Generalized maximum likelihood estimation of the mean of parameters of mixtures. With applications to sampling and to observational studies.

Eitan Greenshtein  
Israel Census Bureau of Statistics; e-mail: eitan.greenshtein@gmail.com  

Ya’acov Ritov  
University of Michigan; e-mail: yaacov.ritov@gmail.com  

Abstract: Let \( f(y \mid \theta), \theta \in \Omega \) be a parametric family, \( \eta(\theta) \) a given function, and \( G \) an unknown mixing distribution. It is desired to estimate \( E_G(\eta(\theta)) \equiv \eta_G \) based on independent observations \( Y_1, ..., Y_n \), where \( Y_i \sim f(y \mid \theta_i) \), and \( \theta_i \sim G \) are iid.

We explore the Generalized Maximum Likelihood Estimators (GMLE) for this problem. Some basic properties and representations of those estimators are shown. In particular we suggest a new perspective, of the weak convergence result by Kiefer and Wolfowitz (1956), with implications to a corresponding setup in which \( \theta_1, ..., \theta_n \) are fixed parameters. We also relate the above problem, of estimating \( \eta_G \), to nonparametric empirical Bayes estimation under a squared loss.

Applications of GMLE to sampling problems are presented. The performance of the GMLE is demonstrated both in simulations and through a real data example.

Keywords: GMLE, Mixing distribution, Nonparametric Empirical Bayes, Sampling.

1. Introduction

Let \( f_\theta(y) \equiv f(y \mid \theta), \theta \in \Omega \), be a parametric family of densities with respect to some measure \( \mu \). Let \( G \) be an unknown mixture distribution. We observe \( Y_1, ..., Y_n \), which are realizations of the following process. Let \( \theta_i \sim G, i = 1, ..., n \), be independent, where conditional on \( \theta_1, ..., \theta_n, Y_i \sim f(y \mid \theta_i), i = 1, ..., n \), are independent. Given a function \( \eta(\theta) \), let \( \eta_G \equiv \eta(G) \equiv E_G(\eta(\theta)) \). Based on our observations, it is desired to estimate \( \eta_G \).
Consider for example the case where \( Y_i \sim N(\theta_i, 1) \) and \( \theta_i \sim G \) where \( G \) is supported away from 0. If \( \eta(\theta) = E_\theta(Y) \), then the natural estimator for \( \eta_G \) is \( \frac{1}{n} \sum Y_i \), which is consistent and efficient. On the other hand, deriving a consistent estimator for \( \eta_G \) is less obvious if \( \eta(\theta) = 1/\theta \). Yet, the Generalized Maximum Likelihood Estimator (GMLE) defined below, see (2), yields a simple consistent estimator for both functionals: If \( \hat{G} \) is a GMLE estimator for \( G \) a GMLE estimator of \( \eta_G \equiv E_G \eta(\theta) \) is defined by
\[
\hat{\eta}_G \equiv E_{\hat{G}} \eta(\theta)
\] (1)

**GMLE for \( G \).** Given a distribution \( G \) and a dominated family of distributions with densities \( \{ f(y \mid \theta) \mid \theta \in \Omega \} \), define
\[
f_G(y) = \int f(y \mid \vartheta) dG(\vartheta).
\]

Given observations \( Y_1, \ldots, Y_n \), a GMLE \( \hat{G} \) for \( G \) (Kiefer and Wolfowitz, 1956) is defined as:
\[
\hat{G} = \operatorname{argmax}_G \Pi f_G(Y_i);
\] (2)
the maximization is with respect to all probability distributions. Note, GMLE for \( G \) is also referred in the literature as Nonparametric Maximum Likelihood Estimator (NPMLE). We use GMLE as a tribute to its originators, Kiefer and Wolfowitz, (1956). Traditionally, a GMLE estimator \( \hat{G} \) for a mixture \( G \) is approximated via EM algorithm on a finite subset of the parameter space, see the seminal paper, Laird (1978). Koenker and Mizera (2014) suggested exploitation of convex optimization techniques.

Much of the research on GMLE is about characterization of the GMLE \( \hat{G} \) in terms of the size of its support, see, e.g., Lindsay (1995). Asymptotics and efficiency of the estimator \( \hat{\eta}_G \) for \( \eta_G \) is studied in semiparametric theory, see e.g.,
Bickel, et.al (1995). One goal of this paper is to present some new problems and applications where GMLE estimators are useful: we elaborate in particular on examples of ‘sampling with no response’, ‘post-stratification’, and ‘observational studies’.

We also relate GMLE estimators $\hat{\eta}_G$ to nonparametric empirical Bayes and compound decision problems, and provide useful and insightful representations. In nonparametric empirical Bayes the goal is to estimate the specific values $\eta_i \equiv \eta(\theta_i), \ i = 1, ..., n$, under a given loss, mainly, as in our case, a squared loss. A mixing distribution $G$ induces a joint distribution of $(Y, \theta)$. We denote the corresponding conditional expectation of $\theta$ conditional on $Y$, by $E_G(\theta | Y)$. The nonparametric empirical Bayes approach is to estimate $\eta_i, \ i = 1, ..., n$ by the plug-in estimator $\hat{\eta}_i = E_{\hat{G}}(\eta(\theta) | Y_i)$, see, the general approach in the seminal papers of Robbins (1951, 1956, 1965), see the plug-in approach in, e.g., Jiang and Zhang (2009), Koenker and Mizera (2014), and $g$-modeling in Efron(2014).

In subsection 3.2 we show that $\hat{\eta}_G$ may be represented as $\hat{\eta}_G = \frac{1}{n} \sum \hat{\eta}_i = \frac{1}{n} \sum E_{\hat{G}}(\eta(\theta) | Y_i)$. In the case of $Y_i \sim N(\theta_i, 1)$ with $\eta(\theta) = E_{\theta} Y$, $\theta_i \sim G$ are iid (and more generally) we show, that $\hat{\eta}_G = E_{\hat{G}}\eta(\theta) = \frac{1}{n} \sum Y_i$.

Asymptotics of GMLE estimators $\hat{\eta}_G$, mainly in terms of consistency, is explored both theoretically and through simulations. There are cases where $\hat{G}$ is not unique, e.g., in situations where $G$ is non-identifiable. When $\hat{G}$ is not unique, we consider as GMLE any $\hat{\eta}_G = E_{\hat{G}}\eta(\theta)$ that corresponds to any GMLE $\hat{G}$. In such cases, as $n \to \infty$, often, not every sequence of GMLE is consistent; we argue that GMLE estimators are still plausible and worthwhile, and suggest “GMLE related” Confidence-Intervals for $\eta_G$.

Given realizations $Y_1, ..., Y_n$ as above, a related problem, studied in Zhang
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(2005), is the prediction of the realized $\sum \eta(\theta_i)$, where $\theta_i \sim G$ are iid. His approach is also related to GMLE. Zhang elaborates on the difference between the estimation and the prediction problems in terms of efficiency. He also presents interesting applications where it is desired to estimate the realized $\sum \eta(\theta_i)$.

Our problem may be formalized without considering the parameters, $\theta_i$s, as random and appealing to a mixture distribution $G$. We consider $\theta_1, \ldots, \theta_n$ as unknown parameters and $Y_i \sim f(y \mid \theta_i)$, $i = 1, \ldots, n$, as independent (but not identically distributed) observations. The goal is to estimate $\sum \eta(\theta_i)$. Let $G^n$ be the empirical distribution of $\theta_1, \ldots, \theta_n$. Let $\hat{G}^n$ be the GMLE based on $Y_1, \ldots, Y_n$, as defined in (2) (i.e., the GMLE pretending $\theta_1, \ldots, \theta_n$ are i.i.d. random variables sampled from a distribution $G$). Under suitable triangular array formulation and conditions, the sequence of signed measures $(\hat{G}^n - G^n)$, converges weakly to the zero-measure. However, a reference to such a general result is not known to us, and we give a proof in Section 3.3. This weak convergence result motivates estimating $\frac{1}{n} \sum \eta(\theta_i) = E_{G^n} \eta(\theta) = \eta_{G^n}$ by $E_{\hat{G}^n} \eta(\theta)$.

As in most of the literature, we often appeal to a mixing distribution $G$ and random $\theta_i \sim G$, which makes the formulation more convenient. The two approaches of appealing to a mixing distribution $G$, versus avoiding it, are analogous to empirical Bayes versus compound decision approaches. See, e.g., Zhang (1997), Brown and Greenshtein (2009), and the fore mentioned seminal papers by Robbins. It is also related to the distinction between incidental and random nuisance parameters, see, e.g., Pfanzagl (1993).

In Section 2 we present some motivating examples. In Section 3 we present some theoretical results concerning representations of $\hat{\eta}_G$, and some asymptotics. In Section 4 we present simulations. In Section 5 we sketch a ‘GMLE related’
Confidence Interval for $\eta_G$. In Section 6 we present a real data example taken from the Israeli Social Survey.

2. Examples.

A main motivation for our study of GMLE are the following sampling models and problems in the context of stratification, and observational studies.

In stratified sampling and in post-stratification, it is desired to have a very fine stratification. Then, conditional on a fine stratum it appeals that missing observations are missing completely at random. Similarly in observational study (or, in convenience sampling), fine stratification makes the assumption that conditional on the strata the success or failure of a treatment are independent of the, often unknown, allocation mechanism to treatment and control. However, under very fine natural stratification many strata are likely to be randomly empty, i.e., with no sampled observations. In practice, when the number of observations is high the stratification used is fine to the level of existence of a meaningful number of empty cells. The GMLE method is a natural way of handling the resulting difficulties of empty strata.

We elaborate on the above. Suppose that it is desired to estimate the proportion of unemployed in the population. A sampled subject will respond to the survey with unknown probability. The proportion of unemployed among the responders in the survey is strictly a biased estimator of their proportion in the population, since there is a strong observed correlation between employment status and response. It is known that in the relevant surveys, subgroups with higher unemployment rates have lower response rates. We may hope, however, that if we consider small enough and homogenous stratum, the willingness to
answer and the answer will be, conditionally on the stratum, practically independent. Thus, the missing responses would be practically missed completely at random (MCAR) within each stratum. See, e.g., Little and Rubin (2002).

Let $K_i$ be the number of full observations in Stratum $i$ (i.e., $K_i$ were sampled and responded). Let $X_i$ be the number of unemployed among the observations from Stratum $i$. For simplicity, we consider a situation of $n$ strata with equal weights. Our observations are $Y_i = (X_i, K_i)$, $i = 1, \ldots, n$, where the conditional distribution of $X_i$ conditional on $K_i$ is $B(K_i, p_i)$. It is desired to estimate the population’s proportion $p = n^{-1} \sum p_i$.

Similar considerations apply in observational studies. Suppose that conditional on a fine stratification, within each stratum the probability of success or failure, under treatment or control, is independent of the unknown allocation mechanism to treatment or control. Given a treatment, we thinks of $X_i$, the number of successes of the treatment in stratum $i$, and $K_i$, the number of times the treatment was applied in Stratum $i$. We observe $Y_i = (X_i, K_i)$. One approach of analyzing such observational data is to pair observations based on their propensity score. Using our method we do not need pairing, and we may handle even the extreme case, where treatment was applied in one subset of strata, while control was applied in another, disjoint, subset of strata.

2.1. Sampling Models

We consider the following two realistic scenarios, where the sample size $K_i$ is random. In light of Section 3.3, the following motivating models apply to both setups of random and fixed $\theta_1, \ldots, \theta_n$. We use the notations and formulation of random $\theta_i$, $i = 1, \ldots, n$, for convenience.
Model (i): *Stratified sampling with non-response.* A random sample of $\kappa_i$ subjects from stratum $i$ is sampled. The probability of a random subject from stratum $i$ to respond is $\pi_i \leq 1$. Thus, the number of actual responses, $K_i$, $K_i \sim B(\kappa_i, \pi_i)$ is a random variable.

Model (ii): *Post-Stratification.* We assume that $K_i$ has a Poisson($\lambda_i$) distribution, $i = 1, 2, \ldots, n$. This scenario is reasonable whenever we have a convenience sample, observational studies, or, very low response rate.

In Model (i), $\theta_i = (\theta_{i1}, \theta_{i2}) \equiv (\pi_i, p_i)$, where $\pi_i$ is the probability of response, while $p_i$ is the proportion (of, say, unemployed) in stratum $i$. Conditional on $\theta_i$, $X_i \mid K_i \sim B(K_i, p_i)$ and $K_i \sim B(\kappa_i, \pi_i)$. In Model (ii), $\theta_i = (\theta_{i1}, \theta_{i2}) \equiv (\lambda_i, p_i)$. Given $\theta_i$, $X_i \mid K_i \sim B(K_i, p_i)$ as before, and $K_i \sim \text{Poisson}(\lambda_i)$.

Suppose we are interested in estimating $\eta_G = E_G \eta(\theta)$, for $\eta(\theta_i) = p_i$. If there are no empty strata, i.e., $K_i > 0$, $i = 1, \ldots, n$ then the obvious and naive estimator:

$$\hat{p}_N = \frac{1}{n} \sum_{i=1}^{n} \frac{X_i}{K_i},$$

is applicable. The problem with the above estimator is that many strata may be empty, i.e., with corresponding $K_i = 0$. When there are some empty strata, a common ad-hoc approach is of ‘collapsing strata’ where after the data is observed some empty strata are unified with non-empty ones. An extreme collapsing is to a single stratum, which yields the Extreme Collapsing estimator:

$$\hat{p}_{EC} = \frac{\sum X_i}{\sum K_i},$$

which is, in fact, desirable when $p_1 = \ldots = p_n$ but not in general.

Another way of handling the difficulty of empty strata, is to assume that strata are ‘missing at random’, that is $K_i$ is uncorrelated with $p_i$, and strata
with $K_i = 0$ may be simply ignored.

Our approach, of applying GMLE, handles cases with many empty strata in a natural way, which is not ad-hoc and does not rely heavily on missing at random type of assumptions.

Finding a GMLE for a two dimensional distribution $G$, outside of the normal setup, only recently appears in the literature, see, e.g., Gu and Koenker (2017), Feng and Dicker (2018). One reason might be its recent popularity due to Koenker and Mizera’s computational methods. Our Model (ii) is presented in Feng and Dicker, however, their motivation and context is very different than ours, and is not related to sampling. In addition, they study the estimation of the the individual $p_i$, $i = 1,\ldots,n$ via non parametric empirical Bayes, while we study the estimation of $\eta_G = E_G\eta(\theta)$, where $\eta(\theta_i) = p_i$. We also provide consistency results in the estimation of $\eta_G$.

Different related sampling models are studied in Greenshtein and Itskov (2018). In their censored-case, $n$ individuals are attempted to be interviewed, where there are at most $\kappa^0$ attempts for each individual $i$, $i = 1,\ldots,n$. The probability of a response from individual $i$ in any single attempt is $\pi_i$, and attempts are independent. Let $K_i \leq \kappa^0$ be the number of interviewing attempts of individual $i$, then $K_i$ is a truncated geometric variable. Let $Z_i$ be a 0-1 valued r.v., e.g., indicator of unemployment, $p_i = P(Z_i = 1)$. Suppose $\theta_i \equiv (\pi_i, p_i)$ are i.i.d distributed $G$. It is desired to estimate $E_G Z_i = E_G(\eta(\theta))$, where $\eta(\theta) = p$. The possible outcomes, of the attempted interviews of individual $i$, are $(Z_i, K_i) \in \{(1, K_i), \ (0, K_i), \ 1 \leq K_i \leq \kappa^0, \ \& \ \text{NULL}\},$ where NULL indicates non-response in $\kappa^0$ attempts. This scenario is within our setup as one may think of individuals as strata of size one.
The truncated case of Greenshtein and Itskov (2018) is similar to the above censored case, except that we do not know about the existence of items that did not respond. Denote by $G^t$ the conditional distribution of $\theta$, conditional on response. Then: $dG^t(\theta) \propto P_\theta(K_i \leq \kappa^0)dG(\theta)$. Let $\hat{G}^t$ be the GMLE for $G^t$, which is based only on the (non-truncated) observations $\{i : K_i \leq \kappa^0\}$. Then, when $\pi_i > 0$ w.p.1, we suggest the GMLE estimator

$$\hat{\eta} = \int \eta(\theta)\hat{d}G(\theta) = \frac{\int \eta(\theta)P_\theta^{-1}(K_i \leq \kappa^0)d\hat{G}^t(\theta)}{\int P_\theta^{-1}(K_i \leq \kappa^0)dG^t(\theta)}.$$ 

3. Asymptotic Results and equivalent representations of GMLE estimators.

We consider mainly the model in which $\theta_i = (\lambda_i, p_i)$, $\theta_i \sim G$, $i = 1,...,n$ are iid random variables and consider the estimation of the function $\eta(\theta_i) = p_i$. In light of subsection 3.3 below, analogous results may be derived when $\theta_1, ..., \theta_n$ are considered unknown parameters.

3.1. Consistency and Asymptotics in Models (i) and (ii).

We will now argue that under (ii) if $P(\lambda = 0) = 0$, then the expected value of $p$ can be consistently estimated, while under model (i) it cannot. The difference between the two models is that on the one hand, in model (ii) $K_i$ may get the values 0, 1, 2, ..., and as $n \to \infty$, the number of observed outcomes grows to infinity. On the other hand, in model (i), $\kappa_i$ is bounded by some finite $\kappa$, then the total number of outcomes is $(\kappa + 1)(\kappa + 2)/2$, which is not enough for the identification of the distribution or even the mean of of $p_i$. Here are the details.
3.1.1. Consistency in Model (ii)

In Model (ii) we have consistency in the estimation of $\eta_G$, when $P_G(\lambda = 0) \equiv G(\{\lambda = 0\}) = 0$, as proved in the following theorem.

It is convenient to re-parametrize the problem, as follows. Given $X_i \sim B(K_i, p_i)$ conditional on $K_i$, and $K_i \sim \text{Poisson}(\lambda_i)$. Denote $W_{i1} \equiv X_i$, and $W_{i2} = K_i - X_i$. Then $W_{i1}$ and $W_{i2}$ are independent Poissons conditional on $(\lambda_i, p_i)$, with corresponding parameters $\xi_{1i} \equiv p_i \lambda_i$ and $\xi_{2i} \equiv (1 - p_i) \lambda_i$.

Before presenting the theorem, recall that $G$ is identifiable in the model $f_G(y) = \int f_\theta(y)dG(\theta)$ if $f_G = f_{\tilde{G}}$ implies $G = \tilde{G}$.

**Theorem 1.** Let $G_{\xi}$ be the distribution of $(\xi_1, \xi_2)$ and let $\hat{G}_\xi$ be the GMLE based on iid $(W_{i1}, W_{i2})$, $i = 1, ..., n$. then $G_{\xi}$ is identifiable and $\hat{G}_\xi$ converges weakly to $G_{\xi}$.

**Proof.** The proof follows from Kiefer and Wolfowitz (1956). Checking the conditions is standard, the identifiability condition is verified, e.g., by Karlis and Xekalaki (2005). \hfill \Box

**Corollary 1.** for any function $\eta(\theta)$, such that $\eta(\theta) = \psi(\xi_1, \xi_2)$, for $\psi$ which is continuous and bounded on the support of $(\xi_1, \xi_2)$ under $G$, $E_G \eta(\theta) \to E_G \eta(\theta)$.

In particular, if under $G_{\xi}$, $P(\lambda > 0) = 1$, then $E_{G\xi}$ is identifiable.

The condition $P(\lambda > 0) = 1$ is necessary, since otherwise $P(\eta = 0) > 0$ while the distribution of $p$ conditioned on $\lambda = 0$ is unidentified, since there are no observations on $p$ from the atom at $\{\lambda = 0\}$. On the other hand, if $P(\lambda > 0) = 1$ than $\eta_\epsilon(\theta) = \psi_\epsilon(\xi_1, \xi_2) \equiv \frac{\xi_1 \xi_2}{\xi_1 + \xi_2 + \epsilon} 1\{\xi_1 + \xi_2 > 0\}$ is bounded and continuous, $E_G \eta_\epsilon(\theta) \to E_G \eta(\theta)$ for any $\epsilon > 0$. But $\lim_{\epsilon \to 0} E_G \eta_\epsilon = E_G \eta(\theta)$, and hence
We conclude that when confining to the class of distributions in \( \Gamma = \{ G \mid G(\{\lambda = 0\}) = 0 \} \), we have consistency of the estimator \( \hat{E}_G \eta(\theta) \) for \( E_G \eta(\theta) \) for any \( G \in \Gamma \). However, the convergence may be arbitrarily slow and depends heavily on the probability concentration of \( G \) in the neighborhood of 0.

Consider the following example. \( \lambda \sim G_\lambda \), \( p = p_0 + \delta 1(\lambda < \lambda_0) \) for some \( \lambda_0 \). To get a bound on the rate, suppose we know all parameters except \( \delta \) and we are told when \( \lambda < \lambda_0 \). Then the only sample relevant to the information is the sample of size \( O_p(\lambda \lambda_0 G_\lambda(\lambda_0)) \) coming from \( \lambda < \lambda_0 \). Given the sample, \( \sum X_i 1(\lambda < \lambda_0) \) is binomial, hence we can estimate \( \delta \) with accuracy of \( (\lambda \lambda_0 G_\lambda(\lambda_0))^{-1/2} \). This error has a contribution of \( G_\lambda(\lambda_0)(\lambda \lambda_0 G_\lambda(\lambda_0))^{-1/2} \). Since \( \lambda_0 \) can be arbitrarily small \( G_\lambda(\lambda) \) can converge to 0 as slow as we want, the estimating error of \( \delta \) has as slow rate as we want.

3.1.2. Inconsistency in Model (i)

Again, we consider the function \( \eta(\theta_i) = p_i \). Now \( \theta_i = (\pi_i, p_i) \).

In Model (i) there is no consistency in estimating \( G \) for every \( G \), neither a consistency in estimating \( E_G(\eta(\theta)) \). This is a result of the non-identifiability, and it is demonstrated in the following example for \( \kappa_i \equiv 1 \).

**Example 1.** When \( \kappa_i \equiv 1 \) there are \( M = 3 \) possible outcomes of the \( i \)th observation, we list them as: \( X_i = 1 \), \( X_i = 0 \) (while \( K_i = 1 \)), and \( K_i = 0 \); the corresponding probabilities are: \( (\pi_i p_i, \pi_i (1 - p_i), (1 - \pi_i)) \). Suppose the outcomes of \( n \) realization have \( n_1 \) occurrences of \( X_i = 1 \), \( n_2 \) occurrences of \( X_i = 0 \), and \( n_3 \) occurrences of \( K_i = 0 \). Note \( (n_1, n_2, n_3) \) is multinomial. Suppose \( \frac{1}{n}(n_1, n_2, n_3) = (0.25, 0.25, 0.5) \), obviously the following \( \hat{G}_1 \) and \( \hat{G}_2 \) are both
GMLE. Let \( \hat{G}_1 \) be degenerate at \((\pi, p) = (0.5, 0.5)\). Let \( \hat{G}_2 \) be the distribution whose support is \((0,1), (1,0), (1,1)\) with corresponding probabilities 0.5, 0.25, 0.25. Then, obviously both \( \hat{G}_1 \) and \( \hat{G}_2 \) are GMLE, while \( E_{\hat{G}_1} \eta(\theta) = 0.5 \neq E_{\hat{G}_2} \eta(\theta) = 0.75 \).

More generally, if \( \kappa_i \in \mathcal{K}, \mathcal{K} \) a bounded set, then \((\kappa_i, K_i, X_i)\) may get at most \( M \) values, and hence the distribution \( G \) is not identified (as long as it is not constraint to have a finite support with cardinality less the above number of possible values). In fact, one can estimate the expectations

\[
g_G(k, x) = E_G \pi^k (1 - \pi)^{\kappa_i - k} p^x (1 - p)^{k - x}, \quad 0 \leq x \leq k \leq \kappa_i \in \mathcal{K}. \tag{3}
\]

Note that \( \eta(\pi, p) \equiv p \) is not the linear span of the functions under the expectation in right hand side of (3). Consider now the system of equations:

\[
g_{G_0}(k, x) = E_G \pi^k (1 - \pi)^{\kappa_i - k} p^x (1 - p)^{k - x}, \quad 0 \leq x \leq k \leq \kappa_i \in \mathcal{K}
\]

\[
\eta = E_G p
\]

The set of equations (4) is a linear system of \( M \), say, linearly independent equation in \( G \). In fact, fix any \( \kappa_i \) and \( k \), the span of the subset of functions with \( \kappa_i \) and \( k \) is \( \pi^k (1 - \pi)^{\kappa_i - k} p^i \), \( i = 0, 1, \ldots, k \). Similarly span\{\( \pi^k (1 - \pi)^{\kappa_i - k} p^i \): \( k = 1, \ldots, \kappa_i \}\} = span\{\( \pi^j \): \( j = 1, \ldots, \kappa_i \}\}. Thus, span of the functions on the top line of (4) is span\{\( \pi^j / p^i \): \( i \leq j \leq \max \mathcal{K} \}\}. In particular, \( p \) is not in this span, and hence (4) is a set of linearly independent equations. Suppose, for simplicity, that the true distribution \( G_0 \) has \( M \) support points. Then, for any \( \eta \), (4) has as a solution with the same support as \( G_0 \). It is a distribution function, since 1 is in the span of the functions in the right hand side of the top line of (4) and hence \( \int dG = \int dG_0 = 1 \). However, it is not necessarily positive. But, by the inverse function theorem, the solution is continuous in \( \eta \) and hence for any
η in some neighborhood of \( \eta_0 = E_{G_0} p \) there is a positive solution \( G \) which is a probability distribution function.

This argument fails for model (ii), where \( K = \{0, 1, 2, \ldots \} \), since \( p \) is now in the closed linear span of the probability functions. However, this is another explanation for the slow potential convergence. We need to observe the rare large values of \( \kappa_i \) to estimate the functional.

### 3.2. The GMLE and the mean of the empirical Bayes estimates

Given a function \( \eta(\theta) \), suppose it is desired to estimate \( \eta_G = E_G \eta(\theta) \). Again, for convenience we consider the notations of random parameters \( \theta_1, \ldots, \theta_n \), but, in light of Section 3.3 the results apply also to the setup of fixed \( \theta_1, \ldots, \theta_n \). By the following theorem, the estimator \( \hat{\eta}_G = E_{\hat{G}} \eta(\theta) \) equals to the average of \( E_{\hat{G}}(\eta(\theta) \mid Y_i), i = 1, \ldots, n \). The appeal of this fact is that if we estimate the values of the individual \( \eta(\theta_i) \) via nonparametric empirical Bayes under squared loss, specifically by \( E_{\hat{G}}(\eta(\theta) \mid Y_i) \), there is a consistency and agreement between the estimates of the individual parameters and the estimate of their total, or, of their average. In Zhang (2005), the problem of estimating random sums involving a latent variable is explored. One approach in Zhang (2005) is to estimate the random sum, by the sum of the estimated conditional expectations of the summands. This is analogous to estimate \( \sum_i \eta(\theta_i) \) by \( \sum_i E_{\hat{G}}(\eta(\theta) \mid Y_i) \), the last term equals to \( nE_{\hat{G}} \eta(\theta) \) by the following theorem.

**Theorem 2.** Assume \( \eta(\theta) \) is a bounded function, then

\[
\hat{\eta}_G \equiv E_{\hat{G}} \eta(\theta) = \frac{1}{n} \sum_i E_{\hat{G}}(\eta(\theta) \mid Y_i).
\]  (5)
Proof. Let
\[ d\hat{G}_t(\theta) = (1 + t(\eta(\theta) - \hat{\eta}_G))d\hat{G}(\theta). \]
Since \( \eta(\theta) \) is bounded, it follows that for \( \varepsilon > 0 \) small enough, \( \mathcal{G} = \{ G_t, \ t \in (-\varepsilon, \varepsilon) \} \) is a curve of cdf’s (i.e., \( G_t \) is a positive measure with total mass 1).

Since \( \hat{G} \) is a GMLE, it maximizes the likelihood within \( \mathcal{G} \). Hence:
\[
0 = \frac{1}{n} \frac{d}{dt} \sum \log(f(Y_i | \theta)d\hat{G}_t(\theta) | t=0) = \frac{1}{n} \sum \int \frac{f(Y_i | \theta)}{f(Y_i | \hat{G}(\theta))} \left( \eta(\theta) - \hat{\eta}_G \right) \left( \frac{\hat{G}(\theta)}{f(Y_i | \theta)} \right) = \frac{1}{n} \sum \int \eta(\theta)d\hat{G}(\theta | Y_i) - \hat{\eta}_G.
\]
This concludes the proof of the theorem.

\[ \square \]

Theorem 2 represent the GMLE estimator, \( \hat{\eta}_G \) as an average of what may look like of almost i.i.d. random variables, \( n^{-1} \sum \mathbb{E}_G(\eta(\theta) | Y_i) \). The latter is \( \sqrt{n} \) consistent and asymptotically normal. However, this is misleading. All terms in the sum in (5) depend on all other through \( \hat{G} \) and \( \mathbb{E}_G(\eta(\theta) | Y_i) \) is possibly a biased estimator for \( \mathbb{E}_G(\eta(\theta) | Y_i) \). In fact, as we argue, \( \eta_G(\theta) \) may be inconsistent or consistent but converges in a very slow rate. On the other hand, in many other models, the GMLE is actually consistent, see the discussion in Section 7.8 of Bickel et al. (1995). One such a case is described in Section 3.2.1 below. See further discussion there.

3.2.1. GMLE for the mean of exponential mixtures

Let \( Y_i \sim N(\theta_i, 1), \ i = 1, ..., n \) be independent observations, the obvious estimator for \( \sum \theta_i \) is \( \sum Y_i \). One may wonder about a comparison between the trivial estimator \( \sum Y_i \) and the estimator \( nE_G \theta = \sum E_G(\theta | Y_i) \). The nonparametric
empirical Bayes $E_{\hat{G}}(\theta | Y_i)$ is strictly better than $Y_i$ as a point-wise estimator of $\theta_i$. c.f., Brown and Greenshtein (2009), Jiang and Zhang (2009), and Koenker and Mizera (2014). Similarly one may wonder about the similar model, estimating $\sum \lambda_i$ when $Y_i \sim \text{Poisson}(\lambda_i)$. In the following we will show that the two estimators are in fact equal. Thus, GMLE does not improve over the trivial estimator, yet, it does not harm.

In the following theorem the notations are for a one dimensional exponential family, but it applies for a general exponential family.

**Theorem 3.** Let $Y_1, ..., Y_n$, be independent observations, $Y_i \sim f_{\theta}(y)$. Suppose that the density of $Y$ is of the form $f_{\theta}(y) \equiv f(y | \theta) = \exp(\theta y - \psi(\theta)), \theta \in \Omega,$ with respect to some dominating measure $\mu$. Let $\hat{G}$ be a GMLE supported on the interior of the parameter set $\Omega$. Let $\eta(\theta) = E_\theta Y$. Then:

$$nE_{\hat{G}}\eta(\theta) = \sum Y_i.$$

**Proof.** Given the observations $Y_1, ..., Y_n$, let $\hat{G}$ be a GMLE. Define the translation $\hat{G}_\Delta(\cdot) = \hat{G}(\cdot - \Delta)$.

Note:

$$\int \exp(\theta y - \psi(\theta))d\hat{G}_\Delta(\theta) = \int \exp((\theta - \Delta)y - \psi(\theta - \Delta))d\hat{G}(\theta).$$

Since $\hat{G}$ is GMLE,

$$0 = \frac{d}{d\Delta} \sum \log(\int \exp(\theta Y_i - \psi(\theta))d\hat{G}_\Delta(\theta))|_{\Delta=0}$$

$$= \frac{d}{d\Delta} \sum \log(\int \exp((\theta - \Delta)Y_i - \psi(\theta - \Delta))d\hat{G}(\theta))|_{\Delta=0}$$

$$= \sum \frac{f(-Y_i + \psi'(\theta)) \exp(\theta Y_i - \psi(\theta))d\hat{G}(\theta)}{\int \exp(\theta Y_i - \psi(\theta))d\hat{G}(\theta)}$$

$$= - \sum Y_i + \sum E_{\hat{G}}(\psi'(\theta) | Y_i)$$
\[ = - \sum Y_i + n E_G \eta(\theta) \]

The last equality follows by Theorem 2, and since for exponential family 
\( \eta(\theta) = \psi'(\theta) = E_\theta Y \).

The above theorem implies that under an exponential family setup, \( E_G \eta(\theta) \) is unique for \( \eta(\theta) = E_\theta Y \) even if \( \hat{G} \) is not. See the following example.

**Example 2.** Let \( Y_i \sim Bernoulli(p_i) \) be independent where \( p_i \sim G \) are independent, \( i = 1, ..., n \). Suppose \( Y_1 = ... = Y_{\frac{n}{2}} = 0 \) and \( Y_{\frac{n}{2}+1} = ... = Y_n = 1 \). Then, obviously both \( \hat{G}_1 \) and \( \hat{G}_2 \) are GMLE, where \( \hat{G}_1 \) has half of its mass at \( p = 0.1 \) and the other half at \( p = 0.9 \), while \( \hat{G}_2 \) is degenerate at \( p = 0.5 \). Obviously \( \hat{G}_1 \neq \hat{G}_2 \), however, \( E_{\hat{G}_1} p = E_{\hat{G}_2} p = 0.5 \).

In the normal model, \( Y_i \sim N(\theta_i, 1) \) all finite cumulants of \( \theta \) can be estimated in the \( \sqrt{n} \) rate by the corresponding cumulants of \( Y \). For example, estimate \( E_G \theta \) by \( n^{-1} \sum Y_i \). That these estimators are equivalent to the GMLE can be argued by a similar argument as in the proof of Theorem 3 by taking further derivatives. However, other functions cannot be estimated in such a rate, e.g., the deconvolution problem of estimating \( G(\theta_0) = E_G 1(\theta \leq \theta_0) \) has a logarithmic rate see Fan (1991). Even a very smooth function like \( \eta(\theta) = e^{-\alpha^2 \theta^2/2}, \alpha \geq 1 \), cannot be estimated in the parametric rate, see Donoho and Low (1992). However, estimating functions of finite cumulants is efficient, as the tangent space is saturated, and any mean is an efficient estimate of its expectations. An example of a less trivial function that can be estimated efficiently at the \( \sqrt{n} \) is the density at 0 of \( G \ast N(0, \alpha^{-2}) \), \( \alpha^2 < 1 \), which in our framework is \( E_G \eta(\theta) \), where \( \eta(\theta) = \sqrt{\alpha^2/2\pi} e^{-\alpha^2 \theta^2/2} \). This is also, the density at 0 of
$f_G \ast N(0, (1 - \alpha^2)/\alpha^2)$, and hence can be estimated with the unbiased kernel estimator: $n^{-1} \sqrt{(1 - \alpha^2)/2\pi\alpha^2} \sum e^{-\alpha^2 Y_i / 2(1 - \alpha^2)}$. See Bickel et al. Section 4.5 for the tangent space and for the argument that it is efficient.

### 3.3. The compound decision model and weak convergence

The seminal paper of Kiefer and Wolfowitz (1956), appeals to a completely unknown distribution $G$, assuming that $\theta_i \sim G$, $i = 1, 2, ...$ are iid. They prove weak convergence of $\hat{G}$ to $G$. It may be seen in the introduction of that paper, that the authors are somewhat uncomfortable with this (very weak) assumption. The weak convergence in KW (1956) is explored from a broader perspective in Chen, J. H. (2017).

In this subsection we attempt to present a weak convergence result when $\theta_1, ..., \theta_n$ are considered as fixed unknown parameters that may depend on $n$. That is, we consider a general triangular array where at stage $n$ the parameters are some $(\theta^n_1, ..., \theta^n_n)$, as also elaborated in the sequel.

Recently there are results about rates of convergence of $\hat{G}$ and other estimators to $G$. See, e.g., Saha and Guntuboyina (2020) for the multivariate Gaussian case, Philippe and Kahn (2018) for finite mixtures. Those papers do not cover the weak convergence for triangular arrays presented in this subsection. In Philippe and Kahn (2018) a non-GMLE estimator, $\tilde{G}$ for $G$, is studied and they provide minmax rates for the Wasserstein distance between $\tilde{G}$ and $G$, for mixtures with $m$ components. However, those rates do not imply our weak convergence result in the triangular array formulation. Specifically, in our triangular array setup, as the number of observations $n$ increases, the number of components $m$ in the mixture is also increased since we allow $m = n$. 
Given any triangular array sequence of sets \( \{\theta^n_1, ..., \theta^n_n\} \), \( n = 1, 2, ... \), denote by \( G^n \) its corresponding empirical distribution. Given independent observations \( Y^n_1, ..., Y^n_n, Y^n_i \sim f(y \mid \theta^n_i) \), we denote the corresponding GMLE, as defined above, by \( \hat{G}^n \). That is, \( \hat{G}^n \) is the GMLE ignoring the fact that the observations are not i.i.d. from the mixture. In the sequel we may omit the superscript \( n \) and write \( \theta_i, Y_i \).

Note that, in our current setup the true joint likelihood \( f^n_G(Y_1, ..., Y_n) \) does not satisfy \( f^n_G(Y_1, ..., Y_n) = \prod_i f_G(Y_i) \), since \( Y_i \) are not independent.

**Theorem 4.** Assume A1-A5 below, then for any bounded and continuous function \( f \), \( \int f \, d\hat{G}^n - \int f \, dG^n \rightarrow_{a.s.} 0. \)

**Assumptions**

A1 The parameter space \( \Omega \) is compact.

Let

\[
h(y) = \sup_{\theta} |\log(f(y \mid \theta))|,
\]

A2 \( \sup_{\theta \in \Omega} E_\theta h^4(Y) < \infty. \)

A3 The functions \( f(y_0 \mid \theta) \) are uniformly bounded and continuous in \( \theta \) for every \( y_0 \).

A4 The class of densities \( \{f_G(y)\} \), that correspond to the set of all possible mixtures \( \{G\} \), is identifiable.

A5 For every \( G_0 \in \{G\} \), every \( \epsilon > 0 \), and every \( M > 0 \), the set

\[
\{ G \mid \| \log(f_G^{(M)}) - \log(f_{G_0}^{(M)}) \|_\infty < \epsilon \},
\]

is an open set under the weak convergence topology, where \( f_G^{(M)}(Y) = [f_G(Y)]_M \) and \( [x]_M = \max(-M, \min(x, M)) \).
Condition A1 is used to avoid situations where mass can escape. The theorem fails when \((\theta_1, \ldots, \theta_n) = (n, n + 1/n, \ldots, n + (n - 1)/n)\) since we can consider a bounded and continuous \(f\) which is not uniform continuous, for example, \(f(\theta) = \sin(\theta^4)\). This may be considered as a technical issue, as the GMLE is translation equivariant, and in this case we do have convergence to uniform, as could be obtained by restricting to a smaller set of test functions, e.g., to bounded functions with a bounded first derivative. More problematic example is with the parameters at sample size \(n\) being \((1, 2, \ldots, n)\). In this case the convergence fails even when \(f(\theta) = e^{-\theta^2}\). Assumption A1, however, can be replaced by a weaker one that ensures compactness of the measures such as strong moments conditions. We preferred to keep it simple, but strong enough for our models (although it does not cover, for example, a Gaussian mixing distribution).

The purpose of A5 is to ensure, that given an open cover for \(\{G\}\), a finite covering subset may be extracted by compactness. The assumption is implied by A1–A4 if, in addition, it is assumed that for every \(M > 0\), the functions \(\log((f^{(M)}(\theta)), \theta \in \Omega\) are uniformly continuous.

**Proof.** By A1 we trivially have tightness, and hence any sequence \(G^n\) of measures, has a weakly converging sub-sequence. Our plan is to show that every sub-sequence of \(a_n \equiv \int f d\tilde{G}^n - \int f dG^n\) has a further sub-sequence that converges to zero w.p.1. The later implies the assertion of the theorem.

Given any sub-sequence of \(a_n\), take a corresponding further sub-sequence \(n_j\), so that \(G^{n_j} \Rightarrow_w G^0\) and \(\tilde{G}^{n_j} \Rightarrow_w \tilde{G}^0\), for some \(\tilde{G}^0\) and \(G^0\).

Denote by \(G^n_1\) the joint distribution of \(\theta_1, \ldots, \theta_n\) when the parameters are sampled independently from \(G^n\), as if the parameters are sampled with replacement
from the set \{\theta_1, \ldots, \theta_n\}. We denote by \(G_2^n\) the joint distribution of \(\theta_1, \ldots, \theta_n\) when the sampling is done without replacement. Note that, for any function \(\psi\),

\[
E_{G_1^n} \sum_i \psi(Y_i) = E_{G_2^n} \sum_i \psi(Y_i). \tag{6}
\]

We keep the notations that for any function \(\psi\), \(E_{G_0} \phi(Y)\) is the expectation when \(\theta \sim G\) and \(Y \mid \theta \sim f(y \mid \theta)\).

By Lemma 1 in the Appendix, under both sequences of joint measures \(G_1^{n_j}\) and \(G_2^{n_j}\),

\[
\frac{1}{n_j} \sum_{i=1}^{n_j} \log(f_{G_1^{n_j}}(Y_i)) \rightarrow_{a.s.} E_{G_0} \log(f_{G_0}(Y)). \tag{7}
\]

Similarly, under both sequences of joint measures \(G_2^{n_j}\), and \(G_1^{n_j}\)

\[
\frac{1}{n_j} \sum_{i=1}^{n_j} \log(f_{G_2^{n_j}}(Y_i)) \rightarrow_{a.s.} E_{G_0} \log(f_{G_0}(Y)). \tag{8}
\]

By definition of \(\hat{G}_j\) as GMLE, \(\frac{1}{n_j} \sum_i \log(f_{G_0}(Y_i)) \geq \frac{1}{n_j} \sum_i \log(f_{G_0}(Y_i))\), thus under the sequence of measures \(G_2^{n_j}\):

\[
E_{G_0} \log(f_{\hat{G}_0}(Y)) = \lim_{n_j} \frac{1}{n_j} \sum_{i=1}^{n_j} \log(f_{\hat{G}_0}(Y_i)) \\
\geq \lim_{n_j} \frac{1}{n_j} \sum_{i=1}^{n_j} \log(f_{G_0}(Y_i)) \tag{9}
\]

= \(E_{G_0} \log(f_{G_0}(Y))\),

Since \(f_{G_0} = \arg\max_{f_G} E_{G_0} \log(f_G(Y))\), we obtain that:

\[
E_{G_0} \log(f_{\hat{G}_0}(Y)) \leq E_{G_0} \log(f_{G_0}(Y)). \tag{10}
\]

From the last two inequalities we obtain:

\[
E_{G_0} \log(f_{G_0}(Y)) = E_{G_0} \log(f_{\hat{G}_0}(Y)).
\]

By the identifiability assumption A4, and by concavity of the log function, \(\arg\max_{\{f_G\}} E_{G_0} \log(f_G(Y))\) is unique, and \(\hat{G}_0 = G_0\).
This concludes our proof.

\[\square\]

**Remark.** In *compound decision*, where \(\theta_1, ..., \theta_n\) are fixed, it is desired to estimate \(\eta_i = \eta(\theta_i)\). Then, the goal is to approximate the optimal separable estimator, some times also termed optimal simple symmetric estimator. Under a squared loss, the later estimator for \(\eta_i\), given an observation \(Y_i\), is \(E_{G^n}(\eta(\theta_i) \mid Y_i)\), see, e.g., Zhang (1997), Brown and Greenshtein (2009). In our triangular array setup, our weak convergence result, implies under suitable conditions \([E_{G^n}(\eta(\theta_i) \mid Y_i) - E_{\hat{G}_n}(\eta(\theta_i) \mid Y_i)] \to 0\). A simple such possible additional condition in our triangular array setup, is that \(\lim \inf f_{G^n}(Y^n_i) > 0\).

4. **Simulations.**

In this section, we simulate the estimation of \(\eta_G = E_G \eta(\theta)\), where \(\eta(\theta_i) = p_i\), under both models (i) and (ii). In all of the following simulations, for convenience, \(\eta_G\) equals 0.5. For any parameter configuration, the number of repetitions is 50.

We compute the GMLE \(\hat{G}\) via EM algorithm on a grid. As suggested by Koenker and Mizera (2014), we search for a “confined GMLE” where we confine our search to distributions with a finite and specific support. Our choice of a specific support is ad-hoc, see some rigorous treatment in Dicker and Zhao (2014). The search for GMLE among distributions on the specific grid via EM-algorithm, is also the approach in Feng and Dicker (2018), see further elaboration there.

The grids for \((\theta_{i1}, \theta_{i2})\) contain \(40 \times 40 = 1600\) equally spaced grid points in a range that fits the relevant problem. The parametrization in Model (ii) is via a two dimensional Poisson, as explained in Section 3.1.1. The EM algorithm
Table 1

Poisson Simulation discrete $G$. The estimated mean and standard deviations of two estimators.

| Support points of $(\lambda, p)$ | The naive estimator | The GMLE |
|----------------------------------|---------------------|----------|
| $(2, 0.4), (1, 0.6)$             | $0.486$, $(0.014)$  | $0.503$, $(0.020)$ |
| $(2, 0.2), (1, 0.8)$             | $0.453$, $(0.015)$  | $0.496$, $(0.018)$ |
| $(2, 0.2), (0.5, 0.8)$           | $0.385$, $(0.015)$  | $0.505$, $(0.022)$ |

Table 2

Poisson simulation with continuous $G$. The distribution of $\lambda$ is $U(0, 5, 1)$, for the first 500 strata, and $U(0, 5, 2)$ for the remaining strata. The binary probabilities $p^I$ corresponding to the first 500 strata are given the table. The binary probabilities of in the rest of the strata are $1 - p^I$. The table gives estimates of the mean and standard deviation of the two estimators.

| $p^I$ | Naive       | GMLE       |
|-------|-------------|------------|
| 0.4   | $0.513$, $(0.015)$ | $0.500$, $(0.023)$ |
| 0.3   | $0.529$, $(0.017)$ | $0.501$, $(0.027)$ |
| 0.2   | $0.538$, $(0.016)$ | $0.491$, $(0.027)$ |

started with $G$ uniform on the grid and ran for 1000 iterations.

4.1. Poisson sample sizes.

Table 1 presents the results of Model (ii) with 2-points support distribution $G$. There are 500 strata corresponding to each of the two points in the support of $G$. The table presents the mean and the sample standard deviation of the naive and the GMLE estimators for $\eta_G$.

In Table 2 we report on one such set of simulations. In the three cases presented in Table 2 there are again 500 strata of two types. The probability $p^I$ of first 500 strata is fixed while for the rest of the strata the probability is $p^{II} = (1 - p^I)$. The Poisson parameter $\lambda$ is continuous in all the three cases and is chosen from a uniform distribution, $U(0.5, 1)$ in the first 500 strata and from $U(0.5, 2)$ in each of the remaining 500 strata.
Table 3
The mean and standard deviation of two estimators. Binomial Simulation. \( \kappa \equiv 4 \) and \( \pi_i = p_i = 0.5 \pm \delta \).

| \( \delta \) | Naive       | GMLE       |
|------------|-------------|------------|
| 0.3        | 0.569, (0.012) | 0.562, (0.014) |
| 0.2        | 0.522, (0.011) | 0.564, (0.012) |
| 0.1        | 0.604, (0.010) | 0.601, (0.010) |

Table 4
Binomial Simulations with continuous \( G \). \( \kappa = 1, 2, 3, 4, 5 \).

| \( \kappa \) | Naive       | GMLE       |
|------------|-------------|------------|
| 1          | 0.544, (0.019) | 0.530, (0.015) |
| 2          | 0.528, (0.014) | 0.502, (0.021) |
| 3          | 0.522, (0.014) | 0.498, (0.022) |
| 4          | 0.517, (0.012) | 0.499, (0.020) |
| 5          | 0.512, (0.009) | 0.501, (0.013) |

4.2. Binomial sample sizes.

We next study Model (i), where \( K_i \), the realized sample size from stratum \( i \), is distributed \( B(\kappa_i, \pi_i) \). Again, our simulated populations have two types of strata, 500 of each type. In the simulations reported in Table 3, \( \kappa_i = 4 \) for all of the 1000 strata, and \( \eta_G = 0.5 \) throughout the three simulations summarized in the table. In 500 strata \( \pi_i = p_i = 0.5 - \delta \) while in the other 500 strata, \( \pi_i = p_i = 0.5 + \delta \).

The final reported simulations are summarized in Table 4. We study Binomial sampling with various values of \( \kappa \), fixed at \( \kappa = 1, \ldots, 5 \). Again, there are two types of strata, 500 strata for each of the two types; for 500 strata \( p_i \) and \( \pi_i \) are sampled independently from \( U(0.1, 0.6) \), while for the rest of the strata they are i.i.d. from \( U(0.4, 0.9) \).

It is surprising how well the GMLE is doing already for \( \kappa = 2, 3 \), in spite of the non-identifiability of \( G \) and the inconsistency of the GMLE.
5. Confidence Interval for $\eta_G$

Although the GMLE $\hat{\eta}_G$ is an appealing estimator, we do not know how good is its performance, beyond the consistency results we established. This is especially under non-identifiability where consistency is not implied. In the following we suggest an asymptotically level-$(1-\alpha)$ conservative confidence interval for $\eta_G = E_G\eta(\theta)$. We only elaborate on defining the corresponding convex optimization problem. Implementing and solving the sketched convex optimization problem, could be challenging.

Let $Z = Z(Y)$ be a random variable with $M$ possible outcomes, $z_j, j = 1, \ldots, M$. One may think of $M$ “chosen cells” in a goodness of fit test. The considerations for the choice and for the number of cells is beyond the scope of this section, in particular $M(n) \equiv M$ is fixed. Then the densities $f_Y(y | \theta)$ induce densities $f_Z(z | \theta), \theta \in \Omega$. Denote

$$p_j^G = P_G(Z = z_j) = \int f_Z(z | \theta) dG(\theta);$$

denote $\hat{p}_j = \frac{\#(Z = z_j)}{n} = \frac{n_j}{n}, j = 1, \ldots, M$, where $n_j$ is implicitly defined. Recall that:

$$2\left(\sum_j n_j \log(\hat{p}_j) - \sum_j n_j \log(p_j^G)\right) \Rightarrow G \chi^2_{(M-1)}.$$

Let $\chi^2_{(M-1),1-\alpha}$ be the $(1 - \alpha)$ quantile of a $\chi^2_{(M-1)}$ distribution. Define

$$\Gamma_\alpha = \{ G : \sum_j n_j \log(p_j^G) \geq \sum_j n_j \log(\hat{p}_j) - \frac{1}{2} \chi^2_{(M-1),1-\alpha}\}.$$

Observe that $\Gamma_\alpha$ is a convex set of distributions.

The functional $\eta(G) = E_G\eta(\theta)$, is a linear functional.

Let

$$\eta^L = \min_{G \in \Gamma_\alpha} \eta(G); \eta^U = \max_{G \in \Gamma_\alpha} \eta(G).$$
By the above, finding \((\eta^L, \eta^U)\) is a convex problem, and in addition the later interval is a \((1 - \alpha)\) conservative confidence interval for \(E_G\eta(\theta)\).

The above set \(\Gamma_\alpha\) is in the spirit of the F-localization in Ignatiadis and Wager (2021). While our suggestion for the choice of \(Z(Y)\) is ad-hoc, as often done in goodness of fit tests, they carefully elaborate on optimal choices of ‘affine estimators’.

It may be shown that the above is also a conservative \((1 - \alpha)\)-level CI for \(\sum \eta(\theta_i)\) for fixed \(\theta_1, ..., \theta_n\), and also a \((1 - \alpha)\)-level conservative credible set for \(\sum \eta(\theta_i)\) when \(\theta_1, ..., \theta_n\), are independent realizations \(\theta_i \sim G\).

6. Real Data Example.

Our example is based on data from the Social-Survey conducted yearly by Israel Census Bureau (ICB). A random sample representing a \(1/1000\) fraction of the age 20 or older individuals in the registry is drawn. The home addresses of those in the sampled are verified and they are interviewed in person.

We study here the data accumulated for Tel-Aviv, in the surveys collected during 2015–2017. The total sample size in those three years is 1256. There are 156 ‘statistical-areas’ in Tel-Aviv, very roughly of equal size, about 3000 individuals in each. This means that around three individuals are sample from each statistical area. Statistical-areas are considered homogeneous in many respects, and we take them as our strata.

Let \(K_i\) be the sample size in stratum \(i, i = 1, ...156\), then our data satisfy \(K_i > 0, i = 1, ..., 156\). (Admittedly, we neglected a few small statistical-areas that actually had zero sample sizes). In our analysis, \(p_i\) is the proportion of individuals in stratum \(i\) that own their living place (or, it is owned by a member
of their household). The goal is to estimate \( n^{-1} \sum p_i \), roughly the proportion of individuals that own their living-place.

The naive estimator is applicable since \( K_i > 0, \ i = 1, \ldots, 156 \). The estimated proportion obtained by the naive estimator is:

\[
\frac{1}{n} \sum_{i=1}^{n} \frac{X_i}{K_i} = 0.434.
\]

On the other hand, the ‘extreme collapse’ estimator, satisfy:

\[
\frac{\sum X_i}{\sum K_i} = 0.488.
\]

The significant difference between the two estimates has to do also with the fact that strata are, in fact, not of equal size. But, more importantly for us, the ‘extreme collapse’ seems to over estimate the proportion, since owners are over represented in the sample. One reason is that their address in the registry is more accurate and thus it is easier to find them. In other words, individuals are not MCAR (missing completely at random). This phenomena is partially corrected by the stratification, when MAR (missing at random) conditional on the fine strata is approximately right.

In Table 5 we compare the GMLE and the naive estimators in semi-real simulated scenarios in which only a randomly sub-sampled with probability \( \gamma \), of the described sample, is retained. The average estimates based on 25 simulations applied on the real data with \( \gamma = 0.1, 0.2, 0.25 \), are presented in Table 5. The (random) number of simulated strata with zero sample sizes, corresponding to \( \gamma = 0.1, 0.2, 0.25 \), are around 70, 40, and 30, correspondingly.

It is reasonable to assume that the number of observations in the strata are Poisson random variables. Thus, the sample sizes \( K_i \) in the simulated sub-sample are \( \text{Poisson}(\lambda_i) \) too, \( i = 1, \ldots, 156 \). The naive estimate based on the entire data
equals 0.434, which is a reasonable benchmark, since that based on the entire data we have no empty strata.

In the semi-real simulations the GMLE seems to perform much better than the naive estimator.

7. Appendix

Lemma 1. Assume the setup of Section 3.3, Assumptions A1–A5, and the notion of the proof of Theorem 4. Then, under both $G_{1}^{n_{j}}$ and $G_{2}^{n_{j}}$,

i) \[
\frac{1}{n_{j}} \sum_{i=1}^{n_{j}} \log(f_{G_{1}^{n_{j}}}(Y_{i})) \rightarrow_{a.s.} E_{G^0} \log(f_{G^0}(Y)).
\]

ii) \[
\frac{1}{n_{j}} \sum_{i=1}^{n_{j}} \log(f_{G_{2}^{n_{j}}}(Y_{i})) \rightarrow_{a.s.} E_{G^0} \log(f_{G^0}(Y)).
\]

Proof. We prove part i).

First we show that:

\[
\lim E_{G_{1}^{n_{j}}} \log(f_{G_{2}^{n_{j}}}(Y)) = E_{G^0} \log(f_{G^0}(Y)).
\]

Note that by A2 for every $\epsilon > 0$, there exists large enough $m > 0$, such that for every $G', G'' \in \{G\}$,

\[
|E_{G'} \log f_{G''}(Y) \times I(h(Y) \leq m) - E_{G'} \log f_{G''}(Y)| < \epsilon,
\]
here $I$ is an indicator function.

Hence, (13) is implied by the following,

$$
\lim_{n_j} E_{G^{n_j}} \log(f_{G^{n_j}}(Y)) \times I(h(Y) \leq m)
$$

$$
= \lim_{n_j} E_{G^0} \log(f_{G^{n_j}}(Y)) \times I(h(Y) \leq m) \times \frac{f_{G^{n_j}}(Y)}{f_{G^0}(Y)}
$$

$$
= E_{G^0} \lim_{n_j} \log(f_{G^{n_j}}(Y)) \times I(h(Y) \leq m) \times \frac{f_{G^{n_j}}(Y)}{f_{G^0}(Y)}
$$

$$
= E_{G^0} \log(f_{G^0}(Y)) \times I(h(Y) \leq m).
$$

The above is by Lebesgue dominating convergence theorem, applying A2. Note that, the likelihood ratio $\frac{f_{G^{n_j}}(Y)}{f_{G^0}(Y)}$ is bounded when $h(Y) \leq m$. Equation (13) follows by letting $m \to \infty$ and utilizing A2.

Under $G_1^{n_j}$, the proof will follow, by Borel Cantelli, if we show that for every $\epsilon > 0$:

$$
\sum_{n_j} P_{G_1^{n_j}} \left( \left| \frac{1}{n_j} \sum_{i=1}^{n_j} \log(f_{G^{n_j}}(Y_i)) - E_{G^{n_j}} \log(f_{G^{n_j}}(Y_i)) \right| > \epsilon \right) < \kappa/n^2,
$$

(14)

The complication in the above is that in the terms $f_{G^{n_j}}(Y_i)$, the random function $f_{G^{n_j}}$ depends on its argument $Y_i$. We will prove (14) through the following steps.

In fact, the convergence in equation (14) is of a sub-series $n_j$, but also the original series converges. Indeed, in the sequel we consider the original series with a general index $n$. Let $\log(\Psi) \in \{\log(f_G), G \in \{G\}\}$, be a fixed function, in particular, independent of the observations. Then for every $\epsilon > 0$

$$
P_{G^1} \left( \left| \frac{1}{n} \sum_{i=1}^{n} \log(\Psi(Y_i)) - E_{G^1} \log(\Psi(Y)) \right| > \epsilon \right) < \kappa/n^2,
$$

for a suitable large enough $\kappa$. The above is by is by Markov inequality, utilizing the bounded fourth moment assumption A2, and the fact that under $G_1^n$ (sampling with replacement), $Y_i$, $i = 1, ..., n$, are independent.
Note, that by tightness, \( \{G\} \) is compact under the weak convergence topology, since that every sequence \( G^k \) has a converging sub-sequence. For every \( \epsilon > 0 \) and \( M > 0 \), by A5 and compactness of \( \{G\} \), we may find a finite cover of \( L \) open sets, and corresponding functions \( \Psi_1, ..., \Psi_L, \Psi_i \in \{f_G\} \), such that for every \( f_0 \in \{f_G\} \)

\[
\inf_{\Psi \in \{\Psi_1, ..., \Psi_L\}} \| \log(f_0^{(M)}) - \log(\Psi^{(M)}) \|_\infty < \epsilon.
\]  

(15)

Next, since \( L \) is finite, the above considerations coupled with Bonferroni, imply that:

\[
P_{G^n_\mathcal{I}} \left( \max_{\Psi \in \{\Psi_1, ..., \Psi_L\}} \left| \frac{1}{n} \sum_{i=1}^{n} \log(\Psi(Y_i)) p - E_{G^n} \log(\Psi(Y)) \right| > \epsilon \right) < \kappa/n^2,
\]  

(16)

for a suitable large enough \( \kappa \).

Now we show that in Equation (14), for large enough \( M \), we may neglect the terms where \( | \log(f_0^{(M)}(Y_i))| > M \). For every \( \epsilon > 0 \), for large enough \( M \), a.s.,

\[
\limsup \frac{1}{n} \sum_{i: |\log(f_0^{(M)}(Y_i))| \geq M} |\log(f_0^{(M)}(Y_i))| < \epsilon.
\]

This follows since \( |\log(f_0^{(M)}(Y_i))| \leq h(Y_i) \), and by A2, coupled with Borel Cantelli. Hence, in the following, considering large \( M \), we may neglect

\[
\frac{1}{n} \sum_{i: |\log(f_0^{(M)}(Y_i))| \geq M} |\log(f_0^{(M)}(Y_i))|.
\]

By (15) we may conclude, by taking large enough \( M \), that for every \( \epsilon > 0 \),

\[
P_{G^n_\mathcal{I}} \left( \sup_{\Psi \in \{f_G\}} \left| \frac{1}{n} \sum_{i=1}^{n} \log(\Psi(Y_i)) - E_{G^n} \log(\Psi(Y)) \right| > \epsilon \right) < \kappa/n^2,
\]  

(17)

for a suitable large enough \( \kappa \).

Equation (14) follows by the last equation and the above.
The Lemma is now implied by Borel Cantelli, for the case $G_{1}^{n_{j}}$, of sampling with replacement.

The convergence under $G_{2}^{n_{j}}$ is obtained by comparing sampling with replacement to sampling without replacement. Let $S_{n_{j}} = \sum_{i=1}^{n_{j}} \log(\Psi(Y_{i}))$. Then, $E_{G_{1}^{n_{j}}} S_{n_{j}}^{4} \geq E_{G_{2}^{n_{j}}} S_{n_{j}}^{4}$. It may be seen by applying Rao-Blackwell on the convex function $g(S_{n_{j}}) = S_{n_{j}}^{4}$, when conditioning on $k_{i}$, $i = 1, ..., n_{j}$, where $k_{i}$ is the number of times $\theta_{i}$ was sampled in the sampling with replacement process, that defines $G_{1}^{n_{j}}$. The implied bounded 4’th moment of $S_{n_{j}}$ under $G_{2}^{n_{j}}$, coupled with (6), implies (14), along the lines of the above, under $G_{2}^{n_{j}}$. This concludes our proof. □
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