The rate of convergence of the mean curvature flow

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Abstract

We study the flow $M_t$ of a smooth, strictly convex hypersurface by its mean curvature in $\mathbb{R}^{n+1}$. The surface remains smooth and convex, shrinking monotonically until it disappears at a critical time $T$ and point $x^*$ (which is due to Huisken). This is equivalent to saying that the corresponding rescaled mean curvature flow converges to a sphere $S^n$ of radius $\sqrt{n}$. In this paper we will study the rate of exponential convergence of a rescaled flow. We will present here a method that tells us the rate of the exponential decay is at least $\frac{2}{n}$. We can define the "arrival time" $u$ of a smooth, strictly convex $n$-dimensional hypersurface as it moves with normal velocity equal to its mean curvature as $u(x) = t$, if $x \in M_t$ for $x \in \text{Int}(M_0)$. Huisken proved that for $n \geq 2$ $u(x)$ is $C^2$ near $x^*$. The case $n = 1$ has been treated by Kohn and Serfaty, they proved $C^3$ regularity of $u$. As a consequence of obtained rate of convergence of the mean curvature flow we prove that $u$ is not $C^3$ near $x^*$ for $n \geq 2$. We also show that the obtained rate of convergence $2/n$, that comes out from linearizing a mean curvature flow is the optimal one, at least for $n \geq 2$.

1 Introduction

In this paper we study a compact, smooth, strictly convex hypersurface $M_0 \in \mathbb{R}^{n+1}$ that moves with normal velocity equal to its mean curvature. In other words, let $M_0$ be represented locally by a diffeomorphism $F_0$ and let
$F(\cdot, t)$ be a family of maps satisfying the evolution equation

$$\frac{d}{dt} F = -H\nu,$$  \hspace{1cm} (1)

where $H(\cdot, t)$ is the mean curvature and $\nu(\cdot, t)$ is the outer unit normal on $M_t$ and $M_t$ is the surface represented by $F(\cdot, t)$. We often drop the $t$-dependence when no confusion will result. Due to Huisken (see [9]) the surface remains smooth and convex and shrinks to a point. Assume it disappears at time $T$ and that $x^*$ is a point to which it shrinks. Setting $x = F(p, t)$, (1) is then interpreted as

$$\frac{d}{dt} x = -H\nu(x).$$

The induced metric and the second fundamental form on $M$ will be denoted by $g = \{g_{ij}\}$ and $A = \{h_{ij}\}$. They can be computed as follows:

$$g_{ij}(x) = \langle \frac{\partial F(x)}{\partial x_i}, \frac{\partial F(x)}{\partial x_j} \rangle,$$

$$h_{ij}(x) = -\langle \nu(x), \frac{\partial^2 F(x)}{\partial x_i \partial x_j} \rangle,$$

for $x \in \mathbb{R}^n$. The mean curvature is

$$H = g^{ij} h_{ij}.$$

We also use the notation

$$|A|^2 = g^{ij} g^{kl} h_{ik} h_{jl},$$

$$h = \frac{1}{\text{Vol}M} \int_M HdV.$$

In [9] Huisken computed the evolution equations of different curvatures.

**Theorem 1 (Corollary 3.5 of [9]).**

$$\frac{d}{dt} H = \Delta H + |A|^2 H,$$

$$\frac{d}{dt} |A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4,$$

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\[
\frac{d}{dt}(|A|^2 - \frac{1}{n}H^2) = \Delta(|A|^2 - \frac{1}{n}H^2) - 2(\nabla A|^2 - \frac{1}{n}N H^2) + 2|A|^2(|A|^2 - \frac{1}{n}H^2).
\]

In order to prove his shrinking result (see Theorem 1.1 in [9]) Huisken introduced a normalized flow, obtained by reparametrization, keeping the total area of the evolving surface fixed. He established important estimates for a normalized flow

\[
\frac{d}{dt} \bar{F} = -\bar{H}\nu + \frac{1}{n}\bar{h}\bar{F},
\]

where \( \bar{F}(\cdot, t) = \psi(t)F(\cdot, t) \) and \( \psi(t) \) is a function chosen so that the total area of \( \bar{M}_t \) is being fixed and \( \bar{h} = \frac{1}{\text{Vol}(\bar{M})} \int_{\bar{M}} \bar{H}^2 \). Those estimates are

\[
\bar{H}_{\text{max}} - \bar{H}_{\text{min}} \leq C e^{-\delta t}, \quad (2)
\]

\[
|\bar{A}|^2 - \frac{1}{n}\bar{H}^2 \leq C e^{-\delta t}, \quad (3)
\]

\[
|\nabla^m \bar{A}| \leq C_m e^{-\delta m t}, \quad m > 0, \quad (4)
\]

for some \( \delta, \delta_m > 0 \). It is known that convex surfaces are of type 1 singularities (see [9] and [10]).

From now on, when we mention a rescaled flow, we will be thinking of the following rescaling,

\[
\tilde{F}(p, s) = (2(T - t))^{-1/2} F(p, t), \quad (5)
\]

with \( s = -\frac{1}{2} \ln(T - t) \), where \( T \) is a singularity time for the original mean curvature flow. We will denote by \( \tilde{M}_t \) the rescaled surfaces moving by reparametrized flow. The rescaled position vector then satisfies the equation

\[
\frac{d}{dt} \tilde{F} = -\tilde{H}\nu + \tilde{F}.
\]

In [9] and [10] Huisken showed that if the expressions \( P \) and \( Q \), formed from \( g \) and \( A \), satisfy \( \frac{\partial P}{\partial t} = \Delta P + Q \) and if \( \bar{P} = \psi^\alpha P \) and if \( \bar{P} = (2(T - t))^{-\alpha/2} P \),
then $\bar{Q}$ and $\tilde{Q}$ have degree $\alpha - 2$ and
\[
\frac{d\bar{P}}{dt} = \bar{\Delta} \bar{P} + \bar{Q} + \frac{\alpha}{n} \bar{h} \bar{P},
\]
\[
\frac{d\tilde{P}}{dt} = \tilde{\Delta} \tilde{P} + \tilde{Q} + \alpha \tilde{P}.
\]

Equations for $\bar{P}$ and $\tilde{P}$ look quite similar and if one goes carefully through the estimates established in [9], one can see that estimates (2), (3), (4) hold for corresponding quantities $\tilde{A}$, $\tilde{H}$, etc. associated with rescaling (5). In particular this tells us $\tilde{g}(s)$ uniformly converge to a round spherical metric, that is, the surfaces $\tilde{M}_s$ are homothetic expansions of the $M_t$'s and the surfaces $\tilde{M}_s$ converge to a sphere of radius $\sqrt{n}$ in the $C^\infty$ topology as $s \to \infty$.

**Remark 2.** The convergence of $\tilde{M}_s$ in any $C^k$-norm is exponential.

We want to say more about this exponential convergence, that is, we want to prove the following theorem.

**Theorem 3.** If $M_0$ is uniformly convex, meaning that the eigenvalues of the second fundamental form are strictly positive everywhere, then the normalized equation (7) has a solution $x$ that converges to a sphere of radius $\sqrt{n}$ exponentially at the rate at least $2/n$.

Theorem 3 can be used to study the arrival time of a smooth, strictly convex $n$-dimensional hypersurface moving by a normal velocity equal to its mean curvature. Due to Huisken we know the surface remains smooth and convex and shrinks to a point $x^*$ at some finite time $T$. We define the ”arrival time” on the interior of the initial surface ($\partial \Omega = M_0$) as $u(x) = t$ if $x \in M_t$. A point $X^*$ to which a surface shrinks has a unique maximum at $x^*$, $u(x^*) = T$. The smoothness of $u$ is related to a roundness of $M_t$ as it shrinks to a point and it is the best expressed in terms of the estimates for a curvature. Huisken proved that $u$ is at least $C^2$ in $\Omega$ for $n \geq 2$.

The question whether $u$ is at least $C^3$ was raised by Kohn and Serfaty in their recent work on a deterministic-control-based approach to motion by
curvature. Kohn and Serfaty proved that in the case \( n = 1 \), involving convex curves in the plane, \( u \) is \( C^3 \) with \( D^3 u(x^*) = 0 \). The analogue of Huisken’s work was done for curves in the plane by Gage and Hamilton (see [6]). The regularity of the arrival time was studied in this setting by Kohn and Serfaty in [12]. They needed at least \( C^3 \) regularity of \( u \) to draw a connection between a minimum exit time of two-person game (see [12] for more details) and the arrival time for a curve shortening flow (see [6]). Their results would completely extend to higher dimensions (drawing a connection between a minimum exit time of the same game as above in higher dimensions and the arrival time of a mean curvature flow) if we knew \( u \) were \( C^3 \) near \( x^* \). By Theorem 5 we can obtain the following result.

**Theorem 4.** Function \( u \) is not in general \( C^3 \) in \( \Omega \) for \( n \geq 2 \).

We believe Theorem 4 holds in the case \( n = 2 \) as well. In order to prove Theorem 4 we will construct a solution to the rescaled mean curvature flow equation whose behaviour is dictated by the first negative eigenvalue of an operator \( \Delta_{S^n} + 2 \) (that is \( -\frac{2}{n} \)), which we obtain while linearizing the rescaled mean curvature flow equation. As a consequence of Theorem 5 and Theorem 4 we get the optimal rate of convergence of a mean curvature flow starting with a strictly convex hypersurface.

**Theorem 5.** The rate of convergence obtained in Theorem 5 is the optimal one for \( n \geq 2 \).

The organization of the paper is as follows. In section 2 we derive a linearization of a mean curvature flow equation. In section 3 we prove the rate of exponential convergence of a strictly convex hypersurface moving by (1) is at least \( 2/n \), where \( -2/n \) happens to be the biggest negative eigenvalue of a linear operator \( \Delta_{S^n} + 2 \). In section 4 we prove Theorem 4 with a help of Theorem 5. In section 5 we say more about the regularity of \( u \), that is, we give a condition on eigenvalues of \( \Delta_{S^n} + 2 \) (for \( n \geq 2 \)) under which we can guarantee to have some orders of regularity for \( u \).
Acknowledgements: I would like to thank R.Kohn and Sylvia Serfaty for bringing the problem of regularity of $u$ to my attention and for many useful discussions.

2 Linearizing the mean curvature flow equation around a sphere in $\mathbb{R}^{n+1}$

In order to prove Theorem 5 we will study a linearization of the rescaled mean curvature flow equation. It is a standard matter, but for the sake of completeness we will include it here.

**Definition 6.** A family of smoothly embedded hypersurfaces $(M_t)_{t \in I}$ in $\mathbb{R}^{n+1}$ moves by mean curvature if

$$\frac{d}{dt}x = -H(x),$$

for $x \in M_t$ and $t \in I$, $I \subset \mathbb{R}$ an open interval. Here $H(x)$ is the mean curvature vector at $x \in M_t$.

Consider the family of smooth embeddings $F_t = F(\cdot,t) : M^n \rightarrow \mathbb{R}^{n+1}$, with $M_t = F_t(M^n)$ where $M^n$ is an $n$-dimensional manifold. Setting $x = F(p,t)$, (6) is then interpreted as

$$\frac{d}{dt}F(p,t) = -H(F(p,t)),$$

for $p \in M^n$ and $t \in I$. We can write a mean curvature vector as $H(F(p,t)) = H(p,t)\nu(p,t)$, where $H(\cdot,t)$ is the mean curvature and $\nu(\cdot,t)$ is the outer unit normal on $M_t$. We can define the rescaled embeddings $\tilde{F}(p,s) = (2(T-t))^{-1/2}F(p,t)$, with $s(t) = -\frac{1}{2}\ln(T-t)$. The surfaces $\tilde{M}_s = \tilde{F}(\cdot,s)(M)$ are defined for $-\frac{1}{2}\ln T \leq s < \infty$ and satisfy the equation

$$\frac{d}{ds}\tilde{F}(p,s) = -\tilde{H}(p,s) + \tilde{F}(p,s),$$

that is

$$\frac{d}{ds}\tilde{x} = -\tilde{H} + \tilde{x},$$

(7)
if \( \tilde{x} = \tilde{F}(p, s) \). In the rest of the paper we will be considering evolution equation (7) and from now on we will omit symbol \( \tilde{\cdot} \) in symbols denoting the quantities characterizing the rescaled mean curvature flow. If we couple (7) with a normal \( \nu \), we get

\[
\langle \frac{d}{dt} x, \nu \rangle = -H + \langle x, \nu \rangle.
\] (8)

Consider the operator \( L(x) = -H + \langle x, \nu \rangle \). We want to linearize it at a hypersurface given by \( x \). In other words, we want to compute \( \frac{d}{ds} L(x_s) \bigg|_{s=0} \), where \( x_s \) is a small perturbation of \( x \) (at some fixed time \( t \)) and \( x_0 = x \). Let \( u = \frac{d}{ds} x_s \bigg|_{s=0} \).

**Lemma 7.** \( \frac{d}{ds} \nu(x_s) \big|_{s=0} = -\langle \nu, \frac{\partial u}{\partial x_i} \partial F / \partial x_j \rangle g^{ij} \).

**Proof.** This is a straightforward computation:

\[
\frac{d}{ds} \nu(x_s) \big|_{s=0} = \left\langle \frac{d}{ds} \nu(x_s) \big|_{s=0}, \frac{\partial F}{\partial x_i} \partial F / \partial x_j g^{ij} \right\rangle
= -\langle \nu, \frac{\partial u}{\partial x_i} \partial F / \partial x_j \rangle g^{ij}.
\]

**Lemma 8.**

\[
\frac{d}{ds} h_{ij} \big|_{s=0} = \langle -\frac{\partial^2 u}{\partial x_i \partial x_j} + \frac{\partial u}{\partial x_k} \Gamma_{ij}^k, \nu \rangle.
\]

**Proof.** We know that the second fundamental form is given by a matrix

\[
h_{ij} = -\langle \nu, \frac{\partial^2 F}{\partial x_i \partial x_j} \rangle = \langle \frac{\partial F}{\partial x_i}, \frac{\partial \nu}{\partial x_j} \rangle.
\]

We use Gauss-Weingarten relations

\[
\frac{\partial^2 F}{\partial x_i \partial x_j} = \Gamma_{ij}^k \frac{\partial F}{\partial x_k} - h_{ij} \nu,
\]

to conclude

\[
\frac{d}{ds} h_{ij} \big|_{s=0} = -\langle \frac{\partial^2 u}{\partial x_i \partial x_j}, \nu \rangle + \langle \frac{\partial^2 F}{\partial x_i \partial x_j}, \nu, \frac{\partial \nu}{\partial x_k} \partial F / \partial x_l g^{kl} \rangle
= -\langle \frac{\partial^2 u}{\partial x_i \partial x_j} + \frac{\partial u}{\partial x_k} \Gamma_{ij}^k, \nu \rangle,
\]

since \( \langle \frac{\partial F}{\partial x_k}, \frac{\partial F}{\partial x_l} \rangle = g_{kl} \) and \( \langle \frac{\partial F}{\partial x_l}, \nu \rangle = 0. \)

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Lemma 9. The linearization of the mean curvature $H$ is

$$
-\frac{d}{ds}H|_{s=0} = \langle \Delta u, \nu \rangle + h_{ij}g^{ip}g^{jq}\{\langle \frac{\partial u}{\partial x_p}, \frac{\partial F}{\partial x_q} \rangle + \langle \frac{\partial u}{\partial x_q}, \frac{\partial F}{\partial x_p} \rangle\}.
$$

where $u$ is a vector in $\mathbb{R}^{n+1}$ in a direction of a normal vector $\nu$.

Proof. Since $H = g^{ij}h_{ij}$, we have

$$
\frac{d}{ds}H|_{s=0} = \frac{d}{ds}g_{ij}|_{s=0}h_{ij} + g^{ij}\frac{d}{ds}h_{ij}|_{s=0}.
$$

(9)

and

$$
\frac{d}{ds}g_{ij}|_{s=0} = -g^{ip}g^{jq}\frac{d}{ds}g_{pq}|_{s=0},
$$

This together with (9) and Lemma 8 give

$$
-\frac{d}{ds}H|_{s=0} = \langle \Delta u, \nu \rangle + h_{ij}g^{ip}g^{jq}\{\langle \frac{\partial u}{\partial x_p}, \frac{\partial F}{\partial x_q} \rangle + \langle \frac{\partial u}{\partial x_q}, \frac{\partial F}{\partial x_p} \rangle\}.
$$

Proposition 10. Let $u = \frac{d}{dt}x|_{s=0}$ where $x^s$ is a perturbation of $x$ and let $w = \langle u, \nu \rangle$. Then

$$
\frac{d}{dt}w = \Delta w + |A|^2 w + w.
$$

Proof. After taking $\frac{d}{ds}|_{s=0}$ of both sides of the evolution equation $\langle \frac{d}{dt}x, \nu \rangle = -H + x \cdot \nu$ and using Lemma 7 and Lemma 9 we get

$$
\langle \frac{d}{dt}u, \nu \rangle + \langle \frac{d}{dt}x, -\langle \nu, \frac{\partial u}{\partial x_i} \rangle \frac{\partial F}{\partial x_j}g^{ij} \rangle = \langle \Delta u, \nu \rangle + h_{ij}g^{ip}g^{jq}\{\langle \frac{\partial u}{\partial x_p}, \frac{\partial F}{\partial x_q} \rangle + \langle \frac{\partial u}{\partial x_q}, \frac{\partial F}{\partial x_p} \rangle\} + 
$$

$$
+ \langle u, \nu \rangle - \langle \nu, \frac{\partial u}{\partial x_i} \rangle g^{ij}x \cdot \frac{\partial F}{\partial x_j}.
$$

Since $\frac{d}{dt}x = -H\nu + x$, we have

$$
\langle \frac{d}{dt}u, \nu \rangle = \langle \Delta u, \nu \rangle + h_{ij}g^{ip}g^{jq}\{\langle \frac{\partial u}{\partial x_p}, \frac{\partial F}{\partial x_q} \rangle + \langle \frac{\partial u}{\partial x_q}, \frac{\partial F}{\partial x_p} \rangle\} + \langle u, \nu \rangle.
$$

(10)
We will now compute $\Delta w$,
\[
D_i w = \langle D_i u, \nu \rangle + \langle u, D_i \nu \rangle.
\]
\[
D_j D_i w = \langle D_j D_i u, \nu \rangle + \langle D_i u, D_j \nu \rangle + \langle D_j u, D_i \nu \rangle + \langle u, D_j D_i \nu \rangle.
\]
By Gauss-Weingarten relation $\frac{\partial}{\partial x_j} \nu = h_{ij} g^{lm} \frac{\partial F}{\partial x_m}$ we have
\[
D_j D_i w = \langle D_j D_i u, \nu \rangle + h_{ij} g^{lm} \langle D_i u, D_m F \rangle + h_{il} g^{lm} \langle D_j u, D_m F \rangle + \langle u, D_j D_i \nu \rangle,
\]
which gives
\[
\Delta w = \langle \Delta u, \nu \rangle + \langle u, \Delta \nu \rangle + g^{ij} g^{lm} h_{ij} \langle D_i u, D_m F \rangle + g^{ij} g^{lm} h_{il} \langle D_j u, D_m F \rangle.
\] (11)
Since $\frac{d}{dt} \nu = D^T H$, by (10) and (11) we have
\[
\frac{d}{dt} w = \langle u, D^T H \rangle + \Delta w - \langle u, \Delta \nu \rangle + w.
\] (12)

Claim 11. Let $M$ be a hypersurface in $\mathbb{R}^{n+1}$, given by an embedding $F$. Then $\Delta \nu = D^T H - |A|^2 \nu$, where $D^T H$ is a projection of $DH$ onto a tangent space of a surface $M$.

Proof. This is a pointwise computation, so we may assume $g_{ij} = \delta_{ij}$ and $\Gamma^k_{ij} = 0$ at a particular point. Denote by $e_i = \frac{\partial F}{\partial x_i}$. Since $D_j \nu = h_{jj} D_j F$, by Gauss-Weingarten relation $\frac{\partial^2 F}{\partial x_i \partial x_j} = \Gamma^k_{ij} \frac{\partial F}{\partial x_k} - h_{ij} \nu$, we have $\langle D_j \nu, e_i \rangle = h_{ij}$, and therefore
\[
\langle D_j D_j \nu, e_i \rangle + \langle D_j \nu, D_j e_i \rangle = D_j h_{ij} = D_i h_{jj},
\] (13)
by Codazi equations. Moreover, $\langle D_j \nu, D_j e_i \rangle = 0$ at a point at which we are performing our computations. If we sum all equations (13) over $j$ we get
\[
\langle \Delta \nu, e_i \rangle = D^T H.
\] (14)
By a similar computation we have
\[
\langle D_j D_j \nu, \nu \rangle = -\langle D_j \nu, D_j \nu \rangle = -h_{jj} h_{jm} g^{lm}.
\]
If we sum the previous equations over $j$, we get
\[ \langle \Delta \nu, \nu \rangle = -g^{ij} g^{lm} h_{l m} h_{j j} = -|A|^2. \] (15)

By our choice of coordinates at a point and by relations (14) and (15) we have
\[ \Delta \nu = D^T H - |A|^2 \nu. \]

The previous claim together with (12) yield
\[ \frac{d}{dt} w = \Delta w + |A|^2 w + w. \] (16)

Let $x_{S^n}$ be an image of an embedding of a sphere $S^n$ of radius $\sqrt{n}$ into $\mathbb{R}^{n+1}$. Let $u = x - x_{S^n}$ and $w = \langle u, \nu \rangle$.

**Lemma 12.** A scalar function $w$ satisfies the following evolution equation
\[ \frac{d}{dt} w = \Delta_{S^n} w + 2w + Q, \]
where $\Delta_{S^n}$ is a Laplacian with respect to a metric on $S^n$ and $Q$ is a quadratic term in $u$, $w$ and their first and second covariant derivatives.

**Proof.** Since $x_{S^n}$ does not depend on time, it satisfies
\[ \langle \frac{d}{dt} x_{S^n}, \nu \rangle = -H_{S^n} + \langle x_{S^n}, \nu_{S^n} \rangle, \]
because both sides of the previous identity are equal to zero. If we subtract this equation from (3), we get
\[ \langle \frac{d}{dt} (x - x_{S^n}), \nu \rangle = K(x) - K(x_{S^n}), \]
where $K(x) = -H(x) + x \cdot \nu$. By the previous consideration, (16) and somewhat tedious, but standard computation, we have
\[ \frac{d}{dt} w = \Delta w + |A|^2 w + w + Q, \]
where $Q$ is a quadratic term as in the statement of the lemma. Since $|A| = 1$ on a sphere of radius $\sqrt{n}$, we can write the previous equation as

$$\frac{d}{dt}w = \Delta_S^n w + 2w + Q', \quad (17)$$

where $Q'$ is again a quadratic term in the same quantities as above, possibly different from $Q$. Since $x(t) \to x_{S^n}$ exponentially, we can find a radial parametrization of $M_t$ for $t$ sufficiently big, so that we can view $M_t$ as a radial graph over $S^n(\sqrt{n})$ and consider $w$ as a scalar function defined on $S^n$. $\square$

3 The rate of exponential convergence of the mean curvature flow

If $M_0$ is uniformly convex, i.e., the eigenvalues of its second fundamental form are strictly positive everywhere. By results in [9] it follows that the rescaled equation (17) has a solution that exponentially converges to a sphere of radius $\sqrt{n}$. We want to say something more about the rate of that exponential convergence. In order to do that we will analyze the spectrum of $L(w) = \Delta_S^n w + 2w$. It is a standard fact (see [7]) that the spectrum of $L$ is given by $\{-\frac{k(k+n-1)}{n} + 2\}_{k \in \{0\} \cup \mathbb{N}}$, if we adopt the notation that $\Delta_S^n$ is a negative operator. The first negative eigenvalue for $L$ is achieved for $k = 2$ and is equal to $-\frac{2}{n}$ (for $k = 0, 1$ the corresponding eigenvalues are 2, 1 respectively). This implies that $L$ does not have a zero eigenvalue.

**Definition 13.** We will say that $x$ converges to a sphere $x_{S^n}$ exponentially at a rate $\delta$ in $C^k$ norm, if there exist $C(k), t_0$ such that for all $t \geq t_0$,

$$|x - x_{S^n}|_k \leq C(k)e^{-\delta t}.$$ 

We may assume that $x$ converges to $x_{S^n}$ exponentially at rate $\delta$. We want to say more about the rate of exponential convergence.
Theorem 14. If $M_0$ is uniformly convex, meaning that the eigenvalues of the second fundamental form are strictly positive everywhere, then the normalized equation (7) has a solution $x$ that converges to a sphere of radius $\sqrt{n}$ exponentially at the rate at least $\frac{2}{n}$.

The ideas and techniques that we will use to prove Theorem 5 rely on work of Cheeger and Tian (see [1]). Similar approach has been used in [14] to prove the uniqueness of a limit of the Ricci flow.

Assume $\delta < \frac{2}{n}$ since otherwise we are done. In order to prove Theorem 5 we will use that the behaviour of a solution of (17) is modeled on the behaviour of a solution of a linear equation

$$\frac{d}{dt}v = L(v),$$

where $L(v) = \Delta_{S^n}v + 2v$. Let $\{\lambda_k\}$ be the set of all eigenvalues of $L$. We can write $v = v_\uparrow + v_\downarrow + v_0$, where $v_\uparrow(t) = \sum_{\lambda_k < 0} a_k e^{\lambda_k t}$, $v_\downarrow(t) = \sum_{\lambda_k > 0} a_k e^{\lambda_k t}$, and $v_0$ is a projection of $v$ to a kernel of $L$. Since $L$ does not have zero eigenvalue, $v_0 = 0$.

Some of the following consideration is based on the ideas and results whose detailed proofs can be found in [1] (see also [14]) so we will just briefly outline them. The following three lemmas can be found in [1] (see also [14]). The idea of considering three consecutive time intervals is originally due to Simon ([15]).

Lemma 15. There exists $\alpha > 1$ such that

$$\sup_{[K, 2K]} \|v_\uparrow\| \geq \alpha \sup_{[0, K]} \|v_\uparrow\|,$$

(19)

$$\sup_{[K, 2K]} \|v_\downarrow\| \leq \alpha^{-1} \sup_{[0, K]} \|v_\downarrow\|.$$  

(20)

The norms considered above are standard $L^2$ norms. In particular, we can choose $\alpha = e^{\frac{2}{n}}$. 

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Lemma 16. There exists $\beta < \alpha$ such that if

$$\sup_{[K,2K]} ||v|| \geq \beta \sup_{[0,K]} ||v||, \quad (21)$$

then

$$\sup_{[2K,3K]} ||v|| \geq \beta \sup_{[K,2K]} ||v||, \quad (22)$$

and if

$$\sup_{[2K,3K]} ||v|| \leq \beta^{-1} \sup_{[K,2K]} ||v||, \quad (23)$$

then

$$\sup_{[K,2K]} ||v|| \leq \beta^{-1} \sup_{[0,K]} ||v||. \quad (24)$$

Moreover, if $v_0 = 0$ at least one of (22), (24) holds. If also $v\uparrow = 0$, we can choose $\beta = e^{2/n}$.

The basic parabolic estimates (for example similarly as in [15] and [1]) yield the following lemma.

Lemma 17. There exists $\tau > 0$ such that for any solution $w$ of (17) with $|w(t_0)|_{k+2,\alpha} \leq \tau$, we have that

$$\sup_{(t_0,t_0+L)} |w(t)|_{k,\alpha} \leq C \sup_{(t_0,t_0+L)} ||w||,$$

where the first norm is $C^{k,\alpha}$ norm and the last norm is $L^2$ norm.

Let as in the previous section $u = x - x_{S^n}$ and $w = \langle u, \nu \rangle$. It satisfies,

$$\frac{d}{dt}w = \Delta_{S^n}w + 2w + Q,$$

where $Q$ is a quadratic term in $u$, $w$ and their first and second covariant derivatives. Let $|| \cdot ||_{a,b} = \int_a^b | \cdot |$, where $| \cdot |$ is just the $L^2$ norm. Let $\pi$ denote an orthogonal projection on the subspace $\ker(-\frac{d}{dt} + \Delta_{S^n} + 2)S^n(\sqrt{n}) \times [t_0,t_0+K]$, with respect to norm $|| \cdot ||_{t_0,t_0+K}$. Put $\pi w = (\pi w)\uparrow + (\pi w)\downarrow$. The following proposition shows that the behaviour of a solution of a linear equation (18) is modeled on a behaviour of a solution of (17). If $\epsilon > 0$ is any small number, there is some $t_0$ so that $|w(t)|_k < \epsilon$ for $t \geq t_0$. 

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Proposition 18. There exists $\epsilon_0 > 0$, depending on the uniform bounds on the geometries $g(t)$, such that if $\epsilon < \epsilon_0$, then if

$$\sup_{[K,2K]} ||w|| \geq \beta \sup_{[0,K]} ||w||,$$

then

$$\sup_{[2K,3K]} ||w|| \geq \beta \sup_{[K,2K]} ||w||,$$

and if

$$\sup_{[2K,3K]} ||w|| \leq \beta^{-1} \sup_{[K,2K]} ||w||,$$

then

$$\sup_{[K,2K]} ||w|| \leq \beta^{-1} \sup_{[0,K]} ||w||, \quad (28)$$

Moreover, since $(\pi w)_0 = 0$, at least one of (26), (28) holds. If $(\pi w)_\uparrow = 0$ we can choose $\beta = e^{2/n}$.

Proof. Assume there exist a sequence of constants $\tau_i \to 0$, and a sequence of times $t_i \to \infty$ such that $|\eta_i(t))_{k,\alpha} = |w(t_i+t)|_{k,\alpha} \leq \tau_i \to 0$ for all $t \geq 0$, but for which none of the assertions in Proposition 18 holds. Let $\psi_i = \frac{\eta_i}{\sup_{[K,2K]} |\eta_i|}$.

Then in view of Lemma 17, from standard compactness results (see Lemma 5.22 and Proposition 5.49 in [1]) we get that for a subsequence $\psi_i \to \psi$ and

$$\frac{d}{dt} \psi = \Delta_{S^u} \psi + 2\psi,$$

where $\psi$ has a property that contradicts Lemma 16.

Lemma 19. If $v$ is a solution of (18), such that $|v| \leq Ce^{-\delta t}$, then $v_\uparrow = 0$.

Proof. If that is not the case, assume $v = \sum_{\lambda_k < 0} e^{\lambda_k t} + be^{\gamma t} = \tilde{v} + be^{\gamma t}$, where $\gamma > 0$, $b \neq 0$ and $\tilde{v} = \sum_{\lambda_k < 0} e^{\lambda_k t}$. By Lemma 15 we have that

$$\sup_{[K,2K]} ||\tilde{v}_\downarrow|| \leq \alpha^{-1} \sup_{[0,K]} ||\tilde{v}_\downarrow||,$$
for $\alpha = e^{2\pi}$. Applying Lemma 16 inductively to $\sup_{iK,(i+1)K} ||\tilde{v}||$, for every $i$, we get

$$||\tilde{v}|| \leq Ce^{-\alpha t}. \quad (29)$$

The fact that a rate of an exponential decay in (29) is given by the same $\alpha$ as in Lemma 15 follows immediately from the proof of Lemma 5 in [1]. Furthermore,

$$||be^{\gamma t}|| \leq ||v|| + ||\tilde{v}|| \leq Ce^{-\min\{\delta,2/n\}t},$$

and we get a contradiction for big values of $t$ unless $b = 0$.

We know that $w = \langle x - x_{S^n}, \nu \rangle$ solves the evolution equation (17). Since $|w|_{k,\alpha} < Ce^{-\delta t}$, by Lemma 19 we have $(\pi w)_{\uparrow} = 0$. Since $(\pi w)_{0} = 0$, by Proposition 18 at least one of (26), (28) holds. Since $w \to 0$ exponentially as $t \to \infty$, we have (28) holding with a rate of decay at least $2/n$ because $(\pi w)_{\uparrow} = 0$. By using a parabolic regularity theory we can get $C^k$ exponential decay with the rate at least $2/n$. We can now finish the proof of Theorem 5.

**Proof of Theorem 5.** From the previous discussion we know that $|w|_k \leq C(k)e^{-2\gamma t}$. Let $t_0$ be such that $|w(t)|_{k+2} < \epsilon$ for all $t \geq t_0$, where $\epsilon$ is taken from Proposition 18. Assume that $\gamma$ is a maximal rate of decay of $x$ to $x_{S^n}$, that is $\gamma = \max\{\delta \mid \exists C \text{ such that } |w| \leq Ce^{-\delta t}, \forall t \geq t_0\}$. We may assume $\gamma < 2/n$, since otherwise we are done.

$$\langle x - x_{S^n}, \nu_{S^n} \rangle = \langle x - x_{S^n}, \nu_{S^n} - \nu \rangle + \langle x - x_{S^n}, \nu \rangle. \quad (30)$$

Since $x \to x_{S^n}$ as $t \to \infty$ uniformly, we can regard $x$ as a radial graph over $S^n$ and therefore $x - x_{S^n} \perp \nu_{S^n}$, for $t$ sufficiently big, that is, $x - x_{S^n} = |x - x_{S^n}|\nu_{S^n}$. From (30) we get

$$|x - x_{S^n}| \leq Ce^{-2\gamma t} + Ce^{-2\gamma t} = Ce^{-\min\{2\gamma,2/n\}t},$$

which contradicts the maximality of $\gamma$ unless $\gamma = 2/n$. \qed
4 Regularity of the arrival time function

Due to Huisken (see [1]) we know that the arrival time function is at least of class $C^2$ in $\Omega = \text{Int}(M_0)$. In the case of $n = 1$ (where instead of the mean curvature flow we deal with the curve shortening flow) Kohn and Serfaty showed that $u$ is at least $C^3$. The question that remains open is whether $u$ is $C^3$ or more in higher dimensions ($n \geq 2$). It turns out it is not $C^3$ at $x^*$ in a generic case.

Before we start proving Theorem 4, let’s first slightly change the notation from above. Let $x$ satisfy
\[
\frac{d}{dt} x = -\bar{H} \nu,
\]
and $y = (2\tau)^{-1/2} x$, where $\tau = T - t$ and $s = -\frac{1}{2} \ln \tau$. Quantities $\bar{H}$ and $\bar{\nu}$ correspond to the original mean curvature flow. Then $y$ satisfies
\[
\frac{d}{ds} y = -H \nu + y.
\]
We may assume $y(s)$ converges to a sphere $S^n$ of radius $\sqrt{n}$. We have derived in section 2 that $w' = (y - y_{S^n}, \nu_{S^n})$ satisfies
\[
\frac{d}{ds} w' = \Delta_{S^n} w' + 2w' + Q(w'),
\]
where $Q$ is a quadratic term in $w'$ and its first and second covariant derivatives. Let $w = (y - y_{S^n}, \nu_{S^n})$.

Claim 20. A scalar function $w$ satisfies
\[
\frac{d}{ds} w = \Delta_{S^n} w + 2w + \tilde{Q}(w),
\]
where $\tilde{Q}$ is an expression containing the quadratic terms in $w$, $w'$ and their first and second covariant derivatives.

Proof.
\[
\frac{d}{ds} w = \frac{d}{ds} w' + \frac{d}{ds} (y - y_{S^n}, \nu_{S^n} - \nu)
= \Delta_{S^n} w' + 2w' + Qw' + \langle -(H \nu + y) - (-H_{S^n} \nu_{S^n} + y_{S^n}), \nu_{S^n} - \nu \rangle + \\
+ \langle y - y_{S^n}, -\nabla H + \nabla H_{S^n} \rangle
= \Delta_{S^n} w + 2w + \tilde{Q}(w, w').
\]
Let $\mathcal{A}$ be a set of all solutions of (17). Define a map $\psi : \mathcal{A} \to \mathbb{R}$ by $\psi(a) = \alpha$, where $\alpha$ is a coefficient of $\phi e^{-\beta s}$ in $\pi a$.

To prove Theorem ?? we will need the following Proposition that tells us how to construct solutions to the rescaled mean curvature flow with a certain behaviour, dictated by the first negative eigenvalue of $\Delta_{S^n} + 2$.

**Proposition 21.** There exists a solution $y$ to a rescaled mean curvature flow (7) such that $\psi(\langle y - y_{S^n}, \nu_{S^n} \rangle) \neq 0$.

**Proof.** Fix a sphere $S^n$ of radius $\sqrt{n}$ and let $\phi$ be a homogeneous, harmonic polynomial on $S^n$ corresponding to an eigenvalue $-2/n$ of a differential operator $\Delta_{S^n} + 2$. Consider a set of solutions $y_\alpha$ of

$$\begin{align*}
\frac{d}{ds}y_\alpha &= -H\nu + y_\alpha, \\
y_\alpha(0) &= \alpha \phi \nu_{S^n},
\end{align*}$$

for small values of $\alpha$, so that $y_\alpha$ is a strictly convex hypersurface which by Huisken’s result implies that every such solution $y_\alpha(s)$ exponentially converges to a sphere of radius $\sqrt{n}$. Let $w_\alpha = (y_\alpha - y_{S^n}, \nu_{S^n})$. We have seen that $w_\alpha$ satisfies

$$\begin{align*}
\frac{d}{ds}w_\alpha &= \Delta_{S^n} w_\alpha + 2w_\alpha + Q(w_\alpha) \\
w_\alpha(0) &= \alpha \phi.
\end{align*}$$

(31)

Our goal is to show there exists some $\alpha \neq 0$ so that a solution $y_\alpha$ satisfies the property stated in the proposition. The proof of the existence of such an $\alpha$ is given in the few following steps.

**Step 1.** If $w$ is a solution to a nonlinear equation (31) such that $|w|_l \leq C(l)$ for all $l$ and all $s \in [0, L]$, and if $k \geq 0$, there exist a uniform constant $C = C(L, k)$ and $\epsilon = \epsilon(k)$, so that if $\sup_{s \in [0, L]} |w|_k < \epsilon$, then $\sup_{s \in [0, L]} |w|_{k+1} < C\epsilon$. 

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Proof. The assertion tells us that if a solution is small in $C^k$ norm, it will stay comparably small in $C^{k+1}$ norm. Assume without loss of generality that $k = 0$ (consideration for bigger $k$ is analogous). Our goal is to show that $W^{2,l}$ norms of $w$ stay comparably small and then to use Sobolev embedding theorems to draw the conclusion of the assertion. If we multiply (31) by $w$ and integrate it over $S^n$, \[
abla w \cdot \nabla w \leq Qw \cdot w,\]
Moreover, since $|Qw| \leq C(|\nabla w||w| + |\nabla w|^2)$,
\[|\int Qw \cdot w| \leq C \int |w|^2 + C\epsilon \int |\nabla w|^2,\]
and therefore, for small enough $\epsilon$, after integrating in $s \in [0, L]$, we get,
\[\sup_{s \in [0, L]} \int w^2(s) + 2 \int_0^L \int |\nabla w|^2 \leq C\epsilon L = C(L)\epsilon, \quad (32)\]
where we will use the same symbol $C$ to denote different uniform constants and $C(L)$ to denote different uniform constants depending on $L$. Apply a covariant derivative (with respect to an induced metric on a sphere $S^n$) to (31), multiply it by $\nabla w$ and integrate over $S^n$. A simple calculation yields \[
abla (Q(w)) \cdot \nabla w \leq C \int |\nabla w|^2 + \int \nabla (Q(w)) \cdot \nabla w, \quad (33)\]
where we denote by $A \ast B$ any quantity obtained from $A \otimes B$ by one or more of the following operations: summation over pairs of matching upper and lower indices; contraction on upper indices with respect to the metric; contraction on lower indices with respect to the metric inverse; multiplication by uniform constants ([2]) or by uniformly bounded scalar functions (e.g. geometric quantities defined for $S^n$). Since \[
abla (Q(w)) \cdot \nabla w = - \int Q(w) \ast \nabla^2 w \leq C\epsilon \int |\nabla^2 w|^2 + C \int |\nabla w|^2,\]
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integrating (33) in $s$ and choosing $\epsilon$ small enough so that we can absorb $C\epsilon \int |\nabla^2 w|^2$ in a corresponding term appearing on the right hand side of (33), we get

$$\sup_{s \in [0,L]} \int |\nabla w|^2 + \int_0^L \int |\nabla^2 w|^2 \leq C \int_0^L \int |\nabla w|^2 \leq C(L)\epsilon,$$

where we have used (32). By taking more and more derivatives of (31), a similar consideration as above yields that

$$\sup_{s \in [0,L]} \int |\nabla^l w|^2(s) \leq C(L, l, n)\epsilon.$$

By Sobolev embedding theorems we have that $|w|_1 \leq C(L, n)\epsilon$, and more general, if $\sup_{s \in [0,L]} |w|_k < \epsilon$, then $\sup_{s \in [0,L]} |w|_{k+l} < C(L, l, n)\epsilon$.  

**Step 2.** Fix $L > 0$. There exist $\epsilon = \epsilon(L, n)$ and $\delta = \delta(L, n)$ so that if $|\alpha| < \epsilon$, then a solution $w(s)$ exists for all $s \in [0, L]$ and $|w|_{C^0} < \delta$.

**Proof.** A semigroup representation formula for $w_\alpha$ gives

$$w_\alpha(s) = e^{As}w_\alpha(0) + \int_0^s e^{A(s-t)}Q(w_\alpha(t))dt,$$

where $A = \Delta + 2$. We will omit the subscript $\alpha$. The spectrum of $A$ is given by $\{-\frac{k(k+n-1)}{n} + 2\}_{k=0}^\infty$. Denote those eigenvalues by $\lambda_k$. Denote by $\psi_0, \psi_1$ the harmonic, homogeneous polynomials (eigenfunctions of $A$) corresponding to $k = 0$ and $k = 1$, respectively, by $\phi$ and $\phi_i$ the ones corresponding to $k = 2$ and $k = i \geq 3$, respectively. For every $\epsilon$ choose a maximal time $\eta$ so that $w(s)$ exists for $s \in [0, \eta]$ and $|w(s)|_0 < \delta$ (we will see how we choose $\delta$ later). We want to show that for small $\epsilon$ we can take $\eta$ to be at least $L$. Assume that it is not the case, that is, $\eta < L$ no matter how small $\epsilon$ we take. By Step 1 we get that $|w(s)|_2 < C(L)\delta$, for a constant $C(L)$ that linearly depends on $L$, as we can see from the consideration in the previous step. We can write $Q(w(s)) = \alpha_0(s)\psi_0 + \alpha_1(s)\psi_1 + \alpha_3(s)\phi + \sum_{k \geq 3} \alpha_1(s)\phi_k$, 

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where \(|Q(w(s))| \leq C_1(L)\delta^2\), for all \(s \in [0, \eta]\), where \(C_1(L)\) is now a constant that depends on \(L\) quadratically. Then,

\[
w(s) = \alpha e^{-\frac{2s}{\eta}} + \int_0^s (\alpha_0(t)e^{2(s-t)}\psi_0 + \alpha_1(t)e^{s-t}\psi_1 + \sum_{i \geq 3} \phi_i \alpha_i(t)e^{\lambda_i(s-t)})dt.
\]

Notice that all \(\lambda_i < 0\). We have that for all \(s \in [0, \eta]\),

\[
|w(s)|_0 \leq C e^{-\frac{2L}{\eta}} + \delta^2 C_1(L)(Le^{2L} + Le^L + L).
\]

Choose \(\delta \leq \frac{1}{3(C_1(L)L(2e^L + e^L))}\) and let \(C e^{-\frac{2L}{\eta}} < \delta/3\). Then,

\[
|w(s)|_0 < \frac{2\delta}{3} < \delta,
\]

which implies that for sufficiently small initial data (sufficiently small \(\epsilon\)) we can extend \(w(s)\) beyond \(\eta\) so that \(|w(s)|_0 < \delta\) continues holding. This contradicts the maximality of \(\eta\), that is, for sufficiently small \(\epsilon\) we have the conclusion of the step. \(\Box\)

(*Fix some big \(3L\) and choose \(\epsilon\) and \(\delta\) as in Step 2. Our next goal is to show that for sufficiently small \(\epsilon, \delta > 0\) we can actually extend our solution \(w\) (as a scalar function on \(S^n\)) all the way up to infinity, so that \(|w(s)|_0 < 2\delta\). For each small \(\epsilon, \delta\) which satisfy (*), find \(L'\), that is, a maximal time so that \(w(s)\) can be extended all the way to \(L'\), with \(|w(s)|_0 < 2\delta\) holding. Subdivide interval \([0, L']\) into subintervals of length \(L\). We want to show that for some choice of \(\epsilon, L' = \infty\). Assume therefore \(L' < \infty\), no matter which choice for \(\epsilon\) we make. Let \(\pi\) be as before, an orthogonal projection onto a subspace \(\ker(-\frac{d}{ds} + \Delta S^n + 2)|_{S^n \times [iL, (i+1)L]}\) and \(w = (\pi w)_\uparrow + (\pi w)_\downarrow\). Similarly as in [1] we have that a behaviour of a solution of (34) is modeled on a behaviour of a solution of a linear equation \(\frac{d}{dt}F = \Delta S^n F + 2F\) (see Proposition 18 in section 3).

**Step 3.** For sufficiently small \(\epsilon\), where \(|\alpha| < \epsilon\) we can extend a solution \(w_\alpha\) (call it only \(w\)) all the way up to infinity so that \(|w| < 2\delta\).
Proof. Let $N$ be a maximal integer so that $[(N - 1)L, NL] \subset [0, L']$ and let $\gamma = \min\{|\lambda_i| \mid \lambda_i \text{ is an eigenvalue of } \Delta_{S^n} + 2\}$. Since $(\pi w)_0 = 0$ on $S^n \times [(N - 2)L, (N - 1)L]$, by Proposition 18 we have the following two cases.

Case 1. $\sup_{[(N-1)L,NL]} ||w|| \geq e^{L\gamma/2} \sup_{[(N-2)L,(N-1)L]} ||w||$.

This together with standard parabolic regularity imply

$$
\sup_{[(N-2)L,(N-1)L]} |w|_k \leq C(k)e^{-L\gamma/2}\delta \leq \delta e^{-L\gamma/3},
$$

for $L$ sufficiently big, that we fix at the beginning (from the previous estimate we see that its "bigness" depends on uniform constants; it is independent from the choices for $\epsilon$ and $\delta$). By the same proof as in Step 2 that is, by our choice of $\epsilon$ and $\delta$, considering $w((N - 1)L)$ as an initial value, we get that $w$ can be extended to $[(N - 1)L, (N + 2)L]$ so that $|w(s)|_0 < 2\delta$ for all $s \in [0, (N + 2)L]$. To justify that, notice the following two things: (a) as in Step 2 we can see that if $\delta' < \delta$ we can choose smaller $\epsilon' < \epsilon$ so that when $|w(s_0)| < \epsilon'$ then $\sup_{s \in [s_0, s_0 + 2L]} |w(s)| < \delta'$, where everything is independent of the initial time $s_0$; (b) by standard parabolic regularity, as in Step 1 we can get that $\sup_{s \in [s_0, s_0 + L]} |w(s)| < 2\delta$ implies $\sup_{s \in [s_0, s_0 + L]} |w(s)|_k \leq C(k)\delta$. This contradicts the maximality of $L'$, since $L' < (N + 1)L$.

Case 2. $\sup_{[(N-1)L,NL]} ||w|| \leq e^{-L\gamma/2} \sup_{[(N-2)L,(N-1)L]} ||w||$.

In this case (we may assume that $\delta$ is chosen so that $2\delta < \eta$), applying Proposition 18 inductively, we get

$$
\sup_{[(N-1)L,NL]} ||w|| \leq e^{-NL\gamma/2}2\delta.
$$

We can now argue similarly as in the previous case, that is we can again extend solution $w$ past time $L'$ so that $|w(s)| < 2\delta$.

This actually tells us there is an $\epsilon$ such that whenever $|\alpha| < \epsilon$, then $L' = \infty$, that is, for sufficiently small initial data, a solution $w$ to 18 exists.
and $|w(s)| < 2\delta$, where $\delta$ is taken to be small. By Proposition 18, $w(s)$ has either a growing or a decaying type of behaviour. If it had a growing type of behaviour on some interval $[kL, (k + 1)L]$, applying Proposition 18 inductively, we would get that

$$C\delta > \sup_{[NL, (N+1)L]} ||w(s)|| \geq e^{(N-k)L^2/2} \sup_{[kL, (k+1)L]} ||w(s)||,$$

for all $N$, which yields a contradiction when $N \to \infty$. In particular, this means that by using the implication (27) $\Rightarrow$ (28) inductively and standard parabolic estimates we have that

$$|w(s)|_k \leq C(k)\delta e^{-s\gamma/2},$$

for a uniform constant $C(k)$.

**Step 4.** There exists $\epsilon$, so that for $|\alpha| < \epsilon$, a solution $y_\alpha(s)$ of a mean curvature flow

$$\frac{d}{ds} y_\alpha = -H\nu + y_\alpha,$$

$$y_\alpha(0) = \alpha \phi,$$  \hspace{1cm} (34)

converges exponentially to $S^n$ (the one that we have started with).

**Proof.** Let $\epsilon_0$ be such that whenever $|\alpha| \leq \epsilon_0$, then $y_\alpha(0)$ is a strictly convex hypersurface. We know in that case $y_\alpha(s)$ converges in $C^k$ norm, exponentially, to a sphere $S^n_{\alpha}$ of radius $\sqrt{n}$, and a quantity $\sup_{s \in [0, \infty]} |y_\alpha(s) - y_{S^n}|_k$ makes sense. Define a function $G : [-\epsilon_0, \epsilon_0] \to [0, \infty)$ by $G(\alpha) = \sup_{s \in [0, \infty]} |y_\alpha(s) - y_{S^n}|_k$. It is a continuous function and therefore bounded on a compact set $[0, \epsilon_0]$. This implies all solutions $y_\alpha(s)$, for $\alpha \in [0, \epsilon_0]$ lie in a $C^k$ ball of a finite radius, with a center at $y_{S^n}$. The continuity of this map implies that for sufficiently small, say $\epsilon_0$, all solutions $y_\alpha$, for $\alpha \in [-\epsilon_0, \epsilon_0]$ lie in a $C^0$ ball centred at $y_{S^n}$, of radius 1/2. This implies that every limit sphere $S^n_{\alpha}$ of a solution $y_\alpha$ has a nonempty intersection with $S^n$. It is a well known result that if two solutions of a mean curvature flow become disjoint,
they stay disjoint for the remaining time of their existence. That is why our solutions \( y_\alpha(s) \) never become disjoint from \( S^n \).

Denote by \( w = \langle y - y_{S^n}, \nu_{S^n} \rangle \) (we actually mean \( y_\alpha \), but we are omitting the subscripts). An initial hypersurface \( y(0) \) can be written as an entire graph over \( S^n \), that is, for a choice of a unit normal \( \nu \) for \( M \), we have
\[
\langle \nu, \nu \rangle > 0\quad \text{everywhere on } M.
\]
Choose \( \alpha \) small (\(|\alpha| < \epsilon\)), as in Step 3, so that an equation
\[
\frac{d}{ds} \tilde{w}(s) = \Delta_{S^n} \tilde{w} + 2 \tilde{w} + Q(\tilde{w}),
\]
\[
\tilde{w}(0) = \alpha \phi,
\]
has a solution all the way to infinity and \(|\tilde{w}(s)|_k < C(k)\delta e^{-s\gamma/2}\), where \( C(k) \) is a uniform constant. Let \( \epsilon \) and \( \delta \) be very small and let \( \eta < \infty \) be a maximal time such that \( \langle \nu(s), \nu_{S^n} \rangle > 0 \) for \( s \in [0, \eta) \). We can regard \( w(s) \) as a function over \( S^n \) for \( s \in [0, \eta) \), therefore satisfying (35). This implies \( \tilde{w}(s) = w(s) \) and \(|w(s)|_k \leq C(k)\delta e^{-s\nu/2}\), for \( s \in [0, \eta) \).

From (34) we get,
\[
\frac{d}{ds} \langle y - y_{S^n}, \nu_{S^n} \rangle = -H \langle \nu, \nu_{S^n} \rangle + \langle y, \nu_{S^n} \rangle.
\]
This implies
\[
\frac{d}{ds} \langle y - y_{S^n}, \nu_{S^n} \rangle = -H \langle \nu, \nu_{S^n} \rangle + \langle y, \nu_{S^n} \rangle.
\]
We have
\[
\langle y, \nu_{S^n} \rangle = \langle y - y_{S^n}, \nu_{S^n} \rangle + \sqrt{n} = w(s) + \sqrt{n} - C\delta e^{-s\gamma/2},
\]
and
\[
\left| \frac{d}{ds} w \right|_0 = \left| \Delta_{S^n} w + 2w + Q(w) \right|_0 \leq C|w|_2 \leq \tilde{C}\delta e^{-s\gamma/2}.
\]
By (35) and (37),
\[
\frac{d}{ds} w = \frac{d}{ds} \langle y - y_{S^n}, \nu_{S^n} \rangle > -H \langle \nu, \nu_{S^n} \rangle + \sqrt{n} - C\delta e^{-s\gamma/2}.
\]
Combining the last estimate together with (35) yields,
\[
H \langle \nu(s), \nu_{S^n} \rangle > \sqrt{n} - C\delta e^{-s\gamma/2} - \tilde{C}\delta e^{-s\gamma/2}.
\]
Our constants in the previous estimate are uniform and therefore if we make \( \delta \) small enough (which we can achieve by decreasing \( \epsilon \)), since \( \gamma > 0 \), we get

\[
H \langle \nu(s), \nu_{S^n} \rangle > n/2,
\]

for all \( s \in [0, \eta) \). Since \( H \) is bounded from above, we get

\[
\langle \nu(s), \nu_{S^n} \rangle > \beta > 0,
\]

for all \( s \in [0, \eta) \). This implies the property \( \langle \nu(s), \nu_{S^n} \rangle > 0 \) continues holding for our solution \( y(s) \) past time \( \eta \), which contradicts the maximality of \( \eta \), unless \( \eta = \infty \). This together with (***) imply we can consider \( w(s) \) as a function over \( S^n \) for all \( s \in [0, \infty) \), satisfying (**). By uniqueness of solutions, we have \( w(s) = \tilde{w}(s) \) and henceforth \( |w(s)| < Ce^{-s\gamma/2} \), for all \( s \), that is, \( y_\alpha(s) \) converges exponentially to a sphere \( y_{S^n} \) when \( |\alpha| \) is small.

We can now finish the proof of Proposition 21. Once we have a conclusion of Step 4, similarly as in Lemma 19 we can prove there are no growing modes in \( w \), that is we can write

\[
w(s) = \alpha \phi e^{-\frac{\tau}{2}} + \int_0^s \sum_{i \geq 3} \phi_i \alpha_i(t) e^{\lambda_i(s-t)} dt,
\]

where the notation is the same as in Step 2. Similarly as in the Claim 22 we can show \( \int_0^s \sum_{i \geq 3} \phi_i \alpha_i(t) e^{\lambda_i(s-t)} dt \) will decay at least at a rate of \( e^{-4s/n} \), so we can not expect any cancellations and since \( \pi(w)(0) = \alpha \phi \), we have \( \pi w(s) = \alpha \phi e^{-2s/n} \), where \( \alpha \) is small, but can be taken to be different from zero.

We can now finish the proof of Theorem 4.

**Proof of Theorem 4.** Take a solution \( y \) found by Proposition 21. Since \( y - y_{S^n} = (2\tau)^{-1/2} x - y_{S^n} \), where \( \tau = T - t = e^{-2s} \), we have that \( w = (2\tau)^{-1/2} \langle x, \nu_{S^n} \rangle - \sqrt{n} \). We know that \( y \to y_{S^n} \) exponentially and because of this uniform convergence we can consider \( y \) as a radial graph over \( S^n \) for
all sufficiently big values of $s$. That is why we can write $|y| = \langle y, \nu_{S^n} \rangle$ and $|x| = \langle x, \nu_{S^n} \rangle$ and

$$
\tau = \frac{|x|^2}{2(w + \sqrt{n})^2},
$$

which yields

$$
u(x^*) - u(x) = \frac{|x|^2}{2n}(1 - 2 \frac{w}{\sqrt{n}} + 3 \frac{w^2}{n} + o(w^2)).
$$

Let $\pi$ be a projection onto $\ker(-d/ds + \Delta_{S^n} + 2)$ with $\pi(w)(0) = w(0)$. Then $w = \pi w + R$, where $R$ is not in $\ker(-d/ds + \Delta_{S^n} + 2)$ and

$$
\pi w = \alpha \phi e^{-\beta s} + \sum_k \alpha_k \phi_k e^{-\beta_k s},
$$

where $\alpha \neq 0$ (justified by Proposition 21), $\beta = 2/n$, $-\beta_k$ are the remaining negative eigenvalues of $\Delta_{S^n} + 2$ and $\phi, \phi_k$ are harmonic, homogenous polynomials restricted to a sphere $S^n(\sqrt{n})$, of degrees 2 and $k \geq 3$, respectively, and $\alpha, \alpha_k$ are some constants. Because of Lemma 19 in $\pi w$ there are no contributions from eigenfuctions corresponding to positive eigenvalues of $\Delta_{S^n} + 2$. Then

$$
\pi w = \alpha \phi \tau^{\beta/2} + o(\tau^{3\beta/2}).
$$

We may assume $x^*$ is the origin in $\mathbb{R}^{n+1}$. Since $\nabla u(x^*) = 0$ and $\nabla_i \nabla_j u(x^*) = -\frac{1}{n} \delta_{ij}$ (see [11]), we have

$$
\tau = u(0) - u(x) = \frac{1}{n}|x|^2 + O(|x|^3), \quad (39)
$$

which yields

$$
u(0) - u(x) = \frac{|x|^2}{2n}(1 - \alpha \frac{\phi}{\sqrt{n}} \frac{|x|^\beta}{n^{\beta/2}} + O(|x|^{3\beta/2}) + R).
$$

Since $(2\tau)^{1/2} = \frac{|x|}{|y|}$, by (39) we get

$$
\frac{1}{2^{\beta/2}} \frac{|x|^{\beta}}{|y|^{\beta}} = \frac{|x|^{\beta}}{n^{\beta/2}} + O(|x|^{3\beta/2}),
$$

and therefore

$$
u(0) - u(x) = \frac{|x|^2}{2n}(1 - \alpha \frac{\phi}{\sqrt{n}} \frac{1}{2^{\beta/2}} \frac{|x|^\beta}{|y|^\beta} + O(|x|^{3\beta/2}) + R). \quad (40)
$$
Claim 22. There is $\gamma \geq 2\beta$ so that $R = O(|x|^\gamma)$.

Proof. A scalar function $R$ satisfies
\[
\frac{d}{ds}R = \Delta_{S^n}R + 2R + Qw. \tag{41}
\]
If $R = O(|x|^\gamma) = O(|y|^\gamma e^{-\gamma s})$, then $\frac{d}{ds}R = O(|y|^\gamma e^{-\gamma s})$ and $\Delta_{S^n}R = O(e^{-\gamma s})$. On the other hand, $Qw$ is of order $O(e^{-2\beta s})$. Since $Q(w) \neq 0$, we have that $\gamma \geq 2\beta$, because otherwise (41) could not be satisfied for large values of $s$. \hfill $\square$

So far we have found a solution $y(s)$ to a rescaled mean curvature flow, whose existence, together with the asymptotic behaviour of its arrival time given by (40), for $\alpha \neq 0$ is provided by Proposition 21. For $n \geq 3$, since $\beta = 2/n < 1$, from (40) it follows immediately that $u(x)$ can not be $C^3$ at the origin. In the case $n = 2$ we have
\[
u(0) - u(x) = \frac{|x|^2}{4} - \alpha \frac{\phi(x) |x|}{8\sqrt{2} |y|} + O(|x|^{3/2} + 2).
\]
Take any $x$ such that $\phi(x) \neq 0$ and choose a line $tx$, for $t \in \mathbb{R}$. Then, since $\phi$ is a homogeneous polynomial of degree two,
\[
u(0) - u(tx) = t^2 \frac{|x|^2}{4} - \alpha t^2 |t| \frac{\phi(x) |x|}{8\sqrt{2} |y|} + O(|tx|^{7/2}).
\]
If we treat the right hand side as a function of $t$, we can see that it is not $C^3$ at $t = 0$. Henceforth, $u(x)$ can not be $C^3$ at the origin. \hfill $\square$

In section 3 we have proved that a solution to a rescaled mean curvature flow (7), starting with a strictly convex hypersurface $M_0$, converges exponentially to a sphere of radius $\sqrt{n}$ at a rate of at least $2/n$, that is
\[
|y(s) - y_{SS}| \leq Ce^{-\delta s},
\]
for $\delta \geq 2/n$. We will conclude that in a generic case we can not expect to have $\delta > 2/n$. 

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Proof of Theorem 5. The proof goes by contradiction. Assume there is $\delta > 2/n$ so that $\delta$ is the optimal rate of convergence of (7) for $M_0$ being a strictly convex hypersurface. Take a solution $y(s)$ that we have constructed in Proposition 21. From the proof of the Proposition we know that for some $\alpha \neq 0$, we have

$$\langle y(s) - y_{S^n}, \nu_{S^n} \rangle = \alpha \phi e^{-\frac{2s}{n}} + \int_0^s \sum_{i \geq 3} \phi_i \alpha_i(t) e^{\lambda_i(s-t)} dt, \quad (42)$$

where $\lambda_i \leq -(1+6/n)$ and we know that $y(s)$ converges to $y_{S^n}$ exponentially. If $|\langle y(s) - y_{S^n}, \nu_{S^n} \rangle| \leq C e^{-\delta s}$, then (42) would yield a contradiction for big values of $s$. \hfill \Box

5 More on regularity of $u(x)$ for some solutions to the mean curvature flow

If it happens that we have a solution $y$ such that $\psi(\langle y - y_{S^n}, \nu_{S^n} \rangle) = 0$, our "arrival time" $u(x)$ might be $C^3$ in $\Omega$. Moreover, the order of regularity depends on the first term of form $\alpha_k \phi_k e^{-\beta_k s}$, appearing in $\pi(\langle y - y_{S^n}, \nu_{S^n} \rangle)$, which actually determines the rate of exponential convergence of $y$ to $y_{S^n}$. We will below discuss the case of $C^3$-regularity, but the consideration is analogous in the case of $C^k$-regularity, for $k > 3$.

Corollary 23. Let $y$ be a solution to (7) such that $\psi(\langle y - y_{S^n}, \nu_{S^n} \rangle) = 0$ and $\pi w = \sum_{k \geq l} \alpha_k \phi_k e^{-\beta_k s}$, for $l \geq 3$. If $\beta_l > 3$, then $u \in C^3(\Omega)$. This holds for any $n \geq 2$.

If our solution $y$ satisfies a condition in the corollary, as in section 3, we can prove that

$$|y - y_{S^n}|_k \leq C(k) e^{-\beta_l s},$$

where $\beta_l \geq 1 + \frac{6}{n}$, that is, $y$ converges to a sphere $y_{S^n}$ exponentially, at the rate at least $1 + \frac{6}{n}$.
Function $u(x)$ can be viewed as the unique viscosity solution to the non-linear partial differential equation

$$\Delta u - \langle D^2 u \frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} \rangle + 1 = 0,$$

in $\Omega$ and $u = 0$ at $\partial \Omega$. This equation was first studied by Evans and Spruck in [4]. They showed its solution has the property that each level set $u = t$ is the smooth image of $\partial \Omega$ under motion by curvature for time $t$, for any $0 \leq t < T$. That is why the smoothness of $u(x)$ is apparent away from $x^*$. Denote by $Z = |Du|^{-2}D_iuD_juD_iD_ju$. Then we can write the above equation as

$$\Delta u = Z - 1.$$  \hfill (43)

Due to Huisken we know $u \in C^2$ with $\text{Hess}_{ij}u = -\frac{1}{n}\delta_{ij}$. To prove $u \in C^3(\Omega)$ (for our flow $y$ having the properties as in the statement of the corollary) we need to estimate $DZ$. Take $p \in \Omega$ and let $p \in M_t$, for some time $t$. We want to estimate $D_\nu Z(p)$ and $D_\tau Z(p)$, where $D_\nu Z$ is a derivative of $Z$ in normal direction to the level set $M_t$ and $D_\tau Z$ is a tangential derivative at point $p \in M_t$. The estimate for $D_\nu Z$ is reduced to obtaining the estimate for $H^{-\frac{1}{n}} \frac{d}{dt} Z$, leading in particular to a term like $H^{-\frac{1}{n}} \Delta \Delta H$. All our hypersurfaces $M_t$ are embedded in $\mathbb{R}^{n+1}$ and for every function $f$ on $\Omega$ we have that $(\nabla_{\mathbb{R}^{n+1}} f)^T = \nabla_{M_t} f$ at $x \in M_t$. We will use $\nabla$ for $\nabla_{M_t}$. We need to estimate $D_\tau Z$ which is translated to obtaining the estimate for $H^{-\frac{3}{n}} \Delta H$. If $\nu$ is the unit normal to $M_t$ then the derivative of any function $f$ in the normal direction to the level set $M_t$ of $u$ is given by $D_\nu f = H^{-\frac{1}{n}} \frac{d}{dt} f$. We can write

$$Z = D_\nu D_\nu u = D_\nu H^{-1}$$

$$= H^{-\frac{1}{n}} \frac{d}{dt} H^{-1} = -\frac{1}{H^3} \frac{dH}{dt}$$

$$= -\frac{1}{H^3} (\Delta H + |A|^2 H)$$

$$= -\frac{1}{n} \left( \frac{\Delta H}{H^3} + \frac{|A|^2}{H} - \frac{1}{n}\frac{H^2}{H^2} \right).$$

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5.1 Estimates on $D_\nu Z$ and $D_\tau Z$

If we have $\frac{d}{dt} F = -H\nu$, in \cite{9} it was computed that

$$\frac{d}{dt} g_{ij} = -2H h_{ij}.$$ 

It is easy to compute the evolution equation for the Christoffel symbols (see \cite{2})

$$\frac{d}{dt} \Gamma^k_{ij} = -g^{kl}(\nabla_i (H h_{jl}) + \nabla_j (H h_{il}) - \nabla_l (H h_{ij})).$$

We can compute

$$D_\nu Z = -\frac{d}{dt} \Delta H \frac{H^4}{H^4} - \frac{d}{dt} (|A|^2 H) \frac{H^4}{H^4} + 3 \frac{(\Delta H + |A|^2 H)^2}{H^5}.$$ 

Fix $x \in M$ and a corresponding time $t_x$ such that $u(x) = t_x$. Choose normal coordinates around $x$ in metric $g(t_x)$, so that $\Gamma^k_{ij}(x, t_x) = 0$. Since $\Delta H = g^{ij}(\nabla_i \nabla_j H - \Gamma^k_{ij} \nabla_k H)$,

$$\frac{d}{dt} \Delta H = 2g^{ip}g^{jq} h_{pq}(\nabla_i \nabla_j H - \Gamma^k_{ij} \nabla_k H) + \Delta \Delta H + g^{ij} g^{kl} \nabla_k H (\nabla_i (H h_{jl}) + \nabla_j (H h_{il}) - \nabla_l (H h_{ij})).$$

By the curvature evolution equations (see \cite{1}) we get,

$$D_\nu Z = -\frac{\Delta^2 H}{H^4} - \frac{2g^{ip}g^{jq} h_{pq} \nabla_i H \nabla_j H}{H^3} - \frac{g^{ij} g^{kl} \nabla_k H (\nabla_i (H h_{jl}) + \nabla_j (H h_{il}) - \nabla_l (H h_{ij}))}{H^4} - \frac{(\Delta |A|^2 + 2|\nabla A|^2)}{H^5} + \frac{5 |A|^2 \Delta H}{h^4} + \frac{3 (\Delta H)^2}{H^5}.$$ 

We want to discuss the asymptotics of terms appearing on the right hand side of identity (44).

- Terms $\frac{\Delta^2 H}{H^4}$, $\frac{2g^{ip}g^{jq} h_{pq} \nabla_i H \nabla_j H}{H^3}$ can be estimated by a constant times $(T - t)^{(\beta_1 - 1)/2} \to 0$ as $t \to T$.

- Since the eigenvalues of $A$ are strictly positive, $|A|^2 \leq H^2$ and

$$\frac{|g^{ij} g^{kl} \nabla_k H \nabla_l H|}{H^4} \leq C \frac{|\nabla A|^2}{H^4} \leq C (T - t)^{(\beta_1 - 1)/2}.$$
We can similarly estimate the rest of the terms appearing in (44). The conclusion is that \(|D_\nu Z| \leq C(T - t)^\delta\), for some \(\delta > 0\).

As we have mentioned above we will use symbol \(\nabla\) to denote a derivative with respect to the induced metric \(g(t)\) on \(M_t\). We can compute
\[
\nabla Z = -\frac{\nabla \Delta H}{H^3} - \frac{\nabla (|A|^2 H)}{H^4} + 3 \frac{\Delta H \nabla H}{H^4} + 3 \frac{\nabla H |A|^2}{H^3}.
\]
All terms appearing on the right hand side of the above identity are easy to estimate, e.g.
\[
\left| \frac{\nabla \Delta H}{H^3} \right| \leq C(T - t)^{(\beta_l - 1)}.
\]
The conclusion is that \(|D_\tau Z| \leq C(T - t)^\delta\), for some \(\delta > 0\). We can now finish the proof of Corollary 23.

**Proof of Corollary 23**. From (43) we get
\[
\Delta D u = D Z. \quad (45)
\]
We know that
\[
|D Z(x)| \leq C(T - t)^{(\beta_l - 1)/2} = C(u(x^*) - u(x)) \leq \tilde{C}|x - x^*|^{(\beta_l - 1)/2},
\]
since \(u \in C^1(\Omega)\). If \(\beta_l > 3\) we get that \(DZ\) is differentiable at \(x^*\), which means everywhere (\(u\) is smooth everywhere on \(\Omega\) except at \(x^*\)). This tells us \(DZ\) lies in some Hölder space \(C^{0,\alpha}\), for \(\alpha \in (0,1)\), and by elliptic regularity applied to (45) we get \(u \in C^{3,\alpha}(\Omega)\).

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