Desynchronizing Networks Using Phase Resetting

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Abstract

Understanding complex systems which exhibit desynchronization as an emergent property should have important implications, particularly in treating neurological disorders and designing efficient communication networks. Here we demonstrate how, using a system similar to the pulse coupling used to model firefly interactions, phase desynchronization can be achieved in pulse coupled oscillator systems, for a variety of network architectures, with symmetric and non symmetric internal oscillator frequencies and with both instantaneous and time delayed coupling.

Keywords: Desynchronization, Phase resetting, Pulse Coupling, Oscillator Dynamics

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1. Introduction

Synchronization, both full and partial, has been observed and studied in a wide variety of systems and is known to be an essential feature of the collective dynamics of interacting systems [1, 2, 3, 4, 5]. Desynchronization is likewise of considerable interest. This is the case particularly in neuroscience where it has implications for the treatment of Parkinson’s disease [6, 7] and other neurological disorders such as epilepsy [8, 9, 10]. Consequently, research has focussed primarily on breaking synchronization in neural systems via a variety of methods [11, 12, 13, 14, 15, 16] or investigating phase resetting.

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cluster splitting and the stability of cluster dynamics in general \cite{17, 18, 19, 20, 21, 22}. Desynchronization is also a useful concept in designing asynchronous communication systems \cite{23, 24, 25, 26}.

Arguably, desynchronization is a ubiquitous naturally occurring emergent property. Here we demonstrate a general mechanism, similar to that found in the pulse coupling model of synchronization in firefly interactions, which generates desynchronization in a variety of coupled oscillator network architectures, having either uniform or non-uniform distributions of frequencies and either instantaneous or time-delayed coupling. We use an adaptation of the Mirollo-Strogatz model \cite{27}, which is itself an adaptation of the Peskin \cite{28} model for synchronization in heart pacemaker cells. (See also \cite{29, 30, 31, 32, 33} for investigations of coupled systems with time delays.)

2. The Basic Model

Consider a network of $N$ coupled oscillators as an undirected graph, $(V, E)$, where the vertices, $V$, represent individual oscillators and the edges, $E$, the network connections between them. The state of each vertex $v_i \in V$, where $i \in (1, \ldots, N)$ is described by a phase $x_i \in S^1$. In this basic model, it will be assumed that the internal oscillator frequencies are identical and that the coupling is instantaneous. This model will later be extended to cover variable frequencies and networks with time delays. In a manner similar to \cite{27} coupling is through phase resetting. Whenever the phase of an individual oscillator passes through $x_i = 1$ it is reset to $x_i = 0$ ($0 \sim 1$ in $S^1$) and each oscillator connected directly to this oscillator has its phase reset according to a smooth function $f : S^1 \rightarrow S^1$ where $Df(x) > 0$ for all $x \in S^1$. For desynchronization to occur, $f$ will be a nonlinear function having the asynchronous state as a stable (under iteration) fixed point with all other fixed points unstable. For instance

$$f(x) = \frac{1}{4} \left( \ln(1 + (e^2 - 1)x) - \ln(1 + (e^{-2} - 1)x) \right),$$

(1)

(inspired by the rise function of the Mirollo-Strogatz model) has an unstable fixed point at 0 (the synchronous state) and a stable fixed point at 0.5 (see figure 1).

For a network consisting of oscillators with identical frequencies and instantaneous coupling, steady-state behaviour will be determined by the network topology and the form of $f$. In the simplest network, consisting of two
Figure 1: Interaction function $f$ (Equation 1) has a stable fixed point at $x = 0.5$ and an unstable fixed point at $x = 0$.

coupled oscillators with phases $(x_1, x_2)$, the dynamics is a flow on a 2-torus (seen as the unit square with the opposing edges identified). Since the oscillators advance at the same rate, the flow is parallel to the diagonal $x_1 = x_2$. The identification of the opposite edges means that trajectories leaving the right and top edges of the square reappear at the left and bottom edges respectively (equivalent to resetting $x_i = 1$ to $x_i = 0$). Function evaluations $f(x)$ are performed at this time. The form of $f$ yields a repelling closed orbit $x_1 = x_2$ (as the gradient $f'(1) > 1$) and an attracting closed orbit $x_1 = x_2 - 0.5$, since $0 < f'(0.5) < 1$.

Considering the dynamics in the rotating $x_1$ frame, the 2−dimensional system may be reduced to a 1−dimensional map describing the phase difference, $d = x_1 - x_2$. We observe that the phase rotation is an isometry which swaps the upper and right edges of the unit square, where the function evaluations occur. The phase difference after each rotation of $x_1$ is given by a single discontinuous function, $F$, which is the composition of the phase rotation and $f$ (see Figure 2).

$$F(d) = \begin{cases} 
  f(d + 1) & -1 < d < 0 \\
  0 & d = 0 \\
  -f(1 - d) & 0 < d \leq 1
\end{cases}$$

(2)

$F(d)$ has a globally attracting period-2 orbit $d = \pm 0.5$ (the desynchronized state $x_1 = x_2 - 0.5$). The other (isolated) fixed point ($d = 0$) is unstable. For the two oscillator model, therefore, the asymptotic dynamics are a globally stable limit cycle at $x_1 = x_2 - 0.5$; apart from initial synchronization, all initial conditions result in phase desynchronization whereby the difference between the phases of the oscillators is maximal.
Figure 2: Discrete map function for two oscillator model describing phase difference \( d = x_1 - x_2 \) in the rotating \( x_1 \) frame. This represents the second return map for the flow and function evaluations for each phase rotation of \( x_1 \).

For a system of \( N \) globally coupled oscillators—with phases \( x = (x_1, \ldots, x_N) \)—the dynamics can be represented as flow in an \( N \)-torus, seen as an \( N \)-cube with opposing faces identified. The flow is parallel to the \( x_1 = \ldots = x_N \) diagonal and function evaluations occur on the exiting faces. As with the two oscillator model, we determine the steady state behavior by considering a discrete map equivalent of the dynamics constructed from the phase differences in the rotating \( x_1 \) frame i.e. \( d_i = x_1 - x_{i+1} \). The transformation from \( x \) to \( d \) is accomplished by a \( (N - 1) \times N \)-matrix \( M = (e : -1_{N-1}) \) where \( e \) is a column vector with all components equal to unity and \( 1_{N-1} \) is the \( (N-1) \times (N-1) \) unit matrix. Between transitions each component of \( d \) is conserved because the flow is parallel to the diagonal.

If the components of \( x \) are ordered so that \( x_1 > \ldots > x_N \), the first transition occurs when \( x_1 = 1 \) and the effect of the functional mapping is \( d \mapsto G(d) \) where \( G_k(d) = -f(1 - d_k) \). With all-to-all coupling (since \( x_i > x_{i+1} \) implies \( f(x_i) > f(x_{i+1}) \)), following the transition the cyclic permutation \( P_*(x_1, x_2, \ldots, x_N) = (x_2, x_3, \ldots, x_1) \) restores the assumed ordering of the components of \( x \). Given a general permutation \( P \), we would like to find a corresponding transformation \( T \) on \( d \), where \( TM = MP \). Since \( MP \) is a matrix consisting of the permuted columns of \( M \), we write \( MP = (v : M') \) where \( v \) is the column coming first in the permutation, and \( M' \) is the rest of the matrix. Now \( TM = T(e : -1_{N-1}) = (Te : -T) \) which implies that \( T = -M' \). For this to be a solution, it must be true that \( v = -M'e \). This follows from the fact that the row sums of \( M \) are all zero. The transformations \( \{T\} \) inherit the group structure of the set of permutation matrices \( \{P\} \). In particular, if \( \{P\} = \langle P_* \rangle \), the cyclic group of order \( N \) generated by \( P_* \), then the group of transformations, \( \{T\} \), is isomorphic, consisting of the powers of
$T_*$ where $T_*M = MP_*$. Given any initial condition which is consistent with the initial ordering of the phases we can evolve $d$ in time

$$d(t + k) = T_*^{1-k} \circ G \circ (T_* \circ G)^{k-1}(d(t)).$$

When $k = N$, the order of the cyclic group, this reduces to a simple iteration

$$d(t + N) = (T_* \circ G)^N (d(t)).$$

![Diagram](image)

Figure 3: The projection of the 3-cube along its principle diagonal $x_1 = x_2 = x_3$. The plane is parameterised by $d = (x_1 - x_2, x_1 - x_3)$. Points on different faces of the cube with the same vector $d$ are identified by the projection. For example, the shaded region $A$, when interpreted as being in the bottom face of the cube is the set $\{x_1 > x_2 > x_3 = 0\}$; interpreted as the top face it is the set $\{1 = x_1 > x_2 > x_3\}$.

The three oscillator all-to-all network is illustrated in Figure 3 which shows a projection of the 3-cube along its principle diagonal $x_1 = x_2 = x_3$. In this projection the flow reduces to a field of fixed points. Points on different faces of the cube which correspond to the same vector $d$ are identified by the projection. The shaded region $A$, when interpreted as being in the bottom face of the cube is the set $x_1 > x_2 > x_3 = 0$. The flow identifies this with the top face $1 = x_1 > x_2 > x_3$ interpretation of $A$. Resetting $x_1$ to zero and applying $f$ to the other two phases maps this region onto $B$ (interpreted as the back face). The same argument now shows that $B$ maps onto $C$ and thus back to $A$. Region $A$ is therefore invariant under the action of
The closure of $A$ contains synchronized or partially synchronized states in its boundary. However, the choice of $f$ ensures that these are all unstable. Numerical work suggests that given this form of $f$, the interior of $A$ contains a unique attracting fixed point. That is, we find a unique attracting orbit which visits the regions $A$, $B$ and $C$ cyclically. The unshaded regions contain a second attracting orbit which is generated by assuming a different initial ordering of the phases, $x_2 > x_1 > x_3$ for example. In the general $N$ clock setting, assuming the unique attractor suggested by the numerics, the group theoretical underpinning of this system allows us to count the number of distinct orbits corresponding to desynchronized states.

The order of the symmetric group consisting of all permutations of $N$ objects is $N!$ and, by Lagrange's theorem, the number of permutations which are not mapped to each other when acted upon by $< P_* >$ is $(N-1)!$. For each of these initial permutations there is an attracting desynchronized orbit and a repelling synchronized orbit (excluding symmetry).

3. Systems with Non-identical Frequencies

The heart of the previous analysis is that the behaviour of the $N$-oscillator system is based on a cyclic permutation of the initial ordering of the phases. We might suppose that this property is robust to a sufficiently small variation in the frequencies of the oscillators. Consider a system of two oscillators with frequencies $\omega_1$ and $\omega_2$. We can demonstrate graphically that if $\rho = \frac{\omega_1}{\omega_2}$ is less than some critical value $\rho_c$ then limit cycles as described above will occur. Figure 4 demonstrates that if a stable limit cycle exists outside a region determined by the ratio of the oscillators' frequencies then the dynamics will pass through alternate faces of the torus in succession. As the change in flow does not affect the existence of the globally stable limit cycle (as the interaction function $f$ has not changed), it is only required that the limit cycle lie within this region. It is straightforward to derive the orbit of the limit cycle to be $x_i = f(\rho(1 - f(\rho(1 - x_i))))$ for all $x_i$. For $f$ given in equation 1 the limit cycle is within the ‘non-overtaking’ region (see figure 4) for $\rho \leq 1.11$. This value increases as the gradient of equation 1 becomes more pronounced.

Using a similar argument, it can be shown that for any network it is sufficient that the ratio of any two connected oscillators’ frequencies be less than $\rho_c$ for desynchronization to occur. Again, the value of $\rho_c$ is determined by the interaction function $f$. For networks which are not globally connected, we
interaction function case, desynchronization will occur. The size of the hashed region is determined by the interaction function $f(x)$ and the angle of flow.

If the difference in their phases $x_1 = x_2$, the oscillators will alternately pass through $x = 1$, if the limit cycle is outside the hashed region. In this case, desynchronization will occur. The size of the hashed region is determined by the interaction function $f(x)$ and the angle of flow.

can, via an argument identical to that used above, demonstrate that any two connected oscillators will desynchronize. However some consideration must be given to the reducibility of such networks via symmetry arguments (see [34]). For non-globally connected networks, phase coupling of the form proposed will result in local asymptotic desynchronization across the network i.e. each oscillator will desynchronize with each of those to which it is connected.

4. Time Delays

The final modeling assumption that will be relaxed concerns time delays across the network. When considering desynchronization using the previous coupling function (equation 1) the time delay can reverse the stability of the fixed point at $x = 0$. Consider two oscillators separated by a time delay of $\tau$. If the difference in their phases $x_1 - x_2 < \tau$, when the first oscillator sends a pulse on crossing $x = 1$, the second oscillator passes $x = 1$ before receiving the pulse. In this case the second oscillator is perturbed closer to the first and with each cycle the oscillators move closer together.

This difficulty can be overcome by redesigning the interaction function $f$. From the argument described above synchronization will only occur (using a continuous interaction function) if the oscillators' frequencies differ by less than the propagation delay between them. It is required, therefore, that should this occur, the perturbed oscillator (the one receiving the data) should not be perturbed closer to the transmitting oscillator’s time. We can derive a new interaction function which has this property and still retains a stable fixed point at $d = 0.5$ using a shifted cubic curve. Interpolating through the fixed point at $d = 0.5$, the curve can be expressed as follows (see also Figure 4).

Figure 4: When the flow is not parallel to the diagonal $x_1 = x_2$, the oscillators will alternately pass through $x = 1$, if the limit cycle is outside the hashed region. In this case, desynchronization will occur. The size of the hashed region is determined by the interaction function $f(x)$ and the angle of flow.
if $\tau$ is the maximum propagation delay, and $\beta \in (0, 1)$ the gradient at the fixed point:

$$f(x) = a(\tau, \beta)x^3 + b(\tau, \beta)x^2 + c(\tau, \beta)x - \tau \mod 1.$$  \hspace{1cm} (3)

This function is a continuous mapping of the circle $f : S^1 \rightarrow S^1$ but appears discontinuous in the interval $x \in [0, 1)$.

![Figure 5: Discontinuous interaction functions with a stable fixed point at $x = 0.5$ and no fixed point at $x = 0, 1$. Left figure $\tau = 0.1$ and $\beta = 0.5$ and right $\tau = 0.05$ and $\beta = 0.25$.](image)

There are important factors within the transient dynamics, which may have some impact on the functioning of such time delayed systems. For instance, the existence of transient chaos cannot, at this stage, be excluded nor other dynamical effects present in systems of interacting oscillators and particularly the presence of unstable attractors within the dynamics needs to be considered [35]. However, the above arguments can be applied to suggest that the only asymptotically stable dynamics of such systems would be the desynchronized state, where each oscillator pulses in turn.

5. Simulations

The model has been simulated for a variety of networks, both homogeneous and inhomogeneous, using identical internal frequencies, distributed internal frequencies and for networks with small, uniform time delays. In all cases local asymptotic desynchronization was observed, in accordance with the above (see Figure 5). For time delayed systems the duration of transient behaviour grows considerably as the number of oscillators is increased and it is possible to observe clustering if a suitable choice of the interaction function is not made, however, the long term dynamics appear in all cases to converge to the desynchronized limit cycle previously described.
Figure 6: Time series for simulations of 5 globally coupled oscillators (top) and the associated time series for the order parameter $P$ (bottom) with near synchronous initial conditions: (a) identical oscillator frequencies period = 1, no time delays; (b) normally distributed frequencies (mean period = 1 standard deviation = 0.05); (c) identical oscillator frequencies, uniform time delay $\tau = 0.01$; (d) normally distributed frequencies (mean period = 1 standard deviation = 0.05) uniform time delay $\tau = 0.01$.

Figure 6 also shows the time series for an order parameter $P$ which gives a measure of the total coherence in the network [36, 37].

$$P = \frac{1}{N} \left| \sum_{k=1}^{N} e^{2\pi i x_k} \right|,$$

where $P = 1$ corresponds to synchronous oscillation and for any other state $0 \leq P < 1$. As can be observed, with initial conditions near the ordered synchronous state, the oscillator dynamics rapidly desynchronize and the order of each oscillator as it is distributed around the phase remains unchanged.
6. Summary

The analysis as presented here demonstrates how, via pulse coupling, a network of connected oscillators may be forced to achieve phase desynchronization as a collective dynamic. The model is intended to demonstrate ‘proof of principle’ of the design of an emergent property. We conjecture that the method of pulse coupling applied here would be equally applicable to weakly coupled oscillators exhibiting synchronization. The applications of such a concept may be far reaching, particularly when applied to digital communication systems, the design of neural based computers and in the treatment of Parkinson’s disease and epilepsy. Acknowledgements The authors would like to thank J. Shapiro, M. Sorea, S. Furber and L.O.Gowrie for their advice, discussions and encouragement. This work was sponsored by EPSRC grant GR/T11258/01.

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