2D Eigenvalue Problem I: Existence and Number of Solutions

Yangfeng Su† Tianyi Lu† Zhaojun Bai‡

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Abstract

A two dimensional eigenvalue problem (2DEVP) of a Hermitian matrix pair \((A, C)\) is introduced in this paper. The 2DEVP can be viewed as a linear algebra formulation of the well-known eigenvalue optimization problem of the parameter matrix \(H(\mu) = A - \mu C\). We present fundamental properties of the 2DEVP such as the existence, the necessary and sufficient condition for the finite number of 2D-eigenvalues and variational characterizations of 2D-eigenvalues. We use eigenvalue optimization problems from the minmax of two Rayleigh quotients and the computation of distance to instability to show their connections with the 2DEVP and new insights of these problems derived from the properties of the 2DEVP.

Key words. eigenvalue problem, eigenvalue optimization, variational characterization.

AMS subject classifications. 65F15, 65K10

1 Introduction

We are interested in finding scalars \(\mu, \lambda \in \mathbb{R}\) and nonzero vectors \(x \in \mathbb{C}^n\) to satisfy the nonlinear equations

\[
\begin{cases}
(A - \mu C)x = \lambda x, \\
x^H C x = 0, \\
x^H x = 1,
\end{cases}
\]

where \(A, C \in \mathbb{C}^{n \times n}\) are given Hermitian matrices and \(C\) is indefinite. The pair \((\mu, \lambda)\) is called a 2D-eigenvalue, \(x\) is called the corresponding 2D-eigenvector, and the triplet \((\mu, \lambda, x)\) is called a 2D-eigentriplet. We use the term “2D” based on the fact that an eigenvalue has two components, which is a point in the \((\mu, \lambda)\)-plane. The nonlinear equations (1.1) are called a two dimensional eigenvalue problem, 2DEVP in short, of the matrix pair \((A, C)\).

Our interest in studying the 2DEVP (1.1) primarily stems from eigenvalue optimization problems. If we regard \(\mu\) as a parameter in the 2DEVP (1.1), the equation (1.1a) is a parameter eigenvalue problem of \(H(\mu) = A - \mu C\). Since \(A\) and \(C\) are Hermitian, \(H(\mu)\) has \(n\) real eigenvalues \(\lambda_1(\mu), \lambda_2(\mu), \ldots, \lambda_n(\mu)\) for any \(\mu \in \mathbb{R}\). Suppose that these eigenvalues are sorted such that \(\lambda_1(\mu) \geq \lambda_2(\mu) \geq \cdots \geq \lambda_n(\mu)\). When one wants to optimize an eigenvalue \(\lambda_i(\mu)\) with respect to \(\mu\):

\[
\inf_{\mu \in \mathbb{R}} \lambda_i(\mu),
\]
the second equation (1.1b) is actually a stationary condition for (local or global) maxima or minima (see Section 5). This fact has been observed by Overton [26] when \( \lambda_i(\mu^*) \) is a simple eigenvalue of \( H(\mu^*) \) at a stationary point \( \mu^* \). In general, when \( \lambda_i(\mu^*) \) is a multiple eigenvalue, to the best of our knowledge, the connection to the 2DEVP (1.1) as presented in this paper is new.

Different equivalent conditions of the eigenvalue optimization have been discovered in the literature, such as conditions based on the generalized gradient [28] and existence of a special positive semidefinite matrix [8] in the context of minimizing the largest eigenvalue of a multivariable Hermitian matrix. These conditions will lead to a different optimization method. The eigenvalue optimization in the presence of multiplicity is still one of main challenges [8, 14, 20, 22, 23, 24, 26, 28, 31].

Blum and Chang [1] considered the following so-called two-parameter or double eigenvalue problem arising from solving a boundary value problem of ordinary differential equations with double parameters:

\[
\begin{align*}
Ax &= \lambda C_1 x + \mu C_2 x, \\
f(x) &= 0, \\
\|x\| &= 1,
\end{align*}
\]

where \( A, C_1, C_2 \in \mathbb{R}^{n \times n} \) and \( f \) is a real-valued function. \( \lambda, \mu \in \mathbb{R} \) and \( x \in \mathbb{R}^n \) are the eigenvalues and eigenvectors to be found. Khazanov [16] generalized the problem (1.3) to more than two parameters and derived a related eigenvalue problem. Obviously, when \( A \) and \( C \) in (1.1) are real, the 2DEVP (1.1) is a special case of (1.3). The reasons for our study of the 2DEVP (1.1) are two-fold. There is a lack of analysis of the two-parameter eigenvalue problem (1.3) such as the existence, and convergence analysis of proposed algorithms in [1, 16]. More important, only real matrices are considered. The extension to the complex matrices is necessary for applications such as calculating the distance to instability [33] (see Section 2). It is a non-trivial extension since one cannot take the derivatives of the equations (1.1b) and (1.1c) directly.

The objectives of this paper include revealing the relationship between the 2DEVP (1.1) and the eigenvalue optimization of the matrix \( H(\mu) = A - \mu C \), and studying fundamental properties of the 2DEVP (1.1) such as the existence and the necessary and sufficient condition for the finite number of 2D-eigenvalues. This is the first paper of ours in a sequel on the 2DEVP (1.1). In the forthcoming work, we will show that by transforming the eigenvalue optimization (1.2) into the 2DEVP (1.1), we will be able to migrate well-established Rayleigh quotient iteration for solving symmetric eigenvalue problems to the large-scale 2DEVP (1.1).

The rest of this paper is organized as follows. In Section 2 we discuss two eigenvalue optimizations as the origins of the 2DEVP (1.1). In Section 3 we use simple 2-by-2 2DEVPs to reveal some essential features and complexity of the 2DEVP. In Section 4 we study the related parameter eigenvalue problems and introduce the notion of sorted eigencurves and analyticalized eigencurves. In Section 5 we investigate existence and variational characterization of 2D-eigenvalues by exploiting the connection between the 2D-eigenvalues and the stationary points of sorted eigencurves. In Section 6 we provide a necessary and sufficient condition for the existence of finite number of 2D-eigenvalues. In Section 7 we revisit the two eigenvalue optimization problems in Section 2 to show new insights derived from the properties of the 2DEVP (1.1). Concluding remarks are in Section 8.

## 2 Applications

In this section, we discuss two eigenvalue optimization problems that can be formulated as the 2DEVP (1.1).
2.1 Minmax of Rayleigh Quotients

Given Hermitian matrices $A, B \in \mathbb{C}^{n \times n}$, the following minmax problem of Rayleigh quotients (RQminmax):

$$\min_{x \neq 0} \max \left\{ \frac{x^H Ax}{x^H x}, \frac{x^H Bx}{x^H x} \right\},$$

(2.1) arises from quadratic constrained quadratic programs (QCQP) with constraints [10].

We have the following theorem for the characterization of the solution of the RQminmax (2.1).

**Theorem 2.1** ([10]). Let $\lambda_A (\lambda_B)$ be the minimum eigenvalue of $A (B)$, $S_A (S_B)$ be an orthonormal basis of the corresponding eigen-subspace of $\lambda_A (\lambda_B)$, and $(\theta_A, z_A) ((\theta_B, z_B))$ be the minimum eigenpair of $S_B^H A S_B (S_A^H B S_A)$.

I. If $\lambda_A > \theta_B$, then $x^* = S_A z_B$ is a solution of the RQminmax (2.1).

II. If $\lambda_B > \theta_A$, then $x^* = S_B z_A$ is a solution of the RQminmax (2.1).

III. Otherwise, namely $\lambda_A \leq \theta_B$ and $\lambda_B \leq \theta_A$, $x^*$ is a solution of the RQminmax (2.1) if and only if $x^*$ is an eigenvector corresponding to the minimum eigenvalue $\lambda_{\text{min}}(A - \mu_* C)$ of $A - \mu C$ and $x^H C x^* = 0$, where $C = A - B$ and $\mu_*$ is an optimizer of the following eigenvalue optimization problem (EOpt):

$$\max_{\mu \in \mathbb{R}} \lambda_{\text{min}}(A - \mu C).$$

(2.2)

Theorem 2.1 is actually a generalization of the results by Gaurav and Hari [10]. In [10], it is implicitly assumed that the multiplicities of $\lambda_A$, $\lambda_B$ and $\lambda_{\text{min}}(A - \mu_* C)$ are one, while Theorem 2.1 does not need this assumption. In addition, Theorem 2.1 provides a precise description of the relation between the RQminmax (2.1) and the EOpt (2.2). A proof of Theorem 2.1 is in Appendix A.

In Theorem 5.4, we will show that the solution of the EOpt (2.2) is the minimum 2D-eigentriplet of the 2DEVP (1.1). We note that the RQminmax (2.1) also arises from optimal conditions of CDT problems in the trust region methods for nonlinear equality constrained optimization [4, 35], transmit beamforming [11, 15, 36], MIMO relay optimization [30] and cognitive radio networks [37], and is closely related to the well-known S-lemma in control theory and robust optimization [27, 34].

2.2 Distance to instability

A basic problem in studying the stability of linear dynamical systems is to compute the distance to instability, e.g. [32, §49]. In matrix notation, for a stable matrix $\hat{A} \in \mathbb{C}^{m \times m}$, namely all eigenvalues of $\hat{A}$ locate in the left half of the complex plane $\mathbb{C}$, the distance to instability (DTI) $\beta(\hat{A})$ is defined by

$$\beta(\hat{A}) = \min \left\{ \|E\| \mid \hat{A} + E \text{ is unstable}, E \in \mathbb{C}^{m \times m} \right\}.$$

(2.3)

Van Loan [33] showed that the DTI $\beta(\hat{A})$ can be recast as the singular value optimization

$$\beta(\hat{A}) = \min_{\mu \in \mathbb{R}} \sigma_{\text{min}}(\hat{A} - \mu I),$$

(2.4)

where $I = \sqrt{-1}$ and $\sigma_{\text{min}}(X)$ refers to the smallest singular value of the matrix $X$.

By the relation between the singular values of a matrix $X$ and eigenvalues of Hermitian matrix

$$\begin{bmatrix} 0 & X^H \\ X & 0 \end{bmatrix},$$

see e.g. [6, Theorem 3.3], the singular value optimization (2.4) can be transformed to the eigenvalue optimization (EOpt)

$$\beta(\hat{A}) = \min_{\mu \in \mathbb{R}} \lambda_{\text{min}}(A - \mu C),$$

(2.5)
where \( A \) and \( C \) are \( 2m \times 2m \) matrices: 
\[
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} c_1 & c_2 \\ c_2 & c_1 \end{bmatrix}.
\]
and \( \lambda_m(A - \mu C) \) is the smallest positive eigenvalue of \( A - \mu C \).

In Theorem 5.2, we will show that if \( \mu_* \) is an optimizer of (2.5), then \((\mu_*, \lambda_*)\) is a 2D-eigenvalue of the 2DEVP (1.1), where \( \lambda_* = \lambda_m(A - \mu_* C) \).

### 3 2-by-2 2DEVPs

We start from the smallest 2-by-2 2DEVPs, namely \( A \) and \( C \) are 2-by-2 Hermitian matrices to investigate the existence of 2D-eigentriplets, and connection with the stational points of the eigen-curves of the parameter matrix \( A - \mu C \). Without loss of generality, we assume that the indefinite matrix \( C \) is diagonal with diagonal elements \( c_1 > 0 \) and \( c_2 < 0 \), and the 2-by-2 matrices of \( A \) and \( C \) are denoted by
\[
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} c_1 & c_2 \\ c_2 & c_1 \end{bmatrix}.
\]

First we note that up to a scaling any nonzero vector \( x \) satisfying (1.1b) and (1.1c) is of the form
\[
x(\alpha) = \frac{1}{\sqrt{c_1 - c_2}} \left[ \sqrt{-c_2} \alpha^2 \right],
\]
where \( \alpha \in \mathbb{C} \) and \( |\alpha| = 1 \). By multiplying \( x^H(\alpha) \) and \( x^H(\alpha)C \) on the left of (1.1a) respectively, we have
\[
\mu = \frac{x^H(\alpha)CAx(\alpha)}{\|Cx(\alpha)\|^2} \quad \text{and} \quad \lambda = x^H(\alpha)Ax(\alpha),
\]
and the triplet \((\mu, \lambda, x(\alpha))\) satisfies (1.1a).

Since there exist infinitely many \( \alpha \) with \( |\alpha| = 1 \), the 2DEVP (1.1) seems to possess infinite number of 2D-eigenvalues. However, this does not imply that any triplet \((\mu, \lambda, x(\alpha))\) defined in (3.1) and (3.2) is a 2D-eigentriplet of 2DEVP (1.1) since only real pairs \((\mu, \lambda)\) are of interest.

Obviously, \( \lambda \) in (3.2) is always real. By straightforward calculation, we have
\[
\mu = \frac{a_{11} - a_{22} + (c_1 a_{12} + c_2 a_{22})/\sqrt{-c_1 c_2}}{c_1 - c_2}.
\]

Since \( c_1 > 0 \) and \( c_2 < 0 \), \( \mu \) is real if and only if \( \alpha a_{12} \) is real. There are two cases:

- **\( a_{12} \neq 0 \).** By choosing \( \alpha_{1,2} = \pm|a_{12}|/a_{12} \), then we have
  \[
  \mu_{1,2} = \frac{a_{11} - a_{22} \pm |a_{12}|(c_1 + c_2)/\sqrt{-c_1 c_2}}{c_1 - c_2},
  \]
  and
  \[
  \lambda_{1,2} = \frac{a_{11}/c_1 - a_{22}/c_2 \pm 2|a_{12}|/\sqrt{-c_1 c_2}}{(c_1 - c_2)/(c_1 c_2)},
  \]
  and
  \[
  x_{1,2} = x(\alpha_{1,2}).
  \]
  Therefore, the 2DEVP (1.1) has exactly two 2D-eigentriplets \((\mu_1, \lambda_1, x_1)\) and \((\mu_2, \lambda_2, x_2)\). In addition, we note that \( \lambda_1 \) and \( \lambda_2 \) are simple eigenvalues of \( A - \mu_1 C \) and \( A - \mu_2 C \), respectively.

- **\( a_{12} = 0 \).** The 2D-eigentriplets are given by
  \[
  (\mu_1, \lambda_1, x_1(\alpha)) \equiv \left( \frac{a_{11} - a_{22}}{c_1 - c_2}, \frac{a_{22}c_1 - a_{11}c_2}{c_1 - c_2}, x(\alpha) \right)
  \]
  for any \( \alpha \in \mathbb{C} \) with \( |\alpha| = 1 \). There are infinitely many eigentriplets with the same 2D-eigenvalue. In addition, \( \lambda_1 \) is an eigenvalue of \( A - \mu_1 C \) with multiplicity 2.
If we discard the equation (1.1b), the 2DEVP becomes a parameter eigenvalue problem of $H(\mu) = A - \mu C$. There are also two cases:

- **$a_{12} \neq 0$.** In this case, we have two smooth eigencurves $\lambda_1(\mu)$ and $\lambda_2(\mu)$ of $H(\mu)$:

$$
\lambda_{1,2}(\mu) = \frac{1}{2} \left( a_{11} + a_{22} - \mu(c_1 + c_2) \pm \sqrt{[(a_{11} - a_{22}) - \mu(c_1 - c_2)]^2 + 4|a_{12}|^2} \right).
$$

By setting $\lambda'_{1,2}(\mu) = 0$, we have the following two stationary points of eigencurves $\lambda_1(\mu)$ and $\lambda_2(\mu)$:

$$
\mu_{1,2} = \frac{a_{11} - a_{22} \pm |a_{12}|(c_1 + c_2)/\sqrt{-c_1 c_2}}{c_1 - c_2}.
$$

- **$a_{12} = 0$.** In this case, eigencurves $\lambda_1(\mu)$ and $\lambda_2(\mu)$ are

$$
\lambda_{1,2}(\mu) = \frac{1}{2} \left( a_{11} + a_{22} - \mu(c_1 + c_2) \pm |a_{11} - a_{22} - \mu(c_1 - c_2)| \right).
$$

Moreover, $\lambda_1(\mu)$ and $\lambda_2(\mu)$ intersect and are not differentiable at the intersection point $\mu_1 = \mu_2 = \frac{a_{11} - a_{22}}{c_1 - c_2}$. Furthermore, since $|c_1 + c_2| < |c_1 - c_2|$, $\mu_1$ and $\mu_2$ are the minimum and maximum of eigencurves $\lambda_1(\mu)$ and $\lambda_2(\mu)$, respectively. By (3.3), $(\mu_1, \lambda_1(\mu_1))$ and $(\mu_2, \lambda_2(\mu_2))$ are the 2D-eigenvalues.

**Example 1.** Consider the 2DEVP (1.1) with

$$
A = \begin{bmatrix} 1 & a_{12} \\ a_{12} & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.2 & 0 \\ 0 & -0.5 \end{bmatrix},
$$

where $a_{12} \in \mathbb{R}$. Figure 3.1 shows the eigenfunctions $\lambda_1(\mu)$ and $\lambda_2(\mu)$ of $A - \mu C$ with $a_{12} \neq 0$ and $a_{12} = 0$, where $\lambda_1(\mu)$ and $\lambda_2(\mu)$ are sorted such that $\lambda_1(\mu) \geq \lambda_2(\mu)$. The 2D-eigenvalues $(\mu_*, \lambda_*)$ are marked by red circles. We observe that

1. in the case $a_{12} \neq 0$, $\lambda_1(\mu)$ and $\lambda_2(\mu)$ are differentiable and the 2D-eigenvalues correspond to stationary points of $\lambda_1(\mu)$ and $\lambda_2(\mu)$, see Figure 3.1(a),

2. in the case $a_{12} = 0$, $\lambda_1(\mu)$ and $\lambda_2(\mu)$ are not differentiable at $\mu_*$, but the 2D-eigenvalue still corresponds to minimum and maximum of $\lambda_1(\mu)$ and $\lambda_2(\mu)$, see Figure 3.1(b).

In addition, we note that if $\mu$ is not restricted to be real, then the red line in Figure 3.1(a) is a plot $(\text{Re}(\mu), \lambda)$ calculated from (3.2) with $\mu \in \mathbb{C}$. Therefore, if $\mu$ is not restricted to be real, the 2DEVP has continuous spectrum.

### 4 The associated parameter eigenvalue problem

If we discard the equation (1.1b), the remaining two equations of the 2DEVP (1.1) are a parameter eigenvalue problem of $H(\mu) = A - \mu C$ with real parameter $\mu$. In this section, we study the properties of this associated parameter eigenvalue problem.

For $\mu \in \mathbb{R}$, there exist $n$ real eigenvalues $\lambda_i(\mu)$ and corresponding orthonormal eigenvectors $x_i(\mu)$ of $A - \mu C$. If $\lambda_i(\mu)$ are sorted such that $\lambda_1(\mu) \geq \cdots \geq \lambda_n(\mu)$, then we have $n$ sorted eigencurves $\lambda_i(\mu)$ for $i = 1, 2, \ldots, n$. The following theorem is a well-known result on the convexity of the extreme eigencurves $\lambda_1(\mu)$ and $\lambda_n(\mu)$, see e.g. 7 21 25.
Theorem 4.1. $\lambda_1(\mu)$ is convex and $\lambda_n(\mu)$ is concave.

The sorted eigencurves $\lambda_i(\mu)$ are continuous and may be non-differentiable on intersections, see Figure 4.1(a). The following theorem is a direct application of Theorem 4.6.3 in [12] and shows that with proper reordering, the eigencurves $\lambda$ and furthermore, $\lambda^*$ will be called analyticalized eigencurves. Analyticalized eigencurves may be different from sorted eigencurves as illustrated in Figure 4.1. For clarification, in the rest of the paper, we use $\lambda_i(\mu)$ to denote a sorted eigencurve of $A - \mu C$ and $\tilde{\lambda}_i(\mu)$ to denote an analyticalized eigencurve of $A - \mu C$.

Recall that a function $f(\mu)$ is called real analytic [13] p. 3, Definition 1.1.5] on an open set $U \subseteq \mathbb{R}$, if and only if $f(\mu)$ can be expanded into a convergent power series in a neighbour centered at any $\mu_0 \in U$. The following theorem lists some properties of a real analytic function, see [13] pp. 4, 9, 10, 14, 19] for details.

Theorem 4.3 ([17]). Let $f(\mu)$ be real analytic on $U$. Then

(i) $f \in C^\infty(U)$.

(ii) The sum and product of two real analytic functions on $U$ are also real analytic on $U$.

(iii) $f'(\mu)$ is also real analytic on $U$. Furthermore, if $f(\mu) = \sum_{i=0}^{+\infty} \alpha_i(\mu - \mu_0)^i$ is the power series expansion of $f$ at $\mu_0$, then $f'(\mu) = \sum_{i=1}^{+\infty} i \alpha_i(\mu - \mu_0)^{i-1}$ is the power series expansion of $f'$ with the same radius of convergence.

(iv) Assume $U$ is an interval. If $g(\mu)$ is also real analytic on $U$, and there exists a sequence $\{\mu_k\}_{k=1}^{\infty} \subseteq U$, where $u_k$ are distinct and $\lim_{k \to \infty} \mu_k = \mu_* \in U$ such that $f(\mu_k) = g(\mu_k)$, then $f \equiv g$ on $U$. This is known as the identity theorem for analytic functions.
For the sake of completeness, we provide a proof here. We begin with the following identity:

\[ p \leq \mu \]

This implies that \( \tilde{\lambda}_i(\mu) \) is an eigenvalue of \( \tilde{A}_i(\mu) \), i.e., \( \tilde{\lambda}_j(\mu_0) = \lambda_0 \) for \( p \leq j \leq p + k - 1 \) with some integer \( p \geq 1 \). Let \( X_k \) be an orthonormal eigen-subspace corresponding to \( \lambda_0 \). Then by counting multiplicities, \( \tilde{\lambda}_j'(\mu_0) \) has one-to-one correspondence with the eigenvalues of \( -X_k^H C X_k \) for \( p \leq j \leq p + k - 1 \).

**Theorem 4.4** \([29]\). Let \( \tilde{\lambda}_1(\mu), \cdots, \tilde{\lambda}_n(\mu) \) be the analyticalized eigencurves of \( A - \mu C \). Assume \( \lambda_0 \) is an eigenvalue of \( A - \mu_0 C \) with algebraic multiplicity \( k \), i.e., \( \tilde{\lambda}_j(\mu_0) = \lambda_0 \) for \( p \leq j \leq p + k - 1 \) with some integer \( p \geq 1 \). Let \( X_k \) be an orthonormal eigen-subspace corresponding to \( \lambda_0 \). Then by counting multiplicities, \( \tilde{\lambda}_j'(\mu_0) \) has one-to-one correspondence with the eigenvalues of \( -X_k^H C X_k \) for \( p \leq j \leq p + k - 1 \).

**Proof.** For the sake of completeness, we provide a proof here. We begin with the following identity:

\[
(A - \mu_0 C - \lambda_0 I)x_j(\mu) - (\mu - \mu_0)Cx_j(\mu) - (\tilde{\lambda}_j(\mu) - \lambda_0)x_j(\mu) = 0 \tag{4.2}
\]

for \( p \leq j \leq p + k - 1 \). Multiplying \( x_i^H(\mu_0) \) on the left and dividing by \( \mu - \mu_0 \), we have

\[
-x_i^H(\mu_0)Cx_j(\mu) = \frac{\tilde{\lambda}_j(\mu) - \lambda_0}{\mu - \mu_0}x_i^H(\mu_0)x_j(\mu).
\]

Let \( \mu \to \mu_0 \), we obtain

\[
-x_i^H(\mu_0)Cx_j(\mu_0) = \tilde{\lambda}_j'(\mu_0)x_i^H(\mu_0)x_j(\mu_0). \tag{4.3}
\]

Since \( \{x_i(\mu_0)\}_{i=p}^{p+k-1} \) is a basis of \( X_k \), \( \text{(4.3)} \) is equivalent to

\[
-X_k^H C x_j(\mu_0) = \tilde{\lambda}_j'(\mu_0)X_k^H x_j(\mu_0). \tag{4.4}
\]

Furthermore, there exists \( z_j \), such that \( x_j(\mu_0) = X_kz_j \) and \( \{z_j\}_{j=p}^{p+k-1} \) are orthonormal. Thus the equation \( \text{(4.4)} \) is turned into

\[
-X_k^H C X_kz_j = \tilde{\lambda}_j'(\mu_0)z_j.
\]

This implies that \( \tilde{\lambda}_j'(\mu_0) \) for \( p \leq j \leq p + k - 1 \) have one-to-one correspondence with eigenvalues of \( -X_k^H C X_k \), counting multiplicities.
The following theorem is an application of Theorem 4.4 to show that the sorted eigencurves $\lambda_j(\mu)$ have well-defined one-sided derivatives.

**Theorem 4.5.** Assume $(\mu_*, \lambda_*)$ is an intersection of $k$ sorted eigencurves, i.e., $\lambda_j(\mu_*) = \lambda_*$ for $p \leq j \leq p+k-1$ for some integer $p \geq 1$, and $\lambda_{p-1}(\mu_*) > \lambda_* > \lambda_{p+k}(\mu_*)$. Let $X_k$ be an orthonormal eigen-subspace of the eigenvalue $\lambda_*$ of $A-\mu_*C$. Then for $p \leq j \leq p+k-1$, the one-sided derivatives

$$
\lambda_j^{(+)}(\mu_*) \equiv \lim_{t \to 0^+} \frac{\lambda_j(\mu_* + t) - \lambda_j(\mu_*)}{t}
$$

and

$$
\lambda_j^{(-)}(\mu_*) \equiv \lim_{t \to 0^-} \frac{\lambda_j(\mu_* + t) - \lambda_j(\mu_*)}{t},
$$

exist. Furthermore, both multisets \{\lambda_j^{(+)}(\mu_*) | p \leq j \leq p+k-1\} and \{\lambda_j^{(-)}(\mu_*) | p \leq j \leq p+k-1\} have one-to-one correspondence with the multiset of eigenvalues of $-C_k \equiv -X_k^HCX_k$, i.e., if the eigenvalues of $-C_k \equiv -X_k^HCX_k$ are $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_k$, then $\lambda_{p-1}^{(+)}(\mu_*) = \tau_i = \lambda_{p+k-1}^{(-)}(\mu_*)$ for $i = 1, \ldots, k$.

**Proof.** We first prove by contradiction that there exists $r > 0$, such that in the interval $(\mu_*, \mu_* + r)$, for any $i, j$, there are only following two cases between any two analyticalized eigencurves $\tilde{\lambda}_i(\mu)$ and $\tilde{\lambda}_j(\mu)$ of $A - \mu C$:

$$
\tilde{\lambda}_i(\mu) = \tilde{\lambda}_j(\mu) \quad \text{or} \quad \tilde{\lambda}_i(\mu) \neq \tilde{\lambda}_j(\mu)
$$

for any $\mu \in (\mu_*, \mu_* + r)$. If $r$ does not exist, then we can find a fixed pair $(i, j)$ and a sequence $\{\mu_m\}_{m=1}^{\infty}$ such that $\tilde{\lambda}_i(\mu_m) = \tilde{\lambda}_j(\mu_m)$, $\mu_m \to \mu_*$, $\mu_m \neq \mu_*$ but $\tilde{\lambda}_i(\mu) \neq \tilde{\lambda}_j(\mu)$, which contradicts Theorem 4.4.4.

We next prove that in the interval $[\mu_*, \mu_* + r)$, each sorted eigencurve identically equals to an analyticalized eigencurve. In fact, we have proved that in the interval $(\mu_*, \mu_* + r)$, two analyticalized eigencurves that do not equal identically will not intersect. Then by the continuity of analyticalized eigencurves, for any $i, j$, there are exactly the following three cases that hold for all $\mu \in (\mu_*, \mu_* + r)$:

$$
\tilde{\lambda}_i(\mu) < \tilde{\lambda}_j(\mu) \quad \text{or} \quad \tilde{\lambda}_i(\mu) = \tilde{\lambda}_j(\mu) \quad \text{or} \quad \tilde{\lambda}_i(\mu) > \tilde{\lambda}_j(\mu).
$$

This implies in the interval $(\mu_*, \mu_* + r)$, the algebraic order of the analyticalized eigencurves are preserved. Thus we can find a permutation $\{\ell_1, \ell_2, \ldots, \ell_n\}$ of $\{1, 2, \ldots, n\}$, such that $\tilde{\lambda}_{\ell_i}(\mu) = \lambda_i(\mu)$ for $\mu \in (\mu_*, \mu_* + r)$ and $i = 1, \ldots, n$. By continuity, $\tilde{\lambda}_{\ell_i}(\mu) = \lambda_i(\mu)$, for $i = 1, \ldots, n$. Consequently, for $p \leq j \leq p+k-1$, the limit

$$
\lim_{t \to 0^+} \frac{\lambda_j(\mu_* + t) - \lambda_j(\mu_*)}{t}
$$

exist and equals to $\tilde{\lambda}_j^{(+)}(\mu_*)$. By Theorem 4.4, the multiset $\{\lambda_j^{(+)}(\mu_*) | p \leq j \leq p+k-1\}$ have one-to-one correspondence with the multiset of eigenvalues of $-C_k$. By a similar argument, we can show that the limit $\lambda_j^{(-)}(\mu_*)$ exists and has one-to-one correspondence with the eigenvalues of $-C_k$, counting multiplicities.

Furthermore, note that for $t > 0$ and $p \leq j \leq p+k-2$,

$$
\frac{\lambda_j(\mu_* + t) - \lambda_j(\mu_*)}{t} \geq \frac{\lambda_{j+1}(\mu_* + t) - \lambda_{j+1}(\mu_*)}{t}
$$

and

$$
\frac{\lambda_j(\mu_* - t) - \lambda_j(\mu_*)}{-t} \leq \frac{\lambda_{j+1}(\mu_* - t) - \lambda_{j+1}(\mu_*)}{-t}.
$$

Thus for $p \leq j \leq p+k-2$,

$$
\lambda_j^{(+)}(\mu_*) \geq \lambda_{j+1}^{(+)}(\mu_*) \quad \text{and} \quad \lambda_j^{(-)}(\mu_*) \leq \lambda_{j+1}^{(-)}(\mu_*).
$$

(4.5)
Then the equality $\lambda_{p+1}^{(i)}(\mu_s) = \tau_i = \lambda_{p+k}^{(-)}(\mu_s)$ follows from the equation (1.5) and the correspondence between $\{\lambda_{j}^{(+)}(\mu_s) \mid p \leq j \leq p+k-1\}$, $\{\lambda_{j}^{(-)}(\mu_s) \mid p \leq j \leq p+k-1\}$ and $\{\tau_i \mid 1 \leq i \leq k\}$.

When $k = 1$ in Theorem 4.5, the argument in the proof shows the following result.

**Corollary 4.1.** Under the notation in Theorem 4.5, if $k = 1$, $\lambda_k(\mu)$ is differentiable at $\mu_s$ and satisfies $\lambda_k(\mu_s) = -x_p(\mu_s)^H C x_p(\mu_s)$, where $x_p(\mu_s)$ is the corresponding unit eigenvector of $\lambda_p(\mu_s)$.

## 5 Existence and variational characterization of 2D-eigenvalues

In this section, we discuss the existence of the 2D-eigenvalues and their variational characterizations. The following theorem characterizes when $(\mu_s, \lambda_s)$ is a 2D eigenvalue.

**Theorem 5.1.** Let $(\mu_s, \lambda_s)$ be an intersection of $k$ analyticalized eigencurves $\tilde{\lambda}_i(\mu)$ of $A - \mu C$, $i = 1, \cdots, k$.

- If $k = 1$, then $(\mu_s, \lambda_s)$ is a 2D eigenvalue if and only if $\tilde{\lambda}_1(\mu_s) = 0$;
- If $k > 1$, then $(\mu_s, \lambda_s)$ is a 2D eigenvalue if and only if there exist $i, j \leq k$, such that $\tilde{\lambda}_i(\mu_s) \tilde{\lambda}_j(\mu_s) \leq 0$.

**Proof.** Denote $X_k$ as the orthonormal basis of the eigen-subspace of the eigenvalue $\lambda_s$ of $A - \mu C$, and $C_k = X_k^H C X_k$. According to (1.1), $(\mu_s, \lambda_s)$ is a 2D eigenvalue if and only if there exists nonzero vector $x_s \in \text{span}\{X_k\}$, such that $x_s^H C x_s = 0$. Denote $x_s = X_k z$, then $x_s^H C x_s = 0$ becomes $z^H C_k z = 0$. Thus $(\mu_s, \lambda_s)$ is a 2D eigenvalue if and only if $-C_k$ is not a definite matrix.

When $k = 1$, $-C_k$ is not a definite matrix if and only if $C_k = 0$. According to Theorem 4.4, this is equivalent to $\tilde{\lambda}_1(\mu_s) = 0$.

When $k > 1$, $-C_k$ is not a definite matrix if and only if $-C_k$ has both negative and positive eigenvalues or has eigenvalue 0. According to Theorem 4.4, this is equivalent to there existing $i, j \leq k$, such that $\tilde{\lambda}_i(\mu_s) \tilde{\lambda}_j(\mu_s) \leq 0$.

**Theorem 5.2.** If $(\mu_s, \lambda_s)$ is a local minimum or maximum of a sorted eigencurve $\lambda(\mu)$ of $A - \mu C$, then $(\mu_s, \lambda_s)$ must be a 2D-eigenvalue of $(A, C)$.

**Proof.** We prove the case that $(\mu_s, \lambda_s)$ is a local maximum of some sorted eigencurve. The proof for the local minimum is similar. Assume $(\mu_s, \lambda_s)$ is an intersection of $k$ sorted eigencurves $\tilde{\lambda}_j(\mu)$ of $A - \mu C$ for $p \leq j \leq p+k-1$ with some integer $p \geq 1$. Let $\tilde{\lambda}_1, \cdots, \tilde{\lambda}_k$ be $k$ analyticalized eigencurves that satisfies $\tilde{\lambda}(\mu_s) = \lambda_s$. Then according to Theorem 4.4 and Theorem 4.5, both multiset $\{\lambda_{j}^{(-)}(\mu_s) \mid p \leq j \leq p+k-1\}$ and $\{\lambda_{j}^{(+)}(\mu_s) \mid p \leq j \leq p+k-1\}$ have one-to-one correspondence with the multiset $\{\tilde{\lambda}_i(\mu_s) \mid 1 \leq i \leq k\}$.

Since $(\mu_s, \lambda_s)$ is a local maximum, we have $\lambda_{p+k}^{(-)}(\mu_s) \leq 0$ and $\lambda_{p+k}^{(-)}(\mu_s) \geq 0$. By the one-to-one correspondence, $\{\tilde{\lambda}_i(\mu_s) \mid 1 \leq i \leq k\}$ has both non-negative and non-positive element. This implies $\tilde{\lambda}_i = 0$ when $k = 1$, and there exist $i, j \leq k$ such that $\tilde{\lambda}_i(\mu_s) \tilde{\lambda}_j(\mu_s) \leq 0$ when $k > 1$. By Theorem 5.1, $(\mu_s, \lambda_s)$ is a 2D eigenvalue. This completes the proof.

**Remark 5.1.** The proof of Theorem 5.2 is algebraic. An alternative proof is to use Clarke’s generalized directional derivative and generalized gradient in nonsmooth optimization [3]. p.10. Specifically, if $(\mu, \lambda)$ is a stationary point (locally minimum or maximum) of some sorted eigencurve $\lambda_j(\mu)$, then we have the first-order optimality condition

$$0 \in \partial \lambda_j(\mu),$$

(5.1)
where \( \partial \lambda_j(\mu_*) \) is Clarke’s generalized derivative \( \partial \lambda_j(\mu) \) at \( \mu_* \) of the eigencurve \( \lambda_j(\mu) \) \[5\], p.38, Proposition 2.3.2]. Based on Clarke’s generalized derivatives of spectral functions \[21\], p.585 and the chain rule \[5\], p.42, Theorem 2.3.9], we can derive that

\[
\partial \lambda_j(\mu_*) \subseteq \{-x^H C x \mid x \text{ is a unit eigenvector corresponding of } \lambda_* \text{ of } A - \mu_* C.\}
\] (5.2)

Consequently, by \( (5.1) \) and \( (5.2) \), we conclude that if \( (\mu_*, \lambda_*) \) is a stationary point of some sorted eigencurve \( \lambda_j(\mu) \), then there exists a unit eigenvector \( x_* \) corresponding to the eigenvalue \( \lambda_* \) of \( A - \mu_* C \), such that \( 0 = x^H C x_* \). Thus \( (\mu_*, \lambda_*, x_*) \) is a 2D-eigentriplet of the 2DEVP \( (\text{(1)}) \).

Theorem 5.2 shows that if \( (\mu_*, \lambda_*) \) is a local mininum or maximum of some sorted eigencurve \( \lambda(\mu) \), then \( (\mu_*, \lambda_*) \) must be a 2D-eigenvalue. Conversely, a 2D-eigenvalue \( (\mu, \lambda) \) does not necessarily correspond to a local minimum or maximum of a sorted eigencurve \( \lambda(\mu) \) as shown in the following example.

**Example 2.** Let

\[
A_1 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.
\]

The three sorted eigencurves \( \lambda_1(\mu) \geq \lambda_2(\mu) \geq \lambda_3(\mu) \) of \( A_1 - \mu C_1 \) are depicted in Figure 5.1(a) in blue, red and yellow, respectively. \( (\mu, \lambda, x) = (1, 0, e_3) \) is a 2D-eigentriplet and the 2D-eigenvalue \( (\mu, \lambda) = (1, 0) \) is on the eigencurve \( \lambda_3(\mu) \). However, it is neither a local minimum nor a maximum of \( \lambda_2(\mu) \) as shown in the close up plot in Figure 5.1(b).

Let

\[
A_2 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.6 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.6 \end{bmatrix}.
\]

The four sorted eigencurves of \( A_2 - \mu C_2 \) are depicted in Figure 5.1(c). Note that

\[
\det (A_2 - \mu C_2 - \lambda I) = (0.6 - 0.6\mu - \lambda) \det (A_1 - \mu C_1 - \lambda I).
\]

Thus three analyticalized eigencurves of \( A_2 - \mu C_2 \) are identical to ones of \( A_1 - \mu C_1 \). The 2D-eigenvalue \( (\mu, \lambda) = (1, 0) \) of \( (A_2, C_2) \) corresponds to the intersection of eigencurves \( \lambda_2(\mu) \) and \( \lambda_3(\mu) \). However, it is neither a local minimum nor a maximum of \( \lambda_2(\mu) \) and \( \lambda_3(\mu) \) as shown in the closed up plot in Figure 5.1(d).

The following theorem is on the existence of 2D-eigenvalues.

**Theorem 5.3.** The 2DEVP \( (\text{(1)}) \) has at least one 2D-eigenvalue.

**Proof.** We prove by construction. Let \( \lambda_1(\mu) \) be the largest sorted eigencurve of \( A - \mu C \). Then as \( \mu \to -\infty \),

\[
\lambda_1(\mu) = \max_{\|x\|=1} x^H (A - \mu C)x \geq \lambda_{\min}(A) - \mu \max_{\|x\|=1} x^H C x = \lambda_{\min}(A) - \mu \lambda_C^{(1)} \to +\infty,
\]

where \( \lambda_C^{(1)} \) is the algebraically largest eigenvalue of \( C \). Note that \( \lambda_C^{(1)} > 0 \) since \( C \) is indefinite. On the other hand, as \( \mu \to +\infty \),

\[
\lambda_1(\mu) \geq \lambda_{\min}(A) - \mu \min_{\|x\|=1} x^H C x = \lambda_{\min}(A) - \mu \lambda_C^{(2)} \to +\infty,
\]

where \( \lambda_C^{(2)} \) is the algebraically smallest eigenvalue of \( C \). Note that \( \lambda_C^{(2)} < 0 \). Therefore, the minimum of \( \lambda_1(\mu) \) is attainable at some point \( \mu_* \). By Theorem 5.2 \( (\mu_*, \lambda_1(\mu_*)) \) is a 2D-eigenvalue of \( (A, C) \). \( \square \)
Figure 5.1: The 2D-eigenvalues can be neither minima nor maxima: (a) and (b) for the non-intersection case, (c) and (d) for the intersection case.
By Theorem 5.3, we have the following variational characterizations of extreme eigencurves \( \lambda_1(\mu) \) and \( \lambda_n(\mu) \).

**Theorem 5.4.** Let \( \lambda_1(\mu) \geq \cdots \geq \lambda_n(\mu) \) be \( n \) sorted eigenvalues of \( A - \mu C \). Then it holds that

\[
\min_{\mu \in \mathbb{R}} \lambda_1(\mu) = \max_{x \neq 0} \rho_A(x) \quad \text{and} \quad \max_{\mu \in \mathbb{R}} \lambda_n(\mu) = \min_{x \neq 0} \rho_A(x),
\]

where \( \rho_A(x) \) is the Rayleigh quotient of \( A \), \( \rho_A(x) = x^H A x / (x^H x) \).

**Proof.** We only prove the first identity in (5.3). The proof for the second identity is similar. We note that the proof of Theorem 5.3 indicates that the minimum of \( \lambda_1(\mu) \) is attainable at some point \( \mu_* \) and \((\mu_*, \lambda_*) = (\mu_*, \lambda_1(\mu_*))\) is a 2D-eigenvalue. Let \( x_* \) be the corresponding 2D-eigenvector of \((\mu_*, \lambda_*)\), then

\[
x_*^H C x_* = 0 \quad \text{and} \quad \lambda_* = \rho_A(x_*) \leq \max_{x \neq 0} \rho_A(x).
\]

On the other hand,

\[
\lambda_* = \lambda_1(\mu_*) = \max_{x^H x = 1} x^H (A - \mu_* C) x \geq \max_{x^H x = 1} x^H (A - \mu_* C) x = \max_{x^H C x = 0} x^H A x.
\]

This completes the proof. \( \square \)

As a corollary of Theorem 5.4, the following result provides lower and upper bounds of \( \lambda \) of 2D-eigenvalues \((\mu, \lambda)\) on the \((\mu, \lambda)\)-plane.

**Corollary 5.1.** Let \((\mu_*, \lambda_*)\) be a 2D-eigenvalue of \((A, C)\) and \( \lambda_1(\mu) \geq \cdots \geq \lambda_n(\mu) \) be \( n \) sorted eigencurves of \( A - \mu C \). Then

\[
\max_{\mu \in \mathbb{R}} \lambda_n(\mu) \leq \lambda_* \leq \min_{\mu \in \mathbb{R}} \lambda_1(\mu),
\]

where the first equality holds if \( \lambda_n(\mu_*) = \lambda_* \), and the second equality holds if \( \lambda_1(\mu_*) = \lambda_* \).

**Proof.** Let \( x_* \) be the 2D-eigenvector associated with \((\mu_*, \lambda_*)\). Then the inequalities in (5.4) hold by Theorem 5.4 and the identity \( \lambda_* = \rho_A(x_*) \). The equalities hold due to the facts

\[
\max_{\mu \in \mathbb{R}} \lambda_n(\mu) \leq \lambda_* = \lambda_n(\mu_*) \leq \max_{\mu \in \mathbb{R}} \lambda_n(\mu) \quad \text{and} \quad \min_{\mu \in \mathbb{R}} \lambda_1(\mu) \leq \lambda_1(\mu_*) = \lambda_* \leq \min_{\mu \in \mathbb{R}} \lambda_1(\mu).
\]

\( \square \)

By Corollary 5.1 we have the following definition.

**Definition 5.1.** Let \( \lambda_1(\mu_*) = \min_{\mu \in \mathbb{R}} \lambda_1(\mu) \) and \( \lambda_n(\mu_*) = \max_{\mu \in \mathbb{R}} \lambda_n(\mu) \). Then \((\mu_*, \lambda_1(\mu_*))\) and \((\mu_*, \lambda_n(\mu_*))\) are called the maximum and minimum 2D-eigenvalues of \((A, C)\), respectively.

The following theorem provides an upper bound of \(|\mu|\) of 2D-eigenvalues \((\mu, \lambda)\) on the \((\mu, \lambda)\)-plane when \( C \) is nonsingular.

**Theorem 5.5.** Assume \( C \) is nonsingular. Let \( \lambda_C^{(+)}, \lambda_C^{(-)} \) be the minimum positive and maximum negative eigenvalues of \( C \), respectively. If \((\mu_*, \lambda_*, x_*)\) is a 2D-eigentriplet, then

\[
|\mu_*| \leq \|A\|/\sqrt{-\lambda_C^{(-)} \lambda_C^{(+)}}.
\]
Proof. By multiplying $x_*^H C$ on the left of \[(1.1a),\] we have
\[
|\mu_*| = \frac{|x_*^H C A x_*|}{\|C x_*\|^2} \leq \frac{\|A\|\|x_*\|}{\|C x_*\|} = \frac{\|A\|}{\|C x_*\|}
\]
Then an upper bound of $\frac{\|A\|}{\|C x_*\|}$ is from a lower bound of $\|C x_*\|$, which leads to the compute the quantity
\[
\min_{\|x^H x\|=1} \|C x\|.
\]
By substituting $C^2$ for $A$ in the second equation of \[(5.3),\] we have
\[
\min_{x^H C x=0} \|C x\|^2 = \min_{x^H C x=0} \frac{x^H C^2 x}{x^H x} = \max \lambda_n(C^2 - \mu C) = \max_{\mu \in \mathbb{R}} \{ c_i^2 - \mu c_i \mid i = 1, \ldots, n \},
\]
where $c_1, c_2, \ldots, c_n$ are eigenvalues of $C$. Let $\lambda_C^{-} = c_j$ and $\lambda_C^{+} = c_k$ for some $j$ and $k$. Then at the intersection $\bar{\mu}_s = c_j + c_k$ of lines $c_j - \mu c_j$ and $c_k^2 - \mu c_k$, we have
\[
\max_{\mu \in \mathbb{R}} \min_{\mu \in \mathbb{R}} \{ c_i^2 - \mu c_i \mid i = 1, \ldots, n \} \leq \max_{\mu \in \mathbb{R}} \{ c_i^2 - \mu c_i \mid i = j, k \} = \min \{ c_i^2 - \bar{\mu}_s c_i \mid i = j, k \} = -\lambda_C^{-}\lambda_C^{+}.
\]
On the other hand, we can prove $-\lambda_C^{-}\lambda_C^{+} \leq c_i^2 - \bar{\mu}_s c_i$ for $i = 1, \ldots, n$. Without loss of generality, we only consider the case $c_i > 0$. Then
\[
-\lambda_C^{-}\lambda_C^{+} - (c_i^2 - \bar{\mu}_s c_i) = c_k^2 - \bar{\mu}_s c_k - c_i^2 + \bar{\mu}_s c_i = (c_k - c_i)(c_k + c_i - \bar{\mu}_s) = (c_k - c_i)(c_k - c_i) \leq 0,
\]
where the first equation is due to the fact $-\lambda_C^{-}\lambda_C^{+} = c_j^2 - \bar{\mu}_s c_j = c_k^2 - \bar{\mu}_s c_k$ and the last inequality results from the fact that either $c_j \leq c_k \leq c_i$ or $c_i \leq c_j \leq c_k$ holds. Hence we have
\[
\max_{\mu \in \mathbb{R}} \{ c_i^2 - \mu c_i \mid i = 1, \ldots, n \} \geq \min \{ c_i^2 - \bar{\mu}_s c_i \mid i = 1, \ldots, n \} = -\lambda_C^{-}\lambda_C^{+}.
\]
This implies
\[
\min_{x^H x=1} \max_{\mu \in \mathbb{R}} \{ c_i^2 - \mu c_i \mid i = 1, \ldots, n \} = -\lambda_C^{-}\lambda_C^{+}.
\]
This completes the proof. \(\square\)

6 Number of 2D-eigenvalues

In this section, we derive a sufficient and necessary condition for the 2DEVP \[(1.1),\] to have a finite number of 2D-eigenvalues. By speaking the number of 2D-eigenvalues, we mean the number of distinct 2D-eigenvalues.

**Theorem 6.1.** The 2DEVP \[(1.1),\] has a finite number of 2D-eigenvalues if and only if the matrix pair $(A - \sigma I, C)$ is regular for any $\sigma \in \mathbb{R}$.\footnote{A matrix pair $(A, B)$ is called regular if $\det(A - \theta B) \neq 0$ for any $\theta \in \mathbb{C}$. Otherwise, it is called singular.}

Geometrically, the sufficient and necessary condition in Theorem 6.1 is equivalent to the fact that there is no any eigencurve $\lambda(\mu)$ of $A - \mu C$ which is a constant (a horizontal line) with respective to $\mu$. The proof of Theorem 6.1 is built on the following four propositions. The first proposition states a case where there are infinitely many 2D-eigenvalues.
Proposition 6.1. If there exists a shift $\sigma_0 \in \mathbb{R}$, such that the matrix pair $(A - \sigma_0 I, C)$ is singular, then the 2DEVP (1.1) has infinitely many 2D-eigenvalues.

Proof. Let $\tilde{\lambda}_1(\mu), \ldots, \tilde{\lambda}_n(\mu)$ be the analyticalized eigencurves of $A - \mu C$. Let $\mu_k = 1 - \frac{1}{k}$ for $k = 1, 2, \ldots$. For each $k$, since the matrix pair $(A - \sigma_0 I, C)$ is singular, $\det(A - \sigma_0 I - \mu_k C) = 0$. Hence $\sigma_0$ is an eigenvalue of $A - \mu_k C$. Thus there exists at least one $j_k$ such that $\tilde{\lambda}_{j_k}(\mu_k) = \sigma_0$. By selecting subsequences, we can assume $j_k$ are the same for each $k$. Without loss of generality, assume $j_k = 1$. Then by Theorem 4.3[iv], we have $\lambda_1(\mu) \equiv \sigma_0$. This implies that $\tilde{\lambda}_1'(\mu) = 0$. By taking derivation on the equation $(A - \mu C - \tilde{\lambda}_1(\mu))\tilde{x}_1(\mu) = 0$ and multiplying $\tilde{x}_1^H(\mu)$ on the left, we obtain $0 = \tilde{\lambda}_1'(\mu) = -\tilde{x}_1^H(\mu)C\tilde{x}_1(\mu)$. Thus for any $\mu$, $(\mu, \sigma_0)$ corresponds to a 2D-eigenvalue. □

Here is an example to illustrate Proposition 6.1.

Example 3. Consider the matrix pair

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$ 

By the shift $\sigma_0 = -1$, the matrix pair $(A + I, C)$ is singular since the vector $x = [1 \ -1 \ 0]^T$ is in the common nullspace of $A + I$ and $C$. It can be verified that $(\mu, -1)$ is a 2D-eigenvalue of $(A, C)$ for any $\mu \in \mathbb{R}$. Therefore the 2DEVP of $(A, C)$ has infinitely many 2D-eigenvalues. $\lambda(\mu) \equiv -1$ is a horizontal eigencurve.

We next prove a proposition which says that on a finite interval $[a, b]$, the number of 2D-eigenvalues $(\mu, \lambda)$ with $\mu \in [a, b]$ is finite if there exists no such a shift as described in Proposition 6.1.

Proposition 6.2. If the matrix pair $(A - \sigma I, C)$ is regular for any $\sigma \in \mathbb{R}$, then the number of 2D-eigenvalues $(\mu, \lambda)$ with $\mu \in [a, b]$ is finite, where $a, b \in \mathbb{R}$ are finite.

Proof. Let $\tilde{\lambda}_k(\mu), k = 1, \ldots, n$ be the analyticalized eigencurves of $A - \mu C$. According to Corollary 5.1, $\lambda$ of the 2D-eigenvalues $(\mu, \lambda)$ are bounded and we denote the bound by $[lb, ub]$. Assume there exist infinitely many 2D-eigenvalues $(\mu_k, \lambda_k)$ in the close domain $[a, b] \times [lb, ub]$ with $(\mu_k, \lambda_k) \neq (\mu_j, \lambda_j)$ for $j \neq k$, then they must have a convergent subseries. Without loss of generality, we still denote them by $(\mu_k, \lambda_k)$ and assume $(\mu_k, \lambda_k) \to (\mu^*, \lambda^*)$.

If there are infinitely many 2D-eigenvalues that correspond to intersections, then there exist two analyticalized eigencurves, which for convenience we assume to be $\tilde{\lambda}_1, \tilde{\lambda}_2$, and a subseries of $\mu_k$, which we still denote as $\mu_k$, such that $\tilde{\lambda}_1(\mu_k) = \tilde{\lambda}_2(\mu_k) = 0$. Since $\{\mu_k\}_{k=1}^\infty$ converges, using Theorem 4.3[iv] we know $\lambda_1(\mu) = \lambda_2(\mu)$, which further implies $\tilde{\lambda}_1'(\mu) = \tilde{\lambda}_2'(\mu)$. Thus $\tilde{\lambda}_1(\mu_k) = \tilde{\lambda}_2(\mu_k) = 0$. Using Theorem 4.3[iv] again, we obtain $\tilde{\lambda}_1(\mu) = \tilde{\lambda}_2(\mu) = \lambda^*$. So rank $(A - \lambda^* I - \mu C) \leq n - 2$ for any $\mu \in \mathbb{R}$. Note that the singularity of the matrix pair $(A - \lambda^* I, C)$ can be equivalently described as $\det(A - \lambda^* I - \mu C) = 0$, $\forall \mu \in \mathbb{R}$. Thus it contradicts the assumption that $(A - \lambda^* I, C)$ is regular.

Thus we can assume all the $(\mu_k, \lambda_k)$ correspond to non-intersections. There exist one $\tilde{\lambda}_1$, which for convenience we assume to be $\tilde{\lambda}_1$, such that $\tilde{\lambda}_1(\mu_k) = \lambda_k$. Then $\tilde{\lambda}_1'(\mu_k) = 0$ since it is an eigentriplet corresponding to non-intersection. Using Theorem 4.3[iv] we have $\tilde{\lambda}_1'(\mu) = 0$ and $\tilde{\lambda}_1(\mu) = \lambda^*$. Thus rank $(A - \lambda^* I - \mu C) = n - 1$ for any $\mu$ and contradicts the assumption. □

The next proposition shows that under the conditions of Proposition 6.2, all analyticalized eigencurves $\tilde{\lambda}_k(\mu)$ of $A - \mu C$ are strictly monotonous for sufficiently large $\mu$.

Proposition 6.3. If the matrix pair $(A - \sigma I, C)$ is regular for any shift $\sigma \in \mathbb{R}$, then for any analyticalized eigencurve $\tilde{\lambda}_k(\mu)$ of $A - \mu C$, there exists a positive constant $M$ such that $\tilde{\lambda}_k'(\mu) \neq 0$ for $|\mu| > M$. 

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Following two cases:

If Case (i) holds, then the real analytic function

\[ g(\epsilon) \equiv \epsilon \hat{\lambda}_\ell(\frac{1}{\epsilon}) = \epsilon \hat{\lambda}_\ell(A - \frac{1}{\epsilon}C). \]  

(6.1)

We note that

\[ \det(-C + \epsilon A - g(\epsilon)I) = \epsilon^n \det\left( A - \frac{1}{\epsilon}C - \hat{\lambda}_\ell(A - \frac{1}{\epsilon}C)I \right) = 0. \]  

(6.2)

Thus \( g(\epsilon) \) is an eigenvalue of \(-C + \epsilon A\). According to (6.1) and Theorem 6.3, \( g(\cdot) \) is real analytic on \((-\infty, 0)\) and \((0, +\infty)\). Therefore it must identically equal to two analyticalized eigencurves of the parameter eigenvalue problem \((-C + \epsilon A)x = \lambda x\), say \( \hat{\lambda}_1(\epsilon) \) and \( \hat{\lambda}_2(\epsilon) \), for \( \epsilon < 0 \) and \( \epsilon > 0 \), respectively.

For \( \mu = \frac{1}{\epsilon} \neq 0 \), we have

\[ \frac{d\hat{\lambda}_\ell(\mu)}{d\mu} = \frac{d}{d\mu} \left( \mu g(\frac{1}{\mu}) \right) = g(\frac{1}{\mu}) - \frac{1}{\mu} g'(\frac{1}{\mu}) = g(\epsilon) - \epsilon g'(\epsilon). \]  

(6.3)

We first consider the case where \( \mu < 0 \). In such case, (6.3) becomes

\[ \frac{d}{d\mu} \hat{\lambda}_\ell(\mu) = \hat{\lambda}_1(\epsilon) - \epsilon \hat{\lambda}'_1(\epsilon), \quad \forall \mu < 0. \]

Since \( \hat{\lambda}_1(\epsilon) \) is real analytic on \( \mathbb{R} \), \( \hat{\lambda}_1(\epsilon) - \epsilon \hat{\lambda}'_1(\epsilon) \) is also real analytic on \( \mathbb{R} \). Thus we have the following two cases:

(i) \( \hat{\lambda}_1(\epsilon) - \epsilon \hat{\lambda}'_1(\epsilon) \equiv 0 \) on \( \mathbb{R} \).

(ii) \( \hat{\lambda}_1(\epsilon) - \epsilon \hat{\lambda}'_1(\epsilon) \) has only finite number of roots on any finite closed interval.

If Case (i) holds, then the real analytic function \( \frac{d}{d\mu} \hat{\lambda}_\ell(A - \mu C) = 0 \) for \( \mu \neq 0 \), which further implies \( \frac{d}{d\mu} \hat{\lambda}_\ell(A - \mu C) \equiv 0 \) for \( \mu \in \mathbb{R} \). Thus there exists a fixed \( \sigma_0 \) such that \( \hat{\lambda}_\ell(A - \mu C) \equiv \sigma_0 \). Then \( (A - \sigma_0 I, C) \) is singular and it contradicts the assumption.

Therefore, only Case (ii) holds. If \( \hat{\lambda}_1(\epsilon) - \epsilon \hat{\lambda}'_1(\epsilon) \) has no negative roots, we define \( \epsilon_0 = 1 \). Otherwise we define

\[ \epsilon_0 = \min\{|\epsilon| \mid \hat{\lambda}_1(\epsilon) - \epsilon \hat{\lambda}'_1(\epsilon) = 0, \epsilon < 0\}. \]

Then we have \( \hat{\lambda}_1(\epsilon) - \epsilon \hat{\lambda}'_1(\epsilon) \neq 0 \) for \( \epsilon \in (-\epsilon_0, 0) \), which is equivalent to \( \frac{d}{d\mu} \hat{\lambda}_\ell(A - \mu C) \neq 0 \) for \( \mu \in \left(-\infty, \frac{1}{\epsilon_0}\right) \). Similarly, we could prove there exists \( \epsilon_1 > 0 \), such that \( \frac{d}{d\mu} \hat{\lambda}_\ell(A - \mu C) \neq 0 \) for \( \mu \in \left(\frac{1}{\epsilon_1}, +\infty\right) \).

Let \( M = \frac{1}{\min(\epsilon_0, \epsilon_1)} \) and we reach the conclusion.

By Proposition 6.3, we have the following proposition.

**Proposition 6.4.** If the matrix pair \((A - \sigma I, C)\) is regular for any \( \sigma \in \mathbb{R} \), then there exists a positive constant \( \tilde{M} \) such that all 2D-eigenvalues \((\mu, \lambda)\) of \((A, C)\) are bounded by \([-\tilde{M}, \tilde{M}] \times \mathbb{R}\).

**Proof.** Let \( \tilde{\lambda}_1(\mu), \cdots, \tilde{\lambda}_n(\mu) \) be \( n \) analyticalized eigencurves. According to Proposition 6.3, there exists \( M \) such that \( \tilde{\lambda}_i(\mu) \) are all monotonous with nonzero derivatives when \( |\mu| \geq M \). Thus no 2D-eigentriplets corresponding to non-intersections exist for \( |\mu| \geq M \). If there are infinitely many eigentriplets for \( |\mu| > M \) corresponding to intersections, then there exist two eigencurves having infinitely many intersections. However, since they are strictly monotonous for \( |\mu| > M \), this cannot happen. Thus there exist only finitely many 2D-eigentriplets outside \([-M, M] \times [lb, ub] \). \( \tilde{M} \) can be found by increasing \( M \). \( \square \)
Proof of Theorem 6.1. The necessary condition is immediately from Proposition 6.1. We only need to prove the sufficiency. With Proposition 6.4, all 2D-eigenvalues are bounded by \([- \tilde{M}, \tilde{M}] \times \mathbb{R}\). Then by Proposition 6.2, the total number of 2D-eigenvalues is finite. □

We now provide a vivid description of numbers of 2D-eigenvalues on an analyticalized eigencurve. We will see that besides possible trivial 2D-eigenvalues, there are only finite number of 2D-eigenvalues.

**Theorem 6.2.** There are only following two cases on an analyticalized eigencurve \(\tilde{\lambda}(\mu)\) of \(A - \mu C\):

- there are finite number of 2D-eigenvalues on \(\tilde{\lambda}(\mu)\), or
- \(\tilde{\lambda}(\mu)\) is a horizontal line and all points on the horizontal line are 2D-eigenvalues.

**Proof.** We only need to show that there are only finite number of 2D-eigenvalues on analyticalized eigencurves that are not horizontal lines. For these non-horizontal analyticalized eigencurves, the proof of Proposition 6.3 actually shows that their derivatives are non-zero for sufficiently large \(|\mu|\). Utilizing this fact and following the proof in Proposition 6.4, we know all 2D-eigenvalues on them are bounded by \([- \tilde{M}, \tilde{M}] \times \mathbb{R}\) for a large scalar \(\tilde{M}\). Finally, the proof in Proposition 6.2 indicates there could not be infinite number of 2D-eigenvalues on these non-horizontal analyticalized eigencurves among \([- \tilde{M}, \tilde{M}] \times \mathbb{R}\). Thus we reach the conclusion. □

**Remark 6.1.** If \(C\) is nonsingular, then the matrix pair \((A - \sigma I, C)\) is regular for any shift \(\sigma \in \mathbb{R}\). This implies that there are only finite number of 2D eigenvalues of \((A, C)\) when \(C\) is nonsingular.

7 Applications revisited

In this section, we revisit the two origins of the 2DEVP presented in Section 2.

7.1 Minmax of Rayleigh Quotients

We first show that the search interval of the EVopt (2.2) can be reduced from \(\mathbb{R}\) to the interval \([0, 1]\).

**Theorem 7.1.** For Case III of Theorem 2.1, i.e., \(\lambda_A \leq \theta_B\) and \(\lambda_B \leq \theta_A\), the EVopt (2.2) satisfies

\[
\max_{\mu \in \mathbb{R}} \lambda_{\min}(A - \mu C) = \max_{\mu \in [0, 1]} \lambda_{\min}(A - \mu C). \tag{7.1}
\]

Furthermore, if \(\lambda_A < \theta_B\) and \(\lambda_B < \theta_A\), the EVopt (2.2) satisfies

\[
\arg \max_{\mu \in \mathbb{R}} \lambda_{\min}(A - \mu C) = \arg \max_{\mu \in (0, 1)} \lambda_{\min}(A - \mu C). \tag{7.2}
\]

**Proof.** By Theorem 4.1, the minimum eigenvalue \(\lambda_{\min}(\mu) \equiv \lambda_n(\mu)\) of \(A - \mu C\) is concave. Thus to prove the identity (7.1), it is sufficient to prove that

\[
\lambda_n'(-)(1) \equiv \lim_{t \to 0^+} \frac{\lambda_n(1) - \lambda_n(1 - t)}{t} \leq 0, \tag{7.3}
\]

and

\[
\lambda_n'(+) (0) \equiv \lim_{t \to 0^+} \frac{\lambda_n(t) - \lambda_n(0)}{t} \geq 0. \tag{7.4}
\]

For (7.3), if \(\lambda_n(1) = \lambda_B\) is a simple eigenvalue of \(A - C = B\), then \(\lambda_n\) is differentiable at \(\mu = 1\) and \(\lambda_n'(1) = -x^H C x = -x^H (A - B)x\), where \(x\) is the eigenvector of \(B\) associated with \(\lambda_B\). Thus \(\lambda_n'(1) = -x^H A x + x^H B x = \lambda_B - \theta_A \leq 0\) and (7.3) holds. If \(\lambda_n(1)\) is not simple, by Theorem 4.5.
\( \lambda_n^{(-)}(1) \) equals to the maximum eigenvalue of \(-C_k = S_B^H(B - A)S_B\), where \( S_B \) is the eigen-subspace of \( \lambda_n(1) = \lambda_B \) for \( A - C = B \). In Case III of Theorem 2.1 any nonzero \( x \) belonging to \( S_B \) satisfies \( x^HBx = \lambda_B \leq \theta_A \leq x^HAx \). Thus \(-C_k\) is negative or semi-negative definite. This is to say, \( \lambda_n^{(-)}(1) \leq 0 \). The argument for \( \lambda_n^{(-)}(1) \) is similar. Thus the identity (7.1) holds.

To prove (7.2), we only need to prove the inequalities in (7.3) and (7.4) are strict. For the strict inequality in (7.3), it is sufficient to prove \( \lambda_n'(1) < 0 \) when multiplicity of \( \lambda_B \) is one and \( \lambda_n^{(-)}(1) < 0 \) when multiplicity of \( \lambda_B \) is larger than one. This can be proved using the same argument and noting \( \lambda_B < \theta_A \). The argument for the strict inequality in (7.4) is similar.

By combining Theorems 2.1 and (7.1), we establish the following equivalence between the RQminmax and the eigenvalue optimization:

\[
\min_{x \neq 0} \max \left\{ \frac{x^TAx}{x^Tx}, \frac{x^TBx}{x^Tx} \right\} = \max_{\mu \in [0,1]} \lambda_{\min}(A - \mu C).
\]

Specifically, if it is Case II of Theorem 2.1 by using the similar argument in the proof of the identity (7.1), in Theorem 7.1 we can prove that \( \lambda_n'(0) \leq 0 \). Thus with the concavity of \( \lambda_n(\mu) \), for any \( \mu \in [0,1] \), we have

\[
\lambda_n(\mu) \leq \lambda_n(0) + \mu \lambda_n'(0) \leq \lambda_n(0) = \lambda_{\min}(A).
\]

This implies that the equation (7.5) holds in Case II The similar argument shows the equation (7.5) also holds for Case III.

We note that the equivalence identity (7.5) is exactly Yuan’s lemma [35, Lemma 2.3], which is closely related to the well-known S-lemma in control theory and robust optimization [27, 34].

By Definition 5.1 we know that if \( \mu_* \) is an optimizer of the eigenvalue optimization, then \( (\mu_*, \lambda_{\min}(\mu_*)) \) is the minimum 2D-eigenvalue of \((A, C)\). We will demonstrate the advantages of the 2DEVP formulation of the RQminmax in our forthcoming work on algorithms for solving the 2DEVP.

7.2 Distance to instability

From Section 2 we learn that the calculation of the distance to instability \( \beta(\hat{A}) \) can be recast as an eigenvalue optimization problem (2.3), namely \( \beta(\hat{A}) = \lambda_{\min}(\mu_*) \), where \( \mu_* \) is an optimizer of the \( m \)-th eigencurve \( \lambda_m(\mu) \) of \( A - \mu C \). Van Loan [33] proved that the optimizer \( \mu_* \in [-2\|A\|, 2\|A\|] \).

By Theorem 5.2 if \( \mu_* \) is an optimizer of (2.3), then \( (\mu_*, \beta(\hat{A})) \) is a 2D-eigenvalue of the 2DEVP of \((A, C)\). In addition, we have the following list of characterizations of the target 2D-eigentriplet:

- If \( (\mu, \lambda, [u]) \) is a 2D-eigentriplet of \((A, C)\), then \( (\mu, -\lambda, [-u]) \) is also a 2D-eigentriplet. This implies the 2D-eigenvalues are symmetric with regard to \( \lambda = 0 \).

- Based on the ordering of \( 2m \) eigenvalues of \( A - \mu C \):

\[
\lambda_1(\mu) \geq \lambda_2(\mu) \geq \cdots \geq \lambda_m(\mu) > 0 > \lambda_{m+1}(\mu) \geq \cdots \geq \lambda_{2m}(\mu),
\]

we have the following characterization of \( \beta(\hat{A}) \):

\[
\beta(\hat{A}) = \min \{ \lambda \mid (\mu, \lambda) \text{ is a 2D-eigenvalue of } (A, C) \text{ and } \lambda > 0 \}
\]

\[
= -\max \{ \lambda \mid (\mu, \lambda) \text{ is a 2D-eigenvalue of } (A, C) \text{ and } \lambda < 0 \}
\]

\[
= \min \{ |\lambda| \mid (\mu, \lambda) \text{ is a 2D-eigenvalue of } (A, C) \}. \tag{7.7}
\]

Moreover, by Theorem 5.5 on the range of \( \mu \) of the 2D-eigenvalue \( (\mu, \lambda) \), we immediately conclude that the optimal \( \mu_* \) must be in \([-\|A\|, \|A\|]\). This shortens the search interval by a half. Finally, by Theorem 6.1 we know that there is only a finite number of local minima of \( \lambda_m(\mu) \) in \([-\|A\|, \|A\|]\).
Example 4. Consider the following stable matrix from [9, Example 5]:

\[
\hat{A} = \begin{bmatrix}
-0.4 + 6i & 1 & 1 \\
1 & -0.1 + 1i & -1 - 3i \\
1 & 1 & -5 + 1i
\end{bmatrix}.
\]

The distance to instability is

\[
\beta(\hat{A}) = \min_{\mu \in \mathbb{R}} \lambda_4(A - \mu C) \quad \text{with} \quad A = \begin{bmatrix} \hat{A} \hat{A}^H \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -iI_4 & iI_4 \end{bmatrix}.
\]

Figure 7.1 shows the eigencurve \( \lambda_4(\mu) \) on the interval \([-2\|A\|, 2\|A\|]\). As we can see that \( \lambda_4(\mu) \) is monotonic outside the interval \([-\|A\|, \|A\|]\). The optimal \( \mu_* \) locates within \([-\|A\|, \|A\|]\) masked by "**".

Figure 7.1: \( \lambda_4(\mu) \) on the interval \([-2\|A\|, 2\|A\|]\). The vertical lines are \( \pm\|A\| \). Note that \( \lambda_4(\mu) \) is monotonic outside the interval \([-\|A\|, \|A\|]\).

Numerous algorithms have been developed for computing the distance \( \beta(\hat{A}) \) [2, 3, 9, 13, 19, 33]. By reformulating the distance problem to the 2DEVP, it provides an opportunity for a new class of methods to solve the distance problem, in particular, for handling large scale ones. This will be the subject in the sequel of this paper.

8 Conclusion

We introduced a 2DEVP of a Hermitian matrix pair \((A, C)\). We highlighted the relationship between the well-known eigenvalue optimization problem of the parameter matrix \( H(\mu) = A - \mu C \) and the 2DEVP. We presented essential properties of the 2DEVP such as the existence and the necessary and sufficient condition for the finite number of 2D-eigenvalues. In addition, we also provided the variational characterizations of 2D-eigenvalues and the bound of 2D-eigenvalues. We used eigenvalue optimization problems from the quadratic constrained quadratic program and the computation of distance to instability to show new insights of these problems derived from the properties of the 2DEVP.

This paper is the first in a sequel on the study of the 2DEVP. An immediate sequel to this work will focus on numerical algorithms for solving the 2DEVP.
A Proof of Theorem 2.1

For Case I, we first note that for any \( x \neq 0 \),
\[
\max \{ \rho_A(x), \rho_B(x) \} \geq \rho_A(x) \geq \lambda_A.
\]
On the other hand, for \( x_s = S_A z_B \),
\[
\max \{ \rho_A(x_s), \rho_B(x_s) \} = \max \{ z_H^H A x_s, z_B^H B x_s \} = \max \{ \lambda_A, \theta_B \} = \lambda_A,
\]
where for the last equality, we use the condition \( \lambda_A > \theta_B \). Thus \( x_s \) is a solution of the RQminmax (2.1).

Case II is proven by exchanging the roles of \( A \) and \( B \) in the proof of Case I.

For Case III (i.e., \( \lambda_A \leq \theta_B \) and \( \lambda_B \leq \theta_A \)), we first prove by contradiction that in this case, if \( x_s \) is a solution of the RQminmax (2.1), then
\[
x_s^H C x_s = 0.
\]
We prove by contradiction. Assume \( x_s^H C x_s > 0 \), i.e., \( x_s^H A x_s > x_s^H B x_s \). Then \( x_s \) does not belong to \( S_A \) since otherwise \( \lambda_A = \rho_A(x_s) > \rho_B(x_s) \geq \theta_B \) and contradicts the condition \( \lambda_A \leq \theta_B \). Consider \( x(t) = x_s + t \mathrm{sign}(x_s^H x_s) x_A \) with \( t > 0 \), where by convention \( \mathrm{sign}(0) = 1 \). A straightforward calculation shows
\[
\rho_A(x(t)) = \frac{||x_s||^2 \rho_A(x_s) + (t^2 ||x_A||^2 + 2t |x_s^H x_A|) \lambda_A}{||x_s||^2 + t^2 ||x_A||^2 + 2t |x_s^H x_A|} < \rho_A(x_s).
\]
On the other hand, by the continuity of \( \rho_A(x(t)) \) and \( \rho_B(x(t)) \) with respect to \( t \), \( \rho_B(x(t)) < \rho_A(x(t)) \) holds for a sufficiently small \( t \). This implies for such \( t \) we have
\[
\max \{ \rho_A(x(t)), \rho_B(x(t)) \} = \rho_A(x(t)) < \rho_A(x_s) = \max \{ \rho_A(x_s), \rho_B(x_s) \},
\]
which contradicts the condition that \( x_s \) is the solution of the RQminmax (2.1). Hence \( x_s^H C x_s \leq 0 \).

A similar argument leads to \( x_s^H C x_s \geq 0 \) and therefore we have the identity (A.1).

The identity (A.1) implies that in Case III we have
\[
\arg \min_{x \neq 0} \max \{ \rho_A(x), \rho_B(x) \} = \arg \min_{x \neq 0} \max \{ \rho_A(x), \rho_B(x) \} = \arg \min_{x \neq 0} \rho_A(x).
\]
Thus \( x_s \) is the optimizer of the RQminmax (2.1) if and only if \( x_s \) is the optimizer of the following constrained Rayleigh quotient optimization problem:
\[
\min_{x \neq 0} \rho_A(x).
\]
Now according to Theorem 5.4 and the definition of \( \mu_s \), we have
\[
\min_{x \neq 0} \rho_A(x) = \max_{\mu \in \mathbb{R}} \lambda_{\min}(A - \mu C) = \lambda_{\min}(A - \mu_s C).
\]
Thus, by (A.3) and (A.4), we have
\[
\arg \min_{x \neq 0} \max \{ \rho_A(x), \rho_B(x) \} = \arg \min_{x \neq 0} \rho_A(x)
\]
\[
= \{ x \neq 0 \mid \rho_A(x) = \lambda_{\min}(A - \mu_s C), x^H C x = 0 \}
\]
\[
= \{ x \neq 0 \mid \rho_{A-\mu_C}(x) = \lambda_{\min}(A - \mu_s C), x^H C x = 0 \},
\]
Note that for any \( x \neq 0 \), the identity \( \rho_{A-\mu_s C}(x) = \lambda_{\min}(A - \mu_C) \) is equivalent to the fact that \( x \) is an eigenvector corresponding to \( \lambda_{\min}(A - \mu_C) \).

This completes the proof that for Case III \( x_s \) is the solution of the RQminmax (2.1) if and only if \( x_s \) is an eigenvector of \( A - \mu_s C \) corresponding to \( \lambda_{\min}(A - \mu_s C) \) and \( x_s^H C x_s = 0 \). \( \square \)
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