Contractivity of positive and trace preserving maps under $L_p$ norms

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We provide a complete picture of contractivity of trace preserving positive maps with respect to $p$-norms. We show that for $p > 1$ contractivity holds in general if and only if the map is unital. When the domain is restricted to the traceless subspace of Hermitian matrices, then contractivity is shown to hold in the case of qubits for arbitrary $p \geq 1$ and in the case of qutrits if and only if $p = 1, \infty$. In all non-contractive cases best possible bounds on the $p$-norms are derived.

I. INTRODUCTION

This paper deals with the following question:

Given a positive and trace preserving linear map $T$ between matrix spaces, when is $T$ contractive with respect to the $p$-norm, with $1 \leq p \leq \infty$?

This problem has come up in several contexts in recent years. For instance, Olkiewicz [7], in his investigation of the superselection structure of dynamical semigroups, needs as a starting point the fact that a 2-positive map that is contractive with respect to both the trace and operator norm is also contractive with respect to the Hilbert-Schmidt norm. The same result is needed by Raginsky in [10] in the study of entropy production of a quantum channel. In the context of quantum information this question arose again in [12], in the study of entanglement measures. It is shown there that any distance (in the space of matrices) that is contractive under completely positive
trace preserving maps gives rise to a “suitable” entanglement measure. Their conjecture that the Hilbert-Schmidt norm is such a distance was disproved soon later by Ozawa in [8]. In [6], Nielsen stated (without proof) that the Hilbert-Schmidt distance is contractive in the space of qubits, with respect to any completely positive trace preserving map. He also encouraged further study of this problem. Recently, the fact that a completely positive trace preserving map is contractive with respect to the trace norm was used in [13] in the context of condensed matter theory in a theoretical justification for the high accuracy of renormalization group algorithms.

Motivated by the appearance of the above question in so many different areas of physics, we will try in this note to give a complete picture of the solution. We will first study the general case and then restrict the domain of the maps to the traceless hyperplane.

II. THE GENERAL CASE

In the following $\mathcal{M}_n$ will denote the space of $n \times n$ matrices. A linear map $T : \mathcal{M}_n \to \mathcal{M}_r$ is called positive if it maps positive semi-definite matrices to positive semi-definite matrices, trace-preserving if $\text{tr} T(A) = \text{tr} A$ for all $A \in \mathcal{M}_n$, and unital if $T(1) = 1$. It is easy to see that $T$ is trace-preserving if and only if its adjoint $T^* : \mathcal{M}_r \to \mathcal{M}_n$ is unital, and that $T$ is positive if and only if $T^*$ is positive.

The $p$-norm (we will assume always $1 \leq p \leq \infty$) of a matrix $A$ is defined as $(\text{tr} |A|^p)^{1/p} = (\sum_i \lambda_i^p)^{1/p}$, where the $\lambda_i$ are the singular values of $A$ (i.e., the eigenvalues of $|A| \equiv \sqrt{A^*A}$). We write $\mathcal{S}^n_p$ for $\mathcal{M}_n$ endowed with the $p$-norm. For $T : \mathcal{M}_n \to \mathcal{M}_r$, we use $\|T\|_{p-p}$ to denote the operator norm of $T$ when we consider the $p$-norm in both the original and the final space, i.e., $\|T\|_{p-p} = \sup_{A \in \mathcal{M}_n} \|T(A)\|_p / \|A\|_p$. $T$ is called contractive under the $p$-norm if $\|T\|_{p-p} \leq 1$. Our first result is

**Theorem II.1.** If $T : \mathcal{M}_n \to \mathcal{M}_r$ is positive and trace preserving, then $\|T\|_{p-p} \leq n^{1-1/p}$.

Moreover, the bound $n^{1-1/p}$ is attained when $T$ is the the trace operator $\text{tr} : \mathcal{M}_n \to \mathbb{C}$ (which is completely positive and trace preserving).

The main ingredient in the proof is a non-commutative version of the Riesz-Thorin Theorem. (See [4] or Section IX.4 of [11].) We will also use a theorem of Russo and Dye [9, Corollary 2.9].

**Theorem II.2 (Non-commutative Riesz-Thorin).** If $T : \mathcal{M}_n \to \mathcal{M}_r$ is a linear map, then

$$\|T\|_{p-p} \leq \|T\|_{1-1}^{1/p} \|T\|_{\infty-\infty}^{1-1/p}.$$
Theorem II.3 (Russo-Dye). If $T : \mathcal{M}_n \rightarrow \mathcal{M}_r$ is positive, then $\|T\|_{\infty-\infty} = \|T(1)\|_{\infty}$.

Proof. To prove Theorem II.3, first note that under its hypotheses, $T^*$ is positive and unital. Then Theorem II.3 implies that $\|T^*\|_{\infty-\infty} = \|T^*(1)\|_{\infty} = \|1\|_{\infty} = 1$. Hence, using the duality $(S^p)^* = S_n^p$, we can conclude that $\|T\|_{1-1} = 1$. Moreover, $\|T\|_{\infty-\infty} = \|T(1)\|_{\infty} \leq \|T(1)\|_1 = \|1\|_1 = n$. Combining these bounds with Theorem II.2 gives the result claimed result.

We used the fact that when $T$ is trace preserving, then $T$ positive implies $\|T\|_{1-1} = 1$. In [9, Proposition 2.11] it is shown that for $T$ trace preserving, $T$ is positive if and only if $\|T\|_{1-1} = 1$.

When $T$ is positive, trace preserving and unital the argument used to prove Theorem II.1 shows that $\|T\|_{1-1} = \|T\|_{\infty-\infty} = 1$. Then Theorem II.2 implies that $T$ is contractive for all $p$-norms.

The next Theorem shows that this is an equivalence.

Theorem II.4. If $T : \mathcal{M}_n \rightarrow \mathcal{M}_n$ is positive and trace preserving, the following are equivalent:

i) $\|T(1)\|_p \leq n^{\frac{1}{p}}$ for some $1 < p \leq \infty$.

ii) $T$ is unital.

iii) $T$ is contractive for the $p$-norm for every $1 \leq p \leq \infty$.

iv) $T$ is contractive for the $p$-norm for some $1 < p \leq \infty$.

Proof. It only remains to prove that (i) $\Rightarrow$ (ii). To do this, let $(\lambda_i)_{i=1}^n$ denote the eigenvalues of $T(1)$. Since $T$ is positive, $\lambda_i \geq 0$; and since $T$ is trace-preserving, $\sum_i \lambda_i = \text{tr}T(1) = \text{tr}1 = n$. Hölder’s inequality can then be used to conclude that $\sum_i \lambda_i^p \geq n$ with equality if and only if $\lambda_i = 1$ for all $i$. But, by assumption, $\|T(1)\|_p \leq n^{\frac{1}{p}}$ for some $p > 1$. Thus, we must have equality so that $T(1) = 1$.

The hypothesis that $T$ is both unital and trace-preserving can only be satisfied when $r = n$. In that case, when $T$ is trace-preserving, but not unital, it follows that $\|T\|_{p-p} > 1$. When $n \neq r$, this does not hold, i.e., there are non-unital trace-preserving completely positive maps $T : \mathcal{M}_n \rightarrow \mathcal{M}_r$ for which $\|T\|_{p-p} < 1$. To see this one needs Jencova’s result [4] that $\|T\|_{p-p} = \omega_p(T^C)$ where $\omega_p(T)$ is the completely bounded $1 \rightarrow p$ norm studied in [2] and $T^C$ denotes the conjugate or complementary channel defined in [3] and [4]. From the results in [2] one can find depolarizing channels $T_{\text{dep}}$ such that $\omega_p(T_{\text{dep}}) < 1$. To see this let $\mu = \frac{1}{n+1}$ in eq. (5.4) in [2]. Since $\mu = \frac{1}{n+1}$ is the boundary between depolarizing channels which are entanglement-breaking and those which are not, this yields examples in both classes. Since the conjugate $T_{\text{dep}}^C$ is not unital [5], we have
explicit examples of non-unital trace-preserving completely positive maps $T : \mathcal{M}_n \rightarrow \mathcal{M}_{n^2}$ for which $\|T\|_{p-p} < 1$.

The implication (ii) $\Rightarrow$ (iii) was proved using complex interpolation. For $p = 2$, one can obtain an elementary proof by using that $\|T\|_{2-2}$ is the largest eigenvalue value of $T^* \circ T$ considered as an operator on the Hilbert space $\mathcal{M}_n$ with inner product $(A,B) = \text{tr} A^* B$. When $T$ is both trace-preserving and unital, $1$ is an eigenvector with eigenvalue $1$, and the orthogonality of eigenvectors implies that $\text{tr} G = \text{tr} 1 G = 0$ for any other eigenvector $G$ (which we can assume is Hermitian without loss of generality). Now let $G$ be one of these eigenvectors and let $\omega$ be the largest real number for which $1 + \omega G$ is positive semi-definite. Since $T^* \circ T$ is also positive, $(T^* \circ T)(1 + \omega G) = 1 + \lambda \omega G \geq 0$. But this implies that $\lambda \leq 1$ by the definition of $\omega$ so that $\|T^* \circ T\|_\infty = 1$.

III. THE TRACELESS HYPERPLANE

Using the $p$-norm to measure the distance between density matrices, gives expressions of the form $\|\rho - \rho'\|_p$, where $\rho - \rho'$ is a Hermitian matrix with trace $0$. In this section we investigate the behavior of such distances under positive and trace preserving maps. Let $T|_{\mathcal{H}_0}$ denote the restriction of $T$ to the hyperplane $\mathcal{H}_0$ of traceless Hermitian matrices.

**Theorem III.1.** Let $T : \mathcal{M}_n \rightarrow \mathcal{M}_n$ be a positive trace preserving linear map. Then

$$\|T|_{\mathcal{H}_0}\|_{p-p} \leq \begin{cases} \left(\frac{n}{2}\right)^{1-\frac{1}{p}}, & n \text{ even} \\ \left(\frac{2^{2-p} \cdot (n-1)^2 \cdot 2 + (n+1)^2 \cdot \tau}{(n-1)^2 + 2 + (n+1)^2 \cdot \tau}\right)^{1/p}, & n \text{ odd} \end{cases}$$

Moreover, this bound is optimal, since there exists a completely positive trace preserving map that saturates the inequality.

**Proof.** We begin by proving the upper bound. For an arbitrary positive trace preserving map $T : \mathcal{M}_n \rightarrow \mathcal{M}_n$, consider $A$ Hermitian, traceless and with $\|A\|_p \leq 1$. We can write $A = A_+ - A_-$ with $A_+$ and $A_-$ both positive semi-definite and $A_+A_- = 0$. Since $T$ is positive, $[T(A)]_+ \leq T(A_+)$ and $[T(A)]_- \leq T(A_-)$. Then,

$$\|T(A)\|_p^p = \text{tr} |T(A)|^p = \text{tr} ([T(A)]_+)^p + \text{tr} ([T(A)]_-)^p \\
= \|[T(A)]_+\|_p^p + \|[T(A)]_-\|_p^p \leq \|T(A_+)\|_p^p + \|T(A_-)\|_p^p.$$  

Call $r = \text{range}(A_+)$ and $s = \text{range}(A_-)$ and denote the eigenvalues of $A_+$ and $A_-$ by $\lambda_1, \ldots, \lambda_r$ and $\mu_1, \ldots, \mu_s$ respectively. It follows from Theorem III.4 that $\|T(A_+)\|_p^p \leq r^{p-1}\|A_+\|_p^p$ and $\|T(A_-)\|_p^p \leq
Using Lagrange multipliers in the problem

\[
\text{maximize} \left\{ r^{p-1} \sum_{i=1}^{r} \lambda_i^p + s^{p-1} \sum_{i=1}^{s} \mu_i^p \right\} \quad \text{restricted to}
\]

\[
\sum_{i=1}^{r} \lambda_i^p + \sum_{i=1}^{s} \mu_i^p = 1
\]
\[
\sum_{i=1}^{r} \lambda_i - \sum_{i=1}^{s} \mu_i = 0
\]

one finds that at least one of the following two conditions is satisfied: \( \lambda_i = \lambda_j \) and \( \mu_i = \mu_j \) for every \( i, j \), or \( s = r \).

In the first case we have that (assuming now w.l.o.g. that \( \text{tr}(A_+)=1 \))

\[
\frac{\|T(A)\|_p^p}{\|A\|_p^p} \leq \frac{2}{r^{1-p} + s^{1-p}}.
\]

This is in turn maximized and leads to the inequality in Theorem III.1 if \( s = n - r \) and \( r = n/2 \) for even \( n \), and \( r = (n+1)/2 \) for odd \( n \) respectively. In the second case \( r = s \) we have that

\[
\|T(A)\|_p^p \leq r^{p-1} \left( \|A_+\|_p^p + \|A_-\|_p^p \right),
\]

yielding to the sought inequality for \( r = n/2 \) (even \( n \)) whereas \( r < n/2 \) does not lead to a new inequality.

To prove optimality of the bound above, consider the completely positive and trace preserving map \( T : \mathcal{M}_n \rightarrow \mathcal{M}_n \) given by

\[
T(A) = |0\rangle \langle 0| \text{tr}(PA) + |1\rangle \langle 1| \text{tr}((1 - P)A),
\]

where \( P \) is a projector of dimension \( d = \text{tr}P \). If we apply this map to a traceless Hermitian operator of the form \( A = P - \frac{d}{n-d}(1 - P) \) we obtain

\[
\frac{\|T(A)\|_p}{\|A\|_p} = \left( \frac{2d^p}{d^p(n-d)^{1-p}} \right)^{1/p}.
\]

This achieves the above bound if \( d = n/2 \) (\( d=(n+1)/2 \)) for \( n \) even (odd).

Any trace-preserving map can be written uniquely in the form \( T(A) = N \text{tr}(A) + T_1(A) \) where \( T_1(A) \) is a unital trace-preserving map and \( N = \frac{1}{d}[T(1) - 1] \) is traceless. If \( T_1 \) is also positive, it follows from Theorem III.4 that \( \|T\}_{\text{tr=0}}\|_{p-p} \leq 1 \) and we can drop the restriction to Hermitian matrices. Unfortunately, the results above demonstrate that even when \( T \) is positive and trace-preserving, \( T_1 \) need not be positive.
A. Maps on qubits

When \( n = 2 \), Theorem III.1 implies contractivity in the traceless subspace of Hermitian matrices, i.e., \( \|T|_{\mathcal{H}_0}\|_{p-p} = 1 \). Here, however, there is no need to restrict to Hermitian matrices. For qubits, \( T \) positive and trace-preserving implies that the map \( T_1 \) above is always positive.

**Theorem III.2.** For any positive trace preserving linear map \( T : \mathcal{M}_2 \rightarrow \mathcal{M}_2 \) and \( 1 \leq p \leq \infty \) we have that

\[
\max_{\text{tr}(A)=0,\|A\|_p=1} \|T(A)\|_p \leq 1.
\]

**Proof.** The theorem is proved by showing that the \( T_1 \) defined above is indeed positive. Consider the action of \( T \) on a density operator \( \rho = \frac{1}{2}(1 + w \cdot \sigma) \) represented as a vector \( w \in \mathbb{R}^3 \) on the Bloch sphere. Any trace preserving and positive linear map acts as

\[
T(1 + w \cdot \sigma) = 1 + [r + Rw] \cdot \sigma,
\]

where \( r \in \mathbb{R}^3 \) and \( R \) is a real \( 3 \times 3 \) matrix. \( T \) is positive iff \( \|w\|_2 \leq 1 \) implies \( \|r + Rw\|_2 \leq 1 \). Let \( \lambda \) be the largest singular value of \( R \). Then there are unit vectors \( u, w \in \mathbb{R}^3 \) such that \( Rw = \lambda u \). Since \( R(-w) = \lambda(-u) \), one can choose the sign of \( w \) such that \( r \cdot u \geq 0 \), and thus \( 1 \geq \|r + \lambda u\|_2 \geq \lambda \). This implies that the unital trace preserving map \( T_1(1 + w \cdot \sigma) := 1 + [Rw] \cdot \sigma \) is indeed positive, and the result follows from Theorem II.4.

B. The case of qutrits

Theorem III.1 still implies contractivity in \( \mathcal{H}_0 \) for the case \( n = 3 \) if \( p = 1 \) or \( p = \infty \) (while this fails for \( 1 < p < \infty \)). As in the case of qubits one might expect that the result for \( p = \infty \) also extends to non Hermitian matrices. This is, however, not the case. A simple counterexample is given by the map

\[
T(A) = \sum_{i=0}^{1} (i |A|i) |0\rangle \langle 0| + (2 |A|2) |1\rangle \langle 1|.
\]

acting on \( A = a_0 |0\rangle \langle 0| + a_1 |1\rangle \langle 1| + a_2 |2\rangle \langle 2| \), where \( a_0, a_1, a_2 \) are the 3 complex cubic roots of unity. In this case we have that \( \text{tr}(A) = 0 \), \( \|A\|_\infty = 1 \), but \( \|T(A)\|_\infty > 1 \).

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