NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS AT RESONANCE
WITH NONLINEAR WENTZELL BOUNDARY CONDITIONS

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Dedicated to the 70th birthday of Jerome A. Goldstein

ABSTRACT. In the first part of the article, we give necessary and sufficient conditions for the solvability of a class of nonlinear elliptic boundary value problems with nonlinear boundary conditions involving the \( q \)-Laplace-Beltrami operator. In the second part, we give some additional results on existence and uniqueness and we study the regularity of the weak solutions for these classes of nonlinear problems. More precisely, we show some global a priori estimates for these weak solutions in an \( L^\infty \)-setting.

1. INTRODUCTION

Let \( \Omega \subset \mathbb{R}^N, N > 1 \), be a bounded domain with a Lipschitz boundary \( \partial \Omega \) and consider the following nonlinear boundary value problem with nonlinear second order boundary conditions:

\[
\begin{aligned}
-\Delta_p u + \alpha_1(u) &= f(x), & \text{in } \Omega, \\
\beta(x)|\nabla u|^{p-2}\nabla u - \rho \beta(x)\Delta_q u + \alpha_2(u) &= g(x), & \text{on } \partial \Omega,
\end{aligned}
\]

where \( \beta \in L^\infty(\partial \Omega) \), \( \beta(x) \geq b_0 > 0 \), for some constant \( b_0 \), \( \rho \) is either 0 or 1, and \( \alpha_1, \alpha_2 \in C(\mathbb{R}, \mathbb{R}) \) are monotone nondecreasing functions such that \( \alpha_i(0) = 0 \). Moreover, \( \Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u) \) is the \( p \)-Laplace operator, \( p \in (1, +\infty) \) and \( f \in L^2(\Omega, dx) \), \( g \in L^2(\partial \Omega, \sigma) \) are given real-valued functions. Here, \( dx \) denotes the usual \( N \)-dimensional Lebesgue measure in \( \Omega \) and \( \sigma \) denotes the restriction to \( \partial \Omega \) of the \((N-1)\)-dimensional Hausdorff measure. Recall that \( \sigma \) coincides with the usual Lebesgue surface measure since \( \Omega \) has a Lipschitz boundary, and \( \partial_u \) denotes the normal derivative of \( u \) in direction of the outer normal vector \( \mathbf{n} \). Furthermore, \( \Delta_{q, \Gamma} \) is defined as the generalized \( q \)-Laplace-Beltrami operator on \( \partial \Omega \), that is, \( \Delta_{q, \Gamma} u = \text{div}_\Gamma(|\nabla_{\Gamma} u|^{q-2}\nabla_{\Gamma} u), q \in (1, +\infty) \). In particular, \( \Delta_2 = \Delta \) and \( \Delta_{2, \Gamma} = \Delta_{\Gamma} \) become the well-known Laplace and Laplace-Beltrami operators on \( \Omega \) and \( \partial \Omega \), respectively. Here, for any real valued function \( v \),

\[
\text{div}_\Gamma v = \sum_{i=1}^{N-1} \partial_{\tau_i} v,
\]

where \( \partial_{\tau_i} v \) denotes the directional derivative of \( v \) along the tangential directions \( \tau_i \) at each point on the boundary, whereas \( \nabla_{\Gamma} v = (\partial_{\tau_1} v, ..., \partial_{\tau_{N-1}} v) \) denotes the tangential gradient at \( \partial \Omega \). It is worth mentioning again that when \( \rho = 0 \) in (1.1), the boundary conditions are...
of lower order than the order of the $p$-Laplace operator, while for $p = 1$, we deal with boundary conditions which have the same differential order as the operator acting in the domain $\Omega$. Such boundary conditions arise in many applications, such as phase-transition phenomena (see, e.g., [13, 14] and the references therein) and have been studied by several authors (see, e.g., [2][12][16][24][28]).

In a recent paper [12], the authors have formulated necessary and sufficient conditions for the solvability of (1.1) when $p = q = 2$, by establishing a sort of "nonlinear Fredholm alternative" for such elliptic boundary value problems. We shall now state their main result. Defining two real parameters $\lambda_1, \lambda_2 \in \mathbb{R}_+$ by

$$\lambda_1 = \int_\Omega dx, \lambda_2 = \int_{\partial \Omega} \frac{d\sigma}{b},$$  \hfill (1.2)

this result reads that a necessary condition for the existence of a weak solution of (1.1) is that

$$\int_\Omega f(x)dx + \int_{\partial \Omega} g(x) \frac{d\sigma}{b(x)} \in (\lambda_1 \mathcal{R}(\alpha_1) + \lambda_2 \mathcal{R}(\alpha_2)), \hfill (1.3)$$

while a sufficient condition is

$$\int_\Omega f(x)dx + \int_{\partial \Omega} g(x) \frac{d\sigma}{b(x)} \in \text{int}(\lambda_1 \mathcal{R}(\alpha_1) + \lambda_2 \mathcal{R}(\alpha_2)), \hfill (1.4)$$

where $\mathcal{R}(\alpha_j)$ denotes the range of $\alpha_j$, $j = 1, 2$ and $\text{int}(G)$ denotes the interior of the set $G$.

Relation (1.3) turns out to be both necessary and sufficient if either of the sets $\mathcal{R}(\alpha_1)$ or $\mathcal{R}(\alpha_2)$ is an open interval. This particular result was established in [12, Theorem 3], by employing methods from convex analysis involving subdifferentials of convex, lower semicontinuous functionals on suitable Hilbert spaces. As an application of our results, we can consider the following boundary value problem

$$\begin{cases} -\Delta u + \alpha_1(u) = f(x), & \text{in } \Omega, \\ b(x) \partial_n u = g(x), & \text{on } \partial \Omega, \end{cases} \hfill (1.5)$$

which is only a special case of (1.1) (i.e., $p = 0, \alpha_2 \equiv 0$ and $p = 2$). According to [12, Theorem 3] (see also (1.4)), this problem has a weak solution if

$$\int_\Omega f(x)dx + \int_{\partial \Omega} g(x) \frac{d\sigma}{b(x)} \in \text{int}(\lambda_1 \mathcal{R}(\alpha_1)), \hfill (1.6)$$

which yields the result of Landesman and Lazer [17] for $g \equiv 0$. This last condition is both necessary and sufficient when the interval $\mathcal{R}(\alpha_1)$ is open. This was put into an abstract context and significantly extended by Brezis and Haraux [8]. Their work was much further extended by Brezis and Nirenberg [9]. The goal of the present article is comparable to that of [12] since we want to establish similar conditions to (1.4) and (1.6) for the existence of solutions to (1.1) when $p, q \neq 2$, with main emphasis on the generality of the boundary conditions.

Recall that $\lambda_1$ and $\lambda_2$ are given by (1.2). Let $I$ be the interval $\lambda_1 \mathcal{R}(\alpha_1) + \lambda_2 \mathcal{R}(\alpha_2)$. Our first main result is as follows (see Section 4 also).

**Theorem 1.1.** Let $\alpha_j : \mathbb{R} \to \mathbb{R}$ ($j = 1, 2$) be odd, monotone nondecreasing, continuous function such that $\alpha_j(0) = 0$. Assume that the functions $\Lambda_j(t) := \int_0^t \alpha_j(s)ds$ satisfy

$$\Lambda_j(2t) \leq C_j \Lambda_j(t), \text{ for all } t \in \mathbb{R}, \hfill (1.7)$$
for some constants $C_j > 1$, $j = 1, 2$. If $u$ is a weak solution of \( (1.1) \) (in the sense of Definition 2.10 below), then

\[
\int_{\Omega} f(x) \, dx + \int_{\partial \Omega} g(x) \frac{d\sigma}{b(x)} \in 1.
\]

(1.8)

Conversely, if

\[
\int_{\Omega} f(x) \, dx + \int_{\partial \Omega} g(x) \frac{d\sigma}{b(x)} \in \text{int}(1),
\]

then \( (1.1) \) has a weak solution.

Our second main result of the paper deals with a modified version of \( (1.1) \) which is obtained by replacing the functions $\alpha_1(t)$, $\alpha_2(t)$ in \( (1.1) \) by $\overline{\alpha}_1(t) + |t|^{p-2}t$ and $\overline{\alpha}_2(t) + \rho b|t|^{q-2}t$, respectively, and also allowing $\overline{\alpha}_1$, $\overline{\alpha}_2$ to depend on $x \in \Omega$. Under additional assumptions on $\overline{\alpha}_1$, $\overline{\alpha}_2$ and under higher integrability properties for the data $(f, g)$, the next theorem provides us with conditions for unique solvability results for solutions to such boundary value problems. Then, we obtain some regularity results for these solutions. In addition to these results, the continuous dependence of the solution to \( (1.1) \) with respect to the data $(f, g)$ can be also established. In particular, we prove the following

**Theorem 1.2.** Let all the assumptions of Theorem 1.1 be satisfied for the functions $\overline{\alpha}_1$, $\overline{\alpha}_2$. Moreover, for each $j = 1, 2$, assume that $\overline{\alpha}_j(t)/t \to 0$, as $t \to 0$ and $\overline{\alpha}_j(t)/t \to \infty$, as $t \to \infty$, respectively.

(a) Then, for every $(f, g) \in L^{p_1}(\Omega) \times L^{q_1}(\partial \Omega)$ with

\[
p_1 > \max \left\{ 1, \frac{N}{p} \right\}, \quad q_1 > \begin{cases} \max \left\{ 1, \frac{N-1}{p-1} \right\}, & \text{if } \rho \in (0, 1), \\
\max \left\{ 1, \frac{N-1}{p} \right\}, & \text{if } \rho = 1 \text{ and } p = q,
\end{cases}
\]

there exists a unique weak solution to problem \( (1.1) \) (in the sense of Definition 5.3 below) which is bounded.

(b) Let $\overline{\alpha}_j$, $j = 1, 2$, be such that

\[
c_j |\overline{\alpha}_j(\xi - \eta)| \leq |\overline{\alpha}_j(\xi) - \overline{\alpha}_j(\eta)|, \text{ for all } \xi, \eta \in \mathbb{R},
\]

for some constants $c_j \in (0, 1]$. Then, the weak (bounded) solution of problem \( (1.1) \) depends continuously on the data $(f, g)$. Precisely, let us indicate by $u_j$ the unique solution corresponding to the data $F_j := (f_j, g_j) \in L^{p_1}(\Omega) \times L^{q_1}(\partial \Omega)$, for each $j = 1, 2$. Then, the following estimate holds:

\[
\|u_{j1} - u_{j2}\|_{L^q(\Omega)} + \|u_{j1} - u_{j2}\|_{L^q(\partial \Omega)} \leq Q(\|f_1 - f_2\|_{L^{p_1}(\Omega)} + \|g_1 - g_2\|_{L^{q_1}(\partial \Omega)}),
\]

for some nonnegative function $Q : \mathbb{R}^2_+ \to \mathbb{R}_+$, $Q(0, 0) = 0$, which can be computed explicitly.

We organize the paper as follows. In Section 2 we introduce some notations and recall some well-known results about Sobolev spaces, maximal monotone operators and Orlicz type spaces which will be needed throughout the article. In Section 3 we show that the subdifferential of a suitable functional associated with problem \( (1.1) \) satisfies a sort of "quasilinear" version of the Fredholm alternative (cf. Theorem 3.5), which is needed in order to obtain the result in Theorem 1.1. Finally, in Sections 4 and 5 we provide detailed proofs of Theorem 1.1 and Theorem 1.2. We also illustrate the application of these results with some examples.
2. Preliminaries and Notations

In this section we put together some well-known results on nonlinear forms, maximal monotone operators and Sobolev spaces. For more details on maximal monotone operators, we refer to the monographs [4, 7, 20, 21, 27]. We will also introduce some notations.

2.1. Maximal monotone operators. Let $H$ be a real Hilbert space with scalar product $(\cdot, \cdot)_H$.

**Definition 2.1.** Let $A : D(A) \subset H \to H$ be a closed (nonlinear) operator. The operator $A$ is said to be:

(i) **monotone**, if for all $u, v \in D(A)$ one has 
$$(Au - Av, u - v)_H \geq 0.$$ 

(ii) **maximal monotone**, if it is monotone and the operator $I + A$ is invertible.

Next, let $V$ be a real reflexive Banach space which is densely and continuously embedded into the real Hilbert space $H$, and let $V'$ be its dual space such that $V \hookrightarrow H \hookrightarrow V'$.

**Definition 2.2.** Let $\mathcal{A} : V \times V \to \mathbb{R}$ be a continuous map.

(a) The map $\mathcal{A} : V \times V \to \mathbb{R}$ is called a nonlinear form on $H$ if for all $u \in V$ one has $\mathcal{A}(u, \cdot) \in V'$, that is, if $\mathcal{A}$ is linear and bounded in the second variable.

(b) The nonlinear form $\mathcal{A} : V \times V \to \mathbb{R}$ is said to be:

(i) **monotone** if $\mathcal{A}(u, u - v) - \mathcal{A}(v, u - v) \geq 0$ for all $u, v \in V$;

(ii) **hemicontinuous** if $\lim_{t \downarrow 0} \mathcal{A}(u + tv, w) = \mathcal{A}(u, w)$, $\forall u, v, w \in V$;

(iii) **coercive**, if $\lim_{\|v\|V \to +\infty} \frac{\mathcal{A}(v, v)}{\|v\|^V} = +\infty$.

Now, let $\varphi : H \to (-\infty, +\infty]$ be a proper, convex, lower semicontinuous functional with effective domain 
$$D(\varphi) := \{u \in H : \varphi(u) < \infty\}.$$ 

The subdifferential $\partial \varphi$ of the functional $\varphi$ is defined by
$$\left\{ \begin{array}{ll}
D(\partial \varphi) & := \{u \in D(\varphi) : \exists w \in H \forall v \in D(\varphi) : \varphi(v) - \varphi(u) \geq (w, v - u)_H\}; \\
\partial \varphi(u) & := \{w \in H : \forall v \in D(\varphi) : \varphi(v) - \varphi(u) \geq (w, v - u)_H\}.
\end{array} \right.$$ 

By a classical result of Minty [20] (see also [7, 21]), $\partial \varphi$ is a maximal monotone operator.

2.2. Functional setup. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary $\partial \Omega$. For $1 < p < \infty$, we let $W^{1,p}(\Omega)$ be the first order Sobolev space, that is,
$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \nabla u \in (L^p(\Omega))^N\}.$$ 

Then $W^{1,p}(\Omega)$, endowed with the norm
$$\|u\|_{W^{1,p}(\Omega)} := \left(\|u\|_{L^p(\Omega)}^p + \|
abla u\|_{L^p(\Omega)}^p\right)^{1/p}$$ 

is a Banach space, where we have set
$$\|u\|_{L^p(\Omega)}^p := \int_{\Omega} |u|^p \, dx.$$ 

Since $\Omega$ has a Lipschitz boundary, it is well-known that there exists a constant $C > 0$ such that
$$\|u\|_{L^p(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}$$ 
for all $u \in W^{1,p}(\Omega), \quad (2.1)$$
where $p_s = \frac{pN}{N-p}$ if $p < N$, and $1 \leq p_s < \infty$ if $N = p$. Moreover the trace operator $\text{Tr}(u) := u|_{\partial \Omega}$ initially defined for $u \in C^1(\overline{\Omega})$ has an extension to a bounded linear operator from $W^{1,p}(\Omega)$ into $L^q(\partial \Omega)$ where $q_s := \frac{p(N-1)}{N-p}$ if $p < N$, and $1 \leq q_s < \infty$ if $N = p$. Hence, there is a constant $C > 0$ such that
\[
\|u\|_{\partial \Omega, q_s} \leq C \|u\|_{W^{1,p}(\Omega)}, \text{ for all } u \in W^{1,p}(\Omega). \tag{2.2}
\]
Throughout the remainder of this article, for $1 < p < N$, we let
\[
p_s := \frac{pN}{N-p} \quad \text{and} \quad q_s := \frac{p(N-1)}{N-p}. \tag{2.3}
\]
If $p > N$, one has that
\[
W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{N}{p}}(\overline{\Omega}), \quad \tag{2.4}
\]
that is, the space $W^{1,p}(\Omega)$ is continuously embedded into $C^{0,1-\frac{N}{p}}(\overline{\Omega})$. For more details, we refer to [23] Theorem 4.7 (see also [19] Chapter 4).

For $1 < q < \infty$, we define the Sobolev space $W^{1,q}(\partial \Omega)$ to be the completion of the space $C^1(\partial \Omega)$ with respect to the norm
\[
\|u\|_{W^{1,q}(\partial \Omega)} := \left( \int_{\partial \Omega} |u|^q \, d\sigma + \int_{\partial \Omega} |\nabla_T u|^q \, d\sigma \right)^{1/q},
\]
where we recall that $\nabla_T u$ denotes the tangential gradient of the function $u$ at the boundary $\partial \Omega$. It is also well-known that $W^{1,q}(\partial \Omega)$ is continuously embedded into $L^q(\partial \Omega)$ where $q_s := \frac{q(N-1)}{N-1-q}$ if $1 < q < N-1$, and $1 \leq q_s < \infty$ if $q = N-1$. Hence, for $1 < q \leq N-1$, there exists a constant $C > 0$ such that
\[
\|u\|_{q_s, \partial \Omega} \leq C \|u\|_{W^{1,q}(\partial \Omega)}, \text{ for all } u \in W^{1,q}(\partial \Omega). \tag{2.5}
\]
Let $\lambda_N$ denote the $N$-dimensional Lebesgue measure and let the measure $\mu := \lambda_N|\Omega \cap \sigma$ on $\overline{\Omega}$ be defined for every measurable set $A \subset \overline{\Omega}$ by
\[
\mu(A) := \lambda_N(\Omega \cap A) + \sigma(A \cap \partial \Omega).
\]
For $p, q \in [1, \infty]$, we define the Banach space
\[
X^{p,q}(\overline{\Omega}, \mu) := \{ F = (f, g) : f \in L^p(\Omega) \text{ and } g \in L^q(\partial \Omega) \}
\]
endowed with the norm
\[
\|F\|_{X^{p,q}(\overline{\Omega})} = \|F\|_{L^p(\Omega)} + \|F\|_{\partial \Omega, q},
\]
if $1 \leq p, q < \infty$, and
\[
\|F\|_{X^{\infty,\infty}(\overline{\Omega}, \mu)} = \|F\|_{L^\infty} := \max\{ \|f\|_{\Omega, \infty}, \|g\|_{\partial \Omega, \infty} \}.
\]
If $p = q$, we will simply denote $\|F\|_{L^p(\Omega)} = \|F\|_{L^p(\partial \Omega)}$.

Identifying each function $u \in W^{1,p}(\Omega)$ with $U = (u, u|_{\partial \Omega})$, we have that $W^{1,p}(\Omega)$ is a subspace of $X^{p,p}(\overline{\Omega}, \mu)$.

For $1 < p, q < \infty$, we endow
\[
\mathcal{Y}_1 := \{ U := (u, u|_{\partial \Omega}) : u \in W^{1,p}(\Omega), u|_{\partial \Omega} \in W^{1,q}(\partial \Omega) \}
\]
with the norm
\[
\|U\|_{\mathcal{Y}_1} := \|u\|_{W^{1,p}(\Omega)} + \|u\|_{W^{1,q}(\partial \Omega)}.
\]
while
\[
\mathcal{Y}_0 := \{ U := (u, u|_{\partial \Omega}) : u \in W^{1,p}(\Omega) \}
\]
is endowed with the norm
\[ \|U\|_{\mathcal{X}_0} := \|u\|_{W^{1,p}(\Omega)}. \]

It follows from (2.1)-(2.2) that \( \mathcal{X}_0 \) is continuously embedded into \( X^{p,q}(\Omega, \mu) \), with \( p_s \) and \( q_s \) given by (2.3), for \( 1 < p < N \). Moreover, by (2.1) and (2.5), \( \mathcal{X}_1 \) is continuously embedded into \( X^{p,q}(\Omega, \mu) \).

2.3. Musielak-Orlicz type spaces. For the convenience of the reader, we introduce the Orlicz and Musielak-Orlicz type spaces and prove some properties of these spaces which will be frequently used in the sequel (see Section 5).

**Definition 2.3.** Let \((X, \Sigma, \nu)\) be a complete measure space. We call a function \( B : X \times \mathbb{R} \to [0, \infty] \) a Musielak-Orlicz function on \( X \) if

(a) \( B(x, \cdot) \) is non-trivial, even, convex for \( \nu \)-a.e. \( x \in X \);
(b) \( B(x, \cdot) \) is vanishing and continuous at 0 for \( \nu \)-a.e. \( x \in X \);
(c) \( B(x, \cdot) \) is left continuous on \( [0, \infty) \);
(d) \( B(\cdot, t) \) is \( \Sigma \)-measurable for all \( t \in [0, \infty) \);
(e) \( \lim_{t \to \infty} \frac{B(x,t)}{t} = \infty. \)

The complementary Musielak-Orlicz function \( \widetilde{B} \) is defined by
\[ \widetilde{B}(x,t) := \sup \{s|t| - B(x,s) : s > 0\}. \]

It follows directly from the definition that for \( t, s \geq 0 \) (and hence for all \( t, s \in \mathbb{R} \))
\[ st \leq B(x,t) + \widetilde{B}(x,s). \]

**Definition 2.4.** We say that a Musielak-Orlicz function \( B \) satisfies the \((\Delta^\alpha_0)\)-condition \((\alpha > 1)\) if there exists a set \( X_0 \) of \( \nu \)-measure zero and a constant \( C_\alpha > 1 \) such that
\[ B(x, \alpha t) \leq C_\alpha B(x,t), \]
for all \( t \in \mathbb{R} \) and every \( x \in X \setminus X_0. \)

We say that \( B \) satisfies the \((\nabla^2_0)\)-condition if there is a set \( X_0 \) of \( \nu \)-measure zero and a constant \( c > 1 \) such that
\[ B(x,t) \leq \frac{1}{2c} B(x,ct), \]
for all \( t \in \mathbb{R} \) and all \( x \in X \setminus X_0. \)

**Definition 2.5.** A function \( \Phi : \mathbb{R} \to [0, \infty] \) is called an \( \mathcal{N} \)-function if

- \( \Phi \) is even, strictly increasing and convex;
- \( \Phi(t) = 0 \) if and only if \( t = 0 \);
- \( \lim_{t \to 0^+} \frac{\Phi(t)}{t} = 0 \) and \( \lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty. \)

We say that an \( \mathcal{N} \)-function \( \Phi \) satisfies the \((\nabla_2)\)-condition if there exists a constant \( C_2 > 1 \) such that
\[ \Phi(2t) \leq C_2 \Phi(t), \quad \text{for all } t \in \mathbb{R}, \]
and it satisfies the \((\nabla_2)\)-condition if there is a constant \( c > 1 \) such that
\[ \Phi(t) \leq \Phi(ct)/(2c), \quad \text{for all } t \in \mathbb{R}. \]

For more details on \( \mathcal{N} \)-functions, we refer to the monograph of Adams [11, Chapter VIII] (see also [25, Chapter I], [26, Chapter I]).
Moreover, if On this space we consider the Luxemburg norm Corollary 2.8. Hence, \( \Phi(t) = \int_0^{|t|} \varphi(s) \, ds \), \( \Psi(t) := \int_0^{|t|} \psi(s) \, ds = \sup\{ |t|s - \Phi(s) : s > 0 \}. \)

As before for all \( s, t \in \mathbb{R} \),

\[
\forall \leq \Phi(t) + \Psi(s).
\]

Moreover, if \( s = \varphi(t) \) or \( t = \psi(s) \) then we have equality, that is,

\[
\Psi(\varphi(t)) = t\varphi(t) - \Phi(t) .
\]

The function \( \Psi \) is called the complementary \( \mathcal{N} \)-function of \( \Phi \). It is also known that an \( \mathcal{N} \)-function \( \Phi \) satisfies the \((\triangle_0)\)-condition if and only if

\[
ct\varphi(t) \leq \Phi(t) \leq t\varphi(t) ,
\]

for some constant \( c \in (0, 1] \) and for all \( t \in \mathbb{R} \), where \( \varphi \) is the left-sided derivative of \( \Phi \).

**Lemma 2.7.** Let \( \Phi \) be an \( \mathcal{N} \)-function which satisfies the \((\triangle_0)\)-condition with the constant \( C_2 > 1 \) and let \( \Psi \) be its complementary \( \mathcal{N} \)-function. Then \( \Psi \) satisfies the \((\triangledown_0)\)-condition with the constant \( c := 2^{C_2^{-1}} \).

**Proof.** We have

\[
t\varphi(t) \leq \int_t^{2t} \varphi(s) \, ds \leq \int_0^{2t} \varphi(s) \, ds = \Phi(2t) \leq C_2\Phi(t).
\]

Since \( \varphi(\psi(s)) \geq s \) for all \( s \geq 0 \) and \( s/\Psi(s) \) and \( s/(s - 1) \) are decreasing, we get for \( t := \psi(s) \), that

\[
\frac{s\psi(s)}{\Psi(s)} \geq \frac{\varphi(\psi(s))}{\Psi(\varphi(s))} \geq \frac{t\varphi(t)}{\Psi(\varphi(t))} = \frac{t\varphi(t)}{t\varphi(t) - \Phi(t)} \geq \frac{C_2}{C_2 - 1}.
\]

Now let \( c := 2^{C_2^{-1}} \). Then for \( t \geq 0 \),

\[
\ln \left( \frac{\Psi(ct)}{\Psi(t)} \right) = \int_t^{ct} \frac{\psi(s)}{\Psi(s)} \, ds \geq \int_t^{ct} \frac{C_2}{s(C_2 - 1)} \, ds.
\]

\[
= \frac{C_2}{C_2 - 1} \ln(c) = C_2 \ln(2) = \ln(2 \cdot 2^{C_2^{-1}}).
\]

Hence, \( \Psi(t)2^{c} \leq \Psi(ct) \).

**Corollary 2.8.** Let \( B \) be a Musielak-Orlicz function such that \( B(x, \cdot) \) is an \( \mathcal{N} \)-function for \( \nu \)-a.e. \( x \). If \( B \) satisfies the \((\triangle_0)\)-condition, then \( \tilde{B} \) satisfies the \((\triangledown_0)\)-condition.

**Definition 2.9.** Let \( B \) be a Musielak-Orlicz function. Then the Musielak-Orlicz space \( L^B(X) \) associated with \( B \) is defined by

\[
L^B(X) := \{ u : X \to \mathbb{R} \text{ measurable} : \rho_B(u/\alpha) < \infty \text{ for some } \alpha > 0 \},
\]

where

\[
\rho_B(v) := \int_X B(x, v(x)) \, d\nu(x).
\]

On this space we consider the Luxemburg norm \( \| \cdot \|_{X,B} \) defined by

\[
\| u \|_{X,B} := \inf\{ \alpha > 0 : \rho_B(u/\alpha) \leq 1 \}.
\]
Proof. If $B$ satisfies the $(\nabla^0_N)$-condition, then there exists a set $X_0 \subset X$ of measure zero such that for every $\varepsilon > 0$ there exists $\alpha = \alpha(\varepsilon) > 0$, 

$$B(x, \alpha t) \leq \alpha \varepsilon B(x, t),$$

for all $t \in \mathbb{R}$ and all $x \in X \setminus X_0$. Let $\lambda \in (0, \infty)$ be fixed. For $\varepsilon := 1/\lambda$ there exists $\alpha > 0$ satisfying the above inequality. We will show that $\rho_B(u) \geq \lambda \|u\|_{X,B}$ whenever $\|u\|_{X,B} > 1/\alpha$. Assume that $\|u\|_{X,B} > 1/\alpha$ and let $\delta > 0$ be such that $\alpha = (1 + \delta)/\|u\|_{X,B}$. Then 

$$\rho_B(\alpha u) = \int_X B(x, u(1 + \delta)/\|u\|_{X,B}) \, d\mu$$

$$\geq (1 + \delta)^{1-1/n} \int_X B(x, u(1 + \delta)^{1/n}/\|u\|_{X,B}) \, d\mu \geq (1 + \delta)^{1-1/n},$$

for all $n \in \mathbb{N}$. If we assume that the last inequality does not hold, then 

$$\|u\|_{X,B}/(1 + \delta) \in \{ \alpha > 0 : \rho(u/\alpha) \leq 1 \},$$

and this clearly contradicts the definition of $\|u\|_{X,B}$. Therefore, we must have 

$$\rho_B(\alpha u) \geq 1 + \delta = \alpha \|u\|_{X,B}. \quad (2.10)$$

From (2.9), (2.10), we obtain 

$$\rho_B(u) = \int_X B(x, u(x)) \, d\mu \geq \frac{\lambda}{\alpha} \int_X B(x, \alpha u(x)) \, d\mu = \frac{\lambda}{\alpha} \rho_B(\alpha u) \geq \lambda \|u\|_{X,B}.$$

The proof is finished. \qed

Corollary 2.11. Let $B$ be a Musielak-Orlicz function such that $B(x, \cdot)$ is an $N$-function for $\nu$-a.e. $x$. If its complementary $N$-function $\tilde{B}$ satisfies the $(\Delta^0_N)$-condition, then $B$ satisfies the $(\nabla^0_N)$-condition and 

$$\lim_{\|u\|_{X,B} \to +\infty} \frac{\rho_B(u)}{\|u\|_{X,B}} = +\infty.$$ 

2.4. Some tools. For the reader’s convenience, we report here below some useful inequalities which will be needed in the course of investigation.

Lemma 2.12. Let $a, b \in \mathbb{R}^N$ and $p \in (1, \infty)$. Then, there exists a constant $C_p > 0$ such that 

$$\left( |a|^{p-2} a - |b|^{p-2} b \right) (a - b) \geq C_p (|a| + |b|)^{p-2} |a - b|^2 \geq 0. \quad (2.11)$$

If $p \in [2, \infty)$, then there exists a constant $c_p \in (0, 1]$ such that 

$$\left( |a|^{p-2} a - |b|^{p-2} b \right) (a - b) \geq c_p |a - b|^p. \quad (2.12)$$

Proof. The proof of (2.11) is included in [10] Lemma I.4.4. In order to show (2.11), one only needs to show that the left hand side is non-negative, which follows easily. \qed

The following result which is of analytic nature and whose proof can be found in [22], Lemma 3.11] will be useful in deriving some a priori estimates of weak solutions of elliptic equations.
Lemma 2.13. Let \( \psi : [k_0, \infty) \to \mathbb{R} \) be a non-negative, non-increasing function such that there are positive constants \( c, \alpha \) and \( \delta \) (\( \delta > 1 \)) such that
\[
\psi(h) \leq c(h-k)^{-\alpha} \psi(k)^\delta, \quad \forall k > h \geq k_0.
\]
Then \( \psi(k_0 + d) = 0 \) with \( d = c^{1/\alpha} \psi(k_0)^{(\delta-1)/\alpha \delta (\delta-1)} \).

3. The Fredholm Alternative

In what follows, we assume that \( \Omega \subset \mathbb{R}^N \) is a bounded domain with Lipschitz boundary \( \partial \Omega \). Let \( b \in L^\infty(\partial \Omega) \) satisfy \( b(x) \geq b_0 > 0 \) for some constant \( b_0 \). Let \( X_2 \) be the real Hilbert space \( L^2(\Omega, dx) \oplus L^2(\partial \Omega, d\sigma) \). Then, it is clear that \( X_2 \) is isomorphic to \( X^{2,2}(\overline{\Omega}, \lambda_N \oplus \sigma) \) with equivalent norms.

Next, let \( \rho \in \{0, 1\} \) and \( p, q \in (1, \infty) \) be fixed. We define the functional \( J_\rho : X_2 \to [0, +\infty] \) by setting
\[
J_\rho(U) = \begin{cases} \int_\Omega |\nabla u|^p \, dx + \int_{\partial \Omega} \rho \, |\nabla \Gamma u|^q \, d\sigma, & \text{if } U = (u, u|_{\partial \Omega}) \in D(J_\rho), \\ +\infty, & \text{if } U \in X_2 \setminus D(J_\rho), \end{cases}
\]
where the effective domain is given \( D(J_\rho) = Y_\rho \cap X_2 \).

Throughout the remainder of this section, we let \( \mu := \lambda_N \oplus \frac{d\sigma}{b} \). The following result can be obtained easily.

Proposition 3.1. The functional \( J_\rho \) defined by (3.1) is proper, convex and lower semicontinuous on \( X_2 = X^{2,2}(\overline{\Omega}, \mu) \).

The following result contains a computation of the subdifferential \( \partial J_\rho \) for the functional \( J_\rho \).

Remark 3.2. Let \( U = (u, u|_{\partial \Omega}) \in D(\partial J_\rho) \) and let \( F := (f, g) \in \partial J_\rho(U) \). Then, by definition, \( F \in X_2 \) and for all \( V = (v, v|_{\partial \Omega}) \in D(J_\rho) \), we have
\[
\int_\Omega F(V-U) \, d\mu \leq \int_\Omega \frac{1}{p} \left| \nabla v \right|^p \, dx + \int_{\partial \Omega} \frac{1}{q} \left| \nabla \Gamma v \right|^q \, d\sigma.
\]
Let \( W = (w, w|_{\partial \Omega}) \in D(J_\rho) \), \( 0 < t \leq 1 \) and set \( V := tW + U \) above. Dividing by \( t \) and taking the limit as \( t \downarrow 0 \), we obtain that
\[
\int_\Omega FW \, d\mu \leq \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla w \, dx + \int_{\partial \Omega} |\nabla \Gamma u|^{q-2} \nabla \Gamma u \cdot \nabla \Gamma w \, d\sigma,
\]
where we recall that
\[
\int_{\Omega} F \, d\mu = \int_{\Omega} f \, dx + \int_{\partial \Omega} g \, d\sigma - b.
\]
Choosing \( w = \pm \psi \) with \( \psi \in \mathcal{D}(\Omega) \) (the space of test functions) and integrating by parts in (3.2), we obtain
\[
-\Delta u = f \quad \text{in } \mathcal{D}'(\Omega)
\]
and
\[
g = b(x) |\nabla u|^{p-2} \partial u - \rho b(x) \Delta u \quad \text{weakly on } \partial \Omega.
\]
Therefore, the single valued operator \( \partial J_\rho \) is given by
\[
D(\partial J_\rho) = \{ U = (u, u|_{\partial \Omega}) \in D(J_\rho), (\Delta u, b(x) |\nabla u|^{p-2} \partial u - \rho b(x) \Delta u) \in X_2 \}.
\]
\[ \partial \mathcal{J}_\rho(U) = \left( -\Delta_{\rho} u, b(x) |\nabla u|^{p-2} \partial_{\eta} u - \rho b(x) \Delta_{\rho} u \right). \] (3.3)

Since the functional \( \mathcal{J}_\rho \) is proper, convex and lower semicontinuous, it follows that its subdifferential \( \partial \mathcal{J}_\rho \) is a maximal monotone operator.

In the following two lemmas, we establish a relation between the null space of the operator \( A_\rho := \partial \mathcal{J}_\rho \) and its range.

**Lemma 3.3.** Let \( \mathcal{N}(A_\rho) \) denote the null space of the operator \( A_\rho \). Then

\[ \mathcal{N}(A_\rho) = C1 = \{ C = (c, c) : c \in \mathbb{R} \}, \]

that is, \( \mathcal{N}(A_\rho) \) consists of all the real constant functions on \( \Omega \).

**Proof.** We say that \( U \in \mathcal{N}(A_\rho) \) if and only if (by definition) \( U = (u, u|_{\partial \Omega}) \) is a weak solution of

\[ \begin{cases}
-\Delta_{\rho} u = 0, & \text{in } \Omega, \\
\quad b(x) |\nabla u|^{p-2} \partial_{\eta} u - \rho b(x) \Delta_{\rho} u = 0, & \text{on } \partial \Omega.
\end{cases} \] (3.4)

A function \( U = (u, u|_{\partial \Omega}) \in \mathcal{V}_\rho \cap \mathcal{K}_2 \) is said to be a weak solution of (3.4), if for every \( V = (v, v|_{\partial \Omega}) \in \mathcal{V}_\rho \cap \mathcal{K}_2 \), there holds

\[ \mathcal{A}_\rho(U, V) := \int_\Omega |\nabla u|^p \nabla u \cdot \nabla v \, dx + \rho \int_{\partial \Omega} |\nabla u|^{p-2} \nabla_{\rho} u \cdot \nabla v \, d\sigma = 0. \] (3.5)

Let \( C := (c, c) \) with \( c \in \mathbb{R} \). Then it is clear that \( C \in \mathcal{N}(A_\rho) \).

Conversely, let \( U = (u, u|_{\partial \Omega}) \in \mathcal{N}(A_\rho) \). Then, it follows from (3.5) that

\[ \mathcal{A}_\rho(U, U) := \int_\Omega |\nabla u|^p \, dx + \rho \int_{\partial \Omega} |\nabla u|^{p-2} \nabla_{\rho} u \cdot \nabla u \, d\sigma = 0. \] (3.6)

Since \( \Omega \) is bounded and connected, this implies that \( u \) is equal to a constant. Therefore, \( U = C1 \) and this completes the proof. \( \square \)

**Lemma 3.4.** The range of the operator \( A_\rho \) is given by

\[ \mathcal{R}(A_\rho) = \left\{ F := (f, g) \in \mathcal{X}_2 : \int_{\Omega} F \, d\mu := \int_{\Omega} f \, dx + \int_{\partial \Omega} g \frac{d\sigma}{b(x)} = 0 \right\}. \]

**Proof.** Let \( F \in \mathcal{R}(A_\rho) \subset \mathcal{X}_2 \). Then there exists \( U = (u, u|_{\partial \Omega}) \in \mathcal{D}(A_\rho) \) such that \( A_\rho(U) = F \). More precisely, for every \( V = (v, v|_{\partial \Omega}) \in \mathcal{V}_\rho \cap \mathcal{K}_2 \), we have

\[ \mathcal{A}_\rho(U, V) = \int_{\Omega} |\nabla u|^p \nabla u \cdot \nabla v \, dx + \rho \int_{\partial \Omega} |\nabla u|^{p-2} \nabla_{\rho} u \cdot \nabla v \, d\sigma \]
\[ = \int_{\Omega} FV \, d\mu. \] (3.7)

Taking \( V = (1, 1) \in \mathcal{V}_\rho \cap \mathcal{K}_2 \), we obtain that \( \int_{\Omega} F \, d\mu = 0 \). Hence,

\[ \mathcal{R}(A_\rho) \subseteq \left\{ F \in \mathcal{X}_2 : \int_{\Omega} F \, d\mu = 0 \right\}. \]
Let us now prove the converse. To this end, let $F \in \mathcal{X}_2$ be such that $\int_{\Omega} F \, d\mu = 0$. We have to show that $F \in \mathcal{R}(A_P)$, that is, there exists $U \in \mathcal{Y}_\rho \cap \mathcal{X}_2$ such that (3.7) holds, for every $V \in \mathcal{Y}_\rho \cap \mathcal{X}_2$. To this end, consider

$$\mathcal{Y}_{\rho,0} := \left\{ U = (u,u,\sigma) \in \mathcal{Y}_\rho \cap \mathcal{X}_2 : \int_{\Omega} U \, d\mu := \int_{\Omega} u \, dx + \int_{\partial \Omega} u \frac{d\sigma}{b} = 0 \right\}.$$  

It is clear that $\mathcal{Y}_{\rho,0}$ is a closed linear subspace of $\mathcal{Y}_{\rho} \cap \mathcal{X}_2 \hookrightarrow \mathcal{X}_2$, and therefore is a reflexive Banach space. Using [18, Section 1.1], we have that the norm

$$\|U\|_{\mathcal{Y}_{\rho,0}} := \|\nabla u\|_{\rho,\Omega} + \rho \|\nabla u\|_{q,\partial \Omega}$$

defines an equivalent norm on $\mathcal{Y}_{\rho,0}$. Hence, there exists a constant $C > 0$ such that for every $U \in \mathcal{Y}_{\rho,0}$,

$$\|U\|_{2} \leq C \|U\|_{\mathcal{Y}_{\rho,0}} := \|\nabla u\|_{\rho,\Omega} + \rho \|\nabla u\|_{q,\partial \Omega}.$$  

(3.8)

Define the functional $\mathcal{F}_\rho : \mathcal{Y}_{\rho,0} \to \mathbb{R}$ by

$$\mathcal{F}_\rho(U) = \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx + \frac{\rho}{q} \int_{\partial \Omega} |\nabla u|^q \, d\sigma - \int_{\Omega} F \, d\mu.$$  

It is easy to see that $\mathcal{F}_\rho$ is convex and lower-semicontinuous on $\mathcal{X}_2$ (see Proposition 3.1). We show now that $\mathcal{F}_\rho$ is coercive. By exploiting a classical Hölder inequality and using (3.8), we have

$$\left| \int_{\Omega} F \, d\mu \right| \leq C \|F\|_2 \|U\|_2 \leq C \|F\|_2 \|U\|_{\mathcal{Y}_{\rho,0}}$$

$$= C \|F\|_2 \left( \|\nabla u\|_{\rho,\Omega} + \rho \|\nabla u\|_{q,\partial \Omega} \right).$$

Obviously, this estimate yields

$$- \int_{\Omega} F \, d\mu \geq -C \|F\|_2 \left( \|\nabla u\|_{\rho,\Omega} + \rho \|\nabla u\|_{q,\partial \Omega} \right).$$  

(3.9)

Therefore, from (3.9), we immediately get

$$\frac{\mathcal{F}_\rho(U)}{\|U\|_{\mathcal{Y}_{\rho,0}}} \geq \frac{\frac{1}{p} \|\nabla u\|^p_{\rho,\Omega} + \frac{\rho}{q} \|\nabla u\|^q_{q,\partial \Omega}}{\|\nabla u\|_{\rho,\Omega} + \rho \|\nabla u\|_{q,\partial \Omega}} - C \|F\|_2.$$  

This inequality implies that

$$\lim_{\|U\|_{\mathcal{Y}_{\rho,0}} \to +\infty} \frac{\mathcal{F}_\rho(U)}{\|U\|_{\mathcal{Y}_{\rho,0}}} = +\infty,$$

and this shows that the functional $\mathcal{F}_\rho$ is coercive. Since $\mathcal{F}_\rho$ is also convex, lower-semicontinuous, it follows from [3 Theorem 3.3.4] that, there exists a function $U^* \in \mathcal{Y}_{\rho,0}$ which minimizes $\mathcal{F}_\rho$. More precisely, for all $V \in \mathcal{Y}_{\rho,0}$, $\mathcal{F}_\rho(U^*) \leq \mathcal{F}_\rho(V)$; this implies that for every $0 < t \leq 1$ and every $V \in \mathcal{Y}_{\rho,0}$,

$$\mathcal{F}_\rho(U^* + tV) - \mathcal{F}_\rho(U^*) \geq 0.$$

Hence,

$$\lim_{t \downarrow 0} \frac{\mathcal{F}_\rho(U^* + tV) - \mathcal{F}_\rho(U^*)}{t} \geq 0.$$
Using the Lebesgue Dominated Convergence, an easy computation shows that

\[
0 \leq \lim_{t \to 0} \frac{\mathcal{F}_\rho(U^* + tV) - \mathcal{F}_\rho(U^*)}{t} = \int_\Omega |\nabla u^*|^{p-2} \nabla u^* \cdot \nabla v \, dx + \rho \int_{\partial \Omega} |\nabla \Gamma u^*|^{q-2} \nabla \Gamma u^* \cdot \nabla \Gamma v \cdot d\sigma - \int_{\Omega} FV \, d\mu.
\]

Changing \( V \) to \(-V\) into (3.10) gives that

\[
\int_\Omega |\nabla u^*|^{p-2} \nabla u^* \cdot \nabla v \, dx + \rho \int_{\partial \Omega} |\nabla \Gamma u^*|^{q-2} \nabla \Gamma u^* \cdot \nabla \Gamma v \cdot d\sigma = \int_{\Omega} FV \, d\mu,
\]

for every \( V \in \mathcal{V}_{\rho,0} \). Now, let \( V \in \mathcal{V}_{\rho} \cap \mathcal{X}_2 \). Writing \( V = -C + C \) with \( C = (c,c) \),

\[
c := \frac{1}{(\lambda_1 + \lambda_2)} \left( \int_\Omega v \, dx + \int_{\partial \Omega} v \, d\sigma \right),
\]

and using the fact that \( \int_{\Omega} F \, d\mu = 0 \), we obtain, for every \( V \in \mathcal{V}_{\rho} \cap \mathcal{X}_2 \), that

\[
\int_\Omega |\nabla u^*|^{p-2} \nabla u^* \cdot \nabla v \, dx + \rho \int_{\partial \Omega} |\nabla \Gamma u^*|^{q-2} \nabla \Gamma u^* \cdot \nabla \Gamma v \cdot d\sigma = \int_{\Omega} FV \, d\mu.
\]

Therefore, \( A_\rho(U) = F \). Hence, \( F \in \mathcal{R}(A_\rho) \) and this completes the proof of the lemma. \( \square \)

The following result is a direct consequence of Lemmas 3.3 and 3.4. This is the main result of this section.

**Theorem 3.5.** The operator \( A_\rho = \partial \mathcal{F}_\rho \) satisfies the following type of “quasi-linear” Friedrichs alternative:

\[
\mathcal{R}(A_\rho) = \mathcal{N}(A_\rho)^\perp = \left\{ F \in \mathcal{X}_2 : \langle F, 1 \rangle_{\mathcal{X}_2} = 0 \right\}.
\]

4. **NECESSARY AND SUFFICIENT CONDITIONS FOR EXISTENCE OF SOLUTIONS**

In this section, we prove the first main result (cf. Theorem 1.1) for problem (1.1). Before we do so, we will need the following results from maximal monotone operators theory and convex analysis.

**Definition 4.1.** Let \( \mathcal{H} \) be a real Hilbert space. Two subsets \( K_1 \) and \( K_2 \) of \( \mathcal{H} \) are said to be almost equal, written, \( K_1 \simeq K_2 \), if \( K_1 \) and \( K_2 \) have the same closure and the same interior, that is, \( \overline{K_1} = \overline{K_2} \) and \( \text{int}(K_1) = \text{int}(K_2) \).

The following abstract result is taken from [8] Theorem 3 and Generalization in p.173–174.

**Theorem 4.2 (Brezis-Haraux).** Let \( A \) and \( B \) be subdifferentials of proper convex lower semicontinuous functionals \( \varphi_1 \) and \( \varphi_2 \), respectively, on a real Hilbert space \( \mathcal{H} \) with \( D(\varphi_1) \cap D(\varphi_2) \neq \emptyset \), and let \( C \) be the subdifferential of the proper, convex lower semicontinuous functional \( \varphi_1 + \varphi_2 \), that is \( C = \partial(\varphi_1 + \varphi_2) \). Then

\[
\mathcal{R}(A) + \mathcal{R}(B) \subset \mathcal{R}(C) \quad \text{and} \quad \text{Int}(\mathcal{R}(A) + \mathcal{R}(B)) \subset \mathcal{R}(C)
\]

In particular, if the operator \( A + B \) is maximal monotone, then

\[
\mathcal{R}(A + B) \simeq \mathcal{R}(A) + \mathcal{R}(B),
\]

and this is the case if \( \partial(\varphi_1 + \varphi_2) = \partial \varphi_1 + \partial \varphi_2 \).
4.1. Assumptions and intermediate results. Let us recall that the aim of this section is to establish some necessary and sufficient conditions for the solvability of the following nonlinear elliptic problem:

\[
\begin{aligned}
-\Delta_p u + \alpha_1(u) &= f, & \text{in } \Omega, \\
& \quad b(x)|\nabla u|^{p-2} \partial_t u - \rho b(x) \Delta_p u + \alpha_2(u) &= g, & \text{on } \partial\Omega,
\end{aligned}
\]  

(4.1)

where \( p, q \in (1, +\infty) \) are fixed. We also assume that \( \alpha_j : \mathbb{R} \to \mathbb{R} \) \((j = 1, 2)\) satisfy the following assumptions.

**Assumption 4.3.** The functions \( \alpha_j : \mathbb{R} \to \mathbb{R} \) \((j = 1, 2)\) are odd, monotone nondecreasing, continuous and satisfy \( \alpha_j(0) = 0 \).

Let \( \tilde{\alpha}_j \) be the inverse of \( \alpha_j \). We define the functions \( \Lambda_j, \tilde{\Lambda}_j : \mathbb{R} \to \mathbb{R}_+ \) \((j = 1, 2)\) by

\[
\Lambda_j(t) := \int_0^{|t|} \alpha_j(s) ds \quad \text{and} \quad \tilde{\Lambda}_j(t) := \int_0^{|t|} \tilde{\alpha}_j(s) ds.
\]

Then it is clear that \( \Lambda_j, \tilde{\Lambda}_j \) are even, convex and monotone increasing on \( \mathbb{R}_+ \), with \( \Lambda_j(0) = \tilde{\Lambda}_j(0) \), for each \( j = 1, 2 \). Moreover, since \( \alpha_j \) are odd, we have \( \Lambda_j(t) = \alpha_j(t) \), for all \( t \in \mathbb{R} \) and \( j = 1, 2 \), with a similar relation holding for \( \tilde{\Lambda}_j \) as well. The following result whose proof is included in \([25, \text{Chap. I, Section 1.3, Theorem 3}]\) holds.

**Lemma 4.4.** The functions \( \Lambda_j \) and \( \tilde{\Lambda}_j \) \((j = 1, 2)\) satisfy \((2.6)\) and \((2.7)\). More precisely, for all \( s, t \in \mathbb{R} \),

\[
st \leq \Lambda_j(s) + \tilde{\Lambda}_j(t).
\]

If \( s = \alpha_j(t) \) or \( t = \tilde{\alpha}_j(s) \), then we also have equality, that is,

\[
\tilde{\Lambda}_j(\alpha_j(s)) = s\alpha_j(s) - \Lambda_j(s), \quad j = 1, 2.
\]

We note that in \([25]\), the statement of Lemma 4.4 assumed that \( \Lambda_j, \tilde{\Lambda}_j \) are \( \mathcal{N} \)-functions in the sense of Definition 2.5. However, the conclusion of that result holds under the weaker hypotheses of Lemma 4.4.

Define the functional \( \mathcal{J}_2 : \mathcal{Y}_2 \to [0, +\infty] \) by

\[
\mathcal{J}_2(u, v) := \begin{cases} 
\int_\Omega \Lambda_1(u) dx + \int_{\partial\Omega} \Lambda_2(v) \frac{d\sigma}{\partial}, & \text{if } (u, v) \in D(\mathcal{J}_2), \\
+\infty, & \text{if } (u, v) \notin \mathcal{Y}_2 \setminus D(\mathcal{J}_2),
\end{cases}
\]

with the effective domain

\[
D(\mathcal{J}_2) := \left\{ (u, v) \in \mathcal{Y}_2 : \int_\Omega \Lambda_1(u) dx + \int_{\partial\Omega} \Lambda_2(v) \frac{d\sigma}{\partial} < \infty \right\}.
\]

**Lemma 4.5.** Let \( \alpha_j \) \((j = 1, 2)\) satisfy Assumption 4.3. Then the functional \( \mathcal{J}_2 \) is proper, convex and lower semicontinuous on \( \mathcal{Y}_2 \).

**Proof.** It is routine to check that \( \mathcal{J}_2 \) is convex and proper. This follows easily from the convexity of \( \Lambda_j \) and the fact that \( \alpha_j(0) = 0 \). To show the lower semicontinuity on \( \mathcal{Y}_2 \), let \( U_n = (u_n, v_n) \in D(\mathcal{J}_2) \) be such that \( U_n \to U := (u, v) \) in \( \mathcal{Y}_2 \) and \( \mathcal{J}_2(U_n) \leq C \) for some constant \( C > 0 \). Since \( U_n \to U \) in \( \mathcal{Y}_2 \), then there is a subsequence, which we also denote by \( U_n = (u_n, v_n) \), such that \( u_n \to u \) a.e. on \( \Omega \) and \( v_n \to v \) \( \sigma \)-a.e. on \( \Gamma \). Since \( \alpha_j(\cdot) \) are continuous (thus, lower-semicontinuous), we have

\[
\Lambda_1(u) \leq \liminf_{n \to +\infty} \Lambda_1(u_n) \quad \text{and} \quad \Lambda_2(v) \leq \liminf_{n \to +\infty} \Lambda_2(v_n).
\]
By Fatou’s Lemma, we obtain
\[
\int_{\Omega} \Lambda_1(u) \, dx \leq \int_{\Omega} \liminf_{n \to \infty} \Lambda_1(u_n) \, dx \leq \liminf_{n \to \infty} \int_{\Omega} \Lambda_1(u_n) \, dx
\]
and
\[
\int_{\partial \Omega} \Lambda_2(v) \frac{d\sigma}{b} \leq \int_{\partial \Omega} \liminf_{n \to \infty} \Lambda_2(v_n) \frac{d\sigma}{b} \leq \liminf_{n \to \infty} \int_{\partial \Omega} \Lambda_2(v_n) \frac{d\sigma}{b}.
\]
Hence, \( \mathcal{J}_2 \) is lower semicontinuous on \( \mathcal{X}_2 \).

We have the following result whose proof is contained in [25, Chap. III, Section 3.1, Theorem 2].

**Lemma 4.6.** Let \( \alpha_j \) (\( j = 1, 2 \)) satisfy Assumption 4.3 and assume that there exist constants \( C_j > 1 \) (\( j = 1, 2 \)) such that
\[
\Lambda_j(2t) \leq C_j \Lambda_j(t), \text{ for all } t \in \mathbb{R}.
\]
Then \( D(\mathcal{J}_2) \) is a vector space.

Let the operator \( B_2 \) be defined by
\[
\begin{align*}
D(B_2) &= \{ U := (u,v) \in \mathcal{X}_2 : (\alpha_1(u), \alpha_2(v)) \in \mathcal{X}_2 \}, \\
B_2(U) &= (\alpha_1(u), \alpha_2(v)).
\end{align*}
\]

We have the following result.

**Lemma 4.7.** Let the assumptions of Lemma 4.6 be satisfied. Then the subdifferential of \( \mathcal{J}_2 \) and the operator \( B_2 \) coincide, that is, for all \( (u,v) \in D(B_2) = D(\partial \mathcal{J}_2) \),
\[
\partial \mathcal{J}_2(u,v) = B_2(u,v).
\]

**Proof.** Let \( U = (u,v) \in D(\mathcal{J}_2) \) and \( F = (f,g) \in \partial \mathcal{J}_2(u,v) \). Then by definition, \( F \in \mathcal{X}_2 \) and, for every \( V = (u_1,v_1) \in D(\mathcal{J}_2) \), we get
\[
\int_{\Omega} F(V-U) \, d\mu \leq \mathcal{J}_2(V) - \mathcal{J}_2(U).
\]
Let \( V = U + tW \), with \( W = (u_2,v_2) \in D(\mathcal{J}_2) \) and \( 0 < t \leq 1 \). Then by Lemma 4.6 \( V = U + tW \in D(\mathcal{J}_2) \). Now, dividing by \( t \) and taking the limit as \( t \downarrow 0 \), we obtain
\[
\int_{\Omega} FW \, d\mu \leq \int_{\Omega} \alpha_1(u)u_2 \, dx + \int_{\partial \Omega} \alpha_2(v)v_2 \frac{d\sigma}{b}.
\]
Changing \( W \) to \( -W \) in (4.5) gives that
\[
\int_{\Omega} FW \, d\mu = \int_{\Omega} \alpha_1(u)u_2 \, dx + \int_{\partial \Omega} \alpha_2(v)v_2 \frac{d\sigma}{b}.
\]
In particular, if \( W = (u_2,0) \) with \( u_2 \in \mathcal{P}(\Omega) \), we have
\[
\int_{\Omega} fu_2 \, dx = \int_{\Omega} \alpha_1(u)u_2 \, dx,
\]
and this shows that \( \alpha_1(u) = f \). Similarly, one obtains that \( \alpha_2(v) = g \). We have shown that \( U \in D(B_2) \) and
\[
B_2(U) := B_2(u,v) = (\alpha_1(u), \alpha_2(v)) = (f,g).
\]
Conversely, let \( U = (u,v) \in D(B_2) \) and set \( F = (f,g) := B_2(u,v) = (\alpha_1(u), \alpha_2(v)) \). Since \( (\alpha_1(u), \alpha_2(v)) \in \mathcal{X}_2 \), from (4.2) and (4.3), it follows that
\[
\int_{\Omega} \Lambda_1(u) \, dx + \int_{\partial \Omega} \Lambda_2(v) \frac{d\sigma}{b} < \infty.
\]
Hence, \( U = (u, v) \in D(\mathcal{J}_2) \). Let \( V = (u_1, v_1) \in D(\mathcal{J}_2) \). Using Lemma 4.4 we obtain
\[
\alpha_1(u)(u_1 - u) = \alpha_1(u)u_1 - \alpha_1(u)u \\
\leq \Lambda_1(u_1) + \Lambda_1(\alpha_1(u)) - \alpha_1(u)u \\
= \Lambda_1(u_1) - \Lambda_1(u)
\] (4.6)
and similarly,
\[
\alpha_2(v)(v_1 - v) \leq \Lambda_2(v_1) - \Lambda_2(v).
\]
Therefore,
\[
\int_{\Omega} F(V - U) \, d\mu = \int_{\Omega} \alpha_1(u)(u_1 - u) \, dx + \int_{\partial\Omega} \alpha_2(v)(v_1 - v) \frac{d\sigma}{b} \\
\leq \mathcal{J}_2(V) - \mathcal{J}_2(U).
\]
By definition, this shows that \( F = (\alpha_1(u), \alpha_2(v)) = B_2(U) \in \partial \mathcal{J}_2(U) \). We have shown that \( U \in D(\partial \mathcal{J}_2) \) and \( B_2(U) \in \partial \mathcal{J}_2(U) \). This completes the proof of the lemma. \( \Box \)

Next, we define the functional \( \mathcal{J}_{3,\rho} : X_2 \to [0, +\infty) \) by
\[
\mathcal{J}_{3,\rho}(U) = \begin{cases} 
\mathcal{J}_\rho(U) + \mathcal{J}_2(U) & \text{if } U \in D(\mathcal{J}_{3,\rho}) := D(\mathcal{J}_\rho) \cap D(\mathcal{J}_2), \\
+\infty & \text{if } U \in X_2 \setminus D(\mathcal{J}_{3,\rho}). 
\end{cases}
\]
(4.7)
Note that for \( \rho = 0 \),
\[
D(\mathcal{J}_{3,0}) = \{ U = (u, u|_{\partial\Omega}) \in D(\mathcal{J}_2) : u \in W^{1, p}(\Omega) \cap L^2(\Omega), u|_{\partial\Omega} \in L^2(\partial\Omega) \},
\]
while for \( \rho = 1 \),
\[
D(\mathcal{J}_{3,1}) = \{ U = (u, u|_{\partial\Omega}) \in D(\mathcal{J}_2) : u \in W^{1, p}(\Omega) \cap L^2(\Omega), u|_{\partial\Omega} \in W^{1, 2}(\partial\Omega) \cap L^2(\partial\Omega) \}.
\]

We have the following result.

**Lemma 4.8.** Let the assumptions of Lemma 4.6 be satisfied. Then the subdifferential of the functional \( \mathcal{J}_{3,\rho} \) is given by
\[
D(\partial \mathcal{J}_{3,\rho}) = \{ U = (u, u|_{\partial\Omega}) \in D(\mathcal{J}_{3,\rho}) : -\Delta_p u + \alpha_1(u) \in L^2(\Omega) \\
\text{and } b(x)|\nabla u|^p - 2\Delta u - b(x)\rho \Delta_{q,1} u + \alpha_2(u) \in L^2(\partial\Omega, d\sigma/b) \}
\]
and
\[
\partial \mathcal{J}_{3,\rho}(U) = \left( -\Delta_p u + \alpha_1(u), b(x)|\nabla u|^p - 2\Delta u - b(x)\rho \Delta_{q,1} u + \alpha_2(u) \right).
\]
(4.10)
In particular, if for every \( U = (u, u|_{\partial\Omega}) \in D(\mathcal{J}_{3,\rho}) \), the function \( (\alpha_1(u), \alpha_2(u)) \in X_2 \), then
\[
\partial \mathcal{J}_{3,\rho} := \partial (\mathcal{J}_\rho + \mathcal{J}_2) = \partial \mathcal{J}_\rho + \partial \mathcal{J}_2.
\]
**Proof.** We calculate the subdifferential \( \partial \mathcal{J}_{3,\rho} \). Let \( F = (f, g) \in \partial \mathcal{J}_{3,\rho}(U) \), that is, \( F \in X_2, U \in D(\mathcal{J}_{3,\rho}) = D(\mathcal{J}_\rho) \cap D(\mathcal{J}_2) \) and for every \( V \in D(\mathcal{J}_{3,\rho}) \), we have
\[
\int_{\Omega} F(V - U) \, d\mu \leq \mathcal{J}_{3,\rho}(V) - \mathcal{J}_{3,\rho}(U).
\]
Proceeding as in Remark 3.2 and the proof of Lemma 4.7, we obtain that
\[
-\Delta_p u + \alpha_1(u) = f \text{ in } \mathcal{D}(\Omega)',
\]
and
\[
b(x)|\nabla u|^p - 2\Delta u - b(x)\rho \Delta_{q,1} u + \alpha_2(u) = g \text{ weakly on } \partial\Omega.
\]
Noting that $\partial J_{f,p}$ is also a single-valued operator (which follows from the assumptions on $\alpha_j$ and $\Lambda_j$), we easily obtain (4.10), and this completes the proof of the first part.

To show the last part, note that it is clear that $\partial J_{f,p} + \partial J_2 \subset \partial J_{f,p}$ always holds. To show the converse inclusion, let assume that for every $U = (u,u_{\partial \Omega}) \in D(J_{f,p})$, the function $(\alpha_1(u),\alpha_2(u)) \in \mathcal{X}_2$. Then it follows from (3.3), (4.3) (since $\partial J_2 = B_2$) and (4.10), that $D(\partial J_{f,p}) = D(\partial J_{f,p}) \cap D(\partial J_2)$ and

$$
\partial J_{f,p}(U) = (-\Delta_p u + \alpha_1(u), b(x)|\nabla u|^{p-2}\partial_p u - b(x)\rho \partial \Lambda u + \alpha_2(u))
$$

$$
= (-\Delta_p u, b(x)|\nabla u|^{p-2}\partial_p u - b(x)\rho \partial \Lambda u + (\alpha_1(u),\alpha_2(u))
$$

$$
= \partial J_{f,p}(U) + \partial J_2(U).
$$

This completes the proof. \hfill \Box

The following lemma is the main ingredient in the proof of Theorem 4.11 below.

**Lemma 4.9.** Let $B_1 := A_0$ and set $B_3 := \partial J_{f,p}$. Then

$$
\mathcal{R}(B_1) + \mathcal{R}(B_2) \subset \overline{\mathcal{R}(B_3)} \quad \text{and} \quad \text{Int} \mathcal{R}(B_1) + \mathcal{R}(B_2) \subset \mathcal{R}(B_3). \quad (4.11)
$$

In particular, if for every $U = (u,u_{\partial \Omega}) \in D(J_{f,p})$, the function $(\alpha_1(u),\alpha_2(u)) \in \mathcal{X}_2$, then

$$
\mathcal{R}(B_1) := \mathcal{R}(B_1 + B_2) \simeq \mathcal{R}(B_1) + \mathcal{R}(B_2). \quad (4.12)
$$

**Proof.** By Remark 4.2 and Lemmas 4.7 and 4.8 the operators $B_1$, $B_2$ and $B_3$ are subdifferentials of proper, convex and lower semicontinuous functionals $J_f$, $J_2$ and $J_{f} + J_2$, respectively, on $\mathcal{X}_2$. Hence, $B_1$, $B_2$ and $B_3$ are maximal monotone operators. In particular, if $(\alpha_1(u),\alpha_2(u)) \in \mathcal{X}_2$, for every $U = (u,u_{\partial \Omega}) \in D(J_{f,p})$, then by Lemma 4.8 one has $B_3 = B_1 + B_2$. Now, the lemma follows from the celebrated Brezis-Haraux result in Theorem 4.2. \hfill \Box

4.2. Statement and proof of the main result. Next, let $\mathcal{Y}_0 := D(J_{f,p})$ be given by (4.8) if $\rho = 0$ and by (4.9) if $\rho = 1$.

**Definition 4.10.** Let $F = (f,g) \in \mathcal{X}_2$. A function $u \in W^{1,p}(\Omega)$ is said to be a weak solution of (4.1), if $\alpha_1(u) \in L^1(\Omega)$, $\alpha_2(u) \in L^1(\partial \Omega)$, $u_{\partial \Omega} \in W^{1,q}(\partial \Omega)$, if $\rho > 0$ and

$$
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla vdx + \rho \int_{\partial \Omega} |\nabla \Gamma u|^{q-2} \nabla \Gamma u \cdot \nabla v d\sigma
$$

$$
+ \int_{\Omega} \alpha_1(u)vdx + \int_{\partial \Omega} \alpha_2(u)v \frac{d\sigma}{b} = \int_{\Omega} fvdx + \int_{\partial \Omega} gvd\sigma,
$$

for every $v \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ with $v_{\partial \Omega} \in W^{1,q}(\partial \Omega)$, if $\rho > 0$.

Recall that $\lambda_1 := \int_{\Omega} dx$ and $\lambda_2 := \int_{\partial \Omega} \frac{d\sigma}{b}$. We also define the average $\langle F \rangle_{\Omega}$ of $F = (f,g)$ with respect to the measure $\mu$, as follows:

$$
\langle F \rangle_{\Omega} := \frac{1}{\mu(\Omega)} \int_{\Omega} Fd\mu = \frac{1}{\mu(\Omega)} \left( \int_{\Omega} fdx + \int_{\partial \Omega} g \frac{d\sigma}{b} \right),
$$

where $\mu(\overline{\Omega}) = \lambda_1 + \lambda_2$. Now, we are ready to state the main result of this section.

**Theorem 4.11.** Let $\alpha_j$, $(j = 1,2)$ satisfy Assumption 4.3 and assume that the functions $\Lambda_j$ $(j = 1,2)$ satisfy (4.3). Let $F = (f,g) \in \mathcal{X}_2$. The following hold:

(a) Suppose that the nonlinear elliptic problem (4.1) possesses a weak solution. Then

$$
\langle F \rangle_{\Omega} \in \frac{\lambda_1 \mathcal{R}(\alpha_1) + \lambda_2 \mathcal{R}(\alpha_2)}{\lambda_1 + \lambda_2}. \quad (4.14)
$$
(b) Assume that
\[
\{F\}_\Omega \in \text{int} \left( \frac{\lambda_1 \mathscr{R}(\alpha_1) + \lambda_2 \mathscr{R}(\alpha_2)}{\lambda_1 + \lambda_2} \right) .
\]  
(4.15)

Then the nonlinear elliptic problem (4.14) has at least one weak solution.

Proof. We show that condition (4.14) is necessary. Let \( F := (f, g) \in \mathcal{X}_2 \) and let \( U = (u, u_{\partial \Omega}) \in D(B_3) \subset \mathcal{Y}_\rho \) be a weak solution of \( B_3 U = F \). Then, by definition, for every \( V = (v, v|_{\partial \Omega}) \in \mathcal{Y}_\rho \), (4.13) holds. Taking \( v = 1 \) in (4.13) yields
\[
\int_\Omega f \, dx + \int_{\partial \Omega} g \frac{d\sigma}{b} = \int_\Omega \alpha_1(u) \, dx + \int_{\partial \Omega} \alpha_2(u) \frac{d\sigma}{b} .
\]

Hence,
\[
\int_\Omega f \, dx + \int_{\partial \Omega} g \frac{d\sigma}{b} \in (\lambda_1 \mathscr{R}(\alpha_1) + \lambda_2 \mathscr{R}(\alpha_2)) ,
\]
and so (4.14) holds. This completes the proof of part (a).

We show that the condition (4.15) is sufficient.

(i) First, let \( C \subseteq \mathcal{C} \), where
\[
C := \{ C = (c_1, c_2) : (c_1, c_2) \in \mathscr{R}(\alpha_1) \times \mathscr{R}(\alpha_2) \} .
\]
By definition, one has that \( C \subseteq \mathscr{R}(B_3) \) since \( c_1 = \alpha_1(d_1) \) for some constant function \( d_1 \) on \( \Omega \), and \( c_2 = \alpha_2(d_2) \) for some constant function \( d_2 \) on \( \partial \Omega \). Let \( F \in \mathcal{X}_2 \) be such that (4.15) holds. We must show \( F \in \mathscr{R}(B_3) \).

By (4.15), we may choose \( C = (c_1, c_2) \in C \) such that
\[
\langle F \rangle_\Omega = \frac{\lambda_1 c_1 + \lambda_2 c_2}{\lambda_1 + \lambda_2} \in \text{int} \left( \frac{\lambda_1 \mathscr{R}(\alpha_1) + \lambda_2 \mathscr{R}(\alpha_2)}{\lambda_1 + \lambda_2} \right) .
\]

Then, for \( F \in \mathcal{X}_2 \), we have \( F = F_1 + F_2 \) with
\[
F_1 := F - C \quad \text{and} \quad F_2 = C.
\]

First, \( F_1 \in \mathscr{R}(B_1) = \mathcal{N}(B_1) \uparrow = 1^4 \), since
\[
\int_\Omega F_1 \, d\mu = \int_\Omega (F - C) \, d\mu = \int_\Omega f \, dx + \int_{\partial \Omega} g \frac{d\sigma}{b} - (\lambda_1 c_1 + \lambda_2 c_2) = (\lambda_1 + \lambda_2) \langle F \rangle_\Omega - (\lambda_1 c_1 + \lambda_2 c_2) = 0.
\]

Obviously, \( F_2 = C \subseteq \mathscr{R}(B_2) \). Hence, it is readily seen that
\[
F \in (\mathscr{R}(B_1) + \mathscr{R}(B_2)) .
\]

(ii) Next, denote by \( B_R(x, r) \) the open ball in \( \mathbb{R} \) of center \( x \) and radius \( r > 0 \). Since
\[
\langle F \rangle_\Omega \in \text{int} \left( \frac{\lambda_1 \mathscr{R}(\alpha_1) + \lambda_2 \mathscr{R}(\alpha_2)}{\lambda_1 + \lambda_2} \right) ,
\]
there exists \( \delta > 0 \) such that the open ball
\[
B_R(\langle F \rangle_\Omega, \delta) \subset \left( \frac{\lambda_1 \mathscr{R}(\alpha_1) + \lambda_2 \mathscr{R}(\alpha_2)}{\lambda_1 + \lambda_2} \right) .
\]

Since the mapping \( F \mapsto \langle F \rangle_\Omega \) from \( \mathcal{X}_2 \) into \( \mathbb{R} \) is continuous, then there exists \( \varepsilon > 0 \) such that
\[
\langle G \rangle_\Omega \in B_R(\langle F \rangle_\Omega, \delta) \subset \left( \frac{\lambda_1 \mathscr{R}(\alpha_1) + \lambda_2 \mathscr{R}(\alpha_2)}{\lambda_1 + \lambda_2} \right) ,
\]
for all $G \in \mathcal{X}_2$ satisfying $\|F - G\|_2 < \varepsilon$. It finally follows from part (i) above that $(\mathcal{R}(B_1) + \mathcal{R}(B_2))$ contains an $\varepsilon$-ball in $\mathcal{X}_2$ centered at $F$. Therefore,

$$F \in \text{int}(\mathcal{R}(B_1) + \mathcal{R}(B_2)) \subset \mathcal{R}(B_2).$$

Consequently, problem (4.1) is (weakly) solvable for every function $F = (f, g) \in \mathcal{X}_2$, if (4.15) holds. This completes the proof of the theorem.

Remark 4.12. It is important to remark that in order to prove Theorem 4.11, we do not require that $(\alpha_1(u), \alpha_2(u))$ should belong to $\mathcal{X}_2$, for every $U = (u, u_\Gamma) \in \mathcal{D}(\mathcal{F}_3, \rho)$. In particular, only the assumption (4.11) was needed. However, if this happens, then we get the much stronger result in (4.12) which would require that the nonlinearities $\alpha_1, \alpha_2$ satisfy growth assumptions at infinity.

We conclude this section with the following corollary and some examples.

Corollary 4.13. Let the assumptions of Theorem 4.11 be satisfied. Let $F = (f, g) \in \mathcal{X}_2$. Assume that at least one of the sets $\mathcal{R}(\alpha_1), \mathcal{R}(\alpha_2)$ is open. Then the nonlinear elliptic problem (4.1) possesses a weak solution if and only if (4.15) holds.

Remark 4.14. Similar results to Theorem 4.11 and Corollary 4.13 were also obtained in [12, Theorem 4.4], but only when $p = q = 2$.

4.3 Examples. We will now give some examples as applications of Theorem 4.11. Let $p, q \in (1, +\infty)$ be fixed.

Example 4.15. Let $\alpha_1(s) = \alpha_2(s)$ be equal to $\alpha(s) = c |s|^r - 1$, where $c, r > 0$. Note that $\mathcal{R}(\alpha) = \mathbb{R}$. It is easy to check that $\alpha$ satisfies all the conditions of Assumption 4.3 and that the function $\Lambda(t) = \int_0^t \alpha(s)ds$ satisfies (4.3). Then, it follows that problem (4.1) is solvable for any $f \in L^2(\Omega), g \in L^2(\partial\Omega)$.

Example 4.16. Consider the case when $p = 2 \equiv 0$ in (4.1), that is, consider the following boundary value problem:

$$\begin{cases}
-\Delta u + \alpha_1(u) = f & \text{in } \Omega, \\
b(x) |\nabla u|^{p-2} \partial_n u = g & \text{on } \Gamma.
\end{cases}$$

Then, by Theorem 4.11, this problem has a weak solution if

$$\int_\Omega f \, dx + \int_{\partial\Omega} g \frac{d\sigma}{b} \in \lambda_1 \text{int}(\mathcal{R}(\alpha_1)),$$

which yields the classical Landesman-Lazer result (see (1.6)) for $g \equiv 0$ and $p = 2$.

Example 4.17. Let us now consider the case when $\alpha_1 \equiv \alpha$ and $\alpha_2 \equiv 0$, where $\alpha$ is a continuous, odd and nondecreasing function on $\mathbb{R}$ such that $\alpha(0) = 0$. The problem

$$\begin{cases}
-\Delta u + \alpha(u) = f, & \text{in } \Omega, \\
b(x) |\nabla u|^{p-2} \partial_n u - pb(x) \Delta_\Gamma u = g, & \text{on } \partial\Omega,
\end{cases}$$

has a weak solution if

$$\int_\Omega f \, dx + \int_{\partial\Omega} g \frac{d\sigma}{b} \in \lambda_2 \text{int}(\mathcal{R}(\alpha)).$$

(4.17) Let us now choose $\alpha(s) = \arctan(s)$ in (4.17). Then, it is easy to check that

$$\Lambda(t) := \int_0^t \alpha(s)ds = |t| \arctan(|t|) - \frac{1}{2} \ln \left(1 + t^2\right), t \in \mathbb{R}.$$
is monotone increasing on $\mathbb{R}_+$ and that it satisfies $\Lambda(2t) \leq C_2 \Lambda(t)$, $\forall t \in \mathbb{R}$, for some constant $C_2 > 1$. Therefore, (4.17) becomes the necessary and sufficient condition

$$\frac{1}{\lambda_2} \left( \int_{\Omega} f \, dx + \int_{\partial\Omega} g \, \frac{d\sigma}{b} \right) < \frac{\pi}{2}. \tag{4.18}$$

5. A PRIORI ESTIMATES

Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain with boundary $\partial\Omega$. Recall that $1 < p, q < \infty$, $\rho \in \{0, 1\}$ and $b \in L^\infty(\partial\Omega)$ with $b(x) \geq b_0 > 0$, for some constant $b_0$. We consider the nonlinear elliptic boundary value problem formally given by

$$\begin{cases}
-\Delta_p u + \alpha_1(x,u) + |u|^{p-2}u = f, & \text{in } \Omega \\
-\rho b(x)\Delta_{q,1} u + \rho b(x)|u|^{q-2}u + b(x)|\nabla u|^{p-2}\partial_{t}u + \alpha_2(x,u) = g, & \text{on } \partial\Omega,
\end{cases} \tag{5.1}$$

where $f \in L^{p_1}(\Omega)$ and $g \in L^{q_1}(\partial\Omega)$ for some $1 \leq p_1, q_1 \leq \infty$. If $\rho = 0$, then the boundary conditions in (5.1) are of Robin type. Existence and regularity of weak solutions for this case have been obtained in [3] for $p = 2$ (see also [29] for the linear case) and for general $p$ in [6]. Therefore, we will concentrate our attention to the case $\rho = 1$ only; in this case, the boundary condition in (5.1) is a generalized Wentzell-Robin boundary condition. For the sake of simplicity, from now on we will also take $b \equiv 1$.

5.1. General assumptions. Throughout this section, we assume that the functions $\alpha_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\alpha_2 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:

Assumption 5.1.

$$\begin{cases}
\alpha_j(x, \cdot) & \text{is odd and strictly increasing}, \\
\alpha_j(x, 0) = 0 & \text{is continuous}, \\
\lim_{t \to 0} \frac{\alpha_j(x,t)}{t} = 0, & \lim_{t \to +\infty} \frac{\alpha_j(x,t)}{t} = +\infty,
\end{cases}$$

for $\lambda_{N}$-a.e. $x \in \Omega$ if $j = 1$ and $\sigma$-a.e. $x \in \partial\Omega$ if $j = 2$.

Since $\alpha_j(x, \cdot)$ are strictly increasing for $\lambda_{N}$-a.e. $x \in \Omega$ if $j = 1$ and $\sigma$-a.e. $x \in \partial\Omega$ if $j = 2$, then they have inverses which we denote by $\tilde{\alpha}_j(x, \cdot)$ (cf. also Section 4). We define the functions $\Lambda_1, \Lambda_1 : \Omega \times \mathbb{R} \to [0, \infty)$ and $\Lambda_2, \Lambda_2 : \partial\Omega \times \mathbb{R} \to [0, \infty)$ by

$$\Lambda_j(x,t) := \int_0^{|t|} \alpha_j(x,s) \, ds \quad \text{and} \quad \Lambda_j(x,t) := \int_0^{|t|} \tilde{\alpha}_j(x,s) \, ds.$$

Then, it is clear that, for $\lambda_{N}$-a.e. $x \in \Omega$ if $j = 1$ and $\sigma$-a.e. $x \in \partial\Omega$ if $j = 2$, $\Lambda_j(x, \cdot)$ and $\Lambda_j(x, \cdot)$ are differentiable, monotone and convex with $\Lambda_j(x, 0) = \Lambda_j(x, 0) = 0$. Furthermore, $\Lambda_j(x, \cdot)$ is a $\nu$-function and $\Lambda_j(x, \cdot)$ is its complementary $\nu$-function. The function $\Lambda_j$ is then the complementary Musielak-Orlick function of $\alpha_j$ in the sense of Young (see Definition 2.3).

Assumption 5.2. We assume, for $\lambda_{N}$-a.e. $x \in \Omega$ if $j = 1$ and $\sigma$-a.e. $x \in \partial\Omega$ if $j = 2$, that $\Lambda_j(x, \cdot)$ and $\Lambda_j(x, \cdot)$ satisfy the $(\lambda_2)$-condition in the sense of Definition 2.3.

It follows from Assumption 5.2 that there exist two constants $c_1, c_2 \in (0, 1]$ such that for $\lambda_{N}$-a.e. $x \in \Omega$ if $j = 1$ and $\sigma$-a.e. $x \in \partial\Omega$ if $j = 2$ and for all $t \in \mathbb{R}$,

$$c_1 t \alpha_j(x, t) \leq \Lambda_j(x, t) \leq t \alpha_j(x, t). \tag{5.2}$$
Lemma 5.4. Assume Assumptions 5.1 and 5.2. Let the functional \( V \) and \( \tilde{V} \) be fixed. Then for every \( V \) such that \( \| u \|_{\Lambda_1, \Omega} < \infty \), it follows from [1, Theorem 8.19], that \( L_{\Lambda_1}(\Omega) \) and \( L_{\Lambda_2}(\partial \Omega) \), endowed respectively with the norms

\[
\| u \|_{\Lambda_1, \Omega} := \inf \left\{ k > 0 : \int_{\Omega} \Lambda_1 \left( x, \frac{u(x)}{k} \right) dx \leq 1 \right\},
\]

and

\[
\| u \|_{\Lambda_2, \partial \Omega} := \inf \left\{ k > 0 : \int_{\partial \Omega} \Lambda_2 \left( x, \frac{u(x)}{k} \right) d\sigma \leq 1 \right\},
\]

are reflexive Banach spaces. Moreover, by [1, Section 8.11, p.234], the following generalized versions of Hölder’s inequality will also become useful in the sequel,

\[
\left| \int_{\Omega} uv \, dx \right| \leq 2 \| u \|_{\Lambda_1, \Omega} \| v \|_{\Lambda_1, \Omega},
\]

and

\[
\left| \int_{\partial \Omega} uv \, d\sigma \right| \leq 2 \| u \|_{\Lambda_2, \partial \Omega} \| v \|_{\Lambda_2, \partial \Omega}.
\]

5.2. Existence and uniqueness of weak solutions of perturbed equations. Let

\( \mathcal{Y} := \{ U := (u, u|_{\partial \Omega}) : u \in W^{1,p}(\Omega) \cap L_{\Lambda_1}(\Omega), u|_{\partial \Omega} \in W^{1,q}(\partial \Omega) \} \).

Then for every \( 1 < p, q < \infty \), \( \mathcal{Y} \) endowed with the norm

\[
\| U \|_{\mathcal{Y}} = \| u \|_{W^{1,p}(\Omega)} + \| u \|_{\Lambda_1, \Omega} + \| u \|_{W^{1,q}(\partial \Omega)} + \| u \|_{\Lambda_2, \partial \Omega}
\]

is a reflexive Banach space. Recall that \( p = 1 \). Throughout the following, we denote by \( \mathcal{Y}' \) the dual of \( \mathcal{Y} \).

Definition 5.3. A function \( U = (u, u|_{\partial \Omega}) \in \mathcal{Y} \) is said to be a weak solution of

\[
(5.1)
\]

if for every \( V = (v, v|_{\partial \Omega}) \in \mathcal{Y}' \),

\[
\mathcal{A}(U, V) = \int_{\Omega} fv \, dx + \int_{\partial \Omega} gvd\sigma,
\]

provided that the integrals on the right-hand side exist. Here,

\[
\mathcal{A}(U, V) := \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx + \int_{\Omega} |u|^{p-2} uv \, dx
\]

\[
+ \int_{\Omega} \alpha_1(x,u)vdx + \int_{\partial \Omega} |\nabla v| \cdot \nabla u \cdot \nabla v \, d\sigma
\]

\[
+ \int_{\partial \Omega} |u|^{q-2} uv \, d\sigma + \int_{\partial \Omega} \alpha_2(x,u)vd\sigma.
\]

Lemma 5.4. Assume Assumptions 5.1 and 5.2. Let \( 1 < p, q < \infty \) and \( U \in \mathcal{Y}' \) be fixed. Then the functional \( V \mapsto \mathcal{A}(U, V) \) belongs to \( \mathcal{Y}' \). Moreover, \( \mathcal{A} \) is strictly monotone, hemicontinuous and coercive.
Proof. Let $U = (u, u|_{\partial \Omega}) \in \mathcal{Y}$ be fixed. It is clear that $\mathcal{A}(U, \cdot)$ is linear. Let $V = (v, v|_{\partial \Omega}) \in \mathcal{Y}$. Then, exploiting (5.3) and (5.4), we obtain

$$|\mathcal{A}(U, V)| \leq \|u\|_{W^{1, p}(\Omega)}^{p-1} \|v\|_{W^{1, p}(\Omega)} + \|u\|_{W^{1, q}(\partial \Omega)}^{q-1} \|v\|_{W^{1, q}(\partial \Omega)}$$

(5.6)

$$+ 2 \max \left\{ 1, \int_{\Omega} \tilde{A}_1(x, \alpha_1(x, u)) \, dx \right\} \|v\|_{L_1} \Omega$$

$$+ 2 \max \left\{ 1, \int_{\partial \Omega} \tilde{A}_2(x, \alpha_2(x, u)) \, d\sigma \right\} \|v\|_{L_2} \partial \Omega,$$

where

$$K(U) := \|u\|_{W^{1, p}(\Omega)}^{p-1} + 2 \max \left\{ 1, \int_{\Omega} \tilde{A}_1(x, \alpha_1(x, u)) \, dx \right\}$$

$$+ \|u\|_{W^{1, q}(\partial \Omega)}^{q-1} + 2 \max \left\{ 1, \int_{\partial \Omega} \tilde{A}_2(x, \alpha_2(x, u)) \, d\sigma \right\}.$$ This shows $\mathcal{A}(U, \cdot) \in \mathcal{Y}'$, for every $U \in \mathcal{Y}$.

Next, let $U, V \in \mathcal{Y}$. Then, using (2.11) and the fact that $\alpha_j(x, \cdot)$ are monotone non-decreasing, that is, $(\alpha_j(x, t) - \alpha_j(x, s))(t - s) \geq 0$, for all $t, s \in \mathbb{R}$, we obtain

$$\mathcal{A}(U, U - V) - \mathcal{A}(V, U - V)$$

$$= \int_{\Omega} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \cdot (\nabla u - \nabla v) dx + \int_{\Omega} (|u|^{p-2} u - |v|^{p-2} v) (u - v) dx$$

$$+ \int_{\partial \Omega} (\alpha_1(x, u) - \alpha_1(x, v)) (u - v) d\sigma + \int_{\partial \Omega} (|u|^{q-2} u - |v|^{q-2} v) (u - v) d\sigma$$

$$\geq \int_{\Omega} (|\nabla u| + |\nabla v|)^{p-2} \nabla (u - v)^2 dx + \int_{\Omega} (|u| + |v|)^{p-2} (u - v)^2 dx$$

$$+ \int_{\partial \Omega} (|\nabla u| + |\nabla v|)^{p-2} \nabla (u - v)^2 d\sigma + \int_{\partial \Omega} (|u| + |v|)^{p-2} (u - v)^2 d\sigma \geq 0.$$

This shows that $\mathcal{A}$ is monotone. The estimate (5.7) also shows that

$$\mathcal{A}(U, U - V) - \mathcal{A}(V, U - V) > 0,$$

for all $U, V \in \mathcal{V}$ with $U \neq V$, that is, $u \neq v$ or $u|_{\partial \Omega} \neq v|_{\partial \Omega}$. Thus, $\mathcal{A}$ is strictly monotone.

The continuity of the norm function and the continuity of $\alpha_j(x, \cdot)$, $j = 1, 2$ imply that $\mathcal{A}$ is hemicontinuous.

Finally, since $\Lambda_j$ and $\bar{\Lambda}_j$ satisfy the $\langle \triangle_0 \rangle$-condition, from Proposition 2.10 and Corollary 2.11 it follows

$$\lim_{\|u\|_{\Lambda_j} \to +\infty} \frac{\int_{\Omega} u \alpha_1(x, u) \, dx}{\|u\|_{\Lambda_j}} = +\infty, \quad \text{and} \quad \lim_{\|u\|_{\Lambda_2 \partial \Omega} \to +\infty} \frac{\int_{\partial \Omega} u \alpha_2(x, u) \, d\sigma}{\|u\|_{\Lambda_2 \partial \Omega}} = +\infty.$$

Consequently, we deduce

$$\lim_{\|U\|_{\mathcal{Y}} \to +\infty} \frac{\mathcal{A}(U, U)}{\|U\|_{\mathcal{Y}}} = +\infty,$$

(5.8)

which shows that $\mathcal{A}$ is coercive. The proof of the lemma is finished. \qed
The following result is concerned with the existence and uniqueness of weak solutions to problem (5.1).

**Theorem 5.5.** Assume Assumptions [5.1] and [5.5]. Let $1 < p, q < \infty$, $p_1 \geq p^*$ and $q_1 \geq q^*$, where $p^* := p/(p - 1)$ and $q^* := q/(q - 1)$. Then for every $(f, g) \in X^{p_1,q_1}(\Omega, \mu)$, there exists a unique function $U \in \mathcal{V}$ which is a weak solution to (5.1).

**Proof.** Let $\langle \cdot, \cdot \rangle$ denote the duality between $\mathcal{V}$ and $\mathcal{V}'$. Then, from Lemma [5.4] it follows that for each $U \in \mathcal{V}$, there exists $A(U) \in \mathcal{V}'$ such that

$$\langle A(U), V \rangle = \langle U, V \rangle,$$

for every $V \in \mathcal{V}$. Hence, this relation defines an operator $A : \mathcal{V} \rightarrow \mathcal{V}'$, which is bounded by (5.6). Exploiting Lemma [5.4] once again, it is easy to see that $A$ is monotone and coercive.

Therefore, for every $F \in \mathcal{V}'$ there exists $U \in \mathcal{V}$ such that $A(U) = F$, that is, for every

$$\langle A(U), V \rangle = \langle U, V \rangle = \langle V, F \rangle.$$

Since $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ and $W^{1,q}(\partial \Omega) \hookrightarrow L^q(\partial \Omega)$ with dense injection, by duality, we have $X^{p_1,q_1}(\Omega, \mu) \hookrightarrow \mathcal{V}'$. Since $\Omega$ is bounded and $\sigma(\partial \Omega) < \infty$, we obtain that

$$X^{p_1,q_1}(\Omega, \mu) \hookrightarrow X^{p_1,q_1}(\Omega, \mu) \hookrightarrow \mathcal{V}'.$$

This shows the existence of weak solutions. The uniqueness follows from the fact that $\mathcal{A}$ is strictly monotone (cf. Lemma [5.4]). This completes the proof of the theorem. \hfill \Box

**Corollary 5.6.** Let the assumptions of Theorem 5.5 be satisfied. Let

$$p_h := \frac{Np}{N(p - 1)} + p, \quad q_h := \frac{p(N - 1)}{N(p - 1)}, \quad \text{and} \quad q_k := \frac{q(N - 1)}{N(q - 1) + 1}.$$  \hfill (5.9)

(a) Let $1 < p < N, 1 < q < p(N - 1)/N, p_1 \geq p_h$ and $q_1 \geq q_k$. Then for every $(f, g) \in X^{p_1,q_1}(\Omega, \mu)$, there exists a function $U \in \mathcal{V}$ which is the unique weak solution to (5.1).

(b) Let $1 < q < N - 1, 1 < p < Nq/(N - 1), p_1 \geq p_h$ and $q_1 \geq q_k$. Then for every $(f, g) \in X^{p_1,q_1}(\Omega, \mu)$, there exists a function $U \in \mathcal{V}$ which is the unique weak solution to (5.1).

**Proof.** We first prove (1). Let $1 < p < N$ and $1 < q < p(N - 1)/N$ and let $p_1 \geq p_h$ and $q_1 \geq q_k$, where $p_h$ and $q_k$ are given by (5.9). Let $p_s := Np/(N - p)$ and $q_s := (N - 1)q/(N - 1 - q)$. Since $W^{1,p}(\Omega) \hookrightarrow L^{p_s}(\Omega)$ and $W^{1,q}(\partial \Omega) \hookrightarrow L^{q_s}(\partial \Omega)$ with dense injection, then by duality, $X^{p_s,q_s}(\Omega, \mu) \hookrightarrow \mathcal{V}'$, where $1/p_s + 1/p_h = 1$ and $1/q_s + 1/q_k = 1$. Since $\mu(\Omega) < \infty$, we have that

$$X^{p_1,q_1}(\Omega, \mu) \hookrightarrow X^{p_s,q_s}(\Omega, \mu) \hookrightarrow \mathcal{V}'.$$

Hence, for every $F := (f, g) \in X^{p_1,q_1}(\Omega, \mu) \hookrightarrow \mathcal{V}'$, there exists $U \in \mathcal{V}$ such that for every $V \in \mathcal{V}$,

$$\langle A(U), V \rangle = \mathcal{A}(U, V) = \int_{\Omega} f v \, dx + \int_{\partial \Omega} g v \, d\sigma.$$

The uniqueness of the weak solution follows again from the fact that $\mathcal{A}$ is strictly monotone.

In order to prove the second part, we use the embeddings $W^{1,p}(\Omega) \hookrightarrow L^{p_s}(\Omega)$, $W^{1,q}(\partial \Omega) \hookrightarrow L^{q_s}(\partial \Omega)$ and proceed exactly as above. We omit the details. \hfill \Box
5.3. Properties of the solution operator of the perturbed equation. In the sequel, we establish some interesting properties of the solution operator $A$ to problem (5.1). We begin by assuming the following.

**Assumption 5.7.** Suppose that $\alpha_j$, $j = 1, 2$, satisfy the following conditions:

$$
\begin{align*}
&\text{there are constants } c_j \in (0, 1) \text{ such that } \\
&c_j |\alpha_j(x, \xi) - \eta| \leq |\alpha_j(x, \xi) - \alpha_j(x, \eta)| \quad \text{for all } \xi, \eta \in \mathbb{R}.
\end{align*}
$$

(5.10)

**Theorem 5.8.** Assume Assumptions 5.1, 5.2, and 5.7. Let $p, q \geq 2$ and let $A : \mathcal{V} \to \mathcal{V}'$ be the continuous and bounded operator constructed in the proof of Theorem 5.5. Then $A$ is injective and hence, invertible and its inverse $A^{-1}$ is also continuous and bounded.

**Proof.** First, we remark that, since

$$(\alpha_j(x, t) - \alpha_j(x, s))(t - s) \geq 0, \text{ for all } t, s \in \mathbb{R},$$

for $\lambda_N$-a.e. $x \in \Omega$ if $j = 1$ and $\sigma$-a.e. $x \in \partial\Omega$ if $j = 2$, it follows from (5.10) that, for all $t, s \in \mathbb{R},$

$$(\alpha_j(x, t) - \alpha_j(x, s))(t - s) \geq c_j\alpha_j(x, t - s) \cdot (t - s).$$

(5.11)

Let $U, V \in \mathcal{V}$ and $p, q \in [2, \infty)$. Then, exploiting (2.12), (5.11) and the $(\triangle_2)$-condition, we obtain

$$
\langle A(U) - A(V), U - V \rangle = \mathcal{A}(U, U - V) - \mathcal{A}(V, U - V)
$$

(5.12)

$$
= \int_{\Omega} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot \nabla (u - v) dx + \int_{\Omega} (|u|^{p-2}u - |v|^{p-2}v) (u - v) dx
$$

$$
+ \int_{\Omega} (\alpha_1(x, u) - \alpha_1(x, v)) (u - v) dx + \int_{\partial\Omega} \left( |\nabla u|^{q-2}\nabla u - |\nabla v|^{q-2}\nabla v \right) \cdot \nabla (u - v) d\sigma
$$

$$
+ \int_{\partial\Omega} (|u|^{q-2}u - |v|^{q-2}v) (u - v) d\sigma + \int_{\partial\Omega} (\alpha_2(x, u) - \alpha_2(x, v)) (u - v) d\sigma
$$

$$
\geq ||u - v||_{W_0^{1,p}(\Omega)}^{p} + c_1 \int_{\Omega} \Lambda_1(x, u - v) dx + ||u - v||_{W^{1,q}(\partial\Omega)}^{q} + c_2 \int_{\partial\Omega} \Lambda_2(x, u - v) d\sigma.
$$

This implies that $(A(U) - A(V), U - V) > 0$, for all $U, V \in \mathcal{V}$ with $U \neq V$ (that is, $u \neq v$, or $u|_{\partial\Omega} \neq v|_{\partial\Omega}$). Therefore, the operator $A$ is injective and hence, $A^{-1}$ exists. Since for every $U \in \mathcal{V},$

$$
\mathcal{A}(U, U) = \langle A(U), U \rangle \leq ||A(U)||_{\mathcal{V}'} ||U||_{\mathcal{V}},
$$

from the coercivity of $\mathcal{A}$ (see (5.8)), it is not difficult to see that

$$
\lim_{||U||_{\mathcal{V}} \to +\infty} ||A(U)||_{\mathcal{V}'} = +\infty.
$$

(5.13)

Thus, $A^{-1} : \mathcal{V}' \to \mathcal{V}$ is bounded.

Next, we show that $A^{-1} : \mathcal{V}' \to \mathcal{V}$ is continuous. Assume that $A^{-1}$ is not continuous. Then there is a sequence $F_n \in \mathcal{V}'$ with $F_n \to F$ in $\mathcal{V}'$ and a constant $\delta > 0$ such that

$$
||A^{-1}(F_n) - A^{-1}(F)||_{\mathcal{V}} \geq \delta,
$$

(5.14)

for all $n \in \mathbb{N}$. Let $U_n := A^{-1}(F_n)$ and $U = A^{-1}(F)$. Since $\{F_n\}$ is a bounded sequence and $A^{-1}$ is bounded, we have that $\{U_n\}$ is bounded in $\mathcal{V}$. Thus, we can select a subsequence, which we still denote by $\{U_n\}$, which converges weakly to some function $V \in \mathcal{V}$. Since $A(U_n) - A(V) \to F - A(V)$ strongly in $\mathcal{V}$ and $U_n - V$ converges weakly to zero in $\mathcal{V}$, we deduce

$$
\lim_{n \to \infty} \langle A(U_n) - A(V), U_n - V \rangle = 0.
$$

(5.15)
From (5.12) and (5.15), it follows that
\[
\lim_{n \to \infty} \|u_n - v\|_{W^1,p(\Omega)} = 0 \quad \text{and} \quad \lim_{n \to \infty} \int_\Omega A_1(x, u_n - v) \, dx = 0,
\]
while
\[
\lim_{n \to \infty} \|u_n - v\|_{W^1,q(\partial \Omega)} = 0 \quad \text{and} \quad \lim_{n \to \infty} \int_{\partial \Omega} A_2(x, u_n - v) \, d\sigma = 0.
\]
Therefore, \(U_n \to V\) strongly in \(\mathcal{Y}\). Since \(A\) is continuous and
\[
F_n = A(U_n) \to A(V) = F = A(U)
\]
it follows from the injectivity of \(A\), that \(U = V\). This shows that
\[
\lim_{n \to \infty} \|A^{-1}(F_n) - A^{-1}(F)\|_{\mathcal{Y}} = \lim_{n \to \infty} \|U_n - U\|_{\mathcal{Y}} = 0,
\]
which contradicts (5.14). Hence, \(A^{-1}: \mathcal{Y}' \to \mathcal{Y}\) is continuous. The proof is finished. \(\square\)

**Corollary 5.9.** Let the assumptions of Theorem 5.8 be satisfied. Let \(p_h, q_h\), and \(q_h\) be as in (5.9) and let \(A: \mathcal{Y} \to \mathcal{Y}'\) be the continuous and bounded operator constructed in the proof of Theorem 5.5.

(a) If \(2 \leq p < N, 2 \leq q < p(N-1)/N, p_1 \geq p_h\) and \(q_1 \geq q_h\), then \(A^{-1}: X^{p_1,q_1}(\Omega, \mu) \to X^{p,q}(\Omega, \mu)\) is continuous and bounded. Moreover, \(A^{-1}: X^{p_1,q_1}(\Omega, \mu) \to \mathcal{Y} \cap X^{r,e}(\Omega, \mu)\) is compact for every \(r \in (1, p_h)\) and \(s \in (1, q_h)\).

(b) If \(2 \leq q < N - 1, 2 \leq p < qN/(N-1), p_1 \geq p_h\) and \(q_1 \geq q_h\), then the operator \(A^{-1}: X^{p_1,q_1}(\Omega, \mu) \to \mathcal{Y}'\) is continuous and bounded. Moreover, \(A^{-1}: X^{p_1,q_1}(\Omega, \mu) \to \mathcal{Y}' \cap X^{r,e}(\Omega, \mu)\) is compact for every \(r \in (1, p_h)\) and \(s \in (1, q_h)\).

**Proof.** We only prove the first part. The second part of the proof follows by analogy and is left to the reader. Let \(2 \leq p < N, 2 \leq q < p(N-1)/N, p_1 \geq p_h\) and \(q_1 \geq q_h\) and let \(F \in X^{p,q}(\Omega, \mu)\). Proceeding exactly as in the proof of Theorem 5.8 we obtain
\[
\|A^{-1}(F)\|_{p_1,q_1} \leq C_1 \|A^{-1}(F)\|_{\mathcal{Y}} \leq C \|F\|_{p_1,q_1}.
\]
Hence, the operator \(A^{-1}: X^{p_1,q_1}(\Omega, \mu) \to X^{p,q}(\Omega, \mu)\) is bounded. Finally, using the facts that \(X^{p_1,q_1}(\Omega, \mu) \to \mathcal{Y}'\), \(A^{-1}: \mathcal{Y}' \to \mathcal{Y}\) is continuous and \(\mathcal{Y} \to X^{p,q}(\Omega, \mu)\), we easily deduce that \(A^{-1}: X^{p_1,q_1}(\Omega, \mu) \to \mathcal{Y}\) is continuous.

Now, let \(1 < r < p_h\) and \(1 < s < q_h\). Since the injection \(\mathcal{Y} \hookrightarrow X^{r,e}(\Omega, \mu)\) is compact, then by duality, the injection \(X^{r,e}(\Omega, \mu) \hookrightarrow (\mathcal{Y}')^*\) is compact for every \(r' > p'_h = p_h - 1\) and \(s' > q'_s = q_h - 1\). This, together with the fact that \(A^{-1}: (\mathcal{Y}')^* \to \mathcal{Y}\) is continuous and bounded, imply that \(A^{-1}: X^{r,e}(\Omega, \mu) \to \mathcal{Y}\) is compact for every \(p_1 > p_h\) and \(q_1 > q_h\).

It remains to show that \(A^{-1}\) is also compact as a map into \(X^{r,e}(\Omega, \mu)\) for every \(r \in (1, p_h)\) and \(s \in (1, q_h)\). Since \(A^{-1}\) is bounded, we have to show that the image of every bounded set \(B \subset X^{p,q}(\Omega, \mu)\) is relatively compact in \(X^{r,e}(\Omega, \mu)\) for every \(r \in (1, p_h)\) and \(s \in (1, q_h)\). Let \(U_n\) be a sequence in \(A^{-1}(B)\). Let \(F_n = A(U_n) \in B\). Since \(B\) is bounded, then the sequence \(F_n\) is bounded. Since \(A^{-1}\) is compact as a map into \(\mathcal{Y}'\), it follows that there is a subsequence \(F_{n_k}\) such that \(A^{-1}(F_{n_k}) \to U \in \mathcal{Y}\). We may assume that \(U_n = A^{-1}(F_{n_k}) \to U \in \mathcal{Y}\) and hence, in \(X^{r,e}(\Omega, \mu)\). It remains to show that \(U_n \to U\) in \(X^{r,e}(\Omega, \mu)\). Let \(r \in (p_h, p)\) and \(s \in (p, q_h)\). Since \(U_n := (u_n, u_n|_{\partial \Omega})\) is bounded in \(X^{p_1,q_1}(\Omega, \mu)\), a standard interpolation inequality shows that there exists \(\tau \in (0, 1)\) such that
\[
\|U_n - U_m\|_{r,s} \leq \|U_n - U_m\|_{p,p}^{\tau} \|U_n - U_m\|_{p_1,q_1}^{1-\tau} \leq C \|U_n - U_m\|_{p,p}^{\tau}.
\]
As \(U_n\) converges in \(X^{p,q}(\Omega, \mu)\), it follows from the preceding inequality that \(U_n\) is a Cauchy sequence in \(X^{r,e}(\Omega, \mu)\) and therefore converges in \(X^{r,e}(\Omega, \mu)\). Hence, \(A^{-1}: X^{p_1,q_1}(\Omega, \mu) \to \mathcal{Y}\).

\[\text{(Continued)}\]

From (5.12) and (5.15), it follows that
\[
\lim_{n \to \infty} \|u_n - v\|_{W^1,p(\Omega)} = 0 \quad \text{and} \quad \lim_{n \to \infty} \int_\Omega A_1(x, u_n - v) \, dx = 0,
\]
while
\[
\lim_{n \to \infty} \|u_n - v\|_{W^1,q(\partial \Omega)} = 0 \quad \text{and} \quad \lim_{n \to \infty} \int_{\partial \Omega} A_2(x, u_n - v) \, d\sigma = 0.
\]
Therefore, \(U_n \to V\) strongly in \(\mathcal{Y}\). Since \(A\) is continuous and
\[
F_n = A(U_n) \to A(V) = F = A(U)
\]
it follows from the injectivity of \(A\), that \(U = V\). This shows that
\[
\lim_{n \to \infty} \|A^{-1}(F_n) - A^{-1}(F)\|_{\mathcal{Y}} = \lim_{n \to \infty} \|U_n - U\|_{\mathcal{Y}} = 0,
\]
which contradicts (5.14). Hence, \(A^{-1}: \mathcal{Y}' \to \mathcal{Y}\) is continuous. The proof is finished. \(\square\)
\( \mathcal{V} \cap X^{r,q}(\Omega, \mu) \) is compact for every \( r \in [p, p_s) \) and \( s \in [p, q_s) \). The case \( r, s \in (1, p) \) follows from the fact that \( X^{p,p}(\Omega, \mu) \hookrightarrow X^{r,q}(\Omega, \mu) \) and the proof is finished \( \Box \)

5.4. Statement and proof of the main result. We will now establish under what conditions the operator \( A^{-1} \) maps \( X^{p,q_1}(\Omega, \mu) \) boundedly and continuously into \( X^{r,q}(\Omega, \mu) \). The following is the main result of this section.

Theorem 5.10. Let the assumptions of Theorem 5.8 be satisfied.

(a) Suppose \( 2 \leq p < N \) and \( 2 \leq q < \infty \). Let

\[
1 > \frac{p_s}{p_s - p} = \frac{N}{p} \quad \text{and} \quad \frac{q_s}{q_s - p} = \frac{N - 1}{p - 1}.
\]

Let \( f \in L^{p_1}(\Omega) \), \( g \in L^{p_1}(\partial \Omega) \) and \( U, V \in \mathcal{V} \) be such that for every function \( \Phi = (\varphi, \varphi|_{\partial \Omega}) \in \mathcal{V}' \),

\[
\mathcal{A}(U, \Phi) - \mathcal{A}(V, \Phi) = \int_{\Omega} f \varphi \, dx + \int_{\partial \Omega} g \varphi \, d\sigma. \tag{5.16}
\]

Then there is a constant \( C = C(N, p, q, \Omega) > 0 \) such that

\[
\|U - V\|_{\infty}^{p-1} \leq C(\|f\|_{p_1, \Omega} + \|g\|_{q_1, \partial \Omega}).
\]

(b) Suppose \( 2 \leq p = q < N - 1 \). Let

\[
1 > \frac{p_s}{p_s - p} = \frac{N}{p} \quad \text{and} \quad \frac{q_s}{q_s - p} = \frac{N - 1}{p - 1}.
\]

Let \( f \in L^{p_1}(\Omega) \), \( g \in L^{p_1}(\partial \Omega) \) and \( U, V \in \mathcal{V} \) satisfy (5.16). Then there is a constant \( C = C(N, p, q, \Omega) > 0 \) such that

\[
\|U - V\|_{\infty}^{p-1} \leq C(\|f\|_{p_1, \Omega} + \|g\|_{q_1, \partial \Omega}).
\]

Proof. Let \( U, V \in \mathcal{V} \) satisfy (5.16). Let \( k \geq 0 \) be a real number and set

\[
w_k := (|u - v| - k)^+ \text{ sgn}(u - v), \quad W_k := (w_k, w_k|_{\partial \Omega}) \quad \text{and} \quad w := |u - v|.
\]

Let \( A_k := \{ x \in \overline{\Omega} : |w(x)| \geq k \} \), and

\[
A_k^+ := \{ x \in \overline{\Omega} : w(x) \geq k \}, \quad A_k^- := \{ x \in \overline{\Omega} : w(x) \leq -k \}.
\]

Clearly \( W_k \in \mathcal{V} \) and \( A_k = A_k^+ \cup A_k^- \). We claim that there exists a constant \( C > 0 \) such that

\[
\mathcal{A}(W_k, W_k) \leq C(\mathcal{A}(U, W_k) - \mathcal{A}(V, W_k)), \tag{5.17}
\]

for all \( U, V \in \mathcal{V} \). Using the definition of the form \( \mathcal{A} \), we have

\[
\mathcal{A}(U, W_k) - \mathcal{A}(V, W_k) = \int_{\Omega} ((\mathbf{\nabla} u)^{p-2}\mathbf{\nabla} u - |\mathbf{\nabla} u|^{p-2}|\mathbf{\nabla} v|^{p-2}|\mathbf{\nabla} v|) \cdot \mathbf{\nabla} w_k \, dx + \int_{\Omega} (|u|^{p-2}u - |v|^{p-2}v)w_k \, dx
\]

\[
+ \int_{\Omega} (\alpha_1(x,u) - \alpha_2(x,v))w_k \, dx + \int_{\partial \Omega} (|u|^{q-2}u - |v|^{q-2}v)w_k \, d\sigma
\]

\[
+ \int_{\partial \Omega} ((\mathbf{\nabla} \Gamma u)^{p-2}\mathbf{\nabla} \Gamma u - |\mathbf{\nabla} \Gamma u|^{p-2}|\mathbf{\nabla} \Gamma v|^{p-2}|\mathbf{\nabla} \Gamma v|) \cdot \mathbf{\nabla} \Gamma w_k \, d\sigma + \int_{\partial \Omega} (\alpha_2(x,u) - \alpha_2(x,v))w_k \, d\sigma.
\]
Since $\nabla W_k = \begin{cases} \nabla(u - v) & \text{in } A(k), \\ 0 & \text{otherwise,} \end{cases}$ we can rewrite (5.18) as follows:

$$\mathcal{A}(U, W_k) = \mathcal{A}(V, W_k) = \int_{A(k) \cap \Omega} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \cdot \nabla(u - v) \, dx$$

$$+ \int_{A(k) \cap \partial \Omega} (|\nabla u|^q - |\nabla v|^q) \, d\sigma$$

$$+ \lambda \int_{A(k) \cap \Omega} (|u|^{p-2} u - |v|^{p-2} v) w_k \, dx$$

$$+ \int_{A(k) \cap \partial \Omega} (\alpha_1(x, u) - \alpha_1(x, v)) w_k \, d\sigma.$$

Exploiting inequality (2.12), from (5.19) and (5.11), we deduce

$$\mathcal{A}(U, W_k) - \mathcal{A}(V, W_k) \geq \int_{A(k) \cap \Omega} (|\nabla w_k|^p + |w_k|^p) \, dx + \int_{A(k) \cap \partial \Omega} c_1 \alpha_1(x, w_k) w_k \, dx$$

$$+ \int_{A(k) \cap \partial \Omega} (|u|^{p-2} u w_k - |v|^{p-2} v w_k - |w_k|^p) \, dx$$

$$+ \int_{A(k) \cap \partial \Omega} \alpha_1(x, u) - \alpha_1(x, v) - c_1 \alpha_1(x, w_k) w_k \, dx$$

$$+ \int_{A(k) \cap \partial \Omega} (|\nabla w_k|^q + |w_k|^q) \, d\sigma + \int_{A(k) \cap \partial \Omega} c_2 \alpha_2(x, w_k) w_k \, d\sigma$$

$$+ \int_{A(k) \cap \partial \Omega} (|u|^{q-2} u w_k - |v|^{q-2} v w_k - |w_k|^q) \, d\sigma$$

$$+ \int_{A(k) \cap \partial \Omega} (\alpha_2(x, u) - \alpha_2(x, v) - c_2 \alpha_2(x, w_k)) w_k \, d\sigma.$$
Hence, multiplying this inequality by $w_k(x) \leq 0$, we get
\[
(\alpha_j(x,u(x)) - \alpha_j(x,v(x)) - \epsilon_j(x,w_k(x)))w_k(x) \geq 0,
\]
for all $x \in A^-$. Hence, on account of (5.21) and (5.22), from (5.20) we obtain the required estimate of (5.17).

(a) To prove this part, note that from Definition 5.3 it is clear that,
\[
\|w_k\|_{L^p(\Omega)}^p \leq \mathcal{A}(W_k, W_k).
\]
Let $f \in L^p(\Omega)$ and $g \in L^p(\partial\Omega)$ with
\[
p_1 > \frac{p_s}{p_s - p} = \frac{N}{p} \quad \text{and} \quad q_1 > \frac{q_s}{q_s - p} = \frac{N - 1}{p - 1},
\]
and let $B \subset \Omega$ be any $\mu$-measurable set. We claim that there exists a constant $C \geq 0$ such that, for every $F \in X^{p_1,q_1}(\Omega, \mu)$ and $\varphi \in W^{1,p}(\Omega)$, we have
\[
\|F\varphi|_B\|_{1,1} \leq C\|F\|_{p_1,q_1}\|\varphi\|_{W^{1,p}(\Omega)}\|\chi_B\|_{p_3,q_3},
\]
where $p_3$ and $q_3$ are such that $1/p_3 + 1/p_1 + 1/p_s = 1$ and $1/q_3 + 1/q_1 + 1/q_s = 1$. In fact, note that if $n \in \mathbb{N}$ and $p_i, q_i \in [1, \infty]$ ($i = 1, \ldots, n$) are such that
\[
\sum_{i=1}^n \frac{1}{p_i} = \sum_{i=1}^n \frac{1}{q_i} = 1,
\]
and, if $F_i \in X^{p_i,q_i}(\Omega, \mu)$, $(i = 1, \ldots, n)$, then by H"older’s inequality,
\[
\|\prod_{i=1}^n F_i\|_{1,1} \leq \prod_{i=1}^n \|F_i\|_{p_i,q_i}.
\]
Since $W^{1,p}(\Omega) \hookrightarrow X^{p_1,q_1}(\Omega, \mu)$, (5.24) follows immediately from (5.25) and the claim (5.24) is proved. Next, it follows from (5.24), that
\[
\int_{\Omega} F W_k d\mu = \|FW_k\|_{1,1} = \|FW_k\chi_{A_k}\|_{1,1} \leq \|F\|_{p_1,q_1}\|w_k\|_{W^{1,p}(\Omega)}\|\chi_{A_k}\|_{p_3,q_3},
\]
where we recall that $1/p_3 = (1 - 1/p_s - 1/p_1)/(p - 1)/p_s$ and $q_3 < q_s/(p - 1)$. Therefore, for every $k \geq 0$,
\[
\mathcal{A}(U, W_k) - \mathcal{A}(V, W_k) \leq \|F\|_{p_1,q_1}\|w_k\|_{W^{1,p}(\Omega)}\|\chi_{A_k}\|_{p_3,q_3},
\]
which together with estimate (5.17) yields the desired inequality
\[
C\mathcal{A}(W_k, W_k) \leq \mathcal{A}(U, W_k) - \mathcal{A}(V, W_k) \leq \|F\|_{p_1,q_1}\|w_k\|_{W^{1,p}(\Omega)}\|\chi_{A_k}\|_{p_3,q_3},
\]
It follows from (5.23) and (5.26), that for every $k > 0$,
\[
C\|w_k\|_{L^p(\Omega)}^p \leq \mathcal{A}(W_k, W_k) \leq \mathcal{A}(U, W_k) - \mathcal{A}(V, W_k) \leq \|F\|_{p_1,q_1}\|w_k\|_{W^{1,p}(\Omega)}\|\chi_{A_k}\|_{p_3,q_3},
\]
Hence, for every $k > 0$, $\|w_k\|_{L^p(\Omega)}^p \leq C\|\chi_{A_k}\|_{p_3,q_3}$. Using the fact $W^{1,p}(\Omega) \hookrightarrow X^{p_1,q_1}(\Omega, \mu)$, we obtain for every $k > 0$, that
\[
\|w_k\|_{L^p(\Omega)}^p \leq C\|F\|_{p_1,q_1}\|\chi_{A_k}\|_{p_3,q_3},
\]
Let $h > k$. Then $A_h \subset A_k$ and on $A_h$ the inequality $|w_k| \geq (h - k)$ holds. Therefore,
\[
\|(h - k)\chi_{A_k}\|_{p_1,q_1} \leq C\|F\|_{p_1,q_1}\|\chi_{A_k}\|_{p_3,q_3},
\]
which shows that
\[
\|X_{A_k}\|_{p,q_1}^{p-1} \leq C\|F\|_{p_1,q_1}(h-k)^{-(p-1)}\|X_{A_k}\|_{p_3,q_3}. 
\] (5.27)

Let \( C_3 := \|1\|_{p,q_1} \), and
\[
\delta := \min \left\{ \frac{p_3}{p}, \frac{q_3}{p_3} \right\} > p - 1, \delta_0 := \frac{\delta}{p-1} > 1.
\]

Then
\[
\|C_3^{-\frac{p}{p_3}}X_{A_k}\|_{\Omega,p_3} = \|C_3^{-\frac{p}{p_3}}X_{A_k}\|_{\Omega,p_3} \leq \|C_3^{-\frac{\delta}{\delta_0}}X_{A_k}\|_{\Omega,p_3} \leq \|X_{A_k}\|_{p_3,q_3}C_3^\delta. 
\] (5.28)

and
\[
\|C_3^{-\frac{q_3}{q_3}}X_{A_k}\|_{\partial\Omega,q_3} = \|C_3^{-\frac{q_3}{q_3}}X_{A_k}\|_{\partial\Omega,q_3} \leq \|C_3^{-\frac{\delta}{\delta_0}}X_{A_k}\|_{\partial\Omega,q_3} \leq \|X_{A_k}\|_{p_3,q_3}C_3^\delta. 
\] (5.29)

Choosing \( C_\Omega := C_3^{-\frac{p_3}{p_3}} + C_3^{-\frac{q_3}{q_3}} \), from (5.28)–(5.29) we have
\[
\|X_{A_k}\|_{p_3,q_3} \leq C_\Omega \|X_{A_k}\|_{p_3,q_3}^\delta. 
\] (5.30)

Therefore, combining (5.27) with (5.30), we get
\[
\|X_{A_k}\|_{p_3,q_3}^{p-1} \leq C\|F\|_{p_1,q_1}(h-k)^{-(p-1)}\|X_{A_k}\|_{p_3,q_3}^\delta 
\]
\[
= C\|F\|_{p_1,q_1}(h-k)^{-(p-1)}\|X_{A_k}\|_{p_3,q_3}^{p-1}\delta^\frac{1}{\delta}. 
\] (5.31)

Setting \( \psi(h) := \|X_{A_k}\|_{p_3,q_3}^{p-1} \) in Lemma 2.13 on account of (5.31), we can find a constant \( C_2 \) (independent of \( F \)) such that
\[
\|X_{A_k}\|_{p_3,q_3}^{p-1} = 0 \text{ with } K := C_2\|F\|_{p_1,q_1}^{1/(p-1)}.
\]

This shows that \( \mu(A_K) = 0 \), where \( A_K = \{ x \in \Omega : |u - v|(x) \geq K \} \). Hence, we have \( |u - v| \leq K \), \( \mu \)-a.e. on \( \Omega \) so that
\[
\|U - V\|_{p_3,q_3}^{p-1} \leq C_2\|F\|_{p_1,q_1} = C_2 \left( \|f\|_{p_1,\Omega} + \|g\|_{q_1,\Omega} \right),
\]
which completes the proof of part (a).

(b) To prove this part, instead of (5.23) and (5.24), one uses \( \|W_k\|_{p_1,\Omega} \leq C\|W_k\|_{p_1,\Omega} \) and \( \|F\|_{p_1,q_1} \leq C\|F\|_{p_1,q_1} \|\Omega\|_{p_3,q_3} \), (where \( p_3 \) and \( q_3 \) are such that \( 1/p_3 + 1/p_1 + 1/p_3 = 1 \) and \( 1/q_3 + 1/q_1 + 1/p_3 = 1 \) and the embedding \( \nu' \hookrightarrow \nu' \hookrightarrow X_{p_3,p_3}(\Omega,\mu) \).

The remainder of the proof follows as in the proof of part (a).

We conclude this section with the following example.

**Example 5.11.** Let \( p \in [2,\infty), b : \partial\Omega \to (0,\infty) \) be a strictly positive and \( \sigma \)-measurable function and let
\[
\beta(x,\xi) := b(x)|\xi|^{p-2}\xi, \quad \xi \in \mathbb{R}.
\]

Then, it is easy to verify that \( \beta \) satisfies Assumptions 5.2, 5.3 and 5.7 (see, e.g., Example 4.17).
REFERENCES

[1] R. A. Adams. Sobolev Spaces. Pure and Applied Mathematics, Vol. 65. Academic Press, New York, 1975.
[2] S. Agmon, A. Douglis and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I. Comm. Pure Appl. Math. 12 (1959), 623–727.
[3] H. Attouch, G. Buttazzo and G. Michaille. Variational Analysis in Sobolev and BV Spaces. Applications to PDEs and optimization. MPS/SIAM Series on Optimization. 6. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2006.
[4] Ph. Bénilan and M.G. Crandall. Completely accretive operators. Semigroup theory and evolution equations (Delft, 1989), 41–75, Lecture Notes in Pure and Appl. Math., 135, Dekker, New York, 1991.
[5] M. Biegert and M. Warma. The heat equation with nonlinear generalized Robin boundary conditions. J. Differential Equations 247 (2009), 1949–1979.
[6] M. Biegert and M. Warma. Some Quasi-linear elliptic Equations with inhomogeneous generalized Robin boundary conditions on “bad” domains. Adv. Differential Equations 15 (2010), 893–924.
[7] H. Brézis. Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert. American Elsevier Publishing Co., Inc., New York, 1973.
[8] H. Brézis and A. Haraux. Image d’une somme d’opérateurs monotones et applications. Israel J. Math. 23 (1976), 165–186.
[9] H. Brézis and L. Nirenberg. Image d’une somme d’opérateurs non linéaires et applications. C. R. Acad. Sci. Paris Sér. A-B 284 (1977), no. 21, A1365–A1368.
[10] E. DiBenedetto. Degenerate Parabolic Equations. Springer, New York, 1993.
[11] P. Drabek and J. Milota. Methods of Nonlinear Analysis. Applications to Differential Equations. Birkhäuser Advanced Texts, Birkhäuser, Basel, 2007.
[12] C.G. Gal, G.R. Goldstein, J.A. Goldstein, S. Romanelli and M. Warma. Fredholm alternative, semilinear elliptic problems, and Wentzell boundary conditions. Preprint.
[13] C.G. Gal, M. Grasselli and A. Miranville. Nonisothermal Allen-Cahn equations with coupled dynamic boundary conditions. Nonlinear phenomena with energy dissipation, GAKUTO Internat. Ser. Math. Sci. Appl., 29 (2008), 117–139.
[14] C.G. Gal and A. Miranville. Uniform global attractors for non-isothermal viscous and non-viscous Cahn–Hilliard equations with dynamic boundary conditions. Nonlinear Analysis: Real World Applications 10 (2009), 1738–1766.
[15] J.A. Goldstein. Nonlinear Semigroups. Lecture Notes.
[16] L. Hörmander. Linear Partial Differential Operators. Springer-Verlag, Berlin, 1976.
[17] E.M. Landesman and A.C. Lazer. Nonlinear perturbations of linear elliptic boundary value problems at resonance. J. Math. Mech. 19 (1969/1970), 609–623.
[18] V. Maz’ya. Sobolev Spaces. Springer-Verlag, Berlin, 1985.
[19] V.G. Maz’ya and S.V. Poborchii. Differentiable Functions on Bad Domains. World Scientific Publishing, 1997.
[20] G. J. Minty. Monotone (nonlinear) operators in Hilbert space. Duke Math. J. 29 (1962), 341–346.
[21] G. J. Minty. On the solvability of nonlinear functional equations of monotonic type. Pacific J. Math. 14 (1964), 249–255.
[22] M. K. V. Murthy and G. Stampacchia. Boundary value problems for some degenerate-elliptic operators. Ann. Mat. Pura Appl. 80 (1968), 1–122.
[23] J. Necas. Les Méthodes Directes en Théorie des Équations Elliptiques. Masson et Cie, Éditeurs, Paris; Academia, Éditeurs, Prague, 1967.
[24] J. Peetre. Another approach to elliptic boundary value problems. Comm. Pure Appl. Math. 14 (1961), 711–734.
[25] M. M. Rao and Z. D. Ren. Theory of Orlicz Spaces. Monographs and Textbooks in Pure and Applied Mathematics, 146. Marcel Dekker, Inc., New York, 1991.
[26] M. M. Rao and Z. D. Ren. Applications of Orlicz Spaces. Monographs and Textbooks in Pure and Applied Mathematics, 250. Marcel Dekker, Inc., New York, 2002.
[27] R. E. Showalter. Monotone Operators in Banach Space and Nonlinear Partial Differential Equations. Amer. Math. Soc., Providence, RI, 1997.
[28] M.I. Vishik. On general boundary problems for elliptic differential equations. Trudy Moskov. Math. Obsc. 1 (1952), 187–246.
[29] M. Warma. The Robin and Wentzell-Robin Laplacians on Lipschitz domains. Semigroup Forum 73 (2006), 10–30.
