Upper Bounds on the Automorphism Group of a Graph
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Abstract
We give upper bounds on the order of the automorphism group of a simple graph

In this note we present some upper bounds on the order of the automorphism group of a graph, which is assumed to be simple, having no loops or multiple edges. Somewhat surprisingly, we did not find such bounds in the literature and the goal of this paper is to fill this gap. As a matter of fact, implicitly such bounds were contained in works dealing with the edge reconstruction conjecture and are the corollaries of a simple theorem which is presented below (Theorem 1). Therefore we bring together a few results spread in different, sometimes in difficult to reach, sources (see Theorem 2 below). In Theorem 3
we derive a new bound, based on the notion of a greedy spanning tree. This new bound improves, in many cases, the bounds (1) and (2) of Theorem 2.

We will use the following notation. Let $F$ be a spanning subgraph of a fixed copy of a graph $G$. The number of embeddings of $F$ in $G$, that is the number of labeled copies of $F$ in $G$, is denoted by $|F \to G|$. Clearly $|F \to G| = s(F \to G)\text{aut}(F)$, where $s(F \to G)$ is the number of subgraphs of $G$ isomorphic to $F$ and $\text{aut}(F)$ is the order of the automorphism group of $F$. We also use $n = n(G)$ for the number of vertices and $e = e(G)$ for the number of edges of $G$. As usual, $\Delta_G$, $\delta_G$ and $d_G$ stand for the maximum, the minimum and the average degree of $G$ respectively. The degree of a vertex $v \in G$ is denoted by $d_G(v)$.

**Theorem 1** Let $F$ be a spanning subgraph of a graph $G$, Then

$$\text{aut}(G) \leq |F \to G| = s(F \to G)\text{aut}(F).$$

**Proof.** Let $\phi : G \to G$ be an automorphism of $G$ and let $F_1$ be a fixed copy of $F$ in $G$. Then, as $F$ is a spanning subgraph of $G$, $\phi$ is completely determined by the knowledge of $\phi(F_1)$. Since the number of different images $\phi(F_1)$ does not exceed $|F \to G|$, the result follows.

Some relevant estimates of $|F \to G|$, $s(F \to G)$ and $\text{aut}(F)$ for graphs in general and for special families of graphs are known and have been obtained mainly in connection with the edge reconstruction conjecture. We try to collect them in the following

**Theorem 2** Let $G$ be a connected graph, then

$$\text{aut}(G) \leq n(\Delta_G)! (\Delta_G - 1)^{n - \Delta_G - 1}$$

(1)

Let $T$ be a spanning tree in $G$, then

$$\text{aut}(G) \leq \frac{\Delta_T}{\Delta_G} (d_G)^n \prod_{v \in V(G)} (d_T(v) - 1)!$$

(2)

Let $p = p(G)$ be the path covering number of a graph, i.e. the minimum number of vertex-disjoint paths containing all vertices of $G$. Then

$$\text{aut}(G) \leq 2 p n^2 (2^{7/8}6^{1/24})^{e-n}$$

(3)

$$\text{aut}(G) \leq (d_G)^n ((\Delta_G - 1)!)^{\frac{e-n+3 - 2\delta_G}{\Delta_G - \delta_G}},$$

(4)
provided \( \delta_G \geq 2, \Delta_G \geq 3 \).

Let \( G \) be either a square of a graph or a three-connected planar graph, then

\[
aut(G) \leq 3 \frac{2^{\frac{n-2}{2}} (d_G)^n}{\Delta_G}
\]

(5)

Let \( G \) be a \( K_{1,m} \)-free graph, then

\[
aut(G) \leq \frac{(m - 1)!((m - 2)!)^{\frac{n}{m-2}} (d_G)^n}{\Delta_G}
\]

(6)

If \( G \) has a hamiltonian path then

\[
aut(G) \leq n(\frac{e}{n - 1})^{n-1}
\]

(7)

\[
aut(G) \leq 2n^2(2^{7/8}6^{1/24})^{e-n}
\]

(8)

**Proof.** Everywhere in the sequel \( T \) is a spanning tree in \( G \). The bound (1) is just Caunter and Nash-Williams’ estimate for \( |T \rightarrow G| \), see [4] and [3, 8].

It has been shown in [5] that

\[
s(T \rightarrow G) \leq \frac{\prod_{v \in G} d_G(v)}{\Delta_G} \leq \frac{d^n_G}{\Delta_G},
\]

(9)

and

\[
aut(T) \leq \Delta_T \prod_{v \in T} (d_T(v) - 1)!,
\]

(10)

giving (2), see also [2].

If \( G \) satisfies \( \delta_G \geq 2, \Delta_G \geq 3 \), than there is a spanning tree \( T \) in \( G \) such that [5]

\[
aut(T) \leq \Delta_G (((\Delta_G - 1)!)^{\frac{\Delta_G-1}{(\Delta_G-2)}}^{\frac{e-n+1-2\Delta_G}{2}}
\]

This gives (3) by (9).

Concerning (5) notice that in both cases the corresponding graphs have a spanning tree of maximum degree at most 3. For the square of a graph this has been proved in [6] and for three-connected planar graphs this is a classical result of Barnette [1]. These yield (5) by (9) and (10) since the maximum of the product in (10) is attained then the tree has the maximal possible number \( \frac{n-2}{2} \) of vertices of degree 3.

The required estimates for \( K_{1,m} \)-free graphs giving (6) has been established in [5]. Namely,
a $K_{1,m}$-free graph has a spanning tree of maximum degree at most $m$. Moreover, such a tree can be modified to have $aut(T) \leq (m-1)!(\frac{m}{m-2})^{m-2}$.

The inequality (3) due to Pyber [12]. If $P$ is a hamiltonian path Lovász proved [11] (see also [3]) $s(P \rightarrow G) \leq \frac{n}{2}(\frac{m}{m-1})^{n-1}$. Since $aut(P) = 2$ this yields (7). Finally (8) follows from (3) with $p = 1$.

We derive now another bound on $aut(G)$ (see Theorem 3 below). First, we shall define the notion of a greedy spanning tree, $T(v_0, v_1, \ldots, v_s)$, of a connected graph $G$ by the following construction:

We shall define the sequence of vertices $v_0, v_1, v_2, \ldots, v_s$ of $G$ and the corresponding sequence $T_0, T_1, \ldots, T_s$ of trees as follows: Let $v_0$ be any vertex of $G$ and let $T_0$ be the tree containing $v_0$ and all the edges of $G$ which are adjacent to $v_0$ (we mean that if a subgraph contains an edge, then it contains also its end vertices). Note that $T_0$ is actually a star with central vertex $v_0$. In order to construct $T_1$ choose any leaf $v_1$ of $T_0$ having at least one adjacent edge which is not adjacent to any vertex of $T_0 - \{v_1\}$, and add to $T_0$ all the edges adjacent to $v_1$ which are not adjacent to any vertex in $V(T_0) - \{v_1\}$. Denote the resulting tree by $T_1$. Continue this construction inductively: given $T_{i-1}$, let $v_i$ be a leaf of $T_{i-1}$ having an adjacent edge which is not adjacent to any vertex of $T_{i-1} - \{v_i\}$, and add to $T_{i-1}$ all the edges which are adjacent to $v_i$ and which are not adjacent to any vertex of $V(T_{i-1}) - \{v_i\}$. Denote the resulting graph by $T_i$. This construction is completed at step $s$, when for every leaf $v$ of $T_s$, each edge of $G$ which is adjacent to $v$, is also adjacent to a vertex in $V(T_s) - \{v\}$.

It is easy to see that for a connected graph $G$, the above (greedy) construction results in a spanning tree $T_s$ of $G$. This spanning tree will be called a greedy spanning tree of $G$ and denoted by $T = T(v_0, v_1, \ldots, v_s)$, where $v_0, v_1, \ldots, v_s$ is the sequence of vertices used in the above construction of $T$.

Using the above notation, we derive the following bound for $autG$.

**Theorem 3** Let $G$ be a connected simple graph with $n$ vertices and let $T = T(v_0, v_1, \ldots, v_s)$ be a greedy spanning tree of $G$. Denote by $n_1$ the length of the orbit of $v_0$ under the action of the automorphism group of $G$. Then

$$autG \leq n_1(d(v_0))! \prod_{i=1}^{s}(d_T(v_i) - 1)!$$
In particular, for any greedy spanning tree $T$ of $G$ we have:

$$autG \leq n(d(v_0))! \prod_{v \in V(G)} (d_T(v) - 1)!$$

Proof. Let $\Gamma$ be the automorphism group of $G$. Given vertices $u_1, u_2, \ldots, u_r$ of $G$, denote by $C_T(u_1, u_2, \ldots, u_r)$ the subgroup of $\Gamma$ which fixes $u_1, u_2, \ldots, u_r$. Then we have $aut(G) = n_1|C_T(v_0)|$. Since $C_T(v_0)$ acts on the set $N(v_0)$ (the set of all neighbors of $v_0$ in $G$), and since $v_1$ is a neighbor of $v_0$, we have $|C_T(v_0)| \leq d(v_0)|C_T(v_0, v_1)|$ (equality holds if and only if $C_T(v_0)$ is transitive on $N(v_0)$). Denote $N(v_0) = \{v_1, u_2, \ldots, u_{d(v_0)}\}$. Then, we have:

$$|C_T(v_0, v_1)| \leq (d(v_0) - 1)|C_T(v_0, v_1, u_2)| \leq (d(v_0) - 1)(d(v_0) - 2)|C_T(v_0, v_1, u_2, u_3)| \leq \ldots \leq (d(v_0) - 1)!|C_T(v_0, v_1, u_2, \ldots, u_{d(v_0)})|.$$  

Whence $aut(G) \leq n_1(d(v_0))!|C_T(\{v_0\} \cup N(v_0))|$. Since $v_1$ is adjacent to $v_0$ in $T$, we have that $C_T(\{v_0\} \cup N(v_0))$ acts on $N_T(v_1) - \{v_0\}$. It follows by the arguments used in the preceding paragraph that $|C_T(\{v_0\} \cup N(v_0))| \leq (d_T(v_1) - 1)!|C_T(N(v_0) \cup N_T(v_1))|$, and consequently, $aut(G) \leq n_1(d(v_0))!(d_T(v_1) - 1)!|C_T(N(v_0) \cup N_T(v_1))|$. The theorem now follows by repeating the above arguments for the vertices $v_2, v_3, \ldots, v_r$.

The following corollary is a straightforward result of Theorem 3.

Corollary 1 Denote $r = \left\lfloor \frac{n - \Delta_G - 1}{\Delta_G - 1} \right\rfloor$, $\alpha = n - r(\Delta_G - 1)$ (clearly $\alpha < \Delta_G - 1$). Then

$$autG \leq n\alpha!\Delta_G![(\Delta_G - 1)!]^r.$$  

Remark It is easily verified that $aut(K_n) = n!$, $aut(K_{m,n}) = 2(m!)^2$ and $aut(K_{p,q}) = p!q!$ for $p \neq q$. Applying Theorem 3 for these graphs, we have that the bound of Theorem 3 is exact (i.e., the corresponding inequality is actually an equality). On the other hand, except for the case of formula (1) applied for $aut(K_n)$, the inequalities of Theorem 2 are not exact in the above cases.
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