Conservation laws in coupled multiplicative random arrays lead to $1/f$ noise

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Abstract

We consider the dynamic evolution of a coupled array of $N$ multiplicative random variables. The magnitude of each is constrained by a lower bound $w_0$ and their sum is conserved. Analytical calculation shows that the simplest case, $N = 2$ and $w_0 = 0$, exhibits a Lorentzian spectrum which gradually becomes fractal as $w_0$ increases. Simulation results for larger $N$ reveal fractal spectra for moderate to high values of $w_0$ and power-law amplitude fluctuations at all values. The results are applied to estimating the fractal exponents for cochlear-nerve-fiber action-potential sequences with remarkable success, using only two parameters.

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1 Introduction

Over the past decade it has become apparent that power-law behavior is ubiquitous in physical and biological phenomena alike\cite{1, 2, 3, 4}. In spite of the importance of developing models to describe these phenomena, the literature is rather sparse when it comes to first-principle approaches that deal with the origins of fractal power-law (1/f) phenomena. Fractal point processes in particular have received short shrift in this connection; this important class of models is suitable for describing phenomena such as trapping times in amorphous semiconductors, neurotransmitter exocytosis, ion-channel opening times, and nerve-fiber action-potential occurrences\cite{5, 6}.

In this paper we develop a plausible dynamic multiplicative stochastic model, with well-understood underpinnings, that yields fractal behavior under certain specified conditions. The model comprises interchangeable variables and an explicit constraint on the allowed values of its individual elements. The system is further restricted by a global conservation law. We find that this simple construct leads to fractal $1/f$ correlations both in the time evolution of the individual variables and in the asymptotic distribution of all of the variables. We demonstrate that the fractal exponents that emerge from the model are controlled solely by the constraints on the individual variables and the size of the array (number of elements). The model is applied to a biological system that is in fact subject to just such constraints and conservation laws. Fractal exponents associated with peripheral mammalian neural-spike trains\cite{7, 8, 9} that carry auditory information to higher centers in the brain are estimated. The model requires only two parameters, both of which can be determined physiologically: the number of innervated nerve fibers per inner hair cell and the minimum neurotransmitter flux per afferent nerve fiber.

We anticipate that the results developed here will find application in a broad range of problems in the physical and biological sciences.
2 Model

2.1 Coupled Array in the Asymptotic Limit

We consider a system comprising of a set of $N$ elements, each denoted $i$ and characterized by a time-dependent real-valued variable $w_i(t)$ (we henceforth refer to this as the array of variables). The discrete time evolution of this array of stochastic variables is prescribed by

$$w_i(t+1) = \lambda_i w_i(t),$$

(1)

where $\lambda$ is a random variable drawn from a probability density $\pi(\lambda)$ with compact support. This probability density depends on neither on $i$ nor on the actual value of $w_i$, and $\lambda$ is independently drawn from $\pi(\lambda)$ for each element $i$. Each variable therefore characterizes a branching process [10].

The elements $i$ are now coupled by imposing a normalization condition on the sum of their values:

$$\sum_{i=1}^{N} w_i(t) = N,$$

(2)

so that the mean value of $w_i(t)$ is unity. We further impose the condition that $w$ is bounded from below for all $i$:

$$w_i(t) > w_0,$$

(3)

with $w_0 \geq 0$.

In the limit of large times and large $N$, the distribution of the variables $w$, denoted $P(w)$, can be determined with the help of the corresponding master equation. The result turns out
to exhibit power-law behavior \[11\]

\[ P(w) \sim w^{-1-1/T} = w^{-\alpha_{\text{asym}}}, \]

where \( T = 1 - w_0 \) and \( \alpha_{\text{asym}} = 1 + 1/T \); interestingly it is independent of the nature of \( \pi(\lambda) \).

The validity of Eq. 4 was verified using simple computer simulations of Eq. (1). We considered \( N \) elements (usually 1024) and for \( \pi(\lambda) \), either a uniform distribution \( \pi_{\text{unif}} = \Theta(\lambda - s) \cdot \Theta(s + 1 - \lambda) \) or an exponential distribution \( \pi_{\exp} = e^{c\lambda} \cdot \Theta(\lambda - s) \cdot \Theta(s + 1 - \lambda) \), where \( \Theta(x) \) is the Heaviside function, and \( s \) represents a non-negative shift parameter. The set of variables \( w_i(t) \), initially set equal to unity, was updated at the time step \( (t' = t + 1) \) in the following way: each of the \( N \) stochastic variables \( w_i(t) \) was multiplied by a random variable \( \lambda_i \) drawn from \( \pi(\lambda) \) (the \( \lambda_i \) were different samples drawn from a single distribution), and were then subjected to normalization such that their sum was equal to the total number of elements in the array, \( N \). In this new sequence all \( w_i \) values that did not obey the restriction \( w_i > w_0 \) were multiplied by a new random variable \( \lambda \), and the whole set \( i \) was then normalized again. This procedure was iterated as long as there were cases for which \( w_i \leq w_0 \). The final set \( w_i \) was then considered to be the sequence at the time \( t' = t + 1 \). The free parameters of the simulations were the lower bound \( w_0 \), the form of \( \pi \), the shift parameter \( s \), and the array size \( N \).

For relatively short time sequences (3 000 time steps), a power-law amplitude histogram \( P(w) \) emerges and the expected dependence \( \alpha(w_0) = 1 + 1/(1 - w_0) \) is qualitatively reproduced. We confirmed that the form of the distribution \( \pi(\lambda) \) indeed has a minimal effect on the outcome. We therefore restricted our consideration to \( \pi = \pi_{\text{unif}} \) in the remainder of this paper. Moreover, we found that bias effects associated with the value of \( s \) remained small if \( s \approx 1 \); we therefore use \( s = 1 \) throughout.

The interesting character of this asymptotic result prompts us to examine the time evolution of this system, which we proceed to do in the next section.
2.2 Dynamical Evolution in the Coupled Array Model

The dynamical nature of the multiplicative random model, constrained by normalization and a minimum value \( w_0 \) for all variables, is elucidated by rewriting Eq. (1) as a set of coupled stochastic differential equations,

\[
\begin{align*}
\dot{w}_1(t) &= (\lambda_1 - 1)w_1(t) \\
\dot{w}_2(t) &= (\lambda_2 - 1)w_2(t) \\
\vdots & \quad \vdots \\
\dot{w}_N(t) &= (\lambda_N - 1)w_N(t)
\end{align*}
\]

\[
\sum_{i=1}^{N} w_i(t) = N, \quad \text{for all } t \quad \text{and } i,
\]

for all \( t \) and \( i \), (5)

where all \( \lambda_i \) are drawn from a single distribution \( \pi(\lambda) \).

In the next subsection we demonstrate that these Langevin equations can be analytically solved for an array of size \( N = 2 \) (when \( w_0 = 0 \)), and that the correlations are exponential. However, power-law behavior emerges as the level of the constraint \( w_0 \) increases.

2.2.1 \( N = 2 \), Analytical Solution

For a system comprising two elements \((i = 1, 2)\), an analytical solution can be found for \( w_0 = 0 \) by iteratively solving a master equation. The set of equations (5) can be reduced to a single equation that incorporates the normalization condition. The transition probability matrix for the system, which provides the conditional probability of obtaining the random
variable $w(t + 1)$, when the starting value is $w(t)$ at the previous time step, is given by

$$W(w(t + 1), t + 1|w(t), t) = \frac{\lambda_1 w(t)}{\lambda_1 w(t) + \lambda_2 (2 - w(t))}$$

(6)

where $\lambda_{1,2}$ is a random variable drawn from the probability distribution $\pi(\lambda)$. Equation (6) is readily rewritten in the form

$$W(w(t + 1), t + 1|w(t), t) = \frac{1}{x},$$

(7)

where $x$ is a random variable drawn from the distribution $f_x = 1 + f(z)(2/w(t) - 1)$. The quantity $f(z)$ is the probability distribution of the quotient of the two random variables, $z = \lambda_2/\lambda_1$. Using basic relations for the products of random variables [12], it can be shown that

$$f(z) = \begin{cases} \frac{1}{2z^2} & \text{for } z > 1 \\ \frac{1}{2} & \text{for } z < 1 \end{cases}.$$  

(8)  

(9)

Finally, setting $y = 1/x$, the probability matrix takes the form

$$W(w(t + 1), t + 1|w(t), t) = \frac{1}{y^2} \cdot f_x\left(\frac{1}{y}\right)$$

(10)

where $y$ is a functional of $f(z)$ and $w(t)$. Computed results are shown in Fig. 1 for uniform probability distribution $\pi(\lambda)$.

Given knowledge of the probability matrix $W$, all of the conditional probabilities $P(\mid)$ can be computed as a function of time by using the time-evolution equation

$$P(w(t + 2), t + 2|w(t), t) = \sum_{w(t+1)} W(w(t + 2), t + 2|w(t + 1), t + 1)P(w(t + 1), t + 1|w(t), t),$$

(11)
which is seen to be an iterative solution of a discrete master equation. The conditional probability converges quite nicely to a function that is constant in time, as is understood by recognizing that the matrix $W$ has three degenerate eigenvalues, $E_1 = E_2 = E_3 = 1$. All of the other eigenvalues are smaller than unity and vanish under repeated applications of $W$. Two of the relevant corresponding eigenvectors are trivial; the third is the asymptotic probability $P(w, t \to \infty)$.

These probabilities permit the correlation functions to be computed by carrying out the integral

$$< w(0) w(t) > = \int dw dw' P(w, t, w', 0) w w', \quad (12)$$

where $P(,, ,)$ is the joint probability

$$P(w, t, w', 0) = P(w, t | w', 0) \cdot P(w', 0). \quad (13)$$

The associated correlation function is exponential with correlation length $\xi$, corresponding to a Lorentzian power spectral density (PSD)

$$\text{PSD}(f) = \frac{1}{f^2 + \xi^2}, \quad (14)$$

where $f$ is the frequency (arbitrary units). The tails of the Lorentzian decay, of course, as $f^{-\alpha}$ with $\alpha = 2$.

In the more difficult situation when $w_0 > 0$, simulations were used to solve Eq. (5). The power-law exponent $\alpha$ was estimated by means of a straight-line fit of the power spectral density (represented on doubly logarithmic coordinates) after 1024 iterations of a given value of $w_i$. 100 such runs were carried out for each value of $w_0$. The average values (dots) and standard deviations (error bars) of $\alpha$ are presented in Fig. 2 as a function of $w_0$. The exponent clearly decreases with increasing $w_0$, revealing the onset of fractal behavior. It
assumes a maximum value of about $1.75 \pm 0.25$ at $w_0 = 0$, which is about one standard deviation below its expected value of 2. We expect that the discrepancy results from finite data length.

Finally, in Fig. 3 we illustrate the simulated amplitude histograms for this multiplicative process ($N = 2$) with the bound $w_0$ as a parameter. The distributions gradually move from an arcsine-like form for $w_0 = 0$ to a rather peaked form for $w_0 = 0.775$. They are clearly non-Gaussian for all $w_0$, leaving no doubt that the resulting process is not equivalent to (fractal) Gaussian noise.

### 2.2.2 $N > 2$, Numerical Results

We now consider interactions involving more than two coupled elements. We have simulated Eq. (5) to obtain spectral densities for various values of $w_0$ and $N$. The power spectral densities for $w_0 = 0.6, 0.7, 0.8$ are shown in the three panels of Fig. 4, for $N = 2, 10, 50$ elements respectively. On these doubly logarithmic coordinates, it is clear that the processes all exhibit power-law spectra. Estimates of the fractal exponents $\alpha$ over the frequency range $10 < f < 512$ (arbitrary units) are provided in Table I. The exponents displayed in the table clearly decrease with increasing $w_0$, revealing the onset of fractal behavior. The results for $N = 10, 50$ do not differ substantially from those for $N = 2$ (also shown in Fig. 2). It is also apparent from Table I that $\alpha(w_0 = 0)$ assumes a maximum value of about $1.75 \pm 0.25$ which, just as for $N = 2$, is about one standard deviation below its expected value of 2. We expect that the discrepancy here too results from finite data length.

The results embodied in Fig. 4 constitute a key finding of our work: coupled arrays of multiplicative random processes that are subject to constraints exhibit power-law behavior in the power spectral density.

The amplitude histograms for $w_0 = 0.0, 0.2, 0.4$ are shown in the two panels of Fig. 5,
for $N = 10, 50$ elements, respectively. On these doubly logarithmic coordinates $P(w)$ is seen to exhibit power-law behavior, in agreement with the asymptotic result given in Eq. (4) [11]. This is true even for $w_0 = 0$. In no case was power-law amplitude behavior evident for two elements (the amplitude histograms for $N = 2$ are displayed in Fig. 3). It is therefore clear that the emergence of fractal amplitude behavior arises from the presence of a sizeable number of interacting elements. This is another key finding of our work.

The behavior of $\alpha$, over a range of $w_0$ that is of interest for the example provided in the next section, is plotted in Fig. 6 for several values of $N$. These curves can be well fit by a three-parameter function of the form

$$\alpha(w_0) = c_1 - (w_0 + c_2)^{c_3};$$ (15)

the parameter values $c_1, c_2, c_3$ are provided in Table II. It is clear from Fig. 6 that the exponent $\alpha(w_0)$ depends strongly on the value of the bound $w_0$.

A crucial observation to be gleaned from Table I and Fig. 6 is the decrease in the power-law exponent, and the concomitant departure of the correlations from exponential form, that herald the onset of fractal behavior as the lower bound $w_0$ increases, whatever the value of $N$. Therein resides the origin of the $1/f - \alpha$ behavior in a coupled multiplicative system with conservation constraints.

3 Example: Estimation of Fractal Exponents for Sequences of Cochlear Nerve-Fiber Action Potentials

The transmission of auditory information from the mammalian hair-cell transducer to higher centers in the brain is mediated by a flux of neurotransmitter molecules. These molecules are produced in the inner hair cell at a certain limited rate. After exocytosis and diffusion
across the synaptic cleft, they are distributed among the roughly 10-20 primary cochlear-nerve fibers (CNFs) that innervate each hair cell, and result in the firing of sequences of nerve spikes that travel on up the auditory pathway [13].

Assuming that the model presented here is applicable for describing this process, we associate the amplitude $w_i$ in Eq. (5) with the neurotransmitter flux reaching one of the $N$ nerve fibers that synapse on a particular inner hair cell. Since the neural firing rate is proportional to the neurotransmitter flux, we can estimate $w_0$ by determining the lowest local firing rate from a CNF spike train. A simple way to achieve this is to divide the spike train into contiguous segments of $T$ sec, and then to define $w_0$ as the ratio of the minimum firing rate observed over the entire data set, to the average firing rate $\rho$. Given $w_0$ and $N$ (which determines $c_1$, $c_2$, and $c_3$ in accordance with Table II), the expected fractal exponent $\alpha_{th}$ is provided by Eq. (15) (see Fig. 6).

We carried out this procedure for cat CNF spike trains obtained in nine experiments [8, 14], using a time window of $T = 10$ sec and $N = 10$ interacting elements. The results for $\alpha_{th}$ are given in Table III, along with the measured fractal exponent $\alpha_{exp}$ obtained from the spectra and Allan factors of the spike trains themselves [8, 9]. The results are in substantial, and surprising, agreement indicating that it is worthwhile to further pursue this line of reasoning.

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Table 1: Fractal exponents $\alpha$ for the multiplicative stochastic model with $N = 2, 10, 50$ coupled variables, for various values of the lower bound $w_0$ operative on the individual variables. The numbers in parentheses indicated standard deviations. The top-most row ($N = 2$) corresponds to the data plotted in Fig. 2. For $w_0 = 0$, a nominal value $\alpha = 1.72 \pm 0.25$ emerges independently of the number of elements $N$, suggesting that the most salient feature of the model responsible for fractal spectral behavior is the constraint.

| Array size | $\alpha(w_0 = 0.0)$ | $\alpha(w_0 = 0.1)$ | $\alpha(w_0 = 0.2)$ | $\alpha(w_0 = 0.3)$ | $\alpha(w_0 = 0.4)$ |
|------------|---------------------|----------------------|----------------------|----------------------|----------------------|
| $N = 2$    | 1.73(25)            | 1.74(9)              | 1.73(7)              | 1.69(8)              | 1.66(7)              |
| $N = 10$   | 1.70(23)            | 1.75(12)             | 1.69(11)             | 1.71(9)              | 1.70(10)             |
| $N = 50$   | 1.72(26)            | 1.74(18)             | 1.74(18)             | 1.70(14)             | 1.69(15)             |

| Array size | $\alpha(w_0 = 0.5)$ | $\alpha(w_0 = 0.6)$ | $\alpha(w_0 = 0.7)$ | $\alpha(w_0 = 0.8)$ |
|------------|---------------------|----------------------|----------------------|----------------------|
| $N = 2$    | 1.64(7)             | 1.52(7)              | 1.27(7)              | 0.71(6)              |
| $N = 10$   | 1.63(10)            | 1.50(9)              | 1.20(10)             | 0.43(8)              |
| $N = 50$   | 1.63(12)            | 1.52(11)             | 1.13(11)             | 0.13(8)              |

Table 2: Parameters $c_1, c_2, c_3$ that provide the best fits of Eq. (15) to the curves shown in Figure 6.

| $N$ | $c_1$ | $c_2$ | $c_3$ |
|-----|-------|-------|-------|
| 2   | 1.74  | 0.201 | 7.3   |
| 10  | 1.75  | 0.235 | 8.1   |
| 50  | 1.72  | 0.250 | 10.0  |
Table 3: Comparison between the predicted fractal exponents $\alpha_{\text{th}}$ obtained using the multiplicative stochastic model with $N = 10$ coupled, constrained, and interchangeable variables; and the fractal exponents $\alpha_{\text{exp}}$ estimated directly from the spike trains. The agreement is unexpectedly good.
FIGURE CAPTIONS

Figure 1. Computed form for the probability matrix $W$ when the random variable $\lambda$ is drawn from a uniform probability distribution. Only values of $w$ constrained from below by $w > 0$ are permitted.

Figure 2. Exponent of the power spectral density $\alpha$ as a function of the lower bound constraint on the amplitude $w_0$ for a constrained two-element stochastic multiplicative process. The decrease of the exponent from its nominal value of 2 at $(w_0 = 0)$ indicates a transition from Lorentzian to fractal behavior.

Figure 3. Simulated amplitude histograms $P(w)$ for various values of $w_0$ when $N = 2$. The curves are symmetrical about unity and are clearly non-Gaussian. Power-law behavior of the amplitude histogram $P(w)$ [as provided in Eq. (4)], emerges only as $N$ increases, as will become clear from Fig. 5.
Figure 4. Power spectral densities (PSDs) with \( w_0 \) as a parameter for three values of \( N \): \( N = 2, 10, 50 \) (upper, middle, and lower panels respectively). Power-law behavior is observed over the frequency range of \( 10 < f < 512 \) (arbitrary units), even in the case when there are only two elements.

Figure 5. Amplitude histograms with \( w_0 \) as a parameter for two values of \( N \): \( N = 10, 50 \) shown in different panels. The presence of power-law behavior in this figure contrasts with its absence when there are only two interacting elements (see Fig. 3).

Figure 6. \( \alpha \) as a function of \( w_0 \), over a limited range, for several values of \( N \). \( \alpha \) was estimated by means of a straight-line fit of the power spectral density (represented on doubly logarithmic coordinates) after 1024 iterations of a given value of \( w_i \). 100 such runs were carried out for each value of \( w_0 \). Average values are shown as dots and standard deviations as error bars. The curve for \( N = 2 \) is a portion of the one plotted in Fig. 2. The lines are to guide the eye.
Figure 1
Figure 2
Figure 3
Figure 4
Figure 5
Fractal Exponent $\alpha(w_0)$

Figure 6