\textbf{\Delta-filtrered Modules and Nilpotent Orbits of a Parabolic Subgroup in $O_N$}

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\textbf{Abstract.} We study certain $\Delta$-filtered modules for the Auslander algebra of $k[T]/T^n \rtimes C_2$ where $C_2$ is the cyclic group of order two. The motivation of this lies in the problem of describing the $P$-orbit structure for the action of a parabolic subgroup $P$ of an orthogonal group. For any parabolic subgroup of an orthogonal group we construct a map from parabolic orbits to $\Delta$-filtered modules and show that in the case of the Richardson orbit, the resulting module has no self-extensions.

\section{Introduction}

Let $k$ be an algebraically closed field of characteristic different from 2, and let $G$ be a reductive algebraic group over $k$, $P \subset G$ a parabolic subgroup. Now $P$ acts on its unipotent radical $U$ by conjugation and on the nilradical $\mathfrak{n} = \text{Lie}U$ by the adjoint action. By a fundamental result of Richardson (R74), this action has an open dense orbit, the so-called Richardson orbit of $P$. But in general, the number of orbits is not finite and it is a very hard problem to understand the orbit structure. The question of deciding whether $P$ has a finite number of orbits in $\mathfrak{n}$ has been asked by Popov and Röhrle in [PV97]. For groups in characteristic zero or in good characteristic, the parabolic subgroups with finitely many orbits on their nilradical have been classified in a sequence of papers, [R96], [HR97] and [HR99].

If $P$ is a parabolic subgroup of $\text{SL}_N$, then there is an explicit description of the $P$-orbits in work of Hille-Röhrle [HR99], and Brüstle et al. [BHRR99], via a connection with a quasi-hereditary algebra, namely the Auslander algebra $A_n$ of the truncated polynomial ring $R_n := k[T]/T^n$. They have shown that the $P$-orbits are in bijection with the isomorphism classes of certain $\Delta$-filtered modules of $A_n$ with no self-extensions. This list has finitely many indecomposable modules, parametrized as $\Delta(I)$ where $I$ runs through the subsets of $\{1, 2, \ldots, n\}$.

Our main goal in this paper is to establish an analogous correspondence between $P$-orbits for parabolic subgroups of the special orthogonal groups $SO_N$ and certain $\Delta$-filtered modules for the Auslander algebra of $k[T]/T^n \rtimes C_2$, the skew group ring of one considered by [BHRR99], where $C_2$ is a cyclic group of order two. This article establishes the Auslander algebra of $k[T]/T^n \rtimes C_2$ as the correct candidate for such a correspondence. There is one major difference as compared with [BHRR99]. In our situation, a list of all $\Delta$-filtered modules with no self extensions is difficult to obtain and may even be infinite. This must be expected however, as the construction of the Richardson elements for $SO_N$ involves symmetric diagrams and hence gives rise to symmetric $\Delta$-dimension vectors. Here, we use signed sets $I$, that is certain subsets of $\{\pm 1, \pm 2, \ldots, \pm n\}$. We associate to each signed set $I$ another set $J$ (a symmetric complement) and an extension $E(I, J)$ that is $\Delta$-filtered with no self extensions and has the required symmetric $\Delta$-dimension vector. These extensions may then be used to construct a $\Delta$-filtered module that corresponds to the
Richardson element, using the work of Baur [Ba06] and Baur and Goodwin [BG08] on orthogonal Lie algebras.

We now summarise our paper. The first section describes the problem of determining the parabolic orbits in the nilradical and introduce the notation from the side of $P$-orbits. We also recall the correspondence between $P$-orbits and $\Delta$-filtered modules for the Auslander algebra of $k[T]/T^n$ in case $P$ is a subgroup of $SL_N$ as given in [HR99] and [BHRR99].

The key ingredient of our approach is a quasi-hereditary algebra $D_n$ which is the Auslander algebra of $S_n = k[T]/T^n \rtimes C_2$. The algebra $S_n$ and its Auslander algebra have very similar properties as $R_n$ and the Auslander algebra of $R_n$ (cf. Appendix, here we also show that $S_n$ is a skew group ring over $R_n$ which is an essential tool for us.) In Section 2 we explicitly describe the standard, costandard, projective and tilting modules.

In Section 3 we study $\Delta$-filtered modules for the Auslander algebra of $S_n$. We continue this in Section 4 where we define a certain special class of $\Delta$-modules. This class will be essential in the construction of representatives of $P$-orbits. We denote these modules by $\Delta(I)$ where $I$ is a subset of $\{1, \ldots, n\}$. They can be seen as weighted versions of the modules introduced in Section 2 of [BHRR99].

In order to construct the Richardson orbit using $D_n$-modules we need to understand the extension groups between the $\Delta(I)$ introduced in Section 3. In general, this is a very hard problem but we are able to determine them under certain conditions which are always satisfied in the setup we use.

The Auslander algebra of $S_n$, (denoted $D_n$) is isomorphic to a skew group algebra of the Auslander algebra of $R_n$ (denoted $A_n$). Every $D_n$-module is relative $A_n$-projective, and inducing and restricting preserves modules with $\Delta$-filtrations, as is explained in Section 5. In Proposition 5.3 we give an explicit result relating extension groups for $A_n$ to extension groups for $D_n$. Then the extensions between different modules of the form $\Delta(I)$ are discussed in Section 6. In particular, we present a combinatorial way to compute the dimension of the groups of homomorphisms for the algebras $A_n$ and $D_n$ using so-called initial segments of the subsets of $\{1, \ldots, n\}$ in Proposition 8.4 and Lemma 8.6.

Let us emphasise that in one sense, there are far more $\Delta$-filtered modules with no self-extensions for $D_n$ than there are for algebras $A_n$. In Section 9 we construct such modules for $D_n$ which do not exist for $A_n$, and we call these “type II modules” (and our results produce examples of these). They arise as the indecomposable extensions between two modules $\Delta(I)$ and $\Delta(J)$ where $J$ is dependent on $I$. As a preparation for this, we calculate the dimensions of the Ext groups of such pairs $\Delta(I)$, $\Delta(J)$ and show that they can grow arbitrarily large, cf. Lemmata 7.1 and 7.2. From these extensions, we obtain a module $E(I, J)$ without self-extensions in Proposition 7.3. $E(I, J)$ is called a type I module if it is of the form $\Delta(I_1) \oplus \Delta(I_2)$ for some subsets $I_i$. Otherwise, $E(I, J)$ is said to be of type II. We study these type II modules in Section 8. In particular, we explain that type II modules occur roughly in half of the cases of modules obtained as extensions between such $\Delta(I)$, $\Delta(J)$. The characterisation of the other cases, namely of the decomposable ones is presented in Theorem 8.3.

Finally, in Section 9 we explain how to construct a module without self-extensions starting from the dimension vector $d$ for a parabolic subgroup of $SO_N$. The resulting module, $M(d)$, has no self-extension and its restriction to $A_n$ has $\Delta$-support $d$ (Proposition 11.6). The construction uses modules of the form
\(\Delta(I)\) and also the type II modules defined before. We observe that the type II modules can be viewed as module theoretic analogues of the branched line diagrams appearing in the construction of Richardson elements in \cite{Ba06} and of the diagrams \(D(d)\) with branched arrows in Section 2 of \cite{BG08}, both for the orthogonal Lie algebras: In the same way as these branched diagrams are needed to obtain the dense orbit (i.e. the Richardson orbit), the type II modules are essential in the construction of a module without self-extension.

1. Motivation

Our ultimate goal is to find a bijection between \(P\)-orbits for a parabolic subgroup of an orthogonal group and isomorphism classes of certain \(\Delta\)-filtered modules for an algebra \(D_n\). What we present in this article, is the first step towards this. Namely, we produce a module theoretic analogue of the Richardson orbit for \(P\): a \(D_n\)-module without self-extension whose \(\Delta\)-dimension vector (after restriction to \(A_n\)) is equal to the composition \(d\) which determines the parabolic subgroup \(P\).

Let us start by explaining the Lie algebra side of the problem. Let \(P \subset G\) be a parabolic subgroup of a reductive algebraic group \(G\) over an algebraically closed field \(k\). Let \(\mathfrak{p} \subset \mathfrak{g}\) be the corresponding Lie algebras and let \(\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}\) be a Levi decomposition of \(\mathfrak{p}\), i.e. \(\mathfrak{l}\) is a Levi factor of \(\mathfrak{p}\) and \(\mathfrak{n}\) is the corresponding nilradical. It is a very ambitious goal to understand the \(P\)-orbit structure in \(n\). A first step towards this is a fundamental theorem of Richardson, cf. \cite{R74}. It states that \(P\) has an open dense orbit in \(n\), the so-called Richardson orbit of \(P\). Its elements are called Richardson elements for \(P\). Observe that the existence of a dense orbit does not imply that there are only finitely many, in general, there are infinitely many \(P\)-orbits in the nilradical \(n\).

For classical groups (in zero or good characteristic) there exists a classification of parabolic subgroups with a finite number of orbits due to Hille and Röhrle, cf. \cite{HR99}.

For \(G = \text{SL}_N\), the special linear group, we can actually say more. In order to explain this, we need some notation. For all classical Lie groups we will choose the Borel subgroup \(B\) to be the upper triangular matrices in \(G\) and the maximal torus to be the diagonal matrices in \(G\). Let \(\mathfrak{b}\) and \(\mathfrak{h}\) be the corresponding Lie algebras. Then \(P\) is called standard, if \(P \supset B\). Similarly, we call \(\mathfrak{p}\) and \(\mathfrak{l}\) standard, if \(\mathfrak{p} \supset \mathfrak{b}\) and \(\mathfrak{l} \supset \mathfrak{h}\) respectively. After suitable conjugation, we can assume that \(P\), \(\mathfrak{p}\) and \(\mathfrak{l}\) are standard. Then \(\mathfrak{l}\) consists of the matrices whose non-zero entries only lie in a sequence of square blocks on the diagonal. So \(\mathfrak{l} = \mathfrak{l}(d)\) where \(d = (d_1, \ldots, d_n)\) is a composition of \(N\), i.e. \(\sum d_i = N\), describing the sizes of these square blocks. And \(\mathfrak{p} = \mathfrak{p}(d)\) consists of the matrices with zeroes below the sequence of square matrices of size \(d_1, \ldots, d_n\) on the diagonal. We call \(d\) a dimension vector.

From now on we will always assume that \(P\), \(\mathfrak{p}\) and \(\mathfrak{l}\) are standard.

We are ready to formulate the result of Hille and Röhrle, cf. \cite{HR99}: Let \(P = P(d) \subset \text{SL}_N\). Then there is a bijection:

\[
\{ \text{\(P\)-orbits in } n \} \overset{1:1}{\longleftrightarrow} \{ M \in \mathcal{F}(\Delta) \mid \dim_M M = d \} / \sim
\]

where \(\mathcal{F}(\Delta)\) are the \(\Delta\)-filtered modules for the algebra \(A_n\) described in the Appendix. Furthermore, the Richardson orbit is mapped to the unique \(M\) with \(\Delta\)-dimension vector \(d\) which has no self-extensions. In other words, for \(P = P(d)\) in \(\text{SL}_N\) there is a description of \(P\)-orbits in \(n\) via \(\Delta\)-filtered modules (for
An) of dimension vector d.
So far, it has remained an open problem to find an analogous result for the other classical types. In this paper, we show that the algebra Dn as defined in Section 2 is the right candidate to describe P-orbits in n for G = SO_N, the special orthogonal group. Thus we are providing the first part of an analogue of the correspondence above for classical groups: Using the knowledge of the algebra Dn (see Appendix) and our results about the extensions between ∆-filtered modules for Dn (cf. Section 7 below) we will construct certain ∆-filtered modules without self-extensions of ∆-dimension vector d. There is an interesting new phenomenon appearing in this case: the construction of the modules of a given dimension vector leads to a new class of ∆-filtered modules which does not exist for SL_N. In that aspect, the situation clearly differs from the case of SL_N.

To be more precise: to any given dimension vector d we associate a ∆-filtered Dn-module M(d) which has no self-extensions and thus this is a module theoretic counterpart of the dense orbit. The construction is given Section 9 below, once we have all the material needed.

2. The quasi-hereditary algebra Dn

Let Dn be the Auslander algebra of Sn. It is given by the quiver \( \Gamma_n \) with vertices \( i^\pm \), arrows \( \alpha_{i^\pm} \) and \( \beta_{i^\pm} \) and by the relations in Definition A.1 (as defined in the appendix).

Here we describe the standard, costandard, projective and tilting modules for Dn. We use \( L(i^\pm) \) to denote the simple Dn-module corresponding to the indecomposable module \( M(i^\pm) \). From the definition, it is straightforward to write down the projective Dn-modules \( P(i^\pm) = e_{i^\pm}D_n \) (where \( e_{i^\pm} \) is the primitive idempotent at \( i^\pm \)). The indecomposable projective modules embed into each other as follows:

\[
P(1^\pm) \supset P(2^\mp) \supset \cdots \supset P((2k)^\mp) \supset \cdots
\]

The projective module \( P(1^\pm) \) is the injective hull of \( L(1^\pm) \) if \( n \) is odd and of \( L((1^\mp)^\pm) \) if \( n \) is even.

Using the indecomposable projective modules, we can define the standard modules \( \Delta(i^\pm) \) as their successive quotients, we set

\[
\Delta(i^+) = P(i^+)/P((i+1)^-) \quad \Delta(i^-) = P(i^-)/P((i+1)^+)
\]

So \( \Delta(i^\pm) \) has socle \( 1^\pm \). In particular, the \( P(i^\pm) \) are filtered by standard modules, and the quotients can be read off from (*).

For \( 1 \leq i \leq n \), the costandard Dn-module \( \nabla(i^+) \) is the serial module of length \( i \) with socle \( L(i^+) \) and top \( L(1^+) \) where \( * = + \) for odd \( i \) and \( * = - \) when \( i \) is even, the composition factors are labelled by \( i^+, (i-1)^-, (i-2)^+, \ldots \). The costandard module \( \nabla(i^-) \) is described similarly. The costandard module \( \nabla(i^\pm) \) has socle \( i^\pm \). The top of \( \nabla(i^\pm) \) is \( 1^\pm \) if \( i \) is odd and is \( 1^\mp \) otherwise.

**Proposition 2.1.** The algebra Dn is quasi-hereditary with weight set

\( \{1^+, 1^-, 2^+, 2^-, \ldots, n^+, n^-\} \) and order

\[
i^+ < j^\pm \iff i < j,
\]

\[
i^- < j^\pm \iff i < j.
\]
In general, the standard $D_n$-module $\Delta(i^\pm)$ is the serial module of length $i$ with socle $L(1^\pm)$, and the signs of the labels of its composition factors are either all $+$ or all $−$. The $\nabla(i^\pm)$ are also uniserial and have alternating signs on their composition factors.

Now we define the tilting modules. Recall that a module has a $\Delta$-filtration if it has a filtration whose successive quotients are isomorphic to standard modules, similarly one defines a $\nabla$-filtration. Recall also that for each $i^\pm$ there is a unique indecomposable module, which we denote by $T(i^\pm)$, which has both a $\Delta$-filtration and a $\nabla$-filtration, with one composition factor of the form $L(i^\pm)$ and all other composition factors of the form $L(j^\pm)$ with $j < i$ (both $L(j^+)$ and $L(j^-)$ may appear). We refer to direct sums of $T(i^\pm)$ as tilting modules. Then for $n$ odd we have $T(n^\pm) = P(1^\pm)$, and for $n$ even $T(n^\pm) = P(1^\mp)$, and these two are both projective and injective.

**Remark 2.2.** Note that $T((i−1)^\pm)$ is a quotient of $P(1^\epsilon)$ by $P(i^\mp)$, where $\epsilon = +$ if $i$ is even and $\epsilon = −$ if $i$ is odd. The same is true for $T((i−1)^\mp)$ with signs exchanged. So we have a short exact sequence

\[ 0 \to P(i^\mp) \to P(1^\epsilon) \to T((i−1)^\pm) \to 0 \]

(with appropriate sign $\epsilon$).

We will also use $Q(1^\pm)$ to denote the injective hull of $L(1^\pm)$. Note that $Q(1^\pm) \cong P(1^\pm)$ if $n$ is odd and $Q(1^\pm) \cong P(1^\mp)$ if $n$ is even.

**3. $\Delta$-filtered modules for $D_n$**

Let $\mathcal{F}(\Delta)$ be the class of all $\Delta$-filtered $D_n$-modules. In this section, we describe the properties of $\mathcal{F}(\Delta)$ extending the results of [BHRR99] to the algebra $D_n$.

Recently, R. Tan has studied the category of $\Delta$-filtered modules for the Auslander algebra $E$ of a self-injective Nakayama algebra, in particular the submodules of the projective $E$-modules. In Section 3 of [T], she obtains similar results to those we explain here.

**Lemma 3.1.** $\mathcal{F}(\Delta)$ is closed under taking submodules.

**Proof.** Since submodules of standard modules are also standard modules, this follows easily by induction over filtration length. \qed

We say that the socle of a $D_n$-module $M$ is generated by $L(1^\pm)$ if the only simples appearing in the socle are $L(1^+)$ and/or $L(1^-)$. Similarly, we say that the top of $M$ is generated by $L(1^\pm)$ if the only simples appearing in the top are $L(1^+)$ and/or $L(1^-)$.

**Lemma 3.2.** $\mathcal{F}(\Delta)$ is the set of all modules with socle generated by $L(1^\pm)$.

**Proof.** This is similar to Lemma 7.1 of [DR90].

It is clear that the socle of any module in $\mathcal{F}(\Delta)$ is of that form because all standard modules have $L(1^\pm)$ as socle. Now we show that any $D_n$-module with socle generated by $L(1^\pm)$ is in $\mathcal{F}(\Delta)$.

Assume that

\[ \text{soc}(M) = \bigoplus_{j \in I_1} L(1^+) \oplus \bigoplus_{j \in I_2} L(1^-). \]
Then we have an embedding
\[ M \hookrightarrow \bigoplus_{j \in J_1} I(1^+) \oplus \bigoplus_{j \in J_2} I(1^-). \]
Now \( I(1^+) \) and \( I(1^-) \) are tilting modules, so in particular, they are \( \Delta \)-filtered and hence \( M \) is a submodule of a \( \Delta \)-filtered module. Therefore, \( M \in \mathcal{F}(\Delta) \) by Lemma 3.1. \( \square \)

Similarly, the class of \( \nabla \)-filtered modules \( \mathcal{F}(\nabla) \) is the set of modules with top generated by \( L(1^\pm) \).

**Lemma 3.3.** The modules in \( \mathcal{F}(\Delta) \) are the \( D_n \)-modules with projective dimension \( \leq 1 \).

**Proof.** This follows in a similar fashion to [BHRR99, Lemma 1]. \( \square \)

Next we observe that we can describe the submodules of \( P(1^\pm) \) similarly as in Lemma 2 of [BHRR99].

**Lemma 3.4.** Let \( M \) be a \( D_n \)-module. Then the following are equivalent:

(i) \( M \) is a nonzero submodule of \( P(1^\pm) \).

(ii) \( \text{soc}(M) = \begin{cases} L(1^\pm) & \text{if } n \text{ is odd} \\ L(1^\mp) & \text{if } n \text{ is even.} \end{cases} \)

**Proof.** Follows from the fact that for odd \( n \), \( P(1^\pm) \) is the injective envelope of \( L(1^\pm) \) and for even \( n \) it is the injective envelope of \( L(1^\mp) \). \( \square \)

**Lemma 3.5.** Any nonzero submodule of \( P(1^\pm) \) is indecomposable and belongs to \( \mathcal{F}(\Delta) \).

**Proof.** Let \( M \) be a nonzero submodule of \( P(1^\pm) \). Since \( P(1^\pm) \) is equal to the injective envelope of \( L(1^\epsilon) \) (for \( \epsilon = \pm \) if \( n \) is odd and \( \epsilon = \mp \) if \( n \) is even), \( M \) is indecomposable. By Lemma 3.1, any submodule of \( P(1^\pm) \) is in \( \mathcal{F}(\Delta) \). \( \square \)

4. **Submodules and quotients of the projective modules**

In this section, we are going to describe certain indecomposable \( \Delta \)-filtered modules, the \( \Delta(I) \). They are submodules of \( P(1^+) \) or \( P(1^-) \) and do not have self-extensions. These form the key components of our extensions \( E(I, J) \).

From now, we abbreviate \( \{1, 2, \ldots, n\} \) by \( [n] \) and \( \{1^+, 1^-, \ldots, n^+, n^-\} \) by \( [n]^\pm \). Unless mentioned otherwise we will always assume that a subset \( I = \{i_1, i_2, \ldots, i_k\} \subset [n] \) is decreasingly ordered, i.e.

\[ i_1 > i_2 > \cdots > i_k. \]

We call a subset \( I \) of \( [n]^\pm \) signed if there is no \( 1 \leq i \leq n \) with both \( i^+ \in I \) and \( i^- \in I \), i.e. \( I = \{i_1^+, \ldots, i_k^\pm\} \), for some subset \( \{i_1, \ldots, i_k\} \) of \([n]\) and \( k, 1 \leq k \leq n \) with \( \epsilon_l \in \{+,-\} \) for \( l = 1, \ldots, k \). Now let \( I = \{i_1^+, \ldots, i_k^\pm\} \) be a signed subset of \([n]^\pm\).

- If for \( j = 1, \ldots, k - 1 \) we have \( \epsilon_j \neq \epsilon_{j+1} \) we say that the signs \( \epsilon_1, \ldots, \epsilon_k \) of \( I \) are alternating and we also call \( I \) an alternatingly signed subset.
- If \( \epsilon_j \neq \epsilon_{j+1} \) if and only if \( i_{j+1} - i_j \) is even, we say that the signs are step-alternating and we also call \( I \) step alternatingly signed.

Recall that, for any sign \( \epsilon, \bar{\epsilon} \) is the sign opposite to \( \epsilon \). Let \( s(a) \) be the sign of a number \( a \), i.e. \( s(1) = + \) and \( s(-1) = - \). We can now define the modules \( \Delta(I) \):

**Definition 4.1.** Let \( I = \{i_1^+, i_2^\pm, \ldots, i_k^\pm\} \) be a signed subset of \([n]^\pm\) with \( i_1 > i_2 > \cdots > i_k \).
(i) Assume that the signs are alternating. Then we set \( \Delta(I) \) to be the submodule of
\[
\begin{cases}
P(1^+) \text{ with } \Delta\text{-support } I & \text{if } s((-1)^n) = \epsilon_k \\
P(1^-) \text{ with } \Delta\text{-support } I & \text{if } s((-1)^n) = \epsilon_k.
\end{cases}
\]

(ii) Assume that the signs are step-alternating. Then we set \( \nabla(I) \) to be the factor module of
\[
\begin{cases}
P(1^+) \text{ with } \nabla\text{-support } I & \text{if } s((-1)^i) = \epsilon_k \\
P(1^-) \text{ with } \nabla\text{-support } I & \text{if } s((-1)^i) = \epsilon_k.
\end{cases}
\]

It is clear that \( \Delta(I) \) is unique (the existence can be seen using the quiver and relations), i.e. there can be no two different submodules of \( P(1^+) \) with same \( \Delta\)-support. Is it also clear that a submodule of \( T(n^\pm) \) is uniserial in its \( \Delta \)-filtration.

**Remark 4.2.** In other words, for any signed subset \( I \) with signs \( \{\epsilon_1, \ldots, \epsilon_k\} \) we have the following:

(i) If \( I \) is alternatingly signed, then \( \Delta(I) \) is a submodule of \( P(1^\epsilon) \) for odd \( n \) and of \( P(1^\epsilon) \) if \( n \) is even. In all other cases, we do not define \( \Delta(I) \). There will be other \( \Delta \)-filtered modules with \( \Delta\)-support equal to \( I \) but these modules will not be submodules of a single projective module.

(ii) We similarly only define \( \nabla(I) \) if \( I \) is step-alternatingly signed.

From now on we will use the following convention: If we say that \( I \) is signed and we are working with a module \( \Delta(I) \) then we most of the time tacitly assume that the set \( I \) is alternatingly signed. Similarly if we work with \( \nabla(I) \) then the set is step alternatingly signed.

Now we describe the relation between submodules of \( P(1^\pm) \) and subsets of \( [n] \) and between factor modules of \( P(1^\pm) \) and subsets of \( [n] \).

**Lemma 4.3.**

(i) The map sending a submodule \( M \) of \( P(1^+) \) (or \( P(1^-) \)) to its \( \Delta\)-support induces a bijection between the submodules of \( P(1^+) \) (or \( P(1^-) \)) and the subsets of \( [n] \) (ignoring signs).

(ii) The map sending a quotient module \( N \) of \( P(1^+) \) (or \( P(1^-) \)) to its \( \nabla\)-support induces a bijection between the factor modules of \( P(1^+) \) (or \( P(1^-) \)) and the subsets of \( [n] \) (ignoring signs).

In particular, \( P(1^+) \) and \( P(1^-) \) each have precisely \( 2^n \) submodules and \( 2^n \) factor modules.

**Proof.** It is enough to consider (i). Let \( M \) be a submodule of \( P(1^+) \). By Remark 4.2 the map induces a bijection between \( P(1^+) \) and the (alternatingly) signed subsets \( \{i_1^\epsilon, \ldots, i_k^\epsilon\} \) of \( [n] \) with \( \epsilon_k = s((-1)^n) \).

But this is in bijection to the subsets \( \{i_1, \ldots, i_k\} \subset [n] \). \( \square \)

In what follows, we will need to go from a subset of \( [n] \) to a signed subset of \( [n]^\pm \): If we associate to \( I_0 = \{i_1, \ldots, i_k\} \subset [n] \) a \( k \)-tuple \( \epsilon_* = \{\epsilon_1, \ldots, \epsilon_k\} \) of signs, we will call the resulting \( I = \{i_1^\epsilon, \ldots, i_k^\epsilon\} \) a signed version of \( I_0 \) and we say that \( I_0 \) is the unsigned version of \( I \).

**Lemma 4.4.** Let \( I_0 \) be a non-empty subset of \( [n] \). Then there are unique signed versions \( I \) and \( I' \) of \( I_0 \) such that \( \Delta(I) \) is a submodule of \( P(1^+) \) and \( \nabla(I') \) is a factor module of \( P(1^+) \).

Note that the same statements hold for \( P(1^-) \) with “opposite” signs. We leave the (easy) proof to the reader.
Let $I$ and $J$ be signed subsets of $[n]^\pm$. By abuse of terminology we say that $J$ is a complement to $I$ if their unsigned versions $I_0$ and $J_0$ are such that $J_0 = [n] \setminus I_0$. Clearly, the complement to a signed subset is not unique. But we have the following, which is easy to prove, and is analogous to [page 298][BHRR99].

**Lemma 4.5.** Let $I$ be a signed subset of $[n]^\pm$. Assume that $\Delta(I)$ is a submodule of $P(1^+)$. Then there is a unique complement $I^c$ of $I$ such that there is a short exact sequence

$$0 \to \Delta(I) \to P(1^+) \to \nabla(I^c) \to 0.$$  

5. **Results relating $\text{Ext}^*_{A_n}$ to $\text{Ext}^*_{D_n}$**

We have seen that $D_n$ is a skew group ring over $A_n$ which allows us to relate the $\Delta$-filtered modules of these two algebras. In this section, we use induction and restriction to relate $\text{Ext}_{A_n}$ and $\text{Ext}_{D_n}$.

Since $D_n$ is free as module over $A_n$, the adjoint functors given by the $A_n, D_n$ bimodule $D_n$, that is, inducing and corestricting, have good properties: They preserve projectives, so we have Shapiro’s Lemma, $\text{Ext}^*_D(X \otimes_{A_n} D_n, Y) \cong \text{Ext}^*_A(X, Y \downarrow_{A_n})$. (see for example [Be91, 2.8.4]). Furthermore, every $D_n$-module $X$ is relative $A_n$-projective, that is, the multiplication map $X \otimes_{A_n} D_n \to X$ splits (by a ‘Maschke-type’ argument), using char$(k) \neq 2$.

In Section 2 we have defined a partial order on the labels for the simple modules, and have seen that $D_n$ is quasi-hereditary with respect to this order.

**Lemma 5.1.** For each $i^\epsilon$ with $\epsilon = +$ or $\epsilon = -$ we have

(a) $\Delta(i^\epsilon) \downarrow A_n \cong \Delta(i)$ and $\nabla(i^\epsilon) \downarrow A_n \cong \nabla(i)$.

(b) $\Delta(i) \otimes_{A_n} D_n \cong \Delta(i^+) \oplus \Delta(i^-)$ and $\nabla(i) \otimes_{A_n} D_n \cong \nabla(i^+) \oplus \nabla(i^-)$.

(c) Suppose $X$ is any $A_n$-module. Then $X \in \mathcal{F}(\Delta_{A_n})$ if and only if $X \otimes_{A_n} D_n$ belongs to $\mathcal{F}(\Delta_{D_n})$.

**Proof.** Part (a) is easily seen directly.

(b) We know that the multiplication map $\Delta(i^\epsilon) \otimes_{A_n} D_n \to \Delta(i^\epsilon)$ splits. Using part (a), we get that $\Delta(i) \otimes_{A_n} D_n$ has a direct summand isomorphic to $\Delta(i^+)$ and also a direct summand isomorphic to $\Delta(i^-)$. Hence by dimensions, $\Delta(i) \otimes_{A_n} D_n$ must be the direct sum as stated in (b).

(c) For a module $X$ of any quasi-hereditary algebra $\Lambda$, it is known that $X \in \mathcal{F}(\Delta) = \mathcal{F}(\Delta_{\Lambda})$ if and only if $\text{Ext}^1(X, \nabla(j)) = 0$ for all $j$ ([D98 appendix A]).

Now take $\Lambda$ to be $A_n$ or $D_n$, and use Shapiro’s Lemma and part (a),

$$\text{Ext}^1_{A_n}(X, \nabla(j)) \cong \text{Ext}^1_{D_n}(X \otimes_{A_n} D_n, \nabla(j^-))$$

So $X$ has a $\Delta$-filtration if and only if the induced module $X \otimes_{A_n} D_n$ has a $\Delta$-filtration. \hfill \Box

**Corollary 5.2.**

$$\text{Ext}^*_{A_n}(\Delta(i), \Delta(j)) \cong \text{Ext}^*_{D_n}(\Delta(i^+), \Delta(j^-)) \oplus \text{Ext}^*_{D_n}(\Delta(i^-), \Delta(j^+))$$

for a sign $\epsilon$.

**Proof.** Since $\Delta(j^\epsilon) \downarrow A_n \cong \Delta(j)$, by applying Shapiro’s lemma: $\text{Ext}^*_{A_n}(\Delta(i), \Delta(j)) \cong \text{Ext}^*_{D_n}(\Delta(i) \otimes_{A_n} D_n, \Delta(j^-))$. This is isomorphic to $\text{Ext}^*_{D_n}(\Delta(i^+), \Delta(j^\epsilon)) \oplus \text{Ext}^*_{D_n}(\Delta(i^-), \Delta(j^\epsilon))$ using Lemma 5.1(b). \hfill \Box
Suppose \( I \) is an (alternatingly) signed subset of \([n]^{\pm}\). We let \(-I\) denote the signed set which has the same underlying unsigned set \( I_0 \) as \( I \) but with opposite signs to \( I \). That is, \( i^c \in I \) if and only if \( i^c \in -I \).

Recall that we use \( I_0 \) for the unsigned version of \( I \).

**Proposition 5.3.** For \( I \) and \( J \) signed subsets of \([n]^{\pm}\) we have:

(a) \( \Delta(I) \downarrow_{A_n} \cong \Delta(I_0) \)

(b) \( \Delta(I_0) \otimes_{A_n} D_n \cong \Delta(I) \oplus \Delta(-I) \)

(c) \( \text{Ext}^1_{A_n}(\Delta(I_0), \Delta(J_0)) \cong \text{Ext}^1_{D_n}(\Delta(I), \Delta(J)) \oplus \text{Ext}^1_{D_n}(\Delta(-I), \Delta(J)) \).

**Proof.** (a) The module \( \Delta(I) \downarrow_{A_n} \) has a \( \Delta \)-filtration by Lemma 5.1 (a) and induction on filtration length. It also has \( \Delta \)-support equal to \( I_0 \) as an \( A_n \)-module. Since restriction is exact \( \Delta(I) \downarrow_{A_n} \) remains a submodule of \( P(1') \downarrow_{A_n} \) (\( \epsilon \) of the appropriate sign) and hence \( \Delta(I) \downarrow_{A_n} \) is a submodule of \( P(1) \). Thus \( \Delta(I) \downarrow_{A_n} \cong \Delta(I_0) \) as this is the only submodule of \( P(1) \) with the same \( \Delta \)-support.

(b) Using part (a) we may argue similarly to the proof of Lemma 5.1 (b) to show that both \( \Delta(I) \) and \( \Delta(-I) \) are direct summands of \( \Delta(I_0) \otimes_{A_n} D_n \) and hence, by dimensions, this tensor product must be equal to the direct sum.

(c) This result follows as in the proof of the previous corollary. \( \square \)

6. Extensions between the \( \Delta(I) \)

We are now ready to study the extensions between two \( \Delta \)-filtered modules \( \Delta(I) \) and \( \Delta(J) \). In this section, we give a formula for the dimension of \( \text{Ext}^1(\Delta(I), \Delta(J)) \) and \( \text{Hom}(\Delta(I), \Delta(J)) \) over both \( A_n \) and \( D_n \).

Unless stated otherwise, homomorphism and extension spaces are taken over \( D_n \). We will furthermore write \( \text{hom}(A, B) \) for \( \text{dim} \text{Hom}(A, B) \) and \( \text{ext}^1(A, B) \) for \( \text{dim} \text{Ext}^1(A, B) \).

**Lemma 6.1.** In \( A_n \): for \( I_0 \) and \( J_0 \) unsigned subsets of \([n] \), we have

\[
\text{ext}^1_{A_n}(\Delta(I_0), \Delta(J_0)) = \text{hom}_{A_n}(\Delta(I_0), \Delta(J_0)) - \text{hom}_{A_n}(\Delta(I_0), P(1)) + \text{hom}_{A_n}(\Delta(I_0), \nabla(J'_0)).
\]

And in \( D_n \): for \( I \) and \( J \) signed subsets of \([n]^{\pm} \), we have

\[
\text{ext}^1_{D_n}(\Delta(I), \Delta(J)) = \text{hom}_{D_n}(\Delta(I), \Delta(J)) - \text{hom}_{D_n}(\Delta(I), P(1')) + \text{hom}_{D_n}(\Delta(I), \nabla(J'^{\epsilon})).
\]

where \( \epsilon \) is the sign of the largest element in \( J \) if \( n \) is odd and the opposite sign if \( n \) is even.

**Proof.** We prove the signed version — the unsigned version follows similarly. This lemma follows by applying \( \text{Hom}_{D_n}(\Delta(I), -) \) to the following short exact sequence

\[
0 \to \Delta(J) \to P(1') \to \nabla(J^\epsilon) \to 0
\]

from Lemma 4.5 and noting that \( \text{Ext}^1_{D_n}(\Delta(I), P(1')) = 0 \) as \( P(1') \) is a tilting module. \( \square \)

All the terms on the right hand side of the expression in 6.1 are calculable. We will get the first term in Proposition 6.6.
The second term is the sum ([D98 appendix A])

$$\sum_{i' \in [n]^{\geq}} \dim \Delta(I)_{i'} \dim_{\nabla} P(1')_{i'}.$$ 

The $\nabla$-support of $P(1')$ is $\{n', (n-1)', \ldots, 1'\}$ for $n$ odd and $\{n', (n-1)', \ldots, 1'\}$ for $n$ even and so this sum is given by $|I \cap \{n', (n-1)', \ldots, 1'\}|$ if $n$ is odd and $|I \cap \{n', (n-1)', \ldots, 1'\}|$ for $n$ even.

The third term is given by the sum

$$\sum_{i' \in [n]^{\geq}} \dim \Delta(I)_{i'} \dim_{\nabla} \nabla(J')_{i'}$$

and this is equal to the number of elements that are both in $I$ and in $J'$, i.e. $|I \cap J'|$.

**Proposition 6.2.** $\text{ext}^1_{\Delta_n}(\Delta(I), \Delta(J)) = 0$ for all $I_0 \subset J_0$ or $J_0 \subset I_0$.

**Proof.** This follows from the result for the unsigned sets $I_0$ and $J_0$ in [BHRR99] and Proposition 5.3(c).

**Corollary 6.3.** Suppose $M \in F(\Delta)$ with $\Delta$-support $J$. Then $\text{ext}^1_{\Delta_n}(\Delta(I), M) = 0$ and $\text{ext}^1_{\Delta_n}(M, \Delta(I)) = 0$ if $J_0 \subset I_0$.

**Proof.** By the previous lemma $\text{ext}^1_{\Delta_n}(\Delta(I), \Delta(J')) = 0$ and $\text{ext}^1_{\Delta_n}(\Delta(j'), \Delta(I)) = 0$ for $j \in I_0$. Induction on the $\Delta$-filtration length of $M$ then gives the result.

We now focus on calculating $\text{Hom}_{\Delta_n}(\Delta(I), \Delta(J))$ and $\text{Hom}_{\Delta_n}(\Delta(I), \Delta(J))$.

Let us start introducing the necessary notation first. Let $I_0$, $J_0$ be subsets of $[n]$, with $I_0 = \{i_1 > i_2 > \ldots > i_r\}$ and similarly $J_0 = \{j_1 > j_2 > \ldots > j_s\}$.

Call a subset $K$, of $I_0$ an *initial segment* if it is of the form $K := \{i_{r-u} > i_{r-u+1} > \ldots > i_r\}$ for some $u \leq |I|$ (so in total there are $|I|$ nonempty initial segments).

Now define an *order* $\leq$ on the subsets of $[n]$. Let $V$, $W$ be such subsets, say $V = \{v_1 > v_2 > \ldots > v_x\}$ and $W = \{w_1 > w_2 > \ldots > w_y\}$. Then set $V \leq W$ if and only if $x \leq y$ (ie $|V| \leq |W|$)

and $v_1 \leq w_1, v_2 \leq w_2, \ldots, v_x \leq w_x$.

**Proposition 6.4.** Let $I_0 = \{i_1 > i_2 > \ldots > i_r\}$ and $J_0 = \{j_1 > j_2 > \ldots > j_s\}$ be subsets of $[n]$. The dimension of $\text{Hom}_{\Delta_n}(\Delta(I_0), \Delta(J_0))$ is equal to the number of initial segments $K$ of $I_0$ such that $K \leq J_0$.

**Proof.** For the moment we write $I = I_0$ and $J = J_0$. To obtain a homomorphism from $\Delta(I)$ to $\Delta(J)$ we need to map a factor module of $\Delta(I)$ to a submodule of $\Delta(J)$. Factor modules of $\Delta(I)$ which also embed in $\Delta(J)$ must have a $\Delta$-filtration by Lemma 3.1 and hence are given by initial segments $I_u$. Now $\Delta(I_u), I_u = \{i_{r-u+1} > i_{r-u+2} > \ldots > i_r\}$ embeds as a submodule in $\Delta(J)$ if and only if $i_{r-u+1} \leq j_1, i_{r-u+2} \leq j_2, \ldots, i_r \leq j_u$.

**Example 6.5.** If $J_0 = \{n, n-1, \ldots, 1\}$ then $\Delta(J_0) = P(1)$ and all initial segments have the required property and so the dimension of $\text{Hom}_{\Delta_n}(\Delta(I_0), P(1))$ is $|I_0|$.

The signed version of Proposition 6.4 is then:
Proposition 6.6. Let $I = \{i_1 > \cdots > i_s\}$ and $J = \{j_1 > \cdots > j_s\}$ be signed subsets of $[n]^\pm$. The dimension of $\text{Hom}(\Delta(I), \Delta(J))$ is equal to the number of initial segments $K$ of $I$ such that $K \leq J$ and such that the sign of $i_{r-u+1}$, the first element in $K$, is equal to the sign of $j_1$.

Proof. This follows from Proposition 6.4 and the fact that there is a homomorphism $\Delta(i_r^e) \to \Delta(j_s^\delta)$ if and only if $e = \delta$ and $i_{r-u+1} \leq j_1$. □

Example 6.7. If $i < j$ for all $i \in I_0$ and all $j \in J_0$ and $\Delta(J)$ is a submodule of $P(1^\gamma)$ then we have

$$\text{hom}(\Delta(I), \Delta(J)) = \text{hom}(\Delta(I), P(1^\gamma)).$$

7. Ext-result with m gaps

In this section we calculate the extension group between $\Delta(I)$ and $\Delta(J)$ where the underlying unsigned sets for $I$ and $J$ have “$m$ gaps”. We also find a $\Delta$-filtered module with no self-extensions that is an extension of $\Delta(I)$ by $\Delta(J)$. This is the module we will use to build up the $M(d)$ in Section 9. Since we are interested in associating such modules to Richardson orbits we may assume that all the “gaps” only occur on one side of $I$ and that $I$ and $J$ satisfy a symmetry condition, so that $I = \Phi(J)$ defined below.

We will continue to use the notation $I_0$ and $J_0$ for the unsigned versions of the signed subsets $I$ and $J$ of $[n]^\pm$. We now define a map $\Phi$ on both signed and unsigned sets. Let $I_0$ be a unsigned subset of $[n]$. We define

$$\Phi(I_0) = \{n-i+1 \mid i \in I_0\}.$$ 

We now define $\Phi(I)$ to be $\Phi(I_0)$ with signs chosen so that the largest element of $\Phi(I)$ has opposite sign to that of the largest element of $I$.

We now fix a signed subset $I$ of $[n]^\pm$ where the sign of the largest element in $I$ is $+$ so that $\Delta(I)$ has simple socle $L(1^+)$. We set $J = \Phi(I)$ and note that the sign on the largest element in $J$ is $-$. Let $[n] \setminus I_0 = \{a_1, a_2, \ldots, a_m\}$ (in decreasing order) and let $b_j = n + 1 - a_j$ for $1 \leq j \leq m$ so that $[n] \setminus J_0 = \{b_m, b_{m-1}, \ldots, b_1\}$. We note that if $i \in I_0 \setminus J_0$ then $n+1-i \in J_0 \setminus I_0$. We impose a further condition that if $i \in I_0 \setminus J_0$ then $i \geq \frac{n+1}{2}$.

We then choose signs $\epsilon_i$ and $\delta_i$ so that $I^c = \{a_1^\epsilon, a_2^\epsilon, \ldots, a_m^\epsilon\}$ and $J^c = \{b_m^\delta, b_{m-1}^\delta, \ldots, b_1^\delta\}$. We thus have short exact sequences

$$0 \to \Delta(I) \to Q(1^+) \to \nabla(I^c) \to 0, \quad \text{and} \quad 0 \to \Delta(J) \to Q(1^-) \to \nabla(J^c) \to 0$$

where $Q(1^\pm)$ is the injective hull of $L(1^\pm)$.

Lemma 7.1. We have $\text{ext}_{A_n}(\Delta(I_0), \Delta(J_0)) = 0$ and $\text{ext}_{A_n}(\Delta(J_0), \Delta(I_0)) = |J_0 \cap I_0^c|$. 

Proof. This is a matter of calculating the right hand side in Lemma 6.1.

Now, $\text{hom}_{A_n}(\Delta(I_0), \Delta(J_0))$ is the number of overlapping segments. We claim we have $|I_0 \cap J_0|$ overlapping segments. We let $l$ be minimal such that $a_l \in J_0 \setminus I_0$. The assumptions of $I_0$ and $J_0$ imply
that $a_l \leq \frac{n+1}{2}$. Now the last such overlapping segment is:

$$\ldots, i_{s-1}, \ldots, i_s, \ldots, i_{n-m}$$

$$j_1, \ldots, j_t, a_t, \ldots, j_{u-1}, j_u, \ldots, j_{n-m}$$

where $s, t, u$ are appropriate integers and $i_{s-1} > a_t > i_s$ (note we cannot have equality as $a_t \notin I_0$). It is clear that we cannot get any more overlapping segments as $i_s > a_1$. The total number of overlapping segments is thus the amount of overlap in the above diagram. For the calculation, let $r = |J_0 \cap I_0^c|$, the number of gaps in $I_0$ which are not gaps in $J_0$, which is also equal to $|J_0 \cap J_0^c|$. The amount of overlap in the above diagram is equal to:

$$|\{i \in I_0 \mid i < a_l\}| + |\{j \in J_0 \mid j > a_l\}|$$

$$= a_l - 1 - \#\text{gaps after } a_l \text{ in } I_0 + n - a_l - \#\text{gaps before } a_l \text{ in } J_0$$

$$= n - 1 - (m - l) - |\{i \in [n] \mid i \notin I_0 \text{ and } i \notin J_0 \text{ and } i > a_l\}| - |\{i \in [n] \mid i \in I_0 \text{ and } i \notin J_0 \text{ and } i > a_l\}|$$

$$= n - 1 - m + l - (l - 1) - r$$

$$= n - m - r$$

(we have used that $I_0 \cap J_0$ has only weights $<(n+1)/2$). Now $|I_0 \cap J_0| = |I_0| - |I_0 \setminus J_0| = n - m - |I_0 \cap J_0^c| = n - m - r$. Thus $\text{hom}_{A_n}(\Delta(I_0), \Delta(J_0)) = |I_0 \cap J_0|$. Now $\text{hom}_{A_n}(\Delta(I_0), P(1)) = |I_0| = n - m$ and $\text{hom}_{A_n}(\Delta(I_0), \nabla(J_0^c)) = |J_0 \cap J_0^c| = r$. Thus

$$\text{ext}^{1}_{A_n}(\Delta(I_0), \Delta(J_0)) = n - m - r - (n - m) + r = 0.$$

To calculate the other Ext group we consider, $\text{hom}_{A_n}(\Delta(J_0), \Delta(I_0))$ which is the number of overlapping segments. By construction, $j_s \leq i_s$ for all $s$, thus $\Delta(J_0)$ in fact embeds in $\Delta(I_0)$ and the number of overlapping segments is $|J_0| = n - m$. Thus $\text{hom}_{A_n}(\Delta(J_0), \Delta(I_0)) = n - m$. We also have $\text{hom}_{A_n}(\Delta(J_0), P(1)) = |J_0| = n - m$. Hence $\text{hom}_{A_n}(\Delta(J_0), \Delta(I_0)) = \text{hom}_{A_n}(\Delta(J_0), P(1))$ and $\text{ext}^{1}_{A_n}(\Delta(J_0), \Delta(I_0)) = \text{hom}_{A_n}(\Delta(J_0), \nabla(J_0^c))$. Now $\text{hom}_{A_n}(\Delta(J_0), \nabla(I_0^c)) = |J_0 \cap I_0^c| = |I_0 \cap J_0^c|$ thus

$$\text{ext}^{1}_{A_n}(\Delta(J_0), \Delta(I_0)) = |I_0 \cap J_0^c|. \quad \square$$

We now prove the following signed version of the above lemma.

**Lemma 7.2.** Let $|J_0 \cap I_0^c| = r$. We have $\text{ext}^{1}_{D_n}(\Delta(I), \Delta(J)) = 0 = \text{ext}^{1}_{D_n}(\Delta(I), \Delta(-J))$,

$$\text{ext}^{1}_{D_n}(\Delta(J), \Delta(I)) = \begin{cases} \frac{r}{2} & \text{if } r \text{ even} \\ \frac{r+1}{2} & \text{if } r \text{ odd} \end{cases}$$

and

$$\text{ext}^{1}_{D_n}(\Delta(-J), \Delta(I)) = \begin{cases} \frac{r}{2} & \text{if } r \text{ even} \\ \frac{r-1}{2} & \text{if } r \text{ odd.} \end{cases}$$

**Proof.** Since

$$\text{ext}^{1}_{D_n}(\Delta(I), \Delta(J)) + \text{ext}^{1}_{D_n}(\Delta(I), \Delta(-J)) = \text{ext}^{1}_{A_n}(\Delta(J_0), \Delta(J_0)) = 0$$

using Proposition 5.3 (c) and the previous lemma, we have the first part of the lemma.
For the second part, recall from the proof of Lemma 9.1 that

$$\text{Ext}^1(\Delta(J_0), \Delta(I_0)) \cong \text{Hom}(\Delta(J_0), \nabla(I_0^c))$$

Therefore we have, using Lemma 7.3 that

$$\text{Ext}^1_{D_n}(\Delta(J) \oplus \Delta(-J), \Delta(I)) \cong \text{Hom}_{D_n}(\Delta(J) \oplus \Delta(-J), \nabla(I^c))$$

and this has dimension $|I_0 \cap J_0^c| = r$.

We have a surjection $\text{Hom}(\Delta(J), \nabla(I^c)) \rightarrow \text{Ext}^1(\Delta(J), \Delta(I))$, and a similar surjection for $-J$, and by the dimensions these must both be isomorphisms. So we need to find $|J \cap I^c|$, that is, to consider the signs on the $a_i$ in $J$. We start by considering $a_{k_r}$. Note that it must be $< (n+1)/2$.

Assume first that there are elements in $I_0$ which are $< a_{k_r}$, let $i$ be the largest such element. Then in $I$, this $i$ has sign $\epsilon_{k_r}$. Then $i$ is in $J$ (since $i < (n+1)/2$) and has sign $\epsilon_{k_r}$ in $J$. Therefore $a_{k_r}$ has sign $\epsilon_{k_r}$ in $J$. Now assume there is no element in $I_0$ which is $< a_{k_r}$. Then take $i \in I_0$ to be the smallest element, this is then $> a_{k_r}$ and has sign $\epsilon_{k_r}$. Moreover, we must have that $a_{k_r}$ is the smallest element of $J_0$ and then in $J$ its sign is $\epsilon_{k_r}$, and so it again belongs to $J \cap I^c$.

Now if $|a_{k_r} - a_{k_{r+1}}|$ is odd then they have the same sign in $I^c$ and opposite signs in $J$, thus exactly one of them will be in $J \cap I^c$. If $|a_{k_r} - a_{k_{r+1}}|$ is even then they have the opposite sign in $I^c$ and the same signs in $J$, thus again, exactly one of them will be in $J \cap I^c$.

Since $a_{k_r}$ is in $J \cap I^c$ we must have $J \cap I^c = \{a_{k_r-4}, a_{k_r-2}, a_{k_r} \}$ hence

$$\text{ext}^1_{D_n}(\Delta(J), \Delta(I)) = \begin{cases} \frac{r}{2} & \text{if } r \text{ even,} \\ \frac{r-1}{2} & \text{if } r \text{ odd.} \end{cases}$$

We may now use Proposition 5.3 (c) and the previous lemma to obtain:

$$\text{ext}^1_{D_n}(\Delta(-J), \Delta(I)) = \begin{cases} \frac{r}{2} & \text{if } r \text{ even,} \\ \frac{r-1}{2} & \text{if } r \text{ odd.} \end{cases}$$

We now construct an extension of $\Delta(I)$ by $\Delta(\pm J)$ which has no self extensions. Consider the long exact sequence used to calculate the Ext group:

$$0 \rightarrow \text{Hom}(\Delta(\pm J), \Delta(I)) \rightarrow \text{Hom}(\Delta(\pm J), Q(1^+)) \rightarrow \text{Hom}(\Delta(\pm J), \nabla(I^c)) \rightarrow \text{Ext}^1(\Delta(\pm J), \Delta(I)) \rightarrow 0.$$

Thus using the definition of the long exact sequence, we must have that all extensions of $\Delta(I)$ by $\Delta(\pm J)$ are constructed by taking the pullback of an appropriate map from $\Delta(\pm J)$ to $\nabla(I^c)$.

Of course, in general there will be many non-split extensions of $\Delta(I)$ by $\Delta(\pm J)$. We will construct an extension of $\Delta(I)$ by $\Delta(\pm J)$ with no self extensions.

**Proposition 7.3.** Let $I$ be a signed subset of $[n]^\pm$ such that $\Delta(I) \subset Q(1^+)$. We let $J = \Phi(I)$ and impose the further condition that if $i \in I_0 \setminus J_0$ then $i \geq \frac{m+1}{2}$. Then $\text{Hom}_{D_n}(\Delta(J), \nabla(I^c))$ is cyclic as a $\Gamma$-module and $\text{Hom}_{D_n}(\Delta(-J), \nabla(I^c))$ is cyclic as a $\Gamma$-module.
Proof. We prove this for the $J$ case, the $-J$ case is similar. We suppose that $J = \{ j_1^{a_1}, j_2^{a_2}, \ldots, j_{n-m}^{a_{n-m}} \}$ where $a_i$ is of appropriate sign. (In fact $a_i = +$ if $i$ is even and $-$ if $i$ is odd.)

In the proof of Lemma 7.2 we showed that $J \cap I^c = \{ a_{k_1}^{\ell_1}, \ldots, a_{k_r}^{\ell_r}, a_{k_{r+1}} \}$ where $e = 1$ if $r$ is odd, and $e = 2$ otherwise. For each $a_{k_i}^{\ell_i} \in J \cap I^c$ there is a corresponding map $\theta_i$ which restricts to the unique map (up to scalars) $\Delta(a_{k_i}^{\ell_i}) \to \nabla(a_{k_i}^{\ell_i})$ with image $L(a_{k_i}^{\ell_i})$ on the subquotients $\Delta(a_{k_i}^{\ell_i})$ of $\Delta(J)$ and $\nabla(a_{k_i}^{\ell_i})$ of $\nabla(I^c)$. The $\theta_i$’s in fact form a basis for $\text{Hom}_{D_a}(\Delta(J), \nabla(I^c))$ as a $k$-vector space.

We will construct such $\theta_i$ in sufficient detail, and then show that the map $\theta_e$ is a cyclic generator for the hom space as as $\Gamma$ module.

(1) We start with the construction. We fix $i$ and write $a := a_{k_i}$ whose signed version belongs to $J \cap I^c$. For any signed set $K$ we write

$$K_{>a} := \{ j^* \in K : j > a \}, \quad K_{\leq a} := \{ j^* \in K : j \leq a \}.$$ 

Then there are short exact sequences

$$0 \to \Delta(J_{>a}) \to \Delta(J) \xrightarrow{\pi} \Delta(J_{\leq a}) \to 0$$

$$0 \to \nabla(I^c_{\leq a}) \xrightarrow{\kappa} \nabla(I^c) \to \nabla(I^c_{>a}) \to 0.$$

The signed version of $a$ belongs to both $J_{\leq a}$ and to $I^c_{\leq a}$. We will construct a homomorphism $\theta'_i : \Delta(J_{\leq a}) \to \nabla(I^c_{\leq a})$ which comes as before to a non-zero map from $\Delta(a_{k_i}^{\ell_i})$ to $\nabla((a_{k_i}^{\ell_i})$, and then take

$$\theta_i := \kappa \circ \theta'_i \circ \pi.$$

We have inclusions

$$\Delta(I^c_{<a}) \subset \Delta(J_{\leq a}) \subset T(a^*)$$

(with $* = \ell_k$). To see the first inclusion, note that (with $\leq$ the partial order as defined before 8.4) we have $I^c_{\leq a} \leq J_{\leq a}$.

Furthermore, $T(a^*)/\Delta(I^c_{<a})$ is isomorphic to $\nabla(I^c_{<a})$ (which for example one can see working with the factor algebra $D_a$ of $D_a$, for which $T(a^*)$ is $Q(1^*)$, ie is a projective-injective module). Therefore we take for $\theta'_i$ the composition of

$$\Delta(J_{\leq a}) \to \Delta(J_{\leq a})/\Delta(I^c_{<a}) \to T(a^*)/\Delta(I^c_{<a}) \cong \nabla(I^c_{<a})$$

where the first map is the canonical surjection, and the second map is the inclusion. (Each of the modules in this construction has $a^*$ as the unique highest weight with multiplicity one, and the map is non-zero on a vector of this weight so this is a map as required).

Then the kernel of $\theta_i$ has the exact sequence

$$0 \to \Delta(J_{>a}) \to \text{Ker}(\theta_i) \to \Delta(I^c_{<a}) \to 0$$

Furthermore, the kernel is a submodule of $\Delta(J)$ and therefore it has a simple socle. This means

(2) $\ker(\theta_i) = \Delta(N_i)$ where $N_i = J_{>a} \cup I^c_{<a}$ for $a = a_i = a_{k_i}^{\ell_i}$.

Now let $\theta := \theta_e$, then we claim that the kernel of $\theta_i$ is contained in the kernel of $\theta$, for all $i$: We use (2) for $a = a_e$ and also for $a = a_i$. The sets $N_e$ and $N_i$ are both appropriately signed. So to see that $\Delta(N_e) \subseteq \Delta(N_i)$ we only need that the unsigned set $(N_e)_0$ is contained in the unsigned set $(N_i)_0$. 

To show this, we only need the following. If \( j \in I_0 \) and \( a_e > j \geq a_i \) then \( j \in J_0 \). But this holds by the general hypothesis, since we have \( a_e < (n + 1)/2 \).

(3) We can now prove the cyclicity, that is, to show that \( \theta_i = \psi \circ \theta_e \) for some \( \psi \in \Gamma \).

Since \( \ker(\theta_e) \subseteq \ker(\theta_i) \) it follows that \( \theta_i \) maps the kernel of \( \theta_e \) to zero. So there is a homomorphism \( \psi : \nabla(I^c) \to \nabla(I^c) \) with \( \theta_i = \psi \circ \theta_e \).

**Proof.** Using the defining sequence for \( \nabla(I^c) \) we see that for any \( D_n \)-module \( M \) that \( \text{Ext}^1_{D_n}(M, \Delta(I)) \cong \text{Hom}_{D_n}(M, \nabla(I^c)) \) where \( \text{Hom} \) denotes the Hom space modulo homomorphisms that factor through a projective module. Thus as \( \text{Ext}^1_{D_n}(\Delta(J), \Delta(I)) \) is isomorphic to the cyclic \( \Gamma \)-module \( \text{Hom}_{D_n}(\Delta(J), \nabla(I^c)) \), via the induced homomorphism from the long exact sequence and this morphism is compatible with the action of \( \Gamma \), it itself must be cyclic.

The following general lemma from homological algebra is well-known.

**Lemma 7.5.** Assume \( 0 \to A \xrightarrow{\theta} B \xrightarrow{\phi} C \to 0 \) is a short exact sequence of finite-dimensional modules, and let \( \xi \in \text{Ext}^1(C, A) \) represent this sequence. Let also \( \pi^* : \text{Ext}^1(C, A) \to \text{Ext}^1(B, A) \) be the map induced by \( \pi \). Then \( \pi^*(\xi) = 0 \).

We now assume that \( I \) and \( J \) are (alternatingly) signed subsets of \( [n] \) as in the beginning of this section. I.e. \( |I| = n - m \), \( J = \Phi(I) \) and the smallest element of \( J^c \) is larger than \( I^c \).

Let \( \xi \) be a generator of \( \text{Ext}^1_{D_n}(\Delta(\pm J), \Delta(I)) \) as a \( \Gamma \)-module. Now \( \xi \) is the image of some map \( \theta : \Delta(\pm J) \to \nabla(I^c) \) from the long exact sequence. Thus the extension \( \xi \) represents may be taken as the pullback of this map \( \theta \).

We let \( E(I, \pm J) \) be the extension \( \xi \). I.e. it denotes the module with short exact sequence:

\[
0 \to \Delta(I) \to E(I, \pm J) \to \Delta(\pm J) \to 0
\]

constructed by taking the pullback of \( \theta \). We claim the following:

**Proposition 7.6.** The module \( E(I, \pm J) \) has no self-extensions. That is, \( \text{Ext}^1_{D_n}(E(I, \pm J), E(I, \pm J)) = 0 \).

**Proof.** We prove this for the case where \( E(I, J) \) is an extension of \( \Delta(I) \) by \( \Delta(J) \), the \( \Delta(-J) \) case follows similarly.

It is clear that \( \text{ext}^1_{D_n}(E(I, J), \Delta(J)) = 0 \), since both \( \text{ext}^1_{D_n}(\Delta(I), \Delta(J)) \) and \( \text{ext}^1_{D_n}(\Delta(J), \Delta(J)) \) are zero. So it is enough to show that \( \text{ext}^1_{D_n}(E(I, J), \Delta(I)) = 0 \).

The map \( \theta \) is chosen so that its image \( \xi \) is a generator for \( \text{Ext}^1_{D_n}(\Delta(J), \Delta(J)) \) as a module for \( \text{End}_{D_n}(\nabla(I^c)) \).

Apply \( \text{Hom}_{D_n}(\Delta(I)) \) to the short exact sequence defining \( E(I, J) \), this gives

\[
\ldots \to \text{Ext}^1_{D_n}(\Delta(J), \Delta(I)) \xrightarrow{\phi^*} \text{Ext}^1_{D_n}(E(I, J), \Delta(I)) \to \text{Ext}^1_{D_n}(\Delta(I), \Delta(I)) = 0.\]

We can view these \( \text{Ext} \) groups as \( \text{End}_{D_n}(\nabla(I^c)) \) \( \Gamma \)-modules as in the proof of corollary 7.4. The map \( \pi^* \) is a homomorphism of \( \Gamma \)-modules. It takes \( \xi \) to zero. Thus as \( \text{Ext}^1_{D_n}(\Delta(J), \Delta(I)) \) is cyclic as module for \( \Gamma \) with generator \( \xi \) it follows that \( \pi^* = 0 \) and that \( \text{Ext}^1_{D_n}(E(I, J), \Delta(I)) = 0 \).
8. A characterisation of $E(I, \pm J)$

We will split the modules $M \in \mathcal{F}(\Delta)$ as follows.

(1) We say that $M \in \mathcal{F}(\Delta)$ is a type I module if it is a direct sum of modules of the form $\Delta(I_j)$ for some signed sets $I_j$.

(2) We say that $M \in \mathcal{F}(\Delta)$ is a type II module if it is not a direct sum of modules of the form $\Delta(I_j)$ for some signed sets $I_j$.

The indecomposable type I modules are already classified. Potentially there may be indecomposable type II modules with $L(1\pm)$ occurring more than twice in their socles. To classify the Richardson orbits however, we will only need indecomposable type II modules with at most two simples in their socles.

The extension $E(I, \pm J)$ from the previous section is constructed as the pullback of a map $\theta$ as in the previous section. Thus we have the following pullback diagram:

$$
\begin{array}{c}
\begin{array}{ccc}
0 & \rightarrow & \Delta(I) \\
\ker \theta & \rightarrow & \ker \theta \\
\downarrow & & \downarrow \\
0 & \rightarrow & \Delta(I) \rightarrow E(I, \pm J) \rightarrow \Delta(\pm J) \rightarrow 0
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
\downarrow & & \downarrow \\
0 & \rightarrow & \Delta(I) \rightarrow Q(1^+) \rightarrow \nabla(I^c) \rightarrow 0
\end{array}
\end{array}
$$

where $E(I, \pm J) = \{(a, b) \in Q(1^+) \oplus \Delta(\pm J) \mid \theta(b) = \pi(a)\}$ (M63). Note that this implies that $\ker \theta$ is a submodule of $E(I, \pm J)$.

We continue with the case $\Delta(J)$, which is the one of interest for the application, and we use the notation as in 7.3 (the other case is similar). We have seen there that $\ker \theta = \Delta(\tilde{J})$, $\tilde{J} := N_e = J_{>a_k} \cup I_{<a_k}$. We can also identify the cokernel of $\theta$ from the construction in 7.3; it is $\nabla(I_{>a_k} \cup J_{<a_k})$. It is isomorphic to the cokernel of $\tilde{\theta}$, and hence the $\Im \tilde{\theta} = \Delta(\tilde{I})$, $\tilde{I} = I_{>a} \cup J_{<a}$. Thus $E(I, J)$ also has a short exact sequence

$$0 \rightarrow \Delta(\tilde{J}) \rightarrow E(I, J) \rightarrow \Delta(\tilde{I}) \rightarrow 0.$$

Sometimes this sequence will split and $E(I, J)$ will decompose. There are cases, however, where this sequence does not split and $E(I, J)$ is in fact indecomposable. This extension $E(I, J)$ is then not of type I. This is in contrast to the results of [BHRR99] where all $\Delta$-filtered modules were of this type. We now want to show that $E(I, J)$ is in half of the cases indecomposable, i.e. that it really is type II.

Recall that $J \cap I^c = \{a_{k_e}, \ldots, a_{k_{e-2}}, a_{k_r}^e\}$ where $e = 1$ if $r$ is odd and $e = 2$ if $r$ is even.

It is clear that $\tilde{J}_0 = \{i \in I_0 \mid i > a_{k_e}\} \cup \{j \in J_0 \mid j < a_{k_r}\}$ and $\tilde{J}_0 = \{j \in J_0 \mid j > a_{k_e}\} \cup \{i \in I_0 \mid i < a_{k_r}\}$. Thus $\tilde{J}_0 = J_0 \cap I_0$ if $a_{k_e}^e \in J \cap I^c$, i.e. if $e = 1$. Otherwise $\tilde{J}_0 \setminus I_0 = \{a_{k_e}^e\}$.

**Lemma 8.1.** $\text{Ext}^1_{A_n}(\Delta(\tilde{I}_0), \Delta(\tilde{J}_0)) = 0$ if and only if $I_0 = I_0 \cup J_0$ and $\tilde{J}_0 = J_0 \cap I_0$.

**Proof.** Clearly, if $I_0 = I_0 \cup J_0$ and $\tilde{J} = I_0 \cap J_0$ then $\text{Ext}^1_{A_n}(\Delta(\tilde{I}), \Delta(\tilde{J})) = 0$ by Proposition 6.2. To prove the converse, we actually calculate the Ext group directly. Now, $\text{Hom}_{A_n}(\Delta(\tilde{I}_0), \Delta(\tilde{J}_0))$ can be calculated.
Lemma 8.2.

Proof. and ˜ segment in the previous proof. The sign on the last element of ˜ is + if

The sign on the last element of ˜ is + if J is even and − if J is odd.

We also know that the parities of [J] and [J] are equal. Thus the signs on the last two elements of ˜ and ˜ are always opposite. Hence hom_{Dn}(∆( ˜ I), ∆( ˜ J)) is [−m−r+1].

Now hom_{Dn}(∆( ˜ I), P(1−)) = [−m−r+1]. Also hom_{Dn}(∆( ˜ I), ∇( ˜ Jc)) = [ ˜ I ∩ ˜ Jc].

with appropriate signs by a similar argument to the one in the proof of lemma 7.2. Thus hom_{Dn}(∆( ˜ I), ∇( ˜ Jc)) = r and ext^1_{Dn}(∆( ˜ I), ∆( ˜ J)) = [−m−r+1] − [−m−r+1] + r = 1.

We collect information on a possible direct sum decomposition of E(I, ±J). We take E(I, J) in the form E(I, J) = {(x, y) ∈ Q(1^+) ⊕ Δ(J) | π(x) = θ(y)}, the case with − J is similar.

Suppose E(I, J) is a direct sum. The socle of E(I, J) is contained in soc(Δ(I) ⊕ Δ(J)), and it follows that each summand has a simple socle L(1±) and their socles are not isomorphic. Moreover, the summands have ∆-filtration, so E(I, J) = Δ(L1) ⊕ Δ(L2) for some signed sets L1, L2 such that L1 ∪ L2 = I ∪ J.

It follows then that Δ(I) is isomorphic to a submodule of one of Δ(L1) or Δ(L2). Consider the inclusion j : Δ(I) → E(I, J), it is 1-1 and the socle of Δ(I) is simple. Let p_i : E(I, J) → Δ(Li) be the projection onto the summand Δ(Li) then one of p_i o j must be non-zero on the socle of Δ(I) and if so then p_i o j is 1-1. We pick our indices so that Δ(I) is a submodule of Δ(L1), then Δ(L2) is a submodule of Δ(J), as the map q must be 1-1 restricted to the socle of Δ(L2).

Similarly, using the other short exact sequence for E(I, J), Δ( ˜ J) is a submodule of one of the summands Δ(L1) or Δ(L2) of E(I, J). By matching the socles, we see that Δ( ˜ J) ⊂ Δ(L2) and Δ(L1) ⊂ Δ( ˜ I).
We remark that in the special case that $\theta$ is injective, then $E(I, J) \cong \Delta(\tilde{I})$ and hence is indecomposable. When $\theta$ is not injective, we have the following theorem describing exactly when $E(I, J)$ decomposes.

**Theorem 8.3.** Assuming that $\theta$ is not injective, then the following are equivalent.

(i) $E(I, J)$ is decomposable;

(ii) $E(I, J) = \Delta(I_1) \oplus \Delta(I_2)$ for some signed subsets, $I_1$ and $I_2$;

(iii) $E(I, J) = \Delta(\tilde{I}) \oplus \Delta(\tilde{J})$;

(iv) $r$ is odd.

**Proof.** Clearly (iii) $\Rightarrow$ (ii) and (i) $\Leftrightarrow$ (ii) by the discussion about the socle of $E(I, J)$.

Also (iv) $\Rightarrow$ (iii) as then $\text{Ext}^1_{D_n}(\Delta(\tilde{I}), \Delta(\tilde{J})) = 0$ and the sequence for $E(I, J)$ splits.

Next (iii) $\Rightarrow$ (iv). If (iv) is not true then $\text{Ext}^1_{D_n}(E(I, J), E(I, J)) = \text{Ext}^1_{D_n}(\Delta(\tilde{I}), \Delta(\tilde{I})) \oplus \text{Ext}^1_{D_n}(\Delta(\tilde{J}), \Delta(\tilde{I})) \oplus \text{Ext}^1_{D_n}(\Delta(\tilde{J}), \Delta(\tilde{J})) \neq 0$ contradicting that $E(I, J)$ has no self extensions.

Thus it remains to prove that (ii) $\Rightarrow$ (iii). Recall that $\Delta(\tilde{J}) \subset \Delta(L_2) \subset \Delta(J)$ and $\Delta(I) \subset \Delta(L_1) \subset \Delta(\tilde{I})$ by the discussion preceding this proof. We will now show that $\Delta(L_2) \subset \ker \theta \cong \Delta(\tilde{J})$. Now $E(I, J) \cong \Delta(L_1) \oplus \Delta(L_2)$, and $\text{Ext}^1_{D_n}(E(I, J), \Delta(I)) = 0$ we must have $\text{Ext}^1_{D_n}(\Delta(L_2), \Delta(I)) = 0$. Hence we have a short exact sequence

$$0 \to \text{Hom}_{D_n}(\Delta(L_2), \Delta(I)) \to \text{Hom}_{D_n}(\Delta(L_2), P(1^+)) \to \text{Hom}_{D_n}(\Delta(L_2), \nabla(I^c)) \to 0.$$ 

Now $\Delta(L_2) \subset \Delta(J)$. As a $A_n$-module $\Delta(J)$ embeds in $\Delta(I)$ thus so does $\Delta(L_2)$. Hence

$$\text{hom}_{D_n}(\Delta(L_2), \Delta(I)) = \left\lfloor \frac{|L_2|}{2} \right\rfloor.$$ 

Also,

$$\text{hom}_{D_n}(\Delta(L_2), P(1^+)) = \left\lfloor \frac{|L_2|}{2} \right\rfloor.$$ 

Hence by dimensions $\text{Hom}_{D_n}(\Delta(L_2), \nabla(I^c)) = 0$. Since there are no homomorphisms from $\Delta(L_2)$ to $\nabla(I^c)$ and $\Delta(L_2)$ is a submodule of $\Delta(J)$, $\Delta(L_2)$ must be contained in the kernel of $\theta$. Thus $\Delta(L_2) \subset \ker \theta \cong \Delta(\tilde{J})$. As $\Delta(\tilde{J}) \subset \Delta(L_2)$, by dimensions we must have $\Delta(\tilde{J}) \cong \Delta(L_2)$. Thus $\tilde{J} = L_2$. Since $L_1 = I \cup J \setminus L_2 = \tilde{I}$ as a multiset we also have $\Delta(L_1) \cong \Delta(\tilde{I})$. □

**Example 8.4.** Let $I = \{8^+, 7^-, 6^+, 5^-, 4^+, 1^+\}$ and $J = \{8^-, 5^+, 4^-, 3^+, 2^-, 1^+\}$. This is an example with $m = 2$ gaps. We have $I^c = \{3^-, 2^-\}$ and $J^c = \{7^+, 6^+\}$. If we take $\theta : \Delta(J) \to \nabla(I^c)$ which has image $\nabla(2^-)$, then the pullback of $\theta$, the extension $E(I, J)$ has short exact sequence:

$$0 \to \Delta(\tilde{J}) \to E(I, J) \to \Delta(\tilde{I}) \to 0$$

where $\tilde{I} = \{8^+, 7^-, 6^+, 5^-, 4^+, 2^-, 1^+\}$ and $\tilde{J} = \{8^-, 5^+, 4^-, 3^+, 1^+\}$. This sequence is non-split, if it did split then $L(3^+)$, which is in the head of $\Delta(\tilde{J})$ would be in the head of $E(I, J)$. But $L(3^+)$ is not in the head of either $\Delta(\tilde{I})$ nor $\Delta(J)$ and so it cannot be in the head of $E(I, J)$. Thus $E(I, J)$ is an example of a type II extension.
9. Constructing \( M = M(\mathbf{d}) \) with \( \text{Ext}^1_{D_n}(M, M) = 0 \)

Let \( \mathbf{d} = (d_1, \ldots, d_n) \) be a symmetric dimension vector (i.e. \( d_i = d_{n+1-i} \)). Let \( \sum d_i = N \). Then \( P = P(\mathbf{d}) \) is a parabolic subgroup of \( \text{SO}_N \). The goal of this section is to construct a \( \Delta \)-filtered module \( M \). In Theorem 9.6 we show that the \( \Delta \)-dimension vector of \( M \) when restricted to \( A_n \) is \( \mathbf{d} \) and that \( \text{Ext}^1_{D_n}(M, M) = 0 \).

We remark that in general, there are infinitely many (isomorphism classes of) \( A_n \)-modules having \( \Delta \)-support \( \mathbf{d} \). In particular, the representation type of \( \mathcal{F}(\Delta) \) for \( A_n \) is wild for \( n \geq 6 \), [DR90] Proposition 7.2.

If we have an arbitrary parabolic subgroup \( P = P(\mathbf{d}) \) in \( \text{SO}_N \) we want to associate to it a module \( M = M(\mathbf{d}) \) for \( D_n \) which has no self-extensions.

In order to construct such a module \( M \) without self-extension, we use the knowledge of Richardson orbits for parabolic subgroups of \( \text{SO}_N \). Constructions of Richardson elements for parabolic subgroups of the classical groups have been given in [Ba06] (under certain restrictions) and in [BG08] for all parabolic subgroups of \( \text{SO}_{2N}, \text{SO}_{2N+1} \) and \( \text{Sp}_{2N} \) (the symplectic group). Our construction here is similar to the one in Definition 3.1 in [BG08].

The idea is to start with a symmetric dimension vector \( \mathbf{d} \) and construct from it a finite sequence of dimension vectors \( \mathbf{e}^k \) \( (k = 1, 2, \ldots \) if \( N \) is even, \( k \geq 0 \) if \( N \) is odd) such that the sum of the \( \mathbf{e}^k \) is equal to \( \mathbf{d} \). Then to each \( \mathbf{e}^k \) we associate a \( \Delta \)-filtered module \( M(\mathbf{e}^k) \) which has no self-extensions. The modules \( M(\mathbf{e}^k), k \geq 1 \), are either type I modules - in this case the direct sum of two modules \( \Delta(I) \) and \( \Delta(J) \) for some subsets \( I \) and \( J \) or indecomposable type II modules as in the Section 8. The module \( M(\mathbf{e}^0) \), if present, is \( P(1^+) \).

In a third step, we add all the \( M(\mathbf{e}^k) \) and show that the sum \( M(\mathbf{d}) := \oplus_k M(\mathbf{e}^k) \) has the desired property, i.e. (i) that the \( \Delta \)-dimension vector of \( M(\mathbf{d}) \) is \( \mathbf{d} \) and (ii) that \( M(\mathbf{d}) \) has no self-extensions. The precise definition is given in Definition 9.3.

Let \( \mathbf{d} = (d_1, \ldots, d_n) \) be a symmetric dimension vector. The algorithm to obtain a module with \( \Delta \)-support \( \mathbf{d} \) is the following. For \( k = 1, 2, \ldots, m \) (where \( m = m(\mathbf{d}) \in \mathbb{N} \) is roughly half of the maximal entry of \( \mathbf{d} \)) we define dimension vectors \( \mathbf{f}^k, \mathbf{g}^k, \mathbf{e}^k \) using \( \mathbf{d}^k \). To start we let \( k = 0 \) and set \( \mathbf{d}^0 = \mathbf{d} \).

(0) If \( \sum d_i = N \) is odd, let \( \mathbf{e}^0 = (e_1^0, \ldots, e_n^0) = (1, 1, \ldots, 1) \) and replace \( \mathbf{d}^0 \) by \( \mathbf{d}^0 - \mathbf{e}^0 \).

(1) Assume that \( \mathbf{d}^0, \mathbf{d}^1, \ldots, \mathbf{d}^k \) are defined.

(2) Define \( \mathbf{f}^{k+1} = ((f^{k+1})_1, \ldots, (f^{k+1})_n) \) and \( \mathbf{g}^{k+1} = ((g^{k+1})_1, \ldots, (g^{k+1})_n) \) by setting

\[
(f^{k+1})_i := \begin{cases} 1 & \text{if } (d^k)_i \geq 2 \\ 1 & \text{if } (d^k)_i = 1 \text{ and } i < \frac{n+1}{2} \\ 0 & \text{else}; \end{cases}
\]

\[
(g^{k+1})_i := \begin{cases} 1 & \text{if } (d^k)_i \geq 2 \\ 1 & \text{if } (d^k)_i = 1 \text{ and } i > \frac{n+1}{2} \\ 0 & \text{else}. \end{cases}
\]
Example 9.4. We illustrate the construction with the examples $J_{\Delta}$.

Let $d^{k+1} := (e^{k+1})_1, \ldots, (e^{k+1})_n$ where $(e^{k+1})_i := (f^{k+1})_i + (g^{k+1})_i$. And then set $d^{k+1} := d^k - e^{k+1}$. If $d^{k+1} = (0, 0, \ldots, 0)$ we are done. Otherwise, we continue this procedure by going back to step (1).

This gives a sequence of dimension vectors $e^k$, $k \geq 1$, with entries at most 2 and such that $\sum e^k = d$ (coordinate-wise sum). If $N$ is odd, there is in addition a dimension vector $e^0$ consisting only of 1’s. Also, we obtain two decreasing sequences $I_0^k$, $J_0^k$ of subsets of $[n]$ as follows:

For $k \geq 1$ define $I_0^k$ and $J_0^k$ to be the support of $f^k$ and of $g^k$ respectively i.e. $I_0^k := \{i \mid (f^k)_i \neq 0\}$ and let $J_0^k := \{i \mid (g^k)_i \neq 0\}$, always in decreasing order. I.e. if the first entry of $f^k$ (or of $g^k$) is non-zero, $I_0^k$ (or $J_0^k$, respectively) contains $n$. If the second entry of $f^k$ is non-zero, $I_0^k$ contains $n - 1$, etc. For odd $N$ set $I_0^0 = [n] = \{n, n - 1, \ldots, 1\}$.

**Remark 9.1.** Note that we have $I_0^0 \supset I_0^{k+1}$ and $J_0^0 \supset J_0^{k+1}$ for $k = 1, 2, \ldots$, and if $N$ is odd, $I_0^0 \supset I_0^1$, $I_0^1 \supset J_0^1$. Furthermore, the support of $d$ (i.e. the set of indices of the nonzero entries) is equal to $I_0^1 \cup J_0^1$ if $N$ is even and equal to $I_0^0$ if $N$ is odd.

**Lemma 9.2.** We have $I_0^k \subset J_0^{k-1}$ and $J_0^k \subset I_0^{k-1}$ for $k \geq 2$.

**Proof.** Consider the vectors $f^k$ and $g^k$. By definition $(f^k)_i = (g^k)_i$ unless $(d^k)_i = 1$. Thus $I_0^k \cap J_0^k = \{i \mid (f^k)_i = (g^k)_i = 1\}$. Also, by construction, $(f^k)_i \neq 0$ implies that $(f^{k+1})_i \neq 0$. (This is why $I_0^k \subset I_0^{k-1}$ and $J_0^k \subset J_0^{k-1}$.) Thus if $i \in I_0^k \cap J_0^k$ then $i$ is in both $I_0^{k-1}$ and $J_0^{k-1}$.

If $i \in I_0^k \setminus J_0^k$ then $(f^k)_i = 1$ and $(g^k)_i = 0$. Thus $(d^k)_i = 1$ and $(d^{k+1})_i = 3$. Hence $(f^{k+1})_i = (g^{k-1})_i$, and $i$ is in both $I_0^{k-1}$ and $J_0^{k-1}$.

Hence $I_0^k \subset J_0^{k-1}$. Similarly $J_0^k \subset I_0^{k-1}$. \hfill $\Box$

For each of the $I_0^k$, $J_0^k$, $k \geq 1$ we let $I^k$ and $J^k$ be signed versions such that $\Delta(I^k) \subset Q(1^+)$ and $\Delta(J^k) \subset Q(1^-)$. In particular, the largest entry of $I^k$ has a positive sign and the largest entry of $J^k$ has negative sign. If $N$ is odd, let $I^0$ be the signed subset such that $\Delta(I^0) = P(1^+)$. 

**Definition 9.3.** Let $d$ be a symmetric dimension vector and $e^k$ the vectors as defined above. The module $M(d)$ is then defined as follows: In the case where $N$ is odd, we set $M(e^0) := Q(1^+)$. For $k \geq 1$:

If $I_0^k = J_0^k$ we define $M(e_k) := \Delta(I^k) \oplus \Delta(J^k)$. Otherwise, $M(e^k)$ is defined to be the unique module obtained from $\text{Ext}^1_{\mathcal{O}_N}(\Delta(I^k), \Delta(J^k))$ with no self-extensions as in Proposition 7.7. Then the module $M(d)$ is set to be the sum of all $M(e^k)$:

$$M(d) \quad \begin{cases} = \bigoplus \limits_{k \geq 1} M(e^k) & \text{if } N \text{ is even} \\ = \bigoplus \limits_{k \geq 0} M(e^k) & \text{if } N \text{ is odd} \end{cases}$$

**Example 9.4.** We illustrate the construction with the examples $d = (1, 3, 5, 4, 5, 3, 1)$ and $d = (1, 3, 5, 3, 5, 3, 1)$.

(a) $d = (1, 3, 5, 4, 5, 3, 1)$.

In a first step,

$f^1 = (1, 1, 1, 1, 1, 1, 0)$ and $g^1 = (0, 1, 1, 1, 1, 1, 1)$,

$f^2 = (0, 1, 1, 1, 0, 0)$ and $g^2 = (0, 0, 1, 1, 1, 0)$,
\[ \mathbf{f}^3 = (0,0,1,0,0,0,0) \text{ and } \mathbf{g}^3 = (0,0,0,0,1,0,0). \]

From this we get
\[ \mathbf{e}^1 = (1,2,2,2,2,2,1), \]
\[ \mathbf{e}^2 = (0,1,2,2,2,1,0), \]
\[ \mathbf{e}^3 = (0,0,1,0,1,0,0). \]

The signed subsets are
\[ I^1 = \{ 7^+, 6^-, 5^+, 4^-, 3^+, 2^- \} \text{ and } J^1 = \{ 6^-, 5^+, 4^-, 3^+, 2^-, 1^+ \}, \]
\[ I^2 = \{ 6^+, 5^-, 4^+, 3^- \} \text{ and } J^2 = \{ 5^-, 4^+, 3^-, 2^+ \}, \]
\[ I^3 = \{ 5^+ \} \text{ and } J^3 = \{ 3^- \}. \]

From this, \[ M(\mathbf{d}) = M(\mathbf{e}^1) \oplus M(\mathbf{e}^2) \oplus M(\mathbf{e}^3). \] All the \( M(\mathbf{e}^k) \) are obtained as extensions.

(b) \( \mathbf{d} = (1,3,5,3,5,3,1) \).

In a first step,
\[ \mathbf{e}^0 = (1,1,1,1,1,1,1), \]
\[ \mathbf{f}^1 = (0,1,1,1,1,1,0) \text{ and } \mathbf{g}^1 = (0,1,1,1,1,1,0), \]
\[ \mathbf{f}^2 = (0,0,1,0,1,0,0) \text{ and } \mathbf{g}^2 = (0,0,1,0,1,0,0). \]

So this gives
\[ \mathbf{e}^1 = (0,2,2,2,2,2,0), \]
\[ \mathbf{e}^2 = (0,0,2,0,2,0,0), \]

The signed subsets are
\[ I^0 = \{ 7^+, 6^-, 5^+, 4^-, 3^+, 2^- \}, \]
\[ I^1 = \{ 6^+, 5^-, 4^+, 3^- \} \text{ and } J^1 = \{ 6^-, 5^+, 4^-, 3^+, 2^- \}, \]
\[ I^2 = \{ 5^+, 3^- \} \text{ and } J^2 = \{ 5^-, 3^+ \}. \]

From this, \[ M(\mathbf{d}) = M(\mathbf{e}^0) \oplus M(\mathbf{e}^1) \oplus M(\mathbf{e}^2) \] with \( M(\mathbf{e}^0) = P(1^+) \), \( M(\mathbf{e}^1) = \Delta(I^1) \oplus \Delta(J^1) \) and \( M(\mathbf{e}^2) = \Delta(I^2) \oplus \Delta(J^2) \).

**Lemma 9.5.** We have \( \text{Ext}^1_{\Delta_n}(M(\mathbf{e}^k), M(\mathbf{e}^k)) = 0 \) for all \( k \).

**Proof.** If \( I^k_0 = I^k_0 \) then \( M(\mathbf{e}^k) \) is a sum of two type I modules with identical support: \( M(\mathbf{e}^k) = \Delta(I^k) \oplus \Delta(J^k) \) with \( I^k_0 = J^k_0 \). By Proposition 6.2 this has no self extensions. In the other case the claim follows from Proposition 7.6. \( \square \)

**Theorem 9.6.** Let \( M(\mathbf{d}) \) be the module as constructed above. Then we have

(i) \( M(\mathbf{d}) \) has no self-extensions;

(ii) The \( \Delta_{A_n} \)-support of \( M(\mathbf{d}) \downarrow_{A_n} \) is \( \mathbf{d} \).

**Proof.** (i) Using Lemma 9.2 and applying Proposition 6.2 we see that \( \text{Ext}^1_{\Delta_n}(\Delta(I^k), \Delta(J^l)) = 0 \) and \( \text{Ext}^1_{\Delta_n}(\Delta(J^k), \Delta(I^l)) = 0 \) for all \( k \neq l \). Thus using the short exact sequences for \( M(\mathbf{d}^k) \) and \( M(\mathbf{d}^l) \) we see that \( \text{Ext}^1_{\Delta_n}(M(\mathbf{d}^k), M(\mathbf{d}^l)) = 0 \) for all \( k \neq l \). Since, by construction \( \text{Ext}^1_{\Delta_n}(M(\mathbf{d}^k), M(\mathbf{d}^l)) = 0 \), it follows that \( \text{Ext}^1_{\Delta_n}(M(\mathbf{d}), M(\mathbf{d})) = 0 \).

(ii) follows from the construction. \( \square \)
Remark 9.7. We observe that in the case where for some \( k \geq 1 \) the subsets \( I^k_i \) and \( J^k_i \) are different then this procedure involves a choice of signs: the signs of \( I^k \) and \( J^k \) are given by the requirement that \( \Delta(I^k) \) is a submodule of \( Q(1^+) \) and that \( \Delta(J^k) \subset Q(1^-) \). We could equally well take \(-I^k\) and \(-J^k\) instead.

This ambiguity arises unless all \( d_i \) are even. If all the \( d_i \) are even, the module \( M(d) \) is induced by the module \( \Delta(d) \) from Section 2 of [BHRR99], \( M(d) = \Delta(d) \otimes_{A_n} D_n \).

Lemma 9.8. Let \( d = (d_1, \ldots, d_n) \) be a symmetric dimension vector. Let \( s_{\text{odd}} \) be the number of different odd entries of \( d \) and \( s_{\text{even}} \) the number of different even entries of \( d \).

If \( N \) is even then there are \( 2^{s_{\text{odd}}} \) different \( D_n \)-modules without self-extensions such that their restriction to \( A_n \) has \( \Delta \)-support \( d \).

If \( N \) is odd then there are \( 2^{k+s_{\text{even}}} \) different \( D_n \)-modules without self-extensions such that their restriction to \( A_n \) has \( \Delta \)-support \( d \).

Proof. We will proof the case of even \( N \). For the odd case, observe that \( M(e^0) = Q(1^+) \) and \( Q(1^-) \) both have the same \( \Delta \)-support \([n]\) when restricted to \( A_n \). Thus we are left to understand \( \oplus_k M(e^k) \) for \( k > 0 \). This is equivalent to consider the module \( M(d - e^0) \) and \( N - n \). This number is even, since \( n \) is odd for odd orthogonal groups. The number of odd entries of \( d - e^0 \) is just the number of even entries of \( d \). So the case of odd \( N \) reduces to the even case after subtracting 1 from all the \( d_i \).

Let now \( N \) be even. Set \( \tilde{m} \) to be the smallest entry of \( d \). If it is even, let \( m := \tilde{m}/2 \). Then the algorithm obtains all \( e^1, \ldots, e^m \) dimension vectors consisting only of 2's and the corresponding modules \( M(e^k) \) are \( \Delta(I^k) \oplus \Delta(J^k) = \Delta(I^k) \oplus \Delta(-I^k) \), \( 1 \leq k \leq m \). So the first \( m \) summands of \( M(d) \) are uniquely determined and we can ignore them: W.l.o.g. let the minimal entry \( \tilde{m} \) of \( d \) be odd. Then \( I^k_i \neq J^k_i \), i.e. \( \Delta(I^k) \neq \Delta(-J^k) \). In particular, if we set \( \tilde{M}(e^1) \) to be the unique module obtained from \( \text{Ext}_{D_n}^1(\Delta(-I^k), \Delta(-J^k)) \) with no self-extensions as in Proposition 7.6, then the restrictions to \( A_n \) of \( M(e^1) \) and of \( \tilde{M}(e^1) \) are identical, but \( M(e^1) \neq \tilde{M}(e^1) \).

Now by step (3) of the algorithm, the remaining dimension vector is \( d^2 \). Let \( \tilde{m}_2 \) be its minimal entry. If it is even, the algorithm produces \( \tilde{m}_2/2 \) vectors \( e^k \) whose entries are only 0s and 2s. The corresponding \( I^k \) are equal to \(-J^k\) and thus again we have \( M(e^k) = \Delta(I^k) \oplus \Delta(J^k) = \Delta(I^k) \oplus \Delta(-I^k) \). So the only interesting thing happens when \( \tilde{m}_2 \) is odd. In that case, the same reasoning as above shows that there are two \( D_n \)-modules with identical restriction to \( A_n \).

Therefore, each different odd entry of \( d \) produces a pair of \( D_n \)-modules with no self-extensions and with identical \( \Delta \)-support when restricted to \( A_n \). \( \square \)

As an illustration of this: in Example 9.3 part (a) for each of the summands \( M(e^k) \) there is another module with same \( \Delta \)-support when restricted to \( A_n \). In part (b), only for \( M(e^0) \) there is another module with the same \( \Delta \)-support when restricted to \( A_n \), namely \( Q(1^-) \).
Appendix A. $R_n$, $S_n$ and their Auslander Algebras

We fix a field $k$ of characteristic $\neq 2$. Recall that we have defined $R_n := k[T]/T^n$. The algebra $R_n$ has precisely $n$ indecomposable modules, of dimensions $1, 2, \ldots, n$. Following [BHRR99] we write $M(i)$ for the indecomposable module of dimension $n - i + 1$.

We work with right modules, and we write maps to the left. The Auslander algebra of $R_n$ is then by definition the algebra $A_n := \text{End}(M)$ where $M := \bigoplus_{i=1}^n M(i)$. In [BHRR99], a presentation by quiver and relations is given. Since we will need it in the proof of Proposition A.2, we give explicit generators for $A_n$.

For each $i$ with $1 \leq i \leq n$ take a basis of $M(i)$,

$$\{b_j^{(i)} : 1 \leq j \leq n - i + 1\}$$

such that $(b_j^{(i)})T = b_{j+1}^{(i)}$ (with the convention that $b_{n+2}^{(i)} = 0$). Then $M$ has basis $B$, the union of all these bases. The algebra $A_n$ is generated, as an algebra, by inclusion maps $\alpha_{a-1} : M(a) \rightarrow M(a-1)$ together with maps $\beta_a : M(a) \rightarrow M(a+1)$, which are surjections. We fix such maps explicitly, as

$$\alpha_{a-1}(b_j^{(a)}) := b_{j+1}^{(a-1)}, \quad \beta_a(b_j^{(a)}) := b_j^{(a+1)}$$

(for $1 \leq i \leq n - a + 1$ and with the obvious conventions).

This gives directly the quiver and relations. $A_n$ is given by a quiver $Q_n$ with $n$ vertices $\{1, 2, \ldots, n\}$ and $2n - 2$ arrows between them, $\alpha_i : i \rightarrow i + 1$ for $i = 1, \ldots, n - 1$ and $\beta_i : i \rightarrow i - 1$ for $i = 2, \ldots, n$, subject to the relations $\beta_i \alpha_{a-1} = \alpha_i \beta_{a+1}$ for $1 < i < n$ and $\beta_a \alpha_{n-1} = 0$.

The algebra $S_n$ also is of finite type (for details, see below). We let $D_n$ be its Auslander algebra, this is of main interest to us. We will first state its presentation by quiver and relations, and below we will show that it is isomorphic to a skew group ring of $A_n$ with a cyclic group of order 2 (which then will also prove the presentation).

We start by defining a quiver of cylindrical shape.

Definition A.1. For $n \geq 2$, let $\Gamma_n$ be the quiver with vertices $\{1^+, 1^-, 2^+, 2^-, \ldots, n^+, n^-\}$ and arrows $\alpha_{i\pm}$ going from $i^k$ to $(i+1)^\mp$, and $\beta_{i\pm}$ going from $i^k$ to $(i-1)^\pm$, that is

$$
\begin{align*}
\alpha_{i^+} & : i^+ \rightarrow (i+1)^-, \quad \alpha_{i^-} : i^- \rightarrow (i+1)^+, & 1 \leq i < n \\
\beta_{i^+} & : i^+ \rightarrow (i-1)^+, \quad \beta_{i^-} : i^- \rightarrow (i-1)^-, & 1 \leq i < n
\end{align*}
$$

Then the Auslander algebra of $S_n$ is given by the quiver $\Gamma_n$ subject to the relations

$$
\begin{align*}
\beta_{i^-} \alpha_{(i-1)^-} &= \alpha_{i^+} \beta_{(i+1)^-}, & 1 < i < n \\
\beta_{i^+} \alpha_{(i-1)^+} &= \alpha_{i^-} \beta_{(i+1)^+}, & 1 < i < n \\
\beta_{n^-} \alpha_{(n-1)^-} &= \beta_{n^+} \alpha_{(n-1)^+} = 0.
\end{align*}
$$

One way to see that this is a presentation of $D_n$ is via Auslander-Reiten theory. If we let $\tau$ be the map sending $i^k$ to $i^{\mp}$ for $i = 2, \ldots, n$ then $(\Gamma_n, \tau)$ is a translation quiver with projective vertices $\{1^+, 1^-, 2^+, 2^-, \ldots, n^+, n^-\}$. It is precisely the Auslander-Reiten quiver of the algebra $S_n$, and the relations are the ‘mesh relations’. 

\[\Delta\text{-filtered modules and } \mathcal{P}\text{-orbits}\]
A.1. The indecomposable $S_n$-modules. We have defined $S_n = R_n \langle g \rangle$, the skew group algebra, where $g(T) = -T$ (see the introduction). We identify the subalgebra $R_n \otimes 1$ of $S_n \cong R_n \otimes k(\langle g \rangle)$ with $R_n$ and the subalgebra $1 \otimes k(\langle g \rangle)$ with $k(\langle g \rangle)$. The algebra $S_n$ has orthogonal idempotents $e_0, e_1$ with $1 = e_0 + e_1$, where
\[ e_0 = \frac{1}{2}(1 + g), \quad e_1 := \frac{1}{2}(1 - g). \]
Then $S_n = e_0 S_n \oplus e_1 S_n$ as $S_n$-modules. A basis for $e_i S_n$ is given by (the cosets of)
\[ e_i, \ e_i T, \ e_i T^2, \ldots, e_i T^{n-1}. \]
In particular the $e_i S_n$ are uniserial of length $n$, and are indecomposable. One checks that $e_0 T = Te_1$ and $e_1 T = Te_0$, which implies that the composition factors of $e_i S_n$ alternate. We write $L^+$ for the simple top of $e_0 S_n$, and $L^-$ for the simple top of $e_1 S_n$. Then $g$ has eigenvalue $1$ on $L^+$ and eigenvalue $-1$ on $L^-$. This shows that the quiver of $S_n$ has two vertices which we denote by $+$ and $-$, and two arrows, one from $+$ to $-$ and one from $-$ to $+$. The relations are that any path of length $\geq n$ is zero.

$S_n$ is also a self-injective Nakayama algebra. In particular it has finite type, there are $2n$ indecomposable modules (up to isomorphism) and each indecomposable module is uniserial.

Since $g$ has order 2 and the field has characteristic not equal to 2, every $S_n$-module $X$ is relative $R_n$-projective, that is, the multiplication map $X \otimes_{R_n} S_n \to X$ splits.

It is easy to construct the indecomposable $S_n$-modules by inducing from $R_n$. Using the explicit basis for the $R_n$-module $M(i)$ given above, $M(i) \otimes_{R_n} S_n$ has basis \( \{ b_j \otimes e_0, \ b_j \otimes e_1 : 1 \leq j \leq n - i + 1 \} \) (omitting $(-)^{(i)}$ since we fix $i$ for the moment). An easy check gives
\[ b_j \otimes e_0 T = b_j T \otimes e_1 = b_{j+1} \otimes e_1, \]
and similarly $b_j \otimes e_1 T = b_{j+1} \otimes e_0$. Since $T$ generates the algebra $R_n$, this shows that the induced module is the direct sum of $M(i^+)$ with $M(i^-)$ where $M(i^+)$ has basis
\[ \{ b_1 \otimes e_0, b_2 \otimes e_1, b_3 \otimes e_0, b_4 \otimes e_1, \ldots \} \]
and similarly for $M(i^-)$. The top of $M(i^+)$ is $L(i^+)$, similarly for $M(i^-)$. This gives $2n$ uniserial modules for $S_n$ which clearly are pairwise non-isomorphic. Hence this is a full set of the indecomposable $S_n$-modules. It follows that
\[ D_n = \text{End}_{S_n}(M \otimes_{R_n} S_n). \]

With the explicit description of the modules $M(i^\pm)$ it is easy to write down maps which lead to the relations stated in [A]. By using standard methods, it is not difficult to prove the following, and we omit details.

**Proposition A.2.** The algebra $D_n$ is isomorphic to a skew group algebra $A_n \langle \bar{g} \rangle$ where $\bar{g}$ has order 2.

The Auslander-Reiten quiver $\Gamma_n$ of $S_n$ (hence the quiver of $D_n$) is isomorphic to $\mathbb{Z}A_n/r^2$, the irreducible maps are precisely the inclusions of radicals, and taking the socle quotients. One checks that with appropriate labelling, it gives precisely the quiver of the algebra $D_n$. 

The Auslander-Reiten quiver of $S_n$ looks like a cylinder (as does the AR quiver of any self-injective Nakayama algebra). In general, for $k$ algebraically closed not of characteristic 2, the quiver for the Auslander algebra of an algebra of finite type is the Auslander-Reiten quiver of this algebra, and the relations are the ‘mesh relations’ (see [ARS97, p.232]). This gives us:

**Proposition A.3.** $\Gamma_n$ is the Auslander-Reiten quiver of $S_n$.

**References**

[ARS97] M. Auslander, I. Reiten, S. Smalø, *Representation Theory of Artin Algebras*, Cambridge University Press 1997.

[ASS06] I. Assem, D. Simson, A. Skowroński, *Elements of the Representation Theory of Associative Algebras, 1: Techniques of Representation Theory*, L.M.S. Student Texts 65, Cambridge University Press, 2006.

[Ba06] K. Baur, *Richardson elements for classical Lie algebras*, J. Algebra 297 (2006), no. 1, 168–185.

[BG08] K. Baur, S. Goodwin, *Richardson elements for parabolic subgroups of classical groups in positive characteristic*, Algebr. Represent. Theory 11 (2008), no. 3, 275–297.

[Be91] D. Benson, *Representations and Cohomology, Vol. 1: Basic representation theory of finite groups and associative algebras* Cambridge Studies in Advanced Mathematics 30, CUP 1991.

[BHRR99] T. Brüstle, L. Hille, C.M. Ringel, G. Röhrle, *The $\Delta$-filtered modules without self-extensions for the Auslander Algebra of $k[T]/(T^n)$*, Algebras and Representation Theory 2 295–312, 1999.

[DR90] V. Dlab, C.M. Ringel, *The Module Theoretical Approach to Quasi-hereditary Algebras*, Representations of algebras and related topics (Kyoto, 1990), 200–224, LMS Lecture Note Ser., 168, Cambridge Univ. Press, Cambridge, 1992.

[D98] S. Donkin, *The $q$–Schur Algebra*, London Math. Soc. Lecture Note Ser., vol. 253, Cambridge University Press, Cambridge, 1998.

[HR97] L. Hille, G. Röhrle, *On parabolic subgroups of classical groups with a finite number of orbits on the unipotent radical*, C.R. Acad. Sci. Paris, Série I, 465–470, 1997.

[HR99] L. Hille, G. Röhrle, *A classification of parabolic subgroups of classical groups with a finite number of orbits on the unipotent radical*, Transformation Groups, Volume 4, 1, 1999, 35–52.

[M63] S. MacLane, *Homology*, Springer, 1963.

[PV97] V. Popov, G. Röhrle, *On the number of orbits of a parabolic subgroup on its unipotent radical*, in: G. Lehrer (editor), *Algebraic Groups and Lie Groups*, Australian Mathematical Society Lecture Series, Vol. 9.

[R96] G. Röhrle, *Parabolic subgroups of positive modality*, Geom. Dedicata, 50, 163–186, 1996.

[R74] R.W. Richardson, *Conjugacy classes in parabolic subgroups of semisimple algebraic groups*, Bull. London Math. Soc. 6 (1974), 21–24.

[T] R. Tan, *Auslander algebras of self-injective Nakayama algebras*, preprint.