Semianalytical Approach for the Approximate Solution of Delay Differential Equations

Xiankang Luo, Mustafa Habib, Shazia Karim, and Hanan A. Wahash

1Faculty of Science, Yibin University, Yibin 644000, China
2Department of Mathematics, University of Engineering and Technology, Lahore 54890, Pakistan
3Department of Basic Sciences, UET Lahore, FSD Campus 54800, Lahore, Pakistan
4Department of Mathematics, Albayda University, Al Bayda, Yemen

Correspondence should be addressed to Xiankang Luo; xkluo1978@163.com and Hanan A. Wahash; hawahash@baydauniv.net

Received 17 May 2022; Accepted 8 June 2022; Published 25 June 2022

Abstract

In this analysis, we develop a new approach to investigate the semianalytical solution of the delay differential equations. Mohand transform coupled with the homotopy perturbation method is called Mohand homotopy perturbation transform method (MHPTM) and performs the solution results in the form of series. The beauty of this approach is that it does not need to compute the values of the Lagrange multiplier as in the variational iteration method, and also, there is no need to implement the convolution theorem as in the Laplace transform. The main purpose of this scheme is to reduce the less computational work and the error analysis of the problems than others studied in the literature. Some illustrated examples are interpreted to confirm the accuracy of the newly developed scheme.

1. Introduction

Many physical phenomena of differential equations play an important role in various branches of science and engineering such as physics, chemical energy, biology, and medicine involving time delay [1–4]. In most cases, time delay appears in everywhere of physical study of the reality. A delay differential equation is one of the most famous equations where the derivative of an unknown function is given at a specific time as far as the results of the function at the past time. Numerous authors have demonstrated various approaches to find the approximate solution of delay differential equations in different fields of science; Rihan and Velmurugan [5] suggested a delay differential model with fractional order for the tumor-immune system with external treatments. Luks et al. [6] and Ogunfidiitimi [7] used the Adomian decomposition method for the numerical solution of delay differential equations. Shakeri and Dehghan [4] obtained the solution of delay differential equations via a homotopy perturbation method. Evans and Raslan [8] applied the Adomian method to solve particular ordinary delay differential equations in which the delay is located in the linear or nonlinear part. Jane and Robert [9] used a computer algebra system to solve some very simple linear delay differential equations by combining Laplace transform method and a novel least-squares method. Barati and Ivaz [10] used the variational iteration method for delay differential equations whereas Mohyud-Din and Yıldırım [11] combined the variational iteration method with He’s polynomials to obtain the solution of delay differential equations. We recommend the readers to study the new developments in time-delay differential equations [12–14].

Recently, many integral transformations and strategies have been introduced to find the approximate solution of ordinary and partial differential equations such as Elzaki transform [15, 16], Sumudu transform [17], Aboodh transformation [18], spline methods [19], finite difference method [20], but it is still quite difficult to get the exact solutions for these problems. The homotopy perturbation method was developed by He [21–23] to obtain the solution of ordinary and partial differential equations involving nonlinear terms. Mishra and Tripathi [24] used the
homotopy perturbation method of delay differential equation using He’s polynomial with Laplace transform. HPM gives the solution in the form of a rapid and consecutive series toward the exact solution.

In this paper, we develop a hybridization scheme where the Mohand transform is coupled with the homotopy perturbation method for obtaining the approximate solution of linear and nonlinear delay differential equations. This scheme derives the results in the aspect of series without any linearization, variation, and limiting expectations. In addition, this study is organized as follows: in Section 2, we present some basic definitions and preliminary concepts of the Mohand transform which help us to construct the idea of the semianalytical approach. In Section 3, we formulate the idea of MHPTM for obtaining the solution of delay differential equations. We illustrate two examples to show the accuracy and validity of this approach in Section 4. We demonstrate the results and discussion in Section 5, and finally, the conclusion is presented in Section 6.

2. Fundamental Concepts of Mohand Transform

In this section, we introduce some basic definitions and preliminary concepts of the Mohand transform which reveals the idea of its implementations to functions.

**Definition 1.** Mohand and Mahgoub [25] presented a new scheme Mohand transform $M(t)$ in order to gain the results of ordinary differential equations and are defined as

$$M[\vartheta(t)] = R(w) = w^2 \int_0^\infty \vartheta(t)e^{-w^2 t} dt, \quad k_1 \leq w \leq k_2.$$  

(1)

On the other hand, if $R(w)$ is the Mohand transform of a function $\vartheta(t)$, then $M^{-1}[R(w)] = \vartheta(t)$. $M^{-1}$ is inverse Mohand operator.     

**Definition 2.** If $\vartheta(t) = t^n$, then $R(w) = \frac{n!}{w^{n+1}}.$  

(3)

**Definition 3.** If $M[\vartheta(t)] = R(w)$, then it has the following differential properties [26]:

(i) $M[\vartheta'(t)] = w^2 R(w) - w^2 F(0)$  
(ii) $M[\vartheta''(t)] = w^2 R(w) - w^2 F(0) - w^2 F'(0)$  
(iii) $M[\vartheta^{(n)}(t)] = w^n R(w) - w^{n+1} F(0) - w^{n+1} F'(0) - \ldots - w^n F^{(n-1)}(0)$

3. Formulation of MHPTM for Delay Differential Equations

This segment presents the construction of the Mohand homotopy perturbation transform method (MHPTM) for obtaining the approximate solution of linear and nonlinear delay differential equations. Let’s consider a nonlinear second-order differential equation of the form [24]

$$\vartheta''(t) + \vartheta'(t) + \vartheta(t) + g(\vartheta) = g(t)$$  

(4)

with the following conditions:

$$\vartheta(0) = a,$$

$$\vartheta'(0) = b,$$  

(5)

where $\vartheta$ is a function in the time domain $t$, $g(\vartheta)$ represents a nonlinear term, $g(t)$ is a source term whereas $a$ and $b$ are constants. Rewrite (4) again

$$\vartheta''(t) + \vartheta'(t) = -\vartheta(t) - g(\vartheta) + g(t).$$  

(6)

Now, taking MT on both sides of (6), we obtain

$$M[\vartheta''(t) + \vartheta'(t)] = M[-\vartheta(t) - g(\vartheta) + g(t)].$$  

(7)

Applying the differential properties of MT, we get

$$w^2 R(w) - w^2 \vartheta(0) - w^2 \vartheta'(0) + w^2 \vartheta(0) = M[-\vartheta(t) - g(\vartheta) + g(t)].$$  

(8)

Thus, $R(w)$ can be obtained from (8) such as

$$R(w) = \frac{w^2 a + w^2 b + w^2 a}{(w + w^2)} - M\left[ \frac{\vartheta(t) + g(\vartheta)}{(w + w^2)} \right].$$  

(9)

Operating inverse Mohand transform, on (9), we get

$$\vartheta[t] = G(t) - \frac{1}{M^{-1}} \left[ \frac{\vartheta(t) + g(\vartheta)}{(w + w^2)} \right].$$  

(10)

Now, we apply HPM on (10). Let

$$\vartheta(t) = \sum_{i=0}^{\infty} p^i \vartheta_i (n)$$  

(12)

where $p$ is the homotopy parameter, and thus, the nonlinear term $g(\vartheta)$ in (10) can be calculated by using the formula

$$g(\vartheta) = \sum_{i=0}^{\infty} p^i H_i(\vartheta)$$  

(13)

where $H_n s$ is the He’s polynomial, which may be computed using the following procedure.

$$H_n(u_0 + u_1 + \ldots + u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[ g\left( \sum_{i=0}^{\infty} p^i \vartheta_i \right) \right]_{p=0}, \quad n = 0, 1, 2, \ldots$$  

(14)
Complexity

Put (12), (13), and (14) in (10) and comparing the similar factors of $p$, we get the following consecutive elements

\[ p^0: \vartheta_0(t) = G(t), \]

\[ p^1: \vartheta_1(t) = -M^{-1}\left[ \frac{1}{w + w^3} M \{ \vartheta_0 + H_0(\vartheta) \} \right], \]

\[ p^2: \vartheta_2(t) = -M^{-1}\left[ \frac{1}{w + w^3} M \{ \vartheta_1 + H_1(\vartheta) \} \right], \]  

(15)

\[ p^3: \vartheta_3(t) = -M^{-1}\left[ \frac{1}{w + w^3} M \{ \vartheta_2 + H_2(\vartheta) \} \right], \]  

\[ \vdots. \]

On continuing the similar process, we can summarize this series to get the approximate solution such as

\[ \vartheta(t) = \vartheta_0 + \vartheta_1 + \vartheta_2 + \cdots \]

(16)

Thus, (16) is to be considered as an approximate solution to delay differential equations of (4).

4. Numerical Examples

In this part, we test two examples for the authenticity and validity of MHPTM. We also demonstrate 2D plots for a better understanding of this strategy where we see that the solution graphs of the approximate solution and the exact solution coincide with each other only after few iterations.

4.1. Example 1. Consider a nonlinear delay differential equation of order first

\[ \vartheta'(t) = 1 - 2\vartheta^2\left(\frac{t}{2}\right), \]  

(17)

with the initial condition

\[ \vartheta(0) = 0. \]  

(18)

Applying MT on (17) together with the differential property as defined in (3), we get

\[ wR(w) - w^3(0) = w - M\left[ 2\vartheta^2\left(\frac{t}{2}\right) \right]. \]  

(19)

Using (18) into (19) for solving $R(w)$, it yields

\[ R(w) = 1 - \frac{1}{w} M\left[ 2\vartheta^2\left(\frac{t}{2}\right) \right]. \]  

(20)

Using inverse Mohand transform on (17), we get

\[ \vartheta(t) = t - M^{-1}\left[ \frac{1}{w} M\left[ 2\vartheta^2\left(\frac{t}{2}\right) \right] \right]. \]  

(21)

Applying MHPTM to get the He's polynomials

\[ \sum_{i=0}^{\infty} p^i \vartheta_i(t) = t - M^{-1}\left[ \frac{2}{w} M\left\{ \sum_{i=0}^{\infty} p^i \vartheta_i^2\left(\frac{t}{2}\right) \right\} \right]. \]  

(22)

Observing the similar powers of $p$, we get

\[ p^0: \vartheta_0(t) = t, \]

\[ p^1: \vartheta_1(t) = -M^{-1}\left[ \frac{2}{w} M\left\{ \vartheta_0^2\left(\frac{t}{2}\right) \right\} \right] \]

\[ = \frac{t^3}{6}, \]

\[ p^2: \vartheta_2(t) = -M^{-1}\left[ \frac{2}{w} M\left\{ 2\vartheta_0\vartheta_1\left(\frac{t}{2}\right) + \vartheta_1^2\left(\frac{t}{2}\right) \right\} \right] \]

\[ = -\frac{t^5}{120}, \]

\[ p^3: \vartheta_3(t) = -M^{-1}\left[ \frac{2}{w} M\left\{ 2\vartheta_0\vartheta_2\left(\frac{t}{2}\right) + 2\vartheta_1\vartheta_1\left(\frac{t}{2}\right) + \vartheta_2^2\left(\frac{t}{2}\right) \right\} \right] \]

\[ = \frac{t^7}{362880}, \]

\[ \vdots. \]  

(23)

On continuing this process, the results of obtained series can be summarized as

\[ \vartheta(t) = \vartheta_0(t) + \vartheta_1(t) + \vartheta_2(t) + \vartheta_3(t) + \vartheta_4(t) + \cdots \]  

(24)

\[ = t - \frac{t^3}{3!} - \frac{t^5}{5!} - \frac{t^7}{7!} - \frac{t^9}{9!} + \cdots. \]

This series converges to the exact solution

\[ \vartheta(t) = \sin(t). \]  

(25)

4.2. Example 2. Consider a linear delay differential equation of 2nd order

\[ \vartheta''(t) = \frac{3}{4} \vartheta(t) + \vartheta\left(\frac{t}{2}\right) - t^2 + 2, \]  

(26)

with the initial condition

\[ \vartheta(0) = 0, \quad \vartheta'(0) = 0. \]  

(27)

Applying MT on (26) together with the differential property as defined in (3), we get
\[ w^2 R(w) - w^3 \theta(0) - w^3 \theta'(0) = M \left[ \frac{3}{4} \theta(t) + \theta \left( \frac{t}{2} \right) - t^2 + 2 \right]. \] (28)

Using (27) into (28) for solving \( R(w) \), it yields
\[ \bar{R}(w) = -\frac{2}{w} + \frac{2}{w^2} M \left[ \frac{3}{4} \theta(t) + \theta \left( \frac{t}{2} \right) \right]. \] (29)

Using inverse Mohand transform on (29), we get
\[ \theta(t) = -\frac{t^4}{12} + t^2 + M^{-1} \left[ \frac{1}{w^2} M \left[ \frac{3}{4} \theta(t) + \theta \left( \frac{t}{2} \right) \right] \right]. \] (30)

Applying MHPTM to get the He’s polynomials
\[ \sum_{i=0}^{\infty} p^i \theta_i(n) = t^2 - \frac{t^4}{12} + M^{-1} \left[ \frac{1}{w^2} M \left[ \frac{3}{4} \sum_{i=0}^{\infty} p^i \theta_i(t) + \sum_{i=0}^{\infty} p^i \theta_i \left( \frac{t}{2} \right) \right] \right]. \] (31)

Observing the similar powers of \( p \), we get
\[ p^0: \theta_0(t) = M^{-1} \left[ \frac{1}{w^2} M \left[ \frac{3}{4} \theta_0(t) + \theta_0 \left( \frac{t}{2} \right) \right] \right] \]
\[ = t^2 - \frac{t^4}{12}. \]
\[ p^1: \theta_1(t) = M^{-1} \left[ \frac{1}{w^2} M \left[ \frac{3}{4} \theta_1(t) + \theta_1 \left( \frac{t}{2} \right) \right] \right] \]
\[ = \frac{t^4}{12} - \frac{13t^6}{5760}. \]
\[ p^2: \theta_2(t) = M^{-1} \left[ \frac{1}{w^2} M \left[ \frac{3}{4} \theta_2(t) + \theta_2 \left( \frac{t}{2} \right) \right] \right] \]
\[ = \frac{13t^6}{5760} - \frac{91t^8}{2949120}. \]
\[ p^3: \theta_3(t) = M^{-1} \left[ \frac{1}{w^2} M \left[ \frac{3}{4} \theta_3(t) + \theta_3 \left( \frac{t}{2} \right) \right] \right] \]
\[ = \frac{91t^8}{2949120} - \frac{17563t^8}{67947724800}. \]

On continuing this process, the results of obtained series can be summarized as
\[ \theta(t) = \theta(t) + \theta_0(t) + \theta_1(t) + \theta_2(t) + \theta_3(t) + \theta_4(t) + \cdots \]
\[ = t^2 - \frac{17563t^8}{67947724800} + \cdots. \] (33)

This series converges to the exact solution
\[ \theta(t) = t^2. \] (34)

5. Results and Discussion

In this section, we discuss some results obtained by MHPTM for linear and nonlinear delay differential equations. We calculate only four iterations to test the validity and accuracy of this new strategy in both examples. It can be seen that we need only few iterations to show the exact solution. We may extend the series of (22) and (31) for better performance and rapid convergence. Figures 1 and 2 show the error solution between the approximate solution and the exact solution for \( 0 \leq t \leq 4 \) and \( 0 \leq t \leq 6 \), respectively. We also present the
The obtained results are in full agreement with the exact solution and the approximate solution obtained by MHPTM in Tables 1 and 2, respectively. Moreover, this absolute error declares that MHPTM is a valid and authentic tool that does not require any heavy calculation for the computation of the approximate solution of linear and nonlinear delay differential equations. The obtained results are in full agreement with [11, 24].

### 6. Conclusion

In this analysis, we have employed an innovative scheme Mohand homotopy perturbation transform method (MHPTM) to achieve an approximate solution of linear and nonlinear delay differential equations. Since MT is limited to deal with nonlinear terms, so we introduced HPM to overcome this drawback and presented the results in the form of series solutions. The solution plots demonstrated the accuracy and validity of MHPTM and showed the error distribution between the approximate solution and the exact solution. We propose that MHPTM is applicable for future work both in the linear and nonlinear partial differential equation in science and engineering applications.

### Data Availability

All the data are available within the article.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

### Acknowledgments

This work was supported by the Foundation of Yibin University, China (Grant no. 2019QD07).

### References

[1] H. Khan, S.-J. Liao, R. Mohapatra, and K. Vajravelu, “An analytical solution for a nonlinear time-delay model in biology,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 7, pp. 3141–3148, 2009.

[2] I. R. Epstein, “Delay effects and differential delay equations in chemical kinetics,” *International Reviews in Physical Chemistry*, vol. 11, no. 1, pp. 135–160, 1992.

[3] K.-J. Wang, H.-C. Sun, C.-L. Li, G.-D. Wang, and H. W. Zhu, “Thermal management of the hotspots in 3-d integrated circuits,” *Thermal Science*, vol. 22, no. 4, pp. 1685–1690, 2018.

[4] F. Shakeri and M. Dehghan, “Solution of delay differential equations via a homotopy perturbation method,” *Mathematical and Computer Modelling*, vol. 48, no. 3–4, pp. 486–498, 2008.

[5] F. Rihan and G. Velmurugan, “Dynamics of fractional-order delay differential model for tumor-immune system,” *Chaos, Solitons & Fractals*, vol. 132, Article ID 109592, 2020.

[6] L. B. Cocom, A. G. Estrella, and E. A. Vales, “Solving delay differential systems with history functions by the adomian decomposition method,” *Applied Mathematics and Computation*, vol. 218, no. 10, pp. 5994–6011, 2012.

[7] F. Ogundimiti, “Numerical solution of delay differential equations using the adomian decomposition method (adm),” *International Journal of Engineering Science*, vol. 4, no. 5, pp. 18–23, 2015.

[8] D. J. Evans and K. R. Raslan, “The adomian decomposition method for solving delay differential equation,” *International Journal of Computer Mathematics*, vol. 82, no. 1, pp. 49–54, 2005.

[9] J. M. Heffernan and R. M. Corless, “Solving some delay differential equations with computer algebra,” *The Mathematical Scientist*, vol. 31, no. 1, pp. 21–34, 2006.

[10] S. Barati and K. Ivaz, “Variational iteration method for solving systems of linear delay differential equations,” *International Journal of Computational and Mathematical Sciences*, vol. 6, pp. 132–135, 2012.

[11] S. T. Mohyud-Din and A. Yıldırım, “Variational iteration method for delay differential equations using he’s polynomials,” *Zeitschrift für Naturforschung A*, vol. 65, no. 12, pp. 1045–1048, 2010.

[12] R. Shah, H. Khan, P. Kumam, M. Arif, and D. Baleanu, “Natural transform decomposition method for solving fractional-order partial differential equations with proportional delay,” *Mathematics*, vol. 7, no. 6, p. 532, 2019.

[13] F. A. Rihan, *Delay Differential Equations and Applications to Biology*, Springer, Berlin, Germany, 2021.

[14] L. Dugard and E. I. Verriest, *Stability and control of time-delay systems*, Springer, Berlin, Germany, 1998.

[15] T. M. Elzaki, “The new integral transform elzaki transform,” *Global Journal of Pure and Applied Mathematics*, vol. 7, no. 1, pp. 57–64, 2011.

[16] N. Anjum, M. Suleman, D. Lu, J.-H. He, and M. Ramzan, “Numerical iteration for nonlinear oscillators by elzaki transform,” *Journal of Low Frequency Noise, Vibration and Active Control*, vol. 39, no. 4, pp. 879–884, 2020.

[17] M. Rana, A. Siddiqui, Q. Ghori, and R. Qamar, “Application of he’s homotopy perturbation method to sumudu transform,” *International Journal of Nonlinear Sciences and Numerical Stimulation*, vol. 8, no. 2, pp. 185–190, 2007.

[18] A. Aboudi, R. Farah, I. Almardy, and A. Osman, “Solving delay differential equations by aboudi transformation method,” *International Journal of Applied Mathematics & Statistical Sciences*, vol. 7, no. 2, pp. 55–64, 2018.

[19] T. Aziz, A. Khan, and J. Rashidinia, “Spline methods for the solution of fourth-order parabolic partial differential equations,” *Applied Mathematics and Computation*, vol. 167, no. 1, pp. 153–166, 2005.

[20] M. Dehghan, “Finite difference procedures for solving a problem arising in modeling and design of certain optoelectronic devices,” *Mathematics and Computers in Simulation*, vol. 71, no. 1, pp. 16–30, 2006.
[21] J.-H. He, “Homotopy perturbation technique,” *Computer Methods in Applied Mechanics and Engineering*, vol. 178, no. 3-4, pp. 257–262, 1999.

[22] J.-H. He, “Homotopy perturbation method: a new nonlinear analytical technique,” *Applied Mathematics and Computation*, vol. 135, no. 1, pp. 73–79, 2003.

[23] J.-H. He, “Comparison of homotopy perturbation method and homotopy analysis method,” *Applied Mathematics and Computation*, vol. 156, no. 2, pp. 527–539, 2004.

[24] H. K. Mishra and R. Tripathi, “Homotopy perturbation method of delay differential equation using he’s polynomial with laplace transform,” *Proceedings of the National Academy of Sciences, India, Section A: Physical Sciences*, vol. 90, no. 2, pp. 289–298, 2020.

[25] M. Mohand and A. Mahgoub, “The new integral transform mohand transform,” *Applied Mathematical Sciences*, vol. 12, no. 2, pp. 113–120, 2017.

[26] S. Aggarwal and R. Chaudhary, “A comparative study of mohand and laplace transforms,” *Journal of Emerging Technologies and Innovative Research*, vol. 6, no. 2, pp. 230–240, 2019.