Asymptotics for Fermi curves: small magnetic potential

Gustavo de Oliveira
Department of Mathematics,
University of British Columbia, Canada
goliveira5d@gmail.com
March 1, 2010

Abstract

We consider complex Fermi curves of electric and magnetic periodic fields. These are analytic curves in $\mathbb{C}^2$ that arise from the study of the eigenvalue problem for periodic Schrödinger operators. We characterize a certain class of these curves in the region of $\mathbb{C}^2$ where at least one of the coordinates has “large” imaginary part. The new results in this work extend previous results in the absence of magnetic field to the case of “small” magnetic field. Our theorems can be used to show that generically these Fermi curves belong to a class of Riemann surfaces of infinite genus.

1 Introduction

In [1], the authors introduced a class of Riemann surfaces of infinite genus that are “asymptotic to” a finite number of complex lines joined by infinite many handles. These surfaces are constructed by pasting together a compact submanifold of finite genus, plane domains, and handles. All these components satisfy a number of geometric/analytic hypotheses stated in [1] that specify the asymptotic holomorphic structure of the surface. The class of surfaces obtained in this way yields an extension of the classical theory of compact Riemann surfaces that has analogues of many theorems of the classical theory. It was proven in [1] that this new class includes quite general hyperelliptic surfaces, heat curves (which are spectral curves associated to a certain “heat-equation”), and Fermi curves with zero magnetic potential. In order to verify the geometric/analytic hypotheses for the latter the authors proved two “asymptotic” theorems similar to the ones we prove below. This is the main step needed to verify these hypotheses. In this work we extend their results to Fermi curves with “small” magnetic potential.

There are two immediate applications of our results. First, as we have already mentioned, one can use our theorems for verifying the geometric/analytic hypotheses of [1] for Fermi curves with small magnetic potential. This would show that these curves belong to the class of Riemann surfaces mentioned above. Secondly, one can prove that a class of these curves
are irreducible (in the usual algebraic-geometrical sense). Both these applications were done in [1] for Fermi curves with zero magnetic potential.

Complex Fermi curves (and other similar spectral curves) have been studied, in different perspectives, in the absence of magnetic field [1, 2, 3, 4, 5], and in the presence of magnetic field [6]. Some results on the real Fermi curve in the high-energy region were obtained in [7]. There one also finds a short description of the existing results on periodic magnetic Schrödinger operators. An even more general review is presented in [8]. To our knowledge our work provides new results on complex Fermi curves with magnetic field. At this moment we are only able to handle the case of “small” magnetic potential. The asymptotic characterization of Fermi curves with arbitrarily large magnetic potential remains as an open problem. In order to prove our theorems we follow the same strategy as [1]. The presence of magnetic field makes the analysis considerably harder and requires new estimates. As it was pointed out in [7, 8], the study of an operator with magnetic potential is essentially more complicated than the study of the operator with just an electric potential. This seems to be the case in this problem as well.

Before we outline our results let us introduce some definitions. Let $\Gamma$ be a lattice in $\mathbb{R}^2$ and let $A_1, A_2$ and $V$ be real-valued functions in $L^2(\mathbb{R}^2)$ that are periodic with respect to $\Gamma$. Set $A := (A_1, A_2)$ and define the operator

$$H(A, V) := (i\nabla + A)^2 + V$$

acting on $L^2(\mathbb{R}^2)$, where $\nabla$ is the gradient operator in $\mathbb{R}^2$. For $k \in \mathbb{R}^2$ consider the following eigenvalue-eigenvector problem in $L^2(\mathbb{R}^2)$ with boundary conditions,

$$H(A, V)\varphi = \lambda\varphi,$$

$$\varphi(x + \gamma) = e^{ik \cdot \gamma} \varphi(x)$$

for all $x \in \mathbb{R}^2$ and all $\gamma \in \Gamma$. Under suitable hypotheses on the potentials $A$ and $V$ this problem is self-adjoint and its spectrum is discrete. It consists of a sequence of real eigenvalues

$$E_1(k, A, V) \leq E_2(k, A, V) \leq \cdots \leq E_n(k, A, V) \leq \cdots$$

For each integer $n \geq 1$ the eigenvalue $E_n(k, A, V)$ defines a continuous function of $k$. From the above boundary condition it is easy to see that this function is periodic with respect to the dual lattice

$$\Gamma^\# := \{b \in \mathbb{R}^2 \mid b \cdot \gamma \in 2\pi \mathbb{Z} \text{ for all } \gamma \in \Gamma\},$$

where $b \cdot \gamma$ is the usual scalar product on $\mathbb{R}^2$. It is customary to refer to $k$ as the crystal momentum and to $E_n(k, A, V)$ as the $n$-th band function. The corresponding normalized eigenfunctions $\varphi_{n, k}$ are called Bloch eigenfunctions.

The operator $H(A, V)$ (and its three-dimensional counterpart) is important in solid state physics. It is the Hamiltonian of a single electron under the influence of magnetic field with vector potential $A$, and electric field with scalar potential $V$, in the independent electron
model of a two-dimensional solid [9]. The classical framework for studying the spectrum of a differential operator with periodic coefficients is the Floquet (or Bloch) theory [9, 10, 11]. Roughly speaking, the main idea of this theory is to “decompose” the original eigenvalue problem, which usually has continuous spectrum, into a family of boundary value problems, each one having discrete spectrum. In our context this leads to decomposing the problem $H(A, V)\varphi = \lambda \varphi$ (without boundary conditions) into the above $k$-family of boundary value problems.

Let $U_k$ be the unitary transformation on $L^2(\mathbb{R}^2)$ that acts as

$$U_k : \varphi(x) \mapsto e^{ik \cdot x} \varphi(x).$$

By applying this transformation we can rewrite the above problem and put the boundary conditions into the operator. Indeed, if we define

$$H_k(A, V) := U^{-1}_k H(A, V) U_k$$

and $\psi := U^{-1}_k \varphi$, then the above problem is unitarily equivalent to

$$H_k(A, V)\psi = \lambda \psi \quad \text{for} \quad \psi \in L^2(\mathbb{R}^2/\Gamma).$$

Furthermore, a simple (formal) calculation shows that

$$H_k(A, V) = (i\nabla + A - k)^2 + V.$$

The real “lifted” Fermi curve of $(A, V)$ with energy $\lambda \in \mathbb{R}$ is defined as

$$\hat{F}_{\lambda, R}(A, V) := \{k \in \mathbb{R}^2 \mid (H_k(A, V) - \lambda)\varphi = 0 \text{ for some } \varphi \in \mathcal{D}_{H_k(A,V)} \setminus \{0\}\},$$

where $\mathcal{D}_{H_k(A,V)} \subset L^2(\mathbb{R}^2/\Gamma)$ denotes the (dense) domain of $H_k(A, V)$. The adjective “lifted” indicates that $\hat{F}_{\lambda, R}(A, V)$ is a subset of $\mathbb{R}^2$ rather than $\mathbb{R}^2/\Gamma^\#$. As we may replace $V$ by $V - \lambda$, we only discuss the case $\lambda = 0$ and write $\hat{F}_R(A, V)$ in place of $\hat{F}_{0, R}(A, V)$ to simplify the notation. Let $|\Gamma| := \int_{\mathbb{R}^2/\Gamma} dx$ and $\hat{A}(0) := |\Gamma|^{-1} \int_{\mathbb{R}^2/\Gamma} A(x) \, dx$. Since $H_k(A, V)$ is equal to $H_{k - \hat{A}(0)}(A - \hat{A}(0), V)$, if we perform the change of coordinates $k \mapsto k + \hat{A}(0)$ and redefine $A - \hat{A}(0) \rightarrow A$ we may assume, without loss of generality, that $\hat{A}(0) = 0$. The dual lattice $\Gamma^\#$ acts on $\mathbb{R}^2$ by translating $k \mapsto k + b$ for $b \in \Gamma^\#$. This action maps $\hat{F}_R(A, V)$ to itself because for each $n \geq 1$ the function $k \mapsto E_n(k, A, V)$ is periodic with respect to $\Gamma^\#$. In other words, the real lifted Fermi curve “is periodic” with respect to $\Gamma^\#$. Define

$$F_R(A, V) := \hat{F}_R(A, V)/\Gamma^\#.$$ We call $F_R(A, V)$ the real Fermi curve of $(A, V)$. It is a curve in the torus $\mathbb{R}^2/\Gamma^\#$.

The above definitions and the real Fermi curve have physical meaning. It is useful and interesting, however, to study the “complexification” of these curves. Knowledge about the complexified curves may provide information about the real counterparts. For complex-valued functions $A_1$, $A_2$ and $V$ in $L^2(\mathbb{R}^2)$ and for $k \in \mathbb{C}^2$ the above problem is no longer self-adjoint.
Its spectrum, however, remains discrete. It is a sequence of eigenvalues in the complex plane. From the boundary condition in the original problem it is easy to see that the family of functions \( k \mapsto E_n(k, A, V) \) remains periodic with respect to \( \Gamma^\# \). Moreover, the transformation \( U_k \) is no longer unitary but it is still bounded and invertible and it still preserves the spectrum, that is, we can still rewrite the original problem in the form \( H_k(A, V)\psi = \lambda\psi \) for \( \psi \in L^2(\mathbb{R}^2/\Gamma) \) without modifying the eigenvalues. Thus, it makes sense to define

\[
\hat{F}(A, V) := \{ k \in \mathbb{C}^2 \mid H_k(A, V)\varphi = 0 \text{ for some } \varphi \in D_{H_k(A, V)} \setminus \{0\}\},
\]

\[
F(A, V) := \hat{F}(A, V)/\Gamma^\#.
\]

We call \( \hat{F}(A, V) \) and \( F(A, V) \) the complex “lifted” Fermi curve and the complex Fermi curve, respectively. When there is no risk of confusion we refer to either simply as Fermi curve.

We are now ready to outline our results. When \( A \) and \( V \) are zero the (free) Fermi curve can be found explicitly. It consists of two copies of \( \mathbb{C} \) with the points \(-b_2 + ib_1\) (in the first copy) and \(b_2 + ib_1\) (in the second copy) identified for all \((b_1, b_2) \in \Gamma^\# \) with \(b_2 \neq 0\). In this work we prove that in the region of \( \mathbb{C}^2 \) where \( k \in \mathbb{C}^2 \) has “large” imaginary part the Fermi curve (for nonzero \( A \) and \( V \)) is “close to” the free Fermi curve. In a compact form, our main result (that will be stated precisely in Theorems 1 and 2) is essentially the following.

**Main result.** Suppose that \( A \) and \( V \) have some regularity and assume that (in a suitable norm) \( A \) is smaller than a constant given by the parameters of the problem. Write \( k \) in \( \mathbb{C}^2 \) as \( k = u + iv \) with \( u \) and \( v \) in \( \mathbb{R}^2 \) and suppose that \( |v| \) is larger than a constant given by the parameters of the problem. (Recall that the free Fermi curve is two copies of \( \mathbb{C} \) with certain points in one copy identified with points in the other one.) Then, in this region of \( \mathbb{C}^2 \), the Fermi curve of \( A \) and \( V \) is very close to the free Fermi curve, except that instead of two planes we may have two deformed planes, and identifications between points can open up to handles that look like \( \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1z_2 = \text{constant}\} \) in suitable local coordinates.

The proof of our results has basically three steps:

- We first derive very detailed information about the free Fermi curve (which is explicitly known). Then, to compute the interacting Fermi curve we have to find the kernel of \( H \) in \( L^2(\mathbb{R}^2) \) with the above boundary conditions.

- In the second step of the proof we derive a number of estimates for showing that this kernel has finite dimension for small \( A \) and \( k \in \mathbb{C}^2 \) with large imaginary part. Our strategy here is similar to the Feshbach method in perturbation theory [12]. Indeed, we prove that in the complement of the kernel of \( H \) in \( L^2(\mathbb{R}^2) \), after a suitable invertible change of variables in \( L^2(\mathbb{R}^2) \), the operator \( H \) multiplied by the inverse of the operator that implements this change of variables is a compact perturbation of the identity that is invertible for such \( A \) and \( k \). This reduces the problem of finding the kernel to finite dimension and thus we can write local defining equations for the Fermi curve.
In the third step of the proof we use these equations to study the Fermi curve. A few more estimates and the implicit function theorem gives us the deformed planes. The handles are obtained using a quantitative Morse lemma from [13] that is available in the Appendix A.

Steps two and three contain most of the novelties in this work. The critical part of the proof is the second step. The main difficulty arises due to the presence of the term $A \cdot i\nabla$ in the Hamiltonian $H(A, V)$. When $A$ is large, taking the imaginary part of $k \in \mathbb{C}^2$ arbitrarily large is not enough to control this term—it is not enough to make its contribution small and hence have the interacting Fermi curve as a perturbation of the free Fermi curve. (The term $V$ in $H(A, V)$ is easily controlled by this method.) However, the proof can be implemented by assuming that $A$ is small.

This work is organized as follows. In §2 we collect some properties of the free Fermi curve and in §3 we define $\varepsilon$-tubes about it. In §4 we state our main results and in §5 we describe the general strategy of analysis used to prove them. Subsequently, we implement this strategy by proving a number of lemmas and propositions in §6 to §10, which we put together later in §11 and §12 to prove our main theorems. The proof of the estimates of §9 and §10 are left to the Appendices B and C.

Acknowledgments. I would like to thank Professor Joel Feldman for suggesting this problem and for the many discussions I have had with him. I am also grateful to Alessandro Michelangeli for useful comments about the manuscript. This work is part of the author’s Ph.D. thesis [13] defended at the University of British Columbia in Vancouver, Canada.

2 The free Fermi curve

When the potentials $A$ and $V$ are zero the curve $\tilde{F}(A, V)$ can be found explicitly. In this section we collect some properties of this curve. For $\nu \in \{1, 2\}$ and $b \in \Gamma^#$ set

\[ N_{b,\nu}(k) := (k_1 + b_1) + i(-1)^\nu(k_2 + b_2), \]
\[ \mathcal{N}_\nu(b) := \{ k \in \mathbb{C}^2 \mid N_{b,\nu}(k) = 0 \}, \]
\[ N_b(k) := N_{b,1}(k)N_{b,2}(k), \]
\[ \mathcal{N}_b := \mathcal{N}_{1}(b) \cup \mathcal{N}_{2}(b), \]
\[ \theta_\nu(b) := \frac{1}{2}((-1)^\nu b_2 + ib_1). \]

Observe that $\mathcal{N}_\nu(b)$ is a line in $\mathbb{C}^2$. The free lifted Fermi curve is an union of these lines. Here is the precise statement.

Proposition 1 (The free Fermi curve). The curve $\tilde{F}(0, 0)$ is the locally finite union

\[ \bigcup_{b \in \Gamma^#} \bigcup_{\nu \in \{1, 2\}} \mathcal{N}_\nu(b). \]

In particular, the curve $F(0, 0)$ is a complex analytic curve in $\mathbb{C}^2/\Gamma^#$. 
The proof of this proposition is straightforward. It can be found in [13]. Here we only give its first part.

**Proof of Proposition 1 (first part).** For all \( k \in \mathbb{C}^2 \) the functions \( \{ e^{ib \cdot x} \mid b \in \Gamma^\# \} \) form a complete set of eigenfunctions for \( H_k(0,0) \) in \( L^2(\mathbb{R}^2/\Gamma) \) satisfying

\[
H_k(0,0)e^{ib \cdot x} = (i\nabla - k)^2 e^{ib \cdot x} = (b + k)^2 e^{ib \cdot x} = N_b(k)e^{ib \cdot x}.
\]

Hence,

\[
\hat{F}(0,0) = \{ k \in \mathbb{C}^2 \mid N_b(k) = 0 \text{ for some } b \in \Gamma^\# \} = \bigcup_{b \in \Gamma^\#} N_b = \bigcup_{b \in \Gamma^\#} \bigcup_{\nu \in \{1,2\}} N\nu(b).
\]

This is the desired expression for \( \hat{F}(0,0) \).

Figure 1: Sketch of \( \hat{F}(0,0) \) and \( F(0,0) \) when both \( ik_1 \) and \( k_2 \) are real.

The lines \( N\nu(b) \) have the following properties (see [13] for a proof).

**Proposition 2 (Properties of \( N\nu(b) \)).** Let \( \nu \in \{1,2\} \) and let \( b, c, d \in \Gamma^\# \). Then:

(a) \( N\nu(b) \cap N\nu(c) = \emptyset \) if \( b \neq c \);
(b) \( \text{dist}(N\nu(b), N\nu(c)) = \frac{1}{\sqrt{2}}|b - c| \);
(c) \( N1(b) \cap N2(c) = \{(i\theta_1(c) + i\theta_2(b), \theta_1(c) - \theta_2(b))\} \);
(d) the map \( k \mapsto k + d \) maps \( N\nu(b) \) to \( N\nu(b - d) \);
(e) the map \( k \mapsto k + d \) maps \( N1(b) \cap N2(c) \) to \( N1(b - d) \cap N2(c - d) \).
Let us briefly describe what the free Fermi curve looks like. In the Figure 1 there is a sketch of the set of \((k_1, k_2) \in \hat{\mathcal{F}}(0, 0)\) for which both \(i k_1\) and \(k_2\) are real, for the case where the lattice \(\Gamma\) has points over the coordinate axes, that is, it has points of the form \((b_1, 0)\) and \((0, b_2)\). Observe that, in particular, Proposition 2 yields

\[
\mathcal{N}_1(0) \cap \mathcal{N}_2(b) = \{(i \theta_1(b), \theta_1(b))\},
\]
\[
\mathcal{N}_1(-b) \cap \mathcal{N}_2(0) = \{(i \theta_2(-b), \theta_2(b))\},
\]

the map \(k \mapsto k + b\) maps \(\mathcal{N}_1(0) \cap \mathcal{N}_2(b)\) to \(\mathcal{N}_1(-b) \cap \mathcal{N}_2(0)\).

Recall that points in \(\hat{\mathcal{F}}(0, 0)\) that differ by elements of \(\Gamma\) correspond to the same point in \(\mathcal{F}(0, 0)\). Thus, in the sketch on the left, we should identify the lines \(k_2 = -b_2/2\) and \(k_2 = b_2/2\) for all \(b \in \Gamma\) with \(b_2 \neq 0\), to get a pair of helices climbing up the outside of a cylinder, as illustrated by the figure on the right. The helices intersect each other twice on each cycle of the cylinder—once on the front half of the cylinder and once on the back half. Hence, viewed as a “manifold” (with singularities), the pair of helices are just two copies of \(\mathbb{R}\) with points that corresponds to intersections identified. We can use \(k_2\) as a coordinate in each copy of \(\mathbb{R}\) and then the pairs of identified points are \(k_2 = b_2/2\) and \(k_2 = -b_2/2\) for all \(b \in \Gamma\) with \(b_2 \neq 0\). So far we have only considered \(k_2\) real. The full \(\hat{\mathcal{F}}(0, 0)\) is just two copies of \(\mathbb{C}\) with \(k_2\) as a coordinate in each copy, provided we identify the points \(\theta_1(b) = \frac{1}{2}(b_2 + i b_1)\) (in the first copy) and \(\theta_2(b) = \frac{1}{2}(b_2 + i b_1)\) (in the second copy) for all \(b \in \Gamma\) with \(b_2 \neq 0\).

### 3 The \(\varepsilon\)-tubes about the free Fermi curve

We now introduce real and imaginary coordinates in \(\mathbb{C}^2\) and define \(\varepsilon\)-tubes about the free Fermi curve. We derive some properties of the \(\varepsilon\)-tubes as well. For \(k \in \mathbb{C}^2\) write

\[
k_1 = u_1 + iv_1 \quad \text{and} \quad k_2 = u_2 + iv_2,
\]

where \(u_1, u_2, v_1\) and \(v_2\) are real numbers. Then,

\[
N_{b,v}(k) = (k_1 + b_1) + i(-1)^v(k_2 + b_2)
= i(v_1 + (-1)^v(u_2 + b_2)) - (-1)^v(v_2 - (-1)^v(u_1 + b_1)),
\]

so that

\[
|N_{b,v}(k)| = |v + (-1)^v(u + b)^\perp|,
\]

where \((y_1, y_2)^\perp := (y_2, -y_1)\). Since \(N_b(k) = N_{b,1}(k)N_{b,2}(k)\), we have \(N_b(k) = 0\) if and only if

\[
v - (u + b)^\perp = 0 \quad \text{or} \quad v + (u + b)^\perp = 0.
\]

Let \(2\Lambda\) be the length of the shortest nonzero “vector” in \(\Gamma\). Then there is at most one \(b \in \Gamma\) with \(|v + (u + b)^\perp| < \Lambda\) and at most one \(b \in \Gamma\) with \(|v - (u + b)^\perp| < \Lambda\) (see [13] for a proof).
Let \( \varepsilon \) be a constant satisfying \( 0 < \varepsilon < \Lambda / 6 \). For \( \nu \in \{1, 2\} \) and \( b \in \Gamma^\# \) define the \( \varepsilon \)-tube about \( \mathcal{N}_\nu(b) \) as

\[
T_\nu(b) := \{ k \in \mathbb{C}^2 \mid |N_{b,\nu}(k)| = |v + (-1)^\nu (u + b)^\perp| < \varepsilon \},
\]

and the \( \varepsilon \)-tube about \( \mathcal{N}_b = \mathcal{N}_1(b) \cup \mathcal{N}_2(b) \) as

\[
T_b := T_1(b) \cup T_2(b).
\]

Since \( (v + (u + b)^\perp) + (v - (u + b)^\perp) = 2v \), at least one of the factors \( |v + (u + b)^\perp| \) or \( |v - (u + b)^\perp| \) in \( |N_0(k)| \) must always be greater or equal to \( |v| \). If \( k \not\in T_b \) both factors are also greater or equal to \( \varepsilon \). If \( k \in T_b \) one factor is bounded by \( \varepsilon \) and the other must lie within \( \varepsilon \) of \( |2v| \). Thus,

\[
k \not\in T_b \implies |N_0(k)| \geq \varepsilon |v|, \quad (1)
\]

\[
k \in T_b \implies |N_0(k)| \leq \varepsilon (2|v| + \varepsilon). \quad (2)
\]

Finally, denote by \( \overline{T}_b \) the closure of \( T_b \). The intersection \( \overline{T}_b \cap \overline{T}'_b \) is compact whenever \( b \neq b' \), and \( \overline{T}_b \cap \overline{T}'_b \cap \overline{T}''_b \) is empty for all distinct elements \( b, b', b'' \in \Gamma^\# \) (see [13] for details).

If a point \( k \) belongs to the free Fermi curve the function \( N_0(k) \) vanishes for some \( b \in \Gamma^\# \). We now give a lower bound for this function when \( (b, k) \) is not in the zero set.

**Proposition 3** (Lower bound for \( |N_0(k)| \)).

(a) If \( |b + u + v^\perp| \geq \Lambda \) and \( |b + u - v^\perp| \geq \Lambda \), then \( |N_0(k)| \geq \frac{\Lambda}{2} (|v| + |u + b|) \).

(b) If \( |v| > 2\Lambda \) and \( k \in T_0 \), then \( |N_0(k)| \geq \frac{\Lambda}{2} (|v| + |u + b|) \) for all \( b \neq 0 \) but at most one \( b \neq 0 \). This exceptional \( \tilde{b} \) obeys \( |\tilde{b}| > |v| \) and \( |u + \tilde{b} - |v|| < \Lambda \).

(c) If \( |v| > 2\Lambda \) and \( k \in T_0 \cap T_d \) with \( d \neq 0 \), then \( |N_0(k)| \geq \frac{\Lambda}{2} (|v| + |u + b|) \) for all \( b \notin \{0, d\} \). Furthermore we have \( |d| > |v| \) and \( |u + d - |v|| < \Lambda \).

**Proof.** (a) By hypothesis, both factors in \( |N_0(k)| = |v + (u + b)^\perp| |v - (u + b)^\perp| \) are greater or equal to \( \Lambda \). We now prove that at least one of the factors must also be greater or equal to \( \frac{\Lambda}{2} (|v| + |u + b|) \). Suppose that \( |v| \geq |u + b| \). Then, since \( (v + (u + b)^\perp) + (v - (u + b)^\perp) = 2v \), at least one of the factors must also be greater or equal to \( |v| = \frac{\Lambda}{2} (|v| + |v|) \geq \frac{\Lambda}{2} (|v| + |u + b|) \). Now suppose that \( |v| < |u + b| \). Then similarly we prove that \( |u + b| > \frac{\Lambda}{2} (|v| + |u + b|) \). All this together implies that \( |N_0(k)| \geq \frac{\Lambda}{2} (|v| + |u + b|) \), which proves part (a).

(b) By hypothesis \( \varepsilon < \Lambda / 6 < |v| \). Let \( k \in T_0 \). Then, by (2),

\[
|N_0(k)| \leq \varepsilon (2|v| + \varepsilon) < 3\varepsilon |v| < \frac{\Lambda}{2} |v|.
\]

Thus we have either \( |u + v^\perp| < \Lambda \) or \( |u - v^\perp| < \Lambda \) (otherwise apply part (a) to get a contradiction). Suppose that \( |u + v^\perp| < \Lambda \) and there is no \( b \in \Gamma^\# \setminus \{0\} \) with \( |b + u + v^\perp| < \Lambda \) and there is at most one \( \tilde{b} \in \Gamma^\# \setminus \{0\} \) satisfying \( |\tilde{b} + u - v^\perp| < \Lambda \). This inequality implies \( |u + \tilde{b} - |v|| < \Lambda \). Furthermore, for this \( \tilde{b} \),

\[
|\tilde{b}| = |2v^\perp - (u + v^\perp) + (\tilde{b} + u - v^\perp)| > 2|v| - 2\Lambda > |v|,
\]
since $-2\Lambda > -|v|$. Now suppose that $|u - v^+| < \Lambda$. Then similarly we prove that $|\tilde{b}| > |v|$. Finally observe that, if $b \not\in \{0, \tilde{b}\}$ then $|b + u + v^+| > \Lambda$ and $|b + u - v^+| > \Lambda$. Hence, applying part (a) it follows that $|N_b(k)| \geq \frac{\Lambda}{2}(|v| + |u + b|)$. This proves part (b).

(c) As in the proof of part (b), if $k \in T_0 \cap T_d$ then in addition to (3) we have $|N_d(k)| < \frac{\Lambda}{2}|v|$. Thus, applying part (b) we conclude that $d$ must be the exceptional $\tilde{b}$ of part (b). The statement of part (c) follows then from part (b). This completes the proof.

4 Main results

The Riemann surfaces introduced in [1] can be decomposed into

$$X^\text{com} \cup X^\text{reg} \cup X^\text{han},$$

where $X^\text{com}$ is a compact submanifold with smooth boundary and finite genus, $X^\text{reg}$ is a finite union of open “regular pieces”, and $X^\text{han}$ is an infinite union of closed “handles”. All these components satisfy a number of geometric/analytic hypotheses stated in [1] that specify the asymptotic holomorphic structure of the surface. Below we state two “asymptotic” theorems that essentially characterize the $X^\text{reg}$ and $X^\text{han}$ components of Fermi curves with small magnetic potential. Before we move to the theorems let us introduce some definitions.

For any $\varphi \in L^2(\mathbb{R}^2/\Gamma)$ define $\hat{\varphi} : \Gamma^\# \to \mathbb{C}$ as

$$\hat{\varphi}(b) := (\mathcal{F}\varphi)(b) := \frac{1}{|\Gamma|} \int_{\mathbb{R}^2/\Gamma} \varphi(x) e^{-ib \cdot x} \, dx,$$

where $|\Gamma| := \int_{\mathbb{R}^2/\Gamma} dx$. Then,

$$\varphi(x) = (\mathcal{F}^{-1}\hat{\varphi})(x) = \sum_{b \in \Gamma^\#} \hat{\varphi}(b) e^{ib \cdot x},$$

$$\|\varphi\|_{L^2(\mathbb{R}^2/\Gamma)} = |\Gamma|^{1/2}\|\hat{\varphi}\|_{L^2(\Gamma^\#)}.$$

Recall that $k = u + iv$ with $u, v \in \mathbb{R}^2$, let $\rho$ be a positive constant, and set

$$\mathcal{K}_\rho := \{k \in \mathbb{C}^2 \mid |v| \leq \rho\}.$$

Finally, consider the projection

$$pr : \mathbb{C}^2 \to \mathbb{C},$$

$$(k_1, k_2) \mapsto k_2,$$

and define

$$q := (i \nabla \cdot A) + A^2 + V.$$

It is easy to construct a holomorphic map $E : \hat{\mathcal{F}}(A, V) \to \mathcal{F}(A, V)$ [13]. The precise form of this map is irrelevant here. For our purposes it is enough to think of it simply as a “projection” (or “exponential map”).

9
We are ready to state our results. Clearly, the set $K_\rho$ is invariant under the action of $\Gamma^\#$ and $K_\rho/\Gamma^\#$ is compact. Hence, the image of $\tilde{\mathcal{F}}(A,V) \cap K_\rho$ under the holomorphic map $E$ is compact in $\mathcal{F}(A,V)$. This image set will essentially play the role of $X^{\text{com}}$ in the decomposition of $\mathcal{F}(A,V)$. Our first theorem characterizes the regular piece $X^{\text{reg}}$ of $\mathcal{F}(A,V)$.

**Theorem 1 (The regular piece).** Let $0 < \varepsilon < \Lambda/6$ and suppose that $A_1, A_2$ and $V$ are functions in $L^2(\mathbb{R}^2/\Gamma)$ with $\|b^2 \hat{q}(b)\|_{L^2(\Gamma^\#)} < \infty$ and $\|(1 + b^2) \hat{A}(b)\|_{L^2(\Gamma^\# \setminus \{0\})} < 2\varepsilon/63$. Then there is a constant $\rho = \rho_{\Lambda, \varepsilon, q, A}$ such that, for $\nu \in \{1, 2\}$, the projection $\text{pr}$ induces a biholomorphic map between

$$\left(\tilde{\mathcal{F}}(A,V) \cap T_\nu(0) \right) \setminus \left( K_\rho \cup \bigcup_{b \in \Gamma^\# \setminus \{0\}} T_b \right)$$

and its image in $\mathbb{C}$. This image component contains

$$\left\{ z \in \mathbb{C} \mid 8|z| > \rho \text{ and } |z + (-1)^\nu \theta_\nu(b)| > \varepsilon \text{ for all } b \in \Gamma^\# \setminus \{0\} \right\}$$

and is contained in

$$\left\{ z \in \mathbb{C} \mid |z + (-1)^\nu \theta_\nu(b)| > \frac{1}{2} \left( \varepsilon - \frac{\varepsilon^2}{40\Lambda} \right) \text{ for all } b \in \Gamma^\# \setminus \{0\} \right\},$$

where $\theta_\nu(b) = \frac{1}{2}((-1)^\nu b_2 + ib_1)$. Furthermore,

$$\text{pr}^{-1} : \text{Image}(\text{pr}) \longrightarrow T_\nu(0),$$

$$y \mapsto (-\beta_2^{(1,0)} - i(-1)^\nu y - r(y), y),$$

where $\beta_2^{(1,0)}$ is a constant given by (24) that depends only on $\rho$ and $A$,

$$|\beta_2^{(1,0)}| < \frac{\varepsilon^2}{100\Lambda} \quad \text{and} \quad |r(y)| \leq \frac{\varepsilon^3}{50\Lambda^2} + \frac{C}{\rho},$$

where $C = C_{\Lambda, \varepsilon, q, A}$ is a constant.

Now observe that, since $T_b + c = T_{b+c}$ for all $b, c \in \Gamma^\#$, the complement of $E\left(\tilde{\mathcal{F}}(A,V) \cap K_\rho\right)$ in $\mathcal{F}(A,V)$ is the disjoint union of

$$E\left(\tilde{\mathcal{F}}(A,V) \cap T_0 \right) \setminus \left( K_\rho \cup \bigcup_{b \in \Gamma^\# \setminus \{0\}} T_b \right)$$

and

$$\bigcup_{b \in \Gamma^\# \setminus \{0\}} E(\tilde{\mathcal{F}}(A,V) \cap T_0 \cap T_b).$$

Basically, the first of the two sets will be the regular piece of $\mathcal{F}(A,V)$, while the second set will be the handles. The map $\Phi$ parametrizing the regular part will be the composition of the map $E$ with the inverse of the map discussed in the above theorem. The detailed information about the handles $X^{\text{han}}$ in $\mathcal{F}(A,V)$ comes from our second main theorem.
Theorem 2 (The handles). Let $0 < \varepsilon < \Lambda/6$ and suppose that $A_1, A_2$ and $V$ are functions in $L^2(\mathbb{R}^2/\Gamma)$ with $\|b^2 \hat{q}(b)\|_{L^1(\Gamma^\#)} < \infty$ and $\|(1 + b^2) \hat{A}(b)\|_{L^1(\Gamma^\# \setminus \{0\})} < 2\varepsilon/63$. Then, for every sufficiently large constant $\rho$ and for every $d \in \Gamma^\# \setminus \{0\}$ with $2|d| > \rho$, there are maps

$$
\phi_{d,1} : \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| \leq \frac{\varepsilon}{2} \text{ and } |z_2| \leq \frac{\varepsilon}{2}\} \rightarrow T_1(0) \cap T_2(d),
$$

$$
\phi_{d,2} : \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| \leq \frac{\varepsilon}{2} \text{ and } |z_2| \leq \varepsilon\} \rightarrow T_1(-d) \cap T_2(0),
$$

and a complex number $t_d$ with $|t_d| \leq \frac{C}{|d|}$ such that:

(i) For $\nu \in \{1, 2\}$ the domain of the map $\phi_{d,\nu}$ is biholomorphic to its image, and the image contains

$$
\left\{ k \in \mathbb{C}^2 \mid |k_1 + i(-1)^\nu k_2| \leq \frac{\varepsilon}{8} \text{ and } |k_1 + (-1)^\nu d_1 - i(-1)^\nu(k_2 + (-1)^\nu+1 d_2)| \leq \frac{\varepsilon}{8}\right\}.
$$

Furthermore,

$$
D\hat{\phi}_{d,\nu} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -i(-1)^\nu & i(-1)^\nu \end{pmatrix} \left( I + O\left( \frac{1}{|d|^2}\right) \right)
$$

and

$$
\hat{\phi}_{d,\nu}(0) = (i\theta_{\nu}(d), (-1)^{\nu+1} \theta_{\nu}(d)) + O\left( \frac{\varepsilon}{900} \right) + O\left( \frac{1}{\rho} \right).
$$

(ii) $\phi_{d,1}^{-1}(T_1(0) \cap T_2(d) \cap \hat{P}(A, V)) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid z_1z_2 = t_d, \ |z_1| \leq \frac{\varepsilon}{2} \text{ and } |z_2| \leq \frac{\varepsilon}{2}\right\},$

$\phi_{d,2}^{-1}(T_1(-d) \cap T_2(0) \cap \hat{P}(A, V)) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid z_1z_2 = t_d, \ |z_1| \leq \frac{\varepsilon}{2} \text{ and } |z_2| \leq \frac{\varepsilon}{2}\right\}$.

(iii) $\phi_{d,1}(z_1, z_2) = \phi_{d,2}(z_2, z_1) - d.$

These are the main results in this paper. In the next section we outline the strategy for proving them. The proofs are presented in the subsequent sections divided in many steps.

5 Strategy outline

Below we briefly describe the general strategy of analysis used to prove our results. We first introduce some notation and definitions. Observe that

$$
H_k(A, V)\varphi = ((i\nabla + A - k)^2 + V)\varphi = ((i\nabla - k)^2 + 2A \cdot (i\nabla - k) + (i\nabla \cdot A) + A^2 + V)\varphi,
$$

and write

$$
H_k(A, V) = \Delta_k + h(k, A) + q(A, V)
$$
Its matrix elements are
\[ \Delta_k := (i\nabla - k)^2, \quad h(k, A) := 2A \cdot (i\nabla - k) \quad \text{and} \quad q(A, V) := (i\nabla \cdot A) + A^2 + V. \]

For each finite subset \( G \) of \( \Gamma^\# \) set
\[
G' := \Gamma^\# \setminus G \quad \text{and} \quad C^2_G := C^2 \setminus \bigcup_{b \in G'} \mathcal{N}_b,
\]
\[
L^2_G := \text{span}\{e^{ib \cdot x} | b \in G\} \quad \text{and} \quad L^2_{G'} := \text{span}\{e^{ib \cdot x} | b \in G'\}.
\]

To simplify the notation write \( L^2 \) in place of \( L^2(\mathbb{R}^2/\Gamma) \). Let \( I \) be the identity operator on \( L^2 \), and let \( \pi_G \) and \( \pi_G' \) be the orthogonal projections from \( L^2 \) onto \( L^2_G \) and \( L^2_{G'} \), respectively. Then,
\[
L^2 = L^2_G \oplus L^2_{G'} \quad \text{and} \quad I = \pi_G + \pi_{G'}.
\]

For \( k \in C^2_G \) define the partial inverse \( (\Delta_k)^{-1}_G \) on \( L^2 \) as
\[
(\Delta_k)^{-1}_G := \pi_G + \Delta_k^{-1} \pi_{G'}.
\]

Its matrix elements are
\[
((\Delta_k)^{-1}_G)_{b,c} := \left\langle \frac{e^{ib \cdot x}}{|G|^{1/2}}, (\Delta_k)^{-1}_G \frac{e^{ic \cdot x}}{|G|^{1/2}} \right\rangle_{L^2} = \begin{cases} \delta_{b,c} & \text{if } c \in G, \\ \frac{1}{\delta_{b,c} N_G(k)} & \text{if } c \notin G, \end{cases}
\]
where \( b, c \in \Gamma^\# \).

Here is the main idea. By definition, a point \( k \) is in \( \hat{\mathcal{F}}(A, V) \) if \( H_k(A, V) \) has a nontrivial kernel in \( L^2 \). Hence, to study the part of the curve in the intersection of \( \bigcup_{d' \in G} T_{d'} \) with \( C^2 \setminus \bigcup_{b \in G'} T_b \) for some finite subset \( G \) of \( \Gamma^\# \), it is natural to look for a nontrivial solution of
\[
(\Delta_k + h + q)(\psi_G + \psi_{G'}) = 0,
\]
where \( \psi_G \in L^2_G \) and \( \psi_{G'} \in L^2_{G'} \). Equivalently, if we make the following (invertible) change of variables in \( L^2 \),
\[
(\psi_G + \psi_{G'}) = (\Delta_k)^{-1}_G (\varphi_G + \varphi_{G'}),
\]
where \( \varphi_G \in L^2_G \) and \( \varphi_{G'} \in L^2_{G'} \), we may consider the equation
\[
(\Delta_k + h + q)\varphi_G + (I + (h + q)\Delta_k^{-1})\varphi_{G'} = 0. \tag{4}
\]

The projections of this equation onto \( L^2_{G'} \) and \( L^2_G \) are, respectively,
\[
\pi_{G'}(h + q)\varphi_G + \pi_{G'}(I + (h + q)\Delta_k^{-1})\varphi_{G'} = 0, \tag{5}
\]
\[
\pi_G(\Delta_k + h + q)\varphi_G + \pi_G(h + q)\Delta_k^{-1}\varphi_{G'} = 0. \tag{6}
\]

Now define \( R_{G'G'} \) on \( L^2 \) as
\[
R_{G'G'} := \pi_{G'}(I + (h + q)\Delta_k^{-1})\pi_{G'}. \tag{7}
\]

12
Observe that \( R_{G'G'} \) is the zero operator on \( L^2_G \). Then, if \( R_{G'G'} \) has a bounded inverse on \( L^2_G \), the equation (5) is equivalent to
\[
\varphi_{G'} = -R_{G'G'}^{-1}\pi_{G'}(h+q)\varphi_G.
\]
Substituting this into (6) yields
\[
\pi_G(\Delta_k + h+q-(h+q)\Delta_k^{-1}R_{G'G'}^{-1}\pi_{G'}(h+q))\varphi_G = 0.
\]
This equation has a nontrivial solution if and only if the (finite) \(|G| \times |G|\) determinant
\[
\det \left[ \pi_G(\Delta_k + h+q-(h+q)\Delta_k^{-1}R_{G'G'}^{-1}\pi_{G'}(h+q))\pi_G \right] = 0
\]
or, equivalently, expressing all operators as matrices in the basis \( \{|\Gamma\}^{-1/2}e^{ib.x} | b \in \Gamma' \} \),
\[
\det \left[ N_{d'}(k)\delta_{d',d''} + w_{d',d''} - \sum_{b,c\in G'} \frac{w_{d',b}}{N_b(k)} R_{G'G'}^{-1} \frac{w_{c,d''}}{N_c(k)} \right]_{d',d''\in G} = 0, \tag{7}
\]
where
\[
w_{b,c} := h_{b,c} + \hat{q}(b-c) = -2(c+k) \cdot \hat{A}(b-c) + \hat{q}(b-c).
\]
Therefore, if \( R_{G'G'} \) has a bounded inverse on \( L^2_G \)—which is in fact the case under suitable conditions—in the region under consideration we can study the Fermi curve in detail using the (local) defining equation (7).

6 Invertibility of \( R_{G'G'} \)

The following notation will be used whenever we consider vector-valued quantities. Let \( X \) be a Banach space and let \( A, B \in \mathcal{A}^2 \), where \( A = (A_1,A_2) \) and \( B = (B_1,B_2) \). Then,
\[
\|A\|_X := (\|A_1\|_X^2 + \|A_2\|_X^2)^{1/2} \quad \text{and} \quad A \cdot B := A_1B_1 + A_2B_2.
\]
Furthermore, we will denote by \( \| \cdot \| \) the operator norm on \( L^2(\mathbb{R}^2/\Gamma) \).

In general, for any \( B, C \subset \Gamma' \) (\( C \) such that \( \Delta_k^{-1}\pi_C \) exists) define the operator \( R_{BC} \) as
\[
R_{BC} := \pi_B(I+(h+q)\Delta_k^{-1})\pi_C
= \pi_B\pi_C + \pi_B q\Delta_k^{-1}\pi_C + \pi_B(2A\cdot i\nabla)\Delta_k^{-1}\pi_C - \pi_B(2k\cdot A)\Delta_k^{-1}\pi_C. \tag{8}
\]
Its matrix elements are
\[
(R_{BC})_{b,c} = \delta_{b,c} + \frac{\hat{q}(b-c)}{N_c(k)} - \frac{2c \cdot \hat{A}(b-c)}{N_c(k)} - \frac{2k \cdot \hat{A}(b-c)}{N_c(k)}, \tag{9}
\]
where \( b \in B \) and \( c \in C \). We first estimate the norm of the last three terms on the right hand side of (8). We begin with the following proposition.
Proposition 4. Let \( k \in \mathbb{C}^2 \) and let \( B, C \subset \Gamma^\# \) with \( C \subset \{ b \in \Gamma^\# \mid N_b(k) \neq 0 \} \). Then,

\[
\| \pi_B q \Delta_k^{-1} \pi_C \| \leq \| \hat{q} \|_1 \frac{1}{\| N_c(k) \|},
\]

\[
\| \pi_B(A \cdot i\nabla) \Delta_k^{-1} \pi_C \| \leq \| \hat{A} \|_1 \frac{|c|}{\| N_c(k) \|},
\]

\[
\| \pi_B(k \cdot A) \Delta_k^{-1} \pi_C \| \leq \| \hat{A} \|_1 |k| \frac{1}{\| N_c(k) \|}.
\]

To prove this proposition we apply the following well-known inequality (see [13]).

Proposition 5. Consider a linear operator \( T : L^2_C \to L^2_B \) with matrix elements \( T_{b,c} \). Then,

\[
\| T \| \leq \max \left\{ \sup_{c \in C} \sum_{b \in B} |T_{b,c}|, \sup_{b \in B} \sum_{c \in C} |T_{b,c}| \right\}.
\]

Proof of Proposition 4. We only prove the first inequality. The proof of the other ones is similar. Write \( T := \pi_B q \Delta_k^{-1} \pi_C \). Then, in view of (8) and (9),

\[
\sup_{b \in B} \sum_{c \in C} |T_{b,c}| \leq \sup_{c \in C} \frac{1}{\| N_c(k) \|} \| \hat{q} \|_1 |c|,
\]

\[
\sup_{b \in B} \sum_{c \in C} |T_{b,c}| \leq \sup_{c \in C} \frac{1}{\| N_c(k) \|} \| \hat{q} \|_1 |k|.
\]

By Proposition 5, these estimates yield the desired inequality.

The key estimate for the existence of \( R_{G,G'}^{-1} \), is given below.

Proposition 6 (Estimate of \( \| R_{SS} - \pi_S \| \)). Let \( k \in \mathbb{C}^2 \) with \( |u| \leq 2|v| \) and \( |v| > 2\Lambda \). Suppose that \( S \subset \{ b \in \Gamma^\# \mid |N_b(k)| \geq \varepsilon|v| \} \). Then,

\[
\| R_{SS} - \pi_S \| \leq \| \hat{q} \|_1 \frac{1}{\varepsilon|v|} + \frac{14}{\varepsilon} \| \hat{A} \|_1. \quad (10)
\]

If \( A = 0 \), the right hand side of (10) can be made arbitrarily small for any \( V \) by taking \( |v| \) sufficiently large (recall that \( q(0,V) = V \)). If \( A \neq 0 \), however, we need to take \( \| \hat{A} \|_1 \) small to make that quantity less than 1. The term \( \frac{14}{\varepsilon} \| \hat{A} \|_1 \) in (10) comes from the estimate we have for \( \| \pi_{G'} h \Delta_k^{-1} \pi_{G'} \| \).

Proof of Proposition 6. By hypothesis, for all \( b \in S \),

\[
\frac{1}{|N_b(k)|} \leq \frac{1}{\varepsilon|v|}. \quad (11)
\]

We now show that, for all \( b \in S \),

\[
\frac{|b|}{|N_b(k)|} \leq \frac{4}{\varepsilon}. \quad (12)
\]
First suppose that $|b| \leq 4|v|$. Then,
\[
\frac{|b|}{|N_b(k)|} \leq \frac{4|v|}{\varepsilon|v|} = \frac{4}{\varepsilon}.
\]

Now suppose that $|b| \geq 4|v|$. Again, by hypothesis we have $|u| \leq 2|v|$ and $|v| > 2\Lambda > \varepsilon$. Hence,
\[
|v + (u + b)^\perp| \geq |b| - |u| - |v| \geq |b| - 3|v| \geq |b| - \frac{3}{4}|b| = \frac{|b|}{4}.
\]

Consequently,
\[
\frac{|b|}{|N_b(k)|} = \frac{|b|}{|v + (u + b)^\perp|} \leq \frac{|b|}{|b|} \frac{4}{4} = \frac{4}{|v|} \leq \frac{4}{\varepsilon}.
\]

This proves (12).

The expression for $R_{SS} - \pi_S$ is given by (8). Observe that $|k| \leq |u| + |v| \leq 3|v|$. Then, applying Proposition 4 and using (11) and (12) we obtain
\[
\|R_{SS} - \pi_S\| \leq (6|v| \|\hat{A}\|_\mu + \|\hat{q}\|_\mu) \sup_{b \in S} \frac{1}{|N_b(k)|} \|A\|_\mu \sup_{b \in S} \frac{|c|}{|N_b(k)|} + 2 \|\hat{A}\|_\mu \sup_{b \in S} \frac{|c|}{|N_b(k)|} \\
\leq (6|v| \|\hat{A}\|_\mu + \|\hat{q}\|_\mu) \frac{1}{\varepsilon|v|} + 8 \varepsilon \|\hat{A}\|_\mu = \|\hat{q}\|_\mu \frac{1}{\varepsilon|v|} + \frac{14}{\varepsilon} \|\hat{A}\|_\mu.
\]

This is the desired inequality. \qed

From the last proposition it follows easily that $R_{SS}$ has a bounded inverse for large $|v|$ and weak magnetic potential.

**Lemma 1** (Invertibility of $R_{SS}$). Let $k \in \mathbb{C}^2$,
\[
|u| \leq 2|v|, \quad |v| > \max \left\{ 2\Lambda, \|\hat{q}\|_\mu \frac{2}{\varepsilon} \right\}, \quad \|\hat{q}\|_\mu < \infty \quad \text{and} \quad \|\hat{A}\|_\mu < \frac{2}{63}\varepsilon.
\]

Suppose that $S \subset \{ b \in \Gamma^\# \mid |N_b(k)| \geq \varepsilon|v| \}$. Then the operator $R_{SS}$ has a bounded inverse with
\[
\|R_{SS} - \pi_S\| < \|\hat{q}\|_\mu \frac{1}{\varepsilon|v|} + \|\hat{A}\|_\mu \frac{14}{\varepsilon} < \frac{17}{18},
\]
\[
\|R_{SS}^{-1} - \pi_S\| < 18\|R_{SS} - \pi_S\|.
\]

**Proof.** Write $R_{SS} = \pi_S + T$ with $T = R_{SS} - \pi_S$. Then, by Proposition 6,
\[
\|T\| = \|R_{SS} - \pi_S\| \leq \|\hat{q}\|_\mu \frac{1}{\varepsilon|v|} + \|\hat{A}\|_\mu \frac{14}{\varepsilon} < \frac{1}{2} + \frac{4}{9} = \frac{17}{18} < 1.
\]

Hence, the Neumann series for $R_{SS}^{-1} = (\pi_S + T)^{-1}$ converges (and is a bounded operator). Furthermore,
\[
\|R_{SS}^{-1} - \pi_S\| = \|(\pi_S + T)^{-1} - \pi_S\| = \|((\pi_S + T)^{-1} - (\pi_S + T)^{-1}(\pi_S + T))\| \\
= \|(\pi_S + T)^{-1}T\| \leq (1 - \|T\|^{-1})\|T\| < 18\|R_{SS} - \pi_S\|,
\]
as was to be shown. \qed

15
Lemma 1 says that if $G$ is such that $G' \subset \{ b \in \Gamma^\# \mid |N_b(k)| \geq \varepsilon|v| \}$ the operator $R_{GG'}$ has a bounded inverse on $L^2_{G'}$ for $|u| \leq 2|v|$, large $|v|$, and weak magnetic potential. We are now able to write local defining equations for $\hat{F}(A,V)$ under such conditions.

7 Local defining equations

In this section we derive local defining equations for the Fermi curve. We begin with a simple proposition.

**Proposition 7.** Suppose either (i) or (ii) or (iii) where:

(i) $G = \{0\}$ and $k \in T_0 \cup \{b \in \Gamma^\# \mid 0\}$

(ii) $G = \{0,d\}$ and $k \in T_0 \cap T_d$

(iii) $G = \emptyset$ and $k \in C^2 \setminus \{b \in \Gamma^\# \mid 0\}$

Then $G' = \Gamma^\# \setminus G = \{b \in \Gamma^\# \mid |N_b(k)| \geq \varepsilon|v|\}$.

**Proof.** The proposition follows easily if we observe that $G' = \Gamma^\# \setminus G$ and recall from (1) that $k \notin T_b \implies |N_b(k)| \geq \varepsilon|v|$.

We now introduce some notation. Let $B$ be a fundamental cell for $\Gamma^\# \subset \mathbb{R}^2$ (see [9, p 310]). Then any vector $u \in \mathbb{R}^2$ can be written as $u = \xi + u$ for some $\xi \in \Gamma^\#$ and $u \in B$. Define

$$\alpha := \sup\{|u| \mid u \in B\}, \quad R := \max\left\{\alpha, 2\Lambda, \|q\|_\mathbb{R}, \frac{2}{\varepsilon}\right\}, \quad \mathcal{K}_R := \{k \in C^2 \mid |v| \leq R\}.$$ 

We first show that in $C^2 \setminus \mathcal{K}_R$ the Fermi curve is contained in the union of $\varepsilon$-tubes about the free Fermi curve.

**Proposition 8** ($\hat{F}(A,V) \setminus \mathcal{K}_R$ is contained in the union of $\varepsilon$-tubes).

$$\hat{F}(A,V) \setminus \mathcal{K}_R \subset \bigcup_{b \in \Gamma^\#} T_b.$$

**Proof.** Without loss of generality we may consider $k \in C^2$ with real part in $B$. We now prove that any point outside the region $\mathcal{K}_R$ and outside the union of $\varepsilon$-tubes does not belong to $\hat{F}(A,V)$. Suppose that $k \in C^2 \setminus (\mathcal{K}_R \cup \bigcup_{b \in \Gamma^\#} T_b)$ and recall that $k$ is in $\hat{F}(A,V)$ if and only if (4) has a nontrivial solution. If we choose $G = \emptyset$ then $G' = \Gamma^\#$ and this equation reads

$$R_{GG'}\varphi_{G'} = 0.$$ 

By Proposition 7(iii) we have $G' = \Gamma^\# = \{b \in \Gamma^\# \mid |N_b(k)| \geq \varepsilon|v|\}$. Furthermore, since $u \in B$ and $|v| > R \geq \alpha$, it follows that $|u| \leq \alpha < |v| < 2|v|$. Consequently, the operator $R_{GG'}$ has a bounded inverse by Lemma 1. Thus, the only solution of the above equation is $\varphi_{G'} = 0$. That is, there is no nontrivial solution of this equation and therefore $k \notin \hat{F}(A,V)$.
We are left to study the Fermi curve inside the ε-tubes. There are two types of regions to consider: intersections and non-intersections of tubes. To study non-intersections we choose \( G = \{0\} \) and consider the region \((T_0 \setminus \cup_{b \in \Gamma_\# \setminus \{0\}} T_b) \setminus \mathcal{K}_R\). For intersections we take \( G = \{0, d\} \) for some \( d \in \Gamma_\# \setminus \{0\} \) and consider \((T_0 \cap T_d) \setminus \mathcal{K}_R\). Observe that, since the tubes \( T_b \) have the following translational property, \( T_b + c = T_{b+c} \) for all \( b, c \in \Gamma_\# \), and the curve \( \widehat{\mathcal{F}}(A, V) \) is invariant under the action of \( \Gamma_\# \), there is no loss of generality in considering only the two regions above. Any other part of the curve can be reached by translation.

Recall that \( G' = \Gamma_\# \setminus G \) and for \( d', d'' \in G \) and \( i, j \in \{1, 2\} \) set

\[
B_{ij}^{d'd''}(k; G) := -4 \sum_{b,c \in G'} \frac{\hat{A}_i(d' - b)}{N_b(k)} (R_{G'G})_{b,c} \hat{A}_j(c - d''),
\]

\[
C_i^{d'd''}(k; G) := -2\hat{A}_i(d' - d'') + 2 \sum_{b,c \in G'} \frac{\hat{q}(d' - b) - 2b \cdot \hat{A}(d' - b)}{N_b(k)} (R_{G'G'})_{b,c} \hat{A}_i(c - d'')
\]

\[
+ 2 \sum_{b,c \in G'} \frac{\hat{A}_i(d' - b)}{N_b(k)} (R_{G'G'})_{b,c} (\hat{q}(c - d'') - 2d'' \cdot \hat{A}(c - d'')) - \sum_{b,c \in G'} \frac{\hat{q}(d' - b) - 2b \cdot \hat{A}(d' - b)}{N_b(k)} (R_{G'G'})_{b,c} (\hat{q}(c - d'') - 2d'' \cdot \hat{A}(c - d'')).
\]

Then,

\[
D_{d'd''}(k; G) := w_{d'd''} - \sum_{b,c \in G'} \frac{w_{d'd''}}{N_b(k)} (R_{G'G'})_{b,c} w_{c,d''}
\]

\[
= B_{11}^{d'd''} k_1^2 + B_{22}^{d'd''} k_2^2 + (B_{12}^{d'd''} + B_{21}^{d'd''}) k_1 k_2 + C_1^{d'd''} k_1 + C_2^{d'd''} k_2 + C_0^{d'd''}.
\]

These functions have the following property.

**Proposition 9.** For \( d', d'' \in G \) and \( i, j \in \{1, 2\} \), the functions \( B_{ij}^{d'd''}, C_i^{d'd''}, C_0^{d'd''} \) (and consequently \( D_{d'd''} \)) are analytic on \((T_0 \setminus \cup_{b \in \Gamma_\# \setminus \{0\}} T_b) \setminus \mathcal{K}_R\) and \((T_0 \cap T_d) \setminus \mathcal{K}_R\) for \( G = \{0\} \) and \( G = \{0, d\} \), respectively.

**Sketch of the proof.** It suffices to show that \( B_{ij}^{d'd''}, C_i^{d'd''} \) and \( C_0^{d'd''} \) are analytic functions. This property follows from the fact that all the series involved in the definition of these functions are uniformly convergent sums of analytic functions. The argument is similar for all cases. See [13] for details.

Using the above functions we can write (local) defining equations for the Fermi curve.

**Lemma 2** (Local defining equations for \( \widehat{\mathcal{F}}(A, V) \)).

(i) Let \( G = \{0\} \) and \( k \in (T_0 \setminus \cup_{b \in \Gamma_\# \setminus \{0\}} T_b) \setminus \mathcal{K}_R\). Then \( k \in \widehat{\mathcal{F}}(A, V) \) if and only if

\[
N_0(k) + D_{0,0}(k) = 0.
\]
(ii) Let \( G = \{0,d\} \) and \( k \in (T_0 \cap T_d) \setminus \mathcal{K}_R \). Then \( k \in \tilde{\mathcal{F}}(A,V) \) if and only if
\[
(N_0(k) + D_{0,0}(k))(N_d(k) + D_{d,d}(k)) - D_{0,0}(k)D_{d,0}(k) = 0.
\]

**Proof.** We only prove part (i). The proof of part (ii) is similar. First, by Proposition 7(i) we have \( G' = \Gamma^* \setminus \{0\} = \{b \in \Gamma^* \mid |N_b(k)| \geq \varepsilon|v|\} \). Furthermore, since \( k \in T_0 \), we have either \(|v - u^+| < \varepsilon \) or \(|v + u^+| < \varepsilon \). In either case this implies \(|u| < \varepsilon + |v| < 2\Lambda + |v| < 2|v|\). Hence, the operator \( R_{G,G'} \) has a bounded inverse by Lemma 1. Thus, in the region under consideration \( \tilde{\mathcal{F}}(A,V) \) is given by (7):
\[
0 = N_0(k) + w_{0,0} - \sum_{b,c \in G'} \frac{w_{0,b}}{N_b(k)}(R^{-1}_{G,G'})_{b,c}w_{c,0} = N_0(k) + D_{0,0}(k).
\]
This is the desired expression. \( \square \)

To study in detail the defining equations above we shall estimate the asymptotic behaviour of the functions \( B^d_{i,j}, C^d_0, C^d_0 \), and \( D_{d',d''} \) for large \(|v|\). (We sometimes refer to these functions as coefficients.) Since all these functions have a similar form it is convenient to prove these estimates in a general setting and specialize them later. This is the contents of §9 and §10. We next introduce a change of variables in \( \mathbb{C}^2 \) that will be useful for proving these bounds.

### 8 Change of coordinates

Define the (complementary) index \( \nu' \) as \( \nu' := \nu - (-1)\nu \). Observe that \( \nu' = 2 \) if \( \nu = 1, \nu' = 1 \) if \( \nu = 2 \), and \((-1)\nu' = -(1)\nu'\). The following change of coordinates in \( \mathbb{C}^2 \) will be useful for our analysis. For \( \nu \in \{1,2\} \) and \( d', d'' \in G \) define the functions \( w_{\nu,d'}, z_{\nu,d'} : \mathbb{C}^2 \rightarrow \mathbb{C} \) as
\[
\begin{align*}
    w_{\nu,d'}(k) &:= k_1 + d'_{1} + i(1)^\nu(k_2 + d'_{2}), \\
    z_{\nu,d'}(k) &:= k_1 + d'_{1} - i(1)^\nu(k_2 + d'_{2}).
\end{align*}
\] (14)

Observe that, the transformation \((k_1, k_2) \mapsto (w_{\nu,d'}, z_{\nu,d'})\) is just a translation composed with a rotation. Furthermore, if \( k \in T_{\nu}(d') \setminus \mathcal{K}_R \) then \(|w_{\nu,d'}(k)|\) is “small” and \(|z_{\nu,d'}(k)|\) is “large”. Indeed, \(|w_{\nu,d'}(k)| = |N_{d',\nu}(k)| < \varepsilon \) and \(|z_{\nu,d'}(k)| = |N_{d',\nu}(k)| \geq |v| > R \). Define also
\[
\begin{align*}
    J_{\nu,d'} := & \frac{1}{4}(B^d_{11} - B^d_{22} + i(-1)^\nu(B^d_{12} + B^d_{21})), \\
    K_{\nu,d'} := & \frac{1}{2}(B^d_{11} + B^d_{22}'), \\
    L_{\nu,d'} := & -d'_{1}B^d_{11} - i(1)^\nu d'_{2}B^d_{22} - \frac{1}{2}(d'_{2} + i(-1)^\nu d'_{1})(B^d_{12} + B^d_{21}') \\
    & + \frac{1}{2}(C^d_{11} + i(-1)^\nu C^d_{22}'), \\
    M_{\nu,d'} := & d'^2_{1}B^d_{11} + d'^2_{2}B^d_{22} + d'^{1}_{1}d'^{1}_{2}(B^d_{12} + B^d_{21}') - d'^{1}_{1}C^d_{11} - d'^{1}_{2}C^d_{22} + C^d_0,
\end{align*}
\]
where \( J_{\nu,d'}, K_{\nu,d'}, L_{\nu,d'} \) and \( M_{\nu,d'} \) are functions of \( k \in \mathbb{C}^2 \) that also depend on the choice of \( G \subset \Gamma^* \). Using these functions we can express \( N_{d'}(k) + D_{d',\nu}(k) \) and \( D_{d',\nu}(k) \) as follows.
Proposition 10. Let $\nu \in \{1, 2\}$ and let $d', d'' \in G$. Then,
\[
N_{d'} + D_{d', d''} = J_{\nu}^d d' w_{\nu, d'}^2 + J_{\nu}^d d' \zeta_{\nu, d'}^2 + (1 + K^d d') w_{\nu, d'} z_{\nu, d'} + L_{\nu}^d d' w_{\nu, d'} + L_{\nu}^d d' z_{\nu, d'} + M^d d',
\]
\[
D_{d', d''} = J_{\nu}^d d'' w_{\nu, d''}^2 + J_{\nu}^d d'' \zeta_{\nu, d''}^2 + K^d d'' w_{\nu, d''} z_{\nu, d''} + L_{\nu}^d d'' w_{\nu, d''} + L_{\nu}^d d'' z_{\nu, d''} + M^d d''.
\]
Furthermore,
\[
J_{\nu}^d d''(k) = - \sum_{b, c \in G'} \frac{(1, -i(-1)^{\nu}) \cdot \hat{A}(d' - b)}{N_b(k)} (R_{G'G'}^{-1})_{b, c} (1, -i(-1)^{\nu}) \cdot \hat{A}(c - d''),
\]
\[
K^d d''(k) = -2 \sum_{b, c \in G'} \frac{\hat{A}(d' - b) \cdot \hat{A}(c - d'')} (R_{G'G'}^{-1})_{b, c} (1, -i(-1)^{\nu}) \cdot \hat{A}(c - d'')
\]
\[
L_{\nu}^d d''(k) = \sum_{b, c \in G'} \frac{\hat{q}(d' - b) + 2(d' - b) \cdot \hat{A}(d' - b)}{N_b(k)} (R_{G'G'}^{-1})_{b, c} (1, -i(-1)^{\nu}) \cdot \hat{A}(c - d'')
\]
\[
+ \sum_{b, c \in G'} \frac{(1, -i(-1)^{\nu}) \cdot \hat{A}(d' - b)}{N_b(k)} (R_{G'G'}^{-1})_{b, c} (\hat{q}(c - d'') + 2(d' - d'') \cdot \hat{A}(c - d''))
\]
\[
- (1, -i(-1)^{\nu}) \cdot \hat{A}(d' - d''),
\]
\[
M^d d''(k) = - \sum_{b, c \in G'} \frac{\hat{q}(d' - b) + 2(d' - b) \cdot \hat{A}(d' - b)}{N_b(k)} (R_{G'G'}^{-1})_{b, c} (\hat{q}(c - d'') + 2(d' - d'') \cdot \hat{A}(c - d''))
\]
\[
+ \hat{q}(d' - d'') + 2(d' - d'') \cdot \hat{A}(d' - d'').
\]

Proof. To simplify the notation write $w = w_{\nu, d'}$, $z = z_{\nu, d'}$, $B_{ij} = B_{ij}^{d''}$ and $C_i = C_i^{d''}$. First observe that, in view of (14),
\[
N_{d'} = (k_1 + d'_1 + i(-1)^{\nu}(k_2 + d'_1))(k_1 + d'_1 + i(-1)^{\nu}(k_2 + d'_1)) = wz.
\]
Furthermore,
\[
k_1 = \frac{1}{2} (w + z - d'_1),
\]
\[
k_2 = \frac{(-1)^{\nu}}{2} (w - z) - d'_2,
\]
\[
k_1^2 = \frac{1}{4} (w^2 + z^2) + \frac{1}{2} wz - d'_1 (w + z) + d'_1^2,
\]
\[
k_2^2 = \frac{1}{4} (w^2 + z^2) + \frac{1}{2} wz + i(-1)^{\nu} d'_2 (w + z) + d'_2^2,
\]
\[
k_1 k_2 = \frac{i(-1)^{\nu}}{4} (z^2 - w^2) - \frac{1}{2} (d'_2 - i(-1)^{\nu} d'_1) w - \frac{1}{2} (d'_2 + i(-1)^{\nu} d'_1) + d'_1 d'_2.
\]

Hence,
\[
D_{d', d''} = B_{11} k_1^2 + B_{22} k_2^2 + (B_{12} + B_{21}) k_1 k_2 + C_1 k_1 + C_2 k_2 + C_0
\]
\[
= \frac{1}{4} (B_{11} - B_{22} - i(-1)^{\nu} (B_{12} + B_{21})) (w^2 + \frac{1}{4} (B_{11} - B_{22} + i(-1)^{\nu} (B_{12} + B_{21})) z^2
\]
\[
+ ( - d'_1 B_{11} + i(-1)^{\nu} d'_1 (B_{12} + B_{21}) - \frac{1}{2} (d'_2 - i(-1)^{\nu} d'_1) (B_{12} + B_{21}) + \frac{1}{2} (C_1 - i(-1)^{\nu} C_2)) w
\]
\[
+ ( - d'_1 B_{11} + i(-1)^{\nu} d'_1 (B_{12} - B_{21}) - \frac{1}{2} (d'_2 - i(-1)^{\nu} d'_1) (B_{12} + B_{21}) + \frac{1}{2} (C_1 + i(-1)^{\nu} C_2)) z
\]
\[
+ d'_1 B_{11} + d'_2 B_{22} + d'_1 d'_2 (B_{12} + B_{21}) - d'_1 C_1 - d'_2 C_2 + C_0 + \frac{1}{4} (B_{11} + B_{22}) wz
\]
\[
= J_{\nu}^d d' w^2 + J_{\nu}^d d' z^2 + K^d d'' w z + L_{\nu}^d d' w + L_{\nu}^d d' z + M^d d''.
\]
This proves the first claim. Consequently,
\[ N'_d + D_{d',d''} = J_{d'}^{d''} w^2 + J_{d'}^{d''} z^2 + (1 + K^{d''}) w z + I_{d'}^{d''} w + I_{d'}^{d''} z + M^{d''}, \]
which proves the second claim.

Now, again to simplify the notation write
\[ f g = \sum_{b,c \in G'} \hat{f}(b, d') (R^{-1}_{G' G})_{b,c} \hat{g}(c, d''), \]
that is, to represent sums of this form suppress the summation and the other factors. Note that \( f g \neq g f \) according to this notation. Then, substituting (13) into the definition of \( J_{d'}^{d''} \) we have
\[
J_{d'}^{d''} = \frac{1}{7} (B_{11} - B_{22} + i(-1)^{\nu}(B_{12} + B_{21})) = -A_1 A_1 + A_2 A_2 - i(-1)^{\nu}(A_1 A_2 + A_2 A_1) = (A_1 - i(-1)^{\nu} A_2)(-A_1 + i(-1)^{\nu} A_2) = -(1, -i(-1)^{\nu}) \cdot A ((1, -i(-1)^{\nu}) \cdot A) = - \sum_{b,c \in G'} (1, -i(-1)^{\nu}) \cdot A(d' - b) (R^{-1}_{G' G})_{b,c} (1, -i(-1)^{\nu}) \cdot A(c - d'').
\]

Similarly, substituting (13) into the definitions of \( K_{d'}^{d''}, L_{d'}^{d''} \) and \( M_{d''}^{d''} \) we derive the other expressions. \( \square \)

9 Asymptotics for the coefficients

Let \( f \) and \( g \) be functions on \( \Gamma^\# \) and for \( k \in \mathbb{C}^2 \) and \( d', d'' \in G \) set
\[
\Phi_{d',d''}(k; G) := \sum_{b,c \in G'} \frac{f(d' - b)}{N_b(k)} (R^{-1}_{G' G})_{b,c} g(c - d''). \quad (15)
\]

In this section we study the asymptotic behaviour of the function \( \Phi_{d',d''}(k) \) for \( k \) in the union of \( \varepsilon \)-tubes with large \( |v| \). Here we only give the statements. See Appendix B for the proofs. Reset the constant \( R \) as
\[
R := \max \left\{ 1, \alpha, 2\Lambda, 140 \| \hat{A} \|_{L^1}, \| (1 + b^2) \hat{q}(b) \|_{L^1} \frac{4}{\varepsilon} \right\}, \quad (16)
\]
and make the following hypothesis.

Hypothesis 1.
\[
\| b^2 \hat{q}(b) \|_{L^1} < \infty \quad \text{and} \quad \| (1 + b^2) \hat{A}(b) \|_{L^1} < \frac{2}{63} \varepsilon.
\]

Our first lemma provides an expansion for \( \Phi_{d',d''}(k) \) “in powers of \( 1/|z_{\nu, d'}(k)| \).”

Lemma 3 (Asymptotics for \( \Phi_{d',d''}(k) \)). Under Hypothesis 1, let \( \nu \in \{1, 2\} \) and let \( f \) and \( g \) be functions on \( \Gamma^\# \) with \( \| b^2 f(b) \|_{L^1} < \infty \) and \( \| b^2 g(b) \|_{L^1} < \infty \). Suppose either (i) or (ii) where:
(i) $G = \{0\}$ and $k \in (T_{\nu}(0) \setminus \cup_{b \in G} T_{b}) \setminus K_{R};$
(ii) $G = \{0, d\}$ and $k \in (T_{\nu}(0) \cap T_{\nu'}(d)) \setminus K_{R}.$

Then, for $(\mu, d') = (\nu, 0)$ if (i) or $(\mu, d') \in \{(\nu, 0), (\nu', d)\}$ if (ii),

$$\Phi_{d',d'}(k) = \alpha_{\mu,d'}^{(1)}(k) + \alpha_{\mu,d'}^{(2)}(k) + \alpha_{\mu,d'}^{(3)}(k),$$

where for $1 \leq j \leq 2,$

$$|\alpha_{\mu,d'}^{(j)}(k)| \leq \frac{C_{j}}{(2|z_{\mu,d'}(k)| - R)^{j}} \quad \text{and} \quad |\alpha_{\mu,d'}^{(3)}(k)| \leq \frac{C_{3}}{|z_{\mu,d'}(k)|R^{2}},$$

where $C_{j} = C_{j;\Lambda A,q,f,g}$ and $C_{3} = C_{3;\Lambda A,q,f,g}$ are constants. Furthermore, the functions $\alpha_{\mu,d'}^{(j)}(k)$ are given by (66) and (69) and are analytic in the region under consideration.

Below we have more information about the function $\alpha_{\mu,d'}^{(1)}(k)$.

**Lemma 4** (Asymptotics for $\alpha_{\mu,d'}^{(1)}(k)$). Consider the same hypotheses of Lemma 3. Then, for $(\mu, d') = (\nu, 0)$ if (i) or $(\mu, d') \in \{(\nu, 0), (\nu', d)\}$ if (ii),

$$z_{\mu,d'}(k) \alpha_{\mu,d'}^{(1)}(k) = \alpha_{\mu,d'}^{(1,0)}(w(k)) + \alpha_{\mu,d'}^{(1,1)}(w(k)) + \alpha_{\mu,d'}^{(1,2)}(k) + \alpha_{\mu,d'}^{(1,3)}(k),$$

where $\alpha_{\mu,d'}^{(1,0)}$ is a constant given by (80), and the remaining functions $\alpha_{\mu,d'}^{(1,j)}$ are given by (79). Furthermore, for $0 \leq j \leq 2,$

$$|\alpha_{\mu,d'}^{(1,j)}| \leq C_{j} \quad \text{and} \quad |\alpha_{\mu,d'}^{(1,3)}| \leq \frac{C_{3}}{2|z_{\mu,d'}(k)| - R},$$

where $C_{j} = C_{j;\Lambda A,q,f,g}$ and $C_{3} = C_{3;\Lambda A,q,f,g}$ are constants given by (81).

The next lemma estimates the decay of $\Phi_{d',d''}(k)$ with respect to $z_{\nu',d}(k)$ for $d' \neq d''$.

**Lemma 5** (Decay of $\Phi_{d',d''}(k)$ for $d' \neq d''$). Under Hypothesis 1, let $\nu \in \{1, 2\}$ and let $f$ and $g$ be functions on $\Gamma$ with $\|b^{2}f(b)\|_{1} < \infty$ and $\|b^{2}g(b)\|_{1} < \infty.$ Suppose further that $G = \{0, d\}$ and $k \in (T_{\nu}(0) \cap T_{\nu'}(d)) \setminus K_{R}.$ Then, for $d', d'' \in G$ with $d' \neq d'',$

$$|\Phi_{d',d''}(k)| \leq \frac{C_{1,0,\nu',\nu}}{|z_{\nu',d}(k)|^{3-10}},$$

where $C_{1,0,\nu',\nu}$ is a constant.

The next proposition relates the quantities $|v|, |k_{2}|, |z_{\nu',d}(k)|$ and $|d|$ for $k$ in the $\varepsilon$-tubes with large $|v|$.

**Proposition 11.** For $\nu \in \{1, 2\}$ we have:

(i) Let $k \in T_{\nu}(0) \setminus K_{R}.$ Then,

$$\frac{1}{|z_{\nu,0}(k)|} \leq \frac{1}{|v|} \leq \frac{3}{|z_{\nu,0}(k)|} \quad \text{and} \quad \frac{1}{|v|} \leq \frac{1}{|k_{2}|} \leq \frac{8}{|v|}.$$

(ii) Let $k \in (T_{\nu}(0) \cap T_{\nu'}(d)) \setminus K_{R}.$ Then,

$$\frac{1}{|z_{\nu,0}(k)|} \leq \frac{1}{|v|} \leq \frac{3}{|z_{\nu,0}(k)|}, \quad \frac{1}{|v|} \leq \frac{1}{|z_{\nu,d}(k)|} \leq \frac{3}{|v|}, \quad \frac{1}{2|z_{\nu,d}(k)|} \leq \frac{1}{|d|} \leq \frac{2}{|z_{\nu,d}(k)|}.$$
10 Bounds on the derivatives

In the last section we expressed \( \Phi_{d',d''}(k) \) as a sum of certain functions \( \alpha^{(j)}_{\mu,d'}(k) \) for \( k \) in the \( \varepsilon \)-tubes with large \( |v| \). In this section we provide bounds for the derivatives of all these functions. Here we only give the statements. See Appendix C for the proofs.

Our first lemma concerns the derivatives of \( \Phi_{d',d''}(k) \).

Lemma 6 (Derivatives of \( \Phi_{d',d''}(k) \)). Under Hypothesis 1, let \( f \) and \( g \) be functions in \( l^1(\Gamma^\#) \) and suppose either (i) or (ii) where:

(i) \( G = \{0\} \) and \( k \in (T_0 \setminus \cup_{b \in G} T_b) \setminus K_R; \)

(ii) \( G = \{0,d\} \) and \( k \in (T_0 \cap T_d) \setminus K_R. \)

Then, for any integers \( n \) and \( m \) with \( n + m \geq 1 \) and for any \( d',d'' \in G \),

\[
\frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \Phi_{d',d''}(k) \leq \frac{C}{|v|},
\]

where \( C \) is a constant with \( C = C_{\varepsilon,A,f,g,m,n} \) if (i) or \( C = C_{\Lambda,A,f,g,m,n} \) if (ii).

We now improve the estimate of Lemma 6(ii) for \( d' \neq d'' \).

Lemma 7 (Derivatives of \( \Phi_{d',d''}(k) \) for \( d' \neq d'' \)). Consider a constant \( \beta \geq 2 \) and suppose that \( \|b|^{\beta} \hat{q}(b)\|_{l^1} < \infty \) and \( \|(1 + |b|^{\beta})A(b)\|_{l^1} < 2\varepsilon/63. \) Let \( \nu \in \{1,2\} \) and let \( f \) and \( g \) be functions on \( \Gamma^\# \) obeying \( \|b|^{\beta} f(b)\|_{l^1} < \infty \) and \( \|b|^{\beta} g(b)\|_{l^1} < \infty. \) Suppose further that \( G = \{0,d\} \) and \( k \in T_0 \cap T_d \) with \( |v| > \frac{2}{\varepsilon} \|b|^{\beta} \hat{q}(b)\|_{l^1}. \) Then, for any integers \( n \) and \( m \) with \( n + m \geq 0 \) and for any \( d',d'' \in G \) with \( d' \neq d'' \),

\[
\frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \Phi_{d',d''}(k) \leq \frac{C}{|d|^{1+\beta}},
\]

where \( C = C_{\varepsilon,A,\mu,g,m,n} \) is a constant.

Observe that, in particular, this lemma with \( m = n = 0 \) generalizes Lemma 5. We next have bounds for the derivatives of \( \alpha^{(j)}_{\mu,d'}(k) \).

Lemma 8 (Derivatives of \( \alpha^{(j)}_{\mu,d'}(k) \)). Under Hypothesis 1, let \( \nu \in \{1,2\} \) and let \( f \) and \( g \) be functions in \( l^1(\Gamma^\#) \). Suppose either (i) or (ii) where:

(i) \( G = \{0\} \) and \( k \in (T_\nu(0) \setminus \cup_{b \in G} T_b) \setminus K_R; \)

(ii) \( G = \{0,d\} \) and \( k \in (T_\nu(0) \cap T_\nu(d)) \setminus K_R. \)

Then, there is a constant \( \rho = \rho_{\varepsilon,A,q,m,n} \) with \( \rho \geq R \) such that, for \( |v| \geq \rho \) and for \( (\mu,d') = (\nu,0) \) if (i) or \( (\mu,d') \in \{(\nu,0), (\nu',d)\} \) if (ii), for any integers \( n \) and \( m \) with \( n + m \geq 1 \) and for \( 1 \leq j \leq 2 \),

\[
\frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \alpha^{(j)}_{\mu,d'}(k) \leq \frac{C_j}{(2|z_{\mu,d'}(k)| - R)^2} \quad \text{and} \quad \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \alpha^{(3)}_{\mu,d'}(k) \leq \frac{C_j}{|z_{\mu,d'}(k)| R^2},
\]

where \( C_l = C_{l,f,g,A,A,q,m,n} \) for \( 1 \leq l \leq 3 \) are constants. Furthermore,

\[
C_{1,f,g,A,A,1,0}; C_{1,f,g,A,A,0,1} \leq 13 \Lambda^{-2} \|f\|_{l^1} \|g\|_{l^1} \quad \text{and} \quad C_{1,f,g,A,A,1,1} \leq 65 \Lambda^{-3} \|f\|_{l^1} \|g\|_{l^1}.
\]
11 The regular piece

Proof of Theorem 1. Step 1 (defining equation). We first derive a defining equation for the Fermi curve. Without loss of generality we may assume that $\hat{A}(0) = 0$. Let $G = \{0\}$, recall that $G' = \Gamma^# \setminus \{0\}$, and consider the region $(T_{\nu}(0) \setminus \bigcup_{b \in G'} T_b) \setminus K$, where $\rho$ is a constant to be chosen sufficiently large obeying $\rho \geq R$. By Proposition 7(i) we have $G' = \{ b \in \Gamma^# \mid |N_b(k)| \geq \varepsilon |v| \}$. To simplify the notation write

$$M_{\nu} := \left( \hat{F}(A, V) \cap T_{\nu}(0) \right) \setminus \left( K \cup \bigcup_{b \in \Gamma^# \setminus \{0\}} T_b \right).$$

By Lemma 2(i), a point $k$ is in $M_{\nu}$ if and only if

$$N_0(k) + D_{0,0}(k) = 0.$$

By Proposition 10, if we set

$$w(k) := w_{\nu,0}(k) = k_1 + i(-1)^{\nu} k_2 \quad \text{and} \quad z(k) := z_{\nu,0}(k) = k_1 - i(-1)^{\nu} k_2,$$

this equation becomes

$$\beta_1 w^2 + \beta_2 z^2 + (1 + \beta_3) wz + \beta_4 w + \beta_5 z + \beta_6 + \hat{q}(0) = 0, \quad (17)$$

where

$$\beta_1 := J_{\nu}^{00}, \quad \beta_2 := J_{\nu}^{00}, \quad \beta_3 := K^{00}, \quad \beta_4 := L_{\nu}^{00}, \quad \beta_5 := L_{\nu}^{00}, \quad \beta_6 := M^{00} - \hat{q}(0),$$

with $J_{\nu}^{00}$, $K^{00}$, $L_{\nu}^{00}$ and $M^{00}$ given by Proposition 10. Observe that all the coefficients $\beta_1, \ldots, \beta_6$ have exactly the same form as the function $\Phi_{0,0}(k)$ of Lemma 3(i) (see (15)). Thus, by this lemma, for $1 \leq i \leq 6$ we have

$$\beta_i = \beta_i^{(1)} + \beta_i^{(2)} + \beta_i^{(3)}, \quad (18)$$

where the function $\beta_i^{(j)}$ is analytic in the region under consideration with

$$|\beta_i^{(j)}(k)| \leq \frac{C}{(2|z(k)| - \rho)^j} \leq \frac{C}{|z(k)|^{j+2}} \quad \text{for} \quad 1 \leq j \leq 2 \quad \text{and} \quad |\beta_i^{(3)}(k)| \leq \frac{C}{|z(k)|^{j+2}},$$

where $C = C_{\varepsilon, \Lambda, \rho, A}$ is a constant. The exact expression for $\beta_i^{(j)}$ can be easily obtained from the definitions and from Lemma 3(i). Substituting (18) into (17) and dividing both sides of the equation by $z$ yields

$$w + \beta_1^{(1)} z + g = 0, \quad (19)$$

where

$$g := \frac{\beta_1 w^2}{z} + (\beta_2^{(2)} + \beta_2^{(3)}) z + \beta_3 w + \frac{\beta_4 w}{z} + \beta_5 + \frac{\beta_6}{z} + \hat{q}(0) \quad (20)$$

23
Now observe that, if $k \neq 0$, that is, if $k \neq 0$ this point and the second coordinate of a point differ by at most $\varepsilon$.

Sufficient condition on the first and second coordinates of a point $k$ is

$$F(k) = 0,$$

where

$$F(k) := w(k) + \beta_2^{(1)}(k) z(k) + g(k)$$

is an analytic function (in the region under consideration).

**Step 2 (candidates for a solution).** Let us now identify which points are candidates to solve the equation $F(k) = 0$. First observe that, by Proposition 2(c) the lines $N'_\nu(0)$ and $N'_\nu(d)$ intersect at $N'_\nu(0) \cap N'_\nu(d) = \{ (i\theta_\nu(d),(-1)^\nu \theta_\nu(d)) \}$. Hence, the second coordinate of this point and the second coordinate of a point $k$ differ by

$$pr(k) - pr(N'_\nu(0) \cap N'_\nu(d)) = k_2 - (-1)^\nu \theta_\nu(d) = k_2 + (-1)^\nu \theta_\nu(d).$$

Now observe that, if $k \in T_\nu(0) \cap T_\nu(d)$ then $|k_1 + i(-1)^\nu k_2| < \varepsilon$ and

$$|k_2 + (-1)^\nu \theta_\nu(d)| = \left| \frac{k_1}{2}(k_1 + i(-1)^\nu k_2) - \frac{k_1}{2}(k_1 - i(-1)^\nu k_2 + d_2) \right|$$

$$\leq \frac{k_1}{2} |N_{0,\nu}(k) - N_{d,\nu}(k)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

That is, the second coordinate of $k$ and the second coordinate of $N'_\nu(0) \cap N'_\nu(d)$ must be apart from each other by at most $\varepsilon$. This gives a necessary condition on the second coordinate of a point $k$ for being in $M_\nu$. Conversely, if a point $k$ is in the $(\varepsilon/4)$-tube inside $T_\nu(0)$, that is, $|k_1 + i(-1)^\nu k_2| < \frac{\varepsilon}{4}$, and its second coordinate differ from the second coordinate of $N'_\nu(0) \cap N'_\nu(d)$ by at most $\varepsilon/4$, that is, $|k_2 + (-1)^\nu \theta_\nu(d)| < \frac{\varepsilon}{4}$, then

$$|N_{d,\nu}(k)| = |N_{0,\nu}(k) - 2(k_2 + (-1)^\nu \theta_\nu(d))| \leq \frac{\varepsilon}{4} + 2 \frac{\varepsilon}{4} < \varepsilon,$$

that is, the point $k$ is also in $T_\nu(d)$ and hence lie in the intersection $T_\nu(0) \cap T_\nu(d)$. This gives a sufficient condition on the first and second coordinates of a point $k$ for being in $T_\nu(0) \cap T_\nu(d)$.

For $y \in C$ define the set of candidates for a solution of $F(k) = 0$ as

$$M_\nu(y) := pr^{-1}(y) \cap \left( T_\nu(0) \setminus \bigcup_{b \in \Gamma^\# \setminus \{0\}} T_b \right) = pr^{-1}(y) \cap \left( T_\nu(0) \setminus \bigcup_{b \in \Gamma^\# \setminus \{0\}} T_\nu(b) \right).$$

Observe that, if $|y + (-1)^\nu \theta_\nu(b)| \geq \varepsilon$ for all $b \in \Gamma^\# \setminus \{0\}$ then

$$M_\nu(y) = pr^{-1}(y) \cap T_\nu(0) = \{ (k_1, y) \in C^2 \mid |k_1 + i(-1)^\nu y| < \varepsilon \}.$$  \hspace{1cm} (22)

On the other hand, if $|y + (-1)^\nu \theta_\nu(d)| < \varepsilon$ for some $d \in \Gamma^\# \setminus \{0\}$, then there is at most one such $d$ and consequently

$$M_\nu(y) = pr^{-1}(y) \cap (T_\nu(0) \setminus T_\nu(d))$$

$$= \{ (k_1, y) \in C^2 \mid |k_1 + i(-1)^\nu y| < \varepsilon \text{ and } |k_1 + d_1 + i(-1)^\nu (y + d_2)| \geq \varepsilon \}.$$  \hspace{1cm} (23)
Indeed, suppose there is another \( d' \neq 0 \) such that \(|y + (-1)^{\nu} \theta_{\nu}(d')| < \varepsilon\). Then,

\[
|d - d'| = |2(-1)^{\nu} \theta_{\nu}(d - d')| = |y + (-1)^{\nu} \theta_{\nu}(d) - (y + (-1)^{\nu} \theta_{\nu}(d'))| \leq 2\varepsilon < 2\Lambda,
\]

which contradicts the definition of \( \Lambda \). Thus, there is no such \( d' \neq 0 \).

**Step 3 (uniqueness).** We now prove that, given \( k_2 \), if there exists a solution \( k_1(k_2) \) of \( F(k_1, k_2) = 0 \), then this solution is unique and it depends analytically on \( k_2 \). This follows easily using the implicit function theorem and the estimates below, which we prove later.

**Proposition 12.** Under the hypotheses of Theorem 1 we have

\[
|F(k) - w(k)| \leq \frac{\varepsilon}{900} + \frac{C_1}{\rho}, \quad (a)
\]

\[
\left| \frac{\partial F}{\partial k_1}(k) - 1 \right| \leq \frac{1}{7 \cdot 3^4} + \frac{C_2}{\rho}, \quad (b)
\]

where the constants \( C_1 \) and \( C_2 \) depend only on \( \varepsilon \), \( \Lambda \), \( q \) and \( A \).

Now suppose that \( (k_1, y) \in M_{\nu}(y) \). Then,

\[
\left| \frac{\partial F}{\partial k_1}(k_1, y) - 1 \right| \leq \frac{1}{7 \cdot 3^4} + \frac{C_2}{\rho}.
\]

Hence, by the implicit function theorem, by choosing the constant \( \rho \geq R \) sufficiently large, if \( F(k_1^*, y) = 0 \) for some \( (k_1^*, y) \in M_{\nu}(y) \), then there is a neighbourhood \( U \times V \subset \mathbb{C}^2 \) which contains \( (k_1^*, y) \), and an analytic function \( \eta : V \to U \) such that \( F(k_1, k_2) = 0 \) for all \( (k_1, k_2) \in U \times V \) if and only if \( k_1 = \eta(k_2) \). In particular this implies that the equation \( F(k_1, k_2) = 0 \) has at most one solution \( (\eta(y), y) \) in \( M_{\nu}(y) \) for each \( y \in \mathbb{C} \). We next look for conditions on \( y \) to have a solution or have no solution in \( M_{\nu}(y) \).

**Step 4 (existence).** We first state an improved version of Proposition 12(a).

**Proposition 13.** Under the hypotheses of Theorem 1 we have

\[
F(k) - w(k) = \beta_{2}^{(1,0)} + \beta_{2}^{(1,1)}(w(k)) + \beta_{2}^{(1,2)}(k) + h(k),
\]

where

\[
\beta_{2}^{(1,0)} = -2i \sum_{h,c \in G_1} \frac{\theta_{\nu'}(\hat{A}(b))}{\theta_{\nu'}(b)} \left[ \frac{\theta_{\nu'}(\hat{A}(b - c))}{\theta_{\nu'}(c)} \right] \theta_{\nu'}(\hat{A}(c)) \quad (24)
\]

is a constant that depends only on \( \rho \) and \( A \) and

\[
h := \beta_{2}^{(1,3)} + g.
\]

Furthermore,

\[
|\beta_{2}^{(1,0)}| < \frac{1}{100\Lambda} \varepsilon^2, \quad |\beta_{2}^{(1,1)}(k)| < \frac{1}{40\Lambda^2} \varepsilon^3, \quad |\beta_{2}^{(1,2)}(k)| < \frac{1}{7^4\Lambda^3} \varepsilon^4, \quad |h(k)| \leq C_{\varepsilon,\Lambda,q,A} \frac{1}{\rho}.
\]
We now derive conditions for the existence of solutions. Suppose that \( F(\eta(y), y) = 0 \). Then, since \( \eta(y) + i(-1)^\nu y = w(\eta(y), y) \) and \( \varepsilon < \Lambda/6 \), using the above proposition we obtain

\[
|\eta(y) + i(-1)^\nu y| = |w(\eta(y), y)| = |F(\eta(y), y) - w(\eta(y), y)|
\leq \frac{\varepsilon^2}{100\Lambda} + \frac{\varepsilon^3}{40\Lambda^2} + \frac{\varepsilon^4}{74\Lambda^3} + \frac{C}{\rho} \leq \frac{\varepsilon^2}{50\Lambda} + \frac{C}{\rho}.
\]

Hence, by choosing the constant \( \rho \) sufficiently large we find that

\[
|\eta(y) + i(-1)^\nu y| < \frac{\varepsilon^2}{40\Lambda}.
\]

In view of (23), there is no solution in \( M_\nu(y) \) if for some \( d \in \Gamma^\# \setminus \{0\} \) we have

\[
|y + (-1)^\nu \theta_\nu(d)| < \varepsilon \quad \text{and} \quad |\eta(y) + d_1 + i(-1)^\nu (y + d_2)| < \varepsilon.
\]

This happens if

\[
|y + (-1)^\nu \theta_\nu(d)| \leq \frac{1}{2} \left( \varepsilon - \frac{\varepsilon^2}{40\Lambda} \right)
\]

because in this case

\[
|\eta(y) + d_1 + i(-1)^\nu (y + d_2)| = |\eta(y) + i(-1)^\nu y - 2i(-1)^\nu y + d_1 - i(-1)^\nu d_2|
\leq |\eta(y) + i(-1)^\nu y| + 2|y + (-1)^\nu \theta_\nu(d)| < \varepsilon.
\]

Therefore, the image set of \( pr \) is contained in

\[
\Omega_1 := \left\{ z \in \mathbb{C} \mid |z + (-1)^\nu \theta_\nu(b)| > \frac{1}{2} \left( \varepsilon - \frac{\varepsilon^2}{40\Lambda} \right) \text{ for all } b \in \Gamma^\# \setminus \{0\} \right\}.
\]

On the other hand, in view of (22), there is a solution in \( M_\nu(y) \) if \( |y + (-1)^\nu \theta_\nu(b)| > \varepsilon \) for all \( b \in \Gamma^\# \setminus \{0\} \). Recall from Proposition 11(a) that \( \rho < |v| < 8|k_2| \). Thus, the image set of \( pr \) contains the set

\[
\Omega_2 := \left\{ z \in \mathbb{C} \mid 8|z| > \rho \text{ and } |z + (-1)^\nu \theta_\nu(b)| > \varepsilon \text{ for all } b \in \Gamma^\# \setminus \{0\} \right\}.
\]

**Step 5.** Summarizing, we have the following biholomorphic correspondence:

\[
\mathcal{M}_\nu \ni k \xrightarrow{pr} k_2 \in \Omega,
\]

\[
\mathcal{M}_\nu \ni (\eta(y), y) \xleftarrow{pr^{-1}} y \in \Omega,
\]

where

\[
\Omega_2 \subset \Omega \subset \Omega_1 \quad \text{and} \quad \eta(y) = -\beta_2^{(1,0)} - i(-1)^\nu y - r(y),
\]

with the constant \( \beta_2^{(1,0)} \) given by (24),

\[
|\beta_2^{(1,0)}| < \frac{\varepsilon^2}{100\Lambda} \quad \text{and} \quad |r(y)| \leq \frac{\varepsilon^3}{50\Lambda^2} + \frac{C}{\rho}.
\]

This completes the proof of the theorem. \( \Box \)
Proof of Proposition 12. (a) Recall that $\beta_2 = J_{\nu}^{00}$. First observe that, by Proposition 10, Lemma 3, and (66), we have
\[
\beta_2^{(1)}(k) = (J_{\nu}^{00})^{(1)}(k) = \sum_{b,c \in G_1^\prime} \frac{(1, i(-1)^\nu) \cdot \hat{A}(b)}{N_0(k)} S_{b,c} (1, -i(-1)^\nu) \cdot \hat{A}(c). \tag{25}
\]
Thus, by (94) and (99),
\[
|\beta_2^{(1)}(k)| \leq \sqrt{2}||\hat{A}||_{t_1} \frac{2}{\Lambda(2|z(k)| - R)} \frac{45}{44} \sqrt{2}||\hat{A}||_{t_1} \leq \frac{4}{\Lambda|z(k)|} \frac{44}{45} \left(\frac{2\varepsilon}{63}\right)^2 \leq \varepsilon \frac{1}{900} \frac{1}{|z(k)|}. \tag{26}
\]
Now recall that $|g(k)| \leq C_{\varepsilon, \Lambda, q, A} \frac{1}{\rho}$. Hence,
\[
|F(k) - w(k)| = |\beta_2^{(1)}(k)z(k) + g(k)| \leq \frac{\varepsilon}{900} + C_{\varepsilon, \Lambda, q, A} \frac{1}{\rho}.
\]
This proves part (a).

(b) We first compute
\[
\frac{\partial g}{\partial k_1} = \frac{\partial \beta_1}{\partial k_1} \frac{w^2}{z} + \beta_1 \frac{2wz - w^2}{z^2} + \left(\frac{\partial \beta_2^{(2)}}{\partial k_1} + \frac{\partial \beta_2^{(3)}}{\partial k_1}\right) z + \beta_2^{(2)} + \beta_2^{(3)} + \frac{\partial \beta_3}{\partial k_1} w + \beta_3,
\]
\[
+ \frac{\partial \beta_4}{\partial k_1} \frac{w}{z} + \beta_4 \frac{z - w}{z^2} + \frac{\partial \beta_5}{\partial k_1} + \frac{\partial \beta_6}{\partial k_1} \frac{1}{z} - \frac{\beta_6}{z^2} - \frac{\hat{q}(0)}{z^2}.
\tag{27}
\]
Now observe that, since $k \in T_\nu(0) \setminus K_\rho$ we have $|w(k)| < \varepsilon$, $3|v| \geq |z|$ and $\rho < |v| \leq |z|$.
Furthermore, by Lemmas 3(i), 6(i) and 8(i), for $1 \leq i \leq 6$ and $1 \leq j \leq 2$,
\[
|\beta_i(k)| \leq \frac{C}{|z(k)|}, \quad |\beta_i^{(j)}(k)| \leq \frac{C}{|z(k)|^j}, \quad |\beta_i^{(3)}(k)| \leq \frac{C}{|z(k)|^3} \tag{28}
\]
where $C = C_{\varepsilon, \Lambda, q, A}$ in all cases. Hence,
\[
\left|\frac{\partial g}{\partial k_1}\right| \leq \frac{C_{\varepsilon, \Lambda, q, A}}{\rho}. \tag{29}
\]
By Lemma 8(i) with $f = g = (1, -i(-1)^\nu) \cdot \hat{A}$, we obtain
\[
\left|z(k) \frac{\partial \beta_2^{(1)}}{\partial k_1}\right| \leq |z(k)| \frac{13}{\Lambda^2 |z(k)|} \|(1, -i(-1)^\nu) \cdot \hat{A}\|^2_{t_1} \leq \frac{26}{\Lambda^2} \|\hat{A}\|^2_{t_1} \leq \frac{1}{7 \cdot 3^4}. \tag{30}
\]
Therefore,
\[
\left|\frac{\partial F}{\partial k_1}(k) - 1\right| = \left|\frac{\partial}{\partial k_1} (F(k) - w(k))\right| = \left|\frac{\partial}{\partial k_1} (\beta_2^{(1)}(k)z(k) + g(k))\right|
\]
\[
= \left|\frac{\partial \beta_2^{(1)}}{\partial k_1}(k)z(k) + \beta_2^{(1)}(k) + \frac{\partial g}{\partial k_1}(k)\right| \leq \frac{1}{7 \cdot 3^4} + C_{\varepsilon, \Lambda, q, A} \frac{1}{\rho}.
\]
This proves part (b) and completes the proof of the proposition.
Proof of Proposition 13. First observe that

\[(1, i(-1)^\nu) \cdot A = A_1 + i(-1)^\nu A_2 = A_1 - i(-1)^\nu A_2 = -2i\theta_\nu(A).\]

Thus, recalling (25),

\[\beta_2^{(1)}(k) = (J_\nu^{(0)})^{(1)}(k) = \sum_{b,c \in G'_1} \frac{2i\theta_\nu(\hat{A}(b))}{N_b(k)} S_{b,c} 2i\theta_\nu(\hat{A}(c)).\]

Now, by Lemma 4 we have

\[z(k)\beta_2^{(1)}(k) = \beta_2^{(1,0)} + \beta_2^{(1,1)}(w(k)) + \beta_2^{(1,2)}(k) + \beta_3^{(1,3)}(k),\]

where

\[\beta_2^{(1,0)} = -2i \sum_{b,c \in G'_1} \frac{\theta_\nu(\hat{A}(b))}{\theta_\nu(b)} \left[ \frac{\theta_\nu(\hat{A}(b-c))}{\theta_\nu(c)} \right] \theta_\nu(\hat{A}(c)).\]

and

\[|\beta_3^{(1,3)}(k)| \leq C_{\Lambda,A} \frac{1}{|z(k)|} < C_{\Lambda,A} \frac{1}{\rho}.\]

Hence,

\[F(k) - w(k) = z(k)\beta_2^{(1)}(k) + g(k) = \beta_2^{(1,0)} + \beta_2^{(1,1)}(w(k)) + \beta_2^{(1,2)}(k) + h(k)\]

with \(h := \beta_3^{(1,3)} + g\). Furthermore, in view of (21),

\[|h(k)| \leq |\beta_3^{(1,3)}(k)| + |g(k)| < C_{\varepsilon,\Lambda,q,A} \frac{1}{\rho}.\]

This proves the first part of the proposition. Finally, by (81), since \(\|\hat{A}\|_I < 2\varepsilon/63\) and \(\varepsilon < \Lambda/6\), we find that

\[|\beta_2^{(1,0)}| \leq \frac{1}{2\Lambda} \left(1 + \frac{1}{2\Lambda}\right) \|\theta_\nu(\hat{A})\|_I \|2i\theta_\nu(\hat{A})\|_I \|2i\theta_\nu(\hat{A})\|_I \leq \frac{4}{\Lambda} \|\hat{A}\|_I^2 < \frac{1}{100\Lambda^2} \varepsilon^2,\]

\[|\beta_2^{(1,1)}| \leq \frac{\varepsilon}{\Lambda^2} \left(1 + \frac{7}{6\Lambda}\right) \|\theta_\nu(\hat{A})\|_I \|2i\theta_\nu(\hat{A})\|_I \|2i\theta_\nu(\hat{A})\|_I \leq \frac{8}{\Lambda^2} \varepsilon \|\hat{A}\|_I^3 < \frac{1}{40\Lambda^2} \varepsilon^3,\]

and

\[|\beta_2^{(1,2)}| \leq \frac{64}{\Lambda^3} \|\theta_\nu(\hat{A})\|_I^2 \|2i\theta_\nu(\hat{A})\|_I \|2i\theta_\nu(\hat{A})\|_I \leq \frac{256}{\Lambda^3} \|\hat{A}\|_I^4 < \frac{1}{74\Lambda^2} \varepsilon^4.\]

This completes the proof. \(\square\)

12 The handles

Proof of Theorem 2. Step 1 (defining equation). Let \(G = \{0, d\}\) and consider the region \((T_\nu(0) \cap T_\nu(d)) \setminus K_\rho\), where \(\rho\) is a constant to be chosen sufficiently large obeying \(\rho \geq R\). Observe that, this requires \(d\) being sufficiently large for \((T_\nu(0) \cap T_\nu(d)) \setminus K_\rho\) being not empty.
In fact, by Proposition 11(ii), for \( k \) in this region we have \( \rho < |v| \leq 2|d| \). Now, recall from Proposition 7(ii) that \( G' = \{ b \in \Gamma^\# \mid |N_6(k)| \geq \varepsilon|v| \} \), and to simplify the notation write

\[ \mathcal{H}_\nu := \tilde{\mathcal{F}}(A, V) \cap (T_\nu(0) \cap T_{\nu'}(d)) \setminus \mathcal{K}_\nu. \]

By Lemma 2(ii), a point \( k \) is in \( \mathcal{H}_\nu \) if and only if

\[ (N_0(k) + D_{0,0}(k))(N_d(k) + D_{d,d}(k)) - D_{0,d}(k)D_{d,0}(k) = 0. \]  
(31)

Define

\[ w_1(k) := w_{\nu,0} = k_1 + i(-1)^\nu k_2, \]
\[ z_1(k) := z_{\nu,0} = k_1 - i(-1)^\nu k_2, \]
\[ w_2(k) := w_{\nu',d} = k_1 + d_1 + i(-1)^{\nu'}(k_2 + d_2), \]
\[ z_2(k) := z_{\nu',d} = k_1 + d_1 - i(-1)^{\nu'}(k_2 + d_2). \]  
(32)

Note that, by Proposition 11(ii),

\[ |v| \leq |z_1| \leq 3|v|, \quad |v| \leq |z_2| \leq 3|v| \quad \text{and} \quad |d| \leq |z_2| \leq 2|d|. \]

By Proposition 10,

\[ N_0 + D_{0,0} = \beta_1 w_1^2 + \beta_2 z_1^2 + (1 + \beta_3)w_1 z_1 + \beta_4 w_1 + \beta_5 z_1 + \beta_6 + \hat{q}(0), \]
\[ N_d + D_{d,d} = \eta_1 w_2^2 + \eta_2 z_2^2 + (1 + \eta_3)w_2 z_2 + \eta_4 w_2 + \eta_5 z_2 + \eta_6 + \hat{q}(0), \]  
(33)

where

\[ \beta_1 := J_{\nu,0}^{00}, \quad \beta_2 := J_{\nu,0}^{00}, \quad \beta_3 := K^{00}, \]
\[ \beta_4 := L_{\nu',0}^{00}, \quad \beta_5 := L_{\nu',0}^{00}, \quad \beta_6 := M^{00} - \hat{q}(0), \]

and

\[ \eta_1 := J_{\nu,d}^{dd}, \quad \eta_2 := J_{\nu,d}^{dd}, \quad \eta_3 := K^{dd}, \]
\[ \eta_4 := L_{\nu,d}^{dd}, \quad \eta_5 := L_{\nu,d}^{dd}, \quad \eta_6 := M^{dd} - \hat{q}(0), \]

with \( J_{\nu,d}^{dd}, K^{dd}, L_{\nu,d}^{dd} \) and \( M^{dd} \) given by Proposition 10. Observe that all the coefficients \( \beta_1, \ldots, \beta_6 \) and \( \eta_1, \ldots, \eta_6 \) have exactly the same form as the function \( \Phi_{d',d'}(k) \) of Lemma 3(ii) (see (15)). Thus, by this lemma, for \( 1 \leq i \leq 6 \) we have

\[ \beta_i = \beta_i^{(1)} + \beta_i^{(2)} + \beta_i^{(3)} \quad \text{and} \quad \eta_i = \eta_i^{(1)} + \eta_i^{(2)} + \eta_i^{(3)}, \]  
(34)

where the functions \( \beta_i^{(j)} \) and \( \eta_i^{(j)} \) are analytic in the region under consideration with

\[ |\beta_i^{(j)}(k)| \leq \frac{C}{|2z_1(k)|^j - \rho^j} \leq \frac{C}{|z_1(k)|^j} \quad \text{for} \quad 1 \leq j \leq 2 \quad \text{and} \quad |\beta_i^{(3)}(k)| \leq \frac{C}{|z_1(k)|^{j \rho^j}}, \]
\[ |\eta_i^{(j)}(k)| \leq \frac{C}{|2z_2(k)|^j - \rho^j} \leq \frac{C}{|z_2(k)|^j} \quad \text{for} \quad 1 \leq j \leq 2 \quad \text{and} \quad |\eta_i^{(3)}(k)| \leq \frac{C}{|z_2(k)|^{j \rho^j}}. \]
where $C = C_{\varepsilon, \Lambda, q, A}$ is a constant. The exact expressions for $\beta_i^{(j)}$ and $\eta_i^{(j)}$ can be easily obtained from the definitions and from Lemma 3(ii). Substituting (34) into (33) yields

$$
\frac{1}{z_1}(N_0 + D_{0,0}) = w_1 + \beta_2^{(1)} z_1 + g_1,
$$

$$
\frac{1}{z_2}(N_d + D_{d,d}) = w_2 + \eta_2^{(1)} z_2 + g_2,
$$

where

$$
g_1 := \frac{\beta_1 w_1^2}{z_1} + (\beta_2^{(2)} + \beta_2^{(3)}) z_1 + \beta_3 w_1 + \frac{\beta_4 w_1}{z_1} + \beta_5 + \frac{\beta_6}{z_1} + \frac{\hat{q}(0)}{z_1},
$$

$$
g_2 := \frac{\eta_1 w_2^2}{z_2} + (\eta_2^{(2)} + \eta_2^{(3)}) z_2 + \eta_3 w_2 + \frac{\eta_4 w_2}{z_2} + \eta_5 + \frac{\eta_6}{z_2} + \frac{\hat{q}(0)}{z_2}
$$

obey

$$
|g_1(k)| \leq \frac{C}{\rho} \quad \text{and} \quad |g_2(k)| \leq \frac{C}{\rho},
$$

with a constant $C = C_{\varepsilon, \Lambda, q, A}$. This gives us more information about the first term in (31). We next consider the second term in that equation.

Write

$$
D_{0,d} = c_1(d) + p_1 \quad \text{and} \quad D_{d,0} = c_2(d) + p_2
$$

with

$$
c_1(d) := \hat{q}(-d) - 2d \cdot \hat{A}(-d), \quad p_1 := D_{0,d} - \hat{q}(-d) + 2d \cdot \hat{A}(-d),
$$

$$
c_2(d) := \hat{q}(d) + 2d \cdot \hat{A}(d), \quad p_2 := D_{d,0} - \hat{q}(d) - 2d \cdot \hat{A}(d).
$$

We have the following estimates.

**Proposition 14.** Under the hypotheses of Theorem 2 we have, for any integers $n$ and $m$ with $n + m \geq 0$ and for $1 \leq j \leq 2$,

$$
\left| \frac{\partial^{n+m}}{\partial k^n \partial k'^m} p_j(k) \right| \leq \frac{C_1}{|d|} \quad \text{and} \quad |c_j(d)| \leq \frac{C_2}{|d|},
$$

where the constants $C_1$ and $C_2$ depend only on $\varepsilon$, $\Lambda$, $q$ and $A$.

Thus, by dividing both sides of (31) by $z_1 z_2$ and substituting (35) and (38) we find that

$$
0 = \frac{1}{z_1 z_2} \left[ (N_0 + D_{0,0})(N_d + D_{d,d}) - D_{0,d} D_{d,0} \right]
$$

$$
= (w_1 + \beta_2^{(1)} z_1 + g_1)(w_2 + \eta_2^{(1)} z_2 + g_2) - \frac{1}{z_1 z_2}(c_1(d) + p_1)(c_2(d) + p_2).
$$

We now introduce a (nonlinear) change of variables in $\mathbb{C}^2$. Set

$$
x_1(k) := w_1(k) + \beta_2^{(1)}(k) z_1(k) + g_1(k),
$$

$$
x_2(k) := w_2(k) + \eta_2^{(1)}(k) z_2(k) + g_2(k).
$$

This transformation obeys the following estimates.
Proposition 15. Under the hypotheses of Theorem 2 we have:

(i) For \(1 \leq j \leq 2\) and for \(\rho\) sufficiently large,
\[
|x_j(k) - w_j(k)| \leq \frac{\varepsilon}{900} + \frac{C}{\rho} < \frac{\varepsilon}{8}.
\]

(ii)
\[
\left(\frac{\partial x_1}{\partial k_1}, \frac{\partial x_2}{\partial k_2}\right) = \left(\begin{array}{cc} 1 & i(-1)^\nu \\ 1 & i(-1)^{\nu'} \end{array}\right) (I + M)
\]
and
\[
\left(\frac{\partial k_1}{\partial x_1}, \frac{\partial k_2}{\partial x_2}\right) = \frac{1}{2} \left(\begin{array}{cc} 1 & 1 \\ i(-1)^{\nu'} & i(-1)^\nu \end{array}\right) (I + N)
\]
with
\[
\|M\| \leq \frac{4}{7} \cdot 3^4 + \frac{C}{\rho} < \frac{1}{2} \quad \text{and} \quad \|N\| \leq 4\|M\|.
\]
Furthermore, for all \(m, i, j \in \{1, 2\},\)
\[
\left|\frac{\partial^2 k_m}{\partial x_i \partial x_j}\right| \leq \frac{3}{\Lambda^3} \varepsilon^2 + \frac{C}{\rho}.
\]
Here, all the constants \(C\) depend only on \(\varepsilon, \Lambda, q\) and \(A\).

By the inverse function theorem, these estimates imply that the above transformation
is invertible. Therefore, by rewriting the equation (39) in terms of these new variables, we
conclude that a point \(k\) is in \(\mathcal{H}_\nu\) if and only if \(x_1(k)\) and \(x_2(k)\) satisfy the equation
\[
x_1x_2 + r(x_1, x_2) = 0, \quad (41)
\]
where
\[
r(x_1, x_2) := -\frac{1}{x_1 x_2} (c_1(d) + p_1)(c_2(d) + p_2).
\]
In order to study this defining equation we need some estimates.

Step 2 (estimates). Using the above inequalities we have, for \(i, j, l \in \{1, 2\},\)
\[
\left|\frac{\partial p_j}{\partial x_i}\right| \leq \sum_{m=1}^2 \left|\frac{\partial p_j}{\partial k_m}\right| \left|\frac{\partial k_m}{\partial x_i}\right| \leq \frac{C}{|d|},
\]
and
\[
\left|\frac{\partial^2 p_j}{\partial x_i \partial x_l}\right| \leq \sum_{m,n=1}^2 \left|\frac{\partial^2 p_j}{\partial k_m \partial k_n}\right| \left|\frac{\partial k_m}{\partial x_i}\right| \left|\frac{\partial k_n}{\partial x_l}\right| + \sum_{m=1}^2 \left|\frac{\partial p_j}{\partial k_m}\right| \left|\frac{\partial^2 k_m}{\partial x_i \partial x_l}\right| \leq \frac{C}{|d|},
\]
so that
\[
|r(x)| \leq C \frac{1}{|d|} \frac{1}{|d|} \frac{1}{|d|} \leq \frac{C}{|d|^4},
\]
\[
\left|\frac{\partial r}{\partial x_i}\right| \leq C \frac{1}{|d|^3} \frac{1}{|d|} \frac{1}{|d|} + C \frac{1}{|d|^2} \frac{1}{|d|} \frac{1}{|d|} \leq \frac{C}{|d|^4}
\]

31
\[ \left| \frac{\partial^2}{\partial x_i \partial x_j} r(x) \right| \leq \frac{C}{|d|^4}. \]

Here, all the constants depend only on \( \varepsilon, \Lambda, q \) and \( A \).

**Step 3 (Morse lemma).** We now apply the quantitative Morse lemma in Appendix A for studying the equation (41). We consider this lemma with \( a = b = \frac{C}{|d|^4}, \delta = \varepsilon, \) and \( d \) sufficiently large so that \( b < \max \left\{ \frac{\varepsilon}{25}, \frac{\varepsilon}{4} \right\} \). Observe that, under this condition we have

\[ (\delta - a)(1 - 19b) > \frac{\varepsilon}{2} \quad \text{and} \quad (\delta - a)(1 - 55b) > \frac{\varepsilon}{4}. \]

According to this lemma, there is a biholomorphism \( \Phi_\nu \) defined on

\[ \Omega_1 := \left\{ (z_1, z_2) \in \mathbb{C}^2 \left| |z_1| < \frac{\varepsilon}{2} \text{ and } |z_2| < \frac{\varepsilon}{2} \right. \right\} \]

with range containing

\[ \left\{ (x_1, x_2) \in \mathbb{C}^2 \left| |x_1| < \frac{\varepsilon}{4} \text{ and } |x_2| < \frac{\varepsilon}{4} \right. \right\} \] (42)

such that

\[ \| \mathcal{D}\Phi_\nu - I \| \leq \frac{C}{|d|^4}, \]

\[ ((x_1x_2 + r) \circ \Phi_\nu)(z_1, z_2) = z_1z_2 + t_d, \]

\[ |t_d| \leq \frac{C}{|d|^4}, \]

\[ |\Phi_\nu(0)| \leq \frac{C}{|d|^4}, \] (43)

where \( D\Phi_\nu \) is the derivative of \( \Phi_\nu \) and \( t_d \) is a constant that depends on \( d \). Hence, if for \( \nu = 1 \) we define

\[ \phi_{d,1} : \Omega_1 \to T_1(0) \cap T_2(d) \]

as

\[ \phi_{d,1} (z_1, z_2) := (k_1(\Phi_1(z_1, z_2)), k_2(\Phi_1(z_1, z_2))), \]

where \( k(x) \) is the inverse of the transformation (40), we obtain the desired map. Note that the conclusion (i) of the theorem is immediate. We next prove (ii) and (iii).

**Step 4 (proof of (i)).** By Proposition 15(i), for \( 1 \leq j \leq 2 \) we have \( |x_j(k) - w_j(k)| \leq \frac{\varepsilon}{8} \). Now, recall from (32) the definition of \( w_1(k) \) and \( w_2(k) \). Then, since

\[ |x_j(k)| \leq |x_j(k) - w_j(k)| + |w_j(k)| < \frac{\varepsilon}{8} + |w_j(k)|, \]

the set

\[ \left\{ (k_1, k_2) \in \mathbb{C}^2 \left| |w_1(k)| < \frac{\varepsilon}{8} \text{ and } |w_2(k)| < \frac{\varepsilon}{8} \right. \right\} \]

is contained in the set (42). This proves the first part of (i). To prove the second part we use Proposition 15 and (43). First observe that

\[ D\phi_{d,1} = \frac{\partial k}{\partial x} D\Phi_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} (I + N)(I + D\Phi_1 - I) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} (I + N + \mathcal{R}), \]
where
\[ \|N\| \leq \frac{1}{3^3} + \frac{C}{\rho} \quad \text{and} \quad \|R\| \leq \frac{C}{|d|^2}. \]

Furthermore, from (32) and (40) we have
\[ k_1 = i\theta_\nu(d) + \frac{1}{2}(w_1 + w_2) = i\theta_\nu(d) + \frac{1}{2}(x_1 + x_2 + \beta_2^{(1)}z_1 + \eta_2^{(1)}z_2 + g_1 + g_2) \]
and similarly
\[ k_2 = -(1)^\nu\theta_\nu(d) + \frac{(-1)^\nu}{2i}(x_1 - x_2 - \beta_2^{(1)}z_1 + \eta_2^{(1)}z_2 - g_1 + g_2), \]
so that
\[ \phi_{d,1}(0) = k(\Phi_1(0)) = k \left( O \left( \frac{1}{|d|} \right) \right) = (i\theta_\nu(d), -(1)^\nu\theta_\nu(d)) + O \left( \frac{\varepsilon}{900} \right) + O \left( \frac{1}{\rho} \right). \]

**Step 5 (proof of (iii)).** To prove part (iii) it suffices to note that \( T_1(0) \cap T_2(d) \cap \hat{F}(A, V) \) is mapped to \( T_1(-d) \cap T_2(0) \cap \hat{F}(A, V) \) by translation by \( d \) and define \( \phi_{d,2} \) by
\[ \phi_{d,2}(z_1, z_2) := \phi_{d,1}(z_2, z_1) + d. \]
This completes the proof of the theorem. \( \square \)

**Proof of Proposition 14.** It suffices to estimate
\[ c_{d',d''} := \hat{q}(d' - d'') - 2(d' - d'') \cdot \hat{A}(d' - d'') \quad \text{and} \quad p_{d',d''} := D_{d',d''} - c_{d',d''} \]
for \( d', d'' \in \{0, d\} \) with \( d' \neq d'' \). Define \( b_{d',d''} := (1, i(-1)^\nu) \cdot \hat{A}(d' - d'') \). Observe that, since
\[ |\hat{q}(d' - d'')| = \frac{1}{|d' - d''|^2} |d' - d''| \leq \frac{1}{|d' - d''|^2} \sum_{b \in \Gamma} |b|^2 |\hat{q}(b)| \leq \|b^2 \hat{q}(b)\|_{L^1} \frac{1}{|d|^2}, \]
and similarly
\[ |\hat{A}(d' - d'')| \leq \|b^2 \hat{A}(b)\|_{L^1} \frac{1}{|d|^2}, \]
it follows that
\[ |c_{d',d''}| \leq \frac{C_{A,q}}{|d|} \quad \text{and} \quad |p_{d',d''}| \leq \frac{C_A}{|d|^2}. \]
This gives the desired bounds for \( c_1 \) and \( c_2 \).

Now, by Proposition 10 we have
\[ p = J_{d',d''}^{\nu} w_{\nu,d'} + J_{d',d''}^{\nu} b_{\nu,d'} + K_{d',d''}^{\nu} w_{\nu,d'} z_{\nu,d'} + (\tilde{L}_{d',d''}^{\nu} - \tilde{l}_{d',d''}^{\nu}) w_{\nu,d'} + (\tilde{M}_{d',d''}^{\nu} - \tilde{m}_{d',d''}^{\nu}) z_{\nu,d'} + \tilde{M}_{d'}^{d''} \]
with \( \tilde{L}_{d',d''}^{\nu} := L_{d',d''}^{\nu} + l_{d',d''}^{\nu} \) and \( \tilde{M}_{d',d''}^{\nu} := M_{d',d''}^{\nu} - c. \) Observe that all the coefficients \( J_{d',d''}^{\nu}, K_{d',d''}^{\nu}, \tilde{L}_{d',d''}^{\nu} \) and \( \tilde{M}_{d',d''}^{\nu} \) have exactly the same form as the function \( \Phi_{d',d''}(k) \) of Lemma 7 (see Proposition 10 and (15)). Thus, by this lemma with \( \beta = 2 \), for any integers \( n \) and \( m \) with \( n + m \geq 0 \), the absolute value of the \( \frac{\partial^{n+m}}{\partial k_1^{n} \partial k_2^{m}} \)-derivative of each of these functions is bounded
above by $C_{\varepsilon, \Lambda,q,m,n} \frac{1}{|d|}$. Hence, if we recall from Proposition 11(ii) that $|z_1(k)| \leq 6|d|$ and $|z_2(k)| \leq 2|d|$, and apply the Leibniz rule we find that

$$\left| \frac{\partial^{n+m}}{\partial k_j^n \partial k'_2^m} p_{d,d'}(k) \right| \leq C_{m,n} \frac{C}{|d|}.$$  

This yields the desired bounds for $p_1$ and $p_2$ and completes the proof. \hfill \square

**Proof of Proposition 15.** (i) Similarly as in (26) we have

$$|\beta_2^{(1)}(k)| \leq \frac{\varepsilon}{900} \frac{1}{|z_1(k)|} \quad \text{and} \quad |\eta_2^{(1)}(k)| \leq \frac{\varepsilon}{900} \frac{1}{|z_2(k)|}.$$  

Thus, in view of (37), and by choosing $\rho$ sufficiently large,

$$|x_1(k) - w_1(k)| \leq |\beta_2^{(1)}(k) z_1(k) + g_1(k)| \leq \frac{\varepsilon}{900} + C \frac{\rho}{\varepsilon} < \varepsilon,$$

and similarly $|x_2(k) - w_2(k)| < \varepsilon/8$. This proves part (i).

(ii) Recall that $\beta_2 = J_{\nu'}^{00}$ and $\eta_2 = J_{\nu'}^{dd}$. Then, for $1 \leq j \leq 2$,

$$\frac{\partial x_1}{\partial k_j} = \frac{\partial}{\partial k_j} (w_1 + z_1 \beta_2^{(1)} + g_1) = \frac{\partial w_1}{\partial k_j} + z_1 \frac{\partial \beta_2^{(1)}}{\partial k_j} + \frac{\partial z_1}{\partial k_j} \beta_2^{(1)} + \frac{\partial g_1}{\partial k_j},$$

$$\frac{\partial x_2}{\partial k_j} = \frac{\partial}{\partial k_j} (w_2 + z_2 \eta_2^{(1)} + g_2) = \frac{\partial w_2}{\partial k_j} + z_2 \frac{\partial \eta_2^{(1)}}{\partial k_j} + \frac{\partial z_2}{\partial k_j} \eta_2^{(1)} + \frac{\partial g_2}{\partial k_j}.$$  

First observe that the functions $g_1$ and $g_2$ are similar to the function $g$ (see (36) and (20)). Thus, it is easy to see that $\frac{\partial g_1}{\partial k_j}$ and $\frac{\partial g_2}{\partial k_j}$ are given by expressions similar to (27). Since $k \in T_{\nu}(0) \cap T_{\nu'}(d)$ we have $|w_1(k)| < \varepsilon$ and $|w_2(k)| < \varepsilon$. Recall also the inequalities in Proposition 11(ii). Hence, by Lemmas 3(ii), 6(ii) and 8(ii), we obtain (28) with $k_1$ and $z(k)$ replaced by $k_j$ and $z_1(k)$, respectively, and for $k_1$, $z(k)$ and $\beta$ replaced by $k_j$, $z_2(k)$ and $\eta$, respectively. Consequently, similarly as in (29) and using again Lemma 3(ii), for $1 \leq j \leq 2$ we have

$$\left| \frac{\partial z_1}{\partial k_j} \beta_2^{(1)} + \frac{\partial g_1}{\partial k_j} \right| \leq C_{\varepsilon, \Lambda,q,A} \frac{1}{\rho} \quad \text{and} \quad \left| \frac{\partial z_2}{\partial k_j} \eta_2^{(1)} + \frac{\partial g_2}{\partial k_j} \right| \leq C_{\varepsilon, \Lambda,q,A} \frac{1}{\rho}.$$  

Now recall that $\beta_2 = J_{\nu'}^{00}$ and $\eta_2 = J_{\nu'}^{dd}$. Then, by Proposition 10, Lemma 3(ii), and (66), it follows that

$$\beta_2^{(1)}(k) = (J_{\nu'}^{00})^{(1)}(k) = \sum_{b,c \in G'_1} \frac{(1,i(-1)^{\nu}) \cdot \hat{A}(-b)}{N_b(k)} S_{b,c} (1,-i(-1)^{\nu}) \cdot \hat{A}(c),$$

$$\eta_2^{(1)}(k) = (J_{\nu'}^{dd})^{(1)}(k) = \sum_{b,c \in G'_1} \frac{(1,i(-1)^{\nu}) \cdot \hat{A}(d-b)}{N_b(k)} S_{b,c} (1,-i(-1)^{\nu}) \cdot \hat{A}(c-d).$$  

Hence, by Lemma 8(ii), similarly as in (30), for $1 \leq j \leq 2$,

$$\left| z_1(k) \frac{\partial \beta_2^{(1)}}{\partial k_j} \right| \leq \frac{13}{\Lambda^2} \|(1,-i(-1)^{\nu}) \cdot \hat{A}\|_1^2 < \frac{1}{7 \cdot 3^4} \quad \text{and} \quad \left| z_2(k) \frac{\partial \eta_2^{(1)}}{\partial k_j} \right| < \frac{1}{7 \cdot 3^4}.$$  

34
Therefore,

\[
\begin{pmatrix}
\frac{\partial x_1}{\partial k_1} & \frac{\partial x_1}{\partial k_2} \\
\frac{\partial x_2}{\partial k_1} & \frac{\partial x_2}{\partial k_2}
\end{pmatrix} = \begin{pmatrix} 1 & i(-1)^\nu \\ 1 & i(-1)^\nu' \end{pmatrix} + \begin{pmatrix} z_1(k) \frac{\partial \beta_1(1)}{\partial k_1} \\ z_2(k) \frac{\partial \beta_1(1)}{\partial k_2} \end{pmatrix} \begin{pmatrix} 1 & i(-1)^\nu \\ 1 & i(-1)^\nu' \end{pmatrix} + \begin{pmatrix} \beta_2^{(1)}(1) & -i(-1)^\nu \beta_2^{(1)}(1) \\ \eta_2^{(1)} & -i(-1)^\nu' \eta_2^{(1)} \end{pmatrix}
\]

\[
+ \begin{pmatrix} \frac{\partial z_1}{\partial k_1} \frac{\partial \beta_1(1)}{\partial k_1} \\ \frac{\partial z_2}{\partial k_2} \frac{\partial \beta_1(1)}{\partial k_2} \end{pmatrix}
\]

\[
\cdot \begin{pmatrix} 1 & i(-1)^\nu \\ 1 & i(-1)^\nu' \end{pmatrix} (I + M_1 + M_2 + M_3),
\]

where

\[
\|M_1\| \leq 2 \frac{2}{7 \cdot 3^4} \quad \text{and} \quad \|M_2 + M_3\| \leq C_{\epsilon, \Lambda, \rho, \Lambda} \frac{1}{\rho}.
\]

Set \(M := M_1 + M_2 + M_3\). This proves the first claim.

Now, by choosing \(\rho\) sufficiently large we can make \(\|M\| < \frac{1}{2}\). Write

\[
P := \begin{pmatrix} 1 & i(-1)^\nu \\ 1 & i(-1)^\nu' \end{pmatrix}.
\]

Then, by the inverse function theorem and using the Neumann series,

\[
\begin{pmatrix}
\frac{\partial x_1}{\partial k_1} & \frac{\partial x_2}{\partial k_1} \\
\frac{\partial x_2}{\partial k_2} & \frac{\partial x_2}{\partial k_2}
\end{pmatrix}^{-1} = (I + M)^{-1} P^{-1} = (I + \tilde{M}) P^{-1}
\]

\[
=: P^{-1} (I + P \tilde{M} P^{-1}) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i(-1)^\nu & i(-1)^\nu' \end{pmatrix} (I + P \tilde{M} P^{-1}),
\]

with

\[
\|P \tilde{M} P^{-1}\| \leq 2 \|M\| 1 \leq \frac{2 \|M\|}{1 - \|M\|} \leq 4 \|M\|.
\]

Set \(N := P \tilde{M} P^{-1}\). This proves the second claim.

Differentiating the matrix identity \(TT^{-1} = I\) and applying the chain rule we find that

\[
\frac{\partial^2 k_m}{\partial x_i \partial x_j} = -\sum_{l,p=1}^2 \frac{\partial k_m}{\partial x_l} \frac{\partial}{\partial x_i} \left( \frac{\partial x_l}{\partial k_p} \right) \frac{\partial k_p}{\partial x_j} = -\sum_{l,p=1}^2 \frac{\partial k_m}{\partial x_l} \frac{\partial^2 x_l}{\partial k_r \partial x_p} \frac{\partial k_r}{\partial x_i} \frac{\partial k_p}{\partial x_j}.
\]

Furthermore, in view of the above calculations we have

\[
\left| \frac{\partial k_i}{\partial x_j} \right| \leq \frac{1}{2} (1 + \|N\|) \leq \frac{1}{2} (1 + 4 \|M\|) \leq \frac{1}{2} \left( 1 + 4 \frac{1}{2} \right) < \frac{3}{2}.
\]

Thus,

\[
\left| \frac{\partial^2 k_m}{\partial x_i \partial x_j} \right| \leq 4 \left( \frac{3}{2} \right)^3 \sup_{l,r,p} \left| \frac{\partial^2 x_l}{\partial k_r \partial x_p} \right|.
\]

We now estimate

\[
\frac{\partial^2 x_1}{\partial k_i \partial k_j} = \frac{\partial z_1}{\partial k_i} \frac{\partial \beta_2^{(1)}}{\partial k_j} + z_1 \frac{\partial^2 \beta_2^{(1)}}{\partial k_i \partial k_j} + \frac{\partial z_1}{\partial k_j} \frac{\partial \beta_2^{(1)}}{\partial k_i} + \frac{\partial^2 y_1}{\partial k_i \partial k_j} \quad \text{and} \quad \frac{\partial^2 x_2}{\partial k_i \partial k_j}.
\]
From (27) with \( g, w \) and \( z \) replaced by \( g_1, w_1 \) and \( z_1 \), respectively, we obtain
\[
\frac{\partial^2 g_1}{\partial k_1^2} = \frac{\partial^2 \beta_1}{\partial k_1^2} \frac{w_1^2}{z_1^3} + 2 \frac{\partial \beta_1}{\partial k_1} \frac{2w_1 z_1 - w_1^2}{z_1^3} + \beta_1 \frac{2z_1^2 - 6w_1 z_1 + 4w_1^2}{z_1^3} + \left( \frac{\partial^2 \beta_2(2)}{\partial k_1^2} + \frac{\partial^2 \beta_2(3)}{\partial k_1^2} \right) z_1 + 2 \left( \frac{\partial \beta_2}{\partial k_1} + \frac{\partial \beta_2}{\partial k_1} \right) \frac{w_1}{z_1^3} + \frac{\partial^2 \beta_3}{\partial k_1^2} \frac{w_1}{z_1^3} + \frac{\partial \beta_3}{\partial k_1} \frac{w_1}{z_1^3} + \frac{2 \partial \beta_4}{\partial k_1} \frac{z_1 - w_1}{z_1^3} + \frac{\beta_4}{z_1^3} \frac{2(w_1 - z_1)}{z_1^3} + \frac{\partial \beta_5}{\partial k_1} \frac{1}{z_1^3} - \frac{1}{z_1^3} \frac{\partial \beta_6}{\partial k_1} \frac{1}{z_1^3} + \frac{\beta_6}{z_1^3} \frac{1}{z_1^3} + \frac{2 \partial \beta_6}{\partial k_1} \frac{z_1 - w_1}{z_1^3}.
\]
Hence, by Lemmas 3(ii), 6(ii) and 8(ii),
\[
\left| \frac{\partial^2 g_1}{\partial k_1^2} \right| \leq C_{\epsilon, A, q, \Lambda} \frac{1}{\rho}.
\]
Similarily we prove that
\[
\left| \frac{\partial^2 g_1}{\partial k_i \partial k_j} \right| \leq C_{\epsilon, A, q, \Lambda} \frac{1}{\rho}
\]
for all \( l, i, j \in \{1, 2\} \) because all the derivatives acting on \( g_1 \) are essentially the same up to constant factors (see [13]). Furthermore, again by Lemma 8(ii),
\[
\left| \frac{\partial \beta_1}{\partial k_j} \right| \leq C_{\epsilon, A, q, \Lambda} \frac{1}{\rho}, \quad \left| \frac{\partial \beta_2}{\partial k_j} \right| \leq C_{\epsilon, A, q, \Lambda} \frac{1}{\rho},
\]
and
\[
\left| z_1(k) \frac{\partial \beta_2(1)}{\partial k_1 \partial k_j} \right| \leq \frac{65}{\Lambda^3} \| (1, -i(-1)^\nu) \cdot \hat{A} \|_1^2 < \frac{1}{5 \Lambda^3} \epsilon^2, \quad \left| z_2(k) \frac{\partial \beta_2(1)}{\partial k_1 \partial k_j} \right| < \frac{1}{5 \Lambda^3} \epsilon^2.
\]
Hence,
\[
\left| \frac{\partial^2 x_1}{\partial k_i \partial k_j} \right| \leq \frac{1}{5 \Lambda^3} \epsilon^2 + C_{\epsilon, A, q, \Lambda} \frac{1}{\rho}.
\]
Therefore,
\[
\left| \frac{\partial^2 x_m}{\partial x_i \partial x_j} \right| \leq 4 \left( \frac{3}{2} \right)^3 \sup_{i, r, p} \left| \frac{\partial^2 x_1}{\partial k_i \partial x_p} \right| \leq \frac{3}{\Lambda^3} \epsilon^2 + C_{\epsilon, A, q, \Lambda} \frac{1}{\rho}.
\]
This completes the proof of the proposition. \( \square \)

A Quantitative Morse lemma

Lemma 9 (Quantitative Morse lemma [13]). Let \( \delta \) be a constant with \( 0 < \delta < 1 \) and assume that
\[
f(x_1, x_2) = x_1 x_2 + r(x_1, x_2)
\]
is an holomorphic function on \( D_\delta = \{(x_1, x_2) \in \mathbb{C}^2 \mid |x_1| \leq \delta \text{ and } |x_2| \leq \delta\} \). Suppose further that, for all \( x \in D_\delta \) and \( 1 \leq i \leq 2 \), the function \( r \) satisfies
\[
\left| \frac{\partial r}{\partial x_i} (x) \right| \leq a < \delta \quad \text{and} \quad \left\| \left[ \frac{\partial^2 r}{\partial x_i \partial x_j} (x) \right]_{i,j \in \{1,2\}} \right\| \leq b < \frac{1}{55}.
\]
where \(a\) and \(b\) are constants. Then \(f\) has a unique critical point \(\xi = (\xi_1, \xi_2) \in D_\delta\) with \(|\xi_1| \leq a\) and \(|\xi_2| \leq a\). Furthermore, let \(s = \max\{|\xi_1|, |\xi_2|\}\). Then there is a biholomorphic map \(\Phi\) from the domain \(D_{(\delta-s)(1-19b)}\) to a neighbourhood of \(\xi \in D_\delta\) that contains

\[
\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_i - \xi_i| < (\delta - s)(1 - 55b) \text{ for } 1 \leq i \leq 2\}
\]
such that \((f \circ \Phi)(z_1, z_2) = z_1 z_2 + c\), where \(c \in \mathbb{C}\) is a constant fulfilling \(|c - r(0,0)| \leq a^2\). The differential \(D \Phi\) obeys \(\|D \Phi - I\| \leq 18b\). If \(\frac{\partial r}{\partial x_1}(0,0) = 0\) and \(\frac{\partial r}{\partial x_2}(0,0) = 0\), then \(\xi = 0\) and \(s = 0\).

\[\text{B  Asymptotics for the coefficients: proofs}\]

\[\text{Proof of Proposition 11.}\] We first derive a more general inequality and then we prove parts (i) and (ii). First observe that, if \(k \in T_\mu(d') \setminus \mathcal{K}_R\) then

\[|v + (-1)^\mu(u + d')^\perp| = |N_{d',\mu}(k)| < \varepsilon < |v|.
\]

Hence,

\[|v| \leq |2v - (v + (-1)^\mu(u + d')^\perp)| \leq 3|v|.
\]

But

\[|2v - (v + (-1)^\mu(u + d')^\perp)| = |v - (-1)^\mu(u + d')^\perp| = |k_1 + d'_1 - i(-1)^\mu(k_2 + d'_2)| = |z_{\mu,d'}(k)|.
\]

Therefore,

\[\frac{1}{|z_\mu,d'(k)|} \leq \frac{1}{|v|} \leq \frac{3}{|z_\mu,d'(k)|}. \tag{44}\]

We now prove parts (i) and (ii).

(i) The first inequality of part (i) follows from the above estimate setting \((\mu, d') = (\nu, 0)\). To prove the second inequality observe that, since \(|v| > R \geq 2\Lambda \geq 12\varepsilon\) by hypothesis and \(|v| \leq |z_{\nu,0}(k)|\) by (44), on the one hand we have

\[
\frac{1}{4}|v| \leq \frac{11}{12}|v| = |v| - \frac{1}{12}|v| \leq |v| - \frac{1}{6}\Lambda \leq |v| - \varepsilon \leq |z_{\nu,0}(k)| - |k_1 + i(-1)^\nu k_2| \\
\leq |z_{\nu,0}(k)| - k_1 - i(-1)^\nu k_2 = 2|k_2|.
\]

On the other hand, since \(|z_{\nu,0}(k)| < 3|v|\) by (44),

\[
|k_2| = |2i(-1)^\nu k_2| = |k_1 + i(-1)^\nu k_2 - (k_1 - i(-1)^\nu k_2)| \\
= |k_1 + i(-1)^\nu k_2 - z_{\nu,0}(k)| \leq \varepsilon + 3|v| \leq 4|v|.
\]

Combining these estimates we obtain the second inequality of part (i).

(ii) Similarly, in view of (44), if \(k \in T_\mu(d') \setminus \mathcal{K}_R\) for \((\mu, d') \in \{(\nu,0), (\nu',d)\}\) then

\[
\frac{1}{|z_{\nu,0}(k)|} \leq \frac{1}{|v|} \leq \frac{3}{|z_{\nu,0}(k)|} \quad \text{and} \quad \frac{1}{|z_{\nu',d}(k)|} \leq \frac{1}{|v|} \leq \frac{3}{|z_{\nu',d}(k)|}. \tag{45}\]
Furthermore, by (45),

\[ |w_{\nu,0}(k)| < \varepsilon, \text{ and } |d_1 - i(-1)^\nu d_2| = |d|, \]

it follows that

\[ |z_{\nu',d}(k)| - \varepsilon \leq |d| \leq |z_{\nu',d}(k)| + \varepsilon. \]

Furthermore, by (45),

\[ \varepsilon < \frac{\Lambda}{6} \leq \frac{|v|}{12} \leq \frac{|z_{\nu',d}(k)|}{12}. \]

Thus,

\[ \frac{1}{2}|z_{\nu',d}(k)| \leq |d| \leq 2|z_{\nu',d}(k)|. \]

This yields the third inequality of part (ii) and completes the proof.

\[ \square \]

**Proof of Lemma 3.** We consider all cases at the same time. Therefore, we have either hypothesis (i) with \((\mu, d') = (\nu, 0)\) or hypothesis (ii) with \((\mu, d') \in \{(\nu, 0), (\nu', d)\}\). Observe that either \((\nu, \nu') = (1, 2)\) or \((\nu, \nu') = (2, 1)\). **Step 1.** Recall the change of variables (14) and set

\[ G'_1 := \{ b \in G' \mid |b - d'| < \frac{1}{4} R \}, \quad G'_2 := \{ b \in G' \mid |b - d'| \geq \frac{1}{4} R \}. \]

Then \(G' = G'_1 \cup G'_2\) and \(G'_1, G'_2 \subset \{ b \in \Gamma^\# \mid |N_b(k)| \geq \varepsilon |v| \}\) by Proposition 7. Furthermore, by Proposition 11, for \((\mu, d') = (\nu, 0)\) if (i) or \((\mu, d') \in \{(\nu, 0), (\nu', d)\}\) if (ii) we have \(|z_{\mu,d'}| \leq 3|v|\). Thus, observing the definition of \(G'_2\),

\[ |\mathcal{R}_1(k)| := \left| \sum_{b \in G'_1} \sum_{c \in G'_2} \frac{f(d' - b)}{N_b(k)} (R^{-1}_{G''G'})_{b,c} g(c - d') \right| \leq \frac{1}{\varepsilon |v|} \| R^{-1}_{G''G'} \| \sum_{b \in G'_1} |f(d' - b)| \sum_{c \in G'_2} \frac{|c - d'|^2}{|c - d'|^2} |g(c - d')| \leq \frac{1}{\varepsilon |v|} \| R^{-1}_{G''G'} \| \| f \| \frac{16}{R^2} \| e^2 g(c) \| \| t_1 \| \leq \frac{C_{\varepsilon,f,g}}{|z_{\mu,d'}| R^2}, \]

and similarly

\[ |\mathcal{R}_2(k)| \leq \frac{C_{\varepsilon,f,g}}{|z_{\mu,d'}| R^2}. \]

Hence,

\[ \Phi_{d',d}(k) = \left[ \sum_{b,c \in G'_1} + \sum_{b \in G'_1} \sum_{c \in G'_2} + \sum_{b \in G'_2} \sum_{c \in G'} \right] \frac{f(d' - b)}{N_b(k)} (R^{-1}_{G''G'})_{b,c} g(c - d') \]

\[ = \sum_{b,c \in G'_1} \frac{f(d' - b)}{N_b(k)} (R^{-1}_{G''G'})_{b,c} g(c - d') + \mathcal{R}_1(k) + \mathcal{R}_2(k) \]

38
with

\[ |R_1(k) + R_2(k)| \leq \frac{C_{\varepsilon,f,g}}{|z_{\mu,d'}|R^2}. \] (49)

Now, if we set \( T_{G'G'} := \pi_{G'} - R_{G'G'} \) and recall the convergent series expansion

\[ R_{G'G'}^{-1} = (\pi_{G'} - T_{G'G'})^{-1} = \sum_{j=0}^{\infty} T_{j}^{j}_{G'G'}, \]

we can write

\[ \sum_{b,c \in G'_i} f(d' - b) \frac{(R_{G'G'})_{b,c} g(c - d')}{N_b(k)} = \sum_{j=0}^{\infty} \sum_{b,c \in G'_i} f(d' - b) \frac{(T_{j}^{j}_{G'G'})_{b,c} g(c - d')}{N_b(k)}. \] (50)

Note, the above equality is fine because \( G'_i \) is finite set. Let

\[ G'_3 := \{ b \in G' \mid |b - d'| < \frac{1}{2}R \}, \quad G'_4 := \{ b \in G' \mid |b - d'| \geq \frac{1}{2}R \}. \]

Again, observe that \( G' = G'_3 \cup G'_4 \). Thus, we can break \( T_{G'G'} \) into

\[ T_{G'G'} = \pi_{G'}T_{G'} = (\pi_{G'_3} + \pi_{G'_4})T(\pi_{G'_3} + \pi_{G'_4}) = T_{33} + T_{43} + T_{34} + T_{44}, \]

where \( T_{ij} := \pi_{G'_i}T_{G'_j} \) for \( i, j \in \{3, 4\} \). Using this decomposition we prove the following.

**Proposition 16.** Under the hypotheses of Lemma 3 we have

\[ \sum_{j=0}^{\infty} \sum_{b,c \in G'_i} f(d' - b) \frac{(T_{j}^{j}_{G'G'})_{b,c} g(c - d')}{N_b(k)} = \sum_{j=0}^{\infty} \sum_{b,c \in G'_i} f(d' - b) \frac{(T_{33}^{33})_{b,c} g(c - d')} {N_b(k)} + \sum_{j=1}^{3} \mathcal{R}_j(k) \]

with \( \mathcal{R}_3(k) \) given by (75) and

\[ |\mathcal{R}_3(k)| \leq \frac{C_{\lambda,f,g}}{|z_{\mu,d'}|R^2}. \] (51)

This proposition will be proved below. Combining this with (48) and (50) we obtain

\[ \Phi_{d',d''}(k) = \sum_{j=0}^{\infty} \sum_{b,c \in G'_i} f(d' - b) \frac{(T_{j}^{j}_{33})_{b,c} g(c - d')} {N_b(k)} + \sum_{j=1}^{3} \mathcal{R}_j(k). \] (52)

**Step 2.** We now look in detail to the operator \( T_{33} \) and its powers \( T_{33}^{j} \). Recall that \( \theta_{\mu}(b) = \frac{1}{2}((-1)^{\mu}b_2 + ib_1) \) and set \( \mu' := \mu - (-1)^{\mu} \) so that \((-1)^{\mu} = -(-1)^{\mu'}\). Then,

\[ N_b(k) = N_{b,\mu}(k)N_{b,\mu'}(k) \]
\[ = (w_{\mu,d'} - 2i\theta_{\mu'}(b - d'))(z_{\mu,d'} - 2i\theta_{\mu}(b - d')). \]

Extend the definition of \( \theta_{\mu}(y) \) to any \( y \in \mathbb{C}^2 \). Thus,

\[ 2(k + d') \cdot \hat{A}(b - c) = -2i\theta_{\mu}(\hat{A}(b - c)) w_{\mu,d'} - 2i\theta_{\mu'}(\hat{A}(b - c)) z_{\mu,d'}. \]
Hence,

\[
T_{b,c} = \frac{1}{N_c(k)} (2(c + k) \cdot \hat{A}(b - c) - \hat{q}(b - c))
\]

\[
= \frac{2(c - d') \cdot \hat{A}(b - c) - \hat{q}(b - c) + 2(k + d') \cdot \hat{A}(b - c)}{(w_{\mu,d'} - 2i\theta_{\mu'}(c - d'))(z_{\mu,d'} - 2i\theta_{\mu}(c - d'))}
= X_{b,c} + Y_{b,c},
\]

where

\[
X_{b,c} := \frac{2(c - d') \cdot \hat{A}(b - c) - \hat{q}(b - c) - 2i\theta_{\mu}(\hat{A}(b - c)) w_{\mu,d'}}{(w_{\mu,d'} - 2i\theta_{\mu'}(c - d'))(z_{\mu,d'} - 2i\theta_{\mu}(c - d'))},
\]

\[
Y_{b,c} := \frac{-2i\theta_{\mu'}(\hat{A}(b - c)) z_{\mu,d'}}{(w_{\mu,d'} - 2i\theta_{\mu'}(c - d'))(z_{\mu,d'} - 2i\theta_{\mu}(c - d'))},
\]

Let \( X \) and \( Y \) be the operators whose matrix elements are, respectively, \( X_{b,c} \) and \( Y_{b,c} \). Set

\[
X_{33} := \pi_{G'_3}X\pi_{G'_3} \quad \text{and} \quad Y_{33} := \pi_{G'_3}Y\pi_{G'_3}.
\]

We next prove the following estimates,

\[
\|X_{33}\| \leq \left(20\|\hat{A}\| + \frac{4}{3}\|\hat{q}\|\right) \frac{1}{|z_{\mu,d'}|} < \frac{1}{3},
\]

\[
\|Y_{33}\| < \frac{8}{20}\|\theta_{\mu'}(\hat{A})\| < \frac{1}{14},
\]

where

\[
|z_{\mu,d'}| := 2|z_{\mu,d'}| - R.
\]

First observe that the “vector” \( b \in \Gamma^\# \) has the same length as the complex number \( 2i\theta_{\mu}(b) \):

\[
|b| = |(b_1, b_2)| = |b_1 + i(-1)^\mu b_2| = |2i\theta_{\mu}(b)|.
\]

Thus, for \( b \in G'_3 \),

\[
\frac{|2i\theta_{\mu}(b - d')|}{R} = \frac{|b - d'|}{R} < \frac{1}{2}.
\]

Consequently,

\[
\frac{1}{|z_{\mu,d'} - 2i\theta_{\mu}(b - d')|} \leq \frac{1}{|z_{\mu,d'} - |2i\theta_{\mu}(b - d')|} < \frac{1}{|z_{\mu,d'} - \frac{1}{2}R|} = \frac{2}{R}.
\]

Furthermore, for \( b \in G' \),

\[
\frac{1}{|w_{\mu,d'} - 2i\theta_{\mu'}(b - d')|} \leq \frac{1}{|b - d'| - |w_{\mu,d'}|} \leq \frac{1}{|b - d'| - \varepsilon}
\]

\[
\leq \frac{1}{2\Lambda - \Lambda} = \frac{1}{2\Lambda}.
\]

Here we have used that \( |w_{\mu,d'}| < \varepsilon < \Lambda \) and \( |b - d'| \geq 2\Lambda \) for all \( b \in G' \). Using again that \( \varepsilon < \Lambda \leq |c - d'|/2 \) for all \( c \in G' \) we have

\[
\frac{|c - d'|}{|c - d'| - \varepsilon} < 2.
\]
Finally recall that
\[
\varepsilon \frac{1}{\Lambda} < \frac{1}{6} \quad \text{and} \quad \frac{1}{|z_{\mu,d'}|} \leq \frac{1}{|v|} < \frac{1}{R},
\]
where the last inequality follows from Proposition 11 since \(|v| > R\) by hypothesis. Then, using the above inequalities and Proposition 5, the bounds (56) for \(\|X_{33}\|\) and \(\|Y_{33}\|\) follow from the estimates
\[
\left[ \sup_{c \in G_3'} \sum_{b \in G_3} \sum_{c' \in G_3'} |X_{b,c}| + \sup_{c \in G_3'} \sum_{b \in G_3} \sum_{c' \in G_3'} |Y_{b,c}| \right] \leq \frac{2}{|z_{\mu,d'}|} \left[ \sup_{c \in G_3'} \sum_{b \in G_3} \sum_{c' \in G_3'} \frac{2|c - d'| |\hat{A}(b - c)| + |\hat{q}(b - c)| + |2i\theta_{\mu}(\hat{A}(b - c))| |w_{\mu,d'}|}{|w_{\mu,d'} - 2i\theta_{\mu}(c - d')| |z_{\mu,d'} - 2i\theta_{\mu}(c - d')|} \right]
\]
\[
\leq \frac{2}{|z_{\mu,d'}|} \left[ \sup_{c \in G_3'} \sum_{b \in G_3} \sum_{c' \in G_3'} \frac{2|c - d'| |\hat{A}(b - c)| + |\hat{q}(b - c)| + \varepsilon \sqrt{2} |\hat{A}(b - c)|}{|w_{\mu,d'} - 2i\theta_{\mu}(c - d')|} \right] \leq \frac{2}{|z_{\mu,d'}|} \left[ 4 + \varepsilon \sqrt{2} \right] \frac{\|\hat{A}\|_{l_1} + \|\hat{q}\|_{l_1}}{\Lambda} \leq \frac{8}{\Lambda} \frac{\|\hat{A}\|_{l_1} + \|\hat{q}\|_{l_1}}{\Lambda} \frac{1}{|z_{\mu,d'}|} \frac{1}{R} < \frac{1}{7} + \frac{1}{4} = \frac{1}{3}
\]
and similarly
\[
\left[ \sup_{c \in G_3'} \sum_{b \in G_3} \sum_{c' \in G_3'} |X_{b,c}| + \sup_{c \in G_3'} \sum_{b \in G_3} \sum_{c' \in G_3'} |Y_{b,c}| \right] \leq \frac{8}{\Lambda} \frac{\|\hat{A}\|_{l_1} + \|\hat{q}\|_{l_1}}{\Lambda} \frac{1}{|z_{\mu,d'}|} \frac{1}{R} < \frac{1}{14}.
\]

**Step 3.** We now look in detail to \(T_{33}^j\). For each integer \(j \geq 1\) write
\[
T_{33}^j = (X_{33} + Y_{33})^j = Z_j + W_j + Y_{33}^j,
\]
where \(W_j\) is the sum of the \(j\) terms containing only one factor \(X_{33}\) and \(j - 1\) factors \(Y_{33}\),
\[
W_j := \sum_{m=1}^{j} (Y_{33})^{m-1} X_{33} (Y_{33})^{j-m},
\]
\[
Z_j := (X_{33} + Y_{33})^j - W_j - Y_{33}^j.
\]
In view of (56) we have
\[
\|Y_{33}\| \leq \left( \frac{1}{14} \right)^j,
\]
\[
\|W_j\| \leq j \|X_{33}\| \|Y_{33}\|^{j-1} \leq \frac{C_{A,A,q}}{|z_{\mu,d'}|} j \left( \frac{1}{14} \right)^{j-1},
\]
\[
\|Z_j\| \leq (2^j - j - 1) \|X_{33}\|^2 \left( \frac{1}{3} \right)^{j-2} \leq \frac{C_{A,A,q}}{|z_{\mu,d'}|^2} \left( \frac{2}{3} \right)^j.
\]
Hence, the series
\[ S := \sum_{j=0}^{\infty} Y_{33}^j = (I - Y_{33})^{-1}, \quad W := \sum_{j=1}^{\infty} W_j \quad \text{and} \quad Z := \sum_{j=2}^{\infty} Z_j \] (64)
converge, and the operator norm of \( W \) and \( Z \) decay with respect to \( |\mu, \delta'| \). Indeed,
\[
\|S\| \leq \sum_{j=0}^{\infty} \|Y_{33}\|^j \leq \sum_{j=0}^{\infty} \left( \frac{1}{14} \right)^j < C,
\]
\[
\|W\| \leq \sum_{j=1}^{\infty} \|W_j\| \leq \frac{C_{A,A,g}'}{2|\mu, \delta'| - R} \sum_{j=1}^{\infty} j \left( \frac{1}{14} \right)^{j-1} \leq \frac{C_{A,A,g}'}{|\mu, \delta'|_R},
\]
\[
\|Z\| \leq \sum_{j=2}^{\infty} \|Z_j\| \leq \frac{C_{A,A,g}'}{|\mu, \delta'|_R^2} \sum_{j=2}^{\infty} \left( \frac{2}{3} \right)^j \leq \frac{C_{A,A,g}'}{|\mu, \delta'|_R^2}.
\]
Thus, we have the expansion
\[
\sum_{j=0}^{\infty} T_{33}^j = S + W + Z.
\]

**Step 4.** Consequently,
\[
\sum_{j=0}^{\infty} \sum_{b,c \in G_1'} \frac{f(d' - b)}{N_b(k)} (T_{33}^j)_{b,c} g(c - d') = \sum_{b,c \in G_1'} \frac{f(d' - b) (S + W + Z)_{b,c} g(c - d')}{(w_{\mu, \delta'} - 2i\theta_{\mu'} (b - d'))(z_{\mu, \delta'} - 2i\theta_{\mu} (b - d'))}
\]
\[
= \alpha_{\mu, \delta'}^{(1)} + \alpha_{\mu, \delta'}^{(2)} + R_4,
\]
where
\[
\alpha_{\mu, \delta'}^{(1)}(k) := \sum_{b,c \in G_1'} \frac{f(d' - b) S_{b,c}(k) g(c - d')}{(w_{\mu, \delta'}(k) - 2i\theta_{\mu'} (b - d'))(z_{\mu, \delta'}(k) - 2i\theta_{\mu} (b - d'))},
\]
\[
\alpha_{\mu, \delta'}^{(2)}(k) := \sum_{b,c \in G_1'} \frac{f(d' - b) W_{b,c}(k) g(c - d')}{(w_{\mu, \delta'}(k) - 2i\theta_{\mu'} (b - d'))(z_{\mu, \delta'}(k) - 2i\theta_{\mu} (b - d'))},
\]
and
\[
R_4(k) := \sum_{b,c \in G_1'} \frac{f(d' - b) Z_{b,c}(k) g(c - d')}{(w_{\mu, \delta'}(k) - 2i\theta_{\mu'} (b - d'))(z_{\mu, \delta'}(k) - 2i\theta_{\mu} (b - d'))}.
\]

By a short calculation as in (74), using (58) and (60) we find that
\[
|\alpha_{\mu, \delta'}^{(1)}(k)| \leq \frac{1}{\Lambda} \frac{2}{|\mu, \delta'|_R} \|f\|_{L^1} \|g\|_{L^1} \|S\| \leq \frac{C_{A,f,g}}{|\mu, \delta'|_R},
\]
\[
|\alpha_{\mu, \delta'}^{(2)}(k)| \leq \frac{1}{\Lambda} \frac{2}{|\mu, \delta'|_R} \|f\|_{L^1} \|g\|_{L^1} \|W\| \leq \frac{C_{A,A,g,f,g}}{|\mu, \delta'|_R^2},
\]
\[
|R_4(k)| \leq \frac{1}{\Lambda} \frac{2}{|\mu, \delta'|_R} \|f\|_{L^1} \|g\|_{L^1} \|Z\| \leq \frac{C_{A,A,g,f,g}}{|\mu, \delta'|_R^3}.
\]

Hence, recalling (52) we conclude that
\[
\Phi_{d', \delta'} = \alpha_{\mu, \delta'}^{(1)} + \alpha_{\mu, \delta'}^{(2)} + \alpha_{\mu, \delta'},
\]

42
where
\[ \alpha^{(3)}_{\mu,d'}(k) := \sum_{j=1}^{4} R_j(k). \] (69)

Furthermore, in view of (49), (51) and (68), since
\[ \frac{1}{|z_{\mu,d'}|^3_R} = \frac{1}{(2|z_{\mu,d'}| - R)^3} < \frac{1}{|z_{\mu,d'}| R^2}, \]
for \(1 \leq j \leq 2\) we have
\[ |\alpha^{(j)}_{\mu,d'}(k)| \leq \frac{C_j}{|z_{\mu,d'}(k)|^3_R} \quad \text{and} \quad |\alpha^{(3)}_{\mu,d'}(k)| \leq \frac{C_3}{|z_{\mu,d'}(k)| R^2}, \]
where \(C_j = C_{j;A,A,q,f,g}\) and \(C_3 = C_{3;e,A,q,f,g}\) are constants. This proves the main statement of the lemma. Finally observe that, since \(G'_3\) is a finite set, the matrices \(X_{33}\) and \(Y_{33}\) are analytic in \(k\) because their matrix elements are analytic functions of \(k\). (Note, the functions \(w_{\mu,d'}(k)\) and \(z_{\mu,d'}(k)\) are analytic.) Consequently, the matrices \(W_j\) and \(Z_j\) are also analytic and so are \(S_{b,c}\), \(W_{b,c}\) and \(Z_{b,c}\) because the series (64) converge uniformly with respect to \(k\). Thus, all the functions \(\alpha_{\mu,d'}^{(j)}(k)\) are analytic in the region under consideration. This completes the proof of the lemma. \(\square\)

**Proof of Proposition 16.**

**Step 1.** Recall that \(T_{G'_G} = T + T_{43} + T_{44}\) with \(T_{ij} = \pi_{G'_i} T \pi_{G'_j}\) and set \(X_{33}^{(0)} := 0\), \(Y_{34}^{(0)} := T_{34}\), \(W_{43}^{(0)} := T_{43}\), and \(Z_{44}^{(0)} := T_{44}\). It is straightforward to verify that, for any integer \(j \geq 0\),
\[ T_{G'_G}^{j+1} = T_{33}^{j+1} + X_{33}^{(j)} + Y_{34}^{(j)} + W_{43}^{(j)} + Z_{44}^{(j)}, \] (70)
where
\[ X_{33}^{(j)} := T_{33} X_{33}^{(j-1)} + T_{34} W_{43}^{(j-1)} : L_{G'_3}^2 \to L_{G'_3}^2, \]
\[ Y_{34}^{(j)} := T_{34} Y_{34}^{(j-1)} + T_{43} Z_{44}^{(j-1)} : L_{G'_3}^2 \to L_{G'_3}^2, \]
\[ W_{43}^{(j)} := T_{43} W_{43}^{(j-1)} + T_{43} X_{33}^{(j-1)} + T_{44} W_{43}^{(j-1)} : L_{G'_3}^2 \to L_{G'_3}^2, \]
\[ Z_{44}^{(j)} := T_{44} X_{34}^{(j-1)} + T_{44} Z_{44}^{(j-1)} : L_{G'_3}^2 \to L_{G'_3}^2. \] (71)

**Step 2.** Since \(\pi_{G'_1} \pi_{G'_4} = \pi_{G'_4} \pi_{G'_1} = 0\) and \(\pi_{G'_1} \pi_{G'_3} = \pi_{G'_3} \pi_{G'_1} = \pi_{G'_1}\), substituting (70) into the sum below for the terms where \(j \geq 1\) we have, recalling that \(X_{33}^{(0)} = 0\),
\[ \sum_{j=0}^{\infty} \sum_{b,c \in G'_1} \frac{f(d' - b)}{N_b(k)} (T_{G'_G}^j)_{b,c} g(c - d') = \sum_{j=0}^{\infty} \sum_{b,c \in G'_1} \frac{f(d' - b)}{N_b(k)} (T_{33}^j)_{b,c} g(c - d') + \sum_{j=1}^{\infty} \sum_{b,c \in G'_1} \frac{f(d' - b)}{N_b(k)} (X_{33}^{(j)})_{b,c} g(c - d'). \] (72)

Now recall from (58) and (60) that, for all \(b \in G'_3\),
\[ \frac{1}{|N_b(k)|} \leq \frac{2}{\Lambda} \frac{1}{|z_{\mu,d'}| R}. \] (73)
and observe that $G'_1 \subset G'_3$. Let $\mathcal{M}$ be either $T_{G'_1 G'_3}$ or $T_{33}$. Then, the estimate

$$\left| \sum_{b,c \in G'_1} \frac{f(d' - b)}{N_b(k)} (\mathcal{M})_{b,c} g(c - d') \right| = \sum_{b \in G'_1} \frac{f(d' - b)}{N_b(k)} \sum_{c \in G'_1} \left( \frac{\epsilon^{ibx}}{|\Gamma|^{1/2}}, \mathcal{M}^{icx} \frac{|\Gamma|^{1/2}}{} \right) g(c - d') \leq 2 \frac{1}{|z_{\mu,d'}| R} \|f\|_1 \|g\|_1 \|\mathcal{M}\|^j$$

implies that the left hand side and the first term on the right hand side of (72) converge because $\|\mathcal{M}\| < 17/18$. Thus, the last term in (72) also converges. Hence, we are left to show that

$$\mathcal{R}_3(k) := \sum_{j=1}^{\infty} \sum_{b,c \in G'_1} \frac{f(d' - b)}{N_b(k)} (X^{(j)}_{33})_{b,c} g(c - d')$$

obeys

$$|\mathcal{R}_3(k)| \leq \frac{C_{\Lambda,f,g}}{|z_{\mu,d'}| R^2}.$$ 

In order to do this we need the following inequality, which we prove later.

**Proposition 17.** Consider a constant $\beta \geq 0$ and suppose that $\|(1 + |b|^\beta)\hat{q}(b)\|_1 < \infty$ and $\|(1 + |b|^\beta)\hat{A}(b)\|_1 < 2\varepsilon/63$. Suppose further that $|v| > \frac{2}{\varepsilon}(1 + |b|^\beta)\hat{A}(b)\|_1$. Then, for any $B, C \subset G'$ and $m \geq 1$,

$$\|\pi_B T_{G'G'}^m \pi_C \| \leq (1 + (2\Lambda)^{\beta - [\beta]} [\beta] m^{[\beta] - 1}) \left( \frac{17}{18} \right)^m \sup_{b \in B} \sup_{c \in C} \frac{1}{1 + |b - c|^2},$$

where $[\beta]$ is the smallest integer greater or equal than $\beta$.

**Step 3.** Now observe that, if $b \in G'_1$ and $c \in G'_4$ then

$$|b - c| = |b - d' - (c - d')| \geq |c - d'| - |b - d'| \geq \frac{R}{2} - \frac{R}{4} = \frac{R}{4}.$$ 

Thus, applying the last proposition with $\beta = 2$ and recalling that $G'_3 \subset G'$, for $m \geq 0$ we have

$$\|\pi_{G'_1} T_{33}^m T_{34} \| \leq \|\pi_{G'_1} T_{G'G'}^m T_{G'G'}' \| \leq \|\pi_{G'_1} T_{G'G'}^m T_{G'G'}' \| \leq \frac{3(m + 1)}{1 + \frac{1}{16} R^2} \left( \frac{17}{18} \right)^{m+1}.$$ 

Furthermore, since $\pi_{G'_4} \pi_{G'_3} = \pi_{G'_4} \pi_{G'_1} = 0$ and $\pi_{G'_3} \pi_{G'_4} = \pi_{G'_4}$, from (70) we obtain

$$W_{43}^{(j)} \pi_{G'_1} = \pi_{G'_4} T_{G'G'}^{j+1} \pi_{G'_1} = \pi_{G'_4} T_{G'G'}^{j+1} \pi_{G'_1}.$$ 

Hence,

$$\|W_{43}^{(j)} \pi_{G'_1} \| = \|\pi_{G'_4} T_{G'G'}^{j+1} \pi_{G'_1} \| \leq \|T_{G'G'} \|^{j+1} \leq \left( \frac{17}{18} \right)^{j+1}.$$ 

Therefore, for $0 \leq m < j$,

$$\|\pi_{G'_1} T_{33}^m T_{34} W^{(j-m-1)} \pi_{G'_1} \| \leq \|\pi_{G'_1} T_{33}^m T_{34} \| \|W_{43}^{(j-m-1)} \pi_{G'_1} \| \leq \frac{3(m + 1)}{1 + \frac{1}{16} R^2} \left( \frac{17}{18} \right)^{j+1}.$$ 

44
Iterating the first expression in (71) we find that
\[
X_{33}^{(j)} = T_{34}W_{43}^{(j-1)} + T_{33}X_{33}^{(j-1)}
= T_{34}W_{43}^{(j-1)} + T_{33}T_{34}W_{43}^{(j-2)} + T_{33}X_{33}^{(j-2)}
\vdots
\]
\[
= T_{34}W_{43}^{(j-1)} + T_{33}T_{34}W_{43}^{(j-2)} + \cdots + T_{33}^{j-2}T_{34}W_{43}^{(1)} + T_{33}^{j-1}T_{34}W_{43}^{(0)}
= \sum_{m=0}^{j-1} T_{33}^{m}T_{34}W_{43}^{(j-m-1)}.
\]

Thus, using the above inequality,
\[
\|\pi_{G_1'}X_{33}^{(j)}\pi_{G_1'}\| = \left\|\sum_{m=0}^{j-1} \pi_{G_1'}T_{33}^{m}T_{34}W_{43}^{(j-m-1)}\pi_{G_1'}\right\| \leq \sum_{m=0}^{j-1} \|\pi_{G_1'}T_{33}^{m}T_{34}W_{43}^{(j-m-1)}\pi_{G_1'}\| \leq \frac{3}{2} + \frac{3}{2}R^2 \left(1 + \frac{18}{R^2}\right)
\]

Consequently,
\[
\left\|\pi_{G_1'}\left[\sum_{j=1}^{\infty} X_{33}^{(j)}\right]\pi_{G_1'}\right\| \leq \sum_{j=1}^{\infty} \|\pi_{G_1'}X_{33}^{(j)}\pi_{G_1'}\| \leq \frac{3}{2} + \frac{3}{2}R^2 \left(1 + \frac{18}{R^2}\right)
\]

where \(C\) is an universal constant. Finally, using this and (73), since \(|z_{\mu,d'}| \leq 3|v|\) we have
\[
|R_3(k)| = \left|\sum_{b,c \in G_1'} \frac{f(d'-b)}{N_b(k)} \left[\sum_{j=1}^{\infty} X_{33}^{(j)}\right]_{b,c} g(c-d')\right| \leq \frac{6C}{A} \|f\|_1 \|g\|_1 \frac{1}{|z_{\mu,d'}| R^2}.
\]

In view of (72) and (75) this completes the proof. \(\square\)

**Proof of Proposition 17.** For any \(b,c \in \Gamma^\#\) set \(Q_{b,c} := (1 + |b - c|^\beta)T_{b,c}\). We first claim that, for any \(B, C \subset G'\),
\[
\sup_{b \in B} \sum_{c \in C} |Q_{b,c}| < \frac{17}{18} \quad \text{and} \quad \sup_{c \in C} \sum_{b \in B} |Q_{b,c}| < \frac{17}{18}.
\]

In fact, using the bounds (11), (12) and \(|k| \leq 3|v|\), it follows that
\[
\sup_{b \in B} \sum_{c \in C} |Q_{b,c}| = \sup_{b \in B} \sum_{c \in C} (1 + |b - c|^\beta) \left| \frac{\hat{q}(b-c)}{N_c(k)} \right| \leq \|1 + |b|^\beta\|_{\ell^1} \frac{1}{v}\|v\|_1 + \frac{14}{\varepsilon}\|1 + |b|^\beta\|_{\ell^1} \|v\|_1 < \frac{1}{2} + \frac{4}{9} = \frac{17}{18},
\]

and similarly we prove the second bound in (77). Furthermore, since \(|T_{b,c}| \leq |Q_{b,c}|\) for all \(b,c \in \Gamma^\#\), for any integer \(m \geq 1\) we have
\[
\sup_{b \in B} \sum_{c \in C} |(T_{BC}^m)_{b,c}| < \left(\frac{17}{18}\right)^m \quad \text{and} \quad \sup_{c \in C} \sum_{b \in B} |(T_{BC}^m)_{b,c}| < \left(\frac{17}{18}\right)^m.
\]
Now, let $p$ be the smallest integer greater or equal than $\beta$, and for any integer $m \geq 1$ and any $\xi_0, \xi_1, \ldots, \xi_m \in \Gamma^\#$, let $b = \xi_0$ and $c = \xi_m$. Then,

$$|b - c|^{\beta} = (2\Lambda)^{\beta} \left[ \frac{|b - c|}{2\Lambda} \right]^{\beta} \leq (2\Lambda)^{\beta} \left[ \frac{|b - c|}{2\Lambda} \right]^{\beta} = (2\Lambda)^{\beta} \sum_{i_1, \ldots, i_p=1}^{m} |\xi_{i_1-1} - \xi_{i_1}| \cdots |\xi_{i_p-1} - \xi_{i_p}|$$

$$\leq (2\Lambda)^{\beta-p} \sum_{i_1, \ldots, i_p=1}^{m} (|\xi_{i_1-1} - \xi_{i_1}|^p + \cdots + |\xi_{i_p-1} - \xi_{i_p}|^p)$$

$$= (2\Lambda)^{\beta-p} p m^{p-1} \sum_{i=1}^{m} |\xi_{i-1} - \xi_i|^p \leq (2\Lambda)^{\beta-p} p m^{p-1} \prod_{i=1}^{m} (1 + |\xi_{i-1} - \xi_i|^p).$$

(78)

To simplify the notation write $s := \sup_{b \in B, c \in C} \frac{1}{1 + |b - c|^{\beta}}$. Hence,

$$\sup_{b \in B} \sum_{c \in C} |(T_{G'}^m)_{b,c}| \leq \sup_{b \in B} \frac{1}{1 + |b - c|^{\beta}} \sup_{b \in B} \sum_{c \in C} (1 + |b - c|^{\beta}) |(T_{G'}^m)_{b,c}|$$

$$\leq s \left[ \sup_{b \in B} \sum_{c \in C} |(T_{G'}^m)_{b,c}| + (2\Lambda)^{\beta-p} p m^{p-1} \sup_{b \in B} \sum_{\xi_{i} \in G'} (1 + |b - \xi_{i}|^{\beta}) |T_b,\xi_i| \right. $$

$$\times \sum_{\xi_{i} \in G'} (1 + |\xi_{i} - \xi_{i+1}|^{\beta}) |T_{\xi_{i+1},\xi_{i+2}}| \cdots \left. \sum_{c \in C} (1 + |\xi_{m-1} - c|^{\beta}) |T_{\xi_{m-1},c}| \right]$$

$$\leq s \left[ \frac{17}{18} m + (2\Lambda)^{\beta-p} p m^{p-1} \sup_{b \in B} \sum_{\xi_{i} \in G'} (1 + |b - \xi_{i}|^{\beta}) |T_b,\xi_i| \right.$$

$$\times \sup_{\xi_{i} \in G'} \sum_{\xi_{i} \in G'} (1 + |\xi_{i} - \xi_{i+1}|^{\beta}) |T_{\xi_{i+1},\xi_{i+2}}| \cdots \left. \sup_{c \in C} (1 + |\xi_{m-1} - c|^{\beta}) |T_{\xi_{m-1},c}| \right]$$

$$\leq s (1 + (2\Lambda)^{\beta-p} p m^{p-1}) \left( \frac{17}{18} \right)^m,$$

and similarly we prove the other inequality. Therefore, by Proposition 5,

$$\|\pi_B T_{G'}^m \pi_C\| \leq (1 + (2\Lambda)^{\beta-[\beta]} [\beta] m^{[\beta]-1} \left( \frac{17}{18} \right)^m \sup_{b \in B} \frac{1}{1 + |b - c|^{\beta}},$$

where $[\beta]$ is the smallest integer greater or equal than $\beta$. This is the desired estimate. 

Proof of Lemma 4. To simplify the notation write $w = w_{\mu,d'}, z = z_{\mu,d'}$, and $|z|_R = 2|z| - R$. First observe that

$$\frac{1}{w - 2i\theta_{d'}(c - d')} = \frac{-1}{2i\theta_{d'}(c - d')} + \frac{w}{2i\theta_{d'}(c - d')(w - 2i\theta_{d'}(c - d'))},$$

so that

$$\frac{z}{N_c(k)} = \frac{-1}{2i\theta_{d'}(c - d')} + \frac{w}{2i\theta_{d'}(c - d')(w - 2i\theta_{d'}(c - d'))} + \frac{2i\theta_{d'}(c - d')}{w - 2i\theta_{d'}(c - d')} \frac{1}{z - 2i\theta_{d'}(c - d')}$$

$$=: \eta_c^{(0)} + \eta_c^{(w)} + \eta_c^{(z)},$$
where, in view of (58) to (61), since $|w| < \varepsilon$,

$$|\eta_c^{(0)}| \leq \frac{1}{2\Lambda}, \quad |\eta_c^{(w)}| \leq \frac{\varepsilon}{2\Lambda^2} \quad \text{and} \quad |\eta_c^{(z)}| \leq \frac{4}{|z|_R}.$$  

Hence,

$$Y_{b,c} = \frac{-2i\theta_{\mu'}(\hat{A}(b-c))z}{N_c(k)} = -2i\theta_{\mu'}(\hat{A}(b-c))\eta_c^{(0)} - 2i\theta_{\mu'}(\hat{A}(b-c))\eta_c^{(w)} - 2i\theta_{\mu'}(\hat{A}(b-c))\eta_c^{(z)} =: Y_{b,c}^{(0)} + Y_{b,c}^{(w)} + Y_{b,c}^{(z)}.$$  

Let $Y^{(-)}$ be the operator whose matrix elements are $Y_{b,c}^{(-)}$ and set $Y_{33}^{(-)} := \pi_{G_3'} Y^{(-)} \pi_{G_3'}$. Then, similarly as we estimated $\|Y_{33}\|$, using (58) to (61) and Proposition 5, it follows easily that

$$\|Y_{33}^{(0)}\| \leq \frac{1}{2\Lambda} \|\theta_{\mu'}(\hat{A})\|_1, \quad \|Y_{33}^{(w)}\| \leq \frac{\varepsilon}{2\Lambda^2} \|\theta_{\mu'}(\hat{A})\|_1, \quad \|Y_{33}^{(z)}\| \leq \frac{4}{|z|_R} \|\theta_{\mu'}(\hat{A})\|_1.$$  

Furthermore,

$$S = (I - Y_{33})^{-1} = 1 + (1 - Y_{33})^{-1}Y_{33} = 1 + SY_{33}$$

$$= 1 + (1 + SY_{33})Y_{33} = 1 + Y_{33}^{(0)} + Y_{33}^{(w)} + Y_{33}^{(z)} + SY_{33}^2,$$

where, recalling (56),

$$\|SY_{33}^2\| \leq \|(1 - Y_{33})^{-1}\| \|Y_{33}\|^2 \leq \frac{\|Y_{33}\|^2}{1 - \|Y_{33}\|} < \frac{14}{13} \left( \frac{8}{\Lambda} \right)^2 \|\theta_{\mu'}(\hat{A})\|_1^2.$$  

Combining all this we have

$$\frac{zS_{b,c}}{N_{b,c}} = (\eta_b^{(0)} + \eta_b^{(w)})(\delta_{b,c} + Y_{b,c}^{(0)} + Y_{b,c}^{(w)} + Y_{b,c}^{(z)} + (SY_{33}^2)_{b,c}) + \eta_b^{(z)S_{b,c}}$$

$$= \left(\eta_b^{(0)}(\delta_{b,c} + Y_{b,c}^{(0)})\right) + \left[\eta_b^{(w)}(\delta_{b,c} + Y_{b,c}^{(0)})\right] + \left[(\eta_b^{(0)} + \eta_b^{(w)})SY_{33}^2\right] + \left[(\eta_b^{(0)} + \eta_b^{(w)})Y_{b,c}^{(z)} + \eta_b^{(z)S_{b,c}}\right]$$

$$=: K_{b,c}^{(0)} + K_{b,c}^{(1)} + K_{b,c}^{(2)} + K_{b,c}^{(3)}$$

with

$$|K_{b,c}^{(0)}| \leq \frac{1}{2\Lambda} \left(1 + \frac{1}{2\Lambda} \|\theta_{\mu'}(\hat{A})\|_1\right),$$

$$|K_{b,c}^{(1)}| \leq \frac{\varepsilon}{4\Lambda^3} \|\theta_{\mu'}(\hat{A})\|_1 + \frac{\varepsilon}{2\Lambda^2} \left(1 + \frac{1}{2\Lambda} \|\theta_{\mu'}(\hat{A})\|_1 + \frac{\varepsilon}{\Lambda^2} \|\theta_{\mu'}(\hat{A})\|_1\right)$$

$$< \frac{\varepsilon}{2\Lambda^2} \left(1 + \frac{7}{6\Lambda} \|\theta_{\mu'}(\hat{A})\|_1\right),$$

$$|K_{b,c}^{(2)}| \leq \frac{1}{4\Lambda} \left(\frac{8}{\Lambda}\right)^2 \|\theta_{\mu'}(\hat{A})\|_1^2 < \frac{64}{\Lambda^2} \|\theta_{\mu'}(\hat{A})\|_1^2,$$

$$|K_{b,c}^{(3)}| \leq \frac{3}{2\Lambda} \|\theta_{\mu'}(\hat{A})\|_1 \frac{4}{|z|_R} + \frac{14}{13} \frac{4}{|z|_R} < \frac{C_{\Lambda,A}}{|z|_R}.$$
for all \( b, c \in G' \). Here, to estimate \(|K_{b,c}^{(1)}|\) we have used that \( \varepsilon < \Lambda/6 \).

Finally, recalling (66) and using the above estimates we find that

\[
z_{\mu,d'}(k)\alpha_{\mu,d'}^{(1)}(k) = \sum_{b, c \in G'_1} f(d' - b) \frac{z S_{b,c}}{N_0(k)} g(c - d')
\]

\[
= \sum_{b, c \in G'_1} f(d' - b) \left[ 3 \sum_{j=0}^{K_{b,c}} g(c - d') \right]
\]

\[=: \alpha_{\mu,d'}^{(1,0)} + \alpha_{\mu,d'}^{(1,1)}(w(k)) + \alpha_{\mu,d'}^{(1,2)}(k) + \alpha_{\mu,d'}^{(1,3)}(k),
\]

where, in particular,

\[
\alpha_{\mu,d'}^{(1,0)} = - \sum_{b, c \in G'_1} \frac{f(d' - b)}{2i\theta(\mu, d')} \left[ \delta_{b,c} + \frac{\theta(\mu, a(b - c))}{\theta(\mu, c - d')} \right] g(c - d').
\]

Furthermore, for \( 0 \leq j \leq 2 \), it follows easily from (79) that \(|\alpha_{\mu,d'}^{(1,j)}| \leq C_j\) with

\[
C_0 := \frac{1}{2\Lambda} \left( 1 + \frac{1}{2\Lambda} \|\theta(\mu, A)\|_{L^1} \right) \|f\|_{L^1} \|g\|_{L^1},
\]

\[
C_1 := \frac{\varepsilon}{2\Lambda^2} \left( 1 + \frac{7}{6\Lambda} \|\theta(\mu, A)\|_{L^1} \right) \|f\|_{L^1} \|g\|_{L^1},
\]

\[
C_2 := \frac{64}{\Lambda^3} \|\theta(\mu, A)\|_{L^1} \|f\|_{L^1} \|g\|_{L^1},
\]

while for \( j = 3 \),

\[
|\alpha_{\mu,d'}^{(1,3)}| \leq C_{\Lambda, A, f, g} \frac{1}{|\varepsilon|^R}.
\]

This completes the proof of the lemma. \( \square \)

Proof of Lemma 5. To prove this lemma we apply the following (well-known) inequality (see [13] for a proof).

**Proposition 18.** Let \( \alpha \) and \( \delta \) be constants with \( 1 < \alpha \leq 2 \) and \( 1 < \delta \leq 2 \). Suppose that \( f \) is a function on \( \Gamma^# \) obeying \( \|b|^{\alpha} f(b)\|_{L^1} < \infty \). Then, for any \( \xi_1, \xi_2 \in \Gamma^# \) with \( \xi_1 \neq \xi_2 \),

\[
\sum_{b \in \Gamma^# \setminus \{\xi_1, \xi_2\}} \frac{|f(b - \xi_1)|}{|b - \xi_2|^{\beta}} \leq \frac{C}{|\xi_1 - \xi_2|^{\alpha + \delta - 2}} \times \begin{cases} 1 & \text{if } \alpha, \delta < 2, \\ \ln |\xi_1 - \xi_2| & \text{if } \alpha = 2 \text{ or } \delta = 2, \end{cases}
\]

where \( C = C_{\Gamma^#, \alpha, \delta, f} \) is a constant.

First observe that \( \|\pi_{(b)} T_{G'}^{m} \pi_{(c)}\| = |(T_{G'}^{m})_{b,c}|. \) Hence, by Proposition 17 with \( \beta = 2 \), for all \( b, c \in G' \) and \( m \geq 1 \),

\[
|(T_{G'}^{m})_{b,c}| = \|\pi_{(b)} T_{G'}^{m} \pi_{(c)}\| \leq (1 + 2m) \left( \frac{17}{18} \right)^m \frac{1}{1 + |b - c|^2}.
\]
Applying this inequality and (83) to (82) we obtain

$$|\Phi_{d',d''}(k)| = \left| \sum_{m=0}^{\infty} \sum_{b,c \in G'} \frac{f(d' - b)}{N_b(k)} (T_{G'G})_{b,c} g(c - d'') \right|$$

$$\leq \frac{1}{\varepsilon |v|} \left[ \sum_{m=0}^{\infty} (1 + 2m) \left( \frac{17}{18} \right)^m \right] \sum_{b \in G'} |f(d' - b)| \sum_{c \in G'} \frac{|g(c - d'')|}{1 + |b - c|^2} \leq C \frac{\varepsilon}{|v|} \sum_{b \in G'} |f(d' - b)| \left[ \frac{|g(b - d'')|}{|b - d''|} + \sum_{c \in G' \setminus \{b\}} \frac{|g(c - d'')|}{|b - c|^2} \right],$$

where $C$ is an universal constant.

Now, by the triangle inequality, Hölder’s inequality, and since $\| \cdot \|_2 \leq \| \cdot \|_{l_1}$,

$$\sum_{b \in G'} |f(d' - b)| \|g(b - d'')\|_2$$

$$= \sum_{b \in G'} \frac{|d' - d''|^2}{|d' - d''|^2} |f(d' - b)| \|g(b - d'')\|_2$$

$$\leq \frac{4}{|d' - d''|^2} \sum_{b \in G'} (|d' - b|^2 + |b - d''|^2) |f(d' - b)| \|g(b - d'')\|_2$$

$$\leq \frac{4}{|d' - d''|^2} \left( \|b^2 f(b)\|_{l_2} \|g\|_{l_2} + \|f\|_{l_2} \|b^2 g(b)\|_{l_2} \right) \leq \frac{C_{f,g}}{|d' - d''|^2}.$$  

Furthermore, by Proposition 18 with $\alpha = \delta = 2$, for any $0 < \epsilon_1 < 2$,

$$\sum_{c \in G' \setminus \{b\}} \frac{|g(c - d'')|}{|b - c|^2} \leq C_{\Gamma^#, f,g} \ln \frac{|b - d''|}{|b - d'|} \leq \frac{C_{\Gamma^#, f,g,\epsilon_1}}{|b - d''|^2 - \epsilon_1}.$$  

Applying this inequality and (83) to (82) we obtain

$$|\Phi_{d',d''}(k)| \leq \frac{C}{\varepsilon |v|} \left[ \frac{C_{f,g}}{|d' - d''|^2} + C \frac{C_{\Gamma^#, f,g,\epsilon_1}}{|b - d''|^2 - \epsilon_1} \right].$$

Again, by Proposition 18 with $\alpha = 2$ and $\delta = 2 - \epsilon_1$ we conclude that, for any $0 < \epsilon_2 < 2 - \epsilon_1$,

$$|\Phi_{d',d''}(k)| \leq \frac{C}{\varepsilon |v|} \left[ \frac{C_{f,g}}{|d' - d''|^2} + C \frac{C_{\Gamma^#, f,g,\epsilon_1}}{|d' - d''|^2 - \epsilon_1} \right] \leq \frac{C_{\epsilon,\Gamma^#, f,g,\epsilon_1,\epsilon_2}}{|d' - d''|^2 - \epsilon_1 - \epsilon_2}.$$  

Finally, recall from Proposition 11(ii) that $|z_{d',d}| < 3|d|$ and $|z_{d',d}| < 3|v|$, observe that $|d' - d''| = |d|$, and set $\epsilon = \epsilon_1 + \epsilon_2$. Then, for any $0 < \epsilon < 2$,

$$|\Phi_{d',d''}(k)| \leq \frac{C_{\epsilon,\Gamma^#, f,g,\epsilon_1,\epsilon_2}}{|d|} \frac{|d|^{2-\epsilon_1-\epsilon_2}}{|z_{d',d}|^{3-\epsilon}} \leq \frac{C_{\epsilon,\Gamma^#, f,g,\epsilon}}{|z_{d',d}|^{3-\epsilon}}.$$  

Choosing $\epsilon = 10^{-1}$ we obtain the desired inequality. \qed
C  Bounds on the derivatives: proofs

Proof of Lemma 6. Step 0. When there is no risk of confusion we shall use the same notation to denote an operator or its matrix. Define

$$\mathcal{F}_{BC} := [f(b - c)]_{b \in B, c \in C}, \quad \mathcal{G}_{BC} := [g(b - c)]_{b \in B, c \in C}, \quad \Phi_G(k) := [\Phi_{d', d''}(k; G)]_{d', d'' \in G}.$$  

Here $\mathcal{F}_{BC}$ and $\mathcal{G}_{BC}$ are $|B| \times |C|$ matrices and $\Phi_G(k)$ is a $|G| \times |G|$ matrix. First observe that

$$\Phi_G(k) = \left[ \sum_{b, c \in G'} \frac{f(d' - b)}{N_b(k)} (R_{G'}^{-1})_{b,c} g(c - d'') \right]_{d', d'' \in G}$$

can be written as the product of matrices $\mathcal{F}_{G'} \Delta_k^{-1} R_{G'}^{-1} \mathcal{G}_{G'}$. Furthermore, since on $L^2_{G'}$ we have $\Delta_k^{-1} R_{G'}^{-1} = (R_{G'} \Delta_k)^{-1} = H_k^{-1}$, we can write $\Phi_G(k)$ as $\mathcal{F}_{G'} H_k^{-1} \mathcal{G}_{G'}$. Hence,

$$\frac{\partial^{n+m}}{\partial k_1^m \partial k_2^n} \Phi_G(k) = \mathcal{F}_{G'} \frac{\partial^{n+m} H_k^{-1}}{\partial k_1^m \partial k_2^n} \mathcal{G}_{G'}.$$  

This is the quantity we want to estimate.

Step 1. Let $T = T(k)$ be an invertible matrix. Then applying $\frac{\partial^{m_0} T^{-1}}{\partial k_i^{m_0}}$ to the identity $TT^{-1} = I$ and using the Leibniz rule for $\frac{\partial^{m_0} T^{-1}}{\partial k_i^{m_0}}(TT^{-1})$ we find that

$$\frac{\partial^{m_0} T^{-1}}{\partial k_i^{m_0}} = -T^{-1} \sum_{m_1=0}^{m_0-1} \left( \begin{array}{c} m_0 \\ m_1 \end{array} \right) \frac{\partial^{m_0-m_1} T}{\partial k_i^{m_0-m_1}} \frac{\partial^{m_1} T^{-1}}{\partial k_i^{m_1}}.$$  

Iterating this formula $m_0 - 1$ times we obtain

$$\frac{\partial^{m_0} T^{-1}}{\partial k_i^{m_0}} = \left[ \prod_{j=1}^{m_0-1} \sum_{m_j=0}^{m_{j-1}-1} \left( \begin{array}{c} m_{j-1} \\ m_j \end{array} \right) (-T^{-1})^{m_j} \frac{\partial^{m_{j-1}} T}{\partial k_i^{m_{j-1}}} \right] \frac{\partial^{m_0} T^{-1}}{\partial k_i^{m_0}}$$

$$= \left[ \prod_{j=1}^{m_0-1} \sum_{m_j=0}^{m_{j-1}-1} \left( \begin{array}{c} m_{j-1} \\ m_j \end{array} \right) (-T^{-1})^{m_j} \frac{\partial^{m_{j-1}} T}{\partial k_i^{m_{j-1}}} \right] \times \sum_{m_{m_0}=0}^{m_{m_0}-1} \left( \begin{array}{c} m_{m_0} \\ m_{m_0} \end{array} \right) (-T^{-1}) \frac{\partial^{m_{m_0}} T}{\partial k_i^{m_{m_0}}} \frac{\partial^{m_{m_0}} T^{-1}}{\partial k_i^{m_{m_0}}}$$

$$= (-1)^{m_0} \left[ \prod_{j=1}^{m_0-1} \sum_{m_j=0}^{m_{j-1}-1} \left( \begin{array}{c} m_{j-1} \\ m_j \end{array} \right) T^{-1} \frac{\partial^{m_{j-1}} T}{\partial k_i^{m_{j-1}}} \right] T^{-1} \frac{\partial^{m_{m_0}} T^{-1}}{\partial k_i^{m_{m_0}}} T^{-1}.  \hspace{1cm} (85)$$

Step 2. In view of (85), it is not difficult to see that $\frac{\partial^{m} H_k^{-1}}{\partial k_i^{m}}$ is given by a finite linear combination of terms of the form

$$\left[ \prod_{j=1}^{m} H_k^{-1} \frac{\partial^{m_j} H_k}{\partial k_i^{m_j}} \right] H_k^{-1},  \hspace{1cm} (86)$$

50
where $\sum_{j=1}^{m} n_j = m$. Thus, when we compute $\frac{\partial^n}{\partial k_1^j} \frac{\partial^m H_k^{-1}}{\partial k_2^j}$, the derivative $\frac{\partial^n}{\partial k_1^j}$ acts either on $H_k^{-1}$ or $\frac{\partial^m H_k}{\partial k_2^j}$. However, since $(\frac{\partial^m H_k}{\partial k_2^j})_{b,c} = 2(k_2 + b_2)\delta_{b,c} - 2\hat{A}_2(b - c)$, we have $\frac{\partial^n}{\partial k_1^j} \frac{\partial^m H_k}{\partial k_2^j} = 0$ if $n_j \geq 1$ and $\frac{\partial^n}{\partial k_1^j} \frac{\partial^m H_k}{\partial k_2^j} = \frac{\partial^m H_k}{\partial k_2^j}$ if $n_j = 0$. Similarly, using again (85) one can see that $\frac{\partial^n}{\partial k_1^j}$ is given by a finite linear combination of terms of the form (86), with $m$ and $k_2$ replaced by $n$ and $k_1$, respectively, and $\sum_{j=1}^{n} n_j = n$. Therefore, combining all this we conclude that $\frac{\partial^{n+m} H_k^{-1}}{\partial k_1^j \partial k_2^j}$ is given by a finite linear combination of terms of the form

$$\left[ \prod_{j=1}^{n+m} \Delta_k^{-1} R^{-1}_{G'G'} \frac{\partial^{n_j} H_k}{\partial k_1^{n_j}} \right] \Delta_k^{-1} R^{-1}_{G'G'},$$

(87)

where $\sum_{j=1}^{n+m} n_j \delta_{2,j} = m$ and $\sum_{j=1}^{n+m} n_j \delta_{1,j} = n$, that is, where the sum of $n_j$ for which $i_j = 2$ is equal to $m$, and the sum of $n_j$ for which $i_j = 1$ is equal to $n$.

**Step 3.** The first step in bounding (87) is to estimate $\left\| \frac{\partial^{n_j} H_k}{\partial k_1^{n_j}} \Delta_k^{-1} \pi_{G'} \right\|$. A simple calculation shows that

$$\left( \frac{\partial^{n_j} H_k}{\partial k_1^{n_j}} \Delta_k^{-1} \right)_{b,c} = \frac{1}{N_c(k)} \begin{cases} 2(k_{i_j} + b_{i_j})\delta_{b,c} + 2\hat{A}_{i_j}(b - c) & \text{if } n_j = 1, \\ 2\delta_{b,c} & \text{if } n_j = 2, \\ 0 & \text{if } n_j \geq 3. \end{cases}$$

Furthermore, by Proposition 7,

$$\frac{1}{|N_b(k)|} \leq \frac{1}{\varepsilon |v|}$$

for all $b \in G'$, while by Proposition 3 we have

$$\frac{1}{|N_b(k)|} \leq \frac{2}{\Lambda |v|}$$

(88)

and

$$|k_{i} + b_{i}| \leq |u_{i} + b_{i}| + |v_{i}| \leq |v| + |u + b| \leq \frac{2}{\Lambda} |N_b(k)|$$

for all $b \in G'$ if $G = \{0, d\}$, and for all $b \in G' \setminus \{\tilde{b}\}$ if $G = \{0\}$. Furthermore,

$$|\tilde{b}| \leq \Lambda + |u| + |v| < \Lambda + 3|v|,$$

(89)

since $|u| < 2|v|$ because $k \in T_0$. Now, let $1_B(x)$ be the characteristic function of the set $B$. Then, using the above estimates we have

$$\sup_{c \in G} \sum_{b \in G'} \left| \left( \frac{\partial^{n_j} H_k}{\partial k^{n_j}_1} \Delta_k^{-1} \pi_{G'} \right)_{b,c} \right|$$

$$\leq \sup_{c \in G} \sum_{b \in G'} \left[ \frac{2|k_{i_j} + b_{i_j}|\delta_{n_j,1} + 2\delta_{n_j,2}}{|N_b(k)|} \delta_{b,c} + \frac{2|\hat{A}_{i_j}(b - c)|}{|N_b(k)|} \delta_{n_j,1} \right]$$

51
\[ \leq \sup_{c \in G'} \left[ \frac{2|k_i + b_i| + 2}{|N_b(k)|} \delta_{b,c} + \frac{2|\hat{A}_i (\tilde{b} - c)|}{|N_b(k)|} \right] 1_{G'}(\tilde{b}) + \sup_{b \in G' \setminus \{\tilde{b}\}} \left[ \frac{2|k_i + b_i| + 2}{|N_b(k)|} \delta_{b,c} + \frac{2|\hat{A}_i (b - c)|}{|N_b(k)|} \right] \]

\[ \leq \frac{2|k_i + b_i| + 2 + 2\|\hat{A}\|_1}{\varepsilon|v|} 1_{G'}(\tilde{b}) + \sup_{c \in G'} \left[ \frac{4}{\Lambda} + \frac{2}{|N_b(k)|} \right] \delta_{b,c} + \frac{2|\hat{A}_i (b - c)|}{|N_b(k)|} \]

\[ \leq \frac{2}{\varepsilon|v|} (2(|u| + |v| + |\tilde{b}|) + 2 + 2\|\hat{A}\|_1) 1_{G'}(\tilde{b}) + \frac{4}{\Lambda} + \frac{4}{|N_b(k)|} \|\hat{A}\|_1 \]

\[ \leq \frac{2}{\varepsilon|v|} (12|v| + 2\Lambda + 2 + 2\|\hat{A}\|_1) 1_{G'}(\tilde{b}) + \frac{4}{\Lambda} + \frac{4}{|N_b(k)|} \|\hat{A}\|_1 \leq 1_{G'}(\tilde{b}) \varepsilon^{-1} C_{\Lambda,A} + C_{\Lambda,A}. \]

Similarly,

\[ \sup_{b \in G'} \sum_{c \in G'} \left| \left( \frac{\partial^{n_j} H_k}{\partial k_{i_j}^{n_j}} \Delta_k^{-1} \pi_{G'} \right)_{b,c} \right| \leq 1_{G'}(\tilde{b}) \varepsilon^{-1} C_{\Lambda,A} + C_{\Lambda,A}. \]

Hence, by Proposition 5,

\[ \left\| \frac{\partial^{n_j} H_k}{\partial k_{i_j}^{n_j}} \Delta_k^{-1} \pi_{G'} \right\| \leq 1_{G'}(\tilde{b}) \varepsilon^{-1} C_{\Lambda,A} + C_{\Lambda,A}. \]

**Step 4.** By a similar (and much simpler) calculation (using Proposition 5) we get

\[ \left\| \frac{\partial^{n_j} H_k}{\partial k_{i_j}^{n_j}} \Delta_k^{-1} \pi_{G'} \right\| \leq 1_{G'}(\tilde{b}) \frac{1}{\varepsilon|v|} + (1 - 1_{G'}(\tilde{b})) \frac{2}{\Lambda|v|}. \]

From Lemma 1 we have \( (R_{G'}^{-1})^{-1} \leq 18 \). Thus, the operator norm of (87) is bounded by

\[ \left\| \prod_{j=1}^{n+m} \Delta_k^{-1} R_{G'}^{-1} \frac{\partial^{n_j} H_k}{\partial k_{i_j}^{n_j}} \Delta_k^{-1} R_{G'}^{-1} \right\| \leq \left\| \prod_{j=1}^{n+m} \frac{\partial^{n_j} H_k}{\partial k_{i_j}^{n_j}} \Delta_k^{-1} \pi_{G'} \right\| \left\| R_{G'}^{-1} \right\|, \]

which is bounded either by

\[ \frac{1}{\varepsilon|v|} 18 \prod_{j=1}^{n+m} (\varepsilon^{-1} C_{\Lambda,A} + C_{\Lambda,A}) \leq \varepsilon^{-(n+m+1)} C_{\Lambda,A,n,m} \frac{1}{|v|} \]

if \( G = \{0\} \), or by

\[ \frac{1}{\Lambda|v|} 18 \prod_{j=1}^{n+m} C_{\Lambda,A} \leq C_{\Lambda,A,n,m} \frac{1}{|v|} \]

if \( G = \{0, d\} \). Therefore,

\[ \left\| \frac{\partial^{n+m} H_k^{-1}}{\partial k_{i_1}^{n_1} \partial k_{i_2}^{n_2}} \right\| \leq \sum_{\# \text{ of terms depend on } n \text{ and } m} \frac{C'}{|v|} \leq C_{n,m} C' \leq C, \]
with $C = C_{e,A,n,m}$ if $G = \{0\}$ or $C = C_{A,n,m}$ if $G = \{0,d\}$. Finally, recalling (84) and (90) we have
\[
\left| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \Phi_G(k) \right| = \left| \mathcal{F}_{GG'} \frac{\partial^{n+m} H_k^{-1}}{\partial k_1^n \partial k_2^m} \right| \leq \| \mathcal{F}_{GG'} \| \left| \frac{\partial^{n+m} H_k^{-1}}{\partial k_1^n \partial k_2^m} \right| \| G_{G'} \| \leq C_1^\beta |v|,
\]
where $C = C_{e,A,n,m,f,g}$ if $G = \{0\}$ or $C = C_{A,n,m,f,g}$ if $G = \{0,d\}$. This is the desired inequality. The proof of the lemma is complete. \qed

**Proof of Lemma 7.** Let $\mathbb{R}^+$ be the set of non-negative real numbers and let $\sigma$ be a real-valued function on $\mathbb{R}^+$ such that:

1. $\sigma(t) \geq 1$ for all $t \in \mathbb{R}^+$ with $\sigma(0) = 1$;
2. $\sigma(s) \sigma(t) \geq \sigma(s + t)$ for all $s, t \in \mathbb{R}^+$;
3. $\sigma$ increases monotonically.

For example, for any $\beta \geq 0$ the functions $t \mapsto e^{\beta t}$ and $t \mapsto (1 + t)^\beta$ satisfy these properties. Now, let $T$ be a linear operator from $L^2_B$ to $L^2_B$ with $B,C \subseteq \Gamma^\#$ (or a matrix $T = [T_{b,c}]$ with $b \in B$ and $c \in C$) and consider the $\sigma$-norm
\[
\|T\|_\sigma := \max \left\{ \sup_{b \in B} \sum_{c \in C} |T_{b,c}| \sigma(|b - c|), \sup_{c \in C} \sum_{b \in B} |T_{b,c}| \sigma(|b - c|) \right\}.
\]

In [13] we prove that this norm has the following properties.

**Proposition 19 (Properties of $\| \cdot \|_\sigma$).** Let $S$ and $T$ be linear operators from $L^2_B$ to $L^2_B$ with $B,C \subseteq \Gamma^\#$. Then:

1. $\|T\| \leq \|T\|_{\sigma=1} \leq \|T\|_\sigma$;
2. If $B = C$, then $\|ST\|_\sigma \leq \|S\|_\sigma \|T\|_\sigma$;
3. If $B = C$, then $\|(I + T)^{-1}\|_\sigma \leq (1 - \|T\|_\sigma)^{-1}$ if $\|T\|_\sigma < 1$;
4. $|T_{b,c}| \leq \frac{1}{\sigma(b - c)} \|T\|_\sigma$ for all $b \in B$ and all $c \in C$.

Now, by using these properties we prove Lemma 7. We follow the same notation as above. First observe that, similarly as in the last proof we can write
\[
\Phi_{d',d''}(k) = \mathcal{F}_{(d')G'} \Delta_k^{-1} R_{G'}^{-1} G_{G'}(d'') = \mathcal{F}_{(d')G'} H_k^{-1} G_{G'}(d'').
\]
Now, let $\sigma(|b|) = (1 + |b|)^\beta$, and observe that there is a positive constant $C_\beta$ such that $\sigma(|b|) \leq C_\beta (1 + |b|^\beta)$ for all $b \in \Gamma^\#$. Then, it is easy to see that
\[
\| \mathcal{F}_{(d')G'} \|_\sigma = \| f \|_\sigma \leq C_\beta \| (1 + |b|^\beta) f(b) \|_\sigma,
\]
\[
\| G_{G'}(d'') \|_\sigma = \| g \|_\sigma \leq C_\beta \| (1 + |b|^\beta) g(b) \|_\sigma.
\]
Furthermore, by (77) and Proposition 5,

\[ \| R_{G^r G^r}^{-1} \|_\sigma = \| (I + T_{G^r G^r})^{-1} \|_\sigma \leq \sum_{j=0}^{\infty} \| T_{G^r G^r}^j \|_\sigma < 18, \]  

and since for diagonal operators the $\sigma$-norm and the operator norm agree, from (90) we have

\[ \| \Delta_k^{-1} \pi_{G^r} \|_\sigma \leq \frac{2}{\Lambda |v|}. \]

Hence, in view of Proposition 19(b) and Proposition 11(ii),

\[ | \Phi_{d',d''}^{(k)} | \leq \| F\{d'\} \Delta_k^{-1} R_{G^r G^r}^{-1} \| \leq C_{\beta,f,g,A,m,n} \frac{1}{|d'|}, \]

and by repeating the proof of Lemma 6 with the operator norm replaced by the $\sigma$-norm we obtain

\[ \left\| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \Phi_{d',d''}(k) \right\|_\sigma \leq C_{\beta,f,g,A,m,n} \frac{1}{|d'|}. \]

Therefore, by Proposition 19(d), for any integers $n$ and $m$ with $n + m \geq 0$,

\[ \left| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \Phi_{d',d''}(k) \right| \leq \frac{1}{1 + |d' - d''|^\beta} \left\| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \Phi_{d',d''}(k) \right\|_\sigma \leq C_{\beta,f,g,A,m,n} \frac{1}{|d'|^{1+\beta}}. \]

This is the desired inequality. \(\square\)

**Proof of Lemma 8**

Define the operator $M^{(j)} : L^2_{G^r_3} \to L^2_{G^r_3}$ as

\[ M^{(j)} := \begin{cases} S & \text{if } j = 1, \\ W & \text{if } j = 2, \\ Z & \text{if } j = 3, \end{cases} \]

where $S$, $W$ and $Z$ are given by (64). In order to prove Lemma 8 we first prove the following proposition.

**Proposition 20.** Assume the same hypotheses of Lemma 8. Then, for any integers $n$ and $m$ with $n + m \geq 1$ and for $1 \leq j \leq 3$,

\[ \left\| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \Delta_k^{-1} M^{(j)} \right\| \leq \frac{C_j}{(2|z_{n,d'}(k)| - R)^j}, \]

where $C_1 = C_{1;A,n,m}$ and $C_j = C_{j;A,n,n,m}$ for $2 \leq j \leq 3$ are constants. Furthermore,

\[ C_{1;A,1,0} \leq \frac{13}{\Lambda^2}, \quad C_{1;A,0,1} \leq \frac{13}{\Lambda^2} \quad \text{and} \quad C_{1;A,1,1} \leq \frac{65}{\Lambda^3}. \]
Proof. Step 0. To simplify the notation write \( w = w_{\mu, \nu} \), \( z = z_{\mu, \nu} \) and \( |z|_{R} = 2|z| - R \). First observe that, for any analytic function of the form \( h(k) = \tilde{h}(w(k), z(k)) \) we have

\[
\frac{\partial}{\partial k_{1}} h = \left( \frac{\partial}{\partial w} + \frac{\partial}{\partial z} \right) \tilde{h}, \quad \frac{\partial}{\partial k_{2}} h = i(-1)^{\nu} \left( \frac{\partial}{\partial w} - \frac{\partial}{\partial z} \right) \tilde{h}.
\]

Thus,

\[
\left\| \frac{\partial^{n+m}}{\partial k_{1}^{n} \partial k_{2}^{m}} \Delta^{-1}_{k} M^{(j)} \right\| = \left\| (i(-1)^{\nu})^{m} \sum_{p=0}^{m} \sum_{r=0}^{n} \binom{m}{p} \binom{n}{r} (-1)^{m-p} \frac{\partial^{-r+m-p}}{\partial z^{n-r+m-p}} \frac{\partial^{r+p}}{\partial w^{r+p}} \Delta^{-1}_{k} M^{(j)} \right\|
\leq 2^{n+m} \sup_{p \leq r} \sup_{r \leq n} \left\| \frac{\partial^{-r+m-p}}{\partial z^{n-r+m-p}} \frac{\partial^{r+p}}{\partial w^{r+p}} \Delta^{-1}_{k} M^{(j)} \right\|.
\]

Now, by the Leibniz rule,

\[
\left\| \frac{\partial^{n}}{\partial z^{n}} \frac{\partial^{m}}{\partial w^{m}} \Delta^{-1}_{k} M^{(j)} \right\| = \left\| \sum_{p=0}^{m} \sum_{r=0}^{n} \binom{m}{p} \binom{n}{r} \frac{\partial^{-r+m-p}}{\partial z^{n-r+m-p}} \frac{\partial^{r+p}}{\partial w^{r+p}} \Delta^{-1}_{k} M^{(j)} \right\|
\leq 2^{n+m} \sup_{p \leq m} \sup_{r \leq n} \left\| \frac{\partial^{-r+m-p}}{\partial z^{n-r+m-p}} \frac{\partial^{r+p}}{\partial z^{r}} \frac{\partial^{r+p}}{\partial w^{r+p}} \Delta^{-1}_{k} M^{(j)} \right\|.
\]

Furthermore, we shall prove below that

\[
\sup_{p \leq m} \sup_{r \leq n} \left\| \frac{\partial^{-r+m-p}}{\partial z^{n-r}} \frac{\partial^{r+p}}{\partial w^{m-p}} \Delta^{-1}_{k} M^{(j)} \right\| \leq \frac{C_{j,n,m}}{|z|_{R}^{n+j}},
\]

with constants \( C_{1,n,m} = C_{1,n,m;A,A} \) and \( C_{j,n,m} = C_{j,n,m;A,A,q} \) for \( 2 \leq j \leq 3 \). Hence,

\[
\left\| \frac{\partial^{n}}{\partial z^{n}} \frac{\partial^{m}}{\partial w^{m}} \Delta^{-1}_{k} M^{(j)} \right\| \leq 2^{n+m} \frac{C_{j,n,m}}{|z|_{R}^{n+j}}.
\]

Therefore, being careful with the indices,

\[
\left\| \frac{\partial^{n+m}}{\partial k_{1}^{n} \partial k_{2}^{m}} \Delta^{-1}_{k} M^{(j)} \right\| \leq 2^{n+m} \sup_{p \leq m} \sup_{r \leq n} \frac{C_{j,n-r+m-p,r+p}}{|z|_{R}^{n-r+m-p+r+j}} \leq \frac{C_{j}}{|z|_{R}^{j}},
\]

where \( C_{1} = C_{1;A,A,n,m} \) and \( C_{j} = C_{j;A,A,q,n,m} \) for \( 2 \leq j \leq 3 \). This is the desired inequality. We are left to prove (93) and estimate the constants \( C_{1,A,i,j} \) for \( i, j \in \{0,1\} \) to finish the proof of the proposition.

Step 1. The first step for obtaining (93) is to estimate \( \left\| \frac{\partial^{r+p} \Delta^{-1}_{k}}{\partial z^{r} \partial w^{p}} \pi G_{3} \right\| \). Observe that

\[
\left| \frac{\partial^{r+p} \Delta^{-1}_{k}}{\partial z^{r} \partial w^{p}} \right|_{b,c} = \left| \frac{\partial^{r+p}}{\partial z^{r} \partial w^{p}} \left( \Delta^{-1}_{k} \right)_{b,c} \right| = \left| \frac{\partial^{r+p}}{\partial w^{p}} \frac{1}{w - 2i \theta_{\mu'}(b - d')} \frac{\partial^{r}}{\partial z^{r}} \frac{\delta_{b,c}}{z - 2i \theta_{\mu}(b - d')} \right|
\]

\[
= \left| \frac{(-1)^{p} p!}{(w - 2i \theta_{\mu'}(b - d'))^{p+1}} \frac{(-1)^{r} r! \delta_{b,c}}{(z - 2i \theta_{\mu}(b - d'))^{r+1}} \right|
\leq \frac{p! r! \delta_{b,c}}{|w - 2i \theta_{\mu'}(b - d')|^{p+1} |z - 2i \theta_{\mu}(b - d')|^{r+1}}.
\]
and recall from (58) and (59) that, for all \( b \in G'_3 \),
\[
\left| \frac{1}{z - 2i\theta \mu(b - d')} \right| \leq \frac{2}{|z|_R} \quad \text{and} \quad \left| \frac{1}{w - 2i\theta \mu(b - d')} \right| \leq \frac{1}{\Lambda}.
\] (94)

Then,
\[
\left| \left( \frac{\partial^{r+p} \Delta_k^{-1}}{\partial z^r \partial w^p} \right) \right|_{b,c} \leq \frac{p! r! 2^{r+1} \delta_{b,c}}{\Lambda^{p+1} |z|_R^{r+1}},
\]
and consequently,
\[
\left[ \sup_{b \in G'_3} \sum_{c \in G'_3} + \sup_{c \in G'_3} \sum_{b \in G'_3} \right] \left( \frac{\partial^{r+p} \Delta_k^{-1}}{\partial z^r \partial w^p} \right) \leq \frac{p! r! 2^{r+1}}{\Lambda^{p+1} |z|_R^{r+1}} \left[ \sup_{b \in G'_3} \sum_{c \in G'_3} + \sup_{c \in G'_3} \sum_{b \in G'_3} \right] \delta_{b,c}.
\]

Therefore, by Proposition 5,
\[
\left| \frac{\partial^{r+p} \Delta_k^{-1}}{\partial z^r \partial w^p} \right|_{\pi G'_3} \leq \frac{p! r! 2^{r+2}}{\Lambda^{p+1} |z|_R^{r+1}}.
\] (95)

**Step 2.** We now estimate the second factor in (93). Let us first consider the case \( j = 1 \), that is, \( M^{(1)} = S \). Since \( S = (I - Y_{33})^{-1} \), the operator \( S \) is clearly invertible. Thus, by applying (85) with \( T = S^{-1} \), one can see that \( \frac{\partial^{r+p} \Delta^s_k}{\partial z^r \partial w^p} \) is given by a finite linear combination of terms of the form
\[
\left[ \prod_{j=1}^{p} S \frac{\partial^{n_j} \Delta_k^{-1}}{\partial w^{m_j}} \right] S,
\] (96)
where \( \sum_{j=1}^{p} n_j = p \). Hence, when we compute \( \frac{\partial}{\partial z^r} \frac{\partial^{r+p} \Delta^s_k}{\partial w^p} \), the derivative \( \frac{\partial}{\partial z^r} \) acts either on \( S \) or \( \frac{\partial^{r+p} \Delta^s_k}{\partial w^p} \). Similarly, using again (85) with \( T = S^{-1} \), one can see that \( \frac{\partial^{r+p} \Delta^s_k}{\partial z^r \partial w^p} \) is given by a finite linear combination of terms of the form (96), with \( p \) and \( w \) replaced by \( r \) and \( z \), respectively, and \( \sum_{j=1}^{r} m_j = r \). Thus, we conclude that \( \frac{\partial^{r+p} \Delta^s_k}{\partial z^r \partial w^p} \) is given by a finite linear combination of terms of the form
\[
\left[ \prod_{j=1}^{r+p} S \frac{\partial^{m_j+n_j} \Delta_k^{-1}}{\partial z^{m_j} \partial w^{n_j}} \right] S,
\] (97)
where \( \sum_{j=1}^{r+p} m_j = r \) and \( \sum_{j=1}^{r+p} n_j = p \). Indeed, observe that the general form of the terms (97) follows directly from (85) because that identity is also valid for mixed derivatives.

Since \( S = (I - Y_{33})^{-1} \) with \( \| Y_{33} \| < 1/14 \) and
\[
Y_{b,c} = \frac{-2i\theta \mu(\hat{A}(b - c)) z}{(w - 2i\theta \mu(c - d'))(z - 2i\theta \mu(c - d'))},
\] (98)
we have
\[
\| S \| = \| (I - Y_{33})^{-1} \| \leq \frac{1}{1 - \| Y_{33} \|} \leq \frac{14}{13}.
\] (99)
and

\[
\left| \frac{\partial^{j+l}}{\partial z^j \partial w^l S^{-1}} \right|_{b,c} = \left| \frac{\partial^{j+l}}{\partial z^j \partial w^l Y_{b,c}} \right| = \left| \frac{\partial^j}{\partial z^j} \frac{-2i\theta_{j'}(\hat{A}(b - c)) z}{z - 2i\theta_i(c - d')} \frac{\partial^l}{\partial w^l} \frac{1}{w - 2i\theta_{j'}(c - d')} \right|.
\]

Furthermore,

\[
\frac{\partial^j}{\partial z^j} \frac{-2i\theta_{j'}(\hat{A}(b - c)) z}{z - 2i\theta_i(c - d')} = \frac{(z - 2i\theta_i(c - d'))^{j+1}}{(2i\theta_{j'}(c - d'))^{j+1}} \quad \text{for} \quad j \geq 1,
\]

\[
\frac{\partial^l}{\partial w^l} \frac{1}{w - 2i\theta_{j'}(c - d')} = \frac{(-1)^l l!}{(w - 2i\theta_{j'}(c - d'))^{l+1}} \quad \text{for} \quad l \geq 0.
\]

Recall from (59) and (61) that, for all \( c \in G' \),

\[
\frac{|c - d'|}{|w - 2i\theta_{j'}(c - d')|} \leq \frac{|c - d'|}{|c - d'| - \varepsilon} \leq 2. \tag{100}
\]

Then, using this and (94), for \( j \geq 1 \) and \( l \geq 0 \),

\[
\left| \frac{\partial^{j+l}}{\partial z^j \partial w^l S^{-1}} \right|_{b,c} \leq \frac{j! l! \hat{A}(b - c)}{|z - 2i\theta_i(c - d')|^{j+1}|w - 2i\theta_{j'}(c - d')|^{l+1}} \frac{|c - d'|}{|w - 2i\theta_{j'}(c - d')|} \leq \frac{2^{j+2}j! l! \hat{A}(b - c)}{\Lambda |z|^{j+1}}. \tag{101}
\]

while for \( j = 0 \) and \( l \geq 0 \),

\[
\left| \frac{\partial^{j+l}}{\partial z^j \partial w^l S^{-1}} \right|_{b,c} \leq \frac{l! \hat{A}(b - c) |z|}{|z - 2i\theta_i(c - d')| |w - 2i\theta_{j'}(c - d')|^{l+1}} \leq \frac{2 l! \hat{A}(b - c)}{\Lambda^{l+1}}. \tag{102}
\]

Consequently,

\[
\left[ \sup_{b \in G'_3} \sum_{c \in G'_3} \sup_{b \in G'_3} \sum_{c \in G'_3} \left| \frac{\partial^{j+l}}{\partial z^j \partial w^l S^{-1}} \right|_{b,c} \right] \leq \left( 1 - \delta_{0,j} + \frac{|z|_R}{2\Lambda} \delta_{0,j} \right) \frac{2^{j+3}j! l!}{\Lambda |z|^{j+1}} \left[ \sup_{b \in G'_3} \sum_{c \in G'_3} \sup_{b \in G'_3} \sum_{c \in G'_3} \right] \hat{A}(b - c).
\]

Therefore, by Proposition (5),

\[
\left\| \frac{\partial^{j+l}}{\partial z^j \partial w^l S^{-1}} \right\| \leq \left( 1 - \delta_{0,j} + \frac{|z|_R}{2\Lambda} \delta_{0,j} \right) \frac{2^{j+3}j! l!}{\Lambda |z|^{j+1}} \| \hat{A} \|_{l^1}. \tag{103}
\]
Thus, for $r \geq 1$, in view of (97) where $\sum_{j=1}^{r+p} m_j = r$,
\[
\left\| \frac{\partial^{r+p}}{\partial z^r \partial w^p} S \right\| \leq C_{r,p} \left( \prod_{j=1}^{r+p} \left\| \frac{\partial}{\partial z_j} \partial w^p \right\| \right) \left( \frac{\partial^{m_j+n_j}}{\partial z_j^m \partial w^{n_j}} S^{-1} \right) \left( \frac{1}{\prod_{j=1}^{r+p} C_{\Lambda,j} \left( 1 - \delta_{0,m_j} + \frac{|z|_R}{2\Lambda j \delta_{0,m_j}} \right) \frac{1}{|z|_R^{m_j+1}}} \right) \left\| S \right\|
\]
\[
\leq C_{r,p} \left( \prod_{j=1}^{r+p} C_{\Lambda,1} \frac{m_j+3m_j! n_j!}{\Lambda^{n_j}} \right) \left( \prod_{j=1}^{r+p} \frac{1}{\prod_{j=1}^{r+p} C_{\Lambda,j} \left( 1 - \delta_{0,m_j} + \frac{|z|_R}{2\Lambda j \delta_{0,m_j}} \right) \frac{1}{|z|_R^{m_j+1}}} \right) \left\| S \right\|
\]
\[
\leq C_{\Lambda,1,r,p} \frac{1}{|z|_R^{r+p+1}},
\]
since $m_j \geq 1$ for at least one $1 \leq j \leq r+p$. Similarly, if $r = 0$ then
\[
\left\| \frac{\partial^{r+p}}{\partial z^r \partial w^p} S \right\| \leq C_{\Lambda,1,r,p}.
\]
Hence, in view of (95),
\[
\sup_{p \leq m} \sup_{r \leq n} \left\| \frac{\partial^{n-r+m-p} \Delta_{\Lambda}^{-1}}{\partial z^{n-r} \partial w^{m-p}} \right\| \left\| \frac{\partial^{r+p} M^{(1)}}{\partial z^r \partial w^p} \right\| \leq \sup_{p \leq m} \sup_{r \leq n} \frac{(m-p)! (n-r)! 2^{n-r+2}}{\Lambda^{n-p+1} |z|_R^{n-r+1}} C_{\Lambda,1,r,p} \left\| \frac{\partial}{\partial z} \right\| \left( 1 - \delta_{0,r} + \frac{|z|_R}{2\Lambda j \delta_{0,r}} \right) \frac{1}{|z|_R^{r+p+1}} \leq C_{\Lambda,1,n,m} \frac{1}{|z|_R^{r+p+1}}.
\]
This proves (93) for $j = 1$.

**Step 3.** We now estimate the constant $C_{1;\Lambda,i,j}$ for $i, j \in \{0, 1\}$. First observe that
\[
\left| \frac{\partial w}{\partial k_j} \right| = |\delta_{1,j} + i(-1)^{r} \delta_{2,j}| = 1 \quad \text{and} \quad \left| \frac{\partial z}{\partial k_j} \right| = |\delta_{1,j} - i(-1)^{r} \delta_{2,j}| = 1.
\]
Thus, in view of (99) and (103), since $|z| \geq |v| > R \geq 2\Lambda$,
\[
\left\| \frac{\partial S}{\partial k_j} \right\| = \left\| -S \frac{\partial S^{-1}}{\partial k_j} S \right\| = \left\| -S \left( \frac{\partial w}{\partial w} \frac{\partial S^{-1}}{\partial k_j} + \frac{\partial z}{\partial k_j} \frac{\partial S^{-1}}{\partial z} \right) S \right\|
\]
\[
\leq \left\| S \right\| \left( \left\| \frac{\partial S^{-1}}{\partial w} \right\| + \left\| \frac{\partial S^{-1}}{\partial z} \right\| \right) \leq \left( \frac{3}{2} \right)^2 \left( \frac{2^4 \left\| S \right\|}{|z|_R^2} + \frac{2^2 \left\| \frac{\partial}{\partial z} \right\|}{\Lambda^2} \right)
\]
\[
\leq 18 \left\| \frac{\partial}{\partial z} \right\| \frac{1}{\Lambda^2}.
\]
Similarly,
\[
\frac{\partial^2 S}{\partial k_i \partial k_j} = - \frac{\partial S}{\partial k_i} \left( \frac{\partial w}{\partial k_j} \frac{\partial S^{-1}}{\partial w} + \frac{\partial z}{\partial k_j} \frac{\partial S^{-1}}{\partial z} \right) S - \frac{\partial S}{\partial k_i} \left( \frac{\partial w}{\partial k_j} \frac{\partial S^{-1}}{\partial w} + \frac{\partial z}{\partial k_j} \frac{\partial S^{-1}}{\partial z} \right) \frac{\partial S}{\partial k_i}
\]
\[
- \frac{\partial S}{\partial k_i} \left( \frac{\partial w}{\partial k_j} \frac{\partial^2 S^{-1}}{\partial w^2} + \frac{\partial z}{\partial k_j} \frac{\partial^2 S^{-1}}{\partial z^2} \right) \left( \frac{\partial w}{\partial k_i} \frac{\partial S}{\partial w} + \frac{\partial z}{\partial k_i} \frac{\partial S}{\partial z} \right) \frac{\partial z}{\partial k_i}
\]
\[
- \frac{\partial S}{\partial k_i} \left( \frac{\partial w}{\partial k_j} \frac{\partial^2 S^{-1}}{\partial w^2} + \frac{\partial z}{\partial k_j} \frac{\partial^2 S^{-1}}{\partial z^2} \right) \left( \frac{\partial w}{\partial k_i} \frac{\partial S}{\partial w} + \frac{\partial z}{\partial k_i} \frac{\partial S}{\partial z} \right) \frac{\partial z}{\partial k_i}
\]
\[
\leq 18 \left\| \frac{\partial}{\partial k_i} \right\| \frac{1}{\Lambda^2}.
\]

58
so that, using the above inequality as well,

\[
\left\| \frac{\partial^2 S}{\partial k_i \partial k_j} \right\| \leq 2 \|S\| \left\| \frac{\partial S}{\partial k_i} \right\| \left( \left\| \frac{\partial S}{\partial w} \right\| + \left\| \frac{\partial S}{\partial z} \right\| \right) \\
+ \|S\| \left( 2 \left\| \frac{\partial^2 S}{\partial w^2} \right\| + 2 \left\| \frac{\partial^2 S}{\partial z \partial w} \right\| + \left\| \frac{\partial^2 S}{\partial z^2} \right\| \right)
\]

\[
\leq 2 \frac{3}{2} \frac{18 \| \hat{A} \|_{l_1}}{\Lambda^2} + \frac{8 \| \hat{A} \|_{l_1}}{\Lambda^2} \left( \frac{3}{2} \right)^2 \left( \frac{2^3 \| \hat{A} \|_{l_1}}{\Lambda^3} + \frac{2^5 \| \hat{A} \|_{l_1}}{\Lambda |z|_R^2} + \frac{2^6 \| \hat{A} \|_{l_1}}{\Lambda |z|_R^3} \right)
\]

\[
\leq \frac{432}{\Lambda^4} \| \hat{A} \|_{l_1}^2 + \frac{54}{\Lambda^3} \| \hat{A} \|_{l_1} \leq \frac{55 \| \hat{A} \|_{l_1}}{\Lambda^3} \left( \frac{8 \| \hat{A} \|_{l_1}}{\Lambda} + 1 \right).
\]

Furthermore, by (95),

\[
\left\| \frac{\partial \Delta_k^{-1}}{\partial k_j} \right\| \leq \left\| \frac{\partial \Delta_k^{-1}}{\partial w} \right\| + \left\| \frac{\partial \Delta_k^{-1}}{\partial z} \right\| \leq \frac{2^2}{\Lambda^2 |z|_R} + \frac{2^3}{\Lambda |z|_R^2} \leq \frac{8}{\Lambda^2 |z|_R}
\]

and

\[
\left\| \frac{\partial^2 \Delta_k^{-1}}{\partial k_i \partial k_j} \right\| \leq 2 \left\| \frac{\partial^2 \Delta_k^{-1}}{\partial w^2} \right\| + 2 \left\| \frac{\partial^2 \Delta_k^{-1}}{\partial z \partial w} \right\| + \left\| \frac{\partial^2 \Delta_k^{-1}}{\partial z^2} \right\| \leq \frac{2^3}{\Lambda^3 |z|_R} + \frac{2^4}{\Lambda^2 |z|_R^2} + \frac{2^6}{\Lambda |z|_R^3} < \frac{5 \cdot 2^2}{\Lambda^3} \frac{1}{|z|_R}.
\]

Hence, since \( \| \hat{A} \|_{l_1} < 2 \varepsilon / 63 \) and \( \varepsilon < \Lambda/6 \),

\[
\left\| \frac{\partial}{\partial k_j} \Delta_k^{-1} S \right\| \leq \left\| \frac{\partial \Delta_k^{-1}}{\partial k_j} \right\| \|S\| + \| \Delta_k^{-1} \| \left\| \frac{\partial S}{\partial k_j} \right\| \leq \frac{8}{\Lambda^2 |z|_R} + \frac{3}{2} \frac{18 \| \hat{A} \|_{l_1}}{\Lambda^2} \frac{2}{\Lambda |z|_R^2} \leq \frac{13}{\Lambda^2} \frac{1}{|z|_R}
\]

and

\[
\left\| \frac{\partial^2}{\partial k_i \partial k_j} \Delta_k^{-1} S \right\| \leq \left\| \frac{\partial^2 \Delta_k^{-1}}{\partial k_i \partial k_j} \right\| \|S\| + \left\| \frac{\partial \Delta_k^{-1}}{\partial k_i} \right\| \left\| \frac{\partial S}{\partial k_j} \right\| + \left\| \frac{\partial \Delta_k^{-1}}{\partial k_j} \right\| \left\| \frac{\partial S}{\partial k_i} \right\| + \left\| \Delta^{-1} \right\| \left\| \frac{\partial^2 S}{\partial k_i \partial k_j} \right\| \leq \frac{1}{|z|_R} \left( \frac{5 \cdot 2^3}{\Lambda^3} \frac{3}{2} + \frac{8}{\Lambda^2} \frac{18 \| \hat{A} \|_{l_1}}{\Lambda^2} + \frac{2}{\Lambda} \frac{55 \| \hat{A} \|_{l_1}}{\Lambda^3} \left( \frac{8 \| \hat{A} \|_{l_1}}{\Lambda} + 1 \right) \right) < \frac{65}{\Lambda^3} \frac{1}{|z|_R}.
\]

Therefore,

\[
C_{1; \Delta A, 1, 0} \leq \frac{13}{\Lambda^2}, \quad C_{1; \Delta A, 0, 1} \leq \frac{13}{\Lambda^2} \quad \text{and} \quad C_{1; \Delta A, 1, 1} \leq \frac{65}{\Lambda^3},
\]

as was to be shown.
Step 4. To prove (93) for \( j = 2 \) we need to bound \( \| \frac{\partial^{r+p} M(z)}{\partial z^r \partial w^p} \| = \| \frac{\partial^{r+p} W}{\partial z^r \partial w^p} \| \). Recall from (64) that

\[
W = \sum_{j=1}^{\infty} W_j = \sum_{j=1}^{\infty} \sum_{m=1}^{j} (Y_{33})^{m-1} X_{33}(Y_{33})^{j-m},
\]

where \( Y_{b,c} \) is given by (98) and \( \| X_{33} \| \leq C/|z| < 1/3 \) with

\[
X_{b,c} = \frac{(c - d') \cdot \hat{A}(b - c) - \hat{q}(b-c) - 2i \theta_\mu (\hat{A}(b - c)) w}{(w - 2i \theta_\nu (c - d'))(z - 2i \theta_\mu (c - d'))}.
\]

First observe that

\[
\frac{\partial^{r+p}}{\partial z^r \partial w^p} (Y_{33})^{m-1} X_{33}(Y_{33})^{j-m}
\]

is given by a sum of \( j^{r+p} \) terms of the form

\[
\frac{\partial^{l_1+n_1} Y_{33}}{\partial z^{m_1} \partial w^{n_1}} \cdots \frac{\partial^{l_m+n_m} Y_{33}}{\partial z^{m_m} \partial w^{n_m}} \frac{\partial^{j+n_j} Y_{33}}{\partial z^{j} \partial w^{n_j}},
\]

where there are \( j \) factors ordered as in the product \( (Y_{33})^{m-1} X_{33}(Y_{33})^{j-m} \). Furthermore, for each term in the sum we have \( \sum_{i=1}^{j} l_i = r \) and \( \sum_{i=1}^{j} n_i = p \). Thus,

\[
\left\| \frac{\partial^{r+p}}{\partial z^r \partial w^p} W \right\| = \left\| \sum_{j=1}^{\infty} \frac{\partial^{r+p}}{\partial z^r \partial w^p} W_j \right\| \geq \sum_{j=1}^{\infty} \sum_{m=1}^{j} \left\| \frac{\partial^{l_1+n_1} Y_{33}}{\partial z^{m_1} \partial w^{n_1}} \cdots \frac{\partial^{l_m+n_m} Y_{33}}{\partial z^{m_m} \partial w^{n_m}} \frac{\partial^{j+n_j} Y_{33}}{\partial z^{j} \partial w^{n_j}} \right\|.
\]

(104)

\[
\leq \sum_{j=1}^{\infty} \sum_{m=1}^{j} \sum_{i=1}^{n_i} \sup_{I} \left\| \frac{\partial^{l_1+n_1} Y_{33}}{\partial z^{m_1} \partial w^{n_1}} \cdots \frac{\partial^{l_m+n_m} Y_{33}}{\partial z^{m_m} \partial w^{n_m}} \frac{\partial^{j+n_j} Y_{33}}{\partial z^{j} \partial w^{n_j}} \right\|.
\]

(105)

where

\[
I := \left\{ (l_i, n_i) \mid l_i \leq r \text{ and } n_i \leq p \text{ for } 1 \leq i \leq j \text{ with } \sum_{i=1}^{j} l_i = r \text{ and } \sum_{i=1}^{j} n_i = p \right\}.
\]

(106)

Note, we can differentiate the series (104) term-by-term because the sum \( \sum_{j=1}^{\infty} W_j \) converges uniformly and the sum \( \sum_{m=1}^{j} \) is finite. We next estimate the factors in (105).

Combining (101) and (102) we have

\[
\left| \frac{\partial^{l_1+n_1}}{\partial z^{l_1} \partial w^{n_1}} Y_{b,c} \right| \leq \left( 1 - \delta_{0,l_1} + \frac{|z|^R}{2 \lambda} \delta_{0,l_1} \right) \frac{2^{l_1+1} n_1}{\lambda^{n_1} |z|^R} |\hat{A}(b - c)|.
\]

(107)
Furthermore, using (94) and (100),

\[
\frac{\partial^{l_i+n_i}}{\partial z^{l_i} \partial w^{n_i}} X_{b,c}
\]
\[
= \left| \frac{\partial^{l_i}}{\partial z^{l_i}} \frac{1}{w - 2i\theta_{\mu}(c - d')} \frac{\partial^{n_i}}{\partial w^{n_i}} \left( c - d' \cdot \hat{A}(b - c) - \hat{q}(b - c) - 2i\theta_{\mu}(\hat{A}(b - c)) w \right) \right|
\]
\[
= \left| ( -1)^{l_i} l_i ! ( -1)^{n_i} n_i ! (2\theta_{\mu}(\hat{A}(b - c)) 2\theta_{\mu'}(c - d') - (c - d') \cdot \hat{A}(b - c) - \hat{q}(b - c)) \right|
\]
\[
\leq \frac{l_i ! n_i ! (2|\hat{A}(b - c)||c - d'| + |\hat{q}(b - c)|)}{|z - 2i\theta_{\mu}(c - d')|^{l_i+1} |w - 2i\theta_{\mu'}(c - d')|^{n_i+1} w - 2i\theta_{\mu'}(c - d')} \right| (z - 2i\theta_{\mu}(c - d'))^{l_i+1} |w - 2i\theta_{\mu'}(c - d')|^{n_i+1}
\]
\[
\leq \frac{2^{l_i+1} l_i ! n_i ! (4|\hat{A}(b - c)| + \frac{1}{\Lambda} |\hat{q}(b - c)|)}{\Lambda^{n_i+1}}
\]

Hence,

\[
\left[ \sup_{b \in G'_3} \sum_{c \in G'_3} \right] \left[ \sup_{b \in G'_3} \sum_{c \in G'_3} \right] \left[ \frac{\partial^{l_i+n_i}}{\partial z^{l_i} \partial w^{n_i}} Y_{b,c} \right]
\]
\[
\leq \left( 1 - \frac{\delta_{0,l_i}}{2\Lambda} \right) \frac{2^{l_i+2} l_i ! n_i !}{\Lambda^{n_i+1}} \left[ \sup_{b \in G'_3} \sum_{c \in G'_3} \right] \left[ \sup_{b \in G'_3} \sum_{c \in G'_3} \right] |\hat{A}(b - c)|
\]
\[
\leq \left( 1 - \frac{\delta_{0,l_i}}{2\Lambda} \right) \frac{2^{l_i+3} l_i ! n_i !}{\Lambda^{n_i+1}} \|\hat{A}\|_1
\]

and similarly

\[
\left[ \sup_{b \in G'_3} \sum_{c \in G'_3} \right] \left[ \sup_{b \in G'_3} \sum_{c \in G'_3} \right] \left[ \frac{\partial^{l_i+n_i}}{\partial z^{l_i} \partial w^{n_i}} X_{b,c} \right] \leq \frac{2^{l_i+2} l_i ! n_i !}{\Lambda^{n_i+1}} \left( 4\|\hat{A}\|_1 + \frac{\|\hat{q}\|_1}{\Lambda} \right)
\]

Thus, by Proposition (5), since \(|z| \geq |v| > R \geq 2\Lambda,

\[
\left\| \frac{\partial^{l_i+n_i}}{\partial z^{l_i} \partial w^{n_i}} Y_{33} \right\| \leq \left( 1 - \frac{\delta_{0,l_i}}{2\Lambda} \right) \frac{2^{l_i+3} l_i ! n_i !}{\Lambda^{n_i+1}} \|\hat{A}\|_1
\]
\[
\leq \left( \frac{1}{|z|_R} + \frac{1}{2\Lambda} \right) \frac{2^{l_i+3} l_i ! n_i !}{\Lambda^{n_i+1}} \|\hat{A}\|_1 \leq \frac{2^{l_i+3} l_i ! n_i !}{\Lambda^{n_i+1}} \|\hat{A}\|_1
\]

and

\[
\left\| \frac{\partial^{l_i+n_i}}{\partial z^{l_i} \partial w^{n_i}} X_{33} \right\| \leq \frac{2^{l_i+2} l_i ! n_i !}{\Lambda^{n_i+1}} \left( 4\|\hat{A}\|_1 + \frac{\|\hat{q}\|_1}{\Lambda} \right)
\]
\[
= \left( 2\Lambda + \frac{\|\hat{q}\|_1}{2\|\hat{A}\|_1} \right) \frac{2^{l_i+3} l_i ! n_i !}{\Lambda^{n_i+1}} \|\hat{A}\|_1.
\]

61
Applying these estimates to (105) and recalling that \( \sum_{i=1}^{j} l_i = r \) and \( \sum_{i=1}^{j} n_i = p \) we have

\[
\left\| \frac{\partial^{r+p}}{\partial z^r \partial w^p} W \right\| \leq \sum_{j=1}^{\infty} j^{r+p} \sum_{m=1}^{j} \sup_{z} \left\| \frac{\partial^{1+n_1} Y_{33}}{\partial z^{1+n_1} \partial w^{n_1}} \right\| \cdots \left\| \frac{\partial^{m+n_m} X_{33}}{\partial z^{m+n_m} \partial w^{n_m}} \right\| \cdots \left\| \frac{\partial^{j+n_j} Y_{33}}{\partial z^{j+n_j} \partial w^{n_j}} \right\|
\]

\[
\leq \sum_{j=1}^{\infty} j^{r+p} \sum_{m=1}^{j} \sup_{z} \left\{ \left( 2\Lambda + \frac{\|\hat{q}\|_{\lambda}}{2\|A\|_{\lambda}} \right) \frac{1}{|z_R|} \prod_{i=1}^{j} \frac{\|\hat{A}^{i+3} l_i! n_i! \|_{\lambda}}{|z_i| R} \right\} \sup_{z} \left\{ \prod_{i=1}^{j} \frac{l_i! \prod_{m=1}^{j} n_m!}{\lambda} \right\} \sum_{m=1}^{j} 1
\]

\[
= \left( 2\Lambda + \frac{\|\hat{q}\|_{\lambda}}{2\|A\|_{\lambda}} \right) \frac{1}{|z_R|^2} \frac{2^{r} R^2}{\Lambda |z_R|} \sum_{j=1}^{\infty} j^{r+p+1} \left( \frac{1}{21} \right)^j \leq C_{\Lambda, A, q, r, p} \frac{1}{|z_R|^{r+1}}.
\]

This is the inequality we needed to prove (93) for \( j = 2 \). In fact, using (95) we obtain

\[
\sup_{p \leq m \leq n} \sup_{r \leq n} \left\| \frac{\partial^{n-r-m-p} \Delta^{-1} \Lambda}{\partial z^{n-r} \partial w^{m-p}} \right\| \left\| \frac{\partial^{r+p} M^{(2)}}{\partial z^r \partial w^p} \right\| \leq \sup_{p \leq m \leq n} \frac{(m-p)! (n-r)! 2^{n-r+2}}{\Lambda^{m-p+1} |z_R|^{n-r+1}} \frac{1}{|z_R|^{r+1}} \leq C_{\Lambda, A, q, m, n} \frac{1}{|z_R|^{n+2}}.
\]

**Step 5.** To prove (93) for \( j = 3 \) we need to estimate \( \left\| \frac{\partial^{r+p} M^{(3)}}{\partial z^r \partial w^p} \right\| = \left\| \frac{\partial^{r+p} Z}{\partial z^r \partial w^p} \right\| \), where

\[
Z = \sum_{j=2}^{\infty} Z_j = \sum_{j=2}^{\infty} (X_{33} + Y_{33})^j - W_j - Y_{33}^j.
\]

First observe that

\[
\frac{\partial^{r+p}}{\partial z^r \partial w^p} Z_j = \frac{\partial^{r+p}}{\partial z^r \partial w^p} ((X_{33} + Y_{33})^j - W_j - Y_{33}^j)
\]

is given by a sum of \( (2^j - j - 1) \cdot j^{r+p} \) terms of the form

\[
\frac{\partial^{1+n_1} Y_{33}}{\partial z^{1+n_1} \partial w^{n_1}} \cdots \frac{\partial^{m+n_m} X_{33}}{\partial z^{m+n_m} \partial w^{n_m}} \cdots \frac{\partial^{j+n_j} Y_{33}}{\partial z^{j+n_j} \partial w^{n_j}},
\]

where there are \( j \) factors involving \( X_{33} \) or \( Y_{33} \) and two factors containing \( X_{33} \). Furthermore, for each term in the sum we have \( \sum_{i=1}^{j} l_i = r \) and \( \sum_{i=1}^{j} n_i = p \). Thus,

\[
\left\| \frac{\partial^{r+p}}{\partial z^r \partial w^p} Z_j \right\| \leq (2^j - j - 1) j^{r+p} \sup_{z} \left\| \frac{\partial^{1+n_1} Y_{33}}{\partial z^{1+n_1} \partial w^{n_1}} \right\| \cdots \left\| \frac{\partial^{m+n_m} X_{33}}{\partial z^{m+n_m} \partial w^{n_m}} \right\| \cdots \left\| \frac{\partial^{j+n_j} Y_{33}}{\partial z^{j+n_j} \partial w^{n_j}} \right\|,
\]

where the set \( \mathcal{I} \) is given above by (106). Now observe that, the estimate for the derivatives of \( X_{33} \) in (110) is better then the estimate for the derivatives of \( Y_{33} \) in (109) because the
former has an extra factor $C_{A,q}/|z|_R < 1$. Since the product (111) has at least two factors containing $X_{33}$, we can estimate any of these products by considering the worst case. This happens when there are exactly two factors involving $X_{33}$. Hence, by proceeding in this way, for each $j \geq 2$ we have

$$\left\| \frac{\partial^{r+p}}{\partial z^r \partial w^p} Z_j \right\| \leq (2^j - j - 1) j^{r+p} \sup_Z \left\{ \frac{2\Lambda + \|\hat{q}\|_{\|\|}}{2^2} \prod_{i=1}^j \frac{2^{r_i}+3}{\Lambda^{r_i+1} |z|_R^{r_i+1}} \|\hat{A}\|_{\|\|} \right\}$$

$$\leq 2^j j^{r+p} \left( \frac{2\Lambda + \|\hat{q}\|_{\|\|}}{2^2} \prod_{i=1}^j \frac{2^{r_i}+3}{\Lambda^{r_i+1} |z|_R^{r_i+1}} \left( \frac{8\|\hat{A}\|_{\|\|}}{\Lambda} \right) \right)$$

$$\leq C'_{A,q,r,p} j^{r+p} \left( \frac{2}{21} \right) j^2 \frac{1}{|z|_R^{r+2}},$$

since $\|\hat{A}\|_{\|\|} \leq 2\varepsilon/63$ and $\varepsilon < \Lambda/6$. Thus,

$$\left\| \frac{\partial^{r+p}}{\partial z^r \partial w^p} Z \right\| \leq \sup_{j=2}^{\infty} \left\{ \frac{\partial^{r+p}}{\partial z^r \partial w^p} Z_j \right\} \leq \frac{C'_{A,q,r,p}}{|z|_R^{r+2}} \sup_{j=2}^{\infty} \left( \frac{2}{21} \right)^j \leq \frac{C_{A,q,r,p}}{|z|_R^{r+2}}.$$

Therefore, recalling (95),

$$\sup_{p \leq m, r \leq n} \left\| \frac{\partial^{n-r+m-p}}{\partial z^{n-r} \partial w^{m-p}} \Delta_k^{-1} \right\| \leq \sup_{p \leq m, r \leq n} \frac{(m-p)! (n-r)! 2^{n-r+2}}{\Lambda^{m-p+1} |z|_R^{n-r+1}} \frac{C'_{A,q,r,p}}{|z|_R^{r+2}} \leq C_{A,q,m,n} \frac{1}{|z|_R^{n+3}}.$$

This is the desired inequality for $j = 3$. The proof of the proposition is complete. \qed

We can now prove Lemma 8. We first prove it for $1 \leq j \leq 2$ and then for $j = 3$ separately.

**Proof of Lemma 8 for $1 \leq j \leq 2$.** Define the $|B| \times |C|$ matrices

$$\mathcal{F}_{BC} := [f(b-c)]_{b \in B, c \in C} \quad \text{and} \quad \mathcal{G}_{BC} := [g(b-c)]_{b \in B, c \in C},$$

and write $w = w_{\mu,d'}, z = z_{\mu,d'}$ and $|z|_R = 2|z| - R$. First observe that, for $1 \leq j \leq 2$, the functions

$$[\alpha_{\mu,d'}^{(j)}(k)]_{d' \in G} = \sum_{b,c \in G_{d'}^1} \frac{f(d'-b) M_{b,c}^{(j)}(c-d')}{(w - 2i \theta_{\mu}(d'-b))(z - 2i \theta_{\mu}(d'-b))}$$

are the diagonal entries of the matrix $\mathcal{F}_{G_{d'}^1} \Delta_k^{-1} M^{(j)} \mathcal{G}_{G_{d'}^1}$. Thus, similarly as in the proof of Lemma 6, by Proposition 20, for $1 \leq j \leq 2$,

$$\left\| \frac{\partial^{n+m}}{\partial k_{d'}^n \partial k_{d'}^{m}} \alpha_{\mu,d'}^{(j)}(k) \right\| \leq \left\| \mathcal{F}_{G_{d'}^1} \right\| \left\| \frac{\partial^n}{\partial k_{d'}^n} \frac{\partial^m}{\partial k_{d'}^m} \Delta_k^{-1} M^{(j)} \right\| \left\| \mathcal{G}_{G_{d'}^1} \right\| \leq C_{j} \frac{1}{|z|_R^2},$$

63
where $C_1 = C_{1: A, A, n, m, f, g}$ and $C_2 = C_{2: A, A, q, n, m, f, g}$ are constants. Furthermore,

$$C_{1: A, A, 1:0, f, g} \leq \frac{13}{A} \|f\|_1 \|g\|_1, \quad C_{1: A, A, 0, 1, f, g} \leq \frac{13}{A^2} \|f\|_1 \|g\|_1$$

and

$$C_{1: A, A, 1, 1, f, g} \leq \frac{65}{A^3} \|f\|_1 \|g\|_1.$$ 

This proves the lemma for $1 \leq j \leq 2$.

**Proof of Lemma 8 for $j = 3$.** We need to estimate

$$\left| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \Omega_{\mu, \alpha} (k) \right| = \sum_{j=1}^{4} \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \mathcal{R}_j (k),$$

where $\mathcal{R}_1, \ldots, \mathcal{R}_4$ are given by (46), (47), (75) and (67), respectively.

**Step 1.** We begin with the terms involving $\mathcal{R}_1$ and $\mathcal{R}_2$, which are easier. We follow the same notation as above. First observe that, similarly as in the proof of Lemma 6, since $\Delta_k^{-1} R_k^{-1} G' G'' = H_k^{-1}$ on $L^2_{\mu, \nu}$, we have

$$\left| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \mathcal{R}_1 (k) \right| = \left| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \mathcal{F}_{(d')} G_1' \frac{\partial^{n+m} H_k^{-1}}{\partial k_1^n \partial k_2^m} \mathcal{G}_{G_2'} (d') \right| \leq \| \mathcal{F}_{(d')} G_1' \| \left| \frac{\partial^{n+m} H_k^{-1}}{\partial k_1^n \partial k_2^m} \right| \| \mathcal{G}_{G_2'} (d') \|,$n

Furthermore, we have already proved that $\| \mathcal{F}_{(d')} G_1' \| \leq \| f \|_1$ and $\| \mathcal{G}_{G_2'} (d') \| \leq \| g \|_1$ (see (90) and (91)), and since $|z| \leq 3|v|$, by Proposition 11,

$$\left| \frac{\partial^{n+m} H_k^{-1}}{\partial k_1^n \partial k_2^m} \right| \leq \varepsilon^{-(n+m+1)} C_{A, A, n, m} \frac{1}{|z|}.$$ 

Now recall that $G_2' = \{ b \in G' \mid |b - d'| > \frac{1}{4} R \}$. Then,

$$\sup_{b \in \{ d' \}} \sum_{c \in G_2} | f (b - c) | \leq \sum_{c \in G_2} \frac{|d' - c|^2}{|d' - c|^2} | f (d' - c) | \leq \| b^2 f (b) \|_1 \sup_{c \in G_2} \frac{1}{|d' - c|^2} \leq \frac{16}{R^2} \| b^2 f (b) \|_1.$$ 

Therefore, combining all this, for $1 \leq j \leq 2$ we obtain

$$\left| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \mathcal{R}_j (k) \right| \leq \varepsilon^{-(n+m+1)} C_{A, A, n, m, f, g} \frac{1}{|z| R^2}.$$
Step 2. Recall from (67) the expression for $R_4$. Then, similarly as above, by applying Proposition 20 for $j = 3$ we find that

$$\left| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} R_4(k) \right| \leq \| F(x') \| C_{A, A, q, n, m} \frac{1}{|z|^3_R}.$$  

Step 3. To bound the derivatives of $R_3$ (which is given by (75)) we need a few more estimates. Recall from (70) that $W_{43} = \pi G_{44} T_{G_{G'}}^{j+1} \pi G'_{33}$. First observe that

$$\frac{\partial^{r+p}}{\partial k_1^r \partial k_2^p} \pi G_{4} \Delta_k^{-1} T_{33} T_{34} W_{43}^{(j-m-1)} = \frac{\partial^{r+p}}{\partial k_1^r \partial k_2^p} \Delta_k^{-1} \pi G_{4} T_{33} T_{34} T_{G_{G'}}^{j-m} \pi G'_{33}$$

is given by a sum of $(j + 2)^{r+p}$ terms of the form

$$\frac{\partial^{l+m}}{\partial k_1^l \partial k_2^m} \pi G_{4} \Delta_k^{-1} T_{33} T_{34} W_{43}^{(j-m-1)} \cdot \frac{\partial^{l+m}}{\partial k_1^l \partial k_2^m} \pi G_{4} T_{33} T_{34} T_{G_{G'}}^{j-m} \pi G'_{33}$$

Moreover, for each term in the sum we have $\sum_{i=1}^{j+2} l_i = r$ and $\sum_{i=1}^{j+2} n_i = p$. Thus,

$$\left| \frac{\partial^{r+p}}{\partial k_1^r \partial k_2^p} \pi G_{4} \Delta_k^{-1} T_{33} T_{34} W_{43}^{(j-m-1)} \right| \leq (j + 2)^{r+p} \sup_{T'} \left| \left( \prod_{i=1}^{j+2} \frac{\partial^{l+m}}{\partial k_1^l \partial k_2^m} \right) \pi G'_{33} \right|,$$  

(112)

where the set $T'$ is given by (106) with $j$ replaced by $j + 2$ and

$$T_{(i)} := \begin{cases} \Delta_k^{-1} \pi G'_{4} & \text{for } i = 1, \\ T_{33} & \text{for } 2 \leq i \leq m + 1, \\ T_{34} & \text{for } i = m + 2, \\ T_{G_{G'}} & \text{for } m + 3 \leq i \leq j + 2. \end{cases}$$  

(113)

Step 3a. The first step in bounding (112) is to estimate $\left| \frac{\partial^{r+p} \Delta_k^{-1}}{\partial k_1^r \partial k_2^p} \pi G_4 \right|$. We follow the same argument that we have used in the proof of Lemma 6 to bound $\left| \frac{\partial^{n+m} H_{-1}}{\partial k_1^n \partial k_2^m} \right|$. In fact, in view of (85) one can see that

$$\frac{\partial^p \Delta_k^{-1}}{\partial k_2^p} = \sum_{\text{finite sum where } \# \text{ of terms depend on } p} \left[ \prod_{j=1}^{p} \Delta_k^{-1} \frac{\partial^{n_j} \Delta_k}{\partial k_2^{n_j}} \right] \Delta_k^{-1},$$  

(114)

where $\sum_{j=1}^{p} n_j = p$. Hence, when we compute $\frac{\partial^p \Delta_k^{-1}}{\partial k_2^p}$, the derivative $\frac{\partial^p}{\partial k_1^p}$ acts either on $\Delta_k^{-1}$ or $\frac{\partial \Delta_k}{\partial k_2^p}$. However, since

$$\left( \frac{\partial \Delta_k}{\partial k_2^p} \right)_{b,c} = 2(k_2 + c_2) \delta_{b,c},$$

we have

$$\frac{\partial^p \Delta_k}{\partial k_1^p \partial k_2^p} \Delta_k = \frac{\partial^p \Delta_k}{\partial k_1^p} \Delta_k$$

if $n_j = 0$. Similarly, using again (85) one can see that $\frac{\partial^p \Delta_k^{-1}}{\partial k_1^p}$ is given by
a finite sum as in (114), with \( p \) and \( k_2 \) replaced by \( r \) and \( k_1 \), respectively, and \( \sum_{j=1}^{r} n_j = r \).

Thus, combining all this we conclude that

\[
\frac{\partial^{r+p} \Delta_k^{-1}}{\partial k_1^p \partial k_2^p} = \sum_{\text{finite sum where } \# \text{ of terms depend on } r \text{ and } p} \left[ \prod_{j=1}^{r+p} \Delta_k^{-1} \frac{\partial^{n_j} \Delta_k}{\partial k_{ij}^{n_j}} \right] \Delta_k^{-1}, \tag{115}
\]

where \( \sum_{j=1}^{r+p} n_j \delta_{2,ij} = p \) and \( \sum_{j=1}^{r+p} n_j \delta_{1,ij} = r \). If we observe that

\[
\left( \frac{\partial^{n_j} \Delta_k}{\partial k_{ij}^{n_j}} \right)_{b,c} = \begin{cases} 2(k_{ij} + c_{ij}) \delta_{b,c} & \text{if } n_j = 1, \\ 2 \delta_{b,c} & \text{if } n_j = 2, \\ 0 & \text{if } n_j \geq 3,
\end{cases}
\]

and extract the “leading term” from the summation in (115), in a sense that will be clear below, we can rewrite (115) in terms of matrix elements as

\[
\frac{\partial^{r+p}}{\partial k_1^p \partial k_2^p} \frac{1}{N_{\epsilon}(k)} = (-1)^{-r+p}(r+p)! \left[ \frac{2(k_1 + c_1)}{N_{\epsilon}(k)} \right]^r \left( \frac{2(k_2 + c_2)}{N_{\epsilon}(k)} \right)^p + \sum_{\text{finite sum where } \# \text{ of terms depend on } r \text{ and } p} \frac{(2(k_1 + c_1))^{\alpha_j} (2(k_2 + c_2))^{\beta_j}}{N_{\epsilon}(k)^{r+p+1}},
\]

where \( \alpha_j + \beta_j < r + p \) for every \( j \) in the summation. Recall from (88) and (89) that, for all \( c \in G' \setminus \{ \tilde{c} \} \),

\[
\frac{|k_i + c_i|}{|N_{\epsilon}(k)|} \leq \frac{2}{\Lambda} < \frac{1}{3 \varepsilon} < \frac{7}{2 \varepsilon} \quad \text{and} \quad \frac{|k_i + \tilde{c}_i|}{|N_{\tilde{\epsilon}}(k)|} \leq \frac{\Lambda + 3|v|}{\varepsilon|v|} \leq \frac{7}{2 \varepsilon}. \tag{116}
\]

Hence,

\[
\left| \frac{\partial^{r+p}}{\partial k_1^p \partial k_2^p} \frac{1}{N_{\epsilon}(k)} \right| \leq \frac{(r+p)!}{|N_{\epsilon}(k)|} \left( \frac{7}{2 \varepsilon} \right)^{r+p} + \sum_{\text{finite sum where } \# \text{ of terms depend on } r \text{ and } p} \left( \frac{7}{2 \varepsilon} \right)^{\alpha_j + \beta_j} \frac{1}{|N_{\epsilon}(k)|^2} \tag{117}
\]

\[
\leq \frac{(r+p)!}{|N_{\epsilon}(k)|} \left( \frac{7}{2 \varepsilon} \right)^{r+p} + C_{\varepsilon,r,p} \frac{1}{|N_{\epsilon}(k)|^2}.
\]

Thus, by Proposition 5, since \(|N_{\epsilon}(k)| \geq \varepsilon|v| \geq \varepsilon|z|/3\) for all \( c \in G' \), we have

\[
\left\| \frac{\partial^{r+p} \Delta_k^{-1}}{\partial k_1^p \partial k_2^p} \right\|_{G'_{1}} \leq \frac{7^{r+p}(r+p)!}{\varepsilon^{r+p+1}} \frac{3}{|z|} + C_{\varepsilon,r,p} \frac{1}{|z|^2}. \tag{118}
\]

Now, let \( \rho_1 = \rho_{1: \varepsilon, r, p} \) be the constant

\[
\rho_{1: \varepsilon, r, p} := \max_{\begin{subarray}{l} l_1 \leq r \\ n_1 \leq p \end{subarray}} \frac{\varepsilon^{l_1+n_1+1} C_{\varepsilon,l_1,n_1}}{4(l_1 + n_1)! 7^{l_1+n_1}},
\]

66
where \(C_{\varepsilon,t_{1},n_{1}}\) is the constant in (118). Then, for \(|z| > \rho_{1}\) and for any \(l_{1} \leq r\) and any \(n_{1} \leq p\),
\[
\left\| \frac{\partial^{l_{1}+n_{1}} \Delta_{b}^{-1}}{\partial k_{1}^{l_{1}} \partial k_{2}^{n_{1}}} \pi_{G'} \right\| \leq \frac{7l_{1}+n_{1}(l_{1}+n_{1})!}{\varepsilon^{l_{1}+n_{1}+1}|z|} + \frac{7l_{1}+n_{1}(l_{1}+n_{1})!}{\varepsilon^{l_{1}+n_{1}+1}} \frac{4}{|z|} = (l_{1}+n_{1})! \left( \frac{7}{\varepsilon} \right)^{l_{1}+n_{1}+1} \frac{1}{|z|}.
\]

This is the first inequality we need to bound (112). We next estimate the other factors in that expression.

**Step 3b.** Recall from (53) that
\[
T_{b,c} = \frac{1}{N_{c}(k)} (2(c + k) \cdot \hat{A}(b - c) - \hat{q}(b - c)).
\]

By direct calculation we have
\[
\frac{\partial^{r+p} T_{b,c}}{\partial k_{1}^{r} \partial k_{2}^{p}} = \left( \frac{\partial^{r+p}}{\partial k_{1}^{r} \partial k_{2}^{p}} \frac{1}{N_{c}(k)} \right) (2(c + k) \cdot \hat{A}(b - c) - \hat{q}(b - c)) + r \left( \frac{\partial^{r+p-1}}{\partial k_{1}^{r-1} \partial k_{2}^{p}} \frac{1}{N_{c}(k)} \right) 2\hat{A}(b - c) + p \left( \frac{\partial^{r+p-1}}{\partial k_{1}^{r-1} \partial k_{2}^{p-1}} \frac{1}{N_{c}(k)} \right) 2\hat{A}(b - c).
\]

Hence, using (116) and (117), since \(|N_{c}(k)| \geq \varepsilon|v| \geq \varepsilon|z|/3\) for all \(c \in G'\) and \(|v| > 1\),
\[
\left\| \frac{\partial^{r+p} T_{b,c}}{\partial k_{1}^{r} \partial k_{2}^{p}} \right\| \leq \left( r + p \right)! \left( \frac{7}{\varepsilon} \right)^{r+p} \frac{C_{r,p}}{v} \left( \frac{7}{\varepsilon} \right) |\hat{A}(b - c)| + \frac{|\hat{q}(b - c)|}{|z|} \leq \frac{1}{|z|} \left( \frac{7}{\varepsilon} \right)^{r+p+1} \left( |\hat{A}(b - c)| + |\hat{q}(b - c)| \right).
\]

Therefore, by Proposition 5,
\[
\left\| \frac{\partial^{r+p} T_{b,c}}{\partial k_{1}^{r} \partial k_{2}^{p}} \right\| \leq \Theta_{r,p},
\]
where
\[
\Theta_{r,p} := (r + p)! \left( \frac{7}{\varepsilon} \right)^{r+p+1} \frac{C_{r,p}}{|z|} \left( |\hat{A}(b - c)| + |\hat{q}(b - c)| \right).
\]

This is the second estimate we need to bound (112). We next derive one more inequality.

**Step 3c.** Set
\[
Q_{b,c}^{r,p} := (1 + |b - c|^{2}) \frac{\partial^{r+p} T_{b,c}}{\partial k_{1}^{r} \partial k_{2}^{p}}.
\]

We first prove that, for any \(B, C \subseteq G'\),
\[
\sup_{b \in B} \sum_{c \in C} |Q_{b,c}^{r,p}| \leq \Omega_{r,p} \quad \text{and} \quad \sup_{c \in C} \sum_{b \in B} |Q_{b,c}^{r,p}| \leq \Omega_{r,p},
\]
where
\[
\Omega_{r,p} := (r + p)! \left( \frac{7}{\varepsilon} \right)^{r+p+1} \frac{1}{|z|} \left( |(1 + b^{2})\hat{A}(b)|_{l_{1}} + C_{r,p} \frac{|\hat{q}(b - c)|}{|z|} \right).
\]

67
In fact, in view of (120) we have
\[
\sup_{b \in B} \sum_{c \in C} |Q_{b,c}^{r,p}| = \sup_{b \in B} \sum_{c \in C} (1 + |b - c|^2) \left| \frac{\partial^{r+p} T_{b,c}}{\partial k_1^r \partial k_2^p} \right|
\leq \sup_{b \in B} \sum_{c \in C} (1 + |b - c|^2)
\times \left[ (r + p)! \left( \frac{7}{\varepsilon} \right)^{r+p+1} |\hat{A}(b - c)| + C_{\varepsilon,r,p} \left( |\hat{A}(b - c)| + |\hat{q}(b - c)| \right) \right]
\leq (r + p)! \left( \frac{7}{\varepsilon} \right)^{r+p+1} \|(1 + b^2) \hat{A}(b)\|_{L^1} + C_{\varepsilon,A,q,r,p} \frac{1}{|z|},
\]
and similarly we estimate \(\sup_{b \in B} \sum_{c \in C} |Q_{b,c}^{r,p}|.\) Now observe that, as in (78), for any integer \(m \geq 0\) and for any \(\xi_0, \xi_1, \ldots, \xi_{m+2} \in \Gamma^A\), let \(b = \xi_0\) and \(c = \xi_{m+2}\). Then,
\[
|b - c|^2 \leq 2(m + 2) \sum_{i=1}^{m+2} |\xi_{i-1} - \xi_i|^2.
\]
To simplify the notation write \(\partial^{l_i,n_i} = \frac{\partial^{l_i+n_i}}{\partial k_1^{l_i} \partial k_2^{n_i}},\) and recall from (113) and (123) the definition of \(T_{(i)}\) and \(\Omega_{r,p}\). Hence, similarly as in the proof of Proposition 17, since \(|b - c| \geq R/4\) for all \(b \in G_1'\) and \(c \in G_4'\),
\[
\sup_{b \in G_1'} \sum_{c \in G_4'} \left| \left( \prod_{i=2}^{m+2} \partial^{l_i,n_i} T_{(i)} \right) \right| \leq \sup_{b \in G_1'} \sum_{c \in G_4'} \frac{1}{1 + |b - c|^2} \sup_{b \in G_1', c \in G_4'} (1 + |b - c|^2) \left| \left( \prod_{i=2}^{m+2} \partial^{l_i,n_i} T_{(i)} \right) \right| \leq \frac{2(m + 2)}{1 + \frac{1}{16} R^2} \sup_{b \in G_1'} \sum_{c \in G_4'} (1 + |b - \xi_1|^2) \left| \partial^{l_2,n_2} T_{b,\xi_1} \right|
\times \sum_{\xi_2 \in G_3'} (1 + |\xi_1 - \xi_2|^2) \left| \partial^{l_3,n_3} T_{\xi_1,\xi_2} \right| \cdots \sum_{\xi_{m+1} \in G_3'} (1 + |\xi_{m+1} - \xi_{m+2}|) \left| \partial^{l_{m+2},n_{m+2}} T_{\xi_{m+1},\xi_{m+2}} \right|
\leq \frac{2(m + 2)}{1 + \frac{1}{16} R^2} \sup_{b \in G_1'} \sum_{c \in G_4'} (1 + |b - \xi_1|^2) \left| \partial^{l_2,n_2} T_{b,\xi_1} \right| \sup_{\xi_1 \in G_3', \xi_2 \in G_3'} \sum_{\xi_2 \in G_3'} (1 + |\xi_1 - \xi_2|^2) \left| \partial^{l_3,n_3} T_{\xi_1,\xi_2} \right|
\times \sum_{\xi_{m+1} \in G_3', \xi_{m+1} \in G_4'} (1 + |\xi_{m+1} - \xi_{m+2}|) \left| \partial^{l_{m+2},n_{m+2}} T_{\xi_{m+1},\xi_{m+2}} \right|
\leq \frac{2(m + 2)}{1 + \frac{1}{16} R^2} \sup_{b \in G_1'} \sum_{c \in G_3'} |Q_{b,c}^{l_2,n_2}| \cdots \sup_{\xi_{m+1} \in G_3', \xi_{m+1} \in G_4'} \sum_{c \in G_4'} |Q_{\xi_{m+1},c}^{l_{m+2},n_{m+2}}| \leq \frac{2(m + 2)}{1 + \frac{1}{16} R^2} \prod_{i=2}^{m+2} \Omega_{l_i,n_i}
\]
and similarly
\[
\sup_{c \in G_4'} \sum_{b \in G_1'} \left| \left( \prod_{i=2}^{m+2} \partial^{l_i,n_i} T_{(i)} \right) \right| \leq \frac{2(m + 2)}{1 + \frac{1}{16} R^2} \prod_{i=2}^{m+2} \Omega_{l_i,n_i}.
\]
Therefore, by Proposition 5,
\[
\left\| \pi_{G_1'} \prod_{i=2}^{m+2} \frac{\partial^{l_i+n_i} T_{(i)}}{\partial k_1^{l_i} \partial k_2^{n_i}} \right\| \leq \frac{2(m + 2)}{1 + \frac{1}{16} R^2} \prod_{i=2}^{m+2} \Omega_{l_i,n_i}.
\]
We have all we need to bound (112).

**Step 3d.** From (121) and (119) it follows that

\[
\left\| \prod_{i=m+3}^{j+2} \frac{\partial^{n_i}}{\partial k_1^{l_i} \partial k_2^{n_i}} T_{(i)} \right\| \leq \prod_{i=m+3}^{j+2} \Theta_{l_i, n_i}
\]

and

\[
\left\| \frac{\partial^{l_i+n_i} T_{(i)}}{\partial k_1^{l_i} \partial k_2^{n_i}} \right\| \leq (r+p)! \left( \frac{7}{\varepsilon} \right)^{r+p+1} \frac{1}{|z|}.
\]

Thus, recalling (112) we get

\[
\left\| \frac{\partial^{r+p}}{\partial k_1^{l} \partial k_2^{n}} \Delta_k^{-1} \pi G_1 \pi G_3 T_{33} T_{34} W_{43}^{(j-m-1)} \right\| \leq (j+2)^{r+p} \sup_{T'} \left\| \prod_{i=1}^{j+2} \frac{\partial^{l_i+n_i} T_{(i)}}{\partial k_1^{l_i} \partial k_2^{n_i}} \right\| \pi G_3
\]

\[
\leq (j+2)^{r+p} \sup_{T'} \left\{ \frac{1}{|z|^{2R^2}} \left( \frac{2(m+2)}{1 + \frac{1}{9} |z|} \right)^{r+p} \left( \frac{7}{\varepsilon} \right)^{r+p+1} \prod_{i=2}^{m+2} \Omega_{l_i, n_i} \right\}
\]

\[
\leq (j+2)^{r+p}(m+2) \frac{C}{|z|^{2R^2}} \sup_{T'} \left\{ (l_1 + n_1)! \left( \frac{7}{\varepsilon} \right)^{l_1+n_1+1} \prod_{i=2}^{m+2} \Omega_{l_i, n_i} \right\}
\]

where \( C \) is an universal constant. Now, recall the definition of \( \Theta_{r,p} \) and \( \Omega_{r,p} \) in (122) and (123), observe that \( \| \hat{A} \|_1 < \| (1+b^2) \hat{A} \|_1 \), and let \( \rho_2 = \rho_{2; \varepsilon, A_{q,r,p}} \) be a sufficiently large constant such that, for \( |z| > \rho_2 \) and for any \( l_i \leq r \) and any \( n_i \leq p \),

\[
\Theta_{l_i, n_i}, \Omega_{l_i, n_i} \leq 2(l_1 + n_1)! \left( \frac{7}{\varepsilon} \right)^{l_1+n_1+1} \| (1+b^2) \hat{A} \|_1.
\]

Then,

\[
\left\| \frac{\partial^{r+p}}{\partial k_1^{l} \partial k_2^{n}} \Delta_k^{-1} \pi G_1 \pi G_3 T_{33} T_{34} W_{43}^{(j-m-1)} \right\|
\]

\[
\leq (j+2)^{r+p}(m+2) \frac{C}{|z|^{2R^2}} \sup_{T'} \left\{ (l_1 + n_1)! \left( \frac{7}{\varepsilon} \right)^{l_1+n_1+1} \prod_{i=2}^{m+2} \Omega_{l_i, n_i} \right\}
\]

\[
\leq (j+2)^{r+p}(m+2) \frac{C}{|z|^{2R^2}} \left( \sum_{i=1}^{j+2} (l_i + n_i)! < (r+p)! \right)
\]

\[
\leq C(r+p)! \left( \frac{7}{\varepsilon} \right)^{r+p+1} (m+2)(j+2)^{r+p} \left( \frac{14}{\varepsilon} \right) \| (1+b^2) \hat{A} \|_1 \frac{1}{|z|^{2R^2}}
\]

\[
\leq \frac{C_{r,p}}{|z|^{2R^2}} (m+2)(j+2)^{r+p} \left( \frac{4}{9} \right)^{j+1},
\]

since \( \| (1+b^2) \hat{A} \|_1 < 2\varepsilon/63 \). This establishes a bound for (112).
Step 4. We now apply the last inequality for deriving an estimate for the derivatives of \( R_3 \) and complete the proof of the lemma for \( j = 3 \). Recall from (76) that

\[
X_{33}^{(j)} = \sum_{m=0}^{j-1} T_{33}^m T_{34} W_{43}^{(j-m-1)}.
\]

Then,

\[
\left\| \frac{\partial^{r+p}}{\partial k_1^r \partial k_2^p} \pi G_1 \Delta_k^{-1} X_{33}^{(j)} \right\| \leq \sum_{m=0}^{j-1} \left\| \frac{\partial^{r+p}}{\partial k_1^r \partial k_2^p} \Delta_k^{-1} \pi G_1 T_{33}^m T_{34} W_{43}^{(j-m-1)} \right\|
\]

\[
\leq \sum_{m=0}^{j-1} C_{\varepsilon,r,p} (m + 2)(j + 2)^{r+p} \left( \frac{4}{9} \right)^{j+1} \leq C_{\varepsilon,r,p} \frac{(j + 2)^{r+p} \left( \frac{4}{9} \right)^{j+1} \sum_{m=0}^{j-1} (m + 2)}{|z R^2|}
\]

Thus, since \( G_1' \subset G_3' \),

\[
\left\| \pi G_1' \frac{\partial^{r+p}}{\partial k_1^r \partial k_2^p} \left[ \Delta_k^{-1} \sum_{j=1}^{\infty} X_{33}^{(j)} \right] \pi G_1' \right\| \leq \sum_{j=1}^{\infty} \left\| \frac{\partial^{r+p}}{\partial k_1^r \partial k_2^p} \pi G_1' \Delta_k^{-1} X_{33}^{(j)} \right\|
\]

\[
\leq \frac{C_{\varepsilon,r,p}}{|z R^2|} \sum_{j=1}^{\infty} (j + 2)^{r+p} \left( \frac{4}{9} \right)^{j+1} \frac{1}{2} (j^2 + 3j) \leq C C_{\varepsilon,r,p} \frac{1}{|z R^2|},
\]

where \( C \) is an universal constant. Therefore,

\[
\left\| \frac{\partial^{r+p}}{\partial k_1^r \partial k_2^p} R_3(k) \right\| = \left\| F_{(\alpha')} G_1' \right\| \left\| \frac{\partial^{r+p}}{\partial k_1^r \partial k_2^p} \left[ \Delta_k^{-1} \sum_{j=1}^{\infty} X_{33}^{(j)} \right] \pi G_1' \right\| \left\| G_1'(\alpha') \right\|
\]

\[
\leq \left\| F_{\alpha'} G_1' \right\| \left\| \pi G_1' \frac{\partial^{r+p}}{\partial k_1^r \partial k_2^p} \left[ \Delta_k^{-1} \sum_{j=1}^{\infty} X_{33}^{(j)} \right] \pi G_1' \right\| \left\| G_1'(\alpha') \right\|
\]

\[
\leq C C_{\varepsilon,r,p} \frac{1}{|z R^2|}.
\]

Finally, combining all the estimates we have

\[
\left\| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \alpha_{\mu,\nu}^{(3)} (k) \right\| \leq \sum_{j=1}^{4} \left\| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} R_j(k) \right\|
\]

\[
\leq 3 \frac{C}{|z R^2|} + \frac{C}{|z R^2|} \leq \frac{4C}{|z R^2|},
\]

where \( C = C_{\varepsilon,A,q,f,g,m,n} \) is a constant. Set \( \rho_{\varepsilon,A,q,m,n} := \max \{ \rho_{1,\varepsilon,m,n}, \rho_{2,\varepsilon,A,q,m,n} \} \). The proof of the lemma for \( j = 3 \) is complete. \( \square \)
References

[1] J. Feldman, H. Knörrer and E. Trubowitz, *Riemann surfaces of infinite genus*, CRM Monograph Series, Amer. Math. Soc., 2003.

[2] D. Gieseker, H. Knörrer and E. Trubowitz, *The geometry of algebraic Fermi curves*, Perspectives in mathematics 14, 1992.

[3] H. Knörrer and E. Trubowitz, *A directional compactification of the complex Bloch variety*, Comment. Math. Helvetici 65, 114-149 (1990).

[4] I. Krichever, *Spectral theory of two-dimensional periodic operators and its applications*, Russian Math. Surveys 44:2, 145-225 (1989).

[5] H. McKean, *Integrable systems and algebraic curves*, in Global Analysis, Proceedings, 1978, LNM 755, Springer, 1979.

[6] J. Feldman, H. Knörrer and E. Trubowitz, *Asymmetric Fermi surfaces for magnetic Schrödinger operators*, Commun. Part. Diff. Eq. 26, 319-336 (2000).

[7] Y. Karpehina, *Spectral properties of the periodic magnetic Schrödinger operator in the high-energy region. Two-dimensional case*, Commun. Math. Phys. 251, 473-514 (2004).

[8] L. Erdös, *Recent developments in quantum mechanics with magnetic fields*, Proc. of Symposia in Pure Math. 76, 401-428 (2006).

[9] M. Reed and B. Simon, *Methods of modern mathematical physics IV: analysis of operators*, Academic Press, 1978.

[10] P. Kuchment, *Floquet theory for partial differential equations*, Birkhäuser, 1993.

[11] W. Magnus and S. Winkler, *Hill’s equation*, Dover, 2004.

[12] S. Gustafson and I. Sigal, *Mathematical concepts of quantum mechanics*, Springer, 2006.

[13] G. de Oliveira, *Asymptotics for Fermi curves of electric and magnetic periodic fields*, Ph.D. thesis, The University of British Columbia 2009, openly accessible at the digital archive URI: http://hdl.handle.net/2429/11114