Information-theoretic equilibration: the appearance of irreversibility under complex quantum dynamics

Cozmin Ududec\textsuperscript{1,2}, Nathan Wiebe\textsuperscript{2,3}, and Joseph Emerson\textsuperscript{2,4}

\textsuperscript{1}Department of Physics and Astronomy, University of Waterloo, Waterloo, ON, Canada
\textsuperscript{2}Institute for Quantum Computing, 200 University Ave West, Waterloo, ON, Canada
\textsuperscript{3}Department of Combinatics & Opl., University of Waterloo, Waterloo, ON, Canada
\textsuperscript{4}Department of Applied Math, University of Waterloo, Waterloo, ON, Canada

The question of how irreversibility can emerge as a generic phenomena when the underlying mechanical theory is reversible has been a long-standing fundamental problem for both classical and quantum mechanics. We describe a mechanism for the appearance of irreversibility that applies to coherent, isolated systems in a pure quantum state. This equilibration mechanism requires only an assumption of sufficiently complex internal dynamics and natural information-theoretic constraints arising from the infeasibility of collecting an astronomical amount of measurement data. Remarkably, we are able to prove that irreversibility can be understood as typical without assuming decoherence or restricting to coarse-grained observables, and hence occurs under distinct conditions and time-scales than those implied by the usual decoherence point of view. We illustrate the effect numerically in some model systems and prove that the effect is typical under the standard random-matrix conjecture for complex quantum systems.

There has been considerable recent interest in the sufficient conditions for equilibration \cite{1,15}. These approaches normally assume a decoherence mechanism resulting from the entanglement between the system of interest and a larger environment, or else assume highly coarse-grained observables. In this work we describe a mechanism for equilibration that applies to isolated quantum systems in pure states, without assuming decoherence, restricting to subsystems, time-averaging or coarse-graining the observables. The mechanism for equilibration that we describe is an information-theoretic one that requires an assumption of complex internal dynamics coupled with realistic limitations to predicting the detailed evolution of the system and the experimental infeasibility of collecting an astronomically large amount of measurement data. This approach builds on earlier arguments by Peres \cite{15} and Srednicki \cite{17,18} who proposed that the statistical complexity of the system’s eigenvectors could be responsible for equilibration in isolated quantum systems. We show that these conditions are sufficient to account for the effective (microcanonical) equilibration of the measurement statistics for natural choices of (non-degenerate) observable, meaning that, after a finite equilibration time, the dynamical state becomes effectively indistinguishable from the microcanonical state. Hence information-theoretic equilibration (ITE) accounts for microcanonical equilibration in a way that is directly analogous to how classical chaos (mixing) accounts for the microcanonical equilibration of classically chaotic systems \cite{19,20}. Remarkably, we are able to prove that ITE is universal for complex systems under the standard random-matrix conjecture \cite{21,22}. Specifically, we prove that information-theoretic equilibration occurs with high probability for individual Hamiltonians drawn from two physically relevant ensembles: the Gaussian Unitary Ensemble (GUE), which has a successful history predicting universal features of complex quantum systems \cite{21}, and a random local Hamiltonian (RLH) ensemble consisting of many-body systems restricted to two-body interactions. We then illustrate ITE numerically in some surprisingly simple examples of Hamiltonian models under natural choices of (maximally fine-grained) observable: a two-field variant of the many-body Heisenberg Hamiltonian as well as the quantum kicked top \cite{22}, which is a single-body, classically chaotic system.

Consider a pure state evolving under a Hamiltonian $H$, $\rho(t) = \exp(-iHt/\hbar)|\psi(0)\rangle\langle\psi(0)|\exp(iHt/\hbar)$. The dynamical state $\rho(t)$ can not reach the true equilibrium state $\sigma_\infty := \lim_{t\to\infty} \frac{1}{\tau} \int_0^\tau \rho(t)dt$ \cite{11,13} because the state remains pure. In particular, the trace distance $\|\rho(t) - \sigma_\infty\|$, which characterizes the distinguishability under an optimal choice of measurement operator, can be large throughout the evolution. However, for a given complex system $H$ in a large Hilbert space, even a sub-optimal measurement that enables distinguishability of these two states at any time $t$ may neither be known theoretically nor easily engineered experimentally. For example, for a cubic lattice of dipolarly coupled spins, which is an analytically intractable system that has been probed experimentally for decades, only recently was a measurement procedure devised that revealed long-lived (multiple-quantum) coherence after equilibration of the free-induction decay \cite{23}. Conceptually then we see that the appearance of equilibration can and does result from insufficient knowledge of, or control over, choice of observable. Our contribution is to characterize and illustrate conditions under which the signatures of purity and coherence are provably “lost in Hilbert space”, and hence unobservable due to realistic limitations on both theoretical and experimental abilities.

We remark that our assumptions are conceptually similar and yet distinct from those of the usual decoherence argument, in which a system coupled to a reservoir appears to reach equilibrium (due to entanglement between the system and reservoir) although the joint state of system plus reservoir remains pure. That conclusion
holds only if one assumes that one can not predict or perform the kind of (entangling) measurement across the combined system plus reservoir that would readily distinguish the actual state from the equilibrium one; that is, the argument goes through by restricting the set of observables to local ones. In contrast, our observation is that information-theoretic limitations alone are sufficient to account for the appearance of equilibration for accessible observables on complex systems and so, contrary to the usual assumption (see [1,24,25]), decoherence from a reservoir is not necessary from an explanatory point of view. More practically, whereas the time-scale for equilibration under decoherence depends on the strength of the coupling to the reservoir, our mechanism does not and predicts equilibration on a distinct, and potentially shorter, time-scale. Furthermore, our approach is a natural quantum analog of classical microcanonical equilibration [19,20].

We consider a quantum system with some kinematically accessible Hilbert space that is finite-dimensional $\mathcal{H} = C^D$. In order to show that we do not require coarse-graining, we consider a maximally fine-grained (ie, non-degenerate) observable $A$ acting on $\mathcal{H}$, where $A = \sum_{k=1}^{D} a_k \hat{P}_k$ with rank-one orthogonal projectors $\hat{P}_k$. Our argument applies also to local or other coarse-grained observables (which can be represented by degeneracies). For simplicity of analysis we consider the (most adverserial) setting where the system starts in a pure state that is maximally localized with respect to $A$, ie, $\rho_0 = |a_i\rangle\langle a_i|$, and then examine how the pure states spreads out over the eigenbasis of $A$ under time-evolution given a Hamiltonian $H$. The empirical question of whether the system appears to approach (microcanonical) equilibrium given some observable $A$ corresponds to asking whether the experimental measurement statistics for the evolved pure state can be distinguished from those of the equilibrium state. Hence the relevant quantities for this task are the probabilities over distinct outcomes $k$,

$$\Pr(k|\rho(t)) = \text{Tr} \left[ \hat{P}_k U(t) \rho_0 U^\dagger(t) \right], \quad (1)$$

and the goal is to distinguish $\rho(t)$ from $\sigma_{\infty}$ by sampling the distribution in (1). For simplicity we focus on cases where $\sigma_{\infty} = 1/D$, but $\sigma_{\infty}$ may differ from the micro-canonical state $\rho_{\text{mc}} := 1/D$ or any thermal state [23].

**Definition 1.** A Hamiltonian $H$ acting on $\mathcal{H} = C^D$ exhibits information theoretic equilibration (ITE) with respect to an observable $A$ at a time $t$, if the outcome distribution $\Pr(k|\rho(t))$ can only be distinguished from the micro-canonical distribution $\Pr_{\text{mc}}(k)$ with probability at least $1 - O(1/\text{poly}(D))$ by (a) taking a number of samples from $\Pr(k|\rho(t))$ that scales as $O(\text{poly}(D))$ or (b) performing any information processing that requires at least $O(\text{poly}(D))$ arithmetic or logical operations.

This definition emphasizes that although the exact quantum distribution for the system may be in principle distinguishable from the microcanonical distribution, the two are effectively indistinguishable if the resources needed to distinguish them exceed those practically available. We delineate the practical from the impractical by disallowing resources (the number of measurements taken and computational time used in their analysis) that grow polynomially with the Hilbert space dimension (and hence exponentially with the number of subsystems). Of course, for a different physical scenario, a different cut-off may be appropriate. Our condition (b) includes a restriction on computational resources because the two distributions could be distinguished using fewer samples if the $\Pr(k|\rho(t))$ can be pre-computed. In other words, information-theoretic equilibration is relevant precisely when the system is in a sufficiently large Hilbert space that such a pre-computation is infeasible. We represent our ignorance of $\Pr(k|\rho(t))$ by assuming that it is drawn from a distribution that is invariant under permutations of outcome labels. We now show in the following theorem that, without the ability to efficiently predict $\Pr(k|\rho(t))$, ITE with respect to a particular measurement occurs when the outcome variance,

$$V_k \{\Pr(k)\} := D^{-1} \sum_{k=0}^{D-1} [\Pr(k) - D^{-1}]^2, \quad (2)$$

is sufficiently small, which is typical of cases where the underlying dynamics has no constants of motion. Proof is provided in the supplemental material.

**Theorem 1.** Consider an unknown distribution that is promised to be with equal probability either (a) the uniform distribution on the set $S = \{1, \ldots, M\}$ or (b) an unknown distribution $P(k)$ that is drawn from a distribution over probability distributions on $S$ with outcome variances that scale as $O(M^{-2})$ such that $\Pr(P(k))$ is invariant with respect to permutations of $S$. With high-probability, the probability of correctly distinguishing between (a) and (b) after obtaining $N$ samples is at most $1/2 + O(N/M^{1/2})$.

Theorem 1 shows that $N \geq O(M^{1/2})$ samples are needed to distinguish the distributions with probability substantially greater than $1/2$, which is prohibitively expensive in the case of a non-degenerate projective measurement because $M = D$. Similarly, if we consider a
generalized measurement with $M > D$ (as is relevant in the case of SIC POMs), Theorem 1 similarly shows that distinguishing the distributions is hard. Finally, it is straightforward to show that coarse grained measurements with $M < D$ do not provide an advantage under the assumptions of Theorem 1 because the permutation invariance of the prior distribution over $Pr(k)$ prevents such strategies from succeeding with high probability. Another consideration is that $\nabla_k \{Pr(k)\} = O(M^{-2})$ does not imply that the fluctuations are negligible in principle; in fact, it is consistent with $||Pr(k|\rho(t)) - 1/M||_1$ being constant, which implies that an optimal measurement exists that can distinguish the two distributions efficiently [20]. Hence Theorem 1 is only meant to give a hardness result for distinguishing two states given the induced distributions with respect to a fixed measurement, and does not apply to cases where the optimal measurement is both known a priori and experimentally accessible. Indeed the exceptions to our assumptions are relevant, e.g., when the system admits constants of the motion that are simple relative to the selected observable.

Which Hamiltonian systems satisfy the assumptions of Theorem 1, for natural choices of $A$, and hence exhibit information theoretic equilibration? Pure-state fluctuations satisfying the scaling of Theorem 1 were observed already in the two-body, classically chaotic quantum system studied in Refs. [27] [28], which motivated the question: was the behaviour of that complex system exceptional, or was it evidence of a universal equilibration behaviour for closed chaotic systems? If the latter, does this effect carry over from chaotic quantum systems to sufficiently complex many-body quantum systems?

To answer these questions, we take the enormously successful approach of Wigner and Dyson and the army of theoretical physicists following them who have demonstrated that certain features of appropriate random matrix ensembles (RME) can predict typical properties of complex quantum systems. This is known as the random-matrix conjecture, and it has provided accurate predictions of the spectral properties of heavy nuclei [21], spectral and eigenvectors statistics of quantum chaos models [29] [30], and quantum transport in mesoscopic structures [31]. Consider any ensemble that has a mean that equilibrates information theoretically with respect to a fixed observable $A$ and is sufficiently sharply peaked about that mean, then individual systems from the ensemble will satisfy Theorem 1 with (very) high probability. This phenomenon, known as concentration of measure, is central to the random-matrix conjecture, and it is important to note that our averages over the ensemble are not an implicit appeal to decoherence or mixing, but a method for estimating the typical properties of individual systems within the ensemble.

The system must be allowed to evolve for a sufficient amount of time for the state to spread out from a distribution with support on initial eigenstate of $A$ to one that obeys $Pr(k|\rho(t)) \approx 1/D$ for our result to hold (see Fig. 1). We refer to the earliest such time as the equilibration time, which we denote $t_{eq}$. For an individual system, we also require that for most $t \geq t_{eq}$ that $Pr(k|\rho(t))$ is nearly maximally spread out. If the Hamiltonian is drawn from an ensemble, it is then possible to define an equilibration time such that almost all Hamiltonians drawn from the ensemble achieve ITE with respect to $A$ and $t \geq t_{eq}$.

**Lemma 1.** Almost all Hamiltonians sampled from an ensemble of Hamiltonians equilibrate information theoretically with respect to a fixed observable $A$ and time $t \geq t_{eq}$, in the limit as $D \to \infty$, if the ensemble average and variance (denoted $\mathbb{E}_{EH}$ and $\nabla_{EH}$ respectively) of the outcome variance obey for all $t \geq t_{eq}$

$$\mathbb{E}_{EH}\{\nabla_k \{Pr(k|\rho(t))\}\} \leq O(D^{-2}) \quad (3)$$

$$\nabla_{EH}\{\nabla_k \{Pr(k|\rho(t))\}\} \leq O(D^{-1}) \quad (4)$$

Proof is given in the supplemental material.

We now give our first evidence for universality by proving that ITE is typical for the important Gaussian Unitary Ensemble (GUE), which defines an invariant measure on the set of Hamiltonians. The GUE is the appropriate model a highly successful model for many properties of complex physical systems with no hidden symmetries [21].

**Theorem 2.** Take a non-degenerate observable $A$ acting on $\mathcal{H} = \mathbb{C}^D$, and an initial pure state $\rho_0 = |x\rangle\langle x|$ which is an eigenstate of $A$. Almost all Hamiltonians drawn from GUE then equilibrate information theoretically with respect to $A$ and $t \geq t_{eq}$ in the limit as $D \to \infty$ for $t_{eq}(D) = O(D^{-1/6})$.

The proof is in the supplemental material. This theorem tells us the remarkable result that, as $D$ increases, the overwhelming majority of Hamiltonians will cause an initially pure, localized state to spread out over the non-degenerate eigenbasis of $A$ in a sufficiently uniform manner, to become practically indistinguishable from the microcanonical state for any $t \geq t_{eq}$. Thanks to decades
of numerical studies of GUE as a model of complex many-body systems [31] and few-body quantum chaos systems [22], it is known that GUE is a good predictor of short-range spectral fluctuations [32], and low-order moments of eigenvector components [29, 30], but not a good predictor of long-range spectral fluctuations. We now confirm both for GUE is unrealistically short because it depends on the smallness of the fluctuations using GUE (for \( t > t_{eq} \)) depends only on low-order moments of the eigenvector components, i.e., unitary t-design condition with \( t=8 \) [33] (see supplementary material for details). Hence we expect this aspect of the GUE model to be reflected in physically relevant Hamiltonian systems. However, we do not expect the GUE prediction for the equilibration time-scale to be physically relevant (clearly the value of \( t_{eq} \) for GUE is unrealistically short) because it depends on long-range spectral fluctuations. We now confirm both of these expectations for two RMEs consisting of many-body spins with two-body interactions, and conclude by demonstrating ITE with respect to tensor product measurements on a physically relevant time-scale in some example model systems.

We construct an ensemble of random local Hamiltonians (RLH) on \( n \) spins, consisting of 2-body interactions between 2-level quantum systems, as follows:

\[
H = \frac{1}{\|H\|} \left( \sum_{i=1}^{n} \sum_{p} a_{i,p} \sigma_p^{(i)} + \sum_{i<j} \sum_{p,p'} b_{i,j,p,p'} \sigma_p^{(i)} \sigma_{p'}^{(j)} \right),
\]

where \( p, p' \in \{X, Y, Z\} \), and each \( a_{i,p} \) and \( b_{i,j,p,p'} \) is a Gaussian random variable with mean 0 and variance 1. We consider the observable \( A = \sum_{j=0}^{D-1} a_j |j\rangle \langle j| \), corresponding to a non-degenerate projective measurement in the eigenbasis of \( \sigma_{\otimes n} \). RLH is clearly invariant under permutation of qubit labels and local rotations of each qubit, and therefore our results also apply to any \( \mathcal{A}' \) that differs from \( \mathcal{A} \) by local rotations. Fig. 2 shows that pure states evolving under individual elements of RLH approach equilibrium as \( D \) increases. We estimate the equilibration time using the location of the inflection points of the curves in Fig. 2 and find it scales as \( O(\log(D)) \), which is characteristic of quantum chaotic systems [22, 27, 28]. Fig. 3 shows that the outcome variance for a typical Hamiltonian chosen uniformly from the RLH ensemble satisfies the requirements of Lemma 1 which implies that almost all RLH Hamiltonians will equilibrate information theoretically with respect to any non-degenerate measurement in the class \( \mathcal{A}' \) as \( D \to \infty \) for any \( t \geq t_{eq} \). We further strengthen the physical relevance of this result by showing that ITE still holds for \( t \geq t_{eq} \) when the 2-local Hamiltonians are constrained to have nearest-neighbor interactions in one- and two-dimensions (see supplemental material).

We now give two simple examples of individual model systems that exhibit information theoretic equilibration: a many-body system that is a two-field variant of the Heisenberg Hamiltonian and a one-body chaotic model, the quantum kicked top. The two-field variant of the Heisenberg mode consists of \( n \)-spins arranged in a line with periodic boundary conditions:

\[
H = \frac{1}{\|H\|} \left( \sum_{i \leq n/2} \sigma_z^{(i)} + \sum_{i > n/2} \sigma_x^{(i)} + \sum_{i} \sigma^{(i)} \cdot \sigma^{(i+1)} \right).
\]

We choose this Hamiltonian because it is highly structured local Hamiltonian that is not typical of RLH and yet it is unstructured enough to be non-integrable so there are no constants of motion that prevent equilibration on the full Hilbert space (otherwise ITE would be limited to the invariant subspaces fixed by the constants of motion). Figure 4 shows that the outcome variance of the probability distribution indeed scales as \( O(D^{-2}) \) with respect to \( A = \sum_{j=0}^{D-1} a_j |j\rangle \langle j| \), corresponding to read-out of all spins in the computational basis. Hence Theorem 1 implies that information theoretic equilibration occurs for this simple many-body Hamiltonian with respect to a natural observable. This is evidence that our equilibration mechanism is not just a mathematical feature.

FIG. 3: Numerically computed expectation values and variances over the RLH ensemble of the outcome variance computed at \( t = 10 \) where \( t_{eq} \lesssim 2 \) for \( \rho(0) = |0\rangle \langle 0| \). The data was obtained for 250 randomly chosen Hamiltonians, and shows that \( \mathbb{V}_{E,H}(\mathbb{V}_k(\mathbb{P}(k|\rho(t)))) \approx 0.05D^{-4} \) and \( \mathbb{E}_{E,H}(\mathbb{V}_k(\mathbb{P}(k|\rho(t)))) \approx D^{-2} \).

FIG. 4: Evidence of ITE for \( t \geq t_{eq} \) in an extremely simple many-body system with nearest-neighbor interactions for increasing number of spins \( n \) (\( D = 2^n \)). The measurement consists of readout of each spin along the z-axis. The plot shows \( \mathbb{V}_k(\mathbb{P}(k|\rho(t))) \approx 1.6D^{-2} \) for the Hamiltonian [24] with \( t = 20 \) where \( t_{eq} \approx 15 \) (see supplemental material).
of random Hamiltonian ensembles but occurs also in a simple, physically accessible many-body model. We also demonstrate ITE for the quantum kicked top in a regime of global chaos with respect to non-degenerate measurements in the \( J_z \) basis (see supplemental material) and physically accessible times.

**Conclusion.** We have demonstrated a novel mechanism for equilibration that holds very broadly for the probability distributions of even maximally fine-grained measurements on pure quantum states of closed Hamiltonian systems. Remarkably, this information theoretic equilibration is observed to hold without requiring any form of decoherence or restricting to local or otherwise coarse-grained measurements. This is because, in the typical case of a complex system, the dynamical pure state quantum fluctuations, though finite, do not lead to a breakdown of correspondence with the equilibrium state (contrary to a common implicit assumption, see Refs. [11, 23, 25]) because they become unobservably small under purely statistical considerations (in the limit of large \( D \)) after the equilibration time-scale. Our key insight is that although dynamical pure states of complex systems exhibit coherent fluctuations away from true micro-canonical equilibrium, their detection in practice requires extraordinary experimental resources, such as collecting \( O(D^{1/4}) \) measurement outcomes from repetitions of the experiment, or pre-computation of the location of the dynamical state in a \( D \)-dimensional Hilbert space, or performing joint (entangling) measurements on identical copies of the system. In the absence of such resources, by Theorem 1 we see that after some finite time, the empirical probability distributions for dynamical pure states of complex quantum systems cannot be distinguished from the micro-canonical equilibrium state.

**Appendix A: Proof of Theorem 1**

**Proof of Theorem 1.** The first step of the proof is to demonstrate that if we take \( N \ll \sqrt{M} \) samples from the uniform distribution then the probability of obtaining \( N \) distinct outcomes is nearly 1. Since there are \( M!/(M-N)! \) ways \( N \) unique items can be selected from a set of \( M \) and since there are \( M^N \) possible selections of items, we have that the probability of seeing no coincidences if the actual distribution is the uniform distribution is

\[
\frac{M!}{(M-N)!M^N} = 1 - \frac{N(N-1)}{2M} + O(1/M^2). \tag{A1}
\]

This implies that coincidental outcomes are unlikely unless \( N \geq O(\sqrt{M}) \).

Now let us assume that the true probability distribution is not uniform, but rather a distribution with outcome variance \( \mathbb{V}_k\{\text{Pr}(k)\} = O(M^{-2}) \). We compare this to a distribution with \( \mathbb{V}_k\{\text{Pr}(k)\} = O(M^{-2}) \) that has the highest probability of coincidence for some outputs. Using the definition of variance, \( \mathbb{V}_k\{\text{Pr}(k)\} = M^{-1} \sum_k (\text{Pr}(k) - M^{-1})^2 \), expanding the sum and using \( \sum_k \text{Pr}(k) = 1 \), we find

\[
\forall_k \{\text{Pr}(k)\} = O(M^{-2}) \Rightarrow \text{Pr}(k) \leq O(M^{-1/2}), \forall k. \tag{A2}
\]

The probability that no coincidental measurements are observed after \( N \) measurements is therefore at least

\[
\prod_{j=1}^{N-1} \left(1 - O\left(\frac{j}{M^{1/2}}\right)\right) = 1 - O\left(\frac{N^2}{\sqrt{M}}\right). \tag{A3}
\]

Let the event \( \mathcal{M} \) denote the observation that \( N \) unique measurement outcomes are observed, \( \mathcal{M} \) be the event where at least one measurement outcome is repeated, \( m_1 \) be the model that prescribes a uniform probability distribution to the outcomes, \( m_2 \) be the model that the probability distribution has outcome variance \( O(M^{-2}) \), and \( \mathcal{D} \) be the sequence of samples yielded by the device.

Since the underlying distribution \( P(k) \) is drawn from a distribution over distributions on \( \mathcal{S} = \{1, \ldots, M\} \) that is invariant under permutations of \( \mathcal{S} \), \( \text{Pr}(\mathcal{D}|m_2) = \text{Pr}(\mathcal{P}(\mathcal{D})|m_2) \), where \( \mathcal{P} \) is a permutation of \( \mathcal{S} \). We therefore see that all sequences of measurement outcomes that are equivalent up to permutations of labels provide equivalent evidence for model \( m_2 \). Since \( m_1 \) is the uniform distribution, \( \text{Pr}(\mathcal{D}|m_1) = \text{Pr}(\mathcal{P}(\mathcal{D})|m_1) \) for any permutation \( \mathcal{P} \). It is then clear that the labels of the outcomes observed cannot be used to distinguish between model \( m_1 \) and \( m_2 \). We therefore can, without loss of generality, choose the label of the outcomes such that the first outcome observed is outcome 1, the second unique outcome observed is 2 an so forth. From this perspective, it is clear that the differences between both models only become apparent in the distribution of coincidental outcomes. Our proof then follows from Bayes’ theorem and by showing that the probability of a coincidental outcome is small unless \( N \geq O(M^{1/4}) \).

Note that the preceding argument effectively prevents coarse graining from allowing us to distinguish the two distributions with high probability using a small number of measurements because the probability of correctly guessing a coarse graining that assigns high probability to particular coarse-grained outcomes is \( O(1/\text{poly}(M)) \) (given the assumption of permutation invariance).

There are two possible scenarios: either \( \mathcal{D} \) does not contain any repeated sample labels or \( \mathcal{D} \) contains at least one repeated sample label. We will first assume that the entries of \( \mathcal{D} \) are unique, which we denote event \( \mathcal{M} \). Bayes’ Theorem then implies

\[
\text{Pr}(m_1|\mathcal{M}) = \frac{\text{Pr}(\mathcal{M}|m_1) \text{Pr}(m_1)}{\text{Pr}(\mathcal{M}|m_1) \text{Pr}(m_1) + \text{Pr}(\mathcal{M}|m_2) \text{Pr}(m_2)}. \tag{A4}
\]

From our previous discussion, we see that

\[
|1 - \text{Pr}(\mathcal{M}|m_1)| = O\left(\frac{N^2}{M}\right)
\]

\[
|1 - \text{Pr}(\mathcal{M}|m_2)| \leq O\left(\frac{N^2}{\sqrt{M}}\right). \tag{A5}
\]
These results, and the fact that \( \Pr(m_1) + \Pr(m_2) = 1 \) give us
\[
\Pr(m_1|\mathcal{M}) \leq \frac{1 - O\left(\frac{N^2}{\sqrt{M}}\right)\Pr(m_1)}{1 - O\left(\frac{N^2}{\sqrt{M}}\right)}. \tag{A6}
\]
We apply Taylor’s Theorem to the denominator of \( \text{A6} \) and find that
\[
|\Pr(m_1|\mathcal{M}) - \Pr(m_1)| \leq O\left(\frac{N^2}{\sqrt{M}}\right), \tag{A7}
\]
and similarly
\[
|\Pr(m_2|\mathcal{M}) - \Pr(m_2)| \leq O\left(\frac{N^2}{\sqrt{M}}\right). \tag{A8}
\]
This shows us that the support provided by event \( \mathcal{M} \) for either hypothesis is small unless \( N \geq O(M^{1/4}) \).

Our next step is to formally show that a typical data set \( \mathcal{D} \) will be, with high probability, uninformative unless \( N \geq O(M^{1/4}) \). This follows from a concentration of measure argument over \( \mathcal{D} \) for the posterior probability distribution. The average over \( \mathcal{D} \) of the posterior probability is
\[
\mathbb{E}_D(\Pr(m_1|\mathcal{D})) = \Pr(m_1|\mathcal{M})\Pr(\mathcal{M}) + \Pr(m_1|\mathcal{M})\Pr(\mathcal{M}). \tag{A9}
\]
Using \( \Pr(m_1|\mathcal{M}) \leq 1, |1 - \Pr(\mathcal{M})| \leq O(N^2/\sqrt{M}) \) and \( \Pr(\mathcal{M}) \leq O(N^2/\sqrt{M}) \) we find from \( \text{A7} \) that \( \text{A9} \) implies
\[
|\mathbb{E}_D(\Pr(m_1|\mathcal{D})) - \Pr(m_1)| \leq O(N^2/\sqrt{M}). \tag{A10}
\]
A similar calculation gives the variance as
\[
\mathbb{V}_D(\Pr(m_1|\mathcal{D})) = \mathbb{E}_D(\Pr(m_1|\mathcal{D})^2) - (\mathbb{E}_D(\Pr(m_1|\mathcal{D})))^2 \leq O(N^2/\sqrt{M}). \tag{A11}
\]
Therefore, Chebyshev’s inequality implies that the probability of a given data set \( \mathcal{D} \) deviating substantially from the expectation value is
\[
\Pr(|\Pr(m_1|\mathcal{D}) - \mathbb{E}_D(\Pr(m_1|\mathcal{D}))| \geq \epsilon) \leq \frac{N^2}{\epsilon^2\sqrt{M}}. \tag{A12}
\]
Using this result in concert with \( \text{A10} \) implies that, with high probability, the posterior probability distribution after taking \( N \) samples will obey
\[
|\Pr(m_1|\mathcal{D}) - \Pr(m_1)| \leq O\left(\frac{N}{M^{1/4}}\right). \tag{A13}
\]
The result of Theorem 1 then follows from choosing the prior \( \Pr(m_1) = 1/2 \).

Appendix B: Proof of Lemma 1

Proof of Lemma 1. Eqns. (3), (4) and Chebyshev’s inequality imply that in the limit of large \( D \), the outcome variance of \( \Pr(k|\rho(t)) \) for \( t \geq t_{\text{eq}} \) is concentrated around \( O(D^{-2}) \). In particular, for any \( \epsilon > 0 \) we have from Chebyshev’s inequality and (4) that
\[
\Pr_{\mu_D \sim \mathbb{E}_H}\left(\|\mathbb{V}_D\{\Pr(k|\rho(t))\}\| \geq \epsilon\right) \leq \frac{O(D^{-4})}{\epsilon^2}, \tag{B1}
\]
where \( \mu_D \) is the appropriate measure for the ensemble \( \mathbb{E}_H \). In other words, the outcome variance for an individual system are very close to the ensemble average.

We can see that almost all Hamiltonians will have outcome variance \( O(D^{-2}) \) in the limit of large \( D \) via the following argument. To compute the probability that the outcome variance of a particular Hamiltonian scaling as \( O(D^{-2+\gamma}) \) for some \( \gamma > 0 \), we set \( \epsilon = O(D^{-2+\gamma}) \). Equation \( \text{B1} \) then implies that the probability of such an event scales at most as \( O(D^{-2}) \), which vanishes in the limit of large \( D \) unless \( \gamma = 0 \). Almost all Hamiltonians chosen from the ensemble therefore have outcome variance \( O(D^{-2}) \) in the limit of large \( D \) if \( t \geq t_{\text{eq}} \).

Next we will show that this implies information theoretic equilibration with respect to the observable \( A \) and time \( t \geq t_{\text{eq}} \). We know that almost all Hamiltonians drawn from the ensemble \( \mathbb{E}_H \) will satisfy the requirements of Theorem 1. Let us then consider a decision problem where we are maximally ignorant whether the state is a distribution that is uniform or one with outcome variance \( O(D^{-2}) \). This corresponds to taking an equal a priori probability of 1/2 for both outcomes. The theorem then implies that the probability of distinguishing the measurement statistics from those that would be expected from the uniform distribution is at most \( 1/2 + O(N/D^{1/4}) \). Therefore, \( N = O(D^{1/4}) \) samples are needed to distinguish the two possible models with probability greater than \( 1/2 + \delta \) for any fixed \( \delta > 0 \). We then see from Definition 1 that almost all Hamiltonians drawn from this ensemble will equilibrate information theoretically with respect to \( A \) and for any \( t \geq t_{\text{eq}} \) as \( D \to \infty \). \( \square \)

Appendix C: Unitary t-design Condition for Information Theoretic Equilibration

We now discuss how the unitary, \( C \), which transforms the eigenbasis of \( H \) to that of the observable \( A \) can be used to understand the equilibration properties of \( H \). Working in the eigenbasis of \( A \), we can write \( U(t) = CF(t)C^\dagger \), where \( \tilde{F}(t) = \text{diag}(e^{-itE_0}) \), and \( \{E_a\}_{a=1}^D \) are the energy eigenvalues of \( H \). Given an initial pure state \( \rho(0) = |x\rangle\langle x| \), the measurement outcome probabilities can be written as
\[
\Pr(k|\rho(t)) = \text{Tr}\left(|k\rangle\langle k|CF(t)C^\dagger |x\rangle\langle x|CF(t)^\dagger C^\dagger \right). \tag{C1}
\]
Information theoretic equilibration follows from \( \mathbb{V}_k \{ \Pr (|k\rangle \rho(t)) \} \in O(D^{-2}) \), which in turn requires that we know certain properties of \( \Pr (|k\rangle \rho(t))^2 \). It is not difficult to see that \( \Pr (|k\rangle \rho(t)) \) and \( \Pr (|k\rangle \rho(t))^2 \) can be concisely represented by

\[
\Pr (|k\rangle \rho(t)) = \langle L_2 | C^{\otimes 2} \otimes \bar{C}^{\otimes 2} | R_2(t) \rangle, \quad (C2)
\]

\[
\Pr (|k\rangle \rho(t))^2 = \langle L_4 | C^{\otimes 4} \otimes \bar{C}^{\otimes 4} | R_4(t) \rangle, \quad (C3)
\]

where \( \langle L_2 \rangle = \langle k, x, x, k \rangle \), \( \langle L_4 \rangle = \langle k, x, k, x, k, x, k \rangle \), and

\[
| R_2(t) \rangle = \sum_{b,b'} e^{i t(E_b - E_{b'})} | b, b', b, b' \rangle, \quad (C4)
\]

\[
| R_4(t) \rangle = \sum_{b,b',d,d'} e^{i t(E_b - E_{d'} + E_{d} - E_{d'})} | b, b', d, d', b, b', d, d' \rangle. \quad (C5)
\]

We refer to a term of the form \( \langle L_2 | C^{\otimes 2} \otimes \bar{C}^{\otimes 2} | b, b', b, b' \rangle \) in the sum in (C2) as a \( (2,2) \)-term because there are two basis change matrices acting on each factor space. The analogous terms in (C3) will be called \( (4,4) \)-terms; furthermore, \( \mathbb{V}_k \{ \Pr (|k\rangle \rho(t)) \} \) can be expressed as a \( (4,4) \) polynomial (meaning that all terms in the expansion of the outcome variance are at most \( (4,4) \) terms). It can be easily checked using (2) and Chebyshev’s inequality that 3 and (4) hold if the \( (4,4) \)- and \( (8,8) \)-terms scale as \( O(D^{-4}) \) and \( O(D^{-8}) \) respectively. This shows that we can reduce the question of whether typical Hamiltonians drawn from an ensemble equilibrates information theoretically with respect to an observable and time to a question about the properties of these terms.

The scalings given in eqns. 3 and (4) are satisfied if the matrix elements of \( C \), namely the Hamiltonian eigenvector components, satisfy a unitary \( t \)-design condition \[39\], which means that these matrix elements reproduce Haar-randomness for polynomials of degree at most \( t \). A similar connection was identified recently for subsystem equilibration in Refs. \[6\] \[7\] \[11\], which required a unitary 4-design. For microcanonical equilibration, we must also ensure that \( \mathbb{E}_H \{ \mathbb{V}_k \{ \Pr (|k\rangle \rho(t)) \} \} \) is sufficiently small to imply a concentration of measure for \( \mathbb{V}_k \{ \Pr (|k\rangle \rho(t)) \} \) via Chebyshev’s inequality. The resulting expression is an \( (8,8) \)-polynomial and hence a unitary 8-design condition is sufficient to imply that information theoretic equilibration with respect to local qubit measurements and \( t \geq t_{\text{eq}} \) is typical for individual systems from the ensemble (using Theorem 1 and Lemma 1).

Appendix D: Equilibration for Nearest Neighbor Hamiltonians

Previously, we showed numerically that random local Hamiltonians on a complete graph equilibrate information theoretically with respect to observables that are local rotations of \( A = \sum_{j=0}^{D-1} a_j |j\rangle \langle j| \) for non-degenerate \( a_j \) and \( |j\rangle \) an eigenstate of \( \sigma_z^{\otimes n} \). Although some physical systems, such as the Bardeen–Cooper–Schrieffer Hamiltonian for low-temperature superconductivity \[34\], can be represented as random local Hamiltonians on a complete graph, many physically relevant Hamiltonians have interactions that are constrained to nearest neighbors. We consider two relevant cases. First, we consider random local Hamiltonians with nearest neighbor interactions on lines with periodic boundary conditions. We then consider random local Hamiltonians on square lattices with periodic boundary conditions. In both cases, we see compelling numerical evidence for information theoretic equilibration with respect to \( A = \sum_{j=0}^{D-1} a_j |j\rangle \langle j| \) for non-degenerate \( a_j \).

Random local Hamiltonians on a line—We will now consider Hamiltonians of the form

\[
H = \left( \sum_{i=1}^{n} \sum_p a_{i,p} \sigma_p^{(i)} + \frac{1}{2} \sum_{(i,j)} \sum_p b_{i,j,p,p'} \sigma_p^{(i)} \sigma_{p'}^{(j)} \right), \quad (D1)
\]

where the sum over \( (i,j) \) refers to a sum over nearest neighbor \( i \) and \( j \) and \( b_{i,j,p,p'} = b_{j,i,p',p} \). In this case, such that only interactions between qubits \( i \) and \( j \) are permitted if \( |i-j| = 1 \) or \( |i-j| = n-1 \). Similarly to the RLH ensemble, we take \( a_{i,p} \) and \( b_{i,j,p,p'} \) to be Gaussian random variables with mean 0 and variance 1.

Figure 5 shows that typical members drawn from this constrained ensemble of random local Hamiltonians also achieve information theoretical equilibrium with respect to \( A \). We see from the results that the ensemble expectation of the outcome variance scales as \( O(D^{-2}) \) and that the ensemble variance of the outcome variance scales as \( O(D^{-4}) \). We know from the results of Lemma 1 require that if the ensemble average and variance of \( \mathbb{V}_k \{ \Pr (|k\rangle 0/t) \} \) at most as \( O(D^{-2}) \) and \( O(D^{-4}) \) respectively in order to guarantee that a Hamiltonian sampled uniformly from the ensemble will, with high probability, equilibrate information theoretically with respect to the observable. We therefore conclude from this data that information theoretic equilibration with respect to non-degenerate measurements in the eigenbasis of \( \sigma_z^{\otimes n} \) is generic for members of this ensemble of random local Hamiltonians.

Random local Hamiltonians on a square lattice—Next we examine the issue of whether typical members of the ensemble of random local Hamiltonians that are constrained to have only nearest neighbor interactions between qubits on a square lattice equilibrate information theoretically with respect to computational basis measurements. The ensemble of Hamiltonians is similar to that in \[41\] except now we permit two qubits to interact if the two qubits are adjacent vertices on a square lattice. Note that we do not require that the overall shape of the lattice is a square. Specifically, we consider lattices with a number of qubits \( n = 4, 6, 9, 10, 12, 14 \). In the case of \( n = 4 \) the lattice is uniquely a square of \( 2 \times 2 \) qubits. In the case of \( n = 12 \), there is an ambiguity in that the lattice can be expressed as an array of \( 4 \times 3 \) qubits or \( 2 \times 6 \) qubits. We examine the former configuration because it
is less like the 1D case.

Figure 6 shows that Hamiltonians drawn uniformly from this ensemble of Hamiltonians constrained to nearest neighbor interactions on a square lattice, with high probability, equilibrate information theoretically with respect to computational basis measurements for exactly the same reasons as the one–dimensional case discussed above. We also should note that although we have only studied equilibration with respect to a computational basis measurement, the results trivially also hold for local rotations of the computational basis because both the one– and two–dimensional ensembles are invariant with respect to single qubit rotations of any and all qubits.

Appendix E: Equilibration Time for a variant of the Heisenberg Model

In the main body of the text, it was claimed that a variant Heisenberg model:

$$H = \|H\|^{-1} \left( \sum_{i\leq n/2} \sigma_x(i) + \sum_{i>n/2} \sigma_x(i) + \sum_i \sigma_z(i) \cdot \sigma_z(i+1) \right)$$  \hfill (E1)

has an equilibration time of $t_{\text{eq}} \approx O(\log(D))$. This fact can be seen in Figure 7 where we plot the average probability, over $x$, of an initial eigenstate of $|0\rangle^\otimes n$ being measured in the state $|x\rangle \neq |0\rangle^\otimes n$ after evolution under the Hamiltonian for time $t$. We see from the figure strong evidence for equilibration of the measurement outcome.

The equilibration times do not scale as smoothly with $D$ as the data considered for the RLH ensemble. The reason for this discrepancy is that the ratio of spins experiencing a magnetic field in the $X$ direction to those experiencing a field in the $Z$ direction varies with $n$. If $n$ is even, then the ratio will be $1 : 1$; however, if $n$ is odd then there will be an excess of spins experiencing a transverse field in the $X$ direction. This difference causes the equilibration times to vary with the parity of $n$. We do see evidence though in Figure 7 that $t_{\text{eq}} \approx O(\log(D))$; although given the small range for the fit, the precise functional dependence of $t_{\text{eq}}$ on $D$ is not certain.

Appendix F: Equilibration for a One–Body Quantum Chaotic System

Finally we show that information theoretic equilibration of pure states occurs also for an individual system consisting of a one-body dynamical model associated with global classical chaos. In particular we consider a variant of the quantum kicked top, described by the Floquet map [22]:

$$U_F = e^{-iJ_\tau J^T} e^{-iJ_\alpha},$$  \hfill (F1)

where $J = [J_x, J_y, J_z]$ is a vector of angular momentum operators, $\tau$ is the moment of inertia tensor, which is
the system in the macroscopic limit in a regime of global chaos confirming information theoretic equilibration of a one-body system which agrees with the GUE predictions of \( O(\ell) \).

We showed previously that if the Hamiltonian is sufficiently complex, meaning that the change of basis matrix \( C \) satisfies a condition similar to an 8–design condition, then equilibration theoretic equilibration is generic for Hamiltonians drawn uniformly from the ensemble. Here we use this insight to infer from the properties of the \((4,4)\) terms of RLH Hamiltonians that information theoretic equilibration is generic.

In order to show this, we need to estimate the ensemble means and variances of the dominant \((4,4)\) terms (in the limit of large \( D \)). If the initial state preparation is \([x]\) and measurement outcome \( k \) is considered then the relevant sum in computing the value of \( \Pr(k|\rho(t)) \) (which is needed in the computation of \( \mathbb{V}_k \{ \Pr(k) \} \)) is

\[
\Pr(k|\rho(t))^2 = \sum_{b,b',d,d'} \langle k; x, k, x, k, x, k | C \otimes 4 C \otimes 4 e^{i(t(E_b - E_d + E_d - E_b')} | b, b', d, d', b, b', d, d' \rangle.
\]

Here each summand is known as a \((4,4)\) term. It is easy to then see that in the limit of large \( t \) and under the assumption of non–degenerate Hamiltonians, that terms with \( b = b' \) and \( d = d' \) will dominate other terms due to the fact that no phase cancellation appears in the sum over such terms. This means that the value of the outcome variance is dictated by the characteristic magnitude of such terms. In this case, we do not need to compute the \((2,2)\)-terms because \( \mathbb{E}_{\rho(t)} (\Pr(k|\rho(t))) = 1/D \) trivially holds for RLH Hamiltonians.

Fig. 9 shows that the RLH average of the \((4,4)\)–terms agrees with the GUE predictions of \( O(D^{-2}) \) scaling. We also find that the variance of the \((4,4)\) terms is \( O(D^{-8}) \) which suggests that concentration of measure holds for the ensemble. This shows that the equilibration properties observed for the RLH ensemble can also be inferred from the properties of the change of basis matrix \( C \).

**Appendix H: Proof of Theorem 2**

We prove Theorem 2 in two steps:

1. We derive expressions for the Haar expectations of \( \Pr(k|\rho(t)) \) (see \( \text{[H2]} \) below) and the second and fourth moments. See Lemmas \( \text{[2,3]} \) and \( \text{[4]} \)

2. We find the expectations of these expressions over the GUE eigenvalue distribution, and show that there exists a time \( t_{\text{eq}}(D) = O(D^{-1/6}) \) such that for all \( t > t_{\text{eq}}(D) \), eqns. \( \text{(3)} \) and \( \text{(4)} \) hold. See Section \( \text{[H2]} \) for these expectations.
Since the calculations in this subsection require in depth knowledge of the properties of the Gaussian Unitary Ensemble (GUE), we will begin by giving a brief review of its properties [21 22 35]. GUE is the unique unitarily invariant distribution over Hermitian matrices $H$ that factorizes into a product of distributions each over an individual element of $H$. In particular, each independent element of $H$ is an i.i.d. Gaussian random variable. A Hermitian matrix $H$ generated according to the GUE has diagonal elements $h_{aa}$ that are real valued random variables each with distribution $\mathcal{N}(0, \sigma^2)$, and off-diagonal elements $h_{ab}$ with real and imaginary parts that are random variables each with distribution $\mathcal{N}(0, \frac{1}{2} \sigma^2)$. The variance $\sigma^2$ is a free parameter, which is closely related to the expected maximum energy eigenvalue as well as the ensemble average energy level spacing.

While many aspects of GUE have been shown to accurately model complex quantum systems, there are known limitations to using GUE as a model of such systems which deserve mention before we proceed. First, the average level density, which takes the form of a semi-circle, and long-range spectral fluctuations for GUE are not good models for the corresponding properties of physically relevant Hamiltonian systems, even chaotic ones. In particular, for most natural systems, such as those with only two particle interactions, the norm of the Hamiltonian scales polynomially with the number of particles [5]. However, the expected norm of a GUE Hamiltonian scales polynomially in the Hilbert space dimension $D$ [22]. As we will see below, this has a large impact on what might be called the equilibration time for these dynamical systems, which should therefore be taken with a grain of salt. Because of this, and in order to simplify calculations, we will follow the standard practice [7, 21] of taking the variance $\sigma^2 = \frac{1}{2}$.

The joint distribution over all elements of $H$ factorizes into a product of a distribution over eigenvectors, and a distribution over the energy eigenvalues of $H$ (see [21] Theorem 3.3.1, or [22] Chapter 4). Further, the joint probability distribution over eigenvectors of $H$ is the same as that over the change of basis matrix $C$, namely the Haar measure on the unitary group $U(D)$ [21 22]. Letting $C$ be the unitary which takes the eigenbasis of $H$ to that of $A$, and working in the eigenbasis of $A$, we can write

$$ U(t) = CF(t)C^\dagger, $$

where $F(t) := \text{diag}(e^{-itE})$, and $\{E_a\}_{a=1}^D$ are the energy eigenvalues of $H$. We can therefore take separate expectations over eigenvectors and eigenvalues, namely, over the matrices $C$ and $F$. Pr $(k|\rho(t))$ can be expressed as

$$ \text{Pr} (k|\rho(t)) = \text{Tr}[\langle k|CF(t)C^\dagger|x\rangle\langle x|CF(t)^\dagger C^\dagger|k\rangle]. \quad (H2) $$

In the following we will write the expectation over the Haar measure on the unitary group $U(D)$ of change of basis matrices $C$, as $E_C\{\cdot\}$, and $E_{\text{spec}}\{\cdot\}$ for the expectation over the GUE eigenvalue distribution.

1. Expectations over eigenvectors

**Lemma 2.** Take a non-degenerate observable $A$ acting on $\mathcal{H} = \mathbb{C}^D$, an initial pure state $\rho(0) = |x\rangle\langle x|$ which is an eigenstate of $A$, and a unitary $U(t) = e^{-itH}$ where $H$ is drawn uniformly at random from GUE. Then the variance of the measurement outcome probabilities over the Haar measure on the unitary group $U(D)$ of change of basis matrices $C$ is given by:

$$ \forall C \{ \text{Pr} (k|\rho(t)) \} = \frac{1}{D^4} \{ D^2 - 2D + 4 + (7 - 2D)\delta_{xk} + |\mu(t)|^2(2D\delta_{xk} - 10\delta_{xk} - 2) + \delta_{xk}|\mu(2t)|^2 + 2\delta_{xk}\text{Re}[\mu(t)^2\mu(-2t)] \} + O(D^{-5}), \quad (H3) $$

where we have defined $\mu(t) = \text{Tr}[U(t)] = \text{Tr}[F(t)]$ (this is often called the spectral form factor [21 22]).

Proof. First, note that the squares of the outcome probabilities $Pr (k|\rho(t))$ can be written in the form:

$$ \text{Tr}[\langle k|CF(t)C^\dagger|x\rangle\langle x|CF(t)^\dagger C^\dagger|s\rangle] = \sum_{s,s'} \langle s|\langle k|CF(t)C^\dagger|x\rangle\langle x|CF(t)^\dagger C^\dagger|s'\rangle \times \langle s'|k\rangle = (L_4)^{C\otimes \bar{C}} \otimes (\bar{C}^{\otimes 4}) R_4(t), \quad (H4) $$

where $C$ is the complex conjugate of $C$ and

$$ \langle L_4 \rangle = \langle k, x, k, x, k, x, k \rangle, \quad \langle R_4(t) \rangle = \sum_{b,b',d,d'} e^{i(t(E_b - E_{b'}) + E_d - E_{d'})} \times |b,b',d,d', b,b', d,d'\rangle. \quad (H5) $$

The expectation of the expression $(L_4)^{C\otimes \bar{C}} \otimes (\bar{C}^{\otimes 4})$ over Haar measure can be written as the projector onto the subspace spanned by the vectors

$$ |\Phi_\pi\rangle = (V_\pi \otimes 1) |\phi\rangle_{1,5} |\phi\rangle_{2,6} |\phi\rangle_{3,7} |\phi\rangle_{4,8}, \quad (H6) $$

where $|\phi\rangle_{ij} = \sum_{a=1}^D |a\rangle_{i} |a\rangle_{j}$, and the index $\pi$ runs over the 4! permutations of the elements {1, 2, 3, 4}. $V_\pi$ is the unitary permuting the first four factor spaces according to $\pi$. It was shown in [35] that this projector is given by

$$ E_C \{ (L_4)^{C\otimes \bar{C}} \otimes (\bar{C}^{\otimes 4}) \} = \sum_{\pi,\sigma} (M^{-1})_{\pi,\sigma} |\Phi_\pi\rangle \langle \Phi_\sigma|, \quad (H7) $$

where the matrix $M$ has components $M_{\pi,\sigma} = |V_\pi|V_\sigma| = \text{Tr}[V_{\pi^{-1}} V_\sigma] = d^4(\pi^{-1}\sigma)$, and $l(\sigma)$ is number of cycles in the cycle decomposition of the permutation $\pi^{-1}\sigma$. We then have

$$ E_C \{ \langle L_4|\Phi_\pi\rangle (M^{-1})_{\pi,\sigma} |\Phi_\sigma\rangle | R_4(t) \rangle \}, \quad (H8) $$
where the inner products $⟨Φ_σ|R_4(t)⟩$ are given by:

$$⟨Φ(1,2,3,4)|R_4(t)⟩ = |μ(t)|^4,$$

$$⟨Φ(1,2,4,3)|R_4(t)⟩ = ⟨Φ(1,3,2,4)|R_4(t)⟩ = d|μ(t)|^2,$$

$$⟨Φ(2,1,3,4)|R_4(t)⟩ = ⟨Φ(4,2,3,1)|R_4(t)⟩ = d|μ(t)|^2,$$

$$⟨Φ(1,3,4,2)|R_4(t)⟩ = ⟨Φ(1,4,2,3)|R_4(t)⟩ = |μ(t)|^2,$$

$$⟨Φ(2,3,1,4)|R_4(t)⟩ = ⟨Φ(2,4,3,1)|R_4(t)⟩ = |μ(t)|^2,$$

$$⟨Φ(3,1,2,4)|R_4(t)⟩ = ⟨Φ(3,2,1,4)|R_4(t)⟩ = |μ(t)|^2,$$

$$⟨Φ(4,1,3,2)|R_4(t)⟩ = ⟨Φ(4,2,1,3)|R_4(t)⟩ = |μ(t)|^2,$$

$$⟨Φ(3,4,1,2)|R_4(t)⟩ = |μ(2t)|^2,$$

$$⟨Φ(1,4,3,2)|R_4(t)⟩ = d^2,$$

$$⟨Φ(2,1,4,3)|R_4(t)⟩ = ⟨Φ(1,4,2,3)|R_4(t)⟩ = d,$$

$$⟨Φ(2,3,4,1)|R_4(t)⟩ = ⟨Φ(3,4,2,1)|R_4(t)⟩ = d,$$

$$⟨Φ(2,4,1,3)|R_4(t)⟩ = ⟨Φ(3,4,1,2)|R_4(t)⟩ = d,$$

$$⟨Φ(4,1,2,3)|R_4(t)⟩ = ⟨Φ(4,3,1,2)|R_4(t)⟩ = d,$$

Further, $⟨L_4|Φ_π⟩ = 1$ for $π = (4,3,2,1), (4,1,2,3), (2,3,4,1), (2,1,4,3)$, and $⟨L_4|Φ_σ⟩ = δ_{ik}$ for all other $π$.

Taking the sum of the above terms with $M^{-1}$ in eqn. [HS], we find:

$$\mathbb{E}_C\{Pr(k|ρ(t))^2\} = \frac{1}{α}\left\{|μ(t)|^4((D^2 - D - 2)|δ_{kk}| + 2|μ(t)|^2 - 4D^2 - 12D - 16 + (4D^3 + 8D^2 + 4D + 8)|δ_{kk}| + (D^2 - 2)|δ_{kk}| |μ(2t)|^2 + |μ(2t)|^2 |μ(t)|^2 + |μ(t)|^2 |μ(2t)|^2 + 2(D^4 + 8D^3 + 6D^2 - (4D^3 + 12D)|δ_{kk}| + 2|μ(2t)|^2 + 2|μ(2t)|^2 \right\},$$

(9)

where $α = D^2(D - 1)(D + 1)(D + 2)(D + 3)$.

**Lemma 3.** Take a non-degenerate observable $A$ acting on $H = \mathbb{C}^D$, an initial pure state $ρ(0) = |x⟩⟨x|$ which is an eigenstate of $A$, and a unitary $U(t) = e^{-iHt}$ where $H$ is drawn uniformly at random from GUE. Then the fourth moment of the measurement outcome probabilities over the Haar measure on the unitary group $U(D)$ of change of basis matrices $C$ is given by:

$$\mathbb{E}_C\{Pr(k|ρ(t))^4\} \leq \frac{1}{D^8} \sum_π 4^4|⟨Φ_π|R_8(t)⟩|^2 + \sum_π 4^4 |B_π| |⟨Φ_π|R_8(t)⟩|^2,$$

(12)

where $B$ is a matrix with components $≤ O(D^{-9})$.

**Proof.** The fourth power of the outcome probabilities can be written in the form:

$$Tr[(k|k)C(t)^4|x⟩⟨x|C(t)^4] = (L_8|C⊗8 ⊗ C⊗8|R_8(t))^4,$$

(13)

where $(L_8|$ and $|R_8(t)$ are defined analogously to (H5), but with twice the number of tensor factors. Further, the average of the expression $C⊗8 ⊗ C⊗8$ over Haar measure can be written as the projector onto the subspace spanned by the vectors

$$|Φ_π⟩ = (V_π ⊗ \mathbb{1})|φ⟩_{1,9}|φ⟩_{2,10}...|φ⟩_{8,16},$$

(14)

where the index $π$ runs over the 8! permutations of the elements $\{1, 2, 3, ..., 8\}$, and $V_π$ is the unitary permuting the first eight factor spaces according to $π$.

We will now determine the asymptotic scaling of the following expression with $D$:

$$\mathbb{E}_C\{Pr(k|ρ(t))^4\} = \sum_π |L_8|^4|⟨Φ_π|⟨M^{-1}\rangle_π,σ|Φ_σ⟩|^4,$$

(15)

by finding an approximation for $M^{-1}$. Recall that the matrix $M$ has components $M_{π,σ} = ⟨V_π|V_σ⟩ = Tr[V_π^{-1}V_σ] = D^{l(π^{-1}σ)}$, where $l(σ)$ is number of cycles in the cycle decomposition of the permutation $π^{-1}σ$. It is clear that the diagonal components of $M$ are all equal to $D^8$, and all other components of $M$ are strictly $< D^8$, as the identity permutation is the only one with 8 cycles. Letting $A := \mathbb{1} - M/D^8$, we have

$$A := \mathbb{1} - M/D^8,$$

(16)

which converges to $\mathbb{1}$ in the limit $N \to \infty$ if and only if $||A||_{op} < 1$. Because $A$ is of fixed size $8! \times 8!$ and all its elements are $< 1/D$, we have $||A||_{op} < 1$ for $D$ sufficiently
large. Therefore, $(\mathbb{1} - A)^{-1} = \mathbb{1} + A + O(D^{-2})$, which implies that

$$M^{-1} = \frac{\mathbb{1}}{D^8} + O(D^{-9}). \quad (H17)$$

Next, it is not difficult to see that all components of $(L_8|\Phi_\pi)$ are equal to 1 or $\delta_{xk}$. Therefore, we can split the sum in (H15) as

$$E_C\big\{ \Pr(k|\rho(t))^4 \big\} \leq \frac{1}{D^8} \sum_\pi (\Phi_\pi|R_8(t)) + \sum_{\pi,\sigma} (\Phi_\pi|R_8(t)), \quad (H18)$$

where we have used that $(L_8|\Phi_\pi) \leq 1$ for all $\pi$, and $B = M^{-1} - \frac{\mathbb{1}}{D^8}$ has all components $\leq O(D^{-9})$.

2. Expectations over GUE eigenvalues

Our next step towards proving Theorem 2 is to find the GUE average of the the spectral form factor $\mu(t)$, which appears in Lemmas 2 and 3. The joint distribution over the un-ordered energy eigenvalues $\{E_a\}_{a=1}^D$ of a GUE matrix is given by (see [21] Theorem 3.3.1, or [22] Chapter 4):

$$P(\{E_a\}^D) = \frac{1}{C_D} \prod_{1 \leq a < b}^D (E_a - E_b)^2 \exp \left[ -\frac{1}{2\sigma^2} \sum_{a=1}^D E_a^2 \right], \quad (H19)$$

where we define $C_D := (2\pi)^{D/2}\sigma^D D!$. Integration of $P(\{E_a\}^D)$ over $D - m$ variables gives the m-point correlation function (21) 6.1.1)

$$R_m(E_1, \ldots, E_m) = \chi \int_{-\infty}^\infty P(\{E_a\}^D)dx_{m+1} \ldots dx_D, \quad (H20)$$

where $\chi = D!/(D - m)!$. By [21] Theorem 5.1.4, this can be written as:

$$R_m(E_1, \ldots, E_m) = \det[K_D(E_i, E_j)], i,j=1,\ldots,m, \quad (H21)$$

where $K_D(E_i, E_j) = \sum_{k=0}^{D-1} \phi_k(E_i)\phi_k(E_j)$, and the $\phi_k(x)$ are the harmonic oscillator wave-functions $\phi_j(x) = (2^j j! \sqrt{\pi})^{-1/2} e^{-x^2/2}H_j(x)$, with $H_j(x)$ the Hermite polynomials (see [21] section 6.2).

a. Calculation of $E_{\text{spec,C}}\{|\mu(t)|^2\}$

Our next goal is to evaluate the expression $\Gamma(t, D) := E_{\text{spec}}\{|\mu(t)|^2\} = E_{\text{spec}}\{|\text{Tr}[e^{-itH}]|^2\}$. Expanding out the trace and grouping terms, we have

$$\Gamma(t, D) = E_{\text{spec,C}}\left\{ |\text{Tr}[e^{-itH}]|^2 \right\}$$

$$= \int_{-\infty}^\infty \left| \sum_{l=1}^D e^{itE_l} \right|^2 P(\{E_a\}^D) \prod_{j=1}^D dE_j$$

$$= D \int_{-\infty}^\infty P(\{E_a\}^D) \prod_{j=1}^D dE_j$$

$$+ \int_{-\infty}^\infty \sum_{l \neq m} e^{it(E_l - E_m)} P(\{E_a\}^D) \prod_{j=1}^D dE_j. \quad (H22)$$

Using the invariance of the joint distribution $P(\{E_a\}^D)$ under permutations of the energies, and the definition of the 2-point correlation function, this becomes

$$\Gamma(t, D) = D + \int_{-\infty}^\infty e^{it(E_2 - E_1)} R_2(E_1, E_2) dE_1 dE_2. \quad (H23)$$

It is well known that for large $D$ the function $K_D(E, E)$ follows the so called Wigner semi-circle law [21] (with $\sigma^2 = 1/2$):

$$K_D(E, E) \approx \frac{1}{\pi} \sqrt{2D - E^2} \Theta(\sqrt{2D} - |E|), \quad (H24)$$

where $\Theta(x)$ is the heavyside step function. For large $D$ we then have:

$$\Gamma(t, D) = D + 2D \frac{J_1^2(\sqrt{2Dt})^2}{t^2}$$

$$- (\sqrt{2D - t/2}) \Theta(2\sqrt{2D} - t), \quad (H25)$$

where $J_1(x)$ is the first Bessel function of the first kind. It should also be noted that, the limit of the function $K_D(E_1, E_2)$ is generally taken by rescaling (often called ’unfolding’ - see [36] section III.A.1) the energy level density as well as the energies by the local mean spacing [21] [22]. In taking this limit and the integrals for $\Gamma(t, D)$ we have followed the the procedure of [21] Appendices 10, and 11.

b. Equilibration time

Now that we have the GUE expectation of the spectral form factor, we can use Lemma 3 to find the full expectation over eigenvectors and eigenvalues of the outcome probabilities:

$$E_{\text{spec,C}}\{\Pr(k|\rho(t))\} = \frac{D - \frac{1}{2} \Gamma(t, D) + \delta_{xk} (\Gamma(t, D) - 1)}{D^2 - 1}, \quad (H26)$$

Notice that if $\Gamma(t, D) = O(D)$, then

$$E_{\text{spec,C}}\{\Pr(k|\rho(t))\} = \frac{D + \delta_{xk} O(D)}{D^2 - 1} + O(D^{-2}). \quad (H27)$$
In particular, if $D$ is large, then the GUE expectation of the probability distribution from (H27) is essentially the uniform distribution. We therefore take the equilibration time $t_{eq}(D)$ to be defined by the condition

$$t_{eq}(D) := \{ T \mid \Gamma(t, D) = O(D), \forall t > T \}. \quad (H28)$$

It is clear that the second term in (H25) satisfies:

$$\lim_{t \to \infty} \frac{J_1(\sqrt{2Dt})}{2D t^2} = 0. \quad (H29)$$

This shows that there exists a finite time $t_{eq}(D)$ such that $\Gamma(t, D) = O(D)$ for all $t > t_{eq}(D)$.

In order to get a sense of the equilibration time (keeping in mind our previous comments on GUE energy spectra), note that the condition on the equilibration time (H28) is essentially that

$$2D \frac{J_1(\sqrt{2Dt})^2}{t^2} = O(D). \quad (H30)$$

Using the fact that for $x \gg 3/4$ we can approximate $J_1(x) \approx \sqrt{2/\pi x} \cos[x - 3\pi/4]$, it follows that the equilibration time is

$$t_{eq}(D) = O(D^{-1/6}). \quad (H31)$$

c. GUE expectation of first and second power

Our next step towards a proof of Theorem 2 involves evaluating the expectation values over the spectrum that are needed to prove eqns. (3) and (4) for GUE Hamiltonians. First, from the previous section we have that there exists a finite time $t_{eq}(D)$ such that $\Gamma(t, D) = O(D)$ for all $t > t_{eq}(D)$. Using Lemma 3 and the above spectral expectation, we have

$$\mathbb{E}_{\text{spec},C} \{ \Pr \{ k | \rho(t) \} \} = \frac{D + \delta_{zk} O(D)}{D^2 - 1} + O(D^{-2}), \quad (H32)$$

which proves that for all $t > t_{eq}(D)$, eqn. (3) holds for GUE Hamiltonians.

Next, Lemma 2 implies that

$$\mathbb{E}_{\text{spec}} \{ V_C \{ \Pr \{ k | \rho(t) \} \} \} = \frac{1}{D} \{ D^2 - 2D + 4 + (7 - 2D) \delta_{zk} + \Gamma(t, D)(2D \delta_{zk} - 10 \delta_{zk} - 2) + \delta_{zk} \Gamma(2t, D) + 2 \delta_{zk} \mathbb{E}_{\text{spec}} \{ R \epsilon(\mu(t)^2 \mu(-2t)) \} \} + O(D^{-5}). \quad (H33)$$

In the next section we will show that for $t > t_{eq}(D)$, $\mathbb{E}_{\text{spec}} \{ R \epsilon(\mu(t)^2 \mu(-2t)) \} = O(D)$. Using this along with $\Gamma(t, D) = O(D)$ in the above, we find

$$\mathbb{E}_{\text{spec}} \{ V_C \{ \Pr \{ k | \rho(t) \} \} \} = O(D^{-2}), \quad (H34)$$

which proves that for all $t > t_{eq}(D)$, eqn. (3) holds.

d. Bounding $\mathbb{E}_{\text{spec}} \{ \mu(t)^2 \mu(-2t) \}$

In order to simplify the derivation of our upper bounds, we define

$$\Delta(t, D) := \mathbb{E}_{\text{spec}} \{ \mu(t)^2 \mu(-2t) \}$$

$$= \int_{-\infty}^{\infty} \sum_{i,j,k} \epsilon^{it(E_i + E_j - 2E_k)} P \{ \{ E_{\alpha} \}^D \} \prod_{i=1}^{D} dE_i, \quad (H35)$$

We can expand the triple sum into three parts depending on whether $i, j, k$ are: (i) all distinct, (ii) only two equal, or (iii) all equal, which upon integration over the remaining energies, give terms of the form:

(i) $R_3 (E_1, E_2, E_3) e^{it(E_1 + E_2 - 2E_3)}$,
(ii) $R_2 (E_1, E_2) e^{2it(E_1 - E_2)} + 2R_2 (E_1, E_2) e^{it(E_1 - E_3)}$,
(iii) $D$.

By expanding $R_3$ and $R_2$, we find products of integrals of terms of the form $K_D(E_1, E_3) e^{itE_1}$, as well as $K_D(E_1, E_3) e^{2it(E_i - E_j)}$, and

$$K_D(E_1, E_2) K_D(E_2, E_3) K_D(E_3, E_1) e^{it(E_1 + E_2 - 2E_3)}.$$  \quad (H36)

We will bound the magnitude of these terms in a similar fashion as was done in [7]. Note that the integral of $\mathbb{E}^{itE}$ is just

$$\text{Tr} [ P e^{itE_1} P e^{itE_2} P e^{-2itE_3} ], \quad (H37)$$

where $P = \sum_{k=0}^{D-1} | \phi_k \rangle \langle \phi_k |$ is the projector onto the D-dimensional lower-energy subspace spanned by the harmonic oscillator wave-functions, and $X$ is the position operator. Using the Cauchy-Schwartz inequality twice on (H37), we find:

$$| \text{Tr} [ (P e^{itE_1} P e^{itE_2} P e^{-2itE_3}) ] | \leq \sqrt{\text{Tr}[P] \text{Tr}[P] \text{Tr}[P]}. \quad (H38)$$

A similar argument also shows that terms with integrands of the form $K_D(E_1, E_j) e^{it(1 + E_j)}$ are also bounded by $D$. Using these bounds, we then have

$$| \Delta(t, D) | \leq 6D$$

$$+ \left( \int_{-\infty}^{\infty} K_D(E, E) e^{itE} \right)^2 \left( \int_{-\infty}^{\infty} K_D(E, E) e^{-2itE} \right)$$

$$+ D \left( \int_{-\infty}^{\infty} K_D(E, E) e^{-2itE} \right) + 2D \left( \int_{-\infty}^{\infty} K_D(E, E) e^{itE} \right)$$

$$+ \int_{-\infty}^{\infty} K_D(E, E) e^{2itE} \right)^2 + 2 \left( \int_{-\infty}^{\infty} K_D(E, E) e^{itE} \right)^2 \quad (H39)$$

Next recall that

$$\int_{-\infty}^{\infty} K_D(E, E) e^{itE} = \sqrt{2D} \frac{J_1(\sqrt{2Dt})}{t}, \quad (H40)$$
which approaches 0 as $t \to \infty$. The equilibration condition (H28) requires that $\sqrt{2D}J_1(\sqrt{2D} t) = O(\sqrt{D})$ for $t > t_{eq}(D)$, and if this is satisfied, then for all $t > t_{eq}$, we have $|\Delta(t, D)| \leq O(D)$.

e. GUE expectation of fourth power

The final quantity that we need to prove Theorem 2 is the asymptotic scaling of the GUE average of the fourth moment of $\Pr(\{k|\rho(t)\})$ where $\rho(t) = e^{-iHt}|x\rangle\langle x|e^{iHt}$ for $|x\rangle$ an eigenvector of the observable. This quantity shows a concentration of measure for the outcome variance of $\Pr(\{k|\rho(t)\})$, which will allow us to conclude that individual Hamiltonians drawn from the GUE will information theoretically equilibrate with high probability. Specifically, we show that for $t > t_{eq}(D)$ we have that $\mathbb{E}_{\text{spec},C}\{\Pr(\{k|\rho(t)\})^4\} = O(D^{-4})$. Recalling the form of $\mathbb{E}_{\text{C}}\{\Pr(\{k|\rho(t)\})^4\}$ from Lemma 4, we see that it is sufficient to show that for $t > t_{eq}(D)$, we have $\mathbb{E}_{\text{spec}}\{\Phi_\sigma|R_8(t)\} \leq O(D^4)$, for all permutations $\sigma$.

Calculating some explicit examples of

$$\langle \Phi_\sigma|R_8(t)\rangle = \sum_{a,a',b,b',c,c',d,d'} e^{it(E_a - E_{a'} + E_b - E_{b'} + E_c - E_{c'} + E_d - E_{d'})} \times (a,a',b,b',c,c',d,d') |V_\sigma|a,a',b,b',c,c',d,d',$$

(H41)

we see that these are of the general form

$$D^4 \mu(f_1t)\mu(g_1t)^* \ldots \mu(f_4t)\mu(g_4t)^*,$$

(H42)

where $f_j, g_j \in \{0,1,2,3,4\}$, and

$$a + \sum_{j=1}^{4} f_j + \sum_{j=1}^{4} g_j = 8,$$

(H43)

and if $f_j = 0$ then the corresponding $\mu(f_jt)$ does not appear in the product (and similarly for $g_j$). For example, there is a $|\mu(t)|^4$ term arising from $\sigma = \mathbb{1}$, and a $D^4$ term arising from $\sigma = (12)(34)(56)(78)$. The expectations $\mathbb{E}_{\text{spec}}\{D^4 \mu(f_1t)\mu(g_1t)^* \ldots \mu(f_4t)\mu(g_4t)^*\}$ can then be bounded in a similar fashion to Appendix H28. In particular, we can expand the sums in the product of the terms $\mu(f_jt) = \text{Tr}[e^{i f_jtH}]$ into various parts depending on which indices are equal or not equal, just as was done for expression (H35). These will then give a constant $D^b$ factor, and various combinations of integrals of $m$-point correlation functions. Each of the integrals with 2-point or higher order correlation functions can be bounded by $D$, just as was done for eqn. (H36). We will then be left with various powers of integrals of the form

$$\int_0^\infty K_D(E,E)e^{itE}$$

which as we have seen approach 0 as $t \to \infty$, and so are irrelevant for the $t > t_{eq}$ regime. The only remaining question then is power of the constant $D^b$ factor for each term. It is not difficult to see that a power of $D$ arises from each pairing $\mu(f_jt)\mu(g_kt)^*$ with $f_j = g_k$. For example, the expectation of the term $|\mu(t)|^4$ gives a contribution of $D^2$ (this is in fact the infinite time limit). From the constraint (H43), it is not difficult to see that $D^4$ is the highest power of $D$ which can arise. This shows that for $t > t_{eq}$

$$\mathbb{E}_{\text{spec},C}\{\Pr(\{k|\rho(t)\})^4\} = O(D^{-4}).$$

Putting all of the above together, we have:

**Proof of Theorem 2.** Expanding the outcome and ensemble variances and using (H44) and (H44), we find that, for $t > t_{eq}(D)$,

$$\forall_{E_{\text{H}}}\{\forall_k\{\Pr(\{k|\rho(t)\})\}\} = D^{-2} \sum_{j,k} \mathbb{E}_{E_{\text{H}}} \{\Pr(\{k|\rho(t)\})^2 \Pr(\{j|\rho(t)\})^2\} + O(D^{-4}).$$

(H45)

Then using the Cauchy-Schwartz inequality for expectations and (H32), we have:

$$\sum_{j,k} \mathbb{E}_{E_{\text{H}}} \{\Pr(\{k|\rho(t)\})^2 \Pr(\{j|\rho(t)\})^2\} \leq O(D^{-2}).$$

(H46)

This proves that $\forall_{E_{\text{H}}}\{\forall_k\{\Pr(\{k|\rho(t)\})\}\} = O(D^{-4})$. Chebyshev’s inequality then implies (in the same fashion as (H1)) that $\forall_k\{\Pr(\{k|\rho(t)\})\} \in O(D^{-2})$ with high probability over $E_{\text{H}}$. Theorem 1 then implies that $O(D^{1/4})$ samples are required to distinguish the $\Pr(\{k|\rho(t)\})$ from the uniform distribution for $t > t_{eq}(D) = O(D^{-1/6})$, which was shown in (H31). Since a particular GUE Hamiltonian is specified using $O(D^2)$, even drawing a random Hamiltonian from GUE requires $O(\text{poly}(D))$ arithmetic operations. Therefore it follows from Definition 1 and Lemma 1 that almost all GUE Hamiltonians equilibrate information theoretically for $t > t_{eq}$ with high probability.

**Acknowledgements.** We thank Fernando Brandão, Carl Caves, David Cory, Patrick Hayden, Daniel Gottesman, and Victor Veitch for insightful comments. We acknowledge funding from CIFAR, Ontario ERA, NSERC, US ARO/DTO and thank the Perimeter Institute for Theoretical Physics where this work was brought to completion.

[1] S. Popescu, A. J. Short, and A. Winter, Entanglement and the Foundations of Statistical Mechanics, Nature Physics 2, 754 (2006), arXiv:0511225
