LIOUVILLE THEOREMS AND CLASSIFICATION RESULTS FOR A NONLOCAL SCHRÖDINGER EQUATION

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Abstract. In this paper, we study the existence and the nonexistence of positive classical solutions of the static Hartree-Poisson equation

\[-\Delta u = pu^{p-1}(|x|^{2-n} + u^p), \quad u > 0 \quad \text{in} \quad \mathbb{R}^n,\]

where \(n \geq 3\) and \(p \geq 1\). The exponents of the Serrin type, the Sobolev type and the Joseph-Lundgren type play the critical roles as in the study of the Lane-Emden equation. First, we prove that the equation has no positive solution when \(1 \leq p < \frac{n+4}{n-2}\) by means of the method of moving planes to the following system

\[
\begin{align*}
-\Delta u &= \sqrt{p}u^{p-1}v, \quad u > 0 \quad \text{in} \quad \mathbb{R}^n, \\
-\Delta v &= \sqrt{p}u^p, \quad v > 0 \quad \text{in} \quad \mathbb{R}^n.
\end{align*}
\]

When \(p = \frac{n+4}{n-2}\), all the positive solutions can be classified as

\[u(x) = c(t^{2} + |x - x^*|^2)^{\frac{n-2}{2}}\]

with the help of an integral system involving the Newton potential, where \(c, t\) are positive constants, and \(x^* \in \mathbb{R}^n\). In addition, we also give other equivalent conditions to classify those positive solutions. When \(p > \frac{n+4}{n-2}\), by the shooting method and the Pohozaev identity, we find radial solutions for the system. In particular, the equation has a radial solution decaying with slow rate \(2^{1-p}\).

Finally, we point out that the equation has positive stable solutions if and only if \(p \geq 1 + \frac{4}{n-4-2\sqrt{n-1}}\).

1. Introduction. This paper is concerned with the static Hartree-Poisson equation

\[-\Delta u = pu^{p-1}(|x|^{2-n} + u^p), \quad u > 0 \quad \text{in} \quad \mathbb{R}^n, \quad (1.1)\]

where \(n \geq 3\), and \(p \geq 1\).

Such an equation arises in the Hartree-Fock theory of the nonlinear Schrödinger equations (cf. [33]). Equation (1.1) is also helpful in understanding the blow-up or the global existence and scattering of the solutions of the dynamic Hartree equation in the focusing case (cf. [26]). A more general form is the Choquard type equation studied in many papers (such as [14], [22], [31], and [38]), which arises in the study of boson stars and other physical phenomena, and also appears as a continuous-limit
model for mesoscopic molecular structures in chemistry. More related mathematical and physical background can be found in [15], [34], [40] and the references therein. The nonlocal term in (1.1) appears in the example 3.2.8 of the book [4], and it is also related to a simplified model of the Schrödinger-Poisson system (cf. [1], [18], [19] and many others).

The existence of the super-solutions of (1.1) was studied in [39] and several sufficient conditions were listed. However, it seems difficult to investigate directly the existence of positive solutions in view of the convolution term. Write

\[ v(x) = \sqrt{p} \int_{\mathbb{R}^n} \frac{u^p(y)dy}{|x-y|^{n-2}}. \]

Then \( v > 0 \) in \( \mathbb{R}^n \). Noting the relation between the Newton potential and the convolution properties of Dirac function, we see that there holds formally

\[ -\Delta v = \sqrt{p}\delta_x * u^p = \sqrt{p}u^p(x), \quad (1.2) \]

where \( \delta_x \) is the Dirac mass at \( x \).

Thus, the positive solutions of (1.1) must satisfy the following system

\[ \left\{ \begin{array}{ll}
-\Delta u = \sqrt{p}u^{p-1}v, & u > 0 \quad \text{in} \quad \mathbb{R}^n, \\
-\Delta v = \sqrt{p}u^p, & v > 0 \quad \text{in} \quad \mathbb{R}^n.
\end{array} \right. \quad (1.3) \]

Quittner and Souplet [41] studied positive solutions of more general PDE system

\[ \left\{ \begin{array}{ll}
-\Delta u = v^pu^r, & u > 0 \quad \text{in} \quad \mathbb{R}^n, \\
-\Delta v = v^su^q, & v > 0 \quad \text{in} \quad \mathbb{R}^n.
\end{array} \right. \quad (1.4) \]

They proved the following results:

(Rt1) If \( n \geq 3, \ p - s = q - r \geq 0 \) and \( 0 \leq r, s \leq \frac{n}{n-2} \), then positive solutions \( u, v \) of (1.4) satisfy \( u \equiv v \). It is called the symmetry of components in [41].

(Rt2) If \( n \geq 3, \ p - s = q - r \geq 0 \), then nonnegative solutions \( u, v \) satisfy \( u \geq v \) or \( v \geq u \).

In addition, Ma and Liu proved the radial symmetry for the decay solutions of (1.4) by the method of moving planes (cf. [37]).

According to (Rt1), one gets from (1.3) that \( u \equiv v \) when \( 1 \leq p \leq \frac{2n-2}{n-2} \). Now, (1.3) is reduced to the Lane-Emden type equation

\[ -\Delta u = \sqrt{p}u^p, \quad u > 0 \quad \text{in} \quad \mathbb{R}^n. \quad (1.5) \]

The classification of the solutions of this Lane-Emden equation (1.5) has provided an important ingredient in the study of conformal geometry, such as the prescribing scalar curvature problem and the extremal functions of the Sobolev inequalities. It was studied rather extensively. Recall existence results of this single equation. By the Liouville type result in [13], (1.5) has no positive classical solution when \( 1 \leq p < \frac{n+2}{n-2} \). When \( p = \frac{n+2}{n-2} \), Gidas, Ni, and Nirenberg [12] pointed out that all the classical solutions of (1.5) with a reasonable behavior at infinity must be of the form

\[ u(x) = c\left(\frac{t}{t^2 + |x-x^*|^2}\right)^{\frac{n+2}{2}}, \quad (1.6) \]

where the constants \( c, t > 0 \), and \( x^* \in \mathbb{R}^n \). Later, Caffarelli, Gidas, and Spruck [2] removed the decay restriction and obtained the same result. Then Chen and Li simplified their proof (cf. [5] [24]). The method of moving planes comes into play in those work. In addition, the method of moving sphere introduced by Li and Zhu [29] is also an important approach for researching (1.5) (see also [28]).
In this paper, we are concerned with the Liouville type theorems and the classification results on positive solutions of (1.1) via studying (1.3). Our motivation is to answer whether those existence/nonexistence results are analogous to the corresponding conclusions of the Lane-Emden equation. The main challenge is how to handle the incompletely coupling terms of (1.3) (comes from the nonlocal term of (1.1)). The following integral system plays an important role in studying (1.3)

\[
\begin{align*}
  u(x) &= c_1 \int_{\mathbb{R}^n} \frac{u^{p-1}(y)v(y)dy}{|x-y|^{n-2}}, \quad u > 0 \text{ in } \mathbb{R}^n, \\
  v(x) &= c_2 \int_{\mathbb{R}^n} \frac{u^{p}(y)dy}{|x-y|^{n-2}}, \quad v > 0 \text{ in } \mathbb{R}^n,
\end{align*}
\]

with some constants \(c_1, c_2 > 0\) (cf. [7] and [23]).

Remark 1.1. The integral system (1.7) is much close to the classical integral equation of the Hardy-Littlewood-Sobolev type (cf. [7], [27] and [32])

\[ u(x) = \int_{\mathbb{R}^n} \frac{u^{p}(y)dy}{|x-y|^{n-2}}, \quad u > 0 \text{ in } \mathbb{R}^n, \]

and hence its properties are predictable. In addition, the referee pointed out that a more general integral system corresponding to the PDE system (1.4) is also an interesting model. However, differential from the result in [41], the symmetry of components of the integral system seems difficult to be obtained, and hence it cannot be reduced to a single equation. Although this integral system is more complicated, the properties are obviously more abundant and we would study it in the future.

In general, \(p\) is called a subcritical exponent, critical exponent, and supercritical exponent, if \(p < 2^* - 1, p = 2^* - 1, \) and \(p > 2^* - 1\) respectively, where \(2^* = \frac{2n}{n-2}\). We state the main results in three cases.

1.1. Subcritical case \(p \in (1, 2^* - 1)\). When \(n \geq 4\), (1.3) is reduced to (1.5) by (Rt1). So the Liouville theorem of the Lane-Emden equation is already known in subcritical case (cf. [13]). Consider the case of \(n = 3\). In view of the convolution term, the standard profile in [13] does not work in studying the nonexistence result. In Section 2, we apply the method of moving planes employed by Chen-Li [5], Ma-Chen [36] and Ma-Liu [37] to study (1.3) and prove the following theorem.

Theorem 1.1. Let \(n \geq 3\). If \(1 \leq p < 2^* - 1\), then (1.1) has no positive classical solution.

The nonexistence for (1.7) is also considered in Section 2. It indicates that the Liouville type result may be true for other weaker positive solutions of (1.1) (e.g. for some integrable solutions).

Theorem 1.2. Let \(n \geq 3\).

(1) If \(1 < p \leq \frac{2n}{n-2}\), then (1.7) has no positive solution.

(2) If \(1 < p < 2^* - 1\), then (1.7) has no positive solution in \(L^{\frac{n(2^* - 1)}{2}}(\mathbb{R}^n)\).

Remark 1.2. The Serrin exponent \(\frac{n}{n-2}\) is a well known exponent because not only it appears in the trace embedding inequality (cf. [11]), but also it is critical for the existence of super-solutions of (1.5). In addition, the Serrin exponent is also critical for the existence of positive solutions of the Lane-Emden equation with
double bounded coefficient

\[-\Delta w(x) = K(x)w^p(x), \quad x \in \mathbb{R}^n,\]

where \( C^{-1} \leq K(x) \leq C \) with \( C > 1 \). Let \( w = u + v \), then (1.3) implies that \( w \) satisfies the equation above. However \( K(x) \) only satisfies a weaker condition \( 0 < K(x) \leq C \). Now, both the nonexistence in the case of \( 1 \leq p \leq \frac{n}{n-2} \) and the existence in the case of \( p > \frac{n}{n-2} \) are unclear. So the Serrin exponent seems difficult to be used to study (1.3) via investigating the Lane-Emden equation and other semilinear equations. We will introduce a comparison result (cf. Lemma 2.4) implying nonexistence for (1.3) with \( p \leq \frac{n}{n-2} \). For (1.1), this result has been showed by Theorem 1 of [39].

**Remark 1.3.** For the integral system (1.7), the \( L^{\frac{n}{p-1}}(\mathbb{R}^n) \)-solutions are important. In fact, Li and Ma [25] proved that the positive solutions in \( L^{\frac{n}{p-1}}(\mathbb{R}^n) \) are radially symmetric and decreasing. Jin and Li [20] applied a regularity lifting lemma by the contraction maps to obtain the optimal integrability of positive integrable solutions. Based on this result, [42] estimated the fast decay rates when \( |x| \to \infty \). All the results show that those integrable solutions (i.e. \( L^{\frac{n}{p-1}}(\mathbb{R}^n) \)-solutions) of (1.7) have better regular properties (see also [27]). So we keep the focus on this class of solutions in the following argument.

1.2. **Critical case** \( p = 2^{\ast}-1 \). When \( n \geq 4 \), one can classify easily the classical solutions by (Rt1) and the results in [5]. In Section 3, we not only discuss the classification results in the case of \( n = 3 \), but also give other conditions for classifying the positive solutions. Here, the integral system (1.7) comes into play.

**Theorem 1.3.** In critical case \( p = 2^{\ast}-1 \), the classical solutions \( u, v \) of (1.3) must solve (1.7) with some positive constants \( c_1, c_2 \).

**Remark 1.4.** Another integral system related to (1.7) is the following Lane-Emden type equations involving the Riesz potentials

\[
\begin{cases}
u(x) = c_1 \int_{\mathbb{R}^n} \frac{\nu(y)dy}{|x-y|^\lambda}, \quad u > 0 \quad \text{in} \quad \mathbb{R}^n, \\
v(x) = c_2 \int_{\mathbb{R}^n} \frac{\nu(y)dy}{|x-y|^\lambda}, \quad v > 0 \quad \text{in} \quad \mathbb{R}^n.
\end{cases}
\]

(1.9)

It is essential in studying the extremal functions of the Hardy-Littlewood-Sobolev inequality (cf. [32])

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)g(y) |x-y|^\lambda dx dy \leq C(n, s, \lambda) \|f\|_r \|g\|_s
\]

with \( 0 < \lambda < n, \ 1 < s, r < \infty, \ f \in L^r(\mathbb{R}^n) \) and \( g \in L^s(\mathbb{R}^n), \ \frac{1}{r} + \frac{1}{s} + \frac{\lambda}{n} = 2 \).

Define \( Tg(x) = \int_{\mathbb{R}^n} \frac{g(y)dy}{|x-y|^{n-\alpha}} \) with \( \alpha = n - \lambda \). The Hardy-Littlewood-Sobolev inequality becomes

\[
\|Tg\|_p \leq C(n, s, \alpha) \|g\|_s \|g\|_r
\]

(1.10)

This inequality will be used in this paper to research the radial symmetry and the integrability of the solutions of (1.7).
In the critical case, classification results for the single equation of (1.9) can be found in [7] and [27]. We here study the integral system (1.7) and then obtain several necessary and sufficient conditions for the classification results of (1.3).

**Theorem 1.4.** Let \((u, v)\) be a pair of classical solutions of (1.3) with \(p > 1\). Then the following items are equivalent to one another:

1. \(u \in L^\frac{2n}{n+2}(\mathbb{R}^n)\);
2. \(u\) is bounded and decays with the fast rate \(n - 2\);
3. \(u\) belongs to the homogeneous Sobolev space \(D^{1,2}(\mathbb{R}^n)\);
4. \(p = 2^* - 1\);
5. \(u(x) = v(x) \equiv u_\ast\), where \(u_\ast\) is the radial function as the form (1.6).

**Corollary 1.5.** If \(p = 2^* - 1\), then the classical solutions \(u\) of (1.1) must be of the form (1.6).

### 1.3. Supercritical case \(p > 2^* - 1\)

Finally, we consider the supercritical case \(p > 2^* - 1\) in Section 4. The existence and nonexistence are much complicated and not completely understood even for the Lane-Emden equations.

First (1.1) has a singular radial solution in \(\mathbb{R}^n \setminus \{0\}\)

\[
u(x) = U_0(|x|) := c|x|^{\frac{4}{p-2}} \quad (1.11)
\]

with \(c := \left[\frac{2}{\sqrt{p-1}}(n - 2 - \frac{2}{p-1})\right]^{\frac{1}{p-1}}\) (which is a direct corollary of Theorem 4.5), as long as \(p\) is larger than the Serrin exponent \(\frac{n}{n-2}\).

Next, we can find bounded entire solutions.

An example is bounded radial solutions.

**Theorem 1.6.** When \(p > 2^* - 1\), we can find radial solutions of (1.3). In addition, there is a radial solution of (1.1) decaying with the slow rate \(\frac{2}{p-1}\) when \(|x| \to \infty\).

In fact, we can use the shooting method introduced in [35] to find a solution of the following system of ODEs

\[
\begin{aligned}
- (U'' + \frac{n-1}{r} U') &= \sqrt[p]{U^{p-1}}, \\
U'(0) &= V'(0) = 0, \\
U(0) &= 1, \\
v(0) &= a.
\end{aligned}
\]

(1.12)

Interestingly, it seems difficult to show that \(V\) decays to zero when \(r \to \infty\). On the other hand, for the system

\[
\begin{aligned}
- (U'' + \frac{n-1}{r} U') &= \sqrt[p]{U^{p-1}}, \\
V(|x|) &= \sqrt[p]{|x|^{2-n} \ast U^p(|x|)}, \\
U'(0) &= 0, \\
v(0) &= \sigma > 0.
\end{aligned}
\]

(1.13)

we can see that \(V\) goes to zero when \(r \to \infty\) if noting the integral form of \(V\).

**Remark 1.5.** Not all the solutions in the supercritical case are radially symmetric. There is another example to show that some bounded entire solutions are neither radial nor decaying when \(|x| \to \infty\). Introduce a pair of cylinder-shaped solution \((u^*, v^*)\) to (1.3) (cf. [8]). According to Theorem 1.4, \((u_\ast, u_\ast)\) solves (1.3) in the whole space \(\mathbb{R}^n\) in the critical case \(p = \frac{n+2}{n-2}\) where \(u_\ast\) is of the form (1.6). Thus, it is not difficult to see that \(u^*(x, x_{n+1}) = u_\ast(x)\) and \(v^*(x_0, x) = u_\ast(x)\) still solve (1.3) in \(\mathbb{R}^n+1\). In view of \(p > \frac{n+3}{n-1}\), the problem (1.3) (which \(u^*, v^*\) satisfy in \(\mathbb{R}^n+1\)) is equipped with a supercritical exponent. Clearly, this pair of positive solution
(u*, v*) is neither radial nor decaying when |x| \to \infty. We also see u* \neq v* because the generating lines of the cylinders are different.

We also consider the nonexistence of ‘stable’ positive solutions of (1.1). The Joseph-Lundgren type exponent
\[ p_{jl}(n) := 1 + \frac{4}{n - 4 - 2\sqrt{n - 1}} \]
comes into play (cf. [16]), which is also essential to describe how the radial solutions intersect with the singular radial solution (1.11) and with themselves (cf. [21]). In addition, this Joseph-Lundgren exponent can be used to study the Morse index for the sign-changed solutions of the Lane-Emden equation (cf. [10]) and other semilinear elliptic equations with supercritical exponent (cf. [17]).

**Definition 1.7.** A solution \( u \in C^2(R^n) \) of (1.1) is stable, if for any \( \psi \in C_0^1(R^n) \), there holds
\[ \int_{R^n} |\nabla \psi|^2 dx \geq p\sqrt{p} \int_{R^n} u^{p-2}v\psi^2 dx. \] (1.14)
Here \( v = \sqrt{p}|x|^{2-n} * u^p(x) \).

This definition is well-defined. Indeed, (1.14) comes from the fact that the eigenvalue problem
\[ -\Delta \phi = p\sqrt{p}u^{p-2}v\phi + \eta \phi \]
has a first positive eigenvalue \( \eta > 0 \) with corresponding positive eigenfunction \( \phi \).

On the other hand, although (1.14) cannot be deduced directly from the positive definite quadratic form of the functional
\[ E(u) = \frac{1}{2} \int_{R^n} |\nabla u|^2 dx - \frac{1}{2} \int_{R^n} u^p(x)(|x|^{2-n} * u^p)dx, \]
some comparison relation of \( u \) and \( v \) (similar to Lemma 2.4) shows that the right hand side of (1.14) makes sense. More importantly, for such a definition of stable solutions, we can prove that the exponent \( p_{jl}(n) \) is critical on existence and nonexistence.

**Theorem 1.8.** (1) When
\[ \begin{cases} 1 < p < \infty, & \text{if } 3 \leq n \leq 10, \\ 1 < p < p_{jl}(n), & \text{if } n \geq 11, \end{cases} \]
(1.1) has no positive stable solution.

(2) When \( p \geq p_{jl}(n) \), (1.1) has a radial positive stable solution.

2. Subcritical case. In this section, we always assume that \( 1 \leq p < 2^* - 1 \).

2.1. Liouville theorem for integral system.

**Theorem 2.1.** If \( 1 \leq p \leq \frac{n}{n-2} \), then there does not exist any positive solution of (1.7).

**Proof.** When \( p = 1 \), if we write \( w = u + v \), then (1.7) is reduced to
\[ w(x) = \int_{R^n} \frac{u(y)dy}{|x - y|^{n-2}}, \quad w > 0 \text{ in } R^n. \]
According to Theorem 1.4 in [27], we see the conclusion.
Let Theorem 2.2. by Chen-Li-Ou (cf. [7] and [8]), we prove a radial symmetry result.

By (1.7), we have

\[ u(x) \geq c \int_{B_R(0)} \frac{u^{p-1}(y)v(y)dy}{|x-y|^{n-2}} \geq \frac{c}{|x|^{n-2}}, \quad \text{for } |x| > 1. \quad (2.1) \]

In addition, for any \( R > 2 \), from

\[ u(x) \geq c \int_{B_R(0)} \frac{u^{p-1}(y)v(y)dy}{|x-y|^{n-2}} \geq \frac{c}{(R+|x|)^{n-2}} \int_{B_R(0)} u^{p-1}(y)v(y)dy, \]

we deduce that

\[ \int_{B_R(0)} u^p(x)dx \geq c \int_{B_R(0)} \frac{dx}{(R+|x|)^{p(n-2)}} \left( \int_{B_R(0)} u^{p-1}(y)v(y)dy \right)^p \]

\[ \geq cR^{n-p(n-2)} \left( \int_{B_R(0)} u^{p-1}(y)v(y)dy \right)^p, \quad (2.2) \]

where \( c > 0 \) is independent of \( R \).

When \( 1 < p < \frac{n}{n-2} \), we can see a contradiction by letting \( R \to \infty \) in (2.2).

When \( p = \frac{n}{n-2} \), (2.2) implies

\[ \int_{B_R(0)} u^p(x)dx \geq c \left( \int_{B_R(0)} u^{p-1}(y)v(y)dy \right)^p. \quad (2.3) \]

Similarly, from

\[ v(x) \geq \frac{c}{(R+|x|)^{n-2}} \int_{B_R(0)} u^p(y)dy, \quad (2.4) \]

we also deduce

\[ \int_{B_R(0)} u^{p-1}(x)v(x)dx \geq \int_{B_R(0) \setminus B_{R/2}(0)} \frac{cu^{p-1}(x)dx}{(R+|x|)^{n-2}} \int_{B_R(0)} u^p(y)dy. \]

Using (2.1), (2.3) and noting \( p = \frac{n}{n-2} \), we get

\[ \int_{B_R(0)} u^{p-1}(x)v(x)dx \geq c \left( \int_{B_R(0)} u^{p-1}(y)v(y)dy \right)^p, \]

which implies \( u^{p-1}v \in L^1(R^n) \) if we let \( R \to \infty \).

Multiplying (2.4) by \( u^{p-1} \) and integrating on \( A_R := B_R(0) \setminus B_{R/2}(0) \), we still have

\[ \int_{A_R} u^{p-1}(x)v(x)dx \geq c \left( \int_{B_R(0)} u^{p-1}(y)v(y)dy \right)^p. \]

Letting \( R \to \infty \) and noting that \( u^{p-1}v \in L^1(R^n) \), we obtain \( \|u^{p-1}v\|_{L^1(R^n)} = 0 \), which is impossible.

By using the method of moving planes in integral form, which was established by Chen-Li-Ou (cf. [7] and [8]), we prove a radial symmetry result.

**Theorem 2.2.** Let \( p > 1 \). If \( h \geq 0 \) satisfies

\[ \int_{R^n} |y|^{-nh/2}u^{n(p-1)/2}(y)dy < \infty, \quad \int_{R^n} |y|^{-nh/2}v^{n(p-1)/2}(y)dy < \infty. \]

(2.5)
Then the positive continuous solutions of
\[
\begin{cases}
  u(x) = c_1 \int_{\mathbb{R}^n} \frac{u^{p-1}(y)v(y)dy}{|y|^h|x-y|^{n-h-2}}, & u, v > 0 \text{ in } \mathbb{R}^n, \\
  v(x) = c_2 \int_{\mathbb{R}^n} \frac{u^p(y)dy}{|y|^h|x-y|^{n-h-2}}, & c_1, c_2 \text{ are positive constants},
\end{cases}
\]
are radially symmetric and decreasing around \(x^* \in \mathbb{R}^n\). Moreover, \(x^* = 0\) as long as \(h > 0\).

Proof. For some \(\lambda \in \mathbb{R}\), define \(\Sigma_\lambda := \{x = (x_1, \ldots, x_n); x_1 > \lambda\}, x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)\), \(u_\lambda(x) = u(x^\lambda), \Sigma_\lambda^u = \{x \in \Sigma_\lambda|u(x) \leq u_\lambda(x)\}, \Sigma_\lambda^v = \{x \in \Sigma_\lambda|v(x) \leq v_\lambda(x)\}.\) It is not difficult to see that
\[
\begin{align*}
  u_\lambda(x) - u(x) &= c_1 \int_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-2}} - \frac{1}{|x^\lambda-y|^{n-2}}\right) \frac{1}{|y|^h} (v_\lambda u_\lambda^{p-1} - v u^{p-1}) dy \\
  &\leq C \int_{\Sigma_\lambda^u} \frac{1}{|x-y|^{n-2}} \frac{1}{|y|^h} (v_\lambda u_\lambda^{p-1} - v u^{p-1}) dy \\
  &= C \int_{\Sigma_\lambda^u} \frac{1}{|x-y|^{n-2}} \frac{1}{|y|^h} (v_\lambda - v)(y) dy \\
  &\quad + C \int_{\Sigma_\lambda^v} \frac{1}{|x-y|^{n-2}} \frac{1}{|y|^h} v(u_\lambda^{p-1} - u^{p-1}) (y) dy. 
\end{align*}
\]
Using the Hardy-Littlewood-Sobolev inequality \((1.10)\) and the Hölder inequality, we have
\[
\begin{align*}
  &\|u_\lambda - u\|_{L^1(\Sigma_\lambda^u)} \\
  &\leq C \|\|y|^{-h} u_\lambda^{p-1} (v_\lambda - v)\|_{L^{\frac{n}{n-h}}(\Sigma_\lambda^u)} + C \|\|y|^{-h} v u_\lambda^{p-2} (u_\lambda - u)\|_{L^{\frac{n}{n-2}}(\Sigma_\lambda^u)} \\
  &\leq C \|\|y|^{-h} u_\lambda^{p-1}\|_{L^{\frac{n}{n-h}}(\Sigma_\lambda^u)} \|(v_\lambda - v)\|_{L^1(\Sigma_\lambda^u)} \\
  &\quad + C \|\|y|^{-h} u_\lambda^{p-2} v\|_{L^{\frac{n}{n-2}}(\Sigma_\lambda^v)} \|(u_\lambda - u)\|_{L^1(\Sigma_\lambda^u)}. 
\end{align*}
\]
Similarly, we also obtain
\[
\begin{align*}
  &\|v_\lambda - v\|_{L^1(\Sigma_\lambda^v)} \\
  &\leq C \|\|y|^{-h} u_\lambda^{p-1}\|_{L^{\frac{n}{n-h}}(\Sigma_\lambda^v)} \|(u_\lambda - u)\|_{L^1(\Sigma_\lambda^u)} \\
  &\quad + C \|\|y|^{-h} u_\lambda^{p-2} v\|_{L^{\frac{n}{n-2}}(\Sigma_\lambda^v)} \|(u_\lambda - u)\|_{L^1(\Sigma_\lambda^u)}. 
\end{align*}
\]
By \((2.5)\), as \(\lambda \to -\infty\),
\[
C \|\|y|^{-h} u_\lambda^{p-1}\|_{L^{\frac{n}{n-h}}(\Sigma_\lambda^u)} \leq \frac{1}{4}, \quad \|\|y|^{-h} u_\lambda^{p-2} v\|_{L^{\frac{n}{n-2}}(\Sigma_\lambda^v)} \leq \frac{1}{4},
\]
Combining these results, we can see that \(\Sigma_\lambda^u\) and \(\Sigma_\lambda^v\) are empty set as long as \(\lambda\) is near \(-\infty\).

Suppose that at \(\lambda_0 < 0\), we have \(u(x) \geq u_{\lambda_0}(x)\) and \(v(x) \geq v_{\lambda_0}(x)\) but \(u(x) \neq u_{\lambda_0}(x)\) and \(v(x) \neq v_{\lambda_0}(x)\) on \(\Sigma_{\lambda_0}\). By the same argument as above, we can prove that there exists an \(\epsilon > 0\), such that \(u(x) \geq u_\lambda(x)\) and \(v(x) \geq v_\lambda(x)\) on \(\Sigma_\lambda\) for all \(\lambda \in [\lambda_0, \lambda_0 + \epsilon]\). Therefore, we can move the plane \(x_1 = \lambda\) to the right as long as
\[
\begin{align*}
  u(x) &\geq u_\lambda(x) \quad \text{and} \\
  v(x) &\geq v_\lambda(x).
\end{align*}
\]
hold on $\Sigma$. If the plane stops at $x_1 = \lambda_0$ for some $\lambda_0 < 0$, then $u(x)$ and $v(x)$ must be radially symmetric and decreasing about the plane $x_1 = \lambda_0$. Otherwise, we can move the plane all the way to $x_1 = 0$. Since the direction of $x_1$ can be chosen arbitrarily, we obtain that $u(x), v(x)$ are radially symmetric and decreasing about some $x^* \in \mathbb{R}^n$.

If $h \neq 0$, we claim $x^* = 0$. Otherwise, we can find $\lambda_0 < 0$ such that $x_1 = \lambda_0$ is the plane at which we have to stop. From (2.7), we get

$$0 = u_{\lambda_0}(x) - u(x)$$

$$= -c_1 \int_{\Sigma_{\lambda_0}} \left( \frac{1}{|x-y|^{n-2}} \frac{1}{|x_{\lambda_0} - y|^{n-2}} \left( \frac{1}{|y|^{n}} - \frac{1}{|y_{\lambda_0}|^{n}} \right) v u^{p-1} dy \right) \neq 0,$$

which is impossible. \hfill \Box

**Theorem 2.3.** If $1 < p < 2^* - 1$, then there does not exist any positive solution of (1.7) in $L^{(p-1)/2}(\mathbb{R}^n)$.

**Proof.** **Step 1.** Suppose $u, v$ are the $L^{n(p-1)}(\mathbb{R}^n)$-solutions of (1.7). According to Theorem 2.2 with $h = 0$, we see that they are radially symmetric about $x^* \in \mathbb{R}^n$. Since (1.7) is invariant after translation, $x^*$ can be chosen arbitrarily.

**Step 2.** Consider the Kelvin transformation of $u, v$

$$\bar{u}(x) = \frac{1}{|x|^{n-2}} u\left( \frac{x}{|x|^2} \right), \quad \bar{v}(x) = \frac{1}{|x|^{n-2}} v\left( \frac{x}{|x|^2} \right).$$

(2.8)

By (1.7), we see that $\bar{u}, \bar{v}$ solve (2.6) with $h = n + 2 - p(n - 2)$. In view of the fact that $p < 2^* - 1$, it follows $h > 0$. In addition, from $u, v \in L^{n(p-1)}(\mathbb{R}^n)$, we see that (2.5) for $\bar{u}, \bar{v}$ is true. According to Theorem 2.2, $\bar{u}, \bar{v}$ are also radially symmetric but the center point $x^*$ must be the origin. So the translation invariant is absent. By the same argument of Theorem 3 in [8], we can also deduce a contradiction. \hfill \Box

2.2. **Liouville theorem for PDEs.** First, we have a comparison result as (Rt2) and the ideas in [41] are used in the proof.

**Lemma 2.4.** Let $(u, v)$ solve (1.3). Then $u \leq v$ on $\mathbb{R}^n$.

**Proof.** Write the average of $f$ as $f_A(r) := \frac{1}{|\partial B_r|} \int_{\partial B_r} f ds$. We claim

$$\lim_{r \to \infty} \inf \{u^2\}_A(r) = 0. \quad (2.9)$$

Otherwise, there exists $c > 0$ such that $u \geq c$. Thus, $-\Delta v = \sqrt{n} p u^p$ implies

$$-r^{1-n}(p^{n-1} v'_A) \geq c.$$

Integrating from 0 to $R > 0$ yields $v'_A \leq -cR$. Integrating again and letting $R$ sufficiently large, we see that $v'_A$ is negative. It is impossible.

Write $H = (u - v)_+ := \max\{u - v, 0\}$. Then $\Delta H = u^{p-1} H \geq 0$, and

$$\int_{B_R} |\nabla H|^2 dx = \int_{\partial B_R} H n_{\partial B} H ds - \int_{B_R} H \Delta H dx \leq \frac{1}{2} \int_{\partial B_R} \nabla r(H^2) ds \leq CR^{n-1}(H^2)'_A(R).$$

(2.10)

From $0 \leq H \leq u$ and (2.9), we deduce

$$\lim_{r \to \infty} \inf \{H^2\}_A(r) = 0. \quad (2.11)$$
Thus, there exists $r_j \to \infty$ such that $(H^2)'(r_j) \leq 0$. Inserting this into (2.10) with $R = r_j$ implies $H \equiv C$ and hence $H_A \equiv C$. Combining with (2.11) yields $C = 0$. Namely, $H \equiv 0$ and then $u \leq v$ on $\mathbb{R}^n$.

Lemma 2.4 implies that the solution $u$ of (1.3) is also a super-solution of (1.5), i.e. $-\Delta u \geq u^p$ on $\mathbb{R}^n$. Thus, if $p$ is not larger than the Serrin exponent $\frac{n}{n-2}$, (1.3) has no positive classical solution. Moreover, we have the following stronger result.

**Theorem 2.5.** Let $n \geq 3$. If $1 \leq p < \frac{n+2}{n-2}$, then there does not exist any positive classical solution of (1.3).

**Proof.** Here we use the method of moving planes to prove that the positive classical solutions $u, v$ are radially symmetric which implies the nonexistence.

Since $u, v$ are classical solutions, $u(0)$ and $v(0)$ are finite. Therefore, from (2.8) it follows that the Kelvin transformations of $u, v$ satisfy

$$\bar{u}, \bar{v} \simeq |x|^{2-n} \text{ when } |x| \to \infty. \quad (2.12)$$

In addition, $\bar{u}$ and $\bar{v}$ are the positive solutions of

$$\begin{cases} 
-\Delta \bar{u} = \frac{\sqrt{p}}{|x|^{n+2-p(n-2)}} \bar{u}^{p-1}, \\
-\Delta \bar{v} = \frac{\sqrt{p}}{|x|^{n+2-p(n-2)}} \bar{u}^p 
\end{cases}$$

on $\mathbb{R}^n \setminus \{0\}$.

For some $\lambda \in R$, define $\Sigma_\lambda := \{x = (x_1, \ldots, x_n); x_1 > \lambda\} \setminus \{0\}$, $x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$, $\bar{u}_\lambda(x) = \bar{u}(x^\lambda)$ and $\bar{v}_\lambda(x) = \bar{v}(x^\lambda)$. If $\lambda$ is sufficiently negative, we show that on $\Sigma_\lambda$,

$$U_\lambda := \bar{u} - \bar{u}_\lambda > 0, \quad V_\lambda := \bar{v} - \bar{v}_\lambda > 0. \quad (2.13)$$

Once (2.13) is true, we can start to move plane $\{x_1 = \lambda\}$ from $-\infty$ to the origin. Clearly, $|x| \leq |x^\lambda|$ since $\lambda$ is sufficiently negative. Thus,

$$\begin{cases} 
-\Delta U_\lambda = -\Delta \bar{u} - (-\Delta \bar{u}_\lambda) \geq \frac{\sqrt{p}}{|x|^{n+2-p(n-2)}} [(p-1)\psi_1^{p-2}\bar{v}U_\lambda + \bar{u}^{p-1}V_\lambda]; \\
-\Delta V_\lambda = -\Delta \bar{v} - (-\Delta \bar{v}_\lambda) \geq \frac{\sqrt{p}}{|x|^{n+2-p(n-2)}} \psi_2^{p-1}U_\lambda, 
\end{cases} \quad (2.14)$$

where $\psi_i (i = 1, 2)$ are functions between $\bar{u}$ and $\bar{u}_\lambda$.

Write $W_1(x) = |x|^{1/2}U_\lambda(x)$ and $W_2(x) = |x|^{1/2}V_\lambda(x)$.

We claim $U_\lambda > 0$ on $\Sigma_\lambda$ when $\lambda$ is sufficiently negative. Otherwise, by (2.12) we have $\lim_{|x| \to \infty} \inf W_1 = 0$, which together with $W_1|_{\Sigma_\lambda} = 0$ implies that $W_1$ reach its negative minimum at $x_0$. When $\lambda$ is sufficiently negative, $x_0$ is far away from the singularity of $W_1$ by the weak maximum principle. Thus, $\nabla W_1(x_0) = 0$ and

$$0 \geq -\Delta W_1(x_0) \geq -\frac{1}{2} |x_0|^{1/2} \Delta U_\lambda(x_0) - U_\lambda(x_0)(\Delta |x|^{-1/2})_{x_0} |x_0|^{1/2}. $$

Since $\lambda$ is sufficiently negative (which implies that $|x_0|$ sufficiently large), from (2.12) and (2.14) we can find positive constants $c_i (i = 1, 2, 3)$ such that

$$0 \geq -\Delta W_1(x_0) \geq |x_0|^{1/2}[|x_0|^{-4}(c_1 U_\lambda(x_0) + c_2 V_\lambda(x_0))] - c_3 |x_0|^{-2}U_\lambda(x_0)] \quad (2.15)$$

$\geq c_4 |x_0|^{-4}W_2(x_0) - \frac{c_4}{2} |x_0|^{-2}W_1(x_0).$

If $W_2(x_0) \geq 0$, the right hand side is positive if we notice $W_1(x_0) < 0$. This contradiction shows $U_\lambda > 0$.

If $W_2(x_0) < 0$, we can assume that $W_2$ reaches its negative minimum at $y_0$. By the same derivation of (2.15), we also have

$$0 \geq -\Delta W_2(y_0) \geq c_4 |y_0|^{-4}W_1(y_0) - c_5 |y_0|^{-2}W_2(y_0)$$
with positive constants $c_4, c_5$. Combining with (2.15) yields
\[
W_1(x_0) \geq \frac{2c_2}{c_3|x_0|}W_2(x_0) \geq \frac{2c_2}{c_3|x_0|^2}W_2(y_0) \\
\geq \frac{2c_2c_4}{c_3c_5|x_0|^2|y_0|}W_1(y_0) \geq \frac{2c_2c_4}{c_3c_5|x_0|^2|y_0|}W_1(x_0).
\]
In view of $W_1(x_0) < 0$, the result above is impossible for large $|x_0|/|y_0|$. This contradiction shows $U_\lambda > 0$. Similarly, we can also deduce $V_\lambda > 0$, and hence (2.13) is proved.

By an analogous argument above, we can also use the comparing principle to establish that there is $R_0 > 0$ which is independent of $\lambda$ such that if $x_0, y_0$ are the negative minimal value points of $W_1, W_2$ respectively, then $|x_0|, |y_0| \leq R_0$. This conclusion ensures that the plane can move to its right limit $\partial \Sigma_{\lambda_0}$. Here
\[
\lambda_0 = \sup\{\lambda; U_\mu(x) \geq 0, V_\mu(x) \geq 0, \forall x \in \Sigma_\mu, \mu \leq \lambda\}.
\]

The rest proof is standard by means of the method of moving planes (cf. [5] or [6]) and $\bar{u}, \bar{v}$ are radially symmetric and decreasing about the origin. Therefore, $u, v$ are also radially symmetric and decreasing. Since the (1.3) is invariant under translation, the symmetry point can be arbitrary. It also leads to a contradiction and the nonexistence is obtained.

\[\square\]

**Remark 2.1.** It should be pointed out that the elliptic methods in [13] still work if $u, v$ are radially symmetric. In fact, when $n \geq 4$, it follows $p - 1 \leq \frac{n}{n-2}$. Now, we can apply Theorem 1.2 in [41] to get $u \equiv v$ and hence (1.3) becomes (1.5). According to Theorem 1.1 in [13], we see the nonexistence. So we consider the case of $n = 3$ only. When $p = 1$, if we write $w = u + v$, then (1.3) is reduced to (1.5). According to Theorem 1.1 in [13], we also see the nonexistence. When $p \in (1, 2]$, the elliptic method in Section 2 in [13] is still valid. However, when $p \in (2, 5)$, it is difficult to deduce the same conclusion from Lemma 2.4 only, because the partial derivatives of $u, v$ need to be compared. As long as $u, v$ have the radial structure, we can also obtain the estimates by the same argument in [13].

3. Critical case.

**Theorem 3.1.** In critical case $p = \frac{n+2}{n-2}$, the classical solutions $u, v$ of (1.3) solve (1.7) with some positive constants $c_1, c_2$.

**Proof.** Let $u, v$ be positive classical solutions of (1.3). Similar to (1.2), there exists a positive constant $c_2$ such that
\[
\Delta(c_2 \int_{R^n} \frac{u^p(y)dy}{|x-y|^{n+2}}) = -\sqrt{p}u^p(x) = \Delta v(x), \quad \text{in } R^n.
\]
Namely, $c_2 \int_{R^n} \frac{u^p(y)dy}{|x-y|^{n+2}} - v(x)$ is a harmonic function.

In the critical case, (1.3) is conformal invariant. In particular, the Kelvin transformations $\bar{u}, \bar{v}$ of $u, v$ (cf. (2.8)) still solve (1.3) except at the origin. According to the results in [2] or [24], both $\bar{u}(0) := \lim_{|x| \to 0} \bar{u}(x)$ and $\bar{v}(0) := \lim_{|x| \to 0} \bar{v}(x)$ are finite. In view of
\[
\lim_{|x| \to \infty} |x|^{n-2}u(x) = \bar{u}(0), \quad \lim_{|x| \to \infty} |x|^{n-2}v(x) = \bar{v}(0),
\]
the classical solutions $u, v$ are bounded and decay with fast rate $n - 2$. Thus, the harmonic function $c_2 \int_{R^n} \frac{u^p(y)dy}{|x-y|^{n+2}} - v(x)$ has a lower bound. According to the
Liouville theorem on harmonic functions,
\[ c_2 \int_{\mathbb{R}^n} \frac{u^p(y)dy}{|x-y|^{n-2}} - v(x) = \text{Constant} := L. \quad (3.3) \]

We claim that
\[ \lim_{|x| \to \infty} \int_{\mathbb{R}^n} \frac{u^p(y)dy}{|x-y|^{n-2}} = 0. \quad (3.4) \]

In fact, (3.2) implies that there exists a suitably large constant \( R > 0 \) such that
\[ u(y) \leq \frac{C}{|y|^{n-2}}, \text{ when } |y| > R. \]
Thus, for sufficiently large \(|x|\),
\[ \int_{B_{R}(0) \setminus B_{|x|/2}(x)} \frac{u^p(y)dy}{|x-y|^{n-2}} \leq \frac{C}{|x|^{n-2}} \int_{R} \frac{r^{-p(n-2)}dr}{r} \leq \frac{C}{|x|^{n-2}}, \]
and
\[ \int_{B_{|x|/2}(x)} \frac{u^p(y)dy}{|x-y|^{n-2}} \leq \frac{C}{|x|^{n-2}} \int_{B_{|x|/2}(0)} \frac{dy}{|y|^{n-2}} \leq \frac{C}{|x|^n}. \]
In addition, for sufficiently large \(|x|\) we also get easily that
\[ \int_{B_{R}(0)} \frac{u^p(y)dy}{|x-y|^{n-2}} \leq \frac{C}{|x|^{n-2}}. \]
Combining the three estimates above, we obtain (3.4).

Letting \(|x| \to \infty\) in (3.3), and using (3.2) and (3.4), we can see \( L = 0 \). Namely,
\[ u(x) = c_2 \int_{\mathbb{R}^n} \frac{u^p(y)dy}{|x-y|^{n-2}}. \]
Similarly, we can also deduce
\[ u(x) = c_1 \int_{\mathbb{R}^n} \frac{u^{p-1}(y)v(y)dy}{|x-y|^{n-2}}. \]
Theorem 3.1 is complete. \( \Box \)

Theorem 2.1 implies that
\[ p > \frac{n}{n-2} \quad (3.5) \]
is a necessary condition for the existence of positive solutions for (1.7), and hence it still holds for (1.3) by Theorem 3.1. Therefore, we can easily see that
\[ u(x) = v(x) = c(t^2 + |x-x^*|^2)^{\frac{n-2}{2}}, \quad c, t > 0, \quad x^* \in \mathbb{R}^n \]
are positive solutions of (1.3) in \( L^{n(p-1)/2}(\mathbb{R}^n) \). However, it is nontrivial to show that all solutions of (1.3) in \( L^{n(p-1)/2}(\mathbb{R}^n) \) are the above form. In this section, we prove this conclusion.

**Theorem 3.2.** Let \((u, v)\) be a pair of positive solutions of (1.7) with \( p > 1 \). If \( u \in L^{n(p-1)/2}(\mathbb{R}^n) \), then

- \((R1)\) \( u, v \in L^s(\mathbb{R}^n) \) for all \( s > \frac{n}{n-2} \); \( u, v \notin L^s(\mathbb{R}^n) \) for all \( s \leq \frac{n}{n-2} \);
- \((R2)\) \( u, v \) are bounded and
\[ \lim_{|x| \to \infty} u(x) = \lim_{|x| \to \infty} v(x) = 0; \quad (3.6) \]
Proof. (1) By the Hardy-Littlewood-Sobolev inequality (1.10) and $u \in L^{\frac{n(p-1)}{2}}(R^n)$, we can deduce $v \in L^{\frac{n(p-1)}{2}}(R^n)$.

Write $w = u + v$. Then $w \in L^{\frac{n(p-1)}{2}}(R^n)$. From (1.7), it follows that $w$ satisfies

$$w(x) = K(x) \int_{R^n} \frac{w^p(y)dy}{|x-y|^{n-2}},$$  \hspace{1cm}  (3.7)

where $0 < K(x) \leq C$. Set $w_A = w$ as $|x| > A$ or $w > A$; $w = 0$ as $|x| \leq A$ and $w \leq A$. For $f \in L^s(R^n)$ with $s > \frac{n}{n-2}$, define

$$Tf(x) := K(x) \int_{R^n} \frac{w_A^{p-1}(y)f(y)dy}{|x-y|^{n-2}}, \hspace{1cm} F(x) := K(x) \int_{R^n} \frac{(w-w_A)^p(y)dy}{|x-y|^{n-2}}.$$  

Therefore, $w$ solves the operator equation

$$f = Tf + F.$$  

By the Hardy-Littlewood-Sobolev inequality (1.10), we get

$$\|Tf\|_s \leq C\|w_A^{p-1}f\|_{\frac{n}{n-2}} \leq C\|w_A\|^{p-1}_{\frac{n}{p-1}} \|f\|_s,$$

and

$$\|F\|_s \leq C\|w-w_A\|^{p}_{\frac{n(p-1)}{n-2}} < \infty.$$  

Take $A$ suitably large such that $C\|w_A\|^{p-1}_{\frac{n}{p-1}} < 1$. Thus, $T$ is a contraction map from $L^s(R^n)$ to itself. In view of $\frac{n(p-1)}{2} > \frac{n}{n-2}$ (which is implied by (3.5)), $T$ is also a contraction map from $L^{n(p-1)/2}(R^n)$ to itself. By the lifting lemma on the regularity (cf. Theorem 3.3.1 in [6] or Lemma 2.1 in [20]), we obtain $w \in L^s(R^n)$ for $s > \frac{n}{n-2}$. Thus, $u, v \in L^s(R^n)$ for $s > \frac{n}{n-2}$.

On the other hand, if $s \leq \frac{n}{n-2}$, by (2.1) we have

$$\|u\|_s \geq c \int_{R} r^{n-s(n-2)} \frac{dr}{r} = \infty.$$  

Similarly, we also deduce $v \notin L^s(R^n)$ for all $s \leq \frac{n}{n-2}$. (R1) is proved.

(2) For $d > 0$, write

$$w(x) = K(x) \int_{B_d(x)} \frac{w^p(y)dy}{|x-y|^{n-2}} + K(x) \int_{R^n \setminus B_d(x)} \frac{w^p(y)dy}{|x-y|^{n-2}} := K_1 + K_2.$$  

Clearly, for a suitably small $\epsilon > 0$, from (R1) we deduce

$$K_1 \leq C\|w\|^{p}_{\frac{n}{p-1}} \left( \int_{0}^{d} r^{n-\frac{n-2}{2} - \epsilon} \right)^{1-p} \leq C.$$  

On the other hand, in virtue of (3.5) and (R1), we get $w \in L^p(R^n)$. Thus

$$K_2 \leq C\|w\|^{p}_{\frac{n}{p}} \leq C.$$  

Combining the estimates of $K_1$ and $K_2$, we know that $w$ is bounded. Thus, $u, v$ are bounded.

Next, we show that $w$ is decaying. Take $x_0 \in R^n$. By exchanging the order of the integral variables, we have

$$w(x_0) = (n - \alpha)K(x_0) \int_{0}^{\infty} \left( \frac{\int_{B_t(x_0)} w^p(y)dy}{t^{n-\alpha}} \right) \frac{dt}{t}.$$  

Since \( w \in L^\infty(R^n) \), \( \forall \varepsilon > 0 \), there exists \( \delta \in (0, 1/2) \) such that
\[
\int_0^\varepsilon \left[ B_\delta(x_0) \frac{w^p(z)dz}{t^{n-\alpha}} \right] dt \leq C\|w\|^p_\infty \int_0^\varepsilon \frac{t^{\alpha}}{t} d\varepsilon < \varepsilon. \tag{3.8}
\]
as \( |x - x_0| < \delta \),
\[
\int_\delta^\infty \left[ B_\delta(x_0) \frac{w^p(z)dz}{t^{n-\alpha}} \right] dt \leq \int_\delta^\infty \left[ B_{\delta + \varepsilon}(x) \frac{w^p(z)dz}{(t+\delta)^{n-\alpha}} \right] \left[ \frac{(t+\delta)^{n-\alpha+1}}{t+\delta} dt \right] \leq Cw(x).
\]
Combining this result with (3.8), we get
\[
(w(x)_0 < C\varepsilon + Cw(x), \text{ for } |x - x_0| < \delta.
\]
Let \( s = \frac{n(p-1)}{2} \), then
\[
w^s(x_0) = |B_\delta(x_0)|^{-1} \int_{B_\delta(x_0)} w^s(x_0) dx \leq C\varepsilon^s + C|B_\delta(x_0)|^{-1} \|w\|_{L^p(B_\delta(x_0))}. \tag{3.9}
\]
Since \( w \in L^s(R^n) \), \( \lim_{|x| \to \infty} \|w\|_{L^s(B_\delta(x_0))} = 0 \). Inserting this result into (3.9), we have
\[
\lim_{|x| \to \infty} w^s(x_0) = 0.
\]
This result means that \( u \) and \( v \) converge to zero when \( |x| \to \infty \). (R2) is proved. \( \square \)

**Theorem 3.3.** Let \((u, v)\) be a pair of classical solutions of (1.3) with \( p > 1 \). If \( u \in L^{\frac{n(p-1)}{2}}(R^n) \), then \( u^p v \in L^1(R^n) \) and \( u \in D^{1,2}(R^n) \). Moreover,
\[
\int_{R^n} |\nabla u|^2 dx = \int_{R^n} |\nabla v|^2 dx = \sqrt{p} \int_{R^n} u^p v dx. \tag{3.10}
\]

**Proof.** **Step 1.** According to Theorem 3.1, \( u, v \) solve (1.7). By the Hölder inequality, from Theorem 3.2 (R1) and (3.5), we can deduce that \( u^p v \in L^1(R^n) \).

**Step 2.** Take a smooth function \( \zeta(x) \) satisfying
\[
\begin{cases}
\zeta(x) = 1, & \text{for } |x| \leq 1; \\
\zeta(x) \in [0, 1], & \text{for } |x| \in [1, 2]; \\
\zeta(x) = 0, & \text{for } |x| \geq 2.
\end{cases}
\]
Define the cut-off function
\[
\zeta_R(x) = \zeta(\frac{x}{R}). \tag{3.11}
\]

Multiplying the first equation of (1.3) by \( u^2 \zeta_R^2 \) and integrating on \( D := B_{3R}(0) \), we have
\[
- \int_D \zeta_R^2 u \Delta u dx = \sqrt{p} \int_D u^p v \zeta_R^2 dx.
\]
Integrating by parts, we obtain
\[
\int_D |\nabla u|^2 \zeta_R^2 dx + 2 \int_D u \zeta_R \nabla u \nabla \zeta_R dx = \sqrt{p} \int_D u^p v \zeta_R^2 dx. \tag{3.12}
\]
Applying the Cauchy inequality, we get
\[
|\int_D u \zeta_R \nabla u \nabla \zeta_R dx| \leq \delta \int_D |\nabla u|^2 \zeta_R^2 dx + C \int_D u^2 |\nabla \zeta_R|^2 dx \tag{3.13}
\]
By the Hölder inequality, we obtain
\[ \int_D u^2|\nabla \zeta_R|^2 \, dx \leq \left( \int_D u^2 \, dx \right)^{1-2/n} \left( \int_D |\nabla \zeta_R|^n \, dx \right)^{2/n} \leq C. \] (3.14)

Noting that \( u^p \in L^1(R^n) \), from (3.12)-(3.14) we deduce \( \int_D |\nabla u|^2 \zeta_R^2 \, dx \leq C \). Letting \( R \to \infty \) yields \( \int_{R^n} |\nabla u|^2 \, dx \leq \infty \). Similarly, we also obtain \( \int_{R^n} |\nabla v|^2 \, dx < \infty \).

Combining the results above, we can see that
\[ \int_{R^n} (u^2 + v^2 + u^p v + |\nabla u|^2 + |\nabla v|^2) \, dx < \infty. \]

Therefore, we can find \( R_j \) such that
\[ \lim_{R_j \to \infty} R_j \int_{\partial B_{R_j}} (u^2 + v^2 + u^p v + |\nabla u|^2 + |\nabla v|^2) \, ds = 0. \] (3.15)

**Step 3.** Multiplying the first equation of (1.3) by \( u \) and integrating on \( D \), we get
\[ \int_D |\nabla u|^2 \, dx - \int_{\partial D} u \partial_{\nu} u \, ds = \sqrt{p} \int_D u^p v \, dx. \] (3.16)

Here \( \nu \) is the outward unit normal vector on \( \partial D \). By the Hölder inequality, from (3.15) we deduce
\[ |\int_{\partial D} u \partial_{\nu} u \, ds| \leq CR^{n-\frac{2}{p-2}-1/2}(R) \left( \int_{\partial D} |\nabla u|^2 \, ds \right)^{1/2} \left( \int_{\partial D} u^p \, ds \right)^{1/2} \to 0 \]
when \( R = R_j \to \infty \). Letting \( R = R_j \to \infty \) in (3.16), we have \( \|\nabla u\|_p^2 = \sqrt{p}\|u^p v\|_1 \).

Similarly, we can also obtain \( \|\nabla v\|_p^2 = \sqrt{p}\|u^p v\|_1 \). \( \square \)

The following result shows that there does not exist solution in class \( L^{\frac{n(p-1)}{p}}(R^n) \) if \( p \) is not equal to the critical exponent \( 2^* - 1 \).

**Theorem 3.4.** Let \((u, v)\) be a pair of classical solutions of (1.3) with \( p > 1 \). If \( u \in L^{\frac{n(p-1)}{p}}(R^n) \), then \( p = 2^* - 1 \), and hence \( L^{n(p-1)/2}(R^n) = L^2(R^n) \).

**Proof.** Write \( B = B_R(0) \). Multiply two equations of (1.3) by \( x \cdot \nabla u \) and \( x \cdot \nabla v \), respectively. Integrating on \( B \), we get
\[ - \int_B (x \cdot \nabla u) \Delta u \, dx = \frac{1}{\sqrt{p}} \int_B v(x \cdot \nabla u^p) \, dx, \]
\[ - \int_B (x \cdot \nabla v) \Delta v \, dx = \sqrt{p} \int_B u^p(x \cdot \nabla v) \, dx. \]

Integrating by parts yields
\[ -p \int_{\partial B} |x| \partial_{\nu} u \, ds + \frac{p}{2} \int_{\partial B} |x| |\nabla u| \, ds - \frac{n-2}{2} p \int_B |\nabla u|^2 \, dx = \sqrt{p} \int_B v(x \cdot \nabla u^p) \, dx, \]
\[ - \int_{\partial B} |x| \partial_{\nu} v \, ds + \frac{1}{2} \int_{\partial B} |x| |\nabla v| \, ds - \frac{n-2}{2} \int_B |\nabla v|^2 \, dx = \sqrt{p} \int_B u^p(x \cdot \nabla v) \, dx. \]
Adding the two results together and integrating by parts again, we obtain
\[-\int_{\partial B} |x|^2 (p |\partial_{\nu} u|^2 + |\partial_{\nu} v|^2) ds + \frac{1}{2} \int_{\partial B} |x| (p |\nabla u|^2 + |\nabla v|^2) ds\]
\[-\frac{n-2}{2} \int_{B} (p |\nabla u|^2 + |\nabla v|^2) dx\]
\[= \sqrt{p} \int_{B} x \cdot \nabla (u^p v) dx = \sqrt{p} \int_{\partial B} |x| u^p v ds - n \sqrt{p} \int_{B} u^p v dx.\]

Letting \( R = R_j \to \infty \) and using (3.15), we have
\[-\frac{n-2}{2} \int_{B} (p |\nabla u|^2 + |\nabla v|^2) dx = \frac{n}{2} \sqrt{p} \int_{B} u^p v dx.\]

By (3.10) we see \( p = 2^* - 1 \) finally.

According to Theorem 3.1 and Theorem 2.2 with \( h = 0 \), the positive classical solutions of (1.3) are radially symmetric and decreasing about \( x^* \in \mathbb{R}^n \) as long as \( u \in L^{\frac{n}{2^* - 1}}(\mathbb{R}^n) \). Moreover, we have the following stronger result.

**Theorem 3.5.** Let \((u, v)\) be a pair of classical solutions of (1.3) with \( p > 1 \). If \( u \in L^{\frac{n}{2^* - 1}}(\mathbb{R}^n) \), then
\[u(x) = v(x) = c \left( \frac{t}{|x|^2 + |x - x^*|^2} \right)^{\frac{n-2}{2}}.\] (3.17)
with some constant \( c = c(n) \) and for some \( t > 0 \).

**Proof.**

**Step 1.** We claim \( u \equiv v \).

Let \( W = u - v \). By Theorems 3.1-3.3, we see that \( \int_{\mathbb{R}^n} (|W|^2 + |\nabla W|^2) dx < \infty \).
Thus, when \( R = R_j \to \infty \),
\[R \int_{\partial B_R(0)} (|W|^2 + |\partial_{\nu} W|^2) ds \to 0.\] (3.18)

From (1.3), it follows \( \Delta W = \sqrt{p} u^{p-1} W \). Therefore,
\[\int_{B} |\nabla W|^2 dx + \sqrt{p} \int_{B} u^{p-1} W^2 dx = \int_{\partial B} W \partial_{\nu} W ds.\] (3.19)

Here \( B = B_R(0) \). By the Hölder inequality and (3.18), as \( R = R_j \to \infty \),
\[\left| \int_{\partial B} W \partial_{\nu} W ds \right| \leq C \left( \int_{\partial B} |W|^2 ds \right)^{\frac{1}{2}} \left( \int_{\partial B} |\partial_{\nu} W|^2 ds \right)^{\frac{1}{2}} \left( R^{(n-1)(\frac{1}{2} - \frac{1}{2^*}) - \frac{1}{2}} \right) \to 0.
\]

Inserting this into (3.19) with \( R = R_j \to \infty \), we get
\[\int_{\mathbb{R}^n} (|\nabla W|^2 + \sqrt{p} u^{p-1} W^2) dx = 0,
\]
which implies \( u \equiv v \).

**Step 2.** By virtue of \( u \equiv v \) and Theorem 3.4, (1.3) is reduced to the single equation
\[-\Delta u = \sqrt{p} u^{2^* - 1}, \quad u > 0 \text{ in } \mathbb{R}^n.
\]

According to the classification results in [5], the positive solutions must be of the form (3.17) in the critical case. □
The argument above implies that a classical solution \( u \in L^{\frac{n(p-1)}{2}}(\mathbb{R}^n) \) is equivalent to (3.17).

At last, we complete the proof of Theorem 1.4.

**Theorem 3.6.** Let \((u, v)\) be a pair of classical solutions of (1.3) with \( p > 1 \). Then \( u \in L^{\frac{n(p-1)}{2}}(\mathbb{R}^n) \) is equivalent to each of the following items

(C1) \( u \in D^{1,2}(\mathbb{R}^n) \);

(C2) \( p = 2^* - 1 \);

(C3) \( u \in L^\infty(\mathbb{R}^n) \) and \( u(x) \simeq |x|^{2-n} \) as \( |x| \to \infty \).

**Proof.** If \( u \in L^{\frac{n(p-1)}{2}}(\mathbb{R}^n) \), Theorem 3.3 implies (C1).

Next, we claim that (C1) implies (C2). In fact, by the Sobolev inequality we get \( u \in L^{2^*}(\mathbb{R}^n) \). On the other hand, using (3.12)-(3.14) we can deduce \( u^p v \in L^1(\mathbb{R}^n) \) from \( u \in D^{1,2}(\mathbb{R}^n) \). By the same process of (3.12)-(3.14) on the second equation of (1.3), from \( u^p v \in L^1(\mathbb{R}^n) \) we can deduce \( v \in D^{1,2}(\mathbb{R}^n) \). Thus, (3.15) holds, and hence the proof of Theorem 3.4 still works. This shows that (C2) is true.

If (C2) holds, from \( u \in C^2(\mathbb{R}^n) \) and (3.2), it follows that (C3) is true.

Finally, by Theorem 3.1, we know that (1.7) has positive solutions. This implies that (3.5) holds and hence \( n - (n-2)\frac{n(p-1)}{2} < 0 \). By (C3) we get

\[
\int_{\mathbb{R}^n} u^{n-\frac{n(p-1)}{2}} \, dx \leq \int_{B_R(0)} u^{n-\frac{n(p-1)}{2}} \, dx + C \int_{R^+} r^{n-(n-2)\frac{n(p-1)}{2}} \, dr < \infty.
\]

This means \( u \in L^{\frac{n(p-1)}{2}}(\mathbb{R}^n) \). \( \square \)

4. Supercritical case.

4.1. Radial solutions. In order to find the existence of entire solutions in \( \mathbb{R}^n \), we need the following nonexistence result on a bounded domain, which is deduced by the Pohozaev identity.

**Theorem 4.1.** Let \( D \subset \mathbb{R}^n \) be a ball centered at the origin. If

\[
p \geq 2^* - 1,
\]

then the following boundary value problem has no nontrivial nonnegative radial solution in \( C^2(D) \cap C^1(\overline{D}) \)

\[
- \Delta u = \sqrt{p} u^{p-1} v, \quad \text{in} \quad D, \tag{4.2}
\]

\[
- \Delta v = \sqrt{p} u^p, \quad \text{in} \quad D, \tag{4.3}
\]

\[
u = v = 0 \quad \text{on} \quad \partial D. \tag{4.4}
\]

**Proof.** Suppose that \( u, v \) are nontrivial nonnegative radial solutions. Multiply (4.2) and (4.3) by \( u \) and \( v \), respectively. Integrating on \( D \) and using (4.4), we have

\[
\int_D |\nabla u|^2 \, dx = \int_D |\nabla v|^2 \, dx = \sqrt{p} \int_D u^p v \, dx. \tag{4.5}
\]

Since \( u \) has radial structure, \( |\nabla u|^2 = |\partial_r u|^2 \) on \( \partial D \).

Multiplying (4.2) by \( (x \cdot \nabla u) \) and integrating on \( D \), we get

\[
- \int_{\partial D} |x| |\partial_r u|^2 \, ds + \int_D |\nabla u|^2 \, dx + \frac{1}{2} \int_D x \cdot \nabla (|\nabla u|^2) \, dx = \frac{1}{\sqrt{p}} \int_D v(x \cdot \nabla u^p) \, dx.
\]
Integrating by parts and noting (4.4), we obtain
\[ \frac{1}{2} \int_{\partial D} |x|\partial_{\nu}u|^2 \, ds + \frac{n-2}{2} \int_D |\nabla u|^2 \, dx = \frac{n}{\sqrt{p}} \int_D u^p \, v \, dx + \frac{1}{\sqrt{p}} \int_D u^p (x \cdot \nabla v) \, dx. \]

Similarly, from (4.3) we also deduce that
\[ -\frac{1}{2} \int_{\partial D} |x|\partial_{\nu}v|^2 \, ds + \frac{n-2}{2} \int_D |\nabla v|^2 \, dx = \sqrt{p} \int_D u^p (x \cdot \nabla v) \, dx. \]

Combining two results above with (4.5) yields
\[ -\frac{1}{2} \int_{\partial D} |x|((\partial_{\nu}u)^2 + \frac{1}{p}(\partial_{\nu}v)^2) \, ds = \frac{n-2}{2} (\sqrt{p} + \frac{1}{\sqrt{p}}) \int_D u^p \, v \, dx - \frac{n}{\sqrt{p}} \int_D u^p \, dx. \]

If \( u, v \) are nontrivial, then
\[ \frac{n-2}{2} (\sqrt{p} + \frac{1}{\sqrt{p}}) - \frac{n}{\sqrt{p}} < 0, \]
which contradicts with (4.1). \( \square \)

Based on the above Liouville type result, we can search for positive solutions of (1.3) with radial structure. Let \( u, v \) be radially symmetric about \( x^* \in \mathbb{R}^n \). We can write
\[ U(r) = U(|x - x^*|) = u(x - x^*), \quad V(r) = V(|x - x^*|) = v(x - x^*). \]

**Theorem 4.2.** Let \( p > 2^* - 1 \). Then the following ODE system
\[
\begin{align*}
- (U'' + \frac{n-1}{r} U') &= \sqrt{p} U^{p-1} V, & - (V'' + \frac{n-1}{r} V') &= \sqrt{p} V^{p-1}, & r > 0 \\
U'(0) &= V'(0) = 0, & U(0) &= 1, & V(0) &= a,
\end{align*}
\]
has entire solutions for constant \( a > 0 \).

**Proof.** Here we use the shooting method.

**Step 1.** First, we know that \( U \) and \( V \) are not increasing since the right hand sides of equations in (4.6) are positive. By a standard contraction argument, we can see the local existence. We denote the solutions by \( u_a(r), v_a(r) \).

**Step 2.** We claim that either (4.6) has entire solutions for all \( a > 1 \), or for some \( a^* > 1 \), there exists \( R \in (0,1) \) such that \( u_{a^*}(r), v_{a^*}(r) > 0 \) for \( r \in [0, R) \) and \( u_{a^*}(R) = 0 \).

In fact, integrating (4.6) twice yields \[ v_a(r) = v_a(0) - \sqrt{p} \int_0^r \tau^{1-n} \int_0^\tau s^{n-1} u_a^p(s) \, ds \, d\tau, \]
and \[ u_a(r) = u_a(0) - \sqrt{p} \int_0^r \tau^{1-n} \int_0^\tau s^{n-1} u_a^{p-1}(s) v_a(s) \, ds \, d\tau. \]

Thus,
\[ v_a(r) - u_a(r) = (a - 1) + \sqrt{p} \int_0^r \tau^{1-n} \int_0^\tau s^{n-1} u_a^{p-1}(s) (v_a(s) - u_a(s)) \, ds \, d\tau. \]

Let \( a > 1 \). So we can find \( \delta > 0 \) such that \( v_a(r) > u_a(r) \) for \( r \in [0, \delta) \) by the continuity of \( u_a, v_a \).

We claim \( v_a(r) > u_a(r) \) for all \( r \geq 0 \). Otherwise, there exists \( r_0 \geq \delta \) such that \( v(r_0) = u(r_0) \) and \( v(r) > u(r) \) as \( r \in [0, r_0) \). From (4.7) with \( r = r_0 \) we can deduce a contradiction easily.
Therefore, if \( u_a(r) > 0 \) for all \( r \geq 0 \), then (4.6) has entire solutions and the proof is complete. Otherwise, we can find \( R > 0 \) such that \( u_a(r), v_a(r) > 0 \) for \( r \in (0, R) \) and \( u_a(R) = 0 \). We denote the \( a \) in this case by \( a^* \).

**Step 3.** We claim that for \( a < \varepsilon_0 = \frac{1}{n^{2+\frac{1}{p}} \sqrt{p}} \), there exists \( R \in (0, 1] \), such that \( u_a(r), v_a(r) > 0 \) for \( r \in [0, R) \) and \( v_a(R) = 0 \).

In fact, from (4.6) we obtain \( u_a', v_a' < 0 \) for \( r > 0 \). Thus, \( v_a(r) \leq a < \varepsilon_0 \) and \( u_a(r) \leq u_a(0) = 1 \). Therefore,

\[
    u_a(r) = u_a(0) - \sqrt{p} \int_{0}^{r} \tau^{1-n} \int_{0}^{\tau} s^{n-1} u_a^{p-1}(s) v_a(s) \, ds \, d\tau \geq 1 - \frac{\varepsilon_0 \sqrt{p} r^2}{2n} \geq \frac{1}{2},
\]

for \( r \in (0, 1) \), and hence

\[
    v_a(r) = v_a(0) - \sqrt{p} \int_{0}^{r} \tau^{1-n} \int_{0}^{\tau} s^{n-1} u_a^{p-1}(s) v_a(s) \, ds \, d\tau < \varepsilon_0 - \frac{\sqrt{p} r^2}{2n}.
\]

This proves that for \( a < \varepsilon_0 \), we can find \( R \in (0, 1] \) such that \( u_a(r), v_a(r) > 0 \) for \( r \in (0, R) \) and \( v_a(R) = 0 \).

**Step 4.** Let \( g = \sup S \), where

\[
    S := \{ \varepsilon; \quad \text{when } a \in (0, \varepsilon), \quad \exists R_a > 0, \text{ such that } u_a(r) > 0, v_a(r) \geq 0, \text{ for } r \in [0, R_a], v_a(R_a) = 0 \}.
\]

Clearly, \( S \neq \emptyset \) in virtue of \( \varepsilon_0 \in S \). From Step 2, it follows \( \varepsilon \leq a^* \) for \( \varepsilon \in S \). Namely, \( S \) is upper bounded, and hence we see the existence of \( a^* \).

**Step 5.** Write \( \bar{u}(r) = u_a(r) \) and \( \bar{v}(r) = v_a(r) \). We claim that \( u(\bar{r}), v(\bar{r}) > 0 \) for \( r \in (0, \bar{R}) \), and hence they are entire positive solutions of (4.6).

Otherwise, there exists \( \bar{R} > 0 \) such that \( u(\bar{r}), v(\bar{r}) > 0 \) for \( r \in (0, \bar{R}) \) and one of the following consequences holds:

1. \( \bar{u}(\bar{R}) = 0, \bar{v}(\bar{R}) > 0 \;
2. \( \bar{v}(\bar{R}) = 0, \bar{u}(\bar{R}) > 0 \;
3. \( \bar{u}(\bar{R}) = 0, \bar{v}(\bar{R}) = 0 \).

We deduce the contradictions from these consequences above.

(1) By the \( C^1 \)-continuous dependence of \( u_a, v_a \) in \( a \), and the fact \( \bar{u}'(\bar{R}) < 0 \), we see that for all \( |a - a^*| \) small, there exists \( R_a > 0 \) such that

\[
    \bar{u}(r), \bar{v}(r) > 0, \quad \text{for } r \in (0, R_a); \quad \bar{u}(R_a) = 0, \quad \bar{v}(R_a) > 0.
\]

This contradicts the definition of \( a^* \).

(2) Similarly, for \( |a - a^*| \) small, there exists \( R_a > 0 \) such that

\[
    \bar{u}(r), \bar{v}(r) > 0, \quad \text{for } r \in (0, R_a); \quad \bar{u}(R_a) > 0, \quad \bar{v}(R_a) = 0.
\]

This implies that \( a + \delta \in S \) for some \( \delta > 0 \), which contradicts with the definition of \( a^* \).

(3) The consequence implies that \( u(x) = \bar{u}(|x|) \) and \( v(x) = \bar{v}(|x|) \) are solutions of the system

\[
\begin{align*}
-\Delta u &= \sqrt{p} u^{p-1}, & -\Delta v &= \sqrt{p} v^p, \text{ in } B_R, \\
 u, v &> 0 \text{ in } B_R, & u = v = 0 \text{ on } \partial B_R.
\end{align*}
\]

This is impossible by Theorem 4.1.

All the contradiction arguments show that our claim is true. Thus, the entire positive solutions exist.

\[ \square \]
In supercritical case, whether the solution \((u, v)\) of \((1.3)\) satisfy symmetry of components \(u \equiv v\) is not clear (even \((1.3)\) has radial structure). However, if \(u\) is a classical solution of \((1.1)\) and \(v\) is the Newton potential of \(u^p\), the following theorem shows that the symmetry of components \(u \equiv v\) may be true, and hence Theorem 1.6 is a direct corollary.

**Theorem 4.3.** Assume \((4.1)\) holds. Let \(U(r)\) be an entire solution of
\[
\begin{align*}
-(U'' + \frac{n-1}{r}U') &= \sqrt[p]{p}U^{p-1}V, \quad r > 0 \\
V(|x|) &= \sqrt[p]{p}|x|^{2-n} * U^p(|x|), \\
U'(0) &= 0, \quad U(0) = \sigma > 0.
\end{align*}
\]
(4.9)

If we write \(u(x) = U(r) = U(|x|)\) and \(v(x) = V(r) = V(|x|)\), then
\[
\lim_{|x| \to \infty} u(x) = \lim_{|x| \to \infty} v(x) = 0
\]
(4.10)
and \(u \equiv v\). Moreover, either
\[
p = 2^* - 1 \quad \text{and} \quad u = u_*,
\]
where \(u_*\) is the radial function as the form \((1.6)\), or
\[
p > 2^* - 1, \quad \text{and} \quad c_1 \leq u(|x|)^{\frac{2}{p^*}} \leq c_2
\]
when \(|x| \to \infty\), where \(c_1, c_2\) are positive constants.

**Proof.** Step 1. We claim \(u(x) = O(|x|^{\frac{2}{p^*}})\) when \(|x| \to \infty\).

Eq. \((4.9)\) implies \(U'(r) < 0\) for \(r > 0\). So \(U\) is a decreasing positive solution, and hence \(\lim_{r \to 0} U(r)\) exists. By the same argument of the proof of \((2.9)\), we know \(\lim_{|x| \to \infty} u(x) = 0\).

Furthermore, for \(V(|x|) = \sqrt[p]{p}|x|^{2-n} * U^p(|x|)\), from \((1.2)\) it is not difficult to see that
\[
-(r^{n-1}V')' = \sqrt[p]{p}r^{n-1}U^p,
\]
(4.11)
and hence \(V'(r) < 0\) as \(r > 0\). Integrating \((4.9)\) and noting the monotonicity of \(U\) and \(V\), we have
\[
0 < U(r) = U(r/2) - \sqrt[p]{p} \int_{r/2}^{r} \int_{0}^{s} \tau^{n-1}U^{p-1}(\tau)V(\tau)d\tau ds \\
\leq U(r/2) - \frac{3\sqrt[p]{p}}{8n} U^{p-1}(r)V(r)r^2.
\]
Eqs. \((4.9)\) and \((4.11)\) imply that \(u, v\) also solve \((1.3)\). According to Lemma 2.4, the entire solution \(u\) of \((1.1)\) is not larger than \(v\). Therefore, the result above implies \(U^p(r)r^2 \leq \frac{8n}{3\sqrt[p]{p}} U(r/2)\). Set \(G(r) = U(r)r^\frac{2}{p^*}\), then we get an iteration result
\[
G(r) \leq \left(\frac{2^{\frac{n}{p^*}}}{\sqrt[p]{p}} \cdot r^{\frac{2}{p^*}}\right)^{\frac{1}{k}} \leq \left(\frac{2^{\frac{n}{p^*}}}{\sqrt[p]{p}} \cdot r^{\frac{2}{p^*}}\right)^{\frac{1}{k}} G^\frac{k}{p^*} \left(\frac{2}{2^k}\right)= \cdots \leq \left(\frac{2^{\frac{n}{p^*}}}{\sqrt[p]{p}} \cdot r^{\frac{2}{p^*}}\right)^{\frac{1}{k}} G^\frac{k}{p^*} \left(\frac{2}{2^k}\right) = \frac{2}{2^k} r^{\frac{k}{p^*}},
\]
for any \(k = 2, 3, \cdots\). Letting \(k \to \infty\), there holds
\[
G(r) \leq \left(\frac{2^{\frac{n}{p^*}}}{\sqrt[p]{p}} \cdot r^{\frac{2}{p^*}}\right)^{\frac{1}{p^*}}.
\]
Namely, there exists $R > 0$ such that for $|x| > R$,
\[
    u(x) \leq C|x|^{-\frac{2n}{p-1}}.
\]

This shows the claim.

**Step 2.** We claim $\lim_{|x| \to \infty} v(x) = 0$.

Clearly, for large $R > 0$, we obtain from $v = \sqrt{p}|x|^{2-n} * u^p(x)$ that
\[
v(x) = \sqrt{p} \left( \int_{B_R(0)} \frac{u^p(y)dy}{|x-y|^{n-2}} + \int_{R^n \setminus B_R(0)} \frac{u^p(y)dy}{|x-y|^{n-2}} \right) := \sqrt{p}(I_1 + I_2).
\]

First, $u \in C^2(B_R(0))$ implies that as $|x| \to \infty$,
\[
    I_1 \leq C \int_{B_R(0)} \frac{dy}{|x-y|^{n-2}} \leq C|x|^{2-n} \to 0.
\] (4.12)

Next, for large $R$, Step 1 implies $u(y) \leq c|y|^{\frac{2}{p-1}}$ as $|y| \geq R$. Therefore,
\[
    |x|^{\frac{2}{p-1}} I_2 \leq C \int_{B_R(0)} \frac{|x|^{\frac{2}{p-1}} dy}{|x-y|^{n-2} |y|^{\frac{2n}{p-1}}} \leq C \int_{B_R(0)} \frac{dz}{|e-z|^{n-2} |z|^{\frac{2n}{p-1}}}.
\] (4.13)

Clearly, for small $\delta > 0$ and large $\rho > 0$,
\[
    \int_{R^n} \frac{dz}{|e-z|^{n-2} |z|^{\frac{2n}{p-1}}} = C \left( \int_{B_\delta(0)} + \int_{B_\delta(e)} + \int_{B_{\rho}(0) \setminus (B_\delta(0) \cup B_\delta(e))} \right) \frac{dz}{|e-z|^{n-2} |z|^{\frac{2n}{p-1}}} \]
\[
    := C \sum_{j=1}^{4} J_j,
\]
and $J_4 < \infty$. From (4.1) it follows $n > \frac{2n}{p-1}$, and hence
\[
    J_1 \leq C \int_{B_\delta(0)} \frac{dz}{|z|^{\frac{2n}{p-1}}} \leq C \int_{0}^{\delta} r^{n-\frac{2n}{p-1}} \frac{dr}{r} < \infty.
\]

In addition,
\[
    J_2 \leq C \int_{B_\delta(e)} \frac{dz}{|e-z|^{n-2}} < \infty,
\]
and
\[
    J_3 \leq C \int_{0}^{\rho} r^{3-\frac{2n}{p-1}} \frac{dr}{r} < \infty,
\]
Inserting the estimates of $J_j$ ($j = 1, 2, 3, 4$) into (4.14) yields
\[
    \int_{R^n} |e-z|^{2-n} |z|^{\frac{2n}{p-1}} dz < \infty.
\]

Letting $|x| \to \infty$ in (4.13), we get $I_2 \to 0$. Combining with (4.12), we obtain
\[
    \lim_{|x| \to \infty} v(x) = 0.
\]

This result, together with Step 1, implies (4.10).

**Step 3.** We claim $u \equiv v$. The argument in Step 1 of the proof of Theorem 3.5 does not work, since the boundary integral is difficult to handle. Thanks to the work of [25]. Those ideas are powerful to deal with symmetry of components no matter the value of $p$ is critical or not.
For any \( r_0 \geq 0 \), we prove \( U(r_0) = V(r_0) \). Otherwise, by Lemma 2.4 we have
\[
U(r_0) < V(r_0). \tag{4.15}
\]
In view of the continuity of \( U \) and \( V \), we can find \( R > 0 \) such that
\[
U(r) < V(r) \quad \text{as} \quad r \in [r_0, R). \tag{4.16}
\]
Set \( R_0 = \sup \{ R; (4.16) \text{ is true} \} \). Thus, when \( R_0 < \infty \),
\[
U(R_0) = V(R_0). \tag{4.17}
\]
In view of (4.10), (4.17) still holds even if \( R_0 = \infty \).

Integrating (4.9) and (4.11) twice, we see that for \( r > 0 \),
\[
U(R_0) = U(r_0) - \sqrt{p} \int_{r_0}^{R_0} r^{1-n} \int_0^r s^{n-1} U^{p-1}(s) V(s) ds dr,
\]
and
\[
V(R_0) = V(r_0) - \sqrt{p} \int_{r_0}^{R_0} r^{1-n} \int_0^r s^{n-1} U^{p}(s) ds dr.
\]
Using (4.15), (4.16) and (4.17), we obtain
\[
0 > U(r_0) - V(r_0) = \sqrt{p} \int_{r_0}^{R_0} r^{1-n} \int_0^r s^{n-1} U^{p-1}(s) (V(s) - U(s)) ds dr \geq 0,
\]
which is impossible. So (4.15) is not true. Since \( r_0 \) is arbitrary, we know \( u \equiv v \).

**Step 4.** In virtue of \( u \equiv v \), (1.3) is reduced to the single equation
\[
-\Delta u = \sqrt{p} u^p, \quad u > 0 \text{ in } \mathbb{R}^n
\]
with (4.1). According to Theorem 2.41 in [30], we know that the radial solution \( u \) either decays fast
\[
c_1 \leq u(x)|x|^{n-2} \leq c_2,
\]
or decays slowly
\[
c_1 \leq u(x)|x|^2 \leq c_2,
\]
when \( |x| \to \infty \). Here \( c_1, c_2 \) are positive constants.

If \( u \) decays fast, by Theorem 1.4 we know \( p = 2^* - 1 \) and \( u \equiv u_* \). Here \( u_* \) is the radial function in (1.6).

If \( u \) decays slowly, we claim \( p > 2^* - 1 \). Otherwise, from (4.1) we have \( p = 2^* - 1 \). According to Theorem 1.4, \( u \equiv u_* \), which contradicts with the slow decay rate.

**Remark 4.1.** According to the results in [43], we know that the radial solution \( u \) in Theorem 4.3 is of the form
\[
u(x) = \mu^{\frac{2}{2^*}} U(\mu|x|), \quad x \in \mathbb{R}^n,
\]
where \( \mu > 0 \) depends on \( \sigma \) and \( p \), and \( U(r) \) is the unique solution of
\[
\begin{cases}
-(U'' + \frac{n-1}{r} U') = \sqrt{p} U^p, & U(r) > 0, \quad r > 0 \\
U'(0) = 0, & U(0) = 1.
\end{cases}
\]
4.2. Stable solutions and Joseph-Lundgren exponent.

**Theorem 4.4.** Write \( p_{jl}(n) = 1 + \frac{4}{n - 4 - 2\sqrt{n - 1}} \) which is the Joseph-Lundgren exponent. If

\[
\begin{align*}
1 < p < \infty, & \quad \text{if } 3 \leq n \leq 10 \\
1 < p < p_{jl}(n), & \quad \text{if } n \geq 11,
\end{align*}
\]

Then (1.1) has no positive stable solution.

**Proof.** Assume \( u \) is a positive stable solution, we can deduce a contradiction.

Let \( \Omega \subset \mathbb{R}^n \) be an arbitrary ball. For any nonnegative function \( \varphi \in C_0^1(\Omega) \), take \( \psi = u^{\frac{4}{n-4}} \varphi \) in (1.14). Then,

\[
p\sqrt{p} \int_{\Omega} u^{p+\gamma^{-1}} |\nabla \varphi|^2 \, dx 
\]

\[
\leq \int_{\Omega} |\nabla (u^{\frac{4}{n-4}})|^2 \varphi^2 \, dx + \int_{\Omega} u^{\gamma+1} |\nabla \varphi|^2 \, dx + 2 \int_{\Omega} u^{\frac{4}{n-4}} \nabla (u^{\frac{4}{n-4}}) \varphi \nabla \varphi \, dx 
\]

\[
= \int_{\Omega} |\nabla (u^{\frac{4}{n-4}})|^2 \varphi^2 \, dx + \int_{\Omega} u^{\gamma+1} |\nabla \varphi|^2 \, dx - \frac{1}{2} \int_{\Omega} u^{\gamma+1} \Delta (\varphi^2) \, dx. 
\]

Here \( \nu = \sqrt{p} |x|^{2-n} \ast u^p(x) \).

Multiplying (1.1) by \( u^{\gamma} \varphi^2 \) and integrating by parts, we get

\[
\int_{\Omega} |\nabla (u^{\frac{4}{n-4}})|^2 \varphi^2 \, dx 
\]

\[
= \frac{(\gamma + 1)^2}{4} p \int_{\Omega} u^{p+\gamma^{-1}} |\nabla \varphi|^2 \, dx + \frac{\gamma + 1}{4\gamma} \int_{\Omega} u^{\gamma+1} \Delta (\varphi^2) \, dx. 
\]

Combining two results above together, we have

\[
p\sqrt{p} \int_{\Omega} u^{p+\gamma^{-1}} |\nabla \varphi|^2 \, dx 
\]

\[
\leq \int_{\Omega} u^{\gamma+1} |\nabla \varphi|^2 \, dx \left( -\frac{1}{4\gamma} \int_{\Omega} u^{\gamma+1} \Delta (\varphi^2) \, dx + \frac{1}{2} \int_{\Omega} \nabla \varphi^2 \, dx. 
\right) 
\]

Namely, for any \( \gamma \in [1, 2p + 2\sqrt{p(p - 1)} - 1) \),

\[
\alpha \int_{\Omega} u^{p+\gamma^{-1}} |\nabla \varphi|^2 \, dx \leq \beta \int_{\Omega} u^{\gamma+1} \varphi \Delta \varphi \, dx + (\beta + 1) \int_{\Omega} u^{\gamma+1} |\nabla \varphi|^2 \, dx 
\]

(4.19)

with \( \alpha = \sqrt{p} (\frac{(\gamma + 1)^2}{4\gamma} - \frac{1 - \gamma}{2\gamma}) > 0 \) and \( \beta = \frac{1 - \gamma}{2\gamma} \leq 0 \).

For any \( \gamma \in [1, 2p + 2\sqrt{p(p - 1)} - 1), m \geq \max\{\frac{p+\gamma}{p-1}, 2\} \), we take \( \varphi = \psi^m \), where \( \psi \in C_0^1(\Omega) \) is nonnegative. Thus (4.19) implies

\[
\alpha \int_{\Omega} u^{p+\gamma^{-1}} |\nabla \psi|^2 \, dx 
\]

\[
\leq \beta m \int_{\Omega} u^{\gamma+1} \psi^{2m-1} \Delta \psi \, dx + ([\beta + 1]m^2 + \beta m(m - 1)) \int_{\Omega} u^{\gamma+1} \psi^{2m-2} |\nabla \psi|^2 \, dx. 
\]

Setting \( C_1 = \max\{\frac{2m}{\alpha}, \frac{|\beta+1|m^2+\beta m(m-1)|}{\alpha} \} \), we have

\[
\int_{\Omega} u^{p+\gamma^{-1}} \psi |\nabla \psi|^2 \, dx \leq C_1 \int_{\Omega} u^{\gamma+1} |\nabla \psi|^2 \, dx + |\nabla \psi|^2. 
\]

(4.20)
Since \( u \) solves (1.1), then \( u, v \) are positive solutions of (1.3). According to Lemma 2.4, \( u \leq v \) in \( R^n \). Using Holder's inequality and noting \( |\psi| \leq 1 \), we obtain from (4.20) that
\[
\int_{\Omega} u^{p+\gamma}|\psi|^{2m}dx \leq \int_{\Omega} u^{p+\gamma-1}|\psi|^{2m}dx
\]
\[
\leq C(t) \left( \int u^{p+\gamma-1}dx \right)^{\frac{p+\gamma}{p+\gamma-1}} \left( \int \left( |\psi|^2 + |\nabla \psi|^2 \right)^{\frac{p+\gamma}{p+\gamma-1}}dx \right)^{\frac{p+\gamma-1}{p+\gamma}}
\]
\[
\leq C(t) \left( \int u^{p+\gamma}dx \right)^{\frac{p+\gamma}{p+\gamma-1}} \left( \int \left( |\psi|^2 + |\nabla \psi|^2 \right)^{\frac{p+\gamma}{p+\gamma-1}}dx \right)^{\frac{p+\gamma-1}{p+\gamma}}.
\]
Thus, there exists a positive constant \( C(p, m, \gamma) = C(t)^{\frac{p+\gamma}{p+\gamma-1}} \), such that
\[
\int_{\Omega} u^{p+\gamma}|\psi|^{2m}dx \leq C(p, m, \gamma) \left( \int \left( |\psi|^2 + |\nabla \psi|^2 \right)^{\frac{p+\gamma}{p+\gamma-1}}dx \right)^{\frac{p+\gamma-1}{p+\gamma}}.
\]
(4.21)

For any \( R > 0 \), set \( \psi_R(x) \) is the cut-off function as in (3.11). Thus, for any \( \gamma \in [1, 2p+2\sqrt{p(p-1)}-1] \) and any integer \( m \geq \max\{\frac{p+\gamma}{p+\gamma-1}, 2\}, \)
\[
\int_{B_R(0)} u^{p+\gamma}dx \leq C(p, m, \gamma) \int_{R^n} \left( |\nabla \psi_R|^2 + |\psi_R|^2 + |\nabla \psi_R||\Delta \psi_R| \right)^{\frac{p+\gamma}{p+\gamma-1}}dx
\]
\[
\leq CR^{n-\frac{2p+\gamma}{p+\gamma-1}}, \quad \forall R > 0.
\]
(4.22)

Next, we consider the properties of the real-valued function \( f(t) := 2^{t+\gamma(t)} \) with \( t \in (1, +\infty) \), where \( \gamma(t) = 2t + 2\sqrt{t(t-1)} - 1 \). Clearly, \( f \) is a strictly decreasing function since \( f'(t) < 0 \) in \((1, +\infty)\), and satisfies \( \lim_{t \to 1^+} f(t) = +\infty \) and \( \lim_{t \to \infty} f(t) = 10 \). Namely, \( t = 1 \) is the vertical asymptote and \( f(t) = 10 \) is the horizontal asymptote.

Therefore, we only need consider the following two cases:

**Case 1.** if \( 3 \leq n \leq 10 \), for each \( p \in [1, +\infty) \), there exists \( \gamma \in [1, 2p+2\sqrt{p(p-1)}-1] \), such that \( n - 2\frac{p+\gamma}{p+\gamma-1} < 0 \).

**Case 2.** if \( n \geq 11 \), there exists a unique \( p_* > 1 \) such that \( n = 2\left(\frac{p_*+\gamma(p_*)}{p_*-1}\right) \) in view of the monotonicity of \( f \). Therefore, when \( 1 < p < p_* \), we can find \( \gamma \in [1, 2p+2\sqrt{p(p-1)}-1] \) such that \( n - 2\frac{p+\gamma}{p+\gamma-1} < 0 \). Now, from \( n = 2\left(\frac{p_*+\gamma(p_*)}{p_*-1}\right) \) we deduce
\[
p_* = 1 + \frac{4}{n - 2 - 2\sqrt{n - 1}} = p_1(n).
\]

We have proven that, under the conditions of Theorem 4.4, there always exists \( \gamma \in [1, 2p+2\sqrt{p(p-1)}-1] \) such that \( n - 2\frac{p+\gamma}{p+\gamma-1} < 0 \). Therefore, by letting \( R \to +\infty \) in (4.22), we deduce
\[
\int_{R^n} u^{p+\gamma-1}dx = 0,
\]
which contradicts with \( u, v \). The proof of Theorem 4.4 is complete.

The following theorem shows that (1.3) has a radial singular solution, which also implies that (1.1) has a singular solution as the form (1.11).

**Theorem 4.5.** Let \( u(x) = \frac{c_1}{|x|^{t_1}} \) and \( v(x) = \frac{c_2}{|x|^{t_2}} \) solve (1.3) on \( R^n \setminus \{0\} \), where \( c_1, c_2, t_1, t_2 \) are positive constants. If (3.5) holds, then such singular solutions must
be of the form

\[ u(x) = v(x) = \frac{c}{|x|^\frac{2}{p-1}} \]

with \( c = \left[\frac{2n}{\sqrt{p(p-1)}} - \frac{4\sqrt{p}}{(p-1)^2}\right]^{\frac{1}{p-1}} \).

**Proof.** Write \( U(r) = U(|x|) = u(x) \) and \( V(r) = V(|x|) = v(x) \). Thus,

\[
\begin{aligned}
-U'' - \frac{n-1}{r}U' &= \frac{c_1t_1}{r^{t_1+2}}(n - t_1 - 2), \\
-V'' - \frac{n-1}{r}V' &= \frac{c_2t_2}{r^{t_2+2}}(n - t_2 - 2).
\end{aligned}
\]

Since \( u, v \) solve (1.3), it is easy to get

\[
\begin{aligned}
t_1 + 2 &= t_1(p - 1) + t_2, \\
t_2 + 2 &= t_1p,
\end{aligned}
\]

and

\[
\begin{aligned}
c_1t_1(n - t_1 - 2) &= \sqrt{p}c_1^{p-1}c_2, \\
c_2t_2(n - t_2 - 2) &= \sqrt{p}c_2^{p-1}c_1
\end{aligned}
\]
as long as \( n - 2 > \max\{t_1, t_2\} \). Therefore, \( t_1 = t_2 = \frac{2}{p-1} \) and \( c_1 = c_2 = \left[\frac{2n}{\sqrt{p(p-1)}} - \frac{4\sqrt{p}}{(p-1)^2}\right]^{\frac{1}{p-1}} \). In view of (3.5), the condition \( n - 2 > \max\{t_1, t_2\} \) is obviously true. \( \square \)

**Remark 4.2.** Define the intersection number of two functions \( U_1(r) \) and \( U_2(r) \) by

\[ \mathcal{Z}[U_1(r) - U_2(r)] := \{ r \in R_+; U_1(r) = U_2(r) \}. \]

Let \( \sigma_i > 0 \) (i = 1, 2) be the initial values of (4.9). Assume \( U(r, \sigma_i) \) (i = 1, 2) are radial solutions of (1.1) obtained in Theorem 4.3, and \( U_0(r) \) is the singular solution (1.11) of (1.1). By the results in [21], we have the following conclusions.

1. If \( p = 2^* - 1 \), then \( \mathcal{Z}[U(r, \sigma_1) - U(r, \sigma_2)] = 1 \) and \( \mathcal{Z}[U(r, \sigma_1) - U_0(r)] = 2 \).
2. If \( 2^* - 1 < p < p_{jl}(n) \), then \( \mathcal{Z}[U(r, \sigma_1) - U(r, \sigma_2)] = \infty \) and \( \mathcal{Z}[U(r, \sigma_1) - U_0(r)] = \infty \).
3. If \( p \geq p_{jl}(n) \), then \( \mathcal{Z}[U(r, \sigma_1) - U(r, \sigma_2)] = \mathcal{Z}[U(r, \sigma_1) - U_0(r)] = 0 \).

**Theorem 4.6.** If \( p \geq p_{jl}(n) \), then \( U_\sigma(r) \) is a radial stable solution of (1.1), where \( U_\sigma(r) \) is the regular solution in Theorem 4.3.

**Proof.** The Hardy inequality shows that for any \( \psi \in C_0^\infty(R^n) \), there holds

\[
\int_{R^n} |\nabla \psi|^2 dx > \frac{(n-2)^2}{4} \int_{R^n} \frac{\psi^2}{|x|^2} dx.
\]

In view of \( p \geq p_{jl}(n) \), \( \frac{(n-2)^2}{4} \geq pA^{p-1} \). Here \( A := \left(\frac{2}{p-1}\right)^\frac{1}{p-1}(n - 2 - \frac{2}{p-1})^{\frac{1}{p-1}} \).

Therefore, (4.23) implies

\[
\int_{R^n} |\nabla \psi|^2 dx \geq p\sqrt{p} \int_{R^n} U_0^{p-1}(|x|)\psi^2 dx.
\]

Remark 4.2 (3) implies that \( U_\sigma(r) \leq U_0(r) \) for \( r > 0 \). Thus, \( U_\sigma(r) \) is a stable solution of (1.1). \( \square \)

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REFERENCES

[1] A. Ambrosetti, On Schrödinger-Poisson systems, *Milan J. Math.*, 76 (2008), 257–274.
[2] L. Caffarelli, B. Gidas and J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, *Comm. Pure Appl. Math.*, 42 (1989), 271–297.
[3] G. Caristi, L’Ambrosio and E. Mitidieri, Representation formulae for solutions to some classes of higher order systems and related Liouville theorems, *Milan J. Math.*, 76 (2008), 27–67.
[4] T. Cazenave, *Semilinear Schrödinger Equations*, Courant Lecture Notes in Mathematics, 10. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.
[5] W. Chen and C. Li, Classification of solutions of some nonlinear elliptic equations, *Duke Math. J.*, 63 (1991), 615–622.
[6] W. Chen and C. Li, *Methods on Nonlinear Elliptic Equations*, AIMS Book Series on Diff. Equa. Dyn. Sys., Vol. 4, 2010.
[7] W. Chen, C. Li and B. Ou, Classification of solutions for an integral equation, *Comm. Pure Appl. Math.*, 59 (2006), 330–343.
[8] W. Chen, C. Li and B. Ou, Qualitative properties of solutions for an integral equation, *Discrete Contin. Dyn. Syst.*, 12 (2005), 347–354.
[9] W. Chen, C. Li and B. Ou, Classification of solutions for a system of integral equations, *Comm. Partial Differential Equations*, 30 (2005), 59–65.
[10] A. Farina, On the classification of solutions of the Lane-Emden equation on unbounded domains of $R^N$, *J. Math. Pures Appl.*, 87 (2007), 537–561.
[11] F. Gazzola, Critical exponents which relate embedding inequalities with quasilinear elliptic operator, *Proceedings of the Fourth International Conference on Dynamical Systems and Differential Equations*, May 24-27, 2002, Wilmington, NC, USA, 2003, 327–335.
[12] B. Gidas, W.-M. Ni and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $R^n$, *Mathematical Analysis and Applications, Part A, Adv. in Math. Suppl. Stud.*, 7a, Academic Press, New York-London, 1981, 369–402.
[13] B. Gidas and J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, *Comm. Pure Appl. Math.*, 34 (1981), 525–598.
[14] J. Giunti and G. Velo, On a class of non linear Schrödinger equations with non local interaction, *Math. Z.*, 170 (1980), 109–136.
[15] J. Giunti and G. Velo, Long range scattering and modified wave operators for some Hartree-type equations, II, *Annales Henri Poincare*, 1 (2000), 753–800.
[16] C. Gui, W.-M. Ni and X. Wang, On the stability and instability of positive steady states of a semilinear heat equation in $R^n$, *Comm. Pure Appl. Math.*, 45 (1992), 1153–1181.
[17] Z. Guo and J. Wei, Global solution branch and Morse index estimates of a semilinear elliptic equation with super-critical exponent, *Trans. Amer. Math. Soc.*, 363 (2011), 4777–4799.
[18] E. Hebey and J. Wei, Schrödinger-Poisson systems in the 3-sphere, *Calc. Var. Partial Differential Equations*, 47 (2013), 25–54.
[19] L. Jeanjean and T. Luo, Sharp nonexistence results of prescribed $L^2$-norm solutions for some class of Schrödinger-Poisson and quasi-linear equations, *Z. Angew. Math. Phys.*, 64 (2013), 937–954.
[20] C. Jin and C. Li, Qualitative analysis of some systems of integral equations, *Calc. Var. Partial Differential Equations*, 26 (2006), 447–457.
[21] D. Joseph and T. Lundgren, Quasilinear Dirichlet problems driven by positive sources, *Arch. Rational Mech. Anal.*, 49 (1972/73), 241–269.
[22] Y. Lei, On the regularity of positive solutions of a class of Choquard type equations, *Math. Z.*, 273 (2013), 883–905.
[23] Y. Lei, Qualitative analysis for the static Hartree-type equations, *SIAM J. Math. Anal.*, 45 (2013), 388–406.
[24] C. Li, Local asymptotic symmetry of singular solutions to nonlinear elliptic equations, *Invent. Math.*, 123 (1996), 221–231.
[25] C. Li and L. Ma, Uniqueness of positive bound states to Schrödinger systems with critical exponents, *SIAM J. Math. Anal.*, 40 (2008), 1049–1057.
[26] D. Li, C. Miao and X. Zhang, The focusing energy-critical Hartree equation, *J. Differential Equations*, 246 (2009), 1139–1163.
[27] Y. Li, Remark on some conformally invariant integral equations: the method of moving spheres, *J. Eur. Math. Soc.*, **6** (2004), 153–180.

[28] Y. Li and L. Zhang, Liouville type theorems and Harnack type inequalities for semilinear elliptic equations, *J. Anal. Math.*, **90** (2003), 27–87.

[29] Y. Li and M. Zhu, Uniqueness theorems through the method of moving spheres, *Duke Math. J.*, **80** (1995), 383–417.

[30] Y. Li and W.-M. Ni, On conformal scalar curvature equations in $\mathbb{R}^n$, *Duke Math. J.*, **57** (1988), 895–924.

[31] E. Lieb, Existence and uniqueness of the minimizing solution of Choquard’s nonlinear equation, *Studies in Appl. Math.*, **57** (1976/77), 93–105.

[32] E. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, *Ann. of Math.*, **118** (1983), 349–374.

[33] E. Lieb and B. Simon, The Hartree-Fock theory for Coulomb systems, *Comm. Math. Phys.*, **53** (1977), 185–194.

[34] B. Liu and L. Ma, Invariant sets and the blow up threshold for a nonlocal equation of parabolic type, *Nonlinear Anal.*, **110** (2014), 141–156.

[35] J. Liu, Y. Guo and Y. Zhang, Existence of positive entire solutions for polyharmonic equations and systems, *J. Partial Differential Equations*, **19** (2006), 256–270.

[36] L. Ma and D. Chen, A Liouville type theorem for an integral system, *Comm. Pure Appl. Anal.*, **5** (2006), 855–859.

[37] L. Ma and B. Liu, Symmetry results for decay solutions of elliptic systems in the whole Space, *Adv. Math.*, **225** (2010), 3052–3063.

[38] L. Ma and L. Zhao, Classification of positive solitary solutions of the nonlinear Choquard equation, *Arch. Rational Mech. Anal.*, **195** (2010), 455–467.

[39] V. Moroz and J. Van Schaftingen, Nonexistence and optimal decay of supersolutions to Choquard equations in exterior domains, *J. Differential Equations*, **254** (2013), 3089–3145.

[40] K. Nakanishi, Energy scattering for Hartree equations, *Math. Res. Lett.*, **6** (1999), 107–118.

[41] P. Quittner and Ph. Souplet, Symmetry of components for semilinear elliptic systems, *SIAM J. Math. Anal.*, **44** (2012), 2545–2559.

[42] S. Sun and Y. Lei, Fast decay estimates for integrable solutions of the Lane-Emden type integral systems involving the Wolff potentials, *J. Funct. Anal.*, **263** (2012), 3857–3882.

[43] X. Wang, On the Cauchy problem for reaction-diffusion equations, *Trans. Amer. Math. Soc.*, **337** (1993), 549–590.

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