RIMS workshop

“Geometric Aspects of Solutions to Partial Differential Equations”

Abstract
Positive solutions to semi-linear elliptic problems
on metric graphs

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We consider the following semi-linear elliptic equation:

\[-\epsilon^2 \Delta u + u = |u|^{p-1}u\]  \hspace{1cm} (1)

where \( p > 1 \), \( \epsilon > 0 \). Semi-linear equations of this type appear in the study of stationary problems of various equations such as nonlinear Schrödinger equations, and it has been studied by many researchers. In this talk, we consider (1) on a metric graph \( G \). Here, a metric graph \( G \) is a graph \( G = (V, E) \) consisting of a set \( V \) of vertices and a set \( E \) of edges, such that each edge \( e \in E \) is isometric to a closed interval \([0, \ell_e]\).

Similar to (1) in a domain of \( \mathbb{R}^N \), (1) on \( G \) has a variational structure, and we can obtain that there exists a least energy solution \( u_\epsilon \) for each \( \epsilon > 0 \). We discuss the asymptotic behavior of the solution \( u_\epsilon \) as \( \epsilon \to 0 \), and talk about more general positive solutions.

This talk is based on the joint work with Kazuhiro Kurata (Tokyo Metropolitan University).

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On sharpness of the Yang-Yau inequality for the genus two case

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Let \((M, ds^2)\) be an oriented closed surface with the Riemannian metric. For the first positive eigenvalue of the Laplacian \(\lambda_1(ds^2)\) and the area \(\text{area}(ds^2)\), we define the quantity \(\Lambda(ds^2) := \lambda_1(ds^2) \cdot \text{area}(ds^2)\).

Yang and Yau [YY] showed that \(\Lambda(ds^2) \leq 8\pi \cdot \left\lfloor \frac{2\gamma + 3}{2} \right\rfloor\), where \(\gamma\) is the genus of \(M\) and \(\lfloor \cdot \rfloor\) is the integer part. One can see that this inequality is sharp for \(\gamma = 0\) and is not sharp for \(\gamma = 1\). For \(\gamma = 2\), Jakobson-Levitin-Nadirashvili-Nigam-Polterovich [JLNNP] proposed that this inequality is sharp as the conjecture. In [NS], we settled the conjecture in the affirmative, and in this talk, we will introduce the details.

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Weak-strong uniqueness for multiphase mean curvature flow

Tim Laux

Multiphase mean curvature flow has, due to its importance in materials science, received a lot of attention over the last decades. On the one hand, there is substantial recent progress in the construction of weak solutions. On the other hand, strong solutions are - in particular in the planar case of networks - very well understood.

In this talk, I will present a weak-strong uniqueness principle: as long as a strong solution to multiphase mean curvature flow exists, any distributional solution with optimal energy dissipation rate has to coincide with this solution.

In our proof we construct a suitable relative entropy functional, which in this geometric context may be viewed as a time-dependent variant of calibrations. Just like the existence of a calibration guarantees that one has found a global minimum, the existence of a “time-dependent calibration” ensures that the route of steepest descent in the energy landscape is unique and stable.

This is a joint work with Julian Fischer, Sebastian Hensel, and Thilo Simon.
A remark on the asymptotic behavior of solutions of the Allen-Cahn equation

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We are concerned with the behavior of solutions of the Allen-Cahn equation

$$\partial_t u = \Delta u + u(1-u)(u-a) \quad \left(0 < a < \frac{1}{2}\right).$$

For a radially symmetric solution $u$ whose interface moves to infinity, it is known that the radius $R(t)$ of the interface satisfies $0 < kt - R(t) \sim \log t$ and $u(\cdot + R(t)\nu, t)$ converges locally uniformly to a planar traveling wave solution connecting 1 and 0 as $t \to \infty$, where $k$ is the speed of the planar traveling wave solution and $\nu$ is any unit vector. This means that the interface of $u$ gets away from that of a planar traveling wave solution in the opposite direction of travel, although $u$ locally looks like a planar traveling wave solution around its interface. In this talk we give a solution $\tilde{u}$ such that for $t \gg 1$, $\tilde{u}(\cdot + \rho(t)\tilde{\nu}, t)$ locally looks like a planar traveling wave solution for some unit vector $\tilde{\nu}$ and function $\rho(t)$ satisfying $0 < \rho(t) - kt \sim t^\gamma$, where $\gamma \in (0, 1)$. 
Traveling wave solutions of free boundary problems
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Abstract

As one of free boundary problems, I will explain an anisotropic curvature flow with an external driving force depending only on the normal vector in this talk. To study its dynamics, we focus on traveling waves composed of Jordan curves in two-dimensional space. We call them compact traveling waves. The aim of this study is to investigate the condition of the driving force for the existence of compact traveling waves to the anisotropic curvature flow. It is shown that all traveling waves are strictly convex and unstable, and that a compact traveling wave is unique, if they exist. For the existence of compact traveling waves, we have the following three results:

(i) if the driving force is positive, then there exists a compact traveling wave;
(ii) if it is negative, then there is no compact traveling wave;
(iii) if it is sign-changing, then a positive answer is obtained under the assumption called admissible condition.

Lastly, I will explain the related topics such as the inverse problem and non-convex compact traveling waves.

This is a joint work with Harunori Monobe (Okayama University).
Singular limit problem for the Allen-Cahn equation with a zero Neumann boundary condition on non-convex domains

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Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with smooth boundary. We consider the following Allen-Cahn equation with a zero Neumann boundary condition:

\[
\begin{align*}
\partial_t u_\varepsilon &= \Delta u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon^2}, & (x,t) \in \Omega \times (0,\infty) \\
\frac{\partial u_\varepsilon}{\partial \nu} &= 0, & (x,t) \in \partial \Omega \times (0,\infty) \\
u_\varepsilon(x,0) &= u_\varepsilon,0(x), & x \in \Omega.
\end{align*}
\]

Here, \( \varepsilon > 0 \) is a parameter, \( \nu \) is the outer unit normal to the boundary \( \partial \Omega \) and \( W \in C^\infty(\mathbb{R}) \) is a double well potential formed by \( W(s) = (1-s^2)^2 \).

Heuristically, we may expect a diffuse interface area measure \( \mu^t_\varepsilon \) defined as

\[
\mu^t_\varepsilon := \left( \frac{\varepsilon |\nabla u_\varepsilon(\cdot,t)|^2}{2} + \frac{W(u_\varepsilon(\cdot,t))}{\varepsilon} \right) \mathcal{L}^n|_\Omega
\]

behaves more or less like the surface area measure of a mean curvature flow with right angle condition. Mizuno and Tonegawa [2] proved that \( \{\mu^t_\varepsilon\}_{\varepsilon>0} \) converges to Brakke's mean curvature flow with a generalized right angle condition, which is a measure theoretic weak solution to the mean curvature flow, as \( \varepsilon \downarrow 0 \). However, their proof require the convexity of the domain. Accordingly, we expand the convergence theory by [2] on non-convex domains.

In this talk, we assume that initial functions \( u_\varepsilon,0 \in C^1(\overline{\Omega}) \) satisfies

(A1) \( \|u_\varepsilon,0\|_{L^\infty} \leq 1 \),
(A2) there exists \( D_0 > 0 \) such that \( \sup_{y \in \Omega, r > 0} \int_{B_r(y) \cap \Omega} \frac{\varepsilon |\nabla u_\varepsilon,0|^2}{2} + \frac{W(u_\varepsilon,0)}{\varepsilon} \, dx \leq D_0 r^{n-1} \),
(A3) there exists \( c_1 > 0 \) such that \( \sup_{x \in \Omega} \varepsilon |\nabla u_\varepsilon,0| \leq c_1 \),
(A4) there exist \( c_2 > 0 \) and \( \lambda \in [3/5, 1) \) such that \( \sup_{x \in \Omega} \frac{\varepsilon |\nabla u_\varepsilon,0|^2}{2} - \frac{W(u_\varepsilon,0)}{\varepsilon} \leq c_2 \varepsilon^{-\lambda} \),
(A5) \( \frac{\partial u_\varepsilon,0}{\partial \nu} = 0 \) on \( \partial \Omega \).

Although the assumptions are stronger than those of [2], a family of initial functions satisfying the assumptions can be constructed as a perturbation of an arbitrary surface embedded from a disk into \( \Omega \) generating 90 degree angles on \( \partial \Omega \). For this family of initial functions, we can construct a Brakke’s mean curvature flow with a generalized right angle condition starting from the embedded surface by applying our convergence theory.

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UPPER BOUNDS FOR HIGHER-ORDER POINCARÉ CONSTANTS

KEI FUNANO

Abstract

We introduce higher-order Poincaré constants for compact weighted manifolds and estimate them from above in terms of subsets. These estimates imply uniform upper bounds for eigenvalues of the weighted Laplacian and the first nontrivial eigenvalue of the $p$-Laplacian. In the case of the closed eigenvalue problem and the Neumann eigenvalue problem these are related with the estimates obtained by Chung-Grigorya’-Yau and Gozlan-Herry. We also obtain similar upper bounds for Dirichlet eigenvalues and multi-way isoperimetric constants. As an application we give upper bounds for inscribed radii in terms of dimension and the first Dirichlet Poincaré constant. Joint work with Yohei Sakurai (Tohoku university).

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RECENT PROGRESSES IN NONLINEAR POTENTIAL THEORY

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Nonlinear Potential Theory aims at studying the fine properties of solutions to nonlinear, potentially degenerate nonlinear elliptic and parabolic equations in terms of the regularity of the given data. A major model example is here given by the $p$-Laplacian equation

$$-\text{div}(|Du|^{p-2}Du) = \mu, \quad p > 1,$$

where $\mu$ is a Borel measure with finite total mass. When $p = 2$ we find the familiar case of the Poisson equation from which classical Potential Theory stems. Although many basic tools from the classical linear theory are not at hand – most notably: representation formulae via fundamental solutions – many of the classical information can be retrieved for solutions and their pointwise behaviour. In this talk I am going to give a survey of recent results in the field. Especially, I will explain the possibility of getting linear and nonlinear potential estimates for solutions to nonlinear elliptic and parabolic equations which are totally similar to those available in the linear case. I will also draw some parallels with what is nowadays called Nonlinear Calderón-Zygmund theory.

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A limiting case of the Hardy type inequality via extrapolation

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1. Introduction

Let $B_1 \subset \mathbb{R}^N$ be the unit ball, $\omega_{N-1}$ be an area of the unit sphere $\mathbb{S}^{N-1}$, $N \geq 2$, $1 \leq p < N$, $a > 1$, $p^* = \frac{Np}{N-p}$, $u^*$ be the Schwartz symmetrization of $u$, $W_0^{1,p}(B_1)$ is a completion of $C_0^\infty(B_1)$ with respect to $\|\nabla(\cdot)\|_{L^p(B_1)}$, $L^{p,q}(\log L)^r = L^{p,q}(\log L)'(B_1)$ is the Lorentz-Zygmund spaces:

$$L^{p,q}(\log L)' = \{ u : \text{measurable function on } B_1 | \|u\|_{L^{p,q}(\log L)' < \infty} \}$$

$$\|u\|_{L^{p,q}(\log L)'} = \left\{ \begin{array}{ll} \left( \int_{B_1} \left( \frac{\omega_{N-1}}{s} \right)^{\frac{p}{r}} (u^*(s))^p \, ds \right)^\frac{1}{p} & \text{if } 1 \leq q < \infty, \\
\sup_{0 < s < |B_1|} \left( \frac{\omega_{N-1}}{s} \right)^{\frac{p}{r}} u^*(s) & \text{if } q = \infty.
\end{array} \right.$$

If $r = 0$, then the Lorentz-Zygmund space $L^{p,q}(\log L)^0$ becomes the Lorentz space $L^{p,q}$. In the subcritical case where $p < N$, the following embeddings hold for any $q \in (p, \infty)$.

$$W_0^{1,p}(B_1) \hookrightarrow L^{p-p} \hookrightarrow L^{p,q} \hookrightarrow L^{p,\infty}.$$

Note that the subcritical Hardy inequality:

$$\left( \frac{N-p}{p} \right)^p \int_{B_1} \frac{|u|^p}{|x|^p} \, dx \leq \int_{B_1} |\nabla u|^p \, dx,$$

expresses the embedding $W_0^{1,p}(B_1) \hookrightarrow L^p(\mathbb{R}^N) |x|^{-p} \, dx$ which is equivalent to $W_0^{1,p} \hookrightarrow L^{p-p}$ thanks to the rearrangement technique. On the other hand, in the limiting case where $p = N$, the following embeddings hold for any $q \in (N, \infty)$.

$$W_0^{1,N}(B_1) \hookrightarrow L^{N,N}(\log L)^{-1} \hookrightarrow L^{\infty,q}(\log L)^{-1+\frac{1}{q}} \hookrightarrow L^{\infty,\infty}(\log L)^{-1+\frac{1}{q}} \hookrightarrow \text{ExpL}^{\frac{x}{N}}.$$

Note that the critical Hardy inequality:

$$\left( \frac{N-1}{N} \right)^N \int_{B_1} \frac{|u|^N}{|x|^N(\log \frac{|x|}{|a|})^N} \, dx \leq \int_{B_1} |\nabla u|^N \, dx,$$

expresses the embedding $W_0^{1,N}(B_1) \hookrightarrow L^N(\mathbb{R}^N) |x|^{-N}(\log \frac{|x|}{|a|})^{-N} \, dx$ which is equivalent to $W_0^{1,N} \hookrightarrow L^{\infty,N}(\log L)^{-1}$ for large $a > 1$. Trudinger [1] showed the embedding $W_0^{1,N} \hookrightarrow \text{ExpL}^{\frac{x}{N}}$ via some limiting procedure for the Sobolev embedding $W_0^{1,p} \hookrightarrow L^{p} \hookrightarrow L^{p-r}$. In this talk, we consier an analogue of Trudinger’s limiting procedure for the Hardy embedding $W_0^{1,p} \hookrightarrow L^{p-p}$. Concretely, we derive the (non-sharp) critical Hardy inequality from the subcritical Hardy inequality with the best constant $(\frac{N-p}{p})^p$ via some limiting procedure as $p \to N$. Our limiting procedure gives a reason why the Lorentz-Zygmund space appear in the embedding of the critical Sobolev space $W_0^{1,N}$. Our limiting procedure can be also applied in the second order case which is called the Rellich inequality.

References

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Self-similar solutions of
the crystalline mean curvature flow

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It is well-known that a sphere shrinking with a certain rate is a solution of the classical mean curvature flow. More interestingly, there are self-similar embedded solutions of this flow with positive topological genus such as the shrinking torus constructed by S. Angenent or the higher-genus solutions numerically discovered by D. L. Chopp. They are important for the study of singularities of the mean curvature flow and its large time behavior.

It is an interesting question whether analogs of such solutions exist for the anisotropic or crystalline mean curvature flows. These flows are a natural generalization of the isotropic case by considering a gradient flow of the surface energy with anisotropic, possibly singular (crystalline), surface energy density. They are important in modeling the growth of crystals and other material science applications. In recent years there has been a significant progress in understanding of the well-posedness of solutions of the crystalline mean curvature flow in particular. The appearance of flat parts (facets) on the evolving surface where the crystalline curvature is nonlocal makes the analysis of this problem quite challenging in dimensions \( n \geq 3 \), however, it is possible to explicitly construct interesting polyhedral solutions.

In this talk we present a construction of explicit positive genus self-similar solutions resembling those by Angenent and Chopp for the flow with certain types of crystalline-type surface energy density in dimensions \( n \geq 3 \). We discuss the use of such solutions to test numerical methods for the crystalline mean curvature flow in three dimensions.