Outward pointing properties for vectorial hysteresis operators and some applications

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Outward pointing properties for vectorial hysteresis operators and some applications

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Abstract. In a number of papers, the concept of outward pointing hysteresis allowed the application of the method of invariant regions to derive uniform bounds for solutions to evolution equations involving scalar hysteresis operators and input perturbations.

The aim of this note is to generalize the “pointing outward” properties to vectorial hysteresis operators, and to show how these properties can be used to derive uniform estimates for differential equations involving vectorial hysteresis operators and input perturbations.

1. Introduction and statement of the problem
The method of invariant regions is quite often used to derive uniform estimates for solutions to evolution equations involving nonlinear scalar superposition operators and some input perturbations, such that asymptotic stability results can be shown, see, e.g., [16, 17].

In a number of papers, the concept of outward pointing hysteresis allowed the application of the method of invariant regions also to evolution equations involving scalar hysteresis operators and input perturbations, see [2, 3, 7, 8, 9, 10, 11].

The aim of this note is to generalize the “pointing outward” properties to vectorial hysteresis operators, and to show how these properties can be used to derive uniform estimates for differential equations involving vectorial hysteresis operators and input perturbations.

After recalling the definition of hysteresis operators in Section 2, in Section 3 the outward pointing definitions for scalar hysteresis operators will be recalled and some theorems will show that one can derive uniform bounds for evolution equations with hysteresis operators in a similar way as is done with the method of invariant regions for evolution equations with superposition operators.

In Section 4, the definition of the outward pointing condition for vectorial hysteresis operators will be formulated, some examples for vectorial operators satisfying this condition will be presented, and it will be shown how one can derive uniform bounds for ODEs involving this kind of operator. Moreover, there will be a discussion of which problems arise if one also takes input perturbations into account. To overcome these problems, the more restrictive componentwise outward pointing condition for vectorial hysteresis operators is introduced in Section 5, and uniform estimates for ODEs with componentwise outward pointing vectorial hysteresis operators and input perturbations are derived.
2. Hysteresis operators

In this section, let $n$ be a natural number and let $\| \cdot \|$ be the Euclidian norm on $\mathbb{R}^n$.

Denote by Map$(\mathbb{R}_0, \mathbb{R}^n)$ the space of all function from $\mathbb{R}_0 := [0, \infty)$ to $\mathbb{R}^n$ and by $C(\mathbb{R}_0, \mathbb{R}^n) \subset$ Map$(\mathbb{R}_0, \mathbb{R}^n)$ the space of all continuous function from $\mathbb{R}_0$ to $\mathbb{R}^n$, including also the unbounded ones.

The following definitions corresponds to those used in [4, 12, 13, 18]:

**Definition 2.1.** Consider a mapping $H : D(H)(\subseteq C(\mathbb{R}_0, \mathbb{R})) \rightarrow$ Map$(\mathbb{R}_0, \mathbb{R}^n)$.

(i) $H$ is said to be *locally Lipschitz*, if there exists a non-decreasing function $\psi : \mathbb{R}_0 \rightarrow \mathbb{R}_0$ such that the inequality

$$\|H[u](t) - H[v](t)\| \leq \psi \left( \sup_{s \in [0,t]} \|u(s)\| + \sup_{s \in [0,t]} \|v(s)\| \right) \sup_{s \in [0,t]} \|u(s) - v(s)\|$$

holds for every $u, v \in D(H)$ and every $t \geq 0$.

(ii) $H$ is said to be *causal*, if the implication

$$u(s) = v(s) \quad \forall s \in [0, t) \Rightarrow H[u](t) = H[v](t)$$

holds for every $u, v \in D(H)$ and every $t \geq 0$.

(iii) $H$ is said to be *rate-independent*, if

$$H[v \circ \alpha](t) = H[v](\alpha(t))$$

holds for every $v \in D(H)$, every $t \geq 0$, and every continuous non-decreasing function $\alpha : \mathbb{R}_0 \rightarrow \mathbb{R}_0$ with $\alpha(\mathbb{R}_0) = \mathbb{R}_0$ and $v \circ \alpha \in D(H)$.

(iv) $H$ is said to be a *hysteresis operator*, if it is causal and rate-independent.

3. Outward pointing properties for scalar hysteresis operators and uniform estimates

3.1. Outward pointing property for scalar hysteresis operators

**Definition 3.1.** Consider a mapping $H : D(H)(\subseteq C(\mathbb{R}_0, \mathbb{R})) \rightarrow$ Map$(\mathbb{R}_0, \mathbb{R})$.

Let $\delta > 0$, $h \geq 0$, and $A \leq a \leq b \leq B$ be given. $H$ is said to be *pointing outwards with bound $h$ in the $\delta$-neighborhood of $[A, B]$ for initial values in $[a, b]$", if for every $t \geq 0$ and every $u \in D(H)$ such that

$$u(0) \in [a, b], \quad u(s) \in (A - \delta, B + \delta) \quad \forall s \in [0, t],$$

we have

$$\begin{cases}
(\mathcal{H}[u](t) - h)(u(t) - B)^+ \geq 0, \\
(\mathcal{H}[u](t) + h)(u(t) - A)^- \leq 0,
\end{cases}$$

where $z^+ = \max\{z, 0\}$ and $z^- = \max\{-z, 0\}$ for $z \in \mathbb{R}$ denote the positive and negative part of $z$, respectively.

3.2. Uniform estimates for ODE involving outward pointing scalar hysteresis operators

The following theorem provides the uniform estimates used in the sequel.

**Theorem 3.2.** Let $\delta > 0$, $h \geq 0$, $A \leq a \leq b \leq B$ be given. Let $H : D(H)(\subseteq C(\mathbb{R}_0, \mathbb{R})) \rightarrow$ Map$(\mathbb{R}_0, \mathbb{R})$ be a causal operator that is pointing outwards with bound $h$ in the $\delta$-neighborhood of $[A, B]$ for initial values in $[a, b]$. Let $f \in L^\infty(\mathbb{R}_0)$ with $|f(s)| \leq h$ for a.e. $t > 0$, and let $T > 0$ be given.
(i) Let $u^0 \in [a, b]$ be given. If $u \in W^{1,1}_{\text{loc}}(0, T) \cap C([0, T])$ is a solution to

$$ u_t(t) + \mathcal{H}[\lambda, u](t) = f \quad \text{for a.e. } t \in (0, T), \quad u(0) = u^0, \quad (3.3) $$

then

$$ |u(t)| \leq \max \{ |A|, |B| \} \quad \forall t \in [0, T). \quad (3.4) $$

(ii) Let $v \in C(\mathbb{R}_{\geq 0}, \mathbb{R})$ with $\delta > 2|v(t)|$ for all $t > 0$ be given. If $a + \delta < b$ holds and $u \in W^{1,1}_{\text{loc}}(0, T) \cap C([0, T])$ is a solution to

$$ u_t(t) + \mathcal{H}[\lambda, u + v](t) = f \quad \text{for a.e. } t \in (0, T), \quad u(0) = u^0 \quad (3.5) $$

with $u^0 \in [a + \frac{1}{2}\delta, b - \frac{1}{2}\delta]$, then we have

$$ |u(t)| \leq \max \{ |A|, |B| \} + \frac{3}{2}\delta \quad \forall t \in [0, T). \quad (3.6) $$

Proof. (i) Using Lemma 3.4 (see below) with $v \equiv 0$ and every $\delta' \in (0, \delta]$, we deduce that (3.7) holds for all $\delta' \in (0, \delta]$. Hence, we see that (3.4) is proved.

(ii) Applying Lemma 3.4 (see below) with $\delta' := 2\delta$, $a' := a + \frac{1}{2}\delta$ and $b' = b - \frac{1}{2}\delta$, we see that (3.7) holds. The assumption on $v$ implies that (3.6) holds.

\[ \square \]

Remark 3.3. An evolution equation like (3.5) that involves an hysteresis operator with input perturbation arises, for example, if one is dealing with a model for visco-elasticity with hysteresis operators as in [10]. Using the Andrews trick (see [1]) to estimate the time and space dependent displacement, one introduces a function $p$, being the antiderivative with respect to space of the velocity, i.e., of the derivative of the displacement with respect to time. In the proof, one shows that this function is uniformly bounded. Subtracting this function from the derivative of the displacement with respect to space, one gets a function $q$. For each point $x$ in space one has to consider an ODE of the form (3.5) with $u(t) := q(x, t)$ and $v(t) := p(x, t)$. To derive a uniform bound for $q$ depending only on the bounds for $p$ and for the initial data but not on $T$, one can use estimates as those in Theorem 3.9.

Lemma 3.4. Let $\delta' > 0$, $h > 0$, and $A \leq a' < b' \leq B$ be given. Let $\mathcal{H} : D(\mathcal{H})(\subseteq C(\mathbb{R}_{\geq 0}, \mathbb{R})) \to \text{Map}(\mathbb{R}_{\geq 0}, \mathbb{R})$ be a causal operator that is pointing outwards with bound $h$ in the $\delta'$-neighborhood of $[A, B]$ for initial values in $[a', b']$. Let $v \in C(\mathbb{R}_{\geq 0}, \mathbb{R})$ with $\delta' > |v(t) - v(s)|$ for all $t > s \geq 0$, $f \in L^\infty(\mathbb{R}_{\geq 0})$ with $|f(s)| \leq h$ for a.e. $t > 0$, and $u^0 \in [a' - v(0), b' - v(0)]$ be given. Let $T > 0$ be given. If $u \in W^{1,1}_{\text{loc}}(0, T) \cap C([0, T))$ is a solution to (3.5) then

$$ u(t) + v(t) \in (A - \delta', B + \delta') \quad \forall t \in [0, T). \quad (3.7) $$

Proof. To prove (3.7) by contradiction, we assume that the inclusion therein does not hold for all $t \in [0, T)$. Using $u(0) + v(0) \in [a', b'] \subseteq [A, B]$, we see that

$$ t_1 := \inf \{ t \in [0, T) : u(t) + v(t) \in \{ A - \delta', B + \delta' \} \} \quad (3.8) $$

is a well defined number in $(0, T)$ and that (3.7) holds with $T$ replaced by $t_1$.

Since $\mathcal{H}$ is pointing outwards with bound $h$ in the $\delta'$-neighborhood of $[A, B]$ for initial values in $[a', b']$, we have

$$ (\mathcal{H}[u + v](t) - h)(u(t) + v(t) - B)^+ \geq 0, \quad (\mathcal{H}[u + v] + h)(u(t) + v(t) - A)^- \leq 0 \quad (3.9) $$
for every $t \in [0, t_1]$. We consider the case that $u(t_1) + v(t_1) = B + \delta'$. Hence, there is some $t_2 \in (0, t_1)$ such that $u(t_2) + v(t_2) = B$ and $u(t) + v(t) > B$ for all $t \in (t_2, t_1)$. Combining (3.5), (3.9), and the assumption on $f$, we see that $u(t) \leq 0$ holds for a.e. $t \in (t_2, t_1)$. Hence, we have $u(t_2) \geq u(t_1)$, and therefore

$$v(t_1) - v(t_2) \geq v(t_1) + u(t_1) - u(t_2) - v(t_2) > B + \delta - B = \delta.$$ 

But, this contradicts the assumption on $v$. The argument in the case $u(t_1) + v(t_1) = A - \delta$ is analogous. Therefore, we have proved (3.7).

**Corollary 3.5.** Let $\delta > 0$, $h \geq 0$, and $A \leq a \leq b \leq B$ be given. Let $\mathcal{H}$ be a locally Lipschitz causal operator that is pointing outwards with bound $h$ in the $\delta$–neighborhood of $[A, B]$ for initial values in $[a, b]$. Let $v \in C([R_2, R])$ with $\delta \geq |v(t) - v(s)|$ for all $t > s \geq 0$, $f \in L^\infty([R_2, R])$ with $|f(s)| \leq h$ for a.e. $t > 0$, and $u^0 \in [a - v(0), b - v(0)]$ be given. Then there exists a unique classical solution $u \in W^{1,1}_{loc}(0, \infty)$ to

$$u(t) + \mathcal{H}(\lambda, u + v)(t) = f \quad \text{for a.e. } t \in (0, \infty), \quad u(0) = u^0. \quad (3.10)$$

**Proof.** The local existence of a unique classical solution $u$ in an interval $[0, \tau]$ for $\tau > 0$ sufficiently small is obtained by the standard Banach contraction argument. We then extend $u$ into a maximal solution defined in an interval $[0, T]$. Thanks to the inclusion (3.7) proved in Lemma 3.4, we have an a priori bound for this solution, and deduce that the maximal solution is defined on $R_2$.

3.3. Weakly and strongly outward pointing operator and uniform estimates for ODEs involving this kind of operators

**Definition 3.6.** Consider a mapping $\mathcal{H} : D(\mathcal{H})(\subseteq C([R_2, R])) \to \text{Map}(R_2, R)$ and let $h \geq 0$ be given.

i) $\mathcal{H}$ is said to be **weakly pointing outwards with bound $h$**, if for all $R > 0$ one can find $A \leq -R$, $B \geq R$, and $\delta > 0$ such that $\mathcal{H}$ is pointing outwards with bound $h$ in the $\delta$–neighborhood of $[A, B]$ for initial values in $[-R, R]$.

ii) $\mathcal{H}$ is said to be **strongly pointing outwards with bound $h$**, if for all $R > 0$ and all $\delta > 0$ one can find $A \leq -R$ and $B \geq R$ such that $\mathcal{H}$ is pointing outwards with bound $h$ in the $\delta$–neighborhood of $[A, B]$ for initial values in $[-R, R]$.

**Remark 3.7.**

i) If a mapping is strongly pointing outwards with bound $h$, the mapping is also weakly pointing outwards with bound $h$.

ii) While the weak pointing outward property is sufficient to deal with a normal evolution equation involving an hysteresis operator (see Theorem 3.8), one needs in general the strong outward pointing property if the evolution equation involves also an input perturbation (see Theorem 3.9).

iii) Conditions to check the outward pointing properties for the Stop operator, the Play operator, the Prandtl-Ishlinskii operator, and generalized Prandtl-Ishlinskii operators can be found in [10, 11]. In [2, 3] such conditions have been formulated for Preisach operators and for the inverse of Preisach operators one can find corresponding conditions in [7, 8].

Combining Lemma 3.4, Corollary 3.5, and the definition of the weak and the strong outward pointing property, we get the following results:
Theorem 3.8. Let $\mathcal{H} : C(\mathbb{R}_0^+, \mathbb{R}) \to C(\mathbb{R}_0^+, \mathbb{R})$ be a locally Lipschitz causal operator that is weakly pointing outwards with bound $h$ for some given $h \geq 0$.

Then there exists for all $R > 0$ some constant $C_R > 0$ such that for all $u^0 \in [-R, R]$ and all $f \in L^\infty(\mathbb{R}_0^+)$ with $|f(s)| \leq h$ for a.e. $t > 0$, the unique classical solution $u : \mathbb{R}_0^+ \to \mathbb{R}$ to

$$u_t(t) + \mathcal{H}[u](t) = f \quad \text{for a.e. } t \geq 0, \quad u(0) = u^0,$$

satisfies

$$|u(t)| \leq C_R \quad \forall t \geq 0. \quad (3.12)$$

Theorem 3.9. Let $\mathcal{H} : C(\mathbb{R}_0^+, \mathbb{R}) \to C(\mathbb{R}_0^+, \mathbb{R})$ be a locally Lipschitz causal operator that is strongly pointing outwards with bound $h$ for some $h \geq 0$.

Then there exists for all $R > 0$ some constant $C_R > 0$ such that for all $u^0 \in [-R, R]$, all $f \in L^\infty(\mathbb{R}_0^+)$ with $|f(s)| \leq h$ for a.e. $t > 0$, and all $v \in C(\mathbb{R}_0^+, \mathbb{R})$ with $|v(t)| \leq R$ for a.e. $t > 0$, the unique classical solution $u : \mathbb{R}_0^+ \to \mathbb{R}$ to

$$u_t(t) + \mathcal{H}[\lambda, u + v](t) = f \quad \text{for a.e. } t \geq 0, \quad u(0) = u^0,$$

satisfies the estimate (3.12).

Remark 3.10. If the operator $\mathcal{H}$ in Theorem 3.8 or in Theorem 3.9 is not locally Lipschitz, then one is not able to ensure the existence or the uniqueness of the solution to the ODE, but one can still derive the uniform estimates formulated in the theorems. The same holds also, if one is not considering an operator of the form $\mathcal{H} : C(\mathbb{R}_0^+, \mathbb{R}) \to C(\mathbb{R}_0^+, \mathbb{R})$ but of the more general form $\mathcal{H} : D(\mathcal{H}) \subset C(\mathbb{R}_0^+, \mathbb{R})) \to \text{Map}(\mathbb{R}_0^+, \mathbb{R})$, as long as the solution considered belongs to $W_{\text{loc}}^{1,1}(0, \infty)$.

Corresponding estimates have been used to derive uniform estimates for ODE with hysteresis (see [3, 10]), for a model for visco-elasto-plasticity (see [11]), for a model for thermo-visco-elasto-plasticity (see [9, 10]), and for a model for electro-magnetic processes (see [7, 8]).

4. Outward pointing vectorial hysteresis operators and uniform estimates

4.1. Definition of outward pointing vectorial hysteresis operators

In this section, a first extension of the outward pointing property to vectorial operators is presented. Let $n$ be a natural number, let $\|\cdot\|$ be the Euclidian norm on $\mathbb{R}^n$, and let $(\cdot, \cdot)$ be the corresponding inner product.

We recall the usual notation for an open ball around 0 in $\mathbb{R}^n$ and for the sum of subsets of $\mathbb{R}^n$:

$$\mathcal{B}_r(0) := \{x \in \mathbb{R}^n : \|x\| < r\} \quad \forall r > 0, \quad (4.1)$$

$$V + W := \{v + w : v \in V, w \in W\} \quad \forall V, W \subset \mathbb{R}^n. \quad (4.2)$$

The following notation will be used, (c.f., [13, Ch. I.2]):

**Definition 4.1.** Let $\emptyset \neq Z \subset \mathbb{R}^n$ be given. Then let $\text{dist}_Z : \mathbb{R}^n \to \mathbb{R}_0^+$ and $\text{Proj}_Z : \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$\text{dist}_Z(x) := \inf \{\|x - z\| : z \in Z\} \quad \forall x \in \mathbb{R}^n, \quad (4.3)$$

$$\text{Proj}_Z(x) := \{z \in Z : \|x - z\| = \text{dist}_Z(x)\}$$

$$= \{z \in Z : \|x - z\| \leq \|x - z'\| \quad \forall z' \in Z\} \quad \forall x \in \mathbb{R}^n. \quad (4.4)$$

**Remark 4.2.** We see that $\text{dist}_Z$ is a Lipschitz continuous function on $\mathbb{R}^n$, and that $\text{Proj}_Z(z) = \{z\}$ for all $z \in Z$. If $Z$ is closed, we have $\text{Proj}_Z(x) \neq \emptyset$ for all $x \in \mathbb{R}^n$. 


**Definition 4.3.** Consider a mapping $\mathcal{H} : D(\mathcal{H})(\subset C(\mathbb{R}_{\geq 0}, \mathbb{R}^n)) \to \text{Map}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$. Let nonempty sets $W, Z_0, Z \subset \mathbb{R}^n$ with $Z_0 \subset Z$ and $Z$ closed be given. Let $\delta > 0$ be given.

The mapping $\mathcal{H}$ is said to be **pointing outwards in the $\delta$–neighborhood of $Z$ for initial values in $Z_0$ and output modifications in $W$** if and only if for all $u \in D(\mathcal{H})$ and $t \geq 0$ the following holds: If

$$u(0) \in Z_0, \quad u(s) \in Z + B_{\delta}(0) \quad \forall s \in [0, t],$$

then

$$(\mathcal{H}[u](t) + w, z - u(t)) \leq 0 \quad \forall z \in \text{Proj}_Z(u(t)), \ w \in W.$$  \hspace{1cm} (4.6)

The following remark shows connections between the Definition 3.1 of outward pointing for scalar operators and the Definition 4.3.

**Remark 4.4.** Consider a mapping $\mathcal{H} : D(\mathcal{H})(\subset C(\mathbb{R}_{\geq 0}, \mathbb{R})) \to \text{Map}(\mathbb{R}_{\geq 0}, \mathbb{R})$, and let $\delta > 0$, $h \geq 0$, and $A \leq a \leq b \leq B$ be given. Let $Z := [A, B]$, $Z_0 := [a, b]$, and $W := [-h, h]$.

The condition (3.1) in Definition 3.1 is equivalent to

$$u(0) \in Z_0, \quad u(s) \in Z + B_{\delta}(0) \quad \forall s \in [0, t],$$

and the condition (3.2) is equivalent to

$$\begin{cases} \mathcal{H}[u](t) + w \geq 0 & \forall w \in W \text{ if } u(t) > B, \\
\mathcal{H}[u](t) + w \leq 0 & \forall w \in W \text{ if } u(t) < A. \end{cases}$$

Since we have

$$\begin{cases} z - u(t) < 0 & \forall z \in \text{Proj}_Z(u(t)) \text{ if } u(t) > B, \\
 z - u(t) = 0 & \forall z \in \text{Proj}_Z(u(t)) \text{ if } A \leq u(t) \leq B, \\
 z - u(t) > 0 & \forall z \in \text{Proj}_Z(u(t)) \text{ if } u(t) < A, \end{cases}$$

we conclude that (3.2) is equivalent to

$$(\mathcal{H}[u](t) + w) (z - u(t)) \leq 0 \quad \forall z \in \text{Proj}_Z(u(t)), \ w \in W.$$  \hspace{1cm} (4.6)

Hence, we see that $\mathcal{H}$ is pointing outwards with bound $h$ in the $\delta$–neighborhood of $[A, B]$ for initial values in $[a, b]$ in the sense of Definition 3.1 if and only if $\mathcal{H}$ is pointing outwards in the $\delta$–neighborhood of $Z$ and output modifications in $Z_0$ and output modifications in $W$ in the sense of Definition 4.3 considered for $n = 1$.

**Remark 4.5.** The notation “pointing outwards” can be motivated by considering a closed, convex set $Z \subset \mathbb{R}^N$. Assume that (4.6) holds. Considering $\mathcal{H}[u](t) + w$ as a vector fixed at $u(t)$, we see that this vector is not pointing in the direction to the unique element of $\text{Proj}_Z$, see Fig. 1. Hence, $\mathcal{H}[u](t)$ is pointing from $u(t)$ towards the outside of $Z$, in other words, with respect to $Z$ the vector $\mathcal{H}[u](t)$ points outwards.

**Remark 4.6.** Let an open set $B \subset \mathbb{R}^n$ with $0 \in B$ be given. Replacing “$B_{\delta}(0)$” by “$B$” in Definition 4.3, one could introduce the notation of $\mathcal{H}$ to point outwards in the $B$–neighborhood of $Z$ for initial values in $Z_0$ and output modifications in $W$.

**Definition 4.7.** Consider a mapping $\mathcal{H} : D(\mathcal{H})(\subset C(\mathbb{R}_{\geq 0}, \mathbb{R}^n)) \to \text{Map}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$. Let a nonempty set $W \subset \mathbb{R}^n$ be given.

i) $\mathcal{H}$ is said to be **weakly pointing outwards for output modifications in $W$** if for all $R > 0$ one can find some compact set $Z \supset B_R(0)$ and some $\delta > 0$ such that $\mathcal{H}$ is pointing outwards with output modifications in $W$ in the $\delta$–neighborhood of $Z$ for initial values in $B_R(0)$.

ii) $\mathcal{H}$ is said to be **strongly pointing outwards for output modifications in $W$** if for all $R > 0$ and all $\delta > 0$ one can find some compact set $Z \supset B_R(0)$ such that $\mathcal{H}$ is pointing outwards with output modifications in $W$ in the $\delta$–neighborhood of $Z$ for initial values in $B_R(0)$. 
Figure 1. An example for an input \(u(t)\), the corresponding output \(\mathcal{H}[u](t)\) and its modification \(\mathcal{H}[u](t) + w\).

4.2. Examples for outward pointing operators: Vectorial play, relay, and Preisach operator

4.2.1. Vectorial play operator

Lemma 4.8. For \(\emptyset \neq V \subset \mathbb{R}^n\) with \(V\) being closed, convex, and bounded and \(W \subset \mathbb{R}^n\) bounded, the vectorial play \(p_V\) (see, e.g. [13, Theorem 1.9]) is strongly pointing outwards with output modifications in \(W\).

Proof. Let \(R > 0\) and \(\delta > 0\) be given. Since \(V\) and \(W\) are bounded, there is some \(r > 0\) such that for all \(u \in C(\mathbb{R} \geq 0, \mathbb{R}^n)\) and all \(w \in W \ p_V(u(t)) + w \in B_r(u(t))\) holds. Hence, we see that for \(S > R\) sufficiently large the condition (4.6) with \(H := p_V\) and \(Z := B_S(0)\) is satisfied for all \(u \in C(\mathbb{R} \geq 0, \mathbb{R}^n)\). This yields that \(p_V\) is pointing outwards with output modifications in \(W\) in the \(\delta\)-neighborhood of \(B_S(0)\) for initial values in \(B_R(0)\).

Therefore, we have shown that \(p_V\) is strongly pointing outwards with output modifications in \(W\). \(\square\)

4.2.2. Vectorial Preisach operator

Following [6, 15], we define the vectorial relay and the vectorial Preisach operator:

Definition 4.9. For \(x \in \mathbb{R}^n, r > 0,\) and \(\xi \in \mathbb{R}^n\) with \(\|\xi\| = 1\), the vectorial relay \(h_{(x,r,\xi)} : C(\mathbb{R} \geq 0, \mathbb{R}^n) \to \text{Map}(\mathbb{R} \geq 0, \mathbb{R}^n)\) is defined by:

For \(u \in C(\mathbb{R} \geq 0, \mathbb{R}^n)\) and \(t \geq 0,\)

\[
h_{(x,r,\xi)}[u](t) := \begin{cases} 
\frac{u(t) - x}{\|u(t) - x\|} & \text{if } \|u(t) - x\| \geq r, \\
\xi & \text{if } X_{(t,x,r)} = \emptyset, \\
h_{(x,r,\xi)}[u]\left(\max X_{(t,x,r)}\right) & \text{otherwise,}
\end{cases}
\]

(4.7)

with \(X_{(t,x,r)}[u] := \{\tau \in [0, t] : \|u(\tau) - x\| \geq r\}\). (4.8)
Definition 4.10. Let $\omega : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, $\xi : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \partial B_1(0)$ be given. The vectorial Preisach operator $\mathcal{P} : C(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \to \text{Map}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ is defined by

$$\mathcal{P}[u](t) = \int_0^\infty \int_{\mathbb{R}^n} \omega(x,r)h_{(x,r,\xi(x,r))}[u](t) \, dx \, dr$$

for all $u \in C(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$, $t \geq 0$.

Let $\Psi : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$\Psi(y) := \int_0^\infty \int_{\mathbb{R}^n \setminus \{y\}} \omega(x,r) \frac{y - x}{\|y - x\|} \, dx \, dr.$$  

(4.9)

(4.10)

Theorem 4.11. Let $x \in \mathbb{R}^n$, $0 < r \leq s$, and $\xi \in \mathbb{R}^n$ with $\|\xi\| = 1$ be given.

(i) For all $u \in C(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ and all $t \geq 0$, (4.6) holds with $H := h_{(x,r,\xi)}$, $W := B_1(0)$, $Z := B_s(x)$, and $Z_0 := B_{\delta}(x)$.

(ii) For all $\delta > 0$ holds: The vectorial relay $h_{(x,r,\xi)}$ is pointing outwards with output modifications in $B_1(0)$ in the $\delta$–neighborhood of $B_s(x)$ in initial values in $B_s(x)$.

(iii) The vectorial relay $h_{(x,r,\xi)}$ is strongly pointing outwards with output modifications in $B_1(0)$.

Proof. (i) If $u(t) \in Z$ then $\text{Proj}_Z(u(t)) = \{u(t)\}$ and (4.6) is satisfied.

If $u(t) \notin Z = B_s(0)$ then $\|u(t) - x\| > s \geq \delta$. Hence, we have $\text{Proj}_Z(u(t)) = \{z\}$ with $z := x + s\frac{u(t) - x}{\|u(t) - x\|}$ and $H[u](t) = h_{(x,r,\xi)} = \frac{\|u(t) - x\|}{\|u(t) - x\|}$. This yields

$$(H[u](t), z - u(t)) = s - \|u(t) - x\|,$$  

$$\|z - u(t)\| = \|u(t) - x\| - s. \quad (4.11) \quad (4.12)$$

Using this, we obtain for all $w \in B_1(0)$:

$$(H[u](t) + w, u(t) - z) \leq s - \|u(t) - x\| + \|w\| (\|u(t) - x\| - s) < 0.$$

Hence, (4.6) holds for the sets considered.

(ii) Follows directly from assertion (i).

(iii) Follows directly from assertion (ii).

\[\Box\]

Theorem 4.12. Let $\omega : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, $\xi : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \partial B_1(0)$, and $R > 0$ be given such that the support of $\omega$ belongs to $B_R(0) \times [0, R]$. Let $\mathcal{P}$ be the corresponding Preisach operator. For $S > 2R$, $\delta > 0$, and $W \subset \mathbb{R}^n$ holds: $\mathcal{P}$ is pointing outwards with output modifications in $W$ in the $\delta$–neighborhood of $\overline{B_S(0)}$ for initial values in $\overline{B_S(0)}$ if and only if

$$\forall x \in B_{S+\delta}(0) \setminus \overline{B_S(0)}, w \in W : (\Psi_P(x) + w, x) \geq 0. \quad (4.13)$$

Proof. In the light of Lemma 4.13 below and of Definition 4.3, we see that $\mathcal{P}$ is pointing outwards with output modifications in $W$ in the $\delta$–neighborhood of $\overline{B_S(0)}$ for initial values in $\overline{B_S(0)}$ if and only if

$$(\Psi_P(x) + w, z - x) \leq 0 \ \forall z \in \text{Proj}_{\overline{B_S(0)}}(x), w \in W, x \in B_{S+\delta}(0) \setminus \overline{B_S(0)}. \quad (4.14)$$

Since for all pairs of $x$ and $z$ considered in this condition $z - x = \left(\frac{s}{\|x\|} - 1\right)x$ holds, and we have $s < \|x\|$, we deduce that (4.14) and (4.13) are equivalent.

\[\Box\]
Lemma 4.13. Let \( \omega : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R} \), \( \xi : \mathbb{R}^n \times \mathbb{R}_+ \to \partial B_1(0) \), and \( R > 0 \) be given such that the support of \( \omega \) belongs to \( B_R(0) \times [0, R) \). Let \( \mathcal{P} \) be the corresponding Preisach operator. Let \( u \in C(\mathbb{R}_+, \mathbb{R}^n) \) and \( t \geq 0 \) be given. If \( \|u(t)\| \geq 2R \) then \( \mathcal{P}[u](t) = \Psi_P(u(t)) \).

Proof. For \( x \in \mathbb{R}^n \) and \( r > 0 \) such that \( \omega(x, r) \neq 0 \), we have \( \|u(t) - x\| \geq R \geq r \) and therefore \( h_{(x, r, \xi)}[u](t) = \frac{u(t) - x}{\|u(t) - x\|} \). Recalling the definitions of \( \mathcal{P} \) and \( \Psi_P \) in Definition 4.9, we conclude that \( \mathcal{P}[u](t) = \Psi_P(u(t)) \).

Combining Theorem 4.12 and Definition 4.7, we deduce:

Corollary 4.14. Let \( \emptyset \neq W \subset \mathbb{R}^n \) be given. Let \( \omega : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R} \), \( \xi : \mathbb{R}^n \times \mathbb{R}_+ \to \partial B_1(0) \), and \( R > 0 \) be given such that the support of \( \omega \) belongs to \( B_R(0) \times [0, R) \). Let \( \mathcal{P} \) be the corresponding Preisach operator. If for every \( R' > 0 \) one can find some \( S > R' \) such that for all \( x \in B(2S) \setminus \overline{B}(S) \) and all \( w \in W \) \( (\Psi_P(x) + w, x) \geq 0 \) holds, then \( \mathcal{P} \) is strongly pointing outsides with output modifications in \( W \).

4.3. Uniform estimates for differential equations involving outsides pointing mappings
Considering the following initial value problem for a vectorial ordinary differential equation
\[
  u_t = -\mathcal{H}[u](t) - G(t), \quad \text{for a.e. } t > 0, \quad u(0) = u_0,
\]
we can get uniform bounds, if \( \mathcal{H} \) is an outsides pointing mapping. Of course, for proving the existence and regularity of solutions one has to use additional assumptions for \( \mathcal{H} \).

Theorem 4.15. Let \( \mathcal{H} : D(\mathcal{H}) \subset C(\mathbb{R}_+ \times \mathbb{R}^n) \to \text{Map}(\mathbb{R}_+ \times \mathbb{R}^n) \) be given. Consider \( \delta > 0 \), \( \emptyset \neq Z_0 \subset Z \subset \mathbb{R}^n \emptyset \neq W \subset \mathbb{R}^n \), \( \emptyset \neq W \subset \mathbb{R}^n \) and \( \Omega \subset C(\mathbb{R}_+, \mathbb{R}^n) \), such that \( Z \) is closed, \( u_0 \in Z_0 \), and \( \mathcal{H} \) is pointing outsides in the \( \delta \)-neighborhood of \( Z \) for initial values in \( Z_0 \) and with output modifications in \( W \). Let \( T \geq 0 \) be given. If \( u \in W^{1,1}_{\text{loc}}((0,T),\mathbb{R}^n) \cap C([0,T),\mathbb{R}^n) \) satisfies the equation in (4.15) for a.e. \( t \in (0,T) \), then \( u(t) \in Z \) holds for all \( t \in [0,T] \).

Proof. That \( u(t) \in Z \) holds on \([0, \infty) \) is proved by contradiction. Assume that we have \( t' \in [0, T] \) with \( u(t') \notin Z \). We can now consider \( g \in W^{1,1}_{\text{loc}}(\emptyset, \mathbb{R}^n) \subset C([0,T),\mathbb{R}^n) \) defined as in (4.19) with \( w = u \). Recalling now that \( u(0) \in Z \), that \( g(0) = 0 \), that \( u \) and \( g \) are continuous, and that \( Z \) is a closed set, yield the existence of \( t_0, t_1 \in (0,T) \) such that \( t_0 < t_1 \) and
\[
  u(t_1) \notin Z, \quad u(t) \in Z + B_{\delta}(0) \quad \forall t \in [0, t_1],
\]
\[
  0 < g(t_0) < g(t_1), \quad u(t) \notin Z \quad \forall t \in [t_0, t_1].
\]
Hence, (4.5) holds for \( B = B_{\delta}(0) \) and \( t \in [t_0, t_1] \). Using that \( \mathcal{H} \) points outsides in the \( \delta \)-neighborhood of \( Z \) for initial values in \( Z_0 \) and output modifications in \( W \), we conclude that therefore (4.6) holds for all \( t \in [t_0, t_1] \). Applying (4.15) and the definition of \( G \), we see that
\[
  0 \geq (u(t), u(t) - z)_X \quad \forall z \in \text{Proj}_Z(u(t)), \quad \text{a.e. on } (t_0, t_1).
\]
Now, we can use (4.20) (proved in the following Lemma 4.16) with \( w = u \) to show that \( g(t) \leq 0 \) a.e. on \((t_0, t_1) \). Hence, we have \( g(t_0) \geq g(t_1) \), which is a contradiction to the assumptions on \( t_0 \) and \( t_1 \). This yields that \( u(t) \in Z \) holds for all \( t \in [0, T] \).

Lemma 4.16. Let \( T > 0 \), \( \emptyset \neq Z \subset \mathbb{R}^n \), such that \( Z \) closed and \( w \in W^{1,1}_{\text{loc}}((0,T),\mathbb{R}^n) \) be given. For \( g \) defined by
\[
  g(t) := (\text{dist}_Z(w(t)))^2 \quad \forall t \in [0, T],
\]
we have \( g \in W^{1,1}_{\text{loc}}((0,T),\mathbb{R}^n) \) and
\[
  g(t) \leq 2(w(t), w(t) - z) \quad \forall z \in \text{Proj}_Z(w(t)), \quad \text{a.e. on } (0, T).
\]
Proof. Since $\text{dist}_{\mathcal{Z}}$ is a Lipschitz continuous function, we have $g \in W^{1,1}_{\text{loc}}((0,T),\mathbb{R}^n)$. Using the regularities of $g$ and $w$, [13, Prop. 1.22] yields that there is a set $M \subset (0,T)$ of measure zero such that

$$w(t) = \lim_{h \to 0} \frac{1}{h}(w(t+h) - w(t)), \quad g_t(t) = \lim_{h \to 0} \frac{1}{h}(g(t+h) - g(t)) \quad \forall t \in (0,T) \setminus M. \quad (4.21)$$

Now, let $t \in (0,T) \setminus M$ and $\varepsilon > 0$ be arbitrary. Since (4.21) holds, there exists some $h_0 > 0$ such that

$$g_t(t) - \varepsilon < \frac{1}{h}(g(t+h) - g(t)) \quad \forall h \in (0,h_0). \quad (4.22)$$

Since $\mathcal{Z}$ is locally compact, we have some $z \in \text{Proj}_{\mathcal{Z}}(u(t))$. This yields for all $h \in (0,h_0)$:

$$g_t(t)h - \varepsilon h < (\text{dist}_{\mathcal{Z}}(w(s)))^2 - (\text{dist}_{\mathcal{Z}}(w(t)))^2 \leq \|w(s) - z\|^2 - \|w(t) - z\|^2$$

$$= \|w(s) - w(t)\|^2 + 2(w(s) - w(t), u(t) - z).$$

For $s \setminus t$ this yields by (4.21)

$$g_t(t) - \varepsilon \leq 2(u_t(t), u(t) - z).$$

Since this estimates is proved for all $\varepsilon > 0$ and all $t \in (0,T) \setminus M$, we see that (4.20) is proved. \qed

**Corollary 4.17.** Let $\mathcal{H} : C(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \to C(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ be given. Assume that there exists $\emptyset \neq W \subset \mathbb{R}^n$ such that $\mathcal{H}$ is weakly pointing outwards for output modifications in $W$. There there is for every $R > 0$ some $C_R > 0$ such that for all $u_0 \in B_R(0)$ and all $G \in C(\mathbb{R}_{\geq 0}, W)$, holds:

The ordinary differential equation (4.15) has a unique classical solution $u \in W^{1,1}_{\text{loc}}((0,T),\mathbb{R}^n)$ and this solution satisfies

$$\|u(t)\| \leq C_R, \quad \forall t \geq 0. \quad (4.23)$$

**Proof.** Let $R > 0$ be given. Since $\mathcal{H}$ is weakly pointing outwards with output modifications in $W$, we can find some compact set $Z \supset B_R(0)$ and some $\delta > 0$ such that $\mathcal{H}$ is pointing outwards in the $\delta$–neighborhood of $Z$ for initial values in $B_R(0)$ and with output modifications in $W$.

The local existence of a unique classical solution $u$ in an interval $[0, \tau]$ for $\tau > 0$ sufficiently small is obtained by the standard Banach contraction argument. We then extend $u$ into a maximal solution defined in an interval $[0,T)$. Using Theorem 4.15, we get $u(t) \in Z$ for all $t \in [0,T)$. Since $Z$ is compact, we deduce that the maximal solution is defined on $[0,\infty)$ and that (4.23) is satisfied. \qed

**4.4. Considerations for uniform estimates for ODEs with input perturbations**

Considering the following initial value problem for a vectorial ordinary differential equation

$$u_t = -\mathcal{H}[u + v](t) - G(t), \quad \text{for a.e. } t > 0, \quad u(0) = u_0, \quad (4.24)$$

which is a vectorial version of (3.5), one would expect that one gets uniform bounds for the solution, if $\mathcal{H}$ is a strongly outwards pointing mapping and $v$ is bounded, as in the scalar case, see Theorem 3.9.

But this does not work. Consider the following example:

**Remark 4.18.** Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$h((x, y)) = (y, -x) \quad (4.25)$$

and let $\mathcal{H} : C(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \to C(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ defined by $\mathcal{H}[u](t) = h(u(t))$ for all $u \in \mathbb{R}^n$. 

This operator is pointing outwards with output modifications in \( \{0\} \) in the \( \delta \)-neighborhood of \( \overline{B_1(0)} \) for initial values in \( \overline{B_1(0)} \) for all \( \delta > 0 \).

Let \( f \equiv 0 \) and \( u_0 = (-1,0) \).

For \( v : C \left( \mathbb{R}_{\geq 0}, \overline{B_1(0)} \right) \) holds: for all \( t_1 < t_2 \) with \( v \) constant on \((t_1, t_2)\) it holds for all solutions \( u \) to (4.24) that \( v(t) + u(t) \) form a circular arc with constant radius and clockwise orientation for \( t \in (t_1, t_2) \).

Now, for \( \varepsilon > 0 \) arbitrary, we can construct \( v : C \left( \mathbb{R}_{\geq 0}, \overline{B_2(0)} \right) \) such that

- it changes its value from \((0, -\varepsilon)\) to \((0, \varepsilon)\), when \( u + v \) is near to the positive part of the \( y \) axis, and
- changes its values vice-versa, when \( u \) is near to the negative part of the \( y \) axis,
- and is constant for other times,
- such that during each of these changes \( \|u(t) + v(t)\| \) is increased by at least \( \varepsilon \).

Hence, we see that \( v + u \) is unbounded and therefore also \( u \).

Therefore, it seems that the outward property introduced in this section is not appropriate to deal with ODE involving perturbations of the input of the hysteresis operators.

5. Componentwise outward pointing condition for vectorial operators and uniform estimates

5.1. Componentwise outward pointing condition

To overcome the problem arising when trying to derive uniform bounds for ODEs with input perturbations, the following more restrictive outward pointing condition is presented here:

**Definition 5.1.** Let \( \delta = (\delta_1, \delta_2, \ldots, \delta_n), h = (h_1, h_2, \ldots, h_n), A = (A_1, A_2, \ldots, A_n), a = (a_1, a_2, \ldots, a_n), b = (b_1, b_2, \ldots, b_n), B = (B_1, B_2, \ldots, B_n) \in \mathbb{R}^n \) with \( \delta_i > 0, h_i \geq 0 \), and \( A_i \leq a_i \leq b_i \leq B_i \) for all \( 1 \leq i \leq n \) be given.

Let \( \mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n) : D(\mathcal{H})(\subset C(\mathbb{R}_{\geq 0}, \mathbb{R}^n)) \to \text{Map}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \) be given. \( \mathcal{H} \) is said to be componentwise pointing outsides with bound \( h \) in the \( \delta \)-neighborhood of \( \prod_{i=1}^n [A_i, B_i] \) for initial values in \( \prod_{i=1}^n [a_i, b_i] \), if for every \( t \geq 0 \) and every \( u = (u_1, u_2, \ldots, u_n) \in D(\mathcal{H}) \) such that

\[
\forall i \in \{1, \ldots, n\}, u_i(0) \in \prod_{i=1}^n [a_i, b_i], \quad u_i(s) \in \prod_{i=1}^n (A_i - \delta_i, B_i + \delta_i) \quad \forall s \in [0, t],
\]

we have

\[
\begin{cases}
(\mathcal{H}_i[u](t) - h_i)(u_i(t) - B_i)^+ \geq 0 & \forall 1 \leq i \leq n, \\
(\mathcal{H}_i[u](t) + h_i)(u_i(t) - A_i)^- \leq 0 & \forall 1 \leq i \leq n.
\end{cases}
\]  

**Remark 5.2.** Let \( \mathcal{H} : D(\mathcal{H})(\subset C(\mathbb{R}_{\geq 0}, \mathbb{R}^n)) \to C(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \) be given.

Let \( \delta = (\delta_1, \delta_2, \ldots, \delta_n), h = (h_1, h_2, \ldots, h_n) \in \mathbb{R}^n, A = (A_1, A_2, \ldots, A_n), a = (a_1, a_2, \ldots, a_n), b = (b_1, b_2, \ldots, b_n), B = (B_1, B_2, \ldots, B_n) \in \mathbb{R}^n \) with \( \delta_i > 0, h_i \geq 0 \), and \( A_i \leq a_i \leq b_i \leq B_i \) for all \( 1 \leq i \leq n \) be given. Let \( \delta_0 := \min_{1 \leq i \leq n} \delta_i \).

If \( \mathcal{H} \) is componentwise pointing outsides with bound \( h \) in the \( \delta \)-neighborhood of \( \prod_{i=1}^n [A_i, B_i] \) for initial values in \( \prod_{i=1}^n [a_i, b_i] \) (see Definition 5.1) then \( \mathcal{H} \) is pointing outsides with output modifications in \( \prod_{i=1}^n [-h_i, h_i] \) in the \( \delta \)-neighborhood of \( \prod_{i=1}^n [A_i, B_i] \) for initial values in \( \prod_{i=1}^n [a_i, b_i] \) (see Definition 4.3).

On the other hand, assume that for some \( \delta > 0 \), \( \mathcal{H} \) is pointing outsides with output modifications in \( \prod_{i=1}^n [-h_i, h_i] \) in the \( \delta \)-neighborhood of \( \prod_{i=1}^n [A_i, B_i] \) for initial values in \( \prod_{i=1}^n [a_i, b_i] \) (see Definition 4.3). Then for \( \delta^0 := \frac{1}{\sqrt{n}}(\delta_0, \ldots, \delta_n) \in \mathbb{R}^n \), \( \mathcal{H} \) is componentwise pointing outsides with bound \( h \) in the \( \delta^0 \)-neighborhood of \( \prod_{i=1}^n [A_i, B_i] \) for initial values in \( \prod_{i=1}^n [a_i, b_i] \) (see Definition 5.1).
Theorem 5.3. Let \( \delta = (\delta_1, \delta_2, \ldots, \delta_n) \), \( h = (h_1, h_2, \ldots, h_n) \), \( A = (A_1, A_2, \ldots, A_n) \), \( a = (a_1, a_2, \ldots, a_n) \), \( b = (b_1, b_2, \ldots, b_n) \), \( B = (B_1, B_2, \ldots, B_n) \in \mathbb{R}^n \) with \( h_i \geq 0 \), and \( A_i \leq a_i \leq b_i \leq B_i \) for all \( 1 \leq i \leq n \) be given. Let \( \mathcal{H} : D(\mathcal{H}) \subseteq C([\mathbb{R}_0^+, \mathbb{R}^n]) \rightarrow C([\mathbb{R}_0^+, \mathbb{R}^n]) \) be a causal operator that is componentwise pointing outwards with bound \( h \) in the \( \delta \)-neighborhood of \( \prod_{i=1}^n [a_i, b_i] \) for initial values in \( \prod_{i=1}^n [a_i, b_i] \). Let \( v = (v_1, v_2, \ldots, v_n) \in C([\mathbb{R}_0^+, \mathbb{R}^n]) \) with \( \|v_i(t) - v_i(s)\| < \delta_i \) for all \( t > s \geq 0 \) and all \( 1 \leq i \leq n \) be given. Let \( u^0 \in \prod_{i=1}^n [a_i - v_i(0), b_i - v_i(0)] \), and \( f = (f_1, f_2, \ldots, f_n) \in L^\infty([\mathbb{R}_0^+; \mathbb{R}^n]) \) with \( |f_i| \leq h_i \) for a.e. \( t > 0 \) and all \( 1 \leq i \leq n \) be given.

If \( u = (u_1, u_2, \ldots, u_n) \in W^{1,1}_{\text{loc}}((0, T), \mathbb{R}^n) \cap C((0, T), \mathbb{R}^n) \) is a solution to

\[
u(t) + \mathcal{H}[\lambda, u + v](t) = f \quad \text{for a.e. } t \in (0, T), \quad u(0) = u^0, \tag{5.3}\]

then

\[
u(t) + v(t) \in \prod_{i=1}^n (A_i - \delta_i, B_i + \delta_i) \quad \forall t \in (0, T). \tag{5.4}\]

Proof. Using the same argumentation as in Lemma 3.4 for each component, it is shown that (5.4) is satisfied. \( \square \)

Remark 5.4. Generalizing models considered in [5, 16, 17], in [14] a system of equations has been developed to model one-dimensional thermo-visco-plastic developments connected with solid-solid phase transitions taking also into account the hysteresis effects appearing on the macroscopic scale as a consequence of effects on the micro- and/or mesoscale. The system derived involved hysteresis operators \( \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4 \).

In [9], asymptotic results for this system have been derived, and an important step has been the derivation of estimates that are uniform with respect to time by generalizing the method of invariant regions. To be able to do this, in [9, (H8)] appropriate assumptions for \( \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4 \) have been formulated, which will now be reformulated.

Letting \( \mathcal{G}^1 := (\mathcal{H}_1, \mathcal{H}_3), \mathcal{G}^2 := (\mathcal{H}_2, \mathcal{H}_4) \). Let \( h_1 \in \mathbb{R}_0^+ \) be the number on the right-hand side of the first inequality in [9, (2.5)], and let, as in [9, (2.3), (2.4)], the numbers \( \varepsilon_{0, \text{min}}, \varepsilon_{0, \text{max}}, w_{0, \text{min}}, w_{0, \text{max}} \in \mathbb{R} \) be defined as lower and upper bounds for the initial values. Hence, we have \( \varepsilon_{0, \text{min}} \leq \varepsilon_{0, \text{max}} \) and \( w_{0, \text{min}} \leq w_{0, \text{max}} \).

Investigating the assumption [9, (H8)] for \( \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4 \), one observes that this assumption is equivalent to the requirement that for every \( \delta_1 > 0 \) there must exist \( \varepsilon_- \leq \varepsilon_{0, \text{min}}, \varepsilon_+ \geq \varepsilon_{0, \text{max}}, w_- \leq w_{0, \text{min}}, w_+ \geq w_{0, \text{max}} \), and \( \delta_2 > 0 \) such that \( \mathcal{G}^1 \) and \( \mathcal{G}^2 \) are componentwise pointing outwards with bounds \( (h_1, 0) \) and \((0, 0)\), respectively, in the \( (\delta_1, \delta_2) \)-neighborhood of \( [\varepsilon_-, \varepsilon_+] \times [w_-, w_+] \) for initial values in \( [\varepsilon_{0, \text{min}}, \varepsilon_{0, \text{max}}] \times [w_{0, \text{min}}, w_{0, \text{max}}] \).

Having in mind situations as in the above remark, where one needs different conditions for the different unknowns, the following formulation for a strong outward pointing property has been derived.

Definition 5.5. Let \( \mathcal{H} =: D(\mathcal{H}) \subseteq C([\mathbb{R}_0^+, \mathbb{R}^n]) \rightarrow \text{Map}(\mathbb{R}_0^+, \mathbb{R}^n) \) be given.

Let \( h = (h_1, h_2, \ldots, h_n) \in \mathbb{R}^n \), with \( h_i \geq 0 \) for all \( 1 \leq i \leq n \) be given. Let \( k_1, k_2, \ldots, k_m \in \{1, 2, \ldots, n\} \) with \( k_1 < k_2 < \cdots < k_m \) be given. \( \mathcal{H} \) is said to be for the components \( k_1, k_2, \ldots, k_m \) strongly componentwise pointing outwards with bound \( h \) if for all \( R > 0 \) and all \( \delta' > 0 \) one can find some \( \delta = (\delta_1, \delta_2, \ldots, \delta_n) \), \( A = (A_1, A_2, \ldots, A_n) \), \( B = (B_1, B_2, \ldots, B_n) \in \mathbb{R}^n \) with \( \delta_i > 0 \), \( A_i \leq R \), and \( R \leq B_i \) for all \( 1 \leq i \leq n \) and \( \delta_i \geq \delta' \) for all \( 1 \leq j \leq m \) such that \( \mathcal{H} \) is componentwise pointing outwards with bound \( h \) in the \( \delta \)-neighborhood of \( \prod_{i=1}^n [A_i, B_i] \) for initial values in \( \prod_{i=1}^n [-R, R] \).
Theorem 5.6. Let \( h = (h_1, h_2, \ldots, h_n) \in \mathbb{R}^n \) with \( h_i \geq 0 \) for all \( 1 \leq i \leq n \) be given. Let \( \mathcal{H} : D(\mathcal{H})(\subset C(\mathbb{R}_{\geq 0}; \mathbb{R}^n)) \to \text{Map}(\mathbb{R}_{\geq 0}; \mathbb{R}^n) \) be a causal operator and let \( k_1, k_2, \ldots, k_m \in \{1, 2, \ldots, n\} \) with \( k_1 < k_2 < \cdots < k_m \) be given, such that \( \mathcal{H} \) is for the components \( k_1, k_2, \ldots, k_m \) strongly componentwise pointing outwards with bound \( h \). Let
\[
V_{k_1, k_2, \ldots, k_m} := \left\{ (v_1, v_2, \ldots, v_n) \in C(\mathbb{R}_{\geq 0}; \mathbb{R}^n) : \right. \\
v_i(t) = 0 \quad \forall \ t \geq 0, \ i \in \{1, 2, \ldots, n\} \setminus \{k_1, k_2, \ldots, k_m\} \right\}.
\] (5.5)

Then there exists for all \( R > 0 \) some constant \( C_R > 0 \) such that for all \( u^0 \in \prod_{i=1}^{n} [-R, R] \), all \( v \in V_{k_1, k_2, \ldots, k_m} \) with \( \|v(t)\| \leq R \) for all \( t > 0 \), all \( f = (f_1, f_2, \ldots, f_n) \in L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^n) \) with \( |f_i(t)| \leq h_i \) for a.e. \( t > 0 \), all \( T > 0 \) and all solutions \( u \in W^{1,1}_{\text{loc}}((0, T), \mathbb{R}^n) \) \( \cap C([0, T]; \mathbb{R}^n) \) to
\[
u_i(t) + \mathcal{H}[u + v](t) = f \quad \text{for a.e.} \ t \in (0, T), \quad u(0) = u^0,
\] (5.6)
the estimate
\[
\|u(t)\| \leq C_R \quad \forall \ t \in [0, T)
\] (5.7)
is satisfied.

For hysteresis operators that satisfy in addition some regularity assumptions, one can use an argument as in Corollary 3.5, to prove the existence and uniqueness of a classical solution, and can recall Theorem 5.6 to prove the following result.

Corollary 5.7. Let \( \mathcal{H} : C(\mathbb{R}_{\geq 0}; \mathbb{R}^n) \to C(\mathbb{R}_{\geq 0}; \mathbb{R}^n) \) be a locally Lipschitz causal operator and let \( h = (h_1, h_2, \ldots, h_n) \in \mathbb{R}^n \) with \( h_i \geq 0 \) for all \( 1 \leq i \leq n \) be given.

Let \( k_1, k_2, \ldots, k_m \in \{1, 2, \ldots, n\} \) with \( k_1 < k_2 < \cdots < k_m \) be given, let \( V_{k_1, k_2, \ldots, k_m} \) be as in (5.5). Assume that \( \mathcal{H} \) is strongly pointing outwards with bound \( h \) for the components \( k_1, k_2, \ldots, k_m \) and let \( R > 0 \) be given. Then there exists some constant \( C_R > 0 \) such that for all \( u^0 \in \prod_{i=1}^{n} [-R, R] \), all \( v \in V_{k_1, k_2, \ldots, k_m} \) with \( \|v(t)\| \leq R \) for all \( t > 0 \), and all \( f = (f_1, f_2, \ldots, f_n) \in L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^n) \) with \( |f_i(t)| \leq h_i \) for a.e. \( t > 0 \), the following holds: The unique classical solution \( u : \mathbb{R}_{\geq 0} \to \mathbb{R}^n \) to (5.6) satisfies the estimate (5.7).

6. Conclusions
The notation outward point operators has been generalized to vectorial hysteresis operators, and has been used to derive uniform estimates.

The outward pointing formulation introduced in Definition 4.3 is quite elegant and allows us to derive uniform estimates for ODE involving operators with this property, as long as no input perturbations are considered. To be able to deal with these perturbations, the more technical componentwise outward pointing formulation has been introduced in Definition 5.1.

It remains to consider some relevant classes of operators and to formulate conditions that allow the outward pointing properties to be checked.

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