Combating Errors in Propagation of Orbital Angular Momentum Modes of Light in Turbulent Media

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Abstract
There is a wealth of simulation, experimental and analytical studies on propagation of orbital angular momentum (OAM) modes through atmospheric and oceanic turbulence. Using the data of these studies and generalising the framework proposed in [Bala et al., arXiv:2208.04555] for error-immune information transfer, we accomplish two tasks. First, we identify invariants for propagation of OAM modes through atmospheric and oceanic turbulence, in which error-immune information can be encoded. A closer look at the data reveals two universal features: (i) coherence lasts for a much longer distance in turbulence than entanglement, and, (ii) the crosstalk among different OAM modes depends very weakly on the initial OAM mode index in the weak turbulence regime. Keeping these in mind, we next develop a method for combating errors in what we call an idealised crosstalk channel. In an idealised crosstalk channel, the crossover probabilities are independent of the initial mode index (IMI). We lay down a procedure that allows to retrieve full information in a state by identifying invariant quantities. Finally, we construct quantum error correction and rejection codes for idealised crosstalk channels, without any need for multiparty entanglement.

Keywords  Crosstalk · Information retrieval · Atmospheric turbulence · Oceanic turbulence

1 Introduction
Recent times have witnessed an unprecedented surge of interest in quantum information-theoretic protocols [1–4]. However, noise acts as a major impediment in their implementation. Though there have been many proposals [5–10] for mitigating the effect of noise, these
techniques either demand resources such as multiparty entanglement or implementation of interventions at appropriate time intervals [11]. Generation of multiparty entanglement is still a challenge, especially in photonic systems [12].

These challenges in implementation of conventional error mitigation techniques have led us to propose a new information encoding scheme [13]. This scheme rests on encoding information in invariants, due to which information can be transferred in an error-immune manner even through noisy channels.

In parallel, thanks to significant advancements in generation and measurement of OAM states of light, these modes hold the promise of implementation of communication protocols with flying qudits, for both– free-space and underwater communication [14–17]. However, turbulence, being atmospheric or oceanic, acts as a major impediment [18, 19] leading to crossover among different modes and decoherence/ loss of entanglement. The propagation of OAM modes through turbulence is naturally modelled through cross-talk channels— which cause a spillover to neighbouring modes. That is to say, it results in an increase in number of modes in the final state as compared to the initial state.

In this work, we apply the framework proposed in [13] to address the problem of transmission of information with OAM modes through turbulent media. There are two physical situations of interest, viz., atmospheric and oceanic turbulence. To start with, we analyse data from simulation, experimental and analytical studies on propagation of OAM modes of light [19–23]. The data serve a twofold purpose: (i) identification of invariants for transmission of error-free information with OAM modes through turbulent media, (ii) a closer look at the data reveals a weak dependence of crosstalk on the initial mode index (IMI) in weak turbulence regime. Current simulations have studied propagation of a pure entangled biphoton and three-photon OAM state which mimic ‘two-qubit’ and ‘three-qubit’ respectively. We study information retrieval for both the cases Section 2. Interestingly, the invariants identified in the case of atmospheric turbulence also allow for retrieval of full information in the initial state.

We briefly summarise the results of simulation, experimental and analytical studies. It is found [22, 24, 25] that in weak atmospheric turbulence, the crossover probabilities are independent of the IMI under rather simplified assumptions. In a more realistic simulation study, it is observed [20] that for weak turbulence ($C_n^2 \approx 10^{-16}m^{-2/3}$), crosstalk is negligible for all practical purposes. For $C_n^2 = 10^{-15}m^{-2/3}$, the crosstalk is independent of the IMI for $2 \leq \ell \leq 7$. By examining the data presented in [20], we find out that the crosstalk among different modes depends only weakly on the IMI in another regime characterised by $6 \leq \ell \leq 16$ and $C_n^2 = 10^{-14}m^{-2/3}$ (see Section 3 for details).

It is observed in [19] for propagation of ‘two-qubit’ OAM Werner state that coherence lasts much longer than entanglement (the typical distances are of the orders of 50 m and 10 m respectively). Similar conclusions have been drawn in [26] for propagation of ‘two-qubit’ OAM Werner state through atmospheric turbulence.

From these findings, we may conclude that, in weak turbulence regime, the noisy channels are mostly ideal, by which we mean that the crossover probabilities are independent of the IMI. Exploiting this feature, we identify invariants for an IC channel Section 4. We also demonstrate that full information retrieval is possible by employing these invariants Section 5.

Finally, keeping in mind the experimental challenges in generation of multiparty entanglement among OAM modes and its fast degradation through turbulence, in this work, we
also construct *ancilla-free* error-rejection and error-correction codes with appropriate initial modes (resp. Sections 6 and 7) for an IC channel. Section 8 concludes the paper with closing remarks.

2 Propagation of OAM Modes in Crosstalk Channels: Retrieval of Information

There are a number of studies which give final density matrix (in a truncated subspace) after propagation through atmospheric or oceanic turbulence resulting in crossover, decoherence and loss of entanglement. In spite of this, we show that some information can be retrieved in terms of quantities that remain invariant. For this, we consider various simulation and analytical studies of transfer of information in OAM modes through weak turbulence [19, 22, 23]. Employing the results of these studies, we identify invariants which, in some of the cases, lead to retrieval of full information contained in the initial state. To start with, we consider propagation of Werner state propagating through oceanic turbulence.

2.1 Retrieval of Information from a Werner-like State After Passing Through Oceanic Turbulence

In [19], Zhai. et al. have studied the effect of oceanic non-Kolmogorov turbulence on propagation of entangled Laguerre-Gauss beams by chosing the initial state of the photon pair to be in a Werner-like state,

\[ \rho(0) = \left( \frac{1 - \gamma}{4} \right) \mathbb{I} + \gamma |\psi_0\rangle \langle \psi_0| . \]  

(1)

The symbol \( \gamma \) characterizes the purity of the initial state. The symbol \( \mathbb{I} \) represents the identity matrix in the subspace spanned by \( \{|\ell, \ell\rangle, | - \ell, - \ell\rangle, | - \ell, \ell\rangle, | - \ell, - \ell\rangle\} \) and the state \( |\psi_0\rangle \) is chosen to be,

\[ |\psi_0\rangle = \cos \frac{\theta}{2} |\ell, -\ell\rangle + e^{i\phi} \sin \frac{\theta}{2} | - \ell, \ell\rangle. \]  

(2)

In [19], the same turbulence effect has been assumed on both the photons for short distances. They determine the final state in the ordered trucated basis\( \{|\ell, \ell\rangle, | - \ell, - \ell\rangle, | - \ell, \ell\rangle, | - \ell, - \ell\rangle\} \) and report that the non-vanishing elements are given by,

\[ \rho_{11} = \alpha \left( \mu^2 \rho_{11}^{(0)} + \mu \nu \rho_{22}^{(0)} + \mu \nu \rho_{33}^{(0)} + \nu^2 \rho_{44}^{(0)} \right) \]  

(3)

\[ \rho_{22} = \alpha \left( \mu \nu \rho_{11}^{(0)} + \mu^2 \rho_{22}^{(0)} + \nu^2 \rho_{33}^{(0)} + \mu \nu \rho_{44}^{(0)} \right) \]  

(4)

\[ \rho_{33} = \alpha \left( \mu \nu \rho_{11}^{(0)} + \nu^2 \rho_{22}^{(0)} + \mu^2 \rho_{33}^{(0)} + \mu \nu \rho_{44}^{(0)} \right) \]  

(5)

\[ \rho_{44} = \alpha \left( \nu^2 \rho_{11}^{(0)} + \mu \nu \rho_{22}^{(0)} + \mu \nu \rho_{33}^{(0)} + \mu^2 \rho_{44}^{(0)} \right) \]  

(6)

\[ \rho_{14} = \alpha \mu^2 \rho_{14}^{(0)}, \rho_{23} = \alpha \mu^2 \rho_{23}^{(0)}; \quad \alpha = (\mu + \nu)^{-2}. \]  

(7)

The symbols \( \mu \) and \( \nu \) represent the survival and crosstalk amplitudes respectively. We now move on to show that in spite of decoherence, one may retrieve some error-immune information encoded in the initial state.
Retrieval of Information  

The quantity

$$I = \frac{-i(\rho_{23} - \rho_{32})}{\rho_{23} + \rho_{32}} = \frac{-i(\rho_{23}^{(0)} - \rho_{32}^{(0)})}{\rho_{23}^{(0)} + \rho_{32}^{(0)}} \equiv \tan \phi,$$  

(8)

remains invariant during the noisy evolution. The invariant $I$ determines the relative phase between the basis states $|\ell, -\ell\rangle$ and $|-\ell, \ell\rangle$ up to a discrete ambiguity. This ambiguity can be removed by determining the signs of real and imaginary part of $\rho_{23}$.

There are other quantities that remain formally invariant during the noisy evolution,

$$I_1 = \frac{\rho_{11} - \rho_{44}}{\rho_{22} - \rho_{33}} = \frac{\rho_{11}^{(0)} - \rho_{44}^{(0)}}{\rho_{22}^{(0)} - \rho_{33}^{(0)}}, \quad I_2 = \frac{\rho_{14} + \rho_{41}}{\rho_{14} - \rho_{41}} = \frac{\rho_{14}^{(0)} + \rho_{41}^{(0)}}{\rho_{14}^{(0)} - \rho_{41}^{(0)}},$$  

(9)

$$I_3 = \frac{\rho_{14} + \rho_{41}}{\rho_{23} + \rho_{32}} = \frac{\rho_{14}^{(0)} + \rho_{41}^{(0)}}{\rho_{23}^{(0)} + \rho_{32}^{(0)}}.$$  

(10)

However, since for the state (1), $\rho_{11}^{(0)} = \rho_{44}^{(0)}$ and $\rho_{14}^{(0)} = \rho_{41}^{(0)} = 0$, the invariants $I_1, I_2, I_3$ are identically equal to zero.

2.2 Retrieval of Information from a ‘two-qubit’ OAM Entangled State Through Kolmogorov Turbulence

We next study the example of propagation of an entangled state through atmospheric turbulence. In [22], the effect of atmospheric turbulence, following Kolmogorov model of turbulence, on a two-photon OAM entangled state,

$$|\psi_0\rangle = \frac{1}{\sqrt{2}} \left( |\ell, -\ell\rangle + e^{i\gamma} | -\ell, \ell\rangle \right),$$  

(11)

is simulated for a wide range of $\ell$ ($5 - 100$), under the assumption that both the photons undergo the same turbulence effect. The state $|\psi_0\rangle$ has only one free parameter, viz., $\gamma$. The effect of turbulence is described by a map whose non-zero elements are determined in [22] by employing exchange and inversion symmetries. In [22], the final state is determined in the truncated ordered basis $\{|\ell, \ell\rangle, |\ell, -\ell\rangle, |-\ell, \ell\rangle, |-\ell, -\ell\rangle\}$ to be,

$$\rho = \frac{1}{2(a + b)^2} \begin{pmatrix} 2ab & 0 & 0 & 0 \\ 0 & a^2 + b^2 & a^2 e^{-i\gamma} & 0 \\ 0 & a^2 e^{i\gamma} & a^2 + b^2 & 0 \\ 0 & 0 & 0 & 2ab \end{pmatrix}.$$  

(12)

The symbols $a$ and $b$ represent respectively the survival and crosstalk amplitudes. As before, we show that in spite of atmospheric turbulence, one may identify such quantities that remain invariant.

Retrieval of Information  

From (12), we identify the quantity,

$$I = \frac{-i(\rho_{23} - \rho_{32})}{\rho_{23} + \rho_{32}} \equiv \tan \gamma,$$  

(13)

that remains invariant. This quantity determines the relative phase ($\gamma$) between the states $|\ell, -\ell\rangle$ and $|-\ell, \ell\rangle$ up to a discrete ambiguity. As in the previous case, one can remove this discrete ambiguity by determining the signs of the real and the imaginary parts of $\rho_{23}$. In this manner, the parameter $\gamma$ can be uniquely determined. Since this is the only unknown
parameter in the initial state, determination of \( \gamma \) is equivalent to determining the initial state of the system.

### 2.3 Retrieval of Information from ‘three-qubit’ OAM Entangled State Passing Through Atmospheric Turbulence

We now move on to an example of propagation of ‘three-qubit’ OAM entangled state through atmospheric turbulence reported in [23]. It turns out that in spite of a large number of parameters (see (14)), it is possible to extract complete information on the state. For this reason, it merits a brief description.

In [23], the initial state has been assumed to pass through turbulent atmosphere modelled through Kolmogorov turbulence. The authors have assumed a three-photon state initially in the superposition of five modes [23],

\[
|\psi_0\rangle = \alpha_1 |\ell, -\ell, -\ell\rangle + \alpha_2 |\ell, -\ell, \ell\rangle + \alpha_3 |\ell, \ell, -\ell\rangle + \alpha_4 |\ell, \ell, \ell\rangle + \alpha_5 |\ell, -\ell, -\ell\rangle.
\]

The information content in the state is carried by bilinears of the form \( \alpha_i \alpha^*_j \) of which eight are independent. Under the assumption that all the three photons suffer the same turbulence [23], the authors arrive at final state which has additional three modes signifying the effect of turbulence. The final state in the truncated ordered basis \{ \langle \ell, \ell, \ell \rangle, \langle \ell, \ell, -\ell \rangle, \langle \ell, -\ell, \ell \rangle, \langle -\ell, \ell, \ell \rangle, \langle -\ell, -\ell, \ell \rangle, \langle -\ell, -\ell, -\ell \rangle \} is reported to be [23],

\[
\rho_{\text{out}} = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix},
\]

where the block matrices are given by,

\[
A = \begin{pmatrix} G_1 & 0 & 0 & M_{13} \\ 0 & G_2 & 0 & 0 \\ 0 & 0 & G_3 & 0 \\ M_3 & 0 & 0 & G_4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & M_{14} & M_{15} & M_{16} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & M_1 & M_2 & M_4 \end{pmatrix}, \quad C = \begin{pmatrix} G_5 & 0 & 0 & 0 \\ 0 & G_6 & M_6 & M_8 \\ 0 & M_{10} & G_7 & M_{12} \\ 0 & M_{18} & M_{19} & G_8 \end{pmatrix}.
\]

The functional forms of the entries are given explicitly in the Appendix A.

#### 2.3.1 Retrieval of Information with Invariants

It follows from (15) and (16) that there are a large number of invariants, totalling to twenty in number (the complete set of the invariants is given in the Appendix A). This far exceeds the number of independent parameters in the initial state which are only eight in number. Hence, this provides an excellent example where the modelling employed may be checked self-consistently. We employ the following invariants:

\[
I_1 = \frac{-i(M_4 - M_7)}{M_4 + M_7}, \quad I_2 = \frac{-i(M_5 - M_8)}{M_5 + M_8}, \quad I_3 = \frac{-i(M_1 - M_{13})}{M_3 + M_{13}}, \quad I_4 = \frac{-i(M_4 - M_{17})}{M_4 + M_{17}}, \quad I_5 = \frac{M_1}{M_2}, \quad I_6 = \frac{M_1}{M_3}, \quad I_7 = \frac{M_1}{M_4}, \quad I_8 = \frac{M_2}{M_{10}}.
\]

for retrieval of information encoded in the initial state.

**Retrieval of Parameters of Initial State** The initial state chosen in (14) has eight independent parameters. The first four invariants, \( \overline{\text{viz}.} \quad I_1 - I_4 \) determine the relative phase between the successive basis states respectively, up to discrete ambiguities. These ambiguities, as in
the previous cases, can be removed by employing the signs of numerators and denominators of these invariants separately. The next four invariants, \( v_i \), \( I_5 - I_8 \), together with the information of the relative phases between the basis states, provide the magnitude of the co-efficients of the basis states.

This concludes our discussion of retrieval of information for various simulations studied that are already performed for propagation of OAM modes in turbulent media.

We, now, embark on a study of OAM propagation in weakly turbulence regime. Our purpose is to identify possible universal features.

3 Universal Features of Propagation of OAM Modes

There are several studies of OAM propagation in weak atmospheric turbulence regime which are pertinent to our enquiry [19, 20, 24, 25, 27]. In two analytical studies [24, 25], under rather simplified assumptions, it is found that the crosstalk among different modes is independent of the IMI. In a more extensive numerical simulation [20], it is shown that for turbulence strength of the order of \( C_n^2 \approx 10^{-15} \text{m}^{-2/3} \), crosstalk is independent of IMI for small OAM mode indices (2 \( \leq \ell \leq 7 \)). We show that the weak dependence on IMI is not restricted to this regime. A careful examination of the data [20] reveals that the crosstalk is weakly dependent on IMI for 6 \( \leq \ell \leq 16 \) and \( C_n^2 = 10^{-14} \text{m}^{-2/3} \). To demonstrate this, we plot in Fig. 1 the data given in [20] for \( C_n^2 = 10^{-14} \text{m}^{-2/3} \) and for IMI 6 \( \leq \ell \leq 16 \).

The feature that concerns us is small change in the relative intensities for a given value of \( \Delta \ell \) when IMI changes from 6 to 16. For the sake of a better appreciation of this feature, in Fig. 2, we have plotted relative intensities with IMI for different values of \( \Delta \ell \). The slopes of linear fits for the plots of relative intensities vs. \( \Delta \ell \) are of the order of \( 10^{-3} \), shown in Table 1. The table also contains relative change in the intensities for different values of \( \Delta \ell \), which vary from 0.6% – 3%, with \( \Delta \ell = -3 \) being the sole outlier. The weak dependence on the initial OAM mode is also argued by universal entanglement decay law [27] over a wide range of IMI (5 \( \leq \ell \leq 100 \)).

![Fig. 1 Plot of relative intensities for different values of \( \Delta \ell \) for 6 \( \leq \ell \leq 16 \) for \( C_n^2 = 10^{-14} \text{m}^{-2/3} \). The data has been taken from [20]]
Fig. 2  Plot of relative intensities with different initial $\ell$ values for $-3 \leq \Delta \ell \leq 3$ for $C_\ell^2 = 10^{-14} \text{m}^{-2/3}$. It becomes clear from this plot that, in this regime, the change in the relative intensities with initial $\ell$ values, characterised by the slope ($\approx 10^{-3}$) is very small for different values of $\Delta \ell$. The data has been taken from [20]

Furthermore, it has been observed in [19] for propagation of ‘two-qubit’ OAM Werner state that coherence survives up to a distance much larger (50 m), in contrast to entanglement which decays much faster (10 m). Similar conclusions have been shown in [26] for propagation of ‘two-qubit’ OAM Werner state through atmospheric turbulence.

In summary, the weak dependence shows that idealised crosstalk channel is an excellent approximation to realistic weakly turbulent channels. The observation that coherence lasts much longer than entanglement provides a compelling motivation for developing error combating techniques for an IC channel without costly resources such as multiparty entanglement.

3.1 Idealised Crosstalk Channel

Suppose that the set,

$$\mathcal{B} \equiv \{|l_{\min}\rangle, \cdots, |l_{\max}\rangle\}, \quad (18)$$

consists of the modes which are employed for information transfer. The effect of an IC channel, given these modes, can be described in terms of the following Kraus operators:

$$E_{\pm k} = \sqrt{p_k} \sum_{n=l_{\min}}^{l_{\max}} |n \pm k\rangle \langle n|, \quad (19)$$

where $p_k$ is the probability with which the errors (spillovers) $E_{+,k}$ and $E_{-,k}$ corrupt a state. In the next section, we identify invariants that allow to transfer error-free information through an IC channel.

| $\Delta \ell$ | −3 | −2 | −1 | 0 | +1 | +2 | +3 |
|-------------|----|----|----|---|----|----|----|
| Slope       | 0.0022 | 0.0004 | −0.0037 | −0.0065 | −0.0034 | −0.0002 | 0.0012 |
| Relative change in intensity | 0.085 | 0.006 | 0.028 | 0.038 | 0.026 | 0.002 | 0.037 |
4 Identification of Invariants for IC Channel

As is clear from (19), an IC channel is of the kind in which the initial and the final states may involve superpositions of $M$ and $N$ modes respectively [20, 24, 25], i.e., $\mathcal{H}^M \xrightarrow{\mathcal{E}_2} \mathcal{H}^N$. To identify invariants for this channel, we adapt the framework proposed in [13] for noisy evolutions of the kind $\mathcal{H}^M \xrightarrow{\mathcal{E}_1} \mathcal{H}^M$ ($\mathcal{H}^M$ and $\mathcal{H}^N$ denote $M$- and $N$- dimensional Hilbert spaces respectively). This adaptation is accomplished by mapping an IC channel to a generalised flip channel acting on a sufficiently higher-dimensional space. In the following, we first briefly recapitulate the invariants for a generalised flip channel.

4.1 Invariants for a Generalised Flip Channel

We briefly recapitulate the method of determination of invariants for a generalised flip channel obtained in [13]. A state $\rho$, after passing through a generalised flip channel, changes to a state $\rho'$ as follows:

$$\rho \rightarrow \rho' = \sum_{r=0}^{N-1} F_r \rho F_r^\dagger \equiv \sum_{r=0}^{N-1} p_r X_r \rho (X'_r)^\dagger,$$

(20)

where $p_r$ is the probability with which an error $X_r = \sum_{k=0}^{N-1} |k+r\rangle \langle k|$ has corrupted a state (here, the summation is modulo $N$). Employing the cyclicity property of trace, we see that the expectation values of operators,

$$I(m) = \langle X^m \rangle,$$

(21)

are invariants and thus the encoded information remain error–free. These invariants belong to the first family.

Additional invariants are identified by employing the relation $ZX = \omega XZ$, where $Z = \sum_{k=0}^{N-1} \omega^k |k \rangle \langle k |$, and $\omega$ is the $N$th root of unity. As shown in [13], the quantities,

$$I^{(m,l)} = \langle Z^l \rangle \langle X^m Z^l \rangle,$$

(22)

also remain invariants. Hence, information encoded in $I^{(m,l)}$ also remains immune to error. The invariants $I^{(m,l)}$ belong to the second family. Equipped with these results, we, first show that the effect of IC channel on a finite-dimensional space can be modelled by a generalised flip channel in a higher-dimensional space and then, move on to identify invariants for an IC channel.

4.2 Mapping an IC Channel to a Generalised Flip Channel

To start with, we consider an IC channel that causes a spillover by one unit. The corresponding error operators are, $E_{\pm 1} = \sqrt{p} \sum_{k=1}^{M} |k \pm 1\rangle \langle k |$. We, now, consider the state,

$$|\psi\rangle = \sum_{i=1}^{M} \alpha_i |i\rangle \equiv \sum_{i=0}^{M+1} \alpha_i |i\rangle; \quad \alpha_0, \alpha_{M+1} \equiv 0,$$

(23)
that is employed to transfer information. Then, upto normalisation factors,

\[ E_{+1} |\psi\rangle \equiv \sum_{i=1}^{M} \alpha_i |i+1\rangle, \quad E_{-1} |\psi\rangle \equiv \sum_{i=1}^{M} \alpha_i |i-1\rangle. \tag{24} \]

Consider, now, the generalised flip channel acting on \(\mathcal{H}_M^{M+2}\) via the error operators,

\[ F_{+1} = \sqrt{p_1} X, \quad F_{-1} = F_{+1}^\dagger = \sqrt{p_1} X^\dagger, \]

where the flip operator is \(X = \sum_{k=0}^{M+1} |k+1\rangle \langle k|\) (with addition modulo \(M + 2\)). Again, upto normalisation factors, the effect of error operators \(F_{\pm 1}\) on the state \(|\psi\rangle\) is given by,

\[ F_{+1} |\psi\rangle \equiv \sum_{i=1}^{M} \alpha_i |i+1\rangle, \quad F_{-1} |\psi\rangle \equiv \sum_{i=1}^{M} \alpha_i |i-1\rangle, \tag{25} \]

which is no different from (24), thereby establishing that they have the same effect on the family of states chosen in (23). This argument extends to IC channels that shift a mode by \(l\) units provided we employ a generalised flip channel acting on at least \((M+2l)-\) dimensional space.

For an explicit illustration, consider an IC channel that shifts a mode by \(l\) units. The corresponding error operators given in (19) assume the following form,

\[ E_{\pm r} = \sqrt{p_r} \sum_{k=l}^{l+M-1} |k \pm r\rangle \langle k|, \quad 1 \leq r \leq l. \tag{26} \]

The initial state would have the form,

\[ |\phi\rangle = \sum_{j=l}^{l+M-1} \beta_j |j\rangle. \tag{27} \]

Upto normalisation factors, the effect of these operators on the state \(|\phi\rangle\) is,

\[ E_{+r} |\phi\rangle \equiv \sum_{j=l}^{l+M-1} \beta_j |j+r\rangle, \quad E_{-r} |\phi\rangle \equiv \sum_{j=l}^{l+M-1} \beta_j |j-r\rangle, \tag{28} \]

which can be reproduced by the generalised flip channel defined in \((M+2l)-\) dimensional space with operators,

\[ F_{+r} = \sqrt{p_r} X^r, \quad F_{-r} = F_{+r}^\dagger = \sqrt{p_r} (X^r)^\dagger; \quad 1 \leq r \leq l, \tag{29} \]

where \(X^r = \sum_{k=0}^{M+2l-1} |k+r\rangle \langle k|\) (with addition modulo \(M + 2l\)) as may be verified easily.

In summary, the effect of a crosstalk error causing a spillover by \(l\) units is equal to that of a generalised flip errors in \(\mathcal{H}_N^N\) provided the initial state is chosen appropriately and \(N \geq M + 2l\).

Given the equivalence between the two, the invariants of a generalised flip channel identified in [13] also serves as invariants for IC channels (given in (21) and (22)) and hence can be used for error–immune information transfer. In the following section, we describe the scheme that allows for full information retrieval from a state passing through a crosstalk channel that causes a spillover by at most \(l\) units.
5 Complete Information Retrieval from a State After Passing Through an IC Channel

In this section, we lay down a scheme that allows for complete information retrieval from a state by employing the invariants identified in the previous section. Consider a state involving superposition of $M$ modes,

$$\rho = \sum_{i,j=1}^{l+M-1} \rho_{i,j} |i\rangle \langle j|,$$

(30)

that is employed to transfer information through a crosstalk channel. Suppose that the crosstalk channel is such that it can cause spillover to at most $l$ modes. In that case, the final state, after passing through this channel, takes the form,

$$\rho' \equiv \sum_{k=0}^{l} E_{\pm k} \rho E_{\pm k}^\dagger \equiv \sum_{k=0}^{l} p_k \rho_{\pm}^{(k)},$$

(31)

where $\rho_{\pm}^{(0)} \equiv \rho$ and $\rho_{\pm}^{(k)}$ are given by,

$$\rho_{\pm}^{(k)} = \sum_{i,j,l}^{l+M-1} \rho_{i,j} |i \pm k\rangle \langle j \pm k|, \quad \forall k \in \{1, \cdots, l\}.$$  

(32)

From (31) and (32), it is clear that the final state $\rho'$ has a support in an $(M+2l)$–dimensional Hilbert space.

For the purpose of information retrieval, we first fix the the dimension of the space in which the flip operators are to be defined. This requires us to take into account the following point. The dimensionality of the space should be such that the relation,

$$\langle X^{M-1} Z^r \rangle_\rho = f(\omega, r) \langle X^{M-1} \rangle_\rho,$$

(33)

holds where $f(\omega, r)$ is a function of only $\omega$ and $r$. Since the initial state $\rho$ involves superposition of $M$ modes, condition (33) is satisfied iff the operator $X^{M-1}$ is not its own inverse. It holds in the minimum dimension $2M - 1$. This is because in $(2M - 1)$–dimensional space, the operator $X$ satisfies the relation, $X^{2M-1} = \mathbb{I}$ leading to $(X^{M-1})^{-1} = X^M$. On the other hand, since the final state has support in $(M + 2l)$ dimensional space, the minimum dimension for information retrieval is given by,

$$\mathcal{N} \equiv \max\{2M - 1, M + 2l\}.$$  

(34)

Thus, in the $\mathcal{N}$–dimensional space, operators $X^r$ and $Z^r$ have the forms:

$$X^r = \sum_{k=0}^{\mathcal{N}-1} |k + r\rangle \langle k|, \quad Z^r = \sum_{k=0}^{\mathcal{N}-1} \omega^{|k|} |k\rangle \langle k|,$$

(35)

where addition is modulo $\mathcal{N}$ and $\omega$ is $\mathcal{N}^{th}$ root of identity. Following the reasoning in Section 4.2, the invariants of the first family provide the following information on the initial state $\rho$,

$$I_1^{(r)} = \langle X^r \rangle = \sum_{k=l}^{l+M-1-r} \rho_{k,k+r}, \quad 1 \leq r \leq M - 1.$$  

(36)
In a similar manner, the second family of invariants yields the relation,

\[ I^{(r,s)}_2 = \frac{\langle X^r Z^s \rangle}{\langle X^r Z^s \rangle} = \sum_{k=l}^{l+M-1} \omega^{ks} \rho_{k,k} - \sum_{k=l}^{l+M-1-r} \omega^{s-k} \rho_{k,k+r}, \quad 1 \leq r, s \leq M - 1. \]  

(37)

The steps of the scheme that allows full information retrieval of a state are as follows:

- **Step 1:** The first family invariant, \( I^{(M-1)}_1 \) determines the off–diagonal element \( \rho_{l,l+M-1} \).
- **Step 2:** The second family of invariants, \( I^{(M-1,s)}_2 = \sum_{k=l}^{l+M-1} \omega^{ks} \rho_{k,k} \omega^{l+s} \rho_{l,l+M-1} \), \( 1 \leq s \leq M - 1 \), yield the simultaneous set of \( (M - 1) \) equations. These equations, together with the trace condition \( \sum_{i=l}^{l+M-1} \rho_{i,i} = 1 \), allow to retrieve each of the diagonal element \( \rho_{i,i} \). It remains to retrieve the other off–diagonal elements for which the following invariants are employed,

\[ I^{(p)}_1 = \sum_{k=l}^{l+M-1-p} \rho_{k,k+p}, \quad I^{(p,s)}_2 = \frac{\sum_{k=l}^{l+M-1} \omega^{ks} \rho_{k,k}}{\sum_{k=l}^{l+M-1-p} \omega^{s-k} \rho_{k,k+p}}, \quad 1 \leq s \leq M - 1 - p, \]  

(38)

where \( p \) can take values from the set \( \{1, 2, \cdots, M - 2\} \). The index \( p \) corresponds to the retrieval of off–diagonal elements lying on the line \( S_p \) as shown in the Fig. 3. This is explicitly detailed in the Table 2.

As an illustration, we have explicitly shown information retrieval through an example of a state initially in superposition of three modes passing through an IC channel causing a spillover by one unit in Appendix B.

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**Fig. 3** \( \rho_{i,j} \) represents the \( ij \)th element of the initial density matrix \( \rho \), \( D \) represents its diagonal, and \( S_i \) represents its \( i \)th super–diagonal (which we use for convenience)
Table 2  Sets of invariants employed and correspondingly retrieved density matrix elements

| Invariants                              | Elements of density matrix |
|-----------------------------------------|---------------------------|
| $I(M^{M-1})$                            | $\rho_{i,j}; j - i = M - 1$  |
| $I_2^{(M_{-1}), 1 \leq s \leq M - 1}$  | $\rho_{i,i}, \forall i \in \{l, \ldots, l + M - 1\}$  |
| $I_1^{(M-1)}$, $\sum_{i=0}^{l+M-1} \rho_{i,i} = 1.$ | $\rho_{i,j}; j - i = M - 2$  |
| $I_1^{(M-2)}$, $I_2^{(M-2,1)}$         | $\rho_{i,j}; j - i = M - 3$  |
| $I_1^{(M-3)}$, $I_2^{(M-3,1)}$, $I_2^{(M-3,2)}$ | $\rho_{i,j}; j - i = 1$  |
| $\vdots$                                | $\vdots$                  |
| $I_1^{(1)}$, $I_2^{(1,1)}$, $I_2^{(1,2)}$, $\ldots$, $I_2^{(1,M-2)}$ | $\vdots$                  |

In the next section, we construct an ancilla-free error-rejecting codes for an IC channel for propagation of OAM modes.

6 Quantum Error Rejection Code (QERC)

In this section, we determine the additional constraints to be imposed on the choice of basis modes to construct QERC. Recall that, a QERC merely detects whether error has corrupted a state or not [28]. In order to construct a QERC, it is necessary that each error projects the initial state to a space which has no overlap with it. That is,

$$H^M \perp H^E, \quad \text{where } H^E \equiv H^{E+1} \cup \cdots \cup H^{E+l} \cup H^{E-l} \cup \cdots \cup H^{E-l}.$$  (39)

By virtue of this orthogonality, one may employ an observable,

$$O = \Pi^M,$$  (40)

that provides an outcome only if the state is uncorrupted. A more general observable, $O = c_1 \Pi^M + c_2 \Pi^E$, $c_1 \neq c_2$ can be employed for detection of occurrence of errors. Here, $\Pi^M$ and $\Pi^E$ represent the projections having supports in $H^M$ and $H^E$. If the measurement outcome is $c_1$, then, there is no corruption. On the other hand, the outcome $c_2$ corresponds to the case when the post-measurement state is corrupted and hence can be rejected.

6.1 QERC for an IC Channel that Causes a Spillover by at Most $l$ Units

For an IC channel that shifts a mode by $l$ units, the basis modes should be chosen at a distance, $\Delta_r = (l + 1)$ to preserve the condition of orthogonality of the initial state with all the corrupted states. Therefore, if an $M$–dimensional state is to be transferred through an IC channel causing a spillover by $l$ units, the following set of basis modes,

$$\mathcal{B}_r \equiv \{ |l\rangle, |l + \Delta_r\rangle, |l + 2\Delta_r\rangle, \ldots, |l + (M - 1)\Delta_r\rangle \},$$  (41)

can be employed. Thus, any $M$–dimensional state,

$$|\psi\rangle = \sum_{i=0}^{M-1} \alpha_i |l + i\Delta_r\rangle,$$  (42)
can be employed as it remains orthogonal to any of the corrupted states of an IC channel, thus fulfilling the requirement of QERC. For this state, measurement of the observable,

\[ O = \Pi^M = \sum_{j=0}^{M-1} |l + j \Delta_r\rangle\langle l + j \Delta_r|, \]  

\[ (43) \]

gives the outcome only when the state is uncorrupted.

7 Quantum Error Correction Code (QECC)

The demand of error correction codes imposes the most stringent condition that the errors projects the initial state onto mutually orthogonal spaces [29, 30]. That is,

\[ \mathcal{H}^M \perp \mathcal{H}^{E_k}, \mathcal{H}^{E_j} \perp \mathcal{H}^{E_k}; \quad j, k \in \{\pm 1, \pm 2, \cdots, \pm l\}. \]  

\[ (44) \]

By virtue of this orthogonality, an observable can be constructed which can distinguish each of these states and hence, each error can be detected. Such an observable is given as,

\[ O = c_M \Pi^M + \sum_{k=1}^{l} c_{\pm k} \Pi^{E_{\pm k}}, \]  

\[ (45) \]

with distinct eigenvalues \( c_M, c_{\pm k} \) belonging to the respective projections \( \Pi^M, \Pi^{E_{\pm k}} \).

Each outcome of this observable provides information about the subspace to which the post-measurement state belongs. If an error has occurred, its effect can be undone by performing an appropriate transformation on the state.

Example To start with, we consider an IC channel that causes a spillover by one unit. In this case, the error operators are given as \( E_{\pm 1} \). The effect of this crosstalk channel on the basis modes is shown in Fig. 4.

For a state involving superposition of three modes, the basis modes \( \{|1\rangle, |4\rangle, |7\rangle\} \) satisfy the requirement of a QECC. Consider a state,

\[ |\psi\rangle = \alpha |1\rangle + \beta |4\rangle + \gamma |7\rangle, \]  

\[ (46) \]

that is employed to transfer information. After passing through a crosstalk channel, the final state takes the form,

\[ \rho' = p_0 |\psi\rangle\langle \psi| + p_1 \left( |\psi_+\rangle\langle \psi_+| + |\psi_-\rangle\langle \psi_-| \right). \]  

\[ (47) \]

Fig. 4 Choice of modes for a state passing through a crosstalk channel of one unit. Since the channel causes a crosstalk of one unit, the modes are to be chosen at an interval of at least three units, so that the error operators, \( \text{viz.}, E_{\pm 1} \) transform the original state to orthogonal subspaces.
Table 3 Outcomes of stabiliser measurement, error detection and corresponding transformations for its correction

| Measurement outcomes | Error operator | Transformation |
|----------------------|----------------|---------------|
| $c_M$                | $E_0$          | No error      |
| $c_{-1}$             | $E_{-1}$       | $|1\rangle\langle0| + |4\rangle\langle3| + |7\rangle\langle6|$ |
| $c_{+1}$             | $E_{+1}$       | $|1\rangle\langle2| + |4\rangle\langle5| + |7\rangle\langle8|$ |

The states $|\psi_\pm\rangle$ have the forms,

$$
|\psi_+\rangle = \alpha|2\rangle + \beta|5\rangle + \gamma|8\rangle, \quad |\psi_-\rangle = \alpha|0\rangle + \beta|3\rangle + \gamma|6\rangle. \quad (48)
$$

It is to be noted that for this choice of modes, $\langle\psi_\pm|\psi\rangle = 0$ and $\langle\psi_+|\psi_-\rangle = 0$. Detection of errors can be performed by measuring the stabiliser given in (45) for which the projections are:

$$
\Pi^M = |1\rangle\langle1| + |4\rangle\langle4| + |7\rangle\langle7|, \quad \Pi^{E_{+1}} = |2\rangle\langle2| + |5\rangle\langle5| + |8\rangle\langle8|, \quad (49)
\Pi^{E_{-1}} = |0\rangle\langle0| + |3\rangle\langle3| + |6\rangle\langle6|. \quad (50)
$$

Since each outcome of the stabiliser provides information about occurrence of a particular error, effect of which can be undone by performing an appropriate transformation. The measurement outcomes, corresponding errors and required transformations for error correction have been given in Table 3.

For the sake of generalisation, in Appendix C, we have constructed a QECC that allows to retrieve state after passing through an IC channel that causes a spillover by $l$ units.

8 Conclusion

In summary, we have accomplished two tasks in this paper: (i) information retrieval when entangled biphoton and three-photon OAM states propagate through atmospheric turbulence and oceanic turbulence from analytical and simulation data. (ii) We exploit the weak dependence of crossover probability on initial OAM mode index in weak atmospheric turbulence to demonstrate retrieval of information from a state passing through an IC channel if the receiver has a capability of performing measurements in an arbitrarily large subspace of OAM modes. We have also constructed quantum error-correcting and error-rejecting codes for an IC channel, which could be potentially useful.

Appendix A: Complete List of Invariants and the Elements of Final Density Matrix for a ‘three-qubit’ Entangled State

Elements of final density matrix for a ‘three-qubit’ entangled state: The elements of the final state are functions of initial state parameters and survival ($a$) and crosstalk amplitudes
(b) which are reported to be [23]:

\[
\begin{align*}
G_1 & = |\alpha_4|^2 a^3 + \left( |\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 \right) ab^2 + |\alpha_5|^2 b^3, \\
G_2 & = \left( |\alpha_1|^2 + |\alpha_4|^2 + |\alpha_2|^2 \right) a^2 b + |\alpha_5|^2 ab^2 + |\alpha_3|^2 b^3, \\
G_3 & = \left( |\alpha_1|^2 + |\alpha_3|^2 + |\alpha_4|^2 \right) a^2 b + |\alpha_2|^2 b^3 + |\alpha_5|^2 ab^2, \\
G_4 & = |\alpha_1|^2 a^3 + \left( |\alpha_2|^2 + |\alpha_3|^2 + |\alpha_4|^2 \right) ab^2 + |\alpha_5|^2 a^2 b, \\
G_5 & = |\alpha_1|^2 b^3 + \left( |\alpha_2|^2 + |\alpha_3|^2 + |\alpha_4|^2 \right) a^2 b + |\alpha_5|^2 ab^2, \\
G_6 & = |\alpha_2|^2 a^3 + \left( |\alpha_1|^2 + |\alpha_3|^2 + |\alpha_4|^2 \right) ab^2 + |\alpha_5|^2 a^2 b, \\
G_7 & = |\alpha_3|^2 a^3 + \left( |\alpha_1|^2 + |\alpha_2|^2 + |\alpha_4|^2 \right) ab^2 + |\alpha_5|^2 a^2 b, \\
G_8 & = |\alpha_5|^2 a^3 + \left( |\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 \right) a^2 b + |\alpha_4|^2 b^3, \\
M_1 & = M_5^* = \alpha_1 \alpha_2^* a^3, M_2 = M_3^* = \alpha_1 \alpha_4^* a^3, M_3 = M_1^* = \alpha_1 \alpha_5^* a^3 \\
M_6 & = M_{10}^* = \alpha_2 \alpha_5^* a^3, M_7 = M_{14}^* = \alpha_2 \alpha_4^* a^3, M_8 = M_{18}^* = \alpha_2 \alpha_5^* a^3 \\
M_{11} & = M_{15}^* = \alpha_3 \alpha_4^* a^3, M_{12} = M_{19}^* = \alpha_3 \alpha_5^* a^3 \\
M_{16} & = M_{20}^* = \alpha_4 \alpha_5^* a^3
\end{align*}
\]

(A1)

Complete list of invariants: For the sake of completeness, the invariants are given as follows:

\[
\begin{align*}
I_1 & = \frac{i(M_1 - M_5)}{M_1 + M_5}; I_2 = \frac{M_5 + M_9}{M_1 + M_5}; I_3 = \frac{i(M_2 - M_9)}{M_2 + M_9}; I_4 = \frac{i(M_1 - M_{13})}{M_1 + M_{13}}; \\
I_5 & = \frac{M_1 + M_{13}}{M_1 + M_5}; I_6 = \frac{i(M_4 - M_{17})}{M_4 + M_{17}}; I_7 = \frac{M_4 + M_{17}}{M_1 + M_5}; I_8 = \frac{i(M_6 - M_{10})}{M_6 + M_{10}}; \\
I_9 & = \frac{M_6 + M_{10}}{M_1 + M_5}; I_{10} = \frac{i(M_7 - M_{14})}{M_7 + M_{14}}; I_{11} = \frac{M_7 + M_{14}}{M_1 + M_5}; I_{12} = \frac{i(M_8 - M_{18})}{M_8 + M_{18}}; \\
I_{13} & = \frac{M_8 + M_{18}}{M_1 + M_5}; I_{14} = \frac{i(M_11 - M_{15})}{M_{11} + M_{15}}; I_{15} = \frac{M_{11} + M_{15}}{M_1 + M_5}; I_{16} = \frac{i(M_{12} - M_{19})}{M_{12} + M_{19}}; \\
I_{17} & = \frac{M_{12} + M_{19}}{M_1 + M_5}; I_{18} = \frac{i(M_{16} - M_{20})}{M_{16} + M_{20}}; I_{19} = \frac{M_{16} + M_{20}}{M_1 + M_5}; I_{20} = \frac{G_4 - G_6}{G_6 - G_7}.
\end{align*}
\]

(A3)

Appendix B: Information Retrieval from a State Initially in Superposition of Three Modes Passing Through an IC Channel Causing a Spillover by one Unit

This appendix acts as a pedagogic illustration. In this, we show information retrieval from a state initially in superposition of three modes passing through an IC channel causing crosstalk by one unit. The error operators of a crosstalk channel causing a spillover by one unit are given by,

\[
E_0 = \sqrt{p_0} \sum_{k=1}^{3} |k\rangle \langle k|, \quad E_{\pm 1} \equiv \sqrt{p_1} \sum_{k=1}^{3} |k \pm 1\rangle \langle k|.
\]

(B4)

Suppose that a state,

\[
\rho = \sum_{i,j=1}^{3} \rho_{i,j} |i\rangle \langle j|,
\]

(B5)
is employed to transfer information. The resulting state, \( \rho' \), after passing through this channel is given by,

\[
\rho' \equiv E_0 \rho E_0^\dagger + E_{-1} \rho E_{-1}^\dagger + E_{+1} \rho E_{+1}^\dagger \equiv p_0 \rho + p_1 \rho_+ + p_1 \rho_-.
\]  

\( \text{(B6)} \)

The states \( \rho_{\pm}^{(1)} \) would have the forms,

\[
\rho_{\pm}^{(1)} = \sum_{i,j=1}^{3} \rho_{i,j} \langle i \pm 1 | j \pm 1 \rangle.
\]  

\( \text{(B7)} \)

Since the corresponding final state \( \rho' \) involves superpositions of five modes, we define the corresponding flip and phase operators as \( X = \sum_{j=0}^{4} | j + 1 \rangle \langle j | \) and \( Z = \sum_{j=0}^{4} \omega^j | j \rangle \langle j | \) (with addition modulo 5 and \( \omega \) is fifth root of unity).

The invariants of first family are found to be,

\[
I_1^{(1)} = \langle X \rangle = \rho_{1,2} + \rho_{2,3}; \quad I_1^{(2)} = \langle X^2 \rangle = \rho_{1,3}.
\]  

\( \text{(B8)} \)

Similarly, the invariants of second family are found to be,

\[
\begin{align*}
I_2^{(1,1)} &= \langle Z \rangle / \langle XZ \rangle = \frac{\omega \rho_{1,1} + \omega^2 \rho_{2,2} + \omega^3 \rho_{3,3}}{\omega \rho_{1,2} + \omega^2 \rho_{2,3}}, \\
I_2^{(1,2)} &= \langle Z^2 \rangle / \langle XZ \rangle = \frac{\omega^2 \rho_{1,1} + \omega^4 \rho_{2,2} + \omega^5 \rho_{3,3}}{\omega^2 \rho_{1,2} + \omega^4 \rho_{2,3}}, \\
I_2^{(2,1)} &= \langle Z \rangle / \langle XZ \rangle = \frac{\omega \rho_{1,1} + \omega^2 \rho_{2,2} + \omega^3 \rho_{3,3}}{\omega \rho_{1,3}}, \\
I_2^{(2,2)} &= \langle Z^2 \rangle / \langle X^2Z \rangle = \frac{\omega^2 \rho_{1,1} + \omega^4 \rho_{2,2} + \omega^6 \rho_{3,3}}{\omega^2 \rho_{1,3}}.
\end{align*}
\]  

\( \text{(B9)} \)

\( \text{(B10)} \)

It is important to note that the numerators and denominators in \( \text{(B9)} \) and \( \text{(B10)} \) have same dependence on the channel parameters \( p_0 \) and \( p_1 \), that is why their ratios, i.e., \( I_2^{(1,1)}, I_2^{(1,2)}, I_2^{(2,1)} \) and \( I_2^{(2,2)} \) are independent of them. Given these invariants, we next, enumerate the steps to retrieve full information in the state \( \rho \) as follows:

- **Step 1**: The invariant \( I_1^{(2)} \) determines the off–diagonal density matrix element \( \rho_{1,3} \).
- **Step 2**: Having knowledge of \( \rho_{1,3} \), we next employ the invariants \( I_2^{(2,1)} \) and \( I_2^{(2,2)} \), together with the trace condition \( \rho_{1,1} + \rho_{2,2} + \rho_{3,3} = 1 \), to find the values of the diagonal elements of the density matrix \( \rho \), i.e., \( \rho_{1,1}, \rho_{2,2}, \) and \( \rho_{3,3} \).
- **Step 3**: Equipped with the diagonal elements of the density matrix \( \rho \), we employ the invariants \( I_2^{(1,1)} \) and \( I_2^{(1,2)} \) to retrieve the off–diagonal density matrix elements \( \rho_{1,2} \) and \( \rho_{2,3} \).

In this manner, we have determined diagonal as well as off–diagonal elements of the density matrix \( \rho \). Since the equations get undetermined if \( \rho_{1,3} = 0 \), it is necessary that the initial state should be prepared accordingly.

### Appendix C: QECC for IC Channel Causing Spillover by 1 Units

In Section 7, we have constructed a QECC for a state involving superposition of three modes passing through an IC channel causing a spillover by one unit. In this appendix, we generalise it by constructing a QECC for an IC channel causing spillover by \( l \) units. For an IC channel causing a spillover by \( l \) units, the consecutive basis modes should be chosen at a
| Measurement outcomes | Error | Transformation |
|----------------------|-------|----------------|
| $c_0$                | $E_0$ | No error       |
| $c_{\pm 1}$         | $E_{\pm 1}$ | $\sum_{j=0}^{M-1} |l + j\Delta| |l \pm 1 + j\Delta|$ |
| $c_{\pm 2}$         | $E_{\pm 2}$ | $\sum_{j=0}^{M-1} |l + j\Delta| |l \pm 2 + j\Delta|$ |
|                      |       | $\cdots$       |
| $c_{\pm l}$         | $E_{\pm l}$ | $\sum_{j=0}^{M-1} |l + j\Delta| |l \pm l + j\Delta|$ |

distance of $\Delta = (2l + 1)$. So, for an $M$–dimensional system, the following set of basis modes can be employed,

$$B_c \equiv \{|l\rangle, |l + \Delta\rangle, \ldots, |l + (M-1)\Delta\rangle\}. \quad (C11)$$

The most general $M$–dimensional pure state employing these basis modes is given as,

$$|\psi\rangle \equiv \sum_{j=0}^{M-1} \alpha_j |l + j\Delta\rangle. \quad (C12)$$

The projections, for basis modes $B_c$, that determine the stabiliser given in (45) are,

$$\Pi^M \equiv \sum_{j=0}^{M-1} |l + j\Delta\rangle \langle l + j\Delta|; \quad \Delta = 2l + 1, \quad (C13)$$

$$\Pi^{E_{\pm k}} \equiv \sum_{j=0}^{M-1} |l \pm k + j\Delta\rangle \langle l \pm k + j\Delta|, \quad 1 \leq k \leq l, \quad (C14)$$

Depending on the outcome of stabiliser, one may apply an appropriate transformation on the state as given in the Table 4.

**Acknowledgements**  The authors thank the anonymous referee for valuable comments, which have helped us in improving the quality of the manuscript to a large extent. Rajni thanks UGC for funding her research in the initial stages. Sooryansh thanks CSIR (Grant no.: 09/086 (2017)-EMR-I) for funding his research.

**Author Contributions**  R.B. and V.R. conceived the idea and contributed in the entire work. S. A. has contributed significantly in construction of quantum error correction code and in showing the equivalence of generalised flip channels and crosstalk channels. All the authors have written the manuscript collectively.

**Data Availability**  Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

**Declarations**

**Conflict of Interests**  The authors declare no conflicts of interest.

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