Integral equations, quasi-Monte Carlo methods and risk modelling

Dedicated to the 80th anniversary of Ian Sloan

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Abstract

We survey a QMC approach to integral equations and develop some new applications to risk modeling. In particular, a rigorous error bound derived from Koksma-Hlawka type inequalities is achieved for certain expectations related to the probability of ruin in Markovian models. The method is based on a new concept of isotropic discrepancy and its applications to numerical integration. The theoretical results are complemented by numerical examples and computations.

1 Introduction

During the last two decades quasi-Monte-Carlo methods (QMC-methods) have been applied to various problems in numerical analysis, statistical modeling and mathematical finance. In this paper we will give a brief survey on some of these developments and present new applications to more refined risk models involving discontinuous processes. Let us start with Fredholm integral equations of the second kind:

\[ f(x) = g(x) + \int_{[0,1]} K(x,y) f(y) dy, \]

where the kernel is given by \( K(x,y) = k(x - y) \) with \( k(x) \) having period 1 in each component of \( x = (x_1, \ldots, x_s) \). As it is quite common in applications of QMC-methods (see for example [9], [28], [20]) it is assumed that \( g \) and \( k \) belong to a weighted Korobov space. Of course, there exists a vast literature concerning the numerical solution of Fredholm equations, see for instance [18], [5] or [31]. In particular, we want to mention the work of I. Sloan in the late 1980’s where he explored various quadrature rules for solving integral equations and applications to engineering problems ([27], [26] and [34]), which have also, after some modifications, been applied to Volterra type integral equations (see [32] or [33]). In [9] the authors approximate \( f \) using the Nyström method based on QMC rules.

For points \( t_1, \ldots, t_N \) in \([0,1]^{s}\) the \( N \)-th approximation of \( f \) is given by

\[ f_N(x) := g(x) + \frac{1}{N} \sum_{n=1}^{N} K(x,t_n) f_N(t_n), \]

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where the function values \( f_N(t_1), \ldots, f_N(t_N) \) are obtained by solving the linear system

\[
f_N(t_j) = g(t_j) + \frac{1}{N} \sum_{n=1}^{N} K(t_j, t_n) f_N(t_n), \quad j = 1, \ldots, N.
\]

Under some mild conditions on \( K, N \), and the integration points \( t_1, \ldots, t_N \), it is shown in [9] that there exists a unique solution of (3). Furthermore, the authors analyze the worst case error of this, so-called QMC-Nystöm method. In addition, good lattice point sets \( t_1, \ldots, t_N \) are presented, which lead to a best possible worst case error. A special focus of this important paper lies on the study of tractability and strong tractability of the QMC-Nystöm method. For tractability theory in general we refer to the fundamental monograph of [23]. Using ideas of E. Hlawka [16] the third author of the present paper worked on iterative methods for solving Fredholm and Volterra equations, see also Hua-Wang [17].

The idea is to approximate the solution of integral equations by means of iterated (i.e. multi-dimensional) integrals. The convergence of this procedure follows from Banach’s fixed point theorem and error estimates can be established following the proof of the Picard-Lindelöf approximation for ordinary differential equations. To be more precise, let us consider integration points \( t_1, \ldots, t_N \in [0,1]^s \) with star discrepancy \( D_N^s \) defined as usual by

\[
D_N^s = \sup_{J \subset [0,1]^s} \left| \frac{1}{N} \sharp \{ n \leq N : t_n \in J \} - \lambda(J) \right|,
\]

where the supremum is taken over all axis-aligned boxes \( J \) with one vertex in the origin and Lebesgue measure \( \lambda(J) \). In [30] the following system of \( r \) integral equations has been considered for given functions \( g_j \) on \([0,1]^{s+r} \) and \( h_j \) on \([0,1]^s \):

\[
f_j(x) = \int_0^x \cdots \int_0^x g_j(\xi_1, \ldots, \xi_s, f_1(\xi), \ldots, f_r(\xi)) d\xi_s \cdots d\xi_1 + h_j(x), \quad j = 1, \ldots, r
\]

where we have used the notations \( x = (x_1, \ldots, x_s) \in [0,1]^s \) and \( \xi = (\xi_1, \ldots, \xi_s) \). Furthermore, we assume that the partial derivatives up to order \( s \) of the functions \( g_j \) and \( h_j, j = 1, \ldots, r \), are bounded by some constants \( G \) and \( H \), respectively. Then, for a given point set \( t_1, \ldots, t_N \in [0,1]^s \) with discrepancy \( D_N^s \), the solution \( f = (f_1, \ldots, f_r) \) of the system (5) can be approximated by the quantities \( f^{(k)} = (f^{(k)}_1, \ldots, f^{(k)}_r) \), given recursively by

\[
f^{(k+1)}_j(x) = \frac{x_1 \cdots x_s}{N} \sum_{n=1}^{N} g_j(x_1 t_{1,n}, \ldots, x_s t_{s,n}, f^{(k)}_1(x \cdot t_n), \ldots, f^{(k)}_r(x \cdot t_n));
\]

here \( x \cdot t_n \) stands for the inner product \( x_1 t_{1,n} + \ldots + x_s t_{s,n} \), where \( t_n = (t_{1,n}, \ldots, t_{s,n}) \). In [30] it is shown, that based on the classical Koksma-Hlawka inequality the worst case error, i.e., \( \| f^{(k)} - f \|_{\infty} \) (sum of componentwise supremum norms) can be estimated in terms of the bounds \( G \) and \( H \) and the discrepancy \( D_N^s \) of the integration points. This method was also extended to integral equations with singularities, such as Abel’s integral equation. The main focus of the present paper lies on applications in mathematical finance. In Albrecher & Kainhofer [3] the above method was used for the numerical solution of certain Cramér-Lundberg models in risk theory. However,
it turned out that in these models certain discontinuities occur. This means, that one cannot assume bounds for the involved partial derivatives and simply apply the classical Koksma-Hlawka inequality. Moreover, the involved functions are indicator functions of simplices thus not of bounded variation in the sense of Hardy and Krause, see Drmota & Tichy [10] and Kuipers & Niederreiter [19].

Albrecher & Kainhofer [3] considered a risk model with non-linear dividend barrier and made some assumptions to overcome the difficulties caused by discontinuities. For such applications it could help to use a different notion of variation for multivariate functions of simplices thus not of bounded variation in the sense of Hardy and Krause, see Drmota & Tichy [10] and Kuipers & Niederreiter [19].

Aistleitner & Dick [1] considered functions of bounded variation with respect to signed measures and Brandolini et al. [7, 6] replaced the integration domain \([0, 1]^s\) by an arbitrary bounded Borel subset of \(\mathbb{R}^s\) and proved the inequality for piecewise smooth integrands. Based on fundamental work of Harman [15], a new concept of variation was developed for a wide class of functions, see Pausinger & Svane [25] and Aistleitner et al. [2].

In the following we give a brief overview on concepts of multivariate variation and how they can be applied for error estimates in numerical integration. Let \(f(x)\) be a function on \([0, 1]^s\) and \(a = (a_1, \ldots, a_s) \leq b = (b_1, \ldots, b_s)\) points in \([0, 1]^s\), where \(\leq\) denotes the natural componentwise partial order. Following the notation of Owen [24] and Aistleitner et al. [2] for a subset \(u \subseteq \{1, \ldots, s\} \) we denote by \(a^u : b^{-u}\) the point with ith coordinate equal to \(a_i\) if \(i \in u\) and equal to \(b_i\) otherwise. Then for the box \(R = [a, b] \) we introduce the \(s\)-dimensional difference operator

\[
\Delta^{(d)}(f; R) = \Delta(f; R) = \sum_{u} (-1)^{|u|} f(a^u : b^{-u}),
\]

where the summation is extended over all subsets \(u \subseteq \{1, \ldots, s\} \) with cardinality \(|u|\) and complement \(-u\). Next we define partitions of \([0, 1]^s\) as they are used in the theory of multivariate Riemann integrals, which we call here ladder. A ladder \(\mathcal{V}\) in \([0, 1]^s\) is the cartesian product of one-dimensional partitions \(0 = y_{1,j}^1 < \ldots < y_{j}^{1,j} < \ldots < y_{j,k_j}^1 < \ldots < y_{j}^{1,j} < y_{j+1}^j < \ldots < y_{s}^{1,s} < \ldots < y_{s}^{1,s} < 1\) (in any dimension \(j = 1, \ldots, s\)). Define the successor \((y_{i}^{j})_+\) of \(y_{i}^{j}\) to be \(y_{i+1}^{j}\) if \(i < k_j\) and \((y_{k_j}^{j})_+ = 1\). For \(y = (y_1^1, \ldots, y_s^s) \in \mathcal{V}\) we define the successor \(y_+ = (y_1^1)_+, \ldots, (y_s^s)_+\) and have

\[
\Delta(f; [0, 1]^s) = \sum_{y \in \mathcal{V}} \Delta(f; [y, y_+]).
\]

Using the notation

\[
V_\mathcal{V}(f; [0, 1]^s) = \sum_{y \in \mathcal{V}} \Delta(f; [y, y_+])
\]

the Vitali variation of \(f\) over \([0, 1]^s\) is defined by

\[
V(f; [0, 1]^s) = \sup_{\mathcal{V}} V_\mathcal{V}(f; [0, 1]^s). \quad (7)
\]

Given a subset \(u \subseteq \{1, \ldots, s\} \), let

\[
\Delta_u(f; [a, b]) = \sum_{v \subseteq u} (-1)^{|v|} f(a^v : b^{-v})
\]

and set \(0 = (0, \ldots, 0), 1 = (1, \ldots, 1) \in [0, 1]^s\). For a ladder \(\mathcal{V}_u\) there is a corresponding ladder \(\mathcal{V}_u\) on the \(|u|\)-dimensional face of \([0, 1]^s\) consisting of points of the form \(x^u : 1^{-u}\). Clearly,
\[ \Delta_u(f; [0, 1]^s) = \sum_{y \in \mathcal{Y}_u} \Delta_u(f; [y, y_+]). \]

Using the notation
\[ V_{\mathcal{Y}_u}(f; [0, 1]^s) = \sum_{y \in \mathcal{Y}_u} \Delta_u(f; [y, y_+]) \]

for the variation over the ladder \( \mathcal{Y}_u \) of the restriction of \( f \) to the face of \([0, 1]^s \) specified by \( u \), the Hardy-Krause variation is defined as
\[ \mathcal{V}(f) = \mathcal{V}_{HK}(f; [0, 1]^s) = \sum_{\emptyset \neq u \subseteq \{1, \ldots, s\}} \sup_{\mathcal{Y}_u} V_{\mathcal{Y}_u}(f; [0, 1]^s). \]

Assuming that \( f \) is of bounded Hardy-Krause variation, the classical Koksma-Hlawka inequality reads as follows:
\[ \left| \frac{1}{N} \sum_{n=1}^{N} f(x_n) - \int_{[0,1]^s} f(x) \, dx \right| \leq \mathcal{V}(f) \mathcal{D}^*_N, \]
where \( x_1, \ldots, x_N \) is a finite point set in \([0, 1]^s \) with star discrepancy \( \mathcal{D}^*_N \). In the case \( f : [0, 1]^s \to \mathbb{R} \) has continuous mixed partial derivatives up to order \( s \) the Vitali variation is given by
\[ \mathcal{V}(f; [0, 1]^s) = \int_{[0,1]^s} \left| \frac{\partial^s f}{\partial x_1 \cdots \partial x_s}(x) \right| \, dx. \]

Summing over all non-empty subsets \( u \subseteq [0, 1]^s \) immediately yields an explicit formula for the Hardy-Krause variation in terms of integrals of partial derivatives, see Leobacher & Pillichshammer [21, Ch.3, p. 59]. In particular, the Hardy-Krause variation can be estimated from above by an absolute constant if we know global bounds on all partial derivatives up to order \( s \).

In the remaining part of the introduction we briefly sketch a more general concept of multidimensional variation which was recently developed in [25]. Let \( \mathcal{D} \) denote an arbitrary family of measurable subsets of \([0, 1]^s \) which contains the empty set \( \emptyset \) and \([0, 1]^s \). Let \( \mathcal{L}(\mathcal{D}) \) denote the \( \mathbb{R} \)-vectorspace generated by the system of indicator functions \( \mathbb{I}_A \) with \( A \in \mathcal{D} \).

A set \( A \subseteq [0, 1]^s \) is called an algebraic sum of sets in \( \mathcal{D} \) if there exist \( A_1, \ldots, A_m \in \mathcal{D} \) such that
\[ \mathbb{I}_A = \sum_{i=1}^{n} \mathbb{I}_{A_i} - \sum_{i=n+1}^{m} \mathbb{I}_{A_i}, \]
and \( \mathcal{A} \) is defined to be the collection of algebraic sums of sets in \( \mathcal{D} \). As in [25] we define the Harman complexity \( h(A) \) of a non-empty set \( A \in \mathcal{A}, A \neq [0, 1]^s \) as the minimal number \( m \) such there exist \( A_1, \ldots, A_m \) with
\[ \mathbb{I}_A = \sum_{i=1}^{n} \mathbb{I}_{A_i} - \sum_{i=n+1}^{m} \mathbb{I}_{A_i}, \]
for some \( 1 \leq n \leq m \) and \( A_i \in \mathcal{D} \) or \([0, 1]^s \setminus A_i \in \mathcal{D} \). Moreover, set \( h([0, 1]^s) = h(\emptyset) = 0 \) and for \( f \in \mathcal{L}(\mathcal{D}) \)
\( V_D(f) = \inf \left\{ \sum_{i=1}^{m} |\alpha_i| h_{D(A_i)} : f = \sum_{i=1}^{m} \alpha_i \mathbb{1}_{A_i}, \alpha_i \in \mathbb{R}, A_i \in D \right\} \)

Furthermore, let \( V_\infty(D) \) denote the collection of all measurable, real-valued functions on \([0,1]^s\) which can be uniformly approximated by functions in \( L(D) \). Then the \( D \)–variation of \( f \in V_\infty(D) \) is defined by

\[
V_D(f) = \inf \left\{ \liminf_{i \to \infty} V_D^*(f_i) : f_i \in L(D), f = \lim_{i \to \infty} f_i \right\},
\]

and set \( V_D(f) = \infty \) if \( f \notin V_\infty(D) \). The space of functions of bounded \( D \)–variation is denoted by \( \mathcal{V}(D) \). Important classes of sets \( D \) are the class \( \mathcal{K} \) of convex sets and the class \( \mathcal{R}^* \) of axis aligned boxes containing \( 0 \) as a vertex. In Aistleitner et al. \cite{2} it is shown that the Hardy-Krause variation \( V(f) \) coincides with \( V_{\mathcal{R}^*}(f) \). For various applications the \( D \)–variation seems to be a more natural and suitable concept. A convincing example concerning an application to computational geometry is due to Pausinger & Edelsbrunner \cite{11}. Pausinger & Svane \cite{25} considered the variation \( V_\mathcal{K}(f) \) with respect to the class of convex sets. They proved the following version of the Koksma-Hlawka inequality:

\[
\left| \frac{1}{N} \sum_{n=1}^{N} f(x_n) - \int_{[0,1]^s} f(x)dx \right| \leq V_\mathcal{K}(f) \tilde{D}_N,
\]

where \( \tilde{D}_N \) is the isotropic discrepancy of the point set \( x_1, \ldots, x_N \), which is defined as follows

\[
\tilde{D}_N = \sup_{C \subset \mathcal{K}} \left| \frac{1}{N} \sharp\{n \leq N : x_n \in C\} - \lambda(C) \right|.
\]

Pausinger & Svane \cite{25} have shown that twice continuously differentiable functions \( f \) admit finite \( V_\mathcal{K}(f) \), and in addition they gave a bound which will be useful in our context.

Our paper is structured as follows. In Section 2 we introduce specific Markovian models in risk theory where in a natural way integral equations occur. These equations are based on arguments from renewal theory and only in particular cases they can be solved analytically. In Section 3 we develop a QMC method for such equations. We give an error estimates based on Koksma-Hlawka type inequalities for such models. In Section 4 we compare our numerical results to exact solutions in specific instances.

2 Discounted penalties in the renewal risk model

2.1 Stochastic modeling of risks

In the following we assume a stochastic basis \((\Omega, \mathcal{F}, P)\) which is large enough to carry all the subsequently defined random variables. In risk theory the surplus process of an insurance portfolio is modeled by a stochastic process \( X = (X_t)_{t \geq 0} \). In the classical risk model, going back to Lundberg \cite{22}, \( X \) takes the form

\[
X_t = x + ct - \sum_{i=1}^{N_t} Y_i,
\]

where \( N_t \) are the number of claims occurring up to time \( t \), and \( Y_i \) are the claim sizes. The surplus process \( X \) is a compound Poisson process with constant claim rate \( c \).
where the deterministic quantities \( x \geq 0 \) and \( c \geq 0 \) represent the initial capital and the premium rate. The stochastic ingredient \( S_t = \sum_{i=1}^{N_t} Y_i \) is the cumulated claims process which is a compound Poisson process. The jump heights - or claim amounts - are \( \{ Y_i \}_{i \in \mathbb{N}} \) for which \( Y_i \overset{i.d.}{\sim} F_Y \) with \( F_Y(0) = 0 \). The counting process \( N = (N_t)_{t \geq 0} \) is a homogeneous Poisson process with intensity \( \lambda > 0 \). A crucial assumption in the classical model is the independence between \( \{ Y_i \}_{i \in \mathbb{N}} \) and \( N \). A major topic in risk theory is the study of the ruin event. We introduce the time of ruin \( \tau = \inf \{ t \geq 0 \mid X_t < 0 \} \), i.e., the first point in time at which the surplus becomes negative. In this setting \( \tau \) is a stopping time with respect to the filtration generated by \( X_t \), \( \{ F_t^X \}_{t \geq 0} \) with \( F_t^X = \sigma \{ X_s \mid 0 \leq s \leq t \} \). A first approach for quantifying the risk of \( X_t \) is the study of the associated ruin probability

\[
\psi(x) = P_x(X_t < 0 \text{ for some } t \geq 0) = P_x(\tau < \infty),
\]

which is non-degenerate if \( \mathbb{E}_x(X_1) > 0 \), and satisfies the integral equation

\[
\frac{c}{\lambda} \psi(x) = \int_x^{\infty} 1 - F_Y(y) dy + \int_0^x \psi(x-y)(1-F_Y(y)) dy.
\]

In Gerber & Shiu [12, 13] so-called discounted penalty functions are introduced. This concept allows for an integral ruin evaluation and is based on a function \( w : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R} \) which links the deficit at ruin \( |X_\tau| \) and the surplus prior to ruin \( X_{\tau^-} := \lim_{\tau \downarrow \tau} X_\tau \) via the function

\[
V(x) = \mathbb{E}_x \left( e^{-\delta \tau} w(|X_\tau|, X_{\tau^-}) 1_{\{ \tau < \infty \}} \right).
\]

The time of ruin \( \tau \) is included by means of a discounting factor \( \delta > 0 \) which gives more weight to an early ruin event. In this setting specific choices of \( w \) allow for an unified treatment of ruin related quantities.

**Remark 2.1**

*When putting a focus on the study of \( \psi(x) \), the condition \( \mathbb{E}_x(X_1) > 0 \) is crucial. It says that on average premiums exceed claim payments in one unit of time. Standard results, see Asmussen & Albrecher [4], show that under this condition \( \lim_{t \to \infty} X_t = +\infty \) \( P \)-a.s. From an economic perspective the accumulation of an infinite surplus is unrealistic and risk models including shareholder participation via dividends are introduced in the literature. We refer to [2] for model extensions in this direction.*

### 2.2 Markovian risk model

In the following we consider an insurance surplus process \( X = (X_t)_{t \geq 0} \) of the form

\[
X_t = x + \int_0^t c(X_{s^-}) ds - \sum_{i=1}^{N_t} Y_i.
\]

The quantity \( x \geq 0 \) is called the initial capital, the cumulated claims are represented by \( S_t = \sum_{i=1}^{N_t} Y_i \) and the state-dependent premium rate is \( c(\cdot) \). The cumulated claims process \( S = (S_t)_{t \geq 0} \) is given by a sequence \( \{ Y_i \}_{i \in \mathbb{N}} \) of positive, independently and identically distributed (iid) random variables and a counting process \( N = (N_t)_{t \geq 0} \).

For convenience we assume that the claims distribution admits a continuous density \( f_Y : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \). In our setup we model the claim counting process \( N = (N_t)_{t \geq 0} \) as a renewal counting process which is specified by the inter-jump times \( \{ W_i \}_{i \in \mathbb{N}} \) which are positive and iid random variables. Then, the time of the \( i \)-th jump is
We choose, in contrast to classical models, a non-constant premium rate to model the effect of a so-called dividend barrier \( a > 0 \) in a smooth way. A barrier at level \( a > 0 \) has the purpose that every excess of surplus of this level is distributed as a dividend to shareholders which allows to include economic considerations in insurance modeling. Mathematically, this means that the process \( X \) is reflected at level \( a \). Now instead of directly reflecting the process we use the following construction. Fix \( \varepsilon > 0 \) and for some \( \tilde{c} > 0 \), define

\[
e(x) = \begin{cases} \tilde{c}, & x \in [0, a - \varepsilon), \\ f(x), & x \in [a - \varepsilon, a], \\ 0, & x > a, \end{cases}
\] (12)

with some positive and twice continuously differentiable function \( f \) which fulfills \( f(a - \varepsilon) = \tilde{c}, f(a) = 0, f'(a - \varepsilon) = f'(a) = f''(a - \varepsilon) = f''(a) = 0 \). Altogether, we assume \( c(.) \in C^2[0, a] \) with some Lipschitz constant \( L > 0 \) and \( c'(a - \varepsilon) = c'(a) = 0, c''(a - \varepsilon) = c''(a) = 0, c' \leq 0 \) and bounded derivatives \( c', c'' \). Then \( \lim_{x \to a} c(x) = 0 \) and the process always stays below level \( a \) if started in \([0, a]\).

A concrete choice for \( f \) would be

\[
s(x) = \frac{c(a - x)^3 ((15\varepsilon(x - a) + 6(a - x)^2 + 10\varepsilon^2)}{\varepsilon^5}. \] (13)

In the following we do not specify \( f \) any further.

In this setting we add \( X_0 = x \) into the definition of the time of ruin, i.e., \( \tau_x = \inf\{t \geq 0 \mid X_t < 0, X_0 = x\} \).

**Remark 2.2**

In this model setting ruin can only take place at some jump time \( T_k \) and since the process is bounded a.s. we have that \( P_x(\tau_x < \infty) = 1 \). If an approximation to classical reflection of the process at level \( a \) is implemented, then the process virtually started above \( a \) is forced to jump down to \( a - \varepsilon \) and continue from this starting value. Consequently, we put the focus on starting values \( x \in [0, a) \).

In the remainder of this section we will study analytic properties of the discounted value function which in this framework takes the form

\[
V(x) = E_x \left( e^{-\delta \tau_x} w(|X_{\tau_x}|, X_{\tau_x}) \right),
\] (14)

with \( \delta > 0 \) and a continuous penalty function \( w : \mathbb{R}^+ \times [0, a) \to \mathbb{R} \).

To have a well defined function, typically the following integrability condition is used

\[
\int_0^\infty \int_0^\infty w(x, y) f_Y(x + y) dy \ dx < \infty,
\]

see [4]. Since our process is kept below level \( a \) and \( w \) is supposed to be continuous in both arguments we can naturally replace the above condition by

\[
\sup_{z \in [0, a]} \int_0^\infty |w(|z - y|, z)| f_Y(y) dy =: M < \infty, \] (15)
which we will assume in the following. The condition from equation (15) holds true for example, if \(|w(x, y)| \leq (1 + |x| + |y|)^p\) and \(F_Y\) admits a finite \(p\)-th moment for some \(p \geq 1\).

**Remark 2.3**
From the construction of \(X\) we have that \(\tilde{X} = (\tilde{X}_t)_{t \geq 0}\) with \(\tilde{X}_t = (X_t, t'(t), t)\) is a piecewise-deterministic Markov process, see Davis [8]. Since the jump intensity depends on \(t' = t - T_{N_i}\), one needs this additional component for the Markovization of \(X\). But on the discrete time skeleton \(\{T_i\}_{i \in \mathbb{N}}\) with \(T_0 = 0\) the process \(X = \{X_{T_k}\}_{k \in \mathbb{N}}\) has the Markov property.

### 2.3 Analytic properties and a fixed point problem

We start with showing some elementary analytical properties of the function \(V\) defined in (14).

**Theorem 2.1**
The function \(V : [0, \alpha) \to \mathbb{R}\) is bounded and continuous.

**Proof:** The boundedness of \(V\) follows directly from the assumption made in (15).

For proving continuity we split off the expectation defining \(V\) into two parts which we separately deal with. Let \(x > y\) and observe

\[
|V(x) - V(y)| = \mathbb{E} \left[ e^{-\delta \tau_s} \left( w(\{X^x_{\tau_s}\}, X^x_{\tau_s} -) - e^{-\delta \tau_s} w(\{X^y_{\tau_s}\}, X^y_{\tau_s} -) \right) \right] \\
\leq \mathbb{E} \left[ e^{-\delta \tau_s} \left( w(\{X^x_{\tau_s}\}, X^x_{\tau_s} -) - w(\{X^y_{\tau_s}\}, X^y_{\tau_s} -) \right) 1_{\{\tau_s = \tau_\alpha\}} \right] \\
+ \mathbb{E} \left[ e^{-\delta \tau_s} \left( w(\{X^x_{\tau_s}\}, X^x_{\tau_s} -) - e^{-\delta \tau_s} w(\{X^y_{\tau_s}\}, X^y_{\tau_s} -) \right) 1_{\{\tau_s > \tau_\alpha\}} \right] \\
= A + B.
\]

For \(A\) we fix some \(T > 0\) and notice the following bound

\[
A \leq \mathbb{E} \left[ e^{-\delta \tau_s} \left| w(\{X^x_{\tau_s}\}, X^x_{\tau_s} -) - w(\{X^y_{\tau_s}\}, X^y_{\tau_s} -) \right| 1_{\{\tau_s = \tau_\alpha \leq T\}} \right] + 2M P(\tau_x > T) \leq 2M.
\]

(16)

Before going on we need some estimates on the difference of two paths, one starting in \(x\) and the other in \(y\). For fixed \(\omega \in \Omega\) we have that on \((0, T_1(\omega))\) the surplus fulfills \(\frac{\partial X(\omega)}{\partial t} = c(X(\omega))\) with initial condition \(X_0 = 0\). \(T_1(\omega)\) is finite with probability one. Standard arguments on ordinary differential equations, see for instance Stöer & Bulirsch [29], Th. 7.1.1 - 7.1.8, yield that an appropriate solution exists and is continuously differentiable in \(t\) and continuous in the initial value \(x\). We even get the bound \(|X^x_t - X^y_t| \leq e^{L_t} |x - y|\) for fixed \(\omega\), where \(X^x_t\) denotes the path which starts in \(x\) and \(L > 0\) the Lipschitz constant of \(c(\cdot)\). From these results we directly obtain for a given path

\[
|X^x_{T_1} - X^y_{T_1}| \leq e^{LT_1} |x - y|,
\]

which by iteration results in

\[
|X^x_{T_{n+1}} - X^y_{T_{n+1}}| = |X^x_{T_n} - X^y_{T_n}| \leq e^{LT_n} |x - y|,
\]

because \(|X^x_{T_n} - X^y_{T_n}| = |X^x_{T_n} - Y_n - (X^y_{T_n} - Y_n)| + |X^y_{T_n} - X^y_{T_n}|\). Since ruin takes place at some claim occurrence time \(T_k\) we get that on \(\{\omega \in \Omega | \tau_x = \tau_\alpha \leq T\}\) the
quantities $|X_{\tau_1}^x|$ and $X_{\tau_1}^x -$ converge to the corresponding quantities started in $x$, all possible differences are bounded by $e^{LT} |x - y|$. Therefore, sending $y$ to $x$ in (16) and then sending $T$ to infinity, we get that $A$ converges to zero because $P(\tau_x < \infty) = 1$ and bounded convergence. We can repeat the argument for $T$ with $x$ and $y$ interchanged, for which $\tau_x(\omega) > \tau_y(\omega)$, this implies that there is a claim amount $Y_n$, occuring at some point in time $T_n$, for which

$$X_{T_n-}^x(\omega) \geq Y_n(\omega) > X_{T_n-}^y(\omega),$$

i.e., causing ruin for the path started in $y$, $(X_t^y)$, but not causing ruin for the one started in $x$, $(X_t^x)$. From the construction of the drift $c(\cdot)$, it is decreasing to zero, we have that, surpressing the $\omega$ dependence,

$$0 < Y_n - X_{T_n-}^y \leq X_{T_n-}^x - X_{T_n-}^y \leq x - y.$$

Since $X_{T_n-}^y \in [0, \alpha)$ we have

$$P(\tau_x > \tau_y) \leq \sup_{q \in [0, \alpha]} P(0 < Y - q \leq x - y) = \sup_{q \in [0, \alpha]} \{F_Y(x - y + q) - F_Y(q)\},$$

which approaches zero whenever $x$ and $y$ tend to each other since $F_Y$ is continuous. □

Define for functions $f \in C_b([0, \alpha))$ the operator $A$ by

$$A f(x) := \mathbb{E}_x \left( e^{-\delta T_1} f(X_{T_1})I_{\{T_1 < \tau_x\}} + e^{-\delta \tau_x} w(|X_{T_1}|, X_{T_1-})I_{\{\tau_x = T_1\}} \right).$$

(17)

The Markov property of the sequence $\{X_{T_i}\}_{i \in \mathbb{N}}$ and the definition of $V$ in [14] allow us to derive that $V = AV$, or explicitly written

$$V(x) = \mathbb{E}_x \left[ e^{-\delta T_1} V(X_{T_1})I_{\{T_1 < \tau_x\}} + e^{-\delta \tau_x} w(|X_{T_1}|, X_{T_1-})I_{\{\tau_x = T_1\}} \right].$$

We can state the following lemma.

**Lemma 2.2**

If $\delta > 0$, the operator $A : C_b([0, \alpha)) \to C_b([0, \alpha))$ defined in (17) is a contraction with respect to $\| \cdot \|_{\infty}$.

**Proof:** Let $f \in C_b([0, \alpha))$ be bounded by some constant $M'$, then

$$A f(x) = \mathbb{E}_x \left( e^{-\delta T_1} f(X_{T_1})I_{\{T_1 < \tau_x\}} + e^{-\delta \tau_x} w(|X_{T_1}|, X_{T_1-})I_{\{\tau_x = T_1\}} \right),$$

is bounded by $\max\{M, M'\}$. From the integral representation of $A f(x)$ we get continuity in $x$,

$$A f(x) = \int_0^\infty e^{-\delta t_1} f_W(t_1) \left( \int_0^{X_{t_1-}} f(X_{t_1-} - y) dF_Y(y) + \int_{X_{t_1-}}^\infty w(|X_{t_1-} - y|, X_{t_1-}) dF_Y(y) \right) dt_1,$$

where $X_{t_1-}$ is the ODE’s solution up to time $t_1$ with $X_0 = x$. From Stoer & Bulirsch [29, Th. 7.1.4] we have that $X_{t_1-}$ is continuous in its initial value which shows that
\( Af(x) \) is continuous in \( x \).

Let \( f, g \in \mathcal{C}_b([0, a]) \), then we have for all \( x \in [0, a] \) that

\[
|A(f - g)(x)| \leq \int_0^\infty e^{-\delta t_1} f_W(t_1) \int_0^{X_{t_1}} |f(X_{t_1} - y_1) - g(X_{t_1} - y_1)| dF_Y(y_1) dt_1
\]

\[
\leq ||f - g|| \int_0^\infty e^{-\delta t_1} f_W(t_1) dt_1 = ||f - g|| E[e^{-\delta T_1}].
\]

Since \( \delta > 0 \) and \( T_1 > 0 \) \( P \)-a.s., \( A \) is contractive with Lipschitz constant \( \hat{L} = E[e^{-\delta T_1}] < 1 \).

For a possible application of quasi-Monte Carlo techniques we need to examine the structure of \( A \),

\[
A v(x) = \int_0^\infty e^{-\delta t_1} f_W(t_1) \int_0^{X_{t_1}} v(X_{t_1} - y_1) dF_Y(y_1) dt_1 +
\]

\[
\int_0^\infty e^{-\delta t_1} f_W(t_1) \int_{X_{t_1}}^\infty w(y_1 - X_{t_1}, X_{t_1} -) dF_Y(y_1) dt_1
\]

\[=: \mathcal{G} v(x) + \mathcal{H}(x).\]

For \( n \in \mathbb{N} \) the probabilistic interpretation of iterated applications of \( A \) is \( A^n v(x) = \mathbb{E}_x \left( e^{-\delta T_n} v(X_{T_n}) 1_{(T_n < \tau)} + e^{-\delta \tau} w(|X_{\tau_n}|, X_{\tau_n} -) 1_{(\tau_n \leq T_n)} \right) \). Using \( \mathcal{G} \) and \( \mathcal{H} \) we can write

\[
A^n v(x) = \mathcal{G}^n v(x) + \sum_{k=0}^{n-1} \mathcal{G}^k \mathcal{H}(x),
\]

where \( \mathcal{G}^n v(x) = \mathbb{E}_x (e^{-\delta T_n} v(X_{T_n}) 1_{(T_n < \tau)}) \) and

\[
\mathcal{G}^{k-1} \mathcal{H}(x) = \int_0^\infty \cdots \int_0^\infty \int_0^{X_{t_k} -} \cdots \int_0^{X_{t_1} -} \left( \prod_{i=1}^k e^{-\delta t_i} f_W(t_i) \right) w(y_k - X_{t_k}, X_{t_k} -) dF_Y(y_k) \cdots dF_Y(y_1) dt_k \cdots dt_1.
\]

Here, \( \bar{t} := \sum_{i=1}^k t_i \) and represents the time of the \( k \)-th jump. We see that via \( X_{t_k} = X_{t_i} - y_{k-1} + \int_{t_{k-1}}^{t_k} c(X_s) ds \) the path of the process depends on all integration variables \( (t_1, \ldots, t_k, y_1, \ldots, y_k) \).

For dealing with the situation \( \delta = 0 \), i.e., when the contraction argument fails, we can use a probabilistic argument. Since \( \lim_{n \to \infty} T_n = \infty \) and \( P(\tau_n < \infty) = 1 \) we have that \( \lim_{n \to \infty} \mathcal{G}^n v(x) = \mathbb{E}_x \left( e^{-\delta T_n} v(X_{T_n}) 1_{(T_n < \tau)} \right) = 0 \) for \( v \in \mathcal{C}_b([0, a]) \). Using \( |A^n v(x) - V(x)| = |\mathcal{G}^n v(x) - \mathcal{G}^n V(x)| \) we get \( \lim_{n \to \infty} A^n v(x) = V(x) \) pointwise, even in the case if \( \delta = 0 \).

In what follows we put the focus on the determination of \( \mathcal{G}^k \mathcal{H}(x) \).
3 Approximation procedure

For the application of QMC methods we need to transform in a first step the integration domain in

$$G^{k-1}H(x) = \int_0^\infty \cdots \int_0^\infty \int_{X_{t_k-}} \int_0^{X_{t_k-}} \cdots \int_0^{X_{t_1-}} \left( \prod_{i=1}^k e^{-\delta t_i} f_W(t_i) \right) w(y_k - X_{t_k-}, X_{t_k-}) dF_Y(y_k) \cdots dF_Y(y_1) dt_k \cdots dt_1$$

to $[0,1]^{2k}$. This is achieved by use of the following substitutions

$$\alpha_i := e^{-t_i} \Rightarrow t_i = -\log \alpha_i \quad \text{for} \ i \in \{1, \ldots, k\}$$
$$\beta_i := \frac{y_i}{X_{t_i-}} \Rightarrow y_i = X_{t_i-} \beta_i \quad \text{for} \ i \in \{1, \ldots, k-1\}$$
$$\beta_k := e^{X_{t_k-}} e^{-y_k} \Rightarrow y_k = X_{t_k-} - \log \beta_k.$$  

Here it has to be taken into account that the values of the reserve process $X$ have to be calculated recursively, i.e., $X_{t_i-}$ depends on $t_1, \ldots, t_i$ and $y_1, \ldots, y_{i-1}$. Since the Jacobian matrix of this transformation has a lower triangular form, the determinant can easily be found as $\frac{1}{\alpha_1 \cdots \alpha_k} X_{t_1-} \cdots X_{t_k-} = \frac{1}{2^k}$. Altogether, we arrive at

$$G^{k-1}H(x) = \int_{[0,1]^{2k}} \prod_{i=1}^k \alpha_i^\delta f_W(t_i(\alpha_i)) \prod_{i=1}^k f_Y(y_i(\alpha_1, \ldots, \alpha_i, \beta_1, \ldots, \beta_k)) \frac{1}{\alpha_1 \cdots \alpha_k} X_{t_1-} \cdots X_{t_k-} \frac{1}{\beta_k} \frac{1}{\beta_k} d\alpha_1 \cdots d\alpha_k d\beta_1 \cdots d\beta_k.$$  

Consequently, for recovering the Koksma-Hlawka type error bound we need to examine the variation of the integrand:

$$F(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k) = \left( \prod_{i=1}^{k-1} \alpha_i^\delta f_W(-\log(\alpha_i)) \right) \left( \prod_{i=1}^{k-1} f_Y(\beta_i X_{t_i-}) X_{t_i-} \right) \left( \alpha_k^\delta f_W(-\log(\alpha_k)) f_Y(X_{t_k-} - \log(\beta_k)) \frac{1}{\beta_k} w(-\log(\beta_k), X_{t_k-}) \right).$$  

Here we denote by $\phi(t, s)$ the solution to $\frac{d}{dt} x(t) = c(x(t))$ with $x(0) = s$. Consequently, we can write

$$X_{t_i-} = X_{t_{i-1}-} - y_{i-1} + \phi(t_i, X_{t_{i-1}-} - y_{i-1}).$$

Or in terms of $\alpha_i$, putting $\hat{x}_{i-1} = X_{t_{i-1}-} - y_{i-1} = X_{t_{i-1}-} (1 - \beta_{i-1})$ and

$$X_{t_{i-1}-} = \hat{x}_{i-1} + \phi(-\log(\alpha_i), \hat{x}_{i-1}).$$  

In the following proposition we show that with a particular choice of model parameters it is possible to apply results from [25] to show that the integrand in (18) is in some sense of finite variation. Its proof shows that probabilistic and deterministic model ingredients are considerably interconnected.

**Theorem 3.1**

Let $f_W(t) = \lambda e^{-\lambda t} \mathbb{1}_{[t \geq 0]}$ ($\lambda > 0$), $f_Y(y) = \mu e^{-\mu y} \mathbb{1}_{[y \geq 0]}$ ($\mu > 0$), $w \equiv 1$ and $c(\cdot)$ be specified by (13). Then, under the assumption $\lambda + \delta \geq 3$ and $\mu \geq 3$ the variation $V_K(F)$ (see (10) with $D = K$) of $F$, defined in (18), is finite.
Proof: The main idea of the proof is the application of [25 Th. 3.12]. For this purpose we need to show that $M(F) = \sup\{\|H_{\text{ess}}(F, x)\| | x \in [0, 1]^{2k}\}$, sup $F$ and inf $F$ are finite, with the implication

$$V_{\mathcal{K}}(F) \leq \sup F - \inf F + M(F).$$

Since in this theorem the operator (matrix) norm $\|H_{\text{ess}}(F, x)\|$ is arbitrary we use the 2-norm and exploit the relation

$$\|H_{\text{ess}}(F, x)\|_2 \leq \left(\sum_{i=1}^{2k} \sum_{j=1}^{2k} [H_{\text{ess}}(F, x)]_{ij}^2 \right)^{\frac{1}{2}}.$$

We will show that $[H_{\text{ess}}(F, x)]_{ij}$ is finite for all $x \in [0, 1]^{2k}$. At first we observe that when taking derivatives with respect to $\alpha_i$ and $\beta_j$, the structure of (19) implies the appearance of the following terms:

$$\frac{\partial}{\partial t} \phi(t, s) = c(\phi(t, s)), \quad \frac{\partial^2}{\partial t^2} \phi(t, s) = c'(\phi(t, s))c(\phi(t, s)),
$$

$$\frac{\partial}{\partial s} \phi(t, s) =: y(t, s) = e^{\int_0^t c'(\phi(u, s))du}, \quad \frac{\partial^2}{\partial t \partial s} \phi(t, s) = c'(\phi(t, s))y(t, s),
$$

$$\frac{\partial^2}{\partial s^2} \phi(t, s) =: z(t, s) = y(t, s) \int_0^t c''(\phi(u, s))y(u, s)du.$$

The functions $y, z$ correspond to the first and second derivative of the ODE’s solution with respect to the initial value. They can be derived from the associated first and second order variational equations (see [36]). From our assumptions on $c(\cdot)$ we have that $y$ is bounded by one ($c' \leq 0$) and all other derivatives including $z$ are bounded as well. The boundedness of $z$ can be derived from the boundedness of $c''(\cdot)$ and an analysis of the growth behaviour of $y$.

For the structure of $[H_{\text{ess}}(F, x)]_{ij}$ we can derive the following

$$\prod_{i=1}^{k} \alpha_i^{\lambda - \delta} \beta_i^{\mu - b} e^{-\mu(y_{i1} + \cdots + y_{i-1} + x_{i-1})} Q \left( \beta_1, \ldots, \beta_{k-1}, \phi, \frac{\partial}{\partial t} \phi, \frac{\partial^2}{\partial t^2} \phi, \frac{\partial}{\partial s} \phi, \frac{\partial^2}{\partial t \partial s} \phi, \frac{\partial^2}{\partial s^2} \phi \right),$$

where $a, b \in \{1, 2, 3\}$ and a function $Q$. $Q$ is evaluated at the integration points and $\phi$ and its derivatives which themselves are evaluated in points of the form $(- \log(\alpha_i), \tilde{x}_{i-1}) \in (0, \infty) \times [0, a)$ for $i \in \{1, \ldots, k\}$. If $\phi$ and its derivatives are considered to be variables, neglecting their dependence on $\alpha_i$s and $\beta_i$s, then $Q$ is a polynomial of degree $k$. The degree of the polynomial is produced by the recursive structure of the paths and its dependence on all previous jump times and sizes. From this inspection we get that under the conditions $\lambda + \delta \geq 3$ and $\mu \geq 3$ all entries of the Hessian matrix are bounded. Furthermore, the conditions on the parameters $\lambda, \delta, \mu$ ensure that sup $F$ is finite and inf $F = 0$. \hfill \Box

Remark 3.1

We can combine the above result with the convergence rate from Banach’s fixed point theorem and obtain for our specific situation

$$\left\| \sum_{k=0}^{n} \mathcal{G}^k \mathcal{H} - V \right\|_{\infty} \leq \left\| \mathcal{G}^{n} \mathcal{H} - \mathcal{G}^{k} \mathcal{H} \right\|_{\infty} + \|A^n - V\|_{\infty} + \|G^n v\|_{\infty}$$

$$\leq \sum_{k=0}^{n} V_{\mathcal{K}}(F^k) \bar{D}_{N_k} + \frac{\bar{n}}{1 - \bar{L}} \|A v - v\|_{\infty} + M' \left( \frac{\lambda}{\delta + \lambda} \right)^n.$$
Here $F^k$ denotes the integrand from (18) in dimension $2k$, $\hat{D}_{N_k}$ the isotropic discrepancy of a pointset with $N_k$ elements in $[0,1]^{2k}$ and $\hat{G}^k\mathcal{H}$ is the QMC approximation for $\mathcal{G}^k\mathcal{H}$. For the last term we used that $v$ is bounded by some $M' > 0$ and the fact that $T_n$ follows a Gamma distribution $\Gamma(n, \lambda)$.

From the type of arguments we used for the proof of Theorem 3.1, we expect that the result holds true for $\Gamma$-distributed inter-claim times and jump heights and $w(y,z) = y^{\frac{1}{k}}z^{\frac{1}{l}}$ with similar conditions on the parameters. Hence the method is also applicable for this more general situation. A detailed study of this claim is part of future research.

4 Numerical results

In this section, we evaluate the integrals from Section 3 by applying Monte Carlo and quasi-Monte Carlo methods for different choices of the penalty function $w$.

4.1 The discounted time of ruin

Letting $w(y,z) := 1$, we arrive at $V(x) = \mathbb{E}(e^{-\delta \tau} w(|X_\tau|, X_{\tau-})) = \mathbb{E}(e^{-\delta \tau})$ which is the discounted time of ruin. Lin et al. found an analytic expression for this discounted time of ruin if both the inter-arrival times of the claims and the claim sizes are exponentially distributed. To have a reference value, we also adopt these assumptions and denote the parameters of the exponential distributions with $\lambda$ for the parameter of the inter-arrival times and $\mu$ for the parameter of the claim sizes. The premium rate $c(\cdot)$ was chosen as in Section 2.2 with $f$ from equation (13), with $\tilde{c} = 2$, $a = 3$ and $\varepsilon$ was set to 0.001. Note that the results of Lin et al. were proved for a reflected process in the classical sense, which means $c(x) = \tilde{c}$ for $x \leq a$ and $c(x) = 0$ for $x > a$. Since Theorem 3.1 requires a premium rate satisfying certain smoothness conditions, we cannot use a discontinuous $c$ and thus have a methodic error in our simulations. However, we will see that this error is, at least for small $\varepsilon$, very small.

We list the parameters together with the approximation values for increasing numbers of (Q)MC points and $k = 20$ iterations of the algorithm in Table 1, whereas Table 2 shows the approximation values for $k = 100$ iterations of the algorithm. Figure 1 and Figure 2 show the MC points (green) with 95% confidence intervals, together with QMC points from Sobol sequences (blue) and Halton sequences (orange). The red line at height 0.7577 marks the true value. As can be seen in Figure 1, the algorithm has not yet converged for $k = 20$, whereas Figure 2 shows that $k = 100$ already yields a very good approximation.

To illustrate the speed of convergence, we also plotted the absolute error, both for the MC approach as well as for QMC points (again taken from Sobol and Halton sequences) for varying numbers of points $N$. Figures 3 and 4 show the values obtained for $k = 40$ iterations of the algorithm. Obviously, $k = 40$ is also not yet enough to reach the actual value. But notice that the absolute error even for more iterations cannot converge to zero because of the smoothed reflection procedure. For both of the QMC methods, a scramble improved the results. In the Sobol case however, an “unlucky” choice in the scramble and the skip value (i.e. how many elements are dropped in the beginning) can lead to relatively high variation in the output, whereas the Halton set shows a more stable performance (compare Figures 3 and 4).
4.2 The deficit at ruin

If we set \( w(y, z) := y \), and \( \delta = 0 \), we have \( V(x) = \mathbb{E}(|X_{\tau^-}|) \), the expected deficit at ruin. We use the same premium rate \( c(\cdot) \) as before and again choose exponential distributions for the inter-arrival times and claim sizes with parameters \( \lambda \) and \( \mu \) respectively, since also in this case the true value \( \mathbb{E}(|X_{\tau^-}|) = \frac{1}{\mu} \) (for a classically reflected process) can be found in [35]. Figures 6 and 7 show the results for \( k = 20 \).
For Figure 5 we evaluated $k = 40$ iterations of the algorithm with $N = 30000$ (Q)MC points for different starting values $x$, ranging from 0.7 to 2. As expected, the discounted time of ruin decreases for increasing $x$.

Figure 5: Influence of the starting value

and $k = 100$ iterations respectively. The reference value is again shown as a red line, in our case at 1.25. The MC points are drawn in green, the Sobol points blue and the Halton points in orange. Table 3 and Table 4 contain the precise values along with the corresponding parameters.

Note again the difference between Figure 6 and Figure 7, resulting from a different number of iterations $k$.

Remark 4.1

We considered in our numerical examples two test cases for which explicit (approximate) reference values are available. Certainly our approach is not restricted to this particular choice of model ingredients - which are $f_Y$, $f_W$ and the penalty function $w$.

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Figure 6: $k = 20$ iterations of the algorithm.

Figure 7: $k = 100$ iterations of the algorithm.

| $x$ | $\lambda$ | $\mu$ | $w(y,z)$ | $k$ | $\delta$ |
|-----|-----------|-------|----------|-----|---------|
| 1.2 | 1         | 0.8   | $y$      | 20  | 0       |

| $x$ | $\lambda$ | $\mu$ | $w(y,z)$ | $k$ | $\delta$ |
|-----|-----------|-------|----------|-----|---------|
| 1.2 | 1         | 0.8   | $y$      | 100 | 0       |

Table 3:

| $N$   | 10000 | 15000 | 20000 | 25000 | 30000 |
|-------|-------|-------|-------|-------|-------|
| MC    | 1.1952| 1.1991| 1.1986| 1.1987| 1.1939|
| Sobol | 1.2105| 1.2142| 1.2070| 1.2074| 1.1975|
| Halton| 1.2084| 1.2019| 1.1906| 1.1872| 1.1885|

Table 4:

| $N$   | 10000 | 15000 | 20000 | 25000 | 30000 |
|-------|-------|-------|-------|-------|-------|
| MC    | 1.2624| 1.2558| 1.2446| 1.2602| 1.2607|
| Sobol | 1.2669| 1.2704| 1.2624| 1.2373| 1.2398|
| Halton| 1.2487| 1.2404| 1.2344| 1.2273| 1.2199|

Again, we plotted the absolute error for $k = 40$ iterations of the algorithm and a varying number of (Q)MC points $N$. Figure 8 shows the results using the same colorings as before.

Figure 8: The absolute error for the deficit at ruin
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