Generating $G_2$-cosmologies with perfect fluid in dilaton gravity

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Abstract

We present a method for generating exact diagonal $G_2$-cosmological solutions in dilaton gravity coupled to a radiation perfect fluid and with a cosmological potential of a special type. The method is based on the symmetry group of the system of $G_2$-field equations. Several new classes of explicit exact inhomogeneous perfect fluid scalar-tensor cosmologies are presented.

1 Introduction

The generalized scalar-tensor theories of gravity are considered as the most natural generalization of general relativity. Their importance for the current physics is related to the string theory which, in its low energy limit, predicts the existence of a scalar partner of the tensor graviton. A large amount of research has been devoted to the dilaton cosmology [1]–[19] (and references therein).

The interest in studying of inhomogeneous (and anisotropic) cosmological models is motivated by the following reasons. As well known, the present universe is not exactly spacially homogeneous, not even at the large scales. Although the homogeneous models are good approximations of the present universe, there is no reason to assume that such a regular expansion is suitable for description of the early universe. Theoretical explanation of the formation of large scale structures in the universe also necessitates inhomogeneous models. Contrary to the general belief it was shown that the existence of big inhomogeneities in the universe does not necessarily lead to an observable effects left over the spectrum of CMB [20]–[23]. It was also demonstrated the existence of homogeneous but highly anisotropic cosmological models whose CMB is exactly isotropic [24]. In addition, the inhomogeneous cosmological solutions allow us to investigate a number of long standing questions regarding the occurrence of singularities, the behaviour of spacetime in vicinity of a singularity and the possibility of our universe arising from generic initial data.

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In the light of the above reasons the study of inhomogeneous cosmological models is necessary and even imperative. The ideal case is to find general classes of inhomogeneous cosmological solutions of the field equation without any symmetry. However, this seems to be a hopeless task due to the complexity of the field equations. That is why we are forced to assume some simplifications in order to solve the field equations. Usually inhomogeneous models with two spacelike commuting Killing vectors (the so-called $G_2$-cosmologies) are considered. Even for these simple cosmological models few exact perfect fluid solutions are known in general relativity. The first such class of exact solutions was found by Wainwright and Goode [25]. Other classes were later given in [26]–[29]. All solutions were obtained by assuming the separation of variables of the metric components.

With regard to the scalar-tensor theories, there are no known exact inhomogeneous perfect fluid $G_2$-cosmological solutions. The reason is that the scalar-tensor equations are more complex than the Einstein ones and include arbitrary functions of the dilaton field. That is why finding of exact perfect fluid solutions which hold for all scalar-tensor theories is unrealistic in general case. However, for some special equations of state it is possible to find exact solutions which hold for all scalar-tensor theories. In [30], methods for generating scalar-tensor stiff perfect fluid cosmologies were developed and some explicit solutions were presented in [30], [31] and [32].

The other equation of state which is realistic and allows us to solve the field equations for all scalar-tensor theories (with a special form of the dilaton potential) is $\rho = 3p$. It is the purpose of this paper to present a method for generating inhomogeneous perfect fluid diagonal $G_2$-cosmologies with equation of state $\rho = 3p$ in scalar-tensor theories. As an illustration and important consequence of the method, new classes of exact inhomogeneous perfect fluid $G_2$-cosmological solutions are also presented for all scalar-tensor theories.

2 Solution generating

The general form of the extended gravitational action in scalar-tensor theories is

$$S = \frac{1}{16\pi G_*} \int d^4x \sqrt{-\bar{g}} \left( F(\Phi)\bar{R} - Z(\Phi)\bar{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - 2U(\Phi) \right) + S_m[\Psi_m; \bar{g}_{\mu\nu}].$$  

(1)

Here, $G_*$ is the bare gravitational constant, $\bar{R}$ is the Ricci scalar curvature with respect to the space-time metric $\bar{g}_{\mu\nu}$. The dynamics of the scalar field $\Phi$ depends on the functions $F(\Phi)$, $Z(\Phi)$ and $U(\Phi)$. In order for the gravitons to carry positive energy the function $F(\Phi)$ must be positive. The nonnegativity of the energy of the dilaton field requires that $2F(\Phi)Z(\Phi) + 3[dF(\Phi)/d\Phi]^2 \geq 0$. The action of matter depends on the material fields $\Psi_m$ and the space-time metric $\bar{g}_{\mu\nu}$. It should be noted that the stringy generated scalar-tensor theories, in general, admit a direct interaction between the matter fields and the dilaton in the Jordan (string) frame [3]. Here we consider the phenomenological case when the matter action does not involve the dilaton field.
in order for the weak equivalence principle to be satisfied. However, the method we present here holds for the general case since we consider a radiation fluid with a traceless energy-momentum tensor.

It is much more convenient from a mathematical point of view to analyze the scalar-tensor theories with respect to the conformally related Einstein frame given by the metric:

\[ g_{\mu\nu} = F(\Phi) \tilde{g}_{\mu\nu} . \] (2)

Further, let us introduce the scalar field \( \varphi \) (the so called dilaton) via the equation

\[ \left( \frac{d\varphi}{d\Phi} \right)^2 = \frac{3}{4} \left( \frac{d \ln(F(\Phi))}{d\Phi} \right)^2 + \frac{Z(\Phi)}{2F(\Phi)} \] (3)

and define

\[ A(\varphi) = F^{-1/2}(\Phi) \quad 2V(\varphi) = U(\Phi)F^{-2}(\Phi). \] (4)

In the Einstein frame action \( S \) takes the form

\[ S = \frac{1}{16\pi G_*} \int d^4x \sqrt{-g} \left( R - 2g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - 4V(\varphi) \right) + S_m[\Psi_m; A^2(\varphi)g_{\mu\nu}] \] (5)

where \( R \) is the Ricci scalar curvature with respect to the Einstein metric \( g_{\mu\nu} \). The Einstein frame field equations then are

\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_* T_{\mu\nu} + 2\partial_\mu \varphi \partial_\nu \varphi - g_{\mu\nu}g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi - 2V(\varphi)g_{\mu\nu} , \]

\[ \nabla^\rho \nabla_\rho \varphi = -4\pi G_* \alpha(\varphi) T + \frac{dV(\varphi)}{d\varphi} , \]

\[ \nabla_\mu T^\mu_\nu = \alpha(\varphi) T \partial_\nu \varphi . \]

Here \( \alpha(\varphi) = d\ln(A(\varphi))/d\varphi \) and the Einstein frame energy-momentum tensor \( T_{\mu\nu} \) is related to the Jordan frame one \( \tilde{T}_{\mu\nu} \) via \( T_{\mu\nu} = A^2(\varphi)\tilde{T}_{\mu\nu} \). In the case of a perfect fluid one has

\[ \rho = A^4(\varphi)\tilde{\rho} , \]

\[ p = A^4(\varphi)\tilde{p} , \]

\[ u_\mu = A^{-1}(\varphi)\tilde{u}_\mu . \] (7)
In the present paper we consider space-times admitting two hypersurface and mutually orthogonal Killing vectors $K_1 = \frac{\partial}{\partial y}$ and $K_2 = \frac{\partial}{\partial z}$. We also require the dilaton field to satisfy

$$L_{K_1} \phi = L_{K_2} \phi = 0$$

where $L_K$ is the Lie derivative along the Killing vector $K$.

The metric can be presented in the Einstein-Rosen form

$$ds^2 = D(t, x)[-dt^2 + dx^2] + B(t, x)[C(t, x)dy^2 + C^{-1}(t, x)dz^2]$$

and the fluid velocity is given by

$$u = D^{-1/2} \frac{\partial}{\partial t}.$$

In what follows we will consider a scalar potential of the form $V(\phi) = \Lambda = \text{const}$ (i.e. $U(\Phi) = 2\Lambda F^2(\Phi)$).

Under all these assumptions we obtain the following system of partial differential equations:

$$-\partial_t^2 \ln D + \partial_x^2 \ln D + \partial_t \ln D \partial_t \ln B - \partial_t^2 \ln B$$

$$-\frac{\partial_t^2 B}{B} + \partial_x \ln D \partial_x \ln B - (\partial_t \ln C)^2$$

$$= 8\pi G_*(\rho + 3p)D + 4(\partial_t \phi)^2 - 4\Lambda D$$

$$\partial_t^2 \ln D - \partial_x^2 \ln D + \partial_t \ln D \partial_t \ln B$$

$$+ \partial_x \ln D \partial_x \ln B - \partial_x^2 \ln B - \frac{\partial_x^2 B}{B} - (\partial_x \ln C)^2$$

$$= 8\pi G_*(\rho - p)D + 4(\partial_x \phi)^2 + 4\Lambda D$$

$$\partial_t \ln B \partial_x \ln D + \partial_t \ln D \partial_x \ln B + \partial_t \ln B \partial_x \ln B$$

$$-2 \frac{\partial_t \partial_x B}{B} - \partial_t \ln C \partial_x \ln C = 4\partial_t \phi \partial_x \phi$$

$$\frac{\partial_t^2 B}{B} - \frac{\partial_x^2 B}{B} = 8\pi G_*(\rho - p)D + 4\Lambda D$$

$$\frac{1}{B} \partial_t (B \partial_t \ln C) - \frac{1}{B} \partial_x (B \partial_x \ln C) = 0$$
\[ \frac{1}{B} \partial_t (B \partial_t \varphi) - \frac{1}{B} \partial_x (B \partial_x \varphi) = 0 \quad (16) \]

The above system of partial differential equations (11)–(16) is invariant under the group of symmetries \( \text{Iso}(\mathbb{R}^2) \). Let us introduce

\[ X = \left( \ln C \over 2\varphi \right) \in \mathbb{R}^2. \quad (17) \]

The explicit action of the group is given as follows:

\[ X \rightarrow M X + \xi \quad (18) \]

where \( M \in O(2) \) and \( \xi \in \mathbb{R}^2 \).

The group of symmetries can be used to generate new solutions from known ones, especially to generate solutions with nontrivial dilaton field from pure general relativistic \( G_2 \)-cosmologies.

The subgroup of translations corresponds to a constant shift of the dilaton field \( (\varphi \rightarrow \varphi + \text{const}) \) and to a constant rescaling of the metric function \( C \ (C \rightarrow \text{const} \times C) \).

That is why, without loss of generality we shall restrict ourselves to the subgroup \( SO(2) \in \text{Iso}(\mathbb{R}^2) \) consisting of the matrixes:

\[ M = \left( \begin{array}{cc} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{array} \right). \quad (19) \]

The remaining discrete \( Z_2 \) subgroup corresponds to the transformations \( C \rightarrow C^{-1} \) or \( \varphi \rightarrow -\varphi \).

Let us consider an arbitrary solution of the \( G_2 \)-Einstein equations with a radiation perfect fluid and cosmological term:

\[ ds^2_E = D_E(t, x)[-dt^2 + dx^2] + B_E(t, x)[C_E(t, x)dy^2 + C^{-1}_E(t, x)dz^2] \quad (20) \]

\[ \rho_E = \rho_E(t, x) \quad (21) \]

\[ u^\mu_E = u^\mu_E(t, x). \quad (22) \]

The \( SO(2) \)-transformation then generates a new Einstein frame scalar-tensor solution as follows:

\[ ds^2 = D_E(t, x)[-dt^2 + dx^2] + B_E(t, x)[C(t, x)dy^2 + C^{-1}(t, x)dz^2], \quad (23) \]

\[ \rho = \rho_E, \quad (24) \]

\[ u^\mu = u^\mu_E, \quad (25) \]

\[ \varphi = -\frac{1}{2} \sin(\theta) \ln C_E, \quad (26) \]
where

$$\ln C = \cos(\theta) \ln C_E. \quad (27)$$

The $Z_2$-transformations can be used to restrict $^{1}$ the range of the parameter $\theta$ to $0 \leq \theta \leq \pi$. Let us note that for the particular value $\theta = \pi/2$ we obtain plane symmetric solutions.

The Jordan frame solutions are given by

$$F[\Phi(t, x)] = A^2[-\sin(\theta) \ln C_E(t, x)], \quad (28)$$
$$d\tilde{s}^2 = F^{-1}(\Phi) ds^2, \quad (29)$$
$$\tilde{\rho} = F^2(\Phi) \rho_E, \quad (30)$$
$$\tilde{u}^\mu = F^{-1/2}(\Phi) u_E^\mu. \quad (31)$$

In the above considerations the metric of the $(t, x)$-space was taken to be in an isotropic form. It should be noted and it is easy to see that the solution generating method is applicable for an arbitrary form of the $(t, x)$-space metric.

3 Examples of explicit exact inhomogeneous cosmological solutions

As an illustration of the solution generating method we consider some classes of explicit exact inhomogeneous scalar-tensor cosmologies with $\Lambda = 0$.

3.1 Class 1

Let us consider Senovilla’s solution [27] (see also [26]):

$$ds_E^2 = T^4(t) \cosh^2(3ax)[-dt^2 + dx^2]$$
$$+ B_E(t, x)[T^3(t) \sinh(3ax)dy^2]$$
$$+ T^{-3}(t) \sinh^{-1}(3ax)dz^2], \quad (32)$$
$$8\pi G_\star \rho_E = 15a^2 T^{-4}(t) \cosh^{-4}(3ax), \quad (33)$$
$$u_E = T^{-2}(t) \cosh^{-1}(3ax) \frac{\partial}{\partial t} \quad (34)$$

where

$$T(t) = \lambda_1 \cosh(at) + \lambda_2 \sinh(at), \quad (35)$$
$$B_E(t, x) = T(t) \sinh(3ax) \cosh^{-2/3}(3ax), \quad (36)$$

\(^{1}\text{When the coordinates } y \text{ and } z \text{ have the same topology we can restrict the range of } \theta \text{ to } 0 \leq \theta \leq \pi/2 \text{ since the metric is invariant under the simultaneous transformations } C(t, x) \rightarrow C^{-1}(t, x) \text{ and } y \rightarrow z.\)
and $a > 0$, $\lambda_1$ and $\lambda_2$ are arbitrary constants. 

The solution generating method gives the following scalar-tensor solution:

$$
\begin{align*}
\frac{ds^2}{d^2} &= T^4(t) \cosh^2(3ax)[dt^2 + dx^2] \\
&+ B_E(t, x)[T^3 \sinh^{\cos(\theta)}(3ax)dy^2 \\
&+ T^{-3 \sinh(\theta)}(3ax)dz^2], \\
\varphi &= -\frac{1}{2} \sin(\theta) \ln[T^3(t) \sinh(3ax)]
\end{align*}
$$

(37)

3.2 Class 2

Wainwright and Goode's solution [25] is given by:

$$
\begin{align*}
\frac{ds^2}{d^2} &= \sinh^4(2qt) \cosh^2(3qx)[dt^2 + dx^2] \\
&+ B_E(t, x)[\tanh^{3 \cos(\theta)}(qt)dy^2 + \tanh^{-3}(qt)dz^2], \\
8\pi G_s \rho_E &= 15q^2 \sinh^{-4}(2qt) \cosh^{-4}(3qt), \\
u_E &= \sinh^{-2}(2qt) \cosh^{-1}(3qx) \frac{\partial}{\partial t}.
\end{align*}
$$

(39)

(40)

(41)

where

$$
B_E(t, x) = \sinh(2qt) \cosh^{-2/3}(3qx)
$$

(42)

and $q > 0$ is an arbitrary constant.

The corresponding scalar-tensor image of that solution is the following:

$$
\begin{align*}
\frac{ds^2}{d^2} &= \sinh^4(2qt) \cosh^2(3qx)[dt^2 + dx^2] \\
&+ B_E(t, x)[\tanh^{3 \cos(\theta)}(qt)dy^2 + \tanh^{-3 \cos(\theta)}(qt)dz^2], \\
\varphi(t, x) &= -\frac{3}{2} \sin(\theta) \ln[\tanh(qt)].
\end{align*}
$$

(43)

(44)

3.3 Class 3

The solution found by Davidson is the following [29]:

$$
\begin{align*}
\frac{ds^2}{d^2} &= -(1 + x^2)^{6/5} dt^2 + t^{4/3}(1 + x^2)^{2/5} dx^2 \\
&+ B_E(t, x)[(tx)dy^2 + (tx)^{-1} dz^2], \\
8\pi G_s \rho_E &= \frac{12}{5} t^{-4/3}(1 + x^2)^{-12/5}, \\
u_E &= (1 + x^2)^{-3/5} \frac{\partial}{\partial t}
\end{align*}
$$

(45)

(46)

(47)

where
\[ B_E(t, x) = t^{1/3}(1 + x^2)^{-2/5}. \]  

Its scalar-tensor image is given by:

\[
\begin{align*} 
 ds^2 &= -(1 + x^2)^{6/5} dt^2 + t^{4/3}(1 + x^2)^{2/5} dx^2 \\
 &+ B_E(t, x) [(tx)^{\cos(\theta)} dy^2 + (tx)^{-\cos(\theta)} dz^2], \\
 \varphi(t, x) &= -\frac{1}{2} \sin(\theta) \ln(tx). 
\end{align*}
\]  

3.4 Class 4

Here as a seed solution we take Collins’s solution of Bianchi type $VI_h$ [33]:

\[
\begin{align*} 
 ds_E^2 &= -d\tau^2 + \tau^2 dx^2 \\
 &+ B_E(\tau, x) \left[ \tau^{\sqrt{3}b/2} e^{\sqrt{3}\tau x/2} dy^2 + \tau^{-\sqrt{3}b/2} e^{-\sqrt{3}\tau x/2} dz^2 \right], \\
 8\pi G_\ast \rho_E &= \frac{3}{8} \frac{1 - b^2}{\tau^2}, \\
 u_E &= \frac{\partial}{\partial \tau} 
\end{align*}
\]

where $B_E(\tau, x)$ is given by

\[ B_E(\tau, x) = \tau^{1/2} e^{bx/2} \]

and $0 < b < 1$.

The corresponding Einstein frame scalar-tensor solution is the following:

\[
\begin{align*} 
 ds^2 &= -d\tau^2 + \tau^2 dx^2 \\
 &+ B_E(\tau, x) \left[ \tau^{\sqrt{3}b\cos(\theta)/2} e^{\sqrt{3}\cos(\theta)x/2} dy^2 \\
 &+ \tau^{-\sqrt{3}b\cos(\theta)/2} e^{-\sqrt{3}\cos(\theta)x/2} dz^2 \right], \\
 \varphi(\tau, x) &= -\frac{\sqrt{3}}{4} \sin(\theta) (b \ln \tau + x). 
\end{align*}
\]

The Einstein frame metric is homogeneous while the dilaton field is not constant over the surface of homogeneity. So we have "tilted" cosmological solution in the Einstein frame. The Jordan frame solution, however, is inhomogeneous.

We could generate many more examples of exact solutions which are images of the known $G_2$-Einstein cosmologies (see for example the solutions given in [28], [34] and [35]). However, the explicit solutions given here are representable and qualitatively cover the general case.

It should be noted that the properties of the found solutions in the physical Jordan frame depend strongly on the particular scalar-tensor theory and need a separate investigation.
4 Conclusion

In this paper we have presented a simple and effective method for generating exact $G_2$-cosmologies in scalar-tensor theories with a potential of a special form and coupled to perfect fluid with an equation of state $\rho = 3p$. Several classes of explicit exact solutions have been given. These solutions are the only known explicit perfect fluid scalar-tensor $G_2$-cosmologies.

It is worth noting that the solutions can be found by assuming the separation of variables of the metric components [36]. However, the way we derived the solutions here is much more elegant and is applicable to more general case when the metric components are not separable.

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