Decomposition and limit theorems for a class of self-similar Gaussian processes

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Abstract

We introduce a new class of self-similar Gaussian stochastic processes, where the covariance is defined in terms of a fractional Brownian motion and another Gaussian process. A special case is the solution in time to the fractional-colored stochastic heat equation described in Tudor (2013). We prove that the process can be decomposed into a fractional Brownian motion (with a different parameter than the one that defines the covariance), and a Gaussian process first described in Lei and Nualart (2008). The component processes can be expressed as stochastic integrals with respect to the Brownian sheet. We then prove a central limit theorem about the Hermite variations of the process.

1 Introduction

The purpose of this paper is to introduce a new class of Gaussian self-similar stochastic processes related to stochastic partial differential equations, and to establish a decomposition in law and a central limit theorem for the Hermite variations of the increments of such processes.

Consider the $d$-dimensional stochastic heat equation

\[
\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + \dot{W}, \quad t \geq 0, \quad x \in \mathbb{R}^d,
\]

with zero initial condition, where $\dot{W}$ is a zero mean Gaussian field with a covariance of the form

\[
\mathbb{E} \left[ \dot{W}^H(t,x) \dot{W}^H(s,y) \right] = \gamma_0(t-s)\Lambda(x-y), \quad s,t \geq 0, \quad x,y \in \mathbb{R}^d.
\]

We are interested in the process $U = \{U_t, t \geq 0\}$, where $U_t = u(t,0)$.

Suppose that $\dot{W}$ is white in time, that is, $\gamma_0 = \delta_0$ and the spatial covariance is the Riesz kernel, that is, $\Lambda(x) = c_{d,\beta} |x|^{-\beta}$, with $\beta < \min(d,2)$ and $c_{d,\beta} = \pi^{-d/2} 2^\beta \Gamma(\beta/2) / \Gamma((d-
\text{Up to a constant, the covariance (1.2) is the covariance of the bifractional Brownian motion with parameters } H = \frac{1}{2} \text{ and } K = 1 - \frac{\beta}{2}. \text{ We recall that, given constants } H \in (0, 1) \text{ and } K \in (0, 1), \text{ the bifractional Brownian motion } B^{H,K} = \{B^{H,K}_t, t \geq 0\}, \text{ introduced in [3], is a centered Gaussian process with covariance }
\begin{equation}
R_{H,K}(s,t) = \frac{1}{2K} \left( (t^{2H} + s^{2H})^K - |t - s|^{2HK} \right), \quad s, t \geq 0. \tag{1.3}
\end{equation}

When } K = 1, \text{ the process } B^H = B^{H,1} \text{ is simply the fractional Brownian motion (fBm) with Hurst parameter } H \in (0, 1), \text{ with covariance } R_H(s,t) = R_{H,1}(s,t). \text{ In [4], Lei and Nualart obtained the following decomposition in law for the bifractional Brownian motion }
\begin{equation}
B^{H,K} = C_1 B^{HK} + C_2 Y^K, \tag{1.4}
\end{equation}

where } B^{HK} \text{ is a fBm with Hurst parameter } HK, \text{ the process } Y^K \text{ is given by }
\begin{equation}
Y^K_t = \int_0^\infty y^{-\frac{1+K}{2}} (1 - e^{-yt}) dW_y, \tag{1.4}
\end{equation}

with } W = \{W_y, y \geq 0\} \text{ a standard Brownian motion independent of } B^{HK}, \text{ and } C_1, C_2 \text{ are constants given by } C_1 = 2^{\frac{1-K}{2}} \text{ and } C_2 = \sqrt{\frac{2-K}{1-K}}. \text{ The process } Y^K \text{ has trajectories which are infinitely differentiable on } (0, \infty) \text{ and Hölder continuous of order } HK - \epsilon \text{ in any interval } [0, T] \text{ for any } \epsilon > 0. \text{ In particular, this leads to a decomposition in law of the process } U \text{ with covariance (1.2) as the sum of a fractional Brownian motion with Hurst parameter } \frac{1}{2} - \frac{\beta}{4} \text{ plus a regular process.}

The classical one-dimensional space-time white noise can also be considered as an extension of the covariance (1.2) if we take } \beta = 1. \text{ In this case the covariance corresponds, up to a constant, to that of a bifractional Brownian motion with parameters } H = K = \frac{1}{2}.

The case where the noise term } \dot{W} \text{ is a fractional Brownian motion with Hurst parameter } H \in (\frac{1}{2}, 1) \text{ in time and a spatial covariance given by the Riesz kernel, that is, }
\begin{equation}
\mathbb{E} \left[ \frac{1}{1-H} \right] \dot{W}^H(t, x) \dot{W}^H(s, y) = \alpha_H c_{d,\beta} |s - t|^{2H-2} |x - y|^{-\beta}, \tag{1.5}
\end{equation}

where } 0 < \beta < \min(d,2) \text{ and } \alpha_H = H(2H - 1), \text{ has been considered by Tudor and Xiao in [13]. In this case the corresponding process } U \text{ has the covariance }
\begin{equation}
\mathbb{E}[U_tU_s] = D\alpha_H \int_0^t \int_0^s |u - v|^{2H-2} (t + s - u - v)^{-\gamma} dudv. \tag{1.5}
\end{equation}
where $D$ is given in (1.3) and $\gamma = \frac{d-\beta}{2}$. This process is self-similar with parameter $H - \frac{\gamma}{2}$ and it has been studied in a series of papers [1, 7, 11, 12, 13]. In particular, in [13] it is proved that the process $U$ can be decomposed into the sum of a scaled fBm with parameter $H - \frac{\gamma}{2}$, and a Gaussian process $V$ with continuously differentiable trajectories. This decomposition is based on the stochastic heat equation. As a consequence, one can derive the exact uniform and local moduli of continuity and Chung-type laws of the iterated logarithm for this process. In [11], assuming that $d = 1, 2$ or $3$, a central limit theorem is obtained for the renormalized quadratic variation

$$V_n = n^{2H-\gamma-\frac{1}{2}} \sum_{j=0}^{n-1} \left\{ (U_{(j+1)T/n} - U_{jT/n})^2 - \mathbb{E} [(U_{(j+1)T/n} - U_{jT/n})^2] \right\},$$

assuming $\frac{1}{2} < H < \frac{3}{4}$, extending well-known results for fBm (see for example [5, Theorem 7.4.1]).

The purpose of this paper is to establish a decomposition in law, similar to that obtained by Lei and Nualart in [4] for the bifractional Brownian motion, and a central limit theorem for the Hermite variations of the increments, for a class of self-similar processes that includes the covariance (1.5). Consider a centered Gaussian process $\{X_t, t \geq 0\}$ with covariance

$$R(s,t) = \mathbb{E}[X_sX_t] = \mathbb{E}\left[ \left( \int_0^t Z_{t-r}dB^H_r \right) \left( \int_0^s Z_{s-r}dB^H_r \right) \right], \quad (1.6)$$

where

(i) $B^H = \{B^H_t, t \geq 0\}$ is a fBm with Hurst parameter $H \in (0,1)$.

(ii) $Z = \{Z_t, t > 0\}$ is a zero-mean Gaussian process, independent of $B^H$, with covariance

$$\mathbb{E}[Z_sZ_t] = (s+t)^{-\gamma}, \quad (1.7)$$

where $0 < \gamma < 2H$.

In other words, $X$ is a Gaussian process with the same covariance as the process $\{\int_0^t Z_{t-r}dB^H_r, t \geq 0\}$, which is not Gaussian.

When $H \in (\frac{1}{2}, 1)$, the covariance (1.6) coincides with (1.5) with $D = 1$. However, we allow the range of parameters $0 < H < 1$ and $0 < \gamma < 2H$. In other words, up to a constant, $X$ has the law of the solution in time of the stochastic heat equation (1.1), when $H \in (0,1)$ and $d \geq 1$ and $\beta = d - 2\gamma$. Also of interest is that $X$ can be constructed as a sum of stochastic integrals with respect to the Brownian sheet (see the proof of Theorem 1).

1.1 Decomposition of the process $X$

Our first result is the following decomposition in law of the process $X$ as the sum of a fractional Brownian motion with Hurst parameter $\frac{d-\beta}{2} = H - \frac{\gamma}{2}$ plus a process with regular trajectories.
Theorem 1. The process $X$ has the same law as $\{ \sqrt{\kappa} B_t^\alpha + Y_t, t \geq 0 \}$, where

$$\kappa = \frac{1}{\Gamma(\gamma)} \int_0^\infty \frac{z^{\gamma-1}}{1+z^2} \, dz,$$

(1.8)

$B_t^\alpha$ is a fBm with Hurst parameter $\alpha/2$, and $Y$ (up to a constant) has the same law as the process $Y^K$ defined in (1.7), with $K = 2\alpha + 1$, that is, $Y$ is a centered Gaussian process with covariance given by

$$\mathbb{E}[Y_t Y_s] = \lambda_1 \int_0^\infty y^{-\alpha-1}(1-e^{-yt})(1-e^{-ys}) \, dy,$$

where

$$\lambda_1 = \frac{4\pi}{\Gamma(\gamma)\Gamma(2H+1)\sin(\pi H)} \int_0^\infty \frac{\eta^{1-2H}}{1+\eta^2} \, d\eta.$$

The proof of this theorem is given in Section 3.

1.2 Hermite variations of the process

For each integer $q \geq 0$, the $q$th Hermite polynomial is given by

$$H_q(x) = (-1)^q e^{x^2} \frac{d^q}{dx^q} e^{-x^2}.$$

See [5, Section 1.4] for a discussion of properties of these polynomials. In particular, it is well known that the family $\{\frac{1}{\sqrt{n}} H_q, q \geq 0\}$ constitutes an orthonormal basis of the space $L^2(\mathbb{R}, \gamma)$, where $\gamma$ is the $N(0,1)$ measure.

Suppose $\{Z_n, n \geq 1\}$ is a stationary, Gaussian sequence, where each $Z_n$ follows the $N(0,1)$ distribution with covariance function $\rho(k) = \mathbb{E}[Z_n Z_{n+k}]$. If $\sum_{k=1}^\infty |\rho(k)|^q < \infty$, it is well known that as $n$ tends to infinity, the Hermite variation

$$V_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} H_q(Z_j)$$

(1.9)

converges in distribution to a Gaussian random variable with mean zero and variance given by $\sigma^2 = \sum_{k=1}^\infty \rho(k)^q$. This result was proved by Breuer and Major in [2]. In particular, if $B^H$ is a fBm, then the sequence $\{Z_{j,n}, 0 \leq j \leq n-1\}$ defined by

$$Z_{j,n} = n^H \left( B^H_{j+\frac{1}{n}} - B^H_{\frac{j}{n}} \right)$$

is a stationary sequence with unit variance. As a consequence, $H < 1 - \frac{1}{q}$, we have that

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} H_q \left( n^H \left( B^H_{j+\frac{1}{n}} - B^H_{\frac{j}{n}} \right) \right)$$
converges to a normal law with variance given by
\[ \sigma_q^2 = \frac{q!}{2^q} \sum_{m \in \mathbb{Z}} (|m + 1|^{2H} - 2|m|^{2H} + |m - 1|^{2H})^q. \] (1.10)

See [2] and Theorem 7.4.1 of [5].

The above Breuer-Major theorem cannot be applied to our process because \( X \) is not necessarily stationary. However, we have a comparable result.

**Theorem 2.** Let \( q \geq 2 \) be an integer and fix a real \( T > 0 \). Suppose that \( \alpha < 2 - \frac{1}{q} \). For \( t \in [0, T] \), define,
\[ F_n(t) = n^{-\frac{1}{2}} \sum_{j=0}^{\lfloor nt \rfloor - 1} H_q \left( \frac{\Delta X_j}{\|\Delta X_j\|_{L^2(\Omega)}} \right), \]
where \( H_q(x) \) denotes the \( q \)th Hermite polynomial. Then as \( n \to \infty \), the stochastic process \( \{F_n(t), t \in [0, T]\} \) converges in law in the Skorohod space \( D([0,T]) \), to a scaled Brownian motion \( \{\sigma B_t, t \in [0, T]\} \), where \( \{B_t, t \in [0, T]\} \) is a standard Brownian motion and \( \sigma = \sqrt{\sigma^2} \) is given by
\[ \sigma^2 = \frac{q!}{2^q} \sum_{m \in \mathbb{Z}} (|m + 1|^{\alpha} - 2|m|^{\alpha} + |m - 1|^{\alpha})^q. \] (1.11)

The proof of this theorem is given in Section 4.

### 2 Preliminaries

#### 2.1 Analysis on the Wiener space

The reader may refer to [5, 6] for a detailed coverage of this topic. Let \( Z = \{Z(h), h \in \mathcal{H}\} \) be an *isonormal Gaussian process* on a probability space \((\Omega, \mathcal{F}, P)\), indexed by a real separable Hilbert space \( \mathcal{H} \). This means that \( Z \) is a family of Gaussian random variables such that \( \mathbb{E}[Z(h)] = 0 \) and \( \mathbb{E}[Z(h)Z(g)] = \langle h, g \rangle_{\mathcal{H}} \) for all \( h, g \in \mathcal{H} \).

For integers \( q \geq 1 \), let \( \mathcal{H}^\otimes q \) denote the \( q \)th tensor product of \( \mathcal{H} \), and \( \mathcal{H}^\odot q \) denote the subspace of symmetric elements of \( \mathcal{H}^\otimes q \).

Let \( \{e_n, n \geq 1\} \) be a complete orthonormal system in \( \mathcal{H} \). For elements \( f, g \in \mathcal{H}^\otimes q \) and \( p \in \{0, \ldots, q\} \), we define the \( p \)th-order contraction of \( f \) and \( g \) as that element of \( \mathcal{H}^\otimes 2(q-p) \) given by
\[ f \otimes_p g = \sum_{i_1, \ldots, i_p=1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_p} \rangle_{\mathcal{H}^\otimes p} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_p} \rangle_{\mathcal{H}^\otimes p}, \] (2.1)

where \( f \otimes_0 g = f \otimes g \). Note that, if \( f, g \in \mathcal{H}^\otimes q \), then \( f \otimes_q g = \langle f, g \rangle_{\mathcal{H}^\otimes q} \). In particular, if \( f, g \) are real-valued functions in \( \mathcal{H}^\otimes q = L^2(\mathbb{R}^2, \mathcal{B}^2, \mu) \) for a non-atomic measure \( \mu \), then we have
\[ f \otimes_1 g = \int_{\mathbb{R}} f(s, t_1)g(s, t_2) \mu(ds). \] (2.2)
Let $H_q$ be the $q$th Wiener chaos of $Z$, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q(Z(h)), h \in H, \|h\|_H = 1\}$, where $H_q(x)$ is the $q$th Hermite polynomial. It can be shown (see [5, Proposition 2.2.1]) that if $Z, Y \sim N(0, 1)$ are jointly Gaussian, then

$$
\mathbb{E}[H_p(Z)H_q(Y)] = 
\begin{cases} 
p! (\mathbb{E}[ZY])^p & \text{if } p = q \\
0 & \text{otherwise} \end{cases}.
$$

(2.3)

For $q \geq 1$, it is known that the map

$$
I_q(h^\otimes) = H_q(Z(h))
$$

(2.4)

provides a linear isometry between $H^\otimes_q$ (equipped with the modified norm $\sqrt{q!} \cdot \|\cdot\|_{H^\otimes_q}$) and $H_q$, where $I_q(\cdot)$ is the generalized Wiener-Itô stochastic integral (see [5, Theorem 2.7.7]). By convention, $H_0 = \mathbb{R}$ and $I_0(x) = x$.

We use the following integral multiplication theorem from [6, Proposition 1.1.3]. Suppose $f \in H^\otimes_p$ and $g \in H^\otimes_q$. Then

$$
I_p(f)I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \otimes_r g),
$$

(2.5)

where $f \otimes_r g$ denotes the symmetrization of $f \otimes_r g$. For a product of more than two integrals, see Peccati and Taqqu [8].

2.2 Stochastic integration and fBm

We refer to the ‘time domain’ and ‘spectral domain’ representations of fBm. The reader may refer to [9, 10] for details. Let $\mathcal{E}$ denote the set of real-valued step functions on $\mathbb{R}$. Let $B_H$ denote fBm with Hurst parameter $H$. For this case, we view $B_H$ as an isonormal Gaussian process on the Hilbert space $\mathcal{H}$, which is the closure of $\mathcal{E}$ with respect to the inner product $\langle f, g \rangle = \mathbb{E}[I(f)I(g)]$. Consider also the inner product space

$$
\tilde{\Lambda}_H = \left\{f : f \in L^2(\mathbb{R}), \int_{\mathbb{R}} |\mathcal{F}f(\xi)|^2 |\xi|^{1-2H} d\xi < \infty \right\},
$$

where $\mathcal{F}f = \int_{\mathbb{R}} f(x) e^{ix\xi} dx$ is the Fourier transform, and the inner product of $\tilde{\Lambda}_H$ is given by

$$
\langle f, g \rangle_{\tilde{\Lambda}_H} = \frac{1}{C_H^2} \int_{\mathbb{R}} \mathcal{F}f(\xi)\mathcal{F}g(\xi)|\xi|^{1-2H} d\xi,
$$

(2.6)

where $C_H = \left(\frac{2\pi}{1(2H+1)\sin(\pi H)}\right)^{\frac{1}{2}}$. It is known (see [9, Theorem 3.1]) that the space $\tilde{\Lambda}_H$ is isometric to a subspace of $\mathcal{H}$, and $\tilde{\Lambda}_H$ contains $\mathcal{E}$ as a dense subset. This inner product (2.6) is known as the ‘spectral measure’ of fBm. In the case $H \in \left(\frac{1}{2}, 1\right)$, there is another isometry from the space

$$
|\Lambda_H| = \left\{f : \int_0^\infty \int_0^\infty |f(u)||f(v)||u-v|^{2H-2} du \, dv < \infty \right\}
$$
to a subspace of $\mathcal{F}$, where the inner product is defined as

$$\langle f, g \rangle_{\Lambda_H} = H(2H - 1) \int_0^\infty \int_0^\infty f(u)g(v)|u - v|^{2H-2} du \, dv,$$

see [9] or [6, Section 5.1].

### 3 Proof of Theorem 1

For any $\gamma > 0$ and $\lambda > 0$, we can write

$$\lambda^{-\gamma} = \frac{1}{\Gamma(\gamma)} \int_0^\infty y^{\gamma-1} e^{-\lambda y} dy,$$

where $\Gamma$ is the Gamma function defined by $\Gamma(\gamma) = \int_0^\infty y^{\gamma-1} e^{-y} dy$. As a consequence, the covariance (1.7) can be written as

$$\mathbb{E}[Z_sZ_t] = \frac{1}{\Gamma(\gamma)} \int_0^\infty y^{\gamma-1} e^{-(t+s)y} dy. \quad (3.1)$$

Notice that this representation implies the covariance (1.7) is positive definite. Taking first the expectation with respect to the process $Z$, and using formula (3.1), we obtain

$$R(s, t) = \frac{1}{\Gamma(\gamma)} \int_0^\infty \mathbb{E} \left[ \left( \int_0^t e^{yu} dB_u^H \right) \left( \int_0^t e^{yu} dB_u^H \right) \right] y^{\gamma-1} e^{-(t+s)y} dy$$

$$= \frac{1}{\Gamma(\gamma)} \int_0^\infty \langle e^{yu}1_{[0,t]}(u), e^{yu}1_{[0,s]}(v) \rangle_{\mathcal{F}} y^{\gamma-1} e^{-(t+s)y} dy.$$

Using the isometry between $\tilde{\Lambda}_H$ and a subspace of $\mathcal{F}$ (see section 2.2), we can write

$$\langle e^{yu}1_{[0,t]}(u), e^{yu}1_{[0,s]}(v) \rangle_{\mathcal{F}} = C_H^{-2} \int_\mathbb{R} \frac{|\xi|^{1-2H}(\mathcal{F}1_{[0,t]}e^{y}) (\mathcal{F}1_{[0,s]}e^{y})}{y^2 + \xi^2} d\xi$$

$$= C_H^{-2} \int_\mathbb{R} \frac{|\xi|^{1-2H} (e^{yt+i\xi t} - 1) (e^{ys-i\xi s} - 1)}{y^2 + \xi^2} d\xi,$$

where $(\mathcal{F}1_{[0,t]}e^x)$ denotes the Fourier transform and $C_H = \left( \frac{2\pi}{1+(2H+1)\sin(\pi H)} \right)^{\frac{1}{2}}$. This allows us to write, making the change of variable $\xi = \eta y$,

$$R(s, t) = \frac{1}{\Gamma(\gamma)C_H^2} \int_0^\infty \int_\mathbb{R} y^{\gamma-1} \frac{|\xi|^{1-2H}}{y^2 + \xi^2} (e^{yt} - e^{-yt}) (e^{-i\xi s} - e^{-ys}) d\xi \, dy$$

$$= \frac{1}{\Gamma(\gamma)C_H^2} \int_0^\infty \int_\mathbb{R} y^{\alpha-1} \frac{|\eta|^{1-2H}}{1 + \eta^2} (e^{inyt} - e^{-yt}) (e^{-in\eta s} - e^{-ys}) d\eta \, dy, \quad (3.2)$$

where $\alpha = 2H - \gamma$. By Euler’s identity, adding and subtracting 1 to compensate the singularity of $y^{-\alpha-1}$ at the origin, we can write

$$e^{inyt} - e^{-yt} = (\cos(\eta yt) - 1 + i \sin(\eta yt)) + (1 - e^{-yt}). \quad (3.3)$$
Substituting (3.3) into (3.2) and taking into account that the integral of the imaginary part vanishes because it is an odd function, we obtain

\[ R(s, t) = \frac{2}{\Gamma(\gamma) C_H^2} \int_0^\infty \int_0^\infty y^{-\alpha - 1} \eta^{1-2H} \left( (1 - \cos(\eta yt))(1 - \cos(\eta ys)) + \sin(\eta yt) \sin(\eta ys) + (\cos(\eta ys) - 1)(1 - e^{-yt}) + (\cos(\eta yt) - 1)(1 - e^{-ys}) + (1 - e^{-yt})(1 - e^{-ys}) \right) \, d\eta \, dy. \]

Let \( B^{(j)} = \{B^{(j)}(\eta, t), \eta \geq 0, t \geq 0\}, j = 1, 2 \) denote two independent Brownian sheets. That is, for \( j = 1, 2 \), \( B^{(j)} \) is a continuous Gaussian field with mean zero and covariance given by

\[ \mathbb{E} [B^{(j)}(\eta, t)B^{(j)}(\xi, s)] = \min(\eta, \xi) \times \min(t, s). \]

We define the following stochastic processes:

\[ U_t = \frac{\sqrt{2}}{\Gamma(\gamma) C_H} \int_0^\infty \int_0^\infty y^{-\frac{\alpha}{2} - \frac{\gamma}{2}} \eta^{1-2H} (\cos(\eta yt) - 1) B^{(1)}(d\eta, dy), \quad (3.4) \]

\[ V_t = \frac{\sqrt{2}}{\Gamma(\gamma) C_H} \int_0^\infty \int_0^\infty y^{-\frac{\alpha}{2} - \frac{\gamma}{2}} \eta^{1-2H} (\sin(\eta yt)) B^{(2)}(d\eta, dy), \quad (3.5) \]

\[ Y_t = \frac{\sqrt{2}}{\Gamma(\gamma) C_H} \int_0^\infty \int_0^\infty y^{-\frac{\alpha}{2} - \frac{\gamma}{2}} \eta^{1-2H} (1 - e^{-yt}) B^{(1)}(d\eta, dy), \quad (3.6) \]

where the integrals are Wiener-Itô integrals with respect to the Brownian sheet. We then define the stochastic process \( X_t = \{X_t, t \geq 0\} \) by \( X_t = U_t + V_t + Y_t \), and we have \( \mathbb{E} [X_s X_t] = R(s, t) \) as given in (3.2). These processes have the following properties:

(I) The process \( W_t = U_t + V_t \) is a fractional Brownian motion with Hurst parameter \( \alpha \) scaled with the constant \( \sqrt{\kappa} \). In fact, the covariance of this process is

\[ \mathbb{E}[W_t W_s] = \frac{2}{\Gamma(\gamma) C_H^2} \int_0^\infty \int_0^\infty y^{-\alpha - 1} \eta^{1-2H} \left( (\cos(\eta yt) - 1)(\cos(\eta ys) - 1) + \sin(\eta yt) \sin(\eta ys) \right) \, d\eta \, dy \]

\[ = \frac{1}{\Gamma(\gamma) C_H^2} \int_0^\infty \int_0^\infty y^{\gamma - 1} \frac{|\xi|^{1-2H}}{y^2 + \xi^2} (e^{i\xi t} - 1)(e^{-i\xi s} - 1) \, d\eta \, dy. \]

Integrating in the variable \( y \) we finally obtain

\[ \mathbb{E}[W_t W_s] = \frac{c_1}{\Gamma(\gamma) C_H^2} \int_\mathbb{R} \frac{(e^{i\xi t} - 1)(e^{-i\xi s} - 1)}{|\xi|^\alpha + 1} \, d\xi, \]

where \( c_1 = \int_0^\infty \frac{\gamma - 1}{1 + z^2} \, dz = \kappa \Gamma(\gamma) \). Taking into account the Fourier transform representation of fBm (see [10] page 328), this implies \( \kappa^{-\frac{\alpha}{2}} W \) is a fractional Brownian motion with Hurst parameter \( \frac{\alpha}{2} \).
(II) The process $Y$ coincides, up to a constant, with the process $Y^K$ introduced in (1.4) with $K = 2\alpha + 1$. In fact, the covariance of this process is given by

$$
\mathbb{E}[Y_t Y_s] = \frac{2c_2}{\Gamma(\gamma)C_H^{2\alpha}} \int_0^\infty y^{-\alpha-1}(1-e^{-\gamma t})(1-e^{-\gamma s})dy,
$$

where

$$
c_2 = \int_0^\infty \frac{\eta^{1-2H}}{1+\eta^2}d\eta.
$$

Notice that the process $X$ is self-similar with exponent $\frac{\alpha}{2}$. This concludes the proof of Theorem 1.

4 Proof of Theorem 2

Along the proof, the symbol $C$ denotes a generic, positive constant, which may change from line to line. The value of $C$ will depend on parameters of the process and on $T$, but not on the increment width $n^{-1}$.

For integers $n \geq 1$, define a partition of $[0, \infty)$ composed of the intervals \{[$\frac{j}{n}$, $\frac{j+1}{n}$], $j \geq 0$\}. For the process $X$ and related processes $U, V, W, Y$ defined in Section 3, we introduce the notation

$$
\Delta X_{\frac{j}{n}} = X_{\frac{j+1}{n}} - X_{\frac{j}{n}} \text{ and } \Delta X_0 = X_{\frac{1}{n}},
$$

with corresponding notation for $U, V, W, Y$. We start the proof of Theorem 2 with two technical results about the components of the increments.

4.1 Preliminary Lemmas

**Lemma 3.** Using above notation with integers $n \geq 2$ and $j, k \geq 0$, we have

(a) $\mathbb{E} \left[ \Delta W_{\frac{j}{n}} \Delta W_{\frac{k}{n}} \right] = \frac{\kappa}{2} n^{-\alpha} (|j-k-1|^\alpha - 2|j-k|^\alpha + |j-k-1|^\alpha)$, where $\kappa$ is defined in (1.8).

(b) For $j + k \geq 1$,

$$
\left| \mathbb{E} \left[ \Delta Y_{\frac{j}{n}} \Delta Y_{\frac{k}{n}} \right] \right| \leq C n^{-\alpha} (j + k)^{\alpha-2}
$$

for a constant $C > 0$ that is independent of $j$, $k$ and $n$.

**Proof.** Property (a) is well-known for fractional Brownian motion. For (b), we have from (3.7):

$$
\mathbb{E} \left[ \Delta Y_{\frac{j}{n}} \Delta Y_{\frac{k}{n}} \right] = \frac{2c_2}{\Gamma(\gamma)C_H^{2\alpha}} \int_0^\infty y^{-\alpha-1} \left( e^{-yj} - e^{-y(j+1)} \right) \left( e^{-yk} - e^{-y(k+1)} \right) dy
$$

$$
= \frac{2c_2}{\Gamma(\gamma)C_H^{2\alpha}} \int_0^\infty y^{-\alpha+1} \int_0^1 \int_0^1 e^{-y(j+k+u+v)} du \ dv \ dy.
$$
Note that the above integral is nonnegative, and we can bound this with

\[
\left| \mathbb{E} \left[ \Delta Y_j \Delta Y_k \right] \right| \leq C n^{-\alpha} \int_0^\infty y^{-\alpha+1} e^{-y(j+k)} \, dy
\]

\[
= C n^{-\alpha} (j + k)^{-2} \int_0^\infty u^{-\alpha+1} e^{-u} \, du
\]

\[
\leq C n^{-\alpha} (j + k)^{-2}.
\]

\[\square\]

**Lemma 4.** For \( n \geq 2 \) fixed and integers \( j, k \geq 1 \),

\[
\left| \mathbb{E} \left[ \Delta W_n \Delta Y_k \right] \right| \leq C n^{-\alpha} j^{2H-2} k^{-\gamma}
\]

for a constant \( C > 0 \) that is independent of \( j, k \) and \( n \).

**Proof.** From (3.4) - (3.6) in the proof of Theorem 1, observe that

\[
\mathbb{E} \left[ \Delta W_n \Delta Y_k \right] = \mathbb{E} \left[ (\Delta U_n + \Delta V_n) \Delta Y_k \right] = \mathbb{E} \left[ \Delta U_n \Delta Y_k \right].
\]

Assume \( s, t > 0 \). By self-similarity we can define the covariance function \( \psi \) by

\[
\mathbb{E} \left[ U_t Y_s \right] = s^\alpha \mathbb{E} \left[ U_{t/s} Y_1 \right] = s^\alpha \psi(t/s),
\]

where, using the change-of-variable \( \theta = \eta x \),

\[
\psi(x) = \int_0^\infty \int_0^\infty y^{-\alpha-1} \frac{\eta^{1-2H}}{1 + \eta^2} (\cos(\eta y x) - 1) \left( 1 - e^{-y} \right) \, d\eta \, dy
\]

\[
= \int_0^\infty y^{-\alpha-1} \left( 1 - e^{-y} \right) \int_0^\infty \frac{\theta^{1-2H} x^{2H}}{x^2 + \theta^2} \left( \cos(y \theta) - 1 \right) \, d\theta \, dy.
\]

Then using the fact that

\[
\left| \frac{\theta^{1-2H} x^{2H}}{x^2 + \theta^2} \right| \leq |\theta^{-2H}| |x|^{2H-1}, \tag{4.1}
\]

we see that \( |\psi(x)| \leq C x^{2H-1} \), and

\[
\psi'(x) = 2H \int_0^\infty y^{-\alpha-1} \left( 1 - e^{-y} \right) \int_0^\infty \frac{\theta^{1-2H} x^{2H-1}}{x^2 + \theta^2} \left( \cos(y \theta) - 1 \right) \, d\theta \, dy
\]

\[
- 2 \int_0^\infty y^{-\alpha-1} \left( 1 - e^{-y} \right) \int_0^\infty \frac{\theta^{1-2H} x^{2H+1}}{(x^2 + \theta^2)^2} \left( \cos(y \theta) - 1 \right) \, d\theta \, dy.
\]

Using (4.1) and similarly

\[
\left| \frac{\theta^{1-2H} x^{2H+1}}{(x^2 + \theta^2)^2} \right| \leq |\theta^{-2H}| |x|^{2H-2}, \tag{4.2}
\]

we can write

\[
|\psi'(x)| \leq x^{2H-2} |2H - 2| \int_0^\infty y^{-\alpha-1} \left( 1 - e^{-y} \right) \int_0^\infty \theta^{-2H} \left( \cos(y \theta) - 1 \right) \, d\theta \, dy \leq C x^{2H-2}.
\]
By continuing the computation, we can find that $|\psi''(x)| \leq Cx^{2H-3}$. We have for $j, k \geq 1,$

$$
\mathbb{E} \left[ \Delta U_n \Delta Y_n \right] = n^{-\alpha} (k+1)^\alpha \left( \psi \left( \frac{j+1}{k+1} \right) - \psi \left( \frac{j}{k+1} \right) \right) \\
- n^{-\alpha} k^\alpha \left( \psi \left( \frac{j+1}{k} \right) - \psi \left( \frac{j}{k} \right) \right) \\
= n^{-\alpha} ((k+1)^\alpha - k^\alpha) \left( \psi \left( \frac{j+1}{k+1} \right) - \psi \left( \frac{j}{k+1} \right) \right) \\
+ n^{-\alpha} k^\alpha \left( \psi \left( \frac{j+1}{k+1} \right) - \psi \left( \frac{j}{k+1} \right) - \psi \left( \frac{j+1}{k} \right) + \psi \left( \frac{j}{k} \right) \right).
$$

With the above bounds on $\psi$ and its derivatives, the first term is bounded by

$$
n^{-\alpha} |(k+1)^\alpha - k^\alpha| \left| \psi \left( \frac{j+1}{k+1} \right) - \psi \left( \frac{j}{k+1} \right) \right| \\
\leq an^{-\alpha} \int_0^1 (k+u)^{\alpha-1} du \int_0^{k+1} \left| \psi' \left( \frac{j}{k+1} + v \right) \right| dv \\
\leq Cn^{-\alpha} k^{\alpha-2} \left( \frac{j}{k} \right)^{2H-2} \leq Cn^{-\alpha} k^{-\gamma} j^{2H-2},
$$

and

$$
n^{-\alpha} k^\alpha \left| \psi \left( \frac{j+1}{k+1} \right) - \psi \left( \frac{j}{k+1} \right) - \psi \left( \frac{j+1}{k} \right) + \psi \left( \frac{j}{k} \right) \right| \\
= n^{-\alpha} k^\alpha \left| \int_0^{\frac{1}{k+1}} \psi' \left( \frac{j}{k+1} + u \right) du - \int_0^{\frac{1}{k}} \psi' \left( \frac{j}{k} + u \right) du \right| \\
\leq n^{-\alpha} k^\alpha \int_0^{\frac{1}{k+1}} \left| \psi' \left( \frac{j}{k} + u \right) \right| du + \int_0^{\frac{1}{k+1}} \int_{\frac{1}{k+1}}^{\frac{1}{k}} \left| \psi''(u+v) \right| dv \, du \\
\leq Cn^{-\alpha} k^{\alpha-2} \left( \frac{j}{k} \right)^{2H-2} + Cn^{-\alpha} k^{\alpha-3} j \left( \frac{j}{k} \right)^{2H-3} \leq Cn^{-\alpha} k^{-\gamma} j^{2H-2}.
$$

This concludes the proof of the lemma. \hfill \square

### 4.2 Proof of Theorem 2

We will make use of the notation $\beta_{j,n} = \| \Delta X_n \|_{L^2(\Omega)}$. From Lemma 3 and Lemma 4 we have

$$
\beta_{j,n}^2 = \kappa n^{-\alpha} (1 + \theta_{j,n}),
$$

where $|\theta_{j,n}| \leq C j^{\alpha-2}$ if $j \geq 1$. Notice that, in the definition of $F_n(t)$, it suffices to consider the sum for $j \geq n_0$ for a fixed $n_0$. Then, we can choose $n_0$ in such a way that $Cn_0^{\alpha-2} \leq \frac{1}{2}$, which implies

$$
\beta_{j,n}^2 \geq \kappa n^{-\alpha} (1 - C j^{\alpha-2}) \tag{4.3}
$$
for any $j \geq n_0$.

By (2.4),

$$
\beta_{q,j,n}^q H_q \left( \beta_{j,n}^{-1} \Delta X_{j,n} \right) = I_q^X \left( \left( \frac{1}{n} \right)^{\otimes q} \right),
$$

where $I_q^X$ denotes the multiple stochastic integral of order $q$ with respect to the process $X$. Thus, we can write

$$
F_n(t) = n^{-\frac{1}{2}} \sum_{j=n_0}^{\lfloor nt \rfloor - 1} \beta_{j,n}^{-q} I_q^X \left( \left( \frac{1}{n} \right)^{\otimes q} \right).
$$

The decomposition $X = W + Y$ leads to

$$
I_q^X \left( \left( \frac{1}{n} \right)^{\otimes q} \right) = \sum_{r=0}^{q} \binom{q}{r} I_r^W \left( \left( \frac{1}{n} \right)^{\otimes r} \right) I^{q-r}_{q} \left( \left( \frac{1}{n} \right)^{\otimes q-r} \right).
$$

We are going to show that the terms with $r = 0, \ldots, q-1$ do not contribute to the limit. Define

$$
G_n(t) = n^{-\frac{1}{2}} \sum_{j=n_0}^{\lfloor nt \rfloor - 1} \beta_{j,n}^{-q} I_r^W \left( \left( \frac{1}{n} \right)^{\otimes r} \right)
$$

and

$$
\tilde{G}_n(t) = n^{-\frac{1}{2}} \sum_{j=n_0}^{\lfloor nt \rfloor - 1} \left\| \Delta W_{j,n} \right\|^{-q} L^2(\Omega) I_r^W \left( \left( \frac{1}{n} \right)^{\otimes r} \right).
$$

Consider the decomposition

$$
F_n(t) = (F_n(t) - G_n(t)) + (G_n(t) - \tilde{G}_n(t)) + \tilde{G}_n(t).
$$

Notice that all these processes vanish at $t = 0$. We claim that for any $0 \leq s < t \leq T$, we have

$$
\mathbb{E}[|F_n(t) - G_n(t) - (F_n(s) - G_n(s))|^2] \leq \frac{(\lfloor nt \rfloor - |ns|)^{\delta}}{n} \quad (4.4)
$$

and

$$
\mathbb{E}[|G_n(t) - \tilde{G}_n(t) - (G_n(s) - \tilde{G}_n(s))|^2] \leq \frac{(\lfloor nt \rfloor - |ns|)^{\delta}}{n}, \quad (4.5)
$$

where $0 \leq \delta < 1$. By Lemma 3, $\left\| \Delta W_{j,n} \right\|_{L^2(\Omega)} = \kappa n^{-\alpha}$ for every $j$. As a consequence, using (2.4) we can also write

$$
\tilde{G}_n(t) = n^{-\frac{1}{2}} \sum_{j=n_0}^{\lfloor nt \rfloor - 1} H_q \left( \kappa^{-\frac{1}{2}} n^{\frac{\alpha}{2}} \Delta W_{j,n} \right).
$$

Since $\kappa^{-\frac{1}{2}} W$ is a fractional Brownian motion, the Breuer-Major theorem implies that the process $\tilde{G}$ converges in $D([0,T])$ to a scaled Brownian motion $\{\sigma B_t, t \in [0,T]\}$, where
$\sigma^2$ is given in (1.11). By the fact that all the $p$-norms are equivalent on a fixed Wiener chaos, the estimates (4.4) and (4.5) lead to

$$
\mathbb{E}[|F_n(t) - G_n(t) - (F_n(s) - G_n(s))|^2p] \leq \frac{(|nt| - |ns|)^{2p}}{n^p}
$$

(4.6)

and

$$
\mathbb{E}[|G_n(t) - \tilde{G}_n(t) - (G_n(s) - \tilde{G}_n(s))|^2p] \leq \frac{(|nt| - |ns|)^{2p}}{n^p},
$$

(4.7)

for all $p \geq 1$. Letting $n$ tend to infinity, we deduce from (4.6) and (4.7) that the sequences $F_n - G_n$ and $G_n - \tilde{G}_n$ converge to zero in the topology of $D([0, T])$, as $n$ tends to infinity.

**Proof of (4.4):** We can write

$$
\mathbb{E} \left[ |F_n(t) - G_n(t) - (F_n(s) - G_n(s))|^2 \right] \leq C \sum_{r=0}^{q-1} \mathbb{E}[\Phi^2_{r,n}],
$$

where

$$
\Phi_{r,n} = n^{-\frac{r}{2}} \sum_{|nt| = 1}^{\lfloor n^2 \rfloor} \beta_{j,r} W_{r,n} \left( 1_{\left[ \frac{r}{n}, \frac{r+1}{n} \right]} \right) I_{q-r} \left( 1_{\left[ \frac{r}{n}, \frac{r+1}{n} \right]} \right).
$$

We have, using (4.3),

$$
\mathbb{E}[\Phi^2_{r,n}] \leq n^{-1+q^\alpha} \times \sum_{j,k=\lfloor n^2 \rfloor}^{\lfloor n^2 \rfloor} \mathbb{E} \left[ I_{r}^W \left( 1_{\left[ \frac{r}{n}, \frac{r+1}{n} \right]} \right) I_{q-r}^Y \left( 1_{\left[ \frac{r}{n}, \frac{r+1}{n} \right]} \right) I_{r}^W \left( 1_{\left[ \frac{r}{n}, \frac{r+1}{n} \right]} \right) I_{q-r}^Y \left( 1_{\left[ \frac{r}{n}, \frac{r+1}{n} \right]} \right) \right].
$$

Using a diagram method for the expectation of four stochastic integrals (see [8], we find that, for any $j, k$, the above expectation consists of a sum of terms of the form

$$
\left( \mathbb{E} \left[ \Delta W_{\frac{j}{n}}^2 \Delta W_{\frac{k}{n}}^2 \right] \right)^{a_1} \left( \mathbb{E} \left[ \Delta Y_{\frac{j}{n}}^2 \Delta Y_{\frac{k}{n}}^2 \right] \right)^{a_2} \left( \mathbb{E} \left[ \Delta W_{\frac{j}{n}}^2 \Delta Y_{\frac{k}{n}}^2 \right] \right)^{a_3} \left( \mathbb{E} \left[ \Delta Y_{\frac{j}{n}}^2 \Delta W_{\frac{k}{n}}^2 \right] \right)^{a_4},
$$

where the $a_i$ are nonnegative integers such that $a_1 + a_2 + a_3 + a_4 = q$, $a_1 \leq r \leq q - 1$, and $a_2 \leq q - r$. First, consider the case with $a_3 = a_4 = 0$, so that we have the sum

$$
n^{-1+q^\alpha} \sum_{j,k=\lfloor n^2 \rfloor}^{\lfloor n^2 \rfloor} \left( \mathbb{E} \left[ \Delta W_{\frac{j}{n}}^2 \Delta W_{\frac{k}{n}}^2 \right] \right)^{a_1} \left( \mathbb{E} \left[ \Delta Y_{\frac{j}{n}}^2 \Delta Y_{\frac{k}{n}}^2 \right] \right)^{q-a_1},
$$

where $0 \leq a_1 \leq q - 1$. Applying Lemma 3, we can control each of the terms in the above sum by

$$
n^{-q^\alpha} (j - k + 1)^{\alpha} - 2 |j - k|^\alpha + |j - k - 1|^\alpha a_1 (j + k)^{(q-a_1)(\alpha-2)},
$$
which gives

\[ n^{-1+q \alpha} \sum_{j,k=|ns| \vee n_0} \left[ E \left[ \Delta W_{n}^j \Delta W_{n}^k \right] \right]^{q \alpha} \left[ E \left[ \Delta Y_{n}^j \Delta Y_{n}^k \right] \right]^{q \alpha - a_1} \]

\[ \leq C n^{-1} \left( \sum_{j=|ns| \vee n_0} j^{\alpha-2} + \sum_{j,k=|ns| \vee n_0, j \neq k} |j-k|(q-1)(\alpha-2)(j+k)^{\alpha-2} \right) \]

\[ \leq C n^{-1} \sum_{j=|ns| \vee n_0} (j^{\alpha-2} + j^{q(\alpha-2)+1}) \]

\[ \leq C n^{-1} (|nt| - |ns|)^{(\alpha-1)\nu_0} + (|nt| - |ns|)^{q(\alpha-2)+2}\nu_0) \] . \hspace{1cm} (4.8)

Next, we consider the case where \( a_3 + a_4 \geq 1 \). By Lemma 3, we have that, up to a constant \( C \),

\[ E \left[ \Delta Y_{n}^j \Delta Y_{n}^k \right] \leq C E \left[ \Delta W_{n}^j \Delta W_{n}^k \right], \]

so we may assume \( a_2 = 0 \), and have to handle the term

\[ n^{-1+q \alpha} \sum_{j,k=|ns| \vee n_0} \left[ E \left[ \Delta W_{n}^j \Delta W_{n}^k \right] \right]^{q \alpha - a_3 - a_4} \left[ E \left[ \Delta W_{n}^j \Delta Y_{n}^k \right] \right]^{a_3} \left[ E \left[ \Delta Y_{n}^j \Delta W_{n}^k \right] \right]^{a_4} \] \hspace{1cm} (4.9)

for all allowable values of \( a_3, a_4 \) with \( a_3 + a_4 \geq 1 \). Consider the decomposition

\[ n^{-1+q \alpha} \sum_{j,k=|ns| \vee n_0} \left[ E \left[ \Delta W_{n}^j \Delta W_{n}^k \right] \right]^{q \alpha - a_3 - a_4} \left[ E \left[ \Delta W_{n}^j \Delta Y_{n}^k \right] \right]^{a_3} \left[ E \left[ \Delta Y_{n}^j \Delta W_{n}^k \right] \right]^{a_4} \]

\[ = n^{q \alpha - 1} \sum_{j=|ns| \vee n_0} \left[ E \left[ \Delta W_{n}^j \right] \right]^{q \alpha - a_3 - a_4} \left[ E \left[ \Delta W_{n}^j \Delta Y_{n}^k \right] \right]^{a_3} \left[ E \left[ \Delta Y_{n}^j \Delta W_{n}^k \right] \right]^{a_4} \]

\[ + n^{q \alpha - 1} \sum_{j=|ns| \vee n_0} \sum_{k=|ns| \vee n_0} \left[ E \left[ \Delta W_{n}^j \Delta W_{n}^k \right] \right]^{q \alpha - a_3 - a_4} \left[ E \left[ \Delta W_{n}^j \Delta Y_{n}^k \right] \right]^{a_3} \left[ E \left[ \Delta Y_{n}^j \Delta W_{n}^k \right] \right]^{a_4} \]

\[ + n^{q \alpha - 1} \sum_{k=|ns| \vee n_0} \sum_{j=|ns| \vee n_0} \left[ E \left[ \Delta W_{n}^j \Delta W_{n}^k \right] \right]^{q \alpha - a_3 - a_4} \left[ E \left[ \Delta W_{n}^j \Delta Y_{n}^k \right] \right]^{a_3} \left[ E \left[ \Delta Y_{n}^j \Delta W_{n}^k \right] \right]^{a_4} . \]
We have, by Lemma 3 and Lemma 4,

$$n^{-1+q\alpha} \sum_{j,k=[ns] \cap n_0}^{[nt]-1} \left| \mathbb{E} \left[ \Delta W_n \Delta W_q \right] \right|^{q-a_3-a_4} \left| \mathbb{E} \left[ \Delta W_n \Delta W_m \right] \right|^{a_3} \left| \mathbb{E} \left[ \Delta Y_n \Delta W_q \right] \right|^{a_4} \leq C n^{-1} \sum_{j=[ns] \cap n_0}^{[nt]-1} j^{(a_3+a_4)(\alpha-2)} + C n^{-1} \sum_{j=[ns] \cap n_0}^{[nt]-1} j^{a_3(2H-2)-a_4\beta} \sum_{k=[ns] \cap n_0}^{j-1} k^{-a_3\gamma+a_4(2H-2)} \left| j - k \right|^{(q-a_3-a_4)(\alpha-2)} + C n^{-1} \sum_{k=[ns] \cap n_0}^{[nt]-1} k^{-a_3\gamma+a_4(2H-2)} \sum_{j=[ns] \cap n_0}^{k-1} j^{a_3(2H-2)-a_4\beta} \left| k - j \right|^{(q-a_3-a_4)(\alpha-2)} \leq C n^{-1} \left( ([nt] - [ns])^{(a_3+a_4)(\alpha-2)+1}]^{0} + ([nt] - [ns])^{q(\alpha-2)+1}]^{0} \right. + ([nt] - [ns])^{a_3(2H-2)-a_4\gamma+1]}^{0} + ([nt] - [ns])^{a_4(2H-2)-a_3\gamma+1]}^{0} \right). \quad (4.10)

Then (4.8) and (4.10) imply (4.4) because $\alpha < 2 - \frac{1}{q}$.

Proof of (4.5): We have

$$G_n(t) - \tilde{G}_n(t) = n^{-\frac{1}{2}} \sum_{j=n_0}^{[nt]-1} \left( \beta_{j,n}^{q} - \left\| \Delta W_n \right\|_{L^2(\Omega)}^{q} \right) f_{q}^{W} \left( 1_{[\frac{q}{2}, \frac{q+1}{2}]} \right)$$

and we can write, using (4.3) for any $j \geq n_0$,

$$\left| \beta_{j,n}^{q} - \left\| \Delta W_n \right\|_{L^2(\Omega)}^{q} \right| = (\kappa^{-1} n^a)^{\frac{q}{2}} \left( 1 + \theta_{j,n} \right)^{-\frac{3}{2}} - 1 \leq C \left( \kappa^{-1} n^a j^{-2} \right)^{\frac{q}{2}}.$$

This leads to the estimate

$$\mathbb{E} \left[ \left| G_n(t) - \tilde{G}_n(t) - (G_n(s) - \tilde{G}_n(s)) \right|^2 \right] \leq C n^{-1} \times \left( \sum_{j=[ns] \cap n_0}^{[nt]-1} j^{a-2} + \sum_{j,k=[ns] \cap n_0, j \neq k}^{[nt]-1} |j - k|^{q(\alpha-2)} \right) \leq C n^{-1} \left( ([nt] - [ns])^{(\alpha-1)^{0}} + ([nt] - [ns])^{q(\alpha-2)+2]^{0}} \right),$$

which implies (4.3).

This concludes the proof of Theorem 2.

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