NON-RIGIDITY OF SPHERICAL INVERSIVE DISTANCE
CIRCLE PACKINGS

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Abstract. We give a counterexample of Bowers-Stephenson’s conjecture in
the spherical case: spherical inversive distance circle packings are not deter-
mined by their inversive distances.

1. Introduction

In this note we study inversive distance circle packing metrics on a surface \( F \).

1.1. Polyhedral surface. Given a triangulated closed orientable surface \( F \), a Eu-
clidean (resp. spherical or hyperbolic) polyhedral surface is a map \( l : E \to \mathbb{R}^+ \),
where \( E \) is the set of all edges of the triangulation, such that when \( e_1, e_2 \) and \( e_3 \)
are the three edges of a triangle, then \( l(e_1) + l(e_2) > l(e_3) \) (it is also required that
\( l(e_1) + l(e_2) + l(e_3) < 2\pi \) in the spherical case). From this \( l \), there is a polyhedral
metric in \( F \) such that the restriction of the metric to each triangle is isometric
to a triangle in \( \mathbb{E}^2 \) (resp. \( \mathbb{S}^2 \) or \( \mathbb{H}^2 \)) and the length of an edge \( e \) is given by \( l(e) \).
For instance, the boundary of a generic convex polyhedron in \( \mathbb{E}^3 \) (resp. \( \mathbb{S}^3 \) or \( \mathbb{H}^3 \))
admits a natural polyhedral metric.

The discrete curvature \( k \) of a polyhedral surface is the map \( k : V \to \mathbb{R} \), where
\( V \) is the set of all vertices of the triangulation, and for a vertex \( v \in V \), \( k(v) = 2\pi - \sum_{i=1}^{m} \theta_i \), where \( \theta_i \) are the angles at the vertex \( v \).

1.2. Inversive distance circle packings. The notion of inversive distance circle
packing was introduced by Bowers-Stephenson in \([1]\), it is a generalization of An-
dreev and Thurston’s circle packings on a surface, where two circles may intersect
or not. We just give the definition of the spherical inversive distance circle packing,
for Euclidean and hyperbolic cases, see \([1]\) and \([3]\) for more detailed discussions.

For two circles \( C_1 \) and \( C_2 \) centered at \( v_1, v_2 \) of radii \( r_1 \) and \( r_2 \) in \( \mathbb{S}^2 \), so that \( v_1 \)
and \( v_2 \) are of distance \( l \) apart, the inversive distance \( I = I(C_1, C_2) \) between them is

\[
I = \frac{\cos(l) - \cos(r_1) \cos(r_2)}{\sin(r_1) \sin(r_2)}.
\]

When viewed \( B^3 \) as the Klein model of \( \mathbb{H}^3 \), the inversive distance is essentially
the hyperbolic distance(or the intersection angle) between the two totally geodesic
planes in \( \mathbb{H}^3 \) with \( C_1 \) as their ideal boundaries. When those planes intersect, the

\hspace{5in} Date: May 3, 2011.

2000 Mathematics Subject Classification. 52C25, 53B30.

Key words and phrases. inversive distance circle packing, rigidity.

The first author was supported in part by NSFC 10901038 and Shanghai NSF 10ZR1403600.
The second author was supported in part by ANR programs ETTT, ANR-09-BLAN-0116-01,
2009-13, and ACG, ANR-10-BLAN-0105, 2010-14.
inversive distance is the cos of their angle, and if they’re disjoint, it is the cosh of their distance.

In a triangulated surface \( F \), a spherical inversive distance circle packing is given as follows: fix a vector \( I \in [0,\infty)^F \), called the inverse distance vector. For any \( r \in (0,\infty)^V \), called the radius vector, define the edge length by \( l(e) = \sqrt{r(u)^2 + r(v)^2 + 2r(u)r(v)I(e)} \) for an edge \( e \) with \( u \) and \( v \) as its end points. If for any triangle with \( e_1, e_2 \) and \( e_3 \) as its three edges, we have \( l(e_1) + l(e_2) > l(e_3) \) and \( l(e_1) + l(e_2) + l(e_3) < 2\pi \), then the edge length function \( l : E \to \mathbb{R} \) defines a spherical polyhedral metric on \( F \), which is called the spherical inversive distance circle packing metric with inverse distance \( I \).

The geometric meaning is that in \( F \) with this polyhedral metric, if we draw circles with radii \( r \) at the vertices \( V \), then the inverse distance of two circles at the end points of an edge \( e \) is the given number \( I(e) \).

It was conjectured by Bowers and Stephenson [1] that inversive distance circle packings have a global rigidity property: an inversive distance circle packing is determined by its combinatoric, inversive distance vector and discrete curvature at the vertices. Luo [3] proved Bowers-Stephenson’s conjecture in the hyperbolic and Euclidean cases. In this note, we give a counterexample in the spherical case:

**Theorem 2.4.** There is a triangulation of \( S^2 \) and two spherical inversive distance circle packings with the same inverse distance and discrete curvature, but they are not Möbius equivalent.

The example we construct actually have zero discrete curvature at all vertices, so they are inversive distance circle patterns on the (non-singular) sphere.

**Acknowledgements:** This work was done when the first author was visiting Institut de Mathématiques de Toulouse, Université Paul Sabatier (Toulouse III), he would like to thank it for its hospitality. He also would like to thank the China Scholarship Council for financial support.

2. Proof of the theorem

The proof of our theorem uses a well-known infinitesimal flexible Euclidean polyhedron and the Pogorelov map which preserves the relative distances between two points in the configurations in different geometries. We first give a rapid preliminary.

Let \( \langle x, y \rangle = -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 \) be the symmetric 2-form in the Minkowski space \( \mathbb{R}^4_1 \), recall that the hyperbolic space is

\[
\mathbb{H}^3 = \{ x \in \mathbb{R}^4_1 | ||x||^2 = -1, x_0 > 0 \}
\]

with the induced Riemannian metric on it, which is a hyperboloid in \( \mathbb{R}^4_1 \). The totally geodesic planes in \( \mathbb{H}^3 \) are the intersections between \( \mathbb{H}^3 \) and hyperplanes in \( \mathbb{R}^4 \) which pass through the origin.

Let \( B^3 \) be the unit ball in \( \mathbb{R}^3 \), then, there is a projective map \( p_{\mathbb{H}} : \mathbb{H}^3 \to B^3 \) given by \( \rho((x_0, x_1, x_2, x_3)) = (x_1, x_2, x_3)/x_0 \), which is a homeomorphism and which maps geodesic in \( \mathbb{H}^3 \) into geodesic in \( \mathbb{R}^3 \). This map is the projective model (Klein model) of the hyperbolic space.

A hyperideal hyperbolic polyhedron is the image of \( p_{\mathbb{H}}^{-1} : Q \cap B^3 \to \mathbb{H}^3 \), where \( Q \) is a Euclidean polyhedron in \( \mathbb{R}^3 \) such that all vertices of \( Q \) lie out of \( B^3 \) and all edges of \( Q \) intersect with \( B^3 \). For a point \( A \) in \( \mathbb{R}^3 - B^3 \), consider the space \( A^1 \) of the
points in $\mathbb{R}^4$ which are orthogonal to $p_{H}^{-1}(A)$ in the symmetric 2-form. $p_{H}^{-1}(A)$ is a hyperbolic plane in $H^3$. Then take $A^* = p_{H}(A^\perp) \cap B^3$, so it is a hyperbolic plane in the Klein model of the 3-dimensional hyperbolic space $B^3$. Thus its boundary is a round circle $C_A$ in $\partial B^3 = S^2$. $A^*$ is called the hyperbolic plane dual to $A$. By a simple argument, the planes dual to two vertices $A$ and $B$ of $Q$ don’t intersect. The length of an edge of a hyperideal hyperbolic polyhedron is defined as the distance between the dual planes.

The de Sitter space can be defined as
\begin{equation}
S_3^3 = \{ x \in \mathbb{R}_+^4 \mid \|x\|^2 = 1 \}
\end{equation}
with the induced Lorentzian metric on it, which is a one-sheeted hyperboloid in $\mathbb{R}_+^4$. The totally geodesic planes in $S_3^3$ are the intersections of $S_3^3$ with the hyperplanes in $\mathbb{R}^4$ which pass through the origin. Let
\begin{equation}
S_3^3_{1,+} = \{ x \in \mathbb{R}_+^4 \mid \|x\|^2 = 1, x_0 > 0 \}
\end{equation}
be the upper de Sitter space.

As for the hyperbolic space, there is a projective map $p : S_3^3_{1,+} \rightarrow \mathbb{R}^3 - B^3$ given by $\rho((x_0,x_1,x_2,x_3)) = (x_1,x_2,x_3)/x_0$, which is a homeomorphism and which maps geodesic in $S_3^3_{1,+}$ into geodesic in $\mathbb{R}^3$.

In the projective model of $S_3^3_{1,+}$, a geodesic maybe pass through $B^3$, and if it is the case, then the geodesic is time-like. If a geodesic does not pass through the closure of $B^3$, then this geodesic is space-like.

For more details on distances in the de Sitter space, see [5]: for two points $x$ and $y$ in $S^3_{1,+}$, if the geodesic $[x,y]$ is a time-like geodesic, then the distance $d$ between them is the negative number $n$ such that $\cosh(n) = \langle x,y \rangle$; if the geodesic $[x,y]$ is a space-like geodesic, the distance $d$ between them is the unique number in $i[0,\pi]$ such that $\cosh(d) = \langle x,y \rangle$.

There is a duality between points in the de Sitter space and oriented hyperplanes in the hyperbolic 3-space: consider the projective model of the upper de Sitter space $\mathbb{R}^3 - B^3$, when $A$ lies in $\mathbb{R}^3 - B^3$, then the hyperplane $A^*$ constructed above viewed as in hyperbolic 3-space is the dual of $A$.

When $A$ and $B$ are two points in $\mathbb{R}^3 - B^3$, such that the Euclidean line $L$ connecting $A$ to $B$ passes through $B^3$, then the de Sitter distance between $A$ and $B$ is essentially the hyperbolic distance between the two planes $A^*$ and $B^*$: let $l$ be the distance between $A^*$ and $B^*$ in the hyperbolic space, then $l = -d$. It is also essentially the inversive distance between the two circles $C_A$ and $C_B$, where $C_A$ and $C_B$ are the ideal boundaries of the planes $A^*$ and $B^*$ in $S^2 = \partial B_3$.

Lemma 2.1. There is a Euclidean polyhedron $Q$ such that
\begin{enumerate}
\item all vertices of $Q$ lie out of $B^3$,
\item all edges of $Q$ intersect with $B^3$,
\item in any neighborhood of $Q$, there are two Euclidean polyhedra $Q_l$ and $Q_{-l}$ which have the same combinatorics as $Q$ and the same corresponding edge lengths.
\end{enumerate}

Proof: We first recall Schönhardt’s twisted octahedron (see [8] and [2]): let $ABC$ be an equilateral triangle in $\mathbb{R}^3$, and let $L$ be a line that passes through the center of $ABC$ and it is orthogonal to the plane of the triangle. Let $A^0B^0C^0$ be the image of $ABC$ under a screw motion with axis $L$ and rotation angle $\pi/2$. Consider a polyhedron $Q$ bounded by triangles $ABC$, $A^0B^0C^0$, $ABC^0$, $A^0BC$, $AB^0C$, $
Proposition 2.2. There exists a map \( \Phi : S^3_1 + \times S^3_1 + \to \mathbb{R}^3 \times \mathbb{R}^3 \) such that:

1. \( \Phi \) is a homeomorphism from \( S^3_1 + \times S^3_1 + \) to its image in \( \mathbb{R}^3 \times \mathbb{R}^3 \),
2. the restriction of \( \Phi \) to the diagonal \( \triangle \subset S^3_1 + \times S^3_1 + \) is the projective map \( p \)(its image is in the diagonal \( \triangle' \subset \mathbb{R}^3 \times \mathbb{R}^3 \)),
3. let \( \alpha \) be a time-orientation preserving isometry of \( S^3_1 \), and \( x \in S^3_1 + \) with \( \alpha(x) \in S^3_1 + \), we have \( \Phi(x, \alpha(x)) = (y, y') \) in \( \mathbb{R}^3 \), then there is a Euclidean isometry \( \beta \) such that for all \( x \) with \( \alpha(x) \in S^3_1 + \), we have \( y' = \beta(y) \),
4. if \( [x, y] \) and \( [x', y'] \) are two time-like geodesics of the same length in \( S^3_1 + \), and if \( p_1, p_2 \) are the projections of \( \mathbb{R}^3 \times \mathbb{R}^3 \) on the two factors, then \( [p_1 \circ \Phi \circ (x, x'), p_1 \circ \Phi \circ (y, y')] \) and \( [p_2 \circ \Phi \circ (x, x'), p_2 \circ \Phi \circ (y, y')] \) are geodesics of the same length in \( \mathbb{R}^3 \).

\( A^0B^0C, AB^0C^0, \) and \( A^0BC^0 \). The polyhedron \( Q \) is combinatorially isomorphic to an octahedron, and has three edges with dihedral angles bigger than \( \pi \): the edges \( AB^0, BC^0, \) and \( CA^0 \), see Figure 1.

![Figure 1. Schönhardt’s twisted octahedron](image-url)

This polyhedron \( Q \) is infinitesimal flexible (see \( \text{(2)} \): there are vectors \( \eta_{A^0}, \eta_{B^0}, \eta_{C^0} \) in \( \mathbb{R}^3 \) such that the polyhedron \( Q_t \) with the vertices \( A, B, C, A^0 + t\eta_{A^0}, B^0 + t\eta_{B^0}, C^0 + t\eta_{C^0}, \) which is a small deformation of \( Q \), is a non-trivial infinitesimally isometric deformation of \( Q \), where \( \eta_{A^0} \) is a vector orthogonal to the plane \( A^0BC \) of norm 1 and pointing out from \( Q \), and similarly for \( \eta_{B^0} \) and \( \eta_{C^0} \). Then by a direct calculation (or see Lemma 4.1 of \( \text{(2)} \)), the pairs of corresponding edges of \( Q_t \) and \( Q_{-t} \) have the same lengths, for \( 0 < t \) small enough.

Let \( O \) be the center of the polyhedron \( Q \), i.e., \( O \) lies in the line \( L \) and its distances to the planes \( ABC \) and \( A^0B^0C^0 \) are both equal to \( h > 0 \). Let \( a \) be the edge length of the equilateral triangle \( ABC \), a simple calculation shows that conditions (1) and (2) of the lemma are equivalent to

1. \( h^2 + a^2/3 > 1 \),
2. \( h^2 + a^2/12 < 1 \),
3. \( a^2/3 < 1 \).

So, we can assign \( h = 1/2 \) and \( a \) a little bigger than \( 3/2 \), and the lemma follows.

Pogorelov \( \text{(4)} \) has found remarkable maps from \( S^3 \times S^3 \) and \( H^3 \times H^3 \) to \( \mathbb{R}^3 \times \mathbb{R}^3 \), see also \( \text{(5)}, \text{(6)}, \text{(7)} \) for other forms of these maps and their infinitesimal versions. What we really need is the first four properties of the following proposition, we state it along with other properties for the sake of future reference.

Proposition 2.2. There exists a map \( \Phi : S^3_1 + \times S^3_1 + \to \mathbb{R}^3 \times \mathbb{R}^3 \) such that:

1. \( \Phi \) is a homeomorphism from \( S^3_1 + \times S^3_1 + \) to its image in \( \mathbb{R}^3 \times \mathbb{R}^3 \),
2. the restriction of \( \Phi \) to the diagonal \( \triangle \subset S^3_1 + \times S^3_1 + \) is the projective map \( p \)(its image is in the diagonal \( \triangle' \subset \mathbb{R}^3 \times \mathbb{R}^3 \)),
3. let \( \alpha \) be a time-orientation preserving isometry of \( S^3_1 \), and \( x \in S^3_1 + \) with \( \alpha(x) \in S^3_1 + \), we have \( \Phi(x, \alpha(x)) = (y, y') \) in \( \mathbb{R}^3 \), then there is a Euclidean isometry \( \beta \) such that for all \( x \) with \( \alpha(x) \in S^3_1 + \), we have \( y' = \beta(y) \),
4. if \( [x, y] \) and \( [x', y'] \) are two time-like geodesics of the same length in \( S^3_1 + \), and if \( p_1, p_2 \) are the projections of \( \mathbb{R}^3 \times \mathbb{R}^3 \) on the two factors, then \( [p_1 \circ \Phi \circ (x, x'), p_1 \circ \Phi \circ (y, y')] \) and \( [p_2 \circ \Phi \circ (x, x'), p_2 \circ \Phi \circ (y, y')] \) are geodesics of the same length in \( \mathbb{R}^3 \).
∀ to adapt the proof of $f \sim \text{Iso}_-$ of $-\alpha$ hyperbolic space. From this, we have an time-orientation isometry fixed length, which can be see from the duality between the de sitter space and the which is an open neighborhood of the diagonal of $(R[\alpha]) = (a, b)$, so $\rho = 2$, we choose the one similar to [9], which is convenient for us.

Proof. From Section 6 of [9]: in Proposition 6.3 and 6.4 of [9], we should replace $Q[2.3]$.

Or from Section 6 of [9]: in Proposition 6.3 and 6.4 of [9], we should replace $f(a, b) = (a^2 - b^2)^2 - 8(a^2 + b^2 - 2)$ (in de Sitter case and $1 > a, b \geq 0$) to $g(a, b) = -(a^2 - b^2)^2 + 8(a^2 + b^2 - 2)$ (in de Sitter case and $1 < a, b$). Note that $g(a, b) = -(a^2 - b^2)^2 + 8(a^2 + b^2 - 2)$ is not always positive for $1 < a, b$, but this is true for $-4 < a^2 - b^2 < 4$, so $((\xi, \eta) \in (R^3 - B^3) \times (R^3 - B^3)) - 4 < |\xi| - |\eta|^2 < 4 \subset Im(\Phi)$, which is an open neighborhood of the diagonal of $(R^3 - B^3) \times (R^3 - B^3) \subset R^3 \times R^3$.

For the proof of (3) of Proposition 2.2, we just recall that for a time-orientation preserving isometry $\alpha$ of $S^3_1$, in the matrix presentation $A_{4 \times 4}$ of it, the $(1, 1)$-entry of $A$ is positive, and then (3) of Proposition 2.2 follows from arguments similar to Proposition 6.5 of [9].

For the proof of (4) of Proposition 2.2, we need the transitivity of the time-orientation subgroup of $\text{Iso}(S^3_1)$ on the space of time-like geodesic segments of a fixed length, which can be see from the duality between the de sitter space and the hyperbolic space. From this, we have an time-orientation isometry $\alpha$, such that $\alpha([x, y]) = [x', y']$, and then (4) follows from (3). (5) is similar.

The map $\Phi$ in the Minkowski coordinate is given as follows: let $(x, y) \in S^3_1 \times S^3_1$, where $x = (x_0, x_1, x_2, x_3)$ and $y = (y_0, y_1, y_2, y_3)$, then

$$\Phi(x, y) = 2((x_1, x_2, x_3), (y_1, y_2, y_3))/(x_0 + y_0) \in R^3 \times R^3.$$  \hspace{1cm} (5.5)

The converse $\Phi^{-1} : R^3 \times R^3 \to S^3_1 \times S^3_1$ is given by

$$((\xi_1, \xi_2, \xi_3), (\eta_1, \eta_2, \eta_3)) \to (\rho((4 - |\eta|^2 + |\xi|^2, \xi_1, \xi_2, \xi_3)), \rho((4 - |\xi|^2 + |\eta|^2, \eta_1, \eta_2, \eta_3))),$$

where $\rho$ is the linear normalization such that $\rho((4 - |\eta|^2 + |\xi|^2, \xi_1, \xi_2, \xi_3))$ and $\rho((4 - |\xi|^2 + |\eta|^2, \eta_1, \eta_2, \eta_3))$ lie in the hyperboloid $S^3_1$.

Remark 2.3. The Pogorelov’s maps in [4] and [9] are a little different, i.e, up to the multiple constant 2, we choose the one similar to [9], which is convenient for us.

**Theorem 2.4.** There is a triangulation of $S^2$ and two spherical inversive distance circle packings with the same inversive distance and discrete curvature, but they are not Möbius equivalent.

**Proof.** The triangulation of $S^2$ is given from the boundary of the Euclidean octahedron in Lemma 2.1. From Lemma 2.1, we have two Euclidean polyhedra which have the same edge lengths, but they are not congruent, say $Q_t$ and $Q_{-t}$ for a fixed $t > 0$ small enough, which are very near to $Q$. Denoted the vertices of $Q$ by $v^i$ and the corresponding vertices of $Q_t(Q_{-t})$ by $v^i_t(v^i_{-t})$. Note that $(v^i_t, v^j_t) \in Im(\Phi)$ by Proposition 2.2 (2), and from Proposition 2.2 (1), we can assume $(v^i_t, v^j_t) \in Im(\Phi)$.

Since $\Phi^{-1} \circ (v^i_t, v^i_{-t})$ give us a polyhedron in $S^3_1$ such that each of the edges are time-like. Then we use Proposition 2.2 (4), $p_1 \circ \Phi^{-1} \circ (v^i_t, v^i_{-t})$ and $p_2 \circ \Phi^{-1} \circ (v^i_t, v^i_{-t})$
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give us two polyhedra, say $P_t$ and $P_{-t}$, in $S^3_{1,+}$ such that each of the edges are time-like and which have the same corresponding edge lengths. For each vertex of $P_t$, we have a circle in $S^2$, which is the ideal boundary of the hyperbolic plane dual to the vertex. But recall that the de Sitter length here is essentially the inversive distance of the circles corresponding to two ideal vertices of the hyperideal hyperbolic polyhedra. So, we have two spherical inversive distance circle packing metrics, they induced the same standard spherical metric in $S^2$, thus they have the same discrete curvature zero. These two spherical inversive distance circle packing are not Möbius equivalent can be seen also from the Pogorelov map.

Corollary 2.5. There is a hyperideal polyhedron $P$ such that each face of it is a triangle and in any neighborhood of $P$, there are two hyperideal polyhedra $P_t$ and $P_{-t}$ which have the same combinatorics and the corresponding edges of them have the same length.

Proof. Now from Lemma 2.1, we have two Euclidean polyhedra $Q_t$ and $Q_{-t}$, which have the same edge length, but they are not congruent. Then we use Proposition 2.2, we get two polyhedra in $S^3_{1,+}$ such that each of the edges are time-like.

Such polyhedra in $S^3_{1,+}$ can be viewed as hyperideal hyperbolic polyhedra, and the distance in the de Sitter geometry is just the distance of the circles corresponding to two ideal vertices of the hyperideal hyperbolic polyhedra, which is the edge length of the hyperideal hyperbolic polyhedra.

Remark 2.6. Our polyhedra above are not convex, a similar phenomena appears in convex hyperbolic polyhedra, but, some of the faces are not triangle, see Theorem 2’ of [6].

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