ON GRADED DECOMPOSITION NUMBERS
FOR RATIONAL CHEREDNIK ALGEBRAS

C. BOWMAN, A. G. COX, AND L. SPEYER

ABSTRACT. We provide an algorithmic description of a family of graded decomposition numbers for rational Cherednik algebras in terms of affine Kazhdan–Lusztig polynomials.

INTRODUCTION

Double affine Hecke algebras, also known as Cherednik algebras, were first introduced by Cherednik as a tool for proving the MacDonald constant term conjectures. In [GGOR03], the authors introduce a category $\mathcal{O}$ of modules for a given Cherednik algebra, which control the representation theory of the underlying cyclotomic Hecke algebra in much the same way as the classical $q$-Schur algebra does in the case of type $G(1,1,n)$.

Fix a weighting $\Theta \in \mathbb{R}^l$, an $e$-multicharge $\kappa \in (\mathbb{Z}/e\mathbb{Z})^l$, $n \in \mathbb{N}_0$, $g < 0$ and $\epsilon \ll |g|$. In [Web13], Webster defines a finite dimensional (graded) cellular algebra, $A(n, \Theta, \kappa)$, whose module category provides a graded lift of this category $\mathcal{O}$. This algebra is defined by a diagrammatic presentation similar to that of Khovanov and Lauda, [KL09]. The diagrammatic calculus of this algebra captures a great deal of representation theoretic information.

In this paper, we shall study a certain saturated quotient of Webster’s algebra in the case that the weighting $\Theta = (\Theta_1, \ldots, \Theta_l)$ is such that $0 < \Theta_j - \Theta_i < |g| - \epsilon$ for all $1 \leq i < j \leq l$ (we refer to this as a FLOTW weighting, after [FLOTW99]). Here, the set of one-column multipartitions, $\pi$, forms a saturated set; of principal interest in this paper will be the quotient algebra (which we call the quiver Temperley–Lieb algebra of type $G(l,1,n)$) whose module category is the subcategory of representations whose simple constituents lie in this saturated set.

Strikingly, the elements of Webster’s cellular basis in this quotient algebra may be indexed by orbits of paths in a Euclidean space under the action of an affine Weyl group of type $\hat{A}_{l-1}$. Moreover, the graded dimensions of the standard modules are given by running a cancellation-free version of Soergel’s algorithm along the paths in this alcove geometry.

Motivated by this example, we introduce the notion of an algebra with a Soergel-path basis. In Theorem 1.18, we show that (under certain assumptions) the graded decomposition numbers of such an algebra are given by the associated Kazhdan–Lusztig polynomials. Our approach makes use only of elementary linear algebra and is based on Kleshchev and Nash’s proof of the LLT algorithm, [KN10].

In particular, given a Cherednik algebra with FLOTW weighting, the graded decomposition numbers $d_{\lambda\mu}(t)$ for $\lambda, \mu \in \pi$ are given by the affine Kazhdan–Lusztig polynomials of type $\hat{A}_{l-1}$ (see Corollary 3.17). It is shown in [Los13, Web13, RSVV13] that the decomposition numbers of $A(n, \Theta, \kappa)$ can be equated with coefficients of Uglov’s canonical basis of a twisted Fock space. Therefore the decomposition numbers may, in principle, be calculated by running an analogue of the LLT algorithm, see [Jac05], but not in terms of Kazhdan–Lusztig polynomials.

Our alcove-geometric description comes complete with a translation principle; it also allows us to deduce that the decomposition numbers are stable as the rank $n$ tends to infinity. Thus, we conjecture that the algebras considered here are asymptotically related (as the rank tends to infinity) to affine Kac–Moody algebras (see [Kas90]) and in finite rank to the generalised blob algebras (see [MW03]). In the level 2 case, the blob algebra first arose in the study of two-dimensional Potts models, [MS94], and has subsequently been related to the Virasoro algebra [GJSV13] in the limit as $n$ tends to infinity.
In order to clarify the above, let’s consider an example. We will omit technical details and definitions at this stage, and instead concentrate on giving a flavour of the combinatorics that is involved. Let \( l = 3, n = 13, e = 8, \kappa = (0, 4, 6) \). We shall consider a single block/linkage class of the algebra TL\(_{13}(\kappa)\). We embed the one-column multipartitions into Euclidean space via the embedding \((1^{\lambda_1}, 1^{\lambda_2}, 1^{\lambda_3}) \mapsto \sum_{i=1}^{3} \lambda_i \varepsilon_i\) and shall label representations of the algebra by these points (rather than the multipartitions). These points belong to the codimension 1 hyperplane given by \( \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 13 \), which is depicted in Figure 1, below. The affine Weyl group acts on this space fixing the point \(-\rho\) for \( \rho = e(1, 1, 1) - \kappa = (8, 4, 2) \).

![Figure 1](image_url)

**Figure 1.** The black points label the multipartitions of a block of TL\(_{13}(\kappa)\) via the embedding \((1^{\lambda_1}, 1^{\lambda_2}, 1^{\lambda_3}) \mapsto \sum_{i=1}^{3} \lambda_i \varepsilon_i\). The origin is labelled as \(\circ\), the points \(\alpha = (4, 6, 3), \beta = (5, 6, 2)\) and \(\gamma = (5, 8, 0)\) are also marked. The thick black lines denote the hyperplanes for the \(\rho\)-shifted action of the Weyl group.

The labels of the representations in this block in which we are most interested are the points \(\alpha = (4, 6, 3), \beta = (5, 6, 2)\) and \(\gamma = (5, 8, 0)\). The other simple representations in this block are labelled by the black points in Figure 1.

Let \(\lambda, \mu\) be any elements in our block. We wish to calculate the graded dimension of the \(\mu\)-weight space, \(\Delta_\mu(\lambda)\), of a cell-module \(\Delta(\lambda)\) for TL\(_{13}(\kappa)\).

For a given \(\mu\), we fix a distinguished path, \(\omega^{\mu}\), from the origin to \(\mu\), and for each \(\lambda\) in the above set, we look at paths which may be obtained by folding-up the path \(\omega^{\mu}\) along hyperplanes so that it terminates at \(\lambda\) (as illustrated shortly); we denote the set of such paths by \(\text{Path}(\lambda, \mu)\).

Each path has an associated degree which can be calculated by running Soergel’s (cancellation-free) algorithm along this path. The key to working with the quiver Temperley–Lieb algebras is the following observation,

\[
\text{Dim}_t(\Delta_\mu(\lambda)) = \sum_{\omega \in \text{Path}(\lambda, \mu)} t^{\deg(\omega)}.
\]

From this, (and the conditions on our distinguished paths) it is immediate that a necessary condition for \([\Delta(\lambda) : L(\mu)] \neq 0\) is that \(\ell(\mu) > \ell(\lambda)\) in the length function associated to our geometry. For a fixed \(\lambda\), we calculate the decomposition numbers \([\Delta(\lambda) : L(\mu)]\) by running Soergel’s algorithm not once, but many times: we run the algorithm to each point \(\mu\) such that \(\ell(\mu) > \ell(\lambda)\). This is a dual set-up to that usually considered.

As \(n\) tends to infinity, we find that there are infinitely many \(\mu\) such that \(\ell(\mu) > \ell(\lambda)\); the dimension of the cell module \(\Delta(\lambda)\) and the number of composition factors of \(\Delta(\lambda)\) also tend to
infinity as \( n \) becomes arbitrarily large. Fixing a value of \( n \in \mathbb{N} \) truncates the set of weights \( \mu \) in our Euclidean space to a finite set which labels representations of \( TL_n(\kappa) \). We shall see that the decomposition numbers are stable under this limiting behaviour.

For example, if \( \beta = (5, 6, 2) \), then we take the path \( \omega^\beta \) given by
\[
(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_1, \epsilon_2, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_1, \epsilon_2, \epsilon_1, \epsilon_2).
\]
This path passes through a single hyperplane, namely \( x_1 - x_3 = e \) (this is depicted by the thick black line separating \( \alpha \) and \( \beta \) in Figure 1). Reflecting through this hyperplane, we obtain a path
\[
(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_1, \epsilon_2, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_2, \epsilon_2, \epsilon_2)
\]
of degree 1 which terminates at \( \alpha \). Therefore \( \text{Dim}_t(\Delta_\beta(\alpha)) = t^1 \). We will see that there are no removable subpatterns in following Soergel’s procedure in this case, and so (by Theorem 3.15) this path labels a graded decomposition number,
\[
[\Delta(\alpha) : L(\beta)] = t^1.
\]

Now let \( \gamma = (5, 8, 0) \); we wish to calculate the dimension of \( \Delta_\gamma(\lambda) \) for \( \lambda \) in the above set. The distinguished path, \( \omega^\gamma \), in this case is given by
\[
(\epsilon_1, \epsilon_2, \epsilon_1, \epsilon_2, \epsilon_1, \epsilon_2, \epsilon_2, \epsilon_2, \epsilon_2, \epsilon_2, \epsilon_2, \epsilon_2)
\]
and is pictured in Figure 2.

![Figure 2](image2.png)

**Figure 2.** The distinguished path \( \omega^\gamma \) from the origin to \( \gamma = (5, 8, 0) \). (The space has been cropped to only include points less than or equal to \( \gamma \) in the dominance ordering.)

There are a total of \( 2^3 \) distinct paths which may be obtained from this path by a series of reflections (as our path passes through three alcove walls). For brevity, we truncate our diagrams so as to only consider alcoves between the origin and \( \gamma \). The eight paths are listed in Figures 2, 3, 4, and 5.

![Figure 3](image3.png)

**Figure 3.** The paths in \( \text{Path}(\alpha, \gamma) \). These elements are of degrees 1 and 3 respectively.

Those familiar with Soergel’s algorithm will recognise the degrees of the paths listed in the figures, (see Section 1.4 for more details). These paths describe the dimension of \( \Delta_\gamma(\lambda) \) for any point \( \lambda \). In particular,
\[
\text{Dim}_t(\Delta_\gamma(\alpha)) = t^3 + t^1 \quad \text{Dim}_t(\Delta_\gamma(\beta)) = t^2 + t^0.
\]
The leftmost diagram in Figure 4 labels an element of Path(β, γ) of degree zero. This path of degree zero labels a term that would be removed under Soergel’s algorithm. In terms of our basis, this corresponds to the fact that this path labels a vector in the basis of the simple head \(L(\beta)\); in fact
\[
\text{Dim}_t(L(\gamma)) = t^0,
\]
see Section 1.4 for more details. The other path, of degree 2, is not removed under Soergel’s procedure, and therefore (by Theorem 3.15) labels a decomposition number
\[
[\Delta(\beta) : L(\gamma)] = t^2.
\]

Having removed the degree zero path in Path(β, γ) under Soergel’s procedure, we also remove all paths in a certain subpattern labelled by this zero (see Section 1.4). In this case, the only other path in this subpattern is the leftmost path pictured in Figure 3.

The degree 0 path in the subpattern corresponds to Dim_t(\(\Delta(\gamma)\)) = Dim_t(L(\gamma)) = t^0. The degree 1 path in the subpattern corresponds to the degree 1 basis element of \(\Delta(\gamma)\) which occurs in the (degree shifted by 1) copy of \(L(\gamma)(1)\) inside \(\Delta(\alpha)\), (see †). This accounts for the path in Path(α, γ) of degree 1.

That leaves a path of degree 3, which is not removed under Soergel’s procedure and so
\[
[\Delta(\alpha) : L(\gamma)] = t^3.
\]

In general, we will see that at each stage in our algorithm, the removed subpatterns correspond to the weight spaces of simple modules, and the surviving paths correspond to decomposition numbers.

**Acknowledgements.** The authors would like to thank Joe Chuang, Matt Fayers, Daniel Juteau, and Ben Webster for helpful conversations during the preparation of this manuscript.

The authors are grateful for the financial support received from the Royal Commission for the Exhibition of 1851, EPSRC grant EP/L01078X/1, and Queen Mary University of London, respectively. The authors also thank the ICMS in Edinburgh for their hospitality during the early stages of this project.
1. Soergel path algebras

In this section (inspired by [MW03, Section 3]), we define an abstract family of algebras whose bases possess desirable properties. The combinatorics of these algebras is controlled by orbits of paths in a Euclidean space.

1.1. Graded cellular algebras with highest weight theories.

**Definition 1.1.** Suppose that $A$ is a $\mathbb{Z}$-graded $\mathbb{C}$-algebra which is free of finite rank over $\mathbb{C}$. We say that $A$ is a graded cellular algebra with a highest weight theory if the following conditions hold.

The algebra is equipped with a cell datum $(\Lambda, T, C, \deg)$, where $(\Lambda, \geq)$ is the weight poset. For each $\lambda, \mu \in \Lambda$, such that $\lambda \geq \mu$, we have a finite set, denoted $T(\lambda, \mu)$, and we let $T(\lambda) = \bigcup_\mu T(\lambda, \mu)$. There exist maps

$$C : \prod_{\lambda \in \Lambda} T(\lambda) \times T(\lambda) \to A; \quad \text{and} \quad \deg : \prod_{\lambda \in \Lambda} T(\lambda) \to \mathbb{Z}$$

such that $C$ is injective. We denote $C(S, T) = c_{ST}^\lambda$ for $S, T \in T(\lambda)$, and

1. Each element $c_{ST}^\lambda$ is homogeneous of degree

$$\deg(c_{ST}^\lambda) = \deg(S) + \deg(T),$$

for $\lambda \in \Lambda$ and $S, T \in T(\lambda)$.

2. The set $\{c_{ST}^\lambda \mid S, T \in T(\lambda), \lambda \in \Lambda\}$ is a $\mathbb{C}$-basis of $A$.

3. If $S, T \in T(\lambda)$, for some $\lambda \in \Lambda$, and $a \in A$ then there exist scalars $r_{SU}(a)$, which do not depend on $T$, such that

$$ac_{ST}^\lambda = \sum_{U \in T(\lambda)} r_{SU}(a)c_{UT}^\lambda \pmod{A^a\lambda},$$

where $A^a\lambda$ is the $\mathbb{C}$-submodule of $A$ spanned by

$$\{c_{QR}^\mu \mid \mu \geq \lambda \text{ and } Q, R \in T(\mu)\}.$$

4. The $\mathbb{C}$-linear map $*: A \to A$ determined by $(c_{ST}^\lambda)^* = c_{TS}^\lambda$, for all $\lambda \in \Lambda$ and all $S, T \in T(\lambda)$, is an anti-isomorphism of $A$.

5. The algebra $A$ has an identity element, $1_A$, such that $1_A = \sum_{\lambda \in \Lambda} 1_{\lambda}$ is an orthogonal idempotent decomposition.

6. For $S \in T(\lambda, \mu)$, $T \in T(\lambda, \nu)$, we have that $1_{\mu} c_{ST}^\lambda 1_{\nu} = c_{ST}^\nu$. There exists a unique element $T^\lambda \in T(\lambda, \lambda)$, and $c_{TT^T}^\lambda = 1_{\lambda}$.

**Remark.** Notice that the above satisfies the conditions of a graded cellular algebra [HM10]. In addition, such an algebra is quasi-hereditary (as every cell-ideal contains an idempotent). Conditions (5) and (6) allow us to examine standard modules by considering their weight space decompositions.

Unless otherwise stated, all results in this section follow from [HM10]. Suppose that $A$ is a graded cellular algebra with a highest weight theory. Given any $\lambda \in \Lambda$, the graded standard module $\Delta(\lambda)$ is the graded left $A$-module

$$\Delta(\lambda) = \bigoplus_{\mu \in \Lambda} \Delta_\mu(\lambda)_z,$$

where $\Delta_\mu(\lambda)_z$ is the $\mathbb{C}$ vector-space with basis $\{c_{ST}^\lambda \mid S \in T(\lambda, \mu) \text{ and } \deg(S) = z\}$. The action of $A$ on $\Delta(\lambda)$ is given by

$$ac_{ST}^\lambda = \sum_{U \in T(\lambda)} r_{SU}(a)c_{UT}^\lambda,$$

where the scalars $r_{SU}(a)$ are the scalars appearing in condition (3) of Definition 1.1.
Suppose that \( \lambda \in \Lambda \). There is a bilinear form \( \langle \cdot, \cdot \rangle_{\lambda} \) on \( \Delta(\lambda) \) which is determined by
\[
\begin{align*}
c^\lambda_{ST}c^\lambda_{TV} & \equiv \langle c^\lambda_S, c^\lambda_T \rangle \lambda c^\lambda_{UV} \pmod{A^{\lambda}},
\end{align*}
\]
for any \( S, T, U, V \in T(\lambda) \).

Let \( t \) be an indeterminate over \( \mathbb{N}_0 \). If \( M = \bigoplus_{z \in \mathbb{Z}} M_z \) is a free graded \( \mathbb{C} \)-module, then its \textit{graded dimension} is the Laurent polynomial
\[
\dim_t(M) = \sum_{k \in \mathbb{Z}} (\dim_{\mathbb{C}} M_k) t^k.
\]

If \( M \) is a graded \( A \)-module and \( k \in \mathbb{Z} \), define \( M(k) \) to be the same module with \( (M(k))_i = M_{i-k} \) for all \( i \in \mathbb{Z} \). We call this a \textit{degree shift} by \( k \). If \( M \) is a graded \( A \)-module and \( L \) is a graded simple module let \( [M : L(k)] \) be the multiplicity of \( L(k) \) as a graded composition factor of \( M \), for \( k \in \mathbb{Z} \).

Suppose that \( A \) is a graded cellular algebra with a highest weight theory. For every \( \lambda \in \Lambda \), define \( L(\lambda) \) to be the quotient of the corresponding standard module \( \Delta(\lambda) \) by the radical of the bilinear form \( \langle \cdot, \cdot \rangle_{\lambda} \). This module is simple, and every simple module is of the form \( L(\lambda)(k) \) for some \( k \in \mathbb{Z} \), \( \lambda \in \Lambda \). We let \( L_\mu(\lambda) \) denote the \( \mu \)-weight space \( 1_\mu L(\lambda) \). The \textit{graded decomposition matrix} of \( A \) is the matrix \( D_A(t) = (d_{\lambda\mu}(t)) \), where
\[
d_{\lambda\mu}(t) = \sum_{k \in \mathbb{Z}} [\Delta(\lambda) : L(\mu)(k)] t^k,
\]
for \( \lambda, \mu \in \Lambda \). The following proposition is a key ingredient in our proof of the main result of this paper.

\textbf{Proposition 1.2} ([HM10], Proposition 2.18). If \( \mu \in \Lambda \) then \( \dim_t(L(\mu)) \in \mathbb{N}_0[t + t^{-1}] \).

Given \( \lambda, \mu \in \Lambda \) such that \( \lambda \triangleright \mu \), we say that \( \lambda \) and \( \mu \) are \textit{tableau-linked} if the set \( T(\lambda, \mu) \) is non-empty. The equivalence classes of the equivalence relation on \( \Lambda \) generated by this tableau-linkage are called the \textit{tableau-blocks} of \( A \).

\textbf{Proposition 1.3.} [The Linkage Principle] If \( \lambda, \mu \in \Lambda \) label simple modules in the same block of \( A \), then \( \lambda \) and \( \mu \) are tableau-linked.

\textit{Proof.} It is clear that a necessary condition for \( \dim_t(\text{Hom}(P(\lambda), \Delta(\mu))) = [\Delta(\lambda) : L(\mu)] \neq 0 \), is that \( T(\lambda, \mu) \neq \emptyset \). The result then follows from [GL96, (3.9.8)]. \(\square\)

This result inspires the next section, in which we connect tableaux to paths in an alcove geometry.

\textbf{1.2. The alcove geometry.} We shall assume standard facts concerning root systems, see [Bou02]. Let \( \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r\} \) be a set of formal symbols and set
\[
E_r = \bigoplus_{i=1}^r \mathbb{R} \varepsilon_i
\]
to be the \( r \)-dimensional real vector space with basis \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r \). We have an inner product \( \langle \cdot, \cdot \rangle \) given by extending linearly the relations
\[
\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{i,j}
\]
for all \( 1 \leq i, j \leq r \), where \( \delta_{i,j} \) is the Kronecker delta.

Let \( A(\rho, e) \) denote a cellular algebra with a highest weight theory depending on parameters \( \rho \in E_r \) and \( e \in \mathbb{N} \cup \{\infty\} \) and let \( \Lambda \) denote the indexing set of the simple modules. Suppose that there exists an embedding \( \Lambda \hookrightarrow E_r^+ \); we identify an element \( \lambda \in \Lambda \) with its image under this map.

Let \( \Phi \) denote a root system embedded in \( E_r \) as in [Bou02, Plates I to IX] and let \( h \) denote the corresponding Coxeter number. We take \( R^+ \) to be the set of positive roots. For each \( \alpha \in \Phi \)}
there is a unique coroot $\alpha^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$. For $e \in \mathbb{N} \cup \{\infty\}$ we let $W^e$ denote the affine reflection group generated by the reflections $s_{\alpha, me}$ (for $\alpha \in \Phi$, $m \in \mathbb{Z}$) given by

$$s_{\alpha, me}(x) = x - ((x, \alpha^\vee) - me)\alpha$$

for all $x \in E_r$. For $e = \infty$, define $W^\infty$ to be the subgroup generated by the reflections $s_{\alpha, 0}$ for $\alpha \in \Phi$.

Now, given $\rho \in E_r$, we shall always consider the shifted action of $W^e$ by $\rho$ given by

$$w \cdot x = w(x + \rho) - \rho$$

for all $w \in W^e$ and $x \in E_r$. We regard $s_{\alpha, me}$ as a reflection with respect to the hyperplane

$$h_{\alpha, me} = \{ \lambda \in E_r \mid \langle \lambda + \rho, \alpha^\vee \rangle = me \}.$$

The reflection group $W^e$ acting on $E_r$ defines a system of facets. A facet is a non-empty subset of $E_r$ of the form

$$\mathfrak{f} = \{ \lambda \in E_r \mid \langle \lambda + \rho, \alpha^\vee \rangle = ma e \text{ for all } \alpha \in R^0_+(f) \},$$

$$\langle ma - 1, e \rangle < \langle \lambda + \rho, \alpha^\vee \rangle < ma e \text{ for all } \alpha \in R^1_+(f) \},$$

for suitable integers $ma \in \mathbb{Z}$ and a disjoint decomposition $R^+ = R^0_+(f) \cup R^1_+(f)$. A facet, $\mathfrak{f}$, is called an alcove if $|R^0_+(f)| = 0$ and a wall if $|R^1_+(f)| = 1$. A point $x \in \mathfrak{f}$ is called $e$-regular if $|R^0_+(f)| = 0$, and is called $e$-singular if $|R^1_+(f)| \geq 1$. We assume that $\rho_i \neq 0$ modulo $e$ for any $1 \leq i \leq r$, so that the origin is always contained in an alcove, which we refer to as the fundamental alcove. The closure, $\overline{\mathfrak{f}}$, of a facet, $\mathfrak{f}$, is defined as follows

$$\overline{\mathfrak{f}} = \{ \lambda \in E_r \mid (\lambda + \rho, \alpha^\vee) = ma e \text{ for all } \alpha \in R^0_+(f) \},$$

$$\langle ma - 1, e \rangle \leq \langle \lambda + \rho, \alpha^\vee \rangle \leq ma e \text{ for all } \alpha \in R^1_+(f) \}. $$

We define a length function on the set of alcoves as follows. We say that two alcoves, $a_i, a_j$ are adjacent if $\overline{a_i} \cap a_j$ is non-empty. Given any pair of alcoves $a$ and $b$, there exists a chain of adjacent alcoves,

$$a = a_0, a_1, \ldots, a_\ell = b,$$

and we define the length $\ell(a, b)$ to be the minimal number of alcoves in such a chain. We extend this notation to points in alcoves in the obvious manner.

1.3. Paths in an alcove geometry. In this section we fix $e > h$ and consider paths in our alcove geometry. Given $k \in \mathbb{N}$, we let $k$ denote the set $\{1, 2, \ldots, k\}$. Given a map $w : n \to r$ we define points $\omega(k) \in E_r$ by

$$\omega(k) = \sum_{1 \leq i \leq k} \varepsilon_{w(i)},$$

for $1 \leq k \leq n$. We define the associated path of length $n$ in our alcove geometry $E_r$ by $\omega = (\omega(0), \omega(1), \omega(2), \ldots, \omega(n))$, where we fix all paths to begin at the origin, so that $\omega(0) = \emptyset \in E_r$. We let $\omega_{\leq k}$ denote the path of $\omega$ of length $k$ corresponding to $w_{\leq k} : k \to r$.

Definition 1.4. Fix a path $\omega = (\omega(0), \omega(1), \omega(2), \ldots, \omega(n))$ such that $\omega(0) = \emptyset \in E_r$. We define a degree function on $\omega$ by induction. We set $\deg(\omega(0)) = 0$ and set

$$\deg(\omega_{\leq k}) = \deg(\omega_{\leq k - 1}) + \sum_\alpha d_\alpha(\omega, k)$$

where $d_\alpha(\omega, k)$ is defined as follows. Fix $\alpha \in \Phi$, and consider the hyperplanes $h_{\alpha, me}$ for $m \in \mathbb{Z}$. If $\omega(k)$ and $\omega(k + 1)$ both lie on some $h_{\alpha, me}$ or if neither lie on some $h_{\alpha, me}$ for $m \in \mathbb{Z}$, then $d_\alpha(\omega, k) = 0$. Otherwise, exactly one of $\omega(k)$ and $\omega(k - 1)$ lies on some hyperplane $h_{\alpha, me}$.

Removing the hyperplane $h_{\alpha, me}$ leaves two distinct subsets $E^+_r(\alpha, me)$ and $E^-_r(\alpha, me)$ where $\emptyset \in E^-_r(\alpha, me)$. If $\omega(k - 1) \in E^-_r(\alpha, me)$, or $\omega(k) \in E^+_r(\alpha, me)$, then set $d_\alpha(\omega, k) = 0$. If $\omega(k - 1) \in E^+_r(\alpha, me)$, then $d_\alpha(\omega, k) = -1$. If $\omega(k) \in E^-_r(\alpha, me)$, then $d_\alpha(\omega, k) = +1$. 
Figure 6 illustrates the four subcases outlined above. In each case the diagram depicts a hyperplane, labelled by $h_{\alpha,me}$, with the corresponding subsets $E^+_r(\alpha, me)$ and $E^-_r(\alpha, me)$ labelled. The incoming/outgoing arrows labels steps onto and off of the hyperplane and the corresponding $d_{\alpha}(\omega, k)$.

\begin{figure}[h]
\centering
\begin{tabular}{cccc}
- & $h_{\alpha,me}$ & + & - \\
+0 & & & +0 \\
+1 & & & +1 \\
\end{tabular}
\caption{The four subcases for the values of $d_{\alpha}(\omega, k)$ as $\omega$ crosses a wall. The ± indicate the distinct subsets $E^+_r$ and $E^-_r$ of $E_r$. In each case the first (respectively second) step has its degree recorded as a superscript (respectively subscript).}
\end{figure}

Let $\omega$ be a path which passes through a hyperplane $h_{\alpha,me}$ at point $\omega(k)$ (note that $k$ is not necessarily unique). Then, let $\omega'$ be the path obtained from $\omega$ by applying the reflection $s_{\alpha,me}$ to all the steps in $\omega$ after the point $\omega(k)$. In other words, $\omega'(i) = \omega(i)$ for all $1 \leq i \leq k$ and $\omega'(i') = s_{\alpha,me} \cdot \omega(i)$ for $k \leq i \leq n$. We refer to the path $\omega'$ as the reflection of $\omega$ in $h_{\alpha,me}$ at point $\omega(k)$ and denote this by $s_{\alpha,me} \cdot \omega$. We write $\omega \sim \omega'$ if the path $\omega$ can be obtained from $\omega'$ by a series of reflections in $W^e$.

Let $\lambda, \mu \in E_r$. We fix a distinguished path $\omega^\mu$ from the origin to $\mu$ such that $\deg(\omega(k)) = 0$ for all $1 \leq i \leq n$. (It is easy to see that such a path always exists.) We let $\Path(\lambda, \mu)$ denote the set of all paths from the origin to $\lambda$ which may be obtained from $\omega^\mu$ by a series of reflections.

**Example 1.5.** Recall the example from the introduction. Here the geometry is of type $\hat{A}_2$, $n = 13$, $e = 8$ and $\rho = (8, 4, 2)$.

The distinguished path $\omega^\gamma$ is recorded in Figure 2. We clearly have that $d_{\alpha}(\omega, k) = 0$ at all points $1 \leq k \leq n$. Figure 3 contains the two elements of $\Path((4, 6, 3), \gamma)$. Let $\omega$ (respectively $\omega'$) denote the path in the leftmost (respectively rightmost) case. We have that

\[ d_{e_2-e_3}(\omega, 11) = 1, \quad d_{e_1-e_3}(\omega, 12) = -1, \quad d_{e_1-e_3}(\omega, 13) = 1 \]

are the only non-zero values of $d_{\alpha}(\omega, k)$ for $1 \leq k \leq n$, and therefore $\deg(\omega) = 1$. We have that

\[ d_{e_2-e_3}(\omega', 5) = 1, \quad d_{e_1-e_3}(\omega', 12) = 1, \quad d_{e_1-e_3}(\omega', 13) = 1 \]

are the only non-zero values of $d_{\alpha}(\omega', k)$ for $1 \leq k \leq n$, and therefore $\deg(\omega') = 3$.

1.4. **Soergel’s algorithm for paths.** Fix $e > h$, we now recall the classical construction of Soergel’s algorithm with respect to a walk in the geometry. The procedure outlined below is somewhat simpler, as all points in our geometry belong to the dominant chamber [Soe97, Section 4].

**Definition 1.6.** Let $e > h$, and assume $\mu$ is $e$-regular. We say that a path $\omega$ from $\circ$ to $\mu$ of length $n$ is **admissible** if (i) $\deg(\omega(k)) = 0$ for all $1 \leq k \leq n$, and (ii) whenever $\omega(k)$ lies on two hyperplanes $h_{\alpha,me}$ and $h_{\beta,me}$ for some $1 \leq k \leq n$ this implies that $\langle \alpha, \beta^\vee \rangle = 0$ (we say that the hyperplanes are orthogonal).

**Remark.** For $\mu$ an $e$-regular point, and $\omega$ an admissible path from $\circ$ to $\mu$, there exist $2^\ell(\mu)$ paths $\omega'$ such that $\omega' \sim \omega$.

We say that a path, $\omega$, is an **alcove-wall path** if (i) $\deg(\omega(k)) = 0$ for all $1 \leq k \leq n$ and (ii) every step lies either on a wall or in an alcove. It is clear that any alcove-wall path is admissible.

**Definition 1.7.** For a distinguished admissible path $\omega^\mu$, we define

\[ m_\mu(\lambda) = \sum_{\omega \in \Path(\lambda, \mu)} t^{\deg(\omega)}. \]
Given $\omega$ an admissible path of length $n$, we let $f_k$ denote the facet containing the point $\omega(k)$ for $1 \leq k \leq n$.

**Definition 1.8.** We fix an admissible path $\omega$ from $\odot$ to $\mu$ of length $n$. For $1 \leq k \leq n$, we let

$$A_+(\omega, k) = \{(\gamma, m_k \epsilon) \mid \omega(k) \in h_{\gamma,m_k \epsilon}\} \setminus \{(\gamma, m_{k-1}\epsilon) \mid \omega(k-1) \in h_{\gamma,m_{k-1}\epsilon}\},$$

$$A_-(\omega, k) = \{(\gamma, m_{k-1}\epsilon) \mid \omega(k-1) \in h_{\gamma,m_{k-1}\epsilon}\} \setminus \{(\gamma, m_k \epsilon) \mid \omega(k) \in h_{\gamma,m_k \epsilon}\}.$$

The orthogonality condition on the admissible path ensures that for $1 \leq k \leq n$, the set $A_+(\omega, k)$ (respectively $A_-(\omega, k)$) either consists of one element, denoted $\alpha_+(\omega, k)$ (respectively $\alpha_-(\omega, k)$) or is empty.

**Remark.** The $\alpha_+(\omega, k)$ (respectively $\alpha_-(\omega, k)$) record the steps in $\omega$ which are on to (respectively off of) hyperplanes in the geometry.

**Definition 1.9.** We fix an admissible path $\omega$ from $\odot$ to $\mu$ of length $n$. For $1 \leq k \leq n$, we set $A_k$ to be the alcove, minimal in the length ordering, such that $\langle \lambda + \rho, \alpha_+(\omega, i) \rangle > 0$ for all $\lambda \in A_+(\omega, k)$ and all $0 \leq i \leq k$ such that $A_+(\omega, k) \neq \emptyset$. We define the alcove-series of $\omega$ to be the ordered set whose elements are given by the alcoves $A_k$ for $0 \leq k \leq n$ recorded without repeats and in increasing order.

**Example 1.10.** Consider a geometry of type $\tilde{A}_2$ with $\rho = (8, 4, 2)$ and $n = 13$. The path $\omega^7$ in Figure 2 is an alcove-wall path. We let $\gamma^7$ denote the alcove-wall path

$$(\varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2, \varepsilon_2, \varepsilon_2, \varepsilon_2, \varepsilon_2, \varepsilon_1).$$

Both paths pass through (the same) alcoves of length $0, 1, 2, 3$, which we denote by $a(i)$ for $i = 0, 1, 2, 3$. We have that

$$A_k^{\omega^7} = \begin{cases} \{a(0)\} & \text{for } k = 0, 1, 2, \\
\{a(1)\} & \text{for } k = 3, 4, 5, 6, 7, 8, 9, \\
\{a(2)\} & \text{for } k = 10, 11, \\
\{a(3)\} & \text{for } k = 12, 13; \end{cases}$$

$$A_k^{\gamma^7} = \begin{cases} \{a(0)\} & \text{for } k = 0, 1, 2, \\
\{a(1)\} & \text{for } k = 3, 4, 5, 6, 7, 8, \\
\{a(2)\} & \text{for } k = 9, \\
\{a(3)\} & \text{for } k = 10, 11, 12, 13; \end{cases}$$

and so the alcove series in both cases is given by $\{a(0), a(1), a(2), a(3)\}$.

We let $\mathfrak{A}$ denote the set of all alcoves in $E^\circ$. We let $\mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ denote alcoves in our geometry and let $a_0, \ldots, a_{\ell(\mu)}$ denote the alcove series of an admissible path from $\odot$ to $\mu$. We define maps

$$n_{a_{i}} : \mathfrak{A} \to \mathbb{N}_0[t] \quad m_{a_{i}} : \mathfrak{A} \to \mathbb{N}_0[t] \quad e_{a_{i}} : \mathfrak{A} \to \mathbb{N}_0[t + t^{-1}],$$

where $t$ is a formal parameter, as follows. We set

$$n_{a_i}(a_i) = 1, \quad m_{a_i}(a_i) = 1, \quad e_{a_i}(a_i) = 1.$$

We define

$$n_{a_i}(\mathfrak{b}) = 0, \quad m_{a_i}(\mathfrak{b}) = 0, \quad e_{a_i}(\mathfrak{b}) = 0$$

whenever $\ell(\mathfrak{b}) \not\equiv \ell(\mathfrak{a}_i)$. For each adjacent pair of alcoves $a_i$ and $a_{i+1}$, we let $s_i$ denote the reflection in the hyperplane passing through $\overline{a_i} \cap \overline{a_{i+1}}$. The closure, $\overline{\mathfrak{b}}$, of any alcove $\mathfrak{b}$ has one wall which is in the $W^e$-orbit of $s_i$, and we shall write $s_i \cdot \mathfrak{b}$ for the image of $\mathfrak{b}$ in that wall. Then, with $m_{a_i}$ known, we set

$$m_{a_{i+1}}(s_i \cdot \mathfrak{b}) = \begin{cases} m_{a_i}(\mathfrak{b}) + t^{-1}m_{a_i}(s_i \cdot \mathfrak{b}), & \ell(s_i \cdot \mathfrak{b}) > \ell(\mathfrak{b}), \\
m_{a_i}(\mathfrak{b}) + tm_{a_i}(s_i \cdot \mathfrak{b}), & \ell(s_i \cdot \mathfrak{b}) < \ell(\mathfrak{b}). \end{cases}$$

We refer to this procedure as the cancellation-free Soergel algorithm.

**Proposition 1.11.** Given $e > h$, suppose that $\mu$ and $\lambda$ belong to alcoves $\mathfrak{a}$ and $\mathfrak{b}$ respectively, and furthermore that $\mu \in W^e \cdot \lambda$. We let $\mathfrak{a}_0$ denote the fundamental alcove and $\mathfrak{a}_0, \ldots, \mathfrak{a}_{\ell(\mu)} = \mathfrak{a}$ denote the alcove series of an admissible path $\omega^\mu$. We have that $m_\mu(\lambda) = m_\mathfrak{a}(\mathfrak{b})$. 
Proof. For $1 \leq i \leq \ell(\mu)$, note that the $i$th hyperplane $\overline{a}_i \cap \overline{a}_{i+1}$ is the hyperplane given by the $i$th non-trivial $\alpha_{+,m,k}$ This gives the required bijection between paths (obtained from $\omega^\mu$ by a series of reflections through the $h_{\alpha_{+,m,k}}$ for $1 \leq k \leq n$) and terms in Soergel’s cancellation-free algorithm (given by a sequence of alcoves, which are determined by the alcove walls $\overline{a}_i \cap \overline{a}_{i+1}$ through which we reflect).

The $\alpha_{+,m,k}$ and $\alpha_{-,m,k'}$ for $1 \leq k < k' \leq n$ come in pairs (whenever we step on to a hyperplane, we must step off of it at some later point). For a pair $1 \leq k < k' \leq n$, the hyperplanes $h_{\alpha_{+,m,k}}$ and $h_{\alpha_{-,m,k'}}$ for $k < k'' < k'$ are orthogonal to $h_{\alpha_{+,m,k}}$.

Fix two points $\lambda, \lambda + \epsilon_i \in E_r$. Assume that $\lambda + \epsilon_i$ belongs to $E^+_r(\alpha, me)$ or $E^-_r(\alpha, me)$. Let $h_{\beta,me}$ denote a hyperplane orthogonal to $h_{\alpha,me}$ and $s_{\beta,me}$ denote the reflection through this hyperplane. It is clear that $s_{\beta,me} \cdot (\lambda + \epsilon_i)$ still belongs to either $E^+_r(\alpha, me)$ or $E^-_r(\alpha, me)$, respectively. (Compare this with the definition of the degree of a path, Definition 1.4.)

Note that in general, this would not be true for non-orthogonal hyperplanes.

Therefore the contribution $d_{\alpha_{+,m,k}}(\omega, k')$ to the degree given by the step at point $k' - 1$, is the same as if it were taken at point $k + 1$. Thus we can assume that $k' = k + 1$, in other words that our path is an alcove wall path. Folding up an alcove wall path, $\omega^\mu$, so that it terminates at $\lambda$ corresponds to tracing one of the terms in the Soergel cancellation-free algorithm, as follows:

(i) When the path steps from alcove $b$ onto the wall $\overline{a}_i \cap s_i \cdot \overline{b}$ and through to the alcove $s_i \cdot b$, the degree of the path does not change on alcoves (as $-1 + 1 = 0$), as illustrated in the second and third diagrams in Figure 6. This is equivalent to the first term in each of the two cases of equation 1.1.

(ii) When the path steps from alcove $s_i \cdot b$ onto the wall $\overline{a}_i \cap s_i \cdot \overline{b}$ and then returns to the alcove $s_i \cdot b$, the degree either increases or decreases by one, as seen in the first and fourth diagrams in Figure 6, respectively. This is equivalent to the second term in the two cases of equation 1.1.

For ease in the above, we have tacitly assumed that we never simultaneously step off of a hyperplane and on to another hyperplane in the same step (as in $\omega^\beta$ in Example 1.10). In general, this is not the case (as in $\omega^\gamma$ in Example 1.10). Our ignoring of this is justified as the Soergel-degree is given by summing over the Soergel-degrees of the steps from passing through these separate facets (note that in Definition 1.4, the contributions of the $d_\alpha$ for $\alpha \in R^+$ to the sum are independent).

\[ \square \]

Remark. Motivated by the above Proposition, we will omit the bar for $\overline{a}_{\mu}(\lambda)$ for $\mu, \lambda \in E_r$.

Similarly, with $n_{\alpha_i}$ known by induction, we set

\[ n_{\alpha_{i+1}}'(s_i \cdot b) = \begin{cases} n_{\alpha_i}(b) + t^{-1}n_{\alpha_i}(s_i \cdot b), & \ell(s_i \cdot b) > \ell(b); \\ n_{\alpha_i}(b) + tn_{\alpha_i}(s_i \cdot b), & \ell(s_i \cdot b) < \ell(b); \end{cases} \]

and

\[ n_{\alpha_{i+1}}(b) = n_{\alpha_{i+1}}'(b) - \sum_{\{\ell(b) = \ell(s_i \cdot b)\}} (n_{\alpha_{i+1}}'(\overline{b})) \circ \epsilon_0(b). \]

We refer to this procedure as the Soergel algorithm. Importantly for us, it is shown in [Soe97] that this procedure is independent of the path taken. Finally, with $e_{\alpha_i}$ known by induction, we set

\[ e_{\alpha_{i+1}}(s_i \cdot c) = (t + t^{-1})e_{\alpha_i}(s_i \cdot c) + e_{\alpha_i}(c) + (n_{\alpha_{i+1}}'(s_i \cdot c)) \circ \epsilon_0(b) \]

if $\ell(s_i \cdot c) > \ell(c)$, and $e_{\alpha_{i+1}}(s_i \cdot c) = 0$ otherwise. We refer to this procedure as the character algorithm. We extend the $e$ and $n$ functions to $e$-regular points in a given linkage class in the obvious fashion.

Example 1.12. Let $e = 4$ and $\rho = (4, 2)$ and consider the root system of type $\hat{A}_1$. The space $E_1$ can be pictured as $\mathbb{Z}$; however, in order to make the steps $+\epsilon_1$ and $+\epsilon_2$ in a walk of length $n$ clear, we draw a graph with $n$ levels, the ith level featuring the points $\{-i, -(i-1), \ldots, (i-1), i\}$
and let $\omega$ start at level 0 at point $\bullet$ and proceed downwards to level $n$, this is made clear in Figure 7. As pointed out in [PRH14, Pla13], these can be regarded as walks on Pascal’s triangle.

Let $\mu = (0,11)$ and $\lambda = (4,7)$. There are two elements $\omega, \omega' \in \text{Path}_{11}(\lambda, \mu)$, depicted in Figure 7. The former is of degree 2 and the latter of degree 0. In the former case, $d_{e_{1}-e_{2}}(\omega, 7) = 1$. In the latter case, $d_{e_{1}-e_{2}}(\omega', 7) = 1$.

![Figure 7. Two paths $\omega, \omega' \in \text{Path}((4,7),(0,11))$](image)

Let $\nu = (5,6)$, there are two elements of $\omega'', \omega''' \in \text{Path}_{11}(\nu, \mu)$, depicted in Figure 8, of degree 3 and degree 1 respectively. In the former case, $d_{e_{1}-e_{2}}(\omega'', 7) = 1$ and $d_{e_{1}-e_{2}}(\omega'', 11) = 1$. In the latter case, $d_{e_{1}-e_{2}}(\omega''', 7) = 1$.

![Figure 8. Two path $\omega'', \omega''' \in \text{Path}((6,5),(0,11))$](image)

| alcove | $a_3'$ | $a_2'$ | $a_1'$ | $a_0$ | $a_{1R}$ | $a_{2R}$ |
|--------|--------|--------|--------|------|---------|---------|
|        | 1      |        |        | 1    | $t$     |         |
|        |        | 1      |        | $t$  | $t^2$   |         |
|        |        | 1      |        | $t^2$+1 | $t^3+t$ | $t^2$   |

![Figure 9. This table records the result of running the (cancellation-free) Soergel algorithm along the path $\omega''$. The alcoves are labelled by their length and primed (respectively unprimed) if they correspond to an alcove to the left (respectively right) of the origin in the diagrams in Figures 7 and 8](image)

Figure 9 records the result of running the (cancellation-free) Soergel algorithm along the path $\omega''$. Notice that the algorithm produces $m_{\mu}(\lambda) = n'_{\mu}(\lambda) = t^2 + 1$ and $n_{\mu}(\lambda) = t^2$; similarly $m_{\mu}(\nu) = n'_{\mu}(\nu) = t^3 + t$ and $n_{\mu}(\nu) = t^3$. We have that $e_{\mu}(\lambda) = 1$ and $e_{\mu}(\nu) = 0$.

**Proposition 1.13.** Let $\lambda, \mu$ denote points belonging to alcoves in $E_r$. Fix an admissible path $\omega^\mu$. Let $\nu$ vary over all points such that $\text{Path}(\nu, \mu) \neq \emptyset$ and $\text{Path}(\lambda, \nu) \neq \emptyset$. We have that,

$$m_{\mu}(\lambda) = \sum_{\text{Path}(\nu, \mu) \neq \emptyset} n_{\nu}(\lambda) e_{\mu}(\nu).$$
Proof. Let $\lambda, \mu, \nu$ denote points in alcoves $a$, $b$, and $c$, respectively. We have fixed a distinguished walk, $\omega^\mu$, and so both $e_b$ and $m_b$ are well-defined. By [Soe97], $n_c$ is independent of the choice of path from $\odot$ to $\nu$. Therefore the expression above is well-defined.

A subpattern in Soergel’s algorithm is removed if $(n'_a(\delta)|_{t=0}) \neq 0$ for some alcove $\delta$. The subpatterns removed in the $n$-algorithm are (of course) not removed by the $m$-algorithm; the $e$-algorithm will keep track of the leading terms in these subpatterns. The leading term of the subpattern will remain constant unless it is reflected through a hyperplane through the lower closure of the alcove, in which case we multiply the subpattern by $(t + t^{-1})$. This is particularly clear from the alcove-wall path definition of Soergel’s algorithm (see also the singular combinatorics for Soergel’s algorithm developed in [RH06]). The result then follows from the definitions. □

Example 1.14. Let $l = 3$, $n = 13$, $e = 8$, $\rho = (8, 4, 2)$ and consider the root system of type $\hat{A}_2$. Take $\alpha = (4, 6, 3)$, $\beta = (5, 6, 2)$ and $\gamma = (5, 8, 0)$. Let $\omega^\gamma$ be the alcove-wall path depicted in Figure 2 in the introduction. The set of elements in $\text{Path}(-\gamma, \gamma)$, together with their degrees, is depicted across Figures 3, 4, and 5. Figures 10 and 11 depict the four steps of running Soergel’s algorithm along $\omega^\gamma$.

![Figure 10](image10.png)

**Figure 10.** The first four steps of running Soergel’s cancellation-free algorithm along $\omega^\gamma$. We have recorded the powers of the polynomials only (for example, $2 + 0$ should be read as $t^2 + t^0$).

Under the Soergel procedure, we remove the subpattern labelled by the zero in the alcove containing the point $\beta$. The ‘new zero’ is recorded by the character algorithm. We have that

$$m_\gamma(\lambda) = e_\gamma(\gamma)n_\gamma(\lambda) + e_\gamma(\beta)n_\beta(\lambda)$$

for any $\lambda \in E_r$. Here $e_\gamma(\beta) = t^0$ and $e_\gamma(\gamma) = t^0$ and $e_\gamma(\lambda) = 0$ otherwise. This rewriting process is depicted in Figure 11.

![Figure 11](image11.png)

**Figure 11.** Rewriting the $m_\gamma(\lambda)$ in terms of $n_\rho(\lambda)$ and $e_\gamma(\mu)$.

Example 1.15. Let $l = 3$, $n = 21$, $e = 6$, $\rho = (6, 4, 2)$ and consider the root system of type $\hat{A}_2$. We leave it as an exercise for the reader to show that

$$m_{(4,17,0)}(\lambda) = n_{(4,17,0)}(\lambda) + n_{(15,4,2)}(\lambda) + (t + t^{-1})n_{(6,9,0)}(\lambda)$$

for any $\lambda \in E_r$. This is the smallest example where we find a path in negative degree. In this case,

$$e_{(4,17,0)}(6,9,0) = (t + t^{-1}), \quad e_{(4,17,0)}(15,4,2) = t^0, \quad e_{(4,17,0)}(4,17,0) = t^0.$$
1.5. Algebras with Soergel path bases. We shall now define a family of algebras whose representation theory is governed by paths in Euclidean space and show that the decomposition numbers of such an algebra are given by Soergel’s algorithm. Our proof is based on Kleshchev and Nash’s algorithm for computing decomposition numbers (see [KN10]).

Definition 1.16. Let \( A(\rho, e) \) denote a graded cellular algebra with a theory of highest weights with respect to the poset \( \Lambda \). Let \( \Lambda \rightarrow E_r \), where \( E_r \) is equipped with the action of a Weyl group associated to a root system \( \Phi \). We say that the algebra \( A(\rho, e) \) has a Soergel-path basis with respect to \( \Phi \) if there exists a degree preserving bijective map

\[
\omega : T(\lambda, \mu) \rightarrow \text{Path}(\lambda, \mu)
\]

such that \( \omega(T^\mu) = \omega^\mu \) is admissible.

Proposition 1.17. Let \( A(\rho, e) \) denote an algebra with a Soergel-path basis and suppose that \( d_{\lambda \mu}(t) \in t\mathbb{N}_0(t) \) for all \( \lambda \neq \mu \in \Lambda \). Then the following hold:

\[
\begin{align*}
(i) & \text{ we have } \text{Dim}_t(\Delta_\mu(\lambda)) = m_\mu(\lambda) \in \mathbb{N}_0[t, t^{-1}] \text{ and } \text{Dim}_t(L_\mu(\lambda)) \in \mathbb{N}_0[t + t^{-1}]; \\
(ii) & \text{ if } \text{Dim}_t(\Delta_\mu(\lambda)) = 0, \text{ then } d_{\lambda \mu}(t) = 0; \\
(iii) & \text{ we have } \text{Dim}_t(\Delta_\mu(\mu)) = \text{Dim}_t(L_\mu(\mu)) = 1; \\
(iv) & \text{ if } \text{Path}(\lambda, \mu) = \emptyset, \text{ then } \text{Dim}_t(\Delta_\mu(\lambda)) = 0; \\
v & \text{ if } \text{Path}(\lambda, \mu) = \emptyset, \text{ then } \text{Dim}_t(L_\mu(\lambda)) = 0; \\
v & \text{ we have that } \text{Dim}_t(\Delta_\mu(\lambda)) = \sum_{\nu \neq \mu, \text{Path}(\nu, \mu) \neq \emptyset} d_{\lambda \nu}(t) \text{Dim}_t(L_\mu(\nu)) + d_{\lambda \mu}(t).
\end{align*}
\]

Proof. Part (i) is clear by Proposition 1.12. (iii) is a restatement of the condition that \( \omega^\mu \) is the only path in \( \text{Path}(\mu, \mu) \). A necessary condition for \( \text{Dim}_t(\text{Hom}(P(\mu), \Delta(\lambda))) \neq 0 \) is that \( \Delta_\mu(\lambda) \neq 0 \), therefore (ii) follows.

Part (iv) is by definition, and part (v) follows from the cellular structure. Finally, (vi) follows from (i), (iii), (v) and our assumption that \( d_{\lambda \mu}(t) \in t\mathbb{N}_0(t) \) for \( \lambda \neq \mu \).

Theorem 1.18. Let \( A(\rho, e) \) denote an algebra with a Soergel-path basis of type \( \Phi \). Suppose that \( d_{\lambda \mu}(t) \in t\mathbb{N}_0(t) \) for all \( \lambda, \mu \in \Lambda \) such that \( \lambda \neq \mu \). The graded decomposition numbers of an \( e \)-regular block of \( A(\rho, e) \) are given by the Soergel algorithm

\[
d_{\lambda \mu}(t) = n_\mu(\lambda)
\]

and the characters of the \( e \)-regular simple modules are given by the character algorithm

\[
\text{Dim}_t(L_\mu(\lambda)) = e_\mu(\lambda).
\]

Proof. By Proposition 1.17 (ii), we may assume \( \text{Path}(\lambda, \mu) \neq \emptyset \). We now calculate \( d_{\lambda \mu}(t) \) and \( \text{Dim}_t(L_\mu(\lambda)) \) by induction on the length ordering on alcoves. Induction begins when \( \ell(\mu, \lambda) = 0 \), hence \( \mu = \lambda \), and we have \( d_{\lambda \mu}(t) = 1 \) by Proposition 1.17 (iii) and \( \text{Dim}_t(L_\mu(\mu)) = e_\mu(\mu) = 1 \).

Let \( \ell(\mu, \lambda) \geq 1 \). By induction, we know \( d_{\lambda \mu}(t) \) and \( \text{Dim}_t(L_\mu(\nu)) \) for points \( \nu \in E_r \) such that \( \ell(\mu, \nu), \ell(\lambda, \nu) < \ell(\mu, \lambda) \). By Proposition 1.17 (vi) we have

\[
\text{Dim}_t(L_\mu(\lambda)) + d_{\lambda \mu}(t) = \text{Dim}_t(\Delta_\mu(\lambda)) - \sum_{\nu \neq \lambda, \text{Path}(\nu, \mu) \neq \emptyset} d_{\lambda \nu}(t) \text{Dim}_t(L_\mu(\nu)).
\]

By induction and Proposition 1.17 (i), the right-hand side is equal to

\[
m_\mu(\lambda) - \sum_{\nu \neq \lambda, \text{Path}(\nu, \mu) \neq \emptyset} n_\nu(\lambda) e_\nu(\nu).
\]
We know that this final sum is equal to $e_\mu(\lambda)n_\lambda(\lambda) + n_\mu(\lambda)e_\mu(\mu)$ by Proposition 1.13. Our base case for induction showed that $n_\lambda(\lambda) = 1 = e_\mu(\mu)$, therefore
\[
m_\mu(\lambda) - \sum_{\nu \neq \mu, \nu \neq \lambda} n_\nu(\lambda)e_\mu(\nu) = e_\mu(\lambda) + n_\mu(\lambda).
\]

Recall that $\text{Dim}_t(L_\mu(\lambda)) \in \mathbb{N}_0(t + t^{-1})$ and $d_{\lambda\mu} \in t\mathbb{N}(t)$. Therefore there is a unique solution to the equality (see [KN10, Section 4.1: Basic Algorithm] for a general form, or [Soc97] for the interpretation in terms of Kazhdan–Lusztig theory) given by
\[
\text{Dim}_t(L_\mu(\lambda)) = e_\mu(\lambda), \quad d_{\lambda\mu}(t) = n_\mu(\lambda).
\]

\[\text{Corollary 1.19.}\text{ Let } A(\rho, e) \text{ denote an algebra with a Soergel-path basis and suppose } d_{\lambda\mu}(t) \in t\mathbb{N}_0(t) \text{ for } \lambda \neq \mu. \text{ Let } \lambda, \lambda' \in a \text{ and } \mu, \mu' \in b \text{ for some alcoves } a, b \text{ and suppose that } \mu \in W^e \cdot \lambda \text{ and } \mu' \in W^e \cdot \lambda'. \text{ Then }
\]
\[
d_{\lambda\mu}(t) = d_{\lambda'\mu'}(t).
\]

\[\text{Proof.}\text{ This follows as Soergel’s algorithm is well-defined on alcoves.}\]

2. The little Cherednik algebra

Fix integers $l, n \geq 1$, $g < 0$ and $e \in \{3, 4, \ldots\} \cup \{\infty\}$. We let $\Theta \in \mathbb{R}^l$ be any $l$-tuple such that $\Theta_i - \Theta_j$ is not divisible by $g$ for any $i \neq j$. Given any weighting $\Theta = (\Theta_1, \ldots, \Theta_l)$ and $\kappa = (\kappa_1, \ldots, \kappa_l)$ a multicharge, we will define what we refer to as the little Cherednik algebra, $A(n, \Theta, \kappa)$.

This is an example of one of many finite dimensional algebras (reduced steadied quotients of weighted KLR algebras in Webster’s terminology) constructed in [Web13], whose module categories are equivalent to category $\mathcal{O}$ for the Cherednik algebra. For a given $n$, $\Theta$, and $\kappa$, this is the smallest of the algebras he constructs, hence our terminology.

2.1. Combinatorial preliminaries. Fix a weighting $\Theta = (\Theta_1, \ldots, \Theta_l)$. We define the corresponding Russian array as follows. For each $1 \leq m \leq l$, we place a point on the real line at $\Theta_m$ and consider the region bounded by lines at angles $3\pi/4$ and $\pi/4$. We tile the resulting quadrant with a lattice of squares, each with diagonal of length $2|g|$.

The diagram is tilted ever-so-slightly in the clockwise direction so that the top vertex of the box $(r, c, m)$ is placed at $\Theta_m + |g|(c - r) + (r + c)e$ for $e \ll |g|$.

Fix a multicharge $\kappa$. Let $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(l)})$ be a multipartition. The Young diagram $[\lambda]$ is given by the projection of the box in the $r$th row and $c$th column of $\lambda^{(m)}$ to the box $(r, c, m)$ in the Russian array. We define the loading $1_{\lambda}$ to be the projection of the top vertex of each box $(r, c, m)$ to the real number line, and attaching to each point the residue $\kappa_m + c - r \pmod{e}$ of said box.

We let $D_\lambda$ denote the underlying ordered subset of $\mathbb{R}$ given by the points of the loading. Given $1 \leq k \leq n$, we let $D_\lambda(k)$ denote the $k$th element of $D_\lambda$ in the natural ordering on $\mathbb{R}$, we extend this notation to subsets of $\{1, \ldots, n\}$. The residue sequence of $\lambda$ is given by reading the residues of the boxes in $[\lambda]$ according to the ordering given by $D_\lambda$.

Example 2.1. Let $l = 2$, $g = -l$, and $\Theta = (0, 1)$. The partition $((2,1), (1^3))$ has Young diagram and corresponding loading $1_{\lambda}$ given in Figure 12. The residue sequence of $\lambda$ is given by $(\kappa_1 + 1, \kappa_1, \kappa_2, \kappa_1 - 1, \kappa_2 - 1, \kappa_2 - 2)$.

Example 2.2. Let $n = 3$, $l = 2$, $e = 4$, $g = -2$, $\kappa = (0, 2)$, and $\Theta = (0, 1)$. Consider the block with residue $\{0, 1, 2\}$. This block contains 4 multipartitions, $((\emptyset, (1^3)), (1, (1^3)), (2, (1)), (3, (1)))$. We record the diagrams corresponding to these partitions in Figure 13; in the cases where one of the components is empty, we record where it would be, for perspective.
Figure 12. The diagram and loading of the partition $((2,1), (1^3))$ for $l = 2$, $g = -2$, $\Theta = (0, 1)$.

Figure 13. The loadings of the 2-partitions of 3 with residue $\{0, 1, 2\}$ and $\Theta = (0, 1)$, $g = -2$.

The respective sets $D_\mu$ for the 2-partitions $((\emptyset, (1^3)), ((1), (1^2)), ((2), (1)))$, and $((3), \emptyset)$, are as follows:

- $\{1 + \epsilon, 3 + 2\epsilon, 5 + 3\epsilon\}$
- $\{0 + \epsilon, 1 + \epsilon, 3 + 2\epsilon\}$
- $\{-2, 0 + \epsilon, 1 + \epsilon\}$
- $\{-4 - \epsilon, -2, 0 + \epsilon\}$

Definition 2.3. Fix values of $\kappa$ and $\Theta$. Let $\lambda, \mu$ be $l$-partitions of $n$. A $\lambda$-tableau of weight $\mu$ is a bijective map $T : [\lambda] \to D_\mu$ which respects residues. In other words, we fill a given box $(r, c, m)$ of the diagram $[\lambda]$ with a real number $d$ from $D_\mu$ (without multiplicities) so that the residue attached to the real number $d$ in the loading $i_\mu$ is equal to $\kappa_m + c - r \pmod{e}$.

Definition 2.4. A $\lambda$-tableau, $T$, of shape $\lambda$ and weight $\mu$ is said to be semistandard if

- $T(1, 1, m) > \Theta_m$,
- $T(r, c, m) > T(r - 1, c, m) - g$,
- $T(r, c, m) > T(r, c - 1, m) + g$.

We denote the set of all semistandard tableaux of shape $\lambda$ and weight $\mu$ by $\text{SStd}(\lambda, \mu)$. Given $T \in \text{SStd}(\lambda, \mu)$, we write $\text{Shape}(T) = \lambda$.

Definition 2.5. Given $e > 1$, fix a weighting $\Theta$ and a multicharge $\kappa$ and two $l$-partitions $\lambda$ and $\mu$. We write $\lambda \triangleleft_\Theta \mu$ if for any real number $a \in \mathbb{R}$, the number of points in $i_\lambda$ to the left of $a$ and of a fixed residue is less than or equal to the number of points in $i_\mu$ to the left of $a$ of the same residue. If $\lambda \triangleleft_\Theta \mu$ and $\lambda \neq \mu$ then we say that $\mu$ dominates $\lambda$ and write $\lambda \triangleright_\Theta \mu$.

Remark 2.6. In this paper, we only consider examples of multipartitions in which each component is a hook. This means that when drawing diagrams in the Russian convention, no two nodes have the same $x$-coordinate for $\epsilon = 0$, therefore we omit $\epsilon$ from our tableaux and weightings without introducing ambiguity.

Example 2.7. We continue the example above with $n = 3$, $l = 2$, $e = 4$, $g = -2$, $\kappa = (0, 2)$ and $\Theta = (0, 1)$; we consider the block with residue $\{0, 1, 2\}$.
In this case, the dominance order on 2-partitions of residue \{0, 1, 2\} is given by reading the diagrams in Figure 13 from left to right in ascending order. In other words

\[(\emptyset, (1^3)) \prec_\Theta ((1), (1^2)) \prec_\Theta ((2), (1)) \prec_\Theta ((3), \emptyset).\]

Recall the loadings of these 2-partitions from Example 2.2. Recall that we let \(\epsilon \to 0\) for ease of notation. Figure 14 lists all three semistandard tableaux of shape \(\lambda\) and weight \(\mu\) (for \(\mu \neq \lambda\)) for \(\lambda, \mu\) in this block.

Example 2.8. We have the following two important examples of dominance orders. Let \(n = 3\) and \(l = 2\) and take \((\Theta_1, \Theta_2)\) so that (i) \(0 < \Theta_2 - \Theta_1 < |g|\) (ii) \(\Theta_2 - \Theta_1 > n|g|\). We refer to these values of \(\Theta\) as the FLOTW and well-separated weightings, respectively. We specialise \(\kappa\) so that the algebra is non-semisimple. The dominance order on a given block is given by intersecting the posets in Figure 15 with the set of 2-partitions of a given residue class. The leftmost poset in Figure 15 will be of the most interest to us in this paper.

![Figure 14. The tableaux in SStd(((1), (1^2)), (\emptyset, (1^3))), SStd(((2), (1)), ((1), (1^2))), and SStd(((3), \emptyset), ((2), (1))), respectively.](image)

![Figure 15. The Hasse diagrams of the posets corresponding to the FLOTW and well-separated weightings.](image)

2.2. A basis of the little Cherednik algebra. Fix integers \(l, n \geq 1, e \in \{3, 4, \ldots\} \cup \{\infty\}\) and \(g < 0\). Given \(\Theta = (\Theta_1, \ldots, \Theta_l)\) a weighting and \(\kappa = (\kappa_1, \ldots, \kappa_l)\) a multicharge the little Cherednik algebra, \(A(n, \Theta, \kappa)\), is the finite dimensional associative \(\mathbb{C}\)-algebra with basis given by certain diagrams \(\mathcal{C}_T\) which we now define.

Given \(T \in \text{SStd}(\lambda, \mu)\), we have a diagram \(B_T\) consisting of a frame with

- \(l\) double-red vertical strands placed at the points \(\Theta_1, \Theta_2, \ldots, \Theta_l\);
- distinguished black points on the northern and southern boundaries given by the loadings \(i_\mu\) and \(i_\lambda\), respectively;
n black strands each connecting a northern and southern vertex drawn so that they trace out the bijection determined by $T$ in such a way that we use the minimal number of crossings;

- each black strand has a ghost dashed-line drawn $|g|$ units to the left.

Moreover we draw the black strand and their ghosts so that there are no tangencies or triple points. Such a diagram is not unique, but we can pick any such diagram arbitrarily. We do not distinguish between ‘over’ and ‘under’ crossings.

Given a pair of semistandard tableaux of the same shape $(S, T) \in \text{SStd}(\lambda, \mu) \times \text{SStd}(\lambda, \nu)$ we have a diagram $C_{ST} = B_S B_T^*$ where $B_S^*$ is the diagram obtained from $B_S$ by flipping it through the horizontal axis.

Notice that there is a unique element $S \in \text{SStd}(\lambda, \lambda)$ and the corresponding basis element $C_{SS}$ is an idempotent in which all black strands are vertical.

**Example 2.9.** We continue the example above with $n = 3$, $l = 2$, $e = 4$, $g = -2$, and $\kappa = (0,2)$ and let $\Theta = (0,1)$. Consider the block with residue $\{0,1,2\}$. In this case the elements $B_S$ for the semistandard tableaux $S$ of shape $\lambda$ and weight $\mu$ for $\lambda \neq \mu$ are given in Figure 16 below.

![Figure 16](image-url)

**Figure 16.** The basis elements corresponding to the tableaux in $\text{SStd}(((1),(1^2)),((\emptyset),(1^3)))$, $\text{SStd}(((2),(1)),((1),(1^2)))$, and $\text{SStd}(((3),(\emptyset)),((2),(1)))$, respectively (see Figure 14).

### 2.3. The multiplication rule for the little Cherednik algebra

The multiplication of two basis elements in $A(n, \Theta, \kappa)$ is given by concatenation subject to the following relations. Recall that our diagrams do not distinguish between ‘over’ and ‘under’ crossings. We have the following local relations (here a local relation means one that can be applied on a small region of the diagram).

1. Any diagram may be deformed isotopically; that is, by a continuous deformation of the diagram which at no point introduces or removes any crossings of strands (black, ghosts, or double-red). Note that this may change the end-points of strands.

2. For $i \neq j$ we have that dots pass through crossings.
(2.3) For two like-labelled strands we get an error term.

\[
\begin{align*}
\begin{array}{c}
\includegraphics{example1.png} \\
\end{array}
\end{align*}
\]

(2.4) For double crossings of black strands, we have the following.

\[
\begin{align*}
\begin{array}{c}
\includegraphics{example2.png} \\
\end{array}
\end{align*}
\]

(2.5) If \( j \neq i - 1 \), then we can pass ghosts through black strands.

\[
\begin{align*}
\begin{array}{c}
\includegraphics{example3.png} \\
\end{array}
\end{align*}
\]

(2.6) On the other hand, in the case where \( j = i - 1 \), we have the following.

\[
\begin{align*}
\begin{array}{c}
\includegraphics{example4.png} \\
\end{array}
\end{align*}
\]

(2.7) We also have the relation below, obtained by symmetry.

\[
\begin{align*}
\begin{array}{c}
\includegraphics{example5.png} \\
\end{array}
\end{align*}
\]

(2.8) Strands can move through crossings of black strands freely.

\[
\begin{align*}
\begin{array}{c}
\includegraphics{example6.png} \\
\end{array}
\end{align*}
\]

Similarly, this holds for triple points involving ghosts, except for the following relations when \( j = i - 1 \).

(2.9)

\[
\begin{align*}
\begin{array}{c}
\includegraphics{example7.png} \\
\end{array}
\end{align*}
\]

(2.10)

\[
\begin{align*}
\begin{array}{c}
\includegraphics{example8.png} \\
\end{array}
\end{align*}
\]

In the diagrams with crossings in (2.9) and (2.10), we say that the we say that the black (respectively ghost) strand bypasses the crossing of ghost strands (respectively black strands). For \( i \neq j \), the red strands may pass through black strands freely. If the red and black strands have the same label, a dot is added to the black strand when straightening.

(2.11)

\[
\begin{align*}
\begin{array}{c}
\includegraphics{example9.png} \\
\end{array}
\end{align*}
\]

All black crossings and dots can pass through double-red lines, with a correction term.
Finally, we have the following non-local idempotent relation.

\[ (2.15) \text{Any idempotent where the strands can be broken into two groups separated by a blank space of size } |g| \text{ (so no ghost from the right group can be left of a strand in the left group and vice versa) with all double-red strands in the right group is referred to as unsteady and set to be equal to zero.} \]

2.4. The grading on the little Cherednik algebra. This algebra is graded as follows:

- Dots have degree 2;
- The crossing of two strands has degree 0, unless they have the same label, in which case it has degree \(-2\);
- The crossing of a black strand with label \(i\) and a ghost has degree 1 if the ghost has label \(i - 1\) and 0 otherwise;
- The crossing of a black strand with a red strand has degree 0, unless they have the same label, in which case it has degree 1.

In other words,

\[ \deg_i \bullet = 2 \quad \deg_i \bigotimes_j = -2\delta_{i,j} \quad \deg_i \bigotimes_j = \delta_{j,i+1} \quad \deg_i \bigotimes_j = \delta_{j,i-1} \]

Finally,

\[ \deg_i \bigotimes_j = \delta_{i,j} \quad \deg_i \bigotimes_j = \delta_{j,i}. \]

2.5. The structure of the little Cherednik algebra. We now recall the results concerning this algebra which will be needed later on.

**Theorem 2.10** ([Web13], Section 2.6). The little Cherednik algebra \(A(n, \Theta, \kappa)\) is a graded cellular algebra with a theory of highest weights. The cellular basis is given by

\[ \{C_{ST} \mid S \in \text{SStd}(\lambda, \mu), T \in \text{SStd}(\lambda, \nu), \lambda, \mu, \nu \in \Lambda\} \]

with respect to the \(\Theta\)-dominance order on the set of multipartitions, \(\Lambda\).

**Theorem 2.11** ([Web13], Theorem 6.2). The little Cherednik algebra \(A(n, \Theta, \kappa)\) is Koszul. In particular, given \(\lambda \neq \mu \in \Lambda\), the graded decomposition numbers \(d_{\lambda\mu}(t) \in t\mathbb{N}_0(t)\).

2.6. An example. Let \(e = 2, \ l = 1, \ g = -1, \) and \(n = 2\) and \(\kappa = (0)\). In this case we shall see that the algebra \(A(n, \Theta, \kappa)\) is the basic algebra of the Schur algebra of the Hecke algebra of type \(G(1, 1, 2)\) specialised at \(e = 2\). There are two partitions \((2)\) and \((1^2)\) with loadings \((-1 + 2\epsilon, \epsilon)\) and \((\epsilon, 1 + 2\epsilon)\) respectively, depicted in Figure 17. When discussing the combinatorics of tableaux, we will adopt the conventions of Remark 2.6.

There is a unique element \(U \in \text{SStd}((1^2), (1^2))\) given in Figure 18 below, and the corresponding cell module is 1-dimensional. The two elements \(S, T \in \text{SStd}((2), -)\) are also given in Figure 18 and are of weight \((2)\) and \((1^2)\) respectively. The left cell module has basis given by \(B_S\) and \(B_T\), depicted in Figure 19; the full 5-dimensional algebra is given by taking the pairs of flipped elements \(C_{UU}, C_{SS}, C_{ST}, C_{TS}, C_{TT}\).
In $B_T$, the crossing of the ghost strand of residue 1 with the black strand of residue 0 has degree 1. All other crossings in $B_T$ have degree 0 and therefore $\deg(B_T) = 1$. We shall now show that $[\Delta(2) : L(1^2)] = t$. To see this, it is enough to check that $B_T$ is in the radical of $\Delta(2)$, in other words the products $B_T^*B_T$ and $B_T^T B_T$ are zero in the cell module.

The first case is obvious as we are multiplying two non-equal idempotents, in the second case we apply relations (2.6) and (2.11) followed by (2.7) to the product $B_T^*B_T$ as in Figure 20. This product is equal to zero, as the centre of the final diagram in Figure 20 is an unsteady idempotent.

In this section we define the quiver Temperley–Lieb algebra of type $G(l, 1, n)$, which we denote by $\mathcal{T}L_l(\kappa)$ for $\kappa \in (\mathbb{Z}/\mathbb{Z})^l$ and $n \in \mathbb{N}$. Given $l \in \mathbb{N}$, and $g < 0$, we take $\Theta \in \mathbb{R}^l$ such that $0 < \Theta_j - \Theta_i < |g|$ for all $1 \leq i < j \leq l$. 

3. The quiver Temperley–Lieb algebra of type $G(l, 1, n)$
We let $\pi$ denote the set of all multipartitions all of whose components have at most one column. We refer to this as the set of one-column multipartitions. We let

$$e_\pi = \sum_{\lambda \in \pi} C_{T^\lambda T^\lambda}.$$  

We shall see in the proof of Proposition 3.2 below that any such choice of $\Theta$ guarantees that $\pi$ is a saturated subset of $\Lambda$. We shall fix a choice of $\Theta$ below; however, given any such choice the resulting algebras are isomorphic (see Proof of Proposition 3.2, below).

**Definition 3.1.** Fix $e \in \{3, 4, \ldots \} \cup \{\infty\}$ and integers $l \leq e/2$ and $n \geq 1$. Fix a multicharge $\kappa \in (\mathbb{Z}/e\mathbb{Z})^l$ such that $\kappa_i \not\in \{\kappa_j, \kappa_j + 1\}$ for $i \neq j$ and $\Theta$ as above. We define the quiver Temperley–Lieb algebra of type $G(l, 1, n)$, denoted $\text{TL}_n(\kappa)$, to be the algebra

$$\text{TL}_n(\kappa) = A(n, \Theta, \kappa)/(A(n, \Theta, \kappa)e_\pi A(n, \Theta, \kappa)).$$

**Notation:** For the remainder of the paper, given $\text{TL}_n(\kappa)$ with $\kappa \in (\mathbb{Z}/e\mathbb{Z})^l$, we shall fix $g = -l$ and $\Theta = (0, 1, 2, \ldots, l - 1)$ (this choice is easily seen to satisfy the conditions above), and adopt the conventions of Remark 2.6. With this choice made, the loadings of multipartitions in $\pi$ have a very simple form. Namely, any $l$-partition of $n$ is of the form

$$\lambda = (1^{\lambda_1}, 1^{\lambda_2}, \ldots, 1^{\lambda_l}) \in \pi$$

(with $\sum_{i=1}^l \lambda_i = n$) and has loading

$$\{(i - 1) + jl \mid \lambda_i \neq 0 \text{ and } 1 \leq j \leq \lambda_i\}.$$ 

Given $\Theta$ as above, we refer to the $\Theta$-dominance order on multipartitions as the FLOTW dominance order.

**Proposition 3.2.** The quiver Temperley–Lieb algebra of type $G(l, 1, n)$ is a graded cellular algebra with a theory of highest weights. The cellular basis is given by

$$\{C_{ST} \mid S \in \text{SSStd}(\lambda, \mu), T \in \text{SSStd}(\lambda, \nu), \lambda, \mu, \nu \in \pi\},$$

with respect to the FLOTW dominance order on the set of one-column multipartitions, $\pi$. We have that $d_{\lambda\mu}(t) \in t\mathbb{N}_0[t]$ for $\lambda \neq \mu$ elements of $\pi$.

**Proof.** Fix $\Theta$ such that $0 < \Theta_j - \Theta_i < |g| - e$ for $1 \leq i < j \leq l$. We shall show that the set $\pi$ is saturated in the $\Theta$-dominance order. In other words given any $\lambda \in \pi$ and $\mu <_{\Theta} \lambda$, we have that $\mu \in \pi$. This will imply that

$$\langle C_{ST} \mid S \in \text{SSStd}(\lambda, \mu), T \in \text{SSStd}(\lambda, \nu), \lambda, \mu, \nu \not\in \pi \rangle \subset A(n, \Theta, \kappa)$$

is an ideal of $A(n, \Theta, \kappa)$ (the ideal generated by $e_\pi$, in fact) and the resulting quotient has the desired basis (by conditions (2) and (3) of Definition 1.1 and Theorem 2.10). The graded decomposition numbers (as well as dimensions of higher extension groups) are preserved under this quotient, see for example [Don98, Appendix] for the ungraded case. Applying Theorem 2.10 will thus prove the claim about graded decomposition numbers.

Given any two choices $\Theta^{(1)}$ and $\Theta^{(2)}$ satisfying the above, the combinatorics of tableaux are identical. This results in a bijection between the cell-bases of the algebras $A(n, \Theta^{(1)}, \kappa)$ and $A(n, \Theta^{(2)}, \kappa)$. The basis elements identified under this bijection may be obtained from one another by isotopy. Therefore this is an isomorphism of algebras, via relation (2.1) of Section 2.3.

Our choice of $\Theta$ implies that if we add a box to the second column of any component $\lambda^{(m)}$ (that is, add a node $(1, 2, m)$ for some $m$), this box has $x$-coordinate strictly less than all boxes in the first column of all components, and thus the resulting multipartition is more dominant. Therefore the set of one-column multipartitions is saturated. \hfill $\square$

**Definition 3.3.** Let $\lambda$ be a one-column multipartition $(1^{\lambda_1}, 1^{\lambda_2}, \ldots, 1^{\lambda_l})$. A node of $\lambda$ is removable if it can be removed from the diagram of $\lambda$ to leave the diagram of a (one-column) multipartition, while a node not in the diagram of $\lambda$ is an addable node of $\lambda$ if it can be added to the diagram of $\lambda$ to give the diagram of a one-column multipartition.
If the node has residue \( r \in \mathbb{Z}/e\mathbb{Z} \), we say that the node is \( r \)-removable or \( r \)-addable. Given \( \lambda \in \pi \) and \( r \in \mathbb{Z}/e\mathbb{Z} \), we let \( \text{Add}(\lambda, r) \) denote the set of \( 1 \leq j \leq l \) such that there is an \( r \)-addable node in the \( j \)th component of \( \lambda \).

In the previous section, we refrained from defining the degree of a general tableau. This was because of the technicalities in defining addable and removable nodes for such tableaux (see [Web13, Section 2.2]). These difficulties do not appear for tableaux corresponding to one-column multipartitions.

**Definition 3.4.** Suppose \( \lambda \in \pi \) and \( \Box \) is a removable \( A \)-node of \( [\lambda] \). Set
\[
\deg(\lambda(\Box)) = |\{ \text{addable} \ A \text{-nodes of } \lambda \text{ to the right of } \Box \} | - |\{ \text{removable} \ A \text{-nodes of } \lambda \text{ to the right of } \Box \} |.
\]

Given \( 1 \leq k \leq n \) and \( T \in S\text{Std}(\lambda, \mu) \), we let \( T_k \) denote the node of \( [\lambda] \) containing the entry \( D_\mu(k) \) and we let \( T_{\leq k} \) denote the tableau consisting of the nodes with entries in \( D_\mu\{1, 2, \ldots, k\} \).

For \( T \in S\text{Std}(\lambda, -) \) we define the degree of \( T \) recursively, setting \( \deg(T) = 0 \) when \( T \) is the unique \( \emptyset \)-tableau. We set
\[
\deg(T) = \deg(T_n) + \deg(T_{<n}).
\]

**Example 3.5.** Let \( e = 4 \), \( l = 2 \), \( n = 7 \), and \( \kappa = (0, 2) \). By tableau-linkage, it is clear that any residue class decomposes as a sum of blocks of \( \text{TL}_n(\kappa) \). Fix the residue class to be \( \{0, 0, 1, 2, 2, 3, 3\} \). The one-line multipartitions with these residues for our given value \( e \)-multicharge are
\[
\{(1^7, (0)), ((1^6), (1)), ((1^3), (1^4)), ((1^2), (1^2))\}.
\]

Formally, the loading of the multipartition \( \lambda = ((1^7), (0)) \) is
\[
D_\lambda = \{0 + \epsilon, 2 + 2\epsilon, 4 + 3\epsilon, 6 + 4\epsilon, 8 + 5\epsilon, 10 + 6\epsilon, 12 + 7\epsilon\}.
\]

With the conventions of Remark 2.6 in place, our loadings are
\[
(0, 2, 4, 6, 8, 10, 12), \quad (0, 1, 2, 4, 6, 8, 10), \quad (0, 1, 2, 3, 4, 5, 7), \quad (0, 1, 2, 3, 5, 7, 9).
\]

The semistandard tableaux of shape \((1^3, 1^4)\) are given in Figure 21, along with their degrees. For example, the nodes in the rightmost diagram are of degree 0 except for those containing the integers 4 and 12, which are of degree 1. Therefore the rightmost tableau has degree 2.

![Figure 21](image)

**Figure 21.** These semistandard tableaux are of weights \( ((1^3), (1^4)), ((1^2), (1^5)), ((1^6), (1)) \) and \( ((1^7), \emptyset) \) respectively. The tableaux are of degrees 0, 1, 1, and 2 respectively.

### 3.1. The geometry.

Fix integers \( n, l, e \in \mathbb{N} \) and \( \kappa \in (\mathbb{Z}/e\mathbb{Z})^l \). Let \( \Phi_{l-1} \) be a root system of type \( A_{l-1} \) with simple roots
\[
\{ \varepsilon_i - \varepsilon_j : 1 \leq i < j \leq l \},
\]
and let \( W^e_\kappa \) denote the corresponding affine Weyl group, generated by the affine reflections \( s_{i,j,m} \) with \( 1 \leq i < j \leq l \) and \( m \in \mathbb{Z} \) and which acts on \( E_l \) via
\[
s_{i,j,m}(x) = x - (\langle x, \varepsilon_i - \varepsilon_j \rangle - m)(\varepsilon_i - \varepsilon_j).
\]

Given a multicharge \( \kappa = (\kappa_1, \ldots, \kappa_l) \) we let \( \rho = (e - \kappa_1, \ldots, e - \kappa_l) \). Given an element \( w \in W^e_\kappa \) we set
\[
w \cdot \rho x = w(x + \rho) - \rho.
\]

We identify \( \lambda \) an \( l \)-partition of \( n \) with a point in the hyperplane \( V \) of \( E_l \) consisting of all points the sum of which is \( n \). This is done via the map \( (1^{\lambda_1}, \ldots, 1^{\lambda_l}) \mapsto \sum \lambda_i \varepsilon_i \).
Lemma 3.6. Given $\lambda \in \pi$, we have that
\[
\langle \lambda + \rho, \varepsilon_i - \varepsilon_j \rangle = me
\]
for some $m \in \mathbb{Z}$, if and only if the addable nodes in the $i$th and $j$th components of the multipartition $\lambda$ have the same residue.

Proof. To see this, note that both statements are equivalent to\[
(\lambda_i + e - \kappa_i) \equiv (\lambda_j + e - \kappa_j) \pmod{e}.
\]
□

Definition 3.7. Given $T \in SStd(\lambda, \mu)$, we define the component word $R(T)$ of $T$ to be given by reading the entries of the tableau in numerical order and recording the components in which they appear. We define the path $\omega(T)$ to be the associated path in the alcove geometry.

Example 3.8. Let $e = 4$, $l = 2$, $n = 7$, and $\kappa = (0, 2)$. Let $T \in SStd(((1^3, 1^4), ((1^7), \emptyset))$ be the following tableau.

The component word, $R(T)$, is $(1, 2, 2, 2, 1)$. The path $\omega(T)$ is pictured in Figure 22.

Figure 22. The path $\omega(T) \in \text{Path}((3, 4), (7, 0))$.

Given the unique $T^\mu \in SStd(\mu, \mu)$, it is clear that $\omega(T^\mu) = \omega^\mu$ is the path corresponding to the word $w : \{1, \ldots, n\} \rightarrow \{1, \ldots, l\}$ given by
\[
w(1) = \min\{i \mid \mu_i \neq 0\}
\]
and for $i > 1$,
\[
w(i) = (w(i - 1) + j) \pmod{l}
\]
where $j \geq 1$ is minimal such that $\langle \omega^\mu(i - 1) + \rho, \varepsilon_{w(i-1)+j} \rangle < \mu_{w(i-1)+j}$ where the subscripts are also read modulo $l$.

Example 3.9. As in the introduction, let $l = 3$, $n = 13$, $e = 8$, $\kappa = (0, 4, 6)$. For $\mu = (5, 6, 2)$ the component word of $T^\mu$ is
\[
(1, 2, 3, 1, 2, 3, 1, 2, 1, 2, 1, 2, 2).
\]

Proposition 3.10. Given $\mu \in E_l$, we fix the distinguished path $\omega(T^\mu)$ as above. We have that $\omega$ defines a bijective map
\[
\omega : SStd(\lambda, \mu) \rightarrow \text{Path}(\lambda, \mu).
\]

Proof. The map $\omega$ is clearly an injective map, it remains to show that both sets have the same size. The sets $SStd(\mu, \mu)$ and $\text{Path}(\mu, \mu)$ each possess a unique element $T^\mu$, respectively $\omega^\mu$. For $1 \leq k \leq n$, let $r(k)$ denote the residue of the node $T^\mu_k$ and let $t(k)$ denote the component of the $l$-partition in which this node is added. For $1 \leq k \leq n$, it follows by Lemma 3.6 that
\[
\text{Add}(\text{Shape}(T^\mu_{(k-1)}), r(k)) = \{i \mid \omega^\mu(k) \in h_{\varepsilon_i - \varepsilon_{t(k)}}m_{ik}e \text{ for some } m_{ik} \in \mathbb{Z}\}.
\]
We let $d_k$ denote the cardinality of this set.

We construct both $T \in SStd(-, \mu)$ and $\omega \in \text{Path}(-, \mu)$ step-by-step; in the former case, by adding one node at a time to the tableau and in the latter case by taking one step at a time in the geometry.
The number of choices to be made at the $k$th point in the tableau is equal to $d_k$, for $1 \leq k \leq n$. Therefore the number of tableaux of weight $\mu$ is equal to $d_1d_2\ldots d_n$. On the other hand, in the notation of Section 1.3, any path $\omega \in \text{Path}(\cdot, \mu)$ may be written as
\[
\omega = e_{\varepsilon_{i(1)} - \varepsilon_{i(1)}, m_{1}e} \cdots e_{\varepsilon_{i(n)} - \varepsilon_{i(n)}, m_{n}e} \omega^\mu
\]
for $i(k) \in \text{Add}(\text{Shape}(T_{\leq k-1}^\mu), \tau(k))$ and $m_{k} \in \mathbb{Z}$ (of course, if $d_k = 1$, the reflection is necessarily trivial). The number of such paths is equal to the number of distinct possible series of reflections, $d_1 \ldots d_n$. 

**Corollary 3.11.** If $\lambda, \mu \in \pi$, label simple modules in the same $\text{TL}_n(\kappa)$-block, this implies that their images in $E_n$ are in the same $W_{\kappa - 1}^\mu$ orbit under the $\rho$-shifted action.

**Proof.** This follows from Proposition 1.3, as it is easy to see that the equivalence classes of the relation generated by $\lambda \sim \mu$ if $\text{Path}(\lambda, \mu) \neq \emptyset$ are the same as the $W_{\kappa - 1}^\mu$-orbits.

**Lemma 3.12.** Let $\lambda = (1^{\lambda_1}, \ldots, 1^{\lambda_l}) \in \pi$ be such that $\lambda_1 > \lambda_j$ for some $1 \leq i, j \leq l$ and suppose that the residues of the addable nodes in $i$th and $j$th components of $\lambda$ are equal to $r \in (\mathbb{Z}/e\mathbb{Z})$.

Then $\lambda \in E_l$ lies on a hyperplane of the form $x_i - x_j = m_{ij}e$ for some $m_{ij} \in \mathbb{Z}$. We have that $(\lambda + \varepsilon_i) \in E_l^+ (\varepsilon_i - \varepsilon_j, m_{ij}e)$ and $(\lambda + \varepsilon_j) \in E_l^- (\varepsilon_i - \varepsilon_j, m_{ij}e)$.

**Proof.** We have seen that $\lambda$ lies on a hyperplane by Lemma 3.6. We have assumed that $\lambda_i > \lambda_j$, and so
\[
\langle \lambda + \rho, \varepsilon_i - \varepsilon_j \rangle > \langle \lambda + \rho, \varepsilon_i - \varepsilon_j \rangle
\]
as required.

**Proposition 3.13.** The map $\omega : \text{SStd}(\lambda, \mu) \rightarrow \text{Path}(\lambda, \mu)$ is degree preserving.

**Proof.** We fix a tableau $T \in \text{SStd}(\lambda, \mu)$ and let $\omega := \omega(T)$ denote the corresponding element of $\text{Path}(\lambda, \mu)$. For $1 \leq k \leq n$, we truncate to consider the path of length $k-1$ (respectively tableau with $k-1$ nodes), $\omega_{\leq k-1}$ (respectively $T_{\leq k-1}$) and identify this with the multipartition $\text{Shape}(T_{\leq k-1}) \in \pi$.

Let $r_k$ denote the residue of the addable node $T_k$ and let $t(k)$ denote the component in which this node is added. By the definition of the Soergel-degree, we are interested in the cases where $1 \leq i \leq l$ such that
\[
(i) \ \omega(k-1) \in h_{\varepsilon_i - \varepsilon_{t(k)}, m_{ik}e} \text{ and } \omega(k) \in E_l^- (\varepsilon_i - \varepsilon_{t(k)}, m_{ik}e) \text{ for some } m_{ik} \in \mathbb{Z},
\]
\[
(ii) \ \omega(k-1) \in E_l^+ (\varepsilon_i - \varepsilon_{t(k)}, m_{ik}e) \text{ and } \omega(k) \in h_{\varepsilon_i - \varepsilon_{t(k)}, m_{ik}e} \text{ for some } m_{ik} \in \mathbb{Z}.
\]

By Lemma 3.12, the $1 \leq i \leq l$ above label the components of
\[
(i) \text{ the } r_k\text{-addable nodes of } T_{\leq k-1} \text{ to the right of } T_k,
\]
\[
(ii) \text{ the } (r_k - 1)\text{-addable nodes of } T_{\leq k-1} \text{ to the right of } T_k.
\]
We observe that, because of the condition $\kappa_i \notin \{\kappa_j, \kappa_j + 1\}$ for $i \neq j$, the set of $1 \leq i \leq l$ which label $(r_k - 1)$-addable nodes of $T_{\leq k}$ to the right of $T_k$ is equal to the set of $r_k$-removable nodes of $T_{\leq k-1}$ to the right of $T_k$. Therefore the result follows.

**Proposition 3.14.** Given an $e$-regular $\mu \in \Lambda$, the path $\omega^\mu$ is admissible.

**Proof.** It is clear that $\deg(\omega_{\leq k}) = 0$ for $1 \leq k \leq n$. Now assume that $\omega^\mu(k)$ lies on two (or more) hyperplanes $x_i - x_j = m_{1}e$ and $x_i - x_{j'} = m_{2}e$ for some $1 \leq k \leq n$ and $m_{1}, m_{2} \in \mathbb{Z}$. We will show that $i, j, i', j'$ are necessarily distinct.

To prove the claim, we recall our description of $\omega^\mu$. Let $r_k$ denote the residue of the addable node $T_k$ and let $t(k)$ denote the component in which this node is added. It is clear that the result holds for $k = 0$, we proceed by induction. For $1 \leq k \leq n$, assume $\omega^\mu(k)$ lies on the hyperplane $h_{\varepsilon_i - \varepsilon_{t(k)}, m_{ik}e}$ for some $m_{ik} \in \mathbb{Z}$. Our assumption on $\kappa$ ensures that $\varepsilon_{t(k)} \neq \kappa_j, \kappa_j \pm 1$ for any $1 \leq j \leq l$. This implies that if $\langle \omega^\mu(k) + \rho, \varepsilon_{t(k)} - \varepsilon_j \rangle \equiv 0$ (mod $e$) for any $1 \leq j \leq l$, then $\omega^\mu(k)$ lies on two hyperplanes, which is a contradiction.
then \( \langle \omega^\mu(k) + \rho, \varepsilon_j \rangle = \langle \omega^\mu + \rho, \varepsilon_j \rangle \). Our assumption that \( \mu \) is \( e \)-regular implies that there is a maximum of one such value of \( 1 \leq j \leq l \). The result follows. \( \square \)

**Theorem 3.15.** The algebra \( \text{TL}_n(\kappa) \) for \( \kappa \in (\mathbb{Z}/e\mathbb{Z})^l \) has a Soergel-path basis of type \( \hat{A}_{l-1} \). The graded decomposition numbers of an \( e \)-regular block are given by Soergel’s algorithm

\[
d_{\lambda\mu}(t) = n_{\mu}(\lambda),
\]
and the characters of the simple modules are given by the character algorithm

\[
\text{Dim}_t(L_{\mu}(\lambda)) = e_{\mu}(\lambda).
\]

**Proof.** This follows from Theorems 1.18 and 2.11 and Propositions 3.2, 3.10, 3.13, and 3.14. \( \square \)

We also observe the following stability in the decomposition numbers as \( n \) tends to infinity. Fix \( n, l \in \mathbb{N} \). Given \( \lambda \) a one-column multipartition of \( n \) and \( i \geq 0 \), we let \( \lambda + (1^i, \ldots, 1^i) \) denote the one-column multipartition of \( n + il \) obtained by adding \( i \) boxes to every component of \( \lambda \). This defines an injective map from multipartitions of \( n \) to multipartitions of \( n' = n + il \). These points may be identified with points in the hyperplanes \( \varepsilon_1 + \cdots + \varepsilon_l = n \) and \( \varepsilon_1 + \cdots + \varepsilon_l = n + il \) of \( E_l \), respectively. We identify points in these two hyperplanes via the projection in the direction \( \varepsilon_1 + \cdots + \varepsilon_l \).

**Theorem 3.16.** The decomposition numbers of \( \text{TL}_n(\kappa) \) for \( \kappa \in (\mathbb{Z}/e\mathbb{Z})^l \) are stable as \( n \) tends to infinity. To be more precise,

\[
d_{\lambda\mu}(t) = d_{\lambda+(1^i,\ldots,1^i),\mu+(1^i,\ldots,1^i)}(t)
\]
for \( i \geq 0 \).

**Proof.** Given \( \omega \in \text{Path}(\lambda,\mu) \) we let \( \omega' \in \text{Path}(\lambda + (1^i, \ldots, 1^i), \mu + (1^i, \ldots, 1^i)) \) denote the concatenated path

\[
(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l)^t \circ \omega.
\]
It is clear that this map is a degree preserving bijection. The result follows. \( \square \)

**Corollary 3.17.** Fix a multicharge \( \kappa \in (\mathbb{Z}/e\mathbb{Z})^l \) such that \( \kappa_i \notin \{\kappa_j, \kappa_j+1\} \) for all \( i \neq j \). Let \( \Theta \in \mathbb{R}^l \) denote a FLOTW weighting. Let \( \lambda, \mu \) denote a pair of \( e \)-regular one-column multipartitions. The graded decomposition numbers for \( \text{TL}_n(\kappa) \) are

\[
d_{\lambda\mu}(t) = n_{\mu}(\lambda),
\]
where \( n_{\mu}(\lambda) \) is the associated affine Kazhdan–Lusztig polynomial of type \( \hat{A}_{l-1} \). These decomposition numbers are stable as \( n \) tends to infinity as in Theorem 3.16 above.

**Remark 3.18.** This Soergel path basis contains a vast amount of information concerning the representation theory of the quiver Temperley–Lieb algebras of type \( G(l, 1, n) \). We have already seen that it provides a new interpretation for Soergel’s algorithm for computing the decomposition numbers of \( \text{TL}_n(\kappa) \). In the next section we shall consider the \( l = 2 \) case, calculate the full submodule structure of the standard modules of \( \text{TL}_n(\kappa) \) for \( \kappa \in (\mathbb{Z}/e\mathbb{Z})^2 \), and show that the algebra is positively graded.

We have already remarked that our approach to the algebras \( \text{TL}_n(\kappa) \) is heavily inspired by the combinatorics of [MW03]. In [MW03] it is conjectured that the decomposition numbers of the generalised blob algebras are given by the same Kazhdan–Lusztig polynomials as those considered here. Our algebra is a quotient of the little Cherednik algebra, whereas the generalised blob algebra is the corresponding quotient of the Ariki–Koike algebra. For a fixed weighting \( \Theta \), the standard/Specht modules of these algebras have the same labelling set; however, there is no known cellular basis for the Ariki–Koike algebra with respect to the \( \Theta \)-dominance order (except when \( \Theta \) is well-separated, see [GJ11]) and hence no way to relate the representation theories of the generalised blob and Ariki–Koike algebras via an analogue of Proposition 3.2. Moreover, the resulting quotient algebra would not be amenable to our methods as it does not possess a Soergel path basis (for example, for \( l = 2 \) the blob algebra is not positively graded, [Pla13]).
However we do believe that the generalised blob algebras are (graded) Morita equivalent to the corresponding quiver Temperley–Lieb algebras.

3.2. The level two case. For \( l = 2 \), the structure of the standard modules for \( TL_n(\kappa) \) labelled by \( e \)-regular points is particularly simple. The proofs in this section are lightly sketched, but augmented with illustrative examples.

We remark that the submodule lattices obtained here are identical to those computed for the blob algebra in [MW00]. This provides further evidence that the quiver Temperley–Lieb algebras are (graded) Morita equivalent to the generalised blob algebras.

Let \( a_i \) denote the alcove of length \( i = \ell(a_i) \) to the right of the origin and \( a_i' \) denote the alcove of length \( i = \ell(a_i') \) to the left of the origin, as depicted in the examples below. Fix a point \( \lambda_0 \) in the alcove containing the origin. We let \( \lambda_i \) and \( \lambda_i' \) denote the points in alcoves \( a_i \) and \( a_i' \) which are in the same orbit as \( \lambda_0 \). For ease of notation, we often identify \( \lambda_i \) with the subscript \( i \).

**Proposition 3.19.** For \( \kappa \in (\mathbb{Z} / e \mathbb{Z})^2 \), the algebra \( TL_n(\kappa) \) is positively graded. For \( \lambda_i \) and \( \lambda_j \) in \( E_2 \), we have that

\[
d_{i(j)}(t) = n_{j(i)}(i(j)) = \begin{cases} 
\ell^{j-i} & \text{for } i < j, \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof.** Positivity follows as our paths start at \( \odot \) and the root system is of rank 1. The closed form for \( n_{j(i)}(i(j)) \) is well-known (see for example the introduction to [MW03]). \( \square \)

**Remark.** The algebra \( TL_n(\kappa) \) is not positively graded for \( l \geq 3 \), as seen in Example 1.15.

Given any pair \( \lambda_i, \lambda_j \) with \( i < j \), there exists a unique element of \( \text{Path}(\lambda_i, \lambda_j) \) of maximal degree equal to \( j - i \). This is the unique path terminating at \( \lambda_j \) which may be obtained from the distinguished path from \( \odot \) to \( \lambda_j \) using the maximum number of reflections in the hyperplanes \( \overline{a}_0 \cap \overline{a}_1 \) and \( \overline{a}_0 \cap \overline{a}_1' \).

**Example 3.20.** Let \( n = 11, e = 4 \) and \( \kappa = (0, 2) \). Some of the maximal paths in \( \text{Path}((5, 6), -) \) are given in Figures 23 and 24.

![Figure 23](image)

**Figure 23.** Maximal paths in \( \text{Path}(\lambda_0, \lambda_2') \) and \( \text{Path}(\lambda_0, \lambda_2) \), respectively. Both paths have degree 2.

The maximal paths in \( \text{Path}(\lambda_1, \lambda_2') \) and \( \text{Path}(\lambda_1', \lambda_2') \) (which index basis elements of \( \Delta(\lambda_1) \) and \( \Delta(\lambda_1') \) respectively) are depicted in Figure 25 below.
Theorem 3.21. If $l = 2$, the full submodule structure of the $\text{TL}_{n}(\kappa)$-modules $\Delta(\lambda_{i})$ and $\Delta(\lambda_{i'})$ are given by the strong Alperin diagrams (in the sense of [Alp80]) below.

\[
\begin{array}{c|c}
L(\lambda_{i}) & L(\lambda_{i'}) \\
\hline
L(\lambda_{i+1}(1)) & L(\lambda_{i'+1}(1)) \\
\hline
L(\lambda_{i+2}(2)) & L(\lambda_{i'+2}(2)) \\
\end{array}
\]

Therefore $\text{Dim}_{t}(\text{Hom}_{\text{TL}_{n}(\kappa)}(\Delta(\lambda_{j}),\Delta(\lambda_{i}))) = t^{j-i}$ for $i < j$ (in which case this homomorphism is injective) and the dimension is 0 otherwise.

Proof. Fix points $\lambda_{i(0)}, \lambda_{j(0)} \in E_{l}$ such that $i < j$. We have seen that if $\omega(T) \in \text{Path}(\lambda_{i(0)}, \lambda_{j(0)})$ is maximal, then it labels a decomposition number $d_{i(0)j(0)} = t^{j-i}$. Therefore $B_{T}$ generates a simple composition factor $L(\lambda_{j(0)})/\omega(T)$ of the standard module $\Delta(\lambda_{i(0)})$.

Given $\lambda_{i(0)}, \lambda_{j(0)} \in E_{l}$ and $i < j$, we let $1_{i(0)}^{j(0)}$ denote the element $B_{T}$ for $\omega(T)$ the unique maximal path in $\text{Path}(\lambda_{j(0)}, \lambda_{i(0)})$. We shall show that

\[1_{i+1}^{i+2}(0) \circ 1_{i}^{i+1}(0) = \pm 1_{i+1}^{i+2}(0) = 1_{i+1}^{i+2}(0) \circ 1_{i}^{i+1}(0)\]

and the result will follow. First, notice that $\text{deg}(1_{i}^{j(0)}) = j - i$ and this is the unique basis element of $\Delta(i^{(0)})$ of this degree. By comparing degrees, we deduce that

\[1_{i+1}^{i+2}(0) \circ 1_{i}^{i+1}(0) = c 1_{i}^{i+2}(0)\]

for some $c \in \mathbb{C}$. It remains to show that $c = \pm 1$ (note that, for the result to hold, it is enough to show that $c \neq 0$). It is clear that the lefthand side of Equation 3.1 is a diagram.
with distinguished black points on northern and southern boundaries given by the loadings corresponding to the partitions $\lambda_{i+2(0)}$ and $\lambda_{(0)}$, respectively. If the bijection traced out by the strands (after concatenation) uses the minimal number of crossings, then we are done.

Suppose that we are not in the case above, then we must apply the relations to the product to obtain a diagram of the form $c1^{i+2(0)}_{i(0)}$ for some $c \in \mathbb{C}$. This product has a number of ‘extra crossings’ of strands of the same residue (that is, crossings which do not appear in $1_{i(0)}^{i+2(0)}$). The rightmost of these crossings involves a pair of strands of residue $r$, say. This crossing is bypassed by the ghost of the strand of residue $r - 1$ immediately to its right (for an example, see Figure 29). Applying relation (2.10), the product can be written as a sum of two terms: one is zero modulo more dominant terms, the other differs from the original diagram only where we have untied the distinguished crossing (for an example, see Figure 29).

Now suppose that the resulting diagram is not equal to $1_{i(0)}^{i+2(0)}$, in which case it has a rightmost ‘extra crossing’ of residue $r + 1$. Now consider the ghost of the leftmost of the two strands we untied in the previous step; the ghost of this strand bypasses the rightmost ‘extra crossing’. Repeating the above argument for all the crossings, we obtain the result. □

**Example 3.22.** The elements $1^\nu_{\mu}$ for $\nu, \mu \in \lambda_0, \lambda_1, \lambda_1', \lambda_2'$ are depicted in Figures 26 and 27, below. Figure 29 depicts the first use of relation (2.10) on the product $1_{1'}^{2'} \circ 1_{1}^{0}$ to untie a crossing.

![Figure 26](image1)

**Figure 26.** The elements $1_{1'}^{2'}$ and $1_{1}^{2'}$ corresponding to the maximal paths in Figure 25.

![Figure 27](image2)

**Figure 27.** The elements $1_{0}^{1}$ and $1_{0}^{1'}$ corresponding to the maximal paths in Figure 24.
Figure 28. The element $1'_{0}^{2'}$.

Figure 29. The top diagram is obtained by concatenation of the diagram $1'_{0}^{2'}$ above $1'_{1}$. The lower diagram is obtained by applying relation (2.10) to the product $1'_{1}^{2'} \circ 1'_{0}$. We move the ghost 0 strand through the crossing pair of black strands of residue 1 (we do not record the diagram which is zero modulo more dominant terms). We have made emphasised the strands to which we are applying relation (2.10) and we have recorded their residues along the southern edge of the frames. Along the northern edge of the frame of the top diagram, we have recorded the residues of the 3 extra crossings of like-labelled pairs.
References

[Alp80] J. L. Alperin, *Diagrams for modules*, J. Pure Appl. Algebra 16 (1980), no. 2, 111–119. [Page 27.]

[Bou02] N. Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002, Translated from the 1968 French original by Andrew Pressley. [Page 6.]

[Don98] S. Donkin, *The q-Schur algebra*, London Mathematical Society Lecture Note Series, vol. 253, Cambridge University Press, Cambridge, 1998. [Page 21.]

[Fay10] M. Fayers, *An LLT-type algorithm for computing higher-level canonical bases*, J. Pure Appl. Algebra 214 (2010), no. 12, 2186–2198. No citations.

[FLOTW99] O. Foda, B. Leclerc, M. Okado, J.-Y. Thibon, and T. Welsh, *Branching functions of \(A_{n-1}\) and Jantzen–Seitz problem for Ariki–Koike algebras*, Adv. Math. 141 (1999), no. 2, 322–365. [Page 1.]

[GGOR03] V. Ginzburg, N. Guay, E. Opdam, and R. Rouquier, *On the category \(O\) for rational Cherednik algebras*, Invent. Math. 154 (2003), no. 3, 617–651. [Page 1.]

[GJ11] M. Geck and N. Jacon, *Representations of Hecke algebras at roots of unity*, Algebra and Applications, vol. 15, Springer-Verlag London, Ltd., London, 2011. [Page 25.]

[GJSV13] A. Gainutdinov, J. Jacobsen, H. Saleur, and R. Vasseur, *A physical approach to the classification of indecomposable Virasoro representations from the blob algebra*, Nuclear Phys. B 873 (2013), no. 3, 614–681. [Page 1.]

[GJ05] M. Kashiwara, *Kazhdan–Lusztig conjecture for a symmetrizable Kac–Moody Lie algebra*, The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, 1990, pp. 407–433. [Page 1.]

[KL96] P. Martin and H. Saleur, *The blob algebra and the periodic Temperley–Lieb algebra*, Lett. Math. Phys. 30 (1994), no. 3, 189–206. [Page 1.]

[JM03] D. Plaza, *Generalized blob algebras and alcove geometry*, LMS J. Comput. Math. 6 (2003), 249–296. [Pages 1, 5, 25, 26.]

[Pla13] D. Plaza, *Graded decomposition numbers for the blob algebra*, J. Algebra 394 (2013), 182–206. [Pages 11, 25.]

[PRH14] D. Plaza and S. Ryom-Hansen, *Graded cellular bases for Temperley–Lieb algebras of type A and B*, J. Algebraic Combin. 40 (2014), no. 1, 137–177. [Page 11.]

[Rou97] W. Soergel, *Kazhdan–Lusztig polynomials and a combinatoric for tilting modules*, Represent. Theory 1 (1997), 83–114 (electronic). [Pages 8, 10, 12, 14.]

[Web13] B. Webster, *Rouquier’s conjecture and diagrammatic algebra*, arXiv:1306.0074, preprint. [Pages 1, 14, 19, 22.]

E-mail address: Chris.Bowman.2@city.ac.uk

E-mail address: a.g.cox@city.ac.uk

DEPARTMENT OF MATHEMATICS, CITY UNIVERSITY LONDON, NORTHAMPTON SQUARE, LONDON, EC1V 0HB, UK

E-mail address: l.speyer@qmul.ac.uk

QUEEN MARY UNIVERSITY OF LONDON, MILE END ROAD, LONDON E1 4NS, UK