Admissibility and control of switched discrete-time singular systems

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This paper studies the admissibility of switched discrete-time singular systems. Sufficient conditions for this admissibility are derived in a form of a set of strict linear matrix inequalities. Design of both state feedback and static output feedback controllers is given. Numerical examples are presented to illustrate the proposed approaches.

Keywords: singular systems; switched systems; admissibility; state feedback control; static output control; strict L\(\infty\)Is

1. Introduction

Switched systems have received a lot of attention during the last decade. They are dynamical and consist of finite number of subsystems and a logical rule that governs the switching between these subsystems. Mathematically, these subsystems are generally described by a collection of induced differential or difference equations. The motivation for studying switched systems comes from the fact that switched systems have many applications in control of mechanical systems, power systems, aircraft and traffic control (Liberzon, 2003). Many works dealing with the stability analysis and the stabilization of regular or standard switched systems have been developed recently (Daafouz, Riedinger, & Jung, 2002; Ge, Sun, & Lee, 2001; Gelig & Churilov, 1998; Liberzon, 2003; Mancilla-Aguilar, 2003; Stevens & Lewis, 1991).

On the other hand, singular systems (known also as generalized, descriptor or differential algebraic systems) describe a large class of systems which are of great importance in theoretical and practical points of view (Boukas, 2008; Dai, 1989; Mills & Goldenberg, 1989; Newcomb & Dziurla, 1989; Xu & Lam, 2006). They are encountered in chemical, mineral, electronic and economic systems (Luenberger, 1979). In recent years a great deal of works has been devoted to the analysis and design techniques for singular systems (Chadli et al., 2008; Dai, 1989; Masubuchi, Akiyama, & Saeki, 2003; Masubuchi, Kato, Saeki, & Ohara, 2004; Yao, Guan, Chenb, & Hoc, 2006). However, if interesting results on controllers and observers design are developed for singular linear systems (Boukas, 2008; Chadli & Darouach, 2012, 2013; Darouach & Boutayeb, 1995; Xu & Lam, 2006 and references therein), only few results exist for the switched singular systems compared to the standard systems (Daafouz et al., 2002; Ge et al., 2001; Liberzon, 2003; Mancilla-Aguilar, 2003). Then a lot of efforts are still necessary to investigate new approaches and to improve existing results in the standard systems and to extend them to switched singular systems. Recently switched singular systems have been considered in Boukas (2008), this work treats the case of the stability and control design in the framework of systems with Markovian jumps. The systems considered are modeled with mode independent singular matrix \(E\).

In this paper we consider the switched singular systems whose subsystems are discrete-time linear-invariant systems, where the matrix \(E\) is mode dependent. First, the stability for switched singular systems is introduced via a switched quadratic Lyapunov function, then the admissibility is defined and extends the existing admissibility notion for singular systems. Sufficient conditions for switched singular systems to be admissible are given in strict linear matrix inequalities (L\(\infty\)Is) terms (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994). The obtained results are then applied to the stabilization of switched discrete-time singular systems using both state feedback controller and static output feedback controller. The results of this work extend the existing results of switched standard systems presented in Daafouz et al. (2002), where the less conservative approach based on the switched Lyapunov function is introduced.

The paper is organized as follows. In Section 2, the considered class of a discrete-time singular switched systems and the corresponding admissibility concepts are presented. In Section 3, the connection between our results and the existing ones is presented. In Section 4, a static state and output feedback controllers are designed in L\(\infty\)Is formulation. Finally, two numerical examples are provided to illustrate the effectiveness of the obtained results.

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Notation. Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "$^T$" denotes matrix transposition, the notation $X \succeq Y$ (respectively, $X > Y$) where $X$ and $Y$ are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite), the symbol $(*)$ denotes the transpose elements in the symmetric positions, $I$ is the identity matrices with compatible dimensions and $I_N = \{1, 2, \ldots, N\}$.

2. Admissibility of switched discrete-time singular systems

Let us consider the following autonomous switched discrete-time singular system given by

$$E_{\sigma(t)}x(t + 1) = A_{\sigma(t)}x(t),$$

(1)

where $x(t) \in \mathbb{R}^n$ is the state vector and $A_{\sigma(t)} \in \mathbb{R}^{n \times n}$. The matrix $E_{\sigma(t)} \in \mathbb{R}^{r \times n}$ and $\sigma(t)$ is a switching signal taking values in the finite set $I_N$ and assumed to be available in real time.

Let $\xi(t) = [\xi_1(t), \xi_2(t), \xi_3(t), \ldots, \xi_N(t)]^T$ be the indication function such that

$$\xi_i(t) = \begin{cases} 1 & \text{if } \sigma(t) = i; \\ 0 & \text{otherwise.} \end{cases}$$

Then the switched system (1) can be written as

$$\sum_{i=1}^{N} \xi_i(t + 1)E_i x(t + 1) = \sum_{i=1}^{N} \xi_i(t)A_i x(t).$$

(2)

In the sequel of this paper we assume that matrices $E_i$ are of constant and equal rank, i.e. $\text{rank} E_i = r \leq n$. This condition permits to study the singular switched systems with states running in the manifolds with the same dimensions.

For system (1) or (2) we can give the following definitions.

**Definition 1** System (2) or the pair $(E_i, A_i)$ is said to be regular if $\det(zE_i - A_i)$ is not identically zero.

**Definition 2** System (2) or the pair $(E_i, A_i)$ is said to be causal if $\deg(\det(zE_i - A_i)) = \text{rank} E_i$.

**Definition 3** The pair $(E_i, A_i)$ is said to be stable if it is regular and all of $\lambda(E_i, A_i)$ are within $D(0, 1)$, where $\lambda(E_i, A_i) = \{z|\det(zE_i - A_i) = 0\}$ and $D(0, 1)$ represents a disk with center at 0 and radius 1.

**Remark 1** As in the standard switched systems, the stability of the pairs $(E_i, A_i)$ for all $i \in I_N$ does not guarantee the stability of system (2). In fact switching between two stable systems can produce an unstable trajectory.

**Definition 4** System (2) is said to be stable, if it is regular and if there exists a switched quadratic Lyapunov function

$$V(x(t), t) = x(t)^T E_{\sigma(t)} P_{\sigma(t)} E_{\sigma(t)} x(t), \quad P_{\sigma(t)} = \sum_{i=1}^{N} \xi_i(t)P_i$$

(3)

with

$$E_i^T P_i E_i \geq 0, \quad i \in I_N,$$

and whose difference $\Delta V = V(x(t + 1), t + 1) - V(x(t), t)$ is negative.

**Definition 5** System (2) is said to be admissible if it is regular, causal and stable.

**Remark 2**

- When $E_i = I$, the singular system (1) or (2) becomes a standard switched one presented in Daafouz et al. (2002) and the switched Lyapunov function (3) becomes exactly the one given in Daafouz et al. (2002), where $P_i > 0$.
- When $E_i$ and $A_i$ are constant matrices $E_i = E$ and $A_i = A$ in this case the system becomes a non-switched singular system and the Lyapunov function (3) becomes the classical quadratic Lyapunov function (Owens & Debeljkovic, 1985).

Now let the pair $(E_i, A_i)$ be given, then it is always possible to find nonsingular matrices $M_i$ and $N_i$ such that $E_i = M_i \begin{bmatrix} 0 & 0 \\ 0 & N_i \end{bmatrix}$ and $A_i = M_i \begin{bmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{bmatrix} N_i$. This decomposition can be obtained via singular value decomposition of matrix $E_i$ followed by scaling of the bases. Then, we have the following lemmas (Dai, 1989).

**Lemma 1** The pair $(E_i, A_i)$ is causal if $A_{i4}$ is nonsingular.

**Lemma 2** The pair $(E_i, A_i)$ is regular if and only if there exist two nonsingular matrices $M_i$ and $N_i$ such that $E_i = M_i \begin{bmatrix} 0 & 0 \\ 0 & N_i \end{bmatrix}$ and $A_i = M_i \begin{bmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{bmatrix} N_i$, where $N_i$ is a nilpotent matrix.

**Lemma 3** Let the pair $(E_i, A_i)$ be regular, then it is causal if and only if $N_i = 0$.

Let the matrix $A^+$ be any generalized inverse satisfying $AA^+A = A$, then we have the following useful lemma, which can be used in the sequel of the paper (Harville, 1997).

**Lemma 4** Let $A$ and $B$ be two matrices of dimensions $n \times p$ and $p \times m$, respectively, then we have

1. $\text{rank } AB = \text{rank } B$ if and only if $p = \text{rank } \begin{bmatrix} A & B \end{bmatrix}$.

In particular when $B$ is of full row rank, the necessary and sufficient condition for $\text{rank } AB = \text{rank } B$ is that $A$ must be of full column rank.
(2) \( \text{rank}AB = \text{rank}A \) if and only if \( p = \text{rank} \begin{pmatrix} a_1 & \ldots & a_p \end{pmatrix} \).

In particular when \( A \) is of full column rank, the necessary and sufficient condition for \( \text{rank}AB = \text{rank}A \) is that \( B \) must be of full row rank.

The following theorem gives sufficient conditions for system (2) to be admissible.

**Theorem 5** System (2) is admissible if one of the following assertions is satisfied:

1. There exist positive-definite matrices \( P_i \), symmetric matrices \( Q_i \), and matrices \( F_i \) and \( G_i \) such that the following LMIs hold \( \forall (i,j) \in I_N^2 \):

\[
\begin{bmatrix}
-E_i^T P_i E_i + A_i^T F_i + F_i^T A_i & -F_i^T + A_i^T G_i \\
-E_i^T P_i E_i + A_i^T F_i + F_i^T A_i & -G_i - G_i^T
\end{bmatrix} < 0,
\]

where \( E_i^\perp \) is any full row rank matrix such that \( E_i^\perp E_i = 0 \).

2. There exist positive-definite matrices \( X_i \), symmetric matrices \( R_i \), and matrices \( F_i \) and \( G_i \) such that the following LMIs hold \( \forall (i,j) \in I_N^2 \):

\[
\begin{bmatrix}
-E_i(X_i - E_i^\perp R_i E_i^\perp) + A_i^T F_i + F_i^T A_i & -F_i^T + A_i^T G_i \\
-E_i(X_i - E_i^\perp R_i E_i^\perp) + A_i^T F_i + F_i^T A_i & -G_i - G_i^T
\end{bmatrix} < 0,
\]

Proof First, we prove that the conditions of assertion 1 are sufficient for system (2) to be admissible.

Assume that there exist positive-definite matrices \( P_i \), symmetric matrices \( Q_i \), and matrices \( F_i \) and \( G_i \) such that inequalities (4) are satisfied and let \( Y_i = P_i - E_i^\perp Q_i E_i^\perp \) in this case we obtain

\[
E_i^T Y_i E_i = E_i^T P_i E_i \geq 0
\]

since \( P_i > 0 \). Then inequalities (4) become \( \forall (i,j) \in I_N^2 \):

\[
\begin{bmatrix}
-E_i Y_i E_i + A_i^T F_i + F_i^T A_i & -F_i^T + A_i^T G_i \\
-E_i Y_i E_i + A_i^T F_i + F_i^T A_i & -G_i - G_i^T
\end{bmatrix} < 0.
\]

By pre-multiplying (7) by \( [1 \ A_i^T] \) and post-multiplying it by \( [A_i] \) we obtain

\[
A_i^T (P_i - E_i^\perp Q_i E_i^\perp) A_i - E_i^T P_i E_i < 0
\]

or equivalently

\[
A_i^T Y_i A_i - E_i^T Y_i E_i < 0.
\]

Now, let \( M_i \) and \( N_i \) be two nonsingular matrices such that \( E_i = M_i \begin{bmatrix} L & 0 \end{bmatrix} N_i \) and \( A_i = M_i \begin{bmatrix} A_i^0 & A_i^2 \end{bmatrix} N_i \), also let \( M_i^T Y_i M_i = \begin{bmatrix} \tilde{Y}_i & Y_{i2}^T \\ Y_{i2} & Y_{i3} \end{bmatrix} \) and \( M_i^T Y_i M_i = \begin{bmatrix} \tilde{Y}_i & Y_{i2}^T \\ Y_{i2} & Y_{i3} \end{bmatrix} \), then from Equation (9) we obtain

\[
N_i^T \begin{bmatrix} \tau_1 & \bar{A}_2^T \tilde{Y}_i A_i + R_j + R_j^T \\ \bar{A}_2 \end{bmatrix} N_i < 0,
\]

where \( \tau_i \) represents a matrix without any importance and \( R_j = A_i^2 \tilde{Y}_i A_i^2 + \frac{1}{2} A_i \tilde{Y}_i A_i^2 \). Now, since \( A_i^2 \tilde{Y}_i A_i^2 \geq 0 \) and by using Equation (10) we have \( R_j + R_j^T < 0 \), or equivalently \( R_j \) is nonsingular, i.e. \( A_i^2 \tilde{Y}_i A_i^2 + \frac{1}{2} A_i \tilde{Y}_i A_i^2 \) is nonsingular, which leads to \( A_i \) nonsingular. From Lemma 1 the pair \((E_i, A_i)\) is causal. On the other hand, since \( A_i \) is nonsingular, define the following nonsingular matrices, \( \bar{M}_i = \begin{bmatrix} L & -A_i^2 \tilde{Y}_i A_i^2 \\ 0 & A_i \tilde{Y}_i A_i^2 \end{bmatrix} M_i^{-1} \) and \( \bar{N}_i = N_i^{-1} \begin{bmatrix} L & -A_i^2 \tilde{Y}_i A_i^2 \\ 0 & A_i \tilde{Y}_i A_i^2 \end{bmatrix} \), then we have \( \bar{M}_i E_i \bar{N}_i = \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix} \) and \( \bar{M}_i A_i \bar{N}_i = \begin{bmatrix} \tilde{A}_i & 0 \\ 0 & \tilde{A}_i \end{bmatrix} \), where \( \tilde{A}_i = A_i - A_i^2 A_i \). From Lemmas 2 and 3 the pair \((E_i, A_i)\) is regular.

Let \( V(x(t), t) = x^T (t) E_{\sigma(t)} \sigma(t) E_{\sigma(t)} x(t) \), with \( \sigma(t) = \sum_{i=1}^{N} \xi_i(t) Y_i \) and \( E_{\sigma(t)} Y_i E_{\sigma(t)} \geq 0, \forall i \in N \), be a switched Lyapunov function candidate as defined in Definition 4. Then the difference of \( V(x(t), t) \) along the solution of Equation (2) is given by

\[
\Delta V = V(x(t+1), t + 1) - V(x(t), t) = x^T (t+1) E_{\sigma(t+1)} \sigma(t+1) E_{\sigma(t+1)} x(t+1)
\]

\[
- x(t)^T E_{\sigma(t)} \sigma(t) E_{\sigma(t)} x(t)
\]

\[
= x^T (t) \sum_{i=1}^{N} \xi_i(t) A_i^T \sigma(t) \sum_{i=1}^{N} \xi_i(t) A_i x(t)
\]

\[
- x^T (t) \sum_{i=1}^{N} \xi_i(t) E_i^T \sigma(t) \sum_{i=1}^{N} \xi_i(t) E_i x(t)
\]

\[
= x^T (t) \sum_{i=1}^{N} \xi_i(t) (x(t+1))^T (A_i^T Y_i A_i - E_i^T Y_i E_i) x(t)
\]

(11)

Thus, from condition (9) we guarantee that \( \Delta V < 0 \). Consequently, the unforced singular system (2) is admissible since it is regular, causal and stable.

Now, since the pair \((E_i, A_i)\) is regular and causal, there exist two nonsingular matrices \( S_i \) and \( N_i \) such that \( E_i = S_i \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix} N_i \) and \( A_i = M_i \begin{bmatrix} \tilde{A}_i & 0 \\ 0 & \tilde{A}_i \end{bmatrix} N_i \). Let \( Y_i = M_i^{-T} \begin{bmatrix} \tilde{Y}_i & Y_{i2}^T \\ Y_{i2} & Y_{i3} \end{bmatrix} M_i^{-1} \) and \( Y_j = M_j^{-T} \begin{bmatrix} \tilde{Y}_j & Y_{j2}^T \\ Y_{j2} & Y_{j3} \end{bmatrix} M_j^{-1} \) then Equation (9) can be written as

\[
N_i^T \begin{bmatrix} \tilde{A}_i^T \tilde{Y}_i A_i - \tilde{Y}_i & \tilde{A}_i^T Y_{i2} \\ Y_{i2}^T A_i & Y_{i3} \end{bmatrix} N_i < 0
\]

or equivalently, since \( N_i \) is nonsingular matrix,

\[
\begin{bmatrix} \tilde{A}_i^T \tilde{Y}_i A_i - \tilde{Y}_i & \tilde{A}_i^T Y_{i2} \\ Y_{i2}^T A_i & Y_{i3} \end{bmatrix} < 0
\]
and since \( Y_{ij} < 0 \) (see Xu & Yang, 1999) we obtain
\[
\begin{bmatrix}
-\bar{\Phi}_i & \bar{A}_i - \bar{Y}_i
\end{bmatrix} < 0
\]
(12)

Where \( \Phi_i = \bar{Y}_i - Y_{ij} Y_{ij}^{-1} Y_{i} \). Now, the Schur complement applied to Equation (12) gives the following inequality:
\[
\begin{bmatrix}
-\bar{Y}_i & \bar{A}_i - \bar{Y}_i^{-1}
\end{bmatrix} < 0
\]

which is equivalent to
\[
\bar{A}_i \Phi_i \bar{A}_i^{-1} - \Psi_i < 0,
\]
(13)

where matrices \( \Phi_i = \bar{Y}_i^{-1} > 0 \) and \( \Psi_i = \Phi_i^{-1} > 0 \).

Now let us prove that if Equation (5) is satisfied then system (2) is admissible. Assume that there exist positive-definite matrices \( X_i \) and symmetric matrices \( R_i \) such that inequalities (5) are satisfied. and let \( Z_i = X_i - E_i^\top R_i E_i^\top \) in this case inequalities (5) become
\[
\begin{bmatrix}
-\bar{E}_i Z_i E_i^{-1} + A_i F_i + F_i^\top A_i^{-1} & -F_i^\top A_i G_i
\end{bmatrix}
\]
(\#)
\[
\begin{bmatrix}
Z_i - G_i - G_i^\top
\end{bmatrix}
< 0
\]
(14)

\( \forall (i, j) \in I_N^2 \). By pre-multiplying Equation (14) by \( [A_i \, A_i] \) and post-multiplying it by \( [A_i \, A_i]^\top \) we obtain
\[
\begin{bmatrix}
A_i Z_i A_i^{-1} - E_i Z_i E_i^{-1}
\end{bmatrix} < 0
\]
(15)

Let \( M_i \) and \( N_i \) be two nonsingular matrices such that \( E_i = M_i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} N_i \) and \( A_i = M_i \begin{bmatrix} A_i & A_i \\ 0 & A_i \end{bmatrix} N_i \). On the other hand, let \( N_i Z_i N_i^\top = \begin{bmatrix} Z_i & Z_i \\ Z_i & Z_i \end{bmatrix} \) and \( N_i Z_i N_i^\top = \begin{bmatrix} Z_i & Z_i \\ Z_i & Z_i \end{bmatrix} \), then inequalities (15) become
\[
\begin{bmatrix}
\tau_3 \\ \tau_4
\end{bmatrix} \begin{bmatrix}
A_i Z_i A_i^{-1} + Q_i & Q_i
\end{bmatrix} M_i^\top < 0
\]
(16)

where \( \tau_3 \) and \( \tau_4 \) represent matrices without any importance and \( Q_i = (A_i Z_i + \frac{1}{2} A_i Z_i) A_i^{-1} \). As in the above proof we have \( Q_i < 0 \) which implies that matrix \( A_i^{-1} \) is nonsingular or equivalently \( A_i^{-1} \) is nonsingular. Consequently, the pair \( (E_i, A_i) \) is causal and regular. By using the same reasoning as above, we can show that Equation (15) is equivalent to Equations (13) or (9). This proves the theorem.

Remark 3 Note that for \( E_i = E \), conditions (5) are reduced to the existence of matrices \( X_i > 0 \) and \( G_i \) such that the following LMIs hold
\[
\begin{bmatrix}
-\bar{E}_i Z_i E_i^{-1} + A_i F_i + F_i^\top A_i^{-1} & -F_i^\top A_i G_i \\
(* & \begin{bmatrix} X_i - E_i^\top R_i E_i^\top \\ -G_i - G_i^\top
\end{bmatrix}
< 0
\]
(17)

which is only the dual conditions of (4) with \( E_i = E \). This duality is not guaranteed for singular switched systems with matrix \( E \) mode dependent.

3. Connection with the existing results

3.1. Standard switched systems

For the standard case \( E_i = I \), we have \( E_i^\top = E_i \). Then conditions (4) are reduced to the existence of matrices \( P_i > 0 \), \( F_i \) and \( G_i \) such that the following LMIs hold \( \forall (i, j) \in I_N^2 \):
\[
\begin{bmatrix}
-P_i + A_i^\top F_i + F_i^\top A_i & -F_i^\top + A_i^\top G_i \\
(* & P_j - G_i - G_i^\top
\end{bmatrix} < 0
\]
(18)

and conditions (5) are reduced to the existence of matrices \( X_i > 0 \), \( F_i \) and \( G_i \) such that the following LMIs hold
\[
\begin{bmatrix}
-X_i + A_i F_i + F_i^\top A_i & -F_i^\top + A_i G_i \\
(* & X_i - G_i - G_i^\top
\end{bmatrix} < 0.
\]
(19)

Conditions (18) and (19) with \( F_j = 0 \) are exactly the stability conditions given in Daafouz et al. (2002) using poly-quadratic switched Lyapunov function. Note also that the conditions (18) or (19) with \( P_j = P_i \) or \( X_j = X_i \) correspond exactly to the stability condition for uncertain linear system presented in Peaucelle, Arzelier, Bachelier, and Bernussou (2000).

3.2. Singular systems

Singular linear systems. For \( E_i = E \) and \( A_i = A \), the singular system (2) becomes a time-invariant one, in this case condition (8) implies that
\[
A^\top (P - E^\top Q E^\top) A - E^\top P E < 0.
\]
(20)

Which is exactly the condition given in Xu and Lam (2006).

Singular switched systems. When \( E_i = E \), the case where the matrix \( E \) is mode independent, we have \( Q_\sigma = Q \). For \( P_j = P \), this case corresponds to the classical common Lyapunov function, which is less general than the poly-quadratic function presented in this note. In addition, the proposed conditions are in strict LMIs.

4. Stabilization of discrete-time singular switched systems

Let us consider the following controlled discrete-time switched system
\[
E_{\sigma(t+1)} x(t+1) = A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t)
\]
(21a)
\[
y(t) = C_{\sigma(t)} x(t)
\]
(21b)

with \( x(t) \in R^n \) being the state vector, \( y(t) \in R^n \) the output vector, \( u(t) \in R^n \) is the input vector, \( A_{\sigma(t)} \in R^{n \times n}, B_{\sigma(t)} \in R^{n \times m} \) and \( C_{\sigma(t)} \in R^{m \times n} \). The matrix \( E_{\sigma(t)} \in R^{n \times n} \) and \( \sigma(t) \) is a switching signal taking values in the finite set \( I_N \) and assumed available in real time.

In this section, we consider the stabilization of system (21) via state feedback controller and static output feedback control. Sufficient conditions for admissibility are given in a set of LMIs form.
4.1. State feedback controller

The closed loop system of (21) via state feedback controller

\[ u(t) = \sum_{i=1}^{N} \xi_i(t)Kix(t), \]

with \( K_i \in \mathbb{R}^{m \times n} \), can be written as

\[ \sum_{i=1}^{N} \xi_i(t+1)E_ix(t+1) = \sum_{i=1}^{N} \xi_i(t)(A_i + B_iK_i)x(t). \]  

**Definition 6** The closed-loop system (23) is said to be admissible, if there exists a state feedback (22) which guarantees that system (23) is regular, causal and stable.

To derive the admissibility conditions of (23), it suffices to substitute \( A_i \) by \( A_i + B_iK_i \) in conditions (5), conditions more adapted to this problem than conditions (4), with \( F_i = G_i \) and by putting \( L_i = K_iG_i \), in this case we obtain the following theorem.

**Theorem 6** The singular system (23) is admissible if there exist positive-definite matrices \( X_i \), symmetric matrices \( R_i \), matrices \( G_i \) and \( L_i \), such that the following set of LMI's hold \( \forall i \in I_N \):

\[
\begin{bmatrix}
-E_i(X_i - E_j^{T-T}R_jE_j^{T-T})E_i^{T} & -G_i^{T} + Z_i \\
Z_i + Z_i^{T} & X_i - E_j^{T-T}R_jE_j^{T-T} - G_i - G_i^{T}
\end{bmatrix} < 0
\]

\[ (\ast) \]

with \( Z_i = A_iG_i + B_iL_i \). The controller gains are given by

\[ K_i = L_iG_i^{-1}. \]

Remark 4 Assumption is generally used in the control, it means that the measurements are redundant. If this condition is not satisfied we can always find a coordinates change which leads to this condition.

Based on this assumption, sufficient conditions for (27) to be admissible are given by the following lemma.

**Lemma 7** Under Assumption 1, the closed-loop system (27) is admissible if there exist positive-definite matrices \( X_i \), symmetric matrices \( R_i \), matrices \( G_i \), \( S_i \) and \( T_i \), such that the following linear constraints hold \( \forall i \in I_N \):

\[
\begin{bmatrix}
-E_i(X_i - E_j^{T-T}R_jE_j^{T-T})E_i^{T} & -G_i^{T} + \Phi_i \\
\Phi_i + \Phi_i^{T} & X_i - E_j^{T-T}R_jE_j^{T-T} - G_i - G_i^{T}
\end{bmatrix} < 0
\]

\[ (\ast) \]

with \( \Phi_i = A_iG_i + B_iT_iC_i \). In this case the controller gains are given by

\[ K_i = T_iS_i^{-1}. \]

Proof Under Assumption 1, matrices \( C_i \) are of full row rank. Now assume that there exist positive-definite matrices \( X_i \) and matrices \( G_i \), matrices \( S_i \) and \( T_i \) such that Equation (29) are satisfied, then we can see that matrices \( G_i \) are nonsingular and from Lemma 4 and (30) we can deduce that \( S_i \) is nonsingular for all \( i \in I_N \). Now, consider Equation (5) with \( F_i = G_i \) and \( A_i \) replaced by Equation (28). Let \( T_i = K_iS_i \), then by using Equation (30) we obtain Equation (29).

\[ \Box \]

4.2. Static output feedback controller

In this section we consider the admissibility of system (21) via static output feedback control of the form

\[ u(t) = \sum_{i=1}^{N} \xi_i(t)K_iy(t), \]

where \( K_i \in \mathbb{R}^{m \times p} \). The closed-loop system is given by

\[ \sum_{i=1}^{N} \xi_i(t+1)E_ix(t+1) = \sum_{i=1}^{N} \xi_i(t)A_ix(t). \]

**Definition 7** The closed-loop system (27) is said to be admissible, if there exists an output feedback given by Equation (26) which guarantees that system (27) is regular, causal and stable.

Now, without loss of generality, we can make the following assumption which will be used in the sequel.

**Assumption 1** The matrices \( C_i \) are of full row rank for all \( i \in I_N \).

Conditions of Lemma 7 are in \( \mathcal{LMIs} \) form (29) under the linear equalities (30). They can be solved easily by LMI tools (such as the LMITOOL software El Ghaoui &
Commeau, 1999; Vandenberghe & Boyd, 1996) or by elimination of the equality constraints (30) to obtain only strict LMI inequalities, this can be done as follows:

Assume that matrices $C_i$ are of full row rank, this can be always obtained by a some regular transformation. In this case by singular value decomposition, there exist two unitary matrices $U_i$ and $V_i$ such that $U_iC_iV_i^T = \Sigma_i = \begin{bmatrix} \Sigma_{ii} & \Sigma_{i1} \\ \Sigma_{1i} & 0 \end{bmatrix}$. Let $\bar{U}_i = \Sigma_{i1}^{-1}U_i$ and $\Sigma_{i2} = 0$. From the proof of Lemma 7, since $G_i$ is nonsingular, we deduce that $G_{i1}$ is nonsingular.

Now let the partition of matrices $V_i$, $E_i$, $B_i$, $V_iE_iT_iV_i^T$ and $V_iA_iV_i^T$ according to that of $G_i$ be

$V_i = \begin{bmatrix} V_{i2} \\ V_{i1} \end{bmatrix}$, $E_i = \begin{bmatrix} E_{i1} \\ E_{i2} \end{bmatrix}$, $V_iE_iT_iV_i^T = \begin{bmatrix} E_{i1} \\ E_{i2} \end{bmatrix}$, $V_iB_i = \begin{bmatrix} B_{i1} \\ B_{i2} \end{bmatrix}$

and $V_iA_iV_i^T = \begin{bmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{bmatrix}$. Let $\bar{T}_i = T_i\bar{U}_i^{-1}$, then by multiplying (29) by $\begin{bmatrix} v_i^T \\ 0 \end{bmatrix}$ and post-multiplying it by $\begin{bmatrix} 0 \\ v_i \end{bmatrix}$ yields the following theorem.

**Theorem 8** Under Assumption 1, the closed-loop system (27) is admissible if there exist positive-definite matrices $X_i$, symmetric matrices $R_i$, matrices $G_{i1}$, $G_{i3}$, $G_{i4}$ and $\bar{T}_i$ such that the following LMI hold for $i \in i_N$:

$$\begin{bmatrix} \Sigma_{i11} & \Sigma_{i12} & \Sigma_{i13} & \Sigma_{i14} \\ \Sigma_{i12}^T & \Sigma_{i22} & \Sigma_{i23} & \Sigma_{i24} \\ \Sigma_{i13} & \Sigma_{i23}^T & \Sigma_{i33} & \Sigma_{i34} \\ \Sigma_{i14} & \Sigma_{i24}^T & \Sigma_{i34} & \Sigma_{i44} \end{bmatrix} < 0 \quad (32)$$

with

$$\begin{align*}
\Sigma_{i11} &= -E_i(X_i - E_i^T R_i E_i^{T+})E_i^T + A_{i2}G_{i1} + A_{i2}G_{i3} + B_{i1}\bar{T}_i + G_{i1}A_{i1}^T + G_{i1}A_{i3}^T + \bar{T}_i^TB_{i1}, \\
\Sigma_{i12} &= -E_i(X_i - E_i^T R_i E_i^{T+})E_{i2} + A_{i2}G_{i4} + G_{i1}A_{i4}^T + \bar{T}_i^TB_{i2}, \\
\Sigma_{i13} &= -G_{i4}^T + A_{i4}G_{i1} + A_{i4}G_{i3} + B_{i1}\bar{T}_i, \\
\Sigma_{i14} &= -G_{i4}^T + A_{i4}G_{i4}, \\
\Sigma_{i23} &= A_{i3}G_{i1} + A_{i4}G_{i3} + B_{i2}\bar{T}_i, \\
\Sigma_{i24} &= -G_{i4}^T + A_{i4}G_{i4}, \\
\Sigma_{i33} &= V_iX_iV_i^T - \bar{E}_iR_i\bar{E}_i^T - G_{i1}^T - G_{i1}, \\
\Sigma_{i34} &= V_iX_iV_i^T - \bar{E}_iR_i\bar{E}_i^T - G_{i3}, \\
\Sigma_{i44} &= V_iX_iV_i^T - \bar{E}_iR_i\bar{E}_i^T - G_{i4} - G_{i4}^T.
\end{align*}$$

In this case the controller gains are given by

$$K_i = \bar{T}_iG_{i1}^{-1}\bar{U}_i. \quad (33)$$

5. Numerical examples

This section gives two illustrative numerical examples to show the effectiveness of the proposed approach. The first example deals with the state control feedback problem and the second one with the output feedback problem.

### 5.1. Example 1

For the first example, we consider unstable model with two modes:

$$E_1 = \begin{bmatrix} 5.6 & 1.68 & 1.4 & 0 \\ 4.2 & 9.8 & 7.14 & 0.28 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} -4.34 & -9.52 & 5.74 & -8.40 \\ 11.34 & 5.60 & -7.14 & 10.08 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 27.5 & 6.0 & 13.5 & 11.5 \\ -7.6 & -12.5 & 32.5 & 23.5 \\ -31.0 & 14.5 & 15.5 & -14.0 \\ -9.5 & 18.5 & -21.5 & -7.5 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 18.5 & 25.5 & -19.5 & 18.0 \\ -26.5 & -7.5 & 11.0 & 23.0 \\ -12.0 & -29.5 & 11.5 & -9.5 \\ -8.0 & -15.5 & -11.0 & -27.5 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.96 & 2.52 & 2.88 \\ 2.28 & -1.80 & 1.68 \\ -0.72 & 3.00 & 3.36 \\ 1.92 & 4.56 & 1.44 \end{bmatrix},$$

and $B_2 = \begin{bmatrix} 3.00 & 4.44 & 3.48 \\ -4.08 & 5.04 & -3.24 \\ 4.32 & 2.52 & 2.16 \\ 3.24 & 3.00 & -2.8800 \end{bmatrix}$.

It is easy to see that the condition $\text{rank} E_1 = \text{rank} E_2$ is satisfied, also the regularity conditions for these subsystems are satisfied. By solving LMI (24), we obtain the following results:

$$X_1 = \begin{bmatrix} 0.7901 & 0.2433 & -0.8078 & 1.7744 \\ 0.2433 & 6.9993 & -8.3014 & 3.3363 \\ -0.8078 & -8.3014 & 10.3857 & -4.7721 \\ 1.7744 & 3.3363 & -4.7721 & 14.7301 \end{bmatrix},$$

$$X_2 = \begin{bmatrix} 7.0915 & 5.6637 & 5.6332 & -6.2404 \\ 5.6637 & 5.0343 & 5.6973 & -4.4717 \\ 5.6332 & 5.6973 & 15.5431 & 2.3735 \\ -6.2404 & -4.4717 & 2.3735 & 10.5729 \end{bmatrix}.$$
In this case the controller gains obtained from Equation (25) are

\[
K_1 = \begin{bmatrix}
7.3407 & 9.3057 & -13.7690 & -4.5421 \\
81.0161 & -6.3068 & 29.0567 & -38.4358 \\
-71.7610 & -0.2671 & -21.8896 & 25.8279
\end{bmatrix},
\]

\[
K_2 = \begin{bmatrix}
-1.0798 & 11.5894 & -0.9239 & 1.8529 \\
6.1208 & -10.3406 & 0.5433 & 4.3894 \\
5.2579 & -14.5859 & -4.5641 & -0.1930
\end{bmatrix}.
\]

The simulation results under an arbitrary switching law, with the initial value \(x^T = [-3 2 0 -1]\), shows the stability of the considered switched system; i.e. all the state trajectories converge to the origin (Figures 1 and 2).

Next, a second example is proposed to show the effectiveness of the derived static output controller.

### 5.2. Example 2

This example consists of a switched singular system with three modes and described by

\[
E_1 = \begin{bmatrix}
5.59 & 2.47 & 1.963 & 0.026 \\
-14.1328 & 5.2717 & -95.2632 & -93.7725 \\
-43.8234 & 96.7773 & 0.4448 & -106.7421 \\
-24.4442 & 97.3287 & 103.2599 & -3.1477
\end{bmatrix},
\]

\[
E_2 = \begin{bmatrix}
5.2 & 1.56 & 1.3 & 0 \\
3.9 & 9.1 & 6.63 & 0.26 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
E_3 = \begin{bmatrix}
4.81 & 0.65 & 0.637 & -0.026 \\
3.9 & 9.1 & 6.63 & 0.26 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
A_1 = \begin{bmatrix}
3.30 & 0.72 & 1.62 & 1.38 \\
-0.84 & -1.50 & 3.90 & 2.82 \\
2.22 & 3.06 & -2.34 & 2.16 \\
-3.18 & -0.90 & 1.32 & 2.76
\end{bmatrix},
\]
Solving Equations (29)–(30), we obtain the following results:

\[
A_2 = \begin{bmatrix} 2.48 & -1.16 & -1.24 & 1.12 \\ 0.76 & -1.48 & 1.72 & 0.60 \\ 0.96 & 2.36 & -0.92 & 0.76 \\ 0.64 & 1.24 & 0.88 & 2.20 \end{bmatrix},
\]

\[
A_3 = \begin{bmatrix} 2.80 & -2.24 & -1.36 & 0.72 \\ 1.84 & 1.28 & 3.60 & 3.76 \\ 1.36 & 5.20 & 2.56 & 1.92 \\ 0.88 & 1.92 & 2.96 & 3.12 \end{bmatrix},
\]

\[
B_1 = \begin{bmatrix} 0.9600 & 2.5200 & 2.8800 \\ 2.2800 & -1.8000 & 1.6800 \\ -0.7200 & 3.0000 & 3.3600 \\ 1.9200 & 4.5600 & 1.4400 \end{bmatrix},
\]

\[
B_2 = \begin{bmatrix} 3.0000 & 4.4400 & 3.4800 \\ -4.0800 & 5.0400 & -3.2400 \\ 4.3200 & 2.5200 & 2.1600 \\ 3.2400 & 3.0000 & -2.8800 \end{bmatrix},
\]

\[
B_3 = \begin{bmatrix} 2.3800 & 0.7000 & 1.1200 \\ 1.3300 & 0.7700 & 1.8900 \\ 1.8900 & 1.3300 & -2.2400 \\ 1.3300 & 2.3800 & 2.5200 \end{bmatrix},
\]

\[
C_1 = \begin{bmatrix} 0.4000 & -1.0000 & 1.2000 & 1.8000 \\ -14.0000 & -5.0000 & 18.0000 & 17.0000 \end{bmatrix},
\]

\[
C_2 = \begin{bmatrix} 0.6250 & 3.8750 & -0.8750 & -3.7500 \\ 0.3125 & -1.0625 & 0.5625 & 0.6250 \end{bmatrix},
\]

\[
C_3 = \begin{bmatrix} 0.4920 & 0.0900 & -0.3120 & -0.0300 \\ 0.0600 & -0.4500 & 0.8400 & -0.1500 \end{bmatrix}.
\]

Then from Equation (31), we get the following output feedback controller gains:

\[
K_1 = \begin{bmatrix} 0.2893 & -0.0890 \\ 0.0884 & 0.0016 \\ -0.7262 & 0.0546 \end{bmatrix},
\]

\[
K_2 = \begin{bmatrix} -0.1563 & -0.8561 \\ -0.3305 & -1.4876 \\ -0.3023 & -0.6604 \end{bmatrix},
\]

\[
K_3 = \begin{bmatrix} -2.4477 & -0.9075 \\ 2.4108 & 1.7580 \\ -1.0617 & -1.4417 \end{bmatrix}.
\]

The obtained results illustrate the effectiveness of the given sufficient LMI admissibility conditions for a switched singular system.

6. Conclusion

In this paper we have presented a new approach to the stabilization of switched singular discrete-time systems. The properties of stability and admissibility for this class of systems are first introduced, then sufficient conditions for a switched singular system to be admissible are given in strict LMI form. Both state feedback and static output feedback stabilization are presented. Two numerical examples were given to illustrate the obtained results.

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