\textbf{W}-algebras with set of primary fields of dimensions $(3, 4, 5)$ and $(3, 4, 5, 6)$

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Abstract

We show that that the Jacobi-identities for a W-algebra with primary fields of dimensions 3, 4 and 5 allow two different solutions. The first solution can be identified with \(W_4\). The second is special in the sense that, even though associative for general value of the central charge, null-fields appear that violate some of the Jacobi-identities, a fact that is usually linked to exceptional W-algebras. In contrast we find for the algebra that has an additional spin 6 field only the solution \(W_5\).
1 Introduction

Over the last couple of years, a variety of different $W$-algebras has been investigated\footnote{For a recent review on $W$-algebras and extensive list of references see \cite{1}.}.

The efforts of classifying $W$-algebras\cite{2,3,4} led to a better understanding of the origin of $W$-algebras and to an increasing number of $W$-algebras. However, some $W$-algebras still lack an explanation. We shall present here another example, appearing when we consider the possible $W$-algebras with a set of primary fields of dimension 3, 4 and 5. In the following we denote a $W$-algebra with a set of primary fields of dimensions $d_1, d_2, \ldots$ in addition to the stress energy tensor $T$ as $W(2, d_1, d_2, \ldots)$.

We want to construct these $W$-algebras by demanding that the Jacobi-identities are satisfied. We therefore check that the Jacobi-identities that are given for any three simple fields $A$, $B$ and $C$ as the function of double-contractions

\[ \text{JACOBI}[A, B, C] := A(x) B(y) C(z) - \epsilon_{AB} B(y) A(x) C(z) - A(x) B(y) C(z) - O((x-z)^0) - O((y-z)^0) \]

"vanish". Here $\epsilon_{AB} = -1$ if $A$ and $B$ anticommute and +1 else and only poles in $(y - z)$ and $(x - z)$ are considered and the poles in $(x - y)$ that appear in the last contraction of the definition (1.1) are expanded in poles in $(x - z)$ and zeros in $(y - z)$,

\[ \frac{1}{x - y} = \frac{1}{x - z} \sum_{l=0} \left( \frac{y - z}{x - z} \right)^l \]

One has to be cautious with the meaning of the word "vanish". As Zamolodchikov has shown in his original paper\cite{5}, there are $W$-algebras, where the Jacobi-identities are violated by the appearance of null-fields. The simplest example for this is the algebra built by a primary fermionic field of dimension $5/2$, $V$, and the stress energy tensor $T$\cite{5}. It turns out that only for $c = -13/14$ the simple field $V$ itself disappears from the Jacobi-identity $\text{JACOBI}[V, V, V]$ and even then the descendent of $V$ at level 3,

\[ - \frac{4}{9} : T \partial V : + \frac{5}{9} : \partial TV : + \frac{4}{63} \partial^3 V \]

\[ (1.3) \]
still appears. However, this field is a null-field exactly for the same central charge $c = -13/14$. This behaviour is a common characteristic for the exceptional $W$-algebras, i.e. those $W$-algebras that are only defined for a finite set of central charges (though for some of the allowed values of central charges for an exceptional $W$-algebra no null-field might be involved).

Therefore we should use “vanish” in the sense that the Jacobi-identity $\text{JACOBI}[A, B, C]$ is zero modulo null-fields.

The choice (1.1) for the Jacobi-identities has the advantage that they are expressed in terms of fields rather than their modes, so unnecessary combinatoric can be avoided. In our computations we use the Mathematica$^\text{TM}$ package for computing OPEs by K. Thielemans [3, 4] in which the function $\text{JACOBI}[A, B, C]$ can be easily defined.

2 Solutions to the algebra $W(2, 3, 4, 5)$

There are ten non-trivial Jacobi-identities to satisfy: $\text{JACOBI}[W_n, W_m, W_l]$ for $n \geq m \geq l$. The Jacobi-identities $\text{JACOBI}[T, T, W_n]$ are automatically satisfied when the fields $W_n$ are primary and $\text{JACOBI}[T, W_n, W_m]$ when the ansatz for the OPE $W_n(z) W_m(w)$ has already the right conformal structure, i.e. primary fields $P$ appear together with their descendants with the right conformal factors. Unknown parameters are only the structure constants $C_{n,m}^P$. As primary fields will however not only appear the simple fields, but also primary fields that are constructed out of (normal ordered) products of the simple fields. For these composite fields we shall use the notation

$$P_{d}^{n,m} = \frac{1}{(d - n - m)!} : W_n^{(d-n-m)} W_m : + \text{descendents of primary fields (simple or not) of lower dimension} \quad (2.1)$$

In the algebra we are considering here, primary composite fields up to dimension 8 can appear in the OPE of the simple fields. So we take the ansatz (using standard-normalisation and omitting
already those terms that will not contribute by symmetry arguments)

\[ W_3 \star W_3 = \frac{c}{3} I + C_{3,3}^4 W_4 \]  
\( (2.2) \)

\[ W_3 \star W_4 = C_{3,4}^3 W_3 + C_{3,4}^5 W_5 \]  
\( (2.3) \)

\[ W_3 \star W_5 = C_{3,5}^4 W_4 + C_{3,5}^{P_{3,3}} P_6^{3,3} \]  
\( (2.4) \)

\[ W_4 \star W_4 = \frac{c}{4} I + C_{4,4}^4 W_4 + C_{4,4}^{P_{3,3}} P_6^{3,3} \]  
\( (2.5) \)

\[ W_4 \star W_5 = C_{4,5}^3 W_3 + C_{4,5}^5 W_5 + C_{4,5}^{P_{3,4}} P_7^{3,4} + C_{4,5}^{P_{3,4}} P_8^{3,4} \]  
\( (2.6) \)

\[ W_5 \star W_5 = \frac{c}{5} I + C_{5,5}^4 W_4 + C_{5,5}^{P_{3,3}} P_6^{3,3} + C_{5,5}^{P_{3,3}} P_8^{3,3} + C_{5,5}^{P_{3,5}} P_8^{3,5} + C_{5,5}^{P_{4,4}} P_8^{4,4} \]  
\( (2.7) \)

The Jacobi-identities now impose stringent constraints on the structure constants \( C \). Indeed, only two sets of non-equivalent solutions finally obey all Jacobi-identities.

For the set of simple fields we are considering, the constants \( C_{mn}^l \) and \( C_{ml}^n \) are simply related by [8]

\[ C_{mn}^l = \frac{l}{n} C_{ml}^n \]  
\( (2.8) \)

and for both solutions we have therefore

\[ C_{3,3}^3 = \frac{3}{4} C_{3,3}^4 \quad C_{3,3}^4 = \frac{3}{5} C_{3,4}^5 \quad C_{3,4}^3 = \frac{3}{5} C_{3,4}^5 \quad C_{5,5}^4 = \frac{4}{5} C_{3,4}^5 \]  
\( (2.9) \)

The other structure constants for the first solution to the Jacobi-identities are listed in Table I. The squares in the definitions of \( C_{3,3}^4 \) and \( C_{3,4}^5 \) and the dependency of some of the constants on \( C_{3,3}^4 \) and \( C_{3,4}^5 \) reflect the symmetries \( W_n \rightarrow -W_n \).

The existence of this solutions was already known, though the form of the structure constants has not yet been given: It can be identified with the \( \text{WA}_4 \)-algebra that has exactly the set of simple fields we consider. This identification can be made by the structure constants \( C_{3,3}^4 \) and \( C_{4,4}^4 \). In [9], these constants have been derived for \( \text{WA}_{n-1} \) for any \( n > 3 \), using the free field
realisation of Fateev and Lykyanov for these algebras \([10]\):

\[
\left( C_{3,3}^{4,4} | \text{WA}_{n-1} \right) = 64 \frac{n-3}{n-2} \frac{c+2}{5c+22} \frac{c(n+3)+2(4n+3)(n-1)}{c(n+2)+(3n+2)(n-1)} \quad (2.10)
\]

\[
C_{4,4}^{4,4} | \text{WA}_{n-1} | C_{3,3}^{4,4} | \text{WA}_{n-1} = 48 \frac{c^2(n^2-19)+3c(6n^3-25n^2+15)+2(n-1)(6n^2-41n-41)}{n-2} \frac{5c+22}{(c(n+2)+(3n+2)(n-1))} \quad (2.11)
\]

for \(n = 5\) in agreement with our first solution.

For the second solution (see Table II) we find that a difficulty arises: In the course of the computation it turns out that in this solution a ladder of null-fields occur in the embedding algebra for any value of the central charge, starting at dimension 8 and therefore already occurring in the ansatz \((2.2)-(2.7)\). At this dimension we find two null-fields,

\[
N_8^1 = P_8^{3,4} \quad (2.12)
\]

\[
N_8^2 = P_8^{3,3} + \kappa_1 P_8^{3,5} + \kappa_2 P_8^{4,4} \quad (2.13)
\]

with

\[
\kappa_1 = \frac{9}{8} \frac{(-1+2c)(114+7c)}{(10+c)(-870+533c+29c^2)} C_{3,3}^{4,4} C_{5,4}^{5,4} \quad (2.14)
\]

\[
\kappa_2 = \frac{8}{22+5c} \frac{(4-5c)(2+c)(114+7c)}{(-870+533c+29c^2)} \quad (2.15)
\]

We have to take this into account by setting in this solution \(C_{n,m}^{p,q} \) and \(C_{n,m}^{p,q} \) to zero.

The existence of null-fields at any value of the central charge for the second solution leads to a remarkable consequence that is in contrast to the first solution \((\text{WA}_4)\): Even though the null-fields \(N_8^1\) and \(N_8^2\) are taken from the ansatz \((2.2)-(2.7)\), they still appear in some of the Jacobi-identities. Not only these lowest-dimensional null-fields of the algebra violate the Jacobi-identities, but also null-fields of dimension \(> 8\).

This makes the second solution similar to the exceptional \(W\)-algebras. However, the differences are obvious: In exceptional \(W\)-algebras Jacobi-identity violating null-fields are only null for a

\[\text{These constants are for } n \to \infty \text{ identical of those of the } W_\infty\text{-algebra of Pope [11], after changing their basis of simple fields to the one used here.}\]
certain value of the central charge, hence restricting the existence of the algebra to this \( c \)-value. In our case the two null-fields \( N^1_8 \) and \( N^2_8 \) do not specify a central charge.

Exceptional W-algebras can be seen as some “limit” of generic W-algebras (f.e. the exceptional W-algebra of the introduction with \( T \) and a primary field of dimension \( 5/2 \) can be regarded as the “limit” \( c \to -13/14 \) of the super-algebra \( WB(0,2) \); see also [12]). It is not clear whether a similar point of view can be taken here (since in this case there is no obvious meaning in “limit”).

Moreover we see that the second solution has the property that the structure-constants \( c^k_{m,n} \) behave in the large-\( c \) limit as

\[
\left( c^k_{m,n} \right)^2 \sim c 
\]

that shows that the chosen normalisation \( < W_n(z)W_n(w) > = c/n/(z-w)^{2n} \) is not suitable for taking the classical limit. Instead for the second solution one should normalize these fields such that

\[
\left( c^k_{m,n} \right)^2 \sim \text{const.} \quad \text{for } c \to \infty 
\]

showing that this algebra has in the classical limit a central term for the Poisson bracket \( \{ T(z), T(w) \} \) but not for \( \{ W_n(z), W_n(w) \} \).

3 Taking the next step: The WA_5-algebra

Finding two solutions for the algebra \( W(2,3,4,5) \) leads automatically to the question, whether there exist multiple solutions for \( W(2,3,4,\ldots,n) \). We have done the computation for \( W(2,3,4,5,6) \) but with a rather disappointing result: We find that only one solution exists, namely \( WA_5 \).

In this algebra we also have the choice of a basis for the simple field \( W_6 \); we could change the basis by redefining

\[
W_6 \rightarrow \text{norm}(\beta) \left( W_6 + \beta P^3_6 \right) \quad (3.1)
\]
For our calculations we chose a basis in which

\[ < W_6(z) P_6^{3,3}(w) > = 0 \]  

(3.2)

For completeness we give the structure constants in Table 3, with the fields

\[ P_9^{3,3,3} \equiv P_9^{3} P_6^{3,3} \quad \text{and} \quad P_{10}^{3,3,4} \equiv P_{10}^{3} P_7^{3,4} \]  

(3.3)

### 4 Null-fields in the algebra \( W(2,4,6) \)

H. Kausch and G. Watts have shown \[13\] that the \( W \)-algebra with \( T \) and primary fields of dimension 4 and 6, \( W(2,4,6) \), has, similar to the case considered here, four non-equiv alent solutions. Redoing their calculation (that was based on four-point-functions) in our frame-work we find that the similarities between \( W(2,4,6) \) and \( W(2,3,4,5) \) go further.

In the algebra \( W(2,4,6) \) one composite field at dimension 8, \( P_8^{4,4} \) and two at dimension 10, \( P_{10}^{4,4} \) and \( P_{10}^{4,6} \), appear in the irregular part of the OPE of the simple fields:

\[
W_4 \star W_4 = \frac{c}{4} I + \tilde{C}_4 W_4 + \tilde{C}_4^6 W_6 \\
W_4 \star W_6 = \tilde{C}_4 W_4 + \tilde{C}_6 W_6 + \tilde{C}_4^8 P_8^{4,4} \\
W_6 \star W_6 = \frac{c}{6} I + \tilde{C}_6 W_4 + \tilde{C}_6^6 W_6 + \tilde{C}_6^8 P_8^{4,4} + \tilde{C}_6^{10} P_{10}^{4,4} + \tilde{C}_6^{10} P_{10}^{4,6} 
\]  

(4.1)

(4.2)

(4.3)

We use \( \tilde{C} \) for the structure constants to distinguish them from those used for \( W(2,3,4,5) \). We refer to \[13\] for an incomplete list of them.

In one of the solutions (set 1 of ref. \[13\]) emerges a null-field as the combination of the two dimension-10 fields \[14\]

\[ \tilde{N}_{10} = \tilde{P}_{10}^{4,4} + \kappa \tilde{P}_{10}^{4,6} \]  

(4.4)

with

\[
\kappa = \frac{4}{15} \frac{(-16 + c)(24 + c)(21 + 4c)(-7 + 10c)}{(64 - 19c)(-1 + c)(232 + 11c)(-82 + 47c + 10c^2)} \tilde{C}_4^4 \tilde{C}_4^6 
\]  

(4.5)
The field $\tilde{N}_{10}$ is again null for any value of the central charge and as in the case of the $W(2, 3, 4, 5)$-algebra it violates some of the Jacobi-identities, together with higher-dimensional null-fields of this algebra.

This special solution of the $W(2,4,6)$-algebra is the one that can be realized by the Super-Virasoro algebra, identifying (up to normalisations)

$$W_4 \sim P_4^{3/2,3/2} \quad (4.6)$$
$$W_6 \sim P_6^{3/2,3/2} \quad (4.7)$$

the spin-$3/2$ primary field being the supersymmetry generator $G$ \[14, 15\]. In this realization the null-field $\tilde{N}_{10}$ actually is identically zero and all Jacobi-identities are zero.

One might think that due to the freedom of modifying the basic OPEs of the simple fields by adding the null-field(s), that is in the case of the $W(2,3,4,5)$-algebra by

$$W_4 \ (\text{mod}) \ W_5 = W_4 \ (\text{orig}) \ W_5 + \alpha_1 N_8^1 + \alpha_2 N_8^2 \quad (4.8)$$
$$W_5 \ (\text{mod}) \ W_5 = W_5 \ (\text{orig}) \ W_5 + \beta_1 N_8^1 + \beta_2 N_8^2 \quad (4.9)$$

and in the case of the $W(2,4,6)$-algebra by

$$W_6 \ (\text{mod}) \ W_6 = W_6 \ (\text{orig}) \ W_6 + \gamma \tilde{N}_{10} \quad (4.10)$$

(the “(orig)” OPEs refer to eqs. (2.6), (2.7) and (4.3) with $C_{4,5}^{p_{3,3}}$, $C_{4,5}^{p_{3,4}}$, $C_{5,5}^{p_{3,3}}$, $C_{5,5}^{p_{3,4}}$ and $\tilde{C}_{6,6}^{p_{3,4}}$, respectively, set to zero), one could adjust the new parameters such that the Jacobi-identities become zero exactly. It turns out, however, that this is not the case.

5 Conclusions

Interesting questions arising from our calculations are still unsolved:

i) to find a realization for the second solution of the algebra $W(2,3,4,5)$, where the two null-fields $N_8^1$ and $N_8^2$ are identically zero (as it happens in the realizations for the second $W(2,4,6)$-algebra
in terms of the Super-Virasoro-algebra);

ii) even if we have seen that $W(2,3,4,5,6)$ only admits one solution, $W_{A_5}$, it might be that the two possible solutions to $W(2,3,4,5)$ are not a single accident and including even higher spins might again permit more solutions. That leads to the question, given simple fields of dimensions $2, 3, 4, \ldots, n$, how many solutions exist for a given $n$. We have gained the impression that especially for odd $n$ several solutions might be possible. If multiple solutions exist are then all solutions apart from $W_{A_{n-1}}$ “exceptional” in the sense that null-fields appear (in the Jacobi-identities);

iii) if there are null-fields in the envelopping algebra of a general $W$-algebra, do these null-fields always violate the Jacobi-identities (either of the simple fields if their dimension is low enough or the Jacobi-identities of composite fields)?

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| $C_4^3$ | 1024 | $(2 + c) (23 + c)$ | $3 (22 + 5 c) (68 + 7 c)$ |
| $C_3^4$ | 25 | $(116 + 3 c) (22 + 5 c)$ | $(68 + 7 c) (114 + 7 c)$ |
| $C_3^5$ | 9 | $(-1 + 2 c) (68 + 7 c)$ | $40 (2 + c) (23 + c) (116 + 3 c) C_3^4 C_3^3$ |
| $C_4^4$ | 9 | $(-128 + 70 c + c^2)$ | $32 (2 + c) (23 + c) C_3^4 C_3^3$ |
| $C_4^5$ | 9 | $(22 + 5 c)$ | $2 (2 + c) (23 + c)$ |
| $C_5^3$ | $-\frac{15}{64} (70272 + 9340 c + 204 c^2 + 11 c^3)$ | $C_3^4 C_3^3$ |
| $C_5^4$ | $\frac{3}{80} (68 + 7 c) (334 + 37 c)$ | $C_3^4 C_3^5 C_3^3 C_3^3$ |
| $C_5^5$ | $\frac{9}{80} (68 + 7 c)$ | $C_3^4 C_3^5 C_3^3 C_3^3$ |
| $C_5^3$ | $\frac{3}{2} (1507824 + 248948 c + 14880 c^2 + 181 c^3)$ | $C_3^4 C_3^3 C_3^3 C_3^3$ |
| $C_5^3$ | $4 (-13656 - 306 c + 11 c^2)$ | $(2 + c) (116 + 3 c) (114 + 7 c)$ |
| $C_5^3$ | $-\frac{3}{8} (68 + 7 c)$ | $C_3^4 C_3^5 C_3^3 C_3^3$ |
| $C_5^4$ | $64 (114 + 7 c)$ | $(116 + 3 c) (22 + 5 c)$ |

Table I: WA$_4$-solution to W(2,3,4,5)
\begin{align*}
(C_{3,3}^4)^2 & = \frac{16}{3} \frac{(2 + c)(10 + c)^2}{(7 + c)(-1 + 2c)(22 + 5c)} \frac{(-4 + 5c)}{(2 + c)(10 + c)(22 + 5c)} \\
(C_{3,4}^5)^2 & = \frac{25}{(1 + c)(13 + c)(22 + 5c)} \frac{(-1 + 2c)(114 + 7c)}{(2 + c)(10 + c)(22 + 5c)} \\
C_{3,5}^{P_{3,3}} & = \frac{9}{32} \frac{(7 + c)(-1 + 2c)(68 + 7c)}{(-1 + 2c)(2 + c)(13 + c)(-4 + 5c)} C_{3,3}^4 C_{3,4}^5 \\
C_{4,4}^1 & = \frac{9}{4} \frac{(-64 - 6c + 45c^2 + 5c^3)}{(2 + c)(10 + c)(-4 + 5c)} C_{3,3}^4 \\
C_{4,4}^{P_{3,3}} & = \frac{9}{2} \frac{(4 + c)(22 + 5c)}{(2 + c)(-4 + 5c)} \\
C_{4,5}^1 & = \frac{15}{8} \frac{(7 + c)(-2304 - 188c + 1076c^2 + 85c^3)}{(2 + c)(10 + c)(-4 + 5c)(114 + 7c)} C_{3,3}^4 \\
C_{4,5}^{P_{3,4}} & = \frac{3}{10} \frac{(7 + c)(-1 + 2c)(26 + 7c)}{(-1 + c)(2 + c)(10 + c)(-4 + 5c)} C_{3,3}^4 C_{3,4}^5 \\
C_{4,5}^{P_{3,4}} & = 0 \\
C_{5,5}^{P_{3,3}} & = \frac{3}{2} \frac{(7 + c)(-154512 - 26404c + 46628c^2 + 6979c^3 + 259c^4)}{(-1 + c)(2 + c)(13 + c)(-4 + 5c)(114 + 7c)} \\
C_{5,5}^{P_{3,3}} & = 0 \\
C_{5,5}^{P_{3,5}} & = \frac{3}{4} \frac{(7 + c)(-1 + 2c)(-145776 - 32276c + 3222c^2 + 1146c^3 + 49c^4)}{(-1 + c)(2 + c)(13 + c)(-4 + 5c)(-870 + 533c + 29c^2)} C_{3,3}^4 C_{3,4}^5 \\
C_{5,5}^{P_{4,4}} & = \frac{8}{(-1 + c)(22 + 5c)(-870 + 533c + 29c^2)} \frac{(7 + c)(2568 + 2412c + 914c^2 + 35c^3)}{(2 + c)(10 + c)(13 + c)(-4 + 5c)(-870 + 533c + 29c^2)} \\
\end{align*}

Table II: Solution to $W(2,3,4,5)$ involving null-fields
| $C_{3,3}^4$ | $108 (2+c)(30+c)$ $\frac{(25+2c)(22+5c)}{(25+2c)(22+5c)}$ |
| $C_{3,4}^5$ | $125 (51+c)(22+5c)$ $\frac{45 (2+c)(30+c)}{(25+2c)(114+7c)}$ |
| $C_{3,5}^6$ | $3(19+c)(11+2c)(-5+4c)(114+7c)(820+11c)$ $\frac{4(25+c)(-329220+205197c+43943c^2+2052c^3+28c^4)}{4(25+2c)(22+5c)}$ |
| $C_{3,3}^{p,3}$ | $\frac{3(-1+2c)(25+2c)(-8+7c)(68+7c)}{20(2+c)(-329220+205197c+43943c^2+2052c^3+28c^4)} C_{3,3}^{c,3,4}$ |
| $C_{3,3}^{p,4}$ | $\frac{8(23+c)(25+2c)(-4+5c)(-60+29c}{5(19+c)(51+c)(-5+4c)(22+5c)(820+11c)} C_{3,4}^{c,3,5}$ |
| $C_{3,3}^{p,4}$ | $\frac{4(-1+2c)(25+2c)(-4+5c)}{15(19+c)(11+2c)(-5+4c)(22+5c)} C_{3,3}^{c,3,4} C_{3,3}^{c,3,5}$ |
| $C_{4,4}$ | $\frac{-719+1233c+17c^2}{36(2+c)(30+c)} C_{3,3}^{c,3,5}$ |
| $C_{4,4}$ | $\frac{(25+2c)(22+5c)}{135(2+c)(30+c)} C_{3,3}^{c,3,4} C_{3,3}^{c,3,5}$ |
| $C_{4,4}$ | $\frac{15(22+5c)(-17084+11117c+1967c^2+38c^3)}{4(2+c)(-329220+205197c+43943c^2+2052c^3+28c^4)}$ |
| $C_{4,5}$ | $\frac{5(-197580+2104c+3009c^2+11c^3)}{72(2+c)(30+c)(114+7c)} C_{3,3}^{c,3,5}$ |
| $C_{4,5}$ | $\frac{(25+2c)(258+31c)}{15(2+c)(30+c)(91+c)} C_{3,3}^{c,3,5} C_{3,3}^{c,3,4}$ |
| $C_{4,5}$ | $\frac{16(25+2c)}{45(2+c)(30+c)} C_{3,3}^{c,3,4} C_{3,3}^{c,3,4}$ |
| $C_{4,6}$ | $\frac{36895198800-20343407460c-65133194148c^2-573770191c^3-21699119c^4-409786c^5-5096c^6}{452(2+c)(30+c)(-329220+205197c+43943c^2+2052c^3+28c^4)} C_{3,3}^{c,3,5}$ |
| $C_{4,6}$ | $\frac{(-1+2c)(25+2c)(68+7c)(-32+17c)}{20(2+c)(-329220+205197c+43943c^2+2052c^3+28c^4)} C_{3,3}^{c,3,4} C_{3,3}^{c,3,5}$ |
| $C_{4,6}$ | $\frac{4(25+2c)(-12952440+15248466c-4962753c^2-491614c^3-8517c^4+58c^5)}{15(2+c)(91+c)(11+2c)(-5+4c)(114+7c)(820+11c)} C_{3,3}^{c,3,4} C_{3,3}^{c,3,5}$ |
| $C_{4,6}$ | $\frac{5(25+2c)(22+5c)(-1530360+971226c+177569c^2+10851c^3+214c^4)}{6(2+c)(19+c)(30+c)(11+2c)(-5+4c)(114+7c)(820+11c)} C_{3,3}^{c,3,5}$ |
| $C_{4,6}$ | $\frac{64(25+2c)(-4+5c)(-6240+6715c+525c^2+14c^3)}{15(19+c)(51+c)(11+2c)(-5+4c)(22+5c)(820+11c)} C_{3,3}^{c,3,4} C_{3,3}^{c,3,5}$ |
| $C_{4,6}$ | $\frac{5(25+2c)(22+5c)(-5634+3549c+763c^2+22c^3)}{18(2+c)(19+c)(30+c)(11+2c)(-5+4c)(114+7c)} C_{3,3}^{c,3,4} C_{3,3}^{c,3,5}$ |
| $C_{5,5}$ | $\frac{1}{675(2+c)(30+c)(51+c)} C_{3,3}^{c,3,4} C_{3,3}^{c,3,5}$ |

**Table III:** The algebra WA5
\[
\begin{align*}
C_{6,6}^{P_{3,3}} & \left( -11300841937108800 + 14063403584210400 \cdot c - 1352296279092708 \cdot c^2 - 1728077252788212 \cdot c^3 - 296011339188793 \cdot c^4 - \\
& \quad 2118608487121 \cdot c^5 - 709798549133 \cdot c^6 - 8549890955 \cdot c^7 + 100398106 \cdot c^8 + 3404236 \cdot c^9 + 19880 \cdot c^{10} \right) \times \\
& \quad \frac{5}{3 (2 + c) (19 + c) (51 + c) (11 + 2 c) (-5 + 4 c) (11 + 7 c) (820 + 11 c) (-329220 + 205197 c + 43943 c^2 + 2052 c^3 + 28 c^4)} \\
\end{align*}
\]

\[
C_{6,6}^{P_{3,5}} \left( -99289016697600 + 1237084794525600 \cdot c - 121398549036960 \cdot c^2 - 150739618392734 \cdot c^3 - 25893911838127 \cdot c^4 - \\
& \quad 1926099854632 \cdot c^5 - 730523782839 \cdot c^6 - 1436629792 \cdot c^7 - 13963372 \cdot c^8 - 65744 \cdot c^9 \right) \times \\
\end{align*}
\]

\[
\begin{align*}
C_{6,6}^{P_{4,4}} & \left( 28000940079600 - 343027783977600 \cdot c + 266824907278990 \cdot c^2 + 435922843100763 \cdot c^3 + 81243877432609 \cdot c^4 + \\
& \quad 6591440693509 \cdot c^5 + 284635535593 \cdot c^6 + 6791963944 \cdot c^7 + 82361124 \cdot c^8 + 346528 \cdot c^9 \right) \times \\
& \quad \left( 2 (2 + c) (30 + c) (61 + c) (11 + 2 c) (-5 + 4 c) (820 + 11 c) (-329220 + 205197 c + 43943 c^2 + 2052 c^3 + 28 c^4) \\
& \quad \left( 25 + 2 c \right) \\
& \quad \frac{1}{3 (19 + c) (51 + c) (11 + 2 c) (-5 + 4 c) (22 + 5 c) (820 + 11 c) (-329220 + 205197 c + 43943 c^2 + 2052 c^3 + 28 c^4)} \\
& \quad \left( 8034038369665685685485838798361293824000000 - 5845587987803601154130276947855427635200000 \cdot c - \\
& \quad 188934067175351525528357058519820988021760000 \cdot c^2 + 1956891041650431944341575923744286126848000 \cdot c^3 + \\
& \quad 25202285862939634879607240377045553062400 \cdot c^4 - 4821317048567401826860187349857992950753600 \cdot c^5 - \\
& \quad 11396952150620214244678164452942554515040 \cdot c^6 + 353529227897143294253818980480572248483824 \cdot c^7 + \\
& \quad 222176745788240495769330616597587576144 \cdot c^8 + 526049718058270985984216054685844094412 \cdot c^9 + \\
& \quad 7547941989881264360308312799281405011094 \cdot c^{10} - 737016414477363272201642630186076505699 \cdot c^{11} + \\
& \quad 51257520055287620834247277708182057761 \cdot c^{12} + 2573041084930425666898626173718570626 \cdot c^{13} + \\
& \quad 9183243872453971115703756456425728 \cdot c^{14} + 2167939856521534784877436954876615 \cdot c^{15} + \\
& \quad 2416992357723869973741250003557 \cdot c^{16} - 377180583634502280899116259904 \cdot c^{17} - 22947317521687019788768157662 \cdot c^{18} - \\
& \quad 504355207308191419706085270 \cdot c^{19} - 621758810833030815408392 \cdot c^{20} - 36440274767963562583872 \cdot c^{21} + \\
& \quad 109186439856345385184 \cdot c^{22} + 3796118183735011712 \cdot c^{23} + 30125022089651320 \cdot c^{24} + 112091756894208 \cdot c^{25} + 164540952576 \cdot c^{26} \right) \\
& \quad \left( 19 + c) (11 + 2 c) (-5 + 4 c) (11 + 7 c) (820 + 11 c) (-329220 + 205197 c + 43943 c^2 + 2052 c^3 + 28 c^4) d \\
\right)
\end{align*}
\]

Table III: continued
\[
\begin{align*}
C_{P,0,6}^{4,4} &\quad \left(-1910623477092506587046267455127040000000+331239441064584382643548550160512000000 c+ \\
&\quad 366029218735716240430106773341654400000 c^2-12887812922574567064780001686386029760000 c^3- \\
&\quad 114797835700375755720647122150002728000 c^4+3384365305852558483587481922105626192000 c^5+ \\
&\quad 1696486804451652055204070572320801616560 c^6+57550812975399798474941075961950579664 c^7+ \\
&\quad 10423677257560188363999787555626659220 c^8+1231029207538218838631458579145791970 c^9+ \\
&\quad 102477524232932766297633837219414411 c^{10}+626143077111113011762298739022748213 c^{11}+ \\
&\quad 28726145415938738180017221967320 c^{12}+1009996014084890533438605016532 c^{13}+ \\
&\quad 26550179074666874930811033653 c^{14}+5315624656748673642279081393 c^{15}+ \\
&\quad 7837628972315070137145066 c^{16}+800973019723017815394186 c^{17}+469193552633410604736 c^{18}− \\
&\quad 113045237295103696 c^{19}−285973687077593472 c^{20}−2467933562943392 c^{21}−958175859968 c^{22}−14465138688 c^{23} \\
&\quad \frac{2(25+2 c)}{15(2+c)(19+c)(30+c)(51+c)(11+2 c)(−5+4 c)(114+7 c)(820+11 c)} d C_{4,3}^{4,4} C_{3,4}^{5} \\
C_{P,0,6}^{3,3} &\quad\left(-46122621038884440316682121335498240000000+3905461863522664679053857352748324352000000 c+ \\
&\quad 875512424720051537474247941845191577600000 c^2−9082417459342700591916444525630633761280000 c^3− \\
&\quad 163476804170660443812774221187634899110400 c^4+225536250748804812374344358005827906737600 c^5+ \\
&\quad 6559522735481336914011699173328357619520 c^6−1405052481120427031875031022714355283184 c^7− \\
&\quad 1119505301915854909988902979960671241528 c^8−2926016147658516862369783503020464532232 c^9− \\
&\quad 4585807851581531411735449019379019730 c^{10}−49190438118377185304205783171624374582 c^{11}− \\
&\quad 3816519673901649731574126957891146067 c^{12}−220025327779382505664583255505273888 c^{13}− \\
&\quad 9528267248125608129528869969956608 c^{14}−39098198645674568725469534151146 c^{15}− \\
&\quad 7353371892273528389128192703511 c^{16}−12077746188780000687583954832 c^{17}−110653075617955856920760060 c^{18}− \\
&\quad 260834346636895589151328 c^{19}+258142779926122049628784 c^{20}+4215398180665733980032 c^{21}+ \\
&\quad 3846020794174434752 c^{22}+211741709241022880 c^{23}+654042040887936 c^{24}+860675751936 c^{25} \\
&\quad \frac{3(19+c)(11+2 c)(−5+4 c)(22+5 c)(820+11 c)}{−329220+205197 c+439943 c^2+2052 c^3+28 c^4} d \end{align*}
\]

Table III: continued
\[
C_{6, 6}^{P, 10} (225532967564434471048212378368000000 \cdots - 42857521937668998377976174992640000 c^2 + 4287464219781413103921672912915200 c^3 + 133824511393646296312914683697769680 c^4 - 4683209857017419979997438001318800 c^5 - 199141660542332015607283726221744 c^6 - 667741717192240842125277328366428 c^7 - 119745235366313841073795707144234 c^8 - 13976704126513378382531980850322 c^9 - 1145605715563103047367396409525 c^{10} - 685480775341162289713234409538 c^{11} - 3059107974101996553714175784 c^{12} - 1029425846275718856126064434 c^{13} - 26207185688920356479887945 c^{14} - 5030045437743438776506790 c^{15} - 720215789570976947649063721600000 c^+ 1237494012890321944859761438144000 c^2 - 6522795307754095148734901959152000 c^3 - 29513188595189017672429385381920 c^4 + 585813437899245971854775789513472 c^5 + 498747323198824608694778178155032 c^6 + 1233357454353581103842431140526380 c^7 + 17503555444736 c^{10} - 9944782848 c^{21} + 1653350400198059419656974281683 c^9 + 110832656860882971512572109734 c^{10} + 541935801059760743299449996 c^{11} + 200238498730388936199395296 c^{12} + 5527733277081380412011291 c^{13} + 1144613319019157769966 c^{14} + 1758975177341395305738 c^{15} + 19645131058235981664 c^{16} + 153845035701353176 c^{17} + 7947107237490736 c^{18} + 2410313259668 c^{19} + 3164249088 c^{20})
\]

where

\[ d \]

\[
(32264132475165178970764800000 \cdots - 1492265914734533620778816000 c - 48446998713561085019584876800 c^2 - 7185818828253763119712188000 c^3 + 9158179120299264858864287440 c^4 + 41798128787440396145776152 c^5 + 8365375102215865574560012 c^6 + 99103729325671386646055030 c^7 + 7708501783947049512473573 c^8 + 4142347643359833847945 c^9 + 157999358415062911443 c^{10} + 43137397889268752533 c^{11} + 8423092342756373880 c^{12} + 115700518176813704 c^{13} + 1082823894002608 c^{14} + 6506451905744 c^{15} + 22311636480 c^{16} + 32288256 c^{17})
\]

Table III: continued