The energy-critical nonlinear Schrödinger equation on a product of spheres

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Let \((M, g)\) be a compact smooth 3-dimensional Riemannian manifold without boundary. It is proved that the energy-critical nonlinear Schrödinger equation is globally well-posed for small initial data in \(H^1(M)\), provided that a certain tri-linear estimate for free solutions holds true. This estimate is known to hold true on the sphere and tori in 3d and verified here in the case \(S \times S^2\). The necessity of a weak form of this tri-linear estimate is also discussed.

1. Introduction

Burq-Gérard-Tzvetkov [3–6] initiated a line of research on the well-posedness of nonlinear Schrödinger equations on compact manifolds, extending Bourgain’s results on tori [1, 2]. More precisely, on a given compact smooth \(d\)-dimensional Riemannian manifold \((M, g)\) without boundary, the Cauchy-problem

\[
\begin{aligned}
  i\partial_t u + \Delta_g u &= \pm |u|^{p-1}u \\
  u|_{t=0} &= u_0 \in H^s(M)
\end{aligned}
\]

is studied, where \(u_0 \in H^s(M)\) is given initially and the aim is to prove the existence and uniqueness of a solution \(u \in C([0, T), H^s(M, \mathbb{C}))\) and its continuous dependence on \(u_0\). For sufficiently smooth solutions \(u\) the \(L^2(M)\)-norm and the energy

\[
E(u)(t) = \frac{1}{2} \int_M |\nabla u(t, x)|^2 \, dx \pm \frac{1}{p+1} \int_M |u(t, x)|^{p+1} \, dx
\]

are conserved quantities.

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On $M = \mathbb{R}^d$, solutions $u$ of the Equation (1) can be rescaled to solutions $u_\lambda$ by setting

$$u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x) \quad (\lambda > 0).$$

The Sobolev semi-norm $\| \cdot \|_{H^s(\mathbb{R}^d)}$ is invariant under this rescaling iff $s = s_c := \frac{d}{2} - \frac{2}{p-1}$ and we call the range $s > s_c$ sub-critical, $s < s_c$ super-critical and $s = s_c$ critical. In dimension $d = 3$, the quintic problem ($p = 5$) is called energy-critical since $s_c = 1$. In this case, well-posedness in the critical space $H^1(M)$ is a key ingredient in the analysis of global well-posedness: For small initial data this immediately follows from the conservation of the energy $E(u)$, and in the defocusing case it serves as a starting point for a proof of global well-posedness for large initial data.

Recently, first global results for (1) with $p = 5$ in the critical space $H^1(M)$ have been obtained on the specific manifolds $M = T^3$ [11, 13, 19] and $M = S^3$ [10, 16] with standard metrics. These critical results crucially rely on precise spectral information. In this paper, we consider the manifold $M = S \times S^2$ with the standard metric. With regard to concentration of eigenfunctions and localization of the spectrum of $\Delta_g$ this is an intermediate case between $T^3$ and $S^3$, as explained in [5, p. 257, l. 26ff]. We consider this as a toy model for the central question concerning the critical well-posedness on arbitrary smooth compact Riemannian 3-manifold, cp. [5, p. 257, l. 31ff], as it forces us to unify some of the methods developed in [10–12]. On the other hand, its treatment requires new ideas, which we will point out below.

Precisely, we focus on the following Cauchy-problem

\[
(2) \quad \begin{cases}
    i \partial_t u + \Delta_g u = \pm |u|^4 u \\
    u|_{t=0} = u_0 \in H^s(S \times S^2)
\end{cases}
\]

and we will prove the following in the critical case $s = 1$:

**Theorem 1.1.** The Cauchy problem (2) is globally well-posed for small initial data in $H^1(S \times S^2)$.

As usual, this result includes the existence of (mild) solutions $u \in C(\mathbb{R}, H^1(S \times S^2))$, uniqueness in a certain subspace, smooth dependence on the initial data and persistence of higher initial $H^s$-regularity. Our methods also imply local well-posedness for arbitrarily large initial data in $H^1(S \times S^2)$ by standard arguments, which we omit. We refer the reader to [11, Theorem 1.1 and 1.2] for more explanations. In [5] the global well-posedness in $H^1$ has been proved in the sub-quintic case (i.e. $1 < p < 5$), see [5, Theorem 1] for
a more complete statement and [5, Appendix A] for an ill-posedness result in a super-quintic case.

Generally speaking, the method of proof used here is similar to the cases $M = \mathbb{T}^3$ [11] and $M = S^3$ [10] and ideas from [5, 6] are used in order to deal with the fact that the spectral cluster estimates are not optimal on $M = S \times S^2$, see [5, Theorem 3 and Remark 2.1]. However, in the critical case the tri-linear estimate obtained in [5, Proposition 5.1] cannot be used because of the $\varepsilon$-loss, which essentially comes from the number-of-divisor-bound [5, Lemma 4.2]. The main new estimate is a critical tri-linear estimate for free solutions, see Proposition 2.6, which is also known for $M = \mathbb{T}^3$ (both rational [11, Proposition 3.5 and its proof, in particular (26)] and irrational [19, Proposition 4.1]) and $M = S^3$ [10, Proposition 3.6 and its proof, see (20)]. From this estimate we derive the nonlinear estimate which is used for the Picard iteration argument, which is along the lines of [10].

We point out that this reduction of the well-posedness proof to critical tri-linear estimates for free solutions is independent of the specific manifold. Conversely, we find that a weak form ($\delta = 0$) of the estimate in Proposition 2.6 is necessary for a well-posedness result in $H^1(M)$ with a smooth flow map, which again does not depend on the specific manifold $M = S \times S^2$.

This paper is organized as follows: We conclude this section by introducing some notation. In Section 2 we prove the crucial tri-linear estimate for free solutions. In Section 3 we describe how the tri-linear estimate can be extended to a certain function space, which allows us to perform the standard Picard iteration argument. In Section 4 we discuss the necessity of a weak form of the tri-linear estimate for free solutions.

**Notation**

Let $(M, g)$ be a compact smooth 3-dimensional Riemannian manifold without boundary. The spectrum $\sigma(-\Delta_g)$ of the Laplace-Beltrami operator can be listed as $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n \to +\infty$. Let $h_k : L^2(M) \to L^2(M)$ be the spectral projector onto the eigenspace corresponding to the eigenvalue $\lambda_k$. For $f \in L^2(M)$ and a dyadic number $N \in \mathbb{N}$, we define the projector

$$P_N f = \sum_{k \in \mathbb{N}_0 : N \leq \langle \lambda_k \rangle^{\frac{1}{2}} < 2N} h_k(f),$$

where $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$. We define the usual $L^2$-based Sobolev space $H^s(M) = (1 - \Delta_g)^{\frac{s}{2}}L^2(M)$, equipped with the norm
\[ \|f\|_{H^s(M)} = \left( \sum_{k \in \mathbb{N}_0} \langle \lambda_k \rangle^s \|h_k(f)\|_{L^2(M)}^2 \right)^{\frac{1}{2}}. \]

Due to \(L^2\)-orthogonality we have
\[ \|f\|_{H^s(M)}^2 \sim \sum_{N \geq 1} N^{2s} \|P_N f\|_{L^2(M)}^2. \]

Here and in the sequel \(\sum_{N \geq 1}\) indicates that we are summing over all \(N = 1, 2, 4, 8, \ldots\).

In the case \(M = S \times S^2\) we use the same notation for the spectrum and the spectral projectors as in [5, Section 5]: The spectrum of \(-\Delta = -\Delta_g\) is given by
\[ \lambda_{m,n} = m^2 + n^2 + n, \quad (m, n) \in \mathbb{Z} \times \mathbb{N}_0. \]

We denote by \(\Pi_n : L^2(S^2) \to L^2(S^2)\) the spectral projector onto spherical harmonics of degree \(n\) on \(S^2\). For functions \(f\) on \(M\) we write \(S \times S^2 \ni (\theta, \omega) \mapsto f(\theta, \omega)\). The \(m\)-th Fourier-coefficient of \(f(\cdot, \omega)\) is defined by
\[ \Theta_m f(\omega) := \frac{1}{2\pi} \int_0^{2\pi} f(\theta, \omega) e^{-im\theta} \, d\theta. \]

Hence, for \(f \in L^2(M)\), we have
\[ f(\theta, \omega) = \sum_{(m,n) \in \mathbb{Z} \times \mathbb{N}_0} e^{im\theta} \Pi_n \Theta_m (f)(\omega) \]
in the \(L^2\)-sense. For dyadic \(N\) we define the projector
\[ P_N f(\theta, \omega) = \sum_{(m,n) \in \mathbb{Z} \times \mathbb{N}_0: \ N \leq \langle \lambda_{m,n} \rangle^{\frac{1}{2}} < 2N} e^{im\theta} \Pi_n \Theta_m (f)(\omega). \]

We define the Sobolev space \(H^s(M) = (1 - \Delta_g)^{\frac{s}{2}} L^2(M)\), equipped with the norm
\[ \|f\|_{H^s(M)}^2 = \sum_{(m,n) \in \mathbb{Z} \times \mathbb{N}_0} \langle \lambda_{m,n} \rangle^s \|\Pi_n \Theta_m f\|_{L^2(M)}^2 \sim \sum_{N \geq 1} N^{2s} \|P_N f\|_{L^2(M)}^2. \]
2. The tri-linear estimate for free solutions

In this section we are going to prove a new tri-linear Strichartz estimate for free solutions (Proposition 2.6). This proposition is an improvement of the tri-linear estimate [5, Proposition 5.1] of Burq-Gérard-Tzvetkov in the sense that it is critical.

We start this section collecting two known results, which we will rely on later. The following estimate on exponential sums is due to Bourgain [1] and was used to prove Strichartz estimates on the flat torus.

Lemma 2.1 (cp. [1, Formula (3.116)]). Let $p > 4$, then for all $N \geq 1$, $a \in \ell^2(\mathbb{Z}^2)$, $z \in \mathbb{Z}^2$ and $S_N \subseteq z + \{-N, \ldots, N\}^2$ it holds that

$$\left\| \sum_{n \in S_N} e^{-i|n|^2t}e^{in \cdot x}a_n \right\|_{L^p_t(x([0,2\pi]^3))} \lesssim N^{1-\frac{4}{p}} \|a\|_{\ell^2}.$$

Proof/Reference. The desired estimate follows immediately from the Galilean transformation

$$x \cdot (n - z) - t|n - z|^2 = (x + 2tz) \cdot n - t|n|^2 - x \cdot z - t|z|^2,$$

as applied in [1, Formulas (5.7)–(5.8)] and [11, Proposition 3.1], and from [1, Formula (3.116)].

We will also use the succeeding tri-linear spectral cluster estimate of Burq-Gérard-Tzvetkov, which is more generally valid for any compact smooth Riemannian manifold without boundary of dimension two.

Lemma 2.2 ([5, Theorem 3]). For all integers $n_1 \geq n_2 \geq n_3 \geq 0$ and $f_1, f_2, f_3 \in L^2(\mathbb{S}^2)$ the following tri-linear estimate holds true

$$\|\Pi_{n_1}f_1\Pi_{n_2}f_2\Pi_{n_3}f_3\|_{L^2(\mathbb{S}^2)} \lesssim \left(\langle n_2 \rangle \langle n_3 \rangle\right)^{\frac{1}{2}} \prod_{j=1}^3 \|\Pi_{n_j}f_j\|_{L^2(\mathbb{S}^2)}.$$

Throughout this paper, let $\tau_0 = [0,8\pi]$ be the considered time interval. For the purpose of proving Proposition 2.6, we will use following exponential sum estimate. The main idea is to reduce the estimate to Lemma 2.1.
Lemma 2.3. Let $p > 4$. Then, for all $N \geq 1$, $a \in \ell^2(\mathbb{Z}^2)$, $z \in \mathbb{Z}^2$ and $S_N \subseteq z + \{-N, \ldots, N\}^2$ the estimate

$$\left\| \sum_{(m,n) \in S_N} e^{-i\lambda_{m,n} t} e^{im\theta} a_{m,n} \right\|_{L^p_{t,\theta}(\mathbb{R}^2)} \lesssim N^{1-\frac{2}{p}} \|a\|_{\ell^2}$$

holds true.

Proof. We first show that we may replace $\lambda_{m,n}$ by $m^2 + n^2$. We set $4\tilde{t} = t$ and $2\tilde{\theta} = \theta$. Since $4\lambda_{m,n} = (2m)^2 + (2n+1)^2 - 1$, the left hand side is bounded by a constant times

$$\left\| \sum_{(\tilde{m},\tilde{n}) \in \tilde{S}_N} e^{-i(m^2+n^2)t} e^{i\tilde{m}\tilde{n}} a_{\tilde{m},\tilde{n}} \right\|_{L^p_{\tilde{t},\tilde{\theta}}([0,2\pi]^2)},$$

where $\tilde{S}_N := \{(\tilde{m},\tilde{n}) \in \mathbb{Z}^2 : (\tilde{m}/2, (\tilde{n} - 1)/2) \in S_N\}$ is inside a cube of side length $4N$, and

$$a_{\tilde{m},\tilde{n}} := \begin{cases} a_{\tilde{m}/2, (\tilde{n} - 1)/2}, & \tilde{m} \in 2\mathbb{Z}, \tilde{n} \in 2\mathbb{Z} + 1, \\ 0, & \text{otherwise}. \end{cases}$$

Hence, it suffices to prove

$$\left\| \sum_{(m,n) \in S_N} e^{-i(m^2+n^2)t} e^{im\theta} e^{in\nu} a_{m,n} \right\|_{L^p_{t,\theta}([0,2\pi]^2)} \lesssim N^{1-\frac{2}{p}} \|a\|_{\ell^2}.$$ 

In order to apply the exponential sum estimate of Lemma 2.1, we introduce another variable $\nu$. Obviously, the left hand side is bounded by

$$\sup_{\nu \in [0,2\pi]} \left\| \sum_{(m,n) \in S_N} e^{-i(m^2+n^2)t} e^{im\theta} e^{in\nu} a_{m,n} \right\|_{L^p_{t,\theta}([0,2\pi]^2)},$$

which can be further estimated by

$$\left\| \sum_{(m,n) \in S_N} e^{-i(m^2+n^2)t} e^{im\theta} e^{i\nu} a_{m,n} \right\|_{L^p_{t,\theta}([0,2\pi]^2) \cap L^\infty([0,2\pi])}$$

using Minkowski’s inequality. Sobolev’s embedding in $\nu$ allows to bound this by a constant times

$$N^{\frac{2}{p}} \left\| \sum_{(m,n) \in S_N} e^{-i(m^2+n^2)t} e^{im\theta} e^{i\nu} a_{m,n} \right\|_{L^p_{t,\theta,\nu}([0,2\pi]^3)}.$$

Finally, Lemma 2.1 implies the desired result. \qed
Remark 1. One can even lower the exponent w.r.t. \( S \) to 4, if the exponent w.r.t. time is raised to \( p > \frac{16}{3} \). Let \( p > \frac{16}{3} \), then, under the same assumptions on \( a, N, S_N \) as in Lemma 2.3, the following estimate holds true:

\[
\left\| \sum_{(m,n) \in S_N} e^{-i\lambda_{m,n} t} e^{im\theta} a_{m,n} \right\|_{L^p_t(L^{4/3}(S))} \lesssim N^{\frac{3}{4} - \frac{2}{3}} \|a\|_{\ell^2}.
\]

The proof is very similar to Bourgain’s proof of Strichartz estimates on irrational tori [2, Proposition 1.1]. However, it seems that this estimate is not appropriate for studying local existence: We start with a tri-linear \( L^2(\tau_0 \times M) \) estimate and proceed as in the proof of Proposition 2.6 until (5). Then, using Hölder’s inequality to put the two functions with the highest frequencies to \( L^{8/3} L^4_\theta \) and thus the function with the lowest frequency, say \( N_3 \), to \( L^8 L^{4\lambda}_\theta \). We treat the latter term as follows: Applying Sobolev’s embedding to bound it by the \( L^{8/3} L^4_\theta \)-norm gives a factor \( N_3^{\frac{1}{3}} \). The exponential sum estimate (3) gives \( N_3^{\frac{1}{2}} \) and from the spectral cluster estimate we get another \( N_3^{\frac{1}{3}} \) as in (5). All in all we obtain \( N_3^{1-} \), and hence the power on the lowest frequency is too low to conclude local well-posedness.

The subsequent estimate will serve as an \( L^\infty(\tau_0 \times S) \) estimate. It improves the previous lemma, because it takes additional smallness properties of the underlying point set \( S_{N,M} \) into account, which will be induced by almost orthogonality in time.

Lemma 2.4. Let \( a \in \ell^2(\mathbb{Z}^2), N \geq M \geq 1, \) and

\[
S_{N,M} \subseteq \{(m,n) \in z + \{0, \ldots, N\}^2 : \sqrt{\lambda_{m,n}} \in [b, b+M]\}
\]

for some \( z \in \mathbb{Z}^2 \) and \( b \in \mathbb{N}_0 \). Then we have

\[
\sum_{(m,n) \in S_{N,M}} |a_{m,n}| \lesssim M^{\frac{1}{2}} N^{\frac{1}{3}} \|a\|_{\ell^2}.
\]

Proof. By Cauchy-Schwarz, we only have to show \( \#S_{N,M} \lesssim MN \). Since

\[
\#S_{N,M} \leq \#\{(m,n) \in \mathbb{Z} + \{0, \ldots, 2N\}^2 : \sqrt{m^2 + n^2} \in [2b, 2b+4M]\},
\]

where \( \mathbb{Z} = 2z + (0,1) \), we may assume \( \lambda_{m,n} = m^2 + n^2 \). The rest of the proof is motivated by [8, Section 2.7]. Consider all the lattice points in \( S_{N,M} \) as centers of unit squares with sides parallel to the coordinate axes. Obviously,
the number of lattice points in $S_{N,M}$ equals the area of the union of these squares. The diagonal of the unit squares is $\sqrt{2}$. Consequently, the union of the squares is inside a $\frac{1}{\sqrt{2}}$-neighborhood of $S_{N,M}$. This neighborhood can be covered by an annulus of angle $\alpha$, outer radius $R := 2b + 5M$ and inner radius $r := \max\{R - 6M, 0\}$, where $\alpha \in [0, 2\pi]$ is determined as follows: Since the point set is located in a cube of size $N$, the arc length of the annulus sector is bounded by $\sim N$. Thus $\alpha \sim \frac{N}{R}$, and we deduce that the area is bounded by

$$\frac{\alpha}{2}(R^2 - r^2) \lesssim \frac{N}{R} MR \lesssim MN.$$ 

Interpolating Lemma 2.3 and Lemma 2.4, we obtain an $L^p(\tau_0 \times S)$ estimate for $p > 4$ that takes additional smallness properties of the underlying point set into account as Lemma 2.4 does.

**Corollary 2.5.** Let $p > 4$. Then, for all $\varepsilon > 0$, $S_{N,M}$ as in Lemma 2.4, $N \geq M \geq 1$ and $a \in \ell^2(\mathbb{Z}^2)$ we have that

$$\left\| \sum_{(m,n) \in S_{N,M}} e^{-i\lambda_{m,n}t} e^{im\theta} a_{m,n} \right\|_{L^p_{t,\vartheta}(\tau_0 \times S)} \lesssim \left( \frac{N}{M} \right) ^{\frac{p-4}{p}} N^{\frac{1}{2}-\frac{1}{p}} M^{\frac{1}{2}-\frac{2}{p}} \|a\|_{\ell^2}.$$ 

**Proof.** We set $f(t, \theta) := |\sum_{(m,n) \in S_{N,M}} e^{-i\lambda_{m,n}t} e^{im\theta} a_{m,n}|$ for brevity. The estimate is nontrivial only if $\varepsilon \leq \frac{p-4}{2p}$. Furthermore, we set $\varepsilon' = 2p\varepsilon > 0$ and $\vartheta = \frac{4+\varepsilon'}{p} \leq 1$. Then, Hölder’s inequality, Lemma 2.3 and Lemma 2.4 imply

$$\|f\|_{L^p_{t,\vartheta}} = \|f^{1-\vartheta} f^{\vartheta}\|_{L^p_{t,\vartheta}} \leq \|f\|^{\vartheta}_{L^{\infty}_{t,\vartheta}} \|f\|^{1-\vartheta}_{L^p_{t,\vartheta}} \lesssim \left( \frac{N}{M} \right) ^{\frac{p-4}{p}} N^{\frac{1}{2}-\frac{1}{p}} M^{\frac{1}{2}-\frac{2}{p}} \|a\|_{\ell^2}. \Box$$

**Proposition 2.6.** There exists $\delta > 0$ such that for all $\phi_1, \phi_2, \phi_3 \in L^2(M)$ and dyadic numbers $N_1 \geq N_2 \geq N_3 \geq 1$ the estimate

$$\|P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3\|_{L^2(\tau_0 \times M)} \lesssim \left( \frac{N_3}{N_1} + \frac{1}{N_2} \right)^{\delta} N_2 N_3 \prod_{j=1}^3 \|\phi_j\|_{L^2(M)}$$

holds true.

**Proof.** We will exploit almost orthogonality in the first three steps to show that we may assume the highest frequency to be further localized. In the last
step we will estimate the remaining tri-linear estimate using the foregoing
results. First, we recall that for \( t \in \tau_0 \) and \((\theta, \omega) \in S \times S^2\)
\[
P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3(\theta, \omega) = \sum_{\mathcal{N}} \prod_{j=1}^3 e^{-i\lambda_{m_j,n_j} t} e^{im_j \theta} \Pi_{n_j} \Theta_{m_j} \phi_j(\omega),
\]
where \( \mathcal{N} = N_1 \times N_2 \times N_3 \) and
\[
(4) \quad N_j = \{ (m, n) \in \mathbb{Z} \times \mathbb{N}_0 : N_j \leq \langle \lambda_{m,n} \rangle^{\frac{1}{2}} < 2N_j \}, \quad j = 1, 2, 3.
\]
In this proof \( \sum_{\mathcal{N}} \) should be understood as \( \sum_{(m_1,n_1,m_2,n_2,m_3,n_3) \in \mathcal{N}} \).

We apply step 1–3 only if \( N_1 > N_2 \), otherwise we will proceed with step 4
directly (with \( S := \{ N_1, \ldots, 2N_1 - 1 \} \) and \( M := N_1 = N_2 \)).

**Step 1.** Due to spatial almost orthogonality induced by the \( S \) component,
it suffices to prove the desired estimate in the case
\[
P_{\mathcal{R}} P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3,
\]
where \( \mathcal{R} \subseteq [b, b + N_2] \times [0, 2N_1] \) for some \( b \in \mathbb{Z} \). We spell out more details
in the next step.

**Step 2.** Now, we use almost orthogonality that comes from the \( S^2 \) com-
ponent. It is a well-known fact that the product of a spherical harmonic of
degree \( n \) with another of degree \( m \) can be expanded in terms of spherical
harmonics of degree less or equal to \( n + m \). Furthermore, it is well-known
that two spherical harmonics of different degree are orthogonal in \( L^2(\mathbb{S}^d) \),
\( d \in \mathbb{N} \). We finally remark that complex conjugation does not change the
degree of a spherical harmonic. Details may be found in [18, Section VI.2].
Now, we prove that it suffices to consider the case, where \( n_1 \) is located in
an interval of the size of the second highest frequency \( N_2 \). To that purpose,
we define the following partition of \( \mathbb{N}_0 \):
\[
\mathbb{N}_0 = \bigcup_{k \in \mathbb{N}_0} I_k, \quad \text{where} \quad I_k = [kN_2, (k + 1)N_2).
\]
We claim that for fixed \( \theta \in S \) and \( t \in \tau_0 \) it holds that
\[
\| P_{\mathcal{R}} P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3(\theta) \|^2_{L^2(\mathbb{S}^2)} \sim \sum_{k \in \mathbb{N}_0} \| P_{\mathcal{R}_k} P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3(\theta) \|^2_{L^2(\mathbb{S}^2)},
\]
where \( \mathcal{R}_k = \mathcal{R} \cap (\mathbb{Z} \times I_k) \). Let \( k, \tilde{k} \in \mathbb{N}_0 \), then

\[
\langle P_{\mathcal{R}_k} P_{N_1} e^{i t \Delta} \phi_1 P_{N_2} e^{i t \Delta} \phi_2 \rangle_{L^2(\mathbb{S}^2)} = \sum_{\mathcal{R}_k \times N_2 \times N_3} \prod_{j=1}^3 e^{-i(\lambda_{m_j,n_j} - \lambda_{\tilde{m}_j,\tilde{n}_j}) t} e^{i(m_j - \tilde{m}_j) \theta},
\]

where \( m = (m_1, m_2, m_3, \tilde{m}_1, \tilde{m}_2, \tilde{m}_3) \), \( n = (n_1, n_2, n_3, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3) \), and \( I_{m,n} \) is defined by

\[
I_{m,n} = \int_{\mathbb{S}^2} \prod_{j=1}^3 \Theta_{m_j, \phi_1} \Theta_{\tilde{m}_j, \phi_j} d\omega.
\]

Without loss of generality we may assume \( n_1 > \tilde{n}_1 \). Then

\[
Y := \prod_{\tilde{n}_1 \Theta_{\tilde{m}_1, \phi_1}} \prod_{j=2}^3 \Theta_{m_j, \phi_j} \Theta_{\tilde{m}_j, \phi_j} \in L^2(\mathbb{S}^2)
\]

can be expanded in terms of spherical harmonics of degree less or equal to \( \tilde{n}_1 + 8 N_2 \). Hence, if \( |k - \tilde{k}| \gg 1 \), then

\[
I_{m,n} = \langle \prod_{n_1 \Theta_{m_1, \phi_1}} Y \rangle_{L^2(\mathbb{S}^2)} = 0.
\]

**Step 3.** Using almost orthogonality in time, we may gain a small power of \( M := \max\{ N_2^2, 1 \} \). Similar ideas have been used in the proofs of [11, Proposition 3.5] and [10, Proposition 3.6], for instance. We define the partition

\[
\mathbb{N}_0 = \bigcup_{\ell \in \mathbb{N}_0} J_{\ell} \quad \text{where} \quad J_{\ell} = [\ell M, (\ell + 1)M).
\]

We show that we may assume \( \sqrt{\lambda_{m_1,n_1}} \) to vary in an interval of length \( M \):

Fix \( (\theta, \omega) \in \mathbb{S} \times \mathbb{S}^2 \) and set

\[
S_{k,\ell} = \{(m_1,n_1) \in \mathcal{R}_k : \sqrt{\lambda_{m_1,n_1}} \in J_{\ell}\}, \quad k, \ell \in \mathbb{N}_0,
\]
then we claim that
\[
\left\| P_{\mathcal{R}} P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3 (\theta, \omega) \right\|^2_{L_t^2(\tau_0)} \sim \sum_{k, \ell \in N_0} \left\| P_{S_{k, \ell}} P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3 (\theta, \omega) \right\|^2_{L_t^2(\tau_0)}.
\]

We consider the inner product
\[
\langle P_{S_{k, \ell}} P_{N_1} e^{it\Delta} \phi_1 \prod_{j=2}^3 P_{N_j} e^{it\Delta} \phi_j (\theta, \omega),
\]
\[
P_{S_{k, \ell}} P_{N_1} e^{it\Delta} \phi_1 \prod_{j=2}^3 P_{N_j} e^{it\Delta} \phi_j (\theta, \omega) \rangle_{L_t^2(\tau_0)}
\]
\[
= \sum_{S_{k, \ell} \times N_2 \times N_3} I_{m, n} \prod_{j=1}^3 e^{i(m_j - \bar{m}_j) \theta} \Pi_{n_j} \Theta_m \phi_j (\omega) \Pi_{\bar{n}_j} \Theta_{\bar{m}_j} \phi_j (\omega),
\]
where \( m = (m_1, m_2, m_3, \bar{m}_1, \bar{m}_2, \bar{m}_3) \), \( n = (n_1, n_2, n_3, \bar{n}_1, \bar{n}_2, \bar{n}_3) \), and
\[
I_{m, n} = \int_{\tau_0} e^{-i(\lambda_{m_1, n_1} + \lambda_{m_2, n_2} + \lambda_{m_3, n_3} - \lambda_{\bar{m}_1, \bar{n}_1} - \lambda_{\bar{m}_2, \bar{n}_2} - \lambda_{\bar{m}_3, \bar{n}_3}) t} dt.
\]

Assuming \(|\ell - \tilde{\ell}| \gg 1\), we may estimate the modulus of the phase from below by
\[
\left| (\sqrt{\lambda_{m_1, n_1}} + \sqrt{\lambda_{\bar{m}_1, \bar{n}_1}}) (\sqrt{\lambda_{m_1, n_1}} - \sqrt{\lambda_{\bar{m}_1, \bar{n}_1}}) - 16N_2^2 \right| \gg |\ell - \tilde{\ell}| N_2^2,
\]
and since all the eigenvalues are integers, we deduce \( I_{m, n} = 0 \).

**Step 4.** Thanks to the first three steps, we may replace \( P_{N_1} e^{it\Delta} \phi_1 \) by \( P_{S} P_{N_1} e^{it\Delta} \phi_1 \), where \( S = S_{k, \ell} \) for some \( k, \ell \in N_0 \). Recall that for \( t \in \tau_0 \) and \( (\theta, \omega) \in S \times S^2 \)
\[
P_{S} P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3 (\theta, \omega)
\]
\[
= \sum_{\mathcal{M}} \prod_{j=1}^3 e^{-i\lambda_{m_j, n_j} t} e^{im_j \theta} \Pi_{n_j} \Theta_m \phi_j (\omega),
\]
where \( \mathcal{M} := S \times \mathcal{N}_2 \times \mathcal{N}_3 \) and \( \mathcal{N}_j, j = 2, 3 \), are defined in (4). The next step is a nice way to treat the \( L^2(S^2) \)-norm separately without losing oscillations.
in the $S$ component and in time. Note that this was also used by Burq-Gérard-Tzvetkov in the proof of [5, Proposition 5.1]. Plancherel’s identity with respect to $t$ and $\theta$ and the triangle inequality for the $L^2(S^2)$ norm yield

\[ \| P_S P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3 \|_{L^2(\tau_0 \times M)}^2 \]

\[ \leq \sum_{\tau \in \mathbb{N}_0, \xi \in \mathbb{Z}} \left( \sum_{(m_1,n_1,m_2,n_2,m_3,n_3) \in \mathcal{M}} \prod_{j=1}^3 \Pi_{n_j} \Theta_{m_j, \phi_j} \right)^2_{L^2(S^2)} \]

\[ \leq \sum_{\tau \in \mathbb{N}_0, \xi \in \mathbb{Z}} \left( \sum_{(m_1,n_1,m_2,n_2,m_3,n_3) \in \mathcal{M}} \prod_{j=1}^3 \Pi_{n_j} \Theta_{m_j, \phi_j} \right)^2 \]

In contrast to [5, Proposition 5.1], we do not estimate the number of terms of the inner sum, but we go back to the physical space: We set $a_{m_j, n_j}^{(j)} := \| \Pi_{n_j} \Theta_{m_j, \phi_j} \|_{L^2(S^2)}$ for $j = 1, 2, 3$ and apply Lemma 2.2 as well as Plancherel’s identity with respect to $t$ and $\theta$ to obtain

\[ \| P_S P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3 \|_{L^2(\tau_0 \times M)}^2 \]

\[ \lesssim (N_2 N_3)^{\frac{1}{2}} \left( \sum_{\tau \in \mathbb{N}_0, \xi \in \mathbb{Z}} \prod_{j=1}^3 \Pi_{n_j} \Theta_{m_j, \phi_j} \right)^2 \]

\[ \lesssim (N_2 N_3)^{\frac{1}{2}} \left( \sum_{\mathcal{M}} \prod_{j=1}^3 e^{-i \lambda_{m_j,n_j} t} e^{im_j \theta} a_{m_j, n_j}^{(j)} \right)^2_{L^2_t(\tau_0 \times S)} \]

Choose $p_1 > 4$ and $12 < p_3 < \infty$ and let $p_2 > 4$ be defined via the Hölder relation $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}$. We apply Hölder’s estimate to obtain

\[ \| P_S P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3 \|_{L^2(\tau_0 \times M)} \]

\[ \lesssim (N_2 N_3)^{\frac{1}{2}} \left( \sum_{(m_1,n_1) \in S} e^{-i \lambda_{m_1,n_1} t} e^{im_1 \theta} a_{m_1, n_1}^{(1)} \right) \]

\[ \times \prod_{j=2}^3 \left( \sum_{(m_j,n_j) \in N_j} e^{-i \lambda_{m_j,n_j} t} e^{im_j \theta} a_{m_j, n_j}^{(j)} \right) \]
We estimate the first term using Corollary 2.5 and the other terms using Lemma 2.3. Then, we obtain for all \( \varepsilon > 0 \)

\[
\| P_S P_{N_1} e^{it \Delta} \phi_1 P_{N_2} e^{it \Delta} \phi_2 P_{N_3} e^{it \Delta} \phi_3 \|_{L^2(\tau_0 \times M)} 
\lesssim \left( N_2 N_3 \right)^{1/2} M^{1/2 - \frac{1}{p_1} - \varepsilon} N_2^{3/2 - \frac{1}{p_1} + \varepsilon} N_3^{1 - \frac{3}{p_3}} \prod_{j=1}^{3} \| \phi_j \|_{L^2(M)}
\]

Since \( p_1 > 4 \) and \( p_3 > 12 \), this implies the desired estimate provided \( \varepsilon > 0 \) is sufficiently small. \( \square \)

**Remark 2.** The proof of Proposition 2.6 does not extend to the case \( S \times S_\rho^2 \) directly, where \( S_\rho^2 \) is the embedded sphere of radius \( \rho > 0 \) in \( \mathbb{R}^3 \). However, preliminary calculations suggest that Proposition 2.6 may be proved in the more general case by more technical arguments. This will be addressed in the PhD thesis of the second author.

### 3. Function spaces and the nonlinear estimate

We briefly recall the function spaces \( U^p \) and \( V^p \) introduced by Koch-Tataru [14], which have been successfully employed in the context of critical dispersive equations. We refer the reader to [9] or [15] for more details and to [11, Section 2], [10, Section 2], and [12, Section 2] for this machinery in the context of the nonlinear Schrödinger equations on manifolds.

**Definition 3.1.** Let \( 1 \leq p < \infty \).

1) A step function \( a: \mathbb{R} \to L^2 \) is called a \( U^p \)-atom, if

\[
a(t) = \sum_{k=1}^{K} \chi_{(t_{k-1}, t_k)} a_k, \quad \sum_{k=1}^{K} \| a_k \|_{L^2}^p = 1
\]

for a partition \( -\infty < t_0 < \cdots < t_K \leq \infty \). The space \( U^p \) is defined as the corresponding atomic space.
2) The space $V^p$ is the space of right-continuous functions $v: \mathbb{R} \to L^2$ such that
\[
\|v\|_{V^p}^p = \sup_{-\infty < t_0 < \cdots < t_K \leq \infty} \sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_{L^2}^p < +\infty
\]
with the convention $v(+\infty) := 0$, and in addition we require $\lim_{t \to -\infty} v(t) = 0$.

We use the resolution spaces as defined in [10, Definition 2.3]:

**Definition 3.2.** Let $s \in \mathbb{R}$.

1) $X^s$ is defined as the space of all $u: \mathbb{R} \to H^s(M)$ such that $e^{-it\Delta} P_N u \in U^2$ for all dyadic $N \geq 1$ and
\[
\|u\|_{X^s} := \left( \sum_{N \geq 1} N^{2s} \|e^{-it\Delta} P_N u\|_{U^2}^2 \right)^{\frac{1}{2}} < +\infty.
\]

2) $Y^s$ is defined as the space of all $u: \mathbb{R} \to H^s(M)$ such that $e^{-it\Delta} P_N u \in V^2$ for all dyadic $N \geq 1$ and
\[
\|u\|_{Y^s} := \left( \sum_{N \geq 1} N^{2s} \|e^{-it\Delta} P_N u\|_{V^2}^2 \right)^{\frac{1}{2}} < +\infty.
\]

3) For an interval $\tau \subset \mathbb{R}$ we denote by $X^s(\tau)$ resp. $Y^s(\tau)$ the restriction space.

Next, we show how Proposition 2.6 implies Theorem 1.1. We remark that this derivation does not depend on the specifics of $M = S \times S^2$, it is similar to [10, Corollary 3.7], cp. also [11, 12] for corresponding arguments using unit scales instead of dyadic scales.

**Proposition 3.3.** There exists $\delta > 0$ such that for all dyadic numbers $N_1 \geq N_2 \geq N_3 \geq 1$ and $P_{N_j} u_j \in Y^0$ ($j = 1, 2, 3$) the following holds true
\[
\|P_{N_1} \tilde{u}_1 P_{N_2} \tilde{u}_2 P_{N_3} \tilde{u}_3\|_{L^2(\tau_0 \times M)} \lesssim \left( \frac{N_3}{N_1} + \frac{1}{N_2} \right)^\delta N_2 N_3 \prod_{j=1}^{3} \|P_{N_j} u_j\|_{Y^0},
\]
where $\tilde{u}_j$ denotes either $u_j$ or $\overline{u_j}$.
Proof. Since the $L^2$-norm on the left hand side does not change under complex conjugation of any factor, we may ignore possible complex conjugations.

Step 1. We start proving estimate (6) with $Y^0$ replaced by $X^0$. In this case, it suffices to consider $U^2$-atoms $a_1, a_2, a_3$, given as

$$P_N a_j = \sum_{k=1}^{K_j} \chi_{I_{k,j}} e^{it\Delta} P_N \phi_{k,j}, \quad \sum_{k=1}^{K_j} \|\phi_{k,j}\|_{L^2}^2 = 1,$$

with pairwise disjoint right-open intervals $I_{1,j}, I_{2,j}, \ldots, I_{K_j,j}$. Now,

$$\|P_N a_1 P_N a_2 P_N a_3\|_{L^2}^2 \leq \sum_{k_1, k_2, k_3} \|e^{it\Delta} P_N \phi_{k_1,1} e^{it\Delta} P_N \phi_{k_2,2} e^{it\Delta} P_N \phi_{k_3,3}\|_{L^2}^2$$

and Proposition 2.6 implies

$$\|P_N a_1 P_N a_2 P_N a_3\|_{L^2} \leq C_\delta(N_1, N_2, N_3),$$

with the constant $C_\delta(N_1, N_2, N_3)$ from Proposition 2.6, which yields

(7) $\|P_N u_1 P_N u_2 P_N u_3\|_{L^2} \leq C_\delta(N_1, N_2, N_3) \prod_{j=1}^3 \|e^{-it\Delta} P_N u_j\|_{U^2}.$

Step 2. Now, choosing $N_1 = N_2 = N_3 = N$ and $\phi_1 = \phi_2 = \phi_3$ in Proposition 2.6, we obtain

$$\|P_N e^{it\Delta} \phi\|_{L^6} \lesssim N^{\frac{2}{3}} \|P_N \phi\|_{L^2}.$$

As above, the estimate carries over to $U^6$-atoms, hence

$$\|P_N u\|_{L^6} \lesssim N^{\frac{2}{3}} \|e^{-it\Delta} P_N u\|_{U^6},$$

and for general $N_1 \geq N_2 \geq N_3 \geq 1$, by Hölder’s inequality,

(8) $\|P_N u_1 P_N u_2 P_N u_3\|_{L^2} \lesssim (N_1 N_2 N_3)^{\frac{2}{3}} \prod_{j=1}^3 \|e^{-it\Delta} P_N u_j\|_{U^6}.$
Also, by Hölder’s inequality and the Sobolev embedding, see [17, Eq. (2.6)] and [10, Lemma 3.4], we obtain
\[
\|P_{N_1} u_1 P_{N_2} u_2 P_{N_3} u_3\|_{L^2} \leq |\tau_0|^{\frac{1}{2}} \|P_{N_1} u_1\|_{L^\infty_t L^2_x} \|P_{N_2} u_2\|_{L^\infty} \|P_{N_3} u_3\|_{L^\infty} \\
\lesssim (N_2 N_3)^{\frac{3}{2}} \prod_{j=1}^{3} \|P_{N_j} u_j\|_{L^\infty_t L^2_x}.
\]
For any \( p \geq 1 \), using \( U^p \hookrightarrow L^\infty_t L^2_x \), we obtain the bound
\[
(9) \quad \|P_{N_1} u_1 P_{N_2} u_2 P_{N_3} u_3\|_{L^2} \lesssim (N_2 N_3)^{\frac{3}{2}} \prod_{j=1}^{3} \|e^{-it\Delta} P_{N_j} u_j\|_{U^p},
\]
which is not scale invariant, but the constant does not depend on \( N_1 \).

**Step 3.** We distinguish two cases:

**Case a)** \( N_2 N_3 > N_1 \). In this case, we interpolate (7) and (8) using [10, Lemma 2.4] and obtain
\[
\|P_{N_1} u_1 P_{N_2} u_2 P_{N_3} u_3\|_{L^2} \lesssim A_\delta \prod_{j=1}^{3} \|e^{-it\Delta} P_{N_j} u_j\|_{V^2},
\]
where
\[
A_\delta = C_\delta(N_1, N_2, N_3) \left( \ln \frac{(N_1 N_2 N_3)^{\frac{3}{2}}}{C_\delta(N_1, N_2, N_3)} + 1 \right)^3 \\
\lesssim C_\delta(N_1, N_2, N_3) \left( \ln \frac{N_1}{N_3} + 1 \right)^3 \lesssim C_{\delta'}(N_1, N_2, N_3)
\]
for any \( \delta' < \delta \).

**Case b)** \( N_2 N_3 \leq N_1 \). Now, we interpolate (7), (9) using [10, Lemma 2.4] and obtain
\[
\|P_{N_1} u_1 P_{N_2} u_2 P_{N_3} u_3\|_{L^2} \lesssim B_\delta \prod_{j=1}^{3} \|e^{-it\Delta} P_{N_j} u_j\|_{V^2},
\]
where
\[
B_\delta = C_\delta(N_1, N_2, N_3) \left( \ln \frac{(N_2 N_3)^{\frac{3}{2}}}{C_\delta(N_1, N_2, N_3)} + 1 \right)^3 \\
\lesssim C_\delta(N_1, N_2, N_3) (\ln N_2 + 1)^3 \lesssim C_{\delta'}(N_1, N_2, N_3)
\]
for any \( \delta' < \delta \), and the claim follows. \( \square \)
In order to prove Theorem 1.1 we intend to solve the integral equation

\[(10) \quad u(t) = e^{it\Delta}u_0 \mp i\mathcal{I}(|u|^4 u)(t), \quad \mathcal{I}(f)(t) := \int_0^t e^{i(t-s)\Delta} f(s) \, ds,\]

for \(u_0 \in H^1(M)\) by invoking the contraction mapping principle in a small closed ball in the space \(X^1(\tau_0) \cap C(\tau_0, H^1(M))\). For this, it suffices to provide the following estimate, cp. [10, Proposition 4.2] and [11, Proposition 4.1]:

**Proposition 3.4.** For all \(u, v \in X^1(\tau_0)\),

\[\|\mathcal{I}(|u|^4 u) - \mathcal{I}(|v|^4 v)\|_{X^1(\tau_0)} \lesssim (\|u\|_{X^1(\tau_0)}^{\frac{4}{3}} + \|v\|_{X^1(\tau_0)}^{\frac{4}{3}}) \|u - v\|_{X^1(\tau_0)}.\]

**Proof (sketch).** Due to the polynomial structure of the nonlinearity it suffices to prove an estimate for \(\mathcal{I}(\prod_{j=1}^5 \tilde{u}_j)\) where \(\tilde{u}_j\) denotes either \(u_j\) or \(\overline{u_j}\).

This is treated exactly as in [10, Proposition 4.2] (and [11, Proposition 4.1]), where Proposition 3.3 is the replacement for [10, Corollary 3.7]. Note that the contribution \(\Sigma_2\) in [10, pp. 1285–1287] is void in the case \(M = S \times S^2\) (but [10, Lemmas 3.3 and 3.4] hold true on any smooth compact Riemannian 3-manifold \(M\)).

To conclude the proof of Theorem 1.1 one can iterate the local well-posedness to arbitrarily large time intervals \([0, T]\) by using the conservation of the mass and the energy, see [11, pp. 344–347] for more details.

**4. On the necessity of the tri-linear estimate**

As explained above, the tri-linear estimate in Proposition 2.6 on an arbitrary compact boundary-less 3-dimensional Riemannian manifold \(M\) is sufficient to conclude small data global well-posedness in \(H^1(M)\). The proof relies on the contraction mapping principle, which implies that the flow map \(F: u_0 \mapsto u\) is smooth.

Conversely, we can show that the version of the tri-linear estimate in Proposition 2.6 with \(\delta = 0\) is necessary for local well-posedness with a smooth flow. We follow the argument of [4, Remark 2.12], which concerns bi-linear estimates in the context of the cubic NLS.

Fix \(T > 0\) and consider the map

\[F: H^1(M) \to H^1(M), \quad F(u_0) = u(T),\]
where \( u \) is a solution of (1) with initial data \( u(0) = u_0 \). The fifth order differential of \( F \) at the origin is given by

\[
D^5 F(0)(h_1, \ldots, h_5) = 12i \int_0^T e^{i(T-\tau)\Delta_s} \sum_{\sigma} H_{\sigma(1)}(\tau) \overline{H_{\sigma(2)}(\tau)} H_{\sigma(3)}(\tau) \overline{H_{\sigma(4)}(\tau)} H_{\sigma(5)}(\tau) \, d\tau,
\]

where \( H_j(\tau) := e^{i\tau\Delta_s} h_j \) and we sum over the \( 10 = 5! \) of the 5! = 120 permutations \( \sigma \in S_5 \) which give rise to different pairs \((\sigma(2), \sigma(4))\). Indeed, from (10) it follows that

\[
DF(0)(h) = e^{iT\Delta_g} h, \quad D_j F(0) = 0 \quad \text{for} \quad 2 \leq j \leq 4, \quad \text{and we obtain the above formula. If we specify to} \quad h_2 = h_3 = h_4 = h_5 \text{ we obtain}
\]

two contributions

\[
\sum_{\sigma} H_{\sigma(1)} \overline{H_{\sigma(2)}} H_{\sigma(3)} \overline{H_{\sigma(4)}} H_{\sigma(5)} = 6H_1|H_2|^4 + 4\overline{H_1}H_2^3\overline{H_2}.
\]

Now, let us assume that \( D^5 F(0) : (H^1(M))^5 \to H^1(M) \) is bounded. Then, we infer

\[
\left| \int_M D^5 F(0)(h_1, h_2, \ldots, h_2) \overline{H_1}(T) \, dx \right| \lesssim \| h_1 \|_{H^1} \| h_1 \|_{H^{-1}} \| h_2 \|_{H^1}^4.
\]

Because of

\[
\text{Re}\{6|H_1|^2|H_2|^4 + 4\overline{H_1}^2H_2^3\overline{H_2}\} \geq 2|H_1|^2|H_2|^4,
\]

we conclude that

\[
\int_0^T \int_M |H_1|^2|H_2|^4 \, dx \, dt \lesssim \| h_1 \|_{H^1} \| h_1 \|_{H^{-1}} \| h_2 \|_{H^1}^4.
\]

We set \( h_1 = P_{N_1} \phi_1 \), and for \( \phi_2, \phi_3 \in H^1(M) \) we write

\[
e^{it\Delta_s} \phi_2 e^{it\Delta_s} \phi_3 = \frac{1}{4} \left\{ (e^{it\Delta_s} \phi_2 + e^{it\Delta_s} \phi_3)^2 - (e^{it\Delta_s} \phi_2 - e^{it\Delta_s} \phi_3)^2 \right\}
\]

to obtain the bound

\[
\| e^{it\Delta_s} P_{N_1} \phi_1 e^{it\Delta_s} \phi_2 e^{it\Delta_s} \phi_3 \|_{L^2([0,T] \times M)} \lesssim \| P_{N_1} \phi_1 \|_{L^2(M)} \| \phi_2 \|_{H^1(M)} \| \phi_3 \|_{H^1(M)},
\]

which implies the estimate in Proposition 2.6, but only with \( \delta = 0 \).
Remark 3. Patrick Gérard kindly informed us that the necessity of the trilinear estimate with $\delta = 0$ proved here is stated without proof as a special case of Theorem 5.7 i) in the previous work [7].

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References

[1] Jean Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations*. Geom. Funct. Anal., 3(2):107–156, 1993.

[2] Jean Bourgain, *On Strichartz’s inequalities and the nonlinear Schrödinger equation on irrational tori*. In “Mathematical aspects of nonlinear dispersive equations”, volume 163 of “Ann. of Math. Stud.”, pages 1–20. Princeton Univ. Press, Princeton, NJ, 2007.

[3] Nicolas Burq, Patrick Gérard and Nikolay Tzvetkov, *Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds*. Amer. J. Math., 126(3):569–605, 2004.

[4] Nicolas Burq, Patrick Gérard and Nikolay Tzvetkov, *Bilinear eigenfunction estimates and the nonlinear Schrödinger equation on surfaces*. Invent. Math., 159(1):187–223, 2005.

[5] Nicolas Burq, Patrick Gérard and Nikolay Tzvetkov, *Multilinear eigenfunction estimates and global existence for the three dimensional nonlinear Schrödinger equations*. Ann. Sci. École Norm. Sup. (4), 38(2):255–301, 2005.

[6] Nicolas Burq, Patrick Gérard and Nikolay Tzvetkov, *Global solutions for the nonlinear Schrödinger equation on three-dimensional compact manifolds*. In “Mathematical aspects of nonlinear dispersive equations”, volume 163 of “Ann. of Math. Stud.”, pages 111–129. Princeton Univ. Press, Princeton, NJ, 2007.
[7] Patrick Gérard, *Nonlinear Schrödinger equations in inhomogeneous media: wellposedness and illposedness of the Cauchy problem*. In “International Congress of Mathematicians. Vol. III”, pages 157–182. Eur. Math. Soc., Zürich, 2006.

[8] Emil Grosswald, *Representations of integers as sums of squares*. Springer-Verlag, New York, 1985.

[9] Martin Hadac, Sebastian Herr and Herbert Koch, *Well-posedness and scattering for the KP-II equation in a critical space*. Ann. Inst. H. Poincaré Anal. Non Linéaire, 26(3):917–941, 2009. Erratum: ibid. 27 (3) (2010) 971–972.

[10] Sebastian Herr, *The quintic nonlinear Schrödinger equation on three-dimensional Zoll manifolds*. Amer. J. Math., 135(5):1271–1290, 2013.

[11] Sebastian Herr, Daniel Tataru and Nikolay Tzvetkov, *Global well-posedness of the energy-critical nonlinear Schrödinger equation with small initial data in $H^1(T^3)$*. Duke Math. J., 159(2):329–349, 2011.

[12] Sebastian Herr, Daniel Tataru and Nikolay Tzvetkov, *Strichartz estimates for partially periodic solutions to Schrödinger equations in 4d and applications*. J. Reine Angew. Math., 690:65–78, 2014.

[13] Alexandru Ionescu and Benoit Pausader, *The energy-critical defocusing NLS on $T^3$*. Duke Math. J., 161(8):1581–1612, 2012.

[14] Herbert Koch and Daniel Tataru, *Dispersive estimates for principally normal pseudodifferential operators*. Comm. Pure Appl. Math., 58(2):217–284, 2005.

[15] Herbert Koch, Daniel Tataru and Monica Visan, editors, *Dispersive Equations and Nonlinear Waves: Generalized Korteweg-de Vries, Nonlinear Schrödinger, Wave and Schrödinger Maps*, volume 45 of “Oberwolfach Seminars”. Birkhäuser, Basel, 2014.

[16] Benoit Pausader, Nikolay Tzvetkov and Xuecheng Wang, *Global regularity for the energy-critical NLS on $S^3$*. Ann. Inst. H. Poincaré Anal. Non Linéaire, 31(2):315–338, 2014.

[17] Christopher Sogge, *Concerning the $L^p$ norm of spectral clusters for second-order elliptic operators on compact manifolds*. J. Funct. Anal., 77(1):123–138, 1988.

[18] Elias Stein and Guido Weiss, *Introduction to Fourier analysis on Euclidean spaces*. Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.
[19] Nils Strunk, \textit{Strichartz estimates for Schrödinger equations on irrational tori in two and three dimensions}. J. Evol. Equ., 14(4–5):829–839, 2014.

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