The spectral property of hypergraph coverings

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Abstract: Let \( H \) be a connected \( m \)-uniform hypergraph, and let \( \mathcal{A}(H) \) be the adjacency tensor of \( H \) whose spectrum is simply called the spectrum of \( H \). Let \( s(H) \) denote the number of eigenvectors of \( \mathcal{A}(H) \) associated with the spectral radius, and \( c(H) \) denote the number of eigenvalues of \( \mathcal{A}(H) \) with modulus equal to the spectral radius, which are respectively called the stabilizing index and cyclic index of \( H \). Let \( \tilde{H} \) be a \( k \)-fold covering of \( H \) which can be obtained from some permutation assignment in the symmetric group \( \mathfrak{S}_k \) on \( H \). In this paper, we first characterize the connectedness of \( \tilde{H} \) by its incidence graph and the permutation assignment, and then investigate the relationship between the spectral property of \( H \) and that of \( \tilde{H} \). By applying module theory and group representation, if \( \tilde{H} \) is connected, we prove that \( s(H) \mid s(\tilde{H}) \) and \( c(H) \mid c(\tilde{H}) \). In particular, when \( \tilde{H} \) is a 2-fold covering of \( H \), if \( m \) is even, we show that regardless of multiplicities, the spectrum of \( \tilde{H} \) contains the spectrum of \( H \) and the spectrum of a signed hypergraph with \( H \) as underlying hypergraph; if \( m \) is odd, we give an explicit formula for \( s(\tilde{H}) \). We also find some differences on the spectral property between hypergraph coverings and graph coverings by examples.

Keywords: Hypergraph; covering; adjacency tensor; spectrum; stabilizing index; cyclic index

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1 Introduction

A hypergraph \( H = (V, E) \) consists of a vertex set \( V = \{v_1, v_2, \cdots, v_n\} \) denoted by \( V(H) \) and an edge set \( E = \{e_1, e_2, \cdots, e_k\} \) denoted by \( E(H) \), where \( e_i \subseteq V \) for \( i \in [k] := \{1, 2, \ldots, k\} \). If \( |e_i| = m \) for each \( i \in [k] \) and \( m \geq 2 \), then \( H \) is

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called an \( m \)-uniform hypergraph. The hypergraph \( H \) is called simple if there exists no \( i \neq j \) such that \( e_i \subseteq e_j \). In particular, a simple graph is a simple 2-uniform hypergraph. For a vertex \( v \in V(H) \), denote by \( N_H(v) \) or simply \( N(v) \) the vertex neighborhood of \( v \), i.e., the set of vertices of \( H \) adjacent to \( v \); and denote by \( E_H(v) \) or \( E(v) \) the edge neighborhood of \( v \), i.e., the set of edges containing \( v \). Throughout of this paper, all hypergraphs are considered simple.

A homomorphism from a hypergraph \( \tilde{H} \) to \( H \) is a map \( \varpi : V(\tilde{H}) \to V(H) \) such that \( \varpi(e) \in E(H) \) for each \( e \in E(\tilde{H}) \); namely, \( \varpi \) maps edges to edges. So \( \varpi \) induces a map denoted by \( \tilde{\varpi} \) from \( E(\tilde{H}) \) to \( E(H) \), and particularly \( \tilde{\varpi} \) maps \( E_{\tilde{H}}(\tilde{v}) \) to \( E_H(\varpi(\tilde{v})) \) for each vertex \( \tilde{v} \in V(\tilde{H}) \).

**Definition 1.1.** A homomorphism \( \varpi \) from \( \tilde{H} \) to \( H \) is called a covering projection if \( \varpi \) is a surjection, and the induced map \( \tilde{\varpi}|_{E_{\tilde{H}}(\tilde{v})} : E_{\tilde{H}}(\tilde{v}) \to E_H(v) \) is a bijection for each vertex \( v \in V(H) \) and each \( \tilde{v} \in \varpi^{-1}(v) \).

Throughout of this paper, we always assume that the covering projection \( \varpi \) in Definition 1.1 satisfies the following condition: for any edge \( e \in E(\tilde{H}) \), \( \varpi|_e : e \to \varpi(e) \) is a bijection so that \( e \) and \( \varpi(e) \) have the same size. Under this assumption, if \( \tilde{H} \) is \( m \)-uniform, so is \( H \). Suppose that both \( \tilde{H} \) and \( H \) are simple graphs in Definition 1.1. Then \( \tilde{\varpi}|_{N_{\tilde{H}}(\tilde{v})} : N_{\tilde{H}}(\tilde{v}) \to N_H(v) \) as each edge of \( E_{\tilde{H}}(\tilde{v}) \) (respectively, \( E_H(v) \)) contains exactly two vertices: \( \tilde{v} \) and one neighbor of \( \tilde{v} \) (respectively, \( v \) and one neighbor of \( v \))

The covering projection \( \varpi \) is a surjective homomorphism from \( \tilde{H} \) to \( H \) which preserves the local vertex-edge incidences. If \( H \) is connected, then there exists a positive integer \( k \) such that each vertex \( v \) of \( H \) has \( k \) vertices in its preimage \( \varpi^{-1}(v) \), and each edge \( e \) of \( H \) has \( k \) edges in \( \varpi^{-1}(e) \). In this case, \( \tilde{H} \) is called a \( k \)-fold covering (or \( k \)-sheeted covering) of \( H \). We define an equivalence relation \( \sim \) on \( V(\tilde{G}) \) induced by \( \varpi \) such that \( \tilde{u} \sim \tilde{v} \) if \( \varpi(\tilde{u}) = \varpi(\tilde{v}) \), then we have the quotient set \( V(\tilde{G})/\varpi := \{[\tilde{u}] : \tilde{u} \in V(\tilde{G}) \} \), where \([\tilde{u}]\) is an equivalence class of \( \tilde{u} \) under the above relation. The quotient hypergraph of \( \tilde{G} \) by \( \varpi \), denoted by \( \tilde{G}/\varpi \), is the hypergraph for which the vertex set is \( V(\tilde{G})/\varpi \) such that \( \{[\tilde{u}_1], \ldots, [\tilde{u}_t] \} \) forms an edge if there exist \( \tilde{v}_1 \in [\tilde{u}_1], \ldots, \tilde{v}_t \in [\tilde{u}_t] \) such that \( \{\tilde{v}_1, \ldots, \tilde{v}_t\} \in E(\tilde{G}) \).

By the definition of covering projection, \( \tilde{G}/\varpi \) is isomorphic to \( G \).

Gross and Tucker [14] showed that all coverings of simple graphs can be characterized by the derived graphs of permutation voltage graphs. Stark and Terras [28] showed the (Ihara) zeta function of a finite graph divides the zeta function of any covering over the graph. Li and Hou [18] applied Gross and Tucker’s method to generate all hypergraph coverings, and proved that the zeta function of a finite hypergraph divides the zeta function of any covering over the hypergraph. In [28], [18] and related references, the adjacency matrix of a
A graph was used for discussing zeta functions, and the spectra of a graph and its coverings were investigated for zeta function or Ramanujan graphs. In particular, Mizuno and Sato [21] presented a formula for the characteristic polynomial of the derived covering of a simple graph with voltages in any finite group.

In this paper we will investigate the relationship between the spectral property of a uniform hypergraph and that of its coverings. However, the spectrum of a uniform hypergraph here is not referring to the adjacency matrix [12], Laplacian operator [4], or Laplacian matrix [26]. We will use the tensor (also called hypermatrix) for the representation of a uniform hypergraph. Formally, a tensor $\mathcal{A} = (a_{i_1i_2...i_m})$ of order $m$ and dimension $n$ over $\mathbb{C}$ refers to a multiarray of entries $a_{i_1i_2...i_m} \in \mathbb{C}$ for all $i_j \in [n]$ and $j \in [m]$, which can be viewed to be the coordinates of the classical tensor (as a multilinear function) under a certain basis. In 2005 Lim [19] and Qi [23] introduced the eigenvalues of tensors independently. In 2012 Cooper and Dutle [5] introduced the adjacency tensor of a uniform hypergraph, and applied the eigenvalues of the tensor to characterize the structural property of the hypergraph.

**Definition 1.2** [5]. Let $G$ be an $m$-uniform hypergraph on $n$ vertices $v_1, v_2, \ldots, v_n$. The adjacency tensor of $G$ is defined as $\mathcal{A}(G) = (a_{i_1i_2...i_m})$, an $m$-th order $n$-dimensional tensor, where

$$a_{i_1i_2...i_m} = \begin{cases} \frac{1}{(m-1)!}, & \text{if } \{v_{i_1}, \ldots, v_{i_m}\} \in E(H); \\ 0, & \text{else.} \end{cases}$$

Let $\mathcal{A}$ be a weakly irreducible nonnegative tensor of order $m$. By the Perron-Frobenius theorem of nonnegative tensors [2, 13, 30, 31, 32], the spectral radius $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ associated with a unique positive eigenvector up to a scalar, called the Perron vector of $\mathcal{A}$. If $m \geq 3$, including the Perron vector, $\mathcal{A}$ can have more than one eigenvector associated with $\rho(\mathcal{A})$, which is different from the case of nonnegative irreducible matrices (of order $m = 2$). Fan et al. [6] introduced the stabilizing index of a general tensor, and showed that the number of eigenvector of $\mathcal{A}$ associated with $\rho(\mathcal{A})$ is exactly the stabilizing index of $\mathcal{A}$. Recently Fan et al. [7] proved that there are finitely many eigenvectors of $\mathcal{A}$ associated with the spectral radius up to a scalar. If the tensor $\mathcal{A}$ has $k$ eigenvalues with modulus equal to $\rho(\mathcal{A})$, then those $k$ eigenvalues are equally distributed on a circle centered at the origin. The number $k$ is called the cyclic index of $\mathcal{A}$ [3]. For the matrix case, the cyclic index is also called the index of imprimitivity or the index of cyclicity. Fan et al. [8] used the generalized traces of a tensor to give an explicit formula for the cyclic index.

The stabilizing index and the cyclic index of a connected hypergraph $G$, denoted by $s(G)$ and $c(G)$ respectively, are referring to its adjacency tensor. In
this paper, for a connected $m$-uniform hypergraph $G$ and its connected covering $\overline{G}$, we show that $s(G) \mid s(\overline{G})$ and $c(G) \mid c(\overline{G})$. In the situation that $\overline{G}$ is a 2-fold covering of $G$, if $m$ is even, we prove that regardless of multiplicities, the spectrum of $A(\overline{G})$ contains the spectrum of $A(G)$ and the spectrum of a signed hypergraph with $G$ as underlying hypergraph; if $m$ is odd, we give an explicit formula for $s(\overline{G})$. We also find some differences on the spectral property between hypergraph coverings and graph coverings.

2 Preliminaries

2.1 Tensors and hypergraphs

We first introduce some notions of tensors and hypergraphs. Let $A = (a_{i_1i_2\ldots i_m})$ be a real tensor of order $m$ and dimension $n$. The tensor $A$ is *nonnegative* if all of its entries are nonnegative, and is *symmetric* if all entries $a_{i_1i_2\ldots i_m}$ are invariant under any permutation of its indices. The digraph $D(A)$ associated with $A$ is a digraph on vertices $1, 2, \ldots, n$ which has arcs $(i_1, i_2), \ldots, (i_1, i_m)$ for each nonzero entries $a_{i_1i_2\ldots i_m}$ of $A$. The tensor $A$ is called *weakly irreducible* if $D(A)$ is strongly connected [22, 32]. Obviously, the adjacency tensor $A(G)$ is nonnegative and symmetric, and it is weakly irreducible if and only if $G$ is connected [22, 32].

Given a vector $x \in \mathbb{C}^n$, $Ax^{m-1} \in \mathbb{C}^n$, which is defined as follows:

$$(Ax^{m-1})_i = \sum_{i_2, \ldots, i_m \in [n]} a_{i_1i_2\ldots i_m}x_{i_2} \cdots x_{i_m}, i \in [n].$$

Let $I = (i_2, i_2, \ldots, i_m)$ be the identity tensor of order $m$ and dimension $n$, that is, $i_1i_2\ldots i_m = 1$ if $i_1 = i_2 = \cdots = i_m \in [n]$ and $i_1i_2\ldots i_m = 0$ otherwise.

**Definition 2.1** ([19, 23]). Let $A$ be an $m$-th order $n$-dimensional tensor. For some $\lambda \in \mathbb{C}$, if the polynomial system $(\lambda I - A)x^{m-1} = 0$, or equivalently $Ax^{m-1} = \lambda x^{[m-1]}$, has a solution $x \in \mathbb{C}^n \backslash \{0\}$, then $\lambda$ is called an *eigenvalue* of $A$ and $x$ is an *eigenvector* of $A$ associated with $\lambda$, where $x^{[m-1]} := (x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1})$.

The *determinant* of $A$, denoted by $\det A$, is defined as the resultant of the polynomials $A_{x^{m-1}}$ [15], and the *characteristic polynomial* $\varphi_A(\lambda)$ of $A$ is defined as $\det(\lambda I - A)$ [23, 3]. It is known that $\lambda$ is an eigenvalue of $A$ if and only if it is a root of $\varphi_A(\lambda)$. The *spectrum* of $A$, denoted by $\text{Spec}(A)$, is the multi-set of the roots of $\varphi_A(\lambda)$. The *spectral radius* $\rho_A(\lambda)$ of $A$ is the largest modulus of the eigenvalues of $A$. The *spectrum, spectral radius, eigenvalues* and *eigenvectors* of $G$ are referring to its adjacency tensor $A(G)$, and the spectral radius of $G$ is denoted by $\rho(G)$.
Let $\mathbb{P}^{n-1}$ be the complex projective space of dimension $n - 1$, and let $\lambda$ be an eigenvalue of a tensor $A$ with dimension $n$. The projective variety

$$V_\lambda = V_\lambda(A) := \{x \in \mathbb{P}^{n-1} : Ax^{m-1} = \lambda x^{m-1}\}$$

is called the projective eigenvariety of $A$ associated with $\lambda$ [6]. In this paper the number of eigenvectors of $A$ is considered in $V_\lambda(A)$, i.e. only one representative vector of the projective equivalence class is counted.

For a matrix $B \in \mathbb{Z}_m^{k \times n}$, there exist invertible matrices $P \in \mathbb{Z}_m^{k \times k}$ and $Q \in \mathbb{Z}_m^{n \times n}$ such that

$$PBQ = \begin{pmatrix}
d_1 & 0 & 0 & \cdots & 0 \\
0 & d_2 & 0 & \cdots & 0 \\
0 & 0 & \ddots & \vdots \\
\vdots & d_r & \ddots & \vdots \\
0 & \cdots & \cdots & 0
\end{pmatrix}, \quad (2.1)$$

where $0 \leq r \leq \min\{k, n\}$, $d_i \mid d_{i+1}$ for $i \in [r - 1]$, and $d_i \mid m$ for all $i \in [r]$. The matrix in $(2.1)$ is called the Smith normal form of $B$ over $\mathbb{Z}_m$, where $d_1, \ldots, d_r$ are the invariant divisors of $B$ over $\mathbb{Z}_m$.

Let $A = (a_{i_1i_2\cdots i_m})$ be a symmetric tensor of order $m$ and dimension $n$. Set

$$E(A) := \{(i_1, i_2, \cdots, i_m) \in [n]^m : a_{i_1i_2\cdots i_m} \neq 0, 1 \leq i_1 \leq \cdots \leq i_m \leq n\}.$$ 

The incidence matrix [6] of $A$ is defined to be a matrix $Z(A) = (z_{e,j})$ such that

$$z_{e,j} := |\{k : i_k = j, e = (i_1, i_2, \cdots, i_m) \in E(A), k \in [n]\}|, e \in E(A), j \in [n].$$

Let $G$ be a hypergraph. The dual of $G$, denoted by $G^d$, is the hypergraph for which the vertex set is exactly the edge set $E(G)$ of $G$ and edge set is $\{E_G(v) : v \in V(G)\}$. A walk $W$ of length $t$ in $G$ is a sequence of alternate vertices and edges: $v_0e_1v_1e_2\ldots e_tv_t$, where $v_i \neq v_{i+1}$ and $\{v_i, v_{i+1}\} \subseteq e_i$ for $i = 0, 1, \ldots, t - 1$; and $W$ is called closed if $v_0 = v_t$. If $G$ is a simple graph, we simply write $W$ as $v_0v_1\ldots v_t$ as each edge contains exactly two vertices. The hypergraph $G$ is said to be connected if every two vertices are connected by a walk. The incidence matrix of $G$, denoted by $Z(G) = (z_{e,v})$, coincides with that of $A(G)$, that is, $z_{e,v} = 1$ if $v \in e$, and $z_{e,v} = 0$ otherwise.
2.2 Stabilizing index

The Perron-Frobenius theorem was generalized from nonnegative matrices to nonnegative tensors by Chang et al. [2], Yang and Yang [30, 31, 32], and Friedland et al. [13]. Here we list parts of the theorem.

**Theorem 2.2.** Let $A = (a_{i_1i_2⋯i_m})$ be a nonnegative weakly irreducible tensor of order $m$ and dimension $n$.

1. ([13]) The spectral radius $\rho(A)$ is a unique eigenvalue of $A$ associated with positive eigenvectors, and all these positive eigenvectors differ by a scalar.

2. ([32]) If $A$ has $k$ distinct eigenvalues with modulus equal to $\rho(A)$, then these eigenvalues are $\rho(A)e^{\frac{2\pi j}{k}}$, $j = 0, 1, ⋯, k - 1$, and the spectrum of $A$ keeps invariant under a rotation of angle $\frac{2\pi}{k}$ (but not a smaller positive angle) of the complex plane, where $i = \sqrt{-1}$.

3. ([32]) If $B = (b_{i_1i_2⋯i_m})$ is an $m$-th order $n$-dimensional tensors with $|B| \leq A$, namely, $|b_{i_1i_2⋯i_m}| \leq a_{i_1i_2⋯i_m}$ for each $i_j \in [n]$ and $j \in [m]$, then $\rho(B) \leq \rho(A)$. Moreover, if $\rho(B) = \rho(A)$, where $\lambda = \rho(A)e^{i\theta}$ is an eigenvalue of $B$ corresponding to an eigenvector $y$, then $y = (y_1, ⋯, y_n)$ contains no zero entries, and $A = e^{-i\theta}D^{-(m-1)}BD$, where $D = \text{diag}(\frac{y_1}{|y_1|}, \ldots, \frac{y_n}{|y_n|})$.

In Theorem 2.2(3), the tensor (or product) $D^{-(m-1)}BD$ was defined in [27], and has the same spectrum as $B$ which was proven also in [27]. If $A$ is further symmetric, Theorem 2.2(1) can be weakened to some extent.

**Lemma 2.3 ([24]).** If $A$ is a nonnegative symmetric tensor with a positive eigenvector $x$, then $x$ is necessarily associated with $\rho(A)$.

In Theorem 2.2(3), if taking $B = A$ and $y$ an eigenvector of $A$ associated with $\rho(A)$, then

$A = D^{-(m-1)}AD_y$,

where $D_y := \text{diag}(\frac{y_1}{|y_1|}, \ldots, \frac{y_n}{|y_n|})$. In general, let $A$ be a tensor of order $m$ and dimension $n$. Denote

$\mathcal{D}^{(0)}(A) := \{D : D^{-(m-1)}AD = A, d_{11} = 1\}$,

where $D = \text{diag}(d_{11}, \ldots, d_{nn})$ is an $n \times n$ invertible diagonal matrix such that $d_{11} = 1$. It was shown that $\mathcal{D}^{(0)}(A)$ is an abelian group under the usual matrix multiplication, and it is a stabilizer of $A$ under a certain permutation action; see [6, Lemmas 2.5-2.6].
Definition 2.4 ([6]). For a general tensor \( \mathcal{A} \), the cardinality of the abelian group \( \mathcal{D}^{(0)}(\mathcal{A}) \), denoted by \( s(\mathcal{A}) \), is called the stabilizing index of \( \mathcal{A} \).

Suppose that \( \mathcal{A} \) is nonnegative and weakly irreducible. By assigning a quasi-Hadamard product \( \circ \) in \( \mathbb{V}_{\rho(\mathcal{A})} \), \( (\mathbb{V}_{\rho(\mathcal{A})}, \circ) \) is an abelian group isomorphic to \( (\mathcal{D}^{(0)}(\mathcal{A}), \cdot) \); see [6, Lemma 3.1]. So \( s(\mathcal{A}) \) is exactly the number of eigenvectors of \( \mathcal{A} \) associated with \( \rho(\mathcal{A}) \). Assume further that \( \mathcal{A} \) is symmetric. By [6, Lemma 2.5], \( D^m = I \) for each \( D \in \mathcal{D}^{(0)}(\mathcal{A}) \). Then \( \mathbb{V}_{\rho(\mathcal{A})} \) and \( \mathcal{D}^{(0)}(\mathcal{A}) \) both admit \( \mathbb{Z}_m \)-modules and are isomorphic to each other, which are also isomorphic to the following \( \mathbb{Z}_m \)-module:

\[
S_0(\mathcal{A}) := \{ x \in \mathbb{Z}_m^n : \mathcal{A}(\mathcal{A})x = 0 \text{ over } \mathbb{Z}_m, x_1 = 0 \}.
\]

Suppose further \( \mathcal{A} = \mathcal{A}(G) \) for a connected \( m \)-uniform hypergraph \( G \). The stabilizing index of \( G \), denoted by \( s(G) \), is referring to the adjacency tensor \( \mathcal{A}(G) \).

Theorem 2.5 ([6], Lemma 3.3, Theorem 3.4, Theorem 3.6). Let \( G \) be a connected \( m \)-uniform hypergraph on \( n \) vertices. Suppose that the incidence matrix \( Z(G) \) has a Smith normal form over \( \mathbb{Z}_m \) as in (2.1). Then \( 1 \leq r \leq n - 1 \), and as \( \mathbb{Z}_m \)-modules

\[
\mathbb{V}_{\rho(G)}(\mathcal{A}(G)) \cong \mathcal{D}^{(0)}(\mathcal{A}(G)) \cong S_0(\mathcal{A}(G)) \cong \oplus_{i,d_i \neq 1} \mathbb{Z}_{d_i} \oplus (n - 1 - r)\mathbb{Z}_m. \tag{2.3}
\]

In particular, \( s(G) = |S_0(\mathcal{A}(G))| = m^{n-1-r} \Pi_{i=1}^{r} d_i \).

2.3 Cyclic index

Let \( \mathcal{A} \) be a nonnegative weakly irreducible tensor. The number of distinct eigenvalues of \( \mathcal{A} \) with modulus equal to \( \rho(\mathcal{A}) \) is called the cyclic index of \( \mathcal{A} \) by Chang et al. [3], denoted by \( c(\mathcal{A}) \). By Theorem 2.5(2-3),

\[
\text{Spec}(\mathcal{A}) = e^{i \frac{2\pi}{\rho(\mathcal{A})}} \text{Spec}(\mathcal{A}).
\]

So, \( c(\mathcal{A}) \) reflects the spectral symmetry of \( \mathcal{A} \). Fan et al. [8] defined the spectral symmetry for a general tensor.

Definition 2.6 ([8]). Let \( \mathcal{A} \) be a general tensor, and let \( \ell \) be a positive integer. The tensor \( \mathcal{A} \) is called spectral \( \ell \)-symmetric if

\[
\text{Spec}(\mathcal{A}) = e^{i \frac{2\pi}{\ell}} \text{Spec}(\mathcal{A}). \tag{2.4}
\]

The maximum \( \ell \) such that (2.4) holds is called the cyclic index of \( \mathcal{A} \), denoted by \( c(\mathcal{A}) \).
If \( A \) is nonnegative and weakly irreducible, the cyclic index \( c(A) \) in Definition 2.6 is consistent with that defined by Chang et al. [3] by Theorem 2.2(2). It is proved that if \( A \) is spectral \( \ell \)-symmetric, then \( \ell \mid c(A) \) [8]. The spectral symmetry of a connected uniform hypergraph \( G \) is referring to \( A(G) \), which can be characterized by the \((m,\ell)\)-coloring of \( G \).

**Definition 2.7** ([8]). Let \( m \geq 2 \) and \( \ell \geq 1 \) be integers such that \( \ell \mid m \). An \( m \)-uniform hypergraph \( G \) is called \((m,\ell)\)-colorable if there exists a map \( \phi : V(G) \to [m] \) such that if \( \{v_{i_1}, \ldots, v_{i_m}\} \in E(G) \), then

\[
\phi(v_{i_1}) + \cdots + \phi(v_{i_m}) \equiv \frac{m}{\ell} \mod m. \tag{2.5}
\]

**Theorem 2.8** ([8]). Let \( G \) be a connected \( m \)-uniform hypergraph. Then \( G \) is spectral \( \ell \)-symmetric if and only if \( G \) is \((m,\ell)\)-colorable.

Note that Eq. (2.5) is equivalent to

\[
Z(G)\phi = \frac{m}{\ell}1 \text{ over } \mathbb{Z}_m,
\]

where \( \phi = (\phi(v_1), \ldots, \phi(v_n))^\top \) is considered as a column vector, and \( 1 \) is an all-one vector whose size can be implicated by the context. Therefore, Theorem 2.8 can be rewritten as follows.

**Corollary 2.9.** Let \( G \) be a connected \( m \)-uniform hypergraph. Then \( G \) is spectral \( \ell \)-symmetric if and only if the equation

\[
Z(G)x = \frac{m}{\ell}1 \text{ over } \mathbb{Z}_m \tag{2.6}
\]

has a solution.

Hence, the cyclic index \( c(G) \) is exactly the maximum divisor \( \ell \) of \( m \) such that the equation (2.6) has a solution. The stabilizing index and cyclic index of hypergraphs were discussed in [10, 11].

### 3 Hypergraph covering and its connectedness

Let \( G \) be a connected graph, and let \( \bar{G} \) be a \( k \)-fold covering of \( G \). In the work of Gross and Tucker [14], the covering graph \( \bar{G} \) of \( G \) is not required to be connected, though in topology the covering space \( \bar{G} \) and base space \( G \) should be both connected. In this section, we will investigate the connectedness of hypergraph coverings as Theorem 2.5 for the stabilizing index and Corollary 2.9 for the cyclic index require the hypergraphs under discussion to be connected.
3.1 Covering

Gross and Tucker [14] used permutation voltage graphs to characterize the coverings of simple graphs. Let \( S_k \) be the symmetric group on the set \([k]\), and let \( D \) be a digraph possibly with multiple arcs. Let \( \phi : E(D) \to S_k \) which assigns a permutation to each arc of \( D \). The pair \((D, \phi)\) is called a permutation voltage digraph. A derived digraph \( D^\phi \) associated with \((D, \phi)\) is a digraph with vertex set \( V(D) \times [k] \) such that \(((u, i), (v, j))\) is an arc of \( D^\phi \) if and only if \((u, v) \in E(D)\) and \( i = \phi((u, v))(j)\).

Let \( \bar{G} \) be the symmetric digraph of a simple (undirected) graph \( G \), which is obtained from \( G \) by replacing each edge \( \{u, v\} \) by two arcs with opposite directions, written as \( e = (u, v) \) and \( e^{-1} := (v, u) \). Let \( \phi : E(\bar{G}) \to S_k \) be a permutation assignment on \( \bar{G} \) which holds that \( \phi(e)^{-1} = \phi(e^{-1}) \) for each arc \( e \) of \( \bar{G} \). The pair \((G, \phi)\) is called a permutation voltage graph. The derived digraph \( \bar{G}^\phi \), simply written as \( G^\phi \), has symmetric arcs by definition, and is considered as a graph. Gross and Tucker [14] established a relationship between \( k \)-fold coverings and derived graphs.

**Lemma 3.1** ([14]). Let \( G \) be a connected graph and let \( \bar{G} \) be a \( k \)-fold covering of \( G \). Then there exists an assignment \( \phi \) of permutations in \( S_k \) on \( G \) such that \( G^\phi \) is isomorphic to \( \bar{G} \).

**Lemma 3.2.** Let \( G \) be a connected simple graph and let \( \bar{G} \) be a \( k \)-fold covering of \( G \). Then \( G \) has at most \( k \) connected components, each of which is a \( \tilde{k} \)-fold covering of \( G \) for some positive integer \( \tilde{k} \), where \( 1 \leq \tilde{k} \leq k \). If \( \bar{G} \) has exactly \( k \) connected components, then each connected component is isomorphic to \( G \).

**Proof.** By Lemma 3.1 there exists a permutation voltage assignment \( \phi : E(\bar{G}) \to S_k \) such that \( G^\phi \) is isomorphic to \( \bar{G} \). So it suffices to discuss the graph \( G^\phi \).

Let \( G^1, \ldots, G^r \) be all connected components of \( G^\phi \), where \( r \geq 1 \). We assert that each connected component \( G^i \) is a covering of \( G \) for \( i \in [r] \). Let \( \varpi : V(G^\phi) \to V(G) \) be a covering projection of \( G^\phi \) on \( G \), where \( \varpi(v, i) = v \) for each \((v, i) \in V(G) \times [k]\). Consider the map \( \varpi|_{V(G^i)} : V(G^i) \to V(G) \). For each vertex \( v \in \varpi(V(G^i)) \) and each vertex \( \bar{v} \in \varpi^{-1}(v) \cap V(G^i) \), surely \( \varpi|_{E_{G^i}(\bar{v})} : E_{G^i}(\bar{v}) \to E_G(v) \) is a bijection by the definition of covering projection. So, it suffices to prove that \( \varpi|_{V(G^i)} : V(G^i) \to V(G) \) is a surjection, namely, \( \varpi(V(G^i)) = V(G) \). Let \( u \in \varpi(V(G^i)) \) and let \((u, t) \in \varpi^{-1}(u) \cap V(G^i) \). For any vertex \( v \in V(G) \), as \( G \) is connected, there exists a walk of \( G \): \( u_0u_1 \ldots u_s \), which connects \( u \) and \( v \), where \( u_0 = u, u_s = v \). By the definition of \( G^\phi \), there exists a walk in \( G^i \) as follows:

\[
(u_0, t)(u_1, \phi(u_1, u_0)(t)) \ldots (u_s, \phi(u_s, u_{s-1})(t))
\]
Note \( \varpi(u_s, \phi(u_s, u_{s-1}) \phi(u_{s-1}, u_{s-2}) \cdots \phi(u_1, u_0)(l)) = u_s = v. \) So \( v \in \varpi(V(G^i)) \), which implies that \( \varpi|_{V(G_i)} : V(G^i) \to V(G) \) is a surjection. As \( G \) is connected, \( G_i \) is a \( k_i \)-fold covering of \( G \) for some positive integer \( k_i \). As \( G^\phi \) is a \( k \)-fold covering of \( G \), we have \( \sum_{i=1}^r k_i = k \), which implies that \( 1 \leq k_i \leq k \) and \( 1 \leq r \leq k \). So, \( G^\phi \) has at most \( k \) connected components. If \( r = k \), then each \( k_i \) equals 1, and \( G^1, \ldots, G^k \) are all copies of \( G \). \( \square \)

Li and Hou [18] generalized Lemma 3.1 from graphs to hypergraphs by using two kinds of graph representations of hypergraphs. Here we only introduce the incidence graph representation of a hypergraph. The incidence graph \( B_H \) of a hypergraph \( H \) is a bipartite graph with vertex set \( V(H) \cup E(H) \) such that \( v \in V(H) \) (called the vertex-vertex of \( B_H \)) is adjacent to \( e \in E(H) \) (called the edge-vertex of \( B_H \)) if and only if \( v \in e \). Let \( \phi : E(B_H^\phi) \to S_k \) be a permutation voltage assignment on \( B_H \). From the derived graph \( B_H^\phi \), we can construct a hypergraph denoted by \( H^\phi_B \) with vertex set \( V(H) \times [k] \) such that for each \( e \in E(H) \) and each \( i \in [k] \), the set of vertices in \( B_H^\phi \) adjacent to \( (e, i) \) forms a hyperedge, also denoted by \( (e, i) \) in \( H^\phi_B \). Li and Hou [18] proved that any \( k \)-fold covering \( \bar{H} \) of \( H \) is isomorphic to \( H^\phi_B \) for some \( \phi \). We give a proof here to emphasize that if \( H \) and \( \bar{H} \) are both connected, then there is an isomorphism \( \psi \) from \( B_H^\phi \) to \( B_{\bar{H}} \) which sends \( V(H) \times [k] \) to \( V(\bar{H}) \), and hence \( H^\phi_B \) is isomorphic to \( \bar{H} \).

**Lemma 3.3.** Let \( \bar{H} \) be a connected \( k \)-fold covering of a connected hypergraph \( H \). Then there exists a permutation assignment \( \phi \) in \( S_k \) on \( B_H \) such that \( B_H^\phi \) is isomorphic to \( B_{\bar{H}} \) by a map which sends \( V(H) \times [k] \) to \( V(\bar{H}) \), and hence \( H^\phi_B \) is isomorphic to \( \bar{H} \).

**Proof.** By definition, there is a \( k \) to 1 surjective covering projection \( \varpi : V(\bar{H}) \to V(H) \), which induces a \( k \) to 1 surjection \( \varpi : E(\bar{H}) \to E(H) \). So \( \varpi \) induces a \( k \) to 1 surjection \( \hat{\varpi} \) from \( V(B_{\bar{H}}) \) to \( V(B_H) \), namely,

\[
\hat{\varpi} : V(\bar{H}) \cup E(\bar{H}) \to V(H) \cup E(H),
\]

such that \( \hat{\varpi}|_{V(\bar{H})} = \varpi \) and \( \hat{\varpi}|_{E(\bar{H})} = \varpi \).

We assert that \( \hat{\varpi} \) is a covering projection from \( B_{\bar{H}} \) to \( B_H \), which is necessarily a \( k \)-fold covering projection by the definition of \( \hat{\varpi} \). Obviously, \( \hat{\varpi} \) is a homomorphism from \( B_{\bar{H}} \) to \( B_H \), as for any edge \( \{\bar{v}, \bar{e}\} \) of \( B_{\bar{H}} \), surely \( \bar{v} \in \bar{e} \), \( \varpi(\bar{v}) \in \varpi(\bar{e}) \) and hence \( \hat{\varpi}(\{\bar{v}, \bar{e}\}) = \{\varpi(\bar{v}), \varpi(\bar{e})\} \in E(B_H) \). To prove \( \hat{\varpi} \) preserves the local vertex-edge incidences of graphs, we will use the vertex neighborhoods rather than edge neighborhoods as remarked after Definition 1.1. Note that \( N_{B_{\bar{H}}} (\bar{v}) = E_{\bar{H}} (\bar{v}) \) for each \( \bar{v} \in V(\bar{H}) \subseteq V(B_{\bar{H}}) \); and \( N_{B_H} (\bar{e}) = \bar{e} \) for \( \bar{e} \in E(\bar{H}) \subseteq V(B_{\bar{H}}) \), where the first \( \bar{e} \) means an edge-vertex of \( B_{\bar{H}} \) and the second \( \bar{e} \) means the
set of vertices of $\bar{H}$ that are contained in $\bar{e}$. For each $v \in V(H) \subseteq V(B_H)$ and $\bar{v} \in \varpi^{-1}(v)$, the map $\hat{\varphi}|_{N_{B_H}(\bar{v})} : N_{B_H}(\bar{v}) \to N_{B_H}(v)$ is exactly the map $\hat{\varphi}|_{E_{B_H}(\bar{v})} : E_{\bar{H}}(\bar{v}) \to E_{\bar{H}}(v)$, which is a bijection by definition. Similarly, for $e \in E(H) \subseteq V(B_H)$ and $\bar{e} \in \varpi^{-1}(e)$, $\hat{\varphi}|_{N_{B_H}(\bar{e})} : N_{B_H}(\bar{e}) \to N_{B_H}(e)$ is exactly the map $\varpi|_{e} : e \to e$, which is also a bijection by the assumption on covering projection after Definition 1.1.

By Lemma 3.1 there exists a permutation assignment $\phi$ such that $B_H^\phi$ is isomorphic to $B_H$ via a map $\psi : V(B_H^\phi) \to V(B_H)$. Note that $B_H^\phi$ is a $k$-fold covering of $B_H$ by a projection $\varphi$ such that $\varphi[v]_\times [k] = v$ and $\varphi[e]_\times [k] = e$ for all $v \in V(H)$ and $e \in E(H)$. Observe that there exist no two distinct vertex-vertices $(u, i)$ and $(v, j)$ of $B_H^\phi$ such that $\psi(u, i) \in V(\bar{H})$ and $\psi(v, j) \in E(\bar{H})$. Otherwise, as $H$ is connected, $B_H$ and $B_H^\phi$ are both connected, the distance between $(u, i)$ and $(v, j)$ is even while the distance between $\psi(u, i)$ and $\psi(v, j)$ is odd, which yields a contradiction as $\psi$ is an isomorphism. Similarly, there exist no two distinct edge-vertices $(e, i)$ and $(f, j)$ of $B_H^\phi$ such that $\psi(e, i) \in V(\bar{H})$ and $\psi(f, j) \in E(\bar{H})$. So we can divide the discussion into two cases.

Case 1. $\psi(V(H) \times [k]) = V(\bar{H})$ and $\psi(E(H) \times [k]) = E(\bar{H})$. Hence $H_B^\phi$ is isomorphic to $\bar{H}$.

Case 2. $\psi(V(H) \times [k]) = E(\bar{H})$ and $\psi(E(H) \times [k]) = V(\bar{H})$. Then $B_H^\phi$ is isomorphic to $B_H^\bar{d}$ also by $\psi$ which maps the vertex-vertices of $B_H^\phi$ to the vertex-vertices of $B_H^\bar{d}$, where $\bar{d}$ is the dual of $\bar{H}$. So the quotient hypergraph $(B_H^\phi)/\varphi$ is isomorphic to $B_{H^\bar{d}}/\varpi$, which implies that $B_H$ is isomorphic to $B_{H^\bar{d}}$ by a map $\tau$ which sends the vertices (edges) of $H$ to the edges (vertices) of $\bar{H}$. Let $\bar{\phi}$ be another permutation assignment on $B_H$ (or $B_{H^\bar{d}}$) such that $\bar{\phi}((\tau(v), \tau(e))) = \phi((v, e))$ for each $(v, e) \in V(H) \times E(H)$, where $v \in e$. It is easily verified that $B_H^\phi$ is isomorphic to $B_H^\bar{d}$ via a map $\bar{\tau}$ such that $\bar{\tau}(v, i) = (\tau(v), i)$ and $\bar{\tau}(e, j) = (\tau(e), j)$ for all vertices $(v, i)$ and $(e, j)$ of $B_H^\phi$. So, $\psi \circ \bar{\tau}^{-1}$ is an isomorphism from $B_H^\phi$ to $B_{H^\bar{d}}$, and hence an isomorphism from $B_H^\phi$ to $B_H$, which sends $V(H) \times [k]$ to $V(\bar{H})$. The result follows.

\[\square\]

3.2 Connectedness

We start the discussion from the connectedness of the graph coverings, and then get the results on hypergraph coverings. Firstly we introduce some notions for preparation. A gain graph $(G, \mathfrak{G}, \phi)$ (also called voltage graph) is a group $\mathfrak{G}$ and a map $\phi : E(G) \to \mathfrak{G}$ such that $\phi(e^{-1}) = \phi(e)^{-1}$ for each arc $e \in E(G)$. So, if $\mathfrak{G} = \mathbb{S}_k$, the gain graph is exactly the permutation voltage graph; and if $\mathfrak{G} = \{z \in \mathbb{C} : |z| = 1\}$, the gain graph is called the complex unit gain graph. In particular, a signed graph is the gain graph by taking $\mathfrak{G}$ to be the multiplicative subgroup $\{1, -1\}$ of
C. Let \( W = v_0v_1 \ldots v_t \) be a walk of \((G, \mathcal{G}, \phi)\) (in fact the underlying graph \(G\)). The gain value of \( W \) is denoted and defined by \( \phi(W) = \prod_{i=1}^t \phi((v_{i-1}, v_i)) \). A cycle \( C \) of \( G \) is balanced if \( \phi(C) = 1 \). The gain graph \((G, \mathcal{G}, \phi)\) is balanced if each cycle of \( G \) is balanced, and is unbalanced otherwise.

**Theorem 3.4.** Let \( G \) be a connected simple graph and \( \phi \) be a permutation assignment in \( S_k \) on \( G \). Then the following are equivalent.

1. \( G^{\phi} \) is connected.
2. For any \( v \in V(G) \) and any \( i, j \in [k] \), there exists a closed walk \( W \) of \((G, \phi)\) starting from \( v \) such that \( i = \phi(W)(j) \).
3. There exists a vertex \( v \in V(G) \) such that for any \( i, j \in [k] \), \((G, \phi)\) contains a closed walk \( W \) starting from \( v \) satisfying \( i = \phi(W)(j) \).

**Proof.** (1) \( \Rightarrow \) (2). Suppose \( G^{\phi} \) is connected. Then for any \( v \in V(G) \) and any \( i, j \in [k] \), there exists a walk in \( G^{\phi} \) from \((v, i)\) and \((v, j)\), say

\[ W = (v_0, i_0)(v_1, i_1)(v_2, i_2) \ldots (v_t, i_t), \]

where \((v_0, i_0) = (v, i)\) and \((v_t, i_t) = (v, j)\). By definition, \((G, \phi)\) contains a closed walk \( \bar{W} = v_0v_1v_2 \ldots v_t \), and

\[ i = i_0 = \phi((v_0, v_1))(i_1) = \phi((v_0, v_1))\phi((v_1, v_2))(i_2) = \cdots = \phi(W)(i_t) = \phi(W)(j). \]

(2) \( \Rightarrow \) (3). It is obvious.

(3) \( \Rightarrow \) (1). It suffices to prove for any vertex \((u, i)\) and a fixed vertex \((v, j)\), there is a walk in \( G^{\phi} \) from \((u, i)\) to \((v, j)\). As \( G \) is connected, there exists a walk from \( u \) to \( v \), say \( Y = v_0v_1v_2 \ldots v_t \), where \( v_0 = u \) and \( v_t = v \). Then by definition \((u, i)\) is connecting to \((v, s)\) in \( G^{\phi} \) by a walk \( \bar{Y} \), where \( s = \phi(Y)^{-1}(i) \).

By the assumption, \((G, \phi)\) contains a closed walk \( W \) starting from \( v \) such that \( s = \phi(W)(j) \). So \((v, s)\) is connecting to \((v, j)\) in \( G^{\phi} \) by a walk \( \bar{W} \). Hence \((u, i)\) is connecting to \((v, j)\) by joining \( \bar{Y} \) and \( \bar{W} \).

We now discuss the other extreme case in Lemma 3.2 that is, each component of \( G^{\phi} \) is a copy of \( G \). Let \((G, \mathcal{G}, \phi)\) be a gain graph and let \( \alpha \in \mathcal{G} \).

An \( \alpha \)-switching \[^{34}\] at a vertex \( u \) of \((G, \mathcal{G}, \phi)\) means only replacing \( \phi(e) \) by \( \phi^\alpha(e) = \alpha \phi(e) \), and \( \phi(e^{-1}) \) by \( \phi^\alpha(e^{-1}) = \phi(e^{-1})\alpha^{-1} \) for each arc \( e = (u, v) \) of \( G \) starting from \( u \). If \((G, \mathcal{G}, \psi)\) can be obtained from \((G, \mathcal{G}, \phi)\) by a sequence of switchings at some vertices of \( G \), then \((G, \mathcal{G}, \psi)\) is called switching equivalent to \((G, \mathcal{G}, \phi)\).
Lemma 3.5 \([34]\). A gain graph \((G, \mathfrak{G}, \phi)\) is balanced if and only if it is switching equivalent to \((G, \mathfrak{G}, 1)\), where 1 is the map such that \(1(e) = 1\) for each arc \(e \in E(G)\).

In the following we assume that \(\mathfrak{G} = S_k\) and write \((G, \mathfrak{G}, \phi)\) simply as \((G, \phi)\).

Lemma 3.6. Let \(G\) be a simple graph. If \((G, \psi)\) is switching equivalent to \((G, \phi)\), then the derived graph \(G\psi\) is isomorphic to \(G\phi\).

Proof. Without loss of generality, assume that \((G, \psi)\) is obtained from \((G, \phi)\) by applying an \(\alpha\)-switching at a vertex \(v\) of \((G, \phi)\). Define \(f: V(G\phi) \rightarrow V(G\psi)\), which satisfies that \(f(v, i) = (v, \alpha(i))\), \(f(u, i) = (u, i)\) for \(i \in [k], u \neq v\).

It is known that \(f\) is a bijection. If \(((v, i), (u, j)) \in E(G\phi)\), then \((v, u) \in E(G)\) and \(i = \phi(v, u)(j)\). So

\[
\alpha(i) = \alpha(\phi(v, u)(j)) = \psi(v, u)(j),
\]

and hence \(((v, \alpha(i)), (u, j)) \in E(G\psi)\), namely, \(((f(v, i), f(u, j)) \in E(G\psi)\) \(\alpha\)-switching at a vertex \(v\) of \((G, \phi)\).

Conversely, if \(((v, \alpha(i)), (u, j)) \in E(G\psi)\), it is easily verified that \(((v, i), (u, j)) \in E(G\psi)\). The other cases can be shown similarly. \(\square\)

Note that \(G^1\) is a union of \(k\) disjoint copies of \(G\), where 1 is the identity of \(S_k\).

Corollary 3.7. If \((G, \phi)\) is balanced, or equivalently, \((G, \phi)\) is switching equivalent to \((G, 1)\), then \(G\phi\) is a union of \(k\) disjoint copies of \(G\).

Note that for a tree \(T\), \((T, \phi)\) is balanced for any \(\phi\) by definition.

Corollary 3.8. Let \(T\) be a tree. Then \(T\phi\) is a union of \(k\) disjoint copies of \(T\) for any \(\phi\).

We now show the inverse of Corollary \([3.7]\) is also true.

Theorem 3.9. A permutation voltage graph \((G, \phi)\) is balanced if and only if \(G\phi\) is a union of \(k\) disjoint copies of \(G\).

Proof. It is enough to consider the sufficiency. Suppose to the contrary that \((G, \phi)\) is not balanced. Then \(G\) contains a cycle \(C: v_0v_1v_2 \ldots v_t\), where \(v_0 = v_t\), such that \(\phi(C) \neq 1\). So there exist two distinct elements \(i_0, j_0 \in [k]\) such that \(i_0 = \phi(C)(j_0)\). Let \(\tilde{G}\) be a connected component of \(G\phi\) which contains the
vertex \((v_0, i_0)\). By definition, \(\bar{G}\) contains a path \((v_0, i_0)(v_1, i_1)(v_2, i_2) \ldots (v_t, i_t)\), where \(i_{p-1} = \phi(v_{p-1}, v_p)(i_p)\) for \(p \in [t]\). So,

\[
i_0 = \phi((v_0, v_1))\phi((v_1, v_2)) \cdots \phi((v_{t-1}, v_t))(i_t) = \phi(C)(i_t),
\]

which implies that \(i_t = j_0\) and \(\bar{G}\) contains both \((v_0, i_0)\) and \((v_0, j_0)\). Hence, \(\bar{G}\) is not a copy of \(G\); a contradiction.

**Theorem 3.10.** Let \(G\) be a connected simple graph and let \(G^\phi\) be a 2-fold covering of \(G\). Then the following statements are equivalent.

1. \(G^\phi\) is connected.
2. \((G, \phi)\) is unbalanced.
3. \((G, \phi)\) contains a cycle such that it has an odd number of arcs with \((12)\)-permutation assigned by \(\phi\).

**Proof.** (1) \(\Rightarrow\) (2). It follows by Theorem 3.9.

(2) \(\Rightarrow\) (3). By definition, \((G, \phi)\) contains a cycle \(C\) with \(\phi(C) \neq 1\). As \(\phi(C) \in S_2\), \(\phi(C) = (12)\), which implies that \(C\) contains an odd number of arcs with \((12)\)-permutation assigned by \(\phi\).

(3) \(\Rightarrow\) (1). Observe that \((G, \phi)\) contains a cycle \(C\) with \(\phi(C) = (12)\). The assertion follows by Theorem 3.4.

We finally return to the connectedness of hypergraph coverings. Note a hypergraph \(H\) is connected if and only if its incidence graph \(B_H\) is connected. So, by Theorem 3.4 and Theorem 3.10, we easily get the following results.

**Corollary 3.11.** Let \(H\) be a connected \(m\)-uniform hypergraph, and let \(H_B^\phi\) be a \(k\)-fold covering of \(H\), where \(\phi\) is a permutation assignment in \(S_k\) on the incidence graph \(B_H\). Then \(H_B^\phi\) is connected if and only if there exists a vertex \(v\) of \(B_H\) such that for any \(i, j \in [k]\), \((B_H, \phi)\) contains a closed walk \(W\) starting from \(v\) satisfying \(i = \phi(W)(j)\).

**Corollary 3.12.** Let \(H\) be a connected \(m\)-uniform hypergraph, and let \(H_B^\phi\) be a 2-fold covering of \(H\), where \(\phi\) is a permutation assignment in \(S_2\) on the incidence graph \(B_H\). Then \(H_B^\phi\) is connected if and only if \((B_H, \phi)\) contains a cycle such that it has an odd number of arcs with \((12)\)-permutation assigned by \(\phi\).
4 Stabilizing index and cyclic index of covering

4.1 Spectrum

Let $H$ be a connected $m$-uniform hypergraph on $n$ vertices $v_1, \ldots, v_n$, and let $B_H$ be its incidence graph with a permutation assignment $\phi$. From the permutation voltage graph $(B_H, \phi)$, we define a signed hypergraph, denoted by $\Gamma(H, \phi)$, such that for each edge $e \in E(H)$,

$$\text{sgn } e = \prod_{v \in e} \text{sgn } \phi(v, e).$$

The adjacency tensor of $\Gamma(H, \phi)$ is defined as $A(\Gamma(H, \phi)) = (a^\phi_{i_1 i_2 \ldots i_m})$, where

$$a^\phi_{i_1 i_2 \ldots i_m} = \begin{cases} \text{sgn } e \cdot \frac{1}{(m-1)!}, & \text{if } e = \{v_{i_1}, \ldots, v_{i_m}\} \in E(H); \\ 0, & \text{else}. \end{cases}$$

The spectrum, eigenvalues and eigenvectors of $\Gamma(H, \phi)$ are referring to $A(\Gamma(H, \phi))$.

**Theorem 4.1.** Let $H$ be a connected $m$-uniform hypergraph, let $H_B^\phi$ be a $k$-fold covering of $H$, where $\phi$ is a permutation assignment in $S_k$ on $B_H$. Then, regardless of multiplicities, the spectrum of $H_B^\phi$ contains the spectrum of $H$, and $\rho(H_B^\phi) = \rho(H)$.

In particular, if $k = 2$ and $m$ is even, then, regardless of multiplicities, the spectrum of $H_B^\phi$ also contains the spectrum of the signed hypergraph $\Gamma(H, \phi)$.

**Proof.** Define

$$\tau : \mathbb{C}^{V(H)} \rightarrow \mathbb{C}^{V(H) \times [k]}$$

such that $\tau(x)|_{\{v\} \times [k]} = x(v)$ for all $x \in \mathbb{C}^{V(H)}$ and all $v \in V(H)$. Let $x$ be an eigenvector of $A(H)$ associated with an eigenvalue $\lambda$. We assert that $\tau(x)$ is an eigenvector of $A(H_B^\phi)$ also associated with the eigenvalue $\lambda$. By eigenvector equations, for each $v \in V(H)$,

$$\lambda x(v)^{m-1} = \sum_{e \in E_H(v)} \prod_{u \in e \setminus \{v\}} x(u).$$

For each vertex $v \in V(H)$ and each $i \in [k]$, if $(e, j) \in E_{H_B^\phi}((v, i))$, then $v \in e$ and $j = \phi(e,v)(i)$, and $E_H(v)$ is one to one mapping onto $E_{H_B^\phi}((v, i))$ by

$$\eta_{v,i} : e \mapsto (e, \phi(e, v)(i))$$

furthermore, if $(u, t) \in (e, \phi(e, v)(i))$, then $u \in e$ and $t = \phi(u,e)\phi(e,v)(i)$, and $e$ is one to one mapping onto $(e, \phi(e, v)(i))$ by

$$\zeta_{v,i,e} : u \mapsto (u, \phi(u,e)\phi(e,v)(i)).$$
So
\[
\lambda(x(v,i))^{m-1} = \lambda x(v)^{m-1} = \sum_{e \in E_H(v)} \prod_{u \in e \setminus \{v\}} x(u)
\]

which implies \(\tau(x)\) is an eigenvector of \(A(H_B^\phi)\) associated with the eigenvalue \(\lambda\).

In the above discussion, by Theorem 2.2(1), taking \(\lambda = \rho(H)\) and \(x\) be a positive eigenvector associated with \(\rho(H)\), then \(\tau(x)\) is a positive eigenvector of \(A(H_B^\phi)\) associated with the eigenvalue \(\rho(H)\). By Lemma 2.3, \(\rho(H)\) is the spectral radius of \(A(H_B^\phi)\), i.e. \(\rho(H_B^\phi) = \rho(H)\).

Now suppose \(k = 2\) and \(m\) is even. Let \(x\) be an eigenvector of \(\Gamma := \Gamma(H, \phi)\) associated with an eigenvalue \(\lambda\). Define \(y \in C^v(H) \times [2]\) such that \(y(v, 1) = x(v)\) and \(y(v, 2) = -x(v)\) for each \(v \in V(H)\). We will show that \(y\) is an eigenvector of \(A(H_B^\phi)\) also associated with the eigenvalue \(\lambda\).

By eigenvector equations, for each \(v \in V(\Gamma)\),
\[
\lambda x(v)^{m-1} = \sum_{e \in E_v(v)} \text{sgn} e \prod_{u \in e \setminus \{v\}} x(u).
\]

For each vertex \(v \in V(H)\) and each edge \(e \in E_v(v)\), by Eq. (4.2) and (4.3), \(\eta(v,1)(e) = (e,\phi(e,v)(1)) \in E_{H_B^\phi}((v,1))\) and \(\zeta_{v,1,e}(u) = (u,\phi(u,e)\phi(e,v)(1)) \in \eta(v,1)(e)\). By definition,
\[
y(\zeta_1(u)) = (\text{sgn} \phi(u,e)\phi(e,v))x(u),
\]
and since \(m\) is even,
\[
\prod_{\zeta_{v,1,e}(u) \in \eta(v,1)(e) \setminus \{(v,1)\}} y(\zeta_{v,1,e}(u)) = \text{sgn} \phi(e,v)^{m-2} \text{sgn} e \prod_{u \in e \setminus \{v\}} x(u)
\]
\[
= \text{sgn} e \prod_{u \in e \setminus \{v\}} x(u).
\]

So
\[
\lambda y(v,1)^{m-1} = \lambda x(v)^{m-1} = \sum_{e \in E_v(v)} \text{sgn} e \prod_{u \in e \setminus \{v\}} x(u)
\]
\[
= \sum_{\eta(v,1)(e) \in E_{H_B^\phi}((v,1))} \prod_{\zeta_{v,1,e}(u) \in \eta(v,1)(e) \setminus \{(v,1)\}} y(\zeta_{v,1,e}(u)). \tag{4.4}
\]

Similarly, for each vertex \(\zeta_{v,2,e}(u) = (u,\phi(u,e)\phi(e,v)(2)) \in \eta(v,2)(e)\), where \(\eta(v,2)(e) = (e,\phi(e,v)(2)) \in E_{H_B^\phi}((v,2))\),
\[
y(\zeta_{v,2,e}(u)) = -(\text{sgn} \phi(u,e)\phi(e,v))x(u),
\]

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and
\[
\prod_{\zeta_{v,2,e}(u) \in \eta_{v,2}(e) \setminus \{(v,2)\}} \lambda y(v,2)(u) = (-1)^{m-1} \sgn e \prod_{u \in e \setminus \{v\}} x(u) = - \sgn e \prod_{u \in e \setminus \{v\}} x(u).
\]

So
\[
\lambda y(v,2)^{m-1} = -\lambda x(v)^{m-1} = - \sum_{e \in E_{\Gamma}(v)} \sgn e \prod_{u \in e \setminus \{v\}} x(u)
\]
\[
= \sum_{\eta_{v,2}(e) \in E_{\Gamma}(v)} \prod_{\zeta_{v,2,e}(u) \in \eta_{v,2}(e) \setminus \{(v,2)\}} \lambda y(v,2)(u).
\]

By Eq. (4.4) and (4.5), we show that \( y \) is an eigenvector of \( A(H_B^0) \) also associated with the eigenvalue \( \lambda \). \( \square \)

**Corollary 4.2.** Let \( H \) be a connected \( m \)-uniform hypergraph, let \( \hat{H} \) be a connected \( k \)-fold covering of \( H \). Then, regardless of multiplicities, the spectrum of \( \hat{H} \) contains the spectrum of \( H \), and \( \rho(\hat{H}) = \rho(H) \).

In particular, if \( k = 2 \) and \( m \) is even, then, regardless of multiplicities, the spectrum of \( \hat{H} \) also contains the spectrum of a signed hypergraph with \( H \) as underlying hypergraph.

**Proof.** By Lemma 3.3 there exists a permutation assignment \( \phi \) on \( B_H \) such that \( \hat{H} \) is isomorphic to \( H_B^0 \). The result follows by Theorem 4.1. \( \square \)

By Theorem 4.1 if \( H_B^0 \) is connected, then \( s(H) \leq s(H_B^0) \) by the map \( \tau \) defined in Eq. (4.1). In fact, \( (\forall_{\rho(H)}, \circ) \) can be embedded as a \( \mathbb{Z}_m \)-submodule of \( (\forall_{\rho(H_B^0)}, \circ) \) so that \( s(H) \mid s(H_B^0) \). However, it will need more preparations to show the above division relation. We will use another \( \mathbb{Z}_m \)-module involved with incidence matrix to investigate the division relation in next subsection.

The 2-fold covering \( \phi \) of a simple graph \( G \) is also called a 2-lift of \( G \) [20]. The spectrum of a 2-lift of \( G \) is exactly the union of the spectrum of \( G \) and the spectrum of \( \Gamma(G, \phi) \) [1]. However, the above result does not hold for 2-fold coverings of a hypergraph; see the following example.

**Example 4.3.** By using the terminology in [17], denote by \( C_k^{4,2} \) a 4-uniform hypergraph obtained from a cycle \( C_k \) of length \( k \) (as a simple graph) by blowing up each vertex into a 2-set and keeping the adjacency. Consider \( H = C_3^{4,2} \) with vertex set \([6]\) and edge set \( \{e_1 = \{1, 2, 3, 4\}, e_2 = \{3, 4, 5, 6\}, e_3 = \{5, 6, 1, 2\}\} \).

(1) Let \( \phi \) be a permutation assignment in \( S_2 \) on \( B_H \) such that \( \phi(e_1, 3) = \phi(e_1, 4) = (12) \), and \( \phi(e, v) = 1 \) for all other incidences \( (e, v) \). By definition the signed hypergraph \( \Gamma(H, \phi) \) is same as \( H \) as the sign of each edge of \( \Gamma(H, \phi) \) equals 1. The 2-fold covering hypergraph \( H_B^0 \) has the following edges

\[ \{11, 21, 32, 42\}, \{32, 42, 52, 62\}, \{52, 62, 12, 22\}, \]
\{12, 22, 31, 41\}, \{31, 41, 51, 61\}, \{51, 61, 11, 21\},

where a vertex \((v, k)\) in the vertex set \(V(H) \times [2]\) of \(H_B^\phi\) is simply written as \(vk\). Note that \(H_B^\phi\) is isomorphic to \(C_6^k\). By using SAGE\textregistered\textregistered\textsuperscript{package: TensorCharpolyPackage written by Aaron Dutle}\footnote{https://people.math.sc.edu/dutle/spectraresults.html} we get the characteristic polynomial of \(A(H)\) as follows:

\[
\varphi_A(H)(\lambda) = \lambda^{498}(\lambda^2 - 4)^{32}(\lambda^2 - 1)^{208}(\lambda^2 + 1)^{48}(\lambda^2 + \lambda + 2)^{96}.
\]

By Theorem 2.10 in \cite{9}, \(H_B^\phi\) has an eigenvalue \(\sqrt{3}\), which is the largest eigenvalue of a path \(P_3\) on 5 vertices (as an induced subgraph of \(C_6\)). But \(\sqrt{3}\) is not an eigenvalue of \(H\) or \(\Gamma(H, \phi)\).

(2) Let \(\psi\) be another permutation assignment in \(S_2\) on \(B_H\) such that \(\psi(e_1, 3) = \psi(e_2, 5) = (12)\), and \(\psi(e, v) = 1\) for all other incidences \((e, v)\). Then \(\Gamma(H, \psi)\) contains exactly two negative edges, namely \(\{1, 2, 3, 4\}\) and \(\{3, 4, 5, 6\}\). The characteristic polynomial of \(A(\Gamma(H, \psi))\) is

\[
\varphi_A(\Gamma(H, \psi))(\lambda) = \lambda^{1074}(\lambda^4 - 1)^{32}(\lambda^4 - 4)^{64}.
\]

The 2-fold covering hypergraph \(H_B^\psi\) has the following edges

\[
\{11, 21, 32, 41\}, \{31, 41, 52, 61\}, \{51, 61, 11, 21\},
\]

\[
\{12, 22, 31, 42\}, \{32, 42, 51, 62\}, \{52, 62, 12, 22\}.
\]

Let \(K\) be the subhypergraph of \(H_B^\psi\) with edges

\[
\{11, 21, 32, 41\}, \{31, 41, 52, 61\}, \{51, 61, 11, 21\}, \{32, 42, 51, 62\}.
\]

Observe that if \(\lambda\) is an eigenvalue of \(A(K)\) associated with an eigenvector \(x\), then \(\lambda\) is also an eigenvalue of \(A(H_B^\psi)\) associated with an eigenvector \(\tilde{x}\) by setting \(\tilde{x}|_{V(K)} = x\) and \(\tilde{x}(12) = \tilde{x}(22) = 0\). If letting \(x(11) = x(21) = a, x(32) = x(41) = x(51) = x(61) = b\) and \(x(31) = x(42) = x(52) = x(62) = c\), by eigenvector equations, we get that \(K\) and hence \(H_B^\psi\) has an eigenvalue \(\sqrt{3}\), which is neither an eigenvalue of \(H\) nor \(\Gamma(H, \psi)\).

For the 2-fold covering \(H_B^\phi\) of an \(m\)-uniform hypergraph \(H\), if \(m\) is odd, the spectrum of \(H_B^\phi\) can not contain the spectrum of the signed hypergraph \(\Gamma(H, \phi)\); see the following example.

**Example 4.4.** Let \(H\) be a 3-uniform hypergraph with vertex set \([4]\) and edge set \(\{e_1 = \{1, 2, 3\}, e_2 = \{2, 3, 4\}\}\). Let \(\phi\) be a permutation assignment in \(S_2\) such that \(\phi(e_1, 2) = (12)\) and \(\phi(e, v) = 1\) for all other incidences \((e, v)\). Then the
signed hypergraph $\Gamma(H, \phi)$ contains a negative edge $\{1, 2, 3\}$ and a positive edge $\{2, 3, 4\}$, and $H_B^\phi$ is obtained from $C_4$ by inserting an additional vertex into each edge, called the power hypergraph of $C_4$ \cite{[10]}. The characteristic polynomial of $\mathcal{A}(\Gamma(H, \phi))$ is

$$\varphi_{\mathcal{A}(\Gamma(H, \phi))}(\lambda) = \lambda^{14}(\lambda^6 + 4)^3,$$

and the characteristic polynomials of $\mathcal{A}(H_B^\phi)$ is

$$\varphi_{\mathcal{A}(H_B^\phi)}(\lambda) = \lambda^{493}(\lambda^3 - 1)^{24}(\lambda^3 - 4)^{27}(\lambda^3 - 2)^{126}.$$ 

So $\sqrt[3]{2}$ is an eigenvalue of $\Gamma(H, \phi)$ but not an eigenvalue of $H_B^\phi$.

**Corollary 4.5.** Let $H$ be a connected $m$-uniform hypergraph, and let $\bar{H}$ be a connected $k$-fold covering of $H$. Then $c(H) \mid c(\bar{H})$.

**Proof.** By Lemma 3.3 there exists a permutation assignment $\phi$ on $B_H$ such that $\bar{H}$ is isomorphic to $H_B^\phi$. So, it suffices to consider $H_B^\phi$. By Theorem 4.1 regardless of multiplicities, the spectrum of $H_B^\phi$ contains that of $H$, and $\rho(H_B^\phi) = \rho(H)$. So, by Theorem 2.2(2) and the definition of cyclic index, $\rho(H(e^{1/n_H}))$ is an eigenvalue of $\mathcal{A}(H)$, and hence $\rho(H_B^\phi)e^{1/n_H}$ is an eigenvalue of $\mathcal{A}(H_B^\phi)$. By by Theorem 2.2(3), there exists a diagonal matrix $D$ such that

$$\mathcal{A}(H_B^\phi) = e^{-1/n_H}D^{-m-1}\mathcal{A}(H_B^\phi)D.$$

So $\Spec(H_B^\phi) = e^{1/n_H}\Spec(H_B^\phi)$, implying that $H_B^\phi$ is spectral $c(H)$-symmetric, and hence $c(H) \mid c(H_B^\phi)$. $\Box$

4.2 $\mathbb{Z}_m$-Module

We will use $\mathbb{Z}_m$-Module to establish the division relation between $s(H)$ and $s(\bar{H})$, where $\bar{H}$ is a $k$-fold covering of $H$.

**Theorem 4.6.** Let $H$ be a connected $m$-uniform hypergraph, and let $\bar{H}$ be a connected $k$-fold covering of $H$. Then $S_0(H)$ can be embedded as a $\mathbb{Z}_m$-submodule of $S_0(\bar{H})$, and hence $s(H) \mid s(\bar{H})$.

**Proof.** By Lemma 3.3 there exists a permutation assignment $\phi$ on $B_H$ such that $B_H^\phi$ is isomorphic to $B_B$ via a map $\psi$ with $\psi(V(H) \times [k]) = V(\bar{H})$. So, it suffices to consider $H_B^\phi$ whose incidence graph is exactly $B_B^\phi$.

Consider

$$S_0(H) := \{x \in \mathbb{Z}_m^{V(H)} : Z(H)x = \mathbf{0} \text{ over } \mathbb{Z}_m\},$$

and

$$S_0(H_B^\phi) := \{x \in \mathbb{Z}_m^{V(H_B^\phi)} : Z(H_B^\phi)x = \mathbf{0} \text{ over } \mathbb{Z}_m\}.$$
Let
\[ \tau : \mathbb{Z}^V(H) \to \mathbb{Z}^V(H) \times [k] \] (4.6)
such that \( \tau(x)(v) = x(v) \) for all \( x \in \mathbb{Z}^V(H) \) and \( v \in V(H) \). We assert that for each \( x \in S_0(H) \), \( \tau(x) \in S_0(H_B^\phi) \). If \( x \in S_0(H) \), then for each edge \( e = (v_{i_1}, \ldots, v_{i_m}) \in E(H) \),
\[
x(v_{i_1}) + \cdots + x(v_{i_m}) = 0 \mod m.
\]
Now for each edge \( (e, i) \in E(H_B^\phi) \), by definition
\[
(e, i) = \{(v_{i_1}, \phi(v_{i_1}, e)(i)), \ldots, (v_{i_m}, \phi(v_{i_m}, e)(i))\}.
\]
So
\[
\tau(x)(v_{i_1}, \phi(v_{i_1}, e)(i)) + \cdots + \tau(x)(v_{i_m}, \phi(v_{i_m}, e)(i)) = x(v_{i_1}) + \cdots + x(v_{i_m}) = 0 \mod m,
\]
which implies that \( \tau(x) \in S_0(H_B^\phi) \).

Observe that \( \tau : S_0(H) \to S_0(H_B^\phi) \) is an injection and also a \( \mathbb{Z}_m \)-module homomorphism. So \( S_0(H) \) is \( \mathbb{Z}_m \)-isomorphic to \( \tau(S_0(H)) \), the latter of which is a \( \mathbb{Z}_m \)-submodule of \( S_0(H_B^\phi) \). Therefore, \( S_0(H) \) can be embedded as a \( \mathbb{Z}_m \)-submodule of \( S_0(H_B^\phi) \). Note that if \( x \in S_0(H) \) then \( x + t1 \in S_0(H) \) for any \( t \in \mathbb{Z}_m \) as \( Z(H)1 = 0 \mod m \). So \( S_0(H) \cong S_0(H)/(\mathbb{Z}_m1) \), and similarly \( S_0(H_B^\phi) \cong S_0(H_B^\phi)/(\mathbb{Z}_m1) \). Then \( S_0(H) \) is embedded as a \( \mathbb{Z}_m \)-submodule of \( S_0(H_B^\phi) \). So we get \( s(H) \mid s(H_B^\phi) \) by Theorem 2.5.

One may wonder what is the exact expression or value of \( s(H_B^\phi) \). We will discuss the problem by the representation theory of group ring.

### 4.3 Representation

Given a group \( G \) and a ring \( R \), the group ring or group algebra of \( G \) over \( R \), denoted and defined by \( RG = \left\{ \sum_{g \in G} r_g g \mid r_g \in R \right\} \), which is a free \( R \)-module with the elements of \( G \) as a basis.

Let \( M \) be a \( R \)-module. \( M \) is called an \( RG \)-module, if there exists a \( R \)-module homomorphism
\[
\varrho : RG \to \text{End}_R(M),
\]
where \( \text{End}_R(M) \) is the ring of \( R \)-module endomorphism of \( M \). \((M, \varrho)\) is also called an \( R \)-module representation of \( G \). If \( M \) is a free \( R \)-module with a basis \( b_1, \ldots, b_k \), then the above \( R \)-module homomorphism \( \varrho \) is equivalent to a group homomorphism
\[
\hat{\varrho} : G \to \text{GL}_k(R),
\]

where $\text{GL}_k(R)$ is the group of all invertible matrices of order $k$ over $R$. If $gb_j = \sum_{i=1}^k \alpha_{ij} b_i$, then $\tilde{g}(g) = (\alpha_{ij})_{k \times k}$. $\tilde{g}$ is called an $R$-matrix representation of $G$.

Now consider a special case. Take $R = \mathbb{Z}_m$ and $M$ be a free $\mathbb{Z}_m$-module with a basis $b_1, \ldots, b_k$. Let $G$ be a subgroup of $\mathfrak{S}_k$. Consider the permutation representation of $G$ over $M$, i.e. each element $g \in G$ maps $b_i$ to $b_{g(i)}$. Using matrix language,

$$\rho : G \to \text{GL}_k(\mathbb{Z}_m), g \mapsto P_g = (p_{ij}^g),$$

where,

$$p_{ij}^g = \begin{cases} 1, & \text{if } i = g(j); \\ 0, & \text{else.} \end{cases}$$

In this case, we call $\rho$ the $\mathbb{Z}_m$-permutation matrix representation of $G$.

Let $M_1 = \mathbb{Z}_m \left( \sum_{i=1}^k b_i \right)$. It is easily verified that $M_1$ is a subrepresentation of $M$, denoted by $(M_1, \rho_1)$, where $\rho_1$ is the identity. Let

$$M_2 = \left\{ \sum_{i=1}^k \ell_i b_i \mid \ell_i \in \mathbb{Z}_m, \sum_{i=1}^k \ell_i = 0 \right\}.$$

It is easily verified that $M_2$ is a subrepresentation of $M$, denoted by $(M_2, \rho_2)$, with a basis $b_1 - b_i, i = 2, \ldots, k$. Suppose that $\gcd(m, k) = 1$. Then $k$ is invertible in $\mathbb{Z}_m$. As

$$\sum_{i=1}^k \ell_i b_i = \frac{\sum_{i=1}^k \ell_i}{k} \sum_{i=1}^k b_i + \sum_{i=1}^k \left( \ell_i - \frac{\sum_{i=1}^k \ell_i}{k} \right) b_i \in M_1 + M_2,$$

we have $M = M_1 + M_2$. If $c \in M_1 \cap M_2$, then there exist $\ell, \ell_1, \ldots, \ell_k \in \mathbb{Z}_m$ such that

$$c = \ell \sum_{i=1}^k b_i = \sum_{i=1}^k \ell_i b_i,$$

where $\sum_{i=1}^k \ell_i = 0$. So we have $k \ell = 0$, and thus $\ell = 0$ and $c = 0$, which implies that $M = M_1 \oplus M_2$.

**Theorem 4.7.** Let $M$ be a free $\mathbb{Z}_m$-module of rank $k$, and let $G$ be a subgroup of $\mathfrak{S}_k$, where $\gcd(m, k) = 1$. Then as a $\mathbb{Z}_m$-$G$-module by permutation representation, $M$ has a decomposition

$$M = M_1 \oplus M_2,$$

where $M_1$ is a subrepresentation of $M$ with degree 1, and $M_2$ is a subrepresentation of $M$ with degree $k - 1$; or equivalently, there exists an invertible matrix.
$T$ over $\mathbb{Z}_m$ such that for all $g \in G$, the representation matrix $P_g$ in (4.7) holds that
\begin{equation}
T^{-1}P_gT = I_1 \oplus \varrho_2(g),
\end{equation}
where $I_1$ is the identity matrix representation of $G$ with order 1 and $\varrho_2(g)$ is the matrix representation of $G$ with order $k - 1$.

We next give an expression of the incidence matrix of $H^\phi_B$, where $\phi : E(\overset{\rightarrow}{B_H}) \to S_k$ is a permutation assignment on $B_H$. Let $\Phi = \langle \phi(e) \mid e \in E(\overset{\rightarrow}{B_H}) \rangle$, the subgroup of $S_k$ generated by the permutation assigned on $B_H$. For each $g \in \Phi$, define a matrix $Z_g = (z_{ev}^{(g)})$, where
\begin{equation}
Z_{ev}^{(g)} = \begin{cases} 
1, & \text{if } (e,v) \in E(\overset{\rightarrow}{B_H}) \text{ and } \phi(e,v) = g; \\
0, & \text{else.}
\end{cases}
\end{equation}

Then the incidence matrix $Z(H)$ holds that
\begin{equation}
Z(H) = \sum_{g \in \Phi} Z_g,
\end{equation}
and the incidence matrix $Z(H^\phi_B)$ satisfies that
\begin{equation}
Z(H^\phi_B) = \sum_{g \in \Phi} Z_g \otimes P_g.
\end{equation}

**Theorem 4.8.** Let $H$ be a connected $m$-uniform hypergraph on $n$ vertices, let $H^\phi_B$ be a $k$-fold covering of $H$, where $\phi : E(\overset{\rightarrow}{B_H}) \to S_k$ is a permutation assignment on $B_H$, and $\gcd(m,k) = 1$. Let $\Phi = \langle \phi(e) \mid e \in E(\overset{\rightarrow}{B_H}) \rangle$, with a $\mathbb{Z}_m$-permutation matrix representation as defined in (4.7). Then there exists an invertible matrix $T$ over $\mathbb{Z}_m$ such that
\begin{equation}
(I \otimes T^{-1})Z(H^\phi_B)(I \otimes T) = Z(H) \oplus \left( \sum_{g \in \Phi} Z_g \otimes \varrho_2(g) \right),
\end{equation}
where $\varrho_2(g)$ is the matrix representation of $\Phi$ with order $k - 1$.

Moreover, if $H^\phi_B$ is connected, and $Z(H,\phi) := \sum_{g \in \Phi} Z_g \otimes \varrho_2(g)$ has invariant divisors $\hat{d}_1, \ldots, \hat{d}_{\tilde{r}}$ over $\mathbb{Z}_m$, then
\begin{equation}
s(H^\phi_B) = s(H) \left( m^{n-\tilde{r}} \prod_{i=1}^{\tilde{r}} \hat{d}_i \right),
\end{equation}
Proof. By Theorem 4.7, there exists an invertible matrix $T$, such that for any $g \in \Phi$,
\begin{equation}
T^{-1}P_gT = I_1 \oplus \varrho_2(g),
\end{equation}

\[22\]
where $\varrho_2(g)$ is the matrix representation of $\Phi$ with order $k - 1$. By Eq. (4.11), (4.12) and (4.15), we have

$$
(I \otimes T^{-1})Z(H_B^\varphi)(I \otimes T) = \sum_{g \in \Phi} Z_g \otimes (T^{-1} P_g T).
$$

$$
= \sum_{g \in \Phi} Z_g \otimes (I_1 \oplus \varrho_2(g))
$$

$$
= Z(H) \oplus \left( \sum_{g \in \Phi} Z_g \otimes \varrho_2(g) \right).
$$

Suppose that $Z(H)$ has invariant divisors $d_1, \ldots, d_r$ over $\mathbb{Z}_m$. Then $Z(H_B^\varphi)$ has invariant divisors $d_1, \ldots, d_r, \hat{d}_1, \ldots, \hat{d}_r$ over $\mathbb{Z}_m$. The result follows by Theorem 2.5.

In Theorem 4.8, if we know the representation $\varrho_2$ of $\Phi$, then we will get the matrix $Z(H, \varphi)$ explicitly. By calculating the Smith normal forms of $Z(H)$ and $Z(H, \varphi)$, we will get the exact value of $s(H_B^\varphi)$ by Theorem 2.5. In particular, the $\mathbb{Z}_m$-permutation representation of $S_2$ is equivalent to $I_1 \oplus \varrho_2(g)$, where $\varrho_2(1) = 1$ and $\varrho_2((12)) = -1$, where $m$ is odd. So, in this situation, $Z(H, \varphi)$ satisfies that $Z(H, \varphi)(e, v) = \varrho_2(\varphi(e, v)) = \text{sgn} \varphi(e, v)$ if $v \in e$, and $Z(H, \varphi)(e, v) = 0$ else. We call $Z(H, \varphi)$ the signed incidence matrix of $H$ associated with $\varphi$.

**Corollary 4.9.** Let $H$ be a connected $m$-uniform hypergraph on $n$ vertices, let $H_B^\varphi$ be a connected $2$-fold covering of $H$, where $\varphi$ is a permutation assignment in $S_2$ on $B_H$, and $m$ is odd. If the signed incidence matrix $Z(H, \varphi)$ has invariant divisors $\hat{d}_1, \ldots, \hat{d}_r$ over $\mathbb{Z}_m$, then

$$
s(H_B^\varphi) = s(H) \left( m^{n-r} \prod_{i=1}^{\hat{r}} \hat{d}_i \right).
$$

**Example 4.10.** Consider the 3-uniform hypergraph $H$ and permutation assignment $\varphi$ in Example 4.4. By Corollary 3.12, $H_B^\varphi$ is connected as $(B_H, \varphi)$ contains a cycle with only one edge assigned by the permutation (12). Both $Z(H)$ and $Z(H, \varphi)$ have invariant divisors 1, 1 over $\mathbb{Z}_3$. So $s(H) = 3$ by Theorem 2.5 and $s(H_B^\varphi) = 27$ by Corollary 4.9.

We note that Corollary 4.9 does not hold for $m$ being even; see the following example.

**Example 4.11.** Consider the 4-uniform hypergraph $H$ and the permutation assignment $\varphi$ in Example 4.3(1). Then $Z(H)$ has invariant divisors 1, 1, 2 over $\mathbb{Z}_4$, implying $s(H) = 2 \cdot 4^2$ by Theorem 2.5. The matrix $Z(H, \varphi)$ has invariant
divisors 1, 1 over \( \mathbb{Z}_4 \), and \( Z(H^\phi_B) \) has invariant divisors 1, 1, 1, 1, 1 over \( \mathbb{Z}_4 \). We have \( s(H^\phi_B) = 4^6 < (2 \cdot 4^2) \cdot 4^4 \). For the permutation assignment \( \psi \) in Example 4.3.2, the matrix \( Z(H, \psi) \) has invariant divisors 1, 1, 2 over \( \mathbb{Z}_4 \), and \( Z(H^\psi_B) \) has invariant divisors 1, 1, 1, 1, 1 over \( \mathbb{Z}_4 \). We have \( s(H^\psi_B) = 4^5 < (2 \cdot 4^2) \cdot (2 \cdot 4^3) \). So, for the two permutation assignments in Example 4.3, the equality in Corollary 4.9 does not hold.

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