Futaki invariant for Fedosov’s star products.*

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Abstract

We study obstructions to the existence of closed Fedosov’s star products on a given Kähler manifold \((M, \omega, J)\). In our previous paper [11], we proved that the Levi-Civita connection of a Kähler manifold will produce a closed (in the sense of Connes-Flato-Sternheimer [4]) Fedosov’s star product only if it is a zero of a moment map \(\mu\) on the space of symplectic connections. By analogy with the Futaki invariant obstructing the existence of cscK metrics, we build an obstruction for the existence of zero of \(\mu\) and hence for the existence of closed Fedosov’s star product on a Kähler manifold.

Keywords: Symplectic connections, Moment map, Deformation quantization, closed star products, Kähler manifolds.

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1 Introduction

In [3], a moment map \( \mu \) on the space of symplectic connections is introduced. The study of zeroes of \( \mu \) and of the so-called critical symplectic connections was first proposed by D.J. Fox [8] in analogy with the moment map picture for the Hermitian scalar curvature on almost-Kähler manifolds. Recently [11], we give additional motivations for the study of \( \mu \), and its zeroes on Kähler manifolds, coming from the formal deformation quantization of symplectic manifolds.

Our goal is to exhibit an obstruction to the existence of zeroes of \( \mu \) on closed Kähler manifolds in the spirit of Futaki invariants. We will consider closed Kähler manifolds \((M, \omega, J)\) with Kähler class \( \Theta \) and denote by \( \mathfrak{h} \) the Lie algebra of the reduced group of complex automorphisms of the Kähler manifold that is the Lie algebra of holomorphic vector fields of the form \( Z = X_F + JX_H \), where we denoted by \( X_K \) the Hamiltonian vector field defined by \( i(X_K)\omega = dK \) for \( K \in C^\infty(M) \) normalised by \( \int_M K \omega^n = 0 \). Our first result is:

**Theorem 1.** Let \((M, \omega, J)\) be a closed Kähler manifold with Kähler class \( \Theta \) and Levi-Civita connection \( \nabla \). Let \( \mathfrak{h} \) be the Lie algebra of the reduced group of complex automorphisms of the Kähler manifold. Then, the map

\[
\mathcal{F}_\omega : \mathfrak{h} \to \mathbb{R} : Z \mapsto \int_M H_\mu(\nabla) \frac{\omega^n}{n!},
\]

for \( Z = X_F + JX_H \) and \( \mu \) is the Cahen-Gutt moment map on \( \mathcal{E}(M, \omega) \), is a character that does not depend on the choice of a Kähler form in the Kähler class \( \Theta \).

Deformation quantization as defined in [2] is a formal associative deformation of the Poisson algebra \((C^\infty(M), \{\cdot, \cdot\})\) of a Poisson manifold \((M, \pi)\) in the direction of the Poisson bracket. The deformed algebra is the space \( C^\infty(M)[[\nu]] \) of formal power series of smooth functions with composition law \(*\) called star product.
On a symplectic manifold $(M, \omega)$ endowed with a symplectic connection $\nabla$ (i.e. torsion-free connection leaving $\omega$ parallel), one can associate the Fedosov’s star product $*_{\nabla}$, see [4]. The moment map $\mu$ evaluated at $\nabla$ is the first non-trivial term in the expression of a trace density for the star product $*_{\nabla}$, see [11]. So that, if the star product $*_{\nabla}$ is closed (in the sense of Connes-Flato-Sternheimer [4]), then $\mu(\nabla)$ is the zero function which implies the following result.

**Corollary 1.1.** Let $(M, \omega, J)$ be a closed Kähler manifold with Kähler class $\Theta$, such that $\mathcal{F}^\omega$ is not identically zero, then, given any Kähler form $\tilde{\omega} \in \mathcal{M}_\Theta$ with Levi-Civita connection $\nabla$, the Fedosov’s star product $*_{\nabla}$ is not closed.

Finally, we identify the character $\mathcal{F}^\omega$ with one of the so-called higher Futaki invariants. It enables us to exhibit an example of Kähler manifolds [13, 14] admitting non-zero values of $\mathcal{F}$ and hence no closed Fedosov’s star products as considered in the above Corollary 1.1.

## 2 The moment map and Fedosov’s star products

Consider a closed symplectic manifold $(M, \omega)$ of dimension $2n$. A symplectic connection $\nabla$ on $(M, \omega)$ is a torsion-free connection such that $\nabla \omega = 0$. There always exists a symplectic connection on a symplectic manifold and the space $\mathcal{E}(M, \omega)$ of symplectic connections is the affine space

$$\mathcal{E}(M, \omega) = \nabla + \Gamma(S^3 T^* M) \text{ for some } \nabla \in \mathcal{E}(M, \omega),$$

where $S^3 T^* M := \{ A \in \Lambda^1(M) \otimes \text{End}(TM) \mid \omega(A(\cdot, \cdot)) \text{ is completely symmetric} \}$. For $A \in S^3 T^* M$, we set $A(\cdot, \cdot, \cdot)$ for the symmetric 3-tensor $\omega(A(\cdot, \cdot, \cdot))$.

There is a natural symplectic form on $\mathcal{E}(M, \omega)$. For $A, B \in T_{\nabla} \mathcal{E}(M, \omega)$, seen as element of $\Lambda^1(M) \otimes \text{End}(TM, \omega)$, one defines

$$\Omega^\mathcal{E}_\nabla(A, B) := \int_M \text{tr}(A \hat{\lambda} B) \wedge \frac{\omega^{n-1}}{(n-1)!} = -\int_M \Lambda^{kl} \text{tr}(A(e_k)B(e_l)) \frac{\omega^n}{n!},$$

where $\hat{\lambda}$ is the product on $\Lambda^1(M) \otimes \text{End}(TM, \omega)$ induced by the usual $\wedge$-product on forms and the composition on the endomorphism part, $\Lambda^{kl}$ is defined by $\Lambda^{kl} \omega_{lk} = \delta^k_l$ for $\omega_{lk} := \omega(e_l, e_k)$ for a frame $\{e_k\}$ of $T_x M$. The 2-form $\Omega^\mathcal{E}$ is a symplectic form on $\mathcal{E}(M, \omega)$.

**Remark 2.1.** The symplectic form $\Omega^\mathcal{E}$ can be written in coordinate as :

$$\Omega^\mathcal{E}_\nabla(A, B) := \int_M \Lambda^{i1j1} \Lambda^{i2j2} A^{i3j3} A_{i1i2j3} B_{j1j2j3} \frac{\omega^n}{n!},$$

for $A, B \in T_{\nabla} \mathcal{E}(M, \omega)$.

There is a natural symplectic action of the group of symplectomorphisms on $\mathcal{E}(M, \omega)$. For $\varphi$, a symplectic diffeomorphism, we define an action

$$(\varphi, \nabla)_X Y := \varphi_* (\nabla_{\varphi_*^{-1}X} \varphi_*^{-1} Y),$$

(1)
for all $X, Y \in TM$ and $\nabla \in \mathcal{E}(M, \omega)$.

Recall that a Hamiltonian vector field is a vector field $X_F$ for $F \in C^\infty(M)$ such that $i(X_F)\omega = dF$. We denote by $\text{Ham}(M, \omega)$ the group of Hamiltonian diffeomorphisms of the symplectic manifold $(M, \omega)$ with Lie algebra the space $C^\infty_0(M)$ of smooth functions $F$ such that $\int_M F^2 d\omega = 0$.

The action defined in Equation (1) restricts to an action of the group $\text{Ham}(M, \omega)$. Let $X_F$ be a Hamiltonian vector field with $F \in C^\infty_0(M)$, the fundamental vector field on $\mathcal{E}(M, \omega)$ associated to this action is

$$(X_F)^{\mathcal{E}}(Y)Z := \frac{d}{dt}|_{t=0} [\phi_t^F, \nabla] = (\mathcal{L}_{X_F} \nabla)(Y)Z = \nabla^2_{(Y,Z)}X_F + R^\nabla(X_F,Y)Z,$$

where $R^\nabla(U,V)W := [\nabla_U, \nabla_V]W - \nabla_{[U,V]}W$ is the curvature tensor of $\nabla$.

Denote by $Ric^\nabla$ the Ricci tensor of $\nabla$ defined by $Ric^\nabla(X,Y) := \text{tr}[V \mapsto R^\nabla(V,X,Y)]$ for all $X, Y \in TM$.

**Theorem 2.2** (Cahen-Gutt [3]). The map $\mu : \mathcal{E}(M, \omega) \rightarrow C^\infty_0(M)$ defined by

$$\mu(\nabla) := (\nabla^2_{(e_p,e_q)}Ric^\nabla)(e_p,e_q) + P(\nabla) - \mu_0$$

where $\{e_k\}$ is a frame of $T_xM$ and $\{e^l\}$ is the symplectic dual frame of $\{e_k\}$ (that is $\omega(e_k, e^l) = \delta_k^l$) and $P(\nabla)$ is the function defined by $P(\nabla)\frac{\omega^n}{n!} := \frac{1}{2}\text{tr}(Ric^\nabla(.,.) \wedge R^\nabla(.,.)) \wedge \frac{e^n}{(n-2)!}$, with $\int_M P(\nabla)\frac{\omega^n}{n!} =: \mu_0$, is a moment map for the action of $\text{Ham}(M, \omega)$ on $\mathcal{E}(M, \omega)$, i.e.

$$\frac{d}{dt}|_{t=0} \int_M \mu(\nabla + tA)F\frac{\omega^n}{n!} = \Omega^\nabla((X_F)^{\mathcal{E}}, A).$$

In [11], the moment map $\mu$ is related to the notion of trace density for Fedosov’s star products. Also, the closedness (closedness in the sense of Connes-Flato-Sternheimer [2]) of Fedosov’s star product implies $\mu = 0$. Let us recall briefly all those notions and results.

A **star product**, as defined in [2], on $(M, \omega)$ is a $\mathbb{R}[\nu]$-bilinear associative law on the space $C^\infty(M)[[\nu]]$ of formal power series of smooth functions:

$$*: (C^\infty(M)[[\nu]])^2 \rightarrow C^\infty(M)[[\nu]] : (H, K) \mapsto H * K := \sum_{r=0}^{\infty} \nu^r C_r(H,K)$$

where the $C_r$’s are bidifferential operators null on constants such that for all $H, K \in C^\infty(M)[[\nu]] : C_0(H, K) = HK$ and $C_1(H, K) - C_1(K, H) = \{H, K\}$.

In [3], Fedosov gave a geometric construction of star products on symplectic manifolds using a symplectic connection $\nabla$ and a formal series of closed 2-forms $\Omega \in \nu\Omega^2(M)[[\nu]]$. We will only consider Fedosov’s star products build with $\Omega = 0$ and denote them by $*_\nabla$.

Let $*$ be a star product on a symplectic manifold. A **trace** for $*$ is a $\mathbb{R}[\nu]$-linear map

$$\text{tr} : C^\infty(M)[[\nu]] \rightarrow \mathbb{R}[[\nu]],$$

satisfying $\text{tr}(F \ast H) = \text{tr}(H \ast F)$ for all $F, H \in C^\infty(M)[[\nu]]$. 

4
Any star product $\ast$ on a symplectic manifold $(M, \omega)$ admit a trace $[7, 12, 10]$. More precisely, there exists $\kappa \in C^\infty(M)[[\nu]]$ such that
$$\text{tr}(F) := \int_M F \kappa \omega^n/n!$$
for all $F \in C^\infty(M)[[\nu]]$. The function $\kappa$ is called a trace density. Moreover, any two traces for $\ast$ differ from each other by multiplication with a formal constant $C \in \mathbb{R}[\nu^{-1}, \nu]]$.

A star product is called closed [4] if the map $F \mapsto \int_M F \omega^n/n!$ satisfies the trace property:
$$\int_M F \ast H \omega^n/n! = \int_M H \ast F \omega^n/n!,$$
for all $F, H \in C^\infty(M)[[\nu]]$.

In [11], we linked the moment map with the trace density $\kappa$ of the Fedosov’s star product $\ast_\nabla$ by the formula:
$$\kappa_\nabla := 1 + \frac{\mu^2}{24} + O(\nu^3).$$
So that, if $\ast_\nabla$ is closed, then $\mu(\nabla) = 0$.

3 Futaki invariant for $\mu$

3.1 Definition and main Theorem

We consider a closed Kähler manifold $(M, \omega, J)$. Let $\Theta$ be the Kähler class of $\omega$ and denote by $\mathcal{M}_\Theta$ the set of Kähler forms in the class $\Theta$. By the classical $dd^c$-lemma,
$$\mathcal{M}_\Theta := \{ \omega_\phi = \omega + dd^c \phi \text{ s.t. } \phi \in C^\infty_0(M), \, \omega_\phi(\cdot, J \cdot) \text{ is positive definite } \}$$
where $dd^c F := -dF \circ J$ for $F \in C^\infty(M)$.

Consider the functional
$$\omega_\phi \in \mathcal{M}_\Theta \mapsto \mu_\phi(\nabla^\phi) \in C^\infty(M),$$
where $\mu_\phi$ is the moment map on $\mathcal{E}(M, \omega_\phi)$ and $\nabla^\phi$ is the Levi-Civita connection of $g_\phi(\cdot, \cdot) := \omega_\phi(\cdot, J \cdot)$. Using the second Bianchi identity, one can write:
$$\mu_\phi(\nabla^\phi) = \frac{1}{2} \Delta^\phi \text{Scal}^{\nabla^\phi} + P(\nabla^\phi) - \mu_0.$$
Note that $\mu_0$ does not depend on $\phi$ and that $\mu_\phi(\nabla^\phi)$ is normalised with respect to the integral using the Kähler form $\omega_\phi$.

Let $\mathfrak{h}$ be the Lie algebra of all holomorphic vector fields having at least one zero on $M$. For any $Z \in \mathfrak{h}$ and $\omega_\phi \in \mathcal{M}_\Theta$, there are unique $F^\phi, H^\phi \in C^\infty(M)$ (depending on $\omega_\phi$) whose integral with respect to $\omega_\phi^n/n!$ is zero such that $Z = X_{F^\phi}^{\omega_\phi} + JX_{H^\phi}^{\omega_\phi}$, where $X_{K_\phi}^{\omega_\phi}$ denotes the Hamiltonian vector field of $K \in C^\infty(M)$ with respect to the symplectic form $\omega_\phi$.  

Definition 3.1. For $\omega_\phi \in M_\Theta$, we define the map

$$F^{\omega_\phi} : \mathfrak{h}(M) \mapsto \mathbb{R} : Z \mapsto \int_M H^{\phi}(\nabla^{\phi}) \frac{\omega^n}{n!},$$

for $Z = X^{\omega_\phi} + JX^{\omega_\phi}$ as above.

Though, the definition of $F^{\omega_\phi}$ seems a priori to depend on the choice of a point in $M_\Theta$, we will prove it is not the case.

Theorem 1. Let $(M, \omega, J)$ be a closed Kähler manifold with Kähler class $\Theta$ and Levi-Civita connection $\nabla$. Let $\mathfrak{h}$ be the Lie algebra of the reduced group of complex automorphisms of the Kähler manifold. Then, the map

$$F^\omega : \mathfrak{h}(M) \to \mathbb{R} : Z \mapsto \int_M H^\omega(\nabla) \frac{\omega^n}{n!},$$

for $Z = X^\omega + JX^\omega$ and $\mu$ is the Cahen-Gutt moment map on $\mathcal{E}(M, \omega)$, is a character that does not depend on the choice of a Kähler form in the Kähler class $\Theta$.

The Theorem implies that the non-vanishing of $F^\omega$ is an obstruction to the existence of $\omega_\phi \in M_\Theta$ such that $\mu^{\phi}(\nabla^{\phi}) = 0$.

Proof of Corollary 1.1. For $\tilde{\omega} \in M_\Theta$ with Levi-Civita connection $\tilde{\nabla}$, assume the Fedosov’s star product $\ast_{\tilde{\nabla}}$ is closed. Then $\mu^{\tilde{\omega}}(\nabla) = 0$ and hence $F^\omega = 0$. It concludes the proof. \qed

3.2 The space $\mathcal{J}_{int}(M, \omega)$

The goal of this subsection is to state the necessary formulas coming from [11] in order to prove Theorem 1.

Definition 3.2. We denote by $\mathcal{J}_{int}(M, \omega)$ the space of integrable complex structures on $M$ compatible with $\omega$, that is $J \in \mathcal{J}_{int}(M, \omega)$ is a complex structure such that $\omega(J., J.) = \omega(., .)$ and $\omega(., J.)$ is a Riemannian metric.

For $J_t \in \mathcal{J}_{int}(M, \omega)$ a smooth path and $A := \frac{d}{dt}|_0 J_t \in T_J \mathcal{J}_{int}(M, \omega)$. Then, $A$ is a section of the bundle $\text{End}(TM)$ satisfying $AJ_0 + J_0 A = 0$ and the 2-tensor

$$J(\nabla A(X)Y) - (\nabla A)(JX)Y$$

is symmetric in $X, Y$.

Consider the map

$$\text{lc} : \mathcal{J}_{int}(M, \omega) \to \mathcal{E}(M, \omega) : J \mapsto \nabla^J$$

which associates to an integrable complex structure $J$ compatible with $\omega$, the Levi-Civita connection $\nabla^J$ of the Kähler metric $g_J(., .) := \omega(., J.)$.

The map $\text{lc}$ is equivariant with respect to the group of symplectic diffeomorphisms of $(M, \omega)$. That is : for all $\varphi \in \text{Symp}(M, \omega)$ and $J \in \mathcal{J}_{int}(M, \omega)$ with $\varphi.J := \varphi_* J \varphi^{-1}$:

$$\text{lc}(\varphi.J) = \varphi.\text{lc}(J).$$

6
Proposition 3.3. Let $A \in T_JJ\mathfrak{int}(M, \omega)$ and write $B \in T\mathcal{E}(M, \omega)$ such that $B = \text{lc}_sJ(A)$. Then $B$ is the unique solution to the equation

$$B(X)Y + JB(X)JY = -\nabla JA(X)Y.$$ 

and if $JA \in T_JJ\mathfrak{int}(M, \omega)$, then:

$$\text{lc}_s(JA)(X)Y = JB(JX)JY + \frac{1}{2}(J(\nabla A)(JX)Y) + (\nabla A)(X)Y.$$ 

From those equations we obtain [11]:

Lemma 3.4. If $A, A', JA, JA' \in T_JJ\mathfrak{int}(M, \omega)$ then

$$(\text{lc}_s^{\Omega^E}J)(JA, JA') = (\text{lc}_s^{\Omega^E})J(A, A').$$

3.3 Proof of Theorem 1

We will prove Theorem 1 in this section. For this, consider a smooth one-parameter family $\phi : -\epsilon, \epsilon \rightarrow C_0^\infty(M) : t \mapsto \phi(t)$ for some $\epsilon \in \mathbb{R}^+$ such that the 2-form $\omega_{\phi(t)} := \omega + dd^c\phi(t)$ is a smooth path in $\mathcal{M}_\Theta$ passing through $\omega$. To prove the independence of $F_{\omega_{\phi(t)}}$, we will show that for all $Z \in \mathfrak{h}(M)$:

$$\frac{d}{dt}_{|0}F_{\omega_{\phi(t)}}(Z) = 0.$$ 

All the forms $\omega_{\phi(t)}$ are symplectomorphic to each other. Indeed, set $X_t := -\text{grad}^g_{\phi(t)}(\dot{\phi})$ the gradient vector field of $\dot{\phi}(t)$ with respect to $g_{\phi(t)}$ (that is $g_{\phi(t)}(\text{grad}^g_{\phi(t)}(\dot{\phi}), \cdot) = d\phi$). Then the one parameter family of diffeomorphisms $f_t$ integrating the time-dependent vector field $X_t$ satisfies

$$f_t^*\omega_{\phi(t)} = \omega.$$ (3)

Consider $f_t$ as in the above equation (3). Then, the natural action of $f_t^{-1}$ on $J$ produces a path

$$J_t := f_t^{-1}J := f_t^{-1}Jf_t \in J\mathfrak{int}(M, \omega).$$

Define the associated Kähler metric $g_{J_t}(\cdot, \cdot) := \omega(\cdot, J_t\cdot)$ and denote by $\nabla^{J_t}$ its Levi-Civita connection. Then, $\nabla^{J_t}$ and $\nabla^{\phi(t)}$ are related by the following formula:

$$\nabla^{J_t} = f_t^{-1}\nabla^{\phi(t)},$$

where $(f_t^{-1}\nabla^{\phi(t)})_Y Z = f_t^{-1}\nabla^{\phi(t)}_{f_tY} f_tZ$. Then, their image by the moment map is related by:

$$\mu(\nabla^{J_t}) = f_t^*\mu^{\phi(t)}(\nabla^{\phi(t)}).$$ (4)

Note that on the LHS the moment map is taken with respect to a fixed symplectic form while on the RHS $\mu^{\phi(t)}$ is a function on $\mathcal{E}(M, \omega_{\phi(t)})$. 

7
Proof of Theorem 1. We will use the notations introduced above. First, using Equations (2), (3) and (4), we have:

\[ \mathcal{F}_{\omega^\phi(t)}(Z) = \int_M H^{\phi(t)} \mu^{\phi(t)}(\nabla^{\phi(t)}) \frac{\omega^n}{n!} = \int_M f_t^\prime(H^{\phi(t)}) \mu(\nabla^{J_t}) \frac{\omega^n}{n!}. \]

We will differentiate at \( t = 0 \). We will write \( H \) for \( H^{\phi(0)} \):

\[ \frac{d}{dt}|_0 f_t^\prime(H^{\phi(t)}) = X_0(H) + \frac{d}{dt}|_0 H^{\phi(t)} = X_0(H) + Z(\dot{\phi}(0)) = X_F(\dot{\phi}(0)). \]

Using the fact that \( \frac{d}{dt}|_0 J_t = \mathcal{L}_{X_0} J = -\mathcal{L}_J X_{\dot{\phi}(0)} J = -J \mathcal{L}_{X_{\dot{\phi}(0)}} J \), we compute

\[ \frac{d}{dt}|_0 \mathcal{F}_{\omega^\phi(t)}(Z) = \int_M X_F(\dot{\phi}(0)) \mu(\nabla) \frac{\omega^n}{n!} + \frac{d}{dt}|_0 \int_M H \mu(\nabla^{J_t}) \frac{\omega^n}{n!}, \]

\[ = \int_M X_F(\dot{\phi}(0)) \mu(\nabla) \frac{\omega^n}{n!} + \Omega(\mathcal{L}_{X_H} \nabla, lc_{\ast} J(-J \mathcal{L}_{X_{\dot{\phi}(0)}} J)). \]

Using the equivariance of the map \( lc \) and Lemma 3.4, we have

\[ \Omega(\mathcal{L}_{X_H} \nabla, lc_{\ast} J(-J \mathcal{L}_{X_{\dot{\phi}(0)}} J)) = \Omega(lc_{\ast} J(\mathcal{L}_{X_H} J), lc_{\ast} J(-J \mathcal{L}_{X_{\dot{\phi}(0)}} J)), \]

\[ = \Omega(lc_{\ast} J(J \mathcal{L}_{X_H} J), lc_{\ast} J(\mathcal{L}_{X_{\dot{\phi}(0)}} J)). \]

Finally, as \( Z \) is holomorphic, \( J \mathcal{L}_{X_H} J = -\mathcal{L}_{X_{\bar{F}}} J \), so that:

\[ \frac{d}{dt}|_0 \mathcal{F}_{\omega^\phi(t)}(Z) = \int_M X_F(\dot{\phi}(0)) \mu(\nabla) \frac{\omega^n}{n!} - \Omega(lc_{\ast} J(\mathcal{L}_{X_{\bar{F}}} J), lc_{\ast} J(\mathcal{L}_{X_{\dot{\phi}(0)}} J)), \]

\[ = \int_M X_F(\dot{\phi}(0)) \mu(\nabla) \frac{\omega^n}{n!} + \int_M \mu(\nabla) X_{\dot{\phi}(0)}(F) \frac{\omega^n}{n!} = 0. \]

The fact that \( \mathcal{F} \) is a character, is a consequence of the above computations. Indeed, for \( Y, Z \in \mathfrak{h}(M) \), one has \( [Y, Z] = \frac{d}{dt}|_0 \varphi_Y Z, \) for \( \varphi_Y \) the flow of \( Y \). Then, when \( Z = X_{\bar{F}} + J X_H \), one computes \( \varphi_{Y_{\bar{F}}} Z = X^{\varphi_Y}_{\bar{F}} + J X^{\varphi_Y}_{\bar{F}} \). Finally, one has

\[ \mathcal{F}^\omega([Y, Z]) = \frac{d}{dt}|_0 \mathcal{F}^\omega(\varphi_Y Z) = \frac{d}{dt}|_0 \mathcal{F}^{\varphi_Y}_{\bar{F}}(\varphi_Y Z) = \frac{d}{dt}|_0 \mathcal{F}^\omega(Z) = 0 \]

\[ \square \]

4 Generalised Futaki invariants

4.1 \( \mathcal{F}^\omega \) is a generalised Futaki invariant

In [9], Futaki generalised the Futaki invariant obstructing the existence of Kähler-Einstein metrics. One of these so-called generalised Futaki invariants is the invariant we define using the moment map.
Futaki’s construction goes as follows. On a Kähler manifold \((M, \omega, J)\), consider the holomorphic bundle \(T^{(1,0)}M\) of tangent vectors of type \((1, 0)\). Choose any \((1, 0)\)-connection \(\nabla\) on \(T^{(1,0)}M\) with curvature \(\nabla^2\). For \(Z \in \mathfrak{h}(M)\), define \(L(Z^{(1,0)}) := \nabla Z^{(1,0)} - \mathcal{L}_{Z^{(1,0)}}\), it is a section of the bundle \(\text{End}(T^{(1,0)}M)\). Let \(q\) be a \(\text{Gl}(n, \mathbb{C})\)-invariant polynomials on \(\mathfrak{gl}(n, \mathbb{C})\) of degree \(p\), Futaki defined in [9], the map \(F_q : \mathfrak{h} \to \mathbb{C}\) by

\[
F_q(Z) := (n - p + 1) \int_M u_Z q(R^\nabla) \wedge \omega^{(n-p)} + \int_M q(L(Z^{(1,0)}) + R^\nabla) \wedge \omega^{(n-p+1)},
\]

where \(u_Z\) is the complex valued function defined by \(i(Z^{(1,0)})\omega = \bar{\partial} u_Z\).

Futaki shows that \(F_q\) depends neither on the choice of the \((1, 0)\)-connection nor on the choice of the Kähler form in \(\mathcal{M}_\Theta\), see [9]. Moreover, if you take \(q = c_k\) the polynomials defining the \(k\)-th Chern form, it is proved in [9] that one recovers Bando’s obstruction to the harmonicity of the \(k\)th Chern form:

\[
F_{c_k}(Z) = (n - k + 1) \int_M u_Z c_k(R^\nabla) \wedge \omega^{(n-k)}. \quad (5)
\]

**Proposition 4.1.** We have that \(\mathcal{F}^\omega\) is the imaginary part of \(F_{\frac{\pi^2}{(n-1)!}c_2^{-1/2}c_1c_1}\)

**Proof.** The key of the computation is that the Pontryagin 4-form defining \(P(\nabla)\) satisfies:

\[
\text{tr}(R^\nabla \wedge R^\nabla) = 16\pi^2(c_2 - \frac{1}{2}c_1c_1)(R^\nabla).
\]

Then,

\[
\mathcal{F}^\omega(Z) = -\frac{1}{2} \int_M H \Delta \text{Scal} \frac{\omega^n}{n!} + 8\pi^2 \int_M H c_2(R^\nabla) \wedge \frac{\omega^{n-2}}{(n-2)!} - 4\pi^2 \int_M H c_1c_1(R^\nabla) \wedge \frac{\omega^{n-2}}{(n-2)!}.
\]

As \(u_z = F + iH\), Equation (5) tells us that the imaginary part of \(F_{\frac{\pi^2}{(n-1)!}c_2^{-1/2}c_1c_1}(Z)\) is:

\[
8\pi^2 \int_M H c_2(R^\nabla) \wedge \frac{\omega^{n-2}}{(n-2)!}.
\]

It remains to compute de second term of \(F_{\frac{\pi^2}{(n-1)!}c_2^{-1/2}c_1c_1}\):

\[
F_{\frac{\pi^2}{(n-1)!}c_2^{-1/2}c_1c_1}(Z) = 4\pi^2 \left( \int_M u_Z c_1c_1(R^\nabla) \wedge \frac{\omega^{n-2}}{(n-2)!} + \int_M c_1c_1(L(Z^{(1,0)}) + R^\nabla) \wedge \frac{\omega^{n-1}}{(n-1)!} \right)
\]

\[
= 4\pi^2 \int_M u_Z c_1c_1(R^\nabla) \wedge \frac{\omega^{n-2}}{(n-2)!} + 2i \int_M \text{tr}^c(L(Z^{(1,0)})) \rho^\nabla \wedge \frac{\omega^{n-1}}{(n-1)!}.
\]

Since \(\text{tr}^c(L(Z^{(1,0)})) = \frac{i}{2} (\Delta F + i\Delta H)\), we have:

\[
2i \int_M \text{tr}^c(L(Z^{(1,0)})) \rho^\nabla \wedge \frac{\omega^{n-1}}{(n-1)!} = \frac{1}{2} \int_M (\Delta F + i\Delta H) \text{Scal} \frac{\omega^n}{n!},
\]

\[
= \frac{1}{2} \int_M (F + iH) \Delta \text{Scal} \frac{\omega^n}{n!}.
\]

So, \(\mathcal{F}^\omega\) is the imaginary part of \(F_{\frac{\pi^2}{(n-1)!}c_2^{-1/2}c_1c_1}\). \qed
4.2 Example

The computations of generalized Futaki invariants $F_q$ defined by (4.1) is not an easy task. For $q = \text{Td}_p$, the invariant polynomials defining the $p^{th}$ Todd class, methods coming from algebraic geometry are developed to compute $F_{\text{Td}_p}$, see [5] [14], in order to study the asymptotic semi-stability [9] of the manifold. Those methods and this notion of asymptotic semi-stability are beyond the scope of this paper. However, when the manifold is Kähler-Einstein, as it is the case in [14], $F_{\text{Td}_2}$ determines completely $F_{\omega}$.

Observation 4.2. When $(M, \omega, J)$ is Kähler-Einstein, $F_{\omega}$ is the imaginary part of $\frac{8\pi^2}{(n-1)!} F_{\text{Td}_2}$.

Proof. Recall that $\text{Td}_2 = c_2 + c_1.c_1$. Now, because the Ricci form $\rho = \lambda \omega$, from the computations in the proof of Proposition 4.1 one has $F_{c_1.c_1} = 0$. So that, $\frac{8\pi^2}{(n-1)!} F_{\text{Td}_2} = F_{\frac{8\pi^2}{(n-1)!}(c_2 - \frac{1}{2}c_1.c_1)}$ and its imaginary part is $F_{\omega}$ by Proposition 4.1.

In [13], a 7-dimensional (complex dimension) smooth Kähler manifold $(V, \omega, J)$ is constructed, the so-called Nill-Paffenholz example. $V$ is a toric Fano manifold that is Kähler-Einstein, [13]. Moreover, Ono, Sano and Yotsutani [14] showed that, on $V$, $F_{\text{Td}_p} \neq 0$ for $2 \leq p \leq 7$. Combined with the above Observation 4.2, it means $F_{\omega} \neq 0$. Consequently, Corollary 1.1 implies:

Theorem 4.3. Let $(V, \omega, J)$ be the Nill-Paffenholz example [13] and $\Theta = [\omega]$, then there is no closed Fedosov’s star products of the form $*_\nabla$ for $\nabla$ the Levi-Civita connection of some $\tilde{\omega} \in \mathcal{M}_\Theta$.

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