Spin 0 and spin 1/2 quantum relativistic particles in a constant gravitational field

M. Khorrami$^{1,3}$*, M. Alimohammadi$^2$ † and A. Shariati$^{1,3}$

$^1$ Institute for Advanced Studies in Basic Sciences, P.O.Box 159, Zanjan 45195, Iran.
$^2$ Department of Physics, University of Tehran, North Karegar Ave., Tehran, Iran.
$^3$ Institute of Applied Physics, P.O.Box 5878, Tehran 15875, Iran.

Abstract
The Klein-Gordon and Dirac equations in a semi-infinite lab ($x > 0$), in the background metric $ds^2 = u^2(x)(-dt^2 + dx^2) + dy^2 + dz^2$, are investigated. The resulting equations are studied for the special case $u(x) = 1 + gx$. It is shown that in the case of zero transverse-momentum, the square of the energy eigenvalues of the spin-1/2 particles are less than the squares of the corresponding eigenvalues of spin-0 particles with same masses, by an amount of $mg\hbar c$. Finally, for nonzero transverse-momentum, the energy eigenvalues corresponding to large quantum numbers are obtained, and the results for spin-0 and spin-1/2 particles are compared to each other.

1 Introduction
The behaviour of bosons and fermions in a gravitational field, has been of interest for many years, from the simplest case of a nonrelativistic quantum particle in the presence of constant gravity, [1] for example, to more complicated cases of the relativistic spin-1/2 particles in a curved space-time with torsion, [2,3,4] for example. Several experiments have been performed to test theoretical predictions, among which are the experiments of Colella et al. [5], which detected gravitational effects by neutron interferometry, and the recent experiment of Nesvizhevsky et al. [6], in which the quantum energy levels of neutrons in the Earth’s gravity were detected.

Chandrasekhar considered the Dirac equation in a Kerr-geometry background, and separated the Dirac equation into radial and angular parts, [7,8]. In [9], the angular part was solved, and in [10] some semi-analytical results for the radial part were obtained. Similar calculations were performed for the Kerr-Newman geometry and around dyon black holes in refs. [11] and [12], respectively.

*manwad@mailaps.org
†alimohmd@ut.ac.ir
In this article, we investigate relativistic spin-0 and spin-1/2 particles in a background metric $dx^2 = u^2(x)(-dt^2 + dx^2) + dy^2 + dz^2$, where the particles exist in a semi-infinite laboratory ($x > 0$). The wall $x = 0$, which prevents particles from penetrating to the region $x < 0$, corresponds to a boundary condition. For spin-0 particles, this is simply the vanishing of wave function on the wall. For the spin-1/2 particles, it is less trivial and will be discussed in the article. We consider the Hamiltonian-eigenvalue problems corresponding to a general function $u(x)$, and obtain the differential equations and boundary conditions corresponding to the spin-0 and spin-1/2 particles. The special case $u(x) = 1 + gx$, is investigated in more detail. An exact relation between the square of the energy eigenvalues of spin-0 and spin-1/2 particles, with same masses and no transverse momenta, is obtained, namely $E_0^2 = E_{KG}^2 - mg\hbar c$. Finally, the energy eigenvalues for large energies are obtained.

2 Review of the non-relativistic problem

The potential energy of a non-relativistic particle in a constant gravitational field is $V_{grav}(x) = mgx$, where $g$ is the acceleration of gravity, the direction of which is along the $x$ axis, towards the negative values of $x$.

The Schrödinger equation is written as $H\psi = i\hbar \partial_x \psi$, where $H = P^2/(2m) + V_{grav}$, subject to the boundary conditions that $\psi$ does not diverge at $x \rightarrow \pm \infty$.

Writing $\psi_{Sch} = \exp\{(-iEt + ip_2y + ip_3z)/\hbar\}F(x)$, it is easily seen that $F(x)$ must satisfy $F''(x) = (L^{-3}x + L^{-2}\lambda_{Sch})F(x)$, where a prime means differentiation with respect to the argument, $L^{-2}\lambda_{Sch} := (p_2^2 + p_3^2 - 2mE)/(\hbar^2)$, and $L := (2m^2g/\hbar^2)^{-1/3}$. This equation has two linearly independent solutions: the Airy functions $Ai(L^{-1}x + \lambda_{Sch})$ and $Bi(L^{-1}x + \lambda_{Sch})$ (see for example p. 569 of [13]). $Bi$ violates the boundary condition at $+\infty$. So the solutions are $Ai(L^{-1}x + \lambda_{Sch})$. These functions don’t tend to zero at $x \rightarrow -\infty$, and it is expected, since the potential $mgx$ is not bounded from below. But we note that a lab usually has walls. Consider a semi-infinite lab, for which

$$V_{walls} = \begin{cases} +\infty & x < 0 \\ 0 & x \geq 0. \end{cases}$$

In such a lab, the boundary condition at $-\infty$ is replaced with $\lim_{x \rightarrow -\infty} \psi_{Sch} = 0$. So the solutions are $Ai(L^{-1}x + \lambda_{Sch})$, where $\lambda_{Sch}$ must be one of the zeros of $Ai$.

The first four zeros of $Ai$ are approximately $\lambda_1 = -2.3381$, $\lambda_2 = -4.0879$, $\lambda_3 = -5.5206$, and $\lambda_4 = -6.7867$. If the transverse momenta of the particle vanish, the energy levels are these numbers multiplied by $-\hbar^2/3(2m)^{1/3}(g/2)^{2/3}$, the value of which for a neutron, in a lab on the Earth, where $g \cong 9.8 \text{ ms}^{-2}$, is 0.59 peV. Therefore, the first 4 energy levels of a neutron are $E_1 = 1.4 \text{ peV}$, $E_2 = 2.5 \text{ peV}$, $E_3 = 3.3 \text{ peV}$, and $E_4 = 4.0 \text{ peV}$. This result has been recently verified experimentally [3].

3 A relativistic quantum particle

According to general relativity, gravity is represented by a pseudo-Riemannian metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. We use the signature $(- + + +)$ for the metric. The
form of the metric depends on both the gravitational field and the coordinate system used to describe the field.

In a spacetime with metric $g_{\mu\nu}$, the Klein-Gordon equation, the equation for a spinless massive particle, is

$$\left( \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \sqrt{-g} g^{\mu\nu} \frac{\partial}{\partial x^\nu} - m^2 \right) \psi_{\text{KG}} = 0,$$

where $g := \text{det} [g_{\mu\nu}]$. We have used a system of units in which the numerical values of the velocity of light ($c$) and the Planck constant (divided by $2\pi$) are both unity.

To write the Dirac equation in a curved spacetime, or a flat spacetime but in a curvilinear coordinate system, one can employ the Equivalence Principle – see for example pp. 365–373 of [14], but note that our convention for the Dirac Lagrangian, and therefore the Dirac equation, is different from that of Weinberg: we use ‘$-m$’ in the following equation, instead of a ‘$+m$’ in Weinberg’s. It reads as follows:

$$\gamma^a (\partial_a + \Gamma_a) \psi_D - m \psi_D = 0,$$

where $\Gamma_a$'s are spin connections, obtained from the dual tetrad $e^a$, through

$$de^a + \Gamma^a_b \wedge e^b = 0$$

$$\Gamma^a_b := \Gamma^a_{cb} e^c$$

$$\Gamma_a := \frac{1}{2} S_{bc} \Gamma^c_a$$

$$S_{bc} := - \frac{1}{4} [\gamma_b, \gamma_c].$$

We consider a gravitational field which is represented by the metric

$$ds^2 = u^2(x) (-dt^2 + dx^2) + dy^2 + dz^2. \quad (2)$$

One can write the above metric, also like

$$ds^2 = -U^2(X)dt^2 + dX^2 + dy^2 + dz^2,$$

where $(dX)/(dx) = U(X) = u(x)$. In a small region of space (compared to the length in which the gravitational field changes significantly) one can use $u(x) = 1 + gx$, $U(X) = 1 + gX$, and $X = x$. We will investigate the case of a general $u(x)$, but then limit ourselves to the special case $u(x) = 1 + gx$.

For the metric (2), the Klein-Gordon equation reads

$$\left[ \frac{1}{u^2} \left( \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} \right) + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - m^2 \right] \psi_{\text{KG}} = 0.$$

To write the Dirac equation, we need the spin connections. The only non-vanishing spin connections for metric (2) are $\Gamma^0_1 = \Gamma^1_0 = (u'/u^2)e^0$. Thus, $\Gamma^0_1 = -\Gamma^1_0 = u'/u^2$, and $\Gamma_0 = \gamma_0 \gamma_1 u'(2u^2)$. Therefore, the Dirac equation for this metric reads as

$$\left( \gamma^a \partial_a + \frac{1}{2} \gamma_1 \frac{u'}{u^2} - m \right) \psi_D = 0,$$

which means

$$\left[ \frac{1}{u} \left( \gamma^0 \frac{\partial}{\partial t} + \gamma^1 \frac{\partial}{\partial x} \right) + \frac{1}{2} \gamma_1 \frac{u'}{u^2} + \gamma^2 \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z} - m \right] \psi_D = 0. \quad (3)$$
4 Boundary condition at the infinite barrier

In the non-relativistic equation, the infinite potential barrier which prevents particles from penetrating the region \( x < 0 \), leads to the boundary condition \( \lim_{x \to 0^+} \psi_{\text{Sch}} = 0 \). This boundary condition emerges from the fact that the Schrödinger equation is second order in \( x \), from which it follows that \( \psi_{\text{Sch}} \) must be continuous at \( x = 0 \).

The Klein-Gordon equation is also second order in \( x \), so the same boundary condition
\[
\psi_{\text{KG}}(0) = 0 \tag{4}
\]
emerges. But the Dirac equation is of first order, so the four-spinor \( \psi_D \) can be discontinuous at \( x = 0 \), if the potential goes to infinity there. To find the proper boundary condition, one must first find a proper way of confining a Dirac particle to the region \( x > 0 \). The first guess is to add a step potential to the Hamiltonian. But this is the same as adding an electrostatic potential, which is the time component of a four-vector, having opposite effects on particles and anti-particles. Such a potential will not result in a wave function decaying as \( x \to -\infty \). A better way is to add a term \(-V \bar{\psi}_D \psi_D\) to the Lagrangian, where \( V \) is a function of \( t, x, y, \) and \( z \) – something like a Higgs term. As a result of adding this ‘scalar’ potential to the Lagrangian, the mass \( m \) of the Dirac particle is replaced by \( m + V \). So the Dirac equation reads
\[
\left( \gamma^\mu \frac{\partial}{\partial x^\mu} - m - V(x) \right) \psi_D = 0.
\]

Now let’s consider the step potential
\[
V(x) = \begin{cases} 
V, & x < 0 \\
0, & x \geq 0 
\end{cases} \tag{5}
\]
where \( V \) is a positive constant. If the energy and the transverse momenta are finite, and the constant \( V \) is large, one can neglect the parts containing derivatives with respect to \( t, y, \) and \( z \), arriving at
\[
\left( \gamma^1 \gamma \cdot x - m - V(x) \right) \psi_D(x) = 0.
\]
The solution to this equation, which does not diverge as \( x \to -\infty \), is \( \psi_D \propto \exp(\frac{-i}{E}t + i\mathbf{p} \cdot \mathbf{y} + i\mathbf{p} \cdot \mathbf{z}) \psi_{\text{KG}}(x) \).

5 The Klein-Gordon equation

Since \( u \) in metric \( 2 \) does not depend on \( t, y, \) and \( z \), one can seek a solution whose functional form is as \( \psi_{\text{KG}}(t, x, y, z) = \exp(-iEt + ip_2y + ip_3z)\psi_{\text{KG}}(x) \).
We arrive at
\[ E^2 + \frac{d^2}{dx^2} - (p^2 + m^2) u^2(x) \psi_{KG}(x) = 0, \quad x > 0, \] (7)
subject to the boundary condition \( \psi_{KG}(0) = 0 \), where \( p := \sqrt{(p_2^2 + (p_3)^2)} \).

Defining
\[ \varepsilon := \sqrt{p^2 + m^2}, \quad \xi := \varepsilon x, \] (8)
we get
\[ \left[ \left( -\frac{d}{d\xi} + u \right) \left( \frac{d}{d\xi} + u \right) + \frac{du}{d\xi} \right] \psi_{KG} \left( \frac{\xi}{\varepsilon} \right) = \left( \frac{E}{\varepsilon} \right)^2 \psi_{KG} \left( \frac{\xi}{\varepsilon} \right). \] (9)

6 The Dirac equation

The functions in the equation \( \psi \), do not depend on \( t, y, z \); therefore energy \( E \) and transverse momenta \( p_2 \) and \( p_3 \) are constants. By a suitable choice of coordinates (and without loss of generality) one can set \( p_3 = 0 \), and \( p_2 = p \).

So we seek the solution as \( \psi_D(t, x, y, z) = \exp(-iEt + ipy)\psi_D(x) \). Inserting this ansatz in \( \psi \), the resulting equation in terms of the \( \alpha \) and \( \beta \) matrices becomes
\[ \left( \frac{E}{u} \beta + i \frac{\alpha_1}{u} \frac{d}{dx} + \frac{i}{2} \beta \alpha_1 \frac{u'}{u} - p\beta \alpha_2 - m \right) \psi_D = 0. \] (10)

Defining
\[ \hat{\psi} := \sqrt{u}\psi_D, \]
one arrives at
\[ \left( E + i\alpha_1 \frac{d}{dx} - up\alpha_2 - mu\beta \right) \hat{\psi} = 0. \] (11)

In terms of the so called long and short spinors
\[ \phi := \frac{1}{2} (1 + \beta) \hat{\psi}, \quad \tilde{\chi} := \frac{1}{2} (1 - \beta) \hat{\psi}, \] (12)
which are eigenspinors of \( \beta \) with eigenvalues \( +1 \) and \( -1 \), respectively, \( \psi \) becomes
\[ (E - mu) \phi + \left( i\alpha_1 \frac{d}{dx} - up\alpha_2 \right) \tilde{\chi} = 0, \]
\[ \left( i\alpha_1 \frac{d}{dx} - up\alpha_2 \right) \phi + (E + mu) \tilde{\chi} = 0. \]

Defining further \( \chi := \alpha_1 \tilde{\chi} \), which satisfies \( \beta \chi = \chi \), the above equations read as
\[ (E - mu) \phi + \left( i \frac{d}{dx} - up\alpha_2 \alpha_1 \right) \chi = 0, \] (13)
\[ \left( i \frac{d}{dx} - up\alpha_2 \alpha_1 \right) \phi + (E + mu) \chi = 0. \] (14)

The matrix \( S \) defined through
\[ S := -i\beta \alpha_1 \alpha_2, \] (15)
commutes with $\alpha_1$, $\alpha_2$, and $\beta$; and its eigenvalues are $\pm 1$. So one can take $\psi_D$, and hence $\hat{\psi}$, $\hat{\phi}$, $\hat{\chi}$, and $\chi$, eigenspinors of $S$ with the same eigenvalue $\varsigma$, in terms of which equations (13) and (14) can be written as

$$
\begin{bmatrix}
\left( E - \mu u \right) & i \left( \frac{d}{dx} + \varsigma p u \right) \\
\frac{i}{E} \left( \frac{d}{dx} - \varsigma p u \right) & \left( E + \mu u \right)
\end{bmatrix}
\begin{bmatrix}
\phi \\
\chi
\end{bmatrix} = 0,
$$

or

$$
E \begin{bmatrix}
\phi \\
\chi
\end{bmatrix} = \left( \mu u \sigma_3 + \varsigma p u \sigma_2 - i \sigma_1 \frac{d}{dx} \right) \begin{bmatrix}
\phi \\
\chi
\end{bmatrix},
$$

where $\sigma$s are the usual Pauli matrices. Defining

$$
\varepsilon := \sqrt{p^2 + m^2},
\theta := \arctan \frac{p}{m}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},
$$

we have

$$
m\sigma_3 + \varsigma p \sigma_2 = \varepsilon \left( \sigma_3 \cos \theta + \sigma_2 \sin \theta \right) = \varepsilon \exp \left( \frac{i}{2} \sigma_1 \theta \right) \sigma_3 \exp \left( -\frac{i}{2} \sigma_1 \theta \right).
$$

Defining

$$
\begin{bmatrix}
\hat{\phi} \\
\hat{\chi}
\end{bmatrix} := \exp \left( -\frac{i}{2} \sigma_1 \theta \right) \begin{bmatrix}
\phi \\
\chi
\end{bmatrix} = \begin{bmatrix}
\phi \cos \frac{\theta}{2} - i \chi \sin \frac{\theta}{2} \\
- i \phi \sin \frac{\theta}{2} + \chi \cos \frac{\theta}{2}
\end{bmatrix},
$$

one gets

$$
E \begin{bmatrix}
\hat{\phi} \\
\hat{\chi}
\end{bmatrix} = \left( \varepsilon u \sigma_3 - i \sigma_1 \frac{d}{dx} \right) \begin{bmatrix}
\hat{\phi} \\
\hat{\chi}
\end{bmatrix},
$$

which can be written as

$$
\begin{bmatrix}
E - \varepsilon u & i \frac{d}{dx} \\
\frac{i}{E} \frac{d}{dx} & E + \varepsilon u
\end{bmatrix}
\begin{bmatrix}
\hat{\phi} \\
\hat{\chi}
\end{bmatrix} = 0.
$$

Introducing $\phi^\pm := \hat{\phi} \pm i \hat{\chi}$, this equation is transformed to

$$
\begin{bmatrix}
E & \frac{d}{dx} - \varepsilon u \\
\frac{d}{dx} - \varepsilon u & E
\end{bmatrix}
\begin{bmatrix}
\phi^+ \\
\phi^-
\end{bmatrix} = 0,
$$

which, in terms of the variable $\xi = \varepsilon x$, leads to the following second order differential equation for $\phi^-$

$$
\left( -\frac{d}{d\xi} + u \right) \left( \frac{d}{d\xi} + u \right) \phi^- \left( \frac{\xi}{\varepsilon} \right) = \left( \frac{E}{\varepsilon} \right)^2 \phi^- \left( \frac{\xi}{\varepsilon} \right).
$$

$\phi^+$ can be obtained from $\phi^-$ by

$$
\phi^+ \left( \frac{\xi}{\varepsilon} \right) = \frac{\varepsilon}{E} \left( \frac{d}{d\xi} + u \right) \phi^- \left( \frac{\xi}{\varepsilon} \right).
$$

Now turn to the boundary condition. In terms of $\phi$ and $\chi$, eq. (6) is written as

$$
(\phi - i \chi) \big|_{x=0} = 0.
$$
Using (19), we have
\[ \phi - i\chi = \phi^- \cos \frac{\theta}{2} + \phi^+ \sin \frac{\theta}{2}, \]
from which and (23), the boundary condition for \( \phi^- \) follows.
\[ \phi^-(0) \cos \frac{\theta}{2} + \varepsilon \left( \frac{d}{d\xi} + u \right) \phi^-(0) \sin \frac{\theta}{2} = 0. \quad (25) \]
What remains, is to solve equation (22) subject to the boundary condition (25).

7 The special case \( u(x) = 1 + gx \)

7.1 The eigenfunctions
For the special case \( u(x) = 1 + gx \), we introduce
\[ X := \sqrt{\frac{g}{\varepsilon}} \xi + \sqrt{\varepsilon} g, \quad (26) \]
and the Klein-Gordon equation (9) becomes
\[ \left( -\frac{d}{dX} + X \right) \left( \frac{d}{dX} + X \right) f_{KG}(X) = \lambda_{KG}^2 f_{KG}(X), \quad X \geq \sqrt{\varepsilon} g, \quad (27) \]
where
\[ \lambda_{KG}^2 := \frac{E_{KG}^2}{g\varepsilon}, \]
and
\[ f_{KG}(X) := \psi_{KG}(x). \]
The boundary condition reads (see equation (11))
\[ f_{KG} \left( \sqrt{\frac{\varepsilon}{g}} \right) = 0. \quad (28) \]

For the Dirac equation, in this special case, again in terms of the variable \( X \), equation (22) becomes
\[ \left( -\frac{d}{dX} + X \right) \left( \frac{d}{dX} + X \right) f_{D}(X) = \lambda_{D}^2 f_{D}(X), \quad X \geq \sqrt{\varepsilon} g, \quad (29) \]
where
\[ \lambda_{D}^2 := \frac{E_{D}^2}{g\varepsilon}, \]
and
\[ f_{D}(X) := \phi^-(x). \]
The boundary condition (25), is now
\[ f_{D} \left( \sqrt{\frac{\varepsilon}{g}} \right) \cos \frac{\theta}{2} + \frac{1}{\lambda_{D}} \left( \frac{d}{dX} + X \right) f_{D} \left( \sqrt{\frac{\varepsilon}{g}} \right) \sin \frac{\theta}{2} = 0. \quad (30) \]
Comparing (27) and (29), and defining \( \lambda^2 = \lambda_{KG}^2 - 1 \) for the Klein-Gordon equation and \( \lambda^2 = \lambda_D^2 \) for the Dirac equation, we see that both of them are of the same form:

\[
\left( -\frac{d}{dX} + X \right) \left( \frac{d}{dX} + X \right) f(X) = \lambda^2 f(X),
\]

but subject to different boundary conditions (28) and (30).

Note that this final equation is the same as the Schrödinger equation for a one-dimensional simple harmonic oscillator. The difference is that for the simple harmonic oscillator, \(-\infty < X < +\infty\); while in our case \( \sqrt{\varepsilon/g} < X < +\infty \), and that we have the boundary condition (28) or (30) for \( X \to \sqrt{\varepsilon/g} \).

To solve (31), we define \( h(X) \) as

\[
f(X) = h(X) \exp(-\frac{1}{2}X^2),
\]

and obtain

\[
\left( 2X - \frac{d}{dX} \right) \frac{dh}{dX} = \lambda^2 h.
\]

Now we seek a power-series solution for \( h(X) \):

\[
h(X) = \sum_{0}^{\infty} a_n X^n.
\]

Putting this in (33), one obtains

\[
a_{2k} = \frac{4^k \Gamma \left( k - \frac{\lambda^2}{4} \right)}{(2k)! \Gamma \left( -\frac{\lambda^2}{4} \right)} a_0,
\]

\[
a_{2k+1} = \frac{4^k \Gamma \left( k + \frac{1}{2} - \frac{\lambda^2}{4} \right)}{(2k+1)! \Gamma \left( \frac{1}{2} - \frac{\lambda^2}{4} \right)} a_1,
\]

and from that the following two solutions.

\[
h_0(X) = \frac{a_0}{\Gamma \left( -\frac{\lambda^2}{4} \right)} \sum_{k=0}^{\infty} \frac{(2X)^{2k} \Gamma \left( k - \frac{\lambda^2}{4} \right)}{(2k)!},
\]

\[
h_1(X) = \frac{a_1}{2\Gamma \left( \frac{1}{2} - \frac{\lambda^2}{4} \right)} \sum_{k=0}^{\infty} \frac{(2X)^{2k+1} \Gamma \left( k + \frac{1}{2} - \frac{\lambda^2}{4} \right)}{(2k+1)!}.
\]

We have to find a linear combination of these two functions, which remains finite as \( X \to \infty \). To do so, we define the function

\[
S_\alpha(X) := \sum_{k=0}^{\infty} \frac{(2X)^{2k+\alpha} \Gamma \left( k + \frac{1}{2} + \frac{\lambda^2}{4} \right)}{(2k+\alpha)!},
\]

It is seen that \( h_0 \) and \( h_1 \) are proportional to \( S_0 \) and \( S_1 \), respectively. Using a steepest-descent analysis to obtain the large-\( X \) behavior of \( S_\alpha \), it is seen that
this behavior is in fact independent of \( \alpha \). This comes from the fact that if one replaces the summation over \( k \) with an integration, and change the variable \( k + \alpha \) into \( k \), then the dependence of \( S \) on \( \alpha \) comes solely from the lower bound of the integration region. But this bound is unimportant, since the major part of the sum comes from large \( k \)'s. In fact, one can perform the steepest descent analysis and find

\[
S_\alpha(X) \sim X^{-1-\frac{\lambda^2}{4}} \exp(X^2 + 1 + \frac{\lambda^2}{4}).
\]

So, \( h_0(X) + h_1(X) \) remains finite as \( X \to \infty \), provided

\[
\frac{a_1}{2\Gamma\left(\frac{1}{2} - \frac{\lambda^2}{4}\right)} = -\frac{a_0}{\Gamma\left(-\frac{\lambda^2}{4}\right)} =: b.
\]

So the unique (up to normalization) normalizable function which solves (31), is

\[
h(X) := b \sum_{n=0}^{\infty} \frac{(-2X)^n \Gamma\left(\frac{n}{2} - \frac{\lambda^2}{4}\right)}{n!}.
\] (38)

### 7.2 Comparing the energy eigenvalues

To compare the energies of three different Hamiltonians, namely the Schrödinger equation, the Klein-Gordon equation (27), and the Dirac equation (29), we reintroduce the physical constants previously chosen to be equal to 1. One obtains

\[
E_{\text{Sch}} = \frac{p^2}{2m} - 2^{-1/3} \lambda_{\text{Sch}} \left(\frac{\hbar g}{mc^3}\right)^{2/3} mc^2,
\]

\[
E_{\text{KG}} = \lambda_{\text{KG}} \left(\frac{\hbar g}{mc^3}\right)^{1/2} mc^2 \left(1 + \frac{p^2}{m^2c^2}\right)^{1/4},
\]

\[
E_{\text{D}} = \lambda_{\text{D}} \left(\frac{\hbar g}{mc^3}\right)^{1/2} mc^2 \left(1 + \frac{p^2}{m^2c^2}\right)^{1/4}.
\]

To compare \( \lambda_{\text{D}} \) and \( \lambda_{\text{KG}} \), we first note that if \( p = 0 \), then \( \theta = 0 \) (see equation (18)). Therefore, the two boundary conditions (28) and (30) become the same, and eigenvalue \( \lambda^2 \) in (31), becomes the same for the Klein-Gordon and the Dirac case. But \( \lambda^2 \) is equal to \( \lambda_{\text{KG}}^2 - 1 \), and \( \lambda_{\text{D}}^2 \). So,

\[
\lambda_{\text{D}}^2 = \lambda_{\text{KG}}^2 - 1, \quad \text{for } p = 0,
\]

or

\[
E_{\text{D}}^2 = E_{\text{KG}}^2 - mg\hbar c, \quad \text{for } p = 0.
\] (39)

This is an exact result which determines the effect of spin on the gravitational interaction of relativistic particles. A fermion and a boson with same masses, have different quantum energies when fall vertically in a constant gravitational field.

For \( p \neq 0 \), the relation between the eigenvalues is more complicated, because although the differential equations are the same, the boundary conditions are
different. In this case, it is possible to find approximate relations for $\lambda_{KG}$ and $\lambda_{D}$, for large values of them, and compare them in this region.

A WKB analysis of the wave function $h(X)$, performed in appendix A, shows that

$$h(X) \sim \cos \left( \frac{\pi \lambda_{D}^{2}}{4} - \lambda X \right).$$

(40)

Inserting (32) in (30), the boundary condition on $h(X)$ for Dirac particles reads

$$\left[ h(X) \cos \frac{\theta}{2} + \frac{1}{\lambda_{D}} \frac{dh}{dX} \sin \frac{\theta}{2} \right] \bigg|_{X=\sqrt{\varepsilon/g}} = 0,$$

which, using (40), leads to

$$\cos \left( \frac{\pi \lambda_{D}^{2}}{4} - \lambda_{D} \sqrt{\varepsilon/g} \right) = 0,$$

or

$$\frac{\pi \lambda_{D}^{2}}{4} - \lambda_{D} \sqrt{\varepsilon/g} = \left( n + \frac{1}{2} \right) \pi.$$

(41)

For the Klein-Gordon equation, from (28) and (32), it is seen that $h(\sqrt{\varepsilon/g}) = 0$. Using (40), and remembering $\lambda_{2} = \lambda_{KG}^{2} - 1$, we find

$$\frac{\pi \lambda_{KG}^{2}}{4} - \frac{\pi}{4} - \lambda_{KG} \sqrt{\varepsilon/g} = \left( n + \frac{1}{2} \right) \pi,$$

in which $O(1/\lambda_{KG})$ terms have been ignored. Comparing (41) and (42), results in

$$\lambda_{D}^{2} = \lambda_{KG}^{2} - 1 + \frac{2\theta}{\pi},$$

or

$$E_{D}^{2} = E_{KG}^{2} - \sqrt{p^{2} + m^{2}c^{2}} \cos \left( 1 - \frac{2}{\pi} \arctan \frac{p c}{m} \right).$$

(43)

A The asymptotic behaviour of the function $h(X)$

To obtain the asymptotic behaviour of $h(X)$, eq.(38), for $\lambda \gg 1$ and finite $X$, we first write (38) as

$$h(X) = b \sum_{k=0}^{\infty} \frac{(2X)^{2k} \Gamma \left( k - \frac{\lambda_{D}^{2}}{4} \right)}{(2k)!} - b \sum_{k=0}^{\infty} \frac{(2X)^{2k+1} \Gamma \left( k + \frac{1}{2} - \frac{\lambda_{D}^{2}}{4} \right)}{(2k+1)!}.$$

(44)

Using $\Gamma(p)\Gamma(1-p) = \pi/(\sin \pi p)$, we have

$$\Gamma \left( k - \frac{\lambda_{D}^{2}}{4} \right) = \frac{\pi}{\sin \left( \pi \left( k - \frac{\lambda_{D}^{2}}{4} \right) \right)} = -\frac{(-1)^{k} \pi}{\sin \left( \pi \frac{\lambda_{D}^{2}}{4} - k + 1 \right)}. $$

(45)

As $X$ is finite, large $k$'s have negligible contributions in the power series for $h(X)$. So we can use the Stirling’s formula $\Gamma(x+1) \sim \sqrt{2\pi x^{x+1/2}}e^{-x}$, to
obtain
\[
\Gamma\left(\frac{\lambda^2}{4} - k + 1\right) = \sqrt{2\pi} \left(\frac{\lambda^2}{4}\right)^{\frac{\lambda^2}{4} - k + \frac{1}{2}} \left(1 - \frac{k}{\lambda^2/4}\right)^{\frac{\lambda^2}{4} - k + \frac{1}{2}} e^{k - (\lambda^2/4)}
\]
\[\sim \sqrt{2\pi} \left(\frac{\lambda^2}{4}\right)^{\frac{\lambda^2}{4} - k + \frac{1}{2}} e^{-\lambda^2/4}.
\]

Therefore (45) leads to
\[
\Gamma\left(k - \frac{\lambda^2}{4}\right) \sim -\frac{(-1)^k \pi}{\sqrt{2\pi \sin (\pi \lambda^2/4)}} \left(\frac{\lambda^2}{4}\right)^{k - (1/2)} \left(\frac{4e}{\lambda^2}\right)^{\lambda^2/4}. \quad (46)
\]

A similar argument shows that
\[
\Gamma\left(k + \frac{1}{2} - \frac{\lambda^2}{4}\right) \sim \frac{(-1)^k \pi}{\sqrt{2\pi \cos (\pi \lambda^2/4)}} \left(\frac{\lambda^2}{4}\right)^{k} \left(\frac{4e}{\lambda^2}\right)^{\lambda^2/4}. \quad (47)
\]

Inserting (46) and (47) in (44), one obtains
\[
h(X) \sim -b \sqrt{\frac{\pi}{2}} \left(\frac{\lambda^2}{4}\right)^{-1/2} \left(\frac{4e}{\lambda^2}\right)^{\lambda^2/4}
\times \left[\frac{1}{\sin (\pi \lambda^2/4)} \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda X)^{2k}}{(2k)!} + \frac{1}{\cos (\pi \lambda^2/4)} \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda X)^{2k+1}}{(2k + 1)!}\right]
\]
\[= -\frac{2b\sqrt{2\pi}}{\lambda \sin (\pi \lambda^2/2)} \left(\frac{4e}{\lambda^2}\right)^{\lambda^2/4} \cos \left(\pi \frac{\lambda^2}{4} - \lambda X\right). \quad (48)
\]

This is nothing, but the leading-order result of a WKB analysis.
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