ON THE GEOMETRIC PROPERTIES OF THE
BESSEL-STRUVE KERNEL FUNCTION

SAIFUL R. MONDAL

Abstract. This paper introduces the Bessel-Struve kernel functions \( B_\nu \) defined on the unit disk in the complex plane. We study the close-to-convexity of \( B_\nu \) with respect to several starlike functions. Sufficient condition on \( \nu \) for which the function \( zB_\nu \) is starlike is given.

1. Introduction and Preliminaries

1.1. Bessel-Struve Kernel functions. The function \( J_\nu \), known as the Bessel function of the first kind of order \( \nu \), is a particular solution of the second order Bessel differential equation

\[
z^2 y''(z) + zy'(z) + (z^2 - \nu^2)y(z) = 0.
\]

This function has the power series representation

\[
J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu + n + 1)} \left( \frac{z^2}{2} \right)^{2n+\nu}, \quad |z| < \infty. \tag{1.1}
\]

On the other hand, the modified Bessel function \( I_\nu(z) \) is the particular solution of the differential equation

\[
z^2 y''(z) + zy'(z) - (z^2 - \nu^2)y(z) = 0,
\]

and have the series representation

\[
I_\nu(z) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(\nu + n + 1)} \left( \frac{z^2}{2} \right)^{2n+\nu}, \quad |z| < \infty. \tag{1.2}
\]

The Struve function of order \( \nu \) given by

\[
H_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^k}{\Gamma(n + \nu + \frac{3}{2})\Gamma(n + \frac{3}{2})} \left( \frac{z^2}{2} \right)^{2n+\nu+1} \tag{1.3}
\]

2010 Mathematics Subject Classification. 30C45, 33C10, 30C80, 40G05.

Key words and phrases. Bessel functions, Struve functions, Bessel-Struve kernel, Starlike, Close-to-convex.

* The author thanks the Deanship of Scientific Research at King Faisal University for funding his work under project number 150244.
is a particular solution of the non-homogeneous Bessel differential equation
\[ z^2 y''(z) + zy'(z) + (z^2 - \nu^2)y(z) = \frac{4 \left(\frac{z}{2}\right)^{\nu+1}}{\sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right)}. \] (1.4)
A solution of the non-homogeneous modified Bessel equation
\[ z^2 y''(z) + zy'(z) - (z^2 + \nu^2)y(z) = \frac{4 \left(\frac{z}{2}\right)^{\nu+1}}{\sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right)}. \] (1.5)
yields the modified Struve function
\[ L_{\nu}(z) := -ie^{-\frac{i\pi}{2}} H_{\nu}(iz) \nonumber = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n + \nu + \frac{3}{2})\Gamma(n + \frac{3}{2})} \left(\frac{z}{2}\right)^{2n+\nu+1}. \] (1.6)
The Struve functions occur in areas of physics and applied mathematics, for example, in water-wave and surface-wave problems, as well as in problems on unsteady aerodynamics. The Struve functions are also important in particle quantum dynamical studies of spin decoherence and nanotubes.

Consider the Bessel-Struve kernel function \( B_\nu \) defined on the unit disk \( D = \{z : |z| < 1\} \) as
\[ B_\nu(z) := j_\nu(iz) - ih_\nu(iz), \quad \nu > -\frac{1}{2}, \] (1.7)
where, \( j_\nu(z) := 2^\nu z^{-\nu}\Gamma(\nu + 1)J_\nu(z) \) and \( h_\nu(z) := 2^\nu z^{-\nu}\Gamma(\nu + 1)H_\nu(z) \) are respectively known as the normalized Bessel functions and the normalized Struve functions of first kind of index \( \nu \). The Bessel-Struve transformation and Bessel-Struve kernel functions are appeared in many article. In [10], Hamem et. al. studies an analogue of the Cowling’s Price theorem for the Bessel-Struve transform defined on real domain and also provide Hardy type theorem associated with this transform. The Bessel-Struve intertwining operator on \( C \) is considered in [8, 11]. The fock space of the Bessel-Struve kernel functions is discussed in [9]. The kernel \( z \mapsto B_\nu(\gamma z), \gamma \in C \) is the unique solution of the initial value problem
\[ L_\nu u(z) = \lambda^2 u(z), \quad u(0) = 1, u'(0) = \frac{\lambda \Gamma(\nu + 1)}{\sqrt{\pi} \Gamma\left(\nu + \frac{3}{2}\right)}. \] (1.8)
Here \( L_\nu, \nu > -1/2 \) is the Bessel-Struve operator given by
\[ L_\nu(u(z)) := \frac{d^2 u}{dz^2}(z) + \frac{2\nu + 1}{z} \left(\frac{du}{dz}(z) - \frac{du}{dz}(0)\right). \] (1.9)
Now from (1.1) and (1.6), it is evident that $B_{\nu}$ (taking $\gamma = 1$) possesses the power series

$$B_{\nu}(z) := \sum_{n=0}^{\infty} \frac{\Gamma(\nu + 1)\Gamma(\nu + 1/2)}{\sqrt{\pi n!\Gamma(\nu + 1/2 + 1)}} z^n.$$  (1.10)

The kernel $B_{\nu}$ also have the integral representation

$$B_{\nu}(z) := \frac{2\Gamma(\nu + 1)}{\sqrt{\pi}\Gamma(\nu + 1/2)} \int_{0}^{1} (1 - t^2)^{\nu - 1/2} e^{zt} dt.$$  (1.11)

The identity (1.8) and (1.9) together imply that $B_{\nu}$ satisfy the differential equation

$$z^2 g_{\nu}''(z) + (2\nu + 1)z g_{\nu}'(z) - z g_{\nu}(z) = z M,$$  (1.12)

where $M = 2\Gamma(\nu + 1) \left( \sqrt{\pi} \Gamma(\nu + 1/2) \right)^{-1}$.

Another significance is that $B_{\nu}$ can be express as the sum of the modified Bessel and the modified Struve function of first kind of order $\nu$. For the sake of completeness, in the following result we established this relation.

**Proposition 1.1.** For $\nu > 0$, the following identity holds:

$$z^{\nu} B_{\nu}(z) = 2^{\nu} \Gamma(\nu + 1) \left( I_{\nu}(z) + L_{\nu}(z) \right).$$

The function $B_{\nu}$ have the following recurrence relation which is useful in sequel.

**Proposition 1.2.** For $\nu > 0$, the following recurrence relation holds for $B_{\nu}$:

$$z B'_{\nu}(z) = 2\nu B_{\nu-1}(z) - 2\nu B_{\nu}(z).$$  (1.13)

1.2. Starlike and close-to-convex functions. Let $D = \{ z : |z| < 1 \}$ be the unit disk and $A$ be the class of all analytic functions $f$ defined on $D$ such that $f(0) = 0 = f'(0) - 1$. Clearly each $f \in A$ have the power series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$  (1.14)

A function $f \in A$ is said to be starlike if $f(D)$ is starlike with respect to the origin. Now if for any starlike function $g$ and for some real number $\beta$, we have $\text{Re}(e^{i\beta} f'(z)/g(z)) > 0$, then the function $f$ is said to be close-to-convex with respect to the starlike function $g$. A function
$f \in \mathcal{A}$ is convex if $f(\mathbb{D})$ is convex. The starlike and convex functions can be represent analytically as

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0 \quad \text{and} \quad \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0,$$

respectively. Traditionally the class of starlike functions is denoted as $\mathcal{S}^*$, while the class of close-to-convex, and convex functions are denoted respectively as $\mathcal{C}$ and $\mathcal{K}$. These classes also be generalized by order $\lambda \in [0, 1)$ with the analytical formulation as follows:

$$f \in \mathcal{S}^*(\lambda) \Leftrightarrow \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \lambda,$$

$$f \in \mathcal{C}(\lambda) \Leftrightarrow \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \lambda,$$

$$f \in \mathcal{K}(\lambda) \Leftrightarrow \text{Re} \left( \frac{zf'(z)}{g(z)} \right) > \lambda, \quad \text{for some} \ g \in \mathcal{S}^*.$$

According to the Alexander duality theorem [4], the function $f : \mathbb{D} \to \mathbb{C}$ is in $\mathcal{C}(\nu)$ if and only if $z \to zf'(z)$ is starlike of order $\nu$. Here we remark that the definition of $\mathcal{C}(\nu)$ is also valid for non-normalized analytic function $f : \mathbb{D} \to \mathbb{C}$ with the property $f'(0) \neq 0$.

Let introduce another subclass of $\mathcal{S}^*(\lambda)$ consisting of functions $f$ for which

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \lambda, \quad (1.15)$$

and denoted the class as $\mathcal{S}_1(\lambda)$. The Alexander duality theorem can be apply to the class $\mathcal{S}_1(\lambda)$ and a function $f$ is said to be in the class $\mathcal{C}_1(\lambda)$ if $zf'(z) \in \mathcal{S}_1(\lambda)$.

Following result is required in sequel.

**Lemma 1.1.** [16] Let $\lambda \in [0, 1/2]$ be fixed and $\beta \geq 0$. If $f \in \mathcal{A}$ and

$$\left| \frac{zf'(z)}{f(z)} - 1 \right|^{1-\beta} \left| \frac{zf''(z)}{f'(z)} \right|^\beta < (1 - \lambda)^{1-2\beta} \left( 1 - \frac{3\lambda}{2} + \lambda^2 \right)^\beta, \quad (1.16)$$

for all $z \in \mathbb{D}$, then $f \in \mathcal{S}^*(\lambda)$.

Next we state our main results which are proved in Section 2 by using Lemma 1.1.

**Theorem 1.1.** Let the function $\mathcal{B}_\nu$ as defined in (1.8) satisfy the inequality

$$\left| \frac{z\mathcal{B}_\nu'(z)}{\mathcal{B}_\nu(z)} \right| < 1 - \lambda, \quad (1.17)$$

for $\lambda \in [0, 1/2]$. Then $z\mathcal{B}_\nu \in \mathcal{S}^*(\lambda)$.
Now we will introduce a subclass of $S^*(\lambda)$ consisting of functions $f$ satisfying the inequality

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \lambda,$$

is known as the class of the starlike functions with respect to 1 and denoted as $S^*_1(\nu)$.

In our next result, we obtain sufficient condition by which the Bessel-Struve kernel functions is starlike with respect to 1.

**Theorem 1.2.** Let the function $B_\nu$ as defined in (1.8) satisfy the inequality

$$\left| \frac{zB''_\nu(z)}{B'_\nu(z)} \right| < \frac{2 - 3\lambda + 2\lambda^2}{2(1 - \lambda)},$$

for $\lambda \in [0, 1/2]$. Then $B_\nu \in S^*_1(\lambda)$.

Following problem is well known in the literature:

**Problem 1.1.** Find the conditions on $a_n$ such that

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

is close-to-convex, or starlike or convex or any other subclasses of univalent functions.

Now in accordance with the Problem 1.1, we need to find the condition on $\nu$ such that the Bessel-Struve kernel functions or it’s normalization, will be in any of the classes mention above.

There are many results in the literature (see. [2, 5, 12, 17] and reference their in) which answer the above problem. As per requirement for this work, we listed few of them here. Here we would like to remark that the functions

$$z, \frac{z}{1-z}, \frac{z}{1-z^2}, \frac{z}{(1-z)^2}$$

and

$$\frac{z}{1+z}, \frac{z}{1+z^2}, \frac{z}{(1+z)^2}$$

and their particular rotations

$$\frac{z}{1+z}, \frac{z}{1+z^2}, \frac{z}{(1+z)^2}$$

are the only nine functions which have integer coefficients and are starlike univalent in $D$ (See [7]). The sufficient coefficient conditions for which a function $f \in A$ is close-to-convex can be easily obtain at least when the corresponding starlike functions is one of the above listed form. In this article we will consider for $z$, $z/(1-z)$ and $z/(1-z^2)$. 
Lemma 1.2. \[2\] Let \(\{a_n\}_{n=1}^{\infty}\) be a sequence of non-negative real numbers such that \(a_1 = 1, \Delta a_n \geq 0\) when \(n \geq 1\) and \(\Delta^2 a_n \) when \(n \geq 2\). Then the function \(f\), defined in \((1.19)\), is starlike and close-to-convex with respect to the starlike functions \(z\) and \(z/(1-z)\). Here \(\Delta a_n = na_n - (n+1)a_{n+1}\) and \(\Delta^{m+1} a_n = \Delta^m (\Delta a_n), m = 1, 2, \ldots\).

The starlikeness and close-to-convexity of \(zB_\nu\) is obtained by using Lemma 1.2 in the following result.

Theorem 1.3. For \(\nu \geq 1/2\), the normalized Bessel-Struve kernel function \(zB_\nu\) is starlike. The function \(zB_\nu\) is also close-to-convex with respect to the starlike functions \(z\) and \(z/(1-z)\).

It can be observed that Theorem 1.3 can also be proved by using the following lemma given in \(12\).

Lemma 1.3. \(12\) Let \(\{a_n\}_{n \geq 1}\) be a sequence of positive real numbers such that \(a_1 \geq 6a_2 \geq 6a_3\) and \(n(n-2)a_n \geq (n-1)(n+1)a_{n+1}\) for \(n \geq 3\). Then \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n\) is close-to-convex with respect to both the starlike functions \(z\) and \(z/(1-z)\). Further, the function \(f\) is starlike univalent in \(D\).

In our next result we will study the close-to-convexity of \(zB_\nu\) with respect to the starlike functions \(z/(1-z^2)\).

Theorem 1.4. If \(\nu \geq \nu_0 \approx 19.6203\), the function \(zB_\nu\) is close-to-convex with respect to the starlike functions \(z/(1-z^2)\).

The following result is used to prove Theorem 1.4.

Lemma 1.4. \(12\) Theorem 4.4] Let \(\{a_n\}_{n \geq 1}\) be a sequence of positive real numbers such that \(a_1 = 1\). Suppose that \(a_1 \geq 8 a_2\), and \((n-1)a_n \geq (n+1)a_{n+1}\) for all \(n \geq 2\). Then the function \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n\) is close-to-convex with respect to the starlike functions \(z/(1-z^2)\).

2. Proof of the main results

Proof of Proposition 1.1. From \((1.10)\), it follows that

\[
z^\nu B_\nu(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\nu+1)\Gamma(n+1)}{\sqrt{\pi} n! \Gamma(\frac{n+1}{2} + \nu + 1)} z^{n+\nu} \tag{2.20}
= \sum_{m=0}^{\infty} \frac{\Gamma(\nu+1)\Gamma(m+\frac{1}{2})}{\sqrt{\pi} (2m)! \Gamma(m + \nu + 1)} z^{2m+\nu}
+ \sum_{m=0}^{\infty} \frac{\Gamma(\nu+1)\Gamma(m+1)}{\sqrt{\pi} (2m+1)! \Gamma(m + \nu + \frac{3}{2})} z^{2m+1+\nu}.
\]
ON THE GEOMETRIC PROPERTIES OF THE BESSEL-STRUVE KERNEL FUNCTION

The Legendre duplication formula (see [1, 6])

\[ \Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z} \sqrt{\pi} \Gamma(2z) \]

shows that

\[ \frac{\Gamma\left(m + \frac{1}{2}\right)}{\sqrt{\pi}(2m)!} = \frac{1}{2^{2m}m!} \quad \text{and} \quad \frac{\Gamma(m + 1)}{\sqrt{\pi}(2m + 1)!} = \frac{1}{2^{2m+1}\Gamma\left(m + \frac{3}{2}\right)}. \]

This along with (1.2) and (1.6), the identity (2.20) reduce to

\[ z^\nu B^\nu(z) = 2^\nu\Gamma(\nu + 1) \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m+\nu}}{m!\Gamma(m + \nu + 1)} \left(\frac{z}{2}\right)^{2m+\nu} \]

This complete the proof. \(\Box\)

Proof of Proposition 1.2. Differentiating the series (1.10), it follows that

\[ zd\frac{d}{dz}B^\nu(z) = \sum_{n=0}^{\infty} n\Gamma(\nu + 1)\Gamma\left(\frac{n+1}{2}\right)\frac{\left(\frac{z}{2}\right)^{2n+\nu}}{\sqrt{\pi}n!\Gamma\left(n + \nu + 1\right)} z^n \]

\[ = 2\nu\sum_{n=0}^{\infty} \frac{\left(\frac{n}{2} + \nu\right)\Gamma(\nu)\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}n!\Gamma\left(n + \nu + 1\right)} z^n \]

\[ - 2\nu\sum_{n=0}^{\infty} \frac{\Gamma(\nu + 1)\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}n!\Gamma\left(n + \nu + 1\right)} z^n \]

This complete the proof. \(\Box\)

Proof of Theorem 1.1. Denote \(f(z) = zB^\nu(z)\). Then a computation together with the hypothesis (1.18) yield

\[ \left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{zB^\nu_\nu(z)}{B^\nu(z)}\right| < 1 - \lambda, \]

which is equivalent to (1.16) for \(\beta = 0\). The conclusion follows from Lemma 1.1. \(\Box\)

Proof of Theorem 1.2. Define \(h : \mathbb{D} \to \mathbb{C}\) as

\[ h(z) := \frac{2\Gamma(\nu + 3/2)}{\Gamma(\nu + 1)}(B^\nu(z) - 1). \quad (2.21) \]
Then clearly \( h \in A \). Now a computation yields
\[
\left| \frac{zh''(z)}{h'(z)} \right| = \left| \frac{zB''_{\nu}(z)}{B'(z)} \right| < \frac{2 - 3\nu + 2\nu^2}{2(1 - \nu)}.
\]
Taking \( \beta = 1 \), from Lemma 1.1 it follows that \( h \in \mathcal{S}^*(\lambda) \) with respect to origin. Now Theorem 1.2 follows from the definition of \( h \) in (2.21).

\[\square\]

**Proof of Theorem 1.3**: From (1.10), we can express \( z\mathcal{B}_\nu \) as
\[
z\mathcal{B}_\nu(z) = \sum_{n=1}^{\infty} a_n z^n, \quad (2.22)
\]
where
\[
a_1 = 1 \quad \text{and} \quad a_n = \frac{\Gamma(\nu + 1)\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi(n-1)!}\Gamma\left(\frac{n+1}{2} + \nu\right)}. \quad n \geq 2. \quad (2.23)
\]
Define the function \( g_n : [0, \infty) \to \mathbb{R} \) as
\[
g_n(\nu) := \frac{\Gamma\left(\frac{n}{2} + 1 + \nu\right)}{\Gamma\left(\frac{n+1}{2} + \nu\right)}. \quad (2.24)
\]
The logarithmic differentiation with respect to \( \nu \) implies
\[
g'_n(\nu) = g_n(\nu) \left( \Psi\left(\frac{n}{2} + \nu + 1\right) - \Psi\left(\frac{n+1}{2} + \nu\right) \right). \quad (2.25)
\]
Here \( \Psi \) is the well-known digamma function which is an increasing function on \([0, \infty)\), and consequently \( g_n \) is also increasing. Thus for \( \nu \geq 1/2 \),
\[
g_n(\nu) \geq g_n(1/2) = \frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}. \quad (2.26)
\]
Thus for \( n \geq 1 \) and \( \nu \geq 1/2 \), it follows that
\[
\frac{a_n}{a_{n+1}} = \frac{\Gamma(\nu + 1)\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi(n-1)!}\Gamma\left(\frac{n+1}{2} + \nu\right)} \times \frac{\sqrt{\pi(n)!}\Gamma\left(\frac{n+2}{2} + \nu\right)}{\Gamma(\nu + 1)\Gamma\left(\frac{n+1}{2}\right)}
\geq \frac{n\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)} = n + 1.
\quad (2.27)
\]
This implies for \( n \geq 1 \)
\[
\Delta a_n = na_n - (n + 1)a_{n+1} \geq (n^2 - 1)a_{n+1} \geq 0,
\]
and for $n \geq 2$ we have
\[
\Delta^2 a_n = n a_n - 2(n+1)a_{n+1} + (n+2)a_{n+2} \\
\geq (n+1)(n-2)a_{n+1} + (n+2)a_{n+2} \\
\geq (n+2)(n^2 - n - 1)a_{n+2} > 0.
\]
Thus $\{a_n\}$ satisfy the hypothesis of Lemma 1.2 and hence the conclusion. \hfill \Box

**Proof of Theorem 1.4.** The inequality (2.27) yield that for $n \geq 2$, and $\nu \geq 1/2$,
\[
(n-1)a_n - (n+1)a_{n+1} \geq na_{n+1} > 0,
\]
Now from (2.23) it follows that the coefficient $a_n$ satisfy the hypothesis $a_1 \geq 8a_2$ is equivalent to $\sqrt{\pi}\Gamma(\nu+3/2) \geq 8\Gamma(\nu+1)$ which holds when $\nu \geq \nu_0$, where $\nu_0 \approx 19.6203$ is the positive root of the identity
\[
\sqrt{\pi}\Gamma(\nu+3/2) = 8\Gamma(\nu+1).
\]
Now the result follows from the Lemma 1.4. \hfill \Box

**Problem 2.1 (Open).** *Find the sharp lowest bound for $\nu > -1$ so that $zB_\nu$ is starlike in $\mathbb{D}$ and also close-to-convex with respect to the starlike functions $z/(1-z^2)$.*

Acknowledgement

The author thanks the Deanship of Scientific Research at King Faisal University for funding this work under project number 150244.

**References**

[1] M. Abramowitz and I. A. Stegun, *A Handbook of Mathematical Functions*, New York, (1965).
[2] A. P. Acharya, Univalence criteria for analytic functions and applications to hypergeometric functions, Ph.D Diss., University of Würzburg, 1997.
[3] A. R. Ahmadi and S. E. Widnall, Unsteady lifting-line theory as a singular-perturbation problem. J. Fluid Mech. 153 (1985), 59–81.
[4] J. W. Alexander, Functions which map the interior of the unit circle upon simple regions, Ann. of Math. (2) 17 (1915), no. 1, 12–22.
[5] R. M. Ali, S. K. Lee and S. R. Mondal, Coefficient conditions for starlikeness of nonnegative order, Abstr. Appl. Anal. 2012, Art. ID 450318, 13 pp.
[6] G. E. Andrews, R. Askey and R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications, 71, Cambridge Univ. Press, Cambridge, 1999.
[7] B. Friedman, Two theorems on schlicht functions, Duke Math. J. 13 (1946), 171–177.
[8] A. Gasmi, M. Sifi, The Bessel-Struve intertwining operator on $\mathbb{C}$ and mean-periodic functions, *Int. J. Math. Math. Sci.* 57-60 (2004) 3171–3185.
[9] A. Gasmi, F. Soltani, Fock spaces for the Bessel-Stuve kernel, *J. Anal. Appl.* 3 (2005) 91–106.

[10] S. Hamem, L. Kamoun and S. Negzaoui, Cowling-Price type theorem related to Bessel-Struve transform, *Arab J. Math. Sci.* 19 (2013), no. 2, 187–198.

[11] L. Kamoun, M. Sifi, Bessel-Struve intertwining operator and generalized Taylor series on the real line, *Integral Transforms Spec. Funct.* 16 (2005) 39–55.

[12] S. R. Mondal and A. Swaminathan, Coefficient conditions for univalency and starlikeness of analytic functions, *J. Math. Appl.* 31 (2009), 77–90.

[13] T. G. Pedersen, Variational approach to excitons in carbon nanotubes. *Phys. Rev. B* 67 (2003), no. 7, (0734011)–(0734014).

[14] J. Shao, P. Hänggi, Decoherent dynamics of a two-level system coupled to a sea of spins. *Phys. Rev. Lett.* 81 (1998), no. 26, 5710–5713.

[15] D. C. Shaw, Perturbational results for diffraction of water-waves by nearly-vertical barriers. *IMA, J. Appl. Math.* 34 (1985), no. 1, 99–117.

[16] S. Owa and H. M. Srivastava, Univalent and starlike generalized hypergeometric functions, *Canad. J. Math.* 39 (1987), no. 5, 1057–1077.

[17] A. Swaminathan, Univalent polynomials and fractional order differences of their coefficients, *J. Math. Anal. Appl.* 353 (2009), no. 1, 232–238.

Department of Mathematics and Statistics, College of Science, King Faisal University, Al-Hasa 31982, Hofuf, Saudi Arabia.

E-mail address: smondal@kfup.edu.sa