Rigidity estimates for isometric and conformal maps from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$

Stephan Luckhaus\textsuperscript{1}, Konstantinos Zemas\textsuperscript{2}

Abstract

We investigate both linear and nonlinear stability aspects of rigid motions (resp. Möbius transformations) of $\mathbb{S}^{n-1}$ among Sobolev maps from $\mathbb{S}^{n-1}$ into $\mathbb{R}^n$. Unlike similar in flavour results for maps defined on domains of $\mathbb{R}^n$ and mapping into $\mathbb{R}^n$, not only an isometric (resp. conformal) deficit is necessary in this more flexible setting, but also a deficit measuring the distortion of $\mathbb{S}^{n-1}$ under the maps in consideration. The latter is defined as an associated isoperimetric type of deficit. The focus is mostly on the case $n = 3$ (where it is explained why the estimates are optimal in their corresponding settings), but we also address the necessary adaptations for the results in higher dimensions. We also obtain linear stability estimates for both cases in all dimensions. These can be regarded as Korn-type inequalities for the combination of the quadratic form associated with the isometric (resp. conformal) deficit on $\mathbb{S}^{n-1}$ and the isoperimetric one.

2010 MSC Classification: 26D10, 30C70, 49Q20

Keywords: Liouville’s theorem, rigid motions, Möbius transformations, isometric deficit, conformal-isoperimetric deficit, stability

1 Introduction

In this paper we examine stability issues of isometric and conformal maps from $\mathbb{S}^{n-1}$ into $\mathbb{R}^n$ of relatively low regularity, focusing mostly, but not solely, on the case $n = 3$. Since the starting domain is of codimension 1 in $\mathbb{R}^n$, these maps exhibit of course more flexibility than their analogues from open subdomains of $\mathbb{R}^n$ into $\mathbb{R}^n$. On the one hand, isometric and conformal maps are actually rigid when considered from $\mathbb{S}^{n-1}$ into itself, as the following version of the well known theorem by J. Liouville asserts.

Theorem 1.1. (Liouville’s Theorem on $\mathbb{S}^{n-1}$)

(i) Let $n \geq 2$ and $p \in [1, +\infty]$. A generalized orientation-preserving (\textbackslash-reversing) $u \in W^{1,p}(\mathbb{S}^{n-1}; \mathbb{S}^{n-1})$ is isometric iff it is a rigid motion of $\mathbb{S}^{n-1}$, i.e., iff there exists $O \in O(n)$ so that for every $x \in \mathbb{S}^{n-1}$,

$$u(x) = Ox.$$  \hfill (1.1)

(ii) Let $n \geq 3$. A generalized orientation-preserving (\textbackslash-reversing) $u \in W^{1,1}(\mathbb{S}^{n-1}; \mathbb{S}^{n-1})$ of degree 1 (\textbackslash-1) is conformal iff it is a Möbius transformation of $\mathbb{S}^{n-1}$, i.e., iff there exist $O \in O(n)$, $\xi \in \mathbb{S}^{n-1}$ and $\lambda > 0$ so that for every $x \in \mathbb{S}^{n-1}$,

$$u(x) = O\phi_{\xi,\lambda}(x).$$  \hfill (1.2)

Here, $\phi_{\xi,\lambda} := \sigma_{\xi}^{-1} \circ i_{\lambda} \circ \sigma_{\xi}$, where $\sigma_{\xi}$ is the stereographic projection of $\mathbb{S}^{n-1}$ onto $T_\xi\mathbb{S}^{n-1} \cup \{\infty\}$, and $i_{\lambda} : T_\xi\mathbb{S}^{n-1} \mapsto T_\xi\mathbb{S}^{n-1}$ is the dilation in $T_\xi\mathbb{S}^{n-1}$ by factor $\lambda > 0$.

On the other hand however, there is a wide variety of such maps from $\mathbb{S}^{n-1}$ into $\mathbb{R}^n$. In contrast to the classical rigidity in the Weyl problem for isometric embeddings, according to which the only $C^2$ (or even
$C^{1,\alpha}$ for $\alpha > \frac{2}{3}$) isometric embedding of $\mathbb{S}^{n-1}$ into $\mathbb{R}^n$ is the standard one modulo rigid motions (cf. [9, 11]), as a consequence of the celebrated Nash-Kuiper theorem (cf. [16, 20]), the following paradox happens for less regular, say $C^1$ isometric embeddings.

Given any $\delta \in (0, 1)$, in an arbitrarily small $C^0$-neighbourhood of the short homothety $u_\delta : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$, $u_\delta(x) := \delta x$, there exist $C^1$ isometric embeddings, which can be visualized as wrinkling isometrically $\mathbb{S}^{n-1}$ inside the small ball $B_\delta(0)$ in a way that produces continuously changing tangent planes. For the more general case of conformal maps from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$, at least when $n = 3$, other examples that are not Möbius transformations are provided by the Uniformization Theorem and some of them have often been used in cartography, for instance the inverse of Jacobi’s conformal map projection that smoothly and conformally transforms are provided by the Uniformization Theorem and some of them have often been used in cartography, for instance the inverse of Jacobi’s conformal map projection that smoothly and conformally maps $\mathbb{S}^2$ onto the surface of an ellipsoid.

Therefore, Liouville’s rigidity theorem on $\mathbb{S}^{n-1}$ on the one hand, and the aforementioned flexibility phenomena on the other, indicate the following fact. When one seeks stability of the isometry (resp. the conformal) group of $\mathbb{S}^{n-1}$ among Sobolev maps $u : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$, apart from an isometric (resp. conformal) deficit, an extra deficit measuring the deviation of $u(\mathbb{S}^{n-1})$ from being a round sphere is necessary. In this paper we make a connection between stability aspects for these two classes of mappings and the isoperimetric inequality, and this extra deficit should be interpreted in both cases as an isoperimetric type of deficit produced by the maps in consideration.

With the notations that we adopt in Section 2 our main result in the isometric case is the following.

**Theorem 1.2.** There exists $c_1 > 0$ so that for every $u \in W^{1,2}(\mathbb{S}^2; \mathbb{R}^3)$ there exists $O \in O(3)$ such that

$$\int_{\mathbb{S}^2} \left| \nabla_T u - OP_T \right|^2 \leq c_1 \left( \left\| (\sigma_2 - 1)_+ \right\|_{L^2(\mathbb{S}^2)} + \left( 1 - \left| V_3(u) \right| \right)_+ \right),$$

where $0 \leq \sigma_1 \leq \sigma_2$ are the principal stretches of $u$, i.e., the eigenvalues of $\sqrt{\nabla_T u^T \nabla_T u}$, and

$$V_3(u) := \int_{\mathbb{S}^2} \left\langle u, \partial_{\sigma_1} u \wedge \partial_{\sigma_2} u \right\rangle$$

is the signed volume of $u$. \hfill (1.3)

The first term on the right hand side of (1.3) is an $L^2$-isometric deficit of $u$ penalizing local stretches, while the second term (in the definition of which in (1.4) we use the identification between a 2-simple vector and its Hodge dual) represents in this setting the isoperimetric deficit of $u$. Since isometric maps preserve the surface area of $\mathbb{S}^2$, the latter reduces in this situation to the positive part of the excess in the signed volume produced by $u$. The exact analogue of Theorem 1.2 holds true also in dimension $n = 2$ (see Proposition 3.2 in Section 3) and, as long as $u$ satisfies an apriori bound on its homogeneous $W^{1,2(n-2)}$-seminorm, also in dimensions $n \geq 4$, as stated in the following.

**Theorem 1.3.** Let $n \geq 4$ and $M > 0$. There exists $c_{n,M} > 0$ so that for every $u \in \tilde{W}^{1,2(n-2)}(\mathbb{S}^{n-1}; \mathbb{R}^n)$ with $\left\| \nabla T u \right\|_{L^2(n-2)(\mathbb{S}^{n-1})} \leq M$, there exists $O \in O(n)$ such that

$$\int_{\mathbb{S}^{n-1}} \left| \nabla_T u - OP_T \right|^2 \leq c_{n,M} \left( \left\| (\sigma_{n-1} - 1)_+ \right\|_{L^2(\mathbb{S}^{n-1})} + \left( 1 - \left| V_n(u) \right| \right)_+ \right),$$

where $0 \leq \sigma_1 \leq \cdots \leq \sigma_{n-1}$ are again the eigenvalues of $\sqrt{\nabla_T u^T \nabla_T u}$, and the signed volume of $u$ is now

$$V_n(u) := \int_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\sigma_i} u \right\rangle.$$ \hfill (1.6)
Let us clarify that here we are using the identification

\[
\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \rangle := \langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \rangle = \det \begin{pmatrix}
\partial_{\tau_1} u^1 & \cdots & \partial_{\tau_1} u^n \\
\partial_{\tau_2} u^1 & \cdots & \partial_{\tau_2} u^n \\
\vdots & & \vdots \\
\partial_{\tau_{n-1}} u^1 & \cdots & \partial_{\tau_{n-1}} u^n
\end{pmatrix}.
\] (1.7)

The constant in (1.5) depends in principle now both on the dimension and on the apriori bound in the \(L^{2(n-2)}\)-norm of the gradient. The reason why this particular condition is introduced will be explained in Subsection 3.4. As we also justify by examples in Remark 3.3, the estimate is optimal in this setting, in the sense that the exponents with which the two deficits appear cannot generically be improved.

For the conformal case, due to the scaling invariant nature of the problem, the correct notions for the average conformal deficit and the isoperimetric one can be combined together. The main result when \(n = 3\) in this case is the following.

**Theorem 1.4.** There exists a constant \(c_2 > 0\) so that for every \(u \in W^{1,2}(S^2; \mathbb{R}^3)\) with \(V_3(u) \neq 0\) there exist a M"obius transformation \(\phi\) of \(S^2\) and \(\lambda > 0\) such that

\[
\int_{S^2} \frac{1}{\lambda} |\nabla_T u - \nabla_T \phi|^2 \leq c_2 \left( \frac{D_2(u)}{|V_3(u)|} \right)^{\frac{3}{2}} - 1,
\] (1.8)

where \(D_2(u) := \frac{1}{2} \int_{S^2} |\nabla_T u|^2\) is the Dirichlet energy of \(u\), and \(V_3(u)\) is again its signed volume, as in (1.4).

Of course the question is void when \(n = 2\), since conformality is a trivial notion for maps from \(S^1\) to \(\mathbb{R}^2\). One can directly check that the estimate (1.8) is again optimal in its setting, by considering the sequence of maps \(u_\sigma(x) := A_\sigma x : S^2 \mapsto \mathbb{R}^3\), where \(A_\sigma := \text{diag}(1,1,1+\sigma) \in \mathbb{R}^{3 \times 3}\) as \(\sigma \to 0^+\).

The use of this combined conformal-isoperimetric deficit is very natural in this framework. Indeed, generalizing to any dimension \(n \geq 3\) (for \(n = 3\) cf. [26, Theorem 2.4]), for \(u \in W^{1,n-1}(S^{n-1}; \mathbb{R}^n)\) the following inequalities, sometimes referred to as Wente’s isoperimetric inequality for mappings, are known to hold.

\[
\int_{S^{n-1}} \left( \frac{|\nabla_T u|^2}{n-1} \right)^{\frac{n-1}{n}} \geq \int_{S^{n-1}} \sqrt{\det(\nabla_T u^i \nabla_T u) e_1} \geq \int_{S^{n-1}} \langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \rangle \geq |V_n(u)|,
\] (1.9)

where \(D_{n-1}(u), P_{n-1}(u)\) are the first two integral quantities in the first line of the above inequalities, i.e., the \((n-1)\)-Dirichlet energy and the generalized area produced by \(u\) respectively. The first inequality in (1.9) follows from the arithmetic mean-geometric mean inequality for the eigenvalues of \(\sqrt{\nabla_T u^i \nabla_T u}\) and equality is achieved iff these eigenvalues coincide for \(\mathcal{H}^{n-1}\)-a.e. \(x \in S^{n-1}\), or equivalently, iff

\[
(\nabla_T u)^i \nabla_T u = \frac{|\nabla_T u|^2}{n-1} I_x \quad \mathcal{H}^{n-1}\text{-a.e. on } S^{n-1},
\]
i.e., iff $u$ is a generalized conformal map from $S^{n-1}$ to $\mathbb{R}^n$. The second inequality in (1.9) is the functional form of the isoperimetric inequality (cf. [1] inequality (2)), which can be proven first for smooth maps and can then be extended by density in $W^{1,n-1}(S^{n-1}; \mathbb{R}^n)$. Equality is achieved iff the image of $u$ is another round sphere in the $\mathcal{H}^{n-1}$-a.e. sense. In the case of a $C^1$ embedding, the inequality reduces of course to the classical Euclidean isoperimetric inequality for the open bounded set in $\mathbb{R}^n$ whose boundary is $u(S^{n-1})$.

Based on these simple observations, the combined conformal-isoperimetric deficit

$$\mathcal{E}_{n-1}(u) := \left| \frac{D_{n-1}(u)}{|V_n(u)|} \right|^{\frac{n}{n-1}} - 1,$$  \hspace{1cm} (1.10)

considered among maps $u \in W^{1,n-1}(S^{n-1}; \mathbb{R}^n)$ for which $V_n(u) \neq 0$, provides a correct notion of deficit when one seeks stability of the conformal group of $S^{n-1}$ among maps from $S^{n-1}$ into $\mathbb{R}^n$. Indeed, it is immediate that $\mathcal{E}_{n-1}$ is translation, rotation and scaling invariant, as well as invariant under precompositions with Möbius transformations of $S^{n-1}$. Moreover, as we have discussed above, $\mathcal{E}_{n-1}$ is nonnegative and vanishes iff $u$ is a generalized conformal map from $S^{n-1}$ onto another round sphere, which after translation and scaling can be taken to be $S^{n-1}$ again. If $d \in \mathbb{Z}$ would denote the degree of $u \in W^{1,n-1}(S^{n-1}; S^{n-1})$ (following the definitions in [5]), then

$$|d| = |V_n(u)| = \left| D_{n-1}(u) \right|^{\frac{n}{n-1}} \geq |V_n(u)|^{\frac{n}{n-1}} = |d|^{\frac{n}{n-1}}.$$  

Since the degree (for maps from $S^{n-1}$ to itself) takes integer values, we would have that either $d = 0$ or $d = \pm 1$, with the first case being excluded automatically, since by assumption $V_n(u) \neq 0$. Hence, absolute minimizers of $\mathcal{E}_{n-1}$ are degree $\pm 1$ conformal maps from $S^{n-1}$ into itself, up to a translation vector and a scaling factor, i.e., according to Theorem 1.1 Möbius transformations of $S^{n-1}$ up to translation and scaling. In this respect, Theorem 1.4 can be thought of as a sharp quantitative version of the previous statements for $n = 3$. At the core of its proof lies the study of the linearized version of the problem, since by the use of a contradiction compactness argument it is enough to show the theorem for maps that are sufficiently close to the $\text{id}_{S^2}$ in the $W^{1,2}$-topology. In this regime, and after a correct rescaling of $u$, if $w := u - \text{id}_{S^2}$ is the corresponding displacement field, one obtains the formal Taylor expansion

$$\mathcal{E}_2(w) = Q_3(w) + o \left( \int_{S^2} |\nabla_T w|^2 \right),$$  \hspace{1cm} (1.11)

where $Q_3(w)$ is the associated quadratic form, i.e., the second derivative of $\mathcal{E}_2$ at the $\text{id}_{S^2}$, defined explicitly later in (1.7). The next and main step of the proof is to examine the coercivity properties of the quadratic form $Q_3$. This is something that can actually be done in every dimension $n \geq 3$, the main ingredient for doing so being the fine interplay between the Fourier decomposition of a $W^{1,2}(S^{n-1}; \mathbb{R}^n)$-vector field into $\mathbb{R}^n$-valued spherical harmonics and the properties of the linear first order differential operator associated to the second derivative of $V_n$ at the $\text{id}_{S^{n-1}}$.

To be more precise, as we thoroughly examine in Subsection 4.2 for the case $n = 3$, and in Subsection 5.1 for the higher dimensional case, if one rescales $u$ properly, sets $w := u - \text{id}_{S^{n-1}}$ and expands $\mathcal{E}_{n-1}(u)$ in (1.10) around the $\text{id}_{S^{n-1}}$, then the resulting quadratic form

$$Q_n(w) := \frac{n}{2(n-1)} \int_{S^{n-1}} \langle |\nabla_T w|^2 + \frac{n-3}{n-1} (\text{div}_{S^{n-1}} w)^2 \rangle - \frac{n}{2} \int_{S^{n-1}} \langle w, (\text{div}_{S^{n-1}} w)x - \sum_{j=1}^{n} x_j \nabla_T w^j \rangle$$  \hspace{1cm} (1.12)
has finite-dimensional kernel and its dimension actually coincides with that of the conformal group of \( S^n \).

Moreover, when considered in the correct space (see the definitions of the spaces \( H_n, (H_{n,k,i})_{k \geq 1, i=1,2,3} \) in equation (4.15) and Theorem 4.7 in Subsection 4.2), the form \( Q_n \) satisfies the following coercivity estimate.

**Theorem 1.5.** Let \( n \geq 3 \). There exists a constant \( C_n > 0 \) such that for every \( w \in H_n \),

\[
Q_n(w) \geq C_n \int_{S^{n-1}} |\nabla_T w - \nabla_T (\Pi_{n,0} w)|^2,
\]

(1.13)

where \( H_{n,0} := H_{n,1,2} \oplus H_{n,2,3} \) is the kernel of \( Q_n \) in \( H_n \), and \( \Pi_{n,0} : H_n \mapsto H_{n,0} \) is the \( W^{1,2} \)-orthogonal projection of \( H_n \) onto \( H_{n,0} \).

When \( n = 3 \), the optimal constant in (1.13) can actually be calculated explicitly. Since \( H_{n,0} \) turns out to be isomorphic to the Lie algebra of infinitesimal Möbius transformations of \( S^{n-1} \), an application of the Inverse Function Theorem together with a topological argument (given in Lemma 4.13) will finally allow us to infer the nonlinear estimate (1.8) from the linear one (1.13) in the \( W^{1,2} \)-close to the \( \text{id}_{S^2} \)-regime, and hence conclude with Theorem 1.4.

It is maybe worth remarking here that, in contrast to (1.11), in dimensions \( n \geq 4 \) a formal expansion of the combined conformal-isoperimetric deficit around the \( \text{id}_{S^{n-1}} \) yields

\[
E_{n-1}(u) = Q_n(w) + O \left( \int_{S^{n-1}} |\nabla_T w|^3 \right).
\]

(1.14)

Since the higher order term is now cubic in \( \nabla_T w \), the linear estimate (1.13) alone would only imply the nonlinear one (following exactly the same steps of proof as those described in Subsections 4.1, 4.3 and 4.4 for the case \( n = 3 \)) only in the \( W^{1,\infty} \)-close to the \( \text{id}_{S^{n-1}} \)-regime (see Remark 5.5), as stated in the following.

**Corollary 1.6.** Let \( n \geq 4 \). There exist constants \( \theta \in (0,1) \) (sufficiently small) and \( c_{n-1} > 0 \) such that the following statement holds. For every \( u \in W^{1,\infty}(S^{n-1}; \mathbb{R}^n) \) with \( \| \nabla_T u - P_T \|_{L^{\infty}(S^{n-1})} \leq \theta \ll 1 \), there exist a Möbius transformation \( \phi \) of \( S^{n-1} \) and \( \lambda > 0 \) such that

\[
\int_{S^{n-1}} \left| \frac{1}{\lambda} \nabla_T u - \nabla_T \phi \right|^2 \leq c_{n-1} E_{n-1}(u),
\]

(1.15)

where \( E_{n-1} \) is defined in (1.10).

**Remark 1.7.** An interesting question would be if the local statement of the above Corollary can be improved to a global one, possibly via a PDE argument. However, in the case of maps \( u : S^{n-1} \mapsto \mathbb{R}^n \), one cannot perform something like an \( n \)-harmonic replacement trick, as for instance in [22] (or \( F \)-harmonic, harmonic in the setting of [10],[11] respectively), since \( S^{n-1} \) is boundaryless, and there are of course no boundary conditions to relate to the replacement map. It seems that a penalization argument in the spirit of the selection principle devised in [7] (for the optimal quantitative isoperimetric inequality) could be more promising in that direction, which is an interesting question for future investigation.
where

\[ Q_{n, \text{conf}}(w) := \frac{n}{n-1} \int_{S^{n-1}} \left( (P^1_T \nabla T w)_{\text{sym}} - \frac{\text{div}_{S^{n-1}} w}{n-1} I_x \right)^2 \]

is the quadratic form associated to the nonlinear conformal deficit \[\left[ \frac{P_{n-1}(u)}{V_n(u)} \right]^{n} - 1 \geq 0,\]

and

\[ Q_{n, \text{isop}}(w) := \frac{n}{2(n-1)} \left[ \int_{S^{n-1}} |\nabla T w|^2 + (\text{div}_{S^{n-1}} w)^2 - 2 \left( (P^1_T \nabla T w)_{\text{sym}} \right) - Q_V(w) \right] (1.16) \]

is the one associated to the nonlinear isoperimetric deficit \[\left[ \frac{P_{n-1}(u)}{V_n(u)} \right]^{n} - 1 \geq 0.\] Actually, an estimate like \[1 \text{.}13\] holds true for every positive combination of the two forms \(Q_{n, \text{conf}}\) and \(Q_{n, \text{isop}}\).

Finally, as we mention in Subsection 5.3, a similar linear stability phenomenon holds true in the isometric case as well, namely one can prove the following.

**Theorem 1.8.** Let \( n \geq 2 \). For every \( \alpha > 0 \) there exists a constant \( C_{n,\alpha} > 0 \) such that for every map \( w \in W^{1,2}(S^{n-1}; \mathbb{R}^n) \),

\[ \alpha Q_{n, \text{isom}}(w) + Q_{n, \text{isop}}(w) \geq C_{n,\alpha} \int_{S^{n-1}} |\nabla T w|^2 - [\nabla w_h(0)]_{\text{skew}} P_T^1|^2, \]

where,

\[ Q_{n, \text{isom}}(w) := \int_{S^{n-1}} |(P^1_T \nabla T w)_{\text{sym}}|^2 \]

is the quadratic form associated to the full \( L^2 \)-isometric deficit \( \int_{S^{n-1}} |\nabla T u'|^2 - I_x|^2 \), \( Q_{n, \text{isop}} \) is as in \[1 \text{.}16\], and \( w_h : B_1 \mapsto \mathbb{R}^n \) denotes the (componentwise) harmonic continuation of \( w \) in the interior of \( B_1 \).

The structure of the paper is the following. In Section 2 we introduce some notations that we are going to use in the subsequent sections. In Section 3 we give in steps the proof of Theorem 1.2 and remark on the adaptations needed to prove its generalization in higher dimensions, i.e., Theorem 1.3. In Section 4 we give again in steps the proof of Theorem 1.4. Building upon the analysis that we perform in Subsection 4.2 in Section 5 we prove the linear stability estimates stated in Theorems 1.5 and 1.8 in all dimensions. In Appendix A we first exhibit a short, intrinsic and to our knowledge, new proof of Liouville’s Theorem 1.1, as well as a related compactness result that can be proven by a slight perturbation of the idea. In Appendix B we include just for the convenience of the reader a detailed derivation of some integral identities for Jacobians, as well as the Taylor expansions of the geometric quantities that appear in the main body of the paper. Finally, in Appendix C we collect some basic facts from the theory of spherical harmonics that we are using.

## 2 Notation

The following standard notation will be adopted throughout the paper.

- \( \{e_i\}_{i=1}^n \), \( \langle \cdot, \cdot \rangle \), \( |\cdot| \) the Euclidean orthonormal basis, inner product, norm in \( \mathbb{R}^n \)
- \( A^t \) the transpose of a matrix or the adjoint of the corresponding linear map
- \( \text{Sym}(n), Skew(n) \) the space of \( n \times n \) symmetric, skew-symmetric matrices respectively
A_{\text{sym}}, A_{\text{skew}} \quad \text{the symmetric, skew-symmetric part of a matrix } A \in \mathbb{R}^{n \times n} \text{ respectively}

\{\tau_1, \ldots, \tau_{n-1}\} \quad \text{a positively oriented local orthonormal frame for } T_x \mathbb{S}^{n-1}, \text{ so that for every } x \in \mathbb{S}^{n-1}

\{\tau_1(x), \ldots, \tau_{n-1}(x), x\} \quad \text{is a positively oriented orthonormal system of } n \text{ vectors in } \mathbb{R}^n

\omega_n \quad \text{the volume of the unit ball } B_1 \text{ in } \mathbb{R}^n

dv_g \quad \text{the standard } (n-1)\text{-volume form on } \mathbb{S}^{n-1}

\mathcal{H}^k \quad \text{the } k\text{-dimensional Hausdorff measure}

O(n), SO(n) \quad \text{the orthogonal, special orthogonal group of } \mathbb{R}^n \text{ respectively}

Isom_{(+)}(\mathbb{S}^{n-1}) \quad \text{the group of rigid motions of } \mathbb{S}^{n-1} \text{ (the orientation-preserving ones respectively)}

Conf_{(+)}(\mathbb{S}^{n-1}) \quad \text{the group of Möbius transformations of } \mathbb{S}^{n-1} \text{ (the orientation-preserving ones respectively)}

I_x \quad \text{the identity transformation on } T_x \mathbb{S}^{n-1}

\nabla_{T} u \quad \text{the tangential gradient of } u : \mathbb{S}^{n-1} \to \mathbb{R}^n, \text{ represented in local coordinates by the } \n \times (n-1) \text{ matrix with entries } \langle \nabla_T u, \tau_j \rangle

P_T \quad \nabla_T \text{id}_{\mathbb{S}^{n-1}}

d_x u \quad \text{the intrinsic gradient of a map } u : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}, \text{ viewed as a linear map } 
\quad d_x u : T_x \mathbb{S}^{n-1} \to T_{u(x)} \mathbb{S}^{n-1} \text{ with respect to the frame } \{\tau_1, \ldots, \tau_{n-1}\}

\partial_\tau f \quad \text{the radial derivative of a map } f : \mathbb{B}_1 \to \mathbb{R}^n \text{ on } \mathbb{S}^{n-1}

\text{div}_{\mathbb{S}^{n-1}} u, \Delta_{\mathbb{S}^{n-1}} u \quad \text{the tangential divergence, Laplace-Beltrami operator of a map } u : \mathbb{S}^{n-1} \to \mathbb{R}^n

C^k \quad \text{the space of } k\text{-times continuously differentiable maps, } k \in \mathbb{N}

L^p, W^{1,p} \quad \text{the standard Lebesgue or Sobolev spaces (on } \mathbb{S}^{n-1} \text{) respectively, for } 1 \leq p < \infty.

The norms are taken with respect to the normalized } \mathcal{H}^{n-1}\text{-measure, to simplify some dimensional constants appearing later in the content}

W^{1,\infty}(\mathbb{S}^{n-1}; \mathbb{R}^n) \quad \text{the space of Lipschitz maps from } \mathbb{S}^{n-1} \text{ to } \mathbb{R}^n; ||u||_{W^{1,\infty}} := \max \{||u||_{L^\infty}, ||\nabla_T u||_{L^\infty}\}

\sim_{M_1, M_2, \ldots} \quad \text{the corresponding equality, inequality is valid up to a constant that is allowed to vary from line to line but depends only on the parameters } M_1, M_2, \ldots, \text{ or only on the dimension when the subscripts are absent.}

\begin{align*}
c, C > 0 \quad \text{universal constants whose value is allowed to vary from line to line and place to place but depend in any case only on the dimension.}
\end{align*}

3 \text{ The isometric case: Proof of Theorem 1.3}

In what follows, the } L^2\text{-isometric deficit of a map } u \in W^{1,2}(\mathbb{S}^2; \mathbb{R}^3) \text{ that we are using is denoted by}

\delta(u) := ||(\sigma_2 - 1)^+||_{L^2(\mathbb{S}^2)}, \quad (3.1)

where } 0 \leq \sigma_1 \leq \sigma_2 \text{ are the principal stretches of } u, \text{ i.e., the eigenvalues of } \sqrt{\nabla_T u^T \nabla_T u}.

Note that } \delta(u) = 0 \text{ whenever } u \text{ is a short map, i.e., } u \in W^{1,\infty}(\mathbb{S}^2; \mathbb{R}^3) \text{ with } \nabla_T u^T \nabla_T u \leq I_x, \text{ } \mathcal{H}^2\text{-a.e. on } \mathbb{S}^2 \text{ in the sense of quadratic forms. In general,}

\delta(u) \leq ||\sigma_2 - 1||_{L^2(\mathbb{S}^2)} \leq ||\sqrt{\nabla_T u^T \nabla_T u} - I_x||_{L^2(\mathbb{S}^2)} \leq \sqrt{2}||\sigma_2 - 1||_{L^2(\mathbb{S}^2)}, \quad (3.2)
so that (having in mind the Nash-Kuiper Theorem, cf. [16], [20]) the deficit \( \delta(u) \) is sharper than the full \( L^2 \)-isometric deficit

\[
\delta_{\text{isom}}(u) := \left\| \sqrt{\nabla T u' \nabla T u} - I_x \right\|_{L^2(S^2)},
\]

since it only penalizes local stretches under \( u \). The isoperimetric deficit (or the positive part of the excess in volume) in this setting is denoted by

\[
\varepsilon(u) := \left( 1 - |V_3(u)| \right)_+.
\]

Before presenting the proof of the result, let us make some preliminary remarks.

**Remark 3.1.** (i) If \( u \in W^{1,2}(S^2; \mathbb{R}^3) \) is a globally short map, then \( \delta(u) = 0 \). Moreover, since in this case \( |\partial_r u \wedge \partial_r u| \leq 1 \) and \( |\nabla T u| \leq \sqrt{2} \mathcal{H}^2 \)-a.e. on \( S^2 \), by the Cauchy-Schwarz inequality and the sharp Poincare inequality on \( S^2 \) (equality in which is achieved for restrictions on \( S^2 \) of affine maps of \( \mathbb{R}^3 \), see (C.5) in Appendix C),

\[
|V_3(u)| = \int_{S^2} \langle u - f_{S^2} u, \partial_r u \wedge \partial_r u \rangle \leq \int_{S^2} \left| u - f_{S^2} u \right| \leq \left( \int_{S^2} \left| u - f_{S^2} u \right|^2 \right)^{\frac{1}{2}} \leq \left( \frac{1}{2} \int_{S^2} |\nabla T u|^2 \right)^{\frac{1}{2}} \leq 1,
\]

that is, \( \varepsilon(u) = 1 - |V_3(u)| \). This is something that could also be seen just by using the isoperimetric inequality in this case. Hence, for globally short maps only the excess in volume is present in the right hand side of the stability estimate (1.3).

(ii) On the other hand, if \( u \in W^{1,2}(S^2; \mathbb{R}^3) \) is volume-increasing in the sense that \( |V_3(u)| \geq 1 \), then \( \varepsilon(u) = 0 \), and only the isometric deficit \( \delta(u) \) is present in the right hand side of (1.3).

(iii) In all other cases, i.e., if \( u \in W^{1,2}(S^2; \mathbb{R}^3) \) is not globally short and not volume-increasing, both deficits are present in the estimate. It is also immediate that one cannot have simultaneously a globally short map \( u \) that is volume-increasing, unless \( u \) is a rigid motion of \( S^2 \), something that can be directly verified by checking the equality cases in (3.5).

As we also mentioned in the Introduction, (1.3) is optimal in the norm appearing on the left hand side and the deficits on the right hand side, i.e., the exponent 1 with which \( \delta(u) \) and \( \varepsilon(u) \) appear in the estimate cannot generically be improved. Examples showing the optimality of the exponents can easily be constructed even in dimension \( n = 2 \), where the exact analogue of Theorem 1.2 becomes

**Proposition 3.2.** There exists a constant \( c_0 > 0 \) so that for every \( u \in W^{1,2}(S^1; \mathbb{R}^2) \) there exists \( O \in O(2) \) such that

\[
\int_{S^1} |\partial_r (u - O \text{id}_{S^1})|^2 \leq c_0 \left( \left\| (|\partial_r u| - 1)_+ \right\|_{L^2(S^1)} + \left( 1 - \int_{S^1} \langle u, (\partial_r u)^{-1} \rangle \right)_+ \right).
\]

Here, \( \partial_r u \) denotes the tangential derivative of \( u \) along \( S^1 \). The previous proposition can be proven in exactly the same way as Theorem 1.2, following the arguments of the next subsections. As the reader might observe later, the Lipschitz truncation argument of Subsection 3.1 is even simpler in the case \( n = 2 \), because the signed volume \( V_2(u) := \int_{S^1} \langle u, (\partial_r u)^{-1} \rangle \) is of first order in \( \partial_r u \).

Keeping the notation \( \delta(u) \) and \( \varepsilon(u) \) for the isometric and the isoperimetric deficit also when \( n = 2 \), two instructive examples for the optimality of the exponents are given in the next remark.
Remark 3.3. (i) For $0 < \sigma \ll 2\pi$, let $u_\sigma : S^1 \to \mathbb{R}^2$ be defined in polar coordinates via

$$u_\sigma(\theta) := \begin{cases} (\cos \theta, \sin \theta); & 0 \leq \theta < \frac{3\pi}{2} - \frac{\sigma}{2}, \\ (\cos \theta, 2\sin(\frac{3\pi}{2} - \frac{\sigma}{2}) - \sin \theta); & \frac{3\pi}{2} - \frac{\sigma}{2} \leq \theta < \frac{3\pi}{2} + \frac{\sigma}{2}, \\ (\cos \theta, \sin \theta); & \frac{3\pi}{2} + \frac{\sigma}{2} \leq \theta < 2\pi \end{cases}.$$  \hfill (3.7)

For each $\sigma \in [0, 2\pi)$ the map $u_\sigma$ is isometric, being essentially the identity transformation, except for a small circular arc of angle $\sigma$, where it is a flip with respect to the horizontal line at height $y_0 = \sin(\frac{3\pi}{2} - \frac{\sigma}{2})$. Hence, $\delta(u_\sigma) = 0$ for every $\sigma \in [0, 2\pi)$. Obviously, $\partial_\tau u_\sigma \to \partial_\tau \text{id}_{S^1}$ strongly in $L^2(S^1; \mathbb{R}^2)$ as $\sigma \to 0^+$, and one can easily obtain that

$$\int_{S^1} |\partial_\tau u_\sigma - \partial_\tau \text{id}_{S^1}|^2 \sim \int_0^{2\pi} |\partial_\theta u_\sigma - \partial_\theta \text{id}_{S^1}|^2 = \int_{\frac{3\pi}{2} - \frac{\sigma}{2}}^{\frac{3\pi}{2} + \frac{\sigma}{2}} |(-\sin \theta, -\cos \theta) - (-\sin \theta, \cos \theta)|^2
\sim 4 \int_{\frac{3\pi}{2} - \frac{\sigma}{2}}^{\frac{3\pi}{2} + \frac{\sigma}{2}} \cos^2(\theta) = 2(\sigma - \sin \sigma) = O(\sigma^3), \quad \text{for } 0 < \sigma \ll 2\pi.$$  

On the other hand, using elementary plane-geometry formulas for the area of circular triangles, we can compute the area of the double arc-region of the unit disc missed by $u_\sigma$, so that also

$$\varepsilon(u_\sigma) = \frac{2}{\pi} \left( \pi : \frac{\sigma}{2\pi} - \frac{1}{2} \sin \sigma \right) = \frac{1}{\pi} (\sigma - \sin \sigma) = O(\sigma^3), \quad \text{for } 0 < \sigma \ll 2\pi,$$

which reveals the optimality of the exponent of $\varepsilon(u)$ in the estimate $(3.6)$.

(ii) Identify now $S^1$ with the interval $[0, 1]$ by identifying the endpoints. For $0 < \sigma \ll 1$, consider the maps $f_\sigma : [0, 1] \to [0, 1]$, defined as follows.

$$f_\sigma(t) := \begin{cases} t; & 0 \leq t < \sigma, \\ 2\sigma - t; & \sigma \leq t < 2\sigma, \\ -\frac{2\sigma}{1 - 2\sigma} + \frac{1}{1 - 2\sigma}t; & 2\sigma \leq t < 1 \end{cases},$$  \hfill (3.8)

and let $u_\sigma : S^1 \to S^1$ be the corresponding maps defined on the unit circle. Obviously, $\varepsilon(u_\sigma) = 0$ for every $\sigma \in [0, 2\pi)$. Geometrically, the maps $u_\sigma$ travel back and forth, and produce a triple cover of a small $\sigma$-arc, locally stretching $S^1$. With similar calculations as before,

$$\int_{S^1} |\partial_\tau u_\sigma - \partial_\tau \text{id}_{S^1}|^2 \sim \int_0^1 |f_\sigma'(t) - 1|^2 = \int_\sigma^{2\sigma} (-2)^2 + \int_{2\sigma}^1 \left( \frac{1}{1 - 2\sigma} - 1 \right)^2
\sim 4\sigma + \frac{4\sigma^2}{1 - 2\sigma} \sim \sigma + O(\sigma^2) = O(\sigma), \quad \text{for } 0 < \sigma \ll 1.$$  

Moreover,

$$\delta^2(u_\sigma) \sim \int_0^1 |(|f_\sigma'(t)| - 1)|^2 = \int_{2\sigma}^1 \left( \frac{1}{1 - 2\sigma} - 1 \right)^2 \sim \frac{4\sigma^2}{1 - 2\sigma},$$

which reveals the optimality of the exponent of $\delta(u)$ in the estimate $(3.6)$ in the generic setting. The geometric reason behind this, is the fact that the deficit $\delta(u)$ (as well as the full $L^2$-isometric deficit $\delta_{\text{isom}}(u)$) does not penalize changes in the orientation neither extrinsically, i.e., flips in ambient space, nor intrinsically, when $u$ is seen as a map from the sphere onto its image.
When $n = 3$ (and also in higher dimensions) one can construct similar examples as in (3.7), (3.8). For
instance, in the first case one can consider maps that are the identity outside a small geodesic ball of $S^{n-1}$
and inside being again flips in $\mathbb{R}^n$ with respect to the appropriate affine hyperplane. In the second case,
one can rotate the previous one-dimensional example around a fixed axis.

We are now ready to present the proof of Theorem 1.2 in steps. For the most part, by straightforward
modifications that mainly regard the change of some dimensional constants in the estimates and of some
purely algebraic expressions, the arguments are valid in all dimensions, and can be used to prove Theorem
1.3 as well. We will come back to that issue in Subsection 3.4.

3.1 Reduction to Lipschitz mappings

As in the pioneering geometric rigidity result of G. Friesecke, R.D. James and S. Müller (cf. [11, Theorem 3.1]), the first step is to justify why it suffices to work with maps with a universal upper bound
on their Lipschitz constant. This is achieved through the use of the following standard truncation lemma.

**Lemma 3.4.** There exists $c > 0$ so that for every $u \in W^{1,2}(S^2; \mathbb{R}^3)$ and every $M > 0$, there exists
$u_M \in W^{1,\infty}(S^2; \mathbb{R}^3)$ such that

$$
\begin{align*}
(i) \quad & \|\nabla Tu_M\|_{L^\infty} \leq cM, \\
(ii) \quad & \mathcal{H}^2(\{x \in S^2; u(x) \neq u_M(x)\}) \leq \frac{c}{M^2} \int_{\{\|\nabla Tu\| > M\}} |\nabla Tu|^2, \\
(iii) \quad & \int_{S^2} |\nabla Tu - \nabla Tu_M|^2 \leq c \int_{\{\|\nabla Tu\| > M\}} |\nabla Tu|^2.
\end{align*}
$$

The proof of this lemma can be performed as for the corresponding statement in the bulk (cf. [11, Proposition A.1]), since it relies basically on a partition of unity argument. With the use of it we can now
prove the following.

**Lemma 3.5.** For $u \in W^{1,2}(S^2; \mathbb{R}^3)$ let $u_M$ be its Lipschitz truncation of Lemma 3.4 for $M := 2\sqrt{2}$. Then,

$$
\delta(u_M) \lesssim \delta(u) \quad \text{and} \quad \varepsilon(u_M) \lesssim \delta(u) + \varepsilon(u).
$$

**Proof.** If $\delta(u) > 1$, recalling the definitions of the deficits in (3.2)-(3.4), we trivially have

$$
\delta^2(u_M) \leq \delta^2_{\text{isom}}(u_M) \leq 2 \int_{S^2} \left( \left| \sqrt{\nabla Tu_M'} \cdot \nabla Tu_M \right|^2 + |I_2|^2 \right) \leq 2(c^2 M^2 + 2) \lesssim \delta^2(u),
$$

and

$$
\varepsilon(u_M) \leq 1 \leq \delta(u) \leq \delta(u) + \varepsilon(u),
$$

so we may assume without loss of generality that $0 \leq \delta(u) \leq 1$. With the notation we have employed in
(3.1), $|\nabla Tu|^2 = \sigma_1^2 + \sigma_2^2 \leq 2\sigma_2^2$, and therefore in this case we also have the upper bound

$$
\int_{S^2} |\nabla Tu|^2 \leq \frac{1}{3\omega_3} \int_{\{\sigma_2 \leq 1\}} |\nabla Tu|^2 + \frac{1}{3\omega_3} \int_{\{\sigma_2 > 1\}} |\nabla Tu|^2 \leq 2 + \frac{2}{3\omega_3} \int_{\{\sigma_2 > 1\}} \sigma_2^2 \leq 2 + \frac{4}{3\omega_3} \int_{\{\sigma_2 > 1\}} ((\sigma_2 - 1)^2 + 1) \leq 6 + 4\delta^2(u) \leq 10.
$$

By a standard argument, using Lemma 3.4, we also obtain

\[
M^2 \mathcal{H}^2 \left( \{ x \in S^2 : u(x) \neq u_M(x) \} \right) \leq c \int_{\{ |\nabla_T u| > M \}} |\nabla_T u|^2 \lesssim \delta^2(u),
\]

(3.11)

\[
\int_{S^2} |\nabla_T u - \nabla_T u_M|^2 \leq c \int_{\{ |\nabla_T u| > M \}} |\nabla_T u|^2 \lesssim \delta^2(u).
\]

Indeed, in the set \( \{ x \in S^2 : |\nabla_T u(x)| > M := 2\sqrt{2} \} \) we have \( \sigma_2 \geq \frac{1}{\sqrt{2}} |\nabla_T u| > 2, \) so in this set we can estimate pointwise,

\[
(\sigma_2 - 1)_+ = \sigma_2 - 1 \geq \frac{|\nabla_T u|}{\sqrt{2}} - 1 > \frac{|\nabla_T u|}{2\sqrt{2}} > \frac{|\nabla_T u|}{2\sqrt{2}},
\]

and then the final estimates in (3.11) follow immediately.

For the first estimate in (3.9) the argument is now elementary. Labelling \( 0 \leq \sigma_{M,1} \leq \sigma_{M,2} \) the eigenvalues of \( \sqrt{\nabla_T u_M^t \nabla_T u_M} \), using (3.11) and the fact that \( \{ u_M = u \} \subseteq \{ \nabla_T u_M = \nabla_T u \} \) in the \( \mathcal{H}^2 \)-a.e. sense, we can estimate

\[
\delta^2(u_M) \sim \int_{\{ \sigma_{M,2} > 1 \}} (\sigma_{M,2} - 1)^2 = \int_{\{ u = u_M \} \cap \{ \sigma_{M,2} > 1 \}} (\sigma_{M,2} - 1)^2 + \int_{\{ u \neq u_M \} \cap \{ \sigma_{M,2} > 1 \}} (\sigma_{M,2} - 1)^2
\]

\[
\leq \int_{\{ u = u_M \} \cap \{ \sigma_{M,2} > 1 \}} (\sigma_2 - 1)^2 + \int_{\{ u \neq u_M \} \cap \{ \sigma_{M,2} > 1 \}} (\sigma_{M,2}^2 + 1)
\]

\[
\leq \int_{\{ \sigma_2 > 1 \}} (\sigma_2 - 1)^2 + (c^2 M^2 + 1) \mathcal{H}^2 (\{ u_M \neq u \}) \lesssim \delta^2(u).
\]

For the second desired estimate in (3.9) we observe that if \( |V_3(u_M)| > 1 \) then \( \varepsilon(u_M) = 0 \leq \varepsilon(u) \), so we may assume without loss of generality that \( |V_3(u_M)| \leq 1 \). Then,

\[
\varepsilon(u_M) = 1 - |V_3(u_M)| \lesssim \varepsilon(u) + |V_3(u)| - |V_3(u_M)| \leq \varepsilon(u) + |V_3(u) - V_3(u_M)|,
\]

i.e., it suffices to control the absolute value of the difference between the corresponding signed volumes. Towards this end, denoting by

\[
\overline{v} := \int_{S^2} v, \quad \text{for } v \in W^{1,2}(S^2; \mathbb{R}^3),
\]

(3.14)

one can easily verify that

\[
V_3(u) - V_3(u_M) = V_3(u - u_M) + R_1(u, u_M) + R_2(u, u_M) + R_3(u, u_M) + R_4(u, u_M),
\]

(3.15)

where

\[
R_1(u, u_M) := \int_{S^2} \left( (u - u_M) - \overline{u - u_M}, \partial_{r_1} u_M \wedge \partial_{r_2} u \right),
\]

\[
R_2(u, u_M) := \int_{S^2} \left( (u - u_M) - \overline{u - u_M}, \partial_{r_1} (u - u_M) \wedge \partial_{r_2} u_M \right),
\]

\[
R_3(u, u_M) := \int_{S^2} \left( u_M - \overline{u_M}, \partial_{r_1} u_M \wedge \partial_{r_2} (u - u_M) \right),
\]

\[
R_4(u, u_M) := \int_{S^2} \left( u_M - \overline{u_M}, \partial_{r_1} (u - u_M) \wedge \partial_{r_2} u \right).
\]

(3.16)
We can now estimate each term on the right hand side of (3.15) separately. For the first one, by the isoperimetric inequality (see (1.9) for $n = 3$) and the second estimate in (3.11), we obtain

$$|V_3(u - u_M)| \lesssim \left( \frac{1}{2} \int_{S^2} |\nabla_T u - \nabla_T u_M|^2 \right)^{\frac{3}{2}} \lesssim \delta^3(u).$$

(3.17)

To estimate the terms $(R_i(u, u_M))_{i=1,...,4}$ we can now use the properties of the Lipschitz truncation $u_M$ provided by Lemma 3.4, the Cauchy-Schwarz inequality and the sharp Poincare inequality on $S^2$ (see (C.5) in Appendix C), as well as the estimates (3.10) and (3.11), in order to estimate each of the remaining terms in (3.15) as follows.

$$|R_1(u, u_M)| \lesssim M\|u - u_M\|_L^2 \|\nabla_T u\|_L^2 \lesssim \|\nabla_T u - \nabla_T u_M\|_L^2 \lesssim \delta(u),$$

$$|R_2(u, u_M)| \lesssim M\|u - u_M\|_L^2 \|\nabla_T u - \nabla_T u_M\|_L^2 \lesssim \|\nabla_T u - \nabla_T u_M\|_L^2 \lesssim \delta^2(u),$$

$$|R_3(u, u_M)| \lesssim M^2 \int_{u \neq u_M} |\nabla_T u - \nabla_T u_M| \lesssim \sqrt{H^2(\{u \neq u_M\})} \|\nabla_T u - \nabla_T u_M\|_L^2 \lesssim \delta^2(u),$$

$$|R_4(u, u_M)| \lesssim \|u - u_M\|_L^\infty \|\nabla_T u\|_L^1 \|\nabla_T u - \nabla_T u_M\|_L^2 \lesssim M\delta(u) \lesssim \delta(u).$$

(3.18)

By (3.16)-(3.18), and since we have assumed without loss of generality that $0 \leq \delta(u) \leq 1$, the expansion (3.15) implies that

$$|V_3(u) - V_3(u_M)| \lesssim \|\nabla_T u - \nabla_T u_M\|_L^2 \lesssim \delta(u),$$

(3.19)

and then (3.13) yields the desired estimate (3.9) for the isoperimetric deficit. \qed

In view of Lemma 3.5, fixing from now on $M := 2\sqrt{2}$, we easily see that if the estimate (1.3) holds true for the Lipschitz map $u_M$ for some $O \in O(3)$, then it also holds true for $u$ with the same $O$, up to changing the constant in its right hand side. It therefore suffices to prove Theorem 1.2 for maps $u \in W^{1,\infty}(S^2; \mathbb{R}^3)$ whose Lipschitz constant is apriori bounded from above by $cM$, where $c > 0$ is the constant of Lemma 3.4.

### 3.2 Further reduction to maps $W^{1,2}$-close to the id$_{S^2}$

Having reduced our attention to maps that enjoy an apriori Lipschitz bound, we show in this subsection that for our purposes, we can further assume without loss of generality that the maps in consideration are sufficiently close to the id$_{S^2}$ in the $W^{1,2}$-topology. To do so, we first prove a qualitative analogue of Theorem 1.2. Recalling the notations introduced in (3.1), (3.4) and (3.14), we have.

**Lemma 3.6.** Let $(u_k)_{k \in \mathbb{N}} \subset W^{1,\infty}(S^2; \mathbb{R}^3)$ be such that $\sup_{k \in \mathbb{N}} \|\nabla_T u_k\|_{L^\infty} \leq cM$, and suppose that

$$\lim_{k \to \infty} (\delta(u_k) + \varepsilon(u_k)) = 0.$$  

(3.20)

Then, there exists $O \in O(3)$ so that up to a non-relabeled subsequence,

$$u_k - \overline{u}_k \to O\text{id}_{S^2} \ \text{strongly in} \ W^{1,2}(S^2; \mathbb{R}^3).$$

(3.21)

**Proof.** We can obviously assume without loss of generality that $\overline{u}_k := f_{S^2} u_k = 0$ for all $k \in \mathbb{N}$. Hence, the sequence $(u_k)_{k \in \mathbb{N}}$ is uniformly bounded in $W^{1,2}(S^2; \mathbb{R}^3)$, and up to passing to a non-relabeled subsequence,
converges weakly in $W^{1,2}(S^2;\mathbb{R}^3)$ and also pointwise $\mathcal{H}^2$-a.e. to a map $u \in W^{1,2}(S^2;\mathbb{R}^3)$ with $\pi = 0$. By lower semicontinuity of the Dirichlet energy under weak $W^{1,2}$-convergence, we further have that

$$
\frac{1}{2}\int_{S^2} |\nabla_T u|^2 \leq \liminf_{k \to \infty} \frac{1}{2} \int_{S^2} |\nabla_T u_k|^2 \leq 1.
$$

(3.22)

The last inequality in (3.22) is justified by the following estimates.

$$
\frac{1}{2} \int_{S^2} |\nabla_T u_k|^2 = \frac{\int_{\{0 \leq \sigma_{k,1} \leq 1\}} |\nabla_T u_k|^2 + \int_{\{\sigma_{k,1} > 1\}} |\nabla_T u_k|^2}{6\omega_3} \leq \frac{\mathcal{H}^2(\{0 \leq \sigma_{k,1} \leq 1\}) + \int_{\{\sigma_{k,1} > 1\}} \sigma_{k,1}^2}{3\omega_3}
$$

$$
= 1 + \frac{1}{3\omega_3} \int_{\{\sigma_{k,1} > 1\}} (\sigma_{k,1}^2 - 1) \leq 1 + \frac{(\|\sigma_{k,2}\|_{L^\infty} + 1)}{3\omega_3} \int_{\{\sigma_{k,2} > 1\}} (\sigma_{k,2} - 1) +
$$

$$
\leq 1 + (cM + 1) \left( \int_{S^2} (\sigma_{k,2} - 1)_+^2 \right)^{\frac{1}{2}} \leq 1 + c\delta(u_k).
$$

(3.23)

In a similar manner, we can use again the assumption that $\sup_{k \in \mathbb{N}} \|\nabla_T u_k\|_{L^\infty} \leq cM$, and the fact that the determinant is a Lipschitz function, to estimate also

$$
\int_{S^2} |\partial_{\tau_1} u_k \wedge \partial_{\tau_2} u_k|^2 \leq 1 + c\delta(u_k).
$$

(3.24)

Since $0 \leq 1 - \varepsilon(u_k) \leq |V_3(u_k)|$ and $\overline{u_k} = 0$, by the sharp Poincaré inequality on $S^2$ (see again (C.5) in Appendix C) and the estimates (3.23), (3.24), we obtain

$$
0 \leq 1 - \varepsilon(u_k) \leq \int_{S^2} \langle u_k, \partial_{\tau_1} u_k \wedge \partial_{\tau_2} u_k \rangle \leq \left( \int_{S^2} |u_k|^2 \right)^{\frac{1}{2}} \cdot \left( \int_{S^2} |\partial_{\tau_1} u_k \wedge \partial_{\tau_2} u_k|^2 \right)^{\frac{1}{2}}
$$

$$
\leq \left( \int_{S^2} |u_k|^2 \right)^{\frac{1}{2}} \cdot \left( 1 + c\delta(u_k) \right)^{\frac{1}{2}} \leq \left( \frac{1}{2} \int_{S^2} |\nabla_T u_k|^2 \right)^{\frac{1}{2}} \cdot \left( 1 + c\delta(u_k) \right)^{\frac{1}{2}}
$$

$$
\leq 1 + c\delta(u_k).
$$

(3.25)

By the assumption (3.20), and since $u_k \to u$ strongly in $L^2(S^2;\mathbb{R}^3)$, we can let $k \to \infty$ in (3.25), to obtain

$$
\int_{S^2} |u|^2 = \lim_{k \to \infty} \int_{S^2} |u_k|^2 = 1.
$$

(3.26)

Hence, the limiting map $u$ is such that $\int_{S^2} u = 0$, and by (3.22) and (3.26) it also satisfies

$$
1 \geq \frac{1}{2} \int_{S^2} |\nabla_T u|^2 \geq \int_{S^2} |u|^2 = 1.
$$

(3.27)

By the equality case in the sharp Poincaré inequality on $S^2$ (since the first nontrivial eigenfunctions of $-\Delta_{S^2}$ are the coordinate functions, cf. Appendix C), we deduce from (3.27) that $u(x) = Ax$ for some $A \in \mathbb{R}^{3 \times 3}$ with $|A|^2 = 3$. In particular, equalities are achieved in (3.22), and therefore $u_k \to u := \text{id}_{S^2}$ in the strong $W^{1,2}$-topology.

To show that $A \in O(3)$, we argue as follows. Having established the strong $W^{1,2}$-convergence of $(u_k)_{k \in \mathbb{N}}$ towards $u$, up to a further non-relabeled subsequence we can assume now that $\nabla_T u_k \to \nabla_T u$ also pointwise
\( H^2 \)-a.e. on \( S^2 \) and therefore, using the assumption that \( \| \nabla_T u_k \|_{L^\infty} \leq cM \) for every \( k \in \mathbb{N} \),

\[
\langle u_k, \partial_{r_1} u_k \wedge \partial_{r_2} u_k \rangle \rightarrow \langle u, \partial_{r_1} u \wedge \partial_{r_2} u \rangle \quad \text{pointwise } H^2 \text{-a.e.},
\]

\[
|\langle u_k, \partial_{r_1} u_k \wedge \partial_{r_2} u_k \rangle| \leq |\partial_{r_1} u_k \wedge \partial_{r_2} u_k| |u_k| \leq \frac{1}{2} |\nabla_T u_k|^2 |u_k| \leq |u_k| \quad \forall k \in \mathbb{N},
\]

(3.28)

Using a variant of Lebesgue’s Dominated Convergence Theorem in the assumption that \( \lim_{k \to \infty} \varepsilon(u_k) = 0 \) and (3.28), allows us to conclude. Indeed,

\[
0 = \lim_{k \to \infty} \varepsilon(u_k) = \lim_{k \to \infty} \left( 1 - \int_{S^2} \langle u_k, \partial_{r_1} u_k \wedge \partial_{r_2} u_k \rangle \right) = \left( 1 - \int_{S^2} \lim_{k \to \infty} \langle u_k, \partial_{r_1} u_k \wedge \partial_{r_2} u_k \rangle \right) = \left( 1 - \int_{S^2} \langle u, \partial_{r_1} u \wedge \partial_{r_2} u \rangle \right)
\]

\[
= \left( \frac{|A|^2}{3} \right)^\frac{3}{2} \geq |\det A| \geq 1,
\]

i.e., \( |\det A| \geq 1 \). If we now perform the polar decomposition \( A = O\sqrt{A^T A} \), where \( O \in O(3) \), and label \( 0 \leq \alpha_1 \leq \alpha_2 \leq \alpha_3 \) the eigenvalues of \( \sqrt{A^T A} \), by the arithmetic mean-geometric mean inequality we get

\[
1 = \left( \frac{|A|^2}{3} \right)^\frac{3}{2} \geq |\det A| \geq 1,
\]

and equality in this algebraic inequality implies that \( \alpha_1 = \alpha_2 = \alpha_3 = 1 \), i.e., \( O := A \in O(3) \).

As an immediate consequence of the Lemmata 3.5 and 3.6 we obtain the following.

**Corollary 3.7.** It suffices to prove Theorem 1.2 for maps

\[
u \in \mathcal{A}_{M, \theta} := \left\{ u \in W^{1, \infty}(S^2; \mathbb{R}^3): \begin{cases} (i) & \int_{S^2} u = 0 \\ (ii) & \| \nabla_T u \|_{L^\infty} \leq cM \\ (iii) & \| \nabla_T u - P_T \|_{L^2} \leq \theta \end{cases} \right\},
\]

(3.29)

where \( c > 0 \) is the constant of Lemma 3.4. \( M := 2\sqrt{2} \) and \( 0 < \theta \ll 1 \) is a sufficiently small constant that will be suitably chosen later.

**Proof.** The proof is a standard contradiction argument. Indeed, suppose that we have proven Theorem 1.2 for maps in \( \mathcal{A}_{M, \theta} \) for some \( \theta \in (0, 1) \) sufficiently small. According to the Lipschitz truncation argument provided by Lemma 3.5 for the general case it suffices to prove that

\[
\sup_{O \in O(3)} \left\{ \min_{O \in O(3)} \int_{S^2} \frac{|\nabla_T u - OP_T|^2}{\delta(u) + \varepsilon(u)} \right\} \ll +\infty,
\]

whenever the denominator above is non-zero. Arguing by contradiction, suppose that the latter is false. Then, for every \( k \in \mathbb{N} \) there exist \( u_k \) with mean value 0, Lipschitz norm bounded by \( cM \), \( \delta(u_k) + \varepsilon(u_k) > 0 \), and \( O_k \in \text{Argmin}_{O \in O(3)} \int_{S^2} |\nabla_T u_k - OP_T|^2 \), such that

\[
\int_{S^2} |\nabla_T u_k - O_k P_T|^2 \geq k(\delta(u_k) + \varepsilon(u_k)).
\]

(3.30)
In particular,
\[ \delta(u_k) + \varepsilon(u_k) \leq \frac{1}{k} \int_{\mathbb{S}^2} |\nabla_T u_k - O_k P_T|^2 \leq \frac{2\epsilon^2 M^2 + 2}{k}, \]
and letting \( k \to \infty \) we see that along this sequence, \( \lim_{k \to \infty} (\delta(u_k) + \varepsilon(u_k)) = 0 \). By Lemma 3.6 and up to passing to a subsequence, we can find \( O_0 \in O(3) \) so that \( u_k \to O_0 \text{id}_{\mathbb{S}^2} \) strongly in \( W^{1,2}(\mathbb{S}^2; \mathbb{R}^3) \). Without loss of generality (up to considering \( O_0^t u_k \) instead of \( u_k \) if necessary) we can also suppose that \( O_0 = I_3 \), so there exists \( k_0 := k_0(\theta) \in \mathbb{N} \) such that
\[ \|\nabla_T u_k - P_T\|_{L^2} \leq \theta \quad \forall k \geq k_0, \]
i.e., \( u_k \in \mathcal{A}_{M,\theta} \) for all \( k \geq k_0 \). Therefore, by assumption, there should exist \( (R_k)_{k \geq k_0} \subset O(3) \) (and actually in \( SO(3) \)) such that
\[ \int_{\mathbb{S}^2} |\nabla_T u_k - R_k P_T|^2 \lesssim \delta(u_k) + \varepsilon(u_k) \quad \forall k \geq k_0, \]
which contradicts (3.30).

### 3.3 Proof of the local version of Theorem 1.2

By the reductions we have performed in the previous two subsections, we are left with proving a local version of Theorem 1.2. This will be done by perturbing quantitatively the idea of proof of Lemma 3.6.

**Proposition 3.8.** There exists a constant \( \theta \in (0, 1) \) so that for every \( u \in \mathcal{A}_{M,\theta} \) (defined in (3.29)), there exists \( R \in SO(3) \) such that
\[ \int_{\mathbb{S}^2} |\nabla_T u - R P_T|^2 \lesssim \delta(u) + \varepsilon(u). \tag{3.31} \]

**Proof.** First of all, it obviously suffices to prove (3.31) in the regime where both deficits are sufficiently small, say
\[ 0 \leq \delta(u) \leq \delta_0 \ll 1, \quad 0 \leq \varepsilon(u) \leq \varepsilon_0 \ll 1, \tag{3.32} \]
for some absolute constants \( \delta_0, \varepsilon_0 > 0 \) which will also be chosen sufficiently small later. By using (3.23) with \( u \) instead of \( u_k \), we have
\[ \int_{\mathbb{S}^2} |u|^2 = \frac{1}{2} \int_{\mathbb{S}^2} |\nabla_T u|^2 - \left( \frac{1}{2} \int_{\mathbb{S}^2} |\nabla_T u|^2 - \int_{\mathbb{S}^2} |u|^2 \right) \leq 1 + c\delta(u) - \left( \frac{1}{2} \int_{\mathbb{S}^2} |\nabla_T u|^2 - \int_{\mathbb{S}^2} |u|^2 \right), \]
and therefore (3.25), with \( u \) instead of \( u_k \), would now give us
\[ (1 - \varepsilon(u))^2 \leq \left( \int_{\mathbb{S}^2} |u|^2 \right) \left( 1 + c\delta(u) \right) \leq \left( 1 + c\delta(u) - \left( \frac{1}{2} \int_{\mathbb{S}^2} |\nabla_T u|^2 - \int_{\mathbb{S}^2} |u|^2 \right) \right) \left( 1 + c\delta(u) \right). \]
Since by (C.5) we have \( \frac{1}{2} \int_{\mathbb{S}^2} |\nabla_T u|^2 - \int_{\mathbb{S}^2} |u|^2 \geq 0 \), we can rearrange the terms and use (3.32) to arrive at the estimate
\[ \int_{\mathbb{S}^2} |\nabla_T u - \nabla u_h(0) P_T|^2 \lesssim \frac{1}{2} \int_{\mathbb{S}^2} |\nabla_T u|^2 - \int_{\mathbb{S}^2} |u|^2 \lesssim \delta(u) + \varepsilon(u), \tag{3.33} \]
the first inequality in which, is justified as follows. Let \( u_h : \overline{T_1} \to \mathbb{R}^3 \) be the harmonic continuation of \( u \) in the interior of \( B_1 \), being taken componentwise. The quantity in the middle of (3.33) is the
deficit of $u$ in the $L^2$-Poincare inequality for maps with zero average on $S^2$. For every $k \in \mathbb{N}$, let $H_k$ be the subspace of $W^{1,2}(S^2; \mathbb{R}^3)$ consisting of vector fields whose components are all $k$-th order spherical harmonics (see also Appendix C), so that one has the orthogonal (with respect to the $W^{1,2}$-inner product) decomposition $W^{1,2}(S^2; \mathbb{R}^3) = \bigoplus_{k=0}^\infty H_k$. Let also $\Pi_k$ be the corresponding orthogonal projection. In our case of consideration, $\Pi_0 u = \int_{S^2} u = 0$, and it is straightforward to check that $\Pi_1 u = \nabla u_0(0)x$. Since the first non-trivial eigenvalue of the Laplace-Beltrami operator on $S^2$ is $\lambda_1 = 2$ and the second one is $\lambda_2 = 6$ (see (C.2)), by orthogonally decomposing $u = \Pi_1 u + (u - \Pi_1 u)$, we have

$$\frac{1}{2} \int_{S^2} |\nabla T u| - \int_{S^2} |u|^2 = \frac{1}{2} \int_{S^2} |\nabla T u - \nabla u_0(0) P_T|^2 - \int_{S^2} |u - \nabla u_0(0) x|^2$$

$$\geq \frac{1}{2} \int_{S^2} |\nabla T u - \nabla u_0(0) P_T|^2 - \frac{1}{6} \int_{S^2} |\nabla T u - \nabla u_0(0) P_T|^2$$

$$= \frac{1}{3} \int_{S^2} |\nabla T u - \nabla u_0(0) P_T|^2. \quad (3.34)$$

Hence, the only thing that is left to be justified in order to prove (3.31), is why in (3.33) the matrix

$$A := \nabla u_0(0) \quad (3.35)$$

can be replaced by a matrix $R \in SO(3)$. In that respect, observe that by the mean-value property of harmonic functions, the basic $L^2$-estimate C.6 (whose simple proof is given at the end of Appendix C) applied to the function $u - \text{id}_{S^2}$, and (3.29), we obtain

$$|A - I_3|^2 = \left| \int_{B_1} \nabla u_h - I_3 \right|^2 \leq \int_{B_1} |\nabla u_h - I_3|^2 \leq \frac{3}{2} \int_{S^2} |\nabla T u - P_T|^2 \leq \frac{3}{2} \theta^2. \quad (3.36)$$

In particular, if $\theta \in (0, 1)$ is sufficiently small, (3.36) directly implies that

$$2 \leq |A|^2 \leq 4, \quad 2 \leq |A^{-1}|^2 \leq 4, \quad \frac{1}{2} \leq \det A \leq \frac{3}{2}. \quad (3.37)$$

Using the polar decomposition $A = R_0 \sqrt{A^TA}$ for some $R_0 \in SO(3)$, the last inequality in (3.37) (in particular the fact that $\det A > 0$) and (3.36) yield

$$\text{dist}^2(A, SO(3)) = |A - R_0|^2 \leq |A - I_3|^2 \leq \frac{3}{2} \theta^2. \quad (3.38)$$

Labelling $0 < \alpha_1 \leq \alpha_2 \leq \alpha_3$ the eigenvalues of $\sqrt{A^TA}$, and setting

$$\lambda_i := \alpha_i - 1, \quad \lambda := \sum_{i=1}^3 \lambda_i, \quad A := \left( \sum_{i=1}^3 \lambda_i^2 \right)^{\frac{1}{2}} \quad \text{for } i = 1, 2, 3, \quad (3.39)$$

the inequality (3.38) can be rewritten as

$$A^2 = \text{dist}^2(A, SO(3)) = \sum_{i=1}^3 (\alpha_i - 1)^2 = |\sqrt{A^TA} - I_3|^2 \leq \frac{3}{2} \theta^2 \ll 1. \quad (3.40)$$

The key observation now is that when $\theta \in (0, 1)$ is sufficiently small, a map $u \in \mathcal{A}_{M, \theta}$ satisfies the estimate

$$|\det A - 1| \lesssim \delta(u) + \varepsilon(u). \quad (3.41)$$
The proof of (3.41) is a bit more involved, and is therefore presented separately in Lemma 3.9. Let us assume for the moment its validity, and see how to finish the proof of (3.31). With the notations introduced in (3.35) and (3.39), we can write
\[
\det A - 1 = \prod_{i=1}^{3} \alpha_i - 1 = \prod_{i=1}^{3} (\lambda_i + 1) - 1,
\]
expand the polynomial in the eigenvalues and use (3.40), to obtain
\[
\det A - 1 = \lambda + \frac{1}{2}(\lambda^2 - \Lambda^2) + \lambda_1\lambda_2\lambda_3 \leq \lambda + \frac{1}{2}(\lambda^2 - \Lambda^2) + 3^{-3/2}\Lambda^3 \leq \lambda + \frac{1}{2}(\lambda^2 - \Lambda^2) + \frac{\theta}{3\sqrt{2}}\Lambda^2. \tag{3.42}
\]
Since \(0 < \frac{\theta}{3\sqrt{2}} < \frac{1}{3\sqrt{2}} < \frac{1}{4}\), after rearranging terms in (3.42) and using (3.41), we get
\[
\frac{\Lambda^2}{4} \leq (\lambda + \frac{\lambda^2}{2}) + |\det A - 1| \implies \frac{\Lambda^2}{4} \leq \left(\lambda + \frac{\lambda^2}{2}\right) + c\left(\delta(u) + \varepsilon(u)\right). \tag{3.43}
\]
In order to handle the term \(\left(\lambda + \frac{\lambda^2}{2}\right)\), we proceed as follows. Using again the mean value property of harmonic functions, (C.6) applied to \(u\) now, and the outcome of (3.23) with \(u\) instead of \(u_k\) here, we can estimate
\[
\frac{|A|^2}{3} = \frac{1}{3} \left|\int_{B_1} \nabla u_h\right|^2 \leq \frac{1}{3} \int_{B_1} |\nabla u_h|^2 \leq \frac{1}{2} \int_{S^2} |\nabla T u|^2 \leq 1 + c\delta(u). \tag{3.44}
\]
With the notations introduced in (3.35) and (3.39) we have
\[
|A|^2 = \sum_{i=1}^{3} \alpha_i^2 = \sum_{i=1}^{3} (1 + \lambda_i)^2 = 3 + 2\lambda + \Lambda^2,
\]
and the last identity, together with (3.44), implies that
\[
\lambda \leq -\frac{\Lambda^2}{2} + c\delta(u) \leq c\delta(u). \tag{3.45}
\]
Since \(\lambda\) does not necessarily have a sign, we distinguish two cases:

(i) In the case \(\lambda \leq 0\), and since by (3.40) \(|\lambda| \leq \sqrt{3}\Lambda \leq \frac{3}{\sqrt{2}}\theta \ll 1\), the term in the first parenthesis on the right hand side of (3.43) is estimated by
\[
\lambda + \frac{\lambda^2}{2} \leq \lambda + \frac{3\theta}{2\sqrt{2}}|\lambda| = \left(1 - \frac{3\theta}{2\sqrt{2}}\right)\lambda \leq 0,
\]
since by choosing \(\theta \in (0, 1)\) even smaller if necessary, we can also achieve \(1 - \frac{3\theta}{2\sqrt{2}} > 0\). The term \((\lambda + \frac{\lambda^2}{2})\) is therefore nonpositive in this case, and (3.43) gives
\[
\text{dist}^2(A, SO(3)) = \Lambda^2 \leq 4c\left(\delta(u) + \varepsilon(u)\right).
\]

(ii) In the case \(\lambda > 0\), by (3.45) we have \(0 < \lambda \leq c\delta(u)\), so again (3.43) together with (3.32) imply that
\[
\text{dist}^2(A, SO(3)) = \Lambda^2 \lesssim \delta(u) + \delta^2(u) + c(\delta(u) + \varepsilon(u)) \lesssim \delta(u) + \varepsilon(u).
\]
In both cases, we obtain
\[
|A - R_0|^2 = \text{dist}^2(A, SO(3)) \lesssim \delta(u) + \varepsilon(u), \tag{3.46}
\]
and combining (3.46) with (3.33) allows us to deduce (3.31) with \(R := R_0 \in SO(3)\), and conclude. \(\square\)
To complete the arguments, we finally give the proof of the estimate (3.41), which for convenience of the reader we recall in the next lemma.

**Lemma 3.9.** Let $u \in A_{M,\theta}$, defined in (3.29). Then, the matrix $A := \nabla u_h(0)$ (see (3.35)) satisfies

$$|\det A - 1| \lesssim \delta(u) + \varepsilon(u),$$

where $\delta(u)$ and $\varepsilon(u)$ are as always defined by (3.1) and (3.4) respectively, and are here supposed to further satisfy (3.32).

**Proof.** The main trick is to write the signed-volume in the isoperimetric deficit $\varepsilon(u)$ as the corresponding bulk integral in $B_1$. In particular, using the identity (B.1) (which we prove in Appendix B), we have

$$V_3(u) = \int_{B_1} \det \nabla u_h = \det A \int_{B_1} \det(I_3 + \nabla w_h),$$

where

$$w(x) := A^{-1}(u(x) - Ax).$$

By the fact that $u \in A_{M,\theta}$ and (3.37), the map $w$ also satisfies

$$\bar{w} := \int_{\mathbb{S}^2} w = 0, \quad \|\nabla_T w\|_{L^\infty} = \|A^{-1}\nabla_T u - P_T\|_{L^\infty} \leq \bar{c} := 2cM + \sqrt{2},$$

and we will not distinguish further between the universal constants $c$ and $\bar{c}$. Because of (3.33), (3.35) and (3.37), we actually get

$$\int_{\mathbb{S}^2} |\nabla_T w|^2 \leq |A^{-1}|^2 \int_{\mathbb{S}^2} |\nabla_T u - AP_T|^2 \lesssim \delta(u) + \varepsilon(u).$$

Now, in the rightmost hand side of (3.48) we can use the expansion of the determinant around $I_3$, i.e., the identity (B.2) which is proved in Appendix B according to which,

$$\int_{B_1} \det(I_3 + \nabla w_h) = 1 + 3 \int_{\mathbb{S}^2} \langle w, x \rangle + Q_{V_3}(w) + \int_{\mathbb{S}^2} \langle w, \partial_\gamma w \wedge \partial_\gamma w \rangle,$$

where the quadratic form $Q_{V_3}(w)$ is explicitly given in (B.3). Notice that the linear term is vanishing, because the definitions of $A := \nabla u_h(0)$ and $w$ in (3.49), together with the mean value property of harmonic functions, imply that

$$3 \int_{\mathbb{S}^2} \langle w, x \rangle = \int_{B_1} \text{div} w_h = \int_{B_1} \text{Tr}(\nabla w_h) = \text{Tr} \int_{B_1} \nabla w_h = \text{Tr} \left[ A^{-1} \left( \int_{B_1} \nabla u_h(x) - A \right) \right] = 0.$$

For the higher order terms one can argue as follows. Recalling the notation $\Pi_k$ for the projections onto the subspaces $H_k$ of the $k$-th order spherical harmonics (see the comments after (3.33)), we note that (3.49) and (3.50) directly imply that $\Pi_0 w = \Pi_1 w = 0$. Hence, by the Cauchy-Schwarz inequality and the sharp Poincare inequality on $\mathbb{S}^2$ for $w$ (see the comment just below (C.5)), we obtain

$$|Q_{V_3}(w)| = \frac{3}{2} \int_{\mathbb{S}^2} \left| \mathcal{Q}_w \right| \leq \frac{3}{2} \left( \int_{\mathbb{S}^2} |w|^2 + \sum_{j=1}^3 |x_j \nabla_T w^j|^2 \right) \leq \frac{3}{2} \left( \int_{\mathbb{S}^2} |\nabla_T w|^2 + \sum_{j=1}^3 |x_j \nabla_T w^j|^2 \right) \leq \frac{3}{2} \left( \int_{\mathbb{S}^2} |\nabla_T w|^2 \right) \leq \frac{3}{2} \sqrt{2} \int_{\mathbb{S}^2} |\nabla_T w|^2,$$

where

$$\mathcal{Q}_w := \sum_{j=1}^3 x_j \nabla_T w^j$$

and $\mathcal{Q}_w$ is the Poincare inequality on $\mathbb{S}^2$. Therefore, we have

$$|Q_{V_3}(w)| \lesssim \delta(u) + \varepsilon(u),$$

which completes the proof of Lemma 3.9.
and by Wente’s isoperimetric inequality (see \(1.9\) for \(n = 3\)),
\[
\left| \int_{S^2} \langle w, \partial_{\tau_1} w \wedge \partial_{\tau_2} w \rangle \right| \leq \left( \frac{1}{2} \int_{S^2} |\nabla T w|^2 \right)^{\frac{3}{2}}.
\] (3.55)

Therefore, by \((3.54), (3.55)\) and \((3.51)\), together with the assumption \((3.32)\), we estimate
\[
\left| Q_{V_3}(w) + \int_{S^2} \langle w, \partial_{\tau_1} w \wedge \partial_{\tau_2} w \rangle \right| \lesssim \int_{S^2} |\nabla T w|^2 + \left( \int_{S^2} |\nabla T w|^2 \right)^{\frac{3}{2}} \lesssim (\delta(u) + \varepsilon(u)) + (\delta(u) + \varepsilon(u))^{\frac{3}{2}} \lesssim \delta(u) + \varepsilon(u) \ll 1.
\] (3.56)

In particular, since \(\det A > 0\) (see \((3.37)\)), by \((3.48)\), the expansion in \((3.52), (3.53)\) and \((3.56)\), we deduce that \(V_3(u) > 0\), and we can finally consider two cases:

(i) If \(V_3(u) > 1 \implies \varepsilon(u) = 0\), then by combining \((3.48)\) with \((3.52), (3.53)\), rearranging terms, and then using again \((3.37), (3.56)\), the isoperimetric inequality and \((3.23)\) with \(u\) instead of \(u_k\) here, we obtain the estimate
\[
|\det A - 1| \leq (V_3(u) - 1) + \det A \left| Q_{V_3}(w) + \int_{S^2} \langle w, \partial_{\tau_1} w \wedge \partial_{\tau_2} w \rangle \right| \leq \left( \frac{1}{2} \int_{S^2} |\nabla T w|^2 \right)^{\frac{3}{2}} - 1 + c(\delta(u) + \varepsilon(u)) \leq (1 + c\delta(u))^{\frac{3}{2}} - 1 + c\delta(u) \lesssim \delta(u).
\]

(ii) If \(0 \leq V_3(u) \leq 1\), then we can again similarly estimate,
\[
|1 - \det A| \leq |1 - V_3(u)| + \det A \left| Q_{V_3}(w) + \int_{S^2} \langle w, \partial_{\tau_1} w \wedge \partial_{\tau_2} w \rangle \right| \leq (1 - V_3(u)) + c(\delta(u) + \varepsilon(u)) \lesssim \delta(u) + \varepsilon(u).
\]

This finishes the proof of \((3.47)\) in both cases, and allows us to conclude. \(\square\)

### 3.4 The generalization to dimensions \(n \geq 4\): Proof of Theorem 1.3

By following closely the steps of proof of Theorem 1.2 one can also prove its generalization in dimensions \(n \geq 4\), i.e., Theorem 1.3. Regarding the extra assumption on an apriori bound in the \(L^{2(n-2)}\)-norm of \(\nabla T u\) in the latter, let us first make the following short remark. When \(n = 3\), \(n - 1 = 2(n - 2) = 2\) and the assumption that \(\nabla T u\) is apriori bounded in \(L^2\) is obsolete in this case, since we have anyway seen that it suffices to prove Theorem 1.2 for maps \(u \in W^{1,2}(S^2; \mathbb{R}^3)\) for which \(0 < \delta(u) \ll 1\), which trivially implies the bound \(\|\nabla T u\|_{L^2} \leq \sqrt{10}\) (recall \((3.10)\)). In higher dimensions, the assumption is imposed by the growth behaviour of the signed-volume term with respect to \(\nabla T u\).

Indeed, as we will see next, apart from the obvious differences in the proof due to the change in dimension, the only essential difference appears when we are trying to implement the Lipschitz truncation argument of Subsection 3.1, in order to control both the isometric and the isoperimetric deficit of the Lipschitz truncated map in terms of the ones of the original map \(u\).
Proof of Theorem 1.3 Let \( n \geq 4, M > 0 \) and \( u \in \dot{W}^{1,2(n-2)}(\mathbb{S}^{n-1}; \mathbb{R}^n) \) with \( \| \nabla_T u \|_{L^2(n-2)} \leq M \). Applying the analogue of Lemma 3.4 in \( W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^n) \) for \( M := 2\sqrt{n-1} \), we obtain again \( u_{M_n} \in W^{1,\infty}(\mathbb{S}^{n-1}; \mathbb{R}^n) \) with \( \| \nabla_T u_{M_n} \|_{L^\infty} \lesssim M_n \) and for which, exactly as in the estimates (3.12) (with \( \sigma_{M_n,n-1} \) in the place of \( \sigma_{M,2} \)) and (3.13) of Lemma 3.5

\[
\delta(u_{M_n}) \lesssim \delta(u) \quad \text{and} \quad \varepsilon(u_{M_n}) \leq \varepsilon(u) + \left| V_n(u) - V_n(u_{M_n}) \right|.
\]

(3.57)

Since \( V_n \) is now of order \( n-1 > 2 \) in \( \nabla_T u \), it is of course not expected that one can have an estimate of the form of (3.19) without any further assumption, since \( V_n(u) \) is not even finite if \( u \) does not belong to \( W^{1,n-1}(\mathbb{S}^{n-1}; \mathbb{R}^n) \) at least. Nevertheless, under the imposed assumption that \( \| \nabla_T u \|_{L^2(n-2)} \leq M \), the difference of the corresponding signed volumes in (3.57) can be controlled as follows. Assuming again without loss of generality that \( 0 < \delta(u) \leq 1 \), adopting the notation in (3.14) and using the fact that \( \| \nabla_T u_{M_n} \|_{L^\infty} \lesssim M_n := 2\sqrt{n-1} \), as well as (3.11) (in dimension \( n \geq 4 \) now), we can estimate

\[
\left| V_n(u) - V_n(u_{M_n}) \right| = \left| \int_{\mathbb{S}^{n-1}} \nabla_T u - \nabla_T u_{M_n}, \nabla_T u_{M_n} \right| 
\leq \left| \int_{\mathbb{S}^{n-1}} (u_{M_n} - u), \nabla T u_{M_n} \right| + \left| \int_{\mathbb{S}^{n-1}} u - \nabla T u_{M_n}, \nabla_T u \right| 
\lesssim \left( \int_{\mathbb{S}^{n-1}} \| \nabla_T u_{M_n} - \nabla_T u \|_{L^2} \right)^{\frac{1}{2}} + R_n(u) 
\lesssim \delta(u) + R_n(u),
\]

(3.58)

where

\[
R_n(u) := \int_{\mathbb{S}^{n-1}} \left| u - \nabla_T u_{M_n}, \nabla_T u_{M_n} - \nabla_T u \right|.
\]

(3.59)

By the Sobolev embedding and our assumption, we further have

\[
\| u - \nabla T u \|_{L^\infty} \lesssim \| \nabla_T u \|_{L^2(n-2)} \lesssim M.
\]

(3.60)

Therefore, by the fact that \( \{ u_{M_n} = u \} \subseteq \{ \nabla_T u_{M_n} = \nabla_T u \} \) \( H^{n-1} \)-a.e., (3.60), the first inequality in (3.11) (in dimension \( n \geq 4 \)), and the assumption that \( \| \nabla_T u \|_{L^2(n-2)} \leq M \), the remainder term in (3.59) can be estimated further by

\[
R_n(u) \sim \left| \int_{\{ u \neq u_{M_n} \}} u - \nabla_T u_{M_n}, \nabla_T u_{M_n} - \nabla_T u \right| \lesssim M \left| \int_{\{ u \neq u_{M_n} \}} \nabla_T u_{M_n} - \nabla_T u \right| 
\lesssim M \left( H^{n-1} \{ u \neq u_{M_n} \} + \int_{\{ u \neq u_{M_n} \}} | \nabla_T u |^{n-1} \right) 
\lesssim M \left( \delta^2(u) + \| \nabla_T u \|_{L^2(n-2)} \right) \left( \int_{\{ u \neq u_{M_n} \}} | \nabla_T u |^{2} \right)^{\frac{1}{2}} 
\lesssim M \delta^2(u) + \delta(u) \lesssim M \delta(u).
\]

(3.61)
Therefore, under this extra assumption for \( n \geq 4 \), in view of (3.58) and (3.61), (3.57) implies again that

\[
|V_n(u) - V_n(u_{M_n})| \lesssim_M \delta(u) \implies \varepsilon(u_{M_n}) \lesssim_M \varepsilon(u) + \delta(u).
\]

Hence, under the assumption that \( \|\nabla_T u\|_{L^{2(n-2)}} \leq M \) when \( n \geq 4 \), we can again reduce to proving Theorem 1.3 for Lipschitz maps that enjoy an apriori dimensional upper bound on their Lipschitz constant. The proof can then be continued exactly as in Subsections 3.2 and 3.3 with the obvious modifications in the dimensional constants and the algebraic expressions involved.

For instance, and just for the sake of clarity, we note that in this higher dimensional setting, \( \frac{1}{2} \int_{S^2} |\nabla T u|^2 \) should be replaced in the corresponding arguments by \( \frac{1}{n-1} \int_{S^{n-1}} |\nabla T u|^2 \), but the arguments go through exactly in the same way, since \( n-1 \) is both the norm of the gradient of isometric maps from \( S^{n-1} \) to \( \mathbb{R}^n \) and also the first nontrivial eigenvalue of \( -\Delta_{S^{n-1}} \) (the second being \( 2n \), see Appendix C and (3.34)). In this respect, the estimate (3.33) should of course be replaced by

\[
\frac{n+1}{2n(n-1)} \int_{S^{n-1}} |\nabla T u - \nabla u_h(0) P T|^2 \lesssim \frac{1}{n-1} \int_{S^{n-1}} |\nabla T u|^2 - \int_{S^{n-1}} |u|^2 \lesssim_M \delta(u) + \varepsilon(u),
\]

and analogously to (3.52), the expansion of the signed-volume around the identity is now

\[
\int_{B_1} \det(I_n + \nabla w_h) = 1 + n \int_{S^{n-1}} \langle w, x \rangle + Q_{V_n}(w) + o \left( \int_{S^{n-1}} |\nabla T w|^2 \right).
\]

Modulo these changes, the proof remains essentially unchanged, which is left to the reader to verify. \( \square \)

**Remark 3.10.** An interesting question would be whether for \( n \geq 4 \) the apriori bound on the \( L^{2(n-2)} \)-norm of \( \nabla_T u \), imposed as an assumption in Theorem 1.3, can be replaced by one in an \( L^p \)-norm for some \( p \in (n-1, 2(n-2)) \). The previous approach indicates however that the exponent \( 2(n-2) = (n-1) + (n-3) \) in the assumption is the sharpest one.

Indeed, let us assume that \( \|\nabla T u\|_{L^{n-1+\gamma}} \leq M \) for some \( \gamma \in (0, n-3) \) and \( M > 0 \). Then, for \( \alpha \in (0, n-1) \) and \( p > 1 \), we can apply Hölder’s inequality and use again the analogues of the estimates in (3.11) for \( n \geq 4 \), to deduce as before that

\[
\int_{\{u \neq u_{M_n}\}} |\nabla T u|^{n-1} \leq \left( \int_{\{u \neq u_{M_n}\}} |\nabla T u|^{\frac{p(n-1-\alpha)}{p-1}} \right)^{\frac{p-1}{p}} \left( \int_{\{u \neq u_{M_n}\}} |\nabla T u|^{\alpha p} \right)^{\frac{1}{p}} \lesssim_p \|\nabla T u\|^{n-1-\alpha} L^{p(n-1-\alpha)} H^{n-1} \left( \{u \neq u_{M_n}\} \right) + \int_{\{u \neq u_{M_n}\} \cap \{|\nabla T u| > M_n\}} |\nabla T u|^{\alpha p} \right)^{\frac{1}{p}}.
\]

As long as \( \alpha p \leq 2 \), by Hölder’s inequality again,

\[
\int_{\{u \neq u_{M_n}\} \cap \{|\nabla T u| > M_n\}} |\nabla T u|^{\alpha p} \leq \left( H^{n-1} \left( \{u \neq u_{M_n}\} \right) \right)^{2-\alpha p} \left( \int_{\{|\nabla T u| > M_n\}} |\nabla T u|^2 \right)^{\frac{\alpha p}{2}},
\]

and (3.11) (for \( n \geq 4 \)) would finally give us

\[
\int_{\{u \neq u_{M_n}\}} |\nabla T u|^{n-1} \lesssim_{\alpha, p} \|\nabla T u\|^{n-1-\alpha} L^{p(n-1-\alpha)} \cdot \delta^\frac{2}{p}(u) \lesssim_{\alpha, p} M^{n-1-\alpha} \delta^\frac{2}{p}(u),
\]

(3.63)
as long as \( p n − 1 − α \leq n − 1 + γ \). Therefore, by (3.63) and following closely the estimates used to arrive at (3.58) and (3.61), we deduce that the optimal exponent with which \( δ(u) \) can appear in (3.62) through these estimates is exactly
\[
\frac{2}{p'} = \frac{2(γ + α)}{n − 1 + γ}, \quad \text{achieved for } p' := \frac{n − 1 + γ}{γ + α}.
\]
But this value should also satisfy the inequality \( α p' ≤ 2 \), which implies that
\[
α ≤ \frac{2γ}{n − 3 + γ} \quad \Rightarrow \quad \frac{2}{p'} ≤ \frac{2γ}{n − 3 + γ} < 1,
\]
for \( 0 < γ < n − 3 \), i.e., the exponent of \( δ(u) \) in (3.62) would become suboptimal.

4 Proof of Theorem 1.4

4.1 Reduction to maps \( W^{1,2} \)-close to the \( \text{id}_{S^2} \) and linearization of the problem

We recall that for a map \( u ∈ W^{1,2}(S^2; \mathbb{R}^3) \) with \( V_3(u) ≠ 0 \), its combined conformal-isoperimetric deficit is denoted by
\[
E_2(u) := \left[ \frac{D_2(u)}{|V_3(u)|} \right]^{\frac{3}{2}} − 1 ≥ 0, \quad (4.1)
\]
where \( D_2(u), V_3(u) \) are as in the statement of Theorem 1.4 and that \( E_2(u) = 0 \) iff \( u \) is a M"{o}bius transformation of \( S^2 \), up to a translation vector and a dilation factor.

To pass from the nonlinear deficit \( E_2 \) to its linearized version, we make use of the following compactness result, whose proof can be found for instance in [4] or [6, Lemma 2.1] (stated on \( \mathbb{R}^2 \) rather than \( S^2 \) therein).

Lemma 4.1. Let \( (u_k)_{k ∈ \mathbb{N}} ⊂ W^{1,2}(S^2; \mathbb{R}^3) \) be such that \( V_3(u_k) ≠ 0 \), and for which
\[
E_2(u_k) → 0 \quad \text{as } k → ∞.
\]

Then, there exist M"{o}bius transformations \( (ϕ_k)_{k ∈ \mathbb{N}} \) of \( S^2 \), \( (λ_k)_{k ∈ \mathbb{N}} ⊂ \mathbb{R}_+ \) and \( O ∈ O(3) \), such that
\[
\frac{1}{λ_k} \left( u_k ∘ ϕ_k − \int_{S^2} u_k ∘ ϕ_k \right) → O\text{id}_{S^2} \quad \text{strongly in } W^{1,2}(S^2; \mathbb{R}^3).
\]

Using this compactness lemma and the invariances of the combined conformal-isoperimetric deficit, with a contradiction argument as the one we used in the proof of Corollary 3.7, one can now prove the following.

Corollary 4.2. It suffices to prove the \( W^{1,2} \)-local version of Theorem 1.4, i.e., to prove it for maps
\[
u ∈ \mathcal{B}_{θ, \varepsilon_0} := \left\{ u ∈ W^{1,2}(S^2; \mathbb{R}^3) : \begin{array}{l} (i) \ f_{S^2} u = 0 \\ (ii) \ |∇_Tu − P_T|_{L^2} ≤ θ \\ (iii) \ E_2(u) ≤ \varepsilon_0 \end{array} \right\}, \quad (4.2)
\]
where \( θ, \varepsilon_0 ∈ (0, 1) \) are sufficiently small constants that will be suitably chosen later.
Proof. The fact that without loss of generality we can assume (i) is obvious because (1.8) is translation invariant. That we can assume property (iii) is also immediate, because for every \( u \in W^{1,2}(S^2; \mathbb{R}^3) \) with \( V_3(u) \neq 0 \), choosing \( \lambda := \| \nabla_T u \|_{L^2} > 0 \) and \( \phi := \text{id}_{S^2} \) we have that \( \int_{S^2} \left| \frac{1}{\lambda} \nabla_T u - \nabla_T \phi \right|^2 \leq 6 \). Therefore, it suffices to prove the desired estimate in the small-deficit regime.

Suppose now that we have proven Theorem 1.4 for maps in \( B_{\theta_{\varepsilon_0}} \), but for the sake of contradiction the theorem fails to hold globally. Then, for every \( k \in \mathbb{N} \) there exists \( u_k \in W^{1,2}(S^2; \mathbb{R}^3) \) with \( V_3(u_k) \neq 0 \) such that for every pair \( (\lambda, \phi) \in \mathbb{R}_+ \times \text{Conf}(S^2) \),

\[
\int_{S^2} \left| \frac{1}{\lambda} \nabla_T u_k - \nabla_T \phi \right|^2 \geq k \mathcal{E}_2(u_k).
\]

Choosing \( (\lambda, \phi) := (\| \nabla_T u_k \|_{L^2}, \text{id}_{S^2}) \) we obtain \( \mathcal{E}_2(u_k) \leq \frac{6}{k} \to 0 \), as \( k \to \infty \). We can then use Lemma 4.1 and argue as in the end of the proof of Corollary 3.7 to arrive at a contradiction for the corresponding maps \( \tilde{u}_k := \frac{1}{\lambda_k} \Theta^T (u_k \circ \phi_k - \int_{S^2} u_k \circ \phi_k) \).

\( \square \)

Remark 4.3. Note that \( V_3(u) > 0 \) whenever \( u \in B_{\theta_{\varepsilon_0}} \) with \( \theta \in (0,1) \) sufficiently small. Indeed, recalling the expansions and the estimates (3.15)-(3.18), with the \( \text{id}_{S^2} \) in place of \( u_M \) here, we can arrive exactly as in the second half of the proof of Lemma 3.4 (recall (3.19)) at the estimate

\[
|V_3(u) - 1| = |V_3(u) - V_3(\text{id}_{S^2})| \lesssim \| \nabla_T u - P_T \|_{L^2} \leq \theta \ll 1.
\]

Having now reduced to showing Theorem 1.4 for mappings in \( B_{\theta_{\varepsilon_0}} \), where \( V_3(u) > 0 \), we can linearize the initial problem, by making use of the following two lemmata.

Lemma 4.4. Given \( \theta, \varepsilon_0 \in (0,1) \) sufficiently small, there exists \( \tilde{\theta} \in (0,1) \) sufficiently small accordingly, so that after possibly replacing \( \theta \) with \( \tilde{\theta} \), we can assume that every \( u \in B_{\theta_{\varepsilon_0}} \) has the additional property that

\[
\int_{S^2} \langle u, x \rangle = 1. \tag{4.3}
\]

Proof. Let \( u \in B_{\theta_{\varepsilon_0}} \) (defined in (4.2)) with \( 0 < \theta, \varepsilon_0 \ll 1 \), and set

\[
\lambda_u := \int_{S^2} \langle u, x \rangle. \tag{4.4}
\]

By the cancellation property \( \int_{S^2} x = 0 \) and the sharp Poincare inequality on \( S^2 \) (see (C.5)) we have,

\[
|\lambda_u - 1| = \left| \int_{S^2} \langle u - x, \frac{u - x}{\langle u, x \rangle} \rangle \right| \leq \int_{S^2} \left| u - x - \frac{u - x}{\langle u, x \rangle} \right| \\
\leq \left( \int_{S^2} |u - x - \frac{u - x}{\langle u, x \rangle}|^2 \right)^{\frac{1}{2}} \leq \left( \frac{1}{2} \int_{S^2} \| \nabla_T u - P_T \|^2 \right)^{\frac{1}{2}} \leq \frac{\theta}{\sqrt{2}},
\]

i.e.,

\[
0 < 1 - \frac{\theta}{\sqrt{2}} \leq \lambda_u \leq 1 + \frac{\theta}{\sqrt{2}}. \tag{4.5}
\]

Hence, setting \( \tilde{u} := \frac{u}{\lambda_u} \), by (4.2) and (4.4) we have

\[
\int_{S^2} \tilde{u} = 0, \quad \int_{S^2} \langle \tilde{u}, x \rangle = 1, \quad \mathcal{E}_2(\tilde{u}) \leq \varepsilon_0,
\]

23
and by using \((4.5)\),
\[
\|\nabla T\tilde{u} - P_T\|_{L^2} \leq \frac{1}{\lambda_u} \|\nabla Tu - P_T\|_{L^2} + \left| \frac{1}{\lambda_u} - 1 \right| \|P_T\|_{L^2} \leq \tilde{\theta} := \left( \frac{2}{1 - \theta} \right) \theta.
\]

Although the precise value of the new constant \(\tilde{\theta} > 0\) is not of major importance, what is more important is that \(\lim_{\theta \to 0^+} \tilde{\theta} = 0\), so that when we will finally choose \(\theta > 0\) sufficiently small, \(\tilde{\theta} > 0\) will be sufficiently small accordingly.

**Lemma 4.5.** There exists a constant \(\beta := \beta(\theta, \varepsilon_0) > 0\) that tends to 0 as \((\theta, \varepsilon_0) \to (0, 0)\), such that the following holds. If \(u \in B_{\theta, \varepsilon_0}\) satisfies \((4.3)\) and one sets \(w := u - \text{id}_{S^2}\), then
\[
Q_3(w) \leq \mathcal{E}_2(u) + \beta \int_{S^2} |\nabla_T w|^2,
\]
where
\[
Q_3(w) := \frac{3}{4} \int_{S^2} |\nabla_T w|^2 - \frac{3}{2} \int_{S^2} \langle w, (\text{div}_{S^2} w)x - \sum_{j=1}^{3} x_j \nabla_T w^j \rangle.
\]

**Proof.** For \(u\) as in the statement of the lemma, property \((4.3)\) can be rewritten as
\[
\int_{S^2} \text{div}_{S^2} w = 2\int_{S^2} \langle w, x \rangle = 2\left(\int_{S^2} \langle u, x \rangle - 1\right) = 0.
\]
Then,
\[
[D_2(u)]^2 = \left(1 + \int_{S^2} \text{div}_{S^2} w + \frac{1}{2} \int_{S^2} |\nabla_T w|^2\right)^2 = \left(1 + \frac{1}{2} \int_{S^2} |\nabla_T w|^2\right)^2
= 1 + \frac{3}{4} \int_{S^2} |\nabla_T w|^2 + O\left(\left(\int_{S^2} |\nabla_T w|^2\right)^2\right).
\]
Since \(\int_{S^2} |\nabla_T w|^2 \leq 3/4\), we can take \(\theta \in (0, 1)\) small enough so that (by \((4.2)(ii)\)) the higher-order term in the expansion \((4.9)\) is estimated by
\[
\left|O\left(\int_{S^2} |\nabla_T w|^2\right)^2\right| \leq \frac{1}{2} \left(\int_{S^2} |\nabla_T w|^2\right)^2 \leq \frac{\theta^2}{2} \int_{S^2} |\nabla_T w|^2.
\]
Regarding the expansion of the signed volume term \(V_3(u)\), as we calculate in detail in Lemma B.1 and by using \((4.8)\), we have
\[
V_3(u) = V_3(\text{id}_{S^2} + w) = 1 + Q_{V_3}(w) + V_3(w),
\]
where
\[
Q_{V_3}(w) := \frac{3}{2} \int_{S^2} \langle w, (\text{div}_{S^2} w)x - \sum_{j=1}^{3} x_j \nabla_T w^j \rangle.
\]
Hence, by using the expansions \((4.9)\) and \((4.11)\) in the definition \((4.1)\) of the deficit, we obtain
\[
1 + \frac{3}{4} \int_{S^2} |\nabla_T w|^2 + O\left(\int_{S^2} |\nabla_T w|^2\right)^2 = \left(1 + \mathcal{E}_2(u)\right)\left(1 + Q_{V_3}(w) + \int_{S^2} \langle w, \partial_r w \wedge \partial_r w \rangle\right),
\]
and after rearranging terms,

\[ Q_3(w) = E_2(u) + E_2(u) \cdot Q_{V_3}(w) + (1 + E_2(u)) \int_{S^2} \langle w, \partial_{\tau_1} w \wedge \partial_{\tau_2} w \rangle - \mathcal{O}\left(\int_{S^2} |\nabla_T w|^2 \right)^2 \). \]  

(4.13)

Arguing exactly as in (3.54) (with the Poincaré inequality being applied with constant \( \frac{1}{2} \) instead of \( \frac{1}{6} \) in this case) and (3.55), and using again (4.2)(ii), we have

\[ |Q_{V_3}(w)| \leq \sqrt{\frac{27}{8}} \int_{S^2} |\nabla_T w|^2, \quad \left| \int_{S^2} \langle w, \partial_{\tau_1} w \wedge \partial_{\tau_2} w \rangle \right| \leq \left( \frac{1}{2} \right)_{S^2} |\nabla_T w|^2 \leq 2^{-3/2} \theta \int_{S^2} |\nabla_T w|^2. \]  

(4.14)

Therefore, (4.13), the estimates (4.10) and (4.14) for the remainder terms, and (4.2)(iii) imply that

\[ Q_3(w) \leq E_2(u) + E_2(u)|Q_{V_3}(w)| + (1 + E_2(u)) \int_{S^2} \langle w, \partial_{\tau_1} w \wedge \partial_{\tau_2} w \rangle + \mathcal{O}\left(\int_{S^2} |\nabla_T w|^2 \right)^2 \]  

\[ \leq E_2(u) + \beta \int_{S^2} |\nabla_T w|^2, \]

where the precise value of the constant is \( \beta := \sqrt{\frac{27}{8}} \varepsilon_0 + 2^{-3/2}(1 + \varepsilon_0)\theta + \frac{\theta^2}{2}. \)

\[ \square \]

In view of Lemma 4.5, if we thus choose \( \varepsilon_0 \in (0,1) \) sufficiently small and then \( \theta \in (0,1) \) sufficiently small accordingly, the last term on the right hand side of (4.6) can be set to be a sufficiently small multiple of the Dirichlet energy of \( w \). Therefore, we can move our focus of attention on the coercivity properties of the resulting quadratic form \( Q_3 \) defined in (4.7), which is just the second derivative of the nonlinear combined conformal-isoperimetric deficit \( E_2(u) \) at the \( \text{id}_{S^2} \). This will be the content of the next subsection.

4.2 On the coercivity of the quadratic form \( Q_3 \)

For the most part of this subsection the results hold true in every dimension \( n \geq 3 \). Since we will use them also in Section 5, where we prove linear stability estimates in all dimensions, we also denote here the ambient dimension 3 with the general letter \( n \) (in order to avoid the repetition of the arguments in Section 5), and hope that no confusion will be caused to the reader. Our goal is to examine the coercivity properties of the quadratic form \( Q_n \) in (4.7). By the reductions we have performed (see (4.2) and (4.3)), this can be considered in the space

\[ H_n := \left\{ w \in W^{1,2}(S^n; \mathbb{R}^n) : \int_{S^{n-1}} w = 0, \int_{S^{n-1}} \langle w, x \rangle = 0 \right\}. \]  

(4.15)

Similarly to the notation introduced in the proof of Proposition 3.8 in Subsection 3.3, for every \( k \geq 1 \) we define \( H_{n,k} \) to be the linear subspace of \( H_n \) consisting of those maps in \( H_n \), all the components of which are \( k \)-th order spherical harmonics (cf. also Appendix C), and also define

\[ \tilde{H}_{n,k} := \left\{ w_h : B_1 \mapsto \mathbb{R}^n : \Delta w_h = 0 \text{ in } B_1 \right\}, \]  

(4.16)

so that \( \bigoplus_{k=1}^{\infty} \tilde{H}_{n,k} \) is a \( W^{1,2} \)-orthogonal decomposition of the vector space of harmonic maps \( w_h : B_1 \mapsto \mathbb{R}^n \) for which \( w_h(0) = 0 \) and \( \text{Tr}\nabla w_h(0) = 0 \), the last identities following immediately from their equivalent
formulation on $\mathbb{S}^{n-1}$ in (4.15). Actually, for every $k \geq 1$ we can further consider the $W^{1,2}(B_1)$-Helmholtz decomposition

$$\tilde{H}_{n,k} = \tilde{H}_{n,k,\text{sol}} \oplus \tilde{H}_{n,k,\text{sol}}^\perp,$$

(4.17)

where

$$\tilde{H}_{n,k,\text{sol}} := \left\{ w_h \in \tilde{H}_{n,k} : \text{div} w_h \equiv 0 \text{ in } B_1 \right\},$$

(4.18)

and $\tilde{H}_{n,k,\text{sol}}^\perp$ is its orthogonal complement in $W^{1,2}(B_1; \mathbb{R}^n)$. In view of the $k$-homogeneity of the maps in $\tilde{H}_{n,k}$ in (4.16), we can write the equivalent to (4.17) $W^{1,2}$-decomposition also on $\mathbb{S}^{n-1}$, namely

$$H_{n,k} = H_{n,k,\text{sol}} \oplus H_{n,k,\text{sol}}^\perp,$$

(4.19)

where

$$H_{n,k,\text{sol}} := \left\{ w \in H_{n,k} : w_h \in \tilde{H}_{n,k,\text{sol}} \right\},$$

(4.20)

and $H_{n,k,\text{sol}}^\perp$ is its $W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^n)$-orthogonal complement. Hence, adopting from now on all these notations introduced in (4.15)-(4.20), let us also denote

$$N_{n,k} := \dim H_{n,k} < \infty, \quad N_{1,n,k} := \dim H_{n,k,\text{sol}}, \quad N_{2,n,k} := \dim H_{n,k,\text{sol}}^\perp,$$

(4.21)

so that $N_{n,k} = N_{1,n,k} + N_{2,n,k}$.

Recall also that the second derivative of the signed-volume term $V_n$ at the id$_{\mathbb{S}^{n-1}}$ corresponds to the bilinear form

$$Q_{V_n}(v, w) := \frac{n}{2} \int_{\mathbb{S}^{n-1}} \langle v, A(w) \rangle \quad \text{for } v, w \in H_n,$$

(4.22)

where the associated linear first-order differential operator $A$ is defined as

$$A(w) := (\text{div}_{\mathbb{S}^{n-1}} w)x - \sum_{j=1}^{n} x_j \nabla_T w^j \quad \text{for } w \in H_n.$$

(4.23)

For $n = 3$, the expression (4.22) for the bilinear form $Q_{V_n}(\cdot, \cdot) : H_n \times H_n \to \mathbb{R}$ is essentially derived in the proof of Lemma B.1 see (B.2), (B.3) and (B.9) therein. Another intrinsic calculation for its computation in any dimension is also given in Lemma B.3 at the end of Appendix B.

The main feature that we are going to use in this subsection is the fine interplay between the operator $A$ and the above defined spaces, as it is properly described in the following.

**Lemma 4.6.** For every $k \geq 1$, the operator $A$ defined in (4.23) is a linear self-adjoint isomorphism of the spaces $H_{n,k,\text{sol}}$ and $H_{n,k,\text{sol}}^\perp$ defined in and after (4.20), with respect to the $W^{1,2}$-inner product.

**Proof.** First of all, it is immediate that $A$ is self-adjoint with respect to the $L^2$-inner product in $H_n$, since it arises as the second derivative of $V_n$ at the id$_{\mathbb{S}^{n-1}}$, but it is also easy to verify directly after integrating by parts that for any $v, w \in H_n$,

$$\int_{\mathbb{S}^{n-1}} \langle v, A(w) \rangle = \int_{\mathbb{S}^{n-1}} \langle A(v), w \rangle.$$
Indeed, let
\[ w \in H_{n,k} \implies A(w) \in H_{n,k}. \tag{4.24} \]

Indeed, as we mention in the beginning of Appendix C for \( k \geq 1 \) fixed and \( w \in H_{n,k} \), its harmonic extension \( w_h \) in \( B_1 \) is an \( \mathbb{R}^n \)-valued homogeneous harmonic polynomial of degree \( k \), and \( \forall j = 1, 2, \ldots, n, \)
\[ \nabla w_h^j = \nabla_T w^j + k w^j x, \quad \text{div} w_h = \text{div}_{\mathbb{S}^{n-1}} w + k \langle w, x \rangle \text{ on } \mathbb{S}^{n-1}. \]

Therefore, the operator \( A \) can alternatively be rewritten as
\[ A(w) = (\text{div}_{\mathbb{S}^{n-1}} w)x - \sum_{j=1}^{n} x_j \nabla_T w^j = (\text{div} w_h)x - \sum_{j=1}^{n} x_j \nabla w_h^j \text{ on } \mathbb{S}^{n-1}. \tag{4.25} \]

Writing \( A \) in terms of the full gradient and divergence operators on \( \mathbb{S}^{n-1} \) as in (4.25), we see that
\[ \int_{\mathbb{S}^{n-1}} A(w) = \int_{\mathbb{S}^{n-1}} (\text{div} w_h)x - \sum_{j=1}^{n} x_j \nabla w_h^j = \left. \frac{1}{n} \int_{B_1} (\nabla \text{div} w_h - \sum_{j=1}^{n} \partial_j \nabla w_h^j) \right|_{x} = 0, \tag{4.26} \]
and by (4.15),
\[ \int_{\mathbb{S}^{n-1}} \langle A(w), x \rangle = \int_{\mathbb{S}^{n-1}} \text{div}_{\mathbb{S}^{n-1}} w = (n - 1) \int_{\mathbb{S}^{n-1}} \langle w, x \rangle = 0. \tag{4.27} \]

It is also straightforward to verify that
\[ [A(w)]_h = (\text{div} w_h)x - \sum_{j=1}^{n} x_j \nabla w_h^j \text{ in } B_1, \tag{4.28} \]
so \( [A(w)]_h \) is also an \( \mathbb{R}^n \)-valued homogeneous harmonic polynomial of degree \( k \), and therefore its restriction on \( \mathbb{S}^{n-1} \) is an \( \mathbb{R}^n \)-valued \( k \)-th order spherical harmonic. In total, (4.26)-(4.28) yield the implication in (4.24).

Directly from (4.28) one can also verify that \( A \) leaves \( H_{n,k,\text{sol}} \) invariant, i.e.,
\[ w \in H_{n,k,\text{sol}} \implies \text{div}[A(w)]_h \equiv 0 \text{ in } B_1, \]
as well. It remains to be checked that
\[ \ker A = \{0\} \text{ in } H_{n,k,\text{sol}}. \tag{4.29} \]

Indeed, let \( w \in H_{n,k,\text{sol}} \) be such that
\[ A(w) := (\text{div}_{\mathbb{S}^{n-1}} w)x - \sum_{j=1}^{n} x_j \nabla_T w^j = 0 \iff |A(w)|^2 = (\text{div}_{\mathbb{S}^{n-1}} w)^2 + \sum_{j=1}^{n} x_j \nabla_T w^j = 0 \text{ on } \mathbb{S}^{n-1}. \tag{4.30} \]

Note that since \( w \in H_{n,k,\text{sol}} \) is the restriction on \( \mathbb{S}^{n-1} \) of an \( \mathbb{R}^n \)-valued homogeneous harmonic polynomial of degree \( k \) (hence smooth up to the boundary), the above equation holds true in the classical sense. Hence, by orthogonality, both the normal and the tangential part of \( A(w) \) would have to vanish identically, namely,
\[ \text{div}_{\mathbb{S}^{n-1}} w = 0 \text{ and } \sum_{j=1}^{n} x_j \nabla_T w^j = 0 \text{ on } \mathbb{S}^{n-1}. \tag{4.30} \]
By the definition of $H_{n,k,\text{sol}}$ in (4.20) we have that $\text{div}w_h \equiv 0$ in $B_1$, and therefore (4.30) implies that
\begin{equation}
\langle w, x \rangle = \frac{1}{k}(\text{div}w_h - \text{div}n_{-1}w) = 0 \quad \text{on } S^{n-1}.
\end{equation}

Testing now the second one of the equations in (4.30) with the vector field $w$ itself, integrating by parts on $S^{n-1}$ and using (4.31), we obtain
\begin{equation}
0 = -\int_{S^{n-1}} \left( \sum_{j=1}^{n} x_j \nabla_T w^j, w \right) = -\sum_{j=1}^{n} \int_{S^{n-1}} \left( \nabla_T w, x_j x^T \right)
= \int_{S^{n-1}} |w|^2 - n \int_{S^{n-1}} \langle w, x \rangle^2 = \int_{S^{n-1}} |w|^2,
\end{equation}
i.e., $w \equiv 0$ on $S^{n-1}$. This concludes the proof of (4.29), and thus the proof of the fact that $A$ is a self-adjoint linear isomorphism of $H_{n,k,\text{sol}}$. Hence, $A$ leaves $H_{n,k,\text{sol}}^\perp$ invariant as well, and is actually also an isomorphism of it, as we will see next.

As a consequence of Lemma 4.6, each one of the finite-dimensional subspaces $(H_{n,k,\text{sol}})_{k \geq 1}$ and $(H_{n,k,\text{sol}}^\perp)_{k \geq 1}$ admit an eigenvalue decomposition with respect to $A$ (cf. [17, Chapter 8, Theorem 4.3]).

**Theorem 4.7.** The following statements hold.

(i) For every $k \geq 1$, the subspace $H_{n,k,\text{sol}}$ in (4.20), has an eigenvalue decomposition with respect to the operator $A$, defined in (4.23), as
\begin{equation}
H_{n,k,\text{sol}} = H_{n,k,1} \oplus H_{n,k,2},
\end{equation}
where $H_{n,k,1}$ is the eigenspace of $A$ corresponding to the eigenvalue $\sigma_{n,k,1} := -k$ and $H_{n,k,2}$ is the one corresponding to the eigenvalue $\sigma_{n,k,2} := 1$.

(ii) For every $k \geq 1$, the subspace $H_{n,k,3} := H_{n,k,\text{sol}}^\perp$ is an eigenspace with respect to $A$ corresponding to the eigenvalue $\sigma_{n,k,3} := k + n - 2$.

**Proof.** As we have just remarked before the statement of Theorem 4.7, for every $k \geq 1$ there exists a $W^{1,2}$-orthonormal basis of eigenfunctions $\{w_{n,k,1}, \ldots, w_{n,k,N_{n,k}}\}$ for the subspace $H_{n,k,\text{sol}}$ (see also (4.21)) and similarly, $\{w_{n,k,N_{n,k}+1}, \ldots, w_{n,k,N_{n,k}}\}$ for $H_{n,k,\text{sol}}^\perp$, i.e., for $i = 1, \ldots, N_{n,k}$, the map $w_{n,k,i}$ satisfies the eigenvalue equation
\begin{equation}
A(w_{n,k,i}) := (\text{div}S^{n-1}w_{n,k,i})x - \sum_{j=1}^{n} x_j \nabla_T w_{n,k,i}^j = \sigma_{n,k,i}w_{n,k,i} \quad \text{on } S^{n-1}.
\end{equation}

For each such eigenvalue $\sigma_{n,k,i}$ we denote its corresponding eigenspace by $H_{n,k,i}$. If in (4.34) we take the inner product with the unit normal vector field on $S^{n-1}$, we obtain further that each eigenfunction $w_{n,k,i}$ satisfies the equation
\begin{equation}
\text{div}S^{n-1}w_{n,k,i} = \sigma_{n,k,i} \langle w_{n,k,i}, x \rangle \quad \text{on } S^{n-1},
\end{equation}
which in terms of the full divergence can be rewritten as
\begin{equation}
\text{div}(w_{n,k,i}) = \text{div}S^{n-1}w_{n,k,i} + \langle \partial_t(w_{n,k,i}), x \rangle = (\sigma_{n,k,i} + k) \langle w_{n,k,i}, x \rangle \quad \text{on } S^{n-1}.
\end{equation}

We now fix the index $k \geq 1$ and consider all the different possible cases that will allow us to find the eigenvalues of $A$ in the invariant subspaces $H_{n,k,\text{sol}}$ and $H_{n,k,\text{sol}}^\perp$ respectively.
(a1) Let \( w \) be a non-trivial eigenfunction of \( A \) in \( H_{n,k,\text{sol}} \), so

\[
\text{div} w_h = 0 \text{ in } B_1 \iff \text{div} w_h = 0 \text{ on } S^{n-1},
\]

 debido to (4.16), (4.18) and the \((k - 1)\)-homogeneity of \( \text{div} w_h \) in this case. By (4.36) we see that one possibility for (4.37) to hold, is for the eigenvalue \( \sigma = -k \). We thus set \( \sigma_{n,k,1} := -k \) and label its corresponding eigenspace as

\[
H_{n,k,1} := \text{span}\{w_{n,k,1}, \ldots, w_{n,k,p_{n,k}}\},
\]

where \( p_{n,k} := \dim H_{n,k,1} \).

(a2) Let now \( w \) be a non-trivial eigenfunction of \( A \) in \( H_{n,k,\text{sol}} \), with \( w \in H_{n,k,1}^\perp \). Then, in view of (4.36), the only possibility for (4.37) to hold is iff

\[
\langle w, x \rangle \equiv 0 \text{ on } S^{n-1}.
\]

In this case, \( w \) is a tangential vector field and by (4.35) and (4.38), we have \( \text{div}_{S^{n-1}} w \equiv 0 \text{ on } S^{n-1} \), as well. The eigenvalue equation (4.34) reduces then to

\[
\sigma w = -\sum_{j=1}^n x_j \nabla_T w^j \text{ on } S^{n-1}.
\]

With the very same calculations that we performed in the proof of Lemma 4.6 (see (4.32)), we can test (4.39) with \( w \), integrate by parts and use (4.38), to obtain

\[
\sigma \int_{S^{n-1}} |w|^2 = -\int_{S^{n-1}} \langle w, \sum_{j=1}^n x_j \nabla_T w^j \rangle = \int_{S^{n-1}} |w|^2 - n \int_{S^{n-1}} \langle w, x \rangle^2 = \int_{S^{n-1}} |w|^2.
\]

We label this eigenvalue \( \sigma_{n,k,2} := 1 \), and its corresponding eigenspace as

\[
H_{n,k,2} := \text{span}\{w_{n,k,p_{n,k}+1}, \ldots, w_{n,k,N_{1,n,k}}\},
\]

and in this way we are led to the decomposition (4.33).

(b) Let us now look at eigenfunctions \( w \) of \( A \) in \( H^\perp_{n,k,\text{sol}} \), where the divergence of \( w_h \in H^\perp_{n,k} \) in (4.16) does not vanish identically in \( B_1 \). Since \( w_h \) is an \( \mathbb{R}^n \)-valued \( k \)-homogeneous harmonic polynomial, we have that \( \text{div} w_h \) is a scalar \((k - 1)\)-homogeneous harmonic polynomial, and therefore its restriction on \( S^{n-1} \) is a scalar \((k - 1)\)-spherical harmonic. We can then apply the Laplace-Beltrami operator (see (C.1)) on both sides of (4.36) and use again (4.35), to obtain

\[
(k-1)(k+n-3)\text{div} w_h = -\Delta_{S^{n-1}}(\text{div} w_h) = -(\sigma + k)\Delta_{S^{n-1}}(\langle w, x \rangle)
\]

\[
= (\sigma + k) \left( -\Delta_{S^{n-1}} w - 2\nabla_T w : P_T + \langle w, -\Delta_{S^{n-1}} x \rangle \right)
\]

\[
= \left( k(k+n-2) - 2\sigma + (n-1) \right)(\sigma + k)(\langle w, x \rangle)
\]

\[
= \left( k(k+n-2) - 2\sigma + (n-1) \right)\text{div} w_h \text{ on } S^{n-1}.
\]
Since in this case $\text{div}w_h$ does not vanish identically, we conclude that

$$k(k + n - 2) - 2\sigma + (n - 1) = (k - 1)(k + n - 3) \iff \sigma = k + n - 2.$$ 

We label this eigenvalue as $\sigma_{n,k,3} := k + n - 2$ and its corresponding eigenspace as $H_{n,k,3}$. In particular, we have found that $H_{n,k,3}^\perp = H_{n,k,3}$.

\[\square\]

**Remark 4.8.** We have obtained in total the $W^{1,2}$-orthogonal decomposition of our space of interest into eigenspaces of $A$ as

$$H_n := \bigoplus_{k=1}^\infty (H_{n,k,1} \oplus H_{n,k,2} \oplus H_{n,k,3}).$$ \hspace{1cm} (4.40)

It is easy to construct examples showing that except for $H_{n,1,3}$, none of these eigenspaces are apriori trivial. The triviality of $H_{n,1,3}$ is a consequence of the fact that we had already scaled properly our initial maps $u$, so that the corresponding maps $w$ satisfy $\int_{S^{n-1}} w \cdot x = 0$ (recall (4.15)). Indeed, let $w(x) := \Lambda x \in H_{n,1,3}$ for some $\Lambda \in \mathbb{R}^{n \times n}$. By assumption,

$$0 = \int_{S^{n-1}} w \cdot x = \int_{S^{n-1}} (\Lambda x) \cdot x = \frac{1}{n} \text{Tr}\Lambda.$$ 

Therefore, $\text{div}w_h = \text{Tr}\Lambda \equiv 0$ in $\mathcal{B}_1$, i.e., $w \in H_{n,1,\text{sol}} = H_{n,1,3}^\perp$, forcing $w \equiv 0$, and thus, $H_{n,1,3} = \{0\}$.

The eigenvalue decomposition (4.40) of $H_n$ into eigenspaces of $A$ is valid for every $n \geq 3$. In the case of interest of this subsection, i.e., in dimension $n = 3$, it immediately gives the desired coercivity estimate for the quadratic form $Q_3$ defined in (4.12), with optimal constant. For the rest of this subsection we switch back to denoting the ambient dimension by the number 3. As a consequence of Theorem 4.7 we have

**Lemma 4.9.** The following statements hold true.

(i) The forms $Q_{V_3}$ and $Q_3$, defined in (4.12) and (4.7) respectively, diagonalize on each one of the subspaces $(H_{3,k,i})_{k \geq 1, i = 1,2,3}$, i.e., there exist explicit constants $(c_{3,k,i})_{i=1,2,3}$ and $(C_{3,k,i})_{i=1,2,3}$ so that for every $w \in H_{3,k,i}$,

$$Q_{V_3}(w) = c_{3,k,i} \int_{S^2} |\nabla w|^2 \quad \text{and} \quad Q_3(w) = C_{3,k,i} \int_{S^2} |\nabla w|^2.$$ \hspace{1cm} (4.41)

(ii) For every $k, l \geq 1$ and $i, j = 1, 2, 3$ with $(k, i) \neq (l, j)$, the subspaces $H_{3,k,i}$ and $H_{3,l,j}$ are also $Q_{V_3}$- and $Q_3$-orthogonal, i.e., for every $w_{k,i} \in H_{3,k,i}$ and $w_{l,j} \in H_{3,l,j}$,

$$Q_{V_3}(w_{k,i}, w_{l,j}) = 0 \quad \text{and} \quad Q_3(w_{k,i}, w_{l,j}) = 0.$$ \hspace{1cm} (4.42)

**Proof.** The proof is immediate. For part (i) of the lemma, if we denote by $\lambda_{3,k} := k(k+1)$ the eigenvalues of $-\Delta_{S^2}$ (see (C.2)), then

$$\int_{S^2} |w|^2 = \frac{1}{\lambda_{3,k}} \int_{S^2} |\nabla w|^2 \quad \forall w \in H_{3,k}.$$
In particular, for \( i = 1, 2, 3 \), if \( w \in H_{3,k,i} \), by the definition \((4.22)\) (for \( n = 3 \)) and Theorem \((4.7)\) we have
\[
Q_{V_3}(w) = \frac{3}{2} \int_{S^2} \langle w, A(w) \rangle = \frac{3\sigma_{3,k,i}}{2} \int_{S^2} |w|^2 = \frac{3\sigma_{3,k,i}}{2\lambda_{3,k}} \int_{S^2} |\nabla_T w|^2,
\]
which is precisely the first identity in \((4.41)\) for \( c_{3,k,i} := \frac{3\sigma_{3,k,i}}{2\lambda_{3,k}} \), and then
\[
Q_3(w) = C_{3,k,i} \int_{S^2} |\nabla_T w|^2,
\]
where \( C_{3,k,i} := \frac{3}{4} - c_{3,k,i} \). We list below the precise values of the constants, which are important in this case, since we will need to sum up the identities for \( Q_3 \) in the subspaces \( (H_{3,k,i})_{k \geq 1, i = 1,2,3} \), in order to obtain an estimate on the full space \( H_3 \).
\[
c_{3,k,1} = \frac{-3}{2(k+1)}, \quad c_{3,k,2} = \frac{3}{2k(k+1)}, \quad c_{3,k,3} = \frac{3}{2k}, \quad C_{3,k,1} = \frac{3(k+3)}{4(k+1)}, \quad C_{3,k,2} = \frac{3(k-1)(k+2)}{4k(k+1)}, \quad C_{3,k,3} = \frac{3(k-2)}{4k}.
\]
(4.43)

Part \((ii)\) of the lemma is immediate by the mutual \( W^{1,2} \)-orthogonality of \( (H_{3,k,i})_{k \geq 1, i = 1,2,3} \). \( \square \)

As an immediate consequence of Lemma \((4.9)\) we obtain the desired estimate for the quadratic form \( Q_3 \) defined in \((4.7)\).

**Theorem 4.10.** For every \( w \in H_3 \) (given in \((4.15)\)), the following coercivity estimate holds.
\[
Q_3(w) \geq \frac{1}{4} \int_{S^2} |\nabla_T w - \nabla_T(\Pi_{3,0} w)|^2,
\]
(4.44)

where \( H_{3,0} := H_{3,1,2} \oplus H_{3,2,3} \) is the kernel of \( Q_3 \) in \( H_3 \) and \( \Pi_{3,0} \) is the \( W^{1,2} \)-orthogonal projection of \( H_3 \) onto \( H_{3,0} \). The constant \( \frac{1}{4} \) in the previous estimate is sharp.

**Proof.** Having the precise values of the constants \( (C_{3,k,i})_{k \geq 1, i = 1,2,3} \) in \((4.43)\), we see that \( C_{3,1,2} = C_{3,2,3} = 0 \), but otherwise it is easy to verify that
\[
\tilde{C} := \min_{k \geq 1, \, i \in \{1,2,3\} \atop (k,i) \neq (1,2), (1,3), (2,3)} C_{3,k,i} = C_{3,3,3} = \frac{1}{4}.
\]
(4.45)

Now we can express any \( w \in H_3 \) as a Fourier series in terms of the eigenspace decomposition \((4.40)\), i.e.,
\[
w = \sum_{k,i} w_{3,k,i}, \text{ where } w_{3,k,i} \in H_{3,k,i}, \forall k \geq 1, i = 1,2,3.
\]

Note that, as we have justified in Remark \((4.8)\), \( w_{3,1,3} = 0 \). Expanding the quadratic form \( Q_3 \), and using \((4.41), (4.42)\) and \((4.45)\), we indeed obtain
\[
Q_3(w) = \sum_{(k,i) \in \mathbb{N}^* \times \{1,2,3\}} Q_3(w_{3,k,i}) = \sum_{(k,i) \neq (1,2), (1,3), (2,3)} C_{3,k,i} \int_{S^2} |\nabla_T w_{3,k,i}|^2 \geq \frac{1}{4} \sum_{(k,i) \neq (1,2), (1,3), (2,3)} \int_{S^2} |\nabla_T w_{3,k,i}|^2 = \frac{1}{4} \int_{S^2} |\nabla_T w - \nabla_T (w_{3,1,2} + w_{3,2,3})|^2,
\]
(4.46)

which finishes the proof of \((4.44)\). \( \square \)
4.3 Proof of the local version of Theorem 1.4

The presence of the $Q_3$-degenerate space $H_{3,0}$ in the coercivity estimate (4.44) is a small but natural obstacle to overcome in order to complete the proof of Theorem 1.4. As we mentioned after the statement of Theorem 1.5 in the Introduction, since $H_{3,0}$ will eventually turn out to be isomorphic to the Lie algebra of infinitesimal Möbius transformations of $S^2$, at an infinitesimal level this basically means the following. Although, by the reductions we have made, the map $u$ is apriori supposed to be $\theta$-close to the $id_{S^2}$ in the $W^{1,2}$-topology (recall (4.2)(ii)), there might be another Möbius transformation of $S^2$ that is also $\theta$-close to the $id_{S^2}$ and is a better candidate for the nearest Möbius map to $u$ in terms of its combined conformal-isoperimetric deficit $E_2(u)$ in (4.1). Similarly to [10] and [22], an application of the Inverse Function Theorem and a topological argument will allow us to identify this more suitable candidate, see the details in the subsequent Lemma 4.13 and its proof. For this purpose, we will need the following characterization of the $Q_3$-degenerate subspace $H_{3,0}$, which is valid in every dimension $n \geq 3$, and that is why we now switch back to denoting the ambient dimension by $n$.

Lemma 4.11. The following statements hold true.

(i) The subspace $H_{n,1,2}$ in (4.33) can be characterized as

$$H_{n,1,2} = \{ w \in H_n : w(x) = \Lambda x, \text{ where } \Lambda \in \text{Skew}(n) \},$$

and its dimension is $\dim H_{n,1,2} = \frac{n(n-1)}{2}$. The projection on $H_{n,1,2}$ is therefore characterized by

$$\Pi_{H_{n,1,2}} w = 0 \iff \nabla w_h(0) = \nabla w_h(0)^t.$$

(ii) The subspace $H_{n,2,\text{sol}}$ in (4.20) can be characterized as

$$H_{n,2,\text{sol}} = \left\{ w \in H_n : \forall k = 1, \ldots, n : w^k(x) = \langle \Lambda^k x, x \rangle, \right. \left. \Lambda^k \in \text{Sym}(n) : \text{Tr}\Lambda^k = 0, \sum_{l=1}^n \Lambda^l_{lk} = 0 \right\},$$

and thus

$$\dim H_{n,2,3} = \dim H_{n,2} - \dim H_{n,2,\text{sol}} = n.$$

The projection on $H_{n,2,3}$ is therefore characterized by

$$\Pi_{H_{n,2,3}} w = 0 \iff \int_{S^{n-1}} (\text{div} w_h(x)) x = 0.$$

Proof. For part (i) of the lemma, if $w \in H_{n,1,2}$ we can write it as $w(x) = \Lambda x$ for some $\Lambda \in \mathbb{R}^{n \times n}$. In this space, recalling (4.38),

$$\langle w, x \rangle \equiv 0 \text{ on } S^{n-1} \iff \sum_{1 \leq i \leq j \leq n} (\Lambda_{ij} + \Lambda_{ji}) x_i x_j \equiv 0 \text{ on } S^{n-1} \iff \Lambda^t = -\Lambda.$$

The characterization (4.48) of the projection $\Pi_{H_{n,1,2}}$ is then immediate. For part (ii), let $w \in H_{n,2,\text{sol}}$. By (4.18) and (4.20), its harmonic extension is a homogeneous solenoidal harmonic polynomial of degree 2, so for each $k = 1, \ldots, n$, there exists $\Lambda^k \in \text{Sym}(n)$ such that

$$w^k_h(x) = \langle \Lambda^k x, x \rangle \text{ in } B_1.$$
In particular, for each \( k, l = 1, \ldots, n \), we can compute
\[
\partial_i w^k_i (x) = 2 \sum_{i=1}^n \Lambda^k_i x_i \implies \begin{cases} 
0 = \frac{1}{2} \Delta w^k = \text{Tr} \Lambda^k, \\
0 = \frac{1}{2} \text{div} w = \sum_{i=1}^n \left( \sum_{k=1}^n \Lambda^k_i \right) x_k \iff \sum_{i=1}^n \Lambda^k_i = 0
\end{cases}
\]
so (4.49) follows directly from (4.51). For the last characterization, i.e., (4.50), we have
\[
\Pi_{H_{n,2,3}} w = 0 \iff \Pi_{H_{n,2}} w \in H_{n,2,\text{sol}} \iff \sum_{i=1}^n \left( \nabla^2 w^i_0 (0) \right)_{lk} = 0 \text{ for every } k = 1, \ldots, n,
\]
and by the mean value property of harmonic functions again, we can indeed equivalently write
\[
n \int_{\mathbb{S}^{n-1}} \left( \text{div} w (x) \right) x_k = \int_{B_1} \partial_i \left( \text{div} w (x) \right) = \sum_{i=1}^n \int_{B_1} \partial_t \left( \text{div} w (x) \right) = \sum_{i=1}^n \left( \nabla^2 w^i_0 (0) \right)_{lk} = 0. \quad \square
\]

**Remark 4.12.** It is worth noticing here that simply by counting dimensions,
\[
\dim H_{3,0} = \dim H_{3,1,2} + \dim H_{3,2,3} = 6,
\]
which is also the dimension of \( \text{Conf} (\mathbb{S}^2) \) (recall the notation in Section 2 and see also Remark A.1 for some more details on this Lie group and its corresponding Lie algebra). We therefore only need to verify that \( H_{3,0} \) (introduced after (4.44)) is actually isomorphic to the Lie algebra of infinitesimal Möbius transformations of \( \mathbb{S}^2 \), and then the Inverse Function Theorem can be applied. This is the context of the following.

**Lemma 4.13.** Given \( \theta, \varepsilon_0 \in (0, 1) \) sufficiently small, there exists \( \bar{\theta} \in (0, 1) \) that depends only on \( \theta \) and is sufficiently small as well, so that for every \( u \in B_{\theta, \varepsilon_0} \) (as in (4.2)) there exists \( \phi \in \text{Conf}_+ (\mathbb{S}^2) \) such that
\[
\left( u \circ \phi - \int_{\mathbb{S}^2} u \circ \phi \right) \in B_{\bar{\theta}, \varepsilon_0} \quad \text{and also } \Pi_{3,0} (u \circ \phi) = 0. \tag{4.52}
\]

**Proof.** Given \( u \in B_{\theta, \varepsilon_0} \), in view of the characterizations (4.48) and (4.50), let us introduce the map \( \Psi_u : \text{Conf}^+_+(\mathbb{S}^2) \to \mathbb{R}^6 \), via
\[
\Psi_u (\phi) = \left( \frac{1}{3} \left( \partial_j (u \circ \phi) x^i_j (0) - \partial_i (u \circ \phi) x^i_j (0) \right)_{1 \leq i < j \leq 3}, \int_{\mathbb{S}^2} \left( \text{div} (u \circ \phi) x (0) \right) x \right)
\]
\[
= \left( \int_{\mathbb{S}^2} \left( (u \circ \phi) x_j - (u \circ \phi) x_i \right)_{1 \leq i < j \leq 3}, \int_{\mathbb{S}^2} \left( \text{div} (u \circ \phi) x (0) \right) x \right). \tag{4.53}
\]

Our goal for (4.52) is essentially to show that \( 0 \in \text{Im} (\Psi_u) \). To simplify notation, let us also set
\[
\Psi := \Psi_{|_{\text{id}_{\mathbb{S}^2}}} \tag{4.54}
\]
Clearly, \( \Psi (\text{id}_{\mathbb{S}^2}) = 0 \). In order to apply the Inverse Function Theorem, we look at the differential
\[
d\Psi_{|_{\text{id}_{\mathbb{S}^2}}} : T_{\text{id}_{\mathbb{S}^2}} \text{Conf}^+_+(\mathbb{S}^2) \to \mathbb{R}^6
\]
and prove that it is a non-degenerate linear map. The differential of \( \Psi \) at the \( \text{id}_{\mathbb{S}^2} \) is easy to compute. Indeed, by the linearity of all the operations involved, for every \( Y \in T_{\text{id}_{\mathbb{S}^2}} \text{Conf}^+_+(\mathbb{S}^2) \), defined via
\[
Y (x) := Sx + \mu (\langle x, \xi \rangle x - \xi) : \mathbb{S}^2 \to \mathbb{R}^3, \text{ where } S^t = -S, \xi \in \mathbb{S}^2, \mu \in \mathbb{R}, \tag{4.55}
\]
In particular, (4.57) directly implies that in the domain of definition of $\Psi$ in (4.54), we can calculate (see (A.11) in Remark A.1 for the derivation of this representation), and with a slight abuse of notation in the domain of definition of $\Psi$ in (4.54), we can calculate

$$d\Psi_{|_{id_{S^2}}}(Y) : = \left. \frac{d}{dt} \right|_{t=0} \Psi \left( \exp_{id_{S^2}}(tY) \right) = \Psi \left( \left. \frac{d}{dt} \right|_{t=0} \exp_{id_{S^2}}(tY) \right) = \Psi(Y)$$

$$= \left( \int_{S^2} (Y^i(x)x_j - Y^j(x)x_i) \right)_{1 \leq i < j \leq 3} \int_{S^2} \text{div}Y_h(x)x \right).$$

(4.56)

It is clear that the harmonic extension in $B_1$ of $Y$ as in (4.55) is given by the vector field

$$Y_h(x) = Sx + \mu \left( \langle x, \xi \rangle x - \left( \frac{|x|^2 + 2}{3} \right) \xi \right).$$

(4.57)

In particular, (4.57) directly implies that

$$\int_{S^2} (Y^i(x)x_j - Y^j(x)x_i) = \frac{2}{3} S_{ij} \forall 1 \leq i < j \leq 3, \quad \int_{S^2} \text{div}Y_h(x)x = \frac{10}{9} \mu \xi.$$ 

(4.58)

Therefore, in view of (4.56) and (4.58), we indeed obtain $\ker(d\Psi_{|_{id_{S^2}}}) = \{0\}$, i.e.,

$$d\Psi_{|_{id_{S^2}}} \text{ is a linear isomorphism between } T_{id_{S^2}}Conf_+^1(S^2) \text{ and } \mathbb{R}^6.$$ 

From now on, we denote by $D_\sigma$ the open ball in the 6-dimensional vector space $T_{id_{S^2}}Conf_+^1(S^2)$, centered at 0 (or better said, at $id_{S^2}$) and of radius $\sigma > 0$. Since the exponential mapping $\exp_{id_{S^2}}(\cdot)$ is a local diffeomorphism between a neighbourhood of 0 in $T_{id_{S^2}}Conf_+^1(S^2)$ and a neighbourhood of $id_{S^2}$ in $Conf_+^1(S^2)$, we can use the Inverse Function Theorem to find a sufficiently small $\sigma_0 \in (0, 1)$ such that for the open neighbourhood

$$U_0 := \exp_{id_{S^2}}(D_{\sigma_0})$$

(4.59)

of the $id_{S^2}$ in $Conf_+^1(S^2)$, the map

$$\Psi_{|_{U_0}} : U_0 \subseteq Conf_+^1(S^2) \mapsto \Psi(U_0) \subseteq \mathbb{R}^6 \text{ is a } C^1\text{-diffeomorphism } \Rightarrow \text{ deg}(\Psi, 0; U_0) = 1.$$ 

(4.60)

As a next step, we justify that $\Psi$ is homotopic to $\Psi_u$ in $U_0$. Indeed, for every $\phi \in U_0$, we can estimate

$$|\Psi_u(\phi) - \Psi(\phi)|^2 \leq \sum_{i \neq j} \int_{S^2} \left( \left[ (u - id_{S^2}) \circ \phi \right]^2 \right) x_j^2 + \int_{S^2} \left( \text{div} \left[ (u - id_{S^2}) \circ \phi \right] \right)^2 \leq \int_{S^2} \left( (u - id_{S^2}) \circ \phi \right)^2 + 6 \int_{S^2} \left| \nabla_T \left[ (u - id_{S^2}) \circ \phi \right] \right|^2.$$ 

(4.61)

In the last step of (4.61), we used the general estimate

$$\int_{S^2} (\text{div}v)^2 \leq 3 \int_{S^2} |\nabla v|^2 \leq 6 \int_{S^2} |\nabla Tv|^2 \quad \text{for } v \in W^{1,2}(S^2; \mathbb{R}^3),$$

the last inequality following from the second estimate in [C.7] which is proved in Lemma C.2. Note now that by the conformal invariance of the Dirichlet energy in two dimensions and (4.2)(ii),

$$\int_{S^2} \left| \nabla_T \left[ (u - id_{S^2}) \circ \phi \right] \right|^2 = \int_{S^2} |\nabla_T u - P_T|^2 \leq \theta^2.$$ 

(4.62)
By the change of variables formula, (4.2) again, and the Poincare inequality (C.5) (since $\int_{S^2} (u - \text{id}_{S^2}) = 0$), we also have

\[ \int_{S^2} |(u - \text{id}_{S^2}) \circ \phi|^2 \leq C_1(U_0) \int_{S^2} |\nabla_T u - P_T|^2 \leq C_1(U_0) \theta^2, \]  

(4.63)

where

\[ C_1(U_0) \sim \sup_{\phi \in U_0} \inf_{x \in S^2} |\nabla_T \phi(x)|^{-2} > 0. \]  

(4.64)

The strict positivity of the constant $C_1(U_0)$ in (4.64) is ensured by the fact that we can take the neighbourhood $U_0$ to be sufficiently small around $\text{id}_{S^2}$, which amounts to choosing $\sigma_0 \in (0, 1)$ sufficiently small (see (4.59) and (4.60)). Hence, (4.61)-(4.64) imply that

\[ \| \Psi_u - \Psi \|_{L^\infty(U_0)} \leq C(U_0) \theta, \quad \text{where } C(U_0) := \max \left\{ \sqrt{6}, \sqrt{C_1(U_0)} \right\} > 0. \]  

(4.65)

We can now continue as in [10, Proposition 4.7]. For the sake of making the proof self-contained, we present the argument here, adapted to our setting.

Recalling the notation $D_s$ introduced before (4.59), let us consider the family $(\Gamma_s)_{s \in [0, 1]}$ of closed hypersurfaces in $Conf_+ (\mathbb{S}^2)$, defined by

\[ \Gamma_s := \exp_{\text{id}_{S^2}}(\partial D_{s \sigma_0}), \]  

(4.66)

so that $\Gamma_0 = \{ \text{id}_{S^2} \}$, and $\Gamma_1$ is the topological boundary of $U_0$ inside the manifold $Conf_+ (\mathbb{S}^2)$. Let us define

\[ m(s) := \min_{\phi \in \Gamma_s} |\Psi(\phi)| \quad \text{for every } s \in [0, 1]. \]  

(4.67)

This is obviously a continuous function of $s$. Since $\Psi|_{\Gamma_0} \equiv 0$ and (by (4.66)) $\Psi|_{U_0}$ is a homeomorphism onto its image, by (4.67) we deduce that

\[ m(s) > 0 \quad \forall s \in (0, 1] \quad \text{and} \quad \lim_{s \to 0^+} m(s) = 0. \]

Notice that $m(1) > 0$ depends only on $U_0$, or equivalently only on $\sigma_0$. We can thus choose $\theta \in (0, 1)$ sufficiently small (depending on $\sigma_0$), so that for the constant $C(U_0)$ of (4.65), there holds

\[ (C(U_0) + 1) \theta \leq \frac{m(1)}{2}, \]  

(4.68)

and then define

\[ s_\theta := \inf \left\{ s \in [0, 1] : m(s) \geq (C(U_0) + 1) \theta \right\}. \]  

(4.69)

By (4.67) and (4.69) it is clear that $\lim_{s \to 0^+} s_\theta = 0$. Let us now consider the linear homotopy between $\Psi$ and $\Psi_u$. For every $t \in [0, 1]$ and $\phi \in \Gamma_{s_\theta} \subseteq U_0 \subseteq Conf_+ (\mathbb{S}^2)$, by (4.67), (4.69) and (4.65), we have

\[ \left| \left( (1 - t)\Psi + t\Psi_u \right)(\phi) \right| \geq |\Psi(\phi)| - t(|\Psi_u - \Psi(\phi)| \geq \min_{\phi \in \Gamma_{s_\theta}} |\Psi(\phi)| - \|\Psi_u - \Psi\|_{L^\infty(U_0)} \right. \]

\[ \geq m_{s_\theta} - C(U_0) \theta \geq \theta > 0. \]  

(4.70)

In particular, by (4.70) we find that

\[ \left( (1 - t)\Psi + t\Psi_u \right)(\phi) \neq 0 \quad \forall t \in [0, 1] \quad \text{and} \quad \phi \in \Gamma_{s_\theta}. \]
Since the degree around 0 remains constant through this linear homotopy, if $U_{s_0} \subseteq U_0$ is the open neighbourhood around the id$_{S^2}$ in $Conf_+ (S^2)$ with $\partial U_{s_0} = \Gamma_{s_0}$, i.e., $U_{s_0} := \exp_{id_{S^2}} (D_{s_0} \sigma_0)$ (recall $[4.66]$), then
\[
\deg(\Psi_u, 0; U_{s_0}) = \deg(\Psi, 0; U_{s_0}) = 1 .
\] (4.71)
As a consequence of $[4.71]$, $[4.53]$ and the characterizations $[4.48]$, $[4.50]$, there exists a M"obius transformation $\phi \in U_{s_0} \subseteq Conf_+ (S^2)$, so that
\[
\Psi_u (\phi) = 0 \iff \Pi_{3,0} (u \circ \phi) = 0 .
\]
In the same fashion as have estimated in $[4.62]$, and by $[4.2](ii)$ again,
\[
\int_{S^2} |\nabla_T (u \circ \phi) - P_T|^2 \leq 2 (\theta^2 + \int_{S^2} |\nabla_T \phi - P_T|^2) \leq 2 (\theta^2 + C_{s_0}) ,
\]
where
\[
C_{s_0} := \max_{\phi \in U_{s_0}} \|\phi - id_{S^2}\|_{W^{1,2}(S^2)} .
\] (4.72)
Since all topologies in the finite dimensional manifold $Conf_+ (S^2)$ are equivalent and $\lim_{\theta \to 0^+} s_0 = 0$, by $[4.72]$ we also have that $\lim_{\theta \to 0^+} C_{s_0} = 0$. Hence, we can first take $\sigma_0 \in (0, 1)$ small enough and then $\theta \in (0, 1)$ small enough (see also $[4.68]$), so that after replacing $u \circ \phi$ with $(u \circ \phi - \int_{S^2} u \circ \phi)$ if necessary to fix its mean value to 0, we have (by using again the conformal invariance of the deficit) that $u \circ \phi \in B_{\tilde{\theta}, \varepsilon_0}$, where $\tilde{\theta} := \sqrt{2 (\theta^2 + C_{s_0})} > 0$ is again small accordingly. This finishes the proof of $[4.52]$. \hfill \square

4.4 Proof of Theorem 1.4

We can now combine all the previous steps to complete the proof of the main theorem for the conformal case in dimension 3.

**Proof of Theorem 1.4** In view of Corollary $[4.2]$ for $\theta \in (0, 1)$ and $\varepsilon_0 \in (0, 1)$ that will be chosen sufficiently small in the end, let us consider a map $u \in B_{\theta, \varepsilon_0}$. By first using Lemma $[4.13]$ and then Lemma $[4.4]$ we can find a M"obius transformation $\phi \in Conf_+ (S^2)$ such that the map
\[
\tilde{u} := \frac{1}{\int_{S^2} \langle u \circ \phi, x \rangle} \left( u \circ \phi - \int_{S^2} u \circ \phi \right)
\]
has all the desired properties, i.e.,
\[
\tilde{u} \in B_{\theta, \varepsilon_0} , \quad \mathcal{E}_2 (\tilde{u}) = \mathcal{E}_2 (u) , \quad \int_{S^2} \langle \tilde{u}, x \rangle = 1 \quad \text{and} \quad \Pi_{3,0} (\tilde{u} - id_{S^2}) = 0 ,
\] (4.73)
where we have abused notation by not replacing $\theta$ with $\tilde{\theta}$. Setting $\tilde{w} := \tilde{u} - id_{S^2}$, we can again expand the deficit around the id$_{S^2}$ and arrive at $[4.6]$. This estimate, together with $[4.44]$ and the fact that $\tilde{w} \in H_3$ is such that $\Pi_{3,0} \tilde{w} = 0$ (by $[4.15]$ and $[4.73]$) yield
\[
\frac{1}{4} \int_{S^2} |\nabla_T \tilde{w}|^2 \leq Q_3 (\tilde{w}) \leq \mathcal{E}_2 (u) + \int_{S^2} |\nabla_T \tilde{w}|^2 .
\] (4.74)
By Lemma $[4.5]$ we can then choose $\varepsilon_0 \in (0, 1)$ small enough and subsequently $\theta \in (0, 1)$ small enough, so that $\beta \leq \frac{1}{8}$, in order to absorb the last term on the right hand side of $[4.74]$ in the one on the left, and conclude for $\phi$ as above and $\lambda := \int_{S^2} \langle u \circ \phi, x \rangle > 0$. \hfill \square
5 Linear stability estimates for $Isom(S^{n-1})$ and $Conf(S^{n-1})$

As we have seen in the Section 4, the key step in proving Theorem 1.4 consists in establishing the corresponding linear estimate, i.e., Theorem 4.10. What we would like to present in this section, is how the eigenvalue decomposition of $H_n$ into eigenspaces of $A$, as provided by Theorem 4.7, which is valid in every dimension (actually even in dimension $n = 2$), can be used to prove the analogous linear estimate for the quadratic form $Q_n$ introduced in (1.12), also in dimensions $n \geq 4$. As explained in the Introduction, this quadratic form is associated with the combined conformal-isoperimetric deficit $\mathcal{E}_{n-1}$ defined in (1.10) (cf. Appendix B for the precise calculations). We discuss in detail this case first, since the proof of the corresponding estimate for the isometric case, i.e., Theorem 1.8, is essentially the same and is discussed in Subsection 5.3.

5.1 Proof of Theorem 1.5

For $n = 3$, Theorem 1.5 is precisely Theorem 4.10 whose proof was given in Subsection 4.2 and actually the optimal constant for the linear estimate (4.44) was calculated. In the higher dimensional case $n \geq 4$, the quadratic form $Q_n$ in (1.12) has an extra $\int_{S^{n-1}}(\text{div}_{S^{n-1}}w)^2$-term, and the study of its coercivity properties is slightly more complicated than before.

Remark 5.1. An abstract way to obtain the estimate (1.13) of Theorem 1.5 would be to identify the kernel of $Q_n$ in $H_n$ (see (4.15)) (and prove that it is exactly the subspace $H_{n,0}$) and then use a standard contradiction\compactness argument (that we describe in Subsection 5.3 for the corresponding result in the isometric case). Notice that $w \in H_n$ lies in the kernel of the nonnegative quadratic form $Q_n$ iff $Q_n(v, w) = 0 \ \forall v \in H_n$, i.e., iff $w \in H_n$ lies in the kernel of the associated Euler-Lagrange operator

$$\mathcal{L}(w) := -\frac{1}{n-1}\Delta_{S^{n-1}}w - \frac{n-3}{(n-1)^2}((\nabla_T\text{div}_{S^{n-1}}w)-(n-1)(\text{div}_{S^{n-1}}w)x) - (\text{div}_{S^{n-1}}w)x - \sum_{j=1}^{n} x_j\nabla_Tw_j), \quad (5.1)$$

in the sense of distributions. When $n = 3$, the second term in (5.1) is dropping out, and since $\mathcal{L}$ leaves the subspaces $(H_{3,k,\text{sol}})_{k \geq 1}, (H_{3,k,\text{sol}}^\perp)_{k \geq 1}$ invariant in this case, the Euler-Lagrange equation can be solved explicitly, showing that $\ker \mathcal{L} = H_{3,0}$ (as Theorem 4.10 describes quantitatively). Although in slightly hidden form, this was essentially the point of Lemma 4.6 and the subsequent results in Subsection 4.2.

In higher dimensions $n \geq 4$, since the operator $(\nabla_T\text{div}_{S^{n-1}}w - (n-1)(\text{div}_{S^{n-1}}w)x)$ neither commutes with $A$ ((4.23)), nor leaves the subspaces $(H_{n,k,\text{sol}})_{k \geq 1}, (H_{n,k,\text{sol}}^\perp)_{k \geq 1}$ invariant, it is not clear if there is a straightforward argument to solve the equation $\mathcal{L}(w) = 0$ explicitly, and show that indeed $\ker \mathcal{L} = H_{n,0}$ in this case as well. In particular, (comparing with (4.46)), mixed terms of special type are expected to be present in the Fourier decomposition of $Q_n$ into the eigenspaces of $A$, which were identified in Theorem 4.7. Nevertheless, it will turn out that we can still use the latter to show that the presence of the mixed divergence-terms is harmless, i.e., it does not produce any further zeros (other than $H_{n,0}$) in $Q_n$. Simultaneously, we obtain the desired coercivity estimate (1.13) (with an explicit lower bound for the optimal constant) by examining how $Q_n$ behaves in each one of the eigenspaces $(H_{n,k,i})_{k \geq i = 1, 2, 3}$ of $A$ separately.
Following the notation we had in Subsection 4.2, we first present two auxiliary lemmata that entail most of the essential ingredients for the proof of Theorem 1.5 also in dimensions $n \geq 4$.

**Lemma 5.2.** For $n \geq 3$ and $k \geq 1$, let us denote by $\lambda_{n,k} := k(k + n - 2)$ the eigenvalues of $-\Delta_{S^n-1}$ (see (C.2)) and let $i = 1, 2, 3$. For every $w \in H_{n,k,i}$ (as in Theorem 4.7), we have

$$Q_{V_n}(w) = c_{n,k,i} \int_{S^n-1} |\nabla w|^2, \quad \text{where} \quad c_{n,k,i} := \frac{n\sigma_{n,k,i}}{2\lambda_{n,k}},$$

$$\int_{S^n-1} (\text{div}_{S^n-1} w)^2 = \alpha_{n,k,i} \int_{S^n-1} |\nabla w|^2, \quad \text{where} \quad \alpha_{n,k,i} := \frac{\sigma_{n,k,i}^2(2\lambda_{n,k}c_{n,k,i} - n)}{n\lambda_{n,k}(2\sigma_{n,k,i} - n)},$$

$$Q_n(w) = C_{n,k,i} \int_{S^n-1} |\nabla w|^2, \quad \text{where} \quad C_{n,k,i} := \frac{n}{2(n-1)} + \frac{n(n-3)}{2(n-1)^2}\alpha_{n,k,i} - c_{n,k,i}.$$

**Proof.** The first identity is immediate from (4.22), (4.23), Theorem 4.7 and (C.3). For the second one, after integration by parts we see that the quadratic form $Q_{V_n}$ can be equivalently rewritten as

$$Q_{V_n}(w) = \frac{n}{2} \int_{S^n-1} \left(2\text{div}_{S^n-1} w(w, x) - n\langle w, x \rangle^2 + |w|^2\right).$$

(5.3)

Together with the first identity and (4.35), (5.3) yields the desired identity, and then the one for $Q_n$ follows immediately by its definition in (1.12) and the two previous identities that we just checked.

Since they will again play an important role in the sequel (as it was the case for the constants in (4.43)), let us list below the precise values of the previous constants appearing in (5.2). The last set of constants in the following table is considered for $k \geq 2$, because in any case $\Pi_{H_{n,1,i}} w = 0$ for every $w \in H_n$, as we have justified in Remark 4.8

For $k \geq 1$;

$$\left\{ \begin{array}{l}
  c_{n,k,1} = \frac{-n}{2(k+n-2)} \\
  \alpha_{n,k,1} = \frac{k(k+1)}{(k+n-2)(2k+n)} \\
  C_{n,k,1} = \frac{n}{2} \left(1 + \frac{1}{k+n-2} + \frac{(n-3)(k+1)}{(n-1)^2(k+n-2)(2k+n)}\right)
\end{array} \right\},$$

(5.4)

For $k \geq 1$;

$$\left\{ \begin{array}{l}
  c_{n,k,2} = \frac{n}{2k(k+n-2)} \\
  \alpha_{n,k,2} = 0 \\
  C_{n,k,2} = \frac{(k-1)(k+n-1)}{2(n-1)k(k+n-2)}
\end{array} \right\},$$

For $k \geq 2$;

$$\left\{ \begin{array}{l}
  c_{n,k,3} = \frac{n}{2k} \\
  \alpha_{n,k,3} = \frac{(k+n-2)(k+n-3)}{k(2k+n-4)} \\
  C_{n,k,3} = \frac{n(k-2)((3n-5)k+(n^2-6n+7))}{2(n-1)^2k(2k+n-4)}
\end{array} \right\}.$$

The next ingredient we need is the following.
Lemma 5.3. The following statements hold true.

(i) Let \( n \geq 3 \). For every \( k, l \geq 1 \) and \( i, j = 1, 2, 3 \) with \( (k, i) \neq (l, j) \), the subspaces \( H_{n,k,i} \) and \( H_{n,l,j} \) (introduced in Theorem 4.7) are \( Q_{n^*} \) and \( \tilde{Q}_n \)-orthogonal, where

\[
\tilde{Q}_n(w) := \frac{n}{2(n-1)} \int_{S^{n-1}} |\nabla_T w|^2 - Q_{n}(w),
\]

i.e., for every \( w_{n,k,i} \in H_{n,k,i} \) and \( w_{n,l,j} \in H_{n,l,j} \),

\[
Q_{n}(w_{n,k,i}, w_{n,l,j}) = 0 \quad \text{and} \quad \tilde{Q}_n(w_{n,k,i}, w_{n,l,j}) = 0.
\]

(ii) Let \( n \geq 3 \), \( w \in H_n \) written in Fourier series as \( w = \sum_{(k,i) \in \mathbb{N}^* \times\{1,2,3\}}^{(k,i) \neq (1,3)} w_{n,k,i} \), where \( w_{n,k,i} \in H_{n,k,i} \). Then,

\[
Q_{n}(w) = \sum_{(k,i) \in \mathbb{N}^* \times\{1,2,3\}}^{(k,i) \neq (1,3)} Q_{n}(w_{n,k,i}) + \frac{1}{2(n-1)} \sum_{k \geq 1} \int_{S^{n-1}} \text{div}_{S^{n-1}} w_{n,k,1} \text{div}_{S^{n-1}} w_{n,k+2,3} + \frac{n}{2(n-1)^2} \sum_{k \geq 3} \int_{S^{n-1}} \text{div}_{S^{n-1}} w_{n,k,3} \text{div}_{S^{n-1}} w_{n,k+2,1}.
\]

Remark 5.4. Before giving the proof of Lemma 5.3 let us point out an interesting feature in formula (5.7), which will be useful in the proof of Theorem 1.5, namely that the summation in the last term of the expression starts from \( k = 3 \). The reason for this is that in any case \( w_{n,1,3} \equiv 0 \) whenever \( w \in H_n \) (see Remark 4.8), but one can also check that

\[
\int_{S^{n-1}} \text{div}_{S^{n-1}} w_{n,2,3} \text{div}_{S^{n-1}} w_{n,4,1} = 0.
\]

In order to prove (5.8), recall that by definition (see (4.18), (4.20) and (4.33)), \( \text{div}(w_{n,4,1})_h \equiv 0 \) in \( B_1 \), and by also using (4.35) for \( (k,i) = (2,3), (4,1) \) and the divergence theorem, we obtain

\[
\int_{S^{n-1}} \text{div}_{S^{n-1}} w_{n,2,3} \text{div}_{S^{n-1}} w_{n,4,1} = -4 \int_{S^{n-1}} \langle \langle w_{n,2,3}, x \rangle (w_{n,4,1}, x) = -4 \int_{B_1} \langle \nabla \langle (w_{n,2,3}, x), (w_{n,4,1})_h \rangle \rangle.
\]

To justify that the last integral on the right hand side of (5.9) is zero, observe that

\[
\int_{B_1} \langle \nabla \langle (w_{n,2,3}), x \rangle, (w_{n,4,1})_h \rangle \sum_{i=1}^{n} \int_{B_1} (w_{n,4,1})_h^i \langle \partial_i (w_{n,2,3}), x \rangle,
\]

because \( \int_{B_1} \langle (w_{n,4,1})_h, (w_{n,2,3})_h \rangle = 0 \). Moreover, we observe that for every \( i = 1, \ldots, n \),

\[
\Delta \langle \partial_i (w_{n,2,3})_h, x \rangle = 2 \partial_i \text{div}(w_{n,2,3})_h \quad \text{in} \quad B_1.
\]

Since \( (w_{n,2,3})_h \) is an \( \mathbb{R}^n \)-valued 2nd-order homogeneous harmonic polynomial, \( \partial_i \text{div}(w_{n,2,3})_h \) is simply a constant. But then, (5.11) implies that the function

\[
\langle \partial_i (w_{n,2,3})_h, x \rangle - \frac{\partial_i \text{div}(w_{n,2,3})_h |x|^2}{n}
\]
is a homogeneous harmonic polynomial of degree 2, hence also \( L^2 \)-orthogonal to \( (w_{n,4,1})_h^i \). Thus,
\[
\int_{B_1} (w_{n,4,1})_h^i \partial_i (w_{n,2,3})_h (w_{n,2,3})_h \, x = \frac{\partial_i \text{div}(w_{n,2,3})_h}{n} \int_{B_1} (w_{n,4,1})_h^i |x|^2 = \frac{\partial_i \text{div}(w_{n,2,3})_h}{n+6} \int_{S^{n-1}} (w_{n,4,1})_h^i = 0, \tag{5.12}
\]
where we used the fact that the function \( (w_{n,4,1})_h^i |x|^2 \) is 6-homogeneous, so that we can write its integral over \( B_1 \) as an integral over \( S^{n-1} \), up to the correct multiplicative constant. The last integral in \( \text{(5.12)} \) is of course zero for every nontrivial spherical harmonic. Therefore, \( \text{(5.9)}-\text{(5.12)} \) imply \( \text{(5.8)} \). Note that the previous argument relies on the fact that \( \partial_i \text{div}(w_{n,2,3})_h \) is constant, and of course cannot be implemented for the mixed terms of higher order.

**Proof of Lemma 5.3** As in Lemma 4.9 part (i) is an immediate consequence of the fact that the subspaces \( (H_{n,k,i})_{k \geq 1, i = 1,2,3} \) are mutually orthogonal in \( W^{1,2}(S^{n-1}; \mathbb{R}^n) \). For part (ii), having established that the form \( \tilde{Q}_n \) in \( \text{(5.5)} \) splits completely in the eigenspaces \( (H_{n,k,i})_{(k,i) \in \mathbb{N}^* \times \{1,2,3\}\setminus\{1,3\}} \), what remains to be checked is that whenever \( w_{n,k,i} \in H_{n,k,i} \) and \( w_{n,l,j} \in H_{n,l,j} \), there holds
\[
\int_{S^{n-1}} \text{div}_{S^{n-1}} w_{n,k,i} \text{div}_{S^{n-1}} w_{n,l,j} = 0 \tag{5.13}
\]
for all pairs \( \{(k,i), (l,j)\} \in \mathbb{N}^* \times \{1,2,3\}\setminus\{1,3\} \) with \( (k,i) \neq (l,j) \), except those of the form \( \{(k,1), (k+2,3)\} \) and \( \{(k,3), (k+2,1)\} \).

This can be checked again using the different equivalent formulas for \( Q_{V_n} \). Since \( \text{div}_{S^{n-1}} w \equiv 0 \) whenever \( w \in H_{n,k,2} \) (see \( \text{(4.35)} \) and \( \text{(4.38)} \)), we may suppose without loss of generality that \( i, j \in \{1,3\} \), and then by \( \text{(5.3)} \) (written now in its bilinear expression),
\[
Q_{V_n}(w_{n,k,i}, w_{n,l,j}) = \frac{n}{2} \int_{S^{n-1}} \text{div}_{S^{n-1}} w_{n,k,i} \langle w_{n,l,j}, x \rangle + \frac{n}{2} \int_{S^{n-1}} \text{div}_{S^{n-1}} w_{n,l,j} \langle w_{n,k,i}, x \rangle - \frac{n^2}{2} \int_{S^{n-1}} \langle w_{n,k,i}, x \rangle \langle w_{n,l,j}, x \rangle + \frac{n}{2} \int_{S^{n-1}} \langle w_{n,k,i}, w_{n,l,j} \rangle \tag{5.14}.
\]
Actually, as we verified in part (i), \( Q_{V_n}(w_{n,k,i}, w_{n,l,j}) = 0 \), and in view of \( \text{(4.35)} \), \( \text{(5.14)} \) yields
\[
(s_{n,k,i} + s_{n,l,j} - n) \int_{S^{n-1}} \text{div}_{S^{n-1}} w_{n,k,i} \text{div}_{S^{n-1}} w_{n,l,j} = 0,
\]
i.e., \( \text{(5.13)} \) holds, unless the pairs \( (k,i) \neq (l,j) \) are such that \( s_{n,k,i} + s_{n,l,j} = n \). In this respect,

- **(i)** If \( i = j = 1 \), \( s_{n,k,i} + s_{n,l,j} = -k - l < 0 < n \),
- **(ii)** If \( i = j = 3 \), \( s_{n,k,i} + s_{n,l,j} = k + l + 2n - 4 \geq 2n - 2 > n \),
- **(iii)** If \( i = 1, j = 3 \), \( s_{n,k,i} + s_{n,l,j} = n \iff -k + l + n - 2 = n \iff l = k + 2 \),
- **(iv)** If \( i = 3, j = 1 \), \( s_{n,k,i} + s_{n,l,j} = n \iff k + n - 2 - l = n \iff k = l + 2 \),

which proves the desired claim and then the formula \( \text{(5.7)} \) for \( Q_n \) follows by the bilinearity of the expression, and the observation we made in Remark 5.3. 

\[\square\]
We now have all the necessary ingredients to prove Theorem 1.5. As a preliminary remark, let us note that by taking a closer look at the values of the constants (5.4) of Lemma 5.2 we see that in the case $n \geq 4$ one cannot merely neglect the $\int_{S^{n-1}} (\text{div}_{S^{n-1}} w)^2$-term and argue exactly as in the proof of Theorem 4.10 ($n = 3$), something that was also indicated in Remark 5.1. Indeed, although the quadratic form $\tilde{Q}_n(w)$ in (5.5) is splitting among the eigenspaces $(H_{n,k,i})_{k \geq 1, i = 1,2,3}$, for $n \geq 4$ it does not have a sign. Actually, $\tilde{Q}_n(w)$ is negative in $H_{n,k,3}$ for every $k = 2, \ldots, n - 2$, zero in $H_{n,1,2}, H_{n,n-1,3}$ and strictly positive in each one of the other eigenspaces. On the other hand, we see that the quadratic form $Q_n$ vanishes again in the space $H_{n,0} := H_{n,1,2} \oplus H_{n,2,3}$.

**Proof of Theorem 1.5 ($n \geq 4$).** For every $k \geq 1$, we use the Cauchy-Schwartz inequality, the second identity in (5.2), the inequality

$$|ab| \leq \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2, \quad a, b \in \mathbb{R}, \quad \varepsilon > 0,$$

with weight $\varepsilon_{n,k} = \frac{k+n}{k} > 0$, and (5.4), in order to estimate

$$\left| \int_{S^{n-1}} \text{div}_{S^{n-1}} w_{n,k,1} \text{div}_{S^{n-1}} w_{n,k,2,3} \right| \leq \left( \alpha_{n,k,1} \int_{S^{n-1}} |\nabla_T w_{n,k,1}|^2 \right)^{\frac{1}{2}} \left( \alpha_{n,k+2,3} \int_{S^{n-1}} |\nabla_T w_{n,k+2,3}|^2 \right)^{\frac{1}{2}} \leq \frac{\alpha_{n,k,1} \varepsilon_{n,k}}{2} \int_{S^{n-1}} |\nabla_T w_{n,k,1}|^2 + \frac{\alpha_{n,k+2,3} \varepsilon_{n,k}}{2} \int_{S^{n-1}} |\nabla_T w_{n,k+2,3}|^2 \leq \frac{(k+1)(k+n)}{2(k+n-2)(2k+n)} \int_{S^{n-1}} |\nabla_T w_{n,k,1}|^2 + \frac{k(k+n-1)}{2(2k+n)} \int_{S^{n-1}} |\nabla_T w_{n,k,2,3}|^2.$$

(5.15)

For the last summand in (5.7), we shift the summation index to start from $k = 1$ and estimate as before,

$$\left| \int_{S^{n-1}} \text{div}_{S^{n-1}} w_{n,k+2,3} \text{div}_{S^{n-1}} w_{n,k+4,1} \right| \leq \frac{\alpha_{n,k+2,3} \varepsilon_{n,k}}{2} \int_{S^{n-1}} |\nabla_T w_{n,k+2,3}|^2 + \frac{\alpha_{n,k+4,1} \varepsilon_{n,k}}{2} \int_{S^{n-1}} |\nabla_T w_{n,k+4,1}|^2 \leq \frac{k(k+n-1)}{2(2k+n)} \int_{S^{n-1}} |\nabla_T w_{n,k+2,3}|^2 + \frac{(k+4)(k+5)(k+n)}{2k(k+n+2)(2k+n+8)} \int_{S^{n-1}} |\nabla_T w_{n,k+4,1}|^2.$$

(5.16)

Note that the choice of the weights $\varepsilon_{n,k}$ was such that the second term in the last line of (5.15) coincides with the first term in the last equality in (5.16). The series appearing in (5.7) are all absolutely summable, $Q_n(w_{1,2}) = Q_n(w_{2,3}) = 0$, and using (5.15) and (5.16), we can estimate the form $Q_n$ from below by,

$$Q_n(w) \geq \sum_{(k,i) \in \mathbb{N}^+ \setminus \{1,2,3\}} Q_n(w_{n,k,i}) - \frac{1}{4} \frac{n(n-3)}{(n-1)^2} \sum_{k \geq 1} \frac{(k+1)(k+n)}{k(k+n-2)(2k+n)} \int_{S^{n-1}} |\nabla_T w_{n,k,1}|^2 - \frac{1}{4} \frac{n(n-3)}{(n-1)^2} \sum_{k \geq 1} \frac{(k+4)(k+5)(k+n)}{k(k+n+2)(2k+n+8)} \int_{S^{n-1}} |\nabla_T w_{n,k+2,3}|^2$$

$$- \frac{1}{2} \frac{n(n-3)}{(n-1)^2} \sum_{k \geq 1} \frac{k(k+n-1)}{(k+2)(2k+n)} \int_{S^{n-1}} |\nabla_T w_{n,k+4,1}|^2.$$

(5.17)
Moreover, directly from (5.4) and (5.19), one sees that where the new constants are defined as

\[
\begin{align*}
\tilde{C}_{n,k,1} &:= C_{n,k,1} - \frac{n(n-3)(k+1)}{4(n-1)^2(k+n-2)(2k+n)} \left( \frac{k(k+n-4)\chi_{m>5}(k)}{k-4} + (k + n) \right), \quad k \geq 1 \\
\tilde{C}_{n,k,2} &:= C_{n,k,2}, \quad k \geq 2, \\
\tilde{C}_{n,k,3} &:= C_{n,k,3} - \frac{1}{2} \frac{n(n-3)(k-2)(k+n-3)}{(n-1)^2 (2k+n-4)} , \quad k \geq 3.
\end{align*}
\]

(5.19)

By elementary algebraic calculations that we omit here for the sake of brevity, one can verify that

\[
\min_{k \geq 1} \tilde{C}_{n,k,1} =: C_{n,1} > 0, \quad \min_{k \geq 2} \tilde{C}_{n,k,2} =: C_{n,2} > 0, \quad \min_{k \geq 3} \tilde{C}_{n,k,3} =: C_{n,3} > 0.
\]

(5.20)

Indeed, for \(i = 1, 2, 3\), one can write \(\tilde{C}_{n,k,i} := p_{n,i}(k)/q_{n,i}(k)\), where \(p_{n,i}(k), q_{n,i}(k)\) are explicit polynomials of the same degree in \(k\) (of degree 3 when \(i = 1\), and degree 2 when \(i = 2, 3\)) and verify algebraically that

\[
\tilde{C}_{n,k,1} > 0 \quad \text{for} \quad k \geq 1, \quad \tilde{C}_{n,k,2} > 0 \quad \text{for} \quad k \geq 2, \quad \text{and} \quad \tilde{C}_{n,k,3} > 0 \quad \text{for} \quad k \geq 3.
\]

(5.21)

Moreover, directly from (5.4) and (5.19), one sees that

\[
\lim_{k \to \infty} \tilde{C}_{n,k,1} = \lim_{k \to \infty} \tilde{C}_{n,k,2} = \lim_{k \to \infty} \tilde{C}_{n,k,3} = \frac{n}{2(n-1)} > 0,
\]

(5.22)

and then (5.20) follows from (5.21) and (5.22). The precise values of the constants \(C_{n,i}\) could be calculated as well, by examining the monotonicity with respect to \(k\) of the sequences \((\tilde{C}_{n,k,i})\) respectively (or alternatively of the corresponding rational functions with respect to the continuum variable). Labelling

\[
C_n := \min\{C_{n,1}, C_{n,2}, C_{n,3}\} > 0,
\]

(5.23)

we can use (5.18)–(5.23), to further estimate from below,

\[
Q_n(w) \geq C_n \left( \sum_{k \geq 1} \int_{S^{n-1}} |\nabla_T w_{n,k,1}|^2 + \sum_{k \geq 2} \int_{S^{n-1}} |\nabla_T w_{n,k,2}|^2 + \sum_{k \geq 3} \int_{S^{n-1}} |\nabla_T w_{n,k,3}|^2 \right)
\]

\[
= C_n \int_{S^{n-1}} \left| \nabla_T w - (\nabla_T w_{n,1,2} + \nabla_T w_{n,2,3}) \right|^2,
\]

which finishes the proof of (1.13). For \(n \geq 4\), the constant \(C_n\) in (5.23) provides an explicit lower bound for the value of the optimal constant for which (1.13) holds.

\(\square\)

**Remark 5.5.** As we mentioned in the Introduction, the linear estimate (1.13) would then imply Corollary 1.6 by following exactly the same procedure as in Section 4. In this setting, one can of course reduce to showing 1.15 for maps \(u \in W^{1,\infty}(S^{n-1}; \mathbb{R}^n)\) which satisfy the stronger conditions

\[
\int_{S^{n-1}} u = 0, \quad \|\nabla_T u - P_T\|_{L^\infty} \leq \theta \ll 1, \quad \mathcal{E}_{n-1}(u) \leq \varepsilon_0 \ll 1.
\]

With the aid of Lemma B.3 and Remark B.4 the reader can easily verify that the rest of the arguments in the proof of Theorem 1.4 carry over also in dimensions \(n \geq 4\), essentially unchanged.
5.2 Comparison of Theorem 1.5 with Korn’s inequality for the trace-free symmetrized gradient operator in the bulk

Let \( n \geq 3 \) and \( U \) be an open bounded Lipschitz domain of \( \mathbb{R}^n \). If \( v \in W^{1,n}(U; \mathbb{R}^n) \), then

\[
\int_U \left( \frac{\| \nabla v \|^2}{n} \right)^{\frac{n}{n-1}} \geq \int_U \det \nabla v,
\]

with equality iff \( \nabla v \in \mathbb{R}^+SO(n) \) for a.e. \( x \in U \), i.e., according to Liouville’s theorem, iff \( v \) is the restriction of an orientation-preserving Möbius transformation on \( U \). Setting again \( v := w + \text{id}|_U \), and expanding the deficit, one obtains formally

\[
\int_U \left( \frac{\| \nabla v \|^2}{n} \right)^{\frac{n}{n-1}} - \det \nabla v = \int_U \left| (\nabla w)_{\text{sym}} - \frac{\text{div} w}{n} I_n \right|^2 + \int_U O(\| \nabla w \|^3).
\]  

(5.24)

Another well known fact regarding the connection of the quadratic form in the right hand side of \( (5.24) \) to the geometry of \( \mathbb{R}^+SO(n) \) is the following (see [23, Chapters 2 and 3], or [10] for more details). If \( T\mathbb{R}^+SO(n) \) stands for the tangent space to the conformal group \( \mathbb{R}^+SO(n) \) at \( I_n \), it is immediate that

\[
A \in T\mathbb{R}^+SO(n) \iff A_{\text{sym}} = \frac{\text{Tr} A}{n} I_n,
\]

so that the function \( A \mapsto d(A) := |A_{\text{sym}} - \frac{\text{Tr} A}{n} I_n| \) is equivalent to the distance of \( A \) from \( T\mathbb{R}^+SO(n) \). Therefore, the linear subspace

\[
\Sigma_n := \left\{ u \in W^{1,2}(\mathbb{R}^n; \mathbb{R}^n) : (\nabla u)_{\text{sym}} = \frac{\text{div} u}{n} I_n \right\}
\]

can be viewed as the Lie algebra of the Möbius group of \( \mathbb{R}^n \). If \( \Pi_{\Sigma_n} : W^{1,2}(U; \mathbb{R}^n) \mapsto \Sigma_n \) is the \( W^{1,2} \)-projection on this finite-dimensional subspace, the following variant of Korn’s inequality for the trace-free part of the symmetrized gradient is known to hold.

**Theorem 5.6. (Y.G. Reshetnyak, cf. [23, Theorem 3.3, Chapter 3])** Let \( n \geq 3 \) and \( U \) be an open bounded Lipschitz domain of \( \mathbb{R}^n \) that is starshaped with respect to a ball. There exists a constant \( C := C(n,U) > 0 \) such that for every \( w \in W^{1,2}(U; \mathbb{R}^n) \),

\[
\| \nabla w - \nabla (\Pi_{\Sigma_n} w) \|_{L^2(U)} \leq C \left\| (\nabla w)_{\text{sym}} - \frac{\text{div} w}{n} I_n \right\|_{L^2(U)}.
\]

In a certain sense, Theorem 1.5 is the analogue of Theorem 5.6 for maps from \( S^{n-1} \) to \( \mathbb{R}^n \). In particular, as an upshot of it we have encountered the following fact. *Although the kernels of the nonnegative quadratic forms arising as the second derivatives of the conformal deficit \( [D_{n-1}(u)]^{\frac{n}{n-1}} - [P_{n-1}(u)]^{\frac{n}{n-1}} \) and the isoperimetric deficit \( [P_{n-1}(u)]^{\frac{n}{n-1}} - V_n(u) \) at the \( \text{id}_{S^{n-1}} \) are both infinite-dimensional, the intersection of the two kernels is finite-dimensional and actually isomorphic to the Lie algebra of infinitesimal Möbius transformations of \( S^{n-1} \).*

Indeed, by the calculations that we exhibit in Lemma B.3 in Appendix B, we have that

\[
Q_{n,\text{conf}}(w) := \frac{n}{n-1} \int_{S^{n-1}} \left( (P_T \nabla w)_{\text{sym}} - \frac{\text{div} w}{n-1} I_x \right)^2,
\]  

(5.25)
infinite-dimensional. In fact (with the notation of Theorem 4.7), one can directly check that
\[ \phi \]

Since the space of such bunches of maps defined in the bulk, but the kernel of \( Q_{n,\text{conf}} \) is infinite-dimensional. Intuitively, the underlying geometric reason behind this, is the abundance of \( C^2 \) conformal maps from \( S^{n-1} \) into \( \mathbb{R}^n \).

Actually, we observe that for every \( \phi \in W^{1,2}(S^{n-1}; \mathbb{R}) \) with \( f_{S^{n-1}} \phi = 0 \) and \( f_{S^{n-1}} \phi(x) = 0 \), the map \( w_\phi(x) := \phi(x) x \) belongs to \( H_n \) (recall (4.15)) and one can easily verify that
\[ (P_T^I \nabla_T w_\phi)_\text{sym} - \frac{\operatorname{div}_{S^{n-1}} w_\phi}{n-1} I_x = 0. \]

Since the space of such \( \phi \) is infinite-dimensional, we have in particular that \( \dim(\ker(Q_{n,\text{conf}})) = \infty \).

Regarding the quadratic form \( Q_{n,\text{isop}} \) in (5.26), we have that it is also nonnegative and its kernel is also infinite-dimensional. In fact (with the notation of Theorem 4.7), one can directly check that
\[ H_{n,2} := \bigoplus_{k=1}^\infty H_{n,k,2} \subseteq \ker(Q_{n,\text{isop}}) \implies \dim(\ker(Q_{n,\text{isop}})) = \infty. \]

To verify (5.27) we use the following identity, referred to as Korn’s identity, that is interesting in its own right and whose derivation is a simple calculation which is also included at the end of Appendix B.

**Lemma 5.7. (Korn’s identity on \( S^{n-1} \).)** For every \( w \in W^{1,2}(S^{n-1}; \mathbb{R}^n) \) the following identity holds
\[ \int_{S^{n-1}} |(P_T^I \nabla_T w)_\text{sym}|^2 = \frac{1}{2} \int_{S^{n-1}} \left[ |P_T^I \nabla_T w|^2 + (\operatorname{div}_{S^{n-1}} w)^2 \right] - \frac{n-2}{n} Q_{V_n}(w). \]  

The interesting point of this identity is that when \( n \geq 3 \), the quadratic form \( Q_{V_n} \) appears in the right hand side of (5.28) as some short of curvature contribution and it is really a surface identity, in the sense that the corresponding identity in the bulk is
\[ \int_U |(\nabla w)_\text{sym}|^2 = \frac{1}{2} \int_U |(\nabla w|^2 + (\operatorname{div} w)^2) - \frac{1}{2} \int_U ((\operatorname{div} w)^2 - \operatorname{Tr}(\nabla w)^2), \]

but the last term on the right hand side of (5.29) should now be interpreted as a boundary-term contribution.

By using Korn’s identity, the form \( Q_{n,\text{isop}} \) can be rewritten in a simpler form as
\[ Q_{n,\text{isop}}(w) = \frac{n}{2(n-1)} \int_{S^{n-1}} \left( \left| \sum_{j=1}^n x_j \nabla_T w_j \right|^2 - \langle w, A(w) \rangle \right). \]

But if \( w \in H_{n,2} \), then \( -\sum_{j=1}^n x_j \nabla_T w_j = A(w) = w \) on \( S^{n-1} \) (recall that (4.34), (4.39) are then satisfied with \( \sigma = 1 \)), and therefore \( Q_{n,\text{isop}}(w) = 0 \), which proves the implication in (5.27).

### 5.3 Proof of Theorem 1.8

As we have mentioned in the Introduction, if \( u \in W^{1,2}(S^{n-1}; \mathbb{R}^n) \) and \( w := u - \id_{S^{n-1}}, \) then the full \( L^2 \)-isometric deficit of \( u \) is formally expanded around the identity as
\[ \delta^2_{\text{isom}}(u) := \int_{S^{n-1}} \left| \sqrt{\nabla_T u} \nabla_T u - I_x \right|^2 = Q_{n,\text{isop}}(w) + \int_{S^{n-1}} O(|\nabla_T w|^3), \]
For $n = 2$ we have that $\dim(\ker(Q_{2,\text{conf}})) = \infty$. Indeed, for $v(x) := \psi(x)\tau(x) + \phi(x)x : S^1 \to \mathbb{R}^2$, where $\phi, \psi \in C^\infty(S^1, \mathbb{R})$ and $\tau(x) := (-x_2, x_1)$ is the unit tangent vector field on $S^1$, it is an easy calculation to check that $(P_T^1 \nabla T v)_{\text{sym}} = \phi - \partial_{\tau}\psi$, i.e., for every $\psi \in C^\infty(S^1; \mathbb{R})$ the map $v_\psi(x) := \psi(x)\tau(x) + \partial_{\tau}\psi(x)x$ lies in the kernel of $Q_{2,\text{conf}}$.

For $n \geq 3$, and unlike $Q_{n,\text{conf}}$, which as we have seen has infinite-dimensional kernel, $Q_{n,\text{isom}}$ has finite-dimensional kernel and actually

$$\ker(Q_{n,\text{isom}}) \simeq Skew(n) \simeq so(n),$$

this fact being known as the *infinitesimal rigidity of the sphere*. The reader is referred to [24, Chapter 12] for a detailed discussion and further references regarding this well known geometric fact that is the linear analogue of the $C^2$-rigidity of the sphere in the Weyl problem.

**Remark 5.8.** Without referring to the classical proof of the infinitesimal rigidity of $S^{n-1}$, it is very easy to deduce in particular that

$$\ker(Q_{n,\text{isom}}) \cap \ker(Q_{n,\text{isop}}) \simeq so(n).$$

Indeed, if $w \in W^{1,2}(S^{n-1}; \mathbb{R}^n)$ lies in the common null-space of these two nonnegative forms, we have

$$(P_T^1 \nabla T w)_{\text{sym}} = 0 \quad \text{and} \quad Q_{n,\text{isop}}(w) = 0. \quad (5.32)$$

By taking the trace in the first equation in $(5.32)$, we see that $\text{div}_{S^{n-1}} w \equiv 0$ on $S^{n-1}$, and then the second equation in $(5.32)$, in view also of $(5.26)$, reduces to

$$\frac{1}{2} \frac{n}{n-1} \int_{S^{n-1}} |\nabla T w|^2 - Q_{V_n}(w - \int_{S^{n-1}} w) = 0. \quad (5.33)$$

By the alternative formula $(5.3)$, the equation $(5.33)$ results in

$$\left(\frac{1}{n-1} \int_{S^{n-1}} |\nabla T w|^2 - \int_{S^{n-1}} |w - \int_{S^{n-1}} w|^2\right) + n \int_{S^{n-1}} \langle w - \int_{S^{n-1}} w, x \rangle^2 = 0. \quad (5.34)$$

Once again, the quantity in the parenthesis in $(5.34)$ is nonnegative, being the $L^2$-Poincare deficit of $w$, and therefore the only solutions to $(5.34)$ are maps $w$ for which

$$w(x) - \int_{S^{n-1}} w = \Lambda x \quad \text{for} \quad \Lambda \in \mathbb{R}^{n \times n} \quad \text{and} \quad \langle w - \int_{S^{n-1}} w, x \rangle \equiv 0 \quad \text{on} \quad S^{n-1}. \quad (5.35)$$

Hence, by $(5.35)$, $\Lambda^t = -\Lambda$ and reversely any map of the form $w(x) = \Lambda x + b$, where $\Lambda \in Skew(n), b \in \mathbb{R}^n$ is in the null-space of both quadratic forms. *It would be interesting if by some algebraic manipulations one could directly prove that whenever $n \geq 3$, $Q_{n,\text{isom}}(w) = 0 \implies Q_{n,\text{isop}}(w) = 0$. Combined with the argument just presented, this would give an alternative algebraic proof of the infinitesimal rigidity of the sphere, supplementing the differential-geometric one.*
Remark 5.9. Even though for \( n \geq 3 \) we have \( \ker(Q_{n,\text{isom}}) \simeq \mathfrak{so}(n) \), an estimate of the type
\[
\int_{\mathbb{S}^{n-1}} \left| \nabla_T w - FP_T \right|^2 \lesssim Q_{n,\text{isom}}(w) \quad \text{for some } F \in \text{Skew}(n),
\]
does not hold, the obstacle being (loosely speaking) the derivatives of the normal component of \( w \). For example, if one considers purely normal displacements \( w_\phi(x) := \phi(x)x \) with \( \phi \in W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}) \), then by a straightforward computation one can check that
\[
(P_T \nabla_T w_\phi)_{\text{sym}} = \phi I_x \implies Q_{n,\text{isom}}(w_\phi) = (n-1)\int_{\mathbb{S}^{n-1}} |\phi|^2,
\]
whereas the full gradient of \( w_\phi \) also has derivatives of \( \phi \) in it. In coordinates,
\[
(\nabla_T w_\phi)_{ij} = \phi(P_T)_{ij} + x_i \partial_j \phi,
\]
so if the estimate above was to be valid, it would resemble some sort of reverse-Poincare inequality, which is of course generically false. Combining additively \( Q_{n,\text{isom}} \) and \( Q_{n,\text{isop}} \) has though the merit of providing a Korn-type inequality in terms of the full gradient, as described in Theorem 1.8.

**Proof of Theorem 1.8.** For \( \alpha > 0 \) let us call
\[
Q_{n,\alpha}(w) := \alpha Q_{n,\text{isom}}(w) + Q_{n,\text{isop}}(w).
\]
By looking at the formulas (5.26) and (5.30), and rearranging terms, (5.36) gives
\[
Q_{n,\frac{n}{n-1}}(w) = \frac{n}{2(n-1)} \int_{\mathbb{S}^{n-1}} \left( |\nabla_T w|^2 + (\text{div}_{\mathbb{S}^{n-1}} w)^2 \right) - Q_{V_n}(w).
\]
With the notation we had introduced in Subsection 4.2, the estimate to be proven for \( Q_{n,\frac{n}{n-1}} \) reads
\[
Q_{n,\frac{n}{n-1}}(w) \gtrsim \int_{\mathbb{S}^{n-1}} |\nabla_T w - \nabla_T w_{1,2}|^2 \quad \forall w \in W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^n).
\]
Once we have (5.38), the case of general \( \alpha > 0 \) is immediate, since:

For \( \alpha > \frac{n}{n-1} > 0 \): \( Q_{n,\alpha}(w) \geq Q_{n,\frac{n}{n-1}}(w) \gtrsim \int_{\mathbb{S}^{n-1}} |\nabla_T w - \nabla_T w_{1,2}|^2 \).  

For \( 0 < \alpha \leq \frac{n}{n-1} \): \( Q_{n,\alpha}(w) = \left( \frac{n-1}{n} \right) \frac{1}{n} Q_{n,\text{conf}}(w) + \frac{n}{(n-1)} Q_{n,\text{isop}}(w) \)
\[
\geq \left( \frac{n-1}{n} \right) \frac{1}{n} Q_{n,\text{conf}}(w) + \frac{n}{(n-1)} \int_{\mathbb{S}^{n-1}} |\nabla_T w - \nabla_T w_{1,2}|^2.
\]

The proof of (5.38) can now be performed by following exactly the same procedure as in the proof of Theorem 1.5 in Subsection 5.1. The latter could of course also be phrased in terms of any positive combination of \( Q_{n,\text{conf}} \) in (5.25) and \( Q_{n,\text{isop}} \) in (5.26), and not only of \( Q_n = Q_{n,\text{conf}} + Q_{n,\text{isop}} \), as in (1.12).

The arguments after Lemmata 5.2 and 5.3 (handling the mixed div-terms after the expansion in spherical harmonics as in (5.15)-(5.18)) are then modified accordingly. A last trivial comment in this case is that for \( k \geq 1 \) and \( i = 1, 2, 3 \), if \( w \in H_{n,k,i} \) (recall Theorem 4.7 and Lemma 5.2),
\[
Q_{n,\frac{n}{n-1}}(w) = C_{n,k,i} \int_{\mathbb{S}^{n-1}} |\nabla_T w|^2, \quad \text{where } C_{n,k,i} := \frac{n}{2(n-1)} + \frac{n}{2(n-1)} \alpha_{n,k,i} - \alpha_{n,k,i}.
\]
Recalling the values of the constants from (5.2) and the table (5.4), we have that

\[ C'_{n,k,i} > C_{n,k,i} \quad \text{for } i = 1, 3, \quad \text{while } C'_{n,k,2} = C_{n,k,2} \quad \text{(because } \alpha_{n,k,2} = 0). \]  

(5.39)

So for the quadratic form in (5.37), one can directly check from (5.39) that \( C'_{n,1,2} = 0 \), but otherwise \( \min_{(k,i) \neq (1,2)} C'_{n,k,i} > 0 \), which eventually leads to the desired coercivity estimate (5.38). \( \square \)

**Remark 5.10.** Since in this case we had an easy argument to infer that \( \ker(Q_{n,\text{isom}}) \cap \ker(Q_{n,\text{isop}}) \simeq \mathfrak{s}(n) \) (see Remark 5.8), the proof of Theorem 1.8 for \( \alpha = \frac{n}{n-1} \) for example, could also be performed by a standard contradiction\/compactness argument, for an abstract constant though. Indeed, suppose that for \( \alpha := \frac{n}{n-1} \) the estimate (1.17) (or equivalently (5.38)) is false. By translation and scaling invariance of the estimate, there exists a sequence \((w_k)_{k \in \mathbb{N}} \subset W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^n)\) such that for all \( k \in \mathbb{N} \),

\[ \int_{\mathbb{S}^{n-1}} w_k = 0, \quad \Pi_{H_{n,1,2}} w_k = 0, \quad \int_{\mathbb{S}^{n-1}} |\nabla_T w_k|^2 = 1, \]

(5.40)

and

\[ Q_{n,\frac{n}{n-1}}(w_k) \leq \frac{1}{k} \int_{\mathbb{S}^{n-1}} |\nabla_T w_k|^2 = \frac{1}{k}, \]

(5.41)

in particular,

\[ Q_{n,\frac{n}{n-1}}(w_k) \rightarrow 0 \quad \text{as } k \rightarrow \infty. \]

Up to a further nonrelabeled subsequence we can assume that there exists \( w \in W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^n) \) with \( f_{\mathbb{S}^{n-1}} w = 0 \) such that \( w_k \rightharpoonup w \) in \( W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^n) \), and also pointwise \( H^{n-1}\text{-a.e. on } \mathbb{S}^{n-1} \). But then (recalling (4.22) and (4.23)),

\[ Q_{V_n}(w_k) = \frac{n}{2} \int_{\mathbb{S}^{n-1}} \langle w_k, A(w_k) \rangle \rightarrow \frac{n}{2} \int_{\mathbb{S}^{n-1}} \langle w, A(w) \rangle = Q_{V_n}(w) \quad \text{as } k \rightarrow \infty, \]

(5.42)

since \( w_k \rightarrow w \) strongly in \( L^2(\mathbb{S}^{n-1}; \mathbb{R}^n) \) and \( A(w_k) \rightharpoonup A(w) \) weakly in \( L^2(\mathbb{S}^{n-1}; \mathbb{R}^n) \). Thus, by lower semicontinuity of the first two terms in (5.37) under weak convergence in \( W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^n) \), we obtain

\[ 0 \leq Q_{n,\frac{n}{n-1}}(w) \leq \liminf_{k \rightarrow \infty} Q_{n,\frac{n}{n-1}}(w_k) = 0, \]

i.e. \( Q_{n,\frac{n}{n-1}}(w) = 0 \) and therefore, by (5.31),

\[ w \in \ker(Q_{n,\text{isom}}) \cap \ker(Q_{n,\text{isop}}) \simeq \mathfrak{s}(n), \]

i.e., \( w \in H_{n,1,2} \). Moreover, being the \( (H^{n-1}\text{-a.e.}) \) pointwise limit of \((w_k)_{k \in \mathbb{N}} \subset H_{n,1,2}^1 \), we must also have that \( w \in H_{n,1,2}^1 \), since \( H_{n,1,2} \) is finite-dimensional and therefore its orthogonal complement in \( W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^n) \) is a closed subspace. This forces \( w \equiv 0 \) on \( \mathbb{S}^{n-1} \) and in particular, by (5.42),

\[ Q_{V_n}(w_k) \rightarrow 0 \quad \text{as } k \rightarrow \infty. \]

(5.43)

But then (5.37), (5.41) and (5.43) imply that

\[ 0 \leq \frac{n}{2(n-1)} \int_{\mathbb{S}^{n-1}} |\nabla_T w_k|^2 \leq \frac{1}{k} + Q_{V_n}(w_k) \rightarrow 0 \quad \text{as } k \rightarrow \infty, \]

contradicting the assumption in (5.40) that \( \int_{\mathbb{S}^{n-1}} |\nabla_T w_k|^2 = 1 \).
Remark 5.11. As we had mentioned in Remark 5.1, the argument just described could have also been used to prove Theorem 1.5 if we knew already that the kernel of $Q_n$ (defined in (1.12)) is finite-dimensional (and actually equal to $H_{n,0}$). Nevertheless, as we had remarked therein, there does not seem to be a direct argument to show this fact when $n \geq 4$, neither by trying to solve the Euler-Lagrange equation associated to the operator $L$ in (5.1) explicitly, nor by trying to argue as above.

Indeed, if $w \in \ker (Q_n) \iff w \in \ker (Q_{n,\text{conf}}) \cap \ker (Q_{n,\text{isop}})$ (see (5.25), (5.26)), then again the following two equations must be satisfied simultaneously,

$$ (P^T_T \nabla^T w)^\text{sym} = \frac{\text{div}_{S^{n-1}} w}{n-1} I_x \quad \text{and} \quad Q_{n,\text{isop}}(w) = 0. \quad (5.44) $$

Because of the first equation in (5.44), the second one therein results in the equation

$$ \frac{1}{n-1} \int_{S^{n-1}} |\nabla^T w|^2 + \frac{n-3}{(n-1)^2} \int_{S^{n-1}} (\text{div}_{S^{n-1}} w)^2 - \int_{S^{n-1}} \langle w, A(w) \rangle = 0, \quad (5.45) $$

i.e., (5.45) ends up back to the original equation $Q_n(w) = 0$. Arguing directly with the eigenvalue decomposition with respect to $A$ (recall Theorem 4.7) also has the extra benefit of showing explicitly how the form $Q_n$ behaves in each one of the eigenspaces separately (see Lemma 5.2), and also gives an explicit lower bound ((5.23)) for the value of the optimal constant $C_n$ in the coercivity estimate.

Acknowledgements

The second author is supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy EXC 2044-390685587, Mathematics Münster: Dynamics–Geometry–Structure and would also like to thank the Max Planck Institute for Mathematics in the Sciences in Leipzig, where his PhD project was carried out.

A A new simple proof of Liouville’s theorem on $S^{n-1}$ and a qualitative analogue

Proof of Theorem 1.1. We present the proof in the case of generalized orientation-preserving maps, since the case of orientation-reversing ones is identical (or it can be retrieved by the former by composing with the flip $x := (x_1, \ldots, x_{n-1}, x_n) \mapsto (x_1, \ldots, x_{n-1}, -x_n)$).

For part (i) of the theorem, we have that any map of the form $u(x) = Rx$ with $R \in SO(n)$ is of course an orientation-preserving isometry of $S^{n-1}$. Conversely, let $n \geq 2$, $p \in [1, \infty]$ and $u \in W^{1,p}(S^{n-1}; S^{n-1})$ be a generalized orientation-preserving isometric map. By definition, this means that at $H^{n-1}$-a.e. $x \in S^{n-1}$ the intrinsic gradient of $u$ is an orientation-preserving linear map between $T_x S^{n-1}$ and $T_{u(x)} S^{n-1}$, such that

$$ d_x u^t d_x u = I_x, $$
or equivalently, in terms of the extrinsic gradient,

\[(\nabla_T u^t \nabla_T u) (x) = I_x.\]

In particular, for \(H^{n-1}\)-a.e. \(x \in \mathbb{S}^{n-1}\) one has

\[
\frac{|\nabla_T u(x)|^2}{n-1} = 1, \text{ and } u^t(\omega) = \omega \text{ for every } (n-1)\text{-form } \omega \text{ on } \mathbb{S}^{n-1}. \tag{A.1}
\]

By the change of variables formula applied to the vector-valued \((n-1)\)-form \(xdv_g\), we obtain

\[
0 = \int_{\mathbb{S}^{n-1}} xdv_g = \int_{\mathbb{S}^{n-1}} u^t(xdv_g) = \int_{\mathbb{S}^{n-1}} u(x)dv_g(x), \text{ i.e., } \int_{\mathbb{S}^{n-1}} u = 0. \tag{A.2}
\]

Hence, Poincare’s inequality \([C.5]\) on \(\mathbb{S}^{n-1}\), together with \((A.1), (A.2)\), and the fact that \(|u| \equiv 1\), yield

\[
1 = \int_{\mathbb{S}^{n-1}} |\nabla_T u|^2 \frac{1}{n-1} \geq \int_{\mathbb{S}^{n-1}} |u|^2 = 1.
\]

As we have also encountered before (see also Appendix \([C]\) ), the equality case in the Poincare inequality implies that in the Fourier expansion of \(u\) in spherical harmonics, no other spherical harmonics except the first order ones should appear, hence \(u(x) = Rx\) for some \(R \in \mathbb{R}^{n \times n}\). But this linear map would transform \(\mathbb{S}^{n-1}\) into the boundary of an ellipsoid, which after possibly an orthogonal change of coordinates is

\[
u(\mathbb{S}^{n-1}) = \big\{ y = (y_1, \ldots, y_n) \in \mathbb{R}^n : \frac{y_1^2}{\alpha_1^2} + \cdots + \frac{y_n^2}{\alpha_n^2} = 1 \big\},
\]

where \(0 \leq \alpha_1 \leq \cdots \leq \alpha_n\) are the eigenvalues of \(\sqrt{H}R\). By assumption, \(\nu(\mathbb{S}^{n-1}) \equiv \mathbb{S}^{n-1}\) and this forces \(\alpha_1 = \cdots = \alpha_n = 1\), i.e., \(R \in O(n)\) and in particular, since \(u\) is assumed to be orientation-preserving, \(R \in SO(n)\).

For part (ii) we can argue similarly, after making use of the following useful fact.

**Claim.** Given \(u \in W^{1,n-1}(\mathbb{S}^{n-1}; \mathbb{S}^{n-1})\) of degree one, one can always find a M"obius transformation \(\phi_{\xi_0,\lambda_0}\) of \(\mathbb{S}^{n-1}\) (see \((1.2)\)) so that

\[
\int_{\mathbb{S}^{n-1}} u \circ \phi_{\xi_0,\lambda_0} = 0. \tag{A.3}
\]

Indeed, assume first that \(u \in C^\infty(\mathbb{S}^{n-1}; \mathbb{S}^{n-1})\) and has degree one, in particular \(u\) is surjective. If \(b_u := \int_{\mathbb{S}^{n-1}} u = 0\), there is nothing to prove. If \(b_u \neq 0\), consider the map \(F : \mathbb{S}^{n-1} \times [0,1] \to \overline{B_1}\), defined as

\[
F(\xi, \lambda) := \int_{\mathbb{S}^{n-1}} u \circ \phi_{\xi,\lambda} \text{ for } \lambda \in (0,1], \text{ and } F(\xi,0) := \lim_{\lambda \to 0^+} F(\xi,\lambda).
\]

The map \(F\) is continuous with \(F(\xi,0) = u(\xi)\) for every \(\xi \in \mathbb{S}^{n-1}\), i.e., \(F(\mathbb{S}^{n-1},0) = u(\mathbb{S}^{n-1}) = \mathbb{S}^{n-1}\), whereas \(F(\mathbb{S}^{n-1},1) = \{b_u\}\). In other words, \(F\) is a continuous homotopy between \(\mathbb{S}^{n-1}\) and the point \(b_u \in \overline{B_1} \setminus \{0\}\), and therefore there exists \(\lambda_0 \in (0,1)\) such that \(0 \in F(\mathbb{S}^{n-1},\lambda_0)\), i.e., there exists also \(\xi_0 \in \mathbb{S}^{n-1}\) such that \(F(\xi_0,\lambda_0) = 0\).

In the general case of a map \(u \in W^{1,n-1}(\mathbb{S}^{n-1}; \mathbb{S}^{n-1})\) of degree 1, by the approximation property given in \([5]\) Section I.4, Lemma 7], there exists a sequence \((u_j)_{j \in \mathbb{N}} \subset C^\infty(\mathbb{S}^{n-1}; \mathbb{S}^{n-1})\) with the property that

\[
u_j \to u \text{ strongly in } W^{1,n-1}(\mathbb{S}^{n-1}; \mathbb{S}^{n-1}) \text{ and } \deg u_j = \deg u = 1 \quad \forall j \in \mathbb{N}.
\]
Up to passing to a non-relabeled subsequence, we can without loss of generality also suppose that \( u_j \to u \) and \( \nabla_T u_j \to \nabla_T u \) pointwise \( \mathcal{H}^{n-1} \)-a.e. on \( S^{n-1} \). Since the maps \((u_j)_{j \in \mathbb{N}}\) are smooth and surjective, by the previous argument there exist \((\xi_j)_{j \in \mathbb{N}} \subset S^{n-1} \) and \((\lambda_j)_{j \in \mathbb{N}} \subset (0, 1] \) so that for every \( j \in \mathbb{N}, \)

\[
\int_{S^{n-1}} u_j \circ \phi_{\xi_j, \lambda_j} = 0. \tag{A.4}
\]

Up to non-relabeled subsequences, we can suppose further that \( \xi_j \to \xi_0 \in S^{n-1} \) and \( \lambda_j \to \lambda_0 \in [0, 1] \), thus \( \phi_{\xi_j, \lambda_j} \to \phi_{\xi_0, \lambda_0} \) pointwise \( \mathcal{H}^{n-1} \)-a.e. on \( S^{n-1} \), and also weakly in \( W^{1, n-1}(S^{n-1}, S^{n-1}) \).

In fact \( \lambda_0 \in (0, 1] \), i.e., the Möbius transformations \( (\phi_{\xi_j, \lambda_j})_{j \in \mathbb{N}} \) do not converge to the trivial map \( \phi_{\xi_0, 0}(x) \equiv \xi_0 \). Indeed, suppose that this was the case. Then \( u_j \circ \phi_{\xi_j, \lambda_j} \to u(\xi_0) \) pointwise \( \mathcal{H}^{n-1} \)-a.e., and \( |u_j \circ \phi_{\xi_j, \lambda_j}| \equiv 1 \), so we could use the Dominated Convergence Theorem and (A.4), to infer that

\[
u(\xi_0) = \int u(\xi_0) = \lim_{j \to \infty} \int u_j \circ \phi_{\xi_j, \lambda_j} = 0,
\]

\[
|u(\xi_0)| = \int |u(\xi_0)| = \lim_{j \to \infty} \int |u_j \circ \phi_{\xi_j, \lambda_j}| = 1,
\]

and derive a contradiction. Having justified that \( \lambda_0 \in (0, 1] \), what we actually obtain by the Dominated Convergence Theorem and (A.4) is that

\[
\int_{S^2} u \circ \phi_{\xi_0, \lambda_0} = 0. \tag{A.5}
\]

Continuing with the proof of Theorem 1.1(ii), we of course have that all the maps given by (1.2) are orientation-preserving conformal diffeomorphisms of \( S^{n-1} \). Conversely, if \( u \in W^{1, n-1}(S^{n-1}, S^{n-1}) \) is a generalized orientation-preserving conformal map, similarly to (i) we have that \( \mathcal{H}^{n-1} \)-a.e. on \( S^{n-1} \),

\[
(d_x u^t d_x u)(x) = \frac{|d_x u|^2}{n-1} I_x,
\]

or equivalently, in terms of the extrinsic gradient,

\[
\nabla_T u^t \nabla_T u)(x) = \frac{|
abla_T u|^2}{n-1} I_x. \tag{A.6}
\]

By taking the determinant in both sides of (A.6), we get that \( \mathcal{H}^{n-1} \)-a.e. on \( S^{n-1} \),

\[
\sqrt{\det(\nabla_T u^t \nabla_T u)} = \left(\frac{|
abla_T u|^2}{n-1}\right)^{\frac{n-1}{2}}. \tag{A.7}
\]

Precomposing with the Möbius map \( \phi_{\xi_0, \lambda_0} \in \text{Conf}_+(S^{n-1}) \) of the previous claim, we have that the map \( \tilde{u} := u \circ \phi_{\xi_0, \lambda_0} \), whose mean value is 0 by (A.5), is also a generalized orientation-preserving conformal transformation of \( S^{n-1} \) of degree 1, and therefore by (A.7),

\[
\tilde{u}^2(dv_g) = \sqrt{\det(\nabla_T \tilde{u}^t \nabla_T \tilde{u})} dv_g = \left(\frac{|
abla_T \tilde{u}|^2}{n-1}\right)^{\frac{n-1}{2}} dv_g.
\]

By approximation, the analytic formula for the degree in terms of integration of \((n-1)\)-forms on \( S^{n-1} \) holds true for \( \tilde{u} \) as well, i.e.,

\[
\int_{S^{n-1}} \left(\frac{|
abla_T \tilde{u}|^2}{n-1}\right)^{\frac{n-1}{2}} = \int_{S^{n-1}} \tilde{u}^2(dv_g) = \text{deg} \tilde{u} \int_{S^{n-1}} dv_g = 1. \tag{A.8}
\]
We can now use (A.8), Jensen’s inequality, and the sharp Poincare inequality on $S^{n-1}$ (§C.5), to obtain the chain of inequalities
\[ 1 = \int_{S^{n-1}} \left( \frac{\nabla T \bar{u}}{n-1} \right)^{\frac{n+1}{n}} \geq \left( \int_{S^{n-1}} \frac{\nabla T \bar{u}}{n-1} \right)^{\frac{n+1}{n}} \geq \left( \int_{S^{n-1}} |\bar{u}| \right)^{\frac{n+1}{n}} = 1, \quad (A.9) \]
since $\int_{S^{n-1}} \bar{u} = 0$ and $|\bar{u}| \equiv 1$ $H^{n-1}$-a.e. on $S^{n-1}$. Arguing as in part $(i)$, we deduce that $\bar{u} = R\hat{d}_{S^{n-1}}$, i.e., $u = R\hat{\phi}_{\xi,\lambda}$, where $R \in SO(n)$, $\xi := \xi_0 \in S^{n-1}$ and $\lambda := \frac{1}{\lambda_0} > 0$. \hfill $\square$

**Remark A.1.** The Möbius transformations of $S^{n-1}$ given by (1.2) could of course alternatively be described by performing an inversion in $T_{\xi} S^{n-1}$ with respect to some center, say the origin $\xi$ of the affine hyperplane $T_{\xi} S^{n-1}$ of $\mathbb{R}^n$, and some radius, say $\sqrt{\lambda} > 0$. These maps however would correspond exactly to the Möbius transformations produced by dilation in $T_{\xi} S^{n-1}$ by factor $\frac{1}{\lambda}$, composed finally with a flip in $\mathbb{R}^n$, i.e., an orthogonal map that would change back the orientation. By Liouville’s Theorem 1.1, the conformal group of the sphere is given by

\[ Conf(S^{n-1}) = \{ O\phi_{\xi,\lambda} : O \in O(n), \xi \in S^{n-1}, \lambda > 0 \}, \]

and is a actually a Lie group, i.e., a differentiable manifold (of dimension $\frac{n(n+1)}{2}$) with a group structure, given by composition of maps. Analytically, the maps $(\phi_{\xi,\lambda})_{\xi \in S^{n-1}, \lambda > 0}$ are given by the formula

\[ \phi_{\xi,\lambda}(x) := \frac{-\lambda^2(1 - \langle x, \xi \rangle) \xi + 2\lambda(x - \langle x, \xi \rangle) \xi + (1 + \langle x, \xi \rangle) \xi}{\lambda^2(1 - \langle x, \xi \rangle) + (1 + \langle x, \xi \rangle)}. \quad (A.10) \]

Using (A.10), the corresponding Lie algebra of infinitesimal Möbius transformations, i.e., the tangent space of $Conf(S^{n-1})$ at the $id_{S^{n-1}}$, can easily be identified. Indeed, it is then an elementary exercise in differential geometry to check that

\[ T_{id_{S^{n-1}}} Conf(S^{n-1}) \equiv \{ Sx + \mu(\langle x, \xi \rangle x - \xi) : S \in skew(n), \xi \in S^{n-1}, \mu \in \mathbb{R} \}, \quad (A.11) \]

which is the representation that we made use of in the proof of Lemma 4.13.

**Remark A.2.** In the conformal case, the argument for the proof of Theorem 1.1(ii) can easily be modified in order to give a compactness statement for sequences of orientation-preserving (resp. orientation-reversing) approximately conformal maps on $S^{n-1}$ of degree 1 (resp. $-1$), see the subsequent Lemma 

When $n = 3$, the statement therein essentially reduces to a well known compactness result for harmonic maps of degree $\pm 1$ on $S^2$ (cf. [2] Theorem 2.4, Step 1 in the proof], as well as [18]), which was proven using a concentration-compactness argument in the spirit of P.L. Lions [19]. With the observation that the Möbius transformations can be used to “globally invert” a map from $S^{n-1}$ to itself, in the sense of fixing its mean value to be 0, we can give a simpler and more elementary proof of this fact, which can be appropriately generalized in every dimension $n \geq 3$. For further applications of this simple observation the interested reader is also referred to [15, 25] for two different and shorter proofs of [2] Theorem 2.4]. In the following lemma we present again for simplicity the case of orientation-preserving degree 1 maps.
Lemma A.3. Let \( n \geq 3 \) and \((u_j)_{j \in \mathbb{N}} \subset W^{1,n-1}(S^{n-1};S^{n-1})\) be a sequence of generalized orientation-preserving maps of degree 1 which are approximately conformal, in the sense that

\[
\lim_{j \to \infty} \int_{S^{n-1}} \left( \left( \frac{\nabla_T u_j}{n-1} \right) \right)^{\frac{n+1}{n-1}} - \sqrt{\det (\nabla_T u_j \nabla_T u_j)} = 0,
\]

which as a condition is in this case equivalent to

\[
\lim_{j \to \infty} \int_{S^{n-1}} \left( \frac{\nabla_T u_j}{n-1} \right)^{\frac{n+1}{n-1}} = 1. \tag{A.12}
\]

Then, there exist Möbius transformations \((\phi_j)_{j \in \mathbb{N}} \subset \text{Conf}_+(S^{n-1})\) and \(R \in SO(n)\) so that up to a non-relabeled subsequence,

\[
u_j \circ \phi_j \to \text{Rid}_{S^{n-1}} \text{ strongly in } W^{1,n-1}(S^{n-1};S^{n-1}). \tag{A.13}
\]

Proof. By the degree 1 condition, as in (A.3), we can again find \((\xi_j)_{j \in \mathbb{N}} \subset S^{n-1}\) and \((\lambda_j)_{j \in \mathbb{N}} \subset (0,1]\), so that after setting \(\phi_j := \phi_{\xi_j,\lambda_j} \in \text{Conf}_+(S^{n-1})\) and \(\tilde{u}_j := u_j \circ \phi_j\), we have

\[
\int_{S^{n-1}} \tilde{u}_j = 0. \tag{A.14}
\]

Thanks to the conformal invariance of the \((n-1)\)-Dirichlet energy, (A.12) is left unchanged, i.e.,

\[
\lim_{j \to \infty} \int_{S^{n-1}} \left( \frac{\nabla_T \tilde{u}_j}{n-1} \right)^{\frac{n+1}{n-1}} = 1. \tag{A.15}
\]

Because of (A.14) and (A.15), the sequence \((\tilde{u}_j)_{j \in \mathbb{N}}\) is in particular uniformly bounded in \(W^{1,n-1}(S^{n-1};S^{n-1})\), hence up to a non-relabeled subsequence converges weakly in \(W^{1,n-1}(S^{n-1};S^{n-1})\), and up to a further one also pointwise \(H^{n-1}\)-a.e. to a map \(\hat{u} \in W^{1,n-1}(S^{n-1};S^{n-1})\). Since \(\tilde{u}_j \to \hat{u}\) strongly in \(L^{n-1}(S^{n-1};S^{n-1})\), we obtain by (A.14) that in particular,

\[
\int_{S^{n-1}} \hat{u} = \lim_{j \to \infty} \int_{S^{n-1}} \tilde{u}_j = 0, \tag{A.16}
\]

and by lower semicontinuity of the \((n-1)\)-Dirichlet energy under weak convergence and (A.15),

\[
\int_{S^{n-1}} \left( \frac{\nabla_T \hat{u}}{n-1} \right)^{\frac{n+1}{n-1}} \leq \liminf_{j \to \infty} \int_{S^{n-1}} \left( \frac{\nabla_T \tilde{u}_j}{n-1} \right)^{\frac{n+1}{n-1}} = 1. \tag{A.17}
\]

We can then apply the same argument as in (A.9) in the proof of Theorem 1.1(ii), to obtain the chain of inequalities

\[
1 \geq \int_{S^{n-1}} \left( \frac{\nabla_T \hat{u}}{n-1} \right)^{\frac{n+1}{2}} \geq \left( \int_{S^{n-1}} \left| \nabla_T \hat{u} \right|^2 \right)^{\frac{n-1}{2}} \geq \left( \int_{S^{n-1}} \left| \hat{u} \right|^2 \right)^{\frac{n-1}{2}} = 1, \tag{A.18}
\]

and with the same reasoning as in there, we conclude that \(\hat{u}(x) = Rx\) for some \(R \in O(n)\). Finally, by (A.16)-(A.18) we actually obtain that \(\tilde{u}_j \to \hat{u}\) strongly in \(W^{1,n-1}(S^{n-1};S^{n-1})\). Since the degree is stable under this notion of convergence,

\[
1 = \text{deg} \hat{u} = \int_{S^{n-1}} \left\langle Rx, \bigwedge_{i=1}^{n-1} \partial_{\tau_i}(Rx) \right\rangle = \det R,
\]

i.e., indeed \(R \in SO(n)\). \(\square\)
B Integral identities for Jacobians, Taylor expansions of the deficits and proof of Korn’s identity

We start this appendix by collecting and proving some integral identities for Jacobians that we used in the bulk of the paper, and especially in the proof of Lemma 3.9.

Lemma B.1. Let \( u \in W^{1,2}(S^2; \mathbb{R}^3) \) and let \( u_h: \overline{B}_1 \to \mathbb{R}^3 \) be as usual its harmonic continuation in \( B_1 \), taken componentwise. Then (with the notation adopted in (1.7) for \( n = 3 \)),
\[
\int_{B_1} \det \nabla u_h = V_3(u) := \int_{S^2} \langle u, \partial_{\tau_1} u \wedge \partial_{\tau_2} u \rangle. \tag{B.1}
\]
Moreover, if \( w \in W^{1,2}(S^2; \mathbb{R}^3) \) and \( w_h \) is defined analogously, then
\[
\int_{B_1} \det(I_3 + \nabla w_h) = 1 + \int_{B_1} \text{div} w_h + \frac{1}{2} \int_{B_1} ((\text{div} w_h)^2 - \text{Tr}(\nabla w_h)^2) + \int_{B_1} \det \nabla w_h \tag{B.2}
\]
\[
= 1 + 3 \int_{S^2} \langle w, x \rangle + Q_{V_3}(w) + V_3(w),
\]
where the quadratic form \( Q_{V_3} \) is defined as
\[
Q_{V_3}(w) := \frac{3}{2} \int_{S^2} \langle w, (\text{div} w_h)x - \sum_{j=1}^3 x_j \nabla_T w^j \rangle. \tag{B.3}
\]

Remark B.2. As the reader might already know from the theory of null-Lagrangians or notice from the next proof, the above formulas actually hold true with \( B_1 \) being replaced by any other open bounded domain \( U \subset \mathbb{R}^3 \) with sufficiently regular boundary, and \( u_h \) (resp. \( w_h \)) being replaced by any other interior extension of \( u \) (resp. \( w \)), for which the previous bulk integrals are well defined. Since we only used the expressions for the harmonic extension, which is smooth in the interior of the unit ball, we have preferred to state the previous lemma in this particular form.

Proof of Lemma B.1. Regarding the proof of the identity (B.1), the determinant of \( \nabla u_h := (\partial_j u_h^i)_{1 \leq i,j \leq 3} \) can be rewritten as
\[
\det \nabla u_h = \langle \nabla u_h^1, \nabla u_h^2 \times \nabla u_h^3 \rangle = \langle \nabla u_h^2, \nabla u_h^3 \times \nabla u_h^1 \rangle = \langle \nabla u_h^3, \nabla u_h^1 \times \nabla u_h^2 \rangle, \tag{B.4}
\]
where \( a \times b \in \mathbb{R}^3 \) denotes the exterior product of two vectors \( a, b \in \mathbb{R}^3 \). Using the first identity in (B.4), and integrating by parts, we have
\[
\int_{B_1} \det \nabla u_h = \int_{B_1} \langle \nabla u_h^1, \nabla u_h^2 \times \nabla u_h^3 \rangle = \int_{B_1} \text{div}(u_h^1(\nabla u_h^2 \times \nabla u_h^3)) - \int_{B_1} u_h^1 \text{div}(\nabla u_h^2 \times \nabla u_h^3) \tag{B.5}
\]
\[
= \int_{S^2} u_h^1 \langle \nabla u_h^2 \times \nabla u_h^3, x \rangle - \int_{B_1} u_h^1 ((\text{curl} \nabla u_h^2, \nabla u_h^3) - (\nabla u_h^2, \text{curl} \nabla u_h^3))
\]
\[
= \int_{S^2} u^1 \langle ((\partial_{\tau_1} u^2)^2)\tau_1 + (\partial_{\tau_2} u^2)^2)\tau_2 + (\partial_{\tau_3} u^2)^2)\tau_3 \times ((\partial_{\tau_1} u^3)^2)\tau_1 + (\partial_{\tau_2} u^3)^2)\tau_2 + (\partial_{\tau_3} u^3)^2)\tau_3 \rangle, x \rangle
\]
\[
= \int_{S^2} u^1 (\partial_{\tau_1} u^2 \partial_{\tau_2} u^3 - \partial_{\tau_2} u^2 \partial_{\tau_1} u^3). \]
Hence, the second equality in (B.2) follows from (B.8), (B.9) and (B.1) for $w$
and we have written the full gradients $\nabla u^2$ and $\nabla u^3$ on $S^2$ in terms of the local orthonormal oriented frame
$\{\tau_1, \tau_2, x\}$ for $S^2$, which is such that $\tau_1 \times \tau_2 = x, \tau_2 \times x = \tau_1, x \times \tau_1 = \tau_2$. Using the other two expressions from (B.4) and arguing in the same manner, we also obtain
\[
\int_{B_1} \det \nabla u_h = -\int_{S^2} u^2 (\partial_{\tau_1} u^1 \partial_{\tau_2} u^3 - \partial_{\tau_2} u^1 \partial_{\tau_1} u^3), \tag{B.6}
\]
and
\[
\int_{B_1} \nabla \cdot u_h = \int_{S^2} u^3 (\partial_{\tau_1} u^1 \partial_{\tau_2} u^2 - \partial_{\tau_2} u^1 \partial_{\tau_1} u^2). \tag{B.7}
\]
Therefore, summing (B.5)-(B.7), and recalling the notation (1.7) (for $n = 3$), we arrive at (B.1). Regarding (B.2), the first equality is immediate from the expansion of the determinant around the identity matrix
$I_3$. The expression in the second line of (B.2) follows from the fact that the resulting terms can be written as boundary integrals in the following fashion. Using again Stokes’ theorem, we can calculate
\[
\int_{B_1} \text{div} w_h = 3 \int_{S^2} \langle w, x \rangle, \tag{B.8}
\]
and
\[
\int_{B_1} (\text{div} w_h)^2 = \int_{B_1} \text{div} ((\text{div} w_h) w_h) - \int_{B_1} \langle w_h, \nabla \text{div} w_h \rangle = \int_{S^2} \langle w, (\text{div} w_h) x \rangle - \int_{B_1} \langle w_h, \nabla \text{div} w_h \rangle = \int_{S^2} \langle w, (\text{div} w_h) x \rangle + \int_{S^2} \langle w, x \rangle \langle (\nabla w_h) x, x \rangle - \int_{B_1} \langle w_h, \nabla \text{div} w_h \rangle,
\]
and
\[
\int_{B_1} \text{Tr}(\nabla w_h)^2 = \sum_{i,j=1}^3 \int_{B_1} \partial_j w^i_h \partial_i w^j_h = \sum_{j=1}^3 \int_{B_1} \partial_j (\langle w_h, \nabla w^j_h \rangle) - \sum_{i=1}^3 \int_{B_1} \partial_i (\text{div} w_h) w^i_h = \sum_{j=1}^3 \int_{S^2} \langle w, x_j \nabla w^j_h \rangle - \int_{B_1} \langle w_h, \nabla \text{div} w_h \rangle = \int_{S^2} \langle w, \sum_{j=1}^3 x_j \nabla T w^j \rangle + \int_{S^2} \langle w, x \rangle \langle (\nabla w_h) x, x \rangle - \int_{B_1} \langle w_h, \nabla \text{div} w_h \rangle.
\]
Subtracting the last two identities we arrive at
\[
\frac{1}{2} \int_{B_1} ( (\text{div} w_h)^2 - \text{Tr}(\nabla w_h)^2 ) = \frac{3}{2} \int_{S^2} \langle w, (\text{div} w_h) x - \sum_{j=1}^3 x_j \nabla T w^j \rangle = Q_{V_1}(w). \tag{B.9}
\]
Hence, the second equality in (B.2) follows from (B.8), (B.9) and (B.1) for $w$ in the place of $u$ here. \qed

Next, we calculate in detail the Taylor expansions up to second order of the geometric quantities that we used in the main body of the paper. The computations presented in the following lemma are formal, and we assume without further clarification that the maps in consideration are always regular enough so that we can perform the expansions. In the case $n = 3$, we had directly performed the expansion of the
2-Dirichlet energy of $u$ around the $id_{S^{n-1}}$ in the proof of Lemma 4.5, and the one for $V_3(u)$ was performed in the previous Lemma B.1. Thus, the focus in the next lemma is mostly on the case $n \geq 4$. Moreover, notice that in Subsection 4.2 we had already translated and scaled the initial map $u$ properly (recall (4.2) and (4.3)), which we will also assume for convenience next.

**Lemma B.3.** Let $n \geq 3$, $u : S^{n-1} \to \mathbb{R}^n$ (be sufficiently regular) and as always, $w := u - id_{S^{n-1}}$. Assuming that $f_{S^{n-1}} = 0, f_{S^{n-1}} \langle w, x \rangle = 0$, and recalling the notation introduced in (1.9), we can formally write

\[
(i) \quad [D_{n-1}(u)]_{1}^{n-1} := 1 + Q_{[D_{n-1}]}_{n-1}^{n-1}(w) + R_{1,n}(w), \\
(ii) \quad [P_{n-1}(u)]_{1}^{n-1} := 1 + Q_{[P_{n-1}]}_{n-1}^{n-1}(w) + R_{2,n}(w), \\
(iii) \quad V_n(u) := 1 + Q_{V_n}(w) + R_{3,n}(w),
\]

where the corresponding quadratic forms are given by the expressions

\[
(i) \quad Q_{[D_{n-1}]}_{n-1}^{n-1}(w) := \frac{1}{2} \int_{S^{n-1}} \left( |\nabla_T w|^2 + \frac{n-3}{n-1} (\text{div}_{S^{n-1}} w)^2 \right), \\
(ii) \quad Q_{[P_{n-1}]}_{n-1}^{n-1}(w) := \frac{1}{2} \int_{S^{n-1}} \left( |\nabla_T w|^2 + (\text{div}_{S^{n-1}} w)^2 - 2 |(P_{n-1})^{\text{sym}}|_{n-1}^2 \right), \\
(iii) \quad Q_{V_n}(w) := \frac{n}{2} \int_{S^{n-1}} \langle w, (\text{div}_{S^{n-1}} w)x - \sum_{j=1}^{n} x_j \nabla_T w^j \rangle,
\]

and the remainder terms $(R_{i,n}(w))_{i=1,2,3}$ are of higher order.

**Proof.** Regarding the expansion of the $(n-1)$-Dirichlet energy, in the case $n = 3$ the calculation was performed in the proof of Lemma 4.5 (see (4.9)). For $n \geq 4$, by using the fact that

\[
\int_{S^{n-1}} \text{div}_{S^{n-1}} w = (n-1) \int_{S^{n-1}} \langle w, x \rangle = 0,
\]

we can formally calculate

\[
[D_{n-1}(u)]_{1}^{n-1} = \left( \int_{S^{n-1}} \left( |\nabla_T w|^2 - \frac{2}{n-1} \text{div}_{S^{n-1}} w + |\nabla_T w|^2 \right) \frac{n-1}{n-1} \right)^{\frac{n}{n-1}} = \left( \int_{S^{n-1}} \left( 1 + \frac{2}{n-1} \text{div}_{S^{n-1}} w + \frac{1}{2} |\nabla_T w|^2 + \frac{n-3}{2(n-1)} (\text{div}_{S^{n-1}} w)^2 + O(|\nabla_T w|^3) \right) \right)^{\frac{n}{n-1}}
\]

\[
= \left[ 1 + \frac{n}{2(n-1)} \int_{S^{n-1}} |\nabla_T w|^2 + \frac{n(n-3)}{2(n-1)^2} \int_{S^{n-1}} (\text{div}_{S^{n-1}} w)^2 \right. \\
\left. + \int_{S^{n-1}} O(|\nabla_T w|^3) + O\left( \left( \int_{S^{n-1}} |\nabla_T w|^2 \right)^2 \right) \right].
\]

Therefore, (B.13) gives the expansion (B.10)(i), with the formula (B.11)(i) for the quadratic term and the growth behaviour for the higher order term $R_{1,n}(w)$.

For the expansion of the generalized perimeter-term around the $id_{S^{n-1}}$, for every $n \geq 3$ we have,

\[
\int_{S^{n-1}} \sqrt{\det(\nabla_T u \nabla_T u')} = \int_{S^{n-1}} \sqrt{\det(I + K(w))},
\]
where
\[ K(w) := 2(P_T^T \nabla_T w)^{sym} + \nabla_T w^T \nabla_T w. \]  
(B.15)

The Taylor expansion of the determinant around \( I_x \) gives
\[ \det(I_x + K(w)) = 1 + \text{Tr} K(w) + \frac{1}{2} \left( (\text{Tr} K(w))^2 - \text{Tr}(K(w)^2) \right) + O(|K(w)|^3), \]  
(B.16)
and since in our case,
\[ \text{Tr} K(w) = 2 \text{div}_{\mathbb{S}^{n-1}} w + |\nabla_T w|^2, \]  
(B.17)
by the identities (B.14)–(B.17), we obtain the formal expansion
\[ \int_{\mathbb{S}^{n-1}} \sqrt{\det(\nabla_T u^T \nabla_T u)} = \int_{\mathbb{S}^{n-1}} \sqrt{1 + \Theta(w) + O(|\nabla_T w|^3)}, \]  
(B.18)
where
\[ \Theta(w) := 2 \text{div}_{\mathbb{S}^{n-1}} w + |\nabla_T w|^2 + 2 (\text{div}_{\mathbb{S}^{n-1}} w)^2 - 2 \left| (P_T^T \nabla_T w)^{sym} \right|^2. \]  
(B.19)

Since \((\Theta(w))^2 = 4 (\text{div}_{\mathbb{S}^{n-1}} w)^2 + O(|\nabla_T w|^3)\), we can perform a Taylor expansion of the square root inside the integral in (B.18) and use (B.12), (B.19), to get
\[ \int_{\mathbb{S}^{n-1}} \sqrt{\det(\nabla_T u^T \nabla T u)} = 1 + \frac{1}{2} \int_{\mathbb{S}^{n-1}} \left( |\nabla_T w|^2 + (\text{div}_{\mathbb{S}^{n-1}} w)^2 - 2 \left| (P_T^T \nabla_T w)^{sym} \right|^2 \right) + \int_{\mathbb{S}^{n-1}} O \left( |\nabla_T w|^3 \right). \]

A final Taylor expansion of the function \( t \mapsto (1 + t)^{\frac{n}{n-1}} \) around 0 gives,
\[ [P_{n-1}(u)]^{\frac{n}{n-1}} = 1 + \frac{1}{2} \frac{n}{n-1} \int_{\mathbb{S}^{n-1}} \left( |\nabla T w|^2 + (\text{div}_{\mathbb{S}^{n-1}} w)^2 - 2 \left| (P_T^T \nabla_T w)^{sym} \right|^2 \right) + \int_{\mathbb{S}^{n-1}} O \left( |\nabla_T w|^3 \right). \]  
(B.20)

Therefore, (B.20) gives the expansion (B.10) (ii), with the formula (B.11) (ii) for the corresponding quadratic term and the growth behaviour for the higher order term \( R_{2,n}(w) \).

The expansion of the generalized signed-volume \( V_n(u) \) in (1.6), (1.7) around the id \( \mathbb{S}^{n-1} \), for \( n = 3 \) was given in Lemma B.1 An intrinsic way to perform the calculation in every dimension \( n \geq 3 \) is the following.

\[ V_n(u) := \int_{\mathbb{S}^{n-1}} \left( u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \right) = \int_{\mathbb{S}^{n-1}} \left( w + x, \bigwedge_{i=1}^{n-1} (\partial_{\tau_i} w + \partial_{\tau_i} x) \right) \]
\[ = \int_{\mathbb{S}^{n-1}} \sum_{k=0}^{n-1} \sum_{|\alpha|=k} \sigma(\alpha, \bar{\alpha}) \left( w + x, \bigwedge_{\alpha} \partial_{\tau_\alpha} w \right) \]
\[ = I_{n,0}(w) + I_{n,1}(w) + I_{n,2}(w) + I_{n,3}(w). \]  
(B.21)

Here, we have used standard multiindex notation. For every \( k \in \{0,1,\ldots,n-1\} \) and for every multiindex \( \alpha := (\alpha_1, \ldots, \alpha_k) \), where \((a_i)_{i=1,\ldots,k} \in \mathbb{N} \) with \( 1 \leq \alpha_1 < \cdots < \alpha_k \leq n-1 \) we denote \( \bar{\alpha} \) its complementary multiindex (with its entries also in increasing order), \( \sigma(\alpha, \bar{\alpha}) \) the sign of the permutation that maps \((\alpha, \bar{\alpha})\) to the standard ordering \((1, \ldots, n)\) and
\[ \partial_{\tau_\alpha} w := \partial_{\tau_{\alpha_1}} w \wedge \cdots \wedge \partial_{\tau_{\alpha_k}} w. \]
We have also denoted by \((I_{n,i}(w))_{i=0,1,2}\) the zeroth, first and second order terms with respect to \(w\) and \(\nabla_{T}w\) in the expansion of \(V_{n}(u)\) around the \(id_{S^{n-1}}\) respectively, and by \(I_{n,3}(w)\) the remaining term which is a polynomial of order at least 3 and at most \(n\) in \(w\) and its first derivatives. Keeping in mind that \(\partial_{\tau_{i}}x = \tau_{i}\) for \(i = 1, \ldots, n - 1\) and that by an abuse of notation, \(\tau_{1} \wedge \cdots \wedge \tau_{n-1} \equiv x\), we can compute each term separately.

\[
I_{n,0}(w) = \int_{S^{n-1}} (x, \partial_{\tau_{1}}x \wedge \cdots \wedge \partial_{\tau_{n-1}}x) = \int_{S^{n-1}} |x|^2 = 1, \tag{B.22}
\]

\[
I_{n,1}(w) = \int_{S^{n-1}} \langle w, x \rangle + \int_{S^{n-1}} n \|x\|^2 = \int_{S^{n-1}} \langle w, x \rangle + \int_{S^{n-1}} \text{div}_{S^{n-1}} w = 0, \tag{B.23}
\]

the last equality following from \([B.12]\). For the quadratic term, we can write it as

\[
I_{n,2}(w) := I_{n,2,1}(w) + I_{n,2,2}(w), \quad \text{where}
\]

\[
I_{n,2,1}(w) = \sum_{i=1}^{n-1} \int_{S^{n-1}} \langle w, \left( \bigwedge_{l=1}^{i-1} \partial_{\tau_{l}}x \right) \wedge \partial_{\tau_{i}}w \wedge \left( \bigwedge_{m=i+1}^{n-1} \partial_{\tau_{m}}x \right) \rangle
= \sum_{i=1}^{n-1} \int_{S^{n-1}} \langle w, \left( \bigwedge_{l=1}^{i-1} \tau_{l} \right) \wedge \left( \bigwedge_{j=1}^{n-1} \partial_{\tau_{j}}w, \tau_{j} \right) \rangle \langle \partial_{\tau_{i}}w, \partial_{\tau_{i}}w, x \rangle \wedge \left( \bigwedge_{m=i+1}^{n-1} \tau_{m} \right) \rangle
= \int_{S^{n-1}} \left( \text{div}_{S^{n-1}} w \langle w, x \rangle - \sum_{i=1}^{n-1} \langle \partial_{\tau_{i}}w, \partial_{\tau_{i}}w, x \rangle \right) = \int_{S^{n-1}} \langle w, (\text{div}_{S^{n-1}} w)x \rangle - \sum_{j=1}^{n} x_{j} \nabla_{T}w^{j} \rangle. \tag{B.25}
\]

The change of sign in the last line of \((B.25)\) is due to orientation reasons, since we have taken the local orthonormal basis \(\{\tau_{1}, \ldots, \tau_{n-1}\}\) of \(T_{x}S^{n-1}\) in such a way that at every \(x \in S^{n-1}\) the set of vectors \(\{\tau_{1}(x), \ldots, \tau_{n-1}(x), x\}\) is a positively oriented frame of \(\mathbb{R}^{n}\). Similarly,

\[
I_{n,2,2}(w) = \frac{1}{2} \sum_{1 \leq i, j \leq n-1} \langle x, \left( \bigwedge_{k=1}^{i-1} \partial_{\tau_{k}}x \right) \wedge \partial_{\tau_{i}}w \wedge \left( \bigwedge_{l=i+1}^{j-1} \partial_{\tau_{l}}x \right) \wedge \partial_{\tau_{j}}w \wedge \left( \bigwedge_{m=j+1}^{n-1} \partial_{\tau_{m}}x \right) \rangle
= \frac{1}{2} \sum_{1 \leq i, j \leq n-1} \left( \langle \partial_{\tau_{i}}w, \tau_{i} \rangle \langle \partial_{\tau_{j}}w, \tau_{j} \rangle - \langle \partial_{\tau_{i}}w, \tau_{j} \rangle \langle \partial_{\tau_{j}}w, \tau_{i} \rangle \right). \tag{B.26}
\]

After integrating by parts it is easy to see that the first term in the last line of \((B.26)\) is

\[
\int_{S^{n-1}} \sum_{1 \leq i, j \leq n-1} \langle \partial_{\tau_{i}}w, \tau_{i} \rangle \langle \partial_{\tau_{j}}w, \tau_{j} \rangle = \int_{S^{n-1}} \langle w, (n-1)(\text{div}_{S^{n-1}} w)x - \nabla_{T} \text{div}_{S^{n-1}} w \rangle, \tag{B.27}
\]

while the second one therein is

\[
\int_{S^{n-1}} \sum_{1 \leq i, j \leq n-1} \langle \partial_{\tau_{i}}w, \tau_{i} \rangle \langle \partial_{\tau_{j}}w, \tau_{j} \rangle = \int_{S^{n-1}} \langle w, (\text{div}_{S^{n-1}} w)x - \nabla_{T} \text{div}_{S^{n-1}} w + (n-2) \sum_{j=1}^{n} x_{j} \nabla_{T}w^{j} \rangle. \tag{B.28}
\]

Subtracting \((B.27)\) and \((B.28)\) by parts, we infer that \((B.26)\) implies

\[
I_{n,2,2}(w) = \left( \frac{n}{2} - 1 \right) \int_{S^{n-1}} \langle w, (\text{div}_{S^{n-1}} w)x - \sum_{j=1}^{n} x_{j} \nabla_{T}w^{j} \rangle, \tag{B.29}
\]
and therefore, \((B.22)-(B.29)\) give the desired expansion \((B.10)\)(iii), with the formula \((B.11)\)(iii) for the corresponding quadratic term \(Q_{V_n}\). Regarding the remainder term \(R_{3,n}\) it is easy to see from \((B.21)\) that it has the algebraic structure

\[ R_{3,n}(w) = \sum_{k=2}^{n-1} \int_{S^{n-1}} \langle w, A_{n,k}(w) \rangle \]  \hspace{1cm} (B.30)

where for each \(k = 2, \ldots, n - 1\), \(A_{n,k}\) is a nonlinear first order differential operator that is a homogeneous polynomial of order \(k\) in the first derivatives of \(w\).

\[ \square \]

**Remark B.4.** From the growth behaviour of the higher order terms \((R_{i,n})_{i=1,2,3}\) in \((B.10)\), as these can be derived from \((B.13)\), \((B.20)\) and \((B.30)\), it is immediate to deduce the following simple fact. Even in the case \(n \geq 4\), if \(u \in W^{1,\infty}(S^{n-1}; \mathbb{R}^n)\) is such that \(\| \nabla_T u - P_T \|_{L^\infty(S^{n-1})} \leq \theta\) for some \(\theta \in (0,1)\) sufficiently small (as in the setting of Corollary 1.6 see also \((1.14)\)), and moreover \(f_{S^{n-1}} u = 0\), \(f_{S^{n-1}}(u, x) = 1\), then for \(w := u - \text{id}_{S^{n-1}}\), one has

\[ |R_{1,n}(w)| + |R_{2,n}(w)| + |R_{3,n}(w)| \leq c_\theta \int_{S^{n-1}} |\nabla_T w|^2, \]

for some constant \(c_\theta \in (0,1)\), such that \(c_\theta \to 0\) as \(\theta \to 0\).

Let us conclude this appendix by giving a proof of Korn’s identity on \(S^{n-1}\).

**Proof of Lemma 5.7.** We have

\[ \int_{S^{n-1}} \left| (P^T_T \nabla_T w)_{\text{sym}} \right|^2 = \frac{1}{2} \int_{S^{n-1}} |P^T_T \nabla_T w|^2 + \frac{1}{2} \int_{S^{n-1}} \text{Tr}((P^T_T \nabla_T w)^2), \]

and recalling \((B.26)\),

\[ \int_{S^{n-1}} \text{Tr}((P^T_T \nabla_T w)^2) = \int_{S^{n-1}} \sum_{i,j=1}^{n-1} \langle \partial_{\tau_i} w, \tau_j \rangle \langle \partial_{\tau_j} w, \tau_i \rangle = \int_{S^{n-1}} (\text{div}_{S^{n-1}} w)^2 - 2I_{n,2,2}(w). \]

By the definition of \(Q_{V_n}(w)\) in \((B.11)(iii)\) and \((B.29)\) we have that \(I_{n,2,2}(w) = \frac{n-2}{n} Q_{V_n}(w)\), and Korn’s identity \((5.7)\) follows immediately. \[ \square \]

**C Spherical Harmonics**

We first recall that in local coordinates, the spherical Laplace-Beltrami operator \(-\Delta_{S^{n-1}}\) is given for every \(f \in C^2(S^{n-1}; \mathbb{R}^n)\) through the expression

\[ -\Delta_{S^{n-1}} f := -\frac{1}{\sqrt{\det(g)}} \sum_{i,j=1}^{n-1} \frac{\partial}{\partial \tau_i} \left( \sqrt{|g|} g^{ij} \frac{\partial f}{\partial \tau_j} \right). \]  \hspace{1cm} (C.1)

Here, \(g := (g^{ij})_{i,j=1,\ldots,n-1}\) denotes the standard round metric on \(S^{n-1}\), \((g^{ij})_{i,j=1,\ldots,n-1}\) its inverse, and \(\{\tau_1, \ldots, \tau_{n-1}\}\) is the local orthonormal frame on \(S^{n-1}\) as introduced in Section 2. It is well known that \(L^2(S^{n-1}; \mathbb{R}^n)\) admits an orthonormal basis consisting of eigenfunctions of \(-\Delta_{S^{n-1}}\). In particular, for every
k \in \mathbb{N} there exists a finite number (denoted by \( G_{n,k} \)) of mutually \( L^2 \)- (and actually \( W^{1,2} \)) orthogonal functions \( (\psi_{n,k,j})_{j=1}^{\infty}, G_{n,k} \subset W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^n) \), which are called the vector valued \( k \)-th order spherical harmonics, are restrictions on \( \mathbb{S}^{n-1} \) of \( (\mathbb{R}^n\text{-valued}) \) homogeneous harmonic polynomials in \( \mathbb{R}^n \) of degree \( k \) respectively, and satisfy

\[-\Delta_{\mathbb{S}^{n-1}} \psi_{n,k,j} = \lambda_{n,k} \psi_{n,k,j} \quad \forall k \in \mathbb{N} \text{ and } j = 1, 2, \ldots, G_{n,k}, \text{ where } \lambda_{n,k} := k(k + n - 2). \quad (C.2)\]

In particular,

\[
\int_{\mathbb{S}^{n-1}} \left| \nabla_T \psi_{n,k,j} \right|^2 = \lambda_{n,k} \int_{\mathbb{S}^{n-1}} \left| \psi_{n,k,j} \right|^2 \quad \forall k \in \mathbb{N} \text{ and } j = 1, 2, \ldots, G_{n,k}.
\]

(C.3)

In the scalar case, \( G_{n,0} = 1 \) (with trivial eigenfunction the constant 1), \( G_{n,1} = n \) (with the first order spherical harmonics being the coordinate functions \( \psi_{1,1,j}(x) := \frac{x_j}{\sqrt{\omega_n}} \)), and \( G_{n,k} = (n+k-1)(n+k-3) \) for \( k \geq 2 \). The reader can refer to [12], [13] for more information on spherical harmonics.

**Remark C.1.** The following Parseval identities on \( \mathbb{S}^{n-1} \) hold true: If \( u \in W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^n) \) with its Fourier expansion in spherical harmonics being \( u = \sum_{k=0}^{\infty} \sum_{j=1}^{G_{n,k}} \alpha_{n,k,j} \psi_{n,k,j} \), then

\[
\int_{\mathbb{S}^{n-1}} |u|^2 = \sum_{k=0}^{\infty} \sum_{j=1}^{G_{n,k}} (\alpha_{n,k,j})^2 \quad \text{and} \quad \int_{\mathbb{S}^{n-1}} |\nabla_T u|^2 = \sum_{k=1}^{\infty} \sum_{j=1}^{G_{n,k}} \lambda_{n,k} (\alpha_{n,k,j})^2. \quad (C.4)
\]

The sharp Poincaré inequality for maps \( u \in W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^n) \) is then easily deduced. Since \( \lambda_{n,k} \geq n - 1 \) for every \( k \geq 1 \), from \( [C.4] \) we obtain

\[
\int_{\mathbb{S}^{n-1}} |\nabla_T u|^2 \geq (n - 1) \sum_{k=1}^{\infty} \sum_{j=1}^{G_{n,k}} (\alpha_{n,k,j})^2 = (n - 1) \int_{\mathbb{S}^{n-1}} |u - \int_{\mathbb{S}^{n-1}} u|^2. \quad (C.5)
\]

Of course, depending on the number of vanishing first Fourier modes in the expansion of \( u \), the constant in the above inequality can be improved in an obvious way. By expanding a function in spherical harmonics one can often obtain useful estimates. In the next lemma, we mention two of them that we have used earlier in the paper.

**Lemma C.2.** If \( u \in W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^n) \), and \( u_h : \overline{B_1} \mapsto \mathbb{R}^n \) denotes its (componentwise) harmonic extension, the following estimates hold true:

\[
\int_{B_1} |\nabla u_h|^2 \leq \frac{n}{n - 1} \int_{\mathbb{S}^{n-1}} |\nabla_T u|^2, \quad (C.6)
\]

\[
\frac{n}{n - 1} \int_{\mathbb{S}^{n-1}} |\nabla_T u|^2 \leq \int_{\mathbb{S}^{n-1}} |\nabla u_h|^2 \leq 2 \int_{\mathbb{S}^{n-1}} |\nabla_T u|^2. \quad (C.7)
\]

**Proof.** Let us give the proof of these two simple estimates in the case that \( u \) is scalar-valued, the case of vector-valued \( u \) being an immediate consequence. We write again

\[
u = \sum_{k=0}^{\infty} \sum_{j=1}^{G_{n,k}} \alpha_{n,k,j} \psi_{n,k,j treatment}
\]

and therefore, its harmonic extension can be written in polar coordinates \((r, \theta) \in [0, 1] \times \mathbb{S}^{n-1}\) as

\[
u_h(r, \theta) = \sum_{k=0}^{\infty} \sum_{j=1}^{G_{n,k}} r^k \alpha_{n,k,j} \psi_{n,k,j}(\theta).
\]

59
For (C.6), we write
\[
\int_{B_1} |\nabla u_h|^2 = \int_{B_1} \text{div}(u_h \nabla u_h) = n \int_{\mathbb{S}^{n-1}} \partial_{\nu} u_h = \sum_{k=0}^{\infty} \sum_{j=1}^{G_{n,k}} n k (\alpha_{n,k,j})^2 = \sum_{k=1}^{\infty} \sum_{j=1}^{G_{n,k}} \frac{n \lambda_{n,k}}{k + n - 2} (\alpha_{n,k,j})^2,
\]
while for (C.7), we write
\[
\int_{\mathbb{S}^{n-1}} |\nabla u_h|^2 = \int_{\mathbb{S}^{n-1}} |\nabla T u|^2 + \int_{\mathbb{S}^{n-1}} |\partial_{\nu} u_h|^2 = \int_{\mathbb{S}^{n-1}} |\nabla T u|^2 + \sum_{k=1}^{\infty} \sum_{j=1}^{G_{n,k}} \frac{k}{k + n - 2} \lambda_{n,k} (\alpha_{n,k,j})^2.
\]
Since \( \frac{1}{n-1} \leq \frac{k}{k + n - 2} \leq 1 \) for every \( k \geq 1 \), the desired estimates follows immediately by the above identities and (C.4).

\[\square\]

References

[1] F. Almgren. Optimal isoperimetric inequalities. Indiana University Mathematics Journal. 35 (1986), 451–547.

[2] A. Bernand-Mantel, C.B. Muratov, T. Simon. A Quantitative Description of Skyrmions in Ultrathin Ferromagnetic Films and Rigidity of Degree ±1 Harmonic Maps from \( \mathbb{R}^2 \) to \( \mathbb{S}^2 \). Archive for Rational Mechanics and Analysis. 239 (2020), 219–299.

[3] J.F. Borisov. On the connection between the spatial form of smooth surfaces and their intrinsic geometry. Vestnik Leningrad. Univ.. 14 (1959), 20–26.

[4] H. Brezis, J.M. Coron. Convergence of solutions of H-systems or how to blow bubbles. Archive for Rational Mechanics and Analysis. 89 (1985), 21–56.

[5] H. Brezis, L. Nirenberg. Degree theory and BMO; part I: Compact manifolds without boundaries. Selecta Mathematica New Series. 1 (1995), 197–264.

[6] P. Caldiroli, R. Musina. The Dirichlet problem for H-systems with Small Boundary Data: Blowup Phenomena and Nonexistence Results. Archive for Rational Mechanics and Analysis. 181 (2006), 1–42.

[7] M. Cicalese, G.P. Leonardi. A selection principle for the sharp quantitative isoperimetric inequality. Archive for Rational Mechanics and Analysis. 206, 2 (2012), 617–643.

[8] S. Cohn-Vossen. Zwei Sätze über die Starrheit der Einflächen. Nachrichten Göttingen. (1927), 125–137.

[9] S. Conti, C. De Lellis, L. Székelyhidi. h-Principle and Rigidity for \( C^{1,\alpha} \) Isometric Embeddings. Nonlinear Partial Differential Equations. Abel Symposia. 7 (2012).

[10] D. Faraco, X. Zhong. Geometric rigidity of conformal matrices. Annali della Scuola Normale Superiore di Pisa-Classe di Scienze. 4 (2005), 557–585.
[11] G. Friesecke, R.D. James, S. Müller. *A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity*. Communications on Pure and Applied Mathematics. **55** (2002), 1461–1506.

[12] J. Gallier. *Notes on Spherical Harmonics and Linear Representations of Lie Groups*. (2013).

[13] H. Groemer. *Geometric applications of Fourier series and spherical harmonics*. Cambridge University Press. **61** (1996).

[14] G. Herglotz. *Über die Starrheit der Einflächen*. Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg. **15** (1943), 127–129.

[15] J. Hirsch, K. Zemas. *A simple proof of a rigidity estimate for degree ±1 conformal maps on S²*. Bulletin of LMS. **54**, Issue 1 (2022), 256–263.

[16] N.H. Kuiper. *On C¹-isometric imbeddings. I, II*. Nederl. Akad. Wetensch. Proc. Ser. A. **58**= Indag. Math.. **17** (1955), 545–556, 683–689.

[17] S. Lang. *Linear algebra*. Springer Science & Business Media. (3rd Edition) (1987).

[18] F. Lin. *Mapping problems, fundamental groups and defect measures*. Acta Mathematica Sinica. **15** (1999), 25–52.

[19] P.L. Lions. *The concentration-compactness principle in the calculus of variations. The limit case, part 1*. Revista Mathematica Iberoamericana. **1** (1985), 145–201.

[20] J. Nash. *C¹ isometric imbeddings*. Annals of Mathematics. **(2) 60** (1954), 383–396.

[21] Y.G. Reshetnyak. *On the stability of conformal mappings in multidimensional spaces*. Siberian Mathematical Journal. **8** (1967), 69–85.

[22] Y.G. Reshetnyak. *On stability bounds in the Liouville theorem on conformal mappings of multidimensional spaces*. Siberian Mathematical Journal. **11** (1970), 833–846.

[23] Y.G. Reshetnyak. *Stability theorems in geometry and analysis*. **304** (2013).

[24] M. Spivak. *A comprehensive introduction to differential geometry*. V (1979).

[25] P. Topping. *A Rigidity estimate for maps from S² to S² via the harmonic map flow*. Bulletin of LMS (to appear). Arxiv: 2009.10459 (2020).

[26] H.C. Wente. *An existence theorem for surfaces of constant mean curvature*. Journal of Mathematical Analysis and Applications. **26** (1969), 318–344.

---

1 Institut für Mathematik, Universität Leipzig, Germany
(Stephan.Luckhaus@math.uni-leipzig.de)

2 Institut für Numerische und Angewandte Mathematik, Universität Münster, Germany
(konstantinos.zemas@uni-muenster.de)