Figure 1: Asymptotic scaling for lattice gauge theory in connection with a fundamental length quantum gravity theory (to be verified or falsified by computer experiment).
Figure 2: Part of a 2d random lattice. Case (1) emphasizes a triangle whose area is entirely associated with the $V_h$ of its boundary sites, whereas case (2) exhibits an area $V_h$ which intrudes into a next nearest neighbour triangle.
Figure 3: Two-dimensional illustration of the barycentric definition of $V_h$. 
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Asymptotic Freedom and Euclidean Quantum Gravity *

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Abstract

Pure SU(2) gauge theory is the simplest asymptotically free theory in four dimensions. To investigate Euclidean quantum gravity effects in a fundamental length scenario, we simulate $4d$ SU(2) lattice gauge theory on a dynamically coupled Regge skeleton. The fluctuations of the skeleton are governed by the standard Regge-Einstein action. From a small $2 \cdot 4^3$ lattice we report exploratory numerical results, limited to a region of strong gravity where the Planck mass and hadronic masses take similar orders of magnitude. We find a range of the Planck mass where stable bulk expectation values are obtained which vary smoothly with the gauge coupling, and a remnant of the QCD deconfining phase transition is located.

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1. Introduction

Quantization of gravity is the most fundamental unsolved problem in modern physics. Obviously, nature manages to combine gravity and the quantum field theories of strong, weak and electromagnetic interactions consistently, whereas all our theoretical approaches reveal serious inconsistencies at one or the other level. Despite the fundamental character of this problem we are in the unfortunate situation that there are no clear-cut quantum gravity related experimental facts. Theorists can do little more than exploring all promising branches from a tree of alternatives in the hope that arguments of consistency and beauty may lead to the correct final answer. At the first look, a fairly hopeless attempt in view of the many logical alternatives which all seem to be worthwhile to explore. On the other hand, by now the number of theorists all around the world is also large and, maybe, if things are done in some kind of systematic manner one finally stumbles into the correct branch and, hopefully, realizes it. Once a consistent, acceptable theory exists some previously unrealized experimental evidence may also pop out. From this point of view the various approaches like Supersymmetry, String Theory, etc. all all have their own right to be pursued. Less ambitiously, instead of constructing immediately a complete theory, one may first study problems which arise when one tries to quantize the classical Einstein action

\[ \hat{S}_E = m_P^2 \int \sqrt{-g} \, d^4 x \, R. \]

Here \( m_P = 0.17 \cdot 10^{19} \text{ GeV} \) is the Planck mass and \( R \) is the scalar Riemann curvature. Conveniently the Euclidean action, after rotation in the complex plane \([1]\)

\[ S_E = m_P^2 \int \sqrt{g} \, d^4 x \, R, \quad (1) \]

is used to study a variety of problems*.

With our sign convention the Boltzmann factor reads \( \exp(+S_E) \). The conventional wisdom is that the action (1) describes self-interacting spin two massless particles, but its perturbative quantization runs into the well-known trouble that it is non-renormalizable \([2]\) and unbounded \([1]\). Therefore, terms quadratic

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* In contrast to quantum field theory the usefulness of the Euclidean rotation for quantum gravity remains hypothetical at the present state of affairs.
in the Riemann tensor have been introduced which allow reformulation of gravity as an
asymptotically free field theory. However, this theory has problems with unitarity, for
references see [3].

On the other hand, the possibility exists that we are confusing shortcomings of the
perturbative technique with shortcomings of the theory. A non-perturbative approach
would clearly be favourable. To pursue this simulations are presently the only promising
technique at hand. Unfortunately, realistic quantum gravity related problems are computa-
tionally very intensive. Even the rapid advances of modern computer technology have just
barely brought us to the point where some exploratory studies become feasible. Quantum
gravity simulation are clearly still in their infancy. Nevertheless various directions have
already been pursued, see Menotti [3] for a review. Here we follow a line of thought which
is based on the Regge calculus [4]. The Regge calculus replaces the smooth space-time
manifold by a piecewise flat simplicial manifold, the Regge skeleton. A $d$-dimensional simplicial
lattice is constructed by $d$-simplices which are glued together to form a piecewise flat
geometry. Every $d$-simplex consists of $(d+1)$ sites (0-simplices), $d(d+1)/2 = (d+1)/2$
links (1-simplices), $d(d+1)/3$ triangles (2-simplices), $(d+1)/4$ tetrahedras (3-simplices), etc. until
the dimension $d$ is reached. On a $4d$ simplicial manifold the action (1) becomes [4,5] the
Regge-Einstein action

$$S_{RE} = 2m_p^2 R, \text{ where } R = \sum_t \alpha_t A_t$$  \hfill (2)

is the scalar curvature. The sum is over all triangles (in $d$ dimensions $d-2$ simplices which
are conventionally called hinges) of the $4d$ simplicial manifold, $A_t$ is the area of triangle $t$
and $\alpha_t$ the associated deficit angle. The 4-simplices are pentahedra which we label by $p$
and we denote their associated volumes by $v_p$. The Regge skeleton is piecewise flat with
the curvature concentrated on the hinges. This enables to calculate $\alpha_t$, $A_t$, $v_p$ etc. by
the rules of elementary Euclidean geometry. With respect to this action the Euclidean
partition function is

$$Z = \int D[\{l\}] e^{S_{RE}}. \hfill (3a)$$
Here we use \( l \) in a double meaning, as a label as well as to denote the actual link length. The choice of the appropriate measure has been subject of much debate, see [3] for references. Nevertheless, it is fairly unambiguously determined for our present purposes: The dimensionful action \( S_{RE} \) requires a scale invariant measure (otherwise the dilatation \( l \rightarrow l' = \lambda l \) will already blow up the functional integral) and the technical limitation of finite computer speed requires a local measure. Then the ansatz \( D[\{l\}] = \prod_l f(l) \) implies [7]

\[
D[\{l\}] = \prod_l \frac{dl}{l}.
\]  

(3b)

Of course, this measure is not immune of critics [3] and to involve a technical assumption cannot be regarded as satisfactory. On the other hand, the universality assumption discussed in the next section suggests that we may well expect physically reasonable results. Still, to give a meaning to the dimensionful action \( S_{RE} \) one has first of all to define a scale, henceforth called fundamental length. Without such a fundamental scale the value of \( S_{RE} \) is just undefined. By keeping one dimensional expectation value constant, most naturally \( < l > \), \( < A_t > \) or \( < v_p > \), the Regge calculus allows to introduce a fundamental length [6,7]. This matter of setting the scale should not be confused with introducing a cosmological constant. Once the fundamental scale is identified, all other physical quantities may be measured in units of this scale. In particular renormalization is no longer a problem of infinities, but just a problem of normalization and possibly large numbers due to the fact that the fundamental length (of the order of the Planck length) is very small as compared to hadronic scales (of the order of a Fermi). From this viewpoint of the Planck scale, once the natural setting has been identified, it may well happen that normalization does not require any counterterms. We adopt this point of view in the following and the cosmological constant is then zero through the simple fact that it is not present in the action (2). Under the assumption that the theory exists, in the sense made precise below, every physical quantity will be directly calculable in terms of the fundamental units. Let us denote the fundamental length by \( l_0 \) and the corresponding fundamental mass unit by \( m_0 = l_0^{-1} \). A simulation is then carried out for \( m_P = \text{const} \cdot m_0 \), i.e. for a Planck mass which is a number in units of \( m_0 \). It is a remarkable numerical observation [7,8] that the
pure demand of the existence of such a scale $m_0$ seems to ensure a finite range

$$0 \leq m_P^2 < m_{MAX}^2$$

for which the partition function (3) is well defined. In [7] this range was called “entropy dominated”. For a Planck mass outside this range the partition function simply does not exist and, therefore, it may be somewhat misleading to talk about a phase transition (although in a similar scenario for random walks the endpoint of the existing range exhibits indeed critical behaviour).

In lack of any experimental guidance it is a question of aesthetical simplicity to insist on the action (2). Of course one may add terms quadratic in the Riemann tensor and imagine that those term do not contribute in the classical limit. But new parameters are then implied and it is difficult to imagine that these parameters could be determined from theoretical principles alone, although interesting suggestions, like Weinberg’s [9] proposal of gravity as an asymptotically safe theory, exist in the literature. The reader may consult the review [3] for work along such lines. In this paper we embark on a suggestion of one of the authors [10] to explore the possible physical relevance of the entropy dominated region. Empirically [7,8] small values for $m_{MAX}^2$ of equation (4) are obtained. However, this statement is with reliance to the respective fundamental units. The question of physical relevance is whether, within the entropy dominated region, elementary particle masses can be chosen arbitrarily small as compared to the Planck mass or not. In the former case this would hint towards the possibility of a self-consistent theory of quantum gravity, whereas in the latter case the entropy dominated region would just be a mathematical curiosity. To investigate this problem demands to couple gravity to matter field and to look for a point in parameter space where the mass gap can be send to zero as compared to the Planck mass. The issue is still subtle, as the answer may well depend on the details of the interaction and the matter fields chosen. Here we work under two hypothetical assumptions:

i) The world without gravity is described by a (grand unified) asymptotically free quantum field theory.

ii) The only properties of this quantum field theory, which matter for our questions
of relative scales, are asymptotic freedom and the dimensionality of four space-time dimensions.

Therefore, we decided to couple quantum gravity to the computationally simplest 4d asymptotically free field theory and this is pure SU(2) lattice gauge theory. Let $\beta = 4/g^2$ be the coupling constant for the SU(2) gauge theory. Within the entropy dominated range, the hope is then to establish numerical evidence for the following scenario:

a) With increasing $\beta$, on sufficiently large systems, the hadronic masses (glueballs, string tension, deconfinement temperature) can be chosen arbitrarily small as compared to the Planck mass $m_P$ and, asymptotically, the perturbative two loop scaling formula [11] is approached as indicated in figure 1.

b) In the same limit the space becomes flat when averaged over a hadronic length scale (inverse glueball mass etc.) and, in particular, the Riemann curvature scalar approaches zero.

If a) and b) hold, one could imagine that quantum gravity is simply defined by the Regge-Einstein action in the entropy dominated region. As indicated in figure 1, the concept of a fundamental length would then allow us to fix $\beta$ at $\beta_{\text{physical}} = 4/g^2_{\text{physical}}$ such that the correct value for $m_{\text{hadronic}}/m_P \approx 10^{-18}$ is obtained. In contrast, in ordinary lattice gauge theory one would carry out the limit $\beta \rightarrow \infty$, implying $m_{\text{hadronic}} \rightarrow 0$ in units of the lattice spacing $a^{-1}$ and employ for $\beta = 4/g^2$ the notion of a bare coupling constant. Assuming that a) is correct, a rough order of magnitude estimate $g^2_{\text{physical}} \approx 0.5$ is obtained by equating $m_{\text{hadronic}}$ with the two loop $\Lambda_L$ scale, which in our context is conveniently defined by

$$\Lambda_L = m_P \left(6\pi^2\beta/11\right)^{51/121} \exp\left(-3\pi^2\beta/11\right). \quad (5)$$

On the other hand, the requirements a) and b) are precise enough to allow falsification by computer experiments. It may turn out that the entropy dominated range is limited to $m^2_{\text{MAX}} \leq \text{const} m^2_{\text{hadronic}}$ and / or that flat space cannot be approached. In some sense the previously observed “smallness” of $m^2_{\text{MAX}}$ hints towards this scenario. If this happens indeed, it would be difficult to imagine that the entropy dominated range could be of physical relevance.
The rest of this paper is organized as follows: In section 2 we define the model and state additional assumptions. Numerical results are given in section 3 and final conclusions can be found in section 4. Some special problems are relegated to appendixes.

2. The Model

On a 4 Hamiltonian manifold the action for Regge-Einstein gravity (2) coupled to pure SU(2) lattice gauge theory is given by

\[ S = S_{RE} + S_{gauge}, \quad \text{where} \quad S_{gauge} = -\frac{\beta}{2} \sum_t W_t Re\{ Tr (1 - U_t) \}. \quad (6) \]

As in (2) the sum goes over all triangles of a Regge skeleton and \( \beta = 4/g^2 \) is the gauge coupling. SU(2) matrices \( U_{ij} = U_{ji}^\dagger \) are associated with the links of the skeleton, where \( i, j \) are the sites of the link in question. \( U_t \) is the product of the SU(2) matrices around triangle \( t \), with \( i, j, k \) being the sites of this triangle:

\[ U_t = U_{ij}U_{jk}U_{ji}. \quad (7) \]

The dimensionless weight factors \( W_t \) are functions of the link lengths and couple the \( U \)'s to the geometrical structure of the skeleton. In contrast to ordinary lattice gauge theory on the static lattice the 1 in the term \( Tr(1 - U_t) \) is now of importance due to the fact that the weight factors \( W_t \) are dynamical. Without the 1 the lattice would be driven towards spurious \( W_t \to \infty \) configurations. As the skeleton is piecewise flat and its curvature concentrated on the triangles, the flat space random lattice considerations of [12] are still valid. In the continuum limit the lattice gauge action has to converge to the continuum Yang-Mills action. Phrased in a form appropriate for our present purposes, this yields [13] that

\[ \sum_t W_t A_t^2 = \text{const} \, V, \quad (V \text{ Volume of the lattice}) \quad (8) \]

has to hold on each single Regge skeleton. Beyond this restriction, and that they are functions of the link lengths, the weight factors are arbitrary. Clearly (8) is satisfied by the choice

\[ W_t = \text{const} \, \frac{V_t}{(A_t)^2} \quad (9) \]
if one assigns a 4-volume \( V_t \) with each triangle such that

\[ \sum_t V_t = V \]  

(10)
on each skeleton. The obviously most natural definition for \( V_t \) is the closest distance definition: Each point in the Regge skeleton is attributed to its nearest hinge, where the distance to a hinge is defined as the distance to the closest point of this hinge. Up to an irrelevant sub-volume of measure zero, every point in the skeleton is then uniquely assigned to a hinge and for each hinge the associated volume is manifestly positive:

\[ V_t \geq 0 \quad \text{for each } t. \]  

(11)
However, to implement the closest distance definition for simulations faces a number of technical problems which are easiest illustrated in two dimensions. In 2d the hinges are sites and we use the generic notation \( V_h \) for the volume associated with hinges in arbitrary dimensions. The closest distance definition leads now immediately to the dual lattice which is constructed from the bisectors of the original lattice, see for instance [12]. Figure 2 depicts, in flat space, part of a 2d random lattice and its dual. From the viewpoint of the \( d \)-simplex (triangle in 2d) we notice two possibilities:

1) Its volume contributes only to the \( V_h \) volumes of sites on its boundaries.

2) Its volume contributes also to the \( V_h \) volumes associated with further away sites.

It is the second case which causes troubles. Even in two dimensions we are not aware of a possibility to implement this situation efficiently with local formulas, as needed for simulations with finite CPU time. The problem gets worse in higher dimensions and due to curvature (when the geodesics through the boundary is no longer a straight line).

For lattice gauge theory on a 4d random lattice Christ and Lee [14] suggested the weight factors \( W_t = \tau_t / A_t \) (i.e. \( V_t = \tau_t A_t \)), where \( \tau_t \) is the area of the dual of the triangle \( t \). This choice has been shown to fulfil (8), but it is only in 2d identical with the closest distance definition. The dual cell of a site contains all points closest to it, but this is not true for other simplices. For instance from our 2d random lattice of figure 2 it is
obvious that associating with links a volume\( V_l = l_d l/2 \), where \( l_d \) is the length of the link dual to \( l \), does not coincide with the closest distance definition. Instead, applying the closest distance definition to links in 2\( d \) leads to a barycentric division of each triangle as shown in figure 3. Similar considerations apply to the dual objects of triangles in 4\( d \). Furthermore, the proposal of [14] suffers from essentially the same technical difficulties as the closest distance definition. Only for the situation (1) of figure 2 one has efficient closed formulas [15], and it is tempting to extent their application to the general case as equation (10) turns out to be still fulfilled. However, as already noted in [15] one has then to cope with the problem that the equations may lead to negative \( V_h \) contributions when applied to the situation (2) of figure 2. In more details these problems are discussed in appendix A and the conclusion is that it is crucial to enforce the positivity condition (11).

Notable are two construction which avoid negative \( V_t \) values. As discussed in [15] one may be tempted to use the approach of appendix A and to impose (11) by just rejecting updating steps which propose configurations with some \( V_t \) negative. This is in some analogy with implementing the triangle inequalities and their higher dimensional generalizations in the simulation of a Regge skeleton. Nevertheless it seems rather unnatural to reject configurations with perfectly well-defined geometries. The choice which is actually implemented in the simulations reported here relies on a barycentric subdivision of the 4-simplices which, in technical details, is outlined in appendix B. Similarly as with the closest distance definition, each point in each 4-simplex gets uniquely assigned to one of the triangles on its boundary. Figure 3 illustrates this choice for the two dimensional case: Whereas for the links one would reproduce the closest distance definition exactly, this is no longer the case for the sites. Their associated volumes \( V_h \) are obtained by connecting the barycenters of each triangle with the barycenters (midpoints) of the boundary links.

Our discussion of the different options to avoid negative \( V_t \) contributions leads to a fundamental problem: For discrete quantum gravity with a fundamental length the microscopic details of the theory are supposed to matter. How can we then possibly

\* We use the notation \( V_l \) because \( l \) is not a hinge in 2\( d \).
expect any results of physical relevance from a fairly arbitrary construction? The answer is that the details become relevant for true quantum gravity effect on the scale of the Planck length. In the line of investigations proposed here we are not aiming at calculating these effects. Instead we only like to establish the possibility of consistency, in the sense of figure 1, of Euclidean fundamental length quantum gravity with an asymptotically free gauge theory. For this question we work under the conjecture:

iii) The microscopic details of the Euclidean theory of quantum gravity do not matter in the usual sense of universality in lattice gauge theory.

This conjecture adds to the assumptions i) and ii) of the introduction. In lattice gauge theory the microscopic details of the lattice regularization (as well as of the action) are irrelevant for defining the correct quantum field theory as long as one stays within one universality class. The relevant universality class is specified by general symmetry principles and by requesting the correct classical limit for the action, see for instance [16]. The universality principle has turned out to be a rather powerful tool as it adds greatly to the flexibility of quantum field theory investigations, although it may sometimes be a subject of dispute whether two actions are in the same universality class or not. Our conjecture iii) generalizes universality in the sense that we now assume that the microscopic details of the Regge skeleton will at $\beta_{physical}$ only lead to corrections which are suppressed by order $(m_{hadronic}/m_P)$ as compared to the leading term. Of course $m_p/m_0$ ($m_0$ being the fundamental mass unit) may greatly depend on these details, similarly as in ordinary lattice gauge theory the ratio of $\Lambda$-scales depends on the regularization [17].

Exploiting the universality conjecture we assume now that we may define the theory on a hypercubic lattice, without destroying the essential scaling behaviour of figure 1. In this construction each hypercube gets divided into 24 pentahedra (4-simplices), for more details see [10]. Further, following [7,10] we define the fundamental length as $l_0 = (\langle v_p \rangle)^{1/4}$, i.e. by keeping the expectation value $\langle v_p \rangle$ of the volume of a pentahedron fixed. This implies that the total volume becomes proportional to the total number of pentahedra:

$$V = N_p(l_0)^4. \quad (12)$$
Obviously, we can only simulate a very tiny portion of the universe. In our simulation the total volume is also kept constant, but in principle one could allow for expansion or contraction of the universe by allowing for creation and annihilation of additional simplices. Clearly such updating steps would be fairly difficult to implement and the scope of our investigation is restricted to link length fluctuations which are constrained by a fixed set of incidence matrices which define the geometry of a hypercubic lattice.

Other ways to introduce the fundamental length would be to keep $< l >$, $< l^2 >$, $< A_t >$ or similar quantities fixed. It is a remarkable observation [8] that the action stays finite with either of these three choices. In particular it seems to be attractive to define the fundamental length by setting the scale with link expectation values. Pure gravity simulations with $< l^2 > = \text{const}$ have been carried out [8] and, qualitatively, results were found rather similar to those obtained with $< v_p > = \text{const}$.

To conclude this section, it is certainly worthwhile to spent at least a few thoughts on imagining qualitatively how the microscopic details of a fundamental length quantum gravity scenario could possible look like. It seems natural to think of sites as some kind of sources (or sinks) and of links as some kind of flux strings. Obviously, one does not expect any regular geometry, but has to involve some kind of random structure. The random lattice investigations [12-14] in flat space are a good starting point, but the fact that for gravity the links become dynamical variables complicates matters considerably. One may have to think about re-linking when sites (sources) come too close to one another and about implementing transitions between different topologies [18]. However, due to the universality conjecture these details of the fundamental length gravity theory are argued not be important for verifying the scaling behaviour a) of the introduction. In contrast, they may well be of relevance for the flatness property b). The reason is that (due to the conjectured lack of renormalization) the observable macroscopic curvature would just be the expectation value of the fluctuating microscopic curvature. Therefore, the preferred flatness of empty space might be related to entropy in the sense that the functional integral measure sharply enhances the probability of flat space configurations as compared to
curved space configurations. In this connection we like to comment on the observed [7,8] smallness of the scalar curvature expectation value for pure entropy $m_P^2 = 0$ simulations, i.e. relying alone on the measure (3b). For the hypercubic lattice the coordination numbers of the lattice are close to those found in the average for a random lattice [7]. Still, the fixed, regularly repeating incidence matrices are clearly an artificial constraint, just the effect might be expected to be small. For the pure gravity case two of the authors [19] had carried out a more detailed investigation of the action density distribution. Triangles on the hypergeometric lattice fall in two classes: In the first one each triangle is attached to 4 pentahedra and in the second to 6 pentahedra, correspondingly there are topologically distinct incidence matrices. Figure 4 depicts now the scalar curvature distributions constrained to each of these classes. The remarkable feature is that one distribution is centered around a positive mean, whereas the other is centered around a negative mean. We take this as an indication that the small negative over-all expectation value may be an artifact due to the hypercubic constraints and that entropy of the true random space may favour the desired flat space. A random lattice investigation of pure gravity would be of some interest.

3. Numerical Results

As we have discussed, our scale is set by adopting $l_0 = (\langle v_p \rangle)^{1/4}$ as fundamental length. This defines lattice units to which the numbers used by the computer refer. For instance, $V = N_p$ (12) would be stored for the total volume. We fix the normalization in equations (8), (9) and (B4) by the choice

$$\text{const} = 1500/71.$$ (13)

This implies that in flat space

$$\frac{1}{2} \langle W_t Re[Tr(1 - U_t)] \rangle = 1 \quad \text{for} \quad \beta = 0,$$ (14)

and implies a convention for $\beta$ which ensures similar orders of magnitudes for $\Lambda_L$ as typical for flat space SU(2) simulations. Actual simulations are very CPU time intensive due to the complicated action (6), and our present exploratory study has remained limited to a
$2 \cdot 4^3$ lattice. The volume is $V = 24 \cdot 2 \cdot 4^3 = 3,072$, as each hypercube embeds 24 pentahedra. To compare with lattice sizes of conventional hypercubic systems, one may equate their number of plaquettes with the number of triangles in the present case. Conventional lattices have six different plaquettes per hypercube, we have fifty different triangles. This converts into a factor $(50/6)^{1/4} \approx 1.7$. In this sense, our system is as big as a conventional one of size $3.4 \cdot (6.8)^3$.

Our initial (starting) configuration will be in flat space with SU(2) matrices assigned randomly or ordered to the links of the system. Without gravity updates this would just be another SU(2) lattice gauge theory simulation on a somewhat unconventional lattice, similar to calculations [20] done in the earlier days of lattice gauge theory. Indeed, it has been checked [21] that this simulation gives the expected results. In the present paper, we include now gravity with

$$m_{P}^2 = 0.005$$

in our lattice units*. This value of the Planck mass lies within the range [7,8] for which the partition function of pure quantum gravity simulations stays well-defined. We have chosen the asymmetric $L_0 \cdot L^3$, ($L_0 < L$) lattice size, because we are interested in locating remnants of the QCD deconfining phase transition. We define the Polyakov loop $P$ in the usual way as $P = \text{Tr}(U_1...U_n)$, where the path is closed by the periodic boundary conditions. In our numerical calculation the product of SU(2) matrices is only taken along hypercube edges corresponding to a straight line in the starting configuration. The Polyakov loop along the shortest directions ($L_0$) is regarded as order parameter, i.e. in the limit $L \to \infty$, $L_0$ fixed, the disordered phase is given by $<P> = 0$ and the ordered phase by $<P> \neq 0$. In the almost continuum limit $L_0 \to \text{large}$, if such a limit exists, this transition is supposed to become the QCD (more precisely pure SU(2) gauge) deconfining phase transition. We do not have to bother about the interpretation for small $L_0$, essential is that the Polyakov loop stays to be an order parameter even on strongly fluctuating systems. In the almost continuum limit the usual interpretation is ensured, as according to b) the space should

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* Note that our present convention (2) for $m_{P}^2$ differs by a factor of two from [7].
become flat in the average too.

We have simulated the system by performing alternating updating sweeps of the gauge and the gravity action. A gauge sweep is defined by updating each SU(2) matrix once, and a gravity sweep by updating each link length once. Updating the gravity action will change the weight factor $W_t$ and hence influence the subsequent gauge updating. Similarly the actual gauge configuration influences the gravity updating through the weight factors. We simply use the notion sweep for performing a gauge and a gravity sweep in succession.

The CPU time intensive part of the code is the gravity updating. For our $2 \cdot 4^3$ system a gravity sweep takes about 7s on an IBM 340 RISC workstation, whereas a gauge sweep is done in 0.3s. Our present computer resources did not allow for finite size scaling study of the system. Therefore, we decided to be content with a level of rigour which was typical for pioneering lattice gauge theory studies [23], and to employ the method of thermal cycles for gaining an idea of the phase structure. After each $N_1$ sweeps we vary $\beta$ by

$$\Delta \beta = (\beta_{max} - \beta_{min})/N_2.$$  \hspace{1cm} (16)

We start off with $\beta = \beta_{min}$ and perform $N = N_1 \cdot N_2$ sweeps, increasing $\beta$ by $\Delta \beta$ after each $N_1$ sweeps. Then we reach $\beta_{max}$ and perform another $N$ sweeps, decreasing now $\beta$ by $\Delta \beta$ after each $N_1$ sweeps. At each $\beta$ value thermal averages over the $N_1$ sweeps for various physical quantities are measured. For a suitable choice of $\beta_{max}$, $\beta_{min}$ and $N_1$, $N_2$ a phase transition will then show up as a hysteresis in the time series of one, some or all of these quantities.

Here we present results for the following quantities: the Polyakov loop $P$ (closed in the $L_0 = 2$ direction, the gauge action $S_{gauge}$ and $Tr(U_t)/2$, the link lengths $l$, the triangle areas $A_t$ and the weight factors $W_t$, last not least for the deficit angle $\alpha$ and the gravity action $S_{RE}$, which is of course equivalent to the scalar curvature $R$. The normalization of the gauge and gravity actions is per triangle, for all other quantities the normalization is according to their definitions. Note that in contrast to flat space there is no trivial one-to-one correspondence between $S_{gauge}$ and $Tr(U_t)$ anymore, due to the fact that the
weight factors $W_t$ have become dynamical. Also remember that our definition of the gauge action (6) includes the factor $-\beta/2$. To locate the transition in question $\beta_{\text{min}} = 1.3253$, $\beta_{\text{max}} = 1.635$ $N_1 = 300$ and $N_2 = 36$ have turned out to be suitable parameter values. Before going through the thermal cycle, we performed 20,000 sweeps at $\beta_{\text{min}}$. This has turned out to be relevant due to slow equilibration of the gravity part. Figure 5 shows hysteresis behaviour of the Polyakov loop, indicating the existence of a phase transition. As illustrated by figures 6 and 7, the transition shows also up in other gauge quantities such as $\text{Tr}(U_t)$ and the gauge action itself. Further, it is noticeable in geometrical quantities, such as the link lengths, triangle areas and the weight factors, see figures 8, 9 and 10. However, in the gravity action itself it is obscured by the noise of the deficit angle, see figure 11 and 12. Together these graphs present convincing evidence for a phase transition around $\beta \approx 1.5$. It would be premature to comment on the order of this transition. It is kind of interesting that the transition does not only show up in the gauge part, but also in geometrical quantities. However, one should not jump into speculations. If the outlined physical picture of an almost continuum limit is correct, one expects the effects in the geometry to become suppressed by many orders of magnitude. It is only due to our extreme strong coupling limit that gravity and hadronic scale take on the same order of magnitude.

After having located the transition, we performed longer runs to probe the broken phase at selected $\beta$ values: $\beta = 1.5147$, 1.5265 and 1.5383. Each run consists of 64,000 sweeps, where an additional 20,000 initial sweeps are omitted for reaching thermal equilibrium. Averaged over the lattice, Polyakov loop distributions are depicted in figures 13. Lattice averages are taken. With increasing $\beta$ a double peak structure develops which is typical for the ordered phase. The existence of the phase transition is therefore clear. To determine its location more precisely would require to employ finite size scaling techniques ($L = 8, 16, \ldots$), but our present computer resources do not allow for this. Even less they allow for the most interesting increase of $L_0$. Instead, we intend to study first the phase diagram in the $m_P^2 - \beta$ plane, a task possible within our present limitations. To provide some reference, table 1 collects from our long runs the expectation values of the various physical quantities which we have considered.
4. Conclusions

Simulating Regge-Einstein euclidean gravity coupled to pure SU(2) lattice gauge theory on a small $2 \cdot 4^3$ lattice, we find that an “entropy dominated” region still exists where stable bulk expectation values are obtained. In addition, our data support the existence of a phase transition between a disordered and an ordered phase, the Polyakov loop being the order parameter. However, we cannot yet present evidence for property a) of the scenario outlined in the introduction. To provide modest evidence, would require to simulate a $4 \cdot 8^3$ system. Keeping the Planck mass $m_P$ fixed (in the fundamental units), one likes to find that one is still in the well-defined region and that (in the same units) the critical temperature moves to a smaller value $T_c (L_t = 4) < T_c (L_t = 2)$. Concerning property b), we would like to observe a decline of $< R >$, but finally microscopic details of the gravity action may matter for this quantity. Presently, the major stumbling block against simulating the $4 \cdot 8^3$ system is lack of computer power. Already for the $2 \cdot 4^3$ system we needed considerable computing resources and with our present allocation we could not afford a factor $> 16$ as required for simulating the $4 \cdot 8^3$ system. Assuming continuing rapid improvements of computer technology these simulation should, however, be feasible within the next few years.

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The Monte Carlo data were produced on FSU’s CRAY-YMP, and on the SCRI cluster of RISC workstations.

Appendix A

In this appendix we summarize problems encountered with implementing the dual construction locally. We consider $2d$ where the dual vertex is located by the intersection of the bisectors of the links. Figure 14 shows the the case (1) discussed in section 2. The intersection of the bisectors is inside the triangle, and exact formulas for the contributions to the vertices can be calculated in terms of the link lengths. For example, the contribution
to A inside triangle $ABC$ is given by the sum of area $ADG$ and $AGF$:  
\[ V_A = \left( \frac{\hat{e}_z}{2} \right) \cdot \left( \vec{AD} \times \vec{AG} + \vec{AG} \times \vec{AF} \right). \]  
(17) 

Here $\hat{e}_z$ is the unit vector perpendicular to the plane of the triangle. This formula and 4d versions, as needed for the proposal of Christ and Lee [14], can be found in [15]. However, for case (2) of section 2 there are instances when these formulas fail to yield the area closest to the vertex. This is a consequence of applying the local formulas to situations where more complicated considerations are required. In particular hinge contributions can even become negative, as in the example [15] described now.

Consider figure 15 where the perpendicular bisectors intersect at a point $G$ outside the triangle. The area closest to $A$ is $AHF$, whereas equation (17) gives the area $AGF$ minus the area $AGD$. The contribution from $AGF$ is negative, because the orientation of $\vec{AG} \times \vec{AF}$ has changed. In the case of the figure one clearly sees: area $AGF >$ area $AGF$, hence the summed up contribution is negative. Still, equation (17) has an attractive feature: One easily verifies that the sum of all contributions $V_A + V_B + V_C$ is identical to the area of the triangle $ABC$. Therefore, and as it allows for a straightforward computer implementation, we could not resist to try it out. One might imagine constraints such that large negative contributions imply competing large positive contributions. We performed simulations with the 4d analogues of formula (17). The numerical result is that reasonable bulk expectation values are never obtained. The system builds up configuration with large negative contributions, obviously by having $\tau_t$ negative and $A_t$ small in the equation $W_t = \tau_t/A_t$, and never ever escapes from there (terms of order five and more in the exponent are typical magnitudes).

Figure 16 depicts a test run at $m_P^2 = 0.005$ for the relatively strong coupling $\beta = 710/1500$. One sees the gauge action rendered unbounded from above, catalyzed by the weight factors which acquire negative contributions. The system jumps from one plateau to the next, each time increasing its action, and cannot get out again. Each plateau indicates a time period when nearly all new proposed link lengths were rejected. We conclude that the requirement (11), $V_t \geq 0$, is crucial for numerical simulations of the system.
Appendix B

Here we give in details the barycentric definition of the volumes \( V_t \) as used when implementing equation (9) for our present simulations. Figure 17 illustrates the barycentric subdivision of the volume of a typical pentahedron \( ABCDE \). We consider the construction of the sub-volume which adds to the volume associated with the triangle \( ABC \). We proceed by first locating the barycenters of the pentahedron and that of the two tetrahedra that share the triangle \( ABC \). Let the position vectors of the vertices \( A, B, C, D, \) and \( E \) be \( \mathbf{0}, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \) and \( \mathbf{r}_4 \) respectively. The barycenter of the pentahedron \( ABCDE \) is then given by

\[
\mathbf{G} = \frac{1}{5} \left( \mathbf{0} + \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4 \right),
\]

the barycenter of the tetrahedron \( ABCD \) is

\[
\mathbf{F} = \frac{1}{4} \left( \mathbf{0} + \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 \right),
\]

the barycenter of the tetrahedron \( ABCE \) is

\[
\mathbf{F}' = \frac{1}{4} \left( \mathbf{0} + \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_4 \right).
\]

The 4-volume in the pentahedron \( ABCDE \) which is now associated with the triangle \( ABC \) the combined 4-volumes of the sub-pentahedra \( ABCFG \) (constructed in figure B1) and \( ABCF'G \). These contribution are manifestly positive as the barycenter lie always inside the pentahedra. Further, the sum of the contributions to all ten triangles of a pentahedron adds up to the 4-volume of the pentahedron. The weight factor of triangle \( t \) from the barycentric subdivision is consequently

\[
W_t = \text{const} \sum_{p \ni t} \frac{V_t(p)}{(A_t)^2} = \text{const} \frac{V_t}{(A_t)^2},
\]

where the sum is over all pentahedra which contain \( t \) and \( V_t(p) \) is the sub-volume of pentahedra \( p \) which is associated with \( t \).
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Table 1

Expectation values for $Tr(U_t)/2$, $S_{gauge}$, $l$, $A_t$, $W_t$, $\alpha$ and $R$. Each measurement relies on 64,000 sweeps and measurements after spending 20,000 sweeps for reaching thermal equilibrium. Error bars (in parenthesis) refer to the last digit.

| $\beta$ | $Tr(U_t)/2$ | $S_{gauge}$ | $l$ | $A_t$ | $W_t$ | $\alpha$ | $R$   |
|---------|-------------|-------------|-----|-------|------|---------|------|
| 1.5147  | 0.399(3)    | $-0.535(1)$ | 3.341(4) | 4.29(1) | 0.676(2) | 0.0340(6) | $-0.411(5)$ |
| 1.5265  | 0.416(3)    | $-0.528(1)$ | 3.333(4) | 4.27(1) | 0.683(2) | 0.0346(7) | $-0.407(8)$ |
| 1.5383  | 0.442(4)    | $-0.518(1)$ | 3.310(8) | 4.22(1) | 0.695(2) | 0.0308(9) | $-0.399(4)$ |