On Fuzzy Differential Subordination

A. Haydar Eş

Abstract. The theory of differential subordination was introduced by S.S.Miller and P.T.Mocanu in [2], then developed in many papers. In [1] the authors investigate various subordination results for some subclasses of analytic functions in the unit disc. G.I.Oros and G.Oros define the notion of fuzzy subordination and in [3, 4, 5] they define the notion of fuzzy differential subordination. In this paper, we determine sufficient conditions for a multivalent function to be a dominant of the fuzzy differential subordination.

1. Introduction

We introduce some basic notions and results that are used in the sequel.

Definition 1.1 ([6]). Let $X$ be a non-empty set. An application $F : X \to [0,1]$ is called fuzzy subset. An alternate definition, more precise, would be the following: A pair $(A,F_A)$, where $F_A : X \to [0,1]$ and

$$A = \{x \in X : 0 < F_A(x) \leq 1\} = supp(A,F_A),$$

is called fuzzy subset.

Proposition 1.1 ([3]). If $(M,F_M) = (N,F_N)$, then we have $M = N$, where $M = supp(M,F_M), N = supp(N,F_N)$.

Proposition 1.2 ([3]). If $(M,F_M) \subseteq (N,F_N)$, then we have $M \subseteq N$, where $M = supp(M,F_M), N = supp(N,F_N)$.

We also need the following notations and results from the classical complex analysis [5].

For $D \subset \mathbb{C}$, we denote by $\mathcal{H}(D)$ the class of holomorphic functions on $D$, and by $\mathcal{H}_n(D)$ the class of holomorphic and univalent functions on $D$.

In this paper, we denote by $\mathcal{H}(U)$ the set of holomorphic functions in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ with $\partial U = \{z \in \mathbb{C} : |z| = 1\}$ the boundary of the unit disc.

For $a \in \mathbb{C}$ and $n \in \mathbb{N}$ we denote

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\[ \mathcal{H}[a, n] = \{ f \in \mathcal{H}(U) : f(z) = a + a_nz^n + a_{n+1}z^{n+1} + \ldots, z \in U \}, \]
\[ A_n = \{ f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots, z \in U \} \text{ with } A_1 = A, \]
and \( S = \{ f \in A : f \text{ a univalent function in } U \}. \)

Let \( \mathcal{B} = \{ \varphi \in \mathcal{H}(U) : \varphi(0) = 0, |\varphi(z)| < 1, z \in U \} \) denote the class of Schwarz functions.

**Definition 1.2** ([4]). Let \( f, g \in \mathcal{H}(U) \). We say that the function \( f \) is subordinated to \( g \), written \( f \prec g \) or \( f(z) < g(z) \) if there exists a function \( w \in \mathcal{H}(U) \) with \( w(0) = 0 \) and \( |w(z)| < 1, z \in U \), (which means \( w \in \mathcal{B} \)) such that \( f(z) = g(w(z)), z \in U \).

Let \( D \subset \mathbb{C} \) and \( f, g \in \mathcal{H}(D) \) holomorphic functions. We denote by
\[ f(D) = \{ f(z)|0 < F_{f(D)}f(z) \leq 1, z \in D \} = \text{supp}(f(D), F_{f(D)}) \]
and
\[ g(D) = \{ g(z)|0 < F_{g(D)}g(z) \leq 1, z \in D \} = \text{supp}(g(D), F_{g(D)}). \]

**Definition 1.3** ([5]). Let \( D \subset \mathbb{C}, z_0 \in D \) be a fixed point, and let the functions \( f, g \in \mathcal{H}(D) \). The function \( f \) is said to be fuzzy subordinate to \( g \) and write \( f < F g \) or \( f(z) < F g(z), \) if
1. \( f(z_0) = g(z_0), \)
2. \( F_{f(D)}f(z) \leq F_{g(D)}g(z), z \in D. \)

**Proposition 1.3** ([5]). Let \( D \subset \mathbb{C}, z_0 \in D \) be a fixed point, and let the functions \( f, g \in \mathcal{H}(D) \). If \( f(z) < F g(z), z \in D, \) then
1. \( f(z_0) = g(z_0), \)
2. \( f(D) \subseteq g(D), \) where \( f(D) = \text{supp}(f(D), F_{f(D)}), g(D) = \text{supp}(g(D), F_{g(D)}). \)

The equality occurs if and only if \( F_{f(D)}f(z) = F_{g(D)}g(z). \) Denoted by
\[ S^* = \{ f \in A : Re\frac{zf'(z)}{f(z)} > 0, z \in U \} \]
the class of normalized starlike functions in \( U, \)
\[ K = \{ f \in A : Re\frac{zf''(z)}{f'(z)} + 1 > 0, z \in U \} \]
the class of normalized convex functions in \( U \) and by
\[ C = \{ f \in A : \exists \varphi \in K, Re\frac{f(z)}{\varphi(z)} > 0, z \in U \} \]
the class of normalized close-to-convex functions in \( U \) [5].

Let \( J(\alpha, f; z) = (1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha(1 + \frac{zf''(z)}{f'(z)}), z \in U, \) for \( \alpha \) real number and \( f \in A_p \) [2].

Let \( \Omega = \text{supp}(\Omega, F_{\Omega}) = \{ z \in \mathbb{C} : 0 < F_{\Omega}(z) \leq 1 \}, \)
\( \Delta = \text{supp}(\Delta, F_{\Delta}) = \{ z \in \mathbb{C} : 0 < F_{\Delta}(z) \leq 1 \}, p(U) = \text{supp}(p(U), F_{p(U)}) \)
\( = \{ f(z) : 0 < F_{p(U)}(f(z)) \leq 1 \}, z \in U \} \) and
\( \psi(\mathbb{C}^3 \times U) = \text{supp}(\psi(\mathbb{C}^3 \times U), F_{\psi(\mathbb{C}^3 \times U)}) \)
\( = \{ \psi(p(z), zp^2p''(z); z) : 0 < F_{\psi(\mathbb{C}^3 \times U)}(\psi(p(z), zp^2p''(z)), z) \leq 1, z \in U \} \)
[4].
**Definition 1.4** ([4]). Let \( \psi : \mathbb{C}^3 \times U \to \mathbb{C} \) and let \( h \) be univalent in \( U \). If \( p \) is analytic in \( U \) and satisfies the (second-order) fuzzy differential subordination

\[
F_{\psi(\mathbb{C}^3 \times U)}(\psi(p(z), zp'(z), z^2 p''(z); z) \leq F_{h(U)} h(z) \quad (1)
\]

i.e. \( \psi(p(z), zp'(z), z^2 p''(z); z) < F h(z), z \in U \), then \( p \) is called a fuzzy solution of the fuzzy differential subordination. The univalent function \( q \) is called a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, or more simple a fuzzy dominant, if \( p(z) < F g(z), z \in U \), for all \( p \) satisfying (1). A fuzzy dominant \( \tilde{q} \) that satisfies \( \tilde{q}(z) < F q(z), z \in U \), for all fuzzy dominant \( q \) of (1) is said to be the fuzzy best dominant of (1).

**Theorem 1.1** ([5]). Let \( h \) be analytic in \( U \), let \( \phi \) be analytic in domain \( D \) containing \( h(U) \) and suppose

a) \( \Re \phi[h(z)] > 0, z \in U \) and

b) \( h(z) \) is convex.

If \( p \) is analytic in \( U \), with \( p(0) = h(0), p(U) \subset D \) and

\[
\psi(\mathbb{C}^2 \times U) \to \mathbb{C}, \psi(p(z), zp'(z)) = p(z) + zp'(z), \phi[p(z)] \text{ is analytic in } U,
\]

then

\[
F_{\psi(\mathbb{C}^2 \times U)} \psi(p(z), zp'(z)) \leq F_{h(U)} h(z),
\]

implies

\[
F_{p(U)} p(z) \leq F_{h(U)} h(z), z \in U,
\]

where

\[
\psi(\mathbb{C}^2 \times U) = \text{supp}(\mathbb{C}^2 \times U, F_{\psi(\mathbb{C}^2 \times U)} \psi(p(z), zp'(z)) = \{ z \in \mathbb{C} : 0 < F_{\psi(\mathbb{C}^2 \times U)} \psi(p(z), zp'(z)) \leq 1 \},
\]

\[
h(U) = \text{supp}(U, F_{h(U)} h(z)) = \{ z \in \mathbb{C} : 0 < F_{h(U)} h(z) \leq 1 \}.
\]

**Theorem 1.2** ([5]). Let \( h \) be convex in \( U \) and let \( P : U \to \mathbb{C} \), with \( \Re P(z) > 0 \). If \( p \) is analytic in \( U \) and \( \psi : \mathbb{C}^2 \times U \to \mathbb{C} \),

\[
\psi(p(z), zp'(z)) = p(z) + P(z)zp'(z)
\]

is analytic in \( U \), then

\[
F_{\psi(\mathbb{C}^2 \times U)} [p(z) + P(z)zp'(z)] \leq F_{h(U)} h(z),
\]

implies

\[
F_{p(U)} P(z) \leq F_{h(U)} h(z), z \in U.
\]

**Theorem 1.3** ([5]). (Hallenbeck and Ruscheweyh) Let \( h \) be a convex function with \( h(0) = a \), and let \( \gamma \in \mathbb{C}^* \) be a complex number with \( \Re \gamma \geq 0 \). If \( p \in \mathcal{H}[a, n] \) with \( p(0) = a \) and \( \psi : \mathbb{C}^2 \times U \to \mathbb{C}, \psi(p(z) + zp'(z)) = p(z) + \frac{1}{\gamma} zp'(z) \) is analytic in \( U \), then

\[
F_{\psi(\mathbb{C}^2 \times U)} [p(z) + \frac{1}{\gamma} zp'(z)] \leq F_{h(U)} h(z),
\]

implies

\[
F_{p(U)} p(z) \leq F_{q(U)} q(z) \leq F_{h(U)} h(z), z \in U,
\]
where
\[
q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\gamma-1} \, dt.
\]
The function \(q\) is convex and is the fuzzy best \((a, n)\)-dominant.

2. Main Results

**Proposition 2.1.** Let \(q\) be univalent in \(U\) and let \(\theta\) and \(\phi\) be analytic in a domain \(D\) containing \(q(U)\), with \(\phi(w) \neq 0\), when \(w \in q(U)\). Set \(Q(z) = zq'(z)\phi[q(z)]\) and \(h(z) = \theta[q(z)] + Q(z)\) and suppose that either

(i) \(Q\) is starlike, or
(ii) \(h\) is convex.

In addition, assume that

(iii) \(\text{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \text{Re}\left(\frac{\theta'[q(z)]}{\phi[q(z)]} + \frac{zQ'(z)}{Q(z)}\right) > 0\).

If \(p\) is analytic in \(U\), with \(p(0) = q(0), p(U) \subset D\) and \(\psi : \mathbb{C}^2 \times U \to \mathbb{C}, \psi(p(z),zp'(z)) = p(z) +zp'(z)\phi[p(z)]\) is analytic in \(U\), then

\[F_{\psi(\mathbb{C}^2 \times U)}[p(z) +zp'(z)\phi[p(z)]] \leq F_{h(U)}h(z),\]

implies

\[F_{p(U)}p(z) \leq F_{q(U)}q(z), \quad z \in U, \text{ i.e.}\]

\[p(z) <_{\mathcal{F}} q(z), \text{ and } q \text{ is the best dominant, where}\]

\[
\psi(\mathbb{C}^2 \times U) = \text{supp}(\mathbb{C}^2 \times U, F_{\psi(\mathbb{C}^2 \times U)}\psi(p(z),zp'(z)))
\]
\[= \{z \in \mathbb{C} : 0 < F_{\psi(\mathbb{C}^2 \times U)}\psi(p(z),zp'(z)) \leq 1\}, \text{ and}\]

\[h(U) = \text{supp}(U,F_{h(U)}h(z)) = \{z \in \mathbb{C} : 0 < F_{h(U)}h(z) \leq 1\}.\]

**Proof.** The proof of Proposition is similar to Theorem 1.1[5].

**Proposition 2.2.** Let \(q \in \mathcal{H}[p, p]\) be univalent, \(q(z) \neq 0\) and satisfies the following conditions.

(i) \(\frac{zq'(z)}{q(z)}\) is starlike,

(ii) \(\text{Re}(\frac{2q(z)}{\alpha} + 1 + \frac{zq''(z)}{q(z)} - \frac{zq'(z)}{q(z)}) > 0\) for all \(\alpha \neq 0\) and for all \(z \in U\).

For \(p \in \mathcal{H}[p, p]\) with \(p(z) \neq 0\) in \(U\) and

\[
\psi : \mathbb{C}^2 \times U \to \mathbb{C}, \psi(p(z),zp'(z)) = p(z) + \alpha \frac{zp'(z)}{p(z)}
\]
is analytic in \(U\), then

\[F_{\psi(\mathbb{C}^2 \times U)}[p(z) + \alpha \frac{zp'(z)}{p(z)}] \leq F_{\psi(\mathbb{C}^2 \times U)}[q(z) + \alpha \frac{zq'(z)}{q(z)}] = F_{h(U)}h(z),\]

implies

\[F_{p(U)}p(z) \leq F_{q(U)}q(z) \text{ i.e. } p(z) <_{\mathcal{F}} q(z), \quad z \in U,\]

and \(q\) is the best dominant.

**Proof.** Define the function \(\theta\) and \(\phi\) by \(\theta(w) = w, \phi(w) = \frac{\alpha}{w}, D = \{w : w \neq 0\}\) in Proposition 2.1. Then the functions

\[Q(z) = zq'(z)\phi[q(z)] = \alpha \frac{zq'(z)}{q(z)},\]
\[ h(z) = \theta[q(z)] + Q(z) = q(z) + \lambda \frac{zq'(z)}{q(z)}. \]

Since \( \frac{zq'(z)}{q(z)} \) is starlike, we obtain that \( Q \) is starlike in \( U \) and \( Re(\frac{zh'(z)}{Q(z)}) > 0 \) for all \( z \in U \). It follows Proposition 2.1 and

\[ F_{\psi(C^2 \times U)}[p(z) + \alpha \frac{zp'(z)}{p(z)}] \leq F_{h(U)}h(z), \]

\[ F_{p(U)}p(z) \leq F_{q(U)}q(z) \text{ i.e. } p(z) < F q(z), z \in U, \]

and \( q \) is the best dominant.

\[ \square \]

**Proposition 2.3.** Let \( q \in \mathcal{H}[p, p] \) be univalent, \( q(z) \neq 0 \) and satisfies the conditions:

(i) \( \frac{zq'(z)}{q(z)} \) is starlike,

(ii) \( Re(\frac{\frac{q(z)}{\alpha} + 1 + \frac{zq''(z)}{q(z)} - \frac{zq'(z)}{q(z)}}{zq'(z)}) > 0 \)

for \( \alpha \neq 0 \) and for all \( z \in U \). For \( f \in A_p \) with

\[ J(\alpha, f; z) = (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha(1 + \frac{zf''(z)}{f'(z)}), z \in U \]

and \( \psi : C^2 \times U \rightarrow C \)

\[ \psi(q(z), zq'(z)) = q(z) + \alpha \frac{zq'(z)}{q(z)}, \text{ then} \]

\[ F_{\psi(C^2 \times U)}(\frac{zf'(z)}{f(z)}) \leq F_{q(U)}q(z) \]

and \( q \) is the best dominant.

**Proof.** Let us put \( p(z) = \frac{zf'(z)}{f(z)}, z \in U \), where \( p(0) = 0 \).

Then we obtain that

\[ p(z) + \alpha \frac{zp'(z)}{p(z)} = J(\alpha, f; z). \]

Using Proposition 2.1, we have

\[ F_{p(U)}p(z) \leq F_{q(U)}q(z), z \in U, \]

and \( q \) is the best dominant.

\[ \square \]

**Proposition 2.4.** Let \( q \in \mathcal{H}[1, 1] \) be univalent and satisfies the following conditions:

(i) \( q(z) \) is convex,

(ii) \( Re[(\frac{1}{\alpha} + \rho) + \frac{zq''(z)}{q'(z)}] > 0 \) \( \rho \in \mathbb{N} = \{1, 2, 3, ..\} \)

for \( \alpha \neq 0 \) and for all \( z \in U \). For \( p \in \mathcal{H}[1, 1] \) in \( U \) and

\[ \psi : C^2 \times U \rightarrow C, \]

\[ \psi(p(z), zp'(z)) = (1 - \alpha + \alpha \rho)p(z) + \alpha zp'(z) \text{ is analytic in } U, \text{ then} \]

\[ F_{\psi(C^2 \times U)}[(1 - \alpha + \alpha \rho)p(z) + \alpha zp'(z)] \leq F_{\psi(C^2 \times U)}[(1 - \alpha + \alpha \rho)q(z) + \alpha zq'(z)] = F_{h(U)}h(z), \]

implies \( F_{p(U)}p(z) \leq F_{q(U)}q(z) \), and \( q \) is the best dominant.
Proof. For \( \alpha \neq 0 \) real number, we define the functions \( \theta \) and \( \phi \) by
\[
\theta(w) = (1 - \alpha + \alpha \rho)w, \quad \phi(w) = \alpha, \quad D = \{w : w \neq 0\}
\]
in Proposition 2.1. Then we have
\[
(i) \quad Q(z) = zq'(z)\phi[q(z)] = \alpha zq'(z),
(ii) \quad h(z) = \theta[q(z) + Q(z)] = (1 - \alpha + \mu \rho)q(z) + \alpha zq'(z).
\]
By the (i) and (ii), we obtained that \( Q \) is starlike in \( U \) and \( \Re\left(\frac{zh'(z)}{Q(z)}\right) > 0 \)
for all \( z \in U \). Since it satisfies preconditions of Proposition 2.1, it follows
Proposition 2.1,
\[
F_p(U)p(z) \leq F_q(U)q(z), \quad z \in U,
\]
and \( q \) is the best dominant. \( \square \)

**Theorem 2.1.** Let \( q \in H[1, 1] \) be univalent and satisfies the following conditions:

(i) \( q(z) \) is convex,

(ii) \( \Re\left(\frac{1}{\alpha} + \rho + \frac{zq''(z)}{q'(z)}\right) > 0 \) \( (\rho \in \mathbb{N} = \{1, 2, 3, \ldots\}) \)
for \( \alpha \neq 0 \) and for all \( z \in U \). For \( f \in A_p \) with
\[
J(\alpha, f; z) = (1 - \alpha)z\frac{f'(z)}{f(z)} + \alpha(1 + \frac{zf''(z)}{f'(z)}), \quad z \in U
\]
and if \( \psi : \mathbb{C}^2 \times U \rightarrow \mathbb{C} \),
\[
\psi(q(z), zq'(z)) = (1 - \alpha + \alpha \rho)q(z) + \mu zq'(z), \text{ then } F_{\psi(C^2 \times U)}(\frac{f(z)}{z^p}) \leq F_q(U)q(z), \quad z \in U
\]
and \( q \) is the best dominant.

Proof. Let us put \( p(z) = \frac{f(z)}{z^p} \), where \( p(0) = 1 \). Then we have
\[
(1 - \alpha + \alpha \rho)p(z) + \alpha zp'(z) = J_p(\alpha, f; z).
\]
From the Proposition 2.4, we have
\[
F_{p(U)}p(z) \leq F_q(U)q(z), \quad z \in U
\]
and \( q \) is the best dominant. \( \square \)

**Corollary 2.1.** Let \( q \in H[1, 1] \) be univalent and satisfies the following conditions:

(i) \( q(z) \) is convex,

(ii) \( \Re\left(\frac{1}{\alpha} + 1 + \frac{zq''(z)}{q'(z)}\right) > 0 \) \( (\rho \in \mathbb{N} = \{1, 2, 3, \ldots\}) \)

for \( \alpha \neq 0 \) and for all \( z \in U \). For \( p \in H[1, 1] \) in \( U \),
\[
\text{if } \psi : \mathbb{C}^2 \times U \rightarrow \mathbb{C} \\
\psi(p(z), zp'(z)) = p(z) + \alpha zp'(z),
\]
then \( F_{\psi(C^2 \times U)}p(z) \leq F_q(U)q(z), z \in U \), and \( q \) is the best dominant.
Corollary 2.2. Let \( q \in \mathcal{H}[1, 1] \) be univalent, \( q(z) \) is convex for all \( z \in U \). For \( p \in \mathcal{H}[1, 1] \) in \( U \) if
\[
\psi : \mathbb{C}^2 \times U \to \mathbb{C}, \psi(p(z), zp'(z)) = p(z) + zp'(z), \text{ then}
\]
\[
F_{\psi(\mathbb{C}^2 \times U)} p(z) \leq F_{\psi(\mathbb{C}^2 \times U)} q(z), z \in U,
\]
and \( q \) is the best dominant.

Corollary 2.3. Let \( q \in \mathcal{H}[1, 1] \) be univalent, \( q(z) \) is convex for all \( z \in U \). For \( p \in \mathcal{H}[1, 1] \) in \( U \) if
\[
\psi : \mathbb{C}^2 \times U \to \mathbb{C}, \psi(p(z), zp'(z)) = \rho p(z) + zp'(z), (\rho \in \mathbb{N} = \{1, 2, 3, \ldots\}),
\]
then
\[
F_{\psi(\mathbb{C}^2 \times U)} p(z) \leq F_{\psi(\mathbb{C}^2 \times U)} q(z), z \in U,
\]
and \( q \) is the best dominant.

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