A note on periodic differential equations

Mauro Patrão

August 19, 2009

Abstract

Let $F$ be a Banach space and $L(F)$ be the set of all its bounded linear operators. In this note, we are interested in the asymptotic behavior (recurrence and chain recurrence) of the solution of the following initial value problem

$$x'(t) = X(t)x(t), \quad x(0) = x,$$

where $x \in F$ and the map $t \mapsto X(t) \in L(F)$ is a $T$-periodic continuous curve. This asymptotic behavior is related to the asymptotic behavior of the discrete-time flow on $F$ generated by the invertible operator $g \in L(F)$ given by the associated fundamental solution at time $T$.

AMS 2000 subject classification: Primary: 37B35, 34G10. Secondary: 34A30.

Key words: Recurrence, chain transitivity, stable sets, Banach space.

1 Introduction

Denote by $|\cdot|$ the norm of the Banach space $F$. The canonical operator norm in $L(F)$ is also denoted by $|\cdot|$. It is well known that the solution of (1) is given by $x(t) = g(t)x$, where $t \mapsto g(t) \in L(F)$ is a differentiable curve, called the associated fundamental solution, satisfying the following initial value problem

$$g'(t) = X(t)g(t), \quad g(0) = I,$$

where $I$ is the identity operator. The existence and the uniqueness of the fundamental solution defined for all $t \in \mathbb{R}$ follow from the Picard method, since $X(t)$ is uniformly bounded and $L(F)$ is a Banach space. We can use
this fact to show that the inverse of $g(s)$ is given by $h(-s)$, where $t \mapsto h(t)$ is the fundamental solution associated to $t \mapsto X(t+s)$. We can assume that the period $T = 1$, since otherwise we can replace $X(t)$ by $(1/T)X(t/T)$ whose associated fundamental solution is given by $h(t) = g(t/T)$. The existence and the uniqueness of the fundamental solution together with the periodicity of $X(t)$ can be used to prove that
\[ g(t + n) = g(t)g^n, \] for all $t \in \mathbb{R}$ and all $n \in \mathbb{N}$, where $g = g(1)$.

We denote by $S^1$ the quotient $\mathbb{R}/\mathbb{Z}$ and by $\overline{s}$ the class $s + \mathbb{Z}$, where $s \in \mathbb{R}$. Using equation (2), the so called associated skew-product flow in $S^1 \times F$, defined by
\[ \phi^t(\overline{s}, x) = (\overline{s + t}, g(t + s)g(s)^{-1}x), \]
is a well defined continuous map such that $\phi^t(\overline{0}, x) = (\overline{t}, x(t))$. When $F$ is finite dimensional, Floquet theorem (see e.g [2]) ensures that there exists $X \in L(F)$ such that $g^2 = e^X$. In this case, we have that the associated skew-product is conjugated with the flow given by
\[ \psi^t(\overline{s}, x) = (\overline{s + t}, e^{tX}x). \]

On the other hand, when $F$ is infinite dimensional, there is not in general any $X \in L(F)$ such that $g^n = e^X$, for some $n \in \mathbb{N}$. Thus we can not always analyze the asymptotic behavior of the solution $x(t) = g(t)x$ by considering the asymptotic behavior of an exponential flow in the same way which is done by the Floquet theory.

In this note, we provide an approach which overcome this difficult, relating the asymptotic behavior of the associated skew-product flow with the asymptotic behavior (recurrence and chain recurrence) of the discrete-time flow on $F$ generated by the invertible operator $g$. The presented results can be viewed as a sort of extension to the infinite dimensional situation of the Selgrade Theorem (see [7]), which deals with linear flows on finite dimensional vector bundles over chain-transitive flows on compact base spaces. As we observe in the end of this note, all the results remain true if we replace the fiber given by Banach space $F$ by any compact metrizable fiber where the topological group of the bounded invertible linear operators acts continuously. In this way, the results can also be viewed as a sort extension of the results presented in [1] and [6], which deal with flows of automorphisms of flag bundles over chain-transitive flows on compact base spaces.
2 Some technical results

In this section, we prove some technical results concerning the fundamental solution and the associated skew-product flow. We observe that, by the definitions, it follows that

$$\phi^t(s,g(s)x) = (s+t,g(s+t)x).$$  \hspace{1cm} (3)

for all $s, t \in \mathbb{R}$ and $x \in F$.

Lemma 2.1 We have that

$$S^1 \times F = \{ (s,g(s)x) : s \in [0,1), x \in F \}.$$  

Furthermore, for each $s \in [0,1)$ and all $t \geq 0$, there exist $\tau \in (-1,1)$ and $n \in \mathbb{N}$ such that

$$t = \tau + n \quad \text{and} \quad s + \tau \in [0,1).$$

In this case, we have that

$$\phi^t(s,g(s)x) = (s+\tau,g(s+\tau)g^n x).$$  \hspace{1cm} (4)

Proof: For each $r \in \mathbb{R}$, there exists a unique $s \in [0,1)$ such that $\tau = s$. Thus, for each $y \in F$, putting $x = g(s)^{-1}y$, we have that $(\tau,y) = (s,g(s)x)$, showing the first assertion. For the second assertion, let $n$ the greatest natural number such that $n \leq s + t$. Since $n \leq s + t < (n+1)$, setting $\tau = t - n$, it follows that $s + \tau \in [0,1)$. Thus $\tau \in (-1,1)$, since $s \in [0,1)$. For the last assertion, by equations (3) and (2), it follows that

$$\phi^t(s,g(s)x) = (s+\tau+n,g(s+\tau+n)x) = (s+\tau,g(s+\tau)g^n x).$$

We denote by $d$ the metric induced in $F$ by the norm $|\cdot|$ and consider the following metric in $S^1 \times F$ given by

$$\overline{d}((s,x),(\tau,y)) = \min\{|s-r|, 1-|s-r|\} + d(x,y),$$

where $s, r \in [0,1)$ and $x, y \in F$. 

3
**Lemma 2.2** There exists a constant $C \geq 1$ such that
\[
d(g(t)x, g(t)y) \leq C d(x, y)
\]
for each $t \in [0, 1]$ and every $x, y \in F$. We also have that
\[
\overline{d}(\phi^t(s, x), \phi^t(t, y)) \leq C \overline{d}((s, x), (t, y))
\]
for each $r, s, t \in [0, 1]$ and every $x, y \in F$.

**Proof:** Since $t \mapsto |g(t)|$ is a real continuous function, it is bounded by a constant $C \geq 1$ in the compact interval $[0, 1]$. Hence we have that
\[
d(g(t)x, g(t)y) = |g(t)(x - y)| \leq |g(t)||x - y| \leq C d(x, y).
\]
The inequality (6) follows immediately from the inequality (5) and the definitions of $\overline{d}$ and of $\phi^t$. \hfill \square

Given a topological space $U$, we denote by $P(U)$ the set of all positive continuous functions whose domain is the whole space $U$. We also need the following lemma.

**Lemma 2.3** There exists $c \in P(F)$ with $c \geq 1$ such that
\[
d(x, y) \leq c(x)(|s - r| + d(g(s)x, g(r)y))
\]
for all $r, s \in [-2, 2]$ and every $x, y \in F$.

**Proof:** We have that $t \mapsto g(t)^{-1}$ is a differentiable map, since it is a composition of the differentiable maps $t \mapsto g(t)$ and $h \mapsto h^{-1}$. By the mean value inequality, there exists $B \geq 1$ such that $|g(s)^{-1} - g(r)^{-1}| \leq B|s - r|$, for all $r, s \in [-2, 2]$. We also have that there exists $D \geq 1$ such that $|g(r)|, |g(r)^{-1}| \leq D$, for all $r \in [-2, 2]$. Defining $c(x) = \max\{BD|x|, D\}$, we have that $c \in P(F)$ with $c \geq 1$ and that
\[
d(x, y) &= |g(s)^{-1}g(s)x - g(r)^{-1}g(r)y| \\
&\leq |g(s)^{-1}g(s)x - g(r)^{-1}g(s)x| + |g(r)^{-1}g(s)x - g(r)^{-1}g(r)y| \\
&\leq |g(s)^{-1} - g(r)^{-1}| ||g(s)||x| + |g(r)^{-1}||g(s)x - g(r)y| \\
&\leq BD|x||s - r| + D|g(s)x - g(r)y| \\
&\leq c(x)(|s - r| + d(g(s)x, g(r)y)),
\]
for all $r, s \in [-2, 2]$ and every $x, y \in F$. \hfill \square

We end this technical section with the following result.
Lemma 2.4 Let $P : K \times U \to \mathbb{R}$ be a continuous function, where $K$ is a compact space and $U$ is a topological space. Then we have that $p : U \to \mathbb{R}$, given by

$$p(x) = \min\{P(k, x) : k \in K\},$$

is a continuous function.

Proof: Let $x \in U$. Since $P$ is continuous, given $\varepsilon > 0$, for each $k \in K$, there exist an open neighborhood $A_k \subset K$ of $k$ and an open neighborhood $B_k \subset U$ of $x$ such that $|P(l, x) - P(l, y)| < \varepsilon$, for all $l \in A_k$ and all $y \in B_k$. Since $\{A_k : k \in K\}$ is an open cover of the compact set $K$, there exists a finite subcover $\{A_{k_1}, \ldots, A_{k_n}\}$. Defining $B = B_{k_1} \cap \cdots \cap B_{k_n}$, it follows that $|P(l, x) - P(l, y)| < \varepsilon$, for all $l \in K$ and all $y \in B$. Hence we have that $|p(x) - p(y)| < \varepsilon$, for all $y \in B$, showing that $p$ is continuous. \qed

3 Asymptotic behavior

We begin this section recalling the asymptotic concepts which we deal with in this note (see [3], [4] and [5]). Let $\sigma^t$ be a continuous-time ($t \in \mathbb{R}$) or a discrete-time ($t \in \mathbb{Z}$) flow on a metric space $(U, d)$. The $\omega$-limit set of a given point $x \in U$ is the set of points $y \in U$ such that there exists a sequence $t_k \to \infty$ such that $\sigma^{t_k}(x) \to y$. A point $x \in U$ is recurrent if it belongs to its own $\omega$-limit set. The set of all recurrent points is called recurrent set of $\sigma^t$ and denoted by $\mathcal{R}(\sigma^t)$. A subset $V$ of $U$ is $\sigma^t$-invariant if $\sigma^t(V) = V$ for each time $t$. The stable set of $V$, denoted by $\text{st}(V)$, is the set of all points having its $\omega$-limit set contained in $V$. Given $x, y \in U$, $\varepsilon \in \mathcal{P}(U)$ and $t > 0$, an $(\varepsilon, t)$-chain from $x$ to $y$ is given by a set of times $t_i > t$ and a set of points $x_i \in U$, with $x_0 = x$, $x_n = y$ and such that $d(x_{i+1}, \sigma^{t_i}(x_i)) < \varepsilon(\sigma^{t_i}(x_i))$, where $i = 1, \ldots, k$. Two given points $x, y \in U$ are chain equivalent if, for all $\varepsilon \in \mathcal{P}(U)$ and $t > 0$, there exist an $(\varepsilon, t)$-chain from $x$ to $y$ and also from $y$ to $x$. A set $\mathcal{M}$ is chain transitive if every two points of it are chain equivalent. A point is chain recurrent if it is chain equivalent with itself. The set of all chain recurrent points is called chain recurrent set of $\sigma^t$ and denoted by $\mathcal{R}_c(\sigma^t)$. It is easy to verify that the chain equivalence is an equivalence relation in the chain recurrent set. Hence the chain recurrent set can be partitioned into chain equivalence classes which are called the chain transitive components. It can be proved that the recurrent and the chain
Let $E$ be a $g$-invariant subset of $F$. Using equation (2), we have that

$$S^1 \times_g E = \bigcup_{s \in \mathbb{R}} \{s\} \times g(s)E$$

is a well defined $\phi^t$-invariant subset of $S^1 \times F$. We use this construction to relate the asymptotic behavior of the associated skew-product flow $\phi^t$ with the asymptotic behavior of the discrete-time flow on $F$ generated by the invertible operator $g$. We begin relating their recurrent sets.

**Proposition 3.1** We have that

$$\mathcal{R}(\phi^t) = S^1 \times_g \mathcal{R}(g).$$

**Proof:** Let $(\tilde{s}, g(s)x) \in S^1 \times F$, where $s \in [0, 1)$ and $x \in \mathcal{R}(g)$. By definition of $\mathcal{R}(g)$, there exists $n_k \to \infty$ such that $g^{n_k}x \to x$. Using equation (4), we have that

$$\phi^{n_k}(\tilde{s}, g(s)x) = (\tilde{s}, g(s)g^{n_k}x) \to (\tilde{s}, g(s)x),$$

showing that $S^1 \times_g \mathcal{R}(g) \subset \mathcal{R}(\phi^t)$. On the other hand, let $(\tilde{s}, g(s)x) \in \mathcal{R}(\phi^t)$, where $s \in [0, 1)$. By definition of $\mathcal{R}(\phi^t)$, there exists $t_k \to \infty$ such that

$$\phi^{t_k}(\tilde{s}, g(s)x) \to (\tilde{s}, g(s)x).$$

By Lemma 2.1 there exist $\tau_k \in (-1, 1)$ and $n_k \in \mathbb{N}$ such that

$$t_k = \tau_k + n_k \quad \text{and} \quad s + \tau_k \in [0, 1).$$

Thus we have that $n_k \to \infty$, since $t_k \to \infty$. Using equation (4), we have that

$$\phi^{t_k}(\tilde{s}, g(s)x) = (\tilde{\tau}_k + s + \tau_k, g(s + \tau_k)g^{n_k}x) \to (\tilde{s}, g(s)x).$$

Hence $s + \tau_k \to \tilde{s}$ and, since $s$ and $s + \tau_k \in [0, 1)$, we have that $s + \tau_k \to s$. Therefore

$$g^{n_k}x = g(s + \tau_k)^{-1}g(s + \tau_k)g^{n_k}x \to g(s)^{-1}g(s)x = x,$$

showing that $x \in \mathcal{R}(g)$ and that $\mathcal{R}(\phi^t) \subset S^1 \times_g \mathcal{R}(g)$. \hfill \square

Now we consider the invariant and chain transitive sets.
Proposition 3.2 If $\mathcal{M}$ is invariant and chain transitive for $g$, then $S^1 \times_g \mathcal{M}$ is chain transitive for $\phi^t$.

Proof: First we prove that, for all $\varepsilon \in \mathcal{P}(S^1 \times F)$, $t > 0$, $s \in [0, 1)$ and for every $x, y \in \mathcal{M}$, there exists an $(\varepsilon, t)$-chain from $(\overline{0}, x)$ to $(\overline{s}, g(s)y)$. By Lemma 2.4, defining

$$\delta(z) = C^{-1} \min \{\varepsilon(\overline{r}, g(r)z) : r \in [0, 1]\},$$

we have that $\delta \in \mathcal{P}(F)$. Since $\mathcal{M}$ is chain transitive, there exists a $(\delta, t)$-chain from $x$ to $y$ given by $n_i > t$ and $x_i \in F$ such that $d(x_{i+1}, g^n x_i) < \delta(g^n x_i)$, where $i = 1, \ldots, k$. Putting $r = s/k < 1$ and defining

$$t_i = n_i + r > t, \quad \eta_i = (\overline{0}, x) \quad \text{and} \quad \eta_{i+1} = (\overline{ir}, g(ir)x_{i+1}),$$

where $i = 1, \ldots, k$, we claim that this provides an $(\varepsilon, t)$-chain from $(\overline{0}, x)$ to $(\overline{s}, g(s)y)$. In fact, since $\phi^t(\eta_i) = (\overline{ir}, g(ir)g^n x_i)$ and $ir \in [0, 1]$, we have that

$$\overline{d}(\eta_{i+1}, \phi^t(\eta_i)) = d(g(ir)x_{i+1}, g(ir)g^n x_i) \leq C d(x_{i+1}, g^n x_i) < C\delta(g^n x_i)$$

and hence

$$\overline{d}(\eta_{k+1}, \phi^t(\eta_k)) < \varepsilon(\overline{ir}, g(ir)g^n x_i) = \varepsilon(\phi^t(\eta_i)).$$

We also have that $\eta_{k+1} = (\overline{s}, g(s)y)$, since $kr = s$ and $x_{k+1} = y$.

Now we prove that, for all $\varepsilon \in \mathcal{P}(S^1 \times F)$, $t > 0$, for each $u, v \in [0, 1)$ and for every $x, y \in \mathcal{M}$, there exists an $(\varepsilon, t)$-chain from $(\overline{u}, g(u)x)$ to $(\overline{v}, g(v)y)$. We denote $\delta = C^{-1}\varepsilon \circ \phi^u \in \mathcal{P}(S^1 \times F)$. If $v \geq u$, we have that $v - u \in [0, 1)$. Using the first part of the proof, there exists a $(\delta, t)$-chain from $(\overline{0}, x)$ to $(\overline{v - u}, g(v - u)y)$, denoted by $t_i > t$ and $\eta_i \in S^1 \times F$, with $\overline{d}(\eta_{i+1}, \phi^t(\eta_i)) < \delta(\phi^t(\eta_i))$, where $i = 1, \ldots, k$. Taking the same times $t_i > t$ and putting $\xi_i = \phi^u(\eta_i)$, we get an $(\varepsilon, t)$-chain from $(\overline{u}, g(u)x)$ to $(\overline{v}, g(v)y)$. In fact, using inequality (5), we have that

$$\overline{d}(\xi_{i+1}, \phi^t(\xi_i)) = \overline{d}(\phi^u(\eta_{i+1}), \phi^u(\phi^t(\eta_i))) \leq C \overline{d}(\eta_{i+1}, \phi^t(\eta_i)) < \varepsilon(\phi^t(\eta_i))$$

and that

$$\xi_1 = \phi^u(\overline{0}, x) = (\overline{u}, g(u)x) \quad \text{and} \quad \xi_{k+1} = \phi^u(\overline{v - u}, g(v - u)y) = (\overline{v}, g(v)y).$$

If $v < u$, we have that $1 + v - u \in [0, 1)$. Using again the first part of the proof, there exists a $(\delta, t)$-chain from $(\overline{0}, x)$ to $(\overline{1 + v - u}, g(1 + v - u)g^{-1}y)$,
denoted by $t_i > t$ and $\eta_i \in S^1 \times F$, where $i = 1, \ldots, k$. Arguing exactly as before, we obtain an $(\varepsilon, t)$-chain from $(\tau, g(u)x)$ to $(\tau, g(v)y)$, if we take the same times $t_i > t$ and put $\xi_i = \phi^t(\eta_i)$, since

$$
\phi^t(1 + v - u, g(1 + v - u)g^{-1}y) = (1 + v, g(1 + v)g^{-1}y) = (\tau, g(v)y).
$$

In the following, we provide the relation between the chain recurrent sets.

**Theorem 3.3** We have that

$$
R_c(\phi^t) = S^1 \times_g R_c(g).
$$

**Proof:** By Theorem 3 of [3], we have that $R_c(\phi^t) = R_c(\phi^1)$. Hence we just need to prove that $R_c(\phi^1) = S^1 \times_g R_c(g)$. Since every chain recurrent set is partitioned in its chain transitive components, using Proposition 3.2, it follows that $S^1 \times_g R_c(g) \subset R_c(\phi^1)$. Let $(s, g(s)x) \in R_c(\phi^1)$, where $s \in [0, 1)$ and $x \in F$. We claim that $x \in R_c(g)$. Let $\varepsilon \in \mathcal{P}(F)$ and $n \in \mathbb{N}$. By Lemma 2.4, defining

$$
\delta(\tau, y) = 2^{-1} \min\{c(g(t)^{-1}y)g^{-1}\varepsilon(g(t)^{-1}y) : t \in [0, 1]\},
$$

it follows that $\delta \in \mathcal{P}(S^1 \times F)$. Hence there exists an $(\delta, n)$-chain from $(\tau, g(s)x)$ to itself, denoted by $m_i > 2n$ and $\xi_i = (\tau, g(s_i)x_i) \in S^1 \times F$, where $s_i \in [0, 1)$ and $\delta(\xi_{i+1}, \phi^{m_i}(\xi_i)) < \delta(\phi^{m_i}(\xi_i))$, where $i = 1, \ldots, k$. Hence we have that

$$
\min\{|s_{i+1} - s_i|, 1 - |s_{i+1} - s_i|\} < \delta(\phi^{m_i}(\xi_i)).
$$

Thus we get the following three possibilities and the corresponding choices

1. $|s_{i+1} - s_i| < \delta(\phi^{m_i}(\xi_i))$, $n_i = m_i$
2. $(1 + s_i) - s_{i+1} < \delta(\phi^{m_i}(\xi_i))$, $n_i = m_i - 1$
3. $s_{i+1} - (s_i - 1) < \delta(\phi^{m_i}(\xi_i))$, $n_i = m_i + 1$

In each possibility, we have that $n_i > n$ and, using inequality (7), we get that

$$
d(x_{i+1}, g^{m_i}x_i) \leq 2c(g^{m_i}x_i)\delta(\phi^{m_i}(\xi_i)) \leq \varepsilon(g^{m_i}x_i),
$$

showing that $x \in R_c(g)$, since $x_0 = x = x_{k+1}$. \qed
Corollary 3.4  The map $E \mapsto S^1 \times_g E$, where $E \subset F$ is $g$-invariant, is a bijection between chain transitive components of $g$ and of $\phi^t$.

**Proof:** By Proposition 3.2, if $\mathcal{M}$ is a chain transitive component of $g$, then $S^1 \times_g \mathcal{M}$ is chain transitive for $\phi^t$ and hence is contained in a chain transitive component of $\phi^t$. The result follows, since $R_c(\phi^t) = S^1 \times_g R_c(g)$ and since every chain recurrent set is partitioned in its chain transitive components. □

We end this note relating the stable sets of chain transitive components.

**Proposition 3.5** If $\mathcal{M}$ is a chain transitive component of $g$, then $\text{st}(S^1 \times_g \mathcal{M}) = S^1 \times_g \text{st}(\mathcal{M})$.

**Proof:** Since the $\omega$-limit sets are chain transitive, a point belongs to the stable set of a given chain component if and only if its omega limit set intersects the chain component. For every $x \in \text{st}(\mathcal{M})$, there exists $n_k \to \infty$ and $y \in \mathcal{M}$ such that $g^{n_k}x \to y$. In this case, for each $s \in [0,1)$, we have that

$$\phi^{n_k}(s, g(s)x) = (s, g(s)g^{n_k}x) \to (s, g(s)y),$$

showing that $(s, g(s)x) \in \text{st}(S^1 \times_g \mathcal{M})$ and that $S^1 \times_g \text{st}(\mathcal{M}) \subset \text{st}(S^1 \times_g \mathcal{M})$. For the other inclusion, let $(s, g(s)x) \in \text{st}(S^1 \times_g \mathcal{M})$, where $s \in [0,1)$. Thus there exists $t_k \to \infty$, $y \in \mathcal{M}$ and $r \in [0,1)$ such that

$$\phi^{t_k}(s, g(s)x) \to (r, g(r)y).$$

Hence we have that $\phi^{t_k+s-r}(s, g(s)x) \to (s, g(s)y)$. By Lemma 2.3, there exist $\tau_k \in (-1,1)$ and $n_k \in \mathbb{N}$ such that

$$t_k + s - r = \tau_k + n_k \quad \text{and} \quad s + \tau_k \in [0,1).$$

Thus we have that $n_k \to \infty$, since $t_k \to \infty$. Using equation (4), we have that

$$\phi^{t_k+s-r}(s, g(s)x) = (s + \tau_k, g(s + \tau_k)g^{n_k}x) \to (s, g(s)y).$$

Hence $s + \tau_k \to s$ and, since $s$ and $s + \tau_k \in [0,1)$, we have that $s + \tau_k \to s$. Therefore

$$g^{n_k}x = g(s + \tau_k)^{-1}g(s + \tau_k)g^{n_k}x \to g(s)^{-1}g(s)y = y,$$

showing that $x \in \text{st}(\mathcal{M})$ and that $\text{st}(S^1 \times_g \mathcal{M}) \subset S^1 \times_g \text{st}(\mathcal{M})$. □
Remark 3.6 All the results presented in this section remain true if we replace the Banach space $F$ by an arbitrary compact metrizable fiber $F$ where the topological group of bounded invertible linear operators acts continuously. In fact, in the definition of chains, we can replace the positive continuous functions by positive constant functions, once every positive continuous function defined on a compact space has a positive minimum. Hence we can replace the inequalities (5), (6) and (7) by the uniform continuity, respectively, of the map $(s,x) \mapsto g(s)x$ on $[0,1] \times F$, of the map $\phi^u$ on $F$ and of the map $(s,x) \mapsto g(s)^{-1}x$ on $[-2,2] \times F$.

References

[1] C.J.B. Barros and L.A.B. San Martin: Chain transitive sets for flows on flag bundles. Forum Math. 19 (2007), 19-60.

[2] C. Chicone: Ordinary Differential Equations with Applications, Springer, New-York (1999).

[3] S. K. Choi, C. Chu and J. S. Park : Chain Recurrent Sets for Flows on Non-Compact Spaces. J. Dynam. Differential Equations. 14 (2002), 597-611.

[4] C. Conley: Isolated invariant sets and the Morse index. CBMS Regional Conf. Ser. in Math., 38, American Mathematical Society, (1978).

[5] M. Hurley : Chain recurrence, semiflows and gradients. J. Dynam. Differential Equations. 7 (1995), 437-456.

[6] M. Patrão, L. A. B. San Martin, Morse decompositions of semiflows on fiber bundles, Discrete Contin. Dynam. Systems A, 17 (2007), 561-587.

[7] Selgrade, J.: Isolate invariant sets for flows on vector bundles. Trans. Amer. Math. Soc., 203 (1975), 259-390.