A multi item probabilistic inventory model

Dharma Lesmono and Taufik Limansyah
Department of Mathematics, Universitas Katolik Parahyangan
Jalan Ciumberleuit 94 Bandung 40141 Indonesia
E-mail: {jdharma, taufik.limansyah}@unpar.ac.id

Abstract. In this paper, we develop a mathematical model for an inventory problem which consists of several products. Demands for these products are following Gamma distribution. Our objective is to determine the optimal ordering quantity that minimizes the total cost for each product. This is called an individual replenishment policy. The total cost consists of the purchasing cost, ordering cost, handling and shortage costs. Using the same model, we also develop a periodic review model that determines an optimal ordering time for all the products. This is called a joint replenishment policy. We then compare the total cost of the individual and the joint replenishment policy and determine a policy that gives minimum total cost. Our numerical experiments show that in general the joint replenishment policy is preferable than the individual replenishment policy. The joint replenishment policy gives higher shortage cost, but less in ordering and handling costs than the individual replenishment policy.

1. Introduction
Inventory management becomes one important thing that should be considered by the management related to the cash flow of the company especially with the funds contained in the inventory. Having a lot of inventory has a consequence of the unavailability of funds for other sectors in the company since most of the funds are invested in the inventory. On the other hand, having less inventory in hands will increase the probability of the unfilled demand, and this will cost the reputation of the company in the future. Therefore we need a mathematical model to find the optimal inventory to help company dealing with this problem. In order to build a mathematical model, we need to know that there are several costs involved in handling the inventory such as purchasing, ordering, holding and shortage costs. For retailer, two basic questions to be answered related to the inventory are when to order and how many (the optimal quantity). These two questions are the building block in the mathematical model in order to find the optimal replenishment policy.

There have been a lot of mathematical models for inventory problems developed in the last few decades and the first and most simple one is the Economic Order Quantity (EOQ) introduced by Wilson (Tersine [8]). This EOQ model has become the basis for the development of the more advanced and complex inventory models. Some factors that have been incorporated in the models involving multi-item, quantity discount, deterioration rate, deterministic or probabilistic models in terms of demand distributions.

A mathematical model for production system control for one item with uncertain deterioration rate has been developed by Bukhari [1], while Ferguson et.al [2] considers a model for perishable goods by considering time dependent non-linear holding cost. Muckstadt and Sapra [6] and Zhang and Wang [11] have developed a multi-item inventory problem with
storage capacity restriction, while a joint inventory model to determine the optimal inventory replenishment product assortment, shelf space and display area is discussed in Hariga et al [4]. Kasthuri, et.al. [5] developed a fuzzy multi-item inventory model by incorporating storage space and production cost. The case of multi-product newsboy problem for uncertain demand is studied in Zhang and Du [10]. Namit and Chen [7] and Tyworth and Ganeshan [9] discussed the process and method of finding the optimal order quantity and time when demand during lead time is Gamma distributed.

In this paper, we develop a mathematical model for an inventory problem which consists of several products. Demands for these products are following Gamma distribution. Our objective is to determine the optimal ordering quantity that minimizes the total cost for each product. This is called an individual replenishment policy. The total cost consists of the purchasing cost, ordering cost, handling and shortage costs. Using the same model, we also develop a periodic review model that determines an optimal ordering time for all the products. This is called a joint replenishment policy. We then compare the total cost of the individual and the joint replenishment policy and determine a policy that gives minimum total cost. Our numerical experiments show that in general the joint replenishment policy is preferable than the individual replenishment policy. The joint replenishment policy gives higher shortage cost, but less in ordering and handling costs than the individual replenishment policy.

The organization of this paper is as follows. In Section 2 we introduce the mathematical model which consists of the model for individual replenishment policy and joint replenishment policy along with an algorithm to find the optimal solution. Section 3 deals with numerical experiments as an application of the models discussed in Section 2. Here, comparison between joint replenishment policy and individual replenishment policy is performed and also the effect of the joint ordering cost on the optimal policy. Conclusions and further research are relegated in Section 4.

2. The Model

Probabilistic EOQ model is an inventory model that is close to the real situation retailer has to face. In reality, demand will vary from time to time. This probabilistic inventory model will incorporate the variation of the demand and uncertain lead time. Demand variation will cause a shortage especially during lead time when retailer only has a limited amount of goods to cover the demand during lead time and the goods ordered have not been arrived yet. Based on that situation, there are three possibilities that can happen for the probabilistic inventory model. The first one is when demand during lead time is constant but the lead time itself varies. The second is when lead time is constant but demand during lead time is not, and the last possibility is when both lead time and demand during lead time are varies. In this section we will discuss the model for the individual and joint replenishment policy.

2.1. Individual Replenishment Policy

In this subsection we derive a mathematical model for individual replenishment policy by considering that demand during lead time follows a class of Gamma distribution with constant lead time. From this model, we will then determine the optimal order quantity and reorder point for each item. Some assumptions in developing this model are as follows.

(i) Demand during lead time follows Gamma distribution.
(ii) The storage capacity is unlimited.
(iii) Lead time is known and constant.
(iv) Shortages occur when demand exceeds the inventory level during lead time.
(v) Unfilled demand during lead time will be replenished in the next period.
(vi) The order quantity is always the same in every replenishment.
(vii) Holding cost depends on the average goods that are stored.

We use the following notations for developing our model.

- $D$ = Average demand within one planning period.
- $P$ = Purchase cost/unit.
- $Q$ = Optimal order quantity.
- $S$ = Ordering cost per order.
- $h$ = holding cost fraction per unit per planning period.
- $\pi$ = shortage cost/unit.
- $1 - \omega = service level$, retailer confidence level to fulfill customers’ demand. (0 < $\omega$ < 1).
- $R$ = reorder point.
- $f(x) = \frac{1}{\beta \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}}$ is Gamma density function for demand during lead time.
- $TAC$ = Total inventory cost.
- $U$ = Quantity limit where there is price change.

The situation of the problem is depicted in Figure 1 that represents the inventory level from time to time during the planning period. From that figure, we can see that shortage can occur during lead time when the demand during lead time exceeds the reorder point $R$.

The total inventory cost considered in building the model consists of purchasing cost, ordering cost, holding cost and shortage cost.

(i) Purchasing cost is the cost spent by the retailer to buy goods from the supplier. If the annual demand is $D$ units, then the annual purchasing cost is $PD$. 

---

**Figure 1.** Probabilistic EOQ Model (Taha, 2011)
(ii) Ordering cost is the cost occurs every time an order is placed. If the ordering cost is $S$, then the annual ordering cost is $\frac{SD}{Q}$.

(iii) Holding cost usually occurs for storing and maintaining goods, such as rent or insurance cost for goods. It is usually represented as fraction of the purchasing cost per unit, viz. $Ph$. In our model, the annual holding cost is given by $Ph(\frac{Q}{2} + R - E(X))$, where $X$ is random variable for demand during lead time.

(iv) The shortage cost in our model occurs when demand during lead time exceeds the reorder point. The amount of the annual shortage cost can be formulated as $\pi D Q \left[ \int_R^{\infty} (x - R) f(x)dx \right]$.

Therefore, the annual total inventory cost for our model is:

$$TAC(Q, R) = PD + \frac{SD}{Q} + Ph\left( \frac{Q}{2} + R - E(X) \right) + \frac{\pi D}{Q}\left[ \int_R^{\infty} (x - R) f(x)dx \right]$$

(1)

In order to find the minimum total annual inventory cost, conditions $\frac{\partial TAC}{\partial Q} = 0$ and $\frac{\partial TAC}{\partial R} = 0$ must be satisfied. The condition $\frac{\partial TAC}{\partial Q} = 0$ will give:

$$Q = \sqrt{2D(S + \pi \int_R^{\infty} (x - R) f(x)dx)}$$

(2)

and from condition $\frac{\partial TAC}{\partial R} = 0$, we have:

$$\int_R^{\infty} f(x)dx = \frac{PhQ}{\pi D}$$

(3)

The right hand side of (3) can be interpreted as the inability of the retailer to fulfill customers’ demand. We can write (3) as

$$\int_R^{\infty} f(x)dx = \omega$$

(4)

The procedure to obtain the optimal order quantity and reorder point for our model follows the Hadley-Whitin algorithm (Hadley and Whitin [3]) as follows.

(i) First we start with the EOQ, that is $Q = \sqrt{\frac{2DS}{\pi \omega}}$.

(ii) Calculate the value of $R$ using (3).

(iii) Calculate $Q$ using (2) with the value of $R$ found in step 2.

(iv) Repeat step 2 and 3 until we found the optimal value of $Q$ and $R$, that is the difference between each value of $Q$ and $R$ in each iteration is less than 1.

(v) Calculate the optimum $TAC$ using (1).

2.2. Joint Replenishment Policy

For the joint replenishment policy, the decision variable in our model is the optimal time of replenishment. This is also called the periodic review model. In this model we assume that all item are ordered at the same time (joint order). The ordering time interval is fixed but the ordering amount for each item varies.

When items are ordered together, some costs such as ordering cost should be cheaper than when they are ordered separately. We denote the ordering cost for joint replenishment policy
### Table 1. Data

|                   | Product 1 | Product 2 | Product 3 |
|-------------------|-----------|-----------|-----------|
| Annual Demand     | 550       | 400       | 800       |
| Ordering cost ($) | 5         | 7         | 4         |
| Holding cost fraction | 0.01    | 0.015     | 0.01      |
| Shortage cost ($) | 1.5       | 2         | 1.2       |
| Purchase cost ($) | 12        | 15        | 8         |
| $\alpha$         | 5         | 5         | 3         |
| $\beta$          | 2         | 1         | 2         |

by $S^*$ where $S^*$ is less than the summation of all individual ordering cost. Since the decision variable is the optimal time of replenishment, $T$, we use the relation $Q_i = D_i T$ for each item $i$ in the model for the individual replenishment policy and then find the optimal $T$. Our model then becomes:

$$\text{TAC}(T) = \frac{S^*}{T} + \sum_{i=1}^{n} \{P_i D_i + P_i h_i \left( \frac{T D_i}{2} + R_i - E(X_i) \right) + \pi_i \int_{R_i}^{\infty} (x_i - R_i) f(x_i) dx_i \}$$  \hspace{1cm} (5)

where $n$ is the number of items jointly ordered. Taking the first derivative of TAC(T) and equates it to zero will give the optimal time of replenishment as follows.

$$T = \sqrt{\frac{2(S^* + \sum_{i=1}^{n} \pi_i \int_{R_i}^{\infty} (x_i - R_i) f(x_i) dx_i)}{\sum_{i=1}^{n} P_i D_i h_i}}$$  \hspace{1cm} (6)

### 3. Numerical Experiments

In this section we consider a retailer who sells three different products to customer and orders these products from one supplier. The demand of each product is Gamma distributed with different values of parameters ($\alpha$ and $\beta$), and all the data is given in Table 1.

For the individual replenishment policy, using Hadley-Whitin algorithm, we have the following results as in Table 2. The optimal ordering quantity and reorder point occur for each product are also given with the total inventory cost of 19087.54.

For the joint replenishment policy, using the joint ordering cost $S^* = 9$ the results are given in Table 3.

From Table 3, the optimal ordering time $T$ is around 0.29 years, the optimal order quantity for each product are 161, 117 and 234 respectively which are less than the optimal order quantities for the individual replenishment policy. This fact has consequences in the holding cost and shortage cost. The holding cost for the joint replenishment policy is less than the holding cost in the individual joint policy but the shortage cost is larger. These consequences make sense since the more goods we have in hands, the more we have to spend for holding cost but the less we spend for shortage cost. Overall, in this numerical experiments, the total inventory cost for the joint replenishment policy is less than the total inventory cost for the individual replenishment policy.

It is also possible the individual replenishment policy gives lower total inventory cost than the joint replenishment policy in this model. This can happen when the saving from the ordering
Table 2. Results for Individual Replenishment Policy

| Product   | 1     | 2     | 3     |
|-----------|-------|-------|-------|
| Q         | 218   | 160   | 800   |
| R         | 20    | 10    | 15    |
| Total Purchase Cost | 6600 | 6000  | 6400  |
| Total Ordering Cost   | 12.67 | 17.58 | 11.21 |
| Total Holding Cost    | 14.2  | 18.89 | 12.10 |
| Total Shortage Cost   | 0.35  | 0.34  | 0.20  |
| Total Inventory Cost  | 6627.22 | 6036.81 | 6423.51 |
| TAC       | 19087.54 |

Table 3. Results for Joint Replenishment Policy

| Product   | 1     | 2     | 3     |
|-----------|-------|-------|-------|
| T (year)  | 0.29  |
| Q         | 161   | 117   | 234   |
| Total Purchase Cost | 6600 | 6000  | 6400  |
| Total Ordering Cost   | 30.86 |
| Total Holding Cost    | 12.47 | 14.38 | 11.83 |
| Total Shortage Cost   | 0.48  | 0.46  | 0.25  |
| TAC       | 19070.73 |

cost for the joint replenishment policy is not substantial compared with the sum of the individual ordering cost for the individual replenishment policy. In our numerical experiments, the joint ordering cost is 30.84 (Table 3) compared to the sum of the individual ordering cost of 41.46 (Table 2), a saving of 10.62.

4. Conclusion and Further Research
In this paper, we have developed a mathematical model for multi-item probabilistic inventory problem where demand for each product is Gamma distributed with different values of $\alpha$ and $\beta$.

We have derived an optimal ordering quantity and reorder point for the individual replenishment policy and an optimal ordering time for the joint replenishment policy. In our numerical experiments, we then compare the individual replenishment policy and joint replenishment policy to find the optimal policy. From our numerical experiment with three products and different distribution for demand during lead time, we find that the joint replenishment policy gives lower total annual inventory cost compared to the individual replenishment policy.

There are some limitations in our model since it has not considered several factors such as deterioration rate of the products, quantity discounts and perhaps non-linear holding cost involved. These factors are interesting to be pursued in further research, analyze their impact on the optimal policy and the optimal ordering quantity, time and reorder point.

Acknowledgments
The authors would like to thank the Institute of Research and Community Service Universitas Katolik Parahyangan for funding this research.
References

[1] Bukhari F 2011 Adaptive control of a production-inventory model with uncertain deterioration rate, *Appl. Math. 2* pp 1170-74

[2] Ferguson M, Jayaraman V and Souza G C 2007 Note : An application of the EOQ model with nonlinear holding cost to inventory management of perishables, *European J. Oper. Res. 180*(1) pp 485-490

[3] Hadley G and Whitin T M 1963 *Analysis of Inventory Systems* (Englewood Cliffs, N.J.: Prentice Hall Inc.)

[4] Hariga M A, Al-Ahmari A and Abdel-Rahman A and Mohamed 2007 A joint optimisation model for inventory replenishment product assortment, shelf space, and display area allocation decisions *European J. Oper. Res. 181* pp 239-251.

[5] Kasthuri R, Vasanthi P, Ranganayaki S and Seshaih C V 2011 Multi-item Fuzzy inventory model involving three constrains : A Karush-Kuhn-Tucker conditions approach, *Amer. J. Oper. Res. 1* pp 155-159.

[6] Muckstadt J A and Sapra A 2010 *Principles of Inventory Management* (New York: Springer)

[7] Namit K and Chen J 1999 Solution to the $< Q, r >$ inventory model for Gamma Lead-Time Demand *Int. J. Phys. Distrib. Logist. Manag. 29*(2) pp 138-151.

[8] Tersine R J 1994 *Principles of Inventory and Material Management* 4th ed (New Jersey: Prentice Hall)

[9] Tyworth J E and Ganeshan R 2000 Research note: A note on solution to the $< Q, r >$ inventory model for Gamma Lead-Time Demand *Inter. J. Phys. Distrib. Logist. Manag. 30*(6) pp 534-539

[10] Zhang B and Du S 2010 Multi-product newsboy problem with limited capacity and outsourcing, *European J. Oper. Res. 202* pp 107-113

[11] Zhang B and Wang X 2011 Optimal policy and simple algorithm for a deteriorated multi-item EOQ problem, *Amer. J. Oper. Res. 1* pp 46-50