On large maximal partial ovoids of the parabolic quadric $Q(4, q)^*$

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Abstract

We use the representation $T_2(\mathcal{O})$ for $Q(4, q)$ to show that maximal partial ovoids of $Q(4, q)$ of size $q^2 - 1$, $q = p^h$, $p$ odd prime, $h > 1$, do not exist. Although this was known before, we give a slightly alternative proof, also resulting in more combinatorial information of the known examples for $q$ odd prime.

**keywords:** maximal partial ovoid, generalized quadrangle, parabolic quadric.

**MSC (2010):** 05B25, 51D20, 51E12, 51E20, 51E21.

This paper is based on joint work with András Gács, started in the autumn of 2008, as a continuation of the work in [4]. His unfortunate and sudden death prevented us from continuing our joint work on the geometrical interpretation of the results. I would like to dedicate this work to András.

1 Introduction

A (finite) **generalized quadrangle** (GQ) is an incidence structure $S = (\mathcal{P}, \mathcal{B}, I)$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint non-empty sets of objects called points and lines, and for which $I \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$ is a symmetric point-line incidence relation satisfying the following axioms:

(i) each point is incident with $1 + t$ lines ($t \geq 1$) and two distinct points are incident with at most one line;

(ii) each line is incident with $1 + s$ points ($s \geq 1$) and two distinct lines are incident with at most one point;

$^*$final version, appeared in Des. Codes Cryptogr., DOI:10.1007/s10623-012-9629-y

$^\dagger$The author is a postdoctoral research fellow of the Research Foundation Flanders – Belgium (FWO).
(iii) if $x$ is a point and $L$ is a line not incident with $x$, then there is a unique pair $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $xIMyIL$.

The integers $s$ and $t$ are the parameters of the GQ and $\mathcal{S}$ is said to have order $(s, t)$. If $s = t$, then $\mathcal{S}$ is said to have order $s$. If $\mathcal{S}$ has order $(s, t)$, then $|\mathcal{P}| = (s + 1)(st + 1)$ and $|\mathcal{B}| = (t + 1)(st + 1)$ (see e.g. [3]).

An ovoid of a GQ $\mathcal{S}$ is a set $\emptyset$ of points of $\mathcal{S}$ such that every line is incident with exactly one point of the ovoid. An ovoid of a GQ of order $(s, t)$ has necessarily size $1 + st$. A partial ovoid of a GQ is a set $\mathcal{K}$ of points such that every line contains at most one point of $\mathcal{K}$. A partial ovoid $\mathcal{K}$ is called maximal if and only if $\mathcal{K} \cup \{P\}$ is not a partial ovoid for any point $P \in \mathcal{P} \setminus \mathcal{K}$, in other words, if $\mathcal{K}$ cannot be extended. It is clear that any partial ovoid of a GQ of order $(s, t)$ contains $1 + st - \rho$ points, $\rho \geq 0$, with $\rho = 0$ if and only if $\mathcal{K}$ is an ovoid.

It is a natural question to study extendability of partial ovoids, i.e. can one always extend a partial ovoid of size $1 + st - \epsilon$ (e.g. to an ovoid) if $\epsilon$ is not too big? The following theorem is a typical example.

**Theorem 1 ([8, 2.7.1])** Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ be a GQ of order $(s, t)$. Any partial ovoid of size $st - \rho$, $0 \leq \rho < \frac{t}{s}$ is contained in a uniquely defined ovoid of $\mathcal{S}$.

Note that if no ovoids of a particular GQ exist, then Theorem 1 implies an upper bound on the size of partial ovoids. The following theorem deals with the limit situation, and will be of use in Section 3.

**Theorem 2 ([8, 2.7.2])** Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ be a GQ of order $(s, t)$. Let $\mathcal{K}$ be a maximal partial ovoid of size $st - t/s$ of $\mathcal{S}$. Let $\mathcal{B}'$ be the set of lines incident with no point of $\mathcal{K}$, and let $\mathcal{P}'$ be the set of points on at least one line of $\mathcal{B}'$ and let $\mathcal{I}'$ be the restriction of $I$ to points of $\mathcal{P}'$ and lines of $\mathcal{B}'$. Then $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathcal{I}')$ is a subquadrangle of order $(s, \rho = t/s)$.

Consider the parabolic quadric $Q(4, q)$ in the 4-dimensional projective space $PG(4, q)$. This quadric is the set of points and lines that are totally isotropic with relation to a non-singular quadratic form on $PG(4, q)$, which is, up to coordinate transform, unique, and its points and lines constitute an example of a generalized quadrangle of order $q$.

It is known, (see e.g. [3]) that this GQ has ovoids. A particular example of an ovoid is any elliptic quadric $Q^- (3, q)$ contained in it and obtained by a hyperplane section of $Q(4, q)$. When $q$ is prime, these are the only ovoids [1]; when $q$ is a prime power, other examples are known, see e.g. [5] for a list of references. The classification of ovoids of $Q(4, q)$, for $q$ prime, is
essentially due to the computation of intersection numbers (modulo \( p \)) of a hypothetical ovoid with elliptic quadrics, and the use of this information in a combinatorial argument.

Applying Theorem 1 to the GQ \( Q(4, q) \) implies that a partial ovoid of size \( q^2 \) cannot be maximal. It is shown in [4] that maximal partial ovoids of \( Q(4, q) \), \( q = p^h \), \( p \) an odd prime, \( h > 1 \), do not exist. The natural question arises if maximal partial ovoids exist when \( h = 1 \). Curiously, this is the case when \( p \in \{3, 5, 7, 11\} \), but no examples are known for \( q > 11 \), [9]. In this paper we give a slightly alternative proof of the non-existence result for \( h > 1 \). Further, we compute the intersection numbers of a hypothetical maximal partial ovoid of size \( q^2 - 1 \) with elliptic quadrics embedded in \( Q(4, q) \), for \( q \) an odd prime. This yields structural information on the existing examples, and it is our hope that this information could contribute to finally proving their uniqueness and non-existence for \( p > 11 \).

2 Non-existence for \( q > p \)

We follow almost the same approach as in [4]. Therefore we need to introduce an alternative representation of the GQ \( Q(4, q) \).

An oval of \( PG(2, q) \) is a set of \( q+1 \) points \( C \), such that no three points of \( C \) are collinear. When \( q \) is odd, it is known that all ovals of \( PG(2, q) \) are conics. When \( q \) is even, several other examples and infinite families are known, see e.g. [2]. The GQ \( T_2(C) \) is defined as follows. Let \( C \) be an oval of \( PG(2, q) \), embed \( PG(2, q) \) as a plane in \( PG(3, q) \) and denote this plane by \( \pi_\infty \). Points are defined as follows:

(i) the points of \( PG(3, q) \setminus PG(2, q) \);

(ii) the planes \( \pi \) of \( PG(3, q) \) for which \( |\pi \cap C| = 1 \);

(iii) one new symbol (\( \infty \)).

Lines are defined as follows:

(a) the lines of \( PG(3, q) \) which are not contained in \( PG(2, q) \) and meet \( C \) (necessarily in a unique point);

(b) the points of \( C \).

Incidence between points of type (i) and (ii) and lines of type (a) and (b) is the inherited incidence of \( PG(3, q) \). In addition, the point (\( \infty \)) is incident with no line of type (a) and with all lines of type (b). It is straightforward to show that this incidence structure is a GQ of order \( q \). The following theorem (see e.g. [8]) allows us to use this representation.
Theorem 3 The GQs $T_2(C)$ and $Q(4,q)$ are isomorphic if and only if $C$ is a conic of the plane $PG(2,q)$.

From now on we suppose that $C$ is a conic. Let $K$ be a maximal partial ovoid of size $k$ of $T_2(C)$. Since $Q(4,q) \cong T_2(C)$ has a collineation group acting transitively on the points (see e.g. [7]), we can suppose that $(\infty) \in K$. This implies that $K$ contains no points of type (ii). It is clear that no two points of type (i) of $K$ determine a line meeting $\pi_\infty$ in a point of $C$. Hence the existence of $K$ implies the existence of a set $U$ of $k-1$ points of type (i) such that no two points determine a line meeting $\pi_\infty$ in $C$. It is easy to see that the converse is also true: from a set $U$ of $k-1$ points in $PG(3,q) \setminus \pi_\infty$ with the property that all lines joining at least two points of $U$ are disjoint from $C$, we can find a partial ovoid $K$ of $T_2(C)$ of size $k$ by adding $(\infty)$ to $U$. The maximality of $K$ is equivalent to the maximality of $U$.

Hence the existence of a maximal partial ovoid of size $q^2-1$ of $Q(4,q)$, is equivalent with the existence of a set $U$ of $q^2-2$ affine points, not determining the points of a conic at infinity. In [4], it is shown that such a set $U$ can always be extended when $q > p$. In fact, only the assumption that at least $p+2$ points are not determined is used. In this paper we will assume that the points of a conic are not determined and that $U$ is not extendable, compute the range of a certain polynomial, and find a contradiction when $q > p$. In the third section, we will describe the use of this particular polynomial to compute the intersection numbers modulo $p$ of the point set $U$ with planes of $AG(3,q)$. This will yield intersection numbers modulo $p$ of the maximal partial ovoid of size $q^2-1$ with elliptic quadrics embedded in $Q(4,q)$.

From now on, let $K$ denote a partial ovoid of $Q(4,q)$, $q = ph$, $p$ an odd prime and $h \geq 1$. Let $U$ denote the point set of $PG(3,q) \setminus \pi_\infty$ corresponding with the partial ovoid of $Q(4,q)$.

Lemma 1 If a plane $\pi \neq \pi_\infty$ of $PG(3,q)$, meets $C$ in at least one point, then $|\pi \cap U| \leq q$.

Proof. Let $P \in C \cap \pi$. If $|\pi \cap U| > q$, then at least one of the $q$ lines of $\pi$ through $P$ must contain two points of $U$. This contradicts the fact that $U$ does not determine any point of $C$. Hence, $|\pi \cap U| \leq q$. □

We choose $\pi_\infty$ to be the plane with equation $X_3 = 0$. Then any line $l$ of $\pi_\infty$ is determined by the equation $yX_0 + zX_1 + wX_2 = 0$, $(y, z, w) \in F_q^2 \setminus \{(0,0,0)\}$. We denote such a line as $l(y, z, w)$. Any plane $\pi \neq \pi_\infty$ through $l(y, z, w)$ is determined by the equation $yX_0 + zX_1 + wX_2 + xX_3 = 0$. We denote such a plane as $\pi(x, y, z, w)$.
The point set \( U = \{(a_i, b_i, c_i, 1) : i = 1, \ldots, q^2 - 2\} \subset \text{PG}(3, q) \setminus \pi_\infty \). We define the Rédei polynomial associated to the point set \( U \) as follows:

\[
R(X, Y, Z, W) = \prod_{i=1}^{q^2-2} (X + a_iY + b_iZ + c_iW) = X^{q^2-2} + \sum_{i=1}^{q^2-2} \sigma_i(Y, Z, W)X^{q^2-2-i}.
\]

Here \( \sigma_i(Y, Z, W) \) is the \( i \)-th elementary symmetric polynomial of the multi-set \( \{a_iY + b_iZ + c_iZ : i\} \) and is either zero or has degree \( i \).

**Lemma 2** Suppose that the line \( l(y, z, w) \) meets \( \mathcal{C} \) in at least one point. Then \( R(X, y, z, w) \mid (X^q - X)^q \).

**Proof.** If \( x \in \mathbb{F}_q \) is a root of \( R(X, y, z, w) = 0 \), then its multiplicity equals \( |\pi(x, y, z, w) \cap U| \leq q \) (the latter by Lemma 1). Since \( |U| = q^2 - 2 \), each of the \( q \) planes \( \pi(x, y, z, w) \), \( x \in \mathbb{F}_q \), contains points of \( U \), hence \( R(X, y, z, w) = 0 \) has each element \( x \in \mathbb{F}_q \) as root, with multiplicity at most \( q \), and the lemma follows. \( \square \)

Since we suppose that \( q \) is odd and \( |U| = q^2 - 2 \), after the affine translation

\[
a_i \mapsto a_i - \frac{\sum a_i}{q^2 - 2}, \quad b_i \mapsto b_i - \frac{\sum b_i}{q^2 - 2}, \quad c_i \mapsto c_i - \frac{\sum c_i}{q^2 - 2},
\]

not affecting the (non)-determined points at infinity, we may assume that \( \sum a_i = \sum b_i = \sum c_i = 0 \), which is equivalent to \( \sigma_1(Y, Z, W) \equiv 0 \).

**Lemma 3** If a line \( l(y, z, w) \) has at least one common point with \( \mathcal{C} \), then

\[
R(X, y, z, w)(X^2 - \sigma_2(y, z, w)) = (X^q - X)^q.
\]

**Proof.** From Lemma 2 we know that

\[
R(X, y, z, w)(X - S)(X - S') = (X^q - X)^q,
\]

where \( S \) and \( S' \) are not necessarily different and depend on \( y, z, w \). Considering the first three terms on both sides and taking into account that \( \sigma_1(Y, Z, W) \equiv 0 \), we have \( (X - S)(X - S') = X^2 - \sigma_2(y, z, w) \).

**Lemma 4** Suppose that the line \( l(y, z, w) \) meets \( \mathcal{C} \) in at least one point. Then

\[
\sigma_{2l+1}(y, z, w) = 0, 0 \leq l \leq \frac{q^2 - 3}{2},
\]

\[
\sigma_{2l}(y, z, w) = \sigma_{2l}(y, z, w), 0 \leq l \leq \frac{q^2 - q - 2}{2},
\]

\[
\sigma_{q^2 - q^2 + 2k}(y, z, w) = \left( \sigma_{2l}^2 \right)^k (y, z, w) - \sigma_{2l}^k (y, z, w), 0 \leq k \leq \frac{q - 3}{2}.
\]
Proof. Computation of both right-hand and left-hand sides of Equation (1), and the use of $\sigma_1(Y, Z, W) \equiv 0$ proves the lemma. □

Corollary 1

$$\sigma_{2l+1}(Y, Z, W) \equiv 0, \quad 0 \leq l \leq \frac{q-1}{2},$$

$$\sigma_{2l}(Y, Z, W) \equiv \sigma_2^l(Y, Z, W), \quad 0 \leq l \leq \frac{q-1}{2}.$$  

Proof. Consider any line $l(y, z, w)$ meeting $C$ in at least one point. By Lemma 4, the equations of the corollary are true after substituting $Y = y, Z = z, W = w$. But for each point $P \in C$, each line $l(y, z, w)$ on $P$ gives a substitution for which the equations are true. Dually, this means that the points of at least $q + 1$ different lines are a solution of the equations of the corollary. Since the degree of each equation is at most $q$, by the theorem of Bézout, each curve represented by an equation must contain $q + 1$ lines as a component. But then its degree must be at least $q + 1$. Hence, the polynomials are identically zero. □

We define now the polynomials $S_j, j = 0, \ldots, q - 1$ as follows.

$$S_j(Y, Z, W) := \sum_{i=1}^{q^2-2} (a_i Y + b_i Z + c_i W)^j.$$  

The Newton identities describe a relation between the polynomials $S_j(Y, Z, W)$ and $\sigma_i(Y, Z, W)$ as follows:

$$k \sigma_k(Y, Z, W) \equiv \sum_{j=1}^{k} (-1)^{j-1} S_j(Y, Z, W) \sigma_{k-j}(Y, Z, W).$$  

Lemma 5

$$S_{2l+1}(Y, Z, W) \equiv 0, \quad 0 \leq l \leq \frac{q-1}{2},$$

$$S_{2l}(Y, Z, W) \equiv -2 \sigma_2^l(Y, Z, W), \quad 0 \leq l \leq \frac{q-1}{2}.$$  

Proof. Using Corollary 1, the Newton identities, the fact that $S_1(Y, Z, W) \equiv \sigma_1(Y, Z, W), \sigma_0 = 1$, and induction, the lemma follows. □

Lemma 6 If $\sigma_2(Y, Z, W)$ is reducible then the set $U$ is extendable.
Proof. Suppose that $\sigma_2(Y, Z, W)$ is reducible. By equation (1), $\sigma_2(y, z, w)$ must be a square for any $(y, z, w)$ such that $l(y, z, w)$ meets $C$ in at least one point. So there are triples $(y, z, w)$, contained in a line (the dual of the pencil of lines through a point $P \in C$) for which $\sigma_2(y, z, w)$ is a square. It follows that $\sigma_2(Y, Z, W) = (AY + BZ + CW)^2$. Now define $U^* := U \cup \{(A, B, C, 1), (-A, -B, -C, 1)\}$. Consider any point $P \in C$ and any line $l(y, z, w)$ on $P$. From Equation (1) it follows that each plane on $l$ now contains exactly $q$ points of $U^*$. But if $P$ is a point determined by $U^*$, then there exists a line $m$ on $P$ containing $r \geq 2$ points of $U^*$. But all $q + 1$ planes on $m$ contain exactly $q$ points of $U^*$, so $q^2 = |U^*| = r + (q + 1)(q - r)$, a contradiction. Hence, $U^*$ does not determine the points of $C$. □

Theorem 4 If $U$ is not extendable, then $q = p$.

Proof. We define

$$\chi(X, Y, Z, W) := \sum_{i=1}^{q^2-2} (X + a_iY + b_iZ + c_iW)^{q-1}$$

$$= \sum_{i=1}^{q^2-2} \sum_{j=0}^{q-1} \binom{q-1}{j} X^{q-1-j}(a_iY + b_iZ + c_iW)^j$$

$$= \sum_{j=0}^{q-1} (-1)^j X^{q-1-j} S_j(Y, Z, W)$$

$$= -2 \sum_{k=0}^{q-1} X^{q-1-2k} \sigma_2^k(Y, Z, W) = -2 \frac{X^{q+1} + \sigma_2^q(Y, Z, W)}{X^2 - \sigma_2(Y, Z, W)}$$

where we used Lemma 5 to obtain the second last equality. If $U$ is not extendable, then $\sigma_2(Y, Z, W)$ is not reducible. So the range of $\sigma_2(Y, Z, W)$ is the complete field $\mathbb{F}_q$, so for each non-square $\nu \in \mathbb{F}_q$, we can find a triple $(y, z, w)$ such that $\sigma_2(y, z, w) = \nu$. Then $\sigma_2^2(y, z, w) = -\sigma_2(y, z, w)$ and

$$\chi(X, y, z, w) = -2 \frac{X^{q+1} + \sigma_2(y, z, w)}{X^2 - \sigma_2(y, z, w)} \tag{2}$$

It is now easy to see that the range of $\chi(X, Y, Z, W)$ will contain at least $\frac{q^2+1}{2}$ different elements of $\mathbb{F}_q$. On the other hand,

$$\chi(x, y, z, w) = q^2 - 2 - |U \cap \pi(x, y, z, w)| \mod p,$$

for any 4-tuple $(x, y, z, w) \not\in \{(1, 0, 0, 0), (0, 0, 0, 0)\}$. So the right hand side is necessarily an element of $\mathbb{F}_p$, a contradiction with the range of $\chi(X, Y, Z, W)$ if $q > p$. □
3 The intersection numbers for \( q \) a prime

Suppose now that \( q = p \), \( p \) an odd prime. We consider the possibilities of \( \chi(X, Y, Z, W) \). Consider a plane \( \pi(x, y, z, w) \).

(a) Suppose that \( \sigma_2(y, z, w) = 0 \). Then \( \chi(x, y, z, w) = -2X^{q-1} \), hence \( \chi(x, y, z, w) = 0 \) if \( x = 0 \) and \( \chi(x, y, z, w) = -2 \) if \( x \neq 0 \).

(b) Suppose that \( \sigma_2(y, z, w) \) is a square different from 0. If \( x^2 \neq \sigma_2(y, z, w) \) then \( \chi(x, y, z, w) = -2 \). If \( x^2 = \sigma_2(y, z, w) \) then \( \chi(x, y, z, w) = -1 \).

(c) Suppose that \( \sigma_2(y, z, w) \) is a non-square. Then
\[
\chi(x, y, z, w) = -2x^2 + \sigma_2(y, z, w) \\
\frac{x^2 - \sigma_2(y, z, w) \neq 0}{x^2 - \sigma_2(y, z, w) \neq 0}
\]

Lemma 7 The curve \( \sigma_2(Y, Z, W) = 0 \) is the dual of \( C \).

Proof. Theorem 2 ensures that the set of lines of \( Q(4, q) \), not meeting \( K \), is the set of lines of a hyperbolic quadric embedded as a hyperplane section in \( Q(4, q) \). We denote this hyperbolic quadric as \( Q^+ \). Since \( K = \{ (\infty) \} \cup U \), clearly \( (\infty) \notin Q^+ \), and from the proof of Theorem 3 in [8], it follows that \( Q^+ \) is represented in \( T_2(\mathcal{C}) \) as a hyperbolic quadric meeting \( \pi_\infty \) in \( \mathcal{C} \). We denote this quadric as \( Q^+_T \). The hyperbolic quadric \( Q^+_T \) contains exactly \( q + 1 \) points of type (ii). Consider such a point, represented by the plane \( \pi \). The two lines of type (a) of \( Q^+_T \) incident with \( \pi \), are contained in \( \pi \), and do not meet \( U \). But the other \( q - 1 \) lines of \( T_2(\mathcal{C}) \), incident with \( \pi \), do meet \( U \) in exactly one point. Hence the plane \( \pi \) must contain exactly \( q - 2 \) points of \( U \). If \( \pi \) is represented by the 4-tuple \( (x, y, z, w) \), then \( \chi(x, y, z, w) = q^2 - 2 - |\pi \cap U| \mod q \). So if \( |\pi \cap U| = q - 2 \), then \( \chi(x, y, z, w) = 0 \) and by the above overview of the range of \( \chi \), the planes \( \pi(x, y, z, w) \) that represent a point of type (ii) of \( Q^+_T \), are exactly those for which \( \sigma_2(y, z, w) = 0 = x \). But the planes that represent points of type (ii) of \( Q^+_T \) are planes that meet \( \mathcal{C} \) in a tangent line. Hence, \( \sigma_2(y, z, w) = 0 \) if and only if \( l(y, z, w) \) is a tangent line to \( \mathcal{C} \).

Corollary 2 A plane \( \pi(x, y, z, w) \) represents an elliptic quadric containing \( (\infty) \) if and only if \( \sigma_2(y, z, w) \) is a non-square.

Proof. From the proof of Theorem 3 it follows that an elliptic quadric containing \( (\infty) \) is represented in \( T_2(\mathcal{C}) \) by a plane meeting \( \pi_\infty \) in a line external to \( \mathcal{C} \). The Corollary now follows from Lemma 7.
Corollary 3 If an elliptic quadric $Q^- \subset Q(4, q)$ contains a point of $\mathcal{K}$, then

$$|Q^- \cap \mathcal{K}| \mod p \in \{-1 + 2x^2 + \nu \| \nu \text{ running over the non-squares, } x \in \mathbb{F}_q\}$$

Proof. If an elliptic quadric contains a point of $\mathcal{K}$, we can choose it to be the point $(\infty)$. Then

$$|\pi(x, y, z, w) \cap U| \mod q = -2 - \chi(x, y, z, w) = -2 + 2\frac{x^2 + \nu}{x^2 - \nu}, \quad (3)$$

$\nu = \sigma_2(y, z, w)$, which is non-square. \qed

Consider now any point $P \in Q(4, q) \setminus Q^+$. Then $P^\perp \cap Q^+$ is a conic $C_P$, and $C_P^\perp = \{P, P'\}$, $P \neq P' \in Q(4, q) \setminus Q^+$. We call $P'$ the antipode of $P$. Consider now the point $(\infty)$, this is collinear with the points of type (ii) of $Q^+_T$. But for each point of type (ii) of $Q^+_T$, represented by a plane $\pi(x, y, z, w)$, we have seen that $x = 0$. Hence the point $(0, 0, 0, 1)$ is contained in the planes representing the points of type (ii) of $Q^+_T$, so, the points of type (ii) of $Q^+_T$ are collinear with $(0, 0, 0, 1)$. Hence, the point $(0, 0, 0, 1)$ is the antipode of the point $(\infty)$.

Lemma 8 If an elliptic quadric $Q^- \subset Q(4, q)$ contains a point of $\mathcal{K}$ and its antipode, then $|Q^- \cap \mathcal{K}| \equiv -3 \mod q$.

Proof. A point and its antipode are non-collinear, and the collineation group of $Q(4, q)$ acts transitively on the pairs of non-collinear points. So in the $T_2(\mathcal{C})$ representation, if an elliptic quadric contains a point of $\mathcal{K}$, this can be chosen $(\infty)$ while its antipode can be chosen to be the point $(0, 0, 0, 1)$. For a plane $\pi(x, y, z, w)$ containing $(0, 0, 0, 1)$, we have $x = 0$. The lemma now follows from Corollary 3. \qed

We remark that the computed intersection numbers (modulo $q$) do not exclude elliptic quadrics that contain no point of $\mathcal{K}$. We list the range for intersection numbers modulo $q$ found in Corollary 3 for $q \in \{5, 7, 11\}$. Recall that these numbers are valid for elliptic quadrics containing at least one point of $\mathcal{K}$. Hence 0 means a positive multiple of $q$ in reality.

- $q = 5$: $\{0, 2, 3\}$
- $q = 7$: $\{2, 3, 4, 6\}$
- $q = 11$: $\{0, 4, 5, 8, 9, 10\}$.
We used an explicit description of the known examples ([3]) to compute the intersection numbers with all elliptic quadrics. We list the results. In this list, for $q = 5$ and $q = 11$, we see that there are elliptic quadrics containing no point of $K$. However this 0 is not related to a 0 in the above list.

- $q = 5$: $\{0, 2, 3, 5, 8, 12\}$
- $q = 7$: $\{2, 3, 4, 6, 9, 10, 18\}$
- $q = 11$: $\{0, 4, 5, 8, 9, 10, 11, 15, 16, 20, 30\}$.

As a final remark, we notice that the number of different intersection numbers is relatively large compared with $q$. On the other hand, an elliptic quadric containing a point of $K$ and its antipode always meets $K$ in $-3 \mod q$ points. In the above list, we notice for each $q$ only two different intersection numbers corresponding to $-3 \mod q$. This might suggest that pairs point-antipode play a special role, and indeed, for the known examples, it is true that when a point belongs to $K$, then also its antipode belongs to $K$, [3, Theorem 12]. Unfortunately, the above combinatorial information seems too weak to prove such a characterisation. It is our feeling that such a characterisation could be helpful in proving the non-existence for larger $q$. We note that in [3], where a completely different approach is used, a comparable conclusion on the pairs point-antipode is made. Finally, we also mention the work in [6], where the non-existence for larger $q$ is shown under the extra assumption that $(q^2 - 1)^2$ divides the order of the automorphism group of the maximal partial ovoid.

**Acknowledgement**

The author thanks the department of Computer Science at Eötvös Loránd University in Budapest, and especially Péter Sziklai, Tamás Szőnyi and Zsuzsa Weiner for their hospitality.

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