

C\(^0\).TOPOLOGY IN MORSE THEORY

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ABSTRACT. Let \( f \) be a Morse function on a closed manifold \( M \), and \( v \) be a Riemannian gradient of \( f \) satisfying the transversality condition. The classical construction (due to Morse, Smale, Thom, Witten), based on the counting of flow lines joining critical points of the function \( f \) associates to these data the Morse complex \( M_*(f,v) \).

In the present paper we introduce a new class of vector fields (\( f \)-gradients) associated to a Morse function \( f \). This class is wider than the class of Riemannian gradients and provides a natural framework for the study of the Morse complex. Our construction of the Morse complex does not use the counting of the flow lines, but rather the fundamental classes of the stable manifolds, and this allows to replace the transversality condition required in the classical setting by a weaker condition on the \( f \)-gradient (almost transversality condition) which is \( C^0 \)-stable. We prove then that the Morse complex is stable with respect to \( C^0 \)-small perturbations of the \( f \)-gradient, and study the functorial properties of the Morse complex.

The last two sections of the paper are devoted to the properties of functoriality and \( C^0 \)-stability for the Novikov complex \( N_*(f,v) \) where \( f \) is a circle-valued Morse map and \( v \) is an almost transverse \( f \)-gradient.

1. INTRODUCTION

Recall that the Riemannian gradient of a differentiable function \( f : M \to \mathbb{R} \) on a Riemannian manifold \( M \) is defined by the following formula:

\[
\langle \text{grad} f(x), h \rangle = f'(x)(h)
\]

(where \( \langle \cdot, \cdot \rangle \) stands for the scalar product on \( T_xM \), and \( h \in T_xM \)). The function \( f \) is strictly increasing along any non-constant integral curve of \( \text{grad} f \). Thus the properties of \( f \) and of the flow generated by \( \text{grad} f \) (gradient flow) are closely related to each other.

In the Morse theory the use of the gradient flows was initiated by R.Thom in the article [25]. Later on, the techniques of gradient flows in Morse theory were developed in several papers of M.Morse and in the book "Lectures on the \( h \)-cobordism theorem" by J.Milnor [7] which provides an alternative approach to S.Smale’s proof of \( h \)-cobordism conjecture. In this book J.Milnor works with a certain particular class of Riemannian gradients; here is the definition.

\textbf{Definition 1.1} ([7], §3). Let \( M \) be a manifold, \( f : M \to \mathbb{R} \) be a Morse function. A vector field \( v \) is called \textit{gradient-like vector field} for \( f \), if

1) for every \( x \notin S(f) \) we have: \( f'(x)(v(x)) > 0 \),

\[1\] It is well-known (and easy to prove), that each gradient-like vector field is a Riemannian gradient.
2) for every \( p \in S(f) \) there is a chart \( \Psi : U \to V \subset \mathbb{R}^m \) of the manifold \( M \), such that

\[
(f \circ \Psi^{-1})(x_1, \ldots, x_m) = f(p) - \sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{m} x_i^2,
\]

(1)

\[
\Psi^*(v)(x_1, \ldots, x_m) = (-x_1, \ldots, -x_k, x_{k+1}, \ldots x_m) \quad \text{where} \quad k = \text{ind} p.
\]

(2)

(Here \( S(f) \) stands for the set of all critical points of \( f \).) This notion has many advantages from the point of view of the differential topology. To construct and modify such a vector field there is no need for an auxiliary object such as Riemannian metric. Also the local structure of the vector field and its integral curves nearby the critical points of \( f \) is much simpler than for general Riemannian gradients.

On the other hand the class of gradient-like vector fields is "smaller" than one would like it to be; for example, a small perturbation of a gradient-like vector field for \( f \) is not necessarily again a gradient-like vector field for \( f \).

In Section 2 of the present paper we suggest a class of vector fields, which is strictly larger than the class of Riemannian gradients, but its definition, similarly to the definition of gradient-like vector fields, uses only the condition 1) above and a certain local non-degeneracy condition at every critical point (Definition 2.1). We call these vector fields \( f \)-gradients. The subject of Section 3 is the construction of the Morse complex for a Morse function \( f \) and an \( f \)-gradient \( v \). Instead of counting the flow lines joining the critical points our construction is based on the fundamental classes of the descending discs. This allows to weaken the condition of transversality required in the classical definition of the Morse complex. Namely we use the almost transversality condition (see Definition 2.16). Intuitively, the almost transversality condition requires that the stable and unstable discs of two critical points be transverse only for the case when the sum of the dimensions of these discs is not greater than the dimension of the ambient manifold (so that these discs do not intersect). The advantage of this condition is that the set of almost transverse \( f \)-gradients is open in the set of all \( f \)-gradients with respect to \( C^0 \)-topology, and this leads to a natural formulation of the \( C^0 \)-stability property of the Morse complex (see Theorem 3.9). In the same section we study the functorial properties of the Morse complex.

The subject of Section 4 is the Novikov complex. For every Morse map \( f : M \to S^1 \) and every almost transverse \( f \)-gradient \( v \) we construct a chain complex \( N_\ast(f,v) \) of modules over the ring \( \mathbb{Z}((t)) \) of Laurent series. Our main aim here is to construct a canonical chain equivalence between this version of the Novikov complex to the Novikov completion of the singular chain complex of the corresponding infinite cyclic covering. We study the functorial properties of this chain equivalence.

In the section 5 we discuss the \( C^0 \)-stability of the Novikov complex, and a formula which relates the homotopy class of the canonical chain equivalence constructed in the previous section and the Lefschetz zeta function of the gradient flow.

### 2. Gradients of Morse functions and forms

#### 2.1. Definition of \( f \)-gradients. Let \( W \) be a compact cobordism or a manifold without boundary, and let \( f : W \to \mathbb{R} \) be a Morse function. Let \( v \) be a \( C^\infty \)-vector field on \( W \) satisfying the following condition

\[
f'(x)(v(x)) > 0 \quad \text{whenever} \quad x \notin S(f).
\]

(3)
The function \( \phi(x) = f'(x)(v(x)) \) vanishes on \( S(f) \) and is strictly positive on \( W \setminus S(f) \). Therefore every point \( p \in S(f) \) is a point of local minimum of \( \phi \), and \( \phi'(p) = 0 \).

**Definition 2.1.** A \( C^\infty \) vector field \( v \) is called \( f \)-gradient if the condition (3) holds, and every \( p \in S(f) \) is a point of non-degenerate minimum of the function \( \phi(x) = f'(x)(v(x)) \) (that is, the second derivative \( \phi''(p) \) is a non-degenerate bilinear form on \( T_pW \)).

We shall now compute the second derivative \( \phi''(p) \) in terms of the derivatives of \( f \) and \( v \) and give an alternative formulation of the non-degeneracy condition above.

**Lemma 2.2.** Let \( v \) be an \( f \)-gradient, and \( p \in S(f) \). Then \( v(p) = 0 \).

**Proof.** Pick a Morse chart \( \Psi : U \to V \subset \mathbb{R}^n \) for \( f \), that is a chart where the function \( f \circ \Psi^{-1} - f(p) \) equals the quadratic form

\[
Q(x,y) = -||x||^2 + ||y||^2; \quad x \in \mathbb{R}^k, y \in \mathbb{R}^{n-k} \quad \text{(where } k = \text{ind} p),
\]

and let \( w = \Psi_*v \). We have \( Q'(z)(w(z)) > 0 \) for every \( z \neq (0,0) \), and we must prove \( w(0,0) = 0 \). Write \( w(0,0) = (\xi, \eta) \), and assume that \( \xi \neq 0 \). Write

\[
w(x,0) = (\xi + O_1(x), \eta + O_2(x))
\]

where \( ||O_i(x)|| \leq C ||x|| \) nearby 0. We have

\[
(4) \quad Q'(x,0)(w(x,0)) = -(2x, \xi + O_1(x)) \geq 0 \quad \text{for every } x.
\]

Set \( x = t\xi \) with \( t \to 0 \) to deduce that (4) is possible only for \( \xi = 0 \). Similarly we prove that \( \eta = 0 \). \( \square \)

At every point \( p \in S(f) \) the vector field \( v \) vanishes, and therefore there is a well defined linear map \( v'(p) : T_pW \to T_pW \). A simple computation in the local coordinates proves the following lemma.

**Lemma 2.3.** For \( p \in S(f) \) we have

\[
(5) \quad \phi''(p)(h,k) = f''(p)(v'(p)h,k) + f''(p)(v'(p)k,h).
\]

The next proposition follows immediately.

**Proposition 2.4.** A vector field \( v \) on \( W \) is an \( f \)-gradient if and only if

A) for every \( x \notin S(f) \) we have \( f'(x)(v(x)) > 0 \) and

B) for every \( p \in S(f) \) we have

\[
(6) \quad f''(p)(v'(p)h,h) > 0 \quad \text{for every } h \in T_pW, \ h \neq 0.
\]

Both gradient-like vector fields and Riemannian gradients are examples of \( f \)-gradients. This is quite obvious for the gradient-like vector fields and for the Riemannian gradients it is the subject of the next proposition.

**Proposition 2.5.** Every Riemannian gradient is an \( f \)-gradient.

**Proof.** The condition A) of the definition is obviously satisfied. To prove the condition B) it suffices to consider the case when \( f \) is defined in a neighborhood of 0 in \( \mathbb{R}^n \), and 0 is the only critical point of \( f \) in this neighborhood. Let \( \mathcal{G}(x) \) be the matrix of the Riemannian metric with respect to the coordinates in the \( \mathbb{R}^n \). Let \( \nabla f(x) \) denote the Euclidean gradient \((\frac{\partial}{\partial x_1}(x), \ldots, \frac{\partial}{\partial x_n}(x))\) of \( f \). We can assume that in a neighborhood of 0 we have:

\[
\mathcal{G}(x) = \text{Id} + O_1(||x||^2)
\]
where $O_1(||x||^2) \leq C||x||^2$. Applying a linear change of base in $\mathbb{R}^n$, we can assume also that

$$\nabla f(x) = Ax + O_2(||x||^2),$$

where $A$ is a diagonal matrix with non-zero diagonal entries. The Riemannian gradient of $f$ with respect to our metric satisfies

$$\text{grad} f(x) = G^{-1}(x) \cdot \nabla f(x) = Ax + O_3(||x||^3).$$

Therefore

$$f'(x)(\text{grad} f(x)) = \langle \text{grad} f(x), \text{grad} f(x) \rangle = \langle Ax, Ax \rangle + O(||x||^3)$$

and this function has a non-degenerate minimum at $x = 0$. □

2.2. Topological properties of the space of all $f$-gradients. In this subsection $W$ is a cobordism, and $f : W \rightarrow [a, b]$ is a Morse function. The space of all $C^\infty$ vector fields on $W$ endowed with the usual $C^\infty$ topology will be denoted $\mathcal{V}(W)$. In this space consider the subspace $\mathcal{V}(f)$ of all the vector fields, which vanish on $S(f)$. This is a closed subspace of finite codimension. The space $G(f)$ of all $f$-gradients is a subspace of $\mathcal{V}(f)$.

**Proposition 2.6.** The set $G(f)$ is an open convex subset of $\mathcal{V}(f)$.

**Proof.** Convexity is obvious. As for the openness, observe that the condition B) from the definition of $f$-gradients is clearly open with respect to $C^\infty$ topology. Proceed to the condition A). Let $w \in G(f)$. We have to prove that a neighborhood of $w$ in $\mathcal{V}(f)$ is in $G(f)$. The cobordism $W$ is compact, therefore it suffices to prove that for every $a \in W$ the following property holds:

$$(P) \quad \text{There is a neighborhood } U(a) \text{ of } a \text{ and a neighborhood } V \text{ of } w \text{ in } \mathcal{V}(f) \text{ such that for every } u \in V \text{ and every } x \in U(a) \setminus S(f) \text{ we have:}$$

$$f'(x)(u(x)) > 0.$$  \hspace{1cm} (8)

This property is obviously fulfilled for any point $a \notin S(f)$. In the case $a \in S(f)$ the property $(P)$ follows from the next Lemma, which is proved by an easy application of the Taylor development. For a function $g : U \rightarrow E$ defined in an open set $U \subset \mathbb{R}^n$, with values in a normed space $E$, put

$$||g||_U = \sup_{x \in U} ||g(x)||.$$

**Lemma 2.7.** Let $\phi : U \rightarrow \mathbb{R}$ be a $C^\infty$ function defined in a neighborhood $U$ of $0$ in $\mathbb{R}^m$. Assume that $\phi(0) = 0$ and that $\phi$ has a non-degenerate minimum in $0$. Then there is a compact neighborhood $U$ of $0$ and a number $\delta > 0$ such that every $C^\infty$ function $\psi : U \rightarrow \mathbb{R}$ satisfying

$$\psi(0) = 0, \psi'(0) = 0, ||\phi'' - \psi''||_U < \delta, ||\phi''' - \psi'''||_U < \delta,$$

satisfies also $\psi(x) > 0$ if $\in U$ and $x \neq 0$. □

Now we have already three types of vector fields, associated with a given Morse function: Riemannian gradients, gradient-like vector fields, and $f$-gradients. One more notion is useful:
Definition 2.8. A vector field $v$ is called a weak gradient for $f$ if 
\[ f'(x)(v(x)) > 0 \quad \text{for every} \quad x \notin S(f). \]

The set of all weak gradients for $f$ will be denoted by $GW(f)$. We have the inclusions 
\[ GL(f) \subset GR(f) \subset G(f) \subset GW(f) \subset V(f), \]
and the rest of this section is devoted to the study of their topological properties.

Proposition 2.9. The set $G(f)$ is everywhere dense in $GW(f)$.

Proof. Let $v$ be any weak gradient for $f$, and $v_0$ be any $f$-gradient (for example a gradient-like vector field for $f$). The vector field $w = v + \epsilon v_0$ is an $f$-gradient, which converges to $v$ as $\epsilon \to 0$. □

Now let us compare the spaces $GR(f), GL(f)$ and $G(f)$. The reader has certainly noticed that the main difference between $f$-gradients, Riemannian gradients and gradient-like vector fields is in their behavior nearby the critical points. It is easy to show that for a critical point $p \in S(f)$ the linear map $v'(p) : T_pW \to T_pW$ is 1) diagonalizable over $\mathbb{R}$ if $v$ is a Riemannian gradient 2) has only the eigenvalues $\pm 1$ if $v$ is a gradient-like vector field. Using this observation it is not difficult to show that for every Morse function $f$ with $S(f) \neq 0$, the space $GL(f)$ is not dense in $GR(f)$, and if moreover, $\dim W > 2$, the space $GR(f)$ is not dense in $G(f)$.

2.3. Gradients of Morse forms. The techniques of the previous subsections are purely local, even when the statements of the results are of global character. Thus the contents of these subsections is generalized to the case of Morse forms without any difficulty. In this subsection we list the corresponding definitions and results.

We shall consider manifolds without boundary, since only this case will be used in the later applications. The most important for us is the case when the Morse form $\omega$ is of the form $\omega = df$, where $f : M \to S^1$ is a Morse map. Let $M$ be a manifold without boundary, $\omega$ be a Morse form on $M$.

Definition 2.10. We say that $v$ is an $\omega$-gradient if $\omega(x)(v(x)) > 0$ whenever $x \notin S(f)$ and for every $p \in S(f)$ the real-valued function $\phi(x) = \omega(x)(v(x))$ has a non-degenerate local minimum at $p$. The space of all $\omega$-gradients will be denoted $G(\omega)$.

Similarly to the case of the Morse functions, the condition (B) is equivalent to the following condition:
\[ f''(p)(v'(p)h,h) > 0 \quad \text{for every} \quad h \neq 0, h \in T_pM \]
(where $f$ is any function in a neighborhood of $p$ with $df = \omega$).

Definition 2.11. The Riemannian gradient $\text{grad} \omega$ with respect to a given Riemannian metric on $M$ is defined by
\[ \omega(x)(h) = \langle \text{grad} \omega(x), h \rangle \]
where $x \in M$, $h \in T_xM$. The space of all Riemannian gradients for $\omega$ will be denoted $GR(\omega)$.

A Riemannian gradient for $\omega$ is an $\omega$-gradient (similarly to Proposition 2.5).

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2 The notion of weak gradient was introduced by D.Schütz in his paper [23].
Definition 2.12. A vector field $v$ is called gradient-like vector field for $\omega$, if for open set $U \subset M$ and every Morse function $f : M \rightarrow \mathbb{R}$ with $df = \omega|U$ the vector field $v|U$ is a gradient-like vector field for $f|U$. The space of all gradient-like vector fields for $\omega$ will be denoted $GL(\omega)$.

Definition 2.13. A vector field $v$ on $M$ is called weak gradient for $\omega$ if
\[(12) \quad \omega(x)(v(x)) > 0 \quad \text{whenever} \quad x \notin S(\omega)\]
The space of all gradient-like vector fields for $\omega$ will be denoted $GW(\omega)$.

Denote by $\mathcal{V}(\omega)$ the space of all vector fields on $M$ vanishing on $S(\omega)$. This is a closed vector subspace of codimension $m(\omega)$ in $\mathcal{V}(M)$. We have the inclusions
\[GL(\omega) \subset GR(\omega) \subset G(\omega) \subset GW(\omega) \subset \mathcal{V}(\omega),\]

The proofs of the next propositions are completely similar to the proofs of Propositions 2.9 and 2.6.

Proposition 2.14. Let $W$ be a closed manifold, and $\omega$ be a closed 1-form on $W$. The set $G(\omega)$ is a convex open subset of $\mathcal{V}(\omega)$. The set $G(\omega)$ is dense in the space $GW(\omega)$.

Proposition 2.15. If $\omega$ has at least one zero and $\text{dim} \, M > 2$, then the set $GL(\omega)$ is not dense in $GR(\omega)$, and the set $GR(\omega)$ is not dense in $G(\omega)$.

2.4. Transversality properties. Let $v$ be a vector field on a manifold $M$, and let $v(p) = 0$. Recall that the stable set $W^{st}(p,v)$ of $p$ is the set of all points $x \in M$, such that the $v$-trajectory $\gamma(x, t; v)$ is defined for all $t \geq 0$ and converges to $p$ as $t \to \infty$. The unstable set $W^{un}(p,v)$ is by definition the set $W^{st}(p, -v)$.

Let $M$ be a manifold without boundary, $\omega$ be a closed 1-form, $v$ be an $f$-gradient, $p$ be a zero of $f$. It is easy to deduce form the formula (10) that $p$ is an elementary zero of $v$, that is, the linear map $v'(p)$ has no imaginary eigenvalues, and therefore by Hadamard-Perron theorem (see [1], Th. 27) there exist local stable and unstable manifolds for $v$ at $p$. In the case when $M$ is compact, the (global) stable manifold $W^{st}(p,v)$ is an immersed manifold of dimension $\text{ind} p$, and $W^{un}(p,v)$ is an immersed manifold of dimension $n - \text{ind} p$. The stable manifold $W^{st}(p,v)$ will be also called descending disc of $p$ and denoted $D(p,v)$. The stable manifold $W^{st}(p, -v)$ will be also called ascending disc of $p$ and denoted $D(p,-v)$.

In the case when $v$ is an $f$-gradient, where $f : W \rightarrow [a,b]$ is a Morse function on a cobordism, the stable manifold $W^{st}(p,v)$ of a critical point $p \in S(f)$ is a submanifold with boundary of $W$ of dimension $\text{ind} p$, and $W^{st}(p,v)$ is an $(n - \text{ind} p)$-dimensional submanifold with boundary.

In the next definition $v$ is an $f$-gradient where $f : W \rightarrow [a,b]$ is a Morse function, or an $\omega$-gradient, where $\omega$ is a Morse form on a closed manifold.

Definition 2.16. We say that $v$ is transverse or satisfies transversality condition, if
\[(13) \quad (x,y \in S(f)) \Rightarrow (D(x,v) \pitchfork D(y,-v)).\]

We say that $v$ is almost transverse or satisfies Almost Transversality Condition, if
\[(14) \quad (x,y \in S(f) \& \text{ind} x \leq \text{ind} y) \Rightarrow (D(x,v) \pitchfork D(y,-v)).\]
The notion of almost transverse gradients was introduced in [12]. We have the corresponding version of Kupka-Smale theorem (the proof is similar to the classical proof, see [20], [19]).

**Theorem 2.17 (Kupka-Smale theorem).** Let \( \omega \) be a closed 1-form on a closed manifold \( M \). The set of all transverse \( f \)-gradients is residual in \( V(\omega) \) (with respect to \( C^\infty \) topology).

The case of Morse functions on cobordisms is not covered by this theorem, but it can be dealt with by a similar argument.

**Proposition 2.18.** Let \( f : W \to [a,b] \) be a Morse function. The set of all transverse \( f \)-gradients is residual in \( V(f) \) (with respect to \( C^\infty \) topology).

The set of all almost transverse gradients is much larger than the set of transverse gradients, as the next proposition shows (the proof is similar to the proof of Cor. 1.7 of [13]).

**Proposition 2.19.** Let \( f : W \to [a,b] \) be a Morse function on a cobordism \( W \). The set of all almost transverse \( f \)-gradients is dense in \( V(f) \) with respect to \( C^\infty \) topology, and open in \( V(f) \) with respect to \( C^0 \)-topology.

2.5. **Ordered Morse functions and Morse-Smale filtrations.** This section contains a bunch of definitions to be used in the sequel.

**Definition 2.20.** An ordered Morse function is a Morse function \( f : W \to [a,b] \) together with a sequence \( a_0, ..., a_{n+1} \) of regular values such that \( a = a_0 < a_1 < ... < a_{n+1} = b \) and

\[
S_i(f) \subset f^{-1}([a_i, a_{i+1}])
\]

for every \( i \). The sequence \( a_0, ..., a_{n+1} \) satisfying (15) is called ordering sequence for \( f \).

Note that we consider the ordering sequence as a part of the data of the ordered Morse function. Each ordered Morse function generates a filtration of \( W \), defined by

\[
W^{(k)} = f^{-1}([a_0, a_{k+1}]);
\]

**Definition 2.21.** A filtration \( W^{(k)} \) of the cobordism \( W \) (where \( 0 \leq k \leq \dim W \)) is called Morse-Smale filtration if there is an ordered Morse function \( \phi : W \to [a,b] \), such that (16) holds.

**Definition 2.22.** A Morse function \( \phi : W \to [\alpha, \beta] \) is called adjusted to \( (f,v) \), if:

1) \( S(\phi) = S(f) \),
2) the function \( f - \phi \) is constant in a neighborhood of \( \partial_0 W \), in a neighborhood of \( \partial_1 W \), and in a neighborhood of each point of \( S(f) \),
3) \( v \) is also a \( \phi \)-gradient.

An analog of the classical Rearrangement Lemma (see [7], §4, or [14], Prop. 2.23) is also valid in the context of \( f \)-gradients, and this implies the next proposition.

**Proposition 2.23.** Let \( f : W \to [a,b] \) be a Morse function on a cobordism \( W \), and \( v \) be an almost transverse \( f \)-gradient. There is an ordered Morse function \( \phi : W \to [a,b] \), adjusted to \( (f,v) \), such that \( v \) is a \( \phi \)-gradient. \( \square \)
The next proposition is proved by the same argument as Lemma 2.74 of [14]. Here and elsewhere in this paper the symbol $\| \cdot \|$ denotes the $C^0$-norm.

**Proposition 2.24.** Let $g$ be a Morse function on $W$, adjusted to $(f,v)$. Then there is $\delta > 0$ such that for every $f$-gradient $w$ with $\| w - v \| < \delta$ the function $g$ is adjusted to $(f,w)$. \qed

**Definition 2.25.** A Morse-Smale filtration $W^{(k)}$ on $W$ is called adjusted to $(f,v)$ if there is an ordered Morse function $\phi$, adjusted to $(f,v)$, such that (16) holds.

**Corollary 2.26.** Let $W^{(k)}$ be a Morse-Smale filtration of $W$, adjusted to $(f,v)$. Then there is $\delta > 0$ such that for every $f$-gradient $w$ with $\| w - v \| < \delta$ the filtration $W^{(k)}$ is adjusted to $(f,w)$. \qed

3. **Morse complex**

Let $f : W \to [a,b]$ be a Morse function on a cobordism $W$ and $v$ be an $f$-gradient. If $v$ is transverse, then the classical construction of the Morse complex, based on counting of the flow lines joining critical points (see [26]), carries over to the present case without any changes. Our aim in this section is to generalize this construction to the case of almost transverse $f$-gradients, and to study its properties. An $f$-gradient $v$ is called oriented if for every critical point $p \in S(f)$ an orientation of the descending disc $D(p,v)$ is fixed. In the first subsection we construct the Morse complex for oriented almost transverse gradients of ordered Morse functions. In the second subsection we generalize this construction to the case of arbitrary Morse functions and oriented almost transverse gradients. We show that the resulting complex is stable with respect to $C^0$-small perturbations of the gradient. We construct a canonical chain equivalence between the Morse complex and the singular chain complex. In the last subsection we study the functorial properties of these constructions.

3.1. **Ordered Morse functions.** Let $W$ be a cobordism of dimension $n$. Let $\phi : W \to [a,b]$ be an ordered Morse function, $a = a_0,\ldots,a_{n+1} = b$ be the ordering sequence of $\phi$, and $W^{(k)} = \phi^{-1}([a_0,a_{k+1}])$ be the corresponding filtration of $W$. Let $v$ be an almost transverse $\phi$-gradient. As in the standard Morse theory, this filtration is cellular, that is, the homology $H_*(W^{(k+1)},W^{(k)})$ vanishes for every $* \neq k$, and $C_k = H_k(W^{(k+1)},W^{(k)})$ is a free abelian group generated by the homology classes of the descending discs of the critical points of index $k$. The generator, corresponding to the descending disc $D(p,v)$ of a critical point $p \in S_k(f)$ with the chosen orientation will be denoted $\Delta(p,v)$. Let us endow the graded abelian group $C_\ast$ with the boundary operator induced from the exact sequences of triples $(W^{(k)},W^{(k-1)},W^{(k-2)})$; denote the resulting chain complex by $C_\ast(\phi,v)$. This is a chain complex of free abelian groups endowed with a basis $\{ \Delta(p,v) \}$. The next theorem is an immediate consequence of the general properties of cellular filtrations (see [2] Ch. 5).

**Proposition 3.1.** $H_*(C_\ast(\phi,v)) \cong H_*(W,\partial_0W)$. \qed

The aim of the next proposition is to compare the chain complexes associated with an ordered Morse function $\phi$ and different $\phi$-gradients.

**Definition 3.2.** Let $v, w$ be oriented $\phi$-gradients. Their orientations are called similar if for every $p \in S(f)$ the fundamental classes of the manifolds $D(p,v)$ and...
$D(p, w)$ induce the same generator of the group $H_k(W_p, W_p \setminus \{p\})$ (where $k = \text{ind} p$, and $W_p = \phi^{-1}([a, F(p)])$).

Recall that $\| \cdot \|$ stands for the $C^0$-norm.

**Proposition 3.3.** There is $\delta > 0$ such that for every $\phi$-gradient $w$ with $\|w - v\| < \delta$ the chain complexes $C_\ast(\phi, v)$ and $C_\ast(\phi, w)$ are basis-preserving isomorphic if the orientations of $v$ and $w$ are similar.

**Proof.** If all the critical points of $f$ of a given index have the same critical value, then it is not difficult to show that $\Delta_p(v) = \Delta_p(w)$ for every oriented $f$-gradient $w$ endowed with orientation similar to the orientation of $v$. The general case is easily reduced to this particular one by the Rearrangement Lemma and Proposition 2.24.

Now we shall construct a canonical chain homotopy equivalence between $C_\ast(\phi, v)$ and the singular chain complex $\mathcal{S}_\ast(W, \partial_0 W)$. Let $\mathcal{S}_\ast^{(k)} = \mathcal{S}_\ast(W^{(k)} \setminus W^{(k-1)})$ denote the filtration induced in $S_\ast(W, \partial_0 W)$ by the subsets $W^{(k)}$. Let us say that a chain map $\mathcal{E} : C_\ast(\phi, v) \to \mathcal{S}_\ast(W, \partial_0 W)$ conserves filtrations, if $\mathcal{E}(C_k(\phi, v)) \subset \mathcal{S}_k^{(k)}$ for every $k$. A chain map $\mathcal{E}$ conserving filtrations induces for every $k$ a homomorphism $\mathcal{E}_{(k)}^{adj} : C_k(\phi, v) \to H_k(\mathcal{S}_k^{(k)}, \mathcal{S}_k^{(k-1)})$.

**Theorem 3.4.** There is a chain homotopy equivalence $\mathcal{E}_\ast : C_\ast(\phi, v) \to \mathcal{S}_\ast$ which conserves filtrations and satisfies $\mathcal{E}_{(\ast)}^{adj} = \text{id}$. Such homotopy equivalence is homotopy unique.

We shall omit the proof, which repeats almost verbatim the proof of Theorem A.5' of [11].

**Definition 3.5.** This chain equivalence will be denoted $\chi(\phi, v)$.

### 3.2. Morse complexes and $C^0$-stability.

Now we proceed to Morse functions which are not ordered in general. Let $f : W \to [a, b]$ be a Morse function on a cobordism $W$ and $v$ be an almost transverse oriented $f$-gradient. Pick an ordered Morse function $\phi : W \to [a, b]$, such that $v$ is also a $\phi$-gradient. As the next proposition shows, the chain complex $C_\ast(\phi, v)$ associated to $\phi$ and $v$ does not depend on the particular choice of $\phi$, neither does the homotopy class of the corresponding chain equivalence $\chi(\phi, v)$.

**Theorem 3.6.** A. Let $\phi_1, \phi_2$ be two ordered Morse functions, such that $v$ is a gradient for both. There is a homotopy commutative diagram

$$
\begin{array}{ccc}
C_\ast(\phi_1, v) & \xrightarrow{I(\phi_1, \phi_2)} & C_\ast(\phi_2, v) \\
\downarrow{\chi(\phi_1, v)} & & \downarrow{\chi(\phi_2, v)} \\
\mathcal{S}_\ast(W, \partial_0 W) & & \mathcal{S}_\ast(W, \partial_0 W)
\end{array}
$$

where $I(\phi_1, \phi_2)$ is a basis preserving isomorphism of chain complexes.

**Proof.** Let $W_1^{(k)}, W_2^{(k)}$ be the filtrations corresponding to the ordered Morse functions $\phi_1, \phi_2$. In the case when $W_1^{(k)} \subset W_2^{(k)}$ our assertion is easily obtained by functoriality. The general case is obtained from this particular one, since for every two ordered Morse function $\phi_1, \phi_2$ such that $v$ is a gradient for both, there
are basis-preserving isomorphic if the orientations of $v$ have the canonical chain homotopy equivalence (17).

**Definition 3.7.** Let $f : W \to [a, b]$ be any Morse function, and $v$ be an almost transverse $f$-gradient. Pick any ordered Morse function $\phi$, such that $v$ is a $\phi$-gradient. Write
\[ \partial (\Delta(p, v)) = \sum_{q \in S_{k-1}(p)} n(p, q; v) \cdot \Delta(q, v) \]
where $\partial$ is the boundary operator in the chain complex $C_*(\phi, v)$. It follows from Theorem 3.6 that the integer $n(p, q; v)$ does not depend on the particular choice of an ordered function $\phi$. This number will be called *incidence coefficient* corresponding to $p, q$ and $v$.

Now we can give the definition of Morse complex for any Morse function $f : W \to [a, b]$ and its gradient $v$ satisfying almost transversality condition.

**Definition 3.8.** Let $f : W \to [a, b]$ and $v$ be an almost transverse oriented $f$-gradient. Let $\mathcal{M}_k(f, v)$ be the free abelian group generated by the critical points of $f$ of index $k$. Define a homomorphism $\partial_k : \mathcal{M}_k(f, v) \to \mathcal{M}_{k-1}(f, v)$ setting
\[ \partial_k(p) = \sum_{q \in S_{k-1}(f)} n(p, q; v) \cdot q, \]
then $\partial_k \circ \partial_{k-1} = 0$. The resulting chain complex $\mathcal{M}_*(f, v)$ is called *Morse complex* corresponding to $(f, v)$.

Thus the Morse complex $\mathcal{M}_*(f, v)$ is basis-preserving isomorphic to $C_*(\phi, v)$, where $\phi$ is any ordered Morse function such that $v$ is a $\phi$-gradient. It is clear that in the case when $v$ is transverse, the Morse complex $\mathcal{M}_*(f, v)$ equals the Morse complex defined via counting the flow lines joining the critical points.

The next theorem follows immediately from 3.3.

**Theorem 3.9.** Let $f : W \to [a, b]$ be a Morse function on a cobordism $W$. Let $v$ be an oriented almost transverse $f$-gradient. Then there is $\delta > 0$ such that for every $f$-gradient $w$ with $||w - v|| < \delta$ the chain complexes $\mathcal{M}_*(f, v)$ and $\mathcal{M}_*(f, w)$ are basis-preserving isomorphic if the orientations of $v$ and $w$ are similar.

For every ordered Morse function $\phi : W \to \mathbb{R}$, such that $v$ is also a $\phi$-gradient we have the canonical chain homotopy equivalence $\chi(\phi, v) : C_*(\phi, v) \to \mathcal{S}_*(W, \partial_0 W)$. Composing it with the basis preserving isomorphism $\mathcal{M}_*(f, v) \to C_*(\phi, v)$ we obtain a chain equivalence
\[ \mathcal{E} = \mathcal{E}(f, v) : \mathcal{M}_*(f, v) \xrightarrow{\sim} \mathcal{S}_*(W, \partial_0 W); \]
its homotopy class does not depend on the particular choice of an ordered Morse function. It will be called *the Morse chain equivalence*.

### 3.3. Functorial properties

The aim of this subsection is to study functorial properties of the Morse complex and the Morse chain equivalence. Let $f_1 : M_1 \to \mathbb{R}, f_2 : M_2 \to \mathbb{R}$ be Morse functions on closed manifolds, and $v_1, v_2$ be oriented almost transverse gradients for $f_1$, resp. $f_2$. Let $A : M_1 \to M_2$ be a continuous map, which satisfies the following condition:

\[ A(D(p, v_1)) \cap D(q, -v_2) = \emptyset \]
for every $p \in S(f_1), q \in S(f_2)$ with $\text{ind}p < \text{ind}q$. 


It is clear that the set $C(v_1, v_2)$ of the maps satisfying the condition (18) is open in the space $C^0(M_1, M_2)$ of all the continuous maps (endowed with $C^0$-topology). As we shall see later on in this subsection the set $C(v_1, v_2)$ is also dense in $C^0(M_1, M_2)$. Assuming that the condition (18) is fulfilled, we shall now construct a homomorphism

\[(19) \quad A_t : M_*(f_1, v_1) \rightarrow M_*(f_2, v_2).\]

Pick some Morse-Smale filtrations $M_1^{(k)}, M_2^{(k)}$ of $M_1, M_2$, adjusted to $(f_1, v_1)$, resp. to $(f_2, v_2)$. Let us first assume that for some $T \geq 0$ the following condition holds:

\[(20) \quad (\Phi(T, -v_2) \circ A)(M_1^{(k)}) \subset M_2^{(k)} \quad \text{for every} \quad k.\]

(Here $\Phi(\cdot, -v_2)$ stands for the flow generated by $-v_2$.) In this case the continuous map $\Phi(T, -v_2) \circ A$ preserves the Morse-Smale filtrations, and induces a chain map

\[(21) \quad C_*(\phi_1, v_1) \rightarrow C_*(\phi_2, v_2)\]

in the Morse complexes. It is obvious that this chain map does not depend on the particular choice of $T$. We obtain therefore a chain map

\[(22) \quad A_t(\phi_1, \phi_2) : M_*(f_1, v_1) \rightarrow M_*(f_2, v_2).\]

An argument, similar to the proof of Theorem 3.6 shows that the map (22) is independent of the choice of $\phi_1, \phi_2$, satisfying the condition (20).

Observe that for every continuous map $A$ satisfying (18) there are Morse-Smale filtrations adjusted to $(f_1, v_1)$ such that (20) holds. Indeed, let us begin with any Morse-Smale filtration $M_2^{(k)}$, adjusted to $(f_2, v_2)$. Let $D_k(v_1)$ denote the union of all descending discs $D(p, v_1)$ with ind $p \leq k$. The condition (18) implies that for every $T$ sufficiently large, we have

\[(23) \quad (\Phi(T, -v_2) \circ A)(D_k(v_1)) \subset M_2^{(k)} \quad \text{for every} \quad k.\]

Then it suffices to choose the Morse-Smale filtration of $M_1$ with

\[M_1^{(k)} \subset (\Phi(T, -v_2) \circ A)^{-1}(M_2^{(k)}) \quad \text{for every} \quad k,\]

(which is possible by [18], Prop 3.3) and the condition (20) is verified.

Thus for every continuous map $A$, satisfying (18) we have constructed a chain map

\[A_t : M_*(f_1, v_1) \rightarrow M_*(f_2, v_2).\]

Our next aim is to show that this map commutes with the Morse chain equivalences.

**Proposition 3.10.** The following diagram is chain homotopy commutative:

\[
\begin{array}{ccc}
M_*(f_1, v_1) & \xrightarrow{A_t} & M_*(f_2, v_2) \\
\varepsilon_1 \downarrow & & \varepsilon_2 \\
S_*(M_1) & \xrightarrow{A_*} & S_*(M_2)
\end{array}
\]

(where $A_*$ is the chain map in the singular chain complexes induced by $A$, and $\varepsilon_i$ are the Morse chain equivalences).
Proof. Pick any ordered Morse functions \( \phi_i \) and a positive number \( T \) such that (20) holds. In the following diagram we denote by \( \alpha \) the adjoint map induced by the continuous map \( A' = (\Phi(T,-v_2) \circ A) \) preserving filtrations, and by \( A'_* \) the chain map induced by \( A' \) in the singular chain complexes. This diagram is homotopy commutative, by functoriality of the canonical chain equivalences (see Proposition 3.11).

\[
\begin{array}{ccc}
C_*(\phi_1, v_1) & \xrightarrow{\alpha} & C_*(\phi_2, v_2) \\
\downarrow{\chi(\phi_1, v_1)} & & \downarrow{\chi(\phi_2, v_2)} \\
S_*(M_1) & \xrightarrow{A'_*} & S_*(M_2)
\end{array}
\]

It remains to note that the chain maps \( A'_* \) and \( A_* \) are chain homotopic. \( \square \)

In the case when \( A \) is a \( C^\infty \) map, satisfying the condition (26) below, we can obtain an explicit formula for \( A_2 \) in terms of intersection indices.

\[
(26) \quad A|D(p, v_1) : D(p, v_1) \to M_2 \text{ is transverse to } D(q, -v_2) \text{ for every } p \in S(f_1), q \in S(f_2).
\]

A routine argument from transversality theory shows that the set of all maps satisfying the above condition, is residual in \( C^\infty(M_1, M_2) \) (which implies in particular, that this set is dense, since \( C^\infty(M_1, M_2) \) is a Baire space). Observe that the condition (26) implies (18), therefore the set \( \mathcal{C}(v_1, v_2) \) is dense in \( C^0(M_1, M_2) \).

It is easy to show that if \( A \) satisfies (26), then for every \( p \in S_k(f_1) \) and \( q \in S_{n_2-k}(f_2) \) the set \( T(p, q) = D(p, v_1) \cap A^{-1}(D(q, -v_2)) \) is finite (where \( n_2 = \dim M_2 \)) and therefore the intersection index

\[ n(p, q; A) = A(D(p, v_1)) \bigoplus D(q, -v_2) \in \mathbb{Z} \]

is defined. Introduce a homomorphism of graded groups

\[
(27) \quad A_2 : \mathcal{M}_*(f_1, v_1) \to \mathcal{M}_*(f_2, v_2)
\]

by the formula

\[
(28) \quad A_2(p) = \sum_q n(p, q; A) \cdot q,
\]

where the summation in the formula is over \( q \in S_{n_2-\text{indp}}(f_2) \).

**Proposition 3.11.** \( A_2 = A_\flat \).

Proof. Let \( B = \Phi(T,-v_2) \circ A; \) then \( B_\flat = A_\flat, B_\sharp = A_\sharp \). For sufficiently large \( T \) the map \( B \) satisfies \( B(M_1^{(k)}) \subset M_2^{(k)} \) for every \( k \). For \( p \in S_k(f_1) \) let \( \Delta_p \in H_k(M_1^{(k+1)}, M_1^{(k)}) \) be the homology class of the descending disc of \( p \). It is easy to check the following formula for the image of \( \Delta_p \) with respect to \( B_* \)

\[
B_*(\Delta_p) = \sum_q n(p, q; A) \Delta_q \in H_k(M_2^{(k+1)}, M_2^{(k)});
\]

here \( \Delta_q \in H_k(M_2^{(k+1)}, M_2^{(k)}) \) is the homology class of the descending disc of \( q \). Thus by definition we have \( B_\flat = B_\sharp \). \( \square \)
4. Novikov complex

Let $M$ be a closed manifold, $f : M \rightarrow S^1$ a Morse map, and $v$ be an oriented $f$-gradient. Recall that if $v$ satisfies transversality condition, then the Novikov complex is defined (see [9], [10], [11]). This is a chain complex of free finitely generated modules over the ring

$$\overline{L} = Z((t)) = Z[[t]][t^{-1}].$$

The definition of this complex is based on a procedure of counting the flow lines of $v$ joining critical points of $f$. In Subsection 4.1 we construct the Novikov complex in a more general situation, when $v$ is only almost transverse. In Subsection 4.2 we construct a canonical chain equivalence between the Novikov complex and the completed singular chain complex of the infinite cyclic covering corresponding to $f$. In the third subsection we study the functorial properties of the Novikov complex thus obtained.

4.1. Construction of the complex. Let $\tilde{M} \rightarrow M$ be the infinite cyclic covering, induced by $f$ from the universal covering $R \rightarrow S^1$. Lift the function $f$ to a Morse function $F : \tilde{M} \rightarrow R$. Let $t$ be the generator of the structure group $\approx Z$ of the covering, such that $F(tx) = F(x) - 1$ for every $x \in \tilde{M}$. Let

$$P = Z[t], \quad \tilde{P} = Z[[t]], \quad L = Z[t, t^{-1}], \quad P_n = P/t^n P \quad \text{where} \quad n \geq 0.$$

Lift the vector field $v$ to a vector field $\tilde{v}$ on $\tilde{M}$; then $\tilde{v}$ is an oriented almost transverse $F$-gradient, invariant with respect to the action of the structure group of the covering. Let $M_k(F)$ be the free abelian group generated by the critical points of $F$ of index $k$, then $M_k(F)$ has a natural structure of a free $L$-module. Let

$$N_k(F) = M_k(F) \otimes_L \overline{L}.$$

We shall define a boundary operator in the graded group $N_\ast(F)$ thus endowing it with the structure of a chain complex of $\overline{L}$-modules. As a first step we shall define an incidence coefficient $n(p, q; v)$ for every pair $p, q$ of critical points of $F$ with $\text{ind} p = \text{ind} q + 1$. Choose any regular values $\lambda, \mu$ of $F$ such that $\lambda < F(p), F(q) < \mu$.

Let $W = F^{-1}([\lambda, \mu])$: then $\tilde{v}|W$ is an oriented almost transverse $F|W$-gradient, and the incidence coefficient $n(p, q; \tilde{v}|W)$ is defined (see Definition 3.7). It is clear that this integer does not depend on the particular choice of the regular values $\lambda, \mu$; it will be denoted $n(p, q; v)$.

**Proposition 4.1.** The formula

$$\partial_k p = \sum n(p, q; v)q$$

defines a homomorphism $\partial_k : N_k \rightarrow N_{k-1}$ of $\overline{L}$-modules such that $\partial_k \circ \partial_{k-1} = 0$.

Let $\lambda$ be a regular value of $F$, denote by $N_\lambda^\ast$ the set of all $\xi \in N_\ast$ such that $\sup \xi \subset F^{-1}([\lambda, \lambda])$. It is clear that $N_\lambda^\ast$ is a free and finitely generated $\tilde{P}$-submodule of $N_\ast$, which is $\partial_{k-1}$-invariant. The module $N_\lambda^\ast/t^n N_\lambda^\ast$ is a free $P_n$-module, and by definition the truncated boundary operator

$$\partial_k^{(n)} : N_k^\lambda/t^n N_k^\lambda \rightarrow N_{k-1}^\lambda/t^n N_{k-1}^\lambda$$
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is equal to the boundary operator in the Morse complex of the restriction of $F$ and $\tilde{v}$ to the cobordism $W_n = F^{-1}([\lambda - n, \lambda])$. Therefore $\partial_k^{(n)} \circ \partial_{k-1}^{(n)} = 0$, and this implies the assertion. \hfill $\square$

**Definition 4.2.** The graded $L$-module $N_\ast$ endowed with the boundary operator $\partial_\ast$ will be denoted $N_\ast(f,v)$ and called *Novikov complex*.

### 4.2. Canonical chain equivalence

In this subsection we construct a chain equivalence

$$N_\ast(f,v) \to S_\ast(\bar{M} \otimes L),$$

where $S_\ast(\bar{M})$ is the singular chain complex of $\bar{M}$. This construction is quite similar to the case of transverse gradients, and we shall give only an outline of the construction, following the exposition in [11], and [18]. The chain equivalence (30) is built from the Morse equivalences corresponding to the cobordisms $W_n = F^{-1}([\lambda - n, \lambda])$ (where $\lambda$ is any regular value of $F$). The truncated Novikov complex $N^\lambda_\ast / t^nN^\lambda_\ast$ is basis-preserving isomorphic to the Morse complex $M_\ast(F|W_n, \bar{v}|W_n)$ (as chain complexes of $\mathbb{Z}$-modules). The composition of this isomorphism with the Morse chain equivalence will be denoted $\varepsilon_n$:

$$\varepsilon_n : N^\lambda_\ast / t^nN^\lambda_\ast \to S_\ast(W_n, \partial_0 W_n).$$

Let

$$V^\lambda_\ast = F^{-1}([\lambda - \infty, \lambda]).$$

The composition

$$N^\lambda_\ast / t^nN^\lambda_\ast \xrightarrow{\varepsilon_n} S_\ast(W_n, \partial_0 W_n) \xrightarrow{\Delta} S_\ast(V^-_\lambda, t^nV^-_\lambda)$$

is a chain equivalence. Both the source and the target of this chain equivalence are free $P_n$-modules, but the map (31) is apriori only a homomorphism of abelian groups. We shall now prove that the map (31) is chain homotopic to a chain equivalence of $P_n$-modules. To this end recall that the Morse chain equivalence $\varepsilon_n$ is defined using a Morse-Smale filtration of $W_n$, adjusted to $(f,v)$. Let $W_n^{(k)}$ be such a filtration. Define a filtration $X^{(k)}$ of the space $V^-_\lambda$ as follows:

$$X^{(k)} = W_n^{(k)} \cup t^nV^-_\lambda.$$

**Definition 4.3.** A Morse-Smale filtration $W_n^{(k)}$ of $W_n$ is called *equivariant* if

$$tX^{(k)} \subseteq X^{(k)} \quad \text{for every } k.$$  

The following proposition is proved in [18] (Proposition 2.3) for the case when $v$ satisfies transversality condition; the proof is carried over to the case of almost transverse gradients without much changes.

**Proposition 4.4.** For every Morse-Smale filtration $W_n^{(k)}$ of the cobordism $W_n$ there is an equivariant Morse-Smale filtration $\tilde{W}_n^{(k)}$ such that $\tilde{W}_n^{(k)} \subseteq W_n^{(k)}$. \hfill $\square$

If $W_n^{(k)}$ is an equivariant Morse-Smale filtration then the filtration

$$S_\ast(X^{(k)}, t^nV^-_\lambda)$$

of the singular chain complex $S_\ast(V^-_\lambda, t^nV^-_\lambda)$ is a filtration by $P_n$-modules. It is not difficult to show that this filtration is a *cell-like* filtration by $P_n$-modules (that is, the homology of the pair $(X^{(k+1)}, X^{(k)})$ vanishes in all degrees except $k$, and in this
degree the homology is a free $P_n$-module). Therefore the adjoint chain complex $H_*(X^{(s)}, X^{(s)})$ is basis-preserving isomorphic to $N_*^\lambda/t^n N_*^\lambda$, and the canonical chain equivalence

$$E_n : N_*^\lambda/t^n N_*^\lambda \to S_*(V^{-\lambda}, t^n V^{-\lambda})$$

of $P_n$-modules associated with this filtration is homotopic over $\mathbb{Z}$ to the chain equivalence (31).

The proof of the next proposition repeats the proof of Theorem 3.6 with corresponding modifications.

**Proposition 4.5.** The chain equivalence (32) is independent (up to chain homotopy) of the particular choice of the equivariant Morse-Smale filtration adjusted to $(f|W_n, \bar{v}|W_n)$.

**Definition 4.6.** The chain equivalence (32) will be called equivariant Morse equivalence.

Now we shall compare the equivariant Morse equivalences corresponding to different regular values of $F$.

**Proposition 4.7.** Let $\lambda \leq \mu$ be regular values of $F$ and $m \leq n$ be positive integers. Then the following diagram of chain maps of complexes over $P$ is homotopy commutative:

$$
\begin{array}{ccc}
N_*^\lambda/t^n N_*^\lambda & \rightarrow & N_*^\mu/t^m N_*^\mu \\
\downarrow & & \downarrow \\
S_*(V^{-\lambda}, t^n V^{-\lambda}) & \rightarrow & S_*(V^{-\mu}, t^m V^{-\mu})
\end{array}
$$

(here the vertical arrows stand for the equivariant Morse equivalences, and the horizontal arrows are induced by the inclusion $V^{-\lambda} \subset V^{-\mu}$).

**Proof.** See [18], Lemma 3.8.

Now we can proceed to the description of the canonical chain equivalence between the Novikov complex and the completed singular chain complex. For $\lambda \in \mathbb{R}$ put

$$S_*^\lambda = S_*(V^{-\lambda}).$$

Then $S_*^\lambda$ is a chain complex of free $P$-modules. Let

$$\hat{S}_\lambda = \lim_{\leftarrow} S_*^\lambda/t^n S_*^\lambda$$

be the completion of this module, then the natural inclusion

$$S_*^\lambda \otimes \hat{P} \subset \hat{S}_\lambda$$

is a homotopy equivalence. Note also that the module $S_*^\lambda \otimes \hat{P}$ is identified in a natural way with the submodule of all the elements of $S_*^\lambda(M) \otimes \mathcal{L}$ with support below $\lambda$.

**Definition 4.8.** A chain map

$$A_* : N_*(f, v) \to S_*(M) \otimes \mathcal{L}$$
is called compatible with the equivariant Morse equivalences if there is \( \lambda \) such that
\[
A_*(N^\lambda_*) \subset S^\lambda_* \otimes \hat{P}
\]
and for every positive integer \( n \) the quotient chain map
\[
A_*/t^n : N^\lambda_*/t^nN^\lambda_* \to S^\lambda_*/t^nS^\lambda_*
\]
is homotopic over \( P_n \) to the equivariant Morse equivalence \( E_n \).

**Theorem 4.9.** There is a homotopy unique chain equivalence

\[
\mathcal{E}_* : N_*(f,v) \to S_*(\bar{M}) \otimes \hat{T}
\]
compatible with the equivariant Morse equivalences.

**Proof.** 1. **Existence.** Let \( \lambda \) be any regular value of \( F \). As it follows from Proposition 4.7 for every positive \( n \) the following diagram is chain homotopy commutative

\[
\begin{array}{ccc}
N^\lambda_*/t^nN^\lambda_* & \xrightarrow{E_n} & N^\lambda_*/t^{n+1}N^\lambda_* \\
\downarrow & & \downarrow \varepsilon_{n+1} \\
S_*(V^-_\lambda, t^nV^-_\lambda) & \xrightarrow{E_*} & S_*(V^-_\lambda, t^{n+1}V^-_\lambda)
\end{array}
\]

Then one can prove that there is a chain map
\[
\hat{E}_* : N_*(f,v) = \lim_{\leftarrow} N^\lambda_*/t^nN^\lambda_* \to \lim_{\leftarrow} S_*(V^-_\lambda)/t^nS_*(V^-_\lambda) = \hat{S}_*(V^-_\lambda)
\]
such that for every \( n \) the quotient map \( \hat{E}_*/t^n \) is homotopic to \( E_n \) (see [11], §3 B). The composition
\[
N_*(f,v) = N^\lambda_* \otimes \hat{T} \xrightarrow{\hat{E}_* \otimes \text{id}} \hat{S}_*(V^-_\lambda) \otimes \hat{T} \xrightarrow{\sim} S_*(V^-_\lambda \otimes \hat{T} = S_*(\bar{M}) \otimes \hat{T}
\]
is then a chain equivalence compatible with the equivariant Morse equivalences.

2. **Uniqueness.** Let \( \mathcal{E}'_* : N_* \to \mathcal{S}_* \) be another chain map compatible with the equivariant Morse equivalences, and let \( \mu \) be the corresponding regular value of \( F \). Let
\[
A = \mathcal{E}_* | N^\lambda_* , \quad A' = \mathcal{E}'_* | N^\mu_-
\]
Assuming that \( \lambda \leq \mu \), consider the following diagram:

\[
\begin{array}{ccc}
N^\lambda_*/t^nN^\mu_* & \xrightarrow{A/t^n} & N^\mu_*/t^nN^\mu_* \\
\downarrow & & \downarrow \varepsilon \equiv \varepsilon \\
S^\lambda_*/t^nS^\mu_* & \xrightarrow{A'/t^n} & S^\mu_*/t^nS^\mu_*
\end{array}
\]

where the both horizontal arrows are induced by the inclusion \( V^-_\lambda \subset V^-_\mu \). The vertical arrows are by the hypotheses homotopic to the equivariant Morse equivalences, so the diagram is identical with the diagram (33) (with \( n = m \)) and therefore (34) is homotopy commutative. Therefore the compositions

\[
N^\lambda_* \xrightarrow{A} S^\lambda_* \xrightarrow{A} S^\mu_*
\]

and

\[
N^\lambda_* \xrightarrow{A'} N^\mu_* \xrightarrow{A'} S^\mu_*
\]

have the property that their \( t^n \)-quotients are chain homotopic for every \( n \). The proof is then completed by the following lemma ([18], Prop. 2.8).
Lemma 4.10. Let $A_*, B_*$ be chain complexes of $\hat{P}$-modules, which are homotopy equivalent to finitely generated free complexes. Let $\psi, \psi' : A_* \to B_*$ be chain maps. Assume that for every $n$ the quotient chain maps $\psi/t^n, \psi'/t^n : A_* / t^n A_* \to B_* / t^n B_*$ are chain homotopic. Then $\psi$ and $\psi'$ are chain homotopic.

\[ \square \]

Remark 4.11. Replacing the singular chain complex by the simplicial chain complex $\Delta_*(\bar{M})$, associated to a $C^\infty$ triangulation of $\bar{M}$, we obtain a chain equivalence

\[ \mathcal{N}_*(f, v) \xrightarrow{\sim} \Delta_*(\bar{M}) \otimes \mathbb{L}. \]

In [23] D. Schütz gives an explicit formula for the inverse of this chain equivalence (in the case when $v$ is transverse):

\[ (35) \quad \xi(\sigma) = \sum_q (\sigma \iff D(q, -v)) \]

where $\sigma$ stands for a simplex of the triangulation of $\bar{M}$, $\dim \sigma = k$, the summation in the right hand side is over all the critical points of $F : \bar{M} \to \mathbb{R}$ of index $k$, and $\iff$ is the algebraic intersection number. The formula (35) is valid in the assumption that the simplices of the triangulation are transverse to the ascending discs of all critical points of $F$. It is not difficult to show that the formula (35) remains valid if the $f$-gradient $v$ is almost transverse.

4.3. Functorial properties. In this subsection we shall study the functorial properties of the Novikov complex and the canonical chain equivalence $\mathcal{E}_*$. Let $f_1 : M_1 \to S^1, f_2 : M_2 \to S^1$ be circle-valued Morse maps (where $M_i$ are closed manifolds). Let $v_i$ be oriented almost transverse $f_i$-gradients (where $i = 1, 2$). Let $C^\infty_0(M_1, M_2)$ denote the set of all $C^\infty$ maps $A$ such that

\[ (36) \quad f_2 \circ A \sim f_1. \]

This is an open set in $C^\infty(M_1, M_2)$, and we shall assume that it is not empty. Let $A \in C^\infty_0(M_1, M_2)$. We impose the following condition on $A$:

\[ (37) \quad \left. A|D(p, v_1) : D(p, v_1) \to M_2 \text{ is transverse to } D(q, -v_2) \right| \text{ for every } p \in S(f_1), q \in S(f_2). \]

A routine argument from transversality theory proves the next proposition.

Proposition 4.12. The set of the maps $A \in C^\infty_0(M_1, M_2)$ satisfying (37) is residual in $C^\infty_0(M_1, M_2)$.

\[ \square \]

Choose and fix a lift $\hat{A} : \hat{M}_1 \to \hat{M}_2$ of the map $A$ to the infinite cyclic coverings (such lift exists since $f_2 \circ A \sim f_1$). Let $F_1, F_2$ be any lifts of the functions $f_1$, resp. $f_2$ to the coverings $\hat{M}_1, \hat{M}_2$. Set $\dim M_1 = n_1$, $\dim M_2 = n_2$. Let $p \in S_k(F_1), q \in S_{n_2-k}(F_2)$. It is easy to see that if the map $A$ satisfies (37), then the set

\[ T(p, q) = D(p, \bar{v}_1) \cap \hat{A}^{-1}(D(q, -\bar{v}_2)) \]

is finite. Therefore the algebraic intersection index

\[ N(p, q; \hat{A}) = \hat{A}(D(p, \bar{v}_1)) \iff D(q, -\bar{v}_2) \in \mathbb{Z} \]
is defined. Observe that $N(t^k p, t^k q; \bar{A}) = N(p, q; \bar{A})$. For $p \in S_k(F_1)$ put

$$(38) \quad \bar{A}_t(p) = \sum q N(p, q; \bar{A})q$$

(where the summation is over the set of all critical points of $F_2$ of index $k$). It is clear that the expression in the right hand side of (38) is an element of $N_k(f_2, v_2)$, and the formula (38) defines a homomorphism of graded $L$-modules

$$\bar{A}_t : N_*(f_1, v_1) \to N_*(f_2, v_2).$$

**Theorem 4.13.** The graded homomorphism $\bar{A}_t$ is a chain map, and the following diagram is chain homotopy commutative:

$$(39) \quad N_*(f_1, v_1) \xrightarrow{\bar{A}_t} N_*(f_2, v_2)$$

$$(\mathcal{E}_*^{(1)} \downarrow \downarrow \mathcal{E}_*^{(2)})$$

$$(S_*(\bar{M}_1) \otimes_L N_*, \bar{A}_t) \xrightarrow{\bar{A}_*} S_*(\bar{M}_2) \otimes_L N_*$$

(57) where $E_*^{(1)}$ and $E_*^{(2)}$ are the canonical chain equivalences.

**Proof.** Let $\lambda$ be a regular value of $F_1$, and $\mu$ be a regular value of $F_2$. We have the Novikov complexes

$$N_*^{(1)} = N_*^{\lambda}(f_1, v_1), \quad N_*^{(2)} = N_*^{\mu}(f_2, v_2).$$

Put $V^{-}_\lambda = F^{-1}_1([-\infty, \lambda]), \quad U^{-}_\mu = F^{-1}_2([-\infty, \mu])$.

Choose $\lambda$ and $\mu$ in such a way, that

$$\bar{A}_t(V^{-}_\lambda) \subset U^{-}_\mu, \quad \text{so that} \quad \bar{A}_t(N_*^{(1)}) \subset N_*^{(2)}.$$

Consider the following diagram (where $N$ is any positive integer):

$$(40) \quad N_*^{(1)} / t^N N_*^{(1)} \xrightarrow{\bar{A}_t / t^N} N_*^{(2)} / t^N N_*^{(2)}$$

$$(\mathcal{E}_N^{(1)} \downarrow \downarrow \mathcal{E}_N^{(2)})$$

$$(S_*(V^{-}_\lambda, V^{-}_\lambda-N) \xrightarrow{\bar{A}_*} S_*(U^{-}_\mu, U^{-}_\mu-N))$$

(Here $\mathcal{E}_N^{(1)}$, $\mathcal{E}_N^{(2)}$ are the equivariant Morse equivalences). To prove our theorem it suffices to show that the upper horizontal arrow in this diagram is a chain map, and that the diagram is chain homotopy commutative. Pick any $t$-ordered Morse functions on the cobordisms

$$W_{(1,N)} = F^{-1}_1([-N, \lambda]), \quad W_{(2,N)} = F^{-1}_2([\mu - N, \mu])$$

and let $(X^{(k)}, V^{-}_{\lambda-N}), (Y^{(k)}, U^{-}_{\mu-N})$ be the corresponding filtrations of the pairs $(V^{-}_\lambda, V^{-}_{\lambda-N})$, resp. $(U^{-}_\mu, U^{-}_{\mu-N})$. Let us first consider a particular case when the map $\bar{A}$ conserves these filtrations, that is

$$\bar{A}(X^{(k)}, V^{-}_{\lambda-N}) \subset (Y^{(k)}, U^{-}_{\mu-N}).$$
The adjoint complexes of these filtrations are isomorphic to the chain complex \( \mathcal{N}^{(1)}/t^N \mathcal{N}^{(1)} \) resp. \( \mathcal{N}^{(2)}/t^N \mathcal{N}^{(2)} \) via an isomorphism which preserves the bases. The map \( \bar{A} \) induces a chain map
\[
A_* : \mathcal{N}^{(1)}/t^N \mathcal{N}^{(1)} \to \mathcal{N}^{(2)}/t^N \mathcal{N}^{(2)}
\]
of the adjoint complexes. If we replace the map \( \bar{A}/t^n \) by \( A_* \) in the diagram (40) the resulting diagram will be homotopy commutative (by functoriality).

**Lemma 4.14.** \( \bar{A}/t^n = A_* \).

**Proof.** It suffices to check that for every critical point \( p \) of \( f_1 \) belonging to \( V_N^- \setminus V_N^- \) the fundamental class \( \Delta(p,v_1) \) is sent to the same element by \( \bar{A}_1 \) and \( A_2 \). The map \( A_* \) sends this fundamental class to the fundamental class of the manifold \( \bar{A}(D(p,\bar{v}_1)) \) in the pair \( (Y(k), Y(k-1)) \). The set \( Z = Y(k) \setminus \text{Int} \ Y(k-1) \) is a cobordism, endowed with a Morse function \( F_2|Z \), which has only critical points of indices \( k \). A standard computation shows that the \( \bar{A} \)-image of the fundamental class of \( D(p,\bar{v}_1) \) equals to
\[
\sum_q \left( \bar{A}(D(p,\bar{v}_1)) \Delta(q,-v_2) \right) \Delta(q,v_2)
\]
where \( q \in S_k(F_2) \cap Z \), and our assertion follows. \( \square \)

Thus we have proved the homotopy commutativity of the diagram (40) in the case when (41) holds. Let us now proceed to the general case. Let \( B : M_1 \to M_2 \) be the map defined by
\[
B = \Phi(T,-v_2) \circ A,
\]
(where \( T > 0 \)). Lift it to the map of coverings as follows:
\[
\bar{B} = \Phi(T,-\bar{v}_2) \circ \bar{A}
\]
Observe that the map \( B \) satisfies the condition (37) and that
\[
\bar{A}(D(p,-v_1)) \Delta(q,-v_2) = \bar{B}(D(p,-v_1)) \Delta(q,v_2).
\]
If \( T > 0 \) is sufficiently large, the map \( \bar{B} \) satisfies condition (41) and the proof of our theorem is now over. \( \square \)

5. **Applications and further developments**

Here we outline some applications of the techniques developed in the preceding sections.

**5.1. On the \( C^0 \)-stability in Novikov complex.** It is most likely that the Novikov complex \( \mathcal{N}_*(f,v) \) is not stable with respect to the \( C^0 \)-small perturbations of \( v \) (although I know no explicit examples). Still for every Morse function \( f : M \to S^1 \) there is a class of almost transverse gradients for which the Novikov complex is \( C^0 \)-stable; namely the class of gradients satisfying the condition (\( \mathcal{C} \)), introduced in [13], [14].

Without reproducing here the definition we shall just give a geometric description of this class. Let \( f : M \to S^1 \) be a Morse map, \( v \) be an \( f \)-gradient. Let \( V = f^{-1}(\lambda) \) be a regular level surface of \( f \). The map which associates to \( x \in V \) the point of the second intersection with \( V \) of the \((-v)\)-trajectory starting at \( x \) is a partially defined smooth map \( (-v)^- : V \to V \). The condition \( (\mathcal{C}) \) requires an existence of a certain handle-like filtration \( V^{(k)} \) of the manifold \( V \) such that the map \((-v)^- \)
gives rise to a family of continuous maps $V^{(k)}/V^{(k-1)} \to V^{(k)}/V^{(k-1)}$ between the successive quotients of the filtration. See the definition in [14], p. 107, (or in [17], page 317, where the condition $(\mathbb{C}C)$ was denoted $(\mathbb{C}C')$). It is proved in [14], that the set of $f$-gradients satisfying $(\mathbb{C}C)$ is open and dense with respect to $C^0$-topology in the set of all $C^\infty$-gradients. It is also proved there (Theorem 5.7 of [14]) that for every transverse $f$-gradient $\nu$ satisfying $(\mathbb{C}C)$ and for every open neighborhood $U$ of $S(f)$ there is $\delta > 0$ such that for every transverse $f$-gradient $\omega$ with $||\omega - \nu|| < \delta$ the Novikov complexes $N_*(f,\nu)$ and $N_*(f,\omega)$ are basis-preserving isomorphic (we imply here that $\nu$ and $\omega$ are similarly oriented).

It is easy to check that any gradient $\nu$ satisfying $(\mathbb{C}C)$ is almost transverse, so we can apply the results of Subsection 4.1 to construct the Novikov complex $N_*(f,\nu)$. Using the methods of the present work (Subsection 3.2) it is not difficult to extend the Theorem 5.7 of [14] as follows:

**Theorem 5.1.** Let $f : M \to S^1$ be a Morse function, let $\nu$ be an oriented $f$-gradient satisfying $(\mathbb{C}C)$. Then there is $\delta > 0$ such that for every oriented $f$-gradient $\omega$ with $||\omega - \nu|| < \delta$ the Novikov complexes $N_*(f,\nu)$ and $N_*(f,\omega)$ are isomorphic via a basis-preserving isomorphism, if $\nu$ and $\omega$ are similarly oriented.

The details of the proof of the theorem will be published elsewhere.

5.2. **Lefschetz zeta functions of almost transverse gradients.** There is a remarkable relation between the simple homotopy type of the Novikov complex and the Lefschetz zeta function, counting the closed orbits of the gradient flow. This relation was discovered by M.Hutchings and Y.J.Lee in the work [5]. In this paper they established a formula which says that in the case when the Novikov complex is acyclic, its Reidemeister torsion equals the Lefschetz zeta function of the gradient flow. The author proved ([15]) that in the general case, when the Novikov complex is not acyclic, the Lefschetz zeta function of the gradient flow equals the Whitehead torsion of the canonical chain equivalence between the Novikov complex and the completed simplicial chain complex of the infinite cyclic covering. (In the paper [15] I considered only the case of $C^0$-generic gradients, and this restriction was removed in the paper [18], using an approximation argument.) These results were generalized to the case of Lefschetz zeta functions with values in the completions of group rings by the author [17], using the $K$-theoretic techniques developed by A.Ranicki [21], and A.Ranicki and the author [16]. D.Schütz ([22], [23]) generalized the results of [17] to the case of irrational forms (see the papers [22], [23]), using the techniques of 1-parameter fixed point theory by R.Geoghegan and A.Nicas ([3], [4]). In all these works only the case of transverse gradients was considered.

In this subsection we outline a generalization of these results to the case of almost transverse gradients. Set $\text{Wh}(\mathcal{L}) = \mathbb{K}_1(\mathcal{L})/T$, where $T$ is the image in $\mathbb{K}_1(\mathcal{L})$ of the group $\{\pm t^n| n \in \mathbb{Z}\} \subset \mathcal{L}$; it is not difficult to check that $\text{Wh}(\mathcal{L})$ is isomorphic to the multiplicative group $W$ of all power series with integer coefficients with the free term equal to 1. Let $f : M \to S^1$ be a circle-valued Morse function and $\nu$ be an oriented almost transverse $f$-gradient. Choose any $C^1$-triangulation of $M$ and lift it to a $\mathbb{Z}$-invariant triangulation of $M$. Let $\Delta_*(M)$ be the corresponding simplicial chain complex. The composition of the homotopy equivalence $\mathcal{E}_*$ from Theorem 4.9 with the natural chain homotopy equivalence $S_*(M) \otimes \mathcal{L} \xrightarrow{\sim} \Delta_*(M) \otimes \mathcal{L}$ is a
chain equivalence
\[ \mathcal{E}_*: N_*(f, v) \xrightarrow{\sim} \Delta_*(\bar{M}) \otimes L \]
of two finitely generated based chain complexes over \( L \); the image of the Whitehead torsion of this chain equivalence in the group \( W \) will be denoted \( w(f, v) \).

Now to zeta functions. Choose a regular value \( \lambda \) of the lift \( F: \bar{M} \to \mathbb{R} \) of \( f \). The composition of gradient descent from \( V_\lambda = F^{-1}(\lambda) \) to \( V_{\lambda - 1} \) with the map \( t^{-1}: V_{\lambda - 1} \to V_{\lambda} \) determines a (partially defined) diffeomorphism \( \Phi \) of \( V_{\lambda} \) to itself.

Using the almost transversality property it is not difficult to show that for any \( n \in \mathbb{N} \) the set of fixed points of the \( n \)-th iteration \( \Phi^n: V_{\lambda} \to V_{\lambda - n} \) is compact. Let \( L(\Phi^n) \) denote its index, and set
\[
\zeta_L(-v) = \exp \left( \sum_{n \geq 1} \frac{L(\Phi^n)}{n} t^n \right) \in \mathbb{Z}[[t]].
\]

It is easy to check that the power series \( \zeta_L(-v) \) does not depend on the particular choice of the regular value \( \lambda \).

**Theorem 5.2.** For every oriented almost transverse \( f \)-gradient \( v \) we have
\[
w(f, v) = (\zeta_L(-v))^{-1}.
\]

The proof of the theorem follows the lines of the proof of the main theorem of [18]: first we check the theorem for gradients satisfying condition (CC), and then apply a perturbation argument.

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