Two-Player Games for Efficient Non-Convex Constrained Optimization

Andrew Cotter\textsuperscript{1}, Heinrich Jiang\textsuperscript{1}, and Karthik Sridharan\textsuperscript{2}

\textsuperscript{1}Google Inc.
\textsuperscript{2}Cornell University

April 19, 2018

Abstract

In recent years, constrained optimization has become increasingly relevant to the machine learning community, with applications including Neyman-Pearson classification, robust optimization, and fair machine learning. A natural approach to constrained optimization is to optimize the Lagrangian, but this is not guaranteed to work in the non-convex setting. Instead, we prove that, given a Bayesian optimization oracle, a modified Lagrangian approach can be used to find a distribution over no more than $m + 1$ models (where $m$ is the number of constraints) that is nearly-optimal and nearly-feasible w.r.t. the original constrained problem.

Interestingly, our method can be extended to non-differentiable—even discontinuous—constraints (where assuming a Bayesian optimization oracle is not realistic) by viewing constrained optimization as a non-zero-sum two-player game. The first player minimizes external regret in terms of easy-to-optimize “proxy constraints”, while the second player enforces the original constraints by minimizing swap-regret.

1 Introduction

We consider the general problem of inequality constrained optimization, in which we wish to find a set of parameters $\theta \in \Theta$ minimizing an objective function subject to $m$ functional constraints:

$$
\min_{\theta \in \Theta} g_0(\theta) \\
\text{s.t. } \forall i \in [m]. g_i(\theta) \leq 0
$$

To highlight some of the challenges that arise in non-convex constrained optimization, consider the specific example of optimizing fairness metrics. Following [Goh et al. 2016], we cast the fairness problem as that of minimizing some empirical loss subject to one or more fairness constraints. One of the simplest examples of such is the following:

$$
\min_{\theta \in \Theta} \frac{1}{|S|} \sum_{x,y \in S} \ell(f(x;\theta), y) \\
\text{s.t. } \frac{1}{|S|} \sum_{x \in S_{\text{min}}} 1_{f(x;\theta) > 0} \geq \frac{0.8}{|S|} \sum_{x \in S} 1_{f(x;\theta) > 0}
$$

Here, $f(\cdot;\theta)$ is a classification function with parameters $\theta$, $S$ is the training dataset, and $S_{\text{min}} \subseteq S$ represents a minority population. The constraint represents a version of the so-called “80% rule” [e.g. Biddle 2005, Vuolo and Levy].
and forces the resulting classifier to make at least 80% of its positive predictions on the minority population—Goh et al. [2016] discuss a number of such constraints, both on fairness and non-fairness metrics.

Despite the simplicity of this problem, serious difficulties arise when we attempt to optimize it:

1. The constraint is data-dependent, and could therefore be very expensive to check.
2. The classification function $f$ may be a badly-behaving function of $\theta$ (e.g. a deep neural network), resulting in non-convex objective and constraint functions.
3. Worse, the constraint is a linear combination of indicators, hence is not even semidifferentiable with respect to $\theta$.

Counterintuitively, our proposed approach is based on one that does not work—at least not out-of-the-box—in light of Items 2 and 3 above. Namely, formulating the Lagrangian:

**Definition 1.** The Lagrangian $L : \Theta \times \Lambda \to \mathbb{R}$ of Equation 1 is:

$$L(\theta, \lambda) := g_0(\theta) + \sum_{i=1}^{m} \lambda_i g_i(\theta)$$

where $\Lambda \subseteq \mathbb{R}^m_+$. and jointly minimizing over $\theta \in \Theta$ and maximizing over $\lambda \in \Lambda \subseteq \mathbb{R}^m_+$. This approach does, however, have a marked advantage over the alternatives (Section 2.1) in terms of Item 1: we can perform the two optimizations (over $\theta$ and $\lambda$) using an efficient stochastic first-order algorithm (e.g. SGD [Robbins and Monro, 1951, Zinkevich, 2003], AdaGrad [Duchi et al., 2011] or ADAM [Kingma and Ba, 2014]). In particular, it is only necessary to examine training elements one-(minibatch)-at-a-time.

### 1.1 Dealing with non-Convexity

We interpret optimizing the Lagrangian as a two player zero-sum game: the first player chooses $\theta$ to minimize $L(\theta, \lambda)$, and the second player chooses $\lambda$ to maximize it. The essential difficulty is that, without strong duality—equivalently, unless the minimax theorem holds, giving that $\min_{\theta \in \Theta} \max_{\lambda \in \Lambda} L(\theta, \lambda) = \max_{\lambda \in \Lambda} \min_{\theta \in \Theta} L(\theta, \lambda)$—then the $\theta$-player, who is working on the primal (minimax) problem, and the $\lambda$-player, who is working on the dual (maximin) problem, might fail to converge to a solution satisfying both players simultaneously (i.e. a pure Nash equilibrium).

If Equation 1 is a convex optimization problem and the action spaces $\Theta$ and $\Lambda$ are compact and convex, then the minimax theorem holds [von Neumann, 1928], and optimizing the Lagrangian will work. Otherwise it might not, and in fact it’s quite easy to construct a counterexample: Figure 1 shows a case in which a pure Nash equilibrium of the Lagrangian game does not exist.

Under general conditions, however, even when there is no pure Nash equilibrium, a mixed equilibrium (i.e. a pair of distributions over $\theta$ and $\lambda$) does exist. If we could find such an equilibrium, then we could use it to define a stochastic classifier: upon receiving an example $x$ to classify, we would sample $\theta$ from its equilibrium distribution, and then evaluate the classification function $f(x; \theta)$. Furthermore, and this is our first main contribution, this equilibrium can be taken to consist of a discrete distribution over at most $m+1$ distinct $\theta$s ($m$ being the number of constraints), and a single non-random $\lambda$. This is a crucial improvement in practical terms, since a machine learning model consisting...

![Figure 1: The plotted rectangular region is the domain $\Theta$, the contours are those of the strictly concave minimization objective function $g_0$, and the shaded triangle is the feasible region determined by the three linear inequality constraints $g_1, \ldots, g_3$. The red dot is the optimal feasible point. The Lagrangian $L(\theta, \lambda)$ is strictly concave in $\theta$ for any choice of $\lambda$, so the optimal choice(s) for the $\theta$-player will always lie on the four corners of the plotted rectangle. However, these points are infeasible, and therefore suboptimal for the $\lambda$-player (assuming that $\lambda \in \Lambda = \mathbb{R}^3_+$).](image-url)
of e.g. a distribution over thousands (or more) of deep neural networks—or worse, a continuous distribution—would likely be so unwieldy as to be unusable.

In Section 3, we (i) provide an algorithm that, given access to an approximate Bayesian optimization oracle, finds a distribution over (a large number of) \( \theta \)’s that, in expectation, is provably approximately feasible and optimal, (ii) show how such a distribution can be efficiently “shrunk” to one that is at least as good, but is supported on only \( m + 1 \) solutions, and (iii) combine these into a practical algorithm that does not require an oracle, and instead uses the typical stochastic Lagrangian optimization procedure to generate an initial “candidate” distribution (for which we have no guarantees in the non-convex setting), and then shrinks it to one supported on \( m + 1 \) solutions.

### 1.2 Introducing Proxy Constraints

While the “practical” algorithm mentioned above can deal with non-convex constrained problems (albeit without guarantees), it cannot deal with non-(semi)differentiable constraints like that in Equation 2—we need a new approach.

To this end, we introduce the notion of “proxy constraints” by taking \( \tilde{g}_i (\theta) \) to be a sufficiently-smooth upper bound on \( g_i (\theta) \) for \( i \in [m] \), and formulating two functions that we call “proxy-Lagrangians”:

**Definition 2.** Given proxy constraint functions \( \tilde{g}_i (\theta) \geq g_i (\theta) \) for \( i \in [m] \), the proxy-Lagrangians \( \mathcal{L}_\theta, \mathcal{L}_\lambda : \Theta \times \Lambda \to \mathbb{R} \) of Equation 1 are:

\[
\mathcal{L}_\theta (\theta, \lambda) := \lambda_1 g_0 (\theta) + \sum_{i=1}^{m} \lambda_{i+1} \tilde{g}_i (\theta)
\]

\[
\mathcal{L}_\lambda (\theta, \lambda) := \sum_{i=1}^{m} \lambda_{i+1} g_i (\theta)
\]

where \( \Lambda := \Delta^{m+1} \) is the \((m + 1)\)-dimensional simplex.

As one might expect, the \( \theta \)-player wishes to minimize \( \mathcal{L}_\theta (\theta, \lambda) \), while the \( \lambda \)-player wishes to maximize \( \mathcal{L}_\lambda (\theta, \lambda) \). Notice that the \( \tilde{g}_i \)s are only used by the \( \theta \)-player—intuitively, the \( \lambda \)-player chooses how much to weigh the proxy constraint functions, but does so in such a way as to satisfy the original constraints.

Unfortunately, because the two players are optimizing different functions, this is a non-zero-sum game, and finding a (mixed) Nash equilibrium of such games is known to be PPAD-complete even in the finite setting [Chen and Deng, 2006]. We prove, however, that a weaker type of equilibrium (a \( \Phi \)-correlated equilibrium [Rakhlin et al., 2011], i.e. a joint distribution over \( \theta \) and \( \lambda \) w.r.t. which neither player can improve)—one that we can find efficiently—suffices to guarantee a nearly-optimal and nearly-feasible solution to Equation 1 in expectation.

In Section 4, we provide an efficient stochastic algorithm that approximates such an equilibrium, assuming convexity (except of the \( g_i \)s). An analogous algorithm without convexity assumptions, but relying on an oracle, is deferred to Appendix C.3 (since if you have an oracle, then you don’t need proxy constraints). The resulting equilibrium is a distribution over (a large number of) \((\theta, \lambda)\) pairs, but applying the same “shrinking” procedure as before yields a distribution over only \( m + 1 \) points that is at least as good as the original. Finally, for non-convex problems, we propose the natural “practical” variant: using the above algorithm as if we were in the convex setting to find a candidate distribution (with no guarantees), and then using the shrinking procedure.

### 2 Related Work

The interpretation of constrained optimization as a two-player game has a long history: [Arora et al., 2012] surveys some such work. The recent paper of [Chen et al., 2017] addresses a slightly different problem, namely non-convex
robust optimization, i.e. problems of the form:

$$\min_{\theta \in \Theta} \max_{i \in [m]} g_i(\theta)$$

Like us, they model such a problem as a two-player game where one player chooses a mixture of objective functions, and the other player minimizes the loss of the mixture, and again like us, they find a distribution over solutions rather than a pure equilibrium. Indeed, Algorithm 1 of their paper is identical in its essentials to (but superficially very different from) our Algorithm 1 as is demonstrated by the fact that robust optimization can be reformulated as constrained optimization via the introduction of a slack variable $\xi$:

$$\min_{\theta \in \Theta, \xi \in \Xi} \xi$$

s.t. $\forall i \in [m]. \xi \geq g_i(\theta)$

Correspondingly, one can transform a robust problem to a constrained one at the cost of an extra bisection search [e.g. Christiano et al. 2011, Rakhlin and Sridharan 2013]. This connection between the two settings suggests some extensions to the work of Chen et al. [2017], in particular (i) our proposed shrinking procedure can be applied to Equation 3 to yield a distribution over only $m + 1$ solutions, and (ii) one could perform robust optimization over non-(semi)differentiable (e.g. indicator-based) losses using “proxy objectives”, just as we use proxy constraints.

2.1 Alternative Approaches

Given the difficulties involved in using a Lagrangian-like formulation for non-convex problems, it’s natural to ask whether one should instead favor a procedure based on entirely different principles. Unfortunately, the potential alternatives each present their own challenges.

The complexity of the constraints all but rules out approaches based on projections (e.g. projected SGD) or optimization of constrained subproblems (e.g. Frank-Wolfe, as in Hazan and Kale [2012], Jaggi [2013], Garber and Hazan [2013]). Similarly, attempting to penalize violations [e.g. Arora et al. 2012, Rakhlin and Sridharan 2013, Mahdavi et al. 2012, Cotter et al. 2016], for example by adding $\gamma \max \{0, \max_{i \in [m]} g_i(\theta)\}$ to the objective, where $\gamma \in \mathbb{R}^+$ is a hyperparameter, and optimizing the resulting problem using a first order method, fails if the constraint functions are non-(semi)differentiable. Even if they are, the constraints may still be data-dependent, so evaluating $g_i$, or even determining whether it is positive (as is necessary for such methods), requires enumerating over the entire dataset, and therefore is incompatible with the use of a computationally-cheap stochastic optimizer.

In response to the idea of proxy constraints, it’s natural to ask “why not just relax the constraints for both players, instead of just the $\theta$-player?” This is indeed a popular approach, having been proposed e.g. for Neyman-Pearson classification [Davenport et al. 2011, Gasso et al. 2011], more general rate metrics [Goh et al. 2016], and AUC [Eban et al. 2017]. The answer is that in many cases, particularly when constraints are data dependent, they represent real-world restrictions on how the learned model is permitted to behave. For example, the “80% rule” of Equation 2 can be found in the HOPA Act of 1995 [Wikipedia, 2018], and it requires an 80% threshold in terms of the number of positive predictions—not a relaxation—which is precisely the target that the proxy-Lagrangian approach will attempt to hit.

This point, in turn, raises the question of generalization: satisfying the correct un-relaxed constraints on training data does not necessarily mean that they will be satisfied at evaluation time. This issue is outside the scope of this paper, but is vital. In many applications, the post-training correction approach of Woodworth et al. [2017] can improve generalization performance, but there is room for future work.

3 Lagrangian Optimization

Our ultimate interest is in constrained optimization, so before we present our proposed algorithm for optimizing the Lagrangian (Definition 1) in the non-convex setting, we will characterize the relationship between an approximate
Algorithm 1 Optimizes the Lagrangian formulation (Definition\(^1\)) in the non-convex setting via the use of an approximate Bayesian optimization oracle \(O_\rho\) (Definition\(^3\)) for the \(\theta\)-player. The parameter \(R\) is the radius of the Lagrange multiplier space \(\Lambda := \{\lambda \in \mathbb{R}^m_+ : ||\lambda||_1 \leq R\}\), and the function \(\Pi_\Lambda\) projects its argument onto \(\Lambda\) w.r.t. the Euclidean norm.

Algorithm

| Line |
|------|
| 1.   | Initialize \(\lambda^{(1)} = 0\) |
| 2.   | For \(t \in [T]\): |
| 3.   | Let \(\theta^{(t)} = O_\rho\left(L_{\cdot, \lambda^{(t)}}\right)\) \(\text{// }\) Oracle optimization |
| 4.   | Let \(\Delta^{(t)}_\lambda\) be a supergradient of \(L_{\theta^{(t)}, \lambda^{(t)}}\) w.r.t. \(\lambda\) |
| 5.   | Update \(\lambda^{(t+1)} = \Pi_\Lambda\left(\lambda^{(t)} + \eta_\lambda \hat{\Delta}^{(t)}_\lambda\right)\) \(\text{// }\) Projected gradient update |
| 6.   | Return \(\theta^{(1)}, \ldots, \theta^{(T)}\) and \(\lambda^{(1)}, \ldots, \lambda^{(T)}\) |

Nash equilibrium of the Lagrangian game, and a nearly-optimal nearly-feasible solution to the original constrained problem (Equation\(^4\)):

**Theorem 1.** Define \(\Lambda := \{\lambda \in \mathbb{R}^m_+ : ||\lambda||_1 \leq R\}\), and let \(\theta^{(1)}, \ldots, \theta^{(T)} \in \Theta\) and \(\lambda^{(1)}, \ldots, \lambda^{(T)} \in \Lambda\) be sequences of parameter vectors and Lagrange multipliers that comprise an approximate mixed Nash equilibrium, i.e.:

\[
\max_{\lambda^* \in \Lambda} \frac{1}{T} \sum_{t=1}^{T} L_{\theta^{(t)}, \lambda^*} - \inf_{\theta^* \in \Theta} \frac{1}{T} \sum_{t=1}^{T} L_{\theta^*, \lambda^{(t)}} \leq \epsilon
\]

Define \(\bar{\theta}\) as a random variable for which \(\bar{\theta} = \theta^{(t)}\) with probability \(1/T\), and let \(\bar{\lambda} := \left(\sum_{t=1}^{T} \lambda^{(t)}\right) / T\). Then \(\bar{\theta}\) is nearly-optimal in expectation:

\[
\mathbb{E}_{\bar{\theta}} [g_0(\bar{\theta})] \leq \inf_{\theta^* \in \Theta : \forall i, g_i(\theta^*) \leq 0} g_0(\theta^*) + \epsilon
\]

and nearly-feasible:

\[
\max_{i \in [m]} \mathbb{E}_{\bar{\theta}} [g_i(\bar{\theta})] \leq \frac{\epsilon}{R - ||\lambda||_1}
\]

(4)

Additionally, if there exists a \(\theta' \in \Theta\) that satisfies all of the constraints with margin \(\gamma\) (i.e. \(g_i(\theta') \leq -\gamma\) for all \(i \in [m]\)), then:

\[
||\bar{\lambda}||_1 \leq \frac{\epsilon + B_{g_0}}{\gamma}
\]

where \(B_{g_0} \geq \sup_{\theta \in \Theta} g_0(\theta) - \inf_{\theta \in \Theta} g_0(\theta)\) is a bound on the range of the objective function \(g_0\).

**Proof.** This is a special case of Theorem\(^3\) and Lemma\(^6\) in Appendix\(^A\)

This theorem has a few differences from the more typically-encountered equivalence between Nash equilibria and optimal feasible solutions in the convex setting. First, it characterizes mixed equilibria, in that uniformly sampling from the sequences \(\theta^{(t)}\) and \(\lambda^{(t)}\) can be interpreted as defining distributions over \(\Theta\) and \(\Lambda\). A convexity assumption would enable us to eliminate this added complexity by appealing to Jensen’s inequality to replace these sequences with their averages. Second, for the technical reason that we require compact domains in order to prove convergence rates (below), \(\Lambda\) is taken to consist only of sets of Lagrange multipliers with bounded 1-norm\(^4\).

Finally, as a consequence of this second point, the feasibility guarantee of Equation\(^4\) only holds if the Lagrange multipliers are, on average, smaller than the maximum 1-norm radius \(R\). Thankfully, as is shown by the final result of Theorem\(^1\) if there exists a point satisfying the constraints with some margin \(\gamma\), then there will exist \(R\)s that are large enough to guarantee feasibility to within \(O(\epsilon)\).

Our proposed algorithm (Algorithm\(^1\)) follows roughly the same lines as Chen et al.\(^2017\)’s algorithm for robust optimization, in that we assume the existence of an oracle for performing approximate non-convex minimization:

\(^1\)In Appendix\(^A\) this is generalized to \(p\)-norms.
Definition 3. A $\rho$-approximate Bayesian optimization oracle is a function $O_\rho : (\Theta \rightarrow \mathbb{R}) \rightarrow \Theta$ for which:

$$f(O_\rho(f)) \leq \inf_{\theta^* \in \Theta} f(\theta^*) + \rho$$

for any $f : \Theta \rightarrow \mathbb{R}$ that can be written as a nonnegative linear combination of the objective and constraint functions $g_0, g_1, \ldots, g_m$.

with the $\theta$-player using this oracle, and the $\lambda$-player using projected gradient ascent. Notice that, unlike the oracle of Chen et al. [2017], which provides a multiplicative approximation, $O_\rho$ provides an additive approximation. Algorithm 1’s convergence rate is:

Lemma 1. Suppose that $\Lambda$ and $R$ are as in Theorem 1 and define the upper bound $B_\Delta \geq \max_{t \in [T]} \| \hat{\Delta}^{(t)} \|_2$. If we run Algorithm 1 with the step size $\eta_\lambda := R/B_\Delta \sqrt{2/T}$, then the result satisfies the conditions of Theorem 1 for:

$$\epsilon = \rho + RB_\Delta \sqrt{2/T}$$

where $\rho$ is the error associated with the oracle $O_\rho$.

Proof. In Appendix C.3.

Combined with Theorem 1 we therefore have that if $R$ is sufficiently large, then Algorithm 1 will converge to a distribution over $\Theta$ that is, in expectation, $O(\rho)$-far from being optimal and feasible at a $O(1/\sqrt{T})$ rate, where $\rho$ is as in Definition 3.

3.1 Shrinking

The main disadvantage of Algorithm 1 (aside from the unrealistic oracle assumption) is that it results in a mixture of $T$ models, which presumably would be far too large to apply in practice. However, much smaller Nash equilibria exist:

Lemma 2. If $\Theta$ is a compact Hausdorff space, $\Lambda$ is compact, and the objective and constraint functions $g_0, g_1, \ldots, g_m$ are continuous, then the Lagrangian game (Definition 1) has a mixed Nash equilibrium pair $(\theta, \lambda)$ where $\theta$ is a random variable supported on at most $m + 1$ elements of $\Theta$, and $\lambda$ is non-random.

Proof. Follows from Theorem 5 in Appendix B.

Of course, the mere existence of such an equilibrium is insufficient—we need to be able to find it, and Algorithm 1 manifestly does not. If we evaluate the objective and constraint functions on each of the $\theta^{(t)}$s, however, and store the results, then we can re-formulate the problem of finding the optimal $\epsilon$-feasible mixture of the $\theta^{(t)}$s as a linear program (LP) that can be solved to “shrink” the support set:

Lemma 3. Let $\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(T)} \in \Theta$ be a sequence of $T$ “candidate solutions” of Equation 1. Define $\bar{g}_0, \bar{g}_i \in \mathbb{R}^T$ such that $(\bar{g}_0)_i = g_0(\theta^{(i)})$ and $(\bar{g}_i)_i = g_i(\theta^{(i)})$ for $i \in [m]$, and consider the linear program:

$$\min_{p \in \Delta^T} \langle p, \bar{g}_0 \rangle$$

s.t. $\forall i \in [m], \langle p, \bar{g}_i \rangle \leq \epsilon$

where $\Delta^T$ is the $T$-dimensional simplex. Then every vertex $p^*$ of the feasible region—in particular an optimal one—has at most $m^* + 1 \leq m + 1$ nonzero elements, where $m^*$ is the number of active $\langle p^*, \bar{g}_i \rangle \leq \epsilon$ constraints.

Proof. In Appendix B.
Algorithm 2 Optimizes the proxy-Lagrangian formulation (Definition 2) in the convex setting, with the \( \theta \)-player minimizing external regret, and the \( \lambda \)-player minimizing swap regret. The fix \( M \) operation on line 3 results in a stationary distribution of \( M \) (i.e., a \( \lambda \in \Lambda \) such that \( M\lambda = \lambda \), which can be derived from the top eigenvector). The exponentiation and product on line 7 are performed element-wise, and the function \( \Pi_{\Theta} \) projects its argument onto \( \Theta \) w.r.t. the Euclidean norm.

StochasticProxyLagrangian \(
\left( \mathcal{L}_\theta, \mathcal{L}_\lambda : \Theta \times \Delta^{m+1} \to \mathbb{R}, \mathcal{T} \in \mathbb{N}, \eta_\theta, \eta_\lambda \in \mathbb{R}_+ \right) :
\)

1. Initialize \( \bar{\theta}^{(1)} = 0 \), and \( M^{(1)} \in \mathbb{R}^{(m+1)\times(m+1)} \) with \( M_{i,j} = 1 / (m + 1) \) \( \text{ // Assumes } 0 \in \Theta \)
2. For \( t \in [T] \):
   3. Let \( \bar{\lambda}^{(t)} = \text{ fix } M^{(t)} \) \( \text{ // Stationary distribution of } M^{(t)} \)
   4. Let \( \hat{\Delta}_\theta^{(t)} \) be a stochastic subgradient of \( \mathcal{L}_\theta (\bar{\theta}^{(t)}, \bar{\lambda}^{(t)}) \) w.r.t. \( \theta \)
   5. Let \( \hat{\Delta}_\lambda^{(t)} \) be a stochastic supergradient of \( \mathcal{L}_\lambda (\bar{\theta}^{(t)}, \bar{\lambda}^{(t)}) \) w.r.t. \( \lambda \)
   6. Update \( \theta^{(t+1)} = \Pi_{\Theta} \left( \bar{\theta}^{(t)} - \eta_\theta \hat{\Delta}_\theta^{(t)} \right) \) \( \text{ // Projected SGD update } \)
   7. Update \( \tilde{M}^{(t+1)} = M^{(t)} \odot \text{ element-wise-exp} \left( \eta_\lambda \hat{\Delta}_\lambda^{(t)} (\bar{\lambda}^{(t)})^T \right) \) \( \text{ // Stochastic multiplicative update } \)
   8. Project \( M_{\cdot i}^{(t+1)} = \tilde{M}_{\cdot i}^{(t+1)} / \left\| \tilde{M}_{\cdot i}^{(t+1)} \right\|_1 \) for \( i \in [m + 1] \) \( \text{ // Column-wise projection w.r.t. KL divergence } \)
9. Return \( \theta^{(1)}, \ldots, \theta^{(T)} \) and \( \lambda^{(1)}, \ldots, \lambda^{(T)} \)

3.2 Overall Procedure

The pieces are now in place to propose a complete optimization procedure: first apply Algorithm 1 to yield an approximate Nash equilibrium, and then optimize the LP of Lemma 3 to shrink it to have support size \( m + 1 \) (with the \( \epsilon \) parameter to the LP being the RHS of Equation 4 in Theorem 4). Since the uniform distribution over the \( \theta^{(i)} \)'s is feasible for this LP, the resulting distribution will be at least as good in terms of feasibility and optimality.

Practical Procedure: The approach outlined above provably works, but is still unrealistic—we propose doing things differently in practice. First, we’ll dispense with the oracle \( O_\rho \) in favor of the “typical” approach: pretending that the problem is convex, and using SGD (or another cheap stochastic algorithm) on both the \( \theta \) and \( \lambda \)-players. On a non-convex problem, this has no guarantees, but one would still hope that it would result in a “candidate set” of \( \theta^{(i)} \)'s (which can be subsampled to make it a reasonable size) that contains enough good solutions to pass on to the LP of Lemma 2. Since Theorem 1 does not apply, we need to determine the \( \epsilon \) parameter of this LP, which we take to be the smallest \( \epsilon \geq 0 \) for which there exists a feasible solution, found via bisection.

Evaluation: The ultimate result of either of these procedures is a distribution over at most \( m + 1 \) distinct \( \theta \)'s. If the underlying problem is one of classification, with \( f (\cdot; \theta) \) being the scoring function, then this distribution defines a stochastic classifier: at evaluation time, upon receiving an example \( x \), we would sample \( \theta \), and then return \( f (x; \theta) \).

4 Proxy-Lagrangian Optimization

For the proxy-Lagrangian game (Definition 2), we cannot expect to find a Nash equilibrium, at least not efficiently, since it is non-zero-sum. However, the analogous result to Theorem 1 requires a weaker type of equilibrium, namely a joint distribution over \( \Theta \) and \( \Lambda \) w.r.t. which the \( \theta \)-player can only make a negligible improvement compared to the best constant strategy, and the \( \lambda \)-player compared to the best action-swapping strategy (this is a particular type of \( \Phi \)-correlated equilibrium [Rakhlin et al. 2011]):

**Theorem 2.** Define \( \mathcal{M} \) as the set of all left-stochastic \( (m + 1) \times (m + 1) \) matrices, \( \Lambda := \Delta^{m+1} \) as the \( (m + 1) \)-dimensional simplex, and assume that each \( \tilde{g}_i \) upper bounds the corresponding \( g_i \). Let \( \theta^{(1)}, \ldots, \theta^{(T)} \in \Theta \) and

\[2\text{This is Algorithm 3 with Lemma 4 being its convergence guarantee in the convex setting, both in Appendix B.4.}\]
\(\lambda^{(1)}, \ldots, \lambda^{(T)} \in \Lambda\) be sequences satisfying:

\[
\frac{1}{T} \sum_{t=1}^{T} L_\theta \left( \theta^{(t)}, \lambda^{(t)} \right) - \inf_{\theta^* \in \Theta} \frac{1}{T} \sum_{t=1}^{T} L_\theta \left( \theta^*, \lambda^{(t)} \right) \leq \epsilon_\theta
\]

\[
\max_{M^* \in \mathcal{M}} \frac{1}{T} \sum_{t=1}^{T} L_\lambda \left( \theta^{(t)}, M^* \lambda^{(t)} \right) - \frac{1}{T} \sum_{t=1}^{T} L_\lambda \left( \theta^{(t)}, \lambda^{(t)} \right) \leq \epsilon_\lambda
\]

Define \(\tilde{\theta}\) as a random variable for which \(\tilde{\theta} = \theta^{(t)}\) with probability \(\lambda_1^{(t)} / \sum_{s=1}^{T} \lambda_1^{(s)}\), and let \(\bar{\lambda} := \left( \sum_{t=1}^{T} \lambda_1^{(t)} \right) / T\). Then \(\bar{\lambda}\) is nearly-optimal in expectation:

\[
\mathbb{E}_\tilde{\theta} \left[ g_0 \left( \tilde{\theta} \right) \right] \leq \inf_{\theta^* \in \Theta, \forall i : \mathcal{g}_i (\theta^*) \leq 0} g_0 \left( \theta^* \right) + \frac{\epsilon_\theta + \epsilon_\lambda}{\lambda_1}
\]

and nearly-feasible:

\[
\max_{i \in [m]} \mathbb{E}_\tilde{\theta} \left[ g_i \left( \tilde{\theta} \right) \right] \leq \frac{\epsilon_\lambda}{\lambda_1}
\]

Additionally, if there exists a \(\theta' \in \Theta\) that satisfies all of the proxy constraints with margin \(\gamma\) (i.e. \(\mathcal{g}_i (\theta') \leq -\gamma\) for all \(i \in [m]\)), then:

\[
\bar{\lambda}_1 \geq \frac{\gamma - \epsilon_\theta - \epsilon_\lambda}{\gamma + B_{g_0}}
\]

where \(B_{g_0} := \sup_{\theta^* \in \Theta} g_0 \left( \theta^* \right) - \inf_{\theta \in \Theta} g_0 \left( \theta \right)\) is a bound on the range of the objective function \(g_0\).

**Proof.** This is a special case of Theorem 4 and Lemma 7 in Appendix A.

Notice that while Equation 6 guarantees feasibility w.r.t. the original constraints, the comparator in Equation 5 is feasible w.r.t. the proxy constraints. Hence, the overall guarantee is no better than what we would achieve if we took \(g_i = \tilde{g}_i\) for all \(i \in [m]\), and optimized the Lagrangian as in Section 3. However, as will be demonstrated in Section 5.2, the feasible region w.r.t. the original constraints is larger (perhaps significantly so) than that w.r.t. the proxy constraints, the proxy-Lagrangian approach has more “room” to find a better solution in practice.

One key difference between this result and Theorem 1 is that the \(R\) parameter is absent. Instead, its role, and that of \(\|\lambda\|_1\), is played by the first coordinate of \(\bar{\lambda}\). Inspection of Definition 2 reveals that, if one or more of the constraints are violated, then the \(\lambda\)-player would prefer \(\lambda_1\) to be zero, whereas if they are satisfied (with some margin), then it would prefer \(\lambda_1\) to be one. In other words, the first coordinate of \(\lambda^{(t)}\) encodes the \(\lambda\)-player’s belief about the feasibility of \(\theta^{(t)}\), for which reason \(\theta^{(t)}\) is weighted by \(\lambda_1^{(t)}\) in the density defining \(\tilde{\theta}\).

Algorithm 2 is motivated by the observation that, while Theorem 2 only requires that the \(\theta^{(t)}\) sequence suffer low external regret w.r.t. \(L_\theta \left( \cdot, \lambda^{(t)} \right)\), the condition on the \(\lambda^{(t)}\) sequence is stronger, requiring it to suffer low swap regret [Blum and Mansour, 2007]. Hence, the \(\theta\)-player uses SGD to minimize external regret, while the \(\lambda\)-player uses a swap-regret minimization algorithm of the type proposed by Gordon et al. [2008], yielding the convergence guarantee:

**Lemma 4.** Suppose that \(\Theta\) is a compact convex set, \(\mathcal{M}\) and \(\Lambda\) are as in Theorem 2 and that the objective and proxy constraint functions \(g_0, \tilde{g}_1, \ldots, \tilde{g}_m\) are convex (but not \(g_1, \ldots, g_m\)). Define the three upper bounds \(B_{g_0} \geq \max_{\theta \in \Theta} \|\theta\|_2\), \(B_\Delta \geq \max_{t \in [T]} \left\| \Delta^{(t)} \right\|_2\), and \(B_{\Delta} \geq \max_{t \in [T]} \left\| \Delta^{(t)} \right\|_\infty\).

If we run Algorithm 2 with the step sizes \(\eta_\theta := B_{g_0} / B_\Delta \sqrt{2T}\) and \(\eta_\lambda := \sqrt{(m + 1) \ln (m + 1) / TB_\Delta^2}\), then the result satisfies the conditions of Theorem 2 for:

\[
\epsilon_\theta = 2B_{g_0} B_\Delta \sqrt{\frac{1 + 16 \ln 2}{T}}
\]

\[
\epsilon_\lambda = 2B_\Delta \sqrt{\frac{2(m + 1) \ln (m + 1)(1 + 16 \ln 2)}{T}}
\]
with probability $1 - \delta$ over the draws of the stochastic \{sub, super\} gradients.

**Proof.** In Appendix C.3. \qed

Algorithm 2 is designed for the convex setting (except for the $g_i$s)—in other words, it’s the proxy-Lagrangian analogue of the stochastic Lagrangian algorithm that we proposed using in the first phase of the “practical procedure” of Section 3.2. It’s straightforward to design an oracle-based algorithm that, like Algorithm 1, doesn’t require convexity, but the purpose of proxy constraints is to substitute optimizable constraints for unoptimizable ones, and there is no need to do so if you have an oracle.

### 4.1 Shrinking

While Algorithm 2 only finds a correlated equilibrium, it turns out that essentially the same existence result that we provided for the Lagrangian game (Lemma 3)—of a Nash equilibrium—holds for the proxy-Lagrangian:

**Lemma 5.** If $\Theta$ is a compact Hausdorff space and the objective, constraint and proxy constraint functions $g_0, g_1, \ldots, g_m, \tilde{g}_1, \ldots, \tilde{g}_m$ are continuous, then the proxy-Lagrangian game (Definition 2) has a mixed Nash equilibrium pair $(\theta, \lambda)$ where $\theta$ is a random variable supported on at most $m + 1$ elements of $\Theta$, and $\lambda$ is non-random.

**Proof.** In Appendix B. \qed

Furthermore, the exact procedure of Lemma 3 can be applied to the $\theta^{(t)}$s expected by Theorem 2 (with the $\vec{g}_i$s being defined in terms of the original—not proxy—constraints), and works equally well, since $\bar{\theta}$ is still feasible for the LP.

### 4.2 Overall Procedure

Our complete proxy-Lagrangian optimization procedure is just like those of Section 3.2: run Algorithm 2 to find a set of “candidate” $\theta^{(t)}$s, and then pass them on to the LP of Lemma 3 to yield a distribution supported on $m + 1$ points, which, assuming that the underlying problem is one of classification, defines a stochastic classifier.

If the objective $g_0$ and proxy constraint functions $\tilde{g}_1, \ldots, \tilde{g}_m$ are convex (but not necessarily the $g_i$s), then this procedure provably works (assuming that the $\epsilon$ parameter of the LP is the RHS of Equation 6 in Theorem 2).

Otherwise, as in the “practical procedure” of Section 3.2, we propose (optionally) subsampling the set of $\theta^{(t)}$s, and then finding the $\epsilon$ parameter of the LP via bisection. This approach has no guarantees, but we believe that it constitutes a good compromise between theoretical correctness and practicality.

### 5 Experiments

We present two experiments: the first, on the robust MNIST problem of Chen et al. [2017], tests the performance of the “practical procedure” of Section 3.2, while the second, a fairness problem on the UCI Adult dataset, tests the proxy-Lagrangian approach of Section 4.2. Both were implemented in TensorFlow.

For all of algorithms (including baselines) we used the AdaGrad optimizer. Lagrangian optimizations were performed using the “practical procedure” of Section 3.2 (with the norms of the Lagrange multipliers being unbounded, i.e. $R = \infty$), while proxy-Lagrangian optimizations used that of Section 4.2. In both cases, the $\theta$ and $\lambda$-updates both used AdaGrad with the same initial learning rates. In the proxy-Lagrangian case, however, the $\lambda$-update (line 7 of Algorithm 2) was performed in the log domain so that it would be multiplicative. To choose the initial AdaGrad

---

3 This is Algorithm 4, with Lemma 11 being its convergence guarantee, both in Appendix C.3.
Table 1: Error rates on the experiments of Section 5.1. The columns correspond to the four corrupted datasets of Chen et al. [2017].

|                  | Testing |       |       |       |       |       |       |
|------------------|---------|-------|-------|-------|-------|-------|-------|
|                  | Set 1   | Set 2 | Set 3 | Set 4 | Set 1 | Set 2 | Set 3 | Set 4 |
| Baseline ($\theta$) | 2.58    | 2.66  | 2.01  | 2.52  | 1.06  | 1.45  | 0.43  | 1.10  |
| Baseline ($\theta(T)$) | 1.77    | 1.92  | 1.77  | 1.75  | 0.04  | 0.40  | 0.01  | 0.08  |
| Lagrangian ($\tilde{\theta}$) | 2.04    | 2.15  | 1.96  | 2.04  | 0.42  | 0.70  | 0.30  | 0.49  |
| Lagrangian (LP) | 1.66    | 1.67  | 1.63  | 1.62  | 0.00  | 0.01  | 0.00  | 0.00  |

Table 2: Support sizes, test error rates, and “equal opportunity” values for the experiments of Section 5.2. For the constraints, each reported number is the ratio of the positive prediction rate on positively-labeled members of the protected class, to the positive prediction rate on the set of all positively-labeled data. The constraints attempt to force this ratio to be at least 95%—quantities lower than this threshold violate the constraint, and are marked in bold.

|                  |                     |       |       |       |       |       |       |       |       |       |       |       |       |       |       |
|------------------|---------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|                  | Support             | Female | Male | Black | White | Female | Male | Black | White | Female | Male | Black | White | Female | Male | Black | White |
| Baseline ($\theta(T)$) | 1                   | 14.2%  | 89.5% | 102%  | 81.6% | 101%  | 88.9% | 102%  | 82.8% | 101%  |
| Lagrangian ($\tilde{\theta}$) | 100                | 16.3%  | 114%  | 97.5% | 126%  | 99.8% | 113%  | 97.8% | 121%  | 99.7% |
| Lagrangian (LP) | 3                   | 15.5%  | 106%  | 99.0% | 111%  | 101%  | 104%  | 99.4% | 105%  | 101%  |
| Proxy ($\tilde{\theta}$) | 100                | 14.4%  | 94.1% | 101%  | 94.9% | 100%  | 94.7% | 101%  | 94.5% | 100%  |
| Proxy (LP) | 3                   | 14.2%  | 94.4% | 101%  | 94.9% | 100%  | 95.0% | 101%  | 95.0% | 100%  |

learning rate, we performed a grid search over powers-of-two, and chose the best model on a validation set. In all experiments, the optimum was in the interior of the grid.

Our constrained optimization algorithms result in stochastic classifiers, and we report results for both the $\tilde{\theta}$ of Theorems 1 or 2 (in the Lagrangian or proxy-Lagrangian cases, respectively), and the optimal distribution found by the LP of Lemma 3 optimized on the training dataset.

5.1 Robust Optimization

In robust optimization, there are multiple objective functions $g_1, \ldots, g_m : \Theta \to \mathbb{R}$, and the goal is to find a $\theta \in \Theta$ minimizing $\max_{i \in [m]} g_i(\theta)$. As was discussed in Section 2, this can be re-written as a constrained problem by introducing a slack variable, as in Equation (3).

The task is the modified MNIST problem created by Chen et al. [2017], which is based on four datasets, each of which is a version of MNIST that has been corrupted in different ways. One would therefore hope that choosing $g_i$ to be an empirical loss on the $i$th such dataset, and optimizing the corresponding robust problem, will result in a classifier that is “robust” to all four types of corruption.

We used a neural network with one 1024-neuron hidden layer, and ReLu activations. The four objective functions were the cross-entropy losses on the corrupted datasets. All models were trained for 50 000 iterations using a minibatch size of 100, and a $\theta(T)$ was extracted every 500 iterations, yielding a sequence of length $T = 100$.

**Baselines:** For our baselines, we trained the neural network over the union of the four datasets. We report two variants: (i) the “Uniform Distribution Baseline” of Chen et al. [2017] is a stochastic classifier, uniformly sampled over the $\theta(T)$s (like our $\tilde{\theta}$ classifier), and (ii) a non-stochastic classifier taking its parameters from the last iterate $\theta(T)$.

**Results:** Table 1 lists, for each of the corrupted datasets, the error rates of the compared models on both the training and testing datasets. Interestingly, although our proposed shrinking procedure is only guaranteed to give a distribution
over \( m + 1 \) solutions, in these experiments it chose only one. Hence, the “Lagrangian (LP)” model of Table 1 is, like “Baseline \( (\theta^{(T)}) \)”, non-stochastic.

While we did not quite match the raw performance reported by Chen et al. [2017]’s algorithm, our results tell a similar story. In particular, we can see that both of our algorithms outperformed their natural baseline equivalents. In particular, the use of shrinking not only greatly simplified the model, but also significantly improved performance.

### 5.2 Equal Opportunity

These experiments were performed on the UCI Adult dataset, which consists of census data including 14 features such as age, gender, race, occupation, and education. The goal was to predict whether income exceeds 50k/year. The dataset contains 32 561 training examples, from which we split off 20% to form a validation set, and 16 281 testing examples.

We dropped the “fnlwgt” weighting feature, and processed the remaining features as in Platt [1998], yielding 120 binary features, on which we trained linear models. The objective was to minimize the average hinge loss, subject to one 95% equal opportunity [Hardt et al., 2016] constraint in the style of Goh et al. [2016] for each “protected class”: 
\[
g_i(\theta) \leq 0 \text{ iff the positive prediction rate on the set of positively-labeled examples for the associated class was at least 95\% of the positive prediction rate on the set of all positively-labeled examples.}
\]

When using proxy constraints, \( \tilde{g}_i \) was taken to be a version of \( g_i \) with the indicator functions defining the positive prediction rates replaced with hinge upper bounds. When not using proxy constraints, the indicator-based constraints were dropped entirely, with these upper bounds being used throughout.

All models were trained for 5 000 iterations with a minibatch size of 100, with a \( \theta^{(T)} \) being extracted every 50 iterations, yielding a sequence of length \( T = 100 \).

**Baseline:** The baseline classifier was optimized to simply minimize training hinge loss. Since this problem is unconstrained, we took the last iterate \( \theta^{(T)} \).

**“Best-model” Heuristic:** For hyperparameter tuning using a grid search, we needed to choose the “best” model on the validation set. Due to the presence of constraints, however, the “best” model was not necessarily that with the lowest validation error. Instead, we used the following heuristic: the models were each ranked in terms of their objective function value, as well as the magnitude of the \( i \)th constraint violation (i.e. \( \max \{ 0, g_i(\theta) \} \)). The “score” of each model was then taken to be the maximal such rank, and the model with the lowest score was chosen, with the objective function serving as a tiebreaker.

**Results:** Table 2 lists the test error rates, (indicator-based) constraint function values on both the training and testing datasets, and support sizes of the stochastic classifiers, for each of the compared algorithms. The “LP” versions of our models, which were found using the shrinking procedure of Lemma 3, uniformly outperformed their \( \bar{\theta} \)-analogues. We can see, however, that the generalization issue discussed in Section 2.1 caused the proxy-Lagrangian LP model to slightly violate the constraints on the testing dataset, despite satisfying them on the training dataset. The non-proxy algorithms satisfied all constraints, on both the training and testing datasets, because there was sufficient “room” between the hinge upper bound that they actually constrained, and the true constraint, to absorb the generalization error. Inspection of the error rates, however, reveals that the relaxed constraints were so overly-conservative that satisfying them significantly damaged classification performance. In contrast, our proxy-Lagrangian approach matched the classification performance of the unconstrained baseline.
Acknowledgments

We thank Seungil You for initially posing the question of whether constraint functions could be relaxed for only the $\theta$-player, as well as Maya Gupta, Taman Narayan and Serena Wang for helping to develop the heuristic used to choose the “best” model on the validation dataset in Section 5.2.

References

Sanjeev Arora, Elad Hazan, and Satyen Kale. The multiplicative weights update method: a meta-algorithm and applications. *Theory of Computing*, 8(6):121–164, 2012.

A. Beck and M. Teboulle. Mirror descent and nonlinear projected subgradient methods for convex optimization. *Oper. Res. Lett.*, 31(3):167–175, May 2003.

D. Biddle. *Adverse Impact and Test Validation: A Practitioner's Guide to Valid and Defensible Employment Testing*. Gower, 2005.

Avrim Blum and Yishay Mansour. From external to internal regret. *JMLR*, 8:1307–1324, 2007.

HF Bohnenblust, Samuel Karlin, and LS Shapley. Games with continuous, convex pay-off. *Contributions to the Theory of Games*, 1(24):181–192, 1950.

Robert S Chen, Brendan Lucier, Yaron Singer, and Vasilis Syrgkanis. Robust optimization for non-convex objectives. In *Nips '17*, 2017.

Xi Chen and Xiaotie Deng. Settling the complexity of two-player nash equilibrium. In *FOCS’06*, pages 261–272. IEEE, 2006.

Paul Christiano, Jonathan A. Kelner, Aleksander Madry, Daniel A. Spielman, and Shang-Hua Teng. Electrical flows, laplacian systems, and faster approximation of maximum flow in undirected graphs. In *STOC ’11*, pages 273–282, 2011.

A. Cotter, M. R. Gupta, and J. Pfeifer. A Light Touch for heavily constrained SGD. In *29th Annual Conference on Learning Theory*, pages 729–771, 2016.

M. Davenport, R. G. Baraniuk, and C. D. Scott. Tuning support vector machines for minimax and Neyman-Pearson classification. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 2010.

John Duchi, Elad Hazan, and Yoram Singer. Adaptive subgradient methods for online learning and stochastic optimization. *JMLR*, 12(Jul):2121–2159, 2011.

E. Eban, M. Schain, A. Mackey, A. Gordon, R. A. Saurous, and G. Elidan. Scalable learning of non-decomposable objectives. *AISstats’17*, 2017.

D. Garber and E. Hazan. Playing non-linear games with linear oracles. In *FOCS*, pages 420–428. IEEE Computer Society, 2013.

G. Gasso, A. Pappaionannou, M. Spivak, and L. Bottou. Batch and online learning algorithms for nonconvex Neyman-Pearson classification. *ACM Transactions on Intelligent Systems and Technology*, 2011.

I. L. Glicksberg. A further generalization of the Kakutani fixed point theorem with application to Nash equilibrium points. *Amer. Math. Soc.*, 3:170–174, 1952.

Gabriel Goh, Andrew Cotter, Maya Gupta, and Michael P Friedlander. Satisfying real-world goals with dataset constraints. In *NIPS*, pages 2415–2423. 2016.
A Proofs of Sub\{optimality,feasibility\} Guarantees

**Theorem 3. (Lagrangian Sub\{optimality,feasibility\})** Define $\Lambda = \{\lambda \in \mathbb{R}^m_+ : \|\lambda\|_p \leq R\}$, and consider the Lagrangian of Equation\[7\] (Definition\[7\]). Suppose that $\theta \in \Theta$ and $\lambda \in \Lambda$ are random variables such that:

$$\sup_{\lambda^* \in \Lambda} \mathbb{E}_\theta [\mathcal{L}(\theta, \lambda^*)] - \inf_{\theta^* \in \Theta} \mathbb{E}_\lambda [\mathcal{L}(\theta^*, \lambda)] \leq \epsilon$$

(8)

which is the optimality claim.

**Proof.** First notice that $\mathcal{L}$ is linear in $\lambda$, so:

$$\sup_{\lambda^* \in \Lambda} \mathbb{E}_\theta [\mathcal{L}(\theta, \lambda^*)] - \inf_{\theta^* \in \Theta} \mathbb{E}_\lambda [\mathcal{L}(\theta^*, \lambda)] \leq \epsilon$$

Furthermore, if $\lambda$ is in the interior of $\Lambda$, in the sense that $\|\lambda\|_p < R$ where $\bar{\lambda} := \mathbb{E}_\lambda [\lambda]$, then $\theta$ is $\epsilon$-suboptimal:

$$\mathbb{E}_\theta [g_0(\theta)] \leq \inf_{\theta^* \in \Theta, \mathbb{E}_\lambda [g(\theta^*)] \leq 0} g_0(\theta^*) + \epsilon$$

Rearranging terms gives the feasibility claim.

**Optimality:** Choose $\theta^*$ to be the optimal feasible solution in Equation\[8\] so that $g_i(\theta^*) \leq 0$ for all $i \in [m]$, and also choose $\lambda^* = 0$, which combined with the definition of $\mathcal{L}$ (Definition\[1\]) gives that:

$$\mathbb{E}_\theta [g_0(\theta)] - g_0(\theta^*) \leq \epsilon$$

which is the optimality claim.

**Feasibility:** Choose $\theta^* = \theta$ in Equation\[8\] By the definition of $\mathcal{L}$ (Definition\[1\]):

$$\sup_{\lambda^* \in \Lambda} \sum_{i=1}^{m} \lambda_i^* \mathbb{E}_\theta [g_i(\theta)] - \sum_{i=1}^{m} \mathbb{E}_\theta [g_i(\theta)] \leq \epsilon$$

Then by the definition of a dual norm, Hölder’s inequality, and the assumption that $\|\bar{\lambda}\|_p < R$:

$$R (\|\mathbb{E}_\theta [g(\theta)]\|_q) - (\|\bar{\lambda}\|_p (\mathbb{E}_\theta [g(\theta)]) \|_q \leq \epsilon$$

Rearranging terms gives the feasibility claim.

**Lemma 6.** In the context of Theorem\[3\] suppose that there exists a $\theta^* \in \Theta$ that satisfies all of the constraints, and does so with q-norm margin $\gamma$, i.e. $g_i(\theta^*) \leq 0$ for all $i \in [m]$ and $\|g(\theta^*)\|_q \geq \gamma$. Then:

$$\|\bar{\lambda}\|_p \leq \frac{\epsilon + B_{g_0}}{\gamma}$$

where $B_{g_0} \geq \sup_{\theta \in \Theta} g_0(\theta) - \inf_{\theta \in \Theta} g_0(\theta)$ is a bound on the range of the objective function $g_0$. 


Proof. Starting from Equation 7 (in Theorem 3), and choosing \( \theta^* = \theta' \) and \( \lambda^* = 0 \):

\[
\epsilon \geq \mathbb{E}_\theta [g_0 (\theta)] - \mathbb{E}_\lambda \left[ g_0 (\theta') + \sum_{i=1}^{m} \lambda_i g_i (\theta') \right]
\]

\[
\epsilon \geq \mathbb{E}_\theta \left[ g_0 (\theta) - \inf_{\theta' \in \Theta} g_0 (\theta') \right] - \left( g_0 (\theta') - \inf_{\theta' \in \Theta} g_0 (\theta') \right) + \gamma \| \lambda \|_p
\]

\[
\epsilon \geq - B g_0 + \gamma \| \lambda \|_p
\]

Solving for \( \| \lambda \|_p \) yields the claim. \( \square \)

Theorem 4. (Proxy-Lagrangian Suboptimality, feasibility) Let \( M := \{ M \in \mathbb{R}^{(m+1) \times (m+1)} : \forall i \in [m+1], M_{i,i} \in \Delta^{m+1} \} \) be the set of all left-stochastic \( (m+1) \times (m+1) \) matrices, and consider the “proxy-Lagrangians” of Equation 7 (Definition 2). Suppose that \( \theta \in \Theta \) and \( \lambda \in \Lambda \) are jointly distributed random variables such that:

\[
\mathbb{E}_{\theta,\lambda} [L_\theta (\theta, \lambda)] - \inf_{\theta^* \in \Theta} \mathbb{E}_\lambda [L_\theta (\theta^*, \lambda)] \leq \epsilon_\theta
\]

\[
\sup_{M^* \in M} \mathbb{E}_{\theta,\lambda} [L_\lambda (\theta, M^* \lambda)] - \mathbb{E}_{\theta,\lambda} [L_\lambda (\theta, \lambda)] \leq \epsilon_\lambda
\]

Define \( \bar{\lambda} := \mathbb{E}_\lambda [\lambda] \), let \( (\Omega, \mathcal{F}, P) \) be the probability space, and define a random variable \( \bar{\theta} \) such that:

\[
\mathbb{P} \{ \bar{\theta} \in S \} = \frac{\int_{\bar{\theta} \in S} \lambda_1 (x) dP (x)}{\int_{\Omega} \lambda_1 (x) dP (x)}
\]

In words, \( \bar{\theta} \) is a version of \( \theta \) that has been resampled with \( \lambda_1 \) being treated as an importance weight. In particular \( \mathbb{E}_{\bar{\theta}} [f (\bar{\theta})] = \mathbb{E}_{\theta,\lambda} [\lambda_1 f (\theta)] / \lambda_1 \) for any \( f : \Theta \rightarrow \mathbb{R} \). Then \( \bar{\theta} \) is nearly-optimal:

\[
\mathbb{E}_{\bar{\theta}} [g_0 (\bar{\theta})] \leq \inf_{\theta^* \in \Theta, \forall i \in [m], \tilde{g}_i (\theta^*) \leq 0} g_0 (\theta^*) + \frac{\epsilon_\theta + \epsilon_\lambda}{\lambda_1}
\]

and nearly-feasible:

\[
\left\| \left( \mathbb{E}_{\bar{\theta}} [g_i (\bar{\theta})] \right) \right\|_\infty \leq \frac{\epsilon_\lambda}{\lambda_1}
\]

Notice the optimality inequality is weaker than it may appear, since the comparator in this equation is not the optimal solution w.r.t. the constraints \( g_i \), but rather w.r.t. the proxy constraints \( \tilde{g}_i \).

Proof. Optimality: We’ll start off by choosing \( M^* \) to be the matrix with its first row being all-one, and all other rows being all-zero. Then the second line of Equation 3 defines the Lagrangian \( L_\theta \) (Definition 3), and the fact that \( \tilde{g}_i \geq g_i \) gives that:

\[
\mathbb{E}_{\theta,\lambda} \left[ \sum_{i=1}^{m} \lambda_{i+1} \tilde{g}_i (\theta) \right] \geq - \epsilon_\lambda
\]

Notice that \( L_\theta \) is linear in \( \lambda \), so the first line of Equation 3, combined with the above result and the definition of \( L_\theta \) (Definition 3) becomes:

\[
\mathbb{E}_{\theta,\lambda} [\lambda_1 g_0 (\theta)] - \inf_{\theta^* \in \Theta} \left( \tilde{\lambda}_1 g_0 (\theta^*) + \sum_{i=1}^{m} \tilde{\lambda}_{i+1} \tilde{g}_i (\theta^*) \right) \leq \epsilon_\theta + \epsilon_\lambda
\]

Choose \( \theta^* \) to be the optimal solution that satisfies the proxy constraints \( \tilde{g}_i \), so that \( \tilde{g}_i (\theta^*) \leq 0 \) for all \( i \in [m] \). Hence:

\[
\mathbb{E}_{\theta,\lambda} [\lambda_1 g_0 (\theta)] - \tilde{\lambda}_1 g_0 (\theta^*) \leq \epsilon_\theta + \epsilon_\lambda
\]

which is the optimality claim.
Feasibility: We’ll simplify our notation by defining $\ell_1(\theta) := 0$ and $\ell_{i+1}(\theta) := g_i(\theta)$ for $i \in [m]$, so that $L_\lambda(\theta, \lambda) = \langle \lambda, \ell(\theta) \rangle$. Next, we’ll choose $M^* \in M$ so that it maximizes the first term in the LHS of the second line of Equation \(9\):

$$E_{\theta,\lambda} \left[ \sum_{i=1}^{m+1} (M^* \lambda)_i \ell_i(\theta) \right] = \sum_{i=1}^{m+1} \max_{j \in [m+1]} E_{\theta,\lambda} [\lambda_i \ell_j(\theta)]$$

Likewise, for the second term, we can use the fact that $\ell_1(\theta) = 0$:

$$E_{\theta,\lambda} \left[ \sum_{i=2}^{m+1} \lambda_i \ell_i(\theta) \right] \leq \sum_{i=2}^{m+1} \max_{j \in [m+1]} E_{\theta,\lambda} [\lambda_i \ell_j(\theta)]$$

Plugging these two results into the second line of Equation \(9\), the two sums collapse, leaving:

$$\max_{i \in [m+1]} E_{\theta,\lambda} [\lambda_1 \ell_i(\theta)] \leq \epsilon$$

By the definition of $\ell_i$, and the fact that $\ell_1 = 0$:

$$\left\| (E_{\theta,\lambda} [\lambda_1 g_i(\theta)]) \right\|_\infty \leq \epsilon$$

which is the feasibility claim. \(\square\)

Lemma 7. In the context of Theorem 4, suppose that there exists a $\theta' \in \Theta$ that satisfies all of the proxy constraints with margin $\gamma$, i.e. $g_i(\theta') \leq -\gamma$ for all $i \in [m]$. Then:

$$\bar{\lambda}_1 \geq \frac{\gamma - \epsilon - \epsilon_\lambda}{\gamma + B_{g_0}}$$

where $B_{g_0} = \sup_{\theta \in \Theta} g_0(\theta) - \inf_{\theta \in \Theta} g_0(\theta)$ is a bound on the range of the objective function $g_0$.

Proof. Starting from Equation \(10\) (in the proof of Theorem 4), and choosing $\theta^* = \theta'$:

$$\epsilon_\theta + \epsilon_\lambda \geq E_{\theta,\lambda} [\lambda_1 g_0(\theta)] - \bar{\lambda}_1 g_0(\theta') + (1 - \bar{\lambda}_1) \gamma$$

$$\geq E_{\theta,\lambda} \left[ \lambda_1 \left( g_0(\theta) - \inf_{\theta' \in \Theta} g_0(\theta') \right) \right] - \bar{\lambda}_1 \left( g_0(\theta') - \inf_{\theta' \in \Theta} g_0(\theta') \right) + (1 - \bar{\lambda}_1) \gamma$$

$$\geq - \bar{\lambda}_1 B_{g_0} + (1 - \bar{\lambda}_1) \gamma$$

Solving for $\bar{\lambda}_1$ yields the claim. \(\square\)

B Proofs of Existence of Sparse Equilibria

Theorem 5. Consider a two player game, played on the compact Hausdorff spaces $\Theta$ and $\Lambda \subseteq \mathbb{R}^m$. Imagine that the $\theta$-player wishes to minimize $L_\theta : \Theta \times \Lambda \to \mathbb{R}$, and the $\lambda$-player wishes to maximize $L_\lambda : \Theta \times \Lambda \to \mathbb{R}$, with both of these functions being continuous in $\theta$ and linear in $\lambda$. Then there exists a Nash equilibrium $\theta^*, \lambda^*$:

$$E_\theta [L_\theta(\theta, \lambda)] = \min_{\theta^* \in \Theta} L_\theta(\theta^*, \lambda)$$

$$E_{\theta} [L_\lambda(\theta, \lambda)] = \max_{\lambda^* \in \Lambda} E_\theta [L_\lambda(\theta, \lambda^*)]$$

where $\theta$ is a random variable placing nonzero probability mass on at most $m + 1$ elements of $\Theta$, and $\lambda \in \Lambda$ is non-random.
Proof. There are some extremely similar (and in some ways more general) results than this in the game theory literature [e.g. Bohnenblust et al. 1950, Parthasarathy 1975], but for our particular (Lagrangian and proxy-Lagrangian) setting it’s possible to provide a fairly straightforward proof.

To begin with, Glicksberg [1952] gives that there exists a mixed strategy in the form of two random variables \( \tilde{\theta} \) and \( \tilde{\lambda} \):

\[
E_{\tilde{\theta}, \tilde{\lambda}} \left[ \mathcal{L}_\theta \left( \tilde{\theta}, \tilde{\lambda} \right) \right] = \min_{\theta^* \in \Theta} E_{\tilde{\lambda}} \left[ \mathcal{L}_\theta \left( \theta^*, \tilde{\lambda} \right) \right]
\]

\[
E_{\tilde{\theta}, \tilde{\lambda}} \left[ \mathcal{L}_\lambda \left( \tilde{\theta}, \tilde{\lambda} \right) \right] = \max_{\lambda^* \in \Lambda} E_{\tilde{\theta}} \left[ \mathcal{L}_\lambda \left( \tilde{\theta}, \lambda^* \right) \right]
\]

Since both functions are linear in \( \tilde{\lambda} \), we can define \( \lambda := E_{\tilde{\lambda}} \left[ \tilde{\lambda} \right] \), and these conditions become:

\[
E_{\tilde{\theta}} \left[ \mathcal{L}_\theta \left( \tilde{\theta}, \lambda \right) \right] = \min_{\theta^* \in \Theta} \mathcal{L}_\theta \left( \theta^* \right) := \ell_{\min}
\]

\[
E_{\tilde{\theta}} \left[ \mathcal{L}_\lambda \left( \tilde{\theta}, \lambda \right) \right] = \max_{\lambda^* \in \Lambda} E_{\tilde{\theta}} \left[ \mathcal{L}_\lambda \left( \tilde{\theta}, \lambda^* \right) \right]
\]

Let’s focus on the first condition. Let \( p_\epsilon := \Pr \left\{ \mathcal{L}_\theta \left( \tilde{\theta}, \lambda \right) \geq \ell_{\min} + \epsilon \right\} \), and notice that \( p_{1/n} \) must equal zero for any \( n \in \{1, 2, \ldots\} \) (otherwise we would contradict the above), implying by the countable additivity of measures that \( \Pr \left\{ \mathcal{L}_\theta \left( \tilde{\theta}, \lambda \right) = \ell_{\min} \right\} = 1 \). We therefore assume henceforth, without loss of generality, that the support of \( \tilde{\theta} \) consists entirely of minimizers of \( \mathcal{L}_\theta \left( , \lambda \right) \). Let \( S \subseteq \Theta \) be this support set.

Define \( G := \left\{ \nabla_\lambda \mathcal{L}_\lambda \left( \theta', \lambda \right) : \theta' \in S \right\} \), and take \( \bar{G} \) to be the closure of the convex hull of \( G \). Since \( E_{\tilde{\theta}} \left[ \nabla_\lambda \mathcal{L}_\lambda \left( \tilde{\theta}, \lambda \right) \right] \in \bar{G} \subseteq \mathbb{R}^m \), we can write it as a convex combination of at most \( m + 1 \) extreme points of \( \bar{G} \), or equivalently of \( m + 1 \) elements of \( G \). Hence, we can take \( \theta \) to be a discrete random variable that places nonzero mass on at most \( m + 1 \) elements of \( S \), and:

\[
E_{\tilde{\theta}} \left[ \nabla_\lambda \mathcal{L}_\lambda \left( \tilde{\theta}, \lambda \right) \right] = E_{\tilde{\theta}} \left[ \nabla_\lambda \mathcal{L}_\lambda \left( \tilde{\theta}, \lambda \right) \right]
\]

Linearity in \( \lambda \) then implies that \( E_{\tilde{\theta}} \left[ \mathcal{L}_\lambda \left( \tilde{\theta}, \cdot \right) \right] \) and \( E_{\tilde{\theta}} \left[ \mathcal{L}_\lambda \left( \tilde{\theta}, \cdot \right) \right] \) are the same function (up to a constant), and therefore have the same maximizer(s). Correspondingly, \( \theta \) is supported on \( S \), which contains only minimizers of \( \mathcal{L}_\theta \left( , \lambda \right) \) by construction.

Lemma 3. If \( \Theta \) is a compact Hausdorff space and the objective, constraint and proxy constraint functions \( g_0, g_1, \ldots, g_m, \tilde{g}_1, \ldots, \tilde{g}_m \) are continuous, then the proxy-Lagrangian game (Definition 2) has a mixed Nash equilibrium pair \( (\theta, \lambda) \) where \( \theta \) is a random variable supported on at most \( m + 1 \) elements of \( \Theta \), and \( \lambda \) is non-random.

Proof. Applying Theorem 2 directly would result in a support size of \( m + 2 \), rather than the desired \( m + 1 \), since \( \Lambda \) is \( (m + 1) \)-dimensional. Instead, we define \( \tilde{\Lambda} = \left\{ \tilde{\lambda} \in \mathbb{R}^n_+ : \left\| \tilde{\lambda} \right\|_1 \leq 1 \right\} \) as the space containing the last \( m \) coordinates of \( \Lambda \). Then we can rewrite the proxy-Lagrangian functions \( \tilde{\mathcal{L}}_\theta, \tilde{\mathcal{L}}_\lambda : \Theta \times \tilde{\Lambda} \to \mathbb{R} \) as:

\[
\tilde{\mathcal{L}}_\theta \left( \theta, \tilde{\lambda} \right) = \left( 1 - \left\| \tilde{\lambda} \right\|_1 \right) g_0 \left( \theta \right) + \sum_{i=1}^{m} \tilde{\lambda}_i \tilde{g}_i \left( \theta \right)
\]

\[
\tilde{\mathcal{L}}_\lambda \left( \theta, \tilde{\lambda} \right) = \sum_{i=1}^{m} \tilde{\lambda}_i g_i \left( \theta \right)
\]

These functions are linear in \( \tilde{\lambda} \), which is a \( m \)-dimensional space, so the conditions of Theorem 2 apply, yielding the claimed result. \( \square \)
Lemma Let $\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(T)} \in \Theta$ be a sequence of $T$ “candidate solutions” of Equation Define $\tilde{g}_0, \tilde{g}_i \in \mathbb{R}^T$ such that $(\tilde{g}_0)_i = g_0(\theta^{(i)})$ and $(\tilde{g}_i)_i = g_i(\theta^{(i)})$ for $i \in [m]$, and consider the linear program:

$$\min_{\tilde{p} \in \Delta^T} \langle p, \tilde{g}_0 \rangle$$

$$\text{s.t. } \forall i \in [m], \langle p, \tilde{g}_i \rangle \leq \epsilon$$

where $\Delta^T$ is the $T$-dimensional simplex. Then every vertex $p^*$ of the feasible region—in particular an optimal one—has at most $m^* + 1 \leq m + 1$ nonzero elements, where $m^*$ is the number of active $\langle p^*, \tilde{g}_i \rangle \leq \epsilon$ constraints.

Proof. The linear program contains not only the $m$ explicit linearized functional constraints, but also, since $p \in \Delta^T$, the $T$ nonnegativity constraints $p_t \geq 0$, and the sum-to-one constraint $\sum_{t=1}^T p_t = 1$.

Since $p$ is $T$-dimensional, every vertex $p^*$ of the feasible region must include $T$ active constraints. Letting $m^* \leq m$ be the number of active linearized functional constraints, and accounting for the sum-to-one constraint, it follows that at least $T - m^* - 1$ nonnegativity constraints are active, implying that $p^*$ contains at most $m^* + 1$ nonzero elements. \hfill \square

C Proofs of Convergence Rates

C.1 Non-Stochastic One-Player Convergence Rates

Theorem 6. (Mirror Descent) Let $f_1, f_2, \ldots : \Theta \to \mathbb{R}$ be a sequence of convex functions that we wish to minimize on a compact convex set $\Theta$. Suppose that the “distance generating function” $\Psi : \Theta \to \mathbb{R}_+$ is nonnegative and $1$-strongly convex w.r.t. a norm $\| \cdot \|$ with dual norm $\| \cdot \|_\ast$.

Define the step size $\eta = \sqrt{B_{\Psi}/TB_{\Psi}'},$ where $B_{\Psi} \geq \max_{\theta \in \Theta} \Psi(\theta)$ is a uniform upper bound on $\Psi$, and $B_{\Psi}' \geq \| \nabla f_t(\theta^{(i)}) \|_\ast$ is a uniform upper bound on the norms of the subgradients. Suppose that we perform $T$ iterations of the following update, starting from $\theta^{(1)} = \arg\min_{\theta \in \Theta} \Psi(\theta)$:

$$\tilde{g}^{(t+1)} = \nabla \Psi^\ast \left( \nabla \Psi \left( \theta^{(t)} \right) - \eta \nabla f_t \left( \theta^{(t)} \right) \right)$$

$$\theta^{(t+1)} = \arg\min_{\theta \in \Theta} D_{\Psi} \left( \theta \mid \tilde{g}^{(t+1)} \right)$$

where $\nabla f_t(\theta) \in \partial f_t(\theta^{(i)})$ is a subgradient of $f_t$ at $\theta$, and $D_{\Psi} \left( \theta \mid \theta' \right) := \Psi(\theta) - \Psi(\theta') - \langle \nabla \Psi(\theta'), \theta - \theta' \rangle$ is the Bregman divergence associated with $\Psi$. Then:

$$\frac{1}{T} \sum_{t=1}^T f_t \left( \theta^{(t)} \right) - \frac{1}{T} \sum_{t=1}^T f_t \left( \theta^* \right) \leq 2B_{\Psi} \sqrt{\frac{B_{\Psi}}{T}}$$

where $\theta^* \in \Theta$ is an arbitrary reference vector.

Proof. Mirror descent [ Nemirovski and Yudin 1983, Beck and Teboulle 2003] dates back to 1983, but this particular statement is taken from Lemma 2 of [Srebro et al. 2011]. \hfill \square

Corollary 1. (Gradient Descent) Let $f_1, f_2, \ldots : \Theta \to \mathbb{R}$ be a sequence of convex functions that we wish to minimize on a compact convex set $\Theta$.

Define the step size $\eta = B_{\Theta}/\sqrt{2TB_{\psi}},$ where $B_{\Theta} \geq \max_{\theta \in \Theta} \| \theta \|_2$, and $B_{\Psi} \geq \| \nabla f_t(\theta^{(i)}) \|_2$ is a uniform upper bound on the norms of the subgradients. Suppose that we perform $T$ iterations of the following update, starting from $\theta^{(1)} = \arg\min_{\theta \in \Theta} \| \theta \|_2$:

$$\theta^{(t+1)} = \Pi_{\Theta} \left( \theta^{(t)} - \eta \nabla f_t \left( \theta^{(t)} \right) \right)$$

18
where $\nabla f_t(\theta) \in \partial f_t(\theta^{(t)})$ is a subgradient of $f_t$ at $\theta$, and $\Pi_\Theta$ projects its argument onto $\Theta$ w.r.t. the Euclidean norm. Then:

$$
\frac{1}{T} \sum_{t=1}^{T} f_t(\theta^{(t)}) - \frac{1}{T} \sum_{t=1}^{T} f_t(\theta^*) \leq B_\Theta B_\Psi \sqrt{\frac{2}{T}}
$$

where $\theta^* \in \Theta$ is an arbitrary reference vector.

**Proof.** Follows from taking $\Psi(\theta) = \|\theta\|_2^2 / 2$ in Theorem 6.

**Corollary 2.** Let $\mathcal{M} := \{ M \in \mathbb{R}^{m \times \hat{m}} : \forall i \in [\hat{m}], M_{i,i} \in \Delta^m \}$ be the set of all left-stochastic $m \times \hat{m}$ matrices, and let $f_1, f_2, \ldots : \mathcal{M} \rightarrow \mathbb{R}$ be a sequence of concave functions that we wish to maximize.

Define the step size $\eta = \sqrt{\hat{m} \ln \hat{m} / T B^2_\Psi}$, where $B_\Psi \geq \left\| \nabla f_t(M^{(t)}) \right\|_{\infty,2}$ is a uniform upper bound on the norms of the supergradients, and $\left\| . \right\|_{\infty,2} := \sqrt{\sum_{i=1}^{\hat{m}} \| M_{i,i} \|_2^2}$ is the $L_{\infty,2}$ matrix norm. Suppose that we perform $T$ iterations of the following update starting from the matrix $M^{(1)}$ with all elements equal to $1/\hat{m}$:

$$
\hat{M}^{(t+1)} = M^{(t)} \odot \text{element-wise-exp} \left( \eta \nabla f_t(\hat{M}^{(t)}) \right)
$$

$$
\hat{M}_{i,i}^{(t+1)} = \frac{\hat{M}_{i,i}^{(t+1)}}{\left\| \hat{M}_{i,i}^{(t+1)} \right\|_1}
$$

where $-\nabla f_t(M^{(t)}) \in \partial (-f_t(M^{(t)}))$, i.e. $\nabla f_t(M^{(t)})$ is a supergradient of $f_t$ at $M^{(t)}$, and the multiplication and exponentiation in the first step are performed element-wise. Then:

$$
\frac{1}{T} \sum_{t=1}^{T} f_t(M^*) - \frac{1}{T} \sum_{t=1}^{T} f_t(M^{(t)}) \leq 2B_\Psi \sqrt{\frac{\hat{m} \ln \hat{m}}{T}}
$$

where $M^* \in \mathcal{M}$ is an arbitrary reference matrix.

**Proof.** Define $\Psi : \mathcal{M} \rightarrow \mathbb{R} := \hat{m} \ln \hat{m} + \sum_{i,j \in [\hat{m}]} M_{i,j} \ln M_{i,j}$ as $\hat{m} \ln \hat{m}$ plus the negative Shannon entropy, applied to its (matrix) argument element-wise ($\hat{m} \ln \hat{m}$ is added to make $\Psi$ nonnegative on $\mathcal{M}$). As in the vector setting, the resulting mirror descent update will be (element-wise) multiplicative.

The Bregman divergence satisfies:

$$
D_\Psi(M|M') = \Psi(M) - \Psi(M') - \langle \nabla \Psi(M'), M - M' \rangle
$$

$$
= \left\| M' \right\|_{1,1} - \left\| M \right\|_{1,1} + \sum_{i=1}^{\hat{m}} D_{KL}(M_{i,i} || M_{i,i}')
$$

(11)

where $\left\| M \right\|_{1,1} = \sum_{i=1}^{\hat{m}} \| M_{i,i} \|_1$ is the $L_{1,1}$ matrix norm. This incidentally shows that one projects onto $\mathcal{M}$ w.r.t. $D_\Psi$ by projecting each column w.r.t. the KL divergence, i.e. by normalizing the columns.

By Pinsker’s inequality (applied to each column of an $M \in \mathcal{M}$):

$$
\left\| M - M' \right\|_{1,2}^2 \leq 2 \sum_{i=1}^{\hat{m}} D_{KL}(M_{i,i} || M_{i,i}')
$$

where $\left\| M \right\|_{1,2} = \sqrt{\sum_{i=1}^{\hat{m}} \| M_{i,i} \|_2^2}$ is the $L_{1,2}$ matrix norm. Substituting this into Equation 11 and using the fact that $\left\| M \right\|_{1,1} = \hat{m}$ for all $M \in \mathcal{M}$, we have that for all $M, M' \in \mathcal{M}$:

$$
D_\Psi(M|M') \geq \frac{1}{2} \left\| M - M' \right\|_{1,2}^2
$$

which shows that $\Psi$ is 1-strongly convex w.r.t. the $L_{1,2}$ matrix norm. The dual norm of the $L_{1,2}$ matrix norm is the $L_{\infty,2}$ norm, which is the last piece needed to apply Theorem 6 yielding the claimed result. □
Lemma 8. Let $\Lambda := \Delta^{\tilde{m}}$ be the $\tilde{m}$-dimensional simplex, define $\mathcal{M} := \{ M \in \mathbb{R}^{\tilde{m} \times \tilde{m}} : \forall i \in [\tilde{m}], M_{i,i} \in \Delta^{\tilde{m}} \}$ as the set of all left-stochastic $\tilde{m} \times \tilde{m}$ matrices, and take $f_1, f_2, \ldots : \Lambda \to \mathbb{R}$ to be a sequence of concave functions that we wish to maximize.

Define the step size $\eta = \sqrt{\frac{\tilde{m} \ln \tilde{m}}{TB_\Psi^2}}$, where $B_\Psi \geq \| \nabla f_t (\lambda(t)) \|_\infty$ is a uniform upper bound on the $\infty$-norms of the supergradients. Suppose that we perform $T$ iterations of the following update, starting from the matrix $M^{(1)}$ with all elements equal to $1/\tilde{m}$:

$$ f_t (M^{(t)}) = \text{fix} M^{(t)} $$

$$ A^{(t)} = \left( \nabla f_t \left( \lambda^{(t)} \right) \right)^T $$

$$ M^{(t+1)} = M^{(t)} \odot \text{element-wise-exp} \left( \eta A^{(t)} \right) $$

$$ M_{:,i}^{(t+1)} = \frac{\tilde{M}_{:,i}^{(t+1)}}{\| \tilde{M}_{:,i}^{(t+1)} \|_1} $$

where $\text{fix} M$ is a stationary distribution of $M$ (i.e. a $\lambda \in \Lambda$ such that $M \lambda = \lambda$—such always exists, since $M$ is left-stochastic), $-\nabla f_t \left( \lambda^{(t)} \right) \in \partial \left( -f_t (\lambda^{(t)}) \right)$, i.e. $\nabla f_t \left( \lambda^{(t)} \right)$ is a supergradient of $f_t$ at $\lambda^{(t)}$, and the multiplication and exponentiation of the third step are performed element-wise. Then:

$$ \frac{1}{T} \sum_{t=1}^{T} f_t \left( M^* \lambda^{(t)} \right) - \frac{1}{T} \sum_{t=1}^{T} f_t \left( \lambda^{(t)} \right) \leq 2B_\Psi \sqrt{-\frac{\tilde{m} \ln \tilde{m}}{T}} $$

where $M^* \in \mathcal{M}$ is an arbitrary left-stochastic reference matrix.

Proof. This algorithm is an instance of that contained in Figure 1 of [Gordon et al. 2008].

Define $\tilde{f}_t \left( M \right) := f_t \left( M^{(t)} \lambda^{(t)} \right)$. Observe that since $\nabla f_t \left( \lambda^{(t)} \right)$ is a supergradient of $f_t$ at $\lambda^{(t)}$, and $M^{(t)} \lambda^{(t)} = \lambda^{(t)}$,

$$ f_t \left( \tilde{M} \lambda^{(t)} \right) \leq f_t \left( M^{(t)} \lambda^{(t)} \right) + \langle \nabla f_t \left( \lambda^{(t)} \right), \tilde{M} \lambda^{(t)} - M^{(t)} \lambda^{(t)} \rangle $$

$$ \leq f_t \left( M^{(t)} \lambda^{(t)} \right) + A^{(t)} \cdot \left( \tilde{M} - M^{(t)} \right) $$

where the matrix product on the last line is performed element-wise. This shows that $A^{(t)}$ is a supergradient of $\tilde{f}_t$ at $M^{(t)}$, from which we conclude that the final two steps of the update are performing the algorithm of Corollary 2 so:

$$ \frac{1}{T} \sum_{t=1}^{T} \tilde{f}_t \left( M^* \right) - \frac{1}{T} \sum_{t=1}^{T} \tilde{f}_t \left( M^{(t)} \right) \leq 2B_\Psi \sqrt{-\frac{\tilde{m} \ln \tilde{m}}{T}} $$

where the $B_\Psi$ of Corollary 2 is a uniform upper bound on the $L_{\infty, 2}$ matrix norms of the $A^{(t)}$s. However, by the definition of $A^{(t)}$ and the fact that $\lambda^{(t)} \in \Delta^{\tilde{m}}$, we can instead take $B_\Psi$ to be a uniform upper bound on $\| \nabla \tilde{f}_t \|_\infty$. Substituting the definition of $\tilde{f}_t$ and again using the fact that $M^{(t)} \lambda^{(t)} = \lambda^{(t)}$ then yields the claimed result.

C.2 Stochastic One-Player Convergence Rates

Theorem 7. (Stochastic Mirror Descent) Let $\Psi, \| \cdot \|, D_\Psi$ and $B_\Psi$ be as in Theorem 6 and let $f_1, f_2, \ldots : \Theta \to \mathbb{R}$ be a sequence of convex functions that we wish to minimize on a compact convex set $\Theta$.

Define the step size $\eta = \sqrt{B_\Psi / TB_\Delta^2}$, where $B_\Delta \geq \| \Delta^{(t)} \|_*$ is a uniform upper bound on the norms of the stochastic subgradients. Suppose that we perform $T$ iterations of the following stochastic update, starting from
\[ \theta^{(1)} = \arg\min_{\theta \in \Theta} \Psi(\theta) \]
\[ \tilde{\theta}^{(t+1)} = \nabla \Psi^* \left( \nabla \Psi \left( \theta^{(t)} \right) - \eta \Delta^{(t)} \right) \]
\[ \theta^{(t+1)} = \arg\min_{\theta \in \Theta} D_{\Psi} \left( \theta | \tilde{\theta}^{(t+1)} \right) \]

where \( \Delta^{(t)} \Delta \in \partial f_t(\theta^{(t)}) \), i.e. \( \Delta^{(t)} \) is a stochastic subgradient of \( f_t \) at \( \theta^{(t)} \). Then, with probability \( 1 - \delta \) over the draws of the stochastic subgradients:

\[
\frac{1}{T} \sum_{t=1}^{T} f_t \left( \theta^{(t)} \right) - \frac{1}{T} \sum_{t=1}^{T} f_t \left( \theta^* \right) \leq 2B_{\Psi} \sqrt{\frac{\left( 1 + 16 \ln \frac{1}{\delta} \right)}{T}}
\]

where \( \theta^* \in \Theta \) is an arbitrary reference vector.

**Proof.** This is nothing more than the usual transformation of a uniform regret guarantee into a stochastic one via the Hoeffding-Azuma inequality—we include a proof for completeness.

Define the sequence:

\[ \tilde{f}_t(\theta) = f_t(\theta^{(t)}) + \left\langle \Delta^{(t)}, \theta - \theta^{(t)} \right\rangle \]

Then applying non-stochastic mirror descent to the sequence \( \tilde{f}_t \) will result in exactly the same sequence of iterates \( \theta^{(t)} \) as applying stochastic mirror descent (above) to \( f_t \). Hence, by Theorem 6 and the definition of \( \tilde{f}_t \) (notice that we can take \( B_{\Psi} = B_{\Delta} \)):

\[
\frac{1}{T} \sum_{t=1}^{T} \tilde{f}_t \left( \theta^{(t)} \right) - \frac{1}{T} \sum_{t=1}^{T} \tilde{f}_t \left( \theta^* \right) \leq 2B_{\Psi} \sqrt{\frac{B_{\Psi}}{T}}
\]

\[
\frac{1}{T} \sum_{t=1}^{T} f_t \left( \theta^{(t)} \right) - \frac{1}{T} \sum_{t=1}^{T} f_t \left( \theta^* \right) \leq 2B_{\Psi} \sqrt{\frac{B_{\Psi}}{T}} + \frac{1}{T} \sum_{t=1}^{T} \left( \tilde{f}_t \left( \theta^* \right) - f_t \left( \theta^* \right) \right)
\]

\[
\leq 2B_{\Psi} \sqrt{\frac{B_{\Psi}}{T}} + \frac{1}{T} \sum_{t=1}^{T} \left\langle \Delta^{(t)} - \nabla f_t \left( \theta^{(t)} \right), \theta^* - \theta^{(t)} \right\rangle \tag{12}
\]

where the last step follows from the convexity of the \( f_t \)s. Consider the second term on the RHS. Observe that, since the \( \Delta^{(t)} \)s are stochastic subgradients, each of the terms in the sum is zero in expectation (conditioned on the past), and the partial sums therefore form a martingale. Furthermore, by Hölder’s inequality:

\[ \left\langle \Delta^{(t)} - \nabla f_t \left( \theta^{(t)} \right), \theta^* - \theta^{(t)} \right\rangle \leq \left\| \Delta^{(t)} - \nabla f_t \left( \theta^{(t)} \right) \right\| \left\| \theta^* - \theta^{(t)} \right\| \leq 4B_{\Delta} \sqrt{2B_{\Psi}} \]

the last step because \( \left\| \theta^* - \theta^{(t)} \right\| \leq \left\| \theta^* - \theta^{(1)} \right\| + \left\| \theta^{(t)} - \theta^{(1)} \right\| \leq 2 \sup_{\theta \in \Theta} \sqrt{2D_{\Psi} \left( \theta | \theta^{(1)} \right)} \leq 2\sqrt{2B_{\Psi}} \), using the fact that \( D_{\Psi} \) is 1-strongly convex w.r.t. \( \left\| \cdot \right\| \), and the definition of \( \theta^{(1)} \). Hence, by the Hoeffding-Azuma inequality:

\[ \Pr \left\{ \frac{1}{T} \sum_{t=1}^{T} \left\langle \Delta^{(t)} - \nabla f_t \left( \theta^{(t)} \right), \theta^* - \theta^{(t)} \right\rangle \geq \epsilon \right\} \leq \exp \left( -\frac{T \epsilon^2}{64B_{\Psi}^2 B_{\Delta}^2} \right) \]

equivalently:

\[ \Pr \left\{ \frac{1}{T} \sum_{t=1}^{T} \left\langle \Delta^{(t)} - \nabla f_t \left( \theta^{(t)} \right), \theta^* - \theta^{(t)} \right\rangle \geq 8B_{\Delta} \sqrt{\frac{B_{\Psi} \ln \frac{1}{\delta}}{T}} \right\} \leq \delta \]

substituting this into Equation 12 and applying the inequality \( \sqrt{a} + \sqrt{b} \leq \sqrt{2a + 2b} \), yields the claimed result. \( \square \)
Corollary 3. (Stochastic Gradient Descent) Let \( f_1, f_2, \ldots : \Theta \rightarrow \mathbb{R} \) be a sequence of convex functions that we wish to minimize on a compact convex set \( \Theta \).

Define the step size \( \eta = B_\Theta / B_\Delta \sqrt{2T} \), where \( B_\Theta \geq \max_{\theta \in \Theta} \| \theta \|_2 \), and \( B_\Delta \geq \| \hat{\Delta}^{(t)} \|_2 \) is a uniform upper bound on the norms of the stochastic subgradients. Suppose that we perform \( T \) iterations of the following stochastic update, starting from \( \theta^{(1)} = \arg\min_{\theta \in \Theta} \| \theta \|_2 \):

\[
\theta^{(t+1)} = \Pi_\Theta \left( \theta^{(t)} - \eta \hat{\Delta}^{(t)} \right)
\]

where \( \mathbb{E} \left[ \hat{\Delta}^{(t)} \mid \theta^{(t)} \right] \in \partial f_t(\theta^{(t)}) \), i.e. \( \hat{\Delta}^{(t)} \) is a stochastic subgradient of \( f_t \) at \( \theta^{(t)} \), and \( \Pi_\Theta \) projects its argument onto \( \Theta \) w.r.t. the Euclidean norm. Then, with probability \( 1 - \delta \) over the draws of the stochastic subgradients:

\[
\frac{1}{T} \sum_{t=1}^{T} f_t \left( \theta^{(t)} \right) - \frac{1}{T} \sum_{t=1}^{T} f_t \left( \theta_* \right) \leq 2B_\Theta B_\xi \sqrt{\frac{1 + 16 \ln \frac{1}{\delta}}{T}}
\]

where \( \theta_* \in \Theta \) is an arbitrary reference vector.

Proof. Follows from taking \( \Psi(\theta) = \| \theta \|^2_2 / 2 \) in Theorem 7.

Corollary 4. Let \( \mathcal{M} := \{ M \in \mathbb{R}^{\tilde{m} \times \tilde{m}} : \forall i \in [\tilde{m}] \cdot M_{:, i} \in \Delta^{\tilde{m}} \} \) be the set of all left-stochastic \( \tilde{m} \times \tilde{m} \) matrices, and let \( f_1, f_2, \ldots : \mathcal{M} \rightarrow \mathbb{R} \) be a sequence of concave functions that we wish to maximize.

Define the step size \( \eta = \sqrt{\tilde{m} \ln \tilde{m} / T B_\Delta^2} \), where \( B_\Delta \geq \| \hat{\Delta}^{(t)} \|_\infty \) is a uniform upper bound on the norms of the stochastic supergradients, and \( \| \cdot \|_{\infty, 2} := \sqrt{\sum_{i=1}^{\tilde{m}} \| M_{:, i} \|_{\infty}^2} \) is the \( L_{\infty, 2} \) matrix norm. Suppose that we perform \( T \) iterations of the following stochastic update starting from the matrix \( M^{(1)} \) with all elements equal to \( 1 / \tilde{m} \):

\[
M^{(t+1)} = M^{(t)} \odot \text{element-wise-exp} \left( \eta \hat{\Delta}^{(t)} \right)
\]

\[
M_{:, i}^{(t+1)} = M_{:, i}^{(t+1)} / \| M_{:, i}^{(t+1)} \|_1
\]

where \( \mathbb{E} \left[ \hat{\Delta}^{(t)} \mid M^{(t)} \right] \in \partial \left( -f_t(M^{(t)}) \right) \), i.e. \( \hat{\Delta}^{(t)} \) is a stochastic supergradient of \( f_t \) at \( M^{(t)} \), and the multiplication and exponentiation in the first step are performed element-wise. Then with probability \( 1 - \delta \) over the draws of the stochastic supergradients:

\[
\frac{1}{T} \sum_{t=1}^{T} f_t \left( M^{*} \right) - \frac{1}{T} \sum_{t=1}^{T} f_t \left( M^{(t)} \right) \leq 2B_\Delta \sqrt{\frac{2 \left( \tilde{m} \ln \tilde{m} \right) \left( 1 + 16 \ln \frac{1}{\delta} \right)}{T}}
\]

where \( M^{*} \in \mathcal{M} \) is an arbitrary reference matrix.

Proof. The same reasoning as was used to prove Corollary 3 from Theorem 6 applies here (but starting from Theorem 7).

Lemma 9. Let \( \Delta := \Delta^{\tilde{m}} \) be the \( \tilde{m} \)-dimensional simplex, define \( \mathcal{M} := \{ M \in \mathbb{R}^{\tilde{m} \times \tilde{m}} : \forall i \in [\tilde{m}] \cdot M_{:, i} \in \Delta^{\tilde{m}} \} \) as the set of all left-stochastic \( \tilde{m} \times \tilde{m} \) matrices, and take \( f_1, f_2, \ldots : \Delta \rightarrow \mathbb{R} \) to be a sequence of concave functions that we wish to maximize.

Define the step size \( \eta = \sqrt{\tilde{m} \ln \tilde{m} / T B_\Delta^2} \), where \( B_\Delta \geq \| \hat{\Delta}^{(t)} \|_\infty \) is a uniform upper bound on the \( \infty \)-norms of the stochastic supergradients. Suppose that we perform \( T \) iterations of the following update, starting from the matrix...
$M^{(1)}$ with all elements equal to $1/\tilde{m}$:

$$
\begin{align*}
\lambda^{(t)} &= \text{fix } M^{(t)} \\
A^{(t)} &= \Delta^{(t)} \left( \lambda^{(t)} \right)^T \\
\tilde{M}^{(t+1)} &= M^{(t)} \odot \text{element-wise-exp} \left( \eta A^{(t)} \right) \\
M^{(t+1)} &= \tilde{M}^{(t+1)} / \left\| \tilde{M}^{(t+1)} \right\|_1
\end{align*}
$$

where fix $M$ is a stationary distribution of $M$ (i.e. a $\lambda \in \Lambda$ such that $M \lambda = \lambda$—such always exists, since $M$ is left-stochastic), $\mathbb{E} \left[ -\Delta^{(t)} \mid \lambda^{(t)} \right] \in \partial \left( -f_t(\lambda^{(t)}) \right)$, i.e. $\Delta^{(t)}$ is a stochastic supergradient of $f_t$ at $\lambda^{(t)}$, and the multiplication and exponentiation of the third step are performed element-wise. Then with probability $1 - \delta$ over the draws of the stochastic supergradients:

$$
\frac{1}{T} \sum_{t=1}^{T} f_t \left( M^* \lambda^{(t)} \right) - \frac{1}{T} \sum_{t=1}^{T} f_t \left( \lambda^{(t)} \right) \leq 2B\Delta \sqrt{\frac{2 (\tilde{m} \ln \tilde{m}) (1 + 16 \ln \frac{1}{\delta})}{T}}
$$

where $M^* \in M$ is an arbitrary left-stochastic reference matrix.

**Proof.** The same reasoning as was used to prove Lemma 8 from Corollary 2 applies here (but starting from Corollary 4).

---

### C.3 Two-Player Convergence Rates

**Lemma 1 (Algorithm 1)** Suppose that $\Lambda$ and $R$ are as in Theorem 1 and define the upper bound $B_\Delta \geq \max_{t \in [T]} \left\| \Delta^{(t)} \right\|_2$.

If we run Algorithm 1 with the step size $\eta \lambda := R / B_\Delta \sqrt{2T}$, then the result satisfies the conditions of Theorem 1 for:

$$
\epsilon = \rho + RB_\Delta \sqrt{\frac{2}{T}}
$$

where $\rho$ is the error associated with the oracle $O_\rho$.

**Proof.** Applying Corollary 1 to the optimization over $\lambda$ gives:

$$
\frac{1}{T} \sum_{t=1}^{T} \mathcal{L} \left( \tilde{\theta}^{(t)}, \lambda^* \right) - \frac{1}{T} \sum_{t=1}^{T} \mathcal{L} \left( \theta^{(t)}, \lambda^{(t)} \right) \leq B_\Lambda B_\Delta \sqrt{\frac{2}{T}}
$$

By the definition of $O_\rho$ (Definition 3):

$$
\frac{1}{T} \sum_{t=1}^{T} \mathcal{L} \left( \tilde{\theta}^{(t)}, \lambda^* \right) - \inf_{\theta^* \in \Theta} \frac{1}{T} \sum_{t=1}^{T} \mathcal{L} \left( \theta^*, \lambda^{(t)} \right) \leq \rho + B_\Lambda B_\Delta \sqrt{\frac{2}{T}}
$$

Using the linearity of $\mathcal{L}$ in $\lambda$, the fact that $B_\Lambda = R$, and the definitions of $\tilde{\theta}$ and $\lambda$, yields the claimed result. \qed
Algorithm 3 Optimizes the Lagrangian formulation (Definition 1) in the convex setting. The parameter \( R \) is the radius of the Lagrange multiplier space \( \Lambda := \{ \lambda \in \mathbb{R}^m : \| \lambda \|_1 \leq R \} \), and the functions \( \Pi_\Theta \) and \( \Pi_\Lambda \) project their arguments onto \( \Theta \) and \( \Lambda \) (respectively) w.r.t. the Euclidean norm.

**StochasticLagrangian** \((R \in \mathbb{R}_+, \mathcal{L} : \Theta \times \Lambda \to \mathbb{R}, T \in \mathbb{N}, \eta, \eta_\lambda \in \mathbb{R}_+)\):

1. Initialize \( \theta^{(1)} = 0, \lambda^{(1)} = 0 \) // Assumes \( 0 \in \Theta 
2. For \( t \in [T] \):
3. Let \( \Delta_\theta^{(t)} \) be a stochastic subgradient of \( \mathcal{L} (\theta^{(t)}, \lambda^{(t)}) \) w.r.t. \( \theta \)
4. Let \( \Delta_\lambda^{(t)} \) be a stochastic supergradient of \( \mathcal{L} (\theta^{(t)}, \lambda^{(t)}) \) w.r.t. \( \lambda \)
5. Update \( \theta^{(t+1)} = \Pi_\Theta \left( \theta^{(t)} - \eta_\lambda \Delta_\theta^{(t)} \right) \) // Projected SGD updates . . .
6. Update \( \lambda^{(t+1)} = \Pi_\Lambda \left( \lambda^{(t)} + \eta_\lambda \Delta_\lambda^{(t)} \right) \) // . . .
7. Return \( \theta^{(1)}, \ldots, \theta^{(T)} \) and \( \lambda^{(1)}, \ldots, \lambda^{(T)} \)

**Lemma 10.** (Algorithm 3) Suppose that \( \Theta \) is a compact convex set, \( \Lambda \) and \( R \) are as in Theorem 7 and that the objective and constraint functions \( g_0, g_1, \ldots, g_m \) are convex. Define the three upper bounds \( B_\Theta \geq \max_{\theta \in \Theta} \| \theta \|_2 \), \( B_\Lambda \geq \max_{t \in [T]} \left\| \Delta_\theta^{(t)} \right\|_2 \), and \( B_\Lambda \geq \max_{t \in [T]} \left\| \Delta_\lambda^{(t)} \right\|_2 \).

If we run Algorithm 3 with the step sizes \( \eta_\theta := B_\Theta / B_\Lambda \sqrt{2T} \) and \( \eta_\lambda := R / B_\Lambda \sqrt{2T} \), then the result satisfies the conditions of Theorem 7 for:

\[
\epsilon = 2 \left( B_\Theta B_\Lambda + RB_\Lambda \right) \sqrt{\frac{1 + 16 \ln \frac{1}{\delta}}{T}}
\]

with probability \( 1 - \delta \) over the draws of the stochastic \{sub,super\} gradients.

**Proof.** Applying Corollary 3 to the two optimizations (over \( \theta \) and \( \lambda \)) gives that with probability \( 1 - 2\delta' \) over the draws of the stochastic \{sub,super\} gradients:

\[
\frac{1}{T} \sum_{t=1}^T \mathcal{L} (\theta^{(t)}, \lambda^{(t)}) - \mathcal{L} (\theta^*, \lambda^*) \leq 2B_\Theta B_\Lambda \sqrt{\frac{1 + 16 \ln \frac{1}{\delta}}{T}}
\]

Adding these inequalities, taking \( \delta = 2\delta' \), using the linearity of \( \mathcal{L} \) in \( \lambda \), the fact that \( B_\Lambda = R \), and the definitions of \( \tilde{\theta} \) and \( \lambda \), yields the claimed result.

**Lemma 11.** (Algorithm 4) Suppose that \( \mathcal{M} \) and \( \Lambda \) are as in Theorem 2 and define the upper bound \( B_\Delta := \max_{t \in [T]} \left\| \Delta_\lambda^{(t)} \right\|_\infty \).

If we run Algorithm 4 with the step size \( \eta_\lambda := \sqrt{(m + 1) \ln (m + 1) / T B_\Delta^2} \), then the result satisfies the conditions of Theorem 2 for:

\[
\epsilon_\theta = \rho
\]

\[
\epsilon_\lambda = 2B_\Delta \sqrt{\frac{(m + 1) \ln (m + 1)}{T}}
\]

where \( \rho \) is the error associated with the oracle \( O_\rho \).
Algorithm 4 Optimizes the proxy-Lagrangian formulation (Definition 2) in the non-convex setting via the use of an approximate Bayesian optimization oracle \( O_\rho \) (Definition 3) but with \( \hat{g}_i \)s instead of \( g_i \)s in the linear combination defining \( f \) for the \( \theta \)-player, with the \( \lambda \)-player minimizing swap regret. The fix \( M \) operation on line 3 results in a stationary distribution of \( M \) (i.e. a \( \lambda \in \Lambda \) such that \( M\lambda = \lambda \), which can be derived from the top eigenvector). The exponentiation and product on line 6 are performed element-wise.

OracleProxyLagrangian \( (L_\theta, L_\lambda : \Theta \times \Delta^{m+1} \to \mathbb{R}, O_\rho : (\Theta \to \mathbb{R}) \to \Theta, T \in \mathbb{N}, \eta_\theta, \eta_\lambda \in \mathbb{R}_+ \) : 

1. Initialize \( M^{(1)} \in \mathbb{R}^{(m+1) \times (m+1)} \) with \( M_{i,j} = 1/(m+1) \)
2. For \( t \in [T] \):
3. Let \( \lambda^{(t)} = \text{fix } M^{(t)} \) // Stationary distribution of \( M^{(t)} \)
4. Let \( \theta^{(t)} = O_\rho (L_\theta (\cdot, \lambda^{(t)})) \) // Oracle optimization
5. Let \( \hat{\Delta}^{(t)} \) be a supergradient of \( L_\lambda (\theta^{(t)}, \lambda^{(t)}) \) w.r.t. \( \lambda \)
6. Update \( \hat{M}^{(t+1)} = M^{(t)} \circ \text{element-wise-exp} \left( \eta \hat{\Delta}^{(t)} (\lambda^{(t)})^T \right) \) // Multiplicative update
7. Project \( \hat{M}^{(t+1)}_{i,i} = M^{(t+1)}_{i,i} / \|\hat{M}^{(t+1)}_{i,i}\|_1 \) for \( i \in [m+1] \) // Column-wise projection w.r.t. KL divergence
8. Return \( \theta^{(1)}, \ldots, \theta^{(T)} \) and \( \lambda^{(1)}, \ldots, \lambda^{(T)} \)

Proof. Applying Lemma 8 to the optimization over \( \lambda \) (with \( \bar{m} = m + 1 \)) gives:

\[
\frac{1}{T} \sum_{t=1}^{T} L_\lambda (\theta^{(t)}, M^* \lambda^{(t)}) - \frac{1}{T} \sum_{t=1}^{T} L_\lambda (\theta^{(t)}, \lambda^{(t)}) \leq 2B_{\Delta} \sqrt{\frac{(m+1) \ln (m+1)}{T}}
\]

By the definition of \( O_\rho \) (Definition 3):

\[
\frac{1}{T} \sum_{t=1}^{T} L_\theta (\theta^{(t)}, \lambda^{(t)}) - \inf_{\theta^* \in \Theta} \frac{1}{T} \sum_{t=1}^{T} L_\theta (\theta^*, \lambda^{(t)}) \leq \rho
\]

Using the definitions of \( \hat{\theta} \) and \( \hat{\lambda} \) yields the claimed result.

Lemma 4 (Algorithm 2) Suppose that \( \Theta \) is a compact convex set, \( M \) and \( \Lambda \) are as in Theorem 2 and that the objective and proxy constraint functions \( g_0, \hat{g}_1, \ldots, \hat{g}_m \) are convex (but not \( g_1, \ldots, g_m \)). Define the three upper bounds \( B_\theta \geq \max_{\theta \in \Theta} \|\theta\|_2, B_{\Delta} \geq \max_{t \in [T]} \|\hat{\Delta}^{(t)}\|_2, \) and \( B_{\hat{\Delta}} \geq \max_{t \in [T]} \|\hat{\Delta}^{(t)}\|_{\infty} \).

If we run Algorithm 2 with the step sizes \( \eta_\theta : = B_\theta / B_{\Delta} \sqrt{2T} \) and \( \eta_\lambda : = \sqrt{(m+1) \ln (m+1) / TB_{\Delta}^2} \), then the result satisfies the conditions of Theorem 2 for:

\[
\epsilon_\theta = 2B_\theta B_{\Delta} \sqrt{\frac{1 + 16 \ln \frac{n}{\delta}}{T}}
\]
\[
\epsilon_\lambda = 2B_{\Delta} \sqrt{\frac{2 (m+1) \ln (m+1)(1 + 16 \ln \frac{n}{\delta})}{T}}
\]

with probability \( 1 - \delta \) over the draws of the stochastic \{sub,super\} gradients.

Proof. Applying Corollary 3 to the optimization over \( \theta \), and Lemma 9 to that over \( \lambda \) (with \( \bar{m} := m + 1 \)), gives that with probability \( 1 - 2\delta' \) over the draws of the stochastic \{sub,super\} gradients:

\[
\frac{1}{T} \sum_{t=1}^{T} L_\theta (\theta^{(t)}, \lambda^{(t)}) - \frac{1}{T} \sum_{t=1}^{T} L_\theta (\theta^*, \lambda^{(t)}) \leq 2B_\theta B_{\Delta} \sqrt{\frac{1 + 16 \ln \frac{n}{\delta}}{T}}
\]
\[
\frac{1}{T} \sum_{t=1}^{T} L_\lambda (\theta^{(t)}, M^* \lambda^{(t)}) - \frac{1}{T} \sum_{t=1}^{T} L_\lambda (\theta^{(t)}, \lambda^{(t)}) \leq 2B_{\Delta} \sqrt{\frac{2 (m+1) \ln (m+1)(1 + 16 \ln \frac{n}{\delta})}{T}}
\]

25
Taking $\delta = 2\delta'$, and using the definitions of $\bar{\theta}$ and $\bar{\lambda}$, yields the claimed result.