Constructing effective action for gravitational field by effective potential method

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Abstract: The aim of this paper is to construct a quantum effective action for gravitational fields by the effective potential method in quantum field theory. The minimum of the quantum effective action gives an equation of quantum fluctuations. We discuss the quantum fluctuation in the flat spacetime and in the Schwarzschild spacetime. It is shown that a baby spacetime may be created from a classical vacuum through a quantum fluctuation.

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1 Introduction

The main aim of this paper is to construct a quantum effective action and an equation of the quantum fluctuation for gravitational fields by the effective potential method. The quantum effective action is constructed by the effective potential method in quantum field theory [1, 2]. The spectral method in quantum field theory also provides a similar treatment [3–9]. One of the features of this kind of methods is that the classical solution, preferably exact classical solutions, plays an important role. The classical solution determines the effective potential in the equation of quantum fluctuations. This feature is sometimes a disadvantage and sometimes an advantage. The disadvantage is that the method is difficult to use without an exact classical solution. Classical field equations are often nonlinear equations, such as the $\phi^4$-field, which is difficult to solve on the one hand and has infinitely many solutions on the other hand [10]. Different classical solutions will give different effective potentials, and thus giving different equations of quantum fluctuations, so it is difficult to give a general treatment. However, this disadvantage is sometimes an advantage in gravity theory. In gravity theory, years of research have accumulated many important exact solutions of the Einstein equation [11, 12]. Each solution of the Einstein equation, the classical solution, leads to an effective potential for the quantum fluctuation of gravity. In this way, we can calculate the quantum fluctuation of classical spacetimes given by the Einstein equation one by one. Although a one-by-one process has its limitation, it can deal with some important bound-state case, such as the Schwarzschild spacetime.
This method also has an advantage. The method is based on classical solutions which can be either scattering states or bound states. The usual quantum-field-theory method cannot deal with bound states, because it is based on the scattering perturbation theory. Concretely, in quantum field theory, one first quantizes free fields, and then adds interactions adiabatically. The validity of such a treatment is guaranteed by the Gell-mann and Low theorem. The free field is the simplest scattering-state field, so the quantum-field method is not suitable to bound states. This is an important difficulty encountered in quantum field theory at the technical level. It is this difficulty that limits the application of quantum field theory in, for example, hadron physics for hadrons are bound states of quarks. The bound state in hadron physics can only be dealt with by various models, numerical methods, or the Bethe-Salpeter equation [13]. In gravity, bound states become more prominent. Many important classical gravitational solutions are bound states, such as the Schwarzschild spacetime. This method, though cannot give a general treatment like Feynman diagram expansions which can only deal with scattering states, can be used to deal with the bound-state spacetime.

The minimum of the classical action $S$ gives the classical field equation,

$$\delta S = 0. \quad (1.1)$$

The minimum of the quantum effective action $\Gamma$ gives the quantum field equation [1, 2],

$$\delta \Gamma = 0. \quad (1.2)$$

In this paper, we construct a quantum effective action $\Gamma$ for gravity and construct an equation for the quantum fluctuation for gravitational fields.

To calculate a quantum corrected gravitational field $g_{\mu\nu}$, we start from the classical gravitational field $\bar{g}_{\mu\nu}$, whether it is strong or weak, bound-state or scattering-state. We write the quantum corrected metric $g_{\mu\nu}$ as the classical metric $\bar{g}_{\mu\nu}$ which is the solution of the Einstein equation plus a quantum correction $h_{\mu\nu}$ in the form of $g_{\mu\nu} = \bar{g}_{\mu\nu} + \hbar h_{\mu\nu}$. In this way, the information of bound states is embodied in the classical part $\bar{g}_{\mu\nu}$ and the information of quantum corrections is embodied in $h_{\mu\nu}$. We will show that the leading-order contribution of the quantum correction satisfies a linear equation. Of course, if the higher-order quantum correction is taken into account, the equation of quantum correction will be a nonlinear equation.

Technically speaking, we construct the quantum effective action of a gravitational field by the path integral. We first calculate the generating functional $Z[J]$ which gives the generating functional $W[J]$. The quantum effective action $\Gamma$ can be obtained from $W[J]$ through a Legendre transform. The minimum of the effective action $\Gamma$, given by Eq. (1.2), gives the equation of the quantum corrected field.

The quantum effect on gravity is an important and difficult problem [14–16]. One fruitful theory of quantum gravity comes from string theory [17–27]. The quantum gravity is also considered in the frame of loop quantum gravity [28–33]. The quantum effect for gravity, such as the quantum gravity effect in Oppenheimer-Snyder collapse [34], the creation and evaporation of black holes [35–39] are considered. Various methods for quantum
gravity are developed, such as the bootstrap theory [40–44], the renormalization group theory [45], the conformal field theory [46], and the group field theory formalism [47]. The quantum effect for various gravity models are considered, including Gauss–Bonnet gravity [48], two-dimensional de Sitter spacetime [49], the nonlocal gravity model [50], the JT and CGHS dilaton gravity [51], the Einstein-Gauss-Bonnet-Maxwell gravity [52–54], dark energies [55] and universe models [56], and the Minkowskian black hole and white hole [57].

The species problem of the Hawking radiation is discussed [58].

In section 2, we solve the quantum effective action for gravitational fields. In section 3, we consider the equation for quantum fluctuations by the quantum effective action. In section 4, we consider the quantum fluctuation in the flat spacetime. In section 5, we consider the quantum fluctuation in the Schwarzschild spacetime. Conclusions are given in section 6.

2 Quantum effective action

In this section, we construct the quantum effective action for gravitational fields.

2.1 Generating functional: $Z[J]$ and $W[J]$

We first calculate the generating functional $Z[J]$ of gravitational fields.

We write the quantum corrected metric $g_{\mu\nu}$ as the classical metric $\bar{g}_{\mu\nu}$ plus a quantum fluctuation $h_{\mu\nu}$:

$$ g_{\mu\nu} = \bar{g}_{\mu\nu} + \hbar h_{\mu\nu}. \tag{2.1} $$

The classical action of gravitational fields is

$$ S = -\int \sqrt{\bar{g}} R d^4x. \tag{2.2} $$

The minimum of the classical action $S$, given by $\delta S = 0$, leads to the Einstein equation whose solution is the classical gravitational field $\bar{g}_{\mu\nu}$.

Next we expand the action (2.2) around the classical gravitational field $\bar{g}_{\mu\nu}$. To do this, we first expand the determinant $\sqrt{g}$ and the Ricci scalar $R$, respectively.

**Expansion of $\sqrt{g}$.** Expanding $\sqrt{g}$ around the classical gravitational field $\bar{g}_{\mu\nu}$,

$$ \sqrt{g} = \sqrt{\bar{g}} + \int \sqrt{\bar{g}} d^4x_1 \frac{\delta \sqrt{g}}{\delta g_{\mu\nu}} |_{g_{\mu\nu} = \bar{g}_{\mu\nu}} (g_{\mu\nu} - \bar{g}_{\mu\nu}) $$

$$ + \frac{1}{2} \int d^4x_1 d^4x_2 (\sqrt{g})^2 \left. \frac{\delta^2 \sqrt{g}}{\delta g_{\mu\nu} \delta g_{\rho\sigma}} \right|_{g_{\mu\nu} = \bar{g}_{\mu\nu}, g_{\rho\sigma} = \bar{g}_{\rho\sigma}} (g_{\mu\nu} - \bar{g}_{\mu\nu}) (g_{\rho\sigma} - \bar{g}_{\rho\sigma}) + \cdots, \tag{2.3} $$

substituting Eq. (2.1) into Eq. (2.3), and keeping up to second order, the harmonic approximation, we arrive at

$$ \sqrt{g} = \sqrt{\bar{g}} + \hbar \int \sqrt{\bar{g}} d^4x_1 \frac{\delta \sqrt{g}}{\delta g_{\mu\nu}} |_{g_{\mu\nu} = \bar{g}_{\mu\nu}} h_{\mu\nu} $$

$$ + \frac{1}{2} \hbar^2 \int d^4x_1 d^4x_2 (\sqrt{g})^2 \left. \frac{\delta^2 \sqrt{g}}{\delta g_{\mu\nu} \delta g_{\rho\sigma}} \right|_{g_{\mu\nu} = \bar{g}_{\mu\nu}, g_{\rho\sigma} = \bar{g}_{\rho\sigma}} h_{\mu\nu} h_{\rho\sigma} + \cdots. \tag{2.4} $$
The variation $\delta \sqrt{g}$ in Eq. (2.4) is [59]

$$\delta \sqrt{g} = \frac{1}{2} \sqrt{g} g^{\mu\nu} \delta g_{\mu\nu}.$$  \hfill (2.5)

To calculate the functional derivative $\frac{\delta \sqrt{g}}{\delta g_{\mu\nu}}$, we rewrite Eq. (2.5) as [60, 61]

$$\delta \sqrt{g} = \int d^4 x_1 \delta^4 (x_1 - x) \sqrt{\frac{1}{2} g} \delta g_{\mu\nu}$$  \hfill (2.6)

and then

$$\frac{\delta \sqrt{g}}{\delta g_{\mu\nu}} = \delta^4 (x_1 - x) \frac{1}{2} g^{\mu\nu}.$$  \hfill (2.7)

Similarly, to calculate the functional derivative $\frac{\delta^2 \sqrt{g}}{\delta g_{\mu\nu} \delta g_{\rho\sigma}}$, we rewrite Eq. (2.7) as [60, 61]

$$\frac{\delta \sqrt{g}}{\delta g_{\mu\nu}} = \int d^4 x_2 \delta^4 (x_2 - x_1) \delta^4 (x_1 - x) \frac{1}{2} g^{\mu\nu}.$$  \hfill (2.8)

Then

$$\delta \left( \frac{\delta \sqrt{g}}{\delta g_{\mu\nu}} \right) = \int d^4 x_2 \delta^4 (x_2 - x_1) \delta^4 (x_1 - x) \frac{1}{2} \delta g^{\mu\nu}$$

$$= - \int d^4 x_2 \sqrt{g} \delta^4 (x_2 - x_1) \delta^4 (x_1 - x) \frac{1}{2} \frac{1}{2} \sqrt{g} g^{\mu\rho} g^{\nu\sigma} \delta g_{\rho\sigma}.$$  \hfill (2.9)

where [59]

$$\delta g^{\mu\nu} = -g^{\mu\rho} g^{\nu\sigma} \delta g_{\rho\sigma}.$$  \hfill (2.10)

is used. Eq. (2.10) directly gives functional derivative

$$\frac{\delta^2 \sqrt{g}}{\delta g_{\mu\nu} \delta g_{\rho\sigma}} = - \delta^4 (x_2 - x_1) \delta^4 (x_1 - x) \frac{1}{2} \frac{1}{2} \sqrt{g} g^{\mu\rho} g^{\nu\sigma} \delta g_{\rho\sigma}.$$  \hfill (2.11)

Substituting Eqs. (2.8) and (2.12) into Eq. (2.4) and performing the integral give

$$\sqrt{g} = \sqrt{g} \left( 1 + h \frac{1}{2} g^{\mu\nu} h_{\mu\nu} \right).$$  \hfill (2.13)

**Expansion of $g^{\mu\nu}$.** Now we expand the inverse metric $g^{\mu\nu}$.

Similarly, we expand $g^{\mu\nu}$ around the classical gravitational field $\bar{g}^{\mu\nu}$ as

$$g^{\mu\nu} = \bar{g}^{\mu\nu} + \hbar \int \sqrt{g} d^4 x_1 \frac{\delta g^{\mu\nu}}{\delta g_{\rho\sigma} \mid_{\bar{g}_{\rho\sigma} = \bar{g}_{\rho\sigma}}} h_{\rho\sigma}$$

$$+ \frac{1}{2} \hbar^2 \int (\sqrt{g})^2 d^4 x_1 d^4 x_2 \frac{\delta^2 g^{\mu\nu}}{\delta g_{\rho\sigma} \delta g_{\alpha\beta} \mid_{g_{\rho\sigma} = \bar{g}_{\rho\sigma}, g_{\alpha\beta} = \bar{g}_{\alpha\beta}}} h_{\rho\sigma} h_{\alpha\beta}.$$  \hfill (2.14)

To calculate the functional derivative $\frac{\delta g^{\mu\nu}}{\delta g_{\rho\sigma}}$, we rewrite Eq. (2.11) as

$$\delta g^{\mu\nu} = \int d^4 x_1 \delta^4 (x_1 - x) \left( -g^{\mu\rho} g^{\nu\sigma} \delta g_{\rho\sigma} \right).$$  \hfill (2.15)
Then
\[ \frac{\delta g^{\mu\nu}}{\delta g_{\rho\sigma}} = -\delta^4 (x_1 - x) \frac{1}{\sqrt{g}} g^{\mu\rho} g^{\nu\sigma}. \] (2.16)

By Eq. (2.16), we have
\[ \delta \left( \frac{\delta g^{\mu\nu}}{\delta g_{\rho\sigma}} \right) = -\delta^4 (x_1 - x) \left[ \delta \left( \frac{1}{\sqrt{g}} \right) g^{\mu\rho} g^{\nu\sigma} + \frac{1}{\sqrt{g}} \delta (g^{\mu\nu}) g^{\rho\sigma} + \frac{1}{\sqrt{g}} g^{\mu\rho} \delta (g^{\nu\sigma}) \right]. \] (2.17)

Substituting Eqs. (2.5) and (2.11) into Eq. (2.17) gives
\[ \delta \left( \frac{\delta g^{\mu\nu}}{\delta g_{\rho\sigma}} \right) = \delta^4 (x_1 - x) \frac{1}{\sqrt{g}} g^{\mu\rho} g^{\nu\sigma} \delta g_{\alpha\beta} \]
\[ = \int d^4 x_2 \delta^4 (x_2 - x_1) \left[ \delta^4 (x_1 - x) \right. \]
\[ \times \left. \frac{1}{\sqrt{g}} \left( \frac{1}{2} g^{\mu\rho} g^{\nu\sigma} g^{\alpha\beta} + g^{\mu\alpha} g^{\rho\beta} g^{\nu\sigma} + g^{\mu\rho} g^{\nu\alpha} g^{\sigma\beta} \right) \right] \]
\[ \delta g_{\alpha\beta}, \] (2.18)

so
\[ \frac{\delta^2 g^{\mu\nu}}{\delta g_{\alpha\beta} \delta g_{\rho\sigma}} = \delta^4 (x_2 - x_1) \delta^4 (x_1 - x) \frac{1}{(\sqrt{g})^2} \]
\[ \times \left( \frac{1}{2} g^{\mu\rho} g^{\nu\sigma} g^{\alpha\beta} + g^{\mu\alpha} g^{\rho\beta} g^{\nu\sigma} + g^{\mu\rho} g^{\nu\alpha} g^{\sigma\beta} \right). \] (2.19)

Substituting Eqs. (2.16) and (2.19) into Eq. (2.14) and performing the integral give
\[ g^{\mu\nu} = \bar{g}^{\mu\nu} - \hbar \bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} h_{\rho\sigma} \]
\[ + \hbar^2 \left( \frac{1}{2} \bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} \bar{g}^{\alpha\beta} + \bar{g}^{\mu\alpha} \bar{g}^{\rho\beta} \bar{g}^{\nu\sigma} + \bar{g}^{\mu\rho} \bar{g}^{\nu\alpha} \bar{g}^{\sigma\beta} \right) h_{\rho\sigma} h_{\alpha\beta}. \] (2.20)

**Expansion of S.** We rewrite the action (2.2) as
\[ S = -\int d^4 x \sqrt{g} g^{\mu\nu} R_{\mu\nu}, \] (2.21)

where the Ricci tensor
\[ R_{\mu\nu} = \frac{\partial \Gamma^\lambda_{\mu\lambda}}{\partial x^\nu} - \frac{\partial \Gamma^\lambda_{\nu\lambda}}{\partial x^\mu} + \Gamma^\eta_{\mu\lambda} \Gamma^\lambda_{\nu\eta} - \Gamma^\eta_{\mu\nu} \Gamma^\lambda_{\lambda\eta}. \] (2.22)

with the connection
\[ \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} \left( \frac{\partial g_{\kappa\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\kappa}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\kappa} \right). \] (2.23)

First we expand the connection \( \Gamma^\lambda_{\mu\nu} \) up to second order:
\[ \Gamma^\lambda_{\mu\nu} = \bar{\Gamma}^\lambda_{\mu\nu} + \hbar^2 \left[ \frac{1}{2} g^{\lambda\rho} g^{\nu\sigma} h_{\rho\sigma} \right] \left( \bar{D}_\mu h_{\nu\nu} + \bar{D}_\nu h_{\mu\mu} - \bar{D}_\kappa h_{\mu\nu} \right) \]
\[ - \hbar^2 \left[ \frac{1}{2} g^{\lambda\rho} g^{\nu\sigma} h_{\rho\sigma} \right] \left( \bar{D}_\mu h_{\nu\nu} + \bar{D}_\nu h_{\mu\mu} - \bar{D}_\kappa h_{\mu\nu} \right) \]
\[ - \frac{1}{2} \left( \frac{1}{2} g^{\lambda\rho} g^{\nu\sigma} g^{\alpha\beta} \bar{g}^{\mu\sigma} - \bar{g}^{\lambda\rho} g^{\beta\sigma} \bar{g}^{\mu\alpha} \right) h_{\rho\sigma} h_{\alpha\beta}, \] (2.24)
where $\Gamma^\lambda_{\mu\nu}$ is the connection and $\bar{D}_\mu$ is the covariant derivative with the classical metric $\bar{g}^{\mu\nu}$. For convenience we denote

$$P_{\mu\kappa\nu} = \bar{D}_\mu h_{\kappa\nu} + \bar{D}_\nu h_{\mu\kappa} - \bar{D}_\kappa h_{\mu\nu},$$

$$Q^{\lambda\rho\alpha\beta\sigma}_{\mu\nu} = \frac{1}{2} \bar{g}^{\lambda\rho} \bar{g}^{\alpha\beta} \Gamma^\sigma_{\mu\nu} + \bar{g}^{\lambda\alpha} \bar{g}^{\rho\sigma} \bar{\Gamma}^\beta_{\mu\nu} - \bar{g}^{\lambda\rho} \bar{g}^{\beta\sigma} \bar{\Gamma}^\alpha_{\mu\nu},$$

and then Eq. (2.24) becomes

$$\Gamma^\lambda_{\mu\nu} = \bar{\Gamma}^\lambda_{\mu\nu} + \hbar \frac{1}{2} \bar{g}^{\kappa\lambda} P_{\mu\kappa\nu} - \hbar^2 \left( \frac{1}{2} \bar{g}^{\lambda\rho} \bar{g}^{\kappa\sigma} h_{\rho\sigma} P_{\mu\kappa\nu} - \frac{1}{2} Q^{\lambda\rho\alpha\beta\sigma}_{\mu\nu} h_{\rho\sigma} h_{\alpha\beta} \right).$$

Note that $h_{\mu\nu} = h_{\nu\mu}$.

The expansion of the Ricci tensor, by substituting Eq. (2.27) into (2.22), reads

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \hbar \frac{1}{2} \left[ \bar{D}_\nu \left( \bar{g}^{\lambda\kappa} P_{\mu\kappa\lambda} \right) - \bar{D}_\lambda \left( \bar{g}^{\lambda\kappa} P_{\mu\nu\kappa} \right) \right] + \hbar^2 \left\{ \frac{1}{2} \left[ \bar{D}_\nu \left( Q^{\lambda\rho\alpha\beta\sigma}_{\mu\nu} h_{\rho\sigma} h_{\alpha\beta} - \bar{g}^{\lambda\rho} \bar{g}^{\alpha\beta} P_{\mu\nu\rho\sigma} \right) \right. \\
- \bar{D}_\lambda \left( Q^{\lambda\rho\alpha\beta\sigma}_{\mu\nu} h_{\rho\sigma} h_{\alpha\beta} - \bar{g}^{\lambda\rho} \bar{g}^{\alpha\beta} P_{\mu\nu\rho\sigma} \right) + \frac{1}{4} \bar{g}^{\rho\lambda\kappa\nu} ( P_{\mu\kappa\lambda} P_{\nu\chi\nu} - P_{\mu\kappa\nu} P_{\nu\chi\nu} ) \right\}. $$

The expansion of the Ricci scalar, by substituting Eqs. (2.28) and (2.20) into

$$R = g^{\mu\nu} R_{\mu\nu},$$

reads

$$R = \bar{R} + \hbar \left\{ \frac{1}{2} \bar{g}^{\mu\nu} \left[ \bar{D}_\nu \left( \bar{g}^{\lambda\kappa} P_{\mu\kappa\lambda} \right) - \bar{D}_\lambda \left( \bar{g}^{\lambda\kappa} P_{\mu\nu\kappa} \right) \right] - \bar{g}^{\mu\nu} g^{\rho\sigma} h_{\rho\sigma} \bar{R}_{\mu\nu} \right\} + \hbar^2 \left\{ - \frac{1}{2} \bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} h_{\rho\sigma} \left[ \bar{D}_\nu \left( \bar{g}^{\lambda\kappa} P_{\mu\kappa\lambda} \right) - \bar{D}_\lambda \left( \bar{g}^{\lambda\kappa} P_{\mu\nu\kappa} \right) \right] \right. \\
+ \frac{1}{2} \left( \frac{1}{2} \bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} \bar{g}^{\alpha\beta} + \bar{g}^{\mu\alpha} \bar{g}^{\nu\beta} \bar{g}^{\rho\sigma} + \bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} \bar{g}^{\alpha\beta} \right) \bar{h}_{\rho\sigma} h_{\alpha\beta} \bar{R}_{\mu\nu} + \frac{1}{2} \bar{g}^{\mu\nu} \left[ \bar{D}_\nu \left( \bar{g}^{\lambda\kappa} P_{\mu\kappa\lambda} + Q^{\lambda\rho\alpha\beta\sigma}_{\mu\nu} h_{\rho\sigma} h_{\alpha\beta} \right) \right. \\
- \bar{D}_\lambda \left( \bar{g}^{\lambda\rho} \bar{g}^{\rho\sigma} h_{\rho\sigma} P_{\mu\nu\kappa} + Q^{\lambda\rho\alpha\beta\sigma}_{\mu\nu} h_{\rho\sigma} h_{\alpha\beta} \right) + \frac{1}{2} \bar{g}^{\mu\nu} \left( \bar{g}^{\rho\lambda\kappa\nu} P_{\mu\kappa\lambda} P_{\nu\chi\nu} - \bar{g}^{\rho\mu\nu\sigma} \bar{g}^{\lambda\chi \nu} P_{\mu\nu\kappa} P_{\lambda\chi\nu} \right) \right\}. $$

The classical gravitational field $\bar{g}^{\mu\nu}$ satisfies the Einstein equation,

$$\bar{R}_{\mu\nu} = 0.$$  

Substituting Eqs. (2.31), (2.30), and (2.13) into Eq. (2.2), using [59]

$$\bar{D}_\mu V^\nu = \frac{1}{\sqrt{\bar{g}}} \partial_\mu \sqrt{\bar{g}} V^\nu,$$

$$
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$$
and vanishing the surface term, we arrive at the expansion of the action

$$S = \frac{\hbar}{4} \int d^4x \sqrt{\bar{g}} h_{\rho\sigma} M^{\rho\sigma\mu\nu} h_{\mu\nu},$$  \hspace{1cm} (2.33)

where

$$M^{\rho\sigma\mu\nu} = \left( \bar{g}^{\mu\nu} \bar{g}^{\rho\sigma} \bar{g}^{\lambda\kappa} - \frac{1}{2} \bar{g}^{\mu\nu} \bar{g}^{\rho\sigma} \bar{g}^{\lambda\kappa} \right) \bar{D}_\lambda \bar{D}_\kappa \hspace{1cm} (2.34)$$

or

$$M^{\rho\sigma\mu\nu} = \bar{g}^{\mu\nu} \left( \bar{g}^{\rho\sigma} \bar{D}^2 - \frac{1}{2} \bar{D}^\sigma \bar{D}^\rho \right) - \bar{g}^{\mu\rho} \left( \frac{1}{2} \bar{g}^{\nu\sigma} \bar{D}^2 - \bar{D}^\nu \bar{D}^\sigma \right). \hspace{1cm} (2.35)$$

**Generating functional:** $Z[J]$ and $W[J]$. Now we calculate the generating functional for gravitational fields.

We consider an action with a source $J$,

$$S_J = \int \sqrt{\bar{g}} d^4x R + \int \sqrt{\bar{g}} d^4xJ.$$  \hspace{1cm} (2.36)

The generating functional is

$$Z[J] = \int Dg_{\mu\nu} \exp \left[ \frac{i}{\hbar} S[g_{\mu\nu}] + \frac{i}{\hbar} \int \sqrt{\bar{g}} d^4x J \right]. \hspace{1cm} (2.37)$$

Substituting the expansions (2.1), (2.13), and (2.33) into the generating functional (2.37), we have

$$Z[J] = \int Dh_{\mu\nu} \exp \left[ \frac{i\hbar}{4} \int \sqrt{\bar{g}} d^4x h_{\rho\sigma} M^{\rho\sigma\mu\nu} h_{\mu\nu} + \frac{i}{2} \int \sqrt{\bar{g}} d^4x \bar{g}^{\mu\nu} h_{\mu\nu} J + \frac{i}{\hbar} \int d^4x \sqrt{\bar{g}} J \right]. \hspace{1cm} (2.38)$$

We formally rewrite the generating functional (2.38) as

$$Z[J] = \int Dh \exp \left[ \frac{i\hbar}{4} (h, Mh) + \frac{i}{2} \left( J\bar{g}, h \right) + \frac{i}{\hbar} \int d^4x \sqrt{\bar{g}} J \right] \hspace{1cm} (2.39)$$

for convenience.

To perform the path integral, we use the Gaussian integral formula [62]

$$\int D\phi \exp \left\{ -\frac{1}{2} (\phi, A\phi) + (b, \phi) + c \right\} = (\det A)^{-1/2} \exp \left[ \frac{1}{2} (b, A^{-1}b) - c \right]. \hspace{1cm} (2.40)$$

By Eq. (2.40), the generating functional (2.39) becomes

$$Z[J] = \left[ \det \left( -\frac{i\hbar}{2} M \right) \right]^{-1/2} \exp \left[ -\frac{i}{4\hbar} \left( J\bar{g}, M^{-1}J\bar{g} \right) + \frac{i}{\hbar} \int d^4x \sqrt{\bar{g}} J \right], \hspace{1cm} (2.41)$$

or equivalently,

$$Z[J] = \left[ \det \left( -\frac{i\hbar}{2} M \right) \right]^{-1/2} \times \exp \left\{ \frac{1}{4i\hbar} \int \sqrt{\bar{g}} d^4x \sqrt{\bar{g}} d^4x' \bar{g}^{\mu\nu} (x) J(x) (M^{-1})^{\mu\nu\rho\sigma} (x, x') \bar{g}^{\rho\sigma} (x') J(x') + \frac{i}{\hbar} \int d^4x \sqrt{\bar{g}} J \right\}. \hspace{1cm} (2.42)$$

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The generating functional $W [J]$ is defined as

$$Z [J] = e^{i \frac{\hbar}{2} W [J]}.$$  \hfill (2.43)

From $Z [J]$ given by Eq. (2.42) we obtain

$$W [J] = -i \hbar \ln Z [J] = \frac{i}{2} \hbar \text{tr} \ln \left( -i \hbar M \right) + \int d^4 x \sqrt{g} \mathcal{J}.$$ \hfill (2.44)

2.2 Quantum effective action: Legendre transform

The quantum effective action is the Legendre transform of the generating functional $W [J]$: \hfill (2.45)

$$\Gamma [g_{\mu\nu}] = W [J] - \int d^4 x \sqrt{g} h_{\mu\nu} \bar{g}^{\mu\nu} \mathcal{J}$$

with \hfill (2.46)

$$h_{\mu\nu} = \frac{\delta W [J]}{\delta (\bar{g}^{\mu\nu} \mathcal{J}).}$$

Here we rewrite $\Gamma [g_{\mu\nu}] = \Gamma [\bar{g}_{\mu\nu} + h_{\mu\nu}] = \Gamma [h_{\mu\nu}]$ for the classical solution $\bar{g}_{\mu\nu}$ is a known function in the functional $\Gamma$. \hfill (2.47)

Rewrite the generating functional $W [J]$ given by Eq. (2.44) as \hfill (2.48)

$$W [J] = \frac{i \hbar}{2} \text{tr} \ln \left( -i \hbar M \right) + \frac{1}{N} \int d^4 x \sqrt{g} \bar{g}_{\mu\nu} \bar{g}^{\mu\nu} \mathcal{J}$$

where $N$ is the dimension of the spacetime.

Taking variation of $W [J]$ gives \hfill (2.49)

$$\delta W [J] = -\frac{1}{2} \int \sqrt{g} d^4 x \sqrt{g} d^4 x' \delta [J (x) \bar{g}^{\rho\sigma} (x) (M^{-1})_{\rho\sigma\mu\nu} (x, x') J (x') \bar{g}^{\mu\nu} (x')] + \frac{1}{N} \int d^4 x \sqrt{g} \delta (\bar{g}^{\mu\nu} \mathcal{J}) \bar{g}_{\mu\nu}. \hfill (2.48)$$

Substituting Eq. (2.48) into Eq. (2.46) gives \hfill (2.49)

$$2 \left( \frac{1}{N} \bar{g}_{\mu\nu} - h_{\mu\nu} \right) = \int \sqrt{g} d^4 x' (M^{-1})_{\mu\nu\rho\sigma} (x, x') J (x') \bar{g}^{\rho\sigma} (x').$$

Now we solve $\mathcal{J}$. The action with the source $\mathcal{J}$, Eq. (2.36), by Eqs. (2.33) and (2.13), reads \hfill (2.50)

$$S_J = \frac{\hbar^2}{4} \int d^4 x \sqrt{g} h_{\rho\sigma} M^{\rho\sigma\mu\nu} h_{\mu\nu} + \frac{1}{2} \hbar \int \sqrt{g} d^4 x h_{\mu\nu} \bar{g}^{\mu\nu} \mathcal{J} + \int \sqrt{g} d^4 x J.$$
Taking variation of $S_J$ and vanishing the surface term give

$$\delta S = \frac{\hbar}{4} \int d^4x \sqrt{\bar{g}} \left( (\bar{g}^{\rho\sigma} \bar{g}^{\epsilon\mu} \bar{g}^{\kappa\lambda} - \frac{1}{2} \bar{g}^{\rho\sigma} \bar{g}^{\epsilon\mu} \bar{g}^{\lambda\mu} + \bar{g}^{\rho\mu} \bar{g}^{\epsilon\nu} \bar{g}^{\lambda\sigma} - \frac{1}{2} \bar{g}^{\rho\mu} \bar{g}^{\lambda\epsilon} \bar{g}^{\nu\sigma}) \bar{D}_\lambda \bar{D}_\kappa h_{\rho\sigma} ight)$$

$$+ \left( (\bar{g}^{\mu\nu} \bar{g}^{\rho\sigma} \bar{g}^{\lambda\kappa} - \frac{1}{2} \bar{g}^{\mu\nu} \bar{g}^{\rho\sigma} \bar{g}^{\lambda\epsilon} + \bar{g}^{\mu\rho} \bar{g}^{\lambda\sigma} \bar{g}^{\nu\epsilon} - \frac{1}{2} \bar{g}^{\mu\rho} \bar{g}^{\lambda\epsilon} \bar{g}^{\nu\sigma}) \bar{D}_\lambda \bar{D}_\kappa h_{\rho\sigma} \right) + \frac{\hbar}{4} \int d^4x \sqrt{\bar{g}} (\delta h_{\mu\nu}) \bar{g}^{\mu\nu} J + \int d^4x \sqrt{\bar{g}} J. \quad (2.51)$$

The minimum of $S_J$, given by $\delta S_J = 0$, gives

$$\frac{1}{2} (M^{\rho\sigma\mu\nu} + M^{\mu\nu\rho\sigma}) h_{\mu\nu} = -\bar{g}^{\rho\sigma} J. \quad (2.52)$$

Substituting Eqs. (2.49) and (2.52) into the Legendre transform (2.45) gives the effective action

$$\Gamma [h_{\mu\nu}] = \frac{i}{2} \hbar \text{tr} \ln (-i\hbar \frac{1}{2} M) + \frac{1}{4} \int d^4x \sqrt{g} h_{\rho\sigma} (M^{\rho\sigma\mu\nu} + M^{\mu\nu\rho\sigma}) h_{\mu\nu}$$

$$- \frac{1}{4N} \int d^4x \sqrt{g} g_{\rho\sigma} (M^{\rho\sigma\mu\nu} + M^{\mu\nu\rho\sigma}) h_{\mu\nu}. \quad (2.53)$$

### 3 Equation of quantum fluctuation

The minimum of the quantum effective action $\Gamma$ gives the equation of the quantum fluctuation:

$$\delta \Gamma = 0 \quad (3.1)$$

Taking variation of the effective action (2.53)

$$\delta \Gamma = \frac{1}{4} \int d^4x \sqrt{g} (\delta h_{\rho\sigma}) (M^{\rho\sigma\mu\nu} + M^{\mu\nu\rho\sigma}) h_{\mu\nu} + \frac{1}{4} \int d^4x \sqrt{g} h_{\rho\sigma} (M^{\rho\sigma\mu\nu} + M^{\mu\nu\rho\sigma}) (\delta h_{\mu\nu})$$

$$- \frac{1}{4N} \int d^4x \sqrt{g} g_{\rho\sigma} (M^{\rho\sigma\mu\nu} + M^{\mu\nu\rho\sigma}) (\delta h_{\mu\nu}), \quad (3.2)$$
In this section, we consider the quantum fluctuation in flat spacetime. Flat space has the metric \( g_{\mu\nu} = \eta_{\mu\nu} \), where \( \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \). The equation of quantum fluctuation is given by

\[
\delta \Gamma = \frac{1}{4} \int d^4x \sqrt{g} \left( \delta h_{\rho\sigma} \left( g^{\mu\nu} g^{\rho\sigma} g_{\mu\lambda} - \frac{1}{2} g^{\mu\nu} g^{\rho\sigma} g_{\lambda\nu} + g^{\mu\rho} g^{\nu\sigma} g_{\lambda\nu} - \frac{1}{2} g^{\mu\rho} g^{\nu\lambda} g_{\sigma\nu} \right) \right) D_\lambda D_\kappa h_{\mu\nu} \\
+ \frac{1}{4} \int d^4x \sqrt{g} \left( \delta h_{\mu\nu} \left( g^{\rho\sigma} g^{\mu\nu} g_{\rho\lambda} - \frac{1}{2} g^{\rho\sigma} g^{\mu\nu} g_{\lambda\nu} + g^{\rho\mu} g^{\sigma\nu} g_{\lambda\nu} - \frac{1}{2} g^{\rho\mu} g^{\sigma\lambda} g_{\nu\sigma} \right) \right) D_\lambda D_\kappa h_{\rho\sigma} \\
+ \frac{1}{4} \int d^4x \sqrt{g} \left( \delta h_{\mu\nu} \left( g^{\rho\sigma} g^{\mu\nu} g_{\rho\lambda} - \frac{1}{2} g^{\rho\sigma} g^{\mu\nu} g_{\lambda\nu} + g^{\rho\mu} g^{\sigma\nu} g_{\lambda\nu} - \frac{1}{2} g^{\rho\mu} g^{\sigma\lambda} g_{\nu\sigma} \right) \right) D_\lambda D_\kappa h_{\rho\sigma} \\
+ \frac{1}{4} \int d^4x \sqrt{g} \left( \delta h_{\mu\nu} \left( g^{\rho\sigma} g^{\mu\nu} g_{\rho\lambda} - \frac{1}{2} g^{\rho\sigma} g^{\mu\nu} g_{\lambda\nu} + g^{\rho\mu} g^{\sigma\nu} g_{\lambda\nu} - \frac{1}{2} g^{\rho\mu} g^{\sigma\lambda} g_{\nu\sigma} \right) \right) D_\lambda D_\kappa h_{\rho\sigma} \\
+ \frac{1}{4} \int d^4x \partial_\lambda \left[ \sqrt{g} \delta h_{\rho\sigma} \left( g^{\mu\nu} g^{\rho\sigma} g_{\mu\lambda} - \frac{1}{2} g^{\mu\nu} g^{\rho\sigma} g_{\lambda\nu} + g^{\mu\rho} g^{\nu\sigma} g_{\lambda\nu} - \frac{1}{2} g^{\mu\rho} g^{\nu\lambda} g_{\sigma\nu} \right) \right] D_\kappa \left( \delta h_{\mu\nu} \right) \\
- \frac{1}{4} \int d^4x \partial_\kappa \left[ \sqrt{g} \left( D_\lambda h_{\rho\sigma} \right) \left( g^{\rho\sigma} g^{\mu\nu} g_{\rho\lambda} - \frac{1}{2} g^{\rho\sigma} g^{\mu\nu} g_{\lambda\nu} + g^{\rho\mu} g^{\sigma\nu} g_{\lambda\nu} - \frac{1}{2} g^{\rho\mu} g^{\sigma\lambda} g_{\nu\sigma} \right) \right] \left( \delta h_{\mu\nu} \right) \\
+ \frac{1}{4} \int d^4x \partial_\lambda \left[ \sqrt{g} \left( D_\kappa h_{\rho\sigma} \right) \left( g^{\rho\sigma} g^{\mu\nu} g_{\rho\lambda} - \frac{1}{2} g^{\rho\sigma} g^{\mu\nu} g_{\lambda\nu} + g^{\rho\mu} g^{\sigma\nu} g_{\lambda\nu} - \frac{1}{2} g^{\rho\mu} g^{\sigma\lambda} g_{\nu\sigma} \right) \right] \left( \delta h_{\mu\nu} \right) \\
- \frac{1}{4} \int d^4x \partial_\kappa \left[ \sqrt{g} \left( D_\lambda h_{\rho\sigma} \right) \left( g^{\rho\sigma} g^{\mu\nu} g_{\rho\lambda} - \frac{1}{2} g^{\rho\sigma} g^{\mu\nu} g_{\lambda\nu} + g^{\rho\mu} g^{\sigma\nu} g_{\lambda\nu} - \frac{1}{2} g^{\rho\mu} g^{\sigma\lambda} g_{\nu\sigma} \right) \right] \left( \delta h_{\mu\nu} \right) \\
- \frac{1}{4} \int d^4x \partial_\lambda \left[ \sqrt{g} \left( D_\kappa h_{\rho\sigma} \right) \left( g^{\rho\sigma} g^{\mu\nu} g_{\rho\lambda} - \frac{1}{2} g^{\rho\sigma} g^{\mu\nu} g_{\lambda\nu} + g^{\rho\mu} g^{\sigma\nu} g_{\lambda\nu} - \frac{1}{2} g^{\rho\mu} g^{\sigma\lambda} g_{\nu\sigma} \right) \right] \left( \delta h_{\mu\nu} \right) \\
- \frac{1}{4N} \int d^4x \partial_\lambda \left[ \sqrt{g} \left( g^{\mu\nu} g^{\rho\sigma} g_{\mu\lambda} - \frac{1}{2} g^{\mu\nu} g^{\rho\sigma} g_{\lambda\nu} + g^{\mu\rho} g^{\nu\sigma} g_{\lambda\nu} - \frac{1}{2} g^{\mu\rho} g^{\nu\lambda} g_{\sigma\nu} \right) \right] D_\kappa \left( \delta h_{\mu\nu} \right) \\
+ \frac{1}{4N} \int d^4x \partial_\kappa \left[ \sqrt{g} \left( g^{\mu\nu} g^{\rho\sigma} g_{\mu\lambda} - \frac{1}{2} g^{\mu\nu} g^{\rho\sigma} g_{\lambda\nu} + g^{\mu\rho} g^{\nu\sigma} g_{\lambda\nu} - \frac{1}{2} g^{\mu\rho} g^{\nu\lambda} g_{\sigma\nu} \right) \right] D_\lambda \left( \delta h_{\mu\nu} \right) \\
+ \frac{1}{4N} \int d^4x \left[ \sqrt{g} \delta h_{\rho\sigma} \left( g^{\mu\nu} g^{\rho\sigma} g_{\mu\lambda} - \frac{1}{2} g^{\mu\nu} g^{\rho\sigma} g_{\lambda\nu} + g^{\mu\rho} g^{\nu\sigma} g_{\lambda\nu} - \frac{1}{2} g^{\mu\rho} g^{\nu\lambda} g_{\sigma\nu} \right) \right] D_\kappa \left( \delta h_{\mu\nu} \right) \\
+ \frac{1}{4N} \int d^4x \left[ \sqrt{g} \delta h_{\mu\nu} \left( g^{\rho\sigma} g^{\mu\nu} g_{\rho\lambda} - \frac{1}{2} g^{\rho\sigma} g^{\mu\nu} g_{\lambda\nu} + g^{\rho\mu} g^{\sigma\nu} g_{\lambda\nu} - \frac{1}{2} g^{\rho\mu} g^{\sigma\lambda} g_{\nu\sigma} \right) \right] D_\lambda \left( \delta h_{\mu\nu} \right) \\
+ \frac{1}{4N} \int d^4x \left[ \sqrt{g} \delta h_{\mu\nu} \left( g^{\rho\sigma} g^{\mu\nu} g_{\rho\lambda} - \frac{1}{2} g^{\rho\sigma} g^{\mu\nu} g_{\lambda\nu} + g^{\rho\mu} g^{\sigma\nu} g_{\lambda\nu} - \frac{1}{2} g^{\rho\mu} g^{\sigma\lambda} g_{\nu\sigma} \right) \right] D_\kappa \left( \delta h_{\mu\nu} \right). \\
\tag{3.3}
\end{align*}

Vanishing the surface term, we arrive at

\[
\delta \Gamma = \frac{1}{2} \int \sqrt{g} d^4x \left( \delta h_{\mu\nu} \left( M^{\rho\sigma\mu\nu} + M^{\nu\rho\sigma\mu} \right) \right) h_{\rho\sigma}.
\tag{3.4}
\end{align*}

\[
\text{The equation of quantum fluctuation} \ h_{\mu\nu} \text{ by Eq. (3.1) is}
\end{align*}

\[
\left( M^{\rho\sigma\mu\nu} + M^{\nu\rho\sigma\mu} \right) h_{\rho\sigma} = 0,
\tag{3.5}
\end{align*}

or equivalently,

\[
\left[ 2g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} \right] \bar{D}^2 - \frac{1}{2} g^{\mu\nu} \bar{D}^\rho \bar{D}^\sigma - \frac{1}{2} g^{\rho\sigma} \bar{D}^\mu \bar{D}^\nu + g^{\mu\rho} \bar{D}^\sigma \bar{D}^\nu + g^{\nu\rho} \bar{D}^\mu \bar{D}^\sigma \right] h_{\rho\sigma} = 0.
\tag{3.6}
\end{align*}

Eq. (3.5) is the equation of the leading-order quantum fluctuation.

## 4 Quantum fluctuation in flat spacetime

In this section, we consider the quantum fluctuation in flat spacetime. Flat space has the largest symmetry, and the symmetry of spacetime after quantum fluctuation is determined by the symmetry of the quantum fluctuation.
Just like in spontaneous magnetization, the system is isotropic before magnetization. The direction of magnetization is determined by the direction of external disturbance. In the calculation of magnetization, to simulate the disturbance, one first add an external magnetic field to the system, and, at the end of the calculation, take the external magnetic field to zero. That is, the effect of this external magnetic field is to point a magnetization direction.

For the sake of calculation, we take the coordinate of flat spacetime to match the symmetry of the fluctuation. In simple cases, for example, for spherically symmetric spacetime, we use spherical coordinates. In complex cases, we use the following method to choose the coordinates for describing flat spacetime. Let us take the Kerr spacetime as an example. If we want the quantum corrected spacetime to have a similar symmetry to the Kerr spacetime, we can take the metric of flat spacetime in the following way. It is known that the zero-mass Kerr spacetime is a flat spacetime. Therefore, we can write the metric of flat spacetime by first writing down a Kerr metric and then vanishing the mass. The curvatures described by the Riemann tensor is of course zero. Note that here in the following case, the Ricci scalar and tensor are zero for the classical spacetime we considered is the vacuum solution of the Einstein equation.

### 4.1 Spherically symmetric quantum fluctuation

In this section, we consider a flat spacetime with a spherically symmetric quantum fluctuation.

We represent the line element of the flat spacetime in spherical coordinates,

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2,$$

where the metric is written in spherical coordinates. Supposing the quantum corrected spacetime is spherically symmetric, we take the line element of the quantum corrected spacetime as

$$ds^2 = [1 + f(r)] dt^2 - [1 + g(r)] dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2.$$  \hspace{1cm} (4.2)

The quantum fluctuation is determined by the equation of fluctuation (3.6). Substituting Eq. (4.2) into Eq. (3.6) gives the following equations

$$2r f'(r) + r^2 f''(r) + 2r g'(r) + \frac{3}{2} r^2 g''(r) - g(r) = 0,$$

$$r \left( -8 f'(r) - 3r f''(r) - 8g'(r) - 4r g''(r) \right) + 2g(r) = 0,$$

$$r \left( 7f'(r) + 4r f''(r) + 7g'(r) + 3r g''(r) \right) + 2g(r) = 0.$$  \hspace{1cm} (4.3)

Solving these equations gives

$$f(r) = -C_1 r^2 + C_2 - C_1 \frac{1}{r},$$  \hspace{1cm} (4.4)

$$g(r) = C_1 r^2 + C_1 \frac{1}{r}.$$  \hspace{1cm} (4.5)
Requiring that the quantum fluctuation $h_{\mu\nu}$ vanishes at $r \to \infty$, we have $C_1 = 0$ and $C_2 = 0$, i.e.,

\begin{align*}
  f(r) &= -\frac{C}{r}, \\
  g(r) &= \frac{C}{r}.
\end{align*}

The quantum corrected result then reads

\begin{equation}
  ds^2 = \left(1 - \frac{C}{r}\right) dt^2 - \left(1 + \frac{C}{r}\right) dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2.
\end{equation}

A simple analysis shows that this quantum corrected spacetime has singularities at $r = 0$ and $r = C$, which can be checked by calculating the Ricci scalar, the Ricci tensor, the Riemann tensor, and the Weyl tensor in the orthogonal frame. This spacetime also has an event horizon and an infinite redshift surface at $r = C$.

By comparing with a small-mass Schwarzschild spacetime whose metric is the leading term of the expansion around $M = 0$ of the Schwarzschild metric,

\begin{equation}
  ds^2 \simeq \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 + \frac{2M}{r}\right) dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2,
\end{equation}

we can see that $C$ plays a role of mass, i.e., $C = 2M$. The constant $C$ comes from a quantum fluctuation, so the mass $M$ must be very small and should be of the magnitude of the Planck mass.

That is, if there occurs a spherically symmetric quantum fluctuation in a flat spacetime, a Schwarzschild spacetime may be created.

It should be emphasized that the quantum corrected flat spacetime is no long a classical vacuum.

### 4.2 Axially symmetric quantum fluctuation

In this section, we consider a flat spacetime with an axially symmetrical quantum fluctuation.

For convenience of calculations, we write the metric of a flat space as a zero-mass Kerr spacetime who is a flat spacetime with zero curvatures, e.g., a zero Riemann curvature.

More concretely, the Kerr metric is [63]

\begin{equation}
  ds^2 = dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - (r^2 + a^2) \sin^2 \theta d\phi^2 - 2 \frac{mr}{\rho^2} (dt - a \sin^2 \theta d\phi)^2,
\end{equation}

where $\rho^2 = r^2 + a^2 \cos^2 \theta$ and $\Delta = r^2 - 2mr + a^2$ with $a$ the angular momentum per unit mass. By the coordinate transformation $\cos \theta = x$, we arrive at the Kerr metric represented by the coordinate $(t, r, x, \varphi)$:

\begin{equation}
  ds^2 = \left[1 - 2\left(\frac{mr}{r^2 + a^2x^2}\right)\right] dt^2 - 4 \frac{amr}{(r^2 + a^2x^2)} dtd\phi - \frac{r^2 + a^2x^2}{r^2 - 2mr + a^2} dr^2
  - \frac{x^2 + a^2x^2}{1 - x^2} dx - \left[(r^2 + a^2) \left(1 - x^2\right) + 2 \frac{mr}{r^2 + a^2x^2} a^2 \left(1 - x^2\right)^2\right] d\phi^2.
\end{equation}
The zero-mass Kerr metric gives a flat metric:

\[
    ds^2 = dt^2 - \frac{r^2 + a^2}{r^2 + a^2} dx^2 - \frac{r^2 + a^2}{1 - x^2} dx^2 - (r^2 + a^2) (1 - x^2) d\varphi^2. 
\]  

(4.12)

Supposing the quantum corrected spacetime is axisymmetric, we take the line element of quantum corrected spacetime as

\[
    ds^2 = \left[1 + a (r, x)\right] dt^2 - \left[\frac{r^2 + a^2 x^2}{r^2 + a^2} - b (r, x)\right] dr^2 - \left[\frac{r^2 + a^2 x^2}{1 - x^2} - c (r, x)\right] d\varphi^2 + 2 \eta (r, x) dt d\varphi. 
\]  

(4.13)

Substituting Eq. (4.13) into Eq. (3.6), we have

\[
    a (r, x) = \frac{-2rm}{r^2 + a^2 x^2}, \\
    b (r, x) = \frac{-2r (r^2 + a^2 x^2) m}{(r^2 + a^2)^2}, \\
    c (r, x) = 0, \\
    d (r, x) = \frac{-2a^2r (x^2 - 1) m}{r^2 + a^2 x^2}, \\
    \eta (r, x) = \frac{-2ar (x^2 - 1) m}{r^2 + a^2 x^2}. 
\]  

(4.14)

The quantum corrected result then reads

\[
    ds^2 = \left(1 - \frac{2rm}{r^2 + a^2 x^2}\right) dt^2 - \left[\frac{r^2 + a^2 x^2}{r^2 + a^2} + \frac{2r (r^2 + a^2 x^2) m}{(r^2 + a^2)^2}\right] dr^2 - \frac{r^2 + a^2 x^2}{1 - x^2} d\varphi^2 - 2 \frac{2ar (x^2 - 1) m}{r^2 + a^2 x^2} dt d\varphi. 
\]  

(4.15)

The singularity of the metric appears at

\[
    x = \pm 1. 
\]  

(4.16)

The singularity of the Ricci tensor appears at

\[
    x = \pm 1, \\
    r = m \pm \sqrt{m^2 - a^2}, \\
    r = -m \pm \sqrt{m^2 - a^2}. 
\]  

(4.17)

The singularities of the Riemann tensor, the Weyl tensor, and the Ricci scalar appear at

\[
    r = m \pm \sqrt{m^2 - a^2}, \\
    r = -m \pm \sqrt{m^2 - a^2}. 
\]  

(4.18)

The spacetime with the metric (4.15) has an event horizon at \( r = \pm a \) and an infinite redshift surface at \( r = m \pm \sqrt{m^2 - ax} = m \pm \sqrt{m^2 - a \cos^2 \theta} \).

Comparing with the Kerr case, we can see that quantum corrected metric (4.15) is a small-mass Kerr metric, i.e., the leading term of the expansion of the Kerr metric around \( m = 0 \).
4.3 Cylindrically symmetric quantum fluctuation

In this section, we consider a flat spacetime with a cylindrically symmetric quantum fluctuation.

We write the metric of a flat space as a zero-mass Curzon spacetime who is a flat spacetime with zero curvatures.

The 1 + 3-dimensional Curzon metric is \[ ds^2 = \exp \left( -\frac{2m}{\sqrt{\rho^2 + z^2}} \right) dt^2 - \exp \left( \frac{2m}{\sqrt{\rho^2 + z^2}} \right) \left[ \exp \left( -\frac{m^2 \rho^2}{(\rho^2 + z^2)^2} \right) (d\rho^2 + dz^2) + \rho^2 d\varphi^2 \right] \].

By the coordinate transformation \( \rho = (r^2 - 2mr)^{1/2} \sin \theta \) and \( z = (r - m) \cos \theta \), we arrive at the Curzon metric represented by the spherical coordinate \( (t, r, \theta, \varphi) \):

\[
ds^2 = \exp \left( -\frac{2m}{\sqrt{r^2 - 2mr + m^2 \cos^2 \theta}} \right) dt^2 - \exp \left( \frac{2m}{\sqrt{r^2 - 2mr + m^2 \cos^2 \theta}} \right) \times \left[ \exp \left( -\frac{m^2 (r^2 - 2mr) \sin^2 \theta}{(r^2 - 2mr + m^2 \cos^2 \theta)^2} \right) (r^2 - 2mr + m^2 \sin^2 \theta) \left( \frac{1}{r^2 - 2mr} dr^2 + d\theta^2 \right) + (r^2 - 2mr) \sin^2 \theta d\varphi^2 \right].
\]

Next, by the coordinate transformation \( \cos \theta = x \), we arrive at the Curzon metric represented by the coordinate \( (t, r, x, \varphi) \):

\[
ds^2 = \exp \left( -\frac{2m}{\sqrt{r^2 - 2mr + m^2 x^2}} \right) dt^2 - \exp \left( \frac{2m}{\sqrt{r^2 - 2mr + m^2 x^2}} \right) \times \left\{ \exp \left( -\frac{m^2 (r^2 - 2mr) (1 - x^2)}{(r^2 - 2mr + m^2 x^2)^2} \right) \left[ r^2 - 2mr + m^2 (1 - x^2) \right] \right\} dr^2 + \frac{1}{1 - x^2} dx^2 + (r^2 - 2mr) (1 - x^2) d\varphi^2.
\]

The zero-mass Curzon metric gives a flat metric:

\[
ds^2 = dt^2 - dr^2 - \frac{r^2}{1 - x^2} dx^2 - r^2 (1 - x^2) d\varphi^2.
\]

Similar procedure gives

\[
ds^2 = \left( 1 - \frac{C}{r} \right) dt^2 - \left( 1 + \frac{C}{r} \right) dr^2 - \frac{r^2}{1 - x^2} dx^2 - r^2 (1 - x^2) d\varphi^2.
\]

The quantum fluctuation is small. To determine the constant \( C \), we compare Eq. (4.24) with the Curzon metric with a small mass. The comparison gives \( C = 2m \). The quantum corrected metric is

\[
ds^2 = \left( 1 - \frac{2m}{r} \right) dt^2 - \left( 1 + \frac{2m}{r} \right) dr^2 - \frac{r^2}{1 - x^2} dx^2 - r^2 (1 - x^2) d\varphi^2.
\]

This is the Curzon metric with a small mass. The singularity of the metric (4.25) is at \( r = 0 \) and \( x = \pm 1 \). The singularity of the Ricci tensor, the Riemann tensor, the Weyl tensor, and the Ricci scalar is at \( r = 0 \) and \( r = 2m \). The infinite redshift surface is at \( r = 2m \). This spacetime has no event horizon.
5 Quantum fluctuation in Schwarzschild spacetime

The classical Schwarzschild spacetime

\[
 ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{1}{1 - \frac{2m}{r}} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2
\]  

(5.1)

is a bound state of gravitational fields. Suppose there is a quantum fluctuation in the form

\[
 ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{1}{1 - \frac{2m}{r}} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2 + 2 \sin^2 \theta f(r) dt d\varphi
\]  

(5.2)

Substituting Eqs. (5.1) and (5.2) into the equation of quantum fluctuations, Eq. (3.6), we obtain an equation for \( f(r) \):

\[
 (2r - 3m) f(r) - r^2 (r - 2m) f''(r) = 0.
\]  

(5.3)

The solution is

\[
 f(r) = Cf_1(r) + C'f_2(r)
\]  

(5.4)

with

\[
 f_1(r) = r^{\frac{1-\sqrt{7}}{2}} (r - 2m) \ 2 F_1 \left( \frac{5 - \sqrt{7}}{2}, -\frac{1 - \sqrt{7}}{2}, 1 - \sqrt{7}, \frac{r}{2m} \right),
\]  

(5.5)

\[
 f_2(r) = r^{\frac{1+\sqrt{7}}{2}} (r - 2m) \ 2 F_1 \left( \frac{5 + \sqrt{7}}{2}, -\frac{1 + \sqrt{7}}{2}, 1 + \sqrt{7}, \frac{r}{2m} \right).
\]  

(5.6)

Next we determine the constants \( C \) and \( C' \).

For \( 0 < r \leq 2m \), \( f_1(r) \) and \( f_2(r) \) are both real functions. This requires that \( C \) and \( C' \) should be real numbers.

For \( r > 2m \), \( f_1(r) \) and \( f_2(r) \) are both complex functions. We rewrite Eq. (5.4) as

\[
 f(r) = \left( C \text{Re} f_1 + C' \text{Re} f_2 \right) + i \left( C \text{Im} f_1 + C' \text{Im} f_2 \right)
\]  

(5.7)

Requiring \( f(r) \) is a real function, we must have \( C \text{Im} f_1 + C' \text{Im} f_2 = 0 \), i.e.,

\[
 \frac{C'}{C} = \frac{\text{Im} f_1}{\text{Im} f_2} \equiv k.
\]  

(5.8)

To determine the constant \( k = -C' / C \), we expand \( f_1(r) \) and \( f_2(r) \) at \( r \to \infty \):

\[
 k = -\frac{1}{m^{\sqrt{7}}} \frac{(4 + \sqrt{7}) \Gamma \left( 1 - \frac{\sqrt{7}}{2} \right) \Gamma \left( \frac{5}{2} + \frac{\sqrt{7}}{2} \right)}{3 \times 8 \sqrt{7} \Gamma \left( 1 + \frac{\sqrt{7}}{2} \right) \Gamma \left( \frac{5}{2} - \frac{\sqrt{7}}{2} \right)}.
\]  

(5.9)

Then we have \( f(r) = C (\text{Re} f_1 - k \text{Re} f_2) \). That is, \( f(r) \) is real for \( r > 2m \), for \( C \) and \( k \) are real.

Now we rewrite \( f(r) \), by substituting Eq. (5.8) into Eq. (5.4), as

\[
 f(r) = C \left[ f_1(r) - kf_2(r) \right].
\]  

(5.10)
The quantum corrected result then reads

\[ ds^2 = \left( 1 - \frac{2m}{r} \right) dt^2 - \frac{1}{1 - \frac{2m}{r}} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + 2C \sin^2 \theta [f_1(r) - kf_2(r)] dt d\phi. \]

The singularities of the metric and various curvatures, such as the Riemann tensor, are at \( r = 0 \) and \( r = 2m \), as that in the classical Schwarzschild spacetime. This spacetime has an event horizon and infinite redshift surface at \( r = 2m \).

6 Conclusion

In this paper, we construct the quantum effective action for gravitational fields by the effective potential method in quantum field theory. Based on the quantum effective action, we construct the equation for the leading contribution of the quantum fluctuation. The equation of quantum fluctuation is a linear equation. If the higher-order contribution of quantum fluctuations is taken into account, the equation for the quantum fluctuation will become a nonlinear equation.

The premise of the effective potential method is that one must first have a classical solution of the field equation, and then calculate the quantum fluctuation for each classical solution. For matter fields, the solution of the classical field equation serves as an effective potential in the equation of quantum fluctuation. For gravitational fields, the solution of the classical gravitational field equation, the Einstein equation, also serves as an effective potential but in a more complex form. Different classical solutions give different effective potentials. The advantage of this method is that we can deal with the quantum effect by quantum field theory method even for bound states.

Flat spacetime is a classical vacuum. In this paper, we calculate the quantum fluctuation in flat spacetime. When there occurs a quantum fluctuation, a spacetime is created. The spacetime created from quantum fluctuations, although the mass is very small, may be of the magnitude of the Planck mass, is a seed of spacetime, namely, a baby spacetime. Some of the baby spacetime will grow into our present spacetime. With the growth of spacetime, the quantum effect will become less and less obvious, closer and closer to the classical spacetime described by the Einstein equation.

We consider the quantum fluctuations in the Schwarzschild spacetime. The Schwarzschild spacetime is a bound state of gravitational field. The usual quantum-field-theory treatment based on the scattering-state perturbation cannot deal with bound states. We deal with this problem using the effective potential method.

In principle, quantum fluctuations occur randomly and arbitrarily. However, for ease of calculation, only some quantum fluctuations with special symmetries are considered. Quantum fluctuations are of course not limited to special symmetry cases. Like finding the exact solution for the Einstein equation, we can only treat such relatively simple cases.

In this paper, we construct an equation for quantum fluctuation by solving the quantum effective action. The quantum effective action is a spectral function \([64–66]\). In Ref. \([66]\), we construct an equation for one-loop effective action. In future works, we may consider the quantum effect in such a way.
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