GLOBAL MAXIMUM PRINCIPLES AND DIVERGENCE
THEOREMS ON COMPLETE MANIFOLDS WITH
BOUNDARY

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Abstract. In this paper we extend to non-compact Riemannian manifolds with boundary the use of two important tools in the geometric analysis of compact spaces, namely, the weak maximum principle for subharmonic functions and the integration by parts. The first one is a new form of the classical Ahlfors maximum principle whereas the second one is a version for manifolds with boundary of the so called Kelvin-Nevanlinna-Royden criterion of parabolicity. In fact, we will show that the validity of non-compact versions of these tools serve as a characterization of the Neumann parabolicity of the space.

The motivation underlying this study is to obtain new information on the geometry of graphs with prescribed mean curvature inside a Riemannian product of the type $N \times \mathbb{R}$. In this direction two kind of results will be presented: height estimates for constant mean curvature graphs parametrized over unbounded domains in a complete manifold and slice type results for graphs whose superlevel sets have finite volume.

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Introduction

This paper aims at extending to non-compact Riemannian manifolds with boundary the use of two important tools in the geometric analysis of compact spaces, namely, the integration by parts and the weak maximum principle
for subharmonic functions. The motivation underlying this study is mainly the attempt to obtain new information on the geometry of graphs or, more generally, of hypersurfaces with boundary and prescribed mean curvature inside a Riemannian product of the type $N \times \mathbb{R}$.

In the setting of Riemannian manifolds without boundary, it is by now well known that parabolicity represents a good substitute of the compactness of the underlying space, see e.g. the account in [28]. Thus, in order to extend the use of the classical tools alluded to above, we are naturally to a deeper study of parabolicity for manifolds with boundary. As we shall see in Appendix A there are several concepts of parabolicity in this setting and they are in a certain hierarchy, so one has to make a choice. In view of our geometric purposes we decided to follow the more traditional path, [9, 10, 11, 12], that, from the stochastic viewpoint, translates into the property that the reflected Brownian motion be recurrent. This is the strongest of the notions of parabolicity known in the literature, but it is also the one which seems to be more related to the geometry of the space. Thus, for instance, every proper minimal graph over a smooth domain of $\mathbb{R}^2$ is parabolic in our traditional sense because of its area growth property; see Appendix A. In order to put the precise definition of parabolicity we need to recall the notion of weak sub (super) solution subjected to Neumann boundary conditions.

Let $(M, g)$ be an oriented Riemannian manifold with smooth boundary $\partial M \neq \emptyset$ and exterior unit normal $\nu$. By a domain in $M$ we mean a non-necessarily connected open set $D \subseteq M$. We say that the domain $D$ is smooth if its topological boundary $\partial D$ is a smooth hypersurface $\Gamma$ with boundary $\partial \Gamma = \partial D \cap \partial M$. Clearly, if $\partial M = \emptyset$ then the smoothness condition reduces to the usual one. It is a standard fact that every manifold $M$ with (possibly empty) boundary has an exhaustion by smooth pre-compact domains. Simply choose a proper smooth function $\rho : M \to \mathbb{R}_{\geq 0}$ and, according to Sard theorem, take a sequence $\{t_k\} \nearrow +\infty$ such that $t_k$ is a regular value for both $\rho|_{\text{int} M}$ and $\rho|_{\partial M}$. Then $D_k = \{\rho < t_k\}$ defines the desired exhaustion with smooth boundary $\partial D_k = \{\rho = t_k\}$.

Adopting a notation similar to the one in [10], for any domain $D \subseteq M$ we define

$$\partial_0 D = \partial D \cap \text{int} M.$$ 

Note also that $D$ could include part of the boundary of $M$. We therefore set

$$\partial_1 D = \partial M \cap D.$$ 

Now, suppose $D \subseteq M$ is any domain. We put the following
Definition 0.1. By a weak Neumann solution $u \in W^{1,2}_{\text{loc}}(D)$ of the problem

\[
\begin{align*}
\Delta u & \geq 0 \quad \text{on } D \\
\frac{\partial u}{\partial \nu} & \leq 0 \quad \text{on } \partial_1 D,
\end{align*}
\]

we mean that the following inequality

\[
- \int_D \langle \nabla u, \nabla \varphi \rangle \geq 0
\]

holds for every $0 \leq \varphi \in C^\infty_c(D)$. Similarly, by taking $D = M$, one defines the notion of weak Neumann subsolution of the Laplace equation on $M$ as a function $u \in W^{1,2}_{\text{loc}}(M)$ which satisfies (2) for every $0 \leq \varphi \in C^\infty_c(M)$. As usual, the notions of weak supersolution and weak solution can be obtained by reversing the inequality or by replacing the inequality with an equality in (2), and removing the sign condition on $\varphi$.

Remark 0.2. Clearly, in the above definition, it is equivalent to require that (2) holds for every $0 \leq \varphi \in \text{Lip}_c(M)$. Note also that standard density arguments work even for manifolds with boundary and, therefore, (2) extends to all compactly supported $0 \leq \varphi \in W^{1,2}_0(D)$. Here, as usual, $W^{1,2}_0(D)$ denotes the closure of $C^\infty_c(D)$ with respect to the $W^{1,2}$-norm.

Remark 0.3. Note that in the equality case we have the usual notion of variational solution of the mixed problem

\[
\begin{align*}
\Delta u & = 0 \quad \text{on } D \\
\frac{\partial u}{\partial \nu} & = 0 \quad \text{on } \partial_1 D \\
u & = 0 \quad \text{on } \partial_0 D.
\end{align*}
\]

Remark 0.4. If $\partial M = \emptyset$ or, more generally, $D \subseteq \text{int}M$, the Neumann condition disappears and we recover the usual definition of weak sub- (super-)solution. Obviously, in the smooth setting, a classical solution of (1) is also a weak Neumann subsolution as one can verify using integration by parts. Actually, this is true in a more general setting. See Definition 3.3 and Lemma 3.4 in Subsection 3.1.

We are now ready to give the following definition of parabolicity in the form of a Liouville-type result.

Definition 0.5. An oriented Riemannian manifold $M$ with boundary $\partial M \neq \emptyset$ is said to be parabolic if any bounded above, weak Neumann subsolution of the Laplace equation on $M$ must be constant. Explicitly, for every $u \in C^0(M) \cap W^{1,2}_{\text{loc}}(M)$,

\[
\begin{align*}
\Delta u & \geq 0 \quad \text{on } M \\
\frac{\partial u}{\partial \nu} & \leq 0 \quad \text{on } \partial M \\
\sup_M u & < +\infty
\end{align*}
\]

⇒ $u \equiv \text{const}$. 
It is known from [10] that, in case $M$ is complete with respect to the intrinsic distance function $d$, then geometric conditions implying parabolicity rely on volume growth properties of the space. In order to give the precise statement it is convenient to introduce some notation. Having fixed a reference origin $o \in \int M$, we set $B^M_R(o) = \{x \in M : d(x,o) < R\}$ and $\partial B^M_R(o) = \{x \in M : d(x,o) = R\}$, the metric ball and sphere of $M$ centered at $o$ and of radius $R > 0$. We also denote by $r(x) = d(x,o)$ the distance function from $o$. Clearly, $r(x)$ is Lipschitz, hence differentiable a.e. in $\int M$. Moreover, for a.e. $x \in \int M$, differentiating $r$ along a minimizing geodesic from $o$ to $x$ (which exists by completeness) we easily see that the usual Gauss Lemma holds, namely, $|\nabla r| = 1$ a.e. in $\int M$. Therefore, by the co-area formula applied to $r|_{\int M}$ and the fact that $\text{vol} B^M_R(o) = \text{vol} (B^M_R(o) \cap \int M)$, we have

$$ \frac{d}{dR} \text{vol} B^M_R(o) = \text{Area} (\partial_0 B^M_R(o)), $$

for a.e. $R > 0$.

The following result is due to Grigor’yan [10]. For a proof in the $C^2$ case see Theorem 3.6 and Remark 3.8.

**Theorem 0.6.** Let $(M,g)$ be a complete Riemannian manifold with boundary $\partial M \neq \emptyset$. If, for some reference point $o \in M$, either

$$ \frac{R}{\text{vol} B^M_R(o)} \notin L^1(+\infty) $$

or

$$ \frac{1}{\text{Area} (\partial_0 B^M_R(o))} \notin L^1(+\infty) $$

then $M$ is parabolic.

It is a usual consequence of the co-area formula that the area growth condition is weaker than the volume growth condition. On the other hand, the volume growth condition is more stable with respect to (even rough) perturbations of the metric and sometimes it characterizes the parabolicity of the space. Therefore, both are important.

The first main result of the paper is the following maximum principle characterization of parabolicity. It extends to manifolds with boundary a classical result by L.V. Ahlfors.

**Theorem 0.7** (Ahlfors maximum principle). $M$ is parabolic if and only the following maximum principle holds. For every domain $D \subseteq M$ with $\partial_0 D \neq \emptyset$ and for every $u \in C^0(D) \cap W^{1,2}_{\text{loc}}(D)$ satisfying, in the weak Neumann sense,

$$ \begin{cases} 
\Delta u \geq 0 & \text{on } D \\
\frac{\partial u}{\partial \nu} \leq 0 & \text{on } \partial_1 D \\
\sup_D u < +\infty
\end{cases} $$
it holds

\[ \sup_D u = \sup_{\partial_0 D} u. \]

It is worth to observe that, in case \( D = M \), the Neumann boundary condition plays no role and the result takes the following form which is crucial in the applications.

**Theorem 0.8.** Let \( M \) be a parabolic manifold with boundary \( \partial M \neq \emptyset \). If \( u \in C^0 (M) \cap W^{1,2}_{\text{loc}} (\text{int} M) \) satisfies

\[
\begin{cases}
\Delta u \geq 0 & \text{on } \text{int} M \\
\sup_M u < +\infty
\end{cases}
\]

then

\[ \sup_M u = \sup_{\partial M} u. \]

It is not surprising that this global maximum principle proves to be very useful to get height estimates for constant mean curvature hypersurfaces in product spaces. By way of example, we point out the following

**Theorem 0.9 (Height estimate).** Let \( N \) be a Riemannian manifold without boundary and Ricci curvature satisfying \( \text{Ric}_N \geq 0 \). Let \( \Sigma \) be a complete, oriented hypersurface in \( N \times \mathbb{R} \) with boundary \( \partial \Sigma \neq \emptyset \) and satisfying the following requirements:

(i) \( \Sigma \) has quadratic intrinsic volume growth

(4) \( \text{vol} B_R^\Sigma (o) = O (R^2) \), as \( R \to +\infty \);

(ii) \( \partial \Sigma \) is contained in the slice \( N \times \{0\} \);

(iii) For a suitable choice of the Gauss map \( N \) of \( \Sigma \), the hypersurface \( \Sigma \) has constant mean curvature \( H > 0 \) and the angle \( \Theta \) between \( N \) and the vertical vector field \( \partial / \partial t \) is contained in the interval \( \left[ \frac{\pi}{2}, \frac{3\pi}{2} \right] \), i.e.,

\[ \cos \Theta = \left\langle N, \frac{\partial}{\partial t} \right\rangle \leq 0. \]

If \( \Sigma \) is contained in a slab \( N \times [-T, T] \) for some \( T > 0 \), then

\[ \Sigma \subseteq N \times \left[ 0, \frac{1}{H} \right]. \]

We observe explicitly that (4) can be replaced by the stronger extrinsic condition

\[ \text{vol} \left( B^N_R (o) \cap \Sigma \right) = O (R^2) \), as \( R \to +\infty \),

which, in turn, follows from the relation

\[ B^\Sigma_R (o) \subseteq B^N_R (o) \cap \Sigma. \]

We also note that there are important situations where the assumption on the Gauss map is automatically satisfied and the volume growth condition on the hypersurface is inherited from that of the ambient space. The following
height estimate extends previous results for $H$-graphs over non-compact domains ([14], [15], [1], [33]).

**Theorem 0.10** (Height estimate for graphs). Let $(N, g)$ be a complete, Riemannian manifold without boundary satisfying $\text{Ric}_N \geq 0$ and

$$\text{vol} B^N_R (o) = O (R^2), \text{ as } R \to +\infty.$$  

Let $M \subset N$ be a closed domain with smooth boundary $\partial M \neq \emptyset$. Suppose we are given a graph $\Sigma$ over $M$ with boundary $\partial \Sigma \subset M \times \{0\}$ and constant mean curvature $H > 0$ with respect to the downward Gauss map. If $\Sigma$ is contained in a slab, then

$$\Sigma \subseteq M \times \left[0, \frac{1}{H}\right].$$

In the particular case of graphs over a domain of a surface of non-negative Gauss curvature we obtain the following result that extends to non-homogeneous surfaces Theorem 4 in [31].

**Corollary 0.11.** Let $(N, g)$ be a complete 2-dimensional Riemannian manifold without boundary of non-negative Gauss curvature. Let $M \subset N$ be a closed domain with smooth boundary $\partial M \neq \emptyset$. Suppose we are given a graph $\Sigma$ over $M$ with boundary $\partial \Sigma \subset M \times \{0\}$ and constant mean curvature $H > 0$ with respect to the downward Gauss map. Then

$$\Sigma \subseteq M \times \left[0, \frac{1}{H}\right].$$

In the setting of manifolds without boundary, it is well known from a classical work by T. Lyons and D. Sullivan [23] that the validity of an $L^2$-divergence theorem is related, and in fact equivalent, to the parabolicity of the space. We shall complete the picture by extending the $L^2$-divergence theorem to non-compact manifolds with boundary.

**Theorem 0.12** ($L^2$-divergence theorem). Let $M$ be a parabolic Riemannian manifold with boundary $\partial M \neq \emptyset$ and outward pointing unit normal $\nu$. Then $M$ is parabolic if and only if the following holds. Let $X$ be a vector field on $M$ satisfying the following conditions:

$$(a) \ |X| \in L^2 (M)$$

$$(b) \ \langle X, \nu \rangle \in L^1 (\partial M)$$

$$(c) \ \text{div} \ X \in L^1_{\text{loc}} (M), \ (\text{div} \ X)_- \in L^1 (M).$$

Then

$$\int_M \text{div} \ X = \int_{\partial M} \langle X, \nu \rangle.$$  

A weaker version of the $L^2$-divergence theorem, involving solutions $X$ of inequalities of the type $\text{div} \ X \geq f$ with boundary conditions $\langle X, \nu \rangle \leq 0$, will be employed in our investigations on hypersurfaces in product spaces; see
Proposition 3.5. In particular, from this latter we shall obtain the following result for hypersurfaces contained in a half-space of $N \times \mathbb{R}$.

**Theorem 0.13 (Slice theorem).** Let $N$ be a Riemannian manifold without boundary. Let $\Sigma \subset N \times [0, +\infty)$ be a complete, oriented hypersurface with boundary $\partial \Sigma \neq \emptyset$ contained in the slice $N \times \{0\}$ and satisfying the volume growth condition

$$\text{vol} B_R^\Sigma (o) = O (R^2), \quad \text{as } R \to +\infty.$$ 

Assume that, for a suitable choice of the Gauss map $N$ of $\Sigma$, the hypersurface $\Sigma$ has non-positive mean curvature $H(x) \leq 0$ and the angle $\Theta$ between $N$ and the vertical vector field $\partial/\partial t$ is contained in the interval $[\frac{\pi}{2}, \frac{3\pi}{2}]$, i.e.,

$$\cos \Theta = \left\langle N, \frac{\partial}{\partial t} \right\rangle \leq 0.$$

If there exists some half-space $N \times [t, +\infty)$ of $N \times \mathbb{R}$ such that

$$\text{vol} (\Sigma \cap N \times [t, +\infty)) < +\infty,$$

then $\Sigma \subset N \times \{0\}$.

In case $\Sigma$ is given graphically over a parabolic manifold $M$, we shall obtain the following variant of the slice theorem that involves the volumes of orthogonal projections of $\Sigma$ on $M$. Its proof requires a Liouville-type theorem for the mean curvature operator under volume growth conditions; see Theorem 3.6.

**Theorem 0.14 (Slice theorem for graphs).** Let $M$ be a complete manifold with boundary $\partial M \neq \emptyset$, outward pointing unit normal $\nu$, and (at most) quadratic volume growth, i.e.,

$$\text{vol} B_R^M (o) = O (R^2), \quad \text{as } R \to +\infty,$$

for some origin $o \in M$. Let $\Sigma$ be a graph over $M$ with non-positive mean curvature $H(x) \leq 0$ with respect to the orientation given by the downward pointing Gauss map $N(x)$. Assume that $\partial \Sigma \cap M \times \{T\} = \emptyset$ for some $T > 0$ and that at least one of the following conditions is satisfied:

(a) $\partial \Sigma = \partial M \times \{0\}$ and $\Sigma \subset M \times [0, +\infty)$.
(b) $M$ and $\Sigma$ are real analytic.
(c) On $\partial \Sigma$, the Gauss map $N(x)$ of $\Sigma$ and the Gauss map $N_0(x) = (-\nu(x), 0)$ of the boundary $\partial M \times \{t\}$ of any slice form an angle $\theta(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

If the portion of the graph $\Sigma$ contained in some half-space $M \times [t, +\infty)$ has finite volume projection on the slice $M \times \{0\}$, then $\Sigma$ is a horizontal slice of $M \times \mathbb{R}$.

It is worth to point out that, in the setting of manifolds without boundary and for $H = 0$, half-space properties in a spirit similar to our slice-type theorems have been obtained in the very recent paper [32] by H. Rosenberg,
F. Schulze and J. Spruck. More precisely, they are able to show that curvature restrictions and potential theoretic properties (parabolicity) of the base manifold $M$ in the ambient product space $M \times \mathbb{R}$ force properly immersed minimal hypersurfaces and entire minimal graphs in a half-space to be totally geodesic slices. This holds without any further condition on their superlevel sets.

The paper is organized as follows. In Section 1 we recall the link between parabolicity and absolute capacity of compact subsets. We also take the occasion to give a detailed proof of the existence and regularity of the equilibrium potentials of condensers in the setting of manifolds with boundary. These rely on the solution of mixed boundary value problems in non-smooth domains. Section 2 contains the proof of the maximum principle characterization of parabolicity and its applications to obtain height estimates for complete CMC hypersurfaces with boundary into Riemannian products. In Section 3 we relate the parabolicity of a manifold with boundary to the validity of the $L^2$-Stokes theorem. We also provide a weak form of this result that applies to get slice-type results for hypersurfaces with boundary in Riemannian products. Further slice-type results that are based on Liouville-type theorem for graphs are also given. In the final Appendix we survey, and compare, different notions of parabolicity for manifolds with boundary. We also exemplify how the results of this paper can be applied in the setting of minimal surfaces. In particular, we recover, with a deterministic proof, a result by R. Neel on the parabolicity of minimal graphs.

In conclusion of this introductory part we mention that there are natural and interesting applications and extensions of the the results obtained in this paper both to Killing graphs and to the $p$-Laplace operator.

These aspects will be presented in the forthcoming papers [18] and [19], respectively.

1. Capacity & equilibrium potentials

As in the case where $M$ has no boundary, given a compact set $K$ and an open set $\Omega$ containing $K$ the capacity of the condenser $(K, \Omega)$ is defined by

$$\text{cap}(K, \Omega) = \inf \{ \int_{\Omega} |\nabla u|^2 : u \in C^\infty_c(\Omega), u \geq 1 \text{ on } K \}.$$  

When $\Omega = M$, we write $\text{cap}(K, M) = \text{cap}(K)$ and we refer to it as the (absolute) capacity of $K$.

A simple approximation argument shows that the infimum on the right hand side can be equivalently computed letting $u$ range over the set

$$\{ u \in \text{Lip}_c(\Omega) : u = 1 \text{ on } K \}$$

or even over

$$W_0(K, \Omega) = \{ u \in C(\overline{\Omega}) \cap W^{1,2}_0(\Omega) : u = 1 \text{ on } K \},$$
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where \( W^{1,2}_0(\Omega) = C^\infty_c(\Omega) \). We refer to functions in \( W_0(K, \Omega) \) as admissible potentials for the condenser \( (K, \Omega) \).

The usual monotonicity properties of capacity hold, namely, if \( K \subseteq K_1 \) are compact sets and \( \Omega \subseteq \Omega_1 \) are open, then \( \text{cap}(K, \Omega) \leq \text{cap}(K_1, \Omega_1) \leq \text{cap}(K_1, \Omega) \) and this allows to define first the capacity of an open set \( U \subset \Omega \) as \( \text{cap}(U, \Omega) = \sup_{U \supset K, \text{compact}} \text{cap}(K, \Omega) \) and then the capacity of an arbitrary set \( E \subset \Omega \) as \( \text{cap}(E, \Omega) = \inf_{E \subset \Omega} \text{cap}(U, \Omega) \).

We are going to show that the Liouville-type definition of parabolicity given in the introduction is equivalent to the statement that every compact subset has zero capacity. This depends on the construction of equilibrium potentials for capacity, which plays a vital role also in the proof of the \( L^2 \) divergence theorem characterization of parabolicity, Theorem 0.12. It should be pointed out that while these results are in some sense well known, we haven’t been able to find a reference which deals explicitly with matters concerning regularity up to the boundary of these equilibrium potentials.

The following simple lemma will be useful in the proof of the proposition.

**Lemma 1.1.** Let \( D \subset \Omega \) be open sets, and let \( D_n \) and \( \Omega_n \) be a sequence of open sets such that

\[
\overline{D} \subset D_{n+1} \subset D_n \subset \overline{D_n} \subset \Omega_n \subset \Omega_{n+1} \subset \Omega, \quad \cap_n D_n = D, \quad \cup_n \Omega_n = \Omega.
\]

Then

\[
(6) \quad \lim_n \text{cap}(\overline{D_n}, \Omega_n) = \text{cap}(\overline{D}, \Omega).
\]

**Proof.** It follows from monotonicity that, for every \( n \), \( \text{cap}(\overline{D_n}, \Omega_n) \) is monotonically decreasing and greater than or equal to \( \text{cap}(\overline{D}, \Omega) \) so the limit on the left hand side of (6) exists and

\[
\lim_n \text{cap}(\overline{D_n}, \Omega_n) \geq \text{cap}(\overline{D}, \Omega).
\]

For the converse, let \( \phi \in \text{Lip}_c(\Omega) \) with \( \phi = 1 \) on \( \overline{D} \), and for \( \epsilon > 0 \) let

\[
\phi_\epsilon = \min \left\{ 1, \left( \frac{\phi - \epsilon}{1 - 2\epsilon} \right)_+ \right\}.
\]

By assumption, for every sufficiently large \( n \) we have

\[
\overline{D_n} \subset \{ x : \epsilon \leq \phi(x) \leq 1 - \epsilon \} \subset \Omega_n,
\]

and therefore \( \phi_\epsilon \) is an admissible potential for the condenser \( (\overline{D_n}, \Omega_n) \) so that

\[
\int |\nabla \phi_\epsilon|^2 \geq \text{cap}(\overline{D_n}, \Omega_n),
\]

whence, letting \( n \to \infty \),

\[
\lim_n \text{cap}(\overline{D_n}, \Omega_n) \leq \int |\nabla \phi_\epsilon|^2 \quad \forall \epsilon > 0.
\]
On the other hand, by monotone convergence, 
\[ \int |\nabla \phi_\epsilon|^2 = \frac{1}{(1 - 2\epsilon)^2} \int_{\{x : \epsilon \leq \phi(x) \leq 1 - \epsilon\}} |\nabla \phi|^2 \to \int_\Omega |\nabla \phi|^2 \quad \text{as} \quad \epsilon \to 0, \]
and we conclude that 
\[ \lim_n \text{cap}(D_n, \Omega_n) \leq \int_\Omega |\nabla \phi|^2, \]
which in turn implies that 
\[ \lim_n \text{cap}(D_n, \Omega_n) \leq \text{cap}(D, \Omega). \]

\[ \square \]

**Proposition 1.2.** Let \( D \subseteq \Omega \) be relatively compact domains with smooth boundaries \( \partial_0 D \) and \( \partial_0 \Omega \) transversal to \( \partial M \). Then there exists \( u \in W_0^1(D, \Omega) \cap C^\infty((\Omega \setminus D) \cup \partial_1(\Omega \setminus D)) \) such that \( 0 \leq u \leq 1 \) and 
\[ \text{cap}(D, \Omega) = \int_\Omega |\nabla u|^2. \]

**Proof.** Consider the mixed boundary value problem
\[ \begin{cases} 
\Delta u = 0 \quad \text{in} \; \Omega \setminus D \\
\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \; \partial_1(\Omega \setminus D) \\
u = 0 \; \text{on} \; \partial_0 \Omega, \; u = 1 \; \text{on} \; \partial_0 D.
\end{cases} \]
(7)

If follows from [21], and the well known local regularity theory, that (7) has a classical solution \( u \in C(\Omega \setminus D) \cap C^\infty((\Omega \setminus D) \cup \partial_1(\Omega \setminus D)) \). By the strong maximum principle and the boundary point lemma, it follows that \( 0 < u < 1 \) on \( \Omega \setminus D \). We extend \( u \) to \( \Omega \) by setting it equal to 1 on \( D \). To show that \( u \in W^{1,2}(\Omega) \), choose \( \epsilon \in (0, 1) \) such that \( \epsilon \) and \( 1 - \epsilon \) are regular values of \( u \), and let \( \Omega_\epsilon = \{ x : u(x) \geq \epsilon \} \), \( D_\epsilon = \{ x : u(x) < 1 - \epsilon \} \) and
\[ u_\epsilon = \frac{u - \epsilon}{1 - 2\epsilon}, \]
so that \( u_\epsilon \in C^2(\Omega_\epsilon \setminus D_\epsilon) \) satisfies
\[ \begin{cases} \Delta u_\epsilon = 0 \quad \text{in} \; \Omega_\epsilon \setminus D_\epsilon \\
\frac{\partial u_\epsilon}{\partial \nu} = 0 \quad \text{on} \; \partial_1(\Omega_\epsilon \setminus D_\epsilon) \\
u_\epsilon = 0 \; \text{on} \; \partial_0 \Omega_\epsilon, \; u_\epsilon = 1 \; \text{on} \; \partial_0 D_\epsilon.
\end{cases} \]

By the usual Dirichlet principle \( u_\epsilon \) is the equilibrium potential of the capacitor \((D_\epsilon, \Omega_\epsilon)\), and, in particular,
\[ \frac{1}{1 - 2\epsilon} \int_{\Omega_\epsilon \setminus D_\epsilon} |\nabla u|^2 = \int_{\Omega_\epsilon \setminus D_\epsilon} |\nabla u_\epsilon|^2 = \text{cap}(D_\epsilon, \Omega_\epsilon). \]
Indeed, let \( \phi \in \text{Lip}_c(\Omega_\epsilon) \) with \( \phi = 1 \) on \( D_\epsilon \), and let \( v = u_\epsilon - \phi \). Then \( \phi = u_\epsilon - v \) and we have
\[ \int_{\Omega_\epsilon} |\nabla \phi|^2 = \int_{\Omega_\epsilon \setminus D_\epsilon} |\nabla u_\epsilon - v|^2 = \int_{\Omega_\epsilon \setminus D_\epsilon} (|\nabla u_\epsilon|^2 + |\nabla v|^2 - 2(\nabla u_\epsilon, \nabla v)) \]
Since $\Delta u_\epsilon = 0$ on $\Omega_\epsilon \setminus D_\epsilon$ and $v = 0$ on $\partial_0(\Omega_\epsilon \setminus D_\epsilon)$ while $\partial u_\epsilon / \partial \nu = 0$ on $\partial_1(\Omega_\epsilon \setminus D_\epsilon)$,
\[
\int_{\Omega_\epsilon \setminus D_\epsilon} \langle \nabla u_\epsilon, \nabla v \rangle = -\int_{\Omega_\epsilon \setminus D_\epsilon} v \Delta u_\epsilon + \int_{\partial_0(\Omega_\epsilon \setminus D_\epsilon) \cup \partial_1(\Omega_\epsilon \setminus D_\epsilon)} \langle \nabla u_\epsilon, \nu \rangle v = 0,
\]
so that
\[
\int_{\Omega_\epsilon \setminus D_\epsilon} |\nabla \phi|^2 = \int_{\Omega_\epsilon \setminus D_\epsilon} (|\nabla u_\epsilon|^2 + |\nabla v|^2) \geq \int_{\Omega_\epsilon \setminus D_\epsilon} |\nabla u_\epsilon|^2,
\]
as claimed.

Letting $\epsilon \to 0$ $\Omega_\epsilon \setminus D_\epsilon \to \Omega \setminus D$, so that, by monotone convergence, the integral in (8) converges to
\[
\int_{\Omega \setminus D} |\nabla u|^2.
\]
On the other hand, by the previous lemma,
\[
\text{cap}(\overline{D}_\epsilon, \Omega_\epsilon) \to \text{cap}(\overline{D}, \Omega), \quad \text{as } \epsilon \to 0
\]
and we conclude that $u \in W^{1,2}(\Omega)$ so that, in fact, $u \in W_0(\overline{D}, \Omega)$ and
\[
\int_{\Omega} |\nabla u|^2 = \text{cap}(\overline{D}, \Omega),
\]
as required to complete the proof. \hfill \Box

**Remark 1.3.** It is worth to point out that the equilibrium potential $u$ of the capacitor $(\overline{D}, \Omega)$ constructed using Liebermann approach coincides with the one obtained by applying the direct calculus of variations to the energy functional on the closed convex space
\[
W_1^{1,2}(\Omega \setminus \overline{D}) = \{ u \in W^{1,2}(\Omega) : u|_{\partial_0 D} = 0 \text{ and } u|_{\partial_1 \Omega} = 1 \}.
\]
Here, Dirichlet data are understood in the trace sense. Thanks to the global $W^{1,2}$-regularity established in Proposition 1.2, this follows e.g. either from maximum principle considerations or from the convexity of the energy functional.

**Proposition 1.4.** Let $D$ be a relatively compact domain and let $\Omega_j$ be an increasing exhaustion of $\overline{M}$ by relatively compact open domains with $\overline{D} \subset \Omega_1$. Assume that $\partial_0 D$ and $\partial_0 \Omega_j$ are smooth and transversal to $\partial M$, and for every $j$, let $u_j$ be the equilibrium potential of the capacitor $(\overline{D}, \Omega_j)$ constructed in Proposition 1.2. Then $u_j$ converges monotonically to a function $u \in C(M) \cap W^{1,2}_{\text{loc}} \cap C^2(M \setminus \overline{D})$ such that $0 \leq u \leq 1$, $u = 1$ on $\overline{D}$, $u$ is harmonic on $M \setminus \overline{D}$, $\partial u / \partial \nu = 0$ on $\partial_1(M \setminus \overline{D})$ and $u$ is a weak Neumann supersolution of the Laplace equation on $M$. Moreover $\nabla u \in L^2(M)$,
\[
\text{cap}(\overline{D}) = \int_M |\nabla u|^2.
\]
Proof. Extend \( u_j \) to all of \( M \) by setting it equal to zero in \( M \setminus \Omega_j \). It follows by the comparison principle that \( 0 \leq u_j \leq u_{j+1} \leq 1 \) in \( \Omega_j \setminus \overline{D} \), and therefore the sequence \( u_j \) converges monotonically to a function \( u \). Note that since \( u_j(x) \leq u(x) \leq 1 \) and \( u_j(x) \to 1 \) as \( x \to y \in \partial_0 D \) is follows that \( u \) is continuous on \( \overline{D} \) and there it is equal to 1. Moreover, by the Schauder type estimate contained in Lemma 1 in [21], for every \( \alpha \in (0, 1) \), every \( j_o \) and every sufficiently small \( \eta > 0 \) there exists a constant \( C \) depending only on \( \alpha \), \( \eta \), \( j_o \) and on the geometry of \( M \) in a neighborhood of \( B_{j_o, \eta} = \{ x \in \Omega_{j_o} \setminus \overline{D} : \text{dist}(x, \partial_0 D \cup \partial_0 \Omega_{j_o}) \geq \eta \} \) such that, for every \( j \geq j_o \),

\[
\|u_j\|_{C^{2,\alpha}(B_{\eta})} \leq C \sup_{B_{\eta/2}} |u_j(x)|.
\]

It follows immediately that (possibly passing to a subsequence) the sequence \( u_j \) converges in \( C^2(B_{j_o, \eta}) \) for every \( j_o \) and \( \eta > 0 \) so that the limit function \( u \) is harmonic in \( \text{int} M \setminus \overline{D} \) and \( C^2 \) up to \( \partial_1(M \setminus \overline{D}) \) where it satisfies the Neumann boundary condition \( \partial u / \partial \nu = 0 \). Summing up, \( u \in C^0(M \setminus D) \cap C^2((M \setminus \overline{D}) \cup (\partial_1(M \setminus \overline{D})) \) is a classical solution of the mixed boundary problem

\[
\begin{aligned}
\Delta u &\geq 0 \quad \text{on} \quad M \setminus \overline{D} \\
\frac{\partial u}{\partial \nu} &\leq 0 \quad \text{on} \quad \partial_1(M \setminus \overline{D}) \\
u &\in \partial_0 D \\
0 &\leq u \leq 1.
\end{aligned}
\]

On the other hand, since

\[
\int_{\Omega_j} |\nabla u_j|^2 = \text{cap}(\overline{D}, \Omega_j) \setminus \text{cap}(\overline{D}),
\]

the sequence \( u_j \in \in C^0(M) \cap W^{1,2}_c(M) \) converges pointwise to \( u \) and \( \nabla u_j \) is bounded in \( L^2(M) \). It follows easily (see, e.g., Lemma 1.33 in [13]) that \( \nabla u \in L^2(M) \) and \( \nabla u_j \rightharpoonup \nabla u \) weakly in \( L^2 \). By the weak lower semicontinuity of the energy functional, it follows that

\[
\int_M |\nabla u|^2 \leq \liminf_j \int_M |\nabla u_j|^2 = \text{cap}(\overline{D})
\]

On the other hand, By Mazur’s Lemma, a convex combination \( \tilde{u}_j \) of the \( u_j \) is such that \( \nabla \tilde{u}_j \rightharpoonup \nabla u \) strongly in \( L^2(M) \), and since each \( \tilde{u}_j \in C^0 \cap W^{1,2}(M) \) is compactly supported, and equal to 1 on \( \overline{D} \), it admissible for the capacitor \( (\overline{D}, M) \) and we deduce that

\[
\int_M |\nabla u|^2 = \lim \int_M |\nabla \tilde{u}_j|^2 \geq \text{cap}(\overline{D}),
\]

and we conclude that

\[
\int_M |\nabla u|^2 = \text{cap}(\overline{D}),
\]

as required.
Finally, assume that \( u \) is non-constant so that, by the strong maximum principle, \( u < 1 \) in \( M \setminus \overline{D} \). Let \( \eta_n \to 1 \) be a sequence of regular values of \( u \), and set \( \Gamma_n = \{ x : u(x) < \eta_n \} \). Using the fact that \( \Delta u = 0 \) on \( \Gamma_n \subset M \setminus \overline{D} \), \( \partial u / \partial \nu = 0 \) on \( \partial_1 \Gamma_n \) and \( \partial u / \partial \nu \geq 0 \) on \( \partial_0 \Gamma_n \), given \( 0 \leq \rho \in C_c^\infty(M) \), we compute

\[
\int_M \langle \nabla u, \nabla \rho \rangle = \lim_n \int_{\Gamma_n} \langle \nabla u, \nabla \rho \rangle = \lim_n \left\{ -\int_{\Gamma_n} \rho \Delta u + \int_{\partial_0 \Gamma_n \cup \partial_1 \Gamma_n} \rho \langle \nabla u, \nu \rangle \right\} \geq 0,
\]

and \( u \) is a weak Neumann supersolution of the Laplace equation on \( M \).

We then obtain the announced equivalent characterization of parabolicity.

**Theorem 1.5.** Let \( (M, \langle \cdot, \cdot \rangle) \) be a connected Riemannian manifold with (possibly empty) boundary \( \partial M \). The following are equivalent:

(i) The capacity of every compact set \( K \) in \( M \) is zero.

(ii) For every relatively compact open domain \( D \Subset M \) there exists an increasing sequence of functions \( h_j \in C^0(M) \cap W^{1,2}_c(M) \) with \( h_j = 1 \) on \( D \), \( 0 \leq h_j \leq h_{j+1} \leq 1 \), \( h_j \) harmonic in the set \( \{ x : 0 < h_j(x) < 1 \} \cap \text{int}M \), such that

\[
\int_M |\nabla h_j|^2 \to 0 \quad \text{as} \quad j \to +\infty.
\]

(iii) \( M \) is parabolic.

**Proof.** (i) \( \Rightarrow \) (ii). Assume first that \( \text{cap}(K) = 0 \) for every compact set \( K \) in \( M \), let \( D \) be as in (ii) and let \( \Omega_j \) be an increasing exhaustion of \( M \) by relatively compact open set with smooth boundary transversal to \( \partial M \) with \( \overline{D} \subset \Omega_1 \). For every \( j \) let \( u_j \) be the equilibrium potential of the capacitor \( (\overline{D}, \Omega_j) \), and extend \( u_j \) to be 0 off \( \Omega_j \). Then \( u_j \) has the regularity properties listed in (ii), and, by Proposition 1.2.

\[
\int |\nabla u_j|^2 = \text{cap}(\overline{D}, \Omega_j) \to \text{cap}(\overline{D}) = 0.
\]

(ii) \( \Rightarrow \) (i) Conversely, assume that (ii) holds. Clearly it suffices to prove that \( \text{cap}(\overline{D}) = 0 \) for every relatively compact open domain \( D \) with smooth boundary transversal to \( \partial M \). Choose an increasing exhaustion of \( M \) by relatively compact domains \( \Omega_j \) with smooth boundary transversal to \( \partial M \) such that \( \text{supp} u_j \Subset \Omega_j \). Then

\[
\text{cap}(\overline{D}) = \lim_j \text{cap}(\overline{D}, \Omega_j) \leq \lim_j \int_{\Omega_j} |\nabla u_j|^2 \to 0,
\]

as required.

(i) \( \Rightarrow \) (iii) Suppose that \( \text{cap}(K) = 0 \) for every compact set in \( M \), and let \( u \in C^0(M) \cap W^{1,2}_{loc}(M) \) satisfy, in the weak Neumann sense,

\[
\begin{cases}
\Delta u \geq 0 \\
\frac{\partial u}{\partial \nu} \leq 0 \text{ on } \partial M \\
sup_M u < +\infty,
\end{cases}
\]

(9)
Let \( v = \sup_M u - u + 1 \), so that \( v \geq 1 \) and, by definition of weak solution of the differential problem (9), \( v \) satisfies
\[
\int \langle \nabla v, \nabla \rho \rangle \geq 0 \quad \forall \rho \in C^0(M) \cap W^{1,2}_0(M).
\]

Next, for every relatively compact domain \( D \), let \( \varphi \in \text{Lip}_c(M) \) with \( \varphi = 1 \) on \( D \), and \( 0 \leq \varphi \leq 1 \). Using \( \rho = \varphi^2 v^{-1} \in C^0(M) \cap W^{1,2}_c(M) \) as a test function we have
\[
0 \leq \int \langle v, \nabla \rho \rangle = 2 \int \varphi \langle v^{-1} \nabla v, \nabla \varphi \rangle - \int \varphi^2 |v^{-1} \nabla v|^2 \\
\leq 2 \int \varphi |v^{-1} \nabla v| |\nabla \varphi| - \int \varphi^2 |v^{-1} \nabla v|^2.
\]
Rearranging, using Young’s inequality \( 2ab \leq a^2 + \frac{1}{2}b^2 \), and recalling that \( \varphi = 1 \) on \( D \) we obtain
\[
\int_D |v^{-1} \nabla v|^2 \leq 4 \int |\nabla \varphi|^2,
\]
and taking the inf of the right hand side over all \( \text{Lip}_c \) function \( \varphi \) which are equal to 1 on \( D \) we conclude that
\[
\int_D |v^{-1} \nabla v|^2 \leq 4\text{cap}(D) = 0
\]
Thus \( v \) and therefore \( u \) is constant on every relatively compact domain \( D \). Thus \( u \) is constant on \( M \), and \( M \) is parabolic in the sense of Definition 0.5.

(iii) \( \Rightarrow \) (i) Assume by contradiction that there exists compact set \( K \) with nonzero capacity. Without loss of generality we can suppose that \( K \) is the closure of a relatively compact open domain \( D \) with smooth boundary \( \partial_0 D \) transversal to \( \partial M \). Let \( u \) be the equilibrium potential of \( \overline{D} \) constructed in Proposition 1.3 which is non-constant since the capacity of \( \overline{D} \) is positive. But then \( u \in C^0(M) \cap W^{1,2}(M) \) is a non-constant bounded weak Neumann superharmonic function, contradicting the assumed parabolicity of \( M \). \( \square \)

2. Maximum principles & height estimates

It is a classical result by L.V. Ahlfors that a Riemannian manifold \( N \) (without boundary) is parabolic if and only if, for every domain \( D \subseteq N \) with \( \partial D \neq \emptyset \) and for every bounded above, subharmonic function \( u \) on \( D \) it holds that \( \sup_D u = \sup_{\partial D} u \). The result has been extended in the setting of \( p \)-parabolicity in [29]. This section aims to provide a new form of the Ahlfors characterization which is valid on manifolds with boundary. This, in turn, will be used to obtain estimate of the height function of complete hypersurfaces with constant mean curvature (CMC for short) immersed into product spaces of the form \( N \times \mathbb{R} \).
2.1. **Global maximum principles.** We are going to prove the Ahlfors-type characterization of parabolicity stated in Theorem 0.7. Actually, a version of this global maximum principle involving the whole manifold and without any Neumann condition will be crucial in the geometric applications. This is the content of Theorem 0.8 that will be proved at the end of the section.

**Proof (of Theorem 0.7).** Assume first that $M$ is parabolic and suppose, by contradiction, that there exists a domain $D \subseteq M$ and a function $u$ as in the statement of the Theorem, such that

$$\sup_D u > \sup_{\partial_0 D} u.$$

Let $\varepsilon > 0$ be so small that

$$\sup_D u > \sup_{\partial_0 D} u + \varepsilon.$$

Then, the open set $D_\varepsilon = \{ x \in D : u > \sup_D u - \varepsilon \} \neq \emptyset$ satisfies $D_\varepsilon \subset D$ and, therefore,

$$u_\varepsilon = \begin{cases} 
\max \{ u, \sup_D u - \varepsilon \} & \text{on } D \\
\sup_D u - \varepsilon & \text{on } M \setminus D
\end{cases}$$

well defines a $C^0(M) \cap W^{1,2}_{\text{loc}}(M)$-subsolution of the Laplace equation on $M$. Furthermore, $\sup_M u_\varepsilon = \sup_D u < +\infty$. It follows from the very definition of parabolicity that $u_\varepsilon$ is constant on $M$. In particular, if we suppose to have chosen $\varepsilon > 0$ in such a way that $\sup_D u - \varepsilon$ is not a local maximum for $u$, then $u_\varepsilon = \sup_D u - \varepsilon$ on $\partial D_\varepsilon \neq \emptyset$ and we conclude

$$u \equiv \sup_D u - \varepsilon,$$

on $D$,

which is absurd.

Suppose now that, for every domain $D \subseteq M$ with $\partial_0 D \neq \emptyset$ and for every $u \in C^0(\overline{D}) \cap W^{1,2}_{\text{loc}}(D)$ satisfying, in the weak Neumann sense,

$$\begin{aligned}
\Delta u &\geq 0 &\text{on } D \\
\frac{\partial u}{\partial \nu} &\leq 0 &\text{on } \partial_1 D \\
\sup_D u &< +\infty,
\end{aligned}$$

it holds

$$\sup_D u = \sup_{\partial_0 D} u.$$

By contradiction assume that $M$ is not parabolic. Then, there exists a non-constant function $v \in C^0(M) \cap W^{1,2}_{\text{loc}}(M)$ satisfying

$$\begin{aligned}
\Delta v &\geq 0 &\text{on } M \\
\frac{\partial v}{\partial \nu} &\leq 0 &\text{on } \partial M \\
v^* &\equiv \sup_M v < +\infty.
\end{aligned}$$
Given $\eta < v^*$ consider the domain $\Omega_\eta = \{ x \in M : v(x) > \eta \} \neq \emptyset$. We can choose $\eta$ sufficiently close to $v^*$ in such a way that $\text{int} M \subseteq \Omega_\eta$. In particular, $\partial \Omega_\eta \subseteq \{ v = \eta \}$ and $\partial_0 \Omega_\eta \neq \emptyset$. Now, $v \in C^0(\overline{\Omega_\eta}) \cap W^{1,2}_{\text{loc}}(\Omega_\eta)$ is a bounded above weak Neumann subsolution on $\partial_1 \Omega_\eta$. Moreover,

$$\sup_{\partial_0 \Omega_\eta} v = \eta < \sup_{\Omega_\eta} v,$$

contradicting our assumptions. \qed

**Remark 2.1.** If we take $D = M$ in the first half of the above proof then we immediately realize that the Neumann boundary condition plays no role. This suggests the validity of the following restricted form of the maximum principle that was adopted by F.R. De Lima [7] as a definition of a weak notion of parabolicity; see Appendix A.

**Proof (of Theorem 0.8).** If, by contradiction,

$$\sup_M u > \sup_{\partial M} u$$

then, we can choose $\varepsilon > 0$ so small that

$$\sup_M u > \sup_{\partial M} u - 2\varepsilon.$$ 

Define $u_\varepsilon \in C^0(M) \cap W^{1,2}_{\text{loc}}(M)$ by setting

$$u_\varepsilon = \begin{cases} 
\max (u, \sup_M u - \varepsilon) & \text{on } \Omega_{2\varepsilon} \\
\sup_M u - \varepsilon & \text{on } M \setminus \Omega_{2\varepsilon},
\end{cases}$$

where we have set

$$\Omega_{2\varepsilon} = \left\{ x \in M : u(x) > \sup_M u - 2\varepsilon \right\}.$$ 

Since $\Omega_{2\varepsilon} \subset \text{int} M$, we have that $u_\varepsilon$ is constant in a neighborhood of $\partial M$. Since $\Delta u \geq 0$ weakly on $\text{int} M$, it follows that $u_\varepsilon$ is a weak Neumann subsolution on $M$. Moreover, $\sup_M u_\varepsilon = \sup_M u < +\infty$ so that, by parabolicity, $u_\varepsilon \equiv \sup_M u - \varepsilon$, a contradiction. \qed

### 2.2. Height estimates for CMC hypersurfaces in product spaces.

We now present some applications of this global maximum principle to get height estimates both for $H$-hypersurfaces with boundary in product spaces and for $H$-graphs over manifolds with boundary. By an $H$-hypersurfaces of $N \times \mathbb{R}$ we mean and oriented hypersurface $\Sigma$ with constant mean curvature $H$ with respect to a choice of its Gauss map. An $H$-graph over the $m$-dimensional Riemannian manifold $M$ with boundary $\partial M \neq \emptyset$ is an embedded $H$-hypersurfaces given by $\Sigma = \Gamma_u(M)$ where $\Gamma_u : M \to M \times \mathbb{R}$ is defined, as usual, by $\Gamma_u(x) = (x, u(x))$, for some smooth function $u : M \to \mathbb{R}$. The downward (pointing) unit normal to $\Sigma$ is defined by

$$\mathcal{N} = \frac{1}{\sqrt{1 + |\nabla_M u|^2}} (\nabla_M u, -1).$$
With respect to $N$, the mean curvature of the graph writes as

$$H = -\frac{1}{m} \text{div}_{M} \left( \frac{\nabla_{M} u}{\sqrt{1 + |\nabla_{M} u|^{2}}} \right).$$

On the other hand, let $M_{\Sigma}$ be the original manifold $M$ endowed with the metric pulled back from $M \times \mathbb{R}$ via $\Gamma_{u}$. Then, it is well known that the mean curvature vector field of the isometric immersion $\Gamma_{u}$

$$H(x) = H(x)N(x)$$

satisfies

$$\Delta_{\Sigma} \Gamma_{u} = mH,$$

where $\Delta_{\Sigma}$ denotes the Laplacian on manifold-valued maps. Since $\Delta_{\Sigma}$ is linear with respect to the Riemannian product structure in the codomain, from the above we also get

$$\Delta_{\Sigma} u = \frac{1}{\sqrt{1 + |\nabla_{M} u|^{2}}} \text{div}_{M} \left( \frac{\nabla_{M} u}{\sqrt{1 + |\nabla_{M} u|^{2}}} \right)$$

$$= -\frac{m}{\sqrt{1 + |\nabla_{M} u|^{2}}} H(x)$$

(10)

With this preparation, we begin by noting the following version of Lemma 1 in [20].

**Lemma 2.2.** Let $N$ be an $m$-dimensional complete manifold without boundary and let $M \subset N$ be a closed domain with smooth boundary $\partial M \neq \emptyset$. Consider a graph $\Sigma = \Gamma_{u}(M) \subset N \times \mathbb{R}$ over $M$ with smooth boundary

$$\partial \Sigma \subset M \times \{0\}.$$

Assume that

$$\sup_{M} |u| + \sup_{M} |H| < +\infty.$$

Then there exists a constant $C = C(m, \sup_{M} |u|, \sup_{M} |H|) > 0$ such that, for every $\delta > 0$ and $R > 1$,

$$\text{vol} B_{R}^{\Sigma}(\bar{p}) \leq C \left( 1 + \frac{1}{\delta R} \right) \text{vol} \left( M \cap B_{(1+\delta)R}^{N}(\bar{x}) \right),$$

where $\bar{x}$ is a reference point in $N$ and $\bar{p} = (\bar{x}, u(\bar{x}))$. Moreover, the following estimate

$$\text{vol} B_{R}^{\Sigma}(\bar{p}) \leq C \left\{ \text{vol} B_{R}^{N} (\bar{x}) + \text{Area} (\partial B_{R}^{N} (\bar{x})) \right\}$$

holds for almost every $R > 1$.

**Proof.** Note that

$$d_{\Sigma}(\bar{x}, u(\bar{x})), (x, u(x)) \geq d_{N \times \mathbb{R}}((\bar{x}, u(\bar{x})), (x, u(x)))$$

$$\geq \max \{ d_{N}(\bar{x}, x) + |u(\bar{x}) - u(x)| \}. $$
Set \( \tilde{p} = (\bar{x}, u(\bar{x})) \). Therefore
\[
B^\Sigma_R(\tilde{p}) \subseteq \Sigma \cap B^N_R(\tilde{p}) \\
\subseteq (M \cap B^N_R(\bar{x})) \times (-R + u(\bar{x}), R + u(\bar{x}))
\]
and it follows that
\[
\text{vol}B^\Sigma_R(\tilde{p}) = \int_{\Pi_N(B^\Sigma_R(\tilde{p}))} \sqrt{1 + |\nabla u|^2} \, d\text{vol}_N
\leq \int_{M \cap B^N_R(\bar{x})} \frac{|\nabla u|^2}{1 + |\nabla u|^2} \, d\text{vol}_N + \int_{M \cap B^N_R(\bar{x})} \frac{1}{\sqrt{1 + |\nabla u|^2}} \, d\text{vol}_N
\leq \int_{M \cap B^N_R(\bar{x})} \frac{|\nabla u|^2}{1 + |\nabla u|^2} \, d\text{vol}_N + \text{vol}(M \cap B^N_R(\bar{x})).
\]
Here \( \Pi_N : \Sigma \rightarrow N \) denotes the projection on the \( N \) factor. Now, for any \( \delta > 0 \), we choose a cut-off function \( \rho \) as follows:
\[
\rho(x) = \begin{cases} 
1 & \text{on } B_R(\bar{x}) \\
\frac{(1+\delta)R - r(x)}{\delta R} & \text{on } B_{(1+\delta)R}(\bar{x}) \setminus B_R(\bar{x}) \\
0 & \text{elsewhere,}
\end{cases}
\]
where \( r(x) \) denotes the distance function on \( N \) from a reference point \( \bar{x} \).
Since
\[
X = \rho u \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}
\]
is a compactly supported vector field that vanishes on \( \partial M \) and on \( \partial B^N_{(1+\delta)R}(\bar{x}) \), as an application of the divergence theorem we get
\[
0 = \int_{M \cap B^N_{(1+\delta)R}(\bar{x})} \text{div}(X) \, d\text{vol}_N
= -m \int_{M \cap B^N_{(1+\delta)R}(\bar{x})} \rho H u \, d\text{vol}_N + \int_{M \cap B^N_{(1+\delta)R}(\bar{x})} \frac{\rho |\nabla u|^2}{\sqrt{1 + |\nabla u|^2}} \, d\text{vol}_N
- \frac{1}{\delta R} \int_{M \cap (B^N_{(1+\delta)R}(\bar{x}) \setminus B^N_R(\bar{x}))} u \frac{\langle \nabla u, \nabla r \rangle}{\sqrt{1 + |\nabla u|^2}} \, d\text{vol}_N.
\]
Hence
\[
\int_{M \cap B^N_R(\bar{x})} \frac{|\nabla u|^2}{\sqrt{1 + |\nabla u|^2}} \, dvol_N \leq \int_{M \cap B^N_{(1+\delta)}R(\bar{x})} \frac{\rho |\nabla u|^2}{\sqrt{1 + |\nabla u|^2}} \, dvol_N
\]
\[
\leq m \sup_{M} |u| \sup_{M} |\nabla H| \text{vol}(M \cap B^N_{(1+\delta)}R(\bar{x}))
\]
\[
+ \frac{1}{\delta R} \text{vol}(M \cap (B^N_{(1+\delta)}R(\bar{x}) \setminus B^N_R(\bar{x}))).
\]

Inserting this latter into (11) gives, for every \( R > 1, \)
\[
\text{vol}B^\Sigma_R(\bar{p}) \leq C \left\{ \text{vol}(M \cap B^N_R(\bar{x})) + \text{vol}(M \cap B^N_{(1+\delta)}R(\bar{x})) \right\}
\]
\[
+ \frac{1}{\delta R} \text{vol}(M \cap (B^N_{(1+\delta)}R(\bar{x}) \setminus B^N_R(\bar{x}))) \right\}.
\]

To conclude, we let \( \delta \to 0 \) and we use the co-area formula. \( \square \)

**Remark 2.3.** We note that, actually, the somewhat weaker conclusions
\[
\text{vol}B^\Sigma_R(\bar{p}) \leq C \left( 1 + \frac{1}{\delta} \right) \text{vol} \left( M \cap B^N_{(1+\delta)}R(\bar{x}) \right),
\]
and
\[
\text{vol}B^\Sigma_R(\bar{p}) \leq C \left\{ \text{vol}B^N_R(\bar{x}) + R \text{Area} (\partial B^N_R(\bar{x})) \right\}
\]
hold under the assumption
\[
\sup_{M} |uH| < +\infty.
\]

Indeed, to overcome the problem that \( u \) can be unbounded, following the proof in the minimal case \( H \equiv 0, \) one can apply the divergence theorem to the vector field
\[
X = \rho u \sqrt{2R} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}},
\]
where \( u_R \) is defined as
\[
u_R = \begin{cases} 
-R & \text{if } u(x) < -R \\
u(x) & \text{if } |u(x)| < R \\
R & \text{if } u(x) > R.
\end{cases}
\]

**Remark 2.4.** It could be interesting to observe that, in certain situations, an improved version of Lemma 2.2 can be obtained from the a-priori gradient estimates due to N. Koreevar, X.-J. Wang and J. Spruck, [17, 34, 33]. See also [32] where the injectivity radius assumption has been removed. More precisely, we have the next simple result. We explicitly note that, with respect to Lemma 2.2, no assumption on \( \partial \Sigma \) is required. Moreover, the volume estimate involves the same radius \( R > 0 \) without any further contribution.
Lemma 2.5. Let \((N, g)\) be a complete, \(m\)-dimensional Riemannian manifold (without boundary) satisfying \(\text{Sec}_N \geq -K\) and let \(M \subset N\) be a closed domain with smooth boundary \(\partial M \neq \emptyset\). Suppose we are given a vertically bounded graph \(\Sigma = \Gamma_u (U_\varepsilon (M))\) with bounded mean curvature \(H\), parametrized over an \(\varepsilon\)-neighborhood \(U_\varepsilon (M)\) of \(M\). Let \(\Sigma = \Gamma_u (M)\). Then, there exists a constant \(C = C (m, \varepsilon, H, K, \sup_M |u|, \sup_M |H|) > 0\) such that
\[
\text{vol} B^\Sigma_R (\bar{p}) \leq C \text{vol} (M \cap B^N_R (\bar{x})) ,
\]
for every \(R > 0\), where \(\bar{x} \in \text{int} M\) is a reference point and \(\bar{p} = (\bar{x}, u (\bar{x}))\).

Proof. Indeed, since
\[
\text{vol} B^\Sigma_R (\bar{p}) = \int_{\Pi_N (B^\alpha_R (\bar{p}))} \sqrt{1 + |\nabla u|^2} d\text{vol}_N \leq \int_{M \cap B^N_R (\bar{x})} \sqrt{1 + |\nabla u|^2} d\text{vol}_N ,
\]
we have only to show that \(|\nabla u|\) is uniformly bounded on \(M\). To this end, note that \(u : U_\varepsilon (M) \to \mathbb{R}\) is a bounded function defining a bounded mean curvature graph \(\Gamma_u (U_\varepsilon (M))\). Therefore, we can apply Theorem 1.1 in \([33]\) to either \(w (x) = \sup_M u - u (x) \geq 0\) or \(w (x) = u (x) - \inf_M u \geq 0\) and obtain that, in fact, \(|\nabla^M u|\) is uniformly bounded on every ball \(B^N_{\varepsilon/2} (x) \subset U_\varepsilon (M)\), with \(x \in M\). This completes the proof. \(\square\)

Lemma 2.2 allows to prove Theorem 0.10 stated in the Introduction.

Proof (of Theorem 0.10). Observe first that, according to Lemma 2.2, since \(N\) has quadratic volume growth, so has \(\Sigma\). In particular, by Theorem 0.6, if we denote by \(M_\Sigma\) the original domain \(M\) endowed with the metric pulled back from \(\Sigma\) via \(\Gamma_u\), we conclude that \(M_\Sigma\) is parabolic. Consider now the real-valued function \(w \in C^0 (M_\Sigma) \cap C^\infty (\text{int} M_\Sigma)\) defined by
\[
w (x) = Hu (x) - \frac{1}{\sqrt{1 + |\nabla u (x)|^2}} .
\]
Since \(\text{Ric}_N \geq 0\), it is well known that \(w\) is subharmonic; see e.g. \([1]\). Moreover, \(w \leq 0\) on \(\partial M_\Sigma\) and \(\sup_{M_\Sigma} w \leq H \sup_M u < +\infty\). It follows from Theorem 0.8 that
\[
\sup_{M_\Sigma} w = \sup_{\partial M_\Sigma} w \leq 0
\]
and, therefore,
\[
H \sup_M u - 1 \leq \sup_{M_\Sigma} w \leq 0 .
\]
This shows that \(u \leq 1/H\). To conclude the proof, observe that, by (10), \(u \in C^0 (M_\Sigma) \cap C^\infty (\text{int} M_\Sigma)\) is a superharmonic function. Moreover, by assumption, \(u\) is bounded and \(u = 0\) on \(\partial M_\Sigma\). Therefore, using again Theorem 0.8 in the form of a minimum principle, we deduce
\[
\inf_{M_\Sigma} u = \inf_{\partial M_\Sigma} u = 0 ,
\]
proving that \(u \geq 0\). \(\square\)
Remark 2.6. It is well known that, in case $\partial M = \emptyset$, the above volume growth assumption implies that the vertically bounded $H$-graph must be necessarily minimal, $H = 0$. Actually, according to Theorem 5.1 in [30], the same conclusion holds if $\text{vol} B_R \leq C_1 e^{C_2 R^2}$ for some constants $C_1, C_2 > 0$. Indeed, under this condition, the weak maximum/minimum principle at infinity for the mean-curvature operator holds on $M$. Therefore, there exists a sequence $x_k$ along which

\((a)\) $u(x_k) < \inf_M u + 1/k$

\((b)\) $mH \equiv -\text{div}((1 + |\nabla M u(x_k)|^2)^{-1/2} \nabla M u(x_k)) < 1/k.$

This shows that $H \leq 0$. In a similar fashion we obtain the opposite inequality, proving that $H \equiv 0$. The same conclusion was also obtained in [27] by different methods.

On the other hand, if $\partial M = \emptyset$ and the volume growth of $M$ is sub-quadratic then $M$ is parabolic with respect to the mean curvature operator, \[30\]. Therefore, not only the $H$-graph is minimal, but it must be a slice of $M \times \mathbb{R}$.

Remark 2.7. Theorem 0.10 goes in the direction of generalizing Theorem 4 in [31] by A. Ros and H. Rosenberg to non-homogeneous domains. Indeed, assume that $m = 2, 3, 4$ and $\text{Sec}_N \geq 0$. Then, for every $|H| > 0$, an $H$-graph $\Sigma = \Gamma u(M)$ in $N \times \mathbb{R}$ over a domain $M \subseteq N$, is necessarily bounded; \[31, 3, 8\]. Furthermore, in case $m = 2$, it follows by the Bishop-Gromov comparison theorem that, if $\text{Sec}_N \geq 0$, then $N$ has quadratic volume growth, that is

$$\text{vol} B^N_R(\bar{x}) \leq \omega_2 R^2,$$

where $\omega_2$ denotes the area of the unit ball in $\mathbb{R}^2$. Moreover, if $N$ is complete, $\partial N = \emptyset$, then $M$ is a complete parabolic manifold with boundary. Indeed, let $d_M$ and $d_N$ denote the intrinsic distance functions on $M$ and $N$, respectively. Clearly

\[(12)\] $d_M \geq d_N|_{M \times M}$

and $(M, d_M)$ is a complete metric space. Indeed, from (12), any Cauchy sequence $\{x_k\} \subset (M, d_M)$ is Cauchy in the complete space $(N, d_N)$. It follows that $x_k \xrightarrow{d_N} \bar{x} \in N$ as $k \to +\infty$. Actually, since $M$ is a closed subset of $(N, d_N)$, we have $\bar{x} \in M$. To conclude that $x_k \xrightarrow{d_M} \bar{x}$, simply recall that the metric topology on $M$ induced by $d_M$ is the original topology of $M$, i.e., the subspace topology inherited from $N$. Moreover, since, by (12),

$$\text{vol} B^M_R(x) \leq \text{vol} (B^N_R(x) \cap M) \leq \text{vol} (B^N_R(x)),$$

for every $x \in M$, it follows that $M$ enjoys the same volume growth property of $N$.

In light of the considerations above, Corollary 0.11 is now straightforward.

We end this section, by considering the more general case of an oriented CMC hypersurface in the Riemannian product $N \times \mathbb{R}$. Abstracting from the
previous arguments, and up to using more involved computations as in \[1\],
we easily obtain the proof of Theorem 0.9 stated in the Introduction.

Proof (of Theorem 0.9). Let \( f : \Sigma^m \to N^m \times \mathbb{R} \) be a complete, oriented \( H \)-hypersurface isometrically immersed in \( N \times \mathbb{R} \), and denote by \( h \) the projection of the image of \( \Sigma \) on \( \mathbb{R} \) under the immersion, that is, \( h = \pi_{\mathbb{R}} \circ f \).

Note that
\[
\Delta_\Sigma h = n \cos \Theta H \leq 0,
\]
where, we recall, \( \Theta \in \left[ \frac{\pi}{2}, \frac{3\pi}{2} \right] \) stands for the angle between the Gauss map \( N \) and the vertical vector field \( \partial/\partial t \). Since, by Theorem 0.6, \( \Sigma \) is parabolic and \( h \) is a bounded below superharmonic function, we can apply the Ahlfors maximum principle to get
\[
h \geq \inf_{\Sigma} h = \inf_{\partial \Sigma} h = 0.
\]
Consider now the function \( \varphi \) defined as
\[
\varphi = Hh + \cos \Theta.
\]
We know by Theorem 3.1 in \[1\] that \( \varphi \) is subharmonic. Since it is also bounded, applying again the Ahlfors maximum principle we conclude that
\[
Hh - 1 \leq \varphi \leq \sup_{\Sigma} \varphi = \sup_{\partial \Sigma} \varphi \leq 0.
\]
We have thus shown that
\[
0 \leq \pi_{\mathbb{R}} \circ f(x) \leq \frac{1}{H},
\]
as required. \( \square \)

3. The \( L^2 \)-Stokes theorem & slice-type results

In this section we prove the global divergence theorem stated in the Introduction as Theorem 0.12. We also provide a somewhat weaker form of this result which involves differential inequalities of the type \( \text{div} X \geq f \); see Proposition 3.5 below. This latter, together with the Ahlfors maximum principle, is then applied to prove slice-type results for hypersurfaces in product spaces and for graphs; see Theorems 0.13 and 0.14 in the Introduction. Actually, the graph-version of this result also requires a Liouville-type theorem for the mean curvature operator on manifolds with boundary, under volume growth conditions. This is modeled on \[30\].

3.1. Global divergence theorems. Recall that, for a given smooth, compactly supported vector field \( X \) on an oriented Riemannian manifold \( M \) with boundary \( \partial M \neq \emptyset \), the ordinary Stokes theorem asserts that
\[
\int_M \text{div} X = \int_{\partial M} \langle X, \nu \rangle,
\]
where \( \nu \) is the exterior unit normal to \( \partial M \). In particular, this holds for every smooth vector field if \( M \) is compact. The result still holds if we relax
Definition 3.1. Let $X$ be a vector field on $M$ satisfying $X \in L^1_{\text{loc}}(M)$ and $\langle X, \nu \rangle \in L^1_{\text{loc}}(\partial M)$. The distributional divergence of $X$ is defined by

$$
(\text{div } X, \varphi) = -\int_M \langle X, \nabla \varphi \rangle + \int_{\partial M} \varphi \langle X, \nu \rangle,
$$

for every $\varphi \in C^\infty_c(M)$.

Remark 3.2. The above definition extends trivially to $\varphi \in \text{Lip}_c(M)$. Actually, more is true. Recall that, given a domain $D \subseteq M$, $W^{1,p}_0(D)$ denotes the closure of $C_c^\infty(D)$ in $W^{1,p}(D)$. Then, by a density argument, the previous definition extends to every $\varphi \in C^0_c(M) \cap W^{1,2}_0(M)$. Indeed, let $\varphi$ be such a function. Then, we find an approximating sequence $\varphi_n \in C^\infty_c(M)$ such that $\varphi_n \to \varphi$ in $W^{1,2}(M)$, as $n \to +\infty$. Since $\text{supp}(\varphi)$ is compact, we can assume that there exists a domain $\Omega \subset M$ such that $\text{supp}(\varphi_n) \subset \Omega$, for every $n$. Moreover, a subsequence (still denoted by $\varphi_n$) converges pointwise a.e. to $\varphi$. Let $c = \max_M |\varphi| + 1$ and define $\phi_n = f \circ \varphi_n \in \text{Lip}_c(M)$ where

$$
f(t) = \begin{cases} 
 c, & t \geq c \\
 t, & -c < t < c \\
 -c, & t \leq -c.
\end{cases}
$$

Note that $\{\phi_n\}$ is an equibounded sequence, $\text{supp}(\phi_n) \subset \Omega$ and, furthermore, $\phi_n \to f \circ \varphi = \varphi$ in $W^{1,2}(M)$ and pointwise a.e. in $M$. Therefore, evaluating $[15]$ along $\phi_n$, taking limits as $n \to +\infty$ and using the dominated convergence theorem completes the proof.

Now, suppose also that $\text{div } X \in L^1_{\text{loc}}(M)$. Then we can write

$$
(\text{div } X, \varphi) = \int_M \varphi \text{div } X
$$

and, therefore, from $[14]$, we get

$$
\int_M \varphi \text{div } X = -\int_M \langle X, \nabla \varphi \rangle + \int_{\partial M} \varphi \langle X, \nu \rangle.
$$

In particular, if $X$ is compactly supported, by choosing $\varphi = 1$ on the support of $X$, we recover the Stokes formula $[14]$ for every compactly supported vector field $X$ satisfying $X \in L^1_{\text{loc}}(M)$, div $X \in L^1_{\text{loc}}(M)$ and $\langle X, \nu \rangle \in L^1_{\text{loc}}(\partial M)$.

Note that, by similar reasonings, if the vector field $X \in L^1_{\text{loc}}(M)$ has a weak divergence $\text{div } X \in L^1_{\text{loc}}(M)$ and $\langle X, \nu \rangle \in L^1_{\text{loc}}(\partial M)$, then, for every $\rho \in C^0_c(M) \cap W^{1,2}_0(M)$, we have that $\text{div } (\rho X) \in L^1_{\text{loc}}(M)$. Moreover, as in the smooth case,

$$
\int_M \text{div } (\rho X) = \int_M \langle \nabla \rho, X \rangle + \int_M \rho \text{div } X.
$$
To see this, we take $\varphi \in C_\infty^c (M)$ and, using (15) in the form of Remark 3.2, we compute

\[
(\text{div} \ (\rho X), \varphi) = - \int_M \langle \rho X, \nabla \varphi \rangle + \int_{\partial M} \rho \varphi \langle X, \nu \rangle \\
= - \int_M \langle X, \nabla (\rho \varphi) \rangle + \int_{\partial M} \rho \varphi \langle X, \nu \rangle + \int_M \varphi \langle X, \nabla \rho \rangle \\
= (\text{div} \ X, \rho \varphi) + \int_M \varphi \langle X, \nabla \rho \rangle \\
= \int_M (\rho \text{div} \ X + \langle X, \nabla \rho \rangle) \varphi \\
= (\rho \text{div} \ X + \langle X, \nabla \rho \rangle, \varphi).
\]

Whence, we conclude that

\[
\text{div} \ (\rho X) = \rho \text{div} \ X + \langle X, \nabla \rho \rangle \in L^1_{\text{loc}} (M)
\]

as desired.

All these facts will be tacitly employed several times in the rest of the Section.

If $M$ is not compact, we can still prove a global version of Stokes theorem for vector fields with prescribed asymptotic behavior at infinity. This is the content of Theorem 0.12.

**Proof (of Theorem 0.12).** Suppose $M$ is parabolic. According to Theorem (1.5) (ii) there exists an increasing sequence of functions $\varphi_n \in C_c(M) \cap W^{1,2}(M)$ such that $0 \leq \varphi_n \leq 1$ and

\[
\varphi_n \to 1 \text{ locally uniformly on } M \text{ and } \int_M |\nabla \varphi_n|^2 \to 0.
\]

Consider now any vector field $X$ satisfying (15). Since $\varphi_n X$ is compactly supported, applying the usual (weak) divergence theorem we get

\[
\int_M \text{div} \ (\varphi_n X) = \int_{\Omega_n} \text{div} \ (\varphi_n X) = \int_{\partial \Omega_n} \varphi_n \langle X, \nu \rangle.
\]

On the other hand

\[
\int_M \text{div} \ (\varphi_n X) = \int_M \langle \nabla \varphi_n, X \rangle + \int_M \varphi_n \text{div} \ X,
\]

where

\[
\left| \int_M \langle \nabla \varphi_n, X \rangle \right| \leq \left( \int_M |\nabla \varphi_n|^2 \right)^{\frac{1}{2}} \left( \int_M |X|^2 \right)^{\frac{1}{2}} \to 0
\]

as $n \to +\infty$. Moreover

\[
\int_M \varphi_n \text{div} \ X = \int_M \varphi_n (\text{div} \ X)_+ - \int_M \varphi_n (\text{div} \ X)_-
\]
and
\[ \int_M \varphi_n (\text{div } X)_+ \leq \int_M \varphi_n (\text{div } X)_- + \int_{\partial_1 \Omega_n} \varphi_n \langle X, \nu \rangle - \int_M \langle \nabla \varphi_n, X \rangle. \]

Using the monotone convergence theorem and the fact that \(0 \leq \varphi_n \leq 1\), we obtain
\[ \int_M (\text{div } X)_+ \leq \int_M (\text{div } X)_- + \int_{\partial_1 \Omega_n} \varphi_n \langle X, \nu \rangle < +\infty. \]

Hence \(\text{div } X \in L^1(M)\) and taking limits on both sides of \((16)\) completes the first part of the proof.

Conversely, assume that \(M\) is not parabolic so that \(M\) possesses a smooth, finite, positive Green kernel, \([10, 12]\). We shall show that the global Stokes theorem fails. To this end, choose an exhaustion \(\{\Omega_n\}\) of \(M\) by smooth and relatively compact domains. Then, the Neumann Green kernel \(G(x, y)\) of \(M\) is obtained as the limit of the Green functions \(G_n(x, y)\) of \(\Omega_n\) which satisfy
\[ \begin{cases} 
    \Delta G_n (x, y) & = -\delta_x (y) \quad \text{on } \Omega_n \cap \text{int } M \\
    \frac{\partial G_n}{\partial \nu} & = 0 \quad \text{on } \partial_1 \Omega_n \\
    G_n & = 0 \quad \text{on } \partial_0 \Omega_n .
\end{cases} \]

Let \(f \geq 0\) be a smooth function compactly supported in \(\text{int } M\). For each \(n\) define
\[ u_n (x) = \int_{\Omega_n} G_n (x, y) f (y) \, dy. \]

Then, each \(u_n\) is a positive, classical solution of the boundary value problem
\[ \begin{cases} 
    \Delta u_n & = -f \quad \text{on } \Omega_n \cap \text{int } M \\
    \frac{\partial u_n}{\partial \nu} & = 0 \quad \text{on } \partial_1 \Omega_n \\
    u_n & = 0 \quad \text{on } \partial_0 \Omega_n .
\end{cases} \]

By the maximum principle and the boundary point lemma, the sequence is monotonically increasing and converges to a solution \(u\) of
\[ \begin{cases} 
    \Delta u & = -f \quad \text{on } M \\
    \frac{\partial u}{\partial \nu} & = 0 \quad \text{on } \partial M .
\end{cases} \]

Also, using Fatou Lemma,
\[ \int_M |\nabla u_n|^2 \geq \int_M |\nabla u|^2 . \]

Now consider the vector field
\[ X = \nabla u. \]

Clearly \(X\) satisfies all the conditions in \((5)\). On the other hand, we have
\[ \int_M \text{div } X = - \int_M f \neq 0 \]
and
\[ \int_{\partial M} \langle X, \nu \rangle = \int_{\partial M} \frac{\partial u}{\partial \nu} = 0, \]
proving that the global Stokes theorem fails to hold.

Using Definition 3.1 of weak divergence one could introduce the notion of weak solution of a differential inequality like \( \text{div} X \geq f \). We stress that \( \text{div} X \) is not required to be a function.

**Definition 3.3.** Let \( X \in L^1_{\text{loc}}(M) \) be a vector field satisfying \( \langle X, \nu \rangle \in L^1_{\text{loc}}(\partial M) \) and let \( f \in L^1_{\text{loc}}(M) \). We say that \( \text{div} X \geq f \) in the distributional sense on \( M \) if
\[ (\text{div} X, \varphi) \geq \int_M f \varphi, \]
for every \( 0 \leq \varphi \in C^\infty_c(M) \). Actually, according to Remark 3.2, the definition extends to every \( 0 \leq \varphi \in C^0_c(M) \cap W^{1,2}(M) \).

In the special case where \( f = 0 \) and \( X = \nabla u \) for some \( u \in W^{1,2}_{\text{loc}}(M) \) satisfying \( \partial u/\partial \nu \in L^1_{\text{loc}}(\partial M) \), we obtain the corresponding notion of weak solution of \( \Delta u \geq 0 \) on \( M \).

Although elementary, it is important to realize that, as in the smooth setting, the above definition is compatible with that of weak Neumann subsolution given in the Introduction.

**Lemma 3.4.** Let \( u \in W^{1,2}_{\text{loc}}(M) \) satisfy \( \partial u/\partial \nu \in L^1_{\text{loc}}(\partial M) \). Then \( u \) is a weak Neumann subsolution of the Laplace equation provided \( u \) satisfies
\[ \left\{ \begin{array}{lcl} \Delta u & \geq & 0 \quad \text{on} \ M \\ \frac{\partial u}{\partial \nu} & \leq & 0 \quad \text{on} \ \partial M, \end{array} \right. \]
where the differential inequality is interpreted according to Definition 3.3.

**Proof.** Straightforward from the equation
\[ (\Delta u, \varphi) \overset{\text{def}}{=} -\int_M \langle \nabla u, \nabla \varphi \rangle + \int_{\partial M} \frac{\partial u}{\partial \nu} \varphi, \]
with \( 0 \leq \varphi \in C^\infty_c(M) \). \( \square \)

Reasoning as in the proof of Theorem 0.12, we can now prove the following result which extends to manifolds with boundary a result in [16].

**Proposition 3.5.** Let \((M, g)\) be an \( m \)-dimensional, parabolic manifold with smooth boundary \( \partial M \). Let \( X \) be a vector field on \( M \) satisfying:
\( (a) \ |X| \in L^2(M) \); \( (b) \ 0 \geq \langle X, \nu \rangle \in L^1_{\text{loc}}(\partial M) \).
Assume that \( \text{div} X \geq f \) for some \( f \in L^1(M) \) in the sense of distributions. Then
\[ \int_M f \leq \int_{\partial M} \langle X, \nu \rangle. \]
The same conclusion holds if $0 \leq f \in L^1_{loc}(M)$ and yields
\[ f \equiv 0. \]

Moreover, if $\text{div } X \geq 0$ in the distributional sense, then
\[ \int_M \langle X, \nabla \alpha \rangle \leq \int_{\partial M} \alpha \langle X, \nu \rangle \]
for every $0 \leq \alpha \in C^\infty_c(M)$.

**Proof.** Choose a smooth, relatively compact exhaustion $\Omega_n \subset M$ and denote by $\varphi_n$ the equilibrium potential of the capacitor $(\Omega_0, \Omega_n)$. Extend $\varphi_n$ to be identically 1 on $\Omega_0$ and identically 0 on $M \setminus \Omega_n$. Then, by assumption,
\[ \int_M \varphi_n f \leq (\text{div } X, \varphi_n) \]
\[ = -\int_M \langle X, \nabla \varphi_n \rangle + \int_{\partial M} \varphi_n \langle X, \nu \rangle \]
\[ \leq \left( \int_M |X|^2 \right)^{1/2} \left( \int_M |\nabla \varphi_n|^2 \right)^{1/2} + \int_{\partial M} \varphi_n \langle X, \nu \rangle. \]
The first part of the statement follows by taking the lim sup as $n \to +\infty$ and applying the Fatou Lemma and either the monotone convergence theorem if $0 \leq f \in L^1_{loc}(M)$ or the dominated convergence theorem if $f \in L^1(M)$. For what concern the second part, consider the test function $\eta = \varphi_n \alpha$. Then,
\[ 0 \leq (\text{div } X, \alpha \varphi_n) \]
\[ = -\int_M \alpha \langle X, \nabla \varphi_n \rangle - \int_M \varphi_n \langle X, \nabla \alpha \rangle + \int_{\partial M} \alpha \varphi_n \langle X, \nu \rangle \]
\[ \leq \sup_M |\alpha| \left( \int_M |X|^2 \right)^{1/2} \left( \int_M |\nabla \varphi_n|^2 \right)^{1/2} - \int_M \varphi_n \langle X, \nabla \alpha \rangle + \int_{\partial M} \alpha \varphi_n \langle X, \nu \rangle. \]
and the conclusion follows as above computing the lim sup as $n \to +\infty$. \qed

### 3.2. Slice-type theorems for hypersurfaces in a half-space.

This Section is devoted to the proofs of Theorems 0.13 and 0.14 stated in the Introduction. The first one of these results involves a complete hypersurface $\Sigma$ contained in the half-space $N \times [0 + \infty)$ of the ambient product space $N \times \mathbb{R}$. It is assumed that the boundary $\partial \Sigma \neq \emptyset$ lies in the slice $N \times \{0\}$ and that $\Sigma$ has non-positive mean curvature $H \leq 0$ with respect to the “downward” Gauss map. The result states that, under a quadratic area growth assumption on $\Sigma$ and regardless of the geometry of $N$, the portion of the hypersurface $\Sigma$ in any upper-halfspace of $N \times \mathbb{R}$ must have infinite volume unless $\Sigma$ is contained in the totally geodesic slice $N \times \{0\}$. The second result provides a graphical version of this theorem when $\Sigma = \Gamma_u(M)$. If $M$ satisfies a quadratic volume growth assumption, then each superlevel set $M_t = \{u \geq t > 0\} \subseteq M$ has infinite volume unless $\Sigma$ is contained in the
totally geodesic slice $M \times \{0\}$. Note that $M_t$ is the orthogonal projection of $\Sigma \cap [t, +\infty)$ on the slice $M \times \{0\}$.

Let us begin with the

\[ \text{Proof (of Theorem 0.13).} \]

Suppose that $\Sigma$ is not contained in the slice $N \times \{0\}$. If the height function $h$ on $\Sigma$ is bounded from above (for the precise definition of $h$ see the proof of Theorem 0.9 in Subsection 2.2) the parabolicity of $\Sigma$ in the form of the Ahlfors maximum principle implies that

\[ h \leq \sup_{\Sigma} h = \sup_{\partial \Sigma} h = 0. \]

The conclusion is then immediate because, by assumption, $\Sigma$ is contained in the half-space $N \times [0, +\infty)$.

Suppose now that $\sup_{\Sigma} h = +\infty$, so that $\Sigma \cap N \times \{t\} \neq \emptyset$ for an arbitrary $t > 0$. Letting $\Sigma_t = \Sigma \cap N \times [t, +\infty) = \{p \in \Sigma : h(p) \geq t\}$, and since $\text{vol}(\Sigma_s) \geq \text{vol}(\Sigma_s)$, for every $s \geq t$, we can assume that $\text{vol}(\Sigma_t) < +\infty$ for every $t > 0$. Moreover, by Sard theorem we can suppose that $t$ is a regular value of $h|_{\text{int}(\Sigma_t)}$. In particular, $\Sigma_t$ is a smooth complete hypersurface with boundary $\partial \Sigma_t = \{p \in \Sigma : h(p) = t\}$ and exterior unit normal $\nu_t = -\nabla h / |\nabla h|$. Clearly, $\Sigma_t$ is parabolic because it has finite volume. According to (13), $h$ is a subharmonic function on $\Sigma_t$ and satisfies $|\nabla h| \leq 1$. In particular, $|\nabla h| \in L^2(\Sigma_t)$. For any $\varepsilon > 0$ define

\[ h_\varepsilon = \max \{h, t + \varepsilon\}. \]

Then $h_\varepsilon$ is again subharmonic on $M_t$, it has finite Dirichlet energy $|\nabla h_\varepsilon| \in L^2(\Sigma_t)$ and, furthermore, $\partial h_\varepsilon / \partial \nu = 0$ on $\partial \Sigma_t$. Therefore, we can apply Proposition 3.5 and deduce that $h_\varepsilon$ has to be harmonic on $\Sigma_t$. Actually, since $h_\varepsilon$ is bounded from below on the parabolic manifold $\Sigma_t$ it follows that $h_\varepsilon$ is constant on every connected component of $\Sigma_t$. Whence, on noting that $h_\varepsilon = t + \varepsilon$ on $\partial \Sigma_t$ we obtain that $t \leq h \leq t + \varepsilon$ on $\Sigma_t$. Since this holds for every $\varepsilon > 0$ we conclude that $h \equiv t$ on $\Sigma_t$, contradicting the assumption of $h$ being unbounded.

The proof of Theorem 0.14 is completely similar but requires some preparation. The next Liouville-type result for the mean curvature operator is adapted from [30]; see also [5, 2]. We provide a detailed proof for the sake of completeness.

**Theorem 3.6.** Let $(M, g)$ be a complete Riemannian manifold with boundary $\partial M \neq \emptyset$. If, for some reference point $o \in \text{int}M$,

\[ \frac{1}{\text{Area}(\partial_0 B_R(o))} \notin L^1(+\infty), \]

\[ (17) \]
then the following holds. Let $u \in C^1(M)$ be a weak Neumann solution of the problem

\[
\begin{cases}
\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \geq 0 & \text{on } M \\
\frac{\partial u}{\partial \nu} \leq 0 & \text{on } \partial M \\
\sup_M u < +\infty.
\end{cases}
\]

(18)

Then $u \equiv \text{const}.$

**Remark 3.7.** As already pointed out for the Laplace-Beltrami operator, being a weak Neumann solution of $\text{div}((1 + |\nabla u|^2)^{-1/2}\nabla u)) \geq 0$ means that

\[
- \int_M \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, \nabla \varphi \right) \geq 0,
\]

for every $0 \leq \varphi \in C^\infty_c(M).$ Actually, it is obvious that the same definition extends to any elliptic operator of the form $L_\Phi(u) = \text{div}(\Phi(|\nabla u|)\nabla u)$, where $\Phi(t)$ is subjected to certain structural conditions. Moreover, under the assumption $|\nabla u| \in L^1_{\text{loc}}(\partial M)$, this definition is also coherent with the notion of weak divergence. Namely, $u$ satisfies (19) provided $\langle \text{div} X, \varphi \rangle \geq 0$ and $\partial u / \partial \nu \leq 0$, where we have set $X = (1 + |\nabla u|^2)^{-1/2}\nabla u$. This follows immediately from the equation

\[
\langle \text{div} X, \varphi \rangle \overset{\text{def}}{=} - \int_M \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, \nabla \varphi \right) + \int_{\partial M} \frac{\varphi}{\sqrt{1 + |\nabla u|^2}} \frac{\partial u}{\partial \nu}.
\]

**Remark 3.8.** If we take $\Phi(t) = 1$ in the argument below we recover Theorem 0.6 by Grigor’yan, in the form of a Liouville result for $C^1(M)$ subsolutions of the Laplace equation.

**Proof.** Let $u$ be as in the statement of the theorem and assume, by contradiction, that $u$ is non-constant on the ball $B_{R_0}(o)$, for some $R_0 > 0$. Without loss of generality we can suppose that $u \leq 0$ on $M$. Define

\[
\Phi(t) = \frac{1}{\sqrt{1 + t^2}}.
\]

Now, having fixed $R > R_0$ and $\varepsilon > 0$, we choose $\rho = \rho_{\varepsilon,R}$ as follows:

\[
\rho(x) = \begin{cases} 
1 & \text{on } B_R(o) \\
\frac{R + \varepsilon - r(x)}{\varepsilon} & \text{on } B_{R + \varepsilon}(o) \setminus B_R(o) \\
0 & \text{elsewhere},
\end{cases}
\]
Inserting the test function $\varphi = \rho e^u$ into (19) and elaborating we get
\[
0 \leq - \int_M \langle \Phi(|\nabla u|) \nabla u, \nabla (\rho e^u) \rangle = - \int_M \rho e^u \Phi(|\nabla u|) \langle \nabla u, \nabla \rho \rangle - \int_M \rho e^u \Phi(|\nabla u|) |\nabla u|^2.
\]
Then, on noting also that $\partial M$ has measure zero, we have
\[
\varepsilon^{-1} \int_{(B_{R+\varepsilon}(o) \setminus B_R(o)) \cap \text{int} M} e^u \Phi(|\nabla u|) \langle \nabla u, \nabla r \rangle \geq \int_{B_R(o) \cap \text{int} M} e^u \Phi(|\nabla u|) |\nabla u|^2.
\]
Using the co-area formula and letting $\varepsilon \to 0$ we get, for a.e. $R > R_0$,
\[
\int_{\partial_B B_R(o)} e^u \Phi(|\nabla u|) \langle \nabla u, \nabla r \rangle \geq \int_{B_R(o) \cap \text{int} M} e^u \Phi(|\nabla u|) |\nabla u|^2.
\]
On the other hand, using the Cauchy-Schwartz and Hölder inequalities, we obtain
\[
\int_{\partial_B B_R(o)} e^u \Phi(|\nabla u|) \langle \nabla u, \nabla r \rangle \leq \int_{\partial_B B_R(o)} e^u \Phi(|\nabla u|) |\nabla u| \leq \left( \int_{\partial_B B_R(o)} e^u \Phi(|\nabla u|) \right)^{\frac{1}{2}} \left( \int_{\partial_B B_R(o)} e^u \Phi(|\nabla u|) |\nabla u|^2 \right)^{\frac{1}{2}} \leq \text{Area}(\partial_B B_R(o))^{\frac{1}{2}} \left( \int_{\partial_B B_R(o)} e^u \Phi(|\nabla u|) |\nabla u|^{\frac{3}{2}} \right)^{\frac{1}{2}}.
\]
Now, set
\[
H(R) = \int_{B_R(o) \cap \text{int} M} e^u \Phi(|\nabla u|) |\nabla u|^2.
\]
Then, by the co-area formula and the previous inequalities,
\[
\frac{H'(R)}{H(R)^{\frac{3}{2}}} \geq \frac{1}{\text{Area}(\partial_B B_R(o))}.
\]
Integrating this latter on $[R_0, R]$ and letting $R \to +\infty$ we conclude
\[
H(R_0) \leq \frac{1}{\int_{R_0}^{+\infty} \text{Area}(\partial_B B_R(o))^{-1}} = 0,
\]
proving that
\[
\int_{B_R(o) \cap \text{int} M} e^u \Phi(|\nabla u|) |\nabla u|^2 = 0.
\]
Therefore, $u$ must be constant on $B_{R_0}(o)$, leading to a contradiction. \qed

We are now ready to prove the slice theorem for graphs.
Proof (of Theorem 0.14). Let $\Sigma = \Gamma_u(M)$, with $u \in C^0(M) \cap C^\infty(\text{int}M)$, and for every $s \in \mathbb{R}$ define

$$M_s := \{x \in M : u(x) \geq s\}.$$ 

By the assumption on $\partial \Sigma = \Gamma_u(\partial M)$, there exists $t > 0$ such that $M_t \subset \text{int}M$ and $\text{vol}(M_t) < +\infty$. Assume that $M_t \neq \emptyset$ for, otherwise, as in Theorem 0.13, the proof is easier. We claim that $u$ is constant on $M_t$. Indeed, by contradiction, suppose that this is not the case. Then, by Sard Theorem, we can choose $t < c < \sup_M u$ such that $c$ is a regular value of $u|_{\text{int}M}$. Thus, the closed subset $M_c$ is a complete manifold with boundary $\partial M_c \neq \emptyset$ and exterior unit normal $\nu_c = -\nabla u/|\nabla u|$. In particular, as a complete manifold with finite volume, $M_c$ is parabolic. Since the smooth function $u$ satisfies

$$\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = -mH \geq 0, \text{ on } M_c$$

then, having fixed any $\varepsilon > 0$, the same differential inequality holds for

$$u_\varepsilon = \max\{u, c + \varepsilon\};$$

see e.g. [28]. Note also that $\partial u_\varepsilon/\partial \nu = 0$ on $\partial M_c$. Summarizing, the vector field

$$X_\varepsilon = \frac{\nabla u_\varepsilon}{\sqrt{1 + |\nabla u_\varepsilon|^2}}$$

satisfies

$$\begin{cases} 
\text{div}_M X_\varepsilon \geq 0 & \text{on } M_c \\
1 \geq |X_\varepsilon| & \text{in } L^2(M_c) \\
0 = (X_\varepsilon, \nu_c).
\end{cases}$$

By applying Proposition 3.5 we deduce that $\text{div}_M X = 0$ on $M_c$, i.e., $\Sigma_c = \Gamma_u(M_c)$ is a minimal graph. Actually, since $\text{vol}(M_c) < +\infty$, by Theorem 3.6 we get that $u_\varepsilon$ must be constant on every connected component of $M_c$. Since $u_\varepsilon = c + \varepsilon$ on $\partial M_c$ it follows that $c \leq u \leq c + \varepsilon$ on $M_c$. Whence, using the fact that $\varepsilon > 0$ was chosen arbitrarily, we conclude that $u \equiv c$ on $M_c$. This contradicts the fact that $c$ is a regular value of $u$, and the claim is proved.

Since $u$ is constant on $M_t$ we have that $\sup_M u < +\infty$. We now distinguish three cases.

(a) Suppose that $\partial \Sigma = \partial M \times \{0\}$ and $\Sigma \subset [0, +\infty)$. This means that $u \geq 0$ with $u = 0$ on $\partial M$. In this case the conclusion $u \equiv 0$ follows exactly as in proof of Theorem 0.13.

(b) Suppose that $\Sigma$ is real analytic, i.e., it is described by a real analytic function $u$. Since $u$ is constant on the open set $\{u < c\}$ we must conclude that $u$ is constant everywhere.

(c) Suppose that $\cos \tilde{N}_0 \hat{N} \leq 0$ on $\partial \Sigma = \Gamma_u(\partial M)$. This means that $\partial u/\partial \nu \leq 0$ on $\partial M$. The desired conclusion follows by a direct application of Theorem 3.6. □
The following corollary is a straightforward consequence of the above proof.

**Corollary 3.9.** Let \((M, g)\) be a complete manifold with boundary \(\partial M\) and assume that \(\text{vol} M < +\infty\). Let \(\Sigma = \Gamma_u(M)\) be a graph with non-positive mean curvature \(H(x)\) with respect to the downward Gauss map \(\mathcal{N}\). Assume also that the angle \(\theta\) between the Gauss map \(\mathcal{N}\) of the graph \(\Sigma\) and the Gauss map \(\mathcal{N}_0 = (-\nu, 0)\) of \(\partial M \times \{t\} \hookrightarrow M \times \{t\}\) satisfies \(\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]\). Then \(\Sigma\) is a horizontal slice of \(M \times \mathbb{R}\).

**Appendix A. Different notions of parabolicity & some remarks on minimal graphs**

Let \(M\) be a Riemannian manifold without boundary \(\partial M = \emptyset\). Then, from the stochastic viewpoint, \(M\) is called parabolic if the Brownian motion \(X_t\) on \(M\) is recurrent, that is \(X_t\) enters infinitely many times a fixed compact set with probability 1. As recorded in the survey paper [11], the recurrence of the Brownian motion for manifolds without boundary can be characterized in terms of fundamental solutions to the Laplace equation, maximum principles for superharmonic functions, capacities, heat kernel, Liouville properties for certain Schrödinger equations, volume growth conditions, function theoretic tests (Khas'minskii criterion), \(L^2\)-Stokes theorems (Kelvin-Nevanlinna-Royden criterion) and many other geometric and potential-theoretic properties.

If \(M\) has non-empty boundary \(\partial M \neq \emptyset\), a quick check at the literature shows that there are many (non-equivalent) definitions of parabolicity. The most classical one, which is also the one we have adopted throughout the paper, was systematically used by A. Grigor’yan starting from [9, 10], and states that \(M\) is parabolic provided the reflected Brownian motion on \(M\) is recurrent. This is equivalent to require the following Liouville-type property, which imposes Neumann-type boundary conditions on relevant functions. Namely,

**Definition A.1.** A Riemannian manifold \(M\) with \(\partial M \neq \emptyset\) is \(\mathcal{N}\)-parabolic if the only solution of the problem

\[
\begin{align*}
\Delta u &\geq 0 & \text{on } M \\
\frac{\partial u}{\partial \nu} &\leq 0 & \text{on } \partial M \\
\sup_M u &< +\infty
\end{align*}
\]

is the constant function \(u \equiv \sup_M u\).

Most of the geometric and functional-analytic characterizations of \(\mathcal{N}\)-parabolicity of manifolds without boundary have already been extended to the reflected Brownian motion; see [9, 10, 11]. Two remarkable exceptions were represented by the \(L^2\)-Stokes theorem and the Ahlfors-type maximum principles, which are some of the main topics of the present paper.
A second interesting definition can be found in a paper by R. F. De Lima, who was interested in maximum principles at infinity for CMC surfaces. His definition is oriented in the direction of the classical Ahlfors maximum principle characterization of parabolic manifolds without boundary. Apparently there was no further research in this direction. Moreover, note that, a priori, there is no obvious relation between his notion and the behaviour of the Brownian motion on \( M \). Anyway, in the terminology of De Lima, we have the following

**Definition A.2.** A Riemannian manifold \( M \) is \( A \)-parabolic if for every solution of the problem

\[
\begin{cases}
\Delta u \geq 0 & \text{on } M \\
\sup_M u < +\infty
\end{cases}
\]

it holds

\[
\sup_M u = \sup_{\partial M} u.
\]

As we already observed in Section 2.1 it is not difficult to prove that the classical (i.e. Neumann) definition of parabolicity implies the one introduced by De Lima. Namely,

**Proposition A.3.** Assume that \( M \) is a \( N \)-parabolic manifold with boundary \( \partial M \neq \emptyset \) and let \( u \) be a solution of the problem

\[
\begin{cases}
\Delta u \geq 0 & \text{on } M \\
\sup_M u < +\infty
\end{cases}
\]

Then

\[
\sup_M u = \sup_{\partial M} u.
\]

Finally, a third fruitful definition comes from very recent works in the theory of minimal surfaces in the Euclidean space, \[6, 23, 22, 24\]. From the Brownian motion viewpoint, it states that \( M \) is parabolic provided the absorbed Brownian motion is recurrent, i.e., with probability 1 the particle reaches the boundary (and dies) in a finite time. From a deterministic viewpoint, this definition involves Dirichlet boundary conditions on the relevant functions. In this context, a Riemannian manifold is said to be parabolic if bounded harmonic functions are determined by their boundary values. This is equivalent to the following

**Definition A.4.** A Riemannian manifold \( M \) is \( D \)-parabolic if the unique solution of the problem

\[
\begin{cases}
\Delta u = 0 & \text{on } M \\
u = 0 & \text{on } \partial M \\
\sup_M |u| < +\infty
\end{cases}
\]

is the constant function \( u \equiv 0 \).
This notion of parabolicity has been used in the theory of minimal surfaces in $\mathbb{R}^3$ because it turned out to be a powerful tool in order to face the problem of determining which conformal structures are allowed on a minimal surface subjected to some geometric restrictions on its image.

The notion of $\mathcal{D}$-parabolicity is related to the classical Neumann one via the Ahlfors maximum principle. Indeed, the following result follows by applying twice Proposition A.3 to $u$ and to $-u$.

**Proposition A.5.** Assume that $M$ is a $\mathcal{N}$-parabolic manifold with boundary $\partial M \neq \emptyset$ and let $u$ be a solution of the problem

\[
\begin{cases}
\Delta u = 0 & \text{on } M \\
u = 0 & \text{on } \partial M \\
\sup_M |u| < +\infty
\end{cases}
\]

Then $u \equiv 0$.

In the theory of minimal surfaces in the Euclidean space, $\mathcal{D}$-parabolicity is not the only global property of surfaces with boundary that has been studied. Another property of interest is the quadratic area growth with respect to the extrinsic distance (see [6, 26, 24] for more details and applications of this property). To be more precise, we say that a surface $M \subset \mathbb{R}^3$ has quadratic area growth if, for some $C > 0$ and $A > 0$, one has

\[\text{vol}(M \cap \{\sqrt{x_1^2 + x_2^2 + x_3^2} < R\}) \leq CR^2,\]

for all $R > A$.

The notions of $\mathcal{D}$-parabolicity and quadratic area growth seem to be, in general, unrelated concepts. For this reason, this global properties have been studied separately in the theory of minimal surfaces in $\mathbb{R}^3$. However, according to Proposition A.5, the volume condition

\[
R \text{ vol}(B_R(o)) \notin L^1(+\infty),
\]

is sufficient to guarantee that a complete Riemannian manifold $M$ is $\mathcal{D}$-parabolic. Hence, all the results obtained in this setting under geometric conditions on the ambient space and exploiting $\mathcal{D}$-parabolicity can be obtained imposing a volume growth condition on the surface instead. Moreover, since the volume of intrinsic balls is dominated by that of extrinsic balls with the same radius, we conclude also that any complete (e.g. properly immersed) surface in the Euclidean space with quadratic area growth is $\mathcal{D}$-parabolic.

To give an example of how this circle of ideas applies we note that it was conjectured by W. Meeks that any complete (or properly embedded) minimal graph over a proper subdomain of the plane is $\mathcal{D}$-parabolic. In [25], using refined stochastic methods, R. Neel gave a positive answer to this conjecture. Actually, he was able to prove that for a complete, embedded minimal surface with boundary whose Gauss image is eventually contained in
a hyperbolic domain of the sphere, the Brownian motion strikes the boundary almost surely in finite time. However, apparently, no proofs based on analytic techniques of this fact has appeared yet in literature.

Nevertheless, it was observed by P. Li and J. Wang [20, Lemma 1] that minimal graphs in $\mathbb{R}^{n+1}$ supported on a domain $\Omega \subset \mathbb{R}^n$ have the following (extrinsic) volume growth property

$$\text{vol}(M \cap \{ \sqrt{x_1^2 + \cdots + x_n^2} < R \}) \leq (n+1)\omega_n R^n,$$

where $\omega_n$ denotes the volume of the $n$-dimensional unit sphere. In particular, for a complete minimal graph $M$ in the Euclidean 3-space,

$$\text{vol}(B_R(o)) \leq \text{vol}(M \cap \{ \sqrt{x_1^2 + x_2^2 + x_3^2} < R \}) \leq 3\omega_2 R^2,$$

where $B_R(o)$ denotes the geodesic ball in $M$ of radius $R$ centered at a reference point $o \in \text{int}M$. Hence, complete minimal graphs in $\mathbb{R}^3$ have (intrinsic) quadratic volume growth. In view of Proposition A.5, we have then proved the following theorem, that recovers the result by Neel.

**Theorem A.6.** Any complete minimal graph in $\mathbb{R}^3$ supported on a domain of the plane is $D$-parabolic.

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GLOBAL MAXIMUM PRINCIPLES AND DIVERGENCE THEOREMS

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