Dynamics of perfect fluids in nonminimally coupled gravity

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May 18, 2012

Abstract

An alternative theory of gravity exhibiting a nonminimal coupling between curvature and matter is discussed. It is shown that this extension of $f(R)$ theories leads to some new and potentially interesting features, most notably, a modification of the conservation law for the stress-energy tensor. This alteration is explored in the context of perfect fluid matter, and it is argued that its consequences may have relevance both at stellar and galactic level. Regarding the latter, a dark matter mimicking mechanism is introduced.

Keywords: Modified theories of gravity, Perfect fluids, Dark matter, Modified Newtonian dynamics.

1 Introduction

Einstein’s General Relativity (GR) has been around for almost one hundred years now, and, to this date, it is still regarded as the prime model of gravitational interaction [1, 2]. Nevertheless, as more astrophysical and cosmological data is gathered everyday, it is starting to become apparent that the theory has some shortcomings, with dark matter [3, 4, 5, 6] and dark energy [7, 8, 9] as the top challengers.

Facing these mounting puzzles, one has to admit that a description of the Universe relying solely on GR seems somewhat limited. The simplest alternative that adequately fits the experimental observations is the concordance model or Λ-CDM (Λ-Cold Dark Matter), which is the combination of GR with a cosmological constant $\Lambda$, dark matter and inflation, the latter usually based on some scalar field called inflaton. Nevertheless, the theoretical motivation for this theory seems rather poor: it does not explain the origin of inflation
or the nature of dark matter, and the dark energy sector (controlled by the cosmological constant) is burdened with well-known problems [10]. Other hypothesis have been put forward, such as TeVeS or \(f(R)\) theories [11, 12].

More recently, a different model exhibiting a nonminimal coupling (NMC) between the matter Lagrangian density and the curvature has been suggested [13, 14]. This proposal can be regarded as an extension of the typical \(f(R)\) theories, and it has already been shown that it introduces new and potentially interesting phenomenology, such as an extra acceleration in the equation of motion of perfect fluids [13], or the ability to mimic the effects of dark matter [15, 16] and dark energy [17, 18].

This work deals with the dynamics of perfect fluids in the context of the abovementioned theory, and is organized as follows: section 2 presents the general features of the model, with particular emphasizes on those that have direct relevance to the rest of the work. Section 3 contains the main results of the thesis, and follows the ideas introduced in [19]. The case of both static and nonstatic fluids is considered, and some possible applications are discussed. The work concludes in section 4.

## 2 Nonminimally coupled gravity:

### general features

The proposal put forward in [13] generalizes “\(f(R)\) theories” action by introducing a nonminimal coupling between curvature and matter:

\[
S = \int \left[ \frac{1}{2} f_1(R) + f_2(R) \mathcal{L}_m \right] \sqrt{-g} d^4x, \tag{1}
\]

where \(f_i(R)\) are arbitrary functions of the scalar curvature. Setting \(f_2(R) = 1\) one recovers the usual \(f(R)\) theories, while this choice with \(f_1(R) = 2\kappa (R - 2\Lambda)\), where \(\kappa = \frac{1}{16\pi G} (c = 1)\), originates the standard Einstein-Hilbert action (with cosmological constant \(\Lambda\)).

Varying action (1) with respect to the metric coefficients yields the field equations

\[
(F_1 + 2F_2 \mathcal{L}_m) R_{\mu\nu} - \frac{1}{2} f_1 g_{\mu\nu} = \nabla_{\mu} (F_1 + 2F_2 \mathcal{L}_m) + f_2 T_{\mu\nu}, \tag{2}
\]

where one defines \(F_i(R) = f_i'(R)\) for convenience, and the argument of the functions is omitted. The stress-energy tensor is defined, as usual, by

\[
T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} \mathcal{L}_m)}{\delta g^{\mu\nu}}. \tag{3}
\]

Taking the trace of eq. (2), one obtains

\[
(F_1 + 2F_2 \mathcal{L}_m) R - 2f_1 = -3\nabla (F_1 + 2F_2 \mathcal{L}_m) + f_2 T. \tag{4}
\]

The covariant derivative of the field equations (2) may be used to derive the conservation law for \(T^{\mu\nu}\), given by

\[
\nabla_\mu T^{\mu\nu} = \frac{F_2}{f_2} [g^{\mu\nu} \mathcal{L}_m - T^{\mu\nu}] \nabla_\mu R. \tag{5}
\]

This equation, which contrasts starkly with the usual \(\nabla_\mu T^{\mu\nu} = 0\) of GR and \(f(R)\) theories, is certainly one of the most interesting features of the model. Physically, it has been interpreted as an
exchange of energy and momentum between curvature and matter fields [13, 20].

The fact that eq. (5) depends only on the function $f_2(R)$ makes it a prime candidate to study the effects of a NMC. Indeed, this feature enables one to consider the general form of the action (1) in most situations. However, when necessary, it will be assumed that $f_1(R) = 2\kappa R$. This is because generally a joint study of both $f_1(R)$ and $f_2(R)$ is unattainable, as the mathematical treatment would be too involved. With this choice, the field equations (2) reduce to

$$\left(1 + \frac{F_2}{\kappa} L_m\right) R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = \frac{1}{\kappa} \Delta_{\mu \nu} (F_2 L_m) + \frac{1}{2\kappa} f_2 T_{\mu \nu},$$

and the trace reads

$$\left(1 - \frac{F_2}{\kappa} L_m\right) R = \frac{3}{\kappa} \Box (F_2 L_m) - \frac{1}{2\kappa} f_2 T.$$

3 Perfect fluids in nonminimally coupled gravity

3.1 Static, spherically symmetric matter distributions

Hydrostatic equilibrium

To study this case, one starts with the “standard” spherically symmetric metric

$$g_{tt} = -B(r),$$
$$g_{rr} = A(r),$$
$$g_{\theta \theta} = r^2,$$
$$g_{\phi \phi} = r^2 \sin^2 \theta,$$
$$g_{\mu \nu} = 0 \text{ for } \mu \neq \nu,$$

and the perfect fluid stress-energy tensor

$$T_{\mu \nu} = (\rho + p) u_\mu u_\nu + p g_{\mu \nu}.$$  

Since the fluid is at rest, the velocity four-vector has components

$$u_r = u_\theta = u_\phi = 0,$$
$$(u_t)^2 = \frac{1}{-g^{tt}} = B(r).$$

The last equality follows from $g^{tt} u_\mu u_\nu = -1$.

With these assumptions, taking $\nu = r$ (the remaining choices yield trivial $0 = 0$ relations) in eq. (5) yields

$$\frac{B'(r)}{B(r)} = -\frac{2}{\rho + p} p'(r) - \frac{2}{f_2(r)} f_2'(r),$$

where the prime represents differentiation with respect to the radial coordinate $r$. This generalizes the usual hydrostatic equilibrium equation of GR.
by adding \(-\frac{2}{f_2(r)} f_2'(r)\) to the RHS. The extra term bears a strong resemblance to the common pressure gradient acceleration, and can be regarded as a sort of “geometric pressure” arising from the NMC. This may bring interesting consequences to stellar dynamics, including, for example, the possibility of having stars supported by this “geometric pressure dynamics”, or a mechanism to prevent the formation of singularities.

Eq. (16) may be integrated from \(r < R_s\) to \(R_s\), where \(R_s\) is the radius of the star, to yield

\[
B(r) = \frac{B(R_s)}{[f_2(r)]^2} \left[ 2 \int_r^{R_s} \frac{\rho'(s)}{\rho(s) + p(s)} ds \right].
\]

(17)

and, using the boundary conditions \(B(R_s) = 1 - \frac{2GM}{R_s}\) and \(f(R_s) \approx 1\), where \(M\) is the mass of the star, the gravitational redshift of radiation emitted at \(r \leq R_s\) and observed by someone at \(R_c > R_s\) is

\[
1 + z_{NMC}(r) = f_2(r) \left[ \frac{1 - \frac{2GM}{R_s}}{1 - \frac{2GM}{R_c}} \right]^{\frac{1}{2}} \exp \left[ - \int_r^{R_s} \frac{\rho'(s)}{\rho(s) + p(s)} ds \right].
\]

(18)

This result may be compared to GR using the formula

\[
1 + z_{NMC}(r) \bigg/ 1 + z_{GR}(r) = f_2(r).
\]

(19)

Note that at the surface of the star there is no difference between NMC gravity and GR. However, if one can probe radiation from the interior of stellar objects with sufficient precision, this formula may be very useful in constraining the model.

**The weak coupling limit**

Having extended the equation of hydrostatic equilibrium to include the effects of the NMC, one now discusses the weak coupling limit of the theory, that is, its applicability to situations where \(rf_2'(r) \ll f_2(r)\).

Under the assumed simplification, the field equations (6) read

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \approx 8\pi G f_2 T_{\mu\nu},
\]

(20)

where the gravitational constant \(G\) was made explicit. Eq. (20) is identical to Einstein’s field equations with an effective gravitational constant given by \(G_{eff} = G f_2(r)\), which may be related to a violation of the Strong Equivalence Principle. Using this result, one finds that

\[
A(r) = \left[ 1 - \frac{2G_{eff} M^*(r)}{r} \right]^{-1},
\]

(21)

where

\[
M^*(r) = M(r) - \frac{1}{f_2(r)} \int_0^r f_2(s) M(s) ds,
\]

(22)

\[
M(r) = 4\pi \int_0^r \rho(s) s^2 ds,
\]

(23)

and, more importantly,

\[
\rho'(r) = -\frac{G_{eff}}{r^2} \left[ \rho(r) + p(r) \right] \times \left[ \frac{4\pi r^3 \rho(r) + M^*(r)}{1 - \frac{2G_{eff} M^*(r)}{r}} - \frac{G_{eff}'}{G_{eff}} \right].
\]

(24)

This is the central result of stellar dynamics in the weak coupling regime. It states that under the condition \(rf_2'(r) \ll f_2(r)\), the general relativistic Tolman-Oppenheimer-Volkoff equation approximately holds, but with an effective gravitational constant, \(G_{eff}\), an effective mass within a sphere of radius \(r\), \(M^*(r)\), and an added term related to the spatial variation of \(G_{eff}\).
\[
\frac{d}{dr} \left[ r^2 \left( \frac{\rho'(r)}{\rho(r)} + k \right) \right] = -4\pi G r^2 \rho(r). \tag{25}
\]

Using the equation of state for polytropes, \( p = \omega \rho^\gamma \), the initial conditions \( \rho(r = 0) = \rho(0), \rho'(0) = 0 \), and the substitutions
\[
\begin{align*}
r &= \left( \frac{\omega \gamma}{4\pi G (\gamma - 1)} \right)^{\frac{1}{2}} \rho(0)^{(\gamma - 2)/2} \xi, \\
\rho &= \rho(0) \theta^{1/(\gamma - 1)}, \\
k &= \frac{\omega \gamma}{\gamma - 1} \rho(0)^{\gamma - 1} K,
\end{align*}
\]

Eq. (25) takes the form
\[
\frac{1}{\xi^2} \frac{d}{d\xi} \left[ \xi^2 \left( \frac{d\theta}{d\xi} + K \right) \right] + \theta^{1/(\gamma - 1)} = 0, \tag{29}
\]
with initial conditions
\[
\begin{align*}
\theta(0) &= 1, \\
\theta'(0) &= 0. \tag{30, 31}
\end{align*}
\]

This generalizes the usual Lane-Emden equation of GR, recovered when \( K = 0 \). For \( \gamma > \frac{6}{5} \), the solution vanishes\(^1\) for some finite \( \xi_1(K) \), in which case the radius of the star is given by
\[
R_s(K) = \left( \frac{\omega \gamma}{4\pi G (\gamma - 1)} \right)^{\frac{1}{2}} \rho(0)^{(\gamma - 2)/2} \xi_1(K). \tag{32}
\]

The lower panel of fig. (1) shows a graphic representation of the solution \( \theta(\xi) \) near \( \xi_1 \) for \( \gamma = \frac{4}{3} \) and different values of \( K \). Positive (negative) values of \( K \) lead to smaller (larger) radii, as expected from the physical interpretation of eq. (16).

The influence of the NMC may be measured using the parameter
\[
\delta_R = \left( \frac{R_s(K)}{R_s(0)} - 1 \right) = \frac{\xi_1(K)}{\xi_1(0)} - 1. \tag{33}
\]

\(^1\)Rigorously, that is true only if \( K = 0 \). However, for small values of \( K \), the statement is approximately correct.

Figure 1: (Lower panel) Comparison of \( \theta(\xi) \) near \( \xi_1(K) \) for \( \gamma = \frac{4}{3} \) and different values of \( K \). |\( K \)| = 0.01 - Dashed line. |\( K \)| = 0.005 - Dotted line. |\( K \)| = 0.001 - Dotdashed line. \( K = 0 \) - Full line. Graphs above (below) the full line \( (K = 0) \) and in black (gray) correspond to negative (positive) values of \( K \). (Upper panel) Plot of \( \delta_R \) for \( K \) in the range -0.01 to 0.01 and \( \gamma = \frac{4}{3} \).
A plot of $\delta R$ as a function of $K$ for $\gamma = \frac{4}{3}$ is shown in the upper panel of fig. (1). Notice that even small values of $K$ may originate significant changes in the radius. These results underscore the importance that a NMC may have on the dynamics of stars, even when only a weak coupling is considered.

**Arbitrary coupling**

An analytic solution to the field equations of this model seems completely out of reach for the general case. Indeed, even a numerical treatment of the equations is far from being a straightforward matter. Nevertheless, it is important to establish a more suitable mathematical framework to deal with these situations, and that is the main objective of this section.

Defining the auxiliary fields

$$h(r) = -\frac{F_2 \rho(r)}{\kappa}, \quad J_{\mu\nu} = (1 + h) R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \triangle_{\mu\nu} h,$$

it is clear from the field equations (6) that

$$\frac{J_{rr}}{A} = \frac{J_{\theta\theta}}{r^2} = 0. \quad (36)$$

This last equation may be used to show that

$$A(r) = \frac{\exp \left[ \int_r^{R_s} \psi(s) ds \right]}{\left( 1 - \frac{2GM}{R_s} \right) + \int_r^{R_s} \chi(s) \exp \left[ \int_s^{R_s} \psi(q) dq \right] ds}, \quad (37)$$

where $\psi = \frac{\alpha_2}{\alpha_1}$, $\chi = -\frac{\alpha_3}{\alpha_1}$ and

$$\begin{align*}
\alpha_1 &= B \left[ 2rB(1 + h) + r^2 B'(1 + h) + 2r^2 B h' \right], \\
\alpha_2 &= 4B^2(1 + h) + 2rBB'(1 + h) \\
&\quad + r^2 B''(1 + h) + 4rB^2 h' \\
&\quad - 2r^2 B B''(1 + h) - 4r^2 B^2 h'', \\
\alpha_3 &= -4B^2(1 + h).
\end{align*}$$

Note that (37) must be solved self-consistently. To complete the calculation, one inserts (17) and (37) into one of the field equations $J_{\mu\nu} = \frac{1}{2\kappa} f_2 T_{\mu\nu}$, and, provided the coupling function $f_2$, an equation of state relating $p$ and $\rho$, and a set of initial conditions are known, it is possible, at least in principle, to solve for $\rho$. Naturally, for practical purposes, this may well turn out to be unattainable.

Finally, before proceeding to the nonstatic case, a particular type of strong coupling will be qualitatively explored.

Suppose that at some point in spacetime $f_2 \simeq F_2 \simeq 0$. This assumption enables one to write the field equations (6) simply as

$$R_{\mu\nu} \simeq 0. \quad (38)$$

This is just the usual general relativistic vacuum case. The coupling function suppresses the effects of matter. However, as this happens, the curvature scalar tends to 0, and since $f_2 \simeq 1$ in this case, the coupling must bounce back to the weak regime, and, therefore, gets suppressed itself. Naturally, this is possible only if one allows the system to vary in time. Now suppose that the configuration driving $R$ to a value where $f_2 \simeq F_2 \simeq 0$ is
a weak coupling one. Provided the intermediate stage (when $f_2$ is crossing from 1 to 0 or the other way around) does not influence the situation significantly, an oscillatory motion of spacetime will be established. If they exist, these oscillations in the “fabric” of spacetime may be detectable. Naturally, this detection depends heavily on the frequency and amplitude of the waves. Note that this oscillatory mechanism is different from the usual gravitational waves. For example, there is no \textit{a priori} reason to think that it does not apply to spherically symmetric situations, which, according to GR, do not radiate gravitational waves.

3.2 Nonstatic matter distributions

Although the study of static matter distributions is an important one, there are situations in which a departure from this assumption is required. Galaxies and clusters of galaxies are good examples. In what follows, the effects that a NMC may have on these cases will be analyzed.

The equation of motion for a fluid element may be derived from eq. (5):

$$u^\mu \nabla_\mu u^i = \frac{1}{\rho + p} h^{\mu\nu} \nabla_{\mu} p - \frac{1}{f_2} h^{\mu\nu} \nabla_{\mu} f_2, \quad (39)$$

where $h^{\mu\nu} = g^{\mu\nu} + u^{\mu} u^{\nu}$ is the projection tensor, and the chain rule was used. Once again, one notes that this is just the usual equation of GR, but with an added term, $f_{NMC}^{\mu} = -h^{\mu\nu} \nabla_\nu f_2 / f_2$, that resembles a “geometric” pressure gradient.

To explore the Newtonian limit of this model, one will assume that $g_{\mu\nu} = \eta_{\mu\nu} + \epsilon_{\mu\nu}$, where $\eta_{\mu\nu} = \text{diag}(-1,1,1,1)$ and $|\epsilon_{\mu\nu}| \ll 1$, $|\frac{df_2}{d\tau}| \approx 1$ and $|\frac{df_2}{d\tau}| \ll 1$. Under these simplifications, the spatial components of the LHS of eq. (39) read approximately

$$u^\mu \nabla_\mu u^i \approx \frac{d^2 x^i}{dt^2} - \frac{1}{2} \nabla_i \epsilon_{tt}, \quad (40)$$

where the index $i$ runs from 1 to 3. Additionally, the projection tensor may be simplified to $h^{\mu i} \approx \eta^{\mu i} = \delta^{\mu i}$. Combining these results, the equation of motion for a fluid element in the Newtonian limit is given by

$$\frac{d^2 x^i}{dt^2} \approx \nabla_i \left[ g_{tt} - \frac{1}{2} \log (|f_2|) \right] - \frac{\nabla_i p}{\rho + p}, \quad (41)$$

where the identity $\epsilon_{tt} = g_{tt} + 1$ was used. The last term on the RHS of eq. (41) is just the usual pressure gradient force per unit mass. Moreover, the Newtonian gravitational potential is generally defined in GR by

$$\Phi = -\frac{G M}{r^2}. \quad (42)$$

However, there is now an unaccounted term in the equation of motion, related to the NMC. Since this term is geometric in nature (it depends solely on $R$), it seems quite natural to include it in a generalized gravitational potential:

$$\Phi_{NMC} = \Phi_N + \log (|f_2|). \quad (43)$$

Comparing the effects of the usual Newtonian gravitational pull for an object of mass $M$, $g_N = -\frac{G M}{r^2}$, with those of $g_{NMC} = -\frac{\nabla_i f_2}{f_2}$, one has

$$\delta_g = \left| \frac{g_{NMC}}{g_N} \right| = \left| \frac{c^2 r^2}{G M} \frac{\nabla_r f_2}{f_2} \right|, \quad (44)$$

where the speed of light in vacuum $c$ is now explicitly shown. The factor $\frac{c^2 r^2}{G M}$ takes on different values
for distinct astrophysical situations: it is approximately $10^{16}$ m at the surface of Earth, $10^{14}$ m at the surface of the Sun, and $10^{26}$ m for $r = 1$ kpc in a typical galaxy with $M \approx 10^{11} M_\odot$ (not counting dark matter), where $M_\odot$ is the solar mass. If the gradient term $\nabla_r f_2$ is sufficiently small, its effects should be noticeable only at very large scales, such as galaxies or clusters of galaxies. Indeed, if one takes $\nabla_r f_2 \approx 10^{-26}$ m$^{-1}$, with $f_2 \approx 1$, eq. (44) takes the values

\[ \delta_g \approx 10^{-10} \text{ at the surface of Earth, } \]
\[ \delta_g \approx 10^{-12} \text{ at the surface of the Sun, } \]
\[ \delta_g \approx 1 \text{ at } r=1 \text{ kpc in a galaxy. } \]

These results show that the NMC may have an important influence in the dynamics of galaxies and other large-scale astrophysical objects, while, at the same time, having negligible effects at stellar systems level.

One may apply the results derived above to simulate the gravitational effects of dark matter in galaxies. The idea of mimicking dark matter using NMC gravity has already been explored in [15, 16], though using a different approach that focus on power-law couplings of the form $f_2 = 1 + \left( \frac{R}{R_0} \right)^n$. The results presented in this work are, therefore, complementary to those.

The starting point of this derivation is Poisson’s equation for gravity:

\[ \nabla^2 \Phi_N = 4\pi G \rho_v, \quad (48) \]

where $\rho_v$ is the visible matter density (it is assumed that no dark components exist). Suppose now one desires to generalize eq. (48) to include the effects a NMC present in the potential $\Phi_{NMC}$. If it is assumed that the gravitational effects of dark matter are contained in the NMC, a rather straightforward way to do it is adding a mimicked dark matter density profile to the RHS of Poisson’s equation, and connecting the additional term in (43) with it:

\[ \nabla^2 \Phi_{NMC} = 4\pi G (\rho_v + \rho_{DM}), \quad (49) \]
\[ \nabla^2 \log |f_2| = 4\pi G \rho_{DM}, \quad (50) \]

where $\rho_{DM}$ is the replicated dark matter profile.

Considering a spherically symmetric scenario, eq. (50) reads

\[ \nabla_r \left( r^2 \frac{\nabla_r f_2}{f_2} \right) = 4\pi Gr^2 \rho_{DM}, \quad (51) \]

or, provided $\lim_{r \to 0} r^2 \frac{\nabla_r f_2}{f_2} = 0$,

\[ r^2 \frac{\nabla_r f_2}{f_2} = G \int_0^r 4\pi s^2 \rho_{DM}(s) ds = GM_{DM}(r), \quad (52) \]

where $M_{DM}(r)$ is the total mass of simulated dark matter inside a sphere of radius $r$. Eq. (52) is a differential equation for $f_2$ that can be solved rather easily, yielding the solution

\[ f_2(r) = f_2(r_0) \exp \left[ \int_{r_0}^r \frac{GM_{DM}(s)}{s^2} ds \right]. \quad (53) \]

The multiplicative constant $f_2(r_0)$ is irrelevant in most cases, since physically measurable quantities generally depend on $\frac{\nabla_r f_2}{f_2}$.

Given a dark matter profile, eq. (53) can be used to calculate the explicit form of $f_2$. It may be shown that for typical choices of $\rho_{DM}$ [22], $\nabla_r f_2 \approx 10^{-27}$ m$^{-1}$, well within the expected limits.
4 Conclusion

This work discussed the dynamics of perfect fluids in an alternative theory of gravity exhibiting a nonminimal coupling between curvature and matter. It was shown that this feature may introduce new and interesting phenomenology both at stellar and galactic level. Nevertheless, the model is still largely unexplored, and work is under progress to develop it further. Regarding fluid dynamics, there are at least a couple of points that seem worth exploring: firstly, the effects that a strong coupling regime may have on the dynamics of stars. Secondly, the applicability of the dark matter mimicking mechanism introduced, that is, whether or not it can be a serious alternative to the particle models of dark matter.

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