Self-Averaging, Distribution of Pseudo-Critical Temperatures and Finite Size Scaling in Critical Disordered Systems

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Abstract

We evaluate by Monte Carlo simulations various singular thermodynamic quantities $X$, for an ensemble of quenched random Ising and Ashkin-Teller models. The measurements are taken at $T_c$ and we study how the distributions $P(X)$ (and, in particular, their relative squared width, $R_X$) over the ensemble depend on the system size $l$. The Ashkin-Teller model was studied in the regime where bond randomness is irrelevant and we found weak self averaging; $R_X \sim l^{-\alpha} \to 0$, where $\alpha < 0$ and $\nu$ are the exponents (of the pure model fixed point) governing the transition. For the site dilute Ising model on a cubic lattice, known to be governed by a random fixed point, we find that $R_X$ tends to a universal constant independent of the amount of dilution (no self averaging). However this constant is different for canonical and grand canonical disorder. We identify the pseudo-critical temperatures $T_c(i, l)$ of each sample $i$, as the temperature at which the susceptibility reaches its maximal value. The distribution of these $T_c(i, l)$ over the ensemble was investigated; we found that its variance scales as $(\delta T_c(l))^2 \sim l^{-\frac{\nu}{2}}$. These results are in agreement with the recent predictions of Aharony and Harris. Our previously proposed finite size scaling ansatz for disordered systems was tested and found to hold. When we fit the data obtained for many samples of different sizes by a sample-independent form, the resulting scaling function was found to be universal and to behave similarly to pure systems. We did observe that to describe deviations from this universal function we do need sample-dependent scaling functions. These deviations are, however, relatively small and this led us to an interesting side result: sample-to-sample fluctuations of $\chi^{max}$, the susceptibility measured at $T_c(i, l)$, are smaller by a factor of 70 than those of $\chi(T_c)$. This indicates that to obtain a fixed statistical error it may be more computationally efficient to measure $\chi^{max}$. 

05.50.+q, 75.10Nr, 75.40Mg, 75.50.Lk

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I. INTRODUCTION

How is the critical behavior affected by the introduction of disorder (usually dilution or bond–randomness) into a model? This question has been extensively studied experimentally, analytically and numerically for quite some time now. The Harris criterion states that $\phi$, the scaling index of the operator corresponding to randomness at the pure system fixed point (also called the crossover exponent) is equal to $\alpha_p \nu_p$ ($\alpha_p$ and $\nu_p$ are the specific-heat and correlation length exponents of the pure model). Thus the critical behavior of the pure system ($p$) is unaltered by disorder if $\alpha_p < 0$ and a weakly disordered system will have the same critical exponents as the pure one. If $\alpha_p > 0$ even a weakly disordered system will not belong to the same universality class as the pure one. If $\alpha_p = 0$ the situation is marginal.

In Renormalization Group (RG) calculations it is possible to determine to which fixed point a certain disordered model flows and to determine the nature of this fixed point. A disordered fixed point is characterized by a fixed distribution (of finite width) of couplings while a pure fixed point is characterized by a delta function type distribution: a single coupling set. On the other hand, to the best of our knowledge, in the current Monte Carlo state of the art, there is no method that can check directly whether a certain model is governed by a pure or disordered fixed point. Most Monte Carlo studies concentrated on calculating critical exponents of a certain disordered model. If these exponents agreed with an RG calculation, this served as indirect evidence to the nature of the governing fixed point. In such numerical and experimental studies of disordered systems near their critical point finite samples with quenched disorder are used; any sample $i$ is a particular random realization of the quenched disorder. A measurement (or calculation) of any density of an extensive thermodynamic property $X$ (e.g. $X = E, M, C_h$ or $\chi$) would yield a different value $X_i$ for every sample $i$ because of the differences in the realizations of the quenched disorder. Here $X_i$ is the exact thermal average of the sample, which in an experiment or a numerical study can only be estimated by $\overline{X}_i$. In an ensemble of disordered samples of linear size $l$ the values of $X_i$ are governed by a probability distribution $P(X)$. In most MC studies only the ensemble average $\overline{X}$ is studied. As is shown in this study, it is possible to obtain direct evidence, by MC, as to the nature of the governing fixed point by studying $P(X)$ and the factors which govern its shape. As it turns out, this can be done by studying the question of self-averaging, which concerns the behavior of the width of $P(X)$ as the system size is increased. We characterize $P(X)$ by the ensemble average $\overline{X}$ and variance

$$V_X = [X^2] - [X]^2.$$  

Suppose that $X$ is a singular density of an extensive thermodynamic property, such as $M$, $\chi$ or the singular part of $E$ or $C$. The system is said to exhibit self-averaging if

$$R_X(l) = V_X/[X]^2 \to 0 \quad \text{as} \quad l \to \infty,$$  

otherwise, if $R_X$ tends to a constant different from zero, it is said to exhibit lack of self-averaging. In a self-averaging system a single very large sample is a sufficient representative of the ensemble. But in a non self-averaging system a measurement performed on a single sample, no matter how large, does not give a meaningful result and therefore must be
repeated using many samples from the ensemble. In a MC study of a self averaging disordered system the number of samples needed to obtain \( [X] \) to a given relative accuracy decreases with increasing \( l \). On the other hand, in a non self averaging system the number of samples which must be simulated is independent of \( l \) and the total amount of work rises very strongly with \( l \).

Off criticality, where \( l \) is much larger than the correlation length \( \xi \), as first argued by Brout [7], we may divide the sample \( i \) into \( n \) large subsamples (much larger than \( \xi \)). If we assume that the coupling between neighboring subsystems is negligible, then the value of any density of an extensive quantity over the whole sample \( X_i \) is equal to the average of the (independent) values of this quantity over the subsamples. Provided the probability distribution of the \( X \)'s of the subsamples has a finite variance, then according to the Central Limit theorem the value of \( X_i \) is distributed with a Gaussian probability distribution around its mean \( \langle X_i \rangle \). The square of the width of the Gaussian, \( V_X \), is proportional to \( \frac{1}{n} \sim l^{-d} \). In this case (3) is fulfilled, and \( X \) is called self-averaging. In such a case, where \( R_X \sim l^{-d} \), \( X \) is called strongly self averaging [8].

Close to criticality, where \( \xi \sim l \), the Brout argument does not hold, since the average of \( X \) over neighboring subsamples may not be considered as independent. Thus at criticality there is no reason to expect that \( R_X \sim l^{-d} \). In a previous study [9] we considered the question of self-averaging at the critical temperature of the infinite lattice, \( T_c^\infty \), through Monte Carlo simulations of several random-bond Ashkin Teller models. Our findings, which are summarized shortly below, prompted us to suggest a heuristic finite size scaling theory for disordered systems, based on physical considerations similar to those leading to the Harris criterion. We characterized every sample \( i \) of size \( l \) by a pseudo-critical temperature \( T_{c}(i, l) \) and introduced the reduced temperature of each sample

\[
i_i = \frac{T - T_{c}(i, l)}{T_c}.
\]

(3)

\( T_{c}(i, l) \) was assumed to fluctuate around its ensemble average \( T_{c}(l) \) with width \( \delta T_{c}(l) \). We then assumed [for samples with \( T \) close to \( T_{c}(i, l) \)] a sample dependent finite size scaling form

\[
X_i(T, l) = l^\rho \tilde{Q}_i(i, l^{\nu t}) .
\]

(4)

Here \( \rho \) is the exponent characterizing the behavior of \( [X] \) at \( T_c \); e.g. for \( X = \chi \), \( \rho = \frac{\nu}{\nu_t} \). The form of the function \( \tilde{Q}_i(Z) \) (or its coefficients) was assumed to be sample dependent but the critical exponents \( \rho \) and \( y_t = \frac{1}{\nu_t} \) were assumed to be universal or sample independent. Relying on (4) we then related \( R_X \) at \( T_c^\infty \) to \( \delta T_{c}(l) \). By assuming that

\[
(\delta T_{c}(l))^2 \sim l^{-d}
\]

(5)

the theory predicted that when the specific heat exponent of the disordered model \( \alpha < 0 \) then, at criticality,

\[
R_X \sim l^{\frac{\nu}{\nu_t}} .
\]

(6)

Thus, if \( \frac{\nu}{\nu_t} = 0 \), \( X \) is non self-averaging, but if \( -d < \frac{\nu}{\nu_t} < 0 \), \( X \) is called weakly self averaging [8]. Note that according to [11] \( \frac{\nu}{\nu_t} \leq 0 \) in any disordered system, though claims to the contrary exist (e.g. [12] and [13]).
In a subsequent study, Aharony and Harris [13] (AH) used renormalization group analysis to study the dependence of \( P(X) \) on \( l \) and \( \xi \). For \( l \gg \xi \) they recovered strong self-averaging: \( P(X) \) approaches a Gaussian with relative variance \( R_X \sim (l/\xi)^{-d} \). For \( l \ll \xi \) they found two different behaviors. When randomness was irrelevant and the system was governed by a pure fixed point they found

\[
R_X \sim l^{(\alpha_\nu )_p}.
\]  

(7)

In this case the critical exponents of the disordered model are the same as those of a pure model, \( \frac{\alpha}{\nu} = \left( \frac{\alpha}{\nu} \right)_p \), so that (7) is in agreement with (6). On the other hand, when the system is governed by a random fixed point, they found that \( P(X) \) approaches a universal, \( l \) independent shape, and \( R_X \to \text{const} \) as \( l \to \infty \), which implies lack of self-averaging. When \( \alpha \) of the random model is negative, this prediction is in disagreement with (6). As AH point out, this disagreement between the RG result and our scaling ansatz can be reconciled if we assume that in disordered systems governed by a random fixed point, (5) is substituted by

\[
(\delta T_c(l))^2 \sim l^{-\frac{2}{\nu}}.
\]  

(8)

In the Monte Carlo study of [9] several bond-disordered Ashkin-Teller models on a square lattice were simulated. These included the bond-disordered Ising model where \( \left( \frac{\alpha}{\nu} \right)_p = 0_+ \) (\( C \sim \log l \) at \( T_c^\infty \)), and several bond-disordered Ashkin-Teller models where \( \left( \frac{\alpha}{\nu} \right)_p > 0 \). According to renormalization group expansions [14,2,15] around the pure Ising model these models are governed by the pure Ising fixed point, while according to MC results [16] the bond-disordered Ashkin-Teller models may be governed by a different (possibly random) fixed point with \( \frac{\alpha}{\nu} = 0_+ \). It was found that far from criticality all thermodynamic quantities which were examined (energy, magnetization, specific heat, susceptibility) are strongly self averaging, that is \( R_X \sim l^{-d} \). At criticality though, the results indicated that the magnetization \( m \) and the susceptibility \( \chi \) are non self averaging. The energy \( E \) at criticality was clearly weakly self averaging, that is \( V_E \sim l^{-y_v} \) with \( 0 < y_v < d \) (Here \( E \) includes the analytic and singular parts of the energy). The theory (6) is not applicable in the asymptotic limit \( (l \to \infty) \) to the bond-disordered Ashkin-Teller model where \( \frac{\alpha}{\nu} = 0_+ \). Nonetheless in the accessible range of lattice sizes good agreement between the theory and the data for \( V_\chi \) and \( V_E \) was found. In particular \( R_X \) was shown to behave very similarly to the specific heat \( C \), as suggested by (6), for a wide range of lattice sizes and for different degrees of disorder. In a very recent MC study of a mean field Potts glass a lack of self averaging of the order parameter was found as well [17].

In this paper we set out to resolve three issues, neither of which could be investigated by studying the Ashkin Teller model at \( \alpha_p \geq 0 \).

1. So far the prediction (6) has not been tested numerically for the case \( \alpha_p < 0 \). In this case randomness is irrelevant, \( \alpha = \alpha_p \) and the AH prediction (7) coincides with (6).

2. When \( \alpha_p > 0 \) but at the random fixed point \( \alpha < 0 \), the scaling theory (6) predicts weak self averaging, in disagreement with the AH result \( R_X \to \text{const} \). Resolution of this discrepancy will shed light on whether the ansatz (5) or the AH prediction (8) governs the width of the distribution of the pseudocritical temperatures \( \delta T_c(l) \).
3. Testing the scaling form (4): We wish to determine whether it holds and whether the sample dependence enters only via $t_i$ or also through the scaling function $\tilde{Q}_i$.

With these three goals in mind we set out to examine by Monte Carlo simulations the question of self averaging at criticality in two different models. The first is a bond-disordered Ashkin-Teller model at a point where $(\alpha/\nu)_p \approx -0.54$ (a large negative value was chosen to yield unambiguous results). To address the second, more important issue, we simulated the site-dilute Ising model on a cubic lattice. Because the critical exponent $\alpha_p$ is positive for the three dimensional pure Ising model, $\alpha_p \approx 0.11$ [18], the Harris criterion predicts that randomness will lead to a different critical behavior. The disordered model is believed [19,20] to be governed by a random fixed point with $\alpha < 0$. Thus, according to AH, this model should exhibit lack of self averaging while according to our finite size scaling theory it should exhibit weak self averaging as in (6), if the assumption (5) is valid.

Besides calculating $R_X$ at the critical point, we decided to test directly which one of (5) or (8) is correct. To this end we calculated the pseudo-critical temperature of each sample taken from an ensemble of site-diluted 3D Ising samples at different lattice sizes. The pseudo-critical temperature was defined as the temperature of the maxima of the susceptibility of that sample. This calculation was done by using the histogram reweighting method [21–25]. This method allows to use the results of a simulation at one temperature for calculating the value of thermodynamic observables in a whole range of nearby temperatures. We thus obtained for each lattice size distributions of pseudo-critical temperatures $T_c(i,l)$ and were able to calculate their mean, $T_c(l)$, and variance, $(\delta T_c(l))^2$.

To investigate the extent to which the scaling form (4) holds, we studied the relationship between the sample dependent magnetization $m_i(T_c)$ and $T_c(i,l)$ using the data collapse technique. We did find convincing support for the finite size scaling ansatz (4) but also found evidence for sample dependence of the scaling function.

This work is organized as follows. In the first part of Sec. II we define the random bond Ashkin-Teller model which was simulated and summarize its critical behavior as found by the finite size scaling results. In the second part of Sec. II we give our results concerning self-averaging at criticality. The results indicate clearly that $R_X$ is weakly self averaging and are in good agreement with (4). In Sec. III we summarize some relevant properties of the site dilute Ising model on a cubic lattice and give some details of the simulation. Finite size scaling results for some observables at criticality are given as well. In section IV we analyze and discuss our results concerning self averaging at $T_C^\infty$. These results seem to indicate the correctness of the AH scenario, whereby $R_X$ is non self averaging. In Sec. V we study the distributions of the pseudo-critical temperatures of the site dilute Ising model. The scaling of $(\delta T_c(l))^2$ does not agree with (5) but seems to agree with (8), giving additional evidence for the validity of the AH scenario. In Sec. V we also analyze the distributions of the maximal susceptibilities $\chi(T_c(i,l))$, and investigate the extent to which the scaling form (4) holds. The work is summarized in Sec. VI.

II. WEAK SELF AVERAGING IN AN ASHKIN-TELLER MODEL WITH IRRELEVANT DISORDER
A. Definition of the model

The model we study is the random-bond Ashkin-Teller model on a square lattice. On every site of the lattice two Ising spin variables, $\sigma_i$ and $\tau_i$, are placed. Denoting by $\langle ij \rangle$ a pair of nearest neighbor sites, the Hamiltonian is given by

$$H = -\sum_{\langle ij \rangle} [K_{i,j}(\sigma_i\sigma_j + \tau_i\tau_j) + \Lambda_{i,j}\sigma_i\tau_i\sigma_j\tau_j] .$$

(9)

$K_{i,j}$ and $\Lambda_{i,j}$ are chosen according to

$$(K_{i,j}, \Lambda_{i,j}) = \begin{cases} (K^1, \Lambda^1) & \text{with probability } \frac{1}{2} \\ (K^2, \Lambda^2) & \text{with probability } \frac{1}{2} \end{cases} .$$

(10)

The homogeneous (or pure) model $\langle K^1, \Lambda^1 \rangle = \langle K^2, \Lambda^2 \rangle \equiv (K, \Lambda)$ possesses a line of critical points along which critical exponents vary continuously, so that it flows under Renormalization Group (RG) onto a line of fixed points. The scaling exponent corresponding to randomness, $\phi = (\alpha/\nu)$, which is analytically known $\langle 27, 28 \rangle$, also varies continuously along this line. However along the part of this line ($\Lambda \geq 0$) interpolating between the Ising ($\Lambda = 0$) and four state Potts ($\Lambda = K$) models it takes positive values, $(1 \geq \phi \geq 0)$, so that randomness is relevant. Indeed the critical behavior of the disordered model was found to be different from that of the pure one. The self averaging properties of the disordered model were examined in $\langle 9 \rangle$, and as discussed in the introduction, a lack of self averaging was found. Along the other part of this line ($\Lambda < 0$) $\phi = \alpha/\nu$ is negative, so that according to the Harris criterion randomness is irrelevant. Thus a slightly disordered model will exhibit the same behavior as its pure version. It turns out, according to this study (in agreement with RG calculations $\langle 4, 15 \rangle$) that a model with finite disorder will flow under RG onto a pure model that is along the part of the line of fixed points where $\phi = \alpha/\nu < 0$. Thus according to AH as well as according to $\langle 9 \rangle$ we expect to find weak self-averaging at criticality for disordered versions of models corresponding to this part of the Ashkin Teller critical line ($\Lambda < 0$).

The Ashkin Teller model is a convenient paradigm for studying critical behavior of disordered systems both because its scaling exponent corresponding to randomness $\phi = \alpha/\nu$ varies continuously and because part of the critical manifold of the random-bond model can be found exactly through duality. In parts of the coupling space where only two phases exist the self dual manifold

$$(K^2, \Lambda^2) = (\tilde{K}^1, \tilde{\Lambda}^1)$$

is critical. Here $(\tilde{K}^1, \tilde{\Lambda}^1)$ are the dual couplings of $(K^1, \Lambda^1)$ according to the duality transformation of the Ashkin-Teller model; a discussion of this point can be found in $\langle 16 \rangle$. Since the extent of deviation from pure behavior is obviously determined by the difference between the two sets of couplings, we have chosen to study a model with the ratio

$$\frac{K^2}{K^1} = \frac{1}{10} ,$$

(12)

so that randomness will be pronounced. In addition, a ratio of
\[
\frac{\Lambda^1}{K^1} = -f = -\frac{9}{10}
\]

was chosen. Since \(\frac{\partial}{\partial f}\) decreases as \(f\) increases and \(\frac{\partial}{\partial f} = 0\) for \(f = 0\) (Ising model), we have chosen \(f = \frac{9}{10}\) so that \(\frac{\partial}{\partial f}\) will be a pronounced negative number. Equations (11–13) define the couplings of the model simulated, where the temperature \(T\) was absorbed into the couplings \(K_{ij}, \Lambda_{ij}\).

The critical behavior of the disordered model is compared with that of an Anisotropic Ashkin-Teller model which is used as a reference pure model [16]. This model has the same Hamiltonian (3) but with the couplings distributed as follows:

\[
(K_{i,j}, \Lambda_{i,j}) = \begin{cases} 
(K^1, \Lambda^1) & \text{for bonds } (i, j) \text{ in the horizontal direction} \\
(K^2, \Lambda^2) & \text{for bonds } (i, j) \text{ in the vertical direction}
\end{cases}
\]  

(14)

In the Monte Carlo simulations we used a cluster algorithm [29] which is described in [16]. The main idea is to embed into the Ashkin-Teller model an Ising model and simulate it using the Wolff [30] single cluster algorithm for the Ising model. The number of samples simulated was \(n = 2000\) for linear lattice sizes \(4 \leq l \leq 64\), \(n(128) = 1200\) for \(l = 128\), and \(n(256) = 436\). For each sample \(i\) Monte Carlo estimates of various observables \(X_i\) and their errors \(\delta X_i\) were calculated. Next we list results concerning the critical behavior of the estimated bond-disordered ensemble averages \([X_i]\) as a function of lattice size.

**B. Critical behavior of the model**

Here we give the finite size dependences of averages (over all samples) of various observables, defined as in [16], at the critical point \(T^c_{\infty}\) defined through (11–13).

In figure 1 we plot the specific heat of the random bond and the anisotropic models as a function of \(\log l\). The solid lines are fits to the finite size scaling form

\[
C = B_0 + B_1 l^{\beta \nu}.
\]  

(15)

Using lattice sizes of \(16 \leq l \leq 256\), we find \(\frac{\alpha_{\nu}}{\nu} = -0.745(4)\) for the anisotropic model, while using lattice sizes of \(24 \leq l \leq 256\), we find \(\frac{\alpha_{\nu}}{\nu} = -0.536(32)\) for the random bond model (note: \(B_1\) is negative). Thus this strongly disordered model apparently flows under RG onto a pure model with different exponents than its anisotropic version but still one that is along the part of the line of fixed points where \((\alpha_{\nu})_p < 0\).

For both models the magnetization \(m\) and susceptibility \(\chi\) were fitted to the forms

\[
m = A_m l^{-\beta \nu} \quad \chi = A_\chi l^{\gamma \nu}.
\]  

(16)

Similarly the polarization \(P\) (magnetization of the \(\tau\) spins) and susceptibility of the polarization \(\chi^{(p)}\) were fitted to the forms

\[
P = A_p l^{-\beta(p) \nu} \quad \chi^{(p)} = A_{\chi^{(p)}} l^{\gamma(p) \nu}.
\]  

(17)

The estimates for the exponent ratios obtained are listed in Table I. The values of \(\frac{\beta}{\nu}\) and \(\frac{\gamma}{\nu}\) are within errors, for the random bond model or very close, for the anisotropic model, to the Ising exponent ratios \(\frac{\beta}{\nu} = \frac{1}{8}\) and \(\frac{\gamma}{\nu} = \frac{7}{4}\). These exponent ratios are predicted analytically [27,28] to be of this magnitude all along the critical line of the pure Ashkin Teller model.
FIG. 1. The specific heat $C$ of the random bond and the anisotropic Ashkin Teller models as a function of $\log l$. The solid lines are fits to the form (15).

TABLE I. Estimates for critical exponent ratios of the Ashkin Teller models from finite size scaling at $T_c^\infty$, obtained with linear fits to $\log l$ according to equations (15 - 17).

|                  | $\alpha$  | $\beta$  | $\gamma$  | $\rho^{(p)}$ | $\chi^{(p)}$ | fitting interval |
|------------------|------------|----------|------------|--------------|--------------|-----------------|
| random bond      | -0.536(32) | 0.1252(4)| 1.7502(5) | 0.322(1)     | 1.356 6(15)  | 24 $\leq l \leq$ 256 |
| anisotropic      | -0.745(4)  | 0.1262(2)| 1.7488(1) | 0.3312(5)    | 1.339(1)     | 16 $\leq l \leq$ 256 |

C. Calculation of $V_X$

The variance $\sigma_X^2$ of the Monte Carlo estimates $\bar{X}_i$ is the sum of two contributions. The main contribution is due to the variance $V_X$ of the distribution of the true $X_i$. $V_X$ is the quantity we wish to study. The second contribution is due to the errors of the estimated observables, $\delta \bar{X}_i$. Thus, the unbiased estimator [31] of the variance of the $X_i$ is

$$V_X = \sigma_X^2 - [(\delta \bar{X}_i)^2].$$

$\delta \bar{X}_i$ depends on the length of the MC runs and on the autocorrelation time $\tau_X$ of the MC dynamics. To obtain a valid estimate of $V_X$, $[(\delta \bar{X}_i)^2]/V_X$ should be sufficiently small. In the random bond Ashkin Teller model studied here this requirement was not met for the specific heat $C$, whereby we could not study $R_C$. Additional discussion of the practical implications of (18) can be found in section III of [9]. Next we list results concerning the critical behavior of $V_X$ at $T_c^\infty$.

D. The relative variance $R_X$

In Fig. 2 we plot $R_m$, $R_X$, $R_P$ and $R_{\chi^p}$ as a function of $\log l$. The solid lines are linear fits to the form

$$R_X = A_X l^{\rho_X}$$

for $24 \leq l \leq 256$. The estimates obtained for $\rho_X$ are $\rho_X = -0.537(32)$, $\rho_m = -0.546(38)$, $\rho_{\chi^p} = -0.493(37)$ and $\rho_P = -0.509(41)$. Clearly all observables, $m, \chi, P, \chi^{(p)}$ are weakly
FIG. 2. The relative variance of the susceptibility $R_\chi$, of the magnetization $R_m$, of the polarization $R_P$ and of the polarization susceptibility $R_\chi(p)$ of the Ashkin Teller model at $T_c^\infty$ as a function of $\log_{10} l$. The solid lines are linear fits according to (19).

FIG. 3. The variance of the energy $V_E$ of the Ashkin Teller model at $T_c^\infty$ as a function of $\log_{10} l$. The solid line is a linear fit according to $V_E = A_{vE} l^{x_E}$, yielding $x_E = -2.005(26)$.

self averaging. Furthermore $\rho_\chi$ and $\rho_m$ are in very good agreement with the value of $\frac{\alpha}{\nu} = -0.536(32)$, while $\rho_\chi(p)$ and $\rho_P$ are also within errors of $\frac{\alpha}{\nu}$. Thus the results for the scaling of the relative variance of these four observables are in good agreement with the predictions of AH and with (3).

For the energy one cannot separate the singular and the analytic parts. In addition, the singular part decays as $l^\rho$, with $\rho = \frac{\alpha}{\nu} - 1$. Using our estimate of $\frac{\alpha}{\nu} = -0.536(32)$ and the hyperscaling relation $\frac{\alpha}{\nu} = \frac{2}{\nu} - d$, we find $\rho = -1.268(48)$. Thus the variance of the singular part of the energy is expected according to (4) to scale as $l^{-3.072}$. Therefore one would expect $V_E$ to be dominated by the fluctuations of the analytic part of $E$ decaying as $l^{-d}$. In figure 3 the variance of the energy $V_E$ as a function of $\log l$ is plotted. Straight forward linear fits to the form $V_E = A_{vE} l^{x_E}$ in the lattice size range $24 \leq l \leq 256$ yielded $x_E = -2.005(26)$ in good agreement with our expectation $x_E = -d$.

To conclude this part of the study, we found weak self averaging at criticality for a
disordered model governed by a pure fixed point with \( (\alpha / \nu)_p < 0 \). We also found good agreement with the scaling prediction \( (7) \).

III. THE SITE DILUTE ISING MODEL ON A CUBIC LATTICE

The second model we chose to study is the site-dilute Ising model on a cubic lattice (see e.g. \( [3] \) and references therein). On every site of a \( l \times l \times l \) cubic lattice either an Ising magnetic spin \( S_i = \pm 1 \) is placed if \( K_i = 1 \) or a vacancy is placed if \( K_i = 0 \). The \( K_i \) are randomly drawn according to one of the prescriptions given below. The system is governed by the Hamiltonian

\[
\mathcal{H} = -J \sum_{<i,j>} K_i S_i K_j S_j ,
\]

where \( < i,j > \) stands for a pair of nearest neighbors. RG calculations found a dilution independent random fixed point with universal critical exponents. For example, a recent calculation \( [20] \) obtained \( \gamma = 1.313, \beta = 0.342, \nu = 0.666 \) and \( \alpha = 0.002 \). Early MC studies found global effective critical exponents which were found to depend on dilution. This was later interpreted as due to crossover effects. For example, in a most extensive MC study, Heuer \( [32] \) found from finite size scaling in the lattice size range \( 20 \leq l \leq 60 \) values ranging from \( \alpha / \nu (p = 0.95) = 0.12(6) \) with 5% vacancies, to \( \alpha / \nu (p = 0.8) = -0.04(6) \) and \( \alpha / \nu (p = 0.6) = -0.22(6) \). However he argued by analyzing a suitable scaling function, that all models with different amounts of dilution are exhibiting a crossover to the fixed point predicted by RG. His results show that, of the amounts of dilution he studied, the \( p = 0.8 \) model reached the universal behavior at the smallest lattice sizes. Later Janssen Oerding and Sengespeick \( [20] \) showed in their RG calculations that the effective exponent values obtained by Heuer can be related to regions in the space of coupling coefficients away from the fixed point.

A. Details of the simulations

Three site-dilute Ising models were examined, including two types of disorder. In one model disorder was realized in a canonical manner; namely, the number of magnetic sites in each site-diluted sample was fixed at a fraction \( c = 0.6 \) of the number of sites in the lattice. Thus fluctuations among samples occur only in the locations of the magnetic sites but not in their number. In two other models disorder was realized in a grand-canonical manner; namely, each sample was created by assigning to each site of the lattice a magnetic spin (vacancy) with probability \( p (1 - p) \). In one model we used \( p = 0.6 \) and in the second one \( p = 0.8 \). In this case fluctuations among samples include fluctuations in the number of magnetic sites. These fluctuations tend to zero as \( l \to \infty \), but for finite \( l \) they are significant. For this reason we found it of interest, in this study of fluctuations among samples, to compare the two ensembles. We are unaware of any previous findings attesting to differences in the asymptotic critical behavior between the two ensembles. Because of the (spatially) uncorrelated nature of the disorder in the grand canonical ensemble it is favored for its relative simplicity by theoretical studies (see \( [20] \) for references) and by numerical studies \( [33, 37] \).
Table II. Number of site-dilute samples simulated for each model and lattice size \( l \). The last column lists the infinite lattice critical temperature, as estimated by Heuer [32], at which the simulations were performed. For \( c = 0.6, l = 90 \) the pseudocritical temperature was not estimated.

| \( c \) | \( l = 10 \) | \( l = 20 \) | \( l = 40 \) | \( l = 60 \) | \( l = 80 \) | \( l = 90 \) | \( T_c^\infty \) |
|-------|---------|---------|---------|---------|---------|---------|---------|
| \( p = 0.6 \) | 8000 | 26000 | 2000 | 800 | 1000 | 2.4220(6) |
| \( p = 0.8 \) | 72000 | 47000 | 8000 | 950 | 800 | 2.4220(6) |
| \( l = 4 \) | 4000 | 32000 | 4000 | 1479 | 3.4992(5) |
| \( l = 8 \) | 8000 | 47000 | 8000 | 950 | 800 | 2.4220(6) |
| \( l = 16 \) | 10000 | 4000 | 32000 | 4000 | 1479 | 3.4992(5) |
| \( l = 32 \) | 10000 | 4000 | 32000 | 4000 | 1479 | 3.4992(5) |
| \( l = 64 \) | 10000 | 4000 | 32000 | 4000 | 1479 | 3.4992(5) |

Aiming to test them. On the other hand, in studying by Monte Carlo average thermodynamic observables, errors can be reduced by using canonical disorder, as was done in [32]. We note that if one wishes to study by Monte Carlo the fluctuations in the thermodynamic observables due to disorder, the use of grand-canonical disorder is advantageous.

In the Monte Carlo simulations we used the Wolff [30] single cluster algorithm [29] for the Ising model because of its efficiency [38]. Skewed periodic boundary conditions [8] were used in order to speed up the simulations. For each model and lattice size \( n \) site-dilute samples were simulated. Table II summarizes the number of samples \( n \) used for each lattice size for the three models. Simulations were performed at the estimated infinite lattice critical temperatures \( T_c^\infty \), given in table II due to Heuer [32], and taking \( J = 1 \). The procedure for calculating the \( T_c(i, l) \) is described in Sec. V A.

B. Finite size scaling at \( T_c^\infty \)

Here we give the finite size dependence of averages (over all samples) of some observables at the critical point \( [X_i(T_c^\infty)] \).

1. Magnetization \( m \) and susceptibility \( \chi_c \)

Using the abbreviation

\[
M = \sum_i K_i S_i ,
\]

the magnetization density \( m \) is defined as

\[
m = \frac{[\langle |M| \rangle]}{Np} ,
\]

where \( N = l^3 \) and the fraction of magnetic sites was either \( p = 0.8 \) or \( p = 0.6 \). The susceptibility at \( T_c^\infty \) was defined as [8]

\[
\chi_c = \frac{[\langle M^2 \rangle]}{NpT} .
\]

The magnetization density \( m \) and susceptibility \( \chi_c \) were fitted to the finite size scaling forms ([16]).
FIG. 4. The magnetization $m$ at $T_c^\infty$ as a function of $\log_{10} l$. The solid lines are fits to the form (16), yielding estimates for $\frac{\beta}{\nu}$ which are listed in table III.

FIG. 5. The susceptibility $\chi_c$ as defined in (23) at $T_c^\infty$ as a function of $\log_{10} l$. The solid lines are fits to the form (16), yielding estimates for $\frac{\gamma}{\nu}$ which are listed in table III.
TABLE III. Estimates for order parameter critical exponent ratios from finite size scaling at 
$T_c^\infty$. Estimates due to Heuer [32] are listed for comparison. Note that throughout the paper the 
errors given for our results are only statistical while Heuer’s error estimates include the systematic 
errors which could arise from errors in determining $T_c^\infty$.

| $c$  | $\beta/\nu$ | $\beta/\nu$ (Heuer) | $\gamma/\nu$ | $\gamma/\nu$ (Heuer) |
|------|--------------|----------------------|--------------|---------------------|
| 0.6  | 0.438(13)    | 0.45(2)              | 2.110(21)    | 2.09(3)             |
| 0.8  | 0.437(12)    | 0.51(2)              | 1.990(4)     | 1.98(3)             |

The estimates which were obtained for the critical exponents ratios $\beta/\nu$ and $\gamma/\nu$ from the 
fits in figures 4 and 5 are listed in Table III together with the estimates of Heuer [32]. Note 
that exponent ratios and critical temperatures quoted from [32] for $p = 0.8$ were actually 
obtained with canonical disorder $c = 0.8$.

2. $\partial m/\partial t$ and estimation of $\alpha/\nu$

In attempting to estimate the exponent ratio $\alpha/\nu$ directly from the finite size scaling of 
the specific heat $C$ through (15), we encountered two difficulties. First, we found that the 
estimates of the specific heat of each sample $C_i$ were very sensitive to the length of the 
simulations. Shorter simulations biased the specific heat to lower values. e.g. for $p = 0.6$, 
$l = 40$ The value measured for $C$ using average simulation length of $n/(2\tau_E + 1) \approx 120$ was 
two standard deviations smaller than the one measured with a four times longer simulation. 
The systematic underestimation of response functions due to run lengths which are too short 
was studied in [39]. Second, the accuracy in estimating $\alpha/\nu$ from the specific heat behavior is 
rather poor. This is due to the fact that $\alpha/\nu$ is a small negative number so that the singular 
behavior of $C$ is difficult to disentangle from other analytic contributions [40].

In order to overcome the difficulty in estimating the exponent ratio $\alpha/\nu$ we followed Heuer 
[40] and measured the derivative of the magnetization with respect to the reduced temperature 
t = $T - T_c / T_c$. It is equal to the magnetization-energy correlation

$$
\Gamma = -\frac{\partial m}{\partial t} = -\frac{\partial^2 f}{\partial H \partial t} \approx \frac{-1}{NpT^2} \langle \langle |M| - \langle |M| \rangle (H - \langle H \rangle) \rangle \rangle, \quad (24)
$$

where $f$ is the free energy density. From the scaling behavior of the free energy

$$
f(t, h) = b^{-d} f(b^\alpha t, b^\eta h) \quad (25)
$$

one finds that $\Gamma$ diverges as $t^{-\zeta}$, where

$$
\zeta = \frac{y_t - (d - y_h)}{y_t} = 1 - \beta. \quad (26)
$$

Thus we fit $\Gamma$ to the finite size scaling form

$$
\Gamma = C_0 t^{\zeta} . \quad (27)
$$
TABLE IV. Estimates for critical exponent ratios from finite size scaling at $T^\infty_c$. Estimates due to Heuer [32] are listed for comparison. The estimate for $\frac{\alpha}{\nu}$ is based on the relation $\alpha = 2(\beta + \frac{\nu}{\nu}) - d$.

| $c$   | $\frac{\alpha}{\nu}$ (Heuer) | $\frac{\beta}{\nu}$ (Heuer) | $\frac{\lambda}{\nu}$ (Heuer) |
|-------|-------------------------------|-------------------------------|-------------------------------|
| 0.6   | 0.948(6)                      | 0.94(2)                       | -0.228(28)                    |
| 0.6   | 0.958(3)                      |                               | -0.210(30)                    |
| 0.8   | 0.962(4)                      | 0.97(2)                       | -0.066(9)                     |
|       |                               |                               | -0.04(6)                      |

The resulting estimates for $\frac{\zeta}{\nu}$ are given in table IV. Assuming the hyperscaling relation $\alpha = 2\nu - d$ and using (26) the scaling relation $\frac{\alpha}{\nu} = 2(\zeta + \frac{\beta}{\nu}) - d$ is obtained. Using this relation, the results for $m$ and $\Gamma$ are utilized in table IV to give estimates for $\frac{\alpha}{\nu}$ which are much more accurate than those obtained from analysis of the specific heat results.

IV. LACK OF SELF AVERAGING AT $T^\infty_c$

In order to obtain the variance $V_X$ and the relative variance $R_X$ the same procedure and considerations as described in section II C were used.

In figure 6 we plot the relative variance of the magnetization $R_m$ as a function of lattice size on a double-logarithmic scale. Several interesting features are suggested by this figure. First, note that for $p = 0.6$, $R_m$ is decreasing as $l$ increases for the smaller lattice sizes, possibly leveling off for large $l$. $R_m$ of the $p = 0.8$ model first decreases slightly and then seems to tend to a constant. Since it seems plausible that $R_m(p = 0.6) \geq R_m(p = 0.8)$ for any lattice size, these trends seem to imply that for the two grand-canonical models $R_m$ tends to the same constant. Assuming that this constant is bound from above by the $p = 0.6$ model and from below by the $p = 0.8$ model we estimate it as $R_m = 0.055(2)$. The implication of this scenario is that $R_m$ of the weakly diluted $p = 0.8$ model reaches the universal $R_m$ value of the dilute Ising fixed point at smaller system sizes than the highly diluted $p = 0.6$ model. The fluctuations in $m_i$ in the highly diluted $p = 0.6$ model are larger than those of the dilute Ising fixed point model. This finding is in line with Monte Carlo results [32] and RG calculations [20], according to which the critical exponents of the dilute Ising fixed point are closer, in the lattice size range $20 \leq l \leq 60$, to the observed effective critical exponents of the $p = 0.8$ model than to those of the $p = 0.6$ model.

A second feature is the striking difference between the two types of disorder, with the canonically disordered $c = 0.6$ model exhibiting a much smaller relative variance than that of the two grand-canonical disorder models. While $R_m$ of the $c = 0.6$ model is initially increasing with system size it appears to level off to a constant value of $R_m(l = 90) = 0.0227(8)$. Though it is possible that $R_m$ could increase at larger lattice sizes it seems unlikely since the system sizes are already quite large. An indicator to the similarity of the two types of models $p = 0.6$ and $c = 0.6$ is the relative square root mean of the fluctuations in the number of magnetic sites $N = \sum_{j=1}^{N} K_j$ in the $p = 0.6$ systems $\sqrt{\frac{[N - \langle N \rangle]^2}{\langle N \rangle^2}} = \sqrt{\frac{1-p}{Np}}$ which for $l = 80$ is as small as $\sim \frac{1}{1000}$.

If indeed $R_m$ of the $c = 0.6$ model tends to a different constant than that of the models with grand canonical disorder, then according to Aharony and Harris’ very general RG arguments [13] the two types of models do not belong to the same universality class! We
FIG. 6. The relative variance of the magnetization $R_m$ at $T_c^\infty$ as a function of log$_{10} l$.

are not aware of any additional evidence to this effect otherwise. For example our critical exponent estimates for the $c = 0.6$ and $p = 0.6$ are compatible with each other, and our exponents for the $p = 0.8$ model are compatible with those of Heuer [32] for a $c = 0.8$ model. The critical temperatures for both types of models seem also to agree (see table V and References [32,35]). This question is currently under investigation. Preliminary results [41] suggest that the two types of models do flow to the same fixed point and that the difference in $R_m$ will disappear for very large $l$.

The relative variance of the susceptibility $R_{\chi}$ is plotted in Fig. 7. $R_{\chi}$ exhibits the same qualitative behavior as that of $R_m$. $R_{\chi}$ of the grand canonical disorder models tends to $R_{\chi} = 0.156(4)$, while $R_{\chi}$ of the $c = 0.6$ model seems to tend to $R_{\chi}(l = 90) = 0.061(2)$. Aharony and Harris [13,42] found that to leading order in $\epsilon = 4 - d$, $R_M/R_{\chi} = 1/4$. We find that for $p = 0.8 R_M/R_{\chi} = 0.35(2)$ and for $c = 0.6 R_M/R_{\chi} = 0.37(1)$. Possibly terms of higher order in $\epsilon$ would reconcile this discrepancy. It cannot be attributed to the definition of $\chi$. If one defines the susceptibility as in (28) then at $T_c^\infty$ one finds that $R_{\chi}$ becomes smaller by a factor of $\sim 7 - 10$. In this case the ratio $R_M/R_{\chi}$ would become even larger. We did not use this definition for the susceptibility at $T_c^\infty$ because of its large single sample errors $\delta\chi_i$ (see also [8]).

V. SCALING OF PSEUDO CRITICAL TEMPERATURES

A. Calculating $T_c(i,l)$ with the histogram reweighting method

One of the main purposes of this work was to study the distribution of pseudo-critical temperatures $T_c(i,l)$ of the ensemble of site-dilute Ising models. The main aim was to study directly the scaling of $\delta T_c(l)$ with $l$ and test which one of (3) or (8) is correct in the case of a system governed by a disordered fixed point. The inverse pseudo-critical temperature $K_c(i,l) = 1/T_c(i,l)$ of the $i$'th sample was defined as the inverse temperature of the maximum of the susceptibility of that sample, $K_c(i,l) \equiv K_{\text{max}}(i,l)$. Here the definition of the susceptibility was
FIG. 7. The relative variance of the susceptibility $R_{\chi c}$ at $T_{c}\infty$ as a function of $\log_{10} l$.

$\chi_i = \langle M^2 \rangle - \langle |M| \rangle^2 \over NpT$.

(28)

In order to find $K_{\text{max}}(i, l)$ the following iteration procedure was followed for each sample. A first simulation was performed at the infinite lattice critical temperature (as estimated in Ref. 32) $K_1 = K_{c\infty}$ (the index $i$ is omitted from here on). In addition to calculating the observables $m, \chi, \Gamma$, a histogram of the energy and magnetization was generated. Using the single histogram reweighting technique [21–25] (For previous studies of disordered systems utilizing the histogram reweighting technique see [36,43]), this histogram can serve to calculate observables at temperatures close to $K_1$. By calculating $\chi$ at different temperatures a first estimate for the susceptibility maximum $\chi_{\text{max}}^1$ and the temperature at which it occurs $K_{\text{max}}^1$ was obtained. A second simulation was then performed at a temperature somewhat above this estimate $K_2 = K_{\text{max}}^1 - \text{offset}$. Previous studies of the histogram reweighting technique have shown that the errors of observables at $T, \delta X(T)$, are smaller [25,23] when the temperature at which the histogram was generated $T_{\text{sim}}$ is slightly higher, $T_{\text{sim}} > T$. For this reason we chose offset to be a small positive number (see more details below).

Using the energy and magnetization histogram generated at $K_2$ a new estimate for the temperature of the susceptibility maximum, $K_{\text{max}}^2$, was obtained. If the difference between the two estimates was smaller than a predetermined resolution $r$, $|K_{\text{max}}^2 - K_{\text{max}}^1| < r$, the iteration process was stopped. Otherwise the iteration process continued, whereby $K_j = K_{\text{max}}^{j-1} - \text{offset}$, until the condition

$|K_{\text{max}}^j - K_{\text{max}}^{j-1}| < r$

(29)

was met. This iteration process was intended to overcome the problem of systematic errors [23] that occur when the simulation temperature is too far from the true $K_{\text{max}}$. The condition (29) is supposed to ensure that the last two estimates for $K_{\text{max}}$ do not suffer from a systematic error. $r$ was chosen equal to the approximately expected statistical error of $K_{\text{max}}^j$. If the iteration process did not terminate before or with the third estimate $K_{\text{max}}^3$ then the Monte Carlo simulation length at $K_4$ was doubled and the process was continued. It was again doubled if it reached the seventh iteration and again doubled if it reached the tenth iteration.
Non-convergence of the process after twelve iterations was very rare. In those samples the iteration procedure was restarted manually with \( K_1 = K_c^\infty \) but with a larger initial Monte Carlo simulation length. The need to increase the simulation length for some samples occurred because for different samples there were different auto-correlation times (of the Monte Carlo dynamics) and different average cluster sizes, while the simulation length was specified by the number of Wolff cluster flips.

In order to estimate the statistical error and reduce it, a simulation with five times as many Monte Carlo steps (compared to the simulation length of the last iteration) was performed again at the last simulation temperature \( K_j \). The Monte Carlo sequence was broken into five, using each segment to create a separate histogram and calculate a separate estimate of \( K_{\max} \) and \( \chi_{\max} \). Together with the last estimates of \( K_{\max} \) and \( \chi_{\max} \) of the iteration procedure, all together six estimates of \( K_{\max,i} \) and \( \chi_{\max,i} \) were averaged to give final estimates of \( K_{\max,i} \) and \( \chi_{\max,i} \). The variance of these six estimates was used to estimate the error for the two quantities, \( \delta K_{\max,i} \), and \( \delta \chi_{\max,i} \).

The parameter \( offset \) was adjusted for the small system sizes, through trial runs, so as to minimize the errors in \( \chi_{\max} \), while its value for the larger systems was extrapolated from the smaller ones. For \( c = p = 0.6 \) we set \( offset \approx 0.27 l^{-1.66} \), and for \( p = 0.8 \) \( offset \approx 0.12 l^{-1.63} \). The optimal value of \( offset \) was found not to depend strongly on the simulation length. The resolution \( r \) was adjusted so as to be approximately equal to the ensemble average statistical error of \( K_j^{\max} \). Note that the parameters \( offset \) and \( r \) were set once for each model and each lattice size and were not varied for different samples. In some of the larger systems, for a subset of the samples, the simulation with five times as many Monte Carlo steps was not performed, so that error estimates of \( K_{\max} \) and \( \chi_{\max} \) were not obtained. This was done in order to save computer time. For these samples the average squared error of \( K_{\max} \) and \( \chi_{\max} \) was approximated as being six times larger than that of the complementary subset of samples where the error was calculated (from an all together six times longer Monte Carlo sequence). For the \( p = 0.6 \) \( l = 80 \) system the estimated average squared error was extrapolated from the smaller systems.

### B. Scaling of \( t_c(l) \)

In the finite size scaling theory of [8] it was assumed that the average pseudo-critical temperature \( K_{\max}(l) \equiv \langle K_{\max}(i,l) \rangle \) scaled as

\[
K_{\max}(l) - K_c = A_K l^{-\lambda}, \tag{30}
\]

and that the shift exponent \( \lambda = y_t = 1/\nu \). First we assumed the correctness of the critical temperature values \( T_c^\infty \) quoted in table I so that the critical inverse temperature of the infinite sample is assumed to be \( K_c = 1/T_c^\infty = .285781(40) \) for \( p = 0.8 \), and \( K_c = 0.41288(10) \) for \( c = 0.6 \) and \( p = 0.6 \). Fitting \( K_{\max}(l) \) to (30) with \( K_c \) fixed we found for the \( p = 0.8 \) model values of \( \lambda = 1.446(34) \) and \( A_K = 0.040(5) \) from lattice sizes \( 16 \leq l \leq 64 \). For the \( c = 0.6 \) and \( p = 0.6 \) models the results were incompatible with the fixed value of \( K_c = 0.41288 \). In fact in these models \( K_{\max}(l) \) monotonically decreases with \( l \) and for the largest lattices we have \( K_{\max}(80, p = 0.6) - K_c = -0.00026(3) \) and \( K_{\max}(60, c = 0.6) - K_c = -0.000077(35) \). Thus we also fitted \( K_{\max}(l) \) to (30) with \( K_c \) being a free parameter. The
TABLE V. Different parameters related to the average pseudo-critical inverse temperature $K_{\text{max}}(l)$ and its variance $V_{K_{\text{max}}}$. Second column: estimate of the shift exponent $\lambda$ according to (30), where $K_c$ is taken from Heuer [32]. Third and fourth column: same as first column but with $K_c$ being a free parameter. Fifth column: estimate of $y_t$ based on the finite size scaling of $m$ and $\Gamma$. Sixth column: same as fifth according to [32]. Last column: exponent of $V_{K_{\text{max}}}$.  

| $c = 0.6$ | $p = 0.6$ | $p = 0.8$ |
|---|---|---|
| $\lambda$, ($K_c$ fixed) | $\lambda$ | $\lambda$ |
| 0.99(19) | 0.41254(13) | 0.41251(5) |
| 0.41254(13) | 1.398(34) | 1.346(2) |
| 1.386(14) | 1.395(12) | 1.3857609(4) |
| 1.39(4) | $-$ | 1.39(4) |
| 1.41(4) | 1.421(9) | 1.41(4) |
| $y_t = \frac{\xi}{\nu} + \frac{\zeta}{\nu}$ | $y_t$ (Heuer) | $y_t$ (Heuer) |
| 1.467(5) | 1.47(4) | 1.47(4) |
| $\xi/\nu$ | 1.44(2) | 1.44(2) |
| $\zeta/\nu$ | 1.44(2) | 1.44(2) |

values of $\lambda$ and $K_c$ which were found, using lattice sizes $10 \leq l \leq 60$ for $c = 0.6$ and $p = 0.6$, and $8 \leq l \leq 64$ for $p = 0.8$, are given in the third and fourth columns of table IV. For $p = 0.8$ our estimate $K_c = 0.2857609(4)$ is within errors of the estimate of Heuer [32] (with canonical disorder) and of Wang et al [33] (with grand canonical disorder). For $c = 0.6$ and $p = 0.6$ our estimates $K_c = 0.41254(13)$ and $K_c = 0.41251(5)$ are within errors of each other but not within errors of the assumed value $K_c = 0.41288(10)$. A more accurate estimate of $K_c$, which does not require knowledge of $\lambda$, is obtained in section IV C. Variance of pseudo-critical temperatures $V_{K_{\text{max}}}$.

C. Variance of pseudo-critical temperatures $V_{K_{\text{max}}}$

The variance of the pseudo-critical temperatures distribution $V_{K_{\text{max}}}$ was calculated taking into account the errors $[(\delta K_{\text{max}})^2]$. This is done in a manner completely analogous to the discussion of $V_X$ in Sec. [11]. $V_{K_{\text{max}}}$ is plotted in Fig. 8 on a double logarithmic scale. The solid lines are fits to the form $V_{K_{\text{max}}} = A K_l^\rho K$ and the resulting estimates of $\frac{\rho}{\nu}$ are listed in the last column of table IV. As one would expect, $V_{K_{\text{max}}}$ is smaller for $c = 0.6$ than for $p = 0.6$, and is the smallest for $p = 0.8$. We see that for all three models the results for $\rho K$ exclude the possibility [3] that $\rho K = d = 3$. On the other hand $\frac{\rho}{\nu}$ is within errors of $y_t$ for $p = c = 0.6$, and within errors of $\lambda$ (with $K_c$ fixed) for $p = 0.8$, as predicted by Aharony and Harris [6]. Note that the values obtained for $\rho K$ for $p = 0.8$ with lattices $8 \leq l \leq 32$ and $p = 0.6$ with $20 \leq l \leq 60$ are $\rho K = 2.95(6)$ and $\rho K = 3.00(4)$. This behavior of $V_{K_{\text{max}}}$ could
FIG. 8. The variance of the inverse pseudo-critical temperatures $V_{K_{\text{max}}}$ as a function of $l$ on a log – log scale. The lines are linear fits yielding exponents $\rho_K$ listed in the last column of table V.

be a manifestation of a crossover from pure (3) to dilute (8) critical behavior. On the other hand for the model with the canonical disorder the crossover is in the opposite direction since for $c = 0.6$ with $10 \leq l \leq 40$, $\rho_K = 2.77(7)$.

The results for $V_{K_{\text{max}}}$ support the picture implied by AH RG calculations, namely that both the width of the pseudo-critical temperatures $\sqrt{V_{K_{\text{max}}}(l)}$ and the distance of its average from the critical inverse coupling $|K_{\text{max}}(l) - K_c|$ scale as $\sim l^{-y_t}$. This is best visualized in Fig. 9 where the frequency of the scaled pseudo-critical inverse temperatures $(K_{\text{max}}(i, l) - K_c)l^{y_t}$ is plotted for $p = 0.8$ and $l = 16, 32, 64$ with $K_c = 0.285781$ and $y_t = 1.467$. It is evident that the three distributions match well. Their averages are $[(K_{\text{max}}(i, l) - K_c)l^{y_t}] = 0.047(1), 0.0451(28), 0.044(5)$, and their widths are $\sqrt{V_{K_{\text{max}}}}l^{y_t} = 0.172(15), 0.174(28), 0.17(4)$ for $l = 16, 32, 64$ respectively. Note that the average ratio of the width to the average is $\approx 3.8$. Thus, as is evident from Fig. 9, the fluctuations in $K_c(i, l)$ are significantly larger than $|K_c(l) - K_c|$ for any system size. The result is that a measurement of $X$ at the critical temperature $T_c^\infty$ is done in some samples above their pseudo-critical temperature $T_c(i, l)$ and in some samples below $T_c(i, l)$!

1. Estimating $T_c^\infty$ through $V_{K_{\text{max}}}$

Our estimates of $V_{K_{\text{max}}}$ allow us to estimate $T_c^\infty$ by another method (We thank D. Stauffer for bringing this to our attention). Since asymptotically $K_{\text{max}}(l) - K_c \sim l^{-y_t}$ and $\sqrt{V_{K_{\text{max}}}} \sim l^{-y_t}$, one expects that

$$K_{\text{max}}(l) = K_c + B_v\sqrt{V_{K_{\text{max}}}(l)},$$

where $K_c$ and $B_v$ need to be determined. Note that by fitting the data according to (31) (this method was used in percolation studies [13]) it is not necessary to determine $\nu$, and only two fitting parameters are used. Therefore the estimates of $K_c$ obtained in this way are probably more reliable than those given in table V. In figure 10 we plot $K_{\text{max}}$ as a function of $\sqrt{V_{K_{\text{max}}}}$ together with linear fits made according to (31). We find $K_c = 0.285779(2),$
$K_{\text{max}}(i,L) - K_c$ for $p = 0.8$, with $K_c = 0.285781$ and $y_t = 1.467$. Thin dotted line for $l = 16$, thick dashed line for $l = 32$ and thin solid line for $l = 64$. The number of samples used was 32000 for $l = 16$ 4000 for $l = 32$ and 1479 for $l = 64$.

$B_v = 0.257(3)$ for $p = 0.8$, $K_c = 0.41244(4)$, $B_v = 0.22(3)$ for $p = 0.6$, and $K_c = 0.41265(4)$, $B_v = 0.230(15)$ for $c = 0.6$.

D. Maximum of the susceptibility $\chi_{\text{max}}$

1. Scaling of $[\chi_{\text{max}}]$

Another way to study the finite size scaling of the susceptibility is to study the ensemble average of the maximum susceptibility $[\chi_{\text{max}}]$ which is expected to scale with lattice size as in (16) with a scaling exponent $\tilde{\nu}$. In figure 11 $[\chi_{\text{max}}]$ is plotted as a function of $l$ on a double logarithmic scale. The straight lines are linear fits to the form (16). For $p = 0.8$ we find $\tilde{\nu} = 1.987(3)$ which is in agreement with the estimate obtained using the susceptibility $\chi_c$ at $T_c$, $\tilde{\nu} = 1.990(4)$. As can be seen in Fig. 11 the values of $[\chi_{\text{max}}]$ for $p = 0.6$ and $c = 0.6$ are indistinguishable (they are indeed within errors). This is in contrast with the data at $T_c$ of fig. 3 where $\chi_c$ of the two models seem to diverge with a similar exponent but with a different amplitude. We have also calculated $\chi$ at $T_c$ and found the same trend, namely that $\chi(p = 0.6) > \chi(c = 0.6)$, so that this feature is not an artifact of the different definitions, (23) and (25), for $\chi$. For $p = 0.6$ and $c = 0.6$ we found $\tilde{\nu} = 2.027(2)$ and $\tilde{\nu} = 2.034(2)$ respectively. These values are significantly lower than the values found using the susceptibility $\chi_c$ at $T_c$, $\tilde{\nu} = 2.104(20)$ and $\tilde{\nu} = 2.110(21)$. They are also closer to results of RG calculations $\tilde{\nu} = 1.97$ [20,19].
FIG. 10. The average pseudo-critical temperature $K_{\text{max}}$ as a function of the square root of $V_{K_{\text{max}}}$ together with linear fits made according to (31). Fits are made using the three largest system sizes for each model. The fitting parameters are listed in the text.

FIG. 11. The ensemble average of the maximum susceptibility $[\chi_{\text{max}}]$ as a function of $\log_{10} l$. The solid lines are fits to the form (16), yielding estimates of $\frac{\gamma}{\nu} = 2.034(2)$ for $c = 0.6$, $\frac{\gamma}{\nu} = 2.027(2)$ for $p = 0.6$, and $\frac{\gamma}{\nu} = 1.987(3)$ for $p = 0.8$. 

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2. Lack of self averaging of the relative variance $R_{\chi_{\text{max}}}$

In figure 12 we plot the relative variance of the maximal susceptibility $R_{\chi_{\text{max}}}$ as a function of lattice size on a double-logarithmic scale. For $p = 0.8$, $l = 64$ and $p = 0.6$, $l = 80$ we have $[(\delta \chi_{\text{max},i}^2)]/V_{\chi_{\text{max}}} \approx \frac{1}{2}$, 0.15 respectively (see section I(C)). Thus the estimate of $V_{\chi_{\text{max}}}$ is dominated by the estimate of the average squared single sample errors $[(\delta \chi_{\text{max},i})^2]$. For $p = 0.6$ $l = 80$ this estimate was actually extrapolated from the smaller systems estimates (see VA). Thus these two data points should be taken with more than a grain of salt. It is most interesting to compare Fig. 12 to Fig. 7 where the relative variance of the susceptibility at $T_c^\infty$, $R_{\chi_c}$, is plotted. For $p = 0.8$ the behavior of $R_{\chi_{\text{max}}}$ and $R_{\chi_c}$ is qualitatively rather similar with $R_{\chi_{\text{max}}}$ initially decreasing as $l$ increases and tending for larger $l$ to a constant, where $R_{\chi_{\text{max}}}(l = 64) = .00216(16)$. However, in contrast with Fig. 4, this constant is roughly 72 times smaller than the large $l$ value of $R_{\chi_c}$. This is quite a striking difference. It means that in order to obtain the same relative accuracy in $[\chi_c]$ as in $[\chi_{\text{max}}]$ approximately 70 times as many samples are needed. The source of this difference is apparently simple. The susceptibility of each sample is some function $G$ of the temperature with a sharply peaked maximum at $T_c(i,l)$. In fact $G$ is approximately only a function of the difference $T-T_c(i,l)$, $G(T-T_c(i,l))$. Thus, the value of the maximum susceptibility is nearly sample independent, $\chi_{\text{max}} \approx G(0)$. On the other hand, when one measures $\chi$ at $T_c^\infty$, in different samples one is sampling $G$ at different values of its argument. This results in large fluctuations in $\chi$ at $T_c^\infty$.

Our findings suggest that the standard procedure of investing much computation time in finding the $l \rightarrow \infty$ limit of the critical temperature, $T_c^\infty$, and then averaging quantities at this temperature over many samples, is not optimal. A better procedure may be to locate through the single or multiple histogram method the pseudocritical temperature of each sample, and measure quantities at that temperature. In this way sample to sample fluctuations are reduced substantially and better accuracy is achieved.

For $p = 0.6$ $R_{\chi_{\text{max}}}$ monotonically decreases with lattice size, possibly leveling off to a constant for large $l$. In contrast with $R_{\chi_c}$, this constant is different from that of the $p = 0.8$ model. Lastly, for $c = 0.6$, in contrast with $R_{\chi_c}$, we find that $R_{\chi_{\text{max}}}$ is within errors of $R_{\chi_{\text{max}}}$ of the $p = 0.6$ model. In addition, $R_{\chi_{\text{max}}}$ initially decreases as $l$ increases, opposite to the behavior of $R_{\chi_c}$. More explanations to the differences in the behavior of $R_{\chi_{\text{max}}}$ and $R_{\chi_c}$ are given at the end of the next subsection.

E. Dependence of $m(T_c^\infty)$ on $T_c(i,l)$

After examining the behavior of the distribution of $X(i,l)$ at $T_c^\infty$ and the distribution of $T_c(i,l)$ it is imperative to examine the correlation between the two distributions. A good starting point is the finite size scaling ansatz (4), according to which $X_c(T_c^\infty)$ mainly depends on $T_c^\infty-T_c(i,l)$. Fig. 13 is a scatter plot where for each sample $i$ the horizontal axis represents the scaled absolute inverse temperature $|K_c-K_c(i,l)|/l^{\nu}$ and the vertical axis is the scaled magnetization $m_l l^{2\nu}$. This representation is equivalent to the usual data collapse representation which is used to demonstrate finite size scaling. The difference is that here the reference critical temperature is $K_c(i,l)$ instead of $K_c$, and the measurement temperature is always $K_c$ instead of different values of $K$. Points with $K_c > K_c(i,l)$ constitute the higher
FIG. 12. The relative variance of the maximum susceptibility $R_{\chi_{\text{max}}}$ as a function of $\log_{10} l$.

$m$ (lower temperature) branch, whereas points with $K_c < K_c(i,l)$ constitute the lower $m$ (higher temperature) branch. In figure 13 we plot data for $p = 0.8$ and $l = 16, 64$. For the sake of clarity, only 100 points are shown for each system size and each branch, and several points with $|K_c - K_c(i,l)|^{\nu_Y} < 0.001$ were omitted. Figure 13 indicates that to a good approximation the scaled magnetization of the sample at $T_\infty$ is a function of only the scaled reduced temperature of the sample. Thus one may attempt to substitute (4) by a sample independent form for $\tilde{Q}_i(Z)$ so that

$$X_i(T,l) \approx l^n \tilde{Q}(i,l^{\nu_Y}).$$

Note that this is only a good approximation; if (32) were exact, it would mean that $R_{\chi_{\text{max}}} = 0$. Thus in order to describe the magnetization data at $K_c$ we write (the change from temperature to inverse temperature is only for convenience)

$$m_i(K,l) = l^{-\beta} \tilde{Q}_\pm \{ |(K-K_c(i,l)|^{\nu_Y} \}.$$

Here $\tilde{Q}_+(Z)$ is the scaling function for $K < K_c(i,l)$ and $\tilde{Q}_-(Z)$ for $K > K_c(i,l)$. For large $l$, and thus large $Z$, the infinite sample critical behavior, $m_i \sim \{K-K_c(i,l)\}^{\beta}$, must be asymptotically reproduced \textsuperscript{10} for $K > K_c(i,l)$. Thus, for large $Z$, $\tilde{Q}_-(Z) \sim Z^\beta$. For $K < K_c(i,l)$ the shape of $\tilde{Q}_+(Z)$ must reproduce, for large $Z$, the decay of the magnetization to zero as $l \to \infty$. Thus, for large $Z$, $\tilde{Q}_+(Z) \sim Z^{\beta - \frac{d}{2m}}$. For $K = K_c(i,l)$, i.e. $Z = 0$, the finite size scaling behavior $[m_i] \sim l^{-\beta}$ must be asymptotically reproduced, implying $\tilde{Q}_+(Z) \to \text{const}$ as $Z \to 0$. A simple possible form for $\tilde{Q}_+(Z)$ fulfilling these requirements is

$$\tilde{Q}_+(Z) = A_+ Z^{\rho_+} (1 + B_+ Z^{-\rho_+})^{-\frac{\rho_+}{\rho_+}},$$

where $\rho_- = \beta$ and $\rho_+ = \beta - \frac{d}{2m}$ and $A_+, B_+, \rho_\pm$ are free parameters, so that the data of Fig. 13 should be described by (33, 34) with $K = K_c$. Thus, for each lattice size $l = 16, 32, 64$.

\textsuperscript{1}An example of this type of finite size scaling is the scaling we found for $[\chi_{\text{max}}]$. 

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The goodness of the fits is also extremely high. This suggests that equation (32), equations between the three curves for both branches, as seen in Fig. 14, is surprisingly good. The agreement, only 100 points are shown for each system size and each branch, and several points with $|K_c - K_c(i, l)|^{|p_+|} < 0.001$ were omitted.

| $K_c$ $|K_c - K_c(i, l)|^{|p_+|}$
|---|
| $l = 16$ | $0.001$ | $0.01$ | $0.1$ | $1$ |
| $l = 32$ | $0.01$ | $0.1$ | $1$ |
| $l = 64$ | $0.1$ | $1$ |

**FIG. 13.** Scatter plot, where for each sample $i$ the horizontal axis represents the scaled absolute inverse temperature $Z = |K_c - K_c(i, l)|^{|p_+|}$ and the vertical axis is the scaled magnetization $\tilde{Q}_\pm = ml^{p_\pm}$. Points with $K_c > K_c(i, l)$ constitute the higher $m$ (lower temperature) branch, whereas points with $K_c < K_c(i, l)$ constitute the lower $m$ (higher temperature) branch. For the sake of clarity, only 100 points are shown for each system size and each branch, and several points with $|K_c - K_c(i, l)|^{|p_+|} < 0.001$ were omitted.

**TABLE VI.** Parameters of the fitting functions $\tilde{Q}_\pm(Z)$, defined in (34), obtained by fitting the data sets of scaled $\{K_{\max}(i, l), m_i(K_c)\}$ pairs for each lattice size $l = 16, 32, 64$ separately (with $p=0.8$). The number of samples used was 32000, 4000, and 1477 for $l = 16, 32, 64$ respectively. The six fitting functions are plotted in Fig. 14. The crossover lengths, which control the crossover to the large $Z$ behavior are defined as $Z_{\pm}^{\text{cross}} = B_{\pm}^{p_{\pm}}$.

| $A_-$ | $B_-$ | $p_-$ | $Z_{\pm}^{\text{cross}}$ | $A_+$ | $B_+$ | $p_+$ | $Z_{\pm}^{\text{cross}}$ |
|---|---|---|---|---|---|---|---|
| $l = 16$ | $2.387(4)$ | $0.0518(13)$ | $1.45(1)$ | $0.130$ | $0.4695(12)$ | $0.1765(18)$ | $1.299(5)$ | $0.263$ |
| $l = 32$ | $2.380(9)$ | $0.048(3)$ | $1.49(3)$ | $0.130$ | $0.4623(22)$ | $0.1687(34)$ | $1.316(9)$ | $0.259$ |
| $l = 64$ | $2.291(23)$ | $0.07(1)$ | $1.33(6)$ | $0.142$ | $0.4524(75)$ | $0.152(11)$ | $1.368(35)$ | $0.252$ |

and both branches, $K_c < K_c(i, l)$ and $K_c > K_c(i, l)$, the scaled $\{K_{\max}(i, l), m_i(K_c)\}$ pairs (a partial set of which is plotted in figure 13) were fitted to the form (34). The values of $\rho_- = \beta = 0.34295$ and $\rho_+ = \beta - \frac{d}{2y_+} = -0.67565$ which were used rely on the finite size scaling results at $T_c^\infty$ (tables 11 and 11). The six fitting functions which were obtained are plotted in Fig. 14 and their fitting parameters are given in Table VI. The agreement between the three curves for both branches, as seen in Fig. 14, is surprisingly good. The goodness of the fits is also extremely high. This suggests that equation (32), equations similar to (34), and the possibly invariant (as suggested by Fig. 14) distributions of $K_c(i, l)$ provide an excellent description of the scaling behavior of disordered systems.

In figure 15 we show the same data as in figure 13 but for $p = 0.6$ and $c = 0.6$ with system size $l = 40$. The purpose of this analysis is to demonstrate that the magnetization of the two models is governed by the same temperature dependence, and that the main difference is in the distributions of $K_c(i, l)$. For this reason the data were scaled with the same exponents,
FIG. 14. The functions $\tilde{Q}_\pm(Z)$, as defined in (34), obtained from best fits to the scaled magnetization versus temperature scatter plots for $l = 16, 32, 64.$ upper curves according to $\tilde{Q}_-$ ($K_c > K_c(i, l)$) and lower curves according to $\tilde{Q}_+$ ($K_c < K_c(i, l)$). The fitting parameters are given in table VI.

FIG. 15. Same data as in figure 13 but for $p = 0.6$ and $c = 0.6$ with system size $l = 40.$ The data were scaled with exponents taken as the average of the exponents of the two models, $\frac{\beta}{\nu} = 0.4375$ and $\nu = 1.3905.$ For the sake of clarity, only a 100 points for each model and each branch are shown.
FIG. 16. The functions $\tilde{Q}_\pm(Z)$, as defined in (34), obtained from best fits to the scaled magnetization verses temperature scatter plots for $c = 0.6$ (dotted line) and $p = 0.6$, with $l = 40$. Upper curves according to $\tilde{Q}_-$ ($K_c > K_c(i,l)$) and lower curves according to $\tilde{Q}_+$ ($K_c < K_c(i,l)$). Fits made using $\frac{\beta}{\nu} = 0.4375$ and $\gamma_t = 1.3905$. In fact our estimates for $\gamma_t$ and $\frac{\beta}{\nu}$ for the two models are within errors. For the sake of clarity, only 100 points for each model and each branch are shown. As was seen with the $p = 0.8$ data, it is evident that to a good approximation in both models the magnetization at $K_c$ is a function of only the reduced inverse temperature $K_c - K_c(i,l)$. The main difference between the two models is also clear; For $p = 0.6$ there are more points with large $|K_c - K_c(i,l)|$, while for $c = 0.6$ there are more points with small $|K_c - K_c(i,l)|$. Thus larger fluctuations for $p = 0.6$ in $K_c(i,l)$ (see also Fig. 8) together with the large dependence of $m_i(K_c)$ on $K_c - K_c(i,l)$ give rise to the result that $R_m(p = 0.6) > R_m(c = 0.6)$.

In figure 16 we plot the fitting functions $\tilde{Q}_\pm(Z)$, obtained by best fits to the scaled magnetization verses temperature scatter plots for $p = 0.6$ and $c = 0.6$ with $l = 40$ (the full data sets corresponding to figure 15). For the high temperature branch ($K_c < K_c(i,l)$, lower curve) good agreement between the fitting functions $\tilde{Q}_\pm(Z)$ of the two models is found. For the low temperature branch ($K_c > K_c(i,l)$, higher curve) good agreement is found between the functions $\tilde{Q}_\pm(Z)$ for smaller $Z$, while for large $Z \tilde{Q}_-(Z)$ is larger for the grand canonical disorder ($p=0.6$). The fitting functions $\tilde{Q}_\pm(Z)$ for the data for $l = 60$ did not agree with those of $l = 40$. Possibly this is so because the exponents used are not the asymptotic ones.

It is also of interest to contrast the dependence of $\chi_{\text{max}}$ on $K_{\text{max,i}}$ with the dependence of $\chi_c(K_c)$ on $K_{\text{max,i}}$. This is a key to understanding the reasons for the differences between the characteristics of $R_{\chi_c}$ (figure 7) and the characteristics of $R_{\chi_{\text{max}}}$ (figure 12). In figure 17 we show a scatter plot of $(K_{\text{max}} - K_c, \frac{\chi_{\text{max}}}{\chi_{\text{max}}})$ and $(K_{\text{max}} - K_c, \frac{\chi_c(K_c)}{\chi_c(K_c)})$ for $p = 0.6$ and system size $l = 60$ from 950 samples. It is evident that while $\chi_c(K_c)$ shows a strong dependence on
FIG. 17. A scatter plot of $(K_{\text{max}} - K_c, \chi_c(K_c))$ and $(K_{\text{max}} - K_c, \chi_{\text{max}})$, contrasting the dependence of $\chi_c(K_c)$ and $\chi_{\text{max}}$ on $K_{\text{max}}(i,l)$. Data for $p = 0.6$ and system size $l = 60$ from 950 samples.

$K_{\text{max}} - K_c, \chi_{\text{max}}$ shows little dependence on $K_{\text{max}} - K_c$. This qualitative difference persists for all models and all system sizes. This explains why, for any given model, fluctuations in $K_{\text{max},i}$ give rise to fluctuations in $\chi_c(K_c)$ which are much larger than the fluctuations in $\chi_{\text{max}}$. The result is that $R_{\chi_{\text{max}}} \ll R_{\chi_c}$, as we have noted previously.

Fig. 17 is also the key to understanding why $R_{\chi_c}(p = 0.6) > R_{\chi_c}(c = 0.6)$ while $R_{\chi_{\text{max}}}(p = 0.6) \approx R_{\chi_{\text{max}}}(c = 0.6)$. In the first case, since fluctuations in $K_{\text{max}}$ are larger for $p = 0.6$ than for $c = 0.6$ (see figure 8), the strong dependence of $\chi_c(K_c)$ on $K_{\text{max}} - K_c$ gives rise to $R_{\chi_c}(p = 0.6) > R_{\chi_c}(c = 0.6)$. In the second case, despite the fact that fluctuations in $K_{\text{max}}$ are larger for $p = 0.6$ than for $c = 0.6$, the weak dependence of $\chi_{\text{max}}$ on $K_{\text{max}} - K_c$ results in $R_{\chi_{\text{max}}}(p = 0.6) \approx R_{\chi_{\text{max}}}(c = 0.6)$.

VI. SUMMARY AND DISCUSSION

By and large it seems that our MC results confirm the AH scenario. In an Ashkin-Teller model, governed by a pure fixed point, we found that $R_X \sim l^{\frac{\zeta}{\nu}}$, in agreement with (3) and (4). In site dilute Ising models on a cubic lattice, governed by a random fixed point, we found a lack of self averaging for both canonical and grand canonical disorder. One of the aims of our work was to resolve whether at random fixed points our assumption (5), which led to the prediction (6) for the critical width $R_X$, is correct? The alternative $R_X \to \text{const}$ result of AH implies that (8) should replace (5). Our results indicate that the AH result is the correct one. Note though that the absolute value of the exponent ratio $\frac{\zeta}{\nu}$ of the dilute Ising fixed point, either as calculated by RG, $\frac{\zeta}{\nu} = 0.003$, or as indicated by the $p = 0.8$ results $\frac{\zeta}{\nu} = -0.055(8)$, is very small. Thus one could argue that our results for $R_m$ and $R_{\chi_c}$ do not disprove (5). The scaling of $V_{K_{\text{max}}}$ is, however, in agreement with (8) and not with
(3). This therefore rules out (3) since it is based on (3).

We find it appropriate to repeat here the results of Aharony and Harris [13], which we have now validated, with an emphasis on the implication to experiments. In finite size scaling form the relative variance can be written as

$$R_X(\xi, l) = \omega Q(l/\xi).$$  \hspace{1cm} (35)

For a fixed $\xi = \xi_0$ and $l \gg \xi$, and thus large $Z$, strong self averaging, $R_X(\xi_0, l) \sim l^{-d}$, must be asymptotically reproduced. Thus $Q(Z) \sim Z^{-d-\omega}$ for large $Z$. At criticality the correlation length diverges and

$$\lim_{\xi \to \infty} R_X(\xi, l) = \omega Q(0).$$  \hspace{1cm} (36)

When the system is governed by a disordered fixed point $\omega = 0$. When the system is governed by a pure fixed point $\omega = (\alpha/\nu)_p$. Thus the two possible behaviors for $1 \ll \xi \ll l$ are

$$R_X(\xi, l) \sim \begin{cases} (\frac{l}{\xi})^{-d} & \text{for a random fixed point} \\ (\frac{l}{\xi})^{-d} \xi^{\frac{2}{\nu}} & \text{for a pure fixed point} \end{cases}.$$  \hspace{1cm} (37)

In an experiment, since generating many samples is impractical, one studies a single large sample with a particular realization of the quenched disorder of size $l$. For any $\xi$ the value of $X$ measured in the sample is a sampling from a probability distribution with relative variance $R_X(\xi, l)$. Thus $R_X(\xi, l)$ controls the deviation of $X$ from the many samples average. If the system is governed by a random fixed point, as the correlation length is increased, $R_X$ increases as $\sim (\frac{l}{\xi})^{-d}$. $X$ behaves like the average of $(\frac{l}{\xi})^d$ independent measurements on regions of size $\xi^d$. The variance of these measurements does not decrease as $\xi$ increases; it is constant. On the other hand if the disorder is irrelevant and the system is governed by a pure fixed point with $\alpha < 0$, as the correlation length is increased, $R_X$ increases more mildly as $\sim (\frac{l}{\xi})^{-d} \xi^{\frac{2}{\nu}}$. In this case too, $X$ behaves like the average of $(\frac{l}{\xi})^d$ measurements on regions of size $\xi^d$. However, as $\xi$ increases, the variance of these measurements decreases as $\sim \xi^{\frac{2}{\nu}}$.

We have verified that for a disordered system governed by a random fixed point $(\delta T_c(l))^2$ does not scale as $\sim l^{-d}$, but rather that $(\delta T_c(l))^2 \sim l^{-\frac{d}{2}}$. This is an important result, similar to the situation in the purely geometric percolation problem [15]. Recently Pázmándi, Scalalettar and Zimányi [12] claimed that the bound $\nu \geq 2/d$, which was supposed to hold for disordered systems [10], is not valid. As they show, if in systems violating this bound one would have $(\delta T_c(l))^2 \sim l^{-d}$ [our equation (5)], then simulations at $T_c^{\infty}$ would not be able to capture the true critical exponents [12]. In fact in [12] [3] is termed “the most likely scenario” and the conclusion drawn is that “self averaging breaks down”. However, studies of percolation [15], our results, and those of AH [13] imply the contrary. $(\delta T_c(l))^2 \sim l^{-\frac{d}{2}}$ [our equation (5)], and therefore simulations at $T_c^{\infty}$ are able to capture the true critical exponents even if $\nu < 2/d$. This also becomes evident by examining the finite size scaling theory of [9] for $[X_i(T_c^{\infty})]$, assuming that (5) holds versus the consequences of (8).

We’ve shown that fluctuations in $X_i$ at $T_c^{\infty}$ are predominantly due to fluctuations in $\delta T_i = T_c^{\infty} - T_c(i,l)$, and that these fluctuations can be dramatically reduced by measuring
\(X_i\) at \(T_c(i, l)\). This suggests that using the histogram method to obtain \(X_i(T_c(i, l))\) for each sample might be a better strategy for Monte Carlo studies than the current strategy of studying \(X_i(T_c^\infty)\). It was also shown that to a good approximation, fluctuations of \(X_i\) close to criticality can be accounted for by the finite size scaling form \([32]\). We believe that a more extensive study of the finite size scaling of sample to sample fluctuations is both feasible and desired.

One of the surprising results of this work is the difference found between the \(p = 0.6\) model with grand canonical disorder and the \(c = 0.6\) model with canonical disorder. Our results indicate that for \(p = 0.6\) and \(c = 0.6\) \(V_{K_{\text{max}}}\) scales as \(l^{-2\eta}\) and that \(V_{K_{\text{max}}} (p = 0.6)/V_{K_{\text{max}}} (c = 0.6) \approx 3.26\). This is apparently the reason why for these two types of disorder \(R_X\) tends as \(l \to \infty\) to different constants. On the other hand we did not find any difference in the scaling exponents of the two types of disorder.

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