MATRIX COEFFICIENTS OF INTERTWINING OPERATORS AND THE BRUHAT ORDER

DANIEL BUMP AND BÉATRICE CHETARD

Abstract. Let \((\pi_z, V_z)\) be an unramified principal series representation of a reductive group over a nonarchimedean local field, parametrized by an element \(z\) of the maximal torus in the Langlands dual group. If \(v\) is an element of the Weyl group \(W\), then the standard intertwining integral \(A_v\) maps \(V_z\) to \(V_{vz}\). Letting \(\psi^z_w\) with \(w \in W\) be a suitable basis of the Iwahori fixed vectors in \(V_z\), and \(\hat{\psi}^z_v\) a basis of the contragredient representation, we define \(\sigma(u, v, w)\) (for \(u, v, w \in W\)) to be \(\langle A_v \psi^z_u, \hat{\psi}^z_v \rangle\). This is an interesting function and we initiate its study. We show that given \(u\) and \(w\), there is a minimal \(v\) such that \(\sigma(u, v, w) \neq 0\). Denoting this \(v\) as \(v_{\text{min}} = v_{\text{min}}(u, w)\), we will prove that \(\sigma(u, v_{\text{min}}, w)\) is a polynomial of the cardinality \(q\) of the residue field. Indeed if \(v > v_{\text{min}}\), then \(\sigma(u, v, w)\) is a rational function of \(z\) and \(q\), whose denominator we describe. But if \(v = v_{\text{min}}\), the dependence on \(z\) disappears. We will express \(\sigma(u, v_{\text{min}}, w)\) as the Poincaré polynomial of a Bruhat interval. The proof leads to fairly intricate considerations of the Bruhat order.

Thus our results require us to prove some facts that may be of independent interest, relating the Bruhat order \(\leq\) and the weak Bruhat order \(\leq_R\). For example we will prove (for finite Coxeter groups) the following “mixed meet” property. If \(u, w\) are elements of \(W\), then there exists a unique element \(m \in W\) that is maximal with respect to the condition that \(m \leq_R u\) and \(m \leq w\). Thus if \(z \leq_R u\) and \(z \leq w\), then \(x \leq m\). The value \(v_{\text{min}}\) is \(m^{-1}u\).

1. Introduction

The formula of Macdonald [17] for the spherical functions on a reductive \(p\)-adic group \(G(F)\) over a nonarchimedean local field \(F\) plays an important role in the harmonic analysis on \(G(F)\). Let \((\pi, V)\) be a representation of the unramified principal series, and let \((\hat{\pi}, \hat{V})\) be its contragredient. The representation \(\pi\) may be parametrized by Langlands-Satake parameters \(z\) residing in a complex torus \(\hat{T}(\mathbb{C})\), and we will thus write \((\pi, V) = (\pi_z, V_z)\).

The space \(V\) contains a spherical vector \(\phi^o\), unique up to scalar multiple, that is invariant under the standard maximal compact subgroup \(K\); similarly let \(\phi^o\) be the spherical vector in \(\hat{V}\). The Macdonald spherical function is

\[ \varsigma(g) = \langle \pi_z(g) \phi^o, \hat{\phi}^o \rangle. \]

Macdonald’s formula identifies the value \(\varsigma(g)\) with a symmetric function of \(z\) that for \(G = \text{GL}_r\) is a Hall-Littlewood polynomial.

In 1980 Casselman [6] gave a new proof of the Macdonald formula. This proof involved two ingredients:

- Even though the Macdonald formula concerns just \(\phi^o\) in the one-dimensional space \(V^K\) of \(K\)-invariants, it is useful to work with the larger space \(V^J\) of vectors invariant under the Iwahori subgroup \(J\) of \(K\);

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• If \( w \) is an element of the Weyl group \( W \) there is an intertwining integral operator \( \mathcal{A}_w: V_z \to V_z\); systematic use of these and related linear functionals was another key ingredient of the proof.

These two ingredients have played an important role in subsequent theory. Thus it is very natural to investigate

\[
\sigma(u, v, w) = \langle \mathcal{A}_u \psi^u_z, \hat{\psi}^v_z \rangle,
\]

where \( u, v, w \) are elements of \( W \). Here \( \psi^u_z \) runs through a basis of Iwahori fixed vectors \( V^J_z \) and \( \hat{\psi}^v_z \) runs through a basis of the Iwahori fixed vectors \( \hat{V}^J_z \) in the contragredient representation \((\hat{\pi}, \hat{V}^J_z)\).

To describe these bases, let \( \phi^u_z \) be the element of \( V^J_z \) whose restriction to \( K \) is the characteristic function of the double coset \( JwJ \). We may identify \( \hat{V}^J_z \) with \( V_z^{-1} \). (See Section 3 for further information.) Then we have a corresponding basis \( \phi^{-1}_z w \) of \( \hat{V}^J_z \). Then we define

\[
\psi^x_z = \bigcup_{u \geq w} \phi^x_u, \quad \hat{\psi}^y_z = \bigcup_{y \leq w} \phi^{-1}_y.
\]

It may seem inconsistent that in \( \psi^x_z \) we sum over \( u \geq w \) in the Bruhat order while in \( \hat{\psi}^y_z \) we sum over \( y \leq w \). The explanation for this is that it leads to \( \sigma(u, v, w) \) with the nicest combinatorial properties. For example let us consider the simplest case, where \( v = 1 \). Then

\[
\langle \phi^x_z, \phi^y_z \rangle = \delta_{xy} q^{\ell(x)}
\]

where \( q \) is the cardinality of the residue field and \( \ell \) is the length function on the Weyl group. Therefore

\[
\sigma(u, 1, w) = \sum_{x \geq u \leq w} \langle \phi^x_z, \phi^y_z \rangle = \sum_{u \leq x \leq w} q^{\ell(x)}.
\]

This vanishes unless \( u \leq w \), so assume this. We recognize \( \sigma(u, 1, w) \) as the Poincaré polynomial of the Bruhat interval \([u, w] = \{x \in W | u \leq x \leq w\}\).

A second case where \( \sigma \) has already appeared in practice is if \( w = 1 \). We will discuss this next.

Casselman made use of the linear functionals \( \phi \mapsto \mathcal{A}_w \phi(1) \) on \( V^J_z \), and the basis \( f_w \) of \( V^J_z \) is called the Casselman basis. These were used in Casselman [6] and Casselman and Shalika [7] in order to prove the Macdonald formula for the spherical function \( \varsigma \) already mentioned, and the Casselman-Shalika formula; this technique has been applied in many subsequent papers.

As Casselman pointed out, computing the Casselman basis explicitly is a difficult problem; fortunately to prove the Macdonald and Casselman-Shalika formulas, only \( f_{w_0} \) needs to be known explicitly, since the \( f_w \) appear in an integrated form, and once one integral is known, the others may be inferred through functional equations.

Still, the \( f_w \) are interesting objects. To study them, Bump and Nakasuji [3, 4] examined the values of the functionals \( \phi \mapsto \mathcal{A}_w \phi(1) \) introduced by Casselman on the basis \( \psi^u_z \) and defined

\[
(1) \quad m_{u,v} = \mathcal{A}_u \psi_u(1).
\]

About these they found interesting results and conjectures, which were refined by Nakasuji and Naruse [18, 19]; the conjectures were eventually proved using deep methods of algebraic geometry by Aluffi, Mihalcea, Schürmann and Su [1]. See also [16].
To connect this with the present study, note that the functional \( A_w \phi(1) \) in (1) is the vector \( \hat{\psi}_w = \phi_w^{-1} \) in the contragredient representation. Therefore in the notation of the present paper

\[
m_{u,v} = \sigma(u, v, 1).
\]

Now \( m_{u,v} \) vanishes unless \( u \leq v \) in the Bruhat order, and assuming this, Bump and Nakasuji [4] proved that

\[
m_{u,v} = \sum_{u \leq x \leq v} r_{x,v}(z)
\]

where \( r_{x,v}(z) \) are certain deformations of the Kazhdan-Lusztig R-polynomials, and the “bar” means that \( q \) is replaced by \( q^{-1} \) in this formula. A similar formula is also stated in Nakasuji and Naruse [18, 19].

It was conjectured in [3, 4] and proved in [3] that the denominator of \( m_{u,v} \) has the form

\[
\prod_{\alpha \in S(u,v)} (1 - z^{\alpha})
\]

where \( S(u, v) \) is the set of positive roots \( \alpha \) of \( G \) such that \( u \leq v \cdot r_\alpha < v \), where \( r_\alpha \in W \) is the reflection associated to \( \alpha \). It follows from the Deodhar inequality ([8, 5, 20, 9]) that

\[
|S(u, v)| \geq \ell(v) - \ell(u).
\]

Moreover \( |S(u, v)| = \ell(v) - \ell(u) \) provided the inverse Kazhdan Lusztig polynomial \( Q_{u,v} = 1 \); this is the polynomial \( P_{w_0, w_0} \) in the notation of [13].

Bump and Nakasuji conjectured that if \( G \) is simply-laced and the Kazhdan-Lusztig polynomial \( Q_{u,v} = 1 \) then

\[
(3) \quad m_{u,v} = \prod_{\alpha \in S(u,v)} \frac{1 - q^{-1}z^{\alpha}}{1 - z^{\alpha}}.
\]

This is a generalization of the Gindikin-Karpelevich formula [15], which is the case where \( u = 1 \); but it is much harder to prove than the Gindikin-Karpelevich (GK) formula. This conjecture was extended by Nakasuji and Naruse [19, 18] to the non-simply-laced case by replacing the Kazhdan-Lusztig criterion with the a nonsingularity condition for Schubert varieties. In this form, the conjecture was proved by Aluffi, Mihalcea, Schürmann and Su [1] using techniques from algebraic geometry, particularly motivic Chern classes.

We remark also that if \( w = w_0 \) is the long Weyl group element, then \( \sigma(u, v, w_0) \) is connected with the theory of matrix coefficients of the form \( \langle \pi(g) \psi_u, \hat{\psi}_{w_0} \rangle \), where here \( \hat{\psi}_{w_0} \) is the spherical vector. This connects with the theory of nonsymmetric Macdonald polynomials by work of Ion [11].

We found that investigating any question about \( \sigma(u, v, w) \) leads into subtle considerations concerning the Bruhat order \( \leq \) and the weak Bruhat order \( \leq_R \). For example consider the question of whether \( \sigma(u, v, w) \) is nonzero. We are able to give a satisfactory answer to this (Theorem [1] below), but to explain it we must explain a relationship between the two orders.

Recall ([2, Chapter 3]) that \( u \leq_R w \) if \( \ell(w) = \ell(u) + \ell(u^{-1}w) \), which is the same as saying that \( w \) has a reduced expression \( w = s_{i_1} \cdots s_{i_k} \) such that an initial segment \( s_{i_1} \cdots s_{i_k} \) represents \( u \). The left order \( \leq_L \) is defined similarly.
If $u, w \in W$ then there exists a “meet” $x \in W$ for the weak order, making the Weyl group into a meet semilattice. This means that $x \leq_R u$ and $x \leq_R w$, and $x$ is the unique maximal element with this property, so if $z \leq_R u$ and $z \leq_R w$ then $z \leq_R x$.

The strong Bruhat order $\leq$ does not have this property. For example in Cartan type $A_2$ if $u = s_1s_2$ and $w = s_2s_1$ then

$$\{z \in W|z \leq u, w\} = \{1, s_1, s_2\}$$

and this set does not have a unique maximal element.

However we will show (for finite Coxeter groups $W$) that $u, w$ have a mixed meet for the Bruhat and weak orders. We will prove below in Theorem 3 that if $u, w$ are elements of $W$, a finite Coxeter group, then there exists a unique element $m \in W$ that is maximal with respect to the condition that $m \leq_R u$ and $m \leq w$. Thus if $z \leq_R u$ and $z \leq w$, then $z \leq m$. We will call $m$ the mixed meet of $u$ and $w$ with respect to the weak and strong Bruhat orders.

Now let $v_{\min} = v_{\min}(u, w) = m^{-1}u$. We will prove:

**Theorem 1.** We have $\sigma(u, v, w) = 0$ unless $v \geq v_{\min}(u, v)$. If $v = v_{\min}(u, v)$ then $\sigma(u, v, w)$ is a polynomial in $q$, independent of $z$. More precisely,

$$\sigma(u, v_{\min}(u, v), w) = q^{-\ell(u)} \sum_{z \in [u, w]} q^{\ell(z)}.$$  

Let us denote $\sigma_0(u, w) = \sigma(u, v_{\min}(u, w), w)$.

There is one more interesting phenomenon that we only partially understand. For many $(u, v, w)$ we find that $\sigma(u, v, w)$ has a special form. In these cases

$$\sigma(u, v, w) = \sigma_0(u, w) \prod_{\alpha \in S(u, v, w)} \frac{1 - q^{-1}z^\alpha}{1 - z^\alpha}.$$  

where we recall that $\sigma_0(u, w)$ is a polynomial in $q$. Here $S(u, v, w)$ is the set of positive roots $\alpha$ defined in (20) below. This formula generalizes (3), which we have already mentioned resembles the Gindikin-Karpelevich formula which is the case $w = 1$. Therefore if (5) is valid we will say that the triple $(u, v, w)$ is of GK type. In contrast with that special case, we do not know precisely which of $(u, v, w)$ are of GK type. But if $w = 1$, so $\sigma(u, v, 1) = m_{u, v}$, we have already explained that these have been determined in [3, 4, 18, 19, 1].

If $G = GL(3)$, so $W = S_3$, there are 167 triples $(u, v, w)$ with $v \geq v_{\min}(u, w)$. (Recall that these are the cases where $\sigma(u, v, w) \neq 0$.) Of these, all but 20 are of GK type. The exceptional triples $(u, v, w)$ such that $\sigma(u, v, w)$ is not of the form (5) are:

$$(s_1s_2, s_1s_2s_1, s_1s_2) \quad (s_2s_1, s_1s_2s_1, s_2s_1) \quad (s_2s_1s_2, s_1s_2s_1, s_2s_1)$$  

$$(s_1s_2s_1, s_2s_1, s_1s_2s_1) \quad (s_2s_1, s_1s_2s_1, s_2s_1) \quad (s_2s_1s_2, s_1s_2s_1, s_2s_1)$$  

$$(s_1, s_1s_2, s_2s_1s_2, s_1s_2s_1) \quad (s_2, s_1s_2, s_1s_2s_1) \quad (s_2, s_1s_2s_1, s_1)$$  

$$(s_1, s_1s_2s_1, s_2s_1s_2, s_1s_2s_1) \quad (s_2, s_1s_2s_1, s_2s_1s_2, s_1s_2s_1) \quad (s_2, s_1s_2s_1, s_2s_1s_2, s_1s_2s_1)$$

If $G = Sp(4)$ so $W$ is dihedral of order 8, there are 401 triples $(u, v, w)$ such that $v \geq v_{\min}(u, w)$, and of these (3) is satisfied by 305. If $G = GL(4)$, so $S_4$, there are 9597 such triples, and of these, (5) is satisfied by 6281.
Problem 1. Determine for which $(u,v,w)$ equation (5) is satisfied.

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2. The Bruhat order and the weak order

Let $W$ be a Coxeter group. After a preliminary result that is valid for general Coxeter groups we will assume that $W$ is finite. Let $\{s_1, \cdots, s_r\}$ be the simple reflections in $W$. Let $w_0$ denote its longest element. It is probable that some of our results apply more generally to arbitrary Coxeter groups but we will make use of $w_0$ in the proofs.

We will denote by $\leq$ the usual (strong) Bruhat order on $W$. We will also denote by $\leq_R$ the weak Bruhat order. The weak Bruhat order $\leq_R$ can be defined in terms of the length function $\ell$ on $W$ by $u \leq_R v$ if $\ell(u) + \ell(u^{-1}v) = \ell(v)$. An equivalent definition is that $u \leq_R v$ if there exists a reduced word $s_{i_1} \cdots s_{i_n}$ for $v$ such that the initial segment $s_{i_1} \cdots s_{i_k}$ is a reduced word for $u$, for some $k \leq n$. See [2] Proposition 3.1.2 for other characterizations.

We will make use of two versions of the Hecke algebra of $W$. The first has a basis $T_w$ $(w \in W)$ such that if $s = s_i$ is a simple reflection and $sw > w$ then $T_s T_w = T_{sw}$ and $T_s T_{sw} = (q-1)T_{sw} + q T_w$.

See [10] Theorem 7.1 for the construction of $\mathcal{H}_q$. We will denote $T_{s_i} = T_i$.

We will also use a variant $\mathcal{H}_0$ which is the same as $\mathcal{H}_q$ with $q$ specialized to 0. We will use $U_w$ to represent the basis elements, so

$$U_s U_w = \begin{cases} U_{sw} & \text{if } sw > w, \\ U_w & \text{if } sw < w. \end{cases}$$

Again $U_i = U_{s_i}$.

The basis elements $U_w$ of $\mathcal{H}_0$ form a monoid $\mathcal{M}_0$, and $\mathcal{H}_0$ is the algebra of the monoid $\mathcal{M}_0$. We will call $\mathcal{M}_0$ the Demazure monoid. Since $i : W \to \mathcal{M}_0$ defined by $i(w) = U_w$ is a bijection, we may transfer the multiplication in $\mathcal{M}_0$ back to $W$. This multiplication $\circ$ on $W$ is sometimes called the Demazure multiplication.

Define maps $u_i^\vee, u_i^\wedge : W \to W$ by

$$u_i^\vee(v) = \begin{cases} s_i v & \text{if } s_i v > v, \\ v & \text{if } s_i v < v. \end{cases} \quad u_i^\wedge(v) = \begin{cases} v & \text{if } s_i v > v, \\ s_i v & \text{if } s_i v < v. \end{cases}$$

Lemma 1. Suppose that $s_is_j$ has finite order. Then $u_i^\wedge$ and $u_i^\vee$ satisfy the braid relation $u_i u_j \cdots = u_j u_i \cdots$, where the number of terms on both sides is the order of $s_is_j$.

Proof. Let $D_{ij}$ be the dihedral group generated by $s_i$ and $s_j$. Applied to any $v \in W$, it is easy to see that $u_i^\vee u_j^\vee \cdots$ and $u_j^\vee u_i^\vee \cdots$ both produce the shortest element of the coset $D_{ij} v$ in at most $n_{ij}$ steps, where $n_{ij}$ is the order of $s_i s_j$. Hence $u_i^\vee u_j^\vee \cdots (v) = u_j^\vee u_i^\vee \cdots (v)$. The braid relation for $u_i^\vee$ is proved similarly. \hfill \Box

If $u, v \in W$ define

$$U_i \uparrow v = \begin{cases} s_i v & \text{if } s_i v > v, \\ v & \text{if } s_i v < v, \end{cases} \quad U_i \downarrow v = \begin{cases} v & \text{if } s_i v > v, \\ s_i v & \text{if } s_i v < v, \end{cases}$$
these relations are satisfied by the $u, v, v'$.

Proposition 1. The operations $U_i : v \mapsto U_i \uparrow v$, $U_i : v \mapsto U_i \downarrow v$ extend to left actions of the monoid $\mathcal{M}_0$ on $W$, meaning that

$$
(6) \quad (U U') \uparrow v = U \uparrow (U' \uparrow v), \quad (U U') \downarrow v = U \downarrow (U' \downarrow v), \quad U, U' \in \mathcal{M}_0, v \in W.
$$

Similarly $U_i : v \mapsto v \uparrow U_i$, $U_i : v \mapsto v \downarrow U_i$ extend to right actions of $\mathcal{M}_0$ such that

$$
(7) \quad v \uparrow (U U') = (v \uparrow U) \uparrow U', \quad v \downarrow (U U') = (v \downarrow U) \downarrow U', \quad U, U' \in \mathcal{M}_0, v \in W.
$$

Proof. The algebra $\mathcal{H}_0$ may be characterized as the algebra generated by the $U_i$ subject to the quadratic relations $U_i^2 = U_i$ and the braid relations. Taking $u_i = u_i^\nu$ (resp. $u_i = u_i^{\nu'}$) these relations are satisfied by the $u_i$ by Lemma [1] and so we have a representation of $\mathcal{M}_0$ on $W$ satisfying (6). The proof of (7) is similar. \hfill \Box

Since $w \mapsto U_w$ is a bijection $i : W \mapsto \mathcal{M}_0$, the operations $\uparrow$ correspond to the left and right regular operations of $\mathcal{M}_0$ on itself. That is, we can define $U_u \uparrow v = i^{-1}(U_u U_v) = u \uparrow U_v$ and it is clear that this operation has the advertised properties. Moreover $i(U_u \uparrow v) = U_u U_v = i(u \uparrow U_v)$, and therefore in terms of the Demazure multiplication,

$$
(8) \quad U_u \uparrow v = u \uparrow U_v = u \circ v.
$$

For the rest of this section we specialize to the case that $W$ is a finite Coxeter group. In this case $W$ has a longest element $w_0$, then $u \mapsto u w_0$ is an order-reversing involution of $W$. From this it follows that

$$
(9) \quad w_0(u \uparrow U_v) = (w_0 u) \downarrow U_v, \quad (U_u \uparrow v) w_0 = U_u \downarrow (v w_0)
$$

Combining (8) and (9) gives us formulas for $\downarrow$ in terms of the Demazure multiplication:

$$
(10) \quad u \downarrow U_v = w_0((w_0 u) \circ v), \quad U_u \downarrow v = (u \circ (v w_0)) w_0
$$

Let $(i_1, \ldots, i_m)$ be a sequence of indices. We will say that the sequence contains $w \in W$ if $s_{i_1} \cdots s_{i_k}$ is a reduced expression for $w$ for some subsequence $(j_1, \ldots, j_k)$ of $(i_1, \ldots, i_m)$. Knutson and Miller [14] characterized this condition in terms of the Demazure product $\circ$ on $W$ as follows:

Proposition 2. The sequence $(i_1, \ldots, i_m)$ contains $w$ if and only if $w \leq s_{i_1} \cdots \circ s_{i_m}$.

Proof. See [14] Lemma 3.4. \hfill \Box

As a consequence we deduce a monotonicity property of $\circ$.

Proposition 3. If $u, u', v, v' \in W$ and $u \leq u'$, $v \leq v'$ then $u \circ v \leq u' \circ v'$.

Proof. Let $(i_1', \ldots, i_m')$ and $(j_1', \ldots, j_n')$ be reduced words for $u'$ and $v'$. Then we may find subwords $(i_1', \ldots, i_m')$ and $(j_1, \ldots, j_n)$ that are reduced words for $u$ and $v$. By Proposition 2, $u' \circ v'$ is the maximal element of $W$ that is contained in $(i_1', \ldots, i_m', j_1', \ldots, j_n')$ and $u \circ v$ is the maximal element contained in $(i_1, \ldots, i_m, j_1, \ldots, j_n)$. Since the second sequence is a subsequence of the first, we obtain the stated monotonicity property. \hfill \Box

From Proposition 3 we may infer two monotonicity properties of the “down” actions of $\mathcal{H}_0$.

Proposition 4. If $x \geq u$ then $U_w \downarrow x \geq U_w \downarrow u$ and $x \downarrow U_w \geq u \downarrow U_w$.
Proof. Using (10) the inequality that we need for $U_w \downarrow x \supseteq U_w \downarrow u$ is
\begin{equation}
(w \circ (xw_0))w_0 \geq (w \circ (uw_0))w_0.
\end{equation}
We have $xw_0 \leq uw_0$ and so by Proposition 3 we have $w \circ (xw_0) \leq w \circ (uw_0)$. The inequality (11) follows. The second inequality is proved the same way. \hfill \square

**Proposition 5.** If $w \geq v$ then $U_w \downarrow x \subseteq U_v \downarrow x$ and $x \downarrow U_w \leq x \downarrow U_v$ for any $w \in W$.

Proof. By Proposition 3 we have $w \circ xw_0 \geq v \circ xw_0$ and so $(w \circ xw_0)w_0 \leq (v \circ xw_0)w_0$. The first inequality now follows from (10), and the other is proved the same way. \hfill \square

If $u \leq v$ in the Bruhat order let $[u, v]$ denote the Bruhat interval $\{x \in W | u \leq x \leq v\}$.

**Theorem 2.** The translated Bruhat interval $u[1, v]$ has maximal and minimal elements $u \circ v$ and $u \downarrow U_v$. Similarly, $[1, u]v$ has maximal and minimal elements $u \circ v$ and $U_u \downarrow v$. Moreover $[u, w_0]v$ has minimal element $u \downarrow U_v$ and $u[v, w_0]$ has minimal element $U_u \downarrow v$.

Proof. Let us start by showing that $u \circ v \in u[1, v]$. This may be proved by induction on $\ell(v)$: we write $v = v's_i$ where $\ell(v') = \ell(v) - 1$. By induction $u \circ v' \in u[1, v']$. Now $u \circ v$ is either $u \circ v'$ or $(u \circ v')s_i$ and in either case $u \circ v \in u[1, v]$.

We have shown that $u \circ v \in u[1, v]$, but we need to show that it is the maximal element. Let us pick reduced words $(i_1, \ldots, i_k)$ and $(j_1, \ldots, j_l)$ for $u$ and $v$, respectively. If $w \in u[1, v]$ then we may write $w = s_{i_1} \cdots s_{i_k}s_{j_1} \cdots s_{j_l}$ where $(j_1', \ldots, j_l')$ is a subsequence of $(j_1, \ldots, j_l)$ and so the sequence $(i_1, \ldots, i_k, j_1', \ldots, j_l')$ contains $w$. By Proposition 2 we have
\begin{equation}
w \leq s_{i_1} \cdots s_{i_k} \circ s_{j_1} \circ \cdots \circ s_{j_l} = u \circ v,
\end{equation}
where we have used the fact that $(i_1, \ldots, i_k)$ and $(j_1, \ldots, j_l)$ are reduced words, so $u = s_{i_1} \cdots s_{i_k}$ and $v = s_{j_1} \cdots s_{j_l}$. This proves that $u[1, v]$ has the unique maximal element $u \circ v$.

Now we make use of the fact that $u \mapsto w_0u$ is an order reversing bijection of $W$ to deduce that the Bruhat interval $w_0u[1, v]$ has a minimal element $w_0(u \circ v)$. Replacing $u$ by $w_0u$ and using (10) we see that $u[1, v]$ has a minimal element $w_0((w_0u) \circ v) = u \downarrow U_v$.

Now let us look at $u[v, w_0]$. Since right multiplication by $w_0$ is order reversing and maps $[v, w_0]$ to $[1, wv_0]$ we have
\begin{equation}
\min(u[v, w_0]) = (\max(u[v, wv_0]))w_0 = (u \circ (wv_0))w_0 = U_u \downarrow v.
\end{equation}

The remaining statements about $[1, u]v$ and $[u, w_0]v$ are the mirror images of cases already proved and may be established the same way. \hfill \square

**Lemma 2.** Suppose that $u' \leq_R u$ and $w \in W$. Then $u' \downarrow U_w \leq_R U_w$.

Proof. Suppose we know this in the case where $w = s$ is a simple reflection. Then writing $w = s_{i_1} \cdots s_{i_k}$ we have $u' \downarrow U_w = u' \downarrow U_{s_{i_1} \cdots s_{i_k}}$ and so applying the case of a simple reflection repeatedly we get $u' \downarrow U_w \leq u$.

Thus we are reduced to the case where $w = s$ is a simple reflection. If $u's > u'$ then $u' \downarrow U_s = u'$ in which case we are done. Therefore we may assume that $u's < s$. Then we may choose a reduced word for $u'$ of the form $u' = s_{i_1} \cdots s_{i_r}$ where $s_{i_r} = s$. Let $s_{j_1} \cdots s_{j_k}$ be a reduced word for $(u')^{-1}u$. Then since $u' \leq_R u$ the expression $s_{i_1} \cdots s_{i_r}s_{j_1} \cdots s_{j_k}$ is a reduced expression for $u$. So $u' \downarrow U_s = u's = s_{i_1} \cdots s_{i_{r-1}} \leq_R u$. \hfill \square

**Lemma 3.** Let $u, v \in W$. Then $u \downarrow U_v \leq_R u$.
Proof. This is the special case \( u' = u \) of Lemma 2.

Lemma 4. If \( x \leq_R u^{-1} \) then \( ux \leq_R u \).

Proof. We have \( \ell(u^{-1}) = \ell(x) + \ell(x^{-1}u^{-1}) \). We want to deduce \( \ell(u) = \ell(ux) + \ell((ux)^{-1}u) \). But \( \ell(u) = \ell(u^{-1}), \ell((ux)^{-1}u) = \ell(x) \) and \( \ell(x^{-1}u^{-1}) = \ell(x) \), so this is clear.

Lemma 5. If \( u, w \in W \) then \( u(u^{-1} \downarrow U_w) \leq w \).

Proof. Let \( w = s_{i_1} \cdots s_{i_k} \) be a reduced expression. It is clear from the definition of \( u^{-1} \downarrow U_w = u^{-1} \downarrow U_{s_{i_1}} \downarrow \cdots \downarrow U_{s_{i_k}} \) that \( u^{-1} \downarrow U_w = u^{-1}w' \) where \( w' = s_{j_1} \cdots s_{j_k} \) and \( (j_1, \cdots, j_k) \) is a subsequence of \( (i_1, \cdots, i_k) \). Then \( u(u^{-1} \downarrow U_w) = w' \leq w \).

Proposition 6. Suppose that \( w \leq_R wz \) and \( w \leq_R wv \). Moreover assume that \( wz \leq_R wv \). Then \( z \leq v \).

Proof. We argue by induction on \( \ell(w) \). If \( w = 1 \) this is certainly true. Otherwise, let \( s \) be a left descent of \( w \). Since \( w \leq wz \) it is also a left descent of \( wz \) and similarly of \( wv \). Thus we may write \( w = sw' \) and we have \( w'z < sw'z \) and similarly \( w'v < sw'v \). Moreover we are assuming that \( sw'z \leq sw'v \). Using the lifting property of the Bruhat order (Proposition 2.2.7 in [2]), it follows that \( w'z \leq w'v \). By induction we have \( z \leq w \).

It is well-known that left multiplication by \( w_0 \) is order reversing for the Bruhat order. The following result is a generalization, showing that under certain circumstances the Bruhat order is reversed by an arbitrary element \( u \).

Proposition 7. Suppose that \( x \leq_R u^{-1} \) and \( y \leq_R u^{-1} \). Moreover assume that \( x \leq y \). Then \( ux \geq uy \).

Proof. We recall that the map \( a \rightarrow aw_0 \) is order reversing for both the Bruhat order and for the weak order ([2] Propositions 2.3.4 and 3.1.5). Thus if we make the variable changes \( x \rightarrow xw_0, y \rightarrow yw_0 \) and \( u \rightarrow w_0u \) the statement is seen to be equivalent to:

\[
\text{If } u^{-1} \leq_R x \text{ and } u^{-1} \leq_R y \text{ and } y \leq x \text{ then } ux \geq uy.
\]

This follows from Proposition 6 with \( u = w^{-1}, x = wv, y = wz \).

The weak order \( \leq_R \) has the following “meet semilattice” property, which is Theorem 3.2.1 of [2]: If \( u, w \in W \) then the set of \( v \) such that \( v \leq_R u \) and \( v \leq_R w \) has a maximal element. This maximal element is called the meet of \( u \) and \( v \) for the partial order \( \leq_R \).

This property is not shared by the usual (strong) Bruhat order. For example in the \( A_2 \) Weyl group \( s_1s_2 \) and \( s_2s_1 \) do not have a meet for the Bruhat order. However we will prove a “mixed” meet property that combines the weak and strong Bruhat orders.

Theorem 3 (Mixed meet property). Let \( u, w \) be elements of \( W \). Then the set

\[
\{ x \in W | x \leq_R u, x \leq w \}
\]

has a unique maximal element \( m \) for the Bruhat order. This means that \( m \leq_R u, m \leq w \) and if \( v \leq_R u, v \leq w \) then \( v \leq m \). In terms of the \( \downarrow \) operation we have:

\[
m = u(u^{-1} \downarrow U_w).
\]
Proof. It follows from Lemmas 3 and 4 that \( u(u^{-1} \downarrow U_w) \leq_R u \). Moreover by Lemma 5 we have \( u(u^{-1} \downarrow U_w) \leq u \).

Now suppose that \( v \leq_R u, v \leq w \). We must prove that \( v \leq u \). By Proposition 5 we have \( u^{-1} \downarrow U_w \leq u^{-1} \downarrow U_v. \)

Now we note that \( u^{-1} \downarrow U_w, u^{-1} \downarrow U_v \) are both \( \leq_R u^{-1} \) and so we may apply Proposition 7 to see that

\[
m = u(u^{-1} \downarrow U_w) \geq u(u^{-1} \downarrow U_v).
\]

However since \( v \leq_R u \) we have \( u^{-1} \downarrow U_v = u^{-1}v \) proving that \( m \geq v \), as required.

3. MATRIX COEFFICIENTS OF INTERTWINING INTEGRALS

Let \( G \) be a split reductive group over a nonarchimedean local field \( F \). Let \( \mathfrak{o} \) and \( \mathfrak{p} \) be the ring of integers of \( F \), and its maximal ideal, and let \( q = |\mathfrak{o}/\mathfrak{p}| \) be the residue cardinality. Let \( T \) be a split maximal torus and \( B = TN \) a Borel subgroup of \( G \) containing \( T \), with \( N/U \) its unipotent radical. Let \( \Phi \) be the root system of \( G \) with respect to \( T \), with \( \Phi^+ \) the positive roots. Let \( \hat{T} \) be the dual torus in the connected Langlands dual group \( \hat{G} \). Let \( \Lambda = X^*(\hat{T}) \) be the weight lattice of \( \hat{G} \). If \( z \in \hat{T}(\mathbb{C}) \) and \( \lambda \in \Lambda \) we will denote by \( z^\lambda \) the application of \( \lambda \) to \( z \).

The tori \( T \) and \( \hat{T} \) are in duality, so \( \Lambda \) may be identified with the cocharacter group \( X_*(T) \cong T(F)/T(\mathfrak{o}) \). Let \( \varpi \) be a generator of \( \mathfrak{p} \). With \( \lambda \in \Lambda \), the image of \( \varpi \) under the cocharacter \( F^\times \rightarrow T(F) \) corresponding to \( \lambda \) will be denoted \( \varpi^\lambda \). Then the coset \( \varpi^\lambda T(\mathfrak{o}) \) is the image of \( \lambda \) under the isomorphism \( X_*(T) \cong T(F)/T(\mathfrak{o}) \).

Also \( z \) determines an unramified quasicharacter \( \chi_z \) of \( T(F) \) such that \( \chi_z(\varpi^\lambda) = z^\lambda \). The principal series representation \( (\pi_z, V_z) \) is the representation of \( G(F) \) obtained from \( \chi_z \) by parabolic induction. Thus the space \( V_z \) consists of locally constant functions that satisfy

\[
\phi(bg) = (\delta^{1/2} \chi_z)(b) \phi(g), \quad b \in B(F).
\]

Here the function \( \chi_z \) is extended to \( B(F) \) by means of the homomorphism \( B(F) \rightarrow T(F) \) with kernel \( U(F) \), and \( \delta \) is the modular quasicharacter on \( B(F) \).

Let \( K = G(\mathfrak{o}) \) be the standard maximal compact subgroup. Let \( J \) be the Iwahori subgroup that is the preimage of \( B(F_q) \) under the reduction mod \( \mathfrak{p} \) homomorphism \( K \rightarrow G(F_q) \). We will normalize the Haar measures on \( G \) and \( K \) so that \( J \) has volume 1.

The contragredient \( \hat{\pi}_z \) of \( \pi_z \) is \( \pi_{z^{-1}} \). Indeed an invariant nondegenerate bilinear pairing \( V_z \times V_{z^{-1}} \rightarrow \mathbb{C} \) is given by

\[
\langle f_1, f_2 \rangle = \int_K f_1(k) f_2(k) \, dk.
\]

Thus we will denote \( \hat{V}_z = V_{z^{-1}} \).

A basis of \( J \)-fixed vectors in \( V_z \) consists of the functions \( \{ \phi_w^z \} \) with support in a single double coset \( BwJ \) with \( w \) in the Weyl group \( W = N(T)/T \). We choose the representative of a Weyl group element \( w \in W \) to be in \( N(T) \cap K \) and by abuse of notation we denote the representative by the same letter \( w \); this abuse of notation is justified by the fact that in dealing with the unramified principal series, nothing depends on the choice of representative beyond the fact that it is in \( K \). In particular we define

\[
\phi_w^z(bw'k) = \begin{cases} 
(\delta^{1/2} \chi_z)(b) & \text{if } w = w' \text{ in } W, \\
0 & \text{otherwise},
\end{cases}
\]
and this definition does not depend on the choice of representative since $\chi_z$ is unramified.

In view of our identification $\hat{V}_z = V_{z^{-1}}$ we will denote $\hat{\phi}^z_w = \phi^{z^{-1}}_w$. Since $\text{vol}(JwJ) = q^{\ell(w)}$ it follows from the definition (3) that

$$\langle \phi^z_w, \hat{\phi}^z_{w'} \rangle = \text{vol}(JwJ) \delta_{w,w'} = q^{\ell(w)} \delta_{w,w'}.$$  

Following [3, 4, 18, 1] we will not work directly with $\phi^z_w$ but with the basis $\{ \hat{\psi}^z_{w} \}$ defined by

$$\psi^z_w = \sum_{y \geq w} \phi^z_w .$$

In the contragredient we will consider

$$\hat{\psi}^z_w = \sum_{y \leq w} \phi^{z^{-1}}_w .$$

Note the inversion of the Bruhat order.

If $w \in W$ there is an intertwining operator $M_w : V_z \rightarrow V_{wz}$ defined by

$$(M_w f)(g) = \int_{N \cap w N \cdot w^{-1}} f(w^{-1}ng) \, dn.$$  

The integral is convergent if $|z^\alpha| < 1$ for positive roots $\alpha$, and has meromorphic continuation to all $z$. Our purpose is to study the "matrix coefficient"

$$\sigma(u, v, w) = \langle A_v \psi^z_u, \hat{\psi}^{vw}_w \rangle .$$

Define a functional $\Lambda_w$ on $H_q$ by

$$\Lambda_w(T_y) = \begin{cases} q^{\ell(y)} & \text{if } y \leq w, \\ 0 & \text{otherwise.} \end{cases}$$

If $w = w_0$ then $\Lambda_w(T_y) = q^{\ell(y)}$, and it is easy to see that in this case $\Lambda_{w_0}$ is a ring homomorphism $H_q \rightarrow \mathbb{C}$. Define

$$\Theta(x, y, w) = \Lambda_w(T_x T_{y^{-1}}) .$$

The following result was proved in [4].

**Theorem 4.** There exist polynomials $r_{u,v}(z)$ of $z \in \hat{T}(\mathbb{C})$ such that $r_{u,v} = 0$ unless $u \leq v$ and $r_{u,u} = 1$ satisfying the following recursive relations. Let $s$ be a simple reflection such that $sv < v$.

$$r_{u,v}(z) = \begin{cases} \frac{1-q}{1-z^{-v^{-1} \alpha}} r_{u,sv}(z) + r_{su,sv}(z) & \text{if } su < u, \\ (1-q) \frac{z^{-v^{-1} \alpha}}{1-z^{-v^{-1} \alpha}} r_{u,sv}(z) + qr_{su,sv}(z) & \text{if } su > u. \end{cases}$$

The Kazhdan-Lusztig R-polynomial $R_{u,v}$ is the limit of $r_{u,v}(z)$ when $z \rightarrow \infty$ in such a way that $z^\alpha \rightarrow \infty$ for all positive roots $\alpha$. We note that since $sv < v$, $-v^{-1} \alpha$ is a positive root so $z^{-v^{-1} \alpha} \rightarrow \infty$ under this specialization.

The value $r_{u,v}(z)$ is a polynomial in $q$ and $z$. Let us denote by $\overline{r_{u,v}(z)}$ the result of replacing $q$ by $q^{-1}$ in this polynomial.

**Theorem 5.** We have

$$\sigma(u, v, w) = \sum_{y \geq w} q^{-\ell(y)} \Theta(x, y, w) \overline{r_{y,v}(z)} .$$
Proof. The proof follows [3, 4], who made use of a technique from [21] to model the intertwining operators $A_v$ by elements of $H_q$. We note that by results of Iwahori and Matsumoto [12] the convolution algebra of $J$-biinvariant functions on $K$ is isomorphic to $H_q$; in this isomorphism $T_i$ corresponds to the characteristic function of the double coset $Js_iJ$.

The space $V_J^J$ of Iwahori fixed vectors, like $H_q$, has dimension $|W|$. We define a vector space isomorphism $\alpha_z : V_J^J \rightarrow H_q$ as follows. Let $f \in V_J^J$. Define $\alpha_z(f)$ to be the function on $G(F)$ defined by

$$\alpha_z(f)(g) = \begin{cases} f(g^{-1}) & \text{if } g \in K, \\ 0 & \text{otherwise.} \end{cases}$$

The function $\alpha_z(f)$ is clearly constant on double cosets $Jg(K \cap B(F))$ and is supported in $K$; but it is easy to see using the Iwahori factorization of $J$ that if $g \in K$ then $Jg(K \cap B(F)) = JgJ$. Thus $\alpha_z(f)$ is in $H_q$ interpreted as the convolution algebra. It is easy to see that $\alpha_z : V_J^J \rightarrow H_q$ is a linear isomorphism.

Now let $\phi \in H_q$. Then we claim that

$$\alpha_z(\pi(\phi)f) = \phi \ast \alpha_z(f).$$

Indeed, applied to $g \in G$, both sides vanish unless $g \in K$. Assuming this,

$$\alpha_z(\pi(\phi)f)(g) = (\pi(\phi)f)(g^{-1}) = \int_K \phi(k)f(g^{-1}k)dk = \int_K \phi(k)(\alpha_zf)(k^{-1}g)dk$$

from which (15) follows. We may express (15) by saying that $\alpha_z$ intertwines the action of $H_q$ on $V_J^J$ with the left regular representation of $H_q$ on itself.

Now we use the maps $\alpha_z$ to transfer the intertwining operator $A_w : V_z \rightarrow V_v^z$ to a map $A_w : H_q \rightarrow H_q$ by requiring the following diagram to commute:

$$\begin{array}{ccc}
V_z & \xrightarrow{A_w} & V_w^z \\
\downarrow{\alpha_z} & & \downarrow{\alpha_z} \\
H_q & \xrightarrow{A_w} & H_q
\end{array}$$

Using (15) the map $A_w$ is an intertwining operator for the left regular representation of $H_q$, and therefore $A_w(\phi) = A_w(\phi \ast 1_{H_q}) = \phi \ast \mu_z(w)$ where $\mu_z(w) = A_w(1_{H_q}) \in H_q$. This much is in [3]. In [4] it was shown that

$$\mu_z(w) = \sum_{y \leq w} q^{-\ell(y)}T_{y,w}T_{w,1}^{-1}$$

where $r_{u,w}$ is as in Theorem 14. We may now compute the pairing $\langle A_v\psi_x^u, \hat{\psi}_{w}^v \rangle$. This equals

$$\sum_{x \leq u} \langle A_v \phi_x^u, \hat{\psi}_{w}^v \rangle = \sum_{x \leq u} \Lambda_w(T_x \mu_z(v)).$$

Now substituting (16) and using (12) we obtain (14). \[\Box\]

4. Properties of $\sigma(u, v, w)$

In the last section we introduced the linear forms $\Lambda_w$ on the Hecke algebra, and the related values $\Theta(x, y, w)$. These were defined in (13) and appeared in the formula (14) for $\sigma$. Therefore our first task is to prove some properties of $\Theta$. It is obviously a polynomial in $q$. 

Proposition 8. The polynomial $\Theta(x, y, w)$ is divisible by $q^{(\ell(x) + \ell(y) + \ell(xy^{-1}))/2}$. Its degree is bounded by $\ell(x) + \ell(y)$.

Note that $\ell(xy^{-1}) \equiv \ell(x) + \ell(y)$ modulo 2, so the exponent here is an integer.

**Proof.** Let us evaluate $\Theta(x, y, w) = \Lambda_w(T_xT_y^{-1})$ from the definition. Let $r = \ell(x)$, $s = \ell(y)$ and $t = \ell(xy^{-1})$. Taking reduced words $x = s_{i_1} \cdots s_{i_r}$ and $y = s_{j_1} \cdots s_{j_s}$ we have $T_xT_y^{-1} = T_{i_1} \cdots T_{i_r}T_{j_s} \cdots T_{j_1}$. Using the braid relations and the quadratic relations we may express this as a sum of terms of the form $q^a(q-1)^bT_{k_1} \cdots T_{k_a}$ where $z = s_{k_1} \cdots s_{k_a}$ is a reduced expression, as follows. Each time we use a quadratic relation $T_i^2 = (q-1)T_i + q$ we replace $T_i^2$ by either $(q-1)T_i$ or $q$; the number of times we take $(q-1)T_i$ is $b$, and the number of times we take $q$ is $a$. The total number of such quadratic relation applications is $\frac{1}{2}(r+s-t)$, so $a + b = \frac{1}{2}(r+s-t)$. Moreover $u = b + t$ because $t$ of the $T_i$ in the expression $q^a(q-1)^bT_{k_1} \cdots T_{k_a}$ come from the original expression and an additional $b$ come from the $b$ terms $(q-1)T_i$ in the quadratic relations. Now applying $\Lambda_w$ to $q^a(q-1)^bT_{k_1} \cdots T_{k_a}$ we obtain a polynomial in $q$, whose term of lowest degree is $(-1)^aq^{a+b}$. The degree of this is $a + u = a + b + t = \frac{1}{2}(r+s+t)$. So this is the smallest possible exponent for a monomial in $\Theta(x, y, w)$. The degree bound may be obtained similarly. \hfill \Box

If $\xi = \sum_{w \in W} a_wT_w \in \mathcal{H}_q$ we define the support $\text{supp}(\xi)$ of $\xi$ to be $\{w \in W | a_w \neq 0\}$.

Proposition 9. The set $\text{supp}(T_uT_v)$ has minimal element $uv$ and maximal element $u \circ v$.

**Proof.** Let $s$ be a simple reflection that is a left descent of $v$, so $v = sv'$ with $\ell(v') < \ell(v)$. Thus $T_uT_v = T_u T_{sv'}$ and $T_uT_v = T_uT_{sv'}$. If $us > u$ then $T_uT_v = T_uT_{sv'} = T_{us}T_{sv'}$. By induction on $\ell(v)$ the minimal element of $\text{supp}(T_{uv'})$ is $uv' = uw$, and we are done in this case.

If $us < u$ then $T_uT_s = (q-1)T_u + qT_{us}$ and, again using induction on $\ell(v)$:

$$\min(\text{supp}(T_uT_{sv'})) = \min(\text{supp}(T_{us}T_{sv'})) = \min(\text{supp}(T_{ap}T_{bp})) = \min(\text{supp}(T_{uw}T_{uv'})) = \min(\text{supp}(T_{uv'}) = \min(\text{supp}(uv', usv') = usv' = uv).$$

This shows that the minimal element of $\text{supp}(T_uT_v)$ is $uv$. The fact that the maximal element is $u \circ v$ may be proved similarly. \hfill \Box

Proposition 8 gives a lower bound for the exponents of $q$ in the polynomial $\Theta(x, y, w)$. It is not sharp since the actual exponent depends on $w$. For example if $x = y = s$ is a simple reflection, then $\Theta(s, s, 1) = q$ and the bound in Proposition 8 is sharp. However $\Theta(s, s, s) = q^2$ and the bound is not sharp in this case.

It is an empirical observation that the polynomial $\Theta(x, y, w)$ is often a power of $q$. For example:

Proposition 10. (i) If $xy^{-1} \leq w$, then $\Theta(x, y, w)$ is a nonzero polynomial whose value at $q = 1$ equals 1. On the other hand, if $xy^{-1} > w$ then $\Theta(x, y, w) = 0$.

(ii) If $x \circ y^{-1} \leq w$, then $\Theta(x, y, w) = q^{\ell(x) + \ell(y)}$.

**Proof.** To prove (i), the vanishing of $\Theta(x, y, w)$ unless $xy^{-1} \leq w$ follows from Proposition 9. Assume that $xy^{-1} \leq w$. On specializing $q \to 1$, the relations defining $\mathcal{H}_q$ become the Coxeter relations in $W$, so $\Lambda_w(T_xT_y^{-1})$ has the same limit as $\Lambda_w(T_{xy^{-1}}) = q^{\ell(xy^{-1})}$, which is 1. In particular $\Theta(x, y, w)$ is a nonzero polynomial.

We prove (ii). First consider the case where $w = w_0$. Then $\Lambda_{w_0}$ is a homomorphism, so

$$\Theta(x, y, w) = \Lambda_{w_0}(T_xT_y^{-1}) = \Lambda_{w_0}(T_x)\Lambda_{w_0}(T_y^{-1}) = q^{\ell(x)}q^{\ell(y)}.$$
Now in the general case, then by Proposition \ref{prop:9} every element of the support of $T_x T_{y^{-1}}$ is \leq x \circ y^{-1}$, so $\Lambda_w(T_x T_{y^{-1}}) = \Lambda_{uw}(T_x T_{y^{-1}})$, and the statement follows.

See Proposition \ref{prop:13} for another case where $\Theta(x, y, w)$ is known to be a power of $q$. It is not hard to show that assuming $xy^{-1} \leq w$, then when the polynomial $\Theta(x, y, w)$ is evaluated at $q = 1$ the result is 1. In particular $\Theta(x, y, w)$ is nonzero in this case.

**Conjecture 1.** Assume that the Kazhdan-Lusztig polynomial $P_{xy^{-1}, w} = 1$. Assume also that $xy^{-1} \leq w$. (Otherwise $\Theta(x, y, w) = 0$ by Proposition \ref{prop:14}) Then $\Theta(x, y, w)$ is a power of $q$.

This has been verified by computer calculations for Cartan types $A_3$ and $B_3$.

**Lemma 6.** If $x \geq u$, $y \leq v$ and $\Theta(x, y, w) \neq 0$ then

\[(17) \quad U_{w^{-1}} \downarrow u \leq U_{w^{-1}} \downarrow x \leq y \leq v.\]

**Proof.** By Proposition \ref{prop:9} $\Theta(x, y, w) = 0$ unless $w \leq xy^{-1}$. This means that $yx^{-1} \leq w^{-1}$, that is, $y \in [1, w^{-1}] x$. By Theorem \ref{thm:2} the set $[1, w^{-1}] x$ has a unique minimal element $U_{w^{-1}} \downarrow x$. Thus we obtain $U_{w^{-1}} \downarrow x \leq y \leq v$. The remaining inequality $U_{w^{-1}} \downarrow u \leq U_{w^{-1}} \downarrow x$ follows from Proposition \ref{prop:11}

Our next result shows that with $u, w$ fixed, $v = U_{w^{-1}} \downarrow u$ is the smallest $v$ such that $\sigma(u, v, w) \neq 0$, and moreover this value is a polynomial in $q$. We define

$$\sigma_0(u, w) = \sigma(u, U_{w^{-1}} \downarrow u, w).$$

This partially proves Theorem \ref{thm:11}

**Proposition 11.** We have $\sigma(u, v, w) = 0$ unless $v \geq U_{w^{-1}} \downarrow u$. In the minimal case where $v = U_{w^{-1}} \downarrow u$, the value $\sigma_0(u, w)$ is a polynomial in $q$ (independent of $z$).

**Proof.** We make use of (14). With $x \geq u$, $y \leq v$, by Lemma \ref{lem:6} $\Theta(x, y, w) = 0$ unless (17) is satisfied. Therefore $\sigma(u, v, w) = 0$ unless $U_{w^{-1}} \downarrow u \leq v$, proving the first assertion.

Moreover if $v = U_{w^{-1}} \downarrow u$, then (17) implies that $y = v$. Therefore $r_{y, v} = 1$. Also by Proposition \ref{prop:9} $\Theta(x, v, w)$ vanishes unless $xv^{-1} \leq w$. Hence (14) simplifies to

\[(18) \quad \sigma_0(u, u) = q^{-\ell(v)} \sum_{x \geq u, xv^{-1} \leq w} \Theta(x, v, w), \quad v = U_{w^{-1}} \downarrow u.\]

Since $\Theta(x, v, w)$ does not involve $z$, it is a polynomial in $q$. But since we are dividing by $q^{\ell(v)}$ in (18), we must show that $\Theta(x, v, w)$ is divisible by $q^{\ell(v)}$. Actually we will show that it is divisible by $q^{\ell(u)}$, a stronger result since $v \leq_L u$. With $x \geq u$ we have $xv^{-1} \in [u, w_0]v^{-1}$ so by Theorem \ref{thm:2} we have

$$xv^{-1} \geq u \downarrow U_{v^{-1}} = uv^{-1}$$

where the last equality follows since $v \leq_L u$. Now we apply Proposition \ref{prop:8} and use

$$\frac{1}{2}(\ell(x) + \ell(v) + \ell(xv^{-1})) \geq \frac{1}{2}(\ell(u) + \ell(v) + \ell(uv^{-1})) = \ell(u).$$

\[\square\]
If $\alpha$ is a positive root, let $r_{\alpha} \in W$ be the associated reflection. If $u \leq v$ in the Bruhat order, define:

$$S(u, v) = \{\alpha \in \Phi^+ | u \leq vr_{\alpha} < v\}.$$  

We remind the reader of the Deodhar inequality \(^2\), valid if the inverse Kazhdan-Lusztig polynomial $Q_{u,v} := P_{w_0v,w_0u}$ equals 1. We will generalize this definition slightly and denote

$$S(u, v, w) := S(U_{w^{-1}} \downarrow u, v) = \{\alpha \in \Phi^+ | U_{w^{-1}} \downarrow u \leq vr_{\alpha} < v\}.$$  

Bump and Nakasuji conjectured, and Aluffi, Mihalcea, Schürmann and Su proved the following result.

**Theorem 6.** Let $u \leq v$. Then

$$r_{u,v} \prod_{\alpha \in S(u, v)} (1 - z^\alpha)$$

is analytic for all $z \in \hat{T}(\mathbb{C})$.

**Proof.** See [1], Theorem 10.4. \qed

We note the difficulty of Theorem 6. An obvious approach is to prove this recursively from Theorem 4. This is the basis of partial results in [4]. However this does not produce exactly the right set of roots $\alpha$ such that $1 - z^\alpha$ appears in the denominator, because some roots that appear in the terms of the recursive formula actually cancel; these cancellations are not easy to prove.

**Theorem 7.** The function

$$\sigma(u, v, w) \prod_{\alpha \in S(u, v, w)} (1 - z^\alpha)$$

is analytic for all $z \in \hat{T}(\mathbb{C})$.

**Proof.** We make use of (14) and Theorem 6. We see that the possible roots $\alpha$ such that $1 - z^\alpha$ appears in the denominator lie in

$$\bigcup_{x \geq u, y \leq v, \Theta(x,y,w) \neq 0} S(y, v).$$

Now by Lemma 6 we must have $U_{w^{-1}} \downarrow u \leq y$ in this union, so $S(y, v) \leq S(u, v, w)$ from the definition (20). \qed

5. **Proof of Theorem 1**

Let $u, w \in W$. In this section we will take $v = v_{\min}(u,w) = U_{w^{-1}} \downarrow u$ and prove Theorem 1.

We will change our notation slightly for simple reflections. In previous sections we denoted by $\{s_1, \cdots, s_r\}$ the set of simple reflections, so a reduced word for a Weyl group element $v$ would be written $s_{i_1} \cdots s_{i_k}$ where $\{i_1, \cdots, i_k\}$ is some sequence of indices. In the following arguments this notation would be cumbersome, so we dispense with the double subscripts and write a reduced expression for $v$ as $s_1 \cdots s_k$. Thus $k = \ell(v)$.  

Let \( m = wv^{-1} \). Then \( m \) is the “mixed meet” of Theorem 3, that is, the maximal element of \( w \) such that \( m \leq_R u \) and \( m < w \). We will write \( u = m v = m s_1 \cdots s_k \). Because \( v = U_{w^{-1}} u \) we have \( v \leq_L u \), that is \( \ell (u) = \ell (m) + \ell (v) \). Therefore

\[
(21) \quad m < ms_1 < ms_1 s_2 < \cdots < ms_1 \cdots s_k = u.
\]

We will denote

\[
[u, w] = \{ z \in W | u \leq z, z v^{-1} \leq w \}.
\]

![Diagram](image.png)

Figure 1. The Weyl group elements in Proposition 12 form a ladder. Here is the case where \( k = 3 \).

**Proposition 12.** Let \( u, w \in W \) and let \( v = v_{\min}(u, w) \). Then \( \ell (wv) = \ell (w) + \ell (v) \). Moreover \( [u, w] = [u, wv] \). Let \( k = \ell (v) \) and let \( v = s_1 \cdots s_k \) be a reduced expression. Then

\[
(22) \quad w < ws_1 < ws_1 s_2 < \cdots < ws_1 \cdots s_k = wv.
\]

Suppose that \( z \in [u, wv] \). Then

\[
(23) \quad z > zs_k > zs_k s_{k-1} > \cdots > zs_k \cdots s_1 = z v^{-1}.
\]

We have

\[
(24) \quad ws_k \cdots s_{r+1} = ms_1 \cdots s_r \leq zs_k \cdots s_{r+1} \leq ws_1 \cdots s_r
\]

for \( 0 \leq r \leq k \).

**Proof.** The inequalities asserted by the Proposition may be envisioned as forming a ladder, as in Figure 1. We will ascend the ladder to prove the inequalities satisfied by the \( ws_1 \cdots s_r \), then descend the ladder to prove the inequalities satisfied by the \( zs_k s_{k-1} \cdots s_{r+1} \).

Thus we will by induction for \( 0 \leq r \leq k \) that

\[
(25) \quad ms_1 \cdots s_r \leq ws_1 \cdots s_r.
\]

If \( r = 0 \) this is true since \( m = w v^{-1} < w \). Arguing inductively, assume that this is true for \( r < k \); we will prove that it is true for \( r + 1 \). The first step is to show

\[
(26) \quad ws_1 \cdots s_r < ws_1 \cdots s_r s_{r+1}.
\]
If not, \( ws_1 \cdots s_r < w_s_1 \cdots s_r s_{r+1} \). Now since \( ms_1 \cdots s_r \leq w_s_1 \cdots s_r \) and \( ms_1 \cdots s_r < ms_1 \cdots s_r s_{r+1} \), the lifting property of the Bruhat order implies that

\[
ms_1 \cdots s_r s_{r+1} \leq ws_1 \cdots s_r.
\]

Using the monotonicity property Proposition 5 we then have

\[
U(w_s_1 \cdots s_r)^{-1} \downarrow u \leq U(ms_1 \cdots s_r s_{r+1})^{-1} \downarrow u.
\]

However

\[
U(w_s_1 \cdots s_r)^{-1} \downarrow u = U(s_1 \cdots s_r)^{-1} \downarrow U_{w^{-1}} \downarrow u = U(s_1 \cdots s_r)^{-1} \downarrow v = s_{r+1} \cdots s_k,
\]

while

\[
U(ms_1 \cdots s_r s_{r+1})^{-1} \downarrow u = U(ms_1 \cdots s_r s_{r+1})^{-1} \downarrow ms_1 \cdots s_k = s_{r+2} \cdots s_k,
\]

which is interpreted as 1 if \( k = r - 1 \). Thus we have proved that \( s_{r+1} \cdots s_k \leq s_r \cdots s_k \), which is a contradiction. Therefore we have proved (26).

Now using (25) and (26) and the inequality \( ms_1 \cdots s_r < ms_1 \cdots s_r s_{r+1} \) from (21), the lifting property of the Bruhat order implies that \( ms_1 \cdots s_r s_{r+1} \leq w_s_1 \cdots s_r + 1 \), which is (24) for \( r + 1 \). This completes the induction, so we have now proved (25) for all \( r \).

Note that we have also proved (22) from the inequalities (26).

Now let \( z \in [u, w v] \). We now refine (25) to the inequality (24). This time we argue by \textit{downwards} induction, the initial case being \( r = k \), where (24) becomes our assumption \( u \leq z \leq w v \). Assuming (24) for \( r > 0 \), we prove it for \( r - 1 \). First we need to show that

\[
zs_k \cdots s_{r+1} > zs_k \cdots s_r.
\]

If not, \( zs_k \cdots s_{r+1} < zs_k \cdots s_r \). We have also \( zs_k \cdots s_{r+1} \leq w_s_1 \cdots s_r \) by (24) and \( w_s_1 \cdots s_r > w_s_1 \cdots s_{r-1} \), so by the lifting property of the Bruhat order we obtain \( zs_k \cdots s_{r+1} \leq w_s_1 \cdots s_{r-1} \), and using the first inequality in (24) we get \( ms_1 \cdots s_r \leq w_s_1 \cdots s_{r-1} \). Now using Proposition 5 we have

\[
U(w_s_1 \cdots s_{r-1})^{-1} \downarrow u \leq U(ms_1 \cdots s_r) \downarrow u,
\]

that is \( s_r \cdots s_k \leq s_{r+1} \cdots s_k \), which is a contradiction, proving (27). Now using (27), (24) and (22), the lifting property of the Bruhat order implies that \( us_k \cdots s_{r+1} = ms_1 \cdots s_r \leq zs_k \cdots s_{r+1} \) \( \leq w_s_1 \cdots s_r \) which is a contradiction. Therefore we have proved (27).

Given (27), (24), (21) and (22), both inequalities in

\[
us_k \cdots s_r = ms_1 \cdots s_{r-1} \leq zs_k \cdots s_r \leq w_s_1 \cdots s_{r-1}
\]

follow from the lifting property of the Bruhat order. This is (24) for \( r - 1 \), completing the proof of (24) by induction. Note that we have also proved (23).

Taking \( k = 1 \) in (24) gives \( z v^{-1} \leq w \). Therefore \( [u, w] \subseteq [u, w v] \). To prove the opposite inclusion, assume that \( z \in [u, w] \). We must show that \( z \leq w v \). Indeed \( z = (z v^{-1}) v \leq w v \circ v \). But \( w v = w v v \) follows from (22), and so we are done.

\textbf{Lemma 7.} Suppose that \( v = v_{\min}(u, w) \) and that \( z \in [u, w] \). Suppose furthermore that \( v' \leq v \) and that \( z(v')^{-1} \leq w \). Then \( v' = v \).

\textit{Proof.} Since \( v' z^{-1} \leq w^{-1} \) we have \( v' \in [1, w^{-1}] z \) and so by Theorem 2 we have \( U_{w^{-1}} \downarrow z \leq v' \). Since \( u \leq z \), Proposition 4 implies that \( v = U_{w^{-1}} \downarrow u \leq v', \) so \( v' = v \).

\textbf{Proposition 13.} Suppose that \( v = v_{\min}(u, w) \) and that \( z \in [u, w] \). Then \( \Theta(z, v, w) = q^{\ell(z)} \).
Proof. Recall that \( \Theta(z, v, w) = \Lambda_w(T_zT_{v^{-1}}) \). Expanding \( T_zT_{v^{-1}} = T_zT_{s_k} \cdots T_{s_1} \) using the fact that

\[
T_yT_s = \begin{cases} 
T_{ys} & \text{if } ys > y, \\
(q-1)T_y + qT_{ys} & \text{if } ys < y,
\end{cases}
\]

it is clear that

\[
\text{supp}(T_zT_{v^{-1}}) \subseteq \{ T_{z(v')}^{-1} | v' \leq v \}.
\]

By Lemma 7, only \( v' = v \) can contribute to the value of \( \Lambda_w \). Moreover the only way to obtain \( T_zT_{v^{-1}} \) is to select \( qT_{ys} \) in the descent case of (28) each time. Thus

\[
T_zT_{v^{-1}} = q^{\ell(v)}T_{zv^{-1}} + \text{other terms}
\]

where the other terms are linear combinations of \( T_{z(v')}^{-1} \) with \( v' < v \). Note that \( \ell(v) + \ell(zv^{-1}) = \ell(z) \) by (23) in Proposition 12. Thus applying \( \Lambda_w \) we get

\[
\Theta(z, v, w) = \Lambda_w(q^{\ell(v)}T_{zv^{-1}}) = q^{\ell(v) + \ell(zv^{-1})} = q^{\ell(v)}.
\]

\( \square \)

Proof of Theorem 1. A portion of Theorem 1 is contained in Proposition 11. To finish the proof we need to establish (4). By Proposition 12 we may rewrite (18) as

\[
q^{-\ell(v)} \sum_{z \in [u, wv]} \Theta(z, v, w),
\]

and now (4) follows from Proposition 13. \( \square \)

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Department of Mathematics, Stanford University, Stanford, CA 94305-2125

*Email address*: bump@math.stanford.edu

*Email address*: bchetard@gmail.com