Moduli Space Potentials for Heterotic non-Abelian Flux Tubes: Weak Deformation

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Abstract

We consider $\mathcal{N} = 2$ supersymmetric QCD with the U($N$) gauge group (with no Fayet–Iliopoulos term) and $N_f$ flavors of massive quarks deformed by the mass term $\mu$ for the adjoint matter, $\mathcal{W} = \mu \mathcal{A}^2$, assuming that $N \leq N_f < 2N$. This deformation breaks $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$. This theory supports non-Abelian flux tubes (strings) which are stabilized by $\mathcal{W}$. They are referred to as $F$-term stabilized strings. We focus on the studies of such strings in the vacuum in which $N$ squarks condense, at small $\mu$, so that the $Z_N$ strings preserve, in a sense, their BPS nature. We calculate string tensions both in the classical and quantum regimes. Then we translate our results for the tensions in terms of the effective low-energy weighted CP($N_f - 1$) model on the string world sheet. The bulk $\mu$-deformation makes this theory $\mathcal{N} = (0, 2)$ supersymmetric heterotic weighted CP($N_f - 1$) model in two dimensions. We find the deformation potential on the world sheet. This significantly expands the class of the heterotically deformed CP models emerging on the string world sheet compared to that suggested by Edalati and Tong. Among other things, we show that nonperturbative quantum effects in the bulk theory are exactly reproduced by the quantum effects in the world-sheet theory.
1 Introduction

In this paper we will report on further developments in non-Abelian strings, a construction which emerged recently [1, 2, 3, 4] (for detailed reviews see [5, 6, 7, 8]). Originally the non-Abelian strings were discovered [1, 2, 3, 4] in $\mathcal{N} = 2$ supersymmetric QCD with the U($N$) gauge group and $N_f = N$ quark multiplets and the Fayet–Iliopoulos (FI) $D$-term [9]. The role of the FI term is to trigger the quark condensation and provide stabilization for the BPS-saturated flux-tube solitons. The BPS nature of the flux tubes obtained in this way guarantees $\mathcal{N} = (2, 2)$ supersymmetry on the string world sheet.

The next step in this program was breaking $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$ by virtue of the $\mu A^2$ superpotential [10, 11, 12]. As a result of this deformation of the original $\mathcal{N} = 2$ bulk theory, supersymmetry on the string world sheet reduces (classically) from $\mathcal{N} = (2, 2)$ down to $\mathcal{N} = (0, 2)$ [11, 12]. This happens because a superpotential $\omega \sigma^2$ is generated on the world sheet. The parameters $\mu$ and $\omega$ are related by a proportionality formula \[^1\] while the functional dependence of the superpotentials in the bulk and on the world sheet is the same – quadratic – as was suggested in [11] and confirmed in [12] by a direct calculation. Taking account of quantum effects on the string world sheet one observes [12, 13] spontaneous breaking of $\mathcal{N} = (0, 2)$ supersymmetry.

The FI $D$-term (to be denoted $\xi_3$) is not the only way to stabilize the BPS-saturated flux-tube solitons. In $\mathcal{N} = 2$ supersymmetric theories one could alternatively introduce it through $F$ terms of the form $\mathcal{W} = \xi A$. In fact, the FI $D$- and $F$-terms form a triplet of the global SU(2)$_R$ [14, 15]. This explains our notation $\xi_3$ for the coefficient in front of the FI $D$-term. The FI $F$-term coefficient $\xi$ is complex and can be written as $\xi = \xi_1 + i \xi_2$, where $\xi_i$ $(i = 1, 2, 3)$ form an SU(2)$_R$ triplet.

In the past we considered the $F$-term stabilized flux tubes e.g. in [16]. When $\mathcal{N} = 2$ deformations are introduced in the bulk \[^2\] the SU(2)$_R$ symmetry is broken, and the equivalence between the $D$-term and $F$-term stabilized flux tubes disappears. In particular, the $\mathcal{N} = (2, 2)$ -breaking deformation on the world sheet of the $F$-term stabilized strings does not coincide with that determined in [11, 12]. The following question arises: Given an $\mathcal{N} = 2$ bulk theory with no Fayet–Iliopoulos term, deformed by an $\mathcal{N} = 2$ breaking

\[^1\] The proportionality formula obtained in [12] is only valid in the limit of small $\mu$.

\[^2\] We mean such deformations that break $\mathcal{N} = 2$ down to $\mathcal{N} = 1$. 

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superpotential, how can one calculate the corresponding $\mathcal{N} = (2, 2)$ -breaking potential on the $F$-term stabilized string world sheet? In the present paper we address this question in the limit of weak deformations (i.e. in the leading order in the deformation parameters). We consider $U(N)$ gauge theories with $N_f$ matter hypermultiplets where we require

$$N \leq N_f < 2N.$$  \hspace{1cm} (1.1)

The string-stabilizing/deformation terms are introduced via superpotentials, i.e. as $F$-terms. We find, say, for the $U(2)$ theory with $N_f = 3$, that if the bulk deformation is introduced as

$$\mathcal{W}_{3+1} = \frac{1}{2} \left[ \mu_1 \mathcal{A}^2 + \mu_2 (\mathcal{A}^a)^2 \right]$$  \hspace{1cm} (1.2)

(i.e. à la Seiberg–Witten \[17\] \[18\]), the $\mathcal{N} = (2, 2)$ -breaking potential it generates on the string world sheet is

$$V_{1+1}(\sigma) = 4\pi \left| \mu_1 m - \mu_2 \left( \sqrt{2} \sigma - \frac{\Lambda}{2} + m \right) \right|,$$  \hspace{1cm} (1.3)

where $m$ is the average (over three flavors) mass term,

$$m = \frac{1}{3} (m_1 + m_2 + m_3).$$  \hspace{1cm} (1.4)

From (1.3) one can read off vacuum energies for two vacua of the heterotic weighted CP(2) model at hand. These vacua correspond to two strings. They are BPS-saturated in the effective low-energy $U(1)$ theory. However, if considered in the full theory, they are non-BPS. This means that $\mathcal{N} = (0, 2)$ supersymmetry is spontaneously broken in the weighted CP(1) model we deal with already at the classical level (as opposed to the quantum-level breaking in \[11\] \[12\]).

To present things in a proper perspective, let us return for a short while to the $D$-term stabilization. We recall that the FI $D$-term singles out a particular $r$-vacuum, with $r = N$ (i.e. $N$ quark flavors out of $N_f$ develop a

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3This is due to the fact that in $\mathcal{N} = 1$ non-Abelian gauge theories with $\xi_3 = 0$ there is no string central charge \[19\]. This central charge appears in $\mathcal{N} = 2$ theories provided $\xi_{1,2,3} \neq 0$. 

vacuum expectation value (VEV)). For instance, one can choose the quark
condensate as

\[ \langle q^k \rangle = \sqrt{\xi_3} \begin{pmatrix} 1 & \ldots & 0, & 0 & \ldots \\ \ldots & 1 & \ldots, & 0 & \ldots \\ 0 & \ldots & 1, & 0 & \ldots \end{pmatrix}, \]

where the quark fields are represented in the matrix form, as an \(N \times N_f\) matrix in the color and flavor indices. Consider the simplest case with \(N_f = N\). The vacuum field (1.5) results in the spontaneous breaking of both the gauge \(U(N)\) group and flavor (global) \(SU(N)\) group, leaving unbroken a diagonal global \(SU(N)_{C+F}\),

\[ U(N)_{\text{gauge}} \times SU(N)_{\text{flavor}} \rightarrow SU(N)_{C+F}. \] (1.6)

Thus, a color-flavor locking takes place in the vacuum. The presence of the global \(SU(N)_{C+F}\) group is a key reason for the formation of non-Abelian strings whose main feature is the occurrence of orientational zero modes associated with rotations of the flux inside the \(SU(N)_{C+F}\) group. Dynamics of these orientational moduli are described by the effective two-dimensional \(\mathcal{N} = (2,2)\) supersymmetric \(CP(N-1)\) model on the string world sheet.

Next we add the quark mass terms \(m_A (A = 1, 2, \ldots, N)\). If they are unequal, the global \(SU(N)_{C+F}\) group is broken down to \(U(1)^{N-1}\) by VEVs of the adjoint fields \(A^a\). If one assumes that the mass term differences are small, i.e. \(|m_A - m_B| \ll \sqrt{\xi_3}\), the orientational moduli, being lifted, remain as quasimoduli. The two-dimensional low-energy theory that emerges in this case on the world sheet is the \(\mathcal{N} = (2,2)\) supersymmetric \(CP(N-1)\) model with twisted masses. Note, that in this case the \(CP(N-1)\) model still has \(N\) degenerate supersymmetric vacua which are identified with \(N\) elementary \(Z_N\) strings of the bulk theory, see for example our review [7].

The \(\mathcal{N} = 2\) -breaking bulk deformations considered in the literature [10, 11, 20, 12, 13, 21, 22] are as follows: the mass term \(\mu\) for the adjoint matter in the theory with non-zero \(\xi_3\) and \(m_A = 0\) (for all \(A\)) or, more general superpotentials, with the critical points coinciding with the quark masses \(m_A\). The reason is rather obvious. Consider, say, the mass term \(\mu\) for the adjoint matter. If \(m_A = 0\), no FI \(F\)-terms are generated and the bosonic parts of the classical string solutions do not depend on \(\mu\) [10, 12]. The world sheet-theory
changes only in the fermion sector. It becomes $\mathcal{N} = (0,2)$ supersymmetric (heterotic) CP($N - 1$) model [11] [12, 21]. In the gauged formulation the deformation potential on the moduli space has the form

$$V_{1+1} = 4|\omega|^2 |\sigma|^2,$$

(1.7)

where $\sigma$ is a scalar superpartner of the (auxiliary) U(1) gauge field [24], while $\omega$ is a world-sheet deformation parameter determined by the mass $\mu$ of the adjoint fields in the bulk theory,

$$\omega \sim \frac{\mu}{\sqrt{\xi_3}},$$

(1.8)

at small $\mu$ (to the leading order in $\mu$). In other words, up to an overall normalization, $W_{1+1}(\sigma)$ has the same functional form as $W_{3+1}(A)$. A similar situation takes place at $m_A \neq 0$ provided that the critical points of $W_{3+1}(A)$ are at $m_A$.

In this paper we consider $\xi_3 = 0$

(if not stated to the contrary, in some occasional passages), both stabilization and $\mathcal{N} = 2$ breaking are provided by $F$-terms, induced by non-zero $\mu$ times quark masses, which are generically considered to be different. Now the $Z_N$ strings (their number is $N$) become split. Supersymmetry on the string world sheet is spontaneously broken already at the classical level. We calculate the string tensions in the limit

$$|\mu/m_A| \ll 1.$$  

(1.9)

In this limit each of the $Z_N$ strings is still BPS-saturated in the associated U(1) low-energy gauge theory arising from the gauge symmetry breaking $U(N) \to U(1)^N$ through Higgsing. Of course, all $N$ strings are non-BPS in the full U($N$) gauge theory.

We find the potential $V_{1+1}(\sigma)$ induced in the world-sheet heterotic CP model due to the bulk $\mu$-deformation $W_{3+1}(A)$. This potential explicitly exhibits the breaking of $\mathcal{N} = (0,2)$ supersymmetry at the classical level and splitting of the energies of $N$ vacua (the tensions of the $Z_N$ strings).

In our previous works we revealed a number of “protected” quantities, such as the masses of the (confined) monopoles. These parameters are calculable both, in the bulk theory and on the world sheet, with one and
the same result. The first example of this remarkable correspondence was the explanation \cite{3, 4} of the coincidence of the BPS monopole spectrum in four-dimensional $\mathcal{N} = 2$ supersymmetric QCD in the $r = N$ vacuum on the Coulomb branch at $\xi_3 = 0$ (given by the exact Seiberg-Witten solution \cite{18}), on the one hand, with the BPS kink spectrum in the $\mathcal{N} = (2, 2)$ supersymmetric CP model, on the other hand. This coincidence was noted in \cite{25, 26, 27}. The above-mentioned explanation \cite{3, 4} is: (i) the confined monopoles of the bulk theory (represented by two-string junctions) are seen as kinks interpolating between two different vacua in the sigma model on the string world sheet; (ii) the masses of the BPS monopoles cannot depend on the nonholomorphic parameter $\xi_3$.

In this paper we find and analyze another example of such exact correspondence between the bulk and world-sheet theories, namely the tensions of non-Abelian strings stabilized by $F$-terms. We study quantum nonperturbative corrections to the string tensions in the bulk theory and show that they are exactly reproduced by the quantum corrections to vacuum energies in the heterotic CP model model on the string world sheet.

The paper is organized as follows. In Sect. 2 we formulate our theoretical setting – the bulk $\mathcal{N} = 2$ SQCD with a certain superpotential which (a) stabilizes the string solutions; (b) breaks $\mathcal{N} = 2$ down to $\mathcal{N} = 1$. Section 3 is devoted to calculations of the $Z_N$ string tensions in the above bulk theory in the classical limit. Section 4 deals with (nonperturbative) quantum corrections to the string tensions. In Sect. 5 we briefly outline construction of the world-sheet theory in the limit of unbroken $\mathcal{N} = 2$ in the bulk. In Sect. 6 we switch on an $\mathcal{N} = 2$ breaking deformation, and consider its impact on the string world sheet. In Sect. 7 (nonperturbative) quantum effects in the world-sheet theory obtained in Sect. 6 are analyzed. In Sect. 8 we revisit the issue of the monopole confinement, i.e. confinement along the string, in addition to permanent attachment to the strings. This phenomenon is similar to that discussed in \cite{28}. Section 9 is devoted to generic single-trace deformation superpotentials $W_3+1(A)$. Section 10 summarizes our results. In Appendices A and B we discuss details pertinent to particular examples, namely, the U(2) bulk theory with 2 flavors and U(3) theory with 5 flavors, respectively.
2 Bulk theory

We start with the description of the bulk theory with which we will deal throughout the paper. The gauge symmetry of the basic bulk theory is $U(N) = SU(N) \times U(1)$, the number of the matter hypermultiplets in the fundamental representation is $N_f$. With the deformation superpotential switched off this theory has $\mathcal{N} = 2$ supersymmetry. In addition to $N_f$ quark hypermultiplets (with the mass terms $m_A$, $A = 1, 2, ..., N_f$) the theory has gauge bosons, gauginos and their superpartners. We assume $N_f \geq N$ but $N_f < 2N$. The latter inequality ensures asymptotic freedom of the theory. Then we will introduce the deformation superpotential of the type $\mu A^2$ for the adjoint matter breaking $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$. (Later we will consider some other deformation superpotentials too.)

In more detail, the field content is as follows. The $\mathcal{N} = 2$ vector multiplet consists of the $U(1)$ gauge field $A_\mu$ and the $SU(N)$ gauge field $A^a_\mu$, where $a = 1, ..., N^2 - 1$, plus their Weyl fermion superpartners plus complex scalar fields $a$, and $a^a$ and their Weyl superpartners. The $N_f$ quark hypermultiplets of the $U(N)$ theory consist of the complex scalar fields $q^{kA}$ and $\tilde{q}^{Ak}$ (squarks) and their fermion superpartners, all in the fundamental representation of the $SU(N)$ gauge group. Here $k = 1, ..., N$ is the color index while $A$ is the flavor index, $A = 1, ..., N_f$. We will treat $q^{kA}$ and $\tilde{q}^{Ak}$ as rectangular matrices with $N$ rows and $N_f$ columns.

As was mentioned, the undeformed theory has $\mathcal{N} = 2$. The superpotential has the form

$$W_{\mathcal{N}=2} = \sqrt{2} \sum_{A=1}^{N_f} \left( \frac{1}{2} \tilde{q}^{A} A q^{A} + \tilde{q}^{A} A^{a^a} T_{\alpha}^{\beta} q^{A} \right), \quad (2.1)$$

where $A$ and $A^a$ are the adjoint chiral superfields, the $\mathcal{N} = 2$ superpartners of the gauge bosons of the $U(1)$ and $SU(N)$ parts, respectively.

Next, we add the mass term for the adjoint fields which, generally speaking, breaks supersymmetry down to $\mathcal{N} = 1$,

$$W_{3+1} = \sqrt{\frac{N}{2}} \frac{\mu_1}{2} A^2 + \frac{\mu_2}{2} (A^{a^a})^2, \quad (2.2)$$

where $\mu_1$ and $\mu_2$ is some mass parameters for the adjoint chiral superfields, $U(1)$ and $SU(N)$, respectively. The subscript $3+1$ tells us that the
deformation superpotential (2.2) refers to the bulk four-dimensional theory. Clearly, the mass term (2.2) splits $N = 2$ supermultiplets.

The bosonic part of our basic theory has the form (for details see e.g. the
review paper [7])

$$S = \int d^4x \left[ \frac{1}{4g_2^2} (F_{\mu\nu}^a)^2 + \frac{1}{4g_1^2} (F_{\mu\nu})^2 + \frac{1}{g_2^2} |D_\mu a^a|^2 + \frac{1}{g_1^2} |\partial_\mu a|^2 + |\nabla_\mu q_A|^2 + |\nabla_\mu \tilde{q}^A|^2 + V(q^A, \tilde{q}_A, a^a, a) \right].$$ (2.3)

Here $D_\mu$ is the covariant derivative in the adjoint representation of SU($N$), while

$$\nabla_\mu = \partial_\mu - i \frac{1}{2} A_\mu - i A^a_\mu T^a.$$ (2.4)

We suppress the color SU($N$) indices of the matter fields. The normalization
of the SU($N$) generators $T^a$ is as follows

$$\text{Tr} (T^a T^b) = \frac{1}{2} \delta^{ab}.$$ The coupling constants $g_1$ and $g_2$ correspond to the U(1) and SU($N$) sectors, respectively. With our conventions, the U(1) charges of the fundamental matter fields are $\pm 1/2$, see Eq. (2.4). The scalar potential $V(q^A, \tilde{q}_A, a^a, a)$ in the action (2.3) is the sum of the $D$ and $F$ terms,

$$V(q^A, \tilde{q}_A, a^a, a) = \frac{g_2^2}{2} \left( \frac{1}{g_2^2} f^{abc} \tilde{a}^b a^c + \tilde{q}_A T^a q^A - \tilde{q}_A T^a \tilde{q}^A \right)^2$$

$$+ \frac{g_1^2}{8} (\tilde{q}_A q^A - \tilde{q}_A \tilde{q}^A - N \xi_3)^2$$

$$+ 2g_2^2 \left| \tilde{q}_A T^a q^A + \frac{1}{\sqrt{2}} \frac{\partial W_{3+1}}{\partial a^a} \right|^2 + \frac{g_1^2}{2} \left| \tilde{q}_A q^A + \sqrt{2} \frac{\partial W_{3+1}}{\partial a} \right|^2$$

$$+ \frac{1}{2} \sum_{A=1}^{N_f} \left\{ \left| (a + \sqrt{2}m_A + 2T^a a^a) q^A \right|^2 \right.$$}

$$\left. + \left| (a + \sqrt{2}m_A + 2T^a a^a) \tilde{q}^A \right|^2 \right\}. \quad (2.5)$$
Here $f^{abc}$ denote the structure constants of the SU($N$) group, $m_A$ is the mass term for the $A$-th flavor, and the sum over the repeated flavor indices $A$ is implied. For completeness we indicated here the FI $D$-term $\xi_3$. As was mentioned, in the bulk of the paper $\xi_3$ is set at zero. Only occasionally we make digressions and discuss $\xi_3 \neq 0$. In these cases it is clearly stated that $\xi_3 \neq 0$.

The vacuum structure of this theory is as follows. The vacua of the theory (2.3) are determined by zeros of the potential (2.5). In the general case, the theory has many so-called $r$-vacua, i.e. those vacua in which $r$ quarks condense, where $r$ can take any value up to $N$, $r = 0, ..., N$. Say, $N$ vacua with $r = 0$ are always at strong coupling. These are the monopole vacua of Ref. [17, 18].

We focus on a particular set of vacua with the maximal number of condensed quarks, i.e. $r = N$. The reason for this choice is that all U(1) factors of the gauge group are spontaneously broken in these vacua, and they support non-Abelian strings [1, 2, 3, 4].

Let us assume first that our theory is at weak coupling and thus can be analyzed quasiclassically. Below we will explicitly formulate necessary conditions for the quark mass terms and $\mu$ which will guarantee this regime. For generic values of the quark masses we have

$$C_{N_f}^N = \frac{N_f!}{N!(N_f - N)!}$$

isolated $r$-vacua where in our case $r = N$; i.e. $N$ quarks (out of $N_f$) develop vacuum expectation values (VEVs). Consider, say, the (1,2,...,$N$) vacuum in which the first $N$ flavors develop VEVs. In this vacuum the adjoint fields develop VEVs too, namely,

$$\left\langle \left( \frac{1}{2} a + T^a a^a \right) \right\rangle = -\frac{1}{\sqrt{2}} \begin{pmatrix} m_1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & m_N \end{pmatrix}. \quad (2.6)$$

For generic values of the quark mass parameters, the SU($N$) subgroup of the gauge group is broken down to U(1)$^{N-1}$. However, in the special limit

$$m_1 = m_2 = ... = m_{N_f}, \quad (2.7)$$

the SU($N$)$\times$U(1) gauge group remains unbroken by the adjoint field. In this limit the theory acquires also a global flavor SU($N_f$) symmetry.
With all $m_A$’s equal and to the leading order in $\mu$ the mass term for the adjoint matter (2.2) reduces to the FI $F$-term of the U(1) factor $\xi \sim \mu m/\sqrt{N}$ (see below) which does not break $\mathcal{N} = 2$ supersymmetry [14, 15]. In this case the FI $F$-term can be transformed into the FI $D$-term by an SU(2)$_R$ rotation, and the theory reduces to $\mathcal{N} = 2$ supersymmetric QCD described e.g. in the review [7]. Higher orders in parameter (1.9) break $\mathcal{N} = 2$ supersymmetry explicitly splitting $\mathcal{N} = 2$ supermultiplets.

If the values of $m_A$ are unequal, the U($N$) gauge group is spontaneously broken down to U(1)$_N$ by VEVs of $a^a$, see (2.6). To the leading order in $\mu$ the superpotential in (2.2) reduces to $N$ distinct FI terms: one $F$-term for each U(1) gauge factor. Thus, $\mathcal{N} = 2$ supersymmetry in each individual low-energy U(1) sector is unbroken. It gets broken, however, being considered in the full microscopic U($N$) gauge theory.

Assuming that $\xi_3 = 0$ and using (2.2) and (2.6) we get from (2.5) all VEVs of the squark fields. Upon gauge rotation they can be written in the form

$$
\langle q^{kA} \rangle = \langle \bar{q}^{kA} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix}
\sqrt{\xi_1} & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \sqrt{\xi_N} & 0 & \cdots & 0
\end{pmatrix},
$$

$$
k = 1, ..., N, \quad A = 1, ..., N_f,
$$

(2.8)

where the quark fields are represented by matrices carrying color and flavor indices. Here we define the FI $F$-term parameters for each U(1) gauge factor as follows

$$
\xi_P = 2 \left\{ \sqrt{\frac{2}{N}} \mu_1 m + \mu_2 (m_P - m) \right\}, \quad P = 1, ..., N.
$$

(2.9)

Moreover, $m$ is the average value of the first $N$ quark masses,

$$
m = \frac{1}{N} \sum_{P=1}^{N} m_P.
$$

(2.10)

While the adjoint field condensation does not break the SU($N$)$\times$U(1) gauge group in the limit (2.7), in the very same limit the quark condensate (2.8) results in the spontaneous breaking of both gauge and flavor symmetries. A diagonal global SU($N$) combining the gauge SU($N$) and an SU($N$)
subgroup of the flavor SU($N_f$) group survives, however. Below we will refer to this diagonal global symmetry as to color-flavor locked SU($N$)$_{C+F}$.

More exactly, the pattern of the spontaneous breaking of the color and flavor symmetry is as follows:

$$U(N)_{\text{gauge}} \times SU(N_f)_{\text{flavor}} \rightarrow SU(N)_{C+F} \times SU(\tilde{N})_F \times U(1), \quad (2.11)$$

where $\tilde{N} = N_f - N$. Here the SU($\tilde{N}$)$_F$ factor represents the flavor rotation of the “extra” $\tilde{N}$ quarks. The phenomenon of color-flavor locking in the case at hand is slightly different than that in the case $N_f = N$. The presence of the global SU($N$)$_{C+F}$ group is instrumental for formation of the non-Abelian strings (see below).

For unequal quark mass parameters both the adjoint and squark VEVs brake the global symmetry (2.11) down to U(1)$_{N_f-1}$. This should be contrasted with the theory with the FI D-term ($\xi_3 \neq 0$, $\mu_1 = \mu_2 = 0$) in which the squark VEVs are all equal and do not break color-flavor symmetry.

The above quasiclassical analysis is valid if the theory is at weak coupling. This is the case if the mass differences are large,

$$|\Delta m_{AB}| \equiv |m_A - m_B| \gg \Lambda, \quad (2.12)$$

or the quark VEV are large while the mass differences can be not-so-large. In the first case the theory at low energies reduces to U(1)$_N$ gauge theory. In the second case it remains U($N$) gauge theory in which the coupling constant is frozen at the scale equal to the large values of the quark condensate.

From (2.8) we see that the quark condensates are of the order of $\sqrt{\mu m}$, see also [17, 18, 29, 30] (we assume that $\mu_1 \sim \mu_2 \sim \mu$). In this case the weak coupling condition is

$$\sqrt{\mu m} \gg \Lambda. \quad (2.13)$$

We assume that at least one of conditions (2.12) or (2.13) is fulfilled. In particular, the condition (2.13) combined with the condition of small $\mu$ (1.9) ensures that the average quark mass $m$ is very large. In the theory with the FI $D$-term the average quark mass can always assumed to vanish by virtue of a shift of the adjoint U(1) field. In the case under consideration, when $\xi_3 = 0$ and stabilization is achieved through $F$-terms, the presence of the deformation (2.2) forbids this shift; hence, $m$ becomes a physical parameter.

In fact, we can relax both conditions (2.12) and (2.13) and pass to the strong coupling domain at

$$\sqrt{\mu m} \ll \Lambda, \quad |\Delta m_{AB}| \ll \Lambda \quad (2.14)$$
by virtue of duality. We demonstrated \cite{31, 32, 33} that in this passage the theory goes through crossover transitions; in the domain \((2.14)\) it can be described in terms of a weakly coupled (non-asymptotically free) dual theory with with the gauge group

\[ U(\tilde{N}) \times U(1)^{N-\tilde{N}}, \]

and \(N_f\) flavors of light dyons\footnote{This non-Abelian duality is similar to Seiberg’s duality in \(\mathcal{N} = 1\) supersymmetric QCD \cite{34, 35}. Also a dual non-Abelian gauge group \(SU(\tilde{N})\) was identified on the Coulomb branch at the root of a baryonic Higgs branch in the \(\mathcal{N} = 2\) supersymmetric \(SU(N)\) gauge theory with massless quarks \cite{29}.} We will see that our results for non-Abelian string tensions, as well as the effective world-sheet theory obtained at weak coupling, can be analytically continued into the domain \((2.14)\).

3 \(Z_N\) string tensions

In Sect. \cite{2} we argued that the quark fields develop VEVs in the \(r = N\) vacuum which break the gauge group, see \((2.8)\). Therefore, our theory supports strings. In fact, the minimal stings in our theory are the \(Z_N\) strings, progenitors of the non-Abelian strings, having the \(U(1)\) field fluxes reduced by the factor \(1/N\) compared to that of the Abrikosov–Nielsen–Olesen \cite{36} string. In these \(Z_N\) strings the squark fields have windings both in the \(U(1)\) and \(SU(N)\) gauge factors \cite{1, 2, 3, 4}.

We will study these strings applying the same methods as those used in the \(\mathcal{N} = 2\) theory with the FI \(D\)-term, or with one common FI \(F\)-term (which can be transformed into the \(D\)-term by virtue of an \(SU(2)_R\) rotation), see the review \cite{7}. Here we will be interested only in the tensions of the \(Z_N\) strings rather than in full solutions. Therefore, we need to know only the behavior of the gauge and squark fields at infinity. Using the \textit{ansatz}

\[ q^{kA} = \bar{q}^{kA} = \frac{\varphi^{kA}}{\sqrt{2}}, \]

and setting the adjoint scalars at their vacuum values \((2.6)\) (see \cite{7}) we determine the behavior at \(r \to \infty\) of the first of the \(Z_N\) strings. We have

\[ \varphi^{kA} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\xi_1} e^{i\alpha} & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{\xi_2} & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \sqrt{\xi_N} & 0 & \cdots & 0 \end{pmatrix}, \]
for the squark fields and

\[
\left( \frac{1}{2} A_i + T^a A_i^a \right) = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \end{pmatrix} \partial_i \alpha, \quad (3.3)
\]

for the gauge fields, where \( r \) and \( \alpha \) are the polar coordinates in the \((1, 2)\) plane orthogonal to the string axis, \( i = 1, 2 \). Asymptotic behavior of other \( Z_N \) strings is obtained by assigning the winding factor \( e^{i\alpha} \) to any other diagonal element in the matrix \((3.2)\) and putting the corresponding diagonal element of \((3.3)\) to unity. Equation \((3.3)\) implies that the flux of the \( P \)-th string is

\[
\left( \frac{1}{2} F_3^* + T^a F_3^a \right) = 2\pi \begin{pmatrix} 0 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ 0 & \ldots & 0 \end{pmatrix}, \quad (3.4)
\]

where the only nonvanishing element (equal to unity) is located at the diagonal of the matrix above at the \( P \)-th row.

Assuming all \( m_A \)'s to be different and putting all off-diagonal squark and gauge fields to zero we reduce our non-Abelian theory \((2.3)\) to the Abelian \( U(1)^N \) gauge theory. Then, assuming profile functions of the string solutions to be dependent only on \( x_i \) \((i = 1, 2)\), we can write the Bogomol’nyi representation \([37]\) for the \( Z_N \) string tensions. From \((2.3)\) we have (cf. \([7]\))

\[
T = \int d^2 x \left\{ \frac{1}{\sqrt{2}g_2} F_3^{*a} + \frac{g_2}{\sqrt{2}} \left( \varphi_A T^a \varphi^A + \sqrt{2} \mu_2 \langle a^a \rangle \right) \right\}^2 \\
+ \left[ \frac{1}{\sqrt{2}g_1} F_3^* + \frac{g_1}{2\sqrt{2}} \left( |\varphi^A|^2 + 2 \sqrt{N} \mu_1 \langle a \rangle \right) \right]^2 \\
+ |\nabla_1 \varphi^A + i \nabla_2 \varphi^A|^2 - \sqrt{N} \mu_1 \langle a \rangle F_3^* - \sqrt{2} \mu_2 \langle a^a \rangle F_3^{*a} \right\}, \quad (3.5)
\]

where

\[
F_3^* = F_{12} \quad \text{and} \quad F_3^{*a} = F_{12}^a.
\]

In the above expression \( \langle a \rangle \) and \( \langle a^a \rangle \) are just numbers given by \((2.6)\). This is the low-energy approximation which reduces the superpotential \((2.2)\) to
individual FI terms in each of the U(1) gauge. In this approximation the $Z_N$ strings are BPS saturated.

The Bogomol’nyi representation (3.5) leads us to the following first-order equations:

\[
F_3^* + \frac{g_1^2}{2} \left(|\varphi_A|^2 + 2\sqrt{N}\mu_1(a)\right) = 0, \\
F_3^{*a} + g_2^2 \left(\bar{\varphi} A T^a \varphi^A + \sqrt{2}\mu_2(a^a)\right) = 0, \\
(\nabla_1 + i\nabla_2)\varphi^A = 0.
\] (3.6)

Once these equations are satisfied, the energy of the BPS object is given by two last surface terms in (3.5). Substituting (2.6) and the gauge field fluxes (3.4) for each of the $Z_N$ string in the two last surface terms in (3.5) we arrive at the following tensions:

\[
T_{P}^{\text{BPS}} = 2\pi|\xi_P|, \quad P = 1, ..., N,
\] (3.7)

where $T_{P}^{\text{BPS}}$ is the tension of the string associated with the winding of the $P$-th quark (see (3.2)), while $N$ complex FI $F$-terms $\xi_P$ are classically determined by $\mu$’s and $m$’s via Eq. (2.9). We see that the tension of the $P$-th elementary string is determined by the condensate of the very same squark that winds at infinity (cf. (2.8)).

As longs as the string solitons are BPS saturated, their tensions must be given by exact expressions. Equation (2.9) gives the FI parameters in the semiclassical approximation. Later we will see that there are nonperturbative corrections to $\xi_P$’s that are $O(\Lambda/m_A)$. The $\mu$ dependence is a different story, however. The tensions of the BPS saturated $Z_N$ strings are presented in (3.7) only to the leading (linear) order in $\mu$. Higher orders in $\mu$ destroy the property that the superpotential (2.2) is representable as an FI term; they also break $\mathcal{N} = 2$ supersymmetry making strings non-BPS saturated.

4 Quantum effects

This section is devoted to calculation of (nonperturbative) quantum corrections $O(\Lambda/m_A)$ to the $Z_N$ string tensions. The idea is straightforward: we ex-

\footnote{Note that the representation (3.5) can be written also with the opposite sign in front of the flux terms. Then we would get the Bogomol’nyi equations for antistring.}
ploit the exact Seiberg–Witten solution of the theory on the Coulomb branch [17, 18] (more exactly, the SU($N$) generalizations of the Seiberg–Witten solution [38, 39, 40, 41]) to calculate the FI $F$-terms $\partial W_{3+1}/\partial a$ and $\partial W_{3+1}/\partial a{\tilde{a}}$ exactly rather than in the semiclassical approximation, as in Sects. 2 and 3.

Defining $u_k = \langle \text{Tr} \left( \frac{1}{2} a + T^a a^a \right)^k \rangle$, $k = 1, ..., N$, (4.1)

we perform a quantum generalization in the two relevant terms in the third line in (2.5),

$$\frac{\partial W_{3+1}}{\partial a} \rightarrow \mu_2 \frac{\partial u_2}{\partial a}, \quad \frac{\partial W_{3+1}}{\partial a{\tilde{a}}} \rightarrow \mu_1 \sqrt{\frac{2}{N}} \frac{\partial u_2}{\partial a{\tilde{a}}}. \quad (4.2)$$

Then the two last surface terms in (3.5) take the form

$$T^{\text{BPS}} = - \int d^2 x \left\{ \mu_1 \frac{2}{\sqrt{N}} \frac{\partial u_2}{\partial a} F_3^a + \sqrt{2} \mu_2 \frac{\partial u_2}{\partial a{\tilde{a}}} F_3^a \right\}, \quad (4.3)$$

where $\partial u_2/\partial a$ and $\partial u_2/\partial a{\tilde{a}}$ in the $r = N$ vacuum under consideration are some constants depending on $m_A$ and $\Lambda$.

Clearly only the diagonal gauge fluxes do not vanish. Substituting the gauge fluxes from (3.4) in (4.3) we get

$$T^{\text{BPS}}_P = 4\pi \sqrt{2} \left| \frac{2}{\sqrt{N}} \mu_1 e + \mu_2 (e_P - e) \right|, \quad (4.4)$$

where $e_P$ ($P = 1, ..., N$) are the diagonal elements of the $N \times N$ matrix

$$E = \frac{1}{N} \frac{\partial u_2}{\partial a} + T^{\tilde{a}} \frac{\partial u_2}{\partial a^{\tilde{a}}}, \quad (4.5)$$

$T^{\tilde{a}}$ are the Cartan generators of the SU($N$) gauge group ($\tilde{a} = 1, ..., (N - 1)$), while $e$ is their average value,

$$e = \frac{1}{N} \sum_{P=1}^N e_P. \quad (4.6)$$

If $\Delta m_{AB} \gg \Lambda$, we can rely on the quasiclassical expressions namely, the matrix $E$ reduces to the matrix (2.6) and

$$\sqrt{2} e_P \approx -m_P, \quad \sqrt{2} e \approx -m, \quad (4.7)$$
and we then immediately recover the classical result (3.7) from (4.4) where \( \xi_p \)'s are given in Eq. (2.9).

Our current task is to find \( e_p \)'s from the exact solution of the theory on the Coulomb branch (at \( \mu = 0 \)). The Seiberg–Witten curve in the theory under consideration has the form [29]

\[
y^2 = \prod_{p=1}^{N} (x - \phi_p)^2 - 4 \left( \frac{A}{\sqrt{2}} \right)^{N-\tilde{N}} \prod_{A=1}^{N_f} \left( x + \frac{m_A}{\sqrt{2}} \right),
\]

where \( \phi_p \) are gauge invariant parameters on the Coulomb branch. Their relations to the gauge invariant parameters \( u_k \) in (4.1) are as follows:

\[
u_k = \sum_{p=1}^{N} \phi_{kp}.
\]

The curve (4.8) describes the Coulomb branch of the theory for \( N_f < 2N - 1 \). The case \( N_f = 2N - 1 \) (i.e. \( \tilde{N} = N - 1 \)) is special. In this case we must make a shift [29]

\[
m_A \rightarrow \tilde{m}_A = m_A + \frac{A}{N}, \quad N_f = 2N - 1,
\]

in (4.8).

Semiclassically, at large masses

\[
\text{diag} \left( \frac{1}{2} a + T^a a^a \right) \approx [\phi_1, ..., \phi_N].
\]

Therefore, in the \((1, ..., N)\) quark vacuum we have

\[
\phi_p \approx -\frac{m_p}{\sqrt{2}}, \quad P = 1, ..., N,
\]

in the large \( m_A \) limit, see (2.6). In terms of the curve (4.8) this vacuum corresponds to such values of \( \phi_p \) which ensure that the curve has \( N \) double roots and \( \phi_p \)'s are determined by the quark mass parameters in the semiclassical limit, see (4.12). The presence of \( N \) double roots means that \( N \) quark flavors are massless at this singular point on the Coulomb branch. Upon \( \mu \) deformation this singularity becomes the \( r = N \) vacuum, where the \( N \) “former” massless squarks condense.
The curve (4.8) with $N$ double roots has the form

$$y^2 = \prod_{P=1}^{N} (x - e_P)^2,$$  (4.13)

where semiclassically $e_P$'s are given by mass parameters via (4.7). In Appendices A and B we show that the double roots $e_P$ of the Seiberg–Witten curve are precisely given by the diagonal elements of the matrix $E$ (4.5). Thus, the Seiberg–Witten curve which appeared in [17, 18] as a kind of an auxiliary mathematical tool acquires a physical and very transparent meaning in the $r = N$ vacuum: its double roots determine the tensions of the $Z_N$ strings through Eq. (4.4)!

In Appendices A and B we demonstrate that diagonal elements of matrix $E$ are given by $N$ double roots of the Seiberg–Witten curve (4.8) by considering two simple examples: $N_f = N$ and $2N > N_f > N$ theories. More exactly, we analyze there the $N_f = N = 2$ and $N = 3, N_f = 5$ cases. Say, for the simplest $N_f = N = 2$ case, the two double roots of (4.8) are as follows [33]:

$$\sqrt{2} e_{1,2} = -m \mp \sqrt{\frac{\Delta m_{12}^2}{4} + \Lambda^2}.$$  (4.14)

Note that the average value of $e$ is proportional to the average mass $m$,

$$\sqrt{2} e = -m,$$  (4.15)

where $e$ and $m$ are given in (4.6) and (2.10), respectively. In fact, this property is fulfilled for all cases with $N_f \leq 2N - 1$.

Substituting (4.14) into (4.4), we obtain the tensions of two $Z_2$ strings,

$$T_{1,2}^{BPS} = 4\pi \left| \mu_1 m \pm \mu_2 \sqrt{\frac{\Delta m_{12}^2}{4} + \Lambda^2} \right|.$$  (4.16)

This formula takes into account all quantum (instanton) corrections in powers of $\Lambda/\Delta m_{12}$.

As was already mentioned, one can relax the weak-coupling conditions (2.12) and (2.13) and enter the strong-coupling domain (2.14). The theory undergoes a crossover transition, i.e. passes the curves of marginal stability (CMS), where certain states decay, as well as monodromies, which change charges of other states. In this domain the original theory can be better
described by the dual weakly coupled theory of light dyons with the gauge group (2.15), derived in [33].

In Appendices A and B we prove that our result for the $Z_N$-string tensions (4.4) can be analytically continued (as functions of $\Delta m_{AB}$ and $m$) into the strong-coupling domain. This was expected, of course. We encounter a crossover at $\Delta m_{AB} \sim \Lambda$ and $\mu m \sim \Lambda^2$ rather than a phase transition [33, 31, 32]. Therefore, string tensions exhibit a continuous behavior across the crossover.

5 $\mathcal{N} = (2, 2)$ -supersymmetric world sheet theory

In this part of the paper we proceed from the analysis in the bulk to the analysis on the string world sheet, with the intension to demonstrate that both lead to identical consequences. As previously, to establish the appropriate setting, we start from the undeformed case. We briefly review the world-sheet low-energy sigma models on the non-Abelian strings in $\mathcal{N} = 2$ supersymmetric QCD with the FI term [1, 2, 3, 4], see also the review papers [5, 6, 7, 8]. First we will deal with $\mathcal{N} = 2$ QCD with the FI $D$-term. The corresponding Lagrangian is (2.3) with $\xi_3 \neq 0$ and large, and

$$\mu_1 = \mu_2 = 0. \quad (5.1)$$

To begin with, assume that $N_f = N$. The Abelian $Z_N$-string solutions break the $SU(N)_{C+F}$ global group down to $SU(N-1) \times U(1)$. As a result, the non-Abelian strings develop orientational zero modes associated with rotations of their color flux inside the non-Abelian $SU(N)$ group. The moduli space of the non-Abelian string is described by the coset space

$$\frac{SU(N)}{SU(N-1) \times U(1)} \sim \text{CP}(N-1), \quad (5.2)$$

in addition to $C$ spanned by the translational modes. The translational moduli totally decouple. They are sterile free fields which can be ignored in further considerations. Therefore, the low-energy effective theory on the non-Abelian string is the two-dimensional $\mathcal{N} = (2, 2)$ CP$(N-1)$ model [1, 2, 3, 4].
Now let us add “extra” quark flavors, with degenerate masses, increasing $N_f$ from $N$ up to a certain value $N_f > N$ but $N_f \leq 2N - 1$. The strings supported by such theory are semilocal. In particular, the string solutions on the Higgs branches (typical for multiflavor theories) usually are not fixed-radius strings, but, rather, possess radial moduli, a.k.a size moduli, see [42] for a comprehensive review of the Abelian semilocal strings. The transverse size of such a string is not fixed: it can vary without changing the tension.

Non-Abelian semilocal strings in $\mathcal{N} = 2$ SQCD with $N_f > N$ were studied in [1, 4, 43, 44]. The orientational zero modes of the semilocal non-Abelian string are parametrized by a complex vector $n^P$ ($P = 1, ..., N$), while its $\tilde{N} = (N_f - N)$ size moduli are parametrized by another complex vector $\rho^K$ ($K = N + 1, ..., N_f$). The effective two-dimensional theory which describes the internal dynamics of the non-Abelian semilocal string is the $\mathcal{N} = (2, 2)$ weighted CP model on a “toric” manifold, which includes both types of fields. The bosonic part of the action in the gauged formulation (which assumes taking the limit $e^2 \to \infty$) has the form

$$\begin{align*}
S &= \int d^2 x \left\{ |\nabla_\alpha n^P|^2 + |\tilde{\nabla}_\alpha \rho^K|^2 + \frac{1}{4e^2} F_{\alpha\beta}^2 + \frac{1}{e^2} |\partial_\alpha \sigma|^2 \\
&+ 2 \left| \sigma + \frac{m_P}{\sqrt{2}} \right|^2 |n^P|^2 + 2 \left| \sigma + \frac{m_K}{\sqrt{2}} \right|^2 |\rho^K|^2 + \frac{e^2}{2} \left( |n^P|^2 - |\rho^K|^2 - 2\beta \right)^2 \right\},
\end{align*}$$

$$P = 1, ..., N , \quad K = N + 1, ..., N_f.$$ (5.3)

The fields $n^P$ and $\rho^K$ have charges $+1$ and $-1$ with respect to the auxiliary U(1) gauge field; hence, the corresponding covariant derivatives in (5.3) are defined as

$$\nabla_\alpha = \partial_\alpha - i A_\alpha , \quad \tilde{\nabla}_\alpha = \partial_\alpha + i A_\alpha ,$$

respectively. This is the effective low-energy theory on the non-Abelian string in the $r = N$ vacuum, in which the first $N$ squark flavors ($P = 1, ..., N$) condense.

If only the charge $+1$ fields $n$ were present, in the limit $e^2 \to \infty$ we would get a conventional twisted-mass deformed CP $(N - 1)$ model. The presence of the charge $-1$ fields $\rho^K$ converts the CP$(N - 1)$ target space into that of the a weighted CP$(N_f - 1)$ model. In parallel to the CP$(N - 1)$ model, small

---

*Equation (5.3) and similar expressions below are given in the Euclidean notation.*
mass differences $|m_A - m_B|$ lift orientational and size zero modes generating a shallow potential on the modular space. The $D$-term condition

$$|n^P|^2 - |ho^K|^2 = 2\beta$$

(5.5)
is implemented in the limit $e^2 \to \infty$. Moreover, in this limit the gauge field $A_\alpha$ and its $\mathcal{N} = 2$ bosonic superpartner $\sigma$ become auxiliary and can be eliminated through equations of motion.

The two-dimensional coupling constant $\beta$ is related to the four-dimensional one as (e.g. [7])

$$\beta = \frac{2\pi}{g_2^2}.$$  

(5.6)

This relation is obtained at the classical level [2, 3]. In the quantum theory both couplings run. In particular, the model (5.3) is asymptotically free [24] and develops its own scale, which coincides with that of the bulk theory $\Lambda$ [3].

The ultraviolet cut-off in the sigma model on the string world sheet is determined by $g_2\sqrt{\xi_3}$. Equation (5.6) relating the two- and four-dimensional couplings is valid at this scale. At $N \leq N_f < 2N$ the model (5.3) is asymptotically free. Its coupling $\beta$ continues running below $g_2\sqrt{\xi_3}$ until it ceases to run and freezes at the scale of the mass differences $|\Delta m_{AB}|$. If all mass differences are large, $|\Delta m_{AB}| \gg \Lambda$, the model is at weak coupling. From (5.3) we see that in this regime the model has $N$ vacua (i.e. $N$ strings from the standpoint of the bulk theory) at

$$\sqrt{2}\sigma = -m_{P_0}, \quad |n^{P_0}|^2 = 2\beta, \quad n^{P \neq P_0} = \rho^K = 0,$$  

(5.7)

where $P_0 = 1, \ldots, N$.

6 Switching on a weak $\mu$-deformation

Let us break $\mathcal{N} = 2$ supersymmetry in the bulk theory by switching on the deformation superpotential of the type (2.2), assuming that the $\mu$ parameters are small, (1.9). If the parameters $m$ are large enough, we can switch off the FI $D$-term parameter $\xi_3$, keeping the theory in the weak-coupling regime, see (2.13). The string solutions will be stabilized by $F$-terms. Our aim in this section is to find an effective low-energy theory on the world sheet of the non-Abelian strings in the deformed case. If the typical scale of excitations in the world-sheet theory (it is of the order of $\max(\Delta m_{AB}, \Lambda)$)
is much less than the inverse thickness of the string $\sim \sqrt{\mu m}$, we can expect that such low-energy world-sheet description exists and is given by a certain deformation of the $\mathcal{N} = (2, 2)$ supersymmetric CP model (5.3) which breaks $\mathcal{N} = (2, 2)$ down to $\mathcal{N} = (0, 2)$ . As was already mentioned in Sect. 1, this problem is solved in the theories with the $D$-term stabilization. Namely, if we keep $\xi_3$ nonvanishing (and large) and switch on deformation (2.2) putting all masses $m_A = 0$, then the effective theory on the non-Abelian string becomes $\mathcal{N} = (0, 2)$ supersymmetric CP model with the quadratic in $\sigma$ superpotential [11, 12, 21]. After a brief review of this result we move on to consider a motion of the case we are interested in in this paper: $\xi_3 = 0$, while $m \neq 0$ and $\Delta m_{AB} \neq 0$. Switching on $\mu_{1,2} \neq 0$ generates the FI $F$-terms in each of the $U(1)$ factors of the $U(N)$ gauge group and, simultaneously, breaks $\mathcal{N} = (2, 2)$ down to $\mathcal{N} = (0, 2)$ on the world sheet.

6.1 $|m_A - m_B| = 0$

With four supercharges of the deformed $\mathcal{N} = 1$ bulk theory normally the 1/2 BPS-saturated string solution will preserve only two supercharges on the string world sheet. However, it is well-known that the sigma model with the CP (Kähler) target space, when supersymmetrized, automatically yields $\mathcal{N} = (2, 2)$ sigma model; one cannot get $\mathcal{N} = (0, 2)$ . It was pointed out [11] that the target space in the problem at hand is in fact $CP(N_f - 1) \times C$ rather than $CP(N_f - 1)$. Edalati and Tong suggested that the superorientational zero modes can mix with the supertranslational ones. They explicitly constructed an $\mathcal{N} = (0, 2)$ supergeneralization of the sigma model with the target space $CP(N_f - 1) \times C$. In their construction the bosonic part of the $\mathcal{N} = (2, 2)$ model (5.3) is supplemented by the term

$$\delta S_{1+1} = \int d^2 x V_{1+1}(\sigma) = \int d^2 x \left| \frac{\partial W_{1+1}}{\partial \sigma} \right|^2 \quad (6.1)$$

breaking $\mathcal{N} = (2, 2)$ down to $\mathcal{N} = (0, 2)$. Here $W_{1+1}(\sigma)$ is a two-dimensional deformation superpotential.

Later in [12, 21] this conjecture was confirmed. It was shown that the two-dimensional superpotential $W_{1+1}$ is indeed generated on the world sheet of the non-Abelian string in the massless theory. For the bulk deformation (2.2) the world-sheet superpotential is

$$W_{1+1} = \omega \sigma^2,$$ 

(6.2)
where the deformation parameter is proportional to \(\mu\) to the leading order at small \(\mu\). At large \(\mu\) the world-sheet deformation becomes more complicated \[12\ 21\]. For the superpotential (6.2) the scalar potential is \(|\sigma|^2\), see (1.7).

The massless heterotic \(\mathcal{N} = (0,2)\) supersymmetric CP\((N - 1)\) model with the deformation potential (1.7) was solved in \[13\ 22\] in the large-\(N\) approximation. It was shown that, although classically the model has \(\mathcal{N} = (0,2)\) supersymmetry, it gets spontaneously broken by quantum non-perturbative effects. On the other hand, the model has \(N\) strictly degenerate vacua with vacuum energies proportional to \(\Lambda\).

The vacuum energy in the world-sheet theory is obviously identified with the string tensions in the bulk theory. Therefore, our result for the string tensions (3.7) in the \(|m_A - m_B| \neq 0\) theory shows that the vacuum energies obtained in the world-sheet theory cannot be degenerate; they are split in accordance with (3.7). Moreover, we will show below that with \(|m_A - m_B| \neq 0\) and the \(F\)-term stabilization, \(\mathcal{N} = (0,2)\) supersymmetry in the world-sheet theory spontaneously breaks already at the classical level.

### 6.2 \(|m_A - m_B| \neq 0\)

Now we will construct the effective world-sheet theory for a non-Abelian string in \(\mathcal{N} = 2\) SQCD (2.3) with \(|m_A - m_B| \neq 0\) deformed by the mass term for the adjoint matter (2.2). There are two ways of addressing this problem. First, we can start from (2.3) with \(\xi_3 = 0\) and \(|m_A - m_B| = 0\), see (2.7). To the leading order in \(\mu\) the deformation superpotential then reduces to a single FI \(F\)-term – that of the U(1) factor of the U\((N)\) gauge group, with the complex FI parameter

\[
\xi = \xi_1 + i\xi_2 = 2\sqrt{\frac{2}{N}} \mu_1 m, \tag{6.3}
\]

see (2.9). In this limit the theory has unbroken \(\mathcal{N} = 2\) supersymmetry \[14\ 15\], as well as unbroken color-flavor symmetry (2.11). In particular, the FI term (6.3) can be rotated by a global SU(2)\(_R\) transformation to the FI \(D\)-term \(\xi_3\). The world-sheet theory on the non-Abelian string is given in this case by \(\mathcal{N} = (2,2)\) CP model (5.3), where all masses \(m_A\) are equal.

Now we switch on the splittings \(|m_A - m_B| \neq 0\) and ask ourselves: what is the response in the world-sheet theory? Clearly, the world-sheet theory becomes a certain deformation of (5.3), with generic mass parameters \(m_A\).
$\mathcal{N} = (2, 2)$ supersymmetry breaking is expected. To see that this is indeed the case it is worth remembering that the $Z_N$ strings under consideration are BPS saturated only being considered in the $U(1)^N$ Abelian theory, see Sect. 3. In particular, we can introduce $N$ different FI $F$-terms $\xi_P$ which determine central charges of $N$ different BPS strings only in the Abelian $U(1)^N$ theory.

On the other hand, the non-Abelian strings we deal with are, in fact, interpolations between different $Z_N$ strings [1, 2, 3, 4]. They exist only in the full non-Abelian $U(N)$ gauge theory. Therefore, non-Abelian strings in the theory (2.3) are not BPS saturated. To begin with, they all have different tensions, see (3.7). Since the world-sheet theory on the non-Abelian string describes dynamics of the orientational modes interpolating between different $Z_N$ strings, we expect $\mathcal{N} = (2, 2)$ world-sheet supersymmetry to be broken.

Another line of reasoning is to start with a large nonvanishing $\xi_3$ and all $|m_A - m_B| = 0$. Then one deforms the theory by adding the superpotential (2.2). The world-sheet theory becomes massless heterotic $\mathcal{N} = (0, 2)$ CP model. Next, one introduces generic $m_A - m_B \neq 0$, simultaneously decreasing $\xi_3$ and increasing $m$ keeping $\mu m$ large. Eventually one takes the limit $\xi_3 = 0$, see (2.13).

These two approaches combined, suggest that the world-sheet theory we are looking for is a heterotic CP model (5.3), with generic masses deformed by a certain two-dimensional superpotential $W_{1+1}(\sigma)$, which breaks $\mathcal{N} = (2, 2)$ supersymmetry down to $\mathcal{N} = (0, 2)$. We assume this below. To derive the deformation superpotential we have to find, generally speaking, solutions for non-Abelian strings and substitute them in the bulk action (2.3) assuming a slow adiabatic dependence of the moduli $n^P$ and $\rho^K$ on the world-sheet coordinates, cf. [7].

We leave this program for future studies while for the time being we make a crucial shortcut. We determine the deformation potential using the $Z_N$ string tensions (3.7) as an input. As was already mentioned, the $Z_N$ string tensions must be identified with the vacuum energies for $N$ vacua in the world-sheet theory,

$$T_P = E^{1+1}_P, \quad P = 1, ..., N,$$

where $E_P$ are the vacuum energies. For undeformed theory this identification is usually carried out up to a constant shift, see [7]. Namely, the energies of the $N$ vacua in $\mathcal{N} = (2, 2)$ supersymmetric CP model (5.3) all vanish, while
the $Z_N$ string tensions are all equal to $2\pi\xi_3$. In our case we identify vacuum energies of the world-sheet theory with tensions of $Z_N$ strings without any shift.

We use relation (6.4) below to find the deformation potential $V_{1+1}(\sigma)$. To this end we note that at small $\mu$ the potential $V_{1+1}(\sigma)$ is a small perturbation and to the leading order in $\mu$, the expectation values $\sigma_P$ are (classically) given by their unperturbed values in Eq. (5.7). Since the vacuum energies for the undeformed $\mathcal{N} = (2, 2)$ supersymmetric CP model (5.3) all vanish, Eq. (6.4) implies

$$T_P = V_{1+1}(\sigma_P), \quad P = 1, \ldots, N,$$

(6.5)

where $\sigma_P$ are the expectation values of the $\sigma$ field in $N$ vacua of the world-sheet theory.

Combining this with (3.7) and (5.7) we determine the world-sheet potential,

$$V_{1+1}(\sigma) = 4\pi \left| \sqrt{\frac{2}{N}} \mu_1 m - \mu_2 \left( \sqrt{2}\sigma + m \right) \right|,$$

(6.6)

where $m$ is the average of masses of $N$ first quarks, which condense in the $r = N$ vacuum of the bulk theory, see (2.10).

The potential (6.6) gives (to the leading order in $\mu$) the vacuum energies for all $N$ vacua of the world-sheet theory right, i.e. equal to tensions of the $N$ elementary strings. In principle, one could add to (6.6) an arbitrary potential, which vanishes in all $N$ critical points (vacua). For now we assume that this additional potential is zero. A rigorous proof of this assertion will be presented in a future publication.

The deformation potential (6.6) can be written as the modulus squared of the derivative of a certain two-dimensional superpotential $W_{1+1}(\sigma)$,

$$W_{1+1} = \frac{1}{\mu_2} \sqrt{\frac{8\pi}{9}} \left[ \sqrt{\frac{2}{N}} \mu_1 m - \mu_2 \left( \sqrt{2}\sigma + m \right) \right]^{3/2},$$

(6.7)

cf. (6.1). This confirms our initial assumption that the world-sheet theory has $\mathcal{N} = (0, 2)$ supersymmetry at the Lagrangian level (broken spontaneously by the choice of vacua already at the classical level).

The vacuum energies of $N$ vacua in world-sheet theory are nonvanishing and all different for a generic set of $m_A - m_B$. This shows that $\mathcal{N} = (0, 2)$ supersymmetry is broken at the classical level. In [22] we demonstrated,
however, that the masses of the fermion and boson excitations (n’s, ρ’s vs.
their fermion superpartners) are still identical to the leading order in µ. They
split only at the next-to-leading order.

To conclude this section it is instructive to consider the limit of unbroken
maximal supersymmetry. To this end we put µ2 = 0 in the deformation
superpotential of the bulk theory \[2,2\]. To the leading order in µ1 superpo-
tential \[2,2\] reduces in this case to the single FI F-term of the U(1) factor
of U(N) gauge group. This FI term does not break \(\mathcal{N} = 2\) supersymmetry
(to the leading order in µ), and the 1/2 “BPS-ness” of the string solution is
maintained guaranteeing \(\mathcal{N} = (2,2)\) supersymmetry on the world sheet.

In more detail, with µ2 = 0 the deformation potential \[6.6\] reduces to a
constant equal to the common value of the \(Z_N\)-string tensions. This overall
constant does not ruin the \(\mathcal{N} = (2,2)\) supersymmetry of the weighted CP
model \[5.3\] on the string (see the discussion in Sect. 6.3).

If µ2 ≠ 0 (and µ2 ∼ µ1), and the set of masses \(m_P\) is generic (i.e. all
\(|m_P|’s are of the same order of magnitude, none of these masses vanish or are
clustered in a special way) then the split of the string tensions is of the order
of the central value of the tension.\footnote{For simplicity we assume that \(N\) does not grow. Otherwise, we should take into
account the \(N\) dependence of \(\Delta \sigma\).} It is instructive to compare this statement
with the D-term stabilized strings where the central value is proportional to
ξ (see \[12\]) while the split is proportional \[22\] to \(\mu^2 m_P^2 / \xi\) (in the limit of
small deformation). Given the identification \(\xi \sim \mu m\), we conclude that the
situations with the F- and D-term stabilized heterotically deformed strings
are qualitatively similar.

It is curious to mention a special case \(m_P = m_0 \exp \left(\frac{2\pi i P}{N}\right)\) where \(P = 1, 2, \ldots, N\). In this case \(m\) vanishes, and Eq. \[6.6\] reduces to \(4\sqrt{2\pi} |\mu_2 \sigma|\)
(supplemented by \[7.2\] implying for large \(m_P\) that \(E_P = 4\pi |\mu_2 m_0|\)). All
tensions are the same. This is due to the \(Z_N\) symmetry of this example.

To conclude this section, it is worth summarizing our findings regarding
the pattern of supersymmetry breaking in the world-sheet theory. Generally
speaking, there are four mass parameters in the problem at hand, \(m\) (the
average squark mass term), \(\Delta m\) (or, alternatively, \(\Delta m_{AB}\), typical squark
mass differences), \(\Lambda\) (the dynamical scale parameter), and – finally – \(\mu\) (the
deformation parameter). For simplicity let us assume \(\Lambda\) to be very small and
negligible. Then we are left with 3 parameters. In the limit \(\mu \to 0, m \to\)
∞, $(\mu m)$ fixed, and

$$\Delta m = 0$$

do not carry out. The bulk theory has $\mathcal{N} = 2$, the strings are BPS-saturated, and the theory on the world sheet possesses $\mathcal{N} = (2, 2)$ too (to the leading order in $\mu$). All $N$ strings are degenerate. The tension scales as $\mu m$. (The first string studies in this limit were carried out by Bolognesi [23].)

It is worth explicitly verifying $\mathcal{N} = (2, 2)$ supersymmetry in the world-sheet theory at $\Delta m_{AB} = 0$. The field $\sigma$ is not dynamical and can be eliminated by virtue of its equations of motion. In the case $\Delta m_{AB} = 0$ we have then $\sqrt{2}\sigma = -m$. Thus, the second term in the potential (6.6) vanishes, and the potential reduces to a constant which gives common tension to all $Z_N$ strings. This overall constant does not break $\mathcal{N} = (2, 2)$ supersymmetry on the world sheet (see Sect. 6.3).

Now, if we switch on $\Delta m \neq 0$, a superpotential is generated on the world sheet which, generally speaking, breaks the world-sheet supersymmetry down to $\mathcal{N} = (0, 2)$ as far as algebra is concerned (i.e. in the Lagrangian). This world-sheet $\mathcal{N} = (0, 2)$ is further spontaneously broken down to nothing.

Let us stress that if we consider the next-to-leading order in $\mu$, the world-sheet supersymmetry will be explicitly and completely broken. At $O(\mu^2)$ we expect generation of the potential which cannot be presented as the modulus squared (of the derivative) of a certain superpotential. One can expect that this potential may depend directly on the $n$ fields in addition to the $\sigma$ dependence.

### 6.3 Linear $\mathcal{W}_{1+1}(\sigma)$

Formally, the insertion of an “additional” superpotential $\mathcal{W}_{1+1}(\sigma)$ in the weighted $\mathcal{N} = (2, 2)$ CP model (which can be obtained as a dimensional reduction of a super-QED from four to two dimensions) implies that $\mathcal{N} = (2, 2)$ supersymmetry on the world sheet is explicitly broken down to $\mathcal{N} = (0, 2)$ which, in turn, may or may not be spontaneously broken down to nothing. In fact, there is one exception from this rule. Indeed, assume $\mathcal{W}_{1+1}(\sigma)$ to be a linear function of $\sigma$. As is obvious e.g. from Eqs. (4.1) and (4.2) of [12], the part of the heterotically deformed Lagrangian which describes interactions contains only $\partial^2 \mathcal{W}_{1+1}/\partial \sigma^2$. For linear superpotentials it vanishes, and we have exactly the same Lagrangian as that of the $\mathcal{N} = (2, 2)$ model, up to an overall constant shift of energy proportional
to $|\partial W_{1+1}/\partial \sigma|^2$. If we remove this constant shift of the vacuum energy by hand, the remainder satisfies the $\mathcal{N} = (2, 2)$ superalgebra. In other words, the linear deformation superpotential leads to $E_0 + \mathcal{L}_{\mathcal{N}=(2,2)}$, where $E_0$ is a numerical constant. This phenomenon is somewhat similar to a well-known fact in four dimensions. If we consider $\mathcal{N} = 2$ gauge theories, then a generic superpotential $W(\mathcal{A}) \neq 0$ breaks $\mathcal{N} = 2$ down to $\mathcal{N} = 1$ (here $\mathcal{A}$ is the $\mathcal{N} = 2$ photon/photino superpartner). However, linear $W(\mathcal{A})$ does preserve $\mathcal{N} = 2$ supersymmetry. This is an exception too.

7 Quantum effects in the world-sheet theory

In this section we study quantum (nonperturbative) effects in the world-sheet theory and show that quantum corrections to the vacuum energies of $N$ vacua precisely reproduce quantum corrections to string tensions (4.4) obtained in the bulk theory. We start from reviewing the exact superpotential in the undeformed $\mathcal{N} = (2, 2)$ CP model (5.3) and then switch on the deformation potential (6.6).

7.1 Exact superpotential

The $\mathcal{N} = (2, 2)$ supersymmetric CP($N - 1$) models are known to be described by an exact superpotential [45, 46, 24, 25] of the Veneziano–Yankielowicz type [47]. This superpotential was generalized to the case of the weighted CP models in [48, 26]. In this section we will briefly outline this method. Integrating out the fields $n^P$ and $\rho^K$ we can describe the original model (5.3) by the following exact twisted superpotential:

$$
W_{\text{eff}} = \frac{1}{4\pi} \sum_{P=1}^{N} \left( \sqrt{2} \Sigma + m_P \right) \ln \frac{\sqrt{2} \Sigma + m_P}{\Lambda} 

- \frac{1}{4\pi} \sum_{K=N+1}^{N_P} \left( \sqrt{2} \Sigma + m_K \right) \ln \frac{\sqrt{2} \Sigma + m_K}{\Lambda} 

- \frac{N - \tilde{N} \sqrt{2} \Sigma}{4\pi},
$$

(7.1)

$E_0$ can be viewed as a central charge in the $\mathcal{N} = (2, 2)$ superalgebra.
where $\Sigma$ is a twisted superfield \[24\] (with $\sigma$ being its lowest scalar component). Minimizing this superpotential with respect to $\sigma$ we get the vacuum field formula,

$$\prod_{p=1}^{N}(\sqrt{2}\sigma + m_p) = \Lambda^{(N-N)} \prod_{K=N+1}^{N_f}(\sqrt{2}\sigma + m_K). \quad (7.2)$$

Note, that the roots of this equation coincide with the double roots of the Seiberg–Witten curve of the bulk theory \[25, 26\] for all $N_f < 2N - 1$. Below we will see that this fact is crucial for the relation between the tensions of the non-Abelian strings and vacuum energies of the world-sheet theories in the $\mu$-deformed quantum theory. This coincidence is, of course, a manifestation of the coincidence of the Seiberg–Witten solution of the bulk theory in the $r = N$ vacuum with the exact solution of the two-dimensional model (5.3) defined by the superpotential (7.1). As was mentioned in Sec. 1 this coincidence was observed in \[25, 26, 27\] and explained later in \[3, 4\].

In particular, in the example $N_f = N = 2$ considered in Sect. 4 and Appendix A the vacuum equation (7.2) reduces to

$$\sqrt{2}\sigma_1 \approx \sqrt{2}(\sqrt{2}\sigma + m_1) = \Lambda^2. \quad (7.3)$$

It has two solutions

$$\sqrt{2}\sigma_{1,2} = -m \pm \sqrt{\frac{\Delta m_{12}^2}{4} + \Lambda^2}, \quad (7.4)$$

which indeed coincide with double roots (4.14) of the Seiberg–Witten curve,

$$\sigma_p = e_p, \quad N_f < 2N - 1. \quad (7.5)$$

Another example (considered in Appendix B) is the one with $N = 3$ and $N_f = 5$. This is a special example of the case $N_f = 2N - 1$. Restricting ourselves to the mass choice \[3.6\] we easily find VEVs of $\sigma$ for this case from (7.2), namely

$$\sqrt{2}\sigma_1 = -m_1, \quad \sqrt{2}\sigma_2 = -m_2, \quad \sqrt{2}\sigma_3 = -m_3 + \Lambda. \quad (7.6)$$

We see that in this case the relation between VEVs of $\sigma$ and double roots of the Seiberg–Witten curve is modified, namely,

$$\sqrt{2}\sigma_p = \sqrt{2}e_p + \frac{\Lambda}{3}, \quad N = 3, \quad N_f = 5. \quad (7.7)$$
see (B.7). If \( N \) is an arbitrary integer (rather than \( N = 3 \)) this relation takes
the form
\[
\sqrt{2} \sigma_P = \sqrt{2} e_P + \frac{\Lambda}{N}, \quad N_f = 2N - 1, \quad (7.8)
\]
see (4.10).

7.2 Quantum corrections to vacuum energies

Now let us demonstrate that quantum (nonperturbative) corrections to the
vacuum energies in the world-sheet theory precisely reproduce the quantum-
corrected string tensions (4.4). The vacuum energies in the \( N \) vacua of the
world-sheet theory are given by the values of the world-sheet deformation
potential \( V_{1+1}(\sigma) \) calculated at VEVs of the \( \sigma \) field. Much in the same way
as in the classical theory, the VEVs of the \( \sigma \) field are determined by the
undeformed \( \mathcal{N} = (2, 2) \) supersymmetric theory (5.3) to the leading order in \( \mu \).

In quantum theory these VEVs – we denote them as \( \sigma_P \) – are presented
by solutions of the vacuum equation (7.2). Consider the case \( N_f < 2N - 1 \)
first. In this case the potential (6.6) gives
\[
T_P^{\text{BPS}} = V_{1+1}(\sigma_P) = 4\pi \sqrt{\frac{2}{N}} \mu_1 m - \mu_2 \left( \sqrt{2} \sigma_P + m \right), \quad (7.9)
\]
This formula exactly reproduces our result (4.4) obtained in the bulk theory,
provided that the relations (7.5) and (4.15) are taken into account.

As far as the special case \( N_f = 2N - 1 \) is concerned, the deformation
potential (6.6) should be modified. The modification is
\[
V_{1+1}(\sigma) = 4\pi \sqrt{\frac{2}{N}} \mu_1 m - \mu_2 \left( \sqrt{2} \sigma - \frac{\Lambda}{N} + m \right), \quad N_f = 2N - 1. \quad (7.10)
\]
This potential reproduces the string tensions (4.4) once we make use of the
relation (7.8). The superpotential (6.7) is modified accordingly.

8 Monopole confinement

Now we will discuss kinks on the \( F \)-term stabilized strings, which represent
confined monopoles being viewed from the bulk standpoint.
Consider weak coupling regime (2.13) in the bulk theory. Since $N$ squarks are condensed in the $r = N$ vacuum (see (2.8)) and the gauge group is Higgsed the ’t Hooft-Polyakov monopoles are confined. As we know, in the Higgsed U($N$) gauge theories monopoles show up only as junctions of two distinct elementary non-Abelian strings [49, 3, 4]. These strings are in fact represented by different vacua in the effective world-sheet sigma model while the confined monopoles are kinks interpolating between distinct vacua [3, 4, 49].

Given the bulk theory (2.3) let us inspect the domain of small $|\Delta m_{AB}| \ll m$, with $m$ large enough to ensure (2.13) at small $\mu$. In this case the splittings between different vacua of the world-sheet theory are small. We can consider them as “quasivacua.”

This regime is quite similar to the one studied in [28] in non-supersymmetric bulk theory in the Higgs phase, where all quark condensates are equal. In the effective CP($N - 1$) model on the non-Abelian string all vacua are split (on the quantum level), and $N - 1$ would-be vacua become quasivacua, see [7] for a review. The vacuum splitting can be understood as a manifestation of the Coulomb/confining linear potential between the kinks [50, 51] that interpolate between the true vacuum, and say, the lowest quasivacuum. The force is attractive in the kink-antikink pairs, implying formation of weakly coupled bound states (weak coupling is the manifestation of the smallness of the splittings between the vacua). The charged kinks are eliminated from the spectrum, see Fig. 1.

The kink confinement in the two-dimensional CP model can be interpreted [28] as the following phenomenon: the non-Abelian monopoles, in addition to the four-dimensional confinement (which ensures that the monopoles are attached to the strings) acquire a two-dimensional confinement along the string: a monopole–antimonopole forms a meson-like configuration, with necessity, see Fig. 1.

Moreover, as was shown in [52] for the CP($N - 1$) model and in [32] for
the weighted CP model \((5.3)\), the kinks belong to the fundamental \((N, 1) + (1, \bar{N})\) representations of the global group \((2.11)\) in the limit \((2.7)\). The global flavor group is explicitly broken by the mass differences, down to \(U(1)^{N_f - 1}\). The kinks are charged with respect to this group. Therefore, the kink-antikink (monopole-antimonopole) mesons which carry non-trivial charges with respect to the global \(U(1)^{N_f - 1}\) are stable. In other words, if the total global charge of such a meson does not vanish, the kink and antikink cannot annihilate, and the meson they constitute will never decay.

9 Generic single-trace superpotential deformations

In this section we will treat more general deformation of the bulk theory than those considered previously. Namely, instead of \((2.2)\) we will consider a generic single-trace polynomial superpotential of the form

\[
\mathcal{W}_{3+1} = \text{Tr} \sum_{k=1}^{N} \frac{c_k}{k+1} \Phi^{k+1},
\]

where we introduce the adjoint matrix superfield

\[
\Phi = \frac{1}{2} \mathcal{A} + T^a \mathcal{A}^a.
\]

\(\Phi\) in Eq. \((9.2)\) is a matrix from \(U(N)\) rather than \(SU(N)\). The bosonic potential of the bulk theory is still given by \((2.5)\). We will require the coefficients \(c_k\) to be small and study the theory’s response in the leading approximation in \(c_k\). This condition is quite similar to the condition \((1.9)\) of small \(\mu\).

Such more generic deformations were considered in \([11, 12]\) with a special choice of superpotentials: their critical points we supposed to coincide with the quark mass terms. Here we relax this condition and, instead, put \(\xi_3 = 0\), much in the same way as for the deformation \((2.2)\), so the quark condensation is entirely due to the FI \(F\)-terms. The gauge group \(U(N)\) is broken down to \(U(1)^N\) by the adjoint VEVs \((2.6)\).

The squark VEVs can be readily calculated from \((2.5)\). They are still given by Eq. \((2.8)\), where now

\[
(\xi_1, ..., \xi_N) = -\text{diag} \left\{ \sqrt{2} \frac{\partial \mathcal{W}_{3+1}}{\partial \Phi} (\Phi_{\text{VEV}}) \right\}.
\]

30
Here $\Phi_{\text{VEV}}$ is the vacuum expectation value of the matrix field $\Phi$ defined in (2.6).

Consider now the $Z_N$ strings in this theory. If the deformation superpotential (9.1) is weak, these strings are BPS saturated in the low-energy $U(1)^N$ gauge theory. Parallelizing the derivation presented in Sec. 3 we get the same expression (3.7) for their tensions, where $\xi_P$ now are given in Eq. (9.3).

Next we consider the response of the world-sheet theory on the non-Abelian string on the bulk deformation (9.1). The world-sheet theory is still given by the weighted CP model (5.3) deformed by a certain two-dimensional superpotential $W_{3+1}(\sigma)$, which breaks $\mathcal{N} = (2, 2)$ supersymmetry down to $\mathcal{N} = (0, 2)$.

To find this deformation superpotential we again impose the condition (6.5). Since the deformation is weak, by assumption, the $\sigma$ field VEVs can be determined in the undeformed theory (5.3). They are still (classically) given by quark mass parameters via (5.7). This leads us to the following deformation potential

$$V_{1+1}(\sigma) = 2\pi \sqrt{2} \left| \frac{\partial W_{3+1}(\sigma)}{\partial \sigma} \right|. \quad (9.4)$$

Note, that this potential still can be written in terms of certain two-dimensional superpotential $W_{1+1}(\sigma)$ via identification (6.1),

$$W_{1+1} = \sqrt{2\pi} \sqrt{2} \int d\sigma' \sqrt{\frac{\partial W_{3+1}}{\partial \sigma}}. \quad (9.5)$$

This fact shows the presence of $\mathcal{N} = (0, 2)$ supersymmetry in the world-sheet theory at the Lagrangian level, as was expected.

The simplest example is the quadratic superpotential

$$W_{3+1} = \mu_2 \text{Tr} \Phi^2, \quad (9.6)$$

which can be compared to the adjoint mass deformation (2.2) we studied before. For the superpotential (9.6) Eq. (9.4) gives

$$V_{1+1}(\sigma) = 4\pi \sqrt{2} |\mu_2 \sigma|. \quad (9.7)$$

In fact, (2.2) is not a single trace superpotential for generic $\mu_1$ and $\mu_2$. However, if we take

$$\mu_1 = \sqrt{\frac{N}{2}} \mu_2, \quad (9.8)$$
the superpotential (2.2) becomes equal to (9.6). It is worth observing that with this choice of \( \mu_1 \), the potential (9.7) coincides with that in Eq. (6.6).

The results we obtained in Sect. 7 suggest that the potential (9.4) correctly reproduces the quantum corrections \( O(\Lambda/m_A) \) to the classical string tensions. Namely, to reproduce the quantum corrections to (3.7) with \( \xi \)'s given in Eq. (9.3) we calculate (9.4) at the critical points \( \sigma_P \). The latter are given by the solutions of Eq. (7.2) (for \( N_f < 2N - 1 \)).

In the special case \( N_f = 2N - 1 \) we expect that, in order to reproduce the bulk quantum corrections, Eq. (9.4) should be modified as follows:

\[
V_{1+1}(\sigma) = 2\pi \sqrt{2} \left| \frac{\partial W_{3+1}}{\partial \sigma} \left( \sigma - \frac{\Lambda}{\sqrt{2N}} \right) \right|, \quad N_f = 2N - 1, \quad (9.9)
\]

see (7.10).

10 Conclusions

We studied heterotic deformations in the problem of the \( F \)-term stabilized non-Abelian strings. The bulk theory supporting these strings is \( \mathcal{N} = 2 \) SQCD with no Fayet–Iliopoulos \( D \)-term in which \( \mathcal{N} = 2 \) supersymmetry is broken down to \( \mathcal{N} = 1 \) by superpotentials of the type (2.2) (or generic higher order polynomials). In the limit of weak deformation we found the heterotic superpotential appearing in the weighted CP model on the string world sheet which breaks \( \mathcal{N} = (2,2) \) supersymmetry down to \( \mathcal{N} = (0,2) \). The latter world-sheet supersymmetry is further spontaneously broken at the tree level, generally speaking. Our results dramatically expand the class of heterotic models which are generated on the non-Abelian strings in the \( \mathcal{N} = 1 \) bulk theories.

The potential (6.6) linear in \( \mu \) is the leading order-potential. Say, in the case of equality of all squark masses it gives a common tension to all \( N \) strings. The world-sheet supersymmetry is \( \mathcal{N} = (2,2) \). The next-to-leading order terms in the potential are \( O(\mu^2) \). They break supersymmetry completely. Our finding is that if we restrict ourselves to the leading (linear) order in \( \mu \), but consider nondegenerate masses the world-sheet supersymmetry is explicitly broken down to \( \mathcal{N} = (0,2) \).
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Appendix A: U(2) with $N_f = 2$

In this Appendix we consider the simplest example: U(2) gauge theory with $N_f = 2$ flavors. We calculate the diagonal elements of the matrix $E$, see Eq. (4.5), given by

$$E = \frac{1}{2} \frac{\partial u_2}{\partial a} + \frac{\tau^3}{2} \frac{\partial u_2}{\partial a^3}$$

in this particular case.

The exact solution of the theory on the Coulomb branch relates the fields $a$ and $a^3$ to contour integrals running along the contours $\alpha_i$ ($i = 1, 2$) encircling the double roots $e_1$ and $e_2$ (in the anticlockwise direction), see (4.13) for $N = 2$. Using explicit expressions from [38, 39, 40, 41] and generalizing them to the U(N) case\footnote{This amounts to including derivatives with respect to $u_1$ and terms proportional to the average $e$ below (or $m$, which is related to $e$ via (4.15)). These terms are absent in SU(N) case considered in [38, 39, 40, 41].} we write

$$\frac{\partial \Phi^i}{\partial u_2} = \frac{1}{2} \frac{1}{2\pi i} \oint_{\alpha_i} \frac{dx}{y}, \quad \frac{\partial \Phi^i}{\partial u_1} = \frac{1}{2\pi i} \oint_{\alpha_i} \frac{dx}{y} [x - (e_1 + e_2)] ,$$

where we define

$$(\Phi_1, \ldots, \Phi_N) = \text{diag} \left( \frac{1}{2} a + T^{\tilde{a}} a^{\tilde{a}} \right),$$

which gives in the $N = 2$ case

$$a = \Phi_1 + \Phi_2, \quad a^3 = \Phi_1 - \Phi_2.$$
derivatives $\partial a/\partial u_1$, $\partial a^3/\partial u_1$, $\partial a/\partial u_2$ and $\partial a^3/\partial u_2$. Inverting this matrix we obtain the desired derivatives

$$\frac{\partial u_2}{\partial a} = e_1 + e_2, \quad \frac{\partial u_2}{\partial a^3} = e_1 - e_2.$$  \hspace{1cm} (A.5)

Finally, substituting this in (A.1) we get

$$\text{diag } E = (e_1, e_2).$$  \hspace{1cm} (A.6)

Then the result for the string tensions (4.4) ensues. Now $e_1$ and $e_2$ are the two double roots of the curve. For the sake of completeness we present their explicit form (see, for example, [33]),

$$\sqrt{2} e_{1,2} = -m \mp \sqrt{\frac{\Delta m^2_{12}}{4} + \Lambda^2}. \hspace{1cm} (A.7)$$

Substituting (A.7) in (4.4) we get tensions of two $Z_2$ strings in (4.16).

As was already mentioned, we can relax the weak-coupling conditions (2.12) and (2.13) and go to the strong-coupling domain (2.14). The theory undergoes a crossover transition, i.e. crosses curves of marginal stability (CMS), where certain states decay. Also, we pass through monodromies which change charges of other states. In this domain one can exploit a dual description. The dual to the original theory is weakly coupled $U(1)^2$ gauge theory of light dyons, see [33] for details. Two $U(1)$ gauge fields interacting with the light dyons are now $A_\mu$ and $B_\mu = 1/\sqrt{5}(A^3_\mu + 2A^3_D)$, where $A^{3D}_\mu$ is the dual gauge potential. Their scalar superpartners are

$$a, \quad b = \frac{1}{\sqrt{5}}(a^3 + 2a^3_D). \hspace{1cm} (A.8)$$

The charges of two light dyons with respect to these $U(1)$ fields are [33]

$$\left(\frac{1}{2}, \pm \frac{\sqrt{5}}{2}\right). \hspace{1cm} (A.9)$$

Repeating the same steps which led us to (4.4) in the dual theory we get

$$T_{BPS}^{1,2} = 4\pi\sqrt{2} \left| \mu_1 \frac{1}{N} \frac{\partial u_2}{\partial a} \pm \frac{\mu_2}{2\sqrt{5}} \frac{\partial u_2}{\partial b} \right|. \hspace{1cm} (A.10)$$
The monodromies mentioned above change the root pairing but does not change the factorized form of the curve \((4.13)\). Therefore, passing to the dual theory we have

\[
\frac{\partial u_2}{\partial a} \rightarrow \frac{\partial u_2}{\partial a} = e_1 + e_2, \quad \frac{\partial u_2}{\partial a^3} \rightarrow \frac{1}{\sqrt{5}} \frac{\partial u_2}{\partial b} = e_1 - e_2, \quad \tag{A.11}
\]

which gives us the same expressions \((4.4)\) and \((4.16)\) for the string tensions.

We see that our result \((4.4)\) for the \(Z_N\) string tensions can be analytically continued (as functions of \(\Delta m_{AB}\) and \(m\)) in the strong-coupling domain.

**Appendix B: \(U(3)\) with \(N_f = 5\)**

Let us now consider another example pertinent to \(N_f > N\), namely, the \(U(3)\) gauge theory with \(N_f = 5\). The matrix \(E\) now has the form

\[
E = \frac{1}{2} \frac{\partial u_2}{\partial a} + T^3 \frac{\partial u_2}{\partial a^3} + T^8 \frac{\partial u_2}{\partial a^8}, \tag{B.1}
\]

where

\[
T^3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^8 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{B.2}
\]

To determine the derivatives of \(u_2\) with respect to \(a\), \(a^3\) and \(a^8\) we use the exact solution of the theory on the Coulomb branch. Generalizing the \(SU(3)\) solution \[38, 39, 40, 41\] to the \(U(3)\) case we can write

\[
\frac{\partial \Phi^i}{\partial u_3} = \frac{1}{3} \frac{1}{2\pi i} \oint_{\alpha_i} \frac{dx}{y},
\]

\[
\frac{\partial \Phi^i}{\partial u_2} = \frac{1}{2} \frac{1}{2\pi i} \oint_{\alpha_i} \frac{dx}{y} \left[ x - \frac{1}{3} (e_1 + e_2 + e_3) \right],
\]

\[
\frac{\partial \Phi^i}{\partial u_1} = -\frac{1}{\pi i} \oint_{\alpha_i} \frac{dx}{y} \left[ x^2 - \frac{1}{2} x (e_1 + e_2 + e_3) + \frac{1}{9} (e_1 + e_2 + e_3)^2 \right] + 1. \tag{B.3}
\]
The above integrals are readily calculable in the case of the factorized Seiberg–Witten curve (4.13). This gives us the derivatives of \(a, a^3\) and \(a^8\) with respect to \(u_1, u_2\) and \(u_3\). Inverting the matrix of these derivatives is an algebraic albeit tedious calculation. Omitting all details we present the final answer,

\[
\frac{\partial u_2}{\partial a} = e_1 + e_2 + e_3, \quad \frac{\partial u_2}{\partial a^3} = e_1 - e_2, \quad \frac{\partial u_2}{\partial a^8} = \frac{1}{\sqrt{3}}(e_1 + e_2 - 2e_3). \quad (B.4)
\]

Substituting this in Eq. (B.1) we get

\[
\text{diag } E = (e_1, e_2, e_3), \quad (B.5)
\]

which leads us to the final expression (4.4) for the \(Z_3\) string tensions, where now \(e_P\)'s are the double roots of the Seiberg–Witten curve (4.13).

It is instructive to give more explicit expressions for the string tensions in terms of \(m_A\). To this end we consider a particular mass choice,

\[
m_1 = m_4, \quad m_2 = m_5. \quad (B.6)
\]

With this choice, the roots of the curve can be easily found [31],

\[
\sqrt{2}e_1 = -\tilde{m}_1, \quad \sqrt{2}e_2 = -\tilde{m}_2, \quad \sqrt{2}e_3 = -\tilde{m}_3 + \Lambda, \quad (B.7)
\]

where \(\tilde{m}_A\) are defined in (4.10) for the special case \(N_f = 2N - 1\). Substituting this in (4.4) we get the tensions of the three \(Z_3\) strings,

\[
T_1^{BPS} = 4\pi \left| \sqrt{\frac{2}{3}} \mu_1 m + \mu_2 \left( m_1 - m + \frac{\Lambda}{3} \right) \right|, \\
T_2^{BPS} = 4\pi \left| \sqrt{\frac{2}{3}} \mu_1 m + \mu_2 \left( m_2 - m + \frac{\Lambda}{3} \right) \right|, \\
T_3^{BPS} = 4\pi \left| \sqrt{\frac{2}{3}} \mu_1 m + \mu_2 \left( m_3 - m - \frac{2}{3} \Lambda \right) \right|. \quad (B.8)
\]

Note that the relation (4.15) is still fulfilled due to the mass shifts in (4.10) which must be done if \(N_f = 2N - 1\).
Similarly to the $N = N_f = 2$ case we can continue our theory to the strong-coupling domain (2.14), where it is described by the non-Abelian theory with the dual gauge group $U(2) \times U(1)$ and $N_f = 5$ flavors of light dyons [31]. The latter is not asymptotically free. Using the low-energy effective action of this theory found previously in [31] it is straightforward to demonstrate that, although the light state charges are changed due to monodromies in the strong-coupling domain, the resulting expressions (B.8) for the string tensions experience no change. Thus, the string tensions have analytic behavior in $m_A$ across the crossover lines, much in the same as as in the example we dealt with in Appendix A.

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