Regular Turán numbers of complete bipartite graphs

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Abstract

Let \( rex(n, F) \) denote the maximum number of edges in an \( n \)-vertex graph that is regular and does not contain \( F \) as a subgraph. We give lower bounds on \( rex(n, F) \), that are best possible up to a constant factor, when \( F \) is one of \( C_4, K_{2,t}, K_{3,3} \) or \( K_{s,t} \) for \( t > s \).

1 Introduction

For a fixed graph \( F \), the Turán number of \( F \) is the maximum number of edges in an \( n \)-vertex graph that does not contain \( F \) as a subgraph and is denoted by \( ex(n, F) \). Turán numbers for various graphs or families of graphs are the central functions in extremal graph theory. In this paper, we study a related function, where one restricts to regular graphs.

Let \( rex(n, F) \) be the maximum number of edges in an \( n \)-vertex regular \( F \)-free graph. Following [11] and [20], we call this the regular Turán number of \( F \). By the definitions, we have the trivial inequality

\[
rex(n, F) \leq ex(n, F),
\]

for all \( F \) and \( n \). However, unlike \( ex(n, F) \), the function \( rex(n, F) \) is not necessarily monotone in \( n \). For example, Mantel’s theorem shows that \( rex(2k, K_3) = k^2 \), but Andrásfai [3] proved that \( rex(2k + 1, K_3) \leq (2k + 1)^2/5 \).

Most of the previous work on regular Turán numbers is given by the extensive study of cages (see [17] for a survey), where one forbids all cycles up to a fixed length. For other graphs, regular Turán numbers were introduced and studied systematically in [20]. The regular Turán problem was motivated by Caro and Tuza’s work on singular Turán numbers [12]. In [20], Gerbner, Patkós, Tuza, and Vizer showed that for non-bipartite \( F \) with odd girth \( g \), one has \( rex(n, F) \geq n^2/(g + 6) - O(n) \) and asked to determine \( \lim \inf_{n \to \infty} rex(n, F)/n^2 \) for non-bipartite \( F \). This problem was solved independently in

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and [13] for graphs $F$ with chromatic number at least 4, and the authors proved partial results for graphs with chromatic number 3. Following [11] and [13], exact results on regular Turán numbers of trees and complete graphs were obtained in [19].

In both [11] and [20] it is acknowledged that we do not know much about $\text{rex}(n, F)$ when $F$ is a bipartite graph with a cycle. This will be the focus of the current paper.

Given a bipartite graph $F$, the quantity we will be particularly interested in is

$$Q(F) := \limsup_{n \to \infty} \frac{\text{ex}(n, F)}{\text{rex}(n, F)}$$

which is a measure of how close $\text{rex}(n, F)$ is to $\text{ex}(n, F)$. By the trivial inequality, $Q(F)$ is always at least 1. It is natural to ask when $Q(F) = 1$ and when $Q(F) < \infty$. Our main theorems give lower bounds for $\text{rex}(n, F)$, implying that $Q(F)$ is finite for several different bipartite graphs. We begin with $C_4$.

**Theorem 1.1** The regular Turán number of $C_4$ satisfies

$$\text{rex}(n, C_4) \geq \left( \frac{1}{2\sqrt{6}} - o(1) \right) n^{3/2}.$$

Füredi [18] showed that $\text{ex}(K_{2,t+1}) \sim \frac{\sqrt{t}}{2} n^{3/2}$. Since a $C_4$-free graph is also $K_{2,t+1}$-free, Theorem 1.1 shows that for any fixed $t$, $Q(K_{2,t+1})$ is bounded above by a constant that depends on $t$, namely $\sqrt{t/6}$. However, just using $C_4$-free graphs does not show that $\limsup_{t \to \infty} Q(K_{2,t+1})$ is finite. We show that this limit is finite in the following theorem.

**Theorem 1.2** For $t \geq 1$ with $t$ even, the regular Turán number of $K_{2,2t+1}$ satisfies

$$\text{rex}(n, K_{2,2t+1}) \geq \left( \frac{\sqrt{t/20}}{2} - o(1) \right) n^{3/2}.$$

Using the $K_{3,3}$-graphs constructed by Brown [8], together with some number theoretic results on the Waring-Goldbach problem, we can prove a lower bound on the regular Turán number of $K_{3,3}$.

**Theorem 1.3** For large enough $n$, the regular Turán number of $K_{3,3}$ satisfies

$$\text{rex}(n, K_{3,3}) \geq \frac{1}{2\sqrt[3]{14^2}} n^{5/3} - O(n^{3/5}).$$

One comment is that if $n$ is of the form $n = p^3$ where $p$ is an odd prime, then $\text{rex}(n, K_{3,3}) \geq \frac{n^{5/3} - n^{4/3}}{2}$ which is asymptotically best possible since $\text{ex}(n, K_{3,3}) \sim \frac{1}{2} n^{5/3}$. Therefore,

$$\liminf_{n \to \infty} \frac{\text{ex}(n, K_{3,3})}{\text{rex}(n, K_{3,3})} = 1.$$

Finally, we use the norm graphs of Kollár, Rónyai, and Szabó [21] to give lower bounds on the regular Turán number of $K_{s,t}$ when $t > s!$. 

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Theorem 1.4  Let $s \geq 3$ and $t > s!$. Then there is a constant $C$ depending only on $s$ so that for large enough $n$, the regular Turán number of $K_{s,t}$ satisfies

$$\text{re}x(n, K_{s,t}) \geq Cn^{2-1/s}.$$ 

We end the introduction with two open problems that we feel are most natural to try next.

Problem 1.5  Show that $Q(C_6)$ and $Q(C_{10})$ are finite.

Problem 1.6  Determine whether or not $Q(C_4) = 1$.

In Section 2 we give an outline of how the main theorems are proved and establish some necessary lemmas. In Sections 3, 4, 5, and 6 we prove our main theorems.

2 Preliminaries

For $F \in \{C_4, K_{2,t}, K_{3,3}, K_{s,t}\}$ with $t > s!$, there are classical constructions of $F$-free graphs with many edges coming from geometry and algebra. To give an upper bound on the quantity

$$Q(F) = \limsup_{n \to \infty} \frac{\text{ex}(n, F)}{\text{re}x(n, F)},$$

we will give constructions of regular graphs on $n$ vertices that are $F$-free and have many edges. The first difficulty is that to bound the limit superior, we need a lower bound on $\text{re}x(n, F)$ for arbitrary $n$. Since the function $\text{re}x(n, F)$ is not monotone in $n$, it does not suffice to construct a sequence of $F$-free graphs which have number of vertices some function of a prime and then use a density of primes argument, as is common when proving lower bounds for $\text{ex}(n, F)$.

Therefore, given an $n$, our strategy will be to take disjoint unions of $F$-free graphs so that the number of vertices sums to $n$. As the classical constructions of $F$-free graphs are defined algebraically or geometrically, the number of vertices in these graphs is some function of a prime power (see [26] for a survey on algebraically defined graphs). Because of this, we will need the following theorem on the Waring-Goldbach problem when the prime powers are restricted to being almost equal.

Theorem 2.1 (Wei and Wooley [36])  Let $k \geq 3$ be a natural number and let $\theta_k = \frac{4}{5}$ if $k = 3$ and $\frac{5}{6}$ if $k \geq 4$. For a prime $p$, let $\tau = \tau(k, p)$ be the integer such that $p^\tau|k$ but $p^{\tau+1} \nmid k$. Define $\gamma = \gamma(k, p)$ by $\gamma(k, p) = \tau + 2$ when $p = 2$ and $\tau > 0$, and otherwise $\gamma(k, p) = \tau + 1$. Let $R = R(k) = \prod p^\gamma$ where the product is taken over all primes with $(p - 1)|k$. Then for any $\varepsilon > 0$ and $\ell > 2k(k - 1)$, if $n$ is a sufficiently large integer congruent to $\ell$ mod $R$, the equation

$$n = p_1^k + p_2^k + \cdots + p_\ell^k$$

has a solution in prime numbers $p_j$ with $|p_j - (n/\ell)^{1/k}| < ((n/\ell)^{1/k})^{\theta_k + \varepsilon}$. 

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The next difficulty we encounter is that when we take a disjoint union of $F$-free graphs, each component may not be regular of the same degree (indeed, each component may itself not be regular, but we ignore this for the moment). To make the vertices have the same degree, we will remove a regular subgraph from each component. In the case of $K_{3,3}$, each graph in the disjoint union will be regular with all vertices having even degree, and so by Petersen’s 2-factor theorem the edges of each component can be partitioned into 2-factors. Therefore, we may remove a spanning regular subgraph of the appropriate degree from each component in order to make the whole graph regular.

The argument for $K_{s,t}$ with $t > s!$ is more involved because, as mentioned earlier, each component will not be regular. We use Theorem 2.1 to take a disjoint union of norm graphs, which are almost regular but have absolute points of degree 1 fewer than the rest. We would like to remove a spanning regular subgraph of the appropriate degree from each component so that all of the absolute points in the graph have the same degree and all of the other points have degree 1 more. Then we add a matching to the absolute points in a way that preserves $K_{s,t}$-freeness, making the whole graph regular. To do this we will iteratively remove Hamilton cycles from the graph. Removing 2-factors iteratively would be just as good for our purpose, but we explain below why we are doing something seemingly much harder.

Much of the previous work on finding $k$-factors in graphs takes place in the setting where the host graph is regular. The main result of [22] allows the host graph to contain loops, and so applies in our setting. If we use this theorem, we do not know how many loops were contained in the $k$-factor, and so unfortunately we cannot complete the last step in the proof (adding a matching to the absolute points). We also note that a result of Alon, Freidland, and Kalai [4] finds regular subgraphs in almost regular graphs. However, it is crucial for our purpose that these subgraphs be spanning, which is not guaranteed by their theorem.

To overcome these difficulties, we will use a spectral approach. The following result guarantees Hamilton cycles in graphs with a reasonable spectral gap. The combinatorial Laplacian of a graph is the matrix $D - A$ where $D$ is the diagonal degree matrix and $A$ is the adjacency matrix.

**Theorem 2.2 (Butler and Chung [10])** Let $G$ be a graph on $n$ vertices with average degree $d$ and $0 = \mu_1 \leq \mu_1 \leq \cdots \leq \mu_{n-1}$ be the eigenvalues of the combinatorial Laplacian of $G$. There is a constant $c$ so that if

$$|d - \mu_i| \leq c \frac{(\log \log n)^2}{\log n (\log \log \log n)} d,$$

for all $i \neq 0$ and $n$ sufficiently large, then $G$ is Hamiltonian.

We note that much smaller spectral gaps guarantee connectivity of a graph, for example Theorem 4.3 of [24]. It is possible that one could use this connectivity to verify that the proofs of theorems on finding 2-factors in regular graphs will hold in our setting with a graph with loops. However, to keep the proof of Theorem 1.4 transparent, we will use Theorem 2.2 of Butler and Chung even though it is perhaps stronger than what we need.
As a final note on the proof, while our result on the spectral gap of norm graphs (Theorem 6.1) is much stronger than is necessary, we believe that it is interesting in its own right, adding to numerous papers which calculate the eigenvalues of algebraically defined graphs in extremal graph theory (see e.g. [11, 31, 15, 28, 34]).

To compute eigenvalues, we will need the following result on cyclotomic periods. This lemma is known but we provide a proof for completeness.

**Lemma 2.3** Let \( q \) be a prime power, \( \chi \) an additive character of \( \mathbb{F}_q \), and \( H \) a multiplicative subgroup of \( \mathbb{F}_q^* \). Then

\[
\left| \sum_{x \in H} \chi(x) \right| \leq \sqrt{q}.
\]

**Proof.** Let \( \gamma \) generate \( \mathbb{F}_q^* \), and let \( H \) be generated by \( \gamma^h \) where \( h|q-1 \). Consider the multiplicative character \( \theta \) defined by

\[
\theta(\gamma^k) = \exp \left( \frac{2\pi i}{h} \cdot k \right).
\]

Then the linear combination \( 1 + \theta + \theta^2 + \cdots + \theta^{h-1} \) evaluates to \( h \) on \( H \), and to 0 on the complement of \( H \). Therefore,

\[
\sum_{x \in H} \chi(x) = \frac{1}{h} \sum_{x \in \mathbb{F}_q^*} (1 + \theta + \cdots + \theta^{h-1})(x) \cdot \chi(x) = \frac{1}{h} \sum_{j=0}^{h-1} \sum_{x \in \mathbb{F}_q^*} \theta^j(x) \cdot \chi(x).
\]

Standard theorems on Gauss sums (e.g. Theorem 5.11 in [27]) give that each inner sum has modulus bounded by \( \sqrt{q} \). Using the triangle inequality completes the proof. \( \blacksquare \)

### 3 The regular Turán number of \( C_4 \)

In this section we prove Theorem 1.1. Given an \( n \), if \( n \) is even our strategy will be to construct a bipartite Cayley graph that is \( C_4 \)-free. This is the easy case. If \( n \) is odd, we will use a construction from geometry to find a regular \( C_4 \)-free graph on an odd number of vertices. Then we take this graph and a disjoint union of a bipartite Cayley graph to find a \( C_4 \)-free graph on \( n \) vertices. We will choose the size of the generating set of the Cayley graph to ensure that the entire graph is regular. Now for the details.

We begin with a lemma that is most certainly known. It proves that for any large enough integer \( M \), there is an \( M \times M \) \( C_4 \)-free bipartite graph that is \( k \)-regular where \( k \) may be taken asymptotically as large as \( (M/2)^{1/2} \).

**Lemma 3.1** There is an integer \( n_0 \) such that the following holds. For any integer \( M > n_0 \), there is an \( M \times M \) bipartite \( C_4 \)-free graph that is \( k \)-regular for any \( k \) with

\[
1 \leq k \leq (\lfloor M/2 \rfloor + 1)^{1/2} - (\lfloor M/2 \rfloor + 1)^{0.2025}.
\]
**Theorem 3.3**

For all odd \( n \) and \( \epsilon > 0 \), there is an \( n_0 \) such that for all \( n > n_0 \), the interval \( [x - x^{0.525}, x] \) contains a prime. We apply this result to obtain that for \( M > n_0 \), there is a prime \( p \) with

\[
\left( \left\lceil \frac{M}{2} \right\rceil + 1 \right)^{1/2} - \left( \left\lceil \frac{M}{2} \right\rceil + 1 \right)^{0.2625} \leq p \leq \left( \left\lceil \frac{M}{2} \right\rceil + 1 \right)^{1/2}.
\]

**Proof.** By a result of Baker, Harman, and Pintz \([6]\), there is an \( x_0 \) such that for all \( x > x_0 \), the interval \( [x - x^{0.525}, x] \) contains a prime. We apply this result to obtain that for \( M > n_0 \), there is a prime \( p \) with

\[
\left( \left\lfloor \frac{M}{2} \right\rfloor + 1 \right)^{1/2} - \left( \left\lfloor \frac{M}{2} \right\rfloor + 1 \right)^{0.2625} \leq p \leq \left( \left\lceil \frac{M}{2} \right\rceil + 1 \right)^{1/2}.
\]

Let \( A \subset \mathbb{Z}_{p^2-1} \) be a Bose-Chowla Sidon set (see \([9]\)). Thus, \( |A| = p \), and for any \( a_1, a_2, a_3, a_4 \in A \), the equation \( a_1 + a_2 \equiv a_3 + a_4 \pmod{p^2-1} \) implies \( \{a_1, a_2\} = \{a_3, a_4\} \).

Inequality \((\text{II})\) implies that \( p^2 - 1 \leq M/2 \). Since \( A \subset \mathbb{Z}_{p^2-1} = \{1, 2, \ldots, p^2 - 1\} \), we may view \( A \) as a subset of \( \{1, 2, \ldots, M/2\} \). For any \( k \) with \( k \leq (\left\lfloor \frac{M}{2} \right\rfloor + 1)^{1/2} - (\left\lfloor \frac{M}{2} \right\rfloor + 1)^{0.2625} \), we may choose a subset \( A \subseteq A \) with \( |A| = k \). Define an \( M \times M \) bipartite graph with parts \( X = \mathbb{Z}_M \) and \( Y = \mathbb{Z}_M \) where \( x \in X \) is adjacent to \( y \in Y \) if and only if \( x + y \equiv a \pmod{M} \) for some \( a \in A \). This bipartite graph has parts of size \( M \), and is regular of degree \( |A| = k \). We finish the proof of the lemma by showing that this graph is \( C_4 \)-free.

Suppose \( x_1y_1x_2y_2 \) is a 4-cycle with \( x_1, x_2 \in X; y_1, y_2 \in Y \). There are elements \( a, b, c, d \in A \) such that

\[
x_1 + y_1 \equiv a \pmod{M}, \quad x_1 + y_2 \equiv b \pmod{M},
\]
\[
x_2 + y_1 \equiv c \pmod{M}, \quad x_2 + y_2 \equiv d \pmod{M}.
\]

This system of congruences implies

\[
a - b + c - d \equiv 0 \pmod{M} \quad \Rightarrow \quad a + c \equiv b + d \pmod{M}.
\]

Recalling that \( A \), hence \( A \), is contained in \( \{1, 2, \ldots, M/2\} \), this last congruence can be turned into an equality in \( \mathbb{Z} \) so \( a + c = b + d \). Taking this equation modulo \( p^2 - 1 \) and using the fact that \( A \) is a Sidon set gives \( \{a, c\} = \{b, d\} \). If \( a \equiv b \pmod{p^2 - 1} \), then \( a \equiv b \pmod{M} \) which implies \( y_1 \) and \( y_2 \) are the same vertex. A similar contradiction occurs if \( a \equiv d \pmod{M} \).

**Corollary 3.2** There is an \( n_0 \) such that for all \( n > n_0 \) with \( n \) even, there is a \( C_4 \)-free \( n \)-vertex graph that is \((\left\lceil \frac{n}{2} \right\rceil + 1)^{1/2} - (\left\lceil \frac{n}{2} \right\rceil + 1)^{0.2625}\)-regular.

For odd \( n \), our lower bound will be obtained by taking the disjoint union of two \( C_4 \)-free graphs. One of these graphs is an induced subgraph of the Erdős-Rényi orthogonal polarity graph \( ER_q \).

Let \( q \) be a power of an odd prime. Parsons \([31]\) (see also \([37]\)) showed that there is a \( C_4 \)-free \( \frac{q-1}{2} \)-regular graph on \( \binom{q+1}{2} \) vertices, and another on \( \binom{q}{2} \) vertices. We denote these graphs by \( R_{1,q} \) and \( R_{2,q} \), respectively. These are induced subgraphs of \( ER_q \) and more on these subgraphs, and \( ER_q \) in general, can be found in Williford’s Ph.D. thesis \([37]\).

**Theorem 3.3** Let \( 0 < \epsilon < \frac{1}{100} \). There is an \( n_0 = n_0(\epsilon) \) such that the following holds.

For all odd \( n > n_0 \), there is a \( C_4 \)-free \( n \)-vertex graph that is \((\sqrt{1/6 - \epsilon}) n^{1/2}\)-regular.
Proof. Let $0 < \epsilon < \frac{1}{100}$ and let $\delta = \frac{2}{3} - \epsilon$. Let $n$ be large enough so that there is a prime $p$ with
\[
\lfloor \sqrt{\delta n} \rfloor - \lfloor \sqrt{\delta n} \rfloor^{0.525} \leq p \leq \lfloor \sqrt{\delta n} \rfloor.
\]
(2)
Define
\[
N = \begin{cases} 
  n - \binom{p+1}{2} & \text{if } p \equiv 1 \pmod{4}, \\
  n - \binom{p}{2} & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\]
Since $n$ is odd, $N$ is even by definition and we let $N = 2M$. We will now assume that $p \equiv 1 \pmod{4}$ as the proof in the case when $p \equiv 3 \pmod{4}$ is similar.

The graph we construct will be the disjoint union of two graphs, one of which is a bipartite graph obtained from applying Lemma 3.1. The other is $R_{1,p}$, which has $\binom{p+1}{2}$ vertices and is $\frac{p-1}{2}$-regular. We wish to apply Lemma 3.1 to obtain a $\frac{N}{2} \times \frac{N}{2}$ bipartite graph $B_1$ that has $N = n - \binom{p+1}{2}$ vertices and is $\frac{p-1}{2}$-regular. To do so, we need
\[
1 \leq \frac{p-1}{2} \leq \left(\lfloor \frac{N}{4} \rfloor + 1\right)^{1/2} - \left(\lfloor \frac{N}{4} \rfloor \right)^{0.2625}.
\]
(3)
By (2), $p = (1 + o(1))\sqrt{\frac{2}{3} - \epsilon)n}$. By definition of $N$,
\[
N = n - (1 + o(1))\frac{p^2}{2} = n - (1 + o(1))\frac{2/3 - \epsilon}{2}n = \left(\frac{2}{3} + \frac{\epsilon}{2} + o(1)\right)n.
\]
Thus, the right hand side of (3) is
\[
\left(\lfloor \frac{N}{4} \rfloor + 1\right)^{1/2} - \left(\lfloor \frac{N}{4} \rfloor \right)^{0.2625} = (1 + o(1))\frac{1}{2}\sqrt{\frac{2}{3} + \epsilon/2 + o(1)n},
\]
which, with
\[
\frac{p-1}{2} = (1 + o(1))\frac{1}{2}\sqrt{\frac{2}{3} - \epsilon)n},
\]
shows that (3) holds for large enough $n$. By Lemma 3.1 there is a $\frac{p-1}{2}$-regular $\frac{N}{2} \times \frac{N}{2}$ bipartite graph that is $C_4$-free. Taking the disjoint union of this bipartite graph and $R_{1,p}$ proves Theorem 3.3 in the case $p \equiv 1 \pmod{4}$.

4 The regular Turán number of $K_{2,t}$

In this section we prove Theorem 1.2. Similar to the previous section, our strategy will be to use either a bipartite Cayley graph, or the disjoint union of a regular graph on an odd number of vertices and a bipartite Cayley graph. The first step is to construct a graph similar to the $K_{2,t+1}$-free graphs of Füredi [18]. While Füredi’s constructions are algebraically defined graphs, ours will be written as Cayley sum graphs. We note that in [29] and [30] bipartite $K_{2,t}$-free graphs of the same flavor are constructed.

Let $p$ be an odd prime and let $\theta$ be a generator of the multiplicative group $\mathbb{F}_p^\ast$. Suppose $t \geq 1$ is an integer that divides $p - 1$. Let $\Gamma = \mathbb{Z}_{(p-1)/t} \times \mathbb{F}_p$ and $\mu = \theta^{\frac{p-1}{t}}$. Define
\[
S = \{(m, \theta^m\mu^n) : m \in \mathbb{Z}_{(p-1)/t}, 0 \leq n \leq t - 1\}.
\]
The set \( S \) can also be written as
\[
S = \left\{ \left( a \pmod{\frac{p-1}{t}}, \theta^a \pmod{p} \right) : a \in \mathbb{Z}_{p-1} \right\},
\]
where we use the least residues \( \mathbb{Z}_{p-1} = \{0, 1, \ldots, p-2\} \).

Let \( H_{p,t} \) be the graph with vertex set \( \Gamma \), and distinct vertices \((x, y)\) and \((a, b)\) are adjacent if
\[
(x, y) + (a, b) \in S.
\]
Thus, \((x, y)\) and \((a, b)\) are adjacent if and only if there is an \( m \in \mathbb{Z}_{(p-1)/t} \) and \( n \in \{0, 1, \ldots, t-1\} \) such that
\[
x + a \equiv m \pmod{\frac{p-1}{t}} \quad \text{and} \quad y + b \equiv \theta^m \mu^n \pmod{p}.
\]
This graph is a modification of a graph constructed by Ruzsa [32].

**Lemma 4.1** The graph \( H_{p,t} \) is \( K_{2,t+1} \)-free.

**Proof.** Suppose \((x, y)\) and \((u, v)\) are two distinct vertices with \( t + 1 \) common neighbors \((s_i, w_i)\), \( 1 \leq i \leq t + 1 \). There are elements \( a_i, b_i \in \mathbb{Z}_{p-1} \) such that
\[
x + a_i \equiv a_i \pmod{\frac{p-1}{t}}, \quad y + w_i \equiv \theta^{a_i} \pmod{p},
\]
\[
u + s_i \equiv b_i \pmod{\frac{p-1}{t}}, \quad v + w_i \equiv \theta^{b_i} \pmod{p}.
\]
Therefore, \( x - u \equiv a_i - b_i \pmod{\frac{p-1}{t}} \) and \( y - v \equiv \theta^{a_i} - \theta^{b_i} \pmod{p} \). The first congruence implies that there is an integer \( \delta_i \) such that \( x - u = a_i - b_i + \delta_i \left( \frac{p-1}{t} \right) \) in \( \mathbb{Z} \). Hence,
\[
\theta^{x-u} \equiv \theta^{a_i-b_i+\delta_i(p-1)/t} \equiv \theta^{a_i-b_i} \mu^{\delta_i} \pmod{p}.
\]
The exponent \( \delta_i \) may be taken modulo \( t \) since \( \mu = \theta^{(p-1)/t} \), and so we let \( \delta_i^* = \delta_i \pmod{t} \) where \( \delta_i^* \in \{0, 1, \ldots, t-1\} \). Since \( i \) ranges from 1 to \( t + 1 \), there exists \( i, j \) with \( 1 \leq i < j \leq t + 1 \) and \( \delta_i^* = \delta_j^* \). This gives
\[
\theta^{a_i-b_i} \mu^{\delta_i^*} \equiv \theta^{a_j-b_j} \mu^{\delta_j^*} \pmod{p}
\]
so \( \theta^{a_i-b_i} \equiv \theta^{a_j-b_j} \pmod{p} \). Let \( A = \theta^{a_i} \theta^{b_j} = \theta^{a_j} \theta^{b_i} \). Using the fact that \( y - v \equiv \theta^{a_i} - \theta^{b_i} \pmod{p} \), we let
\[
B = \theta^{a_i} + \theta^{b_i} \equiv \theta^{a_j} + \theta^{b_i} \pmod{p}.
\]
The pairs \( \{\theta^{a_i}, \theta^{b_j}\} \) and \( \{\theta^{a_j}, \theta^{b_i}\} \) are the roots of \( X^2 - BX + A \) in \( \mathbb{F}_p \). By unique factorization in \( \mathbb{F}_p[x] \), \( \{\theta^{a_i}, \theta^{b_j}\} = \{\theta^{a_j}, \theta^{b_i}\} \). If \( a_i \equiv a_j \pmod{p} \) and \( b_j \equiv b_i \pmod{p} \), then the vertices \((s_i, w_i)\) and \((s_j, w_j)\) are the same, a contradiction. If \( a_i \equiv b_i \pmod{p} \) and \( b_j \equiv a_j \pmod{p} \), then the vertices \((x, y)\) and \((u, v)\) are the same, another contradiction. This shows \( H_{p,t} \) is \( K_{2,t+1} \)-free.

**Lemma 4.2** The graph \( H_{p,t} \) contains \( p-1 \) vertices of degree \( p-2 \), and all other vertices have degree \( p-1 \).
**Proof.** Let \((x, y)\) be a vertex in \(H_{p, t}\). Then, since \(|S| = p - 1\), the vertex \((x, y)\) has degree \(p - 1\) unless
\[
(x, y) + (x, y) = (m, \theta^m \mu^n)
\]
for some \(m \in \mathbb{Z}_{(p-1)/t}\) and \(n \in \{0, 1, \ldots, t - 1\}\). From \(x + x \equiv m(\mod \frac{p-1}{t})\), we get
\[
2x = m + \delta \left(\frac{p-1}{t}\right)
\]
for some integer \(\delta\). Then
\[
2y \equiv \theta^{2x-\delta(\frac{p-1}{t})} \mu^n \equiv \theta^{2x-\delta} (\mod p).
\]
Therefore,
\[
(x, y) = (x, 2^{-1} \theta^{2x} \mu^{-\delta}).
\]
There are \(\frac{p-1}{t}\) choices for \(x\) and \(t\) choices for \(n - \delta\) (this exponent can be taken modulo \(t\) since \(\mu = \theta^{(p-1)/t}\)) which gives \(p - 1\) vertices of degree \(p - 2\). Conversely, one can check that any vertex of the form \((x, 2^{-1} \theta^{2x} \mu^r)\) with \(r \in \{0, 1, \ldots, t - 1\}\) will have degree \(p - 2\). \(\Box\)

Following the standard terminology, vertices of degree \(p - 2\) in \(H_{p, t}\) are called absolute points. Let \(H^*\) be the supergraph of \(H_{p, t}\) obtained by adding a new vertex \(a\) that is adjacent to all of the absolute points of \(H_{p, t}\). By Lemma 4.1, the graph \(H^*\) has \(1 + \frac{p(p-1)}{t}\) vertices and is \((p - 1)\)-regular. We now show that \(H^*\) is \(K_{2, 2t+1}\)-free.

**Lemma 4.3** The graph \(H^*\) is \(K_{2, 2t+1}\)-free.

**Proof.** By Lemma 4.1, any \(K_{2, 2t+1}\) in \(H^*\) must use the added vertex \(a\). We will show that in \(H_{p, t}\), no vertex is adjacent to \(2t + 1\) absolute points and so \(H^*\) will be \(K_{2, 2t+1}\)-free.

Suppose \((x, y)\) is a vertex in \(H_{p, t}\) that is adjacent to absolute points \((z_i, 2^{-1} \theta^{2z_i} \mu^{r_i})\) for some \(1 \leq i \leq D\), where \(z_i \in \mathbb{Z}_{\frac{p-1}{t}}\) and \(r_i \in \mathbb{Z}_t\). We must show \(D \leq 2t\). Then we have
\[
(x, y) + (z_i, 2^{-1} \theta^{2z_i} \mu^{r_i}) = (m_i, \theta^{m_i} \mu^{n_i})
\]
for some \(m_i \in \mathbb{Z}_{\frac{p-1}{t}}\) and \(n_i \in \mathbb{Z}_t\). Thus, \(x + z_i \equiv m_i(\mod \frac{p-1}{t})\) so \(x + z_i = m_i + \delta_i \left(\frac{p-1}{t}\right)\) for some integer \(\delta_i\). This last equation, with
\[
y + 2^{-1} \theta^{2z_i} \mu^{r_i} - \theta^{m_i} \mu^{n_i} \equiv 0(\mod p),
\]
implies
\[
y + 2^{-1} \theta^{2z_i} \mu^{r_i} - \theta^{x+z_i} \mu^{n_i-\delta_i} \equiv 0(\mod p).
\]
Therefore, \(\theta^{z_i}\) is a root of the degree 2 polynomial
\[
f(X) = 2^{-1} \mu^{r_i} X^2 - \theta^x \mu^{n_i-\delta_i} X + y.
\]
There are \(t\) choices for \(r_i\) and then at most 2 choices for \(\theta^{z_i}\) since \(f(X)\) has at most two roots. Hence, \((x, y)\) is adjacent to at most 2 absolute points, so \(D \leq 2t\). \(\Box\)

**Corollary 4.4** Let \(p\) be a prime and \(t \geq 1\) be an integer for which \(t\) divides \(p - 1\). Then the graph \(H^*\) is a \((p - 1)\)-regular graph with \(\frac{p(p-1)}{t} + 1\) vertices and is \(K_{2, 2t+1}\)-free.
The next step is to prove a version of Lemma 4.1 for $K_{2,2t+1}$. For this we use a set constructed in [35] that was used to solve a bipartite Turán problem in $k$-partite graphs.

Let $q$ be a power of an odd prime and suppose $t \geq 1$ is an integer for which $t$ divides $q - 1$. Let $H$ be the subgroup of $\mathbb{Z}_{(q^2-1)/t}$ generated by $\frac{q^2-1}{t}$, and let $\mathcal{A} = \{ \alpha \in \mathbb{Z}_{(q^2-1)/t} : \theta^a - \theta \in \mathbb{F}_q \}$ be a Bose-Chowla Sidon set [9]. Let $\Gamma$ be the quotient group $\mathbb{Z}_{q^2-1}/H \cong \mathbb{Z}_{(q^2-1)/t}$. Finally, in the quotient group $\Gamma$, let $A$ be the set defined by

$$A = \{ a + H : a \in \mathcal{A} \}.$$  

In [35] it is shown that $|A| = q$, and that for any nonzero $\alpha \in \Gamma$, the number of ordered pairs $(a, b) \in A \times A$ for which $\alpha = a - b$ in $\Gamma$ is at most $t$. We state this as a lemma.

**Lemma 4.5 ([35])** If $t \geq 1$ and $q$ is a power of an odd prime with $q \equiv 1 \pmod{t}$, then there is a set $A \subset \mathbb{Z}_{(q^2-1)/t}$ with $|A| = q$ such that for any $\alpha \in \mathbb{Z}_{(q^2-1)/t} \setminus \{0\}$, there are at most $t$ ordered pairs $(a, b) \in A \times A$ such that

$$a - b \equiv \alpha \left( \begin{array}{c} \mod \frac{q^2-1}{t} \end{array} \right).$$

**Lemma 4.6** Let $t \geq 1$ be an integer and $\epsilon > 0$ be a positive real number. There is an $n_0 = n_0(t, \epsilon)$ such that the following holds. For any $M > n_0$ and $k$ with

$$1 \leq k \leq (1 - \epsilon) \left( \left\lceil \frac{tM}{2} \right\rceil + 1 \right)^{1/2},$$

there is a $k$-regular $M \times M$ bipartite graph that is $K_{2,2t+1}$-free.

**Proof.** Let $t \geq 1$. Choose a prime $p$ with $p \equiv 1 \pmod{t}$ and

$$(1 - \epsilon) \left( \left\lceil \frac{tM}{2} \right\rceil + 1 \right)^{1/2} \leq p < \left( \left\lceil \frac{tM}{2} \right\rceil + 1 \right)^{1/2}.$$ (4)

This can be done by Dirichlet’s Theorem on primes in arithmetic progressions, and in particular the Siegel-Walfisz Theorem. Indeed, the Siegel-Walfisz Theorem gives that for large enough $M$, the number of primes $p$ with $p \equiv 1 \pmod{t}$ satisfying (4) is at least

$$\frac{\epsilon(tM/2)^{1/2}}{2\phi(t) \ln(tM/2)} - O \left( \frac{M^{1/2}}{\ln^2(M)} \right).$$

This is positive for large enough $M$ depending on $\epsilon$ and $t$. Here $\phi(t)$ is Euler’s totient function. Let $A \subset \mathbb{Z}_{(p^2-1)/t}$ be as in Lemma 4.5. By (4), we may view $A \subset \{1, 2, \ldots, \lfloor M/2 \rfloor \}$ since $\frac{p^2-1}{t} < \lfloor \frac{M}{2} \rfloor$. For any $k$ with

$$1 \leq k \leq (1 - \epsilon) \left( \left\lceil \frac{tM}{2} \right\rceil + 1 \right)^{1/2},$$

we may choose a subset $A' \subseteq A$ with $|A'| = k$. Thus, $A'$ is a $k$ element subset of $\mathbb{Z}_M$ that is contained in the “first half” $\{1, 2, \ldots, \lfloor M/2 \rfloor \}$ of $\mathbb{Z}_M$. 

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Define an $M \times M$ bipartite graph with parts $X = \mathbb{Z}_M$ and $Y = \mathbb{Z}_M$ where $x \in X$ is adjacent to $y \in Y$ if and only if

$$x + y \equiv a \pmod{M}$$

for some $a \in A'$. This graph is $k$-regular. We complete the proof by showing that it is $K_{2,2t+1}$-free. Let $x_1, x_2$ be distinct vertices in $X$, say with $1 \leq x_2 < x_1 \leq M$, and suppose this pair of vertices is adjacent $2t + 1$ vertices $y_1, y_2, \ldots, y_{2t+1} \in Y$. Then there are elements $a_i, b_i \in A'$ such that

$$x_1 + y_i \equiv a_i \pmod{M} \quad \text{and} \quad x_2 + y_i \equiv b_i \pmod{M}$$

for $1 \leq i \leq 2t + 1$. Hence,

$$x_1 - x_2 \equiv a_i - b_i \pmod{M} \quad (5)$$

for each $i$. Now $x_1 - x_2 \in \{1, 2, \ldots, M - 1\}$, and since $A' \subset \{1, 2, \ldots, \lfloor M/2 \rfloor\}$, we know $-[M/2] < a_i - b_i < [M/2]$. Thus, from (5) we get

$$x_1 - x_2 = a_i - b_i + \delta_i M$$

where $\delta_i \in \{0, 1\}$. If $\delta_i = 0$ for $t + 1$ distinct $i$, say $1 \leq i \leq t + 1$, then $x_1 - x_2 = a_i - b_i$ (in $\mathbb{Z}$) which gives $x_1 - x_2 \equiv a_i - b_i \pmod{2^t - 1}$. By our assumption on $A$, this forces $x_1 \equiv x_2 \pmod{2^t - 1}$ and so $a_i = b_i$. Combining this with (5) gives $x_1 \equiv x_2 \pmod{M}$ which is a contradiction because $x_1$ and $x_2$ are distinct vertices. Now assume $\delta_i = 1$ for $t + 1$ distinct $i$, again say $1 \leq i \leq t + 1$. This gives $x_1 - x_2 - M = a_i - b_i$ (in $\mathbb{Z}$) and so $x_1 - x_2 - M \equiv a_i - b_i \pmod{2^t - 1}$. This congruence gives a similar contradiction as before. The conclusion is that this bipartite graph is indeed $K_{2,2t+1}$-free.

**Corollary 4.7** Let $t \geq 1$ be an integer and $\epsilon > 0$. There is an $n_0 = n_0(t, \epsilon)$ such that the following holds. For all even $n > n_0$, there is an $n$-vertex $K_{2,2t+1}$-free graph that is $k$-regular where $k \geq (1 - \epsilon)(tn/4)^{1/2}$.

The last result of this section deals with the case when $n$ is odd.

**Theorem 4.8** Let $t \geq 1$ be an even integer and let $\epsilon > 0$. There is an $n_0 = n_0(t, \epsilon)$ such that for all odd $n \geq n_0$, there is a $k$-regular $n$-vertex $K_{2,2t+1}$-free graph with

$$k \geq (1 - 2\epsilon)^{1/2} \sqrt{tn/5}.$$

**Proof.** Let $t \geq 1$ be an even integer and write $t = 2^r s$ where $r \geq 1$ and $s$ is odd. Let $\epsilon > 0$, $n > n_0$ be an odd integer, and $p$ be a prime with

$$(1 - 2\epsilon)^{1/2} \sqrt{tn/5} \leq p \leq (1 - \epsilon)^{1/2} \sqrt{tn/5}$$

and

$$p \equiv 1 + 2^r s \pmod{2^{r+1}s}.$$
Such a prime exists by the Siegel-Walfisz Theorem (note \(\gcd(1 + 2^r s, 2^{r+1} s) = 1\)). Define \(N\) by

\[ n = \frac{p(p - 1)}{t} + N. \]

The assumption on \(p\) implies that there is an integer \(\alpha\) such that \(p - 1 = 2^r s + \alpha 2^{r+1} s\). Then

\[ \frac{p(p - 1)}{t} = \frac{p(2^r s + \alpha 2^{r+1} s)}{2^r s} = p(1 + 2\alpha) \]

which is odd. Therefore, \(N\) is even, say \(2M = N\). We now wish to apply Lemma 4.6 with \(M = N/2\). To do so, we will need

\[ p - 1 \leq (1 - \epsilon)((tN/4] + 1)^{1/2}. \]

Now

\[ N = n - \frac{p(p - 1)}{t} \geq n - \frac{p^2}{t} \geq n - (1 - \epsilon)n/5 = \left(\frac{4}{5} + \epsilon\right)n. \]

Thus,

\[ (1 - \epsilon)((tN/4] + 1)^{1/2} \geq (1 - \epsilon)(tN/4)^{1/2} \geq (1 - \epsilon)(t/4(4/5 + \epsilon)n)^{1/2} \]

\[ = (1 - \epsilon)(1/5 + \epsilon/4)^{1/2} (tn)^{1/2} \]

\[ \geq \sqrt{1 - \epsilon}(tn/5)^{1/2} > p - 1 \]

where the second to last inequality follows since \(\epsilon < \frac{1}{5}\), and the last inequality follows since \(p - 1 < \sqrt{1 - \epsilon}(tn/5)^{1/2}\). We apply Lemma 4.6 to obtain an \(N/2 \times N/2\) bipartite graph that is \(K_{2,2t+1}\)-free and \((p - 1)\)-regular. Taking the disjoint union of this bipartite graph together with the graph \(H^*_{p,t}\) from Corollary 4.4 gives a \((p - 1)\)-regular graph on \(n\) vertices that is \(K_{2,2t+1}\)-free. Finally, observe

\[ p - 1 \geq \sqrt{1 - 2\epsilon}(tn/5)^{1/2}. \]

5 The regular Turán number of \(K_{3,3}\)

In this section we prove Theorem 1.3. The outline of the proof is to take several disjoint copies of regular \(K_{3,3}\)-free graphs constructed by Brown [8] and to remove a regular subgraph from each component so that the entire graph is regular.

Let \(p\) be an odd prime and write \(\eta\) for the quadratic character on \(\mathbb{F}_p\). Brown [8] constructed \(K_{3,3}\)-free that gave an asymptotically tight lower bound on the Turán number of \(K_{3,3}\). This graph, which we denote by \(B(p, \alpha)\), is defined as follows. For an odd prime \(p\) and \(\alpha \in \mathbb{F}_p\) satisfying \(\eta(\alpha) = -\eta(-1)\), let \(B(p, \alpha)\) be the graph with vertex set \(\mathbb{F}_p^3\) where \((x, y, z)\) is adjacent to \((a, b, c)\) if

\[ (x - a)^2 + (y - b)^2 + (z - c)^2 = \alpha. \]
The graph $B(p, \alpha)$ is $(p^2 - p)$-regular. Using Theorem 2.4 gives that for sufficiently large $n$, there are primes $p_1, \ldots, p_k$ where $k = 13$ if $n$ is odd, and $k = 14$ if $n$ is even, such that
\[
n = \sum_{i=1}^{k} p_i^3 \quad \text{and} \quad |p_j - (n/k)|^{1/3} \leq n^{4/15 + \epsilon}
\]
for $1 \leq j \leq k$.

We briefly remark that using a theorem of Kumchev and Liu [25] would give a better error term.

**Theorem 5.1** Let $\epsilon > 0$ be arbitrary. For all sufficiently large $n$, there is an $n$-vertex $K_{3,3}$-free graph that is $k$-regular where $k \geq (n/13)^{2/3} - O(n^{3/5 + \epsilon})$ when $n$ is odd, and $k \geq (n/14)^{2/3} - O(n^{3/5 + \epsilon})$ when $n$ is even.

**Proof.** First assume that $n$ is an odd integer large enough so that there are primes $p_1, \ldots, p_{13}$ for which $n = \sum_{i=1}^{13} p_i^3$ and
\[
|p_i - (n/13)|^{1/3} \leq n^{4/15 + \epsilon}
\]
for $1 \leq i \leq 13$. We may assume that $p_1 \geq p_2 \geq \cdots \geq p_{13}$. Let $G$ be the disjoint union of the Brown graphs $B(p_i, \alpha_i)$ for $1 \leq i \leq 13$ and some choice of $\alpha_i$. Clearly $G$ is $K_{3,3}$-free and has $n$ vertices. We now remove edges from $G$ to obtain a $(p_{13}^2 - p_{13})$-regular graph. This will complete the proof since
\[
p_{13} \geq (n/13)^{1/3} - n^{4/15 + \epsilon} \implies p_{13}^2 - p_{13} \geq \left(\frac{n}{13}\right)^{2/3} - O(n^{3/5 + \epsilon}).
\]

Let $i \in \{1, 2, \ldots, 12\}$. Consider the graph $B(p_i, \alpha_i)$. For any $i$,
\[
|p_i - p_{13}| \leq |p_1 - p_{13}| \leq |p_1 - (n/13)|^{1/3} + |(n/13)|^{1/3} - p_{13} \leq 2n^{4/15 + \epsilon}.
\]
Hence,
\[
|(p_i^2 - p_i) - (p_{13}^2 - p_{13})| \leq 8n^{9/15 + \epsilon}.
\]

Define $k_i$ by $2k_i = (p_i^2 - p_i) - (p_{13}^2 - p_{13})$ (observe $k_i$ is an integer since the right hand side is even). Now each $B(p_i, \alpha_i)$ is $(p_i^2 - p_i)$-regular so by Petersen’s 2-factor theorem we may repeatedly remove 2-factors a total of $k_i$ times, and the result is that each component is $(p_{13}^2 - p_{13})$-regular.

The argument in the case when $n$ is even the same with the exception that we must write $n$ as $\sum_{i=1}^{14} p_i^3$ instead of a sum with 13 terms. 

**6 The regular Turán number of $K_{s,t}$ when $t > s!$**

In this section, let $s$ and $t$ be fixed and $t > s!$. We will use the norm-graphs from [21], defined as follows. For $q$ a prime power and $a \in \mathbb{F}_q^*$, let $N(a)$ be the $\mathbb{F}_q$ norm of $a$, that is
\[
N(a) = a \cdot a^q \cdot a^{q^2} \cdot \cdots \cdot a^{q^{s-1}} = a^{(q^s-1)/(q-1)} \in \mathbb{F}_q.
\]
The norm-graph has vertex set \( \mathbb{F}_q^* \) and \( a \sim b \) if \( N(a + b) = 1 \). If \( N(a + a) = 1 \) we call \( a \) an *absolute point*. Let \( N_{q,s} \) be the norm-graph with the loops removed from the absolute points and \( N_{q,s}^o \) be the norm-graph including the loops. The number of solutions in \( \mathbb{F}_q^* \) to the equation \( N(x) = 1 \) is \( \frac{q^s - 1}{q - 1} \) (see [27] or [21]). Therefore, the graph \( N_{q,s}^o \) is \( \frac{q^s - 1}{q - 1} \)-regular (counting loops as one neighbor), and the graph \( N_{q,s} \) has \( \frac{q^s - 1}{q - 1} \) vertices of degree \( \frac{q^s - 1}{q - 1} \) and \( q^s - \frac{q^s - 1}{q - 1} \) vertices of degree \( \frac{q^s - 1}{q - 1} \). In [21], it is shown that \( N_{q,s} \) is \( K_{s,s!+1} \)-free (and hence \( K_{s,s} \)-free).

The outline of the proof of Theorem 1.4 is as follows. Let \( n \) be fixed and sufficiently large, and we will construct a regular \( K_{s,t} \)-free graph with \( \Omega \left( n^{2 - 1/s} \right) \) edges. We use Theorem 2.1 to write \( n \) as a sum of \( s' \)th powers of primes that are almost equal. We take a disjoint union of norm-graphs whose number of vertices is equal to the \( s' \)th powers of the primes. We use Theorem 2.2 to remove edges from these graphs until most vertices have the same degree and the absolute points have degree 1 fewer. Finally, we add a matching to the absolute points to make the graph regular while making sure that it remains \( K_{s,s} \)-free. We now proceed with the details.

Let \( n \) be fixed. By Theorem 2.1 for \( n \) sufficiently large, there is a constant \( c_s \) which depends only on \( s \) such that we may write

\[
n = p_1^s + p_2^s + \cdots + p_l^s,
\]

where each \( p_j \) is a prime satisfying \( |p_j - (n/\ell)^{1/s}| \leq \left((n/\ell)^{1/s}\right)^{9/10} \) and \( \ell \leq c_s \). Without loss of generality, assume that \( p_1 \geq p_2 \geq \cdots \geq p_l \). Let \( G_1 \) be the graph on \( n \) vertices which is the disjoint union of the norm-graphs \( N_{p_i,s} \) for \( 1 \leq i \leq \ell \). For brevity, call these components \( N_1, \ldots, N_\ell \). Before we can use Theorem 2.2 to equalize the degrees between components, we must show that the norm-graphs have a good spectral gap.

**Theorem 6.1** Let \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{q^s} \) the the eigenvalues of the adjacency matrix of \( N_{q,s}^o \). Then for \( i > 1 \), we have

\[
|\lambda_i| \leq \sqrt{q^s}.
\]

**Proof.** Given an abelian group \( \Gamma \) and a subset \( S \subset \Gamma \), we define the *Cayley sum graph* \( \text{CayS}(\Gamma, S) \) as the graph with vertex set \( \Gamma \) and \( u \sim v \) if and only if \( u + v \in \Gamma \). Let \( S_1 := \{ a \in \mathbb{F}_q^* : N(a) = 1 \} \) be the subset of \( \mathbb{F}_q^* \) with norm 1. Then the norm-graph \( N_{q,s}^o \) can be written as the Cayley sum graph \( \text{CayS}(\mathbb{F}_q^*, +, S_1) \). The eigenvalues of Cayley sum graphs are given by character sums. Given a character \( \chi \) of \( \Gamma \), let

\[
\chi(S) = \sum_{x \in S} \chi(x).
\]

Then all of the eigenvalues of \( \text{CayS}(\Gamma, S) \) are given by \( \chi(S) \) (when \( \chi \) is real valued) or \( \pm|\chi(S)| \) (if \( \chi \) is complex valued) as \( \chi \) ranges over all of the additive characters of \( \Gamma \) (see [2], [14], or [16]).

Note that \( S_1 \) is a multiplicative subgroup of \( \mathbb{F}_q^* \). The largest eigenvalue of \( N_{q,s}^o \) is \( |S_1| \) and corresponds to the trivial additive character. The proof is complete after bounding the remaining eigenvalues by applying the Lemma 2.3.
We use Theorem 6.1 to show that norm-graphs have almost regular spanning subgraphs.

**Theorem 6.2** Let $\epsilon > 0$ and $k \in \mathbb{N}$. For $q$ sufficiently large, if $k < q^{s-1-\epsilon}$ then the norm-graph $N_{q,s}$ contains spanning subgraphs with each of the following degree sequences.

(a) $q^s - \frac{q^s-1}{q-1}$ vertices that have degree

$$\frac{q^s-1}{q-1} - 2k,$$

and the remaining $\frac{q^s-1}{q-1}$ vertices have degree

$$\frac{q^s-1}{q-1} - 2k - 1.$$

(b) $q^s - \frac{q^s-1}{q-1} + 1$ vertices of degree

$$\frac{q^s-1}{q-1} - 2k + 1,$$

and the remaining $\frac{q^s-1}{q-1} - 1$ vertices of degree

$$\frac{q^s-1}{q-1} - 2k.$$

**Proof.** We construct a sequence of graphs $G_0, G_1, \ldots, G_k$ and show that $G_k$ has the degree sequence that we require. Let $G_0 = N_{q,s}$ and note that $G_0$ has $q^s - \frac{q^s-1}{q-1}$ vertices of degree $\frac{q^s-1}{q-1}$ and the remaining vertices (the absolute points) have degree 1 fewer. If each $G_j$ is Hamiltonian, then remove a Hamilton cycle from it to create $G_{j+1}$ fewer. If we reach $G_{k-1}$. For part (a), remove a Hamilton cycle from $G_{k-1}$ and for part (b), remove a matching on $q^s-1$ vertices where the one vertex not incident to an edge in the matching is an absolute point (this must exist if $G_{k-1}$ is Hamiltonian). Since $G_k$ has the correct degree sequence, it suffices to show that each $G_j$ is Hamiltonian.

We prove that each $G_j$ is Hamiltonian using Theorem 2.2 and induction. Let $0 = \mu_0(G_j) \leq \mu_1(G_j) \leq \cdots \leq \mu_{q^s-1}(G_j)$ be the eigenvalues of the combinatorial Laplacian of $G_i$. Let $d(G_j)$ be the average degree of $G_j$ and note that because $k < q^{s-1-\epsilon}$, we have $d(G_j) \sim q^{s-1}$. Therefore, if

$$|d(G_j) - \mu_i(G_j)| \leq q^{s/2} + 1 + 6j = O(q^{s-1-\epsilon}),$$

for all $i \neq 0$, then we may apply Theorem 2.2 to conclude that $G_j$ is Hamiltonian. We prove (7) by induction. Let $i > 0$ be fixed. When $j = 0$, notice that the combinatorial Laplacians of $N_{q,s}$ and $N_{q,s}^\circ$ are in fact the same matrix. That is

$$D(G_0) - A(G_0) = D(N_{q,s}) - A(N_{q,s}) = D(N_{q,s}^\circ) - A(N_{q,s}^\circ) = \left(\frac{q^s-1}{q-1}\right) I - A(N_{q,s}^\circ).$$
By Theorem 6.1, this implies that we have
\[
\left| \frac{q^s - 1}{q - 1} - \mu_i(N_{q,s}) \right| \leq q^{s/2}.
\]
Since the average degree of \( N_{q,s} \) is between \( \frac{q^s - 1}{q - 1} \) and \( \frac{q^s - 1}{q - 1} \), we have that \( |d(G_0) - \mu_i(G_0)| \leq q^{s/2} + 1 \). Now assume that (7) holds for \( G_{j-1} \). Note that \( d(G_j) = d(G_{j-1}) - 2 \) and
\[
(D - A)(G_j) = (D - A)(G_{j-1}) - 2I + A(C_{q^s})
\]
where \( C_{q^s} \) is a cycle on \( q^s \) vertices. By the Courant-Weyl inequalities, we have that \( |\mu_i(G_j) - \mu_i(G_{j-1})| \) is bounded above by the spectral radius of \( 2I - A(C_{q^s}) \) which is less than 4. By the triangle inequality,
\[
|d(G_j) - \mu_i(G_j)| \leq |d(G_{j-1}) - \mu_i(G_{j-1})| + 6.
\]
Applying the induction hypothesis completes the proof. \( \blacksquare \)

We now apply Theorem 6.2 to each component of \( G_1 \), using that the primes \( p_j \) all satisfy \( |p_j - (n/\ell)^{1/s}| \leq \left((n/\ell)^{1/s}\right)^{9/10} \). If \( n \) or \( s \) is even, then apply part (a) to find a subgraph of \( G_1 \) so that all of the non-absolute points have degree \( \frac{p_j^{s-1}}{p_j - 1} \) and all of the absolute points have degree \( \frac{p_j^{s-1}}{p_j - 1} - 1 \). If both \( n \) and \( s \) are odd, then apply part (b) so that in the \( j \)'th component of \( G_1 \), we have \( p_j^s - \frac{p_j^{s-1}}{p_j - 1} + 1 \) vertices of degree \( \frac{p_j^{s-1}}{p_j - 1} - 1 \) and the remaining vertices have degree \( \frac{p_j^{s-1}}{p_j - 1} - 2 \).

Call this graph \( G_2 \). In either case, the number of vertices of minimum degree in \( G_2 \) is even. All that remains is to “fix” the vertices of minimum degree. To do this we will add a matching to the minimum degree vertices of \( G_2 \) such that each edge has one endpoint in \( N_i \) and one endpoint in \( N_j \) for some \( i \neq j \). This is accomplished with the following lemma (see, for example, [33]).

**Lemma 6.3** Let \( n_1 \geq \cdots \geq n_\ell \) be natural numbers satisfying with \( \ell \geq 3 \) and \( n_1 < n_2 + \cdots + n_\ell \) and \( n_1 + \cdots + n_\ell \) even. Then the complete multipartite graph \( K_{n_1,\ldots,n_\ell} \) contains a perfect matching.

Since we have ensured that the number of minimum degree vertices in \( G_2 \) is even and since the number of these vertices in each component is asymptotically equal, we may apply Lemma 6.3 to add a matching to the minimum degree vertices of \( G_2 \) where each edge has endpoints in two components of \( G_2 \). Call this graph \( G_3 \) which is either \( \frac{p_j^{s-1}}{p_j - 1} \) or \( \left(\frac{p_j^{s-1}}{p_j - 1} - 1\right) \)-regular, depending on the parity of \( n \) and \( s \). Since \( \ell \) is upper bounded by a constant depending only on \( s \), we have that the degree of regularity is \( \Omega(n^{1-1/s}) \) where the implicit constant depends only on \( s \).

The proof is complete once we show that \( G_3 \) is \( K_{s,t} \)-free. Since \( G_2 \) was \( K_{s,t} \)-free, any potential \( K_{s,t} \) must contain an edge of the matching that we added. Let \( uv \) be this edge and assume that \( N_u \) and \( N_v \) are the respective components that \( u \) and \( v \) are in in \( G_2 \). Assume that \( u \) is in the part of the \( K_{s,t} \) that has \( s \) vertices. Then because we only added
a matching to $G_2$, the remaining $t-1$ vertices in the part of size $t$ must belong to $N_u$ and the remaining $s-1$ vertices in the part of size $s$ must belong to $N_v$. Since $s-1$ and $t-1$ are at least 2, and there is only a matching between components, this is a contradiction, and the $K_{s,t}$ cannot exist.

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In an earlier version of this paper, the authors overlooked a significantly simpler argument and this was pointed out by Krivelevich [23]. Instead of using the fact that 2-factors exist in regular even degree graphs, a spectral approach similar to the proof of Theorem 1.4 was taken. While we still find the results obtained using that more complicated method (counting copies of $C_4$ in Brown’s $K_{3,3}$-free graph and then showing that it is an expander) interesting, we have removed this unnecessary work. Much thanks to Michael Krivelevich for useful comments regarding our original proofs of Theorems 1.3 and 1.4.

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