Infinitesimal symmetries and conservation laws of the DNLSE hierarchy and the Noether’s theorem

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Abstract

The hierarchy of the integrable nonlinear equations associated with the quadratic bundle is considered. The expressions for the solution of the linearization of these equations and their conservation law in the terms of the solutions of the corresponding Lax pairs are found. It is shown for the first member of the hierarchy that the conservation law is connected with the solution of the linearized equation due to the Noether’s theorem. The local hierarchy and three nonlocal ones of the infinitesimal symmetries and the conservation laws that are explicitly expressed through the variables of the nonlinear equations are derived.

1 Introduction

One of the most effective tools of studying the nonlinear phenomena is the inverse scattering transformation (IST) method [1, 2]. This method reduces the solution of the Cauchy initial value problem of the nonlinear partial differential equations (PDE’s), which admit a representation as the compatibility condition of the overdetermined linear system (Lax pair), to solving linear singular integral equations. It is of a special significance that many of the PDE’s playing an important role in different branches of physics can be investigated in its frameworks. For example, the derivative nonlinear Schrödinger equation (DNLSE) that was originally deduced for the Alfvén waves of finite amplitude [3, 4] and the equations of massive Thirring model (MTM) [5] belong to the class of the PDE’s integrable with the help of the IST method for the quadratic bundle [6–8]. It was revealed that DNLSE describes also the behavior of drifting filamentations in nonlinear electrostatic waves of magnetized plasmas [9], light pulses in the optical fibers [10–12], magnetic holes of space plasmas [13] and large-amplitude magnetohydrodynamics waves [14]. The MTM equations were recently shown to appear in the coherent optics and nonlinear acoustics [15] as a limiting case of the system of long/short-wave coupling (see [16] and references therein).

The integrable nonlinear PDE’s are well known to possess the infinite hierarchies of infinitesimal symmetries and conservation laws. An existence of them was proposed as the integrability test to characterize the equations solvable by IST (see, e.g., [17, 18]). There are different methods of obtaining the infinitesimal symmetries and conservation laws, which originate from the study of KdV equation [19]. Given a set of the scattering data (namely, the time-invariant part of them), infinite hierarchies of the conserved densities are constructed [1, 2, 6, 20–22]. The Bäcklund transformation (BT) of the integrable equation was used to generate the hierarchy of its conservation laws [23, 24]. The approach that exploits the Noether’s theorem was applied in [25, 26] for the derivation of the conservation laws of...
sine–Gordon and KdV equations. To produce the corresponding hierarchy of infinitesimal symmetries, the implicit expressions for the solutions of the linearized equations, which are obtained by means of infinitesimal BT, were expanded in the power series on the parameter of this BT. The infinitesimal version of the dressing method was suggested in [27] to construct the infinitesimal symmetries of integrable PDE’s. Similar expressions for the perturbations of some nonlinear PDE’s and their Lax pairs were presented in [28]. The geometrical approaches that utilize the projective transformations or treat the soliton equations as descriptions of pseudospherical surfaces were developed for nonlinear PDE’s associated with matrix Lax pairs of the second order in [29, 30] (see also [31] and [32], respectively. The hierarchies of local and nonlocal conservation laws for DNLSE were found by means of these methods [30, 33]. The method based on the theory of $\tau$-functions was applied to scalar and two-component KP hierarchies [34].

Although the methods mentioned above appeal to underlying Lax pair to produce the hierarchies of infinitesimal symmetries and conservation laws, they do not entirely cover the class of PDE’s representable as the compatibility condition. This concerns especially the cases of reductions of the nonlinear PDE’s [21] and their integrable deformations (see, e.g., [35, 15, 36]), which are most interesting from physical point of view. The knowledge of the infinitesimal symmetries and conservation densities of the hierarchy allows one to make sure that the PDE given belongs to it. The approach applicable to all integrable nonlinear equations can be based, for instance, on explicit expressions for the solution of linearized equation and the conservation law in the terms of the solutions of corresponding Lax pairs. In the present report, we construct the infinite hierarchies of local and nonlocal infinitesimal symmetries and conservation laws for the DNLSE hierarchy using such the approach.

The paper is organized as follows. The nonlinear equations of the DNLSE hierarchy and their Lax pairs are presented in Sec.II. The formulas for the expansions in series on the spectral parameter powers of the solutions of the Lax pairs are also given there. The solution of the linearization of the nonlinear PDE’s, which is expressed in the terms of the solutions of corresponding Lax pairs, is obtained by means of the infinitesimal version of the binary Darboux transformation (DT) [37] in Sec.III. This technique has been applied to the DNLSE [38–40] and to obtain the infinitesimal symmetries of the nonlinear PDE’s [41–43]. To generate the hierarchies of local and nonlocal infinitesimal symmetries explicitly expressed through the variables of the nonlinear equations under consideration, the expansions in series of the solutions of the Lax pairs or the recursion operator of the hierarchy can be used. In Sec.IV the conservation law for the DNLSE hierarchy is derived. Substitution of the expansion in series of the Lax pairs solutions into this formula yields the hierarchies of local and nonlocal conservation laws. The connection due to the Noether’s theorem of the infinitesimal symmetries and conservation laws we found is shown in this section for the DNLSE case.

2 The DNLSE hierarchy

Let us consider (direct) Lax pair

$$
\psi_x = U(\lambda) \psi, \quad (1)
$$

$$
\psi_t = V(\lambda) \psi, \quad (2)
$$

where $\psi = \psi(x, t, \lambda) = (\psi_1, \psi_2)^T$ is the vector-column solution; $\lambda$ is complex parameter referred to as the spectral parameter in the IST theory; $U(\lambda) = U(x, t, \lambda)$ and $V(\lambda) = V(x, t, \lambda)$ are $2 \times 2$ matrix coefficients. The compatibility condition of the overdetermined
system (12) is

$$U(\lambda)_t - V(\lambda)_x + [U(\lambda), V(\lambda)] = 0.$$  \hspace{1cm} (3)

We suppose in what follows that

$$U(\lambda) = \lambda^2 U^{(2)} + \lambda U^{(1)}$$  \hspace{1cm} (4)

(i.e., Eq.(1) is the quadratic bundle) and

$$U^{(2)} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad U^{(1)} = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}.$$  \hspace{1cm} (5)

If $V(\lambda)$ is chosen in the next form

$$V(\lambda) = \sum_{j=1}^{2m} \lambda^j V^{(j)},$$  \hspace{1cm} (6)

then Eq.(3) gives the expressions for the matrix coefficients of $V(\lambda)$:

$$V^{(2m-2j)} = v^{(2m-2j)}(2m-2j) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad V^{(2m-2j-1)} = \begin{pmatrix} 0 & v^{(2m-2j-1)}_{12} \\ v^{(2m-2j-1)}_{21} & 0 \end{pmatrix},$$  \hspace{1cm} (7)

$$(j = 0, ..., m - 1),$$ where

$$v^{(2m-2j)} = \partial_x^{-1} (qv^{(2m-2j-1)}_{21} - rv^{(2m-2j-1)}_{12}),$$  \hspace{1cm} (8)

$$\begin{pmatrix} v^{(2m-2j-1)}_{12} \\ v^{(2m-2j-1)}_{21} \end{pmatrix} = \hat{R}^{j+1} \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$  \hspace{1cm} (9)

$$\hat{R} = \frac{1}{2} \begin{pmatrix} i \partial_x + q \partial_x^{-1} r \partial_x & q \partial_x^{-1} q \partial_x \\ r \partial_x^{-1} r \partial_x & -i \partial_x + r \partial_x^{-1} q \partial_x \end{pmatrix},$$  \hspace{1cm} (10)

and system of nonlinear equations

$$\begin{pmatrix} q_t \\ r_t \end{pmatrix} = \partial_x \hat{R}^m \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$  \hspace{1cm} (11)

(Note that operators $\partial_x^{-1}$ in Eqs.(8,9) for equal $j$’s add the same time-dependent functions as the constants of integration.) To obtain these formulas we make use the identities

$$R \partial_x = \partial_x \hat{R},$$

$$R^{-1} = 2 \begin{pmatrix} -i + q \partial_x^{-1} r & -q \partial_x^{-1} q \\ -r \partial_x^{-1} r & i + r \partial_x^{-1} q \end{pmatrix} \begin{pmatrix} \partial_x^{-1} & 0 \\ 0 & \partial_x^{-1} \end{pmatrix}$$  \hspace{1cm} (12)

with operator $R$ being defined in the following manner

$$R = \frac{1}{2} \begin{pmatrix} i \partial_x + \partial_x q \partial_x^{-1} r & \partial_x q \partial_x^{-1} q \\ \partial_x r \partial_x^{-1} r & -i \partial_x + \partial_x r \partial_x^{-1} q \end{pmatrix}.$$  \hspace{1cm} (13)

As it will be seen in the next section, $\hat{R}$ and its adjoint $R$ are the squared eigenfunction operator and the recursion one of the hierarchy considered.
The hierarchy of nonlinear equations (11) was found in [22]. It admits under appropriate choice of the constants of integration the next reduction
\[ r = \pm q^*. \] (14)

In this case, the first nontrivial equation of the hierarchy is reduced after rescaling to DNLSE
\[ iq_t + q_{xx} \mp i(|q|^2 q)_x = 0. \] (15)

Let us consider the expansions of the solutions of Eqs. (1,2) in the series on the spectral parameter powers. In the neighborhood of point \( \lambda = \infty \), the vector solutions of the Lax pairs of nonlinear equations (11) are represented as
\[ \psi = \sum_{k=0}^{\infty} \lambda^{-k} A^{(k)} \Lambda |a\rangle. \] (16)

Here \( |a\rangle \) is a constant vector-column,
\[ \Lambda = \begin{pmatrix} \lambda x^2 + \lambda^{2m} v(2m) t & 0 \\ 0 & \lambda x^2 - \lambda^{2m} v(2m) t \end{pmatrix} \]
and coefficients \( A^{(k)} \) solve system of equations
\[
\begin{cases}
[A^{(k)}, U^{(2)}] + A^{(k-2)}_x = U^{(1)} A^{(k-1)} \\
[A^{(k)}, V^{(2m)}] + A^{(k-2m)}_t = \sum_{j=1}^{2m-1} V^{(2m-j)} A^{(k-j)}.
\end{cases}
\]

An expansion in series of the solutions of Lax pairs considered in the neighborhood of point \( \lambda = 0 \) has form
\[ \psi = \sum_{k=0}^{\infty} \lambda^k B^{(k)} \, |a\rangle, \] (17)
where \( B^{(0)} = E \) and coefficients \( B^{(k)} \) \((k \geq 1)\) are determined from equations
\[
\begin{cases}
B^{(k)}_x = U^{(2)} B^{(k-2)} + U^{(1)} B^{(k-1)} \\
B^{(k)}_t = \sum_{j=1}^{2m} V^{(j)} B^{(k-j)}.
\end{cases}
\]

It is seen that the first coefficients of the expansions are
\[ A^{(0)} = \begin{pmatrix} w & 0 \\ 0 & w^{-1} \end{pmatrix}, \quad A^{(1)} = \frac{i}{2} \begin{pmatrix} 0 & -qw^{-1} \\ rw & 0 \end{pmatrix}, \]
\[ A^{(2)} = \frac{1}{4} \begin{pmatrix} \int w^x (qr + iq^2 r^2/2) dx & 0 \\ 0 & w^{x-1} \int (q_x r - iq^2 r^2/2) dx \end{pmatrix}, \]
\[ B^{(1)} = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}, \quad B^{(2)} = \begin{pmatrix} -ix + \int xqv dx & 0 \\ 0 & ix + \int xru dx \end{pmatrix}, \]
where
\[ w = \exp \left( i \int \frac{x}{q} q dx \right), \quad u = \int x q dx, \quad v = \int x r dx. \]
3 Darboux transformation and infinitesimal symmetries

Hierarchy of nonlinear equations (11) follows also from the compatibility condition of dual Lax pair

\[ \xi_x = -\xi U(\alpha), \]
\[ \xi_t = -\xi V(\alpha). \]

Here \( \xi \equiv \xi(x, t, \alpha) \) is a vector-row solution, \( \alpha \) is the spectral parameter of the dual pair. Since matrix coefficients \( U(\lambda) \) and \( V(\lambda) \) defined by Eqs. (4–7) satisfy conditions

\[ \sigma_1 U(-\lambda) + U(\lambda)^T \sigma_1 = 0, \quad \sigma_1 V(-\lambda) + V(\lambda)^T \sigma_1 = 0, \]

where \( \sigma_1 \) is Pauli matrix

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]

the next connection between the solutions of systems (1, 2) and (18, 19) exists:

\[ \xi = \psi^T \sigma_1, \quad \alpha = -\lambda. \]

In the case of reduction (14) the solutions with complex conjugate spectral parameters are also connected. For instance, \( (\psi^*_2, \pm \psi^*_1)^T \) is a solution of direct Lax pair (1, 2) with spectral parameter \( \lambda^* \).

Let vector-column \( \varphi = (\varphi_1, \varphi_2)^T \) and vector-row \( \chi = (\chi_1, \chi_2) \) are solutions of Lax pairs (12) and (18, 19) with spectral parameters \( \mu \) and \( \nu \), respectively. The Lax pairs are covariant with respect to "turned" binary Darboux transformation (BDT) \{\psi, \xi, U(\lambda), V(\lambda)\} \rightarrow \{\psi[1], \xi[1], U(\lambda)[1], V(\lambda)[1]\} of the form

\[ \psi[1] = gT(\lambda)\psi, \quad \xi[1] = \xi T(\alpha)^{-1} g^{-1}, \]

\[ U(\lambda)[1] = \lambda^2 U^{(2)}[1] + \lambda U^{(1)}[1], \quad V(\lambda)[1] = \sum_{j=1}^{2m} \lambda^j V^{(j)}[1], \]

where

\[ T(\lambda) = E - \frac{\mu - \nu}{\lambda - \nu} P = \left(1 - \frac{\nu - \mu}{\lambda - \mu} P\right)^{-1}, \quad P = \frac{\varphi \chi}{\chi \varphi}, \quad g = \sigma_1 T(0)^{-1} \]

and

\[ U^{(2)}[1] = g U^{(2)} g^{-1}, \quad V^{(2m)}[1] = g V^{(2m)} g^{-1}, \]

\[ U^{(1)}[1] = g \left(U^{(1)} + (\mu - \nu)[U^{(2)}, P]\right) g^{-1}, \]

\[ V^{(j)}[1] = g V^{(j)} g^{-1} + (\mu - \nu) \sum_{k=j+1}^{2m} \nu^{k-j-1} \left(V^{(k)}[1] g P - g P V^{(k)}\right) g^{-1} \]

\((j = 1, ..., 2m - 1)\). We call this transformation as "turned" because formulas (22 –26) is a product of usual BDT [41, 42] and additional gauge transformation performed with the help of matrix \( g \). This additional transformation allows us to avoid an appearance of the terms at the zero power of \( \lambda \) in the expressions for \( U(\lambda)[1] \) and \( V(\lambda)[1] \).
Conditions (20, 21) are fulfilled for transformed matrix coefficients $U(\lambda)[1], V(\lambda)[1]$ and solutions $\psi[1], \xi[1]$ of the transformed Lax pairs if we impose restriction

\[ \chi = \varphi^T \sigma_1, \quad \nu = -\mu. \]

In this case, we have

\[ U^{(2)}[1] = U^{(2)}, \]
\[ V^{(2m)}[1] = V^{(2m)}. \]

Then, Eq. (25) gives us expressions for new (transformed) solutions of hierarchy of nonlinear equations (11):

\[ q[1] = r - \frac{1}{\mu} \left( \varphi_2 \right)_x, \]
\[ r[1] = q - \frac{1}{\mu} \left( \varphi_1 \right)_x. \]

The second iteration of the BDT (22–26) keeping conditions (20, 21) yields the following formulas

\begin{align*}
q[2] &= q - \frac{\mu_2 - \mu_2^*}{\mu_1 \mu_2} \left( \frac{\varphi_1^{(1)} \varphi_1^{(2)}}{\mu_1 \varphi_1^{(1)} \varphi_2^{(2)} - \mu_2 \varphi_2^{(1)} \varphi_1^{(2)}} \right)_x, \\
r[2] &= r + \frac{\mu_2 - \mu_2^*}{\mu_1 \mu_2} \left( \frac{\varphi_2^{(1)} \varphi_2^{(2)}}{\mu_2 \varphi_2^{(1)} \varphi_2^{(2)} - \mu_1 \varphi_2^{(1)} \varphi_1^{(2)}} \right)_x,
\end{align*}

where $\varphi_1^{(k)}$ and $\varphi_2^{(k)}$ are the components of vector solution $\varphi^{(k)}$ of the direct Lax pair with spectral parameters $\mu_k$ ($k = 1, 2$). If we put here $\varphi^{(2)} = (\varphi_2^{(1)^*}, \pm \varphi_1^{(1)^*})^T$ and $\mu_2 = \mu_1^*$, then

\[ r[2] = \pm q[2]^*. \]

This way we come to DT for the DNLSE hierarchy. The compact form of $N$-th iteration of this transformation is presented in [40].

Considering limits $\mu_1 \to \mu$ and $\mu_2 \to \mu$ in Eqs. (27, 28), one obtains the next expressions (up to a multiplier) for solution of the linearization of system (11):

\begin{align*}
\delta q &= (\varphi_1^{(1)} \varphi_2^{(2)})_x, \\
\delta r &= -(\varphi_2^{(1)} \varphi_2^{(2)})_x.
\end{align*}

It is checked by straightforward calculation that

\[ R \left( \begin{array}{c} \delta q \\ \delta r \end{array} \right) = \mu^2 \left( \begin{array}{c} \delta q \\ \delta r \end{array} \right). \]

This identity allows us to define in a recurrent manner the coefficients of expansions of the right–hand sides of Eqs. (29, 30) in the power series on the spectral parameter at a neighborhood of the points $\mu = \infty$ and $\mu = 0$. The coefficients of these expansions

\[ \delta q = \sum_{k=0}^{\infty} \mu^{-2k} \delta q^{(k)}, \quad \delta r = \sum_{k=0}^{\infty} \mu^{-2k} \delta r^{(k)} \]

and

\[ \delta q_j = \sum_{k=0}^{\infty} \mu^{2k} \delta q_j^{(k)}, \quad \delta r_j = \sum_{k=0}^{\infty} \mu^{2k} \delta r_j^{(k)} \]
form the infinite hierarchies of infinitesimal symmetries. Operator $R$ satisfying (31) is nothing but the recursion operator of the hierarchy (11). In the case of point $\mu = 0$, there exist three hierarchies of nonlocal infinitesimal symmetries $\delta q_j^{(k)}$, $\delta r_j^{(k)}$ ($j = 1, 2, 3$, $k = 0, 1, \ldots$) that correspond to different choices of the constants of integration in operator $R^{-1}$ (see Eq. (12)). The first nontrivial members of the hierarchies for the points $\mu = \infty$ and $\mu = 0$, respectively, are

$$
\begin{align*}
\delta q_1^{(1)} &= q, \\
\delta r_1^{(1)} &= -r,
\end{align*}
$$

$$
\begin{align*}
\delta q_2^{(1)} &= q, \\
\delta r_2^{(1)} &= -r,
\end{align*}
$$

$$
\begin{align*}
\delta q_3^{(1)} &= 2(qv - i), \\
\delta r_3^{(1)} &= -2(ru + i).
\end{align*}
$$

It is seen from these formulas that $\delta q_j^{(k)} \sim q_j^{(2m - 2k + 1)}$, $\delta r_j^{(k)} \sim q_j^{(2m - 2k + 1)}$ and the infinite hierarchy corresponding to the point $\mu = \infty$ is local. Another way of producing the hierarchies of the infinitesimal symmetries is to substitute expansions (16) and (17) into Eqs. (24-30).

## 4 Conservation laws and Noether’s theorem

Let us consider identity

$$(\xi \psi)_{xt} = (\xi \psi)_{tx}.$$  

Excluding the derivatives of $\psi$ and $\xi$ on $x$ in the left-hand side and the derivatives on $t$ in the right-hand side with the help of Eqs. (12) and (16,17), respectively, and dividing the relation obtained on $\lambda - \omega$, we come to the conservation law of the DNLSE hierarchy

$$T_t + X_x = 0,$$

where

$$T = \xi \left( (\lambda + \omega) U^{(2)} + U^{(1)} \right) \psi, $$

$$X = -\xi \sum_{k=1}^{2m} \sum_{j=0}^{k-1} \lambda^{k-j-1} \omega^j V^{(k)} \psi.$$  

If we put $\lambda = \omega = \mu$, $\psi = \varphi^{(1)}$ and $\xi = (\varphi^{(2)}_2, -\varphi^{(2)}_1)$, where vectors $\varphi^{(k)} = (\varphi^{(k)}_1, \varphi^{(k)}_2)^T$ ($k = 1, 2$), as it was supposed at the end of the previous section, are solutions of Lax pair (12) with spectral parameter $\mu$, then expressions (32) are rewritten in the next manner

$$T = -2i \mu (\varphi^{(1)}_1 \varphi^{(2)}_2 + \varphi^{(1)}_2 \varphi^{(2)}_1) + \varrho^{(1)}_2 \varphi^{(1)}_1 \varphi^{(2)}_1,$$

$$X = -2 \sum_{k=1}^{m} k \mu^{2k-1} v^{(2k-1)} (\varphi^{(1)}_1 \varphi^{(2)}_2 + \varphi^{(1)}_2 \varphi^{(2)}_1) +$$

$$+ \sum_{k=1}^{m} \mu^{2k-2} (v^{(2k-1)}_{21} \varphi^{(1)}_1 \varphi^{(2)}_1 - v^{(2k-1)}_{12} \varphi^{(1)}_2 \varphi^{(2)}_2).$$  

Substitution of expansions (16) and (17) of the Lax pair solutions at the neighborhood of points $\mu = \infty$ and $\mu = 0$ into these formulas leads to the hierarchies of the conservation laws expressed explicitly through the solutions of nonlinear equations (11). In the case of point $\mu = 0$, for example, we have three infinite hierarchies $T_{j,t}^{(k)} + X_{j,x}^{(k)} = 0$ ($j = 1, 2, 3$, $k = 0, 1, \ldots$), whose first conserved densities and currents are

$$T_1^{(0)} = q, \quad X_1^{(0)} = -v^{(1)}_{12}, \quad T_2^{(0)} = r, \quad X_2^{(0)} = -v^{(1)}_{21},$$  

$$T_3^{(0)} = 3q^2 - 2iqr + 3q^2r^2/2,$$  

$$X_3^{(0)} = -2iqr - 3q^2r^2/2.$$  

$$T_4^{(0)} = q^4 - 2iqr + 3q^2r^2/2,$$  

$$X_4^{(0)} = -2iqr - 3q^2r^2/2.$$  

$$T_5^{(0)} = q^5 - 2iqr + 3q^2r^2/2,$$  

$$X_5^{(0)} = -2iqr - 3q^2r^2/2.$$
\( T_3^{(1)} = qv - ur, \quad X_3^{(1)} = uv_{21}^{(1)} - vv_{12}^{(1)} - 2v^{(2)}. \) (37)

The first two conservation laws are immediate consequence of the divergent form of Eqs. (11).

Let us discuss the connection between solutions (29,30) of the linearized equations and the conservation laws found in the case of system of nonlinear equations

\[
\begin{align*}
    iq_t + q_{xx} - i(q^2r)_x &= 0, \\
    ir_t - r_{xx} - i(qr^2)_x &= 0.
\end{align*}
\]

Coefficients of the second equation of Lax pair (1,2) of the system under consideration are

\[
V^{(4)} = 2U^{(2)}, \quad V^{(3)} = 2U^{(1)}, \quad V^{(2)} = qrU^{(2)}, \quad V^{(1)} = \begin{pmatrix} 0 & iq_x + q^2r \\ -i r_x + qr^2 & 0 \end{pmatrix}
\]

DNLSE (15) follows these equations by imposing condition \( r = \pm q^* \).

In the terms of potentials \( u \) and \( v \) the Lagrangian of Eqs. (38,39) reads as

\[
L = i(u_xv_t + v_xu_t) + u_{xx}v_x - v_{xx}u_x - iu_x^2v_x^2.
\]

Using notations for the Euler–Lagrange equations

\[
\begin{align*}
    \Lambda(u) &\equiv -\left( \frac{\partial L}{\partial u_t} \right)_t - \left( \frac{\partial L}{\partial u_x} \right)_x + \left( \frac{\partial L}{\partial u_{xx}} \right)_xx = -2(ir_t - r_{xx} - i(qr^2)_x) = 0, \\
    \Lambda(v) &\equiv -\left( \frac{\partial L}{\partial v_t} \right)_t - \left( \frac{\partial L}{\partial v_x} \right)_x + \left( \frac{\partial L}{\partial v_{xx}} \right)_xx = -2(iq_t + q_{xx} - i(q^2r)_x) = 0,
\end{align*}
\]

the variation of the Lagrangian, which is caused by the infinitesimal transformations of potentials \( u \rightarrow u + \varepsilon \delta u \) and \( v \rightarrow v + \varepsilon \delta v \), is written in a form of Noether’s identity

\[
\delta L = \varepsilon (A_t + B_x + \Lambda(u)\delta u + \Lambda(v)\delta v).
\] (40)

Here

\[
A = \frac{\partial L}{\partial u_t} \delta u + \frac{\partial L}{\partial v_t} \delta v = i(q \delta v + r \delta u),
\]

\[
B = \left( \frac{\partial L}{\partial u_x} - \frac{\partial L}{\partial u_{xx}} \right)_x \delta u + \frac{\partial L}{\partial u_{xx}} \delta u_x + \left( \frac{\partial L}{\partial v_x} - \frac{\partial L}{\partial v_{xx}} \right)_x \delta v + \frac{\partial L}{\partial v_{xx}} \delta v_x = q_x \delta v - r_x \delta u + r \delta q - q \delta r - iqr^2 \delta u - iq^2r \delta v.
\]

Given a symmetry of Eqs. (38,39), a conservation law is derived from Eq. (40) due to the Noether’s theorem. Few examples of the symmetries and associated conservation densities and currents are listed below:

1) \( \delta u = 1, \delta v = 0 \):
\[
\delta L = 0, \quad T_1 = ir, \quad X_1 = iv_t - 2r_x - 2iqr^2.
\] (41)

2) \( \delta u = 0, \delta v = 1 \):
\[
\delta L = 0, \quad T_2 = iq, \quad X_2 = iu_t + 2q_x - 2iq^2r.
\] (42)

3) \( u \rightarrow ue^{i\varepsilon}, v \rightarrow ve^{-i\varepsilon}, \delta u = iu, \delta v = -iv \):
\[
\delta L = 0,
\]
\[T_3 = qv - ur, \quad X_3 = utv - uv_t - 2i(q_xv - qr + ur_x) - 2(qv - ur)qr.\]  

(43)

4) \(x \rightarrow x + \varepsilon, \delta u = q, \delta v = r:\)

\[\delta \mathcal{L} = \varepsilon \mathcal{L}_x,\]

\[T_4 = 2iqr, \quad X_4 = 2(q_xr - qr_x) - 3iq^2r^2.\]  

(44)

5) \(t \rightarrow t + \varepsilon, \delta u = ut, \delta v = vt:\)

\[\delta \mathcal{L} = \varepsilon \mathcal{L}_t,\]

\[T_5 = -q_xr + qr_x + iq^2r^2, \quad X_5 = i(q_{xxx}r - 2q_xr_x + qr_{xx} - 2q^3r^3) + 3(q_xr - qr_x)qr.\]  

(45)

Conservation laws that arise in the first and second cases are trivial. The symmetries of the potentials in the third, fourth and fifth cases correspond, respectively, to infinitesimal symmetries \(\delta q_1^{(1)}, \delta r_1^{(1)}, \delta q^{(1)}, \delta r^{(1)}\) and \(\delta q^{(2)}, \delta r^{(2)}\) presented at the end of previous section. Conserved density \(T_5\) is proportional to the Hamiltonian density of DNLSE [22]. The Noether’s theorem was applied in [44] to obtain \(T_3\) and \(X_3\), which are nothing but \(T_3^{(1)}\) and \(X_3^{(1)}\) [37]. Hence, \(\delta r_1^{(1)}\) and \(\delta q^{(1)}\) are connected by the Noether’s theorem with \(T_3^{(1)}\) and \(X_3^{(1)}\). It will be proven in the sequel that this is valid for all members of the hierarchies of infinitesimal symmetries and conservation laws.

Formulas [29, 30] give us solutions of the linearized equations on potentials

\[\delta u = \varphi_1^{(1)} \varphi_1^{(2)},\]

\[\delta v = -\varphi_2^{(1)} \varphi_2^{(2)}.\]

It is remarkable that we are able to put the corresponding variation of Lagrangian in divergent form:

\[\delta \mathcal{L} = \left( iu \delta r + iv \delta q + 4\mu(\varphi_1^{(1)} \varphi_2^{(2)} + \varphi_2^{(1)} \varphi_1^{(2)}) \right) + \]

\[+ \left( v \delta q_x - u \delta r_x + q \delta r - r \delta q - i(2uvr)q \delta r + i(qv + 2ur)q \right)\]

\[+ 8i\mu^2(q_xr + q \delta v) - 16\mu^3(\varphi_1^{(1)} \varphi_2^{(2)} + \varphi_2^{(1)} \varphi_1^{(2)}) \right) t.\]

Combining this expression with Eq. [10], we come after a cancellation of the terms with potentials \(u\) and \(v\) to the conservation law, whose conserved density \(\tilde{T}\) and current \(\tilde{X}\) are defined in the following manner

\[\tilde{T} = 4\mu(\varphi_1^{(1)} \varphi_2^{(2)} + \varphi_2^{(1)} \varphi_1^{(2)}) + 2i(q\varphi_2^{(1)} \varphi_2^{(2)} - r\varphi_1^{(1)} \varphi_1^{(2)}),\]

\[\tilde{X} = -(16\mu^3 + 4\mu qr)(\varphi_1^{(1)} \varphi_2^{(2)} + \varphi_2^{(1)} \varphi_1^{(2)}) + 12i\mu^2(r\varphi_1^{(1)} \varphi_1^{(2)} - q\varphi_2^{(1)} \varphi_2^{(2)}) + \]

\[+ 2(r_x + iq^2r)\varphi_1^{(1)} \varphi_1^{(2)} + 2(q_x - iq^2r)\varphi_2^{(1)} \varphi_2^{(2)}.\]

These expressions are proportional to ones given by Eqs. [34, 35]. This way, we show that solutions [29, 30] of the linearized equations and conserved densities [34] and currents [35] are connected in the case of DNLSE in accordance with the Noether’s theorem. This connection takes also place between the infinite hierarchies of infinitesimal symmetries \(\delta q_j^{(k)}, \delta r_j^{(k)}\) and \(\delta q_j^{(k)}, \delta r_j^{(k)}\) \((j = 1, 2, 3, k = 0, 1, \ldots)\) and the hierarchies of conservation laws obtained by expansion in formulas [34, 35] of the Lax pair solutions on the spectral parameter powers. First terms of expansions [16] and [17] lead to the conservation laws determined by formulas [14, 15] and [36, 37], respectively, that coincide with ones presented in [22, 29, 33].
5 Conclusion

In the present report, we have found the expressions for the solution of the linearization of the DNLSE hierarchy equations and their conservation law in the terms of the solutions of associated Lax pairs. The approach exploited is based on the Darboux transformation technique. It is shown in the DNLSE case that the conservation law is connected with the solution of the linearized equation accordingly to the Noether’s theorem. The local hierarchy and three nonlocal ones of the infinitesimal symmetries and the conservation laws that are explicitly expressed through the variables of the nonlinear equations are produced using the recursion operator and/or expanding the Lax pair solutions in the series on the spectral parameter powers.

The explicit form of the infinitesimal symmetries and the conservation laws of various hierarchies is useful to determine an integrability of the nonlinear PDE’s given. This is especially important for the cases interesting from the physical point of view, such as the reductions of the PDE’s and their deformations. Recently, it was revealed that some deformations of the well-known nonlinear integrable equations, which have the physical meaning, are also integrable [15, 36]. This opens the problems of a description of the classes of the deformations keeping the integrability and an extension to them of the methods having been developed in the IST theory. The approach suggested here is not specific for the hierarchy considered and can be applied to other integrable hierarchies and their integrable deformations. An investigation of the hierarchy of the deformed nonlinear equations, which is associated with the quadratic bundle and contains as a particular case the following integrable deformation of the DNLSE equation

\[ iq_t + \alpha q_x^* + q_{xx} \pm i(|q|^2) = 0, \]

where \( \alpha \) is an arbitrary parameter, is a subject of the future work.

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