Optimal Number of Facilities in Flow Covering Location Problems

Masashi Miyagawa

Abstract

This paper develops an analytical model for determining the number of flow demand facilities that cover trips. The proportions of covered trips within a specified deviation distance are derived for grid and random patterns of facilities. The analytical expressions for the proportions of covered trips demonstrate how the deviation distance, the number of facilities, and the pattern of facilities affect the level of coverage. The number of facilities required to achieve a certain level of coverage is then obtained. The coverage level is represented as the combination of the deviation distance and the proportion of covered trips. The coverage by the second nearest facility is also considered for the random pattern to deal with the case where facilities are subject to failures. The model incorporates any level of coverage and provides a versatile framework for determining the number of flow demand facilities.

Keywords: Location, Deviation distance, Flow coverage, Proportion of covered trips, Rectilinear distance

1 Introduction

Classical facility location models usually assume that demand for service can be expressed as points. For some types of facilities, however, demand originates from flow in that customers visit facilities on pre-planned trips between origin and destination. Examples of such flow demand facilities include refueling stations, convenience stores, and ATMs.

The optimal location of flow demand facilities has been addressed. Hodgson\(^1\) and Berman et al.\(^2\) formulated the flow interception location model (FILM) for locating facilities so as to maximize the covered flow. The FILM is a flow demand version of the maximal covering location model proposed by Church and ReVelle\(^3\), which aims to maximize the covered points. The FILM has been extended to incorporate the possibility of deviations\(^4\), multi-counting\(^5\), competitive facilities\(^6\), consumers’ preferences\(^7\), and facilities of different sizes and attractions\(^8\). Zeng et al.\(^9\) proposed a generalized flow interception location-allocation model for locating flow demand facilities on a network. Tanaka et al.\(^10\) considered the FILM that covers passenger flow on a railway network. The FILM has been applied to the location of vehicle inspection stations\(^11\), alternative fuel stations\(^12\), and park-and-ride facilities\(^13\). In most of the location models reviewed above, the number of facilities to be located is given and cannot be determined endogenously. Examining how the number of facilities affects the coverage level will therefore supplement further flow covering location models.

\(^*\)University of Yamanashi

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The optimal number of facilities has been studied for facilities with point-based demand\textsuperscript{14–19}. Obviously, these studies cannot deal with flow demand. For alternative fuel stations, the sufficient number of stations has been calculated\textsuperscript{20–22}. These works, however, still failed to deal with flow demand because they assumed that drivers use the nearest station from their home. Although Miyagawa\textsuperscript{23} explicitly considered flow demand in estimating the sufficient number of stations, the focus was on whether limited-range vehicles can make the round trip between origins and destinations.

In this paper, we develop an analytical model for determining the number of flow demand facilities to achieve a certain level of coverage. A trip is called covered if the traveler can visit at least one facility within a specified deviation distance, which is the distance from the shortest path between origin and destination to the facility. The coverage level is then represented as the combination of the deviation distance and the proportion of covered trips. The major characteristics of the model are as follows. First, the model yields analytical expressions for the proportion of covered trips, leading to a fundamental understanding of the flow coverage. Second, the model incorporates any level of coverage and can be applied to various cities with different trip lengths and deviation distances. Finally, the model deals with the coverage by the second nearest facility as well as the first nearest facility. The coverage by two facilities is important when facilities are subject to failures due to accidents, natural disasters, and intentional strikes, and has been addressed in covering location problems\textsuperscript{24–26}. The present model thus provides a versatile framework for determining the number of flow demand facilities.

To obtain analytical expressions for the proportion of covered trips, we use the distribution of deviation distances for grid and random patterns of facilities. Since actual locations of facilities can be regarded as the intermediate between grid and random patterns, the theoretical results of these patterns give an insight into empirical analysis of actual locations. The distribution of deviation distances to the nearest facility was derived by Miyagawa\textsuperscript{27} for the grid and random patterns. We extend this to the distribution of deviation distances to the second nearest facility and the joint distribution of deviation distances to the first and second nearest facilities for the random pattern. This extension enables us to consider the coverage by the second nearest facility.

The remainder of this paper is organized as follows. The next section develops a model for determining the number of flow demand facilities and obtains the optimal number of facilities to achieve a certain level of coverage. The following section examines the effect of the city shape on the optimal number of facilities. The final section presents concluding remarks.

## 2 Model

Assume that origins and destinations of trips are uniformly distributed in an unbounded region. The uniform travel demand serves as a basis for further analysis with more realistic travel demand. For example, the travel demand that depends on the trip length can be described with spatial interaction models\textsuperscript{28}. The distance is measured as the rectilinear distance. The rectilinear distance between two points \((x_1, y_1)\) and \((x_2, y_2)\) is defined as \(|x_1 - x_2| + |y_1 - y_2|\). Miyagawa\textsuperscript{27} demonstrated that the actual deviation distance to visit a facility on a road network can be approximated by the rectilinear deviation distance on a continuous plane.

Let \(t_x\) and \(t_y\) be the horizontal and vertical distances between origin \(O\) and destination \(D\), respectively. The trip length is then \(t_x + t_y\). Assume that \(t_x \geq t_y\) without loss of generality. Since the distance is rectilinear, there exists an infinite number of shortest paths between \(O\) and \(D\). The
set of the shortest paths is expressed as the rectangle with side lengths $t_x$ and $t_y$, which we call the shortest path rectangle. Travelers are assumed to deviate from their shortest paths to visit facilities. Note that if a facility is inside the shortest path rectangle, the traveler can visit the facility without deviation. The deviation distance to a facility is defined as the distance between the facility and the closest point on the shortest path rectangle. The region that the traveler can visit within a deviation distance $r$ is given by an octagon, as shown in Fig. 1. If a facility is inside the octagon with a specified deviation distance, the trip is called covered by the facility.

![Figure 1. Region that the traveler can visit within a deviation distance.](image)

Facilities are represented as points of grid and random patterns. The grid and random patterns are assumed to continue infinitely in an unbounded region. Let $S$ be the area of a study region and $n$ be the number of facilities in the region. The density of facilities (number of facilities per unit area) is then $\rho = n/S$.

### 2.1 Coverage by the nearest facility

Let $F_1(r_1)$ be the cumulative distribution function of the rectilinear deviation distance to the nearest facility, that is, the probability that the deviation distance is less than or equal to $r_1$. Thus, $F_1(r_1)$ represents the proportion of trips covered by the nearest facility. The problem to find the minimum number of facilities required to achieve a certain level of coverage is formulated as follows:

$$\begin{align*}
\text{minimize} \quad & n \\
\text{subject to} \quad & F_1(r_1) \geq \alpha,
\end{align*}$$

where $\alpha$ ($0 < \alpha \leq 1$) is a target level of coverage. If $\alpha = 1$, the complete coverage is required.

Suppose first that facilities are regularly distributed on a square grid with spacing $a$, as shown in Fig. 2. The density of facilities is then $\rho = 1/a^2$. The proportion of covered trips $F_1(r_1)$ for the grid pattern was derived by Miyagawa for $t_y \leq t_x \leq a$. To consider longer trips, we also deal with the case of $t_y < a < t_x$. Note that if $a \leq t_y \leq t_x$, the shortest path rectangle always contains a facility and the traveler can visit the facility without deviation, i.e., $F_1(r_1) = 1$. Since the grid pattern continues infinitely, the study region can be confined to the square centered at the facility with side length $a$. If the deviation distance to a facility is less than or equal to $r_1$, the facility is inside the octagon around the shortest path rectangle. This also means that the center of the shortest path rectangle lies inside the same octagon but centered at the facility, as shown in Fig. 3. Thus, $F_1(r_1)$ is the probability that the center of the shortest path rectangle lies inside the octagon centered at the facility. Since the center of the shortest path rectangle is uniformly distributed, we have

$$F_1(r_1) = \frac{S_1(r_1)}{S},$$

where $S$ is the area of the study region and $S_1(r_1)$ is the area of the octagon with deviation distance $r_1$. This expression is used to solve for the minimum number of facilities required to achieve the target level of coverage.
where $S$ is the area of the square centered at the facility with side length $a$ and $S_1(r_1)$ is the area of the intersection of the octagon with deviation distance $r_1$ and the square, as shown in Fig. 2. The area of the intersection is if $t_y \leq t_x \leq a$,

$$S_1(r_1) = \begin{cases} 2r_1(r_1 + t_x + t_y) + t_xt_y, & 0 < r_1 \leq \frac{a-t_x}{2}, \\ a(2r_1 + t_x + t_y) - \frac{1}{2}(a^2 + t_y^2), & \frac{a-t_y}{2} < r_1 \leq \frac{a-t_x}{2}, \\ a^2 - 2(a - r_1 - \frac{t_x+t_y}{2})^2, & \frac{a-t_y}{2} < r_1 \leq a - \frac{t_x+t_y}{2}, \end{cases}$$  \tag{3}$$

and if $t_y < a < t_x$,

$$S_1(r_1) = a(2r_1 + t_y), \quad 0 < r_1 \leq \frac{a-t_y}{2}. \tag{4}$$

Substituting $S = a^2 (= 1/\rho)$ and $S_1(r_1)$ into (2) yields if $t_y \leq t_x \leq 1/\sqrt{\rho}$,

$$F_1(r_1) = \begin{cases} \rho(2r_1(t_1 + t_y) + t_xt_y), & 0 < r_1 \leq \frac{1}{\sqrt{\rho}} - \frac{t_y}{2}, \\ \sqrt{\rho}(2r_1 + t_x + t_y) - \frac{1}{2}(1 + \rho t_y^2), & \frac{1}{\sqrt{\rho}} - \frac{t_y}{2} < r_1 \leq \frac{1}{\sqrt{\rho}} - \frac{t_x}{2}, \\ 1 - 2\left(1 - \sqrt{\rho}r_1 - \frac{\sqrt{\rho}}{2}(t_x + t_y)\right)^2, & \frac{1}{\sqrt{\rho}} - \frac{t_x}{2} < r_1 \leq \frac{1}{\sqrt{\rho}} - \frac{t_x+t_y}{2}, \end{cases}$$  \tag{5}$$

(Miyagawa\textsuperscript{27}) and if $t_y < 1/\sqrt{\rho} < t_x$,

$$F_1(r_1) = \sqrt{\rho}(2r_1 + t_y), \quad 0 < r_1 \leq \frac{1}{2\sqrt{\rho}} - \frac{t_y}{2}. \tag{6}$$

If $r_1 \geq 1/\sqrt{\rho} - (t_x + t_y)/2$ or $r_1 \geq 1/(2\sqrt{\rho}) - t_y/2$, the complete coverage $F_1(r_1) = 1$ is achieved.

![Figure 2. Grid pattern of facilities.](image1)

![Figure 3. Facility inside the octagon.](image2)

Suppose next that facilities are randomly distributed with density $\rho$. The proportion of covered trips $F_1(r_1)$ for the random pattern was derived by Miyagawa\textsuperscript{27} as follows. Let $P(x, S)$ be the probability that a region of area $S$ contains exactly $x$ facilities. Since $F_1(r_1)$ is the probability that the octagon with deviation distance $r_1$ contains at least one facility, we have

$$F_1(r_1) = 1 - P(0, S_1), \tag{7}$$

where $S_1$ is the area of the octagon with deviation distance $r_1$. The area of the octagon is

$$S_1 = 2r_1(r_1 + t_x + t_y) + t_xt_y \tag{8}$$
and \( P(x, S) \) is expressed as the Poisson distribution
\[
P(x, S) = \frac{(\rho S)^x}{x!} e^{-\rho S},
\] (9)
(Clark and Evans\textsuperscript{29}). Substituting into (7) yields
\[
F_1(r_1) = 1 - \exp[-\rho(2r_1 + tx + ty)],
\] (10)
(Miyagawa\textsuperscript{27}). Since \( F_1(r_1) < 1 \) for all \( r_1 > 0 \), the complete coverage cannot be achieved.

The proportions of covered trips \( F_1(r_1) \) for the grid and random patterns are shown in Fig. 4, where the density of facilities is \( \rho = 0.1 \). Note that \( F_1(r_1) \) increases with the deviation distance \( r_1 \) and the trip length \( tx, ty \). Note also that \( F_1(r_1) \) for \( tx = 5.0, ty = 0 \) is smaller than that for \( tx = 2.5, ty = 2.5 \). This is because the area of the shortest path rectangle for \( tx = 5.0, ty = 0 \) is smaller than that for \( tx = 2.5, ty = 2.5 \). In fact, \( F_1(0) = 0 \) for \( tx = 5.0, ty = 0 \) because the area of the shortest path rectangle is zero. It follows that \( F_1(r_1) \) varies according to the relative position of origin and destination as well as the trip length. For the effect of the pattern of facilities, \( F_1(r_1) \) for the random pattern is smaller than that for the grid pattern.

![Figure 4. Proportion of trips covered by the nearest facility: (a) Grid; (b) Random.](image)

The solution for the problem (1) for the grid and random patterns is shown in Table 1, where the area of the study region is \( S = 100 \). The optimal number of facilities increases with the target coverage level \( \alpha \), and decreases with the deviation distance \( r_1 \) and the trip length \( tx, ty \). The optimal number of facilities also depends on the relative position of origin and destination. If the trip length is the same, the optimal number of facilities for \( tx = ty \) is smaller than that for \( tx \neq ty \). Note that, except for \( tx = 5.0, ty = 0 \), the random pattern requires more facilities than the grid pattern to achieve the same level of coverage.

### 2.2 Coverage by the second nearest facility

Let \( F_2(r_2) \) be the cumulative distribution function of the rectilinear deviation distance to the second nearest facility. Thus, \( F_2(r_2) \) represents the proportion of trips covered by the second nearest facility. The problem to find the minimum number of facilities required to achieve a certain level of coverage is formulated as follows:

\[
\begin{align*}
\text{minimize} & \quad n \\
\text{subject to} & \quad F_2(r_2) \geq \alpha.
\end{align*}
\] (11)
Table 1. Number of facilities required to achieve $F_1(r_1) \geq \alpha$: (a) $t_x = 2.0, t_y = 2.0$; (b) $t_x = 2.5, t_y = 2.5$; (c) $t_x = 5.0, t_y = 0$.

| $\alpha \backslash r_1$ | (a) 0.1 0.2 0.3 | (b) 0.1 0.2 0.3 | (c) 0.1 0.2 0.3 |
|-----------------------|----------------|----------------|----------------|
| Grid                  |                |                |                |
| 0.2                   | 5 4 4          | 3 3 3          | 100 25 12      |
| 0.4                   | 9 8 7          | 6 5 5          | 400 100 45     |
| 0.6                   | 13 11 10       | 9 8 7          | 900 225 100    |
| 0.8                   | 17 15 13       | 12 10 9        | 1600 400 178   |
| 1.0                   | 23 21 19       | 15 14 13       | 2500 625 278   |
| Random                |                |                |                |
| 0.2                   | 5 4 4          | 4 3 3          | 22 11 8        |
| 0.4                   | 11 9 8         | 8 7 6          | 51 25 17       |
| 0.6                   | 20 17 14       | 13 11 10       | 90 45 29       |
| 0.8                   | 34 29 25       | 23 20 18       | 158 78 51      |

Suppose that facilities are randomly distributed with density $\rho$. The proportion of covered trips $F_2(r_2)$ is the probability that the octagon with deviation distance $r_2$ contains at least two facilities, and given by

$$F_2(r_2) = 1 - P(0, S_2) - P(1, S_2),$$

where

$$S_2 = 2r_2(r_2 + t_x + t_y) + t_xt_y$$

is the area of the octagon with deviation distance $r_2$. Using the Poisson distribution (9), we have

$$F_2(r_2) = 1 - [\rho[2r_2(r_2 + t_x + t_y) + t_xt_y] + 1] \exp[-\rho[2r_2(r_2 + t_x + t_y) + t_xt_y]].$$

The proportion of covered trips $F_2(r_2)$ is shown in Fig. 5, where the density of facilities is $\rho = 0.1$. Note that if the deviation distance is the same, $F_2(r_2)$ is smaller than $F_1(r_1)$.

The solution for the problem (11) is shown in Table 2, where the area of the study region is $S = 100$. By comparing with Table 1, we can see that more facilities are required to cover trips by two facilities.
Table 2. Number of facilities required to achieve $F_2(r_2) \geq \alpha$: (a) $t_x = 2.0, t_y = 2.0$; (b) $t_x = 2.5, t_y = 2.5$; (c) $t_x = 5.0, t_y = 0$.

| $\alpha \setminus r_2$ | (a)  | (b)  | (c)  |
|-------------------------|------|------|------|
| 0.2                     | 18   | 12   | 81   |
|                         | 0.2  | 29   | 19   | 135  |
|                         | 0.2  | 42   | 28   | 199  |
|                         | 0.2  | 63   | 42   | 294  |

2.3 Coverage by the first and second nearest facilities

Let $F_{12}(r_1, r_2)$ be the joint cumulative distribution function of the deviation distances to the first and second nearest facilities. Thus, $F_{12}(r_1, r_2)$ represents the proportion of trips covered by the first and second nearest facilities. The joint distribution allows us to deal with the two distances simultaneously. Note that if $r_2 = \infty$, $F_{12}(r_1, r_2)$ reduces to $F_1(r_1)$, and if $r_1 = r_2$, $F_{12}(r_1, r_2)$ reduces to $F_2(r_2)$. The problem to find the minimum number of facilities required to achieve a certain level of coverage is formulated as follows:

$$\begin{align*}
\text{minimize} & \quad n \\
\text{subject to} & \quad F_{12}(r_1, r_2) \geq \alpha.
\end{align*}$$

Suppose that facilities are randomly distributed with density $\rho$. The proportion of covered trips $F_{12}(r_1, r_2)$ is the probability that the octagon with deviation distance $r_1$ contains at least one facility and the octagon with deviation distance $r_2$ contains at least two facilities, and given by

$$F_{12}(r_1, r_2) = 1 - P(0, S_1) - P(1, S_1)P(0, S_2 - S_1).$$

Using the Poisson distribution (9), we have

$$F_{12}(r_1, r_2) = 1 - \exp[-\rho(2r_1(t_x + t_y) + t_xt_y)] - \rho(2r_1(t_x + t_y) + t_xt_y) \exp[-\rho(2r_2(t_x + t_y) + t_xt_y)].$$

The proportion of covered trips $F_{12}(r_1, r_2)$ is shown in Fig. 6, where the density of facilities is $\rho = 0.1$ and $t_x = 2.5, t_y = 2.5$.
numbers of facilities to achieve $F_{12}(0.1, 0.2) \geq \alpha$ is greater than or equal to $F_1(0.1) \geq \alpha$ and $F_2(0.2) \geq \alpha$.

Table 3. Number of facilities required to achieve $F_{12}(r_1, r_2) \geq \alpha$: (a) $t_x = 2.0, t_y = 2.0$; (b) $t_x = 2.5, t_y = 2.5$; (c) $t_x = 5.0, t_y = 0$.

| $r_1$ | (a) | (b) | (c) |
|-------|-----|-----|-----|
| $r_2$ | 0.1 | 0.1 | 0.2 |
| 0.2   | 15  | 10  | 47  |
| 0.4   | 25  | 17  | 80  |
| 0.6   | 37  | 25  | 119 |
| 0.8   | 54  | 37  | 182 |

3 Effect of city shape

The optimal number of facilities depends on the shape of the study region because the shape affects the trip length. In this section, we examine the effect of the shape of the study region on the optimal number of facilities.

Consider three shapes of a rectangular city with the same area, as depicted in Fig. 7, where the side lengths of the city are (a) $a_1 = 10, a_2 = 10$, (b) $a_1 = 10\sqrt{2}, a_2 = 5\sqrt{2}$, (c) $a_1 = 10\sqrt{3}, a_2 = 10/\sqrt{3}$. The aspect ratios of the city $W = a_1/a_2$ are then (a) $W = 1$, (b) $W = 2$, (c) $W = 3$. As a representative trip length, we use the average distance between uniformly distributed origins and destinations in the city. If origins and destinations are uniformly distributed in the rectangle with side lengths $a_1$ and $a_2$, the average horizontal and vertical distances are

$$E(t_x) = \frac{a_1}{3}, \quad E(t_y) = \frac{a_2}{3},$$

respectively.

![Figure 7. Shapes of a rectangular city: (a) $W = 1$; (b) $W = 2$; (c) $W = 3$.](image)

The proportions of covered trips $F_1(r_1)$ for the three shapes of the city are shown in Fig. 8, where the number of facilities is $n = 5$. For the grid pattern, as the aspect ratio of the city increases, $F_1(r_1)$ decreases. For the random pattern, in contrast, the shape of the city has little effect on $F_1(r_1)$.

The solutions for the problem (1) for the three shapes of the city are shown in Table 4. For the grid pattern, as the aspect ratio of the city increases, more facilities are required to achieve a high level of service such as $\alpha \geq 0.6$. If $W = 3$, there exists a small difference in the optimal number of facilities between the grid and random patterns. The grid pattern of facilities is thus effective particularly for a square city but not for a rectangular city. For the random pattern, the optimal
number of facilities is independent of the shape of the city. It follows that the optimal number of facilities can be used without considering the shape of the study region.

Table 4. Number of facilities required to achieve $F_1(r_1) \geq \alpha$ in a rectangular city.

|        | $W = 1$ |           | $W = 2$ |           | $W = 3$ |           |
|--------|---------|-----------|---------|-----------|---------|-----------|
|        | $\alpha$ | 0.1 | 0.2 | 0.3 | 0.1 | 0.2 | 0.3 | 0.1 | 0.2 | 0.3 |
| Grid   |         | 0.2 | 2   | 2   | 2   | 2   | 2   | 2   | 2   | 2   |
|        |         | 0.4 | 2   | 3   | 3   | 4   | 3   | 4   | 3   | 3   |
|        |         | 0.6 | 5   | 5   | 6   | 5   | 8   | 7   | 6   | 6   |
|        |         | 0.8 | 7   | 6   | 10  | 9   | 8   | 15  | 12  | 11  |
|        |         | 1.0 | 9   | 9   | 16  | 14  | 12  | 23  | 19  | 16  |
| Random |         | 0.2 | 2   | 2   | 2   | 2   | 2   | 2   | 2   | 2   |
|        |         | 0.4 | 5   | 4   | 4   | 5   | 4   | 5   | 4   | 4   |
|        |         | 0.6 | 8   | 7   | 6   | 8   | 6   | 8   | 7   | 6   |
|        |         | 0.8 | 13  | 12  | 11  | 13  | 12  | 13  | 12  | 11  |

4 Conclusions

This paper has presented an analytical model for determining the number of flow demand facilities. The proportions of covered trips within a specified deviation distance have been derived for grid and random patterns of facilities. The analytical expressions for the proportions of covered trips demonstrate how the deviation distance, the number of facilities, and the pattern of facilities affect the level of coverage. These relationships provide a fundamental understanding of the flow coverage.

The model supplies building blocks for location analysis of flow demand facilities as follows. First, the model helps planners to estimate the number of facilities required to achieve a certain level of coverage. Note that the estimation requires neither much data nor the computation of the optimal location. The estimated number can be used as an input into location models of flow demand facilities. Second, the optimal number of facilities allows us to evaluate the number of existing facilities, thereby assisting decisions about opening new facilities and closing existing.
facilities. Finally, the coverage by the second nearest facility is useful for the case where facilities are subject to failures.

The model can be extended in future research. First, more realistic travel demand should be incorporated. Although the uniform travel demand is important as the first approximation, the travel demand usually decreases with the trip length. Second, the hierarchy of facilities should be considered. The deviation distance to higher level facilities would be longer than that to lower level facilities. Finally, comparing with other patterns of facilities and other shapes of cities would be interesting.

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