Building Nim

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Abstract The game of NIM, with its simple rules, its elegant solution and its historical importance is the quintessence of a combinatorial game, which is why it led to so many generalizations and modifications. We present a modification with a new spin: BUILDING NIM. With given finite numbers of tokens and stacks, this two-player game is played in two stages (thus belonging to the same family of games as e.g. NINE-MEN’S MORRIS): first BUILDING, where players alternate to put one token on one of the, initially empty, stacks until all tokens have been used. Then, the players play NIM. Of course, because the solution for the game of NIM is known, the goal of the player who starts NIM play is a placement of the tokens so that the Nim-sum of the stack heights at the end of BUILDING is different from 0. This game is trivial if the total

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number of tokens is odd as the Nim-sum could never be 0, or if both the number of tokens and the number of stacks are even, since a simple mimicking strategy results in a Nim-sum of 0 after each of the second player’s moves. We present the solution for this game for some non-trivial cases and state a general conjecture.

**Keywords** Combinatorial game · Nim

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1 Introduction

The game of Nim is believed to have originated in China, but the exact origin is unknown. The earliest references in Europe are in the early 16th century. Bouton completely analyzed the game in 1901 Bouton (1905) and coined the name Nim (thought to be derived from the German word for “to take”). The game is played on a finite number of stacks with a finite number of tokens. Two players alternate in moving, by selecting a stack and taking one or more tokens from that stack, until no further move is possible. A player unable to move loses (also called normal play). Figure 1 shows an example of a position.

Note that there are many variations on the basic game of Nim. A famous modification is the game of Wythoff Wythoff (1907); instead of removing tokens from a single stack, a player can also take the same number of tokens from two stacks. Another modification is the arrangement of stacks, such as in Circular Nim Dufour and Heubach (2013). Other authors have considered Nim on graphs (Duchêne and Renault 2014) or simplicial simplexes (Ehrenborg and Steingrímsson 1996), and in Larsson (2009) it is shown that for the game of Imitation Nim a simple mimicking prevention rule in Nim gives the same $\mathcal{P}$-positions as Blocking Wythoff (Hegarty and Larsson 2006). A standard feature of many such games is that there are only two outcome classes; each game is either an $\mathcal{N}$- or a $\mathcal{P}$-position, that is, a position from which the current or previous player wins, respectively.

Here we present a new variation of Nim by introducing a Building stage before Nim begins. The game of Building Nim, $\text{BN}(n, \ell)$, is played with $n$ tokens on $\ell$ stacks in two stages:

- The first stage is Building: the two players take turns choosing one out of the $\ell$ stacks to place an unused token until all tokens have been used, resulting in a position of the form $s = (s_1, \ldots, s_\ell)$, where $s_i$ denotes the respective stack height, given in canonical form ordered from largest to smallest height (and some stacks may be empty);

![Fig. 1](image-url) The Nim position $(5, 3, 2, 1)$
The second stage is **NIM**: when all BUILDING tokens have been used, the game of NIM starts from position $s$ with the player whose turn it is (that is, the player who did not place the last token in the BUILDING stage).

Obviously, the winning strategy for BUILDING is closely tied to that of NIM. The player who places the last token of BUILDING would like to create a $P$-position of NIM. Such a game having successive stages of play can be considered as a variation of **sequential compounds** of games (Stromquist and Ullman 1993), which consist in playing successive combinatorial games (with the objective of being the last player to move in the last game). The main difference here is that the BUILDING stage is a different type of combinatorial game and also that the opening of the second game depends non-trivially on the closure of the first.

Similar to a BUILDING position, a generic NIM position is represented by the vector of stack heights, $(s_1, s_2, \ldots, s_\ell)$. To describe the set of $P$-positions for NIM, $\mathcal{P}(\text{NIM})$, we define the *Nim-sum* $s_1 \oplus s_2 \oplus \cdots \oplus s_\ell$, as obtained by translating the values into their binary representation and then adding them without carry-over.

**Theorem 1.1** (Bouton 1905) The $P$-positions for NIM are those where the Nim-sum of the stack heights is 0, that is $\mathcal{P}(\text{NIM}) = \{(s_1, s_2, \ldots, s_\ell) \mid s_1 \oplus s_2 \oplus \cdots \oplus s_\ell = 0\}$.

By this elegant formula, perfect NIM play boils down to a simple computation. Hence the BUILDING stage is our only concern. We denote by P1 the player who starts BUILDING NIM, and by P2 the second player. Hence, a phrase like “P1 wins BN$(n, \ell)$” is equivalent to saying that this game is an $N$-position.

We first state the trivial results.

**Theorem 1.2** (Easy cases) In the game BN$(n, \ell)$, the following are true.

1. If $n$ is odd, then P2 wins.
2. If both $n$ and $\ell$ are even, then P2 wins.

**Proof** These statements follow directly from Theorem 1.1. When $n$ is odd, then the Nim-sum of the stack heights can never be zero, and therefore BUILDING ends in a (NIM) $N$-position. P2 starts NIM, and hence wins. If both $n$ and $\ell$ are even, then P2 can always mirror P1’s move in BUILDING, resulting in pairs of stacks that have the same height. Since $a \oplus a = 0$ for any $a$, BUILDING ends in a $P$-position for NIM, and therefore, since P1 starts NIM, again, she loses the game. \hfill $\square$

This leaves the case when $n$ is even and $\ell$ is odd, and we will provide some explicit winning strategies. Specifically, we will prove that in the game of BN$(n, \ell)$, with $n$ even and $\ell$ odd, the following holds.

1. If $\ell = 3$, then P2 wins if and only if $n = 2^k - 2$, for some positive integer $k$;
2. If $\ell > 3$, then P2 wins if $n \leq \ell + 3$;
3. If $\ell = 5$, then P1 wins if and only if $n \geq 10$.

Since the solutions build on particular ideas in the different cases, we will treat these cases as separate results, Theorems 3.1, 3.2, and 3.4 respectively. Let us begin with some preliminary observations.

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1 It is a well-tempered (fixed-length) scoring game as defined by Johnson (2014).
2 Basic strategies and Nim-sum facts

Lemma 2.1  Consider an instance of $BN(n, 3)$ for even $n \geq 1$ in BUILDING play. At each turn $P1$ can force a position of the form $(y, x, x)$, with $y \geq x$, (Strategy I), while $P2$ at each turn can force a position $(z, x, y)$ with $z = x + y$ (Strategy II).

Proof  In each case, we will show the claim by induction, considering two moves, one by each player, for the induction step. We will illustrate the relevant moves by showing the possible moves in a game tree. We use gray tokens for the current position, a white token for a move made by $P1$, a black token for a move made by $P2$, and to indicate the stacks. In the game trees, we usually show only one of two symmetric moves. Note that since $n$ is even, $P2$ makes the last BUILDING move.

The first move for $P1$ is $(1, 0, 0)$ and $P2$ can respond with $(1, 1, 0)$, corresponding to the desired form for the respective strategy. For the induction step for Strategy I, we need to show that if $P1$ has played to a position of the form $(y, x, x)$ with $y \geq x$, then no matter how $P2$ responds, $P1$ can counter to once more create such a position. Figure 2 shows the possible moves of $P2$ and the response by $P1$. In each case, the resulting position is of the desired form. Note that if $P1$ plays this strategy, then the final position after $P2$'s last move is of the form $(y, x, x)$ or $(y, x + 1, x)$, with $y \geq x + 1$.

Now look at the strategy for $P2$ and assume he leaves the stacks in the form $(z, x, y)$ with $z = x + y$ after his move. Figure 3 shows the possible moves of $P1$ and the response of $P2$. Once more, it is possible to obtain a position of the desired type.

Let us present some basic results on the Nim-sum operator.

Lemma 2.2  (Nim-sum facts) Let $x$, $y$, and $t_i$, $i = 1, \ldots, \ell$, be integers. We have the following facts for the Nim-sum:

(NS1) $x \oplus y = 0$ if and only if $x = y$.
(NS2) $x \oplus y \leq x + y$.
(NS3) $x \oplus (x + 1) = 2^k - 1$ for some $k \geq 1$.
(NS4) If $x = 2^h - 1$ for some $h$, then $x \oplus (x + 1) = 2^{h+1} - 1$; otherwise, $x \oplus (x + 1) < x$.
(NS5) $y = 2^k - 1$ for some $k \geq 1$ if and only if $x \oplus (y - x) = y$ for $0 \leq x \leq y$.

\footnote{A colored version is available at \url{http://arxiv.org/abs/1502.04068}.}
(NS6) If \((s_1 + 1) \oplus s_2 \oplus \cdots \oplus s_\ell = 0\), then \(s_1 \oplus s_2 \oplus \cdots \oplus s_\ell = 2^k - 1\) for some \(k \geq 1\).

(NS7) If \(y > s_1 + s_2 + \cdots + s_\ell\), then \(y \oplus s_1 \oplus s_2 \oplus \cdots \oplus s_\ell > 0\).

Proof In what follows we will use the notation \(x = \ldots x_k x_{k-1} \ldots x_1 x_0\) for the binary expansion of \(x = \sum_{i=0}^{\infty} x_i 2^i\), where \(x_i = 0\) or 1, with finitely many values of 1. If we want to put the emphasis on the fact that \(x_i = 1\) (or 0), we will simply write \(1_i\) (or \(0_i\)).

(NS1): If \(x = y\), then \(x\) and \(y\) share the same unique binary expansion, that is, \(x_i = y_i\). As \(x_i + x_i = 0\) (mod 2), we have that \(x \oplus y = 0\). If \(x \oplus y = 0\), then for each \(i\), one has \(x_i = y_i\), so \(x = y\).

(NS2): If for every \(i, x_i\) and \(y_i\) are not both equal to 1, then \(x_i + y_i\) (mod 2) = \(x_i + y_i\), so \(x \oplus y = x + y\). If, on the contrary, \(x_i = y_i = 1\) for some \(i\), then \(x_i\) and \(y_i\) cancel out in the Nim-sum, so \(x \oplus y < x + y\).

(NS3,4): Let \(h\) be the smallest index for which \(x_h = 0\). Then, \(x = \ldots x_{h+1} 0_{h-1} \ldots 1_{10}, x + 1 = \ldots x_{h+1} 1_{h-1} \ldots 0_{10}, \) and \(x \oplus (x + 1) = 1_{h} 1_{h-1} \ldots 1_{10}\). Furthermore, if \(x_j = 0\) whenever \(j > h\), then \(x = 2^h - 1\) and \(x \oplus (x + 1) = x + (x + 1) = 2^h + 1 - 1\). On the other hand, if \(x_j = 1\) for at least one \(j > h\), then \(x \geq 2^h + 1\), and therefore, \(x \oplus (x + 1) = 2^h + 1 - 1\), which proves (NS4).

(NS5): Suppose that \(y = 2^k - 1\), so \(y = 1_{k-1} 1_{k-2} \ldots 1_{10}\). For \(0 \leq x \leq y\) let \(x_{k-1} x_{k-2} \ldots x_1 x_0\) be the binary expansion of \(x\) and \(z_{k-1} z_{k-2} \ldots z_1 z_0\) be the binary expansion of \(y - x\). Then, for each \(i = 0, 1, \ldots, k - 1\), one has \(x_i + z_i = 1\), so \(x \oplus (y - x) = x + (y - x) = y\). Now suppose that \(y \neq 2^k - 1\) for any \(k\). We will show that there is at least one pair of integers \(x, z\) such that \(y = x + z\), but \(x \oplus z \neq y\). If \(y \neq 2^k - 1\) for any \(k\), then the binary expansion of \(y\) is not a string of consecutive ones, so there is at least one 0 immediately to the right of a 1. Let \(h\) be the position of the rightmost such 0, that is, \(y = y_k y_{k-1} \ldots 1_{h+1} 0_{h} y_{h-1} \ldots y_1 y_0\). Define \(x + 1\) as the integer whose binary expansion is \(y_k y_{k-1} \ldots 1_{h+1} 0_{h} y_{h-1} \ldots y_1 y_0 = 2^h + 1 + \sum_{i=h+2}^{\infty} y_i 2^i\), and \(z - 1\) as the integer whose binary expansion is given by \(0_h y_{h-1} \ldots y_1 y_0 = \sum_{i=0}^{h-1} y_i 2^i\). Clearly, \((x + 1) + (z - 1) = x + z = y\) and the binary expansion of \(x + 1\) is given by \(y_k y_{k-1} \ldots 0_{h+1} 1_{h} y_{h-1} \ldots 1_{10}\). As there are only 1s in the \(h + 1\) rightmost digits of \(x\), and because there is also at least one 1 in the (at most) \(h + 1\) digits of \(z\), then at least one pair...
of 1s will cancel out in the Nim-sum of \(x\) and \(z\), so \(x \oplus z < x + z = y\), completing the proof of (NS5). Here is a numerical example that illustrates the proof. Let \(y = 25 = 11001_2\), so \(h = 2\) is the rightmost position of a 0 following a 1. Then \(x + 1 = 11000_2 = 24\) and \(z - 1 = 001_2 = 1\). That makes \(x = 23 = 10111_2\) and \(z = 010_2 = 2\) (using \(h + 1\) digits). The 1s at position \(h = 1\) will cancel out in the Nim-sum, giving \(23 \oplus 2 = 21 < 25\).

(NS6): Let \(y = s_1 \oplus \cdots \oplus s_\ell\). Then \((s_1 + 1) \oplus s_2 \oplus \cdots \oplus s_\ell = (s_1 + 1) \oplus y = 0\) implies that \(y = s_1 + 1\) by (NS1), and therefore, \(s_1 \oplus s_2 \oplus \cdots \oplus s_\ell = s_1 \oplus y = s_1 \oplus (s_1 + 1) = 2^k - 1\) for some \(k \geq 1\) by (NS3) and the proof is complete.

(NS7): Since \(s_1 \oplus s_2 \oplus \cdots \oplus s_\ell \leq s_1 + s_2 + \cdots + s_\ell < y\), we have \(y \oplus s_1 \oplus s_2 \oplus \cdots \oplus s_\ell > 0\) by (NS1).

In the subsequent proofs, we will only use the “only if” part of (NS5). An interesting corollary to (NS6) is currently not used in our proofs, but perhaps it has relevance to the solution of the general conjecture.

**Corollary 2.3** \(P1\) wins if her final move in building play is to a position for which the Nim-sum of the stack sizes is not of the form \(2^h - 1\), for any positive integer \(h\).

**Proof** Indeed, to win, \(P2\) must finish with a Nim-sum of 0. Then, by (NS6), the position before his final move must have a Nim-sum of the form \(2^h - 1\), for some \(h \geq 1\).

Note that (NS6) is not true in the other direction, as for example, \(2 \oplus 5 = 2^3 - 1\), but neither \(3 \oplus 5\) nor \(2 \oplus 6\) equals 0. Thus, Corollary 2.3 is also not an if and only if statement.

On the other hand, we will use (NS7) repeatedly in the proof for \(\ell = 5\) to conclude that \(P1\) wins whenever she manages to build a stack that contains more than half of the tokens. Moreover, as we will see, if two stack heights are equal, then she wins if there is another stack with more than half of the tokens that are not in the matched stacks.

We are now ready to state the main results. We first give the result for who wins on three stacks, as well as a general result that \(P2\) wins when the number of tokens is at most three more than the number of stacks.

### 3 Main results

**Theorem 3.1** If \(n\) is even, then \(P1\) wins \(BN(n, 3)\) if and only if \(n \neq 2^k - 2\) for any \(k\).

**Proof** If \(P1\) follows Strategy I, then building ends in either \((y, x, x)\), or \((y, x + 1, x)\) with \(y \geq x + 1\). In the first case, \(P1\) wins as \(y \oplus x \oplus x = y \oplus 0 > 0\) (so this is not a move \(P2\) should make). In the second case, we need to distinguish between \(x \neq 2^k - 1\) and \(x = 2^k - 1\). If \(x \neq 2^k - 1\), then \(y \oplus (x + 1) \oplus x \neq 0\), as \(x \oplus (x + 1) < x\) by (NS4) and \(x \leq y\) by assumption. On the other hand, if \(x = 2^k - 1\), then we have that

\[
y \oplus (x + 1) \oplus x = y \oplus (2^k + 1 - 1) = 0 \Leftrightarrow n = 2^{k+2} - 2.
\]
Endgame when $n = \ell + 1$

It remains to be shown that $P2$ can force a win in the case where $n = 2^k - 2$ for some $k$, no matter which strategy $P1$ employs. Let $n = 2^k - 2$. If $P2$ follows Strategy II, then the building phase ends in $(x + y, x, y)$. Since $(x + y) + x + y = n = 2^k - 2$, we have that $x + y = 2^{k-1} - 1$, and hence by (NS5), that $x \oplus y = x + y$. This implies that $(x + y) \oplus x \oplus y = (x + y) \oplus (x + y) = 0$, a win for $P2$.

For more than three stacks, the winner depends on the interplay between $n$ and $\ell$, as opposed to depending on the specific value of $n$ only.

**Theorem 3.2** $P2$ wins $BN(n, \ell)$ for odd $\ell > 3$ and even $n \leq \ell + 3$.

**Proof** We consider three cases, namely $n \leq \ell - 1$, $n = \ell + 1$, and $n = \ell + 3$. If $n \leq \ell - 1$, then $P2$ can always mirror the move of $P1$ as there are more stacks than tokens. Pairs of stacks of equal height have a Nim-sum of zero, so the final position has Nim-sum zero in this case. If $n = \ell + 1$ or $n = \ell + 3$, then $P2$ plays the mirroring strategy but adjusts it as needed in the final two moves. To describe how the adjustment is made, we will describe a position as $(x_1, x_2, \ldots, x_\ell; r)$, where the first $\ell$ terms describe stack heights as before and the last term indicates the number of tokens (= number of moves) that remain to be played. Of course, $r = n - x_1 - x_2 - \cdots - x_\ell$, but it will help for the clarity of the proof to emphasize the number of moves that remain.

Specifically, when $n = \ell + 1$, the mirroring strategy does not work when $P1$ always starts a new stack, that is, if the position after the second to last move of $P1$ is $(1, \ldots, 1, 1, 1, 0, 0; 3)$. $P2$ now adjusts his strategy and moves to position $(2, \ldots, 1, 1, 1, 0, 0; 2)$. Figure 4 shows five stacks only (omitting the other pairs of matched stacks at height one) with the possible moves by $P1$ and the response by $P2$ to a position that has either matched stacks or a $1 - 2 - 3$ configuration, each resulting in a zero Nim-sum and a win for $P2$.

Next we look at the case $n = \ell + 3$. Here, there are two positions where the mirroring strategy cannot be played until the end, namely $(2, 2, 1, 1, \ldots, 1, 0, 0; 4)$ or $(1, \ldots, 1, 0; 4)$. In the first case, the end game follows as in the case $n = \ell + 1$ if $P1$ moves to $(2, 2, 1, 1, \ldots, 1, 1, 0, 0; 3)$, or by playing a mirroring strategy if $P1$ plays on a non-empty stack. In the second case, $P2$ adjusts his strategy as shown in Fig. 5 if $P1$ chooses to play on a non-empty stack in move $n - 3$. Figure 6 shows the endgame if $P1$ plays on the empty stack in move $n - 3$.

Once more the final positions consists of either matched stacks or a $1 - 2 - 3$ configuration.
Fig. 5 Endgame when $n = \ell + 3$ and P1 plays on a non-empty stack in move $n - 3$

Fig. 6 Endgame when $n = \ell + 3$ and P1 plays on the empty stack in move $n - 3$

It may seem as if P2 might be able to adjust his strategy earlier and earlier and have a winning strategy also for larger values of $n$. However, one can check (by hand) that P1 has a winning strategy for BN(10, 5) (see also Lemma 3.7) and some other cases. Computer explorations lead to the following conjecture:

**Conjecture 3.3** P1 wins BN($2n$, $\ell$) if $2n > \ell + 3$.

The proof for five stacks is more involved than that for three stacks, and it uses a number of ideas. Before we get into the technical details, we will state the result and discuss the main ideas. Theorem 3.4 shows that Conjecture 3.3 is true for $\ell = 5$. It will be convenient to use $2n$ as the total number of tokens, that is, the players play $n$ tokens each in the building stage.

**Theorem 3.4** P1 wins BN($2n$, 5) if and only if $n \geq 5$.

The strategies of how P1 wins obviously vary depending on P2’s defense attempts, but parts of her ideas are independent of his responses. Item (NS7) of Lemma 2.2 indicates that P1 wins whenever she manages to build a stack that contains more than half of the tokens. Moreover, if some stack heights are equal, then she wins if there is a stack with a height that is more than half of the tokens that are not in the matched stacks. So one of the general strategies for P1 will be to play high. This height strategy consists of playing on the tallest stack (possibly disregarding a pair of matched stacks). Sometimes the height strategy is not appropriate. In such situations, P1 wants to avoid helping P2 match up a tall stack, typically one with a height that is a power of two, and therefore plays low. The low strategy consists of always playing on the minimal
Fig. 7 Reusing a winning strategy: A winning strategy for $2\delta$ is played “on top” of two stacks of height $2^k$.

stack. Note that Strategy I played on three stacks is a combination of the high and low strategies, selected in response to the various moves by P2. A nontrivial variation of this will be true also in the case of five stacks. At the core of the proof of Theorem 3.4 is the idea that P1 can win by playing high, playing low, or by using the winning strategy from a game with fewer tokens for a game with more tokens, thus allowing us to do an inductive proof. We let the computer verify that P1 can win BN($2n$, 5) for several small $n \geq 5$, and then proceed to prove that P1 can win all games for larger values of $n$.

Powers of 2 will play a pivotal role for the players’ BUILDING strategies. Hence we introduce the following terminology. Let $\pi$ be a given power of 2 strictly smaller than the number $n$ of tokens of each player. A game is strategically played in two BUILDING phases:

- the $\pi$-phase: both players play their first $\pi < n$ tokens
- the $\delta$-phase: both players play their remaining $\delta = \delta(n, \pi) = n - \pi > 0$ tokens.

A special case is when the $\pi$-phase results in two matched stacks, and this is the instance where P1 wants to play a winning strategy for the $2\delta$ remaining tokens “on top” of these two stacks if such a strategy exists. Figure 7 illustrates this idea.

Consider an odd integer $\ell \geq 3$ and a positive integer $n$. Let $\delta + \pi = n$, where $\pi$ is a power of 2 and where $0 < \delta < 2\pi$. The following lemma shows that if P1 wins BN($2\delta$, $\ell$) then P1 wins BN($2n$, $\ell$), if the players have built up two matching stacks in the $\pi$-phase.

**Lemma 3.5** Let $\pi$ be a power of 2 and let $\ell \geq 3$ be odd. Further, let $x_1, x_2, \ldots, x_\ell$ be integers with non-zero Nim-sum, but $(x_1 + \pi) \oplus (x_2 + \pi) \oplus x_3 \oplus \cdots \oplus x_\ell = 0$. Then

$$x_1 + x_2 + \cdots + x_\ell \geq 4\pi.$$

*Proof* Since the Nim-sum of the $x_i$ is non-zero, and the addition of $\pi = 2^k$ cannot affect the Nim-sum of the coefficients of $2^r$ with $r < k$, these coefficients already must have a Nim-sum of zero. Therefore, we can disregard those coefficients in the argument, which amounts to proving the result for $k = 0$. Furthermore, without loss of generality one can assume that there are only three stacks, as the stacks $x_3$ to $x_\ell$ can be replaced by a stack of height $x'_3 = x_3 \oplus x_4 \oplus \cdots \oplus x_\ell$, using that $x_3 + x_4 + \cdots + x_\ell \geq x'_3$. So it suffices to prove the following simpler fact:
If
\[ x_1 \oplus x_2 \oplus x_3 > 0 \] (3.1)
\[ (x_1 + 1) \oplus (x_2 + 1) \oplus x_3 = 0 \] (3.2)
then \( x_1 + x_2 + x_3 \geq 4 \).

Suppose that \( x_1 + x_2 + x_3 < 4 \). The smallest configuration for which a Nim-sum of three pairwise distinct numbers is 0 is \((1, 2, 3)\). Hence two of the terms in (3.2) must be equal and the third must equal 0. Notice that \( x_1 = x_2 \) is impossible if both (3.1) and (3.2) are satisfied. Also \( x_i + 1 = x_3 \) forces \( x_{3-i} = -1 \) for \( i = 1, 2 \) which is impossible. \( \square \)

Note that this result does not depend on the particular odd number of stacks. But it does depend on having a winning strategy for smaller games. By Theorem 3.2, P2 wins for the smallest values of \( n \). Therefore we will depend on knowledge of specific ‘initial’ cases for which P1 wins. We can prove manually that P1 wins for the necessary initial cases for five stacks, but we have not yet found any general strategy. Moreover, the next lemma probably generalizes to more than five odd stacks, but we do not yet have a general proof. Lemma 3.6 considers the cases \( 2n = 2^k - 2 \) for \( k > 4 \) (games for which P1 loses when playing on three stacks). In what follows, we will say that a stack has a \( k \)-component when the coefficient of \( 2^k \) in that stack height is not zero. Clearly, whenever Nim begins with an odd number of \( k \)-components, for some \( k \), then P1 wins.

**Lemma 3.6** For \( k > 4 \), P1 wins \( \text{BN}(2^k - 2, 5) \).

**Proof** We show that by playing low, P1 can force an odd number of \( k \)-components, for some \( k \), when Nim-play starts. For ease of describing the argument, we say that a token belongs to the bottom layers if the number of tokens below it is strictly smaller than \( 2^k - 3 \) (see Fig. 8). Note that each player has \( 2^{k-1} - 1 \) tokens to play.

Let us first assume that P2 contributes at least \( 2^{k-3} + 1 \) tokens to the bottom layers. In this case, P1 will be able to ensure that the bottom layers are completely filled by playing low. No matter how many tokens P2 contributes beyond the \( 2^{k-3} + 1 \) tokens, there are a total of \( 2n - 5 \cdot 2^{k-3} = 2^k - 2 - 5 \cdot 2^{k-3} = 3 \cdot 2^{k-3} - 2 \) tokens in the upper layers. How can they be distributed? There are three distinct possibilities how the total configuration of BUILDING can end:

![Fig. 8](image-url)

\( P1 \) plays the bottom layers whereas \( P2 \) plays on stack 1 when \( k = 5 \)
This is the case where $P_2$ contributes $2^{k-3}$ tokens to the bottom layers. When $k = 5$, if he plays strategically on stack 2, he has enough remaining tokens to be able to match the $(k - 2)$-component in $s_1$.

- If $s_1 \geq s_2 \geq 2^{k-2}$, then there are three $(k - 3)$-components, since $s_3 < 2^{k-2}$;
- If $s_1 \geq 2^{k-2}$ and $s_2 < 2^{k-2}$, then there is an unmatched $(k - 2)$-component;
- If $s_1 < 2^{k-2}$ then there are five $(k - 3)$-components.

In all instances, building play ends in a nonzero Nim-sum.

Suppose next that $P_2$ contributes at most $2^{k-3}$ tokens to the bottom layers. Since $P_1$ plays low, $P_2$ has to contribute at least $2^{k-3} - 1$ tokens to the bottom layers, so there are just two different possibilities. Let’s first consider the case when $P_2$ contributes exactly this minimum number of tokens. In this case, it will take until $P_1$ has made $4(2^{k-3} - 1) + 2 = 2^{k-1} - 2$ moves (see Fig. 8) before $P_2$ can play anywhere but on the first stack. At this point, $P_2$ will have made one fewer move, so $P_2$ has two moves left, and $P_1$ has one move left.

With the initial token from $P_1$, the first stack’s height is $s_1 = 4 \cdot 2^{k-3} - 2 = 2^{k-1} - 2$, as shown in Fig. 8 for $k = 5$. Because $k > 4$ by assumption, $s_1$ contains a $(k - 2)$-component. To match this $(k - 2)$-component on stack 2, a total of $2^{k-2} - 2^{k-3} = 2^{k-3} \geq 4$ tokens are needed. In this case, only three tokens remain to be played by the two players together, so $P_2$ cannot build up a $(k - 2)$-component in stack 2 to match the one of stack 1, irrespective of how $P_1$ plays.

Now we look at the remaining case when $P_2$ contributes $2^{k-3}$ tokens to the bottom layers. In this case, $P_2$ can play on the upper layer of stack 2 earlier, after $P_1$ has played $4(2^{k-3} - 2) + 2 = 2^{k-1} - 6$ tokens. At this point, stack 1 contains $2^{k-1} - 6$ tokens, and since $k \geq 5$, it contains a $(k - 2)$-component. After the move by $P_2$ shown in Fig. 9, each player has exactly $(2^{k-1} - 1) - (2^{k-1} - 6) = 5$ moves left, irrespective of $k$.

To build up a matching $(k - 2)$-component in the second stack, $2^{k-3}$ tokens are needed (see above). If $k > 5$, then $P_2$ does not have enough tokens to do so on his own, and $P_1$ can avoid helping in the build-up by playing high. If $k = 5$, then $P_2$ has enough remaining tokens to match the 3-component by playing 4 of his tokens on stack 2. $P_1$ can counter by playing her 5 tokens on stack 3 (and her stack will never be higher than his). The position reached before the last token is placed by $P_2$ is $(10, 8, 7, 2, 2)$, and no matter where he plays, there will be an unmatched component in stack 3—either a 2-component if he does not play on stack 3 or a 3-component if he does. If $P_2$ does not build up the 3-component in stack 2, then that component will be unmatched. Finally, if $P_2$ waits longer to play his 4th token in the bottom layers, then he will not have enough tokens left to match the 3-component in stack 2. In all cases, $P_1$ wins.  

The proof of Theorem 3.4 will make clear why we need to check the following cases.
Lemma 3.7  P1 has a winning strategy for BN(2n, 5) for n = 5, . . . , 12.

Proof  These cases have been checked by a computer program, using the following natural algorithm derived from the definition of $\mathcal{P}$- and $\mathcal{N}$-positions: given a number $2n$ of tokens, we start by computing the outcomes of the positions $(x_1, \ldots, x_5; 0)$. Clearly those which satisfy $x_1 + \cdots + x_5 = 0$ are $\mathcal{P}$, and the other ones are $\mathcal{N}$. Now, for all $\xi$ from 1 to $2n$ we compute the outcomes of all positions $(x_1, \ldots, x_5; \xi)$ with $x_1 + \cdots + x_5 = 2n - \xi$ as follows: a position $(x_1, \ldots, x_5; \xi)$ is $\mathcal{N}$ if at least one of its options is $\mathcal{P}$, otherwise it is $\mathcal{P}$. Note that each position $(x_1, \ldots, x_5; \xi)$ admits five options, namely

$$\left\{ (x_1 + t_1, x_2 + t_2, x_3 + t_3, x_4 + t_4, x_5 + t_5; \xi - 1) : t_i \in \{0, 1\}, \sum_{i=1}^5 t_i = 1 \right\}.$$ 

A computation of the outcomes of positions $(0, \ldots, 0; 2n)$ for $n = 5, \ldots, 12$ shows they are $\mathcal{N}$. \hfill \Box

We are now ready to prove Theorem 3.4.

Proof of Theorem 3.4  We must prove that P1 wins BN(2n, 5) if and only if $n \geq 5$. By Theorem 3.2, P2 wins if $n < 5$. For $n \geq 5$ a given integer, let $k(n)$ be the unique integer such that $2^{k(n)-1} < n \leq 2^{k(n)}$ and let $p(n) = 2^{k(n)} - 1$ be the largest power of 2 strictly less than $n$. For convenience, we will replace $k(n)$ and $p(n)$ by $k$ and $p$ respectively, when the context is clear. We will proceed by induction on $n$. According to Lemma 3.7, P1 wins for all $5 \leq n \leq 12$, and in particular when $k(n) = 3$. Let $n > 12$ and assume that P1 wins for all $m$ with $3 \leq k(m) < k(n)$.

If $n = 2^k - 1$, Lemma 3.6 applies and P1 wins. If $n \neq 2^k - 1$, then one of P1’s strategies will be to reuse the winning strategy of a smaller game if P2 matches a power of two, $\pi$, in the $\pi$-phase; see the respective cases (ii) below. By Lemma 3.5, she has to be careful to choose an appropriate $\pi$, because BN(2$\delta$, 5) is not a first player win for $\delta \leq 4$. Therefore, we consider two cases.

Case 1: $n - p > 4$.

Here P1 chooses $\pi = p$, so $\delta = n - p > 4$ and $k(\delta) \geq 3$. In the $\pi$-phase (which consists of playing $p$ tokens each) P1 plays high, independent of P2’s responses. Then P1 adjusts her strategy for the $\delta$-phase depending on the play of P2 in the $\pi$-phase:

(i) If P2 played at least one of his $\pi$-phase tokens on the tallest stack, then P1 continues to play high in the $\delta$-phase. At the end of BUILDING, the maximum stack height will be larger than the sum of the other stacks, and P1 wins by (NS7) (Lemma 2.2).

(ii) If P2 has matched P1’s play on stack 2, then by the contraposition of Lemma 3.5, P1 wins BN(2$n$, 5) as she has a winning strategy for BN(2$\delta$, 5). Indeed, the conditions of Lemma 3.5 are fulfilled since $\delta < 2\pi$, and by induction hypothesis, P1 can win BN(2$\delta$, 5) since $3 \leq k(\delta) < k(n)$.

(iii) The remaining case is that P2 neither played on the tallest stack nor matched play by P1 on the second stack. This means that $s_1 = \pi$, $s_2 < \pi$, and $s_3 > 0$.
Fig. 10  A P1 strategy for achieving a single
(k − 1)-component

\[ \pi = 2^{k-1} \]

The case \( n = 2^k = 2\pi \) is trivial since P1 continues to play high on stack 1 in the \( \delta \)-phase and wins, because there will be an unmatched \( k \)-component in stack 1 when NIM starts. Since \( n \neq 2^k - 1 \), we may assume that \( n < 2^k - 1 \leq 2\pi - 1 \), that is

\[ \delta < \pi - 1. \] (3.3)

In this case, P1 adjusts her moves in response to P2’s play. The strategy of P1 hinges on whether P2 will be able to build stack 2 to a height of \( \pi \) to match stack 1 or not. To prove that P1 has a winning strategy, we keep track of the stack heights after each pair of moves. After \( \xi \) moves have been played by both players in the \( \delta \)-phase, starting with P1, each player has \( \delta - \xi \) tokens to play, and we denote the number of tokens on the \( i \)th stack by \( s_i(\xi) \). We discuss two ways in which P1 can win.

If at the end of BUILDING, P2 has failed to match stack \( s_1 \) (by not having built up stack \( s_2 \) to a level of \( \pi \) tokens), then there will be an unmatched \( (k - 1) \)-component, so P1 wins. We claim that this type of position will be reachable for P1 if, after \( \xi \) moves by each player in the \( \delta \)-phase, the number of tokens to be played by P2 is insufficient to cover the gap between \( s_2(\xi) \) and \( \pi \) (see Fig. 10).

Suppose that

\[ s_2(\xi) + (\delta - \xi) < \pi \]

or equivalently,

\[ s_2(\xi) - \xi < \pi - \delta, \] (3.4)

holds at some stage in the \( \delta \)-phase. The claim is that P1 can ensure it still holds for the rest of BUILDING, by always playing high. Then

\[ s_2(\xi + 1) - (\xi + 1) \leq s_2(\xi) + 1 - (\xi + 1) < \pi - \delta, \] (3.5)

and inequality (3.4) will hold also for \( \xi + 1 \).

On the other hand, if P2 has enough tokens to complete the second stack to size \( \pi \), then P1 wins if she builds up stack \( s_3 \) to a height of more than \( \delta \) tokens. Indeed, at the end of the \( \delta \)-phase we would have a position of the form \((\pi + x'_1, \pi + x'_2, s_3, s_4, s_5)\)
and then she wins once more by (NS7) of Lemma 2.2, now applied to \( y = s_3 \) and \( x'_1, x'_2, s_4, s_5 \) (see Fig. 11).

P1 can reach such a position if, after each move of P2

\[
 s_3(\xi) > \xi. \tag{3.6}
\]

Note that inequality (3.6) holds at the beginning of the \( \delta \)-phase, as we assumed that \( s_3 = s_3(0) > 0 \). It now remains to be seen whether the inequality can be maintained throughout. We may assume w.l.o.g. that inequality (3.5) does not hold for any \( \xi \), as otherwise P1 wins by playing high. Now assume that \( s_3(\xi) - \xi > 0 \) and that P1 plays on \( s_3 \). Then, since P2 may also play on \( s_3 \), we have that

\[
 s_3(\xi + 1) - (\xi + 1) \geq s_3(\xi) + 1 - (\xi + 1) > 0, \tag{3.7}
\]

unless \( s_2(\xi) = s_3(\xi) \). This case could result in \( s_2(\xi + 1) = s_3(\xi) + 1 \) and \( s_3(\xi + 1) = s_3(\xi) \) (if P2 does not play on either stacks 2 or 3), so inequality (3.7) does not hold any longer. However, since (3.5) does not hold for any \( \xi \) by assumption, we have

\[
 s_3(\xi + 1) - (\xi + 1) = s_3(\xi) - (\xi + 1) = s_2(\xi + 1) - 1 - (\xi + 1) \geq \pi - \delta - 1 > 0 \tag{3.8}
\]

by (3.3).

Thus, either (3.6) can be maintained throughout, or P1 can switch to playing high if (3.4) holds at some point in the \( \delta \)-phase. 

**Case 2:** \( 1 \leq n - p \leq 4 \).

1. If P2 played his first \( p/2 \) tokens on the second stack, P1 chooses \( \pi = p/2 = 2^{k-2} \), so \( \delta = n - \pi = n - 2^{k-2} \). Therefore, \( 2^{k-2} < \delta \leq 2^{k-1} \), so \( k(\delta) = k(n) - 1 \). Note that since \( n > 12 \), we have that \( k(n) \geq 4 \), and by induction hypothesis, BN(\( 2\delta, 5 \)) is winning for P1 since \( 3 \leq k(\delta) = k(n) - 1 < k(n) \). P1 can apply her winning strategy on top of the matched stacks in the \( \delta \)-phase because

\[
x_1 + \cdots + x_5 = 2n - 2\pi = 2n - p \leq n + 4 \leq p + 8 \leq 4 \cdot 2^{k-2} = 4\pi,
\]

so the contraposition of Lemma 3.5 applies and this strategy leads to a win for P1.

2. If P2 did not play his first \( p/2 \) tokens on the second stack, then P1 goes on playing high on the tallest stack until the end of the \( \pi \)-phase, with \( \pi = p \), and then plays the \( \delta \)-phase according to Case 1 (i) or (iii), assuring her win. \( \square \)

**4 Discussion**

We have shown that in the case of three stacks, P2 has a winning strategy for \( n = 2^k - 2 \), which is no longer true for five stacks. With just three stacks, P1 does not have much wiggle room, and P2 can force a win, but with five stacks, P1 gains enough of an advantage in being able to play low. The proof of Lemma 3.6 can most likely be extended to more stacks, but in the proof of the main result, the cases where P1 uses
a winning strategy for a smaller game on top of two stacks of size $2^i$ for some $i$ depends on verification by computer that P1 has winning strategies for a finite number of initial cases. The same would be true for any odd number of stacks $\ell > 5$, with the number of initial cases increasing as the number of stacks increases. The conditions of Lemma 3.5 can be used to precisely define the number of initial games that are needed to use the induction argument. We do not currently have a general argument to prove that P1 can win these initial games but have found manual proofs for several values of $\ell$. For many of P1’s winning strategies that we have checked it suffices for her to respond to P2’s defense attempts by playing ‘high or low’, but we have also encountered cases where such strategies fail, where P1 still wins, but only by departing from ‘high or low’ play. Conjecture 3.3 suggests that P2 rarely wins for the interesting cases of BUILDING NIM ($n$ even and $\ell$ odd), notably fitting the result by Singmaster (1981) that almost all games are first player wins.

In the process of our computer explorations we have also computed Grundy values for all strict BUILDING positions for odd numbers of stacks $5 \leq \ell \leq 19$ and an even number of tokens $\ell + 3 < n \leq 34$. The Grundy function takes only the values 0, 1 or 2. More specifically, P1 moves from positions with Grundy value 0, 1 or 2, and P2 moves from positions with Grundy values 0 or 1. This gives rise to the following questions:

(1) Does this observation hold in general?
(2) Does this observation provide an answer to whether P2 only moves from Grundy value 0 in optimal play?

We note that if the number of tokens is greater than the number of heaps, then the $P$-positions of normal play BUILDING NIM are the same as those of the misère variation (a player who cannot move wins). Indeed, the $P$-positions of Nim and misère Nim are the same, provided there is one heap of size at least 2.

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