THE RADICAL IN A FINITELY GENERATED P.I. ALGEBRA

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Let $R$ be an associative ring over a commutative ring $\Lambda$, $p\{X_1, \ldots, X_e\}$ a polynomial on the free noncommuting variables $X_1, \ldots, X_e$, with coefficients in $\Lambda$ where one of its coefficient is $+1$. We say that $R$ is a P.I. (polynomial identity) ring satisfying $p(r_1, \ldots, r_e) = 0$ for all $r_1, \ldots, r_e$ in $R$.

We have the following

THEOREM A. Let $R = \Lambda\{x_1, \ldots, x_k\}$ be a p.i. ring, where $\Lambda$ is a noetherian subring of the center $Z(R)$ of $R$. Then, $N(R)$, the nil radical of $R$, is nilpotent.

Recall that $N(R) = \bigcap_p P$ where the intersection runs on all prime ideals of $R$.

We obtain, as a corollary, by taking $\Lambda$ to be a field, the following theorem, answering affirmatively the open problem which is posed in [Pr, p. 186].

THEOREM B. Let $R$ be a finitely generated P.I. algebra over a field $F$. Then, $J(R)$, the Jacobson radical of $R$, is nilpotent.

This result, in turn, has the following important consequence.

THEOREM C. Let $R = F\{x_1, \ldots, x_k\}$ be a finitely generated P.I. algebra over the field $F$. Then, $R$ is a subquotient of some $n \times n$ matrix ring $M_n(K)$ where $K$ is a commutative $F$-algebra. Equivalently, there exists an $n$ such that $R$ is a homomorphic image of $G(n, t)$ the ring of $t$, $n \times n$ generic matrices.

Kemer, in [K], announced a proof of Theorem B with the additional assumption that char($F$) = 0. His proof relies on a result of Razmyslov [Ra, Theorem 3] and on certain arguments related to the connection between P.I. ring theory and the theory of representation of the symmetric group $S_n$ over $F$, char $F = 0$. Both results rely heavily on the assumption that char $F = 0$, so they do not seem to generalize directly to arbitrary $F$.

The previously best known results concerning Theorem A are in [Ra, Theorems 1, 3, Se, Theorem 2].

The proof of Theorem C is a straightforward application of Theorem B and a theorem of J. Lewin [Le, Theorem 10].

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We sketch a proof of Theorem B.

A major tool in our proof is the following result of Latyshev [La, Proposition 12]: "Let $R$ be a p.i. $F$-algebra and $I \subseteq N(R)$ a finitely generated two sided ideal. Then, $I$ is nilpotent." Latyshev’s result is originally stated with the additional assumption $\text{char } F = 0$, but it is superfluous.

We sketch the proof of Theorem B; the complete proof will appear elsewhere.

We have $N(R) = P_1 \cap \cdots \cap P_t$, where $P_i$ are the minimal prime ideals of $R$, ordered via

$$\text{p.i.d}(R/P_1) = \cdots = \text{p.i.d}(R/P_m) > \text{p.i.d}(R/P_{m+1}) > \cdots > \text{p.i.d}(R/P_t)$$

where $m \leq t$ (if $m = t$, $P_{m+1} = R$). Here $\text{p.i.d}(S)$ denotes the minimal size of matrices into which $S$ can be embedded.

Let $\pi(R) = \text{p.i.d}(R/P_1)$, $d(R) = \max \{ \text{k.d}((R/P_i)|i = 1, \ldots, m \}$ where $\text{k.d}(S)$ is the classical Krull dimension of $S$. One observes that there exists a $b = b(k, d)$ such that if $S$ is an $F$-algebra satisfying $p(X_1, \ldots, X_d)$ (of degree $d$) and $S = F\{y_1, \ldots, y_k\}$ then $\text{k.d}(S) \leq b < \infty$. We argue on the ordered pair $\langle \pi(R), d(R) \rangle$ ordered lexicographically, that $R$ is a counterexample to the theorem with minimal $\langle \pi(R), d(R) \rangle$. This will imply that there exists a $\lambda \in P_1 \cup \cdots \cup P_m$, a finite sum of evaluations of some central polynomial of $\pi \times \pi$ matrices. Using the result of Latyshev quoted above we may assume that $\lambda \in Z(R)$ and by the minimal choice of $R$, since $\langle \pi(R/\lambda R), d(R/\lambda R) \rangle < \langle \pi(R), d(R) \rangle$, we get that $\lambda^l R \subseteq N(R)$ for some $l$. Using Latyshev’s result once more we may assume that $R_\lambda$, the localization of $R$ with respect to the set $\{\lambda, \lambda^2, \ldots\}$, is Azumaya of rank $\pi^2$ over its center. This in turn implies that $\lambda^e R \subseteq Zb_1 + \cdots + Zb_h \equiv A$, where $Z \equiv Z(R)$, $b_i \in R$, $i = 1, \ldots, h$ and $A$ is a ring, for some $e$.

Consequently, we may assume that $R$ satisfies any preassigned finite set of identities of $\pi \times \pi$ matrices. Finally, an argument mimicking the argument appearing in [Ra, Theorem 3] enables us to settle this case.

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