AXIOMATIC GEOMETRIC FORMULATION OF ELECTROMAGNETISM WITH ONLY ONE AXIOM: THE FIELD EQUATION FOR THE BIVECTOR FIELD $F$ WITH AN EXPLANATION OF THE TROUTON-NOBLE EXPERIMENT

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In this paper we present an axiomatic, geometric, formulation of electromagnetism with only one axiom: the field equation for the Faraday bivector field $F$. This formulation with $F$ field is a self-contained, complete and consistent formulation that dispenses with either electric and magnetic fields or the electromagnetic potentials. *All physical quantities are defined without reference frames*, the absolute quantities, i.e., they are geometric four dimensional (4D) quantities or, when some basis is introduced, every quantity is represented as a 4D coordinate-based geometric quantity comprising both components and a basis. The *new* observer independent, expressions for the stress-energy vector $T(n)$ (1-vector), the energy density $U$ (scalar), the Poynting vector $S$ and the momentum density $g$ (1-vectors), the angular momentum density $M$ (bivector) and the Lorentz force $K$ (1-vector) are directly derived from the field equation for $F$. The local conservation laws are also directly derived from that field equation. The 1-vector Lagrangian with the $F$ field as a 4D absolute quantity is presented; *the interaction term is written in terms of $F$ and not, as usual, in terms of $A$*. It is shown that this geometric formulation is in a full agreement with the Trouton-Noble experiment.

Key words: electromagnetism with bivector field $F$, the Trouton-Noble experiment

1. INTRODUCTION

In the usual Clifford (geometric) algebra treatments of the classical electromagnetism, e.g., with multivectors [1-3] (for a more mathematical treatment
of the Clifford algebra see also [4]), one starts with a single field equation
using the Faraday bivector field $F$ and the gradient operator $\partial$ (1-vector),
see Eq. (1) below. In order to get the more familiar form the bivector field
$F$ is expressed (in [1,2]) in terms of the sum of a relative vector $E_H$ (corre-
sponds to the three-dimensional (3D) electric field vector $E$) and a relative
bivector $\gamma_5B_H$ ($B_H$ corresponds to the 3D magnetic field vector $B$, and $\gamma_5$ is
the (grade-4) pseudoscalar for the standard basis $\{\gamma_\mu\}$) by making a space-
time split in the $\gamma_0$ - frame, which depends on the observer velocity $c\gamma_0$; the
subscript $H$ is for “Hestenes.” Both $E_H$ and $B_H$ are, in fact, bivectors. Then
the following relations (from [1-2])

$$F = E_H + c\gamma_5B_H, \quad E_H = (F \cdot \gamma_0)\gamma_0, \quad \gamma_5B_H = (1/c)(F \wedge \gamma_0)\gamma_0 \quad (1)$$

are understood as that they define $F$ in terms of the sum of $E_H$ and $\gamma_5B_H$
and the components of $F$ are considered to be determined by $E_H$ and $B_H$,
i.e., by the components of the 3D $E$ and $B$. Similarly in [3] $F$ is decomposed
in terms of 1-vector $E_J$ and a bivector $B_J$; the subscript $J$ is for “Jancewicz,”

$$F = \gamma_0 \wedge E_J - cB_J, \quad E_J = F \cdot \gamma_0, \quad B_J = -(1/c)(F \wedge \gamma_0)\gamma_0. \quad (2)$$

It is supposed in [1-3] that the right hand sides ($E_H$, $B_H$ and $E_J$, $B_J$) of the
first equations in (1) and (2) determine the left hand sides ($F$). We remark
that it is generally accepted in the geometric algebra formalism (and in the
tensor formalism as well) that the usual Maxwell equations (ME) with the 3D
vectors $E$ and $B$ and the field equation written in terms of $F$, Eq. (4) below,
are completely equivalent. Further both in the tensor formalism, e.g., [5],
and in the geometric algebra formalism it is assumed that the components
of the 3D $E$ and $B$ define in a unique way the components of $F$ according to
the relations

$$E_i = F^{i0}, \quad B_i = (-1/2c)\varepsilon_{ikl}F_{kl}. \quad (3)$$

In (3) the components of the 3D fields $E$ and $B$ are written with lowered
(generic) subscripts, since they are not the spatial components of the 4D
quantities. This refers to the third-rank antisymmetric $\varepsilon$ tensor too. The
super- and subscripts are used only on the components of the 4D quantities.
Greek indices run from 0 to 3, while latin indices $i, j, k, l, ...$ run from 1 to 3,
and they both designate the components of some geometric object in some
system of coordinates. (It is worth noting that Einstein’s fundamental work
[6] is the earliest reference on covariant electrodynamics and on the identi-
fication of components of $F^{\alpha\beta}$ with the components of the 3D $E$ and $B$.)
All this means that the 3D $\mathbf{E}$ and $\mathbf{B}$ and not the $\mathbf{F}$ field are considered as primary quantities for the whole electromagnetism. Even in the very recent geometric approaches to classical electrodynamics [7,8] the 3D $\mathbf{E}$ and $\mathbf{B}$ are considered as primary quantities. Thus it is stated in [7]: “The electromagnetic field strength $F_{ij} = (\mathbf{E}, \mathbf{B})$ (in [7] $i, j, k, ... = 0, 1, 2, 3$, my remark) is composed of the electric and magnetic 3-vector fields.” In order to get the wave theory of electromagnetism the vector potential $\mathbf{A}$ (1-vector) is usually introduced and $\mathbf{F}$ is defined in terms of $\mathbf{A}$ as $\mathbf{F} = \partial \wedge \mathbf{A}$. In that case the $\mathbf{F}$ field appears as the derived quantity from the potentials. Thence in almost all usual treatments of the electromagnetism, both in the tensor formalism and in the geometric algebra formalism, the theory is presented as that the $\mathbf{F}$ field does not have an independent existence but is defined either by the components of the 3D $\mathbf{E}$ and $\mathbf{B}$ or by the components of the electromagnetic potential $\mathbf{A}$. (An exception is, e.g., [9], in which $\mathbf{F}$ is an independent quantity and the 4D $\mathbf{E}$ and $\mathbf{B}$ are considered as observer dependent functions of $\mathbf{F}$.)

In this paper we present an axiomatic formulation of the classical electromagnetism that uses the bivector field $\mathbf{F}$ (an observer independent 4D quantity) in which only the field equation with $\mathbf{F}$ (Eq. (4) below) is postulated. The presented formulation with the $\mathbf{F}$ field is a self-contained, complete and consistent formulation that does not make use either electric and magnetic fields or the electromagnetic potential $\mathbf{A}$ (thus dispensing with the need for the gauge conditions). In such formulation the $\mathbf{F}$ field is the primary quantity for the whole classical electromagnetism both in the theory and in experiments; $\mathbf{F}$ is a well-defined 4D measurable quantity.

In this geometric approach to electromagnetism physical quantities in the 4D spacetime are represented by Clifford multivectors. They are defined without reference frames (when no basis has been introduced), 4D absolute quantities (AQtis), or, equivalently, they are written as 4D coordinate-based geometric quantities (CBGQts) comprising both components and a basis (when some basis has been introduced). Thus these 4D quantities are independent of the chosen inertial frame of reference and of the chosen system of coordinates in it, i.e., they are observer independent quantities.

In the field view of particle-to-particle interaction the electrodynamic interaction between charges is described as two-steps process; first fields are seen as being generated from their particle sources and then the fields so generated are perceived as interacting with some target particle. The description of the first step in the $\mathbf{F}$ formulation of electrodynamics is given in Sec. 2.2. In Sec. 2.3 the general solution for $\mathbf{F}$ is applied to the determination of the
electromagnetic field $F$ of a point charge. In Sec. 2.4 the integral form of the field equation for $F$ is constructed which is equivalent to the local field equation (4). The second step of the description of the interaction process requires the determination of the Lorentz force in terms of $F$ and its use in Newton’s second law. This is at the same time the way in which the components of $F$ are measured in the chosen reference frame. It is described in Sec. 2.5.

We also give the new expressions for the observer independent stress-energy vector $T(n)$ (1-vector), the energy density $U$ (scalar, i.e., grade-0 multivector), the Poynting vector $S$ (1-vector), the angular momentum density $M$ (bivector) and the Lorentz force $K$ (1-vector). They are all directly derived from the postulated field equation with $F$ (Eq. (4)) and presented in Sec. 2.6.

The local charge-current density and local energy-momentum conservation laws are also directly derived from that field equation with $F$ and there is no need to introduce the Lagrangian and the Noether theorem. These laws are presented in Sec. 2.7. (Of course the integral conservation laws can be similarly derived but it will not be done here.) In contrast to our theory with one postulated equation (Eq. (4)) the recent theory [8] (also a geometric approach) deals with three postulated equations. It will be shown here that all three axioms from [8] simply follow from our axiom (4). Furthermore, it is considered in [7,8], as in almost all other treatments, that $F$ is determined by the 3D $E$ and $B$. As it is said the exposed formulation with the field equation for $F$ does not need the Lagrangian. However in Sec. 3 a brief exposition of the Lagrangian formulation with the $F$ field is presented; the interaction term is also written in terms of $F$ and not, as usual, in terms of $A$. In Sec. 4 we give the comparison with the experiments, particularly with the Trouton-Noble experiment. It is shown that the approach with geometric 4D quantities is in a full agreement with the Trouton-Noble experiment. The explanation for the null result is very simple and natural. Namely in our approach all quantities are invariant 4D quantities, which means that their values are the same in the rest frame of the capacitor and in the moving frame. We have calculated that the torque (as a geometric 4D quantity) is zero for the stationary capacitor. Then automatically it follows that the torque is zero for the moving capacitor as well. In the last section, Sec. 5, the discussion and conclusions are presented.

2. THE $F$ FORMULATION OF ELECTROMAGNETISM

2.1. Generally about Geometric Approach to Electromagnetism
As mentioned in Sec. 1 the presented formulation of electromagnetism with the $F$ field exclusively deals with AQs (thus defined without reference frames) or with the corresponding CBGQs, when some basis has been introduced. Usually it is the standard basis that is introduced, e.g., [1-3]. The generators of the spacetime algebra (the Clifford algebra generated by Minkowski spacetime) are taken to be four basis vectors $\{\gamma_\mu\}, \mu = 0...3$, satisfying $\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = \text{diag}(+ -- -)$. This basis, the standard basis $\{\gamma_\mu\}$, is a right-handed orthonormal frame of vectors in the Minkowski spacetime $M^4$ with $\gamma_0$ in the forward light cone. The $\gamma_k (k = 1, 2, 3)$ are spacelike vectors. The $\gamma_\mu$ generate by multiplication a complete basis for the spacetime algebra: $1, \gamma_\mu, \gamma_\mu \wedge \gamma_\nu, \gamma_\mu \gamma_5, \gamma_5 (2^4 = 16$ independent elements). $\gamma_5$ is the pseudoscalar for the frame $\{\gamma_\mu\}$. For more details about geometric algebra see, e.g., [1-4], or short reviews presented in the second paper in [10] and in [11].

We remark that the standard basis corresponds, in fact, to the specific system of coordinates, i.e., the Einstein system of coordinates, of the chosen inertial frame of reference. (In the Einstein system of coordinates the Einstein synchronization [12] of distant clocks and Cartesian space coordinates $x^i$ are used in the chosen inertial frame of reference.) However different systems of coordinates of an inertial frame of reference are allowed and they are all equivalent in the description of physical phenomena. For example, in [13] (and the second and third paper in [14]) two very different, but completely equivalent systems of coordinates, the Einstein system of coordinates and ”radio” ("r") system of coordinates, are exposed and exploited throughout the paper.

Any Clifford multivector $A$, an AQ, can be written as a CBGQ, thus with components and a basis. Any CBGQ is an invariant quantity upon the Lorentz transformations (LT). In such an interpretation the LT are considered as passive transformations; both the components and the basis vectors are transformed but the whole 4D geometric quantity remains unchanged, e.g., the position 1-vector $x$ can be decomposed in the $S$ and $S'$ (relatively moving) frames and in the standard basis $\{\gamma_\mu\}$ and some non-standard basis $\{e_\mu\}$ as $x = x^\mu \gamma_\mu = x'^\mu \gamma_\mu = .... = x''^\mu e_\mu$. The primed quantities are the Lorentz transforms of the unprimed ones.

However in the usual Clifford algebra formalism, e.g., [1-4], one deals with the multivectors as AQs and the LT are considered as active transformations. If some basis is introduced (for example, the $\{\gamma_\mu\}$ basis) then the components of, e.g., some 1-vector relative to a given inertial frame of reference (with the standard basis $\{\gamma_\mu\}$) are transformed into the components of a new 1-vector
relative to the same frame (the basis \(\{\gamma_\mu\}\) is not changed). (We note that a coordinate-free form for the LT is presented in [13] and [15] and it can be used both in an active way, when there is no basis, or in a passive way, when some basis is introduced.) In this paper, for the sake of brevity and of clearness of the whole exposition, we shall work either with 4D AQs or with 4D CBGQs which are written only in the standard basis \(\{\gamma_\mu\}\), but remembering that the approach with geometric 4D quantities holds for any choice of the basis.

2.2. The Determination of the Electromagnetic Field \(F\)

We start the exposition of the classical electromagnetism by the description of the first step in the field view of particle-to-particle interaction; the determination of \(F\) for the given sources. As it is already said this is an axiomatic formulation of the electromagnetism with only one postulated equation: it is the field equation written in terms of \(F\) [1-3] (a single field equation for \(F\) is first given by M. Riesz [16]). In that equation an electromagnetic field is represented by a bivector-valued function \(F = F(x)\) on the spacetime. The source of the field is the electromagnetic current \(j\) which is a 1-vector field.

Using that the gradient operator \(\partial\) is a 1-vector field this equation can be written as

\[
\partial F = j \varepsilon_0 c, \quad \partial \cdot F + \partial \wedge F = j / \varepsilon_0 c. \tag{4}
\]

The trivector part is identically zero in the absence of magnetic charge.

When (4) is written with CBGQs in the \(\{\gamma_\mu\}\) basis it becomes

\[
\partial_\alpha F^{\alpha \beta} \gamma_\beta = \left(1/2\right) \varepsilon^{\alpha \beta \gamma \delta} \partial_\alpha F_{\gamma \delta} \gamma_\beta = \left(1/\varepsilon_0 c\right) j^{\beta} \gamma_\beta, \tag{5}
\]

where \(\varepsilon^{\alpha \beta \gamma \delta}\) is the totally skew-symmetric Levi-Civita pseudotensor. In AQs from (4) are written as CBGQs in the \(\{\gamma_\mu\}\) basis; \(F = (1/2) F^{\alpha \beta} \gamma_\alpha \wedge \gamma_\beta\) (the basis components \(F^{\alpha \beta}\) are determined as \(F^{\alpha \beta} = \gamma^{\beta} \cdot (\gamma^\alpha \cdot F) = \left(\gamma^{\beta} \wedge \gamma^{\alpha}\right) \cdot F\)). From (5) one easily finds the usual covariant form (thus only the basis components of the 4D geometric quantities in the \(\{\gamma_\mu\}\) basis) of the field equations as

\[
\partial_\alpha F^{\alpha \beta} = j^{\beta} / \varepsilon_0 c, \quad \partial_\alpha F^{\alpha \beta} = 0, \tag{6}
\]

where the usual dual tensor is introduced \(\star F^{\alpha \beta} = (1/2) \varepsilon^{\alpha \beta \gamma \delta} F_{\gamma \delta}\).

The field bivector \(F\) yields the complete description of the electromagnetic field and, in fact, there is no need to introduce either the field vectors or the
potentials. For the given sources the Clifford algebra formalism enables one to find in a simple way the electromagnetic field $F$. Namely the gradient operator $\nabla$ is invertible and (4) can be solved for

$$F = \nabla^{-1}(j/e_0c),$$

(7)

see, e.g., [17] and [1] Spacetime Calculus. We briefly repeat the main points related to (7) from these references. However, the important difference with respect to the usual approaches [1-3] is that for us, as proved in [10] and [11], the field equation (4) is not equivalent to the usual Maxwell equations with the 3D $E$ and $B$ (i.e., with $E_H, B_H$ from (11) or $E_J, B_J$ from (2)). $\nabla^{-1}$ is an integral operator which depends on boundary conditions on $F$ and is an integral form of the field equation (4). If the charge-current density $j(x)$ is the sole source of $F$, then (7) provides the unique solution to the field equation (4). By using Gauss’ Theorem an important formula can be found that allows to calculate $F$ at any point $y$ inside $m$-dimensional manifold $M$ from its derivative $\nabla F$ and its values on the boundary $\partial M$ if a Green’s function $G(y, x)$ is known,

$$F(y) = \int_M G(y, x)\nabla F(x) \mid d^m x \mid - \int_{\partial M} G(y, x)n^{-1}F(x) \mid d^{m-1} x \mid,$$

(8)

$n$ is a unit normal, $n^{-1} = n$ if $n^2 = 1$ or $n^{-1} = -n$ if $n^2 = -1$, and $G(y, x)$ is a solution to the differential equation $\nabla_y G(y, x) = \delta^m(y - x)$. (8) is the relation (4.17) in [17].) If $\nabla F = 0$, i.e., $j = 0$, the first term on the right side of (8) vanishes but not the second term. This general relation can be applied to different examples.

2.3. The Electromagnetic Field of a Point Charge

An example is the determination of the expression for the classical Liénard-Wiechert field that is given, e.g., in [17] and [1] Spacetime Calculus. The usual procedure ([17] and [1]) is to utilize the general relation (8), in which all quantities are defined without reference frames (Geometric calculus), and to specify it to the Minkowski spacetime ($m = 4$). Then a space-time split is introduced by the relation

$$ct = x \cdot n = x \cdot \gamma_0.$$
(n in \( \text{(8)} \) is taken to be \( \gamma_0 \) and \( \text{(9)} \) is the equation for a 1-parameter family of spacelike hyperplanes \( S(t) \) with normal \( \gamma_0 \); \( S(t) \) is a surface of simultaneous \( t \) when the Einstein synchronization is chosen.) For simplicity, \( M \) is taken to be the entire region between the hyperplanes \( S_1 = S(t_1) \) and \( S_2 = S(t_2) \). We shall not discuss this derivation further but we only quote the result for the classical Liénard-Wiechert field. The charge-current density for a particle with charge \( q \) and world line \( z = z(\tau) \) with proper time \( \tau \) is

\[
 j(x) = q \int_{-\infty}^{\infty} d\tau u \delta^4(x - z(\tau)),
\]

where \( u = u(\tau) = dz/d\tau \). Then the classical Liénard-Wiechert retarded field for \( q \) (see, e.g., Sec. 5 in \([17]\)) is

\[
 F(x) = \left( \frac{q}{4\pi \varepsilon_0} \right) \left\{ r \wedge \left[ (u/c) + \left( 1/c^3 \right) r \cdot (u \wedge \dot{u}) \right] \right\} / \left( r \cdot u / c \right)^3,
\]

(10)

where \( r = x - z \) satisfies the light-cone condition \( r^2 = 0 \) and \( z, u, \dot{u} \) are all evaluated at the intersection of the backward light cone (with vertex at \( x \)) and world line of that charge \( q \). It is worth noting that from the general expression \( \text{(8)} \) one can derive not only the retarded interpretation for \( F \) of a charge \( q \) but also the advanced interpretation and the present-time interpretation, i.e., an instantaneous action-at-a-distance interpretation. (This present-time interpretation will be reported elsewhere. In the tensor formalism the expressions for \( F^{ab} \) and the 4-vectors \( E^a \) and \( B^a \) in the present-time interpretation for an uniform and uniformly accelerated motion of a charge \( q \) are given in \([18]\).)

All quantities in \( \text{(11)} \) are geometric 4D quantities, the AQs, and for more practical use they can be written as CBGQs, usually in the \( \{ \gamma_{\mu} \} \) basis. Thus, the general expression for \( F \) for an arbitrary motion of a charge is

\[
 F = (1/2) F^{\alpha\beta} \gamma_\alpha \wedge \gamma_\beta, \quad F^{\alpha\beta} = \left( kq / \zeta^3 \right) \left[ r^2 (r^\alpha u^\beta - r^\beta u^\alpha) \right] \\
+ \left( kq / \zeta^3 \right) \left[ (r^\sigma u_\sigma)(r^\alpha u^\beta - r^\beta u^\alpha) + (r^\sigma u_\sigma)(r^\alpha \dot{u}^\beta - r^\beta \dot{u}^\alpha) \right].
\]

(11)

In \( \text{(11)} \), \( r^\mu = x^\mu - z^\mu(\tau) \), \( x^\mu \) and \( z^\mu(\tau) \) are the field and the source basis components of \( x \) and \( z \) respectively (in the \( \{ \gamma_\mu \} \) basis), \( k = 1/4\pi \varepsilon_0 \) and \( \zeta \equiv r^\sigma u_\sigma \). The right-hand side has to be evaluated at \( \tau_0 \) such that \( x^\mu - z^\mu(\tau_0) \) is light-like, i.e., \( \tau_0 \) is determined by the above mentioned light-cone condition, which in the component form becomes \( (x^\sigma - z^\sigma(\tau_0))(x_\sigma - z_\sigma(\tau_0)) = 0 \), and it holds that \( x^0 - z^0(\tau_0) = |r| \gg 0 \). The expression for \( F^{\alpha\beta} \) from \( \text{(11)} \) is the standard result, e.g., Jackson’s book \([5]\). The first term in \( F^{\alpha\beta} \) \( \text{(11)} \) represents the velocity part and the second one represents the acceleration or radiation part. (However we note that such decomposition of \( F^{\alpha\beta} \) into velocity and
acceleration parts is the consequence of the used retarded representation and it
does not exist for, e.g., the present time representation [18]. We see that
from (10) one can simply find the well-known result for $F^{\alpha\beta}$ given in (11).

The components $F^{\alpha\beta}$ are measurable quantities; they can be measured
using the Lorentz force and Newton’s second law as will be discussed in Sec.
2.5.

Let us specify the above relations to the case of a point charge with constant velocity $u$, see for the comparison Sec. 7.3.2 in [2]. Then the trajectory
is $z(\tau) = u\tau$ (taking that $z(0) = 0$), $v \cdot u/c = |x \wedge (u/c)|$, $r \wedge (u/c) = x \wedge (u/c)$. Substituting these relations into (10) one finds the field strength

$$F(x) = \frac{kq(x \wedge (u/c))}{|x \wedge (u/c)|^3} = D(x \wedge (u/c)), \quad (12)$$

where $D = kq/|x \wedge (u/c)|^3$.

In all usual formulations of electromagnetism both in the Clifford algebra
[1-3] and tensor formalisms [5] the results (10) or $F^{\alpha\beta}$ from (11) are considered only as formal, mathematical results that are necessary to find the
components of the “physical” quantities, the 3D $E$ and $B$. But the relations (7), (8) and (10) show that $F$ has an independent physical reality and the
whole electromagnetism can be treated with $F$ without even mentioning the
3D $E$ and $B$. Consequently, in contrast to the usual approaches [1-3], and
all other previous approaches, we assume that the 4D geometric quantity $F$
can be considered as the primary physical quantity and not the 3D vectors $E$ and $B$. Then from the known $F$ one can find different 4D quantities that
represent the 4D electric and magnetic fields; they are considered in [10,11] and [15].

One of these representations uses the decomposition of $F$ into 1-vectors $E$ and $B$

$$F = (1/c)E \wedge v + (IB) \cdot v, \quad E = (1/c)F \cdot v, \quad B = -(1/c^2)I(F \wedge v), \quad (13)$$

where $I$ is the unit pseudoscalar. ($I$ is defined algebraically without introducing any reference frame, as in [4] Sec. 1.2.) It holds that $E \cdot v = B \cdot v = 0$
(since $F$ is skew-symmetric). $v$ in (13) can be interpreted as the velocity
(1-vector) of a family of observers who measures $E$ and $B$ fields. The velocity $v$ and all other quantities entering into (13) are defined without reference frames; they are AQs. $v$ characterizes some general observer. Thus the relations (13) hold for any observer. (For the equivalent relations with $v$ in
the tensor formalism see also [19].) The relations (13) actually establish the equivalence of the formulation of electromagnetism with the field bivector $F$ (presented here) and the formulation with 1-vectors of the electric $E$ and magnetic $B$ fields (that is presented in [15]). Both formulations, with $F$ and $E$, $B$ fields, are equivalent formulations, but every of them is a complete, consistent and self-contained formulation.

Similarly in [11] (see also the second paper in [10]) it is shown that the relations (1) from [1,2] have to be replaced by the relations

\[ F = E_{Hv} + cB_{Hv}, \quad E_{Hv} = \frac{1}{c^2}(F \cdot v) \wedge v \]
\[ B_{Hv} = -\frac{1}{c^3}I[(F \wedge v) \cdot v]. \]

(14)

Namely in (1) a space-time split in the $\gamma_0$ - frame is used and, as it is already said, it depends on the velocity $c\gamma_0$ of the specific observer, the $\gamma_0$ - observer. Thus in (1) the observer independent quantity $F$ is decomposed into the observer dependent quantities $E_H, B_H$ defined only in the $\gamma_0$ - frame. A space-time split is not a Lorentz invariant procedure. On the other hand in (14) the velocity $c\gamma_0$ of the specific observer is replaced by the velocity $v$ of some general observer. $v$ is an AQ and thence the relations (14), in the same way as the relations (13), hold for any observer. Then, instead of formulating the whole electromagnetism by 1-vectors of the electric and magnetic fields, $E$ and $B$ respectively, one can formulate it by bivectors $E_{Hv}$ and $B_{Hv}$. The only difference is that the decomposition of $F$ into 1-vectors $E$ and $B$ (13) is much simpler and, in fact, closer to the classical formulation of the electromagnetism with the 3D $E$ and $B$, than the decomposition of $F$ into bivectors $E_{Hv}$ and $B_{Hv}$ (14). In contrast to the formulation with $F$, all formulations with the electric and magnetic fields as AQs require the introduction of $v$, the velocity of observers who measure fields. This is the reason why the formulation with the bivector field $F$ is investigated and presented in this paper.

As an example let us apply Eqs. (12) and (14) to determine the electric and magnetic fields $E_{Hv}$ and $B_{Hv}$ for the case of a point charge with constant velocity $u$. We find

\[ E_{Hv} = (D/c^3)[(u \cdot v)(x \wedge v) - (x \cdot v)(u \wedge v)] \]
\[ B_{Hv} = (D/c^4)I[(x \cdot v)(u \wedge v) - (u \cdot v)(x \wedge v) + c^2(x \wedge u)]. \]

(15)

Both $E_{Hv}$ and $B_{Hv}$ from (15) are AQs, i.e., they are defined without reference frames. Remember that $v$ is the velocity (1-vector) of a family of observers.
who measures $E_{Hv}$ and $B_{Hv}$ fields and $u$ is the velocity (1-vector) of a point charge.

Using (13) and $F$ from (12) we get physically equivalent but simpler expressions

$$E = (D/c^2)[(u \cdot v)x - (x \cdot v)u]$$

$$B = (-D/c^3)I(x \wedge u \wedge v),$$

in which the electric and magnetic fields are represented by 1-vectors $E$ and $B$.

We note that $(E, B)$ from (16) or $E_{Hv}, B_{Hv}$ from (15) and $F$ from (12) contain the same amount of physical informations. The expressions with AQs $(E, B)$ from (16) or $E_{Hv}, B_{Hv}$ from (15) were not found in any previous approach including [1-3]. The usual results can be recovered simply taking that the observers who measure $E, B$ or $E_{Hv}, B_{Hv}$ fields are at rest, “fiducial” observers (the $\gamma_0$-frame with the $\{\gamma_\mu\}$ basis), for which $v = c\gamma_0$ in (15) (16). In that case, e.g., the relations (16) become

$$E_f = D(\gamma x - ut), \quad B_f = (-D/c^2)\gamma_5(x \wedge u \wedge \gamma_0),$$

where $f$ stands for “fiducial” and $\gamma_5$ is the unit pseudoscalar $I$ for the $\{\gamma_\mu\}$ basis. Notice that $E^0_f = B^0_f = 0$, which means that in the frame of “fiducial” observers, the $\gamma_0$-frame with the $\{\gamma_\mu\}$ basis, $E_f, B_f$ contain only spatial components. Thence $E_f$ and $B_f$ from (17) are exactly the same as the usual expressions for the 3D electric and magnetic fields of a charge in uniform motion.

2.4. The Integral Form of the Field Equation (11)

Instead of dealing with the axiomatic formulation of electromagnetism that uses only the local form of the field equation (11) one can construct the equivalent integral form. Such form is constructed by Hestenes and nicely presented in [17] and [1] Space-Time Calculus, though, Hestenes does not consider it as an axiomatic formulation. Here only the main results from [17] and [1] will be briefly repeated and applied to the determination of $F$ (and $E, B$) for some simple cases.

The trivector part of (11) $\partial \wedge F = 0$ can be transformed to an equivalent integral form as

$$\int_{\partial M} d^2x \cdot F = 0,$$
where in the directed integral (18), in general, \(d^m x\) is the directed measure and \(\partial M\) is any closed 2-dimensional submanifold in spacetime. \(d^m x\) can be resolved into its magnitude \(|d^m x|\), which is the usual “scalar measure,” and its direction represented by a unit \(m\)-blade \(I_m\): 
\[
d^m x = I_m |d^m x|, 
\]
or in terms of CBGQs 
\[
d^m x = d_1 x \wedge d_2 x \wedge ... \wedge d_m x = e_1 \wedge e_2 \wedge ... e_m dx^1 dx^2 ... dx^m, 
\]
where \(dx^\mu\) is a scalar differential for the scalar variable \(x^\mu\) and \(e_\mu\) is the basis 1-vector for some basis \(\{e_\mu\}\).

In order to find the corresponding integral forms with the electric and magnetic fields Hestenes, [17] and [1], uses the space-time split given by (1) and shows that (18) is equivalent to Faraday’s law or “the absence of magnetic poles,” or a mixture of the two, depending on the choice of \(\partial M\). However, as we have said, the use of such procedure, a space-time split in the \(\gamma_0\)-frame, transforms the integral field equation (18) written in terms of AQs into observer dependent integral field equation written in terms of observer dependent \(E_H\) and \(B_H\) from (1).

Instead of using such procedure we express \(F\) in (18) in terms of AQs \(E_H v, B_H v\) from (14) or \(E, B\) from (13) to find equivalent, coordinate-free, integral forms with electric and magnetic fields as AQs, e.g., with \(E\) and \(B\)

\[
\oint_{\partial M} d^2 x \cdot ((1/c)E \wedge v + (IB) \cdot v) = 0. 
\]

Then going to the frame of “fiducial” observers (for which \(v = c\gamma_0\)), the \(\gamma_0\)-frame with the \(\{\gamma_\mu\}\) basis, \(E\) and \(B\) contain only spatial components, and we recover the usual integral form of Faraday’s law and Gauss’ law for a magnetic field or a mixture of the two, depending on the choice of \(\partial M\).

Hestenes also derived an integral formula for the vector part of the local field equation (1)

\[
\oint_{\partial M} d^2 x \cdot (FI) = (1/\varepsilon_0 c) \int_M j \cdot n |d^3 x|, 
\]

where \(n = n(x)\) is a unit outward normal and \(M\) is any 3-dimensional submanifold in spacetime that is enclosed by \(\partial M\). In [17] and [1] the equation (20) is also written in the less familiar form

\[
\oint_{\partial M} d^2 x \wedge F = (1/\varepsilon_0 c) \int_M (d^3 x) \wedge j, 
\]
which when combined with (18) gives the integral version of the local field equation (4)
\[ \oint_{\partial M} \langle d^2 x F \rangle_I = (1/\varepsilon_0 c) \int_M \langle d^3 x j \rangle_I, \tag{22} \]
where \( \langle ... \rangle_I \) selects only the “invariant (= scalar+pseudoscalar) parts.” Of course the whole discussion that led from (18) to (19) applies in the same measure to the equations (20) (or (21)) and (22) and the corresponding expressions with \( AQs E_{Hv}, B_{Hv} \) from (14) or \( E, B \) from (13). We notice that when \( F \) is decomposed in terms of electric and magnetic fields then the equation (20), or (21), in the frame of “fiducial” observers \( (v = c\gamma_0) \) becomes the Ampère-Maxwell law, the Gauss law for an electric field or a mixture of the two, depending on the choice of \( \partial M \).

In the usual approach with the 3D \( E \) and \( B \) and for some simple cases one can determine, e.g., the 3D \( E \), directly from the integral version of the Gauss law. Similarly instead of to find \( F \) from (8) one can get it, e.g., from the equation (20) (or (21)). Let us consider a flat sheet infinite in extent, with the constant surface charge density \( \sigma \). We can use (13) to connect the \( F \) formulation with the formulation that deals with the electric and magnetic fields \( E \) and \( B \) respectively. For the sake of easier comparison with the common approach that deals with the 3D \( E \) and \( B \) we introduce the \( \gamma_0 \)-frame (with the \( \{\gamma_{\mu}\} \) basis) in which the flat sheet is at rest and situated in the \( \gamma_1 \wedge \gamma_2 \) plane. In the usual application of Gauss’ law with the 3D \( E \) a convenient Gaussian surface in the 3D space is chosen to be a “pill box” piercing a cross-sectional area \( S \) on the flat sheet and whose height is \( 2a \). In the 4D spacetime \( \partial M \) in the equation (20) is chosen to be the same closed 2-dimensional surface as above (a “pill box”) that is instantaneously taken, i.e., it is at rest, in the \( \gamma_0 \)-frame. When \( F \) is written as a CBGQ in the \( \{\gamma_{\mu}\} \) basis it becomes
\[ F = F^{\gamma_0}_{\gamma_3} \gamma_3 \wedge \gamma_0 + (1/2) F^{kl}_{\gamma_3} \gamma_3 \wedge \gamma_0. \tag{23} \]
This is a decomposition of \( F \) into “electriclike” (the first part) and “magneticlike” (the second part), but we note that such decomposition of \( F \) as in (23) is not a Lorentz invariant decomposition. From (13), i.e., from (23) (the chosen \( \gamma_0 \)-frame is the frame of “fiducial” observers, \( v = c\gamma_0 \)), and by analogy with the corresponding 3D formulation, we conclude that only \( F^{30} \neq 0 \) and it is of constant magnitude. Then the equation (20) becomes \( 2SF^{30} = S(\sigma/\varepsilon_0) \) whence
\[ F = F^{30}_{\gamma_3} \gamma_3 \wedge \gamma_0 = (\sigma/2\varepsilon_0) \gamma_3 \wedge \gamma_0. \tag{24} \]
This example will be used in comparison with experiments that is considered in Sec. 4.

In our axiomatic formulation of electromagnetism there is only one postulated equation, either the local field equation (4) or its equivalent integral version (22), whereas, as we have already mentioned, in the recent axiomatic formulation of electromagnetism [8] there are three postulated equations; (1) electric charge conservation, (2) the Lorentz force, (3) magnetic flux conservation. Using these three postulated equations and foliation of spacetime (what is nothing else than the space-time split) the authors of [8] derive the usual form of the Maxwell equations with the 3D $E$ and $B$. Their axiom 3 simply follows from the equation (18) and in the next sections, Secs. 2. 6 and 2. 5, we shall show that not only axiom 3 but also the axioms 1 and 2 from [8] simply follow from our equation (4), or (22).

2.5. The Lorentz Force and the Motion of a Charged Particle in the Electromagnetic Field $F$

As it is said in the field view of particle-to-particle interaction the electrodynamic interaction between charges is described as two-steps process; first fields are seen as being generated from their particle sources and then the fields so generated are perceived as interacting with some target particle. The description of the first step in the $F$ formulation of electrodynamics is given by the above relations (4) (or (22)), (7), (8) and for a point particle with charge $q$ with (10). The second step requires the determination of the Lorentz force in terms of $F$ and its use in Newton’s second law. This will be undertaken below.

In the Clifford algebra formalism one can easily derive the expressions for the stress-energy vector $T(n)$ and the Lorentz force density $K(j)$ directly from the field equation (4) and from the equation for $\tilde{F}$, the reverse of $F$, $\tilde{F}\tilde{\partial} = \tilde{j}/\varepsilon_0c$ ($\tilde{\partial}$ differentiates to the left instead of to the right). Indeed, using (4) and from the equation for $\tilde{F}$ one finds

$$T(\partial) = (-\varepsilon_0/2)(F\partial F) = j \cdot F/c = -K(j),$$

(25)

where in $(F\partial F)$ the derivative $\partial$ operates to the left and to the right by the chain rule. The stress-energy vector $T(n)$ [1-3] for the electromagnetic field is then defined in the $F$ formulation as

$$T(n) = T(n(x), x) = -(\varepsilon_0/2)\langle F n F \rangle_1.$$

(26)
We note that $T(n)$ is a vector-valued linear function on the tangent space at each spacetime point $x$ describing the flow of energy-momentum through a hypersurface with normal $n = n(x)$.

The right-hand side of (25) yields the expression for the Lorentz force density $K_{(j)}$,

$$K_{(j)} = F \cdot j/c.$$  \hspace{1cm} (27)

The Lorentz force in the $F$ formulation for a charge $q$ is $K = (q/c)F \cdot u$, where $u$ is the velocity 1-vector of a charge $q$ (it is defined to be the tangent to its world line).

It is worth noting that the equation (27) is the second postulate, axiom 2, in the axiomatic formulation of electromagnetism [8]. Thus in our approach Eq. (27) simply follows from the field equation (4).

In the approaches [1,2] the Lorentz force is discussed using the space-time split and the corresponding decomposition of $F$ into the electric and magnetic components. However in the analysis of the motion of a charged particle under the action of the Lorentz force we utilize only those parts of the usual approaches [1-3] that are expressed only in terms of $F$ and not those expressed by $E_H, B_H$ or $E_J, B_J$. We shall only quote the main results from [1-3] for the motion of a charged particle in a constant electromagnetic field $F$ but without using $E_H, B_H$ or $E_J, B_J$. Actually, as shown in Sec. 2.3, instead of dealing with the decomposition of $F$ as an AQ into the observer dependent $E_H, B_H$ \(1\) or $E_J, B_J$ \(2\) as in [1-3] one has to make the decomposition of the AQ $F$ into AQs $E_{Hv}, B_{Hv} \(14\)$ or $E, B \(16\)$. Then going to the frame of “fiducial” observers, $v = c\gamma_0$, one recovers in that frame the usual results [1-3] with the electric and magnetic fields.

The particle equation of motion, i.e., Newton’s second law is

$$m\overset{\cdot}{u} = qF \cdot u,$$  \hspace{1cm} (28)

where $\overset{\cdot}{u} = du/d\tau$; the overdot denotes differentiation with respect to proper time $\tau$. Usually [1,2] the equation (28) is not solved directly but solving the rotor equation $\overset{\cdot}{R} = (q/2m)FR$ and using the invariant canonical form for $F$, which is

$$F = f e^{I\varphi} = f(\cos \varphi + I \sin \varphi);$$  \hspace{1cm} (29)

this holds for $F^2 \neq 0$. In that form $f^2 = |f^2|$, which shows that $f$ is a “timelike bivector,” but $If$ is a “spacelike bivector,” since $(If)^2 = -|f^2|$. The equation (29) is the unique decomposition of $F$ into a sum of mutually
commuting timelike and spacelike parts. As shown in, e.g., [1] Space-Time Calculus, both \(f\) and \(\varphi\) can be written in terms of \(F\), i.e., invariants under the LT that are constructed from \(F\); \(\alpha = F \cdot F\), \(I\beta = F \wedge F\). Thus \(e^{I\varphi} = (\alpha + I\beta)^{1/2}/(\alpha^2 + \beta^2)^{1/4}\) and \(f = F(\alpha - I\beta)^{1/2}/(\alpha^2 + \beta^2)^{1/4}\). The same invariant decomposition of \(F\) is given in [3] Chap. 6 par.3, where \(F\) is written as \(F = F_1 + F_2\) and it can be shown that \(F_1\) (\(F_2\)) from [3] is exactly equal to \(f \cos \varphi\) (\(f \sin \varphi\)) from [1]. Note the difference between the decomposition of \(F\) presented in (23) and that one given in (29); the first one is a coordinate-dependent presentation whereas the second one is given in terms of AQs.

Let us consider that \(F\) is an uniform electromagnetic field and let us apply the decomposition (29). Then denoting \(\frac{q}{m}F = \Omega\), \(\Omega_1 = f(\frac{q}{m})\cos \varphi\), \(\Omega_2 = f(\frac{q}{m})\sin \varphi\), and making an invariant decomposition of the initial velocity \(u(0)\) into a component \(u_1\) in the \(f\)-plane and a component \(u_2\) orthogonal to the \(f\)-plane, \(u(0) = f^{-1}(f \cdot u(0)) + f^{-1}(f \wedge u(0)) = u_1 + u_2\), we get

\[
u = e^{(1/2)\Omega_1 \tau} u_1 + e^{(1/2)\Omega_2 \tau} u_2.
\] (30)

As stated in [1], Spacetime Calculus, this is an invariant decomposition of the motion into ”electriclike” and ”magneticlike” components. The particle history is obtained integrating (30)

\[
x(\tau) - x(0) = (e^{(1/2)\Omega_1 \tau} - 1)\Omega_1^{-1} u_1 + e^{(1/2)\Omega_2 \tau} \Omega_2^{-1} u_2.
\] (31)

(For more details see [1-3].) This result applies for arbitrary initial conditions and arbitrary uniform electromagnetic field \(F\). Different special cases of the equation (31) that correspond to the motion of a charge in uniform electric or magnetic fields are already considered, using only \(F\), in, e.g., [3], and will not be considered here. Of course all other special cases, e.g., a charge in an electromagnetic plane wave, can also be investigated exclusively in terms of \(F\) without introducing the electric and magnetic fields. The solutions for the motion of a charged particle in a constant electromagnetic field that are similar to (30) and (31) are already considered in the usual covariant approach (thus with \(F^{\mu\nu}\) and not with AQ \(F\)) in [20].

It is already mentioned that the expression for the Lorentz force in terms of \(F\) determines the way in which \(F\), i.e., the components of \(F\) in some reference frame are measured. First let us assume that in the chosen reference frame with the \(\{\gamma_\mu\}\) basis the considered charge is at rest, \(u = c\gamma_0\) (in components \(u_\mu = (c, 0, 0, 0)\)). Then from the expression for the Lorentz force \(K = (q/c) F \cdot u\) and the decomposition (23) (that holds in the \(\{\gamma_\mu\}\) basis) we
find that only “electriclike” part of $F$ is relevant in that case

$$K_{u_i=0} = qF_{i_0}^{i_0} \gamma_i.$$  

(32)

Thence we see that the relation

$$F_{i_0}^{i_0} \equiv \lim_{q \to 0} K^i_{u_i=0}/q$$  

(33)

defines experimentally the components (in the $\{\gamma_\mu\}$ basis) of “electriclike” part of $F$ as the ratio of the measured force $K_{u_i=0}$ on a stationary charge to the charge in the limit when the charge goes to zero. Having $F_{i_0}^{i_0} \gamma_i$ so defined the charge can be given a convenient uniform velocity $u$ with $u_k \neq 0$, from which the components (in the $\{\gamma_\mu\}$ basis) of “magneticlike” part of $F$ in the decomposition (32) are defined from the limit

$$F^{ik}_{u_k} \equiv \lim_{q \to 0} K^i_{u_i}/q, \quad i \neq k.$$  

(34)

It is worth noting that in the usual approaches the components of the 3D $E$ and $B$ are defined experimentally in exactly the same way through the 3D Lorentz force.

Instead of such coordinate-dependent formulation we can generalize relations (33) and (34) comparing them with the results for measuring $E$ and $B$ from [15]. Let us introduce as in (13) the velocity 1-vector $v$ of a family of observers who measures $F$ field and consider a special case, the Lorentz force acting on a charge as measured by a comoving observer ($v = u$). Then from the definition of $K$ and (13) one finds that

$$K = (q/c)F \cdot u = (q/c)F \cdot v = qE.$$  

Thence we can say that the Lorentz force ascribed by an observer comoving with a charge is purely electric and we define

$$F_E \cdot v/c \equiv \lim_{q \to 0} K_{v=v}/q,$$  

(35)

where $F_E$ is “electriclike” part of $F$. In the $\gamma_0$-frame with the $\{\gamma_\mu\}$ basis (35) reduces to (32) and (33) since $v = c\gamma_0$. Having $F_E$ so defined the charge can be given a convenient uniform velocity $u \neq v$ from which “magneticlike” part of $F$ can be defined from the limit

$$F \cdot u/c \equiv \lim_{q \to 0} K/q.$$  

(36)

In the $\gamma_0$-frame with the $\{\gamma_\mu\}$ basis and when the definitions (32) (or (33)) are also used then the relation (36) reduces to (34). This completely defines the
manner in which \( F \) is measured by an arbitrary observer. All this together explicitly shows that \( F \) is a measurable quantity with a well-defined physical procedure for the measurement.

### 2.6. The Stress-Energy Vector \( T(n) \) and the Quantities Derived from \( T(n) \)

The most important quantity for the momentum and energy of the electromagnetic field is the observer independent stress-energy vector \( T(n) \). It can be written in the following form

\[
T(n) = -\left(\frac{\varepsilon_0}{2}\right) \left[ (F \cdot F)n + 2(F \cdot n) \cdot F \right].
\]  

(37)

We present a new form for \( T(n) \) writing it as a sum of \( n \)-parallel part \((n-\parallel)\) and \( n \)-orthogonal part \((n-\perp)\)

\[
T(n) = -\left(\frac{\varepsilon_0}{2}\right) \left[ (F \cdot F) + 2(F \cdot n)^2 \right] n \\
- \varepsilon_0 \left[ (F \cdot n) \cdot F - (F \cdot n)^2 \right].
\]  

(38)

The first term in (38) is \( n-\parallel \) part and it yields the energy density \( U \). Namely using \( T(n) \) and the fact that \( n \cdot T(n) \) is positive for any timelike vector \( n \) we construct the expression for the observer independent energy density \( U \) contained in an electromagnetic field as \( U = n \cdot T(n) = \langle nT(n) \rangle \), (scalar, i.e., grade-0 multivector). Thus in terms of \( F \) and (38) \( U \) becomes

\[
U = \left(\frac{-\varepsilon_0}{2}\right) \langle FnFn \rangle = -\left(\frac{\varepsilon_0}{2}\right) \left[ (F \cdot F) + 2(F \cdot n)^2 \right].
\]  

(39)

The second term in (38) is \( n-\perp \) part and it is \( (1/c)S \), where \( S \) is the observer independent expression for the Poynting vector (1-vector),

\[
S = -\varepsilon_0 c \left[ (F \cdot n) \cdot F - (F \cdot n)^2 \right],
\]  

(40)

and, as can be seen, \( n \cdot S = 0 \). Thus \( T(n) \) expressed by \( U \) and \( S \) is

\[
T(n) = Un + (1/c)S.
\]  

(41)

Notice that the decompositions of \( T(n) \), (37), (38) and (41), are all observer independent decompositions, thus with AQs. Further the observer independent momentum density \( g \) is defined as \( g = (1/c^2)S \), i.e., \( g \) is \( (1/c) \) of the \( n-\perp \) part from (38)

\[
g = -\left(\frac{\varepsilon_0}{c}\right) \left[ (F \cdot n) \cdot F - (F \cdot n)^2 \right].
\]  

(42)
From \( T(n) \) one finds also the expression for the observer independent angular-momentum density \( M \)

\[
M = \frac{1}{c} T(n) \wedge x = \frac{1}{c} U(n \wedge x) + g \wedge x. \quad (43)
\]

It has to be emphasized once again that all these definitions are the definitions of the quantities that are independent of the chosen reference frame and of the chosen system of coordinates in it; they are all AQs. As I am aware they are not presented earlier in the literature.

All these quantities can be written in some basis \( \{ e_\mu \} \), which does not need to be the standard basis, as CBGQs. The field bivector \( F \) can be written as \( F = (1/2) F^{\alpha \beta} e_\alpha \wedge e_\beta \) where the basis components \( F^{\alpha \beta} \) are determined as \( F^{\alpha \beta} = e^\beta \cdot (e^\alpha \cdot F) = (e^\beta \wedge e^\alpha) \cdot F \). Then the quantities entering into the expressions for \( T(n), U, S, g \) and \( M \) are

\[
F \cdot F = -\frac{1}{2} F^{\alpha \beta} F_{\alpha \beta}, \quad F \cdot n = F^{\alpha \beta} n_\beta e_\alpha, \quad (F \cdot n)^2 = F^{\alpha \beta} F_{\alpha \nu} n_\beta n^\nu \quad \text{and} \quad (F \cdot n) \cdot F = F^{\alpha \beta} F_{\alpha \nu} n_\beta e^\nu. \quad \text{Thence} \quad T(n) \quad \text{becomes}
\]

\[
T(n) = -\frac{\varepsilon_0}{2} \left[ (1/2) F^{\alpha \beta} F_{\beta \alpha} n^\rho e_\rho + 2 F^{\alpha \beta} F_{\alpha \rho} n^\rho n_\beta \right], \quad (44)
\]

the energy density \( U \) is

\[
U = -\frac{\varepsilon_0}{2} \left[ (1/2) F^{\alpha \beta} F_{\beta \alpha} + 2 F^{\alpha \beta} F_{\alpha \rho} n^\rho n_\beta \right], \quad (45)
\]

and the Poynting vector \( S \) becomes

\[
S = -\varepsilon_0 c \left[ F^{\alpha \beta} F_{\alpha \rho} n^\rho e_\beta - F^{\alpha \beta} F_{\alpha \rho} n^\rho n_\beta n^\lambda e_\lambda \right]. \quad (46)
\]

In some basis \( \{ e_\mu \} \) we can write the stress-energy vectors \( T^\mu \) as \( T^\mu = T(e^\mu) = (-\varepsilon_0/2) F e^\mu F \). The components of the \( T^\mu \) represent the energy-momentum tensor \( T^{\mu \nu} \) in the \( \{ e_\mu \} \) basis

\[
T^{\mu \nu} = T^\mu \cdot e^\nu = (-\varepsilon_0/2) \langle F e^\mu F e^\nu \rangle, \quad \text{which reduces to familiar tensor form}
\]

\[
T^{\mu \nu} = \varepsilon_0 \left[ F^{\mu \alpha} g_{\alpha \beta} F^{\beta \nu} + (1/4) F^{\alpha \beta} F_{\alpha \beta} g^{\mu \nu} \right]. \quad (47)
\]

In the usual Clifford algebra approach, e.g., [1,2], one makes the space-time split and considers the energy-momentum density in the \( \gamma_0 \)-system (with the standard basis \( \{ \gamma_\mu \} \) \( T^0 = T(\gamma^0) = T(\gamma_0) \); the split \( T^0 \gamma^0 = T^0 \gamma_0 = T^{00} + T^0 \), separates \( T^0 \) into an energy density \( T^{00} = T^0 \cdot \gamma^0 \) and a momentum density \( T^0 = T^0 \wedge \gamma^0 \). Then from the expression for \( T^\mu \) and the relations [1] one finds [1,2] the familiar results for the energy density \( T^{00} = (\varepsilon_0/2)(E_x^2 + c^2 B_y^2) \) and
the Poynting vector $T^0 = \varepsilon_0 (E_H \times c B_H)$, where the commutator product $A \times B$ is defined as $A \times B \equiv (1/2)(AB - BA)$. However, as already said, the space-time split and the introduction of the electric and magnetic fields $E_H$ and $B_H$ are not only unnecessary but, as shown above and in [10,11], they are not equivalent to our general formulation with AQs. The space-time split is not a Lorentz invariant procedure and if one wants to use the electric and magnetic fields instead of the bivector field $F$ then the decompositions of $F$ into AQs, e.g., $E_{Hv}, B_{Hv}$ from (15), or $E, B$ from (13) have to be used and not the decompositions of $F$ into the observer dependent quantities $E_H, B_H$ from (1), or $E_J, B_J$ from (2).

2.7. The Local Conservation Laws in the $F$-Formulation

It is well-known that from the field equation in the $F$-formulation (4) one can derive a set of conserved currents. Thus, for example, in the $F$-formulation one derives in the standard way that $j$ from (4) is a conserved current. Simply, the vector derivative $\partial$ is applied to the field equation (4) which yields

$$\left(1/\varepsilon_0 c\right) \partial \cdot j = \partial \cdot (\partial \cdot F).$$

Using the identity $\partial \cdot (\partial \cdot M(x)) \equiv 0$ ($M(x)$ is a multivector field) one obtains the local charge conservation law

$$\partial \cdot j = 0.$$  (48)

In the axiomatic formulation [8] the equation (48), the electric charge conservation, is axiom 1. We again see that in the axiomatic formulation with the $F$ field the electric charge conservation is not an independent axiom but it simply follows from the single axiom for our theory, the field equation (4).

In a like manner we find from (25) (which is obtained from (1)) that

$$\partial \cdot T(n) = 0.$$  (49)

for the free fields. This is a local energy-momentum conservation law. In the derivation of (25) we used the fact that $T(a)$ is symmetric, i.e., that $a \cdot T(b) = T(a) \cdot b$. Namely using accents the expression for $T(\partial)$ ($T(\partial) = (\varepsilon_0/2)(F \partial F)$, where $\partial$ operates to the left and to the right by the chain rule) can be written as $T(\partial) = T(\partial) = (-\varepsilon_0/2)(\dot{F} \partial F + F \dot{\partial} F) = 0$, since in
the absence of sources $\partial F = \hat{F}\hat{\partial} = 0$ (the accent denotes the multivector on which the derivative acts). Then from the above mentioned symmetry of $T$ one finds that $\hat{T}(\hat{\partial}) \cdot a = \partial \cdot T(a) = 0$, $\forall$ const. $a$, which proves the equation (49).

Inserting the expression (41) for $T(n)$ into the local energy-momentum conservation law (49) we find

$$(n \cdot \partial)U + (1/c) \partial \cdot S = 0.$$  

(50)

The relation (50) is the well-known Poynting’s theorem but now completely written in terms of the observer independent quantities. Let us introduce the standard basis $\{\gamma_\mu\}$, i.e., an inertial frame of reference with the Einstein system of coordinates, and in the $\{\gamma_\mu\}$ basis we choose that $n = \gamma_0$, or in the component form it is $n^\mu(1,0,0,0)$. Then the familiar form of Poynting’s theorem is recovered in such coordinate system

$$\partial U/\partial t + \partial_i S^i = 0, \quad i = 1, 2, 3.$$  

(51)

It is worth noting that although $U$ (39) and $S$ (38), taken separately, are well-defined observer independent quantities, the relations (41), (49) and (50) reveal that only $T(n)$ (41), as a whole quantity, i.e., the combination of $U$ and $S$, enters into a fundamental physical law, the local energy-momentum conservation law (49). Thence one can say that only $T(n)$ (41), as a whole quantity, does have a real physical meaning, or, better to say, a physically correct interpretation. An interesting example that emphasizes this point is the case of an uniformly accelerated charge. In the usual (3D) approach to the electrodynamics ([5]; Jackson, Classical Electrodynamics, Sec. 6.8.) the Poynting vector $S$ is interpreted as an energy flux due to the propagation of fields. In such an interpretation it is not clear how the fields propagate along the axis of motion since for the field points on the axis of motion one finds that $S = 0$ (there is no energy flow) but at the same time $U \neq 0$ (there is an energy density). Our approach reveals that the important quantity is $T(n)$ and not $S$ and $U$ taken separately. $T(n)$ is $\neq 0$ everywhere on the axis of motion and the local energy-momentum conservation law (49) holds everywhere.

In the same way one can derive the local angular momentum conservation law, see [1], Space-Time Calculus.

3. 1-VECTOR LAGRANGIAN WITH $F$
In this section we only briefly consider the Lagrangian formulation. Instead of starting and constructing the whole theory of electromagnetism using the field equation (4) one can formulate the whole theory in terms of the 1-vector Lagrangian written with AQs

\[ L = (F \cdot \partial) \cdot (IF) - (IF \cdot \partial) \cdot F - 2(IF) \cdot j. \]  

(52)

When \( L \) (52) is written in terms of CBGQs in the standard basis \( \{ \gamma_\mu \} \) it becomes

\[ L = L_\alpha \gamma^\alpha, \quad L_\alpha = F^{\mu\nu}(\partial^\nu \ast F_{\mu\alpha}) - \ast F^{\mu\nu}(\partial^\nu F_{\mu\alpha}) + 2\ast F_{\alpha\mu}j^\mu, \]  

(53)

where \( L_\alpha \) is Sudbery’s Lagrangian [21].

The variational principle is applied to

\[ S = S_\alpha \gamma^\alpha = \left( \int L_\alpha |dx^4| \right) \gamma^\alpha; \]  

(54)

all four \( S_\alpha \) should be stationary under variations of \( F_{\mu\nu}, j^\mu \) being fixed. This leads to Euler-Lagrange equations

\[ \left[ \frac{\partial L_\alpha}{\partial F^{\mu\nu}} - \partial_\rho (\partial L_\alpha / \partial (\partial_\rho F^{\mu\nu})) \right. \]  

\[ \left. - (\mu \leftrightarrow \nu) \right] \gamma^\alpha = 0, \]  

(55)

which, as shown in [21], are equivalent to the full set of the covariant ME (6).

The most important fact in such Lagrangian approach is that the interaction term in (52) and (53) is written directly by means of the measurable electromagnetic field \( F \), and not, as usual, in terms of potentials. This will have important consequences in many branches of physics and they will be discussed in future publications.

Using (52) and the relations that connect \( F \) with different 4D AQs that represent electric and magnetic fields, e.g., \( E \) and \( B \) (13) or \( E_{Hv} B_{Hv} \) from (15), one can derive the equivalent Lagrangians with such 4D AQs.

4. COMPARISON WITH EXPERIMENTS. THE EXPLANATION OF THE TROUTON-NOBLE EXPERIMENT
The usual formulation of special relativity (SR) [12], which deals with the “apparent” transformations, the Lorentz contraction and the dilatation of time, and the invariant SR from [13] (given in terms of geometric 4D quantities, abstract tensors) are compared with the experiments in [14]. It is found in [14] that the usual formulation [12] shows only an “apparent” agreement (not the true one) with the traditional and modern experiments that test SR, e.g., the Michelson-Morley type experiments, the “muon” experiments, the Ives-Stilwell type experiments, etc., whereas the invariant SR from [13] is in a complete agreement with all considered experiments. Similarly it is proved in [10] that the standard transformations of the 3D $E$ and $B$ are also “apparent” transformations and that they significantly differ from the correct LT of 4D quantities representing the electric and magnetic fields. The comparison with experiments on motional electromotive force (emf) given in the second paper in [10] and also in [11] (the Faraday disk) shows that the geometric approach with geometric 4D quantities, 4D AQs or equivalently with 4D CBGQs, and with the LT of the 4D quantities representing the electric and magnetic fields always agrees with experiments for all relatively moving observers, whereas it is not the case for the usual approach with the 3D $E$ and $B$ and their standard transformations. Indeed it is obtained in [11] that the standard formulation yields different values for the emf of the Faraday disk for relatively moving inertial observers, see Eqs. (55) and (58) in [11]. (For the description and the picture of the Faraday disk see, e.g., [22] Chap. 18 or the first paper in [23].) On the other hand in our geometric approach the emf is defined as a Lorentz scalar and consequently the same value for that emf is obtained for all relatively moving inertial frames, see Eqs. (61-63) in [11].

It is worth noting that in these proofs, e.g., in the second paper in [10], we could equivalently use the bivector field $F$ through the relation [13] instead of 1-vectors $E$ and $B$, and similarly in [11].

In this paper we shall discuss the Trouton-Noble experiment [24], see also [25], comparing the usual explanations with our geometric approach that explicitly uses AQs, the $F$ field. In the experiment they looked for the turning motion of a charged parallel plate capacitor suspended at rest in the frame of the earth in order to measure the earth’s motion through the ether. The explanations, which are given until now (see, e.g., [26-30] and references therein) for the null result of the experiments [24] ([25]) are not correct from the invariant SR viewpoint, since they use quantities and transformations that are not well-defined in the 4D spacetime; e.g., the Lorentz contraction,
the nonelectromagnetic forces of undefined nature, the standard transformations for the 3D vectors \( \mathbf{E} \) and \( \mathbf{B} \) and for the torque as the 3D vector, etc. In all previous treatments it is correctly found that there is no torque for the stationary capacitor. However, the torque is always obtained for the moving capacitor and then the above mentioned different explanations are offered for the existence of another torque which is equal in magnitude but of opposite direction giving that the total torque is zero. In our approach the explanation for the null result is very simple and natural; all quantities are invariant 4D quantities, which means that their values are the same in the rest frame of the capacitor and in the moving frame. Thus if there is no torque (but now as a geometric, invariant, 4D quantity) in the rest frame then the capacitor cannot appear to be rotating in a uniformly moving frame.

Let us discuss the mentioned fundamental difference between the usual approaches and our geometric approach considering some recent “explanations” of the Trouton-Noble paradox. First we examine the “explanation” for the null result that is given in [26]. It is shown in [26] that no turning moment exists both in the rest frame of the capacitor and in the moving frame. But such correct result is achieved introducing, together with the electromagnetic forces, some nonelectromagnetic forces, the forces of constraint, whose physical nature is undefined. Thus, when using the energy arguments, the null result for the torque in the moving frame is obtained taking into account not only the electromagnetic energy but, [26]: “a contribution of energy from the forces of constraint, due to work done by these forces during a Lorentz contraction of the system.” There are several objections to such treatment [26] from the point of view of the invariant SR and they refer in the same measure to all similar treatments given in, e.g., [27-29]. These objections are the following:

i) The nonelectromagnetic forces and von Laue’s energy current [27] associated with them are not measurable quantities, see also the discussion of the Poincaré stresses and the electromagnetic energy-momentum in [31]. As Aranoff [32] stated in his severe criticism of von Laue’s explanation: ”The energy current idea of von Laue has to go the way of phlogiston, and the ether. It is interesting how man has to invent very fine fluids which carry energy but which are otherwise unobservable.”

ii) The Lorentz contraction is employed in all “explanations” [26-29] but, as shown in [13] and [14], see also [33], the Lorentz contraction has nothing to do with the LT and moreover it cannot be measured.

iii) The standard transformations of the 3D \( \mathbf{E} \) and \( \mathbf{B} \) are considered, e.g.,
[23] and [26-30], to be the LT of these fields. However it is rigorously proved in [10] that the standard transformations drastically differ from the correct LT of 4D quantities representing the electric and magnetic fields.

iv) The transformations of the components of a 3D force and torque are commonly used in the mentioned “explanations,” but the correct LT always refer to the 4D quantities. In the invariant SR the theoretical and experimental meaning is attributed only to geometric 4D quantities and not to their parts. Different relatively moving inertial 4D observers can compare only 4D quantities since they are connected by the LT.

In the recent paper [30] it is argued that the Trouton-Noble paradox is resolved once the electromagnetic momentum of the moving capacitor is properly taken into account. First it is obtained that there is a mechanical 3D torque on the moving capacitor and then it is shown that the rate of change of the angular electromagnetic field momentum associated with the moving capacitor completely balances that mechanical torque. We want to show that the appearance of the 3D torque on the moving capacitor in [30] is a consequence of the above mentioned objection iv) and another one:

v) The use of the principle of relativity for physical laws that are expressed by 3D quantities.

It will be seen below that in the geometric approach with 4D quantities the torque will not appear for the moving capacitor if it does not exist for the stationary capacitor. Thus, actually, the consideration with 4D quantities and their LT will reveal that there is no need at all either for the nonelectromagnetic forces and their torque, [26-29], or for the angular electromagnetic field momentum and its rate of change, i.e., its torque, [30]. Therefore we shall examine in more detail the calculation of the torque that is presented in [30], but not of the angular electromagnetic field momentum. In the rest frame of a thin parallel-plate capacitor, the $S'$ frame, there is no torque. Then it is assumed in [30] that in the $S$ frame the capacitor moves with uniform velocity $\mathbf{V}$ (the 3D vector) in the positive direction of the $x^1$-axis. (Fig. 1. from [30] is actually a projection onto the hypersurface $t'=\text{const.}$, which means that $x$, $y$ and $\Theta$ from that Fig.1. would need to be denoted as $x'^1$, $x'^2$ and $\Theta'$ respectively.) In the $S'$ frame $A$ denotes the surface area of the capacitor’s plates, $a$ is the distance between the capacitor’s plates and $\Theta'$ is the angle between the line joining the axis of rotation (i.e., the middle of the negative plate) with the middle of the positive plate and the $x'^2$ axis. That line is taken to be in the $x'^1$, $x'^2$ plane (see Fig. 1. in [30]). The torque (the 3D vector) experienced by the moving capacitor is determined by using
“relativistic” (my quotation-marks) transformation equations for the torque. These “relativistic” transformation equations for the 3D torque given in [30] are

\[ N_1 = N'_1/\gamma, \quad N_2 = N'_2 + (V^2/c^2)r'_1 K'_{cl,3}, \quad N_3 = N'_3 - (V^2/c^2)r'_1 K'_{cl,2}. \]  

(56)

where \( \gamma = (1 - V^2/c^2)^{-1/2} \), \( K'_{cl,i} \) are the components of the 3D force acting on the positive plate of the stationary capacitor, and \( r'_i \) are the components of the lever arm joining the axis of rotation with the point of application of the resultant force, i.e., the midpoint of the positive plate, see Fig. 1. in [30]. (The equations (56) are the equations (1)-(3) in [30].) As already said the 3D torque on the stationary capacitor is zero, \( N'_i = \varepsilon_{ijk}r'_j K'_{cl,k} = 0 \). (Note that in this equation for \( N'_i \) and in (56) we have used the same notation as in [8], i.e., the components of the 3D quantities are written with lowered (generic) subscripts, since they are not the spatial components of the 4D quantities. This refers to the third-rank antisymmetric \( \varepsilon \) tensor too.)

Taking into account that \( N'_i = 0 \) and \( K'_{cl,3} = 0 \) Jefimenko [30] finds that \( N_3 \) component is different from zero

\[ N_3 = -(V^2/c^2)r'_1 K'_{cl,2}. \]

(57)

This result is commented in [30] in the following way: “We have thus obtained a paradoxical result: contrary to the relativity principle, although our stationary capacitor experiences no torque, the same capacitor moving with uniform velocity along a straight line appears to experience a torque. What makes this result especially surprising is that we have arrived at it by using relativistic transformations that are based on the very same relativity principle with which they now appear to conflict.”

Let us examine the calculation leading to (56) and (57) and the above quoted statements. First \( N'_i \) is defined by means of the 3D quantities. Then the transformation equations (56) for the components of the 3D torque are derived considering that the transformations of the components of the 3D force are the LT. Since in \( S' \) the capacitor is at rest the mentioned transformations of the components of the 3D force are

\[ K_{cl,1} = K'_{cl,1}, \quad K_{cl,2} = K'_{cl,2}/\gamma, \quad K_{cl,3} = K'_{cl,3}/\gamma. \]

(58)

It is assumed in [30], as in many other papers including [26-29], that the transformations (58) (and similarly for (56)) are the relativistic transformations, i.e., the LT, that are based on the principle of relativity. Such opinion
implicitly supposes that 3D quantities, their transformations and physical laws written in terms of them are physically real in the 4D spacetime and in agreement with the principle of relativity. Actually such opinion prevails already from Einstein’s fundamental work on SR [12].

The approach of the invariant SR [13-15] and [10,11] is completely different. There, as already explained, the physical reality in the 4D spacetime is attributed only to geometric 4D quantities, AQS or CBGQs, their LT and physical laws written in terms of them. The principle of relativity is automatically included in such formulation. Thence in the 4D spacetime we are dealing with the Lorentz force

$$K = \left(\frac{q}{c}\right) F \cdot u,$$

where $u$ is the velocity 1-vector of a charge $q$. The torque, as a 4D AQ, is defined as the bivector

$$N = r \wedge K, \quad r = x_P - x_O,$$  \hspace{1cm} (59)

where $r$ is 1-vector associated with the lever arm, $x_P$ and $x_O$ are the position 1-vectors associated with the spatial point of the axis of rotation and the spatial point of application of the force $K$, $P$ and $O$ are the events whose position 1-vectors are $x_P$ and $x_O$.

In general the proper velocity $u$ for a point particle is $u = dx/d\tau$, $\tau$ is the proper time, $p$ is the proper momentum $p = mu$, the proper angular momentum of a particle is the bivector $L = x \wedge p$ and the torque $N$ about the origin is the bivector $N = dL/d\tau = x \wedge K$, where in this relation $K$ is an arbitrary force 1-vector. When $K$ is written as a CBGQ in the standard basis $\{\gamma_{\mu}\}$ then its components are $K^\mu = (\gamma_u K_{cl.1} V_i/c, \gamma_u K_{cl.1}, \gamma_u K_{cl.2}, \gamma_u K_{cl.3})$, and the components of $u$ in the $\{\gamma_{\mu}\}$ basis are $u^\mu = (\gamma_u c, \gamma_u V_1, \gamma_u V_2, \gamma_u V_3)$. $\gamma_u = (1 - V^2/c^2)^{-1/2}$, $K_{cl,i}$ are components of the 3D force and $V_i$ are components of the 3D velocity. We see that only when the considered particle is at rest, i.e., $V_i = 0$, $\gamma_u = 1$ and consequently $u^\mu = (c, 0, 0, 0)$, then $K^\mu$ contains only the components $K_{cl,i}$, i.e., $K^\mu = (0, K_{cl.1}, K_{cl.2}, K_{cl.3})$. However even in that case $u^\mu$ and $K^\mu$ are the components of geometric 4D quantities $u$ and $K$ in the $\{\gamma_{\mu}\}$ basis and not the components of some 3D quantities $V$ and $K_{cl.}$. The LT correctly transform the whole 4D quantity, which means that there is no physical sense in such transformations like (58) and (56): these transformations are not relativistic and they are not based on the principle of relativity. All conclusions derived from such relations as are (58) and (56) have nothing in common with SR as the theory of the 4D spacetime.

When $N$ is written as a CBGQ in the standard basis $\{\gamma_{\mu}\}$ then it becomes $N = (1/2)N^\mu_\nu \gamma_\mu \wedge \gamma_\nu$, where $N^\alpha_\beta = \gamma_\beta \cdot (\gamma^\alpha \cdot N) = r^\alpha K^\beta - r^\beta K^\alpha$. 27
Let us prove that $N$ is zero, $N = 0$, in the rest frame of the capacitor, here the $S'$ frame. In that frame we choose that $r^{0} = x_{p}^{0} - x_{0}^{0} = 0$. The system of coordinates is chosen in such a way that $r^{3} = 0$ (as in Fig. 1. [30]) giving that $r^{\mu} = (0, r_{1}', r_{2}', 0)$. Further, since in $S'$ we have stationary capacitor, $V_{i} = 0$, $\gamma_{u} = 1$, and from the chosen system of coordinates we conclude that $K'_{cl.3} = 0$, which yields $K'^{\mu} = (0, K'_{cl.1}, K'_{cl.2}, 0)$. Thence we find that $N^{\gamma_{0}} = N^{\gamma_{13}} = N^{\gamma_{23}} = 0$ and only remains $N^{\gamma_{12}} = r^{1}K^{2} - r^{2}K^{1}$. Let us prove that $N^{\gamma_{12}}$ is also zero. We shall use the result (24) obtained in Sec. 2.4 for $F$ of a flat sheet with the constant surface charge density $\sigma$, then also the chosen system of coordinates, i.e., Fig. 1. from [30], and the relation (13). Taking that in the relation (13) the velocity $v$ of the observers in the $S'$ frame is $v = c\gamma'_{0}$, i.e., that the $S'$ frame is the frame of “fiducial” observers, we have that $F^{\gamma_{0}} = E^{n}$ and all other $F^{\mu\nu}$ are zero. Then we can employ the discussion from Sec. 2. in [30]. In that discussion the electric field (as a 3D vector) and the 3D force $K'_{cl.}$ are determined. The electric field is produced by the negative plate of the capacitor at the location of the positive plate. The 3D force $K'_{cl.}$ acting on the positive plate is along the line joining the axis of rotation (i.e., the middle of the negative plate) with the middle of the positive plate. All this together yields that $F^{\gamma_{10}} = (\sigma/2\varepsilon_{0})\sin \Theta'$, $F^{\gamma_{10}} = -(\sigma/2\varepsilon_{0})\cos \Theta'$ and $K' = (\sigma A)(F^{\gamma_{10}}T'_{1} + F^{\gamma_{10}}T'_{2})$ and also $r^{1} = -a\sin \Theta'$, $r^{2} = a\cos \Theta'$, where, as already said, $A$ is the surface area of the capacitor’s plates and $a$ is the distance between the capacitor’s plates. Thence we find that $N^{\gamma_{12}} = (\sigma^{2}A/2\varepsilon_{0})a(\sin \Theta' \cos \Theta' - \sin \Theta' \cos \Theta') = 0$. Thus all $N^{\gamma_{\alpha\beta}}$ are zero in the $S'$ frame in which the capacitor is at rest. Since the CBGQ $(1/2)N^{\mu\nu}\gamma'_{\mu} \wedge \gamma'_{\nu}$ is an invariant quantity upon the passive LT we have proved that not only the components $N^{\gamma_{\alpha\beta}}$ are zero but at the same time that the whole torque $N$ is zero

$$N = (1/2)N^{\mu\nu}\gamma'_{\mu} \wedge \gamma'_{\nu} = (1/2)N^{\mu\nu}\gamma'_{\mu} \wedge \gamma'_{\nu} = 0.$$  \hspace{1cm} (60)

Thence the torque is zero not only for the stationary capacitor but for the moving capacitor as well. We see that in the approach with the geometric 4D quantities there is no Trouton-Noble paradox.

5. DISCUSSION AND CONCLUSIONS

The aim of this work is to present an axiomatic, geometric approach to electromagnetism in which the primary quantity is the electromagnetic field
$F$ as an observer independent 4D quantity. The whole theory is deduced from only one axiom: the field equation for $F$. This formulation with the $F$ field is a self-contained, complete and consistent formulation that does not make use either electric and magnetic fields or the electromagnetic potential $A$. Such approach conceptually differs from all previous approaches in several respects.

First, it places the electromagnetic field $F$ in the centre of the theoretical formulation and not, as usual, the 3D $E$ and $B$.

Second, the bivector field $F$ is considered to have an independent physical reality as a measurable 4D quantity, see particularly the end of Sec. 2.5 and Sec. 4.

Third, the whole theory is manifestly Lorentz invariant; it deals only with 4D AQs or 4D CBGQs and the space-time split, i.e., the foliation of the spacetime, is not introduced anywhere.

Fourth, the connection with the usual picture that deals with electric and magnetic fields is given by the relations (13) or (14). All quantities in these relations are 4D AQs in contrast to the common decompositions of $F$, e.g., from [1-3], into observer dependent quantities $E_H$, $B_H$ [1,2], or $E_J$, $B_J$ [3]. Every relation with $F$ can be transformed to to the corresponding relation with electric and magnetic fields as 4D AQs using the mentioned equations (13) or (14); see, for example, Sec. 2.3 in which the electromagnetic field of a point charge is considered.

Fifth, many new results are obtained here, which are not yet presented in the literature. However even in the cases when we used the results already presented in the literature, particularly in [1-3], these results are interpreted and explained in such a way to be in agreement with our axiomatic formulation given in terms of 4D AQs or 4D CBGQs and without any use of the space-time split. This differs from all previous approaches, e.g., [1-3] and [7,8].

The observer independent expressions for the stress-energy vector $T(n)$, the energy density $U$, the Poynting vector $S$, the momentum density $g$, the angular-momentum density $M$ and the Lorentz force $K$ are derived from the field equation (4) and presented in Sec. 2.6 in this paper. Then the second quantization procedure, and the whole quantum electrodynamics, can be constructed using these geometric, invariant, quantities $F$, $T(n)$, $U$, $S$, $g$ and $M$. Note that the standard covariant approaches to quantum electrodynamics, e.g., [34], usually deal with the component form (in the specific, i.e., the Einstein system of coordinates) of the electromagnetic 4-potential.
$A$ (thus requiring the gauge conditions too) and not with geometric quantities, AQs or CBGQs. The local conservation laws are also directly derived from the field equation (1) and written in an invariant way in Sec. 2.7. The observer independent integral field equation (22) corresponding to the field equation (1) is quoted and discussed in Sec. 2.4. In Sec. 3 we have constructed 1-vector Lagrangian $L$ (52), corresponding to the field equation (1), with a specific feature that the interaction term is written in terms of $F$ and not, as usual, in terms of potential $A$. When that $L$ (52) is written in the standard basis $\{\gamma_\mu\}$ it becomes Sudbery’s Lagrangian [21]. Such form of the Lagrangian suggests that in the classical electromagnetism, contrary to the generally accepted opinion, the interaction term can be expressed exclusively by means of measurable quantities, either $F$, or electric and magnetic fields as 4D geometric quantities when, e.g., the relations (13) or (14) are used. The consequences to the quantum mechanics will be examined elsewhere.

Particularly it has to be emphasized that the observer independent approach to the relativistic electrodynamics that is presented in this paper is in a complete agreement with existing experiments that test special relativity, which is not the case with the usual approaches. This is shown in detail in Sec. 4 for the Trouton-Noble experiment.

Furthermore we note that all observer independent quantities introduced here and the field equations written in terms of them hold in the same form both in the flat and curved spacetimes. The formalism presented here will be the basis for the formulation of quantum electrodynamics and, more generally, of the quantum field theory that exclusively deals with AQs or CBGQs.

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REFERENCES

1. D. Hestenes, *Space-Time Algebra* (Gordon and Breach, New York, 1966); *Space-Time Calculus*; available at: [http://modelingnts.la.asu.edu/evolution](http://modelingnts.la.asu.edu/evolution).
2. C. Doran, and A. Lasenby, *Geometric algebra for physicists* (Cambridge University, Cambridge, 2003).

3. B. Jancewicz, *Multivectors and Clifford Algebra in Electrodynamics* (World Scientific, Singapore, 1989).

4. D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus* (Reidel, Dordrecht, 1984).

5. J.D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1977) 2nd edn.; L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1979) 4th edn.; C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation* (Freeman, San Francisco, 1970). W.G.T.V. Rosser, *Clifford Algebra to Geometric Calculus* (Plenum, New York, 1968).

6. A. Einstein, *Ann. Physik* **49**, 769 (1916), tr. by W. Perrett and G.B. Jeffery, in *The Principle of Relativity* (Dover, New York, 1952).

7. Yu.N. Obukhov and F.W. Hehl, *Phys. Lett. A* **311**, 277 (2003).

8. F.W. Hehl, Yu.N. Obukhov, *Foundations of Classical Electrodynamics* (Birkhäuser, Boston, MA, 2003). F.W. Hehl, Yu.N. Obukhov and G.F. Rubilar, physics/9907046. F.W. Hehl, Yu.N. Obukhov, physics/0005084.

9. J.J. Cruz Guzmán, Z. Oziewicz, *Bull. Soc. Sci. Lett. Łódź* **53**, 107 (2003).

10. T. Ivezić, *Found. Phys.* **33**, 1339 (2003); physics/0411166 (to be published in Foundations of Physics Letters).

11. T. Ivezić, physics/0409118 v2 (to be published in Foundations of Physics).

12. A. Einstein, *Ann. Physik*. **17**, 891 (1905), tr. by W. Perrett and G.B. Jeffery, in *The Principle of Relativity* (Dover, New York, 1952).

13. T. Ivezić, *Found. Phys.* **31**, 1139 (2001).

14. T. Ivezić, *Found. Phys. Lett.* **15**, 27 (2002); physics/0103026 physics/0101091.

15. T. Ivezić, hep-th/0207250 hep-ph/0205277

16. M. Riesz, *Clifford Numbers and Spinors*, Lecture Series No. 38 (The Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, 1958).

17. D. Hestenes, in *Clifford Algebras and their Applications in Mathematical Physics*, F. Brackx et al, Eds. (Kluwer, Dordrecht, 1993).

18. T. Ivezić and Lj. Škovrlj, unpublished results. Lj. Škovrlj, *Thesis* (2002) (in Croatian).

19. R.M. Wald, *General Relativity* (Chicago University, Chicago,
1984). M. Ludvigsen, *General Relativity, A Geometric Approach* (Cambridge University, Cambridge, 1999). S. Sonego and M.A. Abramowicz, *J. Math. Phys.* **39**, 3158 (1998). D.A. T. Vanzella, G.E.A. Matsas, H.W. Crater, *Am. J. Phys.* **64**, 1075 (1996).

20. A.T. Hyman, *Am. J. Phys.* **65**, 195 (1997). G. Múnoz, *Am. J. Phys.* **65**, 429 (1997).

21. A. Sudbery, *J. Phys. A: Math. Gen.* **19**, L33-36 (1986).

22. W.K.H. Panofsky and M. Phillips, *Classical electricity and magnetism* (Addison-Wesley, Reading, Mass., 1962) 2nd edn.

23. L. Nieves, M. Rodríguez, G. Spavieri and E. Tonni, *Nuovo Cimento B* **116**, 585 (2001). G. Spavieri and G.T. Gillies, *Nuovo Cimento B* **118**, 205 (2003).

24. F.T. Trouton and H.R. Noble, *Philos. Trans. R. Soc. London Ser. A* **202**, 165 (1903).

25. H.C. Hayden, *Rev. Sci. Instrum.* **65**, 788 (1994).

26. A.K. Singal, *Am. J. Phys.* **61**, 428 (1993).

27. M. von Laue, *Phys. Zeits.* **12**, 1008 (1911).

28. W. Pauli, *Theory of Relativity* (Pergamon, New York, 1958)

29. S. A. Teukolsky, *Am. J. Phys.* **64**, 1104 (1996).

30. O.D. Jefimenko, *J. Phys. A: Math. Gen.* **32**, 3755 (1999).

31. T. Ivezić, *Found. Phys. Lett.* **12**, 105 (1999)

32. S. Aranoff, *Nuovo Cimento B* **10**, 155 (1972).

33. T. Ivezić, *Found. Phys. Lett.* **12**, 507 (1999).

34. J.D. Bjorken and S.D. Drell, *Relativistic Quantum Field* (McGraw-Hill, New York, 1964). F. Mandl and G. Shaw, *Quantum Field Theory* (Wiley, New York, 1995). S. Weinberg, *The Quantum Theory of Fields, Vol. I Foundations* (Cambridge University, Cambridge, 1995).