Measuring the Matter Density of the Galactic Disk Using Stellar Streams

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ABSTRACT

We present a novel method for determining the total matter surface density of the Galactic disk by analysing the kinematics of a dynamically cold stellar stream that passes through or close to the Galactic plane. The method relies on the fact that the vertical component of energy for such stream stars is approximately constant, such that their vertical positions and vertical velocities are interrelated via the matter density of the Galactic disk. By testing our method on mock data stellar streams, with realistic phase-space dispersions and Gaia uncertainties, we demonstrate that it is applicable to small streams out to a distance of a few kilo-parsec, and that the surface density of the disk can be determined to a precision of 6%. This method is complementary to other mass measurements. In particular, it does not rely on any equilibrium assumption for stars in the Galactic disk, and also makes it possible to measure the surface density to good precision at large distances from the Sun. Such measurements would inform us of the matter composition of the Galactic disk and its spatial variation, place stronger constraints on dark disk sub-structure, and even diagnose possible non-equilibrium effects that bias other types of dynamical mass measurements.

Key words: stars: kinematics and dynamics — Galaxy: fundamental parameters — Galaxy: structure

1 INTRODUCTION

Determining the total matter density of the Galactic disk is of great importance for constraining the composition and dynamics of the Galaxy (Dehnen & Binney 1998; Klypin et al. 2002; Widrow et al. 2008; Weber & de Boer 2010; McMillan 2011; Kallf et al. 2014; McMillan 2017; Nitsch et al. 2019; Cautun et al. 2019; Li et al. 2019), has implications for direct and indirect dark matter detection experiments (Jungman et al. 1996; Klasen et al. 2015), and can inform us about the possibility of dark substructures (Read et al. 1996; Klasen et al. 2015), and can inform us about the possibility of dark substructures (Read et al. 2008; Purcell et al. 2009; Fan et al. 2013; Ruchti et al. 2014).

Usually, dynamical mass measurements are performed by isolating a stellar tracer population and fitting a function to its phase-space distribution (Kapteyn 1922; Oort 1932; Bahcall 1984a,b; Kuijken & Gilmore 1989a,b,c, 1991; Crézé et al. 1998; Holmberg & Flynn 2000; Bienayme et al. 2006; Garbari et al. 2012; Bovy & Tremaine 2012; Zhang et al. 2013). Under an assumption of equilibrium, the tracer population’s velocity and number density distributions are interrelated via the gravitational potential. Although many studies on the local matter density of the Galactic disk quote rather small statistical uncertainties (Read et al. 2014; Sivertsson et al. 2018; Schutz et al. 2018; Buch et al. 2019; Widmark & Monari 2019; Benito et al. 2019; Widmark 2019; Karukes et al. 2019; de Salas et al. 2019), there are significant disagreements between studies, and even between different tracer population samples within the same study. These discrepancies could potentially be due to data systematics, modelling differences, or non-equilibrium effects. With Gaia’s second data release (Gaia Collaboration et al. 2016, 2018), it has become all the more clear that the Galaxy is perturbed, potentially by bar (Hunt et al. 2018; Khoperskov et al. 2019), spiral structures (Michchenko et al. 2017; Binney & Schönrich 2018), satellites and past mergers (Antoja et al. 2018; Necib et al. 2019; Bennett & Bovy 2019; López-Corredoira & Sylos Labini 2019; Lopez-Corredoira et al. 2020). It is an open question how and to what extent such non-equilibrium effects can bias dynamical mass measurements of our Galaxy.

In this work, we examine an alternative way of inferring the matter density of the Galactic disk through the use of stellar streams. A stellar stream is formed by the tidal stripping of a stellar body, such as a globular cluster or a dwarf galaxy, as it orbits the gravitational potential of its host galaxy (Dehnen et al. 2004). A key feature of stream structures is that their locus approximates an orbit, rendering the distribution of stream stars tightly constrained in phase-space. This property of streams is often exploited in order to measure the gravitational potential of the Milky Way (Koposov et al. 2010) and probe the dark matter distribution of the Galactic halo (Law...
& Majewski 2010; Bovy et al. 2016; Malhan & Ibata 2019). These measurements are generally achieved by analysing streams with intermediate to large galactocentric radii (see Grillmair & Carlin 2016 and references therein for a list of known stellar streams).

Our aim is to test if low-mass, dynamically cold stellar streams passing through or close to the Galactic plane can provide useful constraints to the matter density of the Galactic disk. This method is highly complementary to using a tracer population of the Galactic potential and solar velocities. We demonstrate that for a small stream of no more than 300 observed stars, we can constrain the surface density within 240 pc of the Galactic mid-plane to within an uncertainty of 6%. This level of precision is competitive with other methods using disk tracer stars.

This paper is organised as follows. In Section 2, we present the Galactic model used to simulate the streams and define our system of coordinates. In Section 3, we demonstrate the principles and some limitations to our method by applying it on an ideal stream. This is followed by a description of our full model, applicable to realistic streams, described in Section 4. In Section 5, we describe how our mock data stellar streams are generated, and in Section 6 we present our inferred results. Finally, we conclude in Section 7. Some technical aspects of our method is described in detail in Appendix A.

2 GALACTIC MODEL AND COORDINATE SYSTEM

For the purposes of simulating mock stellar streams, we adopt the Milky Way model from Bovy (2015) and use the gala package (Price-Whelan 2017). This mass model of the Galaxy consists of a Hernquist bulge and nucleus, a Miyamoto-Nagai disk and an NFW halo (Hernquist 1990; Miyamoto & Nagai 1975; Navarro et al. 1997).

Throughout the paper, we use a solar rest frame coordinate system with spatial coordinates $X = \{X, Y, Z\}$, where the spatial origin ($X = 0$) is located at the Sun’s position. The direction of positive $X$ is towards the Galactic centre, positive $Y$ is in the direction of the Galaxy’s rotation and positive $Z$ is in the direction of the Galactic north, giving a right-handed coordinate system. Their respective time-derivatives give the velocities $V = \{U, V, W\} = \{\dot{X}, \dot{Y}, \dot{Z}\}$, whose origin is the Sun’s velocity. In our model, the Sun is located 15 pc above the Galactic mid-plane (Yao et al. 2017; Widmark & Monari 2019), and has a velocity of $V_0 = \{11, 12, +232, 7.2\}$ km/s with respect to the Galaxy, corresponding to the Sun’s peculiar motion (Schäferich et al. 2010) plus the rotational velocity at the solar radius of our Galactic model (Eilers et al. 2019).

We also define an additional “stream frame” coordinate system, in which the phase-space distribution of a stellar stream will be expressed. The spatial coordinates are written as $X = \{\tilde{X}, \tilde{Y}, \tilde{Z}\}$, and are analogous to $X$, although they correspond to the directions of the Galactic centre and disk rotation as seen from the stream’s position. Their respective time-derivatives give the velocities $V = \{\tilde{U}, \tilde{V}, \tilde{W}\}$, whose origin corresponds to that of the Galactic rest frame.

While we use a global potential when generating the mock data stellar streams, our model of statistical inference is local, and independent of assumptions about the global structure of the Galaxy. In this work, we consider a nearby stream passing through the stellar disk, and we limit ourselves to a distance of ~400 pc from the Galactic mid-plane. The gravitational potential in our model of inference is assumed to be locally axisymmetric, and separable in the vertical ($\tilde{Z}$) and horizontal ($\tilde{X}$ and $\tilde{Y}$) directions. Close to the Galactic disk, it can be approximated as

$$\Phi(X) = \Phi_\perp(\tilde{Z}) - F_X \tilde{X},$$

where $F_X = -\partial \Phi / \partial \tilde{X}$ is a constant force per mass in the direction of the Galactic centre. The variation of the potential as a function of height is related to the total matter density of the Galactic disk, written $\rho(\tilde{Z})$, according to Poisson’s equation,

$$\frac{\partial^2 \Phi_\perp}{\partial \tilde{Z}^2} + \mathcal{R} = 4\pi G \rho(\tilde{Z}).$$

The vertical potential component $\Phi_\perp$ is normalised to zero in the mid-plane ($\tilde{Z} = 0$), both in function value and first order derivative. The quantity $\mathcal{R}$ is the “radial term”, and is equal to

$$\mathcal{R} = \frac{1}{\tilde{R}} \frac{\partial}{\partial \tilde{R}} \left( \tilde{R} \frac{\partial \Phi}{\partial \tilde{R}} \right),$$

where $\tilde{R}$ is the Galactic radius. The quantity $\tilde{R} \partial \Phi / \partial \tilde{R}$ is equal to the square of the circular velocity, whose derivative with respect to radius is small. For the global Galactic model with which the mock data stellar streams are generated, the radial term can be approximated as a constant, constituting only ~2% of the total mid-plane matter density. It is assumed to be known to good precision, and is corrected for in our model of inference.

3 PERFECT ORBIT

In this section, we first demonstrate the general principles and proof of concept of our method by applying a simplified version of our model of inference to a stellar stream consisting of stars lying on a perfect orbit. This idealised example is useful for demonstrating the merits and limitations of our method, in terms of probing the total matter density distribution of the Galactic disk.

The perfect orbit is generated using the Galactic potential model described in the beginning of Section 2. The orbit is perpendicular to the Galactic plane; it has a vertical velocity of $W = -200$ km/s and its velocities parallel to the plane ($\tilde{U}$ and $\tilde{V}$) are zero in the Galactic mid-plane. We generate a set of 100 stars, randomly placed at heights smaller than 400 pc from the Galactic mid-plane ($|\tilde{Z}| < 400$ pc). In this idealised case, we assume to have perfect knowledge of the height of all stars, but we introduce a small uncertainty of $\sigma_{\tilde{U}} = 0.1$ km/s to the vertical velocities. These vertical velocity uncertainties can be interpreted as observational uncertainties, or as an intrinsic velocity dispersion of the stream itself.

In our model of inference, we express the vertical velocity $\tilde{W}$ of stream stars as a function of height above the plane $\tilde{Z}$, according to

$$\tilde{W}^2(\tilde{Z}) = W_0^2 + 2\Phi_\perp(\tilde{Z}),$$

where $W_0$ is the vertical velocity in the Galactic mid-plane. For this ideal orbit, we consider a model with four free parameters (encapsulated in $\Psi$: the mid-plane velocity ($W_0$), and three parameters that describe the total matter density as a function of height ($\rho_A$, $\rho_B$, $h_A$), according to

$$\rho(\tilde{Z}) = \frac{\rho_A}{\cosh^2(\tilde{Z}/h_A)} + \rho_B.$$  

This functional form for the total matter density is quite free to
vary in shape, and can also be very close to the true shape of the underlying Galaxy model. Following equation (2), this sets the gravitational potential to the underlying Galaxy model. Following equation (2), this sets the density distribution is poor. The legend applies to both the panels. The solid line represents the best fit curve, which is in good agreement with the underlying model. The fainter grey lines are randomly drawn realisations from the posterior density, demonstrating that even with a very small velocity dispersion (in this case 0.1 km/s), information about the shape of the density distribution is poor. The legend applies to both the panels.

Figure 1. The inferred vertical velocity distribution ($\tilde{W}$, upper panel) and total matter density ($\rho$, lower panel), for stars on a perfect orbit. The horizontal axis shows height ($\tilde{Z}$) and is shared between both panels. The dots represent the stream stars’ intrinsic vertical velocity (upper panel), and the true values of the matter density at their respective heights (lower panel). The solid line represents the best fit curve, which is in good agreement with the underlying model. The fainter grey lines are randomly drawn realisations from the posterior density, demonstrating that even with a very small velocity dispersion (in this case 0.1 km/s), information about the shape of the density distribution is poor. The legend applies to both the panels.

\[
\Phi_\tilde{Z}(\tilde{Z}, \Psi) = 4\pi G \rho_A h_A^2 \ln \left[ \cosh \left( \frac{\tilde{Z}}{h_A} \right) \right] + \frac{4\pi G \rho_B - R}{2} \tilde{Z}^2. \tag{6}
\]

The posterior density is proportional to

\[
\Pr(\Psi | \tilde{Z}_i=(1,...,100), \tilde{W}_i=(1,...,100)) \propto \Pr(\Psi) \prod_{i=1}^{100} N \left| \tilde{W}_i - \sqrt{\frac{\tilde{W}_0^2}{2} + 2\Phi_\tilde{Z}(\tilde{Z}_i, \Psi), \sigma_{\tilde{W}}} \right|. \tag{7}
\]

where $\Pr(\Psi)$ is a flat box prior that takes a non-zero value for $\rho_A, \rho_B \in [0.0, 2] M_\odot pc^{-3}$, $h_A \in [100, 400]$ pc, and $\tilde{W}_0 \in [-500, 500]$ km/s. The quantities $\tilde{Z}_i$ and $\tilde{W}_i$ are the height and vertical velocity of the $i$th star, and $N$ is the normal distribution defined as

\[
N(x, \sigma) = \frac{\exp \left( \frac{x^2}{2\sigma^2} \right)}{\sqrt{2\pi}\sigma^2}. \tag{8}
\]

The best fit of the posterior density, which is the curve found in the limit of zero uncertainties, is visualised by the solid line in Fig. 1. This demonstrates that inference on $\rho(\tilde{Z})$ is precise in the fully idealised case. The fainter grey lines in Fig. 1 are randomly drawn realisations of the posterior density, which is sampled using a Markov chain Monte Carlo algorithm. Even though the velocity uncertainties are very small ($\sigma_{\tilde{W}} = 0.1$ km/s), information about the shape of the density distribution is poor. While the shape of the matter density distribution is poorly constrained, there is a strong degeneracy between the mid-plane matter density and the distribution’s scale height (i.e. between the maximum value of $\rho$ and how quickly $\rho$ decreases with $|\tilde{Z}|$). This degeneracy corresponds well to a constant integrated surface density. In Fig. 2, we show the inferred surface density within 240 pc from the Galactic mid-plane, defined as

\[
\Sigma_{240} \equiv \int_{-240}^{240} \rho(\tilde{Z})d\tilde{Z}. \tag{9}
\]

which is calculated from the posterior density over the model parameters. The marginalised posterior has a mean and standard deviation of $\Sigma_{240} = (36.91 \pm 0.88) M_\odot pc^{-2}$, corresponding to an uncertainty of 2.4% with respect to its true value of 37.07 $M_\odot pc^{-2}$. In summary, it is difficult to infer the shape of matter density distribution, but the integrated surface density can still be inferred with good precision.

4 STREAM MODEL

In this section, we describe the model of inference used in the rest of this work. Unlike the idealised case presented in the previous section, here we will consider realistic stellar streams that possess an intrinsic phase-space dispersion, as well as realistic observational uncertainties and incomplete radial velocity information. In order to model such a stream, it becomes necessary to describe its full six-dimensional phase-space density, rather than just its vertical components.

As demonstrated in Section 3, it is difficult to constrain the shape of matter density distribution, even with a very small velocity dispersion. For this reason, we fix the shape of the total matter density to model such a stream, it becomes necessary to describe its full six-dimensional phase-space density, rather than just its vertical components.
distribution in our model of inference, and parametrize it in terms of the surface density within 240 pc from the Galactic plane ($\Sigma_{240}$), as defined in equation (9). The matter density as a function of height, in units $M_\odot$ pc$^{-3}$, is equal to

$$\rho(Z) = \frac{\Sigma_{240}}{382.77 \text{ pc}} \left[ \frac{0.8}{\cosh^2(Z/h_A)} + 0.2 \right],$$

(10)

where $h_A = 230$ pc. The value 382.77 pc is a normalisation that is given by the fixed shape of the matter density distribution. This function is close to the true shape of the matter distribution of the Galactic model (visible in the lower panel of Fig. 1).

Our model contains 16 free parameters, which are also listed in Table 1: the surface density ($\Sigma_{240}$), as mentioned above; the gravitational force per mass ($F_X$) acting on the stream in the direction of the Galactic centre, see equation (1); the three-dimensional velocity of the stream as it passes through the Galactic mid-plane ($\bar{U}_0$, $\bar{V}_0$, $\bar{W}_0$); five parameters that describe the stream’s intrinsic phase-space dispersion ($\sigma_U$, $\sigma_V$, $\sigma_W$, $\sigma_X$, $\sigma_Y$); six parameters that describe the Sun’s position and velocity relative to the stream ($d_{0x}$, $d_{0y}$, $d_{0z}$, $U_\odot$, $V_\odot$, $W_\odot$). The full set of free parameters are split into two sets: the first ten parametrize the stellar stream’s intrinsic phase-space distribution ($\Psi_S$); the latter six parametrize the stream’s position and velocity with respect to the Sun ($\Psi_\odot$).

The orbit of the stellar stream is parametrized by its height with respect to the plane ($Z$); all other phase-space coordinates ($X$, $Y$, $\bar{U}$, $\bar{V}$, $\bar{W}$) are functions of $Z$, using the “stream frame” coordinate system defined in Section 2. Given that the gravitational potential is modelled according to equation (1), the velocity of the stream’s orbit follows

$$\bar{U}(Z) = \bar{U}_0 + F_X \tilde{U}(Z),$$

$$\bar{V}(Z) = \bar{V}_0,$$

$$\bar{W}^2(Z) = \bar{W}_0^2 + 2\Phi_2(\tilde{Z}),$$

where

$$\tilde{U}(Z) = \int_0^Z \frac{d\bar{Z}'}{W(\bar{Z}')},$$

(12)

is the time that has passed between a stars current position and its mid-plane passage (which is negative if the star is approaching the mid-plane). Integrating this with respect to time gives the spatial positions

$$X(Z) = X_0 + \bar{U}_0 \tilde{U}(Z) + \frac{1}{2} F_X \tilde{U}^2(\tilde{Z}),$$

$$Y(Z) = Y_0 + \bar{V}_0 \tilde{V}(\tilde{Z}),$$

where $X_0$ and $Y_0$ are the positions where the orbit of the stream passes the mid-plane (given by the model parameters $\Psi_\odot$, see Appendix A3 for details).

The stream stars have an intrinsic scatter in phase-space around this ideal orbit, assumed to be Gaussian and modelled by dispersions $\sigma(U, V, W, X, Y)$. The full phase-space density for a stream star, denoted with an index $i$, is thus proportional to

$$f(\bar{X}_i, \bar{V}_i | \Psi_S) \propto N[\bar{U}_i - \bar{U}(\tilde{Z}_i), \sigma_U] \times N(\bar{V}_i - \bar{V}_0, \sigma_V) \times N(\bar{W}_i - \bar{W}_0, \sigma_W) \times N[\bar{X}_i - \bar{X}(\tilde{Z}_i), \sigma_X] \times N[\bar{Y}_i - \bar{Y}(\tilde{Z}_i), \sigma_Y].$$

(14)

Given a set of $N$ stream stars, labelled by the index $i$, the posterior density of the model is written

$$\Pr(\Psi | d_{1:N}) \propto \Pr(\Psi) \prod_{i=1}^N \Pr(d_i | \bar{X}_i, \bar{V}_i, \Psi_S) \Pr(\Psi_S) \Pr(\Psi_\odot)$$

(15)

where $\Pr(\Psi)$ is a prior on the 16 model parameters, $\Pr(d_i | \bar{X}_i, \bar{V}_i, \Psi_S)$ is the likelihood of the data of the $i$th star, and $\Pr(\Psi_S) \Pr(\Psi_\odot)$ is the phase-space density of the stream, as expressed in equation (14). The posterior contains $N$ six-dimensional phase-space integrals, and also depends on quite a high number of free parameters. However, the posterior can be significantly simplified by reducing the six-dimensional integrals analytically, to numerical integration in only one dimension. This posterior is then implemented in TensorFlow (Abadi et al. 2015), which enables auto-differentiation of the posterior with respect to its free parameters. This allows for efficient minimisation as well as Hamiltonian Monte-Carlo (HMC) sampling of the posterior density, despite its high dimensionality. The posterior and sampling strategy is described in greater detail in Appendix A. All the code that is used to produce the results in this paper is open source and can be found online.\footnote{\url{https://github.com/AxelWidmark/wool}}

The parameter of interest is the surface density parameter $\Sigma_{240}$, and all other parameters of the model are largely to be regarded as nuisance parameters. The parameters that describe the Sun’s position and velocity ($Z_\odot$, $U_\odot$, $V_\odot$, $W_\odot$) and the gravitational force acting on the stream in the direction of the Galactic centre ($F_X$) are more precisely and accurately determined by other methods. However, they are still included as free parameters, as the uncertainty with respect to these parameters can inflate the uncertainty of $\Sigma_{240}$.

Knowledge about these parameters is reflected by the prior, which is described in more detail in Appendix A3.

5 MOCK DATA STELLAR STREAMS

Stellar streams are formed by the tidal disruption and dissolution of satellites, where the escaping stars are lifted out of the gravitational potential well of their progenitor. In doing so, the stars end up with slightly different energies and momenta, causing an intrinsic phase-space dispersion of the stream. In addition to this, the measurements of phase-space positions of stars is accompanied by observational uncertainties, and sometimes missing information (in this case incomplete radial velocity measurements). In this section, we describe how we generate our mock data stellar streams, employing realistic phase-space dispersions and observational uncertainties.

We generate four mock data stellar streams containing 300 stars each, using the Galactic potential described in the beginning of Section 2. The streams are generated as stars on a perfect orbit, randomly placed along its spatial trajectory. The stars are then displaced in phase-space, using dispersions of 20 pc in all three spatial directions, and 1 km/s for all three velocities. These values are adopted in accordance with the internal phase-space dispersions of some of the known dynamically cold streams of the Milky Way (Grillmair & Carlin 2016), and also match the values obtained in some recent studies (Malhan & Ibata 2019).

In order to be able to make simple comparisons of the inferred surface density using the different mock data stellar streams, they
Table 1. Model parameters and data of an observed star. The 16 model parameters are split into two subsets, \( \Psi = \{ \Psi_S, \Psi_F \} \): the former parametrizes the stellar stream’s intrinsic phase-space distribution; the latter parametrizes the stream’s position and velocity relative to the Sun.

| \( \Psi_S \) | Model parameter subset describing the stream’s phase-space density |
|----------------|--------------------------------------------------|
| \( \Sigma_{240} \) | Total matter surface density within 240 pc of the Galactic mid-plane |
| \( F_X \) | Force per mass acting on the stream in the direction of the Galactic centre |
| \( \mathbf{U}_0, \mathbf{V}_0, \mathbf{W}_0 \) | Velocity of the stream orbit’s intersection with the Galactic plane |
| \( \sigma_{(U, V, W)} \) | Three intrinsic velocity dispersions |
| \( \sigma_{(X, Y)} \) | Intrinsic dispersion in \( X \) and \( Y \), respectively |

| \( \Psi_F \) | Model parameter subset describing the Sun’s relative position and velocity |
|----------------|--------------------------------------------------|
| \( d_0 \) | Distance to the stream from the Sun |
| \( l_0 \) | Galactic longitude of the stream at its intersection with the Galactic plane |
| \( Z_0 \) | Height of the Sun with respect to the Galactic mid-plane |
| \( U_0, V_0, W_0 \) | Velocity of the Sun with respect to the Galaxy |

| \( d \) | Data |
|----------------|--------------------------------------------------|
| \( \hat{l}, \hat{b} \) | Galactic longitude and latitude (with negligible uncertainties) |
| \( \hat{\sigma}, \sigma_{\alpha \star} \) | Observed parallax and associated uncertainty |
| \( \hat{\mu}_l, \hat{\mu}_b, \sigma_{\mu} \) | Observed proper motions and associated uncertainty |
| \( \hat{v}_{\text{RV}}, \sigma_{\text{RV}} \) | Observed radial velocity and associated uncertainty (potentially available) |

are all located at the same Galactic radius of 8.1 kpc, where the rotational velocity is roughly 232 km/s (thus we move the position from which the stream is observed, rather than the stream itself). The vertical velocity of all streams is negative, such that their direction of motion is towards the Galactic south. All streams are observed at a distance of about 1 kpc. The four streams are named S1–S4, and their properties are described below.

Stream S1 has velocities \( \mathbf{V} \approx \{0, 0, -100\} \) km/s. It is somewhat idealised, in the sense that it passes perpendicularly through the Galactic disk, with a rather small vertical velocity, such that the stream is at the apocentre of an eccentric orbit. Its total extent is \( \sim 800 \) pc, with heights between \( Z \in [-400, 400] \) pc. In the observer’s frame, the median position of its stars is located at \( \{X, Y\} = \{0, 1\} \) kpc.

Stream S2 is on a prograde orbit, with velocities \( \mathbf{V} \approx \{15, 220, -45\} \) km/s. Its total extent is \( \sim 1 \) kpc, with heights between \( Z \in [-450, -200] \) pc. In the observer’s frame, the median position of its stars is located at \( \{X, Y\} = \{-1, 0\} \) kpc.

Stream S3 is also on a prograde orbit, with velocities \( \mathbf{V} \approx \{0, 200, -70\} \) km/s. Its total extent is \( \sim 500 \) pc, with heights between \( Z \in [200, 400] \) pc. In the observer’s frame, the median position of its stars is located at \( \{X, Y\} = \{-1, 0\} \) kpc.

Stream S4 has velocities \( \mathbf{V} \approx \{-160, 160, -70\} \) km/s. Its orbit is inclined with respect to both the Galactic plane and the direction of rotation. Its total extent is \( \sim 500 \) pc, with heights between \( Z \in [200, 400] \) pc. In the observer’s frame, the median position of its stars is located at \( \{X, Y\} = \{0, 1\} \) kpc.

The intrinsic phase-space coordinates of these streams are shown in Fig. B1–B4, found in Appendix B.

The intrinsic phase-space coordinates of the streams are transformed into observables, given by the relative phase-space position of the Sun. An individual stream star has the following observables: angular position on the sky \( (\hat{l}, \hat{b}) \), parallax \( (\hat{\sigma}) \), proper motions \( (\hat{\mu}_l, \hat{\mu}_b) \), and (for a subset of sufficiently bright stars) radial velocity \( (\hat{v}_{\text{RV}}) \). They all have their associated uncertainties, with the exception of the angular position on the sky, whose uncertainties are negligible. This constitutes the data \( (d) \) of an observed star, which is listed in Table 1.

We use the expected *Gaia* end-of-mission uncertainties, as a function of the apparent G-band magnitude \( m_G \). In order to simulate a realistic distribution of apparent magnitudes, we synthesise a typical globular cluster population with an iron abundance of \( \text{[Fe/H]} = -2.0 \) and an age of 12 Gyr, using Isochrone (Morton 2015). When generating our mock data stellar streams, each star is randomly assigned an absolute magnitude, from which the apparent magnitude is given by its distance modulus. The synthesised distribution of apparent magnitudes, for 4000 stars at a distance of 1 kpc, are visible in the top panel of Fig. 3.

The expected parallax and proper motion uncertainties \( (\sigma_{\sigma_{\alpha \star}} \text{ and } \sigma_{\mu}) \) are shown in the bottom panel of Fig. 3.\(^2\) For the spectrograph that measures radial velocity, the expected limit in apparent magnitude is \( m_G < 14.5 \) (Recio-Blanco et al. 2016), although this is a function of position on the sky. We are interested in streams passing through the Galactic disk, where stellar crowding can impede the spectrograph’s performance. For this reason we set a slightly more conservative limit of \( m_G < 14 \). We set the radial velocity

\(^2\) The interpolation points are taken from this table: https://www.cosmos.esa.int/web/gaia/sp-table1
The intrinsic velocity dispersion also dominates over the radial velocity dispersion of 1 km/s, corresponding to a median observational uncertainty of $\sigma_{\text{RV}} = 0.3$ km/s (Katz et al. 2019). The number of stars with available radial velocities for S1–S4 are 60, 61, 58, and 64, respectively.

6 RESULTS

We explore the posterior density of our four mock data stellar streams using Hamiltonian Monte-Carlo (HMC) sampling, described in Appendix A5.

The inferred posterior for the integrated surface density within 240 pc of the mid-plane ($\Sigma_{240}$), for all four streams, is shown in Fig. 4. The orbits of these streams are quite varied, but the inference of the surface density is accurate for all streams. The posterior width of the inferred surface density varies from roughly 6–12 % of the true value. Other mock data realisations of the same stellar streams produce similar results, with small shifts to the median value of $\Sigma_{240}$.

In Fig. 5, we plot the data and model fit for stream S1. The model fit in this figure corresponds to the respective median values of all 16 model parameters of the inferred posterior. For most proper motion measurements, the observational uncertainty is largely negligible; the spread in proper motion is dominated by the intrinsic velocity dispersion of 1 km/s, corresponding to a $\sim 0.2$ mas yr$^{-1}$ at the distance of the stream, which is an order of magnitude larger than the median observational uncertainty of $\sigma_\mu = 0.028$ mas yr$^{-1}$.

The intrinsic velocity dispersion also dominates over the radial velocity uncertainty ($\sigma_{\text{RV}} = 0.3$ km/s). The most significant uncertainties are associated with the parallax, with a median value of $\sigma_\varpi = 0.052$ mas. However, the spatial extent of the stellar stream is well constrained by the subset of bright stars for which the measurements are more precise, which in turn constrains the position of stars for which parallax information is poor. This is an effect of Bayesian deconvolution (similar in principle to for example Anderson et al. 2018). The inferred model parameters of the other mock data stellar streams produce similarly good fits with respect to the data.

In Fig. 6, we show a correlation matrix for the model’s 16 free parameters, as inferred for streams S1–S4. The parameter of interest, the surface density $\Sigma_{240}$, is not strongly correlated with other parameters, with the exception of some anti-correlation with the stream’s vertical velocity $W_\parallel$ (the vertical velocity is negative, so a greater $\Sigma_{240}$ correlates with a greater absolute value of $W_\parallel$). Other strong degeneracies exist between $U_\parallel$ and $U_\perp$, and between $V_\parallel$ and $V_\perp$, which parametrizes the solar and stream velocities parallel to the Galactic plane. If there was a strong correlation between $\Sigma_{240}$ and model parameters for which we have a highly constrained prior ($F_X$, $Z_\odot$, $U_\odot$, $V_\odot$, $W_\odot$, see Appendix A1 for more details), then this would indicate high sensitivity to systematics associated with these parameters. In such a case, a prior density that is offset from its true value would translate to a bias in the inferred value for $\Sigma_{240}$. Fortunately, such strong degeneracies are not present in the posterior densities of samples S1–S4.

The precision of inference in terms of the matter density is highly contingent on the stream’s orbit. The four mock streams considered in this work have different vertical velocities, ranging from roughly 45 to 100 km/s. With greater vertical velocities the relative change to the vertical energy is smaller ($\delta W \propto \dot{W}$), resulting in lower precision. On the other hand, very low vertical velocities mean that the stream does not vary much in $Z$. In order to have good precision of inference, a stream with such low
inclination would have to be very long, and in such a case the disk density that it is probing is no longer very local. Furthermore, such a stream would be more sensitive to perturbations from structures in the Galactic disk, given its low vertical energy.

The inference discussed above is for an observational distance of roughly 1 kpc, but this method is applicable also for streams at larger distances, as long as the distance to the stream can be accurately constrained. Given the same intrinsic phase-space dispersion, the velocity dispersion is dominant over the proper motion uncertainty out to a distance of several kilo-parsecs. We have used a realistic phase-space dispersion, similar to what have been observed for some streams in the Milky Way. There exists also the possibility of finding dynamically colder streams, which would improve the precision.

When analysing an actual stream, it is important to examine how well the data agrees with the inferred model, especially in terms of systematic offsets in phase-space with respect to the stream’s ideal orbit. Such offsets are not currently included in the model, but can bias the result in the event that such offsets are significant with respect to the intrinsic phase-space dispersion of the stream.
7 CONCLUSION

In this work, we have demonstrated that the total matter density of the Galactic disk can be constrained by analysing the kinematics of a stellar stream that passes through or close to the Galactic mid-plane. This is possible because the vertical energy of stream stars is approximately constant, such that the stars’ heights and vertical velocities are interrelated via the gravitational potential. The method was demonstrated on realistic mock data stellar streams, with projected end-of-mission *Gaia* uncertainties. We found that a small stream containing 300 stars with realistic phase-space dispersion, situated at a distance of approximately 1 kpc, can be used to constrain the Galactic disk surface density to within an uncertainty of 6%. This level of precision is competitive with other methods using disk tracer stars. Our method accounts for the full error covariance matrix of all individual stars, as well as for the uncertainties associated with the Galactic potential and solar velocities. Our method is highly complementary to the methods using disk tracer stars, because it does not rely on any steady-state assumption for any stellar population of the disk itself. Furthermore, it does not rely on stellar number density information, and is therefore insensitive to stellar number count biases arising from, for example, a mismodeled selection function. Our method also makes it possible to make precise measurements of the surface density at different locations in the Galactic disk, wherever a suitable stream can be found. This could give us insights with regards to the disk’s dynamical evolution, for instance if the disk is perturbed by the passage of the Sagittarius dwarf galaxy (Laporte et al. 2019) or due to buckling of the stellar bar (Khoperskov et al. 2019). Such perturbations could cause spatial variations in the matter density of the Galactic disk, and potentially break axisymmetry in a way that increases the importance of the radial and azimuthal terms of Poisson’s equation.

Our method is best suited for low-mass dynamically cold stellar streams, much like those produced from the tidal disruption of globular clusters or very low-mass dwarf galaxies. Streams produced by more massive dwarf galaxies are more diffuse in phase-space, resulting in higher uncertainties in the inferred surface matter density. *Gaia* DR2 has unveiled a large amount of stellar streams in the solar neighbourhood (Koppelman et al. 2018; Myeong et al. 2018; Koppelman et al. 2019; Yuan et al. 2019; Borsato et al. 2020), although most of these streams have rather large phase-space dispersions, as is expected from tidal debris of more massive progenitors. An
exception is Meingast et al. (2019), who found a small stream only 100 pc from the Sun. The stream extends at least 400 pc in distance, but is not useful for our purposes, as it has a very low vertical velocity (∼ 11 km/s) and varies only minimally in height with respect to the Galactic plane.

Most stream finding algorithms mask the region surrounding the disk, in order to bypass the computational challenge of processing a very large number of stars (Malhan et al. 2018; Borsato et al. 2020). However, several globular clusters are known to orbit the Milky Way close to the disk (Arca Sedda & Mastrobuono-Battisti 2019), and many of them could be associated with stellar streams. This motivates the hunt for dynamically cold stellar streams in the Galactic plane. The detection of stellar streams crossing the disk plane is accessible, especially, given the continued improvements in quality of the Gaia data expected over the coming years, which will extend the detection horizon for stellar substructures. We envision applying our method to a suitable stellar stream in the near future.

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Stellar Streams and the Galactic Disk Matter Density

9

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APPENDIX A: POSTERIOR PROBABILITY DENSITY

The posterior, which is formulated in equation (15), is a product of a prior probability of the model parameters, and $N$ six-dimensional integrals over a star’s phase-space probability density and data likelihood, where $N$ is the number of observed stars in a stream. In order to make the posterior computationally tractable, the integrals over a star’s phase-space coordinates can be analytically reduced, such that only a one-dimensional numerical integration over a star’s distance remains. This is described in details below.

A1 Prior

We use a wide flat box prior for the following parameters, with lower and upper bounds:

- $\Sigma_{240} \in [0, 200] \, M_\odot \, \text{pc}^{-2}$
- $U_0, V_0, W_0 \in [-500, 500] \, \text{km/s}$
- $\sigma_U, \sigma_V, \sigma_W \in [0, 10] \, \text{km/s}$
- $\sigma_\mu, \sigma_b \in [0, 100] \, \text{pc}$
- $d_0 \in [0, 5] \, \text{kpc}$
- $l_0 \in [0, 2\pi]$.

The following parameters have Gaussian priors centred on their true value (with standard deviations):

- $F_\odot (2.5\% \, \text{of its true value})$
- $Z_\odot (5 \, \text{pc})$
- $U_\odot, V_\odot (5 \, \text{km/s})$
- $W_\odot (0.1 \, \text{km/s})$

See Section 2 for more details. The uncertainties are representative for how well these parameters are constrained for our own Sun and Galaxy (Sch"{u}nrich et al. 2010; Eilers et al. 2019).

A2 Data likelihood

Given a star with index $i$, with intrinsic position and velocity formulated in terms of the solar rest frame coordinates $X_i$ and $V_i$, the likelihood of its data is

$$
\text{Pr}(d_i \mid X_i, V_i) \propto \delta[l_i - l(X_i)] \times \delta[b_i - b(X_i)]
\times N[\delta \mu_l \times \mu_l(X_i, V_i), \sigma_{\mu_l}] \times N[\delta \mu_b \times \mu_b(X_i, V_i), \sigma_{\mu_b}] \times N[\delta \sigma_l \times \sigma_l(X_i), \sigma_{\sigma_l}]
$$

(A1)

where the observables without hats ($l, b, \mu_l, \mu_b$) are given directly by $X_i$ and $V_i$, assuming no observational errors (for example $\sigma_l(X_i) = |X_i|^{-1} \, \text{kpc} \times \text{mas}$). The quantities with hats are the actual data observables, with associated uncertainties, as discussed in Section 5. The likelihood above is written in terms of solar coordinates, but can equally well be formulated in terms of $X_i$, $V_i$, and $V_\odot$. For the angular parameters, observational uncertainties are neglected, which is why they are written in terms of Dirac delta functions. If radial velocity information is available, we add an additional factor $N[\delta V_\text{RV,i} \times V_\text{RV}(X_i, V_i), \sigma_{\delta V_\text{RV,i}}]$ to the likelihood.

A3 Coordinate system transformations

The subset of model parameters denoted $\Psi_\odot$ describe the transformation from “stream frame” coordinates to solar coordinates. The position of the stream’s mid-plane passage relative to the Sun is equal to

$$
X_0 = \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} = \begin{bmatrix} d_0 \cos l_0 \\ d_0 \sin l_0 \\ -Z_\odot \end{bmatrix}.
$$

(A2)

The transformation from spatial coordinates $\tilde{X}$ to $X$ is

$$
\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \cos \lambda_{\text{diff}} \sin \lambda_{\text{diff}} & 0 & \tilde{X} - X_0 \\ -\sin \lambda_{\text{diff}} \cos \lambda_{\text{diff}} & 0 & \tilde{Y} - Y_0 \\ 0 & 0 & 1 \end{pmatrix} + X_0.
$$

(A3)

where $\lambda_{\text{diff}}$ corresponds to the angular difference between the direction of $X$ and $\tilde{X}$ (the direction of the Galactic centre as seen from the Sun, and as seen from the stream’s position). This angle is given by $d_0, l_0$, and the Galactic radius of the Sun $R_\odot$, according to the relation

$$
\sin(\lambda_{\text{diff}}) = \frac{d_0 \sin l_0}{\sqrt{d_0^2 + R_\odot^2 - 2d_0 R_\odot \cos l_0}}
$$

(A4)

The transformation from velocities $\tilde{V}$ to $V$ is

$$
\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} \cos \lambda_{\text{diff}} \sin \lambda_{\text{diff}} & 0 & \tilde{U} \\ -\sin \lambda_{\text{diff}} \cos \lambda_{\text{diff}} & 0 & \tilde{V} \\ 0 & 0 & 1 \end{pmatrix} + V_\odot.
$$

(A5)

A4 Analytic reduction

The six-dimensional integrals can in large part be computed analytically. The angular coordinates ($\tilde{l}$ and $\tilde{b}$) have negligible uncertainties, such that only a numerical integral over the distance of a star with respect to the Sun is necessary. We denote this distance $s$. For the integral over spatial positions, it can be rewritten as

$$
\int \mathcal{P}(d_i \mid \tilde{l}_i, \tilde{b}_i, \Psi_\odot) \mathcal{L}(\tilde{l}_i, \tilde{b}_i, \Psi_\odot) \, d^3\tilde{l}_i = \int d(l_i - l(X_i)) \delta(l_i - l(X_i)) \norm\tilde{b}_i - b(X_i)\norm \mathcal{L}(\tilde{l}_i, \tilde{b}_i, \Psi_\odot) \mathcal{L}(\tilde{l}_i, \tilde{b}_i, \Psi_\odot) \, d^3\tilde{l}_i
$$

$$
\times \mathcal{N}(\tilde{\sigma}_l \times \sigma_l(X_i), \tilde{\sigma}_\sigma_l \times \sigma_{\sigma_l}) \times \mathcal{N}(\tilde{\sigma}_b \times \sigma_b(X_i), \tilde{\sigma}_\sigma_b \times \sigma_{\sigma_b})
$$

(A6)

where the intrinsic position of a star relative to the Sun is now completely fixed by $l_i, b_i$, and $s$. In this new formulation, a star’s position in $\tilde{X}$ coordinates also depends on the relative position of the stream with respect to the Sun, which is encoded in $\Psi_\odot$. The factor $s^2$ is part of the Jacobian of spherical coordinates, which appears when integrating the $d$ functions.

For the integral over velocity, the observational uncertainties and the stream’s intrinsic dispersion both follow Gaussian distributions, and can therefore be computed analytically. For a fixed spatial position, the velocity part of the integral in equation (15) is equal to

$$
\int \mathcal{P}(d_i \mid \tilde{V}_i, \Psi_\odot) \mathcal{L}(\tilde{V}_i, \Psi_\odot) \, d^3\tilde{V}_i = \mathcal{M}(\mu(d_i) - \mu(l_i, b_i, s, \Psi_\odot), \Sigma_{\text{sun}}(d_i, s, \Psi_\odot))
$$

(A7)

where $\mu(d_i) - \mu(l_i, b_i, s, \Psi_\odot)$ is the mean of the observational uncertainties, and $\Sigma_{\text{sun}}(d_i, s, \Psi_\odot)$ is the covariance matrix of the observed velocities.

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where $\mathcal{M}$ is the multivariate normal distribution defined

$$
\mathcal{M}(x, \Sigma) = \frac{\exp \left( -\frac{1}{2} x^T \Sigma^{-1} x \right)}{(2\pi)^{\frac{q}{2}} |\Sigma|^{\frac{1}{2}}},
$$

(A8)

where $q$ is the dimension of $x$. The quantity

$$
\mu(d_i) = \begin{bmatrix} \hat{\mu}_l \\ \hat{\mu}_b \\ \hat{\mu}_{RV} \end{bmatrix},
$$

(A9)

is a vector with the observed proper motions and radial velocity. The quantity $\mu$ is the corresponding vector as given by the ideal orbit of the stream model, which is defined

$$
\mu(l_i, b_i, s, \Psi) =
$$

$$(A10)
$$

the transformation from velocities to velocities in the latitudinal, longitudinal, and radial directions ($v_l, v_b, v_{RV}$). The quantity $\ell_{\text{diff.}}$ corresponds to the angular difference between the direction of $X$ and $\tilde{X}$, as defined in equation (A4). The diagonal matrix

$$(A11)
$$

transforms the solar coordinate system velocities ($U, V, W$) to velocities in the latitudinal, longitudinal, and radial directions ($v_l, v_b, v_{RV}$). The quantity $\ell_{\text{diff.}}$ corresponds to the angular difference between the direction of $X$ and $\tilde{X}$, as defined in equation (A4). The diagonal matrix

$$(A12)
$$

where $k_{\mu} = 4.74057 \text{kpc}^{-1} \times \text{mas}^{-1} \times \text{yr} \times \text{km/s}$, corresponds to the transformation from velocities $v_U$ and $v_{RV}$ to proper motions $\mu_l$ and $\mu_b$.

The covariance matrix $\Sigma_{\text{sum}}$ is a sum of covariance matrices associated with observational uncertainties and the stream’s intrinsic velocity dispersion around it’s ideal orbit, like

$$
\Sigma_{\text{sum}}(d_i, s, \Psi) = \Sigma_{\mu}(d_i) + \Sigma_{\Sigma}(d_i, s, \Psi),
$$

(A13)

where

$$
\Sigma_{\mu}(d_i) = \begin{bmatrix} \sigma_{\mu}^2 \\ \sigma_{\mu}^2 \\ \sigma_{\mu}^2 \end{bmatrix},
$$

(A14)

and

$$
\Sigma_{\Sigma}(d_i, s, \Psi) = M(s) \times R(l_i + \ell_{\text{diff.}}, \tilde{b}_i)
$$

$$(A15)
$$

The matrices $M$ and $R$ are the same as in equation (A10). Although we have not included error correlations or separate uncertainties for the two proper motions when testing our model on mock data, this is easily incorporated in $\Sigma_{\mu}(d_i)$ when working with actual Gaia data.

Using the simplifications described above, the posterior density becomes

$$
\Pr(\Psi | d_i = \{1,...,N\}) =
$$

$$(A16)
$$

now containing $N$ one-dimensional integrals, which is significantly less intensive computationally.

### Appendix B: Stream Phase-Space Coordinates

In Fig. B1–B4 below, we show the intrinsic phase-space coordinates of streams S1–S4. The $X$ and $Y$ values are normalised to their median values in these figures.

This paper has been typeset from a \TeX/L\LaTeX file prepared by the author.
Figure B1. Intrinsic phase-space coordinates of stream S1.
Figure B2. Intrinsic phase-space coordinates of stream S2.
Figure B3. Intrinsic phase-space coordinates of stream S3.
Figure B4. Intrinsic phase-space coordinates of stream S4.