Nonextensive random-matrix theory based on Kaniadakis entropy

A.Y. Abul-Magd
Department of Mathematics, Faculty of Science,
Zagazig University, Zagazig, Egypt

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Abstract

The joint eigenvalue distributions of random-matrix ensembles are derived by applying the principle maximum entropy to the Rényi, Abe and Kaniadakis entropies. While the Rényi entropy produces essentially the same matrix-element distributions as the previously obtained expression by using the Tsallis entropy, and the Abe entropy does not lead to a closed form expression, the Kaniadakis entropy leads to a new generalized form of the Wigner surmise that describes a transition of the spacing distribution from chaos to order. This expression is compared with the corresponding expression obtained by assuming Tsallis’ entropy as well as the results of a previous numerical experiment.

1 Introduction

In 1957 Jaynes [1] proposed a rule, based on information theory, to provide a constructive criterion for setting up probability distributions on the basis of partial knowledge. This leads to a type of statistical inference which is called the maximum-entropy principle (MaxEnt). It is the least biased estimate possible on the given information. Jaynes showed in particular how his rule, when applied to statistical mechanics, leads to the usual Gibbs’ canonical distribution. The core of the MaxEnt method resides in interpreting entropy, through the Shannon axioms, as a measure of the “amount of uncertainty” or of the “amount of information that is missing” in a probability distribution. This was an important step forward because it extended the applicability of the notion of entropy far beyond its original roots in thermodynamics. In this paper we consider the application of MaxEnt to the random-matrix theory (RMT), which is often used to describe quantum systems whose classical counterparts have chaotic dynamics [2, 3]. This is the statistical theory of random matrices $H$ whose entries fluctuate as independent Gaussian random numbers. Dyson [4] showed that there are three generic ensembles of random matrices, defined in terms of the symmetry properties of the Hamiltonian. The most popular of these is the
Gaussian orthogonal ensemble (GOE) which successfully describes many time-reversal-invariant quantum systems whose classical counterparts have chaotic dynamics. Balian [5] derived the weight functions $P(H)$ for the random-matrix ensembles from MaxEnt postulating the existence of a second moment of the Hamiltonian. He applied the conventional Shannon definition for the entropy to ensembles of random matrices as

$$S_{Sch} = - \int dH P(H) \ln P(H)$$

and maximized it under the constraints of normalization of $P(H)$ and fixed mean value of $\text{Tr}(H^T H)$, where the superscript $T$ denotes the transpose. The latter constraint ensures basis independence, which is a property of the trace of a matrix. Then, the distribution $P(H)$ is determined from the extremum of the functional

$$F_{Sch} = S_{Sch} - \xi \int dH \ P(H) - \eta \int dH \ P(H) \text{Tr}(H^T H),$$

where $\xi$ and $\eta$ are Lagrange multipliers. Its maximum is obtained equating its functional derivative to zero. He obtained

$$P(H) \propto \exp \left( -\eta \text{Tr}(H^T H) \right)$$

which is a Gaussian distribution with inverse variance $1/2\eta$. Most of the interesting results are obtained for $N \times N$ matrices in the limit of $N \rightarrow \infty$. A complete discussion of the level correlations for large matrices is a difficult task. Analytical results have long ago been obtained for the case of $N = 2$ [6]. The distribution of nearest-neighbor-spacings (NNS) of levels for GOE of 2 dimensional matrices,

$$P(s) = \left( \frac{\pi}{2} \right) s \exp \left( -\frac{\pi s^2}{4} \right),$$

is known as Wigner’s surmise. Here $s$ is scaled so that the mean spacing equal one. Analogous expression are obtained for the Gaussian unitary and symplectic ensembles. To demonstrate the accuracy of the Wigner surmise, we expand this distribution in powers of $s$ to obtain

$$P(s) = \frac{32}{\pi^2} s^2 \left( 1 - \frac{4}{\pi} s^2 + \cdots \right) \approx 3.242 s^2 - 4.128 s^4 + \cdots,$$

while the power-series expansion of the corresponding exact distribution [2] yields

$$P_{\text{exact}}(s) = \frac{\pi^2}{3} s^2 - \frac{2\pi^4}{45} s^4 + \cdots \approx 3.290 s^2 - 4.329 s^4 + \cdots.$$
RMT to describe such mixed systems are numerous; for a review please see [3].
MaxEnt can contribute to the generalization of RMT by either introducing additional constraints or modifying the entropy. An example of the first approach is the work of Hussein and Pato [7], who use MaxEnt to construct "deformed" random-matrix ensembles by imposing different constraints for the diagonal and off-diagonal elements.

The second approach to include mixed systems in RMT is to apply MaxEnt to entropic measures other than Shannon’s. Recently, the Tsallis statistical mechanics [8, 9] has been applied to include systems with mixed regular-chaotic dynamics in a nonextensive generalization of RMT [10, 11, 12, 13, 14, 15]. Different entropic forms have been introduced, like Tsallis’, by appropriately deforming the logarithm in the expression for the Shannon entropy [10]. Among the many generalizations, one can find the entropies considered by Rényi [17], by Abe [18, 19], by Landsberg and Vedral [20], and by Kaniadakis (κ-entropy) [21, 22, 23]. The present paper is a natural continuation of the previous work on nonextensive RMT, by considering other forms of generalized entropy.

2 Nonextensive RMT

In this section, we review the main results of non-extensive RMT based on Tsallis entropy, and show that replacing it with other generalized entropies does not always produce new results. The matrix-element distribution \( P(H) \) is obtained by extremizing Tsallis’ \( q \)-entropy

\[
S_{Ts}[q, P(q,H)] = \left( 1 - \int dH \left[ P_{Ts}(q,H) \right]^q \right) / (q - 1).
\]

rather than Shannon’s, but again subject to the same constraints of normalization and existence of the expectation value of \( \text{Tr}(H^T H) \). This can be done by replacing \( S_{Sch} \) in Eq. (2) by \( S_{Ts} \) and then equating the functional derivative in the resulting expression to zero. The extremization then yields

\[
P_{Ts}(q,H) = Z_q^{-1} \left[ 1 + (q - 1) \eta_q \text{Tr}(H^T H) \right]^{\frac{1}{q-1}},
\]

where \( Z_q \) and \( \eta_q \) are expressed in terms of the Lagrange multipliers. The case of 2-dimensional ensembles belonging to the three symmetry classes is considered in Ref. [14]. For ensembles with orthogonal symmetry, the resulting NNS distribution is given by

\[
P_{Ts}(s) = \left( \frac{1}{q-1} - \frac{3}{2} \right) \eta_q s \left[ 1 + \frac{1}{2} (q - 1) \eta_q s^2 \right]^{-1/(q-1)+1/2},
\]

for \( q > 1 \), where

\[
\eta_q = \pi \left( \frac{\Gamma[1/(q-1) - 2]}{\Gamma[1/(q-1) - 3/2]} \right)^{2},
\]
and $\Gamma[x]$ is Euler’s gamma function \[24\]. The extension of these results to the case on ensembles with arbitrarily large $N$ amounts to replacing $q$ in Eq. (9) by another $N$-dependent parameter \[15\].

It is well known that the Shannon entropy of the couple of two independent random variables is the sum of their respective Shannon entropies. We have firstly attempted to extremize Rényi’s entropy for the matrix-element probability density function $P_{Rn}(H)$, which is defined by

$$S_{Rn}[\alpha, P_{Rn}(H)] = \frac{1}{1-\alpha} \ln \left( \int dH \left[ P_{Rn}(H) \right]^\alpha \right),$$  \hspace{1cm} (11)

where the index $\alpha$ varies between 1 and 2. The obtained distribution NNS is very similar to that in Eq. (9), which is obtained by the maximization of Tsallis’ entropy, if we replace $1-q$ by $\alpha-1$. In other words, mere fittings of observed data by the NNS distributions in Eq. (9) do not tell us anything about which of Tsallis’ or Rényi’s entropy is the underlying physical quantity. This result agrees with the known fact \[25\] that the maximum entropy principle in combination with Rényi’s entropy reproduces the equilibrium probability distributions of Tsallis’ non-extensive thermostatistics. In fact, the Rényi entropy is, in all cases, a monotonically increasing function of the Tsallis entropy

$$P_{Rn} = \ln \left[ 1 + (1-q)P_{Ts} \right] \bigg|_{q=\alpha}. \hspace{1cm} (12)$$

Therefore the probability density function that extremizes one of these entropies automatically extremizes the other.

Abe’s entropy \[18\] is another interesting example of non-extensive entropies. It applies a $q$-calculus, which is invariant under the exchange $q \rightarrow q^{-1}$. This is a symmetry which plays a central role in the physical context of quantum groups \[27\]. This entropy has been introduced in \[18, 19\] and has been applied there to generalized statistical-mechanical study of $q$-deformed oscillators. Abe’s entropy, generated for the distribution function of a random-matrix ensemble, is defined by

$$S_{Abe}[q, P(q, H)] = -\int dH \left( [P_{Abe}(q, H)]^q - [P_{Abe}(q, H)]^{-q} \right) / (q - q^{-1}),$$  \hspace{1cm} (13)

and also reduces to Shannon’s entropy in the $q \rightarrow 1$ limit. Due to the $q \rightarrow q^{-1}$ symmetry the range of $q$ may be restricted to $0 < q < 1$. Now, replacing $S_{Sch}$ in Eq. (2) by $S_{Abe}$ and then equating the functional derivative in the resulting expression to zero, one obtains after some reorganizations

$$-\frac{qP_{Abe}^{q-1} - q^{-1}P_{Abe}^{-q-1}}{q - q^{-1}} - \xi_{Abe} - \eta_{Abe} \text{Tr} \left( H^T H \right) = 0. \hspace{1cm} (14)$$

Unfortunately, it is not possible to solve this equation with respect to $P_{Abe}$ analytically. The analogous equation is solved numerically in Ref. \[18\]. When
$q = 1$, one obtains the standard GOE distribution, whereas for $0 < q < 1$, the distribution comes to exhibit the power-law behavior similar to similar to the distributions obtained by extremizing the Tsallis and Renyi entropies. Indeed, in the asymptotic region where $P_{\text{Abe}} \ll 1$, one neglects the term in which $P_{\text{Abe}}$ is raised to a positive power and obtains

$$P_{\text{Abe}}(q, H) \sim \left[ \xi_{\text{Abe}} + \eta_{\text{Abe}} \text{Tr} \left( H^T H \right) \right]^{-1/(1-q)}.$$  \tag{15}$$

### 3 Kaniadakis’ entropy

In this section, we consider a possible generalization of RMT based on an extremization of Kaniadakis’ $\kappa$-entropy \cite{21, 22, 23}. This entropy shares the same symmetry group of the relativistic momentum transformation and has applications in cosmic-ray and plasma physics. For the matrix-element probability distribution function, it reads

$$S_{\text{K}} [\kappa, P_{\text{K}} (\kappa, H)] = -\frac{1}{2\kappa} \int dH \left( \frac{\alpha}{1 + \kappa} [P_{\text{K}} (\kappa, H)]^{1+\kappa} - \frac{\alpha - \kappa}{1 - \kappa} [P_{\text{K}} (\kappa, H)]^{1-\kappa} \right).$$  \tag{16}$$

with $\kappa$ a parameter with value between 0 and 1; the case of $\kappa = 0$, corresponds to the Schannon entropy. Here, $\alpha$ is a real positive parameter. Kaniadakis has considered two choices of $\alpha$, namely $\alpha = 1$ and $\alpha = Z$, where $Z$ is the generalized partition function. We here adopt the second choice. The matrix-element distribution $P_{\text{K}} (\kappa, H)$ is obtained by extremizing the functional

$$F_{\text{K}} = S_{\text{K}} - \eta_{\text{K}} \int dH \ P_{\text{K}} (\kappa, H) \text{Tr} \left( H^T H \right),$$  \tag{17}$$

where $\eta_{\text{K}}$ is a Lagrange multiplier. One arrives to the following distribution

$$P_{\text{K}} (\kappa, H) = \frac{1}{Z} \exp_{\{\kappa\}} \left[ -\eta_{\text{K}} \text{Tr} \left( H^T H \right) \right],$$  \tag{18}$$

where

$$Z = \int dH \ \exp_{\{\kappa\}} \left[ -\eta_{\text{K}} \text{Tr} \left( H^T H \right) \right].$$  \tag{19}$$

Here $\exp_{\{\kappa\}} [x]$ is the $\kappa$-deformed exponential \cite{21} which is defined by

$$\exp_{\{\kappa\}} [x] = \left( \sqrt{1 + \kappa^2 x^2} + x \right)^{1/\kappa} = \exp \left( \frac{1}{\kappa} \text{arcsinh} \ \kappa x \right).$$  \tag{20}$$

and has the properties $\exp_{\{0\}} [x] = \exp(x)$, $\exp_{\{\kappa\}} [0] = 1$ and $\exp_{\{-\kappa\}} [x] = \exp_{\{\kappa\}} [x]$, and obeys the scaling law $\exp_{\{\kappa\}} [\lambda x] = (\exp_{\{\kappa/\lambda\}} [x])^\lambda$. The asymptotic behavior of the $\kappa$-exponential is $\exp_{\{\kappa\}} [x] \sim 2\kappa x |x|^{1/|\kappa|}$.

To calculate $Z$, we note that $\text{Tr} \left( H^T H \right) = \sum_{i=1}^N H_{ii}^2 + 2 \sum_{i>j} H_{ij}^2$. We introduce the new coordinates $r = \{r_1, \cdots, r_n\}$, where $n = N(N + 1)/2$, and
\( r_i^2 \) stand for the square of the diagonal elements or twice the square of the non-diagonal elements, respectively. With these new variables, Eq. (19) becomes

\[
Z = 2^{-N(N-1)/4} \int d^n r \exp \left( -\frac{1}{\kappa} \arcsinh \kappa \eta_K r^2 \right) \\
= 2^{-N(N-1)/4} \left( \frac{\pi}{2 \kappa \eta_K} \right)^{n/2} \frac{\Gamma \left( \frac{1}{\kappa} \right) - \frac{n}{2} \Gamma \left( \frac{1}{\kappa} + \frac{n}{2} \right)}{(1 + n\kappa/2) \Gamma \left( \frac{1}{\kappa} + \frac{n}{2} \right)}
\]

for \( \kappa < 2/n \). Here we use the result obtained by Kaniadakis in evaluating an analogous integral that appears in his treatment of Brownian particles (Eq. (71) of Ref. [21]).

We now calculate the joint probability density for the eigenvalues of the Hamiltonian \( H \). With \( H = U^{-1}EU \), where \( U \) is the global unitary group, we introduce the elements of the diagonal matrix of eigenvalues \( E = \text{diag}(E_1, \cdots, E_N) \) of the eigenvalues and the independent elements of \( U \) as new variables. Then the volume element \( dH \) has the form

\[
dH = |\Delta_N(E)| dEd\mu(U),
\]

where \( \Delta_N(E) = \prod_{n>m}(E_n - E_m) \) is the Vandermonde determinant and \( d\mu(U) \) the invariant Haar measure of the unitary group [2, 3]. The probability density \( P_K(\kappa, H) \) depends on \( H \) through \( \text{Tr}(H^\dagger H) \) and is therefore invariant under arbitrary rotations in the matrix space. Integrating over the "angular variables" \( U \) yields the joint probability density of eigenvalues in the form

\[
P_K(\kappa; E_1, \ldots, E_N) = C \prod_{n>m}(E_n - E_m) \exp(\kappa \sum_{i=1}^N E_i^2) \left[ -\eta_K \sum_{i=1}^N E_i^2 \right].
\]

where \( C \) is a normalization constant.

In order to obtain a generalization of the Wigner surmise, we consider the case of two-dimensional random-matrix ensemble where \( N = 2 \) and \( n = 3 \). In this case, Eq. (23) reads

\[
P_K(\kappa; \varepsilon, s) = \frac{2 \left( 1 + 3\kappa/4 \right)}{B \left( \frac{\pi}{2 \kappa} - \frac{3}{2} \right) \left( \kappa \eta_K s^2 \right)^{3/2}} \exp(\kappa \eta_K \frac{s^2}{2}) \left[ -\eta_K \left( 2 \varepsilon^2 + \frac{1}{2} s^2 \right) \right],
\]

where \( \varepsilon = (E_1 + E_2)/2 \), \( s = |E_1 - E_2| \), and \( B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b) \) is the Beta function [24]. The NNS distribution is obtained by integrating (24) over \( \varepsilon \) from \(-\infty \) to \( \infty \). This can be done by changing the variable \( \varepsilon \) into \( x = \exp(-\frac{1}{\kappa} \arcsinh(\kappa \eta_K s^2/2)) \), integrating by parts, and then replacing the variable \( x \) by another new variable, \( y = \exp(\kappa x) \). The resulting integral can be solved by using the following identity [24]

\[
\int_0^\infty y^{-\lambda}(y + \beta)^{\nu} (y - u)^{\mu-1} dy = u^{\nu+\lambda-\lambda} B(\lambda - \mu - \nu, \mu) 2F_1 \left(-\nu, \lambda - \mu - \nu; \lambda - \nu; -\frac{\beta}{u} \right),
\]
for $|\beta/u| < 1$ and $0 < \mu < \lambda - \nu$, where $\, _2F_1(\nu, \mu; \lambda; x)$ is the hypergeometric function. Thus, after straightforward calculations we can express the NNS distribution as

$$P_K(s) = -2 \left( 1 + \frac{3}{4}\kappa \right) \eta_K s e^{(1/2-1/\kappa) \arcsinh(\kappa \eta_K s^2/2)} \frac{B \left( \frac{1}{2} - \frac{1}{\kappa}, \frac{1}{2} \right)}{B \left( \frac{1}{2} - \frac{3}{4}, \frac{1}{2} \right)} 
\, _2F_1 \left( -\frac{1}{2}, \frac{1}{\kappa} - \frac{1}{2}, \frac{1}{\kappa} + 1; -e^{-2\arcsinh(\kappa \eta_K s^2/2)} \right). \quad (26)$$

The condition of unit mean spacing defines the quantity $\eta_K$ as

$$\eta_K = \left[ \frac{\pi k^{3/2} \left( 1 + \frac{3}{4}\kappa \right)}{(1 - \kappa^2) B \left( \frac{1}{2} - \frac{3}{4}, \frac{1}{2} \right)} \right]^2. \quad (27)$$

We note that the function $B \left( \frac{1}{2} - \frac{1}{\kappa}, \frac{1}{2} \right)$ diverges at $\kappa = \kappa_c = 1/2$, which serves as an upper limit for the range of variation of $\kappa$. We also note that the mean square spacing diverges at $\kappa = \kappa_{\infty} = 2/5$.

We now compare the spacing distributions $P_K(s)$ and $P_{Ts}(s)$. Each of the two distributions coincides with the Wigner distribution for the smallest values of the entropic indices $\kappa = 0$ and $q = 1$, but essentially differs from the Poisson distribution and practically behaves as a delta function $\delta(s)$ at the maximum allowed values of $\kappa = 0.5$ and $q = 1.5$. Therefore, neither $P_K(s)$ nor $P_{Ts}(s)$ can describe the initial stage of the transition from chaos to integrability. In fact, it is hard to believe that nearly integrable systems can be described by a base-independent random-matrix model like the one considered in the present paper. Both distributions increase linearly with $s$ near the origin. The Kaniadakis distribution $P_K(s)$ decays at large $s$ as $s^{2-2/\kappa}$. The NNS in Eq. (9) obtained by the Tsallis statistics asymptotically behaves as $P_{Ts}(s) \sim s^{2-2/(q-1)}$. Both distributions have similar asymptotic behavior when $\kappa = q - 1$. However, distributions $P_K(s)$ and $P_{Ts}(s)$ satisfying this relation have different behavior at intermediate values of $s$. The evolution of $P_K(s)$ as $\kappa$ varies from the GOE value of 0 to $\sim 0.5$ and $P_{Ts}(s)$ as $q$ varies from the GOE value of 1 to $\sim 1.5$ is demonstrated in Fig.1. The figure shows two sets of NNS distributions, one set for each statistics, with entropic indices related by $\kappa = q - 1$ so that the corresponding distribution have similar asymptotic behavior. While the peak of the Tsallis distribution steadily moves towards the origin as $q$ increases, the peak of the Kaniadakis distribution remains almost in the same position as $\kappa$ increases from 0 and only starts to move towards smaller $s$ when $\kappa$ exceeds a value of $\sim 0.4$. We remind that the mean square spacing diverges in this latter domain of $\kappa$.

### 4 Comparison with numerical experiment

This section offers a comparison of the generalized expressions of the Wigner surmise derived from the Tsallis and Kaniadakis entropies with the NNS obtained...
in the numerical experiment done by Życzkowski and Kuś [28]. This experiment imitates a one-parameter (denoted by δ) transition between an ensemble of diagonal matrices with independently and uniformly distributed elements, corresponding to δ = 0, and a circular orthogonal ensemble for δ = 1. These authors were able to achieve reliable statistics by constructing numerically 500 matrices of size N = 100 for each considered value of the transition parameter δ. Diagonalizing these matrices yielded NNS distributions consisting of 50 000 spacings for each value of δ. Three of these distributions intermediate between the Wigner and Poisson distributions, corresponding to δ = 0.9, 0.5 and 0.1, are reported in Ref. [28]. These distributions are compared in Fig. 1 with the spacing distributions in Eqs. (9) and (26), which have been obtained by using the Tsallis and Kaniadakis statistics, respectively. The figure suggests that both entropies yield reasonable description of the stochastic transition although the agreement with the numerical results is not complete. The slight difference in the behavior of the two non-extensive NNS distributions is a positive aspects since it is well known that the transition from order to chaos does not necessarily proceed through the same passage. It remains an open question to find which systems are better described by either entropy.

5 Conclusion

Tsallis’ entropy has been considered by several authors as a starting point for constructing a generalization of RMT. However, there are several other generalized entropies. This paper derives random-matrix ensembles by maximizing the generalized entropies proposed by Rényi, Abe and Kaniadakis under the constraints on normalization of the distribution and the expectation value of \( \text{Tr} \left( H^T H \right) \). The distribution functions obtained for the Rényi entropy are analogous to the corresponding results, previously obtained assuming Tsallis’ entropy, which is not a surprise since these entropies are monotonic functions of each other. The use of Abe’s entropy did not lead to closed-form expressions for the matrix-element distribution. On the other hand, we obtain a new analytical formula for the distribution function in the matrix-element space when the entropy is given by the Kaniadakis measure. This leads to a new expression for the NNS distribution of the eigenvalues in the special case of two-dimensions random-matrix ensembles. This special case is the one that leads to the Wigner surmise when the entropy is given by the Shannon measure. The high accuracy of Wigner’s distribution describing chaotic systems justifies the use of the two-dimensional ensembles for obtaining NNS distributions using other entropies. As in the case of Tsallis’ entropy, the NNS distribution obtained here for Kaniadakis’ entropy has an intermediate shape between the Wigner distribution and the Poissonian that describes generically integrable systems. However, neither of these distributions reaches the Poissonian form for any value of the entropic index. It is generally believed that the route from integrability to chaos is not unique. Therefore, we hope that the NNS distribution obtained in this paper as well as the one, previously obtained with the Tsallis statistics, are good candi-
dates for modelling systems with mixed regular-chaotic dynamics at least when they are not far from the state of chaos. We have tested these expressions by comparing their predictions with the NNS distributions obtained in a numerical experiment by Życzkowski and Kuś.

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Figure caption

FIG. 1. NNS distributions obtained by using the Tsallis (lower panel) and Kaniadakis (upper panel) statistics. The entropic indices used, being \(q = 1\) (GOE), 1.1, 1.2, 1.3, 1.4, 1.45, 1.47, 1.49 in the case of the Tsallis statistics and \(\kappa = 0\) (GOE), 0.1, 0.2, 0.3, 0.4, 0.45, 0.47, 0.49 in the Kaniadakis’, are chosen so that the corresponding curves in the two statistics have the same asymptotic behavior.

FIG. 2. NNS distributions obtained in a numerical simulation \[28\] of the transition from the GOE statistics to the Poissonian (histograms) compared to NNS distributions obtained by using the Tsallis (dashed lines) and Kaniadakis (solid lines) statistics.
