The Polyakov relation for the sphere and higher genus surfaces

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Abstract

The Polyakov relation, which in the sphere topology gives the changes of the Liouville action under the variation of the position of the sources, is also related in the case of higher genus to the dependence of the action on the moduli of the surface. We write and prove such a relation for genus 1 and for all hyperelliptic surfaces.

Keywords: Liouville theory, higher genus, accessory parameters

1. Introduction

On the sphere topology the Polyakov relation connects the dependence of the action on the position of the sources with the accessory parameters of the related Riemann-Hilbert problem. Such a relation was originally conjectured by Polyakov exploiting the semiclassical limit of quantum operator product expansion [1].

The relation plays a key role in several fields related to Liouville theory like the Hamiltonian formulation of 2 + 1-dimensional gravity [2–4] and the study the conformal block expansion of the quantum correlation functions [5–8]. The accessory parameters appear in the generalized monodromy problem [9, 10], and also in connection with the Nekrasov-Shatashvili limit of super Yang-Mills theory [6, 11] and the AGT conjecture [9, 12].

In the simplest case of the sphere topology the Polyakov relation tells us that

$$\frac{\partial S}{\partial z_K} = -\beta_K / 2$$

where $S$ is the on-shell Liouville action, $z_K$ is the position of the source and $\beta_K$ is the related accessory parameter.

The proof of Polyakov relation in the presence of only parabolic singularities was given in [14] using fuchsian mapping techniques. For the sphere in the presence of both parabolic and elliptic singularities, using the potential theory technique, the proof was given in [2, 3] and in [15].

In this paper we shall extend these kinds of relation to higher genus surfaces, showing that such a relation takes a different meaning: not only does it relate the change of the action
under the motion of the sources, but also the change of the action under the change of the moduli of the surface. Here we give the proof of the relation in the case of the torus and in the case of all hyperelliptic surfaces with an arbitrary number of sources.

In the proof of the Polyakov relation it is essential to exploit the property of the accessory parameters to be real-analytic functions of the position of the singularities and of the moduli of the surface. This is not a trivial problem. In [16] it was proven that for the sphere the real-analytic dependence of the accessory parameters on the position of the singularities holds everywhere in the restricted case of parabolic and elliptic singularities of finite order; these are the singularities with strength \( \eta = (1 - 1/n)/2 \).

In [2, 3] it was proven for the sphere that the accessory parameters are real-analytic functions of the positions of the singularities, and also on the strength \( \eta \) of the singularities in an everywhere dense open set for any collection of elliptic and/or parabolic singularities without the restriction on the elliptic singularities having to be of finite order. For the torus with one source a much stronger result was proven in [17], i.e. that the accessory parameter is a real-analytic function of the coupling and of the modulus everywhere except for a zero measure set.

The proofs of the real-analyticity that we shall give in sections 5, 6 rely heavily on the existence and the uniqueness of the solution of the Liouville equation given the strength, the positions of the singularities and the moduli of the surface. Starting from the papers of Picard [18], which apply only to elliptic singularities, there appeared various proofs of the existence and uniqueness of the solution of the Liouville equation [19–22]. The existence proofs are somewhat lengthy and technical; on the other hand, the uniqueness proof is rather straightforward.

The proof of the almost-everywhere real-analytic property of the accessory parameter for the sphere with four sources and for the torus with one source is obtained by applying results and techniques related to analytic varieties [23–25], even though here we are in the presence of a problem of real-analytic varieties [26]. This is dealt with using the techniques of polarization, i.e. by doubling in the intermediate steps of the proof the number of complex variables.

The paper is structured as follows. In section 2 we give the general discussion of the problem. In section 3 we give the action on higher genus surfaces in two different coordinate systems. In section 4 we give the auxiliary differential equation for the torus and all hyperelliptic surfaces with an arbitrary number of sources. In section 5 we give the counting of the degrees of freedom of the parameters appearing in the problem and write the implicit monodromy relations to which the accessory parameters are subject. In section 6 we give a shortened versions of the proof of the real-analyticity property of the accessory parameters in an everywhere dense set in the general case and in the case of the torus or of the four-point function on the sphere, we give a shortened proof of the real-analyticity of the accessory parameter everywhere except for a zero measure set.

In section 7, exploiting the results of the previous sections, we give the proof of the Polyakov relation for the sphere, the torus and all hyperelliptic surfaces. In section 8 we give a short discussion of the results of the paper.

In the appendix, using results obtained in [23], we derive the analytic properties of zeros of Weierstrass polynomials which are required in section 6.
2. General discussion

First we outline the semiclassical argument which leads to the Polyakov relation. It is also useful to lay down the notation and fix the normalization of the Liouville action which we shall choose as in [5].

The Liouville action, boundary terms apart, is given by

$$ A_L = \frac{1}{\pi} \int (\partial_z \phi \partial_{\bar{z}} \phi + \pi i e^{2b\phi}) \, dz \wedge d\bar{z}^i \frac{i}{2} $$

and with $z = x + iy$. The holomorphic energy momentum tensor is

$$ T_{zz} = T(z) = - (\partial_z \phi)^2 + Q \partial_{\bar{z}}^2 \phi, \quad Q = \frac{1}{b} + b $$

and for the vertex functions and their dimensions we have

$$ V_\alpha(w_1) \ldots V_\alpha(w_n) = \int V_\alpha(w_1) \ldots V_\alpha(w_n) e^{-\Delta_0 \phi} D[\phi] . $$

From the operator product expansion we have

$$ T(z)V_\alpha(w) = \frac{\Delta_0}{(z-w)^2} V_\alpha(w) + \frac{1}{z-w} \partial_{\bar{z}} V_\alpha(w) + \ldots $$

To explore the semiclassical limit $b \to 0$ one sets $\varphi = 2b \phi$ and $\alpha = \frac{2}{b}$. The action and the dimension $\Delta_0$ become

$$ A_L[\phi] = \frac{1}{b^2} S_\eta[\varphi], \quad \Delta_0 \approx \frac{1}{b^2} \eta(1 - \eta) $$

where, after performing a constant shift in $\varphi$

$$ S_\eta[\varphi] = \frac{1}{2\pi} \int \left( \frac{1}{2} \partial_z \varphi \partial_{\bar{z}} \varphi + e^\varphi \right) \, dz \wedge d\bar{z}^i \frac{i}{2} $$

and the energy momentum tensor becomes

$$ T(z) \approx \frac{1}{b^2} \left[ \frac{1}{2} \partial_z^2 \varphi - \frac{1}{4} (\partial_z \varphi)^2 \right] = - \frac{1}{b^2} e^\varphi \partial_z^2 e^{-\varphi} . $$

Then in the semiclassical limit $b \to 0$

$$ \langle V_\alpha(w_1) \ldots V_\alpha(w_n) \rangle = c \; e^{-S_{\text{cl}}[\alpha]} $$

where $S_{\text{cl}}(w_1,\ldots,w_n)$ is the classical action computed in presence of the sources of strength $\eta_i$ at the points $w_i$. As in the semiclassical limit, the field is frozen on the classical solution we also have

$$ \langle T(z)V_\alpha(w_1)\ldots \rangle = \frac{c}{b^2} \left( \frac{1}{2} \partial_z^2 \varphi(z) - \frac{1}{4} (\partial_z \varphi(z))^2 \right) e^{-\frac{S_{\text{cl}}[\alpha]}{b^2}} $$

where

$$ \frac{1}{2} \partial_z^2 \varphi(z) - \frac{1}{4} (\partial_z \varphi(z))^2 = Q(z) = \sum_i \frac{1 - \lambda_i^2}{4(z-w_i)^2} + \frac{\beta_i}{2(z-w_i)} . $$
Comparing with the result obtained using the operator product expansion (5) we have

\[ \eta_i(1 - \eta) = \frac{1}{4} - \frac{\lambda_1^2}{4}, \quad \frac{\partial S_1(w_1, \ldots, w_n)}{\partial w_1} = -\frac{\beta_i}{2} \]  

As discussed in the introduction, proofs of (12) have been given in [2, 3, 14, 15] for the topology of the sphere.

In the case of the torus we have two simple representations of the manifold. One is the quotient of the complex \( z \)-plane by the group of discrete translations with generators \( 2\omega_1, 2\omega_2 \) and the other is the Weierstrass representation via the variable \( u = \wp(z) \). One can use the modulus \( \tau = \omega_2/\omega_1 \) as a parameter classifying the torus, as done in [17], but both for the torus and for higher genus it will be simpler to use the position of the branch points of the two sheet representation of the elliptic or hyperelliptic surface.

For \( g = 2 \) the analogue of the \( \wp \) function was given in [27]. For an approach to the \( g = 3 \) problem see [28].

On the other hand, we know that for any genus \( g \geq 2 \) we can represent the Riemann surface as the quotient of the \( z \)-upper half-plane by a fuchsian group i.e. by a standard fundamental curvilinear polygon [29]. Elliptic and hyperelliptic surfaces of any genus can be represented by a two sheet cut \( u \)-plane. Even though the transformation between the two representations is not known explicitly except for \( g = 1 \) and \( g = 2 \), in section 3 we find general properties of the Jacobian relating the \( z \)-representation with the two sheet \( u \)-representation of the hyperelliptic surface. This will be sufficient to relate the actions in the two representations.

Accessory parameters appear through the auxiliary ordinary differential equation associated with the Liouville problem. For elliptic and hyperelliptic surfaces they are in number \( n + 2g + 1 \) for \( g \leq 2 \), \( n \) being the number of sources and \( g \) the genus of the surface, and for \( g \geq 3 \) they are in number \( 3g + n - 1 \). However among them we have relations, the fuchsian relations, with the final result that for any \( g \) we have \( n + 3g - 3 \) independent accessory parameters. This is also true in the general case of non-hyperelliptic surfaces [16].

As mentioned, a hyperelliptic surface can be represented in several forms. It turns out that the simplest choice is to use for the moduli the locations \( u_l \) of the branch points of the two sheet representation of the manifold; for this choice the Polyakov relation takes the form

\[ \frac{\partial S}{\partial u_l} = -B_l \]  

which is similar to (12). \( B_l \) is the accessory parameter at the branch point \( u_l \).

In the process of taking the derivative of the classical action with respect to the locations of the sources \( u_K \) or to the moduli \( u_l \), one has to keep in mind that the classical solutions depend on such positions and, through the auxiliary equation, also on the values of the \( \beta \)'s and of a real weight parameter \( \kappa \) which are fixed by the monodromy conditions. Here is where the real-analyticity of the \( \beta \)'s as functions of the \( u_K, u_l \) enters the problem.

The normalization of the action \( S \) we use in this paper is the one adopted in [5]; it is related to the one used in [17, 30, 31] which we call \( S_f \) by \( S_f = 2\pi S \), and to the one used in [2, 3] and in [14, 15] which we call \( S_{CMS} \) by \( S_{CMS} = 4\pi S \).

### 3. The action on higher genus surfaces

For completeness we start by recalling the action on a surface with the topology of the sphere:

\[ g = 0. \]
The sphere is described by $C \cup \infty$ and the action is given by

\[
S = \frac{1}{2\pi} \int_{D_1} \left( \frac{1}{2} \partial \phi \wedge \bar{\partial} \phi + e^\phi dz \wedge d\bar{z} \right) \frac{i}{2} - \frac{\eta_k}{4\pi i} \oint_{z_K} \phi \left( \frac{dz}{z - z_K} - \frac{d\bar{z}}{\bar{z} - \bar{z}_K} \right) - \eta_k^2 \log \epsilon_k^2 + \frac{1}{4\pi i} \oint_{R} \phi \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) + \log R^2
\]

in the limit $\epsilon_k \to 0$, $R \to \infty$ where $D_1$ is the disk of radius $R$ in the complex plane from which disks of radius $\epsilon_k$ around $u_K$ have been removed. We use the notation $\partial f \equiv \partial_z f dz$, $\bar{\partial} f \equiv \partial_{\bar{z}} f d\bar{z}$. Variation of such an action, with $f$ satisfying at $z_K$ the boundary conditions

\[
\phi(z) = -2\eta_k \log(z - z_K)(\bar{z} - \bar{z}_K) + r_K
\]

and at $z = \infty$ the boundary condition

\[
\phi(z) = -2\log z \bar{z} + r_{\infty}
\]

where $r_K$, $r_{\infty}$ are bounded continuous functions, gives rise to the Liouville equation

\[
-\partial_z \partial_{\bar{z}} \phi + e^\phi = 0
\]

in $C \setminus \{u_K\}$. We shall write for the solution of Liouville equation

\[
r_K = X_K + o(z - z_K), \quad r_{\infty} = X_{\infty} + o \left( \frac{1}{z} \right)
\]

The $\eta_k$ are subject to the restrictions $\eta_k < \frac{1}{2}$ (local finiteness of the area) and to the topological restriction $\sum_k 2\eta_k > 2(1 - g) = \chi = 2$, where $g$ is the genus and $\chi$ the Euler characteristic.

In the case of parabolic singularities the behavior of the field at the singularities is

\[
\phi(z) = -\log(z - z_p)(\bar{z} - \bar{z}_p) - \log(\log(z - z_p)(\bar{z} - \bar{z}_p))^2 + r_p
\]

and in the action (14) and in the previous topological relation $\eta_k$ has to be replaced by $\frac{1}{2}$. $g = 1$

The torus is described by the quotient of the complex plane by the discrete translation group with generators $2\omega_1$, $2\omega_2$ and the Liouville equation is given by equation (17) with periodic boundary conditions in $z$ and $\phi$ behaving as equation (15, 19) at the singularities and $\sum_k 2\eta_k + \sum_p 1 > 2(1 - g) = \chi = 0$.

In such a $z$-representation the action is given by

\[
S_z = \frac{1}{2\pi} \int_{D_1} \left( \frac{1}{2} \partial \phi \wedge \bar{\partial} \phi + e^\phi dz \wedge d\bar{z} \right) \frac{i}{2} - \frac{\eta_k}{4\pi i} \oint_{z_K} \phi \left( \frac{dz}{z - z_K} - \frac{d\bar{z}}{\bar{z} - \bar{z}_K} \right) - \eta_k^2 \log \epsilon_k^2
\]

where the index $K$ runs on the sources. Due to the periodic boundary conditions on $\phi$ we have no boundary terms. Working with periodic boundary conditions is not very simple. It is useful to go over to the Weierstrass representation of the torus given by the equation

\[
w^2 = 4(v - e_1)(v - e_2)(v - e_3), \quad e_1 + e_2 + e_3 = 0.
\]
To connect to the general hyperelliptic case it is useful to maintain a more general formalism in which
\[ v = u - (u_1 + u_2 + u_3)/3 \]
and
\[ \varphi(z) = v = u - \frac{u_1 + u_2 + u_3}{3}. \]

From the well known differential equation satisfied by \( \varphi(z) \)
\[ (\varphi'(z))^2 = 4(\varphi(z) - e_1)(\varphi(z) - e_2)(\varphi(z) - e_3) \]
we have
\[ J = \frac{dz}{du} = \frac{1}{\sqrt{4(u - u_1)(u - u_2)(u - u_3)}}, \]
\[ z = \int_{u_1}^{u} \frac{du}{\sqrt{4(u - u_1)(u - u_2)(u - u_3)}}. \]

A point \( p \) of the surface is given by the couple of numbers \((u, w)\) where \( w \) satisfies equation (22) and thus it can assume two values.

For the torus we have for the half-periods
\[ \omega_1 = \frac{1}{\sqrt{u_1 - u_2}} K \left( \frac{|u_3 - u_2|}{u_1 - u_2} \right) \]
\[ \omega_2 = \frac{i}{\sqrt{u_1 - u_2}} K \left( \frac{|u_1 - u_3|}{u_1 - u_2} \right) \]
and the modulus is \( \tau = \omega_2/\omega_1 \). In studying the dependence of the action on the moduli, one can use for the torus \( \tau \) as done in [17, 30, 31]. On the other hand both for the torus and the general hyperelliptic surface it is simpler to classify the surfaces in terms of the positions of the branch points of the map from the fundamental standard polygon to the two sheeted \( u \)-plane.

Due to the invariance of the area i.e. \( e^\phi du \wedge d\bar{u} = e^\phi dz \wedge d\bar{z} \), in the \( u \)-representation the field is given by
\[ \varphi(u) = \phi(z) + \log J\bar{J}, \quad J = \frac{dz}{du}. \]

From the behavior of the field \( \phi(z) \) at the sources
\[ \phi(z) = -2\eta_K \log(z - z_K)(\bar{z} - \bar{z}_K) + X_K + o(z - z_K) \]
we have that the behavior of \( \varphi(u) \) at the sources is
\[ \varphi(u) = -2\eta_K \log(u - u_K)(\bar{u} - \bar{u}_K) + X^u_K + o(u - u_K) \]
with
\[ X^u_K = X_K - (1 - 2\eta_K)\log|4(u_K - u_1)(u_K - u_2)(u_K - u_3)|. \]
At \( u = \infty \) being \( \phi(0) \) finite we have
\[
\varphi(u) = \phi(0) - \log 4 - \frac{3}{2} \log uu + o\left(\frac{1}{u}\right).
\] (33)

In the following we shall use the convention to denote the dynamical singularities, i.e. the sources by \( u_K \) with upper case index, while the kinematical singularities describing the Riemann surface in the \( u \)-representation will be denoted by \( u_l \), with lower case index.

The action in the \( u \)-representation \( S_u \) taking into account the behavior \((33)\) is given by
\[
S_u = \frac{1}{2\pi} \int_{\mathcal{D}_u} \left( \frac{1}{2} \partial \varphi \wedge \partial \varphi + e^\varphi du \wedge d\bar{a} \right) + \frac{\eta_k}{4\pi^2} f_{\varepsilon_k} \frac{du}{u - u_K} \frac{d\bar{a}}{\bar{a} - \bar{a}_K} - \frac{\eta_k^2}{8} \log \varepsilon_k^2
\]
\[
- \frac{1}{16\pi^2} f_{\varepsilon_l} \frac{du}{u - u_l} \frac{d\bar{a}}{\bar{a} - \bar{a}_l} - \frac{1}{8} \log \varepsilon_l^2
\]
\[
+ \frac{1}{8\pi^2} f_{\varepsilon_l} \frac{du}{u} \frac{d\bar{a}}{\bar{a}} + \frac{1}{2} \left( \frac{3}{2} \right)^2 \log R_u^2
\] (34)

where \( \mathcal{D}_u \) is the double sheeted plane and the index \( d \) on the contour integrals means that a double turn has to be taken around the kinematical singularities \( u_l, l = 1, 2, 3 \) and at \( \infty \) in order to come back to the starting point.

For the actions \( S_z \) and \( S_u \) the general relations \([2, 3, 5]\) hold
\[
\frac{\partial S_z}{\partial \eta_k} = -X_k, \quad \frac{\partial S_u}{\partial \eta_k} = -X_k^u
\] (35)

which are easily proven from the form \((14, 20, 34)\) of the actions.

The relation between the two actions is obtained by replacing in \( S_z, \phi \) in terms of \( \varphi \) as given by equation \((29)\). We find
\[
S_z = S_u - \sum_k \eta_k (1 - \eta_k) \log [4|u_K - u_l||u_K - u_2||u_K - u_3|]
\]
\[
- \frac{1}{2} \log [|u_1 - u_2||u_2 - u_3||u_3 - u_1|].
\] (36)

We notice that equation \((36)\) is consistent with the general relation \((35)\) combined with \((32)\).

The difference between the two actions is of a dynamical nature as it involves the source strengths \( \eta_k \).

In the above equation one recognizes the classical dimensions of the sources \( \eta_k (1 - \eta_k) \) multiplied by the logarithm of the Jacobian of the transformation.

We know that a compact Riemann surface of genus \( g \geq 2 \) can be represented by a standard fundamental domain of the complex upper half-plane. Such a domain is a curvilinear \( 4g \)-gon which is the analog of the parallelogram \( T \) belonging to \( C \) which describes the torus. Surfaces of genus \( g = 2 \) are all hyperelliptic. For these, \( g = 2 \), Komori \([27]\) gave an explicit representation in terms of the analogue of the Weierstrass function \( \wp(z) \), which we shall call \( h(z) \), as the ratio of two 6-forms
\[
h(z) = \frac{f(z)}{g(z)}
\] (37)
where \( f(z) \) and \( g(z) \) are explicitly written in terms of Poincaré series on a fuchsian group \( G \). Then we have the representation

\[
 w^2 = 4(u - h(z_1))(u - h(z_2))(u - h(z_3))(u - h(z_4)) (u - h(z_5)) \tag{38}
\]

with \( u = h(z) \).

\[
 f(z) = \sum_{\gamma \in G} \frac{1}{\gamma z - p_0} P(\gamma z) \gamma'(z)^3 \tag{39}
\]

\[
 g(z) = \sum_{\gamma \in G} P(\gamma z) \gamma'(z)^3 \tag{40}
\]

and \( f(z) \) has simple poles on the orbit of the point \( p_0 \). \( P(z) \) is a properly constructed rational function of \( z \) holomorphic in the upper half-plane [27].

In the following we shall enucleate the general features of the transformation between the \( z \) and \( u \) coordinates for hyperelliptic surfaces of any genus. This we be sufficient to relate \( S_z \) with \( S_u \).

The structure of the Jacobian of the transformation

\[
 J = \frac{dz}{du} \tag{41}
\]

can be extracted as follows. The surface is described by

\[
 w^2 = 4(u - u_1) \cdots (u - u_{2g+1}) \tag{42}
\]

\((u, w)\) is a faithful representation of our Riemann surface and thus to each such point there correspond a point in the standard fundamental polygon in the \( z \)-upper-half-plane; \( z \) is a locally conformal (analytic invertible) representation of the Riemann surface.

In a domain around a point of \( M \), described by \((u, w)\) with \( u = u_i \), \( M \) is represented by \((u, w)\) with \( w \) a determination of \( \sqrt[4]{4(u - u_i)} \cdots (u - u_{2g+1}) \). In a domain around the point of \( M \), described by \((u_i, 0)\), \( M \) is faithfully represented by \( w \). In the first case \( z \) is an analytic (locally invertible) function of \( u \), while in the second case we have

\[
 z = z_i = w f_i(w) \tag{43}
\]

with \( f_i \) analytic and \( f_i(0) \neq 0 \) and \( u \) function of \( w \) according to

\[
 u - u_i = \frac{w^2}{4(u - u_i) \cdots (u - u_i) \cdots (u - u_{2g+1})} \tag{44}
\]

where the term in \( \{ \} \) has to be removed.

The Jacobian is given by

\[
 J = \frac{dz}{du} = \frac{dz}{dw} \frac{dw}{du} = (f_i(w) + w f_i'(w)) \frac{(u - u_i) \cdots (u - u_i) \cdots (u - u_{2g+1}) + O(u - u_i)}{\sqrt[4]{(u - u_i) \cdots (u - u_{2g+1})}} \tag{45}
\]

\[
 = 2(f_i(w) + w f_i'(w)) \frac{(u - u_i) \cdots (u - u_i) \cdots (u - u_{2g+1}) + O(u - u_i)}{w}
\]
so that

\[
\log J = -\frac{1}{2} \log(u - u_i) + \log f_j(0) + \frac{1}{2} \log [(u_t - u_i) \ldots (u_l - u_i)] \ldots (u_l - u_{2^p+1})]
+ O(\sqrt{u - u_i})
\]
\[= -\frac{1}{2} \log(u - u_i) + j_i + O(\sqrt{u - u_i})
\]

(46)

where

\[j_i = \log f_j(0) + \frac{1}{2} \log [(u_t - u_i) \ldots (u_l - u_i)] \ldots (u_l - u_{2^p+1})].
\]

(47)

With regard to the fields we have:

at \(u_K\) from

\[\phi(z) = -2\eta_k \log(z - z_K)(\xi - \xi_K) + X_K + o(z - z_K)
\]

(48)

we deduce

\[\varphi(u) = -2\eta_k \log(u - u_K)(\bar{u} - \bar{u}_K) + (1 - 2\eta_k)\log J_K \bar{J}_K + X_K + O(u - u_K)
\]
\[= -2\eta_k \log(u - u_K)(\bar{u} - \bar{u}_K) + X_K^u + O(u - u_K)
\]

(49)

with

\[
X_K^u = X_K + (1 - 2\eta_k)\log J_K \bar{J}_K.
\]

(50)

At \(u_l\) we have

\[\varphi(u) = -\frac{1}{2} \log(u - u_l)(\bar{u} - \bar{u}_l) + j_l + \bar{j}_l + \phi(z_l) + O(\sqrt{u - u_l})
\]
\[= -\frac{1}{2} \log(u - u_l)(\bar{u} - \bar{u}_l) + X_l^u + O(\sqrt{u - u_l})
\]

(51)

with

\[
X_l^u = j_l + \bar{j}_l + \phi(z_l).
\]

(52)

We shall also need the behavior of \(\varphi\) at infinity in \(u\). The local uniformizing variable in the \(u\) cut-plane at infinity is \(v\) with \(v^2 = 1/u\). Then being \(z\) in a neighborhood of \(z_\infty\) (i.e. of the point which is projected to \(u = \infty\)) a regular representation of the manifold we have

\[z - z_\infty = \alpha_\infty v + O(v^2).
\]

(53)

Thus

\[
J = \frac{dz}{du} = \frac{dz}{dv} \frac{dv}{du} = -\frac{\alpha_\infty}{2} u^{-\frac{1}{2}}(1 + O(v))
\]

(54)

and

\[
\log J = -\frac{3}{2} \log u + j_\infty, \quad j_\infty = \log \left(\frac{\alpha_\infty}{2}\right)
\]

(55)

Then as \(\varphi = \phi + \log JJ\) we have

\[\varphi(u) = \phi(z_\infty) - \frac{3}{2} \log uu + j_\infty + \bar{j}_\infty = -\frac{3}{2} \log uu + X_\infty^u, \quad X_\infty^u = \phi(z_\infty) + j_\infty + \bar{j}_\infty.
\]

(56)
Integrating
\[ \partial_u \partial_\nu \varphi = e^\varphi, \quad e^\varphi du \wedge d\bar{u} \frac{i}{2} > 0 \] (57)
we obtain the topological inequality for the source strengths \( \eta_K \), the number of parabolic singularities and the genus \( g \)
\[
0 < \frac{i}{2} \left( -\int_{\partial K} \partial_\nu \varphi - \int_{\partial K} \partial_{\bar{\nu}} \varphi + \int_{\partial K} \partial_{\nu} \partial_{\bar{\nu}} \varphi \right) = \pi \left( \sum_K 2\eta_K - 3 + \sum_p 1 + \sum_{i=1}^{2g+1} 1 \right)
= \pi \left( \sum_K 2\eta_K + \sum_p 1 + 2(g-1) \right). \tag{58}
\]
For \( g \geq 2 \) the compact Riemann surface is represented by the quotient of the upper \( z \)-plane by a Fuchsian group \([29]\). We refer to a standard fundamental polygon \( Dz \). It is a curvilinear polygon with \( 4g \) sides lying in the upper \( z \) plane with all vertices identified. The sides lie in the order \( A_1B_1A_1^{-1}B_1^{-1}...A_gB_g^{-1} \). There exist one and only one element \( \Gamma^A_{j1} \) of the fuchsian group which maps \( A_j \) into \( A_j^{-1} \) and one and only one element \( \Gamma^B_{j1} \) of the fuchsian group which maps \( B_j \) into \( B_j^{-1} \) \([29]\).

The side \( A_j \) is identified with the side \( A_j^{-1} \) and when one runs along the perimeter of the \( 4g \)-gon the image of \( A_j \), \( \Gamma_{j1}A_j = A_j^{-1} \) is traveled in the opposite direction as \( A_j \). Thus the contour \( A_jA_j^{-1} \) is a closed loop on the Riemann surface.

The action in the \( z \)-representation is given by
\[
S_z = \frac{1}{2\pi} \int_{\partial D_z} \left( \frac{1}{2} \partial_\phi \wedge \partial_\phi + e^\phi dz \wedge d\bar{z} \right) \frac{i}{2} - \frac{\eta_K}{4\pi i} \oint_{\partial K} \phi \left( \frac{dz}{z-z_k} - \frac{dz}{\bar{z}-\bar{z}_k} \right) - \eta_K^2 \log \epsilon_k
+ \frac{i}{8\pi} \int_{A_j} \phi (\partial \log s^A_j - \partial \log s^A_{j1}) + \frac{i}{8\pi} \int_{B_j} \phi (\partial \log s^B_j - \partial \log s^B_{j1}). \tag{59}
\]
The one-dimensional integrals in the last line of the above equation are boundary terms and they are present due to the fact that \( \phi \) is not a scalar but a conformal field, i.e. the periodic boundary conditions are on \( e^\phi dz \wedge d\bar{z} \) and not on \( \phi \). The \( s \) are given by \( s = dz/d\bar{z} \). If the transformation \( \Gamma \) which relates two identified sides \( A \) and \( A^{-1} \) is given by
\[
\Gamma(z) = z' = \frac{az + b}{cz + d}
\tag{60}
\]
we have \( s = (cz+d)^2 \) and \( \partial \log s = 2c (dz)/(cz+d) \).

In addition \( D_z \) excludes small circles of radius \( \epsilon_K \) around the sources \( z_k \) and, as an intermediate step, small circles around \( z_k \) the images of the \( u_k \) and around \( \bar{z}_\infty \), the image of \( u = \infty \). The field dependent boundary terms of the last line in (59) are absent for the torus due to the linear nature of the \( \Gamma \) is such a case.

Substituting in the above equation \( \phi = \varphi - \log J \) and using the information on \( J \) derived previously in this section, we obtain the relation between the action \( S_z \) and the action in the \( u \)-representation, \( S_u \)
\[
S_z = S_u + \eta_K (1 - \eta_K) \log(J_K\bar{J}_K) + \frac{1}{4}(\bar{J}_H + \bar{J}_h) - \frac{3}{4}(J_{\infty} + \bar{J}_\infty). \tag{61}
\]
\[-\frac{i}{8\pi} \int_{B_j} \log(J\tilde{j})(\partial \log \tilde{s}_j^A - \partial \log s_j^A) - \frac{i}{8\pi} \int_{B_j} \log(J\tilde{j})(\partial \log \tilde{s}_j^B - \partial \log s_j^B) \]

\[-\frac{i}{8\pi} \oint (\bar{\partial}) \log(J\tilde{j}) \frac{d \log J}{dz} \]

where the last term is the contour integral along the boundary of the standard fundamental domain. Sums over \(K, l\) and \(j\) are understood.

The action \(S_u\) is given by

\[ S_u = \frac{1}{2\pi} \int_{D_u} \left( \frac{1}{2} \partial \varphi \wedge \tilde{\partial} \varphi + e^\varphi du \wedge d\tilde{u} \right) \frac{i}{2} - \frac{\eta K}{4\pi i} \oint_{\partial \tilde{u}} \varphi \left( \frac{du}{u - u_K} - \frac{d\tilde{u}}{\tilde{u} - \tilde{u}_K} \right) \]

\[ - \eta K^2 \log \varepsilon_k^2 - \frac{1}{16\pi i} \oint_{\partial \tilde{u}} \varphi \left( \frac{du}{u - u_l} - \frac{d\tilde{u}}{\tilde{u} - \tilde{u}_l} \right) - \frac{1}{8} \log \varepsilon_j^2 \]

\[ + \frac{1}{8\pi i} \oint_{\partial \tilde{u}} \varphi \left( \frac{du}{u} - \frac{d\tilde{u}}{\tilde{u}} + \frac{3}{2} \frac{1}{2} \right) \log R_u^2. \]  

(63)

Such an action with the boundary conditions \((49, 51, 56)\) is finite. Its variation, again with the boundary conditions \((49, 51, 56)\), gives rise to the equation of motion

\[ -\partial \tilde{\partial} \varphi + e^\varphi du \wedge d\tilde{u} = 0 \]  

(64)
on the two-sheeted cut \(u\)-plane with the singular points \((u_K, w_K), (u_l, 0)\) removed. The field dependent boundary terms appearing in the last line of equation \((59)\) are canceled when performing the above described transition from \(\phi\) to \(\varphi\).

The general relations \((35)\) hold also for the actions \((59, 63)\) and they are consistent with the term \(\eta K (1 - \eta K) \log(J K \tilde{J}_K)\) appearing in equation \((61)\) and the relation

\[ X_u^w - X_K = (1 - 2\eta K) \log(J K \tilde{J}_K). \]  

(65)

4. The auxiliary differential equation

Given the field \(\phi(z)\) we know that in virtue of the Liouville equation

\[ e^\varphi \partial^2 \varphi - \frac{\varphi}{\partial z^2} \equiv -Q_u(z) \]  

(66)
is analytic in \(z\) except for first and second order poles. Under a change of coordinates e.g. from \(z\) to \(u\), the \(Q\) transforms as follows

\[ Q_u(u) = Q_u(z) \left( \frac{dz}{du} \right)^2 - \{z, u\} \]  

(67)

where \(\{z, u\}\) is the Schwarz derivative

\[ \{z, u\} = \left( \frac{dz}{du} \right) \frac{d^2}{du^2} \left( \frac{dz}{du} \right)^{-\frac{1}{2}}. \]  

(68)

Given the differential equation

\[ f''(u) + Q_u(u) f(u) = 0 \]  

(69)
we know (see e.g. [3, 30, 31]) that the conformal factor can be expressed as
\[ e^{v(u)} = \frac{2w_{12}w_{12}}{[\kappa^2 f'_1(u)f_1(u) - \kappa^2 f'_2(u)f_2(u)]^2} \] (70)
where \( f'_1, f'_2 \) are properly chosen solutions of equation (69) and \( w_{12} \) their Wronskian.

The accessory parameters appear in the ordinary differential equation (69) associated with the Liouville problem. We give in the following the structure of the differential equation in canonical form.

For the torus with a single source at \( z = z_1 \) we have the equation [32]
\[ f''(z) + \epsilon (\phi(z - z_1) + \beta f(z)) = 0 \] (71)
but we are interested in the case with \( n \) sources and of the general hyperelliptic surface for which the \( u \) representation is simpler. The general form of \( Q(u) \) for any hyperelliptic surface with \( n \) sources is
\[ Q_u = \frac{3}{16} \left( \frac{1}{(u - u_1)^2} + \cdots + \frac{1}{(u - u_{2g+1})^2} \right) + \frac{\beta_1}{2(u - u_1)} + \cdots + \frac{\beta_{2g+1}}{2(u - u_{2g+1})} + \sum_K \left( \frac{\epsilon_K (w + w_K)^2}{4(u - u_K)^2w^2} + \frac{\beta_K (w + w_K)}{4(u - u_K)w} \right) + \frac{\beta_{01}^{(0)}}{w} + \cdots + \frac{\beta_{03}^{(225)}w^{-3\beta(\epsilon-3)}}{w} \] (72)
with \( \epsilon_K = (1 - \lambda_K^2)/4 = \eta_K (1 - \eta_K) \), \( w = \sqrt{4(u - u_1) \cdots (u - u_{2g+1})} \) and \( w_K = \sqrt{4(u_K - u_1) \cdots (u_K - u_{2g+1})} \) where the last line is present only for \( g \geq 3 \).

The structure of \( Q_u \) has the following origin. To each kinematical singularity at \( u_l \) with \( l = 1 \ldots 2g + 1 \) there corresponds an accessory parameter \( \beta_l \). To each dynamical singularity at \( (u_K, w_K) \) there corresponds an accessory parameter \( \beta_K \). The factors \( (w + w_K)/(2w) \) and their squares project the singularity on the correct sheet. With regard to the kinematical singularities \( u_l \) we notice that to the total accessory parameter contribute not only \( \beta_1 \) but also the terms \( 1/w^2 \), as explicitly given in section 7.

The function \( \frac{1}{w} \) does not introduce a singularity at \( w = 0 \) as seen going over to the local uniformizing variable \( s^2 = u - u_l \) and computing the related \( Q_v \).

At infinity the uniformizing variable is \( v \) given by \( u = v^{-2} \) and we have for the related \( Q_v \)
\[ Q_v = Q_u \left( \frac{du}{dv} \right)^2 - [u, v], \quad \frac{du}{dv} = -2v^{-3}, \quad [u, v] = \frac{3}{4v^2}. \] (73)
The terms of the last line in equation (72) are allowed provided that when combined with the other contributions, they leave the \( Q_v \) free of singularity at \( v = 0 \). The contribution of the term \( \frac{u^p}{w} \) to \( Q_v \) is
\[ \frac{u^p}{w} \left( \frac{du}{dv} \right)^2 \sim v^{2g - 2p - 5} \quad \text{for} \ v \approx 0 \] (74)
and thus they are consistent with the regularity at infinity only for \( 2g - 2p - 5 \geq 0 \) and this happens for \( g \geq 3 \). The \( \beta_3 \) appearing in equation (72) are subject to the conditions of absence of sources at infinity. The structure is special for \( g = 1 \) and \( g = 2 \) while it becomes systematic for \( g \geq 3 \).
Explicitly:
for \( g = 1 \) the number \( \beta \)'s is \( n + 3 \) and we have the three conditions from the regularity at infinity.

\[
2(\beta_1 + \beta_2 + \beta_3) + \sum_{K} \beta_K = 0, \quad (75)
\]

\[
\frac{3}{2} + \sum_{K}(\epsilon_K + u_K \beta_K) + 2(u_1 \beta_1 + u_2 \beta_2 + u_3 \beta_3) = 0, \quad (76)
\]

\[
\sum_{K} w_K \beta_K = 0, \quad (77)
\]

thus leaving \( n \) free \( \beta \)'s.

For \( g = 2 \) the number of \( \beta \)'s is \( n + 5 \) and we have only two conditions given by

\[
2(\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5) + \sum_{K} \beta_K = 0 \quad (78)
\]

\[
3 + \sum_{K}(\epsilon_K + u_K \beta_K) + 2(u_1 \beta_1 + u_2 \beta_2 + u_3 \beta_3 + u_4 \beta_4 + u_5 \beta_5) = 0 \quad (79)
\]

leaving us with \( n + 3 \) \( \beta \)'s.

For \( g = 3 \) the number of \( \beta \)'s is \( 8 + n \) and we have the two conditions

\[
2(\beta_1 + \cdots + \beta_7) + \sum_{K} \beta_K = 0 \quad (80)
\]

\[
\frac{9}{2} + \sum_{K}(\epsilon_K + u_K \beta_K) + 2(u_1 \beta_1 + \cdots + u_7 \beta_7) = 0 \quad (81)
\]

which leaves us with \( n + 6 \) independent \( \beta \)'s.

From now on, increasing the genus by one we introduce three more \( \beta \)'s while the constraints remain always two. Thus we have recovered from the study of \( Q_n \) for the number of independent accessory parameters the general formula \( 3g - 3 + n \).

From expression (70) we have in a neighborhood of a dynamical singularity \( u_K \) with \( \zeta = u - u_K \)

\[
\varphi = -2 \eta_K \log(\tilde{\zeta}) - 2 \log[f(\zeta)\tilde{f}(\tilde{\zeta}) - \kappa^4(\tilde{\zeta})^4 g(\zeta)\tilde{g}(\tilde{\zeta})] + 2 \log |\kappa|^2 + \log(2w_{12}\bar{w}_{12}) \quad (82)
\]

where \( f(\zeta) \) and \( g(\zeta) \) are given by a locally convergent power expansions. Around parabolic singularities we have the expression \([2, 3]\)

\[
\varphi = -\log \zeta^2 - \log \log^2(\zeta) - 2 \log \left[ g(\zeta)\tilde{g}(\tilde{\zeta}) + \frac{f(\zeta)\tilde{g}(\tilde{\zeta}) + \tilde{f}(\zeta)g(\zeta)}{\log \kappa^2} \right] + \text{const.} \quad (83)
\]

Around a kinematical singularity \( u_I \) the local uniformizing variable is \( s \) with \( s^2 = u - u_I \); in equation (82) \( \eta_K \) has to be replaced by \( 1/4 \) and \( f \) and \( g \) become power expansions in \( s \). The detailed form is given in section 7.

At infinity we have for the sphere \( \phi = -2 \log \zeta + h\left(\frac{1}{\zeta}, \frac{1}{\zeta}\right) \) and for higher genus \( \varphi = -\frac{1}{2} \log u + h\left(\frac{1}{u}, \frac{1}{u}\right) \) with \( h \) analytic function in the two variables. This information can be used to give a very simple proof of the uniqueness of the solution of the Liouville equation.
on the sphere and hyperelliptic surfaces of any genus in presence of any collection of elliptic and parabolic singularities.

Consider two solutions $\varphi_1$ and $\varphi_2$ of equation (64) satisfying the above boundary conditions. Then we have

$$0 \leq \frac{1}{2} \int \partial(\varphi_2 - \varphi_1)\tilde{\partial}(\varphi_2 - \varphi_1) - \frac{1}{2} \int (\varphi_2 - \varphi_1)\tilde{\partial}(\varphi_2 - \varphi_1) = 0 - \int (\varphi_2 - \varphi_1)(e^{\varphi_2} - e^{\varphi_1})d^2u.$$  

(84)

The contour integral is around the singularities $u_K, u_l$ and at infinity and due to the behavior of $\varphi_2 - \varphi_1$ it vanishes. Thus we have $\varphi_2 = \varphi_1$. Picard’s uniqueness argument [18] is more complicated because he did not use the information about the non leading terms appearing in equations (82, 83) provided by the auxiliary differential equation (69).

5. Realization of the $SU(1, 1)$ monodromies

The existence and uniqueness proofs for the solutions of the Liouville equations [18–22] give us information on the accessory parameters. In fact given the solution $\varphi(u)$ we have

$$e^{\varphi/2}\partial_u^2 e^{-\varphi/2} = -Q(u).$$  

(85)

The accessory parameters $\beta$ appear explicitly in the expression of $Q(u)$ (72). In fact one can simply extract each of them by means of a contour integral as presented e.g. in [17].

On the other hand if we find a set of accessory parameters and of the real parameter $\kappa$ such that the monodromies along all cycles and around all singularities are $SU(1, 1)$ then expression (70) provides a single valued solution of the Liouville which we know to be unique. The above reasoning shows that we can replace the problem of solving the Liouville equation with the one of finding a set (which we know to be unique) of accessory parameters which make all monodromies $SU(1, 1)$.

In this section we shall give a minimal set of the relations which determine the $\beta$s and the $\kappa$. All these parameters are necessary to determine the solution. We shall first find a set of relations which are sufficient to determine the $\beta$s and do not involve the $\kappa$. Then we give a relation which determines the $\kappa$. We also remark that $\kappa$ always intervenes in the combination $\kappa\bar{\kappa}$ and thus it counts only as one real parameter.

We saw that the number of independent $\beta$ appearing in $Q_u$ are $n + 3g - 3$. These correspond to $2n + 6g - 6$ real degrees of freedom. After choosing the elliptic monodromy at $u_K$ for $K = 1$ diagonal, $q_1 = D_1$ we have an additional real degree of freedom given by $\kappa$ which describes the remnant $SL(2, C)$ transformation.

We have now to use such $2n + 6g - 5$ real degrees of freedom to make all monodromies $SU(1, 1)$. We know from the existence and uniqueness theorem that this can be done, and in a unique way.

Here we want to examine how this comes about. For clearness we start from the case of genus $g = 0$ (the sphere).

For $n = 3$ there is no freedom of choice and from the explicit solution in terms of hypergeometric functions (see e.g. [33]) we know that a proper choice of the value of $\kappa$ makes $q_2 SU(1, 1)$. Then using $q_1 q_2 q_3 = 1$ we also deduce that $q_3 \in SU(1, 1)$. For $n = 4$ we have one $\beta$ and we can exploit the $\kappa$ and one real degree of freedom of $\beta$ to reduce $q_2$ to the form
\[ q_2 = \begin{pmatrix} m_{11} & m_{12} \\ \bar{m}_{12} & m_{22} \end{pmatrix} \]  

(86)

We have from the SL(2, C) and the elliptic nature of the transformation

\[ m_{11}m_{22} + 1 + m_{12}\bar{m}_{12} \geq 1, \quad m_{11} + m_{22} = -2\cos \alpha_2 = \text{real}, \quad |2\cos \alpha_2| \leq 2 \]  

(87)

which give \( m_{22} = m_{11} \) i.e. \( q_2 \in SU(1, 1) \). We can now use the remaining real degree of freedom to have in \( q_3, m_{11} = \rho_1 e^{i\phi}, \ n_{22} = \rho_2 e^{-i\phi} \) and from the reality of the trace we derive \( n_{22} = \bar{n}_{11} \). On now imposing

\[ \text{tr}D_1q_2q_3 = -2\cos \alpha_4 = \text{real} \]  

(88)

we obtain for \( q_3 \ n_{21} = \bar{n}_{12} \) i.e. \( q_3 \in SU(1, 1) \). Finally, due to \( q_1q_2q_3q_4 = 1 \) we also have \( q_4 \in SU(1, 1) \). Increasing \( n \), i.e. the number of sources, by 1 we gain a further \( \beta \), i.e. two real degrees of freedom and we proceed as above.

For \( g > 0, n > 0 \) we have \( n + 2g \) cycles related by the algebraic relation [29]

\[ q_1 \ldots q_n a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1. \]  

(89)

After fixing \( q_1 = D_1 \) diagonal, the \( n + 3g - 3 \beta \)s and \( \kappa \) give us \( 2n + 6g - 5 \) real degrees of freedom to impose the \( SU(1, 1) \) nature to the remaining \( q_2 \ldots q_n a_1 b_1 \ldots a_g b_g \). We spend \( 2(n - 1) \) of them to make all the elliptic \( q_2 \ldots q_n \) \( SU(1, 1) \) and we spend the \( 3(2g - 1) \) remaining degrees of freedom to make \( a_1 b_1 \ldots a_g \) (but not \( b_g \)) \( SU(1, 1) \). The three degrees of freedom to make each such element \( SU(1, 1) \) are spent as follows: two degrees for making \( m_{21} = \bar{m}_{12} \) and one for obtaining \([m_{11}] = [m_{22}]\). Then from

\[ m_{11}m_{22} = m_{12}m_{21} + 1 \]  

(90)

we obtain \( m_{22} = \bar{m}_{11} \). Then using equation (89) we reach the equation

\[ b_n a_n^{-1} b_n^{-1} \in SU(1, 1) \]  

(91)

which being already \( a_n \) in \( SU(1, 1) \), imposes three real constraints on \( b_n \) turning it from \( SL(2, C) \) into \( SU(1, 1) \).

A different way to proceed is the following: instead of fixing \( q_1 \) diagonal, leave it undetermined. In this way we have in addition to the \( 2(n + 3g - 3) \) degrees of freedom of the \( \beta \)s the three real degrees of freedom of \( SL(2, C)/SU(1, 1) \) in total \( 2n + 6g - 3 \) degrees of freedom. Spend \( 2(n - 1) \) of them to make the elliptic \( q_2 \ldots q_n \) \( SU(1, 1) \) and \( 3(2g - 1) \) to make \( a_1 b_1 \ldots a_g \) (but not \( b_g \)) \( SU(1, 1) \). Use the two left over parameters to make in \( b_n, m_{21} = \bar{m}_{12} \) after which \( b_n \) assumes the form

\[ b_n = \begin{pmatrix} m_{11}/\rho & m_{12} \\ \bar{m}_{12} & \bar{m}_{11}\rho \end{pmatrix} \]  

(92)

where \( \rho \) is a real parameter. The relation

\[ \text{tr}(q_2 \ldots q_n a_1 b_1 \ldots b_g a_g^{-1} b_g^{-1}) = -2\cos \alpha_1 \]  

(93)

is a fourth order equation in \( \rho \). From the existence and uniqueness theorem we know that one solution is \( \rho = 1 \) and such a value makes \( b_n \) and thus also \( q_1 \in SU(1, 1) \), due to equation (89).

We come now to the writing of the relations which determine the \( \beta \)s without involving the parameter \( \kappa \). The relations which solve the monodromies at \((u_k, w_k)\) (dynamical singularities) are given by
because due to the elliptic nature of the monodromy we have
\[
m_{11}(q_K) + m_{22}(q_K) = -2 \cos \alpha_K = \text{real}, \quad |2 \cos \alpha_K| \leq 2 \tag{95}
\]
and
\[
m_{11}(q_K)m_{22}(q_K) = m_{12}(q_K)\tilde{m}_{12}(q_K) + 1 \geq 1. \tag{96}
\]

The relations assuring the monodromy along the cycles \(a_i\) are
\[
m_{12}(a_i) = \tilde{m}_{21}(a_i) \tag{97}
\]
\[
m_{11}(a_i)m_{11}(a_i) = m_{22}(a_i)m_{22}(a_i) \tag{98}
\]
and the same for the \(b_j\) with \(l = 1 \ldots g\).

Below we denote by \(M_k\) the \(SU(1, 1)\) monodromy transformation of the two independent solutions \(f_1, f_2\) appearing in equation (70) along the various cycles. Each matrix element \(M_k\) is an analytic functions of the \(p_j, \beta_j\) where \(p_j = (a_j, w_j)\) with \(j\) running over the \(n\) dynamical and the \(2g + 1\) kinematical singularities and \(s\) runs on \(3g - 3 + n\) values. This is the outcome of the solution of the auxiliary differential equation given by the convergent Volterra series.

After starting canonical at \((u_K, w_K)\) with \(K = 1\) we have \(q_1 = D_1\) and we spend one degree of freedom to have
\[
M_{12}(q_2) = \text{real} \times \tilde{M}_{21}(q_2) \tag{99}
\]
which can be written as
\[
M_{12}(q_2)M_{21}(q_2) = \tilde{M}_{12}(q_2)\tilde{M}_{21}(q_2). \tag{100}
\]
We spend now \(2(n - 2 + 2g - 1)\) real degrees of freedom to impose
\[
\frac{M_{12}(q_2)}{M_{21}(q_2)} = \frac{M_{12}(x)}{M_{21}(x)} \tag{101}
\]
where \(x = q_3, \ldots, q_n, a_1, b_1 \ldots a_g,\) while \(2g - 1\) are needed to have in the \(a_1, b_1 \ldots a_g\)
\[|M_{11}| = |M_{22}|. \tag{102}\]
For satisfying (100), (101) and (102) we need \(2n + 6g - 6\) real parameters which are furnished by the \(n + 3g - 3\) complex \(\beta\)'s. Then \(\beta\) becomes \(SU(1, 1)\) through the relation (89). We notice that \(\kappa\) does not intervene in the above relations and it is determined by
\[
\frac{1}{\kappa^2 R^2} M_{12}(q_2) = 1. \tag{103}
\]

The relations (100, 101, 102) are not pure analytic relations as in all of them the complex conjugate of an analytic function appear. In technical terms it means that the equations (100, 101, 102) i.e.
\[
M_{12}(q_2)M_{21}(q_2) = \tilde{M}_{12}(q_2)\tilde{M}_{21}(q_2) \tag{104}
\]
\[
\frac{M_{12}(q_2)}{M_{12}(x)} = \frac{M_{21}(q_2)}{M_{21}(x)} \tag{105}
\]
\[
\frac{M_{11}(a_i)}{M_{22}(a_i)} = \frac{M_{22}(a_i)}{M_{11}(a_i)} \quad l = 1 \ldots g, \quad \frac{M_{11}(b_l)}{M_{22}(b_l)} = \frac{M_{22}(b_l)}{M_{11}(b_l)} \quad l = 1 \ldots g - 1 \tag{106}
\]
define a real analytic variety [26]. In order to deal with this, it is useful to promote the real variables \( \Re \beta, \Im \beta, \Re \eta, \Im \eta \) to complex variables. An equivalent procedure, which is formally more useful, is the polarization process [24, 26] which consists in considering the variables \( \beta, \bar{\beta}, u, \bar{u} \) and promoting the \( \beta \) and \( \bar{u} \) to the independent complex variables \( \beta, \bar{u} \). Then the results relative to the original problem are obtained for \( u^2 = \bar{u}, \beta^2 = \bar{\beta} \).

Each equation of the type (105) gives rise to two independent relations of the type

\[
A(\beta, u) = \bar{B}(\beta, u') \\
B(\beta, u) = \bar{A}(\beta, u').
\]

On the other hand relations of the type (104, 106) are self-conjugate in the sense that they give rise to the single equation

\[
C(\beta, u) = \bar{C}(\beta, u').
\]

Finally we notice that two self-conjugate relations are equivalent to one 'complex' relation e.g.

\[
C(\beta, u) = \bar{C}(\beta, u') \\
D(\beta, u) = \bar{D}(\beta, u')
\]

can be written as

\[
F(\beta, u) = G(\beta, u') \\
G(\beta, u) = F(\beta, u')
\]

with \( F(\beta, u) = C(\beta, u) + iD(\beta, u), \ G(\beta, u) = C(\beta, u) - iD(\beta, u) \). In this way the equations (104), (105) and (106) can be rewritten as \( n + 3g - 3 \) pairs of complex relations of the type (107).

6. The real-analyticity of the accessory parameters

In proving the Polyakov relation it is necessary to exploit the real analyticity of the dependence of the accessory parameters \( \beta \) and of the parameter \( \kappa \) on the moduli \( u_K, u_l \). In fact, due to relation (103), it is sufficient to prove the real-analyticity of the \( \beta \).

On the sphere for any collection of parabolic singularities and of finite order elliptic singularities it was proven by Kra [16] that the accessory parameters are actually real-analytic functions of \( u_K, u_l \). Finite order elliptic singularities are the discrete set with source strength \( h = -(n-1/2) \). For \( n \to \infty \) they accumulate to the parabolic limit. We are, however, interested in the case in which the elliptic singularities are arbitrary. However in this case we have no proof of real-analyticity everywhere and thus our analysis will be of local nature.

A prerequisite in the proof of real-analyticity of the \( \beta \) exploiting the monodromy conditions of section 5 is the continuity of the \( \beta \) on the moduli \( u_K, u_l \).

In [3] it was proven using Green function technique that, as expected, the functions \( \phi, \partial_u \phi, \partial^2_{u} \phi \) are uniformly bounded in any region of the \( u \) plane, obtained by excluding finite disks around the singularities, with bounds which depends continuously on \( u_l \). Thus taking contour integrals of equation (72) at a finite distance from the singularities we have that the \( \beta \) are bounded functions of the \( u_K, u_l \) when \( u_K, u_l \) vary in a small polidisk. Such a result combined with continuity of equations (104)–(106) and the uniqueness of the solution implies that the \( \beta \)s are continuous functions of the \( u_K, u_l \). Continuity is the basic requirement to translate the equation of the previous section into the local analysis of analytic varieties [23].

In the papers [2, 3] for the sphere topology it was proven that the \( \beta \)s are real-analytic function of the \( u_K \) in an everywhere dense open set in the space of the parameters \( u_K \).

For clearness we illustrate the proof in the case of one accessory parameter \( \beta \), the extension to any number of accessory parameters and moduli being straightforward.
In the following W stands for Weierstrass and WPT for Weierstrass preparation theorem. By a Picard solution we understand the unique values $\beta_R(u_0)\beta_I(u_0)$ which solves the monodromy problem (also in presence of parabolic singularities). The subscripts $R, I$ stay for the real and imaginary part. We denote by $\Delta^{0}$ the set of relations assuring the $SU(1, 1)$ nature of all monodromies.

Given a value $u_0$ we have $\Delta^0(\beta_R(u_0), \beta_I(u_0), u_0) = 0$. Let $\Delta^{0}(\beta_R, \beta_I(u_0), u_0)$ be non identically zero in a neighborhood of $u_0$ then we can solve for $\beta_R$ for values of $u$ which lie as near as we want to $u_0$ and these form a open subset $\mathcal{O}_1$ of $\mathcal{O}_0$.

We now consider $\beta_R - \beta_R(u_0) = 0$ and proceed as above. The result is that $\beta_R, \beta_I$ are real-analytic functions of $u$ in an everywhere dense set.

By iteration, the above procedure works also when $u$ is any collection $u_1, u_2, \ldots$ of parameters and we have any number of $\beta$. In fact the existence and uniqueness result tell us that the $\Delta^{0}$ fix completely the solutions.

In the case of a single accessory parameter like the torus with one source or the four point problem on the sphere, a stronger result can be obtained i.e. that the $\beta$ is a real-analytic function everywhere except for a zero measure set in the $u$ plane [17].

Through polarization [24, 26] i.e. promoting $\bar{u}$ and $\bar{\beta}$ to new independent complex variables $u^c, \beta^c$, the single complex equation which imposes the monodromy $A(\beta, u) = B(\beta, u)$ is promoted to a system of two equations

$$A(\beta, u) = B(\beta^c, u^c)$$
$$B(\beta, u) = A(\beta^c, u^c).$$

In the end we will be interested only in the self conjugate solutions of the system (114) i.e. those which for $u^c = \bar{u}$ give $\beta^c = \bar{\beta}$. We know such a solution to exist and be unique. Applying WPT to the two equations we have

$$P_1(\beta^c - \beta(u_0)|\beta, u, u^c) = 0$$
$$P_2(\beta^c - \beta(u_0)|\beta, u, u^c) = 0.$$
A common solution of the two equations implies the vanishing of the resolvent of the two polynomials

\[ R(P_1, P_2) \equiv f(\beta, u, u') = 0. \]  

(117)

If \( f(\beta, u_0, \bar{u}_0) \) vanish identically in \( \beta \) the system has solutions \( \beta^c \) for any choice of \( \beta \) near \( \beta(u_0) \), but most importantly it can be easily proven [17] that we have infinite self-conjugate solutions for \( u = u_0, u' = \bar{u}_0 \) and \( \beta \) near \( \beta(u_0) \). This violates Picard’s uniqueness result. Thus \( f(\beta, u, \bar{u}) \) has to depend on \( \beta \) and we can apply WPT reducing it to the equation

\[ P(\beta - \beta(u_0))[u, u'] = 0. \]  

(118)

All the solutions of equation (118), and in particular the Picard solution, are analytic in \( u \) and \( u' \) i.e. real-analytic in \( u \) except a zero measure set as shown in the appendix. Thus for the case of a single accessory parameter, we have real-analyticity of the Picard solution \( \beta(u, \bar{u}) \) not only in an everywhere dense open set, but almost everywhere in the space of the moduli.

In the general case of \( N \) accessory parameters, as we have shown above, real-analyticity holds in an everywhere dense open set, but we are not aware of a proof of real-analyticity almost everywhere.

7. Derivation of the Polyakov relation

To derive the Polyakov relation we shall go over to a finite form for the action \( S_\phi \) i.e. a form which does not contain \( \varepsilon \to 0 \) limits as in equation (63). This is achieved by decomposing the field \( \phi \) in a regular and singular part similarly to what originally done in [2, 3]. Starting from

\[ e^{\varphi} = \frac{2w_{12}w_{12}}{[\kappa^2f_2f'^2_2]\beta^2} \]  

(119)

where \( f_1, f_2 \) are solutions of

\[ f'' + Q_a f = 0 \]  

(120)

we have near a singularity \( u_K \) with \( \zeta = u - u_K \)

\[ f'' + \left(1 - \frac{\lambda_K}{4\zeta^2} + \frac{\beta_K}{2\zeta} + \text{regular terms}\right)f = 0 \]  

(121)

\[ f_{12} = \zeta^{1/2}y_{12}(\zeta) \]  

(122)

\[ y_1'' + \frac{1 - \lambda_K}{\zeta}y_1' + \left(\frac{\beta_K}{2\zeta} + \text{regular terms}\right)y_1 = 0 \]  

(123)

\[ y_1 = 1 + a\zeta + ...; \quad a = -\frac{\beta_K}{2(1 - \lambda_K)} = -\frac{\beta_K}{4\eta_K}. \]  

(124)

Thus

\[ e^{\varphi} = \text{const}(\zeta\bar{\zeta})^{\lambda K - 1}[(1 + a\zeta + ...)(1 + \bar{a}\bar{\zeta} + ...) - \kappa^4(\zeta\bar{\zeta})^{\lambda K}(1 + b\zeta + ...)(1 + \bar{b}\bar{\zeta} + ...)]^{-2} \]  

(125)

and then

\[ \varphi = \text{const} - (1 - \lambda_K)\log \zeta\bar{\zeta} - 2[a\zeta + \bar{a}\bar{\zeta} + ... - \kappa^4(\zeta\bar{\zeta})^{\lambda K}(1 + b\zeta + \bar{b}\bar{\zeta} + ...)] \]  

(126)

where \( 1 - \lambda_K = 2\eta_K \).
Let $\Omega$ be a real field which is equal to $-2\eta_k \log(u - u_k)(\bar{u} - \bar{u}_k)$ and to $-\frac{1}{2} \log(u - u)(\bar{u} - \bar{u}_l)$ in finite non-overlapping disks around the singularities $u_K, u_l$ and equal to $-\frac{3}{2} \log u \bar{u}$ outside a disk of radius $R_\Omega$ which includes all singularities. We shall call the union of these regions $C$. Elsewhere $\Omega$ is defined as a smooth field which connects smoothly with the field in the described regions. Notice that $\Omega$ depends on the $u_K, u_l$.

In this way in the decomposition

$$\varphi = \varphi_M + \Omega$$

(127)

$\varphi_M$ is finite and regular both at infinity and at the singularities. Substituting such a decomposition in the action $S_u$ we obtain

$$S_u = \frac{1}{2\pi} \int \left( \frac{1}{2} \partial_\varphi \varphi_M \wedge \partial_\varphi \varphi_M - \varphi_M \partial_\varphi \Omega - \frac{1}{2} \Omega \partial_\varphi \partial_\varphi \Omega + \varphi \partial_\varphi \Omega + \bar{\varphi} \partial_{\bar{\varphi}} \Omega \right) \frac{1}{2}.$$  

(128)

Varying $\varphi_M$ in (128) we derive the equations of motion for $\varphi_M$

$$\partial_\varphi \varphi_M + \partial_{\bar{\varphi}} \Omega = \varphi \partial_\varphi \Omega + \bar{\varphi} \partial_{\bar{\varphi}} \Omega.$$  

(129)

Due to the real-analytic dependence of the $\beta$s and of $\kappa$ on the parameter $u_K$ the integrand in (128) is continuous in $u$ and $u_K$ and uniformly bounded by an integrable function as $u_K$ varies is a small domain and the derivative of the integrand w.r.t. $u_K$ is continuous and bounded by an integrable function as $u_K$ varies is a small domain. Thus we can take the derivative under the integral symbol

$$\frac{\partial S_u}{\partial u_K} = \frac{i}{4\pi} \int -\varphi_M \partial_\varphi \Omega_K + \Omega_K \partial_\varphi \varphi_M + \frac{1}{2} \Omega_K \partial_{\bar{\varphi}} \Omega + \frac{1}{2} \Omega \partial_\varphi \partial_{\bar{\varphi}} \Omega_K$$  

(130)

where the subscript $K$ stays for $\frac{\partial}{\partial u_K}$. Notice that no $\partial_\varphi \varphi_M$ appears due to the equation of motion (129). We notice that in the disk around $u_K$ we have

$$\Omega_K = \frac{2\eta_k}{u - u_k}, \quad \partial_\varphi \Omega_K = 0$$  

(131)

while in the remainder of $C$ we have $\Omega_K = 0$ but not necessarily so in the complement of $C$. The integral in equation (130) is finite and to perform the integration by parts below it is useful to write it as the limit for $\varepsilon$ going to zero of the integral where a disk of radius $\varepsilon$ around $u_K$ is excluded. Integrating by parts we have

$$\int \Omega_K \partial_\varphi \varphi_M = -\int \partial_\varphi \Omega_K \partial_\varphi \varphi_M + \int \partial_\varphi \varphi_M \wedge \partial_\varphi \Omega_K$$

$$= -\oint_{u_K} \Omega_K \partial_\varphi \varphi_M - \oint_{u_K} \varphi_M \partial_\varphi \Omega_K + \int \varphi_M \partial_{\bar{\varphi}} \Omega_K$$  

(132)

where the last term cancels the first term in equation (130) and

$$-\frac{i}{4\pi} \oint_{u_K} \Omega_K \partial_\varphi \varphi_M = -\frac{i}{4\pi} \oint_{u_K} \frac{2\eta_k}{u - u_K} (-2\beta_\varphi) du = -\frac{\beta_K}{2}.$$  

(133)

The minus sign is due to the fact that we are integrating on the boundary of an inner domain. Moreover we have
\[ \int \Omega_K \partial \bar{\Omega} = - \oint_{u_K} \Omega_K \partial \Omega + \int \bar{\Omega} \wedge \partial \Omega = \int \bar{\Omega} \wedge \partial \Omega \quad (134) \]

and

\[ -\int \Omega \partial \bar{\Omega} \partial \Omega = - \oint_{u_K} \Omega_\bar{K} \partial \bar{\Omega} + \int \bar{\Omega} \wedge \partial \Omega = -\int \partial \Omega \wedge \partial \Omega \quad (135) \]

which cancels (134). Summarizing

\[ \frac{\partial S_u}{\partial u_K} = -\frac{\beta_K}{2}. \quad (136) \]

Parabolic singularities are treated in the same way with the same result.

We come now to the variation of the action under the variation of the modulus \( u_l \). The local uniformizing variable around \( (u_l, 0) \) is \( s \) with \( s^2 = u - u_l \). For \( Q_s \) we have

\[ Q_s = 2B_l + O(s) \quad (137) \]

and

\[ B_l = \beta_l + \sum_K s(u_l - u_K)^2 (u_l - u_l) \ldots \{ (u_l - u_l) \ldots (u_l - u_{2l+1}) \} \quad (138) \]

is the total accessory parameter at \( u = u_l \). The two independent solution of \( f'' + Q_s f = 0 \) around \( s = 0 \) are given by

\[ f_1 = 1 + a_1 s - B_l s^2 + a_3 s^3 + \ldots, \quad f_2 = b_1 s + b_2 s^2 + b_3 s^3 + \ldots \quad (139) \]

For the \( \varphi \) we have

\[ \varphi = -\frac{1}{2} \log(u - u_l)(\bar{u} - \bar{u}_l) - 2[a_1 s + \bar{a}_1 \bar{s} + a_1 \bar{a}_1 \bar{s} \bar{s} - B_l s^2 - B_\bar{\lambda} \bar{s}^2 + O(s^3)] \quad (140) \]

\[ -\kappa^4(b_1 s + b_2 s^2 + \ldots)(\bar{b}_1 \bar{s} + \bar{b}_2 \bar{s}^2 + \ldots)] + \text{const}. \quad (141) \]

Then for the analogue of the integral (133) we have

\[ -\frac{i}{4\pi} \oint_{u_l} d^d \Omega_K \partial \varphi_M = \frac{i}{4\pi} \oint_{0} \frac{1}{s^2} (a_1 ds - 2B_l s ds + a_1 \bar{a}_1 \bar{s} ds) \]

\[ -\kappa^4 b_1 b_\bar{\lambda} \bar{s} ds + O(s^2) ds = -B_l \quad (142) \]

and thus

\[ \frac{\partial S_u}{\partial u_K} = -B_l. \quad (143) \]

The factor two of difference between equations (136) and (143) is due to the fact that the boundary of a disk around \( u_l \) in the \( u \) cut-plane is a double turn.

8. Conclusions

The Polyakov relation plays an important role in several aspects of Liouville theory such as the semiclassical limit of conformal blocks [5–8], the generalized monodromy problem [9, 10] and the Hamiltonian formulation of 2 + 1-dimensional gravity in presence of matter [2–4].

In this paper we have extended the Polyakov relation to all hyperelliptic surfaces with an arbitrary number of sources. For higher genus we have a relation between the accessory
parameters and the change of the action induced not only by the change in the position of the sources but also by the change of the moduli.

After imposing the fuchsian conditions the number of independent accessory parameters is $n + 3 \geq 3$, $n$ being the number of the sources and $g$ the genus of the surface, and they are determined by imposing the monodromy condition around the dynamical singularities and along the fundamental cycles.

In the proof, as it happens already in the simple case of the sphere it is necessary to exploit the real-analyticity of the accessory parameters as functions of the singularities $u_K$ and $u_l$ which represent the position of the sources and the moduli of the surface.

For the case of parabolic and finite order elliptic singularity we know that such real-analyticity property is true everywhere [16]. For a collection of parabolic and arbitrary elliptic singularities we proved that real-analyticity holds in an everywhere dense open set in the space of the parameters $u_K$, $u_l$. For the case of the torus with a single source and for the four point case on the sphere we have the stronger result [17] that real-analyticity holds everywhere except for a zero-measure set in the space of the parameters.

The Polyakov relation is then simply proven after decomposing the field in a background component, which takes into account the singularities and the behavior at infinity of the Liouville field, and a regular part. With such a decomposition the change of the action reduces to the computation of a single contour integral.

Appendix

In this appendix we derive the analytic properties of the solutions of the equation given by the W-polynomial

\[
P(u, u^c) = (\beta - \beta(u_0))^{m-1} a_{m-1}(u, u^c) + \cdots + a_0(u, u^c) = 0
\]

which appears in section 6. The $a_j(u, u^c)$ are analytic functions of $u$ and $u^c$ with $a_j(u_0, u_0) = 0$.

We start by computing the the resultant $R(P, P') = f(u, u^c)$ i.e. the discriminant of $P$. We have two cases:

1. $f(u, u^c) \neq 0$ in the W-neighborhood of $u_0, u_0$. Then $f(u, u^c)$ can vanish only on a 'thin' set [23]. Such a set has zero 4-dimensional Lebesgue measure [25] and the set where $f(u, \bar{a}) = 0$ has zero 2-dimensional Lebesgue measure, as shown at the end of this appendix.

   Thus except for such zero measure set we can apply the analytic implicit function theorem to have $\beta(u, u^c)$ analytic function of $u, u^c$ i.e. $\beta(u, \bar{a})$ real analytic function of $u$.

2. $f(u, u^c)$ is identically zero in the W-neighborhood of $u_0, u_0$. Then by a theorem on polarization [24, 26] we have that $f(u, u^c)$ is identically zero.

In this case we proceed by computing the reduced Gram determinants $D_n$ of the power-vectors of the roots [23].
$$D_n = \begin{vmatrix} s_0 & s_1 & \cdots & s_{n-1} \\ s_1 & s_2 & \cdots & s_{n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_{n} & \cdots & s_{2n-2} \end{vmatrix}$$  \hspace{1cm} (145)$$

where

$$s_i = \xi_1^i + \xi_2^i + \cdots + \xi_m^i$$  \hspace{1cm} (146)$$

\(\xi_k\) being the \(m\) roots of \(P\). Being \(D_n\) a symmetric polynomial of the roots it is a polynomial in the coefficients \(a_k(u, u^c)\) and as such an analytic function of \(u, u^c\). Notice that \(D_n = R(P, P')\) \[23\].

In the present case

$$D_m(u, u^c) \equiv 0$$  \hspace{1cm} (147)$$

and we compute \(D_{m-1}\). If it is not identically zero it means that the maximum number of distinct roots is \(m - 1\) and the set where they are \(m - 1\) is open and given by subtracting from the initial open set the zeros of \(D_{m-1}\) which is a thin set and as such of zero measure. In the region where the maximum number of distinct roots is reached all the solutions of (144) (local sheet) are analytic \[23\], and in particular the Picard solution is analytic.

Suppose now that

$$D_m = D_{m-1} \equiv 0.$$  \hspace{1cm} (148)$$

Then we compute \(D_{m-2}\) an proceed as above.

The procedure ends due to the fact that \(D_1 \equiv m\). It corresponds to the situation where we have only one \(m\)-times degenerate solution i.e.

$$P(\beta - \beta(u_0); u, u^c) = (\beta - \beta(u, u^c))^m = 0$$  \hspace{1cm} (149)$$

from which we have \(\beta(u, u^c) - \beta(u_0) = -\frac{1}{m}a_{m-1}(u, u^c)\) which is analytic in \(u, u^c\) and thus \(\beta(u, u^c)\) real-analytic in \(u\).

Thus the accessory parameter \(\beta\) is an analytic function of \(u, u^c\) everywhere except for a thin set. The thin set in \(u, u^c\) have zero 4-dimensional Lebesgue measure \[25\]. However we are interested in the 2-dimensional measure in the \(u\) for \(u^c = \bar{u}\), i.e. given a function \(f(u, \bar{u})\) analytic in both arguments, we are interested in the measure of the points where it vanishes.

In is simpler to go over to the ‘real’ variables \(x = (u + \bar{u})/2,\ y = -i(u - \bar{u})/2\) and write \(f(u, \bar{u}) = f_r(x, y)\) which is also analytic in \(x\) and \(y\). Given any point \((x_0, y_0)\) it is always possible \[24\] to perform a real linear invertible change of variables as to make the WPT applicable at that point. Then we can write

$$f_r(x, y) = U(x, y)((x - x_0)^k + a_{k-1}(y)(x - x_0)^{k-1} + \cdots a_0(y)) \equiv U(x, y)P(x - x_0|y)$$  \hspace{1cm} (150)$$

with \(a_n(y_0) = 0\) and \(U\) a unit. The polynomial in (150) for each \(y\) can vanish only at a finite number of points (real \(x\)). Then denoting with \(\Xi\) the function which equals 1 where its argument vanishes and zero otherwise we have

$$\mu = \int dy \int dx \Xi[P(x - x_0|y)] = \int dy 0 = 0.$$  \hspace{1cm} (151)$$

We can represent the region of the modulus \(u\) as the union of a denumerable set of open domains. We have a zero-measure set of possible non real-analyticity points in each domain and the union of such infinite zero measure set has zero measure.

We conclude that \(\beta\) is a real-analytic function of \(u\) except for a set of zero 2-dimensional Lebesgue measure in the \(u\) plane.
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