GOUSSAROV-HABIRO THEORY FOR STRING LINKS AND
THE MILNOR-JOHNSON CORRESPONDENCE

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Abstract. We study the Goussarov-Habiro finite type invariants theory for framed string links in homology balls. Their degree 1 invariants are computed: they are given by Milnor’s triple linking numbers, the mod 2 reduction of the Sato-Levine invariant, Arf and Rochlin’s µ invariant. These invariants are seen to be naturally related to invariants of homology cylinders through the so-called Milnor-Johnson correspondence: in particular, an analogue of the Birman-Craggs homomorphism for string links is computed. The relation with Vassiliev theory is studied.

1. Motivations

In the late 90’s, M. Goussarov and K. Habiro independently developed a finite type invariant theory for compact oriented 3-manifolds. The theory makes use of an efficient surgical calculus machinery called calculus of claspers [GH, GGP, H]. In particular the $Y_k$-equivalence, an equivalence relation for 3-manifolds arising from calculus of claspers, plays an important role in the understanding of the invariants. Though it is also well-defined for manifolds with links, this aspect of the theory remains so far almost non-existing in the literature. In the present paper, we study the case of framed $n$-string links in homology balls. For $n = 1$, this is equivalent to studying homology spheres with framed knots. String links play an important role in the study of knots and links [HL] and happen to have nice properties in the theory of claspers. Here, we compute explicitly the degree 1 invariants (in the Goussarov-Habiro sense) for framed string links in homology balls, using some versions of classical invariants, such as Milnor numbers, Sato-Levine, Arf and Rochlin invariants. This is the outcome of the characterization of the $Y_2$-equivalence relation for these objects.

String links are very closely related to homology cylinders [GL, L]. Homology cylinders over a compact connected oriented surface $\Sigma$ can be seen as a generalization of the Torelli group of $\Sigma$. G. Massuyeau and the author explicitly computed their degree 1 invariants [MM]; they are given by the natural extensions of the first Johnson homomorphism and the Birman-Craggs homomorphism, initially defined for the Torelli group [BC, J1, J2]. On the other hand, N. Habegger showed in [H] how homology cylinders are geometrically related to string links in homology balls, such that the extension of the first Johnson homomorphism agrees with Milnor’s triple linking numbers. So the problem which naturally arises is to compute, likewise, the analogue of the Birman-Craggs homomorphism for this so-called Milnor-Johnson correspondence. Our computation of degree 1 invariants of string links in homology balls allows us to answer this question.

Like Goussarov-Habiro theory, the Vassiliev theory for (classical) string links can be defined using claspers. This viewpoint allows us to compare both theories. More precisely, we can relate the computation of degree 1 invariants of string links in homology balls to analogous results obtained by the author on Vassiliev invariants [M]. We also consider the link case, where a similar statement exists [TY].
The paper is organized as follows. We will begin with some necessary preliminary material on clasper theory. We compute in §3 the Goussarov-Habiro degree 1 invariants for framed string links in homology balls. §3.3 is devoted to the proof of this result, and §3.2 contains a precise definition of the invariants it involves. In §4, we introduce homology cylinders and study the Milnor-Johnson correspondence. The last section deals with Vassiliev invariants.

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2. Preliminaries

Throughout this paper, all 3-manifolds will be supposed to be compact, connected and oriented.

2.1. A brief review of the Goussarov-Habiro theory. Let us briefly recall from \[H, GGP, G1\] the basic notions of clasper theory for 3-manifolds with links.

**Definition 1.** Let \( \gamma \) be a \( n \)-component link in a 3-manifold \( M \). A clasper \( G \) for \((M, \gamma)\) is the embedding

\[
G : F \longrightarrow M
\]

of a surface \( F \) which is the thickening of a (non-necessarily connected) uni-trivalent graph having a copy of \( S^1 \) attached to each of its univalent vertices. \( G \) is disjoint from the link \( \gamma \).

The (thickened) circles are called the leaves of \( G \), the trivalent vertices are called the nodes of \( G \) and we still call the thickened edges of the graph the edges of \( G \).

In particular, a tree clasper is a connected clasper obtained from the thickening of a simply connected unitrivalent graph (with circles attached). The degree of a clasper \( G \) is the minimal number of nodes of its connected components.

A clasper \( G \) for \((M, \gamma)\) is the instruction for a modification on this pair. There is indeed a precise procedure to construct, in a regular neighbourhood \( N(G) \) of the clasper, an associated framed link \( L_G \). The *surgery along the clasper* \( G \) is defined to be surgery along \( L_G \). Though the procedure for the construction of \( L_G \) will not be explained here, it is well illustrated by the two examples of Figure 2.1.

![Figure 2.1. A degree 1 and a degree 2 clasper and the associated framed links in their regular neighbourhoods.](image)

We respectively call these two particular types of claspers \( Y \)-graphs and \( H \)-graphs. We denote by \((M, \gamma)_G = (M_G, \gamma_G)\) the result of a surgery move on \((M, \gamma)\) along a

\[^1\]Here and throughout this paper, blackboard framing convention is used.
clasper $G$:
\[
\begin{array}{ll}
\cdot & M_G = (M \setminus \text{int}(N(G_0))) \cup \partial N(G_0)_{L_G}, \\
\cdot & \gamma_G is the link in $M_G$ defined by $\gamma \subset M \setminus \text{int}(N(G_0)) \subset M_G$.
\end{array}
\]

**Definition 2.** Let $k \geq 1$ be an integer, and $\gamma$ be a link in a 3-manifold $M$. A surgery move on $(M, \gamma)$ along a connected clasper $G$ of degree $k$ is called a $Y_k$-move.

The $Y_k$-equivalence, denoted by $\sim_{Y_k}$, is the equivalence relation on 3-manifolds with links generated by the $Y_k$-moves and orientation-preserving diffeomorphisms (with respect to the boundary).

Note that $Y_1$-moves originally appear in [M] under the name of Borromean surgery (as Fig. 2.1 suggests). The next proposition outlines a couple of key facts about this equivalence relation.

**Proposition 3.**

1. Tree claspers do suffice to define the $Y_k$-equivalence.
2. If $1 \leq k \leq n$, the $Y_n$-equivalence relation implies the $Y_k$-equivalence.

We conclude this section with the definition of the Goussarov-Habiro theory, based on the notion of clasper. Consider a link $\gamma_0$ in a 3-manifold $M_0$, and the $Y_1$-equivalence class $\mathcal{M}_0$ of $(M_0, \gamma_0)$.

**Definition 4.** Let $A$ be an Abelian group, and $k \geq 0$ be an integer. A finite type invariant of degree $k$ (in the Goussarov-Habiro sense) on $\mathcal{M}_0$ is a map $f : \mathcal{M}_0 \to A$ such that, for all $(M, \gamma) \in \mathcal{M}_0$ and all family $F = \{G_1, \ldots, G_{k+1}\}$ of $(k+1)$ disjoint $Y$-graphs for $(M, \gamma)$, the following equality holds:
\[
\sum_{F' \subseteq F} (-1)^{|F'|} f((M, \gamma)_{F'}) = 0.
\]

2.2. Vassiliev theory using claspers. Another aspect of the theory of claspers is that it allows to redefine and study Vassiliev invariants of knots and links in a fixed manifold $[H]$ [G2]. Here, for simplicity, we recall the definitions for the case of knots in $S^3$. For more about Vassiliev invariants, see [H].

**Definition 5.** Let $K$ be a knot in $S^3$. A clasper $G$ for $K$ is the embedding
\[
G : F \rightarrow S^3
\]
of a surface $F$ which is the planar thickening of a uni-trivalent tree (a graph without loops). The (thickened) 1-vertices are called the disk-leaves of $G$, and the thickened trivalent vertices and edges of the graph are still called nodes and edges respectively. $K$ is disjoint from $G$, except from the disk-leaves which it may intersect transversely once.

The $C$-degree of a connected clasper $G$ is the number of nodes plus 1.

Again, a clasper $G$ for $K$ is the instruction for a surgical modification: it maps $K$ to a new knot $K_G$ in $S^3$. Examples are given for low $C$-degrees in Fig. 2.2.

**Definition 6.** Let $k \geq 1$ be an integer, and $K$ be a knot in $S^3$. A surgery move on $K$ along a connected $C$-degree $k$ clasper $G$ is called a $C_k$-move.

The $C_k$-equivalence, denoted by $\sim_{C_k}$, is the equivalence relation on knots generated by the $C_k$-moves and isotopies.

As in Prop 3(2), the $C_n$-equivalence relation implies the $C_k$-equivalence if $1 \leq k \leq n$.

**Remark 7.**

1. Note that a $C_1$-move is just a crossing change. As [MN] Fig. 2.2 shows, a $C_2$-move is equivalent to a $\Delta$-move. Moreover, a $C_3$-move is equivalent to a clasp-pass move (see [H] for a definition) [H].
Figure 2.2. A $C_1$-move and a $C_2$-move.

(2). The $C_{k+1}$-equivalence implies the $Y_k$-equivalence, for all $K \geq 1$. More precisely, a $C_{k+1}$-move can be regarded as a special case of $Y_k$-move, where the leaves of the degree $k$ clasper are (0-framed) copies of the meridian of the knot.

A $C_1$-move being equivalent to a crossing change, we can reformulate the notion of Vassiliev invariant in terms of claspers.

**Definition 8.** Let $A$ be an Abelian group, and $k \geq 0$ be an integer. An $A$-valued knot invariant $v$ is a Vassiliev invariant of degree $k$ if, for all knot $K$ and all family $F = \{C_1, \ldots, C_{k+1}\}$ of $(k+1)$ disjoint $C$-degree $1$ claspers for $K$, the following equality holds:

$$\sum_{F' \subseteq F} (-1)^{|F'|} v(K_{F'}) = 0.$$ 

3. **Goussarov-Habiro theory for string links in homology balls.**

Here and throughout the paper, unless said otherwise, by homology we mean integral homology. Thus by homology ball we mean a compact oriented 3-manifold whose integral homology groups are isomorphic to those of the 3-ball.

3.1. **String links in homology balls.**

3.1.1. **Definition and properties.** Let $D^2$ be the standard two-dimensional disk, and $x_1, \ldots, x_n$ be $n$ marked points in the interior of $D^2$.

**Definition 9.** An $n$-component string link in a homology ball $M$, also called $n$-string link, is a proper, smooth embedding

$$\sigma : \bigsqcup_{i=1}^n I_i \longrightarrow M$$

of $n$ disjoint copies $I_i$ of the unit interval such that, for each $i$, the image $\sigma_i$ of $I_i$ runs from $(x_i, 0)$ to $(x_i, 1)$ via the identification $\partial M = \partial (D^2 \times I)$.

$\sigma_i$ is called the $i^{th}$ string of $\sigma$. It is equipped with an (upward) orientation induced by the natural orientation of $I$.

A framed $n$-string link in $M$ is a string link equipped with an isotopy class of non-singular sections of its normal bundle, whose restriction to the boundary is fixed.

We denote by $\mathcal{SL}^{hb}(n)$ the set of framed $n$-string links in homology balls, considered up to diffeomorphisms relative to the boundary (that is, up to diffeomorphisms whose restriction to the boundary is the identity).

Given two elements $(M, \sigma)$ and $(M', \sigma')$ of $\mathcal{SL}^{hb}(n)$, we can define their product as follows. Denote by $M \cdot M'$ the homology ball obtained by identifying $\Sigma \times \{1\} \subset \partial M$ and $\Sigma \times \{0\} \subset \partial M'$. $(M, \sigma) \cdot (M', \sigma')$ is defined by stacking $\sigma'$ over $\sigma$ in $M \cdot M'$.

This product induces a monoid structure on $\mathcal{SL}^{hb}(n)$, with $(D^2 \times I, 1_n)$ as unit element. Here $1_n$ is the trivial $n$-string link.
Figure 3.1. Two 2-string links in $D^2 \times I$, and their product.

**Notations** 10. Throughout this paper, the notation $1_{D^2}$ will be often used for the product $D^2 \times I$.

$D^2_n$ will denote the $n$-punctured disk $D^2 \setminus \{x_1, \ldots, x_n\}$. $H := H_1(D^2_n, \mathbb{Z})$ will denote its first integral homology group, and $H_{(2)} := H_1(D^2_n, \mathbb{Z}_2)$.

$B = \{e_1, \ldots, e_n\}$ denotes the basis of $H$ induced by the $n$ curves $h_1, h_2, \ldots, h_n$ of $D^2_n$ shown in Fig. 3.2.

Similarly, $B_{(2)} = \{\overline{e}_1, \ldots, \overline{e}_n\}$ is the associated basis of $H_{(2)}$.

Let $(M, \sigma) \in \mathcal{SL}^{hb}(n)$. We denote by $\hat{M}$ the homology sphere obtained by pasting a copy of $(D^2 \times I)$ along its boundary, via the identification $\partial M = \partial (D^2 \times I)$.

At the string links level, suitably pasting a copy of $(1_{D^2}, 1_n)$ along the boundary of $M$ maps $\sigma \subset M$ to a framed oriented $n$-component link $\hat{\sigma} \subset \hat{M}$. $(\hat{M}, \hat{\sigma})$ is called the closure of $(M, \sigma)$. In particular, for $M = 1_{D^2}$, it is the usual notion of closure for $\sigma$ as defined in [HL].

Given an element $(\hat{M}, \hat{\sigma})$ of $\mathcal{SL}^{hb}(n)$, let us denote by $T(\sigma)$ a tubular neighbourhood of $\sigma$. We denote by $M^\sigma := M \setminus T(\sigma)$ the exterior of the string link: the boundary of $M^\sigma$ is identified with $\partial (D^2_n \times I)$. Let $i_\epsilon \ (\epsilon = 0, 1)$ be the embeddings

\[ i_\epsilon : D^2_n \to D^2_n \times \{\epsilon\} \subset M^\sigma. \]

We need the following classical result of Stallings.

**Theorem 11.** [St Thm. 5.1] Let $f : A \to B$ be a map between connected CW-complexes that induces an isomorphism on the first homology groups and a surjective homomorphism on the second homology groups. Then for all $k \geq 2$, $f$ induces an isomorphism at the level of each nilpotent quotient of the fundamental group

\[ f_k : \pi_1(A) / (\pi_1(A))_k \cong \pi_1(B) / (\pi_1(B))_k, \]

where, for any group $G$, $G_k$ is the $k^{th}$ term of its lower central series.

So by a standard Mayer-Vietoris calculation and the above theorem, the map $i_\epsilon \ (\epsilon = 0, 1)$ induces an isomorphism

\[ (i_\epsilon)_k : \frac{\pi_1(D^2_n)}{\pi_1(D^2_n)_k} \cong \frac{F}{F_k} \cong \frac{\pi_1(M^\sigma)}{\pi_1(M^\sigma)_k}. \]
for each $k \geq 2$, where $F$ stands for the free group on $n$ generators. So any element $\sigma$ of $\mathcal{SL}^{hb}(n)$ induces an automorphism of $F/F_{k+1}$, called its $k^{th}$ Artin representation, defined by $A_k(\sigma) = (i_1)^{-1}_{k-1} \circ (i_0)_{k+1}$.

Actually, $A_k(\sigma)$ conjugates each generator $x_i$ of $F/F_{k+1}$ by $\lambda_i$, the $i^{th}$ longitude of $\sigma$ mod $F_{k+1}$: the framing on $\sigma$ defines a curve in $M^\sigma$ parallel to $\sigma_i$, which determines an element of $\pi_1(M^\sigma)$. The image in $F/F_{k+1}$ of this element by $(i_1)^{-1}_{k+1}$ is $\lambda_i$.

Denote by $\mathcal{SL}^{hb}(n)[k] := \text{Ker}A_k$ the submonoid of all $n$-string links inducing the identity on $F/F_{k+1}$. Note that $\mathcal{SL}^{hb}(n) = \mathcal{SL}^{hb}(n)[1]$ and that $(M,\sigma) \in \mathcal{SL}^{hb}(n)[2]$ if and only if $\sigma$ has null-homologous longitudes, that is, vanishing framings and linking numbers.

**Remark 14.** The next result characterizes the degree 1 Goussarov-Habiro finite type invariants for string links in homology balls.

**Theorem 13.** Let $(M, \sigma)$ and $(M', \sigma')$ be two $n$-string links in homology balls with vanishing framings and linking numbers (i.e. two elements of $\mathcal{SL}^{hb}(n)$). The following assertions are equivalent:

(a) $(M, \sigma)$ and $(M', \sigma')$ are $Y_2$-equivalent;
(b) $(M, \sigma)$ and $(M', \sigma')$ are not distinguished by degree 1 Goussarov-Habiro finite type invariants;
(c) $(M, \sigma)$ and $(M', \sigma')$ are not distinguished by Milnor’s triple linking numbers, nor the mod 2 reduction of the Sato-Levine invariant, the Arf invariant and Rochlin’s $\mu$-invariant.

See [3.3] for the definitions of the above-mentioned invariants.

**Remark 14.** When considering higher degrees, the implication (a) $\Rightarrow$ (b) remains true (as well as for knots and links in homology spheres). The converse implication is also true when $n = 1$, that is for knots in homology spheres (see [1]), and it is conjectural for string links with $n > 1$ components. This conjecture is to be compared with [1] Conj. 6.13, for Vassiliev invariants of (classical) string links (see also [13]).

The proof of the theorem is given in [3.3] It consists in computing the Abelian group $\mathcal{SL}^{hb}(n)$, in a graphical way. More precisely, we will define in [3.3.1] a Y-shaped diagrams space $A_1(P_n)$ and a surjective surgery map $A_1(P_n) \xrightarrow{\psi} \mathcal{SL}^{hb}(n)$. We will see that $\psi$ turns out to be an isomorphism, with inverse induced by the invariants listed in Thm. [13].
3.1.3. $Y_1$-equivalence for string links: proof of Proposition \[2\] We first prove the inclusion $\mathcal{S}L^\text{hb}_1(n) \subset \mathcal{S}L^\text{hb}_1(n)[2]$; any element of $\mathcal{S}L^\text{hb}_1(n)$ obtained from $(1_D^2, 1_n)$ by a finite sequence of $Y_1$-moves has null homologous longitudes. It suffices to show that, if $(M_2, \sigma_2)$ is obtained from $(M_1, \sigma_1) \in \mathcal{S}L^\text{hb}_1(n)$ by surgery along a degree 1 clasper $G$, these elements have homologous longitudes. Denote by $M_i^\sigma$, the exterior of the string links $(i = 1, 2)$. We have
\[ M_2^\sigma \cong (M_1^\sigma) \setminus \text{int}(N(G)) \cup_{j_1 \circ \partial H_3} (H_3), \]
where $j : H_3 \hookrightarrow 1_D^2 \setminus 1_n$ is the embedding of a genus 3 handlebody onto a regular neighbourhood $N(G)$ of $G$, and where $h$ is an element of the Torelli group of $\Sigma_3 = \partial H_3$ – see [MM1] Lem. 1 for an explicit description of this diffeomorphism. $h$ induces the identity on $\pi_1(\Sigma_3)/\pi_1(\Sigma_3)_2$: it follows (by a Van Kampen type argument) that there is an isomorphism
\[
\frac{\pi_1(M_1^\sigma)}{\left(\pi_1(M_1^\sigma)\right)_2} \cong \frac{\pi_1(M_2^\sigma)}{\left(\pi_1(M_2^\sigma)\right)_2},
\]
which is compatible with the maps $i_\varepsilon : \varepsilon = 0, 1$. The assertion follows.

The other inclusion is essentially due to N. Habegger [Ha]. First, recall that every homology sphere is $Y_1$-equivalent to the 3-sphere $S^3$ [Mi-H]: likewise every homology ball is $Y_1$-equivalent to $B^3 \cong D^2 \times I$. So it suffices to show that a framed string link $\sigma$ in $D^2 \times I$ whose framings and linking numbers are all zero is $Y_1$-equivalent to $(1_D^2, 1_n)$. By a sequence of connected sums on $\sigma$ with copies of the 0-framed Borromean link, we can furthermore suppose that all Milnor’s triple linking numbers are zero: such connected sums are nothing else but $Y_1$-moves (each leaf of the clasper being a meridian of the string on which connected sum is performed). By \[2\] Thm. D), $\sigma$ is thus surgery equivalent to the trivial string link, that is, $\sigma$ is obtained from $1_n$ by a sequence of surgeries on trivial $(\pm 1)$-framed knots $K_i$ in the exterior of $\sigma$, these knots having vanishing linking numbers with $\sigma$. The union $\cup_i K_i$ is a $(\pm 1)$-framed boundary link: surgery on such a link is known to be equivalent to a sequence of $Y_1$-surgeries [Ha Cor. 6.2].

3.2. Classical invariants for string links in homology balls.

3.2.1. Rochlin’s $\mu$-invariant. Let $M$ be a closed 3-manifold endowed with a spin structure $s$, and let $(W, S)$ be a compact spin 4-manifold spin-bounded by $(M, s)$ (that is, $\partial W = M$ and $S$ coincides with $s$ on $M$). Then, the modulo 16 signature $\sigma(W)$ of $W$ is a well-defined closed spin 3-manifolds invariant $R(M, s)$, called the Rochlin invariant of $M$. In the case of homology spheres, there is a unique spin structure $s_0$, and $R(M, s_0)$ is divisible by 8:
\[
\mu(M) := \frac{R(M, s_0)}{8} \in \mathbb{Z}_2
\]
is an invariant of homology spheres called Rochlin’s $\mu$-invariant.

For elements $(M, \sigma)$ of $\mathcal{S}L^\text{hb}_1(n)$, we set
\[
R(M, \sigma) := \mu(\hat{M}),
\]
where the homology sphere $\hat{M}$ is the closure of $M$ as defined in [MM1]. The following result of G. Massuyeau implies that the restriction of $R$ to $\mathcal{S}L^\text{hb}_1(n)$ factors to a homomorphism of Abelian groups
\[
R : \mathcal{S}L^\text{hb}_1(n) \longrightarrow \mathbb{Z}_2.
\]

**Proposition 15.** [M1 Cor. 1] Rochlin’s invariant is a degree 1 finite type invariant (in the Goussarov-Habiro sense) of integral homology spheres.
3.2.2. Milnor Invariants. Let \( \sigma \) be an \( n \)-string link in a homology ball \( M \). Recall from [33] that \( F \) is the free group on \( n \) generators, and that \( F_k \) is the \( k^{th} \) term of its lower central series. Recall also that \( \lambda_i \in F/F_{k+1} \) denotes the \( i^{th} \) longitude of \( \sigma \) mod \( F_{k+1} \).

Denote by \( P(n) \) the ring of power series in the non-commutative variables \( X_1, ..., X_n \). The Magnus expansion [MKS] \( F \longrightarrow P(n) \) is a group homomorphism which maps each generator \( x_i \) of \( F \) to \( 1 + X_i \).

**Definition 16.** The Milnor's \( \mu \)-invariant of length \( l \), \( \mu_{i_1 \ldots i_l} \) of \( \sigma \) is the coefficient of the monomial \( X_{i_1} \ldots X_{i_{l-1}} \) in the Magnus expansion of the longitude \( \lambda_{i_l} \in F/F_k \) for a certain \( k \geq l \).

For example, Milnor’s invariants of length 2 are just the linking number \( s \). Here, we deal with Milnor’s invariants of length 3, also called Milnor’s triple linking number. The following proposition-definition follows from Lemma 19 below.

**Proposition 17.** For all \( i < j < k \in \{1, ..., n\} \), there is a well-defined homomorphism of Abelian groups

\[
\mathbb{S}L^b_1(n) \xrightarrow{\mu_{i,j,k}} \mathbb{Z}
\]

induced by Milnor’s triple linking number.

**Remark 18.** In general, Milnor’s triple linking numbers are not additive on \( \mathbb{S}L(n) \). The homomorphism defect is given by linking numbers, so it vanishes for elements of \( \mathbb{S}L^b_1(n) \).

**Lemma 19.** Let \((M, \sigma)\) be a framed string link in a homology ball. Let also \( G \) be a degree 2 clasper in \( M \) disjoint from \( \sigma \) and let \((M_G, \sigma_G)\) be the result of the surgery along \( G \). Then, there exists an isomorphism

\[
\frac{\pi_1(M^\sigma)}{[\pi_1(M^\sigma)]_3} \cong \frac{\pi_1(M_G^\sigma)}{[\pi_1(M_G^\sigma)]_3}
\]

compatible with the inclusions \( i_{\varepsilon} \); \( \varepsilon = 0, 1 \).

**Proof:** The reader is refered to the proof of [MM Lem. 3.13]. \( \square \)

3.2.3. The Arf Invariant. Let \( K \) be a knot in a homology sphere \( M \), and \( S \) be a Seifert surface for \( K \) of genus \( g \). Denote by \( \cdot \) the mod 2 reduction of the homological intersection form on \( H_1(S, \mathbb{Z}_2) \). Let \( \delta_2 : H_1(S, \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2 \) be the map defined by \( \delta_2(\alpha) = lk(\alpha, \alpha^+) \text{(mod 2)} \), where \( \alpha^+ \) is a parallel copy of \( \alpha \) in the positive normal sense of \( S \) (for a fixed orientation of \( M \)). \( \delta_2 \) is a quadratic form with \( \cdot \) as associated bilinear form: the Arf invariant of the knot \( K \) is the Arf invariant of \( \delta_2 \), that is, for a given symplectic basis \( \{a_1, b_1, ..., a_g, b_g\} \) for \( \cdot \).

\[
Arf(K) = Arf(\delta_2) = \sum_{i=1}^{g} \delta_2(\alpha_i)\delta_2(\beta_i).
\]

**Remark 20.** The fact that the Arf invariant is still well-defined for knots in homology spheres essentially follows from the following fact (see for example [GT] for a proof): two Seifert surfaces \( S_0 \) and \( S_1 \) for a knot \( K \) in an homology sphere \( M \) are related by a sequence of isotopies, additions and removals of tubes \( S^1 \times I \). Indeed, as we will see in the proof of Prop. 22, such tubes do not contribute to the Arf invariant.

For elements of \( \mathbb{S}L^b_1(n) \), the Arf invariant is defined in the obvious way: for an integer \( 1 \leq i \leq n \), denote by \( a_i(M, \sigma) \) the Arf invariant of \( \sigma_i \), the \( i^{th} \) component of the link \( \tilde{\sigma} \in \tilde{M} \). We clearly have the following proposition-definition:
Proposition 21. For any integer $1 \leq i \leq n$, the map $a_i : SL_{hb_1}^b(n) \rightarrow \mathbb{Z}_2$ is a homomorphism of monoids, called the $i^{th}$ Arf invariant of $(M, \sigma)$.

Further, this invariant happens to behave well under a $Y_2$-move.

Proposition 22. The Arf invariant of knots in homology spheres is invariant under a $Y_2$-move. As a consequence, for any $1 \leq i \leq n$, the $i^{th}$ Arf invariant of string links in homology balls factors through a homomorphism of Abelian groups $a_i : SL_{hb_1}^b(n) \rightarrow \mathbb{Z}_2$.

Proof: Let $K$ be a knot in a homology sphere $M$, and let $S$ be a Seifert surface for $K$. Let $G$ be a degree 2 clasper for $(M, K)$; thanks to Prop. 3(2), we can suppose that $G$ is a $H$-graph. It suffices to show that $\text{Arf}(M, K) = \text{Arf}(M_G, K_G) \in \mathbb{Z}_2$.

Denote by $N$ a regular neighbourhood of $G$, which is a genus 4 handlebody. The 10-component surgery link associated to $G$, depicted in Figure 2.1, is Kirby-equivalent to the 2-component link $L$ depicted in Fig. 3.3. This can be checked by using moves 2, 9 and 1 of [H, Prop. 2.7] (see also [L1, pp. 254]).

Figure 3.3. The 2-component link $L$.

Now observe that such an addition of tube doesn’t affect the Arf invariant of $K$: if we denote by $(m, l)$ a meridian/longitude pair for this tube, we have indeed $\delta_2(m) = 0$, such a meridian $m$ having vanishing self-linking.

We must also show that this pair does not contribute to the Arf invariant of $(K)_G$. In other words, if we denote by $(m', l')$ the image of $(m, l)$ after surgery on $L$, we must show that $\delta_2(m')\delta_2(l') = 0$. Observe that the meridian $m$ can be isotoped in a small ball $B$ of $N$ where the crossing between $L_1$ and $L_2$ occurs - see Fig. 3.4(a). Thus, surgery on $L$ sends $m$ to a curve $m'$, which is a parallel copy of $L_2$ outside of $B$, as shown in Fig. 3.4(b): we have $\delta_2(m') = lk(m', (m')^+) = 0$.

3.2.4. The Sato-Levine invariant. Let $L = L_1 \cup L_2$ be a 2-component oriented link such that $lk(L_1, L_2) = 0$. The components of $L$ bound some Seifert surfaces $S_1$ and $S_2$ such that $L_1 \cap S_2 = L_2 \cap S_1 = \emptyset$. $S_1$ and $S_2$ intersect along circles $S_1 \cap S_2 = C_1 \cup ... \cup C_n = C$. The self-linking of $C$ relative to any of both surfaces is called the Sato-Levine invariant of $L$ [Sa]:

$$\beta(L) = lk(C, C^+).$$
The fact that \( \beta \) is still well-defined for links in homology spheres is again a consequence of the fact recalled in Rem. 20. Indeed, if we add a tube \( t \) to (say) \( S_1 \), it will only intersect \( S_2 \) along copies of a meridian of \( t \) (up to isotopy): such a meridian has vanishing self-linking number and links no other component of \( S_1 \cap S_2 \).

The Sato-Levine invariant can also be defined for elements \((M,\sigma)\) of \( SL_{1b}^b(n) \). For any pair of integers \((i,j)\) such that \( 1 \leq i < j \leq n \), we denote by \( \beta_{ij}(M,\sigma) \) the Sato-Levine invariant of the 2-component link of \( \hat{M} \) obtained by closing the \( i \)th and \( j \)th components of \( \sigma \): \( \beta_{ij}(M,\sigma) := \beta(\hat{\sigma}_i \cup \hat{\sigma}_j) \).

Note that this makes sense by Prop. 22 as elements of \( SL_1^{hb}(n) \) have vanishing linking numbers. Moreover, \( \beta_{ij} \) is additive.

**Proposition 23.** \( \forall \ 1 \leq i < j \leq n \), the map \( \beta_{ij} : SL_1^{hb}(n) \longrightarrow \mathbb{Z} \) is a homomorphism of monoids, called the Sato-Levine invariant \( \beta_{ij} \).

Note that the Sato-Levine invariant is not invariant under \( Y_2 \)-moves: for example, it takes value 2 on the string link \( \sigma \) depicted below, obtained by surgery on \((1D_2,1_n)\) along a \( H \) graph whose leaves are meridians of \( 1_n \) as depicted in Fig. 3.5. But it turns out that it is the case for its mod 2 reduction.

**Proposition 24.** The mod 2 reduction of the Sato-Levine invariant of links in homology spheres is invariant under a \( Y_2 \)-move.

In particular, for any \( 1 \leq i < j \leq n \), the Sato-Levine invariant \( \beta_{i,j} \) of string links in homology balls factors through a homomorphism of Abelian groups

\[ \beta_{ij}^{(2)} : \overline{SL}_1^{hb}(n) \longrightarrow \mathbb{Z}_2. \]

**Proof:** Let \( K \cup K' \) be a 2-component oriented link with linking number 0 in a homology sphere \( M \). Let \( G \) be a degree 2 clasper for \((M,K \cup K')\) (which, as in the preceding proof, can be supposed to be a \( H \)-graph), and \( N \) be a regular neighbourhood of \( G \). We must show that

\[ \beta^{(2)}(M,K \cup K') = \beta^{(2)}(M_G, K_G \cup K'_G) \in \mathbb{Z}_2. \]
We denote respectively by $S$ and $S'$ a Seifert surface for $K$ and $K'$: $S \cap S' = C_1 \cup \ldots \cup C_n = C$. Consider in $N$ the 2-component surgery link $L = L_1 \cup L_2$ associated to $G$ depicted in Fig. 3.3. $K$ and $K'$ are supposed to be disjoint from $N$, but $S$ and $S'$ may intersect $N$ (and thus $L$).

When $S$ (resp. $S'$) intersects $L$, we add some tubes to built a new Seifert surface for $K$ (resp. $K'$), which is disjoint from $L$. The procedure for such an addition of tube is the same as the procedure explained in Appendix A for a knot. We denote by $\tilde{C}$ the set of elements of $S \cap S'$ which are possibly created (in $N$) under this addition of tube: $S \cap S' = C \cup \tilde{C}$. A simple example of such a situation is given in Figure 3.6.

Clearly, $\tilde{C}$ is a finite number of copies of small meridians of $L_1$ and $L_2$. We clearly have $\text{lk}(\tilde{C}, C^+) = \text{lk}(C, \tilde{C}^+) = \text{lk}(\tilde{C}, \tilde{C}^+) = 0$. It remains to prove that, after surgery along $L$, the elements of $\tilde{C} \subset S \cap S'$ do also not contribute to $\beta^{(2)}(K_G \cup K'_G)$.

- Suppose that $\tilde{C} = \{m\}$, where $m$ is a meridian of any of both components. Denote by $c$ its image after surgery on $G$: as seen in the proof of Prop. 22 we have $\text{lk}(c, c^+) = 0$.

- Now, consider the case $\tilde{C} = \{m_1, m_2\}$, a pair of meridians of $L_1$ and $L_2$. An example is given by the situation of Fig. 3.7(a).

Again, surgery on $G$ sends $(m_1, m_2)$ to a pair of curves $(c_1, c_2)$, which are parallel copies of $L_1$ and $L_2$ outside of a ball of $N$ where the crossing between $L_1$ and $L_2$ occurs - see Fig. 3.7(b). Thus, $c_1$ and $c_2$ satisfy

\[
\text{lk} \left( c_1 \cup c_2, (c_1 \cup c_2)^+ \right) = \text{lk}(c_1, c_1^+) + \text{lk}(c_2, c_2^+) = 2l_k(c_1, c_2^+) = \pm 2.
\]
It follows that, in these two particular cases, the mod 2 reduction of $\beta$ remains unchanged. The general case, where $C$ consists in several copies of $m_1$ and $m_2$, is proven the same way. □

**Remark 25.** Note that a (less direct) proof of Prop. 23 can be given using a formula of K. Murasugi that expresses the modulo 2 reduction of the Sato-Levine invariant of a link in terms of its Arf invariants [AM]. Indeed, the Arf invariant of a link can be expressed as the Arf invariant of a knot related to $L$, that is (roughly) obtained by performing a connected sum of its components along some band $R$. The result then follows from Prop. 23.

### 3.3. Degree 1 invariants for string links: proof of Theorem 13

As announced in §3.1.2 the proof of Theorem 13 consists in computing the Abelian group $\mathbb{SL}_1^h(n)$. This computation goes in two steps. First we will construct a combinatorial upper bound, by defining a surjective homomorphism $\varphi_1 : \mathcal{A}_1(P_n) \longrightarrow \mathbb{SL}_1^h(n)$, where $\mathcal{A}_1(P_n)$ is a space of diagram. Second, we will show that $\psi$ is actually an isomorphism, with inverse given by the invariants listed in Thm. 13.

The development of the proof, and the objects it involves, are similar to those used in the proof of [MM, Thm 1.4]. We will recall and use several material and facts presented in the latter, to which the reader is referred for more details.

#### 3.3.1. Combinatorial upper bound

Let $P_n$ denote the Abelian group $H \oplus \mathbb{Z}_2$. We denote by $\mathcal{A}_1(P_n)$ the free Abelian group generated by $Y$-shaped unitrivalent graphs, whose trivalent vertex is equipped with a cyclic order on the incident edges and whose univalent vertices are labelled by $P_n$, subject to the two following relations

**Multilinearity:** $Y[z_0 + z_1; z_2; z_3] = Y[z_0; z_2; z_3] + Y[z_1; z_2; z_3]$,

**Slide:** $Y[z_1; z_1; z_2] = Y[s; z_1; z_2]$,

where $z_0, z_1, z_2, z_3 \in P_n$. Here, the notation $Y[z_1, z_2, z_3]$ stands for the graph whose univalent vertices are colored by $z_1, z_2$ and $z_3 \in P_n$ in accordance with the cyclic order. This notation is invariant under cyclic permutation of the $z_i$'s.

**Remark 26.** Note that, as a consequence of the Multilinearity and Slide relations, the Antisymmetry relation

$$Y[z_1; z_2; z_3] = -Y[z_2; z_1; z_3]$$

holds in $\mathcal{A}_1(P_n)$ — for example, apply the Slide relation to $Y[z_1 + z_2; z_1 + z_2; z_3]$.

Consider the map

$$\rho : \mathcal{A}_1(P_n) \longrightarrow \Lambda^3 H \oplus \Lambda^2 H_{(2)} \oplus H_{(2)} \oplus \mathbb{Z}_2$$

defined on the generators of $\mathcal{A}_1(P_n)$ by:

$$\rho(Y[(e_i, 0); (e_j, 0); (e_k, 0)]) = e_i \wedge e_j \wedge e_k \in \Lambda^3 H,$$

$$\rho(Y[(e_i, 0); (e_j, 0); (0, 1)]) = \overline{c_i} \wedge \overline{c_j} \in \Lambda^2 H_{(2)},$$

$$\rho(Y[(e_i, 0); (0, 1); (0, 1)]) = \overline{c_i} \in H_{(2)},$$

$$\rho(Y[(0, 1); (0, 1); (0, 1)]) = 1 \in \mathbb{Z}_2,$$

where $1 \leq i < j < k \leq n$, and where $(e_i)_i$ (resp. $(\overline{c_i})_i$) are the basis elements of $H$ (resp. $H_{(2)}$) defined in Notations 13.

$\rho$ is clearly well-defined and we actually have the following lemma.

**Lemma 27.** The map $\rho$ is an isomorphism.
This is proved in the same way as \[MM\] Lem. 4.24 (see also \[M2\] Lem. 6.3).

We now construct the surgery map

\[ \varphi_1 : A_1(P_n) \rightarrow \mathcal{SL}_{1}^{hb}(n). \]

For each generator \( Y = [z_1; z_2; z_3] \) of \( A_1(P_n) \), where \( z_i := (h_i, \varepsilon_i) \in P_n \), we set

\[ \varphi_1(Y) := (D^2 \times I, 1_n)_{\phi(Y)}, \]

where \( \phi(Y) \) is a degree 1 connected clasper (a \( Y \)-graph) for \( (D^2 \times I, 1_n) \) constructed from the information contained in the diagram \( Y \): For \( i \in \{1, 2, 3\} \), consider an oriented simple closed curve \( c_i \) in \( D^2 \times \{1\} \subset D^2 \times I \) such that \( [c_i] = h_i \in H \), framed along the surface. Then push this framed curves down in the interior of \( (D^2 \times I) \setminus 1_n \cong (D_2^n \times I) \), by adding a \( \varepsilon_i \)-twist. The resulting oriented framed knot is denoted by \( K_i \). Next, pick an embedded 2-disk \( D \) in the interior of \( D^2 \times I \) and disjoint from the \( K_i \)'s, orient it in an arbitrary way, and connect it to the \( K_i \)'s with some bands \( e_i \). These band sums have to be compatible with the orientations, and to be coherent with the cyclic ordering \( (1, 2, 3) \).

\textbf{Proposition 28.} Let \( Y \) be a generator of \( A_1(P_n) \). The \( Y_2 \)-equivalence class of \( (D^2 \times I, 1_n)_{\phi(Y)} \) does not depend on the choice of \( \phi(Y) \) (obtained by the above construction). Hence, we have a well-defined, surjective surgery map

\[ A_1(P_n) \xrightarrow{\varphi_1} \mathcal{SL}_{1}^{hb}(n). \]

The proof is strictly the same as the proof of \[MM\] Thm. 2.11, and essentially uses the calculus of claspers. In particular, the independence on the choice of \( \phi \) follows from facts similar to \[GGP\] Cor. 4.2 and 4.3, Lem. 4.4.

\[ 3.3.2. \text{ Characterization of } Y_2\text{-equivalence for string links.} \]

Set \( V := \Lambda^2 H(2) \oplus H(2) \oplus \mathbb{Z}_2 \), and let \( \tau : \mathcal{SL}_{1}^{hb}(n) \rightarrow \Lambda^3 H(2) \oplus V \) be defined, for any \( (M, \sigma) \in \mathcal{SL}_{1}^{hb}(n) \), by

\[ \tau(M, \sigma) = \sum_{1 \leq i < j < k \leq n} \mu^{(2)}_{ijk}(M, \sigma) \overline{e_i} \wedge \overline{e_j} \wedge \overline{e_k} + \sum_{1 \leq i < j \leq n} \beta^{(2)}_{ij}(M, \sigma) \overline{e_i} \wedge \overline{e_j} + \sum_{1 \leq i \leq n} a_i(M, \sigma) \overline{e_i} + R(M). \]

Here, \( \mu^{(2)}_{ijk} \) denotes the mod 2 reduction of Milnor’s triple linking number \( \mu_{ijk} \). It follows from Propositions 15, 17, 22, and 24 that this well-defined map factors through a homomorphism of Abelian groups

\[ \mathcal{SL}_{1}^{hb}(n) \xrightarrow{\tau} \Lambda^3 H(2) \oplus V. \]

Denote by \( T \) the composition

\[ T : A_1(P_n) \xrightarrow{\rho} \Lambda^3 H \oplus V \xrightarrow{-\otimes \mathbb{Z}_2} \Lambda^3 H(2) \oplus V. \]

\textbf{Lemma 29.} The following diagram commutes

\[ \begin{array}{ccc}
A_1(P_n) & \xrightarrow{\varphi_1} & \mathcal{SL}_{1}^{hb}(n) \\
\downarrow T & & \downarrow \tau \\
\Lambda^3 H(2) \oplus V & & \\
\end{array} \]
Proof: $P$ is generated by $(0, 1)$ and $(e_i, 0)$, $i = 1, ..., n$. So, thanks to the Slide relation, there are four distinct types of generators $Y$ for $\mathcal{A}_1(P_n)$, listed below $(1 \leq i < j < k \leq n)$: we prove that, in these four cases, $\tau(\varphi_1(Y)) = T(Y)$.

1. $Y = Y[(0, 1); (0, 1); (0, 1)]$

In this case, $T(Y) = 1 \in \mathbb{Z}_2$. On the other hand, a representative for $\varphi_1(Y) \in \Sigma_1^{ab}(n)$ is $(1_{D^2}, 1_n)_{G}$, where $G$ is contained in a ball disjoint from $1_n$ and its leaves are three copies of the $(-1)$-framed unknot. It follows that $(1_{D^2}, 1_n)G \cong (P, 1_n)$, where the closure of $P$ is the Poincaré sphere: $R(P, 1_n) = 1$. Moreover,

$$\mu_{rst}(P, 1_n) = \beta^{(2)}_{rs}(P, 1_n) = a_s(P, 1_n) = 0,$$

$\forall r \neq s \neq t \in \{1, ..., n\}$. It follows that $\tau(P, 1_n) = 1 \in \mathbb{Z}_2$.

2. $Y = Y[(e_i, 0); (e_i, 0); (e_j, 0)]$

A representative for $\varphi_1(Y)$ is $(1_{D^2}, 1_n)_{G}$, where the three leaves of $G$ are small meridians of the $i^{th}$ string $(1_n)_i$ of $1_n$. Thus $(1_{D^2}, 1_n)G \cong (1_{D^2}, T_i)$, where $T_i$ only differs from $1_n$ by a copy of the trefoil on the $i^{th}$ string — see the Fig. 3.8(a). We have $a_r(1_{D^2}, T_i) = \delta_{r,i}$, and

$$\mu_{rst}(1_{D^2}, T_i) = \beta^{(2)}_{rs}(1_{D^2}, T_i) = R(1_{D^2}, T_i) = 0, \forall (r, s, t).$$

It follows that $\tau \circ \varphi_1(Y) = \tau T = T(Y)$.

3. $Y = Y[(e_i, 0); (e_j, 0); (e_k, 0)]$

A representative for $\varphi_1(Y)$ is obtained from $(1_{D^2}, 1_n)G$ by surgery along a $Y$-graph $G$ having two copies of a meridian of $(1_n)_j$ and one copy of a meridian of $(1_n)_j$ as leaves: $(1_{D^2}, 1_n)G \cong (1_{D^2}, w_{ij})$, where the $j^{th}$ and $k^{th}$ strings of $w_{ij}$ form a Whitehead link, see Fig. 3.8(b). The Sato-Levine invariant of the Whitehead link being 1, we obtain $\beta^{(2)}_{rs}(1_{D^2}, w_{ij}) = \delta_{(r,s),(i,j)}$, and

$$\mu_{rst}(1_{D^2}, w_{ij}) = a_r(1_{D^2}, w_{ij}) = R(1_{D^2}, w_{ij}) = 0, \forall (r, s, t).$$

It follows that $\tau \circ \varphi_1(Y) = \tau T \wedge \tau T \in \Lambda^2 H(2)$, which coincides with $T(Y)$.

4. $Y = Y[(e_i, 0); (e_j, 0); (e_k, 0)]$

A representative for $\varphi_1(Y)$ is $(1_{D^2}, \sigma_{ijk})$, obtained from $1_n$ by performing a connected sum on strings $\sigma_i$, $\sigma_j$, and $\sigma_k$ with the three components of a Borromean ring, see Fig. 3.8(c). It follows that $\mu_{abc}(\sigma_{ijk}) = 1$ for $(a, b, c) = (i, j, k)$, and 0 otherwise. Moreover,

$$\beta^{(2)}_{rs}(1_{D^2}, \sigma_{ijk}) = a_r(1_{D^2}, \sigma_{ijk}) = R(1_{D^2}, \sigma_{ijk}) = 0, \forall (r, s).$$

We thus obtain $\tau(\varphi_1(Y)) = \tau_i \wedge \tau_j \wedge \tau_k = T(Y)$, which completes the proof.

\[\text{Figure 3.8.}\]

Furthermore, we can define by Prop. a homomorphism of Abelian groups

$$\Sigma_1^{ab}(n) \xrightarrow{\mu_3} \Lambda^3 H.$$

by setting $\mu_3(M, \sigma) = \sum_{1 \leq i < j < k \leq n} \mu_{ijk}(M, \sigma)e_i \wedge e_j \wedge e_k$.

The following lemma is a direct consequence of computations contained in the preceding proof (Case 4).
Lemma 30. The following diagram commutes

\[
\begin{array}{c}
\mathcal{A}_1(P_n) \xrightarrow{\varphi_1} \mathcal{SL}_1^{hb}(n) \\
\mu_3 \searrow \Lambda^3 H
\end{array}
\]

Lemmas 29 and 28 can then be summarized as follows.

Proposition 31. The diagram

\[
\begin{array}{c}
\mathcal{A}_1(P_n) \xrightarrow{\varphi_1} \mathcal{SL}_1^{hb}(n) \\
\mu_3 \searrow ((\mu_3, \tau) \oplus V)
\end{array}
\]

commutes, and all of its arrows are isomorphisms.

More precisely, Lem. 29 and 28 imply the commutativity. The fact that \(\varphi_1\) (and thus \((\mu_3, \tau)\)) is an isomorphism follows.

We are now ready to prove Theorem 13. Assertion (c) \(\implies\) (a) is indeed a direct consequence of Prop. 31. As outlined in Rem. 14, assertion (a) \(\implies\) (b) is a general fact, which follows from the definition of a finite type invariant. Let us prove that (b) implies (c) by showing that in fact any homomorphism of Abelian groups \(\mathcal{SL}_1^{hb}(n) \to A\) gives a degree 1 invariant. Let \((M, \sigma)\) be a \(n\)-string link in a homology ball and let \(G_1, G_2\) be some disjoint \(Y\)-graphs for \((M, \sigma)\). We aim to show that:

\[
(3.1) \quad f(M, \sigma) - f((M, \sigma) G_1) - f((M, \sigma) G_2) + f((M, \sigma) G_1 \cup G_2) = 0.
\]

Let \(G\) be a collection of disjoint \(Y\)-graphs for \((1_{D^2}, 1_n)\) such that \((M, \sigma) = (1_{D^2}, 1_n) G\) (up to \(Y_2\)-equivalence). By possibly isotoping \(G_1\) and \(G_2\) in \(M \setminus \sigma\), they are disjoint from \(G\). We then put \((M_i, \sigma_i) = ((1_{D^2}, 1_n)) G_i\). Up to \(Y_2\)-equivalence, \((M, \sigma) G_i = (M, \sigma) \cdot (M_i, \sigma_i)\) and \((M, \sigma) G_1 \cup G_2 = (M, \sigma) \cdot (M_1, \sigma_1) \cdot (M_2, \sigma_2)\). Equation (3.1) follows then from the additivity of \(f\).

4. On the Milnor-Johnson correspondence

In this section, we study the relation between the Goussarov-Habiro theory for framed string links in homology balls and this theory for homology cylinders. Let us start with a short reminder on the latter.

4.1. Homology cylinders. Let \(\Sigma_{g,1}\) be a compact connected oriented surface of genus \(g\) with 1 boundary component.

A homology cylinder \(M\) over \(\Sigma_{g,1}\) is a homology cobordism with an extra homological triviality condition \([\text{GL}] [\text{H}] [\text{L}]\). Alternatively, it can be defined as follows: a homology cylinder \(M\) over \(\Sigma_{g,1}\) is a 3-manifold obtained from \(\Sigma_{g,1} \times I\) by surgery along some claspers, that is, \(M \sim_{Y_2} \Sigma_{g,1} \times I\).

The set of homology cylinders up to orientation-preserving diffeomorphisms is denoted by \(\mathcal{HC}(\Sigma_{g,1})\). It is equipped with a structure of monoid, with product given by the stacking product and with \(\Sigma_{g,1} \times I\) as unit element.

There is a descending filtration of monoids

\[
\mathcal{HC}(\Sigma_{g,1}) = C_1(\Sigma_{g,1}) \supset C_2(\Sigma_{g,1}) \supset \cdots \supset C_k(\Sigma_{g,1}) \supset \cdots
\]
where $\mathcal{C}_k(\Sigma_{g,1})$ is the submonoid of all homology cylinders which are $Y_k$-equivalent to $1_{\Sigma_{g,1}}$. Moreover, as in the string link case, the quotient monoid $\mathcal{C}_k(\Sigma_{g,1}) := \mathcal{C}_k(\Sigma_{g,1})/Y_{k+1}$ is an Abelian group for every $k \geq 1$.

As mentioned in [GL], the Torelli group $T_{g,1}$ of $\Sigma_{g,1}$ (the isotopy classes of self-diffeomorphisms of $\Sigma_{g,1}$ inducing an isomorphism in homology) naturally imbeds in $\mathcal{H}(\Sigma_{g,1})$ via the mapping cylinder construction, and we can extend classical applications on the Torelli group to the realms of homology cylinders. In particular, we can extend the first Johnson homomorphism $\eta_1$ and the Birman-Craggs homomorphism $\beta$, originally used by D. Johnson in [J1, J2] for the computation of the the Abelianized Torelli group. In [MM], it is shown that these extensions actually are the degree 1 Goussarov-Habiro finite type invariants for homology cylinders.

**Theorem 32 (MM).** Let $M$ and $M'$ be two homology cylinders over $\Sigma_{g,1}$. The following assertions are equivalent:

(a) $M$ and $M'$ are $Y_2$-equivalent;
(b) $M$ and $M'$ are not distinguished by degree 1 Goussarov-Habiro finite type invariants;
(c) $M$ and $M'$ are not distinguished by the first Johnson homomorphism nor the Birman-Craggs homomorphism.

This is proved, as in [33] by computing the abelian group $\mathcal{C}_1(\Sigma_{g,1})$ in a graphical way. More precisely, the authors define (in a strictly similar way) a space of diagrams $A_1(P_{g,1})$ and a surjective surgery map $A_1(P_{g,1}) \xrightarrow{\psi_1} \mathcal{C}_1(\Sigma_{g,1})$, which actually is an isomorphism, with inverse given by $\eta_1$ and $\beta$.

4.2. **From homology cylinders to string links.** This result on homology cylinders over $\Sigma_{g,1}$ looks quite similar to Thm. 33 on framed $n$-string links in homology balls, and suggests a strong analogy between these objects.

This correspondence homology cylinders/string links has been studied by N. Habegger [Ha]: via a certain geometric construction relating these objects, Johnson homomorphisms coincides with Milnor’s numbers. This result is refered to as the Milnor-Johnson correspondence. More precisely, Habegger shows that there exists a bijection between the sets $\mathcal{H}(\Sigma_{g,1})$ and $\mathcal{SL}_1^{bh}(2g)$ which produces an isomorphism of Abelian groups

$$b : \mathcal{C}_1(\Sigma_{g,1}) \xrightarrow{\sim} \mathcal{SL}_1^{bh}(2g)$$

such that the Johnson homomorphism $\eta_1$ corresponds to Milnors invariant $\mu_3$ trough $b$. Proposition 34 allows us to go a bit further.

**Theorem 33.** The homomorphism $\tau$ of Proposition 34 given by the Milnor, Sato-Levine, Arf and Rocklin invariants, is the analogue of the Birman-Craggs homomorphism $\beta$ for the Milnor-Johnson correspondence. In other words, $\beta$ and $\tau$ correspond through the isomorphism $b$.

The proof is given in the next subsection. Actually, we will also give an alternative proof for (part of) Habegger’s result, based on the theory of claspers.

4.3. **Birman-Craggs homomorphism for string links: proof of Theorem 33.** Let us recall from [Ha] the construction on which the Milnor-Johnson correspondence lies. Consider the handle decomposition $A_1, B_1, ..., A_g, B_g$ of $\Sigma_{g,1}$ as in the left part of Fig. 44. Likewise, for the $2g$-punctured disk $D_{2g}^2 \cong \Sigma_{0,2g+1}$, consider the handle decomposition $\{A'_i, B'_i\}_{i=1}^{2g}$ given in the right part of the figure.

We identify $\Sigma_{g,1} \times I$ with $\Sigma_{0,2g+1} \times I$ using the diffeomorphism $F$ defined by the $g$ isotopies exchanging, in $\Sigma_{g,1} \times I$, the second attaching region of the handle $A_i \times I$ and the first attaching region of the handle $B_i \times I$. 
Now, the product \( \Sigma_{0,2g+1} \times I \) can be thought of as (the closure of) the complementary of the 0-framed trivial 2\( g \)-string link \( 1_{2g} \) in \( D^2 \times I \). This defines a bijection between the sets \( \mathcal{C}_1(\Sigma_{g,1}) \) and \( \mathcal{SL}_{1}^{hb}(2g) \).

Indeed, let \( G \) be a degree 1 clasper for \( \Sigma_{g,1} \times I \): the pair \( (\Sigma_{g,1} \times I ; G) \) defines an element of \( \mathcal{C}_1(\Sigma_{g,1}) \). By applying \( F \) to this pair, we obtain a clasper \( G' \) of the same degree for \( (\Sigma_{0,2g+1} \times I) \cong 1_{2g} \setminus 1_{2g} \); the triple \( ((1_{D^2}, 1_{n}); G') \) defines an element of \( \mathcal{SL}_{1}^{hb}(2g) \).

Moreover, though this bijection is not a homomorphism, it produces an isomorphism of Abelian groups

\[
\overline{\mathcal{C}}_1(\Sigma_{g,1}) \xrightarrow{b} \overline{\mathcal{SL}}_1^{hb}(2g).
\]

This follows from the following observation. Let \( M_i \) \((i=1,2)\) be an element of \( \overline{\mathcal{C}}_1(\Sigma_{g,1}) \) obtained from \( \Sigma_{g,1} \times I \) by surgery on the degree 1 clasper \( G_i \).

The product \( M_1 \cdot M_2 \) is mapped by \( b \) to an element which is obtained from \( (1_{D^2}, 1_{2g}) \) by surgery on the union \( G'_1 \cup G'_2 \), where \( G'_i \) is the image of \( G_i \) under the diffeomorphism \( F \) (in particular, \( \text{deg}(G'_i) = 1 \)). Up to \( Y_2 \)-equivalence, we can suppose that these two claspers lie in disjoint portions of the product \( D^2 \times I \); it follows that

\[
(1_{D^2}, 1_{2g}) G'_1 \cup G'_2 \sim_Y (1_{D^2}, 1_{2g}) G'_1 \cdot (1_{D^2}, 1_{2g}) G'_2 = b(M_1) \cdot b(M_2).
\]

Similar arguments show that we actually have an isomorphism of Abelian groups \( \overline{\mathcal{C}}_k(\Sigma_{g,1}) \cong \overline{\mathcal{SL}}_k^{hb}(2g), \) \( \forall \ k \geq 1 \).

At the level of homology, there is an obvious isomorphism between \( H_1(\Sigma_{g,1}; \mathbb{Z}) \) and \( H_1(\Sigma_{0,2g+1}; \mathbb{Z}) \) induced by the diffeomorphism \( F \). We denote by \( H \) these homology groups. This isomorphism allows to identify the diagram spaces \( \mathcal{A}_1(P_{g,1}) \) and \( \mathcal{A}_1(P_{2g}) \). We thus have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{A}_1(P_{g,1}) & \longrightarrow & \mathcal{A}_1(P_{2g}) \\
\psi_1 & \downarrow & \varphi_1 \\
\overline{\mathcal{C}}_1(\Sigma_{g,1}) & \xrightarrow{b} & \overline{\mathcal{SL}}_1^{hb}(2g),
\end{array}
\]

whose arrows are isomorphisms.

Following Notations 111 set \( H_{(2)} = H \otimes \mathbb{Z}_2 \), and \( V = \Lambda^2 H_{(2)} \oplus H_{(2)} \oplus \mathbb{Z}_2 \). By considering the inverse maps (in the sense of [MM] Thm. 1.4) and Prop. 111 of the vertical arrows of (\( D \)), we easily deduce the following commutative diagram

\[
\begin{array}{ccc}
\overline{\mathcal{C}}_1(\Sigma_{g,1}) & \xrightarrow{b} & \overline{\mathcal{SL}}_1^{hb}(2g) \\
\downarrow \cong & \downarrow & \uparrow \cong \\
(\mu_3, \tau) & \cong & \Lambda^3 H \oplus V,
\end{array}
\]
which shows that, via the isomorphism $b$, degree 1 invariants for homology cylinders over $\Sigma_{g,1}$ correspond to those of $2g$-string links in homology balls. More precisely, we deduce from diagram (D) the following result.

**Lemma 34.** The two following diagrams commute.

\[
\begin{array}{ccc}
\mathcal{C}_1(\Sigma_{g,1}) & \xrightarrow{b} & \mathcal{L}_{1}^{2g} \\
\downarrow \psi_1 & & \downarrow \phi \ \\
\Lambda^3 H & & \Lambda^3 H(2) \oplus V.
\end{array}
\]

The first diagram recovers Habegger’s Milnor-Johnson correspondence (at the lowest level). The second one proves Thm. 33.

**Proof of Lemma 34.**
Consider in diagram (D) the projections $p : A_1(P) \xrightarrow{} \Lambda^3 H$, on the one hand, and $T : A_1(P) \xrightarrow{} \Lambda^3 H(2) \oplus V$ on the other hand, where $A_1(P)$ denotes either $A_1(P_{g,1})$ or $A_1(P_{2g})$. Recall from [MM, Lemma 4.22] that the diagram

\[
\begin{array}{ccc}
A_1(P_{g,1}) & \xrightarrow{\psi_1} & \mathcal{C}_1(\Sigma_{g,1}) \\
\downarrow \eta_1 & & \downarrow \eta_1 \\
\Lambda^3 H & & \Lambda^3 H
\end{array}
\]

is commutative. This, together with Lemma 30 implies the commutativity of the first diagram. The second half of the result follows similarly from [MM, Lemma 4.23] and Lemma 29.

\[\square\]

5. **Comparing Goussarov-Habiro and Vassiliev theories**

For several reasons, it is tempting to compare the results of §3 with Vassiliev theory. First, as seen in §2 both Goussarov-Habiro and Vassiliev theories can be defined using claspers (with some slight differences). Second, some results in the literature on Vassiliev invariants have strong similarities with Theorem 13, namely K. Taniyama and A. Yasuhara’s characterization of clasp-pass equivalence for algebraically split links in the 3-sphere [TY], and its analogue for string links [Me].

Recall that the clasp-pass equivalence is the equivalence relation on links generated by isotopies and *clasp-pass moves*, which are local moves as illustrated in Fig. 5.1. As outlined in Rem. 7, the clasp-pass equivalence is actually the same as $C_3$-equivalence, which implies $Y_2$-equivalence.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{clasp-pass-move.png}
\caption{A clasp-pass move.}
\end{figure}
5.1. **Goussarov-Habiro and Vassiliev invariants of string links.** Let us first consider the string link case. Recall that the Casson knot invariant $\varphi(K)$ of a knot $K$ is defined as the $z^2$ coefficient of the Alexander-Conway polynomial of $K$, and that its reduction modulo 2 coincides with the Arf invariant $\alpha$ studied in [3.2.3].

Recall also from [Mc] the definition of the 2-string link invariant $V_2$. Let $\sigma = \sigma_1 \cup \sigma_2$ be a 2-string link. Then

$$V_2(\sigma) := \varphi(p(\sigma)) - \varphi(\sigma_1) - \varphi(\sigma_2),$$

where $p(\sigma)$ denotes the plat-closure of $\sigma$: it is the knot obtained by identifying the upper (resp. lower) endpoints of $\sigma_1$ and $\sigma_2$. Clearly, $V_2$ is a $\mathbb{Z}$-valued Vassiliev invariant of degree two.

We want to relate Thm. 13 to the following:

**Theorem 35 ([Mc]).** Let $\sigma$ and $\sigma'$ be two $n$-component algebraically split string links in $D^2 \times I$ (that is, with all linking numbers zero). Then, the following assertions are equivalent:

(a) $\sigma$ and $\sigma'$ are clasp-pass equivalent;
(b) $\sigma$ and $\sigma'$ are not distinguished by degree 2 Vassiliev invariants;
(c) $\sigma$ and $\sigma'$ are not distinguished by Milnor’s triple linking numbers, nor the invariant $V_2$ and the Casson knot invariant.

We denote by $SL(n)$ the monoid of $n$-string links in $D^2 \times I$ up to isotopy (with fixed endpoints), and by $SL^{ws}(n)$ the submonoid of algebraically split $n$-string links. When considered up to $C_3$-equivalence, the elements of $SL^{ws}(n)$ form an Abelian group, denoted by $\overline{SL}^{ws}(n)$.

**Theorem 36.** The Abelian group $\overline{SL}^{ws}(n)$ is surjectively mapped onto the subgroup $\overline{SL}_1^{(0)}(n) \subset \overline{SL}_1(n)$ of string links in homology balls having vanishing Rochlin’s $\mu$-invariant.

**Proof:** Recall from [Mc] the isomorphism

$$(\mu_3, V_2, \varphi) : \overline{SL}^{ws}(n) \xrightarrow{\cong} \Lambda^3 H \oplus S^2 H$$

given by the formula

$$\sum_{1 \leq i < j < k \leq n} \mu_{ijk}(\sigma), e_i \wedge e_j \wedge e_k + \sum_{1 \leq i < j \leq n} V_2(\sigma_i \cup \sigma_j), e_i \otimes e_j + \sum_{1 \leq i \leq n} \varphi(\sigma_i). e_i.$$

Here, $S^2 H$ is the degree two part of the symmetric algebra of $H$ (we still make use of Notations 10).

On the other hand, we saw in [Mc] the isomorphism of Abelian groups

$$(\mu_3, \tau) : \overline{SL}_1^{ah}(n) \xrightarrow{\cong} \Lambda^3 H \oplus \Lambda^2 H_{(2)} \oplus H_{(2)} \oplus \mathbb{Z}_2,$$

where the $\mathbb{Z}_2$ part is detected by Rochlin’s $\mu$-invariant. We thus have the decomposition $\overline{SL}_1^{ah}(n) = \overline{SL}_1^{(0)}(n) \cup \overline{SL}_1^{(1)}(n)$, where $\overline{SL}_1^{(\epsilon)}(n)$ (for $\epsilon = 0, 1$) is the subset of $\overline{SL}_1(n)$ consisting of elements $(M, \sigma)$ such that $R(M) = \epsilon$.

In particular, $\overline{SL}_1^{(0)}(n)$ is an Abelian subgroup and we clearly have an isomorphism

$$(\mu_3, \beta^{(2)}, \alpha) : \overline{SL}_1^{(0)}(n) \xrightarrow{\cong} \Lambda^3 H \oplus \Lambda^2 H_{(2)} \oplus H_{(2)},$$

given by the formula

$$\sum_{1 \leq i < j < k \leq n} \mu_{ijk}(M, \sigma), e_i \wedge e_j \wedge e_k + \sum_{1 \leq i < j \leq n} \beta_{ij}^{(2)}(M, \sigma), e_i \wedge e_j + \sum_{1 \leq i \leq n} a_i(M, \sigma). e_i.$$

Now, recall that the $C_2$-equivalence is the same as the $\Delta$-equivalence: as in the link case [MN], a $n$-string link $\sigma$ is $C_2$-equivalent to $1_n$ if and only if it has vanishing
linking numbers. So $\mathcal{SL}^{as}(n)$ is just the set of $C_2$-equivalence classes of $n$-string links which are $C_2$-equivalent to $1_n$: given a generator $\sigma$ of $\mathcal{SL}^{as}(n)$, there is a connected $C$-degree 2 clasper $G_{\sigma}$ for $1_n \in \mathcal{D}_2$ such that $\sigma = (1_n)_{G_{\sigma}}$. We define a map

$$\mathcal{SL}^{as}(n) \xrightarrow{\eta} \mathcal{SL}^{(0)}(n)$$

which consists in puncturing each disk-leaf of $G_{\sigma}$, that is removing a small disk $d$ such that $1_n$ intersects the disk-leaf at the interior of $d$; further, equip $1_n$ with 0-framing. As Fig. 5.2 shows, puncturing a disk-leaf of $G_{\sigma}$ produces a leaf. $G_{\sigma}$ becomes a Y-graph $\hat{G}_{\sigma}$, and

$$T(\sigma) := (1_{D^2}, 1_n)_{\hat{G}_{\sigma}}.$$

Note that $\eta$ has a non-trivial kernel: an example is given in Fig. 5.3. It follows from the proofs of Thm. 13 and 35 that we have a commutative diagram

$$\mathcal{SL}^{as}(n) \xrightarrow{\eta} \mathcal{SL}^{(0)}(n)$$

$$(\mu_3, V_2, \varphi) \simeq (\mu_3, \beta^{(2)}, \alpha)$$

$$\Lambda^3 H \oplus S^2 H \xrightarrow{f} \Lambda^3 H \oplus \Lambda^2 H_{(2)} \oplus H_{(2)}$$

where $f$ is the surjective map given by the identity on $\Lambda^3 H$, and by

$$f(e_i \otimes e_j) = \overline{e_i} \wedge \overline{e_j}$$

if $i \neq j$, and $f(e_i \otimes e_i) = \overline{e_i}$ otherwise

on $S^2 H$. It follows that $\eta$ is also surjective. □

Moreover, the maps $(\mu_3, V_2, c_2)$ and $(\mu_3, \beta^{(2)}, a)$ coincide via the surjective map $\eta$ (and $t$). In particular, it follows that

$$V_2 \equiv \beta \pmod{2}.$$ 

However, these invariants are distinct over $\mathbb{Z}$, as mentioned in [Me, Rem 2.7].

Figure 5.2. The $\eta$ map.

Figure 5.3. An element of $\text{Ker}(\eta)$ for $n = 2$. 
5.2. The case of links. In the case of links, we know the following on clasping equivalence.

**Theorem 37** ([LY], Thm. 1.4). Let $L$ and $L'$ be two $n$-component algebraically split links in $S^3$. The following assertions are equivalent:

(a) $L$ and $L'$ are clasping equivalent;
(b) $L$ and $L'$ are not distinguished by Milnor’s triple linking numbers, nor the mod 2 reduction of the Sato-Levine invariant and the Casson knot invariant.

As for $Y_2$-equivalence, one can check (using Thm 13 and its proof) the following corollary, characterizing $Y_2$-equivalence for algebraically split links in homology spheres.

**Corollary 38.** Let $(M, L)$ and $(M', L')$ be two $n$-component algebraically split links in homology spheres. Then, the following assertions are equivalent:

(a) $(M, L)$ and $(M', L')$ are $Y_2$-equivalent;
(b) $(M, L)$ and $(M', L')$ are not distinguished by Milnor’s triple linking numbers, nor the mod 2 reduction of the Sato-Levine invariant, the Arf invariant and Rochlin’s $\mu$-invariant.

This result is related to Thm. 57 in a similar way as Thm. 13 is related to Thm. 56. However, unlike in the string link case, there is no natural group or monoid structure on the sets of $C_k$ or $Y_k$-equivalence classes of links.

**Appendix A. Tubing Seifert surfaces.**

Let us consider the 2-component link $L = L_1 \cup L_2$ in a genus 4 handlebody $N$ depicted in Fig. A.1. We fix an orientation on $N$ and embed it in $S^3$. Let $K$ be an oriented knot in $S^3$ disjoint from $N$, and let $S$ be a Seifert surface for $K$: in general, $S$ may intersect $K$. In this appendix we explain the general procedure to construct, starting from $S$, a new Seifert surface for $K$ which is disjoint from $L$.

First, we fix some more notations. The handlebody $N$ can be regarded as a ball $B$ with 4 handles $D^2 \times I$ attached. The two handles intersecting $L_1$ are denoted by $H_1$ and $H_2$, and we denote by $H_3$ and $H_4$ the other two ; the handles are numbered clockwise in Fig. A.1 so that $H_1$ is in the lower left corner of the figure. Up to isotopy, we can suppose that $S$ is disjoint from $B$, that is $S$ only intersects $N$ at its handles, along copies of $D^2 \times \{t \}$; $t \in I$. When the orientation of $S$ is compatible with the orientation of $N$ along the intersection disk, we call it a positive intersection. Otherwise, we call it a negative intersection. For $1 \leq i \leq 4$, we denote respectively by $p_i$ and $n_i$ the number of positive and negative intersections between the surface $S$ and the handle $H_i$.

In view of the symmetry of the link $L$, we only have to deal with (say) the handles $H_1$ and $H_2$ (the handles $H_3$ and $H_4$ can be treated independently, in a similar way).

First, observe that if $S$ intersects $H_1$ twice, with the opposite orientation, we can add two tubes to $S$ as shown in Fig. A.1(a), so that the new Seifert surface $\hat{S}$ satisfies $|L \cap S| = |L \cap \hat{S}| + 4$.

Likewise, we can always add $|p_i - n_i|$ such pairs of tubes to $S$ in $H_i$ ($i = 1, 2$), by eventually nesting them, so that in each handle the remaining intersections all have the same sign. So we can suppose that $p_1 n_1 = p_2 n_2 = 0$. Suppose further that $n_1 = 0$ (the case $p_1 = 0$ is equivalent, due to the symmetry of $L_1$).

If $p_1 = p_2 = n_2 = 0$, we are done. Otherwise, there are essentially 4 different cases to study.

1. Suppose that $p_2 = n_2 = 0$. In this case $S$ is disjoint from the handle $H_1$. We can thus remove all the elements of $S \cap L_1$ by successively attaching and nesting $p_1$ tubes as depicted in Fig. A.1(b). These tubes will be called tubes of type 1. When
the two attaching circles of a tube \( t \) of type 1 lie in a disk \( D^2 \times \{ t \} \) of the handle \( H_i \), we simply say that \( t \) is attached in \( H_i \) \((i = 1, 2)\).

2. Suppose that \( p_1 = 0 \). This case is equivalent to the first one: \( S \) is disjoint from the handle \( H_2 \), so we can freely attach \( p_2 + n_2 \) tubes of type 1 in \( H_1 \).

3. Suppose that \( p_1 \) and \( p_2 \) are non-zero. In this case, \( S \) always intersects \( N \) with the same sign. Fig. A.1 (a) illustrates the case \( p_1 = 2 \) and \( p_2 = 1 \).

In general, we attach in a similar way \( p \) tubes of type \( a \), \( m \) tubes of type \( b \) and \( m \) tubes of type \( c \) (following the notations of the figure), where \( p := |p_1 - p_2| \) and \( m := \max(p_1, p_2) - p \).

4. Suppose that \( p_1 \) and \( n_2 \) are non-zero. Fig. A.2 (b) illustrates the case \( p_1 = 2 \) and \( n_2 = 1 \). As for the previous case, we deal with the general situation by attaching and nesting the same three types of tubes. Namely, we attach \( |p_1 - n_2| \) tubes of type \( d \) and \( (\max(p_1, n_2) - |p_1 - n_2|) \) tubes of type \( e \) and \( f \).

The obtained surface is the required new Seifert surface for \( K \).

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