HOLOMORPHIC VECTOR FIELDS AND RATIONALITY

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Abstract. We show that a nonsingular complex projective variety admitting a holomorphic vector field with nonempty isolated zeroes, is rational using a key technique by Harvey-Lawson on finite volume flows. This statement was conjectured by J. Carrell. By the same technique, we obtain a uniform upper bound of Betti numbers of nonsingular complex projective variety admitting a holomorphic vector field with exact one zero point. Such an upper bound depends only on the dimension of the variety, which is a stronger version of a result of Akyildiz and Carrell.

1. Introduction

The action of group actions on manifolds is one of long lasting research subject in mathematics. Almost every topic in algebra, geometry and topology, there is a corresponding version with group actions, e.g., for a cohomology theory there is an equivariant cohomology theory. In particular, the action of 1-parameter group on manifolds is one of the simplest and widest concerned interesting branches in the field of group actions on topological spaces.

There are lots of significant results related the structure of a complex analytic space or a complex manifold to its fixed point set under the action of a complex 1-parameter group. The topological and analytic invariants between a compact Kähler manifold or a singular projective variety and those of the fixed point set under an analytic action of a complex multiplicative group can be found in [B-B1], [CG], [Fr], etc. and the references therein.

Bialynicki-Birula showed that any nonsingular projective variety $X$ under a complex multiplicative group $G_m$ action admits a canonical decomposition into $G_m$-invariant locally closed subvarieties. In particular, the algebro-geometric invariants of the variety can be read from those of the fixed point set. When the fixed point set is isolated, all the cohomology theories on $X$ (e.g., Chow groups, algebraic cycles modulo algebraic equivalence, Lawson homology, singular homology) are naturally isomorphic (see [B-B1], [ES], [FT], [LF]). Moreover, such a variety $X$ must be a rational variety.

However, the situation is much more subtle for a projective variety under the action of an algebraic additive group. The fact is that the understanding of the structure of a nonsingular complex projective variety $X$ through the fixed point set under an additive group action amounts to the understanding of the structure of $X$ through the zero locus of a holomorphic vector field. The reason is that the structure of a nonsingular projective variety under the action of multiplicative group is clearly understood through the information of the fixed point set (see [L2]).

Now let $V$ be a holomorphic vector field defined on a nonsingular complex projective algebraic variety $X$. The zero subscheme $Z$ is the subspace of $X$ defined by the ideal generated by $VO_X$.

Contrary to the real case, there are a lot of topological obstructions to the existence of a holomorphic vector field on a complex projective variety. Moreover, the structure of $X$ are closely related to that of $Z$ (see [C §5]). For examples, the rational cohomological
ring of $X$ can be determined by those of $Z$ and the fundamental group of $X$ is perfect if $Z$ is simple (i.e., smooth of dimension zero) (see [CL]).

A fundamental problem in algebraic geometry is to determine which varieties are rational. One of the most famous problems in the direction, formulated by J. Carrell, is the following conjecture.

**Conjecture 1.1** ([CL], p.33). A smooth (or nonsingular) irreducible projective algebraic variety $X$ admitting a holomorphic vector field with nonempty isolated zeroes, is rational.

This conjecture stated in an open problem format, can be found in an earlier literature (see [Li1]).

During last forty years, there have been lots of progress in all kinds of special cases. For surfaces, this conjecture is proved in [CHK], [Li2, p.99]. For $X$ with $\dim X \leq 5$, it was proved by Hwang (see [Hw1]). Hwang also mentioned in his paper that M. Koras had proved the conjecture for dimension less than or equals to 4 in an unpublished paper in a possibly quite different way. Under the assumption that the automorphism group of the manifold is semisimple, it was proved by Deligne (see [C, p.32]). Under the assumption that the induced action on the tangent space at the isolated zero has a single Jordan block, it was proved by Konarski ([K]). Under the assumption that the holomorphic vector field has a totally degenerate zero, it was proved by Hwang ([Hw2]).

This conjecture has been studied by Lieberman (see [Li1], [Li2]), who had shown that it holds if every point in $Z$ is non-degenerate. Recall that a point in $Z$ is called non-degenerate if it is a nonsingular point on $Z$ and the $\mathcal{O}_{X,Z}$-linear map $L_V : I_Z/I_Z^2 \to I_Z/I_Z^2$ has nonzero determinant on $Z$. Note that a holomorphic vector field obtained from the flow of a multiplicative group action is always non-degenerate.

A stronger version of the conjecture (might be also made by Carrell) that every smooth projective variety admitting a holomorphic vector field with nonempty isolated zeroes also admits a $\mathbb{C}^*$-action with nonempty isolated zeros.

Conjecture 1.1 as is well known, is equivalent to the following([Li1], [K]):

**Conjecture 1.2.** If $X$ is a nonsingular projective variety admitting an algebraic additive group action having exactly one fixed point, then $X$ is rational.

We will show below this conjecture to be true by applying techniques developed by Harvey and Lawson to the ring $\mathcal{O}_{X,p}$ of local regular functions of a point on $X$, to get an isomorphism between $\mathcal{O}_{X,p}$ and that of a point on $\mathbb{C}^n$.

As applications of techniques developed by Harvey and Lawson, we obtain a uniform upper bound $C_n = 2^n$ for the Euler number and Hodge numbers of $X$ of dimension $n$, where $X$ is a nonsingular projective variety admitting an algebraic additive group action with exactly one fixed point (see Theorem 3.3).

In particular, we have $b^2(X) \leq \dim X$ for such an $X$. This is a generalization of an “unexpected consequence” of Akyildiz and Carrell (see [AC, Th.2]). In Akyildiz and Carrell theorem, much more stronger conditions for $X$ are assumed. We give a classification of surfaces and a partial classification of threefolds of admitting a $\mathbb{C}$-action with exact one fixed point.

2. Proof of Carrell’s conjecture

In this section we will show Conjecture 1.2 and hence Conjecture 1.1 hold. From now on, let $X$ be a smooth complex projective variety of dimension $n$ admitting a $\mathbb{C}$-action with exactly one fixed point $p$, i.e., $X^C = p$.

Let $m_p \subset \mathcal{O}_{X,p}$ (resp. $m_0 \subset \mathcal{O}_{\mathbb{C}^n,0}$) be the corresponding maximal ideal of local algebras.

Note that the flow $\phi_t$ induced from the $\mathbb{C}$-action generates a finite volume complex graph $\mathcal{G}$ in the sense of Harvey and Lawson (see [HL, §9]), where

$$\mathcal{G} := \{ (t, \phi_t(x), x) \in \mathbb{C} \times X \times X | t \in \mathbb{C} \text{ and } x \in X \} \subset \mathbb{P}^1 \times X \times X$$
and its closure $\mathcal{T} \subset \mathbb{P}^1 \times X \times X$ is a projective variety. According to their theory, this $\mathcal{T}$ provides us an operator equation

$$I - P = d \circ T + T \circ d : \mathcal{E}^k(X) \to \mathcal{D}^k(X)$$

for $k \geq 0$, where $\mathcal{E}^k(X)$ is the space of smooth $k$-forms on $X$, $\mathcal{D}^k(X)$ is the space of $k$-currents on $X$, $I$ is the identity operator, $P : \mathcal{E}^k(X) \to \mathcal{D}^k(X)$ is given by $P\alpha = \lim_{t \to \infty} \phi_t^*(\alpha)$ and $T : \mathcal{E}^k(X) \to \mathcal{D}^{k-1}(X)$ is the operator given by $\mathcal{T}$ (see [3], §2).

When restricting to the local ring of $\mathcal{C}$-algebra $\mathcal{O}_{X,p}$, we get the formula

$$I - P = T \circ d : \mathcal{O}_{X,p} \to \mathcal{O}_{X,p}$$

since $T = 0$ on $\mathcal{E}^0(X) = \mathcal{C}^\infty(X)$ the set of smooth functions on $X$ and hence $T = 0$ on $\mathcal{O}_{X,p}$ since an element $f$ in $\mathcal{O}_{X,p}$ can be extended to a smooth function on $X$. The operator $P$ on $f \in \mathcal{C}^\infty(X)$ is the constant function with the value $f(p)$ since

$$(Pf)(x) = \lim_{t \to \infty} \phi_t^*(f)(x) = \lim_{t \to \infty} f \circ \phi_t(x) = f \lim_{t \to \infty} \phi_t(x) = f(p).$$

The details are given in the following lemma, where Equation (2.2) is applied to local functions in $\mathcal{O}_{X,p}$.

**Lemma 2.3.** One gets the following formula

$$f = f(p) = T \circ df, f \in \mathcal{O}_{X,p}.$$  

**Proof.** Let $F, \tilde{F} \in \mathcal{C}^\infty(X)$ be two extensions of the germ $f \in \mathcal{O}_{X,p}$. Then we have $F - \tilde{F}(p) = T \circ d F$ and $\tilde{F} - F(p) = T \circ d \tilde{F}$ by Equation (2.1). Since $F(p) = \tilde{F}(p) = f(p)$, we get $F - \tilde{F} = T \circ d(F - \tilde{F})$. Since both $F$ and $\tilde{F}$ are the extension of $f$, there exists a neighborhood $U_p$ of $p$ such that $F - \tilde{F} \equiv 0$ on $U_p$. That is, the operator $T \circ d$ applies to $F - \tilde{F}$ is zero on on $U_p$. Hence two different choices of extensions of $f \in \mathcal{O}_{X,p}$ yield the same local function whose class is $f - f(p)$. This shows that $I - P = T \circ d$ holds on $\mathcal{O}_{X,p}$.

Moreover, when restricted on $m_p$, we have

$$f = d \circ df, f \in m_p$$

since $f(p) = 0$ for $f \in m_p$.

Let $\Omega^1_{X,p}$ be the space of local 1-forms with coefficients in $\mathcal{O}_{X,p}$, i.e., $\Omega^1_{X,p} = \{df | f \in \mathcal{O}_{X,p}\}$. Since $X$ is a nonsingular variety, we have $\mathcal{T}^*_p X \cong \Omega^1_{X,p} \otimes \mathbb{C}_{X,p} \cong m_p/m_p^2$ as $\mathbb{C}$-vector spaces, where $\mathcal{T}^*_p X$ is the cotangent space of $X$ at $p$.

Since $p \in X$ is nonsingular, we have $m_p/m_p^2 \cong \mathcal{T}^*_p X$ as $\mathbb{C}$-vector spaces. Since $\dim_{\mathbb{C}} X = n$, there exists a $\mathbb{C}$-linear isomorphism $\psi^* : \mathcal{T}^*_p X \cong \mathbb{C}^n$. Let $\omega_i(1 \leq i \leq n)$ be a basis of $\mathcal{T}^*_p X$ such that $\psi^*(\omega_i) = dz_i$, where $z_i(1 \leq i \leq n)$ is a system of local coordinates of $\mathbb{C}^n$ at the origin.

Set $y_i := T(\omega_i)$, then $y_i \in m_p$ by Equation (2.4). Since $T(dy_i) = y_i$ by Equation (2.5), we have $dy_i - \omega_i \in \ker(T)$. We can choose $dy_1, ..., dy_n$ as the basis of $\mathcal{T}^*_p X$. That is, we give a system of local coordinates $y_1, ..., y_n$ for $X$ at $p$ such that there exists a map $\psi : U_o \to U_p$ is defined from a neighborhood $U_o \subset \mathbb{C}^n$ of $o$ to a neighborhood $U_p \subset X$ of $p$.

For $\omega \in \Omega^1_{X,p}$, we have $\omega = \sum_i f_i dy_i = df$ for some $f \in \mathcal{O}_{X,p}$, $T(\omega) = T(df) = f$, where $f_i = df_i$.

Now we extend the map $\psi^* : \mathcal{T}^*_p X \cong \mathbb{C}^n$ to $\psi^* : \Omega^1_{X,p} \to \Omega^1_{\mathbb{C}^n,o}$ as the following:

$$\psi^*(df)(z_1, ..., z_n) := df(y_1, ..., y_n), \forall df \in \Omega^1_{X,p}.$$

Now we define a map $\psi^* : \mathcal{O}_{X,p} \to \mathcal{O}_{\mathbb{C}^n,o}$ (by the same notation) as $\psi^*(f)(z_1, ..., z_n) := f(y_1, ..., y_n)$ for $f \in \mathcal{O}_{X,p}$. This map $\psi^* : \mathcal{O}_{X,p} \to \mathcal{O}_{\mathbb{C}^n,o}$ is a local ring homomorphism since it is a $\mathbb{C}$-linear map and $\psi^*(f_1 f_2) = \psi^*(f_1) \psi^*(f_2)$. To see the last formula, we get
Lemma 2.6. This map $\psi^* : \mathcal{O}_{X,p} \to \mathcal{O}_{\mathbb{C}^n,o}$ defined by $\psi^*(f)(z_1, \ldots, z_n) = f(y_1, \ldots, y_n)$ for $f \in \mathcal{O}_{X,p}$ is injective.

Proof. Let $f \in \mathcal{O}_{X,p}$ such that $\psi^*(f) = 0$. Then $0 = \psi^*(f)(z_1, \ldots, z_n) = f(y_1, \ldots, y_n)$ for $(y_1, \ldots, y_n)$ in the neighborhood of $p$. This implies that $f$ is zero in $\mathcal{O}_{X,p}$. $\square$

Now we define map $T_o : \Omega^1_{\mathbb{C}^n,o} \to m_o$. $T_o(g) := g$ for $g \in \mathcal{O}_{\mathbb{C}^n,o}$. Then $T_o(dz_i) = z_i$ for $1 \leq i \leq n$.

Lemma 2.7. There exists a commutative diagram

$$\begin{array}{ccc}
\mathcal{O}_{X,p} & \xrightarrow{d} & \Omega^1_{X,p} \\
\downarrow{\psi^*} & & \downarrow{\psi^*} \\
\mathcal{O}_{\mathbb{C}^n,o} & \xrightarrow{d} & \Omega^1_{\mathbb{C}^n,o} \\
\end{array}\quad T \xrightarrow{} \mathcal{O}_{X,p}$$

Moreover, $\psi^*(m_p) \subset m_o$.

Proof. The first square in Equation (2.8) is commutative since $d \circ \psi^*(f) = \psi^* \circ df$ by their definitions, while the second square in Equation (2.8) is commutative since $T \circ d = Id - P$ and the definition of $T_o$. To see this, we take $\alpha = df$ where $f(p) = 0$, then $T_o(\alpha) = f$. Hence $\psi^*T(\alpha) = \psi^*(f) = T_o(df(f)) = T_o(\psi^*(df)) = T_o(\psi^*(\alpha))$ since the map $\psi^* : \mathcal{O}_{X,p} \to \mathcal{O}_{\mathbb{C}^n,o}$ and $\psi^* : \Omega^1_{X,p} \to \Omega^1_{\mathbb{C}^n,o}$ are natural maps. $\square$

Remark 2.9. The key point in Lemma 2.7 is that there exists map $T_o : \Omega^1_{\mathbb{C}^n,o} \to m_o$ such that $T_o \circ d = I$ on $m_o$.

Note that there are natural $\mathbb{C}$-linear isomorphisms

$$\Omega^1_{X,p} \cong \bigoplus_{i=0}^{\infty} \frac{m_{i+1}^p}{m_i^p} \otimes \frac{m_p}{m_p^2}$$

and

$$\mathcal{O}_{X,p} \cong \bigoplus_{i=0}^{\infty} \frac{m_p}{m_i^p}.$$ 

Equation (2.10) follows from Equation (2.11), where the isomorphism of Equation (2.11) follows from Taylor expansion of $f$ in local coordinates $y_1, \ldots, y_n$ at $p, f = f_0 + f_1 + \ldots$, where $f_i \in m_i^p$ is the sum of all the terms of homogeneous degree $i$.

Under these isomorphisms, we see the image of $\frac{m_{i+1}^p}{m_i^p} \otimes \frac{m_p}{m_p^2}$ under the map

$$\Omega^1_{X,p} \xrightarrow{T_o} \mathcal{O}_{X,p}$$

is in $\frac{m_i^p}{m_p^2}$. Moreover,

$$T(\frac{m_{i+1}^p}{m_i^p} \otimes \frac{m_p}{m_p^2}) = \frac{m_{i+1}^p}{m_i^p} \otimes \frac{m_p}{m_p^2}$$

since $m_{i+1}^p \cdot m_p = m_{i+2}^p$.

Now from Equation (2.8), we have the following exact sequence of commutative diagram
Lemma 2.13. There exists

The pullback map \( \psi^* : \Omega^1_{X,p} \to \Omega^1_{n,o} \) is surjective.

Proof. Note that \( \Omega^1_{X,p} \cong \bigoplus_{i=0}^{\infty} \frac{m^i_p}{m^i_p} \otimes \frac{m_p}{m_p} \) and \( \Omega^1_{n,o} \cong \bigoplus_{i=0}^{\infty} \frac{m^i_i}{m^i_i} \otimes \frac{m_i}{m_i} \).

Since \( \dim X = n \), \( \psi^* \) induces an isomorphism \( \psi^* : T^n X \cong T^n_o (\mathbb{C}^n) \) of \( \mathbb{C} \)-vector spaces, i.e., \( \psi^* : \frac{m^i_p}{m^i_p} \otimes \frac{m_p}{m_p} \to \frac{m^i_i}{m^i_i} \otimes \frac{m_i}{m_i} \) is an isomorphism for \( i = 0 \).

Now we use inductions on \( i \). Suppose that \( \psi^* : \frac{m^i_p}{m^i_p} \otimes \frac{m_p}{m_p} \to \frac{m^i_i}{m^i_i} \otimes \frac{m_i}{m_i} \) is surjective for \( i = k \).

Consider the following commutative diagram

\[
\begin{array}{ccccccccc}
& & & 0 & & & 0 & & & \\
& & & \psi^* & & & \psi^* & & & \\
0 & \to & \text{ker}(T) & \to & \Omega^1_{X,p} & \xrightarrow{T} & m_p & \to & 0 \\
& & & \psi^* & & & \psi^* & & & \\
0 & \to & \text{ker}(T_o) & \to & \Omega^1_{n,o} & \xrightarrow{T_o} & m_o & \to & 0.
\end{array}
\]

we get from Equation (2.12) both \( T \) and \( T_o \) are surjective. By the inductive hypothesis, the first column \( \psi^* \) is isomorphism. Hence we know \( \psi^* \) on the second column is surjective. Therefore,

\[
\psi^* : \frac{m^i_p}{m^i_p} \otimes \frac{m_p}{m_p} \to \frac{m^i_i}{m^i_i} \otimes \frac{m_i}{m_i}
\]

is also surjective. This completes the induction and hence the proof of the lemma. \( \square \)

Lemma 2.14. The pullback map \( \psi^* : m_p \to m_o \) is an isomorphism.

Proof. Since \( \psi^* : m_o \to m_p \) is a prior injective as we explained above, we only need to show it is surjective. Note that in Equation (2.12), both \( T \) and \( T_o \) are surjective. Moreover, \( \psi^* \) on the middle column is surjective by Lemma 2.13. Therefore, \( \psi^* \) on the third column is surjective. This completes the proof of the lemma. \( \square \)

Remark 2.15. Harvey and Lawson applied to their techniques to the flow given by a \( \mathbb{C}^* \)-action on a compact Kähler manifold to obtain a complex graph \( T \). Moreover, they obtained that \( T \) is of finite volume, \( T \) are of an analytic subvariety (see [So]).

Theorem 2.16. If \( X \) is a nonsingular irreducible projective variety admitting a \( \mathbb{C} \)-action with nonempty isolated fixed point, then \( X \) is rational.

Proof. Since \( \psi^* : \mathcal{O}_{\mathbb{C}^n,p} \to \mathcal{O}_{X,o} \) is an injective \( \mathbb{C} \)-algebraic homomorphism by Lemma 2.7, by Lemma 2.13, \( \psi^* : m_p \to m_o \) is surjective. Since \( \psi^* \) maps constants in \( \mathcal{O}_{X,p} \) exactly onto constants in \( \mathcal{O}_{\mathbb{C}^n,o} \), we obtain that \( \psi^* : \mathcal{O}_{X,p} \to \mathcal{O}_{\mathbb{C}^n,o} \) is also surjective. Hence \( \psi^* : \mathcal{O}_{X,p} \to \mathcal{O}_{\mathbb{C}^n,o} \) is an isomorphism. Since the rational function field \( K(X) \) (resp. \( K(\mathbb{P}^n) \)) of \( X \) (resp. \( \mathbb{C}^n \)) is the fractional field of \( \mathcal{O}_{X,p} \) (resp. \( \mathcal{O}_{\mathbb{C}^n,o} \)), \( \psi^* : K(X) \to K(\mathbb{C}^n) \) is an isomorphism as a field. This implies that \( X \) is birational to \( \mathbb{C}^n \) (and hence \( \mathbb{P}^n \)) (see [Ha]) and hence \( X \) is rational. This completes the proof of the theorem. \( \square \)
Remark 2.17. Since the condition of regularity of $X$ is a key part in the proof of Theorem 2.16 and in the processing to construct equivariant projections, it obviously does not work for varieties with singularity. Moreover, cones over nonsingular projective varieties are counterexamples which prevent generalizing Theorem 2.16 to the singular case.

3. Uniform Upper bound of Euler numbers and Betti numbers

In this section, let $X$ be a nonsingular complex projective variety of dimension $n$ admitting a $\mathbb{C}$-action with exactly one fixed point. Recall that

$$I - P = d \circ T + T \circ d : \mathcal{E}^k(X) \to D^k(X)$$

For $\alpha \in \mathcal{E}^k(X)$ such that $d\alpha = 0$, we have

$$\alpha - P\alpha = dT\alpha.$$ 

Since $P\alpha = \lim_{t \to \infty} \phi_t^\ast(\alpha)$, the value of $P\alpha$ at $x \in X$ is given by

$$(P\alpha)(x) = \lim_{t \to \infty} \phi_t^\ast(\alpha)(x) = \alpha \circ \phi_t(x) = \alpha(\lim_{t \to \infty} \phi_t(x)) = \alpha(p).$$

That is to say, $P\alpha$ is a constant value form on $X$. Note that the type of $\alpha$ is preserved under the projector operator $P$ from its definition, we get $P\alpha$ is a constant $(p, q)$-form is a closed form of $(p, q)$-type. Recall that Carrell and Lieberman showed that $H^{i,j}(X) = 0$ if $i \neq j$ (see [CL]), there is no constant $(i, j)$-form on $X$ if $i \neq j$. For $i = j$, a constant $(i, i)$-form is the $C$-linear combination of $dz_{n_1} \wedge ... \wedge dz_{n_1} \wedge dz_{n_1} \wedge ... \wedge dz_{n_1}$. The number of such form is $(\alpha)^n$. Hence we get the following upper bound of Hodge numbers $h^{i,j}(X) = \dim H^{i,j}(X)$, Betti numbers $b^i(X)$ and the Euler number $\chi(X)$ of such a nonsingular complex projective variety $X$.

Theorem 3.3. Let $X$ be a nonsingular complex projective variety of $\dim X = n$ admitting a $\mathbb{C}$-action with exactly one fixed point. Then $b^i(X) = h^{i,i}(X) \leq \binom{n}{i}$ and the Euler number $\chi(X)$ of $X$ is upper bounded by $2^n$.

Proof. Note that $b^{n-1}(X) = 0$ and $\dim b^i(X) = \dim H^{i,i}(X)$, which follows from the Hodge decomposition Theorem and Carrell-Lieberman’s theorem ([CL]) for $X$. Moreover, we have $\dim H^{i,i}(X) \leq \binom{n}{i}$ by Lemma 3.4 below.

Then we have

$$\chi(X) = \sum_{i=0}^{n} b^i(X) \leq \sum_{i=0}^{n} \binom{n}{i} = 2^n.$$ 

□

Lemma 3.4. Let $X$ be a nonsingular complex projective variety of $\dim X = n$ admitting a $\mathbb{C}$-action with exactly one fixed point. Then $b^i(X) \leq \binom{n}{i}$.

Proof. Since $b^2(X) = h^{1,1}(X)$, we only need to consider the closed $(1, 1)$-form. Let $\alpha$ be a closed $(1, 1)$-form. Then we have $\alpha$ is homologous to $P\alpha$, which is a $(1, 1)$-form of constant value by Equation (3.2). Since a $(1, 1)$-form $P\alpha$ is a $\mathbb{C}$-linear span of $dz_i \wedge dz_j$. A real $(1, 1)$-form is spanned by $dz_i \wedge dz_j + dz_j \wedge dz_i = d(z_i + z_j) \wedge d(\bar{z}_i + \bar{z}_j) - dz_i \wedge dz_i - dz_j \wedge dz_j$ and an imaginary $(1, 1)$-form is spanned by $dz_i \wedge dz_j - dz_j \wedge dz_i = i\sqrt{-1} \{d(z_i + \sqrt{-1}z_j) \wedge d(\bar{z}_i - \sqrt{-1}z_j) - d(z_i - \sqrt{-1}z_j) \wedge d(\bar{z}_i + \sqrt{-1}z_j)\}$. Hence the class of $\alpha$ is spanned by the form $\xi \wedge \bar{\xi}$, where $\xi$ is a constant 1-form. The space spanned constant 1-forms on $X$ is of dimension at most $n$ since $\dim X = n$. The method works for $i > 2$ to get $b^i(X) \leq \binom{n}{i}$ by induction.

□

Remark 3.5. Lemma 3.4 is a generalization to the “unexpected consequence” of Akyildiz-Carrell on the upper bound of the second Betti number (see [AC Th2]). Note that the condition is much weaker than that in Akyildiz-Carrell’s result.

Remark 3.6. This bound is the optimal upper bound for $b^2(-)$ since $X = (\mathbb{P}^1)^n$ admits a $\mathbb{C}$-action with exact one fixed point and $b^2((\mathbb{P}^1)^n) = \binom{n}{i}$.
Remark 3.7. Since a connected nonsingular projective variety $X$ admits $\mathbb{C}$-action with exact one fixed point is equivalent to that $X$ admits a holomorphic vector field with exact one zero locus by using a result of Bialynicki-Birula [12,11], the condition in Theorem 3.6 can be replaced to be that $X$ is a nonsingular complex projective variety of $\dim X = n$ admitting a holomorphic vector field with exactly one zero locus.

Remark 3.8. Let $X$ admit a holomorphic vector field $V$ with exact one zero locus. Since the upper of Euler number of $X$ can be replaced to be that the exact one fixed point is equivalent to that $\tilde{\chi}(\tilde{X})$ increases in the process but $\dim \tilde{X} = \dim X$. Hence the zero locus of the induced holomorphic vector field on $\tilde{X}$ will not be isolated. Hence the holomorphic vector field $V$ is never “generic” in the sense of Lieberman [Li1].

Proposition 3.9. Let $Y$ admit a holomorphic vector field $V$ with exact one zero locus. Then $Y$ is rational minimal variety in the sense that $Y$ is not the blow up of any smooth projective variety along a codimension at least two subvarieties.

Proof. Assume that $Y = \tilde{X}$, where $\sigma : \tilde{X} \to X$ is a blow up of a smooth projective variety $X$ along a smooth projective subvariety of codimension at least two. Note that a blow up $\sigma : \tilde{X} \to X$ is always equivariant (see [12,2]). The exceptional divisor $E$ is $\mathbb{C}$-invariant since its $\tilde{X} - E \cong X - pt$ is invariant. If there exists a point on $E$ is not fixed point, then the fixed point set must be of codimension bigger than or equal to 2.

Let $O_1, O_2$ be two different orbits in $X$ such that the tangent vectors along the orbits are different. The proper transform $\tilde{O}_1, \tilde{O}_2$ of two different orbits $O_1, O_2$ in $X$ are $\mathbb{C}$-invariant and hence there is a fixed point on each of $\tilde{O}_1, \tilde{O}_2$. By the property of the blow up construction, we know $\tilde{O}_1 \cap \tilde{O}_2 = \emptyset$. Then $\tilde{X}$ have at least two points, contradicts to the assumption that $Y = \tilde{X}$ has exact one fixed point.

4. Surfaces admits an additive group action with exact one fixed point

It is easy to see that $\mathbb{P}^1$ is the only non-singular complex projective curve admitting a holomorphic vector field with exact one zero point. In the section, we give the classification of complex projective surfaces admitting a holomorphic vector field with exact one zero point. We will show that $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are only non-singular complex projective surfaces with this property. Let $X$ be a nonsingular complex projective surface. As we mentioned above, a nonsingular projective surface $X$ admitting a holomorphic vector field with exact one zero point is equivalent to it admits a $\mathbb{C}$-action with exact one fixed point.

By Theorem 3.5 if $\dim X = 2$, then $\chi(X) \leq 4$ and $b_2(X) \leq 2$. Since a prior $b_2(X) \geq 1$. Topologically, there are at most two types of $X$. Both $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ admit a $\mathbb{C}$-action with exact one fixed point. They correspond to $b_2(X) = 1$ and $b_2(X) = 2$.

Proposition 4.1. Let $\phi : \mathbb{C} \times \Sigma_n \to \Sigma_n$ be a $\mathbb{C}$-action over $\Sigma_n$, where $\Sigma_n$ is the Hirzebruch surface $\Sigma_n := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$. If $n \geq 1$, then the fixed point set $X^\mathbb{C}$ cannot be exactly one point set.

Proof. Let $\pi : \Sigma_n \to \mathbb{P}^1$ be the natural projection from the Hirzebruch surface $\Sigma_n$ to $\mathbb{P}^1$. Then the fiber of $\pi$ is also isomorphic to $\mathbb{P}^1$. Let $\phi_t := \phi(t, -) : X \to X$ be the flow given by the action. Then $\phi$ is fiber-preserving (cf. [31, p.267]). Since $n \geq 1$, the zero section $C$ of $\pi$ has negative self-intersection $C^2 = -n < 0$ (cf. [45, Ch. IV]). Therefore, it is $\mathbb{C}$-invariant. Let $p \in C$ be a fixed point of the induced $\mathbb{C}$-action on $C$. Then the fiber $\pi^{-1}(p)$ is $\mathbb{C}$-invariant since the action is fiber-preserving.

Suppose that $X^\mathbb{C}$ contains one point, then $\lim_{t \to \infty} \phi_t(\pi^{-1}(q)) = p$. Note that $\pi^{-1}(q)$ is a fiber, $\phi_t(\pi^{-1}(q))$ is also a fiber for $t \in \mathbb{C}$ since $\phi_t$ is fiber-preserving. Hence $\phi_t(\pi^{-1}(p)) = \pi^{-1}(p)$ for all $t \in \mathbb{C}$.
Let $F$ be a fiber such that $p \notin F$. Since $F \cong \mathbb{P}^1$, we denote that isomorphism by $i : \mathbb{P}^1 \cong F$. Let $\mathbb{C} \times \mathbb{P}^1 \to \Sigma_n$ be defined by

$$\Phi : (t, q) \mapsto \psi_t(q).$$

This map is an injective morphism since $\psi_t$ is a $\mathbb{C}$-action on $\Sigma_n$.

The map $\Phi$ is extended to $\mathbb{P}^1 \times \mathbb{P}^1$ by $\Phi(\infty, q) := \lim_{t \to \infty} \psi_t(q) = p$. Then we get a morphism $\widetilde{\Phi} : \mathbb{P}^1 \times \mathbb{P}^1 \to \Sigma_n$. This is impossible except for the case that $n = 0$ since there does not exist a birational morphism from $\mathbb{P}^1 \times \mathbb{P}^1$ to a Hirzebruch surface $\Sigma_n$ for $n > 0$.

The last statement follows from the fact that their corresponding Betti numbers are the same.

\[ \square \]

**Example 4.2.** In Proposition \[4.1\], $n \geq 1$ is necessary. When $n = 0$, then $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Let $\mathbb{C} \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ be the $\mathbb{C}$-action given by $(t, [u : v], [x : y]) \mapsto ([u + tv : v], [x + ty : y])$.

One can check that the fixed point of this action is $([1 : 0], [1 : 0])$.

**Example 4.3.** It is well-known that there exists a $\mathbb{C}$-action $\mathbb{C} \times \mathbb{P}^2 \to \mathbb{P}^2$ on $\mathbb{P}^2$ given by $(t, [x : y : z]) \mapsto [x + ty + \frac{1}{2}tz^2 : y + tz : z]$ whose fixed point is $[1 : 0 : 0]$.

Therefore, the only nonsingular complex projective surfaces admitting a $\mathbb{C}$-action with exact one fixed point are $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. In summary, we have the following result:

**Theorem 4.4.** Let $X$ be a nonsingular complex projective surface admitting a holomorphic vector field with exact one zero locus. Then $X$ is either isomorphic to $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$.

Clearly, both $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ also admit a $\mathbb{C}^*$-action with isolated fixed points. Hence we also show that the strong version Carrell conjecture in dimension two.

**Corollary 4.5.** Let $X$ be a nonsingular complex projective surface admitting a holomorphic vector field with exact one zero locus. Then $X$ admits a $\mathbb{C}^*$-action with isolated fixed points.

**Remark 4.6.** In Theorem \[4.4\] “nonsingular complex projective surface” can be replaced to be “compact Kähler surface” since such a compact Kähler surface is algebraic (cf. \[AC\]).

5. **Threefolds admits an additive group action with exact one fixed point**

It will be more complicated to classify higher dimensional nonsingular complex projective varieties admitting a $\mathbb{C}$-action with exact one fixed point. For example, there exist rational Fano threefolds admitting $\mathbb{C}$-action with exact one fixed point but not isomorphic to the product of projective spaces. Examples can be found in \[AC\] and references therein. The classification in three dimensional case is closely related to the classification of minimal rational threefolds.

According to Theorem \[4.3\] we see that $b_2(X) \leq 3$ if $X$ is a threefold admits a $\mathbb{C}$-action with exact one fixed point. Equivalently, the Picard number of $X$ is less than or equal to 3.

The following result is about the classification of such an $X$ with $\text{Pic}(X) = 1$.

**Theorem 5.1.** Let $X$ be a nonsingular complex projective threefold admitting a $\mathbb{C}$-action with exact one fixed point. If $\text{Pic}(X) = 1$, then $X$ is isomorphic to one of the following varieties:

1. $\mathbb{P}^3$;
2. A smooth quadric in $\mathbb{P}^4$;
3. A section of the Grassmannian $G(2, 5) \subset \mathbb{P}^9$ by subspace of codimension 3;
4. A Fano threefold $X$ in $\mathbb{P}^{13}$ with $-K_X^3 = 22$, $b_2(X) = 0$ and $g(X) = 12$. 
Proof. First of all, any variety on the list admits a $\mathbb{C}$-action with exact one fixed point. The obvious construction similar to Example 4.3 gives $\mathbb{C}$-action on $X$ such that $X^\mathbb{C}$ is exact one point in Case 1 to Case 3. The fact that the varieties in Case 4 to Case 6 admits a $\mathbb{C}$-action with exact one fixed point can be found in [AC], [K] and the references therein.

Now we will show that only the varieties in the list admits $\mathbb{C}$-action with exact one fixed point. According the classification of smooth Fano threefolds of Picard number one, there are exact four threefolds with $h^{1,2}(X) = 0$ ([PS, p.215]). The following threefolds are in the list:

1. $\mathbb{P}^3$;
2. A smooth quadric in $\mathbb{P}^4$;
3. A section of the Grassmannian $G(2,5) \subset \mathbb{P}^9$ by subspace of codimension 3;
4. A Fano threefold $X$ in $\mathbb{P}^{13}$ with $-K_X^3 = 22$, $b_3(X) = 0$ and $g(X) = 12$.

□

Remark 5.2. Let $X$ be the threefold admitting a $\mathbb{C}$-action with exact one fixed point. We guest that if $\text{Pic}(X) = 2$, then $X$ is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^1$, while if $\text{Pic}(X) = 3$, then $X$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. This could be deduced from the classification of smooth minimal rational threefolds.

If a nonsingular projective variety $X$ admits a $\mathbb{C}$-action with exact one fixed point, then $X$ admits a holomorphic vector field with exact one zero locus. The inverse is also true. This follows from Lieberman that a holomorphic vector field on $X$ generating a $\mathbb{C}^\times$-action of the automorphic group $\text{Aut}(X)$ is the product of $\mathbb{C}^\times$ and at most a copy of $\mathbb{C}$. The fixed point set of a smooth projective variety $X$ with an action of $\mathbb{C}^\times$ has at least two disconnected components. Therefore, Theorem 5.1 can be restated as follows.

Theorem 5.3. Let $X$ be a nonsingular complex projective threefold admitting a holomorphic vector field with exact one zero locus. If $\text{Pic}(X) = 1$, then $X$ is isomorphic to one of the following varieties:

1. $\mathbb{P}^3$;
2. A smooth quadric in $\mathbb{P}^4$;
3. A section of the Grassmannian $G(2,5) \subset \mathbb{P}^9$ by subspace of codimension 3;
4. A Fano threefold $X$ in $\mathbb{P}^{13}$ with $-K_X^3 = 22$, $b_3(X) = 0$ and $g(X) = 12$.

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