Abstract

If $X$ is a topological space then there is a natural homomorphism $\pi_1(X) \to K_1(X)$ from a fundamental group to a $K_1$-homology group. Covering projections depend on fundamental group. So $K_1$-homology groups are interrelated with covering projections. This article is concerned with a noncommutative analogue of this interrelationship.

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1 Introduction

It is known that $K_1(S^1) \approx \mathbb{Z}$. If $x$ is a generator of $K(S^1)$ then there is a natural homomorphism $\varphi_K : \pi_1(X) \to K_1(X)$ given by

$$[f] \mapsto K_1(f)(x)$$

where $f$ is a representative of $[f] \in \pi_1(X)$. This homomorphism does not depend on a basepoint because $K_1(X)$ is an abelian group. So the basepoint is omitted. Let $K_{11}(X) \subset K_1(X)$ be the image of $\varphi_K$. Then $K_{11}(X)$ is a homotopical invariant.

Example 1.1. We have a natural isomorphism $\varphi_K : \pi_1(S^1) \to K_1(S^1)$. From $\pi_1(S^1) = \mathbb{Z}$ it follows that there is a $n$-listed covering projection $f_n : S^1 \to S^1$ for any $n \in \mathbb{N}$.

Example 1.2. Let $f : S^1 \to S^1$ be an $n$ listed covering projection, $C_f$ is the mapping cone of $f$. Then $\pi_1(C_f) \approx K_1(C_f) \approx \mathbb{Z}_n$ and there is a natural isomorphism $\varphi_K : \pi_1(C_f) \to K_1(C_f)$. There is $n$ - listed universal covering projection $f_n : \tilde{C}_f \to C_f$.

Finitely listed covering projections depend of fundamental group. Any epimorphism $\pi_1(X) \to \mathbb{Z}$ (resp. $\pi_1(X) \to \mathbb{Z}_n$) corresponds to the infinite sequence of finitely listed covering projections (resp. an $n$ - listed covering projection). If $\varphi : \pi_1(X) \to G$ is an epimorphism ($G \approx \mathbb{Z}$ or $G \approx \mathbb{Z}_n$) such that $\ker \varphi_K \subset \ker \varphi$ then there is an algebraic construction of these covering projections which is described in this article. A noncommutative analogue of $K_{11}(X)$ is discussed.

This article assumes elementary knowledge of following subjects

1. Algebraic topology [12].

2. $C^*$– algebras and K-theory [1], [4], [9], [10].

Following notation is used.
| Symbol | Meaning |
|--------|---------|
| \(A^+\) | Unitization of \(C^*\)-algebra \(A\) |
| \(A_+\) | A positive cone of \(C^*\)-algebra \(A\) |
| \(A^G\) | Algebra of \(G\) invariants, i.e. \(A^G = \{ a \in A \mid ga = a, \forall g \in G \}\) |
| \(\hat{A}\) | Spectrum of \(C^*\)-algebra \(A\) with the hull-kernel topology (or Jacobson topology) |
| \(\text{Aut}(A)\) | Group \(^*\)-automorphisms of \(C^*\)-algebra \(A\) |
| \(B(H)\) | Algebra of bounded operators on Hilbert space \(H\) |
| \(B_0 = B_0(\{z \in \mathbb{C} \mid |z| = 1\})\) | Algebra of Borel measured functions on the \(\{z \in \mathbb{C} \mid |z| = 1\}\) set. |
| \(\mathbb{C}\) (resp. \(\mathbb{R}\)) | Field of complex (resp. real) numbers |
| \(\mathbb{C}^*\) | \(\{z \in \mathbb{C} \mid |z| = 1\}\) |
| \(\mathbb{C}(X)\) | \(C^*\)-algebra of continuous complex valued functions on topological space \(X\) |
| \(\mathbb{C}^b(X)\) | \(C^*\)-algebra of bounded continuous complex valued functions on topological space \(X\) |
| \(H\) | Hilbert space |
| \(I = [0,1] \subset \mathbb{R}\) | Closed unit interval |
| \(G_{\text{tors}} \subset G\) | The torsion subgroup of an abelian group |
| \(K(H)\) or \(K\) | Algebra of compact operators on Hilbert space \(H\) |
| \(\mathbb{M}_n(A)\) | The \(n \times n\) matrix algebra over \(C^*\)-algebra \(A\) |
| \(\text{Map}(X,Y)\) | The set of maps from \(X\) to \(Y\) |
| \(\mathbb{M}(A)\) | A multiplier algebra of \(C^*\)-algebra \(A\) |
| \(\mathbb{M}^s(A) = \mathbb{M}(A \otimes K)\) | Stable multiplier algebra of \(C^*\)-algebra \(A\) |
| \(\mathbb{M}(N)\) | Monoid of natural numbers |
| \(\mathbb{Q}(A) = \mathbb{M}(A) / A\) | Outer multiplier algebra of \(C^*\)-algebra \(A\) |
| \(\mathbb{Q}(A) = (\mathbb{M}(A \otimes K)) / (A \otimes K)\) | Stable outer multiplier algebra of \(C^*\)-algebra \(A\) |
| \(\mathbb{Q}\) | Field of rational numbers |
| \(\text{sp}(a)\) | Spectrum of element of \(C^*\)-algebra \(a \in A\) |
| \(U(H) \subset B(H)\) | Group of unitary operators on Hilbert space \(H\) |
| \(U(A) \subset A\) | Group of unitary operators of algebra \(A\) |
| \(\mathbb{Z}\) | Ring of integers |
| \(\mathbb{Z}_m\) | Ring of integers modulo \(m\) |
| \(\Omega\) | Natural contravariant functor from category of commutative \(C^*\)-algebras to category of Hausdorff spaces |

### 2 Galois extensions of \(C^*\)-algebras and noncommutative covering projections

#### 2.1 General theory

2.1. **Galois extensions.** Let \(G\) be a finite group, a \(G\)-Galois extensions can be regarded as particular case of Hopf-Galois extensions [3], where Hopf algebra is a commutative alge-
bra \mathcal{C}(G). Let \( A \) be a \( C^* \)-algebra, let \( G \subset \text{Aut}(A) \) be a finite group of \( * \)-automorphisms. Let \( \mathcal{A}M^G \) be a category of \( G \)-equivariant modules. There is a pair of adjoint functors \((F, U)\) given by

\[
F = A \otimes_{A_C} M \rightarrow A^G; \tag{2}
\]
\[
U = (-)^G : A^G \rightarrow A_C^G. \tag{3}
\]

The unit and counit of the adjunction \((F, U)\) are given by the formulas

\[
\eta : N \rightarrow (A \otimes_{A_C} N)^G, \quad \eta(n) = 1 \otimes n; \]
\[
\varepsilon : A \otimes_{A_C} \mathcal{M}_{A^G} \rightarrow \mathcal{M}, \quad \varepsilon(a \otimes m) = am.
\]

Consider a following map

\[
\text{can} : A \otimes_{A_C} A \rightarrow \text{Map}(G, A) \tag{4}
\]

given by

\[
a_1 \otimes a_2 \mapsto (g \mapsto a_1(ga_2)), \quad (a_1, a_2 \in A, \ g \in G).
\]

The can is a \( A_{A_C^G} \) morphism.

**Theorem 2.2.** [3] Let \( A \) be an algebra, let \( G \) be a finite group which acts on \( A \), \((F, U)\) functors given by (2), (3). Consider the following statements:

1. \((F, U)\) is a pair of inverse equivalences;
2. \((F, U)\) is a pair of inverse equivalences and \( A \in A^G_{A_C^G} \) is flat;
3. The can is an isomorphism and \( A \in A^G_{A_C^G} \) is faithfully flat.

These three conditions are equivalent.

**Definition 2.3.** If conditions of theorem 2.2 are hold, then \( A \) is said to be left faithfully flat \( G \)-Galois extension.

**Remark 2.4.** Theorem 2.2 is an adapted to finite groups version of theorem from [3].

In case of commutative \( C^* \)-algebras definition 2.3 supplies finitely listed covering projections of topological spaces. However I think that above definition is not quite good analogue of noncommutative covering projections. Noncommutative algebras contains inner automorphisms. Inner automorphisms are rather gauge transformations [6] than geometrical ones. So I think that inner automorphisms should be excluded. Importance of outer automorphisms was noted by Miyashita [7]. It is reasonably take to account outer automorphisms only. I have set more strong condition.

**Definition 2.5.** [11] Let \( A \) be \( C^* \)-algebra. A \( * \)-automorphism \( \alpha \) is said to be generalized inner if is obtained by conjugating with unitaries from multiplier algebra \( M(A) \).

**Definition 2.6.** [11] Let \( A \) be \( C^* \)-algebra. A \( * \)-automorphism \( \alpha \) is said to be partly inner if its restriction to some non-zero \( \alpha \)-invariant two-sided ideal is generalized inner. We call automorphism purely outer if it is not partly inner.
Instead definitions 2.5, 2.6 following definitions are being used.

**Definition 2.7.** Let $\alpha \in \text{Aut}(A)$ be an automorphism. A representation $\rho : A \to B(H)$ is said to be $\alpha$-invariant if a representation $\rho_\alpha$ given by 
\[
\rho_\alpha(a) = \rho(\alpha(a)) \tag{5}
\]
is unitary equivalent to $\rho$.

**Definition 2.8.** Automorphism $\alpha \in \text{Aut}(A)$ is said to be strictly outer if for any $\alpha$-invariant representation $\rho : A \to B(H)$, automorphism $\rho_\alpha$ is not a generalized inner automorphism.

**Definition 2.9.** Let $A$ be a $C^*$-algebra and $G \subset \text{Aut}(A)$ be a finite subgroup of $*$-automorphisms. An injective $*$-homomorphism $f : A \to B \otimes \mathbb{C}^n$ is said to be a noncommutative finite covering projection (or noncommutative $G$-covering projection) if $f$ satisfies following conditions:

1. $A$ is a finitely generated equivariant projective left and right $A^G \otimes \mathbb{C}^n$-module.
2. If $\alpha \in G$ then $\alpha$ is strictly outer.
3. $f$ is a left faithfully flat $G$-Galois extension.

The $G$ is said to be covering transformation group of $f$. Denote by $G(B|A)$ covering transformation group of covering projection $A \to B$.

**2.10. Irreducible representations of noncommutative covering projections.** Let $f : A^G \to A$ be a noncommutative $G$-covering projection. Let $\rho : A \to B(H)$ be an irreducible representation. Let $g \in G$ and $\rho_g : A \to B(H)$ be such that 
\[
\rho_g(a) = \rho(ga). \tag{6}
\]
So it is an action of $G$ on $\hat{A}$ such that 
\[
g : (\rho \mapsto \rho_g); \forall g \in G, \forall \rho \in \hat{A}. \tag{6}
\]
Let us enumerate elements of $G$ by integers, i.e. $g_1, ..., g_n \in G, n = |G|$ and define action of $\sigma : G \times \{i, ..., n\} \to \{i, ..., n\}$ such that $\sigma(g,i) = j \Leftrightarrow g = g_j$. Let $\rho_\oplus = \oplus_{g \in \text{G}} \rho_g : A \to B(H^n)$ be such that 
\[
\rho_\oplus(a)(h_1, ..., h_n) = (\rho(g_1a)h_1, ..., (\rho(g_n a)h_n). \tag{7}
\]
Let us define such linear action of $G$ on $H^n$ that 
\[
g(h_1, ..., h_n) = (h_{\sigma^{-1}(1)} \cdot \rho_\oplus(g_1), ..., h_{\sigma^{-1}(n)} \cdot \rho_\oplus(g_n)). \tag{8}
\]
From (7), (8) it follows that 
\[
g(ah) = (ga)(gh); \forall a \in A, \forall g \in G, \forall h \in H^n.
\]
i.e. $H^n \in _A \mathcal{M}^G$. Equivariant representation $\rho \circ \otimes \mathbb{C}$ defines representation $\eta : A^G \rightarrow B(K)$. $K = (H^n)^G$. If $\eta$ is not an irreducible then there is a nontrivial $A^G$ - submodule $N \varsubsetneq K$. From $A \mathcal{M}^G \approx _A \mathcal{M}$ it follows that $A \otimes _A N \varsubsetneq H^n$ is a nontrivial $A$ - submodule. If we identify $H$ with first summand of $H^n$ then $(A \otimes _A K) \cap H \varsubsetneq H$ is a nontrivial $A$ - submodule. This fact contradicts with that $\rho$ is irreducible. So $\eta$ is an irreducible representation. In result we have a natural map

$$\hat{f} : \hat{A} \rightarrow \hat{A}^G, \ (\rho \mapsto \eta)$$

and

$$\hat{A}^G \approx \hat{A}/G.$$ (10)

### 2.2 Covering projection of $C^*$-algebras with continuous trace

**Definition 2.11.** [10] A positive element in $C^*$ - algebra $A$ is **abelian** if subalgebra $x Ax \subset A$ is commutative.

**Proposition 2.12.** [10] A positive element $x$ in $C^*$ - algebra $A$ is abelian if $\dim(\pi(x)) \leq 1$ for every irreducible representation $\pi : A \rightarrow B(H)$ of $A$.

2.13. Let $A$ be a $C^*$ - algebra. For each $x \in A_+$ the (canonical) trace $\text{Tr}(\pi(x))$ of $\pi(x)$ depends only on the equivalence class of an irreducible representation $\pi : A \rightarrow B(H)$, so that we may define a function $\hat{x} : \hat{A} \rightarrow [0, \infty]$ by $\hat{x}(t) = \text{Tr}(\pi(x))$ whenever $\pi \in t$. From Proposition 4.4.9 [10] it follows that $\hat{x}$ is lower semicontinuous function on $a$ in Jacobson topology.

**Definition 2.14.** [10] We say that element $x \in A_+$ has **continuous trace** if $\hat{x} \in C^b(\hat{A})$. We say that $A$ is a $C^*$ - algebra with continuous trace if set of elements with continuous trace is dense in $A_+$. We say that a $C^*$ - algebra $A$ is of type $I$ if each non-zero quotient of $A$ contains non-zero abelian element. If $A$ is even generated (as $C^*$ - algebra) by its abelian elements we say that it is of type $I_0$.

**Theorem 2.15.** (Theorem 5.6 [10]) For each $C^*$ - algebra $A$ there is a dense hereditary ideal $K(A)$, which is minimal among dense ideals.

**Proposition 2.16.** [10] Let $A$ be a $C^*$ - algebra with continuous trace Then

1. $A$ is of type $I_0$;
2. $\hat{A}$ is a locally compact Hausdorff space;
3. For each $t \in \hat{A}$ there is an abelian element $x \in A$ such that $\hat{x} \in K(\hat{A})$ and $\hat{x}(t) = 1$.

The last condition is sufficient for $A$ to have continuous trace.

**Remark 2.17.** From [5], Proposition 10, II.9 it follows that a continuous trace $C^*$-algebra is always a CCR-algebra, a $C^*$-algebra where for every irreducible representation $\pi : A \rightarrow B(H)$ and for every element $x \in A$, $\pi(x)$ is a compact operator, i.e. $\pi(A) = \mathcal{K}(H)$.
Lemma 2.18. Let \( A^G \to A \) be a noncommutative covering projection such that \( A \) is a CCR-algebra. Then \( G \) acts freely on \( \hat{A} \).

Proof. Suppose that \( G \) does not act freely on \( \hat{A} \). Then there are \( x \in \hat{A} \) and \( g \in G \) such that \( gt = t \) (\( t \in \hat{A} \)). By definition \( G \) should be strictly outer. Let \( \rho : A \to B(H) \) be representative of \( x \). Then \( \rho_g \) is also representative of \( x \). So \( \rho \) is unitary equivalent to \( \rho_g \), i.e. there is unitary \( U \in U(H) \) such that \( \rho_g(a) = U\rho(a)U^* \) (\( \forall a \in A \)). According to 2.17 \( \rho(A) = K(H), \rho(M(A)) = B(H), \rho(U(M(A))) = U(H) \). So it is \( u \in M(A) \) such that \( \rho(u) = U \) and we have \( \rho_g(a) = \rho(u)\rho(a)\rho(u^*) \). It means that \( g \) is inner with respect to \( \rho \), so action of \( g \) is not strictly outer. This contradiction proves the lemma.

Lemma 2.19. \([10]\) Let \( G \) be a finite group and \( f : A^G \to A \) is a \( G \)-covering projection. If \( A^G \) is a continuous trace \( C^* \)-algebra then \( A \) is also a continuous trace \( C^* \)-algebra.

Proof. From 2.10 it follows that for any irreducible representation \( \rho : A \to B(H) \) there is an irreducible representation \( \eta : A^G \to B(H) \) such that

\[ \rho|_{AC} = \eta \]

(11)

Let \( x \in A^G \) be an abelian element of \( A^G \). From 2.12 it follows that \( \dim \eta(x) \leq 1 \) for any irreducible representation \( \eta : A^G \to B(H) \). From (11) it follows that \( \dim \rho(x) \leq 1 \) for any irreducible representation \( \rho : A \to B(H) \). So any abelian element of \( A^G \) is also an abelian element of \( A \). Let \( t \in \hat{A} \) and \( s = \hat{f}(t) \in \hat{A}^G \) where \( \hat{f} \) is defined by (9). From 2.12 it follows that there is an abelian element \( x \in A^G \) such that \( \tilde{x} \in K(\hat{A}^G) \) and \( \tilde{x}(s) = 1 \). However \( x \) is an abelian element of \( A \), \( \tilde{x} \in K(A) \) and \( \tilde{x}(t) = \tilde{x}(s) = 1 \). From 2.16 it follows that \( A \) is a continuous trace \( C^* \)-algebra.

Proposition 2.20. \([2]\) If a topological group \( G \) acts properly on a topological space then orbit space \( X/G \) is Hausdorff. If \( G \) is Hausdorff, then \( X \) is Hausdorff.

Theorem 2.21. Let \( f : A^G \to A \) be a noncommutative finite covering projection and \( A^G \) is a continuous trace algebra. Then \( \hat{A} \to \hat{A}/G \) is a (topological) covering projection.

Proof. From lemma 2.19 it follows that \( A \) is a continuous trace algebra. From 2.16 it follows that \( \hat{A} \) is Hausdorff. From 2.18 it follows that \( G \) acts freely on \( \hat{A} \). From (10) it follows that \( A^G \approx \hat{A}/G \). It is known \([12]\) that if a finite group \( G \) acts freely on Hausdorff space \( X \) then \( X \to X/G \) is a covering projection.

Remark 2.22. From theorem 2.21 it follows that finite covering projections of commutative algebras are just covering projections of their character spaces. If \( A^G \) is a commutative \( C^* \)-algebra then \( \dim \pi(A^G) = 1 \) for all irreducible \( \pi : A \to B(H) \). If \( f : A^G \to A \) is a noncommutative \( G \) covering projection and \( A^G \) is commutative then \( A^G \) is continuous trace algebra \( \Omega(A^G) \approx \hat{A}^G \). From 2.19 it follows that \( A \) is also a continuous trace \( C^* \)-algebra. If \( \rho : A \to B(H) \) then \( \rho(A) = K(H) \). Let us recall construction from 2.10. Let us enumerate elements of \( G \) by integers, i.e. \( g_1, \ldots, g_n \in G, n = |G| \) and define action of \( \sigma : \)
where \( \theta \) commutative torus \( C \), unitary elements \( u \), homomorphism \( f \) is defined by following way:

\[
G \times \{i, ..., n\} \to \{i, ..., n\} \text{ such that } \sigma(g, i) = j \Leftrightarrow g_j = gg_i \text{ Let } \rho_\oplus = \oplus_{g \in G} \rho_g : A \to B(H^n) \text{ be such that }
\]

\[
\rho_\oplus(a)(h_1, ..., h_n) = (\rho(g_1a)h_1, ..., (\rho(g_na)h_n). \quad (12)
\]

Let us define such linear action of \( G \) on \( H^n \) that

\[
g(h_1, ..., h_n) = (h_{\tau(g^{-1})1}, ..., h_{\tau(g^{-1},n)}). \quad (13)
\]

From (7), (13) it follows that \( G \) is a noncommutative covering projection.

2.3 Covering projections of noncommutative torus

2.23. A noncommutative torus \([13] A_\theta \) is \( C^* \)-norm completion of algebra generated by two unitary elements \( u, v \) which satisfy following conditions

\[
 uu^* = u^*u = vv^* = v^*v = 1;
\]

\[
 uv = e^{2\pi i \theta} vu,
\]

where \( \theta \in \mathbb{R} \). If \( \theta = 0 \) then \( A_\theta = A_0 \) is commutative algebra of continuous functions on commutative torus \( C(S^1 \times S^1) \). There is a trace \( \tau_0 \) on \( A_\theta \) such that \( \tau_0(\sum_{-\infty < i, -\infty < j < \infty} a_{ij}u^i v^j) = a_{00} \). \( C^* \)-norm of \( A_\theta \) is defined by following way \( \|a\| = \sqrt{\tau_0(a^*a)} \). Let us consider *-homomorphism \( f : A_\theta \to A_{\theta'} \), where \( A_{\theta'} \) is generated by unitary elements \( u' \) and \( v' \). Homomorphism \( f \) is defined by following way:

\[
u \mapsto v'^n; \quad u \mapsto u'^m,
\]

It is clear that

\[
\theta' = \frac{\theta + k}{mn}; \quad (k = 0, ..., mn - 1). \quad (14)
\]

Lemma 2.24. Above *-homomorphism \( A_\theta \to A_{\theta'} \) is a noncommutative covering projection.

Proof. We need check conditions of definition 2.29 \( A_{\theta'} \) is a free \( A_\theta \) module generated by monomials \( u^i v^j \) \((i = 0, ..., m - 1; j = 0, ..., n - 1)\), so it is projective finitely generated \( A_\theta \)-module. Commutative \( C^* \)- subalgebras \( C(u') \subset A_\theta \) and \( C(v') \subset A_{\theta'} \) generated by \( u' \) and \( v' \) respectively are isomorphic to algebra \( C(S^1) \), where \( S^1 \) is one dimensional circle. There
are induced by $f$ *-homomorphisms $C(S^1) = C(u) \to C(u') = C(S^1)$, $C(S^1) = C(v) \to C(v') = C(S^1)$. These *-homomorphisms induces $m$ and $n$ listed covering projections respectively. Covering groups of these covering projections are $G_1 \approx \mathbb{Z}_m$ and $G_2 \approx \mathbb{Z}_n$ respectively. Generators of these groups are presented below:

\[ u' \mapsto e^{\frac{2\pi i}{m} u'}; \quad (15) \]
\[ v' \mapsto e^{\frac{2\pi i}{n} v'}; \quad (16) \]

Equations (15), (16) define action of $G = \mathbb{Z}_m \times \mathbb{Z}_n$ on $A_{\theta'}$ and $A_{\theta} = A_{\theta'}^G$. Inner automorphisms of $A_{\theta'}$ are given by

\[ v' \mapsto u' p v' u'^* p = e^{\frac{2\pi i p \theta}{mn}} v'; \]
\[ u' \mapsto v' q u' v'^* q = e^{\frac{2\pi i q \theta}{mn}} u'. \]

These inner automorphisms do not coincide with automorphisms given by (15), (16). Let us show that can : $A_{\theta'} \otimes A_{\theta} \to \text{Map}(G, A_{\theta'})$ is an isomorphism in $A_{\theta}M^G_{\theta}$ category. This fact follows from the set theoretic bijectivity of the can. Homomorphisms of commutative algebras $C(u) \to C(u')$, $C(v) \to C(v')$ correspond to covering projection, it follows that there are elements $x_i \in C(u')$ ($i = 1, ..., r$), $y_j \in C(v')$ ($j = 1, ..., s$) such that

\[ \sum_{1 \leq i \leq r} x_i^2 = 1_{C(u')}; \quad (17) \]
\[ \sum_{1 \leq i \leq r} x_i (g_1 x_i) = 0; \quad g_1 \in G_1; \quad (18) \]
\[ \sum_{1 \leq j \leq s} y_j^2 = 1_{C(v')}; \quad (19) \]
\[ \sum_{1 \leq j \leq s} y_j (g_2 y_j) = 0; \quad g_2 \in G_2, \quad (20) \]

where $g_1$ and $g_2$ are nontrivial elements of $\mathbb{Z}_m$ and $\mathbb{Z}_n$.

Let $a_k, b_k \in A_{\theta}$ be such that

\[ a_k = y_j x_i, \]
\[ b_k = x_i y_j, \]

where $k = 1, ..., rs$.

From (17)- (20) it follows that

\[ \sum_{1 \leq k \leq rs} a_k b_k = 1_{A_{\theta'}}; \]
\[ \sum_{1 \leq k \leq rs} a_k (g b_k) = 0, \]

where $g \in G = \mathbb{Z}_m \times \mathbb{Z}_n$ is a nontrivial element. If $\varphi \in \text{Map}(G, A_{\theta})$ is such that $g_i \mapsto c_i$ ($i = 1, ..., mn$) then

\[ \varphi = \text{can} \left( \sum_{i=1}^{mn} \sum_{k=1}^{rs} a_k \otimes g_i^{-1} b_k c_i \right). \quad (21) \]
So can is a surjective map. Let us show that can is injective. $A_{\theta'}$ is a free left $A_{\theta}$ module, because any element $a \in A_{\theta'}$ has following unique representation
\[
a = \sum_{r=0,s=0}^{m-1,n-1} a_{rs} u^r v^s (a_{rs} \in A_{\theta}). \quad (22)
\]
From (22) it follows that any element $x \in A_{\theta'} \otimes_{A_{\theta}} A_{\theta'}$ has following unique representation
\[
x = \sum_{r=0,s=0}^{m-1,n-1} a_{rs} \otimes u^r v^s (a_{rs} \in A_{\theta'}). \quad (23)
\]
Let us prove that can maps above sum of linearly independent elements of $A_{\theta'} \otimes_{A_{\theta}} A_{\theta'}$ to sum of linearly independent elements of $\text{Map}(\mathbb{Z}_m \times \mathbb{Z}_n, A_{\theta'})$. Really if
\[
\phi = \text{can}(a \otimes u^r v^s) \quad (24)
\]
and $(p, q) \in \mathbb{Z}_m \times \mathbb{Z}_n$ then
\[
\phi((p, q)) = \phi((0, 0)) e^{\frac{2\pi i pr}{m}} e^{\frac{2\pi i qs}{n}}. \quad (25)
\]
i.e. linearly independent elements of (23) correspond to different representations of $G = \mathbb{Z}_m \times \mathbb{Z}_n$, but different representations are linearly independent. So can is injective.

\[\square\]

**Remark 2.25.** Let $\theta \in \mathbb{R}$ be irrational number, $m, n \in \mathbb{N}$, $mn > 1$, $\theta' = \theta/mn$, $\theta'' = (\theta + k)/mn (k \neq 0 \text{ mod } mn)$. Let $u, v \in A_\theta, u', v' \in A_{\theta'}, u'', v'' \in A_{\theta''}$ be unitary generators, $f' : A_\theta \to A_{\theta'}$ (resp. $f'' : A_\theta \to A_{\theta''}$) be $*$ - homomorphism $u \mapsto u'^m$, $v \mapsto v'^n$ (resp. $u \mapsto u''^m$, $v \mapsto v''^n$). We have $A_{\theta'} \not\approx A_{\theta''}$. So this noncommutative covering projections are not isomorphic. However these covering projections can be regarded as equivalent because they are Motita equivalent. Let $U, V \in \mathbb{M}_{m=mn}(\mathbb{C})$ be unitary matrices such that
\[
UV = e^{\frac{2\pi i k}{m^2}} VU.
\]
There is following $G$ equivariant isomorphism $A_{\theta'} \otimes \mathbb{M}_N(\mathbb{C}) \approx A_{\theta''} \otimes \mathbb{M}_N(\mathbb{C})$
\[
u' \otimes 1 \to u'' \otimes U; \; v' \otimes 1 \to v'' \otimes V.
\]
This isomorphism is also $A_{\theta} - A_{\theta}$ bimodule isomorphism. From $K \otimes \mathbb{M}_N(\mathbb{C}) \approx K$ it follows that there exist isomorphism $A_{\theta'} \otimes K \approx A_{\theta''} \otimes K$ and there is following commutative diagram
\[
\begin{array}{ccc}
A_{\theta'} \otimes K & \approx & A_{\theta''} \otimes K \\
\downarrow & & \downarrow \\
A_\theta \otimes K
\end{array}
\]
I find that good theory of noncommutative covering projections should be invariant with respect to Morita equivalence. This theory can replace $C^*$-algebras with their stabilizations (recall that the stabilization of a $C^*$ algebra $A$ is a $C^*$-algebra $A \otimes K$).
3 Covering projections and $K$-homology

3.1 Extensions of $C^*$-algebras generated by unitary elements

Definition 3.1. Let $A$ be a $C^*$-algebra, $A \to B(H)$ is a faithful representation, $u \in U(A^+)$, $v \in U(B(H))$, is such that $v^n = u$ and $v^i \notin U(A^+)$, $(i = 1, \ldots, n - 1)$. A generated by $v$ extension is a minimal subalgebra of $B(H)$ which contains following operators:

1. $v^ia; \ (a \in A, \ i = 0, \ldots, n - 1)$
2. $av^i$.

Denote by $A\{v\}$ a generated by $v$ extension.

Remark 3.2. Sometimes a $*$-homomorphism $A \to A\{v\}$ is a noncommutative covering projection but it is not always true. If the homomorphism is a covering projection then there is a relationship between the covering projection and $K$-homology.

Lemma 3.3. Let $A$ be a $C^*$-algebra, $A \to B(H)$ is a faithful representation, $u \in U(A^+)$ is an unitary element such that $\text{sp}(u) = C^* = \{z \in \mathbb{C} \mid |z| = 1\}$, $\xi, \eta \in B_\infty(\text{sp}(u))$ are Borel measured functions such that $\xi(z)^n = \eta(z)^n = z \ (\forall z \in \text{sp}(u))$. Then there is an isomorphism

$$A\{\xi(u)\} \otimes K \to A\{\eta(u)\} \otimes K$$

which is a left $A$-module isomorphism. The isomorphism is given by

$$\xi(u) \otimes x \mapsto \eta(u) \otimes \xi^{-1}(z)u; \ (x \in K).$$

Proof. Follows from the equality $\xi(u) = \xi\eta^{-1}(\eta(u))$.

Remark 3.4. See remark [2,25]

Definition 3.5. A $n^{\text{th}}$ root of identity map is a Borel-measurable function $\phi \in B_\infty(C^*)$ such that

$$(\phi(z))^n = z \ (\forall z \in U(C(X))).$$

Lemma 3.6. Let $A$ be a $C^*$-algebra, $u \in U((A \otimes K)^+)$ is such that $[u] \neq 0 \in K_1(A)$ then $\text{sp}(u) = C^* = \{z \in \mathbb{C} \mid |z| = 1\}$.

Proof. $\text{sp}(u) \subset C^*$ since $u$ is an unitary. Suppose $z_0 \in \mathbb{C}$ be such that $z_0 \notin \text{sp}(u)$ and $z_1 = -z_0$. Let $\varphi : \text{sp}(u) \times [0, 1] \to C^*$ be such that

$$\varphi(z_1e^{i\phi}, t) = z_1e^{i(1-t)\phi}; \ \phi \in (-\pi, \pi), \ t \in [0, 1].$$

There is a homotopy $u_t = \varphi(u, t) \in U((A \otimes K)^+)$ such that $u_0 = u$, $u_1 = z_1$. From $[z_1] = 0 \in K_1(A)$ it follows that $[u] = 0 \in K_1(A)$. So there is a contradiction which proves this lemma.

\section*{References}
3.2 Universal coefficient theorem

Universal coefficient theorem \cite{1} establishes (in particular) a relationship between $K$-theory and $K$-homology. For any $C^*$-algebra $A$ there is a natural homomorphism

$$\gamma : KK_1(A, C) \to \text{Hom}(K_1(A), K_0(C)) \approx \text{Hom}(K_1(A), \mathbb{Z})$$

which is the adjoint of following pairing

$$KK(C, A) \otimes KK(A, C) \to KK(C, C).$$

If $\tau \in KK_1(A, C)$ is represented by extension

$$0 \to C \to D \to A \to 0$$

then $\gamma$ is given as connecting maps $\partial$ in the associated six-term exact sequence of $K$ theory

$$\begin{array}{ccc}
K_0(C) & \to & K_0(D) \to K_0(A) \\
\downarrow \partial & & \downarrow \partial \\
K_1(A) & \to & K_1(D) \to K_1(C)
\end{array}$$

If $\gamma(\tau) = 0$ for an extension $\tau$ then the six-term $K$-theory exact sequence degenerates into two short exact sequences

$$0 \to K_i(A) \to K_i(D) \to K_i(C) \to 0 \quad (i = 0, 1)$$

and thus determines an element $\kappa(\tau) \in \text{Ext}^1(K_i(A), K_i(C))$. In result we have a sequence of abelian group homomorphisms

$$\text{Ext}^1(K_0(A), K_0(C)) \to KK_1(A, C) \to \text{Hom}(K_1(A), K_0(C))$$

such that composition of the homomorphisms is trivial. Above sequence can be rewritten by following way

$$\text{Ext}^1(K_0(A), \mathbb{Z}) \to K^1(A) \to \text{Hom}(K_1(A), \mathbb{Z}).$$

If $G$ is an abelian group that

$$\text{Ext}^1(G, \mathbb{Z}) = \text{Ext}^1(G_{\text{tors}}, \mathbb{Z}),$$

$$\text{Hom}(G, \mathbb{Z}) = \text{Hom}(G/G_{\text{tors}}, \mathbb{Z}).$$

From \cite{20} it follows that $K^1(A)$ depends on $K_0(A)_{\text{tors}}$ and $K_1(A)/K_1(A)_{\text{tors}}$. We say that dependence \cite{20} on $K_0(A)_{\text{tors}}$ is a torsion special case and dependence \cite{29} of $K^1(A)$ on $K_1(A)/K_1(A)_{\text{tors}}$ is a free special case.
3.3 Free special case

Example 3.7. The $n$-listed coverings of example 3.1 can be constructed algebraically. From (30) it follows that $K_1(C(S^1)) \approx \mathbb{Z}$. Let $u \in U(C(S^1))$ is such that $[u] \in K_1(S^1)$ is a generator of $K_1(S^1)$. Let $C(S^1) \rightarrow B(H)$ be a faithful representation and $\phi$ is an $n^{th}$ root of identity map. If $v = \phi(u) \in B(H)$ then $v^n = u$ and $v \notin C(S^1)$. According to definition 3.1 we have a $\ast$-homomorphism $C(S^1) \rightarrow C(S^1)\{v\}$ which corresponds to $n$ listed covering projection of the $S^1$.

3.8. General construction. Construction of example 3.7 can be generalized. Let $A$ be a C*-algebra such that $K^1(A) \approx G \oplus \mathbb{Z}$. From (30) it follows that

$$K_1(A) = G' \oplus \mathbb{Z}[u]$$

(31)

where $u \in U((A \otimes K)^{+})$. If $\phi$ is an $n^{th}$-root of identity map then we have a generated by $\{\phi(u)\}$ extension $A \rightarrow A\{\phi(u)\}$. Sometimes this extension is a noncommutative covering projection.

Example 3.9. Let $A_\theta$ be a noncommutative torus, $K_1(A_\theta) \approx \mathbb{Z}^2$. Let $u, v \in U(A)$ be representatives of generators of $K^1(A_\theta)$ a sp($u$) = sp($v$) = $\{z \in C \mid |z| = 1\}$. Following $\ast$-homomorphisms

$$A_\theta \rightarrow A_\theta\{\phi(u)\},$$

$$A_\theta \rightarrow A_\theta\{\phi(v)\}$$

are particular cases of noncommutative covering projections which are described in subsection 2.3.

Example 3.10. It is known that $S^3$ is homeomorphic to $SU(2)$, $K_1(C(SU(2))) \approx \mathbb{Z}$ and $K_1(C(SU(2)))$ is generated by unitary $u \in U(C(SU(2) \otimes M_2(C))$. Element $u$ can be regarded as the natural map $SU(2) \rightarrow M_2(C)$ and sp($u$) = $\{z \in C \mid |z| = 1\}$. Denote by $A = C(SU(2)) \otimes M_2(C)$. Let $\phi$ be a 2th -root of identity map, and $v = \phi(u)$. There is an extension $A \rightarrow A\{v\}$. Both $A$ and $A\{v\}$ are continuous trace algebras. The $\mathbb{Z}_2$ group acts on $A\{v\}$ such that action of nontrivial element $g \in \mathbb{Z}_2$ is given by

$$ gv = -v. $$

Let $\rho : A\{v\} \rightarrow B(H)$ be a irreducible representation. Then $V = \rho(v)$ is a $2 \times 2$ unitary matrix. Suppose that $\rho$ is such that by

$$ \rho(v) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. $$

We have

$$ \rho_g(v) = \rho(gv) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. $$

Above matrices are unitary equivalent, i. e.

$$ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. $$

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So the representation $\rho$ is unitary equivalent to the $\rho_\delta$ and action of $g$ is not strictly outer, extension $f : A \to A\{v\}$ does not satisfy definition \[2.9\] i.e. $f : A \to A\{v\}$ is not a noncommutative covering projection. Algebra $A$ does not have nontrivial noncommutative covering projections because

1. $A$ is a continuous trace algebra,
2. $\hat{A} \approx S^3$,
3. $\pi_1(S^3) = 0$, i.e. $S^3$ does not have nontrivial covering projections.

Remark 3.11. This construction supplies a covering projection if $x \in K_1(X)$ belongs to image of $\pi_1(X) \to K_1(X)$.

## 3.4 Torsion special case

### Example 3.12. Universal covering from example \[1.2\] can be constructed algebraically. Let $f : S^1 \to S^1$ be a $n$ listed covering projection of the circle, $C_f$ is the (topological) mapping cone of $f$. $C(f) : C(S^1) \to C(S^1)$ is a corresponding *- homomorphism of $C^*$-algebras $(u \mapsto u^n)$, where $u \in U(C(S^1))$ is such that $[u] \in K_1(C(S^1))$ is a generator. Algebraic mapping cone \[1\] $C_{C(f)}$ of $C(f)$ corresponds to the topological space $C_f$. $C_{C(f)}$ is an algebra of continuous maps $f(0,1) \to U(C)$ such that

$$f(0) = \sum_{k \in \mathbb{Z}} a_k u^{kn}, \ a_k \in C.$$ 

A map $v = (x \mapsto u) \ (\forall x \in [0,1])$ is such that $v^i \notin M(C(C_f)) \ (i = 1, ..., n-1), v^n \in M(C_{C(f)}).$ Homomorphism $C_{C(f)} \to C_{C(f)}\{v\}$ corresponds to a $n$-listed covering projection from the example \[1.2\]

### Example 3.13. General construction. Above construction can be generalized. Let $A$ be a $C^*$ - algebra such that $K^1(A) = G \oplus \mathbb{Z}_n$, where $G$ is an abelian group. From \[3.0\] it follows that $K_0(A) \approx G' \oplus \mathbb{Z}_n$. Let $Q(A) = M(A \otimes K) / (A \otimes K)$ be the stable multiplier algebra of $C^*$ - algebra $A$. Then from \[1\] it follows that $K_1(Q(A)) = K_0(A)$. Let $u \in U(Q(A))$ be such that $K_1(Q(A)) = G' \oplus \mathbb{Z}_n[u]$. Let $\phi$ be a $n^{th}$ root of identity map such that $\phi(u^n) = u$. Let $p : M(A \otimes K) \to M(A \otimes K) / (A \otimes K)$ be a natural surjective *- homomorphism. It is known \[1\] that unitary element $v \in U(Q(A))$ can be lifted to an unitary element $v' \in U(M(A \otimes K))$ (i.e. $v = p(v')$) if and only if $[v] = 0 \in K^1(Q(A))$. From $n[u] = [u^n] = 0$ it follows that there is an unitary $w \in U(M(A \otimes K))$ such that $p(w) = u^n$. Let $M(A \otimes K) \to B(H)$ be a faithful representation, then $\phi(w) \in U(B(H))$. If $\phi(w)$ is a noncommutative covering projection. Example \[3.12\] is a particular case of this general construction.

### Example 3.14. Let $O_n$ be a Cuntz algebra \[1\], $K_0(O_n) = \mathbb{Z}_{n-1}$. Construction \[3.13\] supplies a $\mathbb{Z}_{n-1}$ - Galois extension $f : O_n \to \tilde{O}_n$. However it is not known is $f$ strictly outer.
3.5 A noncommutative generalization of $K_{11}(X)$.

Above construction can generalize $K_{11}(X)$ group. Suppose that $K_{1}(X)$ is group generated by $x_{1},...,x_{n}$. Let $x \in \{x_{1},...,x_{n}\}$ be a generator. Construction of $3.8$, $3.13$ supplies extension of $A$ which is associated with $x$. The element $x$ is said to be proper if the extension is a noncommutative covering projection. Generalization of $K_{11}(X)$ is a generated by proper elements subgroup of $K^{1}(A)$.

4 Conclusion

The presented here theory supplies algebraic construction of covering projections. These projections are well known for commutative case. Example $3.9$ is principally new application of the theory. It is interesting to find other nontrivial examples of this theory.

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