New Transform Fundamental Properties and Its Applications

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Received in:24/April/2018, Accepted in:29/May/2018

Abstract
In this paper, new transform with fundamental properties are presented. The new transform has many interesting properties and applications which make it rival to other transforms. Furthermore, we generalize all existing differentiation, integration, and convolution theorems in the existing literature. New results and new shifting theorems are introduced. Finally, comprehensive list of this transforms of functions will be providing.

Keywords: Functions transform, Integral transform, Distribution space.
**Introduction**

Recently, integral transformations played an important role in many fields of science and engineering [1-4], especially mathematical physics [5], optics [6], engineering mathematics [7, 8], Cryptography [9], image processing [10] and, few others, because they have been successfully used in solving many problems in those fields. The possibility of solving a problem is required transforming the problem from a space to another space where the solving is possible or easy. As well as the possibility of decreasing the independent variables in some problems. Many of these transforms have been introduced which were extensively used and applied on theory and applications, such as Laplace [11, 12], Fourier [2], Sumudu [13, 14], Elzaki [15-17], Aboodh [18], Natural, and ZZ Transforms [19]. Among these the most widely used is Laplace transform. Here, new integral transform is proposed to avoid the complexity of previous transforms.

**Definition of New Transform**

The transform of a function \( f(t) \) is defined by

\[
\tilde{f}(u) = \mathbb{T}\{f(t)\} = \int_{0}^{\infty} e^{-ut} f(\frac{t}{u}) \, dt , \quad (1)
\]

Where \( u \) is a real number, for those values of \( u \) which the improper integral converges.

**Lemma 1**

The improper integral \( \int_{0}^{\infty} e^{-at} \, dt \), converges only for \( a > 0 \).

**Proof**

\[
\int_{0}^{\infty} e^{-at} \, dt = \lim_{N \to \infty} \int_{0}^{N} e^{-at} \, dt = -\frac{1}{a} \lim_{N \to \infty} (e^{-aN} - 1) = \frac{1}{a}, \quad a > 0
\]

In the case that \( a = 0 \), the improper integral is diverging, since we are computing the area of a rectangle with sides equal to one and infinity. In the case \( a < 0 \), holds

\[
\lim_{N \to \infty} -\frac{1}{a} (e^{-aN} - 1) = \infty
\]

Therefore, the improper integral converges only for \( a > 0 \).

**Remark:** Lemma (1) holds when \( t \in [0, \infty) \), if \( t \in (-\infty, 0] \) then the condition of convergence is \( a < 0 \).

Now, we give illustrated examples

**Example 1**

\[
\mathbb{T}\{t^n\} = \frac{n!}{u^n} , \quad u \neq 0 , \quad n = 0,1,2,3, ...
\]

**Proof**

\[
\mathbb{T}\{t^n\} = \int_{0}^{\infty} e^{-t} \left( \frac{t}{u} \right)^n \, dt = u^{-n} \int_{0}^{\infty} e^{-ut} \, dt \quad (2)
\]

Using the rule of integral by part \( n \) times to get

\[
\int_{0}^{\infty} e^{-ut} \, dt = -[e^{-ut} + ne^{-ut} + \cdots + n! e^{-ut}]_{0}^{\infty} = n!
\]

Substituted the value in (3) into (2) to obtain

\[
\mathbb{T}\{t^n\} = \frac{n!}{u^n} , \quad u \neq 0 , \quad n = 0,1,2,3, ...
\]
Example 2

\[ T\{e^{at}\} = \frac{u}{u-a}, \quad u \in \{R \setminus [0, a], a \geq 0 \} \]
\[ T\{e^{at}\} = \left\{ R \setminus [a, 0], a < 0 \right\} \]

Proof

\[ T\{e^{at}\} = \int_{0}^{\infty} e^{-t} e^{at} dt = \int_{0}^{\infty} e^{-(1-a)u} dt = \frac{-u}{u-a} e^{-(u-a)} \bigg|_{0}^{\infty} = \frac{u}{u-a} \]

By Lemma (1), the condition of convergence is

\[ 1 - \frac{a}{u} > 0 \quad \Rightarrow \quad 1 > \frac{a}{u} \]

If \(a \geq 0\) then the condition holds for all \(u > a\) or \(u < 0\), that is \(u \in R \setminus [0, a]\)

If \(a < 0\) then the condition holds for all \(u < a\) or \(u > 0\), that is \(u \in R \setminus [a, 0]\)

Example 3

1. \( T\{\sin(at)\} = \frac{au}{a^2 + u^2}, u \neq 0 \)

2. \( T\{\cos(at)\} = \frac{u^2}{a^2 + u^2}, u \neq 0 \)

Proof

Using the rule of integral by part with \(u \neq 0\), to get

\[ T\{\sin(at)\} = \int_{0}^{\infty} e^{-t} \sin \left( \frac{at}{u} \right) dt = \frac{au}{a^2 + u^2} \]

Similarly \( T\{\cos(at)\} = \frac{u^2}{a^2 + u^2}, u \neq 0 \)

A list of the new transforms for common functions is presented in the Table (1). This can be computed in the same manner of previous examples.
Table (1): New transforms for some common functions

| $f(t)$ | $\hat{f}(u) = T\{f(t)\}$ | $D\hat{f}$ |
|--------|--------------------------|----------|
| $t^n, n=0,1,...$ | $\frac{n!}{u^n}$ | $u \neq 0$ |
| $t^a, \; a>0$ | $\Gamma(a+1)/u^a$ | $u\neq 0$ |
| $e^{at}$ | $\frac{u}{u-a}$ | $u \in \mathbb{R}\setminus[0,a]$ if $a \geq 0$; $u \in \mathbb{R}\setminus[a,0]$ if $a < 0$ |
| $\sin(at)$ | $\frac{au}{a^2 + u^2}$ | $u \neq 0$ |
| $\cos(at)$ | $\frac{u^2}{a^2 + u^2}$ | $u \neq 0$ |
| $\sinh(at)$ | $\frac{-au}{a^2 - u^2}$ | $|u| > |a|$ |
| $\cosh(at)$ | $\frac{-u^2}{a^2 - u^2}$ | $|u| > |a|$ |
| $u_n(t) = u(t-a) = H(t-a)$ | $e^{-au}$ | $u > 0$ |
| $\delta(t-a)$ | $e^{au}/u$ | $u > 0$ |
| $\ln(at), \; a>0$ | $\ln(u/a) - \gamma$ | $u > 0$ |

In the next section, we discuss the existence of the new transform.

**Existence of the New Transform**

Does the new transform always exist? It can be shown that

$$\int_0^\infty e^{-t}e^{\left(\frac{t}{u}\right)^2} dt = \infty$$

For every real number $u$. Hence, the function $e^{-2}$ does not have a transform.

In this section, we will establish the conditions that ensure the existence of the new transform of a function. We first review some relevant definitions from calculus, and for simplicity, we take the following coding

$$f(t_0+) = \lim_{t \to t_0^+} f(t) \quad \text{and} \quad f(t_0-) = \lim_{t \to t_0^-} f(t)$$

**Definition 1 (Limit)**

The limit of a function $f$ at a point $t_0$ exists if and only if $f(t_0^+)$ and $f(t_0^-)$ exist and are equal, that is:

$$\lim_{t \to t_0} f(t) = f(t_0^+) = f(t_0^-)$$

**Definition 2 (Continuous)**

The function $f$ is continuous at a point $t_0$ in an open interval $(a,b)$ if and only if

$$\lim_{t \to t_0} f(t) = f(t_0)$$

[https://doi.org/10.30526/31.2.1954](https://doi.org/10.30526/31.2.1954)  Mathematics | 154
and equivalent to
\[ f(t_0) = f(t_0^+) = f(t_0^-) \]

**Definition 3 (Jump discontinuity)**
If \( f(t_0^+) \neq f(t_0^-) \) then we say that \( f \) has a jump discontinuity at \( t_0 \), and the value \( f(t_0^+) - f(t_0^-) \) is called the jump in \( f \) at \( t_0 \).

**Definition 4 (Removable discontinuity)**
If the limit of a function \( f \) at a point \( t_0 \) exists, but either \( f \) is not defined at \( t_0 \) or it's defined but \( \lim_{t \to t_0} f(t) \neq f(t_0) \)

Then, we say that \( f \) has a removable discontinuity at \( t_0 \).

**Definition 5 (Piecewise continuous)**
- A function \( f \) is said to be piecewise continuous on a closed interval \([0,T]\) if \( f(0^+) \) and \( f(T^-) \) are finite and \( f \) is continuous on the open interval \((0,T)\) except possibly at finitely many points, where \( f \) may have jump discontinuities or removable discontinuities.
- A function \( f \) is said to be piecewise continuous on the interval \([0,\infty)\) if it’s piecewise continuous on \([0,T]\) for every \( T > 0 \).

**Definition 6 (Functions of exponential order)**
A function \( f \) is said to be of exponential order \( s_0 \) if there exist constants \( M \) such that:
\[ |f(t)| < Me^{s_0 t} \quad \text{for all } t \in [0,\infty) \]

**Theorem 1 (Sufficient conditions).**
If a function \( f \) is piecewise continuous on \([0,\infty)\) and of exponential order \( s_0 \), then the transform of \( f \) exists for all \( u > s_0 \).

**Proof**
To proof the existences of the transform of \( f \), we must show \( \tilde{f}(u) \) is well-defined, that is, finite, by bounding its absolute value by a finite number.

\[
|\tilde{f}(u)| = \left| \int_0^\infty e^{-t} f\left( \frac{t}{u} \right) dt \right| = \left| u \int_0^\infty e^{-ux} f(x) dx \right| \quad (4)
\]

We got (4) by putting \( x = \frac{t}{u} \).

Since \( f \) of exponential order \( s_0 \), then there exist \( M > 0 \) such that
\[ |f(x)| < Me^{s_0 x} \quad \text{for all } x \in [0,\infty) \]

then (4) becomes

\[
\left| u \int_0^\infty e^{-ux} f(x) dx \right| \leq \int_0^\infty e^{-ux} |uf(x)| dx \leq |u| \int_0^\infty e^{-ux} Me^{s_0 x} dx = |u|M \int_0^\infty e^{-(u-s_0)x} dx
\]

Then by lemma(1), the condition to this result is \( u-s_0 > 0 \), this implies \( u > s_0 \). Then we have that
\[ |\tilde{f}(u)| \leq \frac{|u|M}{u-s_0} \]

**The General Properties of the New Transform.**

**Theorem 2 (Linear combination)**
If the transforms $T\{f\}$ and $T\{g\}$ of the functions $f$ and $g$ are well-defined and $a, b$ are constants, then the following equation holds:

$$T\{af(t) + bg(t)\} = aT\{f(t)\} + bT\{g(t)\}$$

**Proof**

$$T\{af(t) + bg(t)\} = \int_0^\infty e^{-t} \left( af \left( \frac{t}{u} \right) + bg \left( \frac{t}{u} \right) \right) dt = \int_0^\infty e^{-t} af \left( \frac{t}{u} \right) dt + \int_0^\infty e^{-t} bg \left( \frac{t}{u} \right) dt$$

$$= a \int_0^\infty e^{-t} f \left( \frac{t}{u} \right) dt + b \int_0^\infty e^{-t} g \left( \frac{t}{u} \right) dt = aT\{f(t)\} + bT\{g(t)\} \square$$

**Theorem 3**

If the functions $T\{f\}$ and $T\{tf\}$ are well-defined then:

$$T\{tf\} = -u \frac{d}{du} \left( \frac{1}{u} T\{f\} \right)$$

**Proof**

$$T\{tf\} = \int_0^\infty e^{-t} \left( \frac{t}{u} \right) f \left( \frac{t}{u} \right) dt$$

Using the integral by part we have

$$T\{tf\} = \frac{-t}{u} e^{-t} f \left( \frac{t}{u} \right) \bigg|_0^\infty + \int_0^\infty e^{-t} \left[ \frac{1}{u} f \left( \frac{t}{u} \right) + \frac{t}{u^2} f' \left( \frac{t}{u} \right) \right] dt$$

$$= 0 - 0 + \int_0^\infty e^{-t} \left[ -u \frac{d}{du} \left( \frac{1}{u} f \left( \frac{t}{u} \right) \right) \right] dt = -u \frac{d}{du} \left[ \frac{1}{u} \int_0^\infty e^{-t} f \left( \frac{t}{u} \right) dt \right]$$

$$= -u \frac{d}{du} \left( \frac{1}{u} T\{f\} \right) \quad \therefore \quad T\{tf\} = -u \frac{d}{du} \left( \frac{1}{u} T\{f\} \right) \square$$

**Corollary 1**

If the functions $T\{f\}$ and $T\{t^n f\}$ are well-defined then:

$$T\{t^n f\} = (-1)^n u \frac{d^n}{u^n} \left( \frac{1}{u} T\{f\} \right)$$

**Proof**

The proof is clear by using the Mathematical induction. \square

**Theorem 4(shift)**

If the functions $\tilde{f}(u) = T\{f\}$ is the transform of the function $f(t)$ then:

$$T\{e^{at}f\} = \frac{u}{u - a} \tilde{f}(u - a)$$

**Proof**

$$T\{e^{at}f\} = \int_0^\infty e^{-t} e^{at} f \left( \frac{t}{u} \right) dt = \int_0^\infty e^{-(1-a)t} f \left( \frac{t}{u} \right) dt \quad (5)$$

If we set $x=(1-a)u$ then $t = \frac{x}{1+a}$ and $dt = \frac{dx}{1+a}$, substituted it in (5) we have
The Convolution of Two Functions

Definition 7
Assume that $f$ and $g$ are piecewise continuous functions, or one of them is a Dirac's delta generalized function. The convolution of $f$ and $g$ is a function denoted by $f \ast g$ and given by the following expression.

$$
(f \ast g)(t) = \int_{0}^{t} f(\tau)g(t-\tau)d\tau \quad (6)
$$

In [20] Gabriel Nagy summarized the main properties of the convolution illustrated in the following lemma.

Lemma 2 (Properties)
For every piecewise continuous functions $f$, $g$, and $h$, the following properties hold:

1. Commutativity: $f \ast g = g \ast f$.
2. Associativity: $f \ast (g \ast h) = (f \ast g) \ast h$.
3. Distributivity: $f \ast (g + h) = f \ast g + f \ast h$.
4. Neutral element: $f \ast 0 = 0$.
5. Identity element: $f \ast \delta = f$.

Theorem 5
If the functions $f$ and $g$ have well-defined transforms $\mathcal{T}\{f\}$ and $\mathcal{T}\{g\}$, then

$$
\mathcal{T}\{f \ast g\} = \frac{1}{u} \mathcal{T}\{f\} \mathcal{T}\{g\}
$$

Proof

$$
\mathcal{T}\{f \ast g\} = \int_{0}^{\infty} e^{-ut} (f \ast g) \left(\frac{t}{u}\right) dt = \int_{0}^{\infty} e^{-ut} \left[ \int_{0}^{t} f(\tau)g(t-\tau)d\tau \right] dt \quad (7)
$$

Now let $\lambda = \frac{t}{u}$ then $d\lambda = \frac{dt}{u}$, then we get

$$
\mathcal{T}\{f \ast g\} = \int_{0}^{\infty} e^{-u\lambda} \left[ \int_{0}^{\lambda} f(\tau)g(\lambda-\tau) d\tau \right] u d\lambda
$$

From Figure 1 and Figure 2, It is clear that the regions $R_1=\{(\tau,\lambda) \in \mathbb{R}^2: 0 \leq \tau \leq \lambda, 0 \leq \lambda < \infty\}$ and $R_2=\{(\gamma,\zeta) \in \mathbb{R}^2: \tau \leq \zeta < \infty, 0 \leq \tau < \infty\}$ are equal.
Then we can change the order of integration, i.e., equation (8) becomes:

\[
\mathbb{T}\{f \ast g\} = u \int_0^\infty \int_0^{\infty} e^{-u\lambda} f(\tau) g(\lambda - \tau) \, d\lambda \, d\tau \quad (9)
\]

Now, if \( y = \lambda - \tau \) then \( dy = d\lambda \), since the variable \( \tau \) is constant, hence when we integrate with respect to \( \tau \), we get

\[
\mathbb{T}\{f \ast g\} = u \int_0^\infty \int_0^{\infty} e^{-u(y + \tau)} f(\tau) g(y) \, d\lambda \, d\tau = u \int_0^\infty e^{-u\tau} \left[ \int_0^\infty e^{-uy} g(y) \, dy \right] d\tau
\]

\[
= u \left[ \int_0^\infty e^{-u\tau} f(\tau) d\tau \right] \left[ \int_0^\infty e^{-uy} g(y) \, dy \right]
\]

If \( x = ut \) and \( y = u\tau \) then we have \( dx = udt \) and \( dy = udy \), i.e.,

\[
\therefore \mathbb{T}\{f \ast g\} = u \left[ \int_0^\infty e^{-x} \left( \frac{x}{u} \right) \frac{dx}{u} \right] \left[ \int_0^\infty e^{-y} \left( \frac{y}{u} \right) \frac{dy}{u} \right] = \frac{1}{u} \mathbb{T}\{f\}\mathbb{T}\{g\}
\]

6. The Transform of Derivative and Integral

Theorem 6 (Derivatives)

If the functions \( \mathbb{T}\{f\} \) and \( \mathbb{T}\{f'\} \) are well-defined then

\[
\mathbb{T}\{f'\} = u(\mathbb{T}\{f\} - f(0))
\]

Proof

\[
\mathbb{T}\{f'(t)\} = \int_0^\infty e^{-ut} f' \left( \frac{t}{u} \right) \, dt
\]

Using the integral by part we have

\[
\mathbb{T}\{f'(t)\} = u e^{-t} f \left( \frac{t}{u} \right) \bigg|_0^\infty + u \int_0^\infty e^{-t} f \left( \frac{t}{u} \right) \, dt = 0 - uf(0) + u\mathbb{T}\{f\}
\]

\[
\therefore \mathbb{T}\{f'\} = u\mathbb{T}\{f\} - uf(0)
\]

Corollary 2 (\( n^{th} \) Derivatives)

If the functions \( \mathbb{T}\{f\}, \mathbb{T}\{f'\}, \ldots, \mathbb{T}\{f^{(n)}\} \) are well-defined, \( n=1,2,3,\ldots \) then
\[ T\{f^{(n)}\} = u^n T\{f\} - \sum_{k=0}^{n-1} u^{n-k} f^{(k)}(0) \]

**Proof**

We will use the Mathematical induction

If \( n=1 \) then the corollary (2) holds by theorem (6).

Now suppose it holds for \( n=m \), i.e.,

\[ T\{f^{(m)}\} = u^m T\{f\} - \sum_{k=0}^{m-1} u^{m-k} f^{(k)}(0) \]

and suppose \( n=m+1 \):

\[ T\{f^{(m+1)}\} = T\{(f^{(m)})'\} = u T\{f^{(m)}\} - u f^{(m)}(0) \]

\[ = u \left[ u^m T\{f\} - \sum_{k=0}^{m-1} u^{m-k} f^{(k)}(0) \right] - u f^{(m)}(0) \]

\[ = u^{m+1} T\{f\} - \sum_{k=0}^{m} u^{(m+1)-k} f^{(k)}(0) \]

Then the corollary is performing for any nonnegative integer \( n \).

**Theorem 7 (Derivatives of other variables)**

If the function \( T\{f(t, x)\} \) is well-defined then:

\[ T\left\{ \frac{\partial^n}{\partial x^n} f(t, x) \right\} = \frac{\partial^n}{\partial x^n} T\{f(t, x)\}, \quad n = 1, 2, 3, ... \]

**Proof**

\[ T\left\{ \frac{\partial^n}{\partial x^n} f(t, x) \right\} = \int_0^\infty e^{-t} \frac{\partial^n}{\partial x^n} f\left(\frac{t}{u}, x\right) dt = \frac{\partial^n}{\partial x^n} \left( \int_0^\infty e^{-t} f\left(\frac{t}{u}, x\right) dt \right) \]

\[ \therefore \quad T\left\{ \frac{\partial^n}{\partial x^n} f(t, x) \right\} = \frac{\partial^n}{\partial x^n} T\{f\} \]

**Theorem 8 (Integral)**

If the function \( T\{f(t)\} \) is well-defined then

\[ T\left\{ \int_0^t f(\tau) d\tau \right\} = \frac{1}{u} T\{f\} \]

**Proof**

Suppose that \( g(t-\tau)=1 \), then from theorem (5) and example (1) we have

\[ T\left\{ \int_0^t f(\tau) d\tau \right\} = T\left\{ \int_0^t g(t-\tau)f(\tau) d\tau \right\} = T\{g \ast f\} = \frac{1}{u} T\{f\} T\{g\} = \frac{1}{u} T\{f\} \]
The Inverse of New Transform

Definition 8

Let the functions $\tilde{f}(u) = T\{f\}$ is the transform of the function $f(t)$, then $f(t)$ is called the inverse transform of the function $\tilde{f}(u)$ and we will write it as:

$$f(t) = T^{-1}\{\tilde{f}(u)\}$$

Remark: The inverse transform has the linear combination property, i.e.,

$$T^{-1}\left(\sum_{k=1}^{n} a_k \tilde{f}_k(u)\right) = \sum_{k=1}^{n} a_k T^{-1}\{\tilde{f}_k(u)\}$$

The Duality with Laplace Transform

Theorem 9

Let $f$ satisfy the conditions in theorem 1, and has Laplace transform $L[f(t)] = \hat{f}(s)$. Then the transform $\tilde{f}(u)$ of $f(t)$ is given by

$$\tilde{f}(u) = u\hat{f}(u)$$

Proof

$$\tilde{f}(u) = \int_{0}^{\infty} e^{-ut} f\left(\frac{t}{u}\right) dt \quad (10)$$

If we set $x = \frac{t}{u}$ then $(t = ux$ and $dt = u dx)$, substituted it in equation (10) to get

$$\tilde{f}(u) = \int_{0}^{\infty} e^{-ux} f(x) dx = u \int_{0}^{\infty} e^{-ux} f(x) dx = u\hat{f}(u) \quad (11)$$

Corollary 3

Let $f$ satisfy the conditions in theorem (1) and has the transform $\tilde{f}(u)$. Then the Laplace transform $\hat{f}(s)$ of $f(t)$ is given by

$$\hat{f}(s) = \frac{1}{s} \tilde{f}(u) \quad (12)$$

The Advantages of the New Transform

The new transform has many interesting properties which make it rival to the Laplace transform. Some of these properties are:

1. The domain of the new transform is wider than or equal to the domain of Laplace transform as illustrated in the Table (2). This feature makes the new transform more widely used in problems.
2. From section 8 the new transform has the duality with Laplace transform, hence, that needs a wider domain.
3. The new transform can solve all the problems which would be solved by Laplace transform.
4. The unit step function in the $t$-domain is transformed to unity in the $u$-domain.
5. The differentiation and integration in the $t$-domain are equivalent to multiplication and division of the transformed function $F(u)$ by $u$ in the $u$-domain.
6. By Theorem 2 (Linear combination) and example (1) we have that for any constant $a \in \mathbb{R}, T\{a\} = aT\{1\} = a$, and hence, $T^{-1}\{a\} = a$, that is, we don’t have any problem when we deal with the constant term( the constant with respect to the parameter $u$).
Conclusion

The new transform has some strength points for feature among others transform such that: The new transform easier than the Laplace transform for beginners to understand and use. It can be used to solve problems without resorting to the frequency domain. Especially, with respect to applications in problems with physical dimensions. We showed it to be the theoretical dual to the Laplace transform, and hence ought to rival it in solving intricate problems in engineering mathematics and applied science. Also, the new transform is a convenient tool for solving differential equations in the time domain without the need for performing its inverse. The connection of the new transform with the Laplace transform goes much deeper. We also present many of the new transform properties that make it uniquely qualified to address and solve some applied problems, especially ones in which the units of the problem must be preserved.

Table (2): Illustrates the domain of Laplace and new transform

| \( f(t) \) | Laplace | New Transform |
|-----------|---------|--------------|
| \( t^n \) \( n=0,1,2, \ldots \) | \( n!/s^{n+1} \) Domain | \( n!/u^n \) Domain |
| \( e^{iat} \) | \( 1/(s-a) \) Domain | \( u/(u-a) \) if \( u \geq 0 \) |
| \( \sin(at) \) | \( a/(s^2+a^2) \) Domain | \( au/(u^2+a^2) \) Domain |
| \( \cos(at) \) | \( s/(s^2+a^2) \) Domain | \( u^2/(u^2+a^2) \) if \( u > 0 \) |
| \( \sinh(at) \) | \( a/(s^2-a^2) \) Domain | \( au/(u^2-a^2) \) if \( u > 0 \) |
| \( \cosh(at) \) | \( s/(s^2-a^2) \) Domain | \( u^2/(u^2-a^2) \) if \( u > 0 \) |
| \( u_a(t)=u(t-a)=H(t-a) \) | \( e^{-at}/s \) Domain | \( e^{-at}/u \) Domain |
| \( \delta(t-a) \) | \( e^{-as} \) Domain | \( e^{-at}/u \) Domain |
| \( \ln(at) \) | \( (\ln(at)-\gamma)/s \) Domain | \( ln(a/u)-\gamma \) Domain |

\[ \gamma = -\int_0^\infty e^{-t} \ln(t) \, dt \approx 0.5772 \ldots \]

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