PHASE TRANSITION LAYERS FOR FIFE-GREENLEE
PROBLEM ON SMOOTH BOUNDED DOMAIN

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Abstract. We consider the Fife-Greenlee problem

$$
\epsilon^2 \Delta u + (u - a(y))(1 - u^2) = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega,
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^2$ with smooth boundary, $\epsilon > 0$ is a small parameter, $\nu$ denotes the unit outward normal of $\partial \Omega$. Let $\Gamma = \{y \in \Omega : a(y) = 0\}$ be a simple smooth curve intersecting orthogonally with $\partial \Omega$ at exactly two points and dividing $\Omega$ into two disjoint nonempty components. We assume that $-1 < a(y) < 1$ on $\Omega$, and also some admissibility conditions between the curves $\Gamma$, $\partial \Omega$ and the inhomogeneity $a$ hold at the connecting points. We can prove that there exists a solution $u_\epsilon$ such that: as $\epsilon \to 0$, $u_\epsilon$ approaches $+1$ in one part, while tends to $-1$ in the other part, except a small neighborhood of $\Gamma$.

1. Introduction. We consider the Fife-Greenlee problem

$$
\epsilon^2 \Delta u + (u - a(y))(1 - u^2) = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega,
$$

where $\Omega$ is a smooth and bounded domain in $\mathbb{R}^N$, $-1 < a(y) < 1$ on $\Omega$, $\epsilon$ is a small positive parameter, $\nu$ is the unit outer normal to $\partial \Omega$. The function $u$ represents a continuous realization of the phase present in a material confined to the region $\Omega$ at the point $y$ which, except for a narrow region, is expected to take values close to $+1$ or $-1$. Of interest are of course non-trivial state configurations in which the antiphases coexist.

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1.1. The backgrounds. The case $a(y) \equiv 0$ corresponds to the standard Allen-Cahn equation \[ \epsilon^2 \Delta u + u(1-u^2) = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \] for which extensive literature on transition layer solution is available, see for instance \[3, 4, 8, 21, 22, 24, 25, 28, 29, 30, 31, 33, 34, 35, 36, 39\], and the references therein for these and related issues.

We first mention some works on problem (1.1). Let us assume that $\tilde{\Gamma} = \{ y \in \Omega : a(y) = 0 \}$ is a simple, closed and smooth curve in $\Omega \subset \mathbb{R}^2$ which separates the domain into two disjoint components

\[ \Omega = \Omega_+ \cup \tilde{\Gamma} \cup \Omega_-, \] such that

\[ a(y) < 0 \quad \text{in } \Omega_+, \quad a(y) > 0 \quad \text{in } \Omega_-, \quad \frac{\partial a}{\partial \nu_0} > 0 \quad \text{on } \tilde{\Gamma}, \] where $\nu_0$ is the outer normal of $\partial\Omega_+$, pointing to the interior of $\Omega_-$. By matching asymptotic and bifurcation arguments, Fife and Greenlee \[20\] constructed a solution $u_\epsilon$ to problem (1.1) with the properties

\[ u_\epsilon \to +1 \quad \text{in } \Omega_+ \quad \text{and} \quad u_\epsilon \to -1 \quad \text{in } \Omega_- \quad \text{as } \epsilon \to 0. \] Super-subsolutions were later used by Angenent, Mallet-Paret and Peletier in the one dimensional case (see \[7\]) for construction and classification of stable solutions. Radial solutions were found variationally by Alikakos and Simpson in \[5\]. These results were extended by del Pino in \[11\] for general (even non smooth) interfaces in any dimension, and further constructions have been done by Dancer and Yan \[10\] and Do Nascimento \[17\]. In particular, it was proved in \[10\] that solutions with the asymptotic behavior like (1.5) are typically minimizer of the related Euler functional $J_\epsilon$. Related results can be found in \[1, 2\]. On the other hand, a solution exhibiting a transition layer in the opposite direction, namely

\[ u_\epsilon \to -1 \quad \text{in } \Omega_+ \quad \text{and} \quad u_\epsilon \to +1 \quad \text{in } \Omega_- \quad \text{as } \epsilon \to 0, \] has been believed to exist for many years. Hale and Sakamoto \[23\] established the existence of this type of solution in the one-dimensional case, while this was done for the radial case in \[12\], see also \[9\]. The layer with the asymptotics in (1.6) in this scalar problem is meaningful in describing pattern-formation for reaction-diffusion systems such as Gierer-Meinhardt with saturation, see \[12, 19, 32, 37\] and the references therein. Recently this problem has been completely solved by del Pino-Kowalczyk-Wei \[14\] (in the two dimensional domain case) and Mahmoudi-Malchiodi-Wei \[27\] (in the higher dimensional case), see also \[18\].

1.2. Motivations and results. Note that the results, mentioned in the above, concerned the existence of interior phase transition phenomena (away from $\partial\Omega$) for problem (1.1). On the other hand, phase transition layers, connecting $\partial\Omega$, were found for Allen-Cahn problem (1.2). See \[39, 25, 38, 15\] and the references therein. Hence, in the present paper, we will construct the phase transition layers connecting $\partial\Omega$ for (1.1) on two dimensional smooth bounded domain. It will be shown that the inhomogeneous term $a$ as well as the boundary of $\Omega$ will play an important role in the procedure of the construction. This is the reason for the requirement of the admissibility conditions between $\Gamma$ and $\partial\Omega$, which will be stated in (1.7). Whence we can use suitable methods to decompose the interaction between
the layer, the boundary and also the inhomogeneous term $a$, in such a way that we can construct good local approximate solution and then use the reduction method to get the solutions.

More precisely, in the present paper, for the existence of phase transition layers connecting $\partial \Omega$, we consider problem (1.1) and make the following assumptions:

(A1). Let $\Gamma = \{ y \in \Omega \subset \mathbb{R}^2 : a(y) = 0 \}$ be a simple smooth curve, which is intersecting $\partial \Omega$ at exactly two points, saying $P_1, P_2$, and, at these points $\Gamma \perp \partial \Omega$. In the small neighborhoods of $P_1, P_2$, the boundary $\partial \Omega$ are two curves, say $C_1$ and $C_2$, which can be represented by the graphs of two functions respectively

\[ y_2 = \varphi_1(y_1) \quad \text{with} \quad (0, \varphi_1(0)) = P_1, \]
\[ y_2 = \varphi_2(y_1) \quad \text{with} \quad (0, \varphi_2(0)) = P_2. \]

Without loss of generality, we can assume $\Gamma$ has length 1, and then denote $k_1, k_2, k$ the signed curvatures of $C_1, C_2$ and $\Gamma$ respectively.

(A2). $\Gamma$ separates the domain into two disjoint nonempty components,

\[ \Omega = \Omega_+ \cup \Omega_- \cup \Gamma, \quad \Omega_+ \cap \Omega_- = \emptyset, \]

where \[ a(y) < 0 \quad \text{in} \quad \Omega_+, \quad a(y) > 0 \quad \text{in} \quad \Omega_. \]

(A3). There holds

\[ -1 < a < 1 \quad \text{on} \quad \Omega, \quad \frac{\partial a}{\partial \nu_0} > 0 \quad \text{on} \quad \Gamma, \]

where $\nu_0$ is the outer normal of $\partial \Omega_+$, pointing to the interior of $\Omega_-$.

(A4). We also assume the validity of the admissibility conditions

\[
\begin{align*}
(k_1 k + k') \left. \frac{\partial a}{\partial \nu_0} \right|_{y = P_1} &= k \left. \frac{\partial^2 a}{\partial \nu_0 \partial \tau} \right|_{y = P_1}, \\
(k_2 k + k') \left. \frac{\partial a}{\partial \nu_0} \right|_{y = P_2} &= k \left. \frac{\partial^2 a}{\partial \nu_0 \partial \tau} \right|_{y = P_2},
\end{align*}
\]

(1.7)

where $\tau$ is the unit tangent vector of $\Gamma$, $k'$ denote the derivative of the signed curvatures $k$ of $\Gamma$.

Remark 1.1. An example will be provided to show the validity of the assumptions (A1)-(A4). We can choose $a(y) = y_1 - \frac{1}{2}, \Omega, \Omega_+, \Omega_-$ and $\Gamma$ as the following

![Diagram](image-url)
\[\Omega = \{(y_1, y_2) : |y - \left(\frac{1}{2}, 0\right)| < \frac{1}{2}\}, \quad \Gamma = \{(y_1, y_2) \in \Omega : y_1 = \frac{1}{2}\}\]

\[\Omega_+ = \{(y_1, y_2) \in \Omega : y_1 < \frac{1}{2}\}, \quad \Omega_- = \{(y_1, y_2) \in \Omega : y_1 > \frac{1}{2}\}\]

As \(\Gamma\) is a straight line, the signed curvature \(k\) of \(\Gamma\) is zero. Moreover, the derivative of the signed curvature \(k'\) is zero too. Therefore, the assumption \((A4)\) as well as \((A1)-(A3)\) holds.

Let \(H(x) = \tanh\left(\frac{x}{\sqrt{2}}\right)\) be the unique heteroclinic solution of
\[H'' + H - H^3 = 0, \quad H(0) = 0, \quad H(\pm \infty) = \pm 1. \tag{1.8}\]

It is well known that \(H\) is odd and enjoys the following behavior
\[H(x) - 1 = -A_0 e^{-\sqrt{2}x} + O\left(e^{-\sqrt{2}x}\right), \quad \text{as} \quad x \to +\infty,\]
\[H(x) + 1 = A_0 e^{-\sqrt{2}x} + O\left(e^{-\sqrt{2}x}\right), \quad \text{as} \quad x \to -\infty,\]
\[H'(x) = \sqrt{2}A_0 e^{-\sqrt{2}x} + O\left(e^{-\sqrt{2}x}\right), \quad \text{as} \quad |x| \to +\infty,\]
where \(A_0\) is a universal constant. It is trivial to derive that
\[1 - H^2(x) = \sqrt{2} H_x(x), \quad \int_{\mathbb{R}} H_x^2 \, dx = \frac{2\sqrt{2}}{3}. \tag{1.9}\]

A constant \(\lambda_*\) is defined by
\[\lambda_* = \frac{4}{3\pi^2} \left[\int_{\mathbb{R}} |H'(x)|^2 \, dx\right]^{-1} \left(\int_{\Gamma} \sqrt{\frac{\partial a}{\partial \nu_0}} \right)^2. \tag{1.10}\]

Our main result is as follows.

**Theorem 1.1.** Let \(\Omega\) be a smooth and bounded domain in \(\mathbb{R}^2\). If the assumptions \((A1)-(A4)\) hold, then for given \(c_0 > 0\) there exists \(\epsilon_0 > 0\) such that for all \(\epsilon < \epsilon_0\) satisfying the gap condition
\[|\sqrt{2} \epsilon - \lambda_*| \geq c_0 \sqrt{\epsilon}, \quad \forall j \in \mathbb{N}, \tag{1.11}\]
problem (1.1) has a solution \(u_\epsilon\) with the asymptotics
\[u_\epsilon \to -1 \quad \text{in} \quad \Omega_+, \quad u_\epsilon \to +1 \quad \text{in} \quad \Omega_-, \tag{1.12}\]
as \(\epsilon \to 0\) respectively. Moreover, near \(\Gamma\)
\[u_\epsilon \sim H\left(\frac{t - \epsilon f(\theta)}{\epsilon}\right), \tag{1.13}\]
where \(t\) is the normal coordinate directed along \(\nu_0\), \(\theta\) is the natural arc length parameter of the curve \(\Gamma\) and \(f\) is a bounded function of \(\theta\) as in (4.4).

**Remark 1.2.** As in [14] and [27], to deal with the resonance phenomenon, the phase transition layers with asymptotic behavior in (1.12) can be constructed whenever \(\epsilon\) is small and away from the critical numbers \(\frac{\lambda_*}{\sqrt{2}}\), in the sense that the gap condition (1.11) holds, i.e.
\[\epsilon \notin \left[\frac{\lambda_*}{\sqrt{2}} - \frac{c_0}{\sqrt{3} j} \frac{\lambda_*}{\sqrt{2}} + \frac{c_0}{\sqrt{3} j}\right], \quad \forall j \in \mathbb{N}. \tag{1.14}\]
On the other hand, if the assumptions (A1)-(A4) hold, we can also construct solutions \( \tilde{u} \) to problem (1.1) with asymptotical behavior

\[
\tilde{u}_\epsilon \to +1 \quad \text{in } \Omega_+, \quad \tilde{u}_\epsilon \to -1 \quad \text{in } \Omega_-,
\]
as \( \epsilon \to 0 \) respectively. Moreover, near \( \Gamma \)

\[
\tilde{u}_\epsilon \sim H\left(\frac{-t - \epsilon \tilde{f}(\theta)}{\epsilon}\right),
\]  

(1.15) 

where \( t \) is the normal coordinate directed along \( \nu_0 \) and \( \tilde{f} \) is also a bounded function of \( \theta \). The profile of \( \tilde{u} \) will be given in Remark 4.1. In this case, the problem does not have resonance phenomenon, see Remark 6.1.

In [39], [25], [38], [15], the phase transition layers connecting the boundary \( \partial \Omega \) were found for Allen-Cahn equation in (1.2). On the other hand, in the present paper, the new ingredients are the admissibility conditions in (A4) to decompose the interaction among the term \( a \), the phase transition layer and the boundary \( \partial \Omega \), see (4.11). In fact, this can be done in local coordinate system, called modified Fermi coordinates from [40] with little bit delicate analysis.

Here are some words for further discussions and the organization of the paper. For the convenience of expression, by the rescaling

\[
y = \epsilon \tilde{y}
\]

(1.16) 

in \( \mathbb{R}^2 \), problem (1.1) can be rewritten as

\[
\Delta \hat{u} + \hat{u}(1 - \hat{u}^2) - a(\epsilon \tilde{y})(1 - \hat{u}^2) = 0 \quad \text{in } \Omega_\epsilon, \quad \frac{\partial \hat{u}}{\partial \nu_\epsilon} = 0 \quad \text{on } \partial \Omega_\epsilon,
\]  

(1.17) 

where \( \Omega_\epsilon = \Omega / \epsilon \), \( \nu_\epsilon \) is the unit outer normal of \( \partial \Omega_\epsilon \). We also denote the curve \( \Gamma_\epsilon = \Gamma / \epsilon \) and then write down the local form of (1.17), especially the differential operators in Section 2.2. The outline of the proof will be given in Section 3. Gluing procedure from [13] will be applied to deduce the projected form of (1.17), see (3.46)-(3.49). Section 4 is devoted to the constructing of a local approximate solution in such a way that it solves the nonlinear problem locally up to order \( O(\epsilon^2) \). To get a real solution, the well-known infinite dimensional reduction method [13] will be needed in Sections 5-6. Note that we also need suitable analysis from [14] to deal with the resonance phenomena in Lemma 6.1.

2. Preliminaries: Known facts.

2.1. Recalling the coordinates. Recall the assumptions (A1)-(A4) in Section 1 and the notation therein. For basic notions of curves, such as the signed curvature of a plane curve, the reader can refer to the book by do Carmo [16]. In this section, for the convenience of readers, we recall the coordinate system in the neighborhood of \( \Gamma \) from [40], called modified Fermi coordinates.

**Step 1.** Let the natural parameterization of the curve \( \Gamma \) be as follows:

\[
\gamma_0 : [0, 1] \to \Gamma \subset \bar{\Omega} \subset \mathbb{R}^2.
\]

We can extend \( \gamma_0 \) slightly in a smooth way, i.e., for some small positive numbers \( \sigma_0 \), and define the mapping

\[
\gamma : (-\sigma_0, 1 + \sigma_0) \to \mathbb{R}^2,
\]

such that

\[
\gamma(\tilde{\theta}) = \gamma_0(\tilde{\theta}), \quad \forall \tilde{\theta} \in [0, 1].
\]
We choose the normal \( n \) of \( \Gamma \) with the same direction of \( \nu_0 \), the outer normal of \( \partial \Omega_+ \). There holds the Frenet formula
\[
\gamma'' = kn, \quad n' = -k\gamma',
\]
where \( k \) is the signed curvature of \( \Gamma \). Choosing \( \delta_0 > 0 \) very small, and setting
\[
\mathcal{G}_1 := (\delta_0, \delta_0) \times (-\sigma_0, 1 + \sigma_0),
\]
we construct the following mapping
\[
\mathbb{H} : \mathcal{G}_1 \to \mathbb{H}(\mathcal{G}_1) := \Omega_{\delta_0, \sigma_0} \quad \text{with} \quad \mathbb{H}(t, \theta) = \gamma(\theta) + \tilde{t} n(\theta).
\]
Note that \( \mathbb{H} \) is a diffeomorphism (locally) and \( \mathbb{H}(0, \theta) = \gamma(\theta) \).

**Step 2.** Denote the preimage
\[
\tilde{\mathcal{C}}_1 := \mathbb{H}^{-1}(\mathcal{C}_1), \quad \tilde{\mathcal{C}}_2 := \mathbb{H}^{-1}(\mathcal{C}_2),
\]
which can be parameterized respectively by \((\tilde{t}, \tilde{\varphi}_1(\tilde{t}))\) and \((\tilde{t}, \tilde{\varphi}_2(\tilde{t}))\) for some smooth functions \( \tilde{\varphi}_1(\tilde{t}) \), \( \tilde{\varphi}_2(\tilde{t}) \), and then define a mapping
\[
\tilde{\mathbb{H}} : \mathcal{G}_1 \to \mathcal{G}_2 := \tilde{\mathbb{H}}(\mathcal{G}_1) \subset \mathbb{R}^2,
\]
in the form
\[
t = \tilde{t}, \quad \theta = \frac{\tilde{\theta} - \tilde{\varphi}_1(\tilde{t})}{\tilde{\varphi}_2(\tilde{t}) - \tilde{\varphi}_1(\tilde{t})}.
\]
This transformation will straighten up the curves \( \tilde{\mathcal{C}}_1 \) and \( \tilde{\mathcal{C}}_2 \).

**Step 3.** We define the modified Fermi coordinates
\[
y = F(t, \theta) = \mathbb{H} \circ \tilde{\mathbb{H}}^{-1}(t, \theta) : (\delta_0, \delta_0) \times (-\sigma_0, 1 + \sigma_0) \to \mathbb{R}^2 \tag{2.1}
\]
for the given small positive constants \( \sigma_0 \) and \( \delta_0 \). More precisely,\[
F(t, \theta) = \mathbb{H} \left( t, \left( \tilde{\varphi}_2(t) - \tilde{\varphi}_1(t) \right) \theta + \tilde{\varphi}_1(t) \right)
\]
\[
= \gamma \left( \left( \tilde{\varphi}_2(t) - \tilde{\varphi}_1(t) \right) \theta + \tilde{\varphi}_1(t) \right) + t n \left( \left( \tilde{\varphi}_2(t) - \tilde{\varphi}_1(t) \right) \theta + \tilde{\varphi}_1(t) \right).
\]
For convenience’s sake, in the following, we also denote
\[
\Theta(t, \theta) := (\tilde{\varphi}_2(t) - \tilde{\varphi}_1(t)) \theta + \tilde{\varphi}_1(t),
\]
and so that
\[
F(t, \theta) = \gamma(\Theta(t, \theta)) + t n(\Theta(t, \theta)). \tag{2.2}
\]

The asymptotic expressions of this coordinate system will be given by the following basic facts.

**Lemma 2.1.** (Lemma 2.2 in [40]) The mapping \( F \) in (2.2) has the following properties:

1. \( F(0, \theta) = \gamma(\theta), \quad \frac{\partial F}{\partial t}(0, \theta) = n(\theta), \)
2. \( q_1(\theta) := \frac{\partial^2 F}{\partial t^2}(0, \theta) = \gamma'(\theta) \cdot \Theta_{tt}(0, \theta) \perp n(\theta), \)
3. \( q_2(\theta) := \frac{\partial^3 F}{\partial t^3}(0, \theta) = \gamma'(\theta) \cdot \Theta_{ttt}(0, \theta) + 3n'(\theta) \cdot \Theta_{tt}(0, \theta) \perp n(\theta). \)
2.2. Local forms of the operators. With the aid of the local coordinates \((t, \theta)\) in (2.1), the local form of the differential operator \(\Delta\) in (1.1) are given in (B.4)-(B.6) of [40], i.e.,

\[
\Delta_y = \partial^2_{tt} + \partial^2_{\theta\theta} + \tilde{B}_1(\cdot) + \tilde{B}_0(\cdot),
\]

where

\[
\tilde{B}_1(\cdot) = -(k + k^2 t) \partial_t - 2 \varpi t \partial^2_{\theta\theta} - \varpi \partial_{\theta},
\]

and

\[
\tilde{B}_0(\cdot) = 2kt \partial^2_{\theta\theta} + a_1 t^2 \partial^2_{\theta\theta} + a_2 t^3 \partial^2_{tt} + a_3 t^2 \partial^2_{\theta\theta} + a_4 t \partial_t + a_5 t \partial_{\theta}.
\]

In the above formulas,

\[
\varpi := q_1, \gamma' = \Theta_{tt}(0, \theta),
\]

and \(a_1, \ldots, a_5\) are smooth functions of the variable \(t\). On the other hand, the local expression of \(\partial/\partial \nu\) can be written in (B.7)-(B.9) of [40]. More precisely, for \(\theta = 0\),

\[
k_1 t \partial_t + b_1 t^2 \partial_t + \tilde{D}_0(\cdot) - \partial_{\theta} - k(0) t \partial_{\theta} + b_2 t^2 \partial_{\theta},
\]

where

\[
\tilde{D}_0(u) = \sigma_3(t) u_t + \sigma_4(t) u_{\theta},
\]

and the constants \(b_1\) and \(b_2\) are given by

\[
b_1 = \frac{1}{2} \tilde{\varphi}''(0) - k(0) k_1, \quad b_2 = \frac{1}{2} (k_2 - k_1) - k^2(0) - \frac{1}{2} k^2. \tag{2.9}
\]

On the other hand, for \(\theta = 1\),

\[
k_2 t \partial_t + b_3 t^2 \partial_t + \tilde{D}_1(\cdot) - \partial_{\theta} - k(1) t \partial_{\theta} + b_4 t^2 \partial_{\theta},
\]

with the notation

\[
\tilde{D}_1(u) = \sigma_5(t) u_t + \sigma_6(t) u_{\theta}, \tag{2.11}
\]

\[
b_3 = \frac{1}{2} \tilde{\varphi}''(0) - k(1) k_2, \quad b_4 = \frac{1}{2} (k_2 - k_1) - k^2(1) - \frac{1}{2} k^2. \tag{2.12}
\]

In the above, the functions \(\sigma_3, \ldots, \sigma_6\) are smooth functions of \(t\) with the properties

\[
|\sigma_i(t)| \leq C|t|^3, \quad i = 3, 4, 5, 6.
\]

By recalling the change of coordinates \(y = \epsilon \tilde{y}\) in (1.16), it is useful to locally introduce change of variables

\[
u(t, \theta) = \hat{u}(s, z) \quad \text{with} \quad s = t/\epsilon, \quad z = \theta/\epsilon. \tag{2.13}
\]

It is readily checked that

\[
\epsilon^2 \Delta_{\tilde{y}} = \epsilon^2 \partial^2_{tt} + \epsilon^2 \partial^2_{\theta\theta} + \epsilon^2 \tilde{B}_1(\cdot) + \epsilon^2 \tilde{B}_0(\cdot),
\]

where

\[
\epsilon^2 \tilde{B}_1(\cdot) = - \epsilon^2 (k + k^2 t) \partial_t - 2 \epsilon^2 t \varpi \partial^2_{\theta\theta} - \epsilon \varpi \partial_{\theta}
\]

\[
= -(\epsilon k + 2 \epsilon k^2 s) \partial_s - 2 \epsilon s \varpi \partial^2_{sz} - \epsilon \varpi \partial_z, \tag{2.14}
\]

and

\[
\epsilon^2 \tilde{B}_0(\cdot) = 2\epsilon^2 k t \partial^2_{\theta\theta} + a_1 \epsilon^2 t^2 \partial^2_{\theta\theta} + a_2 \epsilon^2 t^3 \partial^2_{tt} + a_3 \epsilon^2 t^2 \partial^2_{\theta\theta} + a_4 \epsilon^2 t \partial_t + a_5 \epsilon^2 t \partial_{\theta}
\]

\[
= 2\epsilon^2 k s \partial^2_{zz} + a_1 \epsilon^2 s^2 \partial^2_{zz} + a_2 \epsilon^2 s^3 \partial^2_{ss} + a_3 \epsilon^2 s^2 \partial^2_{zz} + a_4 \epsilon^3 s^2 \partial_s + a_5 \epsilon^2 s \partial_z \tag{2.15}
\]
Similarly, the normal derivative $\partial/\partial \nu_{\epsilon}$ can be derived from (2.7) and (2.10). Indeed, if $z = 0$, it becomes the following boundary operator
\[
D_0 := \epsilon k_1 s \partial_s + \epsilon^2 b_1 s^2 \partial_s + \hat{D}_0^0(\cdot) - \partial_z - \epsilon k(0) s \partial_z + \epsilon^2 b_2 s^2 \partial_z,
\]
(2.16)
where
\[
\hat{D}_0^0 = \epsilon \bar{D}_0^0.
\]
And, at $z = 1/\epsilon$, it has the form
\[
D_1 := \epsilon k_2 s \partial_s + \epsilon^2 b_3 s^2 \partial_s + \hat{D}_1^0(\cdot) - \partial_z - \epsilon k(1) s \partial_z + \epsilon^2 b_4 s^2 \partial_z,
\]
(2.17)
where
\[
\hat{D}_1^0 = \epsilon \bar{D}_1^0.
\]

3. Outline of the proof. In this section, the strategy to prove Theorem 1.1 will be provided step by step.

3.1. The gluing procedure. Consider any given approximate solution $H$ (to be chosen later in (4.20)) with the properties:
\[
\min_{\tilde{y} \in \tilde{\Omega}_\epsilon} 2(1 - a(\epsilon \tilde{y})H(\tilde{y})) = \varrho_0 > 0,
\]
(3.1)
and
\[
3(1 - H^2) < \varrho_0 \quad \text{for } |s| > \frac{\delta}{\epsilon},
\]
(3.2)
where $s$ is the normal coordinate to $\Gamma_\epsilon$, $\varrho_0$ and $\delta$ are small constants. For a perturbation term $\Phi$ defined in $\Omega_\epsilon$, the function $u(\tilde{y}) = H(\tilde{y}) + \Phi(\tilde{y})$ satisfies (1.17) if and only if
\[
L(\Phi) = -E + N(\Phi) \quad \text{in } \Omega_\epsilon,
\]
(3.3)
with boundary condition
\[
\frac{\partial \Phi}{\partial \nu_\epsilon} + \frac{\partial H}{\partial \nu_\epsilon} = 0 \quad \text{on } \partial \Omega_\epsilon,
\]
(3.4)
where
\[
L(\Phi) = \Delta \Phi + \left[1 - 3H^2 + 2a(\epsilon \tilde{y})H\right] \Phi, \quad E = \Delta H + (H - a(\epsilon \tilde{y}))(1 - H^2),
\]
\[
N(\Phi) = 3H \Phi^2 + \Phi^3 - a(\epsilon \tilde{y}) \Phi^2.
\]
We further separate $\Phi$ in the following form
\[
\Phi = \eta_3^s(\tilde{s}) \tilde{\phi} + \tilde{\psi},
\]
where $s$ is the normal coordinate to $\Gamma_\epsilon$. In the above formula, the cut-off function is defined as
\[
\eta_3^s(s) = \eta_3^s(|s|),
\]
where $\eta_3(t)$ is also a smooth cut-off function defined as
\[
\eta_3(t) = 1 \quad \text{for } 0 \leq t \leq \delta \quad \text{and} \quad \eta_3(t) = 0 \quad \text{for } t > 2\delta,
\]
(3.5)
for any fixed number $\delta < \delta_0/100$, where $\delta_0$ is a constant defined in (2.1). With this definition, $\Phi$ is a solution of (3.3)-(3.4) if the pair $(\tilde{\phi}, \tilde{\psi})$ satisfies the following coupled system:
\[
\eta_3^s \left[\Delta \tilde{\phi} + \left(1 - 3H^2 + 2aH\right) \tilde{\phi}\right] = \eta_3^s \left[-E + N(\eta_3^s \tilde{\phi} + \tilde{\psi}) - 3(1 - H^2) \tilde{\psi}\right] \quad \text{in } \Omega_\epsilon,
\]
(3.6)
where the linear operator is

\begin{equation}
\eta^\delta \partial \phi + \eta^\delta \partial \mathbf{H} = 0 \quad \text{on } \partial \Omega_e,
\end{equation}

and

\begin{equation}
\Delta \psi - 2(1 - a\mathbf{H}) \psi + 3(1 - \eta^\delta)(1 - \mathbf{H}^2) \psi
\end{equation}

\begin{equation}
= - \epsilon^2 \Delta \eta^\delta \phi - 2\epsilon \Delta \eta^\delta \nabla - (1 - \eta^\delta)E + (1 - \eta^\delta)N(\eta^\delta \phi + \psi) \quad \text{in } \Omega_e,
\end{equation}

\begin{equation}
\partial \psi \partial v - (1 - \eta^\delta) \partial \mathbf{H} \partial v + \epsilon \partial \eta^\delta \partial \phi = 0 \quad \text{on } \partial \Omega_e.
\end{equation}

First, given a small \( \delta \), we solve problems (3.8)-(3.9) for \( \psi \). The problem

\begin{equation}
\Delta \psi - 2(1 - a\mathbf{H}) \psi + 3(1 - \eta^\delta)(1 - \mathbf{H}^2) \psi = \hat{h} \quad \text{in } \Omega_e,
\end{equation}

\begin{equation}
\partial \psi \partial v = -(1 - \eta^\delta) \partial \mathbf{H} \partial v - \epsilon \partial \eta^\delta \partial \phi \quad \text{on } \partial \Omega_e,
\end{equation}

has a unique bounded solution \( \hat{\psi} \) due to (3.1) and (3.2), whenever \( \| \hat{h} \|_\infty < +\infty \). Moreover,

\[ ||\hat{\psi}||_\infty \leq C ||\hat{h}||_\infty. \]

Assume now that \( \hat{\phi} \) satisfies the following decay property

\[ ||\nabla \hat{\phi}|| + ||\hat{\phi}|| \leq e^{-e \delta / \epsilon}, \quad \text{for } |s| > \frac{\delta}{\epsilon}, \]

where \( \rho \) is a very small constant. Note that \( \mathbf{N} \) has a power-like behavior with power greater than one. A direct application of contraction mapping principle yields that (3.8)-(3.9) has a unique (small) solution \( \hat{\psi} = \psi(\hat{\phi}) \) with

\[ ||\psi(\hat{\phi})||_{L^\infty} \leq C \epsilon \left[ ||\hat{\phi}||_{L^\infty(|s|>\frac{\delta}{4})} + ||\nabla \hat{\phi}||_{L^\infty(|s|>\frac{\delta}{4})} \right] + e^{-\frac{s}{\epsilon}}, \]

where \( |s| > \delta / \epsilon \) denotes the complement of \( \Omega_e \) of \( \delta / \epsilon \)-neighborhood of \( \Gamma_e \). Moreover, the nonlinear operator \( \psi \) satisfies a Lipschitz condition of the form

\[ ||\psi(\hat{\phi}_1) - \psi(\hat{\phi}_2)||_{L^\infty(|s|>\delta / \epsilon)} \leq C \epsilon \left[ ||\hat{\phi}_1 - \hat{\phi}_2||_{L^\infty(|s|>\delta / \epsilon)} + ||\nabla \hat{\phi}_1 - \nabla \hat{\phi}_2||_{L^\infty(|s|>\delta / \epsilon)} \right]. \]

After solving (3.8)-(3.9), we can concern (3.6)-(3.7) as a local nonlinear problem involving \( \tilde{\psi} = \psi(\hat{\phi}) \), which can be solved in local coordinates. This is due to the fact that we can decompose the interaction among the boundary and the phase transition layer, and then construct a good approximate solution and also derive the resolution theory of the nonlinear problem by delicate analysis. This is called the gluing procedure in [13].

3.2. Local formulation of the problem. As described in the above, the next step is to consider (3.6)-(3.7) in the neighbourhood of \( \Gamma_e \) so that by the relation \( \tilde{y} = y / \epsilon \) in (1.16) close to \( \Gamma_e \), the variables \( y \) can be represented by the modified Fermi coordinates \((t, \theta)\) in (2.1), which have been prepared in Section 2.

More precisely, in the coordinates \((s, z)\) given by (2.13), the equation in (3.6) becomes

\begin{equation}
\eta^\delta \tilde{\mathcal{L}}(\tilde{\phi}) = \eta^\delta \left( -E + N(\eta^\delta \phi + \psi) - 3(1 - \mathbf{H}^2) \psi \right),
\end{equation}

where the linear operator is

\begin{equation}
\tilde{\mathcal{L}}(\tilde{\phi}) := \tilde{\phi}_{ss} + \tilde{\phi}_{zz} + \tilde{B}_1(\tilde{\phi}) + \tilde{B}_2(\tilde{\phi}) + (1 - 3\mathbf{H}^2) \tilde{\phi} + 2a(\epsilon s, \epsilon z)H \tilde{\phi},
\end{equation}

where
with the expressions of $\hat{B}_0$ and $\hat{B}_1$ given in (2.15) and (2.14). The error is now locally expressed in the form

$$E = H_{ss} + H_{zx} + \hat{B}_1(H) + \hat{B}_0(H) + (1 - H^2)H - a(\epsilon s, \epsilon z) (1 - H^2).$$  \tag{3.16}$$

The boundary condition in (3.7) can also be expressed precisely in the local coordinates. If $z = 0$,

$$\eta^s_{\delta} \mathcal{D}_0(\hat{\phi}) = -\eta^s_{\delta} G_0 \quad \text{with} \quad G_0 = \mathcal{D}_0(H).$$  \tag{3.17}$$

Similarly, at $z = 1/\epsilon$ there holds

$$\eta^s_{\delta} \mathcal{D}_1(\hat{\phi}) = -\eta^s_{\delta} G_1 \quad \text{with} \quad G_1 = \mathcal{D}_1(H).$$  \tag{3.18}$$

Introduce the following change of variables for any function $\tilde{w}$

$$\tilde{w}(s, z) = w(x, z), \quad x = s - f(\epsilon z),$$  \tag{3.19}$$

where $f$ is a parameter to be chosen in (4.4), and then take the Taylor expansion

$$a(t, \theta) = a_i(0, \theta) \cdot t + \frac{1}{2} a_{tt}(0, \theta) \cdot t^2 + a_6(t, \theta) \cdot t^3,$$

where $a_6(t, \theta)$ is a smooth function. In order to express problem (3.14) with boundary conditions (3.17)-(3.18) in terms of these new coordinates, we derive the relations

$$\tilde{w}_x = w_x, \quad \tilde{w}_{ss} = w_{xx}, \quad \tilde{w}_z = -\epsilon f' w_x + w_z,$$

$$\tilde{w}_{sz} = -\epsilon f' w_{xx} + w_{zz}, \quad \tilde{w}_{zz} = \epsilon^2 |f'|^2 w_{xx} + 2 \epsilon f' w_{xz} - \epsilon^2 f'' w_{x} + w_{zz}.$$

By the above change of coordinates and Taylor expansion, we compute

$$\tilde{L}(\hat{\phi}) = \phi_{ss} + \phi_{zz} + B_1(\hat{\phi}) + \hat{B}_0(\hat{\phi}) + (1 - 3H^2)\phi + 2a(\epsilon s, \epsilon z)H\phi$$

$$= \phi_{xx} + \epsilon^2 |f'|^2 \phi_{xx} - 2 \epsilon f' \phi_{xx} - \epsilon^2 f'' \phi_x + \phi_{zz} - \left[ \epsilon k + \epsilon^2 k^2(f + x) \right] \phi_x$$

$$+ 2\epsilon \varpi(f + x) \left( \epsilon f' \phi_{xx} - \phi_{xz} \right) + \epsilon \varpi(\epsilon f' \phi_x - \phi_z) + \hat{B}_0(\hat{\phi})$$

$$+ 2 \left[ a_i(0, \epsilon z) \cdot \epsilon s + \frac{1}{2} a_{tt}(0, \epsilon z) \cdot \epsilon^2 s^2 + a_6(t, \theta) \cdot \epsilon^3 s^3 \right] \mathcal{D}_0(\hat{\phi}) + (1 - 3H^2)\phi$$

$$= \phi_{xx} + \phi_{zz} + (1 - 3H^2)\phi + B_2(\phi) + B_3(\phi)$$

$$:= \tilde{L}(\phi).$$

This gives a new linear operator $\tilde{L}$. The operators are given by

$$B_2(\phi) = \epsilon^2 |f'|^2 \phi_{xx} - 2 \epsilon f' \phi_{xx} - \epsilon^2 f'' \phi_x - \left[ \epsilon k + \epsilon^2 k^2(f + x) \right] \phi_x$$

$$+ 2\epsilon \varpi(f + x) \left( \epsilon f' \phi_{xx} - \phi_{xz} \right) + \epsilon \varpi(\epsilon f' \phi_x - \phi_z),$$  \tag{3.20}$$

and

$$B_3(\phi) = 2 \left[ a_i(0, \epsilon z) \cdot \epsilon s + \frac{1}{2} a_{tt}(0, \epsilon z) \cdot \epsilon^2 s^2 + \epsilon^3 \varpi t \cdot \epsilon^3 s^3 a_6(\epsilon(x + f), \epsilon z) \right] \mathcal{D}_0(\hat{\phi}) + \hat{B}_0(\hat{\phi}).$$  \tag{3.21}$$

In the coordinates $(x, z)$, the boundary conditions in (3.17) and (3.18) can be recast as follows. For $z = 0$,

$$\eta^s_{\delta} (D^0(\phi) - \phi + D^0_0(\phi)) = -\eta^s_{\delta} G_0,$$  \tag{3.22}$$

where

$$D^0_0(\phi) := \epsilon (k_1 x + k_1 f + f') \phi_x + \epsilon^2 \left[ b_1 (x + f)^2 + k_0 (x + f) f' \right] \phi_x$$

$$- \epsilon k_0 (x + f) \phi_z + \epsilon^2 b_2 (x + f)^2 \phi_z,$$  \tag{3.23}$$
and
\[ D_0^0(\phi) := \hat{D}_0^0(\phi) - \epsilon^3 b_2 \left( x + f \right)^2 f' \phi_x. \] (3.24)

Similarly, for \( z = 1/\epsilon \), we have
\[ \eta_3^0 \left( D_3^1(\phi) - \phi_z + D_0^0(\phi) \right) = -\eta_3^0 G_1, \] (3.25)
where
\[ D_3^1(\phi) := \epsilon \left( k_2 x + k_2 f + f' \right) \phi_x + \epsilon^2 \left[ b_3 \left( x + f \right)^2 + k(1) \left( x + f \right) f' \right] \phi_x - \epsilon k(1) \left( x + f \right) \phi_z + \epsilon^2 b_4 \left( x + f \right)^2 \phi_z, \] (3.26)
and
\[ D_0^1(\phi) := \hat{D}_0^1(\phi) - \epsilon^3 b_4 \left( x + f \right)^2 f' \phi_z. \] (3.27)

As a conclusion, it is derived that equations (3.6)-(3.7) become, in local coordinates \((x, z)\),
\[ \eta_3^0 \phi \left[ \eta_3^0 \right] = \eta_3^0 \left[ -E + N(\eta_3^0 \phi + \psi) - 3(1 - H^2)\psi \right], \] (3.28)
\[ \eta_3^0 \left( D_0^0(\phi) - \phi_z + D_0^0(\phi) \right) = -\eta_3^0 G_0, \] (3.29)
\[ \eta_3^0 \left( D_3^1(\phi) - \phi_z + D_0^0(\phi) \right) = -\eta_3^0 G_1. \] (3.30)

Similarly, using the same changing of variables \( \hat{u}(s, z) = v(x, z) \), the equation in (1.17) is locally equivalent to
\[ S(v) := v_{zz} + v_{xx} + v - v^3 + B_2(v) + B_5(v) + B_6(v) = 0, \] (3.31)
where
\[ B_5(v) = -\left( a \phi(0, \epsilon z) (f + x) + \frac{1}{2} a_{tt}(0, \epsilon z) \epsilon^2 (f + x)^2 \right) (1 - v^2), \] (3.32)
\[ B_6(v) = \hat{B}_0(\hat{v}) - \epsilon^3 a_0 \left( \epsilon (x + f), \epsilon z \right) (f + x)^3 (1 - v^2). \] (3.33)

3.3. The projection problem. More words are in order to explain the rest of the strategy. Observe that all functions involved in system (3.28)-(3.30) are expressed in \((x, z)\)-variables, and the natural domain for those variables can be extended to the infinite strip
\[ S = \left\{ (x, z) : -\infty < x < \infty, 0 < z < 1/\epsilon \right\}. \] (3.34)

The boundary components of \( S \) are
\[ \partial_0 S = \left\{ (x, z) : -\infty < x < \infty, z = 0 \right\}, \] (3.35)
\[ \partial_1 S = \left\{ (x, z) : -\infty < x < \infty, z = 1/\epsilon \right\}. \]

One of the left jobs is to find the local form, say \( v_2 \) in (4.19), of the approximate solution \( H \). This will be given in Section 4, see (4.19) and (4.20). As we have done for the equation (3.14), \( E \) can be locally recast in \((x, z)\) coordinate system by the relation
\[ \eta_0^0(s) E = \eta_0^0(s) \mathcal{E}, \] (3.36)
where
\[ \mathcal{E} = S(v_2) \quad \text{with} \quad S(v_2) := v_{z,z} + v_{z,z} + v_2 - v_2^3 + B_2(v_2) + B_5(v_2) + B_6(v_2). \] (3.37)

Moreover, the boundary errors can be expressed in coordinates \((x, z)\) as follows. For \( z = 0 \), there holds
\[ \eta_0^0(s) G_0 = -\eta_3^0(s) g_0 \quad \text{with} \quad g_0 = D_0^0(v_2) - v_{z,z} + D_0^0(v_2), \] (3.38)
and also for \( z = 1/\epsilon \), we have
\[
\eta_\delta(s)G_1 = -\eta_\delta(s)g_1 \quad \text{with} \quad g_1 = D_\delta^1(v_2) - v_{2,z} + D_0^1(v_2).
\] (3.39)
The exact forms of the error terms \( \mathcal{E} \), \( g_0 \) and \( g_1 \) will be given in (4.29) and (4.33). It is of importance that (3.36), (3.38) and (3.39) hold only in a small neighbourhood of \( \Gamma_\epsilon \). Hence we will make extensions and consider \( v_2 \), \( \mathcal{E} \) as functions of the variables \( x \) and \( z \) on \( S \), and also \( g_0 \), \( g_1 \) on \( \partial_0 S \) and \( \partial_1 S \) in the sequel. Moreover, the unknown parameter \( f(\theta) \) in (3.19) will be chosen in the form, (cf. (4.4))
\[
f(\theta) = f_0(\theta) + f(\theta),
\] (3.40)
where \( f_0(\theta) \) is given in (4.4) and \( f(\theta) \) is a new parameter satisfying (4.5).

Now define an operator on the whole strip \( S \) in the form
\[
\mathcal{L}(\phi) := \phi_{zz} + \phi_{xx} + (1 - 3H^2)\phi + \chi(\epsilon|x|)(B_2(\phi) + B_3(\phi))
+ 3(H^2 - \chi(\epsilon|x|)H^2)\phi \quad \text{in} \ S,
\] (3.41)
and also the operators
\[
D_1(\phi) = \chi(\epsilon|x|)D_\delta^1(\phi) - \phi_x + \chi(\epsilon|x|)D_0^1(\phi) \quad \text{on} \ \partial_1 S,
\] (3.42)
\[
D_0(\phi) = \chi(\epsilon|x|)D_\delta^0(\phi) - \phi_z + \chi(\epsilon|x|)D_0^0(\phi) \quad \text{on} \ \partial_0 S,
\] (3.43)
where \( \chi(r) \) is a smooth cut-off function which equals 1 for \( 0 \leq r < 10\delta \) that vanishes identically for \( r > 20\delta \). For the local form of the nonlinear part, we have
\[
\eta_\delta(s)\left[ \mathbf{N}(\eta_\delta(s)\phi + \psi(\phi)) - 3(1 - H^2)\psi(\phi) \right] = \eta_\delta(s)\mathcal{N}(\phi),
\] (3.44)
by the notation
\[
\mathcal{N}(\phi) = \mathbf{N}(\eta_\delta(s)\phi + \psi(\phi)) - 3(1 - H^2)\psi(\phi),
\] (3.45)
where in the right hand side the term \( \psi \) is transformed from \( \tilde{\psi} \) by the relation (3.19). Rather than solving problem (3.28)-(3.30) directly, we deal with the following projected problem: for each \( f \) satisfies the constraint (4.5), finding functions \( \phi \in H^2(S) \), \( c(\epsilon z) \in L^2(0,1) \) and constants \( l_0, l_1 \) such that
\[
\mathcal{L}(\phi) = \eta_\delta(s)\left[ - \mathcal{E} + \mathcal{N}(\phi) \right] + c(\epsilon z)\chi(\epsilon|x|)H_z \quad \text{in} \ S,
\] (3.46)
\[
D_1(\phi) = -\eta_\delta(s)g_1 + l_1\chi(\epsilon|x|)H_z \quad \text{on} \ \partial_1 S,
\] (3.47)
\[
D_0(\phi) = -\eta_\delta(s)g_0 + l_0\chi(\epsilon|x|)H_z \quad \text{on} \ \partial_0 S,
\] (3.48)
\[
\int_\mathbb{R} \phi(x,z)H_z(x) \, dx = 0, \quad 0 < z < \frac{1}{\epsilon}.
\] (3.49)
We will prove that this problem has a unique solution \( \phi \) so that \( \tilde{\phi} \) satisfies the constraint (3.11) due to the relation given the change of variables in (3.19). The result reads

**Proposition 3.1.** There is a number \( C > 0 \) such that for all sufficiently small \( \epsilon \) and all \( f \) satisfying (4.5), problem (3.46)-(3.49) has a unique solution \( \phi = \phi(f) \) which satisfies
\[
\|\phi\|_{H^2(S)} \leq C\epsilon^{1/2}, \quad \|\phi\| + \|\nabla \phi\|_{L^\infty(|x| > \frac{1}{2\epsilon})} \leq \|\phi\|_{H^2(S)} \epsilon^{-\frac{1}{2}}.
\]
Moreover \( \phi \) depends Lipschitz-continuously on \( f \) in the sense of estimate
\[
\|\phi(f_1) - \phi(f_2)\|_{H^2(S)} \leq C\epsilon^{1/2}\|f_1 - f_2\|_*.
\]

**Proof.** The proof is similar as that for Proposition 5.1 in [14]. \( \square \)
After this has been done, our task is to adjust the parameter \( f \) such that the function \( c(\epsilon z) \) and the constants \( l_1, l_0 \) are zero, see (5.1)-(5.3). By the estimates in Section 5, it is equivalent to solving a nonlinear second-order differential equation for \( f \) with suitable boundary conditions, see (6.1)-(6.2). In Section 6, by using the gap condition (1.11) and delicate analysis we will prove that the reduced problem is solvable.

4. Local approximate solutions. The main objective of this section is to construct the approximate solution and then evaluate its error terms \( E, g_0 \) and \( g_1 \) in the coordinate system \((x, z)\).

4.1. The first approximation solution. Recall the function \( H \) in (1.8) and the relation in (1.9). We take \( v_1(x, z) = H(x) \) as the first trial of local approximation to a real solution. Here is the error in the interior of the region,

\[
S(H) = B_2(H) + B_5(H) + B_6(H)
= -\epsilon\sqrt{2}a_t x H_x - \epsilon(k_1 H_x + \sqrt{2}a_t f H_x)
+ \epsilon^2 \left[ (f')^2 H_{xx} - k_2 x H_x - \sqrt{2}a_{tt} f x H_x \right]
- \epsilon^2 \left[ (f')^2 H_{x} - k_2 x H_x + \sqrt{2}a_{tt} f x H_x \right]
+ \epsilon^2 \left[ (2 x H_{xx} + H_x) + B_6(H) \right]
: = \epsilon S_1 + \epsilon S_2 + \epsilon^2 S_3 + \epsilon^2 S_4 + \epsilon^2 S_5 + \epsilon^2 S_6 + B_6(H).
\]

The quantities \( S_1, S_3, S_5 \) are odd functions of \( x \), while \( S_2, S_4, S_6 \) are even. In addition, \( B_6(H) \) turns out to be size of \( O(\epsilon^3) \).

For the first approximate solution \( H \), the boundary errors can be formulated as follows. For \( z = 0 \),

\[
\epsilon(k_1 x + k_1 f + f')H_x + \epsilon^2 \left[ b_1(x + f)^2 + k(0)(x + f)f' \right] H_x + D_0^0(H).
\]

Similarly, for \( z = \frac{1}{\epsilon} \), we have

\[
\epsilon(k_2 x + k_2 f + f')H_x + \epsilon^2 \left[ b_3(x + f)^2 + k(1)(x + f)f' \right] H_x + D_0^1(H).
\]

4.2. The improvements. We now want to construct correction terms and establish a further approximation to a real solution that eliminates the terms of order \( \epsilon \) in the errors.

Let us now choose

\[
f(\theta) = f_0(\theta) + f(\theta) \quad \text{with} \quad f_0(\theta) = -\frac{1}{\sqrt{2}a_t(0, \theta)}.
\]

In all what follows, we will assure the validity of the following constraint on the parameter \( f \)

\[
\|f\|_s = \epsilon \|\nabla f\|_{L^2(0,1)} + \sqrt{\epsilon} \|\nabla f\|_{L^1(0,1)} + \|f\|_{L^\infty(0,1)} \leq \epsilon,
\]

so that

\[
\|f\|_{L^2(0,1)} \leq 1, \quad \|f\|_{L^1(0,1)} \leq \sqrt{\epsilon}, \quad \|f\|_{L^\infty(0,1)} \leq \epsilon.
\]

By interpolation, it also holds that

\[
\|f\|_{L^\infty(0,1)} \leq \sqrt{\epsilon}.
\]
4.2.1. Interior correction layers. The setting (4.4) will give that
\[ \epsilon S_2 = -\epsilon \left[ k H_x + \sqrt{2}a_t f_0 H_x + \sqrt{2}a_t f H_x \right] = -\epsilon \sqrt{2}a_t f H_x. \] (4.8)
To cancel \( \epsilon S_1 \) in (4.1), by the method in [14], the interior correction term can be chosen in the form
\[ \phi_1(x, z) = \epsilon \phi_{11}(x, z) \text{ with } \phi_{11}(x, z) = a_t(0, \epsilon z) H_1(x). \] (4.9)
In the above, the function \( H_1 \) is the unique, odd and decaying solution to the problem
\[ -H_{1,xx} + (3H^2 - 1) H_1 = -\sqrt{2}x H_x, \quad \int_S H_x H_1 \, dx = 0. \] (4.10)
4.2.2. The boundary corrections. In the following, we are in a position to improve the approximate solution so as to remove the boundary error terms of order \( \epsilon \) given in (4.2) and (4.3). The assumption (A4) will imply that
\[ f'_0(0) + k_1 f_0(0) = 0, \quad f'_0(1) + k_2 f_0(1) = 0. \] (4.11)
Combining with (4.4), this will change the first terms in (4.2) and (4.3) into
\[ \epsilon \left( k_1 f + f' \right) H_x + \epsilon k_1 x H_x \quad \text{and} \quad \epsilon \left( k_2 f + f' \right) H_x + \epsilon k_2 x H_x. \]
Whence we will get rid of the terms
\[ \epsilon k_1 x H_x \quad \text{and} \quad \epsilon k_2 x H_x, \]
by using the following lemma to add one more boundary correction term, while the terms involving \( f \) will be concerned by the standard reduction procedure.

**Lemma 4.1.** [15] Let us consider the following problem
\[ (\partial_{xx}^2 + \partial_{zz}^2) \phi^* + (1 - 3H^2) \phi^* = 0 \quad \text{in} \ S, \]
\[ \phi_z^*(x, 0) = x H_x, \quad \phi_z^*(x, 1/\epsilon) = 0. \] (4.12)
The problem has a unique solution \( \phi^* \in H^2(S) \) which is odd in \( x \) for each \( z \). Besides, there is a constant \( C \) such that for all small \( \epsilon \),
\[ \| \phi^* \|_{H^2(S)} \leq C. \]
In addition, there exist constants \( 0 < \zeta < \frac{1}{4}, \mu > 0, C > 0 \) such that
\[ |\phi^*(x, z)| + |D\phi^*(x, z)| + |D^2\phi^*(x, z)| \leq C e^{-\left[ (1-\zeta)\sqrt{2}|x|+\mu z \right]}. \]

We define
\[ \phi_{21}(x, z) = \phi^*(x, z), \quad \phi_{22}(x, z) = -\phi^*(x, \frac{1}{\epsilon} - z). \]
Hence, \( \phi_{21} \) satisfies the following problem
\[ \phi_{21,zz} + \phi_{21,xx} + (1 - 3H^2) \phi_{21} = 0 \quad \text{in} \ S, \]
\[ \phi_{21,z}(x, 0) = x H_x, \quad \phi_{21,z}(x, 1/\epsilon) = 0, \] (4.13)
and also \( \phi_{22} \) satisfies the problem
\[ \phi_{22,zz} + \phi_{22,xx} + (1 - 3H^2) \phi_{22} = 0 \quad \text{in} \ S, \]
\[ \phi_{22,z}(x, 0) = 0, \quad \phi_{22,z}(x, 1/\epsilon) = x H_x. \] (4.14)
Such functions enjoy the following estimates:

\[ |\phi_{21}(x, z)| + |D\phi_{21}(x, z)| + |D^2\phi_{21}(x, z)| \leq C e^{-\left[(1-\epsilon)\sqrt{|x|} + \mu z\right]}, \quad (4.15) \]

and

\[ |\phi_{22}(x, z)| + |D\phi_{22}(x, z)| + |D^2\phi_{22}(x, z)| \leq C e^{-\left[(1-\epsilon)\sqrt{|x|} + \mu(\frac{1}{2}-z)\right]}, \quad (4.16) \]

We define the boundary correction term as follows:

\[ \phi_2(x, z) = \epsilon a_{21}(\epsilon z) \phi_{21}(x, z) + \epsilon a_{22}(\epsilon z) \phi_{22}(x, z), \quad (4.17) \]

where

\[ a_{21}(\theta) = \chi_0(\theta) k_1, \quad a_{22}(\theta) = (1 - \chi_0(\theta)) k_2, \quad (4.18) \]

and the cut-off function \( \chi_0 \) is defined by

\[ \chi_0(\eta) = 1 \text{ if } |\eta| < 1/8 \quad \text{and} \quad \chi_0(\eta) = 0 \text{ if } |\eta| \geq 1/4. \]

Note that \( \phi_2 \) is an odd function in the variable \( x \) for each \( z \in (0, 1/\epsilon) \).

**4.3. Further discussion.** We choose

\[ v_2(x, z) = H(x) + \phi_1(x, z) + \phi_2(x, z), \quad (4.19) \]

as the local approximate solution to (3.31). The change of coordinates in (1.16), (2.1), (2.13) and (3.19) will give the local relation between \((x, z)\) and \(\tilde{y}\). The function \(v_2\) can be extended globally on \(\Omega_\epsilon\), which gives the expression of \(H\) in the form

\[ H(\tilde{y}) = \begin{cases} 
\eta_{3\delta}(s)(v_2(x, z) + 1) - 1 & \text{if } \tilde{y} \in \Omega_\epsilon, \\
\eta_{3\delta}(s)(v_2(x, z) - 1) + 1 & \text{if } \tilde{y} \in \mathbb{R}^2 \setminus \Omega_\epsilon. 
\end{cases} \quad (4.20) \]

It is obvious that the function \(H\) will satisfy (3.1) and (3.2).

**Remark 4.1.** Please note that for the solution \(\tilde{u}_\epsilon\) with profile described in Remark 1.2, we shall define the approximate solution as

\[ \tilde{H}(\tilde{y}) = \begin{cases} 
\eta_{3\delta}(s)(v_2(-x, z) - 1) + 1 & \text{if } \tilde{y} \in \Omega_\epsilon, \\
\eta_{3\delta}(s)(v_2(-x, z) + 1) - 1 & \text{if } \tilde{y} \in \mathbb{R}^2 \setminus \Omega_\epsilon. 
\end{cases} \quad (4.21) \]

In the neighbourhood of \(\Gamma_\epsilon\), \(\tilde{H} = H(-x) + O(\epsilon)\). \(\Box\)

The new error is

\[ \mathcal{E} = S(v_2) = S(H + \phi_1 + \phi_2) \]

\[ = S(H) + L_0(\phi_1) + L_0(\phi_2) + B_2(\phi_1 + \phi_2) + B_5(H + \phi_1 + \phi_2) \]

\[ - B_5(H) + B_6(H + \phi_1 + \phi_2) - B_6(H) + N_0(\phi_1 + \phi_2), \quad (4.22) \]

where

\[ L_0(\phi) = \phi_{zz} + \phi_{xx} + (1 - 3H^2)\phi, \quad (4.23) \]

and

\[ N_0(\phi) = -3H \phi^2 - \phi^3. \quad (4.24) \]

The first objective of this part is to compute the terms in (4.22).

It is easy to compute that

\[ L_0(\phi_1) = \phi_{1,zz} + \phi_{1,xx} + \phi_1 - 3H^2\phi_1 = \phi_{1,zz} + \epsilon\sqrt{2a_0}xH_x = \phi_{1,zz} - \epsilon S_1. \]

Recalling the expression of \(\phi_2\) and using the equation of \(\phi_{21}\) and \(\phi_{22}\) in (4.13)-(4.14), we get

\[ L_0(\phi_2) = 2\epsilon^2 \chi_0'(\epsilon z) \left[ k_1 \phi_{21}^\ast(x, z) - k_2 \phi_{22}^\ast(x, \frac{1}{\epsilon} - z) \right] + O(\epsilon^3) \]

\[ := K_{11}(x, z) + K_{12}(x, z), \quad (4.25) \]
where $K_{12}(x,z)$ is the high order term. According to the expression of $B_2$ in (3.20), it is easy to obtain

$$B_2(\phi_1 + \phi_2) = -\epsilon^2 \left\{ 2\omega x (a_{21}\phi_{21,xz} + a_{22}\phi_{22,xz}) + \omega (a_{21}\phi_{21,z} + a_{22}\phi_{22,z}) \right\}$$

$$+ \epsilon^2 \left\{ -2(f' + \omega f + f_0 + \omega f_0)(a_{21}\phi_{21,xz} + a_{22}\phi_{22,xz}) - k(a_t H_{1,x} + a_{21}\phi_{21,x} + a_{22}\phi_{22,x}) \right\} + O(\epsilon^3)$$

(4.26)

$$:= K_{21}(x,z) + K_{22}(x,z) + K_{23}(x,z).$$

Recalling the definition of $B_5$ and $B_6$ in (3.32), (3.33), we get that

$$B_5(H + \phi_1 + \phi_2) - B_5(H) + B_6(H + \phi_1 + \phi_2) - B_6(H)$$

$$= -\left[ \epsilon a_t (x + f_0 + f) + \frac{1}{2} \epsilon^2 a_{tt} (x + f_0 + f)^2 \right] \times \left[ 1 - (H + \phi_1 + \phi_2)^2 \right]$$

$$+ \left[ \epsilon a_t (x + f_0 + f) + \frac{1}{2} \epsilon^2 a_{tt} (x + f_0 + f)^2 \right] \times \left[ 1 - H^2 \right] + O(\epsilon^3)$$

$$= 2\epsilon^2 a_t (a_t H_1 + a_{21}\phi_{21} + a_{22}\phi_{22}) x H$$

$$+ \left\{ 2\epsilon^2 a_t (a_t H_1 + a_{21}\phi_{21} + a_{22}\phi_{22}) (f + f_0) H \right\} + O(\epsilon^3)$$

$$:= K_{31}(x,z) + K_{32}(x,z) + K_{33}(x,z).$$

Moreover, we can decompose $N_0(\phi_1 + \phi_2)$ as following

$$N_0(\phi_1 + \phi_2) = -3H(\phi_1 + \phi_2)^2 - (\phi_1 + \phi_2)^3$$

$$= -3\epsilon^2 H \left\{ a_t^2 H_1^2 + 2a_t(a_{22}\phi_{22} + a_{21}\phi_{21}) H_1 + (a_{21}\phi_{21} + a_{22}\phi_{22})^2 \right\} + O(\epsilon^3)$$

(4.28)

$$:= K_{41}(x,z) + K_{42}(x,z),$$

where $K_{42}(x,z)$ is the high order term.

So, according to the above rearrangements, we rewrite the expression (4.22) in terms of

$$\mathcal{E} = -\epsilon \sqrt{2} a_t f H_x + \epsilon^2 S_3 + \epsilon^2 S_4 + \epsilon^2 S_5 + \epsilon^2 S_6 + \phi_{1,zz}$$

$$+ K_{11}(x,z) + K_{12}(x,z) + K_{21}(x,z) + K_{22}(x,z) + K_{23}(x,z)$$

$$+ K_{31}(x,z) + K_{32}(x,z) + K_{33}(x,z) + K_{41}(x,z) + K_{42}(x,z),$$

(4.29)

where $S_3, S_5, \phi_{1,zz}, K_{11}(x,z), K_{21}(x,z), K_{31}(x,z)$ and $K_{41}(x,z)$ are the odd terms in the variable $x$. Observe that since $\phi_1, \phi_2$ and $f$ are of size $O(\epsilon)$ all terms in $\mathcal{E}$ carry $\epsilon^2$ in front. Moreover, $K_{12}(x,z), K_{23}(x,z), K_{33}(x,z), K_{42}(x,z)$ are high order terms.

Here are the boundary terms in local forms

$$D_0^0(v_2) - v_{2,z} + D_0^1(v_2) := g_0 \quad \text{on } \partial_0 S,$$

(4.30)

$$D_1^0(v_2) - v_{2,z} + D_1^1(v_2) := g_1 \quad \text{on } \partial_1 S.$$  

(4.31)

In the above, $D_0^0(v_2)$ is defined by

$$D_0^0(v_2) = \epsilon(k_1 x + k_1 f + f') (H + \phi_1 + \phi_2)_x$$

$$+ \epsilon^2 \left[ b_1 (x + f_0 + f)^2 + k(0)(x + f_0 + f)(f_0' + f') \right] v_{2,x}$$
5. Estimates of the projection against $H_x$. As we have mentioned in Section 3, in the next part of the paper, we will set up equations for the parameter $f$ which are equivalent to making $c(\epsilon x), l_1, l_0$ are identically zero in the system (3.46)-(3.49). These equations are obtained by simply integrating the equations (3.46)-(3.49) (only in $x$) against $H_x$. Using the equation of $H$ in (1.8) and the fact $\chi(\epsilon x)$ is an even function in the variable $x$, it is easy to derive the following equations

$$
\int_\mathbb{R} \left[ \eta_0'(s) (E - N(\phi)) + \chi(\epsilon x)|B_2 + B_3| + 3(H^2 - \chi(\epsilon x)|H|^2)\phi \right] H_x \, dx = 0, \quad (5.1)
$$

$$
\int_\mathbb{R} \left[ \eta_0'(s) g_1 + \chi(\epsilon x)|D_1^0(\phi(x, 1/\epsilon)) + \chi(\epsilon|x|)D_0^1(\phi(x, 1/\epsilon)) \right] H_x \, dx = 0, \quad (5.2)
$$
In the following, we are in a position to compute above integrals term by term. It is therefore of crucial importance to carry out computations of the term
\[
\int_{\mathbb{R}} \mathcal{E} H_x \, dx,
\]
and, similarly, some other terms involving \(\phi\).

5.1. **Estimates for projections of the error.** In this section, we carry out some estimates for the term \(\int_{\mathbb{R}} \mathcal{E} H_x \, dx\). We denote \(b_{l\epsilon}, \ l = 1, 2\), generic, uniformly bounded continuous functions of the form
\[
b_{l\epsilon} = b_{l\epsilon}(z, f(\epsilon z), f'(\epsilon z)), \tag{5.4}\]
where additionally \(b_{1\epsilon}\) is uniformly Lipschitz in its last two arguments.

Notice that the odd terms in the variable \(x\) in \(\mathcal{E}\), say \(S_3, S_5, \phi_{1,zz}, K_{11}(x, z), K_{21}(x, z), K_{31}(x, z), K_{41}(x, z)\), do not contribute to the value of the integral since \(H_x\) is even. Therefore,
\[
\int_{\mathbb{R}} \mathcal{E} H_x \, dx = \int_{\mathbb{R}} K_{22}(x, z) H_x \, dx + \int_{\mathbb{R}} K_{32}(x, z) H_x \, dx \tag{5.5}
\]
\[
+ \int_{\mathbb{R}} \left[ K_{12}(x, z) + K_{23}(x, z) + K_{33}(x, z) + K_{42}(x, z) \right] H_x \, dx
\]
\[
:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\]

In the following, we are in a position to compute above integrals term by term.

It is easy to get
\[
I_1 = -\frac{2\sqrt{2}}{3} \epsilon \beta_0(\epsilon z) f, \quad \text{with} \quad \beta_0(\theta) = \sqrt{2a_\epsilon}(0, \theta). \tag{5.6}
\]

According to the expression of \(S_4\) in (4.1), \(f = f_0 + f\), direct computation leads to
\[
I_2 = -\frac{\sqrt{2}}{3} \epsilon^2 \left[ 3 \frac{a_t}{\sqrt{2}} (f'' + f' f) + \frac{\sqrt{2}}{2} a_{tt} f^2 + \frac{\sqrt{2}}{2} a_{tt} x^2 \right] H_x^2 \, dx
\]
\[
- - \frac{2\sqrt{2}}{3} \epsilon^2 \left[ f'' + (k^2 + \sqrt{2} a_{tt} f_0) f \right] + \epsilon^2 \beta_1(\epsilon z), \tag{5.7}
\]
where
\[
\beta_1(\epsilon z) = -\int_{\mathbb{R}} \left[ f_0'' + k^2 f_0 + \frac{\sqrt{2}}{2} a_{tt} (f_0^2 + f^2) + \frac{\sqrt{2}}{2} a_{tt} x^2 \right] H_x^2 \, dx
\]
\[
= -\frac{\sqrt{2}}{3} \left[ f_0'' + k^2 f_0 + \frac{\sqrt{2}}{2} a_{tt} (f_0^2 + f^2) \right] - \frac{\sqrt{2}}{2} a_{tt} \int_{\mathbb{R}} x^2 H_x^2 \, dx.
\]

The identity
\[
\int_{\mathbb{R}} x H_{xx} H_x \, dx = -\frac{1}{2} \int_{\mathbb{R}} H_x^2 \, dx,
\]
gives that the term \(I_3\) is
\[
I_3 = \epsilon^2 \int_{\mathbb{R}} f' \left( 2x H_{xx} + H_x \right) H_x \, dx = 0. \tag{5.8}
\]
Recalling the definition of $K_{22}(x, z)$, it follows that

\[ I_4 = \int_{\mathbb{R}} \left\{ -2e^2(f' + \omega f + f'_0 + \omega f_0)(a_{21}\phi_{21,xz} + a_{22}\phi_{22,xz}) \\
- e^2k(\mathbf{a}_iH_{1,x} + a_{21}\phi_{21,x} + a_{22}\phi_{22,x}) \right\} H_x \, dx \]

\[ = -2e^2(f' + \omega f) \int_{\mathbb{R}} (a_{21}\phi_{21,xz} + a_{22}\phi_{22,xz}) H_x \, dx + e^2[h_1(z) + \beta_2(\epsilon z)], \tag{5.9} \]

where

\[ \beta_2(\epsilon z) = -k \mathbf{a}_i \int_{\mathbb{R}} H_{1,x} H_x \, dx, \]

and

\[ h_1(z) = -\int_{\mathbb{R}} \left[ 2(f'_0 + \omega f_0)(a_{21}\phi_{21,xz} + a_{22}\phi_{22,xz}) + k(a_{21}\phi_{21,x} + a_{22}\phi_{22,x}) \right] H_x \, dx. \]

By the definition of $K_{32}(x, z)$, we obtain

\[ I_5 = 2e^2 \int_{\mathbb{R}} \mathbf{a}_i(f + f_0)(a_{21}H_1 + a_{21}\phi_{21} + a_{22}\phi_{22}) HH_x \, dx \]

\[ = 2e^2\mathbf{a}_i f \int_{\mathbb{R}} (a_{21}H_1 + a_{21}\phi_{21} + a_{22}\phi_{22}) HH_x \, dx + e^2[h_2(z) + \beta_3(\epsilon z)], \tag{5.10} \]

where

\[ \beta_3(\epsilon z) = 2a_i^2 f_0 \int_{\mathbb{R}} H_1 HH_x \, dx, \]

and

\[ h_2(z) = 2\mathbf{a}_i f_0 \int_{\mathbb{R}} (a_{21}\phi_{21} + a_{22}\phi_{22}) HH_x \, dx. \]

Recalling the expression of $K_{12}(x, z), K_{23}(x, z), K_{33}(x, z)$ and $K_{42}(x, z)$, we get

\[ I_6 = e^3 \mathbf{b}_1 f'' + e^3 \mathbf{b}_2. \tag{5.11} \]

In summary, the above estimates will lead to the conclusion of this section

\[ \int_{\mathbb{R}} \mathcal{E}H_x \, dx \]

\[ = -\frac{2\sqrt{2}}{3} \epsilon \beta_0(\epsilon z) f - \frac{2\sqrt{2}}{3} \epsilon^2 \left( f'' + (k^2 + \sqrt{2a_{tt}f_0}) f \right) \]

\[ - 2e^2f' \int_{\mathbb{R}} (a_{21}\phi_{21,xz} + a_{22}\phi_{22,xz}) H_x \, dx \]

\[ - 2e^2 f \int_{\mathbb{R}} [\omega(a_{21}\phi_{21,xz} + a_{22}\phi_{22,xz}) - a_i(a_{21}H_1 + a_{21}\phi_{21} + a_{22}\phi_{22})] H_x \, dx \]

\[ + e^2 \left( \beta_1(\epsilon z) + \beta_2(\epsilon z) + \beta_3(\epsilon z) + h_1(z) + h_2(z) \right) + e^3 \mathbf{b}_1 f'' + e^3 \mathbf{b}_2 \]

\[ := -\frac{2\sqrt{2}}{3} \left[ \epsilon \beta_0(\epsilon z) f + e^2 \left( f'' + \alpha_1(z) f' + (a_2(z) + \beta_4(\epsilon z)) f \right) \right] \]

\[ + e^2(\beta_5(\epsilon z) + \alpha_3(z)) + e^3(\mathbf{b}_1 f'' + \mathbf{b}_2), \]

where $\beta_0$ is given in (5.6) and

\[ \alpha_1(z) = \frac{3\sqrt{2}}{2} \int_{\mathbb{R}} (a_{21}\phi_{21,xz} + a_{22}\phi_{22,xz}) H_x \, dx, \tag{5.12} \]
$$\alpha_2(z) = \frac{3\sqrt{2}}{2} \int_\mathbb{R} \left[ \varpi(a_{21}\phi_{21,xz} + a_{22}\phi_{22,xz}) - a_i(a_{21}\phi_{21} + a_{22}\phi_{22})H \right] H_x \, dx, \quad (5.13)$$

$$\beta_4(\varepsilon) = k^2 + \sqrt{2}a_{tt}f_0 - \frac{3\sqrt{2}}{2}a_i^2 \int_\mathbb{R} H_1 H_x \, dx, \quad (5.14)$$

and

$$\beta_5(\varepsilon) = \beta_1(\varepsilon) + \beta_2(\varepsilon) + \beta_4(\varepsilon), \quad \alpha_3(z) = h_1(z) + h_2(z). \quad (5.15)$$

At the end of this section, we will verify the following Lipschitz dependence of $b_{1\varepsilon}f''$ on $f$

$$c^3 \left\| b_{1\varepsilon}(z, f_1(\varepsilon), f_1'(\varepsilon)) f_1''(\varepsilon) - b_{1\varepsilon}(z, f_2(\varepsilon), f_2'(\varepsilon)) f_2''(\varepsilon) \right\|_{L^2(0,1)} \leq Cc^3 \| f_1 - f_2 \|_\star. \quad (5.16)$$

We only need to prove (5.16) for the term $f f''$, since the other terms can be proved similarly. For any two functions $f_1, f_2$ satisfying (4.5), we have

$$\| f_1 f'' - f_2 f'' \|_{L^2(0,1)} \leq \| f_1 f'' - f_1 f'' \|_{L^2(0,1)} + \| f_1 - f_2 \|_{L^2(0,1)} \| f_2'' \|_{L^2(0,1)} \leq C \| f_1 - f_2 \|_\star.$$ 

Hence (5.16) holds true.

### 5.2. Projection of terms involving $\phi$

In this section, we will estimate the terms involving $\phi$ in (5.1) integrated against $H_x$, which can be decomposed as

$$\Lambda = \int_\mathbb{R} \left[ B_2(\phi) + B_3(\phi) \right] H_x \, dx + \int_\mathbb{R} N(\phi)H_x \, dx + 3 \int_\mathbb{R} (H^2 - \chi|\varepsilon x|)H^2 \phi H_x \, dx \quad (5.17)$$

$$= \sum_{i=1}^3 \Lambda_i.$$ 

It is easy to get

$$\int_0^1 |\Lambda_1(\theta)|^2 \, d\theta \leq C c^3 \left( \| \phi \|_{H^2(S)}^2 + \| \phi \|_{H^2(S)}^6 \right).$$

Hence,

$$\| \Lambda_1 \|_{L^2(0,1)} \leq C c^3. \quad (5.18)$$

In $B_2(\phi) + B_3(\phi)$, we single out two less regular terms. The one whose efficient depends on $f''$ explicit has the form

$$\Lambda_{1\star} = -c^2 f'' \int_\mathbb{R} \phi H_x \, dx = c^2 f'' \int_\mathbb{R} \phi H_{xx} \, dx.$$ 

To estimate Lipschitz dependence of $\Lambda_{1\star}$ on $f$, it is sufficient to evaluate Lipschitz dependence of the following term on $f$. By Proposition 3.1, we get

$$\| \phi(f_1) - \phi(f_2) \|_{L^\infty(S)} \leq C c^2 \| f_1 - f_2 \|_\star,$$

from which it follows

$$\| \Lambda_{1\star}(f_1) - \Lambda_{1\star}(f_2) \|_{L^2(0,1)} = \left\{ \int_0^1 |\Lambda_{1\star}(f_1) - \Lambda_{1\star}(f_2)|^2 \, d\theta \right\}^{1/2} \leq C c^2 \| f_1 - f_2 \|_\star. \quad (5.19)$$
The other less regular one from $\Lambda_1$, which containing $\phi_{zz}$, is the following

$$
\Lambda_{1**} = 2\epsilon k \int_{\mathbb{R}} (f + x) \phi_{zz} H_x \, dx.
$$

In order to get Lipschitz dependence of $\Lambda_{1**}$ on $f$, we can compute

$$
\Lambda_{1**}(f_1) - \Lambda_{1**}(f_2)
= 2\epsilon k \left[ f_1 \int_{\mathbb{R}} \left( \phi_{zz}(f_1) - \phi_{zz}(f_2) \right) H_x \, dx + (f_1 - f_2) \int_{\mathbb{R}} \phi_{zz}(f_2) H_x \, dx \right] + 2\epsilon k \int_{\mathbb{R}} \left( \phi_{zz}(f_1) - \phi_{zz}(f_2) \right) x \, H_x \, dx,
$$

(5.20)

by the standard computation, we obtain

$$
\|\Lambda_{1**}(f_1) - \Lambda_{1**}(f_2)\|_{L^2(0,1)} \leq C \epsilon^3 \|f_1 - f_2\|_*.
$$

The remainder $\Lambda_1 - \Lambda_{1*} - \Lambda_{1**}$ actually defined, for fixed $\epsilon$, a compact operator for $f$ in $L^2(0,1)$. This is a consequence of the fact that weak convergence in $H^2(S)$ implies local strong convergence in $H^1(S)$, and the same as the case for $H^2(0,1)$ and $C^1[0,1]$. If $\{f_k\}$ is a weakly convergent sequence in $H^2(0,1)$, then clearly the functions $\phi(f_k)$ constitute a bounded sequence in $H^2(S)$. In the above remainder one can integrate by parts if necessary once in $x$. Averaging against $H_x$ which decays exponentially localizes the situation and the desired fact follows.

From the definition of $\mathcal{N}$ in (3.45), we obtain

$$
\|\Lambda_2\|_{L^2(0,1)} \leq C \epsilon^{\frac{1}{2}} \|\mathcal{N}(\phi)\|_{L^2(0,1)} \leq C \epsilon^3.
$$

(5.21)

Since

$$
|\phi_1(x, z) + \phi_2(x, z)| \leq C \epsilon (|x|^2 + 1) e^{-C_2|x|},
$$

we easily see that

$$
\|\Lambda_3\|_{L^2(0,1)} \leq C \epsilon^2 \|\phi\|_{H^2(S)} \leq C \epsilon^3.
$$

(5.22)

These terms $\Lambda_2 + \Lambda_3$ define compact operators for $f$ similarly as the remainder $\Lambda_1 - \Lambda_{1*} - \Lambda_{1**}$.

5.3. **Projection of the errors on the boundary.** Without loss of generality, in this section we only compute the projection of error on the boundary $O_S$. The main errors on the boundary integrated against $H_x$ in the variable $x$ can be calculated as the following:

$$
\int_{\mathbb{R}} g_0 H_x \, dx
= \frac{2\sqrt{3}}{3} \epsilon(k_1 f + f') + \epsilon^2 b_1 \int_{\mathbb{R}} [x^2 + (f_0 + f)^2] H_x^2 \, dx
+ \epsilon^2 (k_1 f + f') \int_{\mathbb{R}} (a_{11} H_{1,x} + a_{21} \phi_{21,x} + a_{22} \phi_{22,x}) H_x \, dx
+ \frac{2\sqrt{3}}{3} \epsilon^2 k(0) (f_0 + f) (f'_0 + f') - \epsilon^2 k(0) \int_{\mathbb{R}} (a_{21} \phi_{21,x} + a_{22} \phi_{22,x}) x H_x \, dx + O(\epsilon^3).
$$
Higher order errors can be proceeded as follows:

\[
\int_{\mathbb{R}} D_3^0(\phi(x,0)) H_x \, dx
\]

\[
= \int_{\mathbb{R}} \epsilon (k_1 x + k_1 f + f') \phi_x(x,0) H_x \, dx
\]

\[
+ \epsilon^2 \int_{\mathbb{R}} \left[ b_1 (x + f_0 + f)^2 + k(0)(x + f_0 + f) (f'_0 + f') \right] \phi_x(x,0) H_x \, dx
\]

\[
- \epsilon k(0) \int_{\mathbb{R}} (x + f_0 + f) \phi_x(x,0) H_x \, dx + \epsilon^2 b_2 \int_{\mathbb{R}} (x + f_0 + f)^2 \phi_x H_x \, dx
\]

\[= O(\epsilon^\frac{5}{2}).\]

It is easy to check that

\[
\int_{\mathbb{R}} D_3^0(\phi(x,0)) H_x \, dx = O(\epsilon^\frac{3}{2}).
\]

The other terms \(D_1^0(\phi)\) and \(D_0^0(\phi)\) on the boundary integrated against \(H_x\) in the variable \(x\) is of size \(O(\epsilon^3)\).

6. The reduced equation for \(f\). By the estimates in Section 5, equations (5.1)-(5.3) are equivalent to the problem

\[
L(f) := \epsilon f'' + \epsilon \alpha_1(\theta/\epsilon) f' + \left[ \epsilon \left( \alpha_2(\theta/\epsilon) + \beta_3(0) \right) + \beta_0(\theta) \right] f = \epsilon \beta_0(\theta) + \epsilon^2 M_\epsilon,
\]

with the boundary conditions

\[
f'(1) + k_2 f(1) + M_1^1(f) = 0, \quad f'(0) + k_1 f(0) + M_0^1(f) = 0,
\]

where

\[
M_1^1 = O(\epsilon^\frac{1}{2}), \quad M_0^1 = O(\epsilon^\frac{1}{2}), \quad \beta_0(\theta) = \frac{3\sqrt{2}}{4} (\beta_5(\theta) + \alpha_3(\theta/\epsilon)).
\]

Since \(\beta_0(\theta) = \sqrt{2} a_\epsilon(0, \theta) > 0\), we further set

\[
f(\theta) = \frac{\epsilon \beta_0(\theta)}{\epsilon \left( \alpha_2(\theta/\epsilon) + \beta_3(0) \right) + \beta_0(\theta)} + \hat{f}(\theta).
\]

Then (6.1)-(6.2) becomes

\[
L(\hat{f}) = \epsilon^2 \hat{M}_\epsilon,
\]

with the boundary conditions

\[
\hat{f}'(1) + k_2 \hat{f}(1) + \hat{M}_1^1(\hat{f}) = 0, \quad \hat{f}'(0) + k_1 \hat{f}(0) + \hat{M}_0^1(\hat{f}) = 0.
\]

The operator \(\hat{M}_\epsilon\) can be decomposed into the following form

\[
\hat{M}_\epsilon(\hat{f}) = A_\epsilon(\hat{f}) + K_\epsilon(\hat{f}),
\]

where \(K_\epsilon\) is uniformly bounded in \(L^2(0,1)\) for \(\hat{f}\) satisfying constraint (4.5) and is also compact. The operator \(\epsilon^2 A_\epsilon\) satisfies the following Lipschitz condition

\[
\| \epsilon^2 A_\epsilon(\hat{f}_1) - \epsilon^2 A_\epsilon(\hat{f}_2) \|_{L^2(0,1)} \leq C \epsilon \| \hat{f}_1 - \hat{f}_2 \|_*.
\]

Before solving (6.3)-(6.4), we need to use the gap condition (1.11) to deal with the invertibility of \(L\). Firstly, we consider the following problem

\[
L(\hat{f}) = d, \quad \forall \theta \in (0,1),
\]

\[
\hat{f}'(1) + k_2 \hat{f}(1) = 0, \quad \hat{f}'(0) + k_1 \hat{f}(0) = 0.
\]
We have the following lemma.

**Lemma 6.1.** If \( d \in L^2(0, 1) \), then there exists a constant \( c_0 \) for each \( 0 < c < c_0 \) satisfying (1.11), such that problem (6.7) has a unique solution \( \hat{f} \in H^2(0, 1) \) which satisfies

\[
e\|\hat{f}''\|_{L^2(0, 1)} + \sqrt{c} \|\hat{f}'\|_{L^2(0, 1)} + \|\hat{f}\|_{L^\infty(0, 1)} \leq C\epsilon^{\frac{1}{2}} \|d\|_{L^2(0, 1)}. \tag{6.8}\]

Moreover, if \( d \) is in \( H^2(0, 1) \), then

\[
e\|\hat{f}''\|_{L^2(0, 1)} + \|\hat{f}'\|_{L^2(0, 1)} + \|\hat{f}\|_{L^\infty(0, 1)} \leq C\|d\|_{H^2(0, 1)}. \tag{6.9}\]

**Proof.** Note that

\[
\beta_0(\theta) = \sqrt{2}a_i(0, \theta) > 0, \forall \theta \in (0, 1),
\]

due to the assumption (A3) in Section 1. Hence, delicate analysis from [14] will be used to deal with the resonance phenomena. For the convenience of readers, we here give the complete proof.

It suffices to show Lemma 6.1 with

\[
L_1(\hat{f}) := e\beta^{-2}\hat{f}'' + \epsilon\beta^{-2}\alpha_1(\theta/e)\hat{f}' + \hat{f},
\]

where \( \beta(\theta) = \sqrt{\beta_0(\theta)} \). We make the following Liouville transformation (cf. [26]):

\[
\kappa = \int_0^1 \beta(\theta) \, d\theta, \quad t = \int_0^\theta \frac{\beta(\theta) \, d\theta}{\kappa}, \quad \lambda_0 = \frac{\kappa^2}{\pi^2},
\]

\[
\Psi(\theta) = \beta(\theta)^{-\frac{1}{2}}, \quad e(t) = \frac{\hat{f}(\theta)}{\Psi(\theta)}, \quad \tilde{q}_1(t) = \frac{\alpha_1(\theta/e)\kappa}{\beta(\theta)\pi},
\]

\[
\tilde{q}_2(t) = \frac{\Psi'' + \alpha_1(\theta/e)\Psi'}{\beta^2(\theta)\pi^2}, \quad \tilde{d} = \frac{d\kappa^2}{\beta^2(\theta)\pi^2}.
\]

Then (6.7) with \( L \) replaced by \( L_1 \) is transformed into

\[
L_2(e) := e(e'' + \tilde{q}_1(t)e' + \tilde{q}_2(t)e) + \lambda_0 e = \tilde{d},
\]

\[
e' + k_2 e = 0, \quad e'(0) + \tilde{k}_1 e(0) = 0, \tag{6.10}\]

where

\[
k_2 = \frac{\kappa}{\pi} \beta^{-1}(1) \left( k_2 - \frac{1}{2} \beta^{-1}(1) \beta'(1) \right), \quad \tilde{k}_1 = \frac{\kappa}{\pi} \beta^{-1}(0) \left( k_1 - \frac{1}{2} \beta^{-1}(0) \beta'(0) \right),
\]

and it then suffices to establish the estimates in Lemma 6.1 for the solution of this problem in terms of the corresponding norms of \( \tilde{d} \).

It is standard that the eigenvalue problem

\[
e'' + \tilde{q}_1(t)e' + \tilde{q}_2(t)e + \lambda e = 0,
\]

\[
e'(\pi) + k_2 e(\pi) = 0, \quad e'(0) + \tilde{k}_1 e(0) = 0, \tag{6.11}\]

has an infinite sequence of eigenvalues \( \lambda_j, j \geq 1 \), with an associated orthogonal basis in \( L^2(0, \pi), \{e_j\} \), constituted by eigenfunctions. A result in [26] provides asymptotic expressions as \( j \to +\infty \) for these eigenvalues and eigenfunctions, which turn out to correspond to those for \( \tilde{q}_1 = 0 \) and \( \tilde{q}_2 = 0 \). We have

\[
\sqrt{\lambda_j} = j + \frac{\tilde{k}_2 - \tilde{k}_1}{j} + O\left( \frac{1}{j^3} \right), \quad j \to \infty. \tag{6.12}\]
Problem (6.10) is then solvable if and only if \( \lambda_j \epsilon \neq \lambda_0 \) for all \( j \geq 1 \). In such a case, the solution to (6.10) then can be written as

\[
e(t) = \sum_{j=1}^{\infty} \frac{\tilde{d}_j}{\lambda_0 - \lambda_j \epsilon} e_j(t)
\]

with this series convergent in \( L^2 \). Here, \( \tilde{d}_j \)'s are the Fourier coefficients of \( \tilde{d}(t) \).

Hence

\[
\|e(t)\|_{L^2(0,\pi)}^2 = \sum_{j=1}^{\infty} \frac{\|\tilde{d}_j\|^2}{(\lambda_0 - \lambda_j \epsilon)^2}.
\]

We then choose \( \epsilon \) such that

\[
|j^2 \epsilon - \lambda_0| \geq c_0 \sqrt{\epsilon}
\]

for all \( j \), where \( c_0 \) is small. This corresponds precisely to condition (1.11). From (6.12) we then find that

\[
|\lambda_0 - \lambda_j \epsilon| \geq \frac{c_0}{2} \sqrt{\epsilon},
\]

if \( \epsilon \) is also sufficiently small. By the similar method in [14], we can get the estimates (6.8)-(6.9).

**Remark 6.1.** For the existence of solutions \( \tilde{u}_\epsilon \) in Remark 1.2, the approximation \( \tilde{H} \) is given in Remark 4.1. Careful checking of the error as in Section 4.1 and also the estimates in Section 5 will give that

\[
\beta_0(\theta) = -\sqrt{2} a_1(0, \theta) < 0, \quad \forall \theta \in (0,1).
\]

It is easy to give the proof of Lemma 6.1 due to the coercivity of linear operator \( L \). In other words, we do not need gap condition in (1.11) to deal with resonance of the problem.

Now, we consider the following system

\[
L(\tilde{f}) = d, \quad \forall \theta \in (0,1),
\]

\[
\tilde{f}'(1) + k_2 \tilde{f}(1) = \Gamma_0^1, \quad \tilde{f}'(0) + k_1 \tilde{f}(0) = \Gamma_1^1,
\]

where \( \Gamma_0^1 = \tilde{M}_0^1(\tilde{f}), \Gamma_1^1 = \tilde{M}_1^1(\tilde{f}) \).

**Lemma 6.2.** If \( d \in L^2(0,1) \), then there exists a constant \( \epsilon_0 \) for each \( 0 < \epsilon < \epsilon_0 \) satisfying (1.11), such that the problem (6.16) has a unique solution \( \tilde{f} \in H^2(0,1) \) which satisfies

\[
\epsilon \|\tilde{f}'\|_{L^2(0,1)} + \|\tilde{f}'\|_{L^2(0,1)} + \|\tilde{f}\|_{L^\infty(0,1)} \leq C \left[ \|d''\|_{H^2(0,1)} + \|\Gamma_0^1\| + \|\Gamma_1^1\| \right].
\]

**Proof.** See Lemma 7.3 in [42].

**The completion of the Proof for Theorem 1.1.** From the arguments in Section 3, the left job is to solve problem (6.3)-(6.4). Let us observe now that the linear operator

\[
L(\tilde{f}) = \epsilon \tilde{f}'' + \epsilon \alpha_1(\theta / \epsilon) \tilde{f}' + \left[ \epsilon (\alpha_2(\theta / \epsilon) + \beta_4(\theta)) + \beta_0(\theta) \right] \tilde{f},
\]

is invertible with bounds for \( L(\tilde{f}) = d \) given by

\[
\epsilon \|\tilde{f}'\|_{L^2(0,1)} + \|\tilde{f}'\|_{L^2(0,1)} + \|\tilde{f}\|_{L^\infty(0,1)} \leq C \|d\|_{H^2(0,1)}.
\]
PHASE TRANSITION LAYERS FOR FIFE-GREENLEE PROBLEM

From (6.6), $\epsilon^2 \hat{A}_\epsilon$ is contraction mappings of their arguments. By Banach Contraction Mapping Theorem and Lemma 6.2, we can prove the nonlinear problem

$$[L - \epsilon^2 \hat{A}_\epsilon] (\hat{f}) = d$$

with the boundary conditions defined in (6.4) is uniquely solvable for $\hat{f}$ provided that $\|d\| < \epsilon^3 \delta + \rho$ for some $\rho > 0$. The desired result for full problem (6.3)-(6.4) then follows directly from Schauder’s fixed-point Theorem.

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