On the Genus of the Moonshine Module

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Dedicated to Robert L. Griess on the occasion of his seventy-first birthday

Abstract

We provide a novel and simple description of Schellekens’ seventy-one affine Kac-Moody structures of self-dual vertex operator algebras of central charge 24 by utilizing cyclic subgroups of the glue codes of the Niemeier lattices with roots. We also discuss a possible uniform construction procedure of the self-dual vertex operator algebras of central charge 24 starting from the Leech lattice. This also allows us to consider the uniqueness question for all non-trivial affine Kac-Moody structures. We finally discuss our description from a Lorentzian viewpoint.

1 Introduction

Work of Borcherds [Bo2] and Frenkel, Lepowsky and Meurmann [FLM] established the existence of a vertex operator algebra of central charge 24 with trivial affine Kac-Moody subalgebra called the Moonshine module. Dong [Do] showed that the Moonshine module is self-dual which states that its only irreducible module up to isomorphism is the vertex operator algebra itself. It is an open problem to show the uniqueness of a vertex operator algebra with these properties. The Moonshine module is generated by the Griess algebra [Gr1] and has the monster sporadic group as its automorphism group.

The construction of the Moonshine module utilizes the Leech lattice [Le], the unique positive definite even unimodular lattice with minimum norm 4 of rank 24. It was first shown by Niemeier [Nie] that there are 23 other even unimodular lattices of rank 24 all with minimum norm 2 forming together the genus of the Leech lattice. Later it was observed that there is a one-to-one correspondence between the deep holes in the Leech lattice and the 23 Niemeier lattices with roots [CPS], an observation explained in [Bo1].

Schellekens [Sch1] considered the problem of describing the complete genus of the Moonshine module, i.e. to find all self-dual vertex operator algebras of central charge 24 up to isomorphism. He showed that there are exactly 71 possibilities for the affine Kac-Moody structure, that is the vertex operator subalgebra generated by the weight 1 vectors. The lattice vertex operator algebra construction applied to the Niemeier lattices realizes 24; the \( \mathbb{Z}_2 \)-orbifold construction for the Moonshine module applied to all Niemeier lattices provides a total of 15 further cases [DGM].

Much progress has been made since Schellekens’ work. First, all the properties of vertex operator algebras which he used in his classification approach which was formulated in the physical language of conformal field theory have been mathematically

*Supported by the Simons Foundation, Award ID: 355294
rigorously established [DM04a, DM06]. Secondly, his method in obtaining the list of 71 possible affine Kac-Moody structures has been somewhat refined and confirmed in [EMS]. The basic approach is an analog of Venkov’s [Ven] approach of finding the possible root lattices of the Niemeier lattices. This way one obtains first a list of 223 cases from which all besides the surviving 71 have to be eliminated by various methods.

Of the 71 possible affine Kac-Moody structures, one is trivial, one is abelian and the remaining 69 are of semi-simple type and the Kac-Moody vertex operator subalgebra has already central charge 24. The first case is realized by the Moonshine module. The second case is realized by the Leech lattice vertex operator algebra and its uniqueness follows from the uniqueness of the Leech lattice as the unique even unimodular lattice of rank 24 without roots.

For the 69 non-abelian affine Kac-Moody structures, the problem is reduced to describe all equivalence classes of self-dual extensions. This is a problem which can be completely described in terms of the modular tensor category associated to the Kac-Moody vertex operator subalgebra. However, besides the case of so-called simple current extensions, no well-developed general theory seems to be presently available so that only a few cases have been studied from this viewpoint. In [DGH], Remark 5.4, it was observed that Schellekens case no. 5 with affine Kac Moody structure $A_{16}^{1}$ is equivalent to the existence and uniqueness of a certain Virasoro frame in the $E_8$-lattice vertex operator algebra since the associated modular tensor categories are Galois equivalent. The uniqueness of the corresponding Virasoro frame has later been established in [GH]. Using the language of conformal nets and so-called mirror extensions, F. Xu has shown that Schellekens’ cases no. 18, 27 and 40 exist.

Work of Carnahan, Dong, van Ekeren, Lam, Mason, Miyamoto, Möller, Scheithauer, Shimakura and others over the last twenty-four years have finally established the existence for all remaining cases of Schellekens’ list. Their techniques are generalizations of the original orbifold construction of the Moonshine module: First, $\mathbb{Z}_2$-orbifolds for various outer automorphisms had been considered. Then this approach was generalized to $\mathbb{Z}_p$-orbifolds for other primes $p$, and finally inner automorphisms have been used. The orbifold approach also allows to prove uniqueness in several cases.

The cases where the rank of the Kac-Moody structure is 16 and certain of the rank 12 cases may be treated using Virasoro framed vertex operator algebras (or, alternatively, $\mathbb{Z}_2$-orbifolds of lattice vertex operator algebras) of central charge 8 and 12 by utilizing their associated modular tensor categories together with the classification of certain lattice genera. This approach was discussed in [HS]. It was observed by Carnahan [Ca], that a crucial connection between vertex operator algebras and modular tensor categories was still missing. This was finally established in [EMS, Mo].

In the present paper, we provide a novel and simple description of Schellekens’ 71 affine Kac-Moody structures of self-dual vertex operator algebras of central charge 24 by utilizing cyclic subgroups of the glue codes of the Niemeier lattices with roots.

We also present a new and uniform method (some parts of it already sketched in [HS]) regarding the existence and uniqueness problem of self-dual vertex operator algebras for all 69 non-abelian Kac-Moody structures. We use the theory of even lattices and their automorphism groups together with a more combinatorial viewpoint resulting in a much more uniform and hence conceptually easier to understand construction approach. In particular, we utilize certain orbifold vertex operator algebras associated to automorphism groups of the Leech lattice. We also profit from a more fully developed theory of vertex operator algebras. Certain properties of the used vertex operator algebras remain conjectures in some cases.
Our main results are the following theorems:

**Theorem 1.1.** The possible non-abelian affine Kac-Moody structures of self-dual vertex operator algebras of central charge $24$ are in a natural bijective correspondence with the equivalence classes of cyclic subgroups $Z$ of positive type of the glue codes of the twenty-three Niemeier lattices.

Here, *positive type* means that the frame shape of the elements of the orthogonal group of $\mathbb{R}^{24}$ naturally associated $Z$ has only positive exponents.

Let $\Lambda$ denote the Leech lattice. Under certain assumptions we show:

**Theorem 1.2.** The self-dual vertex operator algebras of central charge $24$ with non-trivial Kac-Moody structure are in natural bijective correspondence to triples $(g, L, [i])$ where $g$ belongs to eleven conjugacy classes of the Conway group, $L$ is an isometry class of lattices in a genus determined by $g$ and $[i]$ describes the isomorphism class of possible gluings between vertex operator algebras $V_L$ and $W = V_{\Lambda g}^\perp$.

Let $M \cong \Lambda \oplus II_{1,1}$ be the unique even unimodular Lorentzian lattice of signature $(25, 1)$.

**Theorem 1.3.** The self-dual vertex operator algebras of central charge $24$ with non-abelian affine Kac-Moody structures are in one-to-one correspondence to $O(M)$-orbits of pairs $(g, v)$ where $g$ is an element in $O(M)$ arising from an element in $O(\Lambda)$ with a frame shape as in cases $A$ to $J$ of Table 4. $v$ is an isotropic vector of $M$ where the Niemeier lattice $v^*/\mathbb{Z}v$ is not the Leech lattice and $g$ fixes $v$. Here, we let $O(M)$ act on the first component of the pair $(g, v)$ by conjugation and use the natural $O(M)$-action on the second.

The paper is organized as follows: Section 2 introduces some of the used notation from lattice and vertex operator algebra theory. We also formulate the results and general conjectures which we will use or assume. Section 3 describes the correspondence between the Niemeier lattices and Schellekens affine Kac-Moody structures. Section 4 contains the uniform construction approach using fixed-point vertex operator algebras related to the Leech lattice and also studies the uniqueness problem. The last Section 5 interprets the results from the previous two sections from a Lorentzian viewpoint although many questions remain.

I would like to thank Richard Borcherds, Sven Möller and Nils Scheithauer for answering questions and Ching Hung, Geoffrey Mason, Hiroki Shimakura and in particular John Duncan for helpful comments. I also like to thank Philipp Höhn in helping to proofread the tables.

# 2 Lattices and vertex operator algebras

## 2.1 Even lattices

We introduce some notation related to integral lattices and their automorphism groups and record some results that we will need.

A lattice $L$ is a finitely generated free $\mathbb{Z}$-module together with a rational-valued symmetric bilinear form $(\cdot, \cdot)$. All lattices in this paper besides in the last section are assumed to be positive-definite. We let $O(L) := \text{Aut}(L)$ be the group of automorphisms (or isometries) of $L$ considered as lattice, i.e., the set of automorphisms of the group $L$ that preserve the bilinear form. It is finite because of the assumed positive-definiteness of the bilinear form. The lattice $L$ is integral if the bilinear form takes values in $\mathbb{Z}$.
and even if the norm \((x, x)\) belongs to \(2\mathbb{Z}\) for all \(x \in L\). An even lattice is necessarily integral.

A finite quadratic space \(A = (A, q)\) is a finite abelian group \(A\) equipped with a quadratic form \(q: A \to \mathbb{Q}/2\mathbb{Z}\). We denote the corresponding orthogonal group by \(O(A)\). This is the subgroup of \(\text{Aut}(A)\) that leaves \(q\) invariant.

The dual lattice of an integral lattice \(L\) is

\[
L^* := \{x \in L \otimes \mathbb{Q} \mid (x, y) \in \mathbb{Z} \text{ for all } y \in L\}.
\]

The discriminant group \(L^*/L\) of an even lattice \(L\) is equipped with the discriminant form \(q_L: L^*/L \to \mathbb{Q}/2\mathbb{Z}\), \(x + L \mapsto (x, x) \pmod{2\mathbb{Z}}\). This turns \(L^*/L\) into a finite quadratic space, called the discriminant space of \(L\) and denoted \(A_L = (A_L, q_L) := (L^*/L, q_L)\). There is a natural induced action of \(O(L)\) on \(A_L\), leading to a short exact sequence

\[
1 \to O_0(L) \to O(L) \to \overline{O}(L) \to 1,
\]

where \(\overline{O}(L)\) is the subgroup of \(O(A_L)\) induced by \(O(L)\) and \(O_0(L)\) consists of the automorphisms of \(L\) which act trivially on \(A_L\).

A sublattice \(K \subseteq L\) is called primitive (in \(L\)) if \(L/K\) is a free abelian group. We set

\[
K^\perp := \{x \in L \mid (x, y) = 0 \text{ for all } y \in K\}.
\]

Assume now that \(L\) is even and unimodular, i.e., \(L^* = L\). If \(K\) is primitive then there is an isomorphism of groups \(i: A_K \xrightarrow{\cong} A_{K^\perp}\) such that \(q_{K^\perp} i(a)) = -q_K(a)\) for \(a \in A_K\). We can recover \(L\) from \(K \oplus K^\perp\) by adjoining the cosets

\[
C := \{(a, i(a)) \mid a \in A_K\} \subseteq A_K \oplus A_{K^\perp}.
\]

See [Nik] for further details. The following is a special case of another result (Propositions 1.4.1 and 1.6.1, loc. cit).

**Proposition 2.1.** The equivalence classes of extensions of \(K \oplus K^\perp\) to an even unimodular lattice \(N\) with \(K\) primitively embedded into \(N\) are in bijective correspondence with double cosets \(\overline{O}(K) \backslash O(A_K)/(i^* \overline{O}(K^\perp))\), where \(i^*: \overline{O}(K^\perp) \to O(A_K)\) is defined by \(g \mapsto i^{-1} \circ g \circ i\).

Suppose that \(G \subseteq O(L)\) is a group of automorphisms of a lattice \(L\). The invariant and coinvariant lattices for \(G\) are

\[
L^G = \{x \in L \mid gx = x \text{ for all } g \in G\},
\]

\[
L_G = (L^G)^\perp = \{x \in L \mid (x, y) = 0 \text{ for all } y \in L^G\}
\]

respectively. They are both primitive sublattices of \(L\). The restriction of the \(G\)-action to \(L_G\) induces an embedding \(G \subseteq O(L_G)\).

We also note that the genus of a positive-definite even lattice \(L\) is determined by the quadratic space \(A_L\) together with the rank of \(L\) [Nik].

A root of \(L\) is a primitive vector \(v\) in \(L\) such that the reflection \(s_v: w \mapsto w - 2\frac{(w, v)}{(v, v)} v\) in \((\mathbb{Z}v)^\perp\) is an isometry of \(L\). The root sublattice \(R\) of \(L\) is the sublattice spanned by all roots. The root lattice \(R\) is the orthogonal direct sum of lattices spanned by certain scalar multiples of the irreducible root systems of type \(A_n\) \((n \geq 1)\), \(B_n\) \((n \geq 2)\), \(C_n\) \((n \geq 3)\), \(D_n\) \((n \geq 4)\), \(E_6\), \(E_7\), \(E_8\), \(F_4\), \(G_2\), cf. [SB]. We write \(X_{n,k}\) for the irreducible root lattice of type \(X_n\) with all norms scaled by a factor \(k\). In the case of even unimodular lattices, only the irreducible root systems of type \(A_n\), \(D_n\) and \(E_n\) with the roots \(v\) having norm \((v, v) = 2\) can occur.
We assume now that the rank of $R$ equals the rank of $L$. Then $L$ can be described uniquely by $R$ and the glue code $C = L/R \subset R^\ast/R$ consisting of rescaled glue vectors of the component root lattices. The automorphism group of $L$ is a semi-direct product $O(L) = W(R)\text{Aut}(C)$. Here, $W(R)$ is the Weyl group of $R$, i.e. the group generated by the reflections $s_v$ at roots $v \in R \subset L$. It is the direct product of the Weyl groups of the root lattice components. The group $\text{Aut}(C) \subset O(L)/W(R)$ are those outer automorphisms of $R$ which map the glue code $C$ onto itself. One has $\text{Aut}(C) = G_1.G_2$, where $G_1$ are the automorphisms of $C$ fixing each root component setwise and $G_2$ is the permutation group of the root components induced by the automorphisms of $C$.

For more details see [56]. We list the possible irreducible root lattices with additional data in Table 1.

There is a certain ambiguity arising when describing extensions $L$ of root lattices $R$ by glue codes $C$. The lattices $D_{n,k}$ and $B_{n,k}$, $A_{2,k}$ and $G_{2,k}$, as well as $B_{4,k}$ and $F_{4,k}$ are equal; the decision how to name a component will depend on the glue code: if no glue vector components $s$ or $c$ are present for $D_{n,k}$ (resp. 1 or 2 for $A_{2,k}$ and 1 for $B_{4,k}$) then $B_{n,k}$ (resp. $G_{2,k}$ and $F_{4,k}$) is used. The extension $L$ of $R$ by $C$ may have additional roots. For example, $F_{4,k}$ equals the root lattice $C_{4,k}$ extended by the glue code $C$ generated by (1). If $R$ is not the full root lattice of an extension $L$ of $R$ by $C$ then $O(L)$ may be strictly larger than $W(R):\text{Aut}(C)$ since the sublattice $R$ may not be stabilized by $O(L)$.

We denote by $\Lambda$ the Leech lattice.

### 2.2 Vertex operator algebras

We assume that the reader is familiar with the general language of vertex operator algebras and modular tensor categories. In the following, we discuss some vertex operator algebra notions and constructions by using the language of modular tensor categories which we need to formulate our approach. We also formulate Schellekens’ result on the classification of self-dual vertex operator algebras of central charge 24 and describe the progress which has been made.

For each finite non-degenerated quadratic space $(A, q)$, there exists a modular tensor category which we denote by $Q(A, q)$. The conjugate of a modular tensor category $\mathcal{T}$ we denote by $\overline{\mathcal{T}}$. One has $\overline{Q}(A, q) \cong Q(A, -q)$.

The vertex operator algebras in this paper are in general assumed to be sufficiently nice, namely, they should be simple, rational, $C_2$-cofinite, self-contragredient and of CFT-type. Under these conditions, it follows from results from Huang [41] that the representation category of a vertex operator algebra forms a modular tensor category which we denote by $\mathcal{T}(V)$.

We often assume in addition that the conformal weights of all irreducible modules besides the vertex operator algebra itself are positive (property P).
We call $W$ an extension of a vertex operator algebra $V$ if $V$ is a vertex operator subalgebra of $W$ with the same Virasoro element. It is possible to describe $W$ and $\mathcal{T}(W)$ in terms of $V$ and the modular tensor category $\mathcal{T}(V)$ (cf. [HI]) by using the notion of a certain type of algebra for $\mathcal{T}(V)$. However, we will not make use of this notation in its full generality. Two extensions $W'$ and $W''$ of $V$ are called equivalent if there exists an automorphism of $V$ which extends to an isomorphism of $W'$ with $W''$.

We will be mostly considering situations where all the irreducible modules are simple currents which is equivalent with the property that the quantum dimensions $T_i$ are integers. This notion generalizes the notion of a certain type of algebra for $Q$-space ($A, q$) where $q$ is a gap. Examples are the lattice vertex operator algebras $V_L$ associated to even lattices $L$. Here one has $\mathcal{T}(V_L) = Q(A_L, q_L)$, the modular tensor category associated to the discriminant space $(L^*/L, q_L)$ of $L$.

One has the following simple current extension theorem, generalizing the corresponding results for even lattices (cf. [Nil]).

**Theorem 2.2** (cf. [HI] and [Mo], Theorem 3.5.1 for the full statement). Let $V$ be a vertex operator algebra satisfying property $P$ for which all modules are simple currents, i.e. one has $\mathcal{T}(V) = Q(A, q)$ for a finite quadratic space $(A, q)$. Then the extensions $W$ of $V$ are up to isomorphism given by the isotropic subspaces $C$ of $(A, q)$. As $V$-module one has $W \cong \bigoplus_{i \in C} V_i$, where the $V_i$, $i \in A$, are representatives of the isomorphism classes of irreducible $V$-modules. Furthermore, one has $\mathcal{T}(W) = Q(C^\perp/C, q_{C^\perp})$.

In [HI], we made the following definition:

**Definition 2.3.** The genus of a vertex operator algebra $V$ is the pair $(\mathcal{T}(V), c)$ consisting of the modular tensor category associated to $V$ and the central charge of $V$.

This notion generalizes the notion of genera for lattices. Given a vertex operator algebra genus $\mathcal{H}$, we mean with “the vertex operator algebras in the genus $\mathcal{H}$" the set of isomorphism types of vertex operator algebras having genus $\mathcal{H}$. Given an even lattice $L$, the vertex operator algebras associated to the lattices in the genus of $L$ belong to the vertex operator algebra genus of $V_L$, but there may be further vertex operator algebras in the genus.

**Definition 2.4.** A self-dual (often called holomorphic) vertex operator algebra is a vertex operator algebra with trivial modular tensor category, i.e. the only simple object up to isomorphism is the vertex operator algebra itself.

Given a vertex operator subalgebra $U$ of a vertex operator algebra $V$ with possibly different Virasoro elements, one can define the coset or commutant $\text{Com}_V(U) = \{ v \in V \mid u_n v = 0 \text{ for all } u \in U \text{ and } n \geq 0 \}$.

**Conjecture 2.5.** Let $U$ be a vertex operator subalgebra of a vertex operator algebra $V$. Then the commutant $\text{Com}_V(U)$ is again a vertex operator algebra, i.e. satisfies all the general assumptions which we made.

We call $U$ a primitive subalgebra of $V$ if $U = \text{Com}_V(\text{Com}_V(U))$. This is again a generalization of the corresponding notion for lattices. In generalization of the lattice situation we expect:

**Conjecture 2.6.** Let $U$ be a primitive vertex operator subalgebra of a self-dual vertex operator algebra $V$. Then one has $\mathcal{T}(\text{Com}_V(U)) \cong \mathcal{T}(U)$.
The map $\text{Aut}(V) \times \text{Irr}(V) \to \text{Irr}(V)$, $(g, M) \mapsto M^g$, where $M^g$ is the vector space $M$ with $V$-module structure $Y_M^g(v, z) = Y_M^g(gv, z)$ defines a permutation action of $\text{Aut}(V)$ on the set of isomorphism classes of irreducible $V$-modules. Thus there is a short exact sequence

$$1 \to \text{Aut}_0(V) \to \text{Aut}(V) \to \overline{\text{Aut}}(V) \to 1,$$

where $\text{Aut}_0(V)$ is the normal subgroup of $\text{Aut}(V)$ consisting of automorphisms which act trivially and $\overline{\text{Aut}}(V)$ is the quotient. If $V$ is a vertex operator algebra with $\mathcal{T}(V) = \mathcal{Q}(A, q)$ for a finite quadratic space $(A, q)$ then $\overline{\text{Aut}}(V)$ is a subgroup of $\text{O}(A, q)$. For a lattice vertex operator algebra $V_L$ one has $\overline{\text{Aut}}(V_L) = \text{O}(L)$ as one can see from the description of $\text{O}(L)$ in [DN].

**Theorem 2.7.** Let $U$ and $V$ be two vertex operator algebras for which all modules are simple currents. Assume that $\mathcal{T}(U) = \mathcal{Q}(A_U, q_U)$ and $\mathcal{T}(V) = \mathcal{Q}(A_V, q_V) \cong \mathcal{Q}(A_U, -q_U) = \mathcal{T}(U)$. Then the equivalence classes of extensions of $U \otimes V$ to a self-dual vertex operator algebra $W$ with $U$ primitively embedded into $W$ are in bijective correspondence with the double cosets $\overline{\text{Aut}}(U) \backslash \text{O}(A_U, q_U) / i^* \overline{\text{Aut}}(V)$ where $i : (A_U, q_U) \to (A_V, q_V)$ is an arbitrary anti-isometry, i.e. an isomorphism of groups such that $q_V(i(a)) = -q_U(a)$ for $a \in A_U$, and $i^* : \overline{\text{Aut}}(V) \to \text{O}(A_U, q_U)$ is defined by $g \mapsto i^-1 \circ g \circ i$.

**Proof:** This is the analog of the corresponding theorem of Nikulin [Nik] for lattices which we formulated in Proposition 2.7. Because of Theorem 2.7 the same argument as given by Nikulin can be used. We note that the number of orbits of the double coset action of $\overline{\text{Aut}}(U) \times i^* \overline{\text{Aut}}(V)$ on $\text{O}(A_U, q_U)$ is independent of the chosen anti-isometry $i : (A_U, q_U) \to (A_V, q_V)$.

**Theorem 2.8** ([CM] [MH]). Let $V$ be a vertex operator algebra and $G$ be a finite solvable group $G$ of automorphisms of $V$. Then the fixed-point space $V^G$ is again a vertex operator algebra, i.e. satisfies all the general assumptions which we made.

We recall that for a simple Lie algebra $g$ and a positive integer level $k$, there exists the affine Kac-Moody vertex operator algebra $V_{g,k}$ with underlying vector space a highest weight module of the affine Kac-Moody algebra for $g$. The irreducible objects of the associated Chern-Simons or quantum group modular tensor category $\mathcal{M}(g, k) = \mathcal{T}(V_{g,k})$ are given by all the level $k$ highest weight modules of the affine Kac-Moody algebra for $g$. The elements in the glue group for $g$ as in Table 1 correspond to the simple currents of $\mathcal{M}(g, k)$. (In addition, there exists a simple current for $g$ of type $E_8$ and level 2 which we will ignore.) The conformal weights of these simple currents depend on the level $k$. For a Cartan algebra $t$ of $g$, the double commutant $\mathcal{T} = \text{Com}_V(\text{Com}_V(T))$ of the (non rational) Heisenberg vertex operator algebra $T$ generated by $t \subset g = (V_{g,k})_1$ is isomorphic to the lattice vertex operator algebra $V_L$ where $L$ is the even root lattice of type $g$ with norms scaled by $k$ as in Table 1.

We call the subalgebra $U$ generated by the degree one component $V_1$ of a vertex operator algebra $V$ the affine Kac-Moody vertex operator subalgebra of $V$.

**Theorem 2.9** ([DM04a] [DM06]). Let $V$ be a self-dual vertex operator algebra satisfying our general assumptions. Then $V_i$ is a reductive Lie algebra, i.e. isomorphic to $g = g_1 \oplus \cdots \oplus g_r \oplus s$ where the $g_i$ are simple Lie algebras and $s$ is abelian. Furthermore, assuming $s = 0$, the affine Kac-Moody vertex operator subalgebra $U$ is a tensor product $U = U_1 \otimes \cdots \otimes U_r$ where each $U_i$ is a highest weight module of the affine Kac-Moody algebra for $g_i$ of level $k_i$.\[7]
The simple currents of each factor \( U_i \cong V_{g_i} \), are given by the glue group of \( g_i \).
The simple currents of \( U \) which appear in the decomposition of \( V \) as \( U \)-module form what we call the *simple current code* and which we denote by \( D \). The extension of \( U \) by the simple currents in \( D \) is a subalgebra \( \tilde{U} \) of \( V \) which we call the *extended* affine Kac-Moody vertex operator subalgebra of \( V \). For a Cartan algebra \( t \) of \( g_1 \oplus \cdots \oplus g_r \), the double commutant \( T = \text{Com}_V(\text{Com}_V(T)) \) of the Heisenberg vertex operator algebra \( T \) generated by \( t \subset g = V_1 \) is isomorphic to the lattice vertex operator algebra \( V_L \), where \( L \) is isometric to the direct sum of the root lattices of type \( g_i \) scaled by the factor \( k_i \) and then extended by the simple current code \( D \).

We refer to the abstract isomorphism type of \( U \) as the *affine Kac-Moody structure* of \( V \).

The following two results were proven by Schellekens using several implicit assumption which are now theorems for vertex operator algebras.

**Theorem 2.10** (Schellekens [Sch1]). The Lie algebra \( V_1 \) of a self-dual vertex operator algebra of central charge 24 is either 0, abelian of rank 24, or one of 69 semisimple Lie algebras. Furthermore, the affine Kac-Moody algebra \( U \) is in the possible 70 non-trivial cases uniquely determined and equals the entry in the table given in [Sch1].

**Theorem 2.11** (Schellekens [Sch1, Sch2]). For any of the possible 71 affine Kac-Moody algebras \( U \), the multiplicities occurring in the decomposition of a self-dual vertex operator algebra \( V \) of central charge 24 into \( U \)-modules depend only on the affine Kac-Moody structure and are the ones given in the table in [Sch1].

Theorem 2.10 reduces the classification of self-dual vertex operator algebras — besides the uniqueness of the moonshine module — to the classification of the extensions of the Kac-Moody vertex operator algebras \( U \), a problem which can be formulated essentially in terms of the modular tensor category \( \mathcal{T}(U) \). Theorem 2.11 solves this problem half-way. In particular, it is enough to consider the modular tensor category \( \mathcal{T}(\tilde{U}) \) of the extended affine Kac-Moody subalgebra \( \tilde{U} \).

Collecting all the previous results [L11, LS12, Mi13, LS15, SS, EMS, LS16a, LS16b] or announced results on the existence and uniqueness one has:

**Theorem 2.12** (Existence and uniqueness). For all 71 Kac-Moody structures found by Schellekens there exists a self-dual vertex operator algebra of central charge 24 realizing this structure. Furthermore, at least for the Kac-Moody structures obtained from the Niemeier lattices and the structures \( A_{1,2}^6, E_8, B_{8,1}, E_6, C_{2,1}^3, A_{2,3}^6 \) or \( A_{5,4}D_{4,3}A_{3,1,1}^1 \), the corresponding self-dual vertex operator algebra is unique.

In the present paper, we will however not make use of this achievement.

We also need the following notation: The *character* of a vertex operator algebra \( V \) of central charge \( c \) we denote by \( \chi_V \). The collection of the characters of all isomorphism classes of irreducible modules of \( V \) we call the *full character* of \( V \) and denote it with \( \Xi_V = (\Xi_a)_{a \in \text{Irr}(V)} \). The full character is a vector valued modular form of weight 0 for the representation of \( \text{SL}_2(\mathbb{Z}) \) defined by \( \mathcal{T}(V) \) and has poles up to order \( c/24 \) under the assumption of property P.

### 3 Niemeier lattices and associated orbit lattices

We give a simple and uniform bijection between certain equivalence classes of cyclic subgroups of the glue codes of the twenty-three Niemeier lattices with roots and the 69 non-abelian Kac-Moody structures of self-dual central charge 24 vertex operator algebras found by Schellekens by considering so-called orbit lattices. The simple current
The even unimodular lattices of rank 24 have first been classified by Niemeier [Nie]. A simplified approach was given by Venkov [Ven]. His approach classifies first the possible root lattices \( R \subset N \) showing that \( R \) has either rank 0 or has rank 24 in which case there are twenty-three possibilities. In the former case, \( N \) can be shown to be isomorphic to the Leech lattice and, in the later case, there exists for each possible root lattices \( R \) glue code. We also study the arising orbit lattices and their genera in detail.

Using this information, it is a straightforward calculation to enumerate the orbits of \( \text{Aut}(C) \) on \( C \) and to determine the equivalence classes of cyclic subgroups of \( C \) in each of the twenty-three cases. For example, in case of the root lattice \( A_1^{24} \) which corresponds to the Golay code of length 24, it is known that the automorphism group which is the Mathieu group \( M_{24} \) acts transitively on the octads and dodecads of the Golay code. Hence there are exactly five orbits of code vectors and thus cyclic subgroups inside the Golay code.

The result for all twenty-three Niemeier lattices are shown in Table 3. The first row in the table gives the root sublattice \( R \) of the Niemeier lattices. For each Niemeier lattice, column two lists a generator for each orbit of cyclic subgroups \( Z \). Column three gives the order of the cyclic group. Column four lists the different minimal norms of the cosets \( R + v \) of \( R \) for all cosets \( (R + v)/R \in Z \). Again we refer to [CCNPW] for the notation for the glue vectors. For a component \( D_n \), we use the notation 0, \( s \), \( v \), and \( c \) instead of 0, 1, 2, and 3 for the cosets. The listed generator may actually belong to an equivalent code.

We associate now to a cyclic subgroup \( Z = \langle c \rangle \) of a Niemeier lattice \( N \) a new lattice \( N(Z) \) which we call the orbit lattice of \( N \) by \( Z \).

Let \( S \) be a simple root system of type \( A_n \), \( D_n \) or \( E_n \). The discriminant group \( A_S \) of the corresponding root lattice can be identified with a normal subgroup of the automorphism group of the corresponding affine Dynkin diagram \( S \). To a cyclic subgroup \( \langle d \rangle \) of \( A_S \) we can associate an orbit diagram \( S/\langle d \rangle \) which is itself the affinization of a Dynkin diagram. We let \( S(d) \) be the corresponding unextended root system. More explicitly, we define \( S(d) \) for \( d \neq 0 \) by the Table 2 where we may assume \( i((n + 1) \) and we also set \( S(0) = S \). The root systems \( B_n \), \( C_n \), \( G_2 \) and \( F_4 \) are scaled as in Table 1 for \( k = 1 \). We also define the type of \( d \) as the formal expression \( \prod v^{n_k} \) listed in row type of Table 2. For the type of \( d = 0 \) we set \( 1^{\text{rank}(S)} \).

Consider a Niemeier lattice \( N \) having a root system with irreducible components

\[
\begin{array}{cccccccc}
S & A_n & D_{2k} & D_{2k} & D_{2k+1} & D_{2k+1} & E_6 & E_7 \\
\hline
d & i & s & v & s & v & 1 & 1 \\
S(d) & \sqrt{\frac{n+1}{i}} A_{i-1} & B_k & C_{2k-2} & B_{k-1} & C_{2k-1} & G_2 & F_4 \\
\hline
\text{induced glue group} & \mathbb{Z}/i\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & - & \mathbb{Z}/2\mathbb{Z} & - & - \\
type & 1^{-1}(\frac{n+1}{i})i & 2^k & 1^{2k-4}2^2 & 1^{-1}2^{k-1}4^1 & 1^{2k-3}2^2 & 3^2 & 1.2^3
\end{array}
\]
$S_i, i = 1, \ldots, r$. A glue vector $c \in C = N/R$ is given by an $r$-tuple $(c_1, \ldots, c_r)$. Let $\ell$ be the order of $c$ in $C$ and $m_i$ be the order of $c_i$ in $A_{S_i}$ for $i = 1, \ldots, r$. We define for a cyclic group $Z = \langle c \rangle$ the orbit root lattice $R(Z)$ as the direct sum of the rescaled simple orbit root lattices $\sqrt{m_i}S(c_i), i = 1, \ldots, r$, where the scaling factors $k_i$ are given by $k_i = \frac{1}{m_i} \alpha$. Here, $\alpha$ is defined as

$$
\alpha = \begin{cases} 
1, & \text{if norm}(Z) = 4, \\
2, & \text{if norm}(Z) = 6, \\
\text{undetermined}, & \text{if norm}(Z) > 6,
\end{cases}
$$

where norm$(Z)$ denotes the largest minimal norm of a coset $R + v$ for all cosets with $R + v/R \in Z$. The orbit lattice $N(Z)$ itself is the root lattice $R(Z)$ extended by the orbit glue code $C(Z) \subset R(Z)^*/R(Z)$ which is defined by the componentwise projection of the elements of $C$ onto the induced glue groups as indicated by the fourth row in Table 2. We note that $R(Z)$ may not be the full root lattice of $N(Z)$. The type of $c$ (or $Z$) is defined as the formal product of the types of the $c_i$, i.e., equals $\prod_i v^{\sum_i c_i}$.

The resulting root lattices $R(Z)$ for all equivalence classes $Z$ are listed in column $R(Z)$ of Table 3. The next column $\dim g$ provides the dimension of the Lie algebra $g$ determined by the root system for $R(Z)$. We see that $g$ has a dimension larger than 24 only if norm$(Z) = 0$, 4 or 6. If $\dim g \leq 24$, then either $\dim g = 24$ or $\dim g = 0$.

Moreover, one observes that one has $\dim g > 24$ if and only if the type $\prod_i v^{\sum_i c_i}$ is of positive type in the sense that $\sum_i c_i > 0$ for all occurring $v$.

Schellekens classified in [Sch1] the affine Kac-Moody structures which can possibly occur for a self-dual vertex operator algebra $V$ of central charge 24, cf. Theorem 2.10. By inspection, we see that the list of possible non-abelian affine Kac-Moody structures provided by Schellekens and the root lattices $R(Z)$ in Table 3 with $\dim g > 24$ agree.

We have listed in Table 3 in the column labelled no. in [Sch1] the corresponding Schellekens case number.

We have established:

**Theorem 3.1.** The possible non-abelian affine Kac-Moody structures of self-dual vertex operator algebras of central charge 24 are in a natural bijective correspondence with the equivalence classes of cyclic subgroups $Z$ of positive type of the glue codes of the twenty-three Niemeier lattices. These are precisely the cases for $Z$ where $R(Z)$ is a root lattice for a Lie algebra $g$ with $\dim g > 24$.

It is suggestive to associate to the $\dim g = 24$ cases the Leech lattice vertex operator algebra $V_{\Lambda}$ (Schellekens case no. 1) and to the $\dim g = 0$ cases the Moonshine module $V^3$ (Schellekens case no. 0). There are 13 cases of $Z$ with $\dim g = 24$ for different 8 types and 6 cases with $\dim g = 0$ for 6 different types.

**Remark 3.2.** We can also associate to $c$ a unique conjugacy class in the Weyl group $W(R)$ of the Niemeier lattice $N$ and hence interpret $c$ as an element of $W(R) < O(N)$. The type of $c$ is then the frame shape of this automorphism.

The orbit lattice $N(Z)$ is related to the fixed-point lattice $N^Z$ but the scaling of the components of $R(Z)$ is different.

One may expect that a suitable lift of $c$ as an element of $W(N) < O(N)$ acting on the lattice vertex operator algebra $V_N$ can be used to provide an orbifold construction of the corresponding Schellekens vertex operator algebra $V$.

Schellekens [Sch1] determined for each of the 71 possible affine Kac-Moody structures of a self-dual vertex operator algebra of central charge 24 the simple current code $D$, cf. Theorem 2.11. One finds again by complete inspection:
Table 3: Isomorphism classes of cyclic subgroups of the Niemeier lattice glue codes and associated orbit lattices $R(Z)$

| Lattice generator | order | norms | $R(Z)$     | dim g | no. in Sch1 | type  |
|-------------------|-------|-------|------------|-------|-------------|-------|
| $D_{24}$          | (0)   | 1     | $D_{24,1}$ | 1128  | 70          | $1^{24}$ |
|                   | (s)   | 2     | $B_{12,2}$ | 300   | 57          | $2^{12}$ |
| $D_{16}E_8$      | (0,0) | 1     | $D_{16,1}E_{8,1}$ | 744   | 69          | $1^{24}$ |
|                   | (s,0) | 2     | $B_{8,1}E_{8,2}$ | 384   | 62          | $1^{8}2^{8}$ |
| $E_{8,1}^3$      | (0,0,0) | 1 | $E_{8,1}^3$ | 744   | 68          | $1^{24}$ |
| $A_{24}$          | (0)   | 1     | $A_{24,1}$ | 624   | 67          | $1^{24}$ |
|                   | (5)   | 5     | $A_{4,2}$  | 24    | –           | $[5^5/1]$ |
| $D_{12}^2$       | (0,0) | 1     | $D_{12,1}^2$ | 552   | 66          | $1^{24}$ |
|                   | (s,v) | 2     | $B_{6,1}C_{10,1}$ | 288   | 56          | $1^{8}2^{8}$ |
|                   | (c,c) | 2     | $B_{6,2}^2$ | 156   | 41          | $2^{12}$ |
| $A_{17}E_7$      | (0,0) | 1     | $A_{17,1}E_{7,1}$ | 456   | 65          | $1^{24}$ |
|                   | (9,1) | 2     | $A_{8,2}F_{4,2}$ | 132   | 36          | $2^{12}$ |
|                   | (6,0) | 3     | $A_{5,1}E_{7,3}$ | 168   | 45          | $1^{6}3^{6}$ |
|                   | (3,1) | 6     | $A_{2,2}F_{4,6}$ | 60    | 14          | $2^{6}3^2$ |
| $D_{10}E_7^2$    | (0,0,0) | 1 | $D_{10,1}E_{7,1}^2$ | 456   | 64          | $1^{24}$ |
|                   | (s,1,0) | 2 | $B_{5,1}F_{4,1}E_{7,2}$ | 240   | 53          | $1^{8}2^{8}$ |
|                   | (v,1,1) | 2 | $C_{8,1}F_{4,1}^2$ | 240   | 52          | $1^{8}2^{8}$ |
| $A_{15}D_9$      | (0,0) | 1     | $A_{15,1}D_{9,1}$ | 408   | 63          | $1^{24}$ |
|                   | (8,0) | 2     | $A_{7,1}D_{9,2}$ | 216   | 50          | $1^{8}2^{8}$ |
|                   | (4,v) | 4     | $A_{3,1}C_{7,2}$ | 120   | 35          | $1^{4}2^{4}3^4$ |
|                   | (2,s) | 8     | $A_{1,1}B_{3,2}$ | 24    | –           | $[2^34.82^1/1^2]$ |
| $D_8^3$          | (0,0,0) | 1 | $D_{8,1}^3$ | 360   | 61          | $1^{24}$ |
|                   | (0,c,c) | 2 | $D_{8,2}B_{4,1}^2$ | 192   | 47          | $1^{8}2^{8}$ |
|                   | (s,v,v) | 2 | $B_{4,1}C_{6,1}^2$ | 192   | 48          | $1^{8}2^{8}$ |
|                   | (s,s,s) | 2 | $B_{4,2}^3$ | 108   | 29          | $2^{12}$ |
| $A_{12}^2$       | (0,0) | 1     | $A_{12,1}^2$ | 336   | 60          | $1^{24}$ |
|                   | (1,5) | 13    | –         | 0     | –           | $[13^2/1^2]$ |

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Table 3: (continued)

| Lattice   | generator | order | norms | $R(Z)$ | dim $g$ | no. in [Sch1] | type   |
|-----------|-----------|-------|-------|--------|---------|---------------|--------|
| $A_{11}D_7E_6$ | (0, 0, 0) | 1     | 0     | $A_{11,1}D_{7,1}E_{6,1}$ | 312 | 59 | $1^{24}$ |
|           | (6, $v$, 0) | 2     | 0, 4  | $A_{5,1}C_{5,1}E_{6,2}$ | 168 | 44 | $1^{82}8$ |
|           | (4, 0, 1)  | 3     | 0, 4  | $A_{3,1}D_{7,1}G_{2,1}$ | 120 | 34 | $1^{63}6$ |
|           | (3, $s$, 0) | 4     | 0, 4  | $A_{2,1}B_{2,1}E_{6,4}$ | 96  | 28 | $1^{42}244$ |
|           | (2, $v$, 2) | 6     | 0, 4  | $A_{1,1}C_{5,3}G_{2,2}$ | 72  | 21 | $1^{2}2^{2}3^{2}6^{2}$ |
|           | (1, $s$, 1) | 12    | 0, 4  | $B_{2,3}G_{2,4}$ | 24  | — | $[2^{2}3^{2}4.12/1^{2}]$ |
| $E_6^4$   | (0, 0, 0, 0) | 1     | 0     | $E_{6,1}^4$ | 312 | 58 | $1^{24}$ |
|           | (0, 1, 1, 1) | 3     | 0, 4  | $E_{6,3}G_{2,1}^3$ | 120 | 32 | $1^{6^{3}}6$ |
| $A_3^2D_6$ | (0, 0, 0) | 1     | 0     | $A_{9,1}^2D_{6,1}$ | 264 | 55 | $1^{24}$ |
|           | (5, $s$, 0) | 2     | 0, 4  | $A_{1,1}A_{9,2}B_{3,1}$ | 144 | 40 | $1^{82}8$ |
|           | (5, 5, $v$)  | 2     | 0, 6  | $A_{4,1}^2C_{4,2}$ | 84  | 22 | $2^{12}$ |
|           | (2, 4, 0)  | 5     | 0, 4  | $A_{1,1}^2D_{6,5}$ | 72  | 20 | $1^{4}5^{4}$ |
|           | (7, 9, $v$) | 10    | 0, 4, 6 | $C_{4,10}$ | 36  | 4 | $2^{2}10^{2}$ |
|           | (2, 9, $c$) | 10    | 0, 4  | $A_{1,2}B_{3,5}$ | 24  | — | $[2^{3}5^{2}10/1^{2}]$ |
| $D_6^4$   | (0, 0, 0, 0) | 1     | 0     | $D_{6,1}^4$ | 264 | 54 | $1^{24}$ |
|           | (v, s, c, 0) | 2     | 0, 4  | $C_{4,1}B_{3,1}^2D_{6,2}$ | 144 | 39 | $1^{82}8$ |
|           | (v, $v$, v, v) | 2     | 0, 4  | $C_{4,1}^2$ | 144 | 38 | $1^{82}8$ |
|           | (s, s, s, s) | 2     | 0, 6  | $B_{3,2}^4$ | 84  | 23 | $2^{12}$ |
| $A_3^3$   | (0, 0, 0) | 1     | 0     | $A_{3,1}^3$ | 240 | 51 | $1^{24}$ |
|           | (6, 3, 0)  | 3     | 0, 4  | $A_{5,1}^2A_{8,3}$ | 96  | 27 | $1^{63}6$ |
|           | (3, 3, 3)  | 3     | 0, 6  | $A_{2,2}^2$ | 24  | — | $[3^{9}/1^{5}]$ |
|           | (1, 1, 4)  | 9     | 0, 4, 6 | $A_{2,2}^2$ | 0   | — | $[9^{3}/1^{5}]$ |
| $A_3^2D_5^2$ | (0, 0, 0, 0) | 1     | 0     | $A_{7,1}^2D_{5,1}^2$ | 216 | 49 | $1^{24}$ |
|           | (4, 4, 0, 0) | 2     | 0, 4  | $A_{3,1}^2D_{5,2}^2$ | 120 | 31 | $1^{82}8$ |
|           | (4, 0, $v$, $v$) | 2     | 0, 4  | $A_{3,1}A_{7,2}C_{3,1}^2$ | 120 | 33 | $1^{82}8$ |
|           | (2, 2, 0, 0) | 4     | 0, 4  | $A_{2,1}^2C_{3,2}D_{5,4}$ | 72  | 19 | $1^{4}22^{4}4$ |
|           | (0, 2, c, s) | 4     | 0, 4  | $A_{7,4}A_{1,1}^4$ | 72  | 18 | $1^{4}22^{4}4$ |
|           | (4, 2, s, c) | 4     | 0, 4, 6 | $A_{3,4}A_{3,1}^3$ | 24  | — | $[2^{6}4^{4}/1^{4}]$ |
|           | (1, 1, s, v) | 8     | 0, 4, 6 | $A_{1,4}C_{3,8}$ | 24  | — | $[2^{4}4.8^{2}/1^{2}]$ |
|           | (5, 1, c, 0) | 8     | 0, 4  | $A_{1,2}D_{5,8}$ | 48  | 10 | $1^{2}2.4.8^{2}$ |
| $A_5^4$   | (0, 0, 0, 0) | 1     | 0     | $A_{6,1}^4$ | 192 | 46 | $1^{24}$ |
|           | (0, 6, 5, 3) | 7     | 0, 4  | $A_{6,7}$ | 48  | 11 | $1^{3}7^{3}$ |
|           | (1, 2, 1, 6) | 7     | 0, 4  | — | 0 | — | $[7^{4}/1^{4}]$ |

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| Lattice generator order norms | $R(Z)$ | $\dim g$ | no. in $\text{Schl}$ | type |
|-----------------------------|--------|---------|----------------|-------|
| $A_2^1 D_4$ | (0, 0, 0, 0, 0) | 1 0 | $A_2^1 D_{4,1}$ | 168 | 43 | $1^{24}$ |
| (3, 3, 3, 3, 0) | 2 0, 6 | $A_2^1 D_{4,4}$ | 60 | 13 | $2^{12}$ |
| (3, 3, 0, 0, s) | 2 0, 4 | $A_2^1 A_5^3 B_{2,1}$ | 96 | 26 | $1^{8}2^8$ |
| (0, 2, 2, 2, 0) | 3 0, 4 | $A_{5,3} A_{4,1}^3 D_{4,3}$ | 72 | 17 | $1^{6}3^6$ |
| (3, 1, 1, 1, 0) | 6 0, 4, 6 | $A_2^1 D_{4,12}$ | 36 | 3 | $2^{3}6^3$ |
| (0, 2, 5, 5, s) | 6 0, 4 | $A_5 A_{1,2} B_{2,3}$ | 48 | 8 | $1^{2}2^{3}3^{2}6^2$ |
| (3, 5, 2, 2, s) | 6 0, 4, 6 | $A_2^1 A_{1,4} A_{2,6}$ | 24 | – | $[5^5 3^6 1^4]$ |
| $D_4^6$ | (0, 0, 0, 0, 0, 0) | 1 0 | $D_{4,1}$ | 168 | 42 | $1^{24}$ |
| (0, 0, v, c, c, v) | 2 0, 4 | $D_{2,2}^1 B_{1,1}^2$ | 96 | 25 | $1^{8}2^8$ |
| (s, s, s, s, s, s) | 2 0, 6 | $B_{2,2}^6$ | 60 | 12 | $2^{12}$ |
| $A_4^6$ | (0, 0, 0, 0, 0, 0) | 1 6 | $A_{4,1}$ | 144 | 37 | $1^{24}$ |
| (0, 4, 3, 2, 1, 0) | 5 0, 4 | $A_{4,5}$ | 48 | 9 | $1^{4}5^3$ |
| (1, 0, 1, 4, 4, 1) | 5 0, 4, 6 | $A_{4,10}$ | 24 | – | $[5^5 1^1]$ |
| $A_3^8$ | (0, 0) | 1 0 | $A_{3,1}$ | 120 | 30 | $1^{24}$ |
| (0, 4, 2^4) | 2 0, 4 | $A_{3,2} A_{4,1}^4$ | 72 | 16 | $1^{8}2^8$ |
| (2^6) | 2 0, 8 | $A_{4,1}^4$ | 24 | – | $[2^{16} 1^8]$ |
| (0, 3, 2^1, (±1)^4) | 4 0, 4 | $A_{3,4} A_{4,1}^2$ | 48 | 7 | $1^{8}2^{24}4^4$ |
| (0, 3, 3^2, (±1)^4) | 4 4, 0, 6, 4 | $A_{3,8} A_{3,4}^2$ | 24 | – | $[2^{9} 4^{14}]$ |
| ((±1)^8) | 4 0, 6, 8 | – | 0 | – | $[4^8 1^8]$ |
| $A_2^{12}$ | (0, 12) | 1 0 | $A_{2,1}^4$ | 96 | 24 | $1^{24}$ |
| (0^6, (±1)^6) | 3 0, 4 | $A_{2,3}^6$ | 48 | 6 | $1^{6}3^6$ |
| (0^3, (±1)^6) | 3 0, 6 | $A_{2,6}^3$ | 24 | – | $[3^9 1^3]$ |
| ((±1)^12) | 3 0, 8 | – | 0 | – | $[3^{12} 1^{12}]$ |
| $A_1^{24}$ | (0, 24) | 1 0 | $A_{1,1}^4$ | 72 | 15 | $1^{24}$ |
| (0^16, 1^8) | 2 0, 4 | $A_{1,2}^1$ | 48 | 5 | $1^{8}2^8$ |
| (0^12, 1^12) | 2 0, 6 | $A_{1,4}^2$ | 36 | 2 | $2^{12}$ |
| (0^8, 1^16) | 2 0, 8 | $A_{1,8}^8$ | 24 | – | $[2^{16} 1^8]$ |
| (1^24) | 2 0, 12 | – | 0 | – | $[2^{24} 1^{24}]$ |
Table 4: Genera associated to self-dual VOAs of central charge 24

| Name | type | rank | \((A, q)\) | class # | mass constant | \(O(A, q)\) |
|------|------|------|-----------|--------|--------------|-------------|
| \(A\) | \(1^{24}\) | 24 | 1 | 24 | \(131,283,593,417,691^{2}, 3617,43867\) | 1 |
| \(B\) | \(1^{8}2^{8}\) | 16 | \(2^{+10}H\) | 17 | \(17,43,127,691\) | \(O_{10}^{+}(2).2\) |
| \(C\) | \(1^{6}2^{6}\) | 12 | \(3^{-8}\) | 6 | \(2,3,11\) | \(O_{5}^{+}(3).2^{2}\) |
| \(D\) | \(2^{12}\) | 12 | \(2^{-10}H, 4^{2}\) | 2 | \(2,3,5,7,11\) | \(2.2^{10}.[6].O_{10}^{+}(2).2\) |
| \(E\) | \(1^{4}2^{4}4^{4}\) | 10 | \(2^{-2}H, 4^{+6}\) | 5 | \(2,3,5\) | \(2.14^{+}2.6.2.7\) |
| \(F\) | \(1^{4}5^{4}\) | 8 | \(5^{+6}\) | 2 | \(3^{1}\) | \(2.5^{+}(5).2\) |
| \(G\) | \(1^{2}2^{2}3^{2}6^{2}\) | 8 | \(2^{-6}H, 3^{-6}\) | 2 | \(3^{1}\) | \(2.5^{+}(3).O_{3}^{+}(2).[2^{2}]\) |
| \(H\) | \(1^{3}7^{3}\) | 6 | \(7^{5}\) | 1 | \(2\) | \(2.5^{+}(7).2\) |
| \(I\) | \(1^{2}2.4.8^{2}\) | 6 | \(2^{-4}H, 4^{+1}8^{+4}\) | 1 | \(2^{1}, 3, 5\) | \(2.2^{1}.O_{5}(2).2\) |
| \(J\) | \(2^{3}6^{3}\) | 6 | \(2^{-2}H, 4^{+2}3^{+5}\) | 1 | \(2^{1}, 3\) | \(2.13^{+}(3).O_{5}(3).2\) |
| \(K\) | \(2^{2}10^{2}\) | 4 | \(2^{-2}H, 5^{-4}\) | 1 | \(2^{1}, 3\) | \(2^{2}.3^{2}.(O_{3}(5) \times O_{3}(5)).[2^{2}]\) |
| \(L\) | \(2^{2}1/2^{2}\) | 0 | \(1\) | 1 | \(1\) | 1 |

Theorem 3.3. The simple current code \(C(Z)\) agrees with the code \(C(Z)\) under the bijective correspondence of Theorem 3.1 for all possible 69 cases of non-abelian affine Kac-Moody structures.

One also checks that the type of a cyclic group \(Z\) uniquely determines the genus of \(N(Z)\). For the eleven occurring positive types one obtains therefore eleven different lattice genera. We add the genus of the trivial 0-dimensional lattice as a twelfth case.

The twelve occurring genera together with additional information are listed in Table 4. The column labeled \((A, q)\) in Table 4 gives the genus symbol, or equivalently (since the lattices are positive definite), the type of the discriminant space associated to the lattices in the genus. The column class \# gives the class number of \(N(Z)\), i.e. the number of lattices in the genus up to isometry. The next column provides the mass constant \(\sum_{L} \frac{1}{|O(L)|}\), where the sum runs over all non isometric lattices of the genus. The last column \(O(A, q)\) provides information about the structure of the orthogonal group of the quadratic space \((A, q)\). Elements of \(A\) of the same order and with the same value of \(q\) form usually one \(O(A, q)\)-orbit. In a few cases, they form two orbits.

We tabulated for each genus the occurring orbit lattices. The result is shown in Tables 5 to 10. The second row in the tables lists the root lattice \(R(Z)\). The next row lists generators of the glue code \(C(Z)\) (cf. [CCNPW], p. 82 and Table 1). The automorphism group of \(N(Z)\) contains the Weyl group \(G_{0} = W(R(Z))\). Its order can be read off from Table 1. In columns \(|G_{1}|\) and \(|G_{2}|\) we list the order of the groups \(G_{1}\) and \(G_{2}\) which describe the induced action on \(R^{+}(Z)/R(Z)\) by the automorphism of \(N(Z)\) fixing \(R(Z)\) setwise. Here, \(G_{1}\) is the normal subgroup fixing the components of \(R(Z)\). Column \(i\) gives the index of the setwise stabilizer of \(R(Z)\) in \(O(N(Z))\). The column labeled \(\text{dim } g\) provides the dimension of the Lie algebra \(\mathfrak{g}\) corresponding to the root system given by \(R(Z)\).

Theorem 3.4. The 69 orbit lattices \(N(Z)\) of Theorem 3.1 together with the Leech lattice and the zero lattice represent together all lattices in their respective genera.

Besides lattices in the two genera \(D\) and \(J\), the orbit lattices \(N(Z)\) are pairwise non isometric. The two lattices in genus \(D\) are isometric to \(D_{12,2}\) and \(E_{8,2} \oplus D_{4,2}\).
| Name | components | Glue code | Order | $|G_1|$ | $|G_2|$ | $i$ | dim $g$ | # in [Sch1] |
|------|------------|-----------|-------|--------|--------|-----|---------|------------|
| A1   | $D_{24,1}$ | $(s)$     | 2     | 1      | 1      | 1   | 1128   | 70         |
| A2   | $D_{16,1}E_{8,1}$ | $(s,0)$ | 2     | 1      | 1      | 1   | 744    | 69         |
| A3   | $E_{8,1}$ | –         | 1     | 1      | 6      | 1   | 744    | 68         |
| A4   | $A_{24,1}$ | $(5)$     | 5     | 2      | 1      | 1   | 624    | 67         |
| A5   | $D_{12,1}^2$ | $(s,v)$ | 4     | 1      | 2      | 1   | 552    | 66         |
| A6   | $A_{17,1}E_{7,1}$ | $(3,1)$ | 6     | 2      | 1      | 1   | 456    | 65         |
| A7   | $D_{10,1}E_{7,1}^2$ | $(s,1,0), (c,0,1)$ | 4 | 1      | 2      | 1   | 456    | 64         |
| A8   | $A_{15,1}D_{9,1}$ | $(2,s)$ | 8     | 2      | 1      | 1   | 408    | 63         |
| A9   | $D_{6,1}^2$ | $(s,v,v)$ | 8     | 1      | 6      | 1   | 360    | 61         |
| A10  | $A_{12,1}D_{9,1}$ | $(1,5)$ | 13    | 2      | 2      | 1   | 336    | 60         |
| A11  | $A_{11,1}D_{7,1}E_{6,1}$ | $(1,s,1)$ | 12    | 2      | 1      | 1   | 312    | 59         |
| A12  | $E_{8,1}^3$ | $(1,[0,1,2])$ | 9     | 2      | 24     | 1   | 312    | 58         |
| A13  | $A_{6,1}^2D_{6,1}$ | $(2,4,0), (5,0,s), (0,5,c)$ | 20    | 2      | 2      | 1   | 264    | 55         |
| A14  | $D_{5,1}^3$ | $(0,s,v,c), (1,1,1,1)$ | 16    | 1      | 24     | 1   | 264    | 54         |
| A15  | $A_{3,1}^4$ | $(1,[1,4])$ | 27    | 2      | 6      | 1   | 240    | 51         |
| A16  | $A_{12,1}^2D_{6,1}^2$ | $(1,1,s,v), (1,7,v,s)$ | 32    | 2      | 4      | 1   | 216    | 49         |
| A17  | $A_{6,1}^2$ | $(1,[2,1,6])$ | 49    | 2      | 12     | 1   | 192    | 46         |
| A18  | $A_{6,1}^2D_{4,1}$ | $(2,[0,2,4],0), (3,3,3,0,0,s), (3,0,3,0,v), (3,0,0,3,c)$ | 72    | 2      | 24     | 1   | 168    | 43         |
| A19  | $D_{5,1}^3$ | $(s,s,s,s,s,s), (0,[0,v,c,c,v])$ | 64    | 3      | 720    | 1   | 168    | 42         |
| A20  | $A_{5,1}^2$ | $(1,[0,1,4,4,1])$ | 125   | 2      | 120    | 1   | 144    | 37         |
| A21  | $A_{5,1}^2$ | $(3,[2,0,0,1,0,1,1])$ | 256   | 2      | 1344   | 1   | 120    | 30         |
| A22  | $A_{5,1}^2$ | $(2,[1,1,2,1,1,2,2,2,1,2])$ | 729   | 2      | $|M_{12}|$ | 1   | 96     | 24         |
| A23  | $A_{5,1}^2$ | $(1,[0,0,0,0,0,1,0,1,0,0,1,1,0,1,0,1,0,1,1,1,1,1,1])$ | 4096  | 1      | $|M_{24}|$ | 1   | 72     | 15         |
| A24  | $U(1)^{24} = \mathbb{R}^{24}/\Lambda$ | – | 1     | 1      | $|\text{Co}_0|$ | 1   | 24     | 1          |

Table 5: Genus $A$ of type $II_{24}(1)$ (Niemeier lattices and Leech lattice)
| Name components | Glue code | Order | $|G_1|$ | $|G_2|$ | $i$ | dim $g$ | # in $[Sch1]$ |
|-----------------|-----------|-------|--------|--------|-------|--------|--------------|
| $B1$ $E_{8,2}B_{8,1}$ | – | 1 | 1 | 1 | 1 | 384 | 62 |
| $B2$ $C_{10,1}B_{6,1}$ | $(1,1)$ | 2 | 1 | 1 | 1 | 288 | 56 |
| $B3$ $C_{8,1}^2F_{4,1}$ | $(1,0,0)$ | 2 | 1 | 2 | 1 | 240 | 52 |
| $B4$ $E_{7,2}B_{5,1}F_{4,1}$ | $(1,1,0)$ | 2 | 1 | 1 | 1 | 240 | 53 |
| $B5$ $D_{9,2}A_{7,1}$ | $(s,2)$ | 4 | 2 | 1 | 1 | 216 | 50 |
| $B6$ $D_{8,2}B_{4,1}$ | $(s,0,0),(v,1,1)$ | 4 | 1 | 2 | 1 | 192 | 47 |
| $B7$ $C_{8,1}^2B_{4,1}$ | $(1,0,1),(0,1,1)$ | 4 | 1 | 2 | 1 | 192 | 48 |
| $B8$ $E_{6,2}C_{5,1}A_{5,1}$ | $(1,1,1)$ | 6 | 2 | 1 | 1 | 168 | 44 |
| $B9$ $A_{9,2}A_{4,1}B_{3,1}$ | $(1,2,1)$ | 10 | 2 | 1 | 1 | 144 | 40 |
| $B10$ $D_{8,2}A_{4,1}B_{3,1}$ | $(s,0,0,1)(0,1,1,1),(v,0,1,1)$ | 8 | 1 | 2 | 1 | 144 | 39 |
| $B11$ $C_{5,1}^2$ | $(1,1,0,0)$ | 8 | 1 | 24 | 1 | 144 | 38 |
| $B12$ $A_{7,2}C_{5,1}^2A_{3,1}$ | $(1,0,1,1),(0,1,1,2)$ | 16 | 4 | 1 | 120 | 33 |
| $B13$ $D_{5,2}^2A_{2,1}$ | $(s,0,1,1),(s,0,3,1)$ | 16 | 8 | 1 | 120 | 31 |
| $B14$ $A_{5,2}^2B_{2,1}A_{3,1}$ | $(1,0,1,1,1),(0,1,1,1,2)$ | 36 | 8 | 1 | 96 | 26 |
| $B15$ $D_{5,2}^2B_{2,1}A_{3,1}$ | $([s,0],1,1,0,0),(v,0,0,1,1,0),(0,0,1,1,1,1)$ | 32 | 48 | 1 | 96 | 25 |
| $B16$ $A_{5,2}^2A_{1,1}^2$ | $(1,[1,0,0],[1,1,0,0]),(2,0,0,0,1,1,1,1)$ | 128 | 192 | 1 | 72 | 16 |
| $B17$ $A_{5,2}^6$ | $H_{16}$ | 2048 | 1 | 322560 | 1 | 48 | 5 |

| Name components | Glue code | Order | $|G_1|$ | $|G_2|$ | $i$ | dim $g$ | # in $[Sch1]$ |
|-----------------|-----------|-------|--------|--------|-------|--------|--------------|
| $C1$ $E_{7,3}A_{5,1}$ | $(1,3)$ | 2 | 2 | 1 | 1 | 168 | 45 |
| $C2$ $D_{2,3}A_{3,1}G_{2,1}$ | $(s,1,0)$ | 4 | 2 | 1 | 1 | 120 | 34 |
| $C3$ $E_{6,3}G_{2,1}^3$ | $(1,0,0,0)$ | 3 | 2 | 6 | 1 | 120 | 32 |
| $C4$ $A_{6,3}A_{2,1}^2$ | $(1,1,1)$ | 9 | 2 | 2 | 1 | 96 | 27 |
| $C5$ $A_{6,3}D_{4,3}A_{1,1}^0$ | $(s,0,0,1,1),(v,0,1,1,0),(1,0,1,1,1)$ | 24 | 2 | 6 | 1 | 72 | 17 |
| $C6$ $A_{5,3}^6$ | $(1,[1,0,0,0,0,0])$ | 243 | 2 | 720 | 1 | 48 | 6 |

| Name components | Glue code | Order | $|G_1|$ | $|G_2|$ | $i$ | dim $g$ | # in $[Sch1]$ |
|-----------------|-----------|-------|--------|--------|-------|--------|--------------|
| $D1a$ $B_{2,2}$ | – | 1 | 1 | 1 | 1 | 300 | 57 |
| $D1b$ $B_{2,2}^3$ | $(1,1)$ | 2 | 2 | 1 | 2 | 462 | 56 |
| $D1c$ $B_{1,2}^4$ | $(1,[1,0])$ | 4 | 1 | 6 | 1 | 5775 | 98 |
| $D1d$ $B_{2,2}^3$ | $(1,[1,0,0])$ | 8 | 1 | 24 | 1 | 15400 | 84 |
| $D1e$ $B_{2,2}^3$ | $(1,[1,0,0,0,0])$ | 32 | 1 | 720 | 1 | 10395 | 60 |
| $D1f$ $A_{1,2}^4$ | $(1,[1,0,0,0,0,0,0,0,0,0,0])$ | 2048 | 1 | 120 | 1 | 36 | 2 |
| $D2a$ $A_{8,2}F_{4,2}$ | $(3,0)$ | 3 | 2 | 1 | 2 | 960 | 36 |
| $D2b$ $C_{4,2}A_{1,2}^4$ | $(1,0,0),(0,1,2)$ | 10 | 2 | 2 | 2 | 36288 | 84 |
| $D2c$ $D_{4,4}A_{4,2}^2$ | $(v,0,0,0,0),(s,0,0,0,0),(0,1,1,1,0),(0,2,1,0,1)$ | 36 | 12 | 24 | 4 | 1400 | 60 |
Table 9: Genus $E$ of type $II_{10}(2^2 4^6)$

| Name components | Glue code | Order | $|G_1|$ | $|G_2|$ | $i$ | dim $g$ | # in Sch1 |
|-----------------|-----------|-------|--------|--------|-----|--------|----------|
| $E1$            | $C_{7,2}A_{3,1}$ | (1,2)  | 2      | 2      | 1   | 1      | 120     | 35       |
| $E2$            | $E_{6,4}B_{2,1}A_{2,1}$ | (1,0,1) | 3      | 2      | 1   | 1      | 96      | 28       |
| $E3$            | $A_{7,4}A_{7,1}^1$ | (1,1,0,0) | 8      | 2      | 2   | 1      | 72      | 18       |
| $E4$            | $D_{5,4}C_{3,3}A_{7,1}^2$ | (0,1,1,1), (s,1,0,0) | 4      | 2      | 2   | 1      | 72      | 19       |
| $E5$            | $A_{3,4}^5A_{1,2}$ | ([1,0,0],1) | 64     | 8      | 6   | 1      | 48      | 7        |

Table 10: Genus $F$ of type $II_8(5^6)$

| Name components | Glue code | Order | $|G_1|$ | $|G_2|$ | $i$ | dim $g$ | # in Sch1 |
|-----------------|-----------|-------|--------|--------|-----|--------|----------|
| $F1$            | $D_{6,3}A_{7,1}^1$ | (s,0,1), (c,1,0) | 4      | 1      | 2   | 1      | 72      | 20       |
| $F2$            | $A_{4,5}^6$ | (1,0), (0,1) | 25     | 4      | 2   | 1      | 48      | 9        |

Table 11: Genus $G$ of type $II_8(2^63^6)$

| Name components | Glue code | Order | $|G_1|$ | $|G_2|$ | $i$ | dim $g$ | # in Sch1 |
|-----------------|-----------|-------|--------|--------|-----|--------|----------|
| $G1$            | $C_{5,3}G_{2,2}A_{1,1}$ | (1,0,1) | 2      | 1      | 1   | 1      | 72      | 21       |
| $G2$            | $A_{5,6}B_{2,3}A_{1,2}$ | (1,0,1), (0,1,1) | 12     | 2      | 1   | 1      | 48      | 8        |

Table 12: Genus $H$ of type $II_6(7^5)$

| Name components | Glue code | Order | $|G_1|$ | $|G_2|$ | $i$ | dim $g$ | # in Sch1 |
|-----------------|-----------|-------|--------|--------|-----|--------|----------|
| $H1$            | $A_{6,7}$ | (1)   | 7      | 2      | 1   | 1      | 48      | 11       |

Table 13: Genus $I$ of type $II_6(2^{1+4}5^{1+8}4^4)$

| Name components | Glue code | Order | $|G_1|$ | $|G_2|$ | $i$ | dim $g$ | # in Sch1 |
|-----------------|-----------|-------|--------|--------|-----|--------|----------|
| $I1$            | $D_{5,8}A_{1,2}$ | (s,0) | 4      | 2      | 1   | 1      | 48      | 10       |

Table 14: Genus $J$ of type $II_6(2^{1+4}4_{11}^25_{11}^2)$

| Name components | Glue code | Order | $|G_1|$ | $|G_2|$ | $i$ | dim $g$ | # in Sch1 |
|-----------------|-----------|-------|--------|--------|-----|--------|----------|
| $J1a$           | $F_{4,6}A_{2,2}$ | –     | 1      | 2      | 1   | 1      | 60      | 14       |
| $J1b$           | $D_{4,12}A_{2,6}$ | (s,0), (v,0), (0,1) | 12     | 12     | 1   | 1      | 36      | 3        |

Table 15: Genus $K$ of type $II_4(2_4^25_4^2)$

| Name components | Glue code | Order | $|G_1|$ | $|G_2|$ | $i$ | dim $g$ | # in Sch1 |
|-----------------|-----------|-------|--------|--------|-----|--------|----------|
| $K1$            | $C_{4,10}$ | (1)   | 2      | 1      | 1   | 3      | 36      | 4        |

Table 16: Genus $L$ of type $II_0(1)$

| Name components | Glue code | Order | $|G_1|$ | $|G_2|$ | $i$ | dim $g$ | # in Sch1 |
|-----------------|-----------|-------|--------|--------|-----|--------|----------|
| $L1$            | –         | –     | 1      | 1      | 1   | 1      | 0        | 0        |
and are presented by 6 and 3 orbit lattices, respectively. The unique lattice in genus \( J \) is represented by 2 orbit lattices.

**Proof:** The result is obtained by comparing Tables 5 to 16 with the computation of all the lattices in each occurring genus using the software MAGMA [Mag].

It is possible for two orbit lattices to be isometric because certain vectors may either be considered as belonging to the root sublattice \( R(Z) \) or to belong to the coset represented by the glue code \( C(Z) \). Thus the root lattice \( R(Z) \) may not be recovered from the isometry type of \( N(Z) \) alone.

We also note that besides possibly for lattices inside the genera \( D \) and \( K \) one has \( O(N(Z)) = W(R(Z)); \text{Aut}(C(Z)) \). There are a few additional cases where \( R(Z) \) is not the full root lattice of the orbit lattice \( N(Z) \).

## 4 Constructions starting from the Leech lattice

In this section, we describe a possible uniform construction of self-dual vertex operator algebras of central charge 24 for all 70 non-trivial Kac-Moody structures by using the Leech lattice. We also discuss the uniqueness question for those cases.

### 4.1 The general approach

We start by describing a general approach in classifying self-dual vertex operator algebras.

Let \( V \) be a vertex operator algebra and \( g = V_1 \) be the weight 1 Lie algebra which we assume to be reductive. Then \( g \) has a Cartan subalgebra \( t \) which is unique up to conjugation under \( \text{Aut}(g) \) and \( \text{Aut}(V) \). The subalgebra \( t \) generates a (non-rational) Heisenberg vertex operator subalgebra \( T \) of \( V \). We consider the two primitive subalgebras \( W := \text{Com}_V(T) \) and \( \overline{T} := \text{Com}_V(W) \) of \( V \). The \( T \)-module decomposition of \( \overline{T} \) defines an even lattice \( L \) in \( t^* \) such that \( \overline{T} \) equals the lattice vertex operator algebra \( V_L \). We assume that the rank of \( L \) equals the rank of \( g \). By construction, the Lie algebra \( W_1 \) of \( W \) is trivial.

The lattice vertex operator algebra \( V_L \) has the modular tensor category \( T(V_L) = \mathcal{Q}(A_L, q_L) \). If we let \( V \) be self-dual, then one expects (cf. Conjecture 2.6) that \( W \) is a vertex operator algebra with modular tensor category \( T(W) \cong \mathcal{Q}(A_L, -q_L) \). We assume that this is the case.

**Theorem 4.1.** Self-dual vertex operator algebras \( V \) of central charge \( c \) satisfying the above assumptions are up to isomorphisms in one-to-one correspondence to quadruples \( (G, L, W, [i]) \) consisting of the following data:

- **(a)** A genus \( G = ((A, q), k) \) of positive definite lattices of rank \( k \).
- **(b)** An isometry class of lattices \( L \) in the genus \( G \).
- **(c)** An isomorphism class of vertex operator algebras \( W \) of central charge \( c - k \) with \( T(W) = \mathcal{Q}(A, -q) \) and \( W_1 = 0 \).
- **(d)** An equivalence class \([i]\) of anti-isometries \( i : (A_L, q_L) \rightarrow (A_W, q_W) \) under the double coset action of \( \text{O}(L) \times \text{i}^* \text{Aut}(W) \) on \( \text{O}(A, q) \).

**Proof:** The primitive vertex operator subalgebras \( V_L \) and \( W \) are up to conjugation canonically associated to \( V \). By Theorem 2.7 the self-dual extensions of \( V_L \otimes W \) are up to equivalence given by the double cosets \([i]\). Thus \( V \) uniquely determines \( (G, L, W, [i]) \). Conversely, a quadruple \((G, L, W, [i])\) defines by this construction a self-dual vertex operator algebra \( V \) of central charge \( c \) satisfying all assumptions. The condition \( W_1 = 0 \) guarantees that the correspondence is a bijection.
Theorem 4.1 classifies certain extensions of $V_L \otimes W$ up to equivalence. Since $V_L$ and $W$ are up to conjugation canonically defined primitive vertex operator subalgebras of $V$, equivalence classes of extensions of $V_L \otimes W$ with $V_L$ primitively embedded are the same as isomorphism classes of self-dual vertex operators $V$ with canonically defined subalgebras isomorphic to $V_L$ and $W$.

The construction of a self-dual vertex operator algebra $V$ from a tuple $(g, L, W, [i])$ does not depend on any unproven assumptions. However, if one likes to use the theorem to classify all self-dual vertex operator algebras of a given charge, one has to be more careful; cf. [Ma14] for a general result in this direction.

In the case of central charge $c = 24$, the work of Schellekens [Sch1] and of Dong and Mason [DM04a, DM04] establish the required properties of $V_L$ and $L$. Unclear are the required conditions on $W$. One probably has not to use Conjecture 2.6 in full generality since $W$ is an extension of para-fermion vertex operator algebras and thus a case by case study seems possible. For example, in the case of $V$ with Kac-Moody vertex operator algebra of type $A_{12}^{16}$ one obtains that $W$ is a framed vertex operator algebra of central charge 16. This allows to use the theory of framed vertex operator algebras to establish the required properties for $W$.

The hardest part in applying Theorem 4.1 is the classification of the vertex operator algebras $W$ and the computation of the image $\text{Aut}(W)$ in $O(A, q)$. We will address both questions in the following. By using the known structure of affine Kac-Moody vertex operator algebras [DL], it is clear that the possible lattices $L$ are the orbit lattices $N(Z)$ described in the previous section. In particular, Theorem 4.1 explains why always all lattices in the genus of an orbit lattice $N(Z)$ occur.

4.2 Existence

We recall from the last section that there are eleven genera of orbit lattices $N(Z)$ occurring and that the genus is determined by the type of $Z = \langle c \rangle$ which we may interpret as an element $g$ of the Weyl group $W(R)$ of the Niemeier lattice $Z$.

The deep hole construction of the Leech lattice [CS] corresponding to a Niemeier lattice $N$ with glue code $C$ allows to embed the group $C$ into $Co_0$ and this embedding is well-defined up to conjugacy. Hence one can associate more directly to a glue vector $c$ in $C$ a conjugacy class $[g]$ of $Co_0$. It follows from the deep hole description, that the frame shape of this conjugacy class agrees with the type of $c$. If the type of $c$ is positive then $[g]$ belongs to a conjugacy class of $2^{12}:M_{24} < Co_0$. In all cases — besides the frame shape $2^4:6^3$ corresponding to genus $J$ — the frame shape is the cycle shape of an element of $M_{24}$.

Let $g$ be an element of $Co_0$. We denote by $\hat{g}$ a lift of $g$ to the automorphism group $\text{Aut}(V_\Lambda)$ of the lattice vertex operator algebra $V_\Lambda$ associated to the lattice $\Lambda$. We can and will assume that $\hat{g}$ acts trivially on the vertex operator subalgebra $V_{\Lambda_8}$. Then the $\text{Aut}(V_\Lambda)$-conjugacy class of $\hat{g}$ is well-defined ([EM2], Prop. 7.1). Let $\Lambda_g = (\Lambda^g)^{\perp}$ be the corresponding coinvariant lattice (cf. [HM2]). Via the induced action, $g$ can also be considered as an element of $O(\Lambda_g) < O(\Lambda)$. We denote by $W = W(g)$ the fixed point vertex operator algebra $V_{\Lambda_8}^{(\hat{g})}$. From the construction it follows that one has $\dim W_1 = 0$.

**Conjecture 4.2.** The fixed-point vertex operator algebras $W = V_{\Lambda_8}^{(\hat{g})}$ associated to elements $g$ of $O(\Lambda)$ with frame shapes equal to the types of lattices $N(Z)$ are vertex operator algebras with a modular tensor category $T(W)$ isomorphic to the modular tensor category $\mathcal{C}(A, \eta = q)$ where $(A, q)$ is the discriminant space of the lattice $N(Z)$.

**Remark 4.3.** From Theorem 2.8, we know that $W$ satisfies the assumptions for Huang's modular category result. It remains to show that all the irreducible modules
of $W$ are simple currents forming the abelian group $A$ under the fusion product, and that the quadratic form on $A$ induced by the conformal weights equals $-q$.

The conjecture is true for genus $A$ since $W = V_A$ and the vertex operator algebra associated to an even unimodular lattice has only one isomorphism class of irreducible modules.

The conjecture is true for genera $B$ and $D$ since in those cases $W$ is a fixed point vertex operator algebra $V_K^\tau$ for a lift $\tau$ of the $-1$ automorphism of a lattice $K$ for which a full description of the irreducible modules [AD] and their fusion rules [ADL] is available. One may also use for these two cases the description of $W$ as a framed vertex operator algebra [DGH] with trivial code $D$, [Mi98], cf. [GH, HS].

For genera $A, B, C, F, G$ and $H$, the conjecture has recently been proven by S. Möller in his Ph.D. thesis [Mi98] under a weak assumption on the group structure of $A$ which can probably be verified by analyzing all possible abelian group structures.

For the last genus $L$, we let $W$ be the Moonshine module $V^\flat$.

**Theorem 4.4.** Assuming Conjecture 4.4, the vertex operator algebras $V$ constructed from $V_{N(Z)}$ and $W$ according to Theorem 4.1 has the affine Kac-Moody structure determined by $N(Z)$.

**Proof:** The vertex operator algebra $V$ constructed from $V_{N(Z)}$ and $W$ according to Theorem 4.1 with the help of an arbitrary orbit of gluing maps $[i]$ is a self-dual vertex operator algebra. It has an extended Kac-Moody vertex operator subalgebra $\tilde{V}$ and its full description of the irreducible modules [AD] and their fusion rules [ADL] was determined in Table 3 of [HS]. It was also shown that $\text{Aut}(W) = 2^{11}.2^{10}.\text{Sym}_{12}.\text{Sym}_3$ which has 7 orbits on $A_W$. We only need that the full character of $\Xi$ is invariant under that group. Only the elements in the orbits no. 1, 6 and 9 of $O(L)$ when combined with $i$ with elements in the orbits no. 1, 4 and 7 of $\text{Aut}(W)$, respectively, can possibly contribute to $\dim V_1$. Indeed, the combination for the first two pairs of corresponding orbits provides a contribution of 12 respectively 24 to the dimension of $V_1$ whereas the third pair may give a contribution between 0 and 24. A double coset enumeration for $O(L) \times i^* \text{Aut}(W)$ in $O(A_L, q_L)$ with MAGMA shows that indeed all the 6 possibilities for $\dim V_1$ can be realized. A similar analysis for $L = E_{8,2} \oplus D_{4,2}$ shows that all the 3 occurring possibilities for $\dim V_1$ can be realized. (The discriminant group of $E_{8,2}$ has three $O(E_{8,2})$-orbits, the discriminant group of $D_{4,2}$ has five $O(D_{4,2})$-orbits.)

We have shown the existence of a self-dual vertex operator algebra of central charge 24 for all Kac-Moody structures for which Conjecture 4.2 is proven.
4.3 Uniqueness

We assume that the assumptions used in Theorem 4.4 hold. This allows us to assign to a self-dual vertex operator algebra $V$ the quadruple $(G, L, W, [i])$. In particular, we assume that the commutant $W$ of $V_L$ is a vertex operator algebra with modular tensor category $Q(A_L, q)$ and $W_1 = 0$.

We consider the case of self-dual vertex operator algebras of central charge 24. We will show that the vertex operator algebra $W$ is unique if its central charge is less than 24. We then consider $\text{Aut}(W)$.

**Theorem 4.5.** The full character of $W$ is uniquely determined.

**Proof:** Let $\Xi$ be the full character of $W$ and $S$ be a maximal torus of $\text{Aut}(V)$ and $\Theta(\tau, z)$ be the full Jacobi form theta series of $L$. The $S$-equivariant character of $V$ is then given by

$$\chi_V(\tau, z) = \sum_{a \in A_L} \frac{\Theta_{L+a}(\tau, z)}{n^k L(\tau)} \cdot \Xi_i(a)(\tau).$$

This character is uniquely determined since there is a unique $U$-module decomposition of $V$, cf. Theorem 2.4. This allows to split the $S$-equivariant character of $V$ into its $A_L$-components and to recover all components of $\Xi$ from $\chi_V(\tau, z)$. \qed

**Corollary 4.6.** Assuming Conjecture 4.2, the full character of $W$ equals the full character of $V_{\Lambda_g}^\partial$.

If we assume Conjecture 4.2, we can choose an isometry $\iota$ between the corresponding quadratic spaces of the modular tensor categories of $W$ and $V_{\Lambda_g}^\partial$, which also respects the full characters.

Let $V_{\Lambda_g} = \bigoplus_{a \in B}(V_{\Lambda_g}^\partial)_a$ with a cyclic isotropic subgroup $B$ of $(A_L, q_L)$ of order $o(g)$. We set $W^+ = \bigoplus_{a \in \iota(B)}(W)_a$. Then $V_{\Lambda_g}$ and $W^+$ are both vertex operator algebras with isomorphic modular tensor categories, have equal full characters and both have an action of a cyclic group $B^* = \langle \hat{h} \rangle$ of order $o(g)$. In particular, $\dim W^+_1 < \dim (V_{\Lambda_g})_1 = 24 - \rk L$.

**Theorem 4.7.** Assuming Conjecture 4.2, the vertex operator algebras $W$ and $V_{\Lambda_g}^\partial$ are isomorphic.

**Proof:** We glue $V_{\Lambda_g}$ and $W^+$ together, mirroring via $\iota$ the gluing of $V_{\Lambda_g}$ and $V_{\Lambda_g}$, which results in $V_\Lambda$. Since $\iota$ respects also the full characters, the resulting self-dual vertex operator algebra $V$ has the same character as $V_\Lambda$ and hence, using the uniqueness of a self-dual vertex operator algebra with this character, must be isomorphic to it (cf. [DMO04b]).

We also see that an extension $\hat{B}^* = \langle \hat{h} \rangle$ of $B^* = \langle \hat{h} \rangle$ with $o(\hat{h}) = o(\hat{g})$ acts on $V$ fixing $V_{\Lambda_g} \otimes W$. We like to show that $\hat{h}$ is conjugated to $\hat{g}$ under the identifications made.

Consider the induced action of $\hat{h}$ on $\text{Com}_W(V_{\Lambda_g}) = W^+$, in particular on $W_1^+$. Since the characters of $W$ and $V_{\Lambda_g}^\partial$ are the same, one has that the traces of the powers $h^k$ of $\hat{h}$ which act like the powers $h^k$ of $h$ on $W_1^+$ are the same as for the powers $\hat{g}^k$ of $\hat{g}$. Since in our eleven cases, one has $\dim W_1 \geq 4$, we can use the observation of [HMI] that under this condition the traces of $\hat{g}$ and $\hat{g}^2$ together with the order of $g$ determine the $O(\Lambda)$-conjugacy class of an element $g$ in $O(\Lambda)$. After the identification and conjugation, we have that $\hat{g}$ and $\hat{h}$ both lift to $\text{Aut}(V_\Lambda)$ of the same element $g \in O(\Lambda)$. Since $\hat{g}$ and $\hat{h}$ both act trivially on $V_{\Lambda_g}$, they are conjugated in $\text{Aut}(V_\Lambda)$ (EMS, Prop. 7.1).

It follows that $W \cong V_{\Lambda_g}^\partial$. \qed
Table 17: The coset vertex operator algebras $W$

| Name  | type | no. in $\text{[HIM2]}$ | $(A_{\Lambda_\gamma}, -q_{\Lambda_\gamma})$ | $i^G$ | c.charge | $\text{Aut}(W)$ | # of VOAs |
|-------|------|-----------------|-----------------|------|---------|-----------------|----------|
| $A$   | $1^{24}$ | 1               | 1               | 1    | 0       | $O_{10}^+(2).2$ | 24       |
| $B$   | $1^{8}2^{8}$ | 2               | $2^+8$           | 2    | 8       | $O_{10}^-(2).2$ | 17       |
| $C$   | $1^{6}3^{6}$ | 4               | $3^+6$           | 1    | 12      | $O_8^-(3).2$    | 6        |
| $D$   | $2^{12}$ | 5               | $2^+12$          | 104448 | 12     | 9          |
| $E$   | $1^{4}2^{2}4^{4}$ | 9               | $2^+2^4$          | 2    | 14      | 5          |
| $F$   | $1^{4}5^{4}$ | 20              | $5^+4$           | 1    | 16      | 2          |
| $G$   | $1^{2}2^{2}3^{2}6^{2}$ | 18              | $2^+3^4$          | 12   | 16      | 2          |
| $H$   | $1^{3}7^{3}$ | 52              | $7^+3$           | 1    | 18      | 1          |
| $I$   | $1^{2}2^{4}4^{2}$ | 55              | $2^+4^2$          | 2    | 18      | 1          |
| $J$   | $2^{3}6^{3}$ | 63              | $2^+6^3$          | 48   | 18      | 2          |
| $K$   | $2^{1}10^{2}$ | 149             | $2^+5^2$          | 6    | 20      | 1          |
| $L$   | $2^{24}/1^{24}$ | 290             | 1               | 1    | 24      | $M$            | ?        |

**Conjecture 4.8.** For each of the eleven vertex operator algebras $W$ as in Conjecture 4.3, the subgroup $\overline{\text{Aut}}(W) < O(A,-q) \cong O(A,q)$ induced by the action of $\text{Aut}(W)$ on the set of irreducible modules has the following property:

Let $K$ be an orbit lattice $N(Z)$ with discriminant space $(A,q)$. Then there is only one orbit under the double coset action of $\overline{O}(K) \times \overline{\text{Aut}}(W)$ on $O(A,q)$ for all genera besides the two genera $D$ and $J$. For the two lattices in the genus $D$, the number of orbits is 6 for the lattice $D_{12}(2)$ and it is 3 for the lattice $E_6(2) \oplus D_4(2)$. For the unique lattice in the genus $J$, the number of orbits is 2.

**Remark 4.9.** The conjecture is trivially true for genus $A$.

The conjecture is true in case of genus $B$: It was shown by Griess [Gr2] that $\text{Aut}(W) \cong O(A,-q) \cong O_{10}^+(2).2$. (Using the Atlas [CCNPW] notation for the groups.) This easily implies that $\text{Aut}(W) = \overline{\text{Aut}}(W)$ since $\text{Aut}(W)$ acts transitively on the Virasoro frames of $W$ with stabilizer equal $\text{Aut}(C) \cong AGL(4,2) \cong \overline{\text{Aut}}(W)$. Then one uses that $O_{10}^+(2)$ is simple. Also, an explicit description of the Virasoro frames of $W$ and the modules of $W$ are available. Since $\overline{\text{Aut}}(W) = O(A,q)$, there is always one orbit under the double coset action.

The conjecture is true for the case of genus $C$: It was shown by [CLS] that $\text{Aut}(W) \cong O_8^-(3).2$ and $\text{Aut}(W) = \overline{\text{Aut}}(W)$. A calculation with MAGMA shows that there is indeed a unique orbit under the double coset action for all lattices $K$ in genus $C$.

For the case of genus $D$, one may use the description of $\overline{\text{Aut}}(W)$ as given in the proof of Theorem 1.2.

In some of the other cases, we were able to construct the subgroup lattice of $O(A,q)$. In those cases, we can explicitly describe the subgroups of $O(A,q)$ having the required property.

We collect some information about $\Lambda_\gamma$, $(A_{\Lambda_\gamma}, q_{\Lambda_\gamma})$, $i^G$ (the index of $\overline{O}(\Lambda_\gamma)$ in $O(A_{\Lambda_\gamma}, q_{\Lambda_\gamma})$) and $\overline{\text{Aut}}(W)$ in Table 17.

We have shown the uniqueness of $W$ in all cases under the assumptions made. Our assumptions are of general nature besides for the structure of $V_{\Lambda_\gamma}$. For uniqueness of
the Schellekens vertex operator algebras we also need the assumptions on $\overline{\text{Aut}}(V^\ominus_N)$ made in Conjecture 4.8. The assumptions are theorems for many of the occurring 69 cases.

5 Discussion

Naturally, a more direct proof of the remarkable correspondence found in Theorem 3.1 and Theorem 5.3 is desirable.

One may expect that a Lorentzian picture — like the one used by Borcherds in his thesis to explain the correspondence between the deep holes of the Leech lattice and the Niemeier lattices with roots — helps to explain the one-to-one correspondence between the self-dual vertex operator algebras of central charge 24 with non-abelian Kac-Moody structure and the Niemeier orbit lattices $N(Z)$.

Let us consider the even Lorentzian lattice $M$ of signature $(25,1)$. It can be obtained by forming the direct sum of any Niemeier lattice $N$ — including the Leech lattice $\Lambda$ — and the two-dimensional hyperbolic plane $II_{1,1}$. Conversely, it is known that there are $24$ $O(M)$-orbits of isotropic vectors $v$ in $M$ corresponding via $N = v^+ / Zv$ to the 24 Niemeier lattices. This allows to identify certain elements of $O(\Lambda) < O(\Lambda \oplus II_{1,1}) \cong \Lambda.O(\Lambda).W(\Lambda \oplus II_{1,1})$ with certain elements of $W(N) < O(N) < O(N \oplus II_{1,1})$ for a Niemeier lattice $N$ with roots. Thus we can interpret $Z$ as a conjugacy class of a cyclic subgroup in $O(\Lambda)$ and hence in $O(M)$. We obtain:

**Theorem 5.1.** The 69 orbit lattices $N(Z)$ are in one-to-one correspondence to $O(M)$-orbits of pairs $(g, v)$ where $g$ is an element in $O(M)$ arising from an element in $O(\Lambda)$ with a frame shape as in cases A to J of Table 4. $v$ is an isotropic vector of $M$ where the Niemeier lattice $v^+ / Zv$ is not the Leech lattice and $g$ fixes $v$. Here, we let $O(M)$ act on the first component of the pair $(g,v)$ by conjugation and use the natural $O(M)$-action on the second.

The orbit of a pair $(g,v)$ recovers the orbit lattices $N(Z)$ together with $R(Z)$ and $C(Z)$ and thus contains more information than just the isometry type of $N(Z)$.

We can now try to find a similar description for the corresponding 69 self-dual vertex operator algebras $V$ of central charge 24.

Given such a vertex operator algebra $V$, we can consider the self-dual “Lorentzian” central charge 26 vertex algebra $X = V \otimes V_{II_{1,1}}$. We also have the analog of Theorem 5.1 for these “Lorentzian” vertex algebras if we request that $L$ is replaced by a Lorentzian lattice $K$. (This is our definition of “Lorentzian vertex algebras.) If we like to recover $V$ from $X$ up to isomorphism, we have to specify a splitting $K = L \oplus II_{1,1}$ together with a selection of an $O(L) \times \text{Aut}(W)$-orbit inside the $O(K) \times \text{Aut}(W)$-orbit [i] describing $V$ as in Theorem 4.1. In all of our eleven cases, one has that $O(K)$ maps subjectively to $O(A_K,q) \cong O(A_L,q)$ (see [Nik], Thm. 1.14.2) and thus there is only one $O(K) \times \text{Aut}(W)$-orbit [i] describing the gluing of $V_K$ and $W$ and thus the specification of [i] can be omitted.

It was conjectured by Borcherds [Bo4] that all “nice” generalized Kac-Moody algebras can be obtained as certain orbit Lie algebras of the fake Monster Lie algebra or the monster Lie algebra. Here we interpret nice that there is a BRST-construction from a Lorentzian self-dual central charge 26 vertex algebra $X$. In particular, he described the procedure for certain elements $\tilde{g}$ in $2^{24}.O(\Lambda)$ arising from automorphisms of the lattice vertex operator algebra $V_M$. The construction of the corresponding new Lorentzian self-dual central charge 26 vertex algebra $X$ was however not specified. Let us assume we have a good definition of a Lorentzian orbit lattice $M(g)$ generalizing
our orbit lattice $N(Z)$. (For our cases, we just set $M(g) = N(Z) \oplus H_{1,1}$.) We can then define $X$ as the up to isomorphism well-defined self-dual extension of $V_{M(g)} \otimes W$ where $W = W_{\Lambda_g}$ and $V_{M(g)}$ remain primitive. To recover $V$, we have first to specify an isotropic vector $v$ in $M(g)$ which can be used to split off an hyperbolic plane. So we have to investigate the $O(M(g))$-orbits of these vectors. Furthermore, we may have to restrict the action to the normalizer of $g$ in $O(M)$. However, the orbit lattices $M(g)$ and $N(Z)$ are only abstractly defined and one would have to construct a natural group action first. Then we have to understand which $\mathcal{O}(N(Z)) \times \mathfrak{Aut}(W)$-orbit $[i]$ one obtains that way. It seems that this fits in principle well with the description of Theorem 6.1.

From the Lorentzian picture, it is clear that for the coinvariant lattices one has $M_g \cong \Lambda_g \cong N_g$. Thus one can define the self-dual vertex operator algebra $V$ belonging to $N(Z)$ by a gluing of $V_{N(Z)}$ with $V_{\Lambda_g} \cong W$. Without a more natural description of $N(Z)$ or $V_{N(Z)}$, it seems somewhat unclear how to define canonically the correct double coset $[i]$ in the cases where it matters.

The problem seems only to occur when the smallest factor of the frame shape of $g$ is 2 and not 1, the case which corresponds to elements $g$ in $M_{23} < M_{24} < 2^{12}M_{24}$. For the $M_{23}$ elements, $\mathfrak{Aut}(W)$ seems nearly as large as $O(A, q)$. When the smallest factor is 2, we seem to be in the situation where the order of any lift $\hat{g}$ of $g$ to $\mathfrak{Aut}(V_M)$ has 2-times the order of $g$. In these cases it seems also be impossible to describe $M(g)$ easily (by direct sums) in terms of the fixed point lattice $\Lambda^g$.

One may also look at the generalized deep holes of $\Lambda^g$. Niemann [Nie] has done this for the $M_{23}$ cases. The generalized affine diagrams of some of those deep holes are equal to the Kac-Moody algebras occurring for the Schellekens vertex operator algebras. But there are further generalized deep holes. The reason seem to be that one has to look at orbits of isotropic vectors in $M(g) \cong \Lambda^g \oplus H_{1,1}(o(g))$ and isotropic vectors split either off a hyperbolic plane or a rescaled hyperbolic plane. Again it is unclear what the correct similar picture for the non $M_{23}$-cases would be.

More mysterious is the result that all vertex operator algebras in the genus of the Moonshine module — assuming the uniqueness of the moonshine module as the unique extremal vertex operator algebra of central charge 24 — can be obtained that way.

This would follow from Borcherds conjecture (in a lifted vertex algebra version) that Lorentzian self-dual vertex operator algebras of central charge 26 are obtained by a certain orbifold procedure from $V_M$ or $V^2 \otimes V_{H_{1,1}}$ if one explains which conjugacy classes of elements on $O(\Lambda)$ should occur. The orbit Kac-Moody algebra should be an honest algebra (not a super algebra) and the resulting lattice $M(g)$ should allow to split off a hyperbolic plane. The last condition for example excludes the $M_{23}$-elements of cycle shape 1.23.

Also, in the $(G, L, W, [i])$ picture of self-dual vertex operator algebras of central charge 24, it follows from Schellekens’ work that the genus $G$ is either unimodular or reflective. Unfortunately, no complete classification of reflective genera is available yet; cf. the work of Esselmann [Es].

The full character of the vertex operator algebra $W$ is a singular automorphic form $F$ for the Weyl representation of $(A, q)$, where the singular part describes the real simple roots of the corresponding generalized Kac-Moody algebra. Together with the condition $\dim W_1 = 0$, it may be possible to classify those forms. Scheithauer has done this under certain conditions [Sch], see also [Ba]. Again this may lead to a classification of totally reflective automorphic forms. Alternatively, one may classify the corresponding automorphic products.

It may be of some interest to consider the vertex operator algebra genera of $W$.
and $V_{N(Z)}$. One may conjecture that in the first case $W$ is the only vertex operator algebra with $W_1 = 0$. This is related to the question of possible automorphic forms $F$ and the uniqueness question considered in the last section.

The algebra $W$ can for $g$ of type $1^8 2^8$ and $1^6 3^6$ embedded into $V^g$. For the other cases, this may not be possible.

The six cases with $\dim g = 0$ in Table 3 correspond to the six conjugacy classes in $\text{Co}_0$ of frame shape $k^{24/(k-1)}/1^{24/(k-1)}$ where $k$ is one of the six numbers 2, 3, 4, 5, 7, 9, 13. The direct generalization of our construction fails since the coinvariant lattice $\Lambda_g$ for the corresponding element $g$ in $O(\Lambda)$ has rank 24. In all of these cases one expects a $\mathbb{Z}/k\mathbb{Z}$-orbifold construction of the Moonshine module generalizing the known orbifold constructions for $k = 2$ and 3.

For the 13 cases with $\dim g = 24$, the direct generalization using $W = V^g_{\Lambda_g}$ also fails. The vertex operator algebra $W$ satisfies $\dim W_1 = 0$ and thus the rank of the Lie algebra $V_1$ obtained by the construction as in Theorem 4.1 cannot be 24. Thus $V$ cannot be the Leech lattice vertex operator algebra. We note that in 5 cases the same conjugacy classes in $\text{Co}_0$ appear twice and the corresponding 4-dimensional orbit lattices $N(Z)$ are rescaled copies of each other. Since the only possibility for a $V$ with $V_1$ of rank 4 has Kac-Moody structure $C_{4,10}$, we conclude that for the $\dim g = 24$ cases the modular tensor category $\mathcal{T}(W)$ cannot be equal to $\mathcal{Q}(A_{N(Z)}, -q_{N(Z)})$.

The uniqueness question for the moonshine module remains open.

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