THE PERIPLECTIC $q$-BRAUER CATEGORY

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Abstract. We introduce the periplectic $q$-Brauer category over an integral domain of characteristic not 2. This is a strict monoidal supercategory and can be considered as a $q$-analogue of the periplectic $q$-Brauer category in [13]. We prove that the periplectic $q$-Brauer category admits a split triangular decomposition in the sense of [6]. When the ground ring is an algebraically closed field, the category of locally finite dimensional right modules for the periplectic $q$-Brauer category is an upper finite fully stratified category in the sense of [6]. We prove that periplectic $q$-Brauer algebras defined in [1] are isomorphic to endomorphism algebras in the periplectic $q$-Brauer category. Furthermore, a periplectic $q$-Brauer algebra is a standardly based algebra in the sense of [13]. We construct a Jucys-Murphy basis for any standard module of the periplectic $q$-Brauer algebra with respect to a family of commutative elements called Jucys-Murphy elements. Via them, we classify blocks for both periplectic $q$-Brauer category and periplectic $q$-Brauer algebras in generic case. Our result shows that both periplectic $q$-Brauer category and periplectic $q$-Brauer algebras are always not semisimple over any algebraically closed field.

1. Introduction

The periplectic Brauer category was introduced by Moon when he studied tensor product of the natural module for the periplectic Lie superalgebra $p_n$ [20]. Later on, Kujawa and Tharp [18] introduced the periplectic Brauer category and proved that any periplectic Brauer algebra appears as an endomorphism algebra of a corresponding object in the periplectic Brauer category. Since then the representation theory of periplectic Brauer algebras have been studied extensively [38] [10], including the classification of blocks, decomposition numbers and the related (weak) categorification. These results have important applications in the representation theory of $p_n$. See [8] for the classification of blocks in the category of finite dimensional $p_n$-modules.

There are many directions to generalize previous results. For example, the affine version of periplectic Brauer algebra (resp., periplectic Brauer category) was introduced in [7] (resp., [2]) and a general monoidal supercategory related to representations of $p_n$ was introduced in [11] and has applications in the study of the Howe duality between $p_n$ and $p_n$ [12]. In [1], Ahmed-Grantcharov-Guay introduced the periplectic $q$-Brauer algebra when they studied the tensor product of the natural module of the quantized enveloping superalgebra $U_q(p_n)$. The aim of this paper is to introduce the periplectic $q$-Brauer category $B$ and to study its representation theory. As expected in the introduction of [11], the periplectic $q$-Brauer category is a $q$-analogue of the periplectic Brauer category and the periplectic $q$-Brauer algebra appears as an endomorphism algebra of a corresponding object in the periplectic $q$-Brauer category. Motivated by [8], we expect that the periplectic $q$-Brauer category will be a useful tool to study finite dimensional $U_q(p_n)$-modules. Details will be given elsewhere.

We briefly explain this paper as follows. After introducing the notion of the periplectic $q$-Brauer category $B$, we prove that any morphism set in $B$ is free over an integral domain $k$ of characteristic not 2. As an application, we prove that the periplectic $q$-Brauer algebra $B_{q,l}$ is isomorphic to the endomorphism algebra of the object $l$ for any natural number $l$. Moreover, we prove that $B$ admits a split triangular decomposition in the sense of [6] Remark 5.32. When the ground ring $k$ is an algebraically closed field, the previous result implies that the category of locally finite dimensional right $B$-module is an upper finite fully stratified category in the sense of [6] Definition 3.34. When $q$ is not a root of unity, the category of locally finite dimensional right $B$-module is an upper finite highest weight category in the sense of [6] Definition 3.34. In this case, we classify blocks of $B$. For this purpose, we construct a standard basis of $B_{q,l}$ in the sense of [13] Definition 1.2.1. By studying...
classical branching rule for $B_{q,l}$, we construct a nice basis, called Jucys-Murphy basis for any standard module in the category of right $B_{q,l}$-modules. We also construct a family of commutative elements of $B_{q,l}$ so that each of them acts upper-triangularly on the Jucys-Murphy basis of a standard module. Therefore, these commutative elements are Jucys-Murphy elements of $B_{q,l}$ with respect to the Jucys-Murphy basis of standard modules for $B_{q,l}$ in the sense of [21, Definition 2.4].

Restricting any standard module to the commutative subalgebra generated by Jucys-Murphy elements, we obtain partial results on the classification of blocks. When $q$ is not a root of unity, we investigate composition factors of certain standard modules for $B_{q,l}$ and finally obtain a classification of blocks for both $B$ and $B_{q,l}$ for any natural numbers $l$. In the later case, $q$ can be a root of unity. However, we need to assume that the quantum characteristic of $q^2$ is bigger than $l$. Our results which can be considered as a counterpart of that for periplectic Brauer algebras in [8] may be used to study finite dimensional $U_q(p_n)$-modules.

The paper is organized as follows. After recalling the notion of monoidal supercategories, we introduce the periplectic $q$-Brauer category $B$ and give a spanning set of any morphism space in section 2. In section 3, we construct a monoidal functor from $B$ to the category of representations of $U_q(p_n)$. Using this we prove that any morphism space of $B$ is free over $k$ with required rank in section 4. As an application, we prove that periplectic $q$-Brauer algebra $B_{q,l}$ is isomorphic to $\text{End}_B(l)$ for any natural number $l$. In section 5, we prove that $B$ admits a split triangular decomposition in the sense of [6]. Consequently, we prove that the category of locally finite dimensional right $B$-modules is an upper finite fully stratified category if $k$ is an algebraically closed field. In section 6, we study the classical branching rule for $B_{q,l}$. Via it, we construct the Jucys-Murphy basis of any standard module in the category of right $B_{q,l}$-modules in section 7. Jucys-Murphy elements act on the Jucys-Murphy basis upper-triangularly. This gives a partial result on the classification of blocks in section 8. Finally, we assume $q$ is not a root of unity (resp., quantum characteristic of $q^2$ is bigger than $l$). We study composition factors of certain standard modules in details and finally obtain a classification of blocks for both periplectic $q$-Brauer category and periplectic $q$-Brauer algebras in generic case.

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2. The periplectic $q$-Brauer category

The aim of this section is to introduce the periplectic $q$-Brauer category. Throughout, $k$ is an integral domain of characteristic not 2. First, we recall the notion of a strict monoidal supercategory in [4]. See also [5] when $k$ is a field.

2.1. A strict monoidal supercategory. A $k$-supermodule $M = M_{\pi} \oplus M_{\pi'}$ is a $\mathbb{Z}_2$-graded $k$-module. Elements in $M_{\pi}$ (resp., $M_{\pi'}$) have parity 0 or are said to be even (resp., parity 1 or odd). Following [4], the parity of a homogeneous element $m \in M$ is denoted by $[m]$. Given two $k$-supermodules $M$ and $N$, the tensor product $M \otimes N$ is again a $k$-supermodule with the $\mathbb{Z}_2$-grading by declaring that $[m \otimes n] = [m] + [n]$ for all homogeneous elements $(m, n) \in M \times N$. Given two homogeneous homomorphisms $f : M \rightarrow M'$ and $g : N \rightarrow N'$ between $k$-supermodules, the tensor product $f \otimes g$ is defined as

$$ (f \otimes g)(m \otimes n) = (-1)^{[g][m]} f(m) \otimes g(n), $$

where $[g]$ is the parity of $g$. The following is the Koszul sign rule where $f, g, h$ and $k$ are homogeneous homomorphisms:

$$ (f \otimes g) \circ (h \otimes k) = (-1)^{[g][h]} (f \circ h) \otimes (g \circ k). $$

A supercategory is a category enriched in $k$-supermodules in the sense that each morphism set is a $k$-supermodule and composition induces an even $k$-homomorphism. A superfunctor $F$ between two supercategories is an even functor enriched in $k$-supermodules.

Definition 2.1. [4, §6] A strict monoidal supercategory is a supercategory $C$ equipped with a bi-superfunctor $- \otimes - : C \times C \rightarrow C$ and a unit object $1$ such that for all objects $a, b, c$ of $C$, we have $(a \otimes b) \otimes c = a \otimes (b \otimes c)$ and $1 \otimes a = a = a \otimes 1$, and for all morphisms $f, g$ and $h$ of $C$, we have $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ and $1_x \otimes f = f = f \otimes 1_x$, where $1_a : a \rightarrow a$ is the identity morphism for any object $a$.

1One can also construct Jucys-Murphy basis of any standard module in the category of left $B_{q,l}$-modules in a similar way. Gluing two kinds of Jucys-Murphy basis together, one will get the Jucys-Murphy basis of $B_{q,l}$ in a standard way. We will not give details since we do not need it in this paper.
There is a well-defined graphical calculus for morphisms in a strict monoidal supercategory $C$ (c.f. [5, §1.2]). For any two objects $a, b$ in $C$, $ab$ represents $a \otimes b$. A morphism $g : a \to b$ is drawn as a coupon label by $g$

$$\begin{array}{c}
\downarrow & \quad \downarrow \\
b & \quad a
\end{array}$$

where $a$ is at the bottom and $b$ is at the top. The identity morphism $1_a : a \to a$ is drawn as a string with no coupon

$$\begin{array}{c}
a \\
\downarrow
\end{array}$$

We often omit the labels of objects (i.e. $a$ and $b$ above) if there is no confusion on the objects. Moreover, the axioms of a strict monoidal supercategory make it natural to omit the identity morphism $1_{1_1}$ of the unit object. So, any morphism $g \in \text{End}(1)$ will be drawn as a free-floating coupon: $g$. The composition (resp., tensor product) of two morphisms is given by vertical stacking (resp., horizontal concatenation) as follows:

$$g \circ h = \begin{array}{c}
\downarrow & \quad \downarrow \\
h & \quad g
\end{array}, \quad g \otimes h = \begin{array}{c}
\downarrow & \quad \downarrow & \quad \downarrow \\
h & \quad g & \quad 1
\end{array}.$$  

Since $- \otimes - : C \times C \to C$ is a superfunctor, there is a super interchange law

$$\left( f \otimes g \right) \circ \left( k \otimes h \right) = (-1)^{[k][g]} \left( f \circ k \right) \otimes \left( g \circ h \right),$$

for any homogeneous morphisms $f, g, k$ and $h$. The left side of (2.4) is drawn as

$$\begin{array}{c}
\downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow \\
f & \quad g & \quad k & \quad h
\end{array},$$

which is not the same as $(f \circ k) \otimes (g \circ h)$ in general. By the super interchange law, we have

$$\left( f \otimes g \right) \circ \left( k \otimes h \right) = (-1)^{[f][g]} \left( f \circ k \right) \otimes \left( g \circ h \right).$$

2.2. The periplectic $q$-Brauer category. Throughout, we assume $q, q^{-1} \in k^\times$ where $k^\times$ is the set of all invertible elements in $k$.

Definition 2.2. The periplectic $q$-Brauer category $B$ is a strict monoidal supercategory generated by a single object $\hbar$ and two even morphisms

$$\begin{array}{c}
\circlearrowleft & \quad \circlearrowright \\
\downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow
\end{array}$$

and two odd morphisms

$$\begin{array}{c}
\circlearrowleft : 1 \to \hbar & \quad \text{and} \quad \circlearrowright : 1 \to \hbar,
\end{array}$$

subject to the following defining relations:

(R1) The braid relations:

$$\begin{array}{c}
\circlearrowleft \circlearrowleft = \circlearrowleft \circlearrowleft = \circlearrowleft \circlearrowleft \quad \text{and} \quad \circlearrowright \circlearrowright = \circlearrowright \circlearrowright \circlearrowright
\end{array}.$$  

(R2) The skein relation:

$$\begin{array}{c}
\circlearrowleft - \circlearrowright = (q - q^{-1})\hbar
\end{array}.$$  

(R3) The snake relations:

$$\begin{array}{c}
\circlearrowleft = \hbar & \quad \text{and} \quad \circlearrowright = \hbar,
\end{array}$$

(R4) The untwisting relations:

$$\begin{array}{c}
\hbar = q & \quad \text{and} \quad \hbar = q
\end{array}.$$  

(R5) The loop removing relation:

$$\hbar = 0.$$
Since \( B \) is generated by a single object \( \mathbb{1} \), the set of objects is \( \mathbb{N} \) where 0 is the unit object \( \mathbb{1}^{\otimes 0} \) and \( m \) is \( \mathbb{1}^{\otimes m} \). This notation is different from that in Definition 2.1 where the unit object is denoted by \( \mathbb{1} \). In the current case, each endpoint at both rows of the diagrams \( \bigotimes, \bigotimes, \bigotimes \) represents the object 1. If there is no endpoints at a row, then the object at this row is the unit object 0. For example, the object at the bottom (resp., top) row of \( \bigotimes \) is 0 (resp., 2). Note that the identity morphism \( \mathbb{1}_m \) is drawn as the object itself when \( m > 0 \). For example, \( \bigotimes \) in (R1) is the identity morphism \( \mathbb{1}_2 \).

Lemma 2.3. In \( B \), we have

\[
\begin{align*}
(1) & \quad \bigotimes = \bigotimes, \\
(2) & \quad \bigotimes = -q, \\
(3) & \quad \bigotimes = -q, \\
(4) & \quad \bigotimes = q^{-1}, \\
(5) & \quad \bigotimes = q = -q.
\end{align*}
\]

Proof. The supercategroy structure means that the height moves satisfy the super interchange law in \( (2.4) \). This proves (1). We have

\[
\begin{align*}
\bigotimes & \overset{(R3)}{=} -\bigotimes \overset{(1)}{=} \bigotimes \overset{(R4)}{=} \bigotimes \overset{(R3)}{=} \bigotimes,
\end{align*}
\]

and (2) follows. The equation in (3) follows from the following equalities:

\[
\begin{align*}
\bigotimes & \overset{(R3)}{=} -\bigotimes \overset{(1)}{=} \bigotimes \overset{(R4)}{=} \bigotimes \overset{(R3)}{=} -q,
\end{align*}
\]

Finally, one can check (4)-(5) via Definition \( (2.2) \) (R2)–(R5) and (2)-(3), easily. \( \square \)

We are going to construct a spanning set of \( \text{Hom}_B(m, s) \) via \((m, s)\)-tangle diagrams. An \((m, s)\)-tangle diagram is a string diagram obtained by tensor products and compositions of the generating morphisms \( \bigotimes, \bigotimes, \bigotimes, \bigotimes \) and the identity morphism \( \mathbb{1} \) such that there are \( m \) (resp., \( s \)) endpoints on the bottom (resp., top) row of the resulting diagram. For example, the diagram in \((2.6)\) is a \((5, 3)\)-tangle diagram:

\[
\begin{align*}
(2.6)
\end{align*}
\]

A strand with no endpoints in a tangle diagram is called a loop, namely a loop is obtained by projecting a connected framed link (i.e., a framed knot [23, Page 15]) in \( \mathbb{R}^3 \). For example there is a loop in the tangle diagram in \((2.6)\).

Let \( \mathcal{B}_{m,s} \) be the set of all \((m, s)\)-tangle diagrams. Obviously, \( \mathcal{B}_{m,s} \) spans \( \text{Hom}_B(m, s) \) when \( m + s \) is even. In the remaining cases, \( \mathcal{B}_{m,s} = \emptyset \) and \( \text{Hom}_B(m, s) = 0 \).

We always assume that \( m + s \) is even later on. In order to obtain a basis of \( \text{Hom}_B(m, s) \), we need to consider special \((m, s)\)-tangle diagrams as follows. We label endpoints at bottom (resp., top) row of an \((m, s)\)-tangle diagram \( d \) as 1, 2, \ldots, \( m \) (resp., \( \overline{1}, \overline{2}, \ldots, \overline{s} \)) from left to right such that

\[
\begin{align*}
1 < 2 < \cdots < m < \overline{1} < \cdots < \overline{s} < \overline{1}.
\end{align*}
\]
For example, $d_1, d_2$ and $d_3$ are three $(4,2)$-tangles, where

\begin{align}
(2.8) \quad & d_1 = \begin{array}{c}
\includegraphics{tangle1.png}
\end{array}, \quad d_2 = \begin{array}{c}
\includegraphics{tangle2.png}
\end{array}, \quad d_3 = \begin{array}{c}
\includegraphics{tangle3.png}
\end{array}.
\end{align}

Any $(m,s)$-tangle diagram $d$ decomposes $\{1, 2, \ldots, m, \underline{\pi}, \underline{\tau}, \underline{\zeta}, \underline{T}\}$ into $\frac{1}{2}(m+s)$ disjoint pairs, say $(i_k, j_k)$'s such that $i_k < j_k$, $1 \leq k \leq \frac{1}{2}(m+s)$ and $i_o < i_l$, $1 \leq o < l \leq \frac{1}{2}(m+s)$. We use $(i_k, j_k)$ to denote the strand of $d$ such that the corresponding endpoints are labelled by $i_k$ and $j_k$. Let

\begin{align}
(2.9) \quad & \text{conn}(d) = \{(i_k, j_k) \mid 1 \leq k \leq \frac{1}{2}(m+s)\}
\end{align}

and call it the $(m,s)$-connector of $d$. So \text{conn}(d_i) = \{(1,3), (2,T), (4,T)\}, 1 \leq i \leq 3$ where $d_i$'s are given in (2.8). Let

\begin{align}
\text{conn}(m,s) = \{\text{conn}(d) \mid d \in \mathbb{B}_{m,s}\}.
\end{align}

A strand connecting a pair on different rows (resp., the same row) is called a vertical (resp., horizontal) arc. Moreover, a horizontal arc connecting a pair on the top (resp., bottom) row is also called a cup (resp., cap). So \includegraphics{cup.png} is a cap and \includegraphics{cup.png} is a cup.

**Definition 2.4.** [22] § 2.3 Suppose $d \in \mathbb{B}_{m,s}$ such that $\text{conn}(d)$ is given in (2.9). Let $k = \frac{1}{2}(m+s)$ and choose $i_1, i_2, \ldots, i_k$ and one point on each loop as a sequence of base-points. Then $d$ is said to be totally descending if on traversing all the strands of $d$, starting from the base point of each component in order, each crossing is first met as an over-crossing.

All tangle diagrams in (2.8) are totally descending whereas all tangle diagrams in (2.10) are not:

\begin{align}
(2.10) \quad & d_4 = \begin{array}{c}
\includegraphics{tangle4.png}
\end{array}, \quad d_5 = \begin{array}{c}
\includegraphics{tangle5.png}
\end{array}, \quad d_6 = \begin{array}{c}
\includegraphics{tangle6.png}
\end{array}.
\end{align}

**Lemma 2.5.** The morphism set $\text{Hom}_\mathbb{B}(m,s)$ is spanned by all totally descending $(m,s)$-tangle diagrams.

**Proof.** One can check the result by arguments similar to those in [22] Theorem 2.6]. More explicitly, these arguments are about induction firstly on the number of crossings and then on the number of non-descending crossings. The only difference is that we need to use Definition 2.2(R2) to replace a non-descending crossing in a tangle diagram and obtain tangle diagrams with fewer non-descending crossings or fewer crossings. \hfill \Box

**Definition 2.6.** A tangle diagram is called reduced if two strands cross each other at most once and neither a strand crosses itself nor there is a loop. A reduced totally descending tangle diagram is both totally descending and reduced.

**Corollary 2.7.** The morphism space $\text{Hom}_\mathbb{B}(m,s)$ is spanned by reduced totally descending $(m,s)$-tangle diagrams.

**Proof.** First, we assume $T$ is a totally descending tangle diagram containing at least one loop. Thanks to Definition 2.2(R1), loops of $T$ are unknotted and lie away from horizontal and vertical arcs. By Lemma 2.3(4)(5) and Definition 2.2(R5), $T$ has to be zero. Next, we consider $T$ without any loop. By Lemma 2.3(4)(5) and Definition 2.2(R1), two strands cross each other at most once and no strand crosses itself. Consequently, by Lemma 2.3 any $(m,s)$-tangle diagram can be expressed as a linear combination of reduced totally descending tangle diagrams. \hfill \Box

Let $\mathbb{R} \mathbb{B}_{m,s}$ be the set of all reduced totally descending diagrams in $\mathbb{B}_{m,s}$. We remark that there are many reduced totally descending diagrams with the same connector $c \in \text{conn}(m,s)$.

**Lemma 2.8.** Suppose $d, d_1 \in \mathbb{R} \mathbb{B}_{m,s}$ such that $\text{conn}(d) = \text{conn}(d_1)$. Then $d = d_1$ up to a sign.
Proof. Since we are assuming that $d, d_1 \in \mathbb{R}_{m,s}$ and $\text{conn}(d) = \text{conn}(d_1)$, $d$ can be obtained from $d_1$ by a sequence of following movements:

1. changing of the order of crossings,
2. $\bigcup \sim \bigcap$,
3. $\bigcap \sim \bigcup$.

Then the result follows by applying Definition 2.2(R1), (R3)-(R4) and Lemma 2.8.

Although there are many reduced totally descending diagrams with the same connector $c$, by Lemma 2.8 it is reasonable to define

$$DB_{m,s} = \{ D_c \mid c \in \text{conn}(m,s) \}.$$ \hfill \(\square\)

Theorem 2.9. If $m + s$ is even then $\text{Hom}_B(m,s)$ is free over $k$ with basis $DB_{m,s}$.

The proof of Theorem 2.9 will be given in section 4. In the remaining part of this section, we illustrate that we can assume $s = 0$ when we prove Theorem 2.10. For any positive integer $m$, define two morphisms $\gamma_m$ and $\eta_m$ in $B$ as follows:

$$\gamma_m = \includegraphics[width=1cm]{gamma_m}, \quad \text{and} \quad \eta_m = \includegraphics[width=1cm]{eta_m}. \quad (2.12)$$

Lemma 2.10. For any positive integer $m$, we have

1. $(1_m \otimes \eta_m) \circ (\gamma_m \otimes 1_m) = (-1)^{\frac{m(m-1)}{2}} 1_m$,
2. $(\eta_m \otimes 1_m) \circ (1_m \otimes \gamma_m) = (-1)^{\frac{m(m-1)}{2}} 1_m$.

Proof. Both (1) and (2) follow immediately from Definition 2.2(R3) and (2.4).

Lemma 2.11. For any $m, s \in \mathbb{N}$ with $2 \mid (m + s)$, $\text{Hom}_B(m,s) \cong \text{Hom}_B(m+s,0)$ as $k$-modules.

Proof. By Lemma 2.10, $\pi_\gamma$ is the two-sided inverse of $\pi_\eta$ up to a sign, where $\pi_\gamma : \text{Hom}_B(m,s) \to \text{Hom}_B(m+s,0)$ is the $k$-homomorphism such that $\pi_\gamma(d) = \eta_s \circ (1_s \otimes d)$ for any $d \in \text{Hom}_B(m,s)$, and $\pi_\eta$ is the $k$-homomorphism such that $\pi_\eta(e) = (1_s \otimes e) \circ (\gamma_s \otimes 1_m)$ for any $e \in \text{Hom}_B(m+s,0)$. \hfill \(\square\)

3. The quantized enveloping superalgebra $U_q(\mathfrak{p}_n)$

In this section, we will recall some basic results of the quantized enveloping superalgebras $U_q(\mathfrak{p}_n)$ in [1].

3.1. The quantized enveloping superalgebras $U_q(\mathfrak{p}_n)$. For a positive integer $n$, we set $I_{n|n} := \{-n, \ldots, -1, 1, \ldots, n\}$, on which we define the parity $[i]$ of $i \in I_{n|n}$ such that $[i] = \overline{i}$ (resp., $\overline{i}$) if $i > 0$ (resp., $i < 0$). Let $V$ be the $C$-superspace with basis $\{ v_i \mid i \in I_{n|n} \}$ such that the parity of $v_i$ is $[i]$. Then the even subspace of $V$ is $V^e = C \{ v_i \mid 1 \leq i \leq n \}$ and the odd subspace of $V$ is $V^o = C \{ v_i \mid -n \leq i \leq -1 \}$. The endomorphism algebra $\text{End}(V)$ is an associative superalgebra with standard basis $E_{i,j}$ whose parity is $[i] + [j]$ for all $i, j \in I_{n|n}$. Under the standard supercommutator, $\text{End}(V)$ is the general linear Lie superalgebra $\mathfrak{gl}_{n|n}$. The super-transpose $(.)^s$ on $\mathfrak{gl}_{n|n}$ is given by the formula

$$(E_{i,j})^{st} = (-1)^{[i][j]+1} E_{j,i}.$$ 

Let $\pi : \mathfrak{gl}_{n|n} \to \mathfrak{gl}_{n|n}$ be the linear map such that $\pi(E_{i,j}) = E_{-i,-j}$. Then the linear map $\ell : \mathfrak{gl}_{n|n} \to \mathfrak{gl}_{n|n}$ sending $x$ to $-\pi(x^s)$ is an involution on $\mathfrak{gl}_{n|n}$. The periplectic Lie superalgebra is

$$\mathfrak{p}_n = \{ x \in \mathfrak{gl}_{n|n} \mid \ell(x) = x \},$$

the subalgebra of $\mathfrak{gl}_{n|n}$ fixed by the involution $\ell$. In terms of matrices,

$$(3.1) \quad \mathfrak{p}_n = \left\{ \begin{pmatrix} A & B \\ C & -At \end{pmatrix} \in \mathfrak{gl}_{n|n} \left| B^t = B, \ C^t = -C \right. \right\},$$

where $t$ is the usual transpose. The periplectic bilinear form is $\beta : V \otimes V \to \mathbb{C}$ such that

$$(3.2) \quad \beta(v_i, v_j) = \delta_{i,-j}(-1)^{[i]}$$

for all $i, j \in I_{n|n}$.\n
Lemma 3.2. It is not difficult to verify in the usual sense. Suppose for homogeneous elements $m$ and antipode $S$ given by

\[
S = 1 + \sum_{i=1}^{n} ((q - 1)E_{i,i} + (q^{-1} - 1)E_{-i,-i}) \otimes (E_{i,i} + E_{-i,-i})
\]

and \( q = \) holds in \( U_q(\mathfrak{p}_n) \). The category \( U_q(\mathfrak{p}_n) \)-mod admits a monoidal structure induced by the coproduct and counit of a Hopf superalgebra \( T^{12}T^{13}S^{23} = S^{23}T^{13}T^{12} \), where \( T = \sum_{|i| \leq |j|} t_{i,j} \otimes E_{i,j} \)

and the relation (3.6) holds in \( U_q(\mathfrak{p}_n) \) such that \( \delta(T) = T^{-1} \).

3.2. The category \( U_q(\mathfrak{p}_n) \)-mod. Given two left \( U_q(\mathfrak{p}_n) \)-supermodules \( M, N \), a homogeneous homomorphism \( f : M \rightarrow N \) is a \( C \)-linear map such that

\[
f(xm) = (-1)^{|x||f|}xf(m)
\]

for homogeneous elements \( m \in M, x \in U_q(\mathfrak{p}_n) \). We allow morphisms to be odd (i.e. to change the par-}

ity of elements they are applied to). Let \( U_q(\mathfrak{p}_n) \)-mod be the category of left \( U_q(\mathfrak{p}_n) \)-supermodules such that \( \text{Hom}_{U_q(\mathfrak{p}_n)}(M, N) \) is the set of all linear combinations of homogeneous homomorphisms. Then \( U_q(\mathfrak{p}_n) \)-mod admits a monoidal structure induced by the coproduct and counit of a Hopf superalgebra in the usual sense. Suppose

\[
P = \sum_{a,b=-n}^{n} (-1)^{|a||b|}E_{a,b} \otimes E_{b,a}.
\]

It is not difficult to verify

\[
PS(v_a \otimes v_b) = (-1)^{|a||b|}v_b \otimes v_a + (q - 1)\delta_{a>0}(\delta_{a,-b} + \delta_{a,b})v_b \otimes v_a + (q^{-1} - 1)\delta_{a<0}(\delta_{a,-b} - \delta_{a,b})v_b \otimes v_a + (q - q^{-1})\delta_{a>0}\delta_{a,-b}v_b \otimes v_a + (q - q^{-1})(-1)^{|a|}\delta_{a,-b} \sum_{1 \leq |j| < |a| \leq n} v_j \otimes v_j.
\]

Lemma 3.2. \( U_q(\mathfrak{p}_n) \)-module homomorphisms \( \vartheta : V_q \otimes V_q \rightarrow \mathbb{C}(q) \) and \( \varepsilon : \mathbb{C}(q) \rightarrow V_q \otimes V_q \) given by \( \vartheta(v_a \otimes v_b) = \delta_{a,-b}(-1)^{|a|} \) and \( \varepsilon(1) = \sum_{a=-n}^{n} v_a \otimes v_a. \)

\(^2\)In \( U_q(\mathfrak{p}_n) \)-module, \( \mathbb{C}(q) \) is assumed to be an odd \( U_q(\mathfrak{p}_n) \)-module.
Lemma 3.5. \( \varepsilon \) satisfies the following relations:

\[
(3.8) \quad t_i = 1^{\otimes i-1} \otimes t \otimes 1^{l-i-1} : V_q^{\otimes l} \rightarrow V_q^{\otimes l} \quad \text{and} \quad c_i = 1^{\otimes i-1} \otimes c \otimes 1^{l-i-1} : V_q^{\otimes l} \rightarrow V_q^{\otimes l}.
\]

3.3. The perliclent \( q \)-Brauer algebra \( B_{q,l} \). The following definition is given in [1 Definition 5.1] when the ground ring \( \mathbb{k} \) is \( \mathbb{C}(q) \).

Definition 3.3. Suppose that \( \mathbb{k} \) is an integral domain of characteristic not 2 and \( l \in \mathbb{N} \). The perliclent \( q \)-Brauer algebra \( B_{q,l} \) is the associative \( \mathbb{k} \)-algebra generated by elements \( T_i \) and \( E_i \) for \( 1 \leq i \leq l-1 \) satisfying the following relations:

\[
\begin{align*}
(1) \quad (T_i - q)(T_i + q^{-1}) &= 0, \quad T_iT_j = T_jT_i, \quad T_kT_{k+1}T_k = T_{k+1}T_kT_k+1, \\
(2) \quad E_i^2 &= 0, \quad E_iE_j = E_jE_i, \quad E_{k+1}E_kE_{k+1} = -E_{k+1}, \quad E_kE_{k+1}E_k = -E_k, \\
(3) \quad T_iE_j &= E_jT_i, \quad E_iT_i = -q^{-1}E_i, \quad T_iE_i = qE_i, \\
(4) \quad T_kE_{k+1}E_k &= -T_{k+1}E_k + (q - q^{-1})E_{k+1}E_k, \quad E_{k+1}E_kT_k &= -E_{k+1}T_k + (q - q^{-1})E_{k+1}E_k, \\
\end{align*}
\]

where \( 1 > |i-j| \) and \( 1 \leq k \leq l-2 \).

By convention, \( B_{q,0} = B_{q,1} = \mathbb{k} \).

Lemma 3.4. There is a \( \mathbb{k} \)-linear anti-involution \( \sigma : B_{q,l} \rightarrow B_{q,l} \) such that \( \sigma(T_i) = -T_i^{-1} \) and \( \sigma(E_i) = -E_i, 1 \leq i \leq l-1 \).

Proof. The result follows directly from Definition 3.3. \( \square \)

Let \( \mathcal{S}_l \) be the symmetric group on \( l \) letters. As Coxeter group, \( \mathcal{S}_l \) is generated by \( s_i, 1 \leq i \leq l-1 \) subject to the relations

\[
s_i^2 = 1, \quad s_is_j = s_js_i \quad \text{and} \quad s_ks_{k+1}s_k = s_{k+1}s_ks_k+1,
\]

for any \( 1 \leq i \leq l-1, |i-j| > 1 \) and \( 1 \leq k \leq l-2 \). Let \( H_l \) be the Hecke algebra associated to \( \mathcal{S}_l \). Then \( H_l \) is the unital associative \( \mathbb{k} \)-algebra generated by \( T_i, 1 \leq i \leq l-1 \), satisfying the following relations

\[
(3.9) \quad (T_i - q)(T_i + q^{-1}) = 0, \quad T_iT_j = T_jT_i, \quad T_kT_{k+1}T_k = T_{k+1}T_kT_{k+1},
\]

where \( 1 > |i-j| \) and \( 1 \leq k \leq l-2 \). Thanks to Definition 3.3 there is a surjective homomorphism from \( B_{q,l} \) to \( H_l \), sending \( T_i \) and \( E_i \) to \( T_i \) and 0, respectively. We will see that \( H_l \) is also a subalgebra of \( B_{q,l} \) after we prove a basis theorem for \( B_{q,l} \) next section.

Lemma 3.5. In \( B_{q,l} \), we have

\[
\begin{align*}
(1) \quad E_{i+1}T_i &= T_iE_{i+1}E_{i+1}T_i^{-1}, \quad \text{for } 1 \leq i < l-1, \\
(2) \quad T_iE_{i+1}T_i^{-1} &= -T_{i+1}E_i, \quad E_{i+1}E_{i+1}T_i = -E_{i+1}T_i, \quad \text{for } 1 \leq i < l-1, \\
(3) \quad E_iE_{i+1}T_i &= E_{i+1}T_i^{-1}, \quad T_{i+1}E_iE_{i+1}T_i = T_{i+1}^{-1}E_{i+1}, \quad \text{for } 1 \leq i < l-1, \\
(4) \quad E_{i+1}T_iE_{i+1} &= q^{-1}E_{i+1}, \quad E_{i+1}T_i^2E_{i+1} = -qE_{i+1}, \quad \text{for } 1 \leq i < l-1, \\
(5) \quad E_iE_{i+2}T_{i+1}E_i &= -E_{i+2}T_{i+1}^{-1}E_iE_{i+2}, \quad T_{i+2}T_{i+1}E_iE_{i+2} = -T_{i+1}^{-1}E_iE_{i+2}, \quad \text{for } 1 \leq i < l-2, \\
(6) \quad E_iE_{i+2}T_{i+1}E_i &= -E_{i+2}T_i^2E_{i+2}, \quad \text{for } 1 \leq i < l-2.
\end{align*}
\]

Proof. Thanks to Definition 3.3 one can check (1)-(4) easily. We verify (5)-(6) as an example. We have

\[
E_{i+2}T_i = -E_{i+2}E_{i+1}T_i = -E_{i+1}T_{i+2}T_i = -E_{i+2}E_{i+1}T_iT_{i+2}^{-1} = -E_{i+2}E_{i+2}T_{i+2}^{-1}T_{i+2}^{-1}.
\]

Applying the anti-involution \( \sigma \) in Lemma 3.4 on the above equation yields the last equation in (5). By (2) and Definition 3.3(2), \( E_iE_{i+2}T_{i+1}E_i = E_iE_{i+2}E_{i+1}E_i = -E_{i+2}E_i \), proving (6). \( \square \)

Theorem 3.6. [1 Theorem 5.5] Suppose \( \mathbb{k} = \mathbb{C}(q) \). There is an algebra epimorphism

\[
\phi : B_{q,l} \rightarrow \text{End}_{U_q(p_n)}(V_q^{\otimes l})
\]

such that \( \phi(T_i) = t_i \) and \( \phi(E_i) = c_i \), \( 1 \leq i \leq l-1 \). When \( n \geq l \), \( \phi \) is injective and hence an isomorphism.
Proposition 3.7. Suppose $k = \mathbb{C}(q)$. There is a monoidal superfunctor $\Phi : \mathcal{B} \to U_q(p_n)\text{-mod}$ such that $\Phi(1) = V_q$, $\Phi\left(\begin{array}{c}
abla
\end{array}\right) = t, \Phi\left(\begin{array}{c}
abla
\end{array}\right) = \varepsilon$ and $\Phi\left(\begin{array}{c}
abla
\end{array}\right) = \vartheta$.

Proof. It is enough to verify that the images of generating morphisms of $\mathcal{B}$ satisfy the defining relations in Definition 2.2 (R1)-(R5). In fact, Definition 2.2 (R1)-(R2) have already been verified in [1 Theorem 5.5]. One can check other relations by straightforward computation. \qed

4. A basis theorem of the periplectic $q$-Brauer category

4.1. Proof of Theorem 2.9 The following definition is motivated by \cite{15} Lemma 3.1.

Definition 4.1. For any $i \in \mathbb{Z}$, define $\mathcal{A}_i = \{ x \in \mathbb{C}(q) \mid ev(x) \geq i \}$, where $ev : \mathbb{C}(q) \to \mathbb{Z} \cup \{\infty\}$ such that $ev(0) = \infty$ and $ev(x) = j$ if $0 \neq x = (q - 1)^j g/h$ and $g, h \in \mathbb{C}[q]$ such that $(g, q - 1) = 1$, $(h, q - 1) = 1$.

Definition 4.2. For any $c \in \mathbb{N}$ and $j = 0, 1$, let $V^{\otimes r}_j$ be the free $\mathcal{A}_0$-submodule of $V_q^{\otimes r}$ with basis \{ $(q - q^{-1})^j v_i \mid i \in I_{n|n}'$ \}, where $v_i = v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_r}$. In particular, $V^{\otimes 0}_j = A_j$ for $j = 0, 1$.

Lemma 4.3. Suppose $j \in \{0, 1\}$ and $a, b \in I_{n|n}'$.
(1) $t^\pm (v_a \otimes v_b) \equiv (-1)^{|a||b|}v_b \otimes v_a (\text{mod } V_1^{\otimes 2})$.
(2) $t^\pm V^{\otimes 2}_j \subseteq V^{\otimes 2}_j$, $\partial V^{\otimes 2}_j \subseteq A_j$ and $\varepsilon A_j \subseteq V^{\otimes 2}_j$.

Proof. Obviously, (1) and the first inclusion in (2) follow from \cite[Definition 2.2 R2]{16} and Proposition 5.7. The remaining inclusions in (2) follow form Lemma 3.2. \qed

Throughout this subsection, we assume $n > r$, where $r$ is any fixed natural number. For any $d \in \mathbb{R}_2 r, 0$ with $\text{conn}(d) = \{(i_1, j_1), \ldots, (i_r, j_r)\}$, define
\begin{equation}
h_d = (h_{i_1}, h_{i_2}, \ldots, h_{i_r})
\end{equation}
such that $h_{i_k} = k$ and $h_{j_k} = -k$ for $k = 1, 2, \ldots, r$. Since $n > r$, $h_d \in I_{n|n}'$. Recall $\mathbb{D}_2 r, 0$ in 2.11.

Lemma 4.4. For any $e_1, e_2 \in \mathbb{D}_2 r, 0$, $e_1 v_{h_{d_2}} \equiv \pm \delta_{\text{conn}(e_1), \text{conn}(e_2)} (\text{mod } A_1)$.

Proof. By Lemma 2.8 we can choose $e_1$ such that all caps of $e_1$ are not at the same height. Then
\begin{equation}
e_1 = \begin{array}{c}
\begin{array}{c}
\cdot
\end{array}
\end{array}
\end{equation}
for some $d_1 \in \mathbb{D}_2 r_{-2, 0}$ and some $k, 1 \leq k \leq r$. So $e_1 = d_1 \circ d'_1$ where $d'_1$ is determined via above diagram in an obvious way. Similarly, we have $e_2 = d_2 \circ d'_2$ where $d_2 \in \mathbb{D}_2 r_{-2, 0}$. Each crossing in $d'_1$ may be over crossing or under crossing, which depends on $e_1$. By Lemma 4.3 we have
\begin{equation}
e_1 v_{h_{d_2}} \equiv \pm \delta_{\text{conn}(d'_1), \text{conn}(d'_2)} d_1 v_{h_{d_2}} (\text{mod } V_1^{\otimes 2_{r-2}})
\end{equation}
where $v_{h_{d_2}} \in V^{\otimes 2_{r-2}}$ is obtained from $v_{h_{d_2}}$ by deleting its $i_k$-th and $j_k$-th tensor factors. Now the results follow from induction assumption for $r - 1$. \qed

Proof of Theorem 2.9: Thanks to Lemma 2.11 we can assume $s = 0$ and $m = 2r$ for some $r \in \mathbb{N}$ when we prove Theorem 2.9. By Corollary 2.7 and Lemma 2.8 it remains to show that $\mathbb{D}_2 r, 0$ is linear independent over $k$. First, we assume that $k = \mathbb{C}(q)$. Suppose
\begin{equation}
\sum_{d \in B} c_d d = 0
\end{equation}
for some $B \subset \mathbb{D}_2 r, 0$ with $0 \neq c_d \in \mathbb{C}(q)$. Without loss of any generality, we can assume $c_d \in A_0$ for all such $d$’s. Otherwise, we can multiply some powers of $q - 1$ on both sides of (4.3) such that all coefficients are in $A_0$.

Let $a$ be the minimal positive integer such that all $c_d \in A_a \setminus A_{a-1}$. Then there exists a $d_0 \in B$ such that $(q - 1)^{-a} c_{d_0} \in A_0 \setminus A_1$. Multiplying both sides of (4.3) by $(q - 1)^{-a}$, we can assume that there exists some $d_0 \in B$ such that $c_{d_0} \in A_0 \setminus A_1$. By Lemma 4.4 $0 = \sum_{d \in B} c_d d h_{d_0} \equiv \pm c_{d_0} (\text{mod } A_1)$, forcing $c_{d_0} = 0$, a contradiction. So $\mathbb{D}_2 r, 0$ is linear independent over $\mathbb{C}(q)$ and hence over $\mathbb{Z}[q, q^{-1}]$. This proves Theorem 2.9 over $\mathbb{Z}[q, q^{-1}]$. By standard arguments on base change, we have Theorem 2.9 over an arbitrary integral domain with characteristic not 2.
**Definition 4.5.** Given an \((m, s)\)-connector \(c\), a reduced lift with respect to \(c\) is either \(D_c \in \mathbb{B}_{m,s}\) or an \((m, s)\)-tangle diagram obtained from \(D_c\) by exchanging some under or over crossings. Let

\[
\mathbb{R}_m = \{ r_c \mid r_c \text{ is any fixed reduced lift of } c, \text{ and } c \in \text{conn}(m, s) \}.
\]

**Corollary 4.6.** The morphism space \(\text{Hom}_\mathbb{B}(m, s)\) has basis given by \(\mathbb{R}_m\).

**Proof.** Thanks to Definition 2.2.2(R2), \(r_c = D_c\) up to a linear combination of some \((m, s)\)-diagrams in \(\mathbb{B}_{m,s}\) with fewer crossings. This shows that the transition matrix between \(\mathbb{R}_m\) and \(\mathbb{D}_{m,s}\) is uni-uppertriangular. Now, the result follows from Theorem 2.9. □

**4.2 A basis of periplectic \(\mathfrak{g}\)-Brauer algebra \(\mathcal{B}_{q,l}\).** From here to the end of this section, \(q\) is always an integral domain with characteristic not 2. Later on, write \(s_{i,j} = s_is_{i+1,j}\) if \(i < j\), and \(s_{i,j} = 1\) if \(i > j\). Define \(D_{0,l} = D_{0,l} = \{ 1 \}\) and

\[
D_{l,l} = \{ s_{2f,i} | i | k | j, k \leq i | k \leq l, 1 \leq k \leq f \},
\]

\[
D_{l,l} = \{ s_{2f,i} | i | k | j, k \leq i | k \leq l, 1 \leq k \leq f \},
\]

if \(0 < f \leq \lfloor l/2 \rfloor\). Let \(B_l(f)\) be the subalgebra of \(\mathcal{B}_{q,l}\) generated by \(E_{2f+1}, T_j, 2f+1 \leq j \leq l-1\). By Lemma 3.3 we see that there is an algebra epimorphism

\[
\psi : B_{q,l-2f} \twoheadrightarrow B_l(f)
\]

sending the generators \(E_i, T_i\) of \(B_{q,l-2f}\) to the generators \(E_{2f+i}\) and \(T_{2f+i}\) of \(B_{q,l}\). To simplify the notation, we write \(T_{i,j} = T_{i+1,j,1}\) if \(j \geq i\), and \(T_{i,j} = 1\) if \(i < j\). Let \(E^0 = 1\) and for any \(1 \leq f \leq \lfloor l/2 \rfloor\), define

\[
E^f = E_1 E_3 \cdots E_{2f-1}.
\]

**Lemma 4.7.** Let \(N^f\) be the left \(B_l(f)\)-module generated by \(V^f = \{ E^f T_d | d \in D_{f,l} \}\), and \(\tilde{N}^f\) be the left \(B_l(f)\)-module generated by \(\tilde{V}^f = \{ E^f T_d | d \in D_{f,l} \}\). Then

1. \(N^f\) is a right \(\mathcal{B}_{q,l}\)-module. 2. \(N^f\) is a right \(\mathcal{B}_{q,l}\)-module. 3. \(\tilde{N}^f = N^f\).

**Proof.** We prove (1) by showing that \(N^f\) is stable under the right action of \(E_1\) and \(T_j\)'s. Write \(d = s_{2i,s_{i+1,j}}\) with \(1 \leq j \leq i\). If \(j = 1\), then \(i < 1\). Thanks to Lemma 3.3(4) and Definition 3.3(2), \(E_j T_d E_1 = -q_j T_{3j} 1 \in N^f\) if \(i_1 \geq 3\) and \(E_1 T_d E_1 = 0\) if \(i_1 = 1\). Suppose \(j > i\). Then \(i_1 \geq 3\). By Lemma 3.3(2)(4) and Definition 3.3(3),

\[
E_1 T_{2i,i} T_{i,1,i} E_1 = \begin{cases} -q^2 T_{3,i,i} E_1 & \text{if } j = 1, \\ -E_2 T_{4,i,i} T_{3,1,i} E_1 & \text{if } j > 1. \end{cases}
\]

In any case, \(N^f 1 \subset N^f\). One can check that \(N^f T_i \subset N^f\) by using Definition 3.3. In this case, it is enough to prove \(E_1 T_{2i,1,j} T_{i,j} \in N^f\) when \(i \leq j\). In fact, since \(s_{2j}s_{i,i-1} \in D_{1,i}\),

\[
E_1 T_{2i,1,j} T_{i,j} = E_1 T_{1,j} T_{i,i-1} = -q^{-1} E_1 T_{2j,i} T_{i,i-1} = N^f.
\]

This completes the proof of (1). Using (1) repeatedly yields (2).

Suppose \(i_1 \geq 2 \geq j_2\) and \(d = s_{i_1,i_2} s_{j_2,j_3} \in D_{2,l}\). We claim \(E^2 T_d \in \tilde{N}^f\), where \(\tilde{N}^f_{i_1}\) is the left \(B_l(2)\)-module of \(\tilde{N}^f\) generated by \(\tilde{V}^f = \{ E^2 T_d | d = s_{i_1,i_2} s_{j_2,j_3} \in D_{2,l}, i_2 < 1 \}\).

In fact, since \(i_1 \geq 2 \geq j_2\), by Lemma 3.3(5) we have

\[
E^2 T_d = E^2 T_{2,i} T_{3,j-1} T_{2,j-1} T_{1,j} = -E^2 T_{3,i} T_{3,j-1} T_{2,j-1} T_{1,j}.
\]

In order to prove our claim, it is enough to verify that the RHS of (4.8) is in \(\tilde{N}^f_{i_1}\).

If \(j_1 \geq j_2 > j_2\) then the RHS of (4.8) is \(-E^2 T_{3,i} T_{3,j-1} T_{2,j-2} T_{1,j-2} \in \tilde{N}^f_{i_1}\). The last inclusion follows from inequalities \(i_1 > i_2 - 2, i_2 - 2 > j_2 - 2\) and \(i_1 > j_1\).

Suppose \(j_2 < j_2\). There are two cases we need to discuss (1) \(j_1 \geq j_2 - 1\), (2) \(j_1 < j_2 - 1\). In case (1), the RHS of (4.8) is \(-E^2 T_{4,i} T_{2} T_{3,i} T_{2,j-1} T_{1,j-2} \in \tilde{N}^f_{i_1}\). So,

\[
\text{RHS of (4.8)} = (q - q^{-1}) E^2 T_{4,i} T_{3,j-1} T_{2,j-2} T_{3,j} T_{1,j-2} - E^2 T_{3,i} T_{2,j-1} T_{3,j} T_{1,j-2}.
\]

Since \(j_2 < j_1 < i_1\), and \(i_2 - 1 < i_1\), the first summand in the RHS of (4.8) is in \(\tilde{N}^f_{i_1}\). Let \(Y\) be the second summand in the RHS of (4.8). It is enough to prove \(Y \in \tilde{N}^f_{i_1}\). Note that we are assuming that \(j_1 < i_2\), we have \(j_1 \leq i_2 - 1\). Suppose \(j_1 < i_2 - 1\). Since \(i_2 - 1 < i_1, j_2 < i_2 - 1\), we
have \( Y = E^2T_{i_1}T_{3,j_1+1}T_{2,i_2-1}T_{1,j_2-2} \in \tilde{N}^2_{i_1} \). Suppose \( j_1 < i_1 \) and \( j_2 - 2 < j_1 - 1 \), \( Y = E^2T_{i_1}T_{3,j_1}T_{2,i_2-1}T^2_{j_1-1}T_{1,j_2-2} \). So
\[
(4.10) \quad Y = (q - q^{-1})E^2T_{i_1}T_{3,j_1}T_{2,j_2-1}T_{1,j_2-2} + E^2T_{i_1}T_{3,j_1}T_{2,j_2}T_{1,j_2-2} \in \tilde{N}^2_{i_1}.
\]
This completes the proof of our claim in case (1). In case (2), we have \( j_1 < j_2 - 1 < i_2 - 1 < i_1 \) and RHS of (4.8) = \(-E^2T_{i_1}T_{3,j_2-1}T_{1,j_1-1}T_{j_2-1} \). So
\[
(4.11) \quad \text{RHS of (4.8)} = -q(q - q^{-1})E^2T_{i_1}T_{3,j_2-1}T_{1,j_1-1} - E^2T_{i_1}T_{2,i_2-1}T_{1,j_2-1}T_{1,j_1-1}.
\]
The first summand in the RHS of (4.11) is \( \tilde{N}^2_{i_1} \) since \( j_2 - 1 < i_1, i_2 - 1 < i_1 \) and \( j_1 < j_2 - 1 \). We have

\[
T_{i_2}T_{i_2-1}T_{1,j_1-1}T_{j_2-1} = (q - q^{-1})T_{i_1}T_{3,j_1}T_{2,i_2-1}T_{1,j_1-1} + T_{i_2}T_{i_2-1}T_{2,i_2-1}T_{1,j_2-1}T_{1,j_1-1}
\]
and \( i_2 - 1 < i_1, j_2 < i_1, j_1 < i_2 - 1, j_1 + 2 < i_1 \) and \( j_2 - 1 < i_2 - 1 \). This proves that the second summand in the RHS of (4.11) is also in \( \tilde{N}^2_{i_1} \). So far, we have proved our claim in any case.

Thanks to (4.6), \( D_{f,l} \subset \tilde{D}_{f,l} \) forcing \( N \supseteq N \). In order to prove (3), it is enough to prove
\[
(4.12) \quad E^fT_d \in N^f \text{ for any } d \in \tilde{D}_{f,l}.
\]
We prove (4.12) by induction on \( f \) and \( i_1 \). When \( f = 1 \) this result is trivial since \( N^1 = \tilde{N}^1 \). In general by induction assumption on \( f \), we can assume that \( i_2 < i_3 < \ldots < i_f \). If \( i_1 = 2 \), then \( i_1 < i_2 \) and hence \( d \in D_{f,l} \) and \( E^fT_d \in N^f \). Suppose \( i_1 > 2 \). It is enough to assume \( i_1 \geq i_2 \). By our previous claim, \( E^fT_d \subseteq M \) where \( M \) is the left \( B(f) \)-module generated by \( \{ E^fT_d \mid d \in \tilde{D}_{f,l} \} \), and \( \tilde{D}_{f,l} \subset D_{f,l} \) consisting of elements \( d' = s_{2f,i'_1}s_{2f-1,i'_2} \cdots s_1,i'_r \) such that \( i'_1 < i_1 \). Using induction assumption on \( i_1 \) yields \( E^fT_d \in N^f \). This completes the proof of (3).

Corollary 4.8. Suppose \( f \leq [l/2] \). Then
(1) \( E^fB_{q,t} \) is the left \( B(f) \)-module generated by \( V^f = \{ E^fT_d \mid d \in D_{f,l} \} \).
(2) \( B_{q,t}E^f \) is the right \( B(f) \)-module generated by \( \{ \sigma(T_d)E^f \mid d \in D_{f,l} \} \) where \( \sigma \) is \( k \)-linear anti-involution of \( B_{q,t} \) in Lemma 4.7.

Proof. (1) follows from Lemma 4.7 and (2) follows from (1) by applying the anti-involution \( \sigma \).

Theorem 4.9. Suppose \( k \) is an integral domain with characteristic not 2.
(1) The algebra \( B_{q,t} \) has basis \( \mathcal{M} \), where \( \mathcal{M} = \{ \sigma(T_d)E^fT_wT_v \mid 1 \leq f \leq [l/2], d, v \in D_{f,l}, w \in \mathcal{S}_{l-2f} \} \).
In particular, the rank of \( B_{q,t} \) is \( (2l - 1)! \).
(2) As \( k \)-algebras, \( B_{q,t} \cong \text{Hom}_{B(l)}(l) \) for any \( l \in \mathbb{N} \).

Proof. Let \( H_{l-2f} \) be the Hecke algebra associated to the symmetric group \( \mathcal{S}_{l-2f} \) on \( l - 2f \) letters \( 2f + 1, 2f + 2, \ldots, l \). There is a \( k \)-algebra epimorphism from \( H_{l-2f} \) to \( B(f)/I \) sending \( T_{2f+i} \)'s to \( T_{2f+i} \)'s, where \( I \) is the two-sided ideal of \( B(f) \) generated by \( E_{2f+1} \). Let
\[
(4.13) \quad B^f_{q,t} = B_{q,t}E^fB_{q,t}.
\]
Thanks to Lemma 4.7(3) and Corollary 4.8 \( B^f_{q,t}/B^f_{q,t}^{l+1} \) is spanned by
\[
\{ \sigma(T_d)E^fT_wT_v \mid d, v \in D_{f,l}, w \in \mathcal{S}_{l-2f} \}.
\]
This proves that \( B_{q,t} \) is spanned by \( \mathcal{M} \) as \( k \)-module. In order to prove that \( \mathcal{M} \) is linear independent over \( k \), it is enough to prove that \( \mathcal{M} \) is linear independent over \( F \), the quotient field of \( k \).

Suppose both \( B_{q,t} \) and \( B \) are defined over \( F \). There is an algebra homomorphism \( \varphi : B_{q,t} \to \text{Hom}_{B(l)}(l) \) sending \( T_i \) and \( E_i \) to \( l_{i-1} \otimes \bigotimes l_{1_i-1} \) and \( l_{1_i-1} \otimes \bigotimes l_{1_i-1} \), respectively. Using induction on the number of crossings for a reduced \( (l, l) \)-tangle diagram, one can write any reduced \( (l, l) \)-tangle diagram as a linear combination of \( T_wT_{w'} \) for some \( w, w' \in \mathcal{S}_1 \), where \( d = (\bigotimes l_{r-2}) \otimes l_{r-2} \) for some \( 1 \leq r \leq l/2 \). This proves that \( \varphi \) is surjective. Thanks to Theorem 2.9 \( \dim_F \text{Hom}_{B(l)}(l) = (2l - 1)! \), and hence \( \dim_F B_{q,t} = (2l - 1)! \), forcing \( \mathcal{M} \) to be linear independent over \( F \). This completes the proofs of (1). Now, (2) follows from (1) and the fact that there is an algebra epimorphism \( \varphi : B_{q,t} \to \text{Hom}_{B(l)}(l) \) over \( k \).
5. REPRESENTATIONS OF THE PERIPLECTIC $q$-BRAUER CATEGORY $\mathcal{B}$

In this section we assume that $\k$ is an algebraically closed field with characteristic not 2. We show that $\mathcal{B}$ admits a split triangular decomposition in the sense of [5]. Consequently, the category of locally finite dimensional right $\mathcal{B}$-modules is an upper finite fully stratified category in the sense of [9, Definition 3.34].

5.1. Split triangular decomposition. Let $A$ be the locally unital associative $\k$-algebra associated to $\mathcal{B}$. Then

\[(5.1)\quad A = \bigoplus_{a, b \in \mathbb{N}} I_a A_1 b, \quad 1_a A_1 b = \text{Hom}_\k(b, a)\]

with multiplication being induced by composition in $\mathcal{B}$. The mutually orthogonal idempotents $\{1_a \mid a \in \mathbb{N}\}$ serves as a family of distinguished local units of $A$. Although $\mathcal{B}$ is a supercategory, we regard $A$ as an ordinary associative $\k$-algebra by forgetting the super structure of $\mathcal{B}$.

**Definition 5.1.** Let $A$ be the locally unital algebra associated to $\mathcal{B}$ in (5.1). Define

1. $A^\circ := \bigoplus_{r \in \mathbb{N}} A_r^\circ$, where $A_r^\circ$ is the $\k$-span of all $(r, r)$-tangle diagrams on which there are neither caps nor cups.
2. $A^\circ$ : $\k$-span of all tangle diagrams on which there are neither caps nor crossings among vertical strands.
3. $A^\circ$ : $\k$-span of all tangle diagrams on which there are neither cups nor crossings among vertical strands.
4. $A^\circ$ : $\k$-span of all tangle diagrams on which there are no caps.
5. $A^\circ$ : $\k$-span of all tangle diagrams on which there are no cups.

Thanks to Definition [2.2], it is not difficult to verify that all tangle diagrams in $A^\circ$ are closed under composition and hence are locally unital subalgebras of $A$ where $\circ \in \{\circ, -, +, \sharp, b\}$. Moreover,

\[(5.2)\quad m - s = 2k \text{ for some } k \in \mathbb{N}, \text{ if either } 1_a A^\circ 1_s \neq 0 \text{ for } \circ \in \{-, \sharp\} \text{ or } 1_s A^* 1_m \neq 0 \text{ for } \star \in \{+, b\}.

Recall $RL_{m,s}$ in (4.3) for all $m, s \in \mathbb{N}$. Define

\[(5.3)\quad RL^\circ = \bigcup_{m,s} RL^\circ_{m,s},\]

where $RL^\circ_{m,s} = RL_{m,s} \cap A^\circ$ and $\circ \in \{\circ, -, +, \sharp, b\}$.

**Lemma 5.2.** The locally unital subalgebra $A^\circ$ has $\k$-basis given by $RL^\circ$ where $\circ \in \{\circ, -, +, \sharp, b\}$. Moreover, the locally unital subalgebra $A^\circ$ has basis given by $RL^\circ_{m,s}$.

**Proof.** Using induction on the number of crossings of tangle diagrams, we see that the result is true if $RL_{m,s}$ is chosen as $RL_{m,s}$. In general, by Definition [2.2](R2) and induction arguments on the number of crossings of tangle diagrams, any reduced lift of a connector $c$ is equal to the corresponding reduced totally descending tangle diagram $D_c$ up to a linear combination of reduced totally descending tangle diagrams with fewer crossing numbers. This implies the result in general case. \(\square\)

**Lemma 5.3.** As $\k$-spaces, $A^- \otimes_\k A^\circ \otimes_\k A^+ \cong A$, $A^- \otimes_\k A^\circ \otimes_\k A^+ \cong A^\circ$ and $A^\circ \otimes_\k A^+ \cong A^b$, where $\k := \oplus_{a \in \mathbb{N}} k_1 a$. Moreover, these isomorphisms are given by the multiplication of $A$.

**Proof.** Fix a $\k$-basis $RL = \bigcup_{m,s} RL_{m,s}$ of $A$ where $RL_{m,s}$ is given in (4.3). By Lemma 5.2 we have $k$-basis of $A^\circ$ for any $\circ \in \{\circ, -, +\}$. Let $\phi : A^- \otimes_\k A^\circ \otimes_\k A^+ \to A$ be the linear map induced by the multiplication of $A$. The image of $(b^-, b^0, b^+)$ $\in RL^-_{m,t} \times RL^\circ_{t,t} \times RL^+_t$, is a reduced lift of $\text{conn}(b^- b^0 b^+)$ and further \(\text{conn}(b^- b^0 b^+) \neq \text{conn}(b^- b^0 b^+)\) if $(\text{conn}(b^-), \text{conn}(b^0), \text{conn}(b^+)) \neq (\text{conn}(b^+_1), \text{conn}(b^+_1), \text{conn}(b^+_1))$ and $(b^-_1, b^0_1, b^+_1) \in RL^-_{m,t} \times RL^\circ_{t,t} \times RL^+_t$. By Lemma 5.2, $\phi$ is injective. Since the dimension of $I_m A_1 s$ is $\sum |RL^-_{m,t}| |RL^\circ_{t,t}| |RL^+_t|$, the image of $\cup RL^-_{m,t} \times RL^\circ_{t,t} \times RL^+_t$ is a $k$-basis of $I_m A_1 s$. This proves that $\phi$ is surjective and hence an isomorphism as required. One can check the second and the third isomorphisms similarly. \(\square\)

We briefly recall the notion of a split triangular decomposition of the locally unital algebra associated to small finite dimensional $\k$-linear category $\mathcal{C}$ whose object set is $I$ [6]. In this case $\dim_k \text{Hom}_\mathcal{C}(a, b) < \infty$ for all $a, b \in I$. Let $C$ be the locally unital algebra associated to $\mathcal{C}$.
Definition 5.4. [6] Remark 5.32] The data \((I, C^-, C^0, C^+)\) is called a \textit{split triangular decomposition} of \(C\) (or \(C\) admits a split triangular decomposition) if:

1. \((I, \preceq)\) is an upper finite poset in the sense that \(\{b \in I \mid a \prec b\}\) is a finite set for any \(a \in I\).
2. \(C^-, C^0\) and \(C^+\) are locally unital subalgebras of \(C\) in the sense that 
   \[ C^\pm = \bigoplus_{b,c \in I} 1_b C^\pm 1_c \]
   and 
   \[ C^0 = \bigoplus_{a \in I} 1_a C^0 1_a. \]
3. \(1_r C^- 1_b = 0\) and \(1_b C^+ 1_a = 0\) unless \(a \preceq b\). Furthermore, \(1_a C^- 1_a = 1_a C^+ 1_a = k 1_a\).
4. \(C^r \otimes_k C^0 \otimes_k C^+ \cong C\) as \(k\)-spaces where \( K = \bigoplus_{b \in I} k 1_b\). The required isomorphism is given by the multiplication on \(C\).

Now, we consider the category \(\mathcal{B}\) whose object set is \(\mathbb{N}\). Let \(\preceq\) be the partial order on \(\mathbb{N}\) such that \(s \preceq m\) if \(s - m \in 2\mathbb{N}\). Then \((\mathbb{N}, \preceq)\) is upper finite.

Theorem 5.5. Let \(A\) be the locally unital algebra associated to \(\mathcal{B}\). Then \(A\) admits a split triangular decomposition. The corresponding date is \((\mathbb{N}, A^-, A^0, A^+)\), where \(A^-, A^0, A^+\) are defined in Definition 5.4.

Proof. Definition 5.4(1) is obvious. By Definition 5.4(2) and Lemma 5.3, we see that the locally unital algebra \(A\) associated to \(\mathcal{B}\) satisfies Definition 5.4(2)–(5). \(\Box\)

5.2. Representations of \(\mathcal{B}\). Recall \(H_l\) is the Hecke algebra associated to symmetric group \(S_l\). Thanks to Lemma 5.5, \(H_l \cong 1_l A^0 1_l\) and the required isomorphism sends \(T_i\) to \(1_{l-i} \otimes \bigwedge 1_{l-i-1}, 1 \leq i \leq l-1\).

Consequently, there is a \(k\)-algebra isomorphism

\[ A^0 \cong \bigoplus_{l \in \mathbb{N}} H_l. \]

Recall that a composition \(\lambda\) of \(l\) is a sequence \((\lambda_1, \lambda_2, \cdots)\) of non-negative integers such that \(|\lambda| := \sum \lambda_i = l\). If \(\lambda_i \geq \lambda_{i+1}\) for all possible \(i\), then \(\lambda\) is called a partition. Given a positive integer \(e\), a partition \(\lambda\) of \(l\) is called \(e\)-restricted if \(\lambda_i - \lambda_{i+1} < e\) for all possible \(i\). When \(e = \infty\), any partition of \(l\) is \(e\)-restricted. Let \(\Lambda(l)\) (resp., \(\Lambda^+(l)\), resp., \(\Lambda^+_c(l)\)) be the set of all compositions (resp., partitions, resp., \(e\)-restricted partitions) of \(l\), where \(e\) is always the quantum characteristic of \(q^2\) (so \(e = \infty\) means that \(q^2\) is not a root of unity). Define

\[ \Lambda = \bigcup_{l \in \mathbb{N}} \Lambda^+(l), \text{ and } \Lambda_c = \bigcup_{l \in \mathbb{N}} \Lambda^+_c(l). \]

For each \(\lambda \in \Lambda^+_c(l)\), let \(S^\lambda\) be the dual Specht module for \(H_l\) (which is not the classical Specht module). This is a right \(H_l\)-module defined via Jucys-Murphy basis of \(H_l\) in Theorem 6.1. At moment, we do not need its explicit construction. It is known that \(S^\lambda\) has the simple head, say \(D^\lambda\) if and only if \(\lambda \in \Lambda^+_c(l)\) and \(\{D^\lambda \mid \lambda \in \Lambda_c\}\) is a complete set of inequivalent irreducible \(A^0\)-modules. For each \(\lambda \in \Lambda^+_c(l)\), let \(Y(\lambda)\) be the projective cover of \(D(\lambda)\).

Let \(A^0\)-fdmod be the category of finite dimensional right \(A^0\)-modules and \(A^0\)-fdmod be the category of locally finite dimensional right \(A\)-modules. Recall \(A^0\) and \(A^\pm\) in Definition 5.4. It follows from Theorem 5.5 and [6] (15.1)-(15.14) [3] that there are two exact functors \(\Delta, \nabla : A^0\)-fdmod \to \(A\)-fdmod called induction and coinduction functors respectively such that

\[ \Delta = \otimes_{A^0} A \text{ and } \nabla = \oplus_{m \in \mathbb{N}} \text{Hom}_{A^0}(1_m A, ?). \]

The following result follows from [6] Theorem 5.38] which is about locally unital and locally finite dimensional \(k\)-algebra admitting a split triangular decomposition. Such a result holds for upper finite weakly triangular categories, a generalized version of split triangular decomposition [10] Theorem 3.5.

Theorem 5.6. [c.f. 6, Theorem 5.38]\]

1. The category \(A\)-fdmod is an upper finite fully stratified category in the sense of Brundan and Stroppel [6] Definition 3.34. In this case, the corresponding stratification is \(\rho : \Lambda_c \to \mathbb{N}\) such that \(\rho(\lambda) = l\) if \(\lambda \in \Lambda^+_c(l)\).
2. The proper standard module \( \overline{\Delta}(\lambda) := \Delta(D(\lambda)) \) has a unique maximal submodule, say \(\text{rad} \overline{\Delta}(\lambda)\), for any \(\lambda \in \Lambda_c\).

---

3Brundan and Stroppel considered the category of locally finite dimensional left \(A\)-modules.
(3) For any $\lambda \in \Lambda_r$, let $L(\lambda)$ be the simple head of $\Sigma(\lambda)$. Then $\{L(\lambda) \mid \lambda \in \Lambda_r\}$ is a complete set of pairwise inequivalent irreducible $A$-modules.

(4) When $q^2$ is not a root of unity, the category $A$-lfdmod is an upper finite highest weight category in the sense of [6, Definition 3.34].

**Proof.** The results (1)-(3) follow immediately from Theorem 5.5 and [6] Theorem 5.38. Finally, (4) follows from [6] Corollary 5.36 since $A^\circ$ is semisimple when $q^2$ is not a root of unity. \hfill \square

### 5.3. Simple modules for periplectic $q$-Brauer algebras.

Thanks to Theorem 1.2, $B_{q,t} \cong 1_t A_1$ where $A$ is the locally unital algebra associated to $B$. The idempotent element $1_t$ gives an exact functor from $A$-lfdmod to $B_{q,t}$-lfdmod, the category of finite dimensional right $B_{q,t}$-modules. In Theorem 5.9 we assume that $k$ is an algebraically closed field of characteristic not 2 although we can use standard basis of $B_{q,t}$ in section 6 to classify simple $B_{q,t}$-modules over an arbitrary field with characteristic not 2.

**Lemma 5.7.** Suppose $M \in \{\Sigma(\lambda), L(\lambda)\}$ where $\lambda \in \Lambda^+(r)$. Then $M 1_t \neq 0$ only if $l - r \leq 2N$.

Moreover, $L(\lambda) 1_r \cong D(\lambda)$. \hfill \square

**Proof.** By (5.2) and Lemma 5.3

$$\Sigma(\lambda) = D(\lambda) \otimes A^t A \cong D(\lambda) \otimes_k A^+ = \bigoplus_{r \in \mathbb{N}} D(\lambda) \otimes_k 1_r A^+ 1_{r + 2k},$$

proving the first statement for $\Sigma(\lambda) 1_t$. Since there is an epimorphism from $\Sigma(\lambda)$ to $L(\lambda)$ and $1_{n}$ is exact for any $n \in \mathbb{N}$, we have the result for $L(\lambda) 1_t$, immediately. By (5.7), $\Sigma(\lambda) 1_r = D(\lambda) \otimes_k 1_r$. Since $D(\lambda)$ is simple, we have $L(\lambda) 1_r \cong \Sigma(\lambda) 1_r$, proving the last statement. \hfill \square

**Lemma 5.8.** For any positive integer $l$, $L(\emptyset) 1_l = 0$ and $\dim L(\emptyset) = 1$.

**Proof.** Let $N = \bigoplus_{i=1}^{\infty} \Sigma(\emptyset) 1_{2r} = \bigoplus_{i=1}^{\infty} 1_0 A^+ 1_{2r}$. Obviously, $N 1_m A_1 \subset N$ if $n \neq 0$. Suppose $n = 0$. Then $N 1_m A_1 0 = 0$ unless $m$ is even and $m \neq 0$. Suppose $m = 2r$ for some positive integer $r$ and $N 1_m A_1 0 = N 1_{2r} A_1 0 \subset 1_0 A^+ 1_{2r} A_1 0$. Since $r > 0$, $1_0 A^+ 1_{2r} A_1 0$ is spanned by diagrams with at least one loop. Using Definition 2.2 (R2), any one of these diagrams can be written as a linear combination of free loops without crossings. Thanks to Definition 2.2 (R5), $1_0 A^+ 1_{2r} A_1 0 = 0$. This proves that $N$ is a submodule of $\Sigma(\emptyset)$ and $\Sigma(\emptyset) 1_r \subset N$ for all positive integers $l$. So $L(\emptyset) 1_l = 0$. Finally, $\dim L(\emptyset) = 1$ since $L(\emptyset) = k$. \hfill \square

For any $l \in \mathbb{N}$ define

$$\Sigma^+ (l) = \bigcup_{0 \leq f \leq \lfloor l/2 \rfloor} \Lambda^+(l - 2f) \quad \text{and} \quad \Sigma^+_e (l) = \bigcup_{0 \leq f \leq \lfloor l/2 \rfloor} \Lambda^+_e (l - 2f),$$

where $e$ is the quantum characteristic of $q^2$.

**Theorem 5.9.** For any positive integer $l$, let $B_{q,t}$ be defined over an algebraically closed field with characteristic not 2 and $\lambda \in \Sigma^+_e (l)$.

(1) If $l$ is odd, then $L(\lambda) 1_l \neq 0$ if and only if $\lambda \in \Sigma^+_e (l)$. Moreover, the set $\{L(\lambda) 1_l \mid \lambda \in \Sigma^+_e (l)\}$ is a complete set of inequivalent irreducible $B_{q,t}$-modules.

(2) If $l$ is even, then $L(\lambda) 1_l \neq 0$ if and only if $\lambda \in \Sigma^+_e (l)$ and $\lambda \neq 0$. Moreover, the set $\{L(\lambda) 1_l \mid \lambda \in \Sigma^+_e (l) \setminus \{0\}\}$ is a complete set of inequivalent irreducible $B_{q,t}$-modules.

**Proof.** By Lemmas 5.7, 5.8 we have $\lambda \in \Sigma^+_e (l)$ (resp., $\Sigma^+_e (l) \setminus \{0\}$) if $L(\lambda) 1_l \neq 0$ and $l$ is odd (resp., even). Conversely, suppose $\lambda \in \Sigma^+_e (l)$. Then $\lambda \in \Lambda^+_e (r)$ and $l - r$ is even. By Lemma 5.7 $L(\lambda) 1_l \cong D(\lambda)$. Since $1_r = (1_l - 1_l \otimes \otimes^{\frac{1}{2}(l-r)} 1_1) \otimes (1_{l-1} \otimes \otimes^{\frac{1}{2}(l-r)} 1_1), (1_{l-1} \otimes \otimes^{\frac{1}{2}(l-r)} 1_1) \in A_1 l$ and $L(\lambda) 1_r \neq 0$, we have $L(\lambda) 1_l \neq 0$. Finally, using the exact functor $1_t : A$-lfdmod $\rightarrow B_{q,t}$-lfdmod sending $M$ to $M 1_t$, we have the remaining statements immediately. \hfill \square

### 5.4. Duality functor.

By Definition 2.2 there is a contravariant auto-equivalence $\phi : B \rightarrow B$ such that $\phi$ fixes any object of $B$ and sends generating morphisms $1_1$ to $1_1$, $-1_1$, $-1_1$, respectively, and $\phi(b \otimes b') = \phi(b) \otimes \phi(b')$ for all morphisms $b, b'$ in $B$.

Standard arguments in [16 §3.3] shows that $\phi$ induces an anti-automorphism of $A$. Restricting it on $1_m A_1$ and $A^\circ$ yield the corresponding anti-involutions on $1_m A_1$ and $A^\circ$, respectively.
For any \( M \in A\text{-}\text{fdmod} \), let \( M^* := \oplus_{a \in \mathbb{N}} \text{Hom}_A(M_1, k) \). Then \( M^* \) is a locally finite dimensional right \( A \)-module such that
\[
(f(x)(m) = f(m\phi(x)), \quad x \in A, f \in M^*, m \in M_{1^n}.
\]
This induces an exact contravariant duality functor on \( A\text{-}\text{fdmod} \). Similarly, we have the exact contravariant duality functors on \( 1_m A_1 \text{-}\text{fdmod} \) and \( A^\circ \text{-}\text{fdmod} \). Let \( * : \Lambda_e \to \Lambda_e \) be the involution map such that \( D(\lambda)^* \equiv D(\lambda^*) \).

**Corollary 5.10.** Suppose \( \lambda \in \Lambda_e \) and \( \mu \in \Lambda \). We have
\[
\begin{align*}
&\text{(1)} \quad \Delta(\lambda)^* \equiv \nabla(\lambda^*), \quad L(\lambda)^* \equiv L(\lambda^*), \quad \text{where} \quad \nabla(\lambda) := \nabla(D(\lambda)).
&\text{(2)} \quad \dim \text{Hom}_A(\Delta(\mu), \Delta(\lambda)^*) = \delta_{\mu, \lambda^*}, \quad \text{where} \quad \Delta(\lambda) := \Delta(Y(\lambda)).
&\text{(3)} \quad \text{If} \quad e > l \quad \text{and} \quad \mu, \lambda \in \sum^+(l), \quad \text{then} \quad (P(\lambda) : \Delta(\mu)) = [\Delta(\mu^t) : L(\lambda^t)], \quad \text{where} \quad P(\lambda) \text{ is the projective cover of } L(\lambda) \text{ and } \lambda^t \text{ is the transpose of } \lambda.
\end{align*}
\]

**Proof.** A category which admits a split triangular decomposition is an upper finite weakly triangular category in the sense of [16] Definition 2.1. This enables us to use [16] Lemma 3.11 to get \( * \circ \Delta \equiv \nabla \circ * \). Applying both of them on \( D(\lambda) \) yields isomorphisms in (1). By general results in [6] Lemma 3.48, \( \dim \text{Hom}_A(\Delta(\mu), \Delta(\lambda)^*) = \delta_{\mu, \lambda^*} \) and hence (2) follows from (1), immediately. Suppose \( e > l \). Then \( H_l \) is semisimple. In this case, \( S(\lambda) = D(\lambda) = Y(\lambda), \Delta(\lambda) = \Delta(\lambda^t) \) and \( \nabla(\lambda) = \nabla(\lambda^t) \), where \( \nabla(\lambda) := \nabla(I(\lambda)) \) and \( I(\lambda) \) is the injective hull of \( D(\lambda) \). Since the Hecke algebra \( H_l \) is a symmetric algebra, \( I(\lambda) = Y(\lambda) \) for all \( \lambda \in \Lambda^+(l) \). Thanks to [19] Exercise 3.14(iii), \( \lambda^t = \lambda \), where \( \lambda \) is the conjugate of \( \lambda \). By [6] Lemma 3.36 or [16] Proposition 3.9(2), \( (P(\lambda) : \Delta(\mu)) = [\nabla(\mu) : L(\lambda)] \). Since \( * \) is an exact functor, by (1) \( [\nabla(\mu) : L(\lambda)] = [\nabla(\mu^t) : L(\lambda^t)] = [\Delta(\mu^t) : L(\lambda^t)] \), proving (3). \( \square \)

## 6. Classical Branching rules for periplectic \( q \)-Brauer algebras

The aim of this section is to give the classical branching rule for right standard modules of \( B_{q,t} \) over \( k \). We will use it to construct Jucys-Murphy basis for the right standard modules of \( B_{q,t} \) next section.

### 6.1. Cell filtration of cell modules and permutation modules for \( \hat{H}_{l-2f} \)

Suppose \( 0 \leq f \leq [l/2] \). Let \( \hat{S}_{l-2f} \) be the symmetric group on letters \( \{2f + 1, 2f + 2, \ldots, l\} \). When \( l \) is even and \( f = l/2 \), we set \( \hat{S}_{l-2f} = 1 \). Let \( \hat{H}_{l-2f} \) be the Hecke algebra associated to \( \hat{S}_{l-2f} \). Then \( \hat{H}_{l-2f} \), a subalgebra of \( H_l \), is isomorphic to \( H_{l-2f} \). The corresponding isomorphism sends \( T_{2f+1} \)'s to \( T_i \)'s in \( 3.9 \).

For any \( \lambda \in \Lambda(l-2f) \), let
\[
\lambda = \sum_{w \in \hat{\lambda}} q^{(w)} T_w,
\]
where \( \hat{\lambda} \) is the Young subgroup of \( \hat{S}_{l-2f} \) with respect to \( \lambda \). Each \( \lambda \) corresponds to the Young diagram \( [\lambda] \) such that there are \( l_i \) boxes in the \( i \)-th row of \( [\lambda] \). A \( \lambda \)-tableau \( t \) is obtained from \( [\lambda] \) by inserting \( 2f + 1, 2f + 2, \ldots, l \) into \( [\lambda] \) without repetition. In this case, we write \( \text{shape}(t) = \lambda \) and call \( \lambda \) the shape of \( t \). Let \( t^\lambda \) be the \( \lambda \)-tableau obtained from \( [\lambda] \) by inserting \( 2f + 1, 2f + 2, \ldots, l \) successively from left to right along the rows of \( [\lambda] \). If the entries in \( t \) increase from left to right along row and down column, \( t \) is called standard. In this case, \( \lambda \in \Lambda^+(l-2f) \). Let \( F^{std}(\lambda) \) be the set of all standard \( \lambda \)-tableaux. Then each \( t \in F^{std}(\lambda) \) satisfies \( t = t^\lambda d(t) \) where \( d(t) \) is a distinguished right coset representative of \( \hat{\lambda} \) in \( \hat{S}_{l-2f} \). The following result gives the Jucys-Murphy basis of \( \hat{H}_{l-2f} \).

**Theorem 6.1.** [19] Theorem 3.20] \( \hat{H}_{l-2f} \) has \( k \)-basis given by \( \{x_{s,t} \mid s, t \in F^{std}(\lambda), \lambda \in \Lambda^+(l-2f)\} \), where \( x_{s,t} = T_{d(t)} x_{t^\lambda} T_{d(t)} \) and \( * \) is the anti-involution on \( \hat{H}_{l-2f} \) fixing the generators \( T_i \)'s. It is a cellular basis in the sense of [17] Definition 1.1.

Suppose \( \lambda \in \Lambda^+(l-2f) \) and \( \mu \in \Lambda^+(l-2f-1) \). We say that \( \mu \) is obtained from \( \lambda \) by removing the removable node \( p = (k, \lambda_k) \) of \( \lambda \) if \( \mu_j = \lambda_j, j \neq k \) and \( \mu_k = \lambda_k - 1 \). In this case, \( \lambda \) is obtained from \( \mu \) by adding the addable node \( p \) of \( \mu \). We write \( \lambda = \mu \cup p \) or \( \mu = \lambda \setminus p \). Let \( R_\lambda \) be the set of all

\[\text{In [19], } (T_i - q)(T_i - 1) = 0 \text{ whereas we use } (T_i - q)(T_i + q^{-1}) = 0. \text{ So our anti-involution induced by } \phi \text{ corresponds to the anti-involution of the Hecke algebra in [19] which sends } T_i \text{ to } -qT_i^{-1}.\]
partitions obtained from \( \lambda \) by removing a removable node and \( A_\lambda \) the set of all partitions obtained from \( \lambda \) by adding an addable node.

For any \( \lambda \in \Lambda^+(l - 2f) \), let \( S^\lambda \) be the cell module of \( H_{l-2f} \) with respect to the Jucys-Murphy basis in Theorem 6.2. When \( f = 0 \), it is the dual Specht module in subsection 5.2. By definition, \( S^\lambda \) is the free \( k \)-module with basis \( \{ x_t \mid t \in \mathcal{T}_{std}(\lambda) \} \) where \( x_t := x_\lambda,1 + H^\lambda_{l-2f} \) and \( H^\lambda_{l-2f} \) is the two-sided ideal of \( H_{l-2f} \) with \( k \)-basis \( \{ x_{u,s} \mid u, s \in \mathcal{T}_{std}(\mu), \mu \triangleright \lambda \} \). Since \( R_\lambda \) is a finite set, we arrange them as \( \mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(a)} \) for some positive integer \( a \) such that \( \mu^{(1)} \triangleright \mu^{(2)} \triangleright \cdots \triangleright \mu^{(a)} \). For any standard \( \lambda \)-tableau \( t \), let \( t_\downarrow = 1 \) be obtained from \( t \) by removing the entry \( l \). Then \( t_\downarrow \in \mathcal{T}_{std}(\mu) \) for some \( \mu \in R_\lambda \). Let \( S_i \) be the free \( k \)-submodule of \( S^\lambda \) spanned by \( \{ x_{t_\downarrow} \mid t \in \mathcal{T}_{std}(\mu^{(j)}), 1 \leq j \leq a \} \). Then \( S_i \) is a right \( H_{l-2f-1} \)-module such that \( S^\lambda = S_0 \supset S_{a-1} \supset \cdots \supset S_1 \supset 0 \), where \( H_{l-2f-1} \) is the Hecke algebra associated to the symmetric group on \( \{ 2f + 1, 2f + 2, \ldots, l - 1 \} \).

**Theorem 6.2.** [19 Proposition 6.1] As \( H_{l-2f-1} \)-modules, \( S_i/S_{i-1} \cong S_{\mu^{(i)}}, 1 \leq i \leq a \), where \( S_{\mu^{(i)}} \) is the cell module with respect to the Jucys-Murphy basis of \( H_{l-2f-1} \) in Theorem 6.1.

Suppose \( (\lambda, \mu) \in \Lambda(l - 2f) \times \Lambda(l - 2f) \). Recall that a \( \lambda \)-tableau of type \( \mu \) is obtained from \( [\lambda] \) by inserting integers \( i \) into \( [\lambda] \) such that \( i \) appears \( \mu_i \) times. A \( \lambda \)-tableau of type \( \mu \) is called row semi-standard if the entries in each row of \( S \) are non-decreasing, \( S \) is semi-standard if \( \lambda \) is a partition and \( S \) is row semi-standard and the entries in each column of \( S \) are strictly increasing. Let \( T(\lambda, \mu) \) be the set of all semi-standard \( \lambda \)-tableaux of type \( \mu \). For any \( t \in \mathcal{T}_{std}(\lambda) \), let \( \mu(t) \) be the \( \lambda \)-tableau obtained from \( t \) by replacing each entry \( i \) in \( t \) with \( k \) if \( i \) appears in the \( k \)-th row of \( t \). By [19 Example 4.2(ii)], \( \mu(t) \) is a row semi-standard \( \lambda \)-tableau of type \( \mu \). Following [19 Chapert 4, §2], write

\[
\sum_{s \in \mathcal{T}_{std}(\lambda), \mu(s) = S} q^{l(d(s))} x_{s,t},
\]

where \((S, t) \in T(\lambda, \mu) \times \mathcal{T}_{std}(\lambda)\).

**Lemma 6.3.** [19 Corollary 4.10] For any \( \mu \in \Lambda(l - 2f) \), the right \( H_{l-2f} \)-module \( M^\mu := x_\mu H_{l-2f} \) has basis \( \{ x_{S,t} \mid S \in T(\lambda, \mu), t \in \mathcal{T}_{std}(\lambda), \lambda \in \Lambda^+(l - 2f) \} \). Arrange all such semi-standard tableaux as \( S_1, S_2, \ldots, S_k \) such that \( S_i \in T(\lambda_i, \mu) \) and \( i > j \) whenever \( \lambda_i \triangleright \lambda_j \). Let \( M_i \) be the \( k \)-submodule of \( M^\mu \) spanned by \( \{ x_{S_i,t} \mid t \in \mathcal{T}_{std}(\lambda_i), i \leq j \leq k \} \). Then \( M^\mu \) has a filtration

\[
M^\mu = M_1 \supset M_2 \supset \cdots \supset M_k \supset 0
\]

such that \( \mu_i/M_{i+1} \cong S^\lambda \). Further, the required isomorphism sends \( x_{S_i,t} + M_{i+1} \) to \( x_{\lambda_i, T_{d(t)} + \tilde{H}_{l-2f}} \).

6.2. A standard filtration of a standard module for \( B_{q,t} \). In this subsection, we prove that \( B_{q,t} \) is a standardly based algebra in the sense of [13]. We give a filtration of a standard module, which will be used to construct its Jucys-Murphy basis.

Recall \( \Sigma^+ (l) \) in [5,3]. There is a partial order \( \preceq \) on \( \Sigma^+ (l) \) such that \( \mu \preceq \lambda \) for \( \mu, \lambda \in \Sigma^+ (l) \) if \( |\mu| \geq |\lambda| \) or \( |\mu| = |\lambda| \) and \( \sum_{j=1}^l \mu_j \leq \sum_{j=1}^l \lambda_j \) for all possible \( k \). For any \( w, v \in D_{f,t}, s \in T_{std}(\lambda) \), and \( \lambda \in \Lambda^+(l - 2f) \), define

\[
C^\lambda_{(w,s),(v,t)} = \sigma(T_w) E^f x_{s,t} T_v,
\]

where \( \sigma \) is the anti-involution of \( B_{q,t} \) in Lemma 3.4 and \( x_{s,t} \)'s are given in Theorem 6.1. In Theorem 6.4, \( k \) can be an arbitrary field with characteristic not 2.

**Theorem 6.4.** Suppose \( B_{q,t} \) is the periplectic \( q \)-Brauer algebra over \( k \). For any \( \lambda \in \Lambda^+(l - 2f) \), define \( I(\lambda) = D_{f,t} \times \mathcal{T}_{std}(\lambda) \).

1. \( B_{q,t} \) has \( k \)-basis \( S = \{ C_{(w,s),(v,t)} \mid (w, s), (v, t) \in I(\lambda), \lambda \in \Lambda^+(l - 2f), 0 \leq f \leq \lfloor l/2 \rfloor \} \).
2. For any \( a \in B_{q,t} \), we have

\[
aC^\lambda_{(w,s),(v,t)} \equiv \sum_{(w_1,s_1) \in I(\lambda)} r_{(w_1,s_1),\lambda} (a, (w, s)) C^\lambda_{(w_1,s_1),(v,t)} \quad (\text{mod } B_{q,t})
\]

\[
C^\lambda_{(w,s),(v,t)} = \sum_{(v_1,t_1) \in I(\lambda)} r_{\lambda, (v_1,t_1)} ((v, t), a) C^\lambda_{(w,s),(v_1,t_1)} \quad (\text{mod } B_{q,t})
\]

where \( B_{q,t}^> \lambda \) is the free \( k \)-submodule generated by all \( C^\mu_{(y,u),(z,u)} \)'s such that \((y, u), (z, u) \in I(\mu)\) and \( \mu \triangleright \lambda \). Further, the coefficient \( r_{(w_1,s_1),\lambda} (a, (w, s)) \) (resp., \( r_{\lambda, (v_1,t_1)} ((v, t), a) \) is independent of \((v, t)\) (resp., \((w, s)\)).
Proof. In fact, (1) follows immediately from Theorems 4.9 and 6.1 and (2) follows from Lemma 4.7 and Theorem 6.11.

The set $S$ is a standard basis of $B_{q,t}$ in the sense of [13] (1.2.1]. It may not be a cellular basis of $B_{q,t}$ in the sense of [17] since we have not found an anti-involution of $B_{q,t}$ sending $C^\lambda_{(w,s),(v,t)}$ to $C^\lambda_{(v,t),(w,s)}$.

**Corollary 6.5.** Let $B_{q,t}$ be the periplectic Brauer algebra over an algebraically closed field of characteristic not 2. Then $B_{q,t}$ is a quasi-hereditary algebra if and only if $e > 1$ and $l$ is odd.

Proof. It is proved in [13] Theorem 3.2.1 that a standardly based algebra over a field is quasi-hereditary if and only if it is “full” in the sense of [13] Definition 1.3.1]. By Theorem 6.4, $B_{q,t}$ is quasi-hereditary if and only if the simple $B_{q,t}$-modules are parameterized by $\Sigma^+(l)$. Now, the result follows from Theorem 5.9.

For any $\lambda \in \Sigma^+(l)$, let $C(\lambda)$ be the (right) standard module of $B_{q,t}$ with respect to the standard basis in Theorem 6.4. Thanks to [13] Definition 2.1.2, up to an isomorphism, $C(\lambda)$ can be considered as the free $k$-module with basis $\{E^1 x_\lambda T_{d(t)} T_v + B_{q,t}^\lambda \ | \ (v,t) \in I(\lambda)\}$. The following result establishes a relationship between $\Delta(S^\lambda)$ in $A$-$\text{ldmod}$ and $C(\lambda)$.

**Lemma 6.6.** Suppose $k$ is an algebraically closed field with characteristic not 2 and $\lambda \in \Sigma^+(l)$.

1. As right $B_{q,t}$-modules, $C(\lambda) \cong \Delta(S^\lambda)I_1$.
2. As right $H_1$-modules, $\Delta(S^\lambda)I_1 \cong \text{Ind}^H_{H_{2f}} \text{Hom}_B(2f,0) \otimes S^\lambda$.

Proof. Thanks to Theorem 5.5 and 5.6, $\Delta(S^\lambda)I_1$ has basis

\[
(6.3) \{x_\lambda T_{d(t)} \otimes E^\lambda T_v \ | \ t \in \mathcal{F}^{std}(\lambda), v \in D_f, t\}
\]

where $E^\lambda = \bigotimes I \otimes 1_{l-2f}$. Obviously, there is a k-linear isomorphism $\phi : C(\lambda) \to \Delta(S^\lambda)I_1$ such that

\[
(6.4) \phi(E^1 x_\lambda T_{d(t)} T_v + B_{q,t}^\lambda) = x_\lambda T_{d(t)} \otimes E^\lambda T_v.
\]

Since $E^\lambda = a_f E^\lambda$, where $a_f = 1 \otimes \bigotimes I \otimes 1_{l-2f-1}$, by Theorem 4.9(2) and Corollary 4.8, both $E^\lambda B_{q,t}$ and $E^\lambda B_{q,t}$ are left $B(f)$-module spanned by $\{E^\lambda T_d \ | \ d \in D_f\}$ and $\{E^\lambda T_d \ | \ d \in D_f\}$, respectively. Recall the two-sided ideal $B_{q,t}^f$ in [4.13]. If we consider $E^\lambda B_{q,t}$ (resp., $E^\lambda B_{q,t}$) up to $B_{q,t}^{f+1}$ (resp., $I_{l+1}$), the subspace of $1_{l-2f}A_1$ spanned by two-dimensional diagrams with at least $f + 1$ cups, we can use Hecke algebra $H_{l-2f}$ to replace $B(f)$. Consequently, the k-linear isomorphism $\phi$ in (6.4) is a right $B_{q,t}$-homomorphism, proving (1).

Obviously, $\text{Ind}^H_{H_{2f}} \text{Hom}_B(2f,0) \otimes S^\lambda$ has basis

\[
\{(\bigotimes (T_{d_1} \otimes x_1) \otimes T_{d_2} \ | \ (t, d_1, d_2) \in \mathcal{F}^{std}(\lambda) \times D_{f,2f} \times D_{f,l-2f})\}
\]

where $D_{2f,l-2f}$ is the distinguished right coset representatives of $\mathcal{G}_{2f} \times \mathcal{G}_{l-2f} \in \mathcal{G}_l$. There is a bijection between $D_{f,t}$ and $D_{f,2f} \times D_{f,l-2f}$ such that any $d \in D_{f,t}$ can uniquely be written as $d_1 d_2$ such that $(d_1, d_2) \in D_{f,2f} \times D_{f,l-2f}$. Consequently, there is a k-linear isomorphism $\phi : \Delta(S^\lambda)I_1 \to \text{Ind}^H_{H_{2f}} \text{Hom}_B(2f,0) \otimes S^\lambda$ such that

\[
(6.5) \phi(x_1 \otimes E^\lambda T_d) = (\bigotimes T_{d_1} \otimes x_1) \otimes T_{d_2}
\]

where $d = d_1 d_2$ and $(d_1, d_2) \in D_{f,2f} \times D_{f,l-2f}$. Finally, it follows from Corollary 4.8 that $\phi$ is a right $H_1$-homomorphism, proving (2).

**Corollary 6.7.** Suppose $\lambda \in \Lambda^+(l-2)$ and $e > l$. As right $H_1$-modules, $C(\lambda) \cong \oplus \mu S^\mu$ where $\mu \in \Lambda^+(l)$ obtained from $\lambda$ by adding two boxes which are not in the same row.

Proof. Suppose $f = 1$. Thanks to Lemma 6.6, we have right $H_1$-isomorphism

\[
(6.6) C(\lambda) \cong \text{Ind}^H_{H_{2f}} \text{Hom}_B(2f,0) \otimes S^\lambda.
\]

However, by Lemma 4.3(3) and Definition 4.3(2), $\text{Hom}_B(2f,0) \cong S^{(1,1)}$ as right $H_2$-modules. Now the result follows from Littlewood-Richardson rule for semi-simple Hecke algebras. In this case, all Littlewood-Richardson coefficients appearing on the decomposition of RHS of (6.6) is 1.
We are going to construct a $B_{q,t-1}$-filtration of $C(\lambda)$. For any $\lambda \in \Sigma^+(l)$, let

$$R_A(\lambda) = \Sigma^+(l-1) \cap (A_\lambda \cup R_\lambda).$$

**Definition 6.8.** Suppose $\lambda \in \Lambda^+(l-2f)$ and $\mu \in R_A(\lambda)$.

1. If $\mu = \lambda \setminus p$ and $p = (k, \lambda_k)$, let $y_\mu^\lambda = E^f x_\lambda T_{a_k,i}$, where $a_k = 2f + \sum_{i=1}^K \lambda_i$.
2. If $\mu = \lambda \setminus p$ and $p = (k, \lambda_k + 1)$, let $y_\mu^\lambda = E_{2f-1} T_{1,2f-1} T_{b_k,2f-1} E_{f-1} x_\mu$, where $b_k = 2f-1 + \sum_{i=1}^K \lambda_i$.

Definition 6.8(2) happens only if $f > 0$. Thanks to Definition 6.8, we rewrite $y_\mu^\lambda$ as follows:

$$y_\mu^\lambda = \begin{cases} \sum_{i=a_k+1}^{a_k} q^{-a_k-i} T_{a_k,i} E^f x_\mu & \text{if } \mu = \lambda \setminus p, \\ E^f x_\lambda T_{1,2f-1} T_{b_k,2f-1} \sum_{i=b_k+1}^{b_k} q^{-b_k-i} T_{b_k,i} & \text{if } \mu = \lambda \setminus p. \end{cases}$$

For any $\lambda \in \Lambda^+(l-2f)$, both $R_\lambda$ and $A_\lambda$ are finite sets. There are two positive integers $a$ and $m$ such that $R_\lambda = \{\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(a)}\}$, $A_\lambda = \{\mu^{(a+1)}, \mu^{(a+2)}, \ldots, \mu^{(m)}\}$ and

$$R_A(\lambda) = \begin{cases} R_\lambda & \text{if } f = 0; \\ R_\lambda \cup A_\lambda & \text{if } f > 0. \end{cases}$$

Moreover, we can arrange elements in $R_A(\lambda)$ such that

$$\mu^{(1)} \triangleright \mu^{(2)} \triangleright \ldots \triangleright \mu^{(a)} \triangleright \mu^{(a+1)} \triangleright \ldots \triangleright \mu^{(m)}$$

with respect the partial order $\triangleright$ on $\Sigma^+(l-1)$. For $1 \leq k \leq m$, define

$$N_k = \sum_{j=1}^k y_{\mu^{(j)}}^\lambda B_{q,t-1} + B_{q,t-1}$$

where $y_{\mu^{(j)}}^\lambda$'s are given in 6.8. Then $N_k$ is a right $B_{q,t-1}$-submodule of $C(\lambda)$. When $k > a$, $N_k$ is defined only if $f > 0$.

**Lemma 6.9.** As $B_{q,t-1}$-modules, $N_k/N_{k-1} \cong C(\mu(k))$, $1 \leq k \leq a$.

**Proof.** Thanks to Corollary 4.8 and Theorem 6.2, any element in $y_{\mu^{(j)}}^\lambda B_{q,t-1} + N_{k-1}$ can be written as a linear combination of elements in

$$M_k := \{y_{\mu^{(j)}}^\lambda T_{d(u)} T_w + N_{k-1} \mid u \in \mathcal{T}^{std}(\mu(k)), w \in D_{f,t-1}\}.$$ 

Further, $M_k$ is $k$-linear independent since $y_{\mu^{(k)}}^\lambda T_{d(u)} T_w + B_{q,t-1}^{\geq \lambda} = E^f x_\lambda T_{d(u)} T_w + B_{q,t-1}^{\geq \lambda}$ with $t \downarrow t-1 = u$, a basis element of $C(\lambda)$. So $N_k/N_{k-1}$ is free over $k$ with basis $M_k$. We have a well-defined $k$-linear isomorphism $\phi: N_k/N_{k-1} \rightarrow C(\mu(k))$ such that

$$\phi(y_{\mu^{(j)}}^\lambda T_{d(u)} T_w + N_{k-1}) = E^f x_{\mu^{(j)}} T_{d(u)} T_w + B_{q,t-1}^{\geq \mu^{(k)}}$$

for all $u \in \text{Std}(\mu(k))$ and $w \in D_{f,t-1}$. Using Corollary 4.8 and Theorem 6.2 again, we see that $\phi$ is a right $B_{q,t-1}$-homomorphism and hence $\phi$ is a $B_{q,t-1}$-isomorphism.

If $\lambda \in \Lambda^+(l)$ (i.e., $f = 0$), then $C(\lambda)$ has a standard filtration in Lemma 6.9. From here to the end of this section, we assume $\lambda \in \Lambda^+(l-2f)$ with $f > 0$ and deal with $N_k/N_{k-1}$, $a + 1 \leq k \leq m$. The following result has already been given in the proof of [14] Corollary 5.4, Lemma 5.5.

**Lemma 6.10.** Suppose that $\mu \in \Lambda^+(l-2f+1)$ and $\mu \triangleright \mu^{(m)}$.

1. Suppose $\mu \notin \{\mu^{(a+1)}, \ldots, \mu^{(m)}\}$. Then $\text{shape}(s_{\downarrow t-2}) \triangleright \lambda$ if $(s, \mu^{(m)}(s)) \in \mathcal{T}^{std}(\mu) \times T(\mu, \mu^{(m)})$.
2. Suppose $\mu = \mu^{(j)} = \lambda \cup (k, \lambda_k + 1)$ for some $a + 1 \leq j \leq m$. Let $s_j = (s_{\downarrow t-1}^{(a)}, s_{\downarrow t-1})$, where $b_j = 2f - 1 + \sum_{i=1}^K \lambda_i$. Then $s_j$ is the unique standard $\mu^{(j)}$-tableau $s$ satisfying $\mu^{(m)}(s) \in T(\mu^{(j)}, \mu^{(m)})$. In this case, $\text{shape}(s_{\downarrow t-2}) = \lambda$.

**Proof.** The arguments in the proof [14] Lemma 5.3 depends only on the braid relations for the Hecke algebras. Therefore, one can imitate arguments there to verify our current result. We leave details to the reader.

**Lemma 6.11.** Suppose $(s, \mu^{(m)}(s)) \in \mathcal{T}^{std}(\mu) \times T(\mu, \mu^{(m)})$. Then $E^f T_{1,2f-1} T_{1-1,2f-1} T_{d(s)} x_n \in B_{q,t-1}^{\geq \lambda}$ whenever $\text{shape}(s_{\downarrow t-2}) \triangleright \lambda$.

**Proof.** The arguments in the proof [14] Lemma 5.3 depends only on the braid relations for the Hecke algebras. Therefore, one can imitate arguments there to verify our current result. We leave details to the reader.

**Lemma 6.12.** As $B_{q,t-1}$-modules, $C(\lambda) = N_m$ where $N_m$ is given in 6.10.
Proof. Recall that the standard module $C(\lambda)$ has basis \{\(E^f x T_{d(t)} T_v + B^\omega_{q,t} \) | \((v, t) \in I(\lambda)\)} If \( v \in D_{f, l-1} \), then
\[
\text{(6.12)} \quad E^f x T_{d(t)} T_v + B^\omega_{q,t} \in E^f x T_{d(t)} B_{q,t-1} + B^\omega_{q,t} \subset N_a.
\]
Otherwise \( T_v = T_{2f} b \) for some \( b \in H_{l-1} \). So,
\[
\text{(6.13)} \quad E^f x T_{d(t)} T_v + B^\omega_{q,t} \in E^f x T_{2f} b B_{q,t-1} + B^\omega_{q,t} \subset \sum_{j=2f+1}^l E^f x T_{j,t} B_{q,t-1} + E^f x T_{1,2f} B_{q,t-1} + B^\omega_{q,t}
\]
By (6.10), the last term in the RHS of (6.13) is in \( N_m \). Thanks to (6.12), the summation in the RHS of (6.13) is in \( N_a \subset N_m \). This implies \( C(\lambda) \subset N_m \) and hence \( C(\lambda) = N_m \) as required. \( \square \)

**Lemma 6.13.** As \( B_{q,t-1} \)-modules, \( N_k/N_{k-1} \cong C(\mu(k)) \) for any \( a + 1 \leq k \leq m \).

**Proof.** First, we find a set which spans \( N_k/N_{k-1} \) as \( k \)-module. Suppose \( h \in B_{q,t-1} \). Thanks to Corollary \( 4.8 \) \( E^f h + B^\omega_{q,t} \) can be written as a linear combination of elements \( h_d E^f T_d + B^\omega_{q,t} \) and \( b + B^\omega_{q,t} \), where \( b \in E^f B_{q,t-1} \cap B^\omega_{q,t}, \) \( d \in D_{f, l-1} \) and \( h_d \in H_{l-2f+1} \), the Hecke algebra associated to the symmetric group on \( \{2f - 1, 2f, \ldots, l - 1\} \). Hence
\[
y_{\mu(k)}^\lambda h = \sum_{d \in D_{f, l-1}} E^f x h_d T_d + E^f T_{1,2f} T_{2f}^{-1} x_{s_k, u^k}^\mu(b) (\text{mod } B^\omega_{q,t}).
\]
By arguments similar to those for the proof of \( [14] \) Claim 5.7 we have
\[
E^f T_{1,2f} T_{2f}^{-1} x_{s_k, u^k}^\mu(b) (\text{mod } B^\omega_{q,t})
\]
for any \( b \in E^f B_{q,t-1} \cap B^\omega_{q,t} \). Write \( x = T_{1,2f} T_{2f}^{-1} x_{s_k, u^k}^\mu(b) \). For \( a + 1 \leq k \leq m \),
\[
\text{(6.14)} \quad y_{\mu(k)}^\lambda h = \sum_{d \in D_{f, l-1}} E^f x h_d T_d = \sum_{d \in D_{f, l-1}} E^f T_{1,2f} T_{2f}^{-1} x_{s_k, u^k}^\mu(h_d T_d) \quad (\text{mod } N_a)
\]
where \( s_k \in \mathcal{S}_{std}(\mu(k)) \) is given in Lemma [6.10](2). Let \( S_k = \mu(m)(s_k) \). By Lemma [6.10](2),
\[
\text{(6.15)} \quad x_{s_k, u} = q^{-\ell(d(s_k))} x_{S_k, u}
\]
for any \( u \in \mathcal{S}_{std}(\mu(k)) \). Rewriting RHS of (6.14) via (6.15) and Lemma 6.3 we have
\[
y_{\mu(k)}^\lambda h = \sum_{d \in D_{f, l-1}} E^f T_{1,2f} T_{2f}^{-1} x_{s_k, u^k}^\mu(h_d T_d) (\text{mod } N_a)
\]
If \( \mu \neq \mu(j) \), by Lemma [6.10](1) and Lemma 6.11 the corresponding terms on the RHS of the above equality is in \( B^\omega_{q,t} \). If \( \mu = \mu(j) \), by Lemma [6.10](2), \( S_j = S_f \). Using (6.15) again, we see that \( N_k/N_{k-1} \) is \( k \)-module spanned by
\[
\text{(6.16)} \quad M_k = \{y_{\mu(k)}^\lambda T_{d(t)} T_d + N_{k-1} | t \in \mathcal{S}_{std}(\mu(k)), d \in D_{f, l-1}\}.
\]
So, \( \dim N_k/N_{k-1} \leq \dim C(\mu(k)) = |M_k| \). By the branching rule for the cell module of BMW algebras in \( [24] \) Theorem 2.2, we have \( \dim C(\lambda) = \sum_{j=1}^m \dim C(\mu(j)) \). By Lemma 6.12 \( \dim N_m = \dim C(\lambda) \), forcing \( \dim N_k/N_{k-1} = \dim C(\mu(k)) = |M_k| \). In particular, \( M_k \) is a basis of \( N_k/N_{k-1} \) for all \( 1 \leq k \leq m \). We have a well-defined \( k \)-linear isomorphism \( \phi : N_k/N_{k-1} \rightarrow C(\mu(k)) \) for any \( a + 1 \leq k \leq m \) such that
\[
\text{(6.17)} \quad \phi(y_{\mu(k)}^\lambda T_{d(t)} T_d + N_{k-1}) = E^f x_{\mu(k)}^\lambda T_{d(t)} T_d + B^\omega_{q,t}
\]
for any \( t \in \mathcal{S}_{std}(\mu(k)) \) and \( d \in D_{f, l-1} \). By arguments similar to those above (more explicitly, using Corollary \( 4.8 \) and Lemma 6.3) we see that the \( k \)-linear isomorphism \( \phi \) in (6.17) is a right \( B_{q,t-1} \)-homomorphism. \( \square \)

**Theorem 6.14.** Suppose \( \lambda \in \Sigma^+(l) \). The standard module \( C(\lambda) \) has a \( B_{q,t-1} \)-filtration \( 0 \subset N_1 \subset \cdots \subset N_m \) such that \( N_k/N_{k-1} \cong C(\mu(k)) \) for \( 1 \leq k \leq m \).

**Proof.** The result follows immediately from Lemmas 6.9 and 6.13. \( \square \)
7. Jucys-Murphy elements and Jucys-Murphy basis

7.1. The Jucys-Murphy basis of any standard module for $B_{q,l}$. Suppose $\lambda \in \Sigma^+(l)$. Write $\mu \to \lambda$ if $\mu \in RA(\lambda)$ (see (6.7)). An up-down tableau $t$ of type $\lambda$ is a sequence of partitions $t = (t_0, t_1, \ldots, t_l)$ such that $t_0 = \emptyset$, $t_i \to t_{i+1}$, $0 \leq i \leq l-1$ and $t_l = \lambda$. Let $\mathcal{R}_{l}^{ud}(\lambda)$ be the set of up-down tableaux of type $\lambda$. There is a partial order $\lessdot$ on $\mathcal{R}_{l}^{ud}(\lambda)$ such that $t \lessdot s$ if $t_k \lessdot s_k$ for some $k$ and $t_j = s_j$ for all $k < j \leq l$.

**Definition 7.1.** Suppose $t \in \mathcal{R}_{l}^{up}(\lambda)$ and $\lambda \in \Lambda^+(l-2f)$. Let $m_{t_1} = 1$. If $m_{t_{l-1}}$ is available, we define $m_t$ inductively as follows:

1. $m_t = \sum a_{t_k} T_{t_i} m_{t_{i-1}}$ if $\lambda = \mu \cup p$ with $p = (k, \mu_k + 1)$, and $a_i = 2f + \sum_{j=1}^i \lambda_j$.
2. $m_t = E_{2f-1} T_{i,2f-1}^{-1} b_{t_{i-1}}$ if $\lambda = \mu \setminus p$ with $p = (k, \mu_k)$, and $b_i = 2f - 1 + \sum_{j=1}^i \lambda_j$.

Later on, we denote $m_{t_1}$ by $m_t$. Thanks to Definition 7.1, $m_t = E^l x_t b_t$ where $b_t = b_{t_1}$ which can be defined inductively as

$$b_t = \begin{cases} T_{a_{t_k}} b_{t_{k-1}} \sum_{j=\mu_{t_{k-1}}}^{\mu_{t_k}} q^{\mu_{t_k} q - j \mu_{t_{k-1}}} & \text{if } \lambda = \mu \cup p \text{ with } p = (k, \mu_k + 1), \\ T_{t_{k-1}}^{-1} b_{t_{k-1}} & \text{if } \lambda = \mu \setminus p \text{ with } p = (k, \mu_k). \end{cases} \quad (7.1)$$

**Theorem 7.2.** Suppose that $\lambda \in \Lambda^+(l-2f)$.

1. $\{m_t + B^{\mu,\lambda}_{q,l} \mid t \in \mathcal{R}_{l}^{up}(\lambda)\}$ is a $k$-basis of $C(\lambda)$.
2. For any $1 \leq j \leq m$, let $M_j$ be the $k$-submodule of $C(\lambda)$ spanned by

$$\{m_t + B^{\mu,\lambda}_{q,l} \mid t \in \mathcal{R}_{l}^{up}(\lambda), t_{m-1} \geq \mu^{(j)}(1)\}$$

where $\mu^{(j)}$s are given in (6.9). Then $M_j$ is a $B_{q,l-1}$-submodule of $C(\lambda)$.
3. As $B_{q,l-1}$-submodules, $M_j/M_{j-1} \cong C(\mu^{(j)})$. The required isomorphism sends $m_t + M_{j-1}$ to $m_{t_{j-1}} + B_{q,l-1}^{\mu^{(j)}(1)}$, where $t_{j-1} = (t_0, t_1, \ldots, t_{j-1}) \in \mathcal{R}_{l}^{ud}(\mu^{(j)})$.
4. $M_j = N_j$ for $1 \leq j \leq m$, where $N_j$’s are given in (6.11).

**Proof.** Using the explicit isomorphisms in the proof of Lemma 6.9 and Lemma 6.13 (i.e. the classical branching rule), we immediately have that (e.g., [13] Theorem 5.9) for BMW algebras) $C(\lambda)$ has a basis

$$\{y_{\mu}^{\lambda} b_u + B^{\mu,\lambda}_{q,l} \mid u \in \mathcal{R}_{l-1}^{up}(\mu), \mu \to \lambda\}$$

if $C(\mu)$ has a basis $\{E^k x_{\mu} b_u + B^{\mu,\lambda}_{q,l-1} \mid u \in \mathcal{R}_{l-1}^{up}(\mu)\}$ for any $\mu \to \lambda$, where $k$ is the integer such that $\mu \in \Lambda^+(l-1-2k)$. Moreover,

$$y_{\mu^{(j)}}^{\lambda} b_u + M_{j-1} \to E^k x_{\mu^{(j)}} b_u + B^{\mu^{(j)}(1)}_{q,l-1}$$

determines an isomorphism $M_j/M_{j-1} \cong C(\mu^{(j)})$ of $B_{q,l-1}$-modules. Note that $m_u = E^k x_{\mu} b_u$ for $u \in \mathcal{R}_{l-1}^{up}(\mu)$. Now (1)–(3) follow from induction on $l$ since $y_{\mu}^{\lambda} b_u = m_t$ with $t_{l-1} = u$ by Definition 7.1.

By Definition 7.1 we see that $m_t + B^{\mu,\lambda}_{q,l} \in N_j$ if $t_{j-1} = \mu^{(k)}$ for $k \leq j$. So, $M_j \subset N_j$ for any $1 \leq j \leq m$. Then (4) follows by comparing dimensions. \hfill $\Box$

The basis in Theorem 7.2(1) is called the Jucys-Murphy basis of $C(\lambda)$. We can give branching rule for standard modules in the category of left $B_{q,l}$-modules and hence to give the Jucys-Murphy basis for $B_{q,l}$. We will not give details since we do not need them in this paper.

7.2. Jucys-Murphy elements of $B_{q,l}$. For any positive integer $i$, $1 \leq i \leq l$ define

$$x_i = \begin{cases} \sum_{j=1}^{i-1} (j,i) + q^{-1}(j,i) & \text{if } i = 1, \\ \sum_{j=1}^{i-1} (j,i) + q^{-1}(j,i) & \text{if } 2 \leq i \leq l, \end{cases} \quad (7.2)$$

where $(j,i) = T_{j,i-1,j,i}$, and $(j,i) = T_{j,i-1} T_{i-1,j} T_{j,i} T_{j,i+1}$. By Definition 3.3, it is easy to verify the following equality:

$$x_i x_{i+1} = T_i x_i T_i + T_i + q^{-1}(i,i+1). \quad (7.3)$$

**Lemma 7.3.** For any $2 \leq i \leq l$ and $y \in B_{q,i-1}$, $yx_i = x_i y$. 

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HEBING RUI AND LINLIANG SONG
Proof. It is known that ∑_{j=1}^{i-1} (j, i) is a Jucys-Murphy element for the Hecke algebra $H_l$ satisfying $T_k \sum_{j=1}^{i-1} (j, i) = T_k$ for any $1 \leq k \leq i - 2$. Further, by direct computation, we have

$$T_k ((k, i) + (k + 1, i)) = ((k, i) + (k + 1, i)) T_k$$

and

$$T_k ((j, i)) = (j, i) T_k$$

if either $j < k$ or $j > k + 1$. So $T_k \sum_{j=1}^{i-1} q^{-1}(j, i) = \sum_{j=1}^{i-1} q^{-1}(j, i) T_k$, proving $T_k x_i = x_i T_k$.

When $i = 1, 2, 3$, it is easy to verify $E_1 x_i = x_i E_1$. It is enough to assume $i \geq 4$ when we prove $E_1 x_i = x_i E_1$. Thanks to Lemma 3.3 (6)-6), we have $E_1 T_3 E_1 = E_1 T_3 E_1 = -E_1 E_2 T_4 T_{1,3} = E_1 E_3$, and $T_{3,1} T_{1,2} E_1 T_{2,4} T_{1,3} = -T_3 T_{1,2} E_1 E_3 = E_1 E_3$. So $E_1 (3, 4) = (3, 4) E_1$. By (7,3), $E_1 x_4 = x_4 E_1$. In general, we have $(i, i + 1) = T_{i-1} T_i (i-1, i) T_{i-1}$ and $E_1 (i, i + 1) = (i, i + 1) E_1$ for $i \geq 4$. Using (7,3) and induction assumption on $i - 1$ yields $E_1 x_i = x_i E_1$ as required. \(\square\)

By Lemma 7.3 \{x_i | 1 \leq i \leq l\} generates an abelian subalgebra $L_l$ of $B_{q,1}$. However, $\sum_{i=1}^l x_i$ may not be a central element of $B_{q,1}$ in general. For example $x_1 + x_2$ is not central in $B_{q,2}$.

**Lemma 7.4.** Suppose $2f + 1 \leq i \leq l - 1$ and $1 \leq j \leq f$.

1. $E^f ((i, l)) \equiv 0 \pmod{B_{q,1}^{f+1}}$,
2. $E^f ((2j, l) + (2j - 1, l) + q^{-1}(2j - 1, l) + (2j, l)) = 0$.

Proof. We have $E^f (2f + 1, l) = T_{1,2f+2} E^f T_{2f+1,1} T_{2f+2,2} E_1 T_{2,1} T_{2f+1}$ by Lemma 3.3 (1)

$$T_{1,2f+2} E^f T_{2f+1,1} T_{2f+2,2} E_1 T_{2,1} T_{2f+1} = T_{1,2f+2} E^f T_{2f+1,1} T_{2f+2,2} E_{1,2} T_{1,2f+1} + E_{1,2} T_{2f+1} + E_{1,2} T_{2f+1} = E_{1,2} T_{2f+1}$$

This proves (1) when $i = 2f + 1$. In general (1) follows from (7,4) and $(i, l) = T_{1,2f+1}(2f + 1, l) T_{2f+1}$. We have $E_1 (2, l) = q^{-1} E_1 (2, l) T_1, E_1 (1, l) = q E_1 (2, l)$ and $E_1 (2, l) = E_1 (2, l) T_1$. Via them, it is easy to verify (1) when $f = 1$. In general, one can check that $E^f (2f - 1, l) = -q E^f (2f, l), E^f (2f, l) = E^f (2f, l) T_{2f-1}$ and $E^f (2f - 1, l) = q E^f (2f, l) T_{2f-1}$. So

$$E^f ((2f, l) + (2f - 1, l) + q^{-1}(2f - 1, l) + (2j, l)) = 0.$$ This proves (2) when $j = f$. In particular, we have $E^f ((2j, l) + (2f - 1, l) + q^{-1}(2f - 1, l) + (2j, l)) = 0$. This implies (2) since $E^f$ is a factor of $E$ for any $1 \leq j \leq f$. \(\square\)

**Lemma 7.5.** Suppose $\lambda \in \Sigma^+(l)$ and $\mu^{(j)} = \lambda \setminus p_j, 1 \leq j \leq a$. Then

$$y_{\mu^{(j)}} x_l \equiv \frac{q^{2(c(p_j))} - 1}{q - q^{-1}} y_{\mu^{(j)}} \pmod{N_{j-1}}$$

where $N_{j-1}$ is given in (6,10) and $c(p_j) = k - i$ if $p_j = (i, k)$.

Proof. By Lemma 7.4 $y_{\mu^{(j)}} x_l \equiv y_{\mu^{(j)}} \sum_{i=1}^{i-1} (i, l) \pmod{B_{q,1}^{f+1}}$. Since $y_{\mu^{(j)}} \equiv E^f x \lambda T_{a,j,l}$ (see Definition 6,8(1)), by [19] Theorem 3.22, we have the formula as required. \(\square\)

**Lemma 7.6.** Suppose $\lambda \in \Lambda^+(l - 2f), 2f \leq i \leq l - 1$ and $2f < j \leq l - 1$.

1. $E^f x_\lambda T_{i,2f}^{-1} (i, l) \in N_a$,
2. $E^f x_\lambda T_{i,2f}^{-1} ((2f - 1, l) + q^{-1}) \in N_a$,
3. $E^f x_\lambda T_{i,2f}^{-1} (2f, l - 1) = 0$,
4. $E^f x_\lambda T_{i,2f}^{-1} ((j, l) - q^{-1}(2f - 1, j)) \in N_a$,

where $N_a$ is given in (6,10).

Proof. Since we are assuming $2f \leq i \leq l - 1$,

$$E^f x_\lambda T_{i,2f}^{-1} (i, l) = E^f x_\lambda T_{i+1,2f}^{-1} = 0 \pmod{N_a},$$

proving (1). We have $E^f x_\lambda T_{i,2f}^{-1} (2f, l) = E^f x_\lambda T_{2f-1, l} = q^{-1} E^f x_\lambda T_{2f, l} = q^{-1} E^f x_\lambda T_{i,2f}^{-1}$ (mod $N_a$), proving (2). Write $x = T_{i,2f}^{-1} T_{2f-1, l} T_{2f, l} T_{2f, l-1}$. Then (3) follows since

$$E^f x_\lambda T_{i,2f}^{-1} (2f - 1, l) = E^f x_\lambda x = E^f x_\lambda T_{2f-1, l} T_{2f, l} T_{2f, l-1} = (1)^{2f-1} x E^f x_\lambda E_1 T_{2f, l} T_{2f, l-1} = 0.$$

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5The Hecke algebra $H_l$ is defined via $(T_i - q)(T_i + 1) = 0$ in [19]. Our $q^{-1} T_i$ (resp., $q^2$) is $T_i$ (resp., $q$) in [19]. Consequently, our $q^{-1} x_i$ is $L_i$ in [19].
where the third equality follows from Lemma 8.5(5). Suppose $2f < j \leq l - 1$, we have

$$E^f x_\lambda T_{1,2j}^{-1}(j,l) = E^f x_\lambda T_{1,2j}^{-1} T_{1,l} T_{1,2j} T_{1,j} E^f x_\lambda T_{1,2j}^{-1} T_{1,j} = E^f x_\lambda T_{1,2j}^{-1} T_{1,j} T_{1,2j}^{-1} E^f x_\lambda T_{1,j}^{-1} T_{1,j} = q^{-1} E^f x_\lambda T_{1,2j}^{-1} T_{1,j} T_{1,2j}^{-1} E^f x_\lambda T_{1,j}^{-1} T_{1,j} = q^{-1} E^f x_\lambda T_{1,2j}^{-1} T_{1,j} T_{1,2j}^{-1} E^f x_\lambda T_{1,j}^{-1} T_{1,j} = (-1)^f q^{-2} E^f x_\lambda T_{1,2j}^{-1} T_{1,j} = (-1)^f q^{-1} E^f x_\lambda T_{1,2j}^{-1} T_{1,j} = (-1)^f q^{-1} E^f x_\lambda T_{1,2j}^{-1} T_{1,j} = (-1)^f q^{-1} E^f x_\lambda T_{1,2j}^{-1} (2f - 1, j) \pmod{N_a},$$

where the seventh and twelfth equalities follow from Lemma 8.5(5). This completes the proof of (4).

\[ \square \]

**Lemma 7.7.** If $\lambda \in \Sigma^+(l)$ and $\mu^{(j)} = \lambda \cup p_j$, $a < j \leq m$, then

$$y^{(}\mu^{(j)}))_{x, l} \equiv \frac{q^{2c(p_j)} - 1}{q - q^{-1}} y^{(}\lambda))_{x, l} \pmod{N_{j-1}}$$

where $N_{j-1}$ is given in (6.10).

**Proof.** Write $p_j = (k, \lambda_k + 1)$ and $h = T_{b_k - 2f, l}^{-1} \sum_{i=b_k-1}^{b_k} q^{b_k - i} T_{b_k, l} \in B_{q, l}$ where $b_k$ is defined in Definition 7.1(2). We have $y^{(}\mu^{(j))}_{x, l} \equiv E^f x_\lambda T_{1,2j}^{-1} x_l h$. Let

$$X' = \sum_{2f+1 \leq i < j \leq l} (i, j) \text{ and } X = \sum_{2f-1 \leq i < j \leq l-1} (i, j)$$

It is well-known that $X$ (resp., $X'$) is a central element in $\tilde{H}_{l-2f+1}$ (resp., $\tilde{H}_{l-2f}$) where $\tilde{H}_{l-2f+1}$ is the Hecke algebra associated to the symmetric group on $\{2f - 1, 2f, \ldots, l - 1\}$. Moreover,

$$x_\lambda X' \equiv \sum_{p \in [A]} \frac{q^{2c(p)} - 1}{q - q^{-1}} x_\lambda \pmod{\tilde{H}_{l-2f}} \quad \text{and} \quad x_\mu^{(j)} X \equiv \sum_{b \in [\mu^{(j)}]} \frac{q^{2c(b)} - 1}{q - q^{-1}} x_\mu^{(j)} \pmod{\tilde{H}_{l-2f+1}}$$

Consequently, by straightforward computation (see arguments on 6.10 for the second equivalence), we have

$$E^f x_\lambda X' \equiv \sum_{p \in [A]} \frac{q^{2c(p)} - 1}{q - q^{-1}} E^f x_\lambda \pmod{\tilde{H}_{l-2f}} \quad \text{and} \quad y^{(}\mu^{(j))}_{x, l} X \equiv \sum_{b \in [\mu^{(j)}]} \frac{q^{2c(b)} - 1}{q - q^{-1}} y^{(}\mu^{(j))}_{x, l} \pmod{N_{j-1}}$$

By (6.8), Lemmas 7.3, 7.4(2) and 7.6 we have

$$E^f x_\lambda T_{1,2f}^{-1} x_l = E^f x_\lambda T_{1,2f}^{-1} \sum_{2f-1 \leq i < j \leq l-1} (i, l) + q^{-1} (i, l) \equiv q^{-2} E^f x_\lambda T_{1,2f}^{-1} \sum_{2f-1 \leq i < j \leq l-1} (2f - 1, j) - q \pmod{N_a}$$

Now, the required formula follows immediately from (7.5)-(7.6).

\[ \square \]

Suppose $t \in \mathcal{F}_{l-2f}^{1}(\lambda)$ and $\lambda \in \Lambda^+(l - 2f)$. For any $1 \leq k \leq l$, define

$$c_t(k) = \begin{cases} 
\frac{q^{2c(p)}}{q - q^{-1}} & \text{if } t_k = t_{k-1} \cup p, \\
\frac{q^{2c(p)-2}}{q - q^{-1}} & \text{if } t_k = t_{k-1} \setminus p.
\end{cases}$$

**Theorem 7.8.** Suppose $\lambda \in \Sigma^+(l)$ and $a_{ij} \cdot k \in k$ for some scalars $a_{ij} \in k$. For any $t \in \mathcal{F}_{l-2f}^{1}(\lambda)$ and $1 \leq i < l$, $m_t x_i = c_t(i) m_t + \sum_{a_{ij} < t} a_{ij} m_s$.

**Proof.** We prove the result by induction on $l$. When $l = 1$, the result is trivial since $x_1 = 0$. In general, by Theorem 7.6(3) and induction assumption for $l - 1$, we have the required formulae on $m_t x_i$, $1 \leq i \leq l - 1$. Finally, the required formula on $m_t x_l$ follows from Lemmas 7.7, 7.8 and Theorem 7.2(4).

\[ \square \]
By Theorem 7.8, \{x_1, x_2, \ldots, x_t\} are Jucys-Murphy elements with respect to the Jucys-Murphy basis \{t \in B_{q,t}^{\lambda} \mid t \in \mathcal{P}_{H}(\lambda)\} of any standard module \(C(\lambda)\) in the sense of [21, Definition 2.4]. Unlike those for Hecke algebras, Brauer algebras and cyclotomic Nazarov-Wenzl algebras etc., \(\sum_{i=1}^{t} x_i\) may not be a central element in \(B_{q,t}\) in general.

**Corollary 7.9.** Suppose \(\lambda \in \Sigma^+(l)\), \(\mu \in \Lambda^+(l)\) and \([C(\lambda) : L(\mu)1_l] \neq 0\).

1. There exist \(t \in \mathcal{P}_{H}(\lambda)\) and \(s \in \mathcal{P}_{H}(\mu)\) with \(c_t(k) = c_s(k)\), \(1 \leq k \leq t\).

2. If \(e > l\), then \(\mu\) is obtained from \(\lambda\) by adding \(2f\) boxes, say \(p_i, q_i\), \(1 \leq i \leq f\) such that \(c(p_i) - c(q_i) = 1\), \(1 \leq i \leq f\).

**Proof.** We have explained that the Jucys-Murphy elements \(x_1, x_2, \ldots, x_t\) generates a subalgebra \(L_1\) of \(B_{q,t}\). Since \([C(\lambda) : L(\mu)1_l] \neq 0\), restricting both \(C(\lambda)\) and \(L(\mu)1_l\) to \(L_1\) yields that a simple \(L_1\)-module of \(L(\mu)1_l\) (and hence of \(C(\mu)\)) has to be a composition factor of right \(L_1\)-module \(C(\lambda)\). Now (1) follows from Theorem 7.8. If \(e > l\), then \(H_1\) is semisimple and the right \(H_2\)-module \(\text{Hom}_{H_1}(2f, 0)\) is a direct sum of certain dual Specht modules. Using Lemma 6.6 and Littlewood-Richardson rule yields that \(\lambda \subset \mu\). By (1) and (7.7), \(\mu\) is obtained from \(\lambda\) by adding \(2f\) boxes, say \(p_i, q_i\), \(1 \leq i \leq f\) such that

\[
\frac{q^{2c(q_i)} - 1}{q - q^{-1}} = \frac{q^{2c(p_i)} - 1}{q - q^{-1}}.
\]

Since \(e > l\), we have \(c(p_i) - c(q_i) = 1\). \(\square\)

**8. Blocks of periplectic \(q\)-Brauer category and its endomorphism algebras**

In this section, \(k\) is always an algebraicly closed field with characteristic not \(2\). We classify blocks of the locally unital algebra \(A\) associated to the periplectic Brauer category over \(k\) when \(q^2\) is not a root of unity. We also classify blocks of \(B_{q,t}\). In this case, \(q^2\) can be a root of unity and the quantum characteristic \(e > l\).

**Lemma 8.1.** Suppose \(e > l\) and \(\lambda \in \Lambda^+(l - 2f)\). Then \([\Delta(\lambda) : L(\mu)] \neq 0\) only if \(\mu \in \Lambda^+(l - 2h)\) with \(h < f\). In this case, \(\lambda \subset \mu\) and \(2(f - h)\) boxes, say \(p_i, q_i\), \(1 \leq i \leq f - h\) in \(\mu \setminus \lambda\) can be paired such that \(c(p_i) - c(q_i) = 1\).

**Proof.** Standard results for upper finite fully stratified category (e.g., [6, Lemma 3.13]) implies \(l - 2f < l - 2h\), proving the first statement of the result. Using the idempotent functor \(1_l - 2h\), Lemma 6.6 and Corollary 7.9 (2) yields the last statement of the result. \(\square\)

**Lemma 8.2.** Suppose \(e > l\) and \((\lambda, \mu) \in \Lambda^+(l - 2) \times \Lambda^+(l)\). If \(\lambda \subset \mu\) and \(\mu\) is obtained from \(\lambda\) by adding two boxes at the same column, then \([\Delta(\lambda) : L(\mu)] = [C(\lambda) : L(\mu)1_l] = 1\).

**Proof.** Since we are assuming \(e > l\), \(L(\mu)1_l \cong S^\mu\). By Corollary 6.7 there is a right \(H_l\)-isomorphism

\[
\text{Res}_{H_l} C(\lambda) \cong S^\mu \oplus \bigoplus_{\nu \neq \mu} S^\nu
\]

where \(\nu \neq \mu\) is obtained from \(\lambda\) by adding two boxes not at the same row. In particular, \([\text{Res}_{H_l} C(\lambda) : S^\mu] = 1\) and \([\Delta(\lambda) : L(\mu)] \leq 1\). We claim that the component \(S^\mu\) in (8.1) is annihilated by \(E_1\). If so, it is a right \(B_{q,t}\)-module and hence \([\Delta(\lambda) : L(\mu)] = 1\).

Thanks to Corollary 4.8, we have \(E_1 B_{q,t} E_1 = E_1 B_{q,t-2}\). Therefore, \(C(\lambda) E_1 \cong S^\lambda\) as right \(H_{l-2}\)-modules and hence \(S^\nu E_1\) can be considered as a right \(H_{l-2}\)-submodule of \(S^\lambda\). Using Littlewood-Richardson rule and Frobenius reciprocity, we have

\[
\text{Res}_{H_{l-2}} \otimes_{H_{l-1}} \text{Res}_{H_{l-1}} S^\mu = S^{(1,1)} \boxtimes S^\lambda \oplus \bigoplus_{\nu \neq \lambda} S^\gamma \boxtimes S^\nu,
\]

where \(\gamma \in \{(1,1), (2)\}\). In particular, \(8.2\) gives a decomposition of right \(H_{l-2}\)-modules. Since \(S^\nu E_1 \subset S^\lambda\), we have \(S^\gamma \boxtimes S^\nu E_1 = 0\) for any \(\nu \neq \lambda\). This proves that \(S^\nu E_1 = S^{(1,1)} \boxtimes S^\lambda E_1\). However, by Definition 3.4, \(T_1 E_1 = qE_1\) and \(T_1\) acts on \(S^{(1,1)}\) as \(-q^{-1}\). Since we are assuming \(e > l\), we have \(q \neq -q^{-1}\), forcing \(S^{(1,1)} \boxtimes S^\lambda E_1 = 0\). This completes the proof of our claim. \(\square\)

Two simple \(A\)-modules \(L(\lambda)\) and \(L(\mu)\) are said to be in the same block if there is a sequence \(\lambda^{(1)} = \lambda, \lambda^{(2)}, \ldots, \lambda^{(k)} = \mu\) in \(\Lambda_e\) such that there is a nontrivial extension between \(L(\lambda^{(i)})\) and \(L(\lambda^{(i+1)})\) for \(i = 1, 2, \ldots, k - 1\). In the current case, it is equivalent to saying that there is a sequence
Lemma 8.3. Suppose that $q^2$ is not a root of unity. If $\lambda$ and $\mu$ have the same 2-core, then right $A$-modules $L(\lambda)$ and $L(\mu)$ are in the same block.

Proof. Suppose $\lambda$ is obtained from $\mu$ by removing a rim 2-hook. If boxes in this 2-hook are at the same column, by Lemma 8.2, $[\Delta(\lambda) : L(\mu)] \neq 0$, forcing $L(\lambda)$ and $L(\mu)$ to be in the same block. If boxes in this 2-hook are at the same row, then $\lambda'$ is obtained from $\mu'$ by two boxes in the same column. Thanks to Lemma 8.2, $[\Delta(\lambda') : L(\mu')] \neq 0$. Using Corollary 8.10(3), we have $[P(\lambda) : L(\mu)] = [\Delta(\lambda') : L(\mu')] \neq 0$. Again, $L(\lambda)$ and $L(\mu)$ are in the same block. The general case follows from this observation and the definition on 2-core of a partition.

Lemma 8.4. Suppose $e > l$. If $\lambda, \mu \in \Sigma^+(l)$ and $\lambda, \mu$ have the same 2-core, then $L(\lambda)1_l$ and $L(\mu)1_l$ are in the same block of $B_{q,1}$.

Proof. We use Lemma 8.2 when we prove this result. It is enough to assume $e > l$. By Theorem 8.9, $L(\lambda)1_l \neq 0$ if $\lambda \neq \emptyset$. Therefore, if the 2-core of $\lambda$ is not $\emptyset$, the result follows immediately from Lemmas 8.2, 8.3 by applying the idempotent functor $1_l$.

Finally, we assume that the 2-core of $\lambda$ is $\emptyset$. In this case, by Lemmas 8.2, 8.3 again, either $L(2)_l$ or $L(1,1)_l$ is in the block containing $L(\lambda)1_l$. By Corollary 8.10(3) and Lemma 8.2, $[P(2) : \Delta(\emptyset)] = [\Delta(\emptyset) : L(1,1)] = 1$. However, by standard results on upper finite fully stratified category (e.g., [6, Lemma 3.36]), each projective cover of a simple module has a finite $\Delta$-flag. In particular,

$$[P(2) : L(1,1)] \geq [P(2) : \Delta(\emptyset)]|\Delta(\emptyset) : L(1,1)| \geq 1.$$

Consequently, $L(\lambda)1_l$, $L(2)_l$, and $L(1,1)_l$ are always in the same block. In any case, $L(\lambda)1_l$ and $L(\mu)1_l$ are in the same block of $B_{q,1}$.

Theorem 8.5. Suppose $\lambda, \mu \in \Lambda$.

1. If $e = \infty$, then $L(\lambda)$ and $L(\mu)$ are in the same block if and only if $\lambda$ and $\mu$ have the same 2-core.
2. If $e > l$ and $\lambda, \mu \in \Sigma^+(l)$, then $L(\lambda)1_l$ and $L(\mu)1_l$ are in the same block if and only if $\lambda$ and $\mu$ have the same 2-core.
3. The locally unital algebra $A$ associated to the periplectic Brauer category is always not semisimple.
4. $B_{q,1}$ is always semisimple and $B_{q, l}$ is always not semisimple if $l \geq 2$.

Proof. Thanks to Lemmas 8.3, 8.4, it remains to show the only if parts of (1) and (2). Let $z\lambda$ be the number of even contents minus the number of odd contents of $\lambda$. By Lemma 8.1, $z\lambda = z\mu$ if $[\Delta(\lambda) : L(\mu)] \neq 0$. It is proved in [8] Lemma 7.3.3 that $z\lambda = z\mu$ if and only if $\lambda$ and $\mu$ have the same 2-core. This proves the only if part of (1). Similarly, we can check the only if part of (2). In this case, we have to use Corollary 7.9(2) instead of Proposition 8.1. Since there is an algebra epimorphism from $B_{q,1}$ to $H_1$, $B_{q,1}$ is not semisimple if $H_1$ is not semisimple. Therefore, we can assume $e > l$ when we prove (4). In this case, (4) follows from (2). Similarly, if $q^2$ is a root of unity, then $A$ is not semisimple. Otherwise, $1_l A_1 \cong B_{q,1}$ is semisimple for any $l \in \mathbb{N}$. In particular, $H_1$ is semisimple when $l > e$, a contradiction. When $q^2$ is not a root of unity, (3) follows from (1). In fact, one can easily find a block which contains at least two simple modules in generic case.

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