I review to which extent the properties of pseudoscalar mesons can be understood in terms of the underlying quark (and eventually gluon) structure. Special emphasis is put on the progress in our understanding of $\eta \rightarrow \eta'$ mixing. Process-independent mixing parameters are defined, and relations between different bases and conventions are studied. Both, the low-energy description in the framework of chiral perturbation theory and the high-energy application in terms of light-cone wave functions for partonic Fock states, are considered. A thorough discussion of theoretical and phenomenological consequences of the mixing approach will be given. Finally, I will discuss mixing with other states ($\pi^0$, $\eta_c$, ...).

1. Introduction

The fundamental degrees of freedom in strong interactions of hadronic matter are quarks and gluons, and their behavior is controlled by Quantum Chromodynamics (QCD). However, due to the confinement mechanism in QCD, in experiments the only observables are hadrons which appear as complex bound systems of quarks and gluons. A rigorous analytical solution of how to relate quarks and gluons in QCD to the hadronic world is still missing. We have therefore developed effective descriptions that allow us to derive non-trivial statements about hadronic processes from QCD and vice versa. It should be obvious that the notion of quark or gluon structure may depend on the physical context. Therefore one aim is to find process-independent concepts which allow a comparison of different approaches.

The simplest example, which essentially reflects our intuitive picture of hadrons, is to assign a particular quark content to each hadron (say, proton $\sim uud$). It enables us to classify the hadrons in the particle data book according to their flavor quantum numbers. This concept is the basis of the numerous versions of constituent quark models which are used as an effective low-energy approximation to QCD. These models are sufficient to obtain a reasonable explanation of global features like the mass hierarchy in the hadronic spectrum, relations among scattering amplitudes and decay widths etc.

A more elaborated approach to the low-energy sector of hadronic physics is based on the symmetry properties of QCD. The (approximate) chiral symmetry of light quarks plays a special role. It appears to be spontaneously broken, and the
light pseudoscalar mesons can be identified as (almost) Goldstone bosons. These properties can be used for a systematic construction of an effective theory, chiral perturbation theory ($\chi$PT), where the non-perturbative information is encoded in the coefficients of operators in an effective Lagrangian. By comparison with experimental data for hadronic low-energy reactions one extracts the values of these coefficients which provide a well-defined measure of the hadronic structure.

At higher energies one resolves the partonic degrees of freedom inside the hadron. The structure functions, measured in deep inelastic scattering experiments, can be successfully described in terms of momentum distributions of valence-, sea-quarks and gluons. These parton distributions are process-independent. They describe the probability to find a certain parton with a specific momentum fraction inside the hadron. They depend on the resolution-scale, but this dependence is controlled solely in terms of perturbative QCD. The enormous amount of data for nucleon structure functions over a wide range of energy and momentum transfer simultaneously allows us to extract the parton distributions and to test QCD to a rather high accuracy. From inclusive pion-nucleon scattering, taking the nucleon structure functions as input, one may also extract the distributions of quarks and gluons in the pion, but with much poorer accuracy. For all other hadrons experimental results of similar quality are not available.

Furthermore, light hadrons provide a rich phenomenology of exclusive reactions with large momentum transfer, e.g. in decays of heavy particles or electroweak form factors. These reactions are expected to be dominated by a finite number of particular parton Fock states of the hadrons being involved. The momentum distribution of the partons in each Fock state defines the so-called distribution amplitudes which behave in a similar way as the parton distributions in inclusive reactions, i.e. they evolve with the resolution scale but are otherwise process-independent. By comparing different exclusive reactions, one is again in the position to obtain important information on the hadron structure and to test our understanding of QCD at the same time. In principle, the parton distributions measured in inclusive reactions can be reconstructed from the light-cone wave functions (from which the distribution amplitudes are obtained by integration over transverse momenta) of each individual Fock state. In practice, however, only a few Fock states are under control, and the connection to the parton distributions can only be exploited at large momentum fraction $x$ (see e.g. Refs. 10, 11, 12).

The main part of this review deals with a subject where already the simple question concerning the quark flavor decomposition turns out to be rather non-trivial, namely for pseudoscalar mesons $\pi^0, \eta, \eta'$ ... with vanishing isospin, strangeness etc. In these cases the strong interaction induces transitions between quarks of different flavors ($u\bar{u}, d\bar{d}, s\bar{s}$) or gluons ($g g$ ...). Both, in the constituent quark model at low energies and in the parton picture at high energies, the physical mesons appear as complicated mixtures of different quark-antiquark combinations. The mixing phenomenon is strongly connected with the $U(1)_A$ anomaly of QCD. I will show below how this fact can be used to define and to quantify reasonable and
process-independent mixing parameters. The investigation of mixing phenomena in the pseudoscalar meson sector has a long tradition. Of particular interest is the $\eta - \eta'$ system: Since the light pseudoscalar mesons approximately fall into multiplets of $SU(3)_F$ flavor symmetry, mainly disturbed by the mass of the strange quark, one conventionally regards $\eta$ and $\eta'$ as linear combinations of octet and singlet basis states, parametrized by a mixing angle $\theta_P$. A determination of its value should be achieved – in principle – from the diagonalization of suitably chosen mass matrices (motivated by e.g. $\chi$PT) or from phenomenology.

Already in 1964 Schwinger\cite{Schwinger1964} used mass formulas to estimate the mass of the $\eta'$ (actually the original prediction, 1600 MeV, is 50% too high due to the neglect of some $SU(3)_F$ corrections). An early estimate of the mixing angle has been given, for instance, by Isgur\cite{Isgur1970} ($\theta_P \simeq -10^\circ$). Kramer et al.\cite{Kramer1972} as well as Fritzsch and Jackson\cite{Fritzsch1974} have emphasized the importance of $SU(3)_F$–breaking effects. Their ansatz for the mass matrix is formulated in the quark-flavor basis ($u\bar{u}, d\bar{d}$, etc.) and takes into account mixing with the $\eta_c$ state. Their mixing angle (corresponding to $\theta_P \simeq -11^\circ$) has also been tested against experimental data. In 1981 Diakonov and Eides\cite{Diakonov1981} presented a quantitative analysis of anomalous Ward identities and quoted a value of $\theta_P \simeq -9^\circ$. In subsequent publications\cite{Diakonov1983,Diakonov1984}, however, very different values for the mixing angles have been found: The incorporation of loop corrections in $\chi$PT to meson masses and decay constants and a comparison with phenomenological data for various decay modes seem to favor values of $\theta_P$ around $-20^\circ$. Ball, Frère and Tytgat\cite{Ball1994} concentrated on processes where the anomalous gluonic content of $\eta$ and $\eta'$, which can be related to the decay constants, is probed. They also prefer values for $\theta_P$ near to $-20^\circ$. On the other hand a quark model calculation of Schechter et al.\cite{Schechter1995} more or less recovers the mixing scenario of Diakonov and Eides with $\theta_P \simeq -13^\circ$. Finally, Bramon et al.\cite{Bramon1997} considered only such processes where the light $(u\bar{u} \text{ or } d\bar{d})$ or strange $(s\bar{s})$ component is probed and found a mixing angle $\theta_P \simeq -15^\circ$.

The situation concerning the actual value of $\theta_P$ thus seemed to be rather dissatisfying with results ranging from $-10^\circ$ to $-20^\circ$. However, it turned out during the last two years, that a big part of the discrepancies between different analyses can be solved by relaxing and correcting some of the implicit assumptions made therein. Considerations of Leutwyler and Kaiser\cite{Leutwyler1996} as well as Kroll, Stech and myself\cite{Kroll1996,Leutwyler1997} have shown that the definition of mixing parameters requires some care. In low-energy effective theories the decay constants relate the physical states with the (bare) octet/singlet fields. It turns out that this connection cannot be a simple rotation. Therefore, $\eta - \eta'$ mixing cannot be adequately described by a single mixing angle $\theta_P$. A new scheme, which can be strictly related to the effective Lagrangian of $\chi$PT, is formulated on the basis of a general parametrization of the octet and singlet decay constants of $\eta$ and $\eta'$ mesons, respectively. This scheme can also be successfully applied to hard exclusive reactions with $\eta$ or $\eta'$ mesons where one is sensitive to the light-cone wave functions ‘at the origin’, which are fixed by the decay constants, too. The main achievement of the new mixing scheme is that the properties of $\eta$ and $\eta'$ mesons in different phenomenological situations and at
different energy scales can be described in a consistent way.

The organization of this article is as follows: In the following section I will shortly recall some important properties of QCD under chiral symmetry transformations and the consequences for the pseudoscalar meson spectrum, which includes a summary of the $U(1)_{A}$ problem and its possible solutions. Section 3, which is the main part of this review, includes a detailed discussion of $\eta$-$\eta'$ mixing: The chiral effective Lagrangian for the pseudoscalar nonet, including the $\eta'$ meson, is presented, and its parameters are related to the octet/singlet decay constants of $\eta$ and $\eta'$ mesons. In the quark-flavor basis, the consequent application of the Okubo-Zweig-Iizuka–rule (OZI-rule) is shown to lead to a scheme with a single mixing angle $\phi$. Phenomenological estimates of mixing parameters from the literature are compared in both, the octet-singlet and quark-flavor basis. This is followed by a discussion of the two-photon decay widths and the $VP\gamma$ coupling constants. The $\eta\gamma$ and $\eta'\gamma$ transition form factors are analyzed within the hard-scattering approach. For this purpose the light-cone wave functions of $\eta$ and $\eta'$ mesons are introduced and compared to the pion one. For the latter also the connection to the parton distribution functions will be illuminated. Interesting relations among the mixing parameters are obtained by considering the matrix elements of pseudoscalar quark currents and of the topological charge density. I will further present improved versions of various mass formulas for the $\eta$-$\eta'$ system. Finally, the consequences of the new mixing scheme for the pseudoscalar coupling constants of the nucleon are investigated. Section 4 is devoted to mixing with $\pi^0$ or $\eta_c$ mesons and includes a comment on mixing with glueballs or excited quarkonium states. A summary is presented in Section 5.

2. Chiral symmetry of light quarks

As a starting point let me recall some important global symmetries of the strong interactions and consider the QCD Lagrangian

$$\mathcal{L}_{\text{QCD}} = \sum_{i=u,d,s,\ldots} \bar{q}_i (i\not\!D - m_i) q_i - \frac{1}{2} \text{tr}[G^{\mu\nu} G_{\mu\nu}] - \theta \omega + \text{FP-ghost + gauge-fixing}.$$  

(1)

Here $G^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu - ig [A^\mu, A^\nu]$ is the gluonic field strength tensor with $A_\mu = A_\mu^a \lambda^a / 2$ denoting the color gauge fields. Furthermore, $q_i$ denotes quark fields of a specific flavor with mass $m_i$, and $i\not\!D = i\not\!\partial - g A$ is the covariant in QCD. Faddeev-Popov ghosts and gauge-fixing terms will be unimportant for the further considerations. A summation over color indices is to be understood. I have also included the $\theta$-term which reflects the non-trivial features connected with the axial $U(1)_A$ anomaly to be discussed below. Here $\omega(x)$ is the topological charge density which can be written as the divergence of a gauge-variant current $K^\mu$ where

$$K_\mu = \frac{\alpha_s}{4\pi} \epsilon_{\mu\nu\rho\sigma} \sum_{a,b,c} A_{\alpha}^\nu \left( \partial^\rho A_{\sigma}^\sigma + \frac{g}{3} f^{abc} A_{\alpha}^a A_{\beta}^b \right),$$
\[ \omega = \partial_{\mu} K^\mu = \frac{\alpha_s}{8\pi} G \tilde{G} . \]  

(2)

The term \( G \tilde{G} \) denotes the product of the gluon field strength and its dual, and \( f^{abc} \) are the anti-symmetric structure constants of \( SU(3), [\lambda^a, \lambda^b] = 2i f^{abc} \lambda^c \). The different topological sectors of QCD are classified by the Pontryagin-Index which is given by the topological charge

\[ \int d^4x \omega(x) = \int d\sigma \mu K^\mu = n \in \mathbb{Z} . \]  

(3)

It is well known that the pseudoscalar mesons \( \pi, K, \eta \) built from the light flavors \( u, d, s \) play a special role in strong interaction physics. This is connected with the behavior of the light quark fields under global chiral transformations

\[ q_L = \frac{1 - \gamma_5}{2} q \rightarrow L q_L , \]

\[ q_R = \frac{1 + \gamma_5}{2} q \rightarrow R q_R . \]

(4)

Here \( q = (u, d, s)^T \), and \( L \in SU(3)_L \) and \( R \in SU(3)_R \) denote unitary \( 3 \times 3 \) matrices acting on left- and right-handed projections of the light quark fields, \( L, R = \exp[i \epsilon^a_{\mu, R} \lambda^a] \), where \( \lambda^a \) are the usual Gell-Mann matrices \((a = 1, ..., 8)\). Obviously, in the limit \( m_{u,d,s} \to 0 \) the Lagrangian (1) is invariant under chiral \( SU(3)_L \times SU(3)_R \) transformations. The corresponding Noether currents,

\[ J^a_{\mu L} = \bar{q} \frac{\lambda^a}{\sqrt{2}} \gamma^\mu q \frac{1 - \gamma_5}{2} , \]

\[ J^a_{\mu R} = \bar{q} \frac{\lambda^a}{\sqrt{2}} \gamma^\mu q \frac{1 + \gamma_5}{2} , \]

(5)

are approximately conserved \((a = 1, ..., 8)\)

\[ \partial^\mu (J^a_{\mu R} \pm J^a_{\mu L}) = \bar{q} \left[ \lambda^a \sqrt{2}, \tilde{m} \right]_{\mp} i\gamma_5 q . \]

(6)

The octet of vector currents \( J^a = J^a_{\mu R} + J^a_{\mu L} \) in Eq. (6) reflects the (approximate) flavor symmetry, which is observed in the physical spectrum. On the other hand, the conservation laws related to the octet of axial-vector currents \( J^a_{\mu 5} = J^a_{\mu R} - J^a_{\mu L} \) in Eq. (6) cannot be observed in the hadronic world. The chiral symmetry of the Lagrangian appears to be spontaneously broken, \( SU(3)_L \times SU(3)_R \rightarrow SU(3)_V \), and \( \pi, K, \eta \) appear in a natural way as an octet of (almost) Goldstone particles which become massless in the limit \( m_{u,d,s} \to 0 \). This is the starting point for the systematic construction of an effective theory for the low-energy hadronic physics, see Section 3.1.

2.1. \( U(1)_A \) anomaly

The QCD Lagrangian (1) has two additional global symmetries: The \( U(1)_V \) symmetry, \( q \to \exp[i \epsilon_V |q|] \), corresponds to the conservation of baryon number. The
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$U(1)_A$ symmetry, $q \rightarrow \exp[i \epsilon A \gamma_5] q$ gives rise neither to a conserved quantum number nor to a ninth Goldstone boson in the mesonic spectrum: the $\eta'$ meson is too heavy to be identified as the Goldstone particle of a spontaneously broken $U(1)_A$ symmetry. The solution of this $U(1)_A$ puzzle is connected with the emergence of an anomaly contributing to the divergence of the singlet axial vector current, and Eq. (6) is to be extended,

$$\partial \mu J^a_{\mu 5} = \bar{q} \left\{ \frac{\lambda^a}{\sqrt{2}}, m \right\} i \gamma_5 q + \delta^{a0} 2\sqrt{3} \omega . \quad (a = 0 \ldots 8) \quad (7)$$

My normalization convention is $\text{tr}[\lambda^a \lambda^b] = 2 \delta^{ab}$ for $a = 0 \ldots 8$, i.e. $\lambda^0 = \sqrt{2/3} 1$.

In Eq. (6) the topological charge density $\omega$ defined in Eq. (2) leads to a non-conservation of the axial-vector current in the flavor-singlet sector even in the limit $\hat{m} \rightarrow 0$.

Anomalies in quantum theories have been studied in several ways, following the pioneering work of Adler, Bell and Jackiw. In perturbation theory the $U(1)_A$ anomaly arises when calculating the quark triangle diagram with the axial-vector current and two gauge fields since it turns out to be impossible to define a regularization prescription which preserves both, gauge invariance and chiral invariance. (In QCD there are no gauge fields coupled to the axial-vector current; thus the anomaly causes no theoretical inconsistencies here.) An elegant derivation of Eq. (7) can be found in the article of Fujikawa.

The anomalous term stems from a non-trivial Jacobian which is related to the transformation of the path integral measure of the fermion fields

$$\Psi(x) \rightarrow \exp[i \epsilon A(x) \gamma_5] \Psi(x) ,$$

$$\bar{\Psi}(x) \rightarrow \bar{\Psi}(x) \exp[i \epsilon A(x) \gamma_5]$$

$$\Rightarrow D\bar{\Psi} D\Psi \rightarrow D\bar{\Psi} D\Psi \exp \left[ 2 i \int dx \epsilon A(x) \omega(x) \right] . \quad (8)$$

It is also interesting to see how the $U(1)_A$ anomaly arises in lattice-QCD. Here, the discretization of space-time on a four-dimensional lattice can be considered as a particular regularization scheme. The lattice action is constructed in such a way that it reproduces the Lagrangian (1) with the lattice spacing approaching zero (continuum limit). It has been shown that the anomalous contribution to the Ward identity (6) is related to an irrelevant operator in the lattice action. The operator itself naively vanishes in the continuum limit, but its contribution to the anomalous Ward identity does not. It has been shown that the correct continuum result (6) is reproduced for any choice of lattice action, as long as very general conditions are fulfilled.

Since the flavor-singlet axial-vector current is not conserved in the presence of the $U(1)_A$ anomaly the $\eta'$ mass does not have to vanish in the limit $\hat{m} \rightarrow 0$. However, in order to understand the experimentally observed mass splitting between octet and singlet pseudoscalar mesons, Eq. (6) alone is not sufficient. In addition one
needs a non-vanishing matrix element of the topological charge density sandwiched between the η' state and the vacuum

$$(0|ω|η') \neq 0.$$  
(9)

Since $ω$ is a total divergence, see Eq. (2), the l.h.s. of Eq. (9) vanishes to any finite order in perturbation theory, i.e. the $U(1)_A$ problem cannot be solved by simply considering quark-antiquark annihilation into (perturbative) gluons. The solution clearly lies in the non-perturbative sector of QCD and is inevitably connected to non-trivial topological features of the theory. 't Hooft suggested instantons as a possible solution to Eq. (9). Kogut/Susskind have argued that the ninth Goldstone field is prevented from being realized in the physical spectrum by the same mechanism that confines colored objects. An alternative approach has been initiated by Witten who proposed to consider QCD from the large-$N_C$ perspective, where $N_C$ is the number of colors. It turns out that in order to obtain a consistent picture for the $θ$-dependence of the pure Yang-Mills theory in the formal limit $N_C → ∞$, the η' mass squared should behave as $O(1/N_C)$. Veneziano has found a realization of Witten's general $1/N_C$ counting rules by introducing a ghost state into the theory (the notion of ghost states in this context has also been used by Kogut and Susskind; it is also close to Weinberg's approach that involves negative metric Goldstone fields). The ghost corresponds to an unphysical massless pole in the correlation function $⟨K^μK_ν⟩$ which generates a non-vanishing topological susceptibility ($mean square winding number per unit volume$)

$$τ_0 = \int d^4x \langle 0|T[ω(x)ω(0)]|0⟩ \neq 0$$

$$⇌ q_μq_ν⟨K^μK_ν⟩_{q→0} \neq 0 \quad (10)$$

which is necessary to fulfill Eq. (9), see also Eq. (12) below. The ghost pole may be viewed as the result of an infinite number of Feynman graphs contributing to $⟨K_μK_ν⟩$, but does not correspond to an observable glueball state, since the currents $K^μ$ are gauge-variant. Diakonov and Eides have discussed the phenomenological consequences of the Witten/Veneziano ansatz by considering the η' meson as a mixture of the Veneziano ghost and a flavor singlet would-be Goldstone boson and investigating the anomalous Ward identities. They found that the gapless excitation given by the Veneziano ghost is a consequence of the periodicity of the QCD potential w.r.t. a certain generalized coordinate which in the gauge $A_0 = 0$ can be chosen as

$$X(t) = \int d^3x K_0(t,x).$$  
(11)

Under gauge transformations it transforms as $X → X + n$, where $n$ is the topological charge (3) of the transformation, while the potential remains unchanged. The quasi-momentum related to $X$ is just the variable $θ$, see Eq. (1), and the connection of the Veneziano ghost to non-trivial topology, as induced by e.g. instantons or other
finite-action field configurations, is obvious. The $U(1)_{A}$ problem has also been studied in lattice-QCD, and both, a non-vanishing $\eta'$ mass and a significant correlation with the topological susceptibility and fermionic zero modes, have been observed.

Finally, the $U(1)_{A}$ problem has been investigated by considering low-energy models of the strong interaction. In the framework of the global color model Frank and Meissner follow the work of Kogut/Susskind and require a certain non-trivial infrared behavior of an effective (i.e. non-perturbative) gluon propagator which leads to quark confinement as well as to a non-vanishing $\eta'$ mass. A low-energy expansion of the effective action following from their ansatz has been shown to reproduce the general results of Witten/Veneziano. Dmitrasinovic investigated different effective $U(1)_{A}$-breaking quark interactions as a low-energy approximation of the t'Hooft or Veneziano/Witten mechanism, which can be used to generate a non-vanishing $\eta'$ mass.

In the ideal world with three massless light quarks and three infinitely heavy quarks the $\eta'$ meson is a pure flavor singlet. However, in the real world the flavor symmetry is not perfect, and the neutral mesons mix among each other. In the isospin limit ($m_u = m_d$) which is a very good approximation to the real world, the $\pi^0$ is still a pure iso-triplet. Without the $U(1)_{A}$ anomaly the two iso-singlet mass eigenstates in the pseudoscalar sector would consist of $u\bar{u} + d\bar{d}$ and $s\bar{s}$, respectively. The $U(1)_{A}$ anomaly mixes these ideally mixed states towards nearly flavor octet or singlet combinations which are to be identified with the physical $\eta$ and $\eta'$ mesons, respectively. From Eq. (8) we see, that the anomalous term in Eq. (7) is independent of the quark masses. Consequently, the $U(1)_{A}$ anomaly also induces mixing with heavier pseudoscalar mesons ($\eta_c, \eta_b$) which is however less important since the non-anomalous terms in Eq. (7) dominate in case of heavy quark masses. Taking into account the mass difference of up- and down-quarks as a source of isospin-violation, also the $\pi^0$ receives a small iso-singlet admixture. The quantification of the mixing parameters is one of the main subjects of this review and will be discussed in Sections 3 and 4.

3. Mixing in the $\eta$-$\eta'$ system

In order to quantify the mixing in the $\eta$-$\eta'$ system, one has to define appropriate mixing parameters which can be related to physical observables. One approach is based on chiral perturbation theory which traditionally leads to a description of $\eta$-$\eta'$ mixing in terms of octet-singlet parameters. Another useful concept to obtain well-defined quantities is to follow e.g. the work of Diakonov/Eides and to consider the operators appearing in the anomaly equation (7) sandwiched between the vacuum and physical meson states. Making consequent use of the OZI-rule, one is led to the quark-flavor basis where these matrix elements are expressed in terms of a single mixing angle $\phi$ which can be determined from theory or phenomenology.

For the sake of clarity, I keep – if not otherwise stated – isospin symmetry to be exact ($m_u = m_d \ll m_s$) and neglect the contributions of heavy quarks ($m_Q \to \infty$).
In this limit, the $\pi^0$ meson is a pure iso-triplet, and the $\eta_c$ meson a pure $c\bar{c}$ state. I will discuss mixing phenomena with $\pi^0$ and $\eta_c$ in sections 4.1 and 4.2, respectively.

3.1. Decay constants in the octet-singlet basis and $\chi$PT

The low-energy physics of light pseudoscalar mesons can be successfully described by an effective Lagrangian which reflects a systematic expansion in powers of small momenta and masses of the (almost) Goldstone bosons $\pi, \eta, K$. This is the basis of chiral perturbation theory (\chiPT). Counter terms, arising from renormalization, can be absorbed into higher order coefficients of the effective Lagrangian. In this sense \chiPT is renormalizable order by order. Since the $\eta'$ meson is not a Goldstone boson and its mass is not small, it is usually not included as an explicit degree of freedom in the Lagrangian. Recently, Leutwyler and Kaiser have discussed how to include the $\eta'$ into the framework of \chiPT in a consistent way, and I shall briefly present their most important results.

Starting point is the observation that in the formal limit $N_C \to \infty$ the anomalous term in Eq. (3) vanishes, and the $\eta'$ formally arises as a ninth Goldstone boson of $U(3)_L \times U(3)_R \to U(3)_V$. One can therefore extend the counting rules for the construction of the effective Lagrangian as follows: $1/N_C = O(\delta)$, $p^2 = O(\delta)$, $m_q = O(\delta)$, where $\delta$ is the small expansion parameter. In the standard framework the octet and singlet pseudoscalar mesons are parametrized in a non-linear way by $U$ which reflects the non-vanishing quark condensate. Finally, $6\tau_0/F^2 = \mathcal{O}(N_C^{-1})$ is the contribution of the $U(1)_A$ anomaly to the singlet mass. Here $\tau_0 = \mathcal{O}(1)$ is the topological susceptibility defined in Eq. (10). In order to account for the experimental fact that the decay constants of the light pseudoscalar mesons differ substantially ($F_K = 1.22 F_\pi$) one can include the relevant terms of the next order in the effective Lagrangian:

$$\mathcal{L}^{(0)} = \frac{F^2}{4} \text{tr}(\partial_\mu U^\dagger \partial^\mu U) + \frac{F^2}{4} \text{tr}(\chi^\dagger U + U^\dagger \chi) - 6\tau_0 \frac{1}{2} (\varphi^0)^2 . \quad (12)$$

To this order the parameter $F = \mathcal{O}(\sqrt{\tau_0})$ is identified with the universal pseudoscalar decay constant $F = F_a \simeq F_\pi = 93$ MeV. This follows immediately if one introduces source terms for the axial-vector currents (3) in a chirally-invariant way. Finite meson masses are induced by the term proportional to $\chi = 2B\hat{m}$. Here $\hat{m}$ is the matrix of (current) quark masses and $B = \mathcal{O}(1)$ a dimensional parameter which reflects the non-vanishing quark condensate. Finally, $6\tau_0/F^2 = \mathcal{O}(N_C^{-1})$ is the contribution of the $U(1)_A$ anomaly to the singlet mass. Here $\tau_0 = \mathcal{O}(1)$ is the topological susceptibility defined in Eq. (10). In order to account for the experimental fact that the decay constants of the light pseudoscalar mesons differ substantially ($F_K = 1.22 F_\pi$) one can include the relevant terms of the next order in the effective Lagrangian:

$$\mathcal{L}^{(1)} = L_5 \text{tr} (\partial_\mu U^\dagger \partial^\mu U (\chi^\dagger U + U^\dagger \chi)) + L_8 \text{tr} ((\chi^\dagger U)^2 + h.c.) + \frac{F^2}{2} \Lambda_1 \partial_\mu \varphi^0 \partial^\mu \varphi^0 + \frac{F^2}{2\sqrt{6}} \Lambda_2 i\varphi^0 \text{tr}(\chi^\dagger U - U^\dagger \chi) + \mathcal{L}_{WZW} + \ldots \quad (13)$$

For convenience I have changed the normalization of the singlet field compared to Leutwyler et al. $\psi \rightarrow \sqrt{\tau_0} \varphi^0$.
Here $L_5 = \mathcal{O}(N_C)$ parametrizes corrections to the decay constants, $L_8 = \mathcal{O}(N_C)$ the ones for the meson masses. $\Lambda_1$ and $\Lambda_2$ only influence the singlet sector and can be attributed to OZI-rule violating contributions, i.e. they are of order $1/N_C$. A possible source for these terms can be provided by e.g. glueball states which are not included in the effective Lagrangian, see Section 4.3. I emphasize that the parameters $\Lambda_i$ are of different origin than the topological susceptibility $\tau_0$. $\mathcal{L}_{WZW}$, finally, denotes the Wess-Zumino-Witten term which describes the anomalous coupling to photons and will be discussed in more detail below. Taking e.g. the pion and kaon masses and decay constants as input, one can – to this order – express the parameters $F$, $L_5$ and $B$, $L_8$ in terms of physical quantities only.

Leutwyler and Kaiser have also discussed the next order, $\mathcal{L}^{(2)}$ see the second paper in Ref. 24. At this level, also loop-corrections calculated from the Lagrangian $\mathcal{L}^{(0)}$ have to be taken into account. They give rise to the typical chiral logs,

$$ \frac{M_P^2}{32 \pi^2 F^2} \ln \frac{M_P^2}{\mu^2}, $$

which contribute at the order $\mathcal{O}(\delta^2)$. For the understanding of the main features of $\eta-\eta'$ mixing the higher order effects of chiral logs and $\mathcal{L}^{(2)}$ are not important. In the following I will therefore concentrate on the contributions from $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(1)}$ and discuss the important consequences for the mixing parameters in the octet-singlet basis, revealed by Leutwyler and Kaiser.

The decay constants in the $\eta-\eta'$ system are defined as matrix elements of axial-vector currents (5)

$$ \langle 0 | J_{\mu}^5 (0) | P (p) \rangle = i f_P^a p_\mu $$

where I have changed the normalization convention to $f_\pi = \sqrt{2} F_\pi = 131$ MeV, and the currents are defined as in Eq. (5). Each of the two mesons $P = \eta, \eta'$ has both, octet and singlet components, $a = 8, 0$. Consequently, Eq. (14) defines four independent decay constants, $f_P^a$. For a given current each pair of decay constants can be used to define a separate mixing angle

$$ \frac{f_\eta^s}{f_\eta'^s} = \cot \theta_8, \quad \frac{f_\eta^0}{f_\eta'^0} = -\tan \theta_0. $$

Here I followed the convention of Ref. 24 and used a parametrization in terms of two basic decay constants $f_8, f_0$ and two angles $\theta_8, \theta_0$

$$ \{f_P^a\} = \begin{pmatrix} f_\eta^s & f_\eta^0 \\ f_\eta'^s & f_\eta'^0 \end{pmatrix} = \begin{pmatrix} f_8 \cos \theta_8 & -f_0 \sin \theta_0 \\ f_8 \sin \theta_8 & f_0 \cos \theta_0 \end{pmatrix}. $$

The angles are chosen in such a way that $\theta_8 = \theta_0 = 0$ corresponds to the $SU(3)_F$ symmetric world.

The matrix $\{f_P^a\}$ defined by Eq. (16) will play a crucial role in the following discussion since it is exactly the quantity that relates the physical fields $P = \eta, \eta'$
(which diagonalize the kinetic and mass terms in the effective lagrangian) to the bare octet or singlet fields $\varphi^a$ in the effective Lagrangian \cite{12} and \cite{13}

\begin{equation}
\varphi^a(x) = \sum_p (f^{-1})^a_p P(x). \tag{17}
\end{equation}

Note that Eq. (17) is unique up to an unimportant overall normalization which can be absorbed into the parameters of the effective Lagrangian. Through the ansatz (17) it is guaranteed that the fields $\varphi^a$ have the proper behavior under renormalization and $SU(3)_F$ transformations. Coupling the fields $\varphi^a$ to external octet and singlet axial-vector currents in an $SU(3)_F$-invariant way, one obtains the decay constants from the matrix elements in Eq. (14). This leads to several important features which shall be emphasized here:

- One can define the matrix product

\begin{equation}
(f^T f)^{ab} = \sum_{P=\eta,\eta'} f^a_P f^b_P. \tag{18}
\end{equation}

The 88- and 08-elements of this matrix are not effected by the parameters $\Lambda_i$ in the effective Lagrangian (13) and can be expressed in terms of the parameters $F$ and $L_5$ only, i.e. to this order they are just fixed in terms of $f_\pi$ and $f_K$.

This leads to the following relations among the decay constants and mixing parameters:

\begin{equation}
\sum_P f^8_P f^8_P = f^2_8 = \frac{4f^2_K - f^2_\pi}{3}, \tag{19}
\end{equation}

\begin{equation}
\sum_P f^8_P f^0_P = f_8 f_0 \sin(\theta_8 - \theta_0) = -\frac{2\sqrt{3}}{3} (f^2_K - f^2_\pi). \tag{20}
\end{equation}

Relation (19) follows from standard $\chi$PT for the members of the pseudoscalar octet alone and has been frequently used. The relation (20) stems from the inclusion of the $\eta'$ meson into the chiral Lagrangian. At first glance, it is surprising since it tells us that the matrix product of the decay constants in the octet-singlet basis is not diagonal, i.e. $\theta_8 \neq \theta_0$. The decay constants of the charged pions and kaons, appearing on the r.h.s. are well known from their leptonic decay ($f_\pi = 130.7$ MeV, $f_K = 1.22 f_\pi$). Thus the strength of the flavor symmetry breaking effects entering Eqs. (19) and (20) is expected to be of the order of 20%. On the other hand, the mixing angles $\theta_8$ and $\theta_0$ themselves are small quantities, too, as long as the size of the anomalous contribution to the singlet mass ($6\tau_0/F^2$) is large compared to the effect of $SU(3)_F$ breaking ($M^2_K - M^2_\pi$). One thus has

\begin{equation}
\left| \frac{\theta_8 - \theta_0}{\theta_8 + \theta_0} \right| \ll 1, \tag{21}
\end{equation}
and the difference between $\theta_8$ and $\theta_0$ should not be neglected.\footnote{It is amusing to note that already 25 years ago Langacker and Pagels have pointed out the inconsistencies resulting from using only one mixing angle in Eqs. (16,17). The strong interactions require a renormalization of the bare octet and singlet fields that is more complicated than a simple rotation.}

The singlet decay constants $f_0$ has an additional contribution from the OZI-rule violating term in Eq. (13) which is proportional to the parameter $\Lambda_1$

$$\sum_P f_P^0 f_P^0 = f_0^2 = \frac{2f_K^2 + f_\pi^2}{3} + f_\pi^2 \Lambda_1.$$  \hspace{1cm} (22)

The value of $\Lambda_1$ has to be determined from phenomenology. Furthermore, the singlet decay constants $f_P^0$ are renormalization-scale dependent\footnote{In the past, in many publications it has often been taken for granted that the two angles $\theta_8$ and $\theta_0$ can be taken as equal, and can be identified with a universal mixing angle $\theta_P$ of the $\eta-\eta'$ system. From the above considerations it should be clear that neither of these assumptions is justified as soon as one includes flavor symmetry breaking effects in a systematic way (i.e. by taking into account corrections to $L(0)$).}

$$\mu \frac{d f_P^0}{d \mu} = \gamma_A(\mu) f_P^0.$$ \hspace{1cm} (23)

The anomalous dimension $\gamma_A$ is of order $\alpha_s^2$

$$\gamma_A(\mu) = -N_F \left( \frac{\alpha_s}{\pi} \right)^2 + O(\alpha_s^3).$$ \hspace{1cm} (24)

A comparison with Eq. (22) reveals that the behavior of $f_0$ under renormalization should be attributed to the scale-dependence of the parameter $\Lambda_1 \rightarrow \Lambda_1(\mu)$. Numerically, the scaling of $f_P^0$, however, is only a sub-leading effect. Varying, for instance, the scale $\mu$ between $M_\eta$ and $M_\eta'$, the value of $f_P^0(\mu)$ changes by less than 10%. Note that the mixing angle $\theta_0$ is not scale-dependent since it is defined as the ratio of two singlet decay constants, see Eq. (15).

• One can construct another matrix product

$$\left( f f^T \right)_{P_1 P_2} = \sum_{a=8,0} f_{P_1}^a f_{P_2}^a = \begin{pmatrix} f_8^2 \cos^2 \theta_8 + f_0^2 \sin^2 \theta_0 & f_8^2 \cos \theta_8 \sin \theta_8 - f_0^2 \cos \theta_0 \sin \theta_0 \\ f_8^2 \cos \theta_8 \sin \theta_8 - f_0^2 \cos \theta_0 \sin \theta_0 & f_8^2 \sin^2 \theta_8 + f_0^2 \cos^2 \theta_0 \end{pmatrix} \neq \text{diag}[f_8^2, f_0^2]$$ \hspace{1cm} (25)

which is defined in the basis of the physical states $P_1, P_2 = \eta, \eta'$. Since it is non-diagonal, the $\eta-\eta'$ decay constants $f_{\eta}, f_{\eta'}$ if mixing is to be taken into account, contrary to what is occasionally claimed in the literature.
3.2. Decay constants in the quark-flavor basis and OZI-rule

The parametrization of the decay constants can (and actually does) look much simpler in another basis, which is frequently used, where the two independent axial-vector currents are taken as

\[
J_{\mu 5}^q = \frac{1}{3} J_{\mu 5}^8 + \frac{2}{3} J_{\mu 5}^0 = \frac{1}{\sqrt{2}} \left( \bar{u} \gamma_\mu \gamma_5 u + \bar{d} \gamma_\mu \gamma_5 d \right),
\]

\[
J_{\mu 5}^s = -\frac{2}{3} J_{\mu 5}^8 + \frac{1}{3} J_{\mu 5}^0 = \bar{s} \gamma_\mu \gamma_5 s.
\]

(26)

In an analogous way one defines new bare fields \( \varphi^q \) and \( \varphi^s \). For obvious reasons, I call this basis the quark-flavor basis for which I take indices \( i,j = q,s \) to distinguish it from the conventional octet-singlet basis with indices \( a,b = 8,0 \). In the quark-flavor basis the matrix \( \chi \), which induces the explicit flavor symmetry breaking in the effective Lagrangian (12) and (13), is diagonal. Without the \( U(1)_A \) anomaly the physical states would thus be close to the fields \( \varphi^q \) and \( \varphi^s \). Consider, for instance, the analogous case in the vector meson sector where the \( \omega \) and \( \phi \) meson are nearly pure \( q\bar{q} \) and \( s\bar{s} \) states, respectively. The smallness of the \( \phi-\omega \) mixing angle (about 3\(^\circ\)) is consistent with the OZI-rule, i.e. amplitudes that involve quark-antiquark annihilation into gluons are suppressed. The OZI-rule becomes rigorous in the formal limit \( N_C \to \infty \) and also for a vanishing strong coupling constant at asymptotically large energies. In the pseudoscalar sector, however, the \( U(1)_A \) anomaly induces a significant mixing between the fields \( \varphi^q \) and \( \varphi^s \). As I have discussed in Section 2.1, the non-trivial effect of the anomaly is connected with the topological properties of the QCD vacuum and is not due to quark-antiquark annihilation. We will see in the following how these observations can be used to obtain a powerful phenomenological scheme for the description of \( \eta-\eta' \) mixing.

For this purpose let me first introduce an analogous parametrization as in Eq. (16)

\[
\{ f_P^i \} = \begin{pmatrix} f_q^q & f_s^s \\ f_q^q & f_s^s \end{pmatrix} = \begin{pmatrix} f_q \cos \phi_q & -f_s \sin \phi_s \\ f_q \sin \phi_q & f_s \cos \phi_s \end{pmatrix}.
\]

(27)

From the effective Lagrangian (12) and (13), I obtain expressions for the basic parameters \( f_q, f_s \) and the difference between the two mixing angles \( \phi_q \) and \( \phi_s \),

\[
\sum_P f_P^q f_P^q = f_q^2 = f_\pi^2 + \frac{2}{3} f_\pi^2 \Lambda_1,
\]

(28)

\[
\sum_P f_P^q f_P^s = f_q f_s \sin(\phi_q - \phi_s) = \frac{\sqrt{2}}{3} f_\pi^2 \Lambda_1,
\]

(29)

\[
\sum_P f_P^s f_P^s = f_s^2 = 2 f_K^2 - f_\pi^2 + \frac{1}{3} f_\pi^2 \Lambda_1,
\]

(30)

which have to be compared with relations (19), (20) and (22). The situation in the quark-flavor basis is different from the one in the octet-singlet one: i) The difference
between the angles $\phi_q$ and $\phi_s$ is determined by an OZI-rule violating contribution ($\Lambda_1 \neq 0$) and not by $SU(3)_F$-breaking ($f_K \neq f_{\pi}$). As already mentioned, the parameter $\Lambda_1$ has to be estimated from phenomenology. Taking typical values for the mixing parameters from the literature (see Tables 1 and 2 below), one obtains the estimate $|\phi_q - \phi_s| < 5^\circ$ which translates into $|\Lambda_1| < 0.3$. ii) The values of $\phi_q$ and $\phi_s$ themselves are not small quantities. In the $SU(3)_F$ symmetry limit they take the ideal value, $\arctan \sqrt{2} \approx 54.7^\circ$. Phenomenological analyses give values around $40^\circ$ (see Table 2). We thus have

$$|\phi_q - \phi_s| \ll 1.$$ (31)

Eq. (31) can be taken as a justification to treat the difference between the parameters $\phi_q$ and $\phi_s$ as a sub-leading correction. The advantage of this procedure is obvious. One has to deal with only one mixing angle $\phi \simeq \phi_q \simeq \phi_s$, now defined in the quark-flavor basis. In this basis the matrix of decay constants is approximately diagonal, and we can write

$$\{f^i_P\} = U(\phi) \text{diag}[f_q, f_s] + O(\Lambda_1)$$ (32)

where I have defined the rotation matrix

$$U(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}.$$ (33)

I stress that assuming an equation like (32) in one basis necessarily leads to a more complicated parametrization like (16) in another.

In the quark-flavor basis the assumption of a common mixing angle is directly related to the OZI-rule. The OZI-rule in this context means that the $1/N_C$-suppressed parameters $\Lambda_i$ in the chiral effective Lagrangian are dropped while e.g. the topological susceptibility $\tau_0$ is kept. It is important to realize that the OZI-rule is a necessary ingredient for the determination of process-independent mixing parameters. Otherwise, we would encounter a problem, since we had to introduce an extra unknown OZI-rule violating parameter for each coupling or decay constant. Turning the argument around this means that process-independent mixing parameters can only be determined up to corrections of order $1/N_C$.

The consequent application of the OZI-rule leads to the scheme which has been advocated for in Ref.26,27. I will refer to it as the FKS scheme in the following. It is based on the following requirements

- All OZI-rule violating parameters $\Lambda_i$ of order $1/N_C$ in the chiral effective Lagrangian are neglected.

- The parameters in the singlet channel ($f_0$ etc.) are not scale-dependent in the FKS scheme, since the additional renormalization effects, which are usually absorbed into the parameters $\Lambda_i$, also violate the OZI-rule.
• All other amplitudes that involve quark-antiquark annihilation but are not
due to topological effects are neglected (e.g. $\phi - \omega$ mixing, glueball admixtures
in $\eta$ and $\eta'$, ...)

For the phenomenological analysis of mixing parameters the FKS scheme provides
the most useful concept. It can also be used as the starting point of a theoretical
analysis of $\eta$-$\eta'$ mixing on the basis of the anomaly equation (7). In an earlier work,
Diakonov/Eides considered the anomalous Ward identities by implicitly assuming
the OZI-rule to apply in the above sense which leads to completely analogous results
as in the FKS scheme.

Of course, if precise data for e.g. the decay constants $f_\eta$ or $f_{\eta'}$ become available,
one will be able to unambiguously determine $\phi_\eta$ and $\phi_{\eta'}$ (and therefore $\Lambda_1$) from that
one source. This in turn could be taken as input in order to determine the OZI-rule
violating parameters in other processes. In Ref. 18 for instance – the estimates
$\Lambda_1 - 2\Lambda_3 \approx 0.25$ and $\Lambda_2 - \Lambda_3 \approx 0.28$ at low renormalization scales have been
obtained from a phenomenological analysis. Varying $\Lambda_1$ in the interval ($-0.3, 0.3$)
this translates into $0 < \Lambda_2 < 0.3$ and $-0.28 < \Lambda_3 < 0.02$.

The connection between the mixing parameters in the FKS scheme and the
mixing angles in the octet-singlet basis reads: 26

$$\theta_8 = \phi - \arctan[\sqrt{2}f_\eta/f_q] + O(\Lambda_1),$$
$$\theta_0 = \phi - \arctan[\sqrt{2}f_{\eta'}/f_s] + O(\Lambda_1).$$  (34)

The deviation from the naive expectation $\theta_P = \phi + \theta_{id}$ where $\theta_{id} = -\arctan\sqrt{2} \approx -54.7^\circ$ is the ideal mixing angle, can be attributed to the deviation of the ratio
$f_\eta/f_q$ from unity which is induced by the parameter $L_5$ in the effective Lagrangian
(13). Eq. (34) is particularly useful since it allows to translate results in the literature
correctly from one basis into another. As we will see below the usage of the
correct relations between $\theta_8$, $\theta_0$ and $\phi$ already resolves a big part of the apparent
discrepancies between the results of different approaches mentioned in the intro-
duction. It also reveals that the notion of a single octet-singlet mixing angle $\theta_P$ is
more confusing than helpful in phenomenological analyses.

### 3.3. Comparison of mixing parameters in the literature

In Tables 1 and 2 I list in chronological order some results for the mixing parameters $f_\eta$ and $\theta_\eta$ in the octet-singlet basis, as well as for the parameters $f_{\eta'}$ and
$\phi_{\eta'}$ in the quark-flavor basis, obtained in various analyses. A key ingredient for the
comparison is the usage of the general parametrization of decay constants, Eqs. (16)
and (22). Note that some of the results of previous articles have not been recog-
nized, simply because one only has concentrated on finding a single mixing angle
$\theta_P$. The importance of determining independently the four decay constants in the
$\eta$-$\eta'$ system (i.e. a set of four mixing parameters) has often not been realized. Ta-
bles 1 and 2 reveal that in most of those cases that are not in conflict with the

\[\text{Of course, due to the long tradition of the subject, the list in Tables 1 and 2 is far from being}
\text{complete. The quoted references, therefore, should be regarded as representative examples.}\]
general parametrization \(\theta_{0}\), there is fair agreement among the results of different approaches. The largest variation is found for the parameter \(\theta_{0}\) which ranges from \(-9^\circ\) to \(0^\circ\).

Table 1. Some results for mixing parameters of the \(\eta-\eta'\) system in the octet-singlet basis. (The entries in parantheses \[\cdots\] have not been quoted in the original literature but have been calculated from information given therein, assuming, for simplicity, that all OZI-rule violating parameters except for \(\Lambda_{1}\) are zero.)

| source | \(f_{s}/f_{\pi}\) | \(f_{0}/f_{\pi}\) | \(\theta_{0}\) | \(\theta_{0}\) |
|--------|----------------|----------------|-------------|-------------|
| Mass matrix and radiative decay \[16\] | 1.3 | 1.2 | \([-20^\circ]\) | \([-1^\circ]\) |
| \(U(1)_{A}\) anomaly from meson masses \[12\] | 1.2 | 1.1 | \([-20^\circ]\) | \([-5^\circ]\) |
| Phenomenology \[26\] | 1.2 - 1.3 | 1.0 - 1.2 | \(-23^\circ - 17^\circ\) | \(-5.5^\circ\) |
| NJL quark model & phenom \[22\] | 1.24 | 1.21 | \([-19.5^\circ]\) | \([-30.7^\circ]\) |
| \(\chi\) mass formula \[40\] | 0.71 | 0.94 | \([-12.2^\circ]\) | \([-7.0^\circ]\) |
| Phenomenology \[26\] | 1.34 | 1.21 | \([-23.2^\circ]\) | \([-7.0^\circ]\) |
| GMO mass formula \[24\] | 1.19 | 1.10 | \(-21.4^\circ\) | \(-9.2^\circ\) |
| \(\chi\)PT & \(1/N_{C}\) expansion & phenom \[24\] | 1.28 | 1.25 | \(-20.5^\circ\) | \(-4^\circ\) |
| FKS scheme & theory \[24\] | 1.28 | 1.15 | \(-21.0^\circ\) | \(-2.7^\circ\) |
| FKS scheme & phenom \[24\] | 1.26 | 1.17 | \(-21.2^\circ\) | \(-9.2^\circ\) |
| Vector meson dominance & phenom \[24\] | 1.36 | 1.32 | \(-20.4^\circ\) | \(-0.1^\circ\) |
| Energy dependent scheme & phenom \[24\] | 1.37 | 1.21 | \(-21.4^\circ\) | \(-7.0^\circ\) |

Table 2. Same as Table 1 but in the quark-flavor basis.

| source | \(f_{q}/f_{\pi}\) | \(f_{s}/f_{\pi}\) | \(\phi_{q}\) | \(\phi_{s}\) |
|--------|----------------|----------------|-------------|-------------|
| Mass matrix and radiative decay \[16\] | 1.6 | 1.4 | 44\(^\circ\) | |
| \(U(1)_{A}\) anomaly from meson masses \[12\] | 1.0 | 1.4 | 42\(^\circ\) | |
| Phenomenology \[26\] | 1.1 - 1.2 | 1.1 - 1.3 | \[28^\circ - 34^\circ\] | \[35^\circ - 41^\circ\] |
| NJL quark model & phenom \[22\] | 1.07 | 1.36 | \[44.1^\circ\] | \[40.6^\circ\] |
| \(\chi\) mass formula \[40\] | 0.98 | 0.66 | \[35.9^\circ\] | \[20.2^\circ\] |
| Phenomenology \[26\] | 1.00 | 1.45 | 39.2\(^\circ\) | |
| GMO mass formula \[24\] | 1.13 | 1.16 | \[31.2^\circ\] | \[35.4^\circ\] |
| \(\chi\)PT & \(1/N_{C}\) expansion & phenom \[24\] | 1.08 | 1.43 | \[44.8^\circ\] | \[40.5^\circ\] |
| FKS scheme & theory \[24\] | 1.00 | 1.41 | 42.4\(^\circ\) | |
| FKS scheme & phenom \[24\] | 1.07 | 1.34 | 39.3\(^\circ\) | |
| Vector meson dominance & phenom \[24\] | 1.09 | 1.55 | \[47.5^\circ\] | \[42.1^\circ\] |
| Energy dependent scheme & phenom \[24\] | 1.10 | 1.46 | \[38.9^\circ\] | \[41.0^\circ\] |

For the following discussion of phenomenological observables I will often refer to the set of mixing parameters obtained in the phenomenogical analysis performed on the basis of the FKS scheme \[24\] 

\[
\begin{align*}
    f_{s} &= (1.26 \pm 0.04) f_{\pi} \quad , \quad \theta_{s} = -21.2^\circ \pm 1.6^\circ \quad , \\
    f_{0} &= (1.17 \pm 0.03) f_{\pi} \quad , \quad \theta_{0} = -9.2^\circ \pm 1.7^\circ \\
\quad \Leftrightarrow \quad f_{q} &= (1.07 \pm 0.02) f_{\pi} \quad , \quad \phi = 39.3^\circ \pm 1.0^\circ , \quad \Lambda_{1} \equiv 0 \quad . \quad (35)
\end{align*}
\]

Note that the quoted errors refer to the experimental uncertainty used in the determination of the mixing parameters, only. A systematical error, arising from
OZI-rule violations or higher orders in the effective Lagrangian, is not included. For given values, say, of the parameters \( f_8, f_0 \) and \( \theta_8 \) which are rather well known this error should be assigned to the angle \( \theta_0 \) which may explain the rather large variation of its value in different phenomenological approaches.

In some of the articles quoted in Tables 1 and 2 one can find different proposals for mixing parameters. Escribano and Frère\(^4\) have suggested an alternative parametrization

\[
\begin{pmatrix}
  f_8^\eta & 0 \\
  f_0^\eta & f_0^{\eta'}
\end{pmatrix}
= \begin{pmatrix}
  \hat{f}_8 \cos \theta_\eta & -\hat{f}_0 \sin \theta_\eta \\
  \hat{f}_8 \sin \theta_\eta' & \hat{f}_0 \cos \theta_\eta'
\end{pmatrix}
\]

(36)

where \( \theta_\eta \) and \( \theta_\eta' \) are interpreted as (energy-dependent) mixing angles of \( \eta \) and \( \eta' \), respectively. It can be mapped onto the one in Eq. (16) via

\[
\tan \theta_8 = \frac{\sin \theta_\eta}{\cos \theta_\eta}, \quad f_8 = \hat{f}_8 \sqrt{\cos^2 \theta_\eta + \sin^2 \theta_\eta'},
\]

\[
\tan \theta_0 = \frac{\sin \theta_\eta}{\cos \theta_\eta'}, \quad f_0 = \hat{f}_0 \sqrt{\sin^2 \theta_\eta + \cos^2 \theta_\eta'}.
\]

(37)

The authors claim that the phenomenological success of the ansatz Eq. (36) with substantially different values of \( \theta_\eta \) and \( \theta_\eta' \) gives evidence for the energy-dependence of the mixing angles. Of course, a similar concept could be introduced for the quark-flavor basis, defining in an analogous way energy-dependent mixing angles \( \phi_\eta \) and \( \phi_\eta' \). The actual values for \( \theta_\eta \) and \( \theta_\eta' \) found in the phenomenological analysis\(^4\) translate into almost equal values for \( \phi_\eta \) and \( \phi_\eta' \). In this basis the apparent energy-dependence cannot be observed. This is in line with the above results: The necessity to use two mixing angles is primordially a consequence of \( SU(3)_F \)-breaking, not of energy-dependence.

Kisselev and Petrov\(^4\) have introduced still another parametrization in terms of an averaged mixing angle \( \bar{\theta} \) and an explicit symmetry breaking parameter \( \varepsilon \). It can be related to the parametrization (14) by

\[
\varepsilon = \sqrt{\frac{\tan \theta_8}{\tan \theta_0}}, \quad \tan \bar{\theta} = -\sqrt{\tan \theta_8 \tan \theta_0}.
\]

However, the authors fix the value of \( \varepsilon \) by model assumptions to values smaller than one, while the analyses based on Eq. (20) find \( \varepsilon > 1 \).

Bramon et al.\(^2\) used constituent quark masses (\( \tilde{m} \)) instead of the decay constants for the parameterization of \( SU(3)_F \)-breaking. Since constituent quarks obey a Goldberger-Treiman relation for the effective quark-meson coupling constant \( g_i f_i = \tilde{m}_i \) with \( g_q \simeq g_s \), one has \( f_q/\tilde{m}_q \simeq f_s/\tilde{m}_s \) which has been used in Tables 1 and 2.

### 3.4. Two-photon decays and \( \mathcal{L}_{WZW} \) Lagrangian

One important source of information about the decay constants are the two-photon decays \( P \rightarrow \gamma \gamma \). They are driven by the chiral anomaly\(^1\) in QED and have
been used in most of the phenomenological analyses. In particular, for the determination of the parameter set \((\ref{16})\), the parameters \(f_q\) and \(f_s\) have been adjusted to the \(\eta(\eta') \to \gamma \gamma\) decay widths for a given value of \(\phi\) (see Table \(\ref{4}\) below). In the effective Lagrangian the chiral anomaly enters via the Wess-Zumino-Witten term in the following way

\[
L_{WZW} = -\frac{N_C}{4\pi} \alpha_{\text{em}} \frac{4}{\pi} F_{\mu\nu} \tilde{F}^{\mu\nu} \text{tr}[Q^2 \phi],
\]

where \(Q = \text{diag}[2/3, -1/3, -1/3]\) denotes the matrix of quark charges. As has been discussed in detail by Leutwyler and Kaiser,\(^24\) in the flavor singlet channel one has again to allow for an OZI-rule violating correction, which essentially corresponds to replacing \(f_0 \to f_0/(1 + \Lambda_3)\) after inserting Eq. \((\ref{17})\) into the WZW Lagrangian. (In the FKS scheme, \(\Lambda_3\) is assumed to be small and neglected.) Using the general parametrization of decay constants \((\ref{16})\) and the connection between basis and physical fields \((\ref{17})\), the chiral anomaly prediction reads

\[
\Gamma[\eta \to \gamma \gamma] = \frac{9\alpha_{\text{em}}^2}{16\pi^3} M_\eta^3 \left\{ \frac{C_8 \cos \theta_0}{f_8 \cos(\theta_8 - \theta_0)} - \frac{(1 + \Lambda_3) C_0 \sin \theta_8}{f_0 \cos(\theta_8 - \theta_0)} \right\}^2,
\]

\[
\Gamma[\eta' \to \gamma \gamma] = \frac{9\alpha_{\text{em}}^2}{16\pi^3} M_{\eta'}^3 \left\{ \frac{C_8 \sin \theta_0}{f_8 \cos(\theta_8 - \theta_0)} + \frac{(1 + \Lambda_3) C_0 \cos \theta_8}{f_0 \cos(\theta_8 - \theta_0)} \right\}^2.
\]

Here \(C_8 = (e_u^2 + e_d^2 - 2e_s^2)/\sqrt{6}\) and \(C_0 = (e_u^2 + e_d^2 + e_s^2)/\sqrt{3}\) are charge factors which are multiplied by the elements of the inverse matrix \(\{f^{-1}\}_P\), obtained from Eq. \((\ref{16})\). In leading order the scale-dependence of \(\Lambda_3\) cancels the one of \(f_0\).

There is an alternative approach to obtain a renormalization group invariant prediction for \(\eta(\eta') \to \gamma \gamma\). In the two-component scheme of Veneziano/Shore\(^46\) the singlet part of the decay amplitude is written as a sum of a contribution from the would-be Goldstone boson, \(g_{\eta\gamma\gamma}\), and a gluonic part which includes the Veneziano ghost, \(g_{\tilde{G}\gamma\gamma}\). The residual effect of the gluonic contribution, after the ghost field has been combined with \(\tilde{\eta}_0\) and \(\eta_8\) to yield the physical \(\eta\) and \(\eta'\) states should be related to the parameter \(\Lambda_3\) in \(\chiPT\).

It is to be stressed, that Eq. \((\ref{39})\) is only rigorously valid in the chiral limit \(m \to 0\). For the \(\pi^0 \to \gamma \gamma\) decay corrections to the chiral limit are indeed small \((m_\pi^2 \ll 1 \text{ GeV}^2)\), and the value of the pion decay constant obtained in this way is compatible with the one measured in \(\pi^\pm \to \mu^\pm \nu_\mu\). In the \(\eta\) and \(\eta'\) case, however, Eq. \((\ref{39})\) may receive additional corrections. The phenomenological success of Eq. \((\ref{33})\), on the other hand, indicates that these corrections cannot be too large.

### 3.5. Radiative transitions between light pseudoscalar and vector mesons

The transitions \(P \to V\gamma\) or \(V \to P\gamma\) with \(P = \eta, \eta', \ldots\) and \(V = \rho, \omega, \phi\ldots\) provide another possibility to investigate the mixing scenario in the pseudoscalar meson sector. A comparison of theory and experimental data may also yield interesting information about the properties of light vector mesons. The relevant
coupling constants are defined by matrix elements of the electromagnetic current

\[ \langle P(p_P)|j_{\mu}^{\text{em}}|V(p_V,\lambda)\rangle|_{q^2=0} = -g_{VP\gamma}\epsilon_{\mu\nu\rho\sigma}p_{P\nu}p_{P\rho}\epsilon^{\sigma}(\lambda) . \] (40)

The decay widths in terms of these coupling constants read

\[ \Gamma[P \rightarrow V\gamma] = \alpha_{\text{em}} g_{PV\gamma}^2 k_V^3 , \quad \Gamma[V \rightarrow P\gamma] = \frac{\alpha_{\text{em}}}{3} g_{PV\gamma}^2 k_V^3 . \] (41)

An early discussion of these reactions in connection with \( \eta-\eta' \) mixing can be found in Refs.\textsuperscript{14,15}. The \( g_{PV\gamma} \) coupling constants have also been used in the analysis of the \( \eta-\eta' \) mixing angle in Refs.\textsuperscript{3,19} The subject has been reconsidered by Ball/Frère/Tytgat\textsuperscript{20} who investigated the coupling constants \( g_{PV\gamma} \) as a function of the mixing angle in the naive octet-singlet scheme, and by Bramon et al.\textsuperscript{21} who used the experimental information on \( g_{PV\gamma} \) coupling constants for a fit of the mixing angle \( \phi \) in the quark-flavor basis. In both analyses a (small) \( \omega-\phi \) mixing angle has been taken into account, too.

The theoretical estimates for the \( g_{PV\gamma} \) coupling constants are obtained by combining the chiral anomaly prediction for the decays \( P \rightarrow \gamma\gamma \) (39) with vector meson dominance. The results look particularly simple in the FKS scheme. Since in many analysis also OZI-rule violating contributions have been (partly) taken into account, I quote here the expressions that include both, the OZI-rule violating contribution to the Wess-Zumino-Witten Lagrangian (\( \Lambda_3 \)) and the \( \omega-\phi \) mixing angle (\( \phi' \)). The following expressions represent a generalization of the formulas quoted, for instance, in Refs.\textsuperscript{3,17}

\[
\begin{align*}
g_{\eta\eta'} &= \frac{3m_{\rho}}{2\pi^2 f_{\rho}} \left( \frac{1}{2} \cos \phi_s - \frac{\Lambda_3}{\sqrt{6}} \sin \theta_s \right), \\
g_{\eta'\eta'} &= \frac{3m_{\rho}}{2\pi^2 f_{\rho}} \left( \frac{1}{2} \sin \phi_s + \frac{\Lambda_3}{\sqrt{6}} \cos \theta_s \right), \\
g_{\eta\eta'} &= \frac{3m_{\omega}}{2\pi^2 f_{\omega}} \left( \frac{1}{2} \cos \phi_s - \frac{\Lambda_3}{\sqrt{6}} \sin \theta_s \right), \\
g_{\eta'\eta'} &= \frac{3m_{\omega}}{2\pi^2 f_{\omega}} \left( \frac{1}{2} \sin \phi_s + \frac{\Lambda_3}{\sqrt{6}} \cos \theta_s \right), \\
g_{\eta\phi} &= \frac{3m_{\phi}}{2\pi^2 f_{\phi}} \left( \frac{1}{2} \cos \phi_s \sin \phi_V + \frac{\Lambda_3}{\sqrt{3}} \sin \theta_s \right), \\
g_{\eta'\phi} &= \frac{3m_{\phi}}{2\pi^2 f_{\phi}} \left( \frac{1}{2} \sin \phi_s \sin \phi_V - \frac{\Lambda_3}{\sqrt{3}} \cos \theta_s \right). \end{align*} \] (42)

Here \( \phi_V \) is defined in the same manner as \( \phi \), see Eq. (55). For simplicity, I have neglected terms of the order \( \Lambda_3 \sin \phi_V \) and have set \( \cos(\phi_q-\phi_s) \), \( \cos(\theta_s-\theta_0) \) and \( \cos \phi_V \) to unity in Eq. (12).

Three recent numerical analyses are summarized in Table 3, referring to a determination in the FKS scheme\textsuperscript{22} and two investigations\textsuperscript{23,24} where different \( \eta-\eta' \) mixing schemes have been used and a non-vanishing \( \phi-\omega \) mixing angle have been
taken into account. The numerical values of the mixing parameters in the three analyses are, however, not too different from each other, see Table 1, and the obtained estimates for \( g_{PV\gamma} \) are very similar and turn out to be in fair agreement with the experimental findings.

### Table 3. Some recent estimates of the coupling constants \( |g_{PV\gamma}| \) compared to experiment.

| PV   | FKS | Escribano/Frère | Benayoun et al. | Experiment |
|------|-----|-----------------|-----------------|------------|
| \( \eta \rho \) | 1.52 | 1.43            | 1.69            | 1.47 ± 0.25 |
| \( \eta'\rho \) | 1.24 | 1.23            | 1.38            | 1.31 ± 0.06 |
| \( \eta\omega \) | 0.56 | 0.54            | 0.58            | 0.53 ± 0.04 |
| \( \eta'\omega \) | 0.46 | 0.55            | 0.44            | 0.45 ± 0.03 |
| \( \eta\phi \) | 0.78 | 0.73            | 0.70            | 0.69 ± 0.02 |
| \( \eta'\phi \) | 0.95 | 0.83            | 0.70            | 1.00 ± 0.29 |

### 3.6. Light-Cone Wave Functions

The decay constants defined in Eq. (14) play an important role in exclusive reactions at large momentum transfer. In this case it is useful to consider an expansion of the physical meson states in terms of Fock states with increasing number of partons. Schematically, for the \( \eta \) and \( \eta' \) mesons, one may write

\[
| \eta \rangle = \sum_{a=8,0} \Psi^a_\eta(x, k_\perp) | a \rangle + \ldots
\]

\[
| \eta' \rangle = \sum_{a=8,0} \Psi^a_{\eta'}(x, k_\perp) | a \rangle + \ldots
\]  

where \(| a \rangle = | \bar{q} \lambda^a q \rangle \) is a partonic quark-antiquark Fock state. Each Fock state has an individual light-cone wave functions \( \Psi^a_\eta \). Here \( x \) denotes the ratio of the quark and meson momenta in the light-cone plus-direction, and \( k_\perp \) is the quark momentum transverse to the meson one. The dots in Eq. (43) stand for the higher Fock states which may include additional gluons or quark-antiquark pairs and, in principle, also a two-gluon component \( |gg\rangle \). Since the matrix elements in Eq. (14) correspond to the annihilation of two quarks at one space-time point, the decay constants \( f^a_\eta \) are related\(^d\) to the values of the light-cone wave functions \( \Psi^a_\eta \) ‘at the origin’

\[
f^a_\eta = 2\sqrt{6} \int dx \, d^2k_\perp 16\pi^3 \Psi^a_\eta .
\]  

An analogous relation is valid for the decay constants \( f^a_\eta \) and light-cone wave functions \( \Psi^a_\eta \) in the quark-flavor basis. The relation (14) underlines the importance of the decay constants for the description of the \( \eta-\eta' \) system: They enter both, the effective Lagrangian relevant for low-energy physics and the light-cone wave functions utilized in high-energy reactions.

\(^d\)I follow the normalization convention of Ref. 3
I remark at this point, that in the flavor singlet channel, quark-antiquark and two-gluon parton states can mix perturbatively. The evolution equations to first order in $\alpha_s$ have been derived by Baier and Grozin. This mixing is a true OZI-rule violating process and should be neglected in the FKS scheme.

At large energies one often integrates out the intrinsic transverse momenta to obtain the (scale-dependent) distribution amplitudes $\Phi^a_P(x; \mu)$ which are defined by non-local matrix elements

$$i f^a_P \Phi^a_P(x; \mu) = \int \frac{dz^-}{2\pi} e^{i x p^+ z^-} (0|\bar{q}(0) \gamma^+ \gamma_5 \frac{\lambda^a}{\sqrt{2}} q(z^-)|P(p)| \mu) .$$  \hspace{1cm} (45)

They can be expanded about Gegenbauer polynomials, which are the eigenfunctions of the QCD evolution equation for mesons

$$\Phi(x; \mu) = 6x(1-x) \left(1 + \sum_{n=2,4,...} B_n(\mu) C_n^{(3/2)}(2x-1) \right) .$$  \hspace{1cm} (46)

In the limit $\mu \to \infty$, the coefficients $B_n$ evolve to zero with anomalous dimensions increasing with $n$, and one is left with the asymptotic distribution amplitude $\phi_{AS}(x) = 6x(1-x)$. Usually one keeps only a finite number of non-zero Gegenbauer coefficients in Eq. (46) which are then determined from phenomenology, QCD sum rules, low-energy models etc. Typical QCD sum rule estimates lead to distribution amplitudes for the pion which are somewhat broader than the asymptotic one ($B_2 \simeq 0.44$ and $B_4 \simeq 0.25$ at $\mu = 1$ GeV). Ball also considered the distribution amplitudes for the $\eta$ meson and finds a smaller value of the first Gegenbauer coefficient, $B_2 \simeq 0.2$, which follows the general trend that heavier mesons have narrower distribution amplitudes. The pion distribution amplitude has also been calculated in the instanton model and values $B_2 \simeq 0.06$ and $B_4 \simeq 0.01$ have been found.

The Fock state expansion is also related to the parton distributions which are extracted from the structure functions measured in deep inelastic scattering. Formally, the parton distributions arise from an infinite sum over all Fock state wave functions squared and integrated over transverse momenta and all but the momentum fraction $x_j$ of the struck quark with a certain flavor, e.g.

$$f_{u/\pi}(x) = \sum_{\beta N} \int [d^2 k]_N [dx]_N |\Psi_{\beta N}|^2 \delta(x - x_j) .$$  \hspace{1cm} (47)

The sum runs over all Fock states with parton number $N$ being in a color/spin/flavor combination labeled by $\beta$. A detailed analysis in the nucleon case has revealed that Fock states higher than the leading $q\bar{q}$ one lead to higher powers of $(1-x)$ in Eq. (47), under the reasonable assumption that the distribution amplitudes of higher Fock states can be described by their asymptotic form multiplied by polynomials in the light-cone momentum fractions $x_i$. Restricting oneself to a few Fock states is thus sufficient to predict the parton distributions at large $x$. 

Quark Structure of Pseudoscalar Mesons
Only for the pion experimental data are available but suffers from rather large errors. The recent analysis by Glück et al.\textsuperscript{6} leads to the following (LO) parametrization of the valence quark distribution inside the pion at 1 GeV\textsuperscript{2}

\[ x f_{u/\pi}(x) = 0.745 (1 - x)^{0.727} (1 - 0.356 \sqrt{x} + 0.379 x) x^{0.506} . \]  

(48)

Despite of the uncertainties, it can be used as a cross-check of the \( x \) distribution in the \( q\bar{q} \) Fock state.\textsuperscript{11,52} In Fig. 1 I plotted the asymptotic distribution amplitude and slight deviations from it (using \( B_2 = \pm 0.1 \)) as a function of \( x \). In the same figure I have shown the contribution of the \( q\bar{q} \) Fock state to the valence quark distribution function of the pion, confronted with the phenomenological parametrization by Glück et al. As one observes, a distribution amplitude \( \Phi_{\pi}(x) \) which is close to the asymptotic form already at low renormalization scales, \( \mu \simeq 1 \text{ GeV} \), is preferred.

Once, the pion distribution amplitude has been determined, it can be applied to other hard exclusive reactions (for instance, soft and hard contributions to the pion electromagnetic form factor\textsuperscript{54,55,56}, charmonium decays into light pseudoscalars\textsuperscript{57}, \( B \) meson decays into light pseudoscalar mesons\textsuperscript{58,59,60}). The most important experimental information on the distribution amplitude of the pion comes from the \( \pi\gamma \) transition form factor at large momentum transfer. In the following paragraph I will compare the \( \pi\gamma \) form factor with the \( \eta\gamma \) and \( \eta'/\gamma \) form factors.

### 3.7. \( \gamma\gamma \) transition form factors

For neutral pseudoscalar mesons the process \( \gamma\gamma^* \to P \) with (at least) one highly

\textsuperscript{5}For the transverse part of the wave function I assumed a Gaussian. Its width is fixed by the \( \pi^0 \to \gamma\gamma \) decay\textsuperscript{53}.\textsuperscript{5}
virtual photon offers a possibility to test the wave functions $\Psi_P^a$ and the decay constants $f_P^a$. The meson-photon transition form factor measured in this process can be expressed as a convolution of the light-cone wave functions with a perturbatively calculable hard-scattering amplitude.

For asymptotically large momentum transfer the form factor is solely determined by the decay constants

$$Q^2 F_{P\gamma}(Q^2) = 6 \sum_a C_a f_P^a \quad (Q^2 \to \infty) \quad (49)$$

where $C_{8,0}$ are defined after Eq. (39) and $C_3 = (e_u^2 - e_d^2)/\sqrt{2}$. If we had experimental data in the asymptotic region of momentum transfer Eq. (49) could be taken as a rigorous way to determine two of four mixing parameters in Eq. (16). However, as the example of the pion (where the decay constant $f_\pi$ is known) reveals, the recent experimental data from CLEO [25] at $Q^2$ values of a few GeV$^2$ are still 15-20% below the asymptotic value. One therefore has to take into account corrections to Eq. (49). A calculation of the $\eta(\eta')\gamma$ transition form factor within the modified hard-scattering approach (which takes into account transverse momenta and Sudakov suppressions) has been performed in Ref. [25]. The usage of the two-angle parametrization (16) turns out to be crucial to obtain a simultaneous description of both, the two-photon decays (depending on the inverse of the decay constant matrix) and the transition form factor at large momentum transfer (depending linearly on the $f_P^a$). A very good description of the CLEO data is obtained if the values of mixing parameters in Eq. (35) and the asymptotic distribution amplitudes are used. The result of that analysis is plotted in Fig. 2a) where I have divided the transition form factors for $\pi^0, \eta, \eta'$ by their asymptotic behavior (49), using the mixing parameters in Eq. (35). As one observes, within the errors the results for the three different mesons nearly fall on top of each other. This indicates, that the octet and singlet pseudoscalar mesons behave very similarly at large energies, or, in other words, the $x$ distributions in the light-cone wave functions of the $q\bar{q}$ Fock state for $\pi, \eta$ and $\eta'$ mesons are not very different from each other. This is to be confronted with the $\eta_c\gamma$ transition form factor which behaves differently [67], see Fig. 2b), due to the suppression with the heavy quark mass at intermediate energies and a different distribution amplitude which can be approximated by a Gaussian around $x_0 = 1/2$,

$$\Phi_{\eta_c}(x) = N x (1-x) \exp \left[ -a_c^2 M_{\eta_c}^2 (x - x_0)^2 \right]. \quad (50)$$

For comparison I have also plotted in Fig. 2a) the result of the standard hard-scattering approach (sHSA), following the work of Brodsky [68] and using the asymptotic distribution amplitude

$$Q^2 F_{\pi\gamma}(Q^2) = \sqrt{2} f_\pi \left( 1 - \frac{5}{3} \frac{\alpha_V(e^{-3/2}Q)}{\pi} \right) \quad (51)$$

Here the deviation from the asymptotic limit (49) is due to the first order QCD correction to the hard-scattering amplitude. Note that the argument of $\alpha_V$ in
Eq. (51) reflects a rather low renormalization scale. One is thus sensitive to the infrared behavior of the strong coupling constant. Brodsky uses a particular choice $\alpha_V(\mu)$ that freezes for $\mu \to 0$. In this special form the sHSA also yields a good description of the data above, say, $3 \, \text{GeV}^2$.

The photon-transition form factors with light pseudoscalar mesons have also been investigated by Anisovich et al. Also in that analysis the universality of the $q\bar{q}$ wave functions of $\pi^0$, $\eta$, and $\eta'$ with a distribution amplitude close to the asymptotic form has been confirmed. The $\eta$-$\eta'$ mixing has been treated in the FKS scheme with a mixing angle $\phi = 37.5^\circ$ which is not too different from the value quoted in Eq. (35). In addition to the HSA analyses a soft hadronic part of the photon has been modeled, which leads to a prescription of the data at low $Q^2$ with a behavior close to vector meson dominance (VDM). The result of that analysis is plotted in Fig. 3.

The $\eta(\eta')\gamma$ transition form factors have also been treated in the conventional mixing scheme, using one common mixing angle in the octet-singlet basis. In this case a decent description of the data can only be achieved by choosing different parameter values for the decay constants and wave functions than the ones favored by other processes and $\chi$PT.

It has often been tried to infer information on the decay constants from fitting a pole formula for the $P\gamma$ transition form factor to experimental data. In the case of the pion this is motivated by an interpolation formula which has been proposed by Brodsky/Lepage:

$$F_{\pi\gamma}^{BL}(Q^2) = \frac{6 C_3 f_\pi}{Q^2 + 4\pi^2 f_\pi^2}. \quad (52)$$
Fig. 3. Results for the photon-transition form factors of light pseudoscalar mesons \( \eta, \eta' \) (Figures taken from Ref. 69). The theoretical result is based on a wave function model taken from Anisovich et al. 69 (Different curves correspond to different parameter sets of the model.) Experimental data are taken from CLEO 70, CELLO 71 and TPC/2 \( \gamma \) 72.

Obviously, it has the correct asymptotic limit (49). Furthermore for \( Q^2 \to 0 \) it coincides with the prediction from the chiral anomaly. It happens to have a similar form as the vector dominance model (VDM) if one identifies \( M_V = 2\pi f_\pi \). Astonishingly, one has the approximate equality \( M_{\rho, \omega} \simeq 2\pi f_\pi \), but there is no theoretical justification to assume that this relation has to be exact. Nevertheless, most of the experiments quote pole mass values extracted from a fit to the data. For the \( \pi \gamma \) form factor this mass comes out to be not too different from \( M_\rho \), but of course it can not be used as a measurement of \( f_\pi \) by requiring Eq. (52) to hold exactly. For the \( \eta \gamma \) and \( \eta' \gamma \) transition form factors the situation is even more complicated due to mixing, and even on the approximate level one does not find a simple relation of the experimental pole-mass fits with the \( \eta \)-\( \eta' \) mixing parameters. Consequently, these fits should be viewed rather as an effective parametrization of experimental data (which even depends on the measured range of momentum transfer) than a determination of process-independent quantities, see the second paper of Ref. 25 and references therein.

There are some additional processes which are similar to the \( P\gamma \) transition form factor and allow for an independent determination of the mixing angles, in particular they may be helpful to fix the value of \( \theta_0 \). One may, for instance, think of central \( \eta \) or \( \eta' \) production in \( pp \) collisions, where the transition form factors for \( g^*g^* \to \eta(\eta') \) are assumed to be relevant. The ratio of these form factors at large momentum transfer exactly gives \( 2 - \tan \theta_0 \). A complementary decay mechanism is provided by \( \gamma\)-Odderon-\( \eta(\eta') \) processes in diffractive \( ep \) scattering as discussed by Kilian/Nachtman. The ratio of form factors in this case turns out to be given by \( 2 \cot \theta_0 \). The experimental determination of the form factors from such processes may, however, be very difficult.
A very academic process is the decay $Z \to \eta(\eta')\gamma$ which is similar to the $P\gamma$ transition form factors. Taking into account the electroweak charges of the involved quarks one obtains\cite{28} \[ \frac{\Gamma[Z \to \eta\gamma]}{\Gamma[Z \to \eta'\gamma]} \simeq \tan^2 \theta_0. \] Due to the smallness of the individual branching ratios experimental data should not be expected in the near future. The same vertex is involved in the decays $\eta(\eta') \to \gamma \mu^+\mu^-$, but at small momentum transfers where the ratio gives a measure for the angle $\theta_8$. As discussed by Bernabeu et al.\cite{78} the $P\gamma Z$ vertex can be measured by extracting the $\gamma Z$ interference term from suitably chosen asymmetries.

### 3.8. Matrix elements with pseudoscalar quark currents

Let me proceed with considering the matrix elements of the pseudoscalar currents, entering the anomaly equation (7). They determine the quark mass contribution to the meson masses. It is natural to take the matrix elements in the quark-flavor basis where the quark mass matrix is diagonal. Let me define the four parameters $h_i^P$ for $i = q, s$ and $P = \eta, \eta'$ as

\[ 2 m_i \langle 0 | j_i^5(0) | P \rangle = h_i^P. \]  

Here the pseudoscalar currents in the quark-flavor basis are given as $j_q^5 = (\bar{u}i\gamma_5 u + \bar{d}i\gamma_5 d)/\sqrt{2}$ and $j_s^5 = \bar{s}i\gamma_5 s$, respectively. Following the chiral effective Lagrangian (12) and (13), we can express the matrix elements in Eq. (53) in terms of the decay constants and the parameters $B, L_8$ and $\Lambda_2$. The values of $B$ and $L_8$ are fixed in terms of the pion and kaon masses and decay constants. The parameter $\Lambda_2$ is needed to cancel the scale-dependence of $\Lambda_1$ (the singlet pseudoscalar current is not renormalized.) In the FKS scheme both, $\Lambda_1$ and $\Lambda_2$, are neglected. In this case one obtains the simple representation

\[ \{ h_i^P \} = \begin{pmatrix} h_q^\eta & h_s^\eta \\ h_q^\eta & h_s^\eta' \end{pmatrix} = U(\phi) \begin{pmatrix} f_q M_\pi^2 & f_s (2M_K^2 - M_\pi^2) \end{pmatrix}. \]  

The validity of this simplification has again to be tested against phenomenology. With the same assumptions that lead to the FKS scheme it makes sense to introduce basis states$^7$ $|\eta_q\rangle$ and $|\eta_s\rangle$. These are connected to the physical states by the same mixing angle $\phi$ as the decay constants (13) and the quark mass contributions (54)

\[ \begin{pmatrix} |\eta_q\rangle \\ |\eta_s\rangle \end{pmatrix} = U(\phi) \begin{pmatrix} |\eta_q\rangle \\ |\eta_s\rangle \end{pmatrix}. \]  

This connection has to be confronted with Eq. (17). The basis states in Eq. (55) have a definite decomposition in terms of quark-antiquark Fock states

\[ |\eta_q\rangle = \Psi_q(x, k_{\perp}) |u\bar{u} + \bar{d}d\rangle/\sqrt{2} + \ldots \\
|\eta_s\rangle = \Psi_s(x, k_{\perp}) |s\bar{s}\rangle + \ldots \]  

$^7$One sometimes finds the notation $|ns\rangle$ and $|s\rangle$ for non-strange and strange $q\bar{q}$ combinations, respectively.
This is to be compared with Eq. (55). In this form (55) the FKS scheme has been utilized in a number of analyses. The separation into strange and non-strange quarks is natural in those reactions where either the $s\bar{s}$ or $u\bar{u} + d\bar{d}$ component is probed, e.g. by light vector mesons, which show nearly ideal mixing, or by Cabbibo-favored weak transitions, $c \to s$. Considering appropriate ratios of observables with $\eta$ and $\eta'$ in such decay modes, one obtains an almost model-independent determination of the mixing angle $\phi$. Such an investigation on the basis of the FKS scheme has been performed in Ref. 24 and led to the result quoted in Table 4 (for the decays $J/\psi \to \eta(\eta')\gamma$ included in that table see Eqs. (42) and (43) below). Braun et al. 26 analyzed even more decay modes, like those of tensor mesons or higher spin-states into pairs of pseudoscalar and the whole class of radiative transitions between vector and pseudoscalar mesons (see Section 3.5). This requires at some stage some additional (but plausible) model-assumptions about $SU(3)_F$–breaking and mixing angles of vector and tensor mesons. Nevertheless, almost the same value, $\phi = 39.2^\circ \pm 1.3^\circ$, as in the FKS analysis has been found.

Table 4. Determination of the mixing angle $\phi$ from different decay channels, according to Ref. 26 and references therein. The quoted error refers to the experimental uncertainties, only.

| Decay Channel | Mixing Angle $\phi$ ± Error |
|---------------|-----------------------------|
| $J/\psi \to \eta(\eta')\rho$ | $39.9^\circ \pm 2.9^\circ$ |
| $D_s \to \eta(\eta')f_0$ | $41.3^\circ \pm 5.3^\circ$ |
| $\eta'\to \rho\gamma$, $\rho \to \eta\gamma$ | $35.3^\circ \pm 5.5^\circ$ |
| $a_2 \to \eta(\eta')\pi$ | $43.1^\circ \pm 3.9^\circ$ |
| $\pi^\pm p \to \eta(\eta')n$ | $36.5^\circ \pm 1.4^\circ$ |
| $p\bar{d} \to \eta(\eta')\pi(\eta, \omega)$ | $37.4^\circ \pm 1.8^\circ$ |
| $J/\psi \to \eta(\eta')\gamma$ | $39.0^\circ \pm 1.6^\circ$ |
| Average | $39.3^\circ \pm 1.0^\circ$ |

Values for the parameters $h_P^q$ can be estimated using the pion and kaon masses and the phenomenological results for the mixing angle $\phi$ and the decay parameters $f_q$ and $f_s$, see Eq. (55)

$$
\begin{pmatrix}
h_\eta^q & h_\eta^s \\
h_\eta'^q & h_\eta'^s
\end{pmatrix}
= 
\begin{pmatrix}
0.0020 \text{ GeV}^3 & -0.053 \text{ GeV}^3 \\
0.0016 \text{ GeV}^3 & 0.065 \text{ GeV}^3
\end{pmatrix}.
$$

(57)

The matrix elements $h_P^q$ are, for example, an important ingredient in the calculation of $B$ meson decays into light mesons in the factorization approach, see e.g. Refs. 24, 26. Occassionally, it has been popular in this context to use a simplified treatment of $\eta$-$\eta'$ mixing. For instance, in the analysis of $B$ decays into $\eta$ or $\eta'$ in Ref. 26 the $\eta$ meson is approximated as $\sim |u\bar{u} + d\bar{d} - s\bar{s}|/\sqrt{3}$ and the $\eta'$ meson as $\sim |u\bar{u} + d\bar{d} + 2s\bar{s}|/\sqrt{6}$. This would correspond to taking a mixing angle $\phi = \arctan \sqrt{2}/2 \approx 35.2^\circ$ and ignoring $SU(3)_F$–breaking effects ($f_s = f_q = f_\pi$) which would lead to significantly different values than in Eq. (55).
3.9. Matrix Elements with the topological charge density

Besides the decay constants, the matrix elements of the topological charge density $\omega$ in Eq. (58)

$$A_P \equiv \langle 0 | 2 \omega | P \rangle \quad (58)$$

play an important role in the understanding of $\eta$-$\eta'$ mixing. They can be used to define yet another mixing angle

$$\frac{A_\eta}{A'_{\eta'}} = -\tan \theta_y \quad (59)$$

Through the anomaly equation (53) the quantities $A_\eta$ and $A'_{\eta'}$ are directly related to the decay constants and the mass parameters $h^i_P$. If one sets the up- and down-quark masses in Eqs. (53) and (58) to zero which is equivalent to neglecting $M^2_\pi$ compared to $M^2_K$, one obtains

$$A_P \simeq \frac{1}{\sqrt{2}} \langle 0 | \partial^\mu J^q_{\mu 5} | P \rangle = M^2_P \frac{f^q_P}{\sqrt{2}} = M^2_P \frac{f^0_P + \sqrt{2} f^q_P}{\sqrt{6}} \quad (60)$$

The remaining pair of equations,

$$\langle 0 | 2 \omega | P \rangle = M^2_P f^q_P - h^q_P \quad (61)$$

when supplied with the ansätze (62) and (64) for the decay constants and the parameters $h^q_P$ in the FKS scheme, can be transformed into an additional relation between the angles $\theta_y$, $\theta_8$ and $\phi$, namely

$$\tan \theta_8 = \tan \theta_y = -\frac{M^2_\eta}{M^2_{\eta'}} \cot \phi \quad (62)$$

Here I have indicated the noteworthy fact that Eq. (62) connects the three different angles $\theta_8$, $\theta_y$ and $\phi$, and thus three different aspects of $\eta$-$\eta'$ mixing. A prominent example where the angle $\theta_y$ enters is the radiative $J/\psi$ decay into $\eta$ or $\eta'$. Novikov et al. have argued that the annihilation of the $J/\psi$ into light quarks is dominated by the anomaly, i.e. the matrix elements in Eq. (60). One then obtains for the ratio of decay widths

$$R(J/\psi) = \frac{\Gamma[J/\psi \to \eta' \gamma]}{\Gamma[J/\psi \to \eta \gamma]} = \frac{\langle 0 | \omega | \eta' \rangle^2}{\langle 0 | \omega | \eta \rangle} \left( \frac{k_{\eta'}}{k_{\eta}} \right)^3 = \cot^2 \theta_y \left( \frac{k_{\eta'}}{k_{\eta}} \right)^3 \quad (63)$$

where $k_P$ is the three-momentum of the final state meson in the rest-frame of the $J/\psi$. The radiative $J/\psi$ decays together with Eq. (62) provide an essential cross-check of the self-consistency of the whole mixing approach. From the experimental measurement of the ratio (63), $5.0 \pm 0.6$, one actually finds the following values of the mixing parameters, $\theta_y = \theta_8 = -22.0^\circ \pm 1.2^\circ$ and $\phi = 39.0^\circ \pm 1.6^\circ$ where
the result for the angle $\phi$ has already been used in Table 4 and turns out to be consistent with the values obtained from other processes. This is to be confronted with the naive but incorrect expectation $\theta_8 = \theta_0 = \theta_P = \phi + \theta_{id}$. This formula is one of the sources of the discrepancies between different determinations of mixing angles which have been mentioned in the introduction.

Taking values for the decay constants $f_0^p$ from Eq. (35) and using Eq. (60), one can give absolute numbers for the matrix elements of the topological charge density

$$A_\eta = \langle 0 | 2 \omega | \eta \rangle = 0.023 \text{ GeV}^3,$$

$$A_{\eta'} = \langle 0 | 2 \omega | \eta' \rangle = 0.058 \text{ GeV}^3.$$ (64)

The ratio $A_{\eta'}/A_\eta$ can also be used to determine the ratios $R(\psi') = 5.8$, analogously to Eq. (63). A recent measurement of the BES collaboration yields $R(\psi') = 2.9^{+5.4}_{-1.8}$.

3.10. Mass formulas

The anomaly equation (8) connects the masses and decay constants of pseudoscalar mesons. Supplied with the ansätze for the decay constants (32) and the quark mass contributions (54) one can obtain several relations that connect the masses of the physical states $\eta$ and $\eta'$ with the parameters in a given basis.

3.10.1. $U(1)_A$ mass shift

First, it is convenient to consider the trace of the physical meson mass matrix

$$M_\eta^2 + M_{\eta'}^2 = 2M_K^2 + \sqrt{3} \frac{\cos \theta_8 A_{\eta'} - \sin \theta_8 A_\eta}{f_0 \cos[\theta_8 - \theta_0]} \equiv 2M_K^2 + M_{U(1)_A}^2.$$ (65)

Here I have defined the mass shift $M_{U(1)_A}$, which parametrizes the deviation from the $U(1)_A$ symmetric world. Note the similarity with the formula for the two-photon decay widths (39): The matrix elements $A_P$ are weighted with the elements of the inverse matrix $(f^{-1})^0_P$. Using Eq. (65) and physical kaon and $\eta, \eta'$ masses or using the phenomenological values for decay constants (35) and gluonic matrix elements (64), respectively, one obtains the value 850 MeV for $M_{U(1)_A}$.

The $U(1)_A$ mass shift has also been calculated on the lattice, using quenched QCD with Wilson fermions for a single massive flavor. The values for $M_{U(1)_A}$ extracted from the $\eta'$ propagator on the lattice show a good approximation a linear rise with decreasing quark mass $m$. Qualitatively, this behavior is expected from the structure of the effective chiral Lagrangian (12) and (13), namely $M_{U(1)_A}^2 \propto F_0^2/F_0^2$ with $F_0^2 = F_0^2 (1 + 2(F_0^2 / F_0^2 - 1) m/m_s)$, where I have expressed the parameter $L_5$ in terms of the kaon and pion decay constants and set $m_s = m_d = 0$. Extrapolating the lattice data to the case of three massless flavors (using $M_{U(1)_A}^2 \propto N_F$) a value of $M_{U(1)_A} = 751 \pm 39$ MeV is found which is somewhat smaller than the phenomenological value. The dependence of the singlet meson mass on the quark masses is stronger than expected from $\chi$PT. At an effective quark mass of
m_s/3, which should be taken for comparison with the real world, the lattice simulation yields only \( M_{U(1)_A} \approx 650 \text{ MeV} \). Calculating on the other hand the mass shift from the topological susceptibility, see Eq. (71) below, a rather large value \( M_{U(1)_A} = 1146 \pm 67 \text{ MeV} \) is found on the lattice. Another lattice calculation has used quenched and unquenched staggered fermions. In that analysis it has been shown that the \( \eta' \) mass is particularly sensitive to fermionic zero-modes, indicating the strong connection with the topological properties of the theory. The result for \( M_{U(1)_A} \) in the limit of massless quarks is found to be \( 876 \pm 16 \text{ MeV} \).

3.10.2. Dashen’s theorem and GMO formula

One is often interested in features of the meson mass matrix in the octet-singlet basis. However, the connection between the physical fields \( P = \eta, \eta' \) and the bare fields \( \varphi^8 \) and \( \varphi^0 \) in the effective chiral Lagrangian is not simple. One has to use Eq. (17) which involves both mixing angles, \( \theta_8 \) and \( \theta_0 \). Thus, instead of considering an octet-singlet mass matrix one is led to considering the following product of decay constants and masses

\[
(f^T M^2 f)^{ab} = \sum_P f^a_P M^2_P f^b_P .
\]  

This result is equivalent to the ideas proposed by several other groups, on the basis of a current algebra theorem proposed by Dashen. With the correct treatment of the decay constants, which is essential to obtaining an object with well-defined octet-singlet quantum numbers, Dashen’s theorem reads

\[
\sum_P f^a_P M^2_P f^b_P = -\langle 0 | [Q^5_a, [Q^5_b, \mathcal{H}_{SB}(0)]] | 0 \rangle \quad (a, b = 1 \ldots 8)
\]  

where \( Q^5_a \) are the pseudoscalar charges and \( \mathcal{H}_{SB} \) the chiral symmetry breaking part of the QCD Hamiltonian. I repeat that single ‘decay constants’ \( f_\eta, f_{\eta'} \), which are sometimes used in this context, have no process-independent interpretation.

In particular, one may consider the diagonal elements of the matrix in Eq. (66). The octet element does not receive contributions from the anomalous or OZI-rule violating terms in the effective chiral Lagrangian and can be expressed in terms of masses and decay constants of the pion and kaon only. In the FKS scheme, using Eqs. (12) and (54), the result can be written as

\[
\sum_P f^8_P M^2_P f^8_P = f_8^2 (\cos^2 \theta_8 M^2_\eta + \sin^2 \theta_8 M^2_{\eta'})
\]

\[
= \frac{f_8^2 M^2_\pi + 2 f_8^2 (2M^2_K - M^2_\pi)}{3} .
\]  

This formula allows to derive an improved Gell-Mann–Okubo (GMO) formula with the angle \( \theta_8 \) playing a distinguished role. In the usual form it reads

\[
\cos^2 \theta_8 M^2_\eta + \sin^2 \theta_8 M^2_{\eta'} \approx \frac{4M^2_K - M^2_\pi}{3} - \Delta_{\text{GMO}} \frac{M^2_K - M^2_\pi}{3} .
\]
With the ansatz (54) one obtains a contribution \( \Delta_{\text{GMO}} = 4 \left( f_q^2 - f_s^2 \right) / \left( 3 f_q^2 \right) \). It is to be stressed that the \( SU(3)_F \) corrections in Eq. (69) enter in second order, and thus additional corrections can be important, e.g. from higher order contributions in \( \chi PT \). In any case, investigations of the GMO formula and its corrections provide an estimate of the mixing parameter \( \theta_8 \). Most of the analyses lead to values of about \(-20^\circ\), which is consistent with the number for \( \theta_8 \) in Eq. (35), see also Table 1.

3.10.3. Topological susceptibility

In the FKS scheme the singlet-singlet matrix element, as defined in Eq. (66), reads

\[
\sum_P f_P^0 M_P^2 f_P^0 = f_0^2 \left( \sin^2 \theta_0 M_q^2 + \cos^2 \theta_0 M_s^2 \right) = \frac{2 f_q^2 M_q^2 + f_s^2 \left( 2 M_K^2 - M_q^2 \right)}{3} + \sqrt{3} f_0 \left( \cos \theta_0 A_{q'} - \sin \theta_0 A_q \right).
\]

\[ \text{(70)} \]

It has a contribution from finite quark masses (i.e. \( M_\pi, M_K \neq 0 \)), similar as the octet-octet matrix element in Eq. (68). Additional contributions appear due to the \( U(1)_A \) anomaly; comparison with Eq. (12) yields an expression for the topological susceptibility \( \tau_0 \)

\[
\tau_0 = \frac{\sqrt{3} f_0}{12} \left( \cos \theta_0 A_{q'} - \sin \theta_0 A_q \right).
\]

\[ \text{(71)} \]

The value of \( \tau_0 \) following from the phenomenological values for \( f_0, \theta_0 \) and \( A_P \) comes out as \((192 \text{ MeV})^4\). Note that for \( \theta_8 \neq \theta_0 \) the connection between the topological susceptibility \( \tau_0 \) and the mass shift \( M_{U(1)_A} \), as defined in Eq. (65), is no longer simple, \( f_0^2 M_{U(1)_A}^2 \neq 12 \tau_0 \). The original Witten-Veneziano formula is recovered for \( \theta_0 = \theta_8 \).

3.10.4. Mass matrix and Schwinger’s formula

One can consider a similar construction as Eq. (68) in the quark-flavor basis. Here, the situation is simplified if one adopts the ansatz (32) for the decay constants in the FKS scheme. In this case (and only then) one can unambiguously define a mass matrix \( M^2 \) in the quark-flavor basis via

\[
(f^T M^2 f)^{ij} = \sum_P f_P^i M_P^2 f_P^j = \sum_P f_i \left( U^i(\phi) \right)_P \left( U(\phi) \right)_P^j f_j \equiv f_i \left( M^2 \right)^{ij} f_j.
\]

\[ \text{(72)} \]

The structure of this matrix follows solely from the anomaly equation (5). Following the notation of Ref. 24 one has

\[
M^2 = \begin{pmatrix} m_{qq}^2 + 2 a^2 & \sqrt{2} y a^2 \\ \sqrt{2} y a^2 & m_{ss}^2 + y^2 a^2 \end{pmatrix}
\]

\[ \text{(73)} \]
with $m_{qq}^2 \simeq M_\pi^2$, $m_{ss}^2 \simeq 2M_K^2 - M_\pi^2$, $y = f_q/f_s$ and $a^2 = M_{U(1),A}^2/(2 + y^2)$. The anomaly contribution $\propto a^2$ is manifestly 'non-democratic' due to the appearance of the flavor symmetry breaking term $y$. The consideration of the mass matrix in Eq. (73) turns out to be completely analogous to the analysis of the anomalous Ward identities performed earlier by Diakonov/Eides. This comes as no surprise since both methods rely on the same assumption about OZI-rule violation and $SU(3)_F$ breaking. (The correspondence between notations used here and in Ref. 18 reads $a^2 = \mu^2/2$ and $y = f_1/f_2$. Diakonov and Eides used the estimates $f_q \simeq f_\pi$ and $f_s \simeq 2f_K - f_\pi$ in order to obtain predictions for the various matrix elements of $\eta$ and $\eta'$. The diagonalization of the mass matrix (73) gives a theoretical estimate of the mixing angle $\phi$, which comes out as about $42^\circ$, see Refs. 18, 26, and Table 2. Within the uncertainties this value is compatible with the phenomenological value quoted in Eq. (35).

The matrix (73) can also be used to obtain an improved version of Schwinger’s mass formula. It is most easily derived by considering the trace and the determinant of the mass matrix in the quark-flavor basis and in the physical one and solving for $M_{\eta'}^2$. A crucial point is that the two matrices have to be connected by a simple rotation which is approximately true for the quark-flavor basis (72) but not for the octet-singlet one. This yields

$$M_{\eta'}^2 = M_\pi^2 + \frac{(M_K^2 - M_\eta^2)(2M_K^2 - M_\eta^2 - M_\pi^2)}{M_K^2 - (2 + y^2)M_\eta^2/4 - (2 - y^2)M_\pi^2/4}. \quad (74)$$

Schwinger’s original formula is recovered in the limit $y \to 1$. It has also been re-derived in another context by Veneziano. It is noticeable that, in the latter analysis, the anomalous mass contribution can be expressed in terms of the Veneziano ghost coupling constant, $a^2 = \lambda_2^2/N_C$. Schwinger used his formula to predict the mass of the $\eta'$ meson. For $y = 1$ he obtains a too high value of about 1600 MeV. As is obvious from Eq. (74), the formula is very sensitive to deviations of the flavor symmetry breaking parameter $y$ from unity since $y^2$ enters in the difference of two terms in the denominator. Indeed, for values of $y$ about 0.8, following from the phenomenological determination of the decay constants (35), the correct value of the $\eta'$ mass is found.

### 3.11. Pseudoscalar coupling constants of the nucleon

The coupling constants of the nucleon with the light pseudoscalar mesons $\pi^0$, $\eta$ and $\eta'$ are basic ingredients for the low-energy description of hadronic physics, especially for the description of nucleon-nucleon scattering data. A phenomenological determination of the coupling constants can be achieved via dispersion relations (see e.g. Grein/Kroll), potential models (see e.g. Nagels et al.), or effective low-energy Lagrangians of the nucleon (see e.g. Stoks/Rijken). While the value for

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*A variant of Schwinger’s formula has recently been discussed by Burakovsky and Goldman. It differs from the one presented in Eq. (74) due to the usage of a simplified treatment of the decay constants.*
g_{\pi NN} \simeq 13 is well-established, the values for \( g_{\eta NN} \) and \( g_{\eta' NN} \) are not so well-known, see e.g. Dumbrajs et al.\(^{[3]}\) and references therein.

An alternative way to estimate the couplings \( g_{P NN} \) is to relate them to the axial-vector coupling constants of the nucleon which are defined as

\[
\langle \mathcal{N}(p, s) | J_{\mu A}^a \mathcal{N}(p, s) \rangle = G_{A}^a s_\mu \tag{75}
\]

where \( s_\mu \) is the spin-vector of the nucleon. A different normalization convention, often found in the literature, is given by \( a_3 = \sqrt{2} G_A^3 \), \( a_8 = \sqrt{6} G_A^8 \), \( a_0 = \sqrt{3} G_A^0 \). The parameter \( a_3 \) can be determined from neutron \( \beta \)-decay by using isospin symmetry. This yields \( a_3 = 1.267 \pm 0.0035 \). Conventionally, one rewrites \( a_3 \) and \( a_8 \) in terms of the coupling constants \( D \) and \( F \), namely \( a_3 = F + D \) and \( a_8 = 3F - D \). The ratio \( F/D = 0.575 \pm 0.016 \) is determined from the hyperon \( \beta \)-decays and \( SU(3)_F \) flavor symmetry relations\(^{[92]}\). Together with the phenomenological value of \( a_3 \) one obtains \( a_8 = 0.58 \pm 0.03 \). Finally, the singlet axial charge \( a_0 \) can be extracted from a measurement of the first moment of the structure function \( g_1^p(x, Q^2) \) measured in polarized lepton-nucleon scattering,

\[
\Gamma_1^p(Q^2) = \int dx \, g_1^p(x, Q^2) = \frac{C_1^{NS}}{12} \left[ a_3 + \frac{a_8}{3} \right] + \frac{C_1^S}{9} a_0(Q^2) , \tag{76}
\]

where \( C_1^{(N)S} \) are known coefficients, calculable in perturbation theory. The latest analysis of the Spin-Muon Collaboration\(^{[93]}\) (SMC) yields \( a_0(5 \text{ GeV}^2) = 0.28 \pm 0.17 \).

The axial charges have recently been determined from a lattice simulation with dynamical Wilson fermions\(^{[94]}\) which leads to slightly smaller values for \( a_3 \), \( a_8 \) and \( a_0 \).

In the parton model the axial charges can be expressed in terms of the integrated polarized parton distributions

\[
\begin{align*}
    a_3 & = \Delta u - \Delta d , \\
    a_8 & = \Delta u + \Delta d - 2\Delta s , \\
    a_0(Q^2) & = \Delta u + \Delta d + \Delta s - 3\alpha_s \frac{\Delta g(Q^2)}{2\pi} .
\end{align*}
\tag{77}
\]

The appearance of the polarized gluon distribution in the expression for the nucleon’s singlet axial charge is a consequence of the \( U(1)_A \) anomaly\(^{[95]}\). In the Adler-Bardeen scheme\(^{[96]}\) the integrated polarized quark distributions \( \Delta q \) are scale-independent. The smallness of the singlet axial charge \( a_0 \) compared to \( a_8 \), which is the origin of the nucleon spin puzzle, can be explained by a substantial (negative) contribution of the polarized gluon distribution, \( \Delta g \).

Let me return to the pseudoscalar coupling constants of the nucleon. For the pion one has the well-known Goldberger-Treiman (GT) relation

\[
2 m_N G_{3}^a = f_\pi g_{\pi NN} \tag{78}
\]
which has been successfully tested against phenomenology. Its derivation is based
on the usual PCAC assumptions. The obvious generalization for $G_A^8$ and $G_A^0$ reads

$$2m_N G_A^a = \sum_{P=\eta,\eta'} f_P^a g_{PNN} \quad (a = 8, 0) \quad (79)$$

Here I have only taken into account the contributions from the physical $\eta$ and $\eta'$
states. Note that both, $G_A^0$ and $f_P^0$, have the same anomalous dimension $\gamma_A$
determined by the renormalization of the singlet axial-vector current $G^\eta$. Consequently,
the quantities $g_{PNN}$ in Eq. (79) are scale-independent as they should be. The intro-
duction of an additional OZI-rule violating parameter besides $\Lambda_1$ (present in $f_P^0$)
is not mandatory for a consistent behavior of the singlet couplings under renormal-
ization. However, higher excited pseudoscalar states or glueballs can be included, in
principle, but – following the FKS scheme – are assumed to be negligible in Eq. (79).

In the literature the GT relation in the singlet channel is often formulated in a
different way. Shore and Veneziano \(97\) (see also Ref. \(98\)) established a two-component
description of the singlet axial-charge which has been investigated in a number of
phenomenological applications (e.g. Refs. \(22\), \(99\); see, however, also Refs. \(100\), \(101\)). In
this picture, the singlet GT relation is modified by an additional direct coupling
of the Veneziano-ghost (more precisely the operator $G_{\tilde{G}NN}$) to the nucleon $(g_{\tilde{G}NN})$.
Neglecting $\eta$-$\eta'$ mixing for the moment, this yields

$$2m_N G_A^0 = \tilde{f} g_{\eta'NN} + \frac{\tilde{f}^2 M_{\eta'}^2 g_{\tilde{G}NN}}{\sqrt{3}} \quad (80)$$

It is to be stressed that the quantity $\tilde{f}$ does not coincide with the decay constant $f_0$.
It is defined via the pseudoscalar singlet current in such a way that it would coincide
with $f_\pi$ if the $U(1)_A$ anomaly were turned off, and it is scale-independent. The cor-
correct behavior of $G_A^0$ under renormalization requires a non-trivial scale-dependence
of the coupling $g_{\tilde{G}NN}$. It has also been shown in the analyses of Shore and Veneziano
that the singlet axial charge $a_0$ can be written as the product of the first moment of
the topological susceptibility and the vertex function of the would-be singlet Gold-
stone boson $\tilde{f}_0$. Let me investigate the phenomenological consequences of the two
alternatives (79) and (80), respectively.

Making use of the GT relation (79) and inserting the phenomenological values
for the meson decay constants (35), one finds the following connection between axial
charges and pseudoscalar coupling constants of the nucleon

$$a_3 = \sqrt{2} G_A^3 = 1.267 \pm 0.004 \quad \Rightarrow \quad g_{\pi NN} = 12.86 \pm 0.06,$$
$$a_8 = \sqrt{6} G_A^8 = 0.58 \pm 0.03 \quad \Rightarrow \quad \left\{ \begin{array}{l}
g_{\eta NN} = 3.4 \pm 0.5, \\
g_{\eta' NN} = 1.4 \pm 1.1. 
\end{array} \right\} \quad (81)$$

The errors take into account the uncertainties w.r.t. the axial charges and the meson
decay constants. This may be compared to the $SU(3)_F$ symmetry prediction $\tilde{f}$.
Quark Structure of Pseudoscalar Mesons

\[ b_{\eta NN} = g_{\pi NN} \frac{3 - 4\alpha}{\sqrt{3}} \simeq 3.4 \text{ with } \alpha = D/(D + F) = 0.635. \]

The value of \( g_{\eta NN} \) in Eq. (81) already saturates the bound obtained from the analysis of \( NN \) forward-scattering data by Kroll/Grein. \[ g_{\eta NN} \leq 3.5. \]

In this case, one would expect \( g_{\eta NN} \) to be close to zero which is in accord with the value quoted in Eq. (81).

Recently, from a measurement of \( \eta' \) production in proton-proton collisions close to threshold at COSY, the bound \[ g_{\eta NN} \leq 2.5 \] has been deduced. On the other hand, fits of \( NN \) data using potential models occasionally lead to significantly larger values for \( g_{\eta NN} \) and \( g_{\eta' NN} \) (see Dumbrajs et al. and references therein).

It is illustrative to rewrite the solution for the pseudoscalar coupling \( s \) of the nucleon from Eqs. (79) and (77) as follows,

\[
\begin{align*}
g_{\pi NN} &= \frac{1}{f_{\pi}} \frac{\Delta u - \Delta d}{\sqrt{2}}, \\
g_{\eta NN} &= \frac{\cos \phi}{f_q} \frac{\Delta u + \Delta d}{\sqrt{2}} - \frac{\sin \phi}{f_s} \frac{\Delta s + \alpha_s}{2\pi} \frac{\Delta g}{f_0} \sqrt{3} \sin \theta_8 \cos[\theta_8 - \theta_0], \\
g_{\eta' NN} &= \frac{\sin \phi}{f_q} \frac{\Delta u + \Delta d}{\sqrt{2}} + \frac{\cos \phi}{f_s} \frac{\Delta s - \alpha_s}{2\pi} \frac{\Delta g}{f_0} \sqrt{3} \cos \theta_8 \cos[\theta_8 - \theta_0],
\end{align*}
\]

where I have used the FKS scheme to express the decay constants in terms of the mixing parameters. The first terms on the r.h.s. in Eq. (82) are the quark contributions which have the form of the standard GT relation. The ratio of the additional gluon contribution to \( g_{\eta' NN} \) and \( g_{\eta NN} \) is given by \(-\cot \theta_8\), the same ratio as e.g. found in the ratio of \( J/\psi \to P\gamma \) decay amplitudes. In the picture based on the GT relation (79), the effect of \( \Delta g \) is thus to reduce both, the singlet axial charge \( a_0 \) and the pseudoscalar coupling \( g_{\eta' NN} \) compared to their nonet symmetry values.

In case of the GT relation (80) one usually assumes that the coupling \( g_{\eta' NN} \) is related to the polarized quark distributions and \( g_{\eta NN} \) to the polarized gluon distribution, respectively.\(^{h}\) This results in a similar representation as in Eq. (81) but without the \( \Delta g \) contributions and \( f_q, f_s \) replaced by \( \bar{f} \). If one assumes \( f \simeq f_{\pi} \), the extracted value for \( g_{\eta' NN} \) in this picture comes out larger and close to its nonet symmetry value. For example, Cheng obtained \( g_{\eta' NN} = 4.7 \) which happens to be closer to the value obtained in potential models but violates the bounds of Ref. 89. The answer to the spin puzzle on the basis of the GT relation (80) remains unchanged, namely that \( a_0 \) is small due to the additional nucleon-ghost coupling which is related to \( \Delta g \).

In summary, both alternatives (79) and (80) give a consistent description of the axial charges of the nucleon, but with significantly different values of the coupling constant \( g_{\eta' NN} \). In any case, the GT relations rely on the PCAC hypothesis. The corresponding uncertainties are on similar footing as the ones for the chiral anomaly.

\(^{h}\)This is to be confronted with Eq. (79) where the contribution from the Veneziano ghost is present in \( \Delta t \) and \( \Delta g \), separately, but is assumed to cancel in the sum for the scheme-independent quantity \( C^\alpha \).
prediction of the $P \to \gamma\gamma$ decay widths, and the same attention concerning corrections is to be paid, see the comments after Eq. (39).

3.12. Summary of $\eta$-$\eta'$ mixing parameters

$\eta$-$\eta'$ mixing can be described by different mixing angles. Let me briefly summarize their definition, their determination from experiment and their relations among each other:

$\theta_8$ and $\theta_0$: These mixing parameters are defined as the ratio of the octet and singlet decay constants of $\eta$ and $\eta'$ mesons through axial-vector currents (13). The decay constants enter the decays $P \to \gamma\gamma$, see Eq. (39), as well as the $P\gamma$ transition form factors at large momentum transfer, see Eq. (49). They also connect the bare fields in the chiral effective Lagrangian with the physical ones (17). Furthermore, the angle $\theta_8$ enters the (improved) Gell-Mann–Okubo formula (68).

$\theta_y$: This angle measures the ratio of matrix elements of the topological charge density $\omega$ with $\eta$ and $\eta'$ mesons. It can easily be extracted from the radiative $J/\psi$ decays (63).

$\phi$: Up to OZI-rule violating corrections of order $1/N_c$, this is the universal mixing angle that parametrizes ratios of matrix elements of quark currents with light $(u,d)$ or strange quarks in the FKS scheme, see Eqs. (52) and (54). There are several reactions where this angle can be probed, see Table 4. It can also be estimated from the diagonalization of the mass matrix in the quark-flavor basis (73).

From a combined expansion in small momenta and masses and in powers of $1/N_C$ in the framework of $\chi$PT one obtains the relation (20) which predicts the difference between the angles $\theta_8$ and $\theta_0$. In the FKS scheme an analysis of the anomaly equation provides another important relation (12) which connects the three angles $\theta_y$, $\theta_8$ and $\phi$ in the limit $m_u, m_d \to 0$. The independent determination of the mixing parameters from phenomenology and theory (12,24) supports the validity of Eqs. (21) and (62) and the internal consistency of the mixing approach.

The reader may wonder why I have not discussed the value of the mixing angle $\theta_P$ between octet and singlet states. I stress again that the correct representation of the physical fields $\eta$ and $\eta'$ in terms of bare octet and singlet fields is given by Eq. (17) which cannot be written as a simple rotation, once the next-to-leading order in the chiral effective Lagrangian (13) is taken into account. Thus the usage of $\theta_P$ is restricted to the leading order Lagrangian (12) which gives only a poor approximation to the real world. Nevertheless, one can define approximate octet and singlet fields by requiring a simple connection with the physical fields via a rotation matrix $U(\theta_P)$. Let me for illustration make the conventional choice

$$\theta_P := \phi - \arctan \sqrt{2} .$$

(83)
The mixing angles $\theta_8$ and $\theta_0$ can then be expanded as
\[ \theta_{8,0} = \theta_P \pm \frac{\sqrt{2}}{3} (1 - y) + \mathcal{O}(1 - y)^2. \] (84)

Thus, in principle, all the results can alternatively be written in terms of a single (approximate) octet-singlet mixing angle $\theta_P \simeq (\theta_8 + \theta_0)/2$ and explicit $SU(3)_F$ symmetry corrections proportional to $(1 - y)$. Such a procedure has been suggested by Benayoun et al.\(^4\) In that approach the transition from pure (bare) octet and singlet fields to the physical ones is accomplished by a non-diagonal matrix which results from a renormalization of the meson fields in an effective Lagrangian and yields an analogous relation to Eq. (17).

The effective octet and singlet states defined implicitly via Eq. (83) have the following Fock state expansion:\(^2\)
\[ |\eta_8\rangle = \frac{\Psi_q + 2 \Psi_s}{\sqrt{3}} |u\bar{u} + d\bar{d} - 2s\bar{s}\rangle + \frac{\sqrt{2}(\Psi_q - \Psi_s)}{\sqrt{3}} |u\bar{u} + d\bar{d} + s\bar{s}\rangle + \ldots \]
\[ |\eta_0\rangle = \frac{\sqrt{2}(\Psi_q - \Psi_s)}{3} |u\bar{u} + d\bar{d} - 2s\bar{s}\rangle + \frac{2 \Psi_q + \Psi_s}{3} |u\bar{u} + d\bar{d} + s\bar{s}\rangle + \ldots \] (85)

which is to be confronted with Eq. (84).

In any case, the value of $\theta_P$ alone is not sufficient to describe $\eta$-$\eta'$ mixing. Since the angles $\theta_8$, $\theta_0$, $\theta_y$ and $\phi$ can be obtained by simple ratios of $\eta$ and $\eta'$ observables and obey the useful relation (62) I prefer to present the results in terms of these angles.

4. Mixing with other states

4.1. Mixing in the $\pi^0$-$\eta$-$\eta'$ system

In the real world isospin is not an exact symmetry, and thus $\pi^0$ is to be viewed as a mixture of pure isospin-triplet and -singlet components. Of course, the magnitude of isospin-symmetry violation which is due to the differences in the masses of up- and down-quarks is small, and we should not expect the same order of accuracy for phenomenological predictions as in the $\eta$-$\eta'$ case. Furthermore, the comparison with experimental data is more difficult due to the interference with isospin-violating effects from the electromagnetic charges which can be important, too.

Nevertheless, it is a straightforward task to generalize the results of Section 3 and consider mixing in the $\pi^0$-$\eta$-$\eta'$ system. As already mentioned, the deviations of $\pi^0$ from a pure isospin-triplet (which I denote as $\varphi_3$) are small and the related mixing angles, $\epsilon$ and $\epsilon'$, can be treated as an expansion parameter. Moreover, a possible difference in basic decay constants for up- and down-quark states should be negligible. Therefore I consider
\[ |\pi^0\rangle = |\varphi_3\rangle + \epsilon |\eta\rangle + \epsilon' |\eta'\rangle \] (86)
where $\epsilon$ and $\epsilon'$ parametrize the $\eta$ and $\eta'$ admixtures in the pion. The mixing parameters $\epsilon$ and $\epsilon'$ receive contributions from the $U(1)_A$ anomaly very similarly as the mixing angles in the $\eta$-$\eta'$ sub-system. An analysis within the FKS scheme leads to the following estimate,

$$
\epsilon = \cos \phi \frac{m_{dd}^2 - m_{uu}^2}{2(M_\eta^2 - M_\pi^2)}, \quad \epsilon' = \sin \phi \frac{m_{dd}^2 - m_{uu}^2}{2(M_\eta'^2 - M_\pi^2)},
$$

(87)

where the difference $m_{dd}^2 - m_{uu}^2$ is defined in an analogous way as in Eq. (73) and can be estimated from $2(M_{K^0}^2 - M_{K^\pm}^2 + M_{\pi^\pm}^2 - M_{\pi^0}^2)$ to amount to 0.0104 GeV$^2$. In the latter formula, the specific combination of meson masses guarantees, that the leading order contributions from electromagnetic self-energy corrections drop out.

Inserting the phenomenological number for the mixing angle $\phi$ (35), one obtains $\epsilon = 0.014$ and $\epsilon' = 0.0037$. Similar values have been obtained by Chao et al. (85). Gardner (86) has recently emphasized the importance of $\pi^0$-$\eta$-$\eta'$ mixing in connection with the determination of the CKM-matrix elements via unitarity triangles from $B$ decays into light pseudoscalar mesons.

It is instructive to consider the ratio $\epsilon'/\epsilon$ and to express it in the following way, using Eq. (62),

$$
\frac{\epsilon'}{\epsilon} = -\tan \theta_8 \tan^2 \phi.
$$

(88)

This formula would coincide with the result obtained earlier in the conventional octet-singlet scheme (84) if one identified $\theta_8 \to \theta_P = \phi - \arctan \sqrt{2}$. However, as we learned, this relation is significantly spoiled by the $SU(3)_F$ corrections arising from $f_q/f_s \neq 1$, which are also relevant in Eq. (88).

It is also possible to estimate the matrix elements of the pion with the topological charge density in the anomaly equation (7),

$$
\langle 0| \omega |\pi^0 \rangle = \epsilon \langle 0| \omega |\eta \rangle + \epsilon' \langle 0| \omega |\eta' \rangle
$$

$$
= \frac{1}{\cos \phi} \frac{m_{dd}^2 - m_{uu}^2}{2M_\eta^2} \langle 0| \omega |\eta \rangle, \quad (89)
$$

where I have used Eq. (62), and $M_\eta^2$ has been neglected compared to $M_\pi^2$ for simplicity. The ratio $\langle 0| \omega |\pi^0 \rangle / \langle 0| \omega |\eta \rangle$ has also been derived in a leading order approach by Leutwyler (87). In that work $\eta$-$\eta'$ mixing has been neglected completely. This corresponds to taking $\cos \phi \simeq \cos \arctan \sqrt{2} = 1/\sqrt{3}$ and $M_\eta^2 \simeq 4/3 M_K^2 \simeq 4/3 m_s B$ in Eq. (88), which results in

$$
r = \frac{\langle 0| \omega |\pi^0 \rangle}{\langle 0| \omega |\eta \rangle} \simeq \frac{3\sqrt{3}}{4} \frac{m_d - m_u}{m_s}.
$$

(90)

This ratio is frequently discussed in connection with a determination of light quark mass ratios from the decays $\psi' \to J/\psi \pi^0$ and $\psi' \to J/\psi \eta$, which are assumed
to be dominated by the gluonic matrix elements in a similar way as the radiative $J/\psi$ decays discussed around Eq. (63)

$$\frac{\Gamma[\psi' \to J/\psi \pi^0]}{\Gamma[\psi' \to J/\psi \eta]} \simeq |r|^2 \frac{k_3}{k_\eta^3}. \quad (91)$$

Taking the experimental result for the decay width, one obtains $r = 0.043 \pm 0.006$ from which Donoghue and Wyler\cite{107} obtain an estimate for the quark mass ratio $(m_d - m_u)/m_s = 0.036 \pm 0.009$. Leutwyler\cite{106} on the other hand, estimated the quark mass ratio from other processes, $(m_d - m_u)/m_s = 0.025$, and predicted $r = 0.032$. Formula (91) leads to an even smaller result $r = 0.022$. One has to conclude\cite{106} that some (higher-order or electro-magnetic) corrections to either Eq. (89) or the decay mechanism for $\psi' \to J/\psi \pi^0$ are still not under control.$^4$

### 4.2. Heavy quark admixtures in light pseudoscalar mesons

Initiated by the CLEO measurement\cite{108} of an unexpectedly large branching fraction of $B \to K\eta'$, the subject of heavy quark components in light pseudoscalar mesons has recently regained interest. Early investigations of this subject already date back to 1976/77, when Kramer et al.\cite{16} and independently Fritzsch and Jackson\cite{17} discussed mixing of light and heavy pseudoscalar mesons in the context of radiative $J/\psi$ decays. A rather exhaustive phenomenological analysis of the mixing parameters in the $\eta$-$\eta'$-$\eta_c$ system has also been performed by Chao\cite{103}. In all cases, a rather small admixture of heavy quarks in light pseudoscalar mesons has been inferred. On the other hand, more recently, Halperin/Zhitnitsky\cite{109} and Cheng/Tseng\cite{110} proposed a prominent intrinsic charm component in $\eta'$ to explain the $B \to K\eta'$ decay width. Their results have been criticized for both theoretical and phenomenological reasons\cite{25, 26, 80, 111}. Also, it has been realized that a modification of the effective parameters in the factorization approach combined with a variation of the $B \to \eta'$ form factor and the usage of the improved $\eta$-$\eta'$ mixing parameters\cite{35} can easily increase the theoretical prediction for $\text{BR}[B \to K\eta']$ by a factor 2 to 3, see the recent report of Cheng et al.\cite{81} and references therein. This means that a large intrinsic charm component in $\eta'$ is not really needed to explain the data.

Let me present the theoretical arguments in favor of a small charm admixture in $\eta$ and $\eta'$. In the FKS scheme the analysis of the anomaly equation (7) can easily be extended to the charmed axial-vector current

$$\partial^\mu \bar{c} \gamma_\mu \gamma_5 c = 2m_c \bar{c} i\gamma_5 c + 2 \omega. \quad (92)$$

In the following it is convenient to treat $1/m_c$ as a small parameter and expand all quantities to first non-trivial order in this parameter. Taking matrix element with

---

$^4$ Note that $e^4$ and $(m_d - m_u)/m_s$ are both of the order of a few percent only; thus a contribution to $\psi' \to J/\psi \pi^0$ via two photons may be of similar size as the isospin-suppressed gluonic mechanism.
light pseudoscalar mesons, $P = \eta, \eta'$, one obtains

$$M_P f_P \uparrow O(1/m_c^2) + A_P \uparrow O(1)$$

(93)

where the notation of Section 3, see Eqs. (53) and (58), has been generalized to include three independent flavor combinations $i = q, s, c$. From Eq. (93) one immediately obtains an estimate for the parameters $h_P$ which provides a measure for the $c\bar{c}$ admixture in $\eta$ or $\eta'$

$$h_P = -A_P(1 + O(1/m_c^2))$$

(94)

In particular, Eq. (62) fixes the ratio $h_P^{\eta'}/h_P^{\eta} = A_{\eta'}/A_{\eta} = -\cot \theta_s$. It is the same quantity that enters the ratio of radiative $J/\psi$ decays (63). Thus the anomaly picture of Novikov et al. 83, see Eq. (63) and the intrinsic charm picture give an equivalent description of these decays. It is convenient to parametrize the quantities $h_P$ as follows

$$h_P^{\eta} = -\theta_c \sin \theta_s f_{\eta c} M_{\eta c}^2,$$

$$h_P^{\eta'} = \theta_c \cos \theta_s f_{\eta c} M_{\eta c}^2.$$  

(95)

Taking the values of $A_P$ quoted in Eq. (64) and $f_{\eta c} \simeq f_{J/\psi} = 410$ MeV, one estimates $\theta_c \simeq -1.0^\circ$. This number is reasonably small, and it can be tested by comparing the radiative $J/\psi$ decays to, say, $\eta'$ or $\eta_c$ mesons

$$\frac{\Gamma[J/\psi \rightarrow \eta'\gamma]}{\Gamma[J/\psi \rightarrow \eta_c\gamma]} = \theta_c^2 \cos^2 \theta_s \left( \frac{k_{\eta'}}{k_{\eta_c}} \right)^3.$$  

(96)

The experimental number for this ratio, $0.33 \pm 0.10$, corresponds to $|\theta_c \cos \theta_s| = 0.014 \pm 0.002$, which is in good agreement with the theoretical estimate (FKS: $\theta_c \cos \theta_s = -0.016$). Chao et al. 103 argue that also higher excitations of the $\eta_c$ meson (e.g. the $\eta_c'$) should be taken into account. In order to quantify the effect a certain hierarchy of the related mixing angles has to be assumed. In total it leads to a slightly smaller value for the mixing angle $\theta_c$ which is still compatible with the radiative $J/\psi$ decays, (Chao: $\theta_c \cos \theta_s = -0.012$).

There is another parameter which can be used to quantify the charm component of a light pseudoscalar, namely the decay constant $f_P$. However, we observe that it enters Eq. (83) only at sub-leading order in the $1/m_c$ expansion, which makes its determination non-trivial. A simple way out is to assume the same mixing behavior as it has been proven successful in the light meson sector. 26 Following Eq. (54), one defines an extended mixing matrix via

$$\begin{pmatrix}
\eta \\
\eta' \\
\eta_c
\end{pmatrix} =
U(\phi, \theta_y, \theta_c)
\begin{pmatrix}
\cos \phi \\
\sin \phi
\end{pmatrix}
\begin{pmatrix}
\sin \phi & -\theta_c \sin \theta_y \\
\cos \phi & \theta_c \cos \theta_y
\end{pmatrix}
\begin{pmatrix}
|\eta_q\rangle \\
|\eta_s\rangle \\
|\eta_0^c\rangle
\end{pmatrix}$$

(97)
with \( |\eta_c^0\rangle = \Psi_c(x, k_\perp) |c\bar{c}\rangle + \ldots \) having no light quark valence components. The ansatz \( \{f_p\} = U(\phi, \theta_p, \theta_\perp) \text{diag}[f_q, f_s, f_u]\) and \( \{h_p\} = U(\phi, \theta_p, \theta_\perp) \text{diag}[h_q, h_s, h_u]\) with \( h_c = f_{\eta_c} M_{\eta_c}^2 \) yields

\[
\begin{align*}
 f_c^c &= -\theta_c \sin\theta_8 f_{\eta_c} \simeq -2.4 \text{ MeV} , \\
 f_c^q &= -\theta_c \cos\theta_8 f_{\eta_c} \simeq -6.3 \text{ MeV} .
\end{align*}
\]

From Eq. (97) one can also read off the light quark admixtures in the \( \eta_c \) meson: 1.5\% of \( \eta_0 \) and 0.8\% of \( \eta_s \). A similar description of \( \eta^\prime-\eta_c \) mixing which leads to comparable results has been given by Petrov[112].

Another estimate of the charm decay constant of \( \eta^\prime \) has been suggested by Franz et al.[113] who integrate out the heavy quarks from the QCD Lagrangian to obtain an approximative \textit{operator} relation between the heavy quark axial-vector current and the topological charge density which is valid at low scales to leading order in \( 1/m_c^2 \). Their final result for the decay constants \( f_{\eta^\prime} \) reads

\[
 f_{\eta^\prime} = \frac{1}{12 m_c^2} A P .
\]

This result is equivalent to a perturbative computation of the \( c\bar{c} \) triangle graph performed by Ali et al.[84]. The so-obtained values for \( f_{\eta^\prime} \) are exactly a factor of three smaller than the ones presented in Eq. (98). The explanation for this discrepancy is not yet clear. Either the ansatz (97) is too naive or the \textit{perturbative} calculation of \( f_{\eta^\prime} \) receives additional non-perturbative contribution which may emerge from \( c\bar{c} \) modes in the \( \eta^\prime \) state itself.

In any case, the charm component inside the \( \eta \) or \( \eta^\prime \) turns out to be rather small, and obviously it is not the dominant contribution to the \( B \to \eta^\prime K \) decay via the elementary weak process \( b \to sc\bar{c} \). The values in Eq. (98) also lie comfortably within the phenomenological bounds, obtained from the consideration of the \( \eta \gamma \) and \( \eta^\prime \gamma \) transition form factors (see Ref. 43 for details), \(-65 \text{ MeV} \leq f_{\eta^\prime} \leq 15 \text{ MeV} \).

The mixing with the even heavier bottomonium states is obtained by scaling Eq. (97) with the quarkonium masses and decay constants, and turns out to be tiny

\[
\theta_b = \theta_c \frac{M_{\eta_c}^2 f_{\eta_c}}{M_{\eta_b}^2 f_{\eta_b}} \simeq -0.06^\circ .
\]

Nevertheless, the small admixture of \( bb \) in \( \eta \) and \( \eta^\prime \) can provide the leading contribution to the radiative \( \Upsilon \) decays in full analogy to Eqs. (63) and (96), and the ratio for the decay widths of \( R(\Upsilon) = \Gamma[\Upsilon \to \eta^\prime \gamma] / \Gamma[\Upsilon \to \eta \gamma] \) is again given in terms of \( \theta_y \) only, and its value can be predicted[44] to amount to 6.5.

One can also combine the results from Section 4.1 about \( \pi-\eta \) mixing with \( \eta-\eta^\prime-\eta_c \) mixing. Let me define (cf. Eq. (86))

\[
|\pi^0\rangle = |\varphi_3\rangle + \epsilon_q |\eta_q\rangle + \epsilon_s |\eta_s\rangle + \epsilon_c |\eta_c\rangle .
\]

Since both, \( m_d - m_u \) and \( 1/m_c^2 \) are small expansion parameters, one can combine Eqs. (96) and (97) and simply obtains

\[
\epsilon_c = \epsilon (-\theta_c \sin\theta_y) + \epsilon' (\theta_c \cos\theta_y) \simeq -1.5 \cdot 10^{-4}
\]
which is compatible with the value found by Chao et al.\cite{103} but probably too small to be of phenomenological relevance.

\section*{4.3. Mixing with pseudoscalar glueballs and/or higher excitations}

In QCD color singlet combinations of only gluon fields may appear as hadronic bound states (glueballs). Lattice calculations, see e.g. Ref.\cite{113}, indeed find some indications for glueballs with masses of a few GeV, and also some experimental candidates are heavily discussed in the literature, see e.g. Refs.\cite{1,74,75} and references therein. In particular, pseudoscalar glueballs with masses of 2 GeV or higher are expected from QCD sum rules and lattice calculations, see e.g. Refs.\cite{114,115}. On the other hand the pseudoscalar state $\eta(1440)$ (the former $\iota$) is often considered as a glueball candidate, too.

So far, the chiral effective Lagrangian presented in Section 3.1 has been the basis of my discussion of $\eta$-$\eta'$ mixing. It does contain neither higher excitations of light pseudoscalar quarkonium states nor glueballs as explicit degrees of freedom. However, implicitly the possible effect of additional states is encoded in the parameters of the effective Lagrangian. Consider, for example, the phenomenological fact that the OZI-rule violating coefficients $A_{1,2,3}$ are small and compatible with zero. From this one may deduce that the glueball admixture in the $\eta$ and $\eta'$ meson is small, say of the order of a few percent, comparable with the mixing in the $\phi$-$\omega$ system. This conclusion has been drawn, for instance, by Anisovich et al.\cite{69} from the analysis of the $\eta(\eta')\gamma$ transition form factors. It is important to understand that such a statement does not necessarily mean that there are no pronounced gluonic effects in the $\eta$ or $\eta'$ meson. Only, as I discussed in Section 2.1, these effects are not to be interpreted as admixtures of conventional glueball fields but rather as topological effects, connected to e.g. instantons. Remember, that the ghost states of the Veneziano- or Kogut/Susskind-type or the negative metric states à la Weinberg – which give rise to a non-vanishing topological susceptibility, the singlet mass shift, $\eta$-$\eta'$ mixing itself, an enhanced $J/\psi \to \eta'\gamma$ decay width etc. – are not physical.

Nevertheless, in order to obtain a complete picture of the pseudoscalar sector one may, of course, include glueballs and higher states in the mixing scenario, e.g. the $\eta(1295)$ which is approximately degenerated with the $\pi(1300)$ and thus likely to be a radial excitation of an $|\eta_q\rangle$ state. In principle, a determination of mixing parameters from phenomenological considerations similar to the ones performed in the analysis of $\eta$-$\eta'$ mixing should be possible. This requires the same attention concerning the definition of mixing angles, inclusion of $SU(3)_F$–breaking effects etc. In particular, the OZI-rule violating contributions have to be under control. Without going into further detail, I refer to the literature (see e.g. Refs.\cite{114,116} and references therein) where one can find some analyses in simplified schemes of mixing between $\eta$ and $\eta'$ mesons, glueballs etc. Due to the reasons mentioned above some of the conclusions therein have to be taken with care, in particular concerning the glueball nature of mesons in the pseudoscalar spectrum.
5. Summary

In the last few years we have achieved a very consistent picture of (strong) mixing phenomena in the pseudoscalar meson sector, in particular for $\eta$ and $\eta'$ mesons. The main progress, which has been discussed in detail in this review, is based on the definition of process-independent mixing parameters that can be used in different phenomenological situations and at different energy scales. The phenomenological determination of these parameters allows us to understand the properties of pseudoscalar mesons in terms of their underlying quark (and eventually gluon) structure. An important role is played by the pseudoscalar meson decay constants. On the one hand, they connect the bare fields in the chiral low-energy Lagrangian to the physical ones \[\eta\]. On the other hand, they enter the light-cone wave functions of quark-antiquark Fock states in the parton picture.

For the description of the decay constants in the $\eta$-$\eta'$ system a universal octet-singlet mixing angle $\theta_P$ has been shown to be not sufficient. The ratios of $\eta$ and $\eta'$ decay constants with octet or singlet axial vector currents define two independent mixing angles $\theta_8$ and $\theta_0$. In chiral perturbation theory the difference between the two angles can be expressed in terms of the parameter $L_5$ which in turn is determined by the difference of pion and kaon decay constants \[20]. This implies that the connection between bare octet and singlet fields and physical $\eta$ and $\eta'$ states is not a simple rotation.

In phenomenological analyses one often does not measure $\theta_8$ or $\theta_0$ directly, but rather extract particular ratios of $\eta$ and $\eta'$ matrix elements: First, there are processes which are induced by the topological charge density $\omega = \frac{\alpha_s}{8\pi} \bar{G}G$, like, for instance, the radiative $J/\psi$ decays into $\eta$ or $\eta'$\[83\]. Matrix elements of $\omega$ between the vacuum and $\eta$ or $\eta'$ fields can be used to define another mixing angle ($\theta_\omega$). Secondly, one considers matrix elements of quark currents with only light ($u$, $d$) or only strange quarks, respectively, for example in the decays $J/\psi \to \rho \eta$ ($\eta'$). A priori, every ratio of independent $\eta$ and $\eta'$ observables defines an independent observable. In order to keep the predictive power of the whole mixing approach one has to apply the usual OZI-rule, i.e. contributions from quark-antiquark annihilation are neglected while the effect of topologically non-trivial field configurations (e.g. instantons) is kept. These assumptions lead to the FKS scheme where mixing in the quark-flavor basis is described by a single mixing angle $\phi$. Moreover, by exploiting the Ward identities, including the $U(1)_A$ anomaly, one obtains the important relation \[12\] which connects the angles $\theta_8$, $\theta_\omega$ and $\phi$. Using these relations instead of the naive one, $\theta_P = \phi - \arctan \sqrt{2}$, resolves a big part of the inconsistencies which have been present in the literature for many years. With the set of mixing parameters \[83\] one reproduces an abundance of phenomenological data and fulfills important theoretical constraints. The uncertainties in the determination of the mixing parameters, arising from the experimental errors, is rather small. The systematical error due to the neglect of OZI-rule violating contributions is of order $1/N_C$. Empirically, the comparison of different phenomenological approaches indicates that the value of the mixing angle $\theta_0$ may be rather sensitive to the treatment of $1/N_C$ corrections.
An important test of the mixing approach is provided by the simultaneous consideration of the decays $\eta(\eta') \rightarrow \gamma\gamma$ on the basis of the Wess-Zumino-Witten term and the $\eta(\eta')\gamma$ transition form factors at large momentum transfer in the hard-scattering approach. The two-angle parameterization of decay constants turned out to be crucial. In this context we have also seen that the light-cone wave functions of the quark-antiquark Fock states of $\pi$, $\eta$ and $\eta'$ mesons can not be very different, which implies that – up to charge factors – the members of the light pseudoscalar nonet behave similarly in hard exclusive reactions.

Moreover, I have considered various mass formulas: The $U(1)_A$ mass shift and its connection to the topological susceptibility has been determined and compared to lattice QCD analyses. An $SU(3)_F$ improved Gell-Mann–Okubo formula on the basis of Dashen’s theorem has been shown to give an estimate of the angle $\theta_8$. Also the influence of flavor symmetry breaking on Schwinger’s mass formula has been discussed. The diagonalization of a mass matrix in the FKS scheme turns out to lead to equivalent results as the analysis of anomalous Ward identities performed in Ref.

The $\eta-\eta'$ mixing approach has been generalized to include also other states, for instance $\pi^0$ and/or $\eta_c$ mesons. In particular, I find a reasonably small intrinsic charm component in the $\eta'$ meson which cannot provide a dominant contribution to the decay $B \rightarrow \eta'K$. Concerning the role of pseudoscalar glueballs and/or higher excitations, I have stressed the importance of distinguishing physical glueball fields from topological effects related to e.g. instantons. In $\chi$PT their effect is encoded in the parameters of the effective Lagrangian. With the present amount of data there is no signal for a sizeable pseudoscalar glueball admixture in $\eta$ or $\eta'$ mesons.

The question of how to treat gluonic contributions has been shown to be also of relevance for the description of the singlet axial charge of the nucleon by means of generalized Goldberger-Treiman relations with the pseudoscalar coupling constants. Saturating the axial charges by the physical $\eta$ and $\eta'$ fields only, I obtained values for $g_{\eta NN}$ and $g_{\eta' NN}$ which are consistent with the bounds from an analysis of $N-N$ scattering data on the basis of dispersion relations. On the other hand, introducing an additional contribution of the Veneziano-ghost to the singlet axial charge of the nucleon and relating it to $\Delta g$ leads to a larger value of $g_{\eta' NN}$ which exceeds the bounds in Ref.

The phenomenological values of mixing parameters have been used to fix several matrix elements involving pseudoscalar mesons which may be useful in future applications. Of present interest are, for instance, the decays of $B$ mesons with $\eta$ or $\eta'$ in the final state. Another topic, which has been discussed recently, is the electro-production of $\pi^0$, $\eta$ and $\eta'$ mesons off nucleons or deuterons. Also, for the description of charmonium decays into light pseudoscalar mesons, a decent knowledge of the mixing parameters is necessary. As I discussed, there are more exotic cases, namely couplings of $\eta$ or $\eta'$ to Pomeron, Odderon or $Z$-bosons, which in principle may provide a direct determination of the mixing angles $\theta_8$ and, in particular, $\theta_0$. The theoretical lessons we learned from the pseudoscalar sector
may also be helpful for our understanding of mixing phenomena in other channels (scalar mesons, vector mesons, ...).

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