Bianchi IX dynamics in bouncing cosmologies: homoclinic chaos and the BKL conjecture

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Abstract

We examine the dynamics of a Bianchi IX model with three scale factors on a 4-dim Lorentzian brane embedded in a 5-dim conformally flat empty bulk with a timelike extra dimension. The matter content is a pressureless perfect fluid restricted to the brane, with the embedding consistently satisfying the Gauss–Codazzi equations. The 4-dim Einstein equations on the brane reduce to a 6-dim Hamiltonian dynamical system with additional terms (due to the bulk–brane interaction) that avoid the singularity and implement nonsingular bounces in the model. We examine the complex Bianchi IX dynamics in its approach to the neighborhood of the bounce which replaces the cosmological singularity of general relativity. The phase space of the model presents (i) two critical points (a saddle-center–center and a center–center–center) in a finite region of phase space, (ii) two asymptotic de Sitter critical points at infinity, one acting as an attractor to late-time acceleration and (iii) a 2-dim invariant plane, which together organize the dynamics of the phase space. The saddle–center–center engenders in the phase space the topology of stable and unstable 4-dim cylinders $R \times S^3$, where $R$ is a saddle direction and $S^3$ is the center manifold of unstable periodic orbits, the latter being the nonlinear extension of the center–center sector. By a proper canonical transformation the degrees of freedom of the dynamics are separated into one degree connected with the expansion/contraction of the scales of the model, and two rotational degrees of freedom associated with the center manifold $S^3$. The typical dynamical flow is thus an oscillatory mode about the orbits of the invariant plane. The stable

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and unstable cylinders are spanned by oscillatory orbits about the separatrix towards the bounce, leading to the homoclinic transversal intersection of the cylinders, as shown numerically in two distinct simulations. The homoclinic intersection manifold has the topology of $R \times S^2$ consisting of homoclinic orbits biasymptotic to the center manifold $S^3$. This behavior defines a chaotic saddle associated with $S^3$, indicating that the intersection points of the cylinders have the nature of a Cantor set with compact support $S^2$. This is an invariant signature of chaos in the model. We discuss the connection between these properties of the dynamics, namely the oscillatory approach to the bounce together with its chaotic behavior, and analogous features present in the BKL conjecture in general relativity.

Keywords: Bianchi IX dynamics, homoclinic chaos, bouncing cosmology

1. Introduction

The general Bianchi IX model has become a paradigm for the behavior of the general relativity dynamics near the cosmological singularity since the seminal papers of Belinskii, Khalatnikov and Lifshitz (BKL) [1–3], and collaborators [4, 5]. They showed that in a Bianchi IX model with three scale factors the approach to the singularity ($t \to 0$) is an oscillatory mode, consisting of an infinite sequence of Kasner eras in each of which two of the scale factors oscillate while the third decreases monotonically. On passing from one era to another (with decreasing time $t$) the monotonic behavior is transferred to another of the three scale factors. It was also shown that (i) the length of each era was determined by a sequence of numbers $x_s, 0 < x_s < 1, s = \text{integer, each of which arises from the preceding one by the map } x_{s+1} = \text{the fractional part of } 1/x_s,$ with the length of the $s$th era given by $k_s = \text{the integral part of } 1/x_s;$ (ii) this map leads to spontaneous stochastization in the sequence of eras on approaching the singularity ($t \to 0$) for arbitrary initial conditions given at $t > 0$. Due to the involved nonintegrable dynamics the evolution of the model had to be actually treated in asymptotic regions of arbitrarily small times together with truncations made to guarantee the validity of the perturbation method, so that ‘in the most general case all details of such regime are not yet fully understood’ [1].

In the past four decades the dynamics of these models has been reexamined and discussed in an extensive literature (see for instance [6–8] and references therein). In particular the advent of powerful numerical resources renewed the interest in the chaotic behavior of these models but—as in the BKL work—the approach consisted basically in obtaining maps which approximate the dynamics of the Einstein equations and which exhibit strong stochastic properties [9]. The examination of how well these discrete maps represent the full nonlinear dynamics of Bianchi IX models in general relativity has been the object of much research, particularly by Berger [10], Rugh [11] and Berger et al [12]. From the point of view of the phase space flow the interest in the chaoticity of Bianchi IX models has been mainly focused on the mixmaster Universe (the vacuum Bianchi IX case with three scale factors [13]). The chaoticity in the mixmaster dynamics has been the object of much dispute in the literature (see for instance the contributions to the section ‘Bianchi IX (Mixmaster) dynamics’ in [6]). Latifi et al [14] and Contopoulos et al [15] have shown the nonintegrability of the mixmaster model in the Painlevé sense, although the question of the generic behavior (chaotic or not) remained unsettled due to the absence of an invariant (or topological) characterization of chaos in the model (standard chaotic indicators as Liapunov exponents being coordinate dependent and hence questionable [6, 16]). Later Cornish and Levin [17] proposed to quantify
chaos in the mixmaster Universe by calculating the dimensions of fractal basin boundaries in initial condition sets for the full dynamics, these boundaries being defined by the code associated with one of the three outcomes on which one of the three axes is collapsing more quickly, as established numerically. Their result had a critical review that nevertheless endorses it [18]. Also [19–21] examined the influence of scalar fields in suppressing or enhancing the mixmaster oscillations. More recently Rendall [22] and Heinzel and Uggla [8] made a detailed critical study of the mixmaster dynamics, there including rigorous results connected to the Bianchi type IX attractor theorem [23, 24] and the Hamiltonian Cosmological Billiard [25] (that approaches the asymptotic mixmaster dynamics as a Hamiltonian billiard for arbitrary $d \geq 4$ spacetime dimensions). Their fine analysis concludes that, although there are indications that the chaotic Kasner map actually describes the asymptotic dynamics of the Einstein equations, so far nothing is rigorously known about dynamical chaotic properties of Bianchi type IX models. Therefore, along with the cosmic censorship conjecture [26], the BKL conjecture still remains one of the major unsolved issues of classical general relativity connected to the presence of a singularity in the dynamics.

Our main purpose in the present paper is to examine the dynamics of a 4-dim Bianchi IX model with three scale factors in the framework of a braneworld formalism (which encompasses general relativity as a classical low-energy limit). Due to an extra timelike dimension, brane-bulk interaction terms correct general relativity substituting the singularity by non-singular bounces in the cosmological dynamics. The dynamics of the approach to the bounces is extremely complex presenting oscillatory and chaotic features of the BKL-type but, as we will show, they are amenable to an exact analytical/numerical treatment so that we may have a more clear picture of what happens in the general relativity limit. We remark that the substitution of the singularity by a nonsingular bounce is a crucial ingredient to our analytical/numerical dynamical system analysis.

One of the main approaches to the problem of the initial singularity and the possible solutions to circumvent this problem lie in the realm of a quantum theory of gravitation. In fact we may consider that the initial conditions of our present expanding Universe were fixed when the early Universe emerged from a Planckian regime and started its classical evolution. However, by evolving back the initial conditions using the Einstein-classical equations the Universe is driven toward a singular point where the classical regime is no longer valid [27]. This is an indication that classical general relativity is not a complete theory and in this domain quantum processes must be taken into account. Among several propositions to describe the dynamics in the semiclassical domain prior to the classical regime are, for instance, loop quantum cosmology [28] and the string based formalism of D-branes [29], both of them leading to corrections to the Einstein equations and to the possibility of bounces, and encompassing general relativity as a classical (low energy) limit. Since the late 90s, braneworld models based on [30] and [31] have been developed as a promising field of research. For example, it has been shown [30] that the compactification of extra dimensions could shed some light on the hierarchy problem.

We should remark that in the domain of pure general relativity there exists a large literature on the realization of bounces without having to appeal to quantum gravity corrections [32]. We mention here some basic references where the bounce is realized (i) with one or two quintom scalar fields [33], (ii) with a scalar field which undergoes ghost condensation [34], (iii) with f(T) gravity [35], (iv) in a non-relativistic gravitational theory [36], (v) with galileon cosmologies [37], (vi) in a extended nonlinear massive gravity [38], (vii) spatial anisotropy in a ekpyrotic scenario [39], (viii) with higher-derivative modifications of (effective) gravity [40], (ix) pure general relativity with radiation plus a negative energy scalar field [41], (x) with a pre-big bang ekpyrotic phase combined with a ghost condensate (which
violates the null energy condition without developing any ghost-like instabilities [42] and finally (xi) with a simple massive scalar field in a general self-interacting potential and closed FRW metric, the bounce realization requiring some amount of fine tuning of the initial conditions [43]. We note that one common feature of these works is that the bounces are realized in a one scale factor (FRW) spacetime geometry and, in its majority, consisting of one bounce solution only. Recently Brechet et al [44] examined the dynamics of a homogeneous and irrotational Weyssenhoff fluid in general relativity, where the spin contributions allow for the presence of bounces in the dynamics, with some particular numerical solutions of the generalized scale factor exhibiting eternal oscillatory nonsingular behavior, while Dechant et al [45] showed that a Bianchi IX model with two scale factors only and a scalar field as source can realize a non-singular, anisotropic pre-big bang bounce in this biaxial case. Therefore there is not enough recurrence in the dynamics via a sufficient number of bounces in the models to make possible the emergence of chaotic properties in the dynamics.

In view of the above results we were led to look for other consistent physical scenarios where the dynamics of a Bianchi IX model with three scale factors could be realized and in which nonsingular bounces could be implemented for a large domain of parameters and initial conditions. In the present paper we adhere to the braneworld scenario with a noncompact extra time dimension. In this context, one extra time dimension is introduced by a bulk space and all the matter in the Universe would be trapped on a brane with three spatial dimensions; only gravitons would be allowed to leave the brane and move into the full bulk [48]. At low energies general relativity is recovered but at high energies significant changes are introduced in the gravitational dynamics. Our interest in this framework comes from the fact that—contrary to the case of extra spatial dimensions—only a timelike extra dimension may provide corrections that are dominant in the neighborhood of the singularity, resulting in a repulsive force which avoids it completely and leads the Universe to undergo nonsingular bounces. Furthermore this braneworld scenario incorporates naturally the dynamics of a three scale factors Bianchi IX model in a brane consistently embedded in a 5-dim conformally flat bulk, as we will see. Bouncing braneworld models were constructed by Shtanov and Sahni [47] based upon a Randall–Sundrum [31] type action with one extra timelike dimension. A complete analysis of bouncing braneworld dynamics embedded in a 5-dim de Sitter spacetime may be found in [49, 50], where both high energy local corrections as well as nonlocal bulk corrections are analyzed on a spatially homogeneous brane.

While spacelike extra dimensions theories have received more attention in the last decades [48], studies involving extra timelike dimensions have been considered [51] despite the fact that propagating tachyonic modes or negative norm states may arise due to timelike extra dimensions. These modes have been regarded as problematic once, as they might violate causality [52] by considering interactions among usual particles. Issues like the exceedingly small lower bound on the size of timelike extra dimensions [53], the imaginary self-energy of charged fermions induced by tachyonic modes (which seems to cause disappearance of fermions into nothing) and the spontaneous decay of stable particles induced by tachyons with negative energy are major difficulties [52]. Nevertheless, in order to address the cosmological constant problem in Kaluza–Klein theories [54] or reconcile a solution of the hierarchy problem with the cosmological expansion of the Universe [55], timelike extra dimensions have been considered. On the other hand, it has been shown in [56] that the appearance of massless ghosts in an effective 4-dim theory can be avoided by considering topological criteria in Kaluza–Klein theories with extra compactified time-like dimensions. Moreover, the avoidance of propagating tachyonic or negative norm states can also be achieved by considering a non-compact timelike extra dimension [57], which is the case in
the models of this paper. In the context of cosmological perturbations, it has also been shown [58] that there are stable bouncing models in braneworld scenario.

In the braneworld scenario considered in this paper, we assume a 5-dim conformally flat empty bulk with a timelike extra dimension, and a 4-dim Lorentzian brane with a Bianchi IX geometry with three scale factors. The matter content of the model is taken as a pressureless perfect fluid (dust) restricted to the brane and an effective nonvanishing cosmological constant is also considered. Given the above assumptions we restrict ourselves to a very particular case in which we show how the Gauss–Codazzi equations—the necessary and sufficient conditions for the embedding—are automatically satisfied, so that the embedding of a Bianchi IX brane in a conformally flat empty bulk is consistent. The modified Einstein equations for the model have a first integral that can be expressed as a Hamiltonian constraint, yielding a three degrees of freedom dynamical system in a 6-dim phase space. The additional correction terms due to the bulk-brane interaction avoid the initial singularity resulting instead in nonsingular bounces in the model.

Our main interest in this model comes from the fact that we can obtain a 6-dim phase space with orbits having a sufficiently large number of bounces (due to the timelike extra dimension only), for an extended domain of parameters and of initial conditions, which are connected to the presence of a saddle-center–center critical point in the phase space. This critical point is associated with a center manifold of unstable periodic orbits having the topology $S^3$. Such configuration allows a different approach to features of the BKL conjecture (oscillatory motion and chaos) as the trajectories in the phase space move towards the bounce, instead moving towards a singularity as in general relativity. We will show that from the center manifold it emerges stable and unstable manifolds with the topology of spherical cylinders $R \times S^3$ (constituted actually of bounded oscillatory orbits) which cross each other transversally in the neighborhood of the bounces. These transversal crossings provide an invariant characterization of homoclinic chaos in the model.

These results are in realm of recent studies in the characterization of homoclinic chaos for Hamiltonian dynamical systems with $n \geq 2$ degrees of freedom. For $n = 2$ the characterization of chaos connected with the presence of homoclinic phenomena in the dynamics has been the object of a large literature (see [59–64] and references therein). The dynamics near homoclinic orbits is very complex, with the homoclinic intersection manifold associated with the presence of the well-known horseshoe structures (see [65–67] and references therein), which is an invariant signature of chaos. Furthermore, invariant Cantor sets associated with a horseshoe construction are connected to chaotic saddles [68–70]. For $n \geq 2$ orbits homoclinic to the center manifold are expected to exist. It has been shown, for instance, that critical points of the type saddle-center–...-center induce reaction type dynamics in the framework of transition state theory (see [71] and references therein). The existence of such homoclinic orbits has been studied in [71, 72]. Although there are no theorems describing the dynamics connected with orbits homoclinic to $S^3$, it has been shown [73] that if there is a transversal intersection of the stable and unstable manifolds, a chaotic saddle, and hence a homoclinic trajectory must exist. An interesting analysis of this feature was given in [72], where the authors provide a computational procedure to detect a chaotic saddle (and thus homoclinic orbits) in the case of Hamiltonian systems with three degrees of freedom. In the present paper we follow an alternative procedure to show the presence of homoclinic connections with the center manifold $S^3$.

We organize the paper as follows. In the next section we present a brief introduction to the braneworld scenario considered in this paper, deriving the modified field equations on the brane. In section 3 we construct a general Bianchi IX cosmological brane model, with an effective cosmological constant and the matter content being dust. In section 4 we study the
structure of the phase space, identifying the constants for the linearized motion. In section 5 the dynamics about the saddle-center–center critical point is examined. Section 6 is devoted to a complete analysis of the nonlinear center manifold, together with the 4-dim stable and unstable cylinders that emanate from it. Finally in section 7 we study the homoclinic transversal intersections of the cylinders that gives an invariant characterization of chaos in the dynamics. Conclusions and future perspectives are presented in the final section.

2. The field equations

For sake of completeness we give here a brief introduction to the braneworld scenario considered, making explicit the specific assumptions used to obtain the dynamics of the model. We refer to [46–48] for a more complete and detailed discussion and our notation closely follows [27]. We start with a 4-dim Lorentzian brane \( \Sigma \) with metric \( (4)g_{ab} \), embedded in a 5-dim conformally flat bulk \( \mathcal{M} \) with metric \( (5)g_{AB} \). Capital Latin indices run from 0 to 4, small Latin indices run from 0 to 3. We regard \( \Sigma \) as a common boundary of two pieces \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) of \( \mathcal{M} \) and \( (4)g_{ab} \) is the induced geometry on the brane by the metrics of the two pieces. These metrics should coincide on \( \Sigma \) although the extrinsic curvatures of \( \Sigma \) with respect to \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) can be different. The action for the theory has the general form

\[
S = \frac{1}{2\kappa_5^2} \left\{ \int_{\mathcal{M}_1} \sqrt{\epsilon} (5)R - 2\Lambda_5 + 2\kappa_5^2 L_5 \right\} d^5x + \int_{\mathcal{M}_2} \sqrt{\epsilon} (5)R - 2\Lambda_5 + 2\kappa_5^2 L_5 \right\} d^5x + 2\epsilon \int_{\Sigma} \sqrt{-\epsilon} (4)g K_2 d^4x - 2\epsilon \int_{\Sigma} \sqrt{-\epsilon} (4)g K_1 d^4x \right\} + \frac{1}{2} \int_{\Sigma} \sqrt{-\epsilon} (4)g \left( \frac{1}{2\kappa_4^2} R - 2\sigma \right) d^4x + \int_{\Sigma} \sqrt{-\epsilon} (4)g L_4 (4)g_{ab}, \rho d^4x.
\]

(1)

In the previous equation, \( \hat{R} \) is the Ricci scalar of the Lorentzian 5-dim metric \( (5)g_{ab} \) on \( \mathcal{M} \), and \( ^4R \) is the scalar curvature of the induced metric \( (4)g_{ab} \) on \( \Sigma \). The parameter \( \sigma \) denotes the brane tension. The unit vector \( n^A \) is normal to the boundary \( \Sigma \) and has norm \( \epsilon \). If \( \epsilon = +1 \) the signature of the bulk space is \( (+, +, −, −, −) \), so that the extra dimension is timelike. The quantity \( K = K_{ab} (4)g^{ab} \) is the trace of the symmetric tensor of extrinsic curvature \( K_{ab} = \Gamma^C_{ab} Y^D, \nabla_C n_D, \) where \( Y^A (x^a) \) are the embedding functions of \( \Sigma \) in \( \mathcal{M} \) [74]. While \( L_4 (4)g_{ab}, \rho \) represents the Lagrangian density of the perfect fluid [75] (with equation of state \( p = \alpha \rho \)), whose dynamics is restricted to the brane \( \Sigma \), \( L_5 \) denotes the Lagrangian of matter in the bulk. All integrations over the bulk and the brane are taken with the natural volume elements \( \sqrt{-\epsilon} (5)g d^5x \) and \( \sqrt{-\epsilon} (4)g d^4x \) respectively. The Einstein constants in five and four dimensions are indicated with \( \kappa_5^2 \) and \( \kappa_4^2 \equiv 8\pi G_N \), respectively (\( G_N \) being Newton’s constant on the brane). Throughout this section we use natural units with \( \hbar = c = 1 \).

Variations that leave the induced metric on \( \Sigma \) intact, result in the equations

\[
(5)G_{AB} + \Lambda_5 (5)g_{AB} = \kappa_5^2 (5)T_{AB}.
\]

(2)
Considering arbitrary variations of $g_{AB}$ and taking into account equation (2), we obtain

$$
(4)G_{ab} + \frac{\kappa_5^2}{\kappa_2^2}(S_{ab}^{(1)} + S_{ab}^{(2)}) = \kappa_4^2(\tau_{ab} - \sigma g_{ab}),
$$

where $S_{ab} \equiv K_{ab} - K^{(4)}g_{ab}$, and $\tau_{ab}$ is the energy–momentum tensor on the brane. In the limit $\kappa_2^2 \gg \kappa_5^2$, equation (3) reduces to the Israel–Darmois junction conditions [76]

$$
(S_{ab}^{(1)} + S_{ab}^{(2)}) = \epsilon \kappa_2^2(\tau_{ab} - \sigma (4)g_{ab}).
$$

(4)

Imposing the $Z_2$-symmetry [46, 48] and using the junction conditions (equation (4)), we determine the extrinsic curvature on the brane

$$
K_{ab} = \frac{\epsilon}{2} \kappa_2^2 \left[ (\tau_{ab} - \frac{1}{3} \tau^{(4)} g_{ab}) + \frac{\sigma}{3} (4)g_{ab} \right].
$$

(5)

Now using Gauss equation

$$
(4)R_{abcd} = (5) R_{MNRS} Y^M_{\cdot a} Y^N_{\cdot b} Y^R_{\cdot c} Y^S_{\cdot d} - \epsilon \left( K_{ac} K_{bd} - K_{ad} K_{bc} \right),
$$

(6)

together with equations (2) and (5) we obtain the induced field equations on the brane

$$
(4) G_{ab} + \Lambda_4 (4) g_{ab} = 8\pi G_N \tau_{ab} - \epsilon \kappa_2^4 \Pi_{ab} + \epsilon E_{ab} + \epsilon F_{ab}.
$$

(7)

In the above $E_{ab} = (5) C_{ABCD} n^A Y^B_{\cdot a} n^C Y^D_{\cdot b}$ is the projection of the 5-dim Weyl tensor, and we have defined

$$
\Lambda_4 = \frac{1}{2} \kappa_2^2 \left[ \frac{\Lambda_5}{\kappa_5^2} - \frac{1}{6} \epsilon \kappa_2^2 \sigma^2 \right], \quad G_N = \epsilon \frac{\kappa_2^4 \sigma}{48\pi}
$$

(8)

$$
\Pi_{ab} = - \frac{1}{4} \tau_{a}^{\cdot a} \tau_{bc} + \frac{1}{12} \tau_{ab} + \frac{1}{8} (4) g_{ab} \tau^{cd} \tau_{cd} - \frac{1}{24} \tau^{2(4)} g_{ab},
$$

(9)

$$
F_{ab} = \frac{2}{3} \kappa_2^2 \left\{ \epsilon (5) T_{BD} Y^B_{\cdot a} Y^D_{\cdot b} - \left( (5) T_{BD} n^B n^D + \frac{1}{4} \epsilon (5) T \right) (4) g_{ab} \right\}.
$$

(10)

Here we stress that the effective 4-dim cosmological constant can be set to zero in the present case of an extra timelike dimension by properly fixing the bulk cosmological constant as $\Lambda_5 = \frac{1}{8} \kappa_5^4 \sigma^2$. It is important to notice that for a 4-dim brane embedded in a conformally flat empty bulk we have the absence of the Weyl conformal tensor projection $E_{ub}$ and of $F_{ab}$ in equation (7).

On the other hand, Codazzi equations imply that

$$
\nabla_a K - \nabla_b K_a^b = \frac{1}{2} \epsilon \kappa_2^2 \nabla_b \tau_{a}^b.
$$

(11)

By imposing that $\nabla_b \tau_{a}^b = 0$, the Codazzi conditions read

$$
\nabla_a E_{ab} = \kappa_5^4 \nabla^a \Pi_{ab} + \nabla^a F_{ab},
$$

(12)

where $\nabla_a$ is the covariant derivative with respect to the induced metric $(4)g_{ab}$. Equations (7) and (12) are the dynamical equations of the gravitational field on the brane. These equations are equivalent to the Gauss–Codazzi equations for the model, which are the sufficient and necessary conditions for the embedding [74] of the 4-dim brane in a 5-dim bulk. In the following section we drop the index (4) in the geometrical quantities on the brane.
3. The model

Inspired by Randall–Sundrum braneworld models, it has been shown \[77, 78\] that in the case of an anisotropic Bianchi I brane, the modified field equations on the brane include components related to projections of the 5-dim Weyl tensor. By solving the 5-dim field equations in the bulk a family of Bianchi braneworlds with anisotropy is obtained. The anisotropy on the brane enforces an anisotropy in the bulk which in turn backreacts on the brane dynamics through the Weyl term \( E_{ab} \) in equation (7). It turns out that in these models \[77, 78\], it is not possible to implement geometric anisotropy for a perfect fluid: the junction conditions induce an anisotropic stress pressure on the brane and hence, an anisotropy in the matter fields. Although this may be a peculiar feature of Bianchi I models adopted, it might be a characteristic of general anisotropic braneworld models.

Regarding the brane only, one cannot find the dynamical equations which govern the projected Weyl components \( E_{ab} \) on the brane. In fact, if the field equations in the full bulk space are not solved, the Weyl term remains unknown and it has to be prescribed ad hoc as done in \[79–83\]. In an analogous way we will use in this paper a braneworld framework where the prescription of a conformally flat empty bulk is adopted, and where the consistency of the embedding of the brane is guaranteed via the Gauss–Codazzi equations. Basically our use of a braneworld formalism with a timelike extra dimension was motivated by the presence of extra terms in the dynamics that allow for a complete examination of the extension of the BKL dynamics in the neighborhood of the bounce, since such terms replace the singularity of the general relativity case by a bounce in the braneworld case. Since the dynamics in the bounce tends to enhance the anisotropy we will not be able to compare our results with the behavior close to the singularity of \[79–83\].

The core of this paper is the analysis of the dynamics of bouncing Bianchi IX cosmological models in the braneworld framework, in which we provide an invariant signature of chaos. The structure of the phase space allows an oscillatory approach to the bounce, with analogous stochastic features present in the BKL conjecture in general relativity. As the field equations are extremely involved, in this first approach we shall not be concerned about 5-dim sources or the Weyl projected terms. That is, we will set \( E_{ab} = 0 = F_{ab} \) from the very beginning. We also assume a pressureless perfect fluid restricted to the brane which, as we shall show, will be supported by an anisotropic brane in our particular model. Although all these prescriptions may seem ad hoc, we will show that the necessary and sufficient embedding conditions—namely the Gauss–Codazzi equations—can still be satisfied, implying a consistent embedding in a particular conformally flat empty bulk.

Let us then consider a Bianchi IX spatially homogeneous geometry on the 4-dim brane embedded in a 5-dim, conformally flat and empty bulk (setting \( E_{ab} = 0 = F_{ab} \)) with a timelike extra dimension (\( \epsilon = 1 \)). In comoving coordinates on the brane, the line element can be expressed as

\[
\text{d}s^2 = \text{d}t^2 - (\theta^1)^2 - (\theta^2)^2 - (\theta^3)^2,
\]

where \( t \) is the cosmological time and

\[
\theta^1 = M(t) \, \omega^1, \quad \theta^2 = N(t) \, \omega^2, \quad \theta^3 = R(t) \, \omega^3.
\]

Here \( M(t), N(t) \) and \( R(t) \) are the scale factors of the model and the \( \omega^i \) (\( i = 1, 2, 3 \)) are Bianchi-type IX one-forms satisfying
\[
\dot{\omega}^i = \frac{1}{2} \epsilon^{ijk} \omega^j \wedge \omega^k, \tag{15}
\]
where \(d\) denotes the exterior derivative.

The matter content of the model is assumed to be dust, whose energy density \(\rho\) is measured by the comoving observers with four-velocity \(u^a = \delta^a_0\). By imposing that the energy–momentum tensor of dust

\[
\tau^{ab} = \rho u^a u^b, \tag{16}
\]
is conserved separately, namely \(\nabla_a \tau^{ab} = 0\), we obtain

\[
\rho = \frac{C_0}{MNR}, \tag{17}
\]
where \(C_0\) is a constant of motion connected to the dust energy. This implies that the equivalent Codazzi equations (12) are automatically satisfied as

\[
\nabla_a \Pi^{ab} = 0. \tag{18}
\]

On the other hand, by setting \(E_{ab} = 0 = F_{ab}\), equation (7) reduces to

\[
G_{ab} + \Lambda g_{ab} = 8\pi G_N \tau_{ab} - \kappa_5^2 \Pi_{ab}. \tag{19}
\]

These are the modified Einstein field equations for the model. As the equivalent Gauss–Codazzi equations (18) and (19) are supposed to hold the necessary and sufficient conditions for the embedding are automatically satisfied together with the assumption of a conformally flat empty bulk. That is, such embedding is consistent under such particular assumptions.

The components of the tensor \(\Pi_{ab}\) are given by

\[
\Pi_{ab} = \frac{1}{12} \rho^2 g_{ab}. \tag{20}
\]

Here we see that as \(\epsilon = 1\) (a timelike extra dimension) the term \(\Pi_{ab}\) in (19) acts as a potential barrier to the dynamics avoiding the singularity.

In terms of the metric functions (13) equations (19) correspond to the modified Friedmann equations of the model, having a first integral that can be expressed as the Hamiltonian constraint

\[
H = \frac{1}{8} \left( -\frac{M^2}{MR} p_M^2 - \frac{N^2}{MN} p_N^2 - \frac{R^2}{MR} p_R^2 + \frac{2}{MN} p_M p_R + \frac{2}{N} p_M p_R + \frac{2}{R} p_M p_N \right) \\
+ \frac{1}{2MNR} \left[ M^4 + N^4 + R^4 - (M^2 - N^2)^2 - (R^2 - M^2)^2 - (R^2 - N^2)^2 \right] \\
- 2\Lambda MNR - 2E_0 + \kappa^2 \frac{E_0^2}{MNR} = 0, \tag{21}
\]

where \(p_M, p_N\) and \(p_R\) are the momenta canonically conjugate to \(M, N\) and \(R\), respectively. \(E_0 \equiv 8\pi G_N C_0\) and \(\kappa^2 \equiv (8\pi G_N)^{-1}|\sigma|^{-1}\). From the Hamilton equations we obtain the following dynamical system.
\[M = \frac{\partial H}{\partial p_M} = \frac{1}{4} \left( \frac{p_N}{R} + \frac{p_R}{N} - \frac{M}{NR} p_M \right),\]
\[N = \frac{\partial H}{\partial p_N} = \frac{1}{4} \left( \frac{p_M}{R} + \frac{p_R}{M} - \frac{N}{MR} p_N \right),\]
\[R = \frac{\partial H}{\partial p_R} = \frac{1}{4} \left( \frac{p_M}{N} + \frac{p_N}{M} - \frac{C}{MN} p_R \right),\]
\[\dot{p}_M = -\frac{\partial H}{\partial M} = \frac{1}{8} \left( \frac{p_M^2}{NR} - \frac{N}{M^2} p_N^2 - \frac{R}{M^2} p_R^2 + 2 \frac{M^2 p_N p_R}{M^2 NR} \right)\]
\[+ \frac{1}{2M^2 NR} \left[ M^4 + N^4 + R^4 - \left( R^2 - N^2 \right)^2 - \left( R^2 - M^2 \right)^2 - \left( M^2 - N^2 \right)^2 \right]\]
\[+ 2\Lambda NR - \frac{2}{MNR} \left[ M^3 + M \left( R^2 - M^2 \right) - M \left( M^2 - N^2 \right) \right] + \frac{\kappa^2 E_0^2}{M^2 NR},\]
\[\dot{p}_N = -\frac{\partial H}{\partial N} = \frac{1}{8} \left( \frac{p_N^2}{MR} - \frac{M}{N^2} p_M^2 - \frac{R}{N^2} p_R^2 + 2 \frac{N^2 p_M p_R}{N^2 MR} \right)\]
\[+ \frac{1}{2MN^2 R} \left[ M^4 + N^4 + R^4 - \left( R^2 - N^2 \right)^2 - \left( R^2 - M^2 \right)^2 - \left( M^2 - N^2 \right)^2 \right]\]
\[+ 2\Lambda MR - \frac{2}{MNR} \left[ N^3 + N \left( R^2 - N^2 \right) - N \left( N^2 - M^2 \right) \right] + \frac{\kappa^2 E_0^2}{MN^2 R},\]
\[\dot{p}_R = -\frac{\partial H}{\partial R} = \frac{1}{8} \left( \frac{p_R^2}{MR} - \frac{M}{NR^2} p_M^2 - \frac{N}{MR^2} p_N^2 + 2 \frac{N^2 p_M p_N}{NR^2 MR} \right)\]
\[+ \frac{1}{2MN^2 R} \left[ M^4 + N^4 + R^4 - \left( R^2 - N^2 \right)^2 - \left( R^2 - M^2 \right)^2 - \left( M^2 - N^2 \right)^2 \right]\]
\[+ 2\Lambda MN - \frac{2}{MNR} \left[ R^3 + R \left( M^2 - R^2 \right) - R \left( R^2 - N^2 \right) \right] + \frac{\kappa^2 E_0^2}{MN^2 R}.\]

Equations (21) and (22) are equivalent to the modified field equations (19).

4. The structure of the phase space

In this section we will examine the basic structures that organize the dynamics of the system in the phase space. The first of these are the set of critical points of the system given, from equation (22), by \( M = N = R = M_0 \) and \( p_M = p_N = p_R = 0 \), where \( M_0 \) satisfies the equation

\[M_0^6 - \frac{M_0^4}{4\Lambda} + \kappa^2 \frac{E_0^2}{2\Lambda} = 0.\]

(23)

We can observe that the critical points, determined by the positive real roots of (23), depend on their respective critical energy appearing in the third term of the left-hand side of the equation, as a consequence of the bulk-brane interaction.

We must also consider the further relation

\[\frac{3}{2} M_0 + \kappa^2 \frac{E_0^2}{M_0^3} - 2\Lambda M_0^3 - 2E_{\text{cf}} = 0,\]

(24)
obtained by evaluating the Hamiltonian constraint (21) at the critical points. Solving (24) for $E_{cr}$ we will restrict ourselves to the root

$$E_{cr} = \frac{M_0^4}{\kappa^2} \left( 1 - \frac{1}{\sqrt{1 - \frac{3\kappa^2}{2M_0^2} + 2\kappa^2\Lambda}} \right)$$

(25)

which yields the correct result in the general relativity limit [84] ($\kappa^2 \to 0$ or equivalently $|\sigma| \to \infty$). Combining equations (23) and (25) we obtain for the critical points the two real positive solutions

$$M_{0,1,2} = \frac{\kappa \left( 3 \pm \sqrt{1 - 16\kappa^2\Lambda} \right)}{2(1 + 2\kappa^2\Lambda)^{1/2}\sqrt{1 - 4\kappa^2\Lambda \pm \sqrt{1 - 16\kappa^2\Lambda}}/(1 + 2\kappa^2\Lambda)}$$

(26)

with $M_0 \leq M_{0,1,2}$. The equality occurs for $\Lambda = 1/16\kappa^2$, the case of just one critical point; for $\Lambda > 1/16\kappa^2$ no critical points exist. In the following we are going to restrict ourselves to the case $\Lambda < 1/16\kappa^2$. As we will see, this condition is necessary for the presence of homoclinic orbits that establish the chaotic behavior of the dynamics. The respective energies associated with the critical points are obtained by substituting $M_{0,1,2}$ in (25) yielding

$$E_{cr,1,2} = \frac{\kappa \left( 3 \pm \sqrt{1 - 16\kappa^2\Lambda} \right)^2}{8(1 + 2\kappa^2\Lambda)^{3/2}\sqrt{1 - 4\kappa^2\Lambda \pm \sqrt{1 - 16\kappa^2\Lambda}}}.$$  

(27)

Much of our understanding of nonlinear systems derives from the linearization about critical points and from the determination of existing invariant submanifolds, which are structures that organize the dynamics in phase space. The system under examination here presents a 2-dim invariant manifold of the dynamics defined by

$$p_M = p_N = p_R, \quad M = N = R.$$  

(28)

This invariant plane is actually the intersection of two 4-dim invariant submanifolds, defined by $(M = N, p_M = p_N)$ and $(N = R, p_N = p_R)$. The critical points obviously belong to the invariant plane.

Finally a straightforward analysis of the infinity of the phase space [85] shows that it has two critical points in this region, one acting as an attractor (stable de Sitter configuration) and the other as a repeller (unstable de Sitter configuration). The scale factors $M, N$ and $R$ approach the de Sitter attractor as $M = N = R \sim \exp\left(\sqrt{\Lambda/3} \ t\right)$, so that the two de Sitter configurations also belong to the invariant plane. The phase picture of the invariant plane is displayed in figure 1, in the variables $(x, p_x)$ defined in section 5.

To proceed let us now linearize the dynamical equations (22) about the critical points $(M = N = R = M_0), \quad p_M = p_N = p_R = 0, \quad i = 1, 2$. Defining

$$X = (M - M_0), \quad W = (p_M - 0),$$
$$Y = (N - M_0), \quad K = (p_N - 0),$$
$$Z = (R - M_0), \quad L = (p_R - 0),$$

(29)
we obtain

\[
\begin{pmatrix}
\dot{X} \\
\dot{Y} \\
\dot{Z} \\
\dot{W} \\
\dot{K} \\
\dot{L}
\end{pmatrix} = 
\begin{pmatrix}
0 & 0 & 0 & -\alpha & \alpha & \alpha \\
0 & 0 & \alpha & -\alpha & \alpha & 0 \\
0 & 0 & 0 & \alpha & -\alpha & 0 \\
\beta & \gamma & \gamma & 0 & 0 & 0 \\
\gamma & \beta & \gamma & 0 & 0 & 0 \\
\gamma & \gamma & \beta & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
X \\
Y \\
Z \\
W \\
K \\
L
\end{pmatrix},
\]

(30)

where

\[
\alpha = \frac{1}{4M_0}, \quad \beta = \frac{3}{M_0} - \frac{2\kappa^2 E_{\text{cyl}}^2}{M_0^5}, \quad \gamma = 2\Lambda M_0 - \frac{3}{2M_0} - \frac{\kappa^2 E_{\text{cyl}}^2}{M_0^5}.
\]

(31)

The associated characteristic polynomial results in

\[
P(\lambda) = \left( \lambda - \sqrt{2}\gamma \alpha + \beta \alpha \right) \left( \lambda + \sqrt{2}\gamma \alpha + \beta \alpha \right)
\times \left( \lambda - \sqrt{2}\alpha (\gamma - \beta) \right)^2 \left( \lambda + \sqrt{2}\alpha (\gamma - \beta) \right)^2,
\]

(32)

with roots

\[
\lambda_{i(\ell)} = \pm i \frac{\sqrt{3}}{M_0},
\]

(33)

\[
\lambda_{i(\ell)} = \pm \sqrt{\frac{3\Lambda - \frac{1}{2M_0^2}}{2}},
\]

(34)

where (23) was used. The pair of imaginary eigenvalues (33) has multiplicity two, characterizing a center–center structure. The analysis of the center–center structure will reveal a manifold of linearized unstable periodic orbits with the topology of \(S^1\). The extension of this
manifold to the nonlinear domain constitutes the center manifold \([60, 69]\) of unstable periodic orbits, parametrized with the constant of motion \(E_0\) (with \(E_0 < E_{cr}\)), which will play a central role in our discussions in the next section.

Using (24) and (26) one can show that the pair of eigenvalues (34) are imaginary for the critical point \(i = 1\) and real for the critical point \(i = 2\). As we shall see below, we have (in the latter case \(i = 2\)) that the critical point \(P_2\) is a saddle-center–center about which the dynamics has the topology \(R \times S^3\). On the other hand, the critical point \(P_1\) is a center–center–center critical point, about which the dynamics has the topology \(S^1 \times S^3\) corresponding to perpetually bounded oscillatory Bianchi IX universes. In fact, as we will see below, the orbits in a finite neighborhood of \(P_1\) belong to the 4-dim tori \(S^1 \times S^3\) corresponding to a dynamics which is the product of the periodic motion in the region I of the invariant plane times the bounded oscillatory motion on the \(S^3\) about \(P_1\). We remark that \(P_1\) corresponds to the configuration of a stable Einstein Universe which has no general relativity analogue.

Finally we should note that, in the limit case of a single critical point (when \(16\kappa^2\Lambda = 1\)), the second pair of eigenvalues (34) are zero and no saddle structure is present in the dynamics. The analysis of this case will not be undertaken here. In the remainder of this section our discussion follows the lines of [84] done for the general relativity case.

To display the structure of the linearized motion, we start by diagonalizing the linearization matrix of (30) with the use of a similarity transformation \(\Phi\) whose columns are composed of six independent eigenvectors of the linearization matrix [86]. A judicious choice of \(\Phi\) yields primed variables defined by the transformation

\[
\begin{align*}
X' &= \frac{1}{3} (X + Y + Z), \\
Y' &= \frac{1}{M_0} (X - Y), \\
Z' &= \frac{1}{M_0} (X + Y - 2Z), \\
W' &= (W + K + L), \\
K' &= \frac{M_0}{2} (W - K), \\
L' &= \frac{M_0}{6} (W + K - 2L).
\end{align*}
\]  

In these new variables, the quadratic Hamiltonian about the \(i\)th critical point is expressed in the form

\[
H = \frac{1}{4} \left( \frac{1}{6} W'^2 - 6qX'^2 \right) - \left( \frac{1}{2M_0^2} K'^2 + M_0 Y'^2 \right) - \left( \frac{3}{2M_0^3} L'^2 + \frac{M_0}{3} Z'^2 \right) + 2 \left( E_{cr} - E_0 \right),
\]

where

\[
q = 6 \left( \Lambda M_0 - \kappa^2 E_{crit}^2 / M_0^2 \right).
\]

These primed variables are conjugated to the pairs according to \([X', W'] = 1, \ [Y', K'] = 1, \ [Z', L'] = 1\). Other Poisson brackets (PB) are zero. The
Hamiltonian (36) is separable, and we can identify the following constants of the linearized motion

\[ E_q = \frac{1}{4} \left( \frac{1}{6} W'^2 - 6qX'^2 \right), \]  
\[ E_{\text{rot}1} = \frac{1}{2M_0} K'^2 + M_0Y'^2, \]  
\[ E_{\text{rot}2} = \frac{3}{2M_0} L'^2 + \frac{M_0}{3} Z'^2, \]  
\[ Q_1 = \frac{M_0}{3} Y'Z' + \frac{1}{2M_0} K'L', \]  
\[ Q_2 = \frac{1}{2M_0} \left( L'Y' - \frac{1}{3} K'Z' \right), \]

in the sense that they all have zero PB with the Hamiltonian (36). The first three constants appear already as separable pieces in the Hamiltonian (36).

The case of \( E_q \) demands a separate analysis for the two critical points. From previous relations we have that \( q > 0 \) for the critical point \( P_2 \), so that \( E_q \) corresponds to the energy associated with the motion in the saddle sector. We remind that this is connected to the fact that the second pair of eigenvalues (34) are real for \( P_2 \). For the critical point \( P_1 \) in which \( q < 0 \), \( E_q \) corresponds to the rotational energy in the additional rotational sector of the dynamics about \( P_1 \) which has the structure \( S^1 \times S^3 \) as mentioned already.

In the following our focus will be the dynamical phenomena connected to the presence of the saddle-center–center critical point \( P_2 \) in the phase space of the model. We remark however that the analysis of the center manifold of unstable periodic orbits can also be applied to the case of the critical point \( P_1 \).

5. The dynamics about the saddle-center–center critical point

We will now proceed to describe the topology of the general dynamics in the linear neighborhood of the saddle-center–center critical point \( P_2 \) for which \( q > 0 \).

If \( E_q = 0 \) two possibilities arise. The first possibility is \( W = 0 = X \). The total energy in this case is \( E_{\text{rot}1} + E_{\text{rot}2} \), the sum of the energies of the rotational motion in the linear neighborhood of the center–center manifold, corresponding to the motion on 2-dim tori [59].

The remaining two constants \( Q_1 \) and \( Q_2 \) are additional symmetries that arise due to the multiplicity two of the imaginary eigenvalues and are connected to the fact that the linearized dynamics in the center–center sector is that of a 2-dim isotropic harmonic oscillator. They are not all independent but related by

\[ 4E_{\text{rot}1}E_{\text{rot}2} = 12Q_1^2 + 6Q_2^2. \]

The motion in the constant energy surfaces \( W = 0 = X \) are periodic orbits of the 2-dim isotropic harmonic oscillator, with Hamiltonian
The above equation shows that \( E_0 - E_{\text{cr}} < 0 \) is necessary for the dynamics in the rotational sector, defining a condition for the existence of the center–center manifold of periodic orbits.

By a proper canonical rescaling of the variables in (44) we can see that these constant energy surfaces are hyperspheres and that the constants of motion \( Q_1, Q_2 \) and \( Q_3 = E_{\text{rot}} - E_{\text{rot}} \) satisfy the algebra of the 3-dim rotation group under the PB operation, namely

\[
\left[ Q_0, Q_j \right] = i \hbar Q_k.
\]  

(45)

The constant of motion \( Q_1 \) considered as a generator of infinitesimal contact transformations has a peculiar significance in characterizing the topology of the underlying group of the algebra (45). While \( Q_2 \) generates infinitesimal rotation of the orbits, \( Q_1 \) generates infinitesimal changes in eccentricity. The action of \( Q_1 \) is to take an orbit—let us say nearly circular—and to transform it into an orbit of higher and higher eccentricity until it collapses into a straight line. Continued application of \( Q_1 \) produces again an elliptic orbit, but now traversed in the opposite sense, so that it takes a 720° to bring the orbit back into itself. The two-valuedness of the mapping arises from the fact that the orbits are oriented. Therefore the group generated by these constants of motion is homomorphic to the unitary unimodular group [87] so that the topology of the center–center manifold is in fact \( S^3 \).

Due to the separate conservation of \( E_{\text{rot}} \) and \( E_{\text{rot}} \) (see (44)) one can show that the center manifold in the linear neighborhood of the critical points is foliated by Clifford 2-dim surfaces in \( S^3 \) [88], namely, 2-tori \( J_{E_0} \) contained in the energy surface \( E_0 = \text{const.} \) Such surfaces, as well as the \( S^3 \) manifold containing them, depend continuously on the parameter \( E_0 \). We remark that these two tori will have limiting configurations \( E_{\text{rot}} = 0 \) or \( E_{\text{rot}} = 0 \), and correspond to the case of maximum eccentricity (for instance, a straight line in the plane \((Y^s, Z^s)\)).

The second possibility to be considered is \( W' = \pm 6 \sqrt{q} X' \). It defines the linear stable \( V_S \) and unstable \( V_U \) manifolds of the saddle sector. \( V_S \) and \( V_U \) limit regions I (\( E_q \equiv E_{\text{hyp}} < 0 \)) and regions II (\( E_q \equiv E_{\text{hyp}} > 0 \)) of motion on hyperbolae which are solutions in the separable saddle sector \( E_{\text{hyp}} = -\frac{1}{6} \frac{1}{\sqrt{q}} (W'^2 - 6qX'^2) \). Note that the saddle sector depicts the structure of the neighborhood of \( P_3 \) in figure 1, with \( V_U \) and \( V_S \) tangent to the separatrices at \( P_3 \). The direct product of \( J_{E_0} \) with \( V_S \) and \( V_U \) generates, in the linear neighborhood of the critical point \( (i = 2) \) the structure of stable \( (J_{E_0} \times V_S) \) and unstable \( (J_{E_0} \times V_U) \) 3-dim tubes which coalesce, with an oscillatory approach to the tori \( J_{E_0} \) for \( t \to \infty \). The energy of any orbit on these tubes is the same as that of the orbits on the tori \( J_{E_0} \). These structures are contained in the 4-dim energy surface \( H = E_0 \) such that \( (E_q - E_{\text{cr}}) < 0 \). We should recall that the tubes constitute a boundary for the general flow and are defined by \( E_q = 0 \) in the linear neighborhood of the critical point. Depending on the sign of \( E_q \) the motion will be confined inside the 4-dim tube (for \( E_q < 0 \)) and will correspond to a flow separated from the one outside the tube (for \( E_q > 0 \)). A detailed examination of the above motion and its extension to the nonlinear regime will be done in the next section.

The extension of our analysis beyond a linear neighborhood of critical points could be made by implementing normal forms [89, 90] and associated coordinates, modulo their radius of convergence. We will instead propose here a suitable canonical transformation which will allows us to obtain an exact analytical form for the center manifold as well as a sufficiently
accurate description of the phase space dynamics in extended regions away from the critical points. In particular we can examine the behavior of the nonlinear extensions $W_S$ and $W_U$ of, respectively, the linear stable ($V_{ES} \times V_{S0}$) and linear unstable ($V_{EU} \times V_{U0}$) manifolds [67] emanating from the neighborhood of the saddle-center–center $P_2$. Let us introduce the canonical transformation with the generating function

$$G = (MNR)^{1/3} p_x + \frac{M}{N} p_y + \frac{MN}{R^2} p_z,$$

where $p_x, p_y$ and $p_z$ are the new momenta, resulting in

$$x = (MNR)^{1/3}, \quad y = \frac{M}{N}, \quad z = \frac{MN}{R^2},$$

and

$$p_M = \frac{1}{3} \frac{NR}{(MNR)^{2/3}} p_x, \quad p_N = \frac{1}{3} \frac{MR}{(MNR)^{2/3}} p_x - \frac{M}{N^2} p_y + \frac{M}{R^2} p_z, \quad p_R = \frac{1}{3} \frac{MN}{(MNR)^{2/3}} p_x - \frac{2MN}{R^3} p_z.$$

Here, the variable $x$ is obviously the average scale factor of the model. In these new canonical variables the equations of the invariant plane reduce to

$$y = 1, \quad z = 1, \quad p_y = 0 = p_z.$$

It is then clear that $(x, p_x)$ are variables defined on the invariant plane. In these variables the phase space picture of the invariant plane is given in figure 1. The separatrices $S$ emerging from the saddle-center–center $P_2$ separate the invariant plane in three disconnected regions, region I of oscillatory universes and regions II and III of one bounce universes. They are constituted of three branches, namely, the separatrix that divides the regions I and II and makes a homoclinic connection with the critical point $P_2$ and two others that approach the de Sitter asymptotic configurations for $t \to \pm \infty$. The first branch will play a fundamental role in our following discussions and will be referred to as separatrix, except where a qualification is needed to avoid confusion. The center–center–center $P_1$ corresponds to a stable Einstein Universe configuration that occurs due to the bulk-brane interaction term proportional to $E_0^2$ in the Hamiltonian (21).

In the variables $(x, p_x, y, p_y, z, p_z)$ the full Hamiltonian (21) assumes the form

$$H(x, p_x, y, p_y, z, p_z; E_0) = \frac{1}{24\chi} p_x^2 - \frac{y^2}{2\chi^3} p_y^2 - \frac{3\chi^2}{2\chi^3} p_z^2 - \frac{x}{2\chi^{4/3}} - \frac{x^2}{2\chi^{2/3}} + \frac{1}{2\chi^{2/3}} + \frac{\kappa^2 E_0^2}{\chi^3} - 2\Lambda \chi^3 - 2E_0 = 0.$$

We remark that the linearization of (47)–(50) about both critical points $P_1$ and $P_2$ of the dynamical system (22) yields exactly the transformation (35), and that the variables $(y, p_y, z, p_z)$ correspond to the primed variables $(k', y', l', z')$ defined on the center–center manifold $S^3$ about a linear neighborhood of $P_2$. 

The new canonical variables are most convenient since they separate the degrees of freedom of the system into pure rotational modes, \((y, p_y)\) and \((z, p_z)\), and the expansion/contraction mode \((x, p_x)\) connected to the invariant plane. This can be illustrated by implementing the expansion of the dynamical system generated from (50) about a linear neighborhood of the invariant plane, producing a linearized Hamiltonian parametrized by the variables \((x(t), p_x(t))\) describing the curves in the invariant plane, for instance, in the region I of periodic orbits bounded by the separatrix \(S\) homoclinic to \(P_2\). This is analogous to the usual expansion of a dynamical system about a periodic orbit. Using (50), we then obtain
\[
H = E_{inv} - \frac{1}{2x^3}\left(p_y^2 + 3p_z^2\right) - x(y - 1)^2 - \frac{1}{3}z(z - 1)^2 = 2E_0, \tag{51}
\]
where
\[
E_{inv} = \frac{1}{24}\pi^2 + \frac{3x}{2} + \frac{\kappa^2E_0^2}{x^3} - 2\Lambda x^3 = \text{const.} \tag{52}
\]
The resulting dynamical equations are
\[
\dot{y} = -\frac{1}{x^3}\delta p_y, \\
\dot{p}_y = 2x\delta y, \\
\dot{z} = -\frac{3}{x^3}\delta p_z, \\
\dot{p}_z = \frac{2x}{3}\delta p_z, \tag{53}
\]
where \(\delta y = (y - 1), \delta z = (z - 1), \delta p_y = (p_y - 0), \) and \(\delta p_z = (p_z - 0). \) The linearization matrix of (53) has imaginary eigenvalues \(\lambda = \pm i\sqrt{2}/x(t)\), both with multiplicity two, corresponding to elliptical modes in the linear neighborhood of the invariant plane so that the motion is oscillatory about the invariant plane.

6. The nonlinear center manifold and the homoclinic cylinders

The nonlinear extension of the center manifold, by continuity, maintains the topology \(S^3\) but it can no longer be decomposable into \(E_{rot1}\) and \(E_{rot2}\), so that now only the 4-dim tubes with the topology \(R \times S^3\) are meaningful for the nonlinear dynamics. Similarly the extension of the structure of the 4-dim tubes away from the neighborhood of the center manifold are to be examined, and our basic interest will reside in the stable and unstable tubes, \(W_S = V_S \times S^3\) and \(W_U = V_U \times S^3\), that leave this neighborhood. The tubes have the structure of 4-dim spherical cylinders (of co-dimension 2), one less dimension than the energy surface, and act therefore as separatrices, separating the energy surface in two dynamically disconnected parts.

The 2-dim invariant plane, defined by (28), is contained in a 6-dim phase space and it is obvious that, contrary to examples in lower dimensional systems, it does not separate the dynamics in disjoint parts. In fact the general motion about the curves of the invariant plane is an oscillatory flow confined in the interior or exterior of 4-dim tubes \(R \times S^3\), so that the invariant plane (or more properly, one of the curves of the invariant plane) can be thought as a structure in the center of the tubes. This latter fact is of crucial importance in the discussion of the transversal crossing of the 4-dim cylinders \(W_S\) and \(W_U\) made in section 7.

The nonlinear extension of the center manifold in the canonical variables \((y, z, p_y, p_z)\) is obtained by substituting \(x = x_0\) and \(p_x = 0\) in (50), yielding after some manipulation the exact
where \( x_{cr} \) and \( E_{cr} \) are respectively the coordinate and the energy of the critical point \( P_2 \). The form (54) adopted above for the center manifold equation makes explicit its dependence on the parameter \( E_0 \). For \( E_0 = E_{cr} \), the center manifold reduces to the critical point. The domain of \( E_0 \) defining the center manifold satisfies the constraint to \( E_0 < E_{cr} \), as already discussed; the case of the linear version (44) corresponds to \( (E_{cr} - E_0) \) sufficiently small. As \( (E_{cr} - E_0) \) increases we have a nonlinear center manifold parametrized by the energy \( E_0 \). In general the center manifold is a 3-dim submanifold of the 6-dim phase space contained in the 5-dim energy hypersurface \( H = E_0 \). In figure 2 we plot the sections \((y = 1, p_y = 10^{-3})\) left panel, and \((z = 1, p_z = 0)\) right panel, of the center manifold for five values of \( E_0 \). We note the deformation of the center manifold \( S^3 \) in the nonlinear domain as \( (E_{cr} - E_0) \) increases. Here \( E_{cr} = 2.5127254138199294 \).

analytical expression

\[
H_c = \frac{y^2}{2x_{cr}^2} + \frac{3z^2}{2x_{cr}^3} p_z^2 + x_{cr}\left(\frac{3}{2} + \frac{1}{2} \frac{E_0^2 - E_{cr}^2}{x_{cr}^3} + \frac{1}{2} \frac{z^2}{y^2} + \frac{z^2}{y^3} - \frac{1}{y^2} \right)
\]

(54)

where \( x_{cr} \) and \( E_{cr} \) are respectively the coordinate and the energy of the critical point \( P_2 \). The form (54) adopted above for the center manifold equation makes explicit its dependence on the parameter \( (E_{cr} - E_0) \). For \( E_0 = E_{cr} \), the center manifold reduces to the critical point. The domain of \( E_0 \) defining the center manifold satisfies the constraint to \( E_0 < E_{cr} \), as already discussed; the case of the linear version (44) corresponds to \( (E_{cr} - E_0) \) sufficiently small. As \( (E_{cr} - E_0) \) increases we have a nonlinear center manifold parametrized by the energy \( E_0 \). In general the center manifold is a 3-dim submanifold of the 6-dim phase space contained in the 5-dim energy hypersurface \( H = E_0 \). In figure 2 we plot the sections \((y = 1, p_y = 10^{-3})\) and \((z = 1, p_z = 0)\) of the \( S^3 \) center manifold (54) showing its deformation in the nonlinear regime as the values of \( (E_{cr} - E_0) \) increase. We adopted the values \( \Lambda = 0.01 \) and \( \kappa^2 = 0.5 \) so that the associated critical energy \( E_{cr} = 2.5127254138199294 \) and \( x_{cr} = 4.974148895632555 \) for the saddle-center–center \( P_2 \). In the figures we selected five values for \( E_0 \).

As we have already seen the canonical coordinates \((y, p_y, z, p_z)\) cover the center manifold \( S^3 \) and therefore we will use them not only to examine the stability of the motion restricted to \( S^3 \) but also to obtain an accurate description of the whole dynamics \((x(t), p_x(t), y(t), p_y(t), z(t), p_z(t))\) emerging from a neighborhood of the center manifold. In the following we will numerically illustrate this behavior. We must remark that we do not make use here of the displacing (in the direction of the unstable cylinder) of initial conditions taken on the invariant center manifold, as the shooting method in [72], but instead we make use of the instability of the motion on the center manifold which computationally conserves the Hamiltonian constraint (50) for all \( t \). Actually in all our numerical simulations the error in the Hamiltonian constraint (50) is checked to remain \( \lesssim 10^{-13} \) for the whole computational domain.
To start let us fix the parameters $\kappa^2 = 0.5$, $\Lambda = 0.01$ as in figure 2. In figure 3 we now show the 2-dim sections $p_z = 10^{-3}$ of the center manifold for $E_0 = 2.512725$ (left). The solid line indicates an orbit with initial conditions obviously satisfying (54). This orbit is evolved with the full dynamics generated from the Hamiltonian (50) and remains on the center manifold for $0 \leq t \leq 2000$. A piece of this center manifold is displayed in figure 3 (middle) where the solid line describes the same previous orbit. Figure 3 (right) displays the section $p_z = 10^{-3}$ of the center manifold for $E_0 = 2.3$. The solid line corresponds to a one-bounce orbit which moves towards large values of $z$ (when $t \sim 37, z(t) \sim 63$) before escaping to the de Sitter attractor when $t \approx 70$. The initial condition for this orbit, $(x_0 = x_{cr}, y_0 = 1, z_0 = 1, p_{x0} = 0, p_{y0} = 10^{-3}, p_{z0} = 5.87927647889785)$ is taken on center manifold. This orbit remains on the center manifold for a time up to $t \sim 12$. Here $E_{cr} = 2.512725413819929$.
we decrease $E_0$. In fact the increase of $(E_{\text{crit}} - E_0)$ makes an orbit, with initial conditions taken on the center manifold, to rapidly leave this neighborhood indicating a dynamical instability (despite the accuracy of the exact dynamics) as shown in figure 3 (right).

We remark that the oscillatory behavior of the orbit in the phase space sectors $(y, p_y)$ and $(z, p_z)$ is typical, even when the orbit tends asymptotically to one of the de Sitter attractors. This is illustrated in figure 5 where we plot the time behavior of $p_y$ and $y$ of the orbit discussed in figure 4. We note a decrease of the amplitude of $p_y$ and an amplification of the amplitude for the conjugated $y$ in a neighborhood of the bounce. This pattern is analogous for the other variables $(p_z, z)$ of the orbit.
Finally we give a numerical illustration of the stable and unstable cylinders emanating from the center manifold which are a nonlinear extension of the VE$^0$ and UE$^0$ with $SE^3_0 \subset I$ defined in a linear neighborhood of the saddle-center–center $P_2$. We must recall that these cylinders are actually composed of orbits that have the same energy $(E_{cr} - E_0)$ of the center manifold and coalesce to it as $t \to \pm \infty$. In figure 6 we display the stable $W_S$ and unstable $W_U$ cylinders emanating from the neighborhood of the center manifold towards the bounce, guided by the separatrix dividing the regions I and II of the invariant plane. We emphasize that the separatrix guiding the cylinders is actually a structure inside the cylinders.

We fixed the parameters $\Lambda = 0.001$ and $\kappa^2 = 0.5$, as in figures 4, and took $E_0 = 7.9096$ so that $(E_{cr} - E_0) \sim 10^{-5}$.

A comment is in order now. Since the cylinders $W_S$ and $W_U$ are 4-dim surfaces they obviously separate the 5-dim energy surface defined by the Hamiltonian constraint (50) in two dynamically disconnected pieces, a fact that will be fundamental in the characterization of chaos in the case of an eventual transversal crossing of $W_S$ and $W_U$ [60, 67]. We remark that in figure 6 the projection of the cylinders on the plane $(x, p_x)$ ‘shadows’ the separatrix of the invariant plane. Here $E_{crit} = 7.909653935314993$.

Finally we give a numerical illustration of the stable and unstable cylinders emanating from the center manifold which are a nonlinear extension of the $J_{E_0} \times V_S$ and $J_{E_0} \times V_U$, with $J_{E_0} \subset S^3$ defined in a linear neighborhood of the saddle-center–center $P_2$. We must recall that these cylinders are actually composed of orbits that have the same energy $(E_{cr} - E_0)$ of the center manifold and coalesce to it as $t \to \pm \infty$. In figure 6 we display the stable $W_S$ and unstable $W_U$ cylinders emanating from the neighborhood of the center manifold towards the bounce, guided by the separatrix dividing the regions I and II of the invariant plane. We emphasize that the separatrix guiding the cylinders is actually a structure inside the cylinders.

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7. The transversal crossings of the cylinders and the homoclinic intersection manifold: a chaotic saddle

The results of the previous sections showed that two 4-dim cylinders, one stable $W_S = R \times S^3$ and one unstable $W_U = R \times S^3$, emerge from a neighborhood of the saddle-center–center $P_2$. The center manifold $S^3$ is the locus of the rotational degrees of freedom of the phase space.
dynamics and is parametrized with the energy $E_0$ ($E_0 < E_{\text{crit}}$). It encloses the critical point $P_2$ and tends to it as $E_0 \to E_{\text{crit}}$. At this limit the cylinders $W_S$ and $W_U$ reduce to the separatrix $S$, which makes a homoclinic connection of $P_2$ to itself in the invariant plane. The separatrix is a structure inside the cylinders, about which the flow with the oscillatory degrees of freedom $(y, p_y, z, p_z)$ proceeds, guiding the cylinders towards the bounce (see figure 1) and leading to their eventual crossing. The first crossing is expected to occur in a neighborhood of the bounce $(x = x_b, p_x = 0)$, where $x_b$ is the scale factor of the bounce for the orbits at $p_x = 0$. In order to detect this first intersection we will adopt as the surface of section [91] the 4-dim surface $\Sigma : (x = x_b, p_x = 0)$. This first transversal crossing of the cylinders will be the main object of the present section.

Due to the conservation of the Hamiltonian constraint (50) we have that at the bounce 

$$
H \left( x = x_b, p_x = 0, y, p_y, z, p_z; E_0 \right) = 0,
$$

which is the equation of a closed surface with the topology of $S^3$. The transversal crossing of the stable $W_S$ and unstable $W_U$ cylinders at the bounce will therefore be a set of points contained in the transversal intersection of two three-spheres defined by (55), then a $S^3$. These points define a set of orbits that are contained both in the stable and the unstable cylinders and are bi-asymptotic (homoclinic) to the center manifold $S^3(E_0)$. They are denoted as homoclinic points and homoclinic orbits. Therefore the set of homoclinic points has the compact support $S^3$. The presence of a homoclinic orbit in the dynamics is an invariant signature of chaos in the model [65, 69]. The homoclinic intersection manifold has the topology $R \times S^2$ and consists of all homoclinic orbits bi-asymptotic to the center manifold. In this sense, a chaotic saddle [72] associated with $S^3$ is defined, indicating that the set of intersection points of the cylinders has the nature of a Cantor-type set with compact support $S^3$ [73].

A complete numerical study of the intersection of the 4-dim cylinders $W_S$ and $W_U$ is beyond the scope of the present paper (it will be considered as the subject for a future publication). Here our numerical simulations will be restricted to the dynamics on the two 4-dim invariant submanifolds of the 6-dim phase space defined by (i) $M = N, p_M = p_N$ (or equivalently $y = 1, p_y = 0$), and (ii) $N = R, p_N = p_R$ (or equivalently $y = z, p_y = 3 p_z$). The denomination invariant submanifolds derives from the fact that each of them is mapped on itself by the general Hamiltonian flow (22), in other words, is invariant under the flow. We will then examine the intersection of 2-dim stable and unstable cylinders in these two 4-dim invariant manifolds, according to the following simulations.

To start we fix the parameters $\kappa^2 = 0.5$, $\Lambda = 0.001$, with corresponding $E_{\text{crit}} = 7.9096533935314993$ and $x_{\text{crit}} = 15.803456796952816$. The total energy of the system is taken $E_0 = 7.9096$, so that the energy available to the rotational modes will be $(E_0 - E_0) \sim 10^{-3}$.

In the first simulation we take $(x_0 = x_{\text{crit}}, p_{x0} = 0)$, and fix initial conditions on the 4-dim invariant submanifol (i), namely, with $(y_0 = 1, p_{y0} = 0)$. That is, we are restricting ourselves to a particular domain of initial conditions in the sector $(z, p_y)$ of the center manifold $S^3$, which has the topology of $S^3$ and is defined by the Hamiltonian constraint (50) as

$$
H \left( x = x_{\text{crit}}, p_x = 0, y = 1, p_y = 0, z, p_z \right) = 0.
$$

By performing the evolution of orbits from a large set (of the order of 1500) of initial conditions in the above domain, the exact dynamics actually evolves a 4-dim invariant subset $(x, p_x, z, p_z)$ of the full 6-dim phase space as expected due to our restriction to the 4-dim invariant manifold $(y = 1, p_y = 0)$. In this particular simulation, we have that under the exact dynamics no motion is present in the sector $(y, p_y)$. We generate one 2-dim stable and one
2-dim unstable cylinders of orbits which initially move towards the first bounce. In order to detect the first intersections of the two cylinders we adopt $\Sigma : (x = x_b, p_x = 0)$ as the surface of section, where $x_b \approx 1.3118$ is the coordinate of the first bounce of the orbits at $p_x = 0$. In figure 7 we plot the points $z_B, p_B$ of the sections of both cylinders in the first cross of $\Sigma$. The points $A, B, C, D$ (contained in the sector $(z, p_z)$ of $\Sigma$) characterize the transversal crossing of the cylinders. An detailed examination of the numerical points of the map shows indeed that all orbits arrive at the first bounce $x_b \approx 1.3118, p_x = 0$ with coordinates $y_B = 1, p_y = 0$, being a further verification of the accuracy of our numerical treatment. The points $A, B, C, D$ in figure 7 therefore define homoclinic orbits, namely, orbits which are in the intersection of the unstable and stable cylinders, and give an invariant characterization of chaos in the model. These homoclinic orbits are contained both in the stable cylinder and the unstable cylinder and are bi-asymptotic to the center manifold $S^3(E_0)$, constituting an invariant signature of chaos in the dynamics. Here $E_{crit} = 7.909653935314993$.

Analogously in the second simulation we maintain the same values for the parameters $E_0, \kappa^2$ and $\Lambda$ together with the initial conditions $x_0 = x_{crit}, p_{x_0} = 0$. However we now fix the remaining initial conditions on the 4-dim invariant submanifold (ii) instead, namely, $(y_0 = z_0, p_y = 3p_z)$. In fact we are restricting ourselves to a particular domain of initial conditions of the center manifold $S^4$ which has the topology of $S^4$ and is defined by the Hamiltonian constraint

$$ H \left( x = x_{crit}, p_x = 0, y = z, p_y = 3p_z \right) = 0. $$

With the exact dynamics we generate one 2-dim stable and one 2-dim unstable cylinders which initially move towards the first bounce. These cylinders are generated from a set of about 1500 orbits, with initial conditions taken in the above domain which actually correspond to a flow in the 4-dim invariant submanifold (ii) of the full 6-dim phase space.
In figure 8 we plot the points \( z_p, b z_b \) of the sections of both cylinders in the first crossing of \( \Sigma \). The four points \( E, F, G, H \) are the unique points of the first transversal crossing of the cylinders (see text). These points define homoclinic orbits which are contained in both unstable and stable cylinders and are bi-asymptotic to the center manifold \( S^3(E_0) \), constituting an invariant signature of chaos in the dynamics of the model.

![Figure 8](image)

**Figure 8.** (Second simulation) Projection on the plane \((z, p_z)\) of the first crossing of the stable cylinder (gray) and unstable cylinder (black) in the surface of section \( \Sigma \) (at the first bounce). In this projection the four points \( E, F, G, H \) are the unique points of the first transversal crossing of the cylinders. These points define homoclinic orbits which are contained in both unstable and stable cylinders and are bi-asymptotic to the center manifold \( S^3(E_0) \), constituting an invariant signature of chaos in the dynamics of the model.

![Figure 9](image)

**Figure 9.** (Second simulation) Projections on the complementary plane \((y, p_y)\) showing the first crossing of the stable cylinder (gray) and unstable cylinder (black) in the surface of section \( \Sigma \) (at the first bounce). The four points \( E, F, G, H \) of figure 8, characterizing the transversal crossing of the cylinders are shown, constituting an invariant signature of chaos in the dynamics of the model.

A detailed examination of the numerical points of the map shows that all orbits arrive at the first bounce \((x \approx 1.3118, p_x = 0)\) with coordinates \((y_b = z_b)\) and \((p_{y_b} = 3p_{z_b})\). This is also
illustrated in figure 10 where the first crossing of the unstable and stable cylinders with the surface of section $\Sigma$, projected on the phase space sections $(y, z)$ and $(p_y, p_z)$. As expected the sections of both cylinders coincide on the straight lines $y = z$ and $p_y = 3p_z$, respectively, as expected. This is also a further verification of the accuracy of our numerical results.

The sets (56) and (57) are two distinct numerical evidences of chaos in the dynamics, and constitute an invariant signature of chaos in the model. The dynamics near homoclinic orbits is very complex associated with the presence of horseshoe structures [60, 65–67, 92]. We must mention that in the neighborhood of each homoclinic orbit there exists an infinite countable set of periodic orbits with arbitrarily long periods. The coordinates of the homoclinic points (56) and (57) satisfy the constraint (55), implying that they are contained in the transversal intersection of two $S^3$ at the bounce, namely, a $S^2$. This fact indicates that the chaotic saddle—connected with the structure of all homoclinic orbits bi-asymptotic to the center manifold $S^3$—is a Cantor-type set having compact support $S^2$ [73].

8. Conclusions and final comments

In this paper we examined the dynamics of a Bianchi IX model, with three scale factors, sourced by a pressureless perfect fluid in the framework of bouncing braneworld cosmology. Assuming a timelike extra dimension and a 5-dim conformally flat empty bulk, the modified Einstein field equations on the 4-dim Lorentzian brane result in a dynamics with correction terms that avoid the singularity and implement nonsingular bounces in the early phase of the Universe. In terms of metric functions the modified Einstein equations have a first integral that can be expressed as a Hamiltonian constraint in a 6-dim phase space, yielding a three degrees of freedom dynamical system which governs the motion in phase space. Due to an
effective cosmological constant on the brane the phase space presents two critical points in a finite region of the phase space, a center–center–center and a saddle-center–center, plus two critical points at infinity corresponding to the de Sitter solution. Together with a 2-dim invariant plane of the dynamics the critical points allow to organize the dynamics of the phase space.

We examine the structure of the dynamics in a linearized neighborhood of the saddle-center–center. We identify constants of motion associated with the saddle sector, which allow to define the linear stable $V_S$ and unstable $V_U$ manifolds. We also identify constants of motion connected to the center–center sector, which define the center manifold of linearized unstable periodic orbits and has the topology of $S^3$. In the linear domain the direct product $V_S \times S^3$ and $V_U \times S^3$ define the structure of stable and unstable cylinders which constitute boundaries in the 5-dim energy surface of the dynamics.

The nonlinear extension of the center manifold of unstable periodic orbits is parametrized by the constant of motion $E_0$ ($E_0 < E_c$) with the topology of $S^3$ maintained. As one decreases the parameter $E_0$ the nonlinearity of the center manifold increases, with a corresponding increasing of the dynamical instability as shown in our numerical simulations. The extension of the 4-dim stable $W_S = V_S \times S^3$ and unstable $W_U = V_U \times S^3$ cylinders away from the neighborhood of the center manifold have the structure of 4-dim spherical cylinders with the topology $R \times S^3$.

By a proper canonical transformation we are able to separate the three degrees of freedom of the dynamics into one degree—connected with the expansion and/or contraction of the scales of the model—isolated from the other two related to pure rotational degrees of freedom associated with the center manifold $S^3$. By expanding the Hamiltonian constraint and the Hamiltonian equations in these coordinates we show that the typical dynamical flow is an oscillatory mode about the orbits of the invariant plane. In particular the stable $W_S$ and unstable $W_U$ cylinders are composed of oscillatory orbits about the separatrices which emerge from the saddle-center–center critical point and guide the cylinders. These cylinders have the same energy $E_0$ of the center manifold and coalesce to it as $t \to \pm \infty$. As these spherical cylinders are 4-dim surfaces they separate the 5-dim energy surface into two dynamically disconnected pieces. This fact is a fundamental feature of the dynamics for characterization of chaos in the case of an eventual transversal intersection of $W_S$ and $W_S$, implying that if the cylinders intersect once they will intersect infinitely many times, a fact that is fundamental to the stretching and folding mechanism that give rise to the horse-shoe structures about the homoclinic points. As the separatrix which divides regions I and II in the invariant plane makes a homoclinic connection to the saddle-center–center critical point, this fact necessarily leads to the transversal crossings of $W_S$ and $W_U$. The transversal intersection of the cylinders consists of homoclinic orbits which are contained both in the stable and the unstable cylinder and are biasymptotic to the center manifold $S^3$. The presence of a homoclinic orbit in the dynamics is an invariant signature of chaos in the model [61, 65, 66, 69]. The homoclinic intersection manifold has the topology of $R \times S^2$ and consists of all homoclinic orbits biasymptotic to the center manifold defining a chaotic saddle [72] associated with $S^3$. We must finally comment that in the neighborhood of each homoclinic orbit there exits an infinite countable set of periodic orbits of very long periods [65] corresponding also to perpetually bouncing universes.

The first transversal crossings of the stable $W_S$ and unstable $W_U$ cylinders are shown numerically in two distinct simulations. For the sake of computational simplicity we restricted ourselves to cylinders generated from initial conditions taken on the center manifolds of the two 4-dim invariant manifolds of the dynamics defined respectively by $(y = 1, p_y = 0)$, and $(y = z, p_y = 3p_z)$. We adopted the surface of section $\Sigma$ at the bounce defined by $(x_b, p_x = 0)$
where $x_b$ is the scale factor of the bounce for the orbits. By performing the evolution of orbits via the 6-dim exact dynamics we generate one 2-dim stable and one 2-dim unstable cylinders of orbits, and detected their transversal intersection corresponding to four homoclinic points in the first crossing of $\Sigma$ by the cylinders, for both simulations. These points define orbits which are homoclinic to the center manifold $\mathcal{J}_{E_0} \subset S^3(E_0)$.

In all our numerical simulations we used the 6-dim exact dynamics, in accordance with (22), and the error in the Hamiltonian constraint (50) is checked to remain $\lesssim 10^{-13}$ for all $t$.

We finally compare some features of the dynamics, namely the oscillatory approach to the bounce and the chaotic behavior of the dynamics, with analogous features present in the BKL conjecture in general relativity. First we note that in both models the oscillatory approach to the bounce/singularity is a key feature of the dynamics. In the general Bianchi IX model discussed here the three degrees of freedom of the dynamics are separated into one degree (connected with the expansion and/or contraction of the scales of the model) plus pure rotational degrees of freedom associated with the center manifold $S^3$. The typical dynamical flow is an oscillatory mode about the orbits of the 2-dim invariant plane; in particular from the center manifold there emerge the stable and unstable 4-dim cylinders of oscillatory orbits that are guided towards the bounce by the separatrix in the invariant plane. In the limit of $\kappa^2 \to 0$ (close to the general relativity dynamics) the motion on, or about the unstable cylinder approximates the oscillatory BKL motion up to a scale $x^3 \approx \kappa^2$. As one can make $\kappa^2$ as small as wanted, a long oscillatory approach towards a neighborhood of $x = 0$ can be developed, with a behavior analogous to one of the Kasner eras of the BKL model. However $\kappa^2$ cannot be made equal to zero as this would correspond to a change of topology of the phase space. The same considerations would apply for the case of a mixmaster Universe, with a nonvanishing cosmological constant and $E_0 = 0$, in general relativity. Second, the chaos in the present model has a homoclinic origin, resulting from the homoclinic transversal intersections of the stable and unstable 4-dim cylinders emerging from the center manifold $S^3$. In contrast the chaotic behavior in the BKL dynamics appears in a map that connects the length of the succeeding Kasner eras in the approach to the singularity of general relativity, for which we have no counterpart. Nevertheless, considering the general relativity limit, we have topological evidence that the 4-dim cylinders—emerging from the center manifold $S^3$ and guided by the separatrix connecting the saddle-center–center to the singularity—should intersect and generate a homoclinic orbit from this intersection.

In a future work we intend to examine the extension of the BKL behavior about a neighborhood of the bounce to a framework with a more general bulk (relaxing the restriction of a conformally flat bulk) and the eventual modifications of the dynamics in the brane due to a more complex gravitational interplay between the bulk and the brane.

Concerning the present work we also intend in the future to examine the transversal intersection of the spherical cylinders $R \times S^3$ in the full 6-dim phase space. We also will examine the chaotic exit to the final accelerated de Sitter stage for initial condition sets (corresponding to initially expanding universes) taken in a small neighborhood about the separatrix $S$ approaching the saddle-center–Sitter for $t > 0$. As in [50], we expect these sets to have fractal basin boundaries connected to the code recollapse/escape leading to a chaotic exit to the de Sitter accelerated phase. We also expect to observe the draining of initial condition basins from recollapse to escape behavior, as time increases. For $t \to \infty$ only the homoclinic intersection manifold is expected to remain in recurrent oscillatory motion.
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References

[1] Belinskii V A, Khalatnikov I M and Lifshitz E M 1970 Adv. Phys. 19 525
[2] Khalatnikov I M and Lifshitz E M 1970 Phys. Rev. Lett. 24 76
[3] Belinskii V A, Khalatnikov I M and Lifshitz E M 1982 Adv. Phys. 31 639
[4] Lifshitz E M, Lifshitz I M and Khalatnikov I M 1970 JETP 59 322
[5] Khalatnikov I M, Lifshitz E M, Khanin K M, Shchur L N and Sinai Ya G 1985 J. Stat. Phys. 38 97
[6] Hobill D, Burd A and Coley A (ed) 1994 Deterministic Chaos in General Relativity (New York: Plenum)
[7] Berger B K 2002 Numerical approaches to spacetime singularities Living Rev. Relativ. 5 6
[8] Heinzle J M and Uggl C 2009 Class. Quantum Grav. 26 075016
[9] Barrow J D 1990 Phys. Rep. 85 1
[10] Chernoff D F and Barrow J D 1983 Phys. Rev. Lett. 50 134
[11] Berger B K 1990 Class. Quantum Grav. 7 203
[12] Berger B K 1991 Gen. Relativ. Gravit. 23 1385
[13] Berger B K 1994 Phys. Rev. D 49 1120
[14] Berger B K 1994 Deterministic Chaos in General Relativity ed D Hobill et al (New York: Plenum)
[15] Rugh S E and Jones B J T 1990 Phys. Lett. A 147 353
[16] Berger B K, Garfinkle D, Isenberg J, Moncrief V and Weaver M 1998 Mod. Phys. Lett. A 13 1565
[17] Misner C W 1969 Phys. Rev. Lett. 22 1071
[18] Latifi A, Musette M and Conte R 1994 Phys. Lett. A 194 83
[19] Contopoulos G, Gammaticos B and Ramani R 1995 J. Phys. A: Math. Gen. 28 5313
[20] Francisco G and Matsas G E A 1998 Gen. Relativ. Gravit. 20 1047
[21] Cornish N J and Levin J J 1997 Phys. Rev. Lett. 78 998
[22] Cornish N J and Levin J J 1997 Phys. Rev. D 55 7489
[23] Cornish N J and Levin J J 1997 Proc. 8th Marcel Grossmann Meeting: On Recent Developments in Theoretical and Experimental General Relativity, Gravitation, and Relativistic Field Theories ed T Piran and R Ruffini pp 616–8
[24] Motter A E and Letelier R S 2001 Phys. Lett. A 285 127
[25] Belinskii V A and Khalatnikov I M 1973 Sov. Phys.—JETP 36 591
[26] Berger B K 1999 Phys. Rev. D 61 023508
[27] Fay S and Lehner T 2005 Gen. Relativ. Gravit. 37 1097
[28] Rendall A D 1997 Class. Quantum Grav. 14 2341
[29] Ringström H 2000 Class. Quantum Grav. 17 713
[30] Ringström H 2001 Ann. Inst. Henri Poincaré 2 405
[31] Heinzle J M and Uggla C 2009 Class. Quantum Grav. 26 075015
[32] Damour T, Henneaux M and Nicolai H 2003 Class. Quantum Grav. 20 R145
[33] Penrose R 1965 Phys. Rev. Lett. 14 57
[34] Wald R M 1984 General Relativity (Chicago, IL: University of Chicago Press)
[35] Bojowald M (Loop Quantum Cosmology Collaboration) 2002 Living Rev. Relativ. 8 11
[36] Bojowald M and Tavakol R 2008 Beyond the Big Bang ed R Vaas (Berlin: Springer)
[37] Dienes K R 1997 String theory and the path to unification: a review of recent developments Phys. Rep. 287 447–525
[38] Kaku M 2000 Strings, Conformal fields and M-theory (New York: Springer)
[39] Rovelli C 1998 Loop quantum gravity Living Rev. Relativ. 1 1–34
[40] Randall L and Sundrum R 1999 Phys. Rev. Lett. 83 3370–3
[41] Randall L and Sundrum R 1999 Phys. Rev. Lett. 83 4690–3
[32] Novello M and Perez Bergliaffa S E 2008 Phys. Rep. 463 127–213
[33] Cai Y-F, Qin T, Piao Y-S, Li M and Zhang X 2007 J. High Energy Phys. JHEP10(2007)071
[34] Lin C, Brandenberger R H and Levasseur L P 2011 J. Cosmol. Astropart. Phys. JCAP04(2011)019
[35] Cai Y-F, Chen S-H, Denti J B, Dutta S and Saridakis E N 2011 Class. Quantum Grav. 28 215011
[36] Cai Y-F and Saridakis E N 2009 J. Cosmol. Astropart. Phys. JCAP10(2009)020
[37] Qin T, Evislin J, Cai Y-F, Li M and Zhang X 2011 J. Cosmol. Astropart. Phys. JCAP10(2011)036
[38] Cai Y-F, Gao C and Saridakis E N 2012 J. Cosmol. Astropart. Phys. JCAP10(2012)048
[39] Cai Y-F, Brandenberger R H and Peter P 2013 Class. Quantum Grav. 30 075019
[40] Biswas T, Mazumdar A and Siegel W 2006 J. Cosmol. Astropart. Phys. JCAP03(2006)009
[41] Peter P and Pinto-Neto N 2002 Phys. Rev. D 66 063509
[42] Buchbinder I E, Khoury J and Ovrut B A 2007 Phys. Rev. D 76 123503
[43] Falciano F T, Lilley M and Peter P 2008 Phys. Rev. D 77 083513
[44] Brechet S D, Hobson M P and Lasenby A N 2008 Class. Quantum Grav. 25 245016
[45] Dechant P P, Lasenby A N and Hobson M P 2009 Phys. Rev. D 79 043524
[46] Shironiziti T, Maeda K and Sasaki M 2000 Phys. Rev. D 62 024012
[47] Shitanov Y V 2000 arXiv:hep-th/0005193
Shitanov Y V 2002 Phys. Lett. B 541 177
Shitanov Y and Sahni V 2003 Phys. Lett. B 557 1
[48] Maartens R 2000 Phys. Rev. D 62 084023
Maartens R 2004 Living Rev. Relativ. 7 1–99
[49] Maier R, Damiao Soares I and Pinto-Neto N 2013 Phys. Rev. D 87 043528
[50] Maier R, Damiao Soares I and Tonini E V 2009 Phys. Rev. D 79 023522
[51] Sakharov A D 1984 Zh. Eksp. Teor. Fiz. 87 375
Sakharov A D 1984 Sov. Phys.—JETP 60 214
Barrett J, Gibbons G W, Perry M J, Pope C N and Ruback P J 1994 Int. J. Mod. Phys. A 9 1457
[52] Dvali G R, Gabadadze G and Senjanovic G 1999 Many Faces of the Superworld: Yuri Golfand Memorial ed Y Golfand et al vols 525–32 (Singapore: World Scientific)
[53] Yndurain F J 1991 Phys. Lett. B 256 15
[54] Arefeva Y, Dragovic B G and Volovich I V 1986 Phys. Lett. B 177 357
[55] Chaichian M and Kobakhidze A B 2000 Phys. Lett. B 488 117
[56] Arefeva Y and Volovich I V 1985 Phys. Lett. B 164 287
[57] Iglesias A and Kukushadze Z 2001 Phys. Lett. B 515 477
[58] Maier R, Pace F and Damiao Soares I 2013 Phys. Rev. D 88 106003
[59] Berry M 1978 Regular and irregular motion in topics on linear dynamics AIP Conf. Proc. 46 16–20
[60] Guckenheimer J and Holmes P 1983 Dynamical Systems and Bifurcations of Vector Fields (New York: Springer)
[61] Conley C 1969 J. Differ. Equ. 5 136–58
[62] Grotta-Ragazzo C 1997 Commun. Pure Appl. Math. 50 105–47
[63] Grotta-Ragazzo C 1997 Commun. Math. Phys. 184 251–72
[64] Llibre J, Martinez R and Simó C 1985 J. Differ. Equ. 58 104–56
[65] Moser J K 1973 Stable and Random Motions in Dynamical Systems (Princeton, NJ: Princeton University Press)
[66] Smale S 1967 Bull. Am. Math. Soc. 73 747–817
[67] Wiggins S 1988 Global Bifurcations and Chaos (Berlin: Springer)
[68] Wiggins S 1994 Normally Hyperbolic Invariant Manifolds in Dynamical Systems (Berlin: Springer)
[69] Wiggins S 2003 Introduction to Applied Nonlinear Dynamical Systems and Chaos (Berlin: Springer)
[70] Nusse H E and Yorke J A 1989 Physica D 36 137
[71] Waalkens H and Wiggins S 2010 Regular Chaotic Dyn. 15 139
[72] Waalkens H, Burbanks A and Wiggins S 2004 J. Phys. A: Math. Gen. 37 257–65
[73] Cresson J 2003 J. Differ. Equ. 187 269
[74] Eisenhart L P 1997 Riemannian Geometry (New Jersey: Princeton University Press)
[75] Taub A H 1954 Phys. Rev. 94 6
[76] Israel W 1966 Nuovo Cimento B 44 1
[77] Campos A, Maartens R, Matravers D and Souputa C F 2003 Phys. Rev. D 68 103520
[78] Fabbri A, Langlois D, Steer D A and Zegers R 2004 J. Cosmol. Astropart. Phys. JHEP09 (2004)025
[79] Maartens R, Sahni V and Saini T D 2001 Phys. Rev. D 63 063509
[80] Campos A and Sopuerta C F 2001 Phys. Rev. D 63 104012
[81] Campos A and Sopuerta C F 2001 Phys. Rev. D 64 104011
[82] Coley A A 2002 Phys. Rev. D 66 023512
[83] Coley A A 2002 Class. Quantum Grav. 19 L45
[84] de Oliveira H P, Ozorio de Almeida A M, Damião Soares I and Tonini E V 2002 Phys. Rev. D 65 083511
[85] Sansone G and Conti R 1964 Non-Linear Differential Equations (Oxford: Pergamon)
Poincaré H 1881 J. Math. Pures Appl. 7 375
[86] Siegel C L and Moser J K 1971 Lectures on Celestial Mechanics (Berlin: Springer)
[87] McIntosh H V 1959 Am. J. Phys. 27 620
[88] Sommerville D M Y 1958 The Elements of Non-Euclidean Geometry (New York: Dover)
[89] Arnold V I, Kozlov V V and Neishtadt A I 1988 Mathematical Aspects of Classical and Celestial Mechanics in Dynamical Systems vol 3 (Berlin: Springer)
[90] Murdock J 2003 Normal Forms and Unfoldings for Local Dynamical Systems (Berlin: Springer)
[91] Lichtenberg A J and Lieberman M A 1992 Regular and Chaotic Dynamics (New York: Springer)
[92] Ozorio de Almeida A M 1993 Hamiltonian Systems Chaos and Quantization (Cambridge: Cambridge University Press)