Quantum Cryptography with Imperfect Apparatus*

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Abstract

Quantum key distribution, first proposed by Bennett and Brassard, provides a possible key distribution scheme whose security depends only on the quantum laws of physics. So far the protocol has been proved secure even under channel noise and detector faults of the receiver, but is vulnerable if the photon source used is imperfect. In this paper we propose and give a concrete design for a new concept, self-checking source, which requires the manufacturer of the photon source to provide certain tests; these tests are designed such that, if passed, the source is guaranteed to be adequate for the security of the quantum key distribution protocol, even though the testing devices may not be built to the original specification. The main mathematical result is a structural theorem which states that, for any state in a Hilbert space, if certain EPR-type equations are satisfied, the state must be essentially the orthogonal sum of EPR pairs.

1 Introduction

In 1984, Bennett and Brassard [7] proposed a revolutionary concept that key distribution may be accomplished through public communications in quantum channels. Hopefully, the privacy of the resulted key is to be guaranteed by quantum physical laws alone, quite independent of how much computational resource is available to the adversary. The primary quantum phase of the proposed protocol is a sequence of single photons produced by Alice (the sender) and detected by Bob (the receiver).

The security proof of the BB84-protocol (or its many variants) for adversaries with unrestricted power is a difficult mathematical problem, and has only been achieved with any generality in the last few years. In brief, the BB84-

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* This research was supported in part by DIMACS, and by DARPA/ITO and the National Science Foundation under Grant CCR-9627819.
the source will be. This is usually sufficient to prove security.

In Section 2, we show how the main mathematical question arises from the security requirement from the BB84-protocol. In Section 3, the precise question is formulated, and the main theorem stated. The proof of the main theorem is given in Section 4.

2 Preliminaries

Ideally, the objective of key distribution is to allow two participants, typically called Alice and Bob, who initially share no information, to share a secret random key (a string of bits) at the end. A third party, usually called Eve, should not be able to obtain any information about the key. In reality, this ideal objective cannot be realized, especially if we give unlimited power to the cheater, but a quantum protocol can achieve something close to it. See [8] (and more recently [28]) for a detailed specification of the quantum key distribution task. One of the greatest challenges in quantum cryptography is to prove that a quantum protocol accomplishes the specified task. One can experimentally try different kinds of attacks, but one can never know in which way the quantum apparatus can be defective. In any case, such experiments are almost never done in practice because it is not the way to establish the security of quantum key distribution. The correct way is a properly designed protocol together with a security proof.

Recently, there has been a growing interest in practical quantum cryptography and systems have been implemented [1, 2, 3, 4, 5, 6]. However, proving the security of quantum key distribution against all attacks turned out to be a serious challenge. During many years, many researchers directly or indirectly worked on this problem [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. Using novel techniques [20, 21], a proof of security against all attacks for the quantum key distribution protocol of Bennett and Brassard was obtained in 1996 [22]. Related results were subsequently obtained [23, 24, 25, 26], but as yet [22] is the only known proof of security against all attacks. A more recent version of the proof with extension to the result is proposed in [28]. Also, the basic ideas of [9, 10, 17] might lead to a complete solution if we accept fault tolerant computation (for example, see [27]), but this is not possible with current technology.

In the quantum transmission, Alice sends \( n \) photons to Bob prepared individually in one of the four BB84 states uniformly picked at random. The BB84 states denoted \( b(0, 2) \), \( b(1, 2) \), \( b(0, 3) \) and \( b(1, 3) \) correspond to a photon polarized at 0, 90, 45 and \(-45\) degrees respectively (see figure 1). (We reserve the states \( b(0, 1) \) and \( b(1, 1) \) for further use: we will have to add two other states in our analysis.) Bob measures each photon using either the rectilinear basis \( \{b(0, 2), b(1, 2)\} \) or the diagonal basis \( \{b(0, 3), b(1, 3)\} \) uniformly chosen at random.

The basic idea of the protocol is the following. Both, Eve and Bob, do not know Alice’s bases until after the quantum transmission. Eve cannot obtain information without creating a disturbance which can be detected. Bob also disturbs the state when he uses the wrong basis, but this is not a problem. After the quantum transmission, Alice and Bob announce their bases. Alice and Bob share a bit when their bases are identical, so they know which bits they share. The key point is that it’s too late for Eve because the photons are on Bob’s side. However, the security of the protocol relies on the fact that the source behaves as specified, and this is the main subject of this paper.

Informally, the source used in the original BB84-protocol [7] can be described as a blackbox with two buttons on it: base2-button and base3-button. When Alice pushes the base2-button, the output is either \( (0, b(0, 2)) \) or \( (1, b(1, 2)) \), where \( b(0, 2) \) and \( b(1, 2) \) form an orthonormal basis of a two-dimensional system \( H_B \), with each possibility occurring with probability 1/2. After the base\( x \)-button is pressed, of the output \( (x, b(x, \alpha)) \), only the vector \( b(x, \alpha) \) goes out to Bob; bit \( x \) is only visible to Alice. Similarly, if Alice pushes the base3-button, the output is either \( (0, b(0, 3)) \) or \( (1, b(1, 3)) \), with each possibility occurring with probability 1/2, where

\[
\begin{align*}
b(0, 3) &= (b(0, 2) + b(1, 2))/\sqrt{2}, \\
b(1, 3) &= (-b(0, 2) + b(1, 2))/\sqrt{2}. 
\end{align*}
\]

The suggested way in [7, 8] to achieve the above is to have the blackbox generates a fixed state, say \( b(0, 2) \), then the bit \( x \in \{0, 1\} \) is uniformly chosen at random and this state is rotated of an appropriate angle to create the desired state \( b(x, \alpha) \) (assuming that the base\( x \)-button is pressed). The security proof of the protocol extends to sources beyond mentioned above. To obtain our self-testing source, we need to consider a different type of sources. A conjugate coding source \( S = (H_A \otimes H_B, \Psi, M_2, M_3) \) consists of a pure state \( \Psi \) in a Hilbert space \( H_A \otimes H_B \), and two measurements (each binary-valued) \( M_2 \), \( M_3 \) defined on

![Figure 1. The BB84 states](image-url)
Now comes the question. If a manufacturer hands over a source and claims that it is a perfect system, how can we check this claims, or at least, makes sure that it is an extended perfect system?

If the source is a perfect system, let $N_2, N_3$ be the measurements operating on $H_B$ in exactly the same way as $M_2, M_3$ on $H_A$. That is, let $R^{±}_2, R^{±}_3$ (where $± \in \{2, 3\}$) be the projection operators to subspaces by $N_n$ with outcome $0, 1$: $R^{±}_2, R^{±}_3$ project to $H_A \otimes |0\rangle, H_A \otimes |1\rangle$, and $R^{±}_3, R^{±}_3$ project to $H_A \otimes |0\rangle, H_A \otimes |1\rangle$, respectively. Now observe that the following are true for $\alpha \neq \beta \in \{2, 3\}, x, y \in \{+, -\}$,

$$
\frac{||P^\alpha_\beta\langle \Psi\rangle||^2}{||P^\alpha_\beta\langle \Psi\rangle||^2} = 1/2, \quad \frac{||R^{±}_2 P^\alpha_\beta\langle \Psi\rangle||^2}{||P^\alpha_\beta\langle \Psi\rangle||^2} = \delta_{x,y}, \quad \frac{||R^{±}_\beta P^\alpha_\beta\langle \Psi\rangle||^2}{||P^\alpha_\beta\langle \Psi\rangle||^2} = 1/2
$$

We can ask the manufacturer to provide in addition two measuring devices outside the blackbox corresponding to $N_2, N_3$. A test can be executed to verify that these equations are satisfied (see the related discussion in the Introduction). Furthermore, as a matter of physical implementation, to make sure that $M’$’s and $N’$’s operate on $H_A, H_B$ respectively, we can further demand that the buttons are replaced by two measuring devices outside the blackbox. Is that sufficient to guarantee that we have at least an extended perfect system?

Unfortunately, the answer is NO. It is not hard to construct examples where (4) is satisfied, but it is not an extended ideal system (and in fact, security is greatly compromised).

However, as we will see, if we add one more measurement appropriately on each side, and perform the corresponding checks, then it guarantees to be an extended perfect system. That will be the main result of this paper.

3 Main Theorem

An object $S = (H_A \otimes H_B, \langle \Psi\rangle, P_1, P_2, P_3)$ is called an ideal source if the following are valid: each of $H_A, H_B$ is a 2-dimensional Hilbert space with $(a(0, 2), a(1, 2)), (a(0, 3), a(1, 3))$ being a pair of orthonormal basis of $H_A$ satisfying equation (1), and $(b(0, 2), b(1, 2)), (b(0, 3), b(1, 3))$ being a pair of orthonormal basis of $H_B$ satisfying equation (1); $\Psi$ is the Bell state $(a(0, 0)|b(0, 2)) + |a(1, 2)|b(1, 2))/\sqrt{2} = (a(0, 3)|b(0, 3)) + |a(1, 3)|b(1, 3))/\sqrt{2}$. $P^\pm_1, P^\pm_2$ are the projection operators on the states $a(0, 2), a(1, 2)$ respectively; $P^\pm_3, P^\pm_3$ are the projection operators on the states $a(0, 3), a(1, 3)$ respectively.

To describe $P^\pm_1$, let $a(x, 1)$
\((x \in \{0, 1\})\) be the state \(a(x, 2) + a(x, 3)\) after being normalized to unit length. The states \(a(x, 1)\) and \(b(x, 1)\) have a particular status in our proof, and we alternatively denote \(a(x, 1) = |x\rangle\) and \(b(x, 1) = |x\rangle'\). Then \(P^+_1, P^-_1\) are respectively the projection operators on the states \(|0\rangle\), \(|1\rangle\). As usual, we consider \(P^\alpha_1\) and \(P^\alpha_1 \otimes I\) as two alternative notations for one and the same projection operators on \(H_A \otimes H_B\). Clearly, \(P^+_1, P^+_2, P^+_3\) are the projection operators on \(H_A \otimes H_B\) corresponding to measuring \(H_A\) with respect to three bases of \(H_A\) (the bases for \(P^+_2, P^+_3\) at an angle of \(-\pi/8, +\pi/8\) with respect to the basis for \(P^+_1\)).

The projection operators \(R_{i1}^+, R_{i2}^+, R_{i3}^\pm\) operate on coordinates in \(H_B\), and are similarly defined as the \(P^\alpha_i\)'s. Let

\[
p_{\alpha,\beta}(x, y) = \|R^\alpha_{\alpha} R^\beta_{\beta} |\Psi\rangle\|^2. \tag{5}
\]

These numbers can be easily computed. For example, \(p_{1,2}(0, 0) = (\cos(\pi/8))^2 / 2\) and \(p_{1,2}(0, 1) = (\sin(\pi/8))^2 / 2\).

A self-checking source

\[
S = (H_A \otimes H_B, |\Psi\rangle, P^+_1, P^+_2, P^+_3, R^+_1, R^+_2, R^+_3) \text{ consists of an initial state } |\Psi\rangle \in H_A \otimes H_B, \text{ three measurements } P^+_1, P^+_2, P^+_3 \text{ acting on coordinates in } H_A, \text{ and three measurements } R^+_1, R^+_2, R^+_3 \text{ acting on coordinates in } H_B, \text{ such that the following conditions are satisfied:}
\]

\[
\|R^\alpha_{\alpha} R^\beta_{\beta} |\Psi\rangle\|^2 = p_{\alpha,\beta}(x, y). \tag{6}
\]

We will see that a self-checking source gives rise to an extended ideal system.

An extended ideal source

\[
S = (H_A \otimes H_B, |\Psi\rangle, P^+_1, P^+_2, P^+_3, R^+_1, R^+_2, R^+_3) \text{ is an orthogonal sum of ideal sources in a similar sense as an extended perfect system in relation to perfect systems. That is, if there is an index set } I, \text{ orthogonal two dimensional subspaces } K_i \subseteq H_A \text{ with } a_i(x) \text{ (or alternatively } a_i(x, 1)\text{) denoting the state } |x\rangle \text{ in } K_i, \text{ orthogonal two dimensional subspaces } H_i \subseteq H_B \text{ with } b_i(x, \alpha) \text{ (or alternatively } b_i(x, 1)\text{) denoting the state } |x\rangle' \text{ in } H_i, \text{ such that for some (possibly complex) numbers } \alpha_i \text{ on } i \in I \text{ with } \sum_{i \in I} |\alpha_i|^2 = 1,
\]

\[
|\Psi\rangle = \sum_{i \in I} \alpha_i (a_i(0) \otimes b_i(0) + a_i(1) \otimes b_i(1)).
\]

Furthermore, for each \(i\), for every projection \(P \in \{P^+_1, P^+_2, P^+_3\}\), \(P\) acts exactly on \(K_i\) like the corresponding projection on \(H_A\) in the ideal source case. That is, if \(P|x\rangle = \lambda_0|0\rangle + \lambda_1|1\rangle\) in the ideal case, we have that \(P a_i(x) = \lambda_0 a_i(0) + \lambda_1 a_i(1)\). The following fact is easy to verify.

**Fact 1** Any extended ideal source is a self-checking source.

Also, it is clear that from any self-checking source, by omitting the measurements \(P^+_1, R^+_1\), one obtains a conjugate coding source.

**Fact 2** The conjugate coding source obtained from an extended ideal source must be an extended perfect system.

The converse of fact 1 is our main theorem.

**Main Theorem** Any self-checking source is an extended ideal source.

It follows from the Main Theorem and Fact 2 that a self-checking source provides an adequate source for the BB84 quantum key distribution protocol [7, 8].

We remark that in our definition of self-checking source, the restriction of the initial state to a pure state \(|\Psi\rangle\) instead of a mixed state \(\rho\) is not a real restriction. Given a source with a mixed state \(\rho\) satisfying equation (6), we can construct one with a pure state \(|\Psi\rangle\) (by enlarging appropriately \(H_A\) satisfying (6)). We can apply the Main Theorem to this new source, and conclude that it also gives rise to an adequate source for the BB84-protocol.

It is well known, from discussions about EPR Experiments (see e.g. [29]), that quantities such as \(\|R^\alpha_{\alpha} R^\beta_{\beta} |\Psi\rangle\|^2\) exhibit behavior characteristic of quantum systems that cannot be explained by classical theories. One may view our main result as stating that such constraints are sometimes strong enough to yield precise structural information about the given quantum system; in this case it has to be an orthogonal sum of EPR pairs.

### 4 Proof of Main Theorem

We give in this Section a sketch of the main steps in the proof.

Let \(S = (H_A \otimes H_B, |\Psi\rangle, P^+_1, P^+_2, P^+_3, R^+_1, R^+_2, R^+_3)\) be a self-checking source. We show that it must be an extended ideal source.

In Section 4.1, we derive some structural properties of the projection operators as imposed by the self-checking conditions, but without considering in details the constraints due to the tensor product nature of the state space. In Sections 4.2 and 4.3, the state is decomposed explicitly in terms of tensor products, and the properties derived in Section 4.1 are used to show that this decomposition satisfies the conditions stated in the Main Theorem.

#### 4.1 Properties of Projections

In this subsection, we present some properties of the projected states (such as \(P^+_1 |\Psi\rangle, P^+_2 R^+_2 |\Psi\rangle\)) as consequences of the constraints put on self-checking sources. The proofs of these lemmas are somewhat lengthy, and will be left to the complete paper.

**Lemma 1** For every \(\alpha \in \{1, 2, 3\}\) and \(x \in \{+, -, 0\}\), we have \(P^\alpha_\alpha |\Psi\rangle = R^\alpha_\alpha |\Psi\rangle\).
Let $v_i \in V, w_i \in W$ for $1 \leq i \leq m$, where $V, W$ are two Hilbert spaces. We say that $(v_1, v_2, \ldots, v_m)$ is isomorphic to $(w_1, w_2, \ldots, w_m)$ if there is an inner-product-preserving linear mapping $f : V \to W$ such that $w_i = f(v_i)$ for all $i$.

Let $\theta = \pi/8$, and $u_1, u_2, \ldots, u_5$ be elements of $C^2$ defined by

$$u_1 = (1, 0),$$
$$u_2 = (\cos^2 \theta, \sin \theta \cos \theta),$$
$$u_3 = (\sin^2 \theta, -\sin \theta \cos \theta),$$
$$u_4 = (\cos^2 \theta, -\sin \theta \cos \theta),$$
$$u_5 = (\sin^2 \theta, \sin \theta \cos \theta).$$

**Lemma 2** $(u_1, u_2, \ldots, u_5)$ is isomorphic to $\sqrt{2}(P_{1+} \Psi, P_{1+} R_{1-} \Psi, P_{1+} R_{3-} \Psi, P_{1+} R_{5-} \Psi, P_{1+} R_{5-} \Psi)$. \[\]

**Lemma 3** Let $h = P_{1+} R_{1-} \Psi - P_{1+} R_{2-} \Psi$. Then $R_1 h = h$.

**Lemma 4** Let $k = P_{1-} P_{1+} \Psi - (\cos \theta)^2 P_{1+} \Psi$. Then $(R_{2-}^2 - R_{3-}^2) k = 2(\sin \theta \cos \theta)^2 P_{1-} \Psi$.

Since there is a symmetry between the projection operators $P$ and $R$, the following is clearly true.

**Lemma 5** Lemmas 2-4 remain valid if the projection operators $P$ and $R$ are exchanged.

### 4.2 The Decomposition

We now prove that the state $\Psi \in H_A \otimes H_B$ can be decomposed into the direct sum of EPR pairs. We begin with a decomposition of $P_{1+} \Psi$, which is equal to $R_{1+} \Psi$ by Lemma 1.

**Lemma 6** One can write

$$P_{1+} \Psi = \sum_{i \in I} \alpha_i a_i(0) \otimes b_i(0)$$

where $I$ is an index set, $\alpha_i$ are complex numbers, and $a_i(0) \in H_A (i \in I)$, $b_i(0) \in H_B (i \in I)$ are two respectively orthonormal sets of eigenvectors of the operators $P_{1+}$ (acting on $H_A$) and $R_{1+}$ (acting on $H_B$).

**Proof** The lemma is proved with the help of Schmidt decomposition theorem [30] [31]. We omit the details here. □

Let $\beta = (2\sin \theta \cos \theta)^{-1}$. Define $a_i(1) = \beta(P_{1+} - P_{2+}) a_i(0)$, and $b_i(1) = \beta(R_{2+} - R_{3+}) b_i(0)$ for $i \in I$. Let $K_i \subseteq H_A$ be the subspace spanned by $a_i(0)$ and $a_i(1)$; Let $H_i \subseteq H_B$ be the subspace spanned by $b_i(0)$ and $b_i(1)$. The plan is to show that

$$\Psi = \sum_{i \in I} \alpha_i (a_i(0) \otimes b_i(0) + a_i(1) \otimes b_i(1)),$$

and that $K_i, H_i$ have all the properties required to satisfy the Main Theorem.

In the remainder of this subsection, we use Lemmas 2-5 to show that each $H_i$ ($K_i$) behaves correctly under the projection operators $R_{i+}$ ($P_{i+}$). In the next subsection, we complete the proof by showing that all $H_i$ ($K_i$) are orthogonal to each other.

By Lemma 2, $(u_1, u_2, \ldots, u_5)$ is isomorphic to $\sqrt{2}(P_{1+} \Psi, P_{1+} R_{1-} \Psi, P_{1+} R_{3-} \Psi, P_{1+} R_{5-} \Psi, P_{1+} R_{5-} \Psi)$. In particular, this implies that any linear relation $\sum_{i} \lambda_i u_i = 0$ must also be satisfied if $u_j$ are replaced by the appropriate projected states. Now

$$P_{1+} \Psi = \sum_{i \in I} \alpha_i a_i(0) \otimes b_i(0),$$
$$P_{1+} R_{1-} \Psi = \sum_{i \in I} \alpha_i a_i(0) \otimes R_{2-} b_i(0),$$
$$P_{1+} R_{3-} \Psi = \sum_{i \in I} \alpha_i a_i(0) \otimes R_{2+} b_i(0),$$
$$P_{1+} R_{5-} \Psi = \sum_{i \in I} \alpha_i a_i(0) \otimes R_{2+} b_i(0).$$

This means that, for each $i \in I$, any linear relation $\sum_{i} \lambda_i u_i = 0$ must also be satisfied if we make the following substitutions:

$$u_1 \leftarrow b_i(0),$$
$$u_2 \leftarrow R_{2-} b_i(0),$$
$$u_3 \leftarrow R_{2+} b_i(0),$$
$$u_4 \leftarrow R_{2-}^2 b_i(0),$$
$$u_5 \leftarrow R_{2+}^2 b_i(0).$$

**Lemma 7** For each $i \in I$, $(u_1, u_2, \ldots, u_5)$ is isomorphic to $(b_i(0), R_{2-} b_i(0), R_{2-} b_i(0), R_{2+} b_i(0), R_{2+} b_i(0))$.

**Proof** Use the preceding observation and the orthogonality between $R_{2-} b_i(0)$ and $R_{2-} b_i(0)$, and the orthogonality between $R_{2+} b_i(0)$ and $R_{2+} b_i(0)$. We omit the details here. □

Note that $b_i(1) = \beta(3 R_{3+} - R_{2+}) b_i(0)$ by definition. From Lemma 7, it is easy to see that $b_i(1)$ is a unit vector perpendicular to $b_i(0)$. In fact, $b_i(1)$ is mapped to the vector $(0, 1)$ under the isomorphism in Lemma 7.

From Lemma 7, for the purpose of vectors in the space $H_i$, the projection operators $R_{i+}, R_{i-}$ correspond to choosing the coordinate system obtained from the system $(b_i(0), b_i(1))$ rotated by the angle $\theta$; similarly, $R_{i+}, R_{i-}$ correspond to choosing a coordinate system obtained from the system $(b_i(0), b_i(1))$ rotated by the angle $-\theta$. It remains to show that $R_{1+}, R_{1-}$ correspond to the coordinate system...
(b_i(0), b_i(1)) itself. By definition R_i^+ b_i(0) = b_i(0). It remains to prove that R_i^- b_i(1) = b_i(1).

To do that, we use Lemma 3. Observe that

\[ h = P_1^+ R_i^- \Psi - P_1^+ R_i \Psi = \sum_{i \in I} \alpha_i a_i(0) \otimes R_i^+ b_i(0) - \sum_{i \in I} \alpha_i a_i(0) \otimes R_i^- b_i(0) = \sum_{i \in I} \alpha_i a_i(0) \otimes (R_i^+ - R_i^-) b_i(0) = \beta_i^{-1} \sum_{i \in I} \alpha_i a_i(0) \otimes b_i(1). \]

Since R_i^- h = h by Lemma 3, we must have R_i^- b_i(1) = b_i(1). This completes the proof that the projection operators R_i^\pm behave as required on the subspace H_i.

As stated explicitly in Lemma 5, we can obtain the symmetric statement that the the projection operators P_i^\mp behave as required on the subspace K_i.

Now that we have determined the behavior of the projection operators on K_i, H_i, we can in principle calculate any polynomial of the projection operators on the state P_1^+ \Psi. By Lemma 4, P_1^- \Psi can be written as

\[ P_1^- \Psi = 2 \beta^2 (R_2^+ - R_3^+)(P_1^+ - \cos^2 \theta)P_1^+ \Psi. \]

This gives

\[ P_1^- \Psi = 2 \beta^2 \sum_{i \in I} \alpha_i (P_2^+ - \cos^2 \theta)a_i(0) \otimes (R_2^+ - R_3^+) b_i(0). \]

After applying the rules and simplifying, we obtain

\[ P_1^- \Psi = \sum_{i \in I} \alpha_i a_i(1) \otimes b_i(1). \]

As \( \Psi = P_1^+ \Psi + P_1^- \Psi \), this proves

\[ \Psi = \sum_{i \in I} \alpha_i (a_i(0) \otimes b_i(0) + a_i(1) \otimes b_i(1)). \]

4.3 Completing the Proof

It remains to show that all \( H_i \) are orthogonal to each other. (A symmetric argument then shows that all \( K_i \) are also orthogonal to each other.) Let \( i \neq j \in I \). Assume that \( H_i \) is not orthogonal to \( H_j \). We derive a contradiction. By definition, \( H_i \) is spanned by \( b_i(0), b_i(1) \), and \( H_j \) is spanned by \( b_j(0), b_j(1) \). Clearly, \( b_i(1) \) and \( b_j(1) \) are not orthogonal to each other, as all the other pairs \((b_i(x), b_j(y))\) are orthogonal.

Choose a coordinate system for the space spanned by the four vectors such that

\[ b_i(0) = (1, 0, 0, 0), \]
\[ b_i(1) = (0, 1, 0, 0), \]
\[ b_j(0) = (0, 0, 1, 0), \]
\[ b_j(1) = (0, s, 0, t), \]

where \( s \neq 0 \). From our knowledge about the behavior of \( R_3 \), we infer that \( R_3^+ b_j(0) = (\cos \theta)w \) where \( w = \cos \theta b_j(0) + \sin \theta b_j(1) = (0, s \sin \theta, \cos \theta, t \sin \theta) \). Similarly, \( R_3^- b_i(0) = (\sin \theta)w' \) where \( w' = -\sin \theta b_i(0) + \cos \theta b_i(1) = (-\sin \theta, \cos \theta, 0, 0) \). As the inner product of \( w \) and \( w' \) is \( s \sin \theta \cos \theta \) which is non-zero, we conclude that \( R_3^+ b_j(0) \) and \( R_3^- b_i(0) \) are not orthogonal. This contradicts the fact that \( R_3^+ \) and \( R_3^- \) are projection operators to orthogonal subspaces. This completes the proof.

5 Concluding Remarks

The security problem for imperfect source is a difficult one to deal with. The present paper is a step in only one possible direction. We have also limited ourselves to the simplest case when the correlation probabilities \( p_{a,b} \) are assumed to be measurable precisely. We leave open as future research topics for extensions to more general models.

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