

\section{Introduction}

A contact manifold is a smooth manifold equipped with a contact structure, i.e., a maximally non-integrable hyperplane distribution. A canonical example of a contact manifold is provided by the space of first jets of hypersurfaces in a given manifold. Accordingly, contact geometry, i.e., the theory of contact structures, is at the foundation of the theory of first order partial differential equations in one dependent variable (see, for instance, \cite{3}). Every contact manifold is naturally equipped with a Jacobi bundle, i.e., a line bundle with a Lie bracket on sections, which is a first order differential operator in each entry. The Lie algebra of sections of the Jacobi bundle of a contact manifold is canonically isomorphic to the Lie algebra of contact vector fields, i.e., infinitesimal symmetries of the contact structure \cite{5,11}.

On another hand, contact manifolds can be understood as odd dimensional analogues of symplectic manifolds and there is a close relationship between contact and symplectic geometry. In particular, every contact manifold can be “extended” in a natural way to a symplectic manifold, its symplectization, encoding all the information about the contact structure. For instance, the Poisson algebra of the symplectization “knows everything” about the Jacobi bundle \cite{11}.

There are higher, or, in a sense, “categorized” versions of symplectic manifolds, namely multisymplectic manifolds. They are smooth manifolds equipped with a multisymplectic structure, i.e., a higher degree, closed, non-degenerate differential form (see, for instance, \cite{4}). A multisymplectic manifold is sometimes called $n$-plectic if its multisymplectic structure is of degree $n$. Thus, 1-plectic manifolds are standard symplectic manifolds. In a similar way as symplectic geometry is at the foundation of classical mechanics, multisymplectic geometry is at the foundation of classical field theory. There is a higher analogue of the Poisson algebra of a symplectic manifold. Namely, every multisymplectic structure gives rise to an $L_\infty$-algebra \cite{12} (see also \cite{14}), i.e., a cochain complex with a bracket satisfying the Jacobi identity only up to (a coherent system of higher) homotopies \cite{10,9}. In the same way as elements in the Poisson algebra of a symplectic manifold are interpreted as observables in mechanics,
elements in the $L_\infty$-algebra of a multisymplectic manifold should be interpreted as observables in field theory \cite{1,7}.

In this paper, I introduce higher versions of contact manifolds. I call them *multicontact manifolds*. They are smooth manifolds equipped with a *multicontact structure*, i.e., a maximally non-integrable distribution of higher codimension. I will call $n$-contact a multicontact manifold whose multicontact structure is $n$-codimensional. Thus, $1$-contact manifolds are standard contact manifolds. Higher order jet spaces are canonical examples of multicontact manifolds. Interestingly, there is a nice relationship between multicontact geometry and multisymplectic geometry: every $n$-contact manifold can be “extended” in a natural way to an $n$-plectic manifold, its multisymplectization, encoding all the information about the $n$-contact structure. Moreover, there is a higher analogue of the Jacobi bundle of a contact manifold. Namely, every multicontact structure gives rise to an $L_\infty$-algebra. The latter, is in the same relation with the $L_\infty$-algebra of the multisymplectization as the Jacobi bundle of a contact manifold is with the Poisson algebra of the symplectization.

Finally, recall that relaxing the non-degeneracy condition in the definition of a symplectic form, one gets the (much more general) notion of pre-symplectic form. A pre-symplectic form is just a closed differential 2-form. Similarly, relaxing the non-degeneracy condition in the definition of a multisymplectic form, one gets the (much more general) notion of pre-multisymplectic form. A pre-multisymplectic form is just a closed differential form. Pre-multisymplectic forms give rise to $L_\infty$-algebras of observables as well \cite{14}. I will show that similar considerations hold in the “contact realm”. Namely, relaxing the maximality condition in the definition of a contact distribution, one gets the (much more general) notion of pre-contact distribution. A pre-contact distribution is just an hyperplane distribution. Similarly, relaxing the maximality condition in the definition of multicontact distribution, one gets the (much more general) notion of pre-multicontact distribution. A pre-multicontact distribution is just a distribution (fulfilling a conceptually unessential, additional, regularity property). As such, it is a very general notion. Indeed, distributions are ubiquitous in differential geometry. The main reason is that any partial differential equation can be understood geometrically as a manifold with a, generically non-integrable, distribution. Solutions then identify with integral submanifolds of a suitable dimension. Below, I show that pre-multicontact distributions give rise to $L_\infty$-algebras as well. In an appendix I also provide coordinate formulas for the higher brackets in the $L_\infty$-algebras of higher order jet spaces.

1.1. **Notations and conventions.** Let $M$ be a smooth manifolds. I denote by $\mathbb{C}^\infty(M)$ the algebra of real-valued smooth functions on $M$. Moreover, I denote by $\mathfrak{X}(M)$ vector fields on $M$. I always understand vector fields as derivations of the algebra $\mathbb{C}^\infty(M)$. I denote by $\Omega(M) = \bigoplus_k \Omega^k(M)$ differential forms on $M$ and by $d : \Omega(M) \rightarrow \Omega(M)$ the exterior differential. I denote by $\iota_X$ and $L_X$ the insertion of a vector field $X$ into and the Lie derivative along $X$ of differential forms respectively. If $V \rightarrow M$ is a vector bundle over $M$, I denote by $V^* \rightarrow M$ the dual bundle. If $v$ is a section of $V$, I denote by $v_x$ its value at $x \in M$. Finally, I adopt the Einstein summation convention on pairs of upper-lower indexes.

2. **Distributions, Contact Manifolds and Symplectization**

In this section I collect my notations, and basic facts, about distributions on manifolds. Let $M$ be a smooth manifold and $C$ a distribution on it, i.e., a linear subbundle of the tangent bundle $TM$ of $M$. I denote by $C_x \subset T_x M$ the fiber of $C$ growing over $x \in M$. The annihilator of $C$ is the linear subbundle $C^\perp$ of the cotangent bundle $T^*M$ consisting of 1-forms vanishing on vectors in $C$. I denote by $\mathfrak{X}_C$ the Lie algebra of infinitesimal symmetries of $C$, i.e., those vector fields on $M$ whose flow preserves $C$. The Lie algebra $\mathfrak{X}_C$ is also the stabilizer of the subspace $\Gamma(C)$ in the Lie algebra $\mathfrak{X}(M)$ of vector fields on $M$. Denote by $N := TM/C$ the normal bundle to $C$. Thus, there is a natural $\mathbb{C}^\infty(M)$-linear
projection $\theta : \mathfrak{X}(M) \to \Gamma(N)$, a short exact sequence

$$0 \to \Gamma(C) \to \mathfrak{X}(M) \xrightarrow{\theta} \Gamma(N) \to 0,$$

and a dual exact sequence

$$0 \leftarrow \Gamma(C^\ast) \leftarrow \Omega^1(M) \leftarrow \Gamma(C^\perp) \leftarrow 0,$$

where I identified $C^\perp$ with the dual bundle $N^\ast$ of $N$. The curvature of $C$ is the well defined skew-symmetric $C^\infty(M)$-bilinear map

$$R : \Gamma(C) \times \Gamma(C) \to \Gamma(N), \quad (X, Y) \mapsto \theta([X, Y]).$$

Clearly, $C$ is integrable, i.e., $\Gamma(C)$ is a Lie subalgebra in $\mathfrak{X}(M)$, iff $R = 0$.

The characteristic distribution $D$ of $C$ consists of tangent vectors $\zeta$ in $C$ such that $R(\zeta, -) = 0$. Notice that $\Gamma(D) \subset \mathfrak{X}_C$. Accordingly, sections of $D$ are also called characteristic symmetries of $C$. A distribution without characteristic symmetries is called maximally non-integrable. It is easy to see that the characteristic distribution is integrable.

**Remark 1** (coordinate formulas). Given a distribution $C$ on a manifold $M$, I will always choose coordinates $x^1, \ldots, x^n, \ldots$ on $M$ which are adapted to $C$, i.e., such that $\Gamma(C)$ is locally spanned by vector fields $C_i := \frac{\partial}{\partial x_i} + C^n_i \frac{\partial}{\partial x^n}$, and $\Gamma(C^\perp)$ is locally spanned by differential 1-forms $\vartheta^a := dx^n - C^n_i dx^i$.

Put $\partial_a := \partial/\partial x^a$. It is easy to see that

$$[\partial_a, C_i] = \partial_a C_i^a \partial_a, \quad [C_i, C_j] = R^a_{ij} \partial_a, \quad R^a_{ij} := C_i(C^a_j) - C_j(C^a_i).$$

Dually,

$$d\vartheta^a = \partial_a C_i^a dx^i \& \vartheta^b - \frac{1}{2} R^a_{ij} dx^i \& dx^j.$$

Sections $\ldots, \vartheta(\partial_a), \ldots$ of the normal bundle $N$ (resp., sections $dx^i|_C$ of the dual bundle of $C$) form a local basis. Locally,

$$\theta = \vartheta^a \otimes \vartheta(\partial_a)$$

and

$$R = \frac{1}{2} R^a_{ij} dx^i|_C \& dx^j|_C \otimes \vartheta(\partial_a).$$

A vector field $Z = Z^i C_i$ in $\Gamma(C)$ sits inside the characteristic distribution $D$ iff $R^a_{ij} Z^j = 0$.

Now, recall that a contact structure, or a contact distribution, is a maximally non-integrable, hyperplane distribution. A contact manifold is a manifold $M$ equipped with a contact distribution $C$. The normal bundle $N = TM/C$ of a contact distribution $C$ is naturally a Jacobi bundle, or Jacobi structure, on $M$, i.e. a line bundle with a Lie bracket on $\Gamma(N)$ which is a differential operator of order 1 in each entry. Indeed, the map $\mathfrak{X}_C \to \Gamma(N)$, $X \mapsto \theta(Y)$, is a vector space isomorphism. In particular, $\Gamma(N)$ inherits from $\mathfrak{X}_C$, a Lie bracket, the Jacobi bracket, with the required bi-differential operator property.

There is a natural way of “producing” a symplectic manifold $(\tilde{M}, \tilde{\omega})$ from a contact manifold $(M, C)$, called the symplectization. Basically, $(\tilde{M}, \tilde{\omega})$ contains a full information about $(M, C)$. Let me recall the construction of $(\tilde{M}, \tilde{\omega})$. First of all, one defines $\tilde{M}$ as $C^\perp$ with the image of the 0 section removed. In particular, the projection $\pi : \tilde{M} \to M$ is a principal bundle with structure group $\mathbb{R}^\times$. Now, notice that $\tilde{M}$ is a symplectic submanifold in $T^*M$, i.e., the canonical symplectic structure on $T^*M$ restricts to a symplectic structure $\tilde{\omega}$ on $\tilde{M}$.

Finally, I describe the relationship between the Jacobi bracket $\{-, -\}$ on $\Gamma(N)$ and the Poisson bracket $\{-, -\}_C$ on $C^\infty(\tilde{M})$ (see, for instance, [10]). First, denote by $\Delta$ the Euler vector field on $C^\perp$. Since $\tilde{M}$ is open in $C$, $\Delta$ restricts to it. I denote again by $\Delta$ the restriction. It is the fundamental vector field corresponding to the canonical generator $1$ in the Lie algebra $\mathfrak{R}$ of the structure group $\mathbb{R}^\times$. 
A function \( f \) on \( \tilde{M} \) is homogeneous if \( \Delta(f) = f \). Now, sections of \( N \) identify with fiberwise linear functions on \( C^1 \), which in their turn, restrict to homogeneous functions on \( \tilde{M} \). Since \( \tilde{M} \) is dense in \( C \), the restriction is injective. Summarizing, a sections \( \nu \) of \( N \) identifies with a homogeneous function \( \tilde{\nu} \) on \( \tilde{M} \). Moreover,
\[
\{\tilde{\nu}_1, \tilde{\nu}_2\} = \{\tilde{\nu}_1, \tilde{\nu}_2\}_{\tilde{M}}.
\]

In fact, one could use Formula (1) as a definition for the Jacobi bracket on \( \Gamma(N) \).

The main aim of this paper is to provide a “higher version” of the content of this section.

### 3. Multicontact Manifolds and Multisymplectization

In this section, I present my proposal of higher (or, “categorified”) contact structures. First recall the definition of higher (pre)symplectic structure. Let \( M \) be a smooth manifold. If \( \sigma \) is a differential form on \( M \), I denote by \( \ker \sigma \subset TM \) the (not necessarily constant dimensional) distribution spanned by vector fields \( X \) such that \( i_X \sigma = 0 \). If \( d \sigma = 0 \) then \( \ker \sigma \) is an integrable distribution.

**Definition 2** \([3, 5]\). A closed differential \((n + 1)\)-form \( \omega \) on \( M \) is a pre-\( n \)-plectic structure (or pre-\( n \)-plectic form) if \( \ker \omega \) is a constant dimensional (integrable) distribution. A pre-\( n \)-plectic structure \( \omega \) such that \( \ker \omega = 0 \) is a \( n \)-plectic structure (or \( n \)-plectic form). A manifold equipped with a (pre-)\( n \)-plectic structure is a (pre-)\( n \)-plectic manifold.

Often, (pre-)\( n \)-plectic structures (resp., forms, manifolds) are collectively referred to as (pre-)multisymplectic structures (resp., forms, manifolds).

**Example 3.** Standard symplectic manifolds are 1-plectic manifolds.

**Example 4.** Let \( M \) be a manifold. On the bundle \( \Lambda^n T^* M \) of \( n \)-forms over \( M \) there is a tautological \( n \)-form \( \theta_M \) defined by
\[
(\theta_M)_a = \text{pr}^* a, \quad a \in \Lambda^n T^* M,
\]
where \( \text{pr} : \Lambda^n T^* M \to M \) being the projection. The exterior differential \( \omega_M := d\theta_M \) of \( \theta_M \) is an \( n \)-plectic structure on \( \Lambda^n T^* M \).

**Definition 5.** An \( n \)-codimensional distribution \( C \) on \( M \) is a pre-\( n \)-contact structure (or pre-\( n \)-contact distribution) if its characteristic distribution \( D \) is constant dimensional. A pre-\( n \)-contact structure such that \( D = 0 \) is an \( n \)-contact structure (or \( n \)-contact distribution). A manifold equipped with a (pre-)\( n \)-contact structure is a (pre-)\( n \)-contact manifold.

I will collectively refer to (pre-)\( n \)-contact structures (resp., forms, manifolds) as (pre-)multicontact structures (resp., forms, manifolds).

**Example 6.** Standard contact manifolds (see Section 2) are 1-contact manifolds.

**Example 7.** The Cartan distribution on a jet space is a multicontact structure (see Appendix B).

**Definition 8** \([12, 8]\). A vector field \( X \) on a pre-multisymplectic manifold \((M, \omega)\) is locally Hamiltonian if it preserves \( \omega \), i.e., \( L_X \omega = 0 \). A differential \((n - 1)\)-form on \((M, \omega)\) is Hamiltonian if there exists a vector field \( Y \) such that \( i_Y \omega = -d\alpha \). Then, \( Y \) is called an Hamiltonian vector field associated to \( \sigma \).

Clearly, Hamiltonian vector fields are locally Hamiltonian. Moreover, on a multisymplectic manifold, every Hamiltonian form possesses a unique associated Hamiltonian vector field. In the following I define the contact analogues of (locally) Hamiltonian vector fields (see Section 4 for the contact analogue of Hamiltonian forms).

**Definition 9.** A vector field \( X \) on a pre-multicontact manifold \((M, C)\) is multicontact if its flow preserves \( C \), i.e., \( X \in \mathfrak{X}C \).
Let \((M, C)\) be a pre-multicontact manifold. As in Section 2, denote by \(N = TM/C\) the normal bundle and by \(\theta : TM \to N\) the projection. Clearly, the kernel of the map \(\theta : \mathfrak{X}_C \to \Gamma(N)\) consists of vector fields sitting in \(D\). In particular, \(\Gamma(D)\) is an ideal in the Lie algebra \(\mathfrak{X}_C\), and the quotient \(\mathfrak{X}_C/\Gamma(D)\) is a Lie algebra.

**Definition 10.** Elements in the image of \(\theta : \mathfrak{X}_C \to \Gamma(N)\) are called Hamiltonian sections. Their collection is denoted by \(\Gamma_{\text{Ham}}(N)\).

Thus, there is a short exact sequence of Lie algebras

\[
0 \to DX \to \mathfrak{X}_C \to \Gamma_{\text{Ham}}(N) \to 0.
\]

When \(C\) is a multicontact structure, i.e., \(D = 0\), there is an isomorphism \(\mathfrak{X}_C \cong \Gamma_{\text{Ham}}(N), X \mapsto \theta(X)\).

**Remark 11.** (Coordinate formulas) A vector field \(X\) on \(M\) locally given by (see Remark 7) \(X = X^a\partial_a + X^iC_i\) is multicontact iff

\[
C_i(X^a) - \partial_bC_i^a X^b + R^a_{ij}X^j = 0.
\]

Since \(X\) projects onto a section \(\theta(X)\) of \(N\) which is locally given by \(\theta(X) = X^a\theta(\partial_a)\), one concludes that a section \(\nu\) of \(N\) locally given by \(\nu = \nu^a\theta(\partial_a)\) is Hamiltonian iff

\[
\partial_bC_i^a \nu^b - C_i(\nu^a) = R^a_{ij}X^j
\]

for some local functions \(X^j\). If \(D = 0\) then the \(X^j\)'s are uniquely defined.

There is a canonical way to how to associate a (pre-)n-plectic manifold to a (pre-)n-contact manifold, generalizing the multisymplectization procedure described in Section 2. Namely, let \((M, C)\) be an \(n\)-contact manifold. In the bundle \(\Lambda^n T^* M\) of \(n\)-forms, with bundle projection \(\text{pr} : \Lambda^n T^* M \to M\), consider the subset \(\tilde{M}\) consisting of \(n\)-forms \(a\) such that

\[
\{\xi \in T_x M : i_\xi a = 0\} = C_x, \quad x = \text{pr}(a).
\]

Since \(C\) is \(n\)-codimensional, \(\tilde{M}\) is a 1-dimensional subbundle of \(\Lambda^n T^* M\). Actually, it coincides with \(\Lambda^n C_\perp\) with the image of the 0 section removed. In particular, it is a principal \(\mathbb{R}^\times\)-bundle. Denote by \(\pi : \tilde{M} \to M\) the projection.

**Remark 12.** Let \(A \in \Omega^n(M)\) be a section of \(\tilde{M}\). A vector field \(X \in \mathfrak{X}(M)\) is multicontact iff \(L_X A = fA\) for a smooth function \(f\) on \(M\). Indeed, \(L_X A\) is proportional to \(A\) iff \(i_Y L_X A = 0\) for all \(Y \in \Gamma(\mathfrak{C})\). But, since \(i_Y A = 0\), then \(i_Y L_X A = i_Y[X, A]\) which vanishes iff \([Y, X] \in \mathfrak{C}X\).

**Remark 13.** Let \(A \in \Omega^n(M)\) be a section of \(\tilde{M}\). Then \(dA|_C = 0\) unless \(n = 1\). In particular, except for the contact case, \(dA|_C\) is not multisymplectic. Instead, \(dA\) has the following degeneracy property. Let \(Y \in \Gamma(\mathfrak{C})\), then

\[
\ker i_Y dA \supset C \iff Y \in \Gamma(D).
\]

Indeed let \(Y \in \Gamma(\mathfrak{C})\). Then \(i_Y dA = L_Y A\). Thus, in view of Remark 12, the condition on the left hand side of (3) is equivalent to \(L_Y A\) being proportional to \(A\), i.e., \(Y\) being a multicontact field. But multicontact fields in \(\Gamma(\mathfrak{C})\) are precisely vector fields sitting in the characteristic distribution \(D\).

**Remark 14** (coordinate formulas). Using the same notations as in Section 2, Remark 11 put

\[
\Theta := \partial^1 \wedge \cdots \wedge \partial^n, \quad \text{and} \quad \Theta_n := i_{\partial_n} \Theta,
\]

and notice that

\[
d\Theta = \partial_a C^a_i dx^i \wedge \Theta - \frac{1}{2} R^a_{ij} dx^i \wedge dx^j \wedge \Theta_a.
\]

A section \(A\) of \(\tilde{M}\) is locally of the form

\[
A = f \Theta, \quad f \in C^\infty(M), \quad f \neq 0.
\]
Then

\[ dA = (C_i(f) + f \partial_a C^a_i) \, dx^i \wedge \Theta - \frac{1}{2} R^a_{bc} \, dx^b \wedge dx^c \wedge \Theta_a. \]

Now, let \( A \in \Gamma(\Lambda^n C^\perp) \), and \( Z \in \Gamma(D) \). Clearly, \( L_Z A \in \Gamma(\Lambda^n C^\perp) \). Moreover \( L_Z A \) is \( C^\infty(M) \)-linear in \( Z \). This shows that \( \Lambda^n C^\perp \) carries a representation of the Lie algebroid \( D \) (see appendix A). In particular, there is a flat linear partial connection in \( \Lambda^n C^\perp \) over \( D \). One can restrict it to the open submanifold \( \tilde{M} \subset \Lambda^n C^\perp \) and get an involutive distribution \( \tilde{D} \) on \( \tilde{M} \) such that \( \dim \tilde{D} = \dim D \), and \( \pi_* (\tilde{D}) = D \). Recall that for \( a \in M \), \( \tilde{D}_a \) can be defined as follows (see Appendix A). Take a section \( A \) of \( \tilde{M} \), locally defined around \( x = \pi(a) \), such that \( A_x = \alpha \), and \( (d_D A)_x = 0 \). Then \( \tilde{D}_a := A_x(D_x) \).

As in Example [4], denote by \( \theta_M \) the tautological \( n \)-form on \( \Lambda^n T^*M \), and by \( \tilde{\theta} \) its restriction to \( \tilde{M} \). Put also \( \omega_M = d\theta_M \), and \( \tilde{\omega} = d\tilde{\theta} = \omega|_{\tilde{M}} \). Finally, recall that \( \omega_M \) is an \( n \)-plectic form.

**Theorem 15.** The canonical form \( \tilde{\omega} \) on \( \tilde{M} \) is pre-\( n \)-plectic and \( \ker \tilde{\omega} = \tilde{D} \).

**Proof.** Notice preliminarily that vertical tangent vectors to \( \Lambda^n T^*M \) identify with points in \( \Lambda^n T^*M \). Similarly, vertical tangent vectors to \( \tilde{M} \) at \( a \) identify with elements \( v \in \Lambda^n T^*M \) such that \( \ker v \supseteq C_x \), \( x = \pi(a) \). In the following, I will understand such identifications. It is easy to see that \( i_v \omega = \pi^*(v) \), for all vertical tangent vectors to \( \tilde{M} \). Now, let \( A \) be a local section of \( \tilde{M} \) such that \( A_x = a \) and \( (d_D A)_x = 0 \), so that \( \tilde{D}_a := A_x(D_x) \). Pick \( \zeta \in D_x \) and \( \omega_M(a) \in \tilde{D}_a \). Show that \( i_{A_x(a)} \tilde{\omega} = 0 \). It is enough to show that

\[ \tilde{\omega}(v, A_x(\zeta), A_x(\xi_1), \ldots, A_x(\xi_{n-1})) = \tilde{\omega}(A_x(\zeta), A_x(\xi_1), \ldots, A_x(\xi_n)) = 0 \]

for all vertical tangent vectors \( v \) to \( \tilde{M} \) at \( a \), and \( \xi_1, \ldots, \xi_n \in T_x \tilde{M} \). Now,

\[ \tilde{\omega}(v, A_x(\zeta), A_x(\xi_1), \ldots, A_x(\xi_{n-1})) = v(\xi_1, \ldots, \xi_{n-1}) = 0 \]

since \( \zeta \in D_x \subset C_x \). Moreover,

\[ \tilde{\omega}(A_x(\zeta), A_x(\xi_1), \ldots, A_x(\xi_n)) = dA(\zeta, \xi_1, \ldots, \xi_n) = (d_D A)(\zeta)(\xi_1, \ldots, \xi_n) = 0. \]

Conversely, let \( \eta \in T_a \tilde{M} \) be such that \( i_\eta \tilde{\omega} = 0 \). In other words,

\[ \tilde{\omega}(\eta_1, \eta_2, \ldots, \eta_n) = 0 \quad \text{for all } \eta_1, \ldots, \eta_n \in T_a \tilde{M}. \]

Now, put \( \xi = \pi_*(\eta) \), and \( \xi_i = \pi_*(\eta_i) \), \( i = 1, \ldots, n \). Then \( \eta = A_\ast(\xi) + v \) and \( \eta_i = A_\ast(\xi_i) + v_i \) for \( v, v_i \) vertical tangent vectors to \( \tilde{M} \), \( i = 1, \ldots, n \). Hence

\[ 0 = \tilde{\omega}(\eta_1, \ldots, \eta_n) = \tilde{\omega}(A_\ast(\xi), \eta_1, \ldots, \eta_n) + v(\xi_1, \ldots, \xi_n) = \tilde{\omega}(A_\ast(\xi), \eta_1, \ldots, \eta_n) - v_1(\xi_2, \ldots, \xi_n) + v(\xi_1, \ldots, \xi_n) \]

Choosing \( \xi_1 = 0 \) one immediately see that \( \xi \in C_x \), and (4) simplifies to

\[ \tilde{\omega}(A_\ast(\xi), \eta_2, \ldots, \eta_n) + v(\xi_1, \ldots, \xi_n) = 0. \]

Now, choosing \( \xi_1 \in C_x \) and \( v_i = 0 \) for \( i > 1 \), one sees that, in view, of Remark [13] \( \xi \in D_x \). Finally, choose \( v_i = 0 \) for \( i > 1 \). Then

\[ 0 = \tilde{\omega}(A_\ast(\xi), A_\ast(\xi_1), \eta_2, \ldots, \eta_n) + v(\xi_1, \ldots, \xi_n) = dA(\xi, \xi_1, \ldots, \xi_n) + v(\xi_1, \ldots, \xi_n) = (d_D A)(\xi)(\xi_1, \ldots, \xi_n) + v(\xi_1, \ldots, \xi_n) = v(\xi_1, \ldots, \xi_n). \]

Hence \( v = 0 \). Concluding, \( \eta = A_\ast(\xi) \), with \( \xi \in D_x \). \( \square \)
In particular $\tilde{\omega}$ is a multisymplectic structure iff $C$ is a multicontact structure.

**Definition 16.** The pair $(\tilde{M}, \tilde{\omega})$ is called the (pre-)multisymplectization of $(M, C)$.

**Remark 17** (coordinate formulas). On $\tilde{M}$ one can choose coordinates $\ldots, x^i, \ldots, z^n, \ldots, p$, where $p$ is implicitly defined by $a = p(u)\Theta \in \tilde{M}$. Notice that $p \neq 0$. Locally, $\tilde{\theta} = p\Theta$,

$$\tilde{\omega} = (dp + p\partial_s C^n dx^i) \wedge \Theta - \frac{1}{2} p R_{ij}^a dx^i \wedge dx^j \wedge \Theta_a,$$

and a direct computation shows that $\tilde{D}$ is locally generated by vector fields of the form

$$Z^i \left( C_i - p \partial_s C^n \frac{\partial}{\partial p} \right), \quad \text{with } R_{ij}^a Z^j = 0.$$

Now, I discuss the relationship between contact vector fields of $(M, C)$ and locally Hamiltonian vector fields on the pre-multisymplectization. Let $X$ be a multicontact vector field, i.e., $X \in \mathfrak{x}_C$. Then $X$ can be naturally lifted to a vector field $\tilde{X}$ on $\tilde{M}$. Namely, $X$ lifts to a unique vector field $X^*$ on $\Lambda^n T^* \tilde{M}$ preserving the tautological $n$-form. It follows from multicontactness that $X^*$ is actually tangent to $\Lambda^n C^\perp$. Clearly, $\tilde{X}$ preserves the tautological $n$-form on $\tilde{M}$ and, therefore, it preserves $\tilde{\omega}$. Conversely, let $Y$ be a locally Hamiltonian vector field on $(\tilde{M}, \tilde{\omega})$. The next proposition shows, in particular, that, if $Y$ is projectable onto $\tilde{M}$, then its projection is a multicontact vector field.

**Proposition 18.** Let $Y$ be a locally Hamiltonian on $(\tilde{M}, \tilde{\omega})$. If $Y$ projects on a vector field $X$ on $M$, then $X$ is multicontact. Moreover, if $R \neq 0$, then $Y = \tilde{X}$.

**Proof.** I need some preliminary remarks. The Euler vector field $\Delta$ on $\Lambda^n C^\perp$ restricts to $\tilde{M}$. Denote again by $\Delta$ the restriction. It is the fundamental vector field corresponding to the canonical generator $1$ of the Lie algebra $\mathbb{R}$ of the structure group $\mathbb{R}^\times$ of the principal bundle $\pi : \tilde{M} \rightarrow M$. Locally $\Delta = p\partial/\partial p$. Notice that

$$i_{\Delta} \tilde{\theta} = 0, \quad \text{and} \quad L_{\Delta} \tilde{\theta} = \tilde{\theta},$$

hence,

$$i_{\Delta} \tilde{\omega} = \tilde{\theta}, \quad \text{and} \quad L_{\Delta} \tilde{\omega} = \tilde{\omega}.$$

Now, prove that $X$ is multicontact. It is enough to show that $Y$ preserves the distribution $\tilde{C} := \pi^{-1}(C)$ on $\tilde{M}$. Clearly, $\tilde{C} = \ker \tilde{\theta}$. Now, since $Y$ is projectable, then $[Y, \Delta] = f\Delta$ for some function $f$ on $\tilde{M}$. From $L_Y \tilde{\omega} = 0$, it follows

$$0 = i_{\Delta} L_Y \tilde{\omega} = i_{[\Delta, Y]} \tilde{\omega} + L_Y i_{\Delta} \tilde{\omega} = -f \tilde{\theta} + L_Y \tilde{\theta}.$$

This shows that $L_Y \tilde{\theta} = f \tilde{\theta}$. Finally, let $Z \in \Gamma(\tilde{C})$. Compute

$$i_{[Y, Z]} \tilde{\theta} = [L_Y, i_Z] \tilde{\theta} = i_Z L_Y \tilde{\theta} = f i_Z \tilde{\theta} = 0.$$

This shows that $[Y, Z] \in \Gamma(\tilde{C})$ for all $Z$, hence $X$ is multicontact. In particular, $Y - \tilde{X}$ is a vertical locally Hamiltonian vector field. But, if $R \neq 0$, then any vertical locally Hamiltonian vector field $V$ is trivial. Indeed, $V = g\Delta$ for some function $g$, and

$$0 = L_{g\Delta} \tilde{\omega} = g \tilde{\omega} + dg \wedge \tilde{\theta}.$$

Assume by absurd that $g \neq 0$ somewhere, hence in an open subset $U$ of $\tilde{M}$. Without loss of generality, let $g > 0$ in $U$. Then

$$\tilde{\omega} = dh \wedge \tilde{\theta}, \quad h = -\log g.$$ (5)
Compute
\[ i_{C_i} \tilde{\omega} = p \partial_a C_i^a \Theta - p R^a_{ij} dz^j \wedge \Theta_a. \] (6)

But
\[ i_{C_i} (dh \wedge \tilde{\theta}) = C_i(h) \tilde{\theta} = p C_i(h) \Theta, \]
which is inconsistent with (6) when \( R \neq 0. \)

\[ \tilde{\omega} = dp \wedge dz^1 \wedge \cdots \wedge dz^m. \] The vector field \( \partial/\partial p \) is locally Hamiltonian and vertical, in particular projectable. However, it projects onto the trivial vector field.

Remark 19. The second part of Proposition 18 cannot be extended to the case \( R = 0. \) Indeed, when \( R = 0, \) in view of Frobenius Theorem, one can choose \( C_i^a = 0, \) and \( \tilde{\omega} \) is locally given by \( \tilde{\omega} = dp \wedge dz^1 \wedge \cdots \wedge dz^m. \) The vector field \( \partial/\partial p \) is locally Hamiltonian and vertical, in particular projectable. However, it projects onto the trivial vector field.

Remark 20 (coordinate formulas). Let \( X \) be a multicontact vector field on \((M, C)\) locally given by \( X = X^a \partial_a + X^i C_i. \) A direct computation shows that
\[ \tilde{X} = X^a \partial_a + X^i C_i - (\partial_a X^a + \partial_a C^a X^i) \Delta. \] (7)

4. Homogeneous de Rham complex

In the algebra \( \Omega(\tilde{M}) \) of differential forms on \( \tilde{M} \) consider the subspace \( \Omega_\sigma(\tilde{M}) \) consisting of homogeneous differential forms, i.e., those differential forms \( \sigma \) such that \( L_\Delta \sigma = \sigma, \) \( \Delta \) being the Euler vector field of \( \Lambda^n C^{-1} \) restricted to \( \tilde{M} \) (see the proof of Proposition 18).

Remark 21. A form \( \sigma \) on \( \tilde{M} \) is homogeneous iff it is locally of the form
\[ \sigma = p \pi^*(\sigma') + dp \wedge \pi^*(\sigma'') \quad \sigma', \sigma'' \in \Omega(M). \]
In particular, \( \tilde{\theta} \) and \( \tilde{\omega} \) are homogeneous.

In general, \( \Omega_\sigma(\tilde{M}) \) is not a subalgebra of \( \Omega(\tilde{M}). \) Nonetheless, since Lie derivatives commute with the exterior differential, it is a subcomplex of the de Rham complex \( (\Omega(\tilde{M}), d). \)

Proposition 22. The homogeneous de Rham complex \((\Omega_\sigma(\tilde{M}), d)\) is acyclic.

Proof. The insertion \( i_\Delta \) is a contracting homotopy for \((\Omega_\sigma(\tilde{M}), d), \) i.e., \([i_\Delta, d] = L_\Delta = id \) on \( \Omega_\sigma(\tilde{M}). \) \( \square \)

It immediately follows from the proof of the above proposition that
\[ \Omega^{n-1}_\sigma(\tilde{M}) = B_\sigma \oplus K \] (8)

where
\[ B_\sigma := \text{im}(d: \Omega^{n-2}_\sigma(\tilde{M}) \rightarrow \Omega^{n-1}_\sigma(\tilde{M})), \]
\[ K := \ker(i_\Delta: \Omega^{n-1}_\sigma(\tilde{M}) \rightarrow \Omega^{n-2}_\sigma(\tilde{M})). \]

Decomposition (8) is implemented as follows. For \( \sigma \in \Omega^{n-1}_\sigma(\tilde{M}), \)
\[ \sigma = L_\Delta \sigma = i_\Delta d \sigma + i_\Delta d \sigma \]
with \( d \sigma \in B_\sigma, \) and \( i_\Delta d \sigma \in K. \)

Now, recall that an \((n-1)\)-form \( \sigma \) on \( \tilde{M} \) is Hamiltonian iff there is a vector field \( X, \) an associated Hamiltonian vector field, not necessarily unique (unless \( \tilde{\omega} \) is multisymplectic), such that \( i_X \omega = -d \sigma. \) Denote by \( \Omega^{n-1}_\text{Ham}(\tilde{M}, \tilde{\omega}) \subset \Omega^{n-1}(\tilde{M}) \) the vector subspace of Hamiltonian forms. Consider the distinguished subspace \( \Omega^{n-1}_\text{Ham}(M, C) \) of \( \Omega^{n-1}(\tilde{M}, \tilde{\omega}) \) consisting of homogenous Hamiltonian forms with an associated Hamiltonian vector field which is projectable on \( M: \)
\[ \Omega^{n-1}_\text{Ham}(M, C) := \{ \sigma \in \Omega^{n-1}_\sigma(\tilde{M}) : \exists Y \in \mathfrak{X}(\tilde{M}) \text{ such that } Y \text{ is projectable and } i_Y \tilde{\omega} = -d \sigma \}. \]
As I will show in the next section elements in $\Omega^{n-1}_{\text{Ham}}(M, C)$ should be understood as the contact analogues of Hamiltonian forms on (pre-) multisymplectic manifolds. For now, notice that if $C$ is 1-contact, then $\tilde{\omega}$ is 1-pectly and every homogeneous function on $\tilde{M}$ is in $\Omega^{n-1}_{\text{Ham}}(M, C)$. Moreover, since $\tilde{M}$ is dense in $C^+$, homogeneous functions on $\tilde{M}$ identify with fiberwise linear functions on $C^+$, i.e., sections of $N$. Thus, if $C$ is 1-contact, $\Omega^{n-1}_{\text{Ham}}(M, C)$ is in one-to-one correspondence with $\Gamma(N)$, sections of the Jacobi bundle. In the general case, I will be interested in the following truncated, homogenous de Rham complex

$$
0 \longrightarrow C^\infty(\tilde{M}) \xrightarrow{d} \Omega^1_{\tilde{M}} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-2}_{\tilde{M}} \xrightarrow{d} \Omega^{n-1}_{\text{Ham}}(M, C) \longrightarrow 0,
$$

which is obviously well defined. In view of Proposition 23 the cohomology of (9) is trivial everywhere except in the last term where it is $\Omega^{n-1}_{\text{Ham}}(M, C)/B_0$. The next proposition describes the quotient $\Omega^{n-1}_{\text{Ham}}(M, C)/B_0$.

**Proposition 23.** There is a canonical isomorphism

$$
\Gamma_{\text{Ham}}(N) \simeq \Omega^{n-1}_{\text{Ham}}(M, C)/B_0.
$$

**Proof.** It follows from (8) that

$$
\Omega^{n-1}_{\text{Ham}}(M, C) = B_0 \oplus K_{\text{Ham}}
$$

where $K_{\text{Ham}} := K \cap \Omega^{n-1}_{\text{Ham}}(M, C)$. It is then enough to show that $K_{\text{Ham}} \simeq \Gamma_{\text{Ham}}(N)$. The isomorphism can be described as follows. First of all, notice that, for all $a \in \tilde{M}$, since ker $a = C_x$, $x = \pi(a)$, there exists a unique linear map $\varphi_a : N_x \longrightarrow \Lambda^{n-1}T_x^*M$ such that $a = \varphi_a \circ \theta$. Now, let $\nu \in \Gamma(N)$. Define an $(n - 1)$-form $\tilde{\nu}$ on $\tilde{M}$ by putting

$$
\tilde{\nu}_a := \pi^*(\varphi_a(\nu)), \quad a \in \tilde{M}.
$$

If $\nu$ is locally given by $\nu = \nu^a \theta(\partial_a)$, then $\tilde{\nu}$ is locally given by $\tilde{\nu} = \nu^a \tilde{\Theta}_a$. This shows that $\tilde{\nu} \in K$. Moreover $\tilde{\nu} = 0$ if $\nu = 0$. Finally, $\tilde{\nu} \in K_{\text{Ham}}$ if $\nu \in \Gamma_{\text{Ham}}(N)$. Indeed, let $\nu$ be a Hamiltonian section. Then $\nu = \theta(X)$ for some multicontact vector field $X$. Lift it to a multisymplectic vector field $\tilde{X}$ on $\tilde{M}$. It is easy to see, for instance in local coordinates, that $i_{\tilde{X}} \tilde{\theta} = \tilde{\nu}$. It follows that

$$
i_{\tilde{X}} \tilde{\omega} = i_{\tilde{X}} d\tilde{\theta} = L_{\tilde{X}} \tilde{\theta} - d i_{\tilde{X}} \tilde{\theta} = -d\tilde{\nu}.
$$

Define the injective map $\Gamma_{\text{Ham}}(N) \longrightarrow K_{\text{Ham}}$ as $\nu \mapsto \tilde{\nu}$. Conversely, let $\sigma \in K_{\text{Ham}}$ and $Y$ be an associated projectable Hamiltonian vector field, i.e., $i_Y \tilde{\omega} = -d\sigma$. Then,

$$
0 = i_{\Delta}(i_Y \tilde{\omega} + d\sigma) = -d i_{\Delta} \tilde{\omega} + L_{\Delta} \sigma = -d\tilde{\nu} + \sigma,
$$

i.e., $\sigma = i_Y \tilde{\theta}$. If $Y$ is locally given by $Y = Y^a \partial_a + Y^t C_i + Y^0 \Delta$, then $\sigma = i_Y \tilde{\theta} = p Y^a \tilde{\Theta}_a$. Now, let $X$ be the projection of $Y$. In view of Proposition 15, $X$ is contact, and $Y = \tilde{X}$. Moreover, $\theta(X) = Y^a \theta(\partial_a)$. Hence $\sigma = \tilde{\theta}(\tilde{X})$. This shows that the map $\Gamma_{\text{Ham}}(N) \longrightarrow K_{\text{Ham}}$ defined above is surjective. \[\square\]

**Remark 24.** If $C$ is multicontact, then $\tilde{\omega}$ is multisymplectic and

$$
\Omega^{n-1}_{\text{Ham}}(M, C) = \Omega^{n-1}_{\text{Ham}}(\tilde{M}, \tilde{\omega}) \cap \Omega^{n-1}_c(\tilde{M}).
$$

Indeed, $\Omega^{n-1}_{\text{Ham}}(M, C) \subset \Omega^{n-1}_{\text{Ham}}(\tilde{M}, \tilde{\omega}) \cap \Omega^{n-1}_c(\tilde{M})$. Now, let $\sigma \in \Omega^{n-1}_{\text{Ham}}(\tilde{M}, \tilde{\omega})$ be homogeneous, and let $Y$ be the (unique) Hamiltonian vector field associated to it. Then

$$
0 = L_{\Delta}(i_Y \tilde{\omega} + d\sigma) = i_{[\Delta, Y]} \tilde{\omega}.
$$

Hence $[\Delta, Y] = 0$ and $Y$ is projectable. This shows that $\sigma \in \Omega^{n-1}_{\text{Ham}}(M, C)$. Thus, $\Omega^{n-1}_{\text{Ham}}(\tilde{M}, \tilde{\omega}) \cap \Omega^{n-1}_c(\tilde{M}) \subset \Omega^{n-1}_{\text{Ham}}(M, C)$. 

5. $L_\infty$-algebras from multicontact geometry

In this section I define a higher analogue of the Jacobi bundle of a contact manifold. Equivalently, I define the contact analogue of the $L_\infty$-algebra of a (pre-)multisymplectic manifold \cite{12, 14}. First of all recall the definition of an $L_\infty$-algebra. I use the “homological convention”.

**Definition 25** \cite{10, 11}. An $L_\infty$-algebra is a pair $(g, \{\lambda_\ell, \ell \in \mathbb{N}\})$, where $g = \bigoplus_\ell \mathfrak{g}_\ell$ is a graded vector space, and the $\lambda_\ell$’s are $\ell$-ary, graded, multilinear, degree $\ell - 2$ operations

\[
\lambda_\ell : \Lambda^\ell g \longrightarrow g, \quad k \in \mathbb{N},
\]

such that

\[
\sum_{i+j=\ell} (-1)^{ij} \sum_\sigma \chi_{\sigma, v} \lambda_{\sigma+1}(v_{\sigma(1)}, \ldots, v_{\sigma(i)}, v_{\sigma(i+1)}, \ldots, v_{\sigma(i+j)}) = 0,
\]

for all $v_1, \ldots, v_\ell \in g$, $\ell \in \mathbb{N}$ (in particular, $(g, \lambda_1)$ is a chain complex and $H(g, \lambda_1)$ is a graded Lie algebra).

In Formula \cite{10}, the sum is over all unshuffles $S_{i,j}$, i.e., permutations $\sigma$ of $\{1, \ldots, \ell\}$ such that $\sigma(1) < \cdots < \sigma(i)$, and $\sigma(i+1) < \cdots < \sigma(\ell)$, and $\chi_{\sigma, v}$ is the sign implicitly defined by

\[
v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(\ell)} = \chi_{\sigma, v} v_1 \wedge \cdots \wedge v_\ell,
\]

where the wedge $\wedge$ indicates the exterior (graded skew-symmetric) product of elements in $g$, which satisfies, by definition, $v \wedge w = -(-)^{|v||w|} w \wedge v$, for $v \in \mathfrak{g}_i$ and $w \in \mathfrak{g}_j$.

If $g$ is concentrated in degree 0, then an $L_\infty$-algebra structure on $g$ is simply a Lie algebra structure. Similarly, if $\lambda_\ell = 0$ for all $\ell > 2$, then $(g, \{\lambda_\ell, \ell \in \mathbb{N}\})$ is a differential graded Lie algebra. More generally, $L_\infty$-algebras are Lie algebras up to homotopy. Indeed, the binary bracket $\lambda_2$ of an $L_\infty$-algebra satisfies the (graded) Jacobi identity only up to an homotopy encoded by $\lambda_3$. Similarly, the higher brackets satisfy higher versions of the Jacobi identity (up to homotopies).

In \cite{12} and \cite{14} the authors show that there is an $L_\infty$-algebra canonically associated to a (pre-)multisymplectic manifold. Such $L_\infty$-algebra plaus a role analogous to that of the Poisson algebra of functions on a symplectic manifold \cite{8, 6, 7}. In the case of the (pre-)multisymplectization $(\tilde{M}, \tilde{\omega})$ of a pre-n-contact manifold, Rogers and Zambon results read as follows.

**Theorem 26.** There is an $L_\infty$-algebra $\mathfrak{g}_\bullet(\tilde{M}, \tilde{\omega}) = \bigoplus_{i=0}^{n-1} \mathfrak{g}_i(\tilde{M}, \tilde{\omega})$, concentrated in degrees $0, \ldots, n-1$, where

\[
\mathfrak{g}_i(\tilde{M}, \tilde{\omega}) := \begin{cases} 
\Omega^{n-1}_{\text{Ham}}(\tilde{M}, \tilde{\omega}) & \text{if } i = 0 \\
\Omega^{n-i-1}(\tilde{M}) & \text{if } 0 < i \leq n-1.
\end{cases}
\]

The operations in $\mathfrak{g}_\bullet(\tilde{M}, \tilde{\omega})$ are defined as follows $(\mathfrak{g}_\bullet(\tilde{M}, \tilde{\omega}), \lambda_1)$ is the truncated de Rham complex

\[
0 \leftarrow \Omega^{n-1}_{\text{Ham}}(\tilde{M}, \tilde{\omega}) \leftarrow \Omega^{n-2}(\tilde{M}) \leftarrow \cdots \leftarrow \Omega^1(\tilde{M}) \leftarrow C^\infty(\tilde{M}) \leftarrow 0,
\]

and, for $\ell > 0$,

\[
\lambda_\ell(\sigma_1, \ldots, \sigma_\ell) = \begin{cases} 
-(-)^{i_{X_{\sigma_1}} \cdots i_{X_{\sigma_\ell}}} \tilde{\omega} & \text{if } \sigma_1 + \cdots + \sigma_\ell = 0 \\
0 & \text{if } \sigma_1 + \cdots + \sigma_\ell > 0,
\end{cases}
\]

where $X_\sigma$ is an Hamiltonian vector field associated to the Hamiltonian form $\sigma$.

Elements of the $L_\infty$-algebra $\mathfrak{g}_\bullet(\tilde{M}, \tilde{\omega})$ should be interpreted as observables of multisymplectic field theories defined on $(\tilde{M}, \tilde{\omega})$ (see, BAEZ et all, SHREIBER et al). I will now present a contact analogue of $\mathfrak{g}_\bullet(\tilde{M}, \tilde{\omega})$. At the same time, it should be a “higher version” of the Jacobi bundle of a standard contact manifold. In order to motivate my definition, I remark that sections of the Jacobi bundle of a contact manifold $(M, C)$ can be understood as homogeneous functions on the symplectization $(\tilde{M}, \tilde{\omega})$.\]
(see Section 2). The Jacobi bracket is then just the restriction to $C^\infty_\omega(\tilde{M}) \cong \Gamma(N)$ of the Poisson bracket on $(\tilde{M}, \tilde{\omega})$. This suggests to look for the contact analogue of $\mathfrak{g}_*(\tilde{M}, \tilde{\omega})$ on the (truncated) homogeneous de Rham complex of Section 3.

Propositions (22) and (23) show that the truncated homogeneous de Rham complex (9) provides a resolution of $\Gamma_{\text{Ham}}(N)$. In its turn, $\Gamma_{\text{Ham}}(N)$ is a Lie algebra. In [2] Barnich, Fulop, Lada, and Stasheff proved that this situation is precisely a source of $L_\infty$-algebras. Namely, whenever the underlying vector space of a Lie algebra is resolved by a chain complex, then there is an $L_\infty$-algebra structure on chains such that 1) the unary operation agrees with the differential, and 2) the binary bracket induces the Lie bracket in cohomology. It immediately follows that there is an $L_\infty$-algebra structure on the underlying graded vector space of (9) such that 1) the unary operation is the de Rham differential, and 2) the binary operation induces the Lie bracket between Hamiltonian sections in cohomology. Actually, such $L_\infty$-algebra can be described in terms of the $L_\infty$-algebra $\mathfrak{g}_*(\tilde{M}, \tilde{\omega})$, at least in the case $R \neq 0$, as shown below.

**Proposition 27.** If $R \neq 0$, the operations $\lambda_\ell$ on $\mathfrak{g}_*(\tilde{M}, \tilde{\omega})$ restrict to the homogeneous truncated de Rham complex.

**Proof.** Recall that $\tilde{\omega}$ is itself homogeneous. Thus, it is enough to prove that, whenever $\sigma \in \Omega_{\text{Ham}}^{n-1}(M, C)$, then the insertion $i_{X_\sigma}$ of an associated projectable Hamiltonian vector field $X_\sigma$ associated to $\sigma$ preserves homogeneous forms. This immediately follows from Proposition 15. Indeed, $X_\sigma$ is the locally Hamiltonian lift of a multicontact vector field on $(M, C)$ and, therefore, $i_{X_\sigma}$ has the required property (see, for instance, coordinate Formula (7)).

Let $R \neq 0$. Collecting the above results, I get the following

**Theorem 28.** There is an $L_\infty$-algebra $\mathfrak{g}_*(M, C) = \bigoplus_{i=0}^{n-1} \mathfrak{g}_i(M, C)$, concentrated in degrees $0, \ldots, n - 1$, where

\[
\mathfrak{g}_i(M, C) := \begin{cases} 
\Omega_{\text{Ham}}^{n-1}(M, C) & \text{if } i = 0 \\
\Omega^{n-i}_\omega(\tilde{M}) & \text{if } 0 < i \leq n - 1 
\end{cases}.
\]

The operations in $\mathfrak{g}_*(M, C)$ are defined as follows $(\mathfrak{g}_*(M, C), \lambda_1)$ is the truncated homogeneous de Rham complex

\[
0 \leftarrow \Omega_{\text{Ham}}^{n-1}(M, C) \leftarrow \Omega^{n-2}_\omega(\tilde{M}) \leftarrow \cdots \leftarrow \Omega^1_\omega(\tilde{M}) \leftarrow C^\infty_\omega(\tilde{M}) \leftarrow 0,
\]

and, for $\ell > 0$,$$
\lambda_\ell(\sigma_1, \ldots, \sigma_\ell) = \begin{cases} 
-(-)^i i_{X_{\sigma_1}} \cdots i_{X_{\sigma_i}} \tilde{\omega} & \text{if } \sigma_1 + \cdots + \sigma_\ell = 0 \\
0 & \text{if } \sigma_1 + \cdots + \sigma_\ell > 0
\end{cases},
\]

where $X_\sigma$ is a projectable Hamiltonian vector field associated to the homogeneous Hamiltonian form $\sigma \in \Omega_{\text{Ham}}^{n-1}(M, C)$.

Moreover, one has the

**Proposition 29.** The binary operation in $\mathfrak{g}_*(M, C)$ induces the Lie bracket on $\Gamma_{\text{Ham}}(N)$ in cohomology.

**Proof.** Let $\sigma_1, \sigma_2 \in \Omega_{\text{Ham}}^{n-1}(M, C)$. Their binary operation $\lambda_2(\sigma_1, \sigma_2)$ in $\mathfrak{g}_*(M, C)$ is a homogeneous, Hamiltonian form with a projectable Hamiltonian vector field $[X_{\sigma_1}, X_{\sigma_2}]$, where $X_{\sigma_1}, X_{\sigma_2}$ are projectable Hamiltonian vector fields associated to $\sigma_1, \sigma_2$ respectively. Indeed, $[X_{\sigma_1}, X_{\sigma_2}]$ is projectable, and, since $d i_{X_{\sigma_2}} \tilde{\omega} = \delta \omega_2 = 0$, and $L_{X_{\sigma_2}} \tilde{\omega} = 0$, one gets

\[
d\lambda_2(\sigma_1, \sigma_2) = d i_{X_{\sigma_1}} i_{X_{\sigma_2}} \tilde{\omega} = L_{X_{\sigma_1}} i_{X_{\sigma_2}} \tilde{\omega} = [L_{X_{\sigma_1}}, i_{X_{\sigma_2}}] \tilde{\omega} = i_{[X_{\sigma_1}, X_{\sigma_2}]} \tilde{\omega}.
\]
Thus, the cohomology class of \( \lambda_2(\sigma_1, \sigma_2) \) in the truncated, homogeneous de Rham complex identify with \( \theta(\pi_\ast [X_{\sigma_1}, X_{\sigma_2}]) \) (see the proof of Proposition 23), which is given by

\[
\theta(\pi_\ast [X_{\sigma_1}, X_{\sigma_2}]) = \theta([\pi_\ast X_{\sigma_1}, \pi_\ast X_{\sigma_2}]) = [\theta(\pi_\ast X_{\sigma_1}), \theta(\pi_\ast X_{\sigma_2})].
\]

\[\square\]

**Appendix A. Lie Algebroids and An Alternative Description of Homogeneous Forms**

Recall that a Lie algebroid over a manifold \( M \) is a vector bundle \( A \to M \) equipped with 1) a \( C^\infty(M) \)-linear map \( \varrho : \Gamma(A) \to \mathfrak{X}(M) \) called the anchor, and 2) a Lie bracket \([\cdot, \cdot]\) on \( \Gamma(A) \) such that

\[
[\alpha, f\beta] = \varrho(\alpha)(f)\beta + f[\alpha, \beta], \quad \alpha, \beta \in \Gamma(A), \quad f \in C^\infty(M).
\]

**Example 30.** An integrable distribution \( D \) on \( M \) (in particular \( TM \)) is a Lie algebroid with anchor \( \Gamma(D) \to \mathfrak{X}(M) \) given by the inclusion and Lie bracket given by the commutator of vector fields.

**Example 31.** Let \( E \to M \) be a vector bundle. An \( \mathbb{R} \)-linear operator \( \square : \Gamma(E) \to \Gamma(E) \) is a derivation of \( \Gamma(E) \) if there exists a (necessarily unique) vector field \( \sigma(\square) \) on \( M \), sometimes called the symbol of \( \square \), such that

\[
\square(f \xi) = \sigma(\square)(f)\xi + f\sigma(\square) \xi, \quad \xi \in \Gamma(E^\ast), \quad f \in C^\infty(M).
\]

Denote by \( \operatorname{Der} E \) the space of derivations of \( \Gamma(E) \). It is a \( C^\infty(M) \)-module with the obvious multiplication, and, a Lie algebra with Lie bracket given by the commutator. Even more, it is the module, and Lie algebra of sections of a Lie algebroid \( \operatorname{der} E \to M \), with anchor \( \operatorname{Der} E \to \mathfrak{X}(M) \), given by \( \square \mapsto \sigma(\square) \).

There is a more geometric description of derivations of \( \Gamma(E) \). Namely, denote by \( \mathfrak{X}_{\operatorname{lin}}(E^\ast) \subset \mathfrak{X}(E^\ast) \) the subspace of linear vector fields on \( E^\ast \), i.e., those vector fields on the dual bundle \( E^\ast \) preserving fiberwise linear functions on \( E^\ast \). Linear vector fields are projectable over \( M \). For \( X \in \mathfrak{X}_{\operatorname{lin}}(E^\ast) \) denote by \( \mathfrak{X}(X) \in \mathfrak{X}(M) \) its projection. It is easy to see that \( \mathfrak{X}_{\operatorname{lin}}(E^\ast) \) is a \( C^\infty(M) \)-submodule and a Lie subalgebra of \( \mathfrak{X}(E^\ast) \). Now, fiberwise linear functions on \( E^\ast \) identify with sections of \( E \), and the map \( \mathfrak{X}_{\operatorname{lin}}(E^\ast) \to \operatorname{Der} E, \ X \mapsto X|_{\Gamma(E)} \) is an isomorphism of \( C^\infty(M) \)-modules and of Lie algebras, such that \( X = \sigma(X|_{\Gamma(E)}) \).

Let \( A \to M \) be a Lie algebroid. Recall that a representation of \( A \) is a vector bundle \( V \to M \) equipped with a flat \( A \)-connection \( \nabla \), i.e., a \( C^\infty(M) \)-linear map \( \nabla : \Gamma(A) \to \operatorname{Der} V \), denoted \( \alpha \mapsto \nabla_\alpha \), such that 1) \( \sigma(\nabla_\alpha) = \varrho(\alpha) \), and 2) \( [\nabla_\alpha, \nabla_\beta] = \nabla_{[\alpha, \beta]} \), \( \alpha, \beta \in \Gamma(A) \). Let \( (V, \nabla) \) be a representation of \( A \). The graded vector space \( \operatorname{Alt}(A, V) := \operatorname{Alt}(\Gamma(A), \Gamma(V)) \) of alternating, \( C^\infty(M) \)-multilinear, \( \Gamma(V) \)-valued forms on \( \Gamma(A) \) is naturally equipped with a homological operator \( d_A \) given by the following Chevalley-Eilenberg formula:

\[
(d_A \varpi)(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_{k+1}) := \sum_i (-)^i \nabla_{\alpha_i} (\varpi(\ldots, \hat{\alpha}_i, \ldots)) + \sum_{i<j} (-)^{i+j} \varpi([\alpha_i, \alpha_j], \alpha_1, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots),
\]

where \( \varpi \in \operatorname{Alt}^k(\Gamma(A), \Gamma(V)) \) is an alternating form with \( k \)-entries, \( \alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_{k+1} \in \Gamma(\Lambda) \), and a hat \( (\hat{\cdot}) \) denotes omission.

**Example 32.** Let \( D \) be an integrable distribution on \( M \). Representations of the Lie algebroid \( D \) are equivalent to vector bundles with flat partial linear connections over \( D \). To see this, notice, preliminarily, that a derivation \( \square \) of the module \( \Gamma(E) \) of sections of a vector bundle \( E \to M \), determines a derivation \( \square^* \) of the module \( \Gamma(E^*) \) of sections of the dual bundle as follows. Identify elements in \( \Gamma(E^*) \) with \( C^\infty(M) \)-linear forms \( \Gamma(E) \to C^\infty(M) \), and put

\[
\square^*(\phi) := \sigma(\square) \circ \phi - \phi \circ \square.
\]
It is easy to see that $\Box^*(\phi) : \Gamma(E) \to C^\infty(M)$ is a well defined $C^\infty(M)$-linear map, and $\Box^*$ is a derivation. It is called the dual of $\Box$. Now, let $\text{pr} : V \to M$ be a vector bundle. A partial Ehresmann connection over $D$ in $V$ is a distribution $\hat{D}$ on $V$ such that 1) $\dim \hat{D} = \dim D$, 2) $\text{pr}_D \hat{D} = D$. Let $V$ be equipped with a partial Ehresmann connection $\hat{D}$. Then vector fields over $M$ sitting in $D$ can be lifted to vector fields over $V$ sitting in $\hat{D}$ in the obvious way. Let $\tilde{X} \in \Gamma(\hat{D})$ denote the lift of $X \in \Gamma(D)$. The partial Ehresmann connection $\hat{D}$ is a flat partial linear connection if 1) $\hat{D}$ is involutive, 2) $\tilde{X}$ is linear for all $X \in \Gamma(D)$. Suppose that $V$ is equipped with a flat partial linear connection $\hat{D}$, and let $X \in \Gamma(D)$. Restricting the linear vector field $\tilde{X}$ to fiberwise linear functions one gets a derivation $\hat{X}|_{\Gamma(V^*)}$ of $\Gamma(V^*)$. Dualizing, one gets a derivation $\hat{\Pi}|_{\Gamma(V^*)}$ of $\Gamma(V^{**}) \cong \Gamma(V)$. The map $\nabla : \Gamma(D) \to \text{Der} V$, $X \mapsto \hat{X}|_{\Gamma(V^*)}$ is a representation of $D$. Conversely, let $(V, \nabla)$ be a representation of the Lie algebroid $D$. For $X \in \Gamma(D)$, consider the derivation $\nabla_X$ of $\Gamma(V)$ and its dual $\nabla_X^*$. The latter is a derivation of $\Gamma(V^*)$ and corresponds to a linear vector field $\hat{X}$ on $V$. Vector fields on $V$ of the form $\hat{X}$ span a distribution $\hat{D}$ which is a flat partial linear connection $\hat{D}$ over $D$. Alternatively, the distribution $\hat{D}$ can be defined as follows. Let $v \in V$, $x = \text{pr}(v)$, and let $v$ be a section of $V$ such that 1) $v_x = v$, and 2) $(d_Dv)_x = 0$. Such a section can be always found. Then $\hat{D}_x := v_*(D_x)$ which is indeed independent of the choice of $v$.

**Example 33.** There is a tautological representation of $\text{der} E \to M$, given by $(E, \nabla)$, with structure flat connection $\nabla : \text{Der} E \to \text{Der} E$ being the identity, i.e., $\nabla_\Box = \Box(\epsilon)$, $\Box \in \text{Der} E$, $\epsilon \in \Gamma(E)$. The associated complex $(\text{Alt}(\text{der} E, E), d_{\text{der} E})$ is sometimes called the $(E$-valued$)$ Der-complex [13]. Let $\Delta$ be the Euler vector field on $E^*$. One can define a subcomplex $(\Omega_\Box(E^*), d)$ of the de Rham complex of $E^*$ exactly as in Section 4. Namely, a differential form $\omega$ on $E^*$ is in $\Omega_\Box(E^*)$ if $L_\Delta \omega = \omega$. In particular elements in $C^\infty_{\Box}(E^*)$ are fiberwise linear functions on $E^*$, i.e., sections of $E$. Elements of $\Omega_\Box(E^*)$ are called linear differential forms on $E^*$ and are preserved by the exterior differential. There is a canonical embedding of graded vector spaces

$$\iota : \Omega_\Box(E^*) \to \text{Alt}(\text{der} E, E), \quad (11)$$

which can be defined as follows. Denote by $\varphi : \mathfrak{X}_{\text{lin}}(E^*) \to \text{Der} E$, $X \mapsto X|_{\Gamma(E)}$ the isomorphism of Example [11]. Notice that $\Delta \in \mathfrak{X}_{\text{lin}}(E^*)$ and $\varphi(\Delta)$ is the identity of $\Gamma(E^*)$. Moreover, for $X \in \mathfrak{X}_{\text{lin}}(E^*)$, the insertion $i_X$ maps linear differential forms to linear differential forms. Thus, let $\sigma \in \Omega_\Box^k(E^*)$. Define $\iota(\sigma) \in \text{Alt}^k(\text{Der} E, \Gamma(E))$ by putting

$$i(\sigma)(\Box_1, \ldots, \Box_k) := \iota_{\varphi^{-1}(\Box_k)} \cdots \iota_{\varphi^{-1}(\Box_1)} \sigma, \quad \Box_1, \ldots, \Box_k \in \text{Der} E$$

It is easy to see that $\iota$ is injective. Moreover, it is a cochain map. Finally, dimension counting proves that $\iota$ is also surjective when $E$ is a line bundle.

The above example provides an alternative description of the homogenous de Rham complex of Section 4. Indeed, use the same notations as in Section 4. Since $\tilde{M}$ is dense in $\Lambda^n C^\perp$, then the restriction of linear forms on $\Lambda^n C^\perp$ to homogeneous forms on $\tilde{M}$ is an isomorphism. Moreover, $\Lambda^n C^\perp \cong L^*$, where $L := \Lambda^n N$ is a line bundle. Collecting previous considerations one gets

$$\Omega_\Box(\tilde{M}), d) \cong (\text{Alt}(\text{der} L, L), d_{\text{der} L}).$$

Finally, recall that $\tilde{\theta} \in \Omega_\Box^k(\tilde{M})$. It is easy to see that the corresponding element $\iota(\tilde{\theta})$ in $\text{Alt}(\text{der} L, L)$ is given by

$$i(\tilde{\theta})(\Box_1, \ldots, \Box_n) := \theta(\sigma(\Box_1)) \wedge \cdots \wedge \theta(\sigma(\Box_n)), \quad \sigma \in \Lambda^n C^\perp, \quad \Box_1, \ldots, \Box_n \in \text{der} L. \quad (12)$$

Using formula (12) one can also find $i(\tilde{\omega}) = i(d_{L^*}(\tilde{\theta}))$, and describe the higher brackets in $g(M, C)$ without reference to the multisymplectization. Details are left to the reader.
Appendix B. Jet Spaces and $L_\infty$-algebras

In this appendix, I provide explicit coordinate formulas for the $L_\infty$-algebras determined by the canonical multicontact structures on jet spaces. Let $E$ be a $(n+m)$-dimensional manifold, and let $J^k = J^k(E, m)$ be the space of $k$-jets of $m$-dimensional submanifolds of $E$, i.e., equivalence classes of tangency of $m$-dimensional submanifolds up to order $k$. There is a tower of fiber bundles

$$E = J^0 \twoheadrightarrow J^1 \twoheadrightarrow \cdots \twoheadrightarrow J^{k-1} \twoheadrightarrow J^k \twoheadrightarrow \cdots.$$ 

If $S \subset E$ is an $m$-dimensional submanifold, its $k$-jet at the point $e \in S$ is denoted by $[S]_e^k$. The $k$-jet prolongation of $S$ is the submanifold $S^{(k)} \subset J^k$ defined as

$$S^{(k)} := \{ [S]_e^k : e \in S \}.$$ 

An $m$-dimensional contact element $R$ in $J^k$ of the form $R = T_y S^{(k)}$, $y \in S^{(k)}$, is called an $R$-plane. Notice that the $R$-plane $T_y S^{(k)}$, $y = [S]_e^k$, is completely determined by $y' := [S]_e^{k+1}$. Accordingly, it is also denoted by $R_y'$. The correspondence $y' \mapsto R_y'$ is bijective. There is a canonical distribution $C$ on $J^k$, the Cartan distribution. For $y \in J^k$, the Cartan plane $C_y$ is spanned by $R$-planes at $y$. It is easy to see, for instance using local coordinates (see below), that $C$ is maximally non-integrable, i.e., the characteristic distribution is trivial, hence it is a canonical multicontact structure on $J^k$. Moreover, there is a canonical isomorphism of vector bundles $N = T J^k / C \simeq V$, where, for $y \in J^k$, and $y := J^{k-1}$, $V_y := T_y J^{k-1} / R_y$.

When $n = k = 1$, $C$ is a contact distribution. In the following, I will assume, for simplicity, $n > 1$. The case $n = 1$ is somewhat exceptional but can be treated in a very similar way. Multicontact vector fields on $J^k$ are characterized by the Lie-Bäcklund theorem. Namely, let $\phi : E \twoheadrightarrow E$ be a (local) diffeomorphism. There exists a unique (local) diffeomorphism $\phi^{(k)} : J^k \twoheadrightarrow J^k$ preserving the Cartan distribution, such that diagram

$$\begin{array}{ccc}
J^k & \phi^{(k)} & J^k \\
\downarrow & & \downarrow \\
E & \phi & E \\
\end{array}$$

commutes. Diffeomorphism $\phi^{(k)}$ is called the $k$-th jet prolongation of $\phi$ and it is defined as follows. For $y = [S]_e^k \in J^k$, $\phi(y) := [\phi(S)]_{\phi(e)}^k$. Infinitesimally, let $X$ be a vector field on $E$. There exists a unique multicontact vector field $X^{(k)}$ on $J^k$ which projects on $X$. The flow of $X^{(k)}$ is the $k$-th jet prolongation of the flow of $X$, and $X^{(k)}$ is called the $k$-th jet prolongation of $X$. Lie-Bäcklund theorem states that (when $n > 1$) every multicontact field on $J^k$ is of the kind $X^{(k)}$. Summarizing, (when $n > 1$) there are Lie algebra isomorphisms $\mathfrak{X}(E) \simeq \mathfrak{X}_C \simeq \Gamma_{\text{Ham}(N)}$, $X \mapsto X^{(k)} \mapsto \theta(X^{(k)})$.

Let $y_0 = [S_0]_{e_0}^k$ be a point in $J^k$. On $E$ choose, around $e_0$, coordinates $x^1, \ldots, x^m, u^1, \ldots, u^n$, $i = 1, \ldots, m$, adapted to $S_0$, i.e., such that $S_0$ is locally given by $S_0 : u^a = f_i^a(x^1, \ldots, x^m)$. There is a neighborhood $U$ of $y_0$ in $J^k$ such that every point $y$ of $U$ is of the form $y = [S]_e^k$ with $S$ locally given by $S : u^a = f^a(x^1, \ldots, x^m)$. One can coordinatize $U$ by jet coordinates $x^1, \ldots, x^m, u^1, \ldots, u^n$ defined as

$$x^i(y) = x^i(e), \quad u^i_\alpha(y) = \frac{\partial |I| f^\alpha}{\partial x^i}(x^1, \ldots, x^m),$$

where $I = i_1 \cdots i_k$ is a multiindex "denoting" multiple partial derivatives, i.e., $\frac{\partial |I|}{\partial x^i} := \frac{\partial^{|I|}}{\partial x^{i_1} \cdots \partial x^{i_k}}$, and $|I| := \ell \leq k$ is the length of the multiindex. The Cartan distribution is then locally spanned by vector fields

$$D_I := \frac{\partial}{\partial x^i} + \sum_{|I| < k} u^i_\alpha \frac{\partial}{\partial u^\alpha}, \quad |K| = k,$$
and its annihilator $C^\perp$ is locally spanned by Cartan forms
\[ \delta u^\alpha_j := du^\alpha_j - u^\alpha_j \partial \alpha \]
Accordingly, the projection $\theta : \mathfrak{X}(J^k) \to \Gamma(N)$ and the curvature $R : \Gamma(C) \times \Gamma(C) \to \Gamma(N)$ are locally given by
\[ \theta = \sum_{|I| < k} \delta u^\alpha_j \otimes \partial \left( \frac{\partial}{\partial u^\alpha_j} \right), \quad \text{and} \quad R = \sum_{|I|=k-1} du^\alpha_j \wedge dx^I \wedge C \otimes \partial \left( \frac{\partial}{\partial u^\alpha_j} \right). \]
If $X$ is a vector field on $M$ locally given by $X = X^i \frac{\partial}{\partial x^i} + X^\alpha \frac{\partial}{\partial u^\alpha}$, the corresponding multicontact vector field $X^{(k)}$ is given by
\[ X^{(k)} = X^i D_i + \sum_{|I| \leq k} D_I \chi_\alpha \frac{\partial}{\partial u^\alpha}, \quad \chi_\alpha := X^\alpha - u^\alpha_i X^i, \tag{13} \]
where $D_{i_1 \cdots i_\ell} := D_{i_1} \circ \cdots \circ D_{i_\ell}$. The Hamiltonian section $\theta(X^{(k)})$ is locally given by
\[ \theta(X^{(k)}) = \sum_{|J| < k} D_J \chi_\alpha \theta \left( \frac{\partial}{\partial u^\alpha_j} \right). \]
The multisymplectic structure of the multisymplectization $(\tilde{M}, \tilde{\omega})$ of $(J^k, C)$ is locally given by
\[ \tilde{\omega} = dp \wedge \Theta - \sum_{|J|=k-1} du^\alpha_j \wedge dx^i \wedge \Theta^I, \tag{14} \]
where $\Theta^I := i_{\partial/\partial u^\alpha_j} \Theta$. Local Formulas (13) and (14) allow one to find coordinate expressions for the higher brackets in $\mathfrak{g}_\ast(J^k, C)$. Namely, let $\sigma_1, \ldots, \sigma_\ell$ be Hamiltonian forms in $K_{\text{Ham}}$, and let $X_1, \ldots, X_\ell$ be associated Hamiltonian vector fields on $J^k$, with $\theta(X_i) = \sum D_J \chi_\alpha \theta(\partial/\partial u^\alpha_j)$, $i = 1, \ldots, \ell$. A straightforward computation shows that
\[ \lambda_t(\sigma_1, \ldots, \sigma_\ell) \]
\[ = \sum_{|I_1|, \ldots, |I_\ell| < k} D_{I_1} \chi_\alpha \cdots D_{I_\ell} \chi_\alpha \left( \left( \sum_{|J|=k-1} \lambda_{J_{i_1 \cdots i_\ell}} \right) dp \wedge \chi_{\alpha_{i_1 \cdots i_\ell}} \right) \]
\[ + \sum_{|I| = \ell} \left( \left( \sum_{|J|=k-1} \lambda_{J_{i_1 \cdots i_\ell}} \right) \chi_{\alpha_{i_1 \cdots i_\ell}} \right) \]
\[ + \sum_{|J|=k-1} \left( X^{i_1} \delta u^\alpha_{j_1} - D_{J_{i_1 \cdots i_\ell}} \chi_\alpha dx^i \right) \wedge \Theta^{I_{i_1 \cdots i_\ell}} \chi_{\alpha_{i_1 \cdots i_\ell}} \]
\[ \wedge \Theta^{I_{i_1 \cdots i_\ell}} \chi_{\alpha_{i_1 \cdots i_\ell}} \]
where $\Theta^{I_{i_1 \cdots i_\ell}} := i_{\partial/\partial u^\alpha_{i_1}} \cdots i_{\partial/\partial u^\alpha_{i_\ell}} \Theta$.

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