Online Starvation Mitigation to Balance Average Flow Time and Fairness

Tung-Wei Kuo
Department of Computer Science
National Chengchi University
twkuo@cs.nccu.edu.tw

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Abstract

In job scheduling, it is well known that Shortest Remaining Processing Time (SRPT) minimizes the average flow time. However, SRPT may cause starvation and unfairness. To balance fairness and average flow time, one common approach is to minimize the $\ell_2$ norm of flow time. All non-trivial algorithms designed for this problem are offline algorithms based on linear programming rounding. For the online setting, all previous works consider standard scheduling algorithms under the assumptions of speed augmentation or certain input distributions. In their seminal paper, Bansal and Pruhs prove that under speed augmentation, fairness is not sacrificed much when SRPT is used [SICOMP 2010]. However, in practice, to achieve better fairness, it is not uncommon to complement SRPT with some starvation mitigation mechanism.

Nonetheless, starvation mitigation inevitably destroys SRPT’s optimality in minimizing the average flow time. Thus, it is not clear whether starvation mitigation can improve SRPT’s performance on minimizing the $\ell_2$ norm of flow time. In this paper, we answer this question in the affirmative. Let $n$ be the number of jobs. We use an estimate of $n$ to carefully mitigate the starvation caused by SRPT. Given a good estimate of $n$, our starvation mitigation mechanism reduces the competitive ratio of SRPT for the $\ell_2$ norm of flow time from $\Omega(n^{1/2})$ to $O(n^{1/3})$. Finally, we remark that all the online algorithms considered previously for this problem have competitive ratios $\tilde{\Omega}(n^{1/2})$. 
1 Introduction

1.1 Background

When a job is submitted to a server, the response time (i.e., the amount of time between job release and job completion) is usually the primary concern. In job scheduling terminology, response time is also termed the flow time. To improve the average quality of service, an obvious approach is to minimize the average flow time. It is well known that Shortest Remaining Processing Time (SRPT) minimizes the average flow time. However, to minimize the average flow time, some jobs may starve and cause inequitable flow time. In their classic textbook, *Operating System Concepts*, Silberschatz, Galvin, and Gagne argue that a system with reasonable and predictable response time may be considered more desirable than a system that is faster on the average, but is highly variable [26]. To avoid starvation and improve fairness, one possible approach is to minimize the maximum flow time, which can be done by First-Come-First-Served (FCFS) [6]. However, to minimize the maximum flow time, the average flow time may increase significantly and degrade the system-wide efficiency.

To balance the average flow time (or equivalently, the $\ell_1$ norm of flow time) and the maximum flow time (or equivalently, the $\ell_\infty$ norm of flow time), a natural approach is to minimize the $\ell_2$ norm of flow time. For the offline setting, there are constant factor approximation algorithms for minimizing the general $\ell_p$ norm of flow time [18, 5, 12]. All the non-trivial offline algorithms for minimizing the $\ell_2$ norm of flow time are based on linear programming rounding [18, 5, 12, 3]. For the online setting, only standard scheduling algorithms, including SRPT, Shortest Job First (SJF), Round Robin (RR, which shares the machine equally among all jobs), and Shortest Execution Time First (SETF), have been studied, and these analyses are based on speed augmentation [4, 17]. In particular, Bansal and Pruhs prove that SRPT is $(1 + \epsilon)$-speed $O(1/\epsilon)$-competitive for all $\epsilon > 0$ [4]. On the other hand, some experimental results indicate that when the job size distribution is heavy tailed, the fear that SRPT may cause starvation is unfounded [2, 9, 15, 25, 4, 7].

1.2 Motivation: The Need to Mitigate Starvation for SRPT in Practice

The above results show that under the assumptions of speed augmentation or special job size distributions, fairness may not be a serious concern for SRPT. However, in practice, it is not uncommon to further improve SRPT’s fairness by some starvation mitigation mechanism. For example, Mangharam et al. observe that SRPT may cause unfairness in multimedia transmission [22], and some SRPT-based schedulers have starvation mitigation mechanisms [22, 27, 10, 21, 11, 23].

Nonetheless, starvation mitigation inevitably destroys the optimality of SRPT for the $\ell_1$ norm of flow time. Because the $\ell_2$ norm of flow time balances the $\ell_1$ norm of flow time and the $\ell_\infty$ norm of flow time, it is unclear whether starvation mitigation is beneficial for SRPT to minimize the $\ell_2$ norm of flow time. As examples, we detail two natural starvation mitigation methods that cannot improve SRPT’s $\ell_2$ norm of flow time in Section 6. Thus, we have the following question, which is of both theoretical and practical interest:

**Question 1.1.** For SRPT, can starvation mitigation improve the $\ell_2$ norm of flow time?

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1 The competitive ratio of an algorithm $A$ for minimizing $\text{obj}$ is $\max I \frac{\text{obj}(A, I)}{\text{obj}(OPT, I)}$, where $\text{obj}(A, I)$ is the objective value achieved by algorithm $A$ on problem instance $I$ and $OPT$ is the optimal algorithm. In addition, $A$ is $s$-speed $c$-competitive if its competitive ratio is at most $c$ when it is running on a machine that is $s$ times faster than the original machine.

2 In fact, SRPT, SJF, SETF, and RR are all $O(1)$-competitive for minimizing the general $\ell_p$ norm of flow time under $O(1)$-speed augmentation.
1.3 Our Result

In this paper, we assume that a good estimate of \( n \) is available. For example, usually job scheduling is only critical during peak hours. Moreover, in many systems, the job arrival process is a Poisson process. Thus, the product of the job arrival rate and the duration of peak hours would be a good estimate of \( n \). Nonetheless, in this study, we do not make any assumptions about the job arrival process or the estimation method. Instead, we assume that an estimate of \( n \) with bounded error is given. Specifically, for any \( \rho \) and \( \sigma \) that satisfy \( 0 < \rho \leq 1 \leq \sigma \), an estimate \( \tilde{n} \) of \( n \) is termed a \((\rho, \sigma)\)-estimate of \( n \) if \( \rho n \leq \tilde{n} \leq \sigma n \). \( \rho \) and \( \sigma \) can be viewed as the estimation error rates.

In this paper, we propose SRPT with Starvation Mitigation (SRPT/SM). We use \( \tilde{n} \) to mitigate the starvation caused by SRPT without sacrificing the \( \ell_1 \) norm of flow time too much. Recall that SRPT is optimal for the \( \ell_1 \) norm of flow time, and FCFS is optimal for the \( \ell_\infty \) norm of flow time. Our algorithm can also be viewed as a combination of SRPT and FCFS. Specifically, when a job has been waiting for a sufficiently long time, it becomes an urgent job. Our algorithm follows SRPT when there is no urgent job. When there is some urgent job, the job that becomes urgent first is served first.

It can be shown easily that the competitive ratio of SRPT for minimizing the \( \ell_2 \) norm of flow time is \( \Omega(n^{\frac{2}{3}}) \). We then answer Question 1.1 in the affirmative. Given a good estimate of \( n \), SRPT/SM is \( O(n^{\frac{1}{3}}) \)-competitive for minimizing the \( \ell_2 \) norm of flow time. Specifically, we prove the following result.

**Theorem 1.2.** SRPT/SM is \( O((\rho^{-\frac{1}{3}} + \sigma^{\frac{1}{3}})n^{\frac{1}{3}}) \)-competitive for minimizing the \( \ell_2 \) norm of flow time.

We remark that all the aforementioned online algorithms for this problem have competitive ratios \( \tilde{\Omega}(n^{\frac{2}{3}}) \).

1.4 Definitions

We consider the online problem of minimizing the \( \ell_2 \) norm of flow time. In this problem, there are \( n \) jobs, \( J_1, J_2, \ldots, J_n \), and one server, and each job \( J_i \) has a processing time \( p_i \) and a release time \( r_i \). As in [3], we assume that \( p_i \) and \( r_i \) are in \( \mathbb{N} \).

For any schedule \( S \), we use \( c_i(S) \) to denote the completion time of job \( J_i \) under \( S \). The flow time of a job \( J_i \) under schedule \( S \), denoted by \( f_i(S) \), is \( c_i(S) - r_i \). Define \( F(S) = \sum_{i=1}^{n} (f_i(S))^2 \). The goal is to compute a schedule \( S \) that minimizes the \( \ell_2 \) norm of flow time, i.e., \( \sqrt{F(S)} \). Thus, any \( c \)-competitive algorithm for minimizing \( F \) is \( c^{1/2} \)-competitive for minimizing the \( \ell_2 \) norm of flow time. Throughout this paper, we use \( ALG, SRPT \), and \( OPT \), to denote the schedule obtained by the proposed algorithm (i.e., SRPT/SM), SRPT, and the optimal solution, respectively. A job is said to be active at time \( t \) under schedule \( S \), if it is released by time \( t \) but has not yet been completed by time \( t \) under \( S \). We use \( A(t, S) \) to denote the index set of the active jobs under \( S \) at time \( t \).

For every \( t \in \mathbb{N} \), the time slot \([t]\) is defined as the time interval between time \( t \) and time \( t+1 \). Thus, we can divide time into time slots \([0],[1],[2], \ldots \). We can view each job \( J_i \) as a chain of tasks \( J_{i,1}, J_{i,2}, \ldots, J_{i,p_i} \), where task \( J_{i,k} \) has unit processing time and a release time \( r_i + k - 1 \). Thus, every schedule must execute tasks of the same job in increasing order of release times. Because all the processing times and release times are in \( \mathbb{N} \), by a simple exchange argument, we can assume that under the optimal schedule, the server never executes more than one task in time slot \([t]\) for any \( t \in \mathbb{N} \) (i.e., the server is either idle or is executing the same task throughout the entire time slot \([t]\)). Thus, in this paper, we view a schedule \( S \) as a function that maps every \( t \in \mathbb{N} \) to some (possibly empty) task executed in time slot \([t]\). If a task is executed in time slot \([t]\), then it is completed at time \( t+1 \). A task is said to be queued at time \( t \) under schedule \( S \),

\(^3\)In this paper, we assume \( \mathbb{N} \) contains 0.
In this paper, we consider functions from a subset of \([1, n]\) to \(\mathbb{N}\). In this paper, we call these functions maps. Thus, \(q_{i,S}(t)\) is a map that maps every \(i \in [1, n]\) to the number of \(J_i\)’s queued tasks at time \(t\) under \(S\). For any map \(f\), \(\text{dom } f\) denotes the domain of \(f\), and \(f^2\) denotes the map such that \(\text{dom } f^2 = \text{dom } f\) and \(f^2(x) = f(x)^2\) for any \(x \in \text{dom } f^2\). For any map \(f\) and any set \(S \subseteq \text{dom } f\), the restriction of \(f\) to \(S\), denoted by \(f|_S\), is a map from \(S\) to \(\mathbb{N}\) such that \(f|_S(i) = f(i)\) for any \(i \in S\).

**Definition 1.3.** For any two maps \(f\) and \(g\), \(f\) dominates \(g\) (or \(g\) is dominated by \(f\)), denoted by \(f \geq g\), if \(\text{dom } f = \text{dom } g\) and \(f(x) \geq g(x)\), \(\forall x \in \text{dom } f\).

**Definition 1.4.** Let \(f\) be any map. Define \(\pi_f\) as a function that maps any \(k \in [1, |\text{dom } f|]\) to the element in \(\text{dom } f\) that has the \(k\)th largest output of \(f\) (ties can be broken arbitrarily). Thus, \(\text{dom } f = \{\pi_f(1), \pi_f(2), \ldots, \pi_f(|\text{dom } f|)\}\) and \(f(\pi_f(1)) \geq f(\pi_f(2)) \geq \cdots \geq f(\pi_f(|\text{dom } f|))\).

**Definition 1.5.** Let \(f\) be any map. Define \(S(f) = \sum_{i \in \text{dom } f} f(i)\). Moreover,

\[
S_k(f) = \begin{cases} 
0 & \text{if } k = 0 \\
\sum_{j=1}^{k} f(\pi_f(j)) & \text{if } k \in [1, |\text{dom } f|] \\
S(f) & \text{if } k > |\text{dom } f| 
\end{cases}
\]

For any \(k \in [1, |\text{dom } f|]\), define

\[
P_k(f) = \sum_{j=|\text{dom } f| - k + 1}^{|\text{dom } f|} f(\pi_f(j)).
\]

1.5 Our Techniques

We consider two lower bounds of \(F(OPT)\). At every time \(t \in \mathbb{N}\), if there is some queued task, any schedule should execute some queued task in time slot \([t]\). A schedule satisfying the above property is called an efficient schedule. All efficient schedules have the same number of queued tasks at any time. Let \(q(t)\) be the number of queued tasks at time \(t\) under any efficient schedule. We have the following simple lower bound of \(F(OPT)\), whose proof can be found in Appendix B.

**Lemma 1.6.**

\[
F(OPT) \geq \sum_{t \in \mathbb{N}} q(t).
\]

On the other hand, observe that if all jobs are released concurrently, jobs should be executed in increasing order of processing times. Fix some time \(t^*\), we focus on the active jobs at time \(t^*\) under \(OPT\). To derive the second lower bound (Lemma 2.3), we set \(r_i = t^*\) for every \(i \in A(t^*, OPT)\), and optimize \(\sum_{i \in A(t^*, OPT)} (f_i(OPT))^2\) using the previous observation. In our analysis, we set \(t^*\) to be the time at which SRPT/SM has the most urgent tasks.5

For each lower bound, there exist instances where the ratio of \(F(OPT)\) to the lower bound is \(\Omega(n)\). Thus, the main challenge is to combine these two lower bounds and to compare the combined lower bound to \(F(ALG)\). In our analysis, we consider a trimmed instance \(\mathcal{I}^*\) of the original instance. To construct \(\mathcal{I}^*\), we remove all the tasks released after time \(t^*\) from the original instance.6 Let \(SRPT^*\) be the schedule obtained by applying SRPT to \(\mathcal{I}^*\). Because

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4In this paper, for any \(a, b \in \mathbb{N}\) with \(a \leq b\), \([a, b]\) is defined as \([i| i \in \mathbb{N}, a \leq i \leq b]\). If \(a > b\), then \([a, b] = \emptyset\).

5If a job becomes urgent, all its remaining tasks become urgent as well.

6Thus, jobs released after time \(t^*\) have zero processing time and zero task in \(\mathcal{I}^*\).
a schedule can be viewed as a function that maps every $t \in \mathbb{N}$ to the (possibly empty) task executed in time slot $[t]$ and $\mathcal{I}^*$ is identical to the original instance by time $t^*$, $\text{SRPT}^*$ is also an efficient schedule for the original instance by time $t^*$. In particular, $q_{t^*, \text{SRPT}}(i)$ is the number of queued tasks of $J_i$ at time $t^*$ under $\text{SRPT}^*$ in the original instance. We will show that $q_{t^*, \text{SRPT}}$ majorizes $q_{t^*, \text{OPT}}$ (Lemma 3.7).

**Definition 1.7.** For any two maps $f$ and $g$, $f$ majorizes $g$, denoted by $f \geq_M g$, if the following two conditions are satisfied.

- $S(f) = S(g)$.
- $S_k(f) \geq S_k(g), \forall k \in \mathbb{N}$.

Observe that if $f \geq_M g$, then $S(f^2) \geq S(g^2)$. The notion of majorization is first studied by Hardy et al. [16], and Golovin et al. use majorization to study all symmetric norms of flow time [14]. The original definition of majorization deals with vectors instead of maps. For example, in [14], the notion of majorization is applied to vectors consisting of the flow time of all jobs. In this paper, we do not directly consider the flow time. Instead, we consider the number of queued tasks and the number of remaining tasks. In addition, we will consider the restriction of a map to some subset of $[1, n]$ (e.g., $A(t, \text{SRPT})$). Thus, we use maps instead of vectors.

In this paper, we show that if $f \geq_M g$, then for any map $f'$ dominated by $f$, there exists a map $m$ such that $g \geq m$, $S(f'^2) \geq S(m^2)$, and $S(m) = \Omega(S(f'))$ (Lemma 3.11). The existence of a map that satisfies the above three constraints is critical in our analysis. In addition to $f \geq_M g$, we give two more sufficient conditions for the existence of a map that satisfies similar constraints (Lemmas 3.5 and 3.9). These two sufficient conditions are used when we consider the relationship between $q_{t^*, \text{ALG}}$ and $q_{t^*, \text{SRPT}}$ and the relationship between $q_{t^*, \text{SRPT}}$ and $q_{t^*, \text{OPT}}$. As a result, we can obtain the relationship between $q_{t^*, \text{ALG}}$ and $q_{t^*, \text{OPT}}$, which will be used to combine the two lower bounds and to compare the combined lower bound $F(\text{ALG})$.

1.6 Related Results

For the $\ell_1$ norm of flow time, SRPT is optimal. For the $\ell_\infty$ norm of flow time, FCFS is optimal [6]. In [4], SRPT, SJF, and SETF are shown to be scalable for the general $\ell_p$ norm of flow time. The results are further extended to all symmetric norms of flow time [14] and identical machines [8, 13, 17]. Specifically, when there are multiple identical machines, SRPT is scalable for the general $\ell_p$ norm of flow time [13], and RR is $O(1)$-speed $O(1)$-competitive for the $\ell_2$ norm of flow time [17]. In [3, 19, 1], more general objective functions are considered. Specifically, for a job $J_i$ with flow time $f_i$, a cost $g_i(f_i)$ is incurred. The only restriction on $g_i$ is that $g_i$ must be non-decreasing. For this general cost minimization problem, there is an $O(1)$-speed $O(1)$-competitive algorithm [19, 1] and an $O(\log \log P)$-approximation algorithm [3]. Finally, for minimizing the general $\ell_p$ norm of flow time, there are $O(1)$-approximation algorithms [18, 12, 5].

2 The Algorithm and the Competitive Analysis

2.1 The Algorithm: SRPT/SM

For brevity, we define $q_t$ as $q_{t, \text{ALG}}$. Let $M_{\text{SRPT}}(t)$ (respectively, $M_{\text{ALG}}(t)$) be the set of tasks executed in time slots $[0], [1], [2], \ldots, [t]$ under SRPT (respectively, SRPT/SM). Define $M_{\text{ALG}}(-1) = \emptyset$. Note that SRPT/SM can obtain $M_{\text{SRPT}}(t)$ by simulating SRPT.

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7 A schedule $\mathcal{S}$ is efficient for an instance $\mathcal{I}$ by time $t$ if it satisfies the following constraint: For any $t' \in [0, t-1]$, if there is a queued task at time $t'$ in $\mathcal{I}$, then $\mathcal{S}$ executes some queued task in $\mathcal{I}$ in time slot $[t']$.

8 A scheduling algorithm is scalable if for all $\epsilon > 0$, it is $(1 + \epsilon)$-speed $c$-competitive for some constant $c$ (which may depend upon $\epsilon$) [24, 20].
To avoid starvation, SRPT/SM categorizes active jobs into two types, urgent and normal. An active job \( J_i \) is said to be \textbf{urgent} at time \( t \) if \( t - (r_i + p_i) \geq \tilde{n}^{2/3} \); otherwise, \( J_i \) is said to be normal. Thus, every job is normal initially. Observe that once a job becomes urgent, the job will always be urgent until completion. In addition, a job can only become urgent after it becomes urgent. That is, \( t_i \) is the smallest \( t \) that satisfies \( t - (r_i + p_i) \geq \tilde{n}^{2/3} \). If \( J_i \) is an FaU job, \( t_i = c_i(\mathcal{ALG}) \).

When there are urgent jobs, SRPT/SM executes the urgent job that has the smallest \( t_i \). In other words, SRPT/SM executes the job that becomes urgent first. If there is no urgent job, SRPT/SM then follows SRPT. More specifically, to decide the task to be executed in time slot \([t] \) when there is no urgent job, we first find an arbitrary task \( J_{i,k} \in MSRPT(t) \setminus M_{\text{ALG}}(t - 1) \). SRPT/SM then executes job \( J_i \) in time slot \([t] \). Notice that if \( MSRPT(t) \setminus M_{\text{ALG}}(t - 1) = \emptyset \) then no job needs to be executed in time slot \([t] \). In the pseudocode of SRPT/SM, we use \( U \) to denote the set of urgent jobs and initially \( U = \emptyset \). We report some simulation results in Appendix A.

\[2.2 \quad \text{Analysis of SRPT/SM}\]

A job \( J_i \)'s \textbf{post-urgent flow time}, denoted by \( \text{post}(i) \), is defined as \( c_i(\mathcal{ALG}) - t_i \), and \( J_i \)'s \textbf{pre-urgent flow time}, denoted by \( \text{pre}(i) \), is defined as \( t_i - (r_i + p_i) \). \( F(\mathcal{ALG}) \) can then be expressed as

\[
F(\mathcal{ALG}) = \sum_{i=1}^{n} (p_i + \text{pre}(i) + \text{post}(i))^2 = O(\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} \text{pre}(i)^2 + \sum_{i=1}^{n} \text{post}(i)^2).
\]

Obviously, \( \sum_{i=1}^{n} p_i^2 \leq F(\text{OPT}) \). Based on Lemma 1.6, we give an upper bound of \( \sum_{i=1}^{n} \text{pre}(i)^2 \) in the following lemma, whose proof can be found in Appendix B.

\[
\text{Lemma 2.1. } \sum_{i=1}^{n} \text{pre}(i)^2 = O((\sigma n)^{\frac{3}{2}})F(\text{OPT}).
\]

By the above lemma, to prove Theorem 1.2, it suffices to show \( \sum_{i=1}^{n} \text{post}(i)^2 = O(\rho^{\frac{1}{2}}n^{\frac{3}{2}})F(\text{OPT}) \). Note that \( \text{post}(i) = 0 \) if \( J_i \) is an FaU job. Next, we give an upper bound of \( \text{post}(i) \) for an FaU job \( J_i \). Once \( J_i \) becomes urgent, all the remaining tasks of \( J_i \) are said to be urgent as well.
Note that at time $t_i$, all the remaining tasks of $J_i$ have been released. Let $u(t)$ be the number of urgent tasks that are queued at time $t$. Under SRPT/SM, at time $t_i$, $J_i$ waits at most $u(t_i)$ time slots before completion. Thus, $c_i(ALG) \leq t_i + u(t_i)$ and $post(i) \leq u(t_i)$. Define $t^* = \arg \max_{t \in \mathbb{N}} u(t)$. We then have $post(i) \leq u(t^*)$ and thus $\sum_{i=1}^n post(i)^2 = O(n \cdot u(t^*)^2)$. As a result, to prove Theorem 1.2, it is sufficient to show

$$\rho^{-\frac{1}{3}} n^{\frac{2}{3}} F(OPT) = \Omega(n \cdot u(t^*)^2).$$

We use the following two lemmas to prove Eq. (1).

**Lemma 2.2.** There exists a map $m_{OPT}$ that satisfies the following properties:

- **P1:** $q_{t^*,OPT} \preceq m_{OPT}$.
- **P2:** $S(m_{OPT}) = \Omega(u(t^*))$.
- **P3:** $F(OPT) \geq \tilde{n}^2 S(m_{OPT})^2$.

**Lemma 2.3.** Let $m_{OPT}$ be the map defined in Lemma 2.2. We then have

$$F(OPT) = \Omega \left( \sum_{i=1}^n P_i(m_{OPT})^2 \right).$$

**Proof of Eq. (1):** For simplicity, we reindex jobs so that $m_{OPT}(1) \leq m_{OPT}(2) \leq \cdots \leq m_{OPT}(n)$. We then have

$$\rho^{-\frac{1}{3}} n^{\frac{2}{3}} F(OPT)
= \Omega(\rho^{-\frac{1}{3}} n^{\frac{2}{3}}) \left( \sum_{i=1}^n \left( \sum_{h=1}^i m_{OPT}(h) \right)^2 + \tilde{n}^{2/3} \sum_{i=1}^n m_{OPT}(i)^2 \right)
= \Omega(\rho^{-\frac{1}{3}} n^{\frac{2}{3}} \tilde{n}^{\frac{1}{3}}) \left( \sum_{i=1}^n \left( \sum_{h=1}^i m_{OPT}(h) \right)^2 \right) \left( \sum_{i=1}^n m_{OPT}(i)^2 \right)
= \Omega(\rho^{-\frac{1}{3}} n^{\frac{2}{3}} (\rho \tilde{n})^{\frac{1}{3}}) \sum_{i=1}^n \left( \sum_{h=1}^i m_{OPT}(h) \right) m_{OPT}(i)
= \Omega(n) \left( \sum_{i=1}^n m_{OPT}(i) \right)^2
= \Omega(n) u(t^*)^2$$

(by P2 in Lemma 2.2).

The proof of Lemma 2.3 can be found in Appendix B. The remainder of this paper is mainly devoted to the proof of Lemma 2.2.

### 2.3 Proof Sketch of Lemma 2.2

To prove Lemma 2.2, we only need to focus on the case where $u(t^*) > 0$. This is because when $u(t^*) = 0$, we can simply set $m_{OPT}(i) = 0$ for every $i \in [1,n]$ to satisfy all the properties in Lemma 2.2. Thus, in the following proof, we assume $u(t^*) \geq 1$. To prove Lemma 2.2, we first construct a map $m_{ALG}$ such that:
\begin{itemize}
  \item $F(OPT) \geq \frac{n^\frac{2}{3}}{m_{\text{ALG}}^2}$.
  \item $S(m_{\text{ALG}}) \geq u(t^*)$.
\end{itemize}

The lower bound $\frac{n^\frac{2}{3}}{m_{\text{ALG}}^2}$ is obtained by Lemma 1.6. For brevity, we introduce the notation $\supseteq$.

**Definition 2.4.** Let $f$ and $g$ be any two maps. We write $f \supseteq g$ if the following two properties hold:
\begin{itemize}
  \item $S(f^2) \geq S(g^2)$.
  \item $S(g) \geq c \cdot S(f)$ for some constant $0 < c < 1$.
\end{itemize}

Thus, to prove Lemma 2.2, it suffices to construct a map $m_{OPT}$ such that:
\begin{itemize}
  \item $q_{*,OPT} \supseteq m_{OPT}$.
  \item $m_{\text{ALG}} \supseteq m_{OPT}$.
\end{itemize}

To this end, we first show that there exists a map $m_{\text{SRPT}}$ such that $q_{*,\text{SRPT}} \supseteq m_{\text{SRPT}}$ and $m_{\text{ALG}} \supseteq m_{\text{SRPT}}$. Recall the schedule $\text{SRPT}^*$ defined in Section 1.5. We then show that there exists a map $m_{\text{SRPT}^*}$ such that $q_{*,\text{SRPT}^*} \supseteq m_{\text{SRPT}^*}$ and $m_{\text{SRPT}} \supseteq m_{\text{SRPT}^*}$. Thus, $m_{\text{ALG}} \supseteq m_{\text{SRPT}^*}$. Finally, we construct the desired map $m_{OPT}$ based on the properties of $m_{\text{SRPT}^*}$ and the fact $q_{*,\text{SRPT}^*} \supseteq q_{*,OPT}$. The constructions of the above four key maps (i.e., $m_{\text{ALG}}, m_{\text{SRPT}}, m_{\text{SRPT}^*}$, and $m_{OPT}$) are given in Section 3.

## 3 The Four Key Maps

### 3.1 The First Key Map: $m_{\text{ALG}}$

**Definition 3.1.** Let $f$ be any map such that $S(f) \geq 1$. Let $c$ be any integer in $[1, S(f)]$. Let $lu(f, c)$ be the least integer such that $S_{lu(f, c)}(f) \geq c$. The **truncation** of $f$ at $c$, denoted by $f_{(c)}$, is a map dominated by $f$ such that
\[
    f_{(c)}(\pi(f)(k)) = \begin{cases} 
    f(\pi(f)(k)) & \text{if } k \in [1, lu(f, c) - 1] \\
    c - S_{lu(f, c) - 1}(f) & \text{if } k = lu(f, c) \\
    0 & \text{if } k \in [lu(f, c) + 1, \lvert \text{dom } f \rvert]
    \end{cases}
\]

Finally, $f_{(c)}$ is said to be a valid truncation if $c \in [1, S(f)]$.

**Example 3.2.** Assume $f(i) = i \cdot 10$ and dom $f = [1, 10]$. We then have $S_3(f) = 100 + 90 + 80 = 270$ and $S_{4}(f) = 100 + 90 + 80 + 70 = 340$. If $c = 300$, then $lu(f, c) = 4$, $f_{(c)}(10) = 100$, $f_{(c)}(9) = 90$, $f_{(c)}(8) = 80$, and $f_{(c)}(7) = 30$. For all $i \in [1, 6]$, $f_{(c)}(i) = 0$.

To construct $m_{\text{ALG}}$, we first introduce a new map $\hat{q}_t$ that dominates $q_t$. For every $t \in \mathbb{N}$ and $i \in [1, n]$, we define $\hat{q}_t(i)$ as follows. If $J_i$ is an FaN job, $\hat{q}_t(i) = q_t(i)$. If $J_i$ is an FaU job and $t \leq t_i$, $\hat{q}_t(i) = q_t(i)$. If $J_i$ is an FaU job and $t > t_i$, $\hat{q}_t(i) = q_t(i)$. Notice that when $t > t_i$, we have $q_{u_t}(i) \geq q_t(i)$ because the final task of $J_i$ is released before time $t_i$. Thus, $\hat{q}_t \geq q_t$. We stress that the definition of $\hat{q}_t(i)$ can be applied to $t > c_i(\text{ALG})$. We have the following lemma, whose proof can be found in Appendix C.

**Lemma 3.3.** Assume $u(t^*) \geq 1$. Let $I_u = \{i | J_i \text{ becomes urgent by time } t^*\}$. Let $f = \hat{q}_{t^*}$. Let $m_{\text{ALG}}$ be a map such that $m_{\text{ALG}}(i) = f_{(q_{(t^*)}(i))}$ if $i \in I_u$ and $m_{\text{ALG}}(i) = 0$ if $i \in [1, n] \setminus I_u$. We then have
\begin{itemize}
  \item $F(OPT) \geq \frac{n^\frac{2}{3}}{m_{\text{ALG}}^2}$.
  \item $S(m_{\text{ALG}}) \geq u(t^*)$.
\end{itemize}
3.2 The Second Key Map: $m_{SRPT}$

The proof of the following lemma can be found in Appendix C.

**Lemma 3.4.** For every $t \in \mathbb{N}$, $\hat{q}_t \succeq q_{*,SRPT}$.

The proof of following lemma can be found in Appendix D.

**Lemma 3.5.** Let $f$ and $g$ be two maps that satisfy the following two constraints:

T1: $f \succeq g$.

T2: $S(g) \geq 1$.

Then for any map $f'$ dominated by $f_{(S(g))}$, there exists a map $m$ such that:

- $g \succeq m$.
- $f' \sqsubseteq m$.

The following lemma proves the existence of $m_{SRPT}$.

**Lemma 3.6.** Let $m_{ALG}$ be the map defined in Lemma 3.3. There exists a map $m_{SRPT}$ such that:

- $q_{t*,SRPT} \succeq M q_{t*,SRPT}$.
- $m_{ALG} \sqsupseteq m_{SRPT}$.

**Proof.** Consider Lemma 3.5 and fix $f = \hat{q}_{t*}$ and $g = q_{t*,SRPT}$. By Lemma 3.4, T1 holds. T2 holds because $S(g) = q(t^*) \geq u(t^*) \geq 1$. By the definition of $m_{ALG}$ in Lemma 3.3, $m_{ALG}$ is dominated by $f_{(S(g))}$. Thus, the proof follows from Lemma 3.5.

3.3 The Third Key Map: $m_{SRPT^*}$

We give the proofs of the following two lemmas in the following sections.

**Lemma 3.7.** Let $S$ be any efficient schedule for the original instance. Then $q_{t*,SRPT^*} \succeq_M q_{t*,S}$.

**Lemma 3.8.** Let $f$ be any map dominated by $q_{t*,SRPT}$. Let $I = \{i \mid f(i) > 2q_{t*,SRPT^*}(i)\}$. If $I \neq \emptyset$, then $\max_{i \in I} f(i) > \frac{1}{2} \sum_{i \in I} f(i)$.

The proof of the following lemma can be found in Appendix D.

**Lemma 3.9.** Let $f$ and $g$ be two maps that satisfy the following three constraints:

D1: $\text{dom } f = \text{dom } g$.

D2: $S_1(g) \geq S_1(f)$.

D3: Let $I = \{i \mid f(i) > 2g(i)\}$. If $I \neq \emptyset$, then $\max_{i \in I} f(i) > \frac{1}{2} \sum_{i \in I} f(i)$.

Then there exists a map $m$ such that:

- $g \succeq m$.
- $f \sqsupseteq m$.

We are now ready to prove the following lemma.

**Lemma 3.10.** Let $m_{SRPT}$ be the map defined in Lemma 3.6. There exists a map $m_{SRPT^*}$ such that:

- $q_{t*,SRPT^*} \succeq m_{SRPT^*}$.
- $m_{SRPT} \sqsupseteq m_{SRPT^*}$.

**Proof.** Consider Lemma 3.9 and fix $f$ as the map $m_{SRPT}$ defined in Lemma 3.6, and fix $g = q_{t*,SRPT}$. Because $\text{dom } f = \text{dom } g = [1, n]$, D1 holds. By Lemma 3.7, $g \succeq_M q_{t*,SRPT}$. Because $q_{t*,SRPT} \succeq f$, D2 holds. By Lemma 3.8, D3 holds. The proof then follows from Lemma 3.9.
3.4 The Final Key Map: \( m_{OPT} \)

The proof of the following lemma can be found in Appendix D.

**Lemma 3.11.** If \( f \succeq_M g \), then for any map \( f' \) dominated by \( f \), there is a map \( m \) such that:

- \( g \succeq m \).
- \( f' \sqsubseteq m \).

The following lemma proves the existence of \( m_{OPT} \) in Lemma 2.2.

**Lemma 3.12.** Let \( m_{SRPT^*} \) be the map defined in Lemma 3.10. There exists a map \( m_{OPT} \) such that:

- \( q_{t^*, SRPT^*} \succeq m_{OPT} \).
- \( m_{SRPT^*} \sqsubseteq m_{OPT} \).

**Proof.** Consider Lemma 3.11 and fix \( f = q_{t^*, SRPT^*} \) and \( g = q_{t^*, OPT} \). By Lemma 3.7, \( f \succeq_M g \). Let \( f' \) be the map \( m_{SRPT^*} \) defined in Lemma 3.10. Thus, \( f \succeq f' \). The proof then follows from Lemma 3.11. \( \square \)

4 The Relationship Between \( q_{t^*, SRPT^*} \) and \( q_{t^*, S} \)

In this section, we prove Lemma 3.7. Recall the trimmed instance \( I^* \) defined in Section 1.5.

**Definition 4.1.** For any \( t \in \mathbb{N} \), any \( i \in [1, n] \), and any schedule \( S \), \( q_{t^*, S}^{trim}(i) \) is defined as the number of queued tasks of \( J_i \) at time \( t \) under schedule \( S \) in \( I^* \).

Because \( I^* \) and the original instance are identical by time \( t^* \), we have the following fact.

**Fact 4.2.** For any schedule \( S \) and any \( k \in \mathbb{N} \), \( S_k(q_{t^*, S}^{trim}) = S_k(q_{t^*, S}^{trim}) \).

**Definition 4.3.** For any \( t \in \mathbb{N} \), any \( i \in [1, n] \), and any schedule \( S \), \( l_{t^*, S}^{trim}(i) \) is defined as the number of remaining tasks of \( J_i \) at time \( t \) under schedule \( S \) in \( I^* \). In other words, \( l_{t^*, S}^{trim}(i) \) is the number of \( J_i \)'s tasks in \( I^* \) that are not completed by time \( t \) under \( S \). Similarly, for any \( t \in \mathbb{N} \), any \( i \in [1, n] \), and any schedule \( S \), \( l_{t^*, S}(i) \) is defined as the number of remaining tasks of \( J_i \) at time \( t \) under schedule \( S \) in the original instance.

Because SRPT always executes the job that has the least remaining tasks, and \( SRPT^* \) is obtained by applying SRPT to \( I^* \), the following lemma should not be too surprising.

**Lemma 4.4.** Let \( t \in \mathbb{N} \). Let \( S \) be any schedule that is efficient for \( I^* \) by time \( t \). Then \( l_{t^*, SRPT^*}^{trim} \succeq_M l_{t^*, S}^{trim} \).

The proof of the above lemma can be found in Appendix E. For any job \( J_i \), because all tasks of \( J_i \) are released by time \( t^* \) in \( I^* \), \( q_{t^*, S}^{trim}(i) = l_{t^*, S}^{trim}(i) \) for any schedule \( S \). As a result, we have the last fact for the proof of Lemma 3.7.

**Fact 4.5.** For any schedule \( S \) and any \( k \in \mathbb{N} \), \( S_k(q_{t^*, S}^{trim}) = S_k(l_{t^*, S}^{trim}) \).
**Proof of Lemma 3.7** First observe that because $SRPT^*$ and $S$ are efficient schedules by time $t^*$ for the original instance, $S(q_{r^*,SRPT}) = S(q_{r^*,S})$. For any $k \in [1, n]$, we have

$$
S_k(q_{r^*,SRPT}) = S_k(q_{r^*,SRPT}^*) \quad \text{(by Fact 4.2)}
$$

$$
= S_k(q_{r^*,SRPT}^*) \quad \text{(by Fact 4.5)}
$$

$$
\geq S_k(q_{r^*,SRPT}^*) \quad \text{(by Lemma 4.4)}
$$

$$
= S_k(q_{r^*,SRPT}^*) \quad \text{(by Fact 4.5)}
$$

$$
= S_k(q_{r^*,S}) \quad \text{(by Fact 4.2)}
$$

## 5 The Relationship Between $q_{r^*,SRPT}$ and $q_{r^*,SRPT}$

In this section, we prove Lemma 3.8.

**Definition 5.1.** A job $J_i$ is **untrimmed** if its last task is released by time $t^*$ in the original instance. Otherwise, $J_i$ is **trimmed**.

In the following definition, we assume that there is a common deadline $t^* + 1$ for every job in the original instance. Let $\tau \in [0, t^*]$. Because any schedule can complete at most $t^* + 1 - \tau$ tasks between time $\tau$ and time $t^* + 1$, $J_i$ cannot meet the deadline under $S$ if $l_{r^*,i}(\tau) > t^* + 1 - \tau$.

**Definition 5.2.** Let $\tau \in [0, t^*]$. A job $J_i$ is a **Definitely Late (DL)** job at time $\tau$ under schedule $S$ if $i \in A(\tau, S)$ and $l_{r^*,i}(\tau) > t^* + 1$. On the other hand, a job $J_i$ is said to be a **Possibly-In-Time (PIT)** job at time $\tau$ under schedule $S$ if $i \in A(\tau, S)$ and $l_{r^*,i}(\tau) \leq t^* + 1 - \tau$.

Define $I_{\tau,S}^{DL}$ as the set of indices of the DL jobs at time $\tau$ under schedule $S$. Similarly, define $I_{\tau,S}^{PIT}$ as the set of indices of the PIT jobs at time $\tau$ under schedule $S$.

Observe that regardless of the schedule, an untrimmed job $J_i$ is a PIT job when it is released, and a trimmed job $J_i$ is a DL job when it is released.

Notice that all the previous results about $SRPT^*$ hold for any tie-breaking rule to choose the job to be executed next. To simplify the proof of Lemma 3.8, we assume that $SRPT^*$ adopts the following tie-breaking rule to maximize the similarity between $SRPT^*$ and $SRPT$. Assume that in time slot $[\tau, \tau^*]$, $SRPT$ executes some task of job $J_i$ in the original instance. In our tie-breaking rule, we assume that if $J_i$ also has the fewest remaining tasks among all active jobs under $SRPT^*$ in $I^*$, then $SRPT^*$ also executes some task of $J_i$ in time slot $[\tau^*]$. We then have the following lemma.

**Lemma 5.3.** Let $\tau \in [0, t^*]$. If $I_{\tau,SRPT}^{PIT} \neq \emptyset$, then $SRPT$ and $SRPT^*$ execute the same task in time slot $[\tau, \tau^*]$.

**Lemma 5.4.** Let $S$ be any schedule. Let $\tau \in [0, t^*]$. Assume that in time slot $[\tau^*, \tau + 1]$, $S$ executes a job $J_i$ with $i \in I_{\tau,S}^{DL}$. Further assume that $S$ executes $s_i$ tasks of $J_i$ between time $\tau$ and time $t^*$. Then $q_{r^*,S}(\tau) \geq t^* + 1 - s_i$.

The proofs of the above two lemmas can be found in Appendix F.

**Proof of Lemma 3.8** For every $i \in I$, define $Q_i$ as the set of queued tasks of $J_i$ under $SRPT$ at time $t^*$. Thus, we have $|Q_i| = q_{r^*,SRPT}(i)$. Because $q_{r^*,SRPT} \geq f$, $|Q_i| - f(i) \geq 0$. We further remove the $|Q_i| - f(i)$ earliest released tasks from $Q_i$ so that $Q_i$ only has $f(i)$ tasks. For every $i \in I$, define $\delta_i$ as

$$
\delta_i = f(i) - q_{r^*,SRPT}(i).
$$
By the definition of $I$, $\delta_i > 0$. Moreover, for each $i \in I$, $\text{SRPT}^*$ completes the $\delta_i$ earliest released tasks in $Q_i$ by time $t^*$. Define $Q^* = \bigcup_{i \in I} Q_i$ and let $J_{i^*,k^*}$ be the task in $Q^*$ that has the earliest completion time under $\text{SRPT}^*$. Because $J_{i^*,k^*} \in Q^*$, we have

$$i^* \in I.$$  \hspace{1cm} (4)

Assume that $\text{SRPT}^*$ executes $J_{i^*,k^*}$ in time slot $[\tau]$. Thus, $\tau \leq t^* - 1$ and for each $i \in I$, $\text{SRPT}^*$ executes the $\delta_i$ earliest released tasks in $Q_i$ between time $\tau$ and time $t^*$. Thus, we have

$$\sum_{i \in I} \delta_i \leq t^* - \tau.$$  \hspace{1cm} (5)

Because $J_{i^*,k^*} \in Q_{i^*}$, $J_{i^*,k^*}$ is not executed in time slot $[\tau]$ under $\text{SRPT}$. By Lemma 5.3, $T^*_{i^*,\text{SRPT}^*} = \emptyset$. Thus, $i^* \in I^*_{\text{SRPT}^*}$. Because $\text{SRPT}^*$ completes $\delta_i$ tasks of $J_i$ between time $\tau$ and time $t^*$, by Lemma 5.4, we then have

$$q_{t^*,\text{SRPT}^*}(i^*) + \delta_i \geq t^* - \tau + 1.$$  \hspace{1cm} (6)

Thus,

$$f(i^*) \text{ by Eq. (4) and Eq. (3)} \geq q_{t^*,\text{SRPT}^*}(i^*) + \delta_i \text{ by Eq. (6)} \geq t^* - \tau + 1 \text{ by Eq. (5)} \geq \sum_{i \in I} \delta_i.$$  \hspace{1cm} (7)

For each $i \in I$, we have $q_{t^*,\text{SRPT}^*}(i) + \delta_i = f(i) > 2q_{t^*,\text{SRPT}^*}(i)$. Therefore, $\delta_i > q_{t^*,\text{SRPT}^*}(i)$ and thus $2\delta_i > f(i)$. By Eq. (7), we then have

$$f(i^*) > \sum_{i \in I} \delta_i > \frac{1}{2} \sum_{i \in I} f(i).$$

By Eq. (4), we then have $\max_{i \in I} f(i) \geq f(i^*) > \frac{1}{2} \sum_{i \in I} f(i)$.

### 6 Conclusion Remarks

In this paper, we answer Question 1.1 by estimating $n$. It is interesting to know whether this question can be solved without knowing any information of $n$. In our algorithm, the only place the estimate $\hat{n}$ is used is when we compare $\frac{\tau - (r_i + p_i)}{q_i(n)}$ (the density) and $\hat{n}^{2/3}$ (the threshold). One possible approach is to dynamically adjust the threshold according to the number of jobs released so far. Specifically, the threshold at time $t$ is $n_t^{2/3}$, where $n_t$ is the number of jobs released by time $t$. Compared to SRPT/SM, this approach mitigates starvation more aggressively because the threshold of becoming an urgent job is lower. However, this approach may prematurely designate some early-arriving large job as an urgent job, and thus the $\ell_1$ norm of flow time is increased significantly. It can be shown that this approach is $\Omega(n^{\frac{2}{3}})$-competitive for the $\ell_2$ norm of flow time.

Another approach is to transform SRPT/SM into a greedy algorithm where the active job with the highest density is processed. Specifically, at time $t$, if there is no active job satisfying $t \geq r_i + p_i + 1$, the algorithm follows SRPT; otherwise, the algorithm executes a task of the active job with the highest density in time slot $[t]$. However, this approach may still prematurely complete some early-arriving large job, and thus the $\ell_1$ norm of flow time is increased significantly. It can be shown that this greedy approach is $\Omega(n^{\frac{2}{3}}/\log n)$-competitive for the $\ell_2$ norm of flow time.
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13
Appendix A Simulation Results

In this section, we compare SRPT and SRPT/SM via simulation. We consider two performance metrics, $R_{\ell_2}$ and $R_{\ell_\infty}$. $R_{\ell_2}$ is the ratio of SRPT/SM’s $\ell_2$ norm of flow time to SRPT’s $\ell_2$ norm of flow time. That is,

$$R_{\ell_2} = \frac{\sqrt{F(ALG)}}{\sqrt{F(SRPT)}}.$$  

$R_{\ell_\infty}$ is the ratio of SRPT/SM’s $\ell_\infty$ norm of flow time to SRPT’s $\ell_\infty$ norm of flow time. That is,

$$R_{\ell_\infty} = \frac{\max_i f_i(ALG)}{\max_i f_i(SRPT)}.$$  

We assume that the job arrival process is a Poisson process with arrival rate $\lambda$, and we schedule jobs that arrive between time 0 and time $10^5$. Thus, we set $\tilde{n} = 10^5 \cdot \lambda$. We consider three
Table 1: Simulation Results.

| ℎ/ λ | Log-Normal + Admission Control | Uniform | Log-Normal + Admission Control | Uniform |
|------|--------------------------------|---------|--------------------------------|---------|
| 10   | 1.01                           | 1.01    | 1.00                           | 0.97    |
| 20   | 1.02                           | 1.02    | 1.03                           | 0.95    |
| 30   | 1.02                           | 1.03    | 1.04                           | 0.92    |
| 40   | 1.02                           | 1.03    | 0.99                           | 0.90    |
| 50   | 1.02                           | 1.02    | 0.99                           | 0.87    |
| 60   | 1.01                           | 1.01    | 0.99                           | 0.85    |
| 70   | 1.00                           | 0.99    | 1.02                           | 0.82    |
| 80   | 0.98                           | 0.97    | 1.00                           | 0.81    |
| 90   | 0.97                           | 0.95    | 0.99                           | 0.77    |
| 100  | 0.95                           | 0.94    | 0.96                           | 0.75    |
| 110  | 0.94                           | 0.91    | 0.94                           | 0.70    |
| 120  | 0.93                           | 0.89    | 0.92                           | 0.67    |
| 130  | 0.92                           | 0.87    | 0.90                           | 0.65    |
| 140  | 0.91                           | 0.85    | 0.88                           | 0.61    |
| 150  | 0.91                           | 0.85    | 0.88                           | 0.62    |
| 160  | 0.92                           | 0.86    | 0.84                           | 0.69    |
| 170  | 0.93                           | 0.86    | 0.84                           | 0.69    |
| 180  | 0.94                           | 0.91    | 0.77                           | 0.70    |
| 190  | 0.96                           | 0.94    | 0.74                           | 0.85    |
| 200  | 0.97                           | 0.95    | 0.72                           | 0.89    |
| 210  | 0.99                           | 0.96    | 0.69                           | 0.91    |
| 220  | 0.98                           | 1.00    | 0.69                           | 1.00    |
| 230  | 0.98                           | 1.00    | 0.67                           | 0.99    |
| 240  | 1.00                           | 1.00    | 0.65                           | 1.00    |
| 250  | 1.00                           | 1.00    | 0.65                           | 1.00    |

The results show that when the mean interarrival time (i.e., \( \frac{1}{\lambda} \)) is between 10 and 50, SRPT/SM slightly increases the \( \ell_2 \) norm of flow time by 1% to 4%. Note that because the average job sizes are approximately 250 (for the log-normal distribution), 200 (for the uniform distribution with admission control), and 300 (for the uniform distribution) in our simulation, the simulated server is extremely busy when the mean interarrival time is between 10 and 50.

When the mean interarrival time is around 150 and the job size follows the log-normal distribution, SRPT/SM can reduce the \( \ell_2 \) norm of flow time by 9%. The reduction is improved to 15% when admission control is adopted. It is interesting to observe that for the log-normal distribution without admission control, even if the \( \ell_\infty \) norm of flow time is only reduced by 1%, the \( \ell_2 \) norm of flow time can be reduced by 9%. With admission control, SRPT/SM can reduce the \( \ell_\infty \) norm of flow time by 10% to 35% when the mean interarrival time is between 50 and 200.

When the job size follows the uniform distribution, the benefit brought by starvation mitigation increases as the mean interarrival time increases. When the mean interarrival time is 250, SRPT/SM can reduce the \( \ell_2 \) norm of flow time and the \( \ell_\infty \) norm of flow time by 35% and

Thus, the average job size is \( e^{5.5} \approx 245 \). Note that this distribution is heavy-tailed. In the second distribution, we assume that to ensure fairness, some admission control is adopted for jobs of a size greater than 10^3 to be rejected. The resulting average job size is approximately 205 in our simulation. In the third distribution, we assume job sizes are uniformly distributed between 100 and 500. The simulation results are shown in Table 1. All results are averaged over 10 independent runs.

different job size distributions. The first is a log-normal distribution with \( \mu = 5 \) and \( \sigma = 1 \). Thus, the average job size is \( e^{5.5} \approx 245 \). Note that this distribution is heavy-tailed. In the second distribution, we assume that to ensure fairness, some admission control is adopted for jobs of a size greater than 10^3 to be rejected. The resulting average job size is approximately 205 in our simulation. In the third distribution, we assume job sizes are uniformly distributed between 100 and 500. The simulation results are shown in Table 1. All results are averaged over 10 independent runs.

Appendix B  Missing Proofs in Section 1 and Section 2

B.1 Proof of Lemma 1.6

For any efficient schedule $S$, we have

$$F(S) \geq \sum_{i=1}^{n} \sum_{t=r_i}^{c_i(S)-1} (t - r_i) = \sum_{i=1}^{n} \sum_{t=r_i}^{c_i(S)-1} (t - r_i + 1) \geq \sum_{i=1}^{n} \sum_{t=r_i}^{c_i(S)-1} q_t S(i) = \sum_{t\in\mathbb{N}} q(t).$$

B.2 Proof of Lemma 2.1

Let $I = \{i | pre(i) \leq 1, i \in [1, n] \}$. Obviously, $\sum_{i\in I} pre(i)^2 \leq F(OPT)$. Thus, we only need to consider job indices that are not in $I$. Let $i \in [1, n] \setminus I$. Define $T_i = [r_i + p_i, t_i - 1]$. Note that because $i \notin I$, $t_i - 1 > r_i + p_i$ and

$$pre(i)^2 = O \left( \sum_{j=0}^{pre(i)-1} j \right) = O \left( \sum_{t\in T_i} (t - (r_i + p_i)) \right).$$

When $t \in T_i$, $\frac{t - (r_i + p_i)}{q_t(i)} < \frac{n}{2}$, and thus $t - (r_i + p_i) < \frac{n}{2} q_t(i)$. As a result, we have

$$pre(i)^2 = O(\frac{n}{2}) \sum_{t\in T_i} q_t(i).$$

We then have

$$\sum_{i\in [1, n] \setminus I} \sum_{i\in [1, n] \setminus I} q_t(i) \text{ by Lemma 1.6 } F(OPT) = O((\frac{n}{2}) F(OPT)) = O((\frac{n}{2}) F(OPT)).$$

B.3 Proof of Lemma 2.3

For simplicity, we reindex jobs so that Eq. (2) holds. Let $i^*$ be the smallest integer such that $m_{OPT}(i^*) > 0$. Thus, $\sum_{i=1}^{n} P_i (m_{OPT})^2 = \sum_{i=i^*}^{n} \left( \sum_{h=i^*}^{i} m_{OPT}(h) \right)^2$. To derive a lower bound of $F(OPT)$, for any $i \geq i^*$ at time $t^*$, we remove some tasks of $J_i$ from the instance so that under $OPT$, $J_i$ has exactly $m_{OPT}(i)$ remaining tasks at time $t^*$, and all these $m_{OPT}(i)$ tasks are queued at time $t^*$. The above modification is achievable because $q_{i^*,OPT} \geq m_{OPT}$. We can further assume that starting from time $t^*$, the goal of $OPT$ becomes to minimize

$$\sum_{i=i^*}^{n} (c_i(OPT) - t^*)^2,$$

where $c_i(OPT)$ is the completion time of $J_i$ under $OPT$ in the modified instance. Observe that for all $i \geq i^*$, because $J_i$ has some queued tasks at time $t^*$, $r_i \leq t^*$. Thus, Eq. (8) is a lower bound of $F(OPT)$. Due to Eq. (2), it is not difficult to see that to minimize Eq. (8), $OPT$ should execute $J_{i^*}, J_{i^*+1}, \cdots, J_n$ in increasing order of job indices. As a result, for any $i \geq i^*$, $c_i(OPT) - t^* = \sum_{h=i^*}^{i} m_{OPT}(h)$.

Appendix C  Properties of $\hat{q}_t$

In this section, we prove Lemma 3.3 and Lemma 3.4.
C.1 Proof of Lemma 3.3

Recall that in Lemma 3.3, we define \( f = \hat{q}_t \). We first prove the following claim.

Claim C.1.

\[
F(OPT) \geq \hat{n}^2 \sum_{i \in I_n} f(i)^2. \tag{9}
\]

Proof. Because we have

\[
\sum_{\text{every FaU job } J_i} q_t(i)^2 \geq \sum_{i \in I_n} q_t(i)^2 = \sum_{i \in I_n} \hat{q}_t(i)^2 = \sum_{i \in I_n} f(i)^2,
\]

it suffices to prove

\[
F(OPT) \geq \hat{n}^2 \sum_{\text{every FaU job } J_i} q_t(i)^2.
\]

Define \( T_i = [r_i + p_i, t_i - 1] \). By Lemma 1.6, we have

\[
F(OPT) \geq \sum_{t \in \mathbb{N}} q(t) \geq \sum_{\text{every FaU job } J_i \in T_i} \sum_{t \in T_i} q_t(i).
\]

Thus, to prove Eq. (9), it suffices to show that for every FaU job \( J_i \), \( \sum_{t \in T_i} q_t(i) \geq \hat{n}^2 q_t(i)^2 \).

Observe that if \( t \in T_i \), \( q_t(i) \geq q_t(i) \). Therefore, \( \sum_{t \in T_i} q_t(i) \geq \sum_{t \in T_i} q_t(i) \). Finally, we argue that \( |T_i| \geq \hat{n}^2 q_t(i) \), which then completes the proof. Because \( J_i \) is an FaU job, \( t_i - (r_i + p_i) \geq \hat{n}^2 q_t(i) \). Thus, \( |T_i| = t_i - (r_i + p_i) \geq \hat{n}^2 q_t(i) \). \( \Box \)

We are now ready to prove Lemma 3.3. First note that \( q(t^*) \in [1, S(f)] \) (since \( q(t^*) \geq u(t^*) \geq 1 \) and \( q(t^*) = S(q_t) \leq S(f) \)). Thus, \( f_{q(t^*)} \) is a valid truncation. By Claim C.1, to prove the first inequality, it suffices to show that \( \sum_{i \in I_n} f(i)^2 \geq S(m_{ALG}^2) \). By the definition of \( m_{ALG}^2 \), we have \( S(m_{ALG}^2) = \sum_{i \in I_n} f_{q(t^*)}(i)^2 \leq \sum_{i \in I_n} f(i)^2 \), where the last inequality follows because \( f_{q(t^*)} \) is dominated by \( f \).

Next, we prove the second inequality. It suffices to show \( \sum_{i \in I_n} f_{q(t^*)}(i) \geq u(t^*) \). Let \( I_{normal} = A(t^*, ALG) \setminus I_n \). We then have

\[
q(t^*) = \sum_{i \in I_{normal}} f(i) + u(t^*) \geq \sum_{i \in I_{normal}} f_{q(t^*)}(i) + u(t^*).
\]

On the other hand, we have

\[
q(t^*) = \sum_{i \in [1,n]} f_{q(t^*)}(i) = \sum_{i \in I_{normal}} f_{q(t^*)}(i) + \sum_{i \in I_n} f_{q(t^*)}(i).
\]

Thus, \( \sum_{i \in I_n} f_{q(t^*)}(i) \geq u(t^*) \).

C.2 Proof of Lemma 3.4

It suffices to show that for all \( i \in [1, n] \), \( \hat{q}_t(i) \geq q_t(SRPT)(i) \).

Case 1: \( J_i \) is an FaN job or \( J_i \) is an FaU job with \( t_i \geq t \). In this case, by Lines 12-14 in Algorithm 1, for any \( t' < t \), if \( ALG \) executes a task \( J_{i,k} \) in time slot \( [t'] \), then it must be the case that \( SRPT \) executes \( J_{i,k} \) in some time slot \( [t''] \) with \( t'' \leq t' \). Thus, \( q_t(SRPT)(i) \leq q_t(J_i) = \hat{q}_t(i) \).

Case 2: \( J_i \) is an FaU job with \( t_i < t \). In this case, we have \( \hat{q}_t(i) = q_t(i) \geq q_t(SRPT)(i) \geq q_t(SRPT)(i) \), where the last inequality follows because all tasks of \( J_i \) are released before time \( t_i \).
Appendix D  Constructions of the Desired Maps

In this section, we first prove Lemma 3.11. We then prove Lemma 3.5 by Lemma 3.11. Finally, we prove Lemma 3.9.

D.1 Proof of Lemma 3.11

For any map $f$, we use $f^+$ to denote the restriction of $f$ to $\{i | f(i) > 0, i \in \text{dom } f\}$. Let $d_f = |\text{dom } f^+|$ and $d_g = |\text{dom } g^+|$. Because $f \geq_M g$, we have

$$S_{d_g}(g^+) = S(g^+) = S(f^+) > S_{d_f-1}(f^+) \geq S_{d_f-1}(g^+).$$

Therefore, $d_g > d_f - 1$ and thus

$$d_g \geq d_f.$$  (10)

We write $\text{dom } f$ as $\{a_1, a_2, \cdots, a_{|\text{dom } f|}\}$ such that $f(a_1) \geq f(a_2) \geq f(a_3) \geq f(a_{|\text{dom } f|})$. Similarly, we write $\text{dom } g$ as $\{b_1, b_2, \cdots, b_{|\text{dom } g|}\}$ such that $g(b_1) \geq g(b_2) \geq g(b_3) \geq g(b_{|\text{dom } g|})$.

We divide $\text{dom } f$ into three sets, $I_0$, $I_1$, and $I_2$, where $I_0 = \text{dom } f \setminus \text{dom } f^+ = \{a_k | k \in [d_f+1,|\text{dom } f|]\}$, $I_1 = \{a_k | f'(a_k) \leq g(b_k), k \in [1,d_f]\}$, and $I_2 = \{a_k | f'(a_k) > g(b_k), k \in [1,d_f]\}$.

Note that by Eq. (10), $b_k$ exists for any $k \in [1,d_f]$. The proof proceeds as follows: For $I_1$ (respectively, $I_2$), we will construct a map $m_1$ (respectively, $m_2$) that is dominated by $g$. The desired map $m$ will be constructed based on $m_1$ and $m_2$.

**Construction and properties of $m_1$.** If $I_1 \neq \emptyset$, we construct a map $m_1$ dominated by $g$. Initially, $m_1(i) = 0$ for all $i \in \text{dom } g$. For each $a_k \in I_1$, set $m_1(b_k)$ as $f'(a_k)$. Thus, $g \geq m_1$. In addition, we have

$$S(m_1) = \sum_{a_k \in I_1} f'(a_k)$$  (11)

and

$$S(m_1^2) = \sum_{a_k \in I_1} f'(a_k)^2.$$  (12)

**Construction of $m_2$.** If $I_2 \neq \emptyset$, we construct another map $m_2$ dominated by $g$. Initially, $m_2(i) = 0$ for any $i \in \text{dom } g$. We rewrite $I_2$ as $\{a_{\kappa(1)}, a_{\kappa(2)}, \cdots, a_{\kappa(|I_2|)}\}$ so that $\kappa(1) \leq \kappa(2) \leq \cdots \leq \kappa(|I_2|)$. For brevity, for any positive integers $x$ and $y$ with $x \leq y \leq d_g$, define

$$S_{x,y}(g) = \sum_{k \in [x,y]} g(b_k).$$

Define $y(0) = \kappa(1) - 1$. The construction of $m_2$ proceeds in rounds. In the $j$th ($j \in [1,|I_2|]$) round, we set

$$x(j) = \max(\kappa(j), y(j-1) + 1).$$  (13)

We set $y(j)$ to be the smallest integer such that

$$S_{x(j),y(j)}(g) \geq \frac{f'(a_{\kappa(j)})}{2}.$$  (14)

We then set

$$m_2(b_k) = g(b_k), \forall k \in [x(j),y(j)].$$  (15)

Thus, $m_2$ is dominated by $g$.

If $x(j)$ is valid (i.e., $x(j) \leq d_g$) and there indeed exists $y(j)$ that satisfies Eq. (14), we then have

$$f'(a_{\kappa(j)}) \geq g(b_{\kappa(j)}) \quad \text{by Eq. (13)}$$

In addition, if $S_{x(j),h}(g) < f'(a_{\kappa(j)})/2$, then $S_{x(j),h+1}(g) < f'(a_{\kappa(j)})$. Thus, by the definition of $y(j)$, we have the following fact.
Fact D.1. If \( x(j) \leq d_g \) and there exists \( y(j) \) that satisfies Eq. (14), then \( S_{x(j),y(j)}(g) < f'(a_{\kappa(j)}) \).

Proof of the validity of \( x(j) \) and the existence of \( y(j) \). We will show that the following statements \( \mathcal{X}(j) \) and \( \mathcal{Y}(j) \) hold for any \( j \in [1, |I_2|] \) by induction on \( j \):

\( \mathcal{X}(j) \): \( x(j) \leq d_g \)

\( \mathcal{Y}(j) \): \( S_{x(j),d_y}(g) \geq \sum_{h=0}^{d_y} f'(a_{\kappa(h)}) \).

Observe that \( \mathcal{Y}(j) \) implies that there exists \( y(j) \) that satisfies Eq. (14). When \( j = 1 \), \( x(1) = \kappa(1) \leq d_f \leq d_g \). Thus, \( \mathcal{X}(1) \) holds. In addition, we have

\[
S_{x(1),d_y}(g) - S_{x(1),d_y}(g) = S(g) - S_{\kappa(1)-1}(g) \geq S(f) - S_{\kappa(1)-1}(f)
\]

\[
= \sum_{h=\kappa(1)}^{d_f} f(a_h) \geq \sum_{h=\kappa(1)}^{d_f} f'(a_{\kappa(h)}) \geq \sum_{h=1}^{d_f} f'(a_{\kappa(h)}).
\]

Thus, \( \mathcal{Y}(1) \) holds.

Assume \( \mathcal{X}(j) \) and \( \mathcal{Y}(j) \) hold when \( j = z \) for some \( z \in [1, |I_2| - 1] \). To prove \( \mathcal{X}(z+1) \) and \( \mathcal{Y}(z+1) \) hold, we first consider the case where \( \kappa(z+1) \geq y_z + 1 \). In this case, \( x(z+1) = \kappa(z+1) \leq d_f \leq d_g \). Thus, \( \mathcal{X}(z+1) \) holds. In addition, we have

\[
S_{x(z+1),d_y}(g) = S_{\kappa(z+1),d_y}(g) = S(g) - S_{\kappa(z+1)-1}(g) \geq S(f) - S_{\kappa(z+1)-1}(f)
\]

\[
= \sum_{h=\kappa(z+1)}^{d_f} f(a_h) \geq \sum_{h=\kappa(z+1)}^{d_f} f'(a_{\kappa(h)}) \geq \sum_{h=z+1}^{d_f} f'(a_{\kappa(h)}).
\]

Thus, \( \mathcal{Y}(z+1) \) holds.

Next, we consider the case where \( y(z) + 1 > \kappa(z+1) \). Thus, \( x(z+1) = y(z) + 1 \). We have

\[
S_{x(z),d_y}(g) - S_{x(z),d_y}(g) = S(g) - S_{\kappa(z)-1}(g) \geq S(f) - S_{\kappa(z)-1}(f)
\]

\[
= \sum_{h=\kappa(z)}^{d_f} f(a_h) \geq \sum_{h=\kappa(z)}^{d_f} f'(a_{\kappa(h)}) \geq \sum_{h=z+1}^{d_f} f'(a_{\kappa(h)}). \tag{16}
\]

Because \( a_{\kappa(z+1)} \in I_2, f'(a_{\kappa(z+1)}) > 0 \). By Eq. (16), \( S_{x(z),d_y}(g) - S_{x(z),d_y}(g) > 0 \). As a result, \( d_g \geq y(z) + 1 = x(z+1) \) and

\[
S_{x(z+1),d_y}(g) = S_{y(z)+1,d_y}(g) = S_{x(z),d_y}(g) - S_{x(z),d_y}(g) > \sum_{h=\kappa(z)+1}^{d_f} f'(a_{\kappa(h)}).
\]

Thus, both \( \mathcal{X}(z+1) \) and \( \mathcal{Y}(z+1) \) hold. By mathematical induction, \( \mathcal{X}(j) \) and \( \mathcal{Y}(j) \) hold for any \( j \in [1, |I_2|] \).

Properties of \( m_2 \). By Eq. (13), \( [x(j), y(j)] \cap [x(j'), y(j')] = \emptyset \) if \( j \neq j' \). In addition, because we have \( m_2(b_k) > 0 \) only when \( k \in [x(j), y(j)] \) for some \( j \in [1, |I_2|] \), we then have

\[
S(m_2) = \sum_{j=1}^{|I_2|} \sum_{k \in [x(j), y(j)]} m_2(b_k) \geq \sum_{j=1}^{|I_2|} \sum_{k \in [x(j), y(j)]} f'(a_{\kappa(j)}) = \frac{1}{2} \sum_{i \in I_2} f'(i). \tag{17}
\]

In addition, we have

\[
\sum_{i \in I_2} f'(i)^2 \geq \sum_{j=1}^{|I_2|} S_{x(j),y(j)}(g)^2 \geq \left( \sum_{k \in [x(j), y(j)]} m_2(b_k)^2 \right)^2 \geq S(m_2^2). \tag{18}
\]
Construction of $m$ based on $m_1$ and $m_2$. If $\sum_{i \in I_1} f'(i) \geq \sum_{i \in I_2} f'(i)$, we set $m = m_1$. Otherwise, we set $m = m_2$. Because both $m_1$ and $m_2$ are dominated by $g$, $g \succeq m$. Observe that $S(f') = \sum_{i \in I_1} f'(i) + \sum_{i \in I_2} f'(i)$. Thus, if $m = m_1$, by Eq. (11), we have $S(m) = S(m_1) = \sum_{i \in I_1} f'(i) \geq \frac{1}{2} S(f')$. By Eq. (12), we have $S(f''_1) \geq \sum_{i \in I_1} f''(i)^2 = S(m_1^2) = S(m)$. Finally, if $m = m_2$, by Eq. (17), we have $S(m) = S(m_2) \geq \frac{1}{2} \sum_{i \in I_2} f''(i) \geq \frac{1}{4} S(f')$. By Eq. (18), we have $S(f''_2) \geq \sum_{i \in I_2} f''(i)^2 > S(m_2^2) = S(m^2)$. Thus, $f' \succeq m$.

D.2 Proof of Lemma 3.5
First note that by T1, $S(g) \leq S(f)$. Thus, by T2, $f_{\overline{S(g)}}$ is a valid truncation. We use Lemma 3.11 to prove Lemma 3.5. It is then sufficient to prove

$$S_k(f_{\overline{S(g)}}) \geq S_k(g), \forall k \in \mathbb{N}. \quad (19)$$

To prove Eq. (19), we consider the following two cases.

**Case 1:** $k \leq lu(f, S(g)) - 1$. The case where $k = 0$ is trivial. Thus, we assume $k \in [1, lu(f, S(g)) - 1]$. By the definition of truncation, we have

$$S_k(f_{\overline{S(g)}}) = S_k(f), \forall k \in [1, lu(f, S(g)) - 1].$$

By T1,

$$S_k(f) \geq S_k(g), \forall k \in [1, \text{dom } f].$$

Thus,

$$S_k(f_{\overline{S(g)}}) \geq S_k(g), \forall k \in [1, lu(f, S(g)) - 1].$$

**Case 2:** $k \geq lu(f, S(g))$. In this case, we have $S_k(f_{\overline{S(g)}}) = S(g) \geq S_k(g)$.

D.3 Proof of Lemma 3.9
Divide $\text{dom } f$ into two sets $I_1 = I$ and $I_2 = \text{dom } f \setminus I$. If $I_1 \neq \emptyset$, we construct a map $m_1$ dominated by $g$. Let $i^* = \arg \max_{i \in I_1} f(i)$. Let $j^* = \arg \max_{j \in \text{dom } g} g(j)$. Let $m_1$ be a map such that $m_1(j^*) = f(i^*)$ and $m_1(j) = 0$ for any $j \in \text{dom } g \setminus \{j^*\}$. Thus, we have

$$\sum_{i \in I_1} f(i)^2 \geq f(i^*)^2 = S(m_1^2). \quad (20)$$

By D2, $g(j^*) \geq f(i^*) = m_1(j^*)$. Thus, $m_1$ is dominated by $g$. Finally, by D3, we have

$$S(m_1) = f(i^*) > \frac{1}{2} \sum_{i \in I_1} f(i). \quad (21)$$

If $I_2 \neq \emptyset$, we construct another map $m_2$ that is dominated by $g$. By D1 and the definition of $I_2$, for every $i \in I_2$, we have $f(i) \leq 2g(i)$. Let $m_2$ be a map such that for any $i \in I_2$, we set $m_2(i) = \min(f(i), g(i))$. As a result, $m_2(i) \geq \frac{f(i)}{2}$ for any $i \in I_2$. For any $i \in \text{dom } g \setminus I_2$, we set $m_2(i) = 0$. Thus, $m_2$ is dominated by $g$. In addition, we have

$$S(m_2) \geq \frac{1}{2} \sum_{i \in I_2} f(i) \quad (22)$$

and

$$\sum_{i \in I_2} f(i)^2 \geq S(m_2^2). \quad (23)$$
We then construct $m$ based on $m_1$ and $m_2$. If $\sum_{i \in I_1} f(i) \geq \sum_{i \in I_2} f(i)$, we set $m$ as $m_1$. Otherwise, we set $m$ as $m_2$. Because both $m_1$ and $m_2$ are dominated by $g$, $g \geq m$. Note that $S(f) = \sum_{i \in I_1} f(i) + \sum_{i \in I_2} f(i)$. Thus, if $m = m_1$, by Eq. (21), we then have $S(m) = S(m_1) > \frac{1}{2} \sum_{i \in I_1} f(i) \geq \frac{1}{2} S(f)$. If $m = m_2$, by Eq. (22), we then have $S(m) = S(m_2) \geq \frac{1}{2} \sum_{i \in I_2} f(i) \geq \frac{1}{2} S(f)$. Finally, by Eq. (20) and Eq. (23), we have $S(F) = \sum_{i \in I_1} f(i)^2 + \sum_{i \in I_2} f(i)^2 \geq S(m_1^2) + S(m_2^2) \geq S(m^2)$. Thus, $f \equiv m$.

**Appendix E Proof of Lemma 4.4**

**Definition E.1.** For any map $f$, any $i \in [1, n] \setminus \text{dom } f$, and any $w \in \mathbb{N}$, define $f \cup (i, w) : \text{dom } f \cup \{i\} \to \mathbb{N}$ as a map such that $(f \cup (i, w))(i') = f(i')$ if $i' \in \text{dom } f$ and $(f \cup (i, w))(i) = w$. In addition, for any two maps $f$ and $g$ such that $\text{dom } f \cap \text{dom } g = \emptyset$, define $f \cup g : \text{dom } f \cup \text{dom } g \to \mathbb{N}$ as a map such that $(f \cup g)|_{\text{dom } f} = f$ and $(f \cup g)|_{\text{dom } g} = g$.

**Example E.2.** Assume $f(1) = 1$, $f(2) = 2$, $g(3) = 3$, and $g(4) = 4$. Further assume $\text{dom } f = \{1, 2\} \text{ and } \text{dom } g = \{3, 4\}$. We then have $\text{dom } (f \cup g) = \{1, 2, 3, 4\}$ and $(f \cup g)(1) = 1$, $(f \cup g)(2) = 2$, $(f \cup g)(3) = 3$, and $(f \cup g)(4) = 4$.

We first prove the following claim.

**Claim E.3.** Let $f$ and $g$ be any two maps such that $S_k(f) \geq S_k(g)$ for any $k \in \mathbb{N}$. Let $i \in [1, n] \setminus (\text{dom } f \cup \text{dom } g)$. Then for any $k, w \in \mathbb{N}$, $S_k(f \cup (i, w)) \geq S_k(g \cup (i, w))$.

**Proof.** Assume that in $\text{dom } f$ (respectively, $\text{dom } g$), there are $p_f$ (respectively, $p_g$) integers $x$ satisfying $f(x) \geq w$ (respectively, $g(x) \geq w$).

**Case 1:** $p_f \leq p_g$.

- If $k \leq p_g$, then $S_k(f \cup (i, w)) \geq S_k(f) \geq S_k(g) = S_k(g \cup (i, w))$.

- If $k \geq p_g + 1$, then $S_k(f \cup (i, w)) = S_{k-1}(f) + w \geq S_{k-1}(g) + w = S_k(g \cup (i, w))$.

**Case 2:** $p_g < p_f$.

- If $k \leq p_g$, then $S_k(f \cup (i, w)) = S_k(f) \geq S_k(g) = S_k(g \cup (i, w))$.

- If $p_g + 1 \leq k \leq p_f$, then $S_k(f \cup (i, w)) = S_k(f) \geq S_{p_g}(f) + w(k - p_g) \geq S_{p_g}(g) + w(k - p_g) \geq S_k(g \cup (i, w))$.

- If $k \geq p_f + 1$, then $S_k(f \cup (i, w)) = S_{k-1}(f) + w \geq S_{k-1}(g) + w = S_k(g \cup (i, w))$.

We then have the following claim.

**Claim E.4.** Let $f$, $g$, and $h$ be any three maps such that $\text{dom } f \cap \text{dom } h = \emptyset$, $\text{dom } g \cap \text{dom } h = \emptyset$, and $S_k(f) \geq S_k(g), \forall k \in \mathbb{N}$. Then $S_k(f \cup h) \geq S_k(g \cup h), \forall k \in \mathbb{N}$.

**Proof.** We prove Claim E.4 by induction on $|\text{dom } h|$. When $|\text{dom } h| = 1$, the claim holds due to Claim E.3. Assume the claim holds when $|\text{dom } h| = z$. When $|\text{dom } h| = z + 1$, pick any $i \in \text{dom } h$. Consider the map $h' = h|_{\text{dom } h \setminus \{i\}}$. Thus, $|\text{dom } h'| = z$ and by the induction hypothesis, we then have $S_k(f \cup h') \geq S_k(g \cup h')$ for any $k \in \mathbb{N}$. By Claim E.3, we then have $S_k((f \cup h') \cup (i, h(i))) \geq S_k((g \cup h') \cup (i, h(i)))$ for any $k \in \mathbb{N}$. The proof then follows from $(f \cup h') \cup (i, h(i)) = f \cup h$ and $(g \cup h') \cup (i, h(i)) = g \cup h$. 

21
We are now ready to prove Lemma 4.4. For simplicity, we prove the following lemma, which is equivalent to Lemma 4.4.

**Lemma E.5.** Let \( t \in \mathbb{N} \). Let \( S \) be any schedule that is efficient for the original instance by time \( t \). Then \( l_t,SRPT \geq_M l_t, S \).

Because both \( S \) and \( SRPT \) are efficient by time \( t \), \( S(l_t, S) = S(l_t, SRPT) \). To compute \( S_k(l_t, S) \), we can ignore jobs that are completed by time \( t \) under \( S \). In addition, jobs released at or after time \( t \) have the same number of remaining tasks at time \( t \) under \( S \) and \( SRPT \). Thus, by Claim E.4, it suffices to prove

\[
S_k(l_t, SRPT|_{A(l_t-1, SRPT)}) \geq S_k(l_t, S|_{A(l_t-1, S)}), \forall k \in \mathbb{N}.
\]

We prove Eq. (24) by induction on \( t \). Obviously, Eq. (24) holds when \( t = 0 \). Assume that Eq. (24) holds for any \( k \in \mathbb{N} \) when \( t = \tau \) for some \( \tau \geq 0 \). To prove Eq. (24) holds for any \( k \in \mathbb{N} \) when \( t = \tau + 1 \), we consider the following two cases:

**Case 1: \( SRPT \) does not execute any task in time slot \([\tau]\).** Because both \( S \) and \( SRPT \) are efficient by time \( t = \tau + 1 \), \( S \) does not execute any task in time slot \([\tau]\) either. We then have \( A(\tau, SRPT) = A(\tau, S) = \emptyset \). Thus, Eq. (24) holds for any \( k \in \mathbb{N} \) when \( t = \tau + 1 \) in this case.

**Case 2: \( SRPT \) executes some task in time slot \([\tau]\).** Let \( I_\tau = \{i | r_i = \tau\} \). We further divide Case 2 into two subcases:

**Case 2A: \( k \leq |A(\tau, SRPT)| - 1 \).

\[
S_k(l_{\tau+1}, SRPT|_{A(\tau, SRPT)|A(\tau, SRPT)}) = S_k(l_\tau, SRPT|_{A(\tau, SRPT)|A(\tau, SRPT)}) \quad \text{(by the property of SRPT and } k \leq |A(\tau, SRPT)| - 1\}
\]

\[
= S_k((l_\tau, SRPT|_{A(l_\tau-1, SRPT)}|_{I_\tau}) \cup (l_\tau, SRPT|_{I_\tau}))
\]

\[
= S_k((l_\tau, SRPT|_{A(l_\tau-1, SRPT)} \cup (l_\tau, SRPT|_{I_\tau}))
\]

Observe that \( l_{\tau+1, SRPT}|_{I_\tau} = l_{\tau, SRPT}|_{I_\tau} \). In addition, \( A(\tau - 1, S) \cap I_\tau = \emptyset \). Thus, by the induction hypothesis and Claim E.4, we then have

\[
S_k((l_\tau, SRPT|_{A(l_\tau-1, SRPT)} \cup (l_\tau, SRPT|_{I_\tau})) \geq S_k(l_\tau, S|_{A(l_\tau-1, S) \cup (l_\tau, S|_{I_\tau})}
\]

\[
\geq S_k(l_{\tau+1}, S|_{A(\tau, S)}).
\]

**Case 2B: \( k \geq |A(\tau, SRPT)| \).

\[
S_k(l_{\tau+1}, SRPT|_{A(\tau, SRPT)}) = S(l_{\tau+1}, SRPT|_{A(\tau, SRPT)})
\]

\[
= S_l(l_{\tau+1}, S|_{A(SRPT)}) \quad \text{(because both } S \text{ and } SRPT \text{ are efficient by time } t = \tau + 1\}
\]

\[
\geq S_k(l_{\tau+1}, S|_{A(\tau, S)}).
\]

**Appendix F  Missing Proofs in Section 5**

**F.1 Proof of Lemma 5.3**

We have the following facts.
Fact F.1. For every untrimmed job \( J_i \) and every time \( \tau \in [0, t^*) \), \( l_{\tau, \text{SRPT}^*}(i) = l_{\tau, \text{SRPT}^*}(i) \).

Fact F.2. Every PIT job is untrimmed. More specifically, for any \( \tau \in [0, t^*] \) and any schedule \( S \), if \( i \in I_{\tau, \text{SRPT}}^{\text{PIT}} \), then \( J_i \) is untrimmed.

We first prove the following claim.

Claim F.3. Let \( \tau \in [0, t^*) \). Let \( i \in I_{\tau, \text{SRPT}^*}^{\text{PIT}} \). Then \( l_{\tau, \text{SRPT}^*}(i) \geq t^* - \tau + 1 \).

Proof. We divide the proof into two cases. Case 1: \( J_i \) is untrimmed. In this case, we have \( l_{\tau, \text{SRPT}^*}(i) = l_{\tau, \text{SRPT}^*}(i) > t^* - \tau + 1 \). Case 2: \( J_i \) is trimmed. Because \( i \in I_{\tau, \text{SRPT}^*}^{\text{PIT}} \), \( i \in A(\tau, \text{SRPT}^*) \). Because \( J_i \) is trimmed and \( i \in A(\tau, \text{SRPT}^*) \), a task of \( J_i \) is released at time \( \tau \), and a task of \( J_i \) is released at time \( t^* \). Thus, by the construction of \( I^* \), \( l_{\tau, \text{SRPT}^*}(i) \geq t^* - \tau + 1 \).

To prove Lemma 5.3, we prove the following stronger statement.

\[ \text{ST}(\tau): \text{If } I_{\tau, \text{SRPT}^*}^{\text{PIT}} \neq \emptyset, \text{ then } \text{SRPT} \text{ and } \text{SRPT}^* \text{ execute the same task } J_{i,k} \text{ in time slot } [\tau] \text{ for some } i \in I_{\tau, \text{SRPT}^*}^{\text{PIT}}. \]

Consider the following two statements:

\[ \text{SP}(\tau): I_{\tau, \text{SRPT}^*}^{\text{PIT}} = I_{\tau, \text{SRPT}^*}^{\text{PIT}}. \]

\[ \text{SL}(\tau): \text{For any } i \in I_{\tau, \text{SRPT}^*}^{\text{PIT}}, l_{\tau, \text{SRPT}^*}(i) = l_{\tau, \text{SRPT}^*}(i). \]

We first show that \( \text{SP}(\tau) \) and \( \text{SL}(\tau) \) imply \( \text{ST}(\tau) \). We then show that \( \text{SP}(\tau) \) and \( \text{SL}(\tau) \) hold for any \( \tau \in [0, t^*] \) by induction on \( \tau \).

Proof of “\( \text{SP}(\tau) \) and \( \text{SL}(\tau) \) imply \( \text{ST}(\tau) \)”.

If \( I_{\tau, \text{SRPT}^*}^{\text{PIT}} \neq \emptyset \), by \( \text{SP}(\tau) \), \( I_{\tau, \text{SRPT}^*}^{\text{PIT}} \neq \emptyset \). Assume that in time slot \( [\tau] \), \( \text{SRPT} \) executes a task of \( J_i \) and \( \text{SRPT}^* \) executes a task of \( J_i \). By the definitions of PIT jobs and DL jobs, as well as the property of SRPT, we have \( i' \in I_{\tau, \text{SRPT}^*}^{\text{PIT}} \) and

\[ l_{\tau, \text{SRPT}^*}(i') \leq t^* - \tau + 1. \tag{25} \]

On the other hand, for every \( i \in I_{\tau, \text{SRPT}^*}^{\text{PIT}} \), \( i \) is untrimmed (by Fact F.2). Thus, for every \( i \in I_{\tau, \text{SRPT}^*}^{\text{PIT}} \), we have

\[ l_{\tau, \text{SRPT}^*}(i) \overset{\text{by Fact F.1}}{=} l_{\tau, \text{SRPT}^*}(i) \overset{\text{by } \text{SP}(\tau) \text{ and } \text{SL}(\tau)}{=} l_{\tau, \text{SRPT}^*}(i). \tag{26} \]

As a result, if \( i^* \in I_{\tau, \text{SRPT}^*}^{\text{PIT}} \), then by \( \text{SP}(\tau) \) and the tie-breaking rule, we have \( i^* = i' \). In addition, because \( l_{\tau, \text{SRPT}^*}(i^*) = l_{\tau, \text{SRPT}^*}(i^*) \) (by \( \text{SL}(\tau) \)), both \( \text{SRPT} \) and \( \text{SRPT}^* \) execute the same task of \( J_i \). \( \text{ST}(\tau) \) holds when \( i^* \in I_{\tau, \text{SRPT}^*}^{\text{PIT}} \).

Next, we show that \( i^* \in I_{\tau, \text{SRPT}^*}^{\text{PIT}} \) must hold due to the tie-breaking rule. For the sake of contradiction, assume \( i^* \in I_{\tau, \text{SRPT}^*}^{\text{DL}} \). By Claim F.3, we have \( l_{\tau, \text{SRPT}^*}(i^*) \geq t^* - \tau + 1 \). Because \( i' \in I_{\tau, \text{SRPT}^*}^{\text{PIT}} \) (by \( \text{SP}(\tau) \)), we have

\[ l_{\tau, \text{SRPT}^*}(i') \overset{\text{by Eq. (26)}}{=} l_{\tau, \text{SRPT}^*}(i') \overset{\text{by Eq. (25)}}{\leq} t^* - \tau + 1. \]

By the tie-breaking rule, \( i^* = i' \) and thus \( i^* \notin I_{\tau, \text{SRPT}^*}^{\text{DL}} \).
Proof of SP(\(\tau\)) and SL(\(\tau\)). Let \(I_t^{UT}\) be the index set of the untrimmed jobs released at time \(t\). Thus, if \(i \in I_t^{UT}\), then \(l_{i,SRPT}(i) = l_{i,SRPT^*}(i) = p_i\). We prove SP(\(\tau\)) and SL(\(\tau\)) by induction on \(\tau\). The first time we have a PIT job is when the first untrimmed job is released (regardless of the schedule). Let the time be \(t_0\). We then have \(I_{t_0,SRPT} = I_{t_0,SRPT^*} = I_0^{UT}\). Thus, SP(\(\tau\)) and SL(\(\tau\)) hold when \(\tau \leq t_0\).

Assume that SP(\(h\)) and SL(\(h\)) hold for some \(h \geq t_0\). Thus, ST(\(h\)) also holds. To prove SP(\(h+1\)) and SL(\(h+1\)), we consider the following two cases:

Case 1: \(I_{h,SRPT}^{PIT} = \emptyset\). By SP(\(h\)), \(I_{h,SRPT}^{PIT} = \emptyset\). Thus, \(I_{h+1,SRPT} = I_{h+1,SRPT^*} = I_{h+1}^{UT}\). Thus, SP(\(h+1\)) and SL(\(h+1\)) hold in Case 1.

Case 2: \(I_{h,SRPT}^{PIT} \neq \emptyset\). Because a DL job cannot become a PIT job in the future, we have \(I_{h+1,SRPT}^{PIT} \subseteq I_{h,SRPT}^{PIT} \cup I_{h+1}^{UT}\) and \(I_{h+1,SRPT^*} \subseteq I_{h,SRPT^*} \cup I_{h+1}^{UT}\). On the other hand, in this case, by ST(\(h\)), SRPT and SRPT* execute the same task of some job whose index is in \(I_{h,SRPT}^{PIT}\). Thus, by SP(\(h\)) and SL(\(h\)), for every \(i \in I_{h,SRPT}^{PIT} \cup I_{h+1}^{PIT}(= I_{h,SRPT}^{PIT} \cup I_{h+1}^{UT})\), \(l_{h,SRPT}(i) = l_{h+1,SRPT^*}(i)\). As a result, SP(\(h+1\)) and SL(\(h+1\)) hold in Case 2.

F.2 Proof of Lemma 5.4

Assume that from time \(\tau + 1\) to time \(t^*\), \(g_i\) tasks of \(J_i\) are released. We then have

\[
q_{r,S}(i) = q_{r,S}(i) + g_i - s_i.
\] (27)

We consider the following two cases. Case 1: The final task of \(J_i\) is released after time \(t^*\). Thus, \(g_i = t^* - \tau\). Because \(S\) executes a task of \(J_i\) in time slot \([\tau]\), \(q_{r,S}(i) \geq 1\). The proof then follows from Eq. (27). Case 2: The final task of \(J_i\) is released by time \(t^*\). Thus, \(l_{r,S}(i) = q_{r,S}(i) + g_i\). Because \(i \in I_{r,S}^{DL}\), \(l_{r,S}(i) > t^* - \tau + 1\). Thus, \(q_{r,S}(i) + g_i > t^* - \tau + 1\) and \(q_{r,S}(i) > t^* - \tau + 1 - s_i\).