ON MODEL-INDEPENDENT PRICING/HEDGING USING SHORTFALL RISK AND QUANTILES

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Abstract. We consider the pricing and hedging of exotic options in a model-independent set-up using shortfall risk and quantiles. We assume that the marginal distributions at certain times are given. This is tantamount to calibrating the model to call options with discrete set of maturities but a continuum of strikes. In the case of pricing with shortfall risk, we prove that the minimum initial amount is equal to the super-hedging price plus the inverse of the utility at the given shortfall level. In the second result, we show that the quantile hedging problem is equivalent to super-hedging problems for knockout options. These results generalize the duality results of [5, 6] to the model independent setting of [1].

1. SET-UP AND THE MAIN RESULTS

We will follow the setting in [1]. Assume that in the market, there is a single risky asset at discrete times $t = 1, \ldots, n$. Let $S = (S_i)_{i=1}^n$ be the canonical process on the path space $\mathbb{R}_+^n$, i.e., for $(s_1, \ldots, s_n) \in \mathbb{R}_+^n$ we have that $S_i(s_1, \ldots, s_n) = s_i$. The random variable $S_i$ represents the price of the risky asset at time $t = i$. We denote the current spot price of the asset as $S_0 = s_0$. In addition, we assume that our model is calibrated to a continuum of call options with payoffs $(S_i - K)^+$, $K \in \mathbb{R}_+$ at each time $t = i$, and price

$$C(i, K) = \mathbb{E}^Q [(S_i - K)^+] .$$

It is well-known that knowing the marginal $S_i$ is equivalent to knowing the prices $C(i, K)$ for all $K \geq 0$; see [3]. Hence, we will assume that the marginals of the stock price $S = (S_i)_{i=1}^n$ are given by $S_i \sim \mu_i$, where $\mu_1, \ldots, \mu_n$ are probability measures on $\mathbb{R}_+$. Let

$$\mathcal{M} := \{Q \text{ probability measure on } \mathbb{R}_+^n : S = (S_i)_{i=1}^n \text{ is a } Q \text{ - martingale};$$

for $i = 1, \ldots, n$, $S_i$ has marginal $\mu_i$ and mean $s_0\} .$$

We make the standing assumption that $\mathcal{M} \neq \emptyset$.

Let us consider the semi-static trading strategies consisting of the sum of a static vanilla portfolio and a dynamic strategy in the stock. We will by $\Delta$ the predictable process corresponding to the

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holdings on the stock. More precisely, the semi-static strategies generate payoffs of the form:

$$\Psi_{(u_i),(\Delta_j)}(s_1, \ldots, s_n) = \sum_{i=1}^{n} u_i(s_i) + \sum_{j=1}^{n-1} \Delta_j(s_1, \ldots, s_j)(s_{j+1} - s_j), \ s_1, \ldots, s_n \in \mathbb{R}_+,$$

where the functions $u_i: \mathbb{R}_+ \to \mathbb{R}$ are $\mu_i$ integrable for $i = 1, \ldots, n$, and the functions $\Delta_j: \mathbb{R}_+^j \to \mathbb{R}$ are assumed to be bounded measurable for $j = 1, \ldots, n-1$. We denote $\Delta = (\Delta_1, \ldots, \Delta_{n-1})$.

Now we are ready to state our main results. We want to point out that similar results also hold for the continuous time version within the framework of [4], and their proofs are the same as in the discrete time case presented here.

**Theorem 1** (Pricing using Shortfall Risk). Let $\Phi: \mathbb{R}_+^n \to \mathbb{R}$ be an upper semi-continuous function such that

$$\Phi(s_1, \ldots, s_n) \leq K : (1 + s_1 + \ldots + s_n)$$

for some constant $K$. Let $U$ be a nondecreasing concave function. Let $\alpha = U(\beta)$. If $U$ is strictly increasing around a neighborhood of $\beta$, then the following duality holds:

$$C := \inf \left\{ \sum_{i=1}^{n} E_{\mu_i}[u_i] : \exists \Delta \text{ s.t. } \inf_{P \in \mathcal{P}} \mathbb{E}^P[U(\Psi_{(u_i),(\Delta_j)} - \Phi)] \geq \alpha \right\}$$

$$= \sup_{Q \in \mathcal{M}} \mathbb{E}^Q \Phi + U^{-1}(\alpha) =: D,$$

where $\mathcal{P}$ is any set of probability measures on $\mathbb{R}_+^n$ containing $\mathcal{M}$. Moreover, the supremum is attained.

**Proof.** Denote

$$A := \left\{ \sum_{i=1}^{n} E_{\mu_i}[u_i] : \exists \Delta \text{ s.t. } \inf_{P \in \mathcal{P}} \mathbb{E}^P[U(\Psi_{(u_i),(\Delta_j)} - \Phi)] \geq \alpha \right\}.$$

For any $\sum_{i=1}^{n} E_{\mu_i}[u_i] \in A$, there exists $\Delta$, such that $\forall Q \in \mathcal{M},$

$$U \left( \sum_{i=1}^{n} E_{\mu_i}[u_i] - \mathbb{E}^Q \Phi \right) = U \left( \mathbb{E}^Q \left[ \Psi_{(u_i),(\Delta_j)} - \Phi \right] \right) \geq \mathbb{E}^Q \left[ U(\Psi_{(u_i),(\Delta_j)} - \Phi) \right] \geq \alpha,$$

due to Jensen’s Inequality. Hence, $\sum_{i=1}^{n} E_{\mu_i}[u_i] \geq \mathbb{E}^Q \Phi + U^{-1}(\alpha), \ \forall Q \in \mathcal{M}$, which implies $C \geq D$.

Applying Corollary 1.2 in [1], we know that the supremum in the definition of $D$ is attained and

$$D = \inf \left\{ \sum_{i=1}^{n} E_{\mu_i}[u_i] : \exists \Delta \text{ s.t. } \Psi_{(u_i),(\Delta_j)} \geq \Phi + U^{-1}(\alpha) \right\}$$

$$= \inf \left\{ \sum_{i=1}^{n} E_{\mu_i}[u_i] : \exists \Delta \text{ s.t. } U(\Psi_{(u_i),(\Delta_j)} - \Phi) \geq \alpha \right\}.$$

Denote the above set inside “inf” by $B$. Fix an arbitrary $\varepsilon > 0$, and let $D + \varepsilon > \sum_{i=1}^{n} E_{\mu_i}[u_i] \in B$. Then there exists $\Delta$, such that $U(\Psi_{(u_i),(\Delta_j)} - \Phi) \geq \alpha$, which implies $\mathbb{E}^P[U(\Psi_{(u_i),(\Delta_j)} - \Phi)] \geq \alpha$ for any probability measure $P$. Therefore, $\sum_{i=1}^{n} E_{\mu_i}[u_i] \in A$. Thus, $D + \varepsilon > \sum_{i=1}^{n} E_{\mu_i}[u_i] \geq C$. As a result we have that $D + \varepsilon > C$ for all $\varepsilon > 0$, which implies $D \geq C$. $\square$
Remark 1. This result is a generalization of Theorem 4.1 in [2] to the framework of [1]. That is we allow the volatility to be uncertain, but restrict the set of probability measures by allowing the hedging strategies to use static trading in options.

Theorem 2 (Pricing by Quantiles). Let $\Phi : \mathbb{R}^n_+ \to \mathbb{R}$ be a continuous function such that

$$0 \leq \Phi(s_1, \ldots, s_n) \leq K \cdot (1 + s_1 + \ldots + s_n),$$

for some constant $K$. Let $\mathcal{P}$ be any set of probability measures on $\mathbb{R}^n_+$ and $\alpha \in [0,1]$. Define

$$A(\mathcal{P}, \alpha) := \left\{ H \in \mathcal{F} \text{ closed} : \inf_{P \in \mathcal{P}} \mathbb{P}(H) \geq \alpha \right\}.$$

We require our semi-static hedging strategies $u_i$, $i = 1, \ldots, n$, and $\Delta_j$, $j = 1, \ldots, n-1$, to be continuous functions. Then the following holds:

$$I := \inf \left\{ \sum_{i=1}^{n} \mathbb{E}_{\mu_i}[u_i] : \exists \Delta, \text{ s.t. } \Psi_{(u_i), (\Delta_j)} \geq 0, \text{ and } \inf_{P \in \mathcal{P}} \mathbb{P}\left( \Psi_{(u_i), (\Delta_j)} \geq \Phi \right) \geq \alpha \right\}$$

$$= \inf_{H \in A(\mathcal{P}, \alpha)} \sup_{Q \in \mathcal{M}} \mathbb{E}^Q[\Phi(1_H)] := J.$$

Proof. For $H \in A(\mathcal{P}, \alpha)$, let $J(H) := \sup_{Q \in \mathcal{M}} \mathbb{E}^Q[\Phi(1_H)]$. Since $H$ is closed and $\Phi$ is upper semi-continuous we can apply in [1, Corollary 1.2] (and the explanation before this result where it is argued that it is sufficient to take $u_i$ and $\Delta_j$ to be continuous), to obtain

$$J(H) = \inf \left\{ \sum_{i=1}^{n} \mathbb{E}_{\mu_i}[u_i] : \exists \Delta, \text{ s.t. } \Psi_{(u_i), (\Delta_j)} \geq \Phi(1_H) \right\}$$

$$\geq \inf \left\{ \sum_{i=1}^{n} \mathbb{E}_{\mu_i}[u_i] : \exists \Delta, \text{ s.t. } \Psi_{(u_i), (\Delta_j)} \geq 0, \text{ and } \inf_{P \in \mathcal{P}} \mathbb{P}\left( \Psi_{(u_i), (\Delta_j)} \geq \Phi \right) \geq \alpha \right\} = I,$$

which implies $J \geq I$.

For $\varepsilon > 0$, let $\sum_{i=1}^{n} \mathbb{E}_{\mu_i}[u_i] \in [I, I + \varepsilon)$ be such that there exists $\Delta$, satisfying $\Psi_{(u_i), (\Delta_j)} \geq 0$ and $\inf_{P \in \mathcal{P}} \mathbb{P}(\Psi_{(u_i), (\Delta_j)} \geq \Phi) \geq \alpha$. Define $H := \{ \Psi_{(u_i), (\Delta_j)} \geq \Phi \}$. By the upper semi-continuity of $\Psi$ and the lower semi-continuity of $\Phi$, we know that $H$ is closed. Then $H \in A(\mathcal{P}, \alpha)$ and $\Psi_{(u_i), (\Delta_j)} \geq \Phi(1_H)$. Hence,

$$I + \varepsilon > \sum_{i=1}^{n} \mathbb{E}_{\mu_i}[u_i] = \sup_{Q \in \mathcal{M}} \mathbb{E}^Q\left[ \Psi_{(u_i), (\Delta_j)} \right] \geq \sup_{Q \in \mathcal{M}} \mathbb{E}^Q[\Phi(1_H)] \geq J.$$

$\square$

Remark 2. In order to solve the “inf sup” problem in the first line of [1] the Neyman-Pearson Lemma needs to be generalized to the setting of [1], the case in which a dominating measure is absent. We leave this for future work.

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