On a two dimensional system associated with the complex of the solutions of the Tetrahedron equation.

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Abstract: A sort of two dimensional linear auxiliary problem for the complex of 3D R – operators associated with the Zamolodchikov – Bazhanov – Baxter statistical model is proposed. This problem resembles the problem of the local Yang – Baxter equation but does not coincide with it. The formulation of the auxiliary problem admits a notion of a “fusion”, and usual local Yang – Baxter equation appears among other results of this “fusion”.

Key words: Tetrahedron equation, 2 + 1 integrability, local Yang – Baxter equation.
1 Introduction

Well known shortcoming of the three dimensional integrability, connected with the tetrahedron equation, is the remarkable poverty of the models. Actually, there exists only one nontrivial complex of solutions of the tetrahedron equation, which has statistical mechanics, field theory and “classical” (i.e. functional) forms. In the statistical mechanics, when the number of states is finite, the model is called Zamolodchikov – Bazhanov – Baxter model \( [1, 2] \). It is known almost everything (except the most complicated aspects, concerning the structure of eigenstates of the transfer matrices and the correlation functions) about this model, namely: the partition function \( [3, 2] \), the vertex – IRF correspondence \( [4, 5] \), affine Toda field theory \( [6] \) and field theory operator \( R \)-matrix \( [7, 8] \), and the transition from the infinite number of the states to finite one \( [9, 8] \), and finally, the functional \( R \)-operator as the sequence of the permutation relations for the field theory \( R \)-matrix \( [8, 10] \). Here one should mention that there exists an hierarchy of the operator \( R \)-matrices which corresponds to several partial specifications of the spectral parameters of ZBB matrix weights. There must exist a reverse way, obtaining of more complicate \( R \)-s in the hierarchy in terms of simplest (some sort of the fusion). Also one may mention the interpretation of the projection of 3D operator \( R \)-matrices in terms of a representation of some specific Drinfeld double \( [11] \).

From the other hand side there are a lot of solutions of the functional tetrahedron equation. Moreover, we’ve got infinitely many such solutions \( [12] \) versus one specific given by the complex mentioned above.

Recall that usually the functional tetrahedron solution appears when one considers so-called local Yang – Baxter equation (LYBE) \( [13] \), which is a proper generalization of the zero curvature condition for two dimensional system. Namely, let \( L_{i,j}(\vec{x}) \) be matrices of weights of the Yang – Baxter type, and \( \vec{x} \) stands for their formal parameters. Then LYBE

\[
L_{12}(\vec{x}) \cdot L_{13}(\vec{y}) \cdot L_{23}(\vec{z}) = L_{23}(\vec{z}) \cdot L_{13}(\vec{y}) \cdot L_{12}(\vec{x}) ,
\]

when it is nondegenerate with respect to \( \vec{x}, \vec{y}, \vec{z} \), gives the functional map from \( \{\vec{x}, \vec{y}, \vec{z}\} \) to \( \{\vec{x}, \vec{y}, \vec{z}\} \) by

\[
\hat{\vec{x}} = \hat{r}_1(\vec{x}, \vec{y}, \vec{z}) , \quad \hat{\vec{y}} = \hat{r}_2(\vec{x}, \vec{y}, \vec{z}) , \quad \hat{\vec{z}} = \hat{r}_3(\vec{x}, \vec{y}, \vec{z}) ,
\]

(2)

Usually the functional operator \( R_{i,j,k} \), giving this map, is introduced:

\[
R_{i,j,k} \cdot \phi(\vec{x}_i, \vec{x}_j, \vec{x}_k) = \phi(\vec{x}_i, \vec{x}_j, \vec{x}_k) ,
\]

(3)

\[
\hat{\vec{x}}_i = \hat{r}_1(\vec{x}_i, \vec{x}_j, \vec{x}_k) , \quad \hat{\vec{x}}_j = \hat{r}_2(\vec{x}_i, \vec{x}_j, \vec{x}_k) , \quad \hat{\vec{x}}_k = \hat{r}_3(\vec{x}_i, \vec{x}_j, \vec{x}_k) ,
\]

(4)

where \( \phi \) is an arbitrary function. Considering the quadrilateral formed by six \( L \)-s

\[
L_{12}(\vec{x}_1) \cdot L_{13}(\vec{x}_2) \cdot L_{23}(\vec{x}_3) \cdot L_{14}(\vec{x}_4) \cdot L_{24}(\vec{x}_5) \cdot L_{34}(\vec{x}_6) =
\]

\[
= L_{34}(\vec{y}_6) \cdot L_{24}(\vec{y}_5) \cdot L_{14}(\vec{y}_4) \cdot L_{23}(\vec{y}_3) \cdot L_{13}(\vec{y}_2) \cdot L_{12}(\vec{y}_1) ,
\]

(5)
one obtains rhs by two different ways, first starting from \( L_{12} L_{13} L_{24} \), second starting from \( L_{23} L_{24} L_{34} \). This gives the equivalence of two successive applications of \( R_{i,j,k} \):

\[
\phi[\vec{y}_1, \vec{y}_2, \vec{y}_3, \vec{y}_4, \vec{y}_5, \vec{y}_6] = \\
R_{123} \cdot \left( R_{145} \cdot \left( R_{246} \cdot \phi[\vec{x}_1, ..., \vec{x}_6] \right) \right) = \\
= R_{356} \cdot \left( R_{246} \cdot \left( R_{145} \cdot \left( R_{123} \cdot \phi[\vec{x}_1, ..., \vec{x}_6] \right) \right) \right).
\]

(6)

This is the functional tetrahedron equation (FTE). 

In the case when the local weights \( L_{a,b}(\vec{x}) \) have the structure of the ferro-electric type free fermion model’s weights, the irreducible part of eq. (6) can be extracted in the form of Korepanov’s equation

\[
X_{12}(\vec{x}) \cdot X_{13}(\vec{y}) \cdot X_{23}(\vec{z}) = X_{23}(\vec{z}) \cdot X_{13}(\vec{y}) \cdot X_{12}(\vec{x}),
\]

(7)

where ‘·’ means the matrix multiplication of

\[
X_{12}(\vec{x}) = \begin{pmatrix}
a(\vec{x}) & b(\vec{x}) & 0 \\
c(\vec{x}) & d(\vec{x}) & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
X_{13}(\vec{x}) = \begin{pmatrix}
a(\vec{x}) & 0 & b(\vec{x}) \\
0 & 1 & 0 \\
c(\vec{x}) & 0 & d(\vec{x})
\end{pmatrix},
\]

\[
X_{23}(\vec{x}) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & a(\vec{x}) \\
0 & c(\vec{x}) & d(\vec{x})
\end{pmatrix}.
\]

(8)

\(a(\vec{x}),...,d(\vec{x})\) are in general some matrix functions. Eq. (7) and its connection with the functional tetrahedron equation was investigated in [12] in most general case. There was proven that there exists a wide class of the solutions of FTE associated with eq. (7). A subset of the solutions of FTE was described in [13], [16] and recently in [17]. These are the cases when \(a, b, c, d\) are some numeric functions of single variable \(x\). Nevertheless, the solution of FTE associated with the ZBB complex is still not described in terms of LYBE (moreover, we suspect, this is impossible at least in terms of the Korepanov’s equation (7)).

So, what we wished to say by this long introduction. From one hand side a plenty of the solutions of FTE exist, these solutions are associated with

\[\text{\footnote{Here one should mention the successful attempt to obtain 3D } R \text{ – matrix directly from the similar consideration of the Zamolodchikov – type algebra for two dimensional } L_{\alpha,i}^{\alpha}, \text{ where } \alpha - \text{ indices of 3D } R \text{ – matrix [4]. This } R \text{ – matrix obtained appeared to be some special case of the } R \text{ – matrix for ZBB model.}}\]
LYBE (LYBE makes FTE obvious), but we have no skill to quantize the m. From the other hand side, only one model is the whole complex, i. e. it exists in the statistical mechanics form, field theory form and a functional transformation form, moreover, there exists a well defined way to obtain the statistical mechanics and the field theory from the functional form, but we do not know a sort of a linear problem (like LYBE) for this.

In this paper we try to give the answer to all these questions. We suggest some formulation of two dimensional system associated with 2D lattice, which is not Yang – Baxter type system (i. e. we do not interpret the vertices as matrices of weights or anything equivalent), but nevertheless the usual graphic pictures remain as well as a sort of zero curvature condition, FTE and so on. Our formulation resembles the electric networks, when the star – triangle equivalence gives the electric network solution of FTE (well described in terms of Korepanov’s equation [15, 17]) but originally this equivalence is not LYBE at all, it is just a resoldering of the resistances. What we suggest, it is just a proper generalization of Kirchhoff’s rules in terms of arbitrary 2D networks formed by rectangular vertices. Such formulation is not equivalent to the usual LYBE-s in general. The functional transformation corresponding to ZBB complex is a partial case of our functional transformation. There is a sort of fusion in the system proposed, and the ferro-electric case is just a subspace of a partial case of fused system.

2 Formulation of the system and \( R \) – operator

Consider 2D graphs formed by the usual oriented rectangular vertices. Each vertex has four adjacent fields. Assign to these fields some variables, \( \textit{currents} \), for the vertex shown in fig. 1 denote them \( J, J', \Phi, \Phi' \). Suppose these current are additive for any field of the 2D network, and for any closed field let the sum of the currents belonging to this field and corresponding to the surrounding vertices be zero (it is supposed that all the currents assigned to any vertex flow into this vertex) – see fig. 3 and considerations for it for an example. Suppose also the currents assigned to the vertex \( V_k \) obey some condition, \( V_k(J_k, J'_k, \Phi_k, \Phi'_k) = 0 \) (obviously, this condition must be a set of linear relations homogeneous with respect to the currents). Thus, for any open network there appears some condition for \( \textit{outer} \) currents. Two networks are equivalent if their such conditions coincide. For example, the evolution of 2D network is given by the condition of the equivalence of the left hand side type and right hand side type graphs, as it is drawn in fig. 2. Solving such equivalence condition with respect to the parameters of \( V'_k \) in rhs, we obtain a solution of FTE.

Now fix the form of \( V(J, J', \Phi, \Phi') \) (a sort of Ohm’s law):

\[
\Phi = i w \cdot J, \quad \Phi' = -i u \cdot J, \\
\Phi = -i \pi \cdot J', \quad \Phi' = i w \cdot J',
\]

(9)
with the primitive “zero – curvature” condition:

\[ u^{-1} \cdot w = \overline{w}^{-1} \cdot \overline{u} \, . \]  

(10)

Surely, invertible elements \( u, w, \overline{u}, \overline{w} \) are treated as the parameters of the vertex \( V \). Note also that this is not unique choice of \( V(J, J', \Phi, \Phi') \), we may for example suppose two of the currents to be independent. Such cases we do not investigate here.

Consider now the Baxter – type correspondence. Let the outer currents be \( x, y, z, x', y', z' \) as it is shown in fig. 2. Left – hand side is drawn in fig. 3. There \( J_k, J'_k, \Phi_k, \Phi'_k \) are connected by (11). Conservation of the currents for closed field is

\[ J_1 + \Phi_2 + J'_3 = 0 \]  

(11)

and given outer currents are

\[ J'_1 = x', \ \Phi'_2 = y, \ J'_3 = z \]  

\[ J_2 + \Phi'_3 = x', \ \Phi_1 + \Phi_3 = y', \ \Phi'_1 + J'_2 = z'. \]  

(12)

These with (11) give the following system:

\[ u_1^{-1} \cdot u_1 \cdot x + \overline{u}_2 \cdot w_2^{-1} \cdot y + w_3^{-1} \cdot u_3 \cdot z = 0 \]  

\[ x' = i \overline{u}_2^{-1} \cdot y - i u_3 \cdot z \]  

\[ y' = -i \overline{u}_1 \cdot x + i \overline{w}_3 \cdot z \]  

\[ z' = i u_1 \cdot x - i w_2^{-1} \cdot y \]  

(13)

Thus two of the currents are independent, and the rest four one can express in terms of the independent.

Analogous consideration for the right – hand side with primed parameters of the vertices dives

\[ \tilde{u}^{-1}_1 \cdot \tilde{u}_1 \cdot x' + \tilde{w}^{-1}_2 \cdot \tilde{w}_2^{-1} \cdot y' + \tilde{u}^{-1}_3 \cdot \tilde{u}_3 \cdot z' = 0 \]  

Thus two of the currents are independent, and the rest four one can express in terms of the independent.
Figure 2: Baxter – type equivalence of two graphs formed by the vertices \( V_{ij} \) in lhs and by \( \hat{V}_{ij} \) in rhs and outer currents \( x, y, z, x', y', z' \).

Figure 3: The left – hand side with the currents attributed to each vertex.
\[ x = i \hat{w}_2^{-1} \cdot y' - i \hat{w}_3 \cdot z', \]
\[ y = -i \hat{u}_1 \cdot x' + i \hat{w}_3 \cdot z', \]
\[ z = i \hat{w}_1 \cdot x' - i \hat{w}_2^{-1} \cdot y'. \]  
(14)

Let now rhs (14) be equivalent to lhs (13). This gives eight equations for the parameters of \( V_k \) and \( \dot{V}_k \). The solution is following: introduce
\[
\begin{align*}
\Lambda_1 &= u_1^{-1} \cdot w_1 + \pi_4^{-1} \cdot \pi_1 + \pi_2 \cdot w_1, \\
\Lambda_2 &= \pi_2 \cdot w_2^{-1} + w_3^{-1} \cdot u_2^{-1} + u_1^{-1} \cdot \pi_2, \\
\Lambda_3 &= w_3^{-1} \cdot u_3 + \pi_2 \cdot u_3 + \pi_1^{-1} \cdot \pi_2, \\
\end{align*}
\]
and let \( \Omega_i \), defined up to an ambiguity \( \Omega_i \rightarrow \omega \cdot \Omega_i \), obey
\[
\begin{align*}
\Omega_1 \cdot [u_2^{-1} \cdot \Lambda_2^{-1} \cdot u_1^{-1}] &= \Omega_2 \cdot [\pi_1 \cdot \Lambda_1^{-1} \cdot \pi_2], \\
\Omega_2 \cdot [\pi_3 \cdot \Lambda_3^{-1} \cdot \pi_2] &= \Omega_3 \cdot [w_2^{-1} \cdot \Lambda_2^{-1} \cdot \pi_3^{-1}], \\
\Omega_3 \cdot [w_1 \cdot \Lambda_1^{-1} \cdot \pi_3^{-1}] &= \Omega_1 \cdot [u_3 \cdot \Lambda_3^{-1} \cdot \pi_3^{-1}],
\end{align*}
\]
(16)
where any of this equations is the rather nontrivial sequence of two other, then
\[
\begin{align*}
\dot{w}_1 &= w_2 \cdot \Omega_3^{-1}, & \dot{u}_1 &= \Lambda_2^{-1} \cdot w_3^{-1}, \\
\dot{w}_1 &= \Lambda_3^{-1} \cdot \pi_2, & \dot{u}_1 &= \pi_3^{-1} \cdot \Omega_2^{-1}, \\
\dot{w}_2 &= \Omega_3 \cdot w_1, & \dot{u}_2 &= \Omega_1 \cdot u_3, \\
\dot{w}_2 &= \pi_1 \cdot \Lambda_3, & \dot{u}_2 &= \pi_3 \cdot \Lambda_1, \\
\dot{w}_3 &= \Lambda_2^{-1} \cdot u_1^{-1}, & \dot{u}_3 &= u_2 \cdot \Omega_1^{-1}, \\
\dot{w}_3 &= \pi_1^{-1} \cdot \Omega_3^{-1}, & \dot{u}_3 &= \Lambda_1^{-1} \cdot \pi_3.
\end{align*}
\]
(17)
(18)
(19)

This map has the gauge degrees of the freedom, one degree in lhs and one in rhs. Namely, the system is unchanged if one change lhs as follows
\[
\begin{align*}
u_1 \rightarrow u_1 \cdot \omega^{-1}, & \quad \pi_1 \rightarrow \pi_1 \cdot \omega^{-1}, \\
u_2 \rightarrow \omega \cdot \pi_2, & \quad \pi_2 \rightarrow \omega \cdot \pi_2, \\
\pi_3 \rightarrow \pi_3 \cdot \omega^{-1}, & \quad w_3 \rightarrow w_3 \cdot \omega^{-1},
\end{align*}
\]
(20)
and rhs (this is the above mentioned shift of \( \Omega-s \))
\[
\begin{align*}
\dot{\pi}_1 \rightarrow \dot{\pi}_1 \cdot \omega^{-1}, & \quad \dot{w}_1 \rightarrow \dot{w}_1 \cdot \omega^{-1}, \\
\dot{\pi}_2 \rightarrow \omega \cdot \dot{\pi}_2, & \quad \dot{w}_2 \rightarrow \omega \cdot \dot{w}_2, \\
\dot{\pi}_3 \rightarrow \dot{\pi}_3 \cdot \omega^{-1}, & \quad \dot{w}_3 \rightarrow \dot{w}_3 \cdot \omega^{-1}.
\end{align*}
\]
(21)

7
Thus we obtain the map \( R(\dot{\omega}, \omega) \),
\[
R_{1,2,3}(\dot{\omega}, \omega) : \{ V_1(\omega), V_2(\omega), V_3(\omega) \} \rightarrow \{ \dot{V}_1(\dot{\omega}), \dot{V}_2(\dot{\omega}), \dot{V}_3(\dot{\omega}) \},
\] (22)
where \( \{ \dot{V}_k \} \) are given by (21) and the gauge ambiguity is expressed via the dependence of \( R \) on \( \omega, \dot{\omega} \). This \( R \) gives the correspondence between two “one dimensional” orbits in the spaces of in- and out state spaces. A fixing of the gauge means a rule comparing points on these orbits. Thus in FTE there exist three-parameters in-orbit and three-parameters out-orbit. Even if the points on in- and out-orbits of the quadrilateral are fixed, there still are one-parameter freedoms in left and right hand sides of FTE.

3 Partial cases

Mention now a possible algebraization of the system (17–19). Suppose the parameters of \( V_{12}, V_{13} \) and \( V_{23} \) commute. Demand that the parameters of \( \dot{V}_{12}, \dot{V}_{13}, \dot{V}_{23} \) also commute. This immediately gives
\[
\overline{w} \cdot w = w \cdot \overline{w} = k^2, \quad \overline{u} \cdot u = u \cdot \overline{u} = q^{-1}k^2,
\] (23)
where \( k^2 \) is a center, and
\[
u \cdot w = q \, w \cdot u.
\] (24)
Expressions for \( \Omega \)-s are
\[
\Omega_1 = f^{-\frac{2}{3}} k_2^2 k_3^{-1} \, \Lambda_1^{-1}, \quad \Omega_2 = f^{-2} \frac{1}{k_1 k_2^2} \cdot \Lambda_2^{-1}, \quad \Omega_3 = f^{-2} \frac{k_1^2}{k_2^2} \cdot \Lambda_3^{-1},
\] (25)
where \( f \) is also a center. When \( k \)-s conserve, i. e. \( f = 1 \), the map (17–19) can be realized by the known complete operator \( R \)– matrix [8, 10].

To make it clear, put \( q = 1 \), i. e. \( u, w, \overline{u}, \overline{w} \) are numbers. Then change
\[
u \rightarrow ku, \quad w \rightarrow kw, \quad \overline{u} \rightarrow k/u, \quad \overline{w} \rightarrow k/w,
\] (26)
Hence
\[
\dot{k}_1 = k_1 f, \quad \dot{k}_2 = \frac{k_2}{f}, \quad \dot{k}_3 = k_3 f,
\] (27)
and
\[
\dot{w}_1 = f \frac{k_3 w_1 w_2 + k_1 w_2 w_3 + k_1 k_2 k_3 u_3 w_3}{w_3},
\]
\[
\dot{u}_1 = f \frac{k_2}{k_1} \frac{u_1 u_2 w_2}{w_3},
\]
\[
\dot{w}_2 = f \frac{k_2}{k_3} \frac{w_1 w_2 w_3}{w_3}.
\]
Figure 4: Vertex $V_{<\alpha,\beta>}$, formed by two usual vertices. The currents $x$ and $x'$ become edge indices.

\[
\begin{align*}
\dot{u}_2 &= \frac{k_2}{f} \frac{u_1 u_2 u_3}{k_3 w_1 u_2 + k_1 u_2 u_3 + k_1 k_2 k_3 u_1 w_1}, \\
\dot{w}_3 &= \frac{k_2}{f} \frac{u_2 w_2 w_3}{k_1 u_1 w_2 + k_3 u_2 w_3 + k_1 k_2 k_3 u_1 w_3}, \\
\dot{u}_3 &= \frac{k_2}{f} \frac{u_3 w_1 u_2 + k_1 u_2 u_3 + k_1 k_2 k_3 u_1 w_1}{u_1}.
\end{align*}
\] (28)

If we choose $f = 1$, so that it is possible to put $k = 1$, then the map (28) is explicitly the complete functional map of ZBB complex (see [10] for a table of two – parameters functional maps, all the examples there are just several specifications of eq. (28) with respect to $k$-s, and the case (iv) there is explicitly (28).)

FTE is the sequence of FTE just for $k$-s. Another possibility of choosing $f$ is the situation when $u_i = w_i = 1$ is the stationary point of the map (28), this gives the electric network form of $f$: 

\[
f = \frac{k_2}{k_1 + k_3 + k_1 k_2 k_3}.
\] (29)

One more possibility is to choose $f$ as it is for the Onsager’s model.

4 A “fusion”

Consider a double vertex formed by two vertices, $V_\alpha$ and $V_\beta$ as it is shown in fig. (4). Denote this object as $V_{<\alpha,\beta>}$. Let the outer currents be $J, J', \Phi, \Phi'$. 
and $x, x'$. They obey four relations,

$$
J' = i \pi^{-1}_\alpha \Phi, \quad \Phi' = -i u_\beta \ J,
$$
$$
x = i w_\beta J - i \pi^{-1}_\alpha \Phi, \quad x' = -w_\beta^{-1} u_\beta J - w_\alpha \pi^{-1}_\alpha \Phi,
$$

(30)

so that only two of the currents are independent. Let them be $J$ and $x$, then

$$
x' = -(\pi_\beta^{-1} + u_\alpha) w_\beta J - i u_\alpha x,
$$
$$
J' = i \pi^{-1}_\alpha w_\alpha \pi_\beta J - \pi^{-1}_\alpha w_\alpha x,
$$
$$
\Phi = w_\alpha \pi_\beta J + i \pi_\alpha x, \quad \Phi' = -i u_\beta J.
$$

(31)

Consider the transformation of a pair of such double vertices, $V_{<2,4>}$ and $V_{<3,5>}$, as it is shown in fig. (5). In terms of $R$ operators, this transformation is given by $R_{123} \cdot R_{145}$. One can impose an invariant condition for left and right hand sides of (5), namely, consider the condition when in $V_{<\alpha,\beta>}$ the "edge" currents, $x$ and $x'$, are proportional, i.e. $\pi_\beta^{-1} + u_\alpha = 0$, so that $x' = -i u_\alpha x$. Obviously, if the "edge" current flows through the left hand side of (5), then it has to flow through the right hand side of (5). This means that if one imposes in the left hand side the conditions

$$
u_2 + \pi_4^{-1} = 0, \quad \nu_3 + \pi_5^{-1} = 0,
$$

(32)

then one obtains

$$
\dot{\nu}_2 + \dot{\pi}_4^{-1} = 0, \quad \dot{\nu}_3 + \dot{\pi}_5^{-1} = 0,
$$

(33)

i.e. (32) is the ideal of $R_{123} \cdot R_{145}$. For this simple system this fact can be easily verified directly.
Consider now more complicated case of the quadrat (see fig. 6). Call this object $V_{\langle \alpha, \beta, \gamma, \delta \rangle}$. In this case three currents are independent, choose them be $J$, $x$ and $y$. Solving the system

$$x = i \overline{\alpha} J - i \overline{\delta} \Phi, \quad x' = i \overline{\gamma} J - i \overline{\delta} \Phi',$$

$$y = i u_{\delta}^{-1} \Phi' - i u_{\alpha} J, \quad y' = i \overline{\beta}^{-1} \Phi - i \overline{\gamma} J',$$

$$w_{\alpha}^{-1} u_{\alpha} J + w_{\beta} \overline{\alpha}^{-1} \Phi + u_{\gamma}^{-1} w_{\gamma} J' + \overline{w}_{\delta} \overline{\delta}^{-1} \Phi' = 0,$$ (34)

we obtain

$$J' = -w_{\gamma}^{-1} u_{\gamma} \left( \chi J + i u_{\beta} x - i \overline{w}_{\delta} y \right),$$

$$\Phi = \overline{w}_{\beta} \overline{w}_{\alpha} J + i \overline{w}_{\beta} x, \quad \Phi' = u_{\delta} u_{\alpha} J - i u_{\delta} y,$$

$$x' = -i \left( w_{\delta}^{-1} u_{\delta} u_{\alpha} + u_{\gamma} \chi \right) J + u_{\gamma} u_{\beta} x - \left( u_{\gamma} + \overline{w}_{\delta}^{-1} \overline{w}_{\delta} y \right),$$

$$y' = i \left( \overline{w}_{\beta} \overline{w}_{\beta} \overline{\alpha} + \overline{w}_{\gamma} \chi \right) J - \left( w_{\beta}^{-1} + \overline{w}_{\alpha} \right) u_{\beta} x + \overline{w}_{\gamma} \overline{w}_{\delta} y,$$ (35)

where

$$\chi = w_{\alpha}^{-1} u_{\alpha} + u_{\beta} \overline{w}_{\alpha} + \overline{w}_{\delta} u_{\alpha}. \quad \text{(36)}$$

Consider the intertwining of three copies of $V_{\langle \alpha, \beta, \gamma, \delta \rangle}$:

$$R_{1,2,3} : V_{\langle \alpha_1, \beta_1, \gamma_1, \delta_1 \rangle}, V_{\langle \alpha_2, \beta_2, \gamma_2, \delta_2 \rangle}, V_{\langle \alpha_3, \beta_3, \gamma_3, \delta_3 \rangle} \rightarrow \hat{V}_{\langle \alpha_1, \beta_1, \gamma_1, \delta_1 \rangle}, \hat{V}_{\langle \alpha_2, \beta_2, \gamma_2, \delta_2 \rangle}, \hat{V}_{\langle \alpha_3, \beta_3, \gamma_3, \delta_3 \rangle}, \quad \text{(37)}$$

so that

$$R_{1,2,3} = R_{\alpha_1, \beta_2, \gamma_3} \cdot R_{\delta_1, \gamma_2, \delta_3} \cdot R_{\beta_1, \beta_2, \beta_3} \cdot R_{\gamma_1, \gamma_2, \gamma_3} \cdot R_{\alpha_1, \alpha_2, \alpha_3} \cdot R_{\delta_1, \delta_2, \delta_3} \cdot R_{\beta_1, \alpha_2, \alpha_3} \cdot R_{\gamma_1, \delta_2, \alpha_3}. \quad \text{(38)}$$
Imposing the condition of independence of \( x', y' \) on \( J \), we obtain the ideal of the corresponding complicated \( \mathcal{R} \):

\[
w_{\delta}^{-1} u_{\alpha} + u_{\gamma} \chi = 0, \quad w_{\beta}^{-1} w_{\beta} w_{\alpha} + w_{\gamma} \chi = 0 . 
\] (39)

Next, ignoring the “edge” currents at all (i.e. putting them zeros), so that

\[
\Phi = w_{\beta} \cdot w_{\alpha} \cdot J = w_{\beta} \cdot w_{\gamma} \cdot J', \\
\Phi' = u_{\delta} \cdot u_{\alpha} \cdot J = w_{\delta} \cdot w_{\gamma} \cdot J',
\] (40)

we obtain on the surface of (39) the morphism

\[
V_{\alpha} \times V_{\beta} \times V_{\gamma} \times V_{\delta} \Rightarrow V
\] (41)

where

\[
\begin{align*}
U &= i u_{\beta} u_{\alpha}, \\
W &= -i w_{\beta} w_{\alpha}, \\
\overline{U} &= i u_{\beta} u_{\alpha}, \\
\overline{W} &= -i w_{\beta} w_{\alpha}.
\end{align*}
\] (42)

Obviously, all these manipulations resemble the fusion for the two dimensional models. For an operator formulation, when \( R \)– operators can be expressed in terms of quantum dilogarithms of \( w_i, u_i \), the ideals \( \mathcal{R} \) mean the operator projectors commuting with \( \mathcal{R} \).

Note that one can ignore the face currents and consider only \( x, y \to x', y' \). Thus one obtains exactly the formulation of the free fermionic 6-vertex type, that was considered in [17]. Most complete formulation is, of course, the formulation with the face–edge currents.

5 Discussion

In this notes we have proposed some algebraical toy which can be interpreted as a intertwining problem for the complex of \( R \)– operators associated with ZBB statistical model. Obvious is only one advantage of this toy: a sort of a “fusion”. A nonsense (or again an advantage) of this toy is also obvious: the gauge ambiguity. One can fix this ambiguity in different ways, so that 3D \( R \)– operators gain some non–quantized functional part.

Nevertheless, we guess that our system would lead to something more general then known set of the solutions of the tetrahedron equation. A naïve way to obtain other \( R \)– matrices is to regard the elements \( u_k, w_k, u_k, w_k \) as matrices of the same structure, and hence noncommutative for different \( k \)-s. Give only one example: consider the case when

\[
u = w = \overline{u} = \overline{w} = \begin{pmatrix} 0 & k \\ 1/s & 0 \end{pmatrix},
\] (43)
that is the simplest generalization of the pure electric network system so as the

gauge ambiguity is canceled, then the rather nontrivial map

\[
\begin{align*}
  k'_1 &= \frac{k_2s_3}{s_1+k_2+s_3}, & s'_1 &= k_3+s_2+s_3, \\
  k'_2 &= k_1+k_3+\frac{k_1k_3}{s_2}, & s'_2 &= \frac{s_1s_3}{s_1+k_2+s_3}, \\
  k'_3 &= \frac{s_1k_2}{s_1+k_2+s_3}, & s'_3 &= k_1+s_2+k_1s_2/k_3
\end{align*}
\] (44)

is obtained.

Acknowledgments: I should like to thank Yu. G. Stroganov, H. E. Boos, V. V. Mangazeev, G. P. Pron’ko, F. W. Nijhoff and especially Rinat Kashaev and Igor Korepanov for many fruitful discussions.

The work was partially supported by the grant of the Russian Foundation for Fundamental research No 95 – 01 – 00249.

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