On Graviton Propagation in Curved Space-Time Background

E. V. Arbuzova\textsuperscript{a,b, *}, A. D. Dolgov\textsuperscript{b,c,**}, and L. A. Panasenko\textsuperscript{b,***}

\textsuperscript{a} Department of Higher Mathematics, Dubna State University, Dubna, 141983 Russia
\textsuperscript{b} Department of Physics, Novosibirsk State University, Novosibirsk, 630090 Russia
\textsuperscript{c} Bogolyubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Moscow oblast, 141980 Russia

\* e-mail: arbuzova@uni-dubna.ru
\** e-mail: dolgov@nsu.ru
\*** e-mail: l.vetoshkina@g.nsu.ru

Received April 2, 2022; revised April 2, 2022; accepted May 17, 2022

Abstract—Equation describing propagation of gravitational waves (GW) over arbitrary curved space-time background is analyzed. New terms, which are absent in the conventional homogeneous and isotropic Friedmann cosmology, are found. Some examples of realistic metric, where these new terms manifest themselves, are presented. Possible implications to very low frequency GW are briefly discussed.

DOI: 10.1134/S1063776122090126

1. INTRODUCTION

Gravitational wave (GW) propagation over Minkowski and curved backgrounds was considered in vast details in the literature, see e.g. textbooks [1–6]. However, in cosmological situation this consideration is confined to the conformally flat Friedmann–Le’Maitre–Robertson–Walker (FLRW) space-time. In the present work we lift this limitation and derive equation of motion of gravitational waves in an arbitrary space-time metric. We show that in the equation for GW propagation over arbitrary space-time there appear additional terms which are absent in the FLRW case.

Formally in the classical book [1] the expansion of the proper curvature tensors up to the first order over small tensor perturbations $h_{\mu\nu}$ is presented in an arbitrary background metric, see Eq. (108.4). However, the equation is immediately reduced to the empty space with vanishing Ricci tensor, $R_{\mu\nu} = 0$, in order to obtain the canonical wave equation $D^2 h_{\mu\nu} = 0$, Eq. (108.7) of this reference. Nevertheless in [1] it is explicitly stated that the non-zero Ricci tensor would modify Eq. (108.7) and may be of interest.

The corresponding Section 1.5 of [4] is the exact replica of [1], see Eq. (1.172). This equation is also applied to the empty space with vanishing energy-momentum tensor $T_{\mu\nu}$. Quoting [4]: “Outside the matter sources $T_{\mu\nu} = 0$... tells us the $R_{\mu\nu} = 0$.” This immediately excludes Eq. (1.172) from application to cosmology, except for the trivial case of asymptotically high frequency. All interesting effects related to the FLRW metric and moreover to deviations from the FLRW space-time in this case are washed out.

It is claimed in [4] that it is impossible to fix the background metric without introduction of the so called “Low” and “High” terms corresponding to slow and fast varying quantities. This condition contradicts the main stream in the literature on GW propagation over the FLRW space-time. The background metric in the world accepted papers is simply taken in the FLRW form and it does not lead to any problem.

Our paper is a minor (though technically more complicated) modification of the description of the GW propagation in space-times which differ from the FLRW one with examples of realistic background metric defined analogously to the definition of the FLRW background. We have found that there exist some new terms in the equations describing the GW propagation and mixing of the tensor modes with the scalar ones which is absent in the FLRW case.

It is important to note that for the space-time, which differs from the FLRW one, it is impossible to impose the standard gauge conditions valid for Minkowski and FLRW metrics. Because of this complication one can not separate propagation of purely tensor mode from scalar and/or vector ones. The fact that there can exist mixing between scalar and tensor modes in the presence of matter and strong anisotropy (like in the Bianchi type I space-time) or inhomogeneity is well known. We thank the referee for this comment. However, in this work we made more general derivation not specifying any concrete form of metric.

As shown in [2], graviton propagation in empty but curved space-time is governed by the equation
\[ D_\alpha D^\alpha h_{\mu\nu} - 2R_{\alpha\nu\beta\delta} h^{\alpha\beta} = 0, \]  
\[ (1) \]

where \( h_{\mu\nu} \) is the tensor perturbation of the total metric:

\[ \bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}, \]
\[ (2) \]

\( g_{\mu\nu} \) is the background metric, covariant derivative \( D_\alpha \) is defined with respect to the background metric, and \( R_{\alpha\nu\beta\delta} \) is the Riemann tensor in the background space-time, which is supposed to be nonzero, while the Ricci tensor vanishes, \( R_{\mu\nu} = 0 \).

It was argued in [7] that the l.h.s. of Eq. (1) was not changed in the case of the graviton propagation in the homogeneous and isotropic universe described by the canonical FLRW-metric and with space-time filled by ideal liquid. Both the background energy-momentum, \( T_{\mu\nu} \), and the first order correction to it, \( T_{\mu\nu}^{(i)} \), as well as the Ricci tensor, \( R_{\mu\nu} \), are supposed to be non-zero. We expand the total energy-momentum tensor \( T_{\mu\nu} \) as:

\[ T_{\mu\nu} = T_{\mu\nu} + T_{\mu\nu}^{(i)}, \]
\[ (3) \]

where \( T_{\mu\nu} \) is the energy-momentum tensor of the background matter, and \( T_{\mu\nu}^{(i)} \) is that induced by perturbations.

The energy density of gravitational waves is of the second order in \( h_{\mu\nu} \) and is neglected, as are all other second order contributions. It is usually assumed that for the mixed components is zero. This condition \( T_{\mu\nu}^{(i)} \) depends upon the type of matter perturbations and, as is known, it can be fulfilled for the ideal fluid. Under this condition Eq. (1) rewritten in terms of mixed components preserves the same form both in empty and filled space.

There is a subtle point however. If one naively lifts or lowers indices in \( T_{\mu\nu}^{(i)} \) with the background metric, one has to conclude that \( T_{\mu\nu}^{(i)} = 0 \). If so, the equations for GW propagation in terms of mixed and lower components would not be equivalent, for arbitrary and not only for FLRW metric. Indeed, we show in what follows that if \( T_{\mu\nu}^{(i)} = 0 \), then \( T_{\mu\nu} \neq 0 \) and an account of this fact leads to equivalent equations in terms of mixed and lower components in arbitrary metric, as shown in this work.

On the other hand, if one assumes that \( T_{\mu\nu}^{(i)} = 0 \), the resulting equation in the filled space would significantly differ, even in FLRW space-time, from the accepted in the literature canonical one written in terms of the mixed components \( h_{\mu}^\mu \), to say nothing about an arbitrary space-time considered in this paper.

In the conventional approach to the description of the gravitational wave propagation over FLRW background the following transversality and traceless conditions on \( h_{\mu}^\mu \) are imposed:

\[ F_{\mu} h_{\nu}^\mu = 0, \quad \text{and} \quad h_{\mu}^\mu = 0. \]
\[ (4) \]

It is verified in numerous works that Eq. (1) allows to impose these conditions in the curved Einstein (empty) space with \( R_{\mu\nu} = 0 \). It is also well known that the same is true also for the filled FLRW space-time. However, as it is found in what follows, these conditions are violated in the space-time with an arbitrary metric different from the FLRW one. Hence the longitudinal or scalar modes of GW may propagate and in the general case tensor and scalar modes are mixed. This situation is similar to the propagation of the longitudinal mode of the electromagnetic wave in plasma and the appearance of non-zero effective photon mass equal to the plasma frequency.

Our results are fully consistent with the results of [8] where a very special case of Ricci-flat metric is considered (i.e. space-time with \( R_{\mu\nu} = 0 \), as well as with the results presented in the numerous existing literature on GW propagation over FLRW space-time with \( R_{\mu\nu} \neq 0 \).

The noticed above an apparent inconsistency between equations describing GW propagation in terms of \( h_{\mu\nu} \) and \( h_{\mu}^\mu \) is related to the fact that \( T_{\mu\nu}^{(i)} \neq \bar{g}_{\mu\nu} T_{\nu}^{(i)\alpha} \). Indeed

\[ T_{\mu\nu} = \bar{g}_{\alpha\nu} T_{\mu\nu}^{(1)} = (\bar{g}_{\alpha\nu} + h_{\alpha\nu})(T_{\mu\nu}^{(i)} + T_{\mu\nu}^{(1)\alpha}), \]
\[ (5) \]

so \( T_{\mu\nu}^{(i)} = h_{\alpha\nu} T_{\mu\nu}^{(i)} + \bar{g}_{\alpha\nu} T_{\mu\nu}^{(1)\alpha} \). The addition of this term into the r.h.s of Eq. (1) allows to impose the transversality condition on this equation, so the extra propagating modes are not excited over FLRW background in contrast to the concern expressed in [8]. For more detail see the discussion below Eq. (28).

In [6] the detailed derivation of the equation describing GW propagation over FLRW background, which is fully equivalent to Eq. (1), is presented for the background interval rewritten in terms of conformal time \( \tau \):

\[ ds^2 = a^2(\tau)(d\tau^2 - \delta_{ij} dx^i dx^j). \]
\[ (6) \]

under assumption that \( a(\tau) \) is a scalar with respect to coordinate transformation. Here \( \delta_{ij} \) is the Kronecker symbol. The transition to conformal time is convenient to make for conformally flat FLRW metric, but in the case of an arbitrary space-time this transition is not particularly useful because a metric, which is not conformally flat, cannot be transformed to that proportional to the Minkowski metric.

In this paper we generalise Eq. (1) for graviton propagation over arbitrary space-time background. We show that in the l.h.s. of this equation there appear additional terms, which vanish in FLRW metric and which can dominate in the low frequency limit.

Remind that the study of metric perturbations was pioneered in [9], see also [10], where it is shown that \( h_{\mu\nu} \) could be separated into three types according to
their properties with respect to 3-dimensional space transformation (rotation): scalar \((h^i_j\) and \(h^j_i\)), vector \((h^i_j, \partial h^i_j = 0\)), and tensor \((h^j_i)\). The coordinate freedom allows to impose the following conditions on \(h^i_j\):

\[
\partial_j h^i_j = 0, \quad h^i_j = 0, \quad (7)
\]
in locally Minkowsky frame. These conditions can be imposed also globally in conformally flat FLRW metric. So the tensor mode \(h^i_j\) has only two independent components describing massless quanta propagation with two, as it should be, transverse helicity states, i.e. gravitational waves. However, in general space-time this might be incorrect and additional propagating degrees of freedom could be excited. The problem of gauge fixing in arbitrary space-time is considered in Section 2.

Usually these perturbations are considered either over vacuum solutions of the Einstein equations or over background FLRW space-time with the metric:

\[
dx^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j. \quad (8)
\]

In what follows we assume for simplicity that the FLRW metric is spatially flat. Transition from this metric to conformal one \((6)\) is given by the linear transformation between time components of the coordinates, \(dt = a(t)\Delta t\).

Gravitational wave production and propagation in FLRW-cosmology was studied in [7, 11–13]. According to the Parker theorem, see [14, 15], mass-less particles production by conformally flat space-time, metric, such as, in particular, FLRW metric is forbidden if the corresponding field equations are conformally invariant. This is true for massless fermions, conformally coupled scalars, and massless vector fields, up to possible breaking of conformal invariance by the trace anomaly [16]. It was discovered by L. Grishchuk that gravitons can be produced in conformally flat space-time since their equation of motion is not conformally invariant [7]. Gravitational waves could be efficiently produced during cosmological inflation [12, 13]. They may be observed by the cosmic interferometer LISA which will presumably reveal information about the mechanism of the primordial inflation.

The paper is organized as follows. Section 2 is devoted to the choice of gauge for four-dimensional tensor modes in arbitrary space-time and comparison with the popular gauges in FLRW metric. In Section 3 we derive equations describing propagation of gravitational waves in an arbitrary space-time making expansion of the exact Einstein equations up to the first order metric perturbations \(h_{\mu\nu}\). In Section 4 the problems with the first order corrections to the energy-momentum tensor are discussed. Further, in Section 5 the equation for the propagation of the mixed components, \(h^{i\mu}_{\nu}\) is derived. Section 6 is devoted to the proof that the transversality condition \(\nabla^\mu h^{i\mu}_{\nu} - \delta^\mu_{\nu}\nabla^\mu h^{i\mu}/2 = 0\) is compatible with the equation of GW propagation and so it can be imposed in arbitrary metric. In Section 7 a few examples of realistic metrics, different from FLRW one, are presented. Lastly, in Section 8 possible implications of our results to low frequency GWs are briefly discussed.

2. CHOICE OF GAUGE

To avoid confusion expressed by one of the referees we stress again that below we consider an arbitrary space-time metric but not only the FLRW one. In the latter case the choice of gauge is heavily based on particular properties of FLRW metric.

We discuss the choice of gauge conditions which can be imposed on \(h_{\mu\nu}\) following the lines presented in [1]. Let us make the coordinate transformation \(x^\mu = x^\mu + \xi^{(1)}\mu\), where \(\xi^{(1)}\mu\) is a small vector. Under such transformation the first order perturbations, \(h_{\mu\nu}\), of the metric \((2)\) changes as:

\[
\tilde{h}_{\mu\nu} = h_{\mu\nu} - D_{\mu}\xi^{(1)}_{\nu} - D_{\nu}\xi^{(1)}_{\mu}. \quad (9)
\]

Using freedom in the choice of four functions \(\xi^{(1)}\mu\) we can impose the following four conditions:

\[
D_{\nu}\psi^\mu_{\nu} = 0, \quad (10)
\]

where \(\psi^\mu_{\nu} = h^{\mu\nu} - \delta^\mu_{\nu}h/2\) and \(h = h^{\alpha\alpha}\). In the flat space-time case condition \((10)\) leads to the wave equation in the classical form, \(D^2h^{\mu\nu} = 0\). The same equation is true the high frequency (eikonal) limit.

There remains freedom in the coordinate transformation \(\tilde{h}^{\mu\nu} = h^{\mu\nu} - D_{\nu}\xi^{(2)}_{\nu} - D_{\nu}\xi^{(2)}_{\nu}\) with the new parameters \(\xi^{(2)}\mu\) which do not violate condition \((10)\). Accordingly parameters \(\xi^{(2)}\mu\) should satisfy the equation:

\[
D^2\xi^{(2)} + R^2\xi^{(2)} = 0. \quad (11)
\]

Thus we are allowed to use four more functions \(\xi^{(2)}\mu\) to fix the gauge.

This freedom was used in [1] to impose the restrictions \(h_{\mu} = 0\) and \(h = 0\), where \(i\) is the space index. We, however, apply this freedom to demand \(h_{\mu\nu} = 0\) for any \(\alpha\). In this case the condition \(h = 0\) may be invalid.

There is still some freedom to make the coordinate transformation with parameter \(\xi^{(3)}\mu\), which, in addition to \((11)\), satisfies the condition:

\[
D_{\mu}\xi^{(3)} = 0. \quad (12)
\]

Evidently, the transformation with the functions \(\xi^{(3)}\mu\) does not change the value of \(h\).

Detailed discussion of different types of perturbations (scalar, vector, and tensor) can be found e.g. in
the book by Mukhanov [3]. However, all that was done there only for the FLRW spacetime.

3. BASIC EQUATIONS

We start from the exact Einstein equations:

\[ \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = T_{\mu\nu}. \]  \hspace{1cm} (13)

Here and in the rest of the paper \( T_{\mu\nu} \) and \( T_{\mu\nu} \) are related to the physical energy-momentum tensor through the constant factor as:

\[ T_{\mu\nu} = \frac{8\pi}{m_p^2} T_{\mu\nu}^{(\text{phys})}. \]  \hspace{1cm} (14)

Overline means that the corresponding exact (total) quantities are calculated in terms of the total metric \( g_{\mu\nu} \) (see Eqs. (2) and (3)). We will consider first-order tensor perturbations \( h_{\mu\nu} \) over background metric \( g_{\mu\nu} \) and expand the total Ricci tensor and energy-momentum tensor as

\[ \mathcal{R}_{\mu\nu} = R_{\mu\nu} + R_{\mu\nu}^{(1)}, \quad T_{\mu\nu} = T_{\mu\nu} + T_{\mu\nu}^{(1)}, \]  \hspace{1cm} (15)

assuming that all background quantities are taken in the background metric \( g_{\mu\nu} \).

Our goal is to derive the first-order perturbation equation governing evolution of \( h_{\mu\nu} \) without any assumptions about the form of the background metric.

From the condition

\[ \mathcal{G}_{\mu\alpha\nu} \mathcal{G}^{\nu\alpha} = \delta^\nu_{\mu}, \]  \hspace{1cm} (16)

it follows that:

\[ \mathcal{G}^{\mu\nu} = g^{\mu\nu} - h^{\mu\nu}. \]  \hspace{1cm} (17)

For the Ricci scalar we have:

\[ \mathcal{R} = \mathcal{G}_{\alpha\beta} R_{\alpha\beta} = (g^{\alpha\beta} - h^{\alpha\beta})(R^{(1)}_{\alpha\beta} + R^{(1)}_{\alpha\beta}) \]

\[ = g^{\alpha\beta} R_{\alpha\beta} - h^{\alpha\beta} R_{\alpha\beta} + g^{\alpha\beta} R_{\alpha\beta}. \]  \hspace{1cm} (18)

Let us note, that the first order correction to the curvature scalar is not simply obtained from the first order correction to the Ricci tensor through the contraction of indices by the background metric, but contains an extra term:

\[ R^{(1)}_{\alpha\beta} = g^{\alpha\beta} R_{\alpha\beta} - h^{\alpha\beta} R_{\alpha\beta}. \]  \hspace{1cm} (19)

Following the book by Landau and Lifshitz [1] we express the perturbation of the Ricci tensor, \( R_{\mu\nu}^{(1)} \), via metric perturbations, \( h_{\mu\nu} \), as

\[ R_{\mu\nu}^{(1)} = \frac{1}{2} (D_\alpha D_\beta h_{\mu\nu}^{\alpha\beta} + D_\alpha D_\beta h_{\mu\nu}^{\alpha\beta}) \]

\[ - \frac{1}{2} (D_\alpha D_\nu h_{\mu\beta}^{\alpha\beta} + D_\alpha D_\nu h_{\mu\beta}^{\alpha\beta}), \]  \hspace{1cm} (20)

where covariant derivatives \( D_\mu \) are taken with respect to the background metric \( g_{\mu\nu} \) and \( D^\nu = g^{\nu\alpha} D_\alpha \). Here and below we use the background metric, \( g^{\mu\nu} \) and \( g_{\mu\nu} \), to move indices up and down.

According to Eq. (10) \( h_{\mu}^{\nu} \) satisfies the condition:

\[ D_t h_{\mu}^{\nu} = \frac{1}{2} \partial_\nu h. \]  \hspace{1cm} (21)

Using the commutation rules of covariant derivatives we arrive to the result:

\[ R_{\mu\nu}^{(1)} = - \frac{1}{2} D_\alpha D^\alpha h_{\mu\nu} + h_{\nu}^{(1)} R_{\mu\alpha\beta} \]

\[ + \frac{1}{2} (h_{\mu\nu} R^\alpha_\alpha + h_{\nu} R^\alpha_\beta). \]  \hspace{1cm} (22)

Substituting Eqs. (22) and (18) into Eq. (13) and keeping only the first-order quantities we obtain the following equation for tensor perturbations of the metric:

\[ D_\alpha D^\alpha h_{\mu\nu} - 2 h_{\nu}^{\alpha\beta} R_{\mu\alpha\beta} - (h_{\mu\nu} R^\alpha_\alpha + h_{\nu} R^\alpha_\beta) \]

\[ + h_{\nu} R - g_{\nu\nu} h^{\alpha\beta} R_{\alpha\beta} - \frac{1}{2} g_{\mu\nu} D^2 h = - 2 T_{\mu\nu}^{(1)}. \]  \hspace{1cm} (23)

Taking trace we arrive to the equation:

\[ D^2 h + 4 h_{\nu}^{\alpha\beta} R_{\alpha\beta} - h R = 2 g^{\mu\nu} T_{\mu\nu}^{(1)}. \]  \hspace{1cm} (24)

In general case \( h^{\alpha\beta} R_{\alpha\beta} \neq 0 \), so one can conclude that \( h = 0 \). On the other hand in FLRW spacetime the condition \( h^{\alpha\beta} R_{\alpha\beta} = 0 \) is fulfilled and thus equations of motion (23) and (24) do not contradict the condition \( h = 0 \), if the source \( T_{\mu\nu}^{(1)} \) is traceless.

Equation (23) coincides with the conventional Eq. (1) in empty space, where \( R_{\mu\nu} = 0 \), but essentially differs from Eq. (1) in filled space even in the Friedmann space-time. Indeed, in FLRW space-time Eq. (23) takes the form:

\[ \left( \partial_t^2 - \frac{1}{a^2} \partial^2 \right) h_{\nu} - H \partial_\nu h_{\nu} - 2 \left( H^2 + 3 \frac{\dot{a}}{a} \right) h_{\nu} = - 2 T_{\nu}^{(1)}, \]  \hspace{1cm} (25)

while the conventional Eq. (1) with an account of the source term, which we denote as \( T_{\mu\nu}^{(\text{lc})} \), becomes

\[ \left( \partial_t^2 - \frac{1}{a^2} \partial^2 \right) h_{\nu} - H \partial_\nu h_{\nu} - 2 \frac{\dot{a}}{a} h_{\nu} = - 2 T_{\nu}^{(\text{lc})}. \]  \hspace{1cm} (26)

Surprisingly both equations are true and the resolution of the evident inconsistency is hidden in a difference between \( T_{\mu\nu}^{(1)} \) and \( T_{\mu\nu}^{(\text{lc})} \), which according to Eq. (5) is

\[ T_{\mu\nu}^{(\text{lc})} = h_{\mu\nu} T_{\mu}^{(\text{lc})} + g_{\mu\alpha} T_{\nu}^{(\text{lc})} = h_{\mu\nu} R_{\mu\nu}^{\alpha} + g_{\mu\alpha} T_{\nu}^{(\text{lc})}. \]  \hspace{1cm} (27)

Keeping in mind that propagation of GWs in FLRW background is described under assumption that \( T_{\nu}^{(\text{lc})} = 0 \), we can check that the condition \( h_{\nu}^{\nu} = 0 \) is fulfilled in this background metric, simply because \( h_{\nu\nu} R^{\nu\nu} = 0 \).
4. FIRST ORDER CORRECTIONS TO ENERGY-MOMENTUM TENSOR

The only possible resolution of the discrepancy between Eqs. (25) and (26) lays in the difference between $T_{\mu \nu}^{(I)}$ and $T_{\mu \nu}^{(lc)}$. The standard derivation of the conventional Eqs. (1) or (26) is heavily based on the condition $T_{\mu \nu}^{(lc)} = 0$, which is incompatible with seemingly equivalent condition $T_{\mu \nu}^{(I)} = 0$. Indeed let us assume, that the first order corrections to the energy-momentum tensor with mixed components are zero, $T_{\mu \nu}^{(I)} = 0$, as it is usually taken in the case of FLRW space-time filled by the ideal fluid. Then, if we consider equation for $h_{\mu \nu}$, we should put down indices using the full metric tensor $\tilde{g}_{\mu \nu}$ and so:

$$T_{\mu \nu} = \tilde{g}_{\mu \alpha} T_{\alpha}^\nu = T_{\mu \nu} + h_{\mu \alpha} T_{\alpha}^\nu + g_{\mu \alpha} T_{\alpha}^{(I)\nu} = T_{\mu \nu} + h_{\mu \alpha} T_{\alpha}^\nu. \quad (28)$$

The last equality is true, if $T_{\mu \nu}^{(I)} = 0$.

Hence

$$T_{\mu \nu}^{(I)} = h_{\mu \alpha} T_{\alpha}^\nu = (1/2)(h_{\alpha \mu} T_{\alpha}^\nu + h_{\alpha \nu} R_{\alpha}^\mu - h_{\mu \nu} R), \quad (29)$$

where the Einstein equation for the background curvature is used:

$$R_{\mu \nu} - g_{\mu \nu} R/2 = T_{\mu \nu}. \quad (30)$$

Substituting expression (29) for $T_{\mu \nu}^{(I)}$ we obtain from Eq. (23)

$$D_{\alpha} D_{\beta} h_{\mu \nu} - 2 h_{\mu \beta} R_{\alpha \nu \beta} - g_{\mu \nu} h_{\alpha \beta} R_{\alpha \beta} - \frac{1}{2} g_{\mu \nu} D^2 h = 0. \quad (31)$$

Since the last two terms in this equation vanish in the FLRW background we arrive to the canonical Eq. (1).

On the other hand, canonical Eq. (1) or Eq. (26), which follows from Eq. (1) for FLRW metric, is obtained originally from the mixed component equation for $h_{\mu \nu}$. In this case the indices are put down with the background metric $g_{\mu \nu}$ and thus starting from $T_{\mu \nu}^{(0)} = 0$ we arrive to $T_{\mu \nu}^{(lc)} = 0$, so both ways ultimately lead to the same result.

Let us note that the condition $T_{\mu \nu}^{(0)} = 0$ is not necessarily true and there are some realistic cases when it is not fulfilled. For example, $T_{\mu \nu}^{(0)} \neq 0$ in the equation describing graviton-to-photon transition in external magnetic field even over the Minkowsky background, see e.g. [17]. Another known example is presented by anisotropic stresses which could be induced e.g. by neutrinos and photons [18, 19]. They all are treated perturbatively, so the background remains the FLRW one.

Let us stress that if the background metric deviates from the Friedmann one, the last two terms in Eq. (31) may essentially change the character of solutions, especially the first of them since it does not vanish in zero frequency limit.

5. EQUATION FOR THE MIXED COMPONENTS

To make the paper self-contained let us now derive equation for the mixed components, $h_{\mu \nu}^{(I)}$. To this end we start from the equation:

$$\tilde{R}_{\mu} - \delta_{\mu}^{\nu} \tilde{R}/2 = T_{\nu}^{\mu}, \quad (32)$$

and decompose it up to the first order:

$$\tilde{R}_{\mu} = R_{\mu}^{(I)} + R_{\mu}^{(I)}. \quad (33)$$

The first correction $R_{\mu}^{(I)}$ is calculated in the book of Landau and Lifshitz [1] and presented here in Eqs. (20) and (22). The indices are raised according to:

$$R_{\alpha \beta}^{(I)} = R_{\alpha \beta}^{(I)} + g_{\mu \alpha} R_{\nu}^{(I)} + h_{\mu \nu} R_{\beta}^{(I)} - \delta_{\alpha \beta} R, \quad (34)$$

Analogously:

$$\tilde{R} = g_{\alpha \beta} \tilde{R}^{(I)} = R - h_{\alpha \beta} R_{\alpha \beta} + g_{\alpha \beta} R_{\alpha \beta}^{(I)}. \quad (35)$$

Finally in the first order we obtain:

$$g_{\mu \alpha} R_{\nu}^{(I)} - h_{\mu \alpha} R_{\mu \beta} + \frac{1}{2} g_{\mu \alpha} h_{\mu \beta} R_{\alpha \beta}^{(I)} = 0, \quad (36)$$

where according to the discussion in the previous section we took $T_{\mu \nu}^{(I)} = 0$.

In FLRW background the product $g_{\alpha \beta} R_{\alpha \beta}^{(I)}$ vanishes due to the conditions $D_{\mu} h_{\mu \nu}^{(I)} = 0$ and $h_{\mu \nu}^{(I)} = 0$, and $h_{\alpha \beta} R_{\alpha \beta}^{(I)} = 0$, since $R_{\nu} = 0$, $R_{\mu \beta} - \delta_{\mu \beta} h_{\mu \beta} = 0$, and $h_{\mu \nu} = 0$. On the other hand, both these products are generally nonzero if the background deviates from the FLRW one. According to Eq. (22) $g_{\alpha \beta} R_{\alpha \beta}^{(I)} = -D^2 h/2$. As for $h_{\alpha \beta} R_{\alpha \beta}^{(I)}$, it is generally nonvanishing for an arbitrary taken Ricci tensor.

Taking expression (22) for $R_{\beta}^{(I)}$ and using the condition that in FLRW background $R_{\mu \nu}^{(I)} \sim \delta_{\mu \nu}$, we conclude that the last term in Eq. (22) is equal to $h_{\mu \beta} R_{\mu \beta}$ and cancels out with the second term in the l.h.s. of Eq. (36). Finally, we obtain the conventional equation for the mixed components:

$$D_{\mu} D_{\alpha} h_{\mu}^{\alpha} - 2 h_{\mu}^{\alpha} R_{\mu \beta}^{(I)} = 0, \quad (37)$$

where the indices are lifted with the background metric tensor $g_{\mu \nu}^{(I)}$. 

JOURNAL OF EXPERIMENTAL AND THEORETICAL PHYSICS Vol. 135 No. 3 2022
This equation in FLRW background turns into:
\[ \left( \partial^2_t - \frac{\Delta}{a^2} + 3H \partial_i \right) \dot{h}^\mu_\nu = 0. \]  
(38)

Making transformation to \( h_{\mu \nu} \) according to
\[ h^\mu_\nu = g^{\mu \alpha} h_{\alpha \nu} = -h_{\alpha \nu} \delta^{\alpha \nu}/a^2 \]  
(39)
we arrive at
\[ \left( \partial^2_t - \frac{\Delta}{a^2} - H \partial_i - \frac{2 \dot{a}}{a} \right) \dot{h}_{\mu \nu} = 0. \]  
(40)
It coincides with the conventional Eq. (26) but not with Eq. (25) if one assumes that the right hand sides of both equations vanish. However, we have shown that taking \( T_{\mu \nu}^{(1)} = 0 \) in FLRW metric for mixed components leads to \( T_{\mu \nu}^{(1)} \neq 0 \) with lower indices.

It may be instructive to present equations in terms of conformal time, since they are frequently analyzed this way. However, if the metric is not conformally flat, transition to conformal time generally does not make much sense.

6. VERIFICATION OF THE TRANSVERSALITY CONDITIONS FOR NON-ZERO \( h^\mu_\nu \)

It is shown here that the action of the covariant divergence \( D_\mu \) on both sides of the equation for \( h^\mu_\nu \) gives self-consistent results. For easier reading we repeat here some equations already presented above.

We start from the exact equations
\[ R^\mu_\nu - \frac{1}{2} \delta^\mu_\nu R = T^\mu_\nu \]  
(41)
and expand \( T^\mu_\nu = T^{(1)}_\mu_\nu + T^{(1)\mu_\nu} \) and \( R^\mu_\nu = R^{(1)}_\mu_\nu + R^{(1)\mu_\nu} \). So we obtain: for the background metric, as expected:
\[ R^\mu_\nu - \frac{1}{2} \delta^\mu_\nu R = T^{(1)}_\mu_\nu. \]  
(42)
However, we use the first order correction to \( R_{\mu \nu} \) as is presented in [1]:
\[ R^{(1)}_{\mu \nu} = \frac{1}{2} (D_\mu D_\nu h^\beta_\alpha + D_\nu D_\mu h^\beta_\alpha - D^2 h_{\mu \nu} - D_{\mu \nu} D_h), \]  
(43)
so we rise one lower index using \( g^{\mu \alpha} \) and find:
\[ D^2 h^\alpha_\mu + 2R^{\alpha \mu \beta \gamma} h^\beta_\gamma + R^{\alpha \mu \beta} h^\beta_\mu \]  
\[ - R^{\alpha \beta \mu \nu} - \delta^\alpha_\mu \left( R^{\beta \mu \nu} h^\beta_\nu + \frac{1}{2} D^2 h \right) = -2T^{(1)\mu_\nu}, \]  
(44)
where \( D^2 = g^{\mu \beta} D_\mu D_\beta \) and it is taken into account that \( h = h^\alpha_\alpha \neq 0 \) and \( D_\mu h^\alpha_\mu = \partial_\nu h^\alpha_\nu /2 \).

Acting by \( D_\mu \) on the first term and applying commutation rules for the covariant derivatives we come to:
\[ D_\mu D^2 h^\alpha_\mu = D_\mu (R^{\alpha \mu \beta \gamma} h^\beta_\gamma + R^{\alpha \mu \beta} h^\beta_\mu) \]  
\[ - R^{\alpha \beta \mu \nu} D_\alpha h^\beta_\nu + \frac{1}{2} D^2 (D_\mu h). \]  
(45)
Substituting Eq. (45) into the divergence of the l.h.s. of Eq. (44) we come to the following result
\[ D_\mu (R^{\alpha \mu \beta \gamma} h^\beta_\gamma + R^{\alpha \mu \beta} h^\beta_\mu) \]  
\[ + \frac{1}{2} D^2 (D_\mu h) + 2D_\mu \left( R^{\alpha \beta \mu \nu} h^\beta_\nu + h^\beta_\gamma D_\nu R^{\alpha \beta \mu} \right) \]  
\[ + \frac{1}{2} R^{\alpha \beta} D_\mu h - D_\nu (R^{\alpha \beta \mu} h^\beta_\nu) - \frac{1}{2} D_\nu D^2 h = -2D_\mu (T^{(1)\mu_\nu}). \]  
(46)
The first and the eighth terms in the l.h.s. of this equation are canceled, the second and the fifth terms are reduced to \( D_\mu (R^{\alpha \beta \mu \nu} h^\beta_\nu) \) and we get:
\[ D_\mu (R^{\alpha \beta \mu \nu} h^\beta_\nu) - R^{\alpha \beta} D_\mu h^\beta_\nu + \frac{1}{2} D^2 (D_\mu h) + h^\beta_\gamma D_\nu R^{\alpha \beta \mu} \]  
\[ + \frac{1}{2} R^{\alpha \beta} D_\mu h - D_\nu (R^{\alpha \beta \mu} h^\beta_\nu) - \frac{1}{2} D_\nu D^2 h = -2D_\mu (T^{(1)\mu_\nu}). \]  
(47)
Using commutation relations of covariant derivatives we find:
\[ D^2 D_\nu h - D_\nu D^2 h = R^{\alpha \beta} D_\mu h. \]  
(48)
Eventually we arrive to
\[ h^{\alpha \beta} D_\mu R^{\alpha \beta \mu \nu} \]  
\[ + h^{\alpha \beta} D_\mu R^{\alpha \beta \mu} - D_\nu (R^{\alpha \beta \mu} h^\beta_\nu) + R^{\alpha \beta} D_\mu h \]  
\[ = -2D_\mu (T^{(1)\mu_\nu}). \]  
(49)
The first term here can be rewritten via the Bianchi identity as:
\[ h^{\alpha \beta} D_\mu R^{\alpha \beta \mu \nu} = \eta^{\alpha \beta} (D_\nu R^{\alpha \beta \mu} - D_\alpha R^{\alpha \beta \nu}) \]  
(50)
leading to:
\[ h^{\alpha \beta} (D_\nu R^{\alpha \beta \mu} - D_\alpha R^{\alpha \beta \nu}) + h^{\alpha \beta} D_\mu R^{\alpha \beta \mu} - h^{\alpha \beta} D_\nu R^{\alpha \beta \nu} \]  
\[ - R^{\alpha \beta} D_\mu h^\beta_\nu + R^{\alpha \beta} D_\mu h = -2D_\mu (T^{(1)\mu_\nu}). \]  
(51)
All the terms containing the derivatives of the Ricci tensor in the equation above neatly cancel out and finally we come to the following equation to be verified:
\[ R^{\alpha \beta} D_\mu h^\beta_\nu - R^{\alpha \beta} D_\mu h = 2D_\mu (T^{(1)\mu_\nu}). \]  
(52)
Now we must check if the l.h.s. and r.h.s. of the Eq. (52) are equal. To this end we will apply the following conservation conditions:
\[ \overline{D}_\mu T^\mu_\nu = 0 \quad \text{and} \quad D_\mu T^\mu_\nu = 0. \]  
(53)
So in the first perturbation order we obtain
\[ \overline{D}_\mu T^\mu_\nu = D_\mu T^\mu_\nu + D_\mu T^{(1)\mu_\nu} + R^{(1)\mu \alpha \beta} T^\alpha_\nu - R^{(1)\mu \alpha \beta} T^\beta_\nu. \]  
(54)
The third term in the r.h.s. of this equation is zero because the first-order perturbation corrections to the Christoffel symbols has the following form

$$\Gamma^{(3)}_{\nu\alpha} = \frac{1}{2} (D_\nu h^\mu_{\alpha} + D_\alpha h^\mu_{\nu} - D^\mu h_{\alpha\nu}),$$  \hspace{1cm} (55)$$

and hence $\Gamma^{(3)}_{\alpha\mu} = D_\alpha h/2$.

From Eqs. (54) and (55) we obtain

$$D_\mu T^{(3)}_{\nu} = (R^\mu_{\nu\mu} D_\nu h^\alpha_{\alpha} - R^\mu_{\nu\mu} D_\mu h)/2.$$  \hspace{1cm} (57)$$

Thus we see that both sides in Eq. (52) are equal. It means that condition (10), $D_\mu \psi^\alpha_{\mu} = 0$, is compatible with Eq. (44).

Let us calculate the trace of Eq. (44). We will find:

$$D^2 h + 2 R^\mu_{\mu} h^\nu = 2 T^{(3)}_{\nu}. \hspace{1cm} (58)$$

Seemingly, this equation is inconsistent with Eq. (24).

However, if we use the relation between $T^{(3)}_{\nu}$ and $g^{\mu\nu} T^{(1)}_{\mu\nu}$, which we can obtain from Eq. (28), we come to:

$$T^{(3)}_{\nu} = g^{\mu\nu} T^{(1)}_{\mu\nu} - h^\nu_{\mu} \left(R^\mu_{\nu} - \frac{1}{2} \delta^\mu_{\nu} R \right) \hspace{1cm} (59)$$

and the consistency is restored.

In the FLRW background $R^\mu_{\nu\mu} D_\nu h^\alpha_{\alpha} = R^\mu_{\nu\mu} h^\nu_{\mu} = 0$ and the usual condition $T^{(3)}_{\nu} = 0$ allows for $h = 0$. However, in arbitrary background it seems generally impossible to impose both conditions of covariant conservation of the source and its zero trace.

7. REALISTIC METRICS DIFFERENT FROM FLRW ONE

Interesting deviations of the cosmological metric from the Friedmann one is induced by density perturbations over the classical FLRW background. They are considered in detail in books [3, 5, 6, 20, 21]. Assume that there exists a cloud of matter with energy density and pressure which are different from the average cosmological ones. Generally speaking the cloud may be anisotropic but the impact of the additional term $h^\mu_{\nu} \delta^\mu \delta^\nu$ (36) would manifest itself even for isotropic distribution of matter in the cloud. So, for simplicity, we confine ourselves to this case.

We choose Schwarzschild-like isotropic coordinates in which the metric takes the form:

$$ds^2 = A dt^2 - B \delta_{ij} dx^i dx^j,$$  \hspace{1cm} (60)$$

where the functions $A$ and $B$ depend upon $r$ and $t$. The corresponding Christoffel symbols are:

$$\Gamma^{\mu}_{\alpha\beta} = \frac{A'}{2A}, \quad \Gamma^{\nu}_{\mu\beta} = \frac{\delta^\nu_{\beta}}{2A}, \quad \Gamma^{\nu}_{\mu\nu} = \frac{\delta^\nu_{\beta}}{2B}, \hspace{1cm} (61)$$

The corresponding Ricci tensor is given by:

$$R^{\mu}_{\mu} = \frac{\Delta A}{2B} B^2 + \frac{3B^2}{4B^2} + \frac{3AB}{4AB} \frac{\partial^\mu A_j B_j}{AB} - \frac{\partial^\mu A_j A_j}{4AB}, \hspace{1cm} (62)$$

$$R^{\nu}_{\nu} = - \frac{\partial A}{B} B + \frac{\partial^\nu A_j B_j}{2AB}, \hspace{1cm} (63)$$

$$R^{\mu}_{\nu} = \frac{\delta^\mu_{\nu}}{2A} \left( \frac{\partial^\nu A_j B_j}{4AB} + \frac{\partial^\nu B_j B_j}{4AB} - \frac{\partial^\mu A_j A_j}{4AB} \right) \hspace{1cm} (64)$$

Here and in what follows the upper space indices are raised with the Kronecker delta, $\partial^\alpha A = \delta^\alpha \partial_\alpha A$.

The space derivatives of an arbitrary function of $r$ are equal to:

$$\partial_r f = \frac{f'}{r}, \quad \partial_i \partial_r f = \left( \frac{\delta^2}{r^2} - \frac{x_i x_i}{r^3} \right) f'' + \frac{x_i x_i}{r^2} f'' \hspace{1cm} (65)$$

where prime means differentiation over $r$. Thus $R_{\nu}$ contains some other terms, except those proportional to $\delta_{ij}$, and consequently the product $h^\mu_{\nu} \delta^\mu \delta^\nu$ is generally nonvanishing.

Usually perturbations are small, so both $A$ and $B$ weakly deviate from unity and linear in $A$ and $B$ terms dominate in $R_{\nu}$. For a particular shape of the perturbations $(A - 1) \sim r^2$ and $(B - 1) \sim r^2$, as is considered in [21], and so $R_{\nu} \sim \delta_{ij}$ and $h^\mu_{\nu} \delta^\mu \delta^\nu = 0$. However, it is not the general case. Moreover, perturbations are known to rise as the cosmological scale factor at matter dominated regime, and so the Ricci tensor may become close by magnitude to its background value or even exceed it.

There are some more physically interesting metrics for which the product $h^\mu_{\nu} \delta^\mu \delta^\nu$ is nonzero. One simple example is presented by a collapse of dust-like matter, described e.g. in [1]. This is the so-called pressureless
ON GRAVITON PROPAGATION

8. CONCLUSIONS

We have found that in an arbitrary curved background it is generally impossible to satisfy both standard conditions $D_\mu h^{\mu \nu} = 0$ and $h^{\mu \nu} = 0$ on tensor perturbations. As we have shown the condition $D_\mu \psi^{\mu \nu} u = 0$ (10) can be imposed in any space-time. Of course tensor modes may propagate in any background but they would not be pure tensor modes or pure transverse modes. According to the equations derived in this work tensor and scalars modes are mixed and propagate together.

In some sense these phenomena resemble the propagation of electromagnetic waves in plasma, when longitudinal modes are excited. Moreover, in inhomogeneous and anisotropic plasma propagating modes may be even more cumbersome. However, in high frequency limit (eikonal approximation) we return to “normality.”

The discovered in this paper additional term in the equation for gravitational waves (GW) propagation in a background, which differs from the FLRW one, may have an essential impact on the low frequency tail of GW spectrum in particular, of GWs which had been produced during inflation, see [12, 13]. Their intensity at low frequencies might be noticeably suppressed. Because of that, the strict limit on very long gravitational waves obtained from the CMB polarization data [22] does not necessarily implies that the traditional inflation induced by a scalar field, inflaton, [23] is excluded. A study of this problem is in progress.

ACKNOWLEDGMENTS

We thank V.A. Rubakov for a very important comment and discussions.

FUNDING

This work was supported by grant 20-42-09010 from the Russian Science Foundation.

REFERENCES

1. L. D. Landau and E. M. Lifshitz, Course of Theoretical Physics, Vol. 2: The Classical Theory of Fields (Nauka, Moscow, 1988; Pergamon, New York, 1971).
2. C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation (W. H. Freeman, San Francisco, 1973).
3. V. Mukhanov, Physical Foundations of Cosmology (Cambridge Univ. Press, New York, 2005).
4. M. Maggiore, Gravitational Waves (Oxford Univ. Press, Oxford, 2008).
5. S. Weinberg, Cosmology (Oxford Univ, Press, Oxford, 2008).
6. D. S. Gorbunov and V. A. Rubakov, Introduction to the Theory of the Early Universe: Cosmological Perturbations and Inflationary Theory (World Scientific, Hackensack, USA, 2011).
7. L. P. Grishchuk, Sov. Phys. JETP 40, 409 (1975).
8. S. Deser, M. Henneaux, Class. Quantum. Grav. 24, 1683 (2007); arXiv: gr-qc/0611157.
9. E. M. Lifshitz, Zh. Eksp. Teor. Phys. 16, 587 (1946).
10. E. M. Lifshitz and I. M. Khalatnikov, Sov. Phys. Usp. 6, 495 (1964).
11. A. V. Zakharov, Sov. Phys. JETP 50, 221 (1979).
12. A. A. Starobinsky, JETP Lett. 30, 682 (1979).
13. V. A. Rubakov, M. V. Sazhin, and A. V. Veryaskin, Phys. Lett. B 115, 189 (1982).
14. L. Parker, Phys. Rev. Lett. 21, 562 (1968).
15. L. Parker, Phys. Rev. 183, 1057 (1969).
16. A. D. Dolgov, Sov. Phys. JETP 54, 223 (1981).
17. A. D. Dolgov and D. Ejlli, J. Cosmol. Astropart. Phys. 12, 003 (2012); arXiv: 1211.0500 [gr-qc]
18. S. Weinberg, Phys. Rev. D 69, 023503 (2003); astro-ph/0306304.
19. S. Saga, K. Ichiki, and N. Sugiyama, Phys. Rev. D 91, 024030 (2015); arXiv: 1412.1081 [astro-ph.CO].
20. C. Bambi and A. D. Dolgov, Introduction to Particle Cosmology (Springer, New York, 2015).
21. E. V. Arbuzova, A. D. Dolgov, and L. Reverberi, Phys. Lett. B 739, 279 (2014); arXiv: 1406.7104 [gr-qc].
22. Planck Collab., Astron. Astrophys. 641, A6 (2020); arXiv: 1807.06209.
23. A. D. Linde, Phys. Rep. 333, 17 (2000).

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.