A Stefan problem on an evolving surface

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Abstract

We formulate a Stefan problem on an evolving hypersurface and study the well-posedness of weak solutions given $L^1$ data. To do this, we first develop some function spaces and results to handle equations on evolving surfaces in order to give a natural treatment of the problem. Then we consider the existence of solutions for $L^\infty$ data; this is done by regularisation of the nonlinearity. The regularised problem is solved by a fixed point theorem and then uniform estimates are obtained in order to pass to the limit. By using a duality method we show uniqueness and continuous dependence which allows us to extend the results to $L^1$ data.

1 Introduction

The Stefan problem is the prototypical time-dependent free boundary problem. It arises in various forms in many models in the physical and biological sciences [1, 2, 3, 4]. In this paper we present the theory of weak solutions associated with the so-called enthalpy approach [1] to the Stefan problem on an evolving curved hypersurface.

Our interest is in the existence, uniqueness and continuous dependence of solutions (in a sufficiently weak sense) to the Stefan problem

\[ \partial^* e(t) - \Delta_{\Omega(t)} u(t) + e(t) \nabla_{\Omega(t)} \cdot w(t) = f(t) \quad \text{in } \Omega(t) \]
\[ e(0) = e_0 \quad \text{on } \Omega(0) \]
\[ e \in \mathcal{E}(u) \]

posed on a moving compact hypersurface $\Omega(t) \subset \mathbb{R}^{n+1}$ evolving with (given) velocity field $w$, where the energy $\mathcal{E} : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is defined by

\[ \mathcal{E}(r) = \begin{cases} 
 r & \text{for } r < 0 \\
 [0,1] & \text{for } r = 0 \\
 r + 1 & \text{for } r > 0.
\end{cases} \]
In (1), $\partial^* e$ means the material derivative of $e$ (which we shall also write as $\dot{e}$) and $\nabla_{\Omega(t)}$ and $\Delta_{\Omega(t)}$ are respectively the surface gradient and Laplace–Beltrami operators on $\Omega(t)$. The novelty of this work is that the Stefan problem itself is formulated on a moving hypersurface and our chosen method to treat this problem, which we believe is naturally suited to equations on moving domains, requires the use of some new function spaces and results that we shall introduce, building upon the spaces and concepts presented in [5, 6]. There is, as alluded to above, a rich literature associated to Stefan-type problems [7, 8, 9, 10, 11, 12]. We will show that arguments similar to those used in the standard setting are also amenable to our problem on a moving hypersurface, thanks in part to the function spaces we decide to use.

First, let us work out a possible pointwise formulation of (1). Start by supposing $\Omega(t) = \Omega_l(t) \cup \Omega_s(t) \cup \Gamma(t)$ where $\Omega_l(t)$ and $\Omega_s(t)$ divide $\Omega(t)$ into a liquid and a solid phase (respectively) with an a priori unknown interface $\Gamma(t)$. The quantity of interest is the temperature $u(t) : \Omega(t) \to \mathbb{R}$, which we suppose satisfies

$$
\begin{align*}
&u(t) > 0 \text{ in } \Omega_l(t) \\
&u(t) = 0 \text{ in } \Gamma(t) \\
&u(t) < 0 \text{ in } \Omega_s(t),
\end{align*}
$$

and (thus) $u = 0$ is the critical temperature where the change of phase occurs. Define

$$
Q_l = \bigcup_{t \in (0,T)} \Omega_l(t) \times \{t\}, \quad S = \bigcup_{t \in (0,T)} \Gamma(t) \times \{t\},
$$

and $Q_s$ similarly. Given $f$ and $u_0$, we formally elucidate in Remark 2.12 the relationship between (1) and the following model describing the temperature $u$:

$$
\begin{align*}
\partial^* u - \Delta_{\Omega} u + (u + 1)\nabla_{\Omega} \cdot w &= f \quad \text{in } Q_l \\
\partial^* u - \Delta_{\Omega} u + u\nabla_{\Omega} \cdot w &= f \quad \text{in } Q_s \\
-(\nabla_{\Omega} u_l - \nabla_{\Omega} u_s) \cdot \mu &= V \quad \text{on } S \\
&u = 0 \quad \text{on } S \\
&u(0) = u_0 \quad \text{on } \Omega(0),
\end{align*}
$$

(2)

where $u_s$ denotes the trace of the restriction $u|_{\Omega_s}$ to the interface $\Gamma$ (likewise with $u_l$), $V(t)$ is the conormal velocity of $\Gamma(t)$ and $\mu(t)$ is the unit conormal vector pointing into $\Omega_l(t)$ (this vector is tangential to $\Omega(t)$ and normal to $\partial \Omega_l(t)$).

We now introduce some notions of a weak solution, similar to [9]. The function spaces $L^p_X$ below will be made precise in §2 but for now can be thought of as generalisations of Bochner spaces $L^p(0,T;X_0)$ where now $u \in L^p_X$ implies $u(t) \in X(t)$ for almost all $t$ (for a suitable family $\{X(t)\}_{t \in [0,T]}$).

1.1 Definition (Weak solution). Given $f \in L^1_{L^1}$ and $e_0 \in L^1(\Omega_0)$, a weak solution of (1) is a pair $(u,e) \in L^1_{L^1} \times L^1_{L^1}$, such that $e \in \mathcal{E}(u)$ and there holds

$$
- \int_0^T \int_{\Omega(t)} \dot{\eta}(t)e(t) - \int_0^T \int_{\Omega(t)} u(t)\Delta_{\Omega}\eta(t) = \int_0^T \int_{\Omega(t)} f(t)\eta(t) + \int_{\Omega_0} e_0\eta(0)
$$

(3)

for all $\eta \in W(L^\infty \cap H^2, L^\infty)$ with $\Delta_{\Omega}\eta \in L^\infty_{L^\infty}$ and $\eta(T) = 0$. 

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1.2 Definition (Bounded weak solution). Given \( f \in L^\infty_1 \) and \( e_0 \in L^\infty(\Omega_0) \), a bounded weak solution of (1) is a pair \((u, e)\) \( \in L^1_{\Omega_0} \times L^1_{\Omega_0} \cap L^\infty_\Omega \) such that \((u, e)\) is a weak solution of (1) satisfying

\[
- \int_0^T \int_{\Omega(t)} \eta(t)e(t) + \int_0^T \int_{\Omega(t)} \nabla u(t) \nabla \eta(t) = \int_0^T \int_{\Omega(t)} f(t) \eta(t) + \int_{\Omega_0} e_0 \eta(0)
\]

for all \( \eta \in W(H^1, L^2) \) with \( \eta(T) = 0 \).

We prove the following results.

Theorem 1.3 (Existence of bounded weak solutions). If \( f \in L^\infty_1 \), \( e_0 \in L^\infty(\Omega_0) \) and \( |\Omega| := \sup_{s \in [0, T]} |\Omega(s)| < \infty \), then there exists a bounded weak solution to (1).

Theorem 1.4 (Uniqueness and continuous dependence of bounded weak solutions). If for \( i = 1, 2 \), \((u^i, e^i)\) are two bounded weak solutions of (1) with data \((f^i, e_0^i) \in L^\infty_1 \times L^\infty(\Omega_0)\), then

\[
\|e^1(t) - e^2(t)\|_{L^1(\Omega(t))} \leq \int_0^t \|f^1(\tau) - f^2(\tau)\|_{L^1(\Omega(\tau))} + \|e_0^1 - e_0^2\|_{L^1(\Omega_0)}.
\]

Theorem 1.5 (Well-posedness of weak solutions). If \( f \in L^1_1 \), \( e_0 \in L^1(\Omega_0) \) and \( |\Omega| := \sup_{s \in [0, T]} |\Omega(s)| \), then there exists a unique weak solution to (1). Furthermore, if for \( i = 1, 2 \), \((u^i, e^i) \in L^1_1 \times L^1_1 \) are two weak solutions of (1) with data \((f^i, e_0^i) \in L^1_1 \times L^1(\Omega_0)\), then

\[
\|e^1 - e^2\|_{L^1_1} \leq C_T \left(\|f^1 - f^2\|_{L^1_1} + \|e_0^1 - e_0^2\|_{L^1(\Omega_0)}\right).
\]

Notation We shall use the notation \( \hookrightarrow \) and \( \overset{\circ}{\hookrightarrow} \) to denote (respectively) a continuous embedding and a compact embedding.

2 Preliminaries

2.1 Abstract evolving function spaces

In [5], we generalised some concepts from [13] and defined the Hilbert space \( L^2_H \) given a sufficiently smooth parametrised family of Hilbert spaces \( \{H(t)\}_{t \in [0, T]} \). We will need a generalisation of this theory to Banach spaces.

For each \( t \in [0, T] \), let \( X(t) \) be a real separable reflexive Banach space with \( X_0 := X(0) \). We informally identify the family \( \{X(t)\}_{t \in [0, T]} \) with the symbol \( X \). Let there also be a linear homeomorphism \( \phi_t : X_0 \to X(t) \) for each \( t \in [0, T] \) (with \( \phi_{-t} : X(t) \to X_0 \) denoting the inverse) such that \( \phi_0 \) is the identity. We will assume that there exists a constant independent of \( t \in [0, T] \) such that

\[
\|\phi_t u\|_{X(t)} \leq C_X \|u\|_{X_0} \quad \forall u \in X_0 \\
\|\phi_{-t} u\|_{X_0} \leq C_X \|u\|_{X(t)} \quad \forall u \in X(t).
\]

We assume for all \( u \in X_0 \) that the map \( t \mapsto \|\phi_t u\|_{X(t)} \) is continuous (in fact, in order to define \( L^p_X \) below, measurability would suffice).
2.1 Definition. Define the Banach spaces
\[ L^p_X = \{ u : [0, T] \rightarrow \bigcup_{t \in [0, T]} X(t) \times \{ t \}, \ t \mapsto (\hat{u}(t), t) \mid \phi_{-t}\hat{u}(\cdot) \in L^p(0, T; X_0) \} \quad \text{for } p \in [1, \infty) \]
\[ L^\infty_X = \{ u \in L^2_X \mid \text{ess sup}_{t \in [0, T]} \| u(t) \|_{X(t)} < \infty \} \]
endowed with the norm
\[ \|u\|_{L^p_X} = \begin{cases} \left( \int_0^T \|u(t)\|_{X(t)}^p \, dt \right)^{\frac{1}{p}} & \text{for } p \in [1, \infty) \\ \text{ess sup}_{t \in [0, T]} \|u(t)\|_{X(t)} & \text{for } p = \infty \end{cases} \tag{6} \]

Note that we made an abuse of notation after the definition of the first space and identified \( u(t) = (\hat{u}(t), t) \) with \( \hat{u}(t) \). That (6) defines a norm is easy to see once one checks that the integrals are well-defined, which can be shown by a straightforward adaptation of the proof of Theorem 2.8 in [5] (see the appendix). The fact that \( L^p_X \) is a Banach space follows from Lemma 2.3 below.

2.2 Important Notation. Given a function \( u \in L^p_X \), the notation \( \hat{u} \) will be used to mean the pullback \( \hat{u}(\cdot) := \phi_{-t}u(\cdot) \in L^p(0, T; X_0) \), and vice-versa.

2.3 Lemma. The spaces \( L^p(0, T; X_0) \) and \( L^p_X \) are isomorphic via \( \phi_{(\cdot)} \) with an equivalence of norms:
\[ \frac{1}{C_X} \|u\|_{L^p_X} \leq \|\phi_{-t}u(\cdot)\|_{L^p(0, T; X_0)} \leq C_X \|u\|_{L^p_X} \quad \text{for all } u \in L^p_X. \]

Proof. We show the case \( p = \infty \) here; an adaptation of the \( p = 2 \) case done in [5] easily proves the lemma for \( p \in [1, \infty) \) (see the appendix). Let \( u \in L^\infty_X \). Measurability of \( \hat{u} \) follows since \( u \in L^2_X \). Now, by definition, we have that for all \( t \in [0, T] \setminus N \), \( \|u(t)\|_{X(t)} \leq A \) where \( N \) is a null set and \( A = \|u\|_{L^\infty_X} \). This means that for all \( t \in [0, T] \setminus N \), \( C_X^{-1} \|\hat{u}(t)\|_{X_0} \leq \|u(t)\|_{X(t)} \leq A \) by the assumption (5), i.e.,
\[ \|\hat{u}\|_{L^\infty(0, T; X_0)} = \text{ess sup}_{t \in [0, T]} \|\hat{u}(t)\|_{X_0} \leq \frac{A}{C_X} \cdot \frac{1}{C_1} \|u\|_{L^\infty_X} \]
so \( \hat{u} \in L^\infty(0, T; X_0) \). Similarly, we conclude that if \( \hat{u} \in L^\infty(0, T; X_0) \) then \( u \in L^\infty_X \).

2.4 Remark. The dual operator \( \phi^*_t : X_0^* \rightarrow X^*(t) \) is also a linear homeomorphism with \( \|\phi^*_t\| = \|\phi_{-t}\| \) and \( (\phi^*_{-t})^{-1} = \phi^*_t \) [14, Theorem 4.5-2 and §4.5], and by separability of \( X_0 \), \( t \mapsto \|\phi^*_t f\|_{X^*(t)} \) is measurable for \( f \in X_0^* \); thus the dual operator also satisfies the same boundedness properties as \( \phi_t \). This means that the spaces \( L^p_X \) are also well-defined Banach spaces given \( \{X(t)\}_{t \in [0, T]} \) (the map \( \phi^*_t \) plays the same role as \( \phi_t \) did for the spaces \( L^p_X \)).

The following subspaces will be of use later:
\[ C^k_X = \{ \xi \in L^2_X \mid \phi_{-t}\xi(\cdot) \in C^k([0, T]; X_0) \} \quad \text{for } k \in \{0, 1, \ldots\} \]
\[ \hat{D}^1_V = \{ u \mid u(t) = \sum_{j=1}^m \alpha_j(t)\chi_j, \ m \in \mathbb{N}, \ \alpha_j \in AC([0, T]) \text{ and } \alpha'_j \in L^2(0, T) \} \]
\[ D_X = \{ \eta \in L^2_X \mid \phi_{-t}\eta(\cdot) \in D((0, T); X_0) \}. \]
2.1.1 Dual spaces

In order to retrieve weakly convergent subsequences from sequences that are bounded in \( L^p_X \), we need \( L^p_X \) to be reflexive. This leads us to consider a characterisation of the dual spaces. We let \( p \in [1, \infty) \) and \((p, q)\) be a conjugate pair in this section.

**Theorem 2.5.** The space \( (L^p_X)^* \) is isometrically isomorphic to \( L^q_X \), and hence we identify \( (L^p_X)^* \equiv L^q_X \) and the duality pairing of \( f \in L^p_X \) with \( u \in L^q_X \) is given by

\[
(f, u)_{L^q_X, L^p_X} = \int_0^T \langle f(t), u(t) \rangle_{X^*(t), X(t)} dt.
\]

To prove this theorem, although we can exploit the fact that the pullback is in a Bochner space, showing that the natural duality map is isometric is not so straightforward because \( \phi(\cdot) \) is not assumed to be an isometry. In fact, we have to back to the foundations and emulate the proof for the dual space identification for Bochner spaces; see [15, §IV].

**2.6 Lemma.** For every \( g \in L^q_X \), the expression

\[
l(f) = \int_0^T \langle g(t), f(t) \rangle_{X^*(t), X(t)} dt
\]

defines a functional \( l \in (L^p_X)^* \) such that \( \|l\| = \|g\|_{L^q_X} \).

**Proof.** Let \( g \in L^q_X \) and define \( l: L^p_X \to \mathbb{R} \) by (7); the integral is well-defined by similar reasoning as in the proof of Lemma 2.13 in [5]. By Hölder’s inequality, we have \( \|l(f)\| \leq \|g\|_{L^q_X} \cdot \|f\|_{L^p_X} \), so \( l \in (L^p_X)^* \) and \( \|l\| \leq \|g\|_{L^q_X} \). We now show the reverse inequality. First suppose \( g \) has the form \( g(t) = \sum x^{*}_{i,t} \chi_{E_i}(t) \) where the \( x^{*}_{i,t} \in X^*(t) \) and the \( E_i \) are measurable, pairwise disjoint and partition \([0, T]\). It is clear that \( ||g(t)||_{X^*(t)} = \sum ||x^{*}_{i,t}||_{X^*(t)} \chi_{E_i}(t) \). Let \( h(t) = ||g(t)||_{X^*(t)}^{q/p}/||g||_{L^q_X}^{q/p} \) which satisfies \( ||h||_{L^p(0,T)} = 1 \) and \( \int_0^T ||g(t)||_{X^*(t)} h(t) = ||g||_{L^q_X} \), hence for any \( \epsilon > 0 \) we have

\[
\int_0^T ||g(t)||_{X^*(t)} h(t) \geq ||g||_{L^q_X} - \epsilon/2.
\]

Now choose \( x_{i,t} \in X(t) \), \( ||x_{i,t}||_{X(t)} = 1 \) such that

\[
||x^{*}_{i,t}||_{X^*(t)} - \langle x^{*}_{i,t}, x_{i,t} \rangle_{X^*(t), X(t)} \leq \frac{\epsilon}{2 \||h||_{L^1(0,T)}}.
\]

Define \( f \in L^p_X \) by \( f(t) = \sum x_{i,t} h(t) \chi_{E_i(t)} \) and note that \( ||f||_{L^p_X} = ||h||_{L^p(0,T)} \). We obtain using (9) and (8) that \( ||f||_{L^p_X} \geq ||g||_{L^q_X} - \epsilon \). This proves that \( ||l|| = ||g||_{L^q_X} \), whenever \( g(t) = \sum x^{*}_{i,t} \chi_{E_i}(t) \) is of the stated form. Now suppose \( g \in L^q_X \) is arbitrary. Then there exist \( \hat{g}_n = \chi_{X^*_n} \) such that \( \hat{g}_n \to \hat{g} \) in \( L^q(0, T; X^*_n) \) and so the sequence \( g_n(t) := \phi_{*,T} \hat{g}_n(t) = \sum \phi_{*,t} \hat{g}_n \chi_{E_i}(t) \) satisfies \( g_n \to g \) in \( L^p_X \). Because the \( \phi_{*,t} \hat{g}_n \chi_{E_i}(t) \), we know by our efforts above that \( l_n: L^p_X \to \mathbb{R} \) defined \( l_n(f) = \int_0^T \langle g_n(t), f(t) \rangle_{X^*(t), X(t)} \) has norm \( ||l_n|| = ||g_n||_{L^q_X} \). We also have

\[
||l_n - l|| \leq ||g_n - g||_{L^q_X} \to 0
\]

which implies \( \lim_{n \to \infty} ||l_n|| = ||l|| \) and also \( \lim_{n \to \infty} ||l_n|| = \lim_{n \to \infty} ||g_n||_{L^q_X} = ||g||_{L^q_X} \). \( \square \)
We have shown that $\mathcal{J} : L^p_{X^*} \to (L^p_X)^*$ defined by $\mathcal{J}(g) := l(\cdot) = \int_0^T \langle g(t), (\cdot)(t) \rangle_{X^*(t), X(t)}$ is isometric: $\|\mathcal{J} g\|_{(L^p_X)^*} = \|l\| = \|g\|_{L^p_X}$. We now show that $\mathcal{J}$ is onto. So given $l \in (L^p_X)^*$, define $\tilde{L} : L^p(0, T; X_0) \to \mathbb{R}$ by $\tilde{L}(\tilde{v}) = l(\tilde{\phi}\circ\tilde{v})(\cdot) = l(v)$ for all $\tilde{v} \in L^p(0, T; X_0)$. It is obvious that $\tilde{L} \in L^p(0, T; X_0)^*$, and by the dual space identification for Bochner spaces, there exists an $\tilde{L}^* \in L^p(0, T; X_0^*)$ such that

$$\langle l, v \rangle_{(L^p_X)^*, L^p_X} = \langle \tilde{L}, \tilde{v} \rangle_{L^p(0, T; X_0^*)} = \int_0^T \langle \phi^* \circ \tilde{L}^*(t), v(t) \rangle_{X^*(t), X(t)},$$

so $\mathcal{J}(\phi^* \circ \tilde{L}^*(\cdot)) = l$ where $\phi^* \circ \tilde{L}^* \in L^p_{X^*}$. Hence $\mathcal{J}$ is onto, and we have proved Theorem 2.5.

### 2.2 Function spaces on evolving surfaces

We now make precise the assumptions on the evolving surface $\Omega(t)$ our Stefan problem is posed on and we discuss function spaces in the context of §2.1. For each $t \in [0, T]$, let $\Omega(t) \subset \mathbb{R}^{n+1}$ be a compact (i.e., no boundary) $n$-dimensional hypersurface of class $C^3$, and assume the existence of a flow $\phi : [0, T] \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that for all $t \in [0, T]$, with $\Omega_0 := \Omega(0)$, the map $\Phi^t_0(\cdot) := \Phi(t, \cdot) : \Omega_0 \to \Omega(t)$ is a $C^3$-diffeomorphism that satisfies $\frac{d}{dt} \Phi^t_0(\cdot) = w(t, \Phi^t_0(\cdot))$ and $\Phi^t_0(\cdot) = \text{Id}(\cdot)$ for a given $C^2$ velocity field $w : [0, T] \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, which we assume satisfies the uniform bound $|\nabla \Phi^t_0 \cdot w(t)| \leq C$ for all $t \in [0, T]$. A $C^2$ normal vector field on the hypersurfaces is denoted by $\nu : [0, T] \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. It follows that the Jacobian $J^t_\nu := \det \Phi^t_0$ is $C^2$ and is uniformly bounded away from zero and infinity.

For $u : \Omega(t) \to \mathbb{R}$ and $v : \Omega(t) \to \mathbb{R}$, define the pushforward $\phi_* u = u \circ \Phi^t_0$ and pullback $\phi^* v = v \circ \Phi^t_0$, where $\Phi^t_0 := (\Phi^t_0)^{-1}$. We showed in [6] that $\phi_* : L^2(\Omega(t)) \to L^2(\Omega(t))$ and $\phi^* : H^1(\Omega(t)) \to H^1(\Omega(t))$ are linear homeomorphisms (with uniform bounds) and (thus) with $L^2 \equiv \{ L^2(\Omega(t)) \}_{t \in [0, T]}$, $H^1 \equiv \{ H^1(\Omega(t)) \}_{t \in [0, T]}$ and $H^{-1} \equiv \{ H^{-1}(\Omega(t)) \}_{t \in [0, T]}$, the spaces $L^2_{L^2}$, $L^2_{H^1}$ and $L^2_{H^{-1}}$ are well-defined (see [6, 16] for an overview of Lebesgue and Sobolev spaces on hypersurfaces) and we let $L^2_{H^1} \subset L^2_{L^2} \subset L^2_{H^{-1}}$ be a Gelfand triple.

A function $u \in C^1_{L^2}$ has a strong material derivative defined by $\dot{u}(t) = \phi_* \left( \frac{\partial}{\partial t} (\phi^* u(t)) \right)$. Given a function $u \in L^2_{H^1}$, we say that it has a weak material derivative $g \in L^2_{H^{-1}}$ if

$$\langle u, \eta \rangle_{L^2_{H^{-1}}} = -\langle g, \eta \rangle_{L^2_{H^{-1}}}, \quad \forall \eta \in D_{H^1}$$

holds, and we write $\dot{u}$ instead of $g$. Define the Hilbert spaces (see [5, 6] for more details)

$$W(H^1(\Omega_0), H^{-1}(\Omega_0)) = \{ u \in L^2(0, T; H^1(\Omega_0)) \ | \ u' \in L^2(0, T; H^{-1}(\Omega_0)) \}$$

$$W(H^1, H^{-1}) = \{ u \in L^2_{H^1} \ | \ \dot{u} \in L^2_{H^{-1}} \}.$$

For subspaces $X \hookrightarrow H^1$ and $Y \hookrightarrow H^{-1}$, we also define the subset $W(X, Y) \subset W(H^1, H^{-1})$ in the natural manner.

### 2.7 Lemma ([5], [6]). Let $X = W(H^1, H^{-1})$ and $X_0 = W(H^1(\Omega_0), H^{-1}(\Omega_0))$, or $X = W(H^1, L^2)$ and $X_0 = W(H^1(\Omega_0), L^2(\Omega_0))$. For such pairs, the space $X$ is isomorphic to $X_0$ via $\phi^* \circ \dot{\cdot}$ with an equivalence of norms:

$$C_1 \| \phi^* \circ \dot{\cdot} \|_{X_0} \leq \| \dot{\cdot} \|_X \leq C_2 \| \phi^* \circ \dot{\cdot} \|_{X_0}.$$
We showed in [5, 6] that for \( u, v \in W(H^1, H^{-1}) \), the map \( t \mapsto (u(t), v(t))_{L^2(\Omega(t))} \) is absolutely continuous, and
\[
\frac{d}{dt} \int_{\Omega(t)} u(t)v(t) = \langle \dot{u}(t), v(t) \rangle + \langle \dot{v}(t), u(t) \rangle + \int_{\Omega(t)} u(t)v(t)\nabla \cdot w(t)
\]
holds for almost all \( t \), where the duality pairing is between \( H^{-1}(\Omega(t)) \) and \( H^1(\Omega(t)) \).

### 2.2.1 Some useful results

In this subsection, \( p \) and \( q \) are not necessarily conjugate. The first part of the following lemma is a particular realisation of Lemma 2.3.

**2.8 Lemma.** For \( p, q \in [1, \infty] \), the spaces \( L^p_{t,q} \) and \( L^p(0, T; L^q(\Omega_0)) \) are isomorphic, and there is an equivalence of norms via the map \( \phi(t) \). If \( q = \infty \) the spaces are isometrically isomorphic. The embedding \( L^\infty_{t,q} \subset L^p_t \) is continuous.

**Proof.** For \( q < \infty \), it is easy to check that \( \phi_t : L^q(\Omega_0) \to L^q(\Omega(t)) \) is a linear homeomorphism satisfying the additional boundedness requirements. Therefore, let us discuss \( q = \infty \). Let \( u \in L^\infty(\Omega(t)) \). We have
\[
\|u\|_{L^\infty(\Omega(t))} = \text{ess sup}_{x \in \Omega(t)} |u(x)| = \text{ess sup}_{y \in \Omega_0} |u(\Phi^0_t(y))| = \|\tilde{u}\|_{L^\infty(\Omega_0)}
\]
because \( \Phi^0_t \) is a diffeomorphism so null sets are mapped to null sets. Now an application of Lemma 2.3 yields the result. The above calculation also shows that the norm is preserved for \( q = \infty \).

The statement regarding the continuous embedding is easy to check. \( \square \)

**2.9 Lemma.** The space \( W(H^1, H^{-1}) \) is compactly embedded in \( L^2_{t,2} \).

**Proof.** That the embedding is continuous is obvious. Let \( w_n \) be a bounded sequence in \( W(H^1, H^{-1}) \). Then \( \phi_{-1}(\cdot)w_n \) is a bounded sequence in \( W(H^1, H^{-1}) \), and by \( W(H^1, H^{-1}) \hookrightarrow L^2(0, T; L^2(\Omega_0)) \), there is a subsequence \( \phi_{-1}(\cdot)w_{n_k} \to \tilde{w} \) that converges in \( L^2(0, T; L^2(\Omega_0)) \). Hence \( w_{n_k} \to \phi(\cdot)w \) in \( L^2_{t,2} \). \( \square \)

**Theorem 2.10** (Dominated convergence theorem for \( L^p_{t,p} \)). Let \( p, q \in [1, \infty) \). Let \( \{w_n\} \) and \( w \) be functions such that \( \{\tilde{w}_n\} \) and \( \tilde{w} \) are measurable (e.g. membership of \( L^1_{t,1} \) will suffice). If for almost all \( t \in [0, T] \),
\[
\exists g \in L^p_{t,q} : |w_n(t)| \leq g(t) \quad \text{almost everywhere in } \Omega(t) \text{ and for all } n,
\]
then \( w_n \to w \) in \( L^p_{t,p} \).

**2.11 Lemma.** If \( u \in W(H^1, H^{-1}) \), then
\[
2 \int_0^T \langle \dot{u}(t), u^+(t) \rangle_{H^{-1}(\Omega(t)), H^1(\Omega(t))} = \int_{\Omega(T)} u^+(T)^2 - \int_{\Omega_0} u^+(0)^2 - \int_0^T \int_{\Omega(t)} u^+(t)^2 \nabla \cdot w. \tag{10}
\]
Proof. By density, we can find \( \{u_n\} \subset W(H^1, L^2) \) with \( u_n \to u \) in \( W(H^1, H^{-1}) \). It follows that \( \partial^*(u_n^+) = \dot{u}_n\chi_{u_n \geq 0} \in L^2_{L^2} \) (this is sensible because \( w \in H^1(\Omega) \) implies \( w^+ \in H^1(\Omega) \)) and therefore (10) holds for \( u_n \). Since \( W(H^1, H^{-1}) \hookrightarrow C^0_\Omega \), it follows that \( u_n^+(t) \to u^+(t) \) in \( L^2(\Omega(t)) \) (for example see [17, Lemma 2.88] or [18, Lemma 1.22]). So we can pass to the limit in the first two terms on the right hand side.

Now we just need to show that \( u_n^+ \to u^+ \) in \( L^2_{L^1} \). It is easy to show the convergence in \( L^2_{L^2} \), so we need only to check the convergence of the gradient. Let \( g(r) = \chi_{\{r > 0\}} \). Then, using \( g \leq 1 \),

\[
|\nabla u_n^+(t, x) - \nabla u^+(t, x)| \leq |\nabla u_n(t, x) - \nabla u(t, x)| + |g(u_n(t, x)) - g(u(t, x))||\nabla u(t, x)|.
\]

For the second term, let us note that since \( u_n \to u \) in \( L^2_{L^1} \), for almost all \( t, u_n(t, x) \to u(t, x) \) almost everywhere in \( \Omega(t) \) for a subsequence (which we have not relabelled). Let us fix \( t \). Then for almost every \( x \in \Omega(t) \), it follows that \( g(u_n(t, x))\nabla u(t, x) \to g(u(t, x))\nabla u(t, x) \) pointwise. Because \( g \leq 1 \), the dominated convergence theorem gives overall \( \nabla u_n^+ \to \nabla u^+ \) in \( L^2_{L^2} \). \( \square \)

2.3 Preliminary results

2.12 Remark. It is well-known in the standard setting that a mushy region (the interior of the set where the temperature is zero) can appear in the presence of heat sources; with no heat sources, the initial data may give rise to mushy regions. We will content ourselves with the following heuristic calculations under the assumptions described below that there is no mushy region.

Let the bounded weak solution of (1) (in the sense of Definition 1.2) have the additional regularity \( u \in W(H^1, L^2) \) and \( \Delta u \in L^2_{L^2} \), and suppose that the sets \( \Omega(t) = \{u > 0\} \) and \( \Omega_u(\Sigma) = \{u < 0\} \) divide \( \Omega(t) \) with a common interface \( \Sigma(t) \), which we assume is a sufficiently smooth \( n \)-dimensional hypersurface (of measure zero with respect to the surface measure on \( \Omega(t) \)). Then the bounded weak solution is also a classical solution in the sense of (2).

To see this, suppose that \( (u, e) \) is a weak solution satisfying the equality in (4). The integration by parts formula on each subdomain of \( \Omega \) implies

\[
\int_0^T \int_{\Omega(t)} \nabla u(t) \nabla \eta(t) = -\int_0^T \int_{\Omega(t)} \eta(t) \Delta u(t) + \int_0^T \int_{\Gamma(t)} \eta(t)(\nabla u \cdot \nu_{\Omega(t)} - \nabla u \cdot \nu_{\Omega_u(t)}) \cdot \mu.
\]

(11)

With \( e(t)\eta(t)\nabla \cdot w = \nabla \cdot (e(t)\eta(t)w) - w \cdot \nabla (e(t)\eta(t)) \) and the divergence theorem [16, §2.2],

\[
\int_0^T \int_{\Omega(t)} e(t)\eta(t)\nabla \cdot w = \int_0^T \int_{\Gamma(t)} e(t)\eta(t)w \cdot \mu + \int_0^T \int_{\Omega_u(t)} w \cdot (e(t)\eta(t)\nu_{\Omega(t)} - \nabla \cdot (e(t)\eta(t))).
\]
We use this result in the formula for integration by parts over time over $\Omega$:

$$
\int_0^T \int_{\Omega(t)} \dot{\eta}(t)e(t) = \int_0^T \frac{d}{dt} \int_{\Omega(t)} e(t)\eta(t) - \int_0^T \int_{\Omega(t)} \dot{\eta}(t)e(t) - \int_0^T \int_{\Gamma(t)} e(t)\eta(t)w \cdot \mu + \int_0^T \int_{\Omega(t)} e(t)\eta(t)w \cdot \nu H + \int_0^T \int_{\Omega(t)} w \cdot \nabla \delta(u(e(t)\eta(t))).
$$

A similar expression over $\Omega$ can also be derived this way, the difference being that the term with $\mu$ has the opposite sign. Then, using $\dot{e} = \partial_\delta(E(u)) = \dot{u}$, $e\eta(t)|_{\Gamma(t)} = 0$, and $e(t)|_{\Gamma(t)} = 1$, we get

$$
\int_0^T \int_{\Omega(t)} \dot{\eta}(t)e(t) = \int_0^T \frac{d}{dt} \int_{\Omega(t)} e(t)\eta(t) - \int_0^T \int_{\Omega(t)} \dot{u}(t)\eta(t) + \int_0^T \int_{\Gamma(t)} \eta(t)w \cdot \mu + \int_0^T \int_{\Omega(t)} e(t)\eta(t)w \cdot \nu H + \int_0^T \int_{\Omega(t)} w \cdot \nabla \delta(u(e(t)\eta(t))). \quad (12)
$$

Since by the partial integration formula $\int_{\Omega(t)} \nabla(g) = \int_{\Omega(t)} g\nu$, we have (with $g = w\eta(t)$) that the fourth term in the right hand side of (12) is

$$
\int_{\Omega(t)} e(t)\eta(t)w \cdot \nu H = \sum_i \int_{\Omega(t)} e(t)\eta(t)w_i\nu_i H = \int_{\Omega(t)} \nabla \delta(u(e(t)\eta(t))) \cdot w + \int_{\Omega(t)} \eta(t)w(e(t)) \nabla \delta \cdot w.
$$

So the calculation (12) becomes

$$
\int_0^T \int_{\Omega(t)} \dot{\eta}(t)e(t) = \int_0^T \frac{d}{dt} \int_{\Omega(t)} e(t)\eta(t) - \int_0^T \int_{\Omega(t)} \dot{u}(t)\eta(t) + \int_0^T \int_{\Gamma(t)} \eta(t)w \cdot \mu + \int_0^T \int_{\varOmega(t)} \eta(t)w \cdot \nu H + \int_0^T \int_{\Omega(t)} w \cdot \nabla \delta(u(e(t)\eta(t))). \quad (13)
$$

Now, taking the weak formulation (4) and substituting (13) together with the expression for the spatial term (11), we get for $\eta$ with $\eta(T) = \eta(0) = 0$

$$
\int_0^T \int_{\Omega(t)} f(t)\eta(t) = -\int_0^T \int_{\Omega(t)} \dot{\eta}(t)e(t) + \int_0^T \int_{\Omega(t)} \nabla \delta(u(t)) \nabla \eta(t) + \int_0^T \int_{\Omega(t)} \dot{u}(t) + \int_0^T \int_{\Omega(t)} e(t)\nabla \delta \cdot w - \Delta u(t) - \nabla \delta \eta(t) + \int_0^T \int_{\Omega(t)} \eta(t) \cdot \nabla \delta(u(t)) - \nabla \delta u(t)) \cdot \mu - (w \cdot \mu). \quad (14)
$$

Taking $\eta$ to be compactly supported in $Q_\gamma$, and afterwards taking $\eta$ compactly supported in $Q_\gamma$, we recover exactly the first two equations in (2). So we may drop the first integral on the left and the right hand side. If we then pick $\eta$ carefully, we will obtain precisely the interface condition in (2).

2.13 Lemma. Given $\xi \in C^1(\Omega_0)$ and $\tilde{\alpha} \in C^2([0,T] \times \Omega_0)$ satisfying $0 < \epsilon \leq \alpha \leq \alpha_0$ a.e., there exists a unique solution $\varphi \in W(H^1, L^2)$ with $\Delta \varphi \in L^2_{\nu}$ to

$$
\varphi - \alpha(x,t) \Delta \varphi = 0 \quad (14)
$$

$$
\varphi(x,0) = \xi(x)
$$
satisfying \( \| \varphi \|_{L^\infty} \leq \| \xi \|_{L^\infty(\Omega_0)} \) and (cf. [19, Chapter V, §9])

\[
\int_0^t \int_{\Omega(\tau)} (\varphi(\tau))^2 + \int_0^t \int_{\Omega(\tau)} a |\Delta \varphi|^2 + \int_{\Omega(\tau)} |\nabla \varphi(t)|^2 \leq (1 + \alpha_0)(1 + e^{2C_\omega(1 + \alpha_0)t}) \int_{\Omega_0} |\nabla \xi|^2.
\]

(15)

Proof. Define the bilinear form \( a(t; \varphi, \eta) = \int_{\Omega(t)} \alpha(x, t) \nabla \varphi \nabla \eta + \int_{\Omega(t)} \nabla \alpha(x, t) \nabla \varphi \eta \) which is clearly bounded and coercive. Split \( a(t; \cdot, \cdot) \) into the forms \( a_s(t; \varphi, \eta) := \int_{\Omega(t)} \alpha(x, t) \nabla \varphi \nabla \eta \) and \( a_n(t; \varphi, \eta) := \int_{\Omega(t)} \nabla \alpha(x, t) \nabla \varphi \eta \). One sees that both \( a_n(t; \cdot, \cdot) : V(t) \times H(t) \to \mathbb{R} \) and \( a_s(t; \cdot, \cdot) : V(t) \times V(t) \to \mathbb{R} \) are bounded, and \( a_s(t; \eta, \eta) \geq 0 \). Also, we have for \( \eta \in \tilde{C}^1_{H_1} \)

\[
\frac{d}{dt} a_s(t; \eta(t), \eta(t)) = 2a_s(t; \eta(t), \eta(t)) + r(t; \eta(t))
\]

where \( r \) is such that \( |r(t; \eta(t))| \leq C \| \eta(t) \|^2_{V(t)} \) (see [16, Lemma 2.1], note that we have \( \tilde{\alpha} \in C^1([0, T]; C^1(\Omega_0)) \)) and thus \( \alpha \in \tilde{C}^1_{H_1} \). Hence by [5, Theorem 3.13] we have the unique existence of \( \varphi \in W(H^1, L^2) \). Rearranging the equation (14) shows that \( \alpha \Delta \Omega \varphi \in L^2_{L^2} \). Since \( \alpha \) is uniformly bounded by positive constants, it follows that \( \Delta \Omega \varphi \in L^2_{L^2} \).

**The \( L^\infty \) bound** Let \( K := \| \xi \|_{L^\infty(\Omega_0)} \). Test the equation with \((\varphi - K)^+\):

\[
\frac{1}{2} \frac{d}{dt} \| (\varphi(t) - K)^+ \|^2_{L^2(\Omega_0)} + \int_{\Omega(t)} \alpha(t) \nabla \varphi ((\varphi(t) - K)^+) \nabla \varphi(t) \]

\[
= \frac{1}{2} \int_{\Omega(t)} ((\varphi(t) - K)^+)^2 \nabla \varphi \cdot w - \int_{\Omega(t)} \nabla \alpha(t) \nabla \varphi(t) (\varphi(t) - K)^+
\]

which becomes, through the use of Young’s inequality with \( \delta \),

\[
\frac{1}{2} \frac{d}{dt} \| (\varphi(t) - K)^+ \|^2_{L^2(\Omega_0)} \leq \left( \frac{C_\omega}{2} + \| \nabla \alpha \|_{L^\infty} C_\delta \right) \| (\varphi(t) - K)^+ \|^2_{L^2(\Omega_0)}.
\]

An application of Gronwall’s inequality and noticing \((\varphi(0) - K)^+ = (\xi - \| \xi \|_{L^\infty})^+ = 0 \) yields \( \varphi(t) \leq \| \xi \|_{L^\infty(\Omega_0)} \). Repeating this process with \((-\varphi(t) - K)^+ \) allows us to conclude.

**The inequality** (15) Multiplying the equation (14) by \( \Delta \Omega \varphi \) and integrate: formally,

\[
\int_0^t \int_{\Omega(\tau)} \alpha |\Delta \varphi|^2 = -\int_0^t \int_{\Omega(\tau)} \nabla \varphi \cdot \nabla \varphi - \int_0^t \frac{1}{2} \frac{d}{d\tau} \int_{\Omega(\tau)} |\nabla \varphi|^2 + \frac{1}{2} \int_0^t \int_{\Omega(t)} |\nabla \varphi|^2 \nabla \cdot w - \int_0^t \int_{\Omega(t)} D(w) \nabla \varphi \cdot \nabla \varphi
\]

\[
\leq \frac{1}{2} \int_{\Omega_0} |\nabla \Omega \xi|^2 - \frac{1}{2} \int_{\Omega(t)} |\nabla \varphi|^2 + C_\omega \int_0^t \int_{\Omega(\tau)} |\nabla \varphi|^2.
\]

(16)
See [16, Lemma 2.1] or [6] for the definition of $D(w)$. This calculation is merely formal because we have not shown that $\varphi(t) \in H^1(\Omega(t))$; however the end result of the calculation is still valid by Lemma 2.14. We also have by squaring (14), integrating and using (16):

$$\int_0^t \int_{\Omega(\tau)} (\varphi(\tau))^2 \leq \alpha_0 \int_0^t \int_{\Omega(\tau)} \alpha(\Delta \varphi)^2 \leq \frac{\alpha_0}{2} \int_{\Omega_0} |\nabla \Omega \xi|^2 + \alpha_0 C_w \int_0^t \int_{\Omega(\tau)} |\nabla \varphi|^2.$$ 

Adding the last two inequalities then we obtain

$$\int_0^t \int_{\Omega(\tau)} (\varphi(\tau))^2 + \int_0^t \int_{\Omega(\tau)} \alpha |\Delta \varphi|^2 + \frac{1}{2} \int_0^t \int_{\Omega(\tau)} |\nabla \varphi|^2 \leq \frac{1}{2} \int_{\Omega_0} |\nabla \Omega \xi|^2 + C_w (1 + \alpha_0) \int_0^t \int_{\Omega(\tau)} |\nabla \varphi|^2.$$ 

Gronwall’s inequality can be used to deal with the last term on the right hand side.

2.14 Lemma. With $\varphi \in W(H^1, L^2)$ from the previous lemma, the following inequality holds:

$$\int_0^t \int_{\Omega(\tau)} \alpha |\Delta \varphi|^2 \leq \frac{1}{2} \int_{\Omega_0} |\nabla \Omega \xi|^2 - \frac{1}{2} \int_0^t \int_{\Omega(\tau)} |\nabla \varphi|^2 + C_w \int_0^t \int_{\Omega(\tau)} |\nabla \varphi|^2. \quad (17)$$

Proof. Let $C^\infty_{H_0^2} := \{ \eta \mid \phi(-\cdot)\eta(\cdot) \in C^\infty([0, T]; H^2(\Omega_0)) \}$. We start with a few preliminary results. Let us show $C^\infty_{H_0^2} \subset W(H^2, H^1)$. Take $\eta \in C^\infty_{H_0^2}$ so that $\tilde{\eta} \in C^\infty([0, T]; H^2(\Omega_0)) \subset W(H^2, H^1)$. By smoothness of $\Phi^{(3)}$, it follows that $\eta = \phi(t)\tilde{\eta} \in L^2_{H_2}$, and $\tilde{\eta} = \partial^* (\phi(t)\tilde{\eta}) = \phi(t) \tilde{\eta}' \in L^2_{H_1}$ because $\tilde{\eta}' \in C^\infty([0, T]; H^2(\Omega_0)) \subset L^2(0, T; H^1(\Omega_0))$. So $\eta \in W(H^2, H^1)$.

Let us also prove that the density of $C^\infty_{H_0^2} \subset W(H^2, L^2)$. Let $w \in W(H^2, L^2)$; then $\tilde{w} \in W(H^2, L^2)$ since $\tilde{w} \in L^2(0, T; H^2(\Omega_0))$ by smoothness of $\Phi^{(3)}$ and since $\tilde{w}' = \phi(-\cdot)\tilde{w} \in L^2(0, T; L^2(\Omega_0))$ because $\tilde{w} \in L^2_{L_2}$. By [20, Lemma II.5.10], there exists a sequence $\tilde{w}_n \in C^\infty([0, T]; H^2(\Omega_0))$ with $\tilde{w}_n \to \tilde{w}$ in $W(H^2, L^2)$. Then, $w_n := \phi(t)\tilde{w}_n \in C^\infty_{H_0^2}$ (by definition) and

$$\|w_n - w\|_{W(H^2, L^2)} \leq C \left( \|\tilde{w}_n - \tilde{w}\|_{L^2(0, T; H^2(\Omega_0))} + \|\tilde{w}'_n - \tilde{w}'\|_{L^2(0, T; L^2(\Omega_0))} \right) \to 0,$$

where we used the smoothness of $\Phi^{(3)}$ and the reasoning behind Assumption 2.37 of [5] (see also [5, Theorem 2.33]).

Given $\varphi \in W(H^2, L^2)$, by the density result, there exists $\varphi_n \in C^\infty_{H_0^2} \subset W(H^2, H^1)$ such that $\varphi_n \to \varphi$ in $W(H^2, L^2)$ with $\varphi_n$ satisfying (17):

$$\int_0^t \int_{\Omega(\tau)} \alpha |\Delta \varphi|^2 \leq \frac{1}{2} \int_{\Omega_0} |\nabla \varphi_n^0|^2 - \frac{1}{2} \int_0^t \int_{\Omega(\tau)} |\nabla \varphi_n(t)|^2 + C_w \int_0^t \int_{\Omega(\tau)} |\nabla \varphi_n|^2. \quad (18)$$

We know that $\varphi_n \to \varphi$ in $W(H^2, L^2)$ (this is just how we construct the sequence $\varphi_n$; see above), and $W(H^2, L^2) \hookrightarrow C^0([0, T]; H^1(\Omega_0))$ [20, Lemma II.5.14] implies $\varphi_n(t) \to \varphi(t)$ in $H^1(\Omega(t))$. Now we can pass to the limit in every term in (18).
3 Well-posedness

We can approximate $\mathcal{E}$ by $C^\infty$ functions $\mathcal{E}_\epsilon$ such that (for example see [11, 12])

\[
\begin{align*}
\mathcal{E}_\epsilon \to \mathcal{E} & \quad \text{uniformly in the compact subsets of } \mathbb{R} \setminus \{0\} \\
\mathcal{E}_\epsilon^{-1} \to \mathcal{E}^{-1} & \quad \text{uniformly in the compact subsets of } \mathbb{R} \\
\mathcal{E}_\epsilon & \text{ is bi-Lipschitz} \\
\mathcal{E}_\epsilon(0) = 0 & \text{ and } \mathcal{E}_\epsilon = \mathcal{E} \text{ on } (-\infty, 0) \cup (\epsilon, \infty) \\
1 \leq \mathcal{E}_\epsilon'(r) & \leq 1 + L_\epsilon \quad \text{for all } r \in \mathbb{R} \\
\frac{1}{1 + L_\epsilon} & \leq (\mathcal{E}_\epsilon^{-1}(r))' \leq 1 \quad \text{for all } r \in \mathbb{R}
\end{align*}
\]

(where $L_\epsilon = O(1/\epsilon)$ is the Lipschitz constant of the approximation to the Heaviside function).

Note that the last property implies for all $r, s \in \mathbb{R}$,

\[
|\mathcal{E}_\epsilon^{-1}(r) - \mathcal{E}_\epsilon^{-1}(s)| \leq |r - s|.
\]

We write $\mathcal{U} := \mathcal{E}^{-1}$ and $\mathcal{U}_\epsilon := \mathcal{E}^{-1}_\epsilon$. In order to prove Theorem 1.3, that of the well-posedness of $L^\infty$ weak solutions given bounded data, we consider the following approximation of (1).

3.1 Definition (The approximate problem using $\mathcal{U}_\epsilon$). Find for each $\epsilon > 0$ a function $e_\epsilon \in W(H^1, H^{-1})$ such that

\[
\partial^\epsilon_\epsilon - \Delta_O(\mathcal{U}_\epsilon e_\epsilon(t)) + e_\epsilon(t)\nabla \Omega \cdot w = f \quad \text{in } L^2_{H^{-1}}
\]

\[
e_\epsilon(0) = e_0.
\]

(P$_\epsilon$)

Theorem 3.2. Given $f \in L^2_{H^{-1}}$ and $u_0 \in L^2(\Omega_0)$, the problem (P$_\epsilon$) has a weak solution $e_\epsilon \in W(H^1, H^{-1})$.

Proof. Using the chain rule on the nonlinearity leads us to consider for fixed $w \in W(H^1, H^{-1})$

\[
\langle \partial^\epsilon_\epsilon(Sw, \eta)_{L^2_{H^{-1}}, L^2_H}, + \mathcal{U}'_\epsilon(w)\nabla \Omega(Sw), \nabla \eta \rangle_{L^2_{H^{-1}}} + \langle Sw, \eta \nabla \Omega \cdot w \rangle_{L^2_{H^{-1}}} = \langle f, \eta \rangle_{L^2_{H^{-1}}, L^2_H}
\]

\[
e_\epsilon(0) = e_0. \quad \text{(P}(w)\)

If $S$ denotes the solution map of (P($w$)) that takes $w \mapsto Sw$, then we seek a fixed point of $S$. First, note that since the bilinear form involving the surface gradient is bounded and coercive, the solution $Sw \in W(H^1, H^{-1})$ of (P($w$)) does indeed exist by [5, Theorem 3.6], and moreover, it satisfies the estimate

\[
\|S(w)\|_{W(H^1, H^{-1})} \leq C \left( \|f\|_{L^2}, + \|u_0\|_{L^2(\Omega_0)} \right) =: C_\star \quad \text{(20)}
\]

where the constant $C$ does not depend on $w$ because $\mathcal{U}'_\epsilon(w(t))$ is uniformly bounded from below (in $w$). Then the set $E := \{w \in W(H^1, H^{-1}) \mid w(0) = e_0, \|w\|_{W(H^1, H^{-1})} \leq C_\star \}$, which is a closed, convex, and bounded subset of $X := W(H^1, H^{-1})$, is such that $S(E) \subset E$ by (20).
We now show that $S$ is weakly continuous. Let $w_n \to w$ in $W(H^1, H^{-1})$ with $w_n \in E$. From the estimate (20), we know that $Sw_n$ is bounded in $W(H^1, H^{-1})$, so for a subsequence

$$Sw_{n_j} \to \chi \text{ in } W(H^1, H^{-1})$$

$$Sw_{n_j} \to \chi \text{ in } L^2$$

by the compact embedding of Lemma 2.9. Now we show that $\chi = Sw$. Due to $W(H^1, H^{-1}) \hookrightarrow C^0_0$. $Sw_{n_j} \to \chi$ in $C^0_0$. This implies $Sw_{n_j}(0) \to \chi(0)$ in $L^2(\Omega_0)$ (to see this consider for arbitrary $f \in L^2(\Omega_0)$ the functional $G \in (C^0_0)^*$ defined $G(u_n) = \int_{\Omega_0} fu_n(0)$). Since $Sw_{n_j}(0) = e_0$, it follows that

$$\chi(0) = e_0. \quad (21)$$

On the other hand, since $w_n$ are weakly convergent in $W^1(H^1, H^{-1})$, they are bounded in the same space. Now, $W(H^1, H^{-1}) \overset{e}{\hookrightarrow} L^2$, hence $w_n \to w$ in $L^2$. It follows that the subsequence $w_{n_j} \to w$ in $L^2$ too, and so there is a subsequence such that for almost every $t \in [0, T]$, $w_{n_j}(t) \to w(t)$ a.e. in $\Omega(t)$. By continuity, for a.a. $t \in [0, T]$, $U'(w_{n_j}(t))\nabla \Omega(t) \to U'(w(t))\nabla \Omega(t)$ a.e., and also we have $|U'(w_{n_j})\nabla \Omega(t)| \leq |\nabla \Omega(t)|$ with the right hand side in $L^2$. Thus we can use the dominated convergence theorem (Theorem 2.10) which tells us that $U'(w_{n_j})\nabla \Omega(t) \to U'(w)\nabla \Omega(t)$ in $L^2$. Now we pass to the limit in the equation $\mathbf{(P(w))}$ with $w$ replaced by $w_{n_j}$ to get

$$\int_0^T (\nabla \chi(t), \eta(t)) + \int_{\Omega(t)} U'(w(t))\nabla \Omega(t)\nabla \Omega(t) + \int_{\Omega(t)} \chi(t)\eta(t)\nabla \Omega \cdot w = \int_0^T (f(t), \eta(t))$$

which, along with (21), shows that $\chi = S(w)$, so $Sw_{n_j} \to S(w)$.

However, we have to show that the whole sequence converges, not just a subsequence. Let $x_n = S(w_n)$ and equip the space $X = W(H^1, H^{-1})$ with the weak topology. Let $x_{n_m} = S(w_{n_m})$ be a subsequence. By the bound of $S$, it follows that $x_{n_m}$ is bounded, hence it has a subsequence such that

$$x_{n_{m_j}} \to x^* \text{ in } X \quad \text{ and } \quad x_{n_{m_j}} \to x^* \text{ in } L^2$$

By a similar reasoning as before, we identify $x^* = S(w)$, and Theorem 3.3 below tells us that indeed $x_n = S(w_n) \to S(w)$. Then by the Schauder–Tikhonov fixed point theorem [21, Theorem 1.4 on p. 118], $S$ has a fixed point.

**Theorem 3.3.** Let $\{x_n\}$ be a sequence in a topological space $X$ such that every subsequence $\{x_{n_k}\}$ has a subsequence $\{x_{n_{k_m}}\}$ converging to $x \in X$. Then the full sequence $x_n$ converges to $x$.

### 3.1 Uniform estimates

We set $u_\epsilon = U(\epsilon x)$. Below we denote by $M$ a constant such that $\|u_\epsilon\|_{L^\infty(\Omega_0)} \leq M$.

**3.4 Lemma.** The following bound holds independent of $\epsilon$:

$$\|u_\epsilon\|_{L^\infty} + \|E(\epsilon u_\epsilon)\|_{L^\infty} \leq 2e^{\|\nabla u\|_{L^\infty}} T \left( T \|f\|_{L^\infty} + \|u_0\|_{L^\infty(\Omega_0)} + 1 \right) + 1.$$
Proof. We substitute \( w(t) = e^{-\lambda t} \epsilon_\epsilon(t) \) in \((P_\epsilon)\) and use \( \partial^* (e^{\lambda t} w(t)) = \lambda e^{\lambda t} w(t) + e^{\lambda t} \dot{w}(t) \) to get
\[
\dot{w}(t) - e^{-\lambda t} \Delta w(\mathcal{U}(e^{\lambda t} w(t))) + \lambda w(t) + w(t) \nabla \cdot w = e^{-\lambda t} f(t).
\]
Let \( \alpha = \|f\|_{L^\infty_T} \) and \( \beta = \|e_0\|_{L^\infty(\Omega_0)} \) and define \( v(t) = \alpha t + \beta \).\footnote{Note that \( \dot{v}(t) = \alpha \) and \( v(0) = \beta \). Subtracting \( \dot{v}(t) \) from the above and testing with \( (w(t) - v(t))^+ \), we get}
\[
\langle \dot{w}(t) - \dot{v}(t), (w(t) - v(t))^+ \rangle_{V^*, V}(t) + \int_{\Omega(t)} e^{-\lambda t} \nabla \mathcal{U}(e^{\lambda t} w(t))) \nabla w(t) - v(t))^+ = \int_{\Omega(t)} (e^{-\lambda t} f(t) - \alpha)(w(t) - v(t))^+.
\]
Note that \( e^{-\lambda t} \nabla \mathcal{U}(e^{\lambda t} w(t))) \nabla w(t) - v(t))^+ = \mathcal{U}(e^{\lambda t} w(t))) \nabla w(t) - v(t))^+| \) because \( \nabla \nabla w(t) = 0 \).\footnote{Set \( \lambda := \nabla \cdot w \|_{L^\infty_T} \), then the last term on the LHS of \((22)\) is non-negative}
\[
\begin{align*}
\int_{\Omega(t)} (\lambda + \nabla \cdot w) w(t)(w(t) - v(t))^+ = \int_{\Omega(t)} (e^{-\lambda t} f(t) - \alpha)(w(t) - v(t))^+.
\end{align*}
\]
Integrating this and using Lemma 2.11, we find
\[
\frac{1}{2} \int_{\Omega(T)} ((w(t) - v(t))^+)^2 \leq \frac{1}{2} \|\nabla \cdot w\| \int_{0}^{T} \int_{\Omega(t)} ((w(t) - v(t))^+)^2
\]
since \( e^{-\lambda t} f(t) - \alpha = e^{-\lambda t} f(t) - \|f(t)\|_{L^\infty(\Omega(t))} \leq 0 \) and \( w(0) - v(0) = e_0 - \|e_0\|_{L^\infty(\Omega_0)} \leq 0 \). The use of Gronwall’s inequality gives \( w(t) \leq T \|f\|_{L^\infty_T} + (1 + M) \) almost everywhere on \( \Omega(t) \). So we have shown that for all \( t \in [0, T] \setminus N_1 \), \( w(t, x) \leq C \) for all \( x \in \Omega(t) \setminus M_1^t \), where \( \mu(N_1) = \mu(M_1^t) = 0 \).\footnote{A similar argument yields for all \( t \in [0, T] \setminus N_2 \), \( w(t, x) \geq -C \) for all \( x \in \Omega(t) \setminus M_2^t \), where \( \mu(N_2) = \mu(M_2^t) = 0 \). Taking these statements together tells us that for all \( t \in [0, T] \setminus N \), \( |w(t, x)| \leq C \) on \( \Omega(t) \setminus M^t \) where \( N = N_1 \cup N_2 \) and \( M^t = M_1^t \cup M_2^t \) have measure zero. This gives \( \|w\|_{L^\infty_T} \leq T \|f\|_{L^\infty_T} + (1 + M) \). From this and \( u_\epsilon = \mathcal{U}(e^{\lambda t} w(\cdot)) \leq e^{\lambda T} |w| \), we obtain the bound on \( u_\epsilon \). The bound on \( \mathcal{E}_\epsilon(u_\epsilon) \) follows from \( \mathcal{E}_\epsilon(u_\epsilon) \leq 1 + |u_\epsilon| \).}

3.5 Lemma. The following bound holds independent of \( \epsilon \):
\[
\|\nabla u_\epsilon\|_{L^4_T^2} + \|\partial^*(\mathcal{E}_\epsilon u_\epsilon)\|_{L^2_{H^{-1}}} \leq C(T, \Omega, M, w, f).
\]
Proof. Testing with \( \mathcal{E}_\epsilon(u_\epsilon) \) in \((P_\epsilon)\), using \( \nabla \nabla u_\epsilon \nabla \mathcal{E}_\epsilon(u_\epsilon) = (\mathcal{E}_\epsilon)'(u_\epsilon)|\nabla \mathcal{E}_\epsilon u_\epsilon|^2 \geq |\nabla \mathcal{E}_\epsilon u_\epsilon|^2 \), integrating over time and using the previous estimate, we find
\[
\begin{align*}
\frac{1}{2} \|\mathcal{E}_\epsilon(u_\epsilon(T))\|^2_{L^2(\Omega(T))} + \int_{0}^{T} \int_{\Omega(t)} |\nabla u_\epsilon|^2 \leq \frac{1}{2} (1 + M)^2 |\Omega_0| + C_1(T, M, w, f).
\end{align*}
\]
The bound on the time derivative follows by taking supremums.\footnote{The bound on the time derivative follows by taking supremums.}

3.6 Lemma. Define \( \tilde{u}_\epsilon = \phi_{-\epsilon(u_\epsilon)} \). The following limit holds uniformly in \( \epsilon \):
\[
\lim_{h \to 0} \int_{0}^{T-h} \int_{\Omega_0} |\tilde{u}_\epsilon(t + h) - \tilde{u}_\epsilon(t)| = 0.
\]
Proof. We follow the proof of Theorem A.1 in [7] here. Fix $h \in (0, T)$ and consider

$$
\int_0^{T-h} (E_c(\tilde{u}_e(t+h)) - E_c(\tilde{u}_e(t)), \tilde{u}_e(t+h) - \tilde{u}_e(t))_{L^2(\Omega_0)} \, dt
$$

$$
= \int_0^{T-h} \int_0^t \frac{d}{d\tau} (E_c(\tilde{u}_e(\tau)), \tilde{u}_e(t+h) - \tilde{u}_e(t))_{L^2(\Omega_0)} \, d\tau \, dt
$$

$$
\leq \sqrt{h} \| (E_c(\tilde{u}_e))' \|_{L^2(0,T;H^{-1}(\Omega_0))} \int_0^{T-h} (\| \tilde{u}_e(t+h) \|_{H^1(\Omega_0)} + \| \tilde{u}_e(t) \|_{H^1(\Omega_0)}) \, dt
$$

$$
\leq C_1(T, \Omega, M, w, f) \sqrt{h} \| (E_c(\tilde{u}_e))' \|_{L^2(0,T;H^{-1}(\Omega_0))} \quad \text{(by the uniform estimates)}
$$

$$
= C_2(T, \Omega, M, w, f) \sqrt{h} \| \partial^* (E_c(\tilde{u}_e)) \|_{L^2_{H^{-1}}(0,T; \Omega)} \quad \text{(see the proof of Theorem 2.33 in [5])}
$$

$$
\leq C_3(T, \Omega, M, w, f) \sqrt{h}.
$$

(25)

with the last inequality by (24). Now, since $\mathcal{U}_e'$ are uniformly bounded above, they are uniformly equicontinuous. Therefore, for fixed $\delta$, there is a $\sigma_\delta$ (depending solely on $\delta$) such that

$$
\text{if } |y - z| < \sigma_\delta, \text{ then } |\mathcal{U}_e(y) - \mathcal{U}_e(z)| < \delta \quad \text{for any } \epsilon.
$$

(26)

So in the set $\{ |\tilde{u}_e(t+h) - \tilde{u}_e(t)| > \delta \} = \{ \mathcal{U}_e(E_c(\tilde{u}_e(t+h))) - \mathcal{U}_e(E_c(\tilde{u}_e(t))) > \delta \}$, we must have $|E_c(\tilde{u}_e(t+h)) - E_c(\tilde{u}_e(t))| \geq \sigma_\delta$ (this is the contrapositive of (26)). This implies from (25) that

$$
\int_0^{T-h} \int_{\Omega_0} (\tilde{u}_e(t+h) - \tilde{u}_e(t)) \chi_{\{|\tilde{u}_e(t+h) - \tilde{u}_e(t)| > \delta \}} \leq \frac{C_3 \sqrt{h}}{\sigma_\delta}.
$$

Writing $\text{Id} = \chi_{\{|\tilde{u}_e(t+h) - \tilde{u}_e(t)| > \delta \}} + \chi_{\{|\tilde{u}_e(t+h) - \tilde{u}_e(t)| \leq \delta \}}$, notice that

$$
\int_0^{T-h} \int_{\Omega_0} |\tilde{u}_e(t+h) - \tilde{u}_e(t)| \leq \int_0^{T-h} \int_{\Omega_0} |\tilde{u}_e(t+h) - \tilde{u}_e(t)| \chi_{\{|\tilde{u}_e(t+h) - \tilde{u}_e(t)| > \delta \}} + \partial |\Omega_0|(T-h)
$$

$$
\leq \frac{C_3 \sqrt{h}}{\sigma_\delta} + \delta |\Omega_0|T.
$$

Taking the limit as $h \to 0$, using the arbitrariness of $\delta > 0$ and the fact that the right hand side of the above does not depend on $\epsilon$ gives us the result.

\[ \square \]

### 3.2 Existence of bounded weak solutions

With all the uniform estimates acquired, we can extract (weakly) convergent subsequences. In fact, we find (we have not relabelled subsequences)

$$
u_e \to \nu \quad \text{in } L^p_{L^4} \text{ for any } p, q \in [1, \infty)
$$

$$
\nabla \Omega u_e \to \nabla \Omega u \quad \text{in } L^2_{L^2}
$$

$$
E_c(u_e) \to \chi \quad \text{in } L^2_{L^2}
$$

(27)

where only the first strong convergence listed requires an explanation. Indeed, the point is to apply [22, Theorem 5] with $H^1(\Omega_0) \hookrightarrow L^1(\Omega_0) \subset L^1(\Omega_0)$, which gives us a subsequence
\( \tilde{u}_{\epsilon_j} \to \tilde{\rho} \) strongly in \( L^1(0, T; L^1(\Omega_0)) \). It follows that \( u_{\epsilon_j} \to \rho \) in \( L^1_{L^1} \), whence for a.a. \( t \), \( u_{\epsilon_j}(t) \to \rho(t) \) a.e. in \( \Omega(t) \). We also know that for a.a. \( t \), \( |u_{\epsilon_j}(t)| \leq C \) a.e. in \( \Omega(t) \) by Lemma 3.4, and so for a.a. \( t \), the limit satisfies \( |\rho(t)| \leq C \) a.e. in \( \Omega(t) \) too. By Theorem 2.10, \( u_{\epsilon_j} \to \rho \) in \( L_p^q \), for all \( p, q \in [1, \infty] \). Since we also know that \( u_{\epsilon_j} \to u \) (subsequences have the same weak limit), it must be the case that \( \rho = u \).

**Proof of Theorem 1.3.** In \( (P_e) \), we can test with a function \( \eta \in W(H^1, L^2) \) with \( \eta(T) = 0 \), integrate by parts and then pass to the limit to obtain

\[
- \int_0^T \int_{\Omega(t)} \dot{\eta}(t) \chi(t) + \int_0^T \int_{\Omega(t)} \nabla u(t) \nabla \eta(t) = \int_0^T \int_{\Omega(t)} f(t) \eta(t) + \int_{\Omega_0} e_0 \eta(0)
\]

and it remains to be seen that \( \chi \in \mathcal{E}(u) \) or equivalently \( u = \mathcal{U}(\chi) \). By monotonicity of \( \mathcal{E}_e \), we have for any \( w \in L^2_{L^2} \)

\[
\int_0^T \int_{\Omega(t)} (\mathcal{E}_e(u_e) - w)(u_e - \mathcal{U}(w)) \geq 0. \tag{28}
\]

Because \( \mathcal{U}_e \to \mathcal{U} \) uniformly, for a.a. \( t \), \( \mathcal{U}_e(w(t)) \to \mathcal{U}(w(t)) \) a.e. in \( \Omega(t) \), and \( |\mathcal{U}_e(w)| \leq |w| \), and the dominated convergence theorem shows that \( \mathcal{U}_e(w) \to \mathcal{U}(w) \) in \( L^2_{L^2} \). Using this and (27), we can easily pass to the limit in this inequality and obtain

\[
\int_0^T \int_{\Omega(t)} (\chi - w)(u_e - \mathcal{U}(w)) \geq 0 \quad \text{for all } w \in L^2_{L^2}.
\]

By Minty’s trick we find \( u = \mathcal{U}(\chi) \). To see why \( \chi \in L^\infty \), we have from the estimate in Lemma 3.4 that for almost all \( t \in [0, T] \), \( \|\mathcal{E}_e(\tilde{u}_e(t))\|_{L^\infty(\Omega_0)} \leq C \), giving \( \mathcal{E}_e(\tilde{u}_e(t)) \rightharpoonup \tilde{\zeta}(t) \) in \( L^\infty(\Omega(t)) \) and (by weak-* lower semicontinuity) \( \|\tilde{\zeta}(t)\|_{L^\infty(\Omega(t))} \leq C \) for almost all \( t \), and we just need to identify \( \tilde{\zeta} \in \mathcal{E}(\tilde{u}) \). It follows from (27) that \( \mathcal{E}_e(u_e) \to \chi \) in \( L^2_{L^{-1}} \) by Lions–Aubin, and so for almost all \( t \) and for a subsequence (not relabelled), \( \mathcal{E}_e(u_e(t)) \to \chi(t) \) in \( H^{-1}(\Omega(t)) \). This allows us to conclude that \( \chi = \tilde{\zeta} \) (the weak-* convergence of \( \mathcal{E}_e(\tilde{u}_e(t)) \) to \( \tilde{\zeta}(t) \) also gives weak convergence in any \( L^p(\Omega(t)) \) to the same limit).

### 3.3 Continuous dependence and uniqueness of bounded weak solutions

We can drop the requirement for our test functions to vanish at time \( T \).

**3.7 Lemma.** If \( (u, e) \) is a bounded weak solution (satisfying (4)), then \( (u, e) \) also satisfies

\[
\int_{\Omega(T)} e(T) \eta(T) - \int_0^T \int_{\Omega(t)} \dot{\eta}(t)e(t) + \int_0^T \int_{\Omega(t)} \nabla u(t) \nabla \eta(t) = \int_0^T \int_{\Omega(t)} f(t) \eta(t) + \int_{\Omega_0} e_0 \eta(0).
\]

**Proof.** To see this, for \( s \in (0, T] \), consider the function \( \chi_{\epsilon,s}(t) = \min (1, \epsilon^{-1}(s - t)^+) \) which has a weak derivative \( \chi'_{\epsilon,s}(t) = -\epsilon^{-1}\chi(s-\epsilon,s)(t) \). Take the test function in (4) to be \( \chi_{\epsilon,T} \eta \) where \( \eta \in W(H^1, L^2) \), send \( \epsilon \to 0 \) and use the Lebesgue differentiation theorem to yield the desired equality. □
Proof of Theorem 1.4. We can prove the continuous dependence like in [19, Chapter V, §9]. As explained in Lemma 3.7, we can drop the requirement \( \eta(T) = 0 \) in our test functions and we now suppose that \( \Delta_\Omega \eta \in L^2_\Omega \). Suppose for \( i = 1, 2 \) that \( (u_i, e_i) \) is the solution to the Stefan problem with data \( (f_i, u_0) \), so

\[
\int_{\Omega(t)} (e_1(t) - e_2(t)) \eta(t) - \int_0^t \int_{\Omega(\tau)} \eta(\tau)(e_1(\tau) - e_2(\tau)) - \int_0^t \int_{\Omega(\tau)} (u_1(\tau) - u_2(\tau)) \Delta_\Omega \eta(\tau)
= \int_0^t \int_{\Omega(\tau)} (f_1(\tau) - f_2(\tau)) \eta(\tau) + \int_{\Omega_0} (e_0^1 - e_0^2) \eta(0). \tag{29}
\]

Define \( a = (u_1 - u_2)/(e_1 - e_2) \) when \( e_1 \neq e_2 \) and \( a = 0 \) otherwise, and note that \( 0 \leq a(x, t) \leq 1 \). Let \( \eta \) solve in \( \cup_{\tau \in (0,t)} \{ \tau \} \times \Omega(\tau) \) the equation

\[
\partial_t \eta(x, \tau) + (a(x, \tau) + \epsilon) \Delta_\Omega \eta(x, \tau) = 0 \tag{30}
\]

with \( \xi \in C^1(\Omega_0) \) and where \( a_e \) satisfies \( \phi(-) a_e \in C^2([0, T] \times \Omega_0) \) and \( 0 \leq a_e \leq 1 \) a.e. and \( \| a_e - a \|_{L^2(Q)} \leq \epsilon \). This is well-posed by Lemma 2.13. Equation (29) can be written in terms of \( a_e \), and if we choose \( \eta = \eta_e \) and use (30), we find

\[
\int_{\Omega(t)} (e_1(t) - e_2(t)) \xi \leq \| e_1 - e_2 \|_{L^\infty} \int_0^t \int_{\Omega(\tau)} |a(x, \tau) - a_e(x, \tau)| + \epsilon|\Delta_\Omega \eta_e(\tau)|
+ \| \xi \|_{L^\infty(\Omega_0)} \int_0^t \| f_1(\tau) - f_2(\tau) \|_{L^1(\Omega(\tau))} + \| \xi \|_{L^\infty(\Omega_0)} \int_{\Omega_0} |e_0^1 - e_0^2| \tag{31}
\]

using the \( L^\infty \) bound from Lemma 2.13. We can estimate the first integral on the right hand side:

\[
\int_0^t \int_{\Omega(\tau)} |a(x, \tau) - a_e(x, \tau)| \| \Delta_\Omega \eta_e(\tau) \| \leq \sqrt{\epsilon} \| a - a_e \|_{L^2_\Omega} \sqrt{(2 + \epsilon)(1 + e^{2C_\Omega(2+\epsilon)t})} \| \nabla_\Omega \xi \|_{L^2(\Omega_0)}
\]

and

\[
\int_0^t \int_{\Omega(\tau)} |\epsilon \Delta_\Omega \eta_e| \leq \sqrt{t|\Omega|} \epsilon(2 + \epsilon)(1 + e^{2C_\Omega(2+\epsilon)t}) \left( \int_{\Omega_0} |\nabla_\Omega \xi|^2 \right)^{1/2}
\]

by the results in Lemma 2.13. Sending \( \epsilon \to 0 \) in the (31) gives us (recalling \( \xi \leq 1 \)),

\[
\int_{\Omega(t)} (e_1(t) - e_2(t)) \xi \leq \int_0^t \int_{\Omega(\tau)} |f_1(\tau) - f_2(\tau)|_{L^1(\Omega(\tau))} + \| e_0^1 - e_0^2 \|_{L^1(\Omega_0)}.
\]

Now pick \( \xi = \xi_n \) where \( \xi_n(x) \to \text{sign}(e_1(t, x) - e_2(t, x)) \in L^2(\Omega(t)) \) a.e. in \( \Omega(t) \). \( \square \)

3.4 Well-posedness of weak solutions

Proof of Theorem 1.5. Suppose \( (u_0, f) \in L^1(\Omega_0) \times L^1\Omega_1 \) are data and consider functions \( u_{0n} \in C^0(\Omega_0) \subset L^\infty(\Omega_0) \) and \( f_n \in L^\infty_{\Omega_0} \) satisfying

\[
(f_n, u_{0n}) \to (f, u_0) \quad \text{in } L^1_{\Omega_1} \times L^1(\Omega_0).
\]
The existence of \( f_n \) holds because by density, there exist \( \tilde{f}_n \in C^0([0, T] \times \Omega_0) \) such that \( \tilde{f}_n \rightarrow \tilde{f} \) in \( L^1((0, T) \times \Omega_0) \equiv L^1(0, T; L^1(\Omega_0)) \). Denote by \((u_n, e_n)\) the respective (bounded weak) solutions to the Stefan problem with the data \((u_{0n}, \tilde{f}_n)\). By virtue of these solutions satisfying the continuous dependence result, it follows that \( \{e_n\}_n \) is a Cauchy sequence in \( L^1_{L^1} \) and thus \( e_n \rightarrow \chi \) in \( L^1_{L^1} \) for some \( \chi \). Recall that \( |u_n| = |\mathcal{U}(e_n)| \leq |e_n| \) so by consideration of an appropriate Nemytskii map, we find \( u_n = \mathcal{U}(e_n) \rightarrow \mathcal{U}(\chi) \). Now we can pass to the limit in

\[
- \int_0^T \int_{\Omega(t)} \dot{\eta}(t)e_n(t) \, dt - \int_0^T \int_{\Omega(t)} u_n(t)\Delta \eta(t) \, dt = \int_0^T \int_{\Omega(t)} f_n(t)\eta(t) \, dt + \int_{\Omega_0} e_n\eta(0)
\]

and doing so gives

\[
- \int_0^T \int_{\Omega(t)} \dot{\eta}(t)\chi(t) - \int_0^T \int_{\Omega(t)} \mathcal{U}(\chi(t))\Delta \eta(t) = \int_0^T \int_{\Omega(t)} f(t)\eta(t) + \int_{\Omega_0} e\eta(0)
\]

and overall this shows that there exists a pair \((\chi, \mathcal{E}^{-1}(\chi)) \in L^1_{L^1} \times L^1_{L^1} \) which is a weak solution of the Stefan problem. For these integrals to make sense, we need \( \eta \in W^1(L^\infty \cap H^2, L^\infty) \) with \( \Delta \eta \in L^\infty_{L^\infty} \).

Now suppose that \((u^1, e^1)\) and \((u^2, e^2)\) are two weak solutions of class \( L^1 \) to the Stefan problem with data \((f^1, e^1_0)\) and \((f^2, e^2_0)\) in \( L^1_{L^1} \times L^1(\Omega_0) \) respectively. We know that there exist approximations \((f^1_n, e^1_{0n}), (f^2_n, e^2_{0n}) \in L^\infty_{L^\infty} \times L^\infty(\Omega_0) \) of the data satisfying

\[
(f^1_n, e^1_{0n}) \rightarrow (f^1, e^1_0) \quad \text{and} \quad (f^2_n, e^2_{0n}) \rightarrow (f^2, e^2_0) \quad \text{in} \quad L^1_{L^1} \times L^1(\Omega_0).
\]

These approximate data give rise to the approximate solutions \( e^1_n \) and \( e^2_n \) both of which are elements of \( L^\infty_{L^\infty} \). It follows from above that \( e^1_n \rightarrow e^1 \) and \( e^2_n \rightarrow e^2 \) in \( L^1_{L^1} \). Now consider the continuous dependence result that \( e^1_n \) and \( e^2_n \) satisfy:

\[
\|e^1_n - e^2_n\|_{L^1_{L^1}} \leq T \left( \|f^1_n - f^2_n\|_{L^1_{L^1}} + \|e^1_{0n} - e^2_{0n}\|_{L^1(\Omega_0)} \right).
\]

Regarding the right hand side, by writing \( e^1_{0n} - e^2_{0n} = e^1_{0n} - e^1_0 + e^1_0 - e^2_0 + e^2_0 - e^2_{0n} \) (and similarly for the \( f^i_n \)) and using triangle inequality, along with the fact that \( e^1_n - e^2_n \rightarrow e^1 - e^2 \) in \( L^1_{L^1} \), we can take the limit in (32) as \( n \to \infty \) and we are left with what we desired.

**A Proofs**

**Proof that \( L^p_X \) is a Banach space.** It is easy to verify that the expressions in (6) define norms if the integrals on the right hand sides are well-defined, which we now check. So let \( u \in L^p_X \). Then \( \hat{u} := \phi_{-1}(u) \in L^p(0, T; X_0) \). Define \( F : [0, T] \times X_0 \to \mathbb{R} \) by \( F(t, x) = \|\phi t x\|_{X(t)} \). By assumption, \( t \mapsto F(t, x) \) is measurable for all \( x \in X_0 \), and if \( x_n \to x \) in \( X_0 \), then by the reverse triangle inequality,

\[
|F(t, x_n) - F(t, x)| \leq \|\phi t(x_n - x)\|_{X(t)} \leq C_X \|x_n - x\|_{X_0} \to 0,
\]

so \( x \mapsto F(t, x) \) is continuous. Thus \( F \) is a Carathéodory function. Due to the condition \( |F(t, x)| \leq C_X \|x\|_{X_0} \), by Remark 3.4.5 of [23], the Nemytskii operator \( N_F \) defined by \((N_F x)(t) := F(t, x(t)) \) maps \( L^p(0, T; X_0) \to L^p(0, T) \), so that

\[
\|N_F \hat{u}\|_{L^p(0, T)} = \int_0^T \|u(t)\|_{X(t)} < \infty.
\]
Proof of Lemma 2.3. First we show that if \( u \in L^p(0, T; X_0) \), then \( \phi_t(u(t)) \in L^p_X \).

Let \( u \in L^p(0, T; X_0) \) be arbitrary. By density, there exists a sequence of simple functions \( u_n \in L^p(0, T; X_0) \) with

\[
\|u_n - u\|_{L^p(0, T; X_0)} \to 0
\]

and thus for almost every \( t \),

\[
\|u_n(t) - u(t)\|_{X_0} \to 0
\]

for a subsequence, which we relabelled. We have that \( \phi_t u_n(t) \to \phi_t u(t) \) in \( X(t) \) by continuity; this implies

\[
\|\phi_t u_n(t)\|_{X(t)} \to \|\phi_t u(t)\|_{X(t)} \quad \text{pointwise a.e. (33)}
\]

Write \( u_n(t) = \sum_{i=1}^{M_n} u_{n,i} \mathbf{1}_{B_i}(t) \) where the \( u_{n,i} \in X_0 \) and the \( B_i \) are measurable, disjoint and partition \([0, T]\). Then

\[
\phi_t u_n(t) = \sum_{i=1}^{M_n} \phi_t(u_{n,i}) \mathbf{1}_{B_i}(t) \in X(t).
\]

Taking norms and exponentiating, we get

\[
\|\phi_t u_n(t)\|^p_{X(t)} = \sum_{i=1}^{M_n} \|\phi_t u_{n,i}\|^p_{X(t)} \mathbf{1}_{B_i}(t),
\]

which is measurable (with respect to \( t \)) since, by assumption, the \( \|\phi_t u_{n,i}\|_{X(t)} \) are continuous and a finite sum of measurable functions is measurable. Thus, by (33), \( \|\phi_t u(t)\|_{X(t)} \), is measurable. Finally,

\[
\int_0^T \|\phi_t u(t)\|^p_{X(t)} \leq \int_0^T C_X^p \|u(t)\|^p_{X_0} = C_X^p \|u\|^p_{L^p(0, T; X_0)},
\]

so \( \phi_t u(\cdot) \in L^p_X \).

So there is a map from \( L^p(0, T; X_0) \) to \( L^p_X \) and vice-versa from the definition of \( L^p_X \). The isomorphism between the spaces is \( T: L^p(0, T; X_0) \to L^p_X \) where

\[
Tu = \phi_t u(\cdot), \quad \text{and} \quad T^{-1}v = \phi_{-t} v(\cdot).
\]

It is easy to check that \( T \) is linear and bijective. The equivalence of norms follows by the bounds on \( \phi_{-t}: X(t) \to X_0 \)

\[
\frac{1}{C_X} \|u(t)\|_{X(t)} \leq \|\phi_{-t} u(t)\|_{X_0} \leq C X \|u(t)\|_{X(t)}.
\]

\[
\square
\]

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