Gravitational waves originated before compactification in Kaluza-Klein theory

Y. Ezawa, H. Iwasaki, A. Nakamura, Y. Ohkuwa† T. Suzuki and T. Yano*

Department of Physics, Ehime University, Matsuyama, 790-8577, Japan
†Section of Mathematical Science, Department of Social Medicine, Faculty of Medicine, University of Miyazaki, Kiyotake, Miyazaki, 889-1692, Japan
*Department of Electrical Engineering, Ehime University, Matsuyama, 790-8577, Japan

Email : ezawa@sci.ehime-u.ac.jp, hirofumi@phys.sci.ehime-u.ac.jp, ohkuwa@med.miyazaki-u.ac.jp and yanota@eng.ehime-u.ac.jp

Abstract

We investigate the propagation of multidimensional gravitational waves generated before compactification and observed today as 4-dimensional gravitational waves and gauge fields by generalizing the investigation of Alesci and Montani(AM) to the case where the internal space is an $n$-sphere. We also derive the 4-dimensional forms of the multidimensional harmonic gauge condition. The primary difference from the case of AM comes from the effects of the curvature of the internal space which prevent the space from being static. The other effects are explicitly shown for propagation of the waves in our 4-dimensional spacetime. Static internal space exists if multidimensional cosmological constant is present. Then the situation is similar to the one-dimensional internal space.

1 Introduction

In unified theories such as Kaluza-Klein(KK) and string theories, extra-dimensional space is necessary. This spacetime may also serve to make hierarchy in energy scales in the brane universe picture through its size[1, 2]. However, it is difficult to detect the existence of the extra-dimensional space directly.

Recently, Alesci and Montani(AM) examined differences between gravitational waves generated before and after compactification of the internal space [3] in the context of original KK theory[4]. AM investigated the propagation of gravitational waves in our 4-dimensional spacetime which were produced in the 5-dimensional spacetime, i.e. before compactification. They also examined the gauge conditions on gravitational waves in the 4-dimensional spacetime imposed originally in the 5-dimensional spacetime. In their investigation, it was assumed that Einstein equations in the 5-dimensional spacetime hold also after compactification if the original metric is replaced by the compactified one. They found that such gravitational waves propagate with the speed of light as the one originated in the 4-dimensional
spacetime. On the other hand, the gauge condition, the 5-dimensional harmonic condition, was found to forbid the 4-dimensional transverse and traceless gauge condition.

As to the observability of these gravitational waves, there may be problems concerning the effects of inflation. Since the inflation dilutes the energy density of the gravitational waves, their detection might be difficult by the present detectors, unless the superposition effects are so large. However, there would not be problems with respect to wavelength which would be so small when the waves were generated, so their wavelength would not be unobservably long. Nevertheless the results of AM are interesting in that observational signals of the extra-dimensional space are not known well.

In this work, we extend their model by taking into account the unified theoretical point of view, from which it is more realistic that the dimension of the extra-space is more than one. Thus we examine the propagation of the gravitational waves generated in (4+n)-dimensional spacetime before compactification. The n-dimensional extra dimensional space is taken to be a sphere as in the multidimensional inflation models. This would be anticipated if we consider that the sphere is maximally symmetric subspace and the breaking of the symmetry would be caused by some kind of excitations. In this case, the extra space cannot be static due to its nonvanishing curvature, contrary to the 5-dimensional case since 1-dimensional space cannot have nonvanishing Riemannian curvature. The sphere would be expected to collapse, so some mechanism for stabilization would be necessary.

We also examine how the harmonic gauge condition imposed in the (4 + n)-dimensional spacetime is expressed in our 4-dimensional spacetime. Four dimensional wave equation and gauge conditions are complicated due to time variation of extra space.

There is a static solution if we introduce the multidimensional cosmological constant. In this case, wave equations take the same form as those in 1-dimensional extra space. Gauge conditions take here the linearized form of the nonabelian gauge theory.

In section 2, we derive wave equations and gauge conditions in (4+n)-dimensional form, i.e., when the waves were generated. In section 3, background spacetime is investigated for cases with and without multidimensional cosmological constant. Section 4 is devoted to derive explicit form of the wave equations and gauge conditions. Summary and discussions are given in section 5. Details of geometric quantities are summarized in the appendix.

2 Wave equations and gauge conditions in
(4+n)-dimensional form

In this section, we write down the equations for the gravitational waves generated in (4+n)(≡ D)-dimensional spacetime before compactification.

Einstein equations in the D-dimensional spacetime take the following form when the cosmological constant is absent

\[ \hat{G}_{AB} = \kappa_D \hat{T}_{AB}, \quad (A, B = 0, 1, 2, ..., D - 1) \]  

(2.1)

where \( \hat{G}_{AB} \) and \( \hat{T}_{AB} \) are D-dimensional Einstein tensor and energy-momentum tensor, respectively. A hat is used to denote quantities defined in D-dimensional spacetime. \( \kappa_D \) is the D-dimensional gravitational constant. Here we investigate the propagation of the gravitational waves in the vacuum, so we put \( \hat{T}_{AB} = 0 \) in the following. Then it is well known that Eqs.(2.1) reduce to the vanishing of the Ricci tensor:

\[ \hat{R}_{AB} = 0. \]  

(2.2)
We denote the background metric by $\hat{g}_{(0)AB}$ and its perturbation by $\hat{h}_{AB}$:

$$\hat{g}_{AB} = \hat{g}_{(0)AB} + \hat{h}_{AB}. \quad (2.3)$$

$\hat{g}_{(0)AB}$ is determined by the zeroth order equations of (2.2), $\hat{R}_{(0)AB} = 0$. For the perturbations, the first order equations of (2.2) are written as

$$\hat{R}_{(1)AB} = \frac{1}{2} \left( \hat{h}_{A|BC} + \hat{h}_{B|AC} - \hat{h}_{AB} |^C |^C - \hat{h}_{|AB} \right) = 0 \quad (2.4)$$

where $\hat{h} \equiv \hat{g}_{(0)}^{AB} \hat{h}_{AB}$. Eqs.(2.4) are the equations for $D$-dimensional gravitational waves. A stroke( | ) denotes the covariant derivative with respect to the $D$-dimensional background metric $\hat{g}_{(0)AB}$. In terms of variables $\psi_{AB}$ defined as

$$\psi_{AB} \equiv \hat{h}_{AB} - \frac{1}{2} \hat{g}_{(0)AB} \hat{h} \quad \text{or} \quad \hat{h}_{AB} = \psi_{AB} - \frac{1}{D-2} \hat{g}_{(0)AB} \psi \quad (2.5)$$

where $\psi \equiv \hat{g}_{(0)AB} \psi_{AB}$, the harmonic gauge condition takes the following form

$$\psi^B_A |_B = 0. \quad (2.6)$$

In this gauge, the wave equations (2.4) are written as

$$\psi_{AB} |^C |^C - 2 \hat{R}_{(0)CABD} \psi^{CD} = 0, \quad (2.7)$$

where $\hat{R}_{(0)ABCD}$ is the Riemann tensor of the background spacetime.

### 3 Background metric

In this section, we examine the background spacetime through which gravitational waves propagate. We assume that, after compactification, extra $n$-dimensional space is maximally symmetric, i.e. $n$-sphere as noted in the introduction. This symmetry is often assumed in multidimensional inflationary models. The breaking of the symmetry can be thought to arise from some excitations. Then the background metric is written as

$$\hat{g}_{(0)AB} = \left( \begin{array}{cc} g_{(0)\mu\nu}(x) & 0 \\ 0 & a_I(x)^2 g_{(0)ab} \end{array} \right), \quad (\mu, \nu = 0, 1, 2, 3; a, b = 1, 2, \ldots, n) \quad (3.1)$$

by a suitable choice of the coordinates of the internal space. Here the scale factor of the extra-dimensional space $a_I(x)$ behaves as a scalar field in the 4-dimensional spacetime and $g_{(0)ab}$ is the metric of the unit $n$-sphere and is given as $g_{(0)ab} \equiv a_I^{-2} \hat{g}_{(0)ab}$. Instead of $a_I$, we will often use a scalar field $\phi$ defined as

$$\phi \equiv \ln(a_I/a_e). \quad (3.2)$$

Here $a_e$ is a constant with dimension of length.

Details of calculations of geometric quantities are given in the appendix.
3.1 The case without the cosmological constant

Einstein equations for the background metric $\hat{R}_{(0)AB} = 0$ are now written as follows

$$R_{(0)\mu\nu} - n (\nabla_\mu \nabla_\nu \phi + \partial_\mu \phi \partial_\nu \phi) = 0 \quad (3.3)$$

and

$$R_{(0)ab} - a_I^2 g_{(0)ab} (\nabla_\phi + n \partial_\lambda \phi \partial^\lambda \phi) = 0 \quad (3.4)$$

where quantities without a hat are defined in 4-dimensional spacetime. $\nabla_\mu$ denotes the covariant derivative with respect to the 4-dimensional background metric $g_{(0)\mu\nu}$ and $\Box \equiv g^{\lambda\rho}_{(0)} \nabla_\lambda \nabla_\rho$.

Equations $\hat{R}_{(0)\mu a} = 0$ are automatically satisfied. Since the extra space is taken to be an $n$-sphere which is maximally symmetric, we have a relation

$$R_{(0)ab} = (n - 1)g_{(0)ab} \quad (3.5)$$

Using (3.5), we have an equation for $a_I$ from (3.4);

$$a_I \Box a_I + (n - 1) \partial^\rho a_I \partial_\rho a_I = n - 1 \quad \text{or} \quad \Box \phi + n \partial^\rho \phi \partial_\rho \phi = (n - 1)a_e^{-2}e^{-2\phi} \quad (3.6a)$$

The righthand sides of these equations come from the curvature of $n$-sphere, $R_I = n(n - 1)$. These equations show that $a_I$ or $\phi$ cannot be constant. This is essentially different from the case of 1-dimensional extra space in which case the extra-dimensional component of the Riemann tensor is vanishing so that $g_{(0)44}$ can be constant as in the original KK-model.

When the 4-dimensional background spacetime is the Robertson-Walker(RW) one, time component of (3.3) (precisely, the time component of $\hat{G}_{(0)\mu\nu} = 0$) becomes as follows

$$H^2 + \frac{(3)R}{6a^2} + nH\dot{\phi} + \frac{1}{6}n(n - 1) \left( \phi^2 + a_e^{-2}e^{-2\phi} \right) = 0 \quad (3.7)$$

where $(3)R$ is the scalar curvature of 3-dimensional space, $a$ is the scale factor of the universe, $H$ is the Hubble parameter $\dot{a}/a$. All the three space components of (3.3) lead to the same equation which is given as

$$H + 3H^2 + nH\dot{\phi} + \frac{(3)R}{3a^2} = 0 \quad (3.8)$$

Equation (3.6a) takes the following form

$$a_I \ddot{a}_I + (n - 1)\dot{a}_I^2 + 3Ha_I\dot{a}_I + n - 1 = 0, \quad \text{or} \quad \ddot{\phi} + n\dot{\phi}^2 + 3H\dot{\phi} + (n - 1)a_e^{-2}e^{-2\phi} = 0 \quad (3.9)$$

Equation (3.7) can be thought of as a constraint as usual. As suggested by observations, and for the sake of simplicity, we put $(3)R = 0$ in the following. Then the scale factor $a$ appears only in the combination $H$. We can solve for $H$ from (3.9) when $\dot{a}_I \neq 0$. Putting the solution into (3.8), we obtain the third order differential equation for $a_I$ which reads as follows:

$$\dot{a}_Ia_I^{(3)} - 2\dot{a}_I^2 + \dot{a}_I^2/a_I - (n - 1) \left[ 3\ddot{a}_I/a_I + (n - 1)(\dot{a}_I/a_I)^2 + (n - 1)/a_I^2 \right] = 0. \quad (3.10)$$

Equation (3.10) has no power-law or exponential type solutions. Considering the multidimensional inflation model, we might expect that $a_I$ would collapse, since it could not expand so large.
3.2 The case with the cosmological constant $\hat{\Lambda}$

It would be desirable that there are static solutions for $a_I$. It is possible if the cosmological constant $\hat{\Lambda}$ term is introduced into the $4+n (\equiv D)$-dimensional gravity, which can be thought of as the least generalization of Einstein gravity. Then Eqs. (2.1) and (2.2) are replaced by

\[ \hat{G}_{AB} - \hat{\Lambda}\hat{g}_{AB} = \kappa_D \hat{T}_{AB} \]  

(3.11)

and

\[ \hat{R}_{AB} - \frac{2}{D-2}\hat{\Lambda}\hat{g}_{AB} = 0 \]  

(3.12)

where we used a relation

\[ \hat{R} = \frac{2}{D-2}[D\hat{\Lambda} - \kappa_D \hat{T}] \]  

(3.13)

and put $\hat{T}_{AB} = 0$.

Concerning the effects of $\hat{\Lambda}$, there are two possibilities. One possibility is that the size of $\hat{\Lambda}$ is the order of fluctuations so that background and the first order field equations are given by

\[ \hat{R}_{(0)AB} = 0 \quad \text{and} \quad \hat{R}_{(1)AB} = \frac{2}{D-2}\hat{\Lambda}\hat{g}_{(0)AB}. \]  

(3.14a)

In this case, the background is unchanged, so that the internal space is not static, which is not interesting here. Another possibility is that the size of $\hat{\Lambda}$ is the order of background curvature:

\[ \hat{R}_{(0)AB} = \frac{2}{D-2}\hat{\Lambda}\hat{g}_{(0)AB} \quad \text{and} \quad \hat{R}_{(1)AB} = \frac{2}{D-2}\hat{\Lambda}\hat{h}_{AB}. \]  

(3.14b)

In this case, the equation for $a_I$, (3.6a), is modified as follows

\[ a_I \Box a_I + (n-1)\partial^\rho a_I \partial_\rho a_I -(n-1) = -\hat{\Lambda}a_I^2 \quad \text{or} \quad \Box \phi + n\partial^\rho \phi \partial_\rho \phi -(n-1)a_I^{-2}e^{-2\phi} = -\hat{\Lambda}. \]  

(3.6b)

Clearly $a_I = [(n - 1)/\hat{\Lambda}]^{1/2}$ is a solution. Similarly both of the right hand sides of (3.7) and (3.8) are replaced by $\hat{\Lambda}$. Note also that the second equation of (3.14b) is the wave equation for this case.

4 Gauge conditions and wave equations

4.1 Description of the perturbations

In this section, we derive the first order gauge conditions and wave equations. As to the perturbations, we restrict here to those which preserve the maximal symmetry of the extra-dimensional space, i.e. the perturbation of $a_I(x)$. So perturbations are denoted in the original $D$-dimensional spacetime as

\[ \hat{h}_{AB} = \begin{pmatrix} \hat{h}_{\mu\nu} & \hat{h}_{\mu a} \\ \hat{h}_{a\nu} & \hat{h}_{ab} \end{pmatrix} \]  

(4.1a)

and we assume

\[ \hat{h}_{ab} = 2a_I \delta a_I \hat{g}_{(0)ab}. \]  

(4.1b)
Here $\delta a_I$ represents the perturbation of $a_I$. That is, we assume that the extra-dimensional space remains to be a sphere and only its radius perturbs. Off-diagonal elements are expressed in terms of the Killing vectors $\xi^{a}_{(\alpha)}$ ($\alpha = 1, 2, \cdots, n(n+1)/2$) on the sphere as

$$\hat{h}_{\alpha\mu} = \hat{g}_{(0)ab}^{\mu} \xi^{b}_{(\alpha)} A^{(\alpha)}_{\mu} = a_{I}^{2} \xi^{a}_{(\alpha)} A^{(\alpha)}_{\mu}$$

(4.2)

where $g_{(0)ab}$ and $\xi^{a}_{(\alpha)}$ are functions of angles $\theta_1, \cdots, \theta_n$ and $A^{(\alpha)}_{\mu}(x)$'s are known to behave as 4-dimensional vector fields and play the role of the gauge fields.

As we choose the harmonic gauge condition, $\psi_{AB}$ are convenient variables. They are expressed, if we use (4.1a,b), (4.2) and

$$\hat{h} = h + nh_{I} \quad h \equiv \hat{g}_{(0)\mu\nu} \hat{h}_{\mu\nu},$$

(4.3)

as follows

$$\begin{cases}
\psi_{\mu\nu} = \hat{h}_{\mu\nu} - \frac{1}{2} \hat{g}_{(0)\mu\nu} (h + nh_{I}), \\
\psi^{a}_{\mu} = \hat{h}^{a}_{\mu}, \\
\psi^{ab} = - \frac{1}{2} \hat{g}_{(0)ab} [h + (n-2)h_{I}]
\end{cases}$$

(4.4)

where $h_{I} \equiv 2a_{I}^{-1}\delta a_{I}$ represents the perturbation of the extra-dimensional space. However, $h_{I}$ does not depend on the coordinates of the extra-dimensional space.

### 4.2 Gauge conditions

Using (3.2), (4.3), (4.4) and (A.2) in (2.6), we have the following gauge conditions. For $A = \mu$, we have the gauge conditions on the 4-dimensional gravitational waves:

$$\nabla_{\rho} \psi^{\rho}_{\mu} + n \partial^{\lambda} \phi \psi_{\mu\lambda} + \left[ h + \frac{1}{2} (n-2) \hat{h} \right] \partial_{\mu} \phi = 0.$$  

(4.5)

For $A = a$, we have the gauge conditions on the 4-dimensional vector(gauge) fields:

$$\nabla^{\lambda} \psi^{a}_{\lambda} + n \partial^{\lambda} \phi \psi^{a}_{\lambda} = 0$$

(4.6a)

which take the following explicit form in terms of $A^{(\alpha)}_{\mu}$

$$\nabla_{\mu} + (n+2) \partial_{\mu} \phi) A^{(\alpha)\mu} = 0.$$  

(4.6b)

These do not appear to be the gauge conditions in non-Abelian gauge theories, since we are dealing with linear perturbations and non-linearity is discarded. The gauge conditions are affected by the existence of $\Lambda$ only through the behavior of $\phi$, and Eqs.(4.5) and (4.6a,b) are unchanged. Therefore if we assume a static $\phi$, these conditions reduce to the transverse condition as in the case of one extra dimension. In the above calculations and in the next section, we use the following relations for the covariant derivative, $\tilde{\nabla}_{a}$, with respect to the internal metric $g_{(0)ab}$:

$$\tilde{\nabla}_{c} \psi^{ab} = 0 \quad \text{and} \quad \tilde{\nabla}_{(a} \psi^{b)} = 0$$  

(4.7)

where ( ) means symmetrization.
4.3 Wave equations

4.3.1 The case of $\tilde{R}_{AB} = 0$

We also obtain the wave equations by using the same equations (3.2), (4.3), (4.4) and (A.2) in (2.7).

For $(AB) = (\mu\nu)$, we have the equations for the gravitational waves

$$\Box \psi_{\mu\nu} + n \partial^\lambda \phi \left[ \psi_{\mu\nu,\lambda} - (\partial_{\mu} \phi \psi_{\lambda\nu} + \partial_{\nu} \phi \psi_{\mu\lambda}) \right] - n [h + (n - 2) h_I] (\nabla_{\mu} \nabla_{\nu} \phi + 2 \partial_{\mu} \phi \partial_{\nu} \phi)$$

$$- 2 R_{(0)\lambda\mu\nu} \psi^{\lambda\rho} = 0. \quad (4.8)$$

For $(AB) = (\mu a)$, we have

$$\Box \psi_{\mu a} + (n - 2) \partial^\lambda \phi \nabla_{\lambda} \psi_{\mu a} - (n + 4) \partial_{\mu} \phi \partial^\lambda \phi \psi_{\lambda a} - (\Box \phi + n \partial^\lambda \phi \partial_{\lambda} \phi) \psi_{\mu a}$$

$$- 2 \nabla_{\mu} \nabla^\lambda \phi \psi_{\lambda a} = 0 \quad (4.9)$$

where $\Box_I \equiv \hat{g}_{ab} ab \nabla_a \nabla_b$.

Finally, for $(AB) = (ab)$, we have

$$\Box \psi_{ab} + (n - 4) \partial^\lambda \phi \partial_{\lambda} \psi_{ab} + 2 \hat{g}_{ab} \left[ (\nabla_{\lambda} \nabla_{\rho} \phi + 2 \partial_{\lambda} \phi \partial_{\rho} \phi) \psi^{\lambda\rho} - \frac{1}{2} n [h + (n - 2) h_I] \partial_{\lambda} \phi \partial^\lambda \phi \right]$$

$$- 2 (\Box \phi + n \partial^\lambda \phi \partial_{\lambda} \phi) \psi_{ab} - 2 R_{(0)\alpha a}^c \psi_{\alpha d} = 0. \quad (4.10)$$

If the extra space is only 1-dimensional, the Riemann tensor vanishes and $\phi$ can be constant, i.e. extra-space can be static. Then only the first terms in the gauge conditions (4.5) and (4.6a,b) and the wave equations (4.8)—(4.10) remain, since other terms come from the time variation and curvature of the internal space.

In terms of the vector fields $A^{(a)}_\mu$, Eqs.(4.10) take the following form describing the propagation of the vector fields $A^{(a)}_\mu$;

$$(\Box + 2 a_I^{-1}) A^{(a)}_\mu + 2 \partial^\lambda \phi \nabla_{\lambda} A^{(a)}_\mu + \partial_{\mu} \phi \nabla^\lambda A^{(a)}_\lambda = 0, \quad (\alpha = 1, 2, \ldots, n(n + 1/2)). \quad (4.11)$$

4.3.2 The case with the cosmological constant $\tilde{\Lambda}$

In this case, the wave equations are given by the second equation of (3.14b) as noted above.

Then $\psi_{|\alpha}^{(a)}$ is nonvanishing but is given by

$$\psi_{|\alpha}^{(a)} = - \frac{4}{D - 2} \tilde{\Lambda} \psi. \quad (4.12)$$

In terms of $\psi_{AB}$, the wave equation (3.14b) takes the same form as (2.7) under the harmonic gauge condition, (2.6). Thus for the case of static background, wave equations (4.8)—(4.10) take the same form as for those in 1-dimensional internal space except for the curvature terms.

5 Summary and discussions

In the framework of Einstein gravity in $D(\equiv 4 + n)$-dimensional spacetime, we investigated the behavior of the 4-dimensional gravitational waves after compactification of the
n-dimensional extra-dimensional space when the gravitational waves were generated before compactification. After compactification, extra-dimensional space which is assumed to be maximally symmetric has nonvanishing Riemannian curvature if its dimension is more than one. Then it cannot be static contrary to the case of the 1-dimensional extra space. The effects of curvature are very complex but appear only through the time variation of the extra-space in the terms other than the first ones in the gauge condition (4.5), (4.6a,b) and wave equations (4.8)–(4.10). These equations are different from those for the ordinary gravitational waves and the differences are very complicated. It is noted that static extra space is possible if the cosmological constant is present in $D$-dimensional gravity. In the static case, equations of the gravitational waves reduce to those of the 1-dimensional extra space.

In our analysis, we used the 4-dimensional components of the original metric $\hat{g}_{\mu\nu}$ as the metric $g_{\mu\nu}$ after compactification without making conformal transformation. In this case, 4-dimensional gravity is not the Einstein one but scalar-tensor gravity type. Then evolutions of the scale factors of the background 3-dimensional and extra spaces are given by (3.8) and (3.10). The latter has no solution which is proportional to $t^\alpha$ or $e^{\beta t}$. With respect to the observation, we point out a possibility that, if the waves were so frequently generated, the interference effects might appear similar to the quantum fluctuations and might have affected structure formations, as they would be expanded during the inflation.

After compactification, 4-dimensional gravity is, at least approximately, the Einstein one. It can be obtained from the dimensionally reduced higher dimensional Einstein gravity by making the following conformal transformation of the metric [7, 8]

$$\bar{g}_{\mu\nu} = \Omega^2 \hat{g}_{\mu\nu}, \quad \Omega = \exp \left[ \frac{n}{2} \ln \left( \frac{a_i}{a_c} \right) \right] = \left( \frac{a_i}{a_c} \right)^{n/2}$$

where $n$ is the dimension of the internal space. Since compactification can be thought of as a kind of phase transition, the conformal transformation could be thought as expressing relations between quantities before and after compactification. Then the propagation of the vector fields and the evolutions of the 4-dimensional spacetime and internal space are affected. In this case, it is possible to have approximate solutions representing growing $a$ and decreasing $a_i$ corresponding to inflation, i.e. $a_i$ playing the role of inflaton, although the probability of inflation is approximately vanishing in the model used in our discussions[8]. To obtain finite probability, at least two internal spaces are required, which may be avoided in higher curvature gravity theories. For the static internal background, inflaton field must be introduced by hand.

From the string theoretical point of view, the original $D$-dimensional spacetime should be a 10-dimensional one and the 4-dimensional spacetime should be replaced by the bulk.

However in the brane universe picture, gauge conditions on fields and propagations of them investigated in this work would be for fields in the bulk. The compactification would be earlier than the appearance of branes in the sense that we do not assume that the brane exists also in internal space.

**Appendix**

**Relations between geometric quantities defined in $D$-dimensional and 4-dimensional spacetimes**
We collect here formulas relating between geometric quantities for the background spacetime defined in $D$-dimensional space and 4-dimensional spacetime.

(i) **metric**

It is evident that (3.1) means the following relation:

\[
\hat{g}_{(0)\mu\nu} = g_{(0)\mu\nu} \quad \text{and} \quad \hat{g}_{(0)ab} = a_l^2 g_{(0)ab}.
\]  

(A.1)

For $(M, N) = (\mu, \nu)$, components of the two metrics have the same values.

(ii) **Christoffel symbol**

Nonvanishing components of the Christoffel symbols satisfy the following relations:

\[
\begin{align*}
\hat{\Gamma}^\lambda_{(0)\mu\nu} &= \Gamma^\lambda_{(0)\mu\nu}, \\
\hat{\Gamma}^a_{(0)ab} &= \delta_a^b \partial_\mu \phi, \quad \hat{\Gamma}^\mu_{(0)ab} = -a_l^2 \partial^\mu \phi g_{(0)ab}, \\
\hat{\Gamma}^c_{(0)ab} &= \Gamma^c_{(0)ab} = \delta_a^c \cot \theta b(a - 1 - b) + \delta_b^c \cot \theta a(b - 1 - a) \\
&\quad - \delta_{ab} \sin \theta c \cos \theta c \prod_{d=c+1}^{b-1} \sin^2 \theta d \theta (b - 1 - c)
\end{align*}
\]

(A.2)

where $\Gamma^\lambda_{(0)\mu\nu}$ and $\Gamma^c_{(0)ab}$ are components of the 4-dimensional and extra-dimensional Christoffel symbols, respectively and $\phi$ and $g_{(0)ab}$ are defined in (3.2).

(iii) **Riemann tensor**

Nonvanishing components of the Riemann tensor satisfy the following relations:

\[
\begin{align*}
\hat{R}^\lambda_{(0)\mu\nu\rho} &= R^\lambda_{(0)\mu\nu\rho}, \\
\hat{R}^a_{(0)\mu\nu} &= -\delta^a_b \left( \nabla_\mu \nabla_\nu \phi + \partial_\mu \phi \partial_\nu \phi \right), \\
\hat{R}^a_{(0)bcd} &= R^a_{(0)bcd} - a_l^2 \partial^\lambda \phi \partial^\lambda \phi \left( \delta^a_c g_{(0)b} - \delta^a_d g_{(0)bc} \right)
\end{align*}
\]

(A.3)

The components of the Riemann tensors $R^\lambda_{(0)\mu\nu\rho}$ and $R^a_{(0)bcd}$ are those of the 4-dimensional and extra-dimensional space respectively.

(iv) **Ricci tensor**

Nonvanishing components of Ricci tensor satisfy the following relations:

\[
\hat{R}_{(0)\mu\nu} = R_{(0)\mu\nu} - n \left( \nabla_\mu \nabla_\nu \phi + \partial_\mu \phi \partial_\nu \phi \right), \quad \hat{R}_{(0)ab} = R_{(0)ab} - a_l^2 g_{(0)ab}(\Box \phi + n \partial_\lambda \phi \partial^\lambda \phi)
\]

(A.4)

(v) **Scalar curvature**

Relations between the scalar curvatures in two spaces are given as

\[
\hat{R}_{(0)} = R_{(0)} + R_T a_l^{-2} - n \left[ 2 \Box \phi + (n + 1) \partial^\rho \phi \partial_\rho \phi \right]
\]

(A.5)

where $R_{(0)}$ is the scalar curvature of the 4-dimensional background and $R_T$ is the scalar curvature of the extra-dimensional space formed from $g_{(0)ab}$.

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