ASYMPTOTIC PROPERTIES OF BANACH SPACES AND COARSE QUOTIENT MAPS

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Abstract. We give a quantitative result about asymptotic moduli of Banach spaces under coarse quotient maps. More precisely, we prove that if a Banach space $Y$ is a coarse quotient of a subset of a Banach space $X$, where the coarse quotient map is coarse Lipschitz, then the $(\beta)$-modulus of $X$ is bounded by the modulus of asymptotic uniform smoothness of $Y$ up to some constants. In particular, if the coarse quotient map is a coarse homeomorphism, then the modulus of asymptotic uniform convexity of $X$ is bounded by the modulus of asymptotic uniform smoothness of $Y$ up to some constants.

1. Introduction

The study of asymptotic geometry of Banach spaces dates back to Milman [14], in which he introduced two asymptotic properties that are now known as asymptotic uniform convexity and asymptotic uniform smoothness (cf. [9]). For a Banach space $X$ and $t > 0$, the modulus of asymptotic uniform smoothness of $X$ is defined by

$$\bar{\rho}_X(t) := \sup_{x \in S_X} \inf_{\dim(X/Y) < \infty} \sup_{y \in S_Y} \|x + ty\| - 1,$$

and the modulus of asymptotic uniform convexity of $X$ is defined by

$$\bar{\delta}_X(t) := \inf_{x \in S_X} \sup_{\dim(X/Y) < \infty} \inf_{y \in S_Y} \|x + ty\| - 1.$$

A Banach space $X$ is said to be asymptotically uniformly smooth (AUS for short) if $\lim_{t \to 0} \bar{\rho}_X(t)/t \to 0$ as $t \to 0$, and it is said to be asymptotically uniformly convex (AUC for short) if $\bar{\delta}_X(t) > 0$ for all $0 < t \leq 1$.

In close relation to AUC and AUS is Rolewicz’s property ($\beta$) that was originally defined using the terminology of “drop” [16]; later Kutzarova [12] gave an equivalent definition, according to which one can define a modulus for the property: for a Banach space $X$ and $t \in (0, a]$, where $a \in [1, 2]$ is a number that depends only on the space $X$, the $(\beta)$-modulus of $X$ is defined by

$$\bar{\beta}_X(t) = 1 - \sup\left\{ \inf_{n \geq 1} \left\{ \frac{\|x - x_n\|}{2} \right\} : x, x_n \in B_X, \text{ sep}(\{x_n\}_{n=1}^\infty) \geq t \right\}.$$ 

Here sep$(\{x_n\}_{n=1}^\infty)$ denotes the separating constant of the sequence $\{x_n\}_{n=1}^\infty$:

$$\text{sep}(\{x_n\}_{n=1}^\infty) := \inf_{i \neq j} \|x_i - x_j\|.$$ 

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A Banach space $X$ is said to have property $(\beta)$ if $\bar{\beta}_X(t) > 0$ for all $t > 0$, and for $p \in (1, \infty)$ we say that the $(\beta)$-modulus of $X$ has power type $p$, or $X$ has property $(\beta_p)$, if there is a constant $C > 0$ so that $\bar{\beta}_X(t) \geq C t^p$ for all $t > 0$.

A reflexive Banach space that is simultaneously AUC and AUS must have property $(\beta)$ [11]. Conversely, if a Banach space $X$ has property $(\beta)$, then it must be reflexive and AUC. More precisely, it was shown in [7] that $\bar{\beta}_X(t) \leq \bar{\delta}_X(2t)$ for all $t \in (0, 1/2]$. However, property $(\beta)$ does not imply AUS isometrically [11], but a Banach space with property $(\beta)$ admits an equivalent norm that is AUS. A complete renorming argument of property $(\beta)$ can be found in the recent paper by Dilworth, Kutzarova, Lancien and Randrianarivony [6].

Bates, Johnson, Lindenstrauss, Preiss and Schechtman first studied nonlinear quotient maps in the Banach space setting [2]. A map $f : X \to Y$ between two metric spaces $X$ and $Y$ is said to be co-uniformly continuous if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x \in X$,

$$ f(B_X(x, \varepsilon)) \supseteq B_Y(f(x), \delta). $$

If $\delta$ can be chosen to be $\varepsilon/C$ for some constant $C > 0$ that is independent of $\varepsilon$, then $f$ is said to be co-Lipschitz. A uniform (resp. Lipschitz) quotient map is a map that is both uniform continuous and co-uniform continuous (resp. Lipschitz and co-Lipschitz), and $Y$ is said to be a uniform (resp. Lipschitz) quotient of $X$ if there exists a uniform (resp. Lipschitz) quotient map from $X$ onto $Y$.

Lima and Randrianarivony [13] showed that for $q > p > 1$, $\ell_q$ is not a uniform quotient of $\ell_p$. Their proof relies on a technical argument called “fork argument”. On the other hand, Baudier and Zhang [3] gave a different proof by estimating the $\ell_p$-distortion of the countably branching trees. The two proofs are based on similar ideas that use the quantification of property $(\beta)$. The theorem below, first appeared in [5], is the quantitative version of the Lima-Randrianarivony result.

**Theorem 1.1** ([5]). Let $X, Y$ be two Banach spaces. $S$ is a subset of $X$ and $f : S \to Y$ is a uniform quotient map that is Lipschitz for large distances. Then there exists constant $C > 0$ that depends only on the map $f$ such that for all $0 < t \leq 1$,

$$ \bar{\beta}_X(Ct) \leq \frac{3}{2} \bar{\rho}_Y(t). $$

The main goal of this paper is to give quantitative results of this kind in the coarse category. It should be note that although property $(\beta)$ is preserved under uniform quotient maps up to renorming (cf. [7] and [6]), one cannot compare $\bar{\beta}_X$ and $\bar{\beta}_Y$ even if $X$ and $Y$ are uniformly homeomorphic. Indeed, [7] gave an example of two uniformly homeomorphic Banach spaces one of which has property $(\beta_p)$, $p \in (1, \infty)$, while the other does not admit any equivalent norm with property $(\beta_p)$.

Throughout this article all Banach spaces are real and of infinite dimension. For a metric space $X$, $B_X(x, r)$ denotes the closed ball centered at $x$ with radius $r$. If $X$ is a Banach space, we denote by $B_X$ and $S_X$ its closed unit ball and unit sphere, respectively.
2. Coarse quotient maps

A map \( f : X \to Y \) between two metric spaces \( X \) and \( Y \) is said to be coarsely continuous if \( \omega_f(t) < \infty \) for all \( t > 0 \), where
\[
\omega_f(t) := \sup \{ d_Y(f(x), f(y)) : d_X(x, y) \leq t \}
\]
is the modulus of continuity of \( f \). If \( X \) is unbounded, one can define for every \( s > 0 \) the Lipschitz constant of \( f \) when distances are at least \( s \) by
\[
\text{Lip}_s(f) := \sup \left\{ \frac{d_Y(f(x), f(y))}{d_X(x, y)} : d_X(x, y) \geq s \right\},
\]
then for all \( t \geq 0 \) and \( s > 0 \),
\[
\omega_f(t) \leq \max \{ \omega_f(s), \text{Lip}_s(f) \cdot t \}.
\]
Let
\[
\text{Lip}_\infty(f) := \inf_{s>0} \text{Lip}_s(f) = \lim_{s \to \infty} \text{Lip}_s(f).
\]
The map \( f \) is said to be coarse Lipschitz if \( \text{Lip}_\infty(f) < \infty \), or equivalently, if \( \text{Lip}_s(f) < \infty \) for some \( s > 0 \).

The following notion of coarse quotient map was introduced by Zhang [17]:

**Definition 2.1 ([17])**. Let \( X, Y \) be two metric spaces. For a constant \( K \geq 0 \), a map \( f : X \to Y \) is said to be co-coarsely continuous with constant \( K \) if for every \( d > K \), there exists \( \delta = \delta(d) > 0 \) such that for all \( x \in X \),
\[
f(B_X(x, \delta))^K \supseteq B_Y(f(x), d).
\]
Here for a subset \( A \) of \( X \), \( A^K \) denotes the \( K \)-neighborhood of \( A \), that is,
\[
A^K := \{ x \in X : d_X(x, a) \leq K \text{ for some } a \in A \}.
\]
If \( f \) is both coarsely continuous and co-coarsely continuous (with constant \( K \)), then we say \( f \) is a coarse quotient map (with constant \( K \)). \( Y \) is said to be a coarse quotient of \( X \) if there exists a coarse quotient map from \( X \) to \( Y \).

Recall that a metric space \( X \) is said to be metrically convex if for every \( x_0, x_1 \in X \) and \( 0 < \lambda < 1 \), there is a point \( x_\lambda \in X \) such that
\[
d(x_0, x_\lambda) = \lambda d(x_0, x_1) \quad \text{and} \quad d(x_1, x_\lambda) = (1 - \lambda)d(x_0, x_1).
\]
It is well-known that a coarsely continuous map defined on a metrically convex space must be Lipschitz for large distances. Similarly, if the range space of a co-coarsely continuous map with constant \( K \) is metrically convex, then the map is “co-Lipschitz for large distances with constant \( K \)” as stated in the Lemma below.

**Lemma 2.2.** Let \( X, Y \) be two metric spaces and assume that \( Y \) is metrically convex. If \( f : X \to Y \) is a co-coarsely continuous map with constant \( K \), then for every \( d > K \), there exists \( c = c(d, K) > 0 \) such that for all \( x \in X \) and \( r \geq d \),
\[
f(B_X(x, cr))^K \supseteq B_Y(f(x), r).
\]
Proof. For $x \in X$ and $r \geq d$, let $n := \lfloor \frac{r}{d-K} \rfloor + 1$. Then for every $y \in B_Y(f(x), r)$, $d_Y(y, f(x)) \leq r < n(d-K)$. By the metric convexity of $Y$, one can find points \{u_i\}_{i=0}^n in $Y$ with $u_0 = f(x)$ and $u_n = y$ such that $d_Y(u_i, u_{i-1}) < d - K$, $i = 1, \ldots, n$. Since $f$ is co-coarsely continuous with constant $K$, we have

$$u_1 \in B_Y(f(x), d) \subseteq f(B_X(x, \delta))^K,$$

where $\delta = \delta(d) > 0$ is given by Definition 2.1, so there exists $z_1 \in B_X(x, \delta)$ so that $d_Y(u_1, f(z_1)) \leq K$. This implies, by the triangle inequality, that $u_2 \in B_Y(f(z_1), d)$. Again the co-coarse continuity of $f$ guarantees that there is $z_2 \in B_X(z_1, \delta)$ that satisfies $d_Y(u_2, f(z_2)) \leq K$. Repeat the procedure $n$ times we get points \{z_i\}_{i=0}^n in $X$, where $z_0 = x$, with the following property: $d_X(z_i, z_{i-1}) \leq \delta$ and $d_Y(u_i, f(z_i)) \leq K$, $i = 1, \ldots, n$. It follows that $z_n \in B_X(x, n\delta)$, hence $y \in f(B_X(x, n\delta))^K$. Note that $n \leq \left(\frac{1}{d-K} + \frac{1}{d}\right) r$, thus (2.1) follows by putting $c = \left(\frac{1}{d-K} + \frac{1}{d}\right) \delta$. \hfill \qed

Remark 2.3. Lemma 2.2 is an improvement of Lemma 3.2 in [17], where $d > 2K$ is required. Also, For $d > K$, it follows from (2.1) that the constant $c = c(d, K) > 0$ satisfies for all $x \in X$ and $r > 0$,

$$f(B_X(x, cr))^d \supseteq B_Y(f(x), r).$$

It means that co-coarsely continuous maps are co-Lipschitz with a slightly larger constant if the range space is metrically convex.

Under the assumption of Lemma 2.2, for $d > K$, let $c_d$ be the infimum of all $c$ that satisfy (2.1) for all $x \in X$ and $r \geq d$. Then \{c_d\}_{d>K} is non-increasing and bounded below by 0, hence converges. Denote $c_{\infty}(f) := \inf_{d>K} c_d = \lim_{d\to\infty} c_d$.

Lemma 2.4. Let $X, Y$ be two metric spaces and assume that $Y$ is metrically convex and unbounded. If $f : X \to Y$ is a coarse quotient map that is coarse Lipschitz, then

$$\text{Lip}_{\infty}(f)c_{\infty}(f) \geq 1.$$  

Proof. Let $f$ be a coarse quotient map with constant $K$. First observe $X$ is unbounded and $\text{Lip}_{s}(f) > 0$ for all $s > 0$, since otherwise $Y = f(X)^K$ is bounded.

Now by Lemma 2.2, for $d > K$, there exists $c = c(d, K) > 0$ such that for all $x \in X$ and $r \geq d$,

$$B_Y(f(x), r) \subseteq f(B_X(x, cr))^K \subseteq B_Y(f(x), \omega_f(cr))^K = B_Y(f(x), \omega_f(cr) + K),$$

and this implies that $r \leq \omega_f(cr) + K$. Since $f$ is coarse Lipschitz, let $s > 0$ be such that $0 < \text{Lip}_s(f) < \infty$. Then for $t \geq s$ one has

$$r \leq \omega_f(cr) + K \leq \max\{\omega_f(t), \text{Lip}_t(f) \cdot cr\} + K.$$

Choose large $r$ so that $\text{Lip}_t(f) \cdot cr > \omega_f(t)$, then $r \leq \text{Lip}_t(f) \cdot cr + K$, so

$$\text{Lip}_t(f) \cdot c \geq \frac{r - K}{r}.$$

Let $r \to \infty$, it follows that $\text{Lip}_t(f) \cdot c \geq 1$, and then we finish the proof by letting $t \to \infty$. \hfill \qed
3. Quantitative results under coarse quotient maps

Before stating our main theorem, we need the following alternative definition for the modulus of AUS that may be known to experts, but we still give a proof here since we could not find one in the literature.

**Proposition 3.1.** Let $X$ be a Banach space and $0 < t \leq 1$. Then
\[
\bar{\rho}_X(t) = \sup_{x \in B_X} \inf_{\dim(X/Y) < \infty} \sup_{y \in S_Y} \|x + ty\| - 1
\]

**Proof.** First we show that for every $x, y \in X$ the function
\[
f(\lambda) = \max\{|\lambda x + y|, |\lambda x - y|\}
\]
is nondecreasing on $(0, \infty)$. Let $0 < \lambda_1 < \lambda_2$, we may assume that $|\lambda_1 x + y| \geq |\lambda_1 x - y|$ and let $x^* \in S_X$, be such that $x^*(\lambda_1 x + y) = |\lambda_1 x + y|$. Then $x^*(x) \geq 0$, since otherwise
\[
|\lambda_1 x + y| \geq |\lambda_1 x - y| \geq (-x^*)(\lambda_1 x - y) > x^*(\lambda_1 x + y) = |\lambda_1 x + y|.
\]
Therefore, $f(\lambda_1) = x^*(\lambda_1 x + y) \geq x^*(\lambda_2 x + y) \leq |\lambda_2 x + y| = f(\lambda_2)$.

Now we prove (3.1). Let $0 < t \leq 1$. If $x = 0$ then
\[
\inf_{\dim(X/Y) < \infty} \sup_{y \in S_Y} \|x + ty\| = t \leq \bar{\rho}_X(t) + 1.
\]
For $x \in B_X\setminus\{0\}$ one has
\[
\inf_{\dim(X/Y) < \infty} \sup_{y \in S_Y} \|x + ty\| = \inf_{\dim(X/Y) < \infty} \sup_{y \in S_Y} \max\{|\lambda x + y|, |\lambda x - y|\}
\]
\[
\leq \inf_{\dim(X/Y) < \infty} \sup_{y \in S_Y} \max \left\{\frac{x}{\|x\|} + ty, \frac{x}{\|x\|} - ty\right\}
\]
\[
= \inf_{\dim(X/Y) < \infty} \sup_{y \in S_Y} \frac{x}{\|x\|} + ty
\]
\[
\leq \bar{\rho}_X(t) + 1,
\]
thus (3.1) follows. \qed

**Theorem 3.2.** Let $X, Y$ be two Banach spaces. $S$ is a subset of $X$ and $f : S \to Y$ is a coarse quotient map that is coarse Lipschitz. Then for all $0 < t \leq 1$,
\[
\bar{\rho}_X \left(\frac{t}{48\text{Lip}_\infty(f)c_\infty(f)}\right) \leq \frac{3}{2} \bar{\rho}_Y(t).
\]

**Proof.** Since $f : S \to Y$ is a coarse quotient map that is coarse Lipschitz, it follows from Lemma 2.4 that $0 < \text{Lip}_\infty(f) < \infty$. Choose $s > 0$ such that $\text{Lip}_s(f) < 2\text{Lip}_\infty(f)$. For $0 < t \leq 1$, one has $0 \leq \bar{\rho}_Y(t) \leq t \leq 1$. Let $\varepsilon > 0$ be small so that
\[
\varepsilon < \min \left\{\frac{1}{2}, \frac{2 - \bar{\rho}_Y(t)}{6\text{Lip}_\infty(f) + 2}, c_\infty(f)\right\},
\]
and choose large $d$ that satisfies
\[
d > \max \left\{\frac{3K}{\varepsilon}, \frac{12(2K + \omega_f(s))}{t}\right\} \quad \text{and} \quad c_{d/3} < c_\infty(f) + \varepsilon.
\]
Since $c_\infty(f) - \varepsilon < c_d$, there exist $z_\varepsilon \in S$ and $R \geq d$ such that

$$B_Y(f(z_\varepsilon), R) \not\subseteq f(B_S(z_\varepsilon, R(c_\infty(f) - \varepsilon))^K,$$

so there is $y_\varepsilon \in Y$ with $0 < \|y_\varepsilon - f(z_\varepsilon)\| \leq R$ such that

$$B_Y(y_\varepsilon, K) \cap f(B_S(z_\varepsilon, R(c_\infty(f) - \varepsilon)) = \emptyset. \tag{3.2}$$

Now cut the line segment $[y_\varepsilon, f(z_\varepsilon)]$ into three equal pieces, namely, let $m, M \in Y$ be such that $m - f(z_\varepsilon) = M - m = y_\varepsilon - M$, then

$$m \in B_Y\left(f(z_\varepsilon), \frac{R}{3}\right) \subseteq f\left(B_S\left(z_\varepsilon, \frac{R}{3}(c_\infty(f) + \varepsilon)\right)\right)^K,$$

so there is $x \in S$ such that

$$\|x - z_\varepsilon\| \leq \frac{R}{3}(c_\infty(f) + \varepsilon) \quad \text{and} \quad \|m - f(x)\| \leq K.$$

By the definition of $\bar{\rho}_Y(t)$ (Proposition 3.1), there exists a finite-codimensional subspace $Z$ of $Y$ so that

$$\sup_{z \in Z} \left\|M - m + \frac{tR}{3} \varepsilon\right\| < \frac{R}{3}(1 + \bar{\rho}_Y(t) + \varepsilon).$$

Set $y_n := M + \frac{tR}{3}e_n$, where $(e_n)$ is a basic sequence in $S_Z$ with basis constant less than 2. Then

$$\|y_n - m\| = \left\|M - m + \frac{tR}{3}e_n\right\| < \frac{R}{3}(1 + \bar{\rho}_Y(t) + \varepsilon),$$

and by the triangle inequality,

$$\|y_n - f(x)\| < \frac{R}{3}(1 + \bar{\rho}_Y(t) + \varepsilon) + K < \frac{R}{3}(1 + \bar{\rho}_Y(t) + 2\varepsilon).$$

Thus

$$y_n \in B_Y\left(f(x), \frac{R(1 + \bar{\rho}_Y(t) + 2\varepsilon)}{3}\right) \subseteq f\left(B_S\left(x, \frac{R(1 + \bar{\rho}_Y(t) + 2\varepsilon)(c_\infty(f) + \varepsilon)}{3}\right)\right)^K,$$

so there exists $z_n \in S$ such that

$$\|z_n - x\| \leq \frac{R}{3}(1 + \bar{\rho}_Y(t) + 2\varepsilon)(c_\infty(f) + \varepsilon) \quad \text{and} \quad \|y_n - f(z_n)\| \leq K.$$

Note that

$$\|y_\varepsilon - y_n\| = \|y_\varepsilon - M - \frac{tR}{3}e_n\| = \left\|M - m - \frac{tR}{3}e_n\right\| < \frac{R}{3}(1 + \bar{\rho}_Y(t) + \varepsilon),$$

so again by the triangle inequality,

$$\|y_\varepsilon - f(z_n)\| < \frac{R}{3}(1 + \bar{\rho}_Y(t) + \varepsilon) + K < \frac{R}{3}(1 + \bar{\rho}_Y(t) + 2\varepsilon),$$

hence

$$y_\varepsilon \in B_Y\left(f(z_n), \frac{R(1 + \bar{\rho}_Y(t) + 2\varepsilon)}{3}\right) \subseteq f\left(B_S\left(z_n, \frac{R(1 + \bar{\rho}_Y(t) + 2\varepsilon)(c_\infty(f) + \varepsilon)}{3}\right)\right)^K,$$
and this gives \( x_n \in S \) that satisfies
\[
\|x_n - z_n\| \leq \frac{R}{3} (1 + \bar{\rho}_Y(t) + 2\varepsilon)(c_{\infty}(f) + \varepsilon) \quad \text{and} \quad \|y_\varepsilon - f(x_n)\| \leq K.
\]

On the other hand, in view of (3.2), one has \( \|z_\varepsilon - x_n\| > R(c_{\infty}(f) - \varepsilon) \), so
\[
\|z_\varepsilon - z_n\| \geq \|z_\varepsilon - x_n\| - \|x_n - z_n\|
\]
\[
> R(c_{\infty}(f) - \varepsilon) - \frac{R}{3} (1 + \bar{\rho}_Y(t) + 2\varepsilon)(c_{\infty}(f) + \varepsilon)
\]
\[
= \frac{R}{3} (c_{\infty}(f) + \varepsilon) \left( \frac{3(c_{\infty}(f) - \varepsilon)}{c_{\infty}(f) + \varepsilon} - 1 - \bar{\rho}_Y(t) - 2\varepsilon \right)
\]
\[
\geq \frac{R}{3} (c_{\infty}(f) + \varepsilon) \left( 1 - \frac{2\varepsilon}{c_{\infty}(f)} - 1 - \bar{\rho}_Y(t) - 2\varepsilon \right)
\]
\[
\geq \frac{R}{3} (c_{\infty}(f) + \varepsilon)(2 - \bar{\rho}_Y(t) - (6\text{Lip}_{\infty}(f) + 2\varepsilon)) > 0.
\]

For \( n, k \in \mathbb{N} \) with \( n \neq k \),
\[
\|y_n - y_k\| = \frac{tR}{3}\|e_n - e_k\| > \frac{tR}{6} > \omega_f(s) + 2K,
\]
and also note that
\[
\|y_n - y_k\| \leq \|y_n - f(z_n)\| + \|f(z_n) - f(z_k)\| + \|y_k - f(z_k)\|
\]
\[
\leq 2K + \omega_f(\|z_n - z_k\|),
\]
so \( \omega_f(\|z_n - z_k\|) > \omega_f(s) \), thus \( \|z_n - z_k\| > s \). It follows that
\[
\frac{tR}{6} < \|y_n - y_k\| \leq \|y_n - f(z_n)\| + \|f(z_n) - f(z_k)\| + \|y_k - f(z_k)\|
\]
\[
\leq 2K + \text{Lip}_s(f)\|z_n - z_k\|
\]
\[
< \frac{tR}{12} + 2\text{Lip}_{\infty}(f)\|z_n - z_k\|,
\]
hence \( \|z_n - z_k\| > \frac{tR}{24\text{Lip}_{\infty}(f)} \).

In summary, for \( n, k \in \mathbb{N} \) with \( n \neq k \) we have the following:
\[
\|z_n - z_k\| = \frac{tR}{24\text{Lip}_{\infty}(f)}; \quad \|z_\varepsilon - z_n\| > \frac{R}{3} (c_{\infty}(f) + \varepsilon)(2 - \bar{\rho}_Y(t) - (6\text{Lip}_{\infty}(f) + 2\varepsilon))
\]
\[
\|z_\varepsilon - x\| \leq \frac{R}{3} (c_{\infty}(f) + \varepsilon), \quad \|z_n - x\| \leq \frac{R}{3} (c_{\infty}(f) + \varepsilon)(1 + \bar{\rho}_Y(t) + 2\varepsilon).
\]

Since
\[
\frac{t}{48\text{Lip}_{\infty}(f)c_{\infty}(f)} \leq \frac{tR}{24\text{Lip}_{\infty}(f)} \frac{1}{\frac{R}{3} (c_{\infty}(f) + \varepsilon)(1 + \bar{\rho}_Y(t) + 2\varepsilon)} \leq \frac{t}{8\text{Lip}_{\infty}(f)c_{\infty}(f)}
\]
and
\[
1 - \frac{1}{2} \frac{R}{\frac{R}{3} (c_{\infty}(f) + \varepsilon)(1 + \bar{\rho}_Y(t) + 2\varepsilon)} \leq \frac{3}{2} \bar{\rho}_Y(t) + (3\text{Lip}_{\infty}(f) + 3)\varepsilon,
\]
it follows from the definition of $(\beta)$-modulus that
\[
\bar{\beta}_X \left( \frac{t}{48\text{Lip}_\infty(f)c_\infty(f)} \right) \leq \frac{3}{2} \bar{\beta}_Y(t) + (3\text{Lip}_\infty(f) + 3)\varepsilon.
\]
The proof is complete by letting $\varepsilon \to 0$. \hfill \Box

It is easy to compute that for $1 < p < \infty$ and $0 < t \leq 1$,
\[
\bar{\delta}_{\ell_p}(t) = \bar{\rho}_{\ell_p}(t) = (1 + t^p)^{\frac{1}{p}} - 1,
\]
and since $\ell_p$ has property $(\beta_p)$ (see [1] for the explicit formula of $(\beta)$-modulus of $\ell_p$), we can recover the main result of [17] as an immediate consequence of Theorem 3.2.

**Corollary 3.3.**

(i) $\ell_q$ is not a coarse quotient of $\ell_p$ for $1 < p < q < \infty$.

(ii) $c_0$ is not a coarse quotient of any Banach space with property $(\beta)$.

4. **Quantitative results under coarse homeomorphisms**

This section is devoted to a special case of Theorem 3.2 when the coarse quotient map $f$ is a coarse homeomorphism. Recall that a coarsely continuous map $f : X \to Y$ between two metric spaces $X$ and $Y$ is called a coarse homeomorphism if there exists another coarsely continuous map $g : Y \to X$ such that
\[
\sup_{x \in X} d_X(g \circ f(x), x) < \infty \quad \text{and} \quad \sup_{y \in Y} d_Y(f \circ g(y), y) < \infty.
\]
It was proved in [17] that a coarse homeomorphism is necessarily a coarse quotient map.

The main tool we need is approximate metric midpoint, which was first used by Enflo (unpublished) to show that $L_1$ is not uniformly homeomorphic to $\ell_1$ (see, e.g., [4]). Given two points $x, y$ in a metric space $X$ and $\delta \in (0, 1)$, the set of $\delta$-approximate metric midpoints between $x$ and $y$ is defined by
\[
\text{Mid}(x, y, \delta) := \left\{ z \in X : \max\{d_X(z, x), d_X(z, y)\} \leq \frac{1 + \delta}{2} d_X(x, y) \right\}.
\]
The lemma below is sometimes known as the “stretching lemma”.

**Lemma 4.1 ([10]).** Let $f : X \to Y$ be a coarse Lipschitz map from an unbounded metric space $X$ to a metric space $Y$. If $\text{Lip}_\infty(f) > 0$ then for any $d > 0$, any $\varepsilon > 0$ and any $0 < \delta < 1$ there exist $x, y \in X$ with $d_X(x, y) \geq d$ such that
\[
f(\text{Mid}(x, y, \delta)) \subseteq \text{Mid}(f(x), f(y), (1 + \varepsilon)\delta).
\]

The next lemma, which can be found in [15], relates the set of approximate metric midpoints in a Banach space with the moduli of AUC and AUS of the space.

**Lemma 4.2 ([15]).** Let $X$ be a Banach space, $x \in S_X$ and $0 < t \leq 1$. 
(i) For every \( \varepsilon > 0 \), there exists a finite-codimensional subspace \( Y \) of \( X \) such that
\[
iB_Y \subseteq \text{Mid}(x, -x, \bar{\rho}_X(t) + \varepsilon).
\]
(ii) If \( \bar{\delta}_X(t) > 0 \), then for every \( 0 < \varepsilon < 1 \), there exists a compact subset \( K \) of \( X \) such that
\[
\text{Mid}(x, -x, (1 - \varepsilon)\bar{\delta}_X(t)) \subseteq K + 3tB_X.
\]

We also need the following easy lemma.

**Lemma 4.3.** Let \( f : X \to Y \) be a map between Banach spaces \( X \) and \( Y \). If there exist a closed ball \( B_r \) of radius \( r \) in \( X \), a closed ball \( B_s \) of radius \( s \) in \( Y \) and a compact set \( K \subseteq Y \) such that
\[
f(B_r) \subseteq K + B_s,
\]
then the compression modulus \( \varphi_f \) of \( f \) satisfies
\[
\varphi_f(r) := \inf\{\|f(x) - f(y)\| : \|x - y\| \geq r\} \leq 2s.
\]

**Proof.** Choose a \( r \)-separating sequence \( \{x_n\}_{n=1}^{\infty} \) in \( B_r \) and let \( f(x_n) = z_n + y_n \), where \( z_n \in K \) and \( y_n \in B_s \) for all \( n \in \mathbb{N} \). For \( \varepsilon > 0 \), since \( K \) is compact, by passing to a subsequence we may assume that \( \|z_n - z_m\| < \varepsilon \) for all \( m, n \in \mathbb{N} \). Then for \( m \neq n \),
\[
2s \geq \|y_n - y_m\| \geq \|f(x_n) - f(x_m)\| - \|z_n - z_m\| \geq \varphi_f(r) - \varepsilon,
\]
and we are done by letting \( \varepsilon \to 0 \). \( \square \)

Theorem 4.4 below is due to Randrianarivony [15]. We present here a proof with improved constants.

**Theorem 4.4.** Let \( X \) and \( Y \) be two Banach spaces and \( f : X \to Y \) a coarse Lipschitz embedding, i.e., there exist \( d \geq 0 \) and \( L, C > 0 \) such that for all \( x, y \in X \) with \( \|x - y\| \geq d \),
\[
\frac{1}{C}\|x - y\| \leq \|f(x) - f(y)\| \leq L\|x - y\|.
\]
Then for all \( 0 < t \leq 1 \),
\[
\bar{\delta}_Y(t/7LC) \leq \bar{\rho}_X(t).
\]

**Proof.** Let \( 0 < t \leq 1 \) be such that \( \bar{\delta}_Y(t/7LC) > 0 \) and \( 0 < \varepsilon < 1/2 \). Note that \( 1/C \leq \text{Lip}_\infty(f) \leq L \), one can apply the stretching lemma to find \( u, v \in X \) with \( \|u - v\| \geq 2d/t \) such that
\[
f\left(\text{Mid}\left(u, v, (1 - 2\varepsilon)\bar{\delta}_Y\left(\frac{t}{7LC}\right)\right)\right) \subseteq \text{Mid}\left(f(u), f(v), (1 - \varepsilon)\bar{\delta}_Y\left(\frac{t}{7LC}\right)\right).
\]
By Lemma 4.2 (ii), there exists a compact set \( K \subseteq Y \) such that
\[
\text{Mid}\left(f(u), f(v), (1 - \varepsilon)\bar{\delta}_Y\left(\frac{t}{7LC}\right)\right) \subseteq K + \frac{3t}{14LC}\|f(u) - f(v)\|B_Y.
\]
\[
\subseteq K + \frac{3t}{14C}\|u - v\|B_Y.
\]
Assume that there exists $\tau > 0$ that satisfies
\[
(1 - 2\varepsilon)\bar{\delta}_Y \left( \frac{t}{7LC} \right) > \bar{\rho}_X(t) + \tau,
\]
then by Lemma 4.2 (i), there exists a finite-codimensional subspace $Z$ of $X$ such that
\[
f \left( \operatorname{Mid} \left( u, v, (1 - 2\varepsilon)\bar{\delta}_Y \left( \frac{t}{7LC} \right) \right) \right) \supseteq f \left( \frac{u + v}{2} + \frac{t\|u - v\|}{2}B_Z \right),
\]
thus
\[
f \left( \frac{u + v}{2} + \frac{t\|u - v\|}{2}B_Z \right) \subseteq K + \frac{3t}{14C}\|u - v\|B_Y.
\]
Now it follows from Lemma 4.3 that
\[
\frac{t\|u - v\|}{2C} \leq \varphi_f \left( \frac{t\|u - v\|}{2} \right) \leq \frac{3t}{7C}\|u - v\|,
\]
a contradiction. Therefore, we must have
\[
(1 - 2\varepsilon)\bar{\delta}_Y \left( \frac{t}{7LC} \right) \leq \bar{\rho}_X(t).
\]
We then finish the proof by letting $\varepsilon \to 0$. □

**Theorem 4.5.** Let $X, Y$ be two Banach spaces. $S$ is a subset of $X$ and $f : S \to Y$ is a coarse homeomorphism that is coarse Lipschitz. Then for all $0 < t \leq 1$,
\[
\bar{\delta}_X \left( \frac{t}{56\text{Lip}_\infty(f)c_\infty(f)} \right) \leq \bar{\rho}_Y(t).
\]

**Proof.** Let $g : Y \to S$ be a coarsely continuous map such that
\[
\sup_{x \in S} d_X(g \circ f(x), x) := M < \infty \quad \text{and} \quad \sup_{y \in Y} d_Y(f \circ g(y), y) := K < \infty.
\]
We claim that $g$ is a coarse Lipschitz embedding from $Y$ into $X$. Indeed, it follows from Proposition 2.5 in [17] that $f$ is a coarse quotient map with constant $K$. Choose $s > 2K$ such that $\text{Lip}_s(f) < 2\text{Lip}_\infty(f)$, and let $d > 6K$ be such that $\varphi_g(d) > s$. For $y_1, y_2 \in Y$ with $\|y_1 - y_2\| \geq d$, one has $\|g(y_1) - g(y_2)\| \geq \varphi_g(d) > s$ and thus
\[
\|f \circ g(y_1) - f \circ g(y_2)\| \leq 2\text{Lip}_\infty(f)\|g(y_1) - g(y_2)\|.
\]
By the triangle inequality,
\[
\|f \circ g(y_1) - f \circ g(y_2)\| \geq \|y_1 - y_2\| - 2K \geq \frac{2}{3}\|y_1 - y_2\|,
\]
so we obtain that
\[
\frac{1}{3\text{Lip}_\infty(f)}\|y_1 - y_2\| \leq \|g(y_1) - g(y_2)\|.
\]
On the other hand, we could make $d$ even larger so that $c_{d-K} < 2c_\infty(f)$ and
\[
\frac{d}{3} \cdot c_\infty(f) > \omega_g(K) + M.
\]
Note that
\[ \|y_1 - y_2\| + K \geq r := \|y_1 - f \circ g(y_2)\| \geq \|y_1 - y_2\| - K \geq d - K > K, \]
it follows from Lemma 2.2 and the definition of \( c_{d-K} \) that
\[ y_1 \in B_Y(f \circ g(y_2), r) \subseteq f(B_S(g(y_2), 2rc_\infty(f)))^K, \]
so there exists \( x \in S \) such that
\[ \|x - g(y_2)\| \leq 2rc_\infty(f) \quad \text{and} \quad \|y_1 - f(x)\| \leq K. \]
Now again by the triangle inequality,
\[ \|g(y_1) - g(y_2)\| \leq \|g(y_1) - g \circ f(x)\| + \|g \circ f(x) - x\| + \|x - g(y_2)\| \leq \omega_g(K) + M + 2rc_\infty(f) \leq \frac{c_\infty(f)}{3} \|y_1 - y_2\| + 2c_\infty(f)(\|y_1 - y_2\| + K) \leq \frac{8c_\infty(f)}{3} \|y_1 - y_2\|. \]
Therefore, for sufficiently large \( d \), one has
\[ \frac{1}{3\text{Lip}_\infty(f)} \|y_1 - y_2\| \leq \|g(y_1) - g(y_2)\| \leq \frac{8c_\infty(f)}{3} \|y_1 - y_2\| \]
whenever \( \|y_1 - y_2\| \geq d \). The desired inequality then follows from Theorem 4.4. \( \square \)

**Remark 4.6.** In connection with the modulus of asymptotic uniform convexity, the asymptotic midpoint uniform convexity modulus of a Banach space \( X \) was introduced in [8] as follows:
\[ \tilde{\delta}_X(t) := \inf_{x \in S_X \dim(X/Y) < \infty} \inf_{y \in S_Y} \max\{\|x + ty\|, \|x - ty\|\} - 1. \]
A Banach space \( X \) is said to be asymptotically midpoint uniformly convex (AMUC for short) if \( \tilde{\delta}_X(t) > 0 \) for all \( 0 < t \leq 1 \). It was implicitly proved in [8] that Lemma 4.2 (ii) still holds true for the AMUC modulus. Therefore, Theorem 4.4 and Theorem 4.5 can be strengthened by replacing \( \bar{\delta}_X \) with \( \delta_X \).

**Corollary 4.7.**
(i) \( \ell_q \) does not coarse Lipschitz embed into \( \ell_p \) for \( 1 < p < q < \infty \).
(ii) \( c_0 \) does not coarse Lipschitz embed into any AMUC Banach space.

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