Surface magnetisation of the 3-state Potts model with free and fixed boundaries on the square lattice

I G Enting
The Centre of Excellence for Mathematics and Statistics of Complex Systems (MASCOS), 139 Barry St.,
The University of Melbourne, VIC 3010, Australia
E-mail: ienting@unimelb.edu.au

Abstract. The finite lattice method is used to derive low-temperature series expansions for surface magnetisations for the 3-state Potts model on the square lattice, for both fixed and free boundary conditions. Combinatorial weights required for surface corrections to the free energy have been generalised to include cases with explicit surface interactions. Analysis of series for the surface correction to the bulk magnetisation is consistent with the generic scaling prediction, assuming a slow convergence comparable to that found in analysis of Potts model series for bulk properties. For free boundaries, analysis of the ordering of the surface layer confirms the specific Potts model result derived from conformal invariance.

1. Introduction
The Potts model [1] has been widely studied in its role as a generalisation of the Ising model, exhibiting diverse forms of critical behaviour. Important questions have included the order of the transition (in three dimensions) and the ability of renormalisation group techniques to correctly reproduce the known order of the transition (in two dimensions). The Potts model has also been important because of special cases such as percolation (the $q \to 1$ limit) and chromatic polynomials (the $T = 0$ antiferromagnetic case). For $q \neq 2$, exact solutions are generally only known at critical points (and only in two dimensions). In this circumstance, series expansions provide a powerful technique for analysing the models. The Finite Lattice Method (FLM) has been particularly powerful technique for obtaining series for the Potts model.

The FLM was described by de Neef and was first applied to the Potts model by de Neef and Enting [2, 3]. Since then, the FLM has been extensively refined by the present author and his co-workers and extended to other problems in lattice statistics as reviewed by Enting [4]. Additional refinements to Potts model calculations have been introduced by Arisue and co-workers [5, 6]. Extensive studies of the Potts model, using series derived by the FLM, were undertaken some years ago [7, 8, 9]. These studies addressed the issue of the order of the transition of the 3-state Potts model in three dimensions. The project included extensive studies of series in two dimensions in order to evaluate, in a known case, the capabilities of series analysis for distinguishing the order of the transition. A subsequent study by Enting and Guttmann [10] analysed critical amplitudes for the square lattice. The present paper uses the FLM to study surface (i.e. edge) properties of the 3-state Potts model on the square lattice. The applicability of the FLM for calculating surface series was demonstrated by Enting and Guttmann [11] in their study of the Ising model above the critical point. However there appears to have been little further development of surface series calculations.

The objectives of the present study are:

- to characterise the surface effects in the Potts model;
• to develop techniques for analysing boundary effects in application areas, especially percolation;
• in particular, to serve as a reference case for comparison with Monte Carlo analysis of surface phenomena in applications such as firn closure [12];
• to develop techniques for exploring the scope for extending FLM calculations through variational approximations based on adjustable boundary conditions. The aim is to combine FLM evaluation techniques with the generic variational formalism described by Vaks and Samolyuk [13]. The computational developments that underly the present paper contribute to such developments.

The contents of the remainder of this paper are: Section 2 gives definitions of the surface properties that are to be calculated and analysed, drawing on the review by Binder [14]. Section 3 gives specific details of the application of the FLM to the calculation of series for surface properties. Section 4 describes the series and the analysis of the critical exponents, noting that the surface exponents are known from a combination of scaling from bulk exponents [14] or specific Potts model results based in conformal invariance [15]. The concluding remarks in Section 5 review the extent to which series of the length obtained can give useful estimates of boundary effects in the critical region.

2. The problem
The general theory of surface effects in critical behaviour was reviewed by Binder [14].

The free energy of a system of size \( L \times L' \) is written (for large \( L, L' \)) in terms of bulk and surface contributions, \( f_b \) and \( f_s \) respectively, as:

\[
F = L \times L' \times f_b(T, H) + 2(L + L')f_s(T, H, H_1)
\]  
(2.1)

Order parameters corresponding to spin expectations are obtained as derivatives of \( F \) with respect to fields. The bulk order parameter is

\[
M = \frac{\partial}{\partial H} f_b(T, H)
\]  
(2.2)

The expectation of a surface site is obtained by differentiating with respect to \( H_1 \):

\[
M_1 = \frac{\partial}{\partial H_1} f_s(T, H, H_1)
\]  
(2.3)

The surface correction to the bulk is

\[
M_s = \frac{\partial}{\partial H} f_s(T, H, H_1)
\]  
(2.4a)

or, if (as in the calculations presented below) the field \( H \) is not applied to surface sites:

\[
M_s = \frac{\partial}{\partial H} f_s(T, H, H_1) + \frac{\partial}{\partial H_1} f_s(T, H, H_1)
\]  
(2.4b)

Binder [14] describes the scaling theory for these surface properties. Restricting attention to the order parameters defined above, one has \( M_s \) and \( M_1 \) exhibiting critical behaviour with exponents \( \beta_s \) and \( \beta_1 \). The surface corrections can be related to finite-size scaling, giving:

\[
\alpha_s = \alpha + \nu
\]  
(2.5a)

\[
\beta_s = \beta - \nu
\]  
(2.5b)

where exponents without subscripts refer to the bulk system. For quantities defined in terms of \( H_1 \), examples given by Binder indicate that various proposals of generic relations connecting \( \beta_1 \) to bulk properties can only be regarded as approximations and that introduction of the field \( H_1 \) introduces an
additional exponent independent of the bulk exponents. However for Potts model there is a specific connection derived from conformal invariance [15].

The Potts model is defined as having a $q$-state variable $n_j$ at each lattice site $j$, so that $n_j = 0, 1, \ldots q-1$ with an energy $J$ associated with each pair of unlike neighbours and an energy $H$ associated with non-zero states. This ordering field, $H$, favours state ‘0’ and we define the spontaneous order through the limit $H \rightarrow 0^+$. Low-temperature/high-field expansions are obtained as perturbations about the fully-aligned ‘all-zero’ state. If the energy is defined as zero for this state, the partition function, $Z$, is denoted $\Lambda$. Working with perturbations means that derivatives with respect to field will give the parameters as a perturbations from the fully-aligned state. The low-temperature/high-field expansion variables are

$$z = \exp(-J/k_B T), \quad \mu = \exp(-H/k_B T) \quad \text{and} \quad \mu_1 = \exp(-H_1/k_B T).$$

The actual calculations work with $z, x = 1 - \mu$ and $x_1 = 1 - \mu_1$. Keeping $x$ and $x_1$ to first order allows calculation of the order parameters described below. Calculation of susceptibilities would require keeping $x$ and $x_1$ to second order.

For Potts model, $\frac{\partial}{\partial \mu} k_B T \ln \Lambda = \Pr(n_j \neq 0) = 1 - \Pr(n_j = 0)$ This quantity goes from 0 to $(q-1)/q$ and so $M = 1 - q/(q-1) \frac{\partial}{\partial \mu} \ln \Lambda$ is defined as an order parameter normalised to the range 1 to 0 and referred to as the magnetisation.

For the remainder of this paper, we restrict consideration to $q = 3$ on the square lattice. For the boundary effects we consider two cases. Firstly, for free boundary conditions, there are no interactions beyond the boundary sites and so the boundary spins will be less ordered than the bulk of the lattice. Secondly, for fixed boundary conditions, the edges interact with a surrounding set of sites fixed in state 0. This means that for $T > 0$ the boundary spins will be more ordered than the bulk. The bulk magnetisation is

$$M = 1 - 3z^4 - \ldots$$

which was evaluated to $z^{47}$ by Briggs et al. [8].

Inspection of the lowest-order perturbations shows that for free boundaries:

$$M_1 = 1 - 3z^3 - 6z^4 - \ldots$$

and

$$M_s = -3z^3 - 3z^4 - \ldots$$

For fixed boundaries:

$$M_1 = 1 - 3z^4 - 9z^6 - \ldots$$

and

$$M_s = 3z^6 + 3z^7 + \ldots$$

The opposite signs of $M_s$ in the two cases reflects the fact that, relative to the bulk ordering, the free boundary conditions reduce the degree of order, while fixed boundary conditions increase the degree of order.

To formalise these definitions we put

$$\Lambda^2_s = \exp[2f_s]$$

and define

$$M_s = \frac{1}{2\Lambda^2_s} \left( \frac{q}{q-1} \right) \frac{\partial}{\partial \mu} \Lambda_s(z, \mu, \mu_1)^2 + \frac{\partial}{\partial \mu_1} \Lambda_s(z, \mu, \mu_1)^2$$

and

$$M_1 = 1 - \frac{1}{2\Lambda^2_s} \left( \frac{q}{q-1} \right) \frac{\partial}{\partial \mu_1} \Lambda_s(z, \mu, \mu_1)^2$$

The difference in the definitions (2.9b,c) comes from the fact that $M_1$ is defined as a (normalised) surface expectation which is 1 in the fully aligned state, while $M_s$ is defined as a ‘correction’ which is 0 in the fully aligned state. The factor of 1/2 arises from the use of $\Lambda^2_s$, which (unlike expansions of $\Lambda_s$) involves only integer weights and integer coefficients in the series.
3. Surface series from the Finite Lattice method

The Finite Lattice Method (FLM) represents a resummation of the free energy in terms of irreducible contributions from finite $m \times n$ rectangular lattices. Away from the boundaries we denote the irreducible contributions by $\tilde{f}_{mn}$. Contributions from $m \times n$ rectangles with $m$ sites along the boundary are denoted $\tilde{g}_{mn}$. (The cases with the $n$ sites along the boundary are obtained using rotational symmetry). Thus:

$$F = \sum_{m,n} \left[ (L - m - 2)(L' - n - 2)\tilde{f}_{mn} + 2(L - m - 2)\tilde{g}_{mn} + 2(L' - n - 2)\tilde{g}_{nm} \right] \quad (3.1)$$

This leads us to put

$$f_b = \sum_{m,n} \tilde{f}_{mn} \quad (3.2a)$$

and

$$2f_s = 2\sum_{mn} \tilde{g}_{mn} - \sum_{mn} (n + 2)\tilde{f}_{mn} \quad (3.2b)$$

The irreducible contributions can be obtained by inverting expressions for the free energies of finite lattices. Approximations for $f_b$ and $f_s$ can be obtained by truncating (3.2a,b) and expressing the irreducible contributions in terms of finite lattice free energies.

The expansions take the form

$$f_b \approx \sum_{m,n \geq k} \tilde{f}_{mn} = \sum_{m,n \geq k} a_{mn}f_{mn} \quad (3.3a)$$

where the coefficients can be obtained by inverting

$$f_{mn} = \sum_{m',n' \leq n} (m + 1 - m')(n + 1 - n')\tilde{f}_{m'n'} \quad (3.3b)$$

For the boundary case,

$$2f_s \approx 2\sum_{m,n \geq k} \tilde{g}_{mn} - \sum_{m,n \geq k} (n + 2)\tilde{f}_{mn} = \sum_{m,n \geq k} b_{mn}g_{mn} + \sum_{m,n \geq k} c_{mn}f_{mn} \quad (3.4a)$$

where the $\tilde{g}_{mn}$ can be obtained by inverting

$$g_{mn} = \sum_{n' \leq n} \sum_{m' \leq m} (m + 1 - m')\tilde{g}_{mn'} + \sum_{m' \leq m,n' < n} (m + 1 - m')(n - n')\tilde{f}_{m'n'} \quad (3.4b)$$

The coefficients in expansion (3.3a, 3.4a) can be readily evaluated numerically for a general lattice (since they involve the inverse of an incidence matrix, i.e. a triangular matrix with diagonal elements equal to 1). On the square lattice there are the explicit results for the $a_{mn}$ [16].

One special case of surface weights was identified by Enting [16], corresponding in the present notation to $H_1 = 0$ with either fixed boundaries at low temperatures or free boundaries at high temperatures. In these cases, $g_{mn} = f_{nm}$ and one can write

$$2f_s \approx \sum_{m,n \geq k} d_{mn}f_{mn} \quad \text{Low-T fixed b.c./High-T, free b.c.} \quad (3.5a)$$

with

$$d_{mn} = b_{mn} + c_{mn} \quad (3.5b)$$

The analysis [16] gives explicit forms of the $a_{mn}$ and $d_{mn}$ and can be readily generalised to give the $b_{mn}$ (and thus $c_{mn}$). Details will be given elsewhere — the explicit form is relatively unimportant since, as noted above, the coefficients can readily be obtained by numerical inversion of relations (3.3b, 3.4b).
As noted at the beginning of this section, the FLM represents a re-summation of a conventional perturbation expansion. The weights (3.4) for the bulk problem represent a resummation that acts to cancel surface contributions from the \( f_{mn} \), i.e. the surface weighted sums \( \sum m a_{mn} \) and \( \sum n a_{mn} \) both vanish. Similarly the surface weights (3.6b) represent a summation in which the bulk contributions cancel so that \( \sum m n d_{mn} = 0 \).

Practical calculations differ from the form above in two important ways: firstly, the relations above are exponentiated so that the partition function is expressed as a product of powers of finite lattice partition functions; secondly rotational symmetry is used when appropriate.

The finite lattice partition functions are calculated using a ‘transfer matrix’ formalism that adds site one at a time. The use of this approach in FLM calculations was first noted in the case of a generalised Potts model [17] and described in detail in connection with the enumeration of self-avoiding polygons [18]. Blöte and Nightingale [19] have described this technique as equivalent to a sparse-matrix decomposition of the conventional row-to-row transfer matrix.

Three sets of transfer matrix calculations are required:

- one for bulk contributions \( f_{mn} \)
- one with surface fields (and free boundaries if applicable) along on edge in the growing direction, giving the \( g_{mn} \) for \( m \geq n \);
- one with surface fields (and free boundaries if applicable) on the initial column of sites, giving the \( g_{mn} \) for \( n > m \);

For the fixed boundaries, the series will be correct to \( z^{2k+1} \) as in the bulk case. For free boundaries, the series are only correct to order \( z^{k+2} \). The truncation of (3.5) at \( m + n \leq k \) is not the most efficient way of calculating series for free boundaries, but is used here because of the simplifications arising from a unified treatment. Alternative groupings of finite lattices to give surface contributions that efficiently calculate high-temperature susceptibilities have been described by Enting and Guttmann [11].

### 4. Results and analysis

Series for \( A_2^q \), \( M_s \) and \( M_1 \) from free and fixed boundaries are presented in the Appendix. For \( q = 3 \), the critical point is known [1] to be \( z_c = (\sqrt{3} - 1)/2 \). The bulk exponents for the \( q = 3 \) Potts model universality class are known from the solution of the hard-hexagon model [20] and are: \( \alpha = \frac{3}{5} \), \( \beta = \frac{1}{9} \), \( \gamma = \frac{13}{3} \), \( \nu = \frac{4}{9} \). This implies that the surface exponents are \( \alpha_s = \frac{7}{9} \), \( \beta_s = -\frac{13}{18} \). As noted above, exponents for quantities defined in terms of \( H_1 \) introduce an independent exponent that lacks any generic connection to bulk exponents. The additional degree of freedom in the scaling relations is often expressed in terms of a correlation exponent \( \eta_{1\perp} \). This is related to the magnetisation exponent by \( \beta_1 = \nu(d - 2 + \eta_{1\perp})/2 \). For the Potts model, Cardy [15] used conformal invariance to derive the relation \( \eta_{1\perp} = 2/(3\nu - 1) \) which for \( q = 3 \) gives \( \eta_{1\perp} = 4/3 \) and thus \( \beta_1 = \frac{5}{9} \).

The series were analysed using the method of differential approximants [21] including Padé approximants [22] as a special case. As with the initial application of the FLM to surface series [11], one of the objectives of the present study is to investigate how well ‘standard’ series analysis techniques can reproduce known results when applied to this type of series. Apart from the specific model \( q = 3 \) rather than \( q = 2 \), the feature that distinguishes the present study from that of Enting and Guttmann [11] is that the low-temperature series analysed here are expected to be less well-behaved than the high-temperature series analysed previously. In particular, no useful results could be obtained from the ratio method.

#### A: Padé approximants

A Padé approximant represents a series as \( \sum_j c_j x^j \approx \left[ \sum_{j=0}^m a_j x^j \right] / \left[ \sum_{j=0}^n b_j x^j \right] \) (where we put \( b_0 = 1 \) to define the normalisation). The ‘standard’ use of Padé approximants in analysing critical phenomena comes from the observation that if \( f(x) \sim A(x_c - x)^-\gamma \) then \( \frac{d}{dx} \ln(f) \sim \gamma/(x_c - x) \). Therefore, if a Padé approximant is fitted to \( \frac{d}{dx} \ln(f) \), the appropriate root of the denominator should
be an estimate of \( x_c \) and the residue \( \) of the approximant at \( x_c \) (and multiplied by \( x_c \)) should give an estimate of \( \gamma \).

The various cases that have been analysed are:

A.1 \( M_s \), free b.c.
The Padé approximants to \( \frac{d}{dz} \ln(M_s/z^3) \) gave reasonable estimates of \( z_c \), with only a small proportion of approximants having competing roots (defined as additional roots in the region \(|\Im(x)| < 0.1 \) and \(-0.1 < x < 0.4 \)) but a slight tendency to overestimate \( z_c \). However the exponent estimates were scattered and generally in the range \(-0.8 \) to \(-0.9 \), compared to the scaling prediction of \(-0.722 \).

A.2 \( M_1 \), free b.c.
The Padé approximants to \( \frac{d}{dz} \ln(M_1) \) gave reasonable estimates of \( z_c \), with a larger proportion of approximants having competing roots (defined as above) and a slight tendency to underestimate \( z_c \). The exponent estimates were in the range 0.52 to 0.54 (c.f. 0.555 from conformal invariance) and tended to be larger for the higher-order approximants, suggesting a convergence towards the predicted value.

A.3 \( M_s \), fixed b.c.
The Padé approximants to \( \frac{d}{dz} \ln(M_s/z^6) \) gave reasonable estimates of \( z_c \), with a high proportion of competing roots (\(|\Im(x)| < 0.1 \) and \(-0.1 < x < 0.4 \)) and a slight tendency to overestimate \( z_c \). However the exponent estimates were scattered and generally in the range \(-0.9 \) to \(-1.0 \), compared to the scaling prediction of \(-0.722 \).

A.4 \( M_1 \), fixed b.c.
Since, for fixed boundaries, \( M_1 \) does not go to zero at the critical point, the analysis was applied to its derivative. The Padé approximants to \( \frac{d}{dz} \ln(z^{-3} \frac{d}{dz} M_1) \) gave estimates of \( z_c \) that were generally too large, but only a small proportion of approximants had competing roots. However the exponent estimates were scattered and generally in the range \(-0.1 \) to \(-0.2 \), compared to the scaling prediction of \( \beta_1-1 = -0.444 \).

B: Differential approximants

The method of differential approximants has been described in detail by Guttmann [21]. It fits a series \( f(x) \) as the solution of a differential equation:

\[
P(x) + \sum_{n=0}^{K} Q_n(x) \frac{d^n}{dx^n} f(x) = 0 \quad (4.1a)
\]

or

\[
P(x) + \sum_{n=0}^{K} x^n Q_n(x) \frac{d^n}{dx^n} f(x) = 0 \quad (4.1b)
\]

where \( P(.) \) and the \( Q_n(.) \) are polynomials of specified degree in \( x \). We use the notation \( [L : N_0, N_1, N_2] \) to denote the numbers of terms in each polynomial (and distinguish the second normalisation as \( [L : N_0, N_1, N_2]^+ \)) — the present analysis is restricted to second- (or lower-) order approximants. Padé approximants appear as a special \([L, N, 0, 0] \) case, and Padé approximants to the logarithmic derivative are \([0, N, D, 0] \) cases. Cases of the form \([0, N_0, N_1, N_2]^+ \) have the additional constraint that the constant term in \( Q_0 \) is zero. Such \([0, N_0, N_1, N_2]^+ \) approximants have regular singularities at the origin [21]. For free boundaries, the applicability of differential approximants was constrained by the shortness of the series.

B.1 \( M_1 \), free b.c.
Biased 2nd order approximants tended to give exponent estimates close to the conformal invariance result, but overall not as close as from biased dlog Padé approximants.

B.2 \( M_s \), free b.c.
Biased approximants gave results that were highly scattered, and generally more negative than the scaling value.

B.3 \( M_1 \), fixed b.c.
In principle, having \( M_1(z_c) \neq 0 \) should not preclude analysis by second order DAs. In practice, the
exponent estimates from biased approximants were scattered, and well above the scaling value of 0.555, values in the range 0.65 ± 0.01 being typical.

B.4 $M_s$, fixed b.c:

As relatively long series with an expected strong singularity, these series would be expected to be most readily analysed. To some extent this is true. Both homogeneous and inhomogeneous DAs gave a spread of exponents that were systematically more negative than the scaling estimate (of order $-0.74$, c.f. $-0.722$ from scaling). Figure 1 plots the estimates of $z_c$ vs $\beta_s$ obtained from 2nd order approximants. The results for $z_c$ and $\beta_s$ are highly correlated and appear to be inconsistent with the scaling value. To some extent, this is reminiscent of early experience with analysis of bulk series for the Potts model [23, 17] where convergence to known exponents was found to be very slow. This slow convergence (of Padé approximants) was later analysed in terms of confluent singularities [24]. However, one characteristic of the DA analyses was the absence of any consistent appearance of multiple roots associated possible confluent singularities in any of the four series.

In order to explore the extent to which the systematic departure from the scaling prediction might be due to slow convergence, the exponents were calculated using approximants biased by forcing a $(z_c - z)$ factor in $Q_2$, and then plotted against the reciprocal of the number of series terms fitted. The results are shown, separately for homogeneous and inhomogeneous approximants in the left and right sections respectively, of Figure 2. While there is a high degree of scatter, in each case a central cluster of estimates appears to indicate a trend that is consistent with having the scaling value as the limit.

5. Conclusions

There are a number of ways in which the present series could be extended:

- most obviously, larger more powerful computers could be used — the series presented here were calculated on an Apple Macintosh G5;
- as noted above, for free boundary conditions an alternative truncation of (3.4a) would lead to longer series;
- finally, Enting et al. [25] have describe an alternative way of evaluating $f_{mn}$ for $m \approx n$ which makes more efficient use of computer memory than the algorithm used here. (The $f_{mn}$ where $m$
Figure 2. Exponent estimates from biased differential approximants — homogeneous at left, inhomogeneous at right — plotted against reciprocal of number of terms fitted. Excludes cases with additional roots of $Q_2$ with $|\Im(z)| < 0.1$ and $-0.1 < \Re(z) < 0.4$.

and $n$ are not approximately equal are evaluated as in this paper). In principle, the memory-efficient algorithm could be generalised to evaluate the $g_{mn}$, but the complexity of the algorithm (involving eight variations of adding sites) makes the task of including surface effects somewhat daunting.

For these reasons, the present calculations must be regarded as exploratory. Some of the main results are:

- The series for free boundary conditions are too short for detailed analysis, and the analysis techniques performed poorly when trying to estimate both critical points and exponents. When using the known $z_c$, dlog Padé approximants (i.e. $[0, N, M, 0, 0]$ differential approximants) to $M_1$ gave consistent estimates of $\beta_1$ close to the value of 0.555 from conformal invariance. The dlog Padés proved very poor at locating $z_c$ using the series for $M_s$. Second order differential approximants proved somewhat better, but, even when biased by using the known $z_c$, the scatter in the exponent estimates was considerable.

- For fixed boundary conditions, the non-zero critical value of $M_1$ made the analysis difficult. Biased approximants gave consistent estimates of an exponent of order 0.65, significantly greater than the value of 0.555 based on conformal invariance. For $M_s$, the surface correction to bulk order, both biased and unbiased approximants gave exponent estimates somewhat above the scaling value of 0.722. with, as expected, greater consistency obtained using biased approximants. The biased approximants show a poorly-defined dependence on the number of series terms fitted, showing a trend that is consistent with the scaling value as the limit.

The present calculations have demonstrated the feasibility of generalising the surface calculations of Enting and Guttmann [11], both to Potts models with $q \neq 2$ and to low temperatures. They have also indicated the importance of optimising the finite lattice calculations to suit the type of boundaries. However, the results also indicate the difficulty of analysing such series, and further confirm the need for longer series.

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Notation

- $a_{mn}$: Weight of $m \times n$ finite lattice, $f_{mn}$, in expansion of $f_b$.
- $b_{mn}$: Weight of $m \times n$ finite lattice $g_{mn}$, in expansion of $f_s$.
- $c_{mn}$: Weight of $m \times n$ finite lattice $f_{mn}$, in expansion of $f_s$ in general case.
- $d_{mn}$: Weight of $m \times n$ finite lattice $f_{mn}$, in expansion of $f_s$ for fixed b.c. and zero surface field.
- $f_b$: Bulk free energy per site.
- $f_s$: Free energy per unit length of boundary.
- $f_{mn}$: Partition function of $m \times n$ finite lattice,
- $\tilde{f}_{mn}$: Irreducible contribution to $f_{mn}$.
- $g_{mn}$: Partition function of $m \times n$ finite lattice with one side of length $m$ along boundary.
- $\tilde{g}_{mn}$: Irreducible contribution to $g_{mn}$.
- $H$: Bulk field.
- $H_1$: Surface field.
- $J$: Interaction strength — energy per pair of unlike neighbours.
- $k_B$: Boltzmann’s constant.
- $L$: Length of large square system.
- $M$: Bulk ‘magnetisation’ $1 - \Pr(n_j \neq 0) \times q/(q - 1)$.
- $M_1$: Surface spin magnetisation.
- $M_s$: Surface correction to $M$.
- $n_j$: Potts model state variable at site $j$.
- $q$: Number of states in Potts model.
- $T$: Temperature.
- $z$: Low-temperature expansion variable: $z = \exp(-J/k_B T)$.
- $Z$: Partition function.
- $\alpha$: Critical exponent for bulk specific heat.
- $\alpha_s$: Critical exponent for surface correction to specific heat.
- $\beta$: Critical exponent for bulk order parameter $M$.
- $\beta_s$: Critical exponent for surface correction to order parameter $M_s$.
- $\beta_1$: Critical exponent for surface order parameter $M_1$.
- $\eta$: Correlation exponent for surface effects.
- $\mu$: Expansion variable for ordering field. $\mu = \exp(-H/k_B T)$.
- $\mu_1$: Expansion variable for surface field. $\mu_1 = \exp(-H_1/k_B T)$.
- $\Lambda$: Reduced partition function (per site).
- $\Lambda_s$: Surface contribution (per unit length of boundary) to reduced partition function.
- $\nu$: Critical exponent for correlation length, $\xi$.
- $\xi$: Correlation length for bulk lattice.
### The series: Fixed boundary

| n  | \( \Lambda^2 \) | \( M_1 \) | \( M_s \) |
|----|----------------|---------|---------|
| 0  | 1              | 1       | 0       |
| 1  | 0              | 0       | 0       |
| 2  | 0              | 0       | 0       |
| 3  | 0              | 0       | 0       |
| 4  | 0              | -3      | 0       |
| 5  | 0              | 0       | 0       |
| 6  | -2             | -9      | 3       |
| 7  | -2             | -9      | 3       |
| 8  | -10            | -12     | 27      |
| 9  | -24            | -60     | 54      |
| 10 | -52            | -48     | 216     |
| 11 | -130           | -153    | 450     |
| 12 | -388           | -504    | 1923    |
| 13 | -620           | -330    | 3369    |
| 14 | -2394          | -2415   | 14841   |
| 15 | -4370          | -3558   | 29919   |
| 16 | -12118         | -7029   | 102192  |
| 17 | -32314         | -28341  | 258912  |
| 18 | -68356         | -32862  | 733956  |
| 19 | -199862        | -134628 | 1999521 |
| 20 | -466754        | -281472 | 5630457 |
| 21 | -1160486       | -564627 | 14781231|
| 22 | -3157190       | -1875285| 42871278|
| 23 | -7370600       | -3433521| 111503781|
| 24 | -19832164      | -9703338| 316807263|
| 25 | -50098568      | -24527004| 85174649|
| 26 | -124733230     | -52280256| 233482861|
| 27 | -333519798     | -152141355| 641489248|
| 28 | -827189010     | -340328889| 1743842444|
| 29 | -2163220720    | -865856643| 47668778127|
| 30 | -5605295194    | -2270118705| 130703758041|
| 31 | -14236838324   | -5261117598| 354352750362|
| 32 | -37525063234   | -1411364374| 973753722000|
| 33 | -96153861050   | -34571651562| 2645183086965|
| 34 | -250010407376  | -86429776635| 7228699923906|
| 35 | -656387531246  | -225963265485| 19732827586044|
| 36 | -1686617537120 | -553507149000| 53736713238600|
| 37 | -4426354756414 | -1437344123532| 146761454268627|
| 38 | -11515945714078| -3648952254324| 400043201310129|
| 39 | -30043995520112 | -9202758880908| 1090373595142140|
| 40 | -78842242959410 | -23907568780911| 2975993658025665|
| 41 | -20562720679564 | -603907850240094| 8106170375325594|
| 42 | -53996216165134 | -155620455564495| 22114015704445812|
| 43 | -1415813195688816 | -400514562312582| 60279224508672003|
| 44 | -3712050559013522 | -1022685297476403| 164296587493395933|
| 45 | -9769216758495434 | -2651300388303702| 448027804527081864|
| 46 | -25653952409425890 | -6806590215847512| 1220988919790522244|
| 47 | -67542769159197940 | -17573483170034106| 3328616271906352083|
The series: Free boundary

| n  | $\Lambda^n$ | $M_1$ | $M_2$ |
|----|------------|------|------|
| 0  | 1          | 1    | 0    |
| 1  | 0          | 0    | 0    |
| 2  | 0          | 0    | 0    |
| 3  | 4          | $-3$ | $-3$ |
| 4  | 0          | $-6$ | $-3$ |
| 5  | 12         | $-18$| $-21$|
| 6  | 14         | $-33$| $-33$|
| 7  | 22         | $-69$| $-108$|
| 8  | 106        | $-141$| $-297$|
| 9  | 80         | $-264$| $-672$|
| 10 | 432        | $-648$| $-2124$|
| 11 | 818        | $-1332$| $-5340$|
| 12 | 1460       | $-3420$| $-14043$|
| 13 | 5368       | $-7668$| $-40341$|
| 14 | 8682       | $-19380$| $-101151$|
| 15 | 24710      | $-44388$| $-280860$|
| 16 | 64226      | $-112725$| $-759630$|
| 17 | 123238     | $-263523$| $-1979973$|
| 18 | 383920     | $-673302$| $-5542941$|
| 19 | 819806     | $-1633077$| $-14537274$|
| 20 | 2065562    | $-4158477$| $-39688650$|
| 21 | 5542422    | $-10382247$| $-783774780$|
| 22 | 12354142   | $-26395017$| $-287271708$|
| 23 | 33865704   | $-66703539$| $-1979973$|
| 24 | 82302072   | $-170292129$| $-5542941$|
| 25 | 203861976  | $-432697869$| $-280860$|

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