Non-Identifiability in Network Autoregressions

Federico Martellosio*

June 1, 2022

Abstract
We study identifiability of the parameters in autoregressions defined on a network. Most identification conditions that are available for these models either rely on the network being observed repeatedly, are only sufficient, or require strong distributional assumptions. This paper derives conditions that apply even when the individuals composing the network are observed only once, are necessary and sufficient for identification, and require weak distributional assumptions. We find that the model parameters are generically, in the measure theoretic sense, identified even without repeated observations, and analyze the combinations of the interaction matrix and the regressor matrix causing identification failures. This is done both in the original model and after certain transformations in the sample space, the latter case being relevant, for example, in some fixed effects specifications.

Keywords: fixed effects, invariance, networks, quasi maximum likelihood estimation.

JEL Classification: C12, C21.

1 Introduction
In a wide range of empirical settings, data are available for an outcome variable and some covariates for each of the nodes of a network, as well as some measure of the pairwise interaction between the nodes. Autoregressive processes offer a simple way to study how the covariates affect the outcome variable, taking into account the network interaction. Models of this type can be traced back at least to Whittle (1954), and have since proved useful in many applications, across many scientific fields. In economics, and the social sciences more generally, they are currently particularly popular in the analysis of peer effects and social networks. The models are known as simultaneous autoregressions in the statistics literature (e.g., Cressie, 1993), spatial autoregressions in the econometrics literature (e.g., LeSage and Pace, 2009), are closely related to linear-in-means models (e.g.,

*School of Economics, University of Surrey, f.martellosio@surrey.ac.uk
Manski, 1993), and have important connections to linear structural equation models (e.g., Drton et al., 2011). To emphasize their wide applicability, we refer to them as network autoregressions.

This paper is concerned with identifiability of the parameters in a network autoregression. We employ a classical notion of identifiability, according to which parameters are identified if they are uniquely recovered from the distribution of the observables. Lack of identification has, of course, serious consequences for inference. For example, identification is necessary for the limiting objective function of an extremum estimator to be uniquely maximized at the true value of the parameter, which is a standard condition for consistency (see, e.g., Newey and McFadden, 1994). In addition, inference is expected to be difficult near the cases in which identification fails. Given how fundamental the problem of identification is, and given that establishing consistency of some extremum estimator is sufficient for identification, it is not surprising that there is a vast amount of work that is relevant for the present study. We mention in particular two very influential papers: Lee (2004) and Bramoullé et al. (2009). Lee (2004), in a rigorous analysis of the asymptotic properties of the quasi maximum likelihood estimator based on the Gaussian distribution, provides conditions that are sufficient for consistency and hence for identifiability. Bramoullé et al. (2009) investigates identifiability by looking at the mapping from the reduced form parameters to the structural parameters, an approach that has become standard in the social network literature.

The present paper studies identifiability directly from the first moment, or the first two moments, of the outcome variable. Compared to the approach via reduced form parameters, identification from moments does not require the nodes of the network being observed over multiple instances (e.g., over time). We show that identification from the first moment is generally possible, and characterize the cases when it is impossible. One class of cases when identifiability from the first moment is impossible is particularly relevant in fixed effects models (for example, the classical linear-in-means model with group fixed effects belongs to this class of cases). In that class, non-identifiability from the first moment is linked to the impossibility of invariant inference; more precisely, the parameters cannot be identified from any statistic that is invariant with respect to a certain group of transformations under which the model itself is invariant. This type of non-identifiability occurs despite the fact that the parameters may be identifiable from the second moment of the outcome variable. Hence, identifiability from the second moment is of very questionable value in these case, because it can only lead to non-invariant inference.

Section 2 sets out the framework. Section 3 studies identifiability from the first and
second moments of the outcome variable, and Section 4 discusses identifiability after reduction by invariance. Section 5 reports simulation evidence on the consequences of being close to non-identifiability. Section 6 briefly concludes. The appendices contain additional material and all proofs. Throughout the paper the results are illustrated by means of several examples.

**Notation.** Matrices are denoted by capital letters, vectors and scalars by lowercase letters. We reserve bold letters for random quantities (scalars, vectors, or matrices), so, for example, \( \mathbf{y} \) denotes a random vector and \( y \) a realization of \( \mathbf{y} \). Throughout the paper, \( \mathbf{1}_n \) denotes the \( n \times 1 \) vector of ones, \( \mathbb{R}^{n \times m} \) denotes the set of real \( n \times m \) matrices, \( \text{col}(A) \) denotes the column space of a matrix \( A \), \( M_A := I_n - A(A' A)^{-1} A' \) for a full column rank matrix \( A \), \( \mu_{\mathbb{R}^n} \) denotes the Lebesgue measure on \( \mathbb{R}^n \), “a.s.” stands for almost surely, with respect to \( \mu_{\mathbb{R}^n} \), and \( A \oplus B \) denotes the direct sum of the matrices \( A \) and \( B \) (that is, if \( A \) is \( n \times m \) and \( B \) is \( p \times q \), \( A \oplus B \) is the \( (n + p) \times (m + q) \) block diagonal matrix with \( A \) as top diagonal block and \( B \) as bottom diagonal block).

## 2 The model

The model of interest is the *network autoregression*

\[
\mathbf{y} = \lambda \mathbf{W} \mathbf{y} + \mathbf{X} \beta + \sigma \mathbf{\varepsilon},
\]

where \( \mathbf{y} \) is the \( n \times 1 \) vector of outcomes, \( \lambda \) is a scalar parameter, \( \mathbf{W} \) is an interaction matrix, \( \mathbf{X} \) is an \( n \times k \) matrix of regressors with full column rank and with \( k \leq n - 2 \), \( \beta \in \mathbb{R}^k \), \( \sigma \) is a positive scale parameter, and \( \mathbf{\varepsilon} \) is an unobservable \( n \times 1 \) random vector with \( \mathbb{E}(\mathbf{\varepsilon}) = 0 \). The matrices \( \mathbf{W} \) and \( \mathbf{X} \) are assumed to be nonstochastic and known as, for instance, in *Lee (2004)*.\(^1\) The entries of \( W \) are supposed to reflect the pairwise interaction between the observational units; in particular, the \( (i, j) \)-th entry of \( W \) is zero if unit \( j \) is not deemed to be a neighbor of unit \( i \). Some of the columns of \( \mathbf{X} \) may be spatial lags of some other columns (the spatial lag of a vector \( x \) being the vector \( \mathbf{W} x \)). That is, in the terminology of social networks, we allow for “contextual effects” or “exogenous spillovers”. We assume that \( \lambda \) is such that the model has a unique reduced form, or, in other words, that \( \mathbf{y} \) is

\(^1\)At the cost of some additional complexity, one could, alternatively, take \( \mathbf{W} \) and/or \( \mathbf{X} \) to be stochastic, and condition on \( \mathbf{W} \) and/or \( \mathbf{X} \) under suitable exogeneity assumptions (see, e.g., *Bramoullé et al.*, 2009; *Gupta*, 2019). In that case the assumption \( \mathbb{E}(\mathbf{\varepsilon}) = 0 \) would be replaced by \( \mathbb{E}(\mathbf{\varepsilon} \mid \mathbf{W}, \mathbf{X}) = 0 \). Allowing for endogeneity of \( \mathbf{W} \) and/or \( \mathbf{X} \) would instead require different methods; see Section 6.
uniquely determined given $X$ and $\varepsilon$.\footnote{Identification analysis when a unique reduced form does not exist would require different tools; see, e.g., Chesher and Rosen (2017).} This requires $S(\lambda) := I_n - \lambda W$ to be nonsingular. We refer to the set $\Lambda_u := \{\lambda \in \mathbb{R} : \det(S(\lambda)) \neq 0\}$ as the unrestricted parameter space for $\lambda$. Note that the values of the (real) parameter $\lambda$ such that $\det(S(\lambda)) = 0$ are $\lambda = \omega^{-1}$ for any nonzero real eigenvalue $\omega$ of $W$, so $\Lambda_u$ is the whole real line minus a number (less or equal to $n$) of isolated points.

When the index set of $y$ has more than one dimension (e.g., individuals and time, or individuals and networks), it is often useful to include in the error term additive unobserved components relative to those dimensions. In that case, we take a fixed effects approach and treat the unobserved effects as parameters to be estimated. Accordingly, for inferential purposes, we incorporate the fixed effects into $\beta$ and the corresponding dummy variables into $X$. Two examples of fixed effects specifications that can be nested into the general model (2.1) are given next.

**Example 1. (Panel data model)** There are $N$ individuals, followed over $T$ time periods. Let $W_t$ be an $N \times N$ matrix describing the interaction between individuals at time $t$, and $\tilde{X}$ an $NT \times \tilde{k}$ regressor matrix. A panel data version of the network autoregression (2.1) is given by $y_{it} = \lambda \sum_{ij} (W_t)_{ij} y_{jt} + \tilde{x}_{it}' \tilde{\beta} + u_{it}$, for $i = 1, \ldots, N$ and $t = 1, \ldots, T$, where $(W_t)_{ij}$ are the entries of $W_t$, and $\tilde{x}_{it}'$ are the $\tilde{k} \times 1$ rows of $\tilde{X}$. The error $u_{it}$ is decomposed into $c_i + \sigma \varepsilon_{it}$ (one-way model) or $c_i + \alpha_t + \sigma \varepsilon_{it}$ (two-way model), where $c_i$ and $\alpha_t$ are, respectively, individual specific effects and time specific effects, and $\varepsilon_{it}$ is an idiosyncratic error. Following a fixed effects approach (i.e., treating the random components $c_i$ and $\alpha_t$ as parameters to be estimated), the model can be written in the notation of equation (2.1), with $W = \bigoplus_{t=1}^{T} W_t$, and, for the two-way model, $X = (\tilde{X}, \iota_T \otimes I_N, I_T \otimes \iota_N)$ and $\beta = (\tilde{\beta}', c', \alpha')'$, where $c$ and $\alpha$ are the vectors with entries $c_i$ and $\alpha_t$, respectively.\footnote{Obviously, for identification of $\beta$, one column of the matrix $(\iota_T \otimes I_N, I_T \otimes \iota_N)$ should be omitted from $X$, or some normalization should be imposed on the fixed effects, and no regressor should be constant over time or over individuals.}

In most applications, $W_t$ is taken to be time invariant, say $W_t = W^*$ for all $t = 1, \ldots, T$, so that $W = I_T \otimes W^*$.

**Example 2. (Network fixed effects)** There are $R$ networks, with network $r$ having $m_r$ individuals. The model is

$$y_r = \lambda W_r y_r + \tilde{X}_r \tilde{\beta} + \alpha_r \iota_{m_r} + \sigma \varepsilon_r, \ r = 1, \ldots, R,$$

where $W_r$ is the $m_r \times m_r$ interaction matrix of network $r$, $\tilde{X}_r$ is the $m_r \times \tilde{k}$ regressor...
matrix of network \( r \), \( \tilde{\beta} \) is a \( \tilde{k} \times 1 \) parameter, and \( \alpha_r \) is a network fixed effect. Stacking the equations in (2.2) vertically, and following a fixed effects approach, the model can be written in the notation of equation (2.1), with \( y = (y'_1, \ldots, y'_R)' \), \( W = \bigoplus_{r=1}^R W_r \), \( \beta = (\tilde{\beta}', \alpha')' \), \( \varepsilon = (\varepsilon_1', \ldots, \varepsilon_R')' \), and \( X = (\tilde{X}, \bigoplus_{r=1}^R t_{mr}) \), where \( \tilde{X} := (\tilde{X}'_1, \ldots, \tilde{X}'_R)' \).

In the rest of the paper, unobserved effects are always treated as parameters. Two specific network autoregressions that will be used to illustrate our results are as follows.

**Example 3. (Group Interaction model)** A particular case of model (2.2), which we refer to as the Group Interaction model, is when all members of a group interact homogeneously, that is, \( W_r = \frac{1}{m_r-1}(t_{m_r}t'_m - I_{m_r}) =: B_{m_r} \), for \( r = 1, \ldots, R \). Following Manski (1993), this specific structure has played a central role in the literature on peer effects. We say that the Group Interaction model is balanced if all group sizes \( m_r \) are the same. In that case, letting \( m \) denote the common group size, \( W = I_R \otimes B_m \).

**Example 4. (Complete Bipartite model)** In a complete bipartite network the \( n \) observational units are partitioned into two groups, of sizes \( p \) and \( q \) say, with all units within a group interacting with all in the other group, but with none in their own group (e.g., Wasserman and Faust, 1994). In economics, such a structure arises commonly when modeling two-sided markets, before any specific matching between the two groups has taken place or when information about matchings is not available. The two groups could be, for instance, buyers and sellers, with each seller interacting with all buyers, and each buyer interacting with all sellers. For \( p = 1 \) or \( q = 1 \) this corresponds to the network known as a star. The adjacency matrix of a complete bipartite network is

\[
A := \begin{pmatrix}
0_{pp} & t_{pt'_q} \\
t_{qt'_p} & 0_{qq}
\end{pmatrix}.
\]

The associated row-normalized interaction matrix is \(^4\)

\[
W = \begin{pmatrix}
0_{pp} & \frac{1}{q} t_{pt'_q} \\
\frac{1}{p} t_{qt'_p} & 0_{qq}
\end{pmatrix}.
\] (2.3)

Alternatively, \( A \) can be rescaled by its largest eigenvalue, yielding the symmetric interaction matrix

\[
W = \frac{1}{\sqrt{pq}} A.
\] (2.4)

\(^4\)For an entrywise nonnegative matrix \( B \) having all row-sums different from zero, the row-normalized version of \( B \) is obtained by dividing each entry of \( B \) by the corresponding row-sum, and is therefore a row-stochastic matrix.
We refer to the network autoregressions with interaction matrix (2.3) or (2.4), as, respectively, the row-normalized Complete Bipartite model and the symmetric Complete Bipartite model.

3 Identifiability

This section explores identifiability of the parameters of a network autoregression in two cases. First, Section 3.1 discusses what can be identified when no probabilistic assumptions beyond the maintained assumption $E(\varepsilon) = 0$ are imposed on the model. Then, Section 3.2 considers adding assumptions on the second moment of $\varepsilon$. Connections to the literature are discussed in Section 3.3.

We employ a classical notion of global identifiability, according to which a parameter is said to be identified if it can be uniquely recovered from the distribution of the observables (see, e.g., Koopmans and Reiersol, 1950; Rothenberg, 1971; Matzkin, 2007). The precise definitions we need are as follows. Consider a statistical model, defined as a family of distributions $\{P_\theta : \theta \in \Theta \subseteq \mathbb{R}^p\}$ for some observable random vector on a certain sample space. A particular value $\theta_1 \in \tilde{\Theta} \subseteq \Theta$ of $\theta$ is said to be identified (from the distribution $P_\theta$) on a set $\tilde{\Theta}$ if there is no other $\theta_2 \in \tilde{\Theta}$ such that $P_{\theta_1} = P_{\theta_2}$. If all values of $\theta$ in $\tilde{\Theta}$ are identified on $\tilde{\Theta}$, we say that the parameter $\theta$ is identified on $\tilde{\Theta}$ (if the set $\tilde{\Theta}$ is the whole parameter space $\Theta$, one often simply says that the model is identified). If all values of $\theta$ in $\tilde{\Theta}$ except for those in a $\mu_{\mathbb{R}^p}$-null set are identified on $\tilde{\Theta}$, we say that the parameter $\theta$ is generically identified on $\tilde{\Theta}$. Identifiability can also be applied to functions of the parameter $\theta$, so that a function $f(\theta)$ is identified if it can be uniquely recovered from $P_\theta$. Formally, the function $f(\theta)$ is said to be identified (on a set $\tilde{\Theta}$) if it can be recovered uniquely from a function $f(\theta)$ that is identified.

Note that identification from a moment is different from the concept, relevant in GMM estimation, of identification from a moment condition. Indeed, the concept of identification

5To see this, take arbitrary $\theta_1, \theta_2$ (in $\Theta$) such that $g(\theta_1) \neq g(\theta_2)$, and assume $g(\theta)$ can be recovered uniquely from $f(\theta)$, so that $f(\theta_1) \neq f(\theta_2)$. Then $P_{\theta_1} \neq P_{\theta_2}$, because $f(\theta)$ is identified, which shows that $g(\theta)$ is identified.
this paper refers to is detached from the choice of an estimator, unless one assumes that the distribution \( P_\theta \) is known (up to \( \theta \)), in which case identification is equivalent to identification based on the (correctly specified) likelihood (Rothenberg, 1971). Identification based on an extremum estimator is sufficient, but in general not necessary, for identification (e.g., Newey and McFadden, 1994).

### 3.1 Identifiability from first moment

We start by studying generic identification, according to the definition just given, of \( \lambda \) and \( \beta \) when no distributional assumptions beyond \( E(\varepsilon) = 0 \) are imposed on the model.\(^6\)

**Proposition 3.1.** In the network autoregression (2.1),

\[
\begin{align*}
(i) & \text{ if } \text{rank}(X, WX) > k, \text{ the parameter } (\lambda, \beta) \text{ is generically identified on } \Lambda_u \times \mathbb{R}^k; \\
(ii) & \text{ if } \text{rank}(X, WX) = k, \text{ no value of the parameter } (\lambda, \beta) \text{ is identified on } \Lambda_u \times \mathbb{R}^k.
\end{align*}
\]

Proposition 3.1 says that the parameters \( \lambda \) and \( \beta \) are generically identified (from the first moment of \( y \)) if the matrices \( X \) and \( W \) are such that \( \text{rank}(X, WX) > k \). Conversely, if \( \text{rank}(X, WX) = k \), \( \lambda \) and \( \beta \) cannot be identified, and hence consistently estimated, without distributional assumptions beyond \( E(\varepsilon) = 0 \). A manifestation of this is that the 2SLS estimators of Kelejian and Prucha (1998) and Lee (2003), which are based on the specification of the first moment only of \( y \), are not defined if \( \text{rank}(X, WX) = k \).\(^7\) Note that if \( \lambda \) is (generically) identified on \( \Lambda_u \) then it is also (generically) identified on any subset of \( \Lambda_u \).

The condition \( \text{rank}(X, WX) = k \) is trivially satisfied when \( k = 0 \); in that case, \( E(y) \) is zero and therefore cannot identify any parameter. When \( k > 0 \), the condition is typically very strong,\(^8\) but specific combinations of \( W \) and \( X \) such that \( \text{rank}(X, WX) = k \) may arise in some cases of interest, particularly in fixed effects models. An important class of cases when \( \text{rank}(X, WX) = k \) is characterized by a failure of the following condition.

---

\(^6\)Of course, the scale parameter \( \sigma \), as well as any parameters that might be used to parametrize the variance of \( \varepsilon \), cannot be identified assuming only \( E(\varepsilon) = 0 \). Identification of \( \sigma \) requires only very weak conditions (or a normalization) on the variance of \( \varepsilon \), whereas identifiability of any parameters affecting the variance of \( \varepsilon \) will depend on the particular parametric specification.

\(^7\)The 2SLS estimator of Kelejian and Prucha (1998) uses as instruments for the \( k + 1 \) variables \((X, Wy)\) a subset of the columns of the linearly independent columns of \((X, WX, W^2 X, \ldots)\). But when \( \text{rank}(X, WX) = k \), there are at most \( k \) such linearly independent columns.

\(^8\)Indeed, for any given \( W \) that is not a scalar multiple of the identity matrix, the set of (full rank) \( n \times k \) matrices \( X \) such that \( \text{rank}(X, WX) = k \) is a \( \mu_{\mathbb{R}^{n\times k}} \)-null set (with \( \mu_{\mathbb{R}^{n\times k}} \) denoting the Lebesgue measure on the set of real \( n \times k \) matrices). Accordingly, Proposition 3.1 could be stated by saying that identification from the first moment of \( y \) is possible for generic parameter values \((\lambda, \beta)\) and for generic regressor matrices \( X \).
**Condition 1.** There is no real eigenvalue $\omega$ of $W$ for which $M_X(\omega I_n - W) = 0$.

Indeed, $M_X(\omega I_n - W) = 0$ implies $M_XWX = 0$, which is equivalent to $\text{rank}(X, WX) = k$. Thus, by Proposition 3.1, any pair of matrices $X$ and $W$ violating Condition 1 gives rise to a model in which $\lambda$ and $\beta$ cannot be identified from $E(y)$. A condition equivalent to $M_X(\omega I_n - W) = 0$ is $\text{col}(\omega I_n - W) \subseteq \text{col}(X)$. That is, a pair $(X, W)$ causes Condition 1 to fail if and only if $\text{col}(X)$ contains the subspace $\text{col}(\omega I_n - W)$, for some real eigenvalue $\omega$ of $W$. Also, note that if, for a given $W$, Condition 1 is violated for some $X = X_1$, then it is also violated for $X = (X_1, X_2)$, for any $X_2$ (such that $X$ is full rank). It is helpful to look at how failures of Condition 1 can arise in the contexts of Examples 3 and 4.

**Example 5.** For the matrix $W = I_R \otimes B_m$ of a Balanced Group Interaction model (Example 3), the smallest eigenvalue is $\omega_{\text{min}} = -\frac{1}{m-1}$, and $\text{col}(\omega_{\text{min}} I_n - W) = \text{col}(I_R \otimes \iota_m)$. Since $I_R \otimes \iota_m$ is the design matrix of the group fixed effects, it follows that Condition 1 is violated if group fixed effects are included in a Balanced Group Interaction model.

**Example 6.** For both the row-normalized Complete Bipartite model and the symmetric Complete Bipartite model (Example 4), $\text{col}(W)$ is spanned by the vectors $(\iota'_p, 0'_q)'$ and $(0'_p, \iota'_q)'$. Hence, for both models, Condition 1 is violated (for $\omega = 0$) if $\text{col}(X)$ contains $(\iota'_p, 0'_q)'$ and $(0'_p, \iota'_q)'$. This is the case if the model contains an intercept for each of the two groups.

Examples 5 and 6 give cases in which Condition 1 is violated, and hence $\text{rank}(X, WX) = k$. Further examples in which Condition 1 fails are given in Appendix A. We now turn our attention to examples in which $\text{rank}(X, WX) = k$ even though Condition 1 is satisfied.

**Example 7.** The condition $\text{rank}(X, WX) = k$ is satisfied in the following instances of model (2.2), when there are at least two groups ($R > 1$):\(^9\)

(i) A Balanced Group Interaction model with contextual effects (e.g., Liu, 2017). The model equation is $y_r = \lambda B_m y_r + \alpha \iota_m + \tilde{X}_r \beta + B_m \tilde{X}_r \delta + \sigma \varepsilon_r$, for $r = 1, \ldots, R$, where $\alpha$ is an intercept and $\tilde{X}_r$ is $m \times \tilde{k}$, with $0 < \tilde{k} < R$. The matrix $X$ is given by $(\iota_m, \tilde{X}, (I_r \otimes B_m) \tilde{X})$, and has therefore $k = 2\tilde{k} + 1$ columns (recall that $\tilde{X}$ is the $n \times \tilde{k}$ matrix obtained by stacking the matrices $\tilde{X}_1, \ldots, \tilde{X}_R$ vertically).\(^{10}\)

---

\(^9\)In case (i), Condition 1 is satisfied for generic matrices $\tilde{X}_1, \ldots, \tilde{X}_R$ if the model is unbalanced, and is violated if the model is balanced (see Example 3). In cases (ii) and (iii), Condition 1 is satisfied for any $\tilde{X}$.

\(^{10}\)The condition $\text{rank}(X, WX) = k$ is satisfied whether the intercept is included in the model or not. Also, note that in this model, group fixed effects cannot be added, because $(\tilde{X}, (I_r \otimes B_m) \tilde{X}, I_R \otimes \iota_m)$ cannot have full column rank (this follows from $(m - 1)^{-1} x + (I_r \otimes B_m) x \in \text{col}(I_R \otimes \iota_m)$ for any $x \in \mathbb{R}^n$).
(ii) The network fixed effects model (2.2) with each $W_r$ being the symmetric or row-normalized adjacency matrix of a complete bipartite network, and with contextual effects. In this case, the matrix $X$ is given by $(\tilde{X}, W \tilde{X}, \bigoplus_{r=1}^{R} \iota_{m_r})$, and has therefore $k = R + 2\tilde{k}$ columns, with $0 \leq \tilde{k} < R$.

(iii) A Group Interaction model with group specific regression coefficients (denoted by $\tilde{\beta}_r$) and group fixed effects. The model equation is $y_r = \lambda B_m y_r + \tilde{X}_r \tilde{\beta}_r + \alpha_r \iota_{m_r} + \sigma \varepsilon_r$, for $r = 1, \ldots, R$, where the regressor matrix $\tilde{X}_r$ is $m_r \times k_r$, with $0 \leq k_r < m_r$. In this case, the matrix $X$ is given by $\bigoplus_{r=1}^{R} (\tilde{X}_r, \iota_{m_r})$, and has therefore $k = R + \sum_{r=1}^{R} k_r$ columns.

Example 8. The condition $\text{rank}(X, WX) = k$ is satisfied in the following network autoregressions with fixed effects and no regressors (i.e., the matrix $X$ contains only the dummies corresponding to the fixed effects):

(i) The one-way model of Example 1 with no regressors and time invariant interaction matrix, as, for instance, in Robinson and Rossi (2015). In this case, letting $W_t = W^*$ for each $t = 1, \ldots, T$, we have $W = I_T \otimes W^*$, and $X$ contains only the individual fixed effects, i.e., $X = \iota_T \otimes I_N$, so that $k = N$. Since $WX = \iota_T \otimes W^*$, it follows that $\text{rank}(X, WX) = \text{rank}(\iota_T \otimes I_N, \iota_T \otimes W^*) = \text{rank}(\iota_T \otimes (I_N, W^*)) = k$.

(ii) The network fixed effects model of Example 2 with no regressors and all matrices $W_r$’s being row-stochastic (a matrix is said to be row-stochastic if all its row sums are 1). In this case, $W = \bigoplus_{r=1}^{R} W_r$ and $X$ contains only the network fixed effects, i.e., $X = \bigoplus_{r=1}^{R} \iota_{m_r}$, so that $k = R$. Since each $W_r$ is row-stochastic, $\text{rank}(X, WX) = \text{rank}(\bigoplus_{r=1}^{R} \iota_{m_r}, \bigoplus_{r=1}^{R} \iota_{m_r}) = k$. Note that, when $R = 1$, this reduces to an intercept-only network autoregression (2.1) with row-stochastic interaction matrix.

(iii) When $r$ is time, case (ii) also covers the case of a panel data model with time fixed effects. Hence, putting cases (i) and (ii) together, another example when $\text{rank}(X, WX) = k$ is the two-way model of Example 1 with no regressors (i.e., $X$ contains $k = N + T - 1$ columns of $(\iota_T \otimes I_N, \iota_T \otimes I_N)$) and all matrices $W_t$’s being row-stochastic. 

Examples 5–8 contain several cases in which $\text{rank}(X, WX) = k$ and therefore $\lambda$ and $\beta$ cannot be identified from $E(y)$ alone. The identification prospects in such cases are

\footnote{In case (i) Condition 1 is satisfied for any $W$. In cases (ii) and (iii) Condition 1 is violated if and only if $W$ equals the interaction matrix of a Balanced Group Interaction model (i.e., if and only if $W = I_R \otimes B_N$ and $W = I_R \otimes B_m$ in cases (ii) and (iii), respectively).}
markedly different depending on whether Condition 1 is satisfied (Examples 7 and 8) or not (Examples 5 and 6). In the former case, identification can be achieved, for example, by imposing higher moments restrictions (see Section 3.2). In the latter case, the identification problem is deeper, and a solution would require more drastic changes to the model (see Section 4).

3.2 Identifiability from first two moments

Proposition 3.1 gives a condition for \( \lambda \) and \( \beta \) to be generically identified when the model only specifies the first moment of \( y \). When that condition fails, identification may be achieved by imposing further restrictions on the model. The simplest of such restrictions is \( \text{var}(\varepsilon) = I_n \), in which case identification can occur via the first two moments of \( y \).

To see this, it is convenient to focus on a parameter space for \( \lambda \) that is smaller than the unrestricted parameter space \( \Lambda_u := \{ \lambda \in \mathbb{R} : \det(S(\lambda)) \neq 0 \} \). Indeed, the parameter space for \( \lambda \) is usually taken to be a much smaller set than \( \Lambda_u \). Consider the case when \( W \) has at least one (real) negative eigenvalue and at least one (real) positive eigenvalue. This is typically satisfied in both applications and theoretical studies. Denote the smallest real eigenvalue of \( W \) by \( \omega_{\min} \), and, without loss of generality, normalize the largest real eigenvalue to 1. The parameter space for \( \lambda \) is often restricted to the largest interval containing the origin in which \( S(\lambda) \) is nonsingular, that is,

\[
\Lambda := (\omega_{\min}^{-1}, 1),
\]

or a subset thereof (possibly independent of \( n \)) such as \((-1, 1)\). Without such parameter space restrictions, the models are believed to be too erratic to be useful in practice, and \( \lambda \) is difficult to interpret.

**Proposition 3.2.** In the network autoregression (2.1) assume that \( \text{var}(\varepsilon) = I_n \) and that \( W \) has at least one negative eigenvalue and at least one positive eigenvalue. The parameter \( (\lambda, \beta, \sigma) \) is identified on \( \Lambda \times \mathbb{R}^k \times (0, \infty) \).

In Proposition 3.2, the parameters \( \lambda \) and \( \sigma \) are identified from \( \text{var}(y) \). Once \( \lambda \) is identified, \( \beta \) can be identified from the first moment \( E(y) = (I_n - \lambda W)^{-1}X\beta \) (because \( X \) has full rank). It should be noted that the restriction \( \text{var}(\varepsilon) = I_n \) is imposed only for simplicity, and one could certainly study identification under more general parametric structures for \( \text{var}(\varepsilon) \).
At this point, it is worth considering the network (or spatial) error model

\[ y = X\beta + u, \quad u = \lambda W u + \sigma \varepsilon, \tag{3.1} \]

even though this specification is considerably less popular than the network autoregression (2.1) in economic applications. The same set of assumptions as in the paragraph after equation (2.1) will be maintained for model (3.1). Proposition 3.2 also applies to the network error model, because equations (2.1) and (3.1) imply the same variance structure for \( y \), and \( \beta \) is trivially identified from the first moment in this model.

On the other hand, in the network error model (3.1) the first moment of \( y \), \( X\beta \), does not depend on \( \lambda \), and hence cannot identify \( \lambda \). Proposition 3.1 can be interpreted as saying that \((\lambda, \beta)\) cannot be identified from \( \mathbb{E}(y) \) in a network autoregression that “behaves” like a network error model. This point is made precise by the following argument. If \( \text{rank}(X, WX) = k \), there exists a unique \( k \times k \) matrix \( A \) such that \( WX =XA \), and hence \( S^{-1}(\lambda)X = X(I_k - \lambda A)^{-1} \), for any \( \lambda \) such that \( S(\lambda) \) is invertible (it is easily seen that the eigenvalues of \( A \) are eigenvalues of \( W \), and therefore \( I_k - \lambda A \) is invertible if \( S(\lambda) \) is). It follows that, when \( \text{rank}(X, WX) = k \), the network autoregression \( y = S^{-1}(\lambda)X\beta + \sigma S^{-1}(\lambda)\varepsilon \) can be written as \( y = X(I_k - \lambda A)^{-1}\beta + \sigma S^{-1}(\lambda)\varepsilon \), which is a network error model with regression coefficients \((I_k - \lambda A)^{-1}\beta\).\(^{12}\)

### 3.3 Connection with existing results

This section discusses connections between Propositions 3.1 and 3.2 and some related results available in the literature.

#### 3.3.1 Identification from reduced form parameter

In the social network literature, identification of the structural parameters in model (2.1) is typically established by verifying that those parameters can be uniquely recovered from the reduced form parameters (e.g., Bramoullé et al., 2009; Blume et al., 2011; Kwok, 2019). Such a strategy relies on the reduced form parameters being identified, which (in the case of a fixed \( W \)) would typically require individuals being observed repeatedly, over time or some other dimension.\(^{13}\) Hence, identification via reduced form parameters may not be

---

\(^{12}\)According to Lemma C.1 in Appendix C, the model \( y = X(I_k - \lambda A)^{-1}\beta + \sigma S^{-1}(\lambda)\varepsilon \) has the same profile quasi log-likelihood \( l(\lambda, \sigma^2) \) as model (3.1), even though, clearly, the MLE of \( \beta \) will be different in the two models.

\(^{13}\)Assuming, for simplicity, that we have only one covariate, \( x \), the reduced form of the network autoregression \( y = \lambda Wy + \beta x + \sigma \varepsilon \) is \( y = \Pi x + \sigma(I - \lambda W)^{-1}\varepsilon \), where \( \Pi := \beta(I - \lambda W)^{-1} \) is an \( n \times n \) matrix and...
appropriate in applications where a single observation of a network is available. Proposition 3.1 can establish identifiability not only when repeated observations are available (in which case \( W \) is block diagonal with identical blocks), but also when a single observation of the network is available. The following example considers a case when parameters can only be identified with repeated observations.

**Example 9.** Consider the row-normalized or symmetric Complete Bipartite model of Example 4 with an intercept, a regressor, and a contextual effect term, so that \( X = (\iota_n, x, W x) \) for some arbitrary vector \( x \in \mathbb{R}^n \) (such that \( X \) is full rank). This model is a particular case of equation (1) in Bramoullé et al. (2009). Because the matrices \( I_n, W, W^2 \) are linearly independent, Proposition 1 in Bramoullé et al. (2009) establishes that the parameters \( \lambda \) and \( \beta \) are identified from an i.i.d. sample of observations from the model, as long as \( \beta_2 \lambda + \beta_3 \neq 0. \)

However, in this model \( \text{rank}(X, WX) = k \), and therefore \( \lambda \) and \( \beta \) cannot be identified from a single observation of the model (without assumptions beyond \( \mathbb{E}(\varepsilon) = 0 \)), by Proposition 3.1.

To understand why, in Example 9, identifiability requires repeated observations of the network it is helpful to apply Proposition 3.1 to the case in which we observe the bipartite network \( R \geq 1 \) times. Note that \( R \) observations of a network autoregression with \( X = (\iota_n, x, W x) \) correspond to a network autoregression with interaction matrix \( W^* = I_R \otimes W \) and regressor matrix \( X^* = (\iota_{nR}, x^*, W^* x^*) \) for some \( x^* \in \mathbb{R}^{nR} \) (the regressor \( x \) is not kept constant across the \( R \) repetitions). When \( W \) is the row-normalized or symmetric Complete Bipartite model, \( \text{rank}(X^*, W^* X^*) > k \) if and only if \( R > 1 \). That is, Proposition 3.1 establishes that identification is achieved if and only if \( R > 1 \).

The applicability to the case of a single observation of a network is the most important difference between Proposition 3.1 and Proposition 1 of Bramoullé et al. (2009). Note also that, contrary to Bramoullé et al. (2009)’s result, Proposition 3.1 does not restrict attention to the case when \( X \) contains contextual effects; our results can be used for that case, but also when no contextual effects are included, or only some contextual effects are included.

---

14 Parameter restrictions such as \( \beta_2 \lambda + \beta_3 \neq 0 \) do not appear in Proposition 3.1, due to the fact that generic identification is considered there.

15 In fact, Condition 1 fails in this model (see Example 19 in Appendix A with a number of partitions equal to two), which implies \( \text{rank}(X, WX) = k \).

---

therefore cannot, without further restrictions, be identified from the distribution of a single realization of the vector \( y \).
3.3.2 Asymptotic identification

The MLE that is typically used for a network autoregression is the one based on the likelihood that would obtain if \( \varepsilon \) were distributed as \( N(0, I_n) \). Following common usage, we refer to this specific quasi MLE (QMLE) simply as the QMLE. Lee (2004) studies asymptotic properties of the QMLE. The condition \( \text{rank}(X, WX) > k \) appearing in Proposition 3.1 can be interpreted as a finite sample counterpart of Assumption 8 in Lee (2004). Indeed, under the latter assumption (and other regularity assumptions) the limit of the Gaussian quasi-likelihood has a unique maximum at the true value of the parameters, which is a sufficient condition for identification; see Newey and McFadden (1994). Similarly, Conditions for \((\lambda, \sigma)\) to be identified from \( \text{var}(y) \) can be seen as finite sample counterparts of Assumption 9 in Lee (2004).

3.3.3 Identification from second moment

Proposition 3.2 complements two results available in the literature that are concerned with identifiability from \( \text{var}(y) \) on a parameter space for \( \lambda \) different from \( \Lambda \). Firstly, Proposition 3.2 extends Lemma 4.2 in Preinerstorfer and Pötscher (2017), which establishes identification of \((\lambda, \sigma)\) from \( \text{var}(y) \) on \((0, 1) \times (0, \infty)\). Secondly, Lemma 4 in Lee and Yu (2016) gives a sufficient condition for \((\lambda, \sigma)\) to be identified from \( \text{var}(y) \) on \( \Lambda_u \times (0, \infty) \), namely that the matrices \( I_n, W + W' \) and \( W'W \) are linearly independent. It is instructive to look at an example in which \((\lambda, \sigma)\) is identified from \( \text{var}(y) \) on \( \Lambda_u \times (0, \infty) \) but not on \( \Lambda_u \times (0, \infty) \), and therefore identification can be established by Proposition 3.2 but not by Lemma 4 in Lee and Yu (2016).

Example 10. Consider a balanced group interaction model (see Example 3) with \( \text{var}(\varepsilon) = I_n \). By Proposition 3.2, \((\lambda, \sigma)\) is identified on \( \Lambda \times (0, \infty) \) (and hence on any subset thereof). On the other hand, Lemma 4 in Lee and Yu (2016) cannot establish identifiability, or non-identifiability, on \( \Lambda \times (0, \infty) \) because the matrices \( I_n, W + W' \) and \( W'W \) are not linearly independent when \( W = I_R \otimes B_m \). Indeed, when \( W = I_R \otimes B_m \), \( \sigma_1^2(S'(\lambda_1)S(\lambda_1))^{-1} = \sigma_2^2(S'(\lambda_2)S(\lambda_2))^{-1} \) if and only if \( \sigma_2^2 = m^2 \sigma_1^2/(2\lambda_1 + m - 2)^2 \) and \( \lambda_2 = -((m - 2)\lambda_1 + 2(1 - m))/(2\lambda_1 + m - 2) \), which shows that \((\lambda, \sigma)\) is not identifiable on \( \Lambda_u \times (0, \infty) \). Note however that \( \lambda_2 \notin \Lambda \) if \( \lambda_1 \in \Lambda \), which confirms that \((\lambda, \sigma)\) is identifiable on the smaller set \( \Lambda \times (0, \infty) \).

Exploiting second moment restrictions to achieve identifiability has been considered in many areas of econometrics (see, e.g., ?, in the context of DSGE models), and in particular in the peer effects literature; see, for instance, Graham (2008), Theorem 3.2 in
Davezies et al. (2009), Rose (2017), and Liu (2017). The latter paper studies identifiability in the context of the model in Example 7(i) above.

4 Invariance

So far, we have considered identifiability of a parameter $\theta$ from the distribution $P_\theta$ of an observable random vector $y$. Sometimes, it may be appropriate to consider identifiability from some transformation of $y$. Such a transformation might be dictated by the desire to eliminate some nuisance parameters, or, more generally, by invariance considerations (see, e.g., Chamberlain and Moreira, 2009). Suppose, for example, that $\theta$ is partitioned as $(\theta'_1, \theta'_2)'$, where $\theta'_2$ is not of direct interest. Particularly when the dimension of $\theta'_2$ is large compared to the sample size, one may want to consider identification from a transformation of $y$ whose distribution does not depend on $\theta'_2$. In general, if $\theta'_1$ is identified from the distribution of $y$ it is also identified from the distribution of the transformation of $y$. However, as we shall see in this section, this is not always the case. To analyze this point, we will need to discuss the full identifiability content of Condition 1. We have seen in Section 3.1 that, in a network autoregression, Condition 1 is not necessary for identifiability from the distribution of $y$; more precisely, it is necessary for identifiability from the first moment of $y$, but not for identifiability from the second moment. Nonetheless, we will show that when Condition 1 fails it is impossible to conduct inference that respects the symmetry properties of the model. More precisely, when Condition 1 fails the model is invariant under certain transformations of the sample space, but identifiability is impossible from any function of the data that is invariant under those transformations. This result requires some general group invariance notions (e.g., Lehmann and Romano, 2005, Chapter 6), which are reviewed in Section 4.1, and then applied to network autoregressions in Section 4.2. Section 4.3 contains the main identifiability result, and Section 4.4 discusses some implications for likelihood inference.

4.1 General invariance notions

Let $\mathcal{G}$ be a group of one-to-one functions (transformations) from a space $\mathcal{S}$ into itself, with the group operation being the composition of functions. The group $\mathcal{G}$ induces a partition of $\mathcal{S}$ into equivalence classes, called orbits, with two elements of $\mathcal{S}$ being in the same orbit if there exists an element of $\mathcal{G}$ transforming one element into the other. The orbit of an element $x \in \mathcal{S}$ is therefore the set $\{g(x) : g \in \mathcal{G}\}$. A function on $\mathcal{S}$ is said to be invariant under $\mathcal{G}$ (or $\mathcal{G}$-invariant) if it is constant on the orbits of $\mathcal{G}$. A function on $\mathcal{S}$ is said to
be a maximal invariant under $\mathcal{G}$ if it is invariant and takes different values on each orbit. A necessary and sufficient condition for a function on $S$ to be invariant under $\mathcal{G}$ is that it depends on $x \in S$ only through a maximal invariant under $\mathcal{G}$.

**Example 11.** (Scale invariance) Consider the group $\mathcal{G} = \{g_\kappa : \kappa > 0\}$, where $g_\kappa$ is the function $y \mapsto \kappa y$ from $\mathbb{R}^n$ to $\mathbb{R}^n$. A maximal invariant under $\mathcal{G}$ is $y/\|y\|$, where $\|y\| := \sqrt{y'y}$, and hence a function on $\mathbb{R}^n$ is $\mathcal{G}$-invariant is and only if it depends on $y$ only through $y/\|y\|$.\[16\]

The notion of group invariance can also be applied to a statistical model, defined to be a family of distributions on a certain sample space. In this case the set $S$ upon which the group $\mathcal{G}$ acts is the sample space, and the functions $g$ in $\mathcal{G}$ are required to be measurable (with respect to the sample space $\sigma$-algebra), so that when $z$ is a random variable with values in $S$, $g(z)$ is too. The family of distributions $\{P_\theta : \theta \in \Theta\}$ is said to be invariant under $\mathcal{G}$ if it is closed under the action of $\mathcal{G}$, that is, if every pair $g, \theta \in \mathcal{G} \times \Theta$ determines a unique element in $\Theta$ denoted by $\bar{g}(\theta)$, such that when $z$ has distribution $P_\theta$, then $g(z)$ has distribution $P_{\bar{g}(\theta)}$. Note that the definition of invariance of a family of distributions requires $\theta$ to be identified. The functions $\theta \mapsto \bar{g}(\theta)$ form a group acting on the parameter space, denoted by $\bar{\mathcal{G}}$.

When a model for $y$ is invariant under a group $\mathcal{G}$, it is natural to require that any inferential conclusion should be the same whether $y$ or $g(y)$ is observed, for any $g \in \mathcal{G}$. Accordingly, the data $y$ can be reduced to any $\mathcal{G}$-invariant function of $y$, with maximal reduction being achieved by reduction to a maximal invariant. This is what the so called principle of invariance advocates. For example, one would typically want to restrict attention to invariant loss functions (and, correspondingly, equivariant estimators) and, when the both the null and the alternative hypotheses are preserved under the group $\mathcal{G}$, to invariant tests. One fundamental results in the theory of invariance says that the distribution of any invariant statistic depends on $\theta$ only through a maximal invariant under $\bar{\mathcal{G}}$. Thus, in addition to a reduction in the sample space (from the dimension of $y$ to the dimension of the maximal invariant under $\mathcal{G}$), the principle of invariance generally also implies a reduction in the parameter space (from the dimension of $\theta$ to that of the maximal invariant under $\bar{\mathcal{G}}$).

**Example 12.** (A simple scale invariant model) Consider the statistical model defined by $y = \sigma \varepsilon$, where the distribution of $\varepsilon$ depends on a parameter $\eta$, and does not depend on $\sigma$.

\[16\] Here we use the convention that $y/\|y\| = 0$ if $y = 0$. Invariance of $y/\|y\|$ holds because $y/\|y\| = \kappa y/\|\kappa y\|$ for any $\kappa > 0$, maximality because, for any $y, \tilde{y} \in \mathbb{R}^n$, $y/\|y\| = \tilde{y}/\|\tilde{y}\|$ if and only if $\tilde{y} = ay$ for some $a > 0$.\[15\]
the parameter $\sigma > 0$. Provided that $\eta$ is identified, the model is invariant under the group of scale transformations $\mathcal{G} = \{ g_\kappa : \kappa > 0 \}$ (Example 11), because if $y$ has distribution $P_{\sigma, \eta}$, $g(y)$ has distribution $P_{\bar{g}(\sigma, \eta)}$, with $\bar{g}(\theta) = (\kappa \sigma, \eta)$, for any $g \in \mathcal{G}$. The maximal invariant under $\mathcal{G}$ is $\eta$, and indeed the distribution of the maximal invariant under $\mathcal{G}$, $y/\|y\|$, depends only on $\eta$.

\[ \square \]

4.2 Invariance of a network autoregression

We start our discussion of the invariance properties of a network autoregression by providing an invariance interpretation of Proposition 3.1. For this we need to introduce a group that is often used for regression models (e.g., Kariya, 1980), and we need a definition of invariance of an expectation. For a given $n \times m$ full column rank matrix $Z$, define the group $\mathcal{G}_Z := \{ g_\kappa, \delta : \kappa > 0, \delta \in \mathbb{R}^m \}$, where $g_\kappa, \delta$ denotes the function $y \mapsto \kappa y + Z \delta$ (a one-to-one transformation of $\mathbb{R}^n$), and its subgroup $\mathcal{G}_Z^1 := \{ g_{1, \delta} : \delta \in \mathbb{R}^m \}$. A maximal invariant under $\mathcal{G}_Z^1$ is $C_Z y$, where $C_Z$ is an $(n - m) \times n$ matrix such that $C_Z C_Z' = I_{n-m}$ and $C_Z' C_Z = M_Z$, and a maximal invariant under $\mathcal{G}_Z$ is $v(y) := C_Z y/\|C_Z y\|$ (with the convention that $v = 0$ if $C_Z y = 0$).\footnote{This can be established by direct verification of the definition of a maximal invariant. We provide the argument for $\mathcal{G}_Z$ (the argument for $\mathcal{G}_Z^1$ is similar). The statistic $v(y)$ is invariant because $v(\kappa y + Z \delta) = v(y)$ for any $\kappa > 0$ and any $\delta \in \mathbb{R}^m$, since $C_Z Z = 0$. It takes on different values on different orbits because, for any $y, \tilde{y} \in \mathbb{R}^n$, $v(y) = v(\tilde{y})$ if and only if $C_Z \tilde{y} = a C_Z y$ for some $a > 0$, that is, $C_Z (\tilde{y} - ay) = 0$, which is equivalent to $\tilde{y} = ay + Z b$ for some $b \in \mathbb{R}^m$.}

\[
\text{We say that the expectation } E_{\theta}(y) \text{ of a family of distributions } \{ P_{\theta} : \theta \in \Theta \} \text{ is } \mathcal{G}-\text{invariant if every pair } g, \theta \in \mathcal{G} \times \Theta \text{ determines a unique } \bar{g}(\theta) \text{ such that } E_{\theta}(g(y)) = E_{\bar{g}(\theta)}(y).\footnote{Note that } \mathcal{G}-\text{invariance of } E_{\theta}(y) \text{ is necessary but not sufficient for } \mathcal{G}-\text{invariance of } \{ P_{\theta} : \theta \in \Theta \} \text{ (the latter requires that every pair } g, \theta \in \mathcal{G} \times \Theta \text{ determines a unique } \bar{g}(\theta) \text{ such that } E_{\theta}(g(y)) = E_{\bar{g}(\theta)}(\varphi(y)), \text{ for every measurable function } \varphi). \]

\[
\text{This type of invariance implies that, when rank}(X, W X) = k, \text{ the expectation } E_{\lambda, \beta}(y) := S^{-1}(\lambda) X \beta \text{ is } \mathcal{G}_X^1-\text{invariant}.\footnote{If rank}(X, W X) = k, \text{ there exists a unique } k \times k \text{ matrix } A \text{ such that } W X = X A, \text{ and hence } S^{-1}(\lambda) X = X(I_k - \lambda A)^{-1}, \text{ for any } \lambda \text{ such that } S(\lambda) \text{ is invertible (note that } I_k - \lambda A \text{ is invertible if } S(\lambda) \text{ is, because the eigenvalues of } A \text{ must be eigenvalues of } W). \text{ Hence } E_{\lambda, \beta}(g_{1, \delta}(y)) = E_{\bar{g}_{1, \delta}(\lambda, \beta)}(y), \text{ with } \bar{g}(\lambda, \beta) = (I_k - \lambda A)^{-1} \beta + \delta.} \text{ This is } \mathcal{G}_X^1-\text{invariant if every pair } \lambda, \beta \in \mathcal{G}_X \times \Theta \text{ determines a unique } \bar{\lambda}, \bar{\beta} \text{ such that } E_{\lambda, \beta}(g_{1, \delta}(y)) = E_{\bar{\lambda}, \bar{\beta}}(g_{1, \delta}(y)), \text{ for every measurable function } \varphi.}
\]

Assumption 1. The distribution of $\varepsilon$ does not depend on the parameters $\lambda$, $\beta$, and $\sigma$.\footnote{This can be established by direct verification of the definition of a maximal invariant. We provide the argument for $\mathcal{G}_Z$ (the argument for $\mathcal{G}_Z^1$ is similar). The statistic $v(y)$ is invariant because $v(\kappa y + Z \delta) = v(y)$ for any $\kappa > 0$ and any $\delta \in \mathbb{R}^m$, since $C_Z Z = 0$. It takes on different values on different orbits because, for any $y, \tilde{y} \in \mathbb{R}^n$, $v(y) = v(\tilde{y})$ if and only if $C_Z \tilde{y} = a C_Z y$ for some $a > 0$, that is, $C_Z (\tilde{y} - ay) = 0$, which is equivalent to $\tilde{y} = ay + Z b$ for some $b \in \mathbb{R}^m$.}
Let $P_\theta$ denote the distribution for $y$ in the network error model, with $\theta := (\lambda, \beta, \sigma, \eta)$, $\eta$ being a parameter indexing the distribution of $\varepsilon$. Under Assumption 1 and provided that $\theta$ is identified, the network error model is $G_X$-invariant, because $g(y)$ has distribution $P_{\bar{g}(\theta)}$, with $\bar{g}(\theta) = (\lambda, \kappa, \beta + \delta, \kappa^2 \sigma^2, \eta)$, for any $g \in G_X$. Using the same parametrization, the result for network autoregression is as follows.

**Lemma 4.1.** Suppose that, in the network autoregression (2.1) Assumption 1 holds and $\theta$ is identified. The model is $G_X$-invariant if and only if $\text{rank}(X, WX) = k$.

Of course, invariance under a certain group implies invariance under a subgroup of that group. A subgroup of $G_X$ that will play an important role in Section 4.3 is $G^{1}_{X^*}$, for some $n \times k^*$ submatrix $X^*$ of $X$ ($k^* \leq k$). This is the group of transformations $y \mapsto y + X^* \delta$. For a network autoregression, it is clear from the proof of Lemma 4.1 that a sufficient condition for the model to be $G^{1}_{X^*}$-invariant (and also $G_X$-invariant) is that $\text{rank}(X^*, WX^*) = k^*$.

Recall now that the principle of invariance says that if a model is invariant under a group $G$ then the data should be reduced to $G$-invariant functions of the data, i.e., to functions of the data that depend on $y$ only through the maximal invariant under $G$. In particular, imposition of invariance under $G^{1}_{X^*}$ implies that $X^*$ is removed from the model, because the maximal invariant under $G^{1}_{X^*}$ is $C_X \cdot y$ and $C_X \cdot X^* = 0$. In the following example, $X^*$ is a matrix of fixed effects.

**Example 13.** Consider the network fixed effects model of Example 2, and let $X_{FE} := \bigoplus_{r=1}^{R} \tau_{m_r}$, the $n \times R$ matrix of network fixed effects. Under Assumption 1, and as long as each $W_r$ is row-stochastic, the model is $G^{1}_{X_{FE}}$-invariant, because $W_{r} \tau_{m_r} = \tau_{m_r}$, and therefore $\text{rank}(X_{FE}, WX_{FE}) = R$. In this case, the principle of invariance suggests that data should be reduced to $G^{1}_{X_{FE}}$-invariant functions of $y$, that is, to functions that depend on $y$ only through the maximal invariant $C_{X_{FE}}y$ under $G_{X_{FE}}$. Since $C_{X_{FE}}X_{FE} = 0$, reduction to $G^{1}_{X_{FE}}$-invariant statistics removes the fixed effects. We note that transformation by $C_{X_{FE}}$ is equivalent to the transformation proposed in Lee et al. (2010) to eliminate the network fixed effects (see Appendix B). Other transformations used to remove fixed effects in this model may or may not satisfy the principle of invariance. For example, the transformation referred to as global differences in Bramoullé et al. (2009) does, whereas that referred to as local differences in the same paper does not.

### 4.3 The role of Condition 1

We are now in a position to discuss the implications of Condition 1. Recall from Section 3.1 that $\text{rank}(X, WX) = k$ if Condition 1 is violated. Thus, by Lemma 4.1 and the principle of
invariance, any failure of Condition 1 is a case in which one would want to reduce the data to $\mathcal{G}_X$-invariant functions of the data. However, the imposition of $\mathcal{G}_X$-invariance causes an identifiability issue when Condition 1 fails. To see this, observe that if Condition 1 fails then $C_X S(\lambda) = (1 - \lambda \omega) C_X$, and therefore premultiplying both sides of the network autoregression equation $S(\lambda) y = X \beta + \sigma \varepsilon$ by $C_X$ yields

$$C_X y = \frac{\sigma}{1 - \lambda \omega} C_X \varepsilon.$$  \hspace{1cm} (4.1)

Note that $\lambda$ and $\sigma$ appear together in the scale factor in front of $C_X \varepsilon$. Thus, when Assumption 1 is satisfied but Condition 1 is not, $(\lambda, \beta, \sigma)$ cannot be separately identified from the distribution of $C_X y$ and hence, since $C_X y$ is a maximal invariant under $\mathcal{G}_X^1$, cannot be identified from the distribution of any $\mathcal{G}_X^1$-invariant statistic. Exactly the same conclusion obtains starting from the network error model $y = X \beta + \sigma S^{-1}(\lambda) \varepsilon$. The result is particularly perverse for the network autoregression: when Condition 1 fails, and under Assumption 1, the model is $\mathcal{G}_X^1$-invariant provided that its parameter $\theta$ is identifiable from the distribution of $y$, and yet $\theta$ cannot be identified from any $\mathcal{G}_X^1$-invariant statistic.

It is possible to be more precise about the cause of this identification failure. Suppose Condition 1 is violated for some eigenvalue $\omega$ of $W$, and let $\gamma_\omega$ be the geometric multiplicity of $\omega$.$^{20}$ Recall from Section 3.1 that a pair $(X, W)$ causes Condition 1 to fail if and only if some of the columns of $X$ span the subspace $\text{col}(\omega I_n - W)$. Observe that this requires $k \geq n - \gamma_\omega$, because the dimension of $\text{col}(\omega I_n - W)$ is $\text{rank}(\omega I_n - W) = n - \text{nullity}(\omega I_n - W) = n - \gamma_\omega$. Let $X_\omega$ be the $n \times (n - \gamma_\omega)$ matrix containing the columns of $X$ that span $\text{col}(\omega I_n - W)$, and reorder the columns of $X$ as in $X = (X_\omega, X^*)$, where $X^*$ is $n \times (k - (n - \gamma_\omega))$, with $k - (n - \gamma_\omega) \geq 0$. Generalizing the argument leading to equation (4.1), if Condition 1 fails then $C_{X_\omega} S(\lambda) = (1 - \lambda \omega) C_{X_\omega}$, and therefore

$$C_{X_\omega} y = \frac{1}{1 - \lambda \omega} C_{X_\omega} X^* \beta^* + \frac{\sigma}{1 - \lambda \omega} C_{X_\omega} \varepsilon,$$  \hspace{1cm} (4.2)

where $\beta^*$ is the component of $\beta$ corresponding to $X^*$. This shows that, under Assumption 1, $(\lambda, \beta, \sigma)$ cannot be identified from the distribution of $C_{X_\omega} y$ if Condition 1 fails. That is, what really causes non-identification when Condition 1 fails is the imposition of invariance with respect to the subgroup $\mathcal{G}_{X_\omega}^1$ of $\mathcal{G}_X$, and what we said above about $\mathcal{G}_X^1$-invariant statistics applies to the (larger) set of $\mathcal{G}_{X_\omega}^1$-invariant statistics. We formally state this

$^{20}$Note that, for fixed $W$ and $X$, the condition $M_X(\omega I_n - W) = 0$ that leads to a violation of Condition 1 can be satisfied at most by one eigenvalue $\omega$. This is because $M_X(\omega_1 I_n - W) = M_X(\omega_2 I_n - W)$ implies $\omega_1 = \omega_2$. Also, note that $M_X(\omega I_n - W) = 0$ implies that $\omega$ is real.
result in the following theorem, and then provide an example.

**Theorem 1.** Suppose that, in the network autoregression (2.1) or in the network error model (3.1), Assumption 1 is satisfied. If Condition 1 fails for some eigenvalue \( \omega \) of \( W \), then \((\lambda, \beta, \sigma)\) cannot be identified from the distribution of any \( G_{X_\omega}^1 \)-invariant statistic.

Theorem 1 says that, for any \( W \), there are matrices of regressors that make invariant inference impossible—these are the matrices leading to a violation of Condition 1, that is, the matrices whose column space contains a subspace \( \text{col}(\omega I_n - W) \), for some eigenvalue \( \omega \) of \( W \). It is worth emphasizing that this result does not require any distributional assumption other than Assumption 1. We provide an illustration of Theorem 1 by revisiting a well-known identification failure in the context of the balanced group interaction model (see Example 3).

**Example 14.** Consider a balanced group interaction model with group fixed effects. We have seen in Example 5 that in this model Condition 1 fails, because the columns of the fixed effects matrix \( I_R \otimes \iota_m \) span \( \text{col}(\omega_{\text{min}} I_n - W) \). That is, in the notation introduced just before equation (4.2), \( X_{\omega_{\text{min}}} = I_R \otimes \iota_m \). Theorem 1 therefore implies that, under Assumption 1, \((\lambda, \beta, \sigma)\) cannot be identified from any statistic that is invariant under \( G_{I_R \otimes \iota_m}^1 \), even though the model is invariant under that group (as shown in Example 13). Also, recall from Example 13 that reducing the data to \( G_{I_R \otimes \iota_m}^1 \)-invariant functions of the data removes the group fixed effects. Thus, in this model, lack of identifiability from \( G_{I_R \otimes \iota_m}^1 \)-invariant statistics corresponds to the well-known identification failure that occurs upon removal of the group fixed effects (Lee, 2007).

Theorem 1 is connected to the results obtained in Section 3. Recall that the parameters of a network autoregression can be identified by suitable restrictions on the variance structure of \( \varepsilon \), regardless of whether Condition 1 holds; for example, this is certainly the case if \( \text{var}(\varepsilon) = I_n \), by Proposition 3.2. According to Theorem 1, however, any result establishing identification from the distribution of \( y \) is pointless when Condition 1 is not satisfied, because in that case the model is invariant under the group \( G_{X_\omega}^1 \), and yet identification from \( G_{X_\omega}^1 \)-invariant functions of the data is impossible. We illustrate this point in the context of Example 14.

**Example 15.** Consider the model in Example 14. Due to the failure of Condition 1, any result establishing identification from the distribution of \( y \) cannot help to achieve inference that respects the invariance properties of the model. This is so, for example, for Proposition 2 in de Paula (2017), which establishes identification from the variance of \( y \).
for the particular case \( R = 1 \), when \( |\lambda| < 1 \). While this identification result is correct, it should be noted that inference based on it cannot respect the invariance properties of the model. Indeed, the model is invariant under the group \( G^1_{\iota n} \) of transformations \( y \mapsto y + \alpha \iota_n \), \( \alpha \in \mathbb{R} \), and yet, by Theorem 1, identification is lost if, as advocated by the principle of invariance, we require inference to satisfy the same symmetries. So, for instance, any invariant test in this model can have only trivial power, and any equivariant estimator will be useless.

**Remark 1.** Theorem 1 is related to some previous results in the literature. Martellosio (2011) considers non-identifiability from \( G_X \)-invariant statistics in a network autoregression (2.1) or spatial error model (3.1) when \( W \) is the matrix \( B_n := \frac{1}{n-1}(\iota_n \iota_n' - I_n) \). Preinerstorfer and Pötscher (2017), p. 30, generalizes Martellosio (2011)’s results to a regression model with correlated errors, which includes the particular case of a spatial error model (3.1) with arbitrary \( W \). Neither of these two papers, however, (i) discusses the relationship between non-identifiability from invariant statistics and identifiability from the first moment or from the first two moments; (ii) considers the set of \( G^1_{X\omega} \)-invariant statistics, which is larger than the set of \( G^1_X \)-invariant statistics.

### 4.4 Likelihood

We now study the consequences of Theorem 1 for likelihood inference. We consider the QMLE based on the original data, as introduced in Section 3.3.2, but also the QMLE after transformation by \( C_{X\omega} \) and the QMLE after transformation by \( C_X \). Transformation by \( C_{X\omega} \) is relevant, for example, when \( X_\omega \) is a matrix of fixed effects and one wishes to remove the fixed effects prior to estimation (Lee, 2007; Lee et al., 2010; Lee and Yu, 2010). Transformation by \( C_X \), on the other hand, is relevant when the model is \( G^1_X \)-invariant, which is certainly the case when Condition 1 fails. Note that, when the model is \( G^1_X \)-invariant, the QMLE after transformation by \( C_X \) is equivalent to the so-called adjusted QMLE, which is obtained from the QMLE by centering the profile score for \((\lambda, \sigma^2)\) (see Yu et al., 2015, and Appendix B). For estimation of \((\lambda, \sigma^2)\), it is well known that the adjusted QMLE usually performs better than the QMLE when the dimension of \( \beta \) is large with respect to the sample size \( n \) (including in fixed effects models, in which case the dimension of \( \beta \) is increasing with \( n \)).

We denote by \( l(\lambda, \beta, \sigma^2; y) \) the Gaussian quasi log-likelihood for \((\lambda, \beta, \sigma^2)\) in a network autoregression or in a network error model, by \( l(\lambda, \beta, \sigma^2; Ay) \) the corresponding log-likelihood obtained after premultiplying the model by a matrix \( A \), and by \( l(\lambda; y) \) the
profile likelihood for $\lambda$. The next result shows that, when Condition 1 fails, likelihood estimation is fruitless, before or after transformation of the data.

**Proposition 4.2.** Consider the network autoregression (2.1) or the network error model (3.1), with $\lambda$ such that $\det(S(\lambda)) \neq 0$, and $y \notin \text{col}(X)$. If Condition 1 is violated for some eigenvalue $\omega$ of $W$, then:

(i) the log-likelihood functions $l(\lambda, \beta, \sigma^2; C_X \omega y)$ and $l(\lambda, \beta, \sigma^2; C_X y)$ do not depend on $(\lambda, \beta, \sigma^2)$;

(ii) the profile score associated with the profile log-likelihood function $l(\lambda; y)$ does not depend on $y$ and $X$.

Part (i) of Proposition 4.2 says that the functions $l(\lambda, \beta, \sigma^2; C_X \omega y)$ and $l(\lambda, \beta, \sigma^2; C_X y)$ are constant. By Proposition 3.2 this cannot be the case for the likelihood based on $y$, $l(\lambda, \beta, \sigma^2; y)$. However, part (ii) of Proposition 4.2 establishes that the first derivative of the profile likelihood $l(\lambda; y)$ does not depend on the data, and consequently that a QMLE based on $l(\lambda; y)$ cannot depend on the data if it exists. This result can be understood in terms of the invariance results in Section 4.3: if Condition 1 is violated for an eigenvalue $\omega$ of $W$, then $l(\lambda; y)$ is a $G^1_{X_\omega}$-invariant loss function (see equation (C.5) in the proof of the proposition), and as such it cannot produce a useful estimator.

**Example 16.** We have seen in Example 13 that, for a network fixed effects model in which all interaction matrices $W_r$ are row-stochastic, transformation by $C_{X_{FE}}$ is equivalent to the transformation proposed in Lee et al. (2010). Thus the QMLE based on the Gaussian likelihood $l(\lambda, \beta, \sigma^2; C_X \omega y)$ is equivalent to the QMLE considered in Lee et al. (2010). In the particular case of a balanced group interaction model (i.e., $W_r = B_{m_r}$, for each $r = 1, \ldots, R$, and $m_1 = m_2 = \ldots = m_R$; see Example 3, Condition 1 fails, with $X_\omega = X_{FE}$; see Example 5. Thus, for the balanced group interaction model, Proposition 4.2 establishes that the Gaussian quasi-likelihood based $C_{X_{FE}} y$ and that based on $C_X y$ are constant functions of the parameters.

## 5 Simulations

Section 3.1 contains several examples in which $\text{rank}(X, WX) = k$, and therefore $\lambda$ and $\beta$ cannot be identified from the first moment of $y$, that is, cannot be identified when the

---

21This is similar to a result contained in Theorem 1 of Kelejian et al. (2006). That result establishes that, in a network autoregression with $W = B_n$, the 2SLS estimator of $\lambda$ considered there does not depend on the data.
only assumption about $\varepsilon$ is $E(\varepsilon) = 0$. As noted in that section, however, the condition \( \text{rank}(X, WX) = k \) is very strong in general. What might be more relevant in typical applications is that the condition is close, in some sense, to being satisfied. In that case, it would be natural to expect that identification from the first moment will be weak. This section analyses, by simulation, the consequences of near non-identification. We consider two Monte Carlo experiments, designed to study what happens close to, respectively, (i) a case when Condition 1 holds but \( \text{rank}(X, WX) = k \), (ii) a case when Condition 1 fails.

In both experiments, we draw 10,000 replications from model (2.1), with errors drawn from either a standard normal distribution or a gamma distribution with shape parameter 1 and scale parameter 1, demeaned by the population mean. Mean, variance, skewness, and kurtosis are 0, 1, 0, and 3 for the former distribution and 0, 1, 2, and 9 for the latter. The main objective of the simulations is to study the behavior of an estimator that uses only first moment information. We focus on the two-stage least squares estimator (2SLSE) with instruments for \((X, W y)\) given by (the linearly independent columns of) \((X, WX, W^2 X)\) (Kelejian and Prucha, 1998). We study two implications of near non-identification: (i) accuracy of the estimator; (ii) adequacy of first-order asymptotic approximation to the distribution of the estimator. Accuracy of the estimator is measured by the median square error. The root median square error is reported rather than the more usual root mean square error because, in the setting we are considering, the variance of the 2SLS estimator does not exist (see Roberts, 1995, Section 7.2.2). Adequacy of the first-order asymptotic approximation is measured by the coverage of 95% Wald confidence intervals.\(^{22}\) As a benchmark, we consider the (quasi) maximum likelihood estimator based on the Gaussian likelihood, abbreviated by (Q)MLE; see Section 4.4). Contrary to the 2SLSE, the QMLE also uses second moment information.

### 5.1 First experiment

In the first experiment, \( n \) is either 100 or 1000, and \( W \) is a row-normalized 2-ahead 2-behind interaction matrix (before row-normalization, this is a matrix with all entries in the two diagonals above and the two diagonals below the main diagonal equal to one, and zero everywhere else), and the model has a single regressor equal to \( \iota_n + b z \), where \( b \in \mathbb{R} \) and \( z \sim \mathcal{N}(0, I_n) \), with \( z \) being generated once, for each \( n \), and then kept fixed across replications. If \( b = 0 \), then \( \text{rank}(X, WX) = k = 1 \) (see Example 8(ii), with \( R = 1 \), and

\(^{22}\)For a parameter \( \phi \), the 95% Wald confidence interval is \( \hat{\phi} \pm 1.96 \sqrt{\hat{v}} \), where \( \hat{\phi} \) is an estimator of \( \phi \) and \( \hat{v} \) its estimated asymptotic variance. Expressions for the asymptotic variance of the 2SLS is standard, and that of the QMLE is given in Lee (2004), Theorem 3.2.
therefore the parameters $\lambda$ and $\beta$ cannot be identified from the first moment. Thus, we expect any estimator of $\lambda$ and $\beta$ that relies entirely on the specification of the first moment of $y$ to perform poorly if $b$ is close to 0 (and to be undefined when $b = 0$). The true values of $\lambda$, $\beta$, $\sigma$ are set to $\lambda_0 = 0$, $\beta_0 = 0.1$, 1, and $\sigma_0 = 1$. Table 1 displays the root median square error of the 2SLE and (Q)MLE of $\lambda$ and $\beta$. Dots indicate nonexistence of the estimator. The 2SLE exploits the (correct) specification of the first moment, whereas the QMLE also exploits the (correct, given that the data are generated under the assumption $\text{var}(\varepsilon) = I_n$) specification of the second moment. Let us look at the case $b_0 = 1$ first. For both $\lambda$ and $\beta$, and for both normal and the gamma distributions, the performance of the 2SLE is satisfactory, compared to the (Q)MLE benchmark, when $b = 1$, but deteriorates rapidly as $b$ gets smaller. Such a deterioration is due to both the bias and the dispersion of the 2SLE growing large as $b$ decreases, for any $n$. When $b = 0$, the 2SLE is not defined. On the contrary, due to the fact that it also exploits second moment information, the (Q)MLE is not much affected by the lack of identifiability from the first moment that occurs when $b = 0$. Indeed, the root median square error of the (Q)MLE is considerably less sensitive to $b$, and the (Q)MLE does well even when $b = 0$. Moving to the case $b_0 = 0.1$, it is natural to expect that identifiability from the first moment to become more difficult when $\beta$ is near zero. Indeed, all values of $(\lambda, \beta)$ in the $\mu_{\text{G2}}$-null set $\Lambda_n \times \{0\}$ cannot be identified from the first moment — see case (a) in the proof of Proposition 3.1. In particular, the Monte Carlo results show that the 2SLE of $\lambda$ performs much worse than the (Q)MLE, even when $b = 1$.

Table 1 confirms that estimation based on the first moment is impossible when $\text{rank}(X, WX) = k$ and difficult in cases close to $\text{rank}(X, WX) = k$. A second consequence of $X$ and $W$ being such that $\text{rank}(X, WX)$ is close to being equal to $k$ is that typical asymptotic approximations may become unreliable. Table 2 displays coverages of 95% two-sided Wald confidence intervals based on asymptotic normality, in the same setting at Table 1. When $b_0 = 1$, the empirical coverages for the 2SLE are close to the nominal one when $b = 1$, but get further and further away as $b$ decreases. When $b_0 = 0.1$, the empirical coverages for the 2SLE are poor even when $b = 1$. The (Q)MLE, on the other hand does well in terms of coverage even when $b = 0$ and even when $b_0 = 0.1$, again due to the fact that, in these simulations, it does not rely on the specification of the first moment only, but also exploits the (correct in these simulations) specification of the second moment.
Table 1: Root median square error of the 2SLS and (Q)ML estimators of $\lambda$ and $\beta$ in the first experiment.

| $\beta_0$ | $n$ | $b$ | Normal $\lambda$ | Normal $\beta$ | Gamma $\lambda$ | Gamma $\beta$ |
|-----------|-----|-----|------------------|----------------|----------------|----------------|
|           |     |     | 2SLS  ML  2SLS  ML | 2SLS  QML 2SLS  QML | 2SLS  QML 2SLS  QML |  
| 1         | 100 | 1   | 0.069 0.056 0.055 0.051 | 0.068 0.055 0.054 0.050 |  
|           |     | 0.1 | 0.489 0.094 0.481 0.115 | 0.479 0.092 0.476 0.110 |  
|           |     | 0.01| 1.464 0.096 1.451 0.119 | 1.472 0.093 1.451 0.114 |  
| 1000      | 1   | 0.025 0.019 0.020 0.018 | 0.026 0.020 0.020 0.018 |  
|           |     | 0.1 | 0.195 0.030 0.194 0.037 | 0.195 0.030 0.194 0.037 |  
|           |     | 0.01| 1.183 0.031 1.183 0.037 | 1.199 0.030 1.194 0.037 |  
| 1         | 100 | 1   | 0.617 0.092 0.056 0.042 | 0.616 0.092 0.055 0.042 |  
|           |     | 0.1 | 1.445 0.095 0.149 0.068 | 1.449 0.093 0.147 0.066 |  
|           |     | 0.01| 1.596 0.096 0.159 0.070 | 1.593 0.093 0.160 0.067 |  
| 1000      | 1   | 0.248 0.030 0.020 0.016 | 0.252 0.030 0.020 0.015 |  
|           |     | 0.1 | 1.190 0.030 0.111 0.022 | 1.205 0.030 0.112 0.021 |  
|           |     | 0.01| 1.623 0.030 0.144 0.022 | 1.669 0.030 0.150 0.021 |  
| 0         |     | 0   |                      |                      | 0.943 0.942 0.952 0.951 |  
| 1         | 100 | 1   | 0.947 0.945 0.948 0.947 | 0.949 0.944 0.946 0.944 |  
|           |     | 0.1 | 0.985 0.945 0.985 0.944 | 0.983 0.950 0.982 0.947 |  
|           |     | 0.01| 0.995 0.943 0.995 0.942 | 0.994 0.951 0.994 0.950 |  
| 1000      | 1   | 0.950 0.948 0.950 0.950 | 0.951 0.951 0.949 0.953 |  
|           |     | 0.1 | 0.962 0.953 0.963 0.952 | 0.961 0.951 0.961 0.951 |  
|           |     | 0.01| 0.995 0.951 0.995 0.952 | 0.994 0.951 0.995 0.950 |  
| 0         |     | 0   |                      |                      | 0.951 0.951 0.951 0.951 |  
| 0.1       | 100 | 1   | 0.986 0.946 0.977 0.945 | 0.990 0.951 0.977 0.937 |  
|           |     | 0.1 | 0.995 0.943 0.994 0.943 | 0.994 0.951 0.990 0.936 |  
|           |     | 0.01| 0.995 0.943 0.994 0.944 | 0.994 0.952 0.992 0.938 |  
| 1000      | 1   | 0.971 0.953 0.962 0.949 | 0.969 0.951 0.960 0.950 |  
|           |     | 0.1 | 0.995 0.951 0.994 0.950 | 0.995 0.951 0.994 0.947 |  
|           |     | 0.01| 0.995 0.951 0.995 0.950 | 0.996 0.951 0.995 0.949 |  
| 0         |     | 0   |                      |                      | 0.951 0.951 0.951 0.951 |  

Table 2: Coverage of 95% confidence intervals for $\lambda$ and $\beta$ in the first experiment.

| $\beta_0$ | $n$ | $b$ | Normal $\lambda$ | Normal $\beta$ | Gamma $\lambda$ | Gamma $\beta$ |
|-----------|-----|-----|------------------|----------------|----------------|----------------|
|           |     |     | 2SLS  ML  2SLS  ML | 2SLS  QML 2SLS  QML | 2SLS  QML 2SLS  QML |  
| 1         | 100 | 1   | 0.947 0.945 0.948 0.947 | 0.949 0.944 0.946 0.944 |  
|           |     | 0.1 | 0.985 0.945 0.985 0.944 | 0.983 0.950 0.982 0.947 |  
|           |     | 0.01| 0.995 0.943 0.995 0.942 | 0.994 0.951 0.994 0.950 |  
| 1000      | 1   | 0.950 0.948 0.950 0.950 | 0.951 0.951 0.949 0.953 |  
|           |     | 0.1 | 0.962 0.953 0.963 0.952 | 0.961 0.951 0.961 0.951 |  
|           |     | 0.01| 0.995 0.951 0.995 0.952 | 0.994 0.951 0.995 0.950 |  
| 0         |     | 0   |                      |                      | 0.951 0.951 0.951 0.951 |  
| 0.1       | 100 | 1   | 0.986 0.946 0.977 0.945 | 0.990 0.951 0.977 0.937 |  
|           |     | 0.1 | 0.995 0.943 0.994 0.943 | 0.994 0.951 0.990 0.936 |  
|           |     | 0.01| 0.995 0.943 0.994 0.944 | 0.994 0.952 0.992 0.938 |  
| 1000      | 1   | 0.971 0.953 0.962 0.949 | 0.969 0.951 0.960 0.950 |  
|           |     | 0.1 | 0.995 0.951 0.994 0.950 | 0.995 0.951 0.994 0.947 |  
|           |     | 0.01| 0.995 0.951 0.995 0.950 | 0.996 0.951 0.995 0.949 |  
| 0         |     | 0   |                      |                      | 0.951 0.951 0.951 0.951 |  

24
5.2 Second experiment

The second data generating process is the Group Interaction model of Example 3 with fixed effects. The model equation is

\[ y_r = \lambda B_{m_r} y_r + \bar{\beta} \bar{x}_r + \alpha_r t_{m_r} + \sigma \varepsilon_r, \ r = 1, \ldots, R. \]  

(5.1)

The number of groups is \( R = 50, 100, 200 \), and for the group sizes we consider 6 cases, all with average group size equal to 10 (so that, corresponding to \( R = 50, 100, 200 \), we have \( n = 500, 1000, 2000 \)), but with various degrees of unbalancedness. Specifically, the sequence of group sizes \( m_1, \ldots, m_R \) is periodic, with period 10 (i.e., \( m_i = m_{i+10} \), for any \( i = 1, \ldots, R - 10 \)), and the first 10 group sizes \( m_1, \ldots, m_{10} \) are given in Table 3. For each \( R \), the single regressor \( \bar{x} = (\bar{x}_1', \ldots, \bar{x}_R')' \) is drawn once from \( N(0, I_n) \) and then kept fixed across replications, and \( \alpha = (\alpha_1, \ldots, \alpha_R)' \) is drawn from \( N(0, I_R) \) in each replication. The true values of \( \lambda, \beta, \sigma \) are set to \( \lambda_0 = 0, \beta_0 = 1 \), and \( \sigma_0 = 1 \).

Table 3: Group sizes in the second experiment.

| case | \( m_1 \) | \( m_2 \) | \( m_3 \) | \( m_4 \) | \( m_5 \) | \( m_6 \) | \( m_7 \) | \( m_8 \) | \( m_9 \) | \( m_{10} \) |
|------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 1    | 2       | 4       | 6       | 8       | 9       | 10      | 12      | 14      | 16      | 18      |
| 2    | 4       | 6       | 8       | 10      | 10      | 12      | 12      | 14      | 14      | 14      |
| 3    | 6       | 8       | 10      | 10      | 10      | 12      | 12      | 14      | 14      | 14      |
| 4    | 8       | 10      | 10      | 10      | 10      | 10      | 10      | 10      | 10      | 10      |
| 5    | 9       | 10      | 10      | 10      | 10      | 10      | 10      | 10      | 10      | 11      |
| 6    | 10      | 10      | 10      | 10      | 10      | 10      | 10      | 10      | 10      | 10      |

Estimation is performed after removal of the fixed effects by premultiplication by \( C_{XFE} \) (see Section 4.4).\(^{23}\) When the model is balanced (case 6), both the 2SLSE and the (Q)MLE do not exist. The 2SLSE does not exist because \( \text{rank}(X, WX) = k = 1 \) and therefore the instrument matrix is singular. The (Q)MLE does not exist because Condition 1 fails and therefore the likelihood based on \( C_{XFE} y \) is constant (see Example 16). Table 4 shows that the root median square error of both 2SLS and (Q)ML estimators increases as the model becomes more balanced. Similarly to the first experiment, it is not surprising that the (Q)MLE performs better than the 2SLSE, given that \( \text{var}(\varepsilon) = I_n \) in the DGP. It is worth noting, however, that in the current experiment, the root median square error of the (Q)MLE, not only that of the 2SLSE, increases substantially close to the non-identifiability case. This is due to the two different types of non-identifiability studied in

\(^{23}\)Note that the 2SLSE of \( (\lambda, \beta) \) after premultiplication by \( C_{XFE} \) is the same as the 2SLSE prior to removal of the fixed effects.
the two experiments.

Table 5 shows that coverages of Wald confidence intervals can be very far from the nominal coverage when the model is close to being balanced. This is true for both the 2SLS and (Q)MLE.

6 Conclusion

This paper has studied identification of an autoregression defined on a general network, under weak distributional assumptions and without requiring repeated observations of the network. In this context, identification is possible for generic parameter values and for generic regressor matrices, whatever the network. Nevertheless, important cases do exist when identification fails, either in the original sample space or after some transformation of the sample space (this could be, for instance, a transformation aimed at removing fixed effects). We have shown that, in particular, there are cases where it is impossible to conduct inference that respects the invariance properties of the model, despite the fact the parameters may be identifiable from the distribution on the original sample space.

For practical purposes, it may be useful to construct a measure of the distance from non-identifiability. This goes beyond the scope of the present paper, but, for example, one may want to have a measure of distance from the non-identifiability condition \( \text{rank}(X, WX) = k \) in Proposition 3.1. One such measure would be the \((k + 1)\)-th largest singular value of \((X, WX)\), or some norm of the matrix \( M_X W X \), possibly upon some normalization of \( X \) and \( W \).

24 Such measures should help model users to avoid not only the cases in which inference based on the first moment is impossible, but also cases close to these, in which inference is likely to be very challenging without additional distributional assumptions.

Finally, it is important to remark that the results in this paper have been derived under the assumption that the network is fully known and exogenous, which may be unrealistic in many applications. The study of identification when the network is (partially) unknown and/or endogenous remains a key challenge in the literature (e.g., Blume et al., 2015; de Paula et al., 2020; ?), and we hope that the results obtained in this paper can prove useful in that setting too.

---

24To justify these two measures note that, since \( \text{rank}(X) = k \), (i) \( \text{rank}(X, WX) = k \) if the \((k + 1)\)-th largest singular value of \((X, WX)\) is zero; (ii) \( \text{rank}(X, WX) = k \) is equivalent to \( \text{col}(WX) \subseteq \text{col}(X) \), or, which is the same, to \( M_X W X = 0 \).
Table 4: Root median square error of the 2SLS and (Q)ML estimators of $\lambda$ and $\tilde{\beta}$ in the second experiment.

| $R$ | 2SLS Normal $\lambda$ | 2SLS $\tilde{\beta}$ | ML Normal $\lambda$ | ML $\tilde{\beta}$ | 2SLS Gamma $\lambda$ | 2SLS QML $\lambda$ | 2SLS $\tilde{\beta}$ | ML $\tilde{\beta}$ |
|-----|-----------------------|-----------------------|--------------------|-------------------|--------------------|----------------------|----------------|-----------------|
| 50  | 2.277                 | 0.152                 | 0.044              | 0.037             | 0.215              | 0.216                | 0.043          | 0.041           |
|     | 4.999                 | 0.328                 | 0.068              | 0.052             | 0.455              | 0.465                | 0.068          | 0.063           |
|     | 0.999                 | 0.662                 | 0.121              | 0.085             | 0.917              | 0.921                | 0.110          | 0.113           |
|     | 2.403                 | 1.453                 | 0.273              | 0.170             | 2.323              | 2.146                | 0.267          | 0.248           |
|     | 4.820                 | 2.917                 | 0.538              | 0.328             | 4.786              | 4.092                | 0.535          | 0.462           |
|     |                      |                       |                    |                   |                    |                      |                |                 |
| 100 |                      |                       |                    |                   |                    | 0.149               | 0.141          | 0.030           |
|     |                      |                       |                    |                   |                    | 0.327               | 0.332          | 0.046           |
|     |                      |                       |                    |                   |                    | 0.702               | 0.697          | 0.084           |
|     |                      |                       |                    |                   |                    | 1.726               | 1.641          | 0.197           |
|     |                      |                       |                    |                   |                    | 3.563               | 3.290          | 0.397           |
| 200 | 0.111                | 0.079                 | 0.020              | 0.018             | 0.102              | 0.097                | 0.021          | 0.020           |
|     | 0.276                | 0.174                 | 0.035              | 0.026             | 0.272              | 0.261                | 0.036          | 0.035           |
|     | 0.551                | 0.317                 | 0.064              | 0.040             | 0.526              | 0.507                | 0.060          | 0.061           |
|     | 1.301                | 0.795                 | 0.145              | 0.088             | 1.292              | 1.220                | 0.144          | 0.137           |
|     | 2.517                | 1.595                 | 0.282              | 0.179             | 2.585              | 2.400                | 0.290          | 0.290           |
| 50  |                      |                       |                    |                   |                    |                      |                |                 |
| 100 |                      |                       |                    |                   |                    |                      |                |                 |
| 200 |                      |                       |                    |                   |                    |                      |                |                 |

Table 5: Coverage of 95% confidence intervals for $\lambda$ and $\tilde{\beta}$ in the second experiment.

| $R$ | 2SLS Normal $\lambda$ | 2SLS $\tilde{\beta}$ | ML Normal $\lambda$ | ML $\tilde{\beta}$ | 2SLS Gamma $\lambda$ | 2SLS QML $\lambda$ | 2SLS $\tilde{\beta}$ | ML $\tilde{\beta}$ |
|-----|-----------------------|-----------------------|--------------------|-------------------|--------------------|----------------------|----------------|-----------------|
| 50  | 0.960                 | 0.959                 | 0.950              | 0.954             | 0.941              | 0.873                | 0.951          | 0.925           |
|     | 0.929                 | 0.947                 | 0.935              | 0.949             | 0.945              | 0.835                | 0.933          | 0.844           |
|     | 0.912                 | 0.951                 | 0.917              | 0.953             | 0.920              | 0.856                | 0.923          | 0.869           |
|     | 0.790                 | 0.928                 | 0.793              | 0.928             | 0.794              | 0.843                | 0.798          | 0.847           |
|     | 0.546                 | 0.882                 | 0.552              | 0.885             | 0.546              | 0.857                | 0.546          | 0.794           |
|     |                      |                       |                    |                   |                    |                      |                |                 |
| 100 | 0.950                 | 0.950                 | 0.938              | 0.950             | 0.946              | 0.857                | 0.949          | 0.933           |
|     | 0.944                 | 0.955                 | 0.945              | 0.944             | 0.941              | 0.824                | 0.938          | 0.870           |
|     | 0.923                 | 0.952                 | 0.927              | 0.949             | 0.934              | 0.818                | 0.932          | 0.828           |
|     | 0.848                 | 0.936                 | 0.849              | 0.935             | 0.845              | 0.832                | 0.845          | 0.834           |
|     | 0.669                 | 0.904                 | 0.668              | 0.909             | 0.675              | 0.803                | 0.675          | 0.808           |
|     |                      |                       |                    |                   |                    |                      |                |                 |
| 200 | 0.946                 | 0.955                 | 0.942              | 0.949             | 0.952              | 0.879                | 0.934          | 0.930           |
|     | 0.933                 | 0.939                 | 0.934              | 0.944             | 0.938              | 0.805                | 0.941          | 0.854           |
|     | 0.934                 | 0.943                 | 0.934              | 0.953             | 0.945              | 0.799                | 0.940          | 0.818           |
|     | 0.883                 | 0.948                 | 0.886              | 0.946             | 0.885              | 0.850                | 0.884          | 0.860           |
|     | 0.770                 | 0.921                 | 0.770              | 0.920             | 0.774              | 0.819                | 0.776          | 0.830           |
|     |                      |                       |                    |                   |                    |                      |                |                 |
Appendix A  Further examples when Condition 1 fails

Further to Examples 5 and 6, other simple cases in which Condition 1 fails are as follows.

Example 17. Consider the modification of the model in Example 3 in which exclusive averaging is replaced by inclusive averaging, meaning that each unit interacts not only with all other units in a group but also with itself. If there are \(R\) groups, each of size \(m_r\), the interaction matrix is \(W = \bigoplus_{r=1}^{R} \frac{1}{m_r} t_{m_r} t_{m_r}'\). Since \(\text{col}(\bigoplus_{r=1}^{R} \frac{1}{m_r} t_{m_r} t_{m_r}') = \text{col}(\bigoplus_{r=1}^{R} t_{m_r})\), Condition 1 is violated (at \(\omega = 0\)) whenever \(X\) contains group intercepts. Thus, in the group interaction model with inclusive averaging and group fixed effects, Condition 1 fails regardless of whether the model is balanced or not. Recall that, in contrast, with exclusive averaging Condition 1 fails only in the balanced case; see Example 5. \(\square\)

Example 18. Example 6 generalizes immediately to complete \(R\)-partite networks, with \(R \geq 2\) (e.g., Wasserman and Faust, 1994). Here, and in the next example, \(R\) denotes the number of partitions (and not the number of bipartite networks as in Example 7 (ii)). Such structures are useful, for instance, to model multi-sided markets in which there are \(R\) types of agents, and each agent interacts with all agents of different type, but with none of the same type. For an autoregression on a complete \(R\)-partite network, Condition 1 is violated (at \(\omega = 0\)) whenever \(\text{col}(\bigoplus_{r=1}^{R} t_{m_r}) \subseteq \text{col}(X)\), where \(m_r\) denotes the size of the \(r\)-th partition, and this is the case if \(X\) contains an intercept for each of the \(R\) partitions. \(\square\)

Example 19. Examples 17 and 18 share important similarities, due to the fact that the networks underlying the two models are complements of each other, in the graph theoretic sense. For both models, in addition to the cases mentioned in Examples 17 and 18, the condition \(\text{col}(\bigoplus_{r=1}^{R} t_{m_r}) \subseteq \text{col}(X)\) leading to a failure of Condition 1 is also satisfied if: (i) \(X\) contains an intercept and \(R-1\) (linearly independent) contextual effect terms \(W x_i\), for some \(x_1, \ldots, x_{R-1} \in \mathbb{R}^n\); (ii) \(X\) contains \(R\) (linearly independent) contextual effect terms \(W x_1, \ldots, W x_R\), for some \(x_1, \ldots, x_R \in \mathbb{R}^n\). Recall that \(R\) here denotes the number of groups for the model of Example 17, and the number of partitions for the model of Example 18, and note that, in order to be full rank, \(X\) can contain at most \(R-1\) contextual effect terms if it contains an intercept, \(R\) contextual effect terms otherwise. \(\square\)
Appendix B  The QMLE

QMLE based on $y$. Omitting additive constants, the quasi log-likelihood corresponding to $\varepsilon \sim N(0, I_n)$ in the network autoregression (2.1) is

$$l(\lambda, \beta, \sigma^2; y) := -\frac{n}{2} \log(\sigma^2) + \log|\text{det}(S(\lambda))| - \frac{1}{2\sigma^2}(S(\lambda)y - X\beta)'(S(\lambda)y - X\beta),$$

(B.1)

for any $\lambda$ such that $S(\lambda)$ is nonsingular. To avoid tedious repetitions, we often omit the ‘quasi’ in front of ‘log-likelihood’. The QMLE in most common use maximizes $l(\lambda, \beta, \sigma^2; y)$ under the condition that $\lambda$ is in $\Lambda$ (or in a subset thereof), the parameter space defined in Section 3.2. That is, the QMLE of $(\lambda, \beta, \sigma^2)$ is

$$(\hat{\lambda}_{\text{ML}}, \hat{\beta}_{\text{ML}}, \hat{\sigma}_{\text{ML}}^2) := \arg\max_{\beta \in \mathbb{R}^k, \sigma^2 > 0, \lambda \in \Lambda} l(\lambda, \beta, \sigma^2; y).$$

Note that if $W$ did not have a negative (resp. positive) eigenvalue, then the left (resp. right) extreme of $\Lambda$ could be taken to be $-\infty$ (resp. $+\infty$). Maximization with respect to $\beta$ and $\sigma^2$ gives $\hat{\beta}_{\text{ML}}(\lambda) := (X'X)^{-1}X'S(\lambda)y$ and $\hat{\sigma}_{\text{ML}}^2(\lambda) := \frac{1}{n}y'S(\lambda)M_{X}S(\lambda)y$. Thus, $\hat{\lambda}_{\text{ML}}$ can be conveniently computed by maximizing over $\Lambda$ the profile likelihood

$$l(\lambda; y) := l(\lambda, \hat{\beta}_{\text{ML}}(\lambda), \hat{\sigma}_{\text{ML}}^2(\lambda)) = -\frac{n}{2} \log(\hat{\sigma}_{\text{ML}}^2(\lambda)) + \log|\text{det}(S(\lambda))|,$$

(B.2)

where additive constants have again been omitted.

QMLE after reduction by invariance. When the model is $G_Z$-invariant, for some $n \times m$ matrix $Z$, the principle of invariance advocates reduction to $G_Z$-invariant statistics, that is, statistics that depend on $y$ only through the maximal invariant $C_{ZY}$. The network model of Example 13, is $G_{XFE}^1$-invariant, and here we show that, for this model, reduction by $G_{XFE}^1$-invariance is equivalent to the transformation approach proposed by Lee et al. (2010) to remove the network fixed effects. Lee et al. (2010) premultiply the model by the $(n - R) \times n$ matrix $E := \bigoplus_{r=1}^R E_r$, where $E_r$ is an $m_{r-1} \times m_r$ matrix such that $E_rE_r' = I_{m_{r-1}}$, and $E_r'E_r = M_{m_{r}}$ (in the notation of Lee et al., 2010, $E_r = F_r'$). Thus $EE' = I_{n-R}$ and $E'E = M_{XFE}$, which shows that $E$ satisfies the defining properties of a matrix $C_{XFE}$, and therefore that premultiplication by $E$ is equivalent to premultiplication by $C_{XFE}$. The transformation approach of Lee et al. (2010), which was given a partial likelihood interpretation in that paper, also admits an invariance interpretation.
Adjusted QMLE. When the dimension of \( \beta \) is large compared to the sample size, the QMLE of \((\lambda, \sigma^2)\) may perform poorly. To tackle this problem, the QMLE of \((\lambda, \sigma^2)\) can be adjusted by recentering the profile score \( s(\lambda, \sigma^2) \) associated to the profile log-likelihood for \((\lambda, \sigma^2)\), \( l(\lambda, \sigma^2) := l(\hat{\beta}_{\text{ML}}(\lambda), \sigma^2, \lambda) \). Under the assumptions \( \mathbb{E}(\varepsilon) = 0 \) and \( \text{var}(\varepsilon) = I_n \), the expectation of \( s(\lambda, \sigma^2) \) over the sample space is available analytically and does not depend on the nuisance parameter \( \beta \). Thus, calculation of the adjusted profile score \( s_a(\lambda, \sigma^2) := s(\sigma^2, \lambda) - \mathbb{E}(s(\lambda, \sigma^2)) \) is straightforward. Given \( s_a(\lambda, \sigma^2) \), one can define the adjusted likelihood \( l_a(\lambda, \sigma^2) \) as the function with gradient equal to \( s_a(\lambda, \sigma^2) \), and hence the adjusted QMLE \((\hat{\lambda}_{\text{aML}}, \hat{\sigma}^2_{\text{aML}})\) as the maximizer of \( l_a(\lambda, \sigma^2) \). Also, letting \( \hat{\sigma}^2_{\text{aML}}(\lambda) \) be the adjusted QMLE of \( \sigma^2 \) for given \( \lambda \), we define the adjusted likelihood for \( \lambda \) only as \( l_a(\lambda) := l_a(\lambda, \hat{\sigma}^2_{\text{aML}}(\lambda)) \). See Yu et al. (2015) for details on these constructions. By standard arguments (available for instance in Rahman and King, 1997), \( l_a(\lambda, \sigma^2) \) corresponds to the density of the maximal invariant \( C_Xy \) under \( G^1_X \), for any network autoregression model violating Condition 1 and for any network error model.

Appendix C  Proofs

Lemma C.1. The network autoregression (2.1) and the network error model (3.1) imply the same profile quasi log-likelihood function for \((\lambda, \sigma^2)\) if and only if \( \text{rank}(X, WX) = k \).

Proof of Lemma C.1. On concentrating the nuisance parameter \( \beta \) out of the likelihood (B.1), the profile quasi log-likelihood for \((\lambda, \sigma^2)\) in a network autoregression is, up to an additive constant,

\[
l(\lambda, \sigma^2) := l(\hat{\beta}_{\text{ML}}(\lambda), \sigma^2, \lambda) = -\frac{n}{2} \log(\sigma^2) + \log|\text{det}(S(\lambda))| - \frac{1}{2\sigma^2} y' S'(\lambda) M_X S(\lambda) y. \quad (C.1)
\]

Similarly, the profile quasi log-likelihood function for \((\lambda, \sigma^2)\) in a network error model, based again on the assumption \( \varepsilon \sim N(0, I_n) \), is

\[
l(\lambda, \sigma^2) := -\frac{n}{2} \log(\sigma^2) + \log|\text{det}(S(\lambda))| - \frac{1}{2\sigma^2} y' S'(\lambda) M_{S(\lambda)X} S(\lambda) y. \quad (C.2)
\]

The two log-likelihood functions are the same if and only if \( M_{S(\lambda)X} = M_X \) for any \( \lambda \) such that \( S(\lambda) \) is invertible. But, for any \( \lambda \) such that \( S(\lambda) \) is invertible, the condition \( M_{S(\lambda)X} = M_X \) is equivalent to \( \text{col}(S(\lambda)X) = \text{col}(X) \), and hence to \( \text{col}(WX) \subseteq \text{col}(X) \), which in turn is the same as \( \text{rank}(X, WX) = k \). \( \square \)
Proof of Proposition 3.1. The parameter \((\lambda, \beta)\) is identified on \(\Lambda_n \times \mathbb{R}^k\) from \(E(y) = S^{-1}(\lambda)X\beta\) if \(S^{-1}(\tilde{\lambda})X\tilde{\beta} = S^{-1}(\hat{\lambda})X\hat{\beta}\) implies \((\tilde{\lambda}, \tilde{\beta}) = (\hat{\lambda}, \hat{\beta})\) for any two values \((\tilde{\lambda}, \tilde{\beta}), (\hat{\lambda}, \hat{\beta})\) of \((\lambda, \beta)\) in \(\Lambda_n \times \mathbb{R}^k\). One immediately has that \(S^{-1}(\tilde{\lambda})X\tilde{\beta} = S^{-1}(\hat{\lambda})X\hat{\beta}\) if and only if

\[
X(\tilde{\beta} - \hat{\beta}) + WX(X\tilde{\beta} - X\hat{\beta}) = 0. \tag{C.3}
\]

We analyze separately three (exhaustive) cases, depending on the rank of the \(n \times 2k\) matrix \((X, WX)\). Recall that \(X\) is assumed to be of full column rank.

(a) \(\text{rank}(X, WX) = 2k\). In this case equation (C.3) is equivalent to \(\tilde{\beta} = \hat{\beta}\) and \(\tilde{\lambda} = \hat{\lambda}\), from which \((\tilde{\lambda}, \tilde{\beta}) = (\hat{\lambda}, \hat{\beta})\) if and only if \(\tilde{\beta} = \hat{\beta} \neq 0\). That is, \((\lambda, \beta)\) is identified on \(\Lambda_n \times \mathbb{R}^k\setminus\{0\}\) from \(E(y)\).

(b) \(k < \text{rank}(X, WX) < 2k\). Partition \(X\) as \((X_1, X_2)\) where \(X_1\) is \(n \times k_1\) and \(X_2\) is \(n \times k_2\), with \(0 < k_1 < k\). The case \(k < \text{rank}(X, WX) < 2k\) may be characterized by assuming \(\text{rank}(X, WX_1) = k + k_1\) and \(WX_2 = XB + WX_1C\), for some \(k \times k_2\) matrix \(B\) and some \(k_1 \times k_2\) matrix \(C\), so that \(\text{rank}(X, WX) = k + k_1\). Replacing \(WX\) with \((WX_1, XB + WX_1C)\) in (C.3), and letting \((\beta_1', \beta_2')'\) be the partition of \(\beta'\) conformable with that of \(X\), we obtain

\[
X(\tilde{\beta} - \hat{\beta} + B(\tilde{\lambda}\tilde{\beta} - \hat{\lambda}\hat{\beta})) + WX_1(\tilde{\lambda}\tilde{\beta}_1 - \hat{\lambda}\hat{\beta}_1 + C(\tilde{\lambda}\tilde{\beta}_2 - \hat{\lambda}\hat{\beta}_2)) = 0,
\]

which is satisfied if and only if \(\tilde{\beta} - \hat{\beta} + B(\tilde{\lambda}\tilde{\beta}_2 - \hat{\lambda}\hat{\beta}_2) = 0\) and \(\tilde{\lambda}\tilde{\beta}_1 - \hat{\lambda}\hat{\beta}_1 + C(\tilde{\lambda}\tilde{\beta}_2 - \hat{\lambda}\hat{\beta}_2) = 0\). As a linear system in the unknowns \(\tilde{\lambda}\) and \(\tilde{\beta}\), these two equations are

\[
M(\tilde{\lambda}, \tilde{\beta}) \begin{pmatrix} \tilde{\lambda} \\ \tilde{\beta} \end{pmatrix} = \begin{pmatrix} \tilde{\beta} \\ 0_{k_1} \end{pmatrix}, \tag{C.4}
\]

where the matrix

\[
M(\tilde{\lambda}, \tilde{\beta}) := \begin{pmatrix} B\tilde{\beta}_2 & I_k - \tilde{\lambda}(0_{k, k_1}, B) \\ \tilde{\beta}_1 + C\tilde{\beta}_2 & -\tilde{\lambda}(I_{k_1}, C) \end{pmatrix}
\]

is of dimension \((k + k_1) \times (1 + k)\). Now, identification of \((\lambda, \beta)\) from \(E(y)\) is equivalent to \((\tilde{\lambda}, \tilde{\beta})\) being the unique solution to system (C.4), and this occurs if and only if \(\text{rank}(M(\tilde{\lambda}, \tilde{\beta})) = 1 + k\), or, equivalently, \(\det(M(\tilde{\lambda}, \tilde{\beta})M(\tilde{\lambda}, \tilde{\beta})) \neq 0\). But \(\det(M(\tilde{\lambda}, \tilde{\beta})M(\tilde{\lambda}, \tilde{\beta}))\) is a polynomial in \((\tilde{\lambda}, \tilde{\beta})\) and hence the set of its zeros is either
the whole \( \mathbb{R}^{k+1} \) or has zero measure with respect to \( \mu_{\mathbb{R}^{k+1}} \). The former case is easily ruled out (e.g., \( M(\tilde{\lambda}, \tilde{\beta}) \) has rank \( k+1 \) for \( (\tilde{\lambda}, \tilde{\beta}) = (0, (1'_{k_1}, 0'_{k_2})) \)), which means that \((\lambda, \beta)\) is generically identified from \( E(y) \).

(c) \( \text{rank}(X, WX) = k \). This happens if and only if there is a \( k \times k \) matrix \( A \) such that \( WX = XA \). In that case, equation (C.3) becomes \( X(\bar{\beta} - \tilde{\beta}) + A(\bar{\lambda} \beta - \tilde{\lambda} \beta) = 0 \), which, since \( \text{rank}(X) = k \), is equivalent to \( \bar{\beta} - \tilde{\beta} + A(\bar{\lambda} \beta - \tilde{\lambda} \beta) = 0 \). Rewrite the last equality as \( (I_k - \bar{\lambda} A) \bar{\beta} - (I_k - \tilde{\lambda} A) \tilde{\beta} = 0 \). Since the eigenvalues of \( A \) are eigenvalues of \( W \), \( I_k - \lambda A \) is invertible for any \( \lambda \in \Lambda_u \), and therefore \( \bar{\beta} = (I_k - \bar{\lambda} A)^{-1}(I_k - \tilde{\lambda} A) \tilde{\beta} \).

This shows that for any \( (\tilde{\lambda}, \tilde{\beta}) \in \Lambda_u \times \mathbb{R}^k \), it is possible to find \( (\bar{\lambda}, \bar{\beta}) \) such that \( S^{-1}(\bar{\lambda})X\bar{\beta} = S^{-1}(\tilde{\lambda})X\tilde{\beta} \).

Summarizing, \((\lambda, \beta)\) is generically identified from \( E(y) \), and hence generically identified on \( \Lambda_u \times \mathbb{R}^k \) in cases (a) and (b), and not identified from \( E(y) \) on \( \Lambda_u \times \mathbb{R}^k \) in case (c).

**Proof of Proposition 3.2.** This proof is similar to that of Lemma 4.2 in Preinerstorfer and Pötscher (2017). Under the assumption that \( \text{var}(\varepsilon) = I_n \), \( \text{var}(y) = \sigma^2(S'(\lambda)S(\lambda))^{-1} \). We want to establish that, if \( \tilde{\sigma}^2 S'(\tilde{\lambda})S(\tilde{\lambda}) = \sigma^2 S'(\bar{\lambda})S(\bar{\lambda}) \) for any two parameter values \((\tilde{\lambda}, \tilde{\sigma}^2), (\bar{\lambda}, \bar{\sigma}^2) \in \Lambda \times (0, \infty)\), then \((\bar{\lambda}, \bar{\sigma}^2) = (\tilde{\lambda}, \tilde{\sigma}^2)\). The maintained assumption that \( W \) has at least one negative eigenvalue and at least one positive eigenvalue guarantees the existence of a nonzero vector \( f \in \text{null}(W - I_n) \) and a nonzero vector \( g \in \text{null}(W - \omega_{\min} I_n) \). Multiplying both sides of the equality \( \tilde{\sigma}^2 S'(\tilde{\lambda})S(\tilde{\lambda}) = \sigma^2 S'(\bar{\lambda})S(\bar{\lambda}) \) by \( f' \) on the left and \( f \) on the right gives \( \tilde{\sigma}^2(1 - \tilde{\lambda})^2 f'f = \sigma^2(1 - \bar{\lambda})^2 f'f \). Since \( 1 - \lambda > 0 \) for any \( \lambda \in \Lambda \), and \( f'f \neq 0 \), the last equality is equivalent to \( \tilde{\sigma}/\sigma = (1 - \tilde{\lambda})/(1 - \bar{\lambda}) \). Repeating with \( g \) in place of \( f \) gives \( \tilde{\sigma}/\bar{\sigma} = (1 - \tilde{\lambda} \omega_{\min})/(1 - \bar{\lambda} \omega_{\min}) \). Thus, we must have \((1 - \tilde{\lambda} \omega_{\min})/(1 - \tilde{\lambda}) = (1 - \bar{\lambda} \omega_{\min})/(1 - \bar{\lambda}) \). Since the function \( \lambda \mapsto (1 - \lambda \omega_{\min})/(1 - \lambda) \) is strictly increasing on \( \Lambda \), we have \( \tilde{\lambda} = \bar{\lambda} \), and hence \( \tilde{\sigma} = \bar{\sigma} \).

**Proof of Lemma 4.1.** For any \( \lambda \) such that \( S(\lambda) \) is nonsingular and under Assumption 1, it is clear from the reduced form \( y = S^{-1}(\lambda)X\beta + \sigma S^{-1}(\lambda)\varepsilon \) that a network autoregression is invariant under \( \mathcal{G}_X \) if and only if \( \text{col}(S^{-1}(\lambda)X) = \text{col}(X) \), or, which is the same, \( \text{col}(S(\lambda)X) = \text{col}(X) \). As noted in the proof of Lemma C.1, the condition \( \text{col}(S(\lambda)X) = \text{col}(X) \) for any \( \lambda \) such that \( S(\lambda) \) is invertible is equivalent to \( \text{rank}(X, WX) = k \).

**Proof of Proposition 4.2.** Since \( C_Z Z = 0 \) for any full column rank matrix \( Z \), it follows that \( l(\lambda, \beta, \sigma^2; C_X y) \) does not depend on \( \beta \), and \( l(\lambda, \beta, \sigma^2; C_{X'} y) \) does not depend
on the component of $\beta$ associated to $X_\omega$. The more general statement (i) follows from Theorem 1. Let us move to part (ii). For any $\lambda$ such that $\text{rank}(S(\lambda)) = n$, and for any $y \notin \text{null}(M_X S(\lambda))$, the profile log-likelihood $l(\lambda; y)$ for a network autoregression is given by equation (B.2). Note that equation (B.2) holds a.s. for any fixed $\lambda$ such that $\text{rank}(S(\lambda)) = n$, because $\text{null}(M_X S(\lambda))$ is a $\mu_{R^n}$-null set when $\text{rank}(S(\lambda)) = n$ (since $k < n$). If Condition 1 is violated for an eigenvalue $\omega$ of $W$, then $M_X(\omega I_n - W) = 0$ and hence $M_X S(\lambda) = (1 - \lambda \omega) M_X$, which substituted into (B.2) gives

$$l(\lambda; y) = \log|\text{det}(S(\lambda))| - n \log|1 - \lambda \omega| - \frac{n}{2} \log(y'M_X y), \quad (C.5)$$

for any $y \notin \text{col}(X)$. Since a violation of Condition 1 implies $\text{rank}(X, W X) = k$, equation (C.5) also applies to a network error model, by Lemma C.1. Part (ii) of the proposition follows on noting that the terms in (C.5) that contain $\lambda$ do not contain $y$.

\[\square\]

**References**

Blume, L. E., Brock, W. A., Durlauf, S. N., Ioannides, Y. M., 2011. Identification of social interactions. Vol. 1 of Handbook of Social Economics. North-Holland, pp. 853–964.

Blume, L. E., Brock, W. A., Durlauf, S. N., Jayaraman, R., 2015. Linear social interactions models. Journal of Political Economy 123 (2), 444–496.

Bramoullé, Y., Djebarri, H., Fortin, B., 2009. Identification of peer effects through social networks. Journal of Econometrics 150 (1), 41–55.

Chamberlain, G., Moreira, M. J., 2009. Decision theory applied to a linear panel data model. Econometrica 77 (1), 107–133.

Chesher, A., Rosen, A. M., 2017. Generalized instrumental variable models. Econometrica 85 (3), 959–989.

Cressie, N., 1993. Statistics for Spatial Data. Wiley, New York.

Davezies, L., D’Haultfoeuille, X., Fougére, D., 2009. Identification of peer effects using group size variation. The Econometrics Journal 12 (3), 397–413.

de Paula, Á., 2017. Econometrics of Network Models. Vol. 1 of Econometric Society Monographs. Cambridge University Press, pp. 268–323.

de Paula, Á., Rasul, I., Souza, P. C., 2020. Identifying network ties from panel data: Theory and an application to tax competition. Working paper.

Drton, M., Foygel, R., Sullivant, S., 2011. Global identifiability of linear structural equation models. Ann. Statist. 39 (2), 865–886.
Graham, B. S., 2008. Identifying social interactions through conditional variance restrictions. Econometrica 76 (3), 643–660.

Gupta, A., 2019. Estimation of spatial autoregressions with stochastic weight matrices. Econometric Theory 35 (2), 417–463.

Kariya, T., 1980. Locally robust tests for serial correlation in least squares regression. The Annals of Statistics 8 (5), 1065–1070.

Kelejian, H. H., Prucha, I. R., 1998. A generalized spatial two-stage least squares procedure for estimating a spatial autoregressive model with autoregressive disturbances. Journal of Real Estate Finance and Economics 17 (1), 99–121.

Kelejian, H. H., Prucha, I. R., Yuzefovich, Y., 2006. Estimation problems in models with spatial weighting matrices which have blocks of equal elements. Journal of Regional Science 46 (3), 507–515.

Koopmans, T. C., Reiersol, O., 1950. The identification of structural characteristics. The Annals of Mathematical Statistics 21 (2), 165–181.

Kwok, H. H., 2019. Identification and estimation of linear social interaction models. Journal of Econometrics 210 (2), 434–458.

Lee, L.-F., 2003. Best spatial two-stage least squares estimators for a spatial autoregressive model with autoregressive disturbances. Econometric Reviews 22 (4), 307–335.

Lee, L.-F., 2004. Asymptotic distributions of quasi-maximum likelihood estimators for spatial autoregressive models. Econometrica 72 (6), 1899–1925.

Lee, L.-F., 2007. Identification and estimation of econometric models with group interactions, contextual factors and fixed effects. Journal of Econometrics 140 (2), 333–374.

Lee, L.-F., Liu, X., Lin, X., 2010. Specification and estimation of social interaction models with network structures. Econometrics Journal 13 (2), 145–176.

Lee, L.-F., Yu, J., 2010. Estimation of spatial autoregressive panel data models with fixed effects. Journal of Econometrics 154 (2), 165–185.

Lee, L.-F., Yu, J., 2016. Identification of spatial Durbin panel models. Journal of Applied Econometrics 31 (1), 133–162.

Lehmann, E. L., Romano, J. P., 2005. Testing Statistical Hypotheses, 3rd Edition. Springer Texts in Statistics. Springer, New York.

LeSage, J., Pace, R., 2009. Introduction to Spatial Econometrics. Chapman and Hall/CRC, New York.

Liu, X., 2017. Identification of peer effects via a root estimator. Economics Letters 156, 168–171.

Manski, C. F., 1993. Identification of endogenous social effects: The reflection problem. The Review of Economic Studies 60 (3), 531–542.

Martelloso, F., 2011. Non-testability of equal weights spatial dependence. Econometric Theory 27 (6), 1369–1375.
Matzkin, R. L., 2007. Nonparametric identification. In: Heckman, J., Leamer, E. (Eds.), Handbook of Econometrics. Vol. 6 of Handbook of Econometrics. Elsevier, Ch. 73.

Newey, W. K., McFadden, D., 1994. Large sample estimation and hypothesis testing. In: Engle, R. F., McFadden, D. L. (Eds.), Handbook of Econometrics. Vol. 4. Elsevier, Ch. 36, pp. 2111–2245.

Preinerstorfer, D., Pötscher, B. M., 2017. On the power of invariant tests for hypotheses on a covariance matrix. Econometric Theory 33 (1), 1–68.

Rahman, S., King, M. L., 1997. Marginal-likelihood score-based tests of regression disturbances in the presence of nuisance parameters. Journal of Econometrics 82 (1), 81–106.

Roberts, L. A., 1995. On the existence of moments of ratios of quadratic forms. Econometric Theory 11 (4), 750–774.

Robinson, P. M., Rossi, F., 2015. Refinements in maximum likelihood inference on spatial autocorrelation in panel data. Journal of Econometrics 189 (2), 447–456.

Rose, C. D., 2017. Identification of peer effects through social networks using variance restrictions. The Econometrics Journal 20 (3), S47–S60.

Rothenberg, T. J., 1971. Identification in parametric models. Econometrica 39 (3), 577–591.

Wasserman, S., Faust, K., 1994. Social Network Analysis: Methods and Applications. Structural Analysis in the Social Sciences. Cambridge University Press.

Whittle, P., 1954. On stationary processes in the plane. Biometrika 41 (3/4), 434–449.

Yu, D., Bai, P., Ding, C., 2015. Adjusted quasi-maximum likelihood estimator for mixed regressive, spatial autoregressive model and its small sample bias. Computational Statistics and Data Analysis 87, 116–135.