Fiber-optical analogue of the event horizon: Appendices

Thomas G. Philbin$^{1,2}$, Chris Kuklewicz$^1$, Scott Robertson$^1$, Stephen Hill$^1$, Friedrich König$^1$, and Ulf Leonhardt$^1$

$^1$School of Physics and Astronomy, University of St Andrews, North Haugh, St Andrews, Fife, KY16 9SS, UK

$^2$Max Planck Research Group of Optics, Information and Photonics, Günther-Scharowsky-Str. 1, Bau 24, D-91058 Erlangen, Germany

February 5, 2008

Abstract

We explain the theory behind our fiber-optical analogue of the event horizon and present the experiment in detail.
A Theory

In this appendix we describe the theory behind our fiber-optical analogue of the event horizon [1]. After a brief summary of Nonlinear Fiber Optics we show how an optical pulse establishes a moving medium and how this medium corresponds to a space-time geometry. This point of view is relatively unusual in the fiber optics community, but, as we demonstrate, it is completely consistent with the established knowledge of this field [2]. We then take advantage of our approach in quantizing the electromagnetic field of a probe in the presence of a pulse. We describe the classical and quantum physics of fiber-optical horizons, explaining the classical frequency shifting and the quantum Hawking effect with as few assumptions on the physics and the prior knowledge of the reader as possible without going into excessive detail.

A.1 Kerr nonlinearity

Optical fibers [2, 3] are metamaterials — materials with optical properties that are dominated by their structure — for the following reason: an optical fiber confines light to narrow transversal regions, usually at the core of the fiber. The electromagnetic field in the transversal plane depends on the frequency and polarization of the light in relation to the transversal structure of the fiber. In this way [2], the waveguide gives rise to an effective dispersion [4] and birefringence [4] that supersedes the natural optical dispersion of silica glass. The confinement of the light in the fiber also enhances the intensity over the fiber length such that the natural optical nonlinearity of glass becomes relevant, in particular the Kerr and Raman nonlinearity caused, respectively, by the electronic and molecular optical response of glass [2]. For making artificial event horizons, one can exploit the custom-designed dispersion and birefringence and the enhanced nonlinearity of optical fibers, in particular of microstructured fibers (also called photonic-crystal fibers) [3].

Let us first describe the optical properties of glass. In the laboratory frame, \( r = (x, y, z) \) describes the position in Cartesian coordinates and \( t \) the time. We use the operator \( \nabla \) to denote spatial derivatives and \( \partial_t \) for time derivatives. We characterize the intrinsic linear optical response of the glass fiber by the susceptibility profile \( \chi_g(t, r) \) and denote the nonlinear polarization vector of the medium by \( P \). The vector \( E \) of the electric field strength obeys the wave equation [5] in SI units,

\[
\nabla \times (\nabla \times E) + \frac{\partial^2}{c^2} \left( E + \int_{-\infty}^{t} \chi_g(t - t') E(t') \ dt' + \frac{P}{\varepsilon_0} \right) = 0
\]

(A1)

with \( \varepsilon_0 \) being the electric permeability of the vacuum. We assume a lossless medium where \( \chi_g \) is real. In isotropic materials such as glass, no second-order nonlinearity exists on symmetry grounds [6]. The lowest-order nonlinearity is proportional to the response from the products of three electric-field components. Such an effect, called Kerr nonlinearity [2] is generated by the electronic excitations of glass on a time scale comparable to the atomic size divided by the speed of light. Assuming instantaneous response, the nonlinear polarization is proportional to a cubic form of the field strength. The effect of optical nonlinearities strongly depends on the
frequency matching of the field components involved [6]. Consider instead of the full electric field $E$ either the positive or negative frequency component oscillating in the optical spectral range. To keep the notation simple, we describe these components by the symbol $E$ as well. The nonlinear polarization of the medium is only effective if $P$ oscillates in a similar spectral range as one of the frequency components of $E$. Consequently, $P$ combines two electric-field components $E_l$ and one complex conjugate $E_m^*$. In isotropic materials such as glass, $P$ is further restricted: isotropy implies for the $P$ components [6]

$$P_m = \frac{2\varepsilon_0}{3} \sum_l \left( \kappa_1 E_m E_l E_l^* + \kappa_2 E_l E_mE_l^* + \kappa_3 E_l E_l E_m^* \right)$$

(A2)

where the $\kappa$ denote the material constants of the Kerr nonlinearity. They correspond to the $\chi^{(3)}$ coefficients in nonlinear optics [6]. In silica, the $\kappa$ constants are identical, to a very good approximation,

$$\kappa_1 = \kappa_2 = \kappa_3 = \kappa.$$ (A3)

In fibers [2], the electric field has two transversal components that we denote by $\pm$. Equations (A2) and (A3) give

$$P_\pm = 2\varepsilon_0\kappa \left( |E_\pm|^2 E_\pm + \frac{2}{3} |E_\mp|^2 E_\pm + \frac{1}{3} E_\mp^2 E_\pm^* \right).$$ (A4)

The first term describes the Self Phase Modulation, the second term the Cross Phase Modulation and the third usually is called Four Wave Mixing [2]. The Self Phase Modulation does not act across polarizations, whereas the other two terms correspond to polarization interactions. For equally polarized fields that carry two distinct frequency bands, the Kerr effect acts across these frequencies, because, writing

$$E_\pm = E_a e^{-i\omega_at} + E_b e^{-i\omega_bt}$$ (A5)

we obtain

$$|E_\pm|^2 E_\pm = (|E_a|^2 + 2|E_b|^2) E_a e^{-i\omega_at} + (|E_b|^2 + 2|E_a|^2) E_b e^{-i\omega_b t}.$$ (A6)

The contribution proportional to $|E_b|^2 E_a$ or $|E_a|^2 E_b$ is also called Cross Phase Modulation [2]. We see that the coupling strength of the Cross Phase Modulation between polarizations is $2/3$ of the Self Phase Modulation and the Cross Phase Modulation within one polarization is twice as strong as the Self Phase Modulation [2].

Another cubic nonlinear effect caused by molecular excitations with longer response time, called Stimulated Raman Scattering [2], contributes to the material polarization for pulses below 100fs duration. Stimulated Raman Scattering is important for forming the shape of ultrashort pulses [2], but, it does not act across polarizations nor significantly different frequency bands, the cases we are interested in; hence we ignore the Raman effect here.
A.2 Waveguides

Consider the effect of the waveguide on the light confined in the $x$ and $y$ direction and propagating along the fiber in the $z$ direction. We assume that the fiber is homogeneous in the $z$ direction and infinitely long, with the Fourier-transformed susceptibility

$$\tilde{\chi}_g = \tilde{\chi}_g(\omega, x, y).$$

(A7)

We represent the Fourier-transformed field strengths as

$$\tilde{E}(\omega, r) = \tilde{E}_\pm(\omega, z) U_\pm(\omega, x, y)$$

(A8)

and require that the fiber modes $U_\pm$ are eigenfunctions of the transversal part of the wave equation for monochromatic light with eigenvalues $\beta^2_\pm(\omega)$,

$$\left(-\nabla \times (\nabla \times U_\pm) + (1 + \tilde{\chi}_g(\omega, x, y))\frac{\omega^2}{c^2} U_\pm \right) = \beta^2_\pm(\omega) U_\pm.$$

(A9)

For single-mode fibers [2], only one eigenvalue $\beta^2_\pm(\omega)$ exists for each optical polarization $\pm$ and frequency $\omega$ (within a limited frequency range). We normalize the mode functions $U_\pm$ such that the integral of $|U_\pm(\omega, x, y)|^2$ over the $(x, y)$ plane is unity.

The optical nonlinearity acts predominantly within a narrow spatial region near the fiber core at $(x, y) = (0, 0)$. Therefore, we obtain, to a very good approximation,

$$[\partial_z^2 + \beta^2_\pm(i\partial_t)] E_\pm = \frac{\partial_t^2 P_\pm}{\varepsilon_0 c^2}$$

(A10)

with the nonlinear polarization (A4). Here we understand $\kappa$ as being averaged over the transversal modes (usually causing almost no difference between the two optical polarizations $\pm$).

The eigenvalues $\beta^2_\pm(\omega)$ of the transversal modes set the effective refractive indices $n_\pm$ of the fiber for light pulses $E_\pm(t, z)$ defined by the relation

$$\beta_\pm = \frac{n_\pm}{c} \omega.$$

(A11)

As a well-known consequence of causality [7] the refractive index must be analytic on the upper half plane of complex $\omega$. In the absence of losses within the frequency range we are considering, the Fourier-transformed $\tilde{\chi}_g(\omega)$ in the longitudinal mode equation (A9), is real on the real axis and the longitudinal mode equation (A9) is Hermitian and positive. Consequently, $n_\pm$ is real on the real axis. Furthermore, since the linear susceptibility $\chi_g(t)$ is real, $\tilde{\chi}_g(\omega)$ must be an even function of $\omega$, which implies that $n^2_\pm$ and $\beta^2$ are even functions of the frequency $\omega$.

When the dielectric structure of the fiber varies over the scale of an optical wavelength, the polarization of light becomes an important issue and the refractive indices differ for $E_+(t, z)$ and $E_-(t, z)$, causing birefringence [4]. As we have seen, microstructured or photonic-crystal fibers [3] allow for some freedom in tailoring the dispersion, nonlinearity, and birefringence for specific applications.
A.3 Effective moving medium

In our case, an intense ultrashort optical pulse interacts with a weak probe field. This probe may be caused by an incident continuous wave of light or by the vacuum fluctuations of the electromagnetic field itself \[8\]. The vacuum fluctuations are carried by modes that behave as weak classical light fields as well. The pulse is polarized along one of the eigen-polarizations of the fiber; the probe field may be co- or cross polarized. Due to Stimulated Raman Scattering \[2\] or the formation of optical shocks \[2\] the pulse will change its shape and velocity; but we assume that the intensity profile \(I(z,t)\) of the pulse uniformly moves with constant velocity \(u\) during the interaction with the probe. Since the probe field is weak or in the vacuum state we can safely neglect the backaction onto the pulse and the nonlinear self interaction of the probe. Since the intensity profile of the pulse is assumed to be fixed and given, we focus attention on the probe field. We describe the probe by the corresponding component \(A\) of the vector potential that generates the electric field \(E\) and the magnetic field \(B\), with

\[
E = -\partial_t A, \quad B = \partial_z A. \tag{A12}
\]

The probe field obeys the wave equation

\[
(c^2 \partial_z^2 + c^2 \beta^2 (i \partial_t) - \partial_t \chi \partial_t) A = 0, \quad \chi \propto I(z,t) \tag{A13}
\]

where \(\chi\) denotes the susceptibility due to the Kerr effect of the pulse on the probe. We take the \(\beta\) of Eq. (A11) that corresponds to the probe polarization and denote the effective refractive index by \(n_0\). Equation (A13) shows that the pulse indeed establishes an effective moving medium \[9\]. It is advantageous \[2\] to use as coordinates the retarded time \(\tau\) and the propagation time \(\zeta\) defined as

\[
\tau = t - \frac{z}{u}, \quad \zeta = \frac{z}{u}, \tag{A14}
\]

because in this case the properties of the effective medium depend only on \(\tau\). Here \(\tau\) plays the role of space and \(\zeta\) of time. In this co-moving frame we replace in the wave equation (A13) the \(z\) and \(t\) derivatives by

\[
\partial_t = \partial_{\tau}, \quad \partial_z = \frac{1}{u} (\partial_{\zeta} - \partial_{\tau}), \tag{A15}
\]

and obtain

\[
(\partial_{\zeta} - \partial_{\tau})^2 A = \partial_{\tau} \frac{u^2}{c^2} n^2 \partial_{\tau} A, \tag{A16}
\]

where the total refractive index \(n\) consists of the effective linear index \(n_0\) and the contribution due the Kerr effect of the pulse,

\[
n^2 = n_0^2 + \chi. \tag{A17}
\]

Since \(\chi \ll n_0\) we approximate

\[
n \approx n_0 + \delta n, \quad \delta n = \frac{\chi}{2n_0}, \tag{A18}
\]

where we can ignore the frequency dependance of \(n_0\) in \(\chi/(2n_0)\), which gives Eq. (1) of our paper \[1\]. Note that Eq. (A14) does not describe a Lorentz transformation, but the \(\tau\) and \(\zeta\) are of course perfectly valid coordinates; they simply do not belong to an inertial system.
A.4 Dispersionless case and metric

Assume, for simplicity, a dispersionless case where the refractive index $n_0$ of the probe does not depend on the frequency. Note that a horizon inevitably violates this condition, because here light comes to a standstill, oscillating at increasingly shorter wavelengths the closer it approaches the horizon. Light waves are dramatically frequency shifted and thus leave any dispersionless frequency window. However, many of the essentials of horizons are still captured within the simplified dispersionless model.

First, we can cast the wave equation (A16) in a relativistic form, introducing a relativistic notation for the coordinates and their derivatives

$$ x^\mu = (\zeta, \tau), \quad \partial_\mu = (\partial_\zeta, \partial_\tau) $$

and the matrix

$$ g^{\mu\nu} = \begin{pmatrix} 1 & -1 \\ -1 & 1 - u^2 n^2 / c^2 \end{pmatrix} $$

that resembles the inverse metric tensor of waves in moving fluids. Adopting these definitions and Einstein’s summation convention over repeated indices the wave equation (A16) appears as

$$ \partial_\mu g^{\mu\nu} \partial_\nu A = 0 $$

which is almost the free wave equation in a curved space-time geometry. (In the case of a constant refractive index the analogy between the moving medium and a space-time manifold is perfect.) The effective metric tensor $g_{\mu\nu}$ is the inverse of $g^{\mu\nu}$. We obtain

$$ g_{\mu\nu} = \frac{c^2}{n^2 u^2} \begin{pmatrix} u^2 n^2 / c^2 - 1 & -1 \\ -1 & -1 \end{pmatrix} $$

In subluminal regions where the velocity $c/n$ of the probe light exceeds the speed of the effective medium, i.e. the velocity $u$ of the pulse, the measure of time $u^2 n^2 / c^2 - 1$ in the metric (A22) is negative. Here both $\partial_\tau$ and $\partial_\zeta$ are timelike vectors. In superluminal regions, however, $c/n$ is reduced such that $u^2 n^2 / c^2 - 1$ is positive. A horizon, where time stands still, is established where the velocity of light matches the speed of the pulse.

A.5 Action

The theory of quantum fields at horizons predicts the spontaneous generation of particles. In our case, the quantum field is light in dielectric media. The quantum theory of light in media at rest has reached a significant level of sophistication, because it forms the foundation of quantum optics and, in particular, the quantum theory of optical instruments, but quantum light in moving media is much less studied. In optical fibers, light is subject to dispersion, which represents experimental opportunities on one side, but poses a theoretical challenge on the
other: we should quantize a field described by a classical wave equation of high order in the retarded time. Moreover, strictly speaking, dispersion is always accompanied by dissipation, which results in additional quantum fluctuations [16]. Here, however, we assume to operate in frequency windows where the absorption is very small. We entirely focus on the dispersive properties of the fibre. To deduce the starting point of the theory, we begin with the dispersionless case in classical optics and then proceed to consider optical dispersion for light quanta.

The classical wave equation of one-dimensional light propagation in dispersionless media follows from the Principle of Least Action [10] with the action of the electromagnetic field in SI units

\[ S = \int \int \frac{\varepsilon_0}{2} \left( n^2 E^2 - c^2 B^2 \right) dzt \]

and hence the Lagrangian density

\[ \mathcal{L} = -\frac{\varepsilon_0}{2} \left[ A \partial_t n^2 \partial_t A + c^2 (\partial_z A)^2 \right] . \]  \hspace{1cm} (A24)

In order to include the optical dispersion in the fiber and the effect of the moving pulse, we express the refractive index in terms of \( \beta(\omega) \) and the effective susceptibility \( \chi(\tau) \) caused by the pulse, using Eqs. (A11) and (A17) with \( \omega = i \partial_t \). We thus propose the Lagrangian density

\[ \mathcal{L} = \frac{\varepsilon_0}{2} \left[ A (c^2 \beta^2 (i \partial_t) - \partial_t \chi \partial_t) A - c^2 (\partial_z A)^2 \right] . \]  \hspace{1cm} (A25)

In the absence of losses, \( \beta^2(\omega) \) is an even function, as we obtained in Sec. A.2. We write down the Euler-Lagrange equation [10] for this case

\[ \partial_\zeta \frac{\partial \mathcal{L}}{\partial (\partial_\zeta A)} - \sum_{\nu=0}^{\infty} (-1)^\nu \partial_\nu \frac{\partial \mathcal{L}}{\partial (\partial_\nu A)} = 0 \]  \hspace{1cm} (A26)

and obtain the wave equation (A16). This proves that the Lagrangian density (A25) is the correct one.

A.6 Quantum field theory

According to the quantum theory of fields [18] the component \( A \) of the vector potential is described by an operator \( \hat{A} \). Since the classical field \( A \) is real, the operator \( \hat{A} \) must be Hermitian. For finding the dynamics of the quantum field we quantize the classical relationship between the field, the canonical momentum density and the Hamiltonian: we replace the Poisson bracket between the field \( A \) and the momentum density \( \partial \mathcal{L} / \partial (\partial_\zeta A) \) by the fundamental commutator between the quantum field \( \hat{A} \)
and the quantized momentum density [18]. We obtain from the Lagrangian (A25) the canonical momentum density

$$\dot{\pi} = -\varepsilon_0 \frac{c^2}{u} \partial_{\zeta} \dot{A}$$

(A27)

and postulate the equivalent of the standard equal-time commutation relation [18, 19]

$$[\hat{A}(\zeta, \tau_1), \hat{\pi}(\zeta, \tau_2)] = \frac{i\hbar}{u} \delta(\tau_1 - \tau_2).$$

(A28)

We obtain the Hamiltonian

$$\hat{H} = \int \left( \hat{\pi} \partial_{\zeta} \hat{A} - \mathcal{L} \right) u d\tau = \varepsilon_0 \frac{c^2}{u} \int \left( \frac{c^2}{u^2} \left((\partial_{\tau} \hat{A})^2 - (\partial_{\zeta} \hat{A})^2\right) - \hat{A} \left( c^2 \beta^2 - \partial_{\tau} \chi \partial_{\tau} \right) \hat{A} \right) u d\tau.$$ 

(A29)

One verifies that the Heisenberg equation of the quantum field $\hat{A}$ is the classical wave equation (A16), as we would expect for fields that obey linear field equations.

### A.7 Mode expansion

Since the field equation is linear and classical, we represent $\hat{A}$ as a superposition of a complete set of classical modes multiplied by quantum amplitudes $\hat{a}_k$. The mode expansion is Hermitian for a real field such as the electromagnetic field,

$$\hat{A} = \sum_k \left( A_k \hat{a}_k + A_k^* \hat{a}_k^\dagger \right).$$

(A30)

The modes $A_k$ obey the classical wave equation (A21) and are subject to the orthonormality relations [14, 9]

$$(A_k, A_{k'}) = \delta_{kk'} \quad (A_k^*, A_{k'}) = 0$$

(A31)

with respect to the scalar product

$$\langle A_1, A_2 \rangle = \frac{\varepsilon_0 c^2}{i\hbar} \int (A_1^* \partial_{\zeta} A_2 - A_2^* \partial_{\zeta} A_1^*) d\tau.$$ 

(A32)

The scalar product is chosen such that it is a conserved quantity for any two solutions $A_1$ and $A_2$ of the classical wave equation (A16),

$$\partial_{\zeta} \langle A_1, A_2 \rangle = 0,$$

(A33)

with a prefactor that turns out to make the commutation relations between the mode operators particularly simple and transparent.

The scalar product serves to identify the quantum amplitudes $\hat{a}_k$ and $\hat{a}_k^\dagger$: the amplitude $\hat{a}_k$ belongs to modes $A_k$ with positive norm, whereas the Hermitian conjugate $\hat{a}_k^\dagger$ is the quantum amplitude to modes $A_k^*$ with negative norm, because

$$\langle A_1^*, A_2^* \rangle = -\langle A_1, A_2 \rangle.$$ 

(A34)
Using the orthonormality relations \( (A31) \) we can express the mode operators \( \hat{a}_k \) and \( \hat{a}_k^\dagger \) as projections of the quantum field \( \hat{A} \) onto the modes \( A_k \) and \( A_k^* \) with respect to the scalar product \( (A32) \),

\[
\hat{a}_k = (A_k, \hat{A}) , \quad \hat{a}_k^\dagger = -(A_k^*, \hat{A}).
\] (A35)

We obtain from the fundamental commutator \( (A28) \) and the orthonormality relations \( (A31) \) of the modes the Bose commutation relations

\[
[\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{kk'} , \quad [\hat{a}_k, \hat{a}_k] = 0 ,
\] (A36)

which justifies the choice of the prefactor in the scalar product \( (A32) \). In agreement with the spin-statistics theorem \([18]\), light consists of bosons and the quantum amplitudes \( \hat{a}_k \) and \( \hat{a}_k^\dagger \) serve as annihilation and creation operators.

The expansion \( (A30) \) is valid for any orthonormal and complete set of modes. Consider stationary modes with frequencies \( \omega'_k \) such that

\[
\partial_\zeta A_k = -i\omega'_k A_k .
\] (A37)

We substitute the mode expansion \( (A30) \) in the Hamiltonian \( (A29) \) and use the wave equation \( (A16) \) and the orthonormality relations \( (A31) \) to obtain

\[
\hat{H} = \hbar \sum_k \omega'_k \left( \hat{a}_k^\dagger \hat{a}_k + \frac{1}{2} \right) .
\] (A38)

Each stationary mode contributes \( \hbar \omega'_k \) to the total energy that also includes the vacuum energy.

The modes with positive norm select the annihilation operators of a quantum field, whereas the negative norm modes pick out the creation operators. In other words, the norm of the modes determines the particle aspects of the quantum field. In the Unruh effect \([20]\), modes with positive norm in the Minkowski space-time consist of superpositions of positive and negative norm modes in the frame of an accelerated observer \([14]\). Consequently, the Minkowski vacuum is not the vacuum as seen in the accelerated frame. Instead, the accelerated observer perceives the Minkowski vacuum as thermal radiation \([20]\). In the Hawking effect \([13]\), the scattering of light at the event horizon turns out to mix positive and negative norm modes, giving rise to Hawking radiation.

A.8 Geometrical optics

A moving dielectric medium with constant refractive index but nonuniform velocity appears to light exactly as an effective space-time geometry \([9, 11]\). Since a stationary \( 1 + 1 \) dimensional geometry is conformally flat \([21]\) a coordinate transformation can reduce the wave equation to describing wave propagation in a uniform medium, leading to plane-wave solutions \([22]\). The plane waves appear as phase-modulated waves in the original frame. Consequently, in this case, geometrical optics is exact. In our case, geometrical optics provides an excellent approximation, because the variations of the refractive index are very small.
Consider a stationary mode $A$. We assume that the mode carries a slowly varying amplitude $A$ and oscillates with a rapidly changing phase $\phi$,

$$A = A \exp(i\varphi). \quad (A39)$$

We represent the phase as

$$\varphi = -\int \omega(\tau) \, d\tau - \omega' \zeta \quad (A40)$$

and obtain from the wave equation (A16) the dispersion relation

$$(\omega - \omega')^2 = \frac{u^2}{c^2} n^2 \omega^2 \quad (A41)$$

by neglecting all derivatives of the amplitude $A$. Here $n$ includes the additional susceptibility $\chi$ due to the Kerr effect of the pulse according to Eq. (A17).

The dispersion relation has two sets of solutions describing waves that are co- or counter-propagating with the pulse in the laboratory frame. Counter-propagating waves will experience the pulse as a tiny transient change of the refractive index, whereas co-propagating modes may be profoundly affected. Consider the solution

$$\omega' = \left(1 - \frac{u}{c} n\right) \omega. \quad (A42)$$

In this case, we obtain outside of the pulse in the laboratory frame $\varphi = n(\omega/c)z - \omega t$, which describes light propagating in the positive $z$ direction. Consequently, the branch (A42) of the dispersion relation corresponds to co-propagating light waves. We also see that $\omega$ is the frequency of light in the laboratory frame, whereas $\omega'$ is the frequency in the frame co-moving with the pulse. Equation (A42) describes how the laboratory-frame and the co-moving frequencies are connected due to the Doppler effect.

In order to find the evolution of the amplitude $A$, we substitute in the exact scalar product (A32) the approximation (A39) with the phase (A40) and the dispersion relation (A42). In the limit $\omega' \to \omega_2$ we obtain

$$(A_1, A_2) = \frac{2 \varepsilon_0 c}{\hbar} \int A^2 n \omega \exp(i \varphi_2 - i \varphi_1) \, d\tau, \quad (A43)$$

which should give $\delta(\omega' - \omega_2)$ according to the normalization (A31). The dominant, diverging contribution to this integral, generating the peak of the delta function, stems from $\tau \to \pm \infty$. Hence, for $\omega' \to \omega_2$, we replace $\varphi$ in the integral by $\varphi$ at $\tau \to \pm \infty$ where $\omega$ does not depend on $\tau$ anymore,

$$(A_1, A_2) = \frac{2 \varepsilon_0 c}{\hbar} \int A^2 n \omega \exp[i(\omega_2 - \omega_1)\tau] \, d\tau, \quad \omega_2 - \omega_1 = \frac{\partial \omega}{\partial \omega'}(\omega'_2 - \omega'_1), \quad (A44)$$

which gives $\delta(\omega'_1 - \omega'_2)$ for

$$|A|^2 = \frac{\hbar}{4 \pi \varepsilon_0 c n \omega} \left| \frac{\partial \omega}{\partial \omega'} \right| \quad (A45)$$
and positive frequencies $\omega$ in the laboratory frame. Note that positive frequencies $\omega'$ in the co-moving frame correspond to negative $\omega$ in superluminal regions where the pulse moves faster than the phase-velocity of the probe light.

Hamilton’s equations \[23\] determine the trajectories of light rays in the co-moving frame, parameterized by the pulse-propagation time $\zeta$. Here $\tau$ plays the role of the ray’s position. Comparing the phase \[A40\] with the standard structure of the eikonal in geometrical optics \[4\] or the semiclassical wave function in quantum mechanics \[24\] we notice that $-\omega$ plays the role of the conjugate momentum here. Therefore, we obtain Hamilton’s equations with a different sign than usual \[23\],

$$\dot{\tau} = -\frac{\partial \omega'}{\partial \omega}, \quad \dot{\omega} = \frac{\partial \omega'}{\partial \tau}.$$  \hspace{1cm} (A46)

We express $\dot{\tau}$ in terms of the group index in the laboratory frame. Here the group velocity $v_g$ is the derivative of the frequency $\omega$ with respect to the wave number $n\omega/c$ or, equivalently, the inverse of the derivative of $n\omega/c$ with respect to $\omega$, which gives for the group index $c/v_g$ the standard expression \[2\]

$$n_g = n + \omega \frac{\partial n}{\partial \omega}.$$ \hspace{1cm} (A47)

We obtain from the first of Hamilton’s equations \[A46\] and the Doppler formula \[A42\]

$$\dot{\tau} = \frac{u}{c} n_g - 1 = -\frac{n_g}{c} v'_g, \quad v'_g = \frac{c}{n_g} - u$$ \hspace{1cm} (A48)

where $v'_g$ denotes the difference between the group velocity of the probe $v_g$ and the pulse speed $u$. We see that the velocity $\dot{\tau}$ in the co-moving frame \[A14\] vanishes when the Kerr susceptibility $\chi$ reduces the group velocity $c/n_g$ such that it matches the speed of the pulse $u$. Since $\dot{\omega}$ does not vanish here in general, the ray does not remain there, but changes direction in the co-moving frame.

At such a turning point we expect a violation of the validity of geometrical optics \[24\]. For example, the amplitude \[A45\] would diverge here. Geometrical optics is an exponentially accurate approximation when

$$\left| \frac{\partial T}{\partial \tau} \right| \ll 1 \quad \text{for} \quad T = \frac{2\pi}{\omega},$$ \hspace{1cm} (A49)

as we see from the analogy to the semiclassical approximation in quantum mechanics \[24\]. Here the cycle $T$ plays the role of the wavelength. We get

$$\frac{\partial T}{\partial \tau} = \frac{\omega T}{\omega \tau}.$$ \hspace{1cm} (A50)

Consequently, geometrical optics indeed is no longer valid near a turning point where

$$n_g = \frac{c}{u}.$$ \hspace{1cm} (A51)

This turning point defines a group-velocity horizon where the pulse has slowed down the probe such that it matches the speed of the pulse. At this horizon the incident
mode is converted into a mode that represents another solution of the dispersion relation; a red-or blue-shifted wave, depending on the dispersion and the sign of the first derivative of $\chi$ with respect to $\tau$ at the group-velocity horizon. White holes correspond to increasing $\chi$ and black holes to decreasing $\chi$. White holes blue-shift, because incident waves freeze in front of the horizon, oscillating with increasing frequency. Black holes red-shift, because they stretch any emerging waves (also because black holes are time-reversed white holes). Due to the effective dispersion of the fiber, the refractive index changes with frequency. In turn, the dispersion limits the frequency shifting by tuning the light out of the grip of the horizon. In particular, the dispersion limits the blue-shifting at white-hole horizons to respectable but finite frequencies, considering the tiny magnitude of $\chi$, as we discuss in Sec. B2.

At the event horizons of astrophysical black holes, similar effects are expected when, due to the wave-number divergence, the wavelength of light is reduced below the Planck length scale $\sqrt{\hbar G/c^3}$ where $G$ is the gravitational constant. The physics beyond the Planck scale is unknown. This trans-Planckian physics should regularize the logarithmic phase singularities of modes at the event horizon. A numerical study of a simple model of trans-Planckian-type physics and a systematic analytical study indicate that the Hawking effect of the black hole is not affected. On the other hand, the quantum radiation of white holes is dominated by trans-Planckian physics, because of the extreme blue shift at white-hole horizons. It has been predicted that black-hole white-hole pairs could act as black hole lasers in a regime of anomalous group-velocity dispersion. From a theoretical point of view, trans-Planckian physics regularizes some of the arcane features of quantum black holes and gives a more natural picture of the physics behind the Hawking effect.

In our case, the optical analogue of trans-Planckian physics, optical dispersion, is known in principle and turns out to be to the advantage of the experiment.

### A.9 Classical Hawking effect

A **phase-velocity horizon** is formed if the pulse has slowed down the probe such that its phase velocity is lower than the speed of the pulse. Here an additional effect occurs: the spontaneous creation of photon pairs, Hawking radiation.

In the near ultraviolet around 300nm, the dispersion of microstructured fibers is dominated by the bare dispersion of glass where $n_0$ rapidly grows with frequency, exceeding the group index $c/u$ of the pulse. For such ultraviolet modes, the medium moves at superluminal speed. According to the Doppler formula these superluminal modes oscillate with negative frequencies $\omega'$ in the co-moving frame for positive frequencies $\omega$ in the laboratory frame, and vice versa. On the other hand, probe modes with phase velocities below $c/n$ oscillate with positive frequencies. Therefore, two waves share a given $\omega'$ in the co-moving frame, a subluminal wave with positive frequency $\omega$ in the laboratory frame and a superluminal wave with negative $\omega$, see Fig. A1. In the case of the astrophysical event horizon, the positive-frequency modes correspond to waves outside the horizon that escape into space, and the negative-frequency modes to waves beyond the horizon that fall into the singularity. The Kerr susceptibility of the pulse may slow down the subluminal modes such that the pulse moves at superluminal speed. As we will show, in this
case sub- and superluminal modes are partially converted into each other. This mode conversion is at the heart of the Hawking effect.

For simplicity, we consider a single white-hole horizon, not the combination of black- and white-hole horizons generated by a moving pulse. We will argue later that in practice the white-hole will dominate the Hawking effect, which \textit{a-posteriori} justifies this simplification. Suppose, without loss of generality, that at $\tau = 0$ the Kerr-reduced phase-velocity of the probe, $c/n$, matches the group velocity of the pulse $u$. We assume that the mode conversion occurs near this point and expand the Kerr susceptibility $\chi$ as a linear function in $\tau$,

\[
\chi(\tau) = \chi_h + \alpha'' \tau, \quad \alpha'' = \frac{\partial \chi}{\partial \tau} \bigg|_{0} .
\] (A52)

The group velocity of the incident probe is much lower than the pulse speed $u$ and so both the sub- and the superluminal probe travels from the front of the pulse to the back, from negative to positive retarded time $\tau$. For a white-hole horizon $\chi$ increases for decreasing retarded time, and so $\dot{\chi}(0) < 0$.

We proceed similar to Ref. \cite{27} and focus on the conversion region where we Fourier-transform with respect to $\tau$ the wave equation (A16) with the refractive index (A17) for stationary waves in the co-moving frame and using the linear expansion (A52). The frequency conjugate to $\tau$ is the laboratory-frame frequency $\omega$. We replace $\tau$ by $-i\partial_\omega$, $\partial_\tau$ by $-i\omega'$ and $\partial_\tau$ by $-i\omega$, denote the Fourier-transformed vector potential by $\tilde{A}$, and obtain

\[
(n_0^2 + \chi_h - i\alpha'' \partial_\omega) \omega \tilde{A} = \left(1 - \frac{\omega'}{\omega} \right)^2 \frac{c^2}{u^2} \omega \tilde{A} .
\] (A53)
This first-order equation has the exact solution
\[
\tilde{A} = \tilde{A}_0 \frac{e^{-i\phi}}{\omega}, \quad \phi = -\frac{1}{\alpha''} \int \left(1 - \frac{\omega'}{\omega}\right)^2 \frac{c^2}{u^2} - n_0^2 - \chi \right) d\omega \tag{A54}
\]
with constant \( \tilde{A}_0 \). We introduce
\[
\alpha' \equiv -\frac{u^2}{c^2} \frac{\alpha''}{2} \approx -\frac{u}{c} \frac{\dot{\chi}(0)}{2n_0} = -\frac{u}{c} \frac{\partial n}{\partial \tau} \bigg|_0 \tag{A55}
\]
in agreement with Eq. (4) of our paper [1]. Note that the phase \( \phi \) contains a logarithmic contribution,
\[
\phi = -\frac{\omega'}{\alpha'} \ln \omega + \phi_0(\omega). \tag{A56}
\]
This logarithmic asymptotics of the phase will lead to the characteristic mode conversion at the group-velocity horizon. In order to see this, we Fourier-transform \( \tilde{A} \) back to the domain of the retarded time,
\[
A = \int_{-\infty}^{+\infty} \tilde{A} e^{-i\omega \tau} d\omega = \int_{-\infty}^{+\infty} \frac{\tilde{A}_0}{\omega} e^{-i\phi - i\omega \tau} d\omega \tag{A57}
\]
and use the saddle-point approximation, \( i.e. \) we quadratically expand the phase \( \phi + \omega \tau \) around the stationary points where \( \partial_{\omega}(\phi + \omega \tau) \) vanishes and perform the integration as Gaussian integrals along the direction of steepest descent. One easily verifies that the stationary points are the solutions of the dispersion relation \( A41 \). We denote the two solutions by \( \omega_{\pm} \) indicating their sign. We obtain for the second derivative in the quadratic expansion
\[
\partial^2_{\omega}(\phi + \omega \tau) = -\frac{2}{\chi(0)} \frac{n}{\omega} \left( \frac{c}{u} - n_g \right). \tag{A58}
\]
The Gaussian integrals at \( \omega_{\pm} \) are proportional to the inverse square root of \( \partial^2_{\omega}(\phi + \omega \tau) \). We see from Eqs. \( A46 \) and \( A48 \) that they are consistent with the amplitudes \( A45 \) of geometrical optics. Consequently, we obtain a superposition of the two waves \( A39 \) that correspond to the two physically-relevant branches of the dispersion relation \( A41 \). We denote the positive-frequency wave by \( A_+ \) and the negative-frequency component by \( A_-^* \). The star indicates that this component resembles the complex conjugate of a mode, because a mode predominantly contains positive laboratory-frame frequencies, according to the normalization \( A45 \). The coefficient of \( A_-^* \) is given by the exponential of the phase integral from the positive branch \( \omega_+ \) to the negative frequency \( \omega_- \) on the complex plane. The amplitude of the coefficient is the exponent of the imaginary part of the phase integral, while the phase of the coefficient is given by the real part. We can incorporate the phase of the superposition coefficient in the prefactor \( A45 \), but not the amplitude. The imaginary part of the phase integral comes from the logarithmic term \( A56 \), giving \( \pi \omega'/\alpha' \). Therefore, the relative weight of the negative-frequency component in the converted mode is \( \exp(-\pi \omega'/\alpha') \). We thus obtain for \( \tau < 0 \)
\[
A \sim Z^{1/2} \left( A_+ + A_-^* e^{-\pi \omega'/\alpha'} \right) \tag{A59}
\]
where $Z$ denotes a constant for given $\omega'$. We determine the physical meaning of $Z$ in Sec. A10, but here we can already work out its value by the following procedure: consider a wavepacket with co-moving frequencies around $\omega'$ that crosses the horizon. Suppose that this wavepacket is normalized to unity. After having crossed the horizon, the norm of the positive-frequency component is $Z$, while the negative-frequency component has the negative norm $-Z \exp(-2\pi \omega'/\alpha')$. The sum of the two components must give unity, and so

$$Z = \left(1 - e^{-2\pi \omega'/\alpha'}\right)^{-1}.$$  \hspace{1cm} (A60)

We represent $Z^{1/2}$ as $\cosh \xi$ and obtain from Eq. (A60) that $\sinh^2 \xi = Z - 1$ gives $Ze^{-2\pi \omega'/\alpha'}$,

$$Z^{1/2} = \cosh \xi, \quad Z^{1/2}e^{-\pi \omega'/\alpha'} = \sinh \xi.$$ \hspace{1cm} (A61)

Consequently, the incident wave $A_\pm$ is converted into the superposition $A_\pm \cosh \xi + A_\mp^* \sinh \xi$ when it crosses the horizon from positive to negative $\tau$. Hence we obtain for this process the mode

$$A_{\pm \text{out}} \sim \begin{cases} A_\pm \\ A_\pm \cosh \xi + A_\mp^* \sinh \xi \end{cases} : \begin{cases} \tau > 0 \\ \tau < 0 \end{cases}.$$ \hspace{1cm} (A62)

Equation (A62) describes the fate of a classical wave that crosses the horizon. A negative-frequency component is generated with weight $\sinh^2 \xi$ relative to the initial wave, but, since $\cosh \xi > 1$, the positive-frequency wave has been amplified. The mode conversion at the horizon is thus an unusual scattering process where the concerted modes are amplified, at the expense of the energy of the driving mechanism, the pulse in our case. (It is also mathematically unusual — the Hawking effect corresponds to scattering without turning points in the complex plane.) Wherever there is amplification of classical waves, \textit{i.e.} stimulated emission of waves, there also is spontaneous emission of quanta [29] — in the case of horizons, Hawking radiation.

### A.10 Hawking radiation

Suppose that no classical probe light is incident; the modes $A_{\pm \text{in}}$ are in the vacuum state. The incident modes are characterized by the asymptotics $A_\pm$ for $\tau > 0$ while outgoing modes are required to approach $A_\pm$ for $\tau < 0$. We perform the superposition

$$A_{\pm \text{out}} = A_{\pm \text{in}} \cosh \xi - A_{\mp \text{in}}^* \sinh \xi$$ \hspace{1cm} (A63)

and see that $A_{\pm \text{out}}$ obeys the asymptotics

$$A_{\pm \text{out}} \sim \begin{cases} A_\pm \cosh \xi - A_\mp^* \sinh \xi \\ A_\pm \\ A_\mp \end{cases} : \begin{cases} \tau > 0 \\ \tau < 0 \end{cases}.$$ \hspace{1cm} (A64)

as required for outgoing modes. The modes (A62) and (A63) describe two sets of mode expansions (A30) of one and the same quantum field; for a given $\omega'$ the sum
of $A_{\pm \text{in}}\hat{a}_{\pm \text{in}}$ and $A^*_{\pm \text{in}}\hat{a}^\dagger_{\pm \text{in}}$ over the two signs $\pm$ of $\omega$ must give the corresponding sum of $A_{\pm \text{out}}\hat{a}_{\pm \text{out}}$ and $A^*_{\pm \text{out}}\hat{a}^\dagger_{\pm \text{out}}$. Consequently,

$$\hat{a}_{\pm \text{in}} = \hat{a}_{\pm \text{out}} \cosh \xi - \hat{a}^\dagger_{\pm \text{out}} \sinh \xi$$  \hspace{1cm} (A65)$$

and by inversion

$$\hat{a}_{\pm \text{out}} = \hat{a}_{\pm \text{in}} \cosh \xi + \hat{a}^\dagger_{\pm \text{in}} \sinh \xi.$$  \hspace{1cm} (A66)$$

The vacuum state $|\text{vac}\rangle$ of the incident field is the eigenstate of the annihilation operators $\hat{a}_{\pm \text{in}}$ with zero eigenvalue (the state that the $\hat{a}_{\pm \text{in}}$ annihilate),

$$\hat{a}_{\pm \text{in}}|\text{vac}\rangle = 0.$$  \hspace{1cm} (A67)$$

To find out whether and how many quanta are spontaneously emitted by the horizon, we express the in-coming vacuum in terms of the out-going modes. We denote the out-going photon-number eigenstates, the out-going Fock states [9], by $|n_+, n_-\rangle$ with the integers $n_\pm$. Using the standard relations for the annihilation and creation operators

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle,$$  \hspace{1cm} (A68)$$

one verifies that $\hat{a}_{\pm \text{in}}$ vanishes for the state

$$|\text{vac}\rangle = Z^{-1/2} \sum_{n=0}^{\infty} e^{-n\pi\omega'/\alpha'}|n, n\rangle.$$  \hspace{1cm} (A69)$$

This is the remarkable result obtained by Hawking [13] for the horizon of the black hole. First, it shows that the event horizon spontaneously generates radiation from the incident quantum vacuum. Second, the emitted radiation consists of correlated photon pairs, each photon on one side is correlated to a partner photon on the other side, because they are always produced in pairs. The total quantum state turns out to be an Einstein-Podolski-Rosen state [9], the strongest entangled state for a given energy [30]. Third, light on either side of the horizon consists of an ensemble of photon-number eigenstates with probability $Z^{-1}e^{-2n\pi\omega'/\alpha'}$. This is a Boltzmann distribution of $n$ photons with energies $n\hbar\omega'$ and temperature $k_B T' = \hbar\alpha'/(2\pi)$. Consequently, the horizon emits a Planck spectrum of black-body radiation with the Hawking temperature (5) of our paper [1]. Fourth, this Planck spectrum is consistent with Bekenstein’s black-hole thermodynamics [15]: black holes seem to have an entropy and a temperature.

In our case, the spectrum of the emitted quanta is a Planck spectrum for the frequencies $\omega'$ in the co-moving frame, as long as a phase-velocity horizon exists. We performed our analysis for the white-hole horizon, but, since black holes are time-reversed white holes, we arrive at the same result for the black hole as well, except that the roles of the incident and outgoing modes are reversed. In the laboratory frame, the spectrum is given by the dependance of $\omega'$ on the laboratory frequency $\omega$ outside of the pulse, i.e. by the dispersion relation (A42) for $\chi = 0$. In our case, $\omega(\omega')$ is single-valued for the spectral region where phase-velocity horizons are established, see Fig. [A1] and so the spectrum of black- and white-hole horizons is identical for identical $\alpha'$. We have shown in Eq. (8) that this mapping from $\omega$ to
\( \omega' \) amounts to a re-definition of the Hawking temperature: in the laboratory frame 
\( k_B T \) is given by the logarithmic derivative of \( \chi \) at the horizon; it is independent 
of the magnitude of the Kerr susceptibility, as long as a phase-velocity horizon is 
established. The particle-production rate depends only on the sharpness of the pulse. 
This important feature makes the experimental observation of Hawking radiation in 
optical fibers feasible using few-cycle pulses \[31\].

### A.11 Optical shock

Another feature increases the Hawking temperature further: the formation of an 
optical shock-front at the trailing end of the pulse. For a sufficiently intense pulse, 
the Kerr effect strongly influences its shape; it counteracts the dispersion of the 
pulse, forming a soliton \[2\] and it may lead to self-steepening \[32\]. Consider the 
propagation of the pulse itself. We describe its electric field \( E \) by the envelope \( \mathcal{E} \) and the phase at the carrier frequency \( \omega_0 \),

\[
E = \mathcal{E} \exp (i\beta_0 z - i\omega_0 t) , \quad \beta_0 = n_0(\omega_0) \frac{\omega_0}{c} , \quad I = |\mathcal{E}|^2 .
\]  

(A70)

Assuming that the envelope varies over longer scales than an optical cycle, we approxi-
mate in the wave equation \( \text{(A10)} \) the differential kernel and the Kerr polariza-
tion \( \text{(A4)} \) by

\[
\begin{align*}
(\partial_z^2 + \beta^2) E & \approx \exp (i\beta_0 z - i\omega_0 t) 2\beta_0 (i\partial_z + \beta - \beta_0) \mathcal{E} , \\
\partial_t^2 P & \approx -2i\varepsilon_0 \kappa \omega_0 E (-i\omega_0 I + 3\partial_t I) .
\end{align*}
\]  

(A71)

The dominant nonlinear term in the resulting wave equation is proportional to the 
intensity \( I \); the term proportional to \( \partial_t I \) only becomes important for sharp features 
and, as we will see, leads to the formation of an optical shock.

First we ignore the shock term and approximate \( \beta(\omega) \) around the carrier fre-
cquency \( \omega_0 \) by a quadratic polynomial,

\[
\beta \approx \beta_0 + \beta_1 (\omega - \omega_0) + \frac{\beta_2}{2} (\omega - \omega_0)^2 .
\]  

(A72)

Here \( \beta_1 \) describes the inverse of the group velocity \( u \) and \( \beta_2 \) the group-velocity 
dispersion. Substituting \( i\partial_t \) for \( \omega - \omega_0 \) wherever it acts on the envelope \( \mathcal{E} \) we obtain 
the usual nonlinear Schrödinger equation of Nonlinear Fiber Optics \[2\]

\[
i \left( \partial_t + \frac{1}{\beta_1} \partial_z \right) \mathcal{E} - \frac{\beta_2}{2\beta_1} \partial_t^2 \mathcal{E} + \frac{\omega_0^2 \kappa}{c^2 \beta_0 \beta_1} |\mathcal{E}|^2 \mathcal{E} = 0 .
\]  

(A73)

This equation is integrable by the Inverse Scattering Method \[33\]. For anomalous 
group-velocity dispersion, where \( \beta_2 < 0 \), it has the fundamental soliton solution

\[
\mathcal{E} = \mathcal{E}_0 \text{sech} \left( \frac{\tau}{T_0} \right) \exp \left( i \frac{|\beta_2| \zeta}{2\beta_1 T_0^2} \right) , \quad \mathcal{E}_0^2 = \frac{c^2 \beta_0 |\beta_2|}{\kappa \omega_0^2 T_0^2}
\]  

(A74)

in terms of the retarded time \( \tau \) and the propagation time \( \zeta \) according to the definition
\( \text{(A14)} \). The constant \( T_0 \) describes the duration of the soliton.
Consider now shock formation. For simplicity, we ignore the group-velocity dispersion and arrive at the nonlinear wave equation

$$\partial_\zeta \mathcal{E} + \gamma \mathcal{E} (-i\omega_0 I + 3\partial_\tau I) = 0, \quad \gamma = \frac{\omega_0 \kappa}{c^2 \beta_0 \beta_1}. \quad (A75)$$

We represent $\mathcal{E}$ in terms of the amplitude $\sqrt{I}$ and a phase, and obtain that the evolution of the phase is completely determined by the amplitude, whereas the intensity obeys the transport equation

$$(\partial_\zeta + 6\gamma I \partial_\tau) I = 0. \quad (A76)$$

The intensity is transported with velocity $6\gamma I$ where the retarded time $\tau$ plays the role of space and the propagation time $\zeta$ the role of time. In a $(\tau, \zeta)$ space-time diagram, the initial intensity profile $I_0(\tau_0)$ is thus transported along a line with steepness $6\gamma I(\tau_0)$. Hence we write down the solution as

$$I = I_0(\tau_0), \quad \tau - 6\gamma I_0(\tau_0) \zeta = \tau_0. \quad (A77)$$

The relationship between $\tau$, $\zeta$ and the initial intensity profile implicitly determines $\tau_0$. At some $\tau$ and $\zeta$, two lines of transported intensity, one belonging to $\tau_0$ and the other to $\tau'_0$, may cross, developing a discontinuity, a shock. Here

$$\frac{1}{\zeta} = \frac{6\gamma [I_0(\tau_0) - I_0(\tau'_0)]}{\tau'_0 - \tau_0} \rightarrow -6\gamma \frac{dI_0}{d\tau_0} \quad \text{for} \quad \tau_0 \rightarrow \tau'_0. \quad (A78)$$

The shock time $\zeta$ is smallest for the largest negative derivative of the initial intensity profile. So the steepest point at the trailing end of the initial pulse is the first to form a shock. Although we ignored the dispersion and Stimulated Raman Scattering \cite{2} in this simple theory of optical shocks \cite{32} ultrashort pulses still form sharp features at their trailing end, features that may easily become comparable to the carrier wavelength, as we and others have seen in numerical simulations using the method of Ref. \cite{34}. Therefore, the white-hole horizon will dominate the Hawking radiation of an optical pulse and the radiation is likely to become strong enough to be detectable.
B  Experiment

In this appendix we describe the experimental observation of frequency shifting of light at the group velocity horizon. To our knowledge, such an experiment has never been carried out before; the cases closest to our scheme are demonstrations of pulse trapping [35, 36] where the dynamics are dominated by the Raman effect or pulse compression in a fiber grating (optical push broom) [37]. Based on the theory of Appendix A, we also derive mathematical expressions for the amount of blue shifting, for the spectral shape, and for estimating the efficiency of this process. We discuss the experimental proceedings and findings and compare them with the theory.

B.1  Dispersion

The creation of artificial event horizons in optical fibers critically depends on the optical properties of these fibers. These properties are described here and summarized in Table B.1. To create an artificial event horizon in our scheme, an intense optical pulse has to be formed inside the fiber. Optical solitons [38, 2] offer a unique possibility for nondispersive stable pulses in fibers. These can be ultrashort, allowing for very high peak powers to drive the nonlinearity of the fiber. Bright solitons only exist for anomalous group velocity dispersion [2]. Microstructured fibers [3] have an arrangement of holes close to the fiber core along the fiber. In the simplest picture, the holes lower the local refractive index in the transverse plane of the fiber, leading to substantially larger index variations compared to conventional fibers. Hence a very wide range of transverse refractive index profiles can be engineered. There are various designs for the shape and location of the holes, leading to a range of different effective dispersions and giving rise to a variety of applications [3]. In particular, the anomalous dispersion required for solitons can be generated at wavelengths reaching the visible.

The dispersion parameter $D$ of optical fibers is defined as the change of group delay per wavelength change and propagation length. Its units are usually ps/(nm km). Since the group delay per propagation length is given by $n_g/c$ and $n_g/c = \partial \beta / \partial \omega$, we have [2]

$$D = \frac{\partial^2 \beta}{\partial \lambda \partial \omega}, \quad \lambda = \frac{2\pi c}{\omega}. \quad \text{(B1)}$$

The group-velocity dispersion is often also characterized by the second derivative of $\beta$ with respect to the frequency $\omega$,

$$\beta_2 = \frac{\partial^2 \beta}{\partial \omega^2} = -\frac{\lambda}{\omega} D. \quad \text{(B2)}$$

The group-velocity dispersion is normal for positive $\beta_2$ and negative $D$, and anomalous for negative $\beta_2$ and positive $D$.

For the creation of a horizon we chose a commercial microstructured fiber, model NL-PM-750B by Crystal Fiber A/S. Figure B.1 shows the dispersion of the particular fiber sample we used. The red curve is a measurement provided by Crystal Fiber; the dotted line was measured for our fiber sample by Alexander Podlipensky and Philip
Figure B1: Two measurements of the dispersion parameter $D$ for the fiber used in the experiments. Red line: company data; dotted line: result of Alexander Podlipensky and Philip Russell at the Max Planck Research Group in Optics, Information and Photonics in Erlangen, Germany. If the total shaded area vanishes, the two wavelengths at either end are group velocity matched.

Russell at the Max Planck Research Group in Optics, Information and Photonics in Erlangen, Germany. The fiber dispersion is anomalous between $\approx 740 \text{nm}$ and $\approx 1235 \text{nm}$ wavelength and normal otherwise. Thus solitons can be created using ultrashort pulses from modelocked Ti:Sapphire lasers. Light that would probe the horizon and experience blue shifting as a result, will have to be slowed down by the Kerr effect of the pulse such that its group velocity matches the speed $u$ of the pulse. The Kerr susceptibility is small (we give an estimate in Sec. B2), and so the initial group velocity of the probe should be only slightly higher than $u$. Integrating Eq. (B1) we obtain

$$\int_{\lambda_0}^{\lambda} D d\lambda = \beta_1(\lambda) - \beta_1(\lambda_0) = \frac{1}{v_g(\lambda)} - \frac{1}{v_g(\lambda_0)}, \quad v_g(\lambda_0) = u. \quad (B3)$$

Here $\lambda_0$ and $\lambda$ denote the center wavelengths of the pulse and the probe light, respectively. Therefore, the probe light travels at the speed of the pulse if the integral of $D$ vanishes, as illustrated by the shaded areas in Fig. B1. This probe wavelength is called the group-velocity-matched wavelength $\lambda_m$ and the corresponding frequency $\omega_m$ the group-velocity-matched frequency. For a pulse carrier-wavelength of 800 nm and the fiber used here we obtain $\lambda_m \approx 1500 \text{nm}$. This value of $\lambda_m$ is useful, because on the one hand it is a standard wavelength for lasers and optical equipment made for fiber-optical telecommunications and on the other hand it is clearly separated from our broadband pulsed light.

The dispersion $D$ essentially describes the effective group index of the fiber as a function of wavelength (or frequency). Integrating Eq. (B1) twice we obtain the
Table 1: Properties of fiber NL-PM-750B. Dispersion data according to Crystal Fiber \ Alexander Podlipensky and Philip Russell, Max Planck Research Group in Optics, Information and Photonics in Erlangen. Nonlinearity according to Crystal Fiber. The birefringence $\Delta n$ and the fiber length $L$ were measured by the authors. The symbols used are defined in the text.

| Property                                               | Crystal Fiber \ Erlangen                  |
|--------------------------------------------------------|------------------------------------------|
| Dispersion $D_0$                                       | 28\ 36 ps/(nm km)                       |
| Dispersion $D_m$                                       | −150\ −180 ps/(nm km)                   |
| Third order dispersion $dD_m/d\lambda$                 | −0.6\ −0.75 ps/(nm$^2$km)               |
| Dispersion $\beta_2(\lambda_0)$                       | −9.5\ −12 ps$^2$/km                    |
| Dispersion $\beta_2(\lambda_m)$                       | 180\ 210 ps$^2$/km                     |
| Group velocity-matched wavelength $\lambda_m$         | 1508\ 1494 nm                           |
| Birefringence $\Delta n_0\ \Delta n_m$               | 7.5\ 5.7 \times 10^{-4}               |
| Nonlinearity $\gamma$ (780 nm)                        | 0.1 W$^{-1}$m$^{-1}$                     |
| Fiber length $L$                                       | 1.5 m                                   |

effective wavenumber (A11)

$$
\beta = \beta(\omega_0) + \frac{\omega - \omega_0}{u} + \int_{\omega_0}^{\omega} \int_{\lambda_0}^{\lambda} D d\lambda d\omega = n \frac{\omega}{c}
$$

$$
= \beta(\omega_0) + \frac{\omega - \omega_0}{u} + \int_{\omega_0}^{\omega} \int_{\omega_0}^{\omega} \beta_2 d\omega d\omega,
$$

(B4)

where $n$ is the linear effective refractive index of the fiber and $\omega_0$ and $\omega$ denote the carrier frequency of pulse and probe, respectively. So, in addition to the measured dispersion curve, only two constants determine $\beta$: the pulse group velocity $u$ at $\omega_0$ and $\beta(\omega_0)$. However, in what follows the most relevant parameters for describing our experiment are independent of these constants.

In general, the two polarization modes of the fiber have slightly different propagation constants $\beta$. This birefringence creates a refractive index difference of $\Delta n$ between the polarization modes. In the fiber chosen in this experiment, the holes form a hexagonal pattern that is slightly distorted to break the hexagonal symmetry. Our fiber exhibits strong birefringence $\Delta n$ of a few times $10^{-4}$. This leads to non-negligible changes in the group velocity.

**B.2 Frequency shifts**

Consider the frequency shifts at a group velocity horizon. During the pulse-probe interaction, the co-moving frequency $\omega'$ is a conserved quantity and so the probe frequency $\omega$ follows a contour line of $\omega'$ as a function of the nonlinear susceptibility $\chi$ induced by the pulse, see Fig. B2. The maximal $\chi$ experienced by the probe is proportional to the maximal nonlinear susceptibility $\chi_0$ experienced by the pulse: assuming perfect mode overlap of pulse and probe, $\chi_{\text{max}}$ reaches $2\chi_0$ when the probe and the pulse are co-polarized and $2\chi_0/3$ when they are cross-polarized, see Sec. A1.
If the pulse is a soliton, we obtain from Eq. \([A74]\) the peak susceptibility
\[
\chi_0 = \frac{2n_0 c \lambda_0 D_0}{(\omega_0 T_0)^2} = \frac{2n_0 c |\beta_2(\lambda_0)|}{\omega_0 T_0^2}
\] (B5)

where \(D_0\) denotes the dispersion parameter at the carrier wavelength \(\lambda_0\). For example, for a soliton \([A74]\) at \(\lambda_0=800\) nm whose full width at half maximum (FWHM) is 70 fs (corresponding to \(T_0=40\) fs), for \(n_0=1.5\), \(D_m=30\) ps/(nm km) the peak susceptibility \(\chi_0\) is as low as \(2 \times 10^{-6}\). Nevertheless, we show that this small variation in the optical properties is sufficient to generate a significant wavelength shift at the horizon.

![Doppler contours](image)

**Figure B2:** Doppler contours. The pulse shifts the laboratory frequency \(\omega\) (or the wavelength \(\lambda\)) along the contour lines of \(\omega'\) as the function \([B6]\) of the refractive-index change \(\delta n = \chi/(2n_0)\). For a sufficiently intense pulse \(\delta n\) reaches the top of a contour. In this case the probe light completes an arch on the diagram while leaving the pulse; it is red- or blue-shifted, depending on its initial frequency.

We obtain the contours of \(\omega'\) from the Doppler formula \([A42]\). We use relations \([A18]\) and \([B4]\), but integrate from the group velocity-matching point,
\[
\omega' = \omega'_m - u \int_{\omega_m}^{\omega} \frac{\beta_2}{\omega} d\omega - \frac{\chi}{2n_0} \frac{u}{c} \omega
\]
\[
\approx \omega'_m - \frac{u}{2} \beta_2(\omega_m)(\omega - \omega_m)^2 - \frac{\chi}{2n_0} \frac{u}{c} \omega_m
\]
(\text{B6})
\[
\approx \omega'_m + \frac{\pi u}{\lambda_m} \left( \frac{D_m c (\lambda - \lambda_m)^2}{\lambda_m} - \frac{\chi}{n_0} \right)
\]
(\text{B7})

The contours of \(\omega'\) do not depend on \(\omega'_m\), nor on the scaling factor \(\pi u/\lambda_m\). Because \(D_m = D(\lambda_m) < 0\), they form inverted parabolas with a maximum at \(\lambda_m\) for the
corresponding $\chi_{\text{max}}$. They intersect the axis of zero $\chi$ at the incident and the emerging wavelengths. Here $|D_m|c(\lambda - \lambda_m)^2/\lambda_m$ equals $\chi_{\text{max}}/n_0$, and so we get

$$\lambda = \lambda_m \pm \delta \lambda, \quad \delta \lambda = \sqrt{\frac{\lambda_m \chi_{\text{max}}}{|D_m| n_0 c}}. \quad (B8)$$

Using again that the pulse is a soliton, we obtain

$$\delta \lambda = \frac{\sqrt{2r \lambda_0}}{T_0 \sqrt{\omega_0 \omega_m}} \sqrt{|D_0|} = \frac{\sqrt{2r \lambda_m \lambda_0}}{T_0 \omega_m} \sqrt{\frac{\beta_2(\lambda_0)}{\beta_2(\lambda_m)}} \quad (B9)$$

with $r=2$ for co-polarized and $r=2/3$ for cross-polarized pulse and probe light. According to Fig. B2 the probe light can maximally be wavelength-shifted from $+\delta \lambda$ to $-\delta \lambda$ over the range $2\delta \lambda$. For the soliton mentioned above the group velocity dispersion $D_0$ is about 30 ps/(nm km). Using $\lambda_m \approx 1500$ nm and $D_m \approx -160$ ps/(nm km), the wavelength shift $2\delta \lambda$ is 20 nm in the co-polarized case and $2\delta \lambda=12$ nm in the cross-polarized case.

We also derive a simple estimate of the efficiency of the frequency shifting from the dispersion data. The probe light that is colliding with the pulse undergoes frequency conversion at the horizon. However, because the group velocities of the probe $v_g$ and of the pulse $u$ are similar, only a small fraction of the total probe light can be converted within the finite length of the fiber. The pulse and the slightly faster probe light travel through the fiber in $t=L/u$ and $t_p=L/v_g$ with $t > t_p$. The time difference multiplied with the probe power $P_{\text{probe}}$ is the energy $E_{\text{coll}}$ converted by pulse collision: $E_{\text{coll}}=P_{\text{probe}} L(1/u - 1/v_g)$. Therefore, the fraction $\eta$ of probe power that is frequency converted is

$$\eta = \nu_{\text{rep}} L \left(1/u - 1/v_g\right) \approx \nu_{\text{rep}} L |D_m| \delta \lambda, \quad (B10)$$

where $\nu_{\text{rep}}$ is the repetition rate of the pulses and $1/v_g=\partial \beta/\partial \omega \approx 1/u + D_m \delta \lambda$ was used. For $L=1.5$ m and $\nu_{\text{rep}}=80$ MHz the maximal conversion efficiency $\eta$ is on the order of $10^{-4}$.

### B.3 Experimental results

The experiment is arranged as displayed in Fig. B3. A modelocked Ti:Sapphire laser (Mai Tai, Spectra Physics) delivers 70-fs pulses (FWHM) in the near infrared (NIR). These linearly polarized pulses are coupled to either one of the principal axes of the microstructured fiber of length $L=1.5$ m. The polarization is rotated by a half-wave plate. Note that the polarizing beam splitter (PBS) only acts on the probe light. At the fiber output temporal autocorrelation traces and spectra are taken to determine the pulse energy necessary to create a fundamental soliton. For the center wavelength of 803 nm, a dispersion $D_0=30$ ps/(nm km) and a nonlinearity $\gamma$ of 0.1 W$^{-1}$m$^{-1}$, 70-fs solitons are generated at 5 pJ pulse energy corresponding to 400 $\mu$W average power for the repetition rate $\nu_{\text{rep}}=80$ MHz.

The output pulse length equalled the 70-fs input pulse length at an input power of approximately 320 $\mu$W. This indicates that a soliton has formed. The observed
power of 320 \( \mu \text{W} \) in comparison with the predicted power of 400 \( \mu \text{W} \) illustrates the uncertainty in the actual fiber dispersion and nonlinearity. The observed Raman-induced soliton self-frequency shift \( \lesssim 4 \text{ nm} \) was. Note that this shift decelerates the pulse and hence is changing the group velocity-matched wavelength \( \lambda_m \) in the infrared (IR). To calculate how much \( \lambda_m \) is shifted, we use Eq. (B3), replacing \( \lambda_0 \) and \( \lambda \) with \( \lambda_0 + \delta \lambda_0 \) and \( \lambda_m + \delta \lambda_r \) and linearize. In this way we get

\[
\delta \lambda_r = \frac{D_0}{D_m} \delta \lambda_0 .
\]  

For the dispersion data shown in Fig. B1, a wavelength change of 4 nm of the pulse changes \( \lambda_m \) by \( \delta \lambda_r = -0.75 \text{ nm} \). Since the probe light is wavelength-shifted symmetrically around \( \lambda_m \), there is a change of the wavelength shift of up to \(-1.5 \text{ nm}\).

The probe light is derived from a tunable external grating diode laser (Lynx Series, Sacher Lasertechnik). It delivers up to 20 mW of continuous-wave light, tunable from 1460 to 1540 nm. The probe light is reflected off a diffraction grating to reduce fluorescence emitted near lasing bandwidth. With another half-wave plate the probe light is coupled into the fiber onto one of the principal axes. Depending on wavelength, 100 to 600 \( \mu \text{W} \) of probe power were coupled through the fiber. After the fiber we use a dichroic optic to filter out all of the pulse light and couple the IR light into a single-mode fiber connected to an optical spectrum analyzer.

Figure B4 shows a typical output spectrum. This spectrum was taken with pulse and probe aligned to the slow axis of the fiber. At \( \lambda = 1506 \text{ nm} \) the diode-laser input
Figure B4: Spectrum of the blue-shifted light for initial probe of wavelength 1506 nm. Pulse and probe are co-polarized along the slow axis of our fiber. Traces with (green) and without (black) probe light are shown on a logarithmic scale. They are subtracted on a linear scale to obtain the normalized signal (red) displayed in parts per million.

A line is visible as a strong signal. From $\lambda = 1502$ nm to $\lambda = 1510$ nm we detect residual weak spontaneous emission from the laser that was not completely eliminated by the diffraction grating. Traces with and without pulses present in the fiber are taken and subtracted, leading to the signal displayed on a linear scale (red color). The signal is normalized by the amount of probe power and by the resolution bandwidth of 0.5 nm and is given in parts per million (ppm). With the pulses present, a clear peak appears on the blue side of the input probe light near 1493 nm. Since the blue-shifted light is generated from the part of the probe light that overlapped with the pulse during fiber propagation, it constitutes itself a pulse of finite length. Hence, this length is determined by the relative group velocity of probe light and the pulse, see for example Eq. (B10). In turn, the unshifted probe light is partially depleted, forming a gap in intensity. These features lead to a spectral broadening of both the shifted and unshifted probe light by a few nanometers.

From the measurements shown in Fig. B4, the efficiency of the blue-shifting is $1.1 \times 10^{-5}$, less than the estimated $10^{-4}$. This indicates that a significant part of the probe light tunnels through the pulse; the pulse is too short to establish a nearly perfect barrier. In the tunneling region of the pulse the laboratory frequency $\omega$ is purely imaginary. In order to estimate the maximal imaginary part of $\omega$ we consider the extreme case where the initial frequency of the probe reaches the group-velocity-matched frequency $\omega_m$ characterized by $\omega' = \omega'_m$. We solve Eq. (B6) for $\omega$ and obtain

$$\text{Im}\omega = \frac{\chi \omega_m}{n_0 \beta_2(\omega_m)c} = \frac{\sqrt{\chi} \omega_m}{\sqrt{n_0 c \lambda_m |D_m|}}.$$  \hspace{1cm} (B12)

Assuming $\chi \approx 2 \times 10^{-6}$ at the soliton peak, $n_0 \approx 1.5$, $\lambda_m = 1500$ nm and $D_m =$
−160 ps/(nm km) the imaginary part of ω reaches about 5 THz. This is insufficient to significantly suppress tunneling through a 70-fs pulse, because the product of Imω and T₀ is much smaller than unity. For longer pulses we would expect perfect frequency conversion at the horizon.

Increasing the probe wavelength further away from λₘ is shifting light further to the blue side of the spectrum, because the wavelength shifts symmetrically around the group velocity-matched wavelength, according to Eq. (B8) and Fig. B2. Figure B5 displays the spectra of shifted light for three detunings of the probe light from the group velocity-matched wavelength λₘ. As expected, the spectra move towards shorter wavelengths by the same amount as the probe laser was tuned towards longer wavelengths.

![Figure B5: Spectra for different input probe wavelengths. Since the probe mode is mirrored around the group-velocity-matched wavelength λₘ, increasing probe wavelengths experience increasing blue shifting, as is also illustrated by the contours of Fig. B2.](image)

Figure B6 shows how the signal strength, the spectrum integrated over the signal peak, evolves with increasing probe power. A clear linear dependence is evident in agreement with our theoretical model and the superposition principle for the probe light. The figure illustrates that the probe indeed is a probe, not influencing the pulses via nonlinear effects.

Changing the input polarizations changes the group velocities of pulse and probe and therefore the group velocity-matched wavelength λₘ shifts by an amount δλₘ. If we change for example the pump polarization from the fast to the slow axis, the
inverse group velocity $\beta_1(\lambda_0)$ increases as $n_{g0}/c$ is replaced by $(n_{g0} + \Delta n_0)/c$. To maintain group velocity matching, $\beta_1(\lambda_m)$ has to change accordingly by $\Delta n_0/c$. We use Eq. (B3), linearizing around $\lambda_m$, and get

$$\Delta n_0/c \approx D_m \delta \lambda_m.$$  \hfill (B13)

For $D_m = -160 \text{ ps}/(\text{nm km})$ and $\Delta n_0 = 7.5 \times 10^{-4}$ we obtain $\delta \lambda_m \approx -16 \text{ nm} \approx 2\delta \lambda$. This means that, when changing polarizations, the probe laser has to be retuned to a wavelength where frequency shifting can be observed.

Figure B7 shows spectra for all four different polarization combinations. As expected, the group velocity-matched wavelength changes. Note that there also is a difference in $\lambda_m$ for the two co-polarized cases, indicating small changes in the dispersion profile for the two polarization axes, a dispersion of the birefringence.

### B.4 Calculation of spectra

Here we derive a functional expression for the spectra of the frequency-shifted probe light, applying the theory developed in Appendix A. We focus on the part of the incident light that interacts with the pulse and ignore the component that tunnels through, because the tunneled light does not contribute to the observed spectrum.

We represent the relevant probe light $A$ as a superposition (A30) of stationary modes $A_R$ and $A_L$ on the right or left side of the horizon that are characterized by the frequency $\omega'$ in the co-moving frame. We replace the mode operators by classical amplitudes $a_R$ and $a_L$ and focus on the positive-frequency component of $A$. In this way we obtain

$$A = \int (a_R A_R + a_L A_L) d\omega'.$$  \hfill (B14)

For the mode functions $A_R$ and $A_L$ we use the theory of geometrical optics in moving media developed in Sec. A.8. We assume that the pulse is infinitely short.
in comparison with variations of the probe light; and write down the modes as

\[
\begin{align*}
A_R &= \Theta(\tau) \left( A_1 e^{-i\omega_1 \tau} + i A_2 e^{-i\omega_2 \tau} \right) e^{-i\omega' \zeta}, \\
A_L &= \Theta(-\tau) \left( A_1 e^{-i\omega_1 \tau} - i A_2 e^{-i\omega_2 \tau} \right) e^{-i\omega' \zeta}.
\end{align*}
\] (B15)

Here \(\omega_1\) denotes the laboratory frequency with respect to \(\omega'\) and \(\omega_2\) the blue-shifted laboratory frequency. The factors of \(\pm i\) describe the \(\pi/2\) phase shifts at turning points \(24\). Their sign depends on the sign of the frequency change and the side of the horizon. We assume that \(\omega_1\) and \(\omega_2\) are sufficiently close to the group velocity-matched frequency \(\omega_m\) such that we can use the quadratic approximation (B6) for \(\omega'\) outside the pulse where \(\chi\) vanishes,

\[
\omega' = \omega'_m - \frac{u}{2} \beta_2 (\omega - \omega_m)^2,
\] (B16)

which implies that the two solutions \(\omega_1\) and \(\omega_2\) for a given \(\omega'\) are symmetric around the group velocity-matched frequency \(\omega_m\), such that

\[
\omega_2 = \omega_m + (\omega_m - \omega_1) = 2\omega_m - \omega_1.
\] (B17)
The amplitudes $A_1$ and $A_2$ are given by the expression (A45). Since $n\omega$ does not vary much with frequency and $\partial \omega_1 / \partial \omega' = -\partial \omega_2 / \partial \omega'$, we approximate

$$A_1 \approx A_2 = A_0 \left| \frac{\partial \omega_1}{\partial \omega'} \right|. \quad (B18)$$

Finally, we write the mode expansion (B14) as an infinite integral over $\omega_1$, assuming that contributions outside the physically relevant range of $\omega_1$ are negligible,

$$A = \int_{-\infty}^{+\infty} \left( a_R(\omega_1)A_R + a_L(\omega_1)A_L \right) d\omega_1. \quad (B19)$$

Formulas (B15-B19) specify our theoretical model.

In our experiment, we measure the modulus squared of the Fourier transform of $A$ with respect to the laboratory time $t$ at the end of the fiber $z=L$. This is identical to the modulus squared of the Fourier transform $\tilde{A}$ with respect to the retarded time $\tau = t - z/u$ at $\zeta = L/u$. Using the standard relations

$$\int_{-\infty}^{0} e^{i\omega \tau} d\tau = i \frac{\pi \delta(\omega)}{\omega}, \quad \int_{0}^{\infty} e^{i\omega \tau} d\tau = -i \frac{\pi \delta(\omega)}{\omega} \quad (B20)$$

we obtain the result

$$\tilde{A} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} A(\tau) e^{i\omega \tau} d\tau$$

$$= A_0 \left( a_R(\omega) + a_L(\omega) \right) e^{-i\omega' \zeta} + iA_0 \left( a_R(2\omega_m - \omega) - a_L(2\omega_m - \omega) \right) e^{-i\omega' \zeta}$$

$$+ \int_{-\infty}^{+\infty} A_0 \left( \frac{ia_R(\omega_1) - ia_L(\omega_1)}{2\pi(\omega - \omega_1)} - \frac{a_R(\omega_1) + a_L(\omega_1)}{2\pi(\omega - \omega_2)} \right) e^{-i\omega' \zeta} d\omega_1 \quad (B21)$$

where the integral is understood as a Principal Value Integral [40] through the poles where $\omega_1$ or $\omega_2$ go through $\omega$. In the first line of Eq. (B21) $\omega'$ is a function of $\omega$ and in the integral in the second line $\omega'$ is understood to be a function of $\omega_1$.

At the entrance of the fiber, the initial probe $A_p$ is a continuous wave with frequency $\omega_p$ and an amplitude we denote as $2A_0$. The coefficients $a_R$ and $a_L$ that describe this situation are given by the expressions

$$a_R = \delta(\omega_1 - \omega_p) + \frac{i}{\pi(\omega_1 - \omega_p)}, \quad a_L = \delta(\omega_1 - \omega_p) - \frac{i}{\pi(\omega_1 - \omega_p)}, \quad (B22)$$

as one verifies by the following procedure: we substitute the coefficients in the integral (B19), extract the contribution of the delta function and apply Cauchy’s Residue Theorem [40] for the remaining integral. For this, we close the integration contours on the complex half planes where the integrand exponentially decreases. For the modes on the right-hand side of the horizon, where $\tau > 0$, we chose the lower half plane for the $\exp(-i\omega_1 \tau)$ term and the upper half plane for $\exp(-i\omega_2 \tau)$, in view of the relationship (B17); on the left-hand side we take the opposite planes. Since we integrate through the poles we obtain half of the residue, similar to the derivation
of Hilbert transformations [40] or Kramers-Kronig relations [7]. The result of this calculation is the incident plane wave $2A_0 \exp(-i\omega_p \tau)$, which justifies the mode coefficients (B22).

In order to calculate the spectrum, we substitute the mode coefficients (B22) into Eq. (B21). The delta functions immediately generate contributions to the spectrum, as we focus on the calculation of the remaining Principal Value Integral through $(\omega - \omega_1)^{-1} (\omega_1 - \omega_p)^{-1}$ with Gaussian $\exp(-i\omega \zeta)$ given by Eq. (B16). We represent $(\omega - \omega_p)(\omega - \omega_1)^{-1} (\omega_1 - \omega_p)^{-1}$ as $(\omega_1 - \omega_p)^{-1} - (\omega_1 - \omega)^{-1}$ and use the Hilbert transform of a Gaussian [31],

$$G(x) = \int_{-\infty}^{\infty} \frac{e^{-\xi^2} d\xi}{\pi(x - \xi)} = e^{-x^2} \text{erfi}(x), \quad \text{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{\xi^2} d\xi. \quad (B23)$$

In this way we find that the spectral field $\tilde{A}$ consists of the sum of the delta peak $A_0 \delta(\omega - \omega_p) \exp(-i\omega_p \zeta)$ and the contribution that describes the frequency shifting of the probe,

$$\tilde{A}_s = \frac{A_0}{\pi(\omega + \omega_p - 2\omega_m)} \left[ \exp\left(iq(\omega - \omega_m)^2\right) - \exp\left(iq(\omega_p - \omega_m)^2\right) \right]$$

$$- \frac{A_0}{\pi(\omega - \omega_p)} \left[ G\left(\sqrt{-iq}(\omega - \omega_m)\right) - G\left(\sqrt{-iq}(\omega_p - \omega_m)\right) \right], \quad (B24)$$

where we ignored the unimportant overall phase of $\omega_p \zeta$ and used the abbreviation

$$q = \frac{\pi c |D_m| L}{\omega_m^2}. \quad (B25)$$

So, up to an overall phase, the spectral field $\tilde{A}$ is given in terms of experimentally accessible parameters, the fiber length $L$, the dispersion $D_m$ and the group velocity-matched frequency $\omega_m$.

As expected, the spectrum gets narrower around the blue-shifted and probe frequencies $2\omega_m - \omega_p$ and $\omega_p$ with increasing propagation distance in the fiber, because both the converted and depleted components of the probe light form longer pulses for longer interaction times. However, the carrier frequency of the pulse is gradually red-shifted due to the soliton self-frequency shift (SFS) [39]. According to Eq. (B11), this leads to a shift $\delta \lambda_r$ in the group velocity-matched wavelength $\lambda_m$ of about $-0.75$ nm along the fiber. The blue-shifted light is created at decreasing wavelengths and also the part of the pulse that interacts with the probe adiabatically follows. In the signal spectrum, both the group-velocity-matched frequency $\omega_m$ and the probe frequency $\omega_p$ appear to be shifted by $\delta \omega = -(\delta \lambda_r / \lambda_m) \omega_m$ and is replaced by $\omega_s$. Since initially the spectrum is broad, we incorporate the SFS effect in a phenomenological form in our formula by a reduced efficiency $\eta_r$ for the blue-shifted part of the spectrum as

$$\tilde{A}_s = \frac{\eta_r A_0}{\pi(\omega + \omega_s - 2\omega_m)} \left[ \exp\left(iq(\omega - \omega_m)^2\right) - \exp\left(iq(\omega_s - \omega_m)^2\right) \right]$$

$$- \frac{A_0}{\pi(\omega - \omega_s)} \left[ G\left(\sqrt{-iq}(\omega - \omega_m)\right) - G\left(\sqrt{-iq}(\omega_s - \omega_m)\right) \right]. \quad (B26)$$
Figure B8 shows the fit of the observed spectrum of Fig. B4 with the theoretical curve (B26). As fitting parameters we used the overall amplitude $A_0$, the shifted probe frequency $\omega_s$ and group-velocity-matched frequency $\omega_m$ (in terms of the corresponding wavelengths), the dispersion $D_m$ and $\eta_r$. We obtain a very good fit for $A_0 = 4.1 \times 10^{12}\sqrt{W s}$, $\lambda_s = 1505.31$ nm, $\lambda_m = 1499.38$ nm, $D_m = -187$ ps/(nm km) and $\eta_r = 0.80$. The shift in $\lambda_p$ is consistent with the effect (B11) of the soliton self-frequency shift. The fitted values for $D_m$ and $\lambda_m$ agree with the independently measured dispersion and the group-velocity-matched frequency calculated from the dispersion curve of Fig. B1.

In conclusion, we have shown that light was blue-shifted by a near group velocity-matched pulse. The measured data was explained by the presence of an optical group velocity horizon inside the fiber. A very good agreement between theory and experiment was achieved. The blue shifting corresponds to the optical analogue of trans-Planckian frequency shifts in astrophysics [25]. In this way, we have demonstrated classical optical effects of the event horizon in our analogue system, a first step towards tabletop astrophysics [42].
References

[1] T. G. Philbin, C. Kuklewicz, S. Robertson, S. Hill, F. König, and U. Leonhardt, arXiv:0711.4796.

[2] G. Agrawal, Nonlinear Fiber Optics (Academic Press, San Diego, 2001).

[3] J. C. Knight, T. A. Birks, P. S. Russell, and D. M. Atkin, Opt. Lett. 21, 1547 (1996); P. Russell, Science 299, 358 (2003).

[4] M. Born and E. Wolf, Principles of Optics (Cambridge University Press, Cambridge, 1999).

[5] L. D. Landau and E. M. Lifshitz, Electrodynamics of Continuous Media (Pergamon, Oxford, 1984).

[6] Y. R. Shen, The Principles of Nonlinear Optics (Wiley, New York, 1984); R. W. Boyd Nonlinear Optics (Academic Press, San Diego, 1992).

[7] J. S. Toll, Phys. Rev. 104, 1760 (1956).

[8] P. W. Milonni, The Quantum Vacuum: An Introduction to Quantum Electrodynamics (Academic Press, San Diego, 1994).

[9] U. Leonhardt, Rep. Prog. Phys. 66, 1207 (2003).

[10] L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields (Pergamon, Oxford, 1975).

[11] The exact wave equation in a curved space time geometry is \( \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu A = 0 \), where \( g \) is the determinant of the metric tensor \([10]\). In the case \([A20]\) \( g \) depends only on the refractive index \( n \) and hence \( g \) is constant for constant \( n \).

[12] W. G. Unruh, Phys. Rev. Lett. 46, 1351 (1981); M. Visser, Class. Quantum Grav. 15, 1767 (1998).

[13] S. M. Hawking, Nature 248, 30 (1974); Commun. Math. Phys. 43, 199 (1975).

[14] N. D. Birrell and P. C. W. Davies, Quantum fields in curved space (Cambridge University Press, Cambridge, 1984); R. Brout, S. Massar, R. Parentani, and Ph. Spindel, Phys. Rep. 260, 329 (1995).

[15] J. D. Bekenstein, Phys. Rev. D 7, 2333 (1973); see also arXiv:gr-qc/0009019 for a review.

[16] See e.g. L. Knöll, S. Scheel, and D.-G. Welsch, QED in dispersing and absorbing media, in Coherence and Statistics of Photons and Atoms ed. by J. Perina (Wiley, New York, 2001), pp.1-63.

[17] See e.g. U. Leonhardt, Quantum Theory of Simple Optical Instruments, PhD thesis, Humboldt University Berlin, 1993.
[18] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, 1999), Volume I.

[19] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics*, (Cambridge University Press, Cambridge, 1995).

[20] G. Moore, J. Math. Phys. 11, 2679 (1970); S. A. Fulling, Phys. Rev. D 7, 2850 (1973); W. G. Unruh, Phys. Rev. D 14, 870 (1976); P. C. W. Davies, J. Phys. A 8, 609 (1975); B. S. DeWitt, Phys. Rep. 19, 295 (1975).

[21] Any two-dimensional Riemannian manifold is conformally flat, see M. Nakahara, *Geometry, Topology and Physics* (Institute of Physics, Bristol, 2003), Sec. 7.6.

[22] U. Leonhardt and T. G. Philbin, New J. Phys. 8, 247 (2006).

[23] L. D. Landau and E. M. Lifshitz, *Mechanics* (Pergamon, Oxford, 1976).

[24] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, Oxford, 1977).

[25] G. t’Hooft, Nucl. Phys. B 256, 727 (1985); T. Jacobson, Phys. Rev. D 44, 1731 (1991).

[26] W. G. Unruh, Phys. Rev. D 51, 2827 (1995).

[27] R. Brout, S. Massar, R. Parentani, and Ph. Spindel, Phys. Rev. D 52, 4559 (1995).

[28] S. Corley and T. Jacobson, Phys. Rev. D 59, 124011 (1999); U. Leonhardt and T. G. Philbin, *Black-hole lasers revisited*, in *Quantum Analogues: From Phase Transitions to Black Holes and Cosmology* edited by W. G. Unruh and R. Schützhold (Springer, Berlin, 2007).

[29] C. M. Caves, Phys. Rev. D 26, 1817 (1982).

[30] S. M. Barnett and S. J. D. Phoenix, Phys. Rev. A 40, 2404 (1989); *ibid.* 44, 535 (1991).

[31] *Few-Cycle Laser Pulse Generation and Its Applications*, edited by F. X. Kärtner (Springer, Berlin, 2004); T. Brabec and F. Krausz, Rev. Mod. Phys. 72, 545 (2000); E. E. Serebryannikov, A. M. Zheltikov, N. Ishii, C. Y. Teisset, S. Köhler, T. Fuji, T. Metzger, F. Krausz, and A. Baltuska, Appl. Phys. B 81, 585 (2005); N. Ishii, C. Y. Teisset, S. Köhler, E. E. Serebryannikov, T. Fuji, T. Metzger, F. Krausz, A. Baltuska, and A. M. Zheltikov, Phys. Rev. E 74, 036617 (2006).

[32] F. DeMartini, C. H. Townes, T. K. Gustafson, and P. L. Kelley, Phys. Rev. 167, 312 (1967).

[33] R. Meinel, G. Neugebauer, and H. Steudel, *Solitonen* (Akademie Verlag, Berlin, 1991).
[34] W. H. Reeves, D. V. Skryabin, F. Biancalana, J. C. Knight, P. S. Russell, F. G. Omenetto, A. Efimov, and A. J. Taylor, Nature 424, 511 (2003).

[35] N. Nishizawa and T. Goto, Opt. Lett. 27, 152 (2002).

[36] For recent theory see A. V. Gorbach and D. V. Skryabin, Nature Photonics 1, 653 (2007); arXiv:0707.1598.

[37] C. N. de Sterke, Opt. Lett. 17, 914 (1992); M. J. Steel, D. G. A. Jackson, and S. M. de Sterke, Phys. Rev. A 50, 3447 (1994); N. G. R. Broderick, D. Taverner, D. J. Richardson, M. Ibsen, and R. I. Laming, Phys. Rev. Lett. 79, 4566 (1997).

[38] A. Hasegawa and F. Tappert, Appl. Phys. Lett. 23, 142 (1973); L. F. Mollenauer, R. H. Stolen, and J. P. Gordon, Phys. Rev. Lett. 45, 1095 (1980).

[39] F. M. Mitschke and L. F. Mollenauer, Opt. Lett. 11, 659 (1986); J. P. Gordon, ibid. 11, 662 (1986).

[40] M. J. Ablowitz and A. S. Fokas, Complex Variables (Cambridge University Press, Cambridge, 1997).

[41] U. Leonhardt, M. Munroie, T. Kiss, Th. Richter, and M. G. Raymer, Opt. Commun. 127, 144 (1996).

[42] P. Ball, Nature 411, 628 (2001).