Identifying Nonconvexity in the Sets of Limited-Dimension Quantum Correlations

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Quantum theory is known to be nonlocal in the sense that separated parties can perform measurements on a shared quantum state to obtain correlated probability distributions which cannot be achieved if the parties share only classical randomness. Here we find that the set of distributions compatible with sharing quantum states subject to some sufficiently restricted dimension is neither convex nor a superset of the classical distributions. We examine the relationship between quantum distributions associated with a dimensional constraint and classical distributions associated with limited shared randomness. We prove that quantum correlations are convex for certain finite dimensional scenarios and that they sometimes offer a dimensional advantage in realizing local distributions. We also consider if there exist Bell scenarios where the set of quantum correlations is never convex with finite dimensionality.

I. INTRODUCTION

A poignant illustration of the non-classicality of quantum mechanics is found in Bell scenarios [1], where space-like-separated parties perform local measurements on a shared resource, as shown in Fig. 1. If the shared resource is an entangled quantum state, the conditional probability distribution resulting from the parties’ measurements may be outside of the set of distributions achievable by sharing classical randomness or, equivalently, through a local hidden variable model (LHVM) [2, 3]. Per the device-independent formalism, we refer to any conditional probability distribution \( p(ab...|xy... \) as a “box” [4–11], where the parties’ choices of measurements \( x, y, z... \) are referred to as “inputs” and their measurement results \( a, b, c... \) as “outputs”. In this framework the quantum nature of the resource is assessed without information regarding the internal mechanism of the box, but rather by determining whether or not the box belongs to the LHVM-compatible set of boxes. Nonlocal boxes are those which cannot be implemented with classical shared randomness, and are highly valued as the resources which drive device-independent quantum communication and quantum cryptography [12–14].

For any given Bell scenario, the set of LHVM-compatible boxes is known as the local polytope; it has a finite number of facets given by Bell inequalities [2]. A box is defined as nonlocal if and only if it violates a Bell inequality. By contrast, we refer to the set of boxes implementable by sharing any quantum state as the quantum ellipsoid, which is also convex [15–18] but cannot be described in terms of linear inequalities [19, see 2, Fig. 4]. Boxes inside the quantum ellipsoid have conditional probabilities of the form

\[
p(ab...|xy...) = \text{Tr}[\rho (\hat{A}_{a|x} \otimes \hat{B}_{b|y} \otimes \hat{C}_{c|z} ...)], \tag{1}
\]

where the dimensions of \( \rho \) and the local measurement operators \( \hat{A}_{a|x}, \hat{B}_{b|y}, ... \) are unconstrained. To enforce no-signalling, i.e. to ensure that no information about the measurement choices of one party can be inferred by the measurement results of the other, the parties are assigned distinct Hilbert spaces [16, 20]. The operational boundary conditions delineating the quantum ellipsoid are notoriously hard to pin down, although various approximations to the quantum ellipsoid have recently become available [20–23]. The quantum ellipsoid strictly contains the local polytope; nevertheless boxes inside it cannot be used for nonlocal signalling [2].

In this letter, we show that convexity and containment of the local polytope are lost when the dimensions of the local Hilbert spaces are restricted. Although dimensionally constrained quantum correlations have already garnered considerable attention, most prior works have considered the parties as additionally having access to unlimited shared randomness. Examples include determining the degree to which Bell inequalities can be violated [24–26], assessing the security of a quantum cryptographic protocol [27–29], and testing if a given probability distribution is achievable [30–32]. The background assumption of unlimited shared randomness imposes convexity automatically, and obscures the fine-grained quantum-to-classical comparison which we seek. In contrast, our approach considers purely quantum systems or systems with limited shared randomness. Foundationally, the non-classical
nature of quantum mechanics is richer in detail if we compare quantum preparations to classical preparations at finite sizes, as opposed to only in the asymptotic limits of the quantum ellipotope and local polytope [11]. The aim of this Letter is to explore this finer characterization.

Some important work has already been done in this framework. For example, Bowles et al. [33] recently reanalyzed prepare-and-measure signalling scenarios without the assumption of unlimited shared randomness [34]. The minimal classical or quantum dimension required to achieve some unconditioned joint probability distribution, i.e. limiting each party to a single input, has recently been equated with the distribution’s non-negative semi-definite rank, respectively [35, 36], and an arbitrary strong quantum dimensional advantage has been noted [37]. For the usual multiple-inputs Bell scenario, the question of minimizing the quantum dimension required to implement a given box has recently been considered in Ref. [38], though see also Refs. [39, 40]. Pál and Vértesi [41] notably called attention to the nonconvexity of some dimensionally constrained quantum correlations. We resolve the open question they posed by showing that sharing merely qubits leads to a non-convex set of correlations in any non-trivial Bell scenario. We then extend the discussion to multipartite and multichotomous measurements. We prove that, in some scenarios, quantum correlations are guaranteed to be convex if the underlying Hilbert space dimension is sufficiently, but finitely, large.

II. NOTATION, DEFINITIONS, AND FUNDAMENTAL HIERARCHIES

We shall examine Bell scenarios where $n$ space-like separated parties choose from among $m$ inputs (measurement settings) each and, in response, observe among $v$ possible outcomes (measurement results), as shown in Fig. 1. We label such Bell scenarios with the index $(n-m-v)$. The $(n-2-2)$ scenarios, for example, are those for which every party has access to two binary-outcome observables, and $(2-2-2)$ is the familiar CHSH nonlocality scenario [42, 43]. For analytical clarity, we consider exclusively symmetric Bell scenarios. As is conventional, for $(2-m-v)$ the two parties are referred to as Alice and Bob. We use $x$ and $y$ to indicate Alice and Bob’s respective apparatus choices (box inputs), and use $a$ and $b$ to indicate their respective measurement outcomes (box outputs), starting all indices from 0.

To avoid over-specifying a no-signalling box, we parameterize boxes such that only the minimum number of conditional probabilities are specified to allow full reconstruction of the distribution. One such parameterization (inspired by Acín et al. [45]) is based on reserving-as-implicit all probabilities involving the outcome 0. For example, since $\sum_{k=1}^{v-1} p(a|x|y) = p(a|x)$ by no-signalling, one finds that $p(a|0|x) = p(a|x) - \sum_{b=1}^{v-1} p(a|b|x|y)$. In this parameterization scheme, each of the $\binom{v}{k}$ possible $k$-partite input tuples one might condition upon has $(v-1)^k$ freely specifiable output probabilities. Consequently, any no-signalling box is specified in terms of precisely

$$F = \sum_{k=1}^{n} \binom{n}{k} m^k (v-1)^k = (m(v-1) + 1)^n - 1 \quad (2)$$

total explicit parameters, regardless of the choice of parameterization scheme [2, 46].

For further simplification, we will only consider quantum measurement scenarios for which the local Hilbert spaces of every party have the same dimension $d$, $\lambda$. Let’s denote the set of quantum boxes achievable with such a dimensional constraint as $Q_d^{\left(n-m-v\right)}$. In this notation, therefore, the quantum ellipotope is identically $Q_d^{\left(n-m-v\right)}$. In this set, the correlation between the parties arises entirely from measurements on a shared quantum state, as shown in Fig. 2(a). We’ll indicate $Q_d^{\left(n-m-v\right)}$ via the shorthand $Q_d$ when the scenario $(n-m-v)$ is clear from context.

It will also be important for us to consider the set of boxes achievable by sharing only a classical random variable $\lambda$ (as opposed to sharing a quantum state $\rho$), as seen in Fig. 2(b). In parallel with the notion of constrained local Hilbert space dimension, we let $L$ denote the set of boxes achievable by sharing only a classical random variable $\lambda$ (as opposed to sharing a quantum state $\rho$). The set of classical boxes achieved using constrained shared randomness is analogously denoted $L_{\lambda}$, where $|\lambda|$ refers to the dimension of the shared classical randomness, i.e. $L_{\lambda}$ indicates that the parties share no more than $\log_2 |\lambda|$ classical bits in common. For example, if the parties share the outcomes of rolling two distinguishable and arbitrarily weighted dice, then $|\lambda| = 36$.

Boxes predicated on classical randomness are identically mixtures of product distributions, i.e. $L_{\lambda}$ is the set of boxes for which

$$p(ab\ldots|xy\ldots) = \sum_{\lambda=0}^{|\lambda|-1} p(\lambda)p(a|x|\lambda)p(b|y|\lambda)\ldots \quad (3)$$

Accordingly, $L_\infty$ may be thought of as the convex hull of $L_1$, where $L_1$ is the set of boxes achievable without actually sharing any randomness. Indeed, that $L_\infty = \text{ConvexHull}[L_1]$ is a corollary of Fine’s theorem [47–49].

While $L_\infty = \text{ConvexHull}[L_1]$ follows from Eq. (3), one can nevertheless span the local polytope without requiring infinite shared randomness. We define the minimum amount of shared randomness which allows every possible local box to be implemented as $|\lambda|$:

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1 The uniform-dimension constraint is insignificant in bipartite scenarios; see Appendix A for details.

2 The notion of sharing classical randomness can be generalized to random variables with “memory”, but our analysis hews to the simplest model; see Ref. [2, Sec. II.C] for more details.
where ConvexHull denotes quantum set of dimension $L$ composed by sharing qudits. Can be constructed by sharing classical dit can also be quantified quantumly by sharing separable qudits with local quantum states are equivalent to boxes arising from separable quantum states of dimension $\lambda$ is contained by the set of boxes implementable by sharing separable qudits with local Hilbert space dimension $d \geq |\lambda|$. Formally, $|\lambda| \equiv \min |\lambda|$ s.t. $L_{\infty}^{(n-m-v)} = L_{|\lambda|}^{(n-m-v)}$ (4)

where $|\lambda|$ intrinsically depends implicitly on $(n-m-v)$. We call $|\lambda|$ the classical Carathéodory number [50, 51] of $L_1$ for every $P \in \text{ConvexHull}(L_1)$ can be decomposed as a convex mixture of at-most $|\lambda|$ boxes from $L_1$, whereas $|\lambda| - 1$ would not be adequate. Details regarding the determination of $|\lambda|$, including the result that $|\lambda| = 4$ for the (2-2-2) scenario, can be found in Appendix B [52].

Note that boxes arising from measurements on separable quantum states are equivalent to boxes arising from shared classical randomness and vice versa. Critically, the set of all classical boxes with degree of shared classical randomness $|\lambda|$ is contained by the set of boxes implemented quantumly by sharing separable qudits with local Hilbert space dimension $d \geq |\lambda|$. Formally, Axiom (1) : $L_{|\lambda|} \subseteq (Q : \text{sep})_{d \geq |\lambda|}$, i.e. any box that can be constructed by sharing a classical dit can also be composed by sharing qudits.

Corollary (1) : $L_{\infty} \subseteq Q_d$ whenever $|\lambda| \leq d$.

Corollary (2) : If $L_{\infty} \not\subseteq Q_d$, then $Q_d$ is not convex.

Proof. Every $P \in L_{|\lambda|}$ can be mapped to a $P' \in (Q : \text{sep})_{d = |\lambda|}$ by the following construction on Eq. (1): $\rho \rightarrow \sum_{\lambda} \rho = |\lambda\rangle \langle \lambda| \otimes^{n}, \hat{A}_{a|x} \rightarrow \sum_{\lambda} \rho(a|x) \lambda \lambda| \lambda\rangle \langle \lambda|$, and so on. Thus, for example, any box in $L_1$ can be implemented quantumly by sharing a product state. Note that the shared randomness required to encode the separable quantum set of dimension $d$ is $|\lambda| \geq d^n$, as shown in Appendix C [53, 54].

The corollaries follow since $L_{\infty} = L_{|\lambda|} = \text{ConvexHull}(L_1)$. It follows from Eq. (4) and Axiom (1) that the local polytope is contained by the quantum set with Hilbert space dimension $d \geq |\lambda|$. Since $L_{\infty} = \text{ConvexHull}(L_1)$, we conclude that $Q_d$ is not convex if it does not contain the local polytope, i.e. whenever $L_{\infty} \not\subseteq Q_d$. Therefore, evidence of any LHVM boxes not achievable by sharing quantum states is evidence of non-convexity.

We conjecture that the shared quantum state must have local dimension $d \geq |\lambda|$ in order to contain the local polytope, i.e. the converse of Corollary (1), that $L_{\infty} \not\subseteq Q_{d < |\lambda|}$.

For that matter, we conjecture that the converse of Axiom (1) is also true, but this is a fundamental unproven open question.

Conjecture 1 : If $L_{|\lambda| \leq |\lambda|} \not\subseteq Q_{d < |\lambda|}$, i.e. the set of boxes which may be realized by sharing (possibly entangled) qudits of local Hilbert space dimension $d$ is conjectured to not contain the set of boxes constructable by sharing a classical random variable of dimension $|\lambda| \leq |\lambda|$ if $d < |\lambda|$.

Note that Conjecture 1 would imply the nonconvexity of $Q_d$ whenever $d < |\lambda|$, per Corollary (2). One can use entropic measures to infer a trivial lower bound on the smallest $d$ for which $L_{|\lambda|} \subseteq Q_d$ by comparing the largest mutual information which can be mediated through the given $|\lambda|$ to the maximum total correlation capacity of quantum states of dimension $d$ [56, Eq. (16)].

III. NON-CONVEX SETS OF QUANTUM CORRELATIONS

With the above definitions in place, we next demonstrate the nonconvexity of the set of quantum distributions explicitly. For this purpose, it suffices to focus on the most trivial example imaginable: a one-dimensional quantum system, ergo the set $Q_1$.

Proposition (1) : $Q_1$ is not convex.

Proof. If $d = 1$ then the only possible quantum state that Alice and Bob can “share” is the product state $|0\rangle \langle 0|_A \otimes |0\rangle \langle 0|_B$. No matter how Alice and Bob choose their POVM elements, their joint probability distributions will always factorize to a product distribution, $p(ab|xy) = \text{Tr[} A_{a|x} \text{]} \text{Tr[} B_{b|y} \text{]} = p(a|x) \times p(b|y)$. Indeed, $Q_1$ is the set of all product distributions, equivalent to...
Figure 3. Three examples of the nonconvexity of the set \( Q_2 \) are shown above. Each triangle represents the set of possible convex combination of its vertex boxes as defined in Table I, where \( c_i \) represents the weight of \( P_i \) in the mixture and \( \sum c_i = 1 \). To emphasize the notion of convex combination we have plotted the regions as equilateral triangles; nevertheless any pair of edges should be thought of as two independent axes such that the third weight is fixed by normalization. Although in each triangle all the vertex boxes are achievable with shared-qubits preparations, only a fraction of possible convex combinations are also achievable, as indicated by the shaded region. Indeed, in (a) only the zero-area edges of the triangle are qubit-achievable. The essential nonconvexity of the set the of boxes achievable with any \( d \) subject to constraining \( \rho \) to be a product state. As such, \( \rho \) isn’t really “shared” at all; the only randomness is local noise. These restrictions are certainly limiting, but because all local deterministic boxes are achievable in \( Q_1 \), we thus have \( L_\infty \subseteq \text{ConvexHull}[Q_1] \).

\[
\begin{array}{c|cccccccc}
| & \langle A_0 \rangle & \langle A_1 \rangle & \langle B_0 \rangle & \langle B_1 \rangle & \langle A_0B_0 \rangle & \langle A_1B_0 \rangle & \langle A_0B_1 \rangle & \langle A_1B_1 \rangle \\
\hline
P_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
P_2 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
P_3 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
P_4 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\
P_{34} & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\
P_{14} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
P_{TB} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Table I. A list of bipartite conditional probability distributions, or boxes, in the (2-2-2) scenario. Per convention [2], we parameterize binary-output boxes in terms of outcome biases, i.e. \( \langle A_i \rangle = p[a=1|x] - p[a=0|x] \) and \( \langle A_iB_j \rangle = p[a=b|xy] - p[a\neq b|xy] \). Note that the five boxes \( P_i \) through \( P_3 \) are product distributions, achievable even absent any shared randomness, i.e. \( |\lambda| = 1 \). \( P_{1,j} \), by contrast, indicates the equally-weighted mixture of boxes \( P_i \) through \( P_j \), requires non-trivial shared randomness. Only \( P_{TB} \), the quantum box which achieves the Tsirelson bound [43, 57], is nonlocal. Every box in this table is achievable with qubits, but many mixtures of these boxes cannot be achieved using qubits, per Fig. 3.

Suppose Alice and Bob both have access to a uniformly-distributed variable \( 0 \leq \lambda \leq v - 1 \), and let their marginal probabilistic dependencies on \( \lambda \) be such that \( p[a|x|\lambda] \rightarrow \delta[a = \lambda] \) and \( p[b|y|\lambda] \rightarrow \delta[b = \lambda] \) regardless of \( x \) or \( y \). The resulting box \( P \in L_\infty \) is such that \( p[\lambda|x|y] = p[\lambda|x] = p[\lambda|y] = v^{-1} \) where \( P \) does not factorize, i.e. \( p[\lambda|x|y] \neq p[\lambda|x] \times p[\lambda|y] \). As such, \( P \in L_\infty \) yet \( P \notin Q_1 \). By Cor. (2), then, \( Q_1 \) is not convex.

In a less trivial example, we also show that nonconvexity persists in the set \( Q_2 \) when the number of inputs \( m \geq 2 \). In Ref. [41] the nonconvexity of \( Q_2 \) was also proven, but only for scenarios with \( m \geq 4 \).

**Proposition (2)**: \( Q_2 \) is not convex (for all \( m \geq 2 \)). For example, in the (2-2-2) scenario, the set of boxes that can be achieved by sharing qubits is nonconvex.

**Proof.** To show this, we considered maximally general two-qubit preparations and measurement schemes, and found that convex combinations of various boxes in \( Q_2 \) listed in Table I were not themselves achievable with qubits\(^3\); see Fig. 3 for illustrative examples. We initially

\( ^3 \) A brute-force proof seemed necessary to the authors. The method of Sikora et al. [38], however, can also be used to prove the essential nonconvexity, by certifying that some of the boxes in the unshaded region of Fig. 3(a) necessitate a quantum description.
identified a few nonconvex boundaries of $Q_2$ via computational maximization, most of which we subsequently recovered analytically. Details of our state and measurements parametrization may be found in Appendix D [17, 55, 58–65]. The full derivation of the boundaries indicated in Fig. 3 may be found in a Mathematica® notebook in the Supplementary Online materials [66].

Proposition (3): \( \mathcal{L}_\infty \setminus \mathcal{L}_{|\lambda| = d} \cap Q_d \neq \emptyset \), i.e. there exist local boxes which classically require shared-randomness of dimension at least $|\lambda'|$, but which admit quantum “shortcuts” through implementations using quantum systems of smaller dimension $d < |\lambda'|$.

Proof. Consider the local box $P_{1:4}$ from Table I. If Alice and Bob choose the same input then their outputs are perfectly correlated, but if they choose different inputs then their outputs are unrelated. To achieve $P_{1:4}$ classically requires flipping two coins: When Alice or Bob input 0 their output is given by the first coin flip. The input 1, however, returns the second coin flip. Thus $P_{1:4}$ requires $|\lambda| \geq 4$, i.e. $P_{1:4} \in \mathcal{L}_\infty \setminus \mathcal{L}_3$. Alternatively, $P_{1:4}$ is quantumly achievable by sharing the entangled pure state $\frac{1}{\sqrt{2}} |00\rangle + |11\rangle$ and letting $A_{1|0} - A_{0|0} = B_{1|0} - B_{0|0} = \hat{\sigma}_Z$ and $A_{1|1} - A_{0|1} = \hat{B}_{1|1} - \hat{B}_{0|1} = \hat{\sigma}_X$. So, $P_{1:4} \in Q_2$. \( \Box \)

Although a box which requires $|\lambda| \geq 4$ is inside $Q_2$, nevertheless $\mathcal{L}_3 \not\subseteq Q_2$. Indeed, the empty interior of Fig. 3(a) shows that $\mathcal{L}_3 \not\subseteq Q_2$, as is expected per Conj. 1. Thus entanglement enables not only non-locality, but also super-locality: quantum measurement schemes can occasionally reproduce boxes of higher corresponding shared randomness dimension, effectively simulating $|\lambda| > d$. See Zhang [37, Sec. 4.1] for a thorough discussion of both the existence and extent of super-locality in the special case (2-1-n). Interestingly, super-locality can occur even in the absence of entanglement [67].

IV. REGAINING CONVEXITY BY ADDING SHARED RANDOMNESS

The quantum ellipote, i.e. $Q_\infty$, is convex. The convexity of the quantum ellipote, i.e. $Q_\infty$, is well established; see for example Refs. [15, Sec. 5C] and [16, Sec. 5]: proof may also be given in terms of properties of C*-algebras [17, 18]. We have shown in Prop. (2), however, that for finite local Hilbert space dimension $Q_d$ is sometimes not convex. The nonconvexity of $Q_d$ was also demonstrated by Pál and Vértesi [41]. There is no contradiction between the convexity of $Q_\infty$ and the nonconvexity of $Q_d$: rather, convexification of quantum boxes requires either classical shared randomness or, equivalently, comes at the expense of increasing the local Hilbert space dimension.

The reason quantum boxes cannot be mixed without increasing Hilbert space dimension is because the measurements are local. The local nature of the measurement operators means that the composite POVM element associated with some global input and output $M_{ab...|xy...} = A_{a|x} \otimes B_{b|y} \otimes ...$ is necessarily a product operator. Mixtures of product operators, sometimes known as separable superoperators [68, 69], are generally no longer product operators.

On the other hand, access to classical randomness allows for the quantum preparations and measurements to co-depend on a shared classical hidden variable. Indeed, any combination of $N$ qudit-based boxes can be implemented by sharing a single qudit, so long as the qudit is prepared according to a classical variable $\lambda$ of dimension $N$, and the variable $\lambda$ remains accessible to the measurements, i.e. per Fig. 2(c). Explicitly, we imagine the quantum preparations and measurements to co-depend deterministically on $\lambda$, i.e. $p(\rho^{(\lambda)}|\lambda) = p(\hat{A}_{a|x}^{(\lambda)}|\lambda = ... = \delta[\gamma - \lambda]$. By this construction, the resulting hybrid quantum-classical boxes yields

\[
\begin{align*}
q[ab...|xy...] = \sum_{\lambda=0}^{|\lambda|-1} p(\lambda) \text{Tr}(\rho^{(\lambda)} \hat{A}_{a|x}^{(\lambda)} \otimes \hat{B}_{b|y}^{(\lambda)} ...).
\end{align*}
\]

Let’s denote the set of boxes achievable using quantum systems of dimension $d$ with the assistance of shared randomness of dimension $|\lambda|$ as $Q_d + \mathcal{L}_{|\lambda|}$. A hybrid box is not only non-local, but also super-local: quantum measurement schemes can occasionally reproduce boxes of higher corresponding shared randomness dimension, effectively simulating $|\lambda| > d$. See Zhang [37, Sec. 4.1] for a thorough discussion of both the existence and extent of super-locality in the special case (2-1-n). Interestingly, super-locality can occur even in the absence of entanglement [67].

in terms of a Hilbert space dimension greater than two, i.e. that qubits would be incompatible. Brute-force feasibility-checking is nevertheless required to generate the nonconvexity illustrations in Fig. 3.
Theorem (1): \( |\lambda^*_Q|_d \leq (m(v-1)+1)^n - 1 \).

Proof. A 1929 theorem of Werner Fenchel \cite{72, 73} states that any point within the convex hull of a not-necessarily-convex \( F \)-dimensional closed and pathwise-connected set can be decomposed as a convex mixture of at-most \( |\lambda^*_Q|_d \) boxes from \( Q_d \), whereas \( |\lambda^*_Q|_d - 1 \) would not be adequate.

Although \( |\lambda^*_Q|_d \) may depend on \( d \), there is a way to upper-bound the quantum Carathéodory number independently of \( d \).

Theorem (2): If \( Q_\infty = \text{ConvexHull}(Q_d) \) means that \( Q_\infty \subseteq Q_{d'} \), from which it follows that \( Q_\infty = Q_{d'} \). Thus, \( Q_{d'} \) is guaranteed to be convex.

The corollary is a consequence of Masanes’ theorem \cite{74, 75}, which states that (projective) measurements on merely shared qubits are capable of achieving all extremal quantum distributions for scenarios involving two binary measurements per party \cite{76}. Masanes’ theorem is often cited when noting that the maximum violation of Bell inequalities for such scenarios can be computed by maximizing over qubit-based boxes.

It is not clear if Cor. (3) can be extended to more general scenarios. For example, the I_{3322} Bell inequality in the (2-m-2) many-measurement-choices scenario is apparently ever-more-violated as \( d \) is increased \cite{77–80}. The boundaries of the quantum set increasing with dimension prevent using the above arguments, and indeed it remains an open question whether \( Q_{d}^{(2-2-2)} \) is convex for any finite \( d \). On the other hand, there is numerical evidence that \( Q_{d}^{(2-n-2)} \) many-measurement-outcomes scenarios, such as the CGLMP scenario \cite{81}, achieve maximum nonlocality at \( d = v \) \cite{22, 82}. The maximum violation of every Bell inequality, however, does not necessarily imply that all quantum extremal distributions have been achieved, so it is not certain that convexity is achieved. See Ref. [2, Sec. III.B] for further details.

Conjecture 2: We conjecture that nonconvexity is nothing more than an artifact of not spanning the local polytope, i.e. that if \( L_{|\lambda|} \subseteq L_{|\lambda|} + Q_d \) then \( L_{|\lambda|} \) is convex.

In stating Conj. 2 we used the equivalence \( L_{\infty} = L_{|\lambda|} \) per Eq. (4). Recall that the converse of Conj. 2 is obviously true: if a set of boxes – quantum, classical, or hybrid – does not contain the local polytope, then the set is not convex per Cor. (2).

Conj. 2 amounts to speculating that \( |\lambda^*_Q|_d = \min \{|\lambda| \} \) such that \( L_{|\lambda|} \subseteq L_{|\lambda|} + Q_d \). If true, this would replace Thm. (1) with the claim \( |\lambda^*_Q|_d \leq \{ |\lambda|/d \} \). This follows from \( L_d \subseteq Q_d \) per Ax. (1), and then by \( L_{|\lambda|} \subseteq L_{|\lambda|/d} + L_d \), which merely notes that a random variable can always be decomposed into multiple constituent parts. Equivalently, any integer \([1, N]\) can be mapped injectively to an ordered tuple \([1, M], [1, [N/M]]\).

To be clear, it is an open question if \( Q_d \) is ever convex for finite \( d \) in scenarios where Thm. (2) does not apply. Conj. 2 not only specifies the affirmative but effectively proposes a minimal value for what finite \( d \) might be. Finally, note that an implication of Conj. 1 and Conj. 2 combined is that \( Q_d \) should be convex if and only if \( d \geq |\lambda| \).

V. DISCUSSION

Questions remain even when still considering the (2-2-2) scenario. We’ve established that \( L_{\infty}^{(2-2-2)} \nsubseteq Q_2 \), but would
qudits be able to span the local polytope, or would $d = 4$ be required?\footnote{4} The insufficiency of qudits is speculated in Conj. 1, but this should certainly be investigated further. Furthermore, although from Cor. (3) is is clear that $\mathcal{Q}_{16}^{(2-2-2)}$ is convex, there’s still a large gap between the non-convex result of $\mathcal{Q}_2$ and the yes-convex result of $\mathcal{Q}_{16}$. The convexity of $\mathcal{Q}_4^{(2-2-2)}$ is speculated in Conj. 2, but this too should certainly be investigated further. For general scenarios where one presumes that Thm. (2) does not apply, such as $(2-3-2)$ \cite{78}, we have noted it is a completely open question if $\mathcal{Q}_d$ is ever convex for finite $d$.

Most generally, given three descriptive elements: 1) an operational description of some Bell scenario such as $(n-m-v)$, 2) the local Hilbert space dimension limit of $d$, and 3) the dimension limit of any shared classical randomness $|\lambda|$, are the resulting correlations $\mathcal{L}_{|\lambda|} + \mathcal{Q}_d$ convex? Such fundamental questions remain generally unanswered despite the broad results of Thms. (1) and (2). The purely classical regime is considered further, however, in the Appendix B.

Quantifying the genuine boundaries of finite-dimensionally-generated quantum correlations, as opposed to the convex hull of such correlations, is an important question for future research. We have evidenced that nonconvexity should be expected in all nonlocality scenarios with sufficiently restrictive constraints on the local Hilbert space dimensions. As many physical systems have implicit bounds on their local Hilbert space dimensions it is all the more important to anticipate nonconvexity of quantum correlations in practical quantum information-theoretic protocols. Purely quantum systems with sufficiently low dimensionality are forbidden from displaying certain classical correlations, a property which may be exploited as a device-authenticating security check in quantum cryptographic implementations.

The nonconvexity of quantum correlations is no less relevant, practically, than other no-go results pertaining to finite Hilbert space dimensions. Quantifying the genuine boundaries of finite-dimensionally-generated quantum correlations, as opposed to the convex hull of such correlations \cite{31, 32}, is therefore an important question for future research. How the nonconvexity of dimensionally limited correlations may clarify the relationship between Hilbert space dimension and degree of classical randomness is an intriguing area for future work \cite{83-85}. Perhaps classical information theory tools for considering limited shared randomness \cite{86} can be adapted and applied to finite dimensional quantum systems. Fuller quantitative operational characterizations of finite dimension quantum systems are thus valuable desiderata for both foundational and practical research.

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Appendix A: Correlations from asymmetric-dimensional quantum states

In Section II, we defined $\mathcal{Q}_d^{(n-m-v)}$ as the set of correlations achievable when all parties share a quantum state with equal local dimensions $d$. Here we note that any result concerning the uniform-dimension case also applies to the asymmetric-dimension case if the scenario in question is bipartite. We are grateful to referee comments for suggesting this extension.

Proposition (4) : If, in some bipartite Bell scenario, Alice and Bob share a quantum system with local dimensions $d_A$ and $d_B$, respectively, the set of distributions they can achieve is identical to the set of distributions achievable if the minimum dimension were common across both parties: $\mathcal{Q}_{d_A, d_B}^{(2-m-v)} = \mathcal{Q}_{d_{\text{min}}}^{(2-m-v)}$, where $d_{\text{min}} = \min\{d_A, d_B\}$.

Proof. In realizing achievable quantum distributions in a specified dimension, it suffices to consider pure states as the mixedness may be included in the local measurement settings; see \cite[Lemma 1]{38}. Any pure state in $\mathcal{H}^{d_A} \otimes \mathcal{H}^{d_B}$ is equivalent under local unitary transformations to its Schmidt decomposition, which has at most $d_{\text{min}} = \min\{d_A, d_B\}$ terms \cite{58-61}. Therefore, any correlation achievable with states in $\mathcal{H}^{d_A} \otimes \mathcal{H}^{d_B}$ may also be achieved with states in $\mathcal{H}^{d_{\text{min}}} \otimes \mathcal{H}^{d_{\text{min}}}$. \hfill $\Box$

Note that this proof relies on the fact that the Schmidt decomposition of a bipartite state can be written as a sum of at most $d_{\text{min}}$ terms. This feature does not extend to general entanglement classes in multipartite scenarios \cite{58}; in these scenarios, asymmetric-dimensional states may lead to unique sets of achievable distributions.

Appendix B: Bounding the Minimum Classical Shared Randomness which Spans the Local Polytope

In this section we establish some upper and lower bounds on $|\lambda|$, as defined per Eq. (4) in the main text. We are grateful to Matt Pusey of the Perimeter Institute for Theoretical Physics for suggesting most of these proofs. The related question of how to find an explicit...
classical implementation of a box while minimizing $|\lambda|$ ("ontological compression") is considered in Ref. [39].

**Proposition (5)**: $|\lambda| \leq v^{m(n-1)}$. In particular for the (2-2-2) scenario $|\lambda| \leq 4$.

**Proof.** Let $n-1$ parties have all their measurements depend deterministically on $\lambda$, and only the $n$th party depending probabilistically on $\lambda$. There are precisely $v^{m(n-1)}$ possible deterministic distributions among the $n-1$ parties, so without loss of generality is suffices to take $|\lambda| = v^{m(n-1)}$. In this manner a reasonable classical correlation is possible.

**Proposition (6)**: $|\lambda| \leq (m(v-1) + 1)^n - 1$.

**Proof.** The proof is analogous to the proof of Thm. (1). $\mathcal{L}_1$ is a pathwise-connected closed compactum of dimension $\mathcal{F}^{(n-m-v)} = (m(v-1) + 1)^n - 1$ by Eq. (2). Fenchel’s theorem [72, 73] then implies that the Carathéodory number of any box $\mathcal{L}_1$ is less than or equal to $\mathcal{F}$; i.e. any $P \in \text{ConvExL}_1$ can be decomposed into a convex combination of at most $\mathcal{F}$ boxes from $\mathcal{L}_1$. $|\lambda|$ is identically that Carathéodory of $\mathcal{L}_1$.

**Proposition (7)**: $|\lambda| \geq (m(v-1) + 1)^{n-1} + \sum_{k=2}^{n} \sum_{j=2}^{k} \left( \begin{array}{c} n-1 \\ k-1 \end{array} \right) \left( \begin{array}{c} m \\ j \end{array} \right) S_2[k,j](j-1)(v-1)^k$, where $S_2$ refers to Stirling number of the second kind [52]. For $n = 2$, therefore, $|\lambda| \geq \frac{(m-1)(v-1)^2}{2} + m(v-1) + 1$ and in particular for the $(2-2-2)$ scenario $|\lambda| \geq 4$.

**Proof.** Consider a box which is as random as possible while still satisfying

$$p[a + b + c... = 0 \mod v | xyz] = 1 \iff x = y = z... \quad \text{(B1)}$$

i.e. the outputs of the parties always (modulo) sum to zero when the inputs are aligned, but this perfect correlation is not detected whenever the inputs are not all aligned. One can then, given $n-1$ of the outputs, perfectly determine the remaining one as $a_x = -b_x - c_x... \mod v$. The instances of perfect correlation enforce that there is no local noise, and thus that every party’s output depends deterministically on the random variable. The degrees of freedom in the outputs of the first $n-1$ parties, given by Eq. (2) as $(m(v-1) + 1)^{n-1} - 1$, are all able to be set independently and yet must be determined solely by the shared random variable. We must give the shared variable an alphabet size equal to the number of degrees of freedom plus one, to account for normalization. Thus, $|\lambda| \geq (m(v-1) + 1)^{n-1}$.

This loose lower bound can be strengthened to the expression in Prop. (7) by counting the remaining degrees of freedom which include the last party’s outcome. There are $\binom{n}{k-1}$ ways to choose a $k$-partite context involving the last party. We then need to assign inputs to the parties; as the case where all inputs are equal is already specified by the definition of the box, we only need to consider distributing $j > 1$ distinct inputs among the $k$ parties. There are $\binom{m}{j}$ ways to choose which $j$ distinct inputs to distribute. The Stirling number $S_2[k,j]$ gives the number of ways $k$ objects can be divided into $j$ (indistinguishable) partitions. Each partition is assigned one input; however, we must consider the various permutations of mapping inputs to partitions. To avoid over-specifying the distribution we take the last party (and any other parties in the partition including the last party) to be associated with largest of the $j$ selected inputs. Thus we only consider $(j-1)!$ permutations of how the smaller inputs can be assigned to the remaining partitions. Finally we specify the outputs, avoiding output 0 per the parameterization scheme discussed prior to Eq. (2). Thus there are $v-1$ distinct outputs considered for each of the $k$ parties considered.

For pedagogical clarity we demonstrate how to obtain $|\lambda| \geq 7$ in this fashion for the $(2-2-3)$ scenario, famous for the Bell inequality $I_{3222}^{78-80}$. The classical box in (2-2-3) which satisfies $b_{x\lambda} = a_x$ has six degrees of freedom, namely $p[a_1 = 1], p[a_2 = 1], p[a_3 = 1], p[1, b_1 = 1], p[0, a_1 = 1, b_2 = 1], p[0, a_2 = 0, b_2 = 1]$. Trivially $p[a_3 = 0] = 1 - p[a_2 = 1]$ and $p[a_2 = 0] = p[a_3 = 1] - p[a_2 = 1]$ etc. Note that $p[b_y = i] = 0$ is fixed as equal to $p[a_y = i]$. Furthermore, $p[a_x = i, b_y = j] = p[a_y = j, b_x = i]$ where $x < y$. Adding one to the six degrees of freedom yields $|\lambda| \geq 7$.

**Appendix C: A Complimentary Fundamental Axiom**

In **Ax. (1)**, it was noted that the set of boxes achievable with classical shared randomness of dimension $|\lambda|$ is achievable with quantum states of dimension $d = |\lambda|$, and furthermore, realizable with separable quantum states. Here, we briefly note that while the inverse is not true, it is true that any box achievable with separable quantum states of dimension $d$ can be achieved with classical shared randomness of finite dimension.

**Axiom (3)**: $(Q, \text{sep})_d \subseteq \mathcal{L}_{|\lambda| \geq d^n}$, i.e. all quantum correlations generated by separable states are also classically achievable given enough shared randomness, where enough means $|\lambda| \geq d^n$.

**Corollary (4)**: Entanglement is required for non-locality.

**Proof.** Recall that by definition separable states can be written as $\rho_{\text{sep}} = \sum_i \rho^{(i)}_A \otimes \rho^{(i)}_B \cdots$, and therefore satisfy $\text{Tr}[\rho_{\text{sep}}(\hat{A}_{a|x} \otimes \hat{B}_{b|y} \cdots)] = \sum_{i=1}^{\text{max}} \left( \rho^{(i)} \text{Tr}[\rho^{(i)}_A \hat{A}_{a|x}] \text{Tr}[\rho^{(i)}_B \hat{B}_{b|y} \cdots] \right)$. Any separable state with local Hilbert space dimension $d$ can be decomposed into a mixture of no-more-than $d^n$ product states [53 Def. 6, 54 Thm. 2] so we can replace $i_{\text{max}}$ with $d^n$ without loss of generality. One can therefore
map every $P \in (Q : \text{sep})_d$ to a $P' \in L_{|\lambda| = d^n}$ by the following construction on Eq. (3): $\lambda \rightarrow i, p[\lambda] \rightarrow c^{(i)}, p[a|x] \rightarrow \text{Tr}[\rho_A^{(i)} \hat{A}_{a|x}]$, etc.

Goh et al. [67] have shown that merely $|\lambda| \geq d$ is not always sufficient to classically simulate the correlations which result from separable states, i.e. that separable states can still manifest super-locality. In particular, they show that the box $p(ab|xy) = \frac{2 + (-1)^{a+b+x+y+2}}{8}$ requires $|\lambda| > 2$, but can nevertheless be achieved by measurements on separable qubits.

Appendix D: Details of the Parameterization of Shared-Qubits Boxes in the (2-2-2) Scenario

We consider qubit-based boxes with explicit representations in terms of two-qubit states of arbitrary entanglement and general two-outcome POVMs. We need to consider exclusively pure states, per Ref. [38, Lemma 1]. As all pure states are equivalent to their Schmidt-decomposed form under local unitary transformations [58–61], it is thus sufficient to consider the state

$$|\psi\rangle = \cos(\frac{\alpha}{2})|00\rangle + \sin(\frac{\alpha}{2})|11\rangle, \quad \alpha \in (0, \pi), \tag{D1}$$

by folding the local degrees of freedom into the measurement operators.

While POVMs can be regarded as projective measurements in a larger Hilbert space, restricting the dimensionality of the quantum systems necessitates the use of general POVMs [41, 55]. Adapting the notation of [62], we express a general binary 0/1 outcome POVM element as

$$\hat{A}_{a|x} = \frac{1}{2} \left[ I + (-1)^{1-a}\kappa_A \cdot \hat{n}_A \cdot \vec{\sigma} \right], \tag{D2}$$

where $\vec{\sigma} = (\hat{\sigma}_X, \hat{\sigma}_Y, \hat{\sigma}_Z)$ is a vector of Pauli matrices and $\hat{n}_A = (\sin \theta_A \cos \phi_A, \sin \theta_A \sin \phi_A, \cos \theta_A)$ is a unit vector defining a direction in the Bloch sphere in spherical coordinates. Bob’s POVM elements are defined similarly in a separate Hilbert space. Technically, the measurement operators of the distinct parties need only commute with each other to form a genuine quantum multiparticle implementation; we relegate each party to a distinct Hilbert space for convenience to ensure appropriate commutativity, which has apparently no loss of generality [17, 63–65]. To ensure positivity of the POVM elements corresponding to both outputs, the following conditions must be met:

$$\forall A_x : \eta_{A_x} - 1 \leq \kappa_{A_x} \leq 1 - \eta_{A_x}. \tag{D3}$$

Note that Eq. (D3) implies $0 \leq \eta_{A_x} \leq 1$. If both bounds in Eq. (D3) are simultaneously saturated then $\kappa_{A_x} = 0$ and $\eta_{A_x} = 1$, and Eq. (D2) represents a projection-valued measurement (PVM).

As we are concerned with two-outcome POVMs, it is conventional to parameterize boxes in terms of bias of the measurement outcome, i.e.

$$\langle A_x \rangle = \langle \hat{A}_{i|x} \rangle - \langle \hat{A}_{0|x} \rangle \tag{D4}$$

$$\langle A_x B_y \rangle = \langle \hat{A}_{i|x} \hat{B}_{j|y} \rangle - \langle \hat{A}_{0|x} \hat{B}_{1|y} \rangle - \langle \hat{A}_{1|x} \hat{B}_{0|y} \rangle + \langle \hat{A}_{0|x} \hat{B}_{0|y} \rangle. \tag{D5}$$

The four marginal and four joint biases (for all $x$ and $y$ options) parameterize the eight-dimensional conditional probability space for the (2-2-2) scenario, and relate to the 17 “backend” parameters of the state and measurements as

$$\langle A_x \rangle = \eta_{A_x} \cos(\alpha) \cos(\theta_{A_x}) + \kappa_{A_x}, \tag{D6}$$

$$\langle A_x B_y \rangle = \eta_{A_x} \eta_{B_y} \cos(\phi_{A_x} + \phi_{B_y}) \sin(\alpha) \sin(\theta_{A_x}) \sin(\theta_{B_y}) + \eta_{A_x} \eta_{B_y} \cos(\alpha) \cos(\theta_{A_x}) \cos(\theta_{B_y})$$

$$+ \eta_{B_y} \kappa_{A_x} \cos(\alpha) \cos(\theta_{B_y}) + \kappa_{A_x} \kappa_{B_y}. \tag{D7}$$

Appendix E: Convexly Combining Quantum Boxes via Direct Sum of Hilbert Spaces

Suppose we wish to take the convex combination of $N$ qudit-based multipartite boxes, i.e.

$$\widetilde{P} = \sum_{i=0}^{N-1} c_i P_i \quad \text{where} \quad P_i \in \mathcal{Q}_d. \tag{E1}$$

Typically, $\widetilde{P} \notin \mathcal{Q}_d$. This can be built as follows:

Index the local measurement operators of each of the $N$ boxes being combined into $\widetilde{P}$ by $\hat{A}_{a|x}^{(i)}, \hat{B}_{b|y}^{(i)}$ etc. Now imagine – although entirely unjustified – that all the $N$ boxes $P_i$ are predicated on sharing the same composite quantum state, i.e. $\forall_i \quad \rho_i = \rho_0$. In order to reproduce the marginal probabilities of $\widetilde{P}$ with a single quantum box (without supplementary shared randomness), we should take $\tilde{\rho} = \rho_0$ and define the new measurement operators $\hat{A}_{a|x} = \sum_{i=0}^{N-1} c_i \hat{A}_{a|x}^{(i)}$. This satisfies the requirement that $\tilde{\rho}[a|x] = \sum_{i=0}^{N-1} c_i \rho_i[a|x]$ etc.

Unfortunately, choosing measurement operators to satisfy the single-partite marginal probabilities does not extend to satisfaction of the required bipartite joint probabilities. Following Eq. (1) we find that

$$p[ab|xy] = \text{Tr}[\tilde{\rho} \hat{A}_{a|x} \otimes \hat{B}_{b|y}]$$

$$= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} c_i c_j \text{Tr}[\rho_0 \hat{A}_{a|x}^{(i)} \otimes \hat{B}_{b|y}^{(j)}] \tag{E2}$$

5 The major exception is when $\mathcal{Q}_d$ is convex, as then $\widetilde{P} \in \mathcal{Q}_d$ by definition. Convexity might be “rare”, however.
which does not remotely match 

\[ \tilde{p}(ab|xy) = \sum_i c_i p_i(ab|xy) = \sum_i c_i \text{Tr}[\rho_0 \hat{A}_{a|x}^{(i)} \otimes \hat{B}_{b|y}^{(i)}]. \]

Therefore quantum boxes predicated on different sets of local measurement operators cannot be combined without increasing the Hilbert space dimension\(^6\); as per Ax. (2), however, \( \tilde{P} \in \mathcal{Q}_{d \times N}. \)

Imagine that each \( P_i \) is implemented in \( \mathcal{H}^{(i)} = (\mathbb{C}^d)^{\otimes n} \) using the (distinct!) quantum states \( \rho^{(i)} \) and local measurement operators \( \hat{A}_x^{(i)}, \hat{B}_y^{(i)} \), etc. Then, to implement \( \tilde{P} \) using a single quantum box, one may take the direct sum of the component Hilbert spaces such that

\[
\tilde{\mathcal{H}} = \bigoplus_{i=0}^{N-1} \mathcal{H}^{(i)}, \quad \tilde{\rho} = \bigoplus_{i=0}^{N-1} c_i \rho_i, \quad \tilde{\hat{A}}_{a|x}^{(i)} = \bigoplus_{i=0}^{N-1} \hat{A}_{a|x}^{(i)}, \tag{E3}
\]

and so on.

To explain how Eq. (E3) automatically satisfies Eq. (E1), and why the new local Hilbert space dimension in Eq. (E3) can be thought of as \( d \times N \), consider how \( \tilde{P} \) is implemented in \( ((\mathbb{C}^d)^{\otimes N})^{\otimes n} \). The idea is to assign an ancilla \( \mathbb{C}^N \) to every party, and to make the new shared state simultaneously diagonalized in all the ancillae. Thus,

\[
\tilde{\rho} = \sum_{i=0}^{N-1} c_i \rho_i \otimes |i_A\rangle \langle i| \otimes |i_B\rangle \otimes \ldots,
\]

\[
\tilde{\hat{A}}_{a|x}^{(i)} = \sum_{i=0}^{N-1} \hat{A}_{a|x}^{(i)} \otimes |i_A\rangle \langle i|,
\]

\[
\tilde{\hat{B}}_{b|y}^{(j)} = \sum_{j=0}^{N-1} \hat{B}_{b|y}^{(j)} \otimes |j_B\rangle \langle j|,
\]

and so on. Indeed, Eq. (E4) amounts to the definition of the direct sum in Eq. (E3). Thus the parties’ new measurement operators now live in distinct Hilbert spaces \( N \) times larger than originally, and the new shared state is a cq-state [40, 44], namely a block-diagonal composition of the component original states in the convex combination per Eq. (E1).

What we find is that direct summation of Hilbert spaces is identically convexification in the sense of Eq. (5). Since \( \bigoplus_{i=0}^{N} \mathbb{C}^d \subset \bigotimes_{i=0}^{N} \mathbb{C}^d \), we have proven that \( \mathcal{L}_N + \mathcal{Q}_d \subseteq \mathcal{Q}_{d \times N}. \)

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\(^6\) Suppose we artificially consider global measurements of the form \( \hat{A}^\dagger \hat{B} = \hat{A}_a \otimes \hat{B}_b \). Such not-local-but-still-separable measurements lack physical meaning, but they can be interpreted as representing the convex combination of boxes acting on the same shared state with different local, i.e. product state, measurements. Such artificial global measurements are known as separable superoperators, and have been studied elsewhere in the context of state discrimination [68, 69].
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