Long colimits of topological groups IV: Spaces with socks

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Abstract

The group of compactly supported homeomorphisms on a Tychonoff space can be topolo-
gized in a number of ways, including as a colimit of homeomorphism groups with a given
compact support, or as a subgroup of the homeomorphism group of its Stone-Čech compac-
tification. A space is said to have the Compactly Supported Homeomorphism Property (CSHP)
if these two topologies coincide. The authors develop techniques for showing that products of
certain spaces with CSHP, such as the Closed Long Ray and the Long Line, have CSHP again.

1. Introduction

Given a compact space $K$, it is well known that the homeomorphism group $\text{Homeo}(K)$ is a topo-
logical group with the compact-open topology ([1]). If $X$ is assumed to be only Tychonoff, then for
every compact subset $K \subseteq X$, the group $\text{Homeo}_K(X)$ of homeomorphisms supported in $K$ (i.e.,
identity on $X \setminus K$) is a topological group with the compact-open topology; however, the full home-

omorphism group $\text{Homeo}(X)$ equipped with the compact-open topology need not be a topological
group ([6]). Nevertheless, $\text{Homeo}(X)$ can be turned into a topological group by embedding it
into $\text{Homeo}(\beta X)$, the homeomorphism group of the Stone-Čech compactification of $X$. The latter
topology has also been studied under the name of zero-cozero topology ([12], [5]).

For a Tychonoff space $X$, let $\mathcal{K}(X)$ denote the poset of compact subsets of $X$ ordered by
inclusion. In light of the foregoing, the group $\text{Homeo}_{\text{cpt}}(X) := \bigcup_{K \in \mathcal{K}(X)} \text{Homeo}_K(X)$ of the com-
pactly supported homeomorphisms of $X$ admits seemingly different topologies:

(a) the finest topology making all inclusions $\text{Homeo}_K(X) \longrightarrow \text{Homeo}_{\text{cpt}}(X)$ continuous (i.e., the
colimit in the category of topological spaces and continuous maps); and

(b) the topology induced by $\text{Homeo}(\beta X)$.

A space $X$ is said to have the Compactly Supported Homeomorphism Property (CSHP) if these
topologies coincide ([3], [4]). In a previous work, the authors gave a complete characterization of a
finite product of ordinals having CSHP ([4, Theorem A]). Having CSHP is not productive, though.

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For example, while $\omega_1$ and $\omega_2$ equipped with the order topology have CSHP ([3], Theorem D(c)), their product $\omega_1 \times \omega_2$ does not ([4], Example 1).

In this paper, we develop techniques allowing one to show that products of certain spaces with CSHP have CSHP again. As an application, we prove the following result.

**Theorem A.** Let $C$ be a compact metrizable space, let $\mathbb{L}_{\geq 0}$ denote the Closed Long Ray, and $\mathbb{L}$ denote the Long Line. For every $n, m, l < \omega$, the product space $\mathbb{L}_0 \times (\mathbb{L}_{\geq 0})^m \times (\omega_1)^l \times C$ has CSHP.

For the definition of $\mathbb{L}_{\geq 0}$ and $\mathbb{L}$, see Section 2. The proof of Theorem A requires introducing the notion of a space having a $\kappa$-sock, which is productive (Proposition 2.3(b)), and under suitable conditions, implies CSHP. To that end, we recall some terminology and notation.

Given a cardinal $\kappa$, a directed set $(\mathbb{I}, \leq)$ is $\kappa$-long if every subset $C \subseteq \mathbb{I}$ with $|C| \leq \kappa$ has an upper bound in $\mathbb{I}$. One says that $\mathbb{I}$ is long if it is $\aleph_0$-long. Recall that a poset $(\mathbb{I}, \leq)$ is an $\omega$-cpo if every non-decreasing sequence $\{\alpha_k\}_{k<\omega}$ in $\mathbb{I}$ has a supremum.

All spaces in this paper are assumed to be Tychonoff. We denote by $\mathcal{C}(X, Y)$ the space of continuous functions from a space $X$ to a space $Y$, equipped with the compact-open topology.

**Definition 1.1.** Let $\kappa$ be an infinite cardinal. A $\kappa$-sock on a space $X$ is a family $\{p_\alpha\}_{\alpha \in \mathbb{I}}$ of retracts on $X$, indexed by a $\kappa$-long $\omega$-cpo $(\mathbb{I}, \leq)$ satisfying the following properties:

(S1) $K_\alpha := p_\alpha(X)$ is compact for every $\alpha \in \mathbb{I}$;
(S2) for every $\alpha, \beta \in \mathbb{I}$ such that $\alpha \leq \beta$, one has $p_\beta p_\alpha = p_\alpha$ (or equivalently, $K_\alpha \subseteq K_\beta$);
(S3) $\{K_\alpha\}_{\alpha \in \mathbb{I}}$ is cofinal in the family $\mathcal{K}(X)$ of compact subsets of $X$ (ordered by inclusion);
(S4) for every non-decreasing sequence $\{\alpha_k\}_{k<\omega}$ with $\gamma = \sup_{k<\omega} \alpha_k$ in $\mathbb{I}$, one has $\lim_{k<\omega} p_{\alpha_k} = p_\gamma$ in $\mathcal{C}(X, K_\gamma)$.

In the special case where $\kappa = \aleph_0$, we say that $\{p_\alpha\}_{\alpha \in \mathbb{I}}$ is a sock. Note that if $\kappa_1 \geq \kappa_2$ are infinite cardinals, then every $\kappa_1$-sock is also a $\kappa_2$-sock. Furthermore, if $X$ has a sock, then $(\mathcal{K}(X), \subseteq)$ is long, and in particular, every countable subset of $X$ is contained in a compact subset, and so $X$ is pseudocompact.

Recall that the tightness $t(X, \mathcal{F})$ of a topological space $(X, \mathcal{F})$ is the smallest cardinal $\kappa$ such that every point $p$ in the closure of a subset $A \subseteq X$ is in the closure of a subset $C \subseteq A$ with $|C| \leq \kappa$. One says that a space is $\kappa$-tight if $t(X, \mathcal{F}) \leq \kappa$. Recall further that the compact weight of a space $X$ is $kw(X) := \sup \{w(K) \mid K \in \mathcal{K}(X)\}$, where $w(K)$ is the weight of $K$ ([2], [8]). (Following [7] 1.7.12), if $K$ is finite, one puts $kw(K) = w(K) = \aleph_0$.

**Theorem B.** Let $\kappa$ be an infinite cardinal. Suppose that $X$ is a locally compact space that has a $\kappa$-sock and $kw(X) \leq \kappa$. Then $X$ has CSHP and $\text{Homeo}_{cpt}(X)$ is $\kappa$-tight.

The paper is structured as follows. In Section 2 we present basic properties of spaces with $\kappa$-socks, such as preservation under finite disjoint unions, arbitrary products, and certain quotients. We also show that a continuous map from a space with a sock into a metrizable space is determined by its values on a given compact subset (Theorem 2.10). In Section 3 we prove Theorems A and B.
2. Basic Properties

**Proposition 2.1.** Let \( \kappa \) be an infinite cardinal, and let \((\mathbb{I}, \leq)\) be a \( \kappa \)-long totally ordered set in which every subset bounded from above has a supremum in \( \mathbb{I} \). Let \( X \) denote \( \mathbb{I} \) with the order topology. Then \( X \) has a \( \kappa \)-sock indexed by \((\mathbb{I}, \leq)\).

**Proof.** Since the empty set is bounded, it has a supremum in \( \mathbb{I} \), and so \( \mathbb{I} \) has a minimum. Put \( 0 := \min \mathbb{I} \). Since \((\mathbb{I}, \leq)\) is \( \kappa \)-long, every countable subset has an upper bound; consequently, by our assumption, every countable subset has a supremum. Thus, \((\mathbb{I}, \leq)\) is an \( \omega \)-cpo. For every \( \alpha \in \mathbb{I} \), put \( p_\alpha(x) = \min(x, \alpha) \). We show that \( \{p_\alpha\}_{\alpha \in \mathbb{I}} \) is a \( \kappa \)-sock on \( X \).

(S1) One has \( p_\alpha(X) = [0, \alpha] \), which is compact ([7] 3.12.3(a)).

(S2) Since \( \min(\min(x, \alpha), \beta) = \min(x, \min(\alpha, \beta)) \), it follows that \( p_\beta p_\alpha = p_\alpha \) whenever \( \alpha \leq \beta \).

(S3) If \( K \subseteq X \) is compact, then \( K \) has an upper bound, say \( \alpha \in \mathbb{I} \), and so \( K \subseteq [0, \alpha] = K_\alpha \).

(S4) It follows from (S1) that \( X \) is locally compact. The map \( \min : X \times X \to X \) is continuous. Thus, \( p : X \to \mathcal{C}(X, X) \), defined by \( p(\alpha) = p_\alpha \), is continuous (cf. [7] 3.4.8). Therefore, if \( \{\alpha_k\}_{k<\omega} \) is a non-decreasing sequence in \( \mathbb{I} \), then \( p(\sup \alpha_k) = p(\lim \alpha_k) = \lim p(\alpha_k) \).

**Corollary 2.2.** Let \( \kappa \) be an infinite cardinal, and let \( \delta \) be an ordinal. If \( \delta \) is \( \kappa \)-long, then the space \( X = \delta \) equipped with the order topology has a \( \kappa \)-sock. In particular, \( \omega_1 \) has a sock.

The product \( \mathbb{L}_{\geq 0} := \omega_1 \times [0, 1) \) equipped with order topology generated by the lexicographic order is called the **Closed Long Ray**. We identify \( \omega_1 \) with the closed cofinal subset \( \omega_1 \times \{0\} \) of \( \mathbb{L}_{\geq 0} \).

**Corollary 2.3.** The **Closed Long Ray** \( \mathbb{L}_{\geq 0} \) with the order topology has a sock and has a countable compact weight.

**Proof.** Since \((\omega_1, \leq)\) is a long poset (with the lexicographic order), so is \((\mathbb{L}_{\geq 0}, \leq)\). Every interval \([0, x)\) in \((\mathbb{L}_{\geq 0}, \leq)\) in the order topology is homeomorphic to the compact interval \([0, 1]\) ([9] 1.10). (In particular, every compact subset of \( \mathbb{L}_{\geq 0} \) has a countable weight.) Therefore, every subset bounded from above has a supremum in \( \mathbb{L}_{\geq 0} \), and the statement follows by Proposition 2.1.

**Proposition 2.4.** Let \( \kappa \) be an infinite cardinal, let \( \{X_j\}_{j \in J} \) be a family of topological spaces, and put \( X := \prod_{j \in J} X_j \). Then:

(a) \( kw(X) = \sup_{j \in J} kw(X_j) \); and

(b) if \( J \) is finite and \( X_j \) has a \( \kappa \)-sock for every \( j \in J \), then \( X \) has a \( \kappa \)-sock.

**Proof.** (a) Clearly, \( kw(X_j) \leq kw(X) \) for every \( j \in J \), and so \( \sup_{j \in J} kw(X_j) \leq kw(X) \). We show now the reverse inequality. Let \( K \subseteq X \) be compact. Then there is a finite subset \( F \subseteq J \) such that \( K \subseteq \prod_{j \in F} X_j \), and \( K = \prod_{j \in F} K \cap X_j \). Therefore,

\[
 w(K) \leq \prod_{j \in F} w(K \cap X_j) \leq \sup_{j \in J} kw(X_j).
\]
(b) For every \( j \in J \), let \( \{ p^{(j)}_\alpha \}_{\alpha \in I_j} \) be a \( \kappa \)-sock on \( X_j \). Put \( \mathbb{I} := \prod_{j \in J} I_j \) equipped with the coordinatewise order, that is, \( (\alpha^{(j)}) \leq (\beta^{(j)}) \) if \( \alpha^{(j)} \leq \beta^{(j)} \) for every \( j \in J \). Then \( (\mathbb{I}, \leq) \) is a \( \kappa \)-long \( \omega \)-cpo, because upper bounds and suprema can be calculated coordinatewise. For \( \alpha = (\alpha^{(j)}) \in \mathbb{I} \), put \( p_\alpha := \prod_{j \in J} p^{(j)}_{\alpha^{(j)}} \). We verify that \( \{ p_\alpha \}_{\alpha \in \mathbb{I}} \) satisfies the conditions of a \( \kappa \)-sock on \( X \).

(S1) For every \( (\alpha^{(j)}) \in \mathbb{I} \), the image \( K_\alpha := p_\alpha(X) = \prod_{j \in J} p^{(j)}_{\alpha^{(j)}}(X_j) \) is compact, because \( p^{(j)}_{\alpha^{(j)}}(X_j) \) is compact for every \( j \in J \) and \( J \) is finite.

(S2) If \( \alpha = (\alpha^{(j)}) \leq \beta = (\beta^{(j)}) \in \mathbb{I} \), then \( \alpha^{(j)} \leq \beta^{(j)} \) for every \( j \in J \), and so \( p^{(j)}_{\beta^{(j)}} p^{(j)}_{\alpha^{(j)}} = p^{(j)}_{\alpha^{(j)}} \) for every \( j \in J \). Thus, \( p_\beta p_\alpha = p_\alpha \).

(S3) If \( K \subseteq X \) is compact, then \( K \cap X_j \) is compact in \( X_j \) for every \( j \in J \), and so there is \( \alpha^{(j)} \in I_j \) such that \( K \cap X_j \subseteq K_{\alpha^{(j)}} \). Thus, for \( \alpha = (\alpha^{(j)}) \), one has \( K \subseteq \prod_{j \in J} K_{\alpha^{(j)}} = K_\alpha \). This shows that \( \{ K_\alpha \}_{\alpha \in \mathbb{I}} \) is cofinal in \( \mathcal{K}(X) \).

(S4) Let \( \{ \alpha_k \}_{k<\omega} \) be a non-decreasing sequence with \( \gamma = \sup_{k<\omega} \alpha_k \) in \( \mathbb{I} \). Since \( \{ p^{(j)}_{\alpha_k} \}_{\alpha_k \in I_j} \) is a \( \kappa \)-sock on \( X_j \), one has \( \lim_{k} p^{(j)}_{\alpha_k} = p^{(j)}_{\gamma^{(j)}} \) in \( \mathcal{C}(X_j, K_{\gamma^{(j)}}) \) for every \( j \in J \). Thus, \( \lim_{k} p^{(j)}_{\alpha_k} = p^{(j)}_{\gamma^{(j)}} \) in \( \mathcal{C}(X_j, K_{\gamma}) \) for every \( j \in J \), and therefore \( \lim_{k} p_\alpha = p_\gamma \) in \( \mathcal{C}(X, K_{\gamma}) \), as desired. \( \square \)

A map \( f : X \to Y \) between spaces is a \( k \)-covering if for every \( C \in \mathcal{K}(Y) \) there exists \( K \in \mathcal{K}(X) \) such that \( C \subseteq f(K) \) ([\( \prod \]) p. 21).

**Proposition 2.5.** Let \( f : X \to Y \) be a continuous map that is a \( k \)-covering. Then \( kw(Y) \leq kw(X) \).

**Proof.** Let \( C \in \mathcal{K}(Y) \). Since \( f \) is a \( k \)-covering, there is \( K \in \mathcal{K}(X) \) such that \( C \subseteq f(K) \). One has \( w(f(K)) \leq w(K) \) ([\( \mathbb{Z} \]) 3.3.7]), and therefore \( w(C) \leq w(K) \leq kw(K) \leq kw(X) \), as desired. \( \square \)

**Proposition 2.6.** Let \( \kappa \) be an infinite cardinal, and let \( q : X \to Y \) be a quotient map that is a \( k \)-covering. The following statements are equivalent for a \( \kappa \)-sock \( \{ p_\alpha \}_{\alpha \in \mathbb{I}} \) on \( X \):

(i) for every \( \alpha \in \mathbb{I} \), \( q p_\alpha \) is constant on the fibres of \( q \);

(ii) there is a unique \( \kappa \)-sock \( \{ \tilde{p}_\alpha \}_{\alpha \in \mathbb{I}} \) on \( Y \) such that \( \tilde{p}_\alpha q = q p_\alpha \) for every \( \alpha \in \mathbb{I} \).

\[
\begin{array}{ccc}
X & \overset{p_\alpha}{\longrightarrow} & X \\
\downarrow{q} & & \downarrow{q} \\
Y & \overset{\exists \tilde{p}_\alpha}{\longrightarrow} & Y
\end{array}
\]

**Proof.** It is clear that (ii) implies (i), and so we prove only that (i) implies (ii). Since for every \( \alpha \in \mathbb{I} \), \( q p_\alpha : X \to Y \) is constant on the fibres of \( q \) (which is a quotient map), \( q p_\alpha \) induces a unique continuous map \( \tilde{p}_\alpha : Y \to Y \) such that \( \tilde{p}_\alpha q = q p_\alpha \). We show that \( \{ \tilde{p}_\alpha \}_{\alpha \in \mathbb{I}} \) is a \( \kappa \)-sock on \( Y \).

(S1) The image \( \tilde{K}_\alpha = \tilde{p}_\alpha(Y) = \tilde{p}_\alpha(q(X)) = q(p_\alpha(X)) = q(K_\alpha) \)
is compact, because \( K_\alpha = p_\alpha(X) \) is compact.

(S2) Let \( \alpha, \beta \in \mathbb{I} \) be such that \( \alpha \leq \beta \). Since \( \{p_\alpha\}_{\alpha \in \mathbb{I}} \) is a \( \kappa \)-sock on \( X \), one has \( p_\beta p_\alpha = p_\alpha \). Thus,
\[
\tilde{p}_\beta \tilde{p}_\alpha q = \tilde{p}_\beta q p_\alpha = q p_\beta p_\alpha = q p_\alpha = \tilde{p}_\alpha q,
\]
and so \( \tilde{p}_\beta \tilde{p}_\alpha = \tilde{p}_\alpha \), because \( q \) is surjective. In particular, \( \tilde{p}_\alpha \) is a retract.

(S3) Let \( C \) be a compact subset of \( Y \). Since \( q \) is a \( k \)-covering, there is a compact subset \( K \) of \( X \) such that \( C \subseteq q(K) \). There is \( \alpha \in \mathbb{I} \) such that \( K \subseteq K_\alpha = p_\alpha(X) \), because \( \{p_\alpha\}_{\alpha \in \mathbb{I}} \) is a \( \kappa \)-sock on \( X \). Thus,
\[
C \subseteq q(K) \subseteq q(p_\alpha(X)) = \tilde{p}_\alpha(q(X)) = \tilde{p}_\alpha(Y) = \tilde{K}_\alpha.
\]

(S4) Let \( \{\alpha_k\}_{k<\omega} \) be a non-decreasing sequence with \( \gamma = \sup \alpha_k \) in \( \mathbb{I} \). Since \( \{p_\alpha\}_{\alpha \in \mathbb{I}} \) is a \( \kappa \)-sock on \( X \), one has \( \lim_{k<\omega} p_{\alpha_k} = p_\gamma \) in \( \mathcal{C}(X, K_\gamma) \). Thus,
\[
\lim_{k<\omega} \tilde{p}_{\alpha_k} q = \lim_{k<\omega} q p_{\alpha_k} = q p_\gamma = \tilde{p}_\gamma q
\]
in \( \mathcal{C}(X, q(K_\gamma)) = \mathcal{C}(X, \tilde{K}_\gamma) \), because composition is continuous in the compact-open topology ([7 3.4.2]). The natural map \( q_* : \mathcal{C}(Y, \tilde{K}_\gamma) \rightarrow \mathcal{C}(X, \tilde{K}_\gamma) \) defined by \( f \mapsto f q \) is not only continuous, but also an embedding, because \( q \) is a \( k \)-covering ([10 2.12(a)]; see also [11 2.2.6(c)]). By (6), \( \lim_{k<\omega} q_*(\tilde{p}_{\alpha_k}) = q_*(\tilde{p}_\gamma) \), and therefore \( \lim_{k<\omega} \tilde{p}_{\alpha_k} = \tilde{p}_\gamma \) in \( \mathcal{C}(Y, K_\gamma) \), as desired.

The Long Line \( \mathbb{L} \) is obtained by gluing together two copies of the Closed Long Ray at the boundary points 0. \textit{Par abus de langage}, we refer to the image of the point 0 in the quotient as 0 again.

**Corollary 2.7.** The Long Line \( \mathbb{L} \) has a sock and has a countable compact weight.

**Proof.** Let \( X \) denote the disjoint union of two copies of the Closed Long Ray \( \mathbb{L}_{\geq 0} \). By definition, there is a quotient map \( q : X \rightarrow \mathbb{L} \), whose fibres are singletons with the exception of the point 0, where \( q^{-1}(\{0\}) \) contains two points. By Corollary 2.3, the Closed Long Ray \( \mathbb{L}_{\geq 0} \) has a sock defined by \( x \mapsto \min(x, \alpha) \). By Proposition 2.4, this induces a sock \( \{p_\alpha\}_{\alpha \in \mathbb{I}} \) on \( X \), and it can easily be checked that \( q p_\alpha \) are constant on the fibres of \( q \). Therefore, by Proposition 2.6, \( \{p_\alpha\}_{\alpha \in \mathbb{I}} \) induces a unique sock \( \{\tilde{p}_\alpha\}_{\alpha \in \mathbb{I}} \) on \( \mathbb{L} \) such that \( \tilde{p}_\alpha q = q p_\alpha \) for every \( \alpha \in \mathbb{I} \). Lastly, by Proposition 2.5, \( kw(\mathbb{L}) \leq kw(X) = \kappa_0 \).

**Proposition 2.8.** Let \( \kappa \) be an infinite cardinal, let \( \{X_j\}_{j \in J} \) be a family of non-empty topological spaces, and put \( X := \prod_{j \in J} X_j \). Then:

(a) \( \sup_{j \in J} kw(X_j) \leq kw(X) \leq |J| \sup_{j \in J} kw(X_j) \); and

(b) if \( X_j \) has a \( \kappa \)-sock for every \( j \in J \), then \( X \) has a \( \kappa \)-sock.

**Proof.** Let \( \pi_j : X \rightarrow X_j \) denote the canonical projection for every \( j \in J \).
(a) Since $X_j$ is non-empty for every $j \in J$, each $X_j$ can be embedded into the product $X$, and so $kw(X_j) \leq kw(X)$. On the other hand, if $K \subseteq X$ is compact, then $K \subseteq \prod_{j \in J} \pi_j(K)$. Thus, by [7, 2.3.13],

$$w(K) \leq |J| \sup_{j \in J} w(\pi_j(K)) \leq |J| \sup_{j \in J} kw(X_j).$$  \hspace{1cm} (7)

(b) For every $j \in J$, let $\{p_{\alpha}^{(j)}\}_{\alpha \in I_j}$ be a $\kappa$-sock on $X_j$. Put $I := \prod_{j \in J} I_j$ equipped with the coordinatewise order, that is, $(\alpha^{(j)}) \leq (\beta^{(j)})$ if $\alpha^{(j)} \leq \beta^{(j)}$ for every $j \in J$. Then $(I, \leq)$ is a $\kappa$-long $\omega$-cpo, because upper bounds and suprema can be calculated coordinatewise. For $\alpha = (\alpha^{(j)}) \in I$, put $p_{\alpha} := \prod_{j \in J} p_{\alpha^{(j)}}^{(j)}$. We verify that $\{p_{\alpha}\}_{\alpha \in I}$ satisfies the conditions of a $\kappa$-sock on $X$.

(S1) For every $(\alpha^{(j)}) \in I$, the image $K_{\alpha} := p_{\alpha}(X) = \prod_{j \in J} p_{\alpha^{(j)}}^{(j)}(X_j)$ is compact, because $p_{\alpha^{(j)}}^{(j)}(X_j)$ is compact for every $j \in J$.

(S2) If $\alpha = (\alpha^{(j)}) \leq (\beta^{(j)}) \in I$, then $\alpha^{(j)} \leq \beta^{(j)}$ for every $j \in J$, and so $p_{\beta^{(j)}}^{(j)} p_{\alpha^{(j)}}^{(j)} = p_{\alpha^{(j)}}^{(j)}$ for every $j \in J$. Thus, $p_{\beta} p_{\alpha} = p_{\alpha}$.

(S3) If $K \subseteq X$ is compact, then $\pi_j(K)$ is compact in $X_j$ for every $j \in J$, and so there is $\alpha^{(j)} \in I_j$ such that $\pi_j(K) \subseteq K_{\alpha^{(j)}}$. Thus, for $\alpha = (\alpha^{(j)})$, one has $K \subseteq \prod_{j \in J} K_{\alpha^{(j)}} = K_{\alpha}$. This shows that $\{K_{\alpha}\}_{\alpha \in I}$ is cofinal in $\mathcal{K}(X)$.

(S4) Let $\{\alpha_k\}_{k < \omega}$ be a non-decreasing sequence with $\gamma = \sup_{k < \omega} \alpha_k$ in $I$. Since $\{p_{\alpha}^{(j)}\}_{\alpha \in I_j}$ is a $\kappa$-sock on $X_j$, one has $\lim_{k} p_{\alpha_k^{(j)}}^{(j)} = p_{\gamma^{(j)}}^{(j)}$ in $\mathcal{C}(X_j, K_{\gamma^{(j)}})$ for every $j \in J$. Thus, $\lim_k \pi_j p_{\alpha_k} = \pi_j p_\gamma$ in $\mathcal{C}(X, K_{\gamma})$ for every $j \in J$, and therefore $\lim_k p_{\alpha_k} = p_\gamma$ in $\mathcal{C}(X, K_\gamma)$, as desired. \hfill \Box

Proposition 2.8 combined with Corollaries 2.2, 2.3, and 2.7 yield the following conclusion.

**Corollary 2.9.** Let $C$ be a compact space. For every $n, m, l < \omega$, the product space

$$\mathbb{L}^n \times (\mathbb{L}_{\geq 0})^m \times (\omega_1)^l \times C$$  \hspace{1cm} (8)

has a sock. Furthermore, if $C$ is metrizable, then $\mathbb{L}^n \times (\mathbb{L}_{\geq 0})^m \times (\omega_1)^l \times C$ has a countable compact weight. \hfill \Box

It is well known that every continuous function from the Closed Long Ray $\mathbb{L}_{\geq 0}$ into the real line is eventually constant; however, only spaces whose Stone-Čech remainder is a singleton can satisfy such a strong property. For example, not every continuous real valued function on the Long Line $\mathbb{L}$ is eventually constant. Nevertheless, spaces that have a sock satisfy the property that every continuous function into a metrizable space is determined by its value on some compact subset.

**Theorem 2.10.** Let $X$ be a locally compact space with a sock $\{p_\alpha\}_{\alpha \in I}$ such that, in addition, $p_\alpha p_\beta = p_\alpha$ for every $\alpha \leq \beta$. Let $g : X \to M$ be a continuous map into a metrizable space $M$. Then there is $\alpha_0 \in I$ such that $g = gp_{\alpha_0}$. 

\hfill \Box
In order to prove Theorem 2.10 we need a lemma.

**Lemma 2.11.** Let $(\mathbb{I}, \leq)$ be a long $\omega$-cpo, and let $M$ be a metrizable space. Suppose that $f : \mathbb{I} \rightarrow M$ is a map such that $f(\sup \alpha_k) = \lim f(\alpha_k)$ whenever $\{\alpha_k\}$ is a non-decreasing sequence in $\mathbb{I}$. Then $f$ is eventually constant, that is, there is $\alpha_0 \in \mathbb{I}$ such that $f(\beta) = f(\alpha_0)$ for every $\beta \geq \alpha_0$.

**Proof.** Step 1. Suppose that $M = [0,1]$ equipped with the real topology. Put $f_+(\alpha) := \sup f(\gamma)$ and $f_-(\alpha) := \inf f(\gamma)$. Since $f_+$ and $f_-$ are monotone and $\mathbb{I}$ is long, there is $\alpha_0 \in \mathbb{I}$ such that $f_+(\beta) = f_+(\alpha_0)$ and $f_-(\beta) = f_-(\alpha_0)$ for every $\beta \geq \alpha_0$ ([7, 2.10]). Put $a := f_-(\alpha_0)$ and $b := f_+(\alpha_0)$. Clearly, $a \leq b$, and it suffices to show that $a = b$.

We define $\{\alpha_k\}_{k<\omega}$ recursively as follows. We pick $\alpha_{k+1} \geq \alpha_k$ such that $f(\alpha_{k+1}) \geq b - \frac{1}{k+1}$ if $k$ is even and $f(\alpha_{k+1}) < a + \frac{1}{k+1}$ if $k$ is odd. (Such $\alpha_{k+1}$ exists, because $f_-(\alpha_k) = a$ and $f_+(\alpha_k) = b$.) Since $\mathbb{I}$ is an $\omega$-cpo, $\sup \alpha_k$ exists, and by our assumption, $\lim f(\alpha_k) = f(\sup \alpha_k)$ also exists. Therefore, $b \leq a$, and $f(\beta) = a$ for every $\beta \geq \alpha_0$.

Step 2. Suppose that $M$ is a discrete space. Assume that $f$ is not eventually constant. Then for every $\beta \in \mathbb{I}$ there is $\alpha \geq \beta$ such that $f(\alpha) \neq f(\beta)$. Thus, we can construct recursively a non-decreasing sequence $\{\alpha_k\}_{k<\omega}$ such that $f(\alpha_{k+1}) \neq f(\alpha_k)$. Since $\mathbb{I}$ is an $\omega$-cpo, $\sup \alpha_k$ exists, and by our assumption, $\lim f(\sup \alpha_k) = f(\sup \alpha_k)$; however, $\lim f(\alpha_k)$ does not exist, because $M$ is discrete and $f(\alpha_{k+1}) \neq f(\alpha_k)$ for every $k$. This contradiction shows that $f$ is eventually constant.

Step 3. Suppose that $M = J(\lambda)$, the hedgehog space of weight $\lambda$, obtained by gluing together $\lambda$ many copies of $[0,1]$ at the point 0, equipped with the metric topology (see [7, 4.1.5]). Let $d$ denote the metric on $J(\lambda)$, and define $g : \mathbb{I} \rightarrow [0,1]$ by putting $g(\alpha) = d(f(\alpha),0)$. Since the metric is continuous, $g(\sup \alpha_k) = \lim g(\alpha_k)$ whenever $\{\alpha_k\}$ is a non-decreasing sequence in $\mathbb{I}$. Thus, by Step 1, there is $\alpha_0 \in \mathbb{I}$ such that $g(\beta) = g(\alpha_0)$ for every $\beta \geq \alpha_0$. Put $r := g(\alpha_0) = d(f(\alpha_0),0)$. Let $D = \{x \in J(\lambda) | d(x,0) = r\}$. Then $D$ is discrete and $f(\beta) \in D$ for every $\beta \geq \alpha_0$. Thus, by Step 2, $f$ is eventually constant.

Step 4. Let $M$ be any metrizable space. Put $\lambda = w(M)$. By Kowalsky’s Hedgehog Theorem, $M$ embeds as a subspace into $J(\lambda)_{\omega}$ (see [7, 4.4.9]). Let $\pi_n : J(\lambda)_{\omega} \rightarrow J(\lambda)$ denote the $n$-th projection. By Step 3, each map $\pi_n f$ is eventually constant, and so there is $\alpha_n \in \mathbb{I}$ such that $\pi_n f(\beta) = \pi_n f(\alpha_n)$ for every $\beta \geq \alpha_n$. Since $\mathbb{I}$ is long, there is $\gamma \in \mathbb{I}$ such that $\gamma \geq \alpha_n$ for every $n$. Therefore, one has $\pi_n f(\beta) = \pi_n f(\gamma)$ for every $\beta \geq \gamma$ and every $n$, that is, $f(\beta) = f(\gamma)$, as desired.

We are now ready to prove Theorem 2.10.

**Proof of Theorem 2.10.** Let $d$ be a metric on $M$, and let $\tilde{d}$ denote the associated uniform metric on the function space $C(X, M)$. Consider $f : \mathbb{I} \rightarrow (C(X, M), \tilde{d})$ defined by $f(\alpha) = gp_{\alpha}$. We show that $f$ satisfies the condition of Lemma 2.11.

Let $\{\alpha_k\}$ be a non-decreasing sequence in $\mathbb{I}$, and put $\gamma = \sup \alpha_k$. By property (S4), $\lim_k p_{\alpha_k} = p_{\gamma}$ in $C(X, K_{\gamma}) \subseteq C(X, X)$ (where the latter is equipped with the compact-open topology). Since composition is continuous with respect to the compact-open topology, $\lim_k gp_{\alpha_k} = gp_{\gamma}$ in $C(X, M)$, and in particular, $gp_{\alpha_k}$ converges uniformly to $gp_{\gamma}$ on $K_{\gamma}$. Therefore, $f(\alpha_k) = gp_{\alpha_k} = gp_{\alpha_k} p_{\gamma}$ converges uniformly to $gp_{\gamma} = f(\gamma)$. This shows that $\lim f(\alpha_k) = f(\sup \alpha_k)$ in the metric space $(C(X, M), \tilde{d})$. Hence, by Lemma 2.11, there is $\alpha_0 \in \mathbb{I}$ such that $f(\alpha) = f(\alpha_0)$ for every $\alpha \geq \alpha_0$. 


Lastly, let \( x \in X \). There is \( \alpha \in \mathbb{I} \) such that \( x \in K_\alpha \). Since \( \mathbb{I} \) is directed, without loss of generality, we may assume that \( \alpha \geq \alpha_0 \). Thus,

\[
g(x) = g(p_\alpha(x)) = f(\alpha)(x) = f(\alpha_0)(x) = g(p_{\alpha_0}(x)),
\]

as desired. \( \square \)

### 3. Homeomorphism groups of spaces with socks

In this section, we prove Theorems \( \mathbf{A} \) and \( \mathbf{B} \). Theorem \( \mathbf{B} \) combined with Corollary \( 2.9 \) yields Theorem \( \mathbf{A} \) so it suffices to prove Theorem \( \mathbf{B} \). We start off with a well-known lemma, whose proof is provided here only for the sake of completeness.

**Lemma 3.1.** If \( C \) and \( K \) are compact spaces, then \( t(\mathcal{C}(C,K)) \leq w(K) \).

**Proof.** Since \( K \) is compact, its uniformity admits a base \( \{U_i\}_{i \in I} \) of entourages of the diagonal such that \( |I| \leq w(K) \) ([7, 8.3.13]). The compact-open topology on \( \mathcal{C}(C,K) \) is induced by uniform convergence on \( C \) ([7, 8.2.7]), and has a base of the form \( \{U_i\}_{i \in I} \), consisting of entourages of the diagonal of the form

\[
\tilde{U}_i := \{(f,g) \in \mathcal{C}(C,K) \times \mathcal{C}(C,K) \mid (f(x),g(x)) \in U_i \text{ for all } x \in C\}.
\]

In particular, the topology of \( \mathcal{C}(C,K) \) has a base of cardinality at most \(|I|\) at each point, and thus the character \( \chi(\mathcal{C}(C,K)) \) of the space \( \mathcal{C}(C,K) \) is at most \( w(K) \). Therefore,

\[
t(\mathcal{C}(C,K)) \leq \chi(\mathcal{C}(C,K)) \leq w(K),
\]

as desired. \( \square \)

Recall that we denote by \( \text{supp} f := \text{cl}_X \{x \in X \mid f(x) \neq x\} \) the support of a homeomorphism of a space \( X \).

**Theorem 3.** Let \( \kappa \) be an infinite cardinal. Suppose that \( X \) is a locally compact space that has a \( \kappa \)-sock and \( kw(X) \leq \kappa \). Then \( X \) has CSHP and \( \text{Homeo}^{\text{cpt}}(X) \) is \( \kappa \)-tight.

**Proof.** We first show that \( \text{Homeo}^{\text{cpt}}(X) \) equipped with the topology induced by \( \text{Homeo}(\beta X) \) is \( \kappa \)-tight. Let \( A \subseteq \text{Homeo}^{\text{cpt}}(X) \) and let \( g \in \overline{A} \). Without loss of generality, we may assume that \( g = \text{id}_X \). We construct \( D \subseteq A \) such that \(|D| \leq \kappa \) and \( \text{id}_X \in \overline{D} \).

Let \( D_{-1} \subseteq A \) be a non-empty subset such that \(|D_{-1}| \leq \kappa \), and let \( \alpha_{-1} \in \mathbb{I} \) be an arbitrary element. We construct inductively a non-decreasing subset \( \{\alpha_m\}_{m \in \omega} \) in \( \mathbb{I} \) such that for every \( m \in \omega \):

(I) \(|D_m| \leq \kappa \);

(II) \( \text{supp} f \subseteq K_{\alpha_m} \) for every \( f \in D_{m-1} \); and

(III) \( \beta p_{\alpha_m} \in (\beta p_{\alpha_m})^*(D_m) \), where \( (\beta p_{\alpha_m})^* : \mathcal{C}(\beta X, \beta X) \to \mathcal{C}(\beta X, K_{\alpha_m}) \) is the map defined by \( h \mapsto \beta p_{\alpha_m} h \), which is continuous ([7, 3.4.2]).
Suppose that $D_m$ and $\alpha_m$ have been constructed. Since the family $\{K_{\alpha}\}_{\alpha \in I}$ is cofinal in $\mathcal{K}(X)$, for every $f \in D_m$ there is $\alpha(f) \in I$ such that $\text{supp} f \subseteq K_{\alpha(f)}$. One has $|\{\alpha(f) \mid f \in D_m\}| \leq |D_m| \leq \kappa$, and so there is $\alpha_{m+1} \in I$ such that $\alpha(f) \leq \alpha_{m+1}$ for every $f \in D_m$ and $\alpha_{m+1} \geq \alpha_m$, because $I$ is $\kappa$-long. We observe that for every $f \in D_m$, one has
\begin{equation}
\text{supp} f \subseteq K_{\alpha(f)} \subseteq K_{\alpha_{m+1}}.
\end{equation}
One has
\begin{equation}
\beta_{\alpha_{m+1}} = (\beta_{\alpha_{m+1}})^*(\text{id}_X) \in (\beta_{\alpha_{m+1}})^*(A) \subseteq \overline{(\beta_{\alpha_{m+1}})^*(A)}.
\end{equation}
By Lemma 3.1 we have
\begin{equation}
t(\mathcal{C}(\beta X, K_{\alpha_{m+1}})) \leq \omega(K_{\alpha_{m+1}}) \leq \omega w(X) \leq \kappa.
\end{equation}
It follows from (13) that there is $E_m \subseteq A$ such that $|E_m| \leq \kappa$ and $\beta_{\alpha_{m+1}} \in (\beta_{\alpha_{m+1}})^*(E_m)$. Put $D_{m+1} := D_m \cup E_m$. It follows from the construction that $|D_{m+1}| \leq \kappa$.

Put $D := \bigcup_{m<\omega} D_m$ and $\gamma = \sup \alpha_m$. (Since $I$ is an $\omega$-cpo, $\gamma$ exists.) It follows from (I) that $|D| \leq \mathfrak{R}_0\kappa = \kappa$. We prove that $\text{id}_X \in \overline{D}$ in $\text{Homeo}_{\text{cpt}}(X)$. The topology of $\text{Homeo}_{\text{cpt}}(X)$ is induced by the topology of $\mathcal{C}(\beta X, \beta X)$, which coincides with the uniform topology; as such, it is generated by pseudometrics of the form
\begin{equation}
\tilde{d}(f, g) := \sup \{d(f(x), g(x)) \mid x \in X\},
\end{equation}
where $d$ is a pseudometric in $\beta X$. Thus, it suffices to show that for every pseudometric $d$ on $\beta X$ and $\varepsilon > 0$, there is $f \in D$ such that $d(f(x), x) < \varepsilon$ for every $x \in X$.

Let $d$ be a pseudometric on $\beta X$ and let $\varepsilon > 0$. Since $\{p_{\alpha}\}_{\alpha \in I}$ is a $\kappa$-sock, one has $\lim_{\alpha \to \gamma} p_{\alpha} = p_{\gamma}$ in $\mathcal{C}(X, K_{\gamma})$; in particular, $\{p_{\alpha}\}_{m<\omega}$ converges uniformly to $p_{\gamma}$ on the compact set $K_{\gamma}$, and so there is $m \in \omega$ such that for every $x \in K_{\gamma}$,
\begin{equation}
d(p_{\alpha_m}(x), x) = d(p_{\alpha_m}(x), p_{\gamma}(x)) < \frac{\varepsilon}{3}.
\end{equation}
By property (III), $\beta_{p_{\alpha_m}} \in (\beta p_{\alpha_m})^*(D_m)$ in $\mathcal{C}(\beta X, K_{\alpha_m})$, which also carries the uniform topology, and so there is $f \in D_m$ such that
\begin{equation}
d(p_{\alpha_m}(f(x)), p_{\alpha_m}(x)) = d(((\beta p_{\alpha_m})^*(f))(x), (\beta p_{\alpha_m})(x)) < \frac{\varepsilon}{3} \text{ for every } x \in X.
\end{equation}
By property (II) sup $f \subseteq K_{\alpha_{m+1}} \subseteq K_{\gamma}$, and so $f(x) \in K_{\gamma}$ for every $x \in K_{\gamma}$. Therefore, by (16) applied to $f(x)$ instead of $x$, one has
\begin{equation}
d(p_{\alpha_m}(f(x)), f(x)) < \frac{\varepsilon}{3} \text{ for every } x \in K_{\gamma}.
\end{equation}
Hence, by (16)–(18), for every $x \in K_{\gamma}$,
\begin{equation}
d(f(x), x) \leq d(f(x), p_{\alpha_m}(f(x))) + d(p_{\alpha_m}(f(x)), p_{\alpha_m}(x)) + d(p_{\alpha_m}(x), x) < \varepsilon.
\end{equation}
Since \( f(x) = x \) for \( x \in X \setminus K_\gamma \) and \( f \in D_m \subseteq D \), this completes the proof that \( \text{id}_X \in \overline{D} \). This shows that \( \text{Homeo}_{\text{cpt}}(X) \) is \( \kappa \)-tight.

Lastly, we prove that \( X \) has CSHP. To that end, we show that the family \( \{ \text{Homeo}_K(X) \}_{K \in \mathcal{K}(X)} \) satisfies the conditions of [3, 2.3]. First, since \( \{ K_\alpha \}_{\alpha \in \mathbb{E}} \) is \( \kappa \)-long and cofinal in \( \mathcal{K}(X) \), it follows that \( \mathcal{K}(X) \) itself is \( \kappa \)-long. Second, the inclusion \( \text{Homeo}_K(X) \to \text{Homeo}_{\text{cpt}}(X) \) is an embedding for every \( K \in \mathcal{K}(X) \). Third, we have just shown that \( t(\text{Homeo}_{\text{cpt}}(X)) \leq \kappa \). Hence, by [3, 2.3], the topology of \( \text{Homeo}_{\text{cpt}}(X) \) coincides with the colimit space topology, as desired.

\[ \square \]

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