Rational Homotopy Type Of Complements Of Submanifold Arrangements

Alexander Zakharov

Abstract

In this work we will provide an explicit model for cdga controlling the rational homotopy type of the complement to a smooth arrangement $X - \bigcup_i Z_i$ in a smooth compact algebraic variety $X$. This generalizes the corresponding result of J. Morgan in case of a divisor with normal crossings to arbitrary smooth arrangements. The model is given in terms of the arrangement $Z_i$.

Introduction

0.1 Main results

Definition 0.1.1. An arrangement $(X, Z_\ast)$ is a smooth compact algebraic variety $X$ over $\mathbb{C}$ and a collection \{ $Z_i \subset X, i \in N$ \} of subvarieties for an ordered index set $N$ such that the set theoretical intersections $Z_I = \bigcap_{i \in I} Z_i, I \subset N$ and $Z_\emptyset = X$ are smooth $C^\infty$-submanifolds. (In other words, the corresponding scheme theoretic intersections with the reduced scheme structure are smooth subschemes.) The complement of the arrangement $(X, Z_\ast)$ is the variety $U = X - \bigcup_i Z_i$.

We study the topology of the arrangement $(X, Z_\ast)$.

Definition 0.1.2. A collection \{ $D_i \subset X$ \} is called an NCD if $D_i$ are smooth and $\bigcup D_i$ is a divisor with normal crossings.

Consider an arrangement $(X, D_\ast)$, where $D_\ast$ is an NCD. Let $U = X - \bigcup_i D_i$ and $j : U \to X$ be the inclusion.

In order to simplify sign issues and making the product structure clear, it will be convienit to use variables $d\nu_I = d\nu_{i_1} \wedge \ldots \wedge d\nu_{i_k}, I = \{i_1, \ldots, i_k\}$ for $i_1 < \ldots < i_k$ viewed as monomials in the Grassmann algebra of degree $-|I|$.

The default coefficient ring for cohomology groups is always $\mathbb{Q}$ and will usually be ommited from notation.

The following theorem is due to J. Morgan [3]:

Theorem. Consider the Leray spectral sequence $E_r$ associated with the inclusion $j : U \to X$ and the constant sheaf $\mathbb{Q}$. Then $E_2^{pq} = H^p(X, R^q j_* \mathbb{Q}) \simeq \bigoplus_{I : |I| = q} H^p(D_I)$ with the differential $d_2$ equal to the Gysin map form a cdga. There is a quasi-isomorphism of dg-algebras:

$$E_\ast^2 \simeq \bigoplus_I H^\ast(X, X - D_I) \cdot d\nu_I \sim C_{\text{sing}}^\ast(U; \mathbb{Q})$$

with the differential on the LHS written as $d_2 = \sum_{i, I \in I} g_{I - \{i\}, I} \otimes \iota_i$

Here $\iota_i d\nu_j = \delta_{ij}$ and $g_{I - \{i\}, I} : H^\ast_{D_I}(X) \to H^\ast_{D_{I - \{i\}}}(X)$ are the natural homomorphisms induced by the inclusion $D_{I - \{i\}} \to D_I$ for $i \in I$.

The quasi-isomorphism of dg-algebras in the theorem is non-canonical, though there is a natural one after tensoring with $\mathbb{C}$.

*xaxa3217 ( •_•) gmail.com
The theorem implies $E_3 = E_{\infty}$. Note that if $(X, Z_*)$ is a smooth arrangement with $Z_i$ of codimension $> 1$, then the Leray spectral sequence does not degenerate at $E_3$. An example is $(CP^2, pt)$. Remarkably the framework developed by Morgan allows us to prove the following generalization for an arbitrary smooth compact algebraic arrangement:

**Theorem 0.1.1** (On the homotopy type of the complement, theorem 3.4.1 in the main text). For a smooth compact algebraic arrangement $(X, Z_*)$ there is a second quadrant multiplicative spectral sequence $E^{pq}_1$ such that:

1. We have $E^{pq}_1 = \bigoplus_{|I|=p} H^{q}_{Z_I}(X) \cdot d_{1, I}$.

2. The differential $d_{1} = \sum_{i, j \in I} g_{I - \{i\}, I} \otimes e_{i, j}$, where $g_{I, I'}$, $I' \subset I$ are the natural homomorphisms:

   $$H^q_{Z_i}(X) \rightarrow H^q_{Z_{I'}}(X)$$

   induced by the inclusions $Z_i \rightarrow Z_{I'}$.

3. The multiplication

   $$H^q_{Z_i}(X) \cdot d_{1} \otimes H^r_{Z_j}(X) \cdot d_{1} \rightarrow H^{q+r}_{Z_{I'}}(X) \cdot d_{1} \wedge d_{1}$$

   with the usual sign rule makes $(E^{pq}_1, d_1)$ into a cdga.

4. There is a natural quasi-isomorphism of cdga's:

   $$E^{pq}_1 \otimes \mathbb{C} \sim C^{\ast}_{\text{sing}}(X - \cup i Z_i; \mathbb{C})$$

   In particular $E_2 = E_{\infty}$.

5. There is a non canonical quasi-isomorphism of the rational homotopy type of $X - \cup_i Z_i$ with $E^{pq}_1$.

   In the case of an NCD-complement this spectral sequence is related to the above Leray spectral sequence via décalage.

   The proof is based on the notion of a mixed Hodge diagram introduced by Morgan in [3]. It is a cdga version of the mixed Hodge complex invented by Deligne [11]. The most useful property of a mixed Hodge diagram is that it has a cofibrant replacement (see also [14]), which produces a mixed Hodge structure on the minimal model of cdga forming the Hodge diagram. In particular its weight filtration $W$ has a natural Deligne splitting (proposition 2.2) over $\mathbb{C}$, thus providing a natural quasi-isomorphism of the corresponding $E_1$-term with the minimal algebra. Then a general result due to Morgan and Sullivan (theorem 12.2.3) implies the existence of a non-canonical quasi-isomorphism over $\mathbb{Q}$.

   Let $j : X - \cup Z_i \rightarrow X$ be the inclusion of the complement. The sheaf $j_!\mathcal{Q}$ has a natural resolution in terms of arrangement, called the Mayer-Vietoris complex (definition 5.1.5). The initial aim of this work is to construct a Hodge diagram structure on this resolution. Once this is done, the main theorem follows.

   It is natural to look for a similar model which better reflects the combinatorics of the arrangement. One way to proceed is to consider for example the corresponding intersection poset. The notion of a locally geometric lattice was applied in [6] to describe the cohomology ring of some arrangement complements in terms of the Orlik-Solomon algebra and the cohomology of smooth intersections. We will prove here similar results without requiring the intersections to be “clean”, i.e. looking locally like a linear subspace arrangement. Further we make weaker assumptions on the poset.

   Let $(L, \leq)$ be an arrangement poset (definition 2.2.1) on $X$ and $M$ be an OS-algebra (definition 2.5.2). Let $U = X - \bigcup_{x \in L} L_x$ be the complement. An example of $(L, \leq)$ is the cubical poset $\mathcal{C}$ of all intersections as in the theorem above. The minimal element $L_0 \in L$ corresponds to $X$ and $x \leq y \in L$ implies $L_x \supset L_y$ in $X$. We require that $(L, \leq)$ is graded by the rank function $r$ such that $r(0) = 0$ and $r(x) = r(y) + 1$ if $x$ lies over $y$. The last condition is denoted by $x :> y$.

   The following result is similar to theorem 1.3 in [6]:

**Theorem 0.1.2** (On the cohomology of the complement, theorem 3.5.3). There is a multiplicative spectral sequence of pure Hodge structures with $E^{pq}_1$ of weight $q$ such that:

1. We have $E^{pq}_1 = \bigoplus_{x \in L, r(x) = -p} H^q(X, X - L_x) \otimes M_x$.
2. The differential \( d_1 = \sum_{y < x} g_{yx} \otimes \partial_{yx} \) is the sum of its \( L \)-components, where \( g \) is the natural homomorphism and \( \partial_{yx} \) are the structure maps on \( M \).

3. We have \( E_2 = E_\infty \).

4. There is a natural multiplicative isomorphism of Hodge structures: \( E_0 \simeq Gr_q^{Dec} H^{p+q}(U) \).

5. The above isomorphism provides a non-natural isomorphism of algebras \( E_2^* \simeq H^*(U; \mathbb{Q}) \) and a natural isomorphism of the complexifications \( E_2^* \otimes \mathbb{C} \simeq H^*(U; \mathbb{C}) \).

**Remark 0.1.1.** Here \( Gr_q^{Dec} H^n(U, \mathbb{C}) \) denotes the usual weight \( q \)-piece of mixed Hodge structure on \( H^n(U, \mathbb{C}) \), see definition [12.1] for \( Dec \ W \).

The spectral sequence is called the Mayer-Vietoris spectral sequence of the arrangement poset \( (L, \leq) \) on \( X \). We will refer to the spectral sequence in the main theorem as the cubical Mayer-Vietoris spectral sequence.

**Remark 0.1.2.** The above theorem is parallel to the fact that the Leray spectral sequence has a mixed Hodge structure (see [13]). Both results agree in the mentioned case of an open embedding of the complement to an NCD arrangement. In general case the weight condition implies that the dg-algebra \( E^*_r, r > 0 \) can be considered as a pure Hodge complex (a Hodge diagram in fact). Considering \( H^*(U) \) as a mixed Hodge complex one may rephrase the statement of the theorem by saying that there is a natural isomorphism of pure Hodge complexes \( E^*_r \simeq Gr^W H^*(U) \). We emphasize, however, that \( H^*(U) \) itself is rarely pure and one cannot, in general, recover the mixed Hodge structure \( H^*(U) \) by our model.

Further, if the arrangement poset \( (L, \leq) \) is a geometric lattice we are able to descend the cubical model of the main theorem to an \( (L, \leq) \)-model:

**Theorem 0.1.3** (The lattice model for the homotopy type of the complement, theorem [5-11]). If the arrangement poset \( (L, \leq) \) is a geometric lattice, then the \( E_1 \)-term of the above spectral sequence naturally describes the \( \mathbb{C} \)-homotopy type of the complement. As above, the rational homotopy type is non canonically quasi-isomorphic to \( E_1 \).

We will prove this by providing an explicit quasi-isomorphism of the lattice model \( E_1(L) \) with the cubical model \( E_1(\mathbb{C}) \).

### 0.2 Applications

As an application we prove the following results. The first one is a generalization of formalism of the complement of a hyperplane arrangement which can be traced back to works of Arnold, Brieskorn, Orlik-Solomon and others, see [7,8]. Consider an arrangement of vector subspaces in \( \mathbb{C}^n \). Let \( (L, \leq) \) be the corresponding intersection poset. Assume that \( (L, \leq) \) is a geometric lattice. Here we give an alternative proof of the following theorem by Feichter and Yuzvinsky [11].

**Theorem 0.2.1** (theorem [5-11]). The complement \( \mathbb{C}^n = \bigcup_{x>0} L_x \) is formal.

The next theorem generalizes the Kriz-Totaro model (see [10]) for the rational homotopy type of the configuration space of \( n \)-tuples of points on a smooth compact algebraic variety \( M \). Namely, for any finite graph \( G \) one can consider the chromatic configuration space \( F(M, G) \) of points \( x_i \in M \) labeled by vertices \( i \in V(G) \) with the condition \( x_i \neq x_j \) if \( (ij) \in E(G) \). Let \( \Delta_{ab} \subset M^n \) be the diagonal \( x_a = x_b \) and \( \Delta_{ab} \) the corresponding cohomology class.

**Theorem 0.2.2** (theorem [5-11]). The rational homotopy type of \( F(M, G) \) is quasi-isomorphic to the cdga equal to the quotient

\[ H^*(M^n) \otimes \Lambda^* \langle \Delta_{ab}, (ab) \in E(G) \rangle / I \]

where \( \deg(\Delta_{ab}) = 2\dim M - 1 \) and \( I \) is the differential graded ideal generated by the relations

1. \( \partial(\Delta_{i_1 i_2} \wedge \ldots \wedge \Delta_{i_k i_{k+1}}) = 0 \), for each cycle \( (i_1, \ldots, i_k = i_1) \) in \( G \).
2. \( p^*_b u \cdot \Delta_{ab} = p^*_b u \cdot \Delta_{ab}, u \in H^*(M) \).

The differential \( d \) is given by \( d\Delta_{ab} = [\Delta_{ab}] \).
0.3 Organization of the paper

The work is organized as follows. In the first section we recall basic facts about Hodge complexes, define Hodge diagrams and state the theorem of Morgan on the existence of the minimal model and the corresponding quasi-isomorphism with $E_r$. Then, in the second section we develop the necessary machinery to treat the general case of our theorem beyond the cubical lattice. There we introduce $(L, \leq)$-chain algebras and define Orlik-Solomon algebras in this context. In the third section we construct the Mayer-Vietoris resolution and corresponding Hodge complexes and Hodge diagrams. In the last section we will prove the stated applications.

0.4 Acknowlegements

I would like to thank my scientific advisor Alexey Gorinov for posing the problem, inspiring support and fruitful suggestions, not to mention comments, remarks and criticism, without which the final work would not be possible. The author is also grateful to Dmitry Kaledin for reviewing a draft of the paper.

The study has been partially supported by the Laboratory Of Algebraic Geometry And Its Applications funded within the framework of the HSE University Basic Research Program.

1 Preliminaries on Hodge Complexes

If $W$ is a decreasing filtration on object $K$, then its $k$-th piece is denoted by $W^K = W_{\leq k}K$, where $W_*$ denotes the corresponding increasing filtration. Likewise we agree to use this notation for associated graded objects, e.g. $Gr^W_k K = W^k / W^{<k} K = Gr^{W}(K, W) = Gr_k(K, W)$ etc. As usual, all cohomological spectral sequences are written with respect to the corresponding decreasing filtrations.

1.1 Filtrations

Here we recall some general preliminaries which can be found in [2],[1]. The reader can skip this subsection and come back latter.

**Lemma 1.1.1.** Let $A,B,C \subset E$ be a subobjects of $E$ in an abelian category, then $(A + B) \cap (B + C) = (A + B) \cap C + B$

**Proof.** Passing to a category of modules via Mitchells theorem, one can prove the equality in an element-wise manner: if $a + b = b' + c \in (A + B) \cap (B + C)$, then $a + b = (a + b' - b') + b' \in (A + B) \cap C + B$. \qed

Let $K^*$ be a complex with filtrations $W,F$. Define

$$Z^{pq}_F := Ker[W^pK^* \to K^*/W^{p+r}K^*]$$

$$B^{pq}_F := dW^{-r+1} + W^{p+1}$$

We note that:

$$dZ^{pq}_F \subset Z^{p+r,q-r+1}_F$$

$$dB^{pq}_F \subset B^{p+r,q-r+1}_F$$

There is a corresponding spectral sequence

$$W^r E^{pq} = \frac{Z^{pq}_F}{Z^{pq}_F \cap B^{pq}_F} \cong \frac{Z^{pq}_F + B^{pq}_F}{B^{pq}_F}$$

with differential $d_r : E^{pq}_r \to E^{p+r,q-r+1}_r$. One has a natural isomorphism:

$$H^*(E_r, d_r) \cong E_{r+1}$$

The non trivial remark is that the realization of $E^{pq}_{r+1}$ as a sub-quotient of $E^{pq}_r$ is neither consistent with the two given descriptions of $E_r$. Namely, $Z_{r+1} \subset Z_r$ and $B_{r+1} \supset B_r$, but it is not true that $Z_{r+1} \cap B_{r+1} \supset Z_r \cap B_r$ or $Z_{r+1} + B_{r+1} \subset Z_r + B_r$.

The two possible ways of identifying $W^r E^{pq}_r$ result in two different filtrations induces by $F$. The first is called $F_{dir}$, the second is $F_{dir}$. On the other hand, since $E_{r+1} = Ker[E_r \xrightarrow{d} E_r]/Im[d_r]$ one can induce a filtration $F_{ind}$ on $E_{r+1}$ by that of $E_r$. 


Lemma 1.1.2.  

\( F^i_{\text{dir}} E_r K^* = \text{Im}[E_r F^i \rightarrow E_r K^*] \)

(b) \( F^i_{\text{dir}} E_r K^* = \text{Ker}[E_r K^* \rightarrow E_r K^*/F^i] \)

Proof. (a) By definition the LHS is

\[ \text{Im}[F^i \cap Z^p_r \rightarrow E^p_r] = \frac{Z^p_r}{Z^p_r \cap B^p_r} \]

On the other hand \( \text{Im}[E_r F^i \rightarrow E_r] \) is equal to

\[ \text{Im}[Z^p_r(F^i) \rightarrow Z^p_r] \]

but

\[ Z^p_r(F^i) = \text{Ker}(F^i \cap W^p \rightarrow K^*/W^{p+r}) = Z^p_r \cap F^i \]

(b) We have:

\[ F^i_{\text{dir}} E_r = \text{Im}[F^i \cap (Z_r + B_r) \rightarrow E_r = \frac{Z_r + B_r}{B_r}] \]

On the other hand:

\[ \text{Ker}[E_r K^* \rightarrow E_r K^*/F^i] = \text{Im}[\text{Ker}(Z_r + B_r \rightarrow \frac{K^*/F^i}{B_r(K^*/F^i)}) \rightarrow E_r] = \]

\[ = \text{Im}[(Z_r + B_r) \cap (B_r + F^i) \rightarrow E_r] = \text{Im}[(Z_r + B_r) \cap F^i + B_r \rightarrow E_r] = \]

\[ = F^i_{\text{dir}} E_r \]

Proposition 1.1. For the filtrations introduced above the following holds:

i) \( F_{\text{dir}} \subset F_{\text{ind}} \subset F_{\text{dir}}^* \)

ii) On \( E_0, E_1 \) all three filtrations coincide.

Proof. Note that by definition the filtrations are equal on \( E_0 K^* = WGr K^* \). Further, since \( B_1 \subset Z_1 \), one has \( F^i_{\text{dir}} E_1 = F^i_{\text{dir}} E_1 \).

We will proceed by induction. Assume \( F^i_{\text{dir}} E_r \subset G^i E_r \), for a filtration \( G \) on \( E_r \). By the previous

\[ F^i_{\text{dir}} E_{r+1} = \text{Im}[(E_r F^i \rightarrow E_r) \rightarrow E_{r+1}] \]

and by definition

\[ G^i E_{r+1} = \text{Im}[(E_r G^i E_r \rightarrow E_r) \rightarrow E_{r+1}] \]

then, since \( \text{Im}[E_r F^i \rightarrow E_r] \subset G^i E_r \), one gets:

\[ F^i_{\text{dir}} E_{r+1} \subset G^i E_{r+1} \]

Putting \( G = F_{\text{ind}} \) proves the claim.

We have \( G^i E_r \subset F^i_{\text{dir}} E_r \). Then:

\[ G^i E_{r+1} = \text{Im}[(E_r G^i E_r \rightarrow E_r) \rightarrow E_{r+1}] \]

\[ F^i_{\text{dir}} E_{r+1} = \text{Ker}[E_{r+1} K^* \rightarrow E_{r+1} K^*/F^i] \]

Since \( E_{r+1} \) is a homology of \( E_r \) and \( E_r K^* \rightarrow E_r K^*/F^i \) is a morphism of complexes, from

\[ G^i E_r \subset F^i_{\text{dir}} E_r = \text{Ker}[E_r K^* \rightarrow E_r K^*/F^i] \]

one immediately have:

\[ G^i E_{r+1} \subset F^i_{\text{dir}} E_{r+1} \]

\[ \square \]
Lemma 1.1.3 (on two filtrations). If \( \forall r \in 0, \ldots, r_0 \) the differential \( d_r \) is strict, then for all \( r' \in 0, \ldots, r_0 + 1 \):

i \( F_{dir} = F_{ind} = F_{dir'} \) on \( E_{r'} \)

ii the sequence \( 0 \to E_r F^i \to E_{r'} K^* \to E_{r'} K^*/F^i \to 0 \) is exact

iii \( F_{Gr^i} E_{r'} \cong E_{r'} W Gr^i K^* \)

Proof. We will proceed by induction. Assume for a given \( r \) the above sequence is exact and all three filtrations coincide. Let us prove that \( E_{r+1} F^i \to E_{r+1} K^* \) is an inclusion with the image equal to \( F_{ind} E_{r+1} \). Consider:

\[
\frac{\text{Ker}[F^i E_r \to E_r]}{\text{Im}[d_r]} = \frac{\text{Ker}[E_r F^i \to E_r F^i]}{\text{Im}[d_r]} = \frac{\text{Im}[d_r]}{\text{Im}[d_r]}
\]

The first equality holds under induction hypothesis, hence the second map is an inclusion since \( d_r \) is \( F \)-strict. Its image in \( E_{r+1} \) is \( \text{Im}[\text{Ker} F_{ind} E_r \to E_r] \to E_{r+1} \), which is by definition \( F_{ind} E_{r+1} \). Thus \( E_{r+1} F^i \to E_{r+1} K^* \) is an inclusion with image \( F_{ind} E_{r+1} \), hence \( F_{dir} = F_{ind} \) on \( E_{r+1} \) as required. To show surjectivity of \( E_{r+1} \to E_{r+1} K^*/F^i \), note that

\[
0 \to F^i E_r \to E_r K^* \to E_r K^*/F^i = E_r K^*/F^i E_r K^* \to 0
\]

is exact, then taking cohomology w.r.t \( d_r \) produces a long exact sequence, hence

\[
F_{ind} E_{r+1} \to \text{Ker}[E_{r+1} \to E_{r+1} K^*/F^i]
\]

is onto, thus \( F_{ind}^i = F_{dir} \) on \( E_{r+1} \). Since \( F^i E_{r+1} \to E_{r+1} \) is into, the same long exact sequence implies also that

\[
E_{r+1} \to E_{r+1} K^*/F^i
\]

is onto.

Finally, since \( E_r F_{r+1} \to E_r F^i \) is equivalent to \( F^{i+1} E_r \to F^i E_r \), one gets a natural isomorphism:

\[
F_{Gr^i} E_r K^* = E_r F_{Gr^i} K^*
\]

\[\square\]

Corollary 1.1.1. If \( d_r \) is \( F_{ind} \)-strict for all \( r \geq 0 \), then \( F E_1 = F E_\infty \)

Proof. By the previous \( F_{Gr^i} E_r K^* = E_r F_{Gr^i} K^* \), in particular for \( r = \infty \) one has:

\[
F_{Gr^i} F W Gr^i H^* (K^*) = F_{Gr^i} W E_\infty = W E_\infty F_{Gr^i} K^* = W Gr^i F E_1 K^*
\]

hence \( F E_\infty K^* = F E_1 K^* \).

\[\square\]

1.2 Mixed Hodge Complexes and Hodge Diagrams

Definition 1.2.1. A data \( (C^*_Q, (C^*_C, F), C_Q \otimes \mathbb{C} \sim C_C) \), where

(i) \( C^*_Q \) is a complex over \( \mathbb{Q} \)

(ii) \( C_Q \otimes \mathbb{C} \sim C_C \) is a quasi-isomorphism.

called a pure Hodge complex of weight \( m \) if

(i) \( F E_1 C^*_C = F E_\infty C^*_C \)

(ii) \( (H^n (C^*_C), F) \) defines a pure Hodge structure of weight \( n + m \). This condition is phrased by saying that \( F \) is \( n + m \)-conjugate.

The data will be denoted simply as \( (C^*, F) \).
Note that if \((C^*, F)\) is a Hodge complex of weight \(m\), then there is a canonical identification:
\[
F E_1^{pq} = F^p H^{p+q} \cap F^{q+m} H^{p+q}
\]
where \(\bar{F}\) is the complex conjugate filtration to \(F\).

**Definition 1.2.2.** A data \(((K^*_Q, W_Q), (K^*_C, W_C, F), (K_Q \otimes C, W_Q \otimes C) \sim (K_C, W_C))\) where

(i) \(K^*_\overline{\cdot}\) is a complex over \(\overline{\cdot}\).

(ii) \((W_Q \otimes C, W_Q \otimes C) \sim (K_C, W_C)\) is a filtered quasi-isomorphism.

called a mixed Hodge complex, if \(Gr^W_p K^* = W Gr^{-p}\) is a pure Hodge complex of weight \(p\). The data will be denoted by \((K^*, W, F)\).

**Corollary 1.2.1.** Assume that \((K^*, F, W)\) is a mixed Hodge complex, then

(a) \(F E_1 K^* = F E_{\infty} K^*\)

(b) \(W E_2 K^* = W E_{\infty} K^*\)

**Proof.** First we note that the filtration \(F\) induces on \(W E_1^{pq} K^* = H^{p+q} \cap W Gr^p K^*\) the pure Hodge structure of weight \((p + q) - p = q\). Since \(d_r\) is compatible with filtration \(F\) all terms \(W E_1^{pq}\) are mixed Hodge structures and \(d_r\) is a morphism of Hodge structures. The purity implies that \(d_{r+1} = 0\), hence \(W E_2 K^* = W E_{\infty} K^*\).

According to the above corollary, in order to show that \(F E_1 = F E_{\infty}\) it is enough to prove that \(d_r\) are \(F_{ind}\)-strict for all \(r \geq 0\). By the previous the only non-trivial differentials are \(d_0, d_1\). Since \(W Gr K^*\) is a pure Hodge complex, \(d_0\) is \(F\)-strict. As we seen \(d_1\) is a morphism of Hodge structures, thus it is automatically \(F\)-strict on \(E_1\).

**Definition 1.2.3.** We denote by \([A \to B] := \text{Cone}(A \to B)]\).

**Proposition 1.2.1** (Mixed cone construction). Assume \(A \to B\) is a morphism of mixed Hodge complexes, then \(K := [A \to B], K^* = A_n \oplus B_{n-1}\) becomes a mixed Hodge complex if we put:

a) \(W^p K := W^p A \oplus W^{p-1} B\)

b) \(F^p K := F^p A \oplus F^{p+1} B\)

**Proof.** It is enough to show that \(W Gr^p K^*\) is a pure Hodge complex of weight \(-p\). Note that \(W Gr^p K^*\) as a complex ignores \(f\) and is equal to the direct sum

\[
W Gr^p A \oplus W Gr^{p-1} B[-1]
\]

hence \(F\) induces a pure Hodge structure of weight \(n - p\) on its \(n\)-th cohomology equal to \(H^n(W Gr^p A) \oplus H^{n-1}(W Gr^{p-1} B)\). Hence \([A \to B]\) is a mixed Hodge complex.

**Definition 1.2.4.** If \((C^*, F)\) is a Hodge complex of weight \(m\), then \((C^*[a], F)\) is a Hodge complex of weight \(m + a\). If \((K^*, F, W)\) is a mixed Hodge complex, then \((K^*[a], F, W[a])\) is a mixed Hodge complex, where: \(W[a]^p = W^{a+p}\).

Similarly one can define the Tate twists of mixed Hodge complexes.

The following will be called iterated cone construction.

**Example 1.2.1** (Iterated cone). Let \(K = [A_0 \to A_1 \to A_2 \to \ldots]\) be a bicomplex formed by complexes \(A_i\). One can consider \(K\) as:

\[
[A_0 \to [A_1 \to [A_2 \to \ldots]]]
\]

If \(A_i\), a mixed Hodge complexes and all differentials are morphisms of mixed Hodge complexes, then \(K\) is a mixed Hodge complex.

The following theorem is the main tool for construction mixed Hodge structures.
Theorem 1.2.1 (Deligne [1]). The cohomology of a mixed Hodge complex \((K, F, W)\) provides a mixed Hodge structure.

Remark 1.2.2. Since we work with weight filtrations on Hodge complexes it is necessary to do the following remark. The mixed Hodge structure weight filtration on \(H^n := H^n(K)\) is given by the décalage, i.e. \((\text{Dec} W)_k H^n = W_{k-n} H^n\) and thus:

\[
Gr^{\text{Dec} W}_k H^n = Gr^k_d W H^n = Gr^{k+n}_W H^n
\]

For example, if the Hodge complex \((K, F, W)\) is pure of weight 0, then \(H^n\) is pure of weight \(n\). In general, since the Hodge filtration \(F\) on \(E^{pq}_W(K, W) = Gr^p_W H^n\) is \(q\)-conjugate, we have \(Gr^{\text{Dec} W}_q H^n = W_q E^{pq}_W\).

For a general complex \(K\) the décalage \(\text{Dec} W\) is defined by:

\[
(\text{Dec} W)^k K^n := \{ x \in W^{k+n} K^n | dx \in W^{k+n+1} K^{n+1} \}
\]

There is a natural quasi-isomorphism \(E_0(K, \text{Dec} W) \sim E_1(K, W)\) and natural isomorphisms \(E^{pq}_W(K, \text{Dec} W) \simeq E^{p+n-p}_W(K, W), n = p + q\). See the original paper [1] by Deligne for details.

Definition 1.2.5. If \((V, F, W)\) is a mixed Hodge structure then \((Gr^*_W V, F, W)\) is a pure Hodge structure called the associated pure Hodge structure.

Definition 1.2.6 (Hodge diagram, see [3][4]). A (mixed) Hodge diagram is a mixed Hodge complex with a multiplication compatible with filtrations and the differential making the rational and the complex parts of the Hodge complex into connected cdga.

Definition 1.2.7. A mixed Hodge dg-algebra \((A^*, F, W)\) is a mixed Hodge complex with \(A_C = A_Q \otimes \mathbb{C}\) such that \((A^*, F, \text{Dec} W)\) is a mixed Hodge structure compatible with the differential and multiplication.

For example, if \((K^*, F, W)\) is a mixed Hodge diagram, then \(E_1^*(K, W)\) is a mixed Hodge dg-algebra with pure terms. In particular the cohomology of a mixed Hodge diagram is a mixed Hodge algebra.

Proposition 1.2.2 (Deligne splitting). A mixed Hodge structure \((V, F, W)\) admits a canonical decomposition \(V = \bigoplus V^{p,q}\) such that \(\bigoplus_{p+q=k} V^{p,q} \rightarrow Gr^k_W V\) is an isomorphism onto Hodge components of the associated pure Hodge structure. Hence there is a canonical isomorphism of vector spaces \(Gr^*_W V_C \simeq V_C\). In particular, if \(V\) is a mixed Hodge dg-algebra, then there is a natural isomorphism of dg-algebras \(V_C \simeq Gr^*_W V_C\).

Remark 1.2.2. In general there is no such decomposition on the level of Hodge complex, though one can lift the canonical splitting on its cohomology.

This implies a powerful corollary:

Corollary 1.2.2. If \((C, F, W)\) is a Hodge complex (diagram), then there is a natural isomorphism \(H^*(C_C) \simeq E_1^* (C_C, W)\) of vector spaces (algebras).

Proof. The cohomology of \(H^*(C)\) form a mixed Hodge structure. If \(C\) in addition is a Hodge diagram, then \(H^*(C)\) is a mixed Hodge algebra. There a chain of natural isomorphisms of vector spaces (algebras):

\[
H^*(C_C) \simeq W Gr^* H^*(C_C) \simeq E_1^*(C_C, W) = E_2^*(C_C, W)
\]

Within the framework introduced by Morgan in [3], then developed in homotopical context by Joana Cirici in [4], the above corollary can be made even stronger, it is the foundation of our work.

Theorem 1.2.2 (Morgan, Joana Cirici). If \((K^*, F, W)\) is a mixed Hodge diagram, then the minimal model \(N^*\) of \(K_Q\) is equipped with a mixed Hodge dg-algebra structure \((N, F, W)\) and a bifiltered quasi-isomorphism \((N, F, W) \rightarrow (K^*, F, W)\) of Hodge diagrams. Moreover, \(d : W_k N \rightarrow W_{k-1}(N^+ \cdot N^+)\).

Corollary 1.2.3 (Morgan, theorem 9.6 in [3]). There is a natural quasi-isomorphism \(N_C \sim E_1(K_C, W)\) of cdga’s, thus there is a natural cdga’s quasi-isomorphism

\[
E_1^{**}(K_C, W) \sim K_C
\]

which non canonically descents to an quasi-isomorphism of algebras over \(Q\):

\[
E_1^{**}(K_Q, W) \sim K_Q
\]
Proof. The morphism \((N, W) \rightarrow (K^*, W)\) is a filtered quasi-isomorphism, thus \(E_1(N, W) \simeq E_1(K^*, W)\). Since \((N, F, W)\) is a mixed Hodge dg-algebra, there is a canonical splitting over \(\mathbb{C}\):

\[
N \simeq \text{Gr}_{Dec W}^* N = E_0(N, Dec W) \simeq E_1(N, W)
\]

Then the Morgan-Sullivan theorem (theorem 10.1 in \([3]\)) implies that such splitting over \(\mathbb{C}\) can be descended over \(\mathbb{Q}\). This completes the proof.

For a general treatment of minimal models in mixed context see the paper \([4]\) by Joana Cirici.

2 Lattices

In this section we introduce the notion of \((L, \leq)\)-chain algebra, show the existence of basic operations such as pullbacks, pushforwards and the projection formula; this will allow us to handle homological combinatorics behind complexes of sheaves related to the arrangement.

2.1 Posets

Here \((L, \leq)\) denotes a poset. For a subset \(S \subseteq L\) we write \(S \leq t, t \in L\) and call \(t\) an upper bound of \(S\) if \(s \leq t, \forall s \in S\).

Definition 2.1.1. The set of maximal elements \(\max(S) \subseteq L\) consists of \(t \in S\) such that for all \(t' \in S, t' \geq t\) implies \(t' = t\).

Dually we define \(\min(S) \subseteq L\).

Definition 2.1.2. If \((L, \leq)\) is a poset and \(S \subseteq L\), then \(t = \sup S \in L\) is called the supremum of \(S\), if \(t \geq S\) is an upper bound and if for any upper bound \(t' \geq S\) we have \(t' \leq t\). In other words, \(\sup S\) is the lowest upper bound of \(S\).

Definition 2.1.3. Denote \(\sup^0 S \subseteq L\) to be the subset of all upper bounds \(t \geq S\) such that for all \(t' \geq S, t \geq t'\) implies \(t' = t\). In other words \(\sup^0 S = \min\{t \in L | t \geq S\}\) is the set of minimal upper bounds.

Dually we define \(\inf S\) and \(\inf^0 S\).

Definition 2.1.4. A poset \((L, \leq)\) is lattice if for any \(S \subseteq L\) there is \(\sup(S)\) and \(\inf(S)\). For any \(a, b \in L\) the operations \(a \lor b := \sup(a, b)\) (join) and \(a \land b := \inf(a, b)\) (meet) are well-defined.

Recall that the order of the lattice is recovered from operations \(\lor, \land\) by \(a \leq b\) if \(a \land b = b\) etc.

Definition 2.1.5. We say that \(x\) covers \(y\) if \(x \geq y\) and there are no other elements between \(x\) and \(y\). Denote this relation as \(x \triangleright y\).

Definition 2.1.6. A poset \((L, \leq)\) is graded, if there is an order-preserving function \(r : L \rightarrow \mathbb{Z}\), called grading such that \(r(x) = r(y) + 1\) if \(x \triangleright y\).

Clearly \(r\), if exists, is determined by its values on \(\min(L)\).

If the poset \((L, \leq)\) has an infimum we denote it by \(0 = \inf(L) \in L\) and call \((L, \leq)\) local. In this case we assume \(r(0) = 0\).

write \(L[0, x] = \{t \in L \mid t \leq x\}\).

Definition 2.1.7. A poset \((L, \leq)\) with the unique minimal element \(0 \in L\), is a local (graded) lattice, if for any \(x \in L, L[0, x]\) is a (graded) lattice. If \(L\) is a graded local lattice, then \(r(x) := r(L[0, x]), r(0) = 0\) is called the grading of \(L\).

Clearly the grading \(r\) of a local lattice \(L\) satisfy \(r(x) = r(y) + 1\) if \(x \triangleright y\).

Definition 2.1.8. Elements \(\text{Atoms}(L) := \{x \in L : r(x) = 1\}\) are called atoms of the graded local lattice \((L, \leq)\).

Definition 2.1.9. A graded lattice \((L, \leq)\) is geometric, if (i) \(x = a_1 \lor \ldots \lor a_k, a_i \in \text{Atoms}(L)\) for any \(x \in L\), (ii) \(r(x \lor y) + r(x \land y) \leq r(x) + r(y)\).

Definition 2.1.10. Elements \(x_i \in L, i \leq n\) in a graded posed are called independent, if \(r(x_1 \lor \ldots \lor x_n) = \sum r(x_i)\). If the lattice is geometric, then \(x, y\) are independent implies \(r(x \lor y) = 0\).
2.2 Arrangement posets

Consider the poset $2^X$ of subsets of $X$ ordered by inclusion.

Let $(L, \leq)$ be a local graded lattice.

**Definition 2.2.1.** An arrangement poset is a $(L, \leq)$-subspace of $X$, i.e. a functor $F : (L, \leq)^{op} \to 2^X$ with values $F(p) =: L_p \subset X, p \in L$ such that:

1. $L_0 = X$.
2. for $I \subset \text{Atoms}(L)$ we have $L_I := \bigcap L_i = \bigcup_{t \in \text{sup}(I)} L_t$
3. For each $x \in X$ the full subposet $\{ p \in L : L_p \ni x \} \subset L$ is a graded lattice of the form $[0, t], t \in L$.
4. $\bigcap_{a \leq x : r(a) = 1} L_a = L_x$

**Definition 2.2.2.** The arrangement poset $(L, \leq)$ on $X$ is good if $L$ is a geometric lattice.

If $(L, \leq)$ is good, then $L_I = L_{\text{sup}(I)} \subset X$.

**Example 2.2.1.** (i) If $(X, Z_\ast)$ is an arrangement, then $Z_I$ form the cubical geometric lattice $C_I = Z_I$ ordered by inclusion of indices. Then the natural map $(C, \leq)^{op} \to 2^X$ defines a good arrangement poset.

(ii) If $X$ is a smooth variety, then the configuration space of $n$-ordered points on $X$ is $X^n - \Delta_n$, where $\Delta$ is the intersection poset of all diagonals in $X^n$. Each diagonal $\Delta_{\{S_1, \ldots, S_k\}} = \{i, j \in S_t \implies x_i = x_j\} \subset X^n$ is parametrized by an unordered collection of disjoint subsets $\{S_1, S_2, \ldots, S_k\}, S_t \subset \{1, \ldots, n\}$. Then $r(\Delta_{\{S_1, \ldots, S_k\}}) = \sum_{i < k} |S_i| - 1$. The atoms a formed by big diagonals $\Delta_{ab} := \Delta_{\{a, b\}}$. The poset $\Delta$ defines a good arrangement poset.

(iii) Consider the arrangement given by two planes and one line $\Pi_1, \Pi_2, l \subset \mathbb{C}^3$ meeting exactly at one point $p$ in general position. The corresponding intersection poset consists of $0 = C^3, \Pi_1, \Pi_2, l, l_{12} = \Pi_1 \cap \Pi_2, p$ is not even ranked. One can always pass to the corresponding cubical lattice to solve the issue. Another way is to duplicate $p$ by adding $p'$ to the poset, then replace $p \leq l_{12}$ by $p' \leq l_{12}$ and $p' \leq p$.

2.3 $(L, \leq)$-objects

Let $(L, \leq)$ be a graded poset.

**Definition 2.3.1 ( $(L, \leq)$-object).** An $(L, \leq)$-object in a category $C$ is a functor $F : (L, \leq)^{op} \to C$. Its values are denoted by $F_s \in C, s \in L$. Corresponding morphisms $g_{y,x}^t : F_x \to F_y, x \geq y \in L$ are called the structure maps.

Assume $(L, \leq)$ is a local lattice. Note that for any $x' \leq x, y' \leq y$ there is a unique map $\gamma^{x'y'}_{x'y} : \overset{\circ}{\text{sup}}(x, y) \to \overset{\circ}{\text{sup}}(x', y')$ such that $\gamma^{x'y'}_{x'y}(t) \leq t$. It is defined by taking $\overset{\circ}{\text{sup}}(x', y')$ in $L[0, t]$.

**Definition 2.3.2.** If $C$ is a monoidal additive category, then an $(L, \leq)$-algebra in $C$ is a functor $A : (L, \leq)^{op} \to C$ with morphisms:

$$m_{x,y} : A_x \otimes A_y \to \bigoplus_{t \in [x, y]} A_t =: A_{x \oplus y}$$

with components $m_{x,y}^t$ such that

1. $m$ is associative, i.e.:

$$m_{t_1, z} \circ (m_{x,y}^{t_1} \otimes \text{id}) = m_{x,t_2}^z \circ (\text{id} \otimes m_{y,z}^{t_2})$$

for all $t_1 \in \overset{\circ}{\text{sup}}(x, y), t_2 \in \overset{\circ}{\text{sup}}(y, z), t_3 \in \overset{\circ}{\text{sup}}(x, y, z)$.  

10
2. $m$ is functorial, i.e. the following square commutes:

\[
\begin{array}{ccc}
A_x \otimes A_y & \xrightarrow{m} & A_{x \vee y} \\
\downarrow g^x \otimes g^y & & \downarrow g \\
A_{x'} \otimes A_{y'} & \xrightarrow{m} & A_{x' \vee y'}
\end{array}
\]

In particular objects $F[x, y] := \bigoplus_{t, x \leq t \leq y \in L} F_t$ are associative algebras in $C$. If $C$ is symmetric we say that $F$ is commutative, if $F[0, x]$ are commutative.

Below $(L, \leq)$ is a graded poset.

**Definition 2.3.3.** An $(L, \leq)$-filtered object $K$ in additive category of complexes $\text{Comp}(C)$ is a collection of complexes $(K_x, d_x) \in \text{Comp}(C)$, $d_x^2 = 0, \forall x \in L$ and morphisms $\partial_{yx} : K_x \to K_y[1], y < x$ such that $\partial := \sum_{y < x} \partial_{yx}$ defines a differential in $\bigoplus_{t \in L} K_t, \forall x \in L$, i.e. $\partial^2 = 0$.

The range of $K$ is $\max (r(x) - r(y))\partial_{yx} \neq 0; x, y \in L \in \mathbb{Z}$.

Morphisms $\partial_{yx}$, the differential $\partial$ and $d = \sum_x d_x$ are called the structure maps of $K$, the combinatorial differential and the inner differential respectively.

By definition $[\partial, d] = 0$. Thus a $(L, \leq)$-filtered object is a $L$-graded complex with an additional combinatorial differential which decreases $L$-grading. We are mostly interested in case when $\partial_{yx} \neq 0$ implies $y < x \in L$.

**Definition 2.3.4** ($(L, \leq)$-chains). $(L, \leq)$-chains are $(L, \leq)$-filtered objects of range 1.

Assume $(L, \leq)$ is a local lattice.

**Definition 2.3.5** ($(L, \leq)$-filtered algebra). An $(L, \leq)$-filtered algebra in monoidal additive category of complexes $\text{Comp}(C)$ is a $(L, \leq)$-filtered complex $K$ with a collection of morphisms:

$$m^t_{x,y} : K_x \otimes K_y \to \bigoplus_{t \in L} K_t$$

with components $m^t_{x,y}$ such that $m$ is associative and obey the Leibniz rule, i.e. the objects $F[0, x] := \bigoplus_{t \in L} K_t$ are dg-algebras in $\text{Comp}(C)$ with differential $\partial + d$.

**Definition 2.3.6.** We say that $(L, \leq)$-filtered algebra or $(L, \leq)$-algebra is well-graded, if the multiplication satisfy $m^t_{xy} = 0, t \in x \vee y$ if $r(t) < r(x) + r(y)$.

**Definition 2.3.7.** If $K$ is $(L, \leq)$-filtered complex and $A$ is $(L, \leq)$-complex, we define the tensor product $A \otimes K$ as $(L, \leq)$-filtered complex given by:

1. $(A \otimes K)_x = A_x \otimes K_x, x \in L$
2. $\partial_{A \otimes K}^t := \partial_{A x}^t \otimes \partial_{K y}^t, y < x \in L$ as a morphism $(A \otimes K)_x \to (A \otimes K)_y$.

Similarly one can define a tensor product of $(L, \leq)$-objects. One can see, that if $K$ is $(L, \leq)$-filtered algebra and $A$ is $(L, \leq)$-algebra, then $A \otimes K$ is $(L, \leq)$-filtered algebra of the same range as $K$. If $K$ or $A$ is well-graded, the so is $A \otimes K$.

**Definition 2.3.8.** A sublattice $L' \subset L$ is called complete if for all $x, y \in L'$ and $t \in L' : x \leq t \leq y$ implies $t \in L$.

Hence the preimage of complete sublattice under an order preserving map is complete. The following notion is clear.

**Proposition 2.3.1.** If $L \subset L'$ is complete and $K'$ is a $(L', \leq)$-filtered complex, then the restriction $K'|_L$ is a $(L, \leq)$-filtered complex.

**Definition 2.3.9.** An order preserving map $f : (L, \leq) \to (L', \leq)$ is called homomorphism of local lattices if
1. \( f(x \lor y) = f(x) \lor f(y) \in L'[0, f(t)], \) for any \( x, y \in L[0, t] \)

2. \( f^{-1}(L'[0, t]) \) is a sublattice in \( L \)

**Definition 2.3.10 (Totalization).** The totalization of \((L, \leq)\)-filtered complex \( K \) is the complex \( \text{Tot}(K) = \bigoplus_{x \in L} K_x \) with differential equal to the sum \( d + \partial \).

**Definition 2.3.11.** An order preserving map \((L, \leq) \to (L', \leq)\) is called contraction, if for any \( y <: x \in L \), \( f(y) <: f(x) \) if \( f(y) = f(x) \).

**Definition 2.3.12 (Pushforward).** Let \( K \) be a \((L, \leq)\)-chains complex in \( \text{Comp}(C) \) and \( f : (L, \leq) \to (L', \leq)\) is a contraction map.

Then the direct image of \( K \) under \( f \) is an \((L', \leq)\)-chains object \( f_*K \) with components:

\[
(f_*K)_{x'} = \text{Tot}(K|_{f^{-1}x'})
\]

and the structure maps:

\[
\partial^{f_*K}_{y', x'} : (f_*K)_{x'} \to (f_*K)_{y'}
\]

for \( y' < x' \) equal to

\[
\sum_{y < x | f(y) = y'} \partial_{yx}
\]

**Proposition 2.3.2.** The above definition is correct and there is natural isomorphism \( \text{Tot}(f_*K) \cong \text{Tot}(K) \). In addition, if \( f \) is a homomorphism and \( K \) is \((L, \leq)\)-chains algebra, then \( f_*K \) is \((L', \leq)\)-chains algebra.

**Proof.** Objects \((f_*K)_{x'}\) are totalizations and thus complexes with its own inner differential \( d'_{x'} = \sum_{x \in f^{-1}x'} d_x + \sum_{y < x | f(y) = y'} \partial_{yx} \).

Let \( \partial' = \partial^{f_*K} \) and \( d' = \sum d'_{x'} \). By the construction

\[
(\bigoplus_{x' \in L'} (f_*K)_{x'}, d' + \partial') = (\bigoplus_{x \in L} K_x, d + \partial)
\]

Thus \( (d' + \partial')^2 = 0 \) and it remains to show \( \partial'_{y', x'}, \partial_{yx} + \partial_{yx} \partial'_{y', x'} \) for all pairs \( y' < x' \in L' \). Take \( x \in f^{-1}(x'), y \in f^{-1}(y') \), then \( yx\)-component of the last expression is equal to:

\[
(\partial^2)_{xy} - \sum_{x < z < y \atop f(x) < f(z) < f(y)} \partial_{zx} \partial_{zy}
\]

Since \( K \) of range 1 and \( f \) is contraction \( \partial'_{y', x'} \neq 0 \) only for \( y' <: x' \). Thus the set \( \{ z : f(x) < f(z) < f(y) \} \) is empty, hence \( \partial'_{y', x'} \) is a morphism of complexes and the claim follows.

If \( K \) is \((L, \leq)\)-chains algebra and \( f \) is a contraction homomorphism of local lattices, then \( \text{Tot}(f_*K|_{L[0, t]}) = \text{Tot}(K|_{f^{-1}L[0, t]}) \) is \( dg \)-algebra. This defines the \((L', \leq)\)-chains algebra structure on \( f_*K \).

Since the composition of two contraction maps is a contraction the pushforward map satisfy \( f_* \circ g_* = (f \circ g)_* \). If \( \pi : (L, \leq) \to (pt, \leq) \) is the projection to the point, then \( \text{Tot}(K) = \pi_*K \).

**Definition 2.3.13 (Pullback).** If \( f : (L, \leq) \to (L', \leq) \) is an order preserving map \( \Lambda' \) is \((L', \leq)\)-object, then its pull-back under \( f \) is \((L, \leq)\)-object \( f^* \Lambda' \) with values \((f^* \Lambda')_x := \Lambda'_{f(x)} \) and obvious structure maps.

The following is clear.

**Proposition 2.3.3.** If \( f \) above is a homomorphism and \( \Lambda' \) is \((L', \leq)\)-algebra, then \( f^* \Lambda' \) has a natural structure of \((L, \leq)\)-algebra.

Note that pushing forward well-graded \((L, \leq)\)-chains algebra or pulling back well-graded \((L, \leq)\)-algebra need not be well-graded.

**Remark 2.3.1.** Note that there is no general recipe for pulling back \((L, \leq)\)-chains objects even for homomorphism of lattices, e.g. the restriction is well-defined only for complete sublattice.
Lemma 2.3.1. If $\phi : L \rightarrow L$ is an order preserving map such that $\phi(x) \geq x, \forall x \in L$ and $A$ is $(L, \leq)$-object, then there is a natural homomorphism $u_{\phi} : \phi^* A \rightarrow A$.

Proof. Define $u_{\phi} : (\phi^* A)_x = A_{\phi(x)} \rightarrow A_x$ as the corresponding structure map. Then for any pair $x' \leq x$ the following diagram commutes:

$$
\begin{array}{ccc}
\phi^*(A)_x & \xrightarrow{u_{\phi}} & A_x \\
\downarrow & & \downarrow \\
\phi^*(A)_{x'} & \xrightarrow{u_{\phi}} & A_{x'}
\end{array}
$$

Lemma 2.3.2 (Projection formula). Let $p : (L, \leq) \rightarrow (L', \leq)$ be a homomorphism of lattices and $A'$ is $(L', \leq)$-complex and $D$ is $(L, \leq)$-chains complex, then there is a natural isomorphism of $(L, \leq)$-chains complexes:

$$p_*(p^* A' \otimes D) \simeq A' \otimes p_* D$$

Proof. We have

$$(p_*(p^* A' \otimes D))_{x'} = \bigoplus_{x \in p^{-1}(x')} (p^* A)_x \otimes D_x = \bigoplus_{x' \in p^{-1}(x')} A_{x'} \otimes D_x = (A' \otimes p_* D)_{x'}$$

Then it is straightforward to check that this identification commutes with structure maps:

$$\partial'_{y,x'} : (p_*(p^* A' \otimes D))_{x'} \rightarrow (p_*(p^* A' \otimes D))_{y'}$$

and

$$\partial''_{y,x'} : (A' \otimes p_* D)_{x'} \rightarrow (A' \otimes p_* D)_{y'}$$

Definition 2.3.14 (Convolution). Let $K$ be a $(L, \leq)$-chains object (algebra) and $A$ is a $(L, \leq)$-object (algebra) in category of complexes $\text{Comp}(C)$ (vector spaces, sheaves etc). Define the convolution

$$A \ast K := \pi_*(A \otimes K) = \text{Tot}(A \otimes K)$$

where $\pi : (L, \leq) \rightarrow (pt, \leq)$ is a morphism to the trivial lattice with one vertex.

Thus $\text{Tot}(K)$ is the convolution of $K$ with the trivial algebra (i.e. with all vertices equal to $\mathbb{Q}$) and

$$K \ast A = \text{Tot}(K \otimes A) = \bigoplus_{x \in L} K_x \otimes A_x, d + \partial$$

Definition 2.3.15 (Rank filtration). If $K$ is $(L, \leq)$-chains complex, then the complex $\text{Tot}(K)$ admits an increasing filtration $W^p \text{Tot}(K) := \bigoplus_{x \in L, x \leq p} K_x$. The filtration $W^p$ is called the rank filtration.

If $K$ is $(L, \leq)$-chains algebra, then the rank filtration $W^p$ is multiplicative. If $K$ is well-graded, then $\text{Tot}(K)$ becomes a bicomplex with $W^p$ corresponding to columns-wise filtration.

2.4 $(L, \leq)$-mixed Hodge complexes

Here we prove two lemmas which allow to produce Hodge complexes from $(L, \leq)$-Hodge complexes.

First we note the existence of the natural Hodge structure on convolution.

Lemma 2.4.1. Assume $(A, F, W)$ is a $(L, \leq)$-Hodge complex of sheaves and $K$ is $(L, \leq)$-chains complex of vector spaces. Then $A \otimes K$ is a $(L, \leq)$-chains complex of Hodge complexes and the convolution $(A \ast K, F, W)$ has a natural structure of a Hodge complex.

Proof. The convolution $A \ast K = \bigoplus_k \bigoplus_{x : \tau(x) = k} A_x \otimes K_x$ can be viewed as a bicomplex with the vertical differential $d_v$ leaving the rank filtration the same and the horizontal $d_h$ which decreases rank by 1. Note that each summand $A_x \otimes K_x$ is a Hodge complex, since it can be viewed as an iterated cone (definition 1.2.1). Thus they are endowed with filtrations $F_x, W_x$. Then we simply take the direct sum of filtrations to obtain $F, W$. After $A_x \otimes K_x$ are endowed with Hodge complex structures there are no additional degree shifts, since $d_v$ is already of degree 1.
On the other hand the rank filtration (definition 2.3.15) can be used to construct an other Hodge complex structure on the convolution:

**Lemma 2.4.2.** Assume \((A, F, W)\) is a \((L, \leq)\)-pure Hodge complex of weight 0 and \(K\) is \((L, \leq)\)-chains complex of vector spaces such that \(H^*(K_x)\) is concentrated in degree \(-r(x)\), then there is the rank Hodge complex \((A*K, F, W')\). Moreover, both the rank and the natural Hodge mixed Hodge structures \(Dec W\) and \(Dec W'\) coincide on \(H^*(A*K)\).

**Proof.** (1) First we shall prove that \(Gr^p(A*K, W')\) is a pure Hodge complex of weight \(-p\). Note that \(Gr^p(A*K, W') = \bigoplus_{x \in L: r(x) = -p} A_x \otimes K_x\) is a sum of complexes, each term is the tensor product \(A_x \otimes K_x\) of complexes. Since \(K_x \sim H^{-r(x)}(K_x)\), we obtain: \(H^n(A_x \otimes K_x) = H^{n-p}(A_x) \otimes H^p(K_x)\). Since \(A_x\) has weight 0, the Hodge filtration \(F\) is \(n-p\)-opposed on \(H^n(Gr^p(A*K, W'))\), thus \((A*K, F, W')\) is a Hodge complex.

(2) Unfortunately we do not have a direct comparison map between two filtrations. To prove that \(W\) and \(W'\) on \(H^* := \mathbb{H}^*(A*K)\) are the same, is to prove \(wGr^p W'.Gr^p H^* = 0\) unless \(p = p'\). We will use lemma 1.1.3 on two filtrations of the first section. Namely, \(W_{dir} on E^p_{p,n-p}'(A*K, W') = wGr^p(H^n)\) is equal to \(W_{conv}\) if \(d_1\) is \(W_{dir}\)-strict on \(E_1(A*K, W')\). Thus it is enough to show that \(wGr^p E^p_{p,n-p}'(A*K, W') = 0\) unless \(p = p'\). The Hodge filtration \(F\) on \(V^n := E^p_{p,n-p}'(A*K, W') = H^n(Gr^p(A*K, W'))\) is \((n-p)\)-conjugate by the previous, thus \((F, W')\)-Hodge complex \(Gr^p(A*K, W')\) has weight \(-p'\), i.e. the corresponding \((F, W')\)-mixed Hodge structure on \(V^n\) admits non trivial weights only in degree \(n-p'\), thus (remark 2.1) \(Gr^p_{\leq p} V^n = Gr^p_{\leq p} V^n = wGr^p V^n \neq 0\) only if \(p' = p\).

**Definition 2.4.1.** The Hodge complex structure provided by the lemma above is called the rank Hodge complex.

### 2.5 Orlik-Solomon algebras

Following Orlik-Solomon’s work [7] we define:

**Definition 2.5.1** (OS-complex). The OS-complex \(M\) of a local graded lattice \((L, \leq)\) is a \((L, \leq)\)-chains complex such that:

1. \(M_0 = \mathbb{Q}\)
2. \(M_s\) sits in degree \(-r(s)\).
3. \(Tot(\mathcal{M}|_{[0,1)})\) is acyclic for all \(s > 0 \in L\). In other words, the negative-graded vector space

\[
\bigoplus_{t \leq s} \mathcal{M}_t
\]

with \(\partial\) defined by:

\[
\partial v_y = \sum_{x: y \geq x} \partial_{xy} v_y
\]

is an acyclic complex for \(s > 0 \in L\).

**Remark 2.5.1.** The OS-complex \(M\) is the categorification of the M obious function \(\mu\) of \(L\), i.e. \(\mu[0, s] = (-1)^{r(s)}dim M_s\).

**Proposition 2.5.1.** For any local graded lattice \((L, \leq)\) there is a natural OS-complex \(M\). It is unique up to a canonical isomorphism.

**Proof.** Since the poset is graded, it is straightforward to prove both claims by induction.

**Definition 2.5.2.** OS-algebra of a local lattice \((L, \leq)\) is an OS-complex \(M\) which is \((L, \leq)\)-chains algebra.

Thus the multiplication in OS-algebra is non-zero only on independent components.
**Theorem 2.5.1** (Orlik-Solomon). If \((L, \leq)\) is a locally geometric lattice, then there is a natural \(OS(L)\) algebra underlying \(L\).

**Proof.** We sketch the construction, the details can be found in the original work \(^2\) by Orlik-Solomon. First we reduce to global geometric lattice by putting \(OS(L) = OS(L[0, x])\). If \(L\) is geometric lattice, then \(OS(L) = \Lambda^\ast \langle dv_i, i \in \text{Atoms}(L) \rangle / I\), where \(I\) is an ideal in the Grassman algebra we want to describe. The Koszul differential \(\partial\) acts on the Grassman algebra and is equal to \(i \sum_i i\). Let \(\langle \text{dep}\rangle\) denotes an ideal spanned by \(v_I\) for \(I \subset \text{Atoms}(L)\) is dependend, i.e. \(r(\sup(I)) < |I|\). Then \(I\) as an ideal is generated by \(\partial(\text{dep})\). One can check that \(I = \langle \text{dep} \rangle + \partial(\text{dep})\) as vector spaces and that \(I\) is \(L\)-graded. The Koszul differential acts on \(OS(L)\) and there is a contraction \(\partial^* = \sum_{i \in \text{Atoms}(L)} dv_i \wedge -\) such that \([\partial, \partial^*]\) is invertible, hence \((OS(L[0, x]), \partial)\) is acyclic unless \(x = 0 \in L\). \(\square\)

**Example 2.5.1.** (i) If \(L = C(X, Z_s)\) is the cubical poset, then the corresponding \(OS(L)\) is the Grassman algebra generated by free variables \(dv_x, \text{deg}(dv_x) = -1\) corresponding to \(Z_s\). The differential \(\partial = \sum_i i\) is the whole Koszul differential. The algebra is clearly \(L\)-graded and \(L\)-filtered according to the differential: the vertice \(I \in C(X, Z_s)\) of the cube corresponds to the element \(dv_I = dv_{t_1} \wedge \ldots \wedge dv_{t_k} \in \mathcal{M}_I\), for \(I_1 < \ldots < I_k\) forming \(I\).

(ii) The intersection poset \(L = (X^n, \Delta_s)\) of all diagonals is a geometric lattice with atoms \(\Delta_{ab}\) given by big diagonals \(x_a = x_b\). The minimal element is \(\infty \in L\) corresponds to \(x_1 = \ldots = x_n\). Then \(M[0, \infty]\) for \(M = OS(L)\) is \(\Lambda(\Delta_{ab}, a \neq b \in I \cup J)\), for \(I \cup J \subset \{1, \ldots, n\}\), \(\text{deg}(\Delta_{ab}) = -1\) with \(\Delta_{ab} = 1\) modulo Arnold’s relation:

\[
\Delta_{ab} \Delta_{bc} + \Delta_{bc} \Delta_{ca} + \Delta_{ca} \Delta_{ab} = 0
\]

Algebra \(M[0, \infty]\) admits a natural \(L\)-grading with \(L\)-pieces \(M[0, \infty], s \in L\) spanned by grassman monomials which define the given diagonal \(L_s\). This defines an \((L, \leq)\)- chains algebra \(M\) with \(M_s = M[0, \infty], s\).

Let \((L, \leq)\) be a locally geometric lattice.

**Definition 2.5.3.** The quasi-cubical lattice of \((L, \leq)\) is a poset \((Q^L, \leq)\) formed by pairs \((x, I)\) for \(I \subset \text{Atoms}(L)\) such that \(\sup(I) = x \in L[0, x]\). Then \((x, I) \leq (y, J)\) iff \(x \leq y\) and \(I \subset J\).

Note that if \((L, \leq)\) is geometric, then \(Q^L = (2^\text{Atoms}(L))^\text{op}\) is the cubical lattice. There is a natural map \(p : (Q^L, \leq) \to (L, \leq)\) with \(p((x, I)) = x\).

**Proposition 2.5.2.** If \(L\) is locally geometric, then \(Q^L\) is a locally geometric lattice and the map \(p\) is contractible homomorphism.

**Proof.** One can check that \((x, I) < (y, J)\) iff \(|J - I| = 1\). Thus \(r(x, I) = |I|\) defines grading and \(p\) is contractible. Then \((x, I) \vee (y, J) = (x \vee y \in L[0, t], I \cup J \subset [0, (t, K)]\) for any \((t, K)\) with \((x, I), (y, J) \leq (t, K)\).

Let \(OS(Q^L)\) denote the natural \(OS\) algebra of \(Q^L\). If \(L\) is geometric it is the usual Grassman algebra \(\Lambda(dv_i, i \in \text{Atoms}(L)), \text{deg}(dv_i) = -1\) with the Koszul differential \(\sum_{i \in \text{Atoms}(L)} i\) and the obvious well-grading by \(L\).

The following is an extension to the local case of the notion of relative atomic complex used by Feichter-Yuzvinsky in \[11\].

**Definition 2.5.4** (Atomic complex). The atomic complex \(\mathcal{D}^L\) of \((L, \leq)\) is \((L, \leq)\)-chains algebra equal to the direct image \(p_*OS(Q^L)\) under natural map \(p : Q^L \to L\).

Explicitly \((\mathcal{D}^L, \partial^*)\) is a complex spanned by \(dv_I(x), I \subset \text{Atoms}(L)\) with \(\sup(I) = x \in L[0, x]\). The differential \(\partial^*\) acts as follows:

\[
\partial^* dv_I(x) = \sum_{i \in I : \sup(I - \{i\}) = x} (\iota_i dv_I)^{(x)}
\]
The structure morphisms $\partial^{D^L}_{xy} : D^L_x \to D^L_y[1], x : y$ are equal to the resting part of the Koszul differential. The multiplication is straightforward.

We have the following homological interpretation of $\text{OS}(L)$. Consider $\text{OS}(L)$ as $(L, \leq)$-chains algebra with trivial inner differential. There is a map $D(L) \to \text{OS}(L)$ of $(L, \leq)$-chains algebras induced by the identity. In other words sending $d\nu$ to $d\nu \in \text{OS}(L) = \Lambda^*(d\nu)/\sim$ is well-defined, i.e. $\partial^*d\nu = 0 \in \text{OS}(L)$. The following result is proved in [11]:

**Theorem 2.5.2 (Formality of $D(L)$).** The morphism $D(L) \to \text{OS}(L)$ is a quasi-isomorphism of $(L, \leq)$-chains algebras. Thus the atomic algebra is formal, one has a natural isomorphism $H^*(D(L)_x) \simeq \text{OS}(L)_x$ commuting with $\partial_{xy}$.

\[\square\]

## 3 Towards the complement

In this section we will define several versions of the Mayer-Vietoris complex: $MV, (MVH_{CQ}, MVH_{C_C})$, $(MVHD_{CQ}, MVHD_{C_C})$ adapted for calculation of the cohomology and the rational homotopy type of the complement $U = X - \bigcup_{x \in L} L_x$.

In subsection (1) we recall basic preliminaries about sheaves. Then, if $i : (L, \leq)^{op} \to 2^X$ is an arrangement poset with a given OS-complex $M$, we define $MV$ as the convolution $i_* i^* G \ast M$ with a flabby $(L, \leq)$-sheaf over $X$, where $G$ is the Godement resolution of $\underline{\mathbb{Z}}$. This provides a natural resolution $MV \sim j_\ast j^*\underline{\mathbb{Z}}$. If $M$ is OS-algebra and $G$ was multiplicative, then $MV$ is dg-algebra of sheaves.

In subsection (2) we construct a multiplicative model for the cohomology of $U$. Though the similar results appeared in [6], we decided to reprove the corresponding theorems under weaker assumptions. Namely, we omit the requirement on $(L, \leq)$ to be locally geometric, only assuming that the local lattice is graded and an OS-algebra $M$ is given, further we do not assume that intersections of varieties in the arrangement are "clean".

The outline is following. The $L$-grading defines the rank filtration (definition 2.3.15) $W'$ on $i_* i^* G \ast M$ and the associated spectral sequence $W'_E1$ is easily expressed in terms of the arrangement. Then, any OS-algebra $M$ of $(L, \leq)$ allows to introduce a Hodge complex $((MVH_{CQ}, W'), (MVH_{C_C}, F, W'))$. By the construction $MVH_{CQ}$ is a dg-algebra, thus the corresponding $W'_E1$-term tensored by $\mathbb{C}$ is canonically isomorphic to $H^*(U, \mathbb{C})$. This also provides the Hodge structure on the associated graded pure Hodge structure of $H^*(U)$.

In the last subsection (3) we define a Hodge diagram $((MVHD_{CQ}, W'), (MVHD_{C_C}, F, W'))$ in case $(L, \leq)$ is geometric. Then Morgan’s machinery provides an expression for $\mathbb{Q}$-homotopy type of $U$ as the corresponding $W'_E1$-term.

### 3.1 Mayer-Vietoris resolution of $j_\ast j^*\underline{\mathbb{Q}}$

If $V_i$ is a collection of subvarieties in $X$, then for a multi-index $I$ we write $V_I = \bigcap_{i \in I} V_i, V_0 = X$ and $V_I = \bigcup_{i \in I} V_i$. Put $U_i = X - Z_i$, then $U_I = X - Z_I$ and $U_I = X - Z_I$. Consider $Z_\bullet$ and $U_\bullet$ as cubical varieties augmented by inclusions $i_* : Z_\bullet \to X$ and $j_* : U_\bullet \to X$.

**Definition 3.1.1 (see [12], §10).** If $X$ is a smooth manifold, we denote by $S(X)_\bullet \in sSet$ the smooth singular simplicial set of $X$.

**Definition-Construction 3.1.1.** If $A_\bullet$ is a simplicial dg-algebra and $X$ is a smooth manifold, then the presheaf

$$X \to A(X)^q = \text{Hom}_{sSet}(X_\bullet, A^\bullet_\bullet)$$

is naturally a dg-algebra with point-wise multiplication.

Thus $A(pt) = A_\bullet$. The basic fact is that if $A_\bullet \to A'_\bullet$ is a quasi-isomorphism, then $A^*(X) \to A'^*(X)$ is a quasi-isomorphism if $A$ and $A'$ are Kan simplicial objects (see [12]).
Example 3.1.1. (i) Take $A_{PL} \bullet$ to be polynomial differential forms over $\mathbb{Q}$ on the standard simplex, i.e. $A_{PL} = \Omega^*_p(\Delta^n)$.

(ii) Taking $CS_{sm} \bullet = C^*_m, sm(\Delta^n)$ gives a non-commutative dg-algebra denoted by $CS^*_m(X)$. Taking $C^*_m$ defines $CS^*(X)$.

(iii) Similarly we can define $A_{sm}(X)$ over $\mathbb{R}$.

Note that $A(pt) = A_\bullet$. The basic fact is that if $A_\bullet \rightarrow A'_\bullet$ is a quasi-isomorphism, then $A^*(X) \rightarrow A'^*(X)$ is a quasi-isomorphism if $A$ and $A'$ are Kan simplicial objects, thus

Proposition 3.1.1 (see [12], §10). The following natural maps are quasi-isomorphism:

$$A_{PL} \rightarrow CS_{sm} \bullet \leftarrow CS^\bullet_\bullet.$$

The corresponding presheaves are naturally quasi-isomorphic to the singular cochains $C^\bullet$ functor.

Definition 3.1.2. The sheaffication of presheaf $X \rightarrow A^*(X)$ is denoted by $\Delta^\bullet$.

Since all this presheaves are quasi-isomorphic to singular cochains $C^\bullet$, the corresponding sheaffications are also quasi-isomorphic as presheaves.

Definition 3.1.3. We call a sheaf $G \in Sh(X)$ adapted, if $G|_U$ is $\Gamma$-acyclic for all open $U \subset X$.

Proposition 3.1.2. Over smooth manifolds there are natural sheaves of commutative dg-algebras $A_{PL}, A_{Sm}$ over $\mathbb{Q}$ and $\mathbb{R}$ respectively. There are natural quasi-isomorphisms of sheaves of dg-algebras:

$$\Delta_{PL} \otimes \mathbb{C} \cong \Delta_{Sm} \otimes \mathbb{C} \cong \Lambda^* \otimes \mathbb{C}$$

where $\Lambda^*$ is the de Rham sheaf. All quasi-isomorphisms holds on the level of presheaves, i.e. all introduced sheaves are adapted. The sheaffication $\mathcal{C}^\bullet_{sm} \otimes \mathbb{Q}$ of singular cochains is naturally quasi-isomorphic to $\Delta_{PL}$.

Definition 3.1.4. Call $F$ a Godement sheaf, if it has the form $F(U) = \prod_{x \in U} \hat{F}_x$, for some sheaf $\hat{F}$.

Fix $(L, \leq)^{op} \rightarrow 2^X$ to be an arrangement poset enhanced with an OS-complex $\mathcal{M}$. Denote by $i_{ts} : L_s \rightarrow L_t \subset X, s \geq t$ and $j : X = \bigcup_{x \in L} L_x \rightarrow X$ the natural inclusions. Put $i_s := i_{0s}$.

Any sheaf $F$ over $X$ defines a $(L, \leq)$-sheaf $(i_s, i^*_s F)_\bullet$, given by $(i_s, i^*_s F)_s := i_s, i^*_s F$.

Definition 3.1.5 (Mayer-Vietoris complex). For any $(L, \leq)$-sheaf $\mathcal{F}_\bullet$, the Mayer-Vietoris complex $MV(\mathcal{F}_\bullet)$ is the convolution $\mathcal{F}_\bullet * \mathcal{M}_\bullet$.

Recall that $MV(\mathcal{F}_\bullet)$ is $L$-filtrated and thus admits the rank filtration $W^i$:

$$W^i_k MV_\bullet(\mathcal{F}) = \bigoplus_{x \in L, r(x) \leq k} MV_x(\mathcal{F})$$

If $\mathcal{F}_\bullet$ is a $(L, \leq)$-algebra of (complexes of) sheaves and $\mathcal{M}$ is an OS-algebra, then $MV(\mathcal{F}_\bullet)$ becomes a dg-algebra of sheaves.

Note that, if $F$ is a sheaf of dg-algebras, then, since $L_x \cap L_y = \bigcup_{t \in x \cap y} (i_s, i^*_s F)_t$ naturally a $(L, \leq)$-algebra of sheaves. For similarity we force $(i_s, i^*_s F)_\bullet$ to be well-graded. Namely, we put the component $m^i_{xy} : (i_s, i^*_s F)_x \otimes (i_t, i^*_t F)_y \rightarrow (i_r, i^*_r F)_t$ for $t \in x \cap y$ to be zero if $r(t) < r(x) + r(y)$ and equal to the product otherwise.

Let $F$ be a sheaf over $X$.

Proposition 3.1.3. There is a natural augmentation $MV_\bullet := MV((i_s, i^*_s)^\bullet F) \rightarrow j_s, j^\ast F$ of complexes mapping $MV_s$ to zero if $s > 0 \in L$. Assume $F$ is a Godement sheaf then

1. There is a natural inclusion $j_s, j^\ast F \rightarrow F = MV_0 \subset MV_\bullet$ of complexes. It is inverse to the augmentation.
2. The augmentation makes $MV_\bullet$ a flabby resolution of $F$. 

17
Proof. Clearly the augmentation is a map of complexes, since $\partial MV_t, t > 0$ is formed by sections supported on the complement $X - \bigcup_{s>0} L_s$.

We want to show it is a quasi-isomorphism. Take an open $U \subset X$.

Since $F(U) = \prod_{x \in U} G_x$ for some sheaf $G$, we have natural identification:

$$MV_\bullet(U) = \prod_{x \in U} \bigoplus_{t \in L, L_t \supset x} M(t) \otimes G_x$$

and

$$j_* j^* F(U) = \prod_{x \in U : x \notin \bigcup_{s>0} L_s} G_x$$

Note that $S_x := \{t \in L : L_t \ni x\} \subset L$, by definition of $L$, is a sublattice with the maximal element $s_x \in L$, thus $S_x = L[0, s_x]$. Moreover, $s_x > 0$ iff $x \notin X - \bigcup_{s>0} L_s$. Hence

$$(\bigoplus_{L_t \ni x} M(t) \otimes G_x, \partial) = (M[0, s_x], \partial) \otimes G_x$$

is naturally quasi-isomorphic to $G_x$ if $x \in X - \bigcup_{s>0} L_s$ and is acyclic otherwise.

Recall that $M[0, 0] = \mathbb{Z}$. Thus $MV_\bullet(F)(U)$ is a product over $x \in U$ of complexes equal to $G_x$ for $x \in U - \bigcup_{s>0} L_s$ and of acyclic complexes for $x \in \bigcup_{s>0} L_s$. This defines the inclusion in the obvious manner. Clearly both maps are quasi-isomorphism. \hfill \qed

If $F$ is a sheaf of dg-algebras, then by the construction the augmentation is a morphism of dg-algebras.

Recall that the Godement cosimplicial construction applied to $\underline{Q}$ provides a canonical multiplicative (non-commutative) resolution of $\underline{Q} \to G$ by Godement sheaves. Consider the sheafification of singular cochains $C^{sing}_\bullet$. Then there is a natural chain of quasi-isomorphisms of presheaves of dg-algebras $C^{sing}_\bullet \to C^{sing}_\bullet \otimes G \leftarrow G$. It is well-known that $(i_* i^!)_\bullet C^{sing}_{\bullet}$ and $i_* i^! G$ both compute cohomology with support, thus are naturally quasi-isomorphic.

If $j_x : X - L_s \to X$ is an open inclusion, then we denote by $(j_* j^!)_\bullet C^{sing}_{\bullet}$ a $(L, \leq)$-sheaf with value at $x \in L$ equal to $(j_x)_* (j^!_x)_\bullet C^{sing}_{\bullet}$. If $C^{sing}_{\bullet}[X]$ denotes the constant $(L, \leq)$-sheaf on $X$, then $(i_* i^!)_\bullet C^{sing}_{\bullet}$ is quasi-isomorphic to the shifted cone (definition [1.21]) $[C^{sing}_{\bullet}[X] \to (j_* j^!)_\bullet C^{sing}_{\bullet}]$.

Then, by naturality of $MV$ along $(L, \leq)$-sheaves we obtain

**Corollary 3.1.1.** Let $M$ be a OS-complex (-algebra) of arrangement poset $(L, \leq)$ on $X$. There is a natural quasi-isomorphism

$$\underline{Q}_J \sim MV((i_* i^!)_\bullet C^{sing}_{\bullet})$$

of complexes (algebras). Similarly one has a natural quasi-isomorphisms of complexes:

$$\underline{Q}_J \sim MV([C^{sing}_{\bullet}[X] \to (j_* j^!)_\bullet C^{sing}_{\bullet}]) \sim MV([\underline{A}_{PL}[X] \to (j_* j^!)_\bullet \underline{A}_{PL}])$$

### 3.2 Čech model for $i_* i^! C^{sing}_{\bullet}$

In course of the proof of the main theorem we will need a multiplicative Čech resolution model for $i_* i^! G$, let us recall the construction. Here $G$ stands for the Godement multiplicative resolution of $\underline{Q}$ or $C^{sing}_{\bullet}$.

Let $U_i \subset X, i \in N$ be a collections of open subsets and $\underline{A}$ denotes one of the natural sheaves $\Omega^\bullet, C^{sing}_{\bullet}$ etc.

Let $i : X = \bigcup_{U_i} U_i \to X$ and $j_t : U_t \to X$ be the natural inclusions and $g_{ij} : (j_* j^!)_t \underline{A} \to (j_* j^!)_i \underline{A}$ denotes the restriction map for $J \supset I$. We define the corresponding Čech model for $i_* i^! G$ as follows.

Let

$$\check{C}(\underline{A}, U_*, N) = \bigoplus_{1 \leq N} (j_* j^!)_t \underline{A} \cdot d^{t}$$
with an additional Čech differential
\[
\delta_{Ch} = \sum_{I,k} g_{I+(k), I} \cdot d\tau^k \wedge -
\]

Let \( U_+ + U' \) denotes the union of collections \( U_+ \) and \( U' \) parametrized by ordered set \( N \sqcup N' \).

With our notation consider the obvious multiplication map:
\[
m_{U,U'} : \check{C}(A, U_+, N) \otimes \check{C}(A, U'_+, N) \to \check{C}(A, V_+, M)
\]

Let \( U' \subset X, j \in N' \) be other collection of opens.

**Proposition 3.2.1.** With notation above

1. the multiplication is associative, i.e. if \( U_+, U'_+, U''_+ \) are three collections of open subsets, then
\[
m_{U + U', U''} \circ m_{U,U'} \otimes id = m_{U,U''} \circ id \otimes m_{U', U''}
\]

2. there is a natural quasi-isomorphism of sheaves:
\[
\check{C}(A, U_+, N) \sim (i_* i')_{X-U} G
\]

Such that the following diagram is commutative:
\[
\begin{array}{ccc}
\check{C}(A, U_+, N) \otimes \check{C}(A, U'_+, N') & \longrightarrow & \check{C}(A, U_+ + U'_+, M) \\
\downarrow \sim & & \downarrow \sim \\
(i_* i')_{X-U} G \otimes (i_* i')_{X-U} G & \longrightarrow & (i_* i')_{X-V} G
\end{array}
\]

**Proof.** Associativity is obvious. Then, using the natural quasi-isomorphism: \( \check{C}(G, U_+, N) \sim \check{C}(A, U_+, N) \) we reduce to case \( A = G \). Since the sheaf \( i_* i' G \) is naturally a multiplicative subcomplex in \( G \), which is the first term of \( \check{C}(G, U_+, N) \).

**3.3 Hodge complex \((MVHC_Q, MVHC_C)\)**

Here \( (L, \leq) \) is an arrangement poset over \( X \) with an OS-complex \( M \). Recall that \( U = X - \bigcup_{s>0} L_s \). We shall construct a functorial replacement for \( i_* i' G \) to define a Hodge complex of sheaves.

**3.3.1 The construction**

Recall that, for a subset \( I \subset Atoms(L) \) we write \( L^I = \bigcup_{x \in I} L_x \) and \( L_I = \bigcap_{x \in I} L_x \).

Let \( C \) be the cubical poset with vertices corresponding to all subsets of \( Atoms(L) \). Consider the open \( C \)-subspace \( U_* \) of \( X \), defined by \( U_I = X - L^I \) and \( U_I \subset Atoms(L) \). Thus \( U_0 = X \) and \( U_0 = X - L^a \to U_0, r(a) = 1 \) etc. For each \( x \in L \) let \( I_x := \{ t \in Atoms(L) : t \leq x \} \). Since \( I_x \subset I_y \) for \( x \leq y \) we may consider \( I : (L, \leq) \to (C, \leq) \) as an order preserving map. By the definition of the arrangement poset \( L_{I_x} = L_x \).

**Proposition 3.3.1.** There is a smooth compact algebraic \( C \)-variety \( \check{U} \), with a morphism \( U_* \rightarrow \check{U} \), which is NCD-embedding and \( \check{U}_0 = X \).

**Proof.** The diagram of spaces \( U_* \) is ranked. So that \( U_0 = X, U_a \to X, r(a) = 1 \) etc. Put \( \check{U}_0 = X \), we construct \( \check{U} \) by induction on \( r(I) = |I| \). If \( \check{U}_I \) for \( r(I) < h \) are given, then for \( J, r(J) = h \) take the diagonal embedding \( j : U_J \to \prod_{I < j} \check{U}_I \). By Hironaka’s theorem there is an extension of \( j \) to a smooth projective \( \check{U}_J \) such that \( U_J \rightarrow \check{U}_J \) is an NCD-embedding.

\( \square \)
Let \( g_1 : \tilde{U}_I \to \tilde{U}_0 = X \) be the natural map. We can consider a \( C^{op} \)-sheaf on \( X \)

\[
\Omega^{C^{op}}_{\log}(X - L^\bullet) : C \to \text{Sh}(X)
\]
defined as

\[
\Omega^{C^{op}}_{\log}(X - L^I) = (g_1)_* \Omega^{C^{op}}_{\tilde{U}_I}(\ln \tilde{U}_I - U_I)
\]
The RHS can be viewed as a subsheaf of \( j_{g_1}^! \Omega_X \) and is formed by forms \( \omega \) on \( X \) with appropriate singularities near \( L^I \), meaning that the pull-back of \( g_1^* \omega \) has logarithmic singularities along \( \tilde{U}_I - U_I \). Since \( \tilde{U}_\bullet \) is a diagram of varieties, the restriction induced by \( X - L^I \to X - L^J, I \subset J \) of a form on \( X \) with appropriate singularity near \( L^I \) has an appropriate singularity near \( L^J \).

Define \( A_P^{op}(X - L^\bullet) : C \to \text{Sh}(X) \) by \( A_P^{op}(X - L^I) = \Delta_P(X - L_\bullet^I) \) and similarly \( \Omega^{C^{op}}(X - L^\bullet) \).

Clearly \( A_P^{op}(X - L^\bullet) \otimes C, \Omega^{C^{op}}(X - L^\bullet) \) and \( \Omega^{C^{op}}(X - L^\bullet) \) are naturally quasi-isomorphic \( C^{op} \)-sheaves. Denote by \( \Omega^{C^{op}}_{\log}(X, X - L_\bullet) \) the \( C \)-sheaf formed by terms \( \Omega^{C^{op}}_{\log}(X, X - L_\bullet) \).

Let \( \Omega^{C^{op}}_{\log}(X, X - L_\bullet) \) denotes a \( C \)-algebra of sheaves on \( X \) defined as the Čech resolution (proposition 3.2.1):

\[
\Omega^{C^{op}}_{\log}(X, X - L_I) = \bigoplus_{K \leq I} \Omega^{C^{op}}_{\log}(X - L^K) \cdot d_I^K = [\Omega^*(X) \to \bigoplus_{a \leq x, r(a) = 1} \Omega^{C^{op}}_{\log}(X - L^a) \to \ldots]
\]

Here \( d_I^K \) are usual fictive Grassmann monomials with \( deg(d_I^K) = |K| \) such that the differential is equal to \( d_{\text{Ch}} = \sum g^{C^{op}}_{K + (k), K} d_I^K \wedge - \), where \( g^{C^{op}}_{K + (k), K} \) is the restriction map which is the structure map of \( C^{op} \)-sheaf \( \Omega^{C^{op}}_{\log}(X - L^\bullet) \).

The structure maps \( g^{C^{op}}_{g_I^J} : \Omega^{C^{op}}_{\log}(X, X - L^I) \to \Omega^{C^{op}}_{\log}(X, X - L_J), I \subset I' \) are defined by mapping the component \( \Omega^{C^{op}}_{\log}(X - L^K), K' \subset I' \) identically if \( K' \subset I \) and to zero otherwise. On the level of Čech resolutions there is a natural map

\[
\Omega^{C^{op}}_{\log}(X, X - L_I) \otimes \Omega^{C^{op}}_{\log}(X, X - L_J) \to \Omega^{C^{op}}_{\log}(X, X - L_{I\cup J}), I \cap J = \emptyset
\]

which acts on components \( \Omega^{C^{op}}_{\log}(X - L^K), \tau^K, K \subset I \) and \( \Omega^{C^{op}}_{\log}(X - L^{K'}), \tau^{K'}, K' \subset I' \) as the inclusion to \( \Omega^{C^{op}}_{\log}(X - L^{KK'}) \) followed by the usual multiplication of forms. Here we used the fact that the log-complex is dg-algebra.

The natural quasi-isomorphisms of the previous section provides the natural quasi-isomorphism \( \Omega^{C^{op}}_{\log}(X, X - L_\bullet) \sim \Omega^{C^{op}}_{\log}(X, X - L_\bullet) \otimes \Omega^{C^{op}}_{\log}(X, X - L_\bullet) \otimes \Omega^{C^{op}}_{\log}(X, X - L_\bullet) \).

Denote by \( \Omega^{L^{op}}_{\log}(X, X - L_\bullet) \) \((L, \leq)\)-sheaf with values \( \Omega^{L^{op}}_{\log}(X, X - L_t) := \Omega^{C^{op}}_{\log}(X, X - L_t), t \in L \).

Similarly for \( A_P^{op}(X - L_\bullet) \), \( \Omega^{L^{op}}_{\log}(X, X - L_\bullet) \) and \( \Omega^{L^{op}}(X, X - L_\bullet) \).

Contrary, we define

\[
\Omega^{L^{op}}_{\log}(X, X - L_\bullet) := \Omega^{C^{op}}_{\log}(X, X - L_\bullet)
\]
as an \((L, \leq)\)-sheaf given by the pulling back along \( I : (L, \leq) \to (C, \leq) \).

Clearly \( \Omega^{L^{op}}_{\log}(X, X - L_\bullet) \otimes \Omega^{C^{op}}_{\log}(X, X - L_\bullet) \) and \( \Omega^{L^{op}}_{\log}(X, X - L_\bullet) \) are quasi-isomorphic.

Thus the data we needed so far is an arrangement poset \((L, \leq)^{op} \to X\), the OS-complex \( \mathcal{M} \) of \((L, \leq)\) and the Hironaka’s compactification \( U_\bullet \to \tilde{U}_\bullet \).

Recall that \( W' \) denotes the rank filtration (definition 2.3.15).

**Definition 3.3.1** (Mayer-Vietoris Hodge complex). Using convolutions define

\[
MVHC_{Q}(L, \mathcal{M}) = (\Omega_{\log}^{L^{op}}(X, X - L_\bullet) \ast \mathcal{M}, W')
\]

\[
MVHC_{C}(L, \mathcal{M}) = (\Omega_{\log}^{L^{op}}(X, X - L_\bullet) \ast \mathcal{M}, F, W')
\]
then the rank Hodge complex

\[
MVHC(L, \mathcal{M}) := ((MVHC_{Q}(L, \mathcal{M}), W'), (MVHC_{C}(L, \mathcal{M}), F, W'))
\]
is called the rank Mayer-Vietoris Hodge complex.
All filtrations are provided by the lemma 4.4.1 on convolutions of Hodge complexes. Similarly, the lemma implies the existence of the natural Hodge complex structure \((MVHC_{\mathcal{Q}}, W), (MVHC_{\mathcal{C}}, F, W)\).

Below we will often omit \((L, \leq)\) and \(\mathcal{M}\) from the notation. Thus \(MVHC_{\mathcal{Q}}\) is equal to \(MV(\mathcal{I}, j_f \mathcal{Q}_{\text{sing}})\) introduced in previous section.

**Remark 3.3.1.** If \(\mathcal{M}\) is an OS-algebra, then \(MVHC_{\mathcal{Q}}(\mathcal{M})\) is dg-algebra. However, since \(I : L \rightarrow \mathcal{C}\) is not a homomorphism in general, the pull-back \(\Omega^k(X, X - L_*) = I^* \Omega^k(X, X - L_*)\) is not \((L, \leq)\)-algebra. Thus \(MVHC_{\mathcal{C}}\) in general does not have a natural algebra structure.

### 3.3.2 An explicit expression for \(MVHC(\mathcal{M})\)

For clarity let us describe \(MVHC\) explicitly. Since \(L_{\mathcal{I}} = L_x\) one can write:

\[
MVHC_{\mathcal{Q}} = \bigoplus_{x \in L} \mathcal{Q}_{\text{sing}}(X, X - L_x) \otimes \mathcal{M}_x
\]

with the differential \(d + \partial\), where \(d\) is the inner differential on cochains and \(\partial = \sum_{x' \leq x} \partial_{x'x} v_x\) is the sum of degree 1. Then \(d, \partial\) clearly commute if we impose the usual sign-rules, e.g.:

\[
\partial(c_x \otimes v_x) = \sum_{x' \leq x} (-1)^{deg(c_{x'})} c_{x'} \otimes \partial_{x'x} v_x
\]

Similarly

\[
MVHC_{\mathcal{C}} = \bigoplus_{x \in L} \mathcal{Q}_{\text{log}}^{\text{op}}(X - I^K) \cdot d\tau^K \otimes \mathcal{M}_x
\]

where \(d\tau^K\) are usual fictive monomials of the corresponding Grassmann algebra, here \(deg(d\tau^K) = |K|\).

The whole differential is \(d_{\text{DR}} + \partial + \delta_{\text{Ch}}\). Here \(d_{\text{DR}}\) is de Rham differential, \(\partial = \sum_{x' \leq x} g_{x'x} \partial_{x'x}\) and \(\delta_{\text{Ch}} = \sum_k (\delta_{K+x} \cdot (k) \cdot d\tau^k \wedge -)\), we consider structure maps \(g\) having degree 0. All maps between different vertices in cubical diagrams of sheaves are geometric and induced by maps of compactified algebraic varieties with NCD-complement.

Thus \(C^{\text{op}}\)-diagram \((\mathcal{Q}_{\text{sing}}(X - L_*), \mathcal{Q}_{\text{log}}(X - L_*))\) form a \(C^{\text{op}}\)-diagram of Hodge complexes with the Hodge and weight filtrations equal to \(F, W\). Then \((\mathcal{Q}_{\text{sing}}(X, X - L_*), \mathcal{Q}_{\text{log}}(X, X - L_*))\) is \((L, \leq)\)-diagram of Hodge complexes.

Since diagram \((L, \leq)\) is graded, one can consider the convolution as an iterated cone, then the mixed cone construction (definition 1.2.1) produces the natural Hodge complex \((MVHC, F, W)\) and the rank Hodge complex \((MVHC, F, W')\) by the lemma 4.4.1.

### 3.3.3 The homotopy type of \(U\)

**Proposition 3.3.2.** The hypercohomology of the Hodge complex \((MVHC_{\mathcal{Q}}, W), (MVHC_{\mathcal{C}}, F, W)\) gives a mixed Hodge structure naturally isomorphic to the cohomology of the complement \(H^\bullet(U, U)\), for \(U = X - \bigcup_{x \geq 0 \in L} L_x\).

**Proof.** Recall that \(\tilde{U}_N, N = \text{Atoms}(L)\) denotes the chosen Hironaki compactification of the complement \(X - L^N = U_N = U\). Since the morphisms between mixed Hodge structures are strict, it is enough to provide a quasi-isomorphism of \((MVHC_{\mathcal{C}}, F, W)\) with the log-complex \((\mathcal{Q}_{\text{log}}^t(\ln \tilde{U}_N - U), F, W)\) compatible with both-filtrations. To do this let \(\Omega_{\mathcal{C}}^{\text{log}}\) be the constant \(C^{\text{op}}\)-sheaf with \(\Omega_{\mathcal{C}}^{\text{log}}(X - L^N)\). We have a natural restriction map \(\Omega_{\mathcal{C}}^{\text{log}}(X - L^N) \rightarrow \Omega_{\mathcal{C}}^{\text{log}}\). Applying the construction of \(\Omega_{\mathcal{C}}^{\text{log}}(X, X - L_*)\) to \(\Omega_{\mathcal{C}}^{\text{log}}\) instead of \(\Omega_{\mathcal{C}}^{\text{log}}(X, X - L_*)\) we get \(\mathcal{C}\)-sheaf \(\Omega_{\mathcal{C}}\). Then we have a natural restriction morphism:

\[
\Omega_{\mathcal{C}}(X, X - L_*) \rightarrow \Omega_{\mathcal{C}}^{\text{log}}
\]

Note that \(\check{\text{C}}\text{ech}\) resolutions \(\Omega_{\mathcal{C}}^{\text{log}}\) are acyclic except for \(I = \emptyset\) with \(\Omega_{\mathcal{C}}^{\text{log}} = \Omega_{\mathcal{C}}^{\text{log}}(\ln \tilde{U} - U)\). Then we have a chain of quasi-isomorphisms compatible with filtrations \(F, W\):

\[
\Omega_{\mathcal{C}}^{\text{log}}(\ln \tilde{U} - U) \sim I^* \Omega_{\mathcal{C}}^{\text{log}} \ast \mathcal{M} \sim I^* \Omega_{\mathcal{C}}^{\text{log}}(X, X - L_*) \ast \mathcal{M}
\]

which follows since all three complexes are natural resolutions of \(j_* j^* \mathcal{C}\).
The following is clear.

**Proposition 3.3.3.** The spectral sequence corresponding to the rank Hodge complex structure $((MVHC_Q(M), W'), (MVHC_C(M), F, W'))$ satisfy the following:

1. \[ W' E_{1}^{pq}(MVHC_Q, W') = \bigoplus_{x \in L: r(x) = -p} H^q(X, X - L_x) \otimes \mathcal{M}_x \]

2. The differential is equal to \( d_1 = \sum g_{x'} \otimes \partial_{x'x} \), where \( g_{x'} : H^1_{x'}(X) \to H^1_{L_{x'}}(X) \) are natural homomorphisms for \( x' : x \in L \) and \( \partial_{x'x} \) are the structure maps of the OS-algebra \( M \).

3. \( W'E_2 = W'E_\infty \)

Thus \( E_1 \) term is simply the convolution of \((L, \leq)-\text{algebra} \ H^*(X, X - L_\bullet) \) with \( M \).

Assume \( M \) is an OS-algebra. Then, according to the remark 3.3.1 \( MVHC_Q(M) \) is a dg-algebra. Since the spectral sequence is multiplicative, it converges to the algebra \( W' Gr \ H^*(U, \mathbb{C}) \).

Moreover, since the weight filtrations \( W, W' \) canonically split over \( \mathbb{C} \), one can use the spectral sequence to describe the algebra \( H^*(U, \mathbb{C}) \). As a direct corollary of what been done, we conclude with the following theorem.

**Theorem 3.3.1** (The cohomology of the complement). Let \( M \) be an OS-complex (algebra) of an arrangement poset \((L, \leq)\) on compact algebraic \( X \). Consider the corresponding Mayer-Vietoris Hodge complex \((MVHC_Q, MVHC_C)\). There is a canonical isomorphism of vector spaces (algebras) \[ E_2^*(MVHC_C, W') \simeq H^*(U, \mathbb{C}) \]

It induces (remark 3.2.1) an isomorphism of pure Hodge structures of weight \( q \):

\[ E_2^{pq}(MVHC_C, W') \simeq Gr_{q}^{Dec} W' H^{p+q}(U, \mathbb{C}) \]

**Proof.** Let \( H^n = H^n(U) \simeq H^n(MVHC_Q) \). Recall that \((H^n, F, Dec W)\) is a mixed Hodge structure with \((Dec W)_q H^n = W^{n-q} H^n\) and thus the weight of \( W' Gr^p H^{p+q} \) is \( q \). The Deligne splitting (proposition 3.2.2) implies the natural isomorphism \[ H^*(U, \mathbb{C}) \simeq W' Gr^* H^*(U, \mathbb{C}) \]

of algebras. Since \( E_2^{pq}(MVHC_C, W) = E_2^{pq}(MVHC_C, W') = W' Gr^p H^{p+q}(U) \) the claim follows.

**Remark 3.3.2.** This result is also proved in [6] by a similar method. We want to note, that the splitting does not holds in general over \( \mathbb{Q} \). Even over \( \mathbb{R} \) the Deligne splitting of the weight filtration is not conjugate-invariant.

Since a non canonical descent of the quasi-isomorphism still exists (theorem 3.2.3), we have:

**Corollary 3.3.1.** If \( M \) is OS-algebra, then there is a non-canonical isomorphism \( E_2(MVHC_Q, W') \simeq H^*(U, \mathbb{Q}) \) of algebras.

**Proof.** Morgan’s theorem 3.2.3 applied to a NCD-compactification of \( U \) implies the existence of a non canonical isomorphism of algebras \( Gr(H^*(U, \mathbb{Q}), W) \simeq H^*(U, \mathbb{Q}) \). The result follows, since \[ E_2(MVHC_Q, W') \simeq E_2(MVHC_Q, W) \simeq Gr(H^n(U, \mathbb{Q}), W) \]
3.4 Hodge diagram \((MV HD_Q, MV HD_C)\)

According to remark 3.3.1 the map \(I : L \rightarrow C\) is not a homomorphism of lattices, thus in order to obtain an algebra structure on a complex part of Mayer-Vietoris complex we have to impose an additional requirements on the arrangement poset \((L, \leq)\).

Assume \((L, \leq)\) is good arrangement poset (definition 2.2.2), i.e. \((L, \leq)\) is geometric lattice and hence \(L_I = L_x\) for \(x = sup(I) \in L\).

In this case the quasi-cubical lattice \(Q^L\) coincides with the cubical lattice \(C\) introduced above and \(OS(C)\) is just the Grassman algebra \(\Lambda^* (dx_I, I \in Atoms(L))\) viewed as \((L, \leq)\)-chains algebra.

Let \(p : (C, \leq) \rightarrow (L, \leq)\) be the homomorphism of lattices defined by \(p(I) = sup(I) \in L\).

Definition 3.4.1. Let

\[
MV HD_Q(L) := \text{Tot} (p_*(A^I_{PL}(X, X - L_\bullet) \otimes OS(C)))
\]

\[
MV HD_C(L) := \text{Tot} (p_*(\Omega^I_{log}(X, X - L_\bullet) \otimes OS(C)))
\]

The pair

\[
((MV HD_Q, W'), (MV HD_C, F, W'))
\]

is called Mayer-Vietoris Hodge Diagram.

Thus \(MV HD\) is a convolution over cubical lattice with the rank filtration induced by that of \(L\). Note that the lemma 2.4.1 on convolutions implies that \(((MV HD_Q, W), (MV HD_C, F, W))\) is a Hodge complex. Despite the definition given above we need an additional argument to prove that the rank filtration \(W'\) indeed induces the Hodge complex structure on \(MV HD\).

Explicitly

\[
MV HD_C = \bigoplus_{I \subset Atoms(L)} \bigoplus_{k < I} \Omega^{I\cap\nu}_r (X - L^K) \cdot d^K \cdot dv_I
\]

All terms with \(r(sup(I)) \leq k\) contributes to \(W'_I MV HD_C\). The whole differential has four components \(d_{dR}, \delta_{CH}, \partial', \partial''\), where \(d_{dR}\) and \(\delta_{CH}\) are de Rham and Čech differentials as for \(MV HC\) respectively, while \(\partial = \partial' + \partial''\) corresponds to the Koszul differential on \(OS(C)\) and decomposes according to \(L\)-filtration:

\[
\partial' = \sum_{i: sup(I) = sup(I - \{i\})} g^r_I - (i) \cdot dv_I.
\]

Recall that \(j : X - \bigcup_{x > 0 \in L} L_x \rightarrow X\) is inclusion of the complement.

Proposition 3.4.1. 1. The pair \(((MV HD_Q(L), W'), (MV HD_C(L), F, W'))\) form a Hodge diagram.

2. There is a natural quasi-isomorphism of Hodge complexes:

\[
(MV HD(L), F, W') \sim (MV HC(L, OS(L)), F, W')
\]

Thus \(MV HD\) is a cdga resolution \(j_* j^* \Omega\).

Proof. Let \(MV HC = MV HC'(L, OS(L))\) denotes a Hodge complex obtained by \(MV HC'_Q = (A^I_{PL}(X, X - L_\bullet) \star M, W')\) and \(MV HC'_C = \Omega^I_{log}(X, X - L_\bullet) \star M, F, W')\). Thus \(MV HC'_C = MV HC_C\).

Since \(MV HD\) is already a filtered algebra all claims will follow if we show an existence of a natural quasi-isomorphisms \(MV HD(L) \sim MV HC'(L, OS(L))\) compatible with the rank filtration \(W'\) and giving an isomorphism of \(W' E_1 \otimes \mathbb{C}\)-terms compatible with the Hodge filtration.

We focus on the complex part, since the rational case is similar. Consider \(I \circ p : C \rightarrow C\). This is a order-preserving selfmap of the lattice such that \((I \circ p)(J) = I(sup(J)) \supset J\), applying lemma 2.3.1 on endomorphism we get a morphism of \((C, \leq)\)-sheaves:

\[
p^* I^* \Omega^I_{log}(X, X - L_\bullet) \rightarrow \Omega^I_{log}(X, X - L_\bullet)
\]

Since in geometric lattice case \(L_I(sup(J)) = L_J\) this map is a quasi-isomorphism.

Then we have a quasi-isomorphism of \((L, \leq)\)-sheaves:

\[
p_*(p^* I^* \Omega^I_{log}(X, X - L_\bullet) \otimes OS(C)) \rightarrow p_*(\Omega^I_{log}(X, X - L_\bullet) \otimes OS(C))
\]
By the projection formula of lemma [2.5.2] this is equivalent to a quasi-isomorphism

$$I^*\Omega^C_{\log}(X, X - L) \otimes \mathcal{D}(L) \to p_* (\Omega^C_{\log}(X, X - L) \otimes OS(C))$$

Since the atomic complex $\mathcal{D}(L) = p_* OS(C)$ is formal (theorem [2.5.2]), i.e. quasi-isomorphic to $OS(L)$, applying $Tot(-)$ to both sides we get a filtered quasi-isomorphism of complexes compatible with the Hodge filtration:

$$(MVHC_C, W) \sim (MVHD_C, W')$$

\[\square\]

**Remark 3.4.1.** Note that our resolution is actually the cubical resolution from the previous section, the role of the lattice $L$ is to control the rank filtration making $E_1$ term more accessible.

Thus $W^1E_1$-term of $MVHD_Q(L)$ have the same description as $MVHC_Q(L, OS(L))$ (proposition 3.3.3).

Recall that $OS(L)_x$ sits in degree $-r(x)$. The natural multiplication is non-zero only on components $x, y \in L$ with $r(x \vee y) = r(x) + r(y)$ and define a cdga structure on $\bigoplus_x OS_x(L)$ by $OS_x(L) \otimes OS_y(L) \to OS_{x \vee y}(L)$ with differential $\partial = \sum_{y < x} \partial g_{yx}$.

As a corollary we get

**Theorem 3.4.1 (The main theorem for lattices).** Assume $(L, \leq)$ is an arrangement poset on $X$ with the complement $X - \bigcup_{x > 0 \in L} L_x$ and $(L, \leq)$ is a geometric lattice. Then there is a multiplicative spectral sequence $E^{pq}_1$ such that:

1. $E^{pq}_1 = \bigoplus_{x \in L, r(x) = -p} H^q(X, X - L_x) \otimes OS(L)_x$
   
   is a cdga with the differential $\partial_1 = \sum_{y < x} g_{yx} \otimes \partial g_{yx}$, where $g_{yx} : H^i_{L_x}(X) \to H^i_{L_y}(X)$ are the natural homomorphisms.

2. $E_2 = E_\infty$.

3. There is a natural quasi-isomorphism of dg-algebras

   $$E^{**}_1 \otimes \mathbb{C} \sim C^*_{\text{sing}}(X - \bigcup_{x > 0 \in L} L_x; \mathbb{C})$$

   thus $E^{**}_1 \otimes \mathbb{C}$ provides a natural model for $\mathbb{C}$-homotopy type of the complement.

4. There is a non canonical quasi-isomorphism of $E^*_1$ with the rational homotopy type of the complement.

**Proof.** Here $E_r = E_r (MVHD_Q, W')$ and since $MVHD$ is a Hodge diagram which is quasi-isomorphic to $MVHC$ as a Hodge complex, the above description of $E_1(MVHC_Q, W')$ and Morgan’s theorem [1.2.3] immediately yields the result. The minor issue is to pass from the Hodge diagram $MVHD$ of sheaves to a Hodge diagram of vector spaces. To do this one can consider, for example, a pair of dg-algebras $((\Gamma MVHD_Q, W'), (\Gamma(MVHD_C \otimes \mathbb{C}, \Omega^{\bullet+}_{\mathbb{C}}), F, W'))$. Here the functor of global sections of the complex part is followed by the Dolbeault resolution (see [1] part 2, 3.2.3). This provides a Hodge diagram to which the theorem [1.2.3] is applied. \[\square\]

Since the cubical lattice is always geometric, the following result is most simple and general result of this type.

**Corollary 3.4.1 (The main theorem).** For a smooth arrangement $Z_1 \subset X$ of projective varieties there is a multiplicative spectral sequence $E^{pq}_1$:

1. $E^{pq}_1 = \bigoplus_{|I| = -p} H^q_{Z_1}(X) \cdot d\nu_I$
2. The differential 
\[ d_1 = \sum g_{I - \{i\}, I} \otimes i, \text{ where } g_{I', I} \subset I \text{ are natural homomorphisms:} \]
\[ H^*_Z(X) \to H^*_Z(X) \]
induced by inclusion \( Z_I \to Z_{I'} \).

3. The multiplication
\[ H^*_Z(X) \cdot d\nu_I \otimes H^*_Z(X) \cdot d\nu_J \to H^{q+q'}_{Z_{IJ}}(X) \cdot d\nu_I \land d\nu_J \]
makes \( E_1^{pq} \) into cdga.

4. There is a natural quasi-isomorphism of cdga
\[ E_1^{*+} \otimes \mathbb{C} \sim C_{sing}^*(X - \bigcup_i Z_i; \mathbb{C}) \]

5. There is a non canonical quasi-isomorphism of rational homotopy type of \( X - \bigcup_i Z_i \) with \( E_1^{*+} \).

4 Applications

4.1 Formality of linear subspace arrangement with geometric intersection poset

Consider an intersection poset of linear subspaces in \( \mathbb{C}^n \). Let \( (L, \leq) \) be the corresponding intersection poset. Assume that \( (L, \leq) \) is geometric. The following result is proved in [11]:

**Theorem 4.1.1 (Feichtner, Yuzvinsky).** The complement \( \mathbb{C}^n - \bigcup_{x>0} L_x \) is formal.

To prove this theorem using our methods let \( X = \mathbb{P}^n \supset \mathbb{C}^n \) and \( \tilde{L} \) be the intersection poset of the compactification of \( L \) together with a hyperplane at infinity \( \tilde{L}_\infty := \mathbb{P}^{n-1} \). Clearly \( U := \mathbb{P}^n - \bigcup_{x \in \tilde{L}} L_x = \mathbb{C}^n - \bigcup_{x \in \tilde{L}} L_x \).

Let us first describe a simple model for the homotopy type of \( U \) without any assumptions on the intersection poset. By our results on the cubical model the following cdga controls the rational homotopy type of \( U \):

\[ ( \bigoplus_{I \subset \text{Atoms}(\tilde{L})} H^*_{\tilde{L}_I}(\mathbb{P}^n) \cdot d\nu_I, \partial) \]

Recall that \( d\nu_I \) has degree \(-|I|\) and the differential \( \partial = \sum_i i_i \) is the whole Koszul differential.

Since \( H^*(\mathbb{P}^n) = \mathbb{Q}[t]/(t^{n+1}), \deg(t) = 2 \) and \( \tilde{L}_I \) are linear one can identify \( H^*_{\tilde{L}_I}(\mathbb{P}^n) \subset H^*(\mathbb{P}^n) \) with \( t^{\text{codim } L_I}(\mathbb{P}^n) \) for \( \tilde{L}_I \neq \emptyset \) and with \( 0 \) otherwise. This description is compatible with the algebra structure, the element \( t^{\text{codim } L_I} \) corresponds to the Thom class \( \tau_I \) of \( L_I \).

Further, since taking the direct product with an affine space does not change the homotopy type of the affine complement and the codimension of subspaces in the arrangement, one can assume that for any \( I \subset \text{Atoms}(\tilde{L}), \tilde{L}_I \subset \mathbb{P}^n \neq \emptyset \).

Consider the following complexes over \( \mathbb{Q}[t] \):

1. \( K = \bigoplus_{I \subset \text{Atoms}(\tilde{L})} t^{\text{codim } L_I} \mathbb{Q}[t]/(t^{n+1} \cdot d\nu_I) \)
2. \( K' = \bigoplus_{I \subset \text{Atoms}(\tilde{L})} t^{\text{codim } L_I} \mathbb{Q}[t] \cdot d\nu_I \)
3. \( K'' = \bigoplus_{I \subset \text{Atoms}(\tilde{L})} d\mu_I \)
The Koszul differential naturally acts on each complex. We consider $K$ and $K'$ as cdga’s with the multiplication $d\nu_1 \cdot d\nu_2 = d\nu_1 \wedge d\nu_2$. By our assumption $K$ controls the rational homotopy type of $U$.

Define $i : K'' \to K'$ by $i(d\mu_t) = t^{n+1}d\nu_1$ and $p : K' \to K$ to be the natural projection. Both maps commute with the Koszul differential and $p$ is a morphism of cdga’s. Clearly, $0 \to K'' \xrightarrow{\partial} K' \xrightarrow{p} K \to 0$ is exact. The Koszul complex $K''$ is acyclic and one can pass from $K$ to $K'$. It is natural to think of $K'$ as a stable version of the model $K$ as $n \to \infty$.

Let $s : K' \to K'[1]$ be the multiplication by $t \cdot d\nu_\infty \in K'$.

**Lemma 4.1.1.**

1. The commutator $[s, \partial] : K' \to K'$ is the multiplication by $t$.

2. The subcomplex $sK' + tK' \subset K'$ is acyclic.

**Proof.** (1) It is straightforward.

(2) By the previous $sK' + tK'$ is a subcomplex. Let $sc_1 + tc_2$ be a cocycle, i.e. $0 = \partial(sc_1 + tc_2) = -sdc_1 + tc_1 + tdc_2$. Then, since $s^2 = 0$, $t(sc_1 + sdc_2) = 0$, thus $sc_1 + sdc_2 = 0$. Hence $\partial(sc_2) = -sdc_2 + tc_2 = sc_1 + tc_2$ is a coboundary.

As a corollary the rational homotopy type of $U$ is quasi-isomorphic to $M := K'/(sK' + tK')$. Factorizing $sK'$ out of $K'$ we reduce the lattice $\tilde{L}$ to $L$. The following factorization by $tK'$ leaves elements spanned by Thom classes. Thus:

$$M = \bigoplus_{I \subset \text{Atoms}(L)} \tau^{\text{codim } \tilde{L}_I} \cdot d\nu_I$$

where $\tau$ is a formal variable of degree 2. The differential $\partial$ acts by

$$\partial d\nu_I = \sum_{i \in I} \nu_i d\nu_I$$

making $K'/(sK' + tK')$ into cdga.

Now we may back to the theorem.

**Proof.** Assume $(L, \leq)$ is a geometric lattice Then $\tilde{L}_I$ is a compactification of $L_I$ and thus $M = \bigoplus_{x \in L} M_x$, where

$$M_x = \bigoplus_{I \subset \text{Atoms}(L)} \tau^{\text{codim } \tilde{L}_I} \cdot d\nu_I = \bigoplus_{I \subset \text{Atoms}(L)} \tau^{\text{codim } L_x} \cdot d\nu_I$$

By formality of the atomic complex (theorem 2.5.2) $M_x$ is naturally quasi-isomorphic to $H^*(M_x) \cong \tau^{\text{codim } L_x} \cdot OS(L)_x$.

This implies formality of $U$:

$$M \sim \bigoplus_{x \in L} \tau^{\text{codim } L_x} \cdot OS(L)_x$$

where the algebra structure on the right hand side is such, that non zero multiplication occurs only for components $x, y \in L$ intersecting transversally, i.e. codim $L_x \cap L_y = \text{codim } L_x + \text{codim } L_y$.

### 4.2 Rational model of configuration spaces

Assume $M$ is a smooth proper algebraic variety over $\mathbb{C}$. Consider the intersection poset $(L, \leq)$ of all diagonals in $X = M^n$ (example 2.2.1). Let $F(M, n) = X - \bigcup_{x > 0 \in L} L_x$ be the configuration space of $n$-tuples of ordered points in $M$. More generally, for any finite graph $G$ with vertices $V(G) = \{1, \ldots, n\}$ one can consider the chromatic configuration space $F(M, G) = \{(x_1, \ldots, x_n) \in M^n : x_i \neq x_j, (ij) \in E(G)\}$. The variety $F(M, n)$ corresponds to the complete graph on $n$ vertices.

It is well-known that $(L, \leq)$ forms a geometric lattice which has an explicit description in terms of $G$, see [5]. Namely, an element $x \in L$ is an unordered collection of disjoint subsets $S_k \subset V(G)$ such that $S_k$
is connected in $G$. The diagonal $L_x$ in $M^n$ is determined by the condition $x_i = x_j$ whenever $i, j \in S_k$ for some $k$. Its codimension is equal to $\dim M \cdot r(x)$. The atoms $\Delta_{ab} \in L$ correspond to big diagonals $L_{\Delta_{ab}} \subset M^n$. An independent set of atoms corresponds to a set of edges in $G$ containing no cycles.

Recall that the corresponding OS-algebra $OS(L)$ is a quotient of the Grassman algebra $\Lambda(\Delta_{ab}, (ab) \in E(G))$ with $\deg(\Delta_{ab}) = -1$, modulo an ideal generated by expressions $\partial(\Delta_{i_1i_2} \wedge \ldots \wedge \Delta_{i_{k-1}i_k})$, where $(i_1, \ldots, i_k = i_1)$ form a cycle in $G$. As usual $\partial = \sum_{i \in \text{Atoms}(L)} \iota_i$ denotes the whole Koszul differential in the Grassman algebra. If $G$ is complete, the corresponding ideal is generated by famous Arnold’s relations

$$\Delta_{ab} \Delta_{bc} + \Delta_{bc} \Delta_{ca} + \Delta_{ca} \Delta_{ab}$$

Thus, the Mayer-Vietoris model controlling the homotopy type of $F(M, G)$ is a cdga

$$\left( \bigoplus_{x \in L} H^*_L(M^n) \otimes OS(L)_x, d \right)$$

provided by our main theorem for lattices.

Let $\tau_x \in H^{2r(x) \cdot \dim M}(M^n)$ be the Thom class corresponding to $L_x \subset M^n$. Denote by $p_i : M^n \to M$ the projection onto $i$-th coordinate. If $x = \Delta_{ab}$ then

$$H^*_L(M^n) = \tau_x \cdot H^*(M^n) \simeq H^*(M^n)/(p^*_a u = p^*_b u, u \in H^*(M))[-2 \dim M]$$

If $x, y \in L$ are independent, then $L_x$ intersects $L_y$ transversally and hence $\tau_x \cdot \tau_y = \tau_x \cup \tau_y$. Thus the model is generated over $H^*(L_0) = H^*(M^n)$ by elements $\Delta_{ab} := \tau_{\Delta_{ab}} \otimes \Delta_{ab}$ of degree $2 \dim M - 1$, subject to relation detemining $OS(L)$. Note that $d\Delta_{ab} = [L_{\Delta_{ab}}] \in H^{2 \dim M}(M^n)$.

As a direct corollary we obtain the chromatic version of the Kriz-Totaro model:

**Theorem 4.2.1.** The rational homotopy type of $F(M, G)$ is a cdga equal to the quotient

$$H^*(M^n) \otimes \Lambda^*(\Delta_{ab}, (ab) \in E(G))/I$$

with $\deg(\Delta_{ab}) = 2 \dim M - 1$, where $I$ is an ideal generated by

1. $\partial(\Delta_{i_1i_2} \wedge \ldots \wedge \Delta_{i_{k-1}i_k})$, for each cycle $(i_1, \ldots, i_k = i_1)$ in $G$.
2. $p^*_a u \cdot \Delta_{ab} = p^*_b u \cdot \Delta_{ab}, u \in H^*(M)$

The differential $d$ is determined by $d\Delta_{ab} = [L_{\Delta_{ab}}] \in H^{2 \dim M}(M^n)$.

$\square$

**References**

[1] Pierre Deligne *Théorie de Hodge: II,III*

[2] Chris A.M. Peters, Joseph H.M. Steenbrink *Mixed Hodge Structures*

[3] John W. Morgan *The algebraic topology of smooth algebraic varieties*

[4] Joana Cirici *Cofibrant models of diagrams: mixed Hodge structures in rational homotopy*

[5] Joana Cirici, Francisco Guillen *E1-formality of complex algebraic varieties*

[6] Junda Chen, Zhi Lü, Jie Wu *Cohomology ring of manifold arrangements*

[7] Peter Orlik, Louis Solomon *Combinatorics and Topology of Complements of Hyperplanes*

[8] Egbert Brieskorn, *Sur les groupes de tresses*

[9] W. Fulton, R. MacPherson, *A compactification of configuration spaces*

[10] Igor Kriz, *On the rational homotopy type of configuration spaces*
[11] E.M. Feichtner, S. Yuzvinsky *Formality of the complements of subspace arrangements with geometric lattices*

[12] Yves Fèlix, Stephen Halperin, Jean-Claude Thomas *Rational Homotopy Theory*

[13] Donu Arapura, *The Leray spectral sequence is motivic*