A note on non-4-list colorable planar graphs

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Abstract

The Four Color Theorem states that every planar graph is properly 4-colorable. Moreover, it is well known that there are planar graphs that are non-4-list colorable. In this paper we investigate a problem combining proper colorings and list colorings. We ask whether the vertex set of every planar graph can be partitioned into two subsets where one subset induces a bipartite graph and the other subset induces a 2-list colorable graph. We answer this question in the negative strengthening the result on non-4-list colorable planar graphs.

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Let $G = (V, E)$ be a simple graph and for every vertex $v \in V$ let $L(v)$ be a set (list) of available colors. A $k$-assignment is a list assignment with $|L(v)| = k$ for all $v \in V(G)$. A graph $G$ is called $L$-colorable if there is a proper coloring $c$ of the vertices with $c(v) \in L(v)$ for all $v \in V(G)$ and $c(v) \neq c(w)$ for all edges $vw \in E(G)$. If $G$ is $L$-colorable for all possible $k$-assignments then $G$ is called $k$-list colorable.

In this note we consider simple planar graphs. Since 1993 it is known by Thomassen [6] and Voigt [7] that every planar graph is 5-list colorable but there are planar graphs that are non-4-list colorable.

Recently, Choi and Kwon [2] introduced the concept of a $t$-common $k$-assignment which is a $k$-assignment satisfying $|\bigcap_{v \in V(G)} L(v)| \geq t$. Using the Four Color Theorem [1, 5], it is easy to see that every planar graph is $L$-colorable for every 3-common 4-assignment $L$. Moreover, Choi and Kwon [2] constructed a planar graph $G$ with a 1-common 4-assignment $L$ such that $G$ is not $L$-colorable and they explicitly asked the following problem.

Problem 1. Is every planar graph $L$-colorable for every 2-common 4-assignment $L$?

Since every proper coloring of the vertices gives a partition of the vertex set we may look for the problem from another point of view.
**Problem 2.** Let $G$ be a planar graph. Is it possible to partition the vertex set of $G$ into two sets in such a way that one partition set induces a bipartite graph and the other one induces a 2-list colorable graph?

If such a partition would always exist for planar graphs, then it would strengthen the Four Color Theorem. Moreover, we have the following relationship to Problem 1.

**Claim 3.** If the vertex set $V$ of a planar graph $G$ can be partitioned into $V_1$ and $V_2$ such that $V_1$ induces a bipartite graph and $V_2$ induces a 2-list colorable graph then $G$ is $L$-colorable for every 2-common 4-assignment $L$.

**Proof.** Let $G$ be a planar graph and $L$ be a 2-common 4-assignment for the vertices of $G$ with $\{\alpha, \beta\} \subseteq L(v)$ for all $v \in V(G)$. Properly color the subgraph induced by $V_1$ with $\alpha$ and $\beta$ and set $L'(v) = L(v) \setminus \{\alpha, \beta\}$ for all $v \in V_2$. Since the subgraph induced by $V_2$ is 2-list colorable it can be colored from the remaining lists $L'(v)$. \hfill \square

Since every acyclic graph is 2-list colorable we may put a stronger question for the partition of $G$.

**Problem 4.** Let $G$ be a planar graph. Is it possible to partition the vertex set $V$ into $V_1$ and $V_2$ such that the subgraph induced by $V_1$ is a bipartite graph and the subgraph induced by $V_2$ is a forest?

Unfortunately, this is not possible for every planar graph as shown by Wegner in 1973 [8].

**Theorem 5.** There is a planar graph $G$ such that in every proper 4-coloring of $G$ the vertices of every two color classes induce a subgraph containing a cycle.

![Figure 1: Subgraph $G_1$ of $G$](image)

The construction of Wegner does not give an answer to Problem 1 but based on his construction, we were able to find our construction.
Theorem 6. There is a planar graph $G$ and a 2-common 4-assignment $L$ such that $G$ is not $L$-colorable.

Proof. We will construct a planar graph $G$ and a 2-common 4-assignment $L$ in two steps. In the first step we consider the subgraph $G_1$ of $G$, which is shown in Figure 1. The structures inside the triangles $D_1$ and $D_2$ are depicted separately outside.

Figure 2: Subgraph $G_1$ with a 2-common 4-list assignment and two precolored vertices

Let $v_1$ be precolored by 1 and $v_2$ be precolored by 2, and consider the list assignment for the other vertices of $G_1$ given in Figure 2. Assume that there is a proper coloring $c$ that assigns every vertex $v$ a color $c(v) \in L(v)$ such that adjacent vertices get different colors.

At first, let $v_{10}$ be colored by $\alpha$. Clearly, one of the vertices $v_4$ and $v_5$ must be colored by 3 since otherwise $v_3$ would not be colorable.

- Case 1: $c(v_4) = 3$

Since $c(v_5) \in \{\alpha, \beta\}$ and $\{c(v_6), c(v_7)\} \subset \{4, \alpha, \beta\}$ for the vertices of the triangle $v_5v_6v_7$ it follows that $c(v_6) = 4$ or $c(v_7) = 4$, which implies $c(v_9) = \beta$ and then $c(v_8) = 4$. Hence, the triangle completely in the interior of $D_1$ must be colored with colors $\alpha$ and $\beta$, a contradiction.
Case 2: $c(v_5) = 3$

Clearly $\{c(v_8), c(v_9)\} = \{4, \beta\}$, which implies successively that $c(v_7) = \alpha$, $c(v_4) = \beta$, $c(v_8) = 4$, $c(v_9) = \beta$, and finally $c(v_6) = 4$. Consequently, the triangle in the interior of $D_2$ must be colored with the two colors $\alpha$ and $\beta$, again a contradiction.

Secondly, let $v_{10}$ be colored by $\beta$. This can be handled using analogous arguments by interchanging the roles of $\alpha$ and $\beta$.

Therefore, the subgraph $G_1$ of $G$ with given precoloring and list assignment as in Figure 2 is not $L$-list colorable.

Next, consider the complete graph $K_4$ where the list of all vertices is $\{1, 2, \alpha, \beta\}$. Construct the graph $G$ as follows. For every edge $xy$ in $K_4$ add two copies of the graph $G_1$ identifying its edge $v_1v_2$ with the edge $xy$, once with $x = v_1$ and $y = v_2$ and the other time with $y = v_1$ and $x = v_2$ (see Figure 3) and then identify the vertices $u$, $v$, and $w$.

Clearly, two vertices of of $K_4$ must be colored by 1 and 2 giving exactly the above precoloring for one of the corresponding subgraphs $G_1$. Hence, $G$ is planar and not $L$-list colorable, where all lists of the list assignment $L$ have length 4 and contain the elements $\alpha$ and $\beta$.

Since the vertices $u$, $v$, and $w$ in Figure 3 are identified, the graph $G$ constructed above has $8 + 12 \cdot 13 = 164$ vertices.

Considering Claim 3 and Theorem 6 we obtain the answer to Problem 2, which also improves the above mentioned result of Wegner. Moreover, in some sense this conclusion is a sharpness result for the Four Color Theorem.

Corollary 7. There is a planar graph $G$ such that in every proper 4-coloring of $G$ the vertices of every two color classes induce a subgraph that is non-2-list colorable.

Finally, let us mention a related concept introduced by Kratochvíl et al. in [4]. A list assignment $L$ for a graph $G = (V, E)$ is called a $(k, c)$-assignment if $L(v) = k$ for all $v \in V(G)$ and $|L(v) \cap L(w)| \leq c$ for all edges $vw \in E(G)$. In [3] it is mentioned that every
planar graph is \(L\)-list colorable for every \((4, 1)\)-assignment \(L\). Moreover, there exist planar graphs \(G\) and corresponding \((4, 3)\)-assignments \(L\) such that \(G\) is not \(L\)-list colorable. So far, there is no result for \((4, 2)\)-assignments and the following problem remains open.

**Problem 8.** Is every planar graph \(L\)-list colorable for every \((4, 2)\)-assignment \(L\)?

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