Reduced Phase Space Approach to the $U(1)^3$ model for Euclidean Quantum Gravity

S. Bakhoda$^{1,2}$; T. Thiemann$^2$

$^1$ Department of Physics, Shahid Beheshti University, Tehran, Iran
$^2$ Institute for Quantum Gravity, FAU Erlangen – Nürnberg, Staudtstr. 7, 91058 Erlangen, Germany

Abstract

If one replaces the constraints of the Ashtekar-Barbero SU(2) gauge theory formulation of Euclidean gravity by their $U(1)^3$ version, one arrives at a consistent model which captures significant structure of its SU(2) version. In particular, it displays a non trivial realisation of the hypersurface deformation algebra which makes it an interesting testing ground for (Euclidean) quantum gravity as has been emphasised in a recent series of papers due to Varadarajan et al.

In this paper we consider a reduced phase space approach to this model. This is especially attractive because, after a canonical transformation, the constraints are at most linear in the momenta. In suitable gauges, it is therefore possible to find a closed and explicit formula for the physical Hamiltonian which depends only on the physical observables.

The corresponding reduced phase space quantisation can be confronted with the constraint quantisation due to Varadarajan et al to gain further insights into the quantum realisation of the hypersurface deformation algebra.

1 Introduction

Using real-valued SU(2) gauge theory variables [1] one can write the Hamiltonian constraint of Lorentzian vacuum General Relativity (GR) as a sum of three terms. The first term can be recognised as the Hamiltonian constraint for Euclidean vacuum GR, the second term corrects for Lorentzian signature and the third is the cosmological constant term. The second term can, in turn, be expressed as a multiple Poisson bracket between the cosmological constant term and the first term [2]. Following the quantisation rule that Poisson brackets be replaced by commutators, the Lorentzian Hamiltonian quantum constraint can thus be written as a linear combination of the Euclidean Hamiltonian constraint, the quantum cosmological constant term, and multiple commutators thereof [2]. Since the quantum cosmological constant term is essentially the volume operator [3] which in Loop Quantum Gravity (LQG) [4] is under complete control, the quantisation of Lorentzian vacuum gravity is reduced essentially to the quantisation of the Euclidean Hamiltonian constraint which has been intensively studied [5].

A critical measure for the success of that quantisation is a faithful representation of the hypersurface deformation algebra which is formed by the Euclidean Hamiltonian constraint together with the spatial diffeomorphism constraint. The quantisation of [5] displays a closed quantum constraint algebra, however, while the commutator of two Euclidean Hamiltonian constraints is a linear combination of spatial diffeomorphism constraints, the quantum coefficients of that linear combination are not quantisations of the coefficients of the corresponding classical linear combination, i.e. the “quantum structure functions” differ from the classical ones.

---

$^*$sbakhoda@sbu.ac.ir, sepideh.bakhoda@gravity.fau.de
$^1$thomas.thiemann@gravity.fau.de

1By the 4-volume resp. 3-volume term we mean $|\det(g)|^{1/2}$ and $\det(q)^{1/2}$ respectively. In any ADM slicing we have the identity $|\det(g)|^{1/2} = N \det(q)^{1/2}$ with the lapse function $N$. Since the constraints concern single moments of the slicing time parameter, the relation between the two terms is really that simple.
In order to find clues for how to improve on this, one can study the same problem in the simpler $U(1)^3$ model [6] which recently was done in the series of works [7]. The $U(1)^3$ model is defined by replacing the SU(2) Gauss constraint and the SU(2) version of the Euclidean Hamiltonian constraint by their corresponding $U(1)^3$ counterparts while the symplectic structure and the spatial diffeomorphism constraints remain unchanged. Thus essentially one replaces the SU(2) covariant derivative and curvature respectively by their $U(1)^3$ versions. The resulting spatial diffeomorphism and Hamiltonian constraints form an algebra isomorphic to its SU(2) version with the same structure functions, i.e. they form a hypersurface deformation algebra. This makes it an ideal testing ground for the actual SU(2) problem under investigation, in particular, the model has two propagating degrees of freedom.

In this paper, we propose to quantise the $U(1)^3$ model using the reduced phase space approach [8] rather than by operator constraint methods [7]. The advantage of the reduced phase space approach is that one can work directly with the physical Hilbert space, the physical observables, and the physical dynamics which in the operator constraint approach have to be determined after representing the constraints on an unphysical, kinematical Hilbert space. The disadvantage of the reduced phase space approach is that generically it involves implicit functions, inversions of phase space-dependent differential operators, and square roots which are difficult to quantise in practice [9]. This problem can be avoided in the presence of suitable matter [10] but in vacuum it is generic.

What makes the reduced phase space approach a practical possibility for the vacuum $U(1)^3$ model is the fact that its constraints are at most linear in the $U(1)^3$ connection while in the SU(2) case the connection dependence is quadratic. Considering the connections as momentum variables after a trivial canonical transformation, this means that no square roots have to be taken to solve the constraints. Even more, the explicit solution of the constraints is not even necessary in order to construct the physical Hamiltonian corresponding to a choice of gauge fixing in terms of the corresponding physical degrees of freedom. Depending on the choice of gauge fixing, that Hamiltonian generically still involves inversions of phase space-dependent differential operators for which we can give explicit perturbative expressions which however will be difficult to quantise. However, we find preferred gauge fixings which avoid this problem. The resulting physical Hamiltonians are spatially non-local, however, they can be written explicitly and non-perturbatively and is not hopeless to be quantisable using LQG methods. The resulting equations of motion for the physical observables can be worked out explicitly. For the electric field these do not involve the connection and are non-polynomial differential equations of second order in time and involve higher order integro-differential operators. By passing to suitable variables one can write these as fourth-order polynomial equations without integral operators at the price of introducing higher-order spatial derivatives. The equations of motion for the connection are linear in the connection and of first-order in time and thus can in principle be easily integrated once the solution for the electric field dynamics has been found. The physical dynamics is such that the two physical degrees of freedom are self-interacting and propagating.

We consider it quite remarkable that a model so closely related to vacuum GR can be cast into a rather manageable form while the symplectic reduction of all constraints has already been carried out. This paper serves as a proof of principle that such a possibility exists. In future publications, we plan to further optimise the choice of gauge fixing in order to simplify the physical Hamiltonian as much as possible and then to find Hilbert space representations thereof. We hope that such a complementary approach to the developments of [7] will shed further light on the actual problem of quantisation of the Euclidean SU(2) Hamiltonian constraint. Another direction of generalisation is the consideration of non-trivial spatial topology, in this paper we restrict to the topology of $\mathbb{R}^3$ supplemented by asymptotically flat boundary conditions. The details of these boundary conditions can be found in [11] which is based on [12] [13].

The architecture of this paper is as follows:

In section 2 we briefly review the classical formulation of the $U(1)^3$ model which mainly serves to fix our notation. We also summarise the asymptotically flat boundary conditions.

In section 3 we prove a simple lemma concerning the construction of the physical Hamiltonian defined by a choice of gauge fixings for a first-class constrained system with constraints which are at most linear in the momenta. The most important conclusion from that lemma is that the constraints do not have to be solved explicitly in order to find the physical Hamiltonian. This result may sound trivial for systems with a finite number

---

2Throughout this paper, by the electric field, we mean a densitised triad.
of degrees of freedom where only a matrix has to be inverted to solve the constraints but in field theory, the result is of major practical importance because the matrix is generically replaced by a system of differential operators with coefficients which depend on the configuration degrees of freedom, thus are not known as explicit functions of the spatial variables and are thus only implicitly invertible.

In section 4 we apply our lemma to the $U(1)^3$ model for various choices of gauge fixings to construct the physical Hamiltonian. The degree of spatial non-locality and non-polynomiality critically depend on that choice. While these choices are classically equivalent (related by a gauge transformation), with regard to quantisation certain choices seem to be preferred. We give an explicit example for which the physical Hamiltonian displays a rather manageable form, find the physical equations of motion and discuss its properties.

In section 5 we conclude and give an outlook to future research.

This paper is the first in a series of three. In our companion papers [11, 14] we study more systematically the asymptotically flat boundary conditions that we chose throughout this paper as we need them in various places in order to invert differential operators (i.e. construct Green functions). We also complement the present canonical analysis by a covariant framework.

2 Review of the $U(1)^3$ model for Euclidean vacuum GR

We consider first the connection formulation of General Relativity [1] in terms of $su(2)$-valued gauge theory variables $(A^a_i, E^a_j)$ where $a, b, c, .. \in \{ x, y, z \}$ are spatial tensor indices and $j, k, l, .. \in \{ 1, 2, 3 \}$ are $su(2)$ Lie algebra indices that are moved with the Kronecker $\delta^a_j$ and so the index position of $j, k, l, ..$ is irrelevant. These variables are canonically conjugate in the sense that the non-vanishing Poisson brackets are

$$\{E^a_j(x), A^a_k(y)\} = \kappa \delta^a_j \delta^a_k \delta(x, y)$$  \hspace{1cm} (2.1)

where $\hbar \kappa = \ell_\text{P}^2$ is the Planck area. Geometrically, $E$ is a density one valued triad for the geometry intrinsic to the spatial hypersurfaces and $A$ differs from the spin connection $\Gamma$ of $E$ by a term linear in their extrinsic curvature.

In what follows we assume for simplicity that the spacetime topology is that of $\mathbb{R}^4$ and that asymptotically flat boundary conditions are imposed in the connection formulation of GR [11, 12, 13]. This in particular implies that

$$E^a_j - \delta^a_j \rightarrow F^a_j(n)/r + O(1/r^2), \hspace{0.5cm} A^a_j \rightarrow G^a_j(n)/r^2 + O(1/r^3)$$  \hspace{1cm} (2.2)

where $r$ is an asymptotic radial variable $r^2 = x^2 + y^2 + z^2$, $n^a = x^a/r$ is an asymptotic angular variable and the functions $F, G$ on the asymptotic $S^2$ are even and odd respectively $F(-n) = F(n), G(-n) = -G(n)$. The meaning of $\delta^a_j$ is that it is non-vanishing only for the pairs $(a, j) = (x, 1), (y, 2), (z, 3)$ and then takes the value one. We will simply copy these boundary conditions as well as the symplectic structure (2.1) into the $U(1)^3$ truncation of (Euclidean) GR, the motivation being the importance of asymptotically flat spacetimes in Lorentzian GR which we thus wish to probe as closely as possible with the $U(1)^3$ model. Further details including the asymptotic symmetry (ADM) charges can be found in [11].

The first class, density one vacuum constraints of Euclidean GR with vanishing cosmological constant can be written in these variables as

$$C_j = D_a E^a_j, \hspace{0.5cm} C^a = F^a_{ab} E^b_j, \hspace{0.5cm} C_0 = F^a_{ab} \delta^a_j \epsilon_{klm} E^a_l E^b_m |\det(E)|^{-1/2}$$  \hspace{1cm} (2.3)

where $D_a$ is the $su(2)$ gauge covariant derivative and $F = 2(dA + A \wedge A)$ the curvature of $A$. The constraints are respectively referred to as the Gauss-, vector- and scalar constraint respectively.

Lorentzian GR is now obtained by adding to $C_0$ a term proportional to [2]

$$\text{Tr}(\{ \int d^3y C_0(y), V \}, A(x)) \wedge \{ \int d^3y C_0(y), V \}, A(x) \wedge \{ V, A(x) \}), \hspace{0.5cm} V = \int d^3y |\det(E)|^{1/2}$$  \hspace{1cm} (2.4)

where $V$ is the total volume of the spatial hypersurfaces. In this paper we will only be concerned with the Euclidean piece which already leads to the hypersurceae deformation algebra.
The truncation of the model is obtained by writing \( A = \kappa \tilde{A} \) so that \( E, \tilde{A} \) are strictly conjugate without a factor of \( \kappa \) and by expanding \((2.3)\) to first non-vanishing order in \( \kappa \) which can be considered as a weak coupling limit with respect to Newtons constant \( \kappa \) [6]. This simply means that \((2.3)\) is replaced by
\[
C_j = \partial_a E_a^j, \quad C_a = F_{ab}^j E_a^b, \quad C_0 = F_{ab}^j \delta^k_{klm} E_a^k E^b_m |\det(E)|^{-1/2} \tag{2.5}
\]
where \( \partial \) is the spatial derivative and \( F = 2dA \). This already shows that the model depends on \( A \) only linearly. Remarkably, as one can check, the constraint algebras defined by \((2.3)\) and \((2.5)\) induced by \((2.1)\) and restricted to the spacetime diffeomorphism sector are identical and form the hypersurface deformation algebra
\[
\{ C(u, f), C(v, g) \} = -\kappa C([u, v] + q^{-1}[f dg - g df], u[g] - v[f]) \tag{2.6}
\]
with
\[
C(u, f) = \int d^3 x \ [u^a C_a + f C_0]; \quad [q^{-1}]^{ab} = E_a^k \delta^j_k |\det(E)|^{-1} \tag{2.7}
\]
Also \( \{ C(\lambda), C(U, f) \} = 0 \) is the same as in the \( su(2) \) theory where \( C(\lambda) = \int d^3 x \lambda C_j \). The only difference is that in \( su(2) \) we have \( \{ C(\lambda), C(\lambda') \} = -\kappa C([\lambda, \lambda']) \neq 0 \) while in \( u(1)^3 \) we have \( \{ C(\lambda), C(\lambda') \} \equiv 0 \).

For our analysis it will prove to be convenient to introduce the following density one valued quadratic combinations
\[
H_a^j = \frac{1}{2} \epsilon_{abc} \epsilon^{jkl} E^b_k E^c_l = \det(E) (E^{-1})^j_a \tag{2.8}
\]
which allows us to write vector and scalar constraint in an equivalent but unified density two valued form
\[
\tilde{C}_j = \epsilon_{jkl} \delta^k_{lm} B_a^m H_a^l, \quad \tilde{C}_0 = B_a^j H_a^j \tag{2.9}
\]
where
\[
B_a^j = \epsilon^{abc} \partial_b A_c^k \delta_{kj} \tag{2.10}
\]
defines the magnetic field of \( A \). To arrive at this form, it was implicitly assumed as usual that the triad is nowhere degenerate. Following [15], it is now apparently possible to solve the constraints algebraically: Let
\[
B_a^j = \epsilon^{ijk} \delta_{kl} E_l^a \tag{2.11}
\]
then the spacetime constraints simply demand that \( c = c^2 \), \( \text{Tr}(c) = 0 \). However, this is not the case as the Bianchi identity and the Gauss constraint in addition implies
\[
E_a^j \partial_a c = 0 \tag{2.12}
\]
which presents a system of three coupled PDE’s of first order which is no longer solvable algebraically. Thus, in order to proceed the gauge fixing analysis of the subsequent section is unavoidable.

We note that the linearity of the constraints in \( A \) implies that the gauge transformations generated by the constraints on \( E_a^j \) have a purely geometric interpretation: Obviously, \( E_a^j \) is Gauss invariant and transforms as a vector density under spatial diffeomorphisms. A scalar gauge transformation has the effect (using the density one version)
\[
\delta_f E_a^j := \{ C_0 (u = 0, f), E_a^j \} = -\epsilon^{abc} \partial_b [f c_c^j] \tag{2.13}
\]
where \( c_c^j \) is the density zero co-triad. It follows that the scalar constraint shifts the divergence free piece of the electric field which is in particular consistent with the Gauss constraint.

This suggests to consider the canonical transformation \((A, E) \mapsto (E, -A)\) and to consider \(-A\) as the momentum conjugate to \( E \). We call this the \((A, E)\) formulation. We consider also a dual \((f, B)\) formulation as follows: It is consistent with the boundary conditions to solve the Gauss constraint in the form \( E_a^j = \delta_a^j + \epsilon^{abc} \partial_b f_c^j \).

When plugging this into the symplectic potential we obtain upon integration by parts
\[
\theta = \int d^3 x \ E_a^j \delta A_a^j = \delta \int d^3 x \ E_a^j A_a^j] - \int d^3 x \ A_a^j \delta [E_a^j - \delta_a^j] = \delta [\int d^3 x \ E_a^j A_a^j] - \int d^3 x \ B_a^j \delta f_a^j \tag{2.14}
\]

\[3\] Throughout this paper, by the magnetic field, we mean the dual of the derivative of the spin connection.
which displays $-B^a_j$ as conjugate to $f^a_j$. Note that the roles of $(f, -B)$ have changed as compared to $(A, E)$: While the Gauss constraint only depends on $E$ and just transforms $A$, there is now no longer a Gauss constraint but instead the Bianchi identity $\partial_a B^a_j = 0$ which, when viewing $B$ as an elementary rather than a derived field, plays the role of a “dual Gauss constraint”. Likewise, the Bianchi identity just depends on $B$ and only transforms $f^a_j$ and shifts its curl free piece. The $(B, f)$ formulation thus has the advantage that all constraints (dual Gauss constraint, spacetime constraints) are linear in $B$ and the spacetime constraints contain the momentum $B$ conjugate to $f$ only algebraically.

3 Physical Hamiltonian for first class constrained systems with certain linearity properties

In this section, we state and prove two simple results on the reduced dynamics of constrained systems which have a certain linear structure in their constraints with respect to the momenta. This linear structure will be tailored to the appearance of the constraints of the $U(1)^3$ model displayed in the previous section. Hence the subsequent results will be applicable to the $U(1)^3$ model both in its $(A, E)$ and its $(B, f)$ description. In what follows, as customary, we distinguish between constraint surface (constraints vanish) and reduced phase space (constraint surface modulo gauge transformation) or (locally) equivalently the gauge cut (constraints and gauge conditions vanish).

3.1 The $(A, E)$ description

We have seen that the $U(1)^3$ model has seven first class constraints: Three Gauss constraints $C_j(x)$, $j = 1, 2, 3$ for each spatial point $x$ which do not depend on the connection $A^a_j$, three density weight two vector constraints $\tilde{C}_j(x)$ linear in the connection and one density weight two Hamiltonian constraint $\tilde{C}_0(x)$ also linear in the connection. We integrate the constraints with respect to a smooth orthonormal basis of test functions $b^\alpha$ on $L_2(d^3x, \sigma)$ where $\alpha$ takes values in a suitable index set thus obtaining $C^\alpha_j = \int d^3x \, C_j(x) \, b^\alpha(x)$ etc. We denote the $C^\alpha_j$ collectively by $C_A$ and the $\tilde{C}^\alpha_j$, $\mu = 0, \ldots, 3$ by $C_I$ for suitable ranges of the indices $A, I$.

We can integrate the canonical pairs $(A^a_j, E^a_j)$ with respect to the same test functions and subdivide them as $w = (u^A, v_A)$, $z = (x^I, y_I)$, $r = (q^a, p_a)$. These are still conjugate pairs and in particular all variables from $w$ have vanishing Poisson brackets with all variables from $z, r$ and all variables from $z$ have vanishing Poisson brackets with variables from $r$. Then the completeness relation for the basis $b^\alpha$ we can write the constraints in the form

$$C_A = C_A(u, x, q), \quad C_I = M^I_J(u, x, q) \, y_J + N^I_A(u, x, q) \, v_A + h_I(u, x, q, p)$$

(3.1)

In our case $C_A$ depends linearly on $u, x, q$ and $h_I$ depends linearly on $p$ but we will not need to use this. Moreover, $M^I_J$, $N^I_A$, $h_I$ are homogeneous and quadratic in $u, x, q$ but we will not need to use this either. What we need is that the subdivision of the canonical pairs into groups is done in such a way that the “matrices” $\sigma_{AB} := \{C_A, v_B\}$ and $M^I_J$ are non-singular and this guides the above subdivision of canonical pairs. This means that $G_A := v_A = 0$ is a suitable gauge fixing condition for $C_A$ and $G^I := F^I(x) - \tau^I$ for $C_I$ where $\tau^I$ are constants on the phase space but possibly functions of physical time $\tau$ and $\Delta_J I := \{y_J, F^I\}$ is non-singular. Indeed, recall [8] that depending on whether the spatial manifold $\sigma$ has a boundary or not, a time dependence of $\tau$ is not necessary or necessary respectively in order that the physical dynamics be non-trivial due to the different boundary conditions on the canonical variables in these two cases. Our assumptions imply, by the implicit function theorem, that $C_A = 0$ can locally be solved for $u^A = g^A(x, q)$ and $G^I = 0$ for $x^I = k^I(\tau)$ for suitable functions $g^A, k^I$.

We thus declare $r$ as our physical degrees of freedom. Note that in field theory the statement that infinite dimensional matrices are non singular is to be taken with care: If for instance differential operators are involved then an inverse of say $M^I_J$ only exists if one specifies a suitable function space possibly accompanied with boundary conditions on them such that otherwise unspecified constants of integration are uniquely fixed.

The first class Hamiltonian is given by

$$H(\lambda, \Lambda) = \lambda^A C_A + \Lambda^I C_I$$

(3.2)
where $\lambda^A$ are the Lagrange multipliers of the Gauss constraint integrated against the basis and likewise $\Lambda^I$ are the density weight minus one lapse and shift functions integrated against that basis, in particular they are phase space independent. We collectively denote them by $l = (\lambda, \Lambda)$. The stability of the gauge conditions under gauge transformations fixes the Lagrange multipliers

$$\dot{G}_A = \{H, G_A\} = \lambda^B \sigma_{BA} + \Lambda^I \{C_I, G_A\} = 0; \quad \dot{G}^I = \{H, G^I\} = \Lambda^I M_J^K \Delta_K^I = \dot{\tau}^I$$

which has the explicit solution

$$\lambda_0^A = -\Lambda_0^I \{C_I, G_B\} (\sigma^{-1})^{BA} + \kappa^A, \quad \Lambda_0^I M_J^K = (\Delta^{-1})^I_J \dot{\tau}^J + \kappa^I =: \delta^I, \quad \Lambda_0^I = (M^{-1})_J^I \delta^J + \tilde{\kappa}^I$$

Here we assumed that both $\sigma, \Delta$ have an unambiguous inverse $\sigma^{-1}, \Delta^{-1}$ on a suitable space of functions and we allow that the space of Lagrange multiplier functions considered is larger and contains a kernel of $\sigma, \Delta$ leading to the “integration constants” $\kappa^A, \kappa^I$. By construction, $\kappa^A$ can only depend on $u, x, q$ and $\kappa^I$ only on $x$. In the $U(1)^3$ model, $\kappa^A$ is in fact phase space independent. As far as $M, M^{-1}$ is concerned, we allow for a similar kernel function $\tilde{\kappa}^I$ which in general may depend on $u, x, q$.

The ambiguities $\kappa^A, \kappa^I, \tilde{\kappa}^I$ are supposed to be fixed by the boundary conditions and we will assume that they imply in particular $\kappa^A = 0$. We emphasise that $l_0 = (\lambda_0, \Lambda_0)$ are phase space-dependent and are defined by (3.4) on the whole phase space and not only on the constraint hypersurface $C_A = C_I = 0$ or the reduced phase space $C_A = G_A = C_I = G^I = 0$. However, note that $\Lambda_0$ does not depend on the momenta $v, y, p$ and $\Lambda_0^I M_J^K = \delta^I$ depends only on $x$.

Let now $F = F(r)$ be a function on the reduced phase space. Its evolution is the gauge motion defined by $H$ restricted to the reduced phase space that is

$$\dot{F} = \{H, F\}_{C=0, G=0, l=l_0}$$

If it exists, the physical Hamiltonian $h$ is a function $h = h(r, \tau, \dot{\tau})$ such that $\dot{F} = \{h, F\}$.

Abstracting from the concrete $U(1)^3$ model we can now state the following general result:

**Theorem.**

Let $C_A, C_I, G_A, G^I$ be as above. Then

$$h = (\Lambda_0^I h_I)_{C=0, G=0}$$

This means that we do not need to solve $C_I = 0$ for $y_I$, we only need to solve for $\Lambda^I$ as in (3.4) and not $\lambda^A$ and only at $C_A = G^I = 0$ which only is a restriction on the configuration degrees of freedom. Given the flexibility in choosing the gauge condition $G^I$ this makes it conceivable that one can arrive at an explicit expression for $h$.

**Proof:**

We have due to the imposition of $C_I = C_A = 0$

$$\dot{f} = (\lambda^A \{C_A, f\} + \Lambda^I \{C_I, f\})_{C=0, G=0, l=l_0} = \lambda_0^A \{C_A, f\}_{C=0, G=0} + \Lambda_0^I \{C_I, f\}_{C=0, G=0} = (\{\lambda_0^A C_A + \Lambda_0^I C_I, f\})_{C=0, G=0}$$

Now by (3.4)

$$\Lambda_0^I C_I = \delta^I y_I + \Lambda_0^I [N_I A v_A + h_I]$$

The first term depends only on $z = (x, y)$, the second depends only on $u, x, q, v$ and is linear in $v_A = G_A$. The first term hence has vanishing Poisson brackets with both $f$ and $v_A$, the second term has a Poisson bracket with both $f$ and $v_A$ which is non vanishing but linear in $G_B$. It thus follows from (3.4) and our assumption $\kappa^A = 0$

$$\{\lambda_0^A C_A, f\}_{C=0, G=0} = -\lambda_0^I \{C_I, G_B\} (\sigma^{-1})^{BA} \{C_A, f\}_{C=0, G=0}$$

$$= -\lambda_0^I \{C_I, G_B\} (\sigma^{-1})^{BA} \{C_A, f\}_{C=0, G=0}$$

$$= -\{h, G_B\} (\sigma^{-1})^{BA} \{C_A, f\}_{C=0, G=0}$$

$$= -\{h, G_B\} (\sigma^{-1})^{BA} \{C_A, f\}_{C=0, G=0}$$

6
where
\[ \tilde{h} = \Lambda_0^I h_I \]  
(3.10)

For the same reason
\[ \{ \Lambda_0^I C_I, f \}_{C=0, G=0} = \{ \tilde{h}, f \}_{C=0, G=0} \]  
(3.11)

Hence both terms (3.9) and (3.11) no longer depend on \( v, y \) so that \( G_A, C_I \) no longer have to be imposed, we only have to impose \( C_A, G^I \). Now we have explicitly from (3.6)
\[ h = \tilde{h}_{C_A=0, G^I=0} = \tilde{h}(u = g(x,q), x, q, p)_{x=k(\tau)} \]  
(3.12)

so that
\[ \{ h, f \} = \{ \tilde{h}, f \}_{C_A=0, G^I=0} = \{ \tilde{h}, f \}_{C_A=0, G^I=0} - \{ \tilde{h}, v_A \} \{ g^A, f \}_{C_A=0, G^I=0} \]  
(3.13)

Since \( C_A(u = g(q,x), x, q) \equiv 0 \) we have by taking the \( q^a \) derivative of this identity
\[ \{ C_A, f \} - \{ C_A, v_B \} \{ g^B, f \}_{C_A=0, G^I=0} = \{ C_A, f \} - \sigma_{AB} \{ g^B, f \}_{C_A=0, G^I=0} = 0 \]  
(3.14)

Thus comparing (3.9) with the second term in (3.13) we arrive at the desired result.
\[ \square \]

### 3.2 The \((B,f)\) description

In the \((B, f)\) reformulation the \( U(1)^3 \) model has seven first class constraints: Three “Bianchi” constraints \( \hat{C}_j \) linear in \( B \) with phase space independent coefficients as well as the already discussed constraints \( \hat{C}_\mu \) also linear in \( B \). This makes the discussion even simpler. Using the basis \( b^a \) from the previous subsection we now subdivide the canonical pairs \( (B^a_j, f^a_j) \) in just two groups \( z = (x^I, y_I) \) and \( r = (q^a, p_a) \) and write the constraints in the form
\[ C_I = M_I^J(x, q) y_J + h_I(x, q, p) \]  
(3.15)

where again the dependence of \( M_I^J \), \( h_I \) on \( x, q \) is at most quadratic and the dependence of \( h_I \) on \( p \) is at most linear, however, we will not need this. Again we assume that the subdivision is such that \( M_I^J \) is invertible on a sufficiently large space of functions as discussed before. We will impose gauge fixings of the form
\[ G^I(x) = F^I(x) - \tau^I = \text{whose stability under } H = \Lambda^I C_I \text{ leads to the solution for } \Lambda \]

\[ \Lambda_0^J M_I^J = (\sigma^{-1})_J^I \tau^I + \kappa^I =: \delta^I, \quad \Lambda_0^I = (M^{-1})_J^I \delta^I + \tilde{\kappa}^I \]  
(3.16)

where \( \sigma_I^J = \{ y_I, G^J \} \) depends only on \( x \), is invertible on a suitable space of functions and \( \kappa^I \) is in its kernel, thus depending only on \( x \). In particular, \( \delta^I \) only depends on \( x \). Likewise \( \tilde{\kappa} \) is in the kernel of \( M \) and may depend in general on \( x, q \).

Abstracting from the \( U(1)^3 \) model, we have the general result:

**Theorem.**

Let \( C_I, G^I \) be as above. Then the physical Hamiltonian reads
\[ h = (\Lambda_0^I h_I)_{G=0} \]  
(3.17)

This means that we can completely forget about the constraints with respect to the dynamics of the physical degrees of freedom \( r \) on the reduced phase space \( C = G = 0 \). We only need to solve \( G = 0 \) and compute \( \Lambda_0 \) as in (3.16) which depending on the choice of \( G \) may be practically conceivable.

**Proof:**

Using that
\[ \Lambda_0^I C_I = \delta^I y_I + \tilde{h}, \quad \tilde{h} = \Lambda_0^I h_I \]  
(3.18)
we have by steps familiar from the previous subsection for \( f = f(r) \)

\[
\dot{f} = \{\Lambda_0 C_I, f\}_{C=G=0} = \{\dot{h}, f\}_{G=0}
\]  

(3.19)

Since the solution of \( G^I = 0 \) is of the form \( x^I = g^I(\tau) \) and is independent of \( r \) it follows that we can set \( G = 0 \) also before computing the Poisson bracket.

\[ \square \]

4 Reduced Phase Dynamics in specific gauges

We construct the reduced phase and its physical Hamiltonian in various gauges which we introduce in the first subsection. In order to keep the technicalities at a minimum, we restrict attention to linear gauges. In the second subsection, we then apply the theorems of the previous section to compute the corresponding physical Hamiltonian.

4.1 Gauge fixing choices

We split the discussion into the \((A,E)\) and \((B,f)\) descriptions respectively.

4.1.1 \((A,E)\) description

As the Gauss constraint only involves \( E \) we must use a gauge fixing condition that involves \( A \). In the literature of Abelian gauge theories, the Coulomb gauge or axial gauge is popular. Here we consider an extension of both of them to fix also the spacetime diffeomorphism gauge symmetry. In what follows we arbitrarily select the \( z \)-coordinate as longitudinal and the \( x,y \) coordinates as transversal directions. Transversal spatial indices are now \( I,J,\cdots \in \{x,y\} \) while the longitudinal index is denoted by \( a = z \). We consider also “transversal” \( u(1)^3 \) indices \( \alpha,\beta,\cdots \in \{1,2\} \) and denote the longitudinal one by 3. The indices \( \alpha,\beta,\cdots \) like \( j,k,\cdots \) are moved with the Kronecker \( \delta_{\alpha\beta} \), so the index position of \( \alpha,\beta,\cdots \) is also irrelevant.

A. transversally magnetic Coulomb - transversally electric anti-Coulomb - longitudinally electric axial gauge (TMC-TEaC-LEA)

The gauge conditions are

\[
G^I = \delta^{IJ}\partial_I A^j_I, \quad \tilde{G}^I_j = \epsilon_{IJ}J^K \partial_K E^j_I, \quad \tilde{G}_0 = E^j_3 - f
\]

(4.1)

where \( f \) is a coordinate dependent function possibly also of physical time. Here \( \epsilon_{IJ} = \epsilon^{IJ} \) is the completely skew symbol in two dimensions and \( \delta_I^J \) is the Kronecker symbol in two dimensions. We will set

\[
\partial^I := \delta^{IJ}\partial_J, \quad \hat{\partial}^I := \epsilon^{IJ}\partial_J, \quad \hat{\partial}_I := \epsilon_{IJ}\hat{\partial}^J
\]

(4.2)

in what follows to simplify the notation. Clearly, these structures break the (spatial) diffeomorphism covariance as they should in order to be appropriate gauge fixings in particular of the spatial diffeomorphism gauge symmetry.

We note that the three sets of gauges affect three different sets of canonical pairs which have mutually vanishing Poisson brackets among each other, namely the transversal Coulomb pair \((\partial^I A^j_I, \Delta^{-1}\partial_I E^j_I)\), the transversal anti-Coulomb pair \((\hat{\partial}^I A^j_I, -\Delta^{-1}\hat{\partial}_I E^j_I)\) and the longitudinal axial pair \((A^3_j, E^j_3)\). Here we introduced the two-dimensional Laplacian

\[
\Delta := \partial^I \partial_I
\]

(4.3)

and \( \Delta^{-1} \) is a Green function, specifically \( \Delta^{-1}(x_1, x_2) = (2\pi)^{-1}\ln(||x_1 - x_2||) \) where we used the flat two dimensional metric. Note that the transformation to these canonical coordinates is canonical despite the fact that \( \Delta \) has a kernel, when inverting that transformation the kernel must be taken into account using the boundary conditions.

We note that the Gauss constraint \( C_j = \partial_a E^a_j = 0 \) together with the electric gauge conditions imply that the transversal curl \( \hat{\partial}_I E^j_I \) vanishes and that the transversal divergence \( \partial_I E^j_I = -\partial_z E^j_3 \) is fixed in terms of \( E^j_3 \). If
we choose the coordinate function $f$ to be independent of $z$ as we will do then in particular $\partial_1 E^l_3 = \partial_1 E^l_3 = 0$. Since $\sigma = \mathbb{R}^3$ is simply connected, $\partial_1 E^l_3 = 0$ implies that $\delta_{ij} E^l_3$ is an exact 1-form, that is, $E^l_3 = \delta_{ij} \partial_i g$ for a certain 0-form $g$ and thus $\partial_1 E^l_3 = \Delta g = 0$, hence the $g$ are harmonic functions with respect to the $x, y$ dependence. By our boundary conditions, $E^l_3$ must decay at infinity and $E^l_3$ is itself harmonic. As is well known, the only smooth harmonic function on $\mathbb{R}^3$ which decays at infinity is the trivial function. Thus our gauge conditions imply $E^l_3 = 0$. By the same reasoning from $\partial_1 E^l_3 = 0$ we infer $E^l_3 = \partial_1 g$ which when plugged into the Gauss constraint yields $\Delta g_\alpha = -\partial_1 E^l_3$. It follows that $g_\alpha = h_\alpha - \Delta^{-1} \partial_1 E^l_3$ where $h_\alpha$ is harmonic and thus $E^l_3 = \partial_1[h_\alpha - \Delta^{-1} \partial_1 E^l_3]$. The second term decays at infinity because $E^l_3$ does, hence the first term must approach $\delta^l_3$ at infinity. It follows that $\partial_1 g_\alpha = \delta^l_3$ is a harmonic function vanishing at infinity, hence must vanish itself. Accordingly $E^l_3 = \delta^l_3 - \partial_1 \Delta^{-1} \partial_1 E^l_3$ is completely expressed in terms of $E^l_3$. This means that the physical degrees of freedom in this gauge correspond to $(A^a_3, E^l_3)$. Indeed the condition $\partial_1 A^j_3 = 0$ together with the boundary conditions implies that $A^j_3$ is a two-dimensional curl $A^j_3 = \partial_k g^j$ for certain $g^j$ and one would solve the constraints for $g^j$ as well as for $A^3_3$ leaving $A^a_3$ unconstrained.

It is important to note that the gauge conditions are not in conflict with the requirement that the density weight two inverse spatial metric $Q^{ab} = E^j_3 E^k_3 \delta^{jk}$ be non-degenerate.

B. Magnetic longitudinal axial - electric tranverse transverse axial (MLA-ETTA) gauge

The gauge conditions are

$$G^j := A^j_3, \ G^l_\alpha := E^l_3 - f^l_\alpha$$

(4.4)

for certain coordinate dependent functions possibly $f^l_\alpha$ also of physical time, e.g. $f^l_\alpha = f^i_\alpha$. This gauge is somewhat simpler in that it does not involve derivatives. If we assume, as we will, that the $f^l_\alpha$ do not depend on $x, y$ then we have from the Gauss constraint $\partial_1 E^l_3 = 0$ which means that $E^l_3$ is independent of $z$. Moving to infinity along the $z-$axis at fixed finite values of $x, y$ this can only decay if $E^l_3$ vanishes identically. The third Gauss constraint $\partial_3 E^l_3 + \partial_1 E^l_3 = 0$ is solved by $E^l_3 = k - \Delta^{-1} \partial_3 E^l_3$ where the Green function $\Delta^{-1}$ is chosen to be $(\delta^{-1}_3)(z_1, z_2) = \frac{1}{4} \text{sgn}(z_1 - z_2)$ which makes it an antisymmetric translation-invariant kernel and $k$ is a function independent of $z$. Since the second term in $E^l_3$ vanishes at infinity, $k$ must approach unity at infinity and $k - 1$ must decay. However, since it is independent of $z$ this is only possible if in fact $k = 1$. It follows that the true degrees of freedom in this gauge are $(A^a_3, E^l_3)$ since one would solve the spacetime diffeomorphism constraints for $A^j_3$.

Note that this gauge is complementary to the previous one as it leaves disjoint sets of canonical pairs as true degrees of freedom. Again it is easy to verify that the gauge conditions are not in conflict with the requirement that the density weight two inverse spatial metric $Q^{ab} = E^j_3 E^k_3 \delta^{jk}$ be non-degenerate.

4.1.2 $(B, f)$ description

First of all, we need to know how the canonical variables $(B, f)$ behave in an asymptotically flat spacetime. By transcribing the boundary conditions imposed on the $(A, E)$ variables, i.e. (2.2), to $(B, f)$, we have

$$e^{abc} \partial_b f^j_a \rightarrow F^a_j(n)/r + O(1/r^2),$$

(4.5)

$$B^j_a \rightarrow \tilde{G}^j_a(n)/r^3 + O(1/r^4)$$

(4.6)

where $\tilde{G}^j_a$ are even functions on the asymptotic sphere and $\tilde{G}^a_j = e^{abc} (\delta^d_b - n_d n^b) \partial_c \tilde{G}^j_a$. Given (4.6) and also taking into account the requirement that the symplectic structure must be well defined, the asymptotic behaviour of $f^a_j$ has to be

$$f^j_a \rightarrow c^j_a + \tilde{F}^j_a(n) + O(1/r)$$

(4.7)

where $c^j_a$ are constants, $\tilde{F}_a^j(n)$ are odd functions on the asymptotic $S^2$ and $F^a_j(n) = e^{abc} (\delta^d_b - n_d n^b) \partial_c \tilde{F}^j_a$. As shown in detail in Appendix A there is no well-defined generator for asymptotic symmetries with these boundary conditions. As a result, the use of them is not acceptable. In Appendix A an alternative to these boundary conditions is considered, which ultimately leads to the conclusion that Hamiltonian constraint and diffeomorphism constraint are well-defined generators for temporal and spatial translations, respectively. Since here we just deal with the spacetime translations, the lack of generators for boosts and rotations does not have
influence on the following calculations. The lack of well-defined generators for boosts and rotations occurs not in the \((B, f)\) description, but also in that of \((A, E)\). The latter has been studied in detail in \[11\]. Note that since the \(U(1)^3\) model is not GR, it is not required to have Poincaré group as its asymptotic symmetries. The appropriate boundary conditions have the fall-off behaviours just the same as (4.6) and (4.7) but with different parity conditions

\[
\bar{F}_a^i(n) = -\bar{G}_a^i(n),\quad \bar{G}_a^i(n) = -\bar{G}_a^i(n).
\]  

(4.8)

Although the chosen boundary conditions do not match those that one would choose in General Relativity, as the \(U(1)^3\) theory is just a toy model for a generally covariant theory with non-trivial dynamics and an infinite number of degrees of freedom, we are allowed to exploit the freedom that is allowed in defining the theory (the choice of boundary condition and choice of polarisation of the phase space is such an element of freedom) in order to learn as much as possible about the actual theory. Note that in the case of topologies without boundary, the choice of boundary conditions is immaterial and the \((A, E)\) description and the \((B, f)\) description are equivalent (see Appendix A).

As the form of Lagrange multiplier \(\Lambda^i\) corresponding to the Bianchi constraint plays an important role in what follows, it is required to find the minimal condition on \(\Lambda^i\) ensuring differentiability and convergence of the Bianchi constraint \(C_i[\Lambda^i] = \int d^3x \Lambda^i \partial_n B^a_i\). It is shown in Appendix A that

\[
\Lambda^i = \lambda^i + O(1)
\]  

(4.9)

where \(\lambda^i\) are odd functions defined on the asymptotic \(S^2\), i.e.

\[
\lambda^i(-n) = -\lambda^i(n)
\]  

(4.10)

The gauge fixings discussed below are consistent with the above asymptotic behaviours. We need also the expression of \(H^j_a\) written in terms of \(f^a_i\) which can be easily derived from combining (2.8) and (4.7).

\[
H^j_a = \delta^j_a + e^{jkl}(\partial_a f^l_k - \partial_b f^l_a)[\delta^b_k + \frac{1}{2} \epsilon^{bde}(\partial_d f^e_k)]
\]  

(4.11)

As in the \((A, E)\) description, we consider \(z\)-coordinate as the longitudinal direction and \(I, J, \cdots \in \{x, y\}\) as the transversal ones. Similarly, in the internal indices, 3 is considered as the longitudinal and \(\alpha, \beta, \cdots \in \{1, 2\}\) as the transversal directions.

In what follows, appearing several differential operators and functions in a row means that each operator acts on all the operators and functions existing in its right side, for instance if we have an operator as \(X := \partial_x g \partial^{-1}_y f + h\) in which \(f, g, h\) are functions, then the action of \(X\) on a function \(u\) would be \(Xu = \partial_x(g \partial^{-1}_y(fu)) + hu\).

A.

**Electric transverse axial - longitudinally electric axial (ETA-LEA) gauge**

The gauge conditions are

\[
G^i_j = f^i_j - \sigma^j_i, \quad G = f^3_z - \sigma
\]  

(4.12)

in which \(\sigma^j_i\) and \(\sigma\) are phase space independent functions which may depend only on the physical time, \(\tau\). If we assume, as we will, that \(\sigma^j_i\) and \(\sigma\) do not depend on the spatial coordinates, then not only will the calculations be simpler but also we can make sure that (4.12) is compatible with (4.7) and (4.8). Applying the gauge fixing (4.12) on \(H^j_a\) results in

\[
\begin{align*}
H^j_x &= \delta^j_x + e^{jkl}(\partial_x f^l_k - \partial_y f^l_x)[\delta^b_k + \frac{1}{2} \epsilon^{bde}(\partial_d f^e_k)] = \delta^j_x + e^{j3l}(\partial_x f^l_3) \\
H^j_y &= \delta^j_y + e^{jkl}(\partial_y f^l_k - \partial_x f^l_y)[\delta^b_k + \frac{1}{2} \epsilon^{bde}(\partial_d f^e_k)] = \delta^j_y + e^{j3l}(\partial_y f^l_3) \\
H^j_z &= \delta^j_z + e^{jkl}(\partial_z f^l_k - \partial_y f^l_z)[\delta^b_k + \frac{1}{2} \epsilon^{bde}(\partial_d f^e_k)] = \delta^j_z - e^{j1l}(\partial_x f^l_3) - e^{j2l}(\partial_y f^l_3) + e^{jl3}(\partial_y f^l_3)(\partial_x f^l_k)
\end{align*}
\]  

(4.13)
Thus, $H_1^3 = H_2^\alpha = 0$ and the non-vanishing ones are

$$
\begin{align*}
H_1^1 &= 1 - \partial_x f_2^3, \\
H_2^1 &= \partial_x f_2^3, \\
H_1^2 &= -\partial_y f_1^2, \\
H_2^2 &= 1 + \partial_y f_1^2, \\
H_3^1 &= 1 - \partial_x f_2^2 + \partial_y f_1^1 - (\partial_x f_2^1)(\partial_y f_1^1) + (\partial_x f_1^1)(\partial_y f_2^1) \\
&= H_1^1 H_2^2 - H_2^2 H_1^1 \\
\end{align*}
$$

(4.14)

With the $H_\alpha^\alpha$ evaluated at the gauge cut, the constraints are much simpler to solve for $B_i^1$ and $B_3^\alpha$. In fact, the matrix representation of (3.15) in this gauge is

$$
\begin{bmatrix}
\partial_x & 0 & 0 & \partial_y & 0 & 0 & 0 \\
0 & \partial_x & 0 & 0 & \partial_y & 0 & 0 \\
0 & 0 & \partial_x & 0 & \partial_y & \partial_z & 0 \\
0 & 0 & -H_2^x & 0 & 0 & -H_3^x & 0 \\
0 & 0 & H_1^x & 0 & 0 & H_1^y & 0 \\
H_1^x & H_2^x & 0 & H_1^y & H_2^y & 0 & H_3^y \\
\end{bmatrix}
\begin{bmatrix}
B_1^i \\
B_2^i \\
B_3^i \\
B_4^i \\
B_5^i \\
B_6^i \\
\end{bmatrix} + \begin{bmatrix}
\partial_z B_1^i \\
\partial_z B_2^i \\
\partial_z B_3^i \\
\partial_z B_4^i \\
\partial_z B_5^i \\
\partial_z B_6^i \\
\end{bmatrix} = 0 \\
(4.15)
$$

Note that the existence of the solutions for this system of equations is essentially dependent on the non-vanishing condition of $H_3^3$ which is guaranteed in the MTA-MLA gauge (see Appendix B). According to (2.8) and the fact that in this gauge $H_1^1 = H_2^2 = 0$, vanishing of $H_3^3$ would lead to a degenerate $E$, which contradicts the requirement that the spatial metric must be non-degenerate. Thus, $H_3^3 \neq 0$ everywhere.

As one can see in detail in Appendix B solving the system (4.15) using the boundary conditions completely gives us $(B_1^1, B_3^\alpha)$ in terms of $B_\alpha^\alpha$. Therefore, the true degrees of freedom in this gauge are $(B_\alpha^\alpha, f_2^\alpha)$.

In this gauge, one can easily see that

$$
E_\alpha^a = \begin{bmatrix}
H_2^y & -H_1^y & 0 \\
-H_2^y & H_2^x & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
$$

(4.16)

Therefore the gauge conditions are not in conflict with the requirement that the density weight two inverse spatial metric $Q^{ab} = E_\alpha^a E_\beta^b \delta^{ij}$ be non-degenerate.

B. Electric transverse transverse axial- electric longitudinal axial (ETTA-ELA) gauge

The gauge conditions are

$$
G_1^\alpha = f_1^\alpha - \sigma_1^\alpha, \quad G_\beta^i = f_2^i - \sigma_\beta^i
$$

(4.17)

where $\sigma_1^\alpha$ and $\sigma_\beta^i$ are phase space independent functions which may depend only on the physical time, $\tau$. In order to make the following calculations simpler we assume that $\sigma_1^\alpha$ and $\sigma_\beta^i$ do not depend on the spatial coordinates. It is assured that (4.12) is totally compatible with (4.7) and (4.8). A straightforward calculation shows that in this gauge $H_\alpha^\alpha$ has a very simple form, that is

$$
\begin{align*}
H_1^1 &= H_2^2 = 1 + \partial_x f_3^y - \partial_y f_3^x \\
H_1^2 &= \partial_x f_3^y - \partial_y f_3^x = \partial_x f_3^y \\
H_2^2 &= \partial_x f_3^y - \partial_y f_3^x = -\partial_x f_3^x \\
H_3^1 &= H_3^2 = H_1^3 = H_2^3 = 0, \quad H_3^3 = 1 \\
\end{align*}
$$

(4.18)

With these $H$’s, we can see that the system of equations that has to be solved here is even simpler than that of
the previous gauge, i.e. (4.15). In fact, the matrix representation of (3.15) in this gauge is

\[
\begin{bmatrix}
\partial_x & 0 & \partial_y & 0 & \partial_z & 0 & 0 \\
0 & \partial_x & 0 & \partial_y & 0 & \partial_z & 0 \\
0 & 0 & 0 & 0 & 0 & \partial_z & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -H^z_x \\
0 & -H^1_x & H^1_x & 0 & H^1_z & -H^1_z & 0 \\
H^1_x & 0 & 0 & H^1_x & H^1_z & H^2_z & 1 \\
\end{bmatrix}
\begin{bmatrix}
B^x_z \\
B^z_z \\
B^y_z \\
B^y_z \\
B^1_z \\
B^1_z \\
B^1_z \\
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix} = 0
\] (4.19)

Note that the existence of the solutions for (4.19) basically depends on the non-vanishing condition of \( H^1_x \) which is guaranteed in the ETTA-ELA gauge (see Appendix B). According to (2.8) and the fact that in this gauge \( H^1_x = H^1_y = 0 \), vanishing of \( H^1_x \) would lead to a degenerate \( E \), which contradicts the requirement that the spatial metric must be non-degenerate. Thus, \( H^1_x \neq 0 \) everywhere.

In Appendix B it is shown that by solving the system (4.19) using the boundary conditions, one can express \((B^I_\alpha, B^j_\beta)\) completely in terms of \( B^3_\beta \). Therefore, the true degrees of freedom in this gauge are \((B^3_\beta, \tilde{f}_I^j)\).

In this gauge, one can easily see that

\[
E^a_i = \begin{bmatrix}
1 & 0 & -H^z_x \\
0 & 1 & -H^z_z \\
0 & 0 & H^1_x \\
\end{bmatrix}
\] (4.20)

Therefore, again, the gauge conditions are not in conflict with the requirement that the density weight two inverse spatial metric \( Q^{ab} = E^a_j E^b_k \delta^j_k \) be non-degenerate.

4.2 Reduced phase space dynamics in various gauges

We will now discuss the reduced phase space dynamics in various gauges and using different pairs of canonical coordinates.

4.2.1 MLA-ETTA gauge in \((A, E)\) description

We verify that the assumptions of theorem 3.1 apply:

First recall that in this gauge condition we have \( w = (E^z_j, A^j_z) \), \( z = (E^I_\alpha, A^\alpha_I) \) and \( r = (E^z_\beta, A^\beta_3) \). We interpret (minus) the \( A^j_z \) as the momenta \( v_A \) and (minus) the \( A^\alpha_I \) as the momenta \( y_I \). Likewise the Gauss constraints are considered as the \( C_A \) and the spacetime constraints as the \( C_I \).

We have

\[
\{C(\lambda), G_j\} = \{C(\lambda), A^j_z\} = -\kappa \partial_z \lambda^j
\] (4.21)

which has as kernel the space of functions that are independent of \( z \). If we consider as space of functions \( \lambda^j \) those that decay at infinity at least as \( 1/r^2 \) this means that the kernel vanishes and the matrix \( \sigma_{AB} \) considered as the integral kernel \( \{C_j(x), G_j(y)\} \) is regular.

Next we have

\[
\bar{C}(U, F) = \int d^3x B^a_j (F H^a_j + e^{ijkl} H^k_a U^l) =: \int d^3x B^a_j u^a_j =: \int d^3x A^a_j v^a_j
\] (4.22)

with

\[
v^a_j = e^{abc} \partial_b u^c_j, \ u^a_j = F H^a_j + e^{ijkl} H^k_a U^l
\] (4.23)

Geometrically, the relation between the density weight minus smearing functions \( F, U^j \) one and the density weight zero lapse and shift functions \( N, N^a \) is that

\[
F = N |\det(E)|^{-1/2}, \quad U^j E^a_j = N^a
\] (4.24)
i.e. $N^a$ is an “electric shift” \footnote{The lapse function $N$ and shift vector $N^a$ can be considered as Lagrange multipliers of the Hamiltonian and diffeomorphism constraints, respectively. Alternatively, one can consider the density weight $-1$ versions of them as the smearing functions of the constraints, i.e., $N|\det(E)|^{-1/2}$ and $N^a E^{ij}_a$, respectively. This is the geometric origin of \eqref{electric_shift}.} if $U, F$ are considered as phase space independent.

Obviously
\[
\{ \tilde{C}(U, F), E_j^\alpha \} = -\kappa v_j^\alpha \tag{4.25}
\]
and we must verify that, given asymptotically flat boundary conditions, $-\kappa v_j^\alpha = \hat{\tau}_j^\alpha$ implies unique values for $F, U$ in order that the gauge fixing be admissible. It will be sufficient to verify this at the gauge cut $C_j = \partial_a E_j^3 = 0$, $E_j^I = \hat{\tau}_j^I$ because then by continuity, it will hold in a neighbourhood of the gauge cut by the usual reasoning of the implicit function theorem (to make this precise, the phase space should be given a suitable (Banach) manifold topology).

To be specific we consider $\tau_j^I = f \delta_j^I$ where $f$ is at most depending on the physical time. In fact this is compatible with asymptotic flatness if we further specify $f \equiv 1$, but let us be more general for the moment. As already mentioned, from the Gauss constraint we then obtain $\partial_a E_j^3 = 0$ which implies $E_j^3 = 0$ if we use the boundary conditions. Furthermore $\partial_a E_j^3 = 0$ is uniquely solved by $E_j^3 = 1 - \partial_z^{-1} \partial_I E_j^3$ so that $E_j^3$ is completely determined by $E_j^I$. We evaluate the $H_j^3$ at the gauge cut
\[
H_j^3 = \det(\{ E_j^\alpha \}) = f^2 \tag{4.26}
\]
and find with $\hat{U}_\alpha = \epsilon_{\alpha\beta} U^\beta$, $\hat{U}^\alpha = \epsilon_{\alpha\beta} U_\beta$
\[
\begin{align*}
F H_j^3 + \epsilon_{\alpha\beta} H_j^\alpha U^\beta &= F H_j^3 + H_j^3 U_\alpha \\
u_j^\alpha &= F H_j^\alpha + \epsilon_{\alpha\beta} (H_j^\beta U^3 - H_j^3 U^\beta) = [F \delta_j^\alpha + U^3 \epsilon_{\alpha\beta}] H_j^3 - H_j^3 U_\alpha \\
\quad &= \Sigma_{\beta} H_j^3 - H_j^3 U_\alpha \tag{4.27}
\end{align*}
\]
The stability condition resulting from \eqref{4.25} is
\[
-\hat{f}/\kappa \delta_j^I = v_j^\alpha = \epsilon^{IJ} [\partial_J u_j^\alpha - \partial_z u_j^\alpha] \tag{4.28}
\]
This can be transformed into
\[
\partial_I u_j^\alpha = \frac{\hat{f}}{\kappa} \epsilon_{IJ} \delta_j^I + \partial_z u_j^\alpha \tag{4.29}
\]
Let us introduce
\[
a := f E_j^3 F, \quad b := f E_j^3 U^3 \tag{4.30}
\]
then using \eqref{4.26}, \eqref{4.27}
\[
u_j^\alpha = a \delta_j^\alpha + b \epsilon_{\alpha\beta} \delta_j^\beta \tag{4.31}
\]
and the four equations \eqref{4.29} can be disentangled
\[
\partial_x u_j^1 = \partial_y u_j^2 = \partial_z a; \quad \partial_y u_j^1 = -\partial_z u_j^2 = \partial_z b - \frac{\hat{f}}{\kappa} \tag{4.32}
\]
We thus find that $(u_j^1, u_j^2)$ forms a Cauchy-Riemann pair, i.e. $w = u_j^1 + i u_j^2$ is a holomorphic function in $x + iy$. On the other hand from \eqref{4.26} and \eqref{4.27} we infer that $u_j^\alpha$ is a bounded function if $U^3, F, E_j^I$ are or that it decays at least as $r^{-1}$ if we allow $N, N^a$ and thus $F, U^3$ (recall \eqref{electric_shift}) to approach asymptotically at most a constant and zero respectively. In the former case, we infer $u_j^\alpha = \text{const.}$ by Liouville’s theorem. In the latter case
we obtain immediately \( u^\alpha_z = 0 \) because a holomorphic function decays at most in \( x^2 + y^2 \) but not in \( r^2 \). In either case
\[
\partial_z a = 0 = \partial_z b - \frac{\dot{f}}{\kappa}
\]
which is solved by
\[
a = a_0(x, y), \quad b = b_0(x, y) + \frac{\dot{f}}{\kappa} z
\]
Comparing with (4.30) we see that \( a \) approaches asymptotically a constant up to \( O(1/r) \) corrections and that \( b \) decays at least as \( 1/r \). Since a function of \( x, y \) cannot decay in \( z \) as \( r \) does, we must have in fact \( a_0 = \text{const.} \) and \( b_0 = \dot{f} = 0 \). Furthermore by comparing the asymptotic values we find \( a_0 = f \) and we conclude \( f = 1 \) from \( E_j^\alpha = \delta_j^\alpha + O(1/r) \). Accordingly
\[
F = \frac{1}{E_3^z}, \quad U^3 = 0
\]
and thus
\[
u^\alpha_z = F H_z^\alpha - H_z^3 \dot{U}^\alpha = \epsilon^\alpha
\]
where \( \epsilon^\alpha = \text{const.} \). However, since \( H_z^3 = 1 \) a non vanishing constant is incompatible with a decaying \( U^\alpha \) so that in fact \( u^\alpha_z = 0 \) and
\[
\dot{U}^\alpha = \frac{H^\alpha_z}{E_3^z} = -\delta^\alpha_j E_j^3 / E_3^z
\]
We conclude that the gauge fixing has resulted in a unique solution for \( F, U^3 \) and thus was admissible.

We now apply theorem 3.1 to compute the physical Hamiltonian \( h \) which depends only on the physical degrees of freedom. We thus decompose
\[
\hat{C}(F, U) = \int d^3x \ [v_\alpha^I A^\alpha_I + v_j^3 A^j_3 + v_3^I A^3_I] \tag{4.38}
\]
and identify the first term with \( \Lambda^I M_I J^J y_J \), the second term with \( \Lambda^I N_I \Lambda^\alpha A_\alpha \) and the third term with \( \Lambda^I h_I \) where \( \Lambda^I \) corresponds to \( (F, U) \). Upon integrations by parts we have
\[
\Lambda^I h_I = \int d^3 x \ \epsilon^{I J} (\partial_J u_z^I - \partial_z u_J^I) A^3_I
\]
\[
= \int d^3 x \ [u_3^J (\epsilon^{I J} \partial_I A_3^J) + u_3^I (\epsilon^{I J} \partial_J A_3^J)]
\]
\[
= \int d^3 x \ [u_3^J B_3^J + u_3^I B_3^I]
\]
\[
= \int d^3 x \ [F(H_z^3 B_3^3 + H_t^3 B_3^3) + \dot{U}^\alpha (H_\alpha^3 B_3^3 + H_\alpha^3 B_3^3)]
\]
\[
= \int d^3 x \ [F H_\alpha^3 B_3^3 + \dot{U}_\alpha H_\alpha^3 B_3^3] \tag{4.39}
\]
where \( B_3^I \) is to be evaluated in the magnetic longitudinal axial gauge \( A_3^I = 0 \). Note that in fact (4.39) does not depend on \( U^3 \) even before using the constraints, the gauge conditions, and the gauge fixed Lagrange multipliers. According to theorem 3.1 we only need to impose the gauge \( E_\alpha^J = \delta_\alpha^J \) and use the solution \( E_3^z = 0, \ E_3^z = 1 - \partial_z^J \partial_I E_3^I \) of the Gauss constraint as well as the solution for \( h_I = (F_0, U_0) \) at these values which are displayed in (4.35), (4.37) and then insert those into (4.39). This yields the final expression
\[
h = \int d^3 x \ [B_3^J (F + H_\alpha^J \dot{U}_\alpha) + B_3^J H_\alpha^J \dot{U}_\alpha]
\]
\[
= \int d^3 x \ [B_3^J \left( \frac{1}{E_3^z} + \delta_{I J} \frac{E_I^J E_3^J}{E_3^z} \right) - B_3^J \delta_{I J} E_3^J]
\]
\[
= \int d^3 x \ A_3^J \epsilon^{I J} \left[ \partial_J \left( \frac{1 + \delta_{I J} E_3^J E_3^J}{E_3^z} \right) - \partial_z \delta_{I J} E_3^J \right] \tag{4.40}
\]
As expected, the physical Hamiltonian (4.40) has the following features:
I. Linearity in momentum $A^I_J$.
II. Non-polynomiality in the configuration variable $E_3^I$.
III. Spatial non-locality.

The latter two properties are due to the appearance of $1/E^3_3$ with $E^3_3 = 1 - \partial^{-1}_z \partial_I E_3^I$. Note that the term $1 + \delta_{JL}E^L_3 E^J_3$ is meaningful from a dimensional point of view because $E^3_3$ is dimension-free and $A^I_J$ has dimension cm$^{-1}$. Thus the Hamiltonian density has dimension cm$^{-2}$ and to turn it into a quantity with dimension of energy we have to divide $h$ by $\kappa$ (if time is multiplied by the speed of light). In fact, we should have worked all the time with constraints rescaled by $1/\kappa$ as they would naturally appear out of an action [14]. The fact that the three individual terms in the Hamiltonian have different density weights is due to the gauge fixing condition $E^3_\alpha = \delta^I_\alpha$ which breaks the density weight of $E^I_3$.

We can now derive the physical equations of motion and indicate strategies for how to solve them, despite the complexity of the Hamiltonian (4.40). Beginning with $E^I_3$ we have, using $\{E^I_3(x), A^J_3(y)\} = \kappa \delta^I_\alpha \delta(x, y)$

$$
\dot{E}^I_3 = \{\frac{\hbar}{\kappa}, E^I_3\} = -\epsilon^{IJK}[\partial_J(\frac{1 + \delta_{IJ}E^L_3 E^J_3}{E^3_3}) - \partial_z \delta_{JK}E^K_3] 
$$

(4.41)

As expected, due to linearity of $h$ in $A$, the equation of motion for $E^I_3$ is of first order in time and closes on itself, i.e. its time derivative no longer involves $A^I_J$ and one can study (4.41) completely independently of the equation of motion for $A^I_J$. Next we have

$$
\dot{A}^I_J (x) = \{\frac{\hbar}{\kappa}, A^I_J(x)\} = (-B^I_J + 2B^I_J E^I_3 E^J_3)(x) - \int d^3y \left( B^I_J \frac{1 + \delta_{JK}E^I_3 E^J_3}{(E^3_3)^2}(y) \frac{\delta E^I_3(y)}{\delta E^I_3(x)} \right) 
$$

(4.42)

We have for any smearing function $s$ using $E^3_3 = -\partial^{-1}_z \partial J E^I_3$, integrations by parts and the antisymmetry of the integral kernel of $\partial^{-1}_z$ as well as $[\partial^{-1}_z, \partial_I] = 0$

$$
\int d^3y \ s(y) \frac{\delta E^3_3(y)}{\delta E^3_3(x)} = \int d^3y \ \partial^{-1}_z \partial Js(y) \frac{\delta E^3_3(y)}{\delta E^3_3(x)} = (\partial^{-1}_z \partial Js)(x) 
$$

(4.43)

whence

$$
\dot{A}^I_J (x) = \{\frac{\hbar}{\kappa}, A^I_J(x)\} = (-B^I_J + 2B^I_J E^I_3 E^J_3 - \partial^{-1}_z \partial_I(B^I_J \frac{1 + \delta_{JK}E^I_3 E^J_3}{(E^3_3)^2})) 
$$

(4.44)

Note the abbreviations $B^I_J := -\epsilon^{IJK} \partial_z A^J_3$, $B^3_3 := \epsilon^{IJK} \partial_I A^3_J$.

Equations (4.41), (4.44) suggest the following solution strategy: First solve (4.41) which is independent of $A$. Then plug that solution into (4.44) which is then a linear integro-differential equation system of first order in all derivatives and the anti-derivative $\partial^{-1}_z$. Turning to the first task we introduce the divergence and curl of $E^I_3$

$$
D := \partial_I E^I_3, \ C := \epsilon^{IJK} \partial_I (\delta_{JK}E^K_3) 
$$

(4.45)

and decompose

$$
E^I_3 = \Delta^{-1} [\delta^{IJ} \partial_j D - \epsilon^{IJK} \partial_J C] 
$$

(4.46)

where $\Delta$ is the transversal Laplacian and we used the boundary conditions ($E^I_3$ must vanish as $1/r$) to exclude a non-vanishing kernel of $\Delta$. Taking the divergence and curl of (4.41) we find

$$
- \dot{D} = -\partial_z C, \quad -\dot{C} = -\Delta \frac{1 + \delta_{IJ}E^I_3 E^J_3}{E^3_3} + \partial_z D 
$$

(4.47)

Note that if we would drop the non-linear interaction term in the second equation of (4.47) and would iterate it we would find

$$
\dot{D} + \partial_z^2 D = \dot{C} + \partial_z^2 C = 0 
$$

(4.48)
which is a “Euclidean” wave operator restricted to the $z$ direction. We interpret this to express the closeness of the model to Euclidean gravity. Continuing with (4.47) we remember that $D = -\partial_z E^z_3$ so that the first equation in (4.47) implies (again possible integration constants must vanish)

$$C = -\dot{E}^z_3$$  \hspace{1cm} (4.49)

This means that all of $E^I_3$ can be written just in terms of $F := E^z_3$

$$E^I_3 = -\Delta^{-1} [\delta^{IJ} \partial_J \partial_z - \epsilon^{IJ} \partial_J \partial_I] F$$  \hspace{1cm} (4.50)

which allows to write the second equation in (4.47) just in terms of $F$

$$\dot{F} + \partial_z^2 F = -\Delta \left[ \frac{1 + \delta^{IJ} E^I_3 E^J_3}{F^2} \right]$$  \hspace{1cm} (4.51)

where (4.50) is to be used on the r.h.s.. Equation (4.51) has the undesirable feature that it is non-polynomial (due to $1/F$) and spatially non-local (due to $\Delta^{-1}$ in (4.50)). We can get rid of the first feature by multiplying (4.51) with $F^3$ because the Laplacian on the r.h.s. produces at most factors of $1/F^3$. We can get rid off the second feature by introducing $G = \Delta^{-1} F$, $F = \Delta G$. Accordingly, we write

$$E^I_3 = -[\delta^{IJ} \partial_J \partial_z - \epsilon^{IJ} \partial_J \partial_I] G$$  \hspace{1cm} (4.52)

and

$$(\Delta G)^3 [\partial_t^2 + \partial_z^2] (\Delta G) = - (\Delta G)^3 \left[ \Delta \left[ \frac{1 + \delta^{IJ} E^I_3 E^J_3}{\Delta G} \right] \right]$$  \hspace{1cm} (4.53)

where (4.52) has to be substituted. Thus (4.53) is a polynomial in $G$ of degree four and involves time derivatives up to second order, spatial derivatives up to order four. To see this we write with $K := \delta_{IK} E^I_3 E^J_3$

$$F^3 \Delta \left( \frac{1 + K}{F} \right) = F^3 \frac{\Delta K}{F} - 2 \frac{\partial_t K}{F^2} \partial^t F + (1 + K) \left[ \frac{\Delta F}{F^2} + 2 \frac{(\partial_t F)(\partial^t F)}{F^3} \right]$$

$$= F^2 (\Delta K) - 2 F(\partial_t K)(\partial^t F) + (1 + K) \left[ - F(\Delta F) + 2(\partial_t F)(\partial^t F) \right]$$  \hspace{1cm} (4.54)

We leave the further analysis of (4.53) for future work. It is remarkable that half of the equations of motion can be encoded just in terms of a single PDE! The fact that the theory is self-interacting is expressed by the fact that this PDE is far from linear and of spatial degree higher than two (namely four) but still of temporal degree at most two. In particular, it is a quasi-linear equation with respect to the highest time derivatives.

As far as Hilbert space representations are concerned that support (some ordering of) $h$ as a densely defined operator as well as the corresponding spectral problem consider the possibility of a representation in which $E^I_3$ acts as a multiplication operator, i.e. $\mathcal{H} = L^2(\mathcal{E}, d\mu)$ where $\mathcal{E}$ is a suitable distributional extension of the set $\mathcal{E}$ of the classical fields $E^I_3$ and $\mu$ a probability measure thereon. Then $A^I_3 = i\ell_P^2 \delta/\delta E^I_3 + D_I(E)$ where $D_I$ is chosen as to make $A^I_3$ a symmetric operator valued distribution and $\ell_P^2 = h\kappa$ is the Planck area. Then for any symmetric ordering of $h$, after reordering in such a way that the functional derivatives are acting directly on the Hilbert space vector $\psi \in \mathcal{H}$ we find that $h$ acts as as

$$(h\psi)[E] = i\ell_P^2 \int d^3 x \ V^I_3(E(x)) \frac{\delta \psi}{\delta E^I_3(x)} + U[E]\psi[E]$$  \hspace{1cm} (4.55)

where the potential term $U[E]$ acts as a multiplication operator and stems from reordering $h$ into the form displayed as well as from the contribution $V^I D_I$ where

$$V^I_3(E(x)) = \epsilon^{IJ} [\partial_J (\frac{1 + K}{F}) - \partial_z E^J_3]$$  \hspace{1cm} (4.56)

The reordering of $h$ in the form displayed produces in general singularities in the form of (derivatives of) $\delta$ distributions evaluated at zero which need to be regularised and which guide the choice of $\mu$ and thus $D_I$ to cancel them. Also the form of $V^I, U$ may raise non-trivial domain questions.
We can now recast the spectral problem for \( h \) into the form
\[
\langle \delta \frac{\delta E}{\delta E^3} \rangle < V^I, \delta \frac{\delta E}{\delta E^3} > \psi = (\lambda - U) \psi
\] (4.57)
where \( \langle . . . \rangle \) denotes the inner product on \( L_2(d^3x, \mathbb{R}^3) \) or equivalently with the WKB Ansatz \( \psi = \exp(-i \frac{S}{\hbar}) \)
\[
< V^I, \delta \frac{\delta S}{\delta E^3} > = (\lambda - U)
\] (4.58)

Equation (4.58) has the form of a linear functional partial differential equation of first order, that is, the infinite-dimensional analogue of the well-known case of a linear partial differential equation of first order. This is the simplest type of first-order FPDE (functional PDE) that one can imagine, it is not even quasi-linear (which would allow \( V^I, U \) to depend on \( S \)) or even non-linear (which would allow \( V^I, U \) to depend on \( S, \delta S/\delta E^3 \)). Note also that in contrast to Hamiltonians with at least quadratic dependence on on the momenta, the WKB Ansatz leads to a Hamilton-Jacobi equation (4.58) which in this case is exact.

One can then try to solve (4.58) by a functional version of the method of characteristics [10], which brings out the closeness of the spectral problem (4.58) to the solution of the classical equations of motion (4.41) in this case. To that end, we solve the equations of motion (4.41), i.e. we determine the integral curves of the vector field \( V \),

\[
\begin{align*}
\text{To that end, we solve the equations of motion (4.41), i.e. we determine the integral curves of the vector field } V.
\end{align*}
\]

Suppose that we found the maximal unique solution \( X^I(t, x; G) \) given prescribed initial data \( X^I(0, x) = G^I(x) \) where \( G \) ranges in some submanifold \( \Sigma \subset \mathcal{E} \) of co-dimension one which is everywhere transversal to the vector field \( V \). We also solve
\[
\dot{s}(t) = \lambda - U[E = X(t, .; G)]
\] (4.59)
with initial condition \( s(0) = s_0[G] \) leading to a unique maximal solution \( s(t; G) \). The transversality of \( \Sigma \) to the flow lines of \( V \) implies that at least for small \( t \) i.e. close to \( \Sigma \) we may invert the equation \( E^3(x) = X^I(t, x; G) \)

\[
\begin{align*}
t &\approx \tau[E], \quad G^I(x) = \sigma^I(x; E)
\end{align*}
\] (4.60)
and then
\[
S[E] := s(t(G), t=\tau, G=\sigma)
\] (4.61)
solves (4.58) with boundary condition \( S[G] = s_0[G] \) i.e. \( S|_{\Sigma} = s_0 \).

While the precise technical implementation of these steps may be quite involved, they are, remarkably, of much lower complexity than one might have feared. We leave the details of this programme for future work.

### 4.2.2 TMC-TEaC-LEA gauge in \( (A, E) \) description

We verify that the assumptions of theorem 3.1 apply:

We interpret (minus) the \( v^j := \partial^I A^j_I \) as the momenta \( v_A \) and (minus) the \( y^j := \hat{\partial}^I A^j_I, A^0_j \) as the momenta \( y_I \). The configuration variables corresponding to these momenta are \( u_j := \Delta^{-1} \partial_I E^I_j, x_j := \Delta^{-1} \hat{\partial}_I E^I_j \) and \( E^3_j, \) respectively. Likewise the Gauss constraints are considered as the \( C_A \) and the spacetime constraints as the \( C_I \).

In what follows, we work with the canonical variables \( w = (u_j, v^j), z = (x_j, y^j), (E^3_j, A^0_j) \) and \( r = (E^3_j, A^0_j) \) where the latter plays the role of our degrees of freedom. Thus, it is required to express \( E^I_j \) and \( A^I_j \) in terms of \( u_j, x_j \) and \( v^j, y^j \), respectively

\[
\begin{align*}
E^I_j &= \delta^I_j + \partial^I u_j + \hat{\partial}^I x_j \\
A^I_j &= \Delta^{-1}(\partial_I v^j + \hat{\partial}_I y^j)
\end{align*}
\] (4.62)

And the constraints in terms of the canonical variables are

\[
\begin{align*}
\tilde{C}_j &\equiv \epsilon_{jkl}(B^k_I H^j_I + B^k_I H^l_I) \\
&\equiv \epsilon_{jkl}(\epsilon^{1j} H^l_I (\partial_j A^3_k - \partial_k (\partial_j v^k + \hat{\partial}_j y^k)) + H^l_I \Delta y^k) \\
&\equiv \epsilon_{jkl}\epsilon^{1j} H^l_I \partial_j A^3_k + \epsilon_{jkl}(H^l_I \Delta - \epsilon^{1j} H^l_I \partial_k \hat{\partial}_j) y^k - \epsilon_{jkl}\epsilon^{1j} H^l_I \partial_k \partial_j v^k + \epsilon_{jkl}\epsilon^{1j} H^l_I \partial_j A^g_k
\end{align*}
\] (4.64)
\[
\tilde{C}_0 = B_1^i H_1^i + B_2^j H_2^j
\]
\[
= \epsilon^{ij} H_1^i \left( \partial_j A_1^i - \partial_z (\partial_j v_1^i + \hat{\partial}_j y_1^i) \right) + H_2^i \Delta y^i \\
= \epsilon^{ij} H_1^i \partial_j A_1^3 + \left( H_2^i \Delta - \epsilon^{ij} H_1^i \partial_z \hat{\partial}_j \right) y^i - \epsilon^{ij} H_1^i \partial_z \partial_j v^j + \epsilon^{ij} H_1^i \partial_j A_1^3
\]
(4.65)

where we have used \( B_1^i = \epsilon^{ij} \left( \partial_j A_1^i - \partial_z (\partial_j v_1^i + \hat{\partial}_j y_1^i) \right) \) and \( B_2^j = \Delta y^j \). It can be read from (4.64) and (4.65) that \( h_I \) introduced in (3.1) is \( (\epsilon_{j0} \epsilon^{ij} H_1^i \partial_j A_1^3, \epsilon^{ij} H_1^i \partial_j A_1^3) \). On the other hand, the gauge conditions (4.1) are translated to these canonical variables as
\[
G^j = v^j, \quad \tilde{G}_j = x_j, \quad \tilde{G}_0 = E_3^z - f
\]
(4.66)

Recall that we assume \( f \) to be independent of \( z \). The non-zero \( H \)'s evaluated at the gauge cut are
\[
H_1^1 = f(1 + \partial_y u^2) - E_2^z (\partial_y u^3) \\
H_2^1 = E_1^z (\partial_y u^3) - f (\partial_y u^1) \\
H_3^1 = E_2^z (\partial_y u^1) - E_1^z (1 + \partial_y u^2) \\
H_1^2 = E_2^z (\partial_x u^3) - f (\partial_x u^2) \\
H_2^2 = f(1 + \partial_x u^1) - E_1^z (\partial_x u^3) \\
H_3^2 = E_2^z (\partial_x u^2) - E_1^z (1 + \partial_x u^1) \\
H_3^3 = (1 + \partial_x u^1)(1 + \partial_y u^2) - (\partial_x u^2) (\partial_y u^1)
\]

and \( H_2^\alpha = 0 \). The stability conditions for \( \tilde{G}_j \) and \( \tilde{G}_0 \) at the gauge cut are
\[
0 = \{ H, x_1 \} = \partial_1 \left[ -\partial_I (\lambda^2 H_2^3) - \partial_z (\lambda^3 H_1^2 - \lambda^2 H_1^3 + \lambda H_1^1) \right] \\
0 = \{ H, x_2 \} = \partial_1 \left[ \partial_I (\lambda^1 H_2^3) - \partial_z (\lambda^3 H_1^2 - \lambda^2 H_1^3 + \lambda H_1^1) \right] \\
0 = \{ H, x_3 \} = \partial_1 \left[ \partial_I (\lambda^2 H_2^3) - \partial_z (\lambda^3 H_1^2 - \lambda^2 H_1^3 + \lambda H_1^1) \right] \\
0 = \{ H, E_3^z \} = \hat{\partial}_j \left[ \lambda^2 H_2^3 - \lambda^1 H_1^2 + \lambda H_1^3 \right]
\]
(4.67) (4.68) (4.69) (4.70)

respectively, which are supposed to be solved for \( \lambda^1 \) and \( \lambda^2 \). From (4.69) and (4.70), one concludes
\[
\partial_I (\lambda H_2^3) - \partial_z (\lambda^2 H_1^2 - \lambda^1 H_1^3 + \lambda H_1^1) = \hat{\partial}_I g_1 \\
\lambda^2 H_1^2 - \lambda^1 H_1^3 + \lambda H_1^1 = \partial_1 g_2
\]
(4.71) (4.72)

respectively, where \( g_1 \) and \( g_2 \) are certain 0-forms. Inserting (4.72) into (4.71) and applying \( \hat{\partial}_j \) on both sides lead to \( \Delta g_1 = 0 \). Since \( g_1 \) is harmonic and also decaying at infinity due to the boundary conditions, then \( g_1 = 0 \). Hence (4.71) turns to
\[
\lambda^2 H_1^2 - \lambda^1 H_1^3 + \lambda H_1^1 = \partial_1 g_2
\]
(4.73)
in which \( g_I \) are functions depending only on \( x, y \). Going to infinity along \( z \)-axis while \( x, y \) are finite and fixed shows that \( g_I = \epsilon_{IJ} \delta_0^I \lambda_0^J \) where \( \lambda_0^J \) are the leading terms of \( \lambda^J \) (recall that \( \lambda^i \) are of \( O(1) \) and the leading terms are constants). Now, (4.73) are two algebraic equations and can be solved for \( \lambda^J \) as
\[
\lambda^J = \frac{1}{H_2^J} \left[ \epsilon^{IJ} H_1^0 \left( \partial_3 \partial^{-1}_z (\lambda H_2^3) - H_3^J \right) + \lambda_0^J \right]
\]
(4.74)

Notice that from plugging (4.74) in (4.67) and (4.68), a system of two integro-differential equations arises which is too complicated to be solved. Even if one could solve the system for \( \lambda^J \) and \( \lambda \), the resulting physical Hamiltonian would be very complicated to be quantized. Hence, we leave further analysis of this gauge and in the \((A, E)\) description, we will continue the quantization process with MLA-ETTA gauge which led to a relatively simple physical Hamiltonian (4.40).
4.2.3 ETA-LEA gauge in \((B,f)\) description

First, recall that in this gauge condition we have \(z = \{(f^i, B^i_1), (f^i_2, B^i_3)\}\) and \(r = (f^a_\alpha, B^a_\alpha)\). The equations of the stability of the gauge conditions are

\[
\dot{\sigma}^I = \{H, f^I_1(y)\} = \int d^3x \left[ \lambda^i \frac{\partial}{\partial x^i} + \tilde{\lambda}^i \epsilon_{ijk} H^j_a + \lambda H^k_a \right] \{B^a_k(x), f^I_1(y)\} = -\partial_x \lambda^i + \tilde{\lambda}^i \epsilon_{ijk} H^j_a + \lambda H^k_a
\]

\[
\dot{\sigma} = \{H, f^2_2(y)\} = \int d^3x \left[ \lambda^i \frac{\partial}{\partial x^i} + \tilde{\lambda}^i \epsilon_{ijk} H^j_a + \lambda H^k_a \right] \{B^a_k(x), f^2_2(y)\} = -\partial_x \lambda^3 + \tilde{\lambda}^3 \epsilon_{ijk} H^j_a + \lambda H^3_a
\]  

(4.75)

which are supposed to be uniquely solved for the Lagrange multipliers. As it is mentioned before, it is sufficient to solve them at the gauge cut where the system of equations (4.75) can be represented as

\[
\begin{pmatrix}
-\partial_x & 0 & 0 & 0 & 0 & H_x^2 & H_x^3 \\
0 & -\partial_y & 0 & 0 & 0 & -H_x^1 & H_x^3 \\
0 & -\partial_y & 0 & 0 & 0 & H_y^3 & H_y^3 \\
-\partial_y & 0 & 0 & 0 & 0 & H_y^3 & H_y^3 \\
0 & -\partial_x & 0 & 0 & 0 & -H_y^1 & H_y^3 \\
0 & 0 & -\partial_z & 0 & 0 & 0 & H_z^3
\end{pmatrix}
\begin{pmatrix}
\lambda^1 \\
\lambda^2 \\
\lambda^3 \\
\tilde{\lambda}^1 \\
\tilde{\lambda}^2 \\
\tilde{\lambda}^3 \\
\sigma
\end{pmatrix} = \Sigma^I
\]

(4.76)

First, the space of the functions needs to be determined in such a way that all integral constants are fixed while solving the system of equations. For this purpose, we will work only with functions that are of the following form

\[
\lambda^1 = \lambda_0^1 \delta_{ij} \delta^i_a x^a + O(r^{-1})
\]

\[
\tilde{\lambda}^1 = \tilde{\lambda}_0^1 + O(r^{-1})
\]

\[
\tilde{\lambda} = \tilde{\lambda}_0 + O(r^{-2})
\]  

(4.77)

where \(\lambda_0^1, \tilde{\lambda}_0^1, \tilde{\lambda}_0\) are arbitrary constants. Note that the first equation of (4.77) is completely consistent with (A.4) and (4.10) and in the second and the third equation rotations and boosts are excluded, respectively, as there are not well-defined generators for them. The reason for the lack of \(r^{-1}\) term in the lapse function is that in the following calculations, when one uses anti-derivatives, it would lead to a logarithmic divergence preventing us from specifying some integration constants. This is exactly why we also use

\[
f^I_1 \rightarrow c^I_a + O(1/r)
\]  

(4.78)

instead of (4.7) in what follows.

Here, we wish to work with \(\dot{\Sigma}_I = 0\). Solving the first equation of (4.76) for \(\lambda^1\) results in \(\lambda^1 = \tilde{\lambda}_0 x + \partial_x^{-1} \left( H_x^1 \lambda + H_x^2 \tilde{\lambda}_0 \lambda^3 \right) + g_1(y,z)\) where we have used \(H_x^1 = 1 + O(r^{-2})\) and \(\lambda = \tilde{\lambda}_0 + O(r^{-2})\) and the fact that all constants are in the kernel of \(\partial_a^{-1}\). Noting that based on (4.77), \(\lambda^1 - \tilde{\lambda}_0 x \rightarrow 0\) asymptotically and moving to infinity along the \(a\)-axis at fixed finite values of \(y, z\), one observes \(g_1 = 0\). Hence,

\[
\lambda^1 = \tilde{\lambda}_0 x + \partial_x^{-1} \left( H_x^1 \lambda + H_x^2 \tilde{\lambda}_0 \lambda^3 \right)
\]

(4.79)

The same argument can be employed to solve the fifth equation of (4.76) for \(\lambda^2\) as \(\lambda^2 = \tilde{\lambda}_0 y + \partial_x^{-1} \left( H_y^2 \lambda - H_y^1 \tilde{\lambda}_0 \lambda^3 \right) + g_2(x,z)\). Going to infinity along \(a\)-axis at fixed finite values of \(x, z\) results in \(g_2 = 0\), because \(\lambda^2 - \tilde{\lambda}_0 y \rightarrow 0\). Therefore,

\[
\lambda^2 = \tilde{\lambda}_0 y + \partial_x^{-1} \left( H_y^2 \lambda - H_y^1 \tilde{\lambda}_0 \lambda^3 \right)
\]  

(4.80)
The last equation can be solved for \( \lambda^3 \) as
\[
\lambda^3 = \tilde{\lambda}_0 z + \partial_z^{-1} \left( H^3_z \tilde{\lambda} \right) + g_3(x,y)
\]
where we used \( H^3_z = 1 + O(r^{-2}) \) again, going to infinity along the \( z \)-axis while \( x, y \) are fixed and finite, we deduce \( g_3 = 0 \). Consequently,
\[
\lambda^3 = \tilde{\lambda}_0 z + \partial_z^{-1} \left( H^3_z \tilde{\lambda} \right) \tag{4.81}
\]
Plugging \( \lambda^1 \) and \( \lambda^2 \) into the second and fourth equations gives us a system of two integro-differential equations for \( \lambda^3, \tilde{\lambda} \)
\[
\begin{align*}
(-\partial_x \partial_y^{-1} H^2_y + H^2_x) \tilde{\lambda} - (-\partial_x \partial_y^{-1} H^1_y + H^1_x) \tilde{\lambda}^3 &= 0 \tag{4.82} \\
(-\partial_y \partial_x^{-1} H^1_x + H^1_y) \tilde{\lambda} + (-\partial_y \partial_x^{-1} H^2_x + H^2_y) \tilde{\lambda}^3 &= 0 \tag{4.83}
\end{align*}
\]
Looking at (4.14), one can easily check that all constants belong to the kernel of both operators \( Y_2 := -\partial_x \partial_y^{-1} H^2_y + H^2_x \) and \( X_1 := -\partial_y \partial_x^{-1} H^1_x + H^1_y \). In fact, if \( u \) is a constant
\[
Y_2 u = -\partial_x \partial_y^{-1} (H^2_y u) + H^2_x u \\
= u (-\partial_x \partial_y^{-1} H^2_y + H^2_x) \\
= u (-\partial_x \partial_y^{-1} (1 + \partial_y f^1_x) + \partial_x f^1_x) \\
= 0
\]
and by a similar argument \( X_1 u = 0 \) as well. On the other hand, both operators \( Y_1 := -\partial_x \partial_y^{-1} H^1_y + H^1_x \) and \( X_2 := -\partial_y \partial_x^{-1} H^2_x + H^2_y \) acts on constants like the identity, in the sense that \( Y_1 u = X_2 u = u \) and the reason would be
\[
Y_1 u = -\partial_x \partial_y^{-1} (H^1_y u) + H^1_x u \\
= u (-\partial_x \partial_y^{-1} H^1_y + H^1_x) \\
= u (-\partial_x \partial_y^{-1} (-\partial_y f^2_x) + (1 - \partial_x f^2_x)) \\
= u
\]
and by the same reasoning \( X_2 u = u \), for all \( u = constant \). Thus, two integro-differential equations (4.82) and (4.83) are equivalent to
\[
\begin{align*}
Y_2(\tilde{\lambda} - \tilde{\lambda}_0) - Y_1(\tilde{\lambda}^3 - \tilde{\lambda}^3_0) - \tilde{\lambda}^3_0 &= 0 \tag{4.84} \\
X_1(\tilde{\lambda} - \tilde{\lambda}_0) + X_2(\tilde{\lambda}^3 - \tilde{\lambda}^3_0) + \tilde{\lambda}^3_0 &= 0 \tag{4.85}
\end{align*}
\]
Since \( \tilde{\lambda} - \tilde{\lambda}_0 \) and \( \tilde{\lambda}^3 - \tilde{\lambda}^3_0 \) are of \( O(r^{-1}) \), the highest order term of both (4.84) and (4.85) is \( \tilde{\lambda}^3_0 \) which must be vanish separately. Hence, \( \lambda^3 = \sum_{n=1}^{\infty} \tilde{\lambda}^3_n r^{-n} \) and (4.84) and (4.85) reduce to
\[
\begin{align*}
Y_2(\tilde{\lambda} - \tilde{\lambda}_0) - Y_1 \tilde{\lambda}^3 &= 0 \tag{4.86} \\
X_1(\tilde{\lambda} - \tilde{\lambda}_0) + X_2 \tilde{\lambda}^3 &= 0 \tag{4.87}
\end{align*}
\]
It follows from (4.86) that \( \tilde{\lambda}^3 = Y_1^{-1} Y_2(\tilde{\lambda} - \tilde{\lambda}_0) + \tilde{\kappa} \), where \( \tilde{\kappa} \) is in the kernel of \( Y_1 \). Since \( \tilde{\lambda}^3 = O(r^{-1}) \), \( \tilde{\kappa} \) has to be of the form \( \tilde{\kappa} = \sum_{n=1}^{\infty} \tilde{\kappa}_n r^{-n} \) where \( \tilde{\kappa}_n \) are functions on the asymptotic sphere. We have
\[
0 = Y_1 \tilde{\kappa} = H^1_x \tilde{\kappa} - \partial_y \partial_x^{-1} (H^1_y \tilde{\kappa}) \tag{4.88}
\]
Since \( H^1_y = \delta^1_y + O(r^{-2}) \) and \( \tilde{\kappa} = O(r^{-1}) \), the first and the second terms of (4.88) are of \( O(r^{-1}) \) and \( O(r^{-3}) \), respectively. Thus, \( \tilde{\kappa} / r \) is the highest order term existing in (4.88) that has to vanish separately, i.e. \( \tilde{\kappa}_1 = 0 \) and consequently \( \tilde{\kappa} = O(r^{-2}) \). Now, the highest order term in (4.88) is \( \tilde{\kappa}_2 / r^2 \) that has to vanish by the same reasoning. By induction one concludes \( \tilde{\kappa}_n = 0 \) for all \( n > 0 \) which means \( \tilde{\kappa} = 0 \). Therefore,
\[
\tilde{\lambda}^3 = Y_1^{-1} Y_2(\tilde{\lambda} - \tilde{\lambda}_0) \tag{4.89}
\]
Plugging (4.89) into (4.87), we have
\[ 0 = (X_2 Y_1^{-1} Y_2 + X_1)(\tilde{\lambda} - \lambda_0) = -\partial_x^\alpha \partial_y^\beta [Y_1 + Y_2 Y_1^{-1} Y_2](\tilde{\lambda} - \lambda_0) \]  
(4.90)
where in writing the second equality we used \( X_\alpha = -\partial_x^\alpha \partial_y Y_\alpha \). (4.90) tells us that \( \partial_y [Y_1 + Y_2 Y_1^{-1} Y_2](\tilde{\lambda} - \lambda_0) = g(y, x) \) in which \( g \) is an arbitrary function depending only on \( y, z \). Since \( \tilde{\lambda} - \lambda_0 = O(r^{-1}) \), moving to infinity along the \( x \)-axis at fixed finite values of \( y, z \) shows that \( g(y, x) = 0 \). Therefore, \([Y_1 + Y_2 Y_1^{-1} Y_2](\tilde{\lambda} - \lambda_0) = h(x, z)\) where \( h \) is an arbitrary function not depending on \( y \). Again, going to infinity along the \( y \)-axis while \( x, z \) have fixed finite values results in \( h(x, z) = 0 \). Thus, (4.90) is equivalent to
\[ [Y_1 + Y_2 Y_1^{-1} Y_2](\tilde{\lambda} - \lambda_0) = 0 \]  
(4.91)
In general, it is easy to show that \((S + P)^{-1} = S^{-1} - S^{-1} P (S + P)^{-1}\) for every two operators \( S, P \). By repeatedly inserting this relation into its r.h.s, one obtains
\[ (S + P)^{-1} = S^{-1} \sum_{n=0}^{\infty} (-PS^{-1})^n \]  
(4.92)
Based on this relation, one gets
\[ Y_1^{-1} = (H_1^x - \partial_y \partial_x^{-1} H_1^y)^{-1} = \frac{1}{H_1^x} \sum_{n=0}^{\infty} (\partial_y \partial_x^{-1} H_1^y)^n \]  
(4.93)
Since \( H_1^y = \delta_1^y + O(r^{-2}) \), \( Y_1^{-1} \) is expanded as \( Y_1^{-1} = 1 + O(r^{-2}) \) and \( Y_2 = -\partial_x \partial_y^{-1} + O(r^{-2}) \). Assuming \( \lambda - \lambda_0 = \sum_{n=1}^{\infty} \lambda_n r^{-n} \) in which \( \lambda_n = \lambda_n(\theta, \varphi) \) in the spherical coordinates, we can extract the highest order term of (4.91) as \( (\partial_x^2 \partial_y^{-2} + 1) \frac{\lambda_1}{r} = 0 \). By applying \( \partial_x^2 \) on both sides of this equation, we see that \( \lambda_1/r \) has to satisfy the 2-dimensional Laplace’s equation, that is
\[ (\partial_x^2 + \partial_y^2) \frac{\lambda_1}{r} = 0 \]  
(4.94)
If one defines \( R = \sqrt{x^2 + y^2} \), it is easy to rewrite the Laplace’s equation in the polar coordinate system in \( x - y \) plane
\[ \left( \frac{\partial^2}{\partial R^2} + \frac{1}{R} \frac{\partial}{\partial R} + \frac{1}{R^2} \frac{\partial^2}{\partial \varphi^2} \right) \frac{\lambda_1}{R^2 + z^2} = 0 \]  
(4.95)
where \( z = r \cos \theta \) and \( r = \sqrt{R^2 + z^2} \). By carrying out the derivatives, (4.95) is reduced to
\[ -\frac{\lambda_1}{(2R^2 + z^2)} + \frac{1}{R^2} \partial^2 \varphi \lambda_1 = 0 \]  
(4.96)
Moving to infinity along \( z \)-axis at fixed finite values of \( x, y \) results in \( \partial^2 \varphi \lambda_1 = 0 \). Inserting this into (4.95) shows that \( \lambda_1 \) has to vanish. Thus, \( \lambda - \lambda_0 = \sum_{n=2}^{\infty} \lambda_n r^{-n} \). Now the highest order term of (4.91) is \((\partial_x^2 \partial_y^{-2} + 1) \frac{\lambda_2}{r} = 0 \) and by a similar argument it is easily concluded that \( \lambda_2 = 0 \) and finally by induction \( \lambda_n = 0 \) for all \( n > 0 \). As a final result of this part, we have \( \lambda - \lambda_0 = 0 \) and consequently \( \lambda^3 = 0 \) (recall (4.89)). It follows from (4.81) that \( \lambda^3 = \lambda_0 z + \lambda_0 \partial_z^{-1} H_2^3 \). Accordingly, the third and sixth equations of (4.76) are a system of two algebraic equations that simply results in
\[ \tilde{\lambda}^\alpha = \frac{\lambda_0}{H_2^3} (H_2^y \partial_x - H_2^x \partial_y) \partial_z^{-1} H_2^3 \]  
(4.97)
for \( \alpha = 1, 2 \) because \( H_2^x H_2^2 - H_2^2 H_2^1 = H_2^3 \neq 0 \). Finally, (4.79) and (4.80) lead to
\[ \lambda^1 = \tilde{\lambda}_0 x + \lambda_0 \partial_x^{-1} H_2^1 = \tilde{\lambda}_0 x - \lambda_0 f_2^1 \]  
\[ \lambda^2 = \tilde{\lambda}_0 y + \lambda_0 \partial_y^{-1} H_2^2 = \tilde{\lambda}_0 y + \lambda_0 f_2^1 \]  
(4.98)
respectively. This ends proving that the solution of the system of PDEs (4.76) with $\Sigma^I = 0$ is of the form

$$\Lambda_0^I = \bar{\lambda}_0 \left( x - f_{z}^2, y + f_{z}^1, z + \partial^{-1} H_3, \frac{1}{H_3^2} (H_2^3 \partial_x - H_1^3 \partial_y) \partial_3^{-1} H_3^2, \frac{1}{H_3^2} (H_2^3 \partial_x - H_1^3 \partial_y) \partial_3^{-1} H_3^2, 0, 1 \right)^T \quad (4.99)$$

where $\bar{\lambda}_0$ is an arbitrary constant. In order to have a unique solution, it is required to fix the asymptotic behaviour of $\bar{\lambda}$. For simplicity, we consider $\bar{\lambda}_0 = 1$ which means that from the scratch one is supposed to work only with those lapse functions in (4.77) which are of the form $\bar{\lambda} = 1 + O(r^{-2})$. Based on the theorem (3.2), to obtain the corresponding physical Hamiltonian for this gauge fixing, it is sufficient to multiply $\Lambda_0^I$ to \((h_f)_{G=0} = (\partial_{z} B_1^x, \partial_{z} B_2^y, 0, B_2^z H_3, -B_1^z H_3, 0, 0)\). Consequently, for this special gauge fixing, one achieves

$$h_{\Sigma^I = 0} = \int d^3x \left( \delta_{\alpha}^3 x^I + \epsilon_{\alpha\beta} f^3_\alpha \right) \partial_\beta B^3_\beta + \epsilon_{\alpha\beta} \epsilon^{IJ} B^2_\alpha H^2_\beta \partial_J \partial_\beta^{-1} H^3_\gamma \right)$$

$$= \int d^3x \epsilon_{\alpha\beta} \left[ f^3_\alpha (\partial_z B^3_\beta) + \epsilon^{IJ} B^2_\alpha H^2_\beta \partial_J \partial_\beta^{-1} H^3_\gamma \right] + \int d^3x \delta_{\alpha}^3 x^I \delta_{\beta}^3 B^3_\beta$$

$$= \int d^3x \epsilon_{\alpha\beta} \left[ f^3_\alpha (\partial_z B^3_\beta) + \epsilon^{IJ} B^2_\alpha H^2_\beta \partial_J \partial_\beta^{-1} H^3_\gamma \right] \quad (4.100)$$

where the surface term has been dropped because $B^2_\beta$ is $O(r^{-3})$ odd.

Two other appropriate choices for gauge fixing turn out to be $\Sigma^I = (0, 0, \tau, 0, 0, 0, 0)^T$ and $\Sigma^I = (0, 0, 0, 0, 0, \tau, 0)^T$ which lead to

$$\Lambda_0^I = \left( x - f_{z}^2, y + f_{z}^1, z + \partial^{-1} H_3, \frac{H_1^3}{H_3^2}, \frac{H_2^3}{H_3^2}, \frac{H_2^3}{H_3^2}, \frac{1}{H_3^2} (H_2^3 \partial_x - H_1^3 \partial_y) \partial_3^{-1} H_3^2, 0, 1 \right) \quad (4.101)$$

$$\Lambda_0^I = \left( x - f_{z}^2, y + f_{z}^1, z + \partial^{-1} H_3, \frac{H_1^3}{H_3^2}, \frac{H_2^3}{H_3^2}, \frac{H_2^3}{H_3^2}, \frac{1}{H_3^2} (H_2^3 \partial_x - H_1^3 \partial_y) \partial_3^{-1} H_3^2, 0, 1 \right) \quad (4.102)$$

respectively. And the corresponding physical Hamiltonian are obtained as

$$h_{\Sigma^I = (0, 0, \tau, 0, 0, 0, 0)^T} = \int d^3x \epsilon_{\alpha\beta} \left[ f^3_\alpha (\partial_z B^3_\beta) + \epsilon^{IJ} B^2_\alpha H^2_\beta \partial_J \partial_\beta^{-1} H^3_\gamma + H^\alpha B^3_\beta \right] \quad (4.103)$$

and

$$h_{\Sigma^I = (0, 0, 0, 0, 0, \tau, 0)^T} = \int d^3x \epsilon_{\alpha\beta} \left[ f^3_\alpha (\partial_z B^3_\beta) + \epsilon^{IJ} B^2_\alpha H^2_\beta \partial_J \partial_\beta^{-1} H^3_\gamma - H^\alpha B^3_\beta \right] \quad (4.104)$$

respectively, where the expressions of $H^\alpha_I$ and $H^3_3$ in terms of $f^3_\alpha$ have been written in (4.14).

The Physical Hamiltonians (4.100), (4.103) and (4.104) have the following features:

I. Linearity in momentum $B^2_\beta$.

II. Polynomialality in the configuration variable $f^3_\alpha$.

III. Spatial non-locality.

Since three physical Hamiltonians obtained here are very similar, in what follows we will only work with (4.100).

At the end of this subsection, we derive the equations of motion using the physical Hamiltonian (4.100) and \( \{ B^\alpha_\alpha(x), f^\beta_\beta(y) \} = \delta_\alpha^3 \delta(x, y) \). For $f^3_\alpha$, one can effortlessly see that

$$f^3_\alpha(x) = \{ h, f^3_\alpha(x) \} = \epsilon_{\alpha\beta} [\partial_z f^3_\beta + \epsilon^{IJ} H^\beta \partial_J \partial_\beta^{-1} H^3_\gamma] \quad (4.105)$$

in which

$$H^\alpha_I = \delta^\alpha_I - \epsilon_{\alpha\beta} \partial_\beta f^3_\gamma$$

$$H^3_3 = 1 + \epsilon_{\alpha\beta} \delta^\alpha_I \partial_I f^\beta_3 + \frac{1}{2} \epsilon_{\alpha\beta} \epsilon^{IJ} (\partial_I f^\alpha_3)(\partial_J f^\beta_3) \quad (4.107)$$

22
Unsurprisingly, the equation of motion for \( f_z^\lambda \) closes on itself thanks to the linearity feature of the physical Hamiltonian in \( B_z^\lambda \). Therefore, (4.105) can be solved separately without the need to refer to the equation of motion of \( B_z^\alpha \). To obtain the time evolution \( B_z^\alpha \), one requires to know the variations of \( H_I^\beta \) and \( H_B^\beta \) with respect to \( f_z^\lambda \) which are straightforwardly derived \[ \frac{\delta^I H_I^\beta}{\delta f_z^\lambda} = e^{\alpha \beta} \partial_I \delta(x,y) \text{ and } \frac{\delta^I H_B^\beta}{\delta f_z^\lambda} = e^{\alpha \beta} \left[ \frac{\partial_I}{\partial x} + e^{IJ} (\partial_J f_z^\lambda) \right] \partial_I \delta(x,y). \]

Employing these equations, we get
\[ \dot{B}_z^\lambda(x) = \{ h, B_z^\lambda(x) \} = -e^{\alpha \beta} \partial_I \delta(x,y) \]
\[ - \int d^3x \epsilon^{\gamma \beta} \left\{ e^{IJ} B_z^\lambda (\partial_I \partial_J - \partial_J \partial_I) H_z^\beta \right\} e^{\alpha \beta} \partial_I \delta(x,y) + e^{KJ} B_z^\lambda \partial_I \partial_J H_z^\beta \left( e^{\alpha \lambda} \left[ \frac{\partial_I}{\partial x} + e^{IL} (\partial_L f_z^\lambda) \right] \partial_I \delta(x,y) \right) \]
\[ = -e^{\alpha \beta} (\partial_I B_z^\lambda) + e^{IJ} (\partial_I B_z^\lambda) (\partial_I \partial_J H_z^\beta) - \int d^3x \epsilon^{\gamma \beta} e^{KJ} e^{\alpha \lambda} \left( \partial_I \partial_J \partial_J H_z^\beta \right) \partial_I \delta(x,y) \]
\[ = -e^{\alpha \beta} (\partial_I B_z^\lambda) + e^{IJ} (\partial_I B_z^\lambda) (\partial_I \partial_J H_z^\beta) + e^{\alpha \beta} e^{KJ} e^{\alpha \lambda} \left( \partial_I \partial_J \partial_J H_z^\beta \right) \partial_I \delta(x,y) \]
\[ = -e^{\alpha \beta} (\partial_I B_z^\lambda) + e^{IJ} (\partial_I B_z^\lambda) (\partial_I \partial_J H_z^\beta) + e^{\alpha \beta} \left( \partial_I + \frac{1}{e^{IL}} (\partial_L f_z^\lambda) \right) \partial_I \partial_J \partial_J H_z^\beta \]  \hspace{1cm} (4.108)

where in the last step we used \( \partial_I \left[ \frac{\partial_I}{\partial x} + e^{IL} (\partial_L f_z^\lambda) \right] = 0 \). As the equations (4.105) and (4.108) are complicated to be solved, we leave the further analysis of them for future work.

### 4.2.4 ETTS-ELA gauge in \((B,f)\) description

First, recall that in this gauge condition we have \( z = \{(f_z^\alpha, B_z^\alpha), (f_z^\beta, B_z^\beta)\} \) and \( r = (f_z^\beta, B_z^\beta) \). The equations of the stability of the gauge conditions are
\[ \dot{\sigma} = \{ H, f_z^\alpha(y) \} = \int d^3x \left[ \lambda^k \partial_a + \lambda^j \epsilon_{akl} H^l_a + \lambda^j H^k \right] \{ B_z^a(x), f_z^\alpha(y) \} \]
\[ = -\partial_I \lambda^a + \lambda^j \epsilon_{akl} H^l_a + \lambda^j H^k \]
\[ \dot{\sigma} = \{ H, f_z^\beta(y) \} = \int d^3x \left[ \lambda^k \partial_a + \lambda^j \epsilon_{akl} H^l_a + \lambda^j H^k \right] \{ B_z^a(x), f_z^\beta(y) \} \]
\[ = -\partial_I \lambda^a + \lambda^j \epsilon_{akl} H^l_a + \lambda^j H^k \] \hspace{1cm} (4.109)

which are supposed to be uniquely solved for the Lagrange multipliers. The system of equations (4.109) can be represented at the gauge cut as
\[ \begin{bmatrix}
-\partial_x & 0 & 0 & 0 & 0 & 0 & H_x^1 \\
0 & -\partial_x & 0 & 0 & 0 & -H_x^1 & 0 \\
0 & -\partial_y & 0 & 0 & 0 & H_y^1 & 0 \\
0 & -\partial_z & 0 & 0 & 0 & 0 & H_z^1 \\
0 & -\partial_x & 0 & 0 & 0 & 1 & 0 & -H_z^1 & H_x^1 & 0 & 0 & -\partial_z & -H_z^2 & H_z^1 & 0 & 0 & -\partial_x & -H_x^2 & H_x^1 & 0 & 0 & -\partial_y & -H_y^2 & H_y^1 \\
\end{bmatrix}
= \begin{bmatrix}
\lambda^1 \\
\lambda^2 \\
\lambda^3 \\
\lambda^4 \\
\lambda^5 \\
\lambda^6 \\
\lambda^7 \\
\end{bmatrix}
= \begin{bmatrix}
\dot{\sigma}_1 \\
\dot{\sigma}_2 \\
\dot{\sigma}_3 \\
\dot{\sigma}_4 \\
\dot{\sigma}_5 \\
\dot{\sigma}_6 \\
\dot{\sigma}_7 \\
\end{bmatrix}
= \dot{\Sigma}^I \] \hspace{1cm} (4.110)

Again, we will work with the space of functions introduced in (4.77). And we consider (4.78), (4.6) and (A.14) as boundary conditions imposed on the canonical variables.

Now, we can solve (4.111) with the assumption \( \dot{\Sigma}^I = 0 \). From the first and the fourth equations of (4.110), it is immediately concluded that \( \lambda^\alpha = \lambda_0 \delta_\alpha^I x^I + \delta_\alpha^I \partial_I (\lambda H_x^1) \) + \( g_\alpha(y,z) \). Since it is assumed that \( H_x^1 = 1 + O(r^{-2}) \), the asymptotic behaviours of (4.77) shows that \( g_1 = 0 \). Thus,
\[ \lambda^\alpha = \lambda_0 \delta_\alpha^I x^I + \delta_\alpha^I \partial_I (\lambda H_x^1) \] \hspace{1cm} (4.111)

Plugging (4.111) into the second and the third equations of (4.110) gives us
\[ -\partial_x \partial_y^{-1} (\lambda H_z^1) - H_z^3 \lambda^3 = 0 \]
\[ -\partial_y \partial_x^{-1} (\lambda H_z^1) + H_z^3 \lambda^3 = 0 \] \hspace{1cm} (4.112)
respectively. One can easily solve the first equation of (4.112) for $\tilde{\lambda}^3$ and gets

$$\tilde{\lambda}^3 = -\frac{1}{H_x} \partial_x \partial_y^{-1}(\lambda H_x^1)$$  \hspace{1cm} (4.113)

Inserting (4.113) into the second equation of (4.112) results in $(\partial_y \partial_x^{-1} + \partial_x \partial_y^{-1}) u = 0$ where $u := \tilde{\lambda} H_x^1 = \lambda_0 + O(r^{-2})$. If one applies the operator $\partial_x \partial_y$ on both sides of this equation, one sees that $u$ has to satisfy the 2-dimensional Laplace’s equation $0 = (\partial_x^2 + \partial_y^2) u = (\partial_x^2 + \partial_y^2)(u - \lambda_0)$ in which $u - \lambda_0$ is of $O(r^{-2})$ and therefore it can be written as $u - \lambda_0 = \sum_{n=2}^{\infty} u_n r^{-n}$. The highest order term of the Laplace’s equation under consideration is $(\partial_x^2 + \partial_y^2)(u_2 r^{-2})$ which can be rewritten in the polar coordinate system in $x - y$ plane as

$$\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R} \frac{\partial}{\partial R} + \frac{1}{R^2} \frac{\partial^2}{\partial \varphi^2}\right) u_2(\theta, \varphi) = 0$$  \hspace{1cm} (4.114)

where $R = \sqrt{x^2 + y^2}$, $z = r \cos \theta$ and $r = \sqrt{R^2 + z^2}$. By carrying out the derivatives, (4.95) is reduced to

$$2u_2 \left(\frac{R^2 - z^2}{(R^2 + z^2)^2}\right) + \frac{1}{R^2} \frac{\partial^2}{\partial \varphi^2} u_2 = 0$$  \hspace{1cm} (4.115)

Moving to infinity along z-axis at fixed finite values of x, y results in $\partial^2_\varphi u_2 = 0$. Inserting this into (4.115) shows that $u_2$ has to vanish. Thus, $u - \lambda_0 = \sum_{n=2}^{\infty} \tilde{\lambda}_n r^{-n}$. By repeating the same argument for the lowest order term of the Laplace’s equation which is now $(\partial_x^2 + \partial_y^2)(u_3 r^{-3})$, we see $u_3 = 0$ and finally by induction $u_n = 0$ for all $n > 1$ which means $\lambda H_x^1 = u = \lambda_0$. Consequently, from (4.111) and (4.113), it follows that

$$\lambda^\alpha = \lambda_0 \delta^\alpha_1 x^1, \quad \tilde{\lambda}^3 = 0$$  \hspace{1cm} (4.116)

respectively. Plugging (4.116) and $\tilde{\lambda} = \frac{\tilde{\lambda}_0}{H_x^2}$ into the fifth and the sixth equations of (4.110), we obtain

$$\tilde{\lambda}^\alpha = -\frac{\lambda_0}{H_x^2} \epsilon^\alpha_3 H^3_z$$  \hspace{1cm} (4.117)

and finally putting all these results in the last equation of (4.110), one gets

$$\lambda^3 = \tilde{\lambda}_0 z + \lambda_0 \partial_z^{-1} \left(\frac{1 + (H^1_x)^2 + (H^2_x)^2}{H^1_x}\right)$$  \hspace{1cm} (4.118)

Note that in the expression of $\lambda^3$ an integration constant that is only dependent on x and y should have appeared, but it must vanish due to the asymptotic behaviour of $\lambda^3$ introduced in (4.77).

This ends showing that the solutions of the system of PDEs (4.110) are of the form

$$\lambda^\alpha = \tilde{\lambda}_0 \left(x, y, z + \partial_z^{-1} \left(\frac{1 + (H^1_x)^2 + (H^2_x)^2}{H^1_x}\right)\right) - \frac{H^2_x}{H^1_x} \frac{H^1_x}{H^1_x} 0, \frac{1}{H^1_x}$$  \hspace{1cm} (4.119)

where $\tilde{\lambda}_0$ is an arbitrary constant.

As explained in the paragraph after (4.99), it is required to fix $\tilde{\lambda}_0 = 1$. Based on the theorem (3.2), to obtain the corresponding physical Hamiltonian for this gauge, $\Lambda^\alpha_0$ should be multiplied to $(h_I)_{G=0} = (0, 0, \partial_1 B_3^1, -H^1_x B_3^1 B_3^1, H^1_x B_3^1, 0, 0)$. Consequently, for this special gauge fixing, one gets

$$h = \int d^3 x \left[\partial_z^{-1} \left(\frac{1 + (H^1_x)^2 + (H^2_x)^2}{H^1_x}\right)\right] \partial_1 B_3^1 + \delta^\gamma_3 H^\gamma_3 B_3^1$$

$$= \int d^3 x \left[\partial_z^{-1} \left(\frac{1 + (H^1_x)^2 + (H^2_x)^2}{H^1_x}\right)\right] \partial_1 B_3^1 + \epsilon^{IJ} B_3^1 (\partial_z f^I_j) + \int dS_o(z \delta^\gamma_3 B_3^1)$$

$$= \int d^3 x \left[\partial_z^{-1} \left(\frac{1 + (H^1_x)^2 + (H^2_x)^2}{H^1_x}\right)\right] \partial_1 B_3^1 + \epsilon^{IJ} B_3^1 (\partial_z f^I_j)$$  \hspace{1cm} (4.120)
Here the surface term has been dropped because according to the boundary conditions (4.6) and (4.8), it is $O(1)$ odd.

The Physical Hamiltonians (4.120) has the following features:
I. Linearity in momentum $B_\alpha$.
II. Non-polynomiality in the configuration variable $f_\alpha$.
III. Spatial non-locality.

In the $(B,f)$ description, comparison of (4.120) and (4.100) tells us that the latter has simpler features, as it is polynomial in the configuration variables. Thus, in the further analysis in future work we will continue with (4.100).

4.2.5 Remark on polynomial degree of the physical Hamiltonian

One may ask why it is the case that in some gauges the physical Hamiltonian is polynomial while in others it is not. The answer lies in the choice of the different polarisations (split between configuration and momentum degrees of freedom) of the phase space and the polynomial degree in which these enter the constraints: In the $(A,E)$ polarisation, the Gauss constraint is in fact independent of the momentum $A$ while in the $(B,f)$ polarisation all constraints are linear in the momentum $B$. This makes it impossible to impose gauge fixings just in terms of configuration coordinates $E$ in the $(A,E)$ polarisation while it is possible for all configuration coordinates $f$ in the $(B,f)$ polarisation. While $(B,f)$ is a linear canonical transformation of $(A,E)$, it is not true that the Gauss constraint $\text{div} E = 0$ in the $(A,E)$ polarisation and the Bianchi constraint $\text{div} B = 0$ in the $(B,f)$ polarisation are simply rewritings of each other, in fact they are not at all: Since $B := \ast dA$ in the $(A,E)$ polarisation is a derived quantity, the relation $\text{div} B \equiv 0$ is considered an identity and not as a constraint. Conversely, since $E := \ast df$ is a derived quantity in the $(B,f)$ polarisation, the relation $\text{div} E \equiv 0$ is considered an identity and not as a constraint. On the other hand $E$ and $B$ respectively are considered as independent quantities in the $(A,E)$ and $(B,f)$ polarisation respectively before imposing the respective Gauss constraints $\text{div} E = 0$ and $\text{div} B = 0$. Hence the Gauss and Bianchi constraint respectively act on disjoint sets of canonical coordinates.

It transpires that this crucial difference has a major impact on the available gauge choices and on the entries of the associated gauge fixing matrix as far as the polynomial degree with respect to $E$ respectively $f$ is concerned.

5 Conclusion and Outlook

In this paper, we have provided various indications for the hope that the $U(1)^3$ truncation model for Euclidean vacuum quantum gravity could have a reduced phase space quantisation with unexpected level of analytical control. The essential feature of the model that makes this possible is the fact that the constraints are at most linear in momentum (understood as the Abelian connection). We have explored various gauges in which the reduced Hamiltonian for the true respective degrees of freedom adopts a manageable algebraic form. We have only sketched the quantisation of the resulting physical degrees of freedom and its physical Hamiltonian in this paper but we could already argue that the essential feature of the model drastically simplifies the spectral problem. It worth mentioning that concerning the relation between gauge-fixing and gauge-invariant formalisms, one can see that there is a one-to-one correspondence between a choice of gauge fixing and a preferred set of gauge invariant functions which generate the full algebra of gauge invariant functions [17]. The two formalisms are therefore equivalent at generic points of the reduced phase space at which the Dirac matrix (which is a non-trivial function on phase space in every interacting theory) is non-singular. In the same sense, different gauge fixing conditions are generically (i.e., locally in phase space) equivalent. As usual, global differences may have an effect on the quantisation in different gauge choices. However, our attitude is that in quantum gravity global non-equivalence of gauge fixed theories is a second order concern, one would be happy to have at least one working quantisation at one’s proposal to start with which then can be further improved. It is in this spirit that the present paper prepares the ground for the reduced phase space quantisation of the $U(1)^3$ test laboratory for LQG.
In the future, we wish to further explore the system in various gauges that may simplify the analysis even further and to develop the quantum theory in much more detail. Our results could shed new light on the operator constraint (or Dirac) approach to the quantisation of the model [7] in the sense that the physical predictions of the two approaches should agree at least semiclassically.

Acknowledgements

S.B. thanks the Ministry of Science, Research and Technology of Iran and FAU Erlangen-Nürnberg for financial support.
A Generators of asymptotic symmetries in the \((B, f)\) description of the \(U(1)^3\) model

In this section we examine two boundary conditions. The first one is just transcription of the boundary conditions (2.2) (see [11] for more details) while in the other one we use opposite parity conditions. It turns out that the latter is useful to obtain a simple physical Hamiltonian in the process of reduced phase space quantization in the \((B, f)\) description.

A. Standard parity conditions

As \(E_i^a = \delta_i^a + \epsilon^{abc} \partial_b f_c^i\) and \(B_i^a = \epsilon^{abc} \partial_b A_c^i\), transcribing the boundary conditions (2.2) imposed on \((A, E)\) to the \((B, f)\) description results in

\[
f_i^a = c_i^a + \bar{F}_i^a + O(r^{-1})
\]

\[
B_i^a = \bar{G}_i^a + O(r^{-4})
\]

(A.1)

where \(c_i^a\) are constants and \(\bar{F}_i^a\) and \(\bar{G}_i^a\) are tensor fields defined on the asymptotic 2-sphere with the following definite parity conditions

\[
\bar{F}_i^a \left(-\frac{x}{r}\right) = -\bar{F}_i^a \left(\frac{x}{r}\right), \quad \bar{G}_i^a \left(-\frac{x}{r}\right) = \bar{G}_i^a \left(\frac{x}{r}\right).
\]

(A.2)

which are direct consequences of the certain parities of the leading terms of \(A\) and \(E\) (recall that the former is odd and the latter is even). By the decay conditions (A.1) and (A.2), it is assured that the symplectic structure is well-defined.

Bianchi Constraint

The Bianchi constraint is

\[
C_i[\Lambda^i] = \int d^3 x \, \Lambda^i \partial_a B_i^a
\]

(A.3)

Since \(\partial_a B_i^a\) fall off as \(O(r^{-4})\) odd, the minimal condition on the multiplier \(\Lambda^i\) ensuring convergence of the integral is

\[
\Lambda^i = \lambda^i r + O(1)
\]

(A.4)

where \(\lambda^i\) are even functions defined on the asymptotic \(S^2\), i.e.

\[
\lambda^i \left(-\frac{x}{r}\right) = \lambda^i \left(\frac{x}{r}\right)
\]

(A.5)

The action of the Bianchi constraint on the phase space variables is

\[
\delta_{\Lambda} f_j^i = \{C_i[\Lambda^i], f_j^i\} = -\partial_a \Lambda^i
\]

\[
\delta_{\Lambda} B_j^a = \{C_i[\Lambda^i], B_j^a\} = 0
\]

(A.6)

Thus one sees that

\[
\delta C_j[\Lambda^i] = \int d^3 x \, \Lambda^i \partial_a \delta B_j^a = \oint dS_a \, \Lambda^a \delta B_j^a - \int d^3 x \, (\partial_a \Lambda^i) \delta B_j^a = -\int d^3 x \, (\partial_a \Lambda^i) \delta B_j^a
\]

\[
= \int d^3 x \, \left[(\delta_{\Lambda} f_j^i) \delta B_j^a - (\delta_{\Lambda} B_j^a) \delta f_j^i\right]
\]

(A.7)

is functionally differentiable. Here the surface term has been dropped because \(\delta B_j^a = O(r^{-3})\) even and \(\Lambda^i = O(r)\) even.
Scalar constraint

It is straightforward to see that

\[
\delta_N f^i_c(x) = \{C[N], f^i_c(x)\} = \epsilon_{ikl}\epsilon_{abc}N(\delta^a_k + \epsilon^{adec}\partial_d f^k_e)(\delta^b_l + \epsilon^{bfjg}\partial_j f^l_g)
\]

\[
\delta_N B^c_i(x) = \{C[N], B^c_i(x)\} = 2\epsilon_{jkl}\epsilon_{abc}\epsilon f^k_d\partial_d \left( NB^c_j(\delta^b_l + \epsilon^{bfjg}\partial_j f^l_g) \right)
\]

(A.8)

Thus the variation of this constraint is

\[
\delta C[N] = \int d^3x \epsilon_{jkl}N \left( \epsilon_{abc}\delta B^c_j(\delta^b_k + \epsilon^{adec}\partial_d f^k_e)(\delta^b_l + \epsilon^{bfjg}\partial_j f^l_g) + 2\epsilon_{abc}B^c_j(\delta^b_k + \epsilon^{bfjg}\partial_j f^l_g)(\epsilon^{adec}\partial_d f^k_e) \right)
\]

\[
= \int d^3x \epsilon_{jkl} \left( \epsilon_{abc}N\delta B^c_j(\delta^b_k + \epsilon^{adec}\partial_d f^k_e)(\delta^b_l + \epsilon^{bfjg}\partial_j f^l_g) - 2\epsilon_{abc}\epsilon^{adec}\delta f^k_e\partial_d \left( NB^c_j(\delta^b_l + \epsilon^{bfjg}\partial_j f^l_g) \right) \right)
\]

\[
+ 2 \oint dS_d \epsilon_{abc}\epsilon^{adec}NB^c_j(\delta^b_k + \epsilon^{bfjg}\partial_j f^l_g)\delta f^k_e
\]

\[
= \int d^3x \left( \delta B^c_j(\delta f^k_d) - (\delta_N B^c_j)\delta f^k_e \right)
\]

(A.9)

where the surface integral vanishes because it is \(O(r^{-1})\) even for a translation and \(O(1)\) odd for a boost. Hence, the scalar constraint is functionally differentiable without the need for modification and now its finiteness has to be checked. The Hamiltonian constraint in terms of \((B, f)\) is

\[
C[N] = \int d^3x \epsilon_{jkl}N \left( \epsilon_{abc}B^c_j(\delta^b_k + \epsilon^{adec}\partial_d f^k_e)(\delta^b_l + \epsilon^{bfjg}\partial_j f^l_g) \right)
\]

(A.10)

that is \(O(r^{-1})\) even for a translation and \(O(1)\) odd for a boost, both of which are divergent. Therefore, neither boosts nor translations have well-defined generators!

Vector constraint

The vector constraint acts on the canonical variables as follows

\[
\delta_N f^i_c(x) = \{C_a[N^a], f^i_c(x)\} = \epsilon_{abc}N^a(\delta^b_j + \epsilon^{bde}\partial_d f^j_e)
\]

\[
\delta_N B^c_i(x) = \{C_a[N^a], B^c_i(x)\} = \epsilon_{abc}\epsilon^{bde}\partial_d(N^a B^c_i)
\]

(A.11)

So, variation of the constraint is

\[
\delta C_a[N^a] = \int d^3x \ N^a \left( \epsilon_{abc}\delta B^c_j(\delta^b_j + \epsilon^{bde}\partial_d f^j_e) + \epsilon_{abc}\epsilon^{bde}B^c_j\partial_d\delta f^j_e \right)
\]

\[
= \int d^3x \left( \epsilon_{abc}N^a\delta B^c_j(\delta^b_j + \epsilon^{bde}\partial_d f^j_e) - \epsilon_{abc}\epsilon^{bde}\partial_d(N^a B^c_j)\delta f^j_e \right) + \int dS_d \epsilon_{abc}\epsilon^{bde}N^a B^c_j\delta f^j_e
\]

(A.12)

Here the surface integral is \(O(1)\) odd for a rotation and \(O(r^{-1})\) even for a translation and so can be put away. Thus, the vector constraint

\[
C_a[N^a] = \int d^3x \epsilon_{abc}N^a B^c_j(\delta^b_j + \epsilon^{bde}\partial_d f^j_e)
\]

(A.13)

is functionally differentiable but not convergent because it is \(O(r^{-1})\) even for a translation and \(O(1)\) odd for a rotation. Consequently, \(C_a[N^a]\) is not a well-defined generator for translations, nor for rotations!

It is worth mentioning that the source of divergences in (A.10) and (A.13) can not be written in terms of the constraints, therefore one can not cure the problem.
\[ F_i^a \left( -\frac{x}{r} \right) = F_i^a \left( \frac{x}{r} \right), \quad G_i^a \left( -\frac{x}{r} \right) = -G_i^a \left( \frac{x}{r} \right). \]

It is obvious that with these new parity conditions still the symplectic structure is well-defined.

The Bianchi constraint is
\[ C_i[\Lambda^i] = \int d^3 x \Lambda^i \partial_3 B_i^a. \]

According to the new boundary conditions \( \partial_3 B_i^a \) fall off as \( O(r^{-4}) \) even, thus the minimal condition on the multiplier \( \Lambda^i \) ensuring convergence of Bianchi constraint is the same as (A.4) except that \( \lambda^i \) has to be an odd function, i.e.
\[ \lambda^i \left( -\frac{x}{r} \right) = -\lambda^i \left( \frac{x}{r} \right). \]

In this case, too, \( C_j[\Lambda] \) is functionally differentiable because the surface term in (A.7) vanishes (recall \( \delta B_j^a = O(r^{-3}) \) odd and \( \Lambda^i = O(r) \) odd).

In this case, \( C[N] \) is functionally differentiable since the surface integral appearing in (A.9) simply vanishes because it is again \( O(r^{-1}) \) even for a translation and \( O(1) \) odd for a boost. But unlike what we observed in the previous case, here by looking at (A.10) one finds that it is \( O(r^{-1}) \) odd for a translation and \( O(1) \) even for a boost. Consequently, translations have a well-defined generator \( C[N] \), but boosts do not! Again \( C_a[N^a] \) is functionally differentiable, since the surface term in (A.12) is \( O(1) \) odd for a rotation and \( O(r^{-1}) \) even for a translation and so it vanishes. It is also convergent for a translation but not for a rotation as (A.13) is \( O(r^{-1}) \) odd for a translation and \( O(1) \) even for a rotation. Hence, \( C_a[N^a] \) is only a well-defined generator for translations but not for rotations.

The question arising here is why, despite the fact that the boundary conditions are exactly transcribed into the \((B,f)\) description, the use of standard parity conditions leads to well-defined generators in the \((A,E)\) description but not in the \((B,f)\) description. To perceive the source of this discrepancy, first consider a general situation in which we have canonical variables \((A,E)\) and a functional \( \mathcal{F}(A,E) \) depending of \( A \) only via \( \partial_3 A \).

For simplicity, all the indices of the fields have been dropped.

The variation of \( \mathcal{F}[\lambda] \), where \( \lambda \) is a test function, would be of the form
\[ \delta \mathcal{F}[\lambda] = \int d^3 x (A \partial_3 \delta A + B \delta E) = \int d^3 x (- (\partial_3 A) \delta A + B \delta E) + \oint dS_3 A \delta A \tag{A.17} \]

If one wants to change the canonical variables \((A,E)\) to \((B,f)\) in which \( B = \partial_3 A \) and \( E = c + \partial_3 f \) (\( c \) is constant), then the variation of \( \mathcal{F} \) would be of the form
\[ \delta \mathcal{F}[\lambda] = \int d^3 x (A \delta B + B \partial_3 f) = \int d^3 x (A \delta B + (\partial_3 B) \delta f) + \oint dS_3 B \delta f \tag{A.18} \]

here \( A, B \) are written in terms of \((B,f)\).

It is obvious from (A.17) and (A.18) that if we worked with a compact space then \( \mathcal{F} \) would be functionally differentiable in terms of both canonical variables. But if one desires to consider boundary conditions, then the story is completely different! Looking at the surface terms in (A.17) and (A.18), we find that they have nothing in common! Thus, it is quite probable that considering boundary conditions which makes (A.17) functionally differentiable keeps (A.18) ill-defined and vice versa.

This is exactly what happens when we want to analyse differentiability of the Hamiltonian and diffeomorphism constraints in the \(U(1)^3\) model. Recall that the Hamiltonian constraint is \( C = 2 \epsilon_{jkl}(\partial_3 A_j^a) E_k^b E_l^c \) which depends...
on $A$ via $\partial_a A$. Therefore,

$$\delta C[N] = \int d^3x \left( 2N\epsilon_{jkl}(\partial_a \delta A^i_k)E^i_l + 4N\epsilon_{jkl}(\partial_a A^i_k)E^l_i \delta E^k_j \right)$$

$$= \int d^3x \left( A^{ab}_j(\partial_a \delta A^i_k) + B^b_\ell \delta E^b_{\ell} \right)$$

$$= \int d^3x \left( -\left(\partial_a A^{ab}_j\right)\delta A^i_k + B^b_\ell \delta E^b_{\ell} \right) + \oint dS_a A^{ab}_j \delta A^i_k$$

(A.19)

(A.20)

where $A^{ab}_j := 2N\epsilon_{jkl}E^i_k E^b_{\ell}$ and $B^b_\ell := 4N\epsilon_{jkl}(\partial_a A^i_k)E^l_i$.

Now, we would like to write $\delta C[N]$ in terms of $(B, f)$ where $B^a_\ell = \epsilon^{abc}\partial_b A^i_c$ and $E^a_i = \delta^a_i + \epsilon^{abc}\partial_b f^i_c$. We start from (A.19) and write anything in terms of $(B, f)$

$$\delta C[N] = \int d^3x \left( A^{ab}_j(\partial_a \delta A^i_k) + B^b_\ell \delta E^b_{\ell} \right)$$

$$= \int d^3x \left( \frac{1}{2} A^{ab}_j \epsilon_{cab} \delta B^c_{\ell} + B^b_\ell \epsilon^{bac}(\partial_a \delta f^i_c) \right)$$

$$= \int d^3x \left( \frac{1}{2} A^{ab}_j \epsilon_{cab} \delta B^c_{\ell} - (\partial_a B^b_\ell) \epsilon^{bac} \delta f^i_c \right) + \oint dS_a B^b_\ell \epsilon^{bac} \delta f^i_c$$

(A.21)

Now if we use the standard boundary conditions in which $E^a_i - \delta^a_i = O(r^{-1})$ even, $A^a_i = O(r^{-2})$ odd and consequently $B^b_\ell = O(r^{-3})$ even, $f^i_c = O(1)$ odd, then it is concluded that for translations $A^{ab}_j = \text{constant} + O(r^{-1})$ even and $B^b_\ell = O(r^{-3})$ even.

Hence, the surface terms in (A.20) and (A.21) are

$$\oint dS_a A^{ab}_j \delta A^i_k = \oint dS_a O(r^{-2}) \text{odd} = \text{divergent}$$

(A.22)

$$\oint dS_a B^b_\ell \epsilon^{bac} \delta f^i_c = \oint dS_a O(r^{-3}) \text{odd} = 0$$

(A.23)

Therefore, $C[N]$ is functionally differentiable in the $(B, f)$ description but not in the $(A, E)$ description!

Hence, to get a well-defined generator for the asymmetric temporal translation, we have to add a term to the Hamiltonian constraint eliminating the divergence appearing in its variation. In this way, one obtains an expression which is functionally differentiable and, as one is lucky, is already finite (for more details see [11]). On the other hand, in the $(B, f)$ description, because there does not exist such a divergence in the variation of $C[N]$, it is already functionally differentiable, and the only factor making the Hamiltonian constraint ill-defined is its own divergence. Since the origin of this divergence cannot be written in terms of the constraints, it is not admissible to subtract it from the Hamiltonian constraint, therefore $C[N]$ remains ill-defined in the $(B, f)$ description. The same happens for the diffeomorphism constraint.

**B Solutions of the constraints in the $(B, f)$ description**

Due to the aforementioned reason in theorem (3.2), the solutions of the constraints are not required to attain the physical Hamiltonian. However, to ensure that the model is consistent, we must answer the question of whether the system of equations under consideration admits solutions satisfying the asymptotic behaviour determined. For this purpose, in this appendix, we exhibit that by considering the boundary conditions (4.6), (4.7) and (A.14), there exist solutions for the systems of equations (4.15) and (4.19).

**B.1 Solutions for the system of equations (4.15)**

Inspecting the system of equations (4.15), one can solve the first and second equations for $B^x_\alpha$ as

$$B^x_\alpha = -\partial^{-1}_x(\partial_y B^y_\alpha + \partial_z B^z_\alpha) + g_\alpha(y, z)$$

where $g_\alpha$ are arbitrary functions depending only on $y, z$. As $B^x_\alpha$ decays at infinity due to (4.6), $g_\alpha$ have to vanish. Therefore,

$$B^x_\alpha = -\partial^{-1}_x(\partial_y B^y_\alpha + \partial_z B^z_\alpha).$$

(B.1)
The third equation of (4.15) is solved for $B_3^y$ as $B_3^y = -\partial_x^{-1}\partial_y B_1^y + g(x, y)$ in which $g$ is an arbitrary function in the kernel of $\partial_x$. Again, using the asymptotic behaviour of $B_3^y$, one concludes $g = 0$. Hence,

$$B_3^y = -\partial_x^{-1}\partial_y B_1^y \quad \text{(B.2)}$$

Since $H_2^y = \text{det}(H_2) \neq 0$, the fourth and fifth equations of (4.15) that simply form an algebraic system of two equations with two unknowns $B_3^I$ can be solved as

$$B_3^I = \epsilon_1^I e^{\alpha_B} B^\alpha H_2^\beta \quad \text{(B.3)}$$

One plugs (B.1)-(B.3) into the sixth and seventh equations of (4.15) and gets

$$-Y_2 B_2^y + Y_1 B_2^y = e^{\alpha_B} H_2^x \partial_x^{-1}\partial_y B_2^\alpha$$

$$-Y_2 B_2^y - Y_2 B_2^y = H_2^x \partial_x^{-1}\partial_y B_2^\alpha + \epsilon_1^J e^{\alpha_B} \partial_x^{-1}\partial_y (H_2^y B_3^z) \quad \text{(B.4)}$$

where $Y_\alpha := H_2^x \partial_x^{-1}\partial_y - H_2^y$. Using the inverse of the operator $Y_2$, we can solve (B.4) for $B_2^y$ as $B_2^y = Y_2^{-1}Y_1 B_2^y - Y_2^{-1} \left(e^{\alpha_B} H_2^x \partial_x^{-1}\partial_y B_2^\alpha\right) + \kappa$ in which $\kappa$ is in the kernel of $Y_2$ and of the form $\kappa = \sum_{n=3}^{\infty} \kappa_n r^{-n}$, since $B_2^y = O(r^{-3})$. As $H_2^y = \delta_1^y + O(r^{-1})$, the highest order term of the defining equation for $\kappa$, that is $0 = Y_2 \kappa = H_2^x \partial_x^{-1}\partial_y \kappa - H_2^y \kappa$, is $\kappa_3/r^3$ which has to vanish individually. Hence, $\kappa_3 = 0$ and $\kappa = O(r^{-4})$. Repeating the same argument, one deduces that $\kappa = 0$. Therefore,

$$B_2^y = Y_2^{-1}Y_1 B_2^y - Y_2^{-1} \left(e^{\alpha_B} H_2^x \partial_x^{-1}\partial_y B_2^\alpha\right) \quad \text{(B.5)}$$

Inserting (B.6) in (B.5) leads to

$$X B_2^y = Y_1 Y_2 Y_1^{-1} \left(e^{\alpha_B} H_2^x \partial_x^{-1}\partial_y B_2^\alpha\right) - X^{-1} (H_2^x \partial_x^{-1}\partial_y B_2^\alpha) - \epsilon_1^J e^{\alpha_B} \partial_x^{-1}\partial_y (H_2^y B_3^z) - X^{-1} \left(e^{\alpha_B} H_2^x \partial_x^{-1}\partial_y B_2^\alpha\right) - \epsilon_1^J e^{\alpha_B} \partial_x^{-1}\partial_y (H_2^y B_3^z) + \kappa, \quad \text{in which } \kappa \text{ is a member of the kernel of } X \text{ and decays at infinity as } O(r^{-3}), \text{ because } B_2^y = O(r^{-3}). \text{ In the following, we will use the same method as described in full detail in the subsection (4.2.3) in order to specify } \kappa. \text{ Note that the decaying behaviour of } H_2^y \text{ tells us that } Y_1 = \partial_x^{-1}\partial_y + O(r^{-1}). \text{ Moreover, if one makes use of (4.92), it is easily concluded that } Y_2^{-1} = -1 + O(r^{-1}). \text{ Thus, the highest order term of the r.h.s. of the defining equation for } \kappa, \text{ i.e. } 0 = X \kappa = (Y_1 Y_2^{-1} Y_1 + Y_2) \kappa, \text{ is } -(\partial_x^{-2} \partial_y^2 + 1) \kappa_3/\tau_3 \text{ that has to vanish separately. Applying } \partial_x^2, \text{ we see that the } \kappa \text{ has to satisfy the } 2\text{-dimensional Laplace equation } \Delta (\kappa_3/\tau_3) = 0 \text{ which means that } \kappa_3 = 0, \text{ since in } \mathbb{R}^3 \text{ the only harmonic function decaying at infinity is the trivial function. Therefore, } \kappa = O(r^{-4}). \text{ Iterating the argument results in } \kappa = 0. \text{ So,}

$$B_2^y = X^{-1} Y_1 Y_2^{-1} \left(e^{\alpha_B} H_2^x \partial_x^{-1}\partial_y B_2^\alpha\right) - X^{-1} (H_2^x \partial_x^{-1}\partial_y B_2^\alpha) - \epsilon_1^J e^{\alpha_B} \partial_x^{-1}\partial_y (H_2^y B_3^z) \quad \text{(B.7)}$$

This ends showing that by solving the constraints, $B_1^I$ and $B_3^2$ can be expressed in terms of $B_2^\alpha$ which are our degrees of freedom.

B.2 Solutions for the system of equations (4.19)

Looking at the system of equations (4.19), we can solve the first and second equations for $B_3^x$ as $B_3^x = -\partial_x^{-1}\partial_y B_1^y + \partial_z B_3^z$ where $g_\alpha$ are arbitrary functions in the kernel of $\partial_x$. Due to the decaying behaviour (4.6), $g_\alpha$ have to vanish. Therefore,

$$B_3^x = -\partial_x^{-1}\partial_y B_1^y + \partial_z B_3^z \quad \text{(B.9)}$$

One solves the third equation of (4.19) for $B_3^2$ as $B_3^2 = -\partial_z^{-1}\partial_1 B_1^2 + g(x, y)$ in which $g$ is an arbitrary function depending only on $x, y$. Again, $g = 0$ follows from the asymptotic behaviour of $B_3^2$. Thus,

$$B_3^2 = -\partial_z^{-1}\partial_1 B_1^2 \quad \text{(B.10)}$$

31
Solving the fourth and fifth equations of (4.19) for \( B^z_\alpha \) simply leads to
\[
B^z_\alpha = H^1_x \delta^z_i B^I_3 - H^\alpha_x \partial^z_1 \partial^I_3
\]  
(B.11)

Since \( H^1_x \neq 0 \), the last two equations of (4.19) can be rewritten as
\[
B^y_1 = B^x_2 - \frac{1}{H^1_x} e^{\alpha \beta} B^z_\alpha H^\beta_x
\]  
(B.12)
\[
B^x_1 + B^y_2 = -\frac{1}{H^1_x} (H^\alpha_x B^z_\alpha + B^z_3)
\]  
(B.13)

If one employs (B.9)-(B.12) in (B.13) and simplifies, the following integro-differential equation arises
\[
\partial^2_x \Delta B^y_2 = -\frac{1}{H^1_x} (H^\alpha_x B^z_\alpha + B^z_3) - \partial^z_1 \partial^y \left( \frac{1}{H^1_x} e^{\alpha \beta} B^z_\alpha H^\beta_x \right) + \partial^z_1 \partial^z B^z_1 - \partial^z_1 \partial^z B^z_2
\]  
(B.14)

which can be solved for \( B^y_2 \) using the inverse of \( \partial^2_x \Delta \). We denote the r.h.s. of (B.14) which depends only on \( B^y_1 \) by \( R(B^y_1) \) and see that \( B^y_2 = \Delta^{-1} \partial^2_x \Delta R + \kappa \) where \( \partial^2_x \Delta \kappa = 0 \) and \( \kappa = O(r^{-3}) \) due to (4.6). The asymptotic behaviour of \( \kappa \) shows that \( \Delta \kappa \) has to vanish and because in \( \mathbb{R}^3 \) the only harmonic function approaching zero at infinity is the trivial function, \( \kappa = 0 \). Therefore,
\[
B^y_2 = \Delta^{-1} \left[ -\partial^2_x \left( \frac{1}{H^1_x} (H^\alpha_x B^z_\alpha + B^z_3) \right) - \partial^z_1 \partial^y \left( \frac{1}{H^1_x} e^{\alpha \beta} B^z_\alpha H^\beta_x \right) \right.
\]
\[
\left. + \partial^z_1 \partial^z B^z_1 - \partial^z_1 \partial^z B^z_2 \right]
\]  
(B.15)

Hence, we exhibited that by solving the constraints, \( B^I_3 \) and \( B^z_i \) can be expressed in terms of \( B^I_3 \) which are our degrees of freedom.

References
[1] A. Ashtekar, “New Variables for Classical and Quantum Gravity”, Physical Review Letters 57 (1986) 2244–2247. J. Barbero, “Real Ashtekar variables for Lorentzian signature space-times”, Physical Review D 51 (1995) 5507–5510. gr-qc/9410014
[2] T. Thiemann, “Quantum Spin Dynamics (QSD)”, Class. Quantum Grav. 15 (1998), 839-873. gr-qc/9606089
[3] C. Rovelli, L. Smolin, “Discreteness of area and volume in quantum gravity”, Nucl. Phys. B 442 (1995) 593-622, Nucl.Phys.B 456 (1995) 753-754 (erratum). gr-qc/9411005 A. Ashtekar, J. Lewandowski, “Quantum theory of geometry. 2. Volume operators”, Adv. Theor. Math. Phys. 1 (1998) 388-429. gr-qc/9711031 T. Thiemann, “Closed formula for the matrix elements of the volume operator in canonical quantum gravity”, J. Math. Phys. 39 (1998) 3347–71, gr-qc/9606091
[4] “Loop Quantum Gravity - The First 30 Years”, A. Ashtekar, J. Pullin (eds.), World Scientific, Singapore, 2017. J. Pullin, R. Gambini, “A First Course in Loop Quantum Gravity”, Oxford University Press, Oxford, 2011. C. Rovelli, “Quantum Gravity”. Cambridge University Press, Cambridge, 2008. T. Thiemann, “Modern Canonical Quantum General Relativity”, Cambridge University Press, Cambridge, 2007.
[5] T. Thiemann, “Quantum Spin Dynamics (QSD): II. The kernel of the Wheeler-DeWitt constraint operator”, Class. Quantum Grav. 15 (1998), 875-905. gr-qc/9606090 T. Thiemann. “Quantum Spin Dynamics (QSD): III. Quantum constraint algebra and physical scalar product in quantum general relativity”, Class. Quantum Grav. 15 (1998), 1207-1247. gr-qc/9705017 T. Thiemann, “Quantum Spin Dynamics (QSD): IV. 2+1 Euclidean quantum gravity as a model to test 3+1 Lorentzian quantum gravity”, Class. Quantum Grav. 15 (1998), 1249-1280. gr-qc/9705018 T. Thiemann, “Quantum Spin Dynamics (QSD): V. Quantum gravity as the natural regulator of the Hamiltonian constraint of matter quantum field theories”, Class. Quantum Grav. 15 (1998), 1281-1314.
T. Thiemann, “Quantum Spin Dynamics (QSD): VI. Quantum Poincaré algebra and a quantum positivity of energy theorem for canonical quantum gravity”, Class. Quantum Grav. 15 (1998), 1463-1485.

T. Thiemann, “Quantum Spin Dynamics (QSD): VII. Symplectic structures and continuum lattice formulations of gauge field theories”, Class. Quant. Grav. 18 (2001) 3293-3338.

T. Thiemann, “Quantum spin dynamics (QSD): VIII. The master constraint”, Class. Quant. Grav. 23 (2006), 2249-2266.

T. Thiemann, “The Phoenix project: master constraint programme for loop quantum gravity”, Class. Quant. Grav. 23 (2006), 2211-2248.

L. Smolin, “The \( \hbar \rightarrow 0 \) limit of Euclidean quantum gravity”, Class.Quant.Grav. 9 (1992) 883-894, hep-th/9202076

J. Samuel. A Lagrangian basis for Ashtekar’s formulation of canonical gravity. Pramana 28 (1987) L429-L432

T. Jacobson, L. Smolin. The Left-Handed Spin Connection as a Variable for Canonical Gravity. Phys.Lett.B 196 (1987) 39-42

T. Jacobson, L. Smolin. Covariant Action for Ashtekar’s Form of Canonical Gravity Class.Quant.Grav. 5 (1988) 583

C. Tomlin, M. Varadarajan, “Towards an Anomaly-Free Quantum Dynamics for a Weak Coupling Limit of Euclidean Gravity” Phys. Rev.D87 (2013) no.4, 044039, [gr-qc/1210.6869]

A. Laddha, “Hamiltonian constraint in Euclidean LQG revisited: First hints of off-shell Closure”, [gr-qc/1401.0931]

A. Henderson, A. Laddha, C. Tomlin, “Constraint algebra in loop quantum gravity reloaded. II. Toy model of an Abelian gauge theory: Spatial diffeomorphisms” Phys.Rev. D88 (2013) no.4, 044029, [gr-qc/1210.3960]

A. Henderson, A. Laddha, C. Tomlin, “Constraint algebra in loop quantum gravity reloaded. I. Toy model of a \( U(1)^3 \) gauge theory”, Phys.Rev. D88 (2013) 4, 044028. [gr-qc/1204.0211]

M. Varadarajan, “On quantum propagation in Smolin’s weak coupling limit of 4d Euclidean Gravity”, Phys. Rev. D 100, 066018 (2019), [gr-qc/1904.02247]

A. Hanson, T. Regge, and C. Teitelboim, “Constrained Hamiltonian Systems”, Accademia Nazionale dei Lincei, Rome, 1976

J. York, “The Initial Value Problem Using Metric and Extrinsic Curvature”, Contribution to: 10th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Gravitation and Relativistic Field Theories (MG X MMIII), gr-qc/0409102

J. Brown and K. Kuchar, “Dust as a standard of space and time in canonical quantum gravity”, Phys. Rev. D51 (1995), 5600-5629. [gr-qc/9409001]

J. Bicak, K. V. Kuchar, “Null dust in canonical gravity”, Phys. Rev. D56 (1997) 4878-4895, [gr-qc/9704053]

K. V. Kuchar, C. G. Torre, “Gaussian reference fluid and interpretation of quantum geometrodynamics” Phys. Rev. D43 (1991) 419-441

S. Bakhoda, H. Shojai, T. Thiemann, “Asymptotically flat boundary conditions for the \( U(1)^3 \) model for Euclidean Quantum Gravity”, Universe 7, no 3 (2021), 68, [gr-qc/2010.16359]

T. Thiemann, “Generalized boundary conditions for general relativity for the asymptotically flat case in terms of Ashtekar’s variables” Class. Quant. Grav. 12 (1995), 181, [gr-qc/9910008]

M. Campiglia, “Note on the phase space of asymptotically flat gravity in Ashtekar-Barbero variables”, Class. Quant. Grav. 32 (2015), 14, [gr-qc/ 1412.5531]

S. Bakhoda, T. Thiemann, “Covariant Origin of the \( U(1)^3 \) model for Euclidean Quantum Gravity”, [gr-qc/2011.00031]

R. Capovilla, J. Dell, T. Jacobson, “General Relativity Without the Metric”, Phys.Rev.Lett. 63 (1989) 2325

R. Capovilla, J. Dell, T. Jacobson, “A Pure spin connection formulation of gravity”, Class.Quant.Grav. 8 (1991) 59-73

L. C. Evans, “Partial Differential Equations”, American Mathematical Society. 2010

K. Giesel, S. Hofmann, T. Thiemann, and Winkler, O., “Manifestly Gauge-Invariant General Relativistic Perturbation Theory. I. Foundations”, Class. Quant. Grav. 27 (2010) 055005