Elementary elliptic \((R, q)\)-polycycles

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Abstract

We consider the following generalization of the decomposition theorem for polycycles. A \((R, q)\)-polycycle is, roughly, a plane graph, whose faces, besides some disjoint holes, are \(i\)-gons, \(i \in R\), and whose vertices, outside of holes, are \(q\)-valent. Such polycycle is called elliptic, parabolic or hyperbolic if \(\frac{1}{q} + \frac{1}{r} - \frac{1}{2}\) (where \(r = \max_{i \in R} i\)) is positive, zero or negative, respectively.

An edge on the boundary of a hole in such polycycle is called open if both its end-vertices have degree less than \(q\). We enumerate all elliptic elementary polycycles, i.e. those that any elliptic \((R, q)\)-polycycle can be obtained from them by agglomeration along some open edges.

1 Introduction

Given \(q \in \mathbb{N}\) and \(R \subset \mathbb{N}\), a \((R, q)\)-polycycle is a non-empty 2-connected plane, locally finite (i.e. any circle contain only finite number of its vertices) graph \(G\) with faces partitioned in two non-empty sets \(F_1\) and \(F_2\), so that:

(i) all elements of \(F_1\) (called proper faces) are combinatorial \(i\)-gons with \(i \in R\);
(ii) all elements of \(F_2\) (called holes, the exterior face(s) \(^1\) are amongst them) are pair-wisely disjoint, i.e. have no common vertices;
(iii) all vertices have degree within \(\{2, \ldots, q\}\) and all interior (i.e. not on the boundary of a hole) vertices are \(q\)-valent.

The plane graph \(G\) can be finite or infinite and some of the faces of the set \(F_2\) can be \(i\)-gons with \(i \in R\). Two \((R, q)\)-polycycles, which are isomorphic as plane graphs, but have different pairs \((F_1, F_2)\), will be considered non-isomorphic in our context. The symmetry group \(\text{Aut}(P)\) of a polycycle \(P\), considered below, consists of all automorphisms of plane graph \(G\) preserving the pair

\(^1\)Any finite plane graph has an unique exterior face; any infinite plane graph can have any number of exterior faces, including 0 and infinity (2-gonal faces are permitted).
$(F_1, F_2)$. Note that it is different from $Aut(G)$, the full automorphism group of the plane graph $G$, i.e. the group of transformations, which preserves the edge-set and the face-set of the plane graph $G$. In fact, $Aut(P)$ is the stabilizer of the pair $(F_1, F_2)$ in $Aut(G)$.

The notion of $(R, q)$-polycycle is a large generalization of the case $|R| = |F_2| = 1$, i.e. $(r, q)$-polycycle introduced by Deza and Shtogrin in [DS98] and studied in their papers [DS98], [DS00a], [DS00b], [DS00c], [DS01], [DS02a], [DS02b], [DS04], [DS05a], [DS05b], [Sh99], [Sh00]. The case $|R| = 1$, i.e. $(r, q)$-polycycles with holes, was considered in [DDS05b].

The notion of $\{r, q\}$-polycycle is a large generalization of the case $|R| = |F_2| = 1$, i.e. $(r, q)$-polycycle introduced by Deza and Shtogrin in [DS98] and studied in their papers [DS98], [DS00a], [DS00b], [DS00c], [DS01], [DS02a], [DS02b], [DS04], [DS05a], [DS05b], [Sh99], [Sh00]. The case $|R| = 1$, i.e. $(r, q)$-polycycles with holes, was considered in [DDS05b].

A boundary of a $(R, q)$-polycycle $P$ is the boundary of any of its holes.

A bridge of a $(R, q)$-polycycle is an edge, which is not on a boundary and goes from a hole to a hole (possibly, the same).

An $(R, q)$-polycycle is called elementary if it has no bridges.

An open edge of a $(R, q)$-polycycle is an edge on a boundary, such that each of its end-vertices have degree less than $q$.

**Theorem 1** Every $(R, q)$-polycycle is uniquely formed by the agglomeration of elementary ones along open edges or, in other words, it can be uniquely cut, along the bridges, into the elementary ones.

Hence, the interesting question is to enumerate those elementary $(R, q)$-polycycles. Call a $(R, q)$-polycycle elliptic, parabolic or hyperbolic if the number $\frac{1}{q} + \frac{1}{r} - \frac{1}{2}$ (where $r = \max_{i \in R} i$) is positive, zero or negative, respectively. The number of elementary $(r, q)$-polycycles is uncountable for any parabolic or hyperbolic pairs $(r, q)$. But in [DS01] and [DS02a] all elliptic elementary $(r, q)$-polycycles were determined. Namely, the countable set of all elementary $(5, 3)$- and $(3, 5)$-polycycles was described; the cases of $(3, 3)$-, $(4, 3)$- and $(3, 4)$-polycycles are easy. In [DDS05b] all elliptic elementary $(\{r\}, q)$-polycycles (i.e. $(r, q)$-polycycles with holes) were determined as a part of a more elaborate classification.

The purpose of this paper is to generalize it for elliptic $(R, q)$-polycycles. In fact, we will consider the main case $R = \{i : 2 \leq i \leq r\}$ and so, all such elliptic possibilities are $(\{2, 3, 4, 5\}, 3)$-, $(\{2, 3\}, 4)$- and $(\{2, 3\}, 5)$-polycycles.

Given a $(R, q)$-polycycle $P$, one can define another $(R, q)$-polycycle $P'$ by removing a face $f$ from $F_1$, i.e. by considering it as a hole. If $f$ has no common vertices with other faces from $F_1$, then removing of it leaves unchanged the plane graph $G$ and only changes the pair $(F_1, F_2)$. If $f$ has some edges in common with a hole, then we remove them. If $f$ has a common vertex with a hole, then we split it in two. See below this operation for a 2-gon, which is incident to a vertex on the boundary.

A $(R, q)$-polycycle $P$ is called extensible if there exists another $(R, q)$-polycycle $P'$, such that the elimination of a face of $P'$ yields $P$.

The same plane graph $G$ can admit several polycycle realizations. For example, Prism$_3$ admits two realizations as a $(\{3, 4\}, 3)$-polycycle (see $(4, 20 - 22, D_{3h})$ below). For elementary $(\{2, 3, 4, 5\}, 3)$-polycycles, we give the list in the form $(n_f, nb_1 - \cdots - nb_p, Aut(G))$ with $n_f$ being
the number of proper faces (from $F_i$), and $nb_1, \ldots, nb_p$ being the position in the corresponding lists of elementary polycycles with $n_f$ faces and $Aut(G)$ being the automorphism group of plane graph $G$: (4, 12−13, $C_s$), (4, 20−22, $D_{3h}$), (4, 23−5th of Lemma 1, $C_{2\nu}$), (5, 26−28−30, $C_{2\nu}$), (5, 27−29, $C_s$), (6, 21−22−31, $C_{3\nu}$), (6, 28−34, $D_{5h}$), (7, 24−36, $D_{3d}$), (7, 29−31, $C_{2\nu}$), (8, 25−29, $D_{3h}$), (9, 11−16, $D_{4d}$). For elementary ($\{2, 3\}, 5$)-polycycles, this concerns the 10th and 20th of Figure 3 ($Aut(G) = C_s$), the 21st of Figure 3 and 1st of Figure 4 ($Aut(G) = D_{3h}$), the 14th and 22nd of Figure 4 ($Aut(G) = D_{2d}$). If $Aut(P) \neq Aut(G)$ and no other polycycle with the same graph $G$ exists, we indicate it by putting the second group in parenthesis.

Theorem 1 together with the determination of the elementary ($R, q$)-polycycles, is especially useful in extremal problems, related to the number of interior/exterior vertices, see [DS01] and [DS02], and in classification of face-regular two-faced maps, see [DD05].

We thank Gil Kalai for putting a question, which lead us to this study.

## 2 Classification of elementary ($\{2, 3, 4, 5\}, 3$)-polycycles

A ($R, 3$)-polycycle is called totally elementary if it is elementary and if, after removing any face adjacent to a hole, one obtains a non-elementary ($R, 3$)-polycycle. So, an elementary ($R, 3$)-polycycle is totally elementary if and only if it is not the result of an extension of some elementary ($R, 3$)-polycycle.

We will classify those polycycles in a number of steps. At first, we find all totally elementary ($\{2, 3, 4, 5\}, 3$)-polycycles. Then, all other elementary ones are obtained by adding faces to the existing elementary or totally elementary polycycles.

**Theorem 2** The list of totally elementary ($\{3, 4, 5\}, 3$)-polycycles consists of:

(i) three isolated $i$-gons, $i \in \{3, 4, 5\}$:

\[
\begin{align*}
&\text{C}_3 \nu \ (D_{3h}) \\
&\text{C}_4 \nu \ (D_{4h}) \\
&\text{C}_5 \nu \ (D_{5h})
\end{align*}
\]

(ii) all ten triples of $i$-gons, $i \in \{3, 4, 5\}$:

\[
\begin{align*}
&\text{C}_3 \nu, \text{ nonext. (}T_d\text{)} \\
&\text{C}_s \ (C_{2\nu}) \\
&\text{C}_s \ (C_{2\nu}) \\
&\text{C}_1 \\
&\text{C}_s \\
&\text{C}_s
\end{align*}
\]

\[
\begin{align*}
&\text{C}_3 \nu, \text{ nonext. (}C_s\text{)} \\
&\text{C}_s \\
&\text{C}_s \\
&\text{C}_3 \nu \\
&\text{C}_3 \nu
\end{align*}
\]

\[2^2\text{In fact, they belong to the infinite series of (}r, 2\text{)-polycycles (monocycles) consisting of isolated r-gons, } r \geq 2.\]
(iii) the following doubly infinite \{(5), 3\}-polycycle, denoted by Barrel$_\infty$:

![Diagram](image)

$pma2$, nonext.

(iv) the infinite series\(^3\) of Barrel$_m$, \(m \geq 2\) (two \(m\)-gonal holes, non-extensible for \(m \neq 3, 4, 5\), with symmetry \(D_{md}\)), represented below for \(m = 2, 3, 4, 5\):

![Diagram](image)

**Proof.** Take a totally elementary \((\{2, 3, 4, 5\}, 3)\)-polycycle \(P\). If \(|F_1| = 1\), then \(P\) is, clearly, totally elementary; so, let us assume that \(|F_1| \geq 2\). If \(|F_1| = 2\), then it is, clearly, not elementary; so, assume \(|F_1| \geq 3\). Of course, \(P\) has at least one interior vertex; let \(v\) be such a vertex. Furthermore, one can assume that \(v\) is adjacent to a vertex \(v'\), which is incident to a hole.

The vertex \(v\) is incident to three faces \(f_1, f_2, f_3\). Let us denote by \(v_{ij}\) unique vertex incident to \(f_i, f_j\) and adjacent to \(v\). Without loss of generality, one can suppose that \(v'\) is incident to the faces \(f_1\) and \(f_2\), i.e. that \(v' = v_{12}\).

The removal of the face \(f_1\) yields a non-elementary polycycle; so, there is at least one bridge separating \(P - f_1\) in two parts. Such bridge should have an end-vertex incident to \(f_1\). The same holds for \(f_2\). The proof consists of a number of cases.

1\(^st\) case: If \(e_1 = \{v, v_{23}\}\) and \(e_2 = \{v, v_{13}\}\) are bridges for \(P - f_1, P - f_2\), respectively, then from the constraint that faces \(f_i\) are \(p\)-gons with \(p \leq 5\), one sees that each face \(f_i\) is adjacent in \(P\) to at most one other face. Furthermore, if \(f_i\) is adjacent to another face, then this adjacency is along a bridge, which is forbidden. Hence, \(F_1 = \{f_1, f_2, f_3\}\).

2\(^nd\) case: Let us assume now that \(e_1 = \{v, v_{23}\}\) is a bridge for \(P - f_1\), but \(e_2 = \{v, v_{13}\}\) is not a bridge for \(P - f_2\). Then, since \(f_2\) is a \(p\)-gon with \(p \leq 5\), it is adjacent to at most one other face and, if so, then along a bridge, which is impossible. So, \(f_2\) is adjacent to only \(f_1\) and \(f_3\) and, since \(e_2\) is not a bridge for \(P - f_2\), one obtains that \(P - f_2\) is elementary, which contradicts the hypothesis.

\(^3\)The Barrel$_m$ is a 3-valent plane graph, consisting of two \(m\)-gons separated by two \(m\)-rings of 5-gons.
3rd case: Let us assume that neither $e_1$, nor $e_2$ are bridges for $P - f_1$ and $P - f_2$. From the consideration of previous two cases, one has that every vertex $v$, adjacent to a vertex on the boundary, is in this 3rd case.

The first subcase, which can happen only if $f_1$ is 5-gon, happens, when the over-lined edge $e'$, in the drawing below, is a bridge.

The face $g$ is adjacent to the faces $h$ and $f_1$ and, possibly, to another face $g'$. But if $g$ is adjacent to such a face $g'$, it is along a bridge of $P$; hence, $g$ is adjacent only to $h$ and $f_1$. So, $P - g$ is elementary, which is impossible.

So, the edge $e'$ is not a bridge and this forces the face $h$ to be 5-gonal. Hence, the vertex $v_{13}$ is in the same situation as the vertex $v$, described in the diagram below:

So, one can repeat the construction. If, at some point, $e_1$ is a bridge, then the construction stops; otherwise, one can continue indefinitely. If one obtains only different vertices, then it means that we have the ($\{5\}, 3$)-polycycle $Barrel_\infty$; otherwise, we obtain a loop of vertices, i.e. a circuit of vertices, which appear over and over, i.e. $Barrel_m$ for some $m \geq 2$.

\begin{proof}
Let $P$ be such polycycle. Clearly, the 2-gon is the only possibility if $|F_1| = 1$. If $|F_1| = 2$, then it is not elementary. If $|F_1| \geq 3$, then the 2-gon should be inside of the structure. So, $P$ contain, as a subgraph, one of three following graphs:

\begin{enumerate}
\item $C_{2\nu} (D_{2h})$
\item $C_{2\nu}, \text{ nonext.} (D_{2h})$
\item $C_{2\nu}$
\item $C_s, \text{ nonext.}$
\item $C_s, \text{ nonext.} (C_{2\nu})$
\item $C_s, \text{ nonext.} (C_{2\nu})$
\item $C_s (C_{2\nu})$
\item $C_{2\nu}, \text{ nonext.} (D_{2d})$
\end{enumerate}
\end{proof}
So, the only possibilities for $P$ are those given in above Lemma.

We will enumerate now all elementary $\{\{3, 4, 5\}, 3\}$-polycycles. If such a polycycle is not totally elementary, then it is obtained from another elementary $\{\{3, 4, 5\}, 3\}$-polycycle (totally elementary or not) by addition of another face.

**Theorem 3** The list of elementary $\{\{2, 3, 4, 5\}, 3\}$-polycycles consists of:

(i) eight $\{\{2, 3, 4, 5\}, 3\}$-polycycles, given in the list of Lemma 4

(ii) the sporadic $\{\{3, 4, 5\}, 3\}$-polycycles, given for 1 and 3 faces in Theorem 2 (i), (ii) and the remainder, after proof of this Theorem.

(iii) the infinite series of $\{\{5\}, 3\}$-polycycles $\text{Barrel}_m$ with $2 \leq m \leq \infty$, given in Theorem 2 (iii), (iv).

(iv) six $\{\{3, 4, 5\}, 3\}$-polycycles, infinite in one direction:

$$\begin{align*}
\alpha &: C_1 \\
\beta &: C_1 \\
\gamma &: C_1, \text{ nonext.}
\end{align*}$$

$$\begin{align*}
\delta &: C_1 \\
\varepsilon &: C_1, \text{ nonext.} \\
\mu &: C_1, \text{ nonext.}
\end{align*}$$

(v) $21 = \binom{6+1}{2}$ infinite series obtained by taking two endings of the infinite polycycles of (iv) above and concatenating them.

For example, merging of $\alpha$ with itself produces the infinite series of elementary $\{\{5\}, 3\}$-polycycles, denoted by $E_n$ in [DS01], $0 \leq n \leq \infty$. For $n = 0, 1, 2, 3, \infty$, it is the 5-gon (Theorem 2 (i)), the triple of 5-gons (Theorem 2 (ii)), 16th in the list for 4 faces below, 22nd in the list for 5 faces below, $\text{Barrel}_\infty$. See Figure 4 for the first 3 members (starting with 6 faces) of two such series: $\alpha \alpha$ and $\beta \varepsilon$.

**Proof.** The proof consists of taking the totally elementary polycycles, given in Theorem 2, adding a face with right number of sides, which preserves the elementarity in all possible ways. Then we reduce, by isomorphism of $(R, q)$-polycycles, and obtain the list of finite elementary $\{\{2, 3, 4, 5\}, 3\}$-polycycles with one hole. If a $\{\{3, 4, 5\}, 3\}$-polycycle has two holes and is not a $\text{Barrel}_m$, then it is not elementary. So, it can be obtained from another elementary polycycle with one hole less, by the addition of one face. It is easy to see that this cannot happen. So, we have the complete list of finite $\{\{2, 3, 4, 5\}, 3\}$-polycycles.

Take now an elementary $\{\{3, 4, 5\}, 3\}$-polycycle $P$, which is infinite. Remove all 3- or 4-gonal faces of it. The result is a $\{\{5\}, 3\}$-polycycle $P'$, which is not necessarily elementary. We will now...
use the classification of elementary \((\{5\},3)\)-polycycles (possibly, infinite) done in [DDS05b]. If the infinite \((\{5\},3)\)-polycycle \(\text{Barrel}_\infty\) appears in the decomposition, then, clearly, \(P\) is reduced to it. If the infinite polycycle \(\alpha\) appear in the decomposition, then there are two possibilities for extending it, indicated below.

If a 3- or 4-gonal face is adjacent on the dotted line, then there should be another face on the over-line edges. So, in any case, there is a face, adjacent on the over-line edges, and we can assume that it is a 3- or 4-gonal face. Then, consideration of all possibilities to extend it, yields \(\beta, \ldots, \mu\). Suppose now, that \(P\) does not contain any infinite \((\{5\},3)\)-polycycles. Then we can find an infinite path \(f_0, \ldots, f_i, \ldots\) of distinct faces of \(P\) in \(F_1\), such that \(f_i\) is adjacent to \(f_{i+1}\) and \(f_{i-1}\) is not adjacent to \(f_{i+1}\). The condition on \(P\) implies that an infinite number of faces are 3- or 4-gons, but the condition of non-adjacency of \(f_{i-1}\) with \(f_{i+1}\) forbids 3-gons. Take now a 4-gon \(f_i\) and assume that \(f_{i-1}\) and \(f_{i+1}\) are 5-gons. The consideration of all possibilities of extension around that face, lead us to an impossibility. If some of \(f_{i-1}\) or \(f_{i+1}\) are 4-gons, then we have a path of 4-gons and the case is even simpler. \(\square\)

List of elementary \((\{3,4,5\},3)\)-polycycles with 4 faces:

List of elementary \((\{3,4,5\},3)\)-polycycles with 5 faces:
List of sporadic elementary ($\{3, 4, 5\}, 3$)-polycycles with 6 faces:
Infinite series $\alpha\alpha$ of elementary $\{(2, 3, 4, 5), 3\}$-polycycles:

Infinite series $\beta\varepsilon$ of elementary $\{(2, 3, 4, 5), 3\}$-polycycles:

Figure 1: The first 3 members (starting with 6 faces) of two infinite series, amongst 21 series of $\{(2, 3, 4, 5), 3\}$-polycycles in Theorem 3 (v)

List of sporadic elementary $\{(3, 4, 5), 3\}$-polycycles with 7 faces:
List of sporadic elementary ($\{3, 4, 5\}, 3$)-polycycles with 8 faces:
List of sporadic elementary \(\{3, 4, 5\}, 3\)-polycycles with 9 faces:
List of sporadic elementary ($\{3, 4, 5\}, 3$)-polycycles with 10 faces:

List of sporadic elementary ($\{3, 4, 5\}, 3$)-polycycles with at least 11 faces:

3 Classification of elementary ($\{2, 3\}, 4$)-polycycles

Theorem 4 Any elementary ($\{2, 3\}, 4$)-polycycle is one of the following eight:
Proof. The list of elementary \((\{3\}, 4)\)-polycycles is determined in \cite{DDS05b} and consists of the first four graphs of this theorem. Let \(P\) be a \((\{2, 3\}, 4)\)-polycycle, containing a 2-gon. If \(|F_1|=1\), then it is the 2-gon. Clearly, the case, in which two 2-gons share one edge, is impossible. Assume that \(P\) contains two 2-gons, which share a vertex. Then we should add triangle on both sides and so, obtain the second above polycycle. If there is a 2-gon, which does not share a vertex with a 2-gon, then \(P\) contains the following pattern:

So, clearly, \(P\) is one of the last two possibilities above. \(\square\)

Note that seventh and fourth polycycles in Theorem 4 are, respectively, 2- and 3-antiprisms; here the exterior face is the unique hole. The \(m\)-antiprism for any \(m \geq 2\) can also be seen as \((\{2, 3\}, 4)\)-polycycle with \(F_2\) consisting of the exterior and interior \(m\)-gons; this polycycle is not elementary.

4 Classification of elementary \((\{2, 3\}, 5)\)-polycycles

Let us consider an elementary \((\{2, 3\}, 5)\)-polycycle \(P\). Assume that \(P\) is not an \(i\)-gon and has a 2-gonal face \(f\). If \(f\) is adjacent to a hole, then the polycycle is not elementary. So, holes are adjacent only to 3-gons. If one remove such a 3-gon \(t\), then the third vertex \(v\) of \(t\), which is necessarily interior in \(P\), becomes non-interior in \(P-t\). The polycycle \(P-t\) is not necessarily elementary. Let us denote by \(e_1, \ldots, e_5\) the edges incident to \(v\) and assume that \(e_1, e_2\) are edges of \(t\). The boundary is adjacent only to 3-gons. The potential bridges in \(P-t\) are \(e_3, e_4\) and \(e_5\). Let us check all five cases:

- If no edge \(e_k\) is a bridge, then \(P-t\) is elementary.
- If only \(e_4\) is a bridge, then two cases can happen:
  - This bridge goes from a hole to the same hole. This means that \(P\) is formed by the merging of two elementary \((\{2, 3\}, 5)\)-polycycles.
  - This bridge goes from a hole to another hole. This means that \(P\) has at least two holes and that \(P-t\) is formed by the merging of two open edges of an elementary \((\{2, 3\}, 5)\)-polycycle, which has one hole less.
- If \(e_3\) or \(e_5\) is a bridge, then \(P-t\) is formed by the agglomeration of an elementary \((\{2, 3\}, 5)\)-polycycle and a \(i\)-gon with \(i = 2\) or \(3\).
- If \(e_4\) is a bridge and \(e_3\) or \(e_5\) is a bridge, then \(P-t\) is formed by the agglomeration of an elementary \((\{2, 3\}, 5)\)-polycycle and two \(i\)-gons with \(i = 2\) or \(3\).
- If all \(e_k\) are bridges, then \(P\) has only one interior vertex.

Given a hole of a \((R, q)\)-polycycle, its \textit{boundary sequence} is the sequence of degrees of all consecutive vertices of the boundary of this hole.
Theorem 5 The list of elementary \((\{2, 3\}, 5)\)-polycycles consists of:

(i) 57 sporadic \((\{2, 3\}, 5)\)-polycycles given on Figures \ref{fig:1} and \ref{fig:2}.

(ii) three following infinite \((\{2, 3\}, 5)\)-polycycles:

\[
\alpha: C_1 \quad \beta: C_1 \quad \gamma: C_1, \text{ nonext.}
\]

(iii) six infinite series of \((\{2, 3\}, 5)\)-polycycles with one hole (they are obtained by concatenating endings of a pair of polycycles, given in (ii); see Figure \ref{fig:3} for the first 5 graphs),

(iv) the following 5-valent doubly infinite \((\{2, 3\}, 5)\)-polycycle, called snub \(\infty\)-antiprism:

\[
pma2, \text{ nonext.}
\]

(v) the infinite series of snub \(m\)-antiprisms\(^4\), \(m \geq 2\) (two \(m\)-gonal holes, non-extensible for \(m \geq 4\), with symmetry \(D_{md}\)), represented below for \(m = 2, 3, 4, 5\):

\[
D_{2d} \quad D_{3d} (I_h) \quad D_{4d} \quad D_{5d}
\]

Proof. Let us take an elementary \((\{2, 3\}, 5)\)-polycycle, which is finite. Then, by removing a triangle, which is adjacent to a boundary, one is led to the situation described above. Hence, the algorithm for enumerating finite elementary \((\{2, 3\}, 5)\) polycycles is the following.

1. Begin with isolated \(i\)-gons with \(i = 2\) or 3.

2. For every vertex \(v\) of an elementary polycycle with \(n\) interior vertices, consider all possibilities of adding 2- and 3-gons incident to \(v\), such that the obtained polycycle is elementary and \(v\) has become an interior vertex.

3. Reduce by isomorphism.

\(^4\text{Snub } m\text{-antiprism is a 5-valent plane graph defined on page 119 of [DGS04]. For } m = 4\text{ it is one of well-known regular-faced polyhedra (Johnson polyhedra). For } m = 3\text{ it has the skeleton of Icosahedron, but it is distinct from the corresponding sporadic one, having only one hole (exterior 3-gonal face).}\)
The above algorithm first finds some sporadic \((\{2,3\},5)\)-polycycles and the first elements of the infinite series and then find only the elements of the infinite series. In order to prove that this is the complete list of all finite \((\{2,3\},5)\)-polycycles with only one hole, one needs to consider the case, in which only \(e_4\) is a bridge going from a hole to the same hole. So, we need to consider all possibilities, where the addition of two elementary \((\{2,3\},5)\)-polycycles and one 3-gon make a larger elementary \((\{2,3\},5)\)-polycycle. Given a sequence \(a_1,\ldots,a_n\), we say that a sequence \(b_1,\ldots,b_p\) with \(p < n\) is a pattern of that sequence if, for some \(n_0\), one has \(a_{n_0+j-1} = b_j\) or \(a_{n_0+1-j} = b_j\) with the addition being modulo \(n\). The \((\{2,3\},5)\)-polycycles, used in that construction, should have the pattern \(3,3,x\) with \(x \leq 4\) in their boundary sequence. Only the polycycles, which belong to the six infinite series, satisfy this and it is easy to see, that the result of the operation is still one of the six infinite series. So, the list of finite elementary \((\{2,3\},5)\)-polycycles with one hole is the announced one.

If a polycycle has more than one hole, then it is obtained by the addition of 2- and 3-gonal faces to a vertex, incident to the boundary (or from another \((\{2,3\},5)\)-polycycle with a smaller number of holes) by the merging of two open edges in a bridge and addition of a 3-gon. This suggest an iterative procedure, where we begin with a \((\{2,3\},5)\)-polycycle with one hole and consider all possibilities of extension to a polycycle with two holes and then continue for more complicated polycycles.

The polycycles, which are suitable for such a construction, should contain the pattern \(3,3,x\) and \(y,3,3\) in their boundary sequence with \(x \leq 4\) and \(y \leq 3\). Amongst all finite \((\{2,3\},5)\)-polycycles with one hole, only members of the infinite series \(\alpha\alpha\) of \((\{2,3\},5)\)-polycycles with one hole, are suitable ones and one obtains the infinite series of the above theorem. Furthermore, the members of this infinite series admit no extension. This proves that there are no other \((\{2,3\},5)\)-polycycles with at least two holes.

Consider now an elementary infinite \((\{2,3\},5)\)-polycycle \(P\). Eliminate all 2-gonal faces of \(P\) and obtain another \((\{3\},5)\)-polycycle \(P'\), which is not necessarily elementary. We do a decomposition of \(P'\) along its elementary components, which are enumerated in [DDS05b]. If snub \(\infty\)-antiprism is one of the components, then we are finished and \(P = P'\) is the snub \(\infty\)-antiprism. If \(\alpha\) is one of the components, then one has two edges, along which to extend the polycycle; they are depicted below:

![Diagram](image)

Clearly, if we extend the polycycle along only one of those edges, then the result is not an elementary polycycle. The consideration of all possibilities yields \(\beta\) and \(\gamma\). Suppose now that \(P'\) has no infinite components. Then \(P\) has at least one infinite path \(f_0,\ldots,f_i,\ldots\), such that \(f_i\) is adjacent to \(f_{i+1}\), but \(f_{i-1}\) is not adjacent to \(f_{i+1}\). The considerations, analogous to the 3-valent case, yield the result. \(\square\)
Infinite series $\alpha\alpha$ of elementary $\{(2,3), 5\}$-polycycles:

\[
\begin{array}{ccccccc}
C_{5\nu} & C_{2\nu} & C_s & C_2 & C_s & C_2 \\
\end{array}
\]

Infinite series $\alpha\beta$ of elementary $\{(2,3), 5\}$-polycycles:

\[
\begin{array}{ccccccc}
C_s & C_s & C_1 & C_1 & C_1 & C_1 \\
\end{array}
\]

Infinite series $\alpha\gamma$ of elementary $\{(2,3), 5\}$-polycycles:

\[
\begin{array}{ccccccc}
C_1 & C_1 & C_1 & C_1 & C_1 & C_1 \\
\end{array}
\]

Infinite series $\beta\beta$ of elementary $\{(2,3), 5\}$-polycycles:

\[
\begin{array}{ccccccc}
C_{2\nu} & C_s & C_2 & C_s & C_2 \\
\end{array}
\]

Infinite series $\beta\gamma$ of elementary $\{(2,3), 5\}$-polycycles:

\[
\begin{array}{ccccccc}
C_1 & C_1 & C_1 & C_1 & C_1 \\
\end{array}
\]

Infinite series $\gamma\gamma$ of elementary $\{(2,3), 5\}$-polycycles:

\[
\begin{array}{ccccccc}
C_2 & C_s & C_2, \text{ nonext.} & C_s, \text{ nonext.} & C_2, \text{ nonext.} \\
\end{array}
\]

Figure 2: The first 5 members of the six infinite series of $\{(2,3), 5\}$-polycycles
Figure 3: Sporadic 5-valent \( (\{2, 3\}, 5) \)-polycycles (first part)
Figure 4: Sporadic 5-valent ($\{2, 3\}, 5$)-polycycles (second part)
5 Possible extensions of the setting

The classification, developed in this paper, is an extension of previous work of Deza and Shtogrin, by allowing various possible number of sides of interior faces and several possible holes. A natural question is, if one can further enlarge the class of polycycles.

We required 2-connectivity and that any two holes do not share a vertex. If one removes those two hypothesis, then many other graphs do appear.

Consider infinite series of $\{\{2\},6\}$-polycycles, $m$-bracelets, $m \geq 2$ (i.e. $m$-circle, but each edge is tripled). The central edge is a bridge for those polycycles, for both 2-gons of the triple of edges. But if one removes those two digons, then the resulting plane graph has two holes sharing a face, i.e. violates the second of the crucial points (i)–(iii) of the definition of $(R,q)$-polycycle. This shows that our hypothesis were necessary.

For even $m$, each even edge (for some order 1, . . . , $m$ of them) can be duplicated $t$ times (for fixed $t$, $1 \leq t \leq 5$), and each odd edge duplicated $6 - t$ times; so, all vertex-valencies will be still 6. On the other hand, two holes ($m$-gons inside and outside of the $m$-bracelet), have common vertices; so, it is again not our polycycle.

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