ULTRAMETRIC FIXED POINTS IN REDUCED AXIOMATIC SYSTEMS

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Abstract. The Brezis-Browder ordering principle [Advances Math., 21 (1976), 355-364] is used to get a proof, in the reduced axiomatic system (ZF-AC+DC), of a fixed point result in the complete axiomatic system (ZF) over Cantor complete ultrametric spaces due to Petalas and Vidalis [Proc. Amer. Math. Soc., 118 (1993), 819-821].

1. Introduction

Throughout this exposition, the axiomatic system in use is Zermelo-Fraenkel’s (in short: ZF), as described by Cohen [9, Ch 2]. The notations and basic facts about its axioms are more or less usual.

Remember that, an outstanding part of it is the Axiom of Choice (abbreviated: AC); which, in a convenient manner, may be written as

\[(AC) \text{ For each nonempty set } X, \text{ there exists a (selective) function } f : (2)^X \to X \text{ with } f(Y) \in Y, \text{ for each } Y \in (2)^X.\]

[Here, \((2)^X\) denotes the class of all nonempty parts in \(X\). There are many logical equivalents of (AC); see, for instance, Moore [21, Appendix 2]. A basic one is the Zorn-Bourbaki Maximal Principle (in short: ZB), expressed as

\[(ZB) \text{ Let the partially ordered set } (X, \leq) \text{ be inductive (any totally ordered part } C \text{ of } X \text{ is bounded above: } C \leq b \text{ (i.e.: } x \leq b, \forall x \in C), \text{ for some } b \in X).\]

Then, for each (starting) \(u \in X\), there exists a maximal element \(v \in X\) (in the sense: \(v \leq z \in X \text{ implies } v = z\), with \(u \leq v\);

for a direct proof of this (avoiding transfinite induction), see Bourbaki [3].

Let \(X\) be a nonempty set. By a sequence in \(X\), we mean any mapping \(x : \mathbb{N} \to X\); where \(\mathbb{N} := \{0, 1, \ldots\}\) is the set of natural numbers. For simplicity reasons, it will be useful to denote it as \((x(n); n \geq 0)\), or \((x_n; n \geq 0)\); moreover, when no confusion can arise, we further simplify this notation as \((x(n))\) or \((x_n)\), respectively. Also, any sequence \((y_n := x_{i(n)}; n \geq 0)\) with

\[(i(n); n \geq 0) \text{ is divergent } [\text{i.e.: } i(n) \to \infty \text{ as } n \to \infty],\]

will be referred to as a subsequence of \((x_n; n \geq 0)\). Call the subset \(Y\) of \(X\), almost singleton (in short: asingleton) provided \([y_1, y_2 \in Y \text{ implies } y_1 = y_2]\); and singleton if, in addition, \(Y\) is nonempty; note that in this case, \(Y = \{y\}, \text{ for some } y \in X\).

Further, let \(d : X \times X \to R_+ := [0, \infty]\) be a metric over \(X\); the couple \((X, d)\) will be termed a metric space. Finally, let \(T \in F(X)\) be a selfmap of \(X\). [Here, for each couple \(A, B\) of nonempty sets, \(F(A, B)\) stands for the class of all functions from \(A\)

2010 Mathematics Subject Classification. 47H10 (Primary), 54H25 (Secondary).

Key words and phrases. Ultrametric space, strict nonexpansive map, fixed point, Brezis-Browder ordering principle, maximal element, Cantor completeness.
to $B$; when $A = B$, we write $F(A)$ in place of $F(A, A)$. Denote $\text{Fix}(T) = \{x \in X; x = Tx\}$; each point of this set is referred to as fixed under $T$. In the metrical fixed point theory, such points are to be determined according to the context below, comparable with the one described in Rus [26, Ch 2, Sect 2.2]:

**pic-1**) We say that $T$ is a Picard operator (modulo $d$) if, for each $x \in X$, the iterative sequence $(T^n x; n \geq 0)$ is $d$-convergent.

**pic-2**) We say that $T$ is a strong Picard operator (modulo $d$) if, for each $x \in X$, $(T^n x; n \geq 0)$ is $d$-convergent with $\lim_n (T^n x) \in \text{Fix}(T)$.

**pic-3**) We say that $T$ is fix-asingleton (resp., fix-singleton) if $\text{Fix}(T)$ is a singleton (resp., singleton).

In this perspective, a basic answer to the posed question is the 1922 one, due to Banach [1]. Given $\alpha \geq 0$, let us say that $T$ is $(d; \alpha)$-contractive, provided

\[(a01) \quad d(Tx, Ty) \leq \alpha d(x, y), \quad \text{for all} \quad x, y \in X.\]

**Theorem 1.** Suppose that $T$ is $(d; \alpha)$-contractive, for some $\alpha \in [0, 1]$. In addition, let $(X, d)$ be complete. Then, $T$ is a strong Picard operator (modulo $d$), and fix-asingleton (hence, fix-singleton).

This result – referred to as Banach’s contraction principle – found a multitude of applications in operator equations theory; so, it was the subject of many extensions. A natural way of doing this is by considering “functional” contractive conditions

\[(a02) \quad d(Tx, Ty) \leq F(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)),\]

for all $x, y \in X$;

where $F : \mathbb{R}_+^5 \to \mathbb{R}_+$ is a function. Some important results in the area have been established by Boyd and Wong [3], Matkowski [19], and Leader [18]. For more details about other possible choices of $F$ we refer to the 1977 paper by Rhoades [24]; some extensions of these to quasi-ordered structures may be found in Turinici [31]. Further, a natural extension of the contractive condition above is

\[(a03) \quad (T \text{ is } d\text{-strictly-nonexpansive}):\]

\[d(Tx, Ty) < d(x, y), \quad \text{for all} \quad x, y \in X, \quad x \neq y.\]

Note that, a fixed point for such maps is to be reached when $(X, d)$ is a compact metric space; cf. Edelstein [11]. Another circumstance when this conclusion holds is that of $(X, d)$ being a (transfinite) Cantor complete ultrametric space; see, for instance, Petalas and Vidalis [23]. In this last case, a basic tool used in authors’ proof is (ZB) (=the Zorn-Bourbaki Maximal Principle); or, equivalently (see above): (AC) (=the Axiom of Choice); hence, this fixed point result is valid in the complete Zermelo-Fraenkel system (ZF). However, since all arguments used there are countable in nature, it is highly expectable that a denumerable version of (ZB) should suffice for the result’s conclusion to hold. It is our aim in the present exposition to prove that this is indeed the case. Precisely, we show that the Zorn-Bourbaki Maximal Principle appearing there may be replaced with a countable version of it – namely, the Brezis-Browder ordering principle [5] – to solve the posed fixed point question; hence, the Petalas-Vidalis result is ultimately deductible in the reduced Zermelo-Fraenkel system (ZF-AC+DC); where (DC) is the Principle of Dependent Choices. Note that, the proposed reasoning is applicable as well to many other statements of this type; such as the ones due to Mishra and Pant [20]. Further aspects will be delineated elsewhere.
2. Brezis-Browder principles

Let $M$ be a nonempty set. Take a quasi-order ($\leq$) [i.e.: a reflexive ($x \leq x$, $\forall x \in X$) and transitive ($x \leq y$, $y \leq z \implies x \leq z$) relation] over it; the pair $(M, \leq)$ will be then referred to as a quasi-ordered structure. Let also $\varphi : M \to R^+$ be a function. Call the point $z \in M$ $(\leq, \varphi)$-maximal when: $z \leq w \in M$ implies $\varphi(z) = \varphi(w)$. A basic result about such points is the 1976 Brezis-Browder ordering principle [3] (in short: BB).

**Proposition 1.** Suppose that the quasi-ordered structure $(M, \leq)$ and the function $\varphi$ (taken as before) fulfill

(b01) $(M, \leq)$ is sequentially inductive:

each ascending sequence has an upper bound (modulo $(\leq)$)

(b02) $\varphi$ is $(\leq)$-decreasing ($x \leq y \implies \varphi(x) \geq \varphi(y)$).

Then, for each $u \in M$ there exists a $(\leq, \varphi)$-maximal $v \in M$ with $u \leq v$.

(A) In particular, assume that (in addition)

$(\leq)$ is antisymmetric ($x \leq y$, $y \leq x \implies x = y$).

We then say that it is a (partial) order on $M$; and the pair $(M, \leq)$ will be called a (partially) ordered structure. In this case, by an appropriate choice of our structure (related to existence of functions $\varphi : M \to R^+$ fulfilling strict versions of (b02)), one gets a countable variant of the Zorn-Bourbaki maximal principle [3]. Some conventions are needed. Let $(<)$ stand for the associated relation

$x < y$ if $x \leq y$ and $x \neq y$

Clearly, $(<)$ is irreflexive ($x < x$ is false, $\forall x \in M$) and transitive ($x < y$ and $y < z$ imply $x < z$); it will be referred to as the strict order attached to $(\leq)$. Call the point $z \in M$, $(\leq)$-maximal, provided

(b03) $w \in M$, $z \leq w \implies z = w$; or, equivalently: $M(z, <)(:= \{x \in M; z < x\})$ is empty.

The following (Zorn-Bourbaki) maximal version of (BB) (denoted, for simplicity, as (BB-Z)) is now available.

**Proposition 2.** Suppose that the (partially) ordered structure $(M, \leq)$ is such that

(b04) $(M, <)$ is sequentially inductive:

each $(<)$-ascending sequence in $M$ has an upper bound in $M$ (modulo $(<)$)

(b05) $(M, <)$ is admissible:

there exists at least one function $\varphi : M \to R^+$ with the $(<)$-decreasing property ($x < y \implies \varphi(x) > \varphi(y)$).

Then, $(\leq)$ is a Zorn order, in the sense: for each $u \in M$ there exists a $(\leq)$-maximal $v \in M$ with $u \leq v$.

**Proof.** There are two steps to be passed.

**Step 1.** We claim that, under these conditions, $(M, \leq)$ is sequentially inductive. In fact, let $(x_n; n \geq 0)$ be a $(\leq)$-ascending sequence in $M$. If the alternative below is in force

there exists $k \geq 0$, such that $x_k = x_n$, for all $n > k$,

we are done; because $y := x_k$ is an upper bound of $(x_n; n \geq 0)$. Suppose that the opposite alternative is true:

for each $k \geq 0$, there exists $h > k$ with $x_k < x_h$.
In this case, we get a \((<)-\)ascending sequence of ranks \((i(n); n \geq 0)\), such that the subsequence \((y_n := x_{i(n)}; n \geq 0)\) is \((<)-\)ascending. By the admitted hypothesis, there exists \(y \in M\) such that \(y_n < y\), for all \(n\). This, along with the \((\leq)-\)ascending property of \((x_n; n \geq 0)\), gives \(x_n < y\), for each \(n\); and the claim follows.

**Step 2.** As \((M, <)\) is admissible, there exists at least one function \(\varphi : M \to R_+\) with the \((<)-\)decreasing property: \(x < y \implies \varphi(x) > \varphi(y)\). Note that, by the very definition of our strict order \((<)\), we have the (converse) representation formula \(x \leq y\) iff either \(x < y\) or \(x = y\).

As a direct consequence of this, one gets that \(\varphi\) is \((\leq)-\)decreasing \((x \leq y \implies \varphi(x) \geq \varphi(y))\).

Putting these together, (BB) is applicable to \((M, \leq)\) and \(\varphi\). From this principle we are assured that, given \(u \in M\), there exists a \((\leq, \varphi)-\)maximal \(v \in M\) with \(u \leq v\). Suppose by contradiction that \(v < w\), for some \(w \in M\). As \(\varphi\) is \((<)-\)decreasing, this gives \(\varphi(v) > \varphi(w)\); in contradiction with the \((\leq, \varphi)-\)maximal property of \(v\). Hence, \(v\) is \((\leq)-\)maximal; and we are done. \(\square\)

Note that, for the moment, \((BB) \implies (BB-Z)\) in the strongly reduced axiomatic system \((ZF-AC)\). On the other hand, this statement includes (see below) Ekeland’s Variational Principle [12] (in short: EVP). As a consequence, many extensions of (BB) were proposed; see, for instance, Hyers, Isac and Rassias [15, Ch 5]. For each (countable) variational principle (VP) of this type, one therefore has \((BB) \implies (EVP)\); so, we may ask whether these inclusions are effective. As we shall see, the answer to this is negative.

**(B)** Let \(M\) be a nonempty set; and \(R \subseteq M \times M\) be a (nonempty) relation over \(M\); for simplicity, we sometimes write \((x, y) \in R\) as \(x R y\). Note that \(R\) may be viewed as a mapping between \(M\) and \(2^M\) (=the class of all subsets in \(M\)). In fact, denote for each \(x \in M\)

\[
M(x, R) = \{y \in M; x R y\} (= \text{the section of } R \text{ through } x); 
\]

then, the mapping representation of \(R\) is \((R(x) = M(x, R); x \in M)\).

Call the relation \(R\) over \(M\), proper when

\[
(b06) \quad M(c, R) \text{ is nonempty, for each } c \in M. 
\]

Clearly, \(R\) may be then viewed as a mapping between \(M\) and \(2^M\) (=the class of all nonempty subsets in \(M\)).

The following ”Principle of Dependent Choices” (in short: DC) is in effect for our future developments.

**Proposition 3.** Suppose that \(R\) is a proper relation over \(M\). Then, for each \(a \in M\) there exists a sequence \((x_n; n \geq 0)\) in \(M\) with \(x_0 = a\) and \(x_n R x_{n+1}\), for all \(n\).

This principle, due to Bernays [2] and Tarski [29], is deductible from AC (= the Axiom of Choice), but not conversely; cf. Wolk [35]. Moreover, the reduced axiomatic system \((ZF-AC+DC)\) seems to be comprehensive enough for a large part of the ”usual” mathematics; see Moore [21, Appendix 2, Table 4].

As an illustration of this assertion, we show that, ultimately, (BB) is contained in the underlying reduced system.

**Proposition 4.** We have \((DC) \implies (BB)\) in the strongly reduced system \((ZF-AC)\); hence, \((BB)\) is deductible in the reduced system \((ZF-AC+DC)\).
Proof. Let the premises of (BB) be admitted; i.e.: the quasi-ordered structure \((M, \leq)\) is sequentially inductive and the function \(\varphi : M \to R_+\) is \((\leq)-\)decreasing. Define the function \(\beta : M \to R_+\) as:

\[
\beta(v) := \inf[\varphi(M(v, \leq))], \quad v \in M.
\]

Clearly, \(\beta\) is increasing; and

\[
\varphi(v) \geq \beta(v), \text{ for all } v \in M. \tag{2.1}
\]

Moreover, \((\varphi=\text{decreasing})\) yields a characterization of maximal elements like

\[
v \text{ is } (\leq, \varphi)-\text{maximal iff } \varphi(v) = \beta(v). \tag{2.2}
\]

Now, assume by contradiction that the conclusion in this statement is false; i.e. [in combination with (2.1)+(2.2)], there must be some \(u \in M\) such that:

\[
\text{(b07) for each } v \in M_u := M(u, \leq), \text{ one has } \varphi(v) > \beta(v).
\]

Consequently (for all such \(v\)), \(\varphi(v) > (1/2)(\varphi(v) + \beta(v)) > \beta(v)\); hence

\[
v \leq w \text{ and } (1/2)(\varphi(v) + \beta(v)) > \varphi(w), \tag{2.3}
\]

for at least one \(w\) (belonging to \(M_u\)). The relation \(\mathcal{R}\) over \(M_u\) introduced via (2.3) is proper on \(M_u\); i.e.:

\[
M_u(v, \mathcal{R}) \neq \emptyset, \text{ for all } v \in M_u.
\]

So, by (DC), there must be a sequence \((u_n)\) in \(M_u\) with \(u_0 = u\) and

\[
u_n \leq u_{n+1}, \quad (1/2)(\varphi(u_n) + \beta(u_n)) > \varphi(u_{n+1}), \text{ for all } n. \tag{2.4}
\]

We have thus constructed an ascending sequence \((u_n)\) in \(M_u\) for which the positive (real) sequence \((\varphi(u_n))\) is (via (b07)) strictly descending and bounded below; hence \(\lambda := \lim_n \varphi(u_n)\) exists in \(R_+\). As \((M, \leq)\) is sequentially inductive, \((u_n)\) is bounded from above in \(M\): there exists \(v \in M\) such that \(u_n \leq v\), for all \(n\) (whence, \(v \in M_u\)). Moreover, since \((\varphi=\text{decreasing})\), we must have (by the properties of \(\beta\))

\[
\text{(j) } \varphi(u_n) \geq \varphi(v), \quad \forall n; \quad \text{(jj) } \varphi(v) \geq \beta(v) \geq \beta(u_n), \quad \forall n.
\]

The former of these relations gives \(\lambda \geq \varphi(v)\) (passing to limit as \(n \to \infty\)). On the other hand, the latter of these relations yields (via (2.4))

\[
(1/2)(\varphi(u_n) + \beta(v)) > \varphi(u_{n+1}), \text{ for all } n.
\]

Passing to limit as \(n \to \infty\) gives

\[
(\varphi(v) \geq) \beta(v) \geq \lambda;
\]

so, combining with the preceding one,

\[
\varphi(v) = \beta(v) = \lambda; \tag{2.5}
\]

Hence, (b07) cannot be accepted; and the conclusion follows. \(\square\)

Note that, a slightly different proof of this may be found in the 2007 monograph by Cărjă et al [8, Ch 2, Sect 2.1]. Further metrical aspects of it may be found in Turinici [32].

(C) In the following, the relationships between (BB) and Ekeland’s variational principle [12] (in short: EVP) are discussed.

Let \((M, d)\) be a metric space; and \(\varphi : M \to R_+\) be a function. Assume that

\[
\text{(b08) } (M, d) \text{ is complete (each } d\text{-Cauchy sequence in } M \text{ is } d\text{-convergent)}
\]

\[
\text{(b09) } \varphi \text{ is } d\text{-lsc: } \liminf_n \varphi(x_n) \geq \varphi(x), \text{ whenever } x_n \xrightarrow{d} x;
\]

or, equivalently: \(\{x \in M; \varphi(x) \leq t\}\) is \(d\)-closed, for each \(t \in R\).
Proposition 5. Let these conditions hold. Then, for each (starting point) \( u \in M \) there exists (another point) \( v \in M \) with
\[
 d(u, v) \leq \varphi(u) - \varphi(v) \quad \text{(hence } \varphi(u) \geq \varphi(v)) \tag{2.5} 
\]
\[
 d(v, x) > \varphi(v) - \varphi(x), \text{ for each } x \in M \setminus \{v\}. \tag{2.6} 
\]

Proof. Let \( (\preceq) \) stand for the relation (over \( M \)):
\[
 x \preceq y \iff d(x, y) \leq \varphi(x) - \varphi(y).
\]
Clearly, \( (\preceq) \) acts as a (partial) order on \( M \); note that, as a consequence of this, its associated relation
\[
 x \prec y \iff 0 < d(x, y) \leq \varphi(x) - \varphi(y).
\]
is a strict order on \( X \). We claim that conditions of (BB-Z) are fulfilled on \((M, \preceq)\). In fact, by this very definition, \( \varphi \) is \((\prec)\)-decreasing on \( M \); so that, \((M, \prec)\) is admissible.

On the other hand, let \((x_n)\) be a \((\prec)\)-ascending sequence in \( M \):
\[
 (b10) \quad 0 < d(x_n, x_m) \leq \varphi(x_n) - \varphi(x_m), \text{ if } n < m.
\]
The sequence \((\varphi(x_n))\) is strictly descending and bounded from below; hence a Cauchy one. This, along with our working hypothesis, tells us that \((x_n)\) is a \( d \)-Cauchy sequence in \( M \); wherefrom by completeness,
\[
 x_n \xrightarrow{d} y \text{ as } n \to \infty, \text{ for some } y \in M.
\]
Passing to limit as \( m \to \infty \) in the same working hypothesis, one derives
\[
 d(x_n, y) \leq \varphi(x_n) - \varphi(y), \text{ (i.e.: } x_n \preceq y)\), \text{ for all } n.
\]
This, combined with \((x_n; n \geq 0)\) being \((\preceq)\)-ascending, gives \( x_n \prec y \), for all \( n \); and shows that \((M, \prec)\) is sequentially inductive. From (BB-Z) it then follows that, for the starting \( u \in M \) there exists some \( v \in M \) with
\[
 h) \quad u \preceq v; \quad hh) \quad v \preceq x \in M \text{ implies } v = x.
\]
The former of these is just \((2.5)\); and the latter one gives at once \((2.6)\). \(\square\)

This principle found some basic applications to control and optimization, generalized differential calculus, critical point theory and global analysis; we refer to the quoted paper for a survey of these. So, it cannot be surprising that, soon after its formulation, many extensions of (EVP) were proposed. For example, the dimensional way of extension refers to the ambient positive halfline \( \mathbb{R}_+ \) of \( \varphi(M) \) being substituted by a convex cone of a (topological or not) vector space. An account of the results in this area is to be found in the 2003 monograph by Göpfert, Riahi, Tammer and Zălinescu [14, Ch 3]; see also Turinici [32]. On the other hand, the (pseudo) metrical one consists in the conditions imposed to the ambient metric over \( M \) being relaxed. Some basic results in this direction were obtained by Kang and Park [16]; see also Tataru [30].

(D) By the developments above, we therefore have the implications:
\[
 (DC) \implies (BB) \implies (BB-Z) \implies (EVP). 
\]
So, we may ask whether these may be reversed. Clearly, the natural setting for solving this problem is (ZF-AC); referred to (see above) as the strongly reduced Zermelo-Fraenkel system.

Let \( X \) be a nonempty set; and \((\leq)\) be a (partial) order on it. We say that \((\leq)\) has the \( \text{inf-lattice} \) property, provided:
\[
 x \land y := \inf(x, y) \text{ exists, for all } x, y \in X.
\]
Remember that \( z \in X \) is a \((\leq)-\text{maximal}\) element if \( X(z, \leq) = \{z\} \); the class of all these points will be denoted as \( \text{max}(X, \leq) \). Call \((\leq)\), a Zorn order when \( \text{max}(X, \leq) \) is nonempty and cofinal in \( X \) (for each \( u \in X \) there exists a \((\leq)\)-maximal \( v \in X \) with \( u \leq v \)).

Further aspects are to be described in a metric setting. Let \( d : X \times X \to R_{+} \) be a metric over \( X \); and \( \varphi : X \to R_{+} \) be some function. Then, the natural choice for \((\leq)\) above is
\[
x \leq_{(d, \varphi)} y \text{ iff } d(x, y) \leq \varphi(x) - \varphi(y);
\]
referred to as the Brøndsted order \([6]\) attached to \((d, \varphi)\). Denote
\[
X(x, \rho) = \{u \in X; d(x, u) < \rho\}, x \in X, \rho > 0
\]
[the open sphere with center \( x \) and radius \( \rho \)]. Call the ambient metric space \((X, d)\), discrete when
for each \( x \in X \) there exists \( \rho = \rho(x) > 0 \) such that \( X(x, \rho) = \{x\} \).

Note that, under such an assumption, any function \( \psi : X \to R \) is continuous over \( X \). However, the \((\text{global}) d\)-Lipschitz property of the same
\[
|\psi(x) - \psi(y)| \leq Ld(x, y), x, y \in X, \text{ for some } L > 0
\]
cannot be assured, in general.

Now, the statement below is a particular case of (EVP):

**Proposition 6.** Let the metric space \((X, d)\) and the function \( \varphi : X \to R_{+} \) satisfy
\[
\begin{align*}
(b11) & \quad (X, d) \text{ is discrete bounded and complete} \\
(b12) & \quad (\leq_{(d, \varphi)}) \text{ has the inf-lattice property} \\
(b13) & \quad \varphi \text{ is } d\text{-nonexpansive and } \varphi(X) \text{ is countable.}
\end{align*}
\]
Then, \((\leq_{(d, \varphi)})\) is a Zorn order.

We shall refer to it as: the discrete Lipschitz countable version of EVP (in short: (EVP-dLc)). Clearly, \((\text{EVP}) \Rightarrow (\text{EVP-dLc})\). The remarkable fact to be added is that this last principle yields \((\text{DC})\); so, it completes the circle between all these.

**Proposition 7.** The inclusion below is holding (in the strongly reduced Zermelo-Fraenkel system): \((\text{EVP-dLc}) \Rightarrow (\text{DC})\). So (by the above),
\[
i) \quad \text{the maximal/variational principles } (\text{BB}), (\text{BB-Z}) \text{ and } (\text{EVP}) \text{ are all equivalent with } (\text{DC}); \text{ hence, mutually equivalent}
\]
\[
ii) \quad \text{each "intermediary" maximal/variational statement } (VP) \text{ with } (\text{DC}) \Rightarrow (VP) \Rightarrow (\text{EVP}) \text{ is equivalent with both } (\text{DC}) \text{ and } (\text{EVP}).
\]

For a complete proof, see Turinici \([33]\). In particular, when the discrete, bounded, inf-lattice and nonexpansive properties are ignored in \((\text{EVP-dLc})\), the last result above reduces to the one in Brunner \([7]\). Note that, in the same particular setting, a different proof of \((\text{EVP}) \Rightarrow (\text{DC})\) was provided in Dodu and Morillon \([10]\). Further aspects may be found in Schechter \([28, \text{Ch 19, Sect 19.51}]\).

### 3. Cantor complete ultrametrics

Let \( X \) be a nonempty set. By an ultrametric (or: non-Archimedean metric) on \( X \), we mean any mapping \( d : X \times X \to R_{+} \) with the properties:
\[
\begin{align*}
(c01) & \quad x = y \text{ iff } d(x, y) = 0 \quad \text{(reflexive sufficient)} \\
(c02) & \quad d(x, y) = d(y, x), \forall x, y \in X \quad \text{(symmetric)} \\
(c03) & \quad d(x, z) \leq \max\{d(x, y), d(y, z)\}, \forall x, y, z \in X \quad \text{(ultra-triangular)};
\end{align*}
\]
in this case, the pair \((X,d)\) will be referred to as an ultrametric space. Note that, any ultrametric is a (standard) metric (on \(X\)), because
\[
d(x,z) \leq \max\{d(x,y), d(y,z)\} \leq d(x,y) + d(y,z), \ \forall x, y, z \in X;
\]
but, the converse is not in general valid. The class of these ultrametrics is nonempty. In fact, the discrete metric on \(X\) introduced as: for each \(x, y \in X\)
\[
d(x,y) = 1 \text{ if } x \neq y; \ d(x,y) = 0, \text{ if } x = y,
\]
is an ultrametric, as it can be directly seen. Further examples may be found in Rooij [25, Ch 3].

Let in the following \((X,d)\) be an ultrametric space. Note that, the presence of ultra-triangular inequality induces a lot of dramatic changes with respect to the standard metrical case; some basic ones will be shown below. [These were stated without proof in Khamsi and Kirk [17, Ch 5, Sect 5.7]; see also Rooij [25, Ch 2]; however, for completeness reasons, we shall provide a proof of them].

**Lemma 1.** Let \(x, y, z \in X\) be such that \(d(x,y) \neq d(y,z)\). Then, necessarily,
\[
d(x,z) = \max\{d(x,y), d(y,z)\};
\]
hence, either \(d(x,z) = d(x,y)\) or \(d(x,z) = d(y,z)\). In other words: each triangle \((x,y,z)\) in \(X\) is \(d\)-isosceles.

**Proof.** Suppose by contradiction that
\[
d(x,z) < \max\{d(x,y), d(y,z)\}.
\]
We have two alternatives to consider:

i) Suppose that \(d(x,y) < d(y,z)\). By the working hypothesis, we then have \(d(x,z) < d(y,z)\). In this case, the ultra-triangular inequality gives
\[
d(y,z) \leq \max\{d(x,y), d(x,z)\} < d(y,z); \text{ contradiction.}
\]

ii) Suppose that \(d(y,z) < d(x,y)\). By the working hypothesis, we then have \(d(x,z) < d(x,y)\); so, again from the ultra-triangular inequality,
\[
d(x,y) \leq \max\{d(x,z), d(y,z)\} < d(x,y); \text{ contradiction.}
\]

Having discussed all possible alternatives, we are done. \(\square\)

By definition, any set of the form
\[
X[a,r] = \{x \in X; d(a,x) \leq r\}, \ a \in X, \ r \in R_+,
\]
will be referred to as a \textit{d-closed sphere} with center \(a \in X\) and radius \(r \in R_+\); note that this is a nonempty subset of \(X\), in view of \(a \in X[a,r]\). In the following, some results involving the family of all \(d\)-closed spheres
\[
\mathcal{M} = \{X[a,r]; a \in X, r \in R_+\} \subseteq (2)^X
\]
will be discussed.

**Lemma 2.** Let \(M_1 := X[a_1,r_1], \ M_2 := X[a_2,r_2]\) be a couple of non-disjoint \(d\)-closed spheres in \(X\). Then,
\[
(i) \ M_1 \subseteq M_2, \text{ whenever } r_1 \leq r_2
\]
\[
(ii) \ M_1 = M_2, \text{ whenever } r_1 = r_2.
\]
Proof. As $M_1 \cap M_2 \neq \emptyset$, there exists at least one element $b \in M_1 \cap M_2$

i): Assume that $r_1 \leq r_2$; and let $x \in M_1$ be arbitrary fixed. From the ultra-
triangular inequality, we have

$$d(x, b) \leq \max\{d(x, a_1), d(b, a_1)\} \leq r_1;$$

and this in turn yields (by the same procedure)

$$d(x, a_2) \leq \max\{d(x, b), d(b, a_2)\} \leq \max\{r_1, r_2\} = r_2; i.e.: x \in M_2.$$ 

As $x \in M_1$ was arbitrarily chosen, one derives $M_1 \subseteq M_2$.

ii): Evident, by the preceding step. □

Lemma 3. Let $a, b \in X$ and $s \geq 0$ be such that $a \in X[b, s]$. Then,

$$X[a, r] \subseteq X[b, s], \text{ for each } r \in [0, s]. \quad (3.1)$$

Proof. Let $r \in [0, s]$ be arbitrary fixed. By the imposed hypothesis,

$$a \in X[a, r] \cap X[b, s]; \text{ whence, } X[a, r] \cap X[b, s] \neq \emptyset;$$

and then, from the previous result, we are done. □

The next statement is, in a certain sense, a reciprocal of the previous one. Denote

$$(Y_1, Y_2 \in 2^X): Y_1 \subset Y_2 \iff Y_1 \subseteq Y_2 \text{ and } Y_1 \neq Y_2.$$ 

Clearly, $(\subset)$ is nothing else than the strict order (i.e.: irreflexive and transitive
relation) attached to the usual (partial) order $(\subseteq)$ over $2^X$.

Lemma 4. Let $M_1 := X[a_1, r_1], M_2 := X[a_2, r_2]$ be two $d$-closed balls in $X$. Then,

$$M_1 \subseteq M_2 \implies r_1 < r_2. \quad (3.2)$$

Proof. Suppose that $M_1 \subset M_2$; but (contrary to the conclusion) $r_2 \leq r_1$. As

$M_1 \cap M_2 = M_1 \neq \emptyset$, one has by a preceding result (and the working hypothesis)

$$M_2 \subseteq M_1 \subset M_2; \text{ contradiction.}$$

This proves our assertion. □

We are now introducing a basic notion. Call the ultrametric space $(X, d)$, Cantor
strongly complete (in short: Cantor s-complete), provided

(c05) each $(\supseteq)$-ascending sequence $(M_n := X[a_n, r_n]; n \geq 0)$ in $\mathcal{M}$

has a nonempty intersection.

A (formally) weaker variant of this definition is as follows. Call the ultrametric
space $(X, d)$, Cantor complete, provided

(c06) each $(\supset)$-ascending sequence $(M_n := X[a_n, r_n]; n \geq 0)$ in $\mathcal{M}$

has a nonempty intersection.

Clearly, we have

$(\forall$ ultrametric structure): Cantor s-complete $\implies$ Cantor complete. \hspace{1cm} (3.3)

The reciprocal inclusion is also true, as results from

Lemma 5. For each ultrametric structure $(X, d)$, we have

$$\text{Cantor complete} \implies \text{Cantor s-complete;}$$

$$\text{hence, Cantor complete} \iff \text{Cantor s-complete.} \quad (3.4)$$

Proof. Suppose that the ultrametric space $(X, d)$ is Cantor complete; and let $(M_n := X[a_n, r_n]; n \geq 0)$ be a $(\supseteq)$-ascending sequence in $\mathcal{M}$. If one has that
$\exists (i \geq 0), \forall (j > i): M_i = M_j$,

we are done; because $\cap \{M_n; n \geq 0\} = M_i$. Suppose now that the opposite alternative is holding:

$\forall (i \geq 0), \exists (j > i): M_i \supset M_j$.

There exists then a strictly ascending sequence of ranks $(i(n); n \geq 0)$, such that the subsequence $(L_n := M_{i(n)}; n \geq 0)$ of $(M_n; n \geq 0)$ fulfills

$(L_n)$ is $(\supset)$-ascending: $p < q \implies L_p \supset L_q$.

By the imposed hypothesis, $L := \cap \{L_n; n \geq 0\}$ is nonempty. This, along with $L = \cap \{M_n; n \geq 0\}$, ends the argument. □

Denote, for simplicity

$\Gamma = X \times R_+; \text{ hence, } \Gamma = \{(a, \rho); a \in X, \rho \in R_+\}$.

A natural relation to be introduced here is the following $(a, \rho) \prec (b, \sigma)$ iff $X[a, \rho] \supset X[b, \sigma]$.

Clearly, $(\prec)$ is irreflexive and transitive; hence, a strict order on $\Gamma$. Let $(\preceq)$ stand for the associated (partial) order $(a, \rho) \preceq (b, \sigma)$ iff either $(a, \rho) \prec (b, \sigma)$ or $(a, \rho) = (b, \sigma)$.

Having these precise, let us introduce the function

$(\varphi : \Gamma \to R_+): \varphi(a, \rho) = \rho$, $(a, \rho) \in \Gamma$.

By a previous result, we have

$\varphi$ is $(\prec)$-decreasing: $(a, \rho) \prec (b, \sigma) \implies \varphi(a, \rho) > \varphi(b, \sigma)$.

This tells us that, necessarily,

$(\Gamma, \prec)$ is admissible; hence, so is $(\Delta, \prec)$, where $\emptyset \neq \Delta \subseteq \Gamma$. (3.5)

As a consequence, the following (relative) maximal result is available.

**Proposition 8.** Let the (nonempty) subset $\Delta$ of $\Gamma$ be such that

$(\Delta, \prec)$ is sequentially inductive:

- each $(\prec)$-ascending sequence in $\Delta$ is bounded above in $\Delta$ (modulo $(\prec)$).

Then, $(\preceq)$ is a Zorn order on $\Delta$; i.e.: for each (starting element) $(a, \rho) \in \Delta$, there exists (another element) $(b, \sigma) \in \Delta$, with

i) $(a, \rho) \preceq (b, \sigma)$; i.e.: either $(a, \rho) \prec (b, \sigma)$ or $(a, \rho) = (b, \sigma)$

ii) $(b, \sigma) \prec (c, \tau)$ is impossible, for each $(c, \tau) \in \Delta$.

**Proof.** By the admissible property for $\Gamma$, we have

the strictly ordered structure $(\Delta, \prec)$ is admissible.

Combining with the admitted hypothesis, it results that the sequential type maximal result (BB-Z) is applicable to $(\Delta, \preceq)$; and, from this, we are done. □

**4. Application (fixed point theorems)**

In the following, an application of the above developments is given to the ultrametric fixed point theory.

Let $(X, d)$ be an ultrametric space. We say that $T \in \mathcal{F}(X)$ is $d$-strictly-nonexpansive, provided

(d01) $d(Tx, Ty) < d(x, y)$, $\forall x, y \in X, x \neq y$. 

Note that, in particular, $T$ is $d$-nonexpansive:
\[(d02) \quad d(Tx, Ty) \leq d(x, y), \quad \text{for all } x, y \in X.\]
The following fixed point theorem over ultrametric spaces is available.

**Theorem 2.** Suppose that $T$ is $d$-strictly-nonexpansive (see above). In addition, let $(X, d)$ be Cantor complete. Then, $T$ is fix-singleton; whence, it has a unique fixed point in $X$.

**Proof.** There are several steps to be followed.

**Step 1.** By the $d$-strict-nonexpansive property, we have
\[\text{Fix}(T) \text{ is asingleton; i.e.: } T \text{ is fix-asingleton.}\]
So, all we have to establish is that Fix$(T)$ appears as nonempty.

**Step 2.** Remember that, over $\Gamma := X \times \mathbb{R}_+$ we introduced the strict ordering \[(a, \rho) \prec (b, \sigma) \iff X[a, \rho] \supset X[b, \sigma];\]
as well as the associated ordering \[(a, \rho) \preceq (b, \sigma) \iff \text{either } (a, \rho) \prec (b, \sigma) \text{ or } (a, \rho) = (b, \sigma).\]
Moreover, we have that $\Gamma, \prec$ is an admissible; hence, so is $\Delta, \prec$, where $\emptyset \neq \Delta \subseteq \Gamma$.

**Step 3.** Denote, for simplicity
\[\Delta = \{(a, d(a, Ta)); a \in X\};\]
this is a nonempty subset of $\Gamma$. By a previous relation, we have that
\[\text{(the strictly ordered structure) } \Delta, \prec \text{ is admissible.}\]
Moreover, we claim that the structure $\Delta, \prec$ is sequentially inductive. In fact, let $((a_n, d(a_n, Ta_n)); n \geq 0)$ be a ($\prec$)-ascending sequence in $\Delta$; i.e.:
\[M_i \supset M_j, \text{ for } i < j; \text{ where } (M_n := X[a_n, d(a_n, Ta_n)]; n \geq 0).\]
As $(X, d)$ is Cantor complete, it follows that
\[L := \cap\{M_n; n \geq 0\} \text{ is nonempty;}\]
let $b \in L$ be some point of it. By the very definition above (and the $d$-nonexpansive property of $T$)
\[d(Tb, Ta_n) \leq d(b, a_n) \leq d(a_n, Ta_n), \quad \forall n \geq 0.\]
Combining with the ultra-triangular inequality, one gets (for the same ranks)
\[d(Tb, Tb) \leq \max\{d(b, a_n), d(Ta_n, Tb)\} \leq d(a_n, Ta_n);\]
and this, by a previous auxiliary fact, yields
\[X[a_n, d(a_n, Ta_n)] \supset X[b, d(b, Tb)], \text{ for all } n;\]
or, equivalently (by definition)
\[(a_n, d(a_n, Ta_n)) \prec (b, d(b, Tb)) \in \Delta, \text{ for all } n;\]
which proves the desired fact.

**Step 4.** Putting these together, it follows that the previous maximal principle is applicable to $\Delta, \preceq$. So, for the starting element $(u, d(u, Tu))$ in $\Delta$, there exists another element $(v, d(v, Tv))$ in $\Delta$, with
i) \[u, d(u, Tu)) \preceq (v, d(v, Tv))\]
ii) for each $w \in X$, $(v, d(v, Tv)) \prec (w, d(w, Tw))$ is impossible.
Suppose by contradiction that 
\[ d(v, T^2v) > 0; \text{ hence, } d(Tv, T^2v) < d(v, Tv). \]
We claim that 
\[ (v, d(v, Tv)) < (Tv, d(Tv, T^2v)); \]
and this, by the previous maximal property of \((v, d(v, Tv))\) yields a contradiction. The desired relation may be written as 
\[ X[v, d(v, Tv)] \supset X[Tv, d(Tv, T^2v)]; \]
to establish it, we may proceed as follows.

\textbf{I)} Let \( y \in X[Tv, d(Tv, T^2v)] \) be arbitrary fixed; hence, 
\[ d(y, Tv) \leq d(Tv, T^2v) < d(v, Tv). \]
By the ultra-triangular inequality, 
\[ d(y, v) \leq \max\{d(y, Tv), d(v, Tv)\} = d(v, Tv); \]
whence, \( y \in X[v, d(v, Tv)] \); this, by the arbitrariness of \( y \), gives 
\[ X[Tv, d(Tv, T^2v)] \subseteq X[v, d(v, Tv)]. \]
\textbf{II)} From the working assumption about \( v \), one must have 
\[ (v \in X[v, d(v, Tv)] \text{ and } v \notin X[Tv, d(Tv, T^2v)]); \]

hence, the above inclusion is strict. The proof is thereby complete. \(\square\)

By the argument above, this fixed point result is a consequence of the Brezis-Browder ordering principle [5]; hence, ultimately, it is deductible in the reduced Zermelo-Fraenkel system (ZF-AC+DC). Note that, similar conclusions are to be derived for the related fixed point results over ultrametric spaces due to Gajić [13] and Pant [22]; see also Wang and Song [34]. Further aspects of this theory concerning fuzzy ultrametric spaces may be found in Sayed [27].

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