Quantum Energy Inequalities and Stability Conditions in Quantum Field Theory

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Summary. We give a brief review of quantum energy inequalities (QEIs) and then discuss two lines of work which suggest that QEIs are closely related to various natural properties of quantum field theory which may all be regarded as stability conditions. The first is based on joint work with Verch, and draws connections between microscopic stability (microlocal spectrum condition), mesoscopic stability (QEIs) and macroscopic stability (passivity). The second direction considers QEIs for a countable number of massive scalar fields, and links the existence and scaling properties of QEIs to the spectrum of masses. The upshot is that the existence of a suitable QEI with polynomial scaling is a sufficient condition for the model to satisfy the Buchholz–Wichmann nuclearity criterion. We briefly discuss on-going work with Ojima and Porrmann which seeks to gain a deeper understanding of this relationship.

1 Introduction

The stress-energy tensor $T_{ab}$ of the real scalar field, in common with those corresponding to most models of classical matter,\(^2\) obeys the dominant energy condition (DEC): for any future-directed timelike vector $u^a$, the contraction $T^a_b u^b$ is itself timelike and future-directed. This may also be stated as the inequality $T_{ab} u^a v^b \geq 0$ for all pairs of future-directed timelike vectors $u^a$ and $v^a$. In the special case $v^a = u^a$, we recover the weak energy condition (WEC), $T_{ab} u^a u^b \geq 0$, i.e., the energy density is nonnegative according to any observer.

In classical general relativity, energy conditions of this type play a key role, guaranteeing the stability of gravitational collapse (singularity theorems \([29]\)) and also excluding certain exotic causal structures. The main exception is the nonminimally coupled scalar field, satisfying the field equation $(\Box + m^2 + \xi R)\phi = 0$ with $\xi \neq 0$.\(^3\)
(see e.g., Hawking’s discussion of chronology protection [28]). However, as has been known for a long time [6], the WEC (and hence DEC) are violated in quantum field theory. It is easy to give a simple proof in Minkowski space: suppose that \( g = T_{\mu\nu}u^\mu u^\nu \), where \( u^\mu \) is now a smooth timelike vector field, and let \( f \) be a nonnegative smooth function of compact support. We need assume only that the smeared field \( \rho(f) \) is (essentially) self-adjoint on the Hilbert space of the theory and that there is a vacuum state \( \Omega \) in the operator domain of \( \rho(f) \) such that \( \langle \Omega | \rho(f) \Omega \rangle = 0 \) but which is not annihilated by \( \rho(f) \), i.e., \( \rho(f)\Omega \neq 0 \). Writing the spectral measure of \( \rho(f) \) as \( d\lambda_\rho \), these last two properties tell us that the probability measure \( \langle \Omega | d\lambda_\rho \Omega \rangle \) on \( \mathbb{R} \) has zero expectation, but that its support is not simply the set \{0\}. Accordingly \( \rho(f) \) must have some negative spectrum.\(^3\) Clearly the same argument applies in many circumstances, and for observables other than energy density.

Thus, the classical pointwise energy conditions are simply incompatible with the structures of quantum field theory. Further analysis of particular models shows that the pointwise energy density is typically unbounded from below as a function of the quantum state, and this can be proved for all theories with a suitable scaling limit [9].

This fact raises questions concerning the applicability of the singularity, positive mass and chronology protection results where quantised matter is concerned. Many authors have also sought to exploit quantum fields to support metrics (including wormhole or warp drive models) which require WEC-violating matter distributions. It is therefore important to understand whether the classical energy conditions are irretrievably lost, or whether one can identify some remnant in the quantum theory. This contribution will discuss a promising candidate: a group of results known as Quantum Energy Inequalities (QEIs), and will in particular focus on their emerging connections with other well-known stability conditions in quantum field theory, namely the microlocal spectrum condition, passivity and nuclearity. The hope is that, by unravelling these connections, further insight is provided into the nature of quantised matter and its (gravitational) stability.

It is a particular pleasure to dedicate this contribution to Jacques Bros, in view of his influential contributions to both microlocal analysis and the description of thermal behaviour in quantum field theory.

2 Quantum Energy Inequalities

As mentioned above, the pointwise energy conditions are unavoidably and severely violated in quantum field theory. However, observations at individual spacetime points are not physically achievable in any case (owing to the uncertainty principle), so it is more natural to consider weighted averages of the stress-energy tensor over a spacetime volume.

\[^3\]The same conclusion is easily drawn by examining the expectation values of \( \Omega + \lambda \rho(f)\Omega \) for small \( \lambda \).
**Definition 1.** Let \( \mathcal{W} \) be a class of second-rank tensors on spacetime, and \( S \) a class of states of the theory. If, for each \( f \in \mathcal{W} \), the averaged expectation values
\[
\int \mathrm{dvol}(x) \langle T_{ab}(x) \rangle_\omega f^{ab}(x) \quad \text{are bounded from below as} \quad \omega \text{ runs over } S,
\]
we say that the theory obeys a Quantum Energy Inequality (QEI) with respect to \( \mathcal{W} \) and \( S \).

One generally aims to find an explicit lower bound \(-\mathcal{Q}[f]\) so that the QEI can be written as an inequality
\[
\int \mathrm{dvol}(x) \langle T_{ab}(x) \rangle_\omega f^{ab}(x) \geq -\mathcal{Q}[f] \quad \forall \omega \in S.
\]

Where \( \mathcal{W} \) consists of tensors of a particular form e.g., \( f^{ab} = u^a u^b \) or \( f^{ab} = u^a v^b \) for timelike vector fields \( u^a, v^a \), we use more specific terms, e.g., Quantum Weak Energy Inequality (QWEI) or Quantum Dominated Energy Inequality (QDEI). Of course a similar approach could be adopted other quantities of interest.

For the most part, QEIs have been developed for averages along timelike curves, rather than over spacetime volumes, in which case the weights may be thought of as being singularly supported on a curve. By threading a spacetime volume by worldlines, these bounds imply the existence of spacetime-averaged QEIs, which may also be obtained directly, as sketched below. It is known that compactly supported weighted averages over spacelike hypersurfaces \[23\] or null lines \[15\] are not generally bounded from below, except for two-dimensional conformal fields \[19, 11\].

QEIs were first proposed by Ford \[21\], who realised that suitable bounds of this type would be sufficient to prevent macroscopic violations of the second law of thermodynamics arising from negative energy phenomena in quantum field theory. They have since been established for the free Klein–Gordon \[22, 23, 26, 35, 10, 18, 19, 17, 20\], Dirac \[47, 17, 12\], Maxwell \[26, 37, 14\] and Proca \[14\] quantum fields in both flat and curved spacetimes, the Rarita–Schwinger field in Minkowski space \[49\], and also for general unitary positive-energy conformal field theories in two-dimensional Minkowski space \[11\]. We will not give a full history of the development of the subject, referring the reader to the recent reviews \[9, 42\]. To give a flavour of the sort of results obtained, we give an example in which the energy density of a scalar field of mass \( m \) is averaged along the inertial trajectory \((t, 0)\) in Minkowski space. It can be obtained by elementary means \[10\] or as a special case of the rigorous result \[8\]. Set \( \rho = T_{ab} u^a u^b \), where \( u = \partial/\partial t \). Then the QWEI
\[
\int \mathrm{d}t \langle \varrho(t, 0) \rangle_\psi g(t) \geq -\mathcal{Q}[g] := -\frac{1}{16\pi^2} \int_0^\infty \mathrm{d}u \ u^4 \vartheta(u - m) |\hat{g}(u)|^2,
\]
holds for all Hadamard states \( \psi \) (see below) and smooth compactly supported \( g \). Here \( \hat{g} \) denotes the Fourier transform \[4\] and \( \vartheta \) is the Heaviside function. The

\[\text{Our convention for the Fourier transform is } \hat{g}(u) = \int \mathrm{d}t e^{iu t} g(t) \text{ etc.}\]
bound is finite, owing to the rapid decay of $\tilde{g}$. In fact the bound given in \[10\]
is slightly tighter than this, but \[11\] will suffice for our present purposes.

For later reference, let us note the scaling behaviour of the bound \[11\]. Replacing $g$ by $g_\tau(t) = \tau^{-1/2}g(t/\tau)$, so that $\tau$ controls the ‘spread’ of the weight, one may show that

$$Q[g_\tau] = \begin{cases} O(\tau^{-4}) & \text{as } \tau \to 0^+ \\ O(\tau^{-\infty}) & \text{as } \tau \to \infty \end{cases}$$

for $m > 0$, where the notation $O(\tau^{-\infty})$ indicates faster-than-inverse-polynomial decay. In the massless case, it turns out that $Q(g_\tau) \propto \tau^{-4}$ for all $\tau > 0$. We note that the $\tau \to 0^+$ limit, which corresponds to sampling at a point, is consistent with the pointwise unboundedness below of the energy density. For intermediate scales, the QWEI allows for a limited violation of the classical WEC; bounds of this type therefore appear to be the natural remnant of the WEC in quantum field theory.

We mention briefly that related bounds appear elsewhere in quantum field theory \[36\] and quantum mechanics \[7\]; QEIs have also been used to place constraints on exotic spacetimes \[25, 39, 42\].

### 3 Stability at Three Scales

The work described in this section, conducted with Verch \[18\] and building on earlier work \[8, 43\], uncovers a circle of connections between stability conditions operating at three different scales: the microscopic (Hadamard condition/microlocal spectrum condition), mesoscopic (QEIs) and macroscopic (thermodynamic stability, expressed by the notion of passivity \[40\]). Each connection takes the form of a rigorous theorem; the reader should be cautioned, however, that the conclusions and hypotheses of successive links do not match perfectly. Moreover, two of the links (mesoscopic to macroscopic, and macroscopic to microscopic) are obtained in greater generality than the particular setting of quantum field theory on curved spacetimes, while the microscopic to mesoscopic link is currently known only for particular models of quantum field theory. Thus the existence of these connections should be regarded as indicative of a close relationship between these three stability conditions, rather than of proving their equivalence. In part, this work gives a precise expression to Ford’s original insight \[21\], that bounds of QEI type would suffice to prevent macroscopic violations of the second law of thermodynamics.

#### 3.1 Microscopic Stability: the Hadamard Condition

Stability of quantum field theory at the microscopic scale is (partly) expressed by the Hadamard condition, which requires that the singular structure of the
two-point function takes a form determined for nearby points by the local geometry of spacetime. As first shown by Radzikowski, this may be reformulated as a condition on the wave-front set of the two-point function. By passing to a Hilbert space representation, however, one obtains a very simple formulation of the Hadamard condition (cf. also): a state of the scalar field on \((M, g)\) is Hadamard if and only if it may be represented by a vector \(\psi\) in some Hilbert space representation of the theory so that \(f \mapsto \Phi(f)\psi\) is a vector-valued distribution whose wave-front set obeys

\[
\text{WF}(\Phi(\cdot)\psi) \subset \mathcal{V}^{-},
\]

where \(\Phi\) is the field and

\[
\mathcal{V}^{-} = \{(x, k) \in T^{*}M : g^{ab}k_{a}k_{b} \geq 0, \ k \text{ past directed}\}
\]

is the bundle of past-pointing causal covectors (our signature convention is \(+ - - -\)). This has the following practical upshot. Suppose \(f\) is smooth and compactly supported within some coordinate patch, with coordinates \(x^{a}\) so that \(\partial/\partial x^{0}\) is future-pointing and timelike. Let \(V\) be any closed cone in \(\mathbb{R}^{4}\) consisting of \(k\) such that the covector field \(k_{a}dx^{a}\) is nowhere causal and past-directed on the coordinate patch. In particular, \(V\) could be the half-space \(V = \{k \in \mathbb{R}^{4} : k_{0} \geq 0\}\). Then

\[
I(k) := \left\| \int d^{4}x e^{ik_{a}x^{a}} f(x)\Phi(x)\psi \right\|
\]

is of rapid decay in \(V\); that is, it decays more rapidly than any inverse polynomial in the Euclidean norm of \(k\) as \(k \to \infty\) in \(V\). Moreover, the same is true if \(f\) is replaced by a partial differential operator with smooth coefficients compactly supported in the coordinate patch.

Microlocal formulations of the Hadamard condition are also known for the Dirac, Maxwell and Proca fields. They may be regarded as local remnants of the spectrum condition, i.e., the Minkowski space requirement that the joint spectrum of the generators \(P_{\mu}\) of spacetime translations should lie in the future causal cone.

3.2 From Microscopic to Mesoscopic

We now show how QEIs may be derived from the Hadamard condition, using an argument based on that of. The classical Klein–Gordon field \(\phi\) obeying \((\Box + m^{2})\phi = 0\) on spacetime \((M, g)\) has stress-energy tensor

\(^{5}\text{Appropriate conditions on higher n-point functions were given in. For non-initiates: the wave-front set }\text{WF}(S)\text{ of a distribution }S\text{ on a manifold }M\text{ is a subset of the cotangent bundle }T^{*}M\text{ which encodes the singular structure of }S.\text{ Singularities are classified in terms of the (lack of) decay of local Fourier transforms of }S\text{ in different directions.}\)

\(^{6}\text{That the forward cone appears in the spectrum condition, but the backward cone in }\text{(2)}\text{, is the result of an unfortunate clash of conventions.}\)
\[ T_{ab} = \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} g^{cd} \nabla_c \phi \nabla_d \phi + \frac{1}{2} g_{ab} m^2 \phi^2 , \]

and obeys the WEC and DEC because the relevant contractions of \( T_{ab} \) can be decomposed as sums of squares. Let us therefore consider – as representing the most general classical energy condition – any tensor field \( f_{ab} \) for which

\[ T_{ab} f^{ab} = \frac{1}{2} \sum_j (P_j \phi)^2 , \quad (3) \]

where the \( P_j \) are finitely many linear partial differential operators (possibly of degree zero) with smooth real coefficients of compact support. Clearly \( T_{ab} f^{ab} \geq 0 \) for classical fields \( \phi \). For simplicity, assume that the supports of the \( P_j \) are contained within a single coordinate patch of \((M,g)\) writing the coordinates as \( x^a \) and assuming as above that \( \partial / \partial x^0 \) is future-pointing and timelike. Write also \( g(x) = | \det g_{\alpha \beta}(x) | \).

Let \( \psi_0 \) be a fixed Hadamard reference state. Defining the stress-energy tensor \( \mathcal{T}_{ab} \) by point-splitting and normal ordering with respect to \( \psi_0 \), we have

\[ \langle \mathcal{T}_{ab}(x) \rangle_{\psi} f^{ab}(x) = \frac{1}{2 \sqrt{g(x)}} F(x,x) \]

for any Hadamard state \( \psi \), where

\[ F(x,y) = (g(x) g(y))^{1/4} \sum_j \left[ \langle (P_j \Phi)(x)(P_j \Phi)(y) \rangle_{\psi} - \langle (P_j \Phi)(x)(P_j \Phi)(y) \rangle_{\psi_0} \right] \]

is smooth (owing to the common singularity structure of Hadamard two-point functions) and symmetric (because two-point functions have a state-independent antisymmetric part). Thus we may write

\[ \int \text{dvol}_g(x) \langle \mathcal{T}_{ab}(x) \rangle_{\psi} f^{ab}(x) = \frac{1}{2} \int \text{d}^4 x \text{d}^4 y \ F(x,y) \delta^{(4)}(x-y) \]

\[ = \int_{k_0 \geq 0} \frac{\text{d}^4 k}{(2\pi)^4} \int \text{d}^4 x \text{d}^4 y \ e^{-i k \cdot (x-y)} F(x,y) , \]

where we have used the Fourier representation of the Dirac-\( \delta \) and symmetry of \( F \) to restrict the outer domain of integration. Now the inner integral is \( A(k; \psi) - A(k; \psi_0) \), where

\[ A(k; \psi) := \sum_j \left\| \int \text{d}^4 x \ e^{ik \cdot x} g(x)^{1/4} P_j \Phi(x) \psi \right\|^2 \geq 0 , \]

and so we obtain the QEI

\[ \int \text{dvol}_g(x) \langle \mathcal{T}_{ab}(x) \rangle_{\psi} f^{ab}(x) \geq - \int_{k_0 \geq 0} \frac{\text{d}^4 k}{(2\pi)^4} A(k; \psi_0) , \quad (4) \]
the left-hand side of which depends on the reference state \( \psi_0 \), but not on \( \psi \).

The key point now is that the microlocal form of the Hadamard condition entails that \( A(k; \psi_0) \) is of rapid decay in the half-space \( k_0 \geq 0 \). Thus the integral on the right-hand side of (4) exists and is finite. We conclude that the real linear scalar field obeys a QEI with respect to the class of weights delineated by \( \mathcal{K} \) and the class of Hadamard states. The same argument would apply to a suitable class of adiabatic states \( \mathcal{A} \) in which one replaces the smooth wave-front set by a wave-front set modulo Sobolev regularity.

Note that this QEI applies to the normal ordered stress-energy tensor, rather than the renormalised tensor.\(^7\) By adding a term to the both sides which depends on the renormalised stress-energy tensor in state \( \psi_0 \) and certain other smooth local geometric terms, this defect can be remedied. (The bound is then typically not a ‘closed form’ expression.)

3.3 Macroscopic Stability: Passivity

Pusz and Woronowicz introduced the notion of passivity in the following way \([40]\). Let \((\mathcal{A}, \alpha_t)\) be a \(C^*\)-dynamical system; that is, \(\mathcal{A}\) is a \(C^*\)-algebra, which we think of as the algebra of observables for some quantum system, while \(\alpha_t\) is the map of evolution through time \(t \in \mathbb{R}\) corresponding to the undisturbed evolution of the system, and has the group property \(\alpha_t \circ \alpha_{t'} = \alpha_{t+t'}\). Provided \(\alpha_t\) is strongly continuous (i.e., the map \(\mathbb{R} \ni t \mapsto \alpha_t(A) \in \mathcal{A}\) is continuous for each \(A \in \mathcal{A}\)) we may define the generator \(\delta\) of the evolution by

\[
\delta(A) = \left. \frac{d}{dt} \alpha_t(A) \right|_{t=0}
\]

for the space of \(\mathcal{A}\) for which the derivative exists (in \(\mathcal{A}\)), which we denote \(D(\delta)\). For example, if \(\mathcal{A}\) is the algebra of bounded operators on the Hilbert space of a quantum mechanical system with Hamiltonian \(H\), then \(\delta(A) = i[H,A]\). We also have \(\delta(A) = \alpha_t^{-1} \left( \frac{d}{dt} \alpha_t(A) \right)\) for any \(t\). The motivating idea of \([40]\) is to understand thermodynamic stability of the dynamical system with respect to cyclical changes of external conditions. One might think of a box of gas which is compressed and then allowed to return to its initial volume. In the current setting, a cyclical process occurring during time interval \([0,T]\) may be modelled by a perturbed time evolution \(\beta_t\) satisfying

\[
\beta_t^{-1} \left( \frac{d}{dt} \beta_t(A) \right) = \delta(A) + i[h_t, A],
\]

and \(\beta_0 = \text{id}\) where \(t \mapsto h_t\) is a differentiable assignment of a self-adjoint element \(h_t \in \mathcal{A}\) to each time \(t\), and \(h_t = 0\) for \(t \notin [0,T]\).

\(^7\)To form the renormalised tensor, we begin by splitting points as above, but then subtract appropriate derivatives of the locally determined Hadamard parametrix, rather than the two-point function of a reference state.
Suppose the system is initially in state \( \omega \). Then the work performed by the external agent driving the cyclical process is

\[
W_h = \int_0^T dt \omega(\beta_t(h_t)),
\]

and the state \( \omega \) is said to be passive if \( W_h \geq 0 \) for any \( h_t \), i.e., if no cyclical process can extract energy from the system. Thus passivity isolates the property characteristic of the second law of thermodynamics in Kelvin's formulation, where we think of the system as a thermal reservoir from which we attempt to extract work.

Pusz and Woronowicz proved

**Theorem 1.** A state \( \omega \) is passive if and only if

\[
i^{-1} \omega(U^* \delta(U)) \geq 0
\]

for all \( U \in \mathcal{U}_1(\delta) := \mathcal{U}_1(A) \cap D(\delta) \), where \( \mathcal{U}_1(A) \) is the identity-connected component of the unitary elements of \( A \).

Particular examples of passive states are provided by ground and KMS states, or mixtures thereof. A key feature of passivity is that it introduces a definite thermodynamic ‘arrow of time’.

### 3.4 From Mesoscopic to Macroscopic

Let us now see how passivity may be obtained from QEIs, giving a simplified and slightly modified version of the discussion in [13]. We begin by introducing an abstract formulation of QEIs for \( C^* \)-dynamical systems, to which end we must first provide a notion of the energy density. Accordingly, we assume that \( A \) is the algebra of observables of a system in a spacetime of the form \( \mathbb{R} \times \Sigma \), for \( \Sigma \) compact and Riemannian, with volume measure \( d\mu(x) \). The evolution \( \alpha_t \) corresponds to time-translations on spacetime and is assumed to be strongly continuous with generator \( \delta \).

As one would not expect the energy density to exist for all states, we must specify a smaller class of states and a class of unitary elements large enough to be dense in \( \mathcal{U}_1(\delta) \), in a suitable sense, but which preserves the state space. Accordingly, let \( \mathcal{O} \) be a *-subalgebra of \( A \) with \( 1 \in \mathcal{O} \subset \bigcap_n D(\delta^n) \), and is large enough that any element of \( \mathcal{U}_1(\delta) \) may be approximated arbitrarily well by unitary elements of \( \mathcal{O} \) with respect to the graph norm of \( \delta \). That is, to any \( U \in \mathcal{U}_1(\delta) \) there is a sequence of unitaries \( U_n \in \mathcal{O} \) with \( U_n \to U \) and \( \delta(U_n) \to \delta(U) \). In addition, let \( \mathcal{S} \) be a convex set of states of \( A \) which is closed under operations in \( \mathcal{O} \).

The energy density \( \varrho(t,x) \) is assumed to obey:

\[8\]
That is, for any \( 0 \neq A \in \mathcal{O} \) and \( \omega \in \mathcal{S} \), we have \( \omega(A^*A) > 0 \) and \( \omega^A(B) = \omega(A^*BA)/\omega(A^*A) \) defines a state \( \omega^A \in \mathcal{S} \).
1. For each $A, B \in \mathcal{O}$ and $\varphi \in S$, $\varphi(\varrho(t, \underline{x})B)$ is a $C^1$ function on $\mathbb{R} \times \Sigma$.

2. The energy density generates the dynamics, and energy is conserved, i.e,

$$\int_{\Sigma} d\mu(\underline{x}) \varphi(A[\varrho(t, \underline{x}), B] C) = \frac{1}{i} \varphi(A \delta(B) C)$$

for arbitrary $A, B, C \in \mathcal{O}, \varphi \in S$ and $t \in \mathbb{R}$.

Here, expressions of the form $\varphi(\varrho(t, \underline{x})B)$ should be taken as a convenient shorthand: what is more precisely meant is the following. Let $F$ be the subspace of continuous linear functionals on $\mathcal{A}$ generated by functionals of the form $C \mapsto \varphi(A \varrho(B) C) := \varphi(ABC)$ (for $A, B \in \mathcal{O}, \varphi \in S$). Then the energy density is a linear map $\varrho : F \to C^1(\mathbb{R} \times \Sigma)$, and our shorthand notation $\varphi(\varrho(t, \underline{x})B)$ means $(\varrho(A \varrho(B)))(t, \underline{x})$.

We are now in a position to define a general type of QWEI in this setting, by analogy with the result (1). Our definition differs slightly from that given in [18].

**Definition 2.** Let $\mathcal{W}$ be a class of nonnegative integrable functions of compact support on $\mathbb{R}$. The system $(A, \alpha_t, \mathcal{O}, S, \varrho)$ obeys a static QWEI (SQWEI) with respect to $\mathcal{W}$ if, for some $\omega \in S$, there exists a map $q_\omega : \mathcal{W} \to L^1(\Sigma)$ such that

$$\int_{\Sigma} d\mu(\underline{x}) \varphi(\varrho(t, \underline{x}):) \geq -q_\omega(f)(\underline{x})$$

$\mu$-a.e. in $\underline{x}$

(6)

for all $\varphi \in S$, where $:\varrho = \varrho - \omega(\varrho)1$. (In this case, the same is true for all $\omega' \in S$, as we may take $q_{\omega'}(f)(\underline{x}) = q_\omega(f)(\underline{x}) + \int dt f(t) \omega'(\varrho(t, \underline{x})):)$)

We now state and prove one of the main results of [18].

**Theorem 2.** If $(A, \alpha_t, \mathcal{O}, S, \varrho)$ obeys a SQWEI then $(A, \alpha_t)$ admits at least one passive state.

**Proof.** Fix a reference state $\omega \in S$ and choose $f \in \mathcal{W}$ with $\int dt f(t) = 1$ (we may assume $\mathcal{W}$ is conic without loss). For unitary $U \in \mathcal{O},$

$$\frac{1}{i} \omega(U^* \delta(U)) = \int_{\Sigma} d\mu(\underline{x}) \omega(U^*[\varrho(t, \underline{x}), U])$$

$$= \int_{\Sigma} d\mu(\underline{x}) \int dt f(t) \omega(U^*[\varrho(t, \underline{x}), U])$$

$$= \int_{\Sigma} d\mu(\underline{x}) \int dt f(t) \omega(U^*: \varrho(t, \underline{x}) U)$$

$$\geq - \int_{\Sigma} d\mu(\underline{x}) q_\omega(f)(\underline{x}),$$

(7)

where we apply (9) with $\varphi$ defined by $\varphi(A) = \omega(U^* A U)$. Because unitary elements of $\mathcal{O}$ provide arbitrarily good approximations to elements of $U_1(\delta)$ we may choose unitaries $U_n \in \mathcal{O}$ such that
\[
\frac{1}{t} \omega(U_n^* \delta(U_n)) \longrightarrow c_\omega := \inf_{U \in \mathcal{U}_1(\delta)} \frac{1}{t} \omega(U^* \delta(U)),
\]
(8)
as \(n \to \infty\), thereby deducing that
\[
c_\omega \geq - \int_{\Sigma} d\mu(x) q_\omega(f)(x) > -\infty.
\]
(9)

If \(c_\omega \geq 0\) then \(\omega\) is passive and we are done, so suppose instead that \(c_\omega < 0\).
By the Banach–Alaoglu Theorem there exists a state \(\omega^p\) on \(\mathcal{A}\) and a subnet \(U_n(\sigma)\) of the \(U_n\) such that
\[
\omega^p(\mathcal{A}) = \lim_{\sigma} \omega(U^*_n(\sigma)\mathcal{A}U_n(\sigma)) = A \in \mathcal{A}.
\]
(10)

To complete the proof, we calculate
\[
\frac{1}{t} \omega^p(U^* \delta(U)) = \lim_{\sigma} \frac{1}{t} \omega(U^*_n(\sigma)U^* \delta(U)U_n(\sigma))
\]
\[
= \lim_{\sigma} \left[ i^{-1} \omega(UU_n(\sigma)) \omega(U^*_n(\sigma)\delta(U_n(\sigma))) - i^{-1} \omega(U^*_n(\sigma)\delta(U_n(\sigma))) \right] \geq c_\omega \to c_\omega
\]
\[
\geq 0,
\]
(11)
so \(\omega^p\) is passive. □

In [18] we also defined the notion of a state \(\omega\) being quiescent, in terms of the behaviour of function \(q_\omega(f_\lambda)\) in the limit \(\lambda \to 0^+\), where \(f_\lambda(t) = f(\lambda t)\).
We showed that quiescent states are passive (and even ground states, under additional clustering assumptions).

Of course, we would like to see that this abstract set-up can be realised in practice, and in particular, that it applies to quantum field theory in static spacetimes with compact spatial section. Here, we encounter a problem with the scalar field because its \(C^*\)-algebraic description in terms of the Weyl algebra with generators \(W(F)\) is not a \(C^*\)-dynamical system with respect to the time-translations
\[
\alpha_t W(F) = W(F_\lambda) \quad \text{where} \quad F_\lambda(\tau, x) = F(\tau - t, x).
\]
(12)
(This problem would not occur with the Dirac field, but less was known about Dirac QEIs when [18] was written!) Instead one can generate \(\mathcal{A}\) from objects of the form
\[
\int dt h(t) \alpha_t W(F) \quad (h \in C_0^\infty(\mathbb{R})),
\]
(13)
formed in quasifree Hadamard Hilbert space representations of the Weyl algebra; as shown in [18], all the requirements of the abstract setting are fulfilled with \(S\) equal to the set of finite convex combinations of Hadamard states occurring as vectors in quasifree Hadamard representations of the Weyl algebra.
(Microlocal techniques turn out to be exactly the right tools for this nontrivial
check.) The \( \ast \)-algebra \( \mathcal{O} \) is generated by operators of the form \( \exp iA \), where \( A = A^\ast \) is a polynomial in objects of the type (13).

A further problem, however, is that the passive state obtained from the Banach–Alaoglu theorem lives on \( \mathcal{A} \), rather than the Weyl algebra itself. Given sufficient regularity (e.g., energy compactness, believed to hold for this theory) we may reconstruct a passive state on the Weyl algebra [18]. Again, this problem would not arise for the Dirac field.

### 3.5 From Macroscopic to Microscopic

Finally, we briefly discuss the last link in our circle of stability conditions. In [43], Sahlmann and Verch considered general topological \( \ast \)-dynamical systems and defined a strictly passive state to be a mixture of ground and KMS states (at possibly different inverse temperatures). Note that this is a stronger requirement than the usual notion of passivity, as employed in [10, 18]. They also introduced the notion of an asymptotic \( n \)-point correlation spectrum which generalises the wave-front set to this setting, and formulated an appropriate generalisation of the microlocal spectrum condition. When applied to linear quantum field theory on stationary spacetimes, with respect to the stationary time evolution, the original microlocal spectrum condition is recovered. They then proved that strictly passive states obey the generalised microlocal spectrum condition: the key ingredient in their argument is that both (strict) passivity and the microlocal spectrum conditions share a common arrow of time.

### 4 Connections with Nuclearity

Quite recently, evidence has emerged to suggest the existence of a connection between QEIs and nuclearity criteria, with possibly far-reaching implications. We will consider the original nuclearity condition of Buchholz and Wichmann [3] (for other closely related criteria see, e.g., [4]). We work within the algebraic approach to quantum field theory [27], and consider a quantum field theory described by a Hilbert space \( \mathcal{H} \), a strongly continuous unitary representation \( g \mapsto U(g) \) on \( \mathcal{H} \) of the universal cover of the proper orthochronous Poincaré group \( \tilde{\mathcal{P}} \), and a net of von Neumann algebras \( \mathcal{R}(\mathcal{O}) \), consisting of bounded operators on \( \mathcal{H} \) and indexed by open bounded contractible spacetime regions \( \mathcal{O} \). The following axioms are assumed to hold: isotony (\( \mathcal{O}' \subset \mathcal{O} \) implies \( \mathcal{R}(\mathcal{O}') \subset \mathcal{R}(\mathcal{O}) \)); covariance \( \mathcal{R}(g\mathcal{O}) U(g)^{-1} = \mathcal{R}(g\mathcal{O}) \) for \( g \in \tilde{\mathcal{P}} \); locality \( \mathcal{R}(\mathcal{O}) \) and \( \mathcal{R}(\mathcal{O}') \) commute if \( \mathcal{O} \) and \( \mathcal{O}' \) are spacelike separated) and the spectrum condition (the generators of spacetime translations, \( P_\mu \), associated with the representation \( U \), are self-adjoint operators such that \( P_0 \) and \( P_0^2 - P_1^2 - P_2^2 - P_3^2 \) are positive). Finally, we assume the existence of a unique vacuum state: namely, that the Hamiltonian \( H = P_0 \) has a simple eigenvalue at zero with normalised eigenvector \( \Omega \).
Given any double cone $\mathcal{O}_r$ based on a ball of radius $r$ and any $\beta > 0$, let

$$\mathcal{N}_{\beta,r} = \{ e^{-\beta H} W \Omega : W \in \mathcal{R}(\mathcal{O}_r) \text{ s.t. } W^* W = 1 \} .$$

(14)

This set may be regarded as the set of local vacuum excitations associated with $\mathcal{O}_r$, damped exponentially in the energy. The theory is said to obey the condition of nuclearity if, firstly, each $\mathcal{N}_{\beta,r}$ is a nuclear subset of $\mathcal{H}$ [see below] and, secondly, there exist positive constants $c, n, r_0$ and $\beta_0$ so that the corresponding nuclearity index $\nu(\mathcal{N}_{\beta,r})$ obeys

$$\nu(\mathcal{N}_{\beta,r}) \leq \exp(c r^3 \beta^n)$$

(15)

for all $0 < \beta < \beta_0$ and $r > r_0$. This condition is therefore a restriction on the number of local degrees of freedom available to the theory.

In the above, a subset $\mathcal{L}$ of $\mathcal{H}$ has nuclearity index $\nu(\mathcal{L}) = \inf \text{Tr}[T]$, where the infimum is taken over the set of trace-class operators $T$ so that $\mathcal{L}$ is contained within the image of the unit ball $\mathcal{H}(1)$ of $\mathcal{H}$ under $T$, and $\mathcal{L}$ is said to be nuclear if it has a finite nuclearity index.\(^9\)

Despite its rather technical definition, the condition of nuclearity is well-motivated from a physical viewpoint as the discussion in [3] makes plain: the nuclearity index can be interpreted as a local partition function, and the form of the nuclearity bound (15) is suggested by the requirement that the associated pressure should remain finite in the thermodynamic limit and scale polynomially with temperature (as is the case, for example, in the Stefan–Boltzmann law).

Buchholz and Wichmann verified in [3] that the massive free scalar field satisfies the condition of nuclearity, and remark that the same is true of the system of countably many fields with masses $m_j$ given suitable conditions on the density of states. Namely, the sets $\mathcal{N}_{\beta,r}$ are nuclear if [3] and only if [5] $\sum_j \exp(-\beta m_j) < \infty$ for all sufficiently small $\beta$; furthermore, the nuclearity index may be estimated from above by

$$\nu(\mathcal{N}_{\beta,r}) \leq \exp\left(c \left(\frac{r}{\beta}\right)^3 \sum_j \left| \log(1 - e^{-\beta m_j/2}) \right| \right)$$

(16)

for all sufficiently large $r$ and small $\beta$, and some constant $c$. It is convenient to introduce $N(u)$, the number of particle species with mass below $u$ by

$$N(u) = \sum_j \vartheta(u - m_j) .$$

(17)

The assumption that $N(u)$ grows polynomially, $N(u) = O(u^p)$ as $u \to \infty$, is sufficient to show (using (16)) that (15) is satisfied, for any $n > 3 + p$.

\(^9\)By convention, an infimum over an empty set is infinite, so this amounts to the assertion that there does exist a trace-class $T$ with $\mathcal{L} \subset T\mathcal{H}(1)$.
It is tempting to conjecture that this condition is also necessary, but this is currently an open question, and relies on finding better lower bounds on the nuclearity index than are currently known. We will return to this point below.

We now present some circumstantial evidence for a connection between nuclearity criteria and QEIs. Fix some inertial frame of reference in Minkowski space and let $\rho_j$ be the energy density of the free field of mass $m_j$ with Hilbert space $H_j$ and vacuum state $\Omega_j$. Let $\text{Had}_j \subset H_j$ be the corresponding space of Hadamard vector states. The Hilbert space of the full theory is the tensor product

$$H = \bigotimes_j \Omega_j \otimes H_j ; \quad (18)$$

that is, the completion with respect to the obvious inner product of the set of finite linear combinations of product states $\bigotimes_j \xi_j$ in which all but finitely many of the $\xi_j$ are equal to $\Omega_j$. We define the space of Hadamard states $\text{Had}$ of the full theory to consist of finite linear combinations of product states $\bigotimes_j \xi_j$ in which each $\xi_j \in \text{Had}_j$ and all but finitely many $\xi_j$ equal $\Omega_j$, and then define the total energy density as follows: for any $\eta = \bigotimes_j \eta_j$ and $\xi = \bigotimes_j \xi_j$ in $\text{Had}$ we set

$$\langle \eta \mid g(x) \xi \rangle = \sum_j \langle \eta_j \mid g_j(x) \xi_j \rangle \prod_{k \neq j} \langle \eta_k \mid \xi_k \rangle \quad (19)$$

(noting that only finitely many terms contribute to the sum, and that each product involves only finitely many terms differing from unity) and then extend by linearity to all $\eta, \xi \in \text{Had}$. The left-hand side should be regarded as a quadratic form on $\text{Had}$, taking values in the space of smooth functions on spacetime; clearly, any normal-ordered quantity could be treated in this way, and no constraints on the $m_j$ have been imposed. Since the $j$'th component of the full theory obeys the QWEI $\Pi$ for each mass $m_j$,

$$\int dt \, |g(t)|^2 \langle \psi_j \mid g_j(t,0) \psi_j \rangle \geq - \frac{\|\psi_j\|_{\mathcal{H}_j}^2}{16\pi^3} \int_0^\infty du \, |\hat{g}(u)|^2 u^4 N(u - m_j) , \quad (20)$$

for all Hadamard states $\psi_j \in \text{Had}_j$, the full theory obeys

$$\int dt \, |g(t)|^2 \langle \psi \mid g(t,0) \psi \rangle \geq - \frac{1}{16\pi^3} \int_0^\infty du \, |\hat{g}(u)|^2 u^4 N(u) , \quad (21)$$

for any normalised $\psi \in \text{Had}$. Accordingly, polynomial growth of $N$ is sufficient for the theory to admit a worldline QWEI with test-functions $g$ drawn from $C_0^\infty(\mathbb{R})$, and it is possible to show that it is a necessary and sufficient condition if certain scaling behaviour is required:

\[10\]

If $N(u)$ grows faster than polynomially, one may still formulate QWEIs, but for weight functions with sufficiently rapid decay in Fourier space. In particular, this would generally exclude compactly supported weights.
**Theorem 3.** Consider a generalised free field with discrete mass spectrum described by $N(u)$. Let $p > 0$. Then the following are equivalent:

1) $N(u) = O(u^p)$ as $u \to \infty$;

2) The generalised free field obeys the QWEI (21) for arbitrary $g \in C_0^\infty(\mathbb{R})$, and the bound has asymptotic behaviour of order $O(\tau^{-(p+4)})$ as $\tau \to 0^+$, if we replace $g$ by $g_\tau(t) = \tau^{-1/2}g(t/\tau)$.

The proof of this result will be reported elsewhere. An immediate corollary is that the existence of a QWEI with polynomial scaling implies that the Buchholz–Wichmann nuclearity condition (15) is satisfied for any $n > p + 3$.

All this raises two questions, which are being pursued in on-going work with Porrmann and Ojima. First, can we show that (15) implies that $N(u)$ is polynomially bounded? If so, we would have an equivalence between QWEIs and nuclearity for this model. This leads to the second question: Can we understand the link at a deeper level, or is it merely a coincidence, with no more significance than that both are manifestations of the uncertainty principle? A suitable understanding of this question might lead to a general framework for establishing QEIs in general quantum field theories. Part of the problem is to identify the right question, of course, and it may be that one or both of nuclearity or QEIs need to be carefully (re)phrased or even replaced. These questions also require consideration of lower bounds on nuclearity indices: here a potential stumbling block is the technical definition of many of the quantities appearing in discussions of nuclearity, which are therefore not easily amenable to direct calculation even in the simplest cases. Indeed this provides pitfalls for the unwary, one of which we have recently noted (13): in the mathematical literature there is a notion of $p$-nuclear map, whose definitions for $p > 1$ and $p \leq 1$ take rather different forms. Although this difference has occasionally been noted in the physics literature (48), one often finds the $p \leq 1$ definition used for all $p$. However, as we show in (13), the corresponding nuclearity index would vanish identically for $p > 1$ according to this definition! Fortunately this confusion does not appear to have adverse consequences in the literature so far, but it serves as a warning.

5 Conclusion

Quantum Energy Inequalities are an expression of the uncertainty principle, and as such are deeply rooted within quantum theory. It is perhaps not surprising that they have connections with other fundamental properties: unravelling these interconnections has the potential to deepen our understanding of the structure of quantum field theory and the nature of quantised matter. Much remains to be done!

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