Research Article
On the Logarithmic Regularity Conditions for the Variable Exponent Hardy Type Inequality

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We discuss a logarithmic regularity condition in a neighborhood of the origin and infinity on the exponent functions \( q(x) \geq p(x) \) and \( \beta(x) \) for the variable exponent Hardy inequality

\[
\left( x^{(\beta^{-1})} \int_0^x f(t) \, dt \right)^{\beta^{-1}} \leq C \left( x^{p^{-1}} \int_0^x f(t) \, dt \right)^{p^{-1}}
\]

(1)

Let \( v, w : \mathbb{R} \rightarrow [0, \infty) \) be positive measurable functions and \( p^- = \inf \{ p(x) : x \in \mathbb{R} \} > 1, p^+ = \sup \{ p(x) : x \in \mathbb{R} \} < \infty \); then inequality (1) and (2) hold under certain conditions on the weight functions \( v, w \) and the exponents \( p(\cdot), q(\cdot) \).

Two types of conditions arise here: a balance condition on the weights and a regularity condition on the exponents (see below). Necessary and sufficient conditions for the validity of general inequality (1) were found in [1] for the case of \( p(x) \leq q(x) \), in [2] for cases \( q(0) \geq p(0) \) and \( q(\infty) \geq p(\infty) \), in [3] for cases \( q(0) < p(0) \) and \( q(\infty) < p(\infty) \), and in [4] for mixed cases \( q(0) \geq p(0) \) and \( q(\infty) < p(\infty) \) \((q(0) < p(0) \) and \( q(\infty) \geq p(\infty) \)). Some special cases of (1) are studied in [5–10] too.

So, the inequality

\[
\left\| x^{(\beta^{-1})} Hf(\cdot) \right\|_{L^{p^-}(0,\infty)} \leq C \left\| x^{p^{-1}} f(\cdot) \right\|_{L^{p^+}(0,\infty)}
\]

(2)

is a particular case of (1) when \( p(x) = q(x), v(x) = x^{\beta(x)-1} \) and \( w(x) = x^{\beta(x)} \), where \( Hf(x) = \int_0^x f(t) \, dt \) is the Hardy operator. For the constant exponents \( p, \beta \) this inequality holds if \( \beta < 1 - (1/p), p > 1 \) (see, e.g., [11]). Necessary and sufficient conditions on the \( \beta(\cdot), p(\cdot) \) for inequality (2) to hold are \( \beta(0) < 1 - (1/p(0)) \) and \( \beta(\infty) < 1 - (1/p(\infty)) \) if the exponents \( p, q \) are continuous near the origin and infinity such that the conditions \( p, \beta \in \Lambda_0 \cap \Lambda_\infty \) are satisfied (see, e.g., [5–7, 9]). In [8] (see also [10]) it was proved that the condition \( p, \beta \in \Lambda_1 \) is necessary for inequality (2) to hold if one of these exponents is a constant. Also, it was proved in [10] that the condition \( \Lambda_1 \) is sufficient for inequality (2) to hold on bounded interval \((0, l)\) if a constant \( \Lambda_1 \) in the condition \( \Lambda_1 \) for the \( p \) satisfies \( A_1 < p(0) \) \((p(0) - 1) \) and the \( \beta \) is zero. The condition \( \Lambda_1 \) is weaker then \( \Lambda_0 \). Also the function \( p(x) = C/\sqrt{\ln(1/x)} \) satisfies the condition \( \Lambda_1 \) but does not satisfy \( \Lambda_0 \).

In this note, we will focus on the results of sufficiency and necessity of regularity conditions \( \Lambda_0, \Lambda_\infty, \) and \( \Lambda_1 \) below for inequality (2) to hold.

The space of functions \( L^{\beta(\cdot)}(0,\infty) \) is introduced as the class of measurable functions \( f(x) \) in \((0,\infty)\), which have a finite \( I_p(f) := \int_0^{\infty} |f(x)|^{p(x)} \, dx \) modular. A norm in \( L^{\beta(\cdot)}(0,\infty) \) is given in the form

\[
\|f\|_{L^{\beta(\cdot)}(0,\infty)} = \inf \left\{ \lambda > 0, I_p \left( \frac{f}{\lambda} \right) \leq 1 \right\}.
\]

(3)

As to the basic properties of spaces \( L^{\beta(\cdot)} \), we refer to [12].
2. Main Results

We will state some sufficiency and necessity assertions concerning inequality (2). Along the way, it will be given a proof for two elementary estimates that we had used.

Let us introduce the following classes of measurable functions. We say, \( s(x) : (0, \infty) \rightarrow \mathbb{R} \) is in the class \( \Lambda_0 \) if

\[
\Lambda_0 := \limsup_{x \to 0} |s(x) - s(0)| \ln \frac{1}{|x|} < \infty, \tag{4}
\]

and in the class \( \Lambda_\infty \) if

\[
\Lambda_\infty := \limsup_{x \to \infty} |s(x) - s(\infty)| \ln |x| < \infty, \tag{5}
\]

and is in the class \( \Lambda_1 \) if

\[
\Lambda_1 := \limsup_{x \to 0} \left| s(x) - s \left( \frac{x}{2} \right) \right| \ln \frac{1}{|x|} < \infty. \tag{6}
\]

Theorem 1 (see [8]). Suppose \( \beta \in \mathbb{R} \) and \( p : (0, l) \rightarrow [1, \infty) \) is an increasing function on \((0, e)\) such that \( p(x) \) is continuous at \( x = 0 \) and \( \beta < 1 - (1/p(0)) \), \( p^* > 1 \); then for the inequality

\[
\left\| x^{\beta(x) - 1} Hf \right\|_{L^p((0,l))} \leq C \left\| x^{1/p(x)} f \right\|_{L^p((0,l))} \tag{7}
\]

to hold it is necessary that \( p(\cdot) \in \Lambda_1 \).

Theorem 2 (see [8]). Suppose \( p \in \mathbb{R} \) and \( \beta : (0, l) \rightarrow \mathbb{R} \) is a decreasing function on \((0, e)\) such that \( \beta(x) \) is continuous at \( x = 0 \) and \( \beta(0) < 1 - (1/p) \), \( p^* > 1 \); then for the inequality

\[
\left\| x^{\beta(x) - 1} Hf \right\|_{L^p((0,l))} \leq C \left\| x^{1/p(x)} f \right\|_{L^p((0,l))} \tag{8}
\]

to hold it is necessary that \( \beta(\cdot) \in \Lambda_1 \).

Theorem 3. Suppose \( p, \beta : (0, \infty) \rightarrow \mathbb{R} \) is measurable functions such that \( 1 < \beta^- \), \( \beta^+ < \infty \), \( 1 < p^- \), \( p^+ < \infty \); then for the inequality

\[
\left\| x^{\beta(x) - 1} Hf \right\|_{L^p((0,\infty))} \leq C \left\| x^{1/p(x)} f \right\|_{L^p((0,\infty))} \tag{9}
\]

to hold it is sufficient that \( \beta, p \in \Lambda_0 \cap \Lambda_\infty \), whenever \( \beta(0) < 1 - (1/p(0)) \) and \( p(\infty) < 1 - (1/p(\infty)) \).

Theorem 4. Let \( \beta \in \mathbb{R} \), \( p : (0, \infty) \rightarrow [1, \infty) \) and let \( p^* > 1 \). There exists a sequence \( f_n \) and a function \( p \), satisfying the conditions \( \beta < 1 - (1/p(0)) \), \( \beta < 1 - (1/p(\infty)) \) and

\[
\lim_{n \to \infty} \left| p(\delta_n) - p(0) \right| \ln \frac{1}{\delta_n} = \infty : \delta_n = 4^{-n}, \tag{10}
\]

or

\[
\lim_{n \to \infty} \left| \beta(p_n) - \beta(\infty) \right| \ln \mu_n = \infty : \mu_n = 4^n, \tag{11}
\]

violating inequality (2).

Theorem 5. Let \( p \in \mathbb{R} \), \( \beta : (0, \infty) \rightarrow \mathbb{R} \) and let \( p^* > 1 \). Then there exists a sequence \( f_n \) and a function \( p \), satisfying the conditions \( \beta_0 < 1 - (1/p) \), \( \beta_\infty < 1 - (1/p) \), and

\[
\lim_{n \to \infty} \left| \beta(\delta_n) - \beta_0 \right| \ln \frac{1}{\delta_n} = \infty : \delta_n = 4^{-n} \tag{12}
\]

or

\[
\lim_{n \to \infty} \left| \beta(\mu_n) - \beta_\infty \right| \ln \mu_n = \infty : \mu_n = 4^n \tag{13}
\]

violating inequality (2).

3. Proof of Main Results

For the proof of Theorems 1 and 2 we refer to [8]. Other proofs of these theorems are given in [10]. The proof of Theorem 3 also is given in [8]. Here we derive an alternative proof of that theorem using the general results of [2, 4].

In the proof of main results we use the following elementary Lemma.

Lemma 6. Suppose \( s : \mathbb{R} \rightarrow (0, \infty) \) is a measurable function such that \( s \in \Lambda_0 \cap \Lambda_\infty \) and \( 0 < s^-, s^+ < \infty \); then it holds the estimate

\[
C_{\beta^{-1}} s^{(x)}(0) \leq s(x) \leq C_{\beta^+} s^{(x)}(0), \tag{14}
\]

for \( 0 < x < 1 \) and the estimate

\[
C_{\beta^{-1}} s^{(x)}(\infty) \leq s^{(x)}(\infty) \leq C_{\beta^+} s^{(x)}(\infty) \tag{15}
\]

for \( x \geq 2 \).

Proof of Theorem 3. To prove Theorem 3 we apply the results of [2, 4], where it was proved that the following conditions

\[
\sup_{x \in (0, 1)} V(x)^{1/(p(0))} W(x)^{(p(0) - 1)/p(0))} < \infty, \tag{16}
\]

\[
\sup_{x \in (2, \infty)} V(x)^{1/(p(\infty))} W(x)^{(p(\infty) - 1)/p(\infty))} < \infty \tag{17}
\]

are necessary and sufficient for inequality (1) to hold if \( q(0) \geq p(0) \), \( q(\infty) \geq p(\infty) \), and the regularity conditions are satisfied:

\[
\lim_{n \to \infty} \left| p(\infty) - p(0) \right| \ln \frac{1}{W(x)} < \infty; \tag{18}
\]

where \( V(x) := \int_{x^0} v(t)dt, W(x) := \int_x^\infty w^{-1/(p(t) - 1)} dt \).

In Theorem 3, we have accepted that \( q(x) = p(x) \) and \( V(x) := V_1(x) = \int_x^\infty t^{(p(t) - 1)/p(t)} dt, W(x) := W_1(x) = \int_x^\infty t^{-(p(t)/p(t))} dt \). It is easy to show that the conditions \( p, \beta \in \Lambda_0 \), \( p \in \Lambda_\infty \) imply \( \beta - 1, 1 \in \Lambda_\infty \). Therefore, it follows from Lemma 6 that

\[
C_{\beta^{-1}} x^{(\beta(0) - 1)/p(0))} \leq x^{(\beta(x) - 1)/p(x))} \leq C_{\beta^+} x^{(\beta(\infty) - 1)/p(\infty))} \tag{19}
\]

for \( x \geq 2 \) by some \( C_{\beta} > 0 \). Also the conditions \( p, \beta \in \Lambda_0 \) imply \( (\beta p/(p - 1)) \in \Lambda_0 \). Therefore, it follows from Lemma 6 that

\[
C_{\beta^{-1}} x^{-(\beta(0) p(0))/(p(0) - 1))} \leq x^{-(\beta(x) p(x)/(p(x) - 1))} \leq C_{\beta^+} x^{-(\beta(\infty) p(\infty))/(p(\infty) - 1))} \tag{20}
\]
for $0 < x < 1$ by some $C_4 > 0$. Integrating these inequalities over the intervals $(x, \infty)$ and $(0, x)$, respectively, we get

$$C_5 x^{(\beta(0) - 1)p(0)}[p(0) + 1] \leq V_1(x) \leq C_5 x^{(\beta(0) - 1)p(0) + 1}, \quad x > 2,$$

$$C_6 x^{-1 - [(\beta(0)p(0) - p(0))/p(0) - 1]} \leq W_1(x) \leq C_6 x^{1 - [(\beta(0)p(0)/p(0) - 1)]}, \quad 0 < x < 1. \quad (20)$$

To complete the proof of Theorem 3 it suffices to apply estimates (20) to verify conditions (16) and (17). Now, Theorem 3 follows from the upper refereed results of the works [2, 4].

**Proof of Lemma 6.** Let $0 < x < \delta$ and let $s(x) \geq s(0)$; then we have $x^{(\beta(x) - s(0))} \leq 1 + \delta^{x - s(0)}$, where $\delta$ is a certain number from the interval $(0, 1)$. If $0 < x \leq \delta$ and $s(x) < s(0)$ then by condition $s \in \Lambda_0$ we have

$$x^{(\beta(x) - s(0))} = \left( \frac{\delta}{x} \right)^{x^{(\beta(x) - s(0))}} \delta^{x^{(\beta(x) - s(0))}}$$

$$\leq \left( \frac{\delta}{x} \right)^{x^{(1/\ln(1/\delta))}} \delta^{x^{(1/\ln(1/\delta))}}$$

$$= e^{\delta x^{(1/\ln(1/\delta))}} \left( 1 + \left( \frac{1}{\delta} \right)^{x^{(1/\ln(1/\delta))}} \right). \quad (21)$$

Therefore, for $0 < x \leq \delta$ we have the estimation

$$x^{(\beta(x) - s(0))} \leq C, \quad (22)$$

where the positive constant $C$ depends on $s^-$, $s^+$, $s(0)$, $\delta$. Same inequality holds for the function $x^{(\delta(x) - s(x))}$. Indeed, for $0 < x \leq \delta$ and $s(x) \geq s(0)$ we have $x^{(\delta(x) - s(x))} \leq 1 + \delta^{x - s(x)}$. If $0 < x \leq \delta$ and $s(x) \geq s(0)$ by condition $s \in \Lambda_0$ we have

$$x^{(\delta(x) - s(x))} = \left( \frac{\delta}{x} \right)^{x^{(\delta(x) - s(x))}} \delta^{x^{(\delta(x) - s(x))}}$$

$$\leq \left( \frac{\delta}{x} \right)^{x^{(1/\ln(1/\delta))}} \delta^{x^{(1/\ln(1/\delta))}}$$

$$= e^{\delta x^{(1/\ln(1/\delta))}} \left( 1 + \left( \frac{1}{\delta} \right)^{x^{(1/\ln(1/\delta))}} \right). \quad (23)$$

By using these inequalities and by the representation

$$x^{(\beta(x))} = x^{(\delta(x))} x^{(\delta(x) - \beta(x))} \quad (24)$$

we have estimate (14).

To show estimate (15) note that for $x \geq M$ and $s(x) \leq s(\infty)$ we have $x^{(\beta(x) - s(\infty))} \leq 1 + (1/M)^{(\beta(x) - s(\infty))}$, where the $M \geq 2$ is a certain number. If $x \geq M$ and $s(x) \geq s(\infty)$ then by the condition $s \in \Lambda_{\infty}$ we have

$$x^{(\beta(x) - s(\infty))} = \left( \frac{x}{M} \right)^{x^{(\beta(x) - s(\infty))}} M^{x^{(\beta(x) - s(\infty))}}$$

$$\leq \left( \frac{x}{M} \right)^{C_2 \ln x} M^{(x^{(\beta(x) - s(\infty))} - s(\infty))}$$

$$= C_2 M^{x^{(\beta(x) - s(\infty))} - s(\infty))} M^{s(\infty) - s(\infty))}$$

$$\leq e^{C_2} \left( 1 + \left( \frac{1}{M} \right)^{C_2 \ln M} \right) \left( 1 + M^{s(\infty) - s(\infty))} \right). \quad (25)$$

Combining the estimates for the functions $x^{(\beta(x) - s(\infty))}$, $x^{(\delta(x) - s(x))}$ for $x \geq M$ by the presentation

$$x^{(\beta(x))} = x^{(\delta(x))} x^{(\delta(x) - \beta(x))} \quad (26)$$

we get estimate (15). To complete the proof of Lemma 6, note that the condition $s \in \Lambda_0$ is equivalent to

$$-C \leq |s(x) - s(0)| \ln \frac{1}{x} \leq C \quad (27)$$

and the $s \in \Lambda_{\infty}$ is equivalent to

$$-C \leq |s(x) - s(\infty)| \ln x \leq C, \quad (28)$$

respectively.

**Proof of Theorem 4.** Let us assume that $f_k(x) = x^{(-1/\beta(x) - 1)} x^{(\alpha_n, \beta_n, \gamma_n)}(x)$; $\alpha_n \in (0, \infty)$. Fix $k \in \mathbb{N}$. We define the step function

$$p(x) = \begin{cases} p_0 + \alpha_n & \text{if } x \in (2\delta_n, 4\delta_n), \\ p_0 & \text{if } x \in (\delta_n, 2\delta_n), \\ 0 & \text{if } x \in (0, \delta_n) \cup (2\delta_n, \infty), \quad n \in \mathbb{N}. \end{cases} \quad (29)$$

Here $\alpha_n$ is a sequence of positive numbers that satisfies the condition $2\alpha_n \to 0$ as $n \to \infty$. Then $\alpha_n \ln(1/\delta_n) \to \infty$ as $n \to \infty$ and condition (10) is fulfilled for the function $p(x)$. We have

$$I_{p_k}(x^{\beta(x)}) \cdot f_k(x) = \int_{\delta_n}^{2\delta_n} \left( t^{\beta(x)} \cdot t^{(1/p(x)) - \beta} \right) dt = \ln 2 < \infty,$$

$$I_{p_k}(x^{\beta(x) - 1}) \cdot H f_k(x) = \int_{2\delta_n}^{4\delta_n} \left( t^{1 - (1/\beta(x))} \right) dt \cdot x^{(\beta(x) - 1) - (p(x))} dx \leq C \cdot \delta_n^{\alpha_n} / p_0 = C \cdot \delta_n^{\alpha_n} \to \infty \quad (30)$$

as $n \to \infty$. The last relation shows violating of inequality (2) for sufficiently large $n$. 
We define \( f_k(x) = x^{-(1/p(x))-\beta} \chi_{(\mu_k,2\mu_k)}(x) \) for \( x \in (0,\infty) \). Fix \( k \in \mathbb{N} \). We also define the step function

\[
p(x) = \begin{cases} 
\rho_0 - a_n & \text{if } x \in (2\mu_n, 4\mu_n) \\
\rho_0 & \text{if } x \in (\mu_n, 2\mu_n), \quad n \in \mathbb{N},
\end{cases}
\]

where \( a_n \to \infty \) as \( n \to \infty \). We have \( a_n \ln \mu_n \to \infty \) as \( n \to \infty \) and condition (11) holds for the function \( p(x) \). Furthermore,

\[
I_{p(\cdot)}(x^\beta \cdot f_k(x)) = \int_{2\mu_n}^{4\mu_n} \left( x^{\beta - (1/p(x))-\beta} \chi_{(\mu_n,2\mu_n)}(x) \right)^p dx = \ln 2 < \infty,
\]

\[
I_{p(\cdot)}(x^{\beta-1} \cdot H f_k(x)) \geq \int_{2\mu_n}^{4\mu_n} \left( \frac{2\mu_n}{\mu_n} \frac{1}{1-(1/p(x))} dt \right)^p (\rho_0 - \alpha_n) \cdot x^{(\beta-1)(\rho_0 - \alpha_n)} dx
\]

\[
\geq C \rho_0^p \rho_0 \cdot C e^{(\rho_0/p_0) \ln \mu_n} \to \infty
\]

(32)
as \( n \to \infty \), which contradicts (2) for sufficiently large \( n \). \( \square \)

**Proof of Theorem 5.** Let us assume that \( f_k(x) = x^{-(1/p(x))-\beta} \chi_{(\delta_n,2\delta_n)}(x) \); \( x \in (0,\infty) \). Fix \( k \in \mathbb{N} \). We define the step function \( \beta \) as

\[
\beta(x) = \begin{cases} 
\beta_0 + a_n & \text{if } x \in (\delta_n, 2\delta_n) \\
\beta_0 & \text{if } x \in (2\delta_n, 4\delta_n), \quad n \in \mathbb{N},
\end{cases}
\]

(33)

where \( a_n \cdot \ln(1/\delta_n) \to \infty \). Then,

\[
I_{p(\cdot)}(x^{\beta(x)} \cdot f_k(x)) = \int_{2\delta_n}^{4\delta_n} \left( x^{\beta_0 + a_n - (1/p(x))-\beta} \chi_{(\delta_n,2\delta_n)}(x) \right)^p dx
\]

\[
= \ln 2 < \infty,
\]

\[
I_{p(\cdot)}(x^{\beta(x)-1} \cdot H f_k(x)) \geq \int_{2\delta_n}^{4\delta_n} \left( \frac{2\delta_n}{\delta_n} \frac{1}{1-(1/p(x))} dt \right)^p \cdot x^{(\beta_0 - 1)p} dx
\]

\[
= C \cdot \delta_n^{p \alpha_n} = C \cdot e^{p \alpha_n \ln \delta_n} \to \infty
\]

as \( n \to \infty \). The last relation contradicts the validity of inequality (2).

We define \( f_k(x) = x^{-(1/x)-\beta(x)} \chi_{(\mu_n,2\mu_n)}(x) \); \( x \in (0,\infty) \). Fix \( k \in \mathbb{N} \), where the function \( \beta \) is defined as

\[
\beta(x) = \begin{cases} 
\beta_\infty - a_n & \text{if } x \in (\mu_n, 2\mu_n) \\
\beta_\infty & \text{if } x \in (2\mu_n, 4\mu_n), \quad n \in \mathbb{N},
\end{cases}
\]

(35)

where \( \lim_{n \to \infty} a_n \cdot \ln \mu_n = \infty \). Then,

\[
I_{p(\cdot)}(x^{\beta(x)} \cdot f_k(x)) = \int_{\mu_n}^{2\mu_n} \left( x^{\rho_0 - \alpha_n - (1/p(x))-\beta} \chi_{(\mu_n,2\mu_n)}(x) \right)^p dx
\]

\[
= \ln 2 < \infty,
\]

\[
I_{p(\cdot)}(x^{\beta(x)-1} \cdot H f_k(x)) \geq \int_{\mu_n}^{2\mu_n} \left( \frac{\mu_n}{\mu_n} \frac{1}{1-(1/p(x))} dt \right)^p \cdot x^{(\beta_\infty - 1)p} dx
\]

\[
\geq C \cdot \mu_n^{p \alpha_n} = C \cdot e^{p \alpha_n \ln \mu_n} \to \infty
\]

as \( n \to \infty \) which contradicts inequality (2). \( \square \)

**Conflict of Interests**

The authors declare that they have no conflict of interests regarding the publication of this paper.

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