Rashba-induced spin electromagnetic fields in the strong sd coupling regime

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Abstract
Spin electromagnetic fields driven by the Rashba spin–orbit interaction, or Rashba-induced spin Berry’s phase, in ferromagnetic metals are theoretically studied using the Keldysh Green’s function method. Considering the limit of strong sd coupling without spin relaxation (adiabatic limit), the spin electric and magnetic fields are determined by calculating transport properties. The spin electromagnetic fields can be expressed in terms of a Rashba-induced effective vector potential, and thus they satisfy Maxwell’s equation. In contrast to the conventional spin Berry’s phase, the Rashba-induced one is linear in the gradient of magnetization profile, and thus can be extremely large even for slowly varying structures. We show that the Rashba-induced spin Berry’s phase exerts a Lorentz force on spin resulting in a giant spin Hall effect in magnetic thin films in the presence of magnetization structures. A Rashba-induced spin magnetic field would be useful to distinguish between topologically equivalent magnetic structures. We propose experimental setups where a Rashba-induced spin magnetic field is identified.

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1. Introduction

The mathematical structure of the electromagnetism emerges due to $U(1)$ gauge symmetry. Such a structure is not unique to the electromagnetism in the vacuum. In solids, several possibilities of $U(1)$ gauge symmetry are known to emerge. A simple example is a metallic magnet. A magnet is an ensemble of many localized spins (magnetization), which are governed quantum mechanically by a non-commutative algebra of $SU(2)$. The conduction electron’s spin is coupled to the localized spin via an sd exchange interaction, whose Hamiltonian in the field-representation reads

$$H_{sd} = -\Delta_{sd} \int d^3 r n \cdot (c^\dagger \sigma c),$$

where $\Delta_{sd}$ is the strength of the sd exchange interaction, $n$ represents the localized spin direction, $c$ and $c^\dagger$ are annihilation and creation operators of the conduction electron respectively and $\sigma$ is a vector of Pauli matrices. This sd exchange interaction specifies the stable direction of the conduction electron spin to be along $n(r)$ at each site $r$. The behavior of the conduction electron spin is then analogous to that of a particle under a gravity force in mechanics. When a magnet has a non-uniform magnetization texture (non-uniform $n$), the ‘gravity force’ electron spin feels changes direction during the electron motion. Such a non-uniform gravity force is described by use of a covariant derivative including a gauge field. Reflecting an $SU(2)$ algebra of spin, the gauge field for the electron spin has three components and is called an $SU(2)$ gauge field. When the sd exchange interaction is strong, the minority spin component of the electron becomes irrelevant due to a large energy splitting, and only one of the three components of the $SU(2)$ gauge field coupling to the majority spin survives. A single component gauge field (a $U(1)$ gauge field) or spin electromagnetism thus emerges from this symmetry breaking [1, 2].

The emergence of $U(1)$ gauge field is understood also by considering a phase attached to the electron spin during an electron’s motion. When a conduction electron travels through a localized spin structure, its spin undergoes a rotation as a result of large sd exchange interaction. This rotation generates an electron’s phase factor, $e^{i\varphi}$, where $\varphi$ depends on the electron path, $C$, as $\varphi = \frac{e}{\hbar} \int_C d\mathbf{r} \cdot \mathbf{A}_s$, where electron charge is $-e$. A vector $\mathbf{A}_s$ appearing here is the $U(1)$ gauge field, which depends on the localized spin texture. Since $\int_C d\mathbf{r} \cdot \mathbf{A}_s$ for a closed path $C$ is written by use of an area integral of $\nabla \times \mathbf{A}_s$, there exists an effective magnetic field, $\mathbf{B}_s \equiv \nabla \times \mathbf{A}_s$. When spin texture is dynamic, voltage $V = -\frac{\hbar}{e} \dot{\varphi}$ arises, resulting in an effective electric field, $\mathbf{E}_s \equiv -\dot{\mathbf{A}}_s$. These effective fields satisfy Faraday’s induction law, $\nabla \times \mathbf{E}_s + \dot{\mathbf{B}}_s = 0$, by definition. The phase here arises from the spin rotation adiabatically following the localized spin direction, and the effective magnetic field, $\mathbf{B}_s$, representing the curvature is in this sense called the spin Berry’s phase [3].

By definition, spin electric field drives spin up and down conduction electrons in opposite directions, inducing a spin current, $j_s$ (figure 1). In ferromagnetic metals, driven spin current is usually associated with a charge current, given by $j = j_s/P$, where $P$ is a parameter representing spin polarization of carrier. Therefore, the spin electric field or spin motive force can be detected as a voltage, as was observed in the case of motion of a domain wall, vortex and skyrmions [4–6]. In the same manner as spin electric field, the spin magnetic field, conventionally called the spin Berry’s phase, exerts a spin-dependent Lorentz force on the conduction electrons, inducing spin Hall effect, and this spin Hall effect is detected as an anomalous Hall effect. To understand the whole structure of the emergent spin electromagnetic field is of essential importance in spintronics.
Figure 1. Schematic figure showing roles of conventional electromagnetic fields, (a) $E$ and $B$, and spin electromagnetic fields, (b) $E_s$ and $B_s$. While $E$ and $B$ drive conduction electrons having two different spins in the same direction, $E_s$ and $B_s$ drive the two spins in opposite directions, inducing spin current and spin Hall effect, respectively.

The motive force on the electron’s spin was theoretically studied in 1986 by Berger, where a voltage generated by a canting of wall plane of a driven domain wall was discussed and an analogy with the Josephson effect in a superconducting junction was pointed out [7]. The emergence of effective electromagnetism coupling to electron’s spin was pointed out by use of a gauge field argument by Volovik in 1987 [1]. Stern discussed the motive force in the context of spin Berry’s phase and the Aharonov–Bohm effect in a ring, and discussed similarity to Faraday’s law [3]. Spin motive force was rederived in [8] in the case of domain wall motion. In [8], however, there are several statements which might mislead readers. For instance, although Barnes and Maekawa [8] claims that the spin electromagnetic field has not been argued from the viewpoint of Faraday’s induction law, it has been in fact clearly done 20 years earlier [1]. Besides, Faraday’s law for the spin motive force was argued in [8] to arise from energy conservation, which is not true; Faraday’s law is simply a result of $U(1)$ gauge symmetry or in other words arises automatically when electromagnetism is described by a vector potential [9] (see also [10]). The topological aspect of spin motive force was discussed in detail in [10]. Spin motive force from the viewpoint of reciprocal relation with current-induced effects was discussed by Saslow [11] and Tserkovnyak and Mecklenburg [12]. Recent theoretical works [8, 10–13] focused solely on spin electric field, which are accessible by studying longitudinal transport or force as a linear response with respect to $E_s$. In contrast, the magnetic counterpart, $B_s$, requires consideration of both $E_s$ and $E$ (or the second order in $E_s$), which makes the calculation more difficult. One of the main aims of this paper is to calculate $B_s$ based on a microscopic formalism.

When spin–orbit interaction is present, $SU(2)$ gauge symmetry is affected resulting in novel contributions to spin electromagnetic fields. The effect of spin–orbit interaction on the spin electric field has been theoretically studied in [13–19]. Duine discussed spin electric field including the effect of spin relaxation by use of non-adiabaticity parameter ($\beta$) [14–16]. The Hall current induced by a spin electric field in the presence of spin–orbit interaction was theoretically studied and an enhancement of the Hall angle was predicted by Shibata and Kohno [13, 17]. Of particular current interest is the effect of Rashba spin–orbit interaction arising from the breaking of inversion symmetry at surfaces and interfaces [20, 21]. Takeuchi et al investigated the weak sd coupling regime and found a spin electric field proportional to $\alpha_R \times (\hat{n} \times \dot{\hat{n}})$ [19, 22], where $\alpha_R$ is the Rashba electric field and $\hat{n}$ is a unit vector representing the local magnetization direction, while Kim et al [18] obtained a different form of $\alpha_R \times \dot{\hat{n}}$ in a strong sd coupling regime. It was shown later [23] that the former expression arises in the strong sd coupling limit when spin relaxation is included. The two contributions thus coexist in general. A unique feature of the Rashba-induced spin electric field is that it arises even from
spatially uniform magnetization, in contrast to the conventional spin electric field induced by inhomogeneous magnetization textures.

For deriving an expression for spin electromagnetic fields in the absence of spin relaxation, a gauge field argument based on a $U(1)$ gauge invariance is useful [1]. The expression can also be calculated directly by evaluating the force, $F$, acting on spin, which is proportional to the time derivative of the current density, $j$, i.e. by use of $F = \frac{\partial}{\partial t} \langle [H, j] \rangle$ ($n$ is electron density, $m$ is electron mass, $H$ is the total Hamiltonian and $\langle \rangle$ represents quantum average) [18, 23]. Spin electromagnetic field can be also derived using transport calculations [22]. In [22], the fields were indeed identified by calculating the induced electric current density and then comparing the result with a general expression resulting from Maxwell’s equation

$$j = \sigma_s E_s + \frac{1}{\mu_s} \nabla \times B_s - D_s \nabla \rho_s, \quad (2)$$

where $\sigma_s$ is spin conductivity, $\mu_s$ is spin magnetic permeability, $D_s$ is diffusion constant for spin and $\rho_s$ is spin density. This approach turned out to be highly useful in the weak sd coupling regime, where the adiabatic component of the spin gauge field cannot be defined. Here we carry out a calculation in a strong sd coupling limit for spatially slowly varying localized spin structures, satisfying $\lambda \gg \sqrt{\ell \hbar / k_F \Delta_d}$ [24], where $\lambda$ is the spatial length scale of the spin structure, $\ell$ is the electron elastic mean free path and $\epsilon_F$ and $k_F$ are Fermi energy and Fermi wavelength, respectively. The spin structure in this case is represented by use of a spin gauge field [24]. We shall show that the results are in agreement with those obtained in a weak coupling limit, but that spin relaxation effects found in [22] do not arise in the present model.

As noted in [19, 22], the information in the pumped current (2), however, is not enough to completely determine the two fields and effective spin electric permittivity and spin magnetic permeability, and additional information is necessary. Besides, it is not clear whether all of the contributions to the current expressed as rotation (the second term of (2)) are always interpreted as due to an effective magnetic field. This issue is answered by investigating the Hall effect; if the contribution is really induced by an effective magnetic field, the field should exert a Lorentz force on the electron spin and induce a spin Hall effect. In this paper, we determine a spin magnetic field uniquely by calculating the spin Hall effect. We shall show that the whole contribution of the pumped current that is written as a rotation is indeed to be identified as an effective magnetic field, at least in the present system of strong sd coupling limit with the Rashba interaction included perturbatively to first order. The spin Hall current thus is written as

$$j^\text{Hall} = \sigma_H (E \times B_s), \quad \text{(3)}$$

where $E$ is the applied (conventional) electric field and $\sigma_H$ is spin Hall conductivity. A different contribution to the Hall current was studied in the absence of the applied electric field and without the Rashba interaction in [13] with a result of $j^\text{Hall} = \sigma_H (E_s \times n)$, where $E_s$ here is the conventional spin electric field without the Rashba interaction. This result indicates that the conventional spin electric field drives the Hall current and that the localized spin, $n$, acts as an effective magnetic field via the sd exchange interaction. The fact that those two studies on different systems lead to the same formula as in the conventional charge electromagnetism, $j = \sigma_H (E \times B)$, indicates that the spin electromagnetic field indeed has the same mathematical structure as the charge one. Those results also provide a justification of spin electromagnetic field as an observable field, in the same manner as the charge electromagnetic field is physical.
The spin electromagnetic fields studied here are generalized spin Berry’s phases including the Rashba effects. Indeed, we shall show that the spin magnetic field is written as a rotation of a Rashba-induced effective vector potential, namely, that it represents a curvature. The effective vector potential (a U(1) gauge field) turns out to be $A_R = \frac{m}{\hbar} (\alpha_R \times \hat{n})$ to linear order in the Rashba interaction. In contrast to the conventional spin Berry’s phase, the Rashba-induced one is linear in the gradient of magnetization profile, and thus it would dominate in the case of slowly varying magnetization. We show that the spin electric and magnetic fields are estimated to be extremely strong up to several kV m$^{-1}$ and kT, respectively, for a strong Rashba interaction induced at surfaces [20] when the frequency and the length scale of magnetization profile are 1 GHz and 1 nm, respectively. Since the spin magnetic field is written in terms of a vector potential, there is no monopole term in the studied system, in contrast to what was observed in [22] in the weak sd coupling regime. In the light of our results and those in [22], spin relaxation seems to be essential for an emergence of monopole, as claimed in [22]. In fact, as we will show below, the spin electric field in the absence of spin relaxation is proportional to $E_s \propto \alpha_R \times \hat{n}$, where $\alpha_R$ is the Rashba field, in agreement with the result of [18]. Its rotation, $\nabla \times E_s$, is thus written as a time derivative of $\nabla \times (\alpha_R \times \hat{n})$. We will show in this paper that it is in fact written totally as a time derivative of the spin magnetic field, $B_s \propto \nabla \times (\alpha_R \times \hat{n})$, i.e. $\nabla \times E_s = -B_s$. The two fields satisfy thus the conventional Faraday’s induction law and there is no monopole term. In contrast, the spin electric field in the presence of spin relaxation is $E_s \propto \alpha_R \times (\hat{n} \times \hat{n})$ [22, 23]. This cannot be written as a time derivative of any local quantity. It therefore follows that $\nabla \times E_s + B_s = -j_m$ is non-vanishing for any local function $B_s$. Since the right-hand side of the equation, $j_m$, is monopole current [9], a monopole current emerges when spin relaxation is taken into account. The monopole was called a spin damping monopole [22]. The emergence of a monopole from spin relaxation, i.e. a monopole distinct from a topological (hedgehog) monopole discussed by Volovik [1], suggests intriguing roles of spin–orbit interaction and spin relaxation in spin-charge conversion. Effects of spin–orbit interaction and spin relaxation on $B_s$ have not been studied so far in the strong sd coupling regime, and this paper is intended to present a general microscopic scheme to calculate $B_s$ in the strong sd coupling regime taking account of spin–orbit interaction. The present analysis is limited to the case of the Rashba spin–orbit interaction without spin relaxation, and a study including both the Rashba interaction and spin relaxation will be carried out in a future publication.

2. Calculation of electric current

In this section, we derive an expression for spin electromagnetic fields by calculating an electric current induced by magnetization dynamics and the Rashba interaction in a metallic ferromagnet. The Lagrangian of the system in the second-quantized representation is

$$L = \int d^3r c^\dagger \left[ i \hbar \frac{\partial}{\partial t} + \left( \frac{\hbar^2}{2m} \nabla^2 + \epsilon_F \right) + \Delta_{sd} (\hat{n} \cdot \sigma) - \frac{i}{2} \alpha_R \cdot (\nabla \times \sigma) - v_i \right] c,$$

where $\alpha_R$ represents the strength and the direction of the Rashba interaction, $c^\dagger \nabla c = c^\dagger (\nabla c) - (\nabla c^\dagger) c$ and $v_i$ is the random potential caused by spin-independent impurities, which give rise to a finite elastic lifetime, $\tau$, in the Green functions. We consider the limit where the sd exchange interaction strength, $\Delta_{sd}$, is large, and performs a local unitary transformation (rotation in spin space) to diagonalize the exchange interaction term. The definition of the
unitary transformation is by $a \equiv Uc$, where $a$ is the annihilation operator in a rotated space, we choose the unitary matrix $U$ ($U^\dagger = U^{-1}$) to satisfy $U^\dagger (\mathbf{n} \cdot \mathbf{\sigma}) U = \sigma_z$. The result is $U \equiv \mathbf{m} \cdot \mathbf{\sigma}$, where $\mathbf{m} \equiv (\sin \theta/2 \cos \phi, \sin \theta/2 \sin \phi, \cos \theta/2)$ ($\theta$ and $\phi$ are polar coordinates of $\mathbf{n}$) [24]. After this transformation, (4) reads

$$\mathcal{L} = \int d^3r \left\{ i\hbar a^\dagger \partial_r a - \hbar a^\dagger A_{s,i} a + a^\dagger \left( \frac{\hbar^2}{2m} \nabla^2 + \epsilon_F \right) a + i\frac{\hbar}{2m} A_{s,j}^\ell (a^\dagger \tilde{\nabla}^n \sigma_i a) - \frac{\hbar}{2m} a^\dagger A_{s,j} a + \Delta_{sd} a^\dagger a - \frac{i}{2} \alpha_{R,j} \epsilon_{jkl} R_{kn} \left( (a^\dagger \sigma_n \nabla_k a) + 2i A_{s,k}^\ell a^\dagger a \right) - v_i a^\dagger a \right\},$$

where $A_{s,kl} \equiv A_{s,\mu}^\ell \sigma_{\mu} \equiv [\mathbf{m} \times (\partial_{\mu} \mathbf{m})]_i \sigma_{\ell}$, is the spin $SU(2)$ gauge field and $R_{kn} \equiv 2m_{\ell} m_{\mu} - \delta_{kn}$ is a rotation matrix element. Summation is assumed for repeated indices ($A_{s,\mu}^\ell \sigma_{\ell} \equiv \sum_{\ell=x,y,z} A_{s,\mu}^\ell \sigma_{\ell}$).

For estimating the effective field, we calculate the electric current induced by the Rashba interaction and compare it to (2), following the approach in [22]. The electric current density written in terms of $a$ and $a^\dagger$ is

$$j_i = \frac{i e \hbar}{2m} \left\{ a^\dagger \tilde{\nabla}^n \sigma_i a \right\} - \frac{e \hbar}{m} A_{s,i}^\ell \sigma_{\ell} \langle a^\dagger \sigma_{\ell} a \rangle - \frac{e}{\hbar} \epsilon_{ijk} R_{k\ell} \sigma_i \langle a^\dagger \sigma_{\ell} a \rangle,$$

where $\langle \rangle$ denotes the expectation value in the ground state. We calculate (6) to first order in the Rashba interaction. (Higher-order contributions would also be accessible analytically, but we here focus on the simplest and essential contribution arising at the linear order.) Generally, electric current pumped by dynamic spins is a sum of local terms and diffusion terms as in (2) [25, 26], but here we study the local terms only, since they represent the local effect of the effective fields. The leading contributions of the local electric current density are diagrammatically depicted in figure 2. The contribution represented by the first two diagrams in figure 2 reads

$$j_i^{1,2}(r, t) = -\frac{i e \hbar^2}{m} \alpha_{R,\ell} \epsilon_{\ell mn} \sum_{k, p, \omega, \Omega} e^{-i pr + i \Omega t} R_{n\alpha}(p, \Omega) \times \text{tr} \left[ k_i k_m g_{k-\xi, \omega}^{-\frac{i}{2}} \sigma_{\alpha} g_{k+\xi, \omega+\frac{i}{2}} + \delta_{im} \frac{m}{n^2} \sigma_{\alpha} g_{k, \omega} \right]^<,$$

where $g_{k, \omega}$ is the counter ordered Green’s function of a conduction electron with wave vector $\mathbf{k}$ and angular frequency $\omega$, and $^<$ represents the lesser component [27]. The external wave number and frequency carried by the magnetization structure are denoted by $\mathbf{q}$ and $\Omega$, respectively. It includes the elastic lifetime due to the impurities, $\tau$, and is a $2 \times 2$ matrix in spin space.
The contribution of the remaining diagrams in figure 2 reads

\[ j_i^{3-6}(r, t) = -\frac{ie^2}{m} a_R i \epsilon_{lmn} \sum_{k, q, p, \omega, \Omega, \bar{\Omega}} e^{-i(q+p)r+i(\Omega+\bar{\Omega})t} R_{mn}(p, \bar{\Omega}) \]

\[ \times \text{tr} \left[ k_i ( \frac{q}{2} ) m \hat{h} J_p ( \frac{p}{2} ) A_3^{\mu}(q, \Omega) \right] \]

\[ \times g_{k-\frac{q}{2}, p-\frac{q}{2}, \omega+\frac{q}{2}} - g_{k+\frac{q}{2}, p+\frac{q}{2}, \omega-\frac{q}{2}} + \text{c.c.} \]

\[ + j_i A_{s,m}(q, \Omega) \sigma \right] \]

\[ + j_i A_{s,m}(q, \Omega) \sigma \right] \]

\[ + \frac{\delta_{lm}}{\hbar^2} \hat{h} J_p (k) A^{\mu}(q, \Omega) \sigma \]

\[ \{ \Omega_1 \} \sigma \]

where \( J_i(k) \equiv 1, J_i(k) \equiv \frac{\hbar}{m} k_i \) (\( p \) and \( \bar{\Omega} \) are external wave number and frequency, respectively).

The total local electric current density is (see appendix A)

\[ j = (\xi_1 + \eta) \left[ \nabla \times \left( \alpha_R \times n \right) \right] + \xi_2 (\alpha_R \times \dot{n}), \]

\[ (9) \]

where \( \xi_1 = \frac{e^2}{2m} (\sum_{j} \sigma \nu_{\sigma}) \), \( \xi_2 = -\frac{2e}{3h} (\sum_{j} \sigma \nu_{F_{\sigma}} \nu_{\sigma}) \), \( \eta = \frac{e^2}{60m \Delta_\perp} (\sum_{j} \sigma (\epsilon_{F_{\sigma}} \nu_{\sigma} - 5 \epsilon_{F_{\sigma}} \epsilon_{F_{-\sigma}} \nu_{\sigma}) \sigma = \pm \) is spin index, \( \nu_{\sigma}, \nu_{F_{\sigma}} \) and \( \tau_{\sigma} \) are spin-dependent density of states, Fermi energy and lifetime, respectively.

Electric current driven by effective electromagnetic fields is generally written in the diffusive regime as (2). Comparing our result, (9), to (2), we see that

\[ \sigma_s E_s = \xi_2 (\alpha_R \times \dot{n}), \]

\[ B_s \]

\[ \mu_s \]

\[ (10) \]

where \( \xi_2 = (\xi_1 + \eta) \left[ \nabla \times (\alpha_R \times n) \right] \].

We know that spin conductivity is given by

\[ \sigma_s = \sum_{\sigma = \pm} \frac{e^2 \hbar^2}{3m^2 k_{F_{\sigma}} \nu_{\sigma} \tau_{\sigma}} \]

\[ (11) \]

and thus the spin electric field reads

\[ E_s = -\frac{m}{e\hbar} \alpha_R \times n. \]

\[ (12) \]

This result agrees with the result of the direct estimate of spin motive force \([18, 23]\). In contrast, to identify spin magnetic field from (10), we need additional information on the permeability. This is accomplished by analyzing the spin Hall effect, which is carried out in the next section.

3. The spin Hall effect induced by spin magnetic field

To determine the effective spin magnetic field, (10) is not sufficient. In this section, we consider the spin Hall effect and calculate the Hall electric current driven by the spin magnetic field when an electric field, \( E \), is applied to uniquely determine the spin magnetic field.

The Lagrangian has now the following additional terms coming from an applied vector potential, \( A \), related to the electric field by \( E = -A \),

\[ \delta \mathcal{L} = \int d^3r \left[ \frac{ie^2}{2m} A_j (a^i \nabla \cdot a) - \frac{e^2}{2m} (a^i A_j^2) - \frac{e^2}{m} (a_i A_j A_{k,j} a) + \frac{e}{\hbar} \alpha_{k,j} \epsilon_{jk\ell} R_{\ell mn} A_k (a^\ell \sigma_n a) \right]. \]

\[ (13) \]
expressed by a standard expression of where

\[ j_i = -\frac{e^2}{m} A_i \langle a^\dagger a \rangle. \]  

(14)

We consider the case where the applied electric field is spatially homogeneous, and calculate the Hall current induced by \( A \) and spin texture (\( A_s \)). The leading contribution to the Hall current density, \( j_i^{\text{Hall}} \), described by the Feynmann diagrams in figure 3, reads

\[
  j_i^{\text{Hall}}(r, t) = -\frac{ie^2}{m} \sum_{k, p, \omega, \Omega} e^{-ipr+i\Omega t} \alpha_{R, j} \epsilon_{jk\ell} \\
  \times \text{tr} \left\{ \frac{\hbar^2}{m} k_i k_\ell A_m(\Omega) R_{\ell n}(p) \left[ \left( k - \frac{p}{2} \right)_m g_{k-\frac{p}{2}, \omega, \Omega} g_{k-\frac{p}{2}, \omega, \Omega} + \delta_{ij} k_m A_m(\Omega) R_{\ell n}(p) \right] \sigma_n g_{k, \omega, \Omega} \right\}.
\]

(15)

The Hall current is calculated as (appendix B)

\[
  j^{\text{Hall}} = -\sum_{\pm} (\pm) \frac{e \tau_\pm}{m} \sigma_{B, \pm} \left( E \times (\nabla \times (\alpha_R \times n)) \right),
\]

(16)

where \( \sigma_{B, \pm} \equiv \frac{2e^2}{5m} \epsilon_{\alpha R} v_\alpha \tau_\alpha \) is the spin-resolved Boltzmann conductivity. Thus the Hall current is expressed by a standard expression of

\[
  j^{\text{Hall}} = \sigma_H (E \times B_s),
\]

(17)

where \( \sigma_H \equiv -\sum_{\pm} (\pm) \frac{e \tau_\pm}{m} \sigma_{B, \pm} \) and

\[
  B_s \equiv \frac{m}{e\hbar} \nabla \times (\alpha_R \times n).
\]

(18)

In terms of the Hall electric field, (16) is written as

\[
  E^{\text{Hall}} = -\frac{1}{ne} j \times B_s,
\]

(19)
where $n$ is electron density, $j$ is the longitudinal electric current driven by an applied electric field. Equations (17) and (19) are our central results. They indicate that the field $B_s$ exerts a Lorentz force $F = -ev \times B_s$ for electron spin and that $B_s$ indeed plays a role as a magnetic field for conduction electrons.

### 4. Rashba-induced effective gauge field

In the preceding sections, we have demonstrated that the effective spin electromagnetic fields are determined from the transport properties. Our results, (12) and (18), indicate that the Rashba-induced effective field is written as

$$ E_s = -A_R, $$
$$ B_s = \nabla \times A_R, $$

where

$$ A_R \equiv \frac{m}{e\hbar} (\alpha_R \times n). $$

The spin electromagnetic fields emerge, therefore, from the standard $U(1)$ gauge theory, as was argued in [18]. This fact is not obvious, since the combination of the Rashba spin–orbit interaction and sd interaction does not necessarily lead to an emergence of $U(1)$ gauge symmetry. Nevertheless, a gauge field scenario holds as far as nonlinear effects of the Rashba interaction are neglected. In fact, the Rashba interaction has the same effect as an $SU(2)$ gauge field if nonlinear effects are neglected. This fact is easily seen from (5), which indicates that the Rashba interaction in a rotated frame is written as

$$ \int d^3 r \frac{i\hbar^2}{2m} A_R^j \cdot (a^j \sigma_i \nabla a), $$

where $A_{R,j} = A_{R,j}^\ell \sigma_\ell \equiv -\frac{m_i}{\hbar} \alpha_R, i_{ijl} R_{ijl} \sigma_\ell$. The kinetic term of (5) thus can be expressed using an $SU(2)$ gauge field defined as $A_R \equiv A_s + A_R$ as

$$ \int d^3 r \frac{\hbar^2}{2m} a^i (\nabla + iA_R^i) a + O((A_R^i)^2). $$

In the strong sd coupling limit, therefore, $A_R^i$ acts as an effective $U(1)$ gauge field and effective electromagnetic fields emerge $\tilde{E}_R = -\tilde{A}_R = E_B + E_s$ and $\tilde{B}_R = \nabla \times \tilde{A}_R = B_B + B_s$, where $E_s$ and $B_s$ are the Rashba contributions in (20) and $E_B \equiv -\tilde{A}_s^z$ and $B_B \equiv \nabla \times \tilde{A}_s^z$ are the Berry’s phase contributions discussed by Volovik [1]. It is generally expected that a novel $U(1)$ gauge field, different from the adiabatic gauge field, emerges when spin–orbit interaction is included, as far as the linear effects are concerned.

### 5. Experimental possibilities

The Rashba-induced spin electromagnetic field we have discussed is an extension of the spin Berry’s phase generalized to include the Rashba interaction. The fields are linear in the space gradient and in time. Thus they become dominant in slowly varying magnetization structures, since the conventional spin Berry’s phase is proportional to field gradients in the second order. Let us estimate the magnitude of the Rashba-induced spin electromagnetic field.
Figure 4. The schematic picture of two vortex structures which are topologically equivalent, i.e. having equal spin Berry’s phase. The magnetization at the center is pointing perpendicular to the plane. The Rashba-induced spin magnetic field vanishes for the structure (a), while it is finite for the structure (b), where the total flux of the spin magnetic field is \( \Phi = \frac{m}{\hbar} \alpha_R L \) (L is perimeter of a vortex).

Rashba interaction is large on the surface of heavy metals in particular when doped with Bi [20]. Choosing \( \alpha_R = 3 \text{ eV Å} \) [18], we obtain \( \frac{m \alpha_R}{\hbar} = 2.5 \times 10^{-6} \text{ V s m}^{-1} \). For a magnetization dynamics with angular frequency \( \omega \) of 1 GHz, we thus expect a large spin electric field of \( E_s = \frac{m \alpha_R}{\hbar} \omega = 2.5 \text{ kV m}^{-1} \). For a magnetization structure having a typical length scale \( \lambda \) of 1 nm, the spin magnetic field corresponds to an extremely high field of \( B_s = \frac{m \alpha_R}{\hbar} \lambda = 2.5 \text{ kT} \). The Rashba interaction is thus useful in creating extremely high effective spin electromagnetic fields.

When a magnetic structure flows, a spin electric field is induced. In the case of a flow without deformation, the magnetization vector depends on the time as \( \mathbf{n}(\mathbf{r}, t) = \mathbf{n}(\mathbf{r} - \mathbf{v} t) \), where \( \mathbf{v} \) is velocity of the flow. The spin electric field then reads

\[
E_s = \frac{m}{\hbar} (\alpha_R \times (\mathbf{v} \cdot \nabla)\mathbf{n}).
\] (24)

5.1. Vortices

A crucial difference between the conventional spin Berry’s phase and Rashba-induced one is that the former is of topological origin while the latter is not. The Rashba-induced field is therefore expected to be useful to discriminate topologically equivalent magnetic structures. An example is a magnetic vortex or skyrmion. Structures shown in figures 4(a) and (b) are topologically equivalent, since the structure (b) is obtained by shifting the in-plane angle \( \phi \) of magnetization by \( \frac{\pi}{2} \). They have, however, a different response to the Rashba-induced field. Choosing \( \alpha \) perpendicular to the plane, the Rashba-induced spin magnetic field measures the divergence of the magnetization structure, \( \nabla \cdot \mathbf{n} \), resulting in a vanishing value for the structure (a) without divergence. In contrast the spin magnetic field for the structure (b) is finite. The total flux for the vortex (b) is \( \Phi \equiv \int_S d\mathbf{S} \cdot \mathbf{B}_s = \int_C d\mathbf{r} \cdot \mathbf{A}_s \) where \( S \) is the area of the vortex and \( C \) is its perimeter. Using (21), the flux is \( \Phi = \frac{m}{\hbar} \alpha_R L = \frac{\hbar}{e} \frac{\alpha_R^2 \lambda}{2e^2} L \), where \( L \) is the length of the perimeter of a vortex.

When a vortex flows without deformation, the Rashba-induced spin electric field (24) vanishes since the average of \( \nabla \mathbf{n} \) inside a vortex vanishes if not deformed. This feature is a significant difference from the conventional Berry’s phase contribution (topological spin motive force), which has been used to detect skyrmion motions [6].
\[ \phi(a) (b) \]

\[
\begin{align*}
\mathbf{B}_s & = m \alpha \mathbf{e}_h \bar{h} \lambda_\phi \frac{\partial n_x}{\partial x} \\
\mathbf{B}_s & = m \alpha \mathbf{e}_h \bar{h} \lambda_\phi \sinh \left( \frac{x}{\lambda} \right) \cosh^2 \left( \frac{x}{\lambda} \right) \hat{z}.
\end{align*}
\]

\textit{Figure 5.} (a) Domain walls with in-plane easy axis and (b) perpendicular easy axis (out-of-plane Neel wall). The spin magnetic field is induced inside the wall proportional to \( \frac{\partial n_x}{\partial x} \), where \( n_x \) is the x component of magnetization. Curves are schematically displayed profiles of spin magnetic field induced by the wall.

5.2. Domain walls

Let us discuss the spin magnetic field for a domain wall in the \( xy \) plane with Rashba field in the \( z \) direction. The magnetization of the domain wall is changing in the \( x \) direction but is uniform in the \( y \) direction. We first consider an in-plane domain wall as in figure 5(a). The domain wall profile is \( \theta = \frac{\pi}{2}, \cos \phi = \tanh \frac{x}{\lambda} \), where \( \lambda \) is thickness of the wall. The spin magnetic field then reads

\[ 
\mathbf{B}_s = m \alpha \mathbf{e}_h \bar{h} \lambda_\phi \frac{\partial n_x}{\partial x} \hat{z} \quad (n_x = \sin \theta \cos \phi \text{ and } \hat{z} \text{ is the unit vector in the } z \text{ direction}).
\]

The spin magnetic field is therefore localized to the wall, and it corresponds to a high field of 250 T if \( \lambda = 10 \text{ nm} \). This localized strong field would be detected as a local spin Hall voltage in the \( y \) direction when current is applied in the \( x \) direction.

When the in-plane domain wall flows in the \( x \) direction with speed \( v_x \), a spin voltage in the \( y \) direction (defined as \( V_y \equiv \int dx \mathbf{E}_{s,y} \), with (24)) is induced;

\[ V_y = \frac{2m \alpha R}{e \hbar} v_x. \tag{25} \]

For \( \alpha_R = 3 \text{ eV Å} \) and \( v_x = 100 \text{ m s}^{-1} \), the voltage is 0.5 mV. This value is 1000 times larger than the conventional Berry’s phase contribution observed in a permalloy, 0.4 \( \mu \text{V} \) at 130 m s\(^{-1} \) [4]. Even for a system having a moderate value of \( \alpha_R = 3 \text{ meVÅ} \), therefore, the Rashba-induced signal is comparable to the conventional signal.

In the case of out-of-plane domain wall of Neel type (figure 5(b)), \( \cos \theta = \tanh \frac{x}{\lambda} \) with magnetization at the center of the wall pointing in the \( x \) direction, we have

\[ 
\mathbf{B}_s = -m \alpha \mathbf{e}_h \bar{h} \lambda_\phi \sinh \left( \frac{x}{\lambda} \right) \cosh^2 \left( \frac{x}{\lambda} \right) \hat{z}.
\]

The field produced by an out-of-plane wall changes sign at the wall center and has a large field gradient. For a flowing out-of-plane wall, the spin voltage, (24), vanishes.

5.3. Effect of gradient of Rashba field

So far we have calculated the spin magnetic fields assuming homogeneous \( \alpha_R \). Equation (20) indicates that it arises also when the Rashba field, \( \alpha_R \), has a gradient, which is expected at the surfaces of thin film [28–30]. The spin magnetic field then becomes

\[ 
\mathbf{B}_s = \frac{m}{e \hbar} (\mathbf{n} \cdot \nabla) \alpha_R - \mathbf{n} (\nabla \cdot \alpha_R). \tag{26} \]

We consider a thin film in the \( xy \) plane with \( \alpha_R = \alpha_R(z) \hat{z} \). When \( \mathbf{n} \) is within the \( xy \) plane, we obtain

\[ 
\mathbf{B}_s = -\frac{m}{e \hbar} (\nabla \cdot \alpha_R) \mathbf{n}. \]

Assuming that the Rashba interaction decays at distance \( d \) from the surface, we approximate \( \nabla \cdot \alpha_R \sim 2 \alpha_R/d \), resulting in a magnetic field of 2.5 kT if \( d = 1 \text{ nm} \).
6. Propagation of spin electromagnetic field

Since the spin electromagnetic field is described by Maxwell’s equation, there are plane-wave solutions describing propagation of spin electromagnetic waves. From our results, (10) and (18), we see that

\[ \frac{1}{\mu_s} = \frac{e\hbar}{m}(\xi_1 + \eta). \]  

(27)

Electric permittivity in a diffusive case is \(\epsilon_s = \sigma_s \tau\). The speed of the spin electromagnetic field is then given by

\[ c_s = \frac{1}{\sqrt{\epsilon_s \mu_s}}. \]  

(28)

It should be noted that the sign of \(\epsilon_s\) or \(\mu_s\) may be negative depending on the material. If the product of the two is positive, the spin electromagnetic wave propagates, while it does not if one of the two is negative.

Let us look into an example of a strong sd coupling limit, \(\nu_s = 0\) (i.e., \(\Delta_{sd} = \epsilon_F\)). In this limit, \(\frac{1}{\mu_s} = \frac{e^2\hbar^2}{60m^2\nu_s}(\frac{\Delta_{sd}^2}{\epsilon_F^2} + 5)\). In our model the energy band is parabolic with \(\epsilon_F^2/\Delta_{sd} = 2\). Approximating \(\frac{1}{\mu_s} \simeq \frac{3e^2\hbar^2}{20m^2\nu} \) and \(\epsilon_s \simeq \frac{e^2h^2}{3m^2k_F^2}\nu^2\nu\tau^2\) (\(\nu, k_F\) and \(\tau\) here are spin-averaged quantities), we obtain

\[ c_s = \frac{3}{2\sqrt{5}} \frac{1}{k_F \nu \tau}. \]  

(29)

For \(k_F^{-1} = 1.5\, \text{Å}\) and \(\tau = 1\, \text{fs}\), the spin electromagnetic wave propagates with a speed of \(c_s = 1 \times 10^5\, \text{m s}^{-1}\). This speed is that of a composite mode of spin texture and electron, and this propagation mode is a novel one, distinct from spin waves. Observation of the propagation speed of spin electric field is an interesting challenge for experimentalists.

7. Conclusions

We have derived expressions for spin electromagnetic fields induced by the Rashba spin–orbit interaction and magnetization structures in the strong sd coupling limit by calculating the pumped current and the spin Hall effect. Spin relaxation was not considered here. We found that the spin electromagnetic fields are completely described by an effective \(U(1)\) gauge field, \(A_R = \frac{m}{e\hbar}(\alpha_R \times n)\) at the linear order in the Rashba interaction strength. Thus a naive picture regarding the Rashba interaction as an effective gauge interaction is valid to linear order in the Rashba interaction strength. The spin electromagnetic field discussed here is a generalized spin Berry’s phase. In contrast to the spin Berry’s phase, the Rashba-induced one is linear in the gradient of the magnetization profile, and can generate extremely high electric and magnetic fields of the order of kV m\(^{-1}\) and kT, respectively, for a frequency of 1 GHz and for a structure of nanometer scale.

Since the spin electromagnetic fields have \(U(1)\) gauge invariance, there is no monopole in the strong sd coupling limit without spin relaxation, in contrast to what was observed in [22] in the weak sd coupling regime. The present result confirms the argument in [22] that spin relaxation is essential for the emergence of a monopole term. In fact, the spin electric field in the presence of spin relaxation, \(E_s \propto \alpha_R \times (n \times \dot{n})\), cannot be written by a time derivative of
any local quantity, resulting in non-vanishing $\nabla \times \mathbf{E}_s + \dot{\mathbf{B}}_s \equiv -j_m$ for any local function $\mathbf{B}_s$. To study the effect of spin relaxation in the present framework is an important future work.

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**Appendix A**

Here we show details of derivation of the total electric current, (9), from (7). The lesser components in (7) are given as

$$g^{<}_{k,\omega} = f_\omega (g^{a}_{k,\omega} - g^{r}_{k,\omega}),$$

(A.2)

where $g^{r}_{k,\omega}$ and $g^{a}_{k,\omega}$ are retarded and advanced Green’s function and $f_\omega = \frac{1}{\beta \epsilon + 1}$ is the Fermi distribution function ($\beta$ is the inverse temperature). Therefore $j^{1,2}$ is calculated as

$$j_{i}^{1,2}(r, t) = \sum_{p, \Omega} e^{-ip \cdot r + i\Omega t} \xi_{ij} \left( \alpha_{R} \times n(p, \Omega) \right)_j,$$

(A.3)

where

$$\xi_{ij} \equiv \frac{ie\hbar^2}{m} \sum_{k, \omega, \sigma} \sum_{\sigma=\pm} \left[ f_{\omega - \frac{\hbar}{2}} k_{ij} \left( g^{a}_{k - \frac{\hbar}{2}, \omega - \frac{\hbar}{2}, \sigma} - g^{r}_{k - \frac{\hbar}{2}, \omega - \frac{\hbar}{2}, \sigma} \right) g^{a}_{k + \frac{\hbar}{2}, \omega + \frac{\hbar}{2}, \sigma} + f_{\omega + \frac{\hbar}{2}} k_{ij} \left( g^{a}_{k + \frac{\hbar}{2}, \omega + \frac{\hbar}{2}, \sigma} - g^{r}_{k + \frac{\hbar}{2}, \omega + \frac{\hbar}{2}, \sigma} \right) g^{a}_{k - \frac{\hbar}{2}, \omega - \frac{\hbar}{2}, \sigma} + \delta_{ij} \frac{m}{\hbar^2} f_\omega \left( g^{a}_{k,\omega,\sigma} - g^{r}_{k,\omega,\sigma} \right) \right],$$

(A.4)

and $g^{r}_{k,\omega,\sigma} \equiv (\hbar \omega - \epsilon_{k,\sigma} + \frac{ie\hbar}{2\epsilon_{k,\sigma}})^{-1}$ ($\sigma = \pm$ is spin index). Expanding with respect to $p$ and $\Omega$, assuming a spatially slowly varying magnetization structure, we obtain

$$\xi_{ij} = \frac{ie\hbar^2}{m} \sum_{k, \omega} \sum_{\sigma} \left[ f_{\omega} k_{ij} \left( (g^{a}_{k,\omega,\sigma})^2 - (g^{r}_{k,\omega,\sigma})^2 \right) + \delta_{ij} \frac{m}{\hbar^2} (g^{a}_{k,\omega,\sigma} - g^{r}_{k,\omega,\sigma}) \right]$$

$$+ f_{\omega} k_{ij} \left[ \frac{\hbar^2}{4m} p^2 \left( (g^{a}_{k,\omega,\sigma})^3 - (g^{r}_{k,\omega,\sigma})^3 \right) + \left( \frac{\hbar^2}{2m} k \cdot p \right)^2 (g^{a}_{k,\omega,\sigma})^4 - (g^{r}_{k,\omega,\sigma})^4 \right]$$

$$- \frac{\Omega}{2} f_{\omega} k_{ij} (g^{a}_{k,\omega,\sigma} - g^{r}_{k,\omega,\sigma})^2 + O(p^3, \Omega^2).$$

(A.5)
Carrying out an integration by parts with respect to $k$ and $\omega$, (A.5) becomes
\[
\xi_{ij} = \frac{ie\hbar^2}{m} \sum_{k,\sigma} \sigma f'_\omega \left[ -\frac{1}{12\hbar} (\delta_{ij} p^2 - p_i p_j) (g^a_{k,\omega,\sigma} - g^i_{k,\omega,\sigma}) - \frac{\Omega}{6} \delta_{ij} k^2 (g^a_{k,\omega,\sigma} - g^i_{k,\omega,\sigma})^2 \right] + O(p^3, \Omega^2). \tag{A.6}
\]
We consider zero temperature, where $f'_\omega = -\delta(\omega)$. The summation over wave vectors in (A.6) is carried out by replacing the summation by an integral over energy as $\sum_k |g^r_{k,\sigma}|^2 = \int d\epsilon \frac{n(\epsilon)}{(\epsilon - \sigma \Delta_{sd})^2 + (\frac{h^2}{2m})^2}$, where $n(\epsilon)$ is density of states at energy $\epsilon (\equiv \frac{hk^2}{2m} - \epsilon_F)$. The integral is then evaluated neglecting terms of higher order in $(\epsilon_F\tau/\hbar)^{-1} (\ll 1)$ as $\sum_k |g^r_{k,\sigma}|^2 \approx \frac{2\pi n_F \tau}{\hbar}$, where $g^r_{k,\sigma} \equiv g^r_{k,\omega = 0, \sigma}$. Similarly we have $\sum_k \epsilon |g^r_{k,\sigma}|^4 \approx \frac{4\pi n_F \epsilon_F \tau^3}{\hbar}$, and the coefficient $\xi_{ij}$ is finally obtained as
\[
\xi_{ij} \approx \sum_\sigma \left[ (\delta_{ij} p^2 - p_i p_j) \frac{e\hbar v_F}{12m} - \delta_{ij} \frac{2ie\epsilon_{F,\sigma} v_\sigma \tau_\sigma}{3\hbar} \right]. \tag{A.7}
\]
Here $v_\sigma$ is the density of states of electron and $\epsilon_{F,\sigma} \equiv \epsilon_F + \sigma \Delta_{sd}$ for spin $\sigma$, respectively. The result of $j^{1,2}$ is
\[
j^{1,2} = \frac{e\hbar}{12m} \left( \sum_\sigma \sigma v_\sigma \right) [\nabla \times [\nabla \times (\alpha_R \times n)]] - \frac{2e}{3h} \left( \sum_\sigma \sigma \epsilon_{F,\sigma} v_\sigma \tau_\sigma \right) (\alpha_R \times n). \tag{A.8}
\]
Next we calculate $j^{3-6}$, (8). Expanding with respect to $q$ and $p$, (8) reduces to
\[
j^{3-6} = -\frac{ie\hbar^2}{m} \alpha_{R,f} \epsilon_{lmn} \sum_{k,q,p,\omega,\Omega,\Omega} e^{-i(q+p)\cdot r+i(\Omega+\Omega)|\Omega|} R_{mn}(p, \Omega) \times f(\omega) \frac{h^2}{2m} k^2 \left[ (q + p)_m A^l_{j,i}(q, \Omega) - \delta_{lm} (q + p)_n A^l_{j,n}(q, \Omega) \right] \times 2\text{Re} \left[ \sigma_j g^a_{k,\omega,\sigma} g^a_{k,\omega,\sigma} - \sigma_o g^a_{k,\omega,\sigma} g^i_{k,\omega,\sigma} \right] + O(q^3, \Omega). \tag{A.9}
\]
We calculate (A.9) by using the following identities:
\[
\sum_{k,\omega} f(\omega) \epsilon \text{Re} \left[ \sigma_j g^a_{k,\omega,\sigma} g^a_{k,\omega,\sigma} - \sigma_o g^a_{k,\omega,\sigma} g^i_{k,\omega,\sigma} \right] = -\frac{2}{5h} \sum_{k,\omega} f'(\omega) \epsilon^2 \text{Re} \left[ \sigma_j g^a_{k,\omega,\sigma} g^a_{k,\omega,\sigma} - \sigma_o g^a_{k,\omega,\sigma} g^i_{k,\omega,\sigma} \right], \tag{A.10}
\]
obtained by integration by parts, and
\[
2\epsilon_{j oc} A^l_{j,i} R_{mn} = \nabla_l n_m, \tag{A.11}
\]
\[
\text{tr} \left[ \sigma_j A \sigma_o B - \sigma_o A \sigma_j B \right] = 2i\epsilon_{j oc} \sum_\sigma \sigma A_{-\sigma} B_\sigma, \tag{A.12}
\]
where $A \equiv \text{diag}(A_+, A_-)$ and $B \equiv \text{diag}(B_+, B_-)$ are any $2 \times 2$ diagonal matrices. The leading contribution of (8) then reads
\[
j^{3-6} = \eta \nabla \times \left[ \nabla \times (\alpha_R \times n) \right], \tag{A.13}
\]
where the coefficient is

\[ \eta = \frac{2e\hbar}{15\pi m} \sum_k \text{Im} \left( \sum_{\sigma} \sigma \epsilon^2 g_{k,-\sigma}(g_{k,\sigma}^a)^2 \right) \]

\[ = \frac{e\hbar}{60m\Delta_{sd}} \sum_{\sigma} \sigma \left( \epsilon_{F,\sigma} v_\sigma - 5\epsilon_{F,\sigma}\epsilon_{F,-\sigma} v_\sigma \right) \quad (A.14) \]

and we used \( \sum_k \epsilon^2 g_{k,-\sigma}(g_{k,\sigma}^a)^2 \simeq \frac{\pi}{8\Delta_{sd}} (2v_\sigma^2 - 5\epsilon_{F,\sigma}^2 + 3\epsilon_{F,\sigma}\epsilon_{F,-\sigma} v_\sigma). \)

Results (A.8) and (A.13) are summarized in (9).

**Appendix B**

Here we show details of calculation of the Hall current, (15). Expanding with respect to the external wave vector and frequency, \( p \) and \( \Omega \), we obtain

\[ j_i^{\text{Hall}}(r, t) = -\frac{i e^2 \hbar}{m} \sum_{k, p, \omega, \Omega} e^{-ip r + i\Omega t} \alpha_R \epsilon_{jkl} \text{Tr} \left\{ \frac{\hbar^2}{2m} k_i k_p p_m A_m(\Omega) R_{\ell n}(p) \right\} \]

\[ \times \left\{ -g_{k-\frac{\epsilon_{k,\omega}}{2},\omega,\sigma} g_{k+\frac{\epsilon_{k,\omega}}{2},\omega,\sigma} \sigma_n g_{k+\frac{\epsilon_{k,\omega}}{2},\omega,\sigma} g_{k+\frac{\epsilon_{k,\omega}}{2},\omega,\sigma} \right\} \]

\[ -k_i (k \cdot p) A_\ell(\Omega) R_{\ell n}(p) g_{k-\frac{\epsilon_{k,\omega}}{2},\omega,\sigma} \sigma_n g_{k+\frac{\epsilon_{k,\omega}}{2},\omega,\sigma} (k \cdot p) \}

+ O(\Omega^2, p^2). \]

(B.1)

The lesser components are calculated as

\[ [g_{k-\frac{\epsilon_{k,\omega}}{2},\omega,\sigma} g_{k+\frac{\epsilon_{k,\omega}}{2},\omega,\sigma}]^\prime = (f_{\omega+\frac{\epsilon_{k,\omega}}{2}} - f_{\omega-\frac{\epsilon_{k,\omega}}{2}}) g_{k-\frac{\epsilon_{k,\omega}}{2},\omega,\sigma} g_{k+\frac{\epsilon_{k,\omega}}{2},\omega,\sigma} \]

\[ + f_{\omega-\frac{\epsilon_{k,\omega}}{2}} g_{k-\frac{\epsilon_{k,\omega}}{2},\omega,\sigma} g_{k+\frac{\epsilon_{k,\omega}}{2},\omega,\sigma} \]

\[ + f_{\omega+\frac{\epsilon_{k,\omega}}{2}} g_{k-\frac{\epsilon_{k,\omega}}{2},\omega,\sigma} g_{k+\frac{\epsilon_{k,\omega}}{2},\omega,\sigma} \]

\[ + f_{\omega-\frac{\epsilon_{k,\omega}}{2}} g_{k-\frac{\epsilon_{k,\omega}}{2},\omega,\sigma} g_{k+\frac{\epsilon_{k,\omega}}{2},\omega,\sigma} \]

\[ + f_{\omega+\frac{\epsilon_{k,\omega}}{2}} g_{k-\frac{\epsilon_{k,\omega}}{2},\omega,\sigma} g_{k+\frac{\epsilon_{k,\omega}}{2},\omega,\sigma} \]

(B.2)

The Hall current density then reduces to

\[ j_i^{\text{Hall}}(r, t) = -\frac{i e^2 \hbar}{3m} \sum_{k, p, \Omega} e^{-ip r + i\Omega t} \epsilon \Gamma_1 \{ A(\Omega) \times [p \times (\alpha_R \times n(p))]) \}, \]

where

\[ \Gamma_1 \equiv \sum_{\omega, \sigma} \sigma [h f_{\omega}(g_{k,\omega,\sigma}^a)^4 - (g_{k,\omega,\sigma}^a)^4] - f_{\omega}'(g_{k,\omega,\sigma}^a)^2 - (g_{k,\omega,\sigma}^a)^2 g_{k,\omega,\sigma}^a)] \]

(B.5)

The summation on \( \omega \) and \( k \) is evaluated as follows assuming \( (\epsilon_F/\hbar)^{-1} \ll 1 \);

\[ \sum_k \epsilon \Gamma_1 = \frac{1}{3} \sum_{k, \omega, \sigma} \sigma f_{\omega}'(g_{k,\omega,\sigma}^a - g_{k,\omega,\sigma}^a)^3 \]

\[ \simeq \frac{2i}{\hbar^2} \sum_{\sigma} \epsilon_{F,\sigma} v_\sigma t_\sigma^2. \]

(B.6)

We therefore obtain (16).
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