{n \neq m} \left( |f(x'_n - x'_m)|^2 - f'(x'_n - x'_m) \right) + \right. \\
\sum_{l \neq m \neq n}^{N} f(x'_n - x'_m)f(x'_n - x'_l) + 2 \sum_{l \neq m}^{N} \sum_{n}^{N+1} f(x'_l - x'_m)f(x_n - x'_l) + \\
\left. \right. \left. \sum_{n}^{N+1} \sum_{m}^{N} \left[ f^2(x_n - x'_m) + f'(x_n - x'_m) + ikf(x_n - x'_m) \right] \right] a_k^\dagger(x,x'),
\]

where we defined \( f(x) = \frac{d}{dx} F(x) \) and \( f'(x) = \frac{d^2}{dx^2} F(x) \). We now define the Hamiltonian \( H_N \) in \( x \) representation, as

\[
\bar{H}_N(x) = -\sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2} + \sum_{n \neq m}^{N} \left( f^2(x_n - x_m) - f'(x_n - x_m) \right) + \\
\sum_{l \neq m}^{N} f(x_n - x_m)f(x_n - x_l),
\]

which comprises all terms in Eqs. (85) and (86), which depend only on one set of variables. Subtracting Eq. (85) from (86) yields

\[
\left[ \bar{H}_{N+1}(x) - \bar{H}_N(x') \right] a_k^\dagger(x,x') = k^2 a_k^\dagger(x,x') + \\
\sum_{l \neq m}^{N+1} \sum_{n}^{N} \left[ 2f(x'_l - x'_m)f(x_n - x'_l) - f(x_n - x'_m)f(x_n - x'_l) \right] + \\
\sum_{n \neq m}^{N+1} \sum_{l}^{N} \left[ f(x_n - x'_l)f(x_m - x'_l) + 2f(x_n - x_m)f(x_n - x'_l) \right] a_k^\dagger(x,x').
\]

If \( f(x) \) fulfills the functional equation \( f(x) = \frac{d}{dx} F(x) \)

\[
f(x)f(y) + f(x)f(z) + f(y)f(z) = \text{const.} \quad x + y + z = 0,
\]

the term in the squared bracket on the left hand side becomes constant\(^4\).

\(^4\)With regard to Eq. (88) one might conclude that the functional equation

\[
f(x_1 - x_2)f(x_1 - y) + f(x_2 - x_1)f(x_2 - y) + f(x_1 - y)f(x_2 - y) = v(x_1, x_2) + u(y)
\]

(90)
We can rewrite Eq. (88) as

$$\left[ \tilde{H}_{N+1}(x) - \tilde{H}_N(x') \right] a_k^\dagger(x, x') = \left[ k^2 - N(N + 1) \text{const.} \right] a_k^\dagger(x, x').$$

(92)

Observing that condition (89) also yields

$$\sum_{l \neq m \neq n} f(x_n - x_m)f(x_n - x_l) = -\frac{1}{3}N(N - 1)(N - 2) \text{const.},$$

(93)

we find

$$k^2 a_k^\dagger(x, x') = \left[ H_{N+1}(x) - H_N(x') \right] a_k^\dagger(x, x')$$

(94)

$$H_{N+1}(x) = -\sum_{n=1}^{N+1} \frac{\partial^2}{\partial x_n^2} + \sum_{n \neq m} \left[ f^2(x_n - x_m) - f'(x_n - x_m) + \text{const.} \right].$$

(95)

This is our assertion Eq. (26) with dispersion relation

$$\epsilon(k) = k^2.$$

(96)

The proof for the annihilation function $a_k(x, x')$ goes along the same lines. This completes the proof of part 1 and of part 2 of Theorem 1.

2. In order to prove part 3 we have to show that the action of $H_{N+1}$ ($H_{N-1}$) on the boundaries of the integration domain $I_{in}^{(N)}$ ($I_{out}^{(N)}$) does not give rise to boundary contributions. In other words, we have to show that

$$H_{N+1}(x) \int_{I_{in}^{(N)}} d^N[x'] a_0^\dagger(x, x') \chi_N(x') =$$

$$\int_{I_{in}^{(N)}} d^N[x'] a_0^\dagger(x, x') H_N(x') \chi_N(x')$$

$$H_{N-1}(x) \int_{I_{out}^{(N)}} d^N[x'] a_0(x, x') \chi_N(x') =$$

$$\int_{I_{out}^{(N)}} d^N[x'] a_0(x, x') H_N(x') \chi_N(x'),$$

(97)

with arbitrary functions $v$ and $u$ would be a sufficient, much weaker, condition on $f$. That this is not the case, is most easily seen by invoking translation invariance. Translation invariance of the left hand side of (90) requires translation invariance of the right hand side

$$v(x_1 + a, x_2 + a) + u(y + a) = v(x_1, x_2) + u(y)$$

$$\left( \frac{d}{dx_1} + \frac{d}{dx_2} \right) v(x_1, x_2) + \frac{d}{dy} u(y) = 0.$$  

(91)

Therefore we obtain $v(x_1, x_2) = v(x_1 - x_2)$ and $u(y) = \text{const.}$. This means, the right hand side of Eq. (90) is independent of $y$. We now invoke permutation symmetry of the left hand side under interchanging $x_1 \rightarrow y$ and $x_2 \rightarrow y$. The same symmetry must hold on the left hand side, and therefore $v(x_1, x_2) = \text{const.}$, too. Eq. (90) reduces to Eq. (89).
where we choose for convenience $k_{N+1} = 0$. Here $\chi_N(x') \in D$ is a test function $D \subset H_N$. The proof of Eq. (97) is given in App. A.

We now show that $a_k$ is an annihilation operator, i.e.

$$a_k|\psi_N(k)\rangle = 0, \quad \text{if } k \neq k_1 \ldots k_N . \quad (98)$$

To this end (for the moment) we assume part 5 of Theorem 1 to be proven and consider the scalar product

$$\langle \psi_N(k')|\psi_N(k)\rangle = \left\{ \begin{array}{ll}
\det[2\pi\delta(k_i' - k_j)]_{1 \leq i,j \leq N} & \\
\det[L\delta(k_i')_{1 \leq i \leq N} & \\
= \int_{\Omega_N} d|x| \psi_N^*(k',x) \psi_N(k,x) & \\
= \int_{\Omega_N} d|x| \int_{i_{in}^{(N)}} d[x'] \int_{i_{out}^{(N)}} d[x''] \left[ a_{k_N}^\dagger(x,x') \right]^* \psi_{N-1}^*(k',x') \int_{i_{in}^{(N)}} d[x'] a_{k_N}(x',x) & \\
& \int_{i_{out}^{(N)}} d[x''] a_{k_N}^\dagger(x,x'') \psi_{N-1}(k,x'') & \\
= \langle \psi_{N-1}(k')|a_{k_N}^\dagger \psi_N(k)\rangle & \\
= 0, \quad \text{if } k_N' \neq k_1 \ldots k_N . & (99)
\right. $$

This proves Eq. (98), since Eq. (99) holds for an arbitrary wave function $|\psi_{N-1}(k')\rangle$.

3. The pieces of part 4 of Theorem 1 which concern the action of the Hamiltonian have been proven already before. To complete the proof we have to show in addition that a differential equation similar to Eq. (21) holds for the center of mass momentum operator $P_N(x)$, defined in Eq. (31), namely

$$[P_{N+1}(x) - P_N(x')] a_k^\dagger(x,x') = k a_k^\dagger(x,x')$$

$$[P_N(x) - P_{N+1}(x')] a_k(x,x') = -k a_k(x,x') . \quad (100)$$

It is straightforward to see that this is true.

4. Finally we have to prove the orthogonality relation (i.e. part 5 of Theorem 1). To this end we observe that the $N$ real numbers $k = \{k_1, \ldots, k_N\}$ are conserved quantities. This means that exactly $N$ mutually commuting selfadjoint operators $I_n$, $1 \leq n \leq N$ can be constructed with eigenvalues $E_n = \sum_{i=1}^N k_i^n$. For the potentials in Table 1 these operators can be
constructed with Dunkl operators $p_m$ \cite{Dum89, Pol92}

\[
I_n = \sum_{m=1}^{N} p_m^n \\
p_m = -i \frac{\partial}{\partial x_m} + \sum_{n \neq m} f(x_m - x_n) P_{nm},
\]

(101)

where $P_{nm}$ is the exchange operator defined in Eq. (59). Therefore, for any two unequal sets $k$ and $k'$ at least one selfadjoint operator $I_n$ can be found for which

\[
I_n \psi_N(k, \mathbf{x}) \neq I_n \psi_N(k', \mathbf{x}), \quad \text{if } k \neq k'.
\]

(102)

Now we can invoke a fundamental theorem of functional analysis for selfadjoint operators: Two eigenfunctions of a selfadjoint operator to different eigenvalues are orthogonal. This completes the proof.

4.2 Proof of Proposition 2

The five potentials of Table 1 have to be treated differently.

Potentials (III)–(V) For the potentials (III)–(V) the normalisation constant $C_N(k)$ is determined by the condition that in the asymptotic regime, where all particles are far apart, $x_i - x_{i+1} \propto M$, $M \to \infty$, $\forall i < N$, the wave function $\psi_{N+1}(k, \mathbf{x})$ becomes the free Fermion wave function $\psi_{N+1}^{(0)}(k, \mathbf{x})$, defined in Eq. (46). For the $\delta$–interaction potential (V) the normalisation has already been calculated in Sec. 3.2.

In the asymptotic regime $M \to \infty$ the cases of hyperbolic interaction potentials, (III) and (IV), can be reduced to the case of $\delta$–interaction. To see this we introduce the rescaled particle coordinates $Mx_i$ and $M\tilde{x}_i = x'_i$ in the recursion formula Eq. (27) and obtain

\[
\psi_{N+1}(k, \mathbf{Mx}) = \frac{M^N C_N(k)}{\sqrt{N+1}} \int_{i_n^{(N)}} d^N [\tilde{x}] a_{k_{N+1}}^\dagger(M \mathbf{x}, M \tilde{\mathbf{x}}) \psi_{N}(k, M\tilde{\mathbf{x}}).
\]

(103)

We now split the integration region into a region where all integration variables $\tilde{x}_i$ are away from the boundaries $x_j$ and the regions where one or more integration variables are close such that $(x'_i - x'_{i+1})/M \to 0$ for $M \to \infty$. Since the integrand has by construction no singularities the second region has integration measure zero for $M \to \infty$ and does not contribute in leading order in $M$. In the first region all integration variables $\tilde{x}_i$ are away from the boundaries $x_j$ and consequently apart from each other $\tilde{x}_i - \tilde{x}_j = \mathcal{O}(M^0)$. Therefore we can substitute $\psi_{N}(k, M\tilde{\mathbf{x}})$ by the free particle wave function (46). On the other hand the creation functions $a_{k_{N+1}}^\dagger(M \mathbf{x}, M\tilde{\mathbf{x}}')$ for the three potentials (III)–(V) become
identical for \( M \to \infty \), since

\[
\begin{align*}
\lim_{M \to \infty} \lambda \ln |\sinh(Ma x)| &= \lim_{M \to \infty} \lambda \ln |\cosh(Ma x)| = M c |x|, \quad c = a \lambda . \tag{104}
\end{align*}
\]

In particular \( \lim_{M \to \infty} a_k^0(Mx, Mx') = 1 \). We directly find, using Eqs. [58] and [59]

\[
\psi_{N+1}(k, Mx) = \frac{C_N(k)}{\prod_{i=1}^{N} (ik_i - i k_{N+1})} \psi_{N+1}^{(0)}(k, Mx), \tag{105}
\]

which is the result of Prop. 2. This completes the proof for potentials (III) to (V).

**Potential (I) and (II)** For the trigonometric potential (I) the normalisation constant

\[
C_N(k) = \left( \frac{L}{2 \pi i} \right)^{-N} \prod_{i=1}^{N} \frac{(2i)^\lambda}{B \left( \frac{L}{2 \pi} (k_i - k_{N+1}), \lambda + 1 \right)}. \tag{106}
\]

has been calculated already in Sec. 3.3. To obtain the normalisation constant for the rational potential (II) one can either follow the route of Sec. 4.2 and perform the integration in the asymptotic regime. But the evaluation of the resulting integral is by no means trivial. Therefore we resort to a different route.

Eigenfunctions for the rational CMS Hamiltonian (II) emerge from the trigonometric case (I) in the limit \( L \to \infty \) keeping \( k \) and \( x \) finite. The same happens to their creation functions

\[
a_k^{(II)}(x, x') = \lim_{L \to \infty} \left( \frac{L}{\pi} \right)^{N \lambda} a_k^{(I)}(x, x'), \tag{107}
\]

where the upper index denotes the type of potential. This allows us to obtain the normalisation for the potential (II) case by taking the limit \( L \to \infty \) of Eq. (106) using the asymptotic expansion of the beta–function

\[
\lim_{x \to \infty} \frac{1}{B(x, y)} = \frac{x^y}{\Gamma(y)} (1 + \text{lower order terms}) . \tag{108}
\]

The result is

\[
C_N(k) = \frac{\sqrt{N + 1}^{N(\lambda + 1)}}{\Gamma^N (\lambda + 1)} \prod_{i=1}^{N} |k_i - k_{N+1}|^\lambda (k_i - k_{N+1}) . \tag{109}
\]

It differs from the normalisation Eq. (5.7) of Ref. [GK02] by a \( k \) independent factor.
5 Conclusions

We constructed creation and annihilation operators for spinless interacting Fermions. Applying the creation operators successively onto the vacuum any \(N\) particle eigenstate can thereby be generated. The eigenstates are given as a \((N - 1)N/2\)-fold integral. For the trigonometric CMS–Hamiltonian the equivalence of these eigenfunctions to other representations of the eigenstates has been demonstrated.

The developed formalism paves a new way of searching and classifying integrable quantum systems, complementary or alternative to Bethe’s Ansatz and to the Yang–Baxter equation. An interacting many–body Hamiltonian is exactly solvable if it can be transformed by a unitary transformation to a Hamiltonian containing one–body operators only. A method to use the present formalism for the calculation of thermodynamical quantities and of correlation functions is yet to be developed.

It has to be stressed that the constructed operators have always Fermionic commutation relations. Therefore they differ fundamentally from the Bosonic operators which appear for instance in the Bosonisation approach. The latter describes the fundamental excitations of the \(N\) particle system by Bosonic operators which act on top of the filled Fermi sea. Thereby the Hamiltonian is effectively diagonalised. In contrast our diagonalisation is exact.

The constructed operators are similar but not equal to the Fadeev–Zamolodchikov operators [EK05]. The latter obey commutation relations which involve the two–body scattering matrix and need not necessarily have the commutation relations (19) with the Hamiltonian. More on the connection of the creation (annihilation) operators, constructed here, to the Fadeev–Zamolodchikov algebra will be given in a subsequent publication [Koh], where spin 1/2 Fermions are to be discussed.

In the present work we focused on non–relativistic spinless Fermions. Moreover translation invariance was assumed. It has to be stressed that the developed formalism does not hinge on these assumptions. The extensions to spin 1/2 Fermions and to Bosons will be given in separate publications. Application to lattice theories and 1–d relativistic field theories present an interesting challenge for the future.

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A Discussion of Boundary terms

In this appendix some technical details concerning boundary contributions are reviewed. In particular the proof of Eq. (97) is given in some more detail.

Scattering Boundary Conditions (SBC) We look at the creation operator, i. e. at the first line of Eq. (97). The action of \(H_{N+1}(x)\) onto the integral
yields after a somewhat cumbersome calculation

\[
H_{N+1}(x) \int_{I_{in}^{(N)}} d^{N}[x'] a_0^\dagger(x, x') \chi_N(x') =
\int_{I_{in}^{(N)}} d^{N}[x'] H_{N+1}(x) a_0^\dagger(x, x') \chi_N(x') + \text{boundary terms}^{(1)}. \quad (110)
\]

We considered for simplicity the operator \( a_0^\dagger \). The results are easily extended to \( a_k^\dagger \). The boundary terms are given by

\[
\text{boundary terms}^{(1)} = - \sum_{i=1}^{N} \int d^{N-1}[x'_{\neq i}] a_0^\dagger(x', x) \frac{\partial}{\partial x_i} \chi_N(x') \bigg|_{x'_i=x_i}
- \sum_{i=1}^{N} \int d^{N-1}[x'_{\neq i}] \frac{\partial}{\partial x_i} a_0^\dagger(x', x) \chi_N(x') \bigg|_{x'_i=x_i}
+ \sum_{i=1}^{N} \int d^{N-1}[x'_{\neq i}] a_0^\dagger(x', x) \frac{\partial}{\partial x_i} \chi_N(x') \bigg|_{x'_i=x_{i+1}}
+ \sum_{i=1}^{N} \int d^{N-1}[x'_{\neq i}] \frac{\partial}{\partial x_{i+1}} a_0^\dagger(x', x) \chi_N(x') \bigg|_{x'_i=x_{i+1}}. \quad (111)
\]

We now use part 1 of Theorem 1 and substitute \( H_{N+1}(x) \) by \( H_N(x') \) in Eq. (110). Integration by parts then yields

\[
\int_{I_{in}^{(N)}} d^{N}[x'] H_N(x') a_0^\dagger(x, x') \chi_N(x') = \int_{I_{in}^{(N)}} d^{N}[x'] a_0^\dagger(x, x') H_N(x') \chi_N(x')
+ \text{boundary terms}^{(2)}, \quad (112)
\]

and altogether

\[
H_{N+1}(x) \int_{I_{in}^{(N)}} d^{N}[x'] a_0^\dagger(x, x') \chi_N(x') =
\int_{I_{in}^{(N)}} d^{N}[x'] a_0^\dagger(x, x') H_N(x') \chi_N(x')
+ \text{boundary terms}^{(1)} + \text{boundary terms}^{(2)}. \quad (113)
\]

This is the assertion of Eq. (28) if the two boundary terms cancel each other. A direct calculation, using the identity

\[
\frac{\partial}{\partial x_i} a_0^\dagger(x', x) \bigg|_{x'_i=x_i} = - \frac{\partial}{\partial x'_j} a_0^\dagger(x', x) \bigg|_{x'_j=x_i} \quad (114)
\]

shows that this is so.
We now look at the action of the annihilation operator, i.e. at the second line in Eq. (97). Acting with $H_{N-1}$ onto the integral yields an expression similar to Eq. (113)

$$H_{N-1}(\mathbf{x}) \int_{I_{\text{out}}^{(N)}} d^N[\mathbf{x}'] a_0(\mathbf{x}, \mathbf{x}') \chi_N(\mathbf{x}') = \int_{I_{\text{out}}^{(N)}} d^N[\mathbf{x}'] a_0(\mathbf{x}, \mathbf{x}') H_N(\mathbf{x}') \chi_N(\mathbf{x}')$$

(115) + boundary terms$^{(1)}$ + boundary terms$^{(2)}$.

However, now the two boundary contributions do not cancel each other identically. The boundary contributions at $\pm \infty$ vanish if the annihilation function $a_0(\mathbf{x}, \mathbf{x}')$ is bound from above such that for $\chi_N(\mathbf{x}') \in D$ the product $a_0(\mathbf{x}, \mathbf{x}') \chi_N(\mathbf{x}')$ is integrable:

$$\exists C \in \mathbb{R} : a_0(\mathbf{x}, \mathbf{x}') < C \quad \text{for} \quad x_1' \to \infty \quad \text{and} \quad x_N' \to -\infty .$$

(116)

For all cases, considered in Table 1, this is indeed the case$^5$.

Periodic Boundary Conditions (PBC) For the action of the creation operator the treatment of the boundary contributions is completely analogous to the treatment for SBC. Here, in addition, we observe that periodicity of the integral

$$I_{N+1}(k, \mathbf{x}) = \int_{I_{\text{in}}^{(N)}} d^N[\mathbf{x}'] a_k^\dagger(\mathbf{x}, \mathbf{x}') \chi_N(\mathbf{x}')$$

(117) in all unprimed variables $x_i$, $1 \leq i \leq N+1$ with period $L$ requires on the one hand $a_k^\dagger(\mathbf{x}, \mathbf{x}')$ to be periodic in $x_i$, $1 \leq i \leq N+1$ and on the other hand

$$\int_0^L dx_i a_k^\dagger(\mathbf{x}, \mathbf{x}') \chi_N(\mathbf{x}') = 0 , \quad \forall 1 \leq i \leq N .$$

(118)

In order to see that for the action of the annihilation operator no boundary terms occur, we rewrite the integration domain $I_{\text{out}}^{(N)}$ in the second line of Eq. (97) as

$$I_{N-1}(k, \mathbf{x}) = \int_{I_{\text{in}}^{(N)}} d^N[\mathbf{x}'] a_k(\mathbf{x}, \mathbf{x}') \chi_N(\mathbf{x}')$$

$$= \int_0^L dx_1' \int_0^{x_1'} dx_2' \cdots \int_0^{x_{N-1}'} dx_N' a_k(\mathbf{x}, \mathbf{x}') \chi_N(\mathbf{x}') .$$

(119)

The second equation was obtained due to the total antisymmetry of the integrand. Now it is seen that periodicity of $I_{N-1}(\mathbf{x})$ in all variables $x_i$, $1 \leq i \leq N+1$

$^5$For particles with $\delta$–interaction this is only so, because we are considering spinless fermions and $a_0(\mathbf{x}, \mathbf{x}') = 1$, see Sec. 3.2.
\( N - 1 \), requires the integral over one period to vanish for all integrations but for the integration over \( x'_1 \)

\[
\int_0^L dx'_1 a_k(x, x')\chi_N(x') = 0, \quad \forall i \neq 1.
\]  
(120)

This allows us to write \( I_{N-1}(k, x) \) as

\[
I_{N-1}(k, x) = \int_0^L dx'_1 \int_0^L dx'_2 \ldots \int_0^L dx'_N \prod_{i=1}^{N-1} \frac{1}{2} \text{sgn}_L(x_i - x'_{i+1}) a_k(x, x') \chi_N(x'),
\]  
(121)

where \( \text{sgn}_L(x) \) is the periodic sign–function \( \text{sgn}_L(x) = \text{sgn}_L(x + L) \). In this form it is clear that

\[
H_{N-1}(x)I_{N-1}(k, x) = \int_0^L dx'_1 \int_0^L dx'_2 \ldots \int_0^L dx'_N \prod_{i=1}^{N-1} \frac{1}{2} \text{sgn}_L(x_i - x'_{i+1}) a_k(x, x') H_N(x') \chi_N(x'),
\]  
(122)

if the integrand is a periodic function and in particular if \( a_k(x, x') \) is periodic with period \( L \) in all \( x'_i \), \( 1 \leq i \leq N \). This is the assertion of Prop. [11] Indeed, for PBC the form (121) serves as a definition of the integration domain of the annihilation operator \( a_k \) alternative to \( I^{(N)}_{\text{out}} \) defined in Eq. (29).

References

[And91] G. W. Anderson. A short proof of Selberg’s generalized beta formula. *Forum Math.*, 3:415, 1991.

[AS72] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions*. Dover, New York, 9th edition, 1972.

[Bet31] H. Bethe. Zur Theorie der Metalle. I. Eigenwerte und Eigenfunktionen der linearen Atomkette. *Zeitschrift f. Physik*, 71:205, 1931.

[BH03] E. Brezin and S. Hikami. An extension of the Harish Chandra–Itzykson–Zuber integral. *Comm. Math. Phys.*, 223:363, 2003.

[Cal75a] F. Calogero. Exactly solvable one–dimensional many–body problems. *Lett. Nuovo Cimento (2)*, 13:411, 1975.

[Cal75b] F. Calogero. One–dimensional many–body problems with pair interaction whose exact ground–state wave function is of product type. *Lett. Nuovo Cimento*, 13:507, 1975.
[CZ04a] V. V. Cheianov and M. B. Zvonarev. Nonunitary spin-charge separation in a one-dimensional fermion gas. *Phys. Rev. Lett.*, 92:176401, 2004.

[CZ04b] V. V. Cheianov and M. B. Zvonarev. Zero temperature correlation functions for the impenetrable fermion gas. *Phys. Rev. A*, 37:2261, 2004.

[Dix05] A. L. Dixon. Generalization of Legendre’s formula $ke' - (k - e)' = \frac{1}{2}\pi$. *Proc. London Math. Soc.*, 3:206, 1905.

[Dun89] C. F. Dunkl. Difference–differential operators associated to reflection groups. *Trans. Amer. Math. Soc.*, 311:167, 1989.

[EK05] F. H. L. Essler and R. M. Konik. Applications of massive integrable quantum field theories to problems in condensed matter physics. Kogan Memorial Volume. World Scientific, 2005.

[For92] P. J. Forrester. Selberg correlation integrals and the $1/r^2$ quantum many body system. *Nucl. Phys. B*, 388:671, 1992.

[For93] P. J. Forrester. Recurrence equations for the computation of correlations in the $1/r^2$ quantum many body system. *J. Stat. Phys.*, 72:39, 1993.

[For95] P. J. Forrester. Integration formulas and exact calculations in the Calogero–Sutherland model. *Phys. Lett. B*, 9:359, 1995.

[FRZ04] J. N. Fuchs, A. Recati, and W. Zwerger. Exactly solvable model of the BCS-BEC crossover. *Phys. Rev. Lett.*, 93:090408, 2004.

[FS80] G. Frobenius and L. Stickelberger. Über die Addition und Multiplikation der elliptischen Functionen. *J. Reine Angew. Math.*, 88:146, 1880.

[Gau66] Gaudin. Un systeme a une dimension de fermions en interaction. *Phys. Lett.*, 24A:55, 1966.

[GK02] T. Guhr and H. Kohler. Recursive construction for a class of radial functions: I ordinary space. *J. Math. Phys.*, 43:2707, 2002.

[GT50] I. M. Gelfand and M. L. Tzetlin. Matrix elements for the unitary groups (russian). *Dokl. Akad. Nauk.*, 71:825, 1950.

[Ha95] Z. N. C. Ha. Fractional statistics in one–dimension: View from an exactly solvable model. *Nucl. Phys. B*, 435:604, 1995.

[Kaw92] N. Kawakami. Asymptotic Bethe–ansatz solution of multicomponent quantum systems with $1/r^2$ long–range interaction. *Phys. Rev. B*, 46:1005, 1992.
[Koh] H. Kohler. Exact diagonalisation of 1-d spin 1/2 Fermions. in preparation.

[LL63] E. Lieb and W. Liniger. Exact analysis of an interacting Bose gas. I. the general solution and the ground state. *Phys. Rev.*, 130:1605, 1963.

[Mac95] I. G. MacDonald. *Symmetric functions and Hall polynomials*. Oxford University Press, Oxford, 2nd edition, 1995.

[OO97] A. Okounkov and G. Olshanski. Shifted Jack polynomials, binomial formula, and applications. *Math. Res. Lett.*, 4:69, 1997.

[OP83] M. A. Olshanetsky, A. M. Perelomov. Quantum Integrable Systems Related to LieAlgebras. *Phys. Rep.*, 94:313, 1983.

[Pol92] A. P. Polychronakos. Exchange operator formalism for integrable systems of particles. *Phys. Rev. Lett.*, 69:703, 1992.

[Sta89] R. P. Stanley. Some combinatorial properties of Jack symmetric functions. *Advances in Math.*, 77:76, 1989.

[Sut71a] B. Sutherland. Quantum many–body problem in one dimension: Ground state. *J. Math. Phys.*, 12:246, 1971.

[Sut71b] B. Sutherland. Quantum many–body problem in one dimension: Thermodynamics. *J. Math. Phys.*, 12:251, 1971.

[Sut75] B. Sutherland. Exact ground-state wave function for a one-dimensional plasma. *Phys. Rev. Lett.*, 34:1083, 1975.

[Yan67] C. N. Yang. Some exact results for the many-body problem in one dimension with repulsive delta-function interaction. *Phys. Rev. Lett.*, 19:1312, 1967.