Effective potential approach to the motion of massive test particles in Kaluza-Klein gravity

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Effective potential for a class of static solutions of Kaluza-Klein equations with three-dimensional spherical symmetry is studied. Test particles motion is analyzed. In attempts to read the obtained results with the experimental data, particular attention is devoted to the Schwarzschild’s limit of the four dimensional counterpart of these electromagnetic free solutions. Massive particles stable circular orbits in particular are studied, and a comparison between the well known results if the Schwarzschild’s case and those found for the static higher dimensional case is performed. A modification of the circular stable orbits is investigated in agreement with the experimental constraints.

Keywords: Kaluza Klein; Generalized Schwarzschild solution (GSS); Circular orbits

1. Generalized Schwarzschild solution

Our aim is to analyze test particles motion in a Generalized Schwarzschild solution (GSS) background by an effective potential approach. First we briefly review the main features of this solution.\textsuperscript{1–3} The metric family

\begin{equation}
5 ds^2 = \Delta^k dt^2 - \Delta^{-\epsilon(k-1)} dr^2 - r^2 \Delta^{1-\epsilon(k-1)} d\Omega^2 - \Delta^{-\epsilon} dx^5^2
\end{equation}

where $\Delta = (1 - 2M/r)$ in the 4D-spherical polar coordinate is a electromagnetic free solution of 5D-Kaluza Klein equation in the vacuum, with 3D-spherical symmetry. The free “metric parameters” $(\epsilon, k)$ are real constants related by $\epsilon^2 (k^2 - k + 1) = 1$. Metric (1) reduces to the Schwarzschild solution on the surface $x^5 = \cos t$ as $\epsilon \to 0$ and $k \to \infty$. In this limit the parameter $M$ is the central body mass. We explored the regions $k \geq 0$ and $\epsilon \geq 0$, analyzing particle motion in $r > 2M$. For these values of the metric parameters, the GSS solution presents a naked singularity behavior: it is a black hole one in the Schwarzschild’s limit for $(\epsilon, k)$. 
2. Timelike circular orbits in the GSS spacetimes.

Particles dynamics has been studied first by standard approach, considering test particles motion by a geodesic in 5D-spacetime, then the analysis has been performed by an approach a l a Papapetrou, therefore considering a 5D-particle described by an energy-momentum tensor picked along the particle 4D-world tube. Finally a comparison of the results obtained into the two different approaches has been made. In both cases we find the effective potential for the circular polar orbits of charges as well as neutral test particles, (for the study of circular orbits in the Schwarzschild case by an effective potential approach see for example Ref. 8).

2.1. Geodesic approach

In the first case, assuming a geodesic motion in 5D-manifold with a constant particle mass the effective potential reads

\[ V_{\text{eff}} \equiv \sqrt{\Delta^k \left[ 1 + r^2 \Delta^{k-1} + r \cdot \mu L^2 / \mu^2 + \Delta \epsilon \cdot \Gamma^2 / \mu^2 \right]} \]

where \( \Gamma \) is the conserved fifth component of the particle momentum, \( L \) is the conserved quantity associated to the azimuthal Killing vector \( \xi_\varphi = \{ 0, 0, 0, 1, 0 \} \) per unit rest mass. The analysis shows that last circular orbits radius, \( r_{lco} = [1 + \epsilon(2k - 1)]M \) is always located under the expected values of \( r_{lco} = 3M \) of the Schwarzschild’s limit: circular orbits (unstable or stable) could be possible also in a region \( r < 3M \).

2.2. Papapetrou analysis

In this section we analyze test particle motion in a GSS background using the Papapetrou revised approach. This is realized by an Papapetrou multipole expansion of a 5D-conservation equation for an energy-momentum tensor supposed picked along the 4D-world tube only. This expansion leads to different equations of motion for a test particle in the 4D-spacetime, where the mass \( m \) is no more constant but \( \frac{\partial m}{\partial x^\mu} = -\frac{1}{A} \frac{\partial A}{\partial x^\mu} \) where \( A \) is a real function and \( \phi^2 = -g_{55} \). An effective potential for a test particle of mass \( m \) can be defined as

\[ V_{\text{eff}} \equiv \sqrt{g_{00} (m^2 - L^2 / g_{\varphi\varphi})} \]

In particular when \( A = 0 \) the equations of motions describe a geodetic motion in the ordinary 4D-spacetime for a test particle of constant mass \( m \), where no scalar field coupling term appears. Formally these are the same equations of motions as those obtained in the standard geodetic approach for neutral particles. We infer \( r_c > [1 + \epsilon(2k - 1)]M \) for the circular orbits radius \( r_c \). Last stable circular orbit radius is \( r_{lsco} = [1 + \epsilon(3k - 2) + \epsilon \sqrt{(4k - 1)(k - 1)}]M \) where in the Schwarzschild’s limit \( r_{lsco} = 6M \) and \( r_{lsco} < 6M \forall k > 0 \). We therefore consider the case \( A = \text{cost} \), where \( m = A / 2\phi^2 \) and in the Schwarzschild’s limit \( m = m_0 \). Equation of motion in this case does not depend on \( A \). Last circular orbit is located also for this case at \( r_{lsco} \equiv [1 + \epsilon(2k - 1)]M \). Last stable circular orbit is in

\[ r_{lsco} / M \equiv \left[ \sqrt{4 + (15k - 8)\epsilon^2 - 5(3k - 8)\epsilon^4 + \epsilon(2 + k - 11\epsilon + 5k\epsilon) + 3} \right] [\epsilon(k + 2)]^{-1} \]

this is a free-A quantity, but it is a function of the only metric parameters \((\epsilon, k)\). Also in this case \(r_{lscO} < 6M\) and in the Schwarzschild’s limit \(r_{lscO} = 6M\). In the case \(A = \beta m\phi^2\), where \(\beta\) is a real number. Mass \(m\) follows the scaling law \(m = m_0 \phi_0^\beta \phi^{-\beta}\) where \(m_0 \phi_0^\beta = \text{cost}\) and the equations of motion do not depend on \(m\) but on the constant \(\beta\). For \(k > -\beta\) last circular orbit is located at \(r_{lco} \equiv M [1 + \epsilon(2k + 1)]\). Last stable circular orbit radius is in

\[
r_{lscO}/M \equiv (k + 2k\beta - 3 - \beta(2 + \beta))\epsilon^2 + (k + \beta)\epsilon + 3 \left[\epsilon(k + \beta)\right]^{-1} + \sqrt{A + \epsilon^2 \left[-3k(1 + 2\beta) (\epsilon^2 - 1) + (2 + \beta) (\beta - 4 + (\beta^2 + 2) \epsilon^2)\right] \left[\epsilon(k + \beta)\right]^{-1}}
\]

note that in the Schwarzschild’s limit \(r_{lscO} = 6M\). Radius of last stable circular orbit depends on two parameters, \(k\) as the independent metric parameter and \(\beta\) as a “dynamical” one. Moreover \(r_{lscO} < 6M\) for \(\beta > 0\), meanwhile for \(\beta < 0\) and \(k > -\beta\), \(r_{lscO} > 6M\) is possible. For \(\beta = 2\) we recover the same physical situation sketched in the case \(A = 0\). More generally it is possible to see that at an increase of \(\beta > 0\) for fixed values of the parameter \(k\), an increase of the difference \(|r_{lscO} - 6M|\) occurs.

3. Conclusion

This work was devoted to the study of massive test particle circular orbits in a GSS solution. We made a comparison between the classical approach to the dynamic and one based on the Papapetrou multipole expansion recently applied in a Kaluza Klein context. By an effective potential approach we found an expression for circular orbits radius. The stability of such orbits has been investigated providing the exact last circular orbits radius. As an interesting result of this analysis we found that test particles in stable orbits should exist also for some \(r < 6M\). This fact, if in agreement with any experimental observation, could lead to some valuable constraints to the hypothesis of multidimensional theory in the Kaluza Klein scenario (see also Ref. 9).

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