An Alternate Approach to Transition Potentials

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Abstract

We analyze transition potentials \( V(r) \rightarrow \alpha r^{-2} \) in non-relativistic quantum mechanics using the techniques of supersymmetry. For the range \(-\frac{1}{4} < \alpha < \frac{3}{4}\), the eigenvalue problem becomes ill-defined (since it is not possible to choose a unique eigenfunction based on square integrability and boundary conditions). It is shown that supersymmetric quantum mechanics (SUSYQM) provides a natural prescription for a unique determination of the spectrum. Interestingly, our SUSYQM based approach picks out the same "less singular" wave functions as the conventional approach, and thus provides a simple justification for the usual practice in the literature. Two examples (the Pöschl-Teller II potential and a two anyon system on the plane) have been worked out for illustrative purposes.
Transition potentials in non-relativistic quantum mechanics are defined by

\[
\lim_{r \to 0} r^2 V(r) = \alpha,
\]

with a finite nonzero \(\alpha\) \cite{1}. For \(\alpha > \frac{3}{4}\), the eigenvalue problem is well defined and can be solved by conventional means. For \(\alpha < -\frac{1}{4}\), both the independent solutions of the Schrödinger differential equation are square integrable at the origin and there is no mechanism available to select any specific linear combination.

However, for the intermediate range \(-\frac{1}{4} < \alpha < \frac{3}{4}\), transition potentials exhibit a very interesting behavior. This range corresponds to the so called "limit circle" case in the literature\cite{1}, and one has to specify another real number \(c = \lim_{r \to 0} \frac{\psi'(r)}{\psi(r)}\) in order to make the Hamiltonian formally self-adjoint. Here, the requirement of square integrability is not sufficient to determine the eigenvalues. Even with the stronger condition that the wave function must vanish at the origin \((c = \infty)\), the non-uniqueness still persists, albeit for a smaller range of \(\alpha\) given by \(-\frac{1}{4} < \alpha < 0\). For values of \(\alpha\) in this interval, eigenvalues are not well defined in the absence of further assumptions. This lack of uniqueness arises from the fact that both linearly independent solutions of the Schrödinger equation are well defined near the origin, and the condition of square integrability does not help us in discarding one of them. In such cases, it is customary\cite{1} to force the coefficient of the term with smaller power of \(r\) to vanish. This conventional approach of retaining the "less singular" wave function then leads to the determination of eigenvalues, and well defined eigenfunctions. Frank et al.\cite{1}, in their comprehensive study, justified the above choice through a regularization procedure. Specifically, the potential is first made regular in a small neighborhood of the singular point, with a radius \(\gamma\). After matching the solutions at the boundary and taking the limit \(\gamma \to 0\), the less singular wave function gets selected.

For any spherically symmetric potential in three dimensions, the \(\frac{1}{r^2}\)-term
arises from the angular momentum term in the Hamiltonian. The requirement of single valuedness constrains the coefficient of $\alpha$ to be positive. However, in two-dimensional quantum mechanical systems, angular momentum can take any real value, and thus the problem of indeterminacy is relevant to the quantum mechanics of anyons\cite{2}. The problem is also relevant for many known physically interesting one dimensional potentials (Rosen-Morse, Eckart, Pöschl-Teller, etc.), which have a $r^{-2}$ behavior at the origin.

In this paper, we provide an alternate way of determining eigenvalues in the critical range $-\frac{1}{4} < \alpha < \frac{3}{4}$. Our method is based on the supersymmetric approach to quantum mechanics\cite{3} (SUSYQM). The key idea is that in situations where the eigenfunctions of a potential $V_-$ are not unambiguously determined, the supersymmetric partner potential $V_+$ has no such problem. Thus, solving for $V_+$ first and then using the degeneracy relation, one can solve the eigenvalue problem for the potential $V_-$. Interestingly, we find that our approach leads to the same answer as the one stated in Ref.\cite{1}. Hence, this paper provides an alternate justification for the prescription of choosing the ”less singular” solution, which obviously works. Alternatively, it can be viewed that supersymmetry is properly realized in a space of eigenstates containing ”less singular” wave functions.

To be complete, we have given a brief review of SUSYQM. For a detailed description of SUSYQM, we refer the reader to Ref.\cite{4} and references therein. We use examples of the Pöschl-Teller potential, and that of a two anyon system to describe how SUSYQM provides a method to resolve the indeterminacy mentioned earlier.

SUSYQM is characterized by a superpotential $W$ and a pair of linear operators $A$ and $A^\dagger$:

$$A = \frac{d}{dr} + W(r), \quad A^\dagger = -\frac{d}{dr} + W(r).$$

(1)
Combining these operators, we can define two Hamiltonians,

\[ H_- = A^\dagger A = -\frac{d^2}{dr^2} + V_-(r), \]

\[ H_+ = AA^\dagger = -\frac{d^2}{dr^2} + V_+(r), \]

\[ V_\pm(r) = W^2(r) \pm W'(r). \]

We have set \( \hbar = 2m = 1 \). The potentials \( V_+ \) and \( V_- \) are called supersymmetric partner potentials. The eigenstates of the Hamiltonians \( H_- \) and \( H_+ \) are \( \psi_n^{(-)} \) and \( \psi_n^{(+)\dagger} \) respectively. \( \psi_n^{(\pm)} \) satisfy the eigenvalue equations

\[ H_- \psi_n^{(-)} = E_n^{(-)} \psi_n^{(-)}, \quad H_+ \psi_n^{(+)\dagger} = E_n^{(+)\dagger} \psi_n^{(+)\dagger}. \]

If the ground state of \( H_- \) has zero energy, i.e. \( E_0^{(-)} = 0 \), then supersymmetry is said to be unbroken and one has \( A \psi_0^{(-)} = 0 \). It then follows from eq. (5) that

\[ \psi_0^{(-)} = \exp \left[-\int r W'(r)dr'\right]. \]

For unbroken supersymmetry, one needs \( \psi_0^{(-)} \) or \( \frac{1}{\psi_0^{(-\dagger)}} \) to be an acceptable wave function, i.e. it must be quadratically integrable and satisfy correct boundary conditions. For a finite domain, the wave function must vanish at the end points. For a normalizable well-defined \( \psi_0^{(-)} \), one gets the energy degeneracy relation

\[ E_{n+1}^{(-\dagger)} = E_n^{(+)} \]

The corresponding eigenfunctions of \( H_- \) and \( H_+ \) are related by

\[ \psi_{n+1}^{(-)} = A^\dagger \psi_n^{(+)\dagger}. \]

The applicability of SUSYQM to lift the ambiguity in the determination of eigenvalues and eigenfunctions that plague transition potentials can be appreciated from the following discussion. If the superpotential \( W(r) \) is given by \(-\frac{l+1}{r}\) near \( r = 0 \), then potentials \( V_-(r) \) and \( V_+(r) \) are
of the form \( \frac{l(l+1)}{r^2} \) and \( \frac{(l+1)(l+2)}{r^2} \) near \( r = 0 \). Their wave functions are given by linear combinations \([c_1 r^{-l} (1 + O(r)) + c_2 r^{l+1} (1 + O(r))]\) and \([c'_1 r^{-l-1} (1 + O(r)) + c'_2 r^{l+2} (1 + O(r))]\) respectively. For \(-\frac{1}{2} < l < \frac{1}{2}\), which corresponds to the problematic range \(-\frac{1}{4} < \alpha < \frac{3}{4}\) for \( V_-\), both solutions are square integrable. For the potential \( V_+ \) one has the range \(-\frac{1}{2} < l < \frac{1}{2}\), which is well defined and one can determine a unique wave function for \( V_+ \). A proper wave function (and from it the eigenvalues) for \( V_- \) is obtained by applying the operator \( A^\dagger \) on the solution of \( V_+ \) as is shown in eq.(8). In what follows, we will use two examples to explicitly describe the working of our approach.

Examples:
(a) Pöschl-Teller Potential

Let us consider the Pöschl-Teller II superpotential

\[ W = A \tanh r - B \coth r \quad (0 < r < \infty). \]  

For \( A > B \) the above superpotential corresponds to a case of unbroken SUSY. The corresponding supersymmetric partner potentials are given by,

\[
V_-(r) = -A(A+1) \text{sech}^2 r + B(B-1) \text{cosech}^2 r + (A-B)^2 \\
V_+(r) = -A(A-1) \text{sech}^2 r + B(B+1) \text{cosech}^2 r + (A-B)^2.
\]  

Without loss of generality we will assume \(-\frac{1}{2} < A < \infty\) and \(\frac{1}{2} < B < \infty\). One should note here that \(r\)-dependent parts of the above potentials are such that one could obtain \( V_+(r) \) from \( V_-(r) \) simply by replacing \( A \) by \( A - 1 \), and \( B \) by \( B + 1 \). To clearly see the ambiguity in the eigenvalue problem, we proceed with the analysis of the Schrödinger equation. The time independent Schrödinger equation for \( V_-(r) \) is given by

\[
\frac{d^2 \psi(-)(r)}{dr^2} + \left[ E + A(A+1) \text{cosh}^2 r - B(B-1) \text{sinh}^2 r - (A-B)^2 \right] \psi(-)(r) = 0.
\]  

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With a change of variables \( y = \cosh^2 r \) and \( \psi(y) = y^{-\frac{1}{2}A} (y - 1)^{\frac{1}{2}B} v(y) \), and also replacing \( y \) by \( 1 - y \), eq.(11) can be cast in the form of a hypergeometric equation, i.e.

\[
y(1 - y) v'' + \left[ \left( \frac{1}{2} - A \right) - (1 - A + B) y \right] v' - \frac{1}{4} E v = 0. \tag{12}
\]

The general solution is:

\[
\psi^{(-)}(r) = \cosh^{-A} r \sinh^B r \left[ c_1 F \left( a', b', c'; -\sinh^2 r \right) + c_2 \sinh^{(1-2B)} r F \left( a' + 1 - c', b' + 1 - c', 2 - c'; -\sinh^2 r \right) \right], \tag{13}
\]

where the constants \( a' \), \( b' \), and \( c' \) are given by

\[
\begin{align*}
a' &= \frac{1}{2} \left( B - A + Q \right), \\
b' &= \frac{1}{2} \left( B - A - Q \right), \\
c' &= B + \frac{1}{2} \\
Q &= \sqrt{(B - A)^2 - E}. \tag{14}
\end{align*}
\]

Near the point \( r \sim 0 \), the solution reduces to

\[
\psi^{(-)} \xrightarrow{r \to 0} \left[ c_1 r^B (1 + O(r)) + c_2 r^{(1-B)} (1 + O(r)) \right]. \tag{15}
\]

Normalizability requires that \( \psi^{(-)} \) be less singular than \( \frac{1}{\sqrt{r}} \) near origin. Thus for \( B \geq \frac{3}{2} \), the wave function becomes non-normalizable unless \( c_2 = 0 \). With \( c_2 = 0 \), a subsequent constraint coming from the requirement of the vanishing of the wave function at infinity (which is demanded by the normalizability) suffices to determine eigenvalue \( E \) in terms of the parameters \( A \) and \( B \). However, if \( \left( \frac{1}{2} < B < \frac{3}{2} \right) \), both terms on the right hand side of eq.(15) are normalizable and hence no condition is imposed upon the coefficients. In such cases, we solve the eigenvalue problem for \( V_+ \) instead, which has a well defined set of eigenfunctions and eigenvalues. Then using the operators \( A \) and \( A^\dagger \), we can determine eigenfunctions and eigenvalues of \( V_- \). As an explicit
example, let us choose $B = \frac{3}{4}$ and determine corresponding eigenvalues of the Hamiltonian $H_+$. The wave function for the Hamiltonian $H_-$ has the following form near the origin:

$$\psi^(-) \xrightarrow{r \to 0} \left[c_1 r^{3/4} (1 + O(r)) + c_2 r^{1/4} (1 + O(r))\right]. \quad (16)$$

We see that both the terms of eq.(16) are well defined near $r \sim 0$, and hence no constraints are placed on their coefficients from requiring normalizability around the origin. The wave function for the Hamiltonian $H_+$ is given by

$$\psi^+ \xrightarrow{r \to 0} \left[c_1 r^{3/4} (1 + O(r)) + c_2 r^{-3/4} (1 + O(r))\right]. \quad (17)$$

Clearly, normalizability of $\psi^+$ requires that we set $c_2 = 0$. To determine the eigenvalues of $H_+$, we have to study the behavior at infinity, and for that one uses an alternate asymptotic form of the hypergeometric function:

$$F(a,b,c;z) \xrightarrow{z \to \infty} \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a} + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}. \quad (18)$$

This leads to

$$\psi^+ \xrightarrow{r \to \infty} \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}e^{-Qr} + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}e^{+Qr} \quad (19)$$

where the constants $a$, $b$, and $c$ are given by

$$a = \frac{1}{2} (B - A + 2 + Q),$$
$$b = \frac{1}{2} (B - A + 2 - Q), \quad \text{and}$$
$$c = B + \frac{1}{2}. \quad (20)$$

The second term on the R.H.S. of eq.(19) must vanish to have a well defined bound state. This can be achieved if $a$ or $(c-b)$ is equal to a negative number (say $-k$). If $a = -k$, then the eigenvalues are given by

$$E_k^(-) = (A - B)^2 - [A - B - 2k - 2]^2 \quad k = 0, 1, \ldots, n.
The integer \( n \) gives the number of bound states that the potential will hold, and is related to the parameters \( A \) and \( B \). It is the largest integer satisfying \( A - B - 2 > 2n \). Under this condition, one can show that \( exp \left[ - \int r W(r')dr' \right] \) is a well defined function, and hence we have a supersymmetric situation. The eigenvalues for the Hamiltonian \( H_- \) will be the same as that for \( H_+ \), except that \( H_- \) will have an additional state (ground state) with zero energy. The eigenfunctions of \( H_+ \) are given by

\[
\psi^{(+)}(r) = (\sinh r)^{(1+\frac{3}{4})} (\cosh r)^{-(A-1)} P_{\frac{1}{2}}^{(1+\frac{3}{4}, -(A-1)-\frac{1}{2})}(\cosh 2r).
\]

Now the eigenfunctions of the Hamiltonian \( H_- \) will be given by applying the operator \( A^\dagger \) [defined in eq.(1)] on the function \( \psi^{(+)} \). Near the origin \( \psi^{(+)} \) is given by

\[
\psi^{(+)}(0, r) \sim r^\frac{1}{4}.
\]

Now operating \( A^\dagger \) on \( \psi^{(+)} \) lowers the power of \( r \) by unity, and hence

\[
\psi^{(-)}(0, r) \sim r^\frac{3}{4}.
\]

Comparing this above expression with eq.(16), we see that SUSYQM automatically chooses the term with higher power of \( r \), which is consistent with the prescription of Ref.[1]. Hence the eigenvalues obtained will also be the same. Thus this method provides a justification for the usual practice of dropping the term with lower power of \( r \) in case of ambiguity.

Instead, if the second condition holds i.e. \( c - b = -k \), the eigenvalues are given by \( E_k^{(-)} = (A - B)^2 - [A + B + 2k + 1]^2 \quad k = 0, 1, \cdots, n \). The condition on \( A \) for \( n \)-bound states in the second case is given by \( A < -B - 2n - 1 \), which can not be satisfied as we have assumed (without loss of generality) \( -\frac{1}{2} < A < \infty \) and \( \frac{1}{2} < B < \infty \).

(b) **Anyons in a Spherically Symmetric Potential**

Here we consider a system of two anyons and proceed along similar lines as above. The motion can be divided into center of mass motion and the
dynamics of the relative coordinate. The two body Hamiltonian is given by
\[ H = - \frac{1}{2m} \left\{ \partial_{1i} - e A_i (\vec{r}_1 - \vec{r}_2) \right\}^2 \]
\[ - \frac{1}{2m} \left\{ \partial_{2i} - e A_i (\vec{r}_2 - \vec{r}_1) \right\}^2 + V (|\vec{r}_1 - \vec{r}_2|). \quad (21) \]

The vector potential is
\[ A_i = \frac{\Phi \epsilon_i j^j}{2\pi r^2}; \quad \Phi = \frac{\theta}{e}, \quad (22) \]
where \( \theta \) is the well known statistics parameter. Now defining the center-of-
mass and relative coordinates as
\[ \vec{R} = \frac{1}{2} (\vec{r}_1 + \vec{r}_2), \quad \vec{r} = (\vec{r}_1 - \vec{r}_2), \]
the Hamiltonian can be written as
\[ H_{c.m} = - \frac{1}{4m} \nabla_R^2, \quad (23) \]
\[ H_{rel} = - \frac{1}{2\mu} \left\{ \partial_i - e A_i \right\}^2 + V (|\vec{r}|), \quad (24) \]
where \( \mu \) represents the reduced mass of the system. The radial part of the
Hamiltonian for the relative coordinate is given by
\[ H_{rel} = - \frac{1}{2\mu} \left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \frac{1}{r^2} \left( m + \frac{\theta}{\pi} \right)^2 \right] + V (r). \quad (25) \]
The Schrödinger equation for the relative coordinate is then given by
\[ - \left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \frac{1}{r^2} \left( m + \frac{\theta}{\pi} \right)^2 \right] \psi + (V (r) - E) \psi = 0, \quad (26) \]
where we have set \( 2\mu = 1 \). Substituting \( \psi = \frac{\phi}{\sqrt{r}} \), we get
\[ - \phi'' + \left[ \left( \nu^2 - \frac{1}{4} \right) - E \right] \phi = 0, \quad (27) \]
where \( \nu = \frac{(m + \frac{\theta}{\pi})^2}{r^2} \). Eq.(27) can now be interpreted as an one dimensional
equation where the domain of the variable \( r \) is given by \( 0 \leq r < \infty \). Now
the important question is the boundary condition. This question has been recently analyzed by Roy and Tarrach,\[7\] who conclude that more general boundary conditions $\phi'(0) = c\phi(0)$, $c \neq 0$, are not allowed because the boundary condition break supersymmetry. It is interesting to observe that in the three-dimensional monopole problem, the above mentioned boundary conditions are also extremely important[8]. If we ask for the overlap of the two particles to be zero, we require $\psi \to 0$ as $r \to 0$. This is equivalent to saying that the configuration space has been reduced to $R^2 \times (R^2 - \{0\})/Z_2$. For $\phi$, that would imply that it goes to zero faster than $\sqrt{r}$.

However, if we only stipulate a need of square integrability, it implies that $\phi$ be less singular than $\frac{1}{\sqrt{r}}$. The solutions of eq.(27) are then of the form $r^{l+1}$ and $r^{-l}$, where $l$ is given by $l(l+1) = (\nu^2 - \frac{1}{4})$. One can show that for $l > \frac{1}{2}$ or $l < -\frac{1}{2}$, only one of the above two solutions is square integrable near the origin, and this leads to the unambiguous determination of eigenvalues and eigenfunctions. However, if $-\frac{1}{2} < l < \frac{1}{2}$ then both solutions vanish at the origin, and are also square integrable. Hence it is not possible to choose one over the other. For the anyon problem one starts with a superpotential of the form \[ W(r) = \frac{l+1}{r} + f(r), \] where $f(r)$ has to be suitably chosen to give the required spherically symmetric potential. One can easily see that in the partner Hamiltonian the singular term is of the form $\frac{(l+1)(l+2)}{r^2}$ and hence just like the Pöschl-Teller case there will be no ambiguity in this sector. As has been worked out in the previous example, the application of the degeneracy theorem will then give the less singular wave function in the $H_-$ sector. Thus, we find that our SUSYQM based formalism gives a clear cut way of finding the eigenvalues and eigenfunctions of transition potentials in the region of ambiguity. Also, the eigenfunctions turn out to be the same ”less singular” type that is commonly chosen in the literature[4].

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