A SHARP EIGENVALUE BOUND FOR QUANTUM GRAPHS IN TERMS OF THEIR DIAMETER

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Abstract. We establish a sharp lower bound on the first non-trivial eigenvalue of the Laplacian on a metric graph equipped with natural (i.e., continuity and Kirchhoff) vertex conditions in terms of the diameter and the total length of the graph. This extends a result of, and resolves an open problem from, [J. B. Kennedy, P. Kurasov, G. Malenová and D. Mugnolo, Ann. Henri Poincaré 17 (2016), 2439–2473, Section 7.2], and also complements an analogous lower bound for the corresponding eigenvalue of the combinatorial Laplacian on a discrete graph.

1. Introduction

Let $\mathcal{G}$ be a connected, compact metric graph and let $-\Delta$ denote the Laplacian operator on $L^2(\mathcal{G})$ with natural (i.e., continuity and Kirchhoff, also known as standard) vertex conditions. Since $-\Delta$ can be shown by standard means to be a self-adjoint operator with compact resolvent, one obtains the existence of a discrete sequence of eigenvalues of this operator, which we think of as eigenvalues of the quantum graph itself, having the form

$$0 = \lambda_0(\mathcal{G}) < \lambda_1(\mathcal{G}) \leq \lambda_2(\mathcal{G}) \leq \ldots \to \infty;$$

the corresponding eigenfunctions may be chosen to form an orthonormal basis of $L^2(\mathcal{G})$.

We refer to the monographs [10, 23] as well as Section 2 for more details.

It is a major preoccupation of spectral geometry to investigate how the sequence of eigenvalues (1.1) of a differential operator such as the Laplacian depends on the structure, be it total size, shape, degree of connectivity etc., of the underlying object on which it is defined. For operators on domains and manifolds, this goes back at least to conjectures of Saint Venant and Lord Rayleigh in the mid-to-late 19th Century (see [26]; we refer also to [15, 16] for more modern overviews of the field). In the case of quantum graphs, that is, metric graphs with a differential operator defined on them, the first work in this

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direction appeared about 30 years ago [25], where it was proved that the first non-trivial eigenvalue $\lambda_1(G)$ of the Laplacian with natural conditions on a graph whose total length, i.e., the sum of all its edge lengths, is $L > 0$ satisfies
\[
\lambda_1(G) \geq \frac{\pi^2}{L^2},
\]
the right-hand side corresponding to the first non-trivial eigenvalue on an interval of the same total length $L$ as $G$. After a lull in the 1990s and early 2000s, in the last few years there seems to have been an explosion of interest in the topic, as witnessed by the long list of works establishing bounds on some or all of the eigenvalues (1.1), for example in terms of the total length, diameter, number of edges or vertices, edge connectivity, . . . of the graph, or establishing properties of extremising graphs realising the bounds, or developing tools with which the eigenvalues can be manipulated, or else considering similar problems for related nonlinear operators: we refer in particular to [1, 2, 3, 4, 6, 7, 8, 9, 13, 17, 18, 19, 20, 27, 28, 29].

The goal of the present contribution is to give a sharp lower bound on $\lambda_1(G)$ in terms of the total length $L \in (0, \infty)$ of the graph $G$ and its diameter
\[
D := \text{diam}(G) := \sup \{ \text{dist}(x, y) : x, y \in G \} \in (0, L],
\]
where the distance is with respect to the canonical (Euclidean) metric in $G$, i.e., the shortest Euclidean path within $G$ connecting the points $x$ and $y$, and the supremum is in fact a maximum since $G$ is assumed to be compact. This problem was first studied in [17, Section 7.2], where a non-trivial but non-sharp lower bound was given, and the question of obtaining the best possible bound was left open (see Remark 7.3(a) there). Here, by using some more advanced tools developed recently in [8] (which we call surgery principles), we can give a complete answer: our main theorem is as follows.

**Theorem 1.1.** Assume that $G$ is a connected, compact metric graph with total length $L > 0$ and diameter $\text{diam}(G) = D \in (0, L)$. Then $\lambda_1(G)$ is larger than the square $\kappa^2$ of the smallest positive solution $\kappa > 0$ of the transcendental equation
\[
(1.2) \quad \cos \left( \frac{\kappa D}{2} \right) = \frac{\kappa(L-D)}{2} \sin \left( \frac{\kappa D}{2} \right).
\]
Equality is never attained on any fixed graph, but there is a sequence of graphs $D_n$ each of length $L$ and diameter $D$ such that $\lambda_1(D_n) \to \kappa^2$ as $n \to \infty$ (see Proposition 1.3).

To describe a concrete sequence $D_n$, and to give an idea of what gives rise to the equation (1.2), we first need to introduce a particular class of graphs which also appeared in [17]; this class will in addition play a role in the proof.

**Definition 1.2.** (1) Fix suitable numbers $n \in \mathbb{N}$ and $\ell_0 > 0$, $\ell_1, \ell_2 \geq 0$. A star dumbbell will for us be a graph consisting of a finite edge (a handle) $e_0$ of length $\ell_0$ connecting two distinct vertices $v_1$ and $v_2$, to each of which are attached $n$ pendant edges each of length $\ell_1$ at $v_1$, and a further $k$ pendant edges of length $\ell_2$ at $v_2$. We will denote such a graph by $D = D(\ell_0, \ell_1, \ell_2, n)$. The set of $n$ pendant edges at $v_i$ will be denoted by $S_i$, $i = 1, 2$.

(2) A symmetric star dumbbell is a star dumbbell with the additional property that $\ell_1 = \ell_2$. 
A star dumbbell is thus a “dumbbell”-type graph where the two “weights” take the form of pendant stars $S_1, S_2$, each with $n$ equal edges; symmetric star dumbbells have the additional property that these two pendant stars are isometric copies of each other. See Figure 1. If $L > 0$ and $D \in (0, L)$ are fixed, for $n \geq 2$ sufficiently large the symmetric star dumbbell of the form

\begin{equation}
D_n = D_n(L, D) := D \left( \frac{nD - L}{n - 1}, \frac{L - D}{2(n - 1)}, \frac{L - D}{2(n - 1)}, n \right)
\end{equation}

has total length $L$ and diameter $D$, and as $n \to \infty$, its two pendant stars contract to “point masses”, that is, subgraphs of infinitesimal diameter but fixed total length $(L - D)/2$ each.

![Figure 1. A star dumbbell with $n = 7$ (left); the symmetric star dumbbell $D_{11}$ (right): the handle has length $(11D - L)/10$, while all 22 short pendant edges have length $(L - D)/20$ each.](image)

We can now describe more exactly where the equation (1.2) comes from, and how it behaves.

**Proposition 1.3.** Fix $L > 0$ and $D \in (0, L)$, and let $\kappa > 0$ be defined by (1.2). Then:

1. for the sequence of graphs $D_n = D_n(L, D)$ given by (1.3), we have $\lambda_1(D_n) \to \kappa^2$;
2. we have the two-sided bound

\begin{equation}
\frac{1}{LD} < \kappa^2 < \frac{3}{LD}:
\end{equation}

in particular, $\lambda_1(G) > 1/(LD)$;
3. for fixed $L > 0$, $\kappa$ is a decreasing function of $D \in (0, L)$.

In fact, the transcendental equation (1.2) corresponds to an eigenvalue problem on an interval $(0, D)$ with the generalised Wentzell-type boundary condition

\begin{equation}
-u''(x) = \kappa^2 u(x) \quad \text{in } (0, D),
\end{equation}

\begin{align*}
u''(0) - \frac{2}{L - D} u'(0) &= 0, \\
u''(D) + \frac{2}{L - D} u'(D) &= 0.
\end{align*}

This condition reflects the concentration of mass at the endpoints of the symmetric star dumbbell $D_n$ as $n \to \infty$. (The term Wentzell boundary condition is usually used to describe the situation where the differential operator, in this case the Laplacian, itself appears in the boundary condition. We refer to [5, 24] for more information in the case of the Laplacian on domains.)

**Remark 1.4.** (a) It is not necessary to consider star dumbbells: any sequence of symmetric dumbbell-type graphs where at each end of the handle there is a subgraph of total
length \((L - D)/2\) but subgraph diameter shrinking to zero should have its eigenvalue converging to \(\kappa^2\).

(b) In [17, Section 7.2], in place of (1.2) the lower bound is the square \(\tilde{\kappa}^2\) of the smallest positive solution of

\[
\cos(2\tilde{\kappa}D) = (L - 2D)\tilde{\kappa}\sin(2\tilde{\kappa}D)
\]

as long as \(D \leq L/2\), which satisfies \(\tilde{\kappa}^2 > 1/(2LD)\). There, proofs of Proposition[1.3](1) and (2) are given, and we refer there for the respective proofs. The derivation of the equation (1.5) from (1.2) is also described there; see [17, Remark 7.3(c)]. However, the proof of Theorem [1.1] uses an essentially different set tools from the proof of the corresponding main result [17, Theorem 7.2]. Indeed, here we will make use of both a new transplantation principle and an Hadamard-type length perturbation formula (see Section 2 for details).

(c) With a somewhat more careful analysis, it may be possible to obtain a corresponding bound for the higher eigenvalues using similar tools: indeed, our proof reduces to comparing each nodal domain (connected component of the set where the eigenfunction does not vanish) with a well-chosen star via the aforementioned transplantation principle, then pasting these stars together to form a star dumbbell and using the perturbation formula to show that the symmetric star dumbbell has the lowest eigenvalue. In principle, these ideas should continue to function for the higher eigenvalues; but it would go well beyond the scope of the current work to explore them here.

Finally, we remark that Theorem [1.1] and Proposition [1.3] recall very much results for the first non-trivial eigenvalue of discrete graph Laplacians (this eigenvalue is often called the algebraic connectivity of the discrete graph), in terms of the number of vertices of the graph − the discrete equivalent of its size, i.e., length − as well as the (now integer-valued) diameter. We reproduce the statements here in our language for ease of comparison.

**Proposition 1.5** ([12], Corollary 3.3). Let \(T\) be any (discrete) tree with diameter \(D \geq 3\) and \(V \geq 4\) vertices. Assume that \(V - D\) is odd. Then the smallest non-trivial discrete Laplacian eigenvalue of \(T\) is at least as large as that of a symmetric star dumbbell each of whose stars has \((V - D + 1)/2\) pendant edges and a handle of length \(D - 2\). (If \(V - D\) is even, one edge must be removed from one of the pendant stars.)

In fact, it seems this result should hold for all graphs on \(V\) vertices, not only trees (just as our result holds independently of the topology of the graph), but to the best of our knowledge this has not been proved. However, there is a slightly older result which very much recalls our estimate from (1.4), and which is at least asymptotically optimal as \(V \to \infty\).

**Proposition 1.6** ([21], Theorem 4.2 and example after it). Let \(G\) be any discrete graph with diameter \(D \geq 1\) and \(V \geq 2\) vertices. Then its smallest non-trivial Laplacian eigenvalue is at least as large as \(4/(DV)\). Equality is achieved in the limit as \(V \to \infty\) for fixed \(D\) by discrete analogues of the symmetric star dumbbells described in Proposition 1.5.

In fact, this bound was extended very recently to infinite discrete graphs equipped with a probability measure in place of the usual one (so that the total “length” is one) and finite diameter; see [22, Corollary 3.7]. We thank Delio Mugnolo for bringing this to our attention.
In Section 2 we will recall from [8] the elementary but powerful technical tools we will need for the proofs; for the sake of readability we will provide proof sketches here but refer to [8] for full details. The proofs of Theorem 1.1 and Proposition 1.3 are in Section 3.

2. Background results: surgical tools

In this section we recall both the formal definition of the Laplacian on a quantum graph and the characterisation of its eigenvalues, as well as the “surgery” tools we shall need from [8].

Formally, the metric graph $G$ is taken to consist of a set of edges $E = \{e_1, \ldots, e_n\}$, each of which may be identified with an interval $e_j \sim [0, \ell_j], j = 1, \ldots, m$, and a set of vertices $V = \{v_1, \ldots, v_m\}$; we write $e_i \sim e_j$ if $e_i$ and $e_j$ are adjacent (share a vertex), and in a slight abuse of notation $e \sim v$ if the vertex $v$ is incident with the edge $e$, and $e \sim vw$ if $e$ runs from $v$ to $w$ (i.e., both are incident with $e$). We always assume our graph to be connected, but we explicitly allow it to have loops ($e \sim v$ for some $v \in V$) and multiple edges running between two given vertices; in the latter case we speak of parallel edges. A pendant edge is any edge which ends at a vertex of degree one.

We consider the operator associated with the bilinear form $a : H^1(G) \times H^1(G) \to \mathbb{R},$

$$a(f, g) := \int_G f'g' \, dx \equiv \sum_{e \in E} \int_e f'g' \, dx,$$

where

$L^2(G) \simeq \bigoplus_{e \in E} L^2(e), \quad H^1(G) = \{f \in L^2(G) : f' \in L^2(G)\}.$

Here $f'$ is to be interpreted in the distributional sense, and the space $H^1(G) \hookrightarrow C(G)$ in particular encodes both the vertex incidence relations and the vertex conditions. Indeed, the corresponding operator is given by the negative Laplacian (negative of the second derivative) on each edge, and its operator domain consists of those $H^1$-functions which, in addition to being automatically continuous across the vertices as members of $H^1(G)$, also satisfy the Kirchhoff condition

$$\sum_{e \sim v} \frac{\partial}{\partial v_e} f(v) = 0$$

at every vertex $v \in V$, where $\frac{\partial}{\partial v_e}$ is the normal derivative of the function along the edge $e$ pointing into $v$. The associated smallest non-trivial eigenvalue $\lambda_1(G)$, often also called the spectral gap since $\lambda_0(G) = 0$, admits the variational characterisation

$$\lambda_1(G) = \inf \left\{ \frac{\int_G |f''|^2 \, dx}{\int_G |f|^2 \, dx} : f \in H^1(G), \int_G f \, dx = 0 \right\}$$

with equality if and only if $f$ is a corresponding eigenfunction, which we will tend to denote by $\psi$.

Note that [8] uses a different eigenvalue numbering convention: there the smallest eigenvalue is $\lambda_1(G) = 0$. Here we use the same numbering as in [17].
If instead we wish to consider the Laplacian with Dirichlet (zero) vertices on a subset \( \mathcal{V}_D \subset \mathcal{V} \), our form is still given by (2.1) but our form domain changes to
\[
H^1_0(\mathcal{G}) := \{ f \in H^1(\mathcal{G}) : f(v) = 0 \text{ for all } v \in \mathcal{V}_D \}
\]
(the set \( \mathcal{V}_D \) being clear from the context). In this case, we will denote the eigenvalues by \( 0 < \mu_1(\mathcal{G}) < \mu_2(\mathcal{G}) \leq \ldots \), where the smallest eigenvalue is given by
\[
\mu_1(\mathcal{G}) = \inf \left\{ \frac{\int_{\mathcal{G}} |f'|^2 \, dx}{\int_{\mathcal{G}} |f|^2 \, dx} : f \in H^1_0(\mathcal{G}) \right\};
\]
again, there is equality if and only if \( f \) is a corresponding eigenfunction. As is standard, we shall call the quotient in (2.3) or (2.2) the Rayleigh quotient (of the function \( f \)). Finally, we set
\[
\nu_1(\mathcal{G}) := \begin{cases} 
\lambda_1(\mathcal{G}) & \text{if } \mathcal{V}_D = \emptyset, \\
\mu_1(\mathcal{G}) & \text{if } \mathcal{V}_D \neq \emptyset
\end{cases}
\]
to denote the smallest non-trivial eigenvalue in general, regardless of the presence or absence of Dirichlet vertices. We refer to both the monographs [10, 23] as well as the introductions and preliminary sections of [8, 9, 17] etc. for more details.

We now collect the tools that we will need in the sequel. These are all based purely on the variational characterisations (2.2), (2.3) of the eigenvalues; most are from [8] although some have appeared in various guises throughout the recent literature. We start with the most elementary.

**Lemma 2.1.** Suppose the graph \( \mathcal{G} \) is formed from \( \mathcal{G} \) by gluing together two vertices \( v_1, v_2 \in \mathcal{V}(\mathcal{G}) \), i.e., every edge that had \( v_1 \) or \( v_2 \) as an endpoint in \( \mathcal{G} \) has a new common vertex \( v_0 \in \mathcal{V}(\mathcal{G}) \). Then \( \nu_1(\mathcal{G}) \geq \nu_1(\tilde{\mathcal{G}}) \), with equality if and only if there is an eigenfunction \( \psi \) of \( \nu_1(\mathcal{G}) \) such that \( \psi(v_1) = \psi(v_2) \). In this case, the image of \( \psi \) under the gluing procedure remains an eigenfunction of \( \nu(\tilde{\mathcal{G}}) \).

**Proof.** The inequality follows from the identification of \( H^1(\tilde{\mathcal{G}}) \) as a subspace of \( H^1(\mathcal{G}) \), but such that the form (2.1) itself is the same. The characterisation of equality follows from the fact that the minimum in (2.2) (corresp. (2.3)) is achieved if and only if the function is a corresponding eigenfunction. This is also a special case of [8, Theorem 3.4]; the inequality itself has appeared in multiple places including [10, Theorem 3.1.8]. \( \square \)

**Lemma 2.2.** Suppose the graph \( \tilde{\mathcal{G}} \) is formed from \( \mathcal{G} \) by lengthening an edge in \( \mathcal{G} \). Then \( \nu_1(\tilde{\mathcal{G}}) \leq \nu_1(\mathcal{G}) \). The inequality is strict if there is a corresponding eigenfunction on \( \mathcal{G} \) which does not vanish on the edge in question.

**Proof.** See, e.g., [17, Lemma 2.3(4)] or [8, Corollary 3.12(1)]. \( \square \)

**Lemma 2.3.** Suppose \( \psi \) is an eigenfunction corresponding to \( \lambda_1(\mathcal{G}) \) and set
\[
\mathcal{G}^+ := \{ x \in \mathcal{G} : \psi(x) > 0 \}, \\
\mathcal{G}^- := \{ x \in \mathcal{G} : \psi(x) < 0 \}.
\]
If we equip each of \( \mathcal{G}^\pm \) with Dirichlet conditions on the (finite) set \( \mathcal{G}^\pm \cap \{ x \in \mathcal{G} : \psi(x) = 0 \} \), then \( \lambda_1(\mathcal{G}) = \mu_1(\mathcal{G}^+) = \mu_1(\mathcal{G}^-) \), and \( \psi|_{\mathcal{G}^\pm} \) is an eigenfunction corresponding to \( \mu_1(\mathcal{G}^\pm) \).
Proof. Since $\psi|_{G^\pm} \in H^1_0(G^\pm)$ is a valid test function on $G^\pm$, whose Rayleigh quotient is seen to be equal to its Rayleigh quotient on $G$, i.e., $\lambda_1(G)$. Thus $\lambda_1(G) \geq \mu_1(G^\pm)$. Conversely, since $\psi|_{G^\pm}$ satisfies the strong form of the eigenvalue equation, including the zero condition, it must be some eigenfunction of $\mu_1(G^\pm)$. But it does not change sign on $G^\pm$, and by standard Perron–Frobenius theory any eigenfunction of the Laplacian which does not change sign corresponds to $\mu_1$. Thus $\psi|_{G^\pm}$ corresponds to $\mu_1(G^\pm)$ and we conclude $\lambda_1(G) = \mu_1(G^\pm)$. \hfill $\Box$

The above lemma does not rule out the possibility that $\psi$ vanishes identically on an edge, whose interior in this case is excluded from $G^\pm$. We also have the following complementary statement.

**Lemma 2.4.** Suppose $G_1, G_2$ form a partition of $G$, that is, $G_1, G_2$ are closed graphs whose intersection is at most a finite set. Assume that $\partial G_i = \{x \in G : x \in G_i \cap G \setminus G_i\}$ is equipped with Dirichlet conditions, $i = 1, 2$. Then $\lambda_1(G) \leq \max\{\mu_1(G_1), \mu_1(G_2)\}$.

Proof. Denote by $\varphi_i \in H^1_0(G_i)$ any eigenfunction corresponding to $\mu_1(G_i)$, $i = 1, 2$. Extend $\varphi_i$ by zero on the rest of $G$ to obtain a function $\tilde{\varphi}_i$ in $H^1(G)$ whose Rayleigh quotient is still $\mu_1(G_i)$. Note that the sets where $\varphi_1 \neq 0$ and $\varphi_2 \neq 0$ are disjoint; so in particular they are linearly independent. Hence we may set $\phi := a_1 \tilde{\varphi}_1 + a_2 \tilde{\varphi}_2 \neq 0$, where the constants $a_1, a_2 \in \mathbb{R}$ are chosen in such that $\int_G \phi \, dx = 0$. Then the Rayleigh quotient of $\phi$ is no larger than $\max\{\mu_1(G_1), \mu_1(G_2)\}$ but at the same time larger than $\lambda_1(G)$, by (2.2). \hfill $\Box$

We now give two surgery lemmata which will be central to the proof of Theorem 1.1. The first shows us that altering a graph by transferring “mass” from where its eigenfunction is smaller to where it is larger lowers the eigenvalue.

**Lemma 2.5** (Transplantation lemma). Suppose $\psi \geq 0$ is an eigenfunction corresponding to $\mu_1(G)$. Suppose there is a vertex $v \in V(G)$ and edges $e_1, \ldots, e_k \in E(G)$ such that

$$\sup\{\psi(x) : x \in e_1 \cup \ldots \cup e_k\} \leq \psi(v),$$

and the total length of these edges is $|e_1| + \ldots + |e_k| = \ell > 0$. Form a new graph $\tilde{G}$ from $G$ by deleting the edges $e_1, \ldots, e_k$ (deleting also any vertices of degree one, without identifying the other endpoints!) and inserting new pendant edges at $v$ and/or lengthening existing edges in $G$ to which $v$ is incident; any Dirichlet vertices in $G$ not deleted should be preserved in $\tilde{G}$. Suppose that the total length of the additions and extensions is equal to or greater than $\ell$. Then $\mu_1(\tilde{G}) \leq \mu_1(G)$. The inequality is strict provided $\psi(v) > 0$.

Proof. This is actually an easy special case of [8, Theorem 3.18(1)], which is also valid for $\lambda_1$, for more general transplantation procedures, and for more general vertex conditions. Here, this follows simply by constructing a test function

$$\varphi(x) := \begin{cases} 
\psi(x) & \text{if } x \in G \cap \tilde{G}, \\
\psi(v) & \text{if } x \in \tilde{G} \setminus G.
\end{cases}$$

Then condition (2.6) guarantees that $\|\varphi\|_{L^2(\tilde{G})} \geq \|\psi\|_{L^2(G)}$; while since $\varphi$ is constant on $\tilde{G} \setminus G$ and identical to $\psi$ elsewhere, we obviously have $\|\varphi\|_{L^2(\tilde{G})} \leq \|\psi\|_{L^2(G)}$. The inequality now follows from (2.3).
The strictness if $\psi(v) > 0$ holds because in this case $\varphi$, being locally equal to a nonzero constant, cannot be an eigenfunction of $\tilde{G}$; hence it has strictly larger Rayleigh quotient than $\mu_1(\tilde{G})$.

We finish with a perturbation formula giving the rate of change of a simple eigenvalue with respect to a perturbation in edge lengths; such a formula is often referred to as being of Hadamard type, by way of analogy with the formulae for the derivative of an eigenvalue on a domain with respect to shape perturbations. The following formula has appeared in the literature multiple times, possibly beginning with [14].

**Lemma 2.6 (Hadamard-type formula).** Let $\lambda$ be a simple eigenvalue of the Laplacian (with either all natural or some natural and some Dirichlet vertices), with eigenfunction $\psi$. Then the quantity

$$E_e := \frac{1}{2} \left( \lambda \psi(x)^2 + \psi'(x)^2 \right), \quad x \in e,$$

is constant on each edge $e \in \mathcal{E}$. Moreover,

1. The derivative of $\lambda$ with respect to the edge length $|e|$ exists and equals

$$\frac{d\lambda}{d|e|} = -E_e.$$

2. In particular, the rate of change of $\lambda$ with respect to lengthening $e_1$ and shortening $e_2$ by the same amount is strictly negative if and only if

$$E_{e_1} > E_{e_2}.$$

The quantity (2.7) is called the Prüfer amplitude.

**Proof.** The formula (1) may be found in [14], [13] Appendix A and [7, Lemma 5.2] (probably among others). Part (2) is an immediate application, and can at any rate be found, together with (1), in [8, Section 3.2].

3. **Proof of the main statements**

We start with Proposition 1.3 since having these properties will be convenient for some of the subsequent proofs. As mentioned in the introduction, (1) and (2) were proved in [7], and to avoid repetition of the somewhat tedious calculations we will not give their proofs again here.

**Proof of Proposition 1.3.**

1. See [17, Lemma 7.6].
2. See [17, Remark 7.3(a)].
3. We introduce the notation

$$F(\kappa, D) := \cos \left( \frac{\kappa D}{2} \right) - \frac{\kappa(L - D)}{2} \sin \left( \frac{\kappa D}{2} \right).$$
The derivative of $\kappa$ with respect to $D$ may be obtained via implicit differentiation as
\[
-\frac{\partial F}{\partial D} / \frac{\partial F}{\partial \kappa} = \frac{\frac{3}{2} \sin \left(\frac{\kappa D}{2}\right) - \frac{3}{2} \sin \left(\frac{\kappa D}{2}\right) + \frac{\kappa^2 L - D}{2} \cos \left(\frac{\kappa D}{2}\right)}{\frac{L}{2} \sin \left(\frac{\kappa D}{2}\right) - \frac{D}{2} \sin \left(\frac{\kappa D}{2}\right) - \frac{D L - D}{2} \cos \left(\frac{\kappa D}{2}\right)}
\]
Replacing $\sin \left(\frac{\kappa D}{2}\right)$ by $\frac{\kappa(L-D)}{2} \cos \left(\frac{\kappa D}{2}\right)$ in accordance with (1.2), our derivative becomes
\[
-\frac{\kappa^3(L-D)^3}{4L + \kappa^2D(L-D)^2} < 0.
\]
\[\square\]

We can now begin with the proof of Theorem 3.1. In fact, in light of Proposition 1.3, it suffices to prove:

**Theorem 3.1.** Suppose $\mathcal{G}$ is any graph of total length $L > 0$ and diameter $D \in (0, L)$. Then there exists some $D_1 \in (0, D]$ and a symmetric star dumbbell $D_n = D_n(L, D_1)$, given by (1.3), for some $n \geq 1$, such that
\[
\lambda_1(D_n) < \lambda_1(\mathcal{G}).
\]

Indeed, since we know by Proposition 1.3 that (1) $\lambda_1(D_n) \rightarrow \kappa^2$ for fixed $D$ and (2) $\kappa$ is a decreasing function of $D$ for fixed $L$, in light of Theorem 3.1 it follows in particular that this convergence can only be from above, $\lambda_1(D_n) \downarrow \kappa^2$, and thus $\kappa^2$ must represent a lower bound on all $D_n$ and so all $\mathcal{G}$ with length $L > 0$ and diameter $D \in (0, L)$. (The case $D = L$, i.e., where $\mathcal{G}$ is a path graph, is not interesting.)

Before we can give the proof of Theorem 3.1 we need a couple of preparatory lemmata.

**Lemma 3.2.** Suppose $\mathcal{D} = \mathcal{D}(\ell_0, \ell_1, \ell_2, n)$ is any star dumbbell, $\ell_0, \ell_1, \ell_2 > 0$, $n \geq 1$. Assume that $\ell_0 > \max\{\ell_1, \ell_2\}$, i.e., the handle is longer than the pendant edges. Then \(\lambda_1(\mathcal{D})\) is simple and its eigenfunction $\psi$, unique up to scalar multiples, is invariant with respect to permutations of the edges within each pendant star $\mathcal{S}_i$, $i = 1, 2$.

**Proof.** Fix one of the $\mathcal{S}_i$. Then we may choose a basis of $L^2(\mathcal{D})$ made of eigenfunctions such that each either takes the value 0 at $\psi_j$ (it is “odd”), or it is invariant with respect to permutations of the edges in $\mathcal{S}_i$ (it is “even”). (Indeed, if $\psi$ is any eigenfunction and $e_1, \ldots, e_n$ are the edges of $\mathcal{S}_i$, then it suffices to consider instead the eigenfunction $(\psi|_{e_1} + \ldots + \psi|_{e_n})/n$ and its orthogonal complement in the span of $\psi$.) We do this for both stars $\mathcal{S}_i$.

Equipped with this basis, we note that the eigenfunctions which are “even” with respect to both stars are all simple within the space of all such “even” eigenfunctions, since their value at any point depends only on that point’s position along any path of $\mathcal{D}$ realising the diameter (and thus they correspond to one-dimensional problems). Hence, to prove the lemma, it is sufficient to show that the smallest non-constant of these has a smaller eigenvalue than any of the “odd” eigenfunctions, each of the latter being supported without loss of generality only on one of the stars $\mathcal{S}_i$. Indeed, under the assumption $\ell_1 \geq \ell_2$, the smallest eigenvalue associated with an odd eigenfunction is $\pi^2/\ell_1^2$, corresponding
Lemma 3.3. For fixed total length $L$, fixed $\ell_0 > \ell > 0$ and fixed $n \geq 1$ consider the family of star dumbbells $\mathcal{D} = \mathcal{D}(\ell_0, \ell_1, \ell - \ell_1, n)$, where $\ell_1 \in [0, \ell]$. Then

(1) $\frac{d}{d\ell_1} \lambda_1(\mathcal{D})$ exists for all $\ell_1 \in (0, \ell)$ and is strictly negative if $\ell_1 \in (0, \ell/2)$ and strictly positive if $\ell_1 \in (\ell/2, \ell)$.

(2) In particular, $\lambda_1(\mathcal{D})$ reaches its unique global minimum over $\ell_1 \in [0, \ell]$ at $\ell_1 = \ell/2$.

In words, a symmetric star dumbbell has the lowest first eigenvalue among all star dumbbells having the same total length, diameter and number of pendant edges at each star.

Proof. (1) The existence of the derivative follows immediately from Lemma 2.6, which is applicable since $\lambda_1(\mathcal{D})$ is always simple by Lemma 3.2. By symmetry, it suffices to restrict attention to $\ell \in (0, \ell/2)$ and prove that

$$\frac{d}{d\ell_1} \lambda_1(\mathcal{D}) < 0 \quad \text{if } \ell \in (0, \ell_1/2).$$

(3.1)

Denote by $\psi$ the corresponding eigenfunction, which is unique up to scalar multiples by Lemma 3.2. Then by Lemma 2.6 $\psi$ is identical on all edges within each of the stars, and to prove (3.1) it suffices to prove that if $e_1$ is an edge in $\mathcal{S}_1$ and $e_2$ is an edge in $\mathcal{S}_2$, then $|e_1| < |e_2|$ implies $\mathcal{E}_{e_1} > \mathcal{E}_{e_2}$. Since our Prüfer amplitudes $\mathcal{E}_{e_i}$ are constant with respect to the point $x \in e_i$ at which they are evaluated (Lemma 2.6), to show this is equivalent to showing

$$\lambda_1(\mathcal{D})|\psi(v_1)|^2 + \left| \frac{\partial \psi}{\partial v_{e_1}}(v_1) \right|^2 > \lambda_1(\mathcal{D})|\psi(v_2)|^2 + \left| \frac{\partial \psi}{\partial v_{e_2}}(v_2) \right|^2,$$

where $\frac{\partial \psi}{\partial v_{e_i}}(v_i)$ denotes the (normal) derivative of $\psi$ on $e_i$ at $v_i$, $i = 1, 2$.

Denote by $e_0$ the handle, so that $e_1$ and $e_0$ are adjacent at $v_1$, and $e_2$ and $e_0$ are adjacent at $v_2$. Note that since $\ell_0 > \max\{\ell_1, \ell - \ell_1\}$ the eigenfunction $\psi$ has exactly one zero, and this is on the handle $e_0$.

Claim: The zero set of $\psi$ is strictly closer to $v_2$ than $v_1$.

To prove the claim: if the claim does not hold, then, supposing $\mathcal{D}^+ = \{\psi \geq 0\}$ to contain $\mathcal{S}_2$, and noting that $\lambda_1(\mathcal{D}) = \mu_1(\mathcal{D}^+)$ by Lemma 2.3 we may reflect $\mathcal{D}^+$ across the set $\{\psi = 0\}$ to obtain a new (symmetric) star dumbbell $\tilde{\mathcal{D}}$ such that, by symmetry, $\lambda_1(\tilde{\mathcal{D}}) = \mu_1(\mathcal{D}^+)$. But the handle of $\tilde{\mathcal{D}}$ is at least as long as $e_0$ and, since $\ell_1 < \ell/2$, the pendant edges of its other star are strictly longer than those of $\mathcal{D}$. But by Lemma 2.2 this means that $\lambda_1(\tilde{\mathcal{D}}) < \lambda_1(\mathcal{D})$, a contradiction. This proves the claim.
It follows from the claim that $|\psi(v_1)|^2 > |\psi(v_2)|^2$; correspondingly, since, again, the Prüfer amplitude is constant on each edge, we also have
\[
\left| \frac{\partial \psi}{\partial \nu_e_0}(v_1) \right|^2 < \left| \frac{\partial \psi}{\partial \nu_e_0}(v_2) \right|^2,
\]
so that
\[
-(n-1) \left| \frac{\partial \psi}{\partial \nu_e_0}(v_1) \right|^2 > -(n-1) \left| \frac{\partial \psi}{\partial \nu_e_0}(v_2) \right|^2.
\]
Now by the Kirchhoff condition and the fact that $\psi$ is identical on all pendant edges within each star,
\[
\lambda_1(D)|\psi(v_1)|^2 + n \left| \frac{\partial \psi}{\partial \nu_e_1}(v_1) \right|^2 = \lambda_1(D)|\psi(v_1)|^2 + \left| \frac{\partial \psi}{\partial \nu_e_0}(v_1) \right|^2
\]
\[
= \lambda_1(D)|\psi(v_2)|^2 + \left| \frac{\partial \psi}{\partial \nu_e_0}(v_2) \right|^2 = \lambda_1(D)|\psi(v_1)|^2 + n \left| \frac{\partial \psi}{\partial \nu_e_1}(v_1) \right|^2.
\]
Adding (3.2) yields
\[
\lambda_1(D)|\psi(v_1)|^2 + \left| \frac{\partial \psi}{\partial \nu_e_1}(v_1) \right|^2 > \lambda_1(D)|\psi(v_2)|^2 + \left| \frac{\partial \psi}{\partial \nu_e_2}(v_2) \right|^2,
\]
as desired.

(2) This follows immediately from (1). \qed

Proof of Theorem 3.1. Step 1. Fix any eigenfunction $\psi$ associated with $\lambda_1(\mathcal{G})$ and glue together the set of zeros
\[
\{ x \in \mathcal{G} : \psi(x) = 0 \text{ and } \exists \text{ neighbourhood of } x \text{ in which } \psi \not\equiv 0 \}
\]
to form a single vertex $v_0$; this is a finite set, and by Lemma 2.1 this does not affect $\lambda_1$. We may likewise assume that $v_{\min} := \arg \min \{ \psi(x) : x \in \mathcal{G} \}$ and $v_{\max} := \arg \max \{ \psi(x) : x \in \mathcal{G} \}$ are single vertices, by gluing together the finitely many points in each of these sets if necessary. Note that gluing points in this fashion does not change $L$ and cannot increase $D$; we will assume without loss of generality that $D$ remains the same.

Set $\mathcal{G}^+$ to be the closure of $\{ \psi > 0 \}$ and $\mathcal{G}^-$ to be the closure of $\{ \psi < 0 \}$ in $\mathcal{G}$ as in (2.5). (If $\psi$ vanishes identically on some edge of $\mathcal{G}$, meaning that this edge is a loop at $v_0$ after the identification, we temporarily remove it: let $c \geq 0$ denote the total length of all edges where $\psi$ vanishes.)

Step 2. We map each of $\mathcal{G}^+$ and $\mathcal{G}^-$ onto certain stars $\mathcal{S}^\pm$ with a Dirichlet condition on one of the vertices of degree one, which will have smaller $\mu_1$.

To this end: by definition of diameter, there exists a (non-intersecting) path $p$ in $\mathcal{G}$ starting at $v_{\min}$ and terminating at $v_{\max}$ whose length does not exceed $D$. We write $p^\pm$ for $p \cap \mathcal{G}^\pm$: then $|p^+| + |p^-| \leq D$.

If $|\mathcal{G}^\pm| = |p^\pm|$, then in fact $\mathcal{G}^\pm = p^\pm$ and we set $\mathcal{S}^\pm := p^\pm$, equipped with a Dirichlet vertex at one endpoint; then $\mu_1(\mathcal{S}^\pm) = \mu_1(\mathcal{G}^\pm) = \lambda_1(\mathcal{G})$ by Lemma 2.3.
But at least one of \( \mathcal{G}^{\pm} \) is not equal to \( \mathbf{p}^{\pm} \), since \( \mathcal{G} \) is not a path graph (for otherwise we would have \( D = L \)). We now fix \( n \geq 1 \) to be specified precisely later, but large enough that \( n \geq \max\{\deg v_{\text{max}}, \deg v_{\text{min}}\} \) and that the shortest edge in each of \( \mathcal{G}^{\pm} \) is longer than
\[
\max\left\{ \left| \frac{\mathcal{G}^{+}}{n} \right| - \left| \frac{\mathbf{p}^{+}}{n} \right|, \left| \frac{\mathcal{G}^{-}}{n} \right| - \left| \frac{\mathbf{p}^{-}}{n} \right| \right\}
\]
(this maximum is positive for any \( n \geq 1 \) since \( \mathcal{G} \) is not a path graph) and such that this maximum itself is smaller than \( \min\{\left| \mathbf{p}^{+} \right|, \left| \mathbf{p}^{-} \right|\} / 2 \).

Suppose now without loss of generality that \( \mathcal{G}^{+} \neq \mathbf{p}^{+} \) and set \( \varepsilon^{+} := (|\mathcal{G}^{+}| + c - |\mathbf{p}^{+}|) / n > 0 \), where \( c \geq 0 \) was defined in Step 1. We form the star \( \mathcal{S}^{+} \) on \( n + 1 \) edges by taking an edge of length \( |\mathbf{p}^{+}| - \varepsilon^{+} \) and attaching at one endpoint \( n \) edges of length \( \varepsilon^{+} \); at the pendant endpoint of the longer edge, we impose a Dirichlet condition. We claim that \( \mu_{1}(\mathcal{G}^{+}) \geq \mu_{1}(\mathcal{S}^{+}) \); we will use the transplantation principle Lemma 2.5 to prove this.

To this end, we look at the value \( m := \max\{\psi(x) : x \in \mathcal{G}^{+} \text{ and } \text{dist}(x, v_{\text{max}}) = \varepsilon^{+}\} \). We glue together all points in \( x \in \mathcal{G}^{+} \) such that \( \psi(x) = m \) (which, again, can only decrease \( D \), without affecting \( \mu_{1}, \psi \) or \( |\mathcal{G}^{+}| \)), creating a vertex \( v_{+} \); then the set \( \{\psi \geq m\} \cap \mathcal{G}^{+} \) consists of a pumpkin (collection of parallel edges) running from \( v_{+} \) to \( v_{\text{max}} \), such that each edge of the pumpkin has length at most \( \varepsilon^{+} \); and of course \( v_{\text{max}} \) lies on \( \mathbf{p}^{+} \) (or more precisely it lies on the image of \( \mathbf{p}^{+} \) under this gluing, which we will still call \( \mathbf{p}^{+} \)). Note that the number of edges of this pumpkin, which is a non-increasing function of \( n \), is less than \( n \) if \( n \) is sufficiently large.

We now, in accordance with Lemma 2.5, remove every edge of \( \mathcal{G}^{+} \) not on \( \mathbf{p}^{+} \) and not between \( v_{+} \) and \( v_{\text{max}} \), and re-attach a set of this total length plus \( c \) at \( v_{\text{max}} \) by first lengthening any edges between \( v_{+} \) and \( v_{\text{max}} \) to have length \( \varepsilon^{+} \) if necessary, and then attaching additional pendant edges of length \( \varepsilon^{+} \) to \( v_{\text{max}} \) until the new graph has the same total length as \( \mathcal{G}^{+} \) did, plus \( c \). We finally de-glue (cut through) \( v_{\text{max}} \) to produce only pendant edges at \( v_{+} \). We have thus transformed \( \mathcal{G}^{+} \) into \( \mathcal{S}^{+} \) in such a way that, by Lemma 2.5 (and Lemma 2.1), we indeed have \( \mu_{1}(\mathcal{G}^{+}) \geq \mu_{1}(\mathcal{S}^{+}) \) as claimed. In fact, \( \mu_{1}(\mathcal{G}^{+}) > \mu_{1}(\mathcal{S}^{+}) \) by Lemma 2.5 since the transplantation was non-trivial and obviously \( \psi(v_{\text{max}}) > 0 \) (since otherwise \( \psi \leq 0 \) everywhere in \( \mathcal{G} \)).

We now do the same for \( \mathcal{G}^{-} \), where now \( \varepsilon^{-} = (|\mathcal{G}^{-}| - |\mathbf{p}^{-}|) / n \), and this is 0 if \( \mathcal{G}^{-} = \mathbf{p}^{-} \); that is, we obtain a (possibly degenerate) star \( \mathcal{S}^{-} \) with a Dirichlet point at the end of the longest edge, such that \( \mu_{1}(\mathcal{G}^{-}) \geq \mu_{1}(\mathcal{S}^{-}) \).

**Step 3.** We now join \( \mathcal{S}^{+} \) and \( \mathcal{S}^{-} \) together at their respective Dirichlet vertices by gluing these, and obtain a star dumbbell which has diameter \( |\mathbf{p}^{+}| + |\mathbf{p}^{-}| \); that is, we obtain the graph \( \mathcal{D} = \mathcal{D}(|\mathbf{p}^{+}| + |\mathbf{p}^{-}|, \varepsilon^{+}, \varepsilon^{-}, n) \). Moreover, by Lemma 2.3 and Lemma 2.4, we have
\[
\lambda_{1}(\mathcal{G}) = \max\{\mu_{1}(\mathcal{G}^{+}), \mu_{1}(\mathcal{G}^{-})\} \geq \max\{\mu_{1}(\mathcal{S}^{+}), \mu_{1}(\mathcal{S}^{-})\} \geq \lambda_{1}(\mathcal{D}(|\mathbf{p}^{+}| + |\mathbf{p}^{-}|, \varepsilon^{+}, \varepsilon^{-}, n)).
\]
Observe that \( |\mathcal{D}| = |\mathbf{p}^{+}| + |\mathbf{p}^{-}| + n \varepsilon^{+} + n \varepsilon^{-} = |\mathcal{G}| = L \), and the diameter \( D_{1} \) of \( \mathcal{D} \) is not larger than \( D \). We finally apply Lemma 3.3 to arrive at
\[
\lambda_{1}(\mathcal{D}_{n}(L, D_{1})) \leq \lambda_{1}(\mathcal{D}(|\mathbf{p}^{+}| + |\mathbf{p}^{-}|, \varepsilon^{+}, \varepsilon^{-}, n)) \leq \lambda_{1}(\mathcal{G}).
\]

**Step 4.** For the strictness of the inequality, we have already seen \( \mu_{1}(\mathcal{G}^{+}) > \mu_{1}(\mathcal{S}^{+}) \). If also \( \mu_{1}(\mathcal{G}^{-}) > \mu_{1}(\mathcal{S}^{-}) \), then there is nothing to prove. If, however, \( \mu_{1}(\mathcal{G}^{-}) = \mu_{1}(\mathcal{S}^{-}) \), then by Lemma 2.5 it must have been the case that the transplantation was trivial, i.e., \( \mathcal{G}^{-} = \mathbf{p}^{-} \).
\[ p^- = S^- \text{ and } \varepsilon^- = 0. \] In this case, the application of Lemma 3.3 yields strictness of the inequality \( \lambda_1(D(|p^+| + |p^-|, \varepsilon^+, 0, n)) > \lambda_1(D_n(L, D_1)), \) since then \( D(|p^+| + |p^-|, \varepsilon^+, 0, n) \) cannot have been a symmetric star graph.

\[ \square \]

**References**

[1] R. Adami, E. Serra and P. Tilli, *Negative energy ground states for the \( L^2 \)-critical NLSE on metric graphs*, Comm. Math. Phys. 352 (2017), 387–406.

[2] R. Adami, E. Serra, and P. Tilli, *Lack of ground state for NLSE on bridge-type graphs*, pp. 1–11 in D. Mugnolo (ed.), Mathematical Technology of Networks (Proc. Bielefeld 2013), volume 128 of Proc. Math. & Stat., Springer-Verlag, New York, 2015.

[3] R. Adami, E. Serra and P. Tilli, *NLS ground states on graphs*, Calc. Var. 54 (2015), 743–761.

[4] M. Aizenman, H. Schanz, U. Smilansky, and S. Warzel, *Edge switching transformations of quantum graphs*, preprint (2017), arXiv:1710.07958.

[5] W. Arendt, G. Metafune, D. Pallara, and S. Romanelli, *The Laplacian with Wentzell–Robin boundary conditions on spaces of continuous functions*, Semigroup Forum 67 (2003), 247–261.

[6] S. Ariturk, *Eigenvalue estimates on quantum graphs*, preprint (2016), arXiv:1609.07471.

[7] R. Band and G. Lévy, *Quantum graphs which optimize the spectral gap*, Ann. Henri Poincaré 18 (2017), 3269–3323.

[8] G. Berkolaiko, J. B. Kennedy, P. Kurasov and D. Mugnolo, *Surgery principles for the spectral analysis of quantum graphs*, preprint (2018).

[9] G. Berkolaiko, J. B. Kennedy, P. Kurasov and D. Mugnolo, *Edge connectivity and the spectral gap of combinatorial and quantum graphs*, J. Phys. A: Math. Theor. 50 (2017), 365201.

[10] G. Berkolaiko and P. Kuchment, *Introduction to quantum graphs*, Math. Surveys and Monographs vol. 186, American Mathematical Society, Providence, RI, 2013.

[11] Y. Colin de Verdière, *Semi-classical measures on quantum graphs*, preprint (2015), arXiv:1608.07471.

[12] S. Fallat and S. Kirkland, *Extremizing algebraic connectivity subject to graph theoretic constraints*, Electron. J. Linear Algebra 3 (1998), 48–74.

[13] L. Friedlander, *Extremal properties of eigenvalues for a metric graph*, Ann. Inst. Fourier (Grenoble) 55 (2005), 199–211.

[14] L. Friedlander, *Genericity of simple eigenvalues for a metric graph*, Israel J. Math. 146 (2005), 149–156.

[15] A. Henrot (ed.), *Shape optimization and spectral theory*, De Gruyter Open, Warsaw-Berlin, 2017.

[16] A. Henrot, *Minimization problems for eigenvalues of the Laplacian*, J. Evol. Equ. 3 (2003), 443–461.

[17] J. B. Kennedy, P. Kurasov, G. Malenová and D. Mugnolo, *On the spectral gap of a quantum graph*, Ann. Henri Poincaré 17 (2016), 2439–2473.

[18] J. B. Kennedy and D. Mugnolo, *The Cheeger constant of a quantum graph*, Conference proceedings of the joint 87th annual meeting of the GAMM and Deutsche Mathematiker-Vereinigung, PAMM 16 (2016), 875–876.

[19] P. Kurasov and S. Naboko, *Rayleigh estimates for differential operators on graphs*, J. Spectr. Theory 4 (2014), 211–219.

[20] P. Kurasov, G. Malenová, and S. Naboko, *Spectral gap for quantum graphs and their edge connectivity*, J. Phys. A: Math. Theor. 46 (2013), 275309.

[21] B. Mohar, *Eigenvalues, diameter, and mean distance in graphs*, Graphs Combin. 7 (1991), 53–64.

[22] D. Lenz, M. Schmidt and P. Stollmann, *Topological Poincaré type inequalities and bounds on the infimum of the spectrum for graphs*, preprint (2018), arXiv:1801.09279.

[23] D. Mugnolo, *Semigroup Methods for Evolution Equations on Networks*, Springer-Verlag, Berlin 2014.

[24] D. Mugnolo and S. Romanelli, *Dynamic and generalized Wentzell node conditions for network equations*, Math. Meth. Appl. Sci. 30 (2007), 681–706.

[25] S. Nicaise, *Spectre des réseaux topologiques finis*, Bull. Sci. Math. (2) 111 (1987), 401–413.
[26] L. E. Payne, *Isoperimetric inequalities and their applications*, SIAM Rev. 9 (1967), 453–488.

[27] L. M. Del Pezzo and J. D. Rossi, *The first eigenvalue of the p-Laplacian on quantum graphs*, Anal. Math. Phys. 6 (2016), 365–391.

[28] J. Rohleder, *Eigenvalue estimates for the Laplacian on a metric tree*, Proc. Amer. Math. Soc. 145 (2017), 2119–2129.

[29] J. Rohleder and C. Seifert, *Spectral monotonicity for Schrödinger operators on metric graphs*, preprint (2018). [arXiv:1804.01827](https://arxiv.org/abs/1804.01827)

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