On the cyclic coloring conjecture

Stanislav Jendrol', Roman Soták

Institute of Mathematics, P. J. Šafárik University, Jesenná 5, 040 01 Košice, Slovakia

Abstract

A cyclic coloring of a plane graph $G$ is a coloring of its vertices such that vertices incident with the same face have distinct colors. The minimum number of colors in a cyclic coloring of a plane graph $G$ is its cyclic chromatic number $\chi_c(G)$. Let $\Delta^*(G)$ be the maximum face degree of a graph $G$.

In this note we show that to prove the Cyclic Coloring Conjecture of Borodin from 1984, saying that every connected plane graph $G$ has $\chi_c(G) \leq \lfloor \frac{3}{2} \Delta^*(G) \rfloor$, it is enough to do it for subdivisions of simple 3-connected plane graphs.

We have discovered four new different upper bounds on $\chi_c(G)$ for graphs $G$ from this restricted family; three bounds of them are tight. As corollaries, we have shown that the conjecture holds for subdivisions of plane triangulations, simple 3-connected plane quadrangulations, and simple 3-connected plane pentagulations with an even maximum face degree, for regular subdivisions of simple 3-connected plane graphs of maximum degree at least 10, and for subdivisions of simple 3-connected plane graphs having the maximum face degree large enough in comparison with the number of vertices of their longest paths consisting only of vertices of degree two.

Keywords: plane graphs, cyclic coloring, vertex-coloring

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1. Introduction

Throughout this note we use graph theory terminology according to the books [22] and [27]. However, we recall the most frequent notions. In this note, $G$ is a connected plane graph with vertex set $V(G)$, edge set $E(G)$, and face set $F(G)$. In what follows, $G$ can have multiple edges but no loops, while a simple graph has no multiple edges. The degree of a vertex $v$, denoted by $\deg_G(v)$, is the number of edges incident with $v$. The degree of a face $f$, denoted by $\deg_G(f)$, is the number of vertices incident with $f$. A $k$-face is any face of degree $k$. We use $\Delta(G)$ and $\Delta^*(G)$ to denote the maximum vertex degree and maximum face degree of $G$, respectively. In a graph $G$, a subdivision of an edge $uv$ is the operation of replacing $uv$ with a paths $u,w,v$ through a new vertex $w$. A subdivision of a graph $G$ is a graph obtained by a sequence of subdivisions of edges of $G$.

For a cycle $C$ (in a plane graph $G$) we denote the set of vertices and edges of $G$ lying inside $C$ and outside $C$ by $\text{int}_G(C)$ and $\text{ext}_G(C)$, respectively. We say that $C$ is a separating cycle if both $\text{int}_G(C)$ and $\text{ext}_G(C)$ are not empty.

Let $G$ be a connected plane graph and $C_n$ be a separating cycle of length $n$ in $G$. Let $G_1$ (resp. $G_2$) be a plane graph obtained from $G$ by deleting the exterior (resp. the interior) of $C_n$. Observe that $\Delta^*(G_i) \leq \max\{\Delta^*(G), n\}, i \in \{1, 2\}$, where $n$ is the degree of the face $g_i$ of $G$, whose boundary is the cycle $C_n$. The face $g_i$ is not a face of $G$ while any other face of $G$ is present in $G_1$ or in $G_2$.

A cyclic coloring of a plane graph is a vertex coloring such that any two different cyclically adjacent vertices, i.e. vertices incident with the same face, receive distinct colors. The minimum number of colors

Email addresses: stanislav.jendrol@upjs.sk (Stanislav Jendrol'), roman.sotak@upjs.sk (Roman Soták)

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needed for a cyclic coloring of a graph $G$, the cyclic chromatic number, is denoted by $\chi_c(G)$. This concept was introduced by Ore and Plummer [23].

It is obvious that any cyclic coloring of a connected plane graph $G$ requires at least $\Delta^*(G)$ colors. As concerns the upper bound, Borodin [5] has conjectured (see also Conjecture 6.1 in [7], or Section 2.5 in [19]):

**Conjecture 1.** [5] If $G$ is a connected plane graph with maximum face degree $\Delta^*(G)$, then

$$\chi_c(G) \leq \lfloor \frac{3}{2} \Delta^*(G) \rfloor.$$ 

As shown in [5], the upper bound in Conjecture [1] whenever true is best possible.

This conjecture is known as the Cyclic Coloring Conjecture, see e.g. [14], [15], or [20].

From the Four Color Theorem [3] and [4] it follows that $\chi_c(G) \leq 4$ if $\Delta^*(G) = 3$. From Borodin’s proof of the Ringel’s conjecture [5] (see also [6]) we have $\chi_c(G) \leq 6$ if $\Delta^*(G) \leq 4$. Both these bounds are tight. Hebdige and Král’ proved this conjecture for $\Delta^*(G) = 6$, see [16]. So in this note we shall study only the cases when $\Delta^*(G) \geq 5$. Ore and Plummer [23] proved that $\chi_c(G) \leq 2\Delta^*(G)$, which was improved to $\lfloor \frac{3}{2} \Delta^*(G) \rfloor$ by Borodin, Sanders, and Zhao [9], and to $\lfloor \frac{3}{2} \Delta^*(G) \rfloor$ by Sanders and Zhao [25]. Moreover, Borodin at al. [9] proved that $\chi_c(G) \leq 8$ for $\Delta^*(G) = 5$. Amini, Esperet, and van den Heuvel [1] proved that the conjecture holds asymptotically in the following sense: for every $\epsilon > 0$, there exists $\Delta_\epsilon$ such that every plane graph with maximum face degree $\Delta^* \geq \Delta_\epsilon$ has a cyclic coloring with at most $(\frac{3}{2} + \epsilon) \Delta^*$ colors.

A beautiful result was obtained by Borodin, Broersma, Glebov, and van den Heuvel [8] using the following parameter $k^* = k^*(G)$, which denotes the maximum number of vertices that two faces of $G$ can have in common.

**Theorem 1.** [8] For every connected plane graph $G$ with $\Delta^*(G) \geq 5$ it holds

$$\chi_c(G) \leq \max\{\Delta^*(G) + 3k^* + 2, \Delta^*(G) + 14\}.$$ 

They also posed the following:

**Conjecture 2.** [8] Every plane graph $G$ with $\Delta^*(G)$ and $k^*$ large enough has a cyclic coloring with $\Delta^*(G) + k^*$ colors.

Better bounds are known for simple 3-connected plane graphs. The first bound for such graphs, $\chi_c(G) \leq \Delta^*(G) + 9$, was obtained by Plummer and Toft [24]. They also conjectured that any simple 3-connected plane graph has $\chi_c(G) \leq \Delta^*(G) + 2$. The best presently known upper bounds for the cyclic chromatic number of simple 3-connected plane graphs are: $\chi_c(G) \leq \Delta^*(G) + 1$ for $\Delta^*(G) \geq 122$ by Borodin and Woodall [10] and for $\Delta^*(G) \geq 60$ by Enomoto, Horňák, and Jendrol’ [13], $\chi_c(G) \leq \Delta^*(G) + 2$ for $\Delta^*(G) \geq 24$ by Horňák and Jendrol’ [17], for $\Delta^*(G) \geq 18$ by Horňák and Žlámalová [18], for $\Delta^*(G) \geq 16$ by Dvořák, Hebdige, Hlásek, Král’, and Noel [11], for $\delta(G) = 4$ and $\Delta^*(G) \geq 6$, or $\delta(G) = 5$ by Žlámalová [28], and for $G$ being locally connected by Kriselell [21]. Azarija, Erman, Král’, Krnc, and Stacho [2] proved that for every plane graph $G$, in which any two faces of degree at least four are vertex disjoint, holds $\chi_c(G) \leq \Delta^*(G) + 1$. Havet, Sereni, and Škrekovski [15] proved that $\chi_c(G) \leq 11$ for $\Delta^*(G) = 7$. The best presently known general upper bound, $\chi_c(G) \leq \Delta^*(G) + 5$, is by Enomoto and Horňák [12]. Summarizing we have:

**Theorem 2.** If $G$ is a simple 3-connected plane graph, then $\chi_c(G) \leq \Delta^*(G) + r$, where

1. $r = 1$ if $\Delta^*(G) \geq 60$, or $\Delta^*(G) = 3$, or any two faces of $G$ of degree at least four are vertex disjoint,
2. $r = 2$ if $\Delta^*(G) \geq 16$, or $\Delta^*(G) = 4$, or $\delta(G) = 4$ and $\Delta^*(G) \geq 6$, or $\delta(G) = 5$, or $G$ is locally connected,
3. $r = 3$ if $5 \leq \Delta^*(G) \leq 6$, or
4. $r = 4$ if $\Delta^*(G) = 7$, and

5. $r = 5$ in the remaining cases.

In this paper we show that to prove Conjecture \[ \] it is enough to do it for subdivisions of simple 3-connected plane graphs. Next we have obtained four different upper bounds on $\chi_c(G)$ for graphs from this family; three of them are tight. As corollaries we show that Conjecture \[ \] holds for large maximum face degree subdivisions of simple 3-connected plane graphs, and for four wide families of plane graphs without restrictions on maximum face degrees.

2. Definitions and some remarks about the structure of plane graphs

Throughout this paper $P_m$ will denote a path of length $m$. By $t = t(G)$ we will denote the number of vertices of a longest path in $G$ all vertices of which have degree 2.

For a 2-connected plane graph $G$ other than cycle the reduction $R(G)$ of $G$ is a graph obtained from $G$ by replacing all maximal $u,v$-paths whose all interior vertices are of degree 2 with the edges $uv$. More precisely, $V(R(G)) = \{v \in V(G) | \deg_G(v) \geq 3\} = V_{\geq 3}(G)$ and $E(R(G)) = P(G)$ which is the set of all maximal paths of $G$ all interior vertices of which have degree 2 and whose ends are vertices of degree at least 3.

Observe, that $|F(G)| = |F(R(G))|$ and that there is a one-to-one correspondence between sets of faces of $F(G)$ and $F(R(G))$, between the set of edges $E(R(G))$ and the set of paths $P(G)$, and between the sets of vertices $V(R(G))$ and $V_{\geq 3}(G)$. Clearly, $R(G)$ is a 2-connected plane graph with minimum degree 3.

Observe, that $R(G)$ has at least one of the following properties:

1. It is a simple 3-connected graph.

2. It contains a 2-face $[u,v]$.

3. It has a vertex-cut $\{u,v\}$ and a component $K$ of $R(G) \setminus \{u,v\}$ such that a 2-connected subgraph $H_R(u,v)$ on the vertex set $V(K) \cup \{u,v\}$ has the following structure: All vertices and edges of $H_R(u,v)$ lie on or in the interior (resp. in the exterior) of a separating cycle $C'$ determined by two internally vertex disjoint $u,v$-paths $P'$ and $P''$ whose all internal (resp. external) vertices are from the set $V(K)$.

Note, that it is enough to choose a vertex-cut $\{u,v\}$ of minimal number of vertices of $V(K)$. Then $H_R(u,v)$ is the subgraph of $R$ consisting of the vertex set $V(K) \cup \{u,v\}$ and the edge set containing $E(K)$ and all edges of $R$ between the vertices of $V(K)$ and the vertices of $\{u,v\}$ (and no edge $uv$). From the minimality of $K$ it follows that $H_R(u,v)$ is 2-connected. The boundary cycle of $H_R(u,v)$ containing both $u$ and $v$ is chosen as a desired cycle $C'$.

3. A structural Theorem

**Theorem 3.** If $G$ is a 2-connected plane graph with maximum face degree $\Delta^*(G)$ and at least four faces, then $G$ is either a subdivision of a simple 3-connected plane graph or contains a separating cycle $C_n$ with $n \leq \Delta^*(G)$.

**Proof.** Let $\Delta^* = \Delta^*(G)$. It is easy to see that the theorem holds if $G$ has exactly four faces or it contains a 2-face. Next we distinguish two cases depending on the structure of the reduction $R = R(G)$.

If $G$ is a subdivision of a simple 3-connected plane graph (i.e. $R$ is a simple 3-connected plane graph), then there is nothing to prove.

Otherwise the reduction $R$ contains either a 2-face $[u,v]$ or has a suitable vertex-cut $\{u,v\}$. Consider $H_G(u,v)$, the subgraph of $G$ corresponding to the 2-face $[u,v]$ or to the subgraph $H_R(u,v)$ of $R$.

Let the cycle $C_{n+4}$, bounding $H_G(u,v)$ in $G$, be defined by the $u,v$-paths $P_0$ and $P_0$ corresponding to the edges $uv$ of the 2-face $[u,v]$ or to the $u,v$-paths $P'$ and $P''$ of the subgraph $H_R(u,v)$, respectively, of $R$. Let $f_1$ and $f_2$ be the faces of $G$ in the exterior of $C_{n+4}$ defined by the paths $P_0$ and $P_0$, respectively. Let $P_c$ and $P_d$ be the other $u,v$-paths bounded the faces $f_1$ and $f_2$, respectively. This is always possible because
in every 2-connected plane graph any face is bounded by a cycle. Observe that \(\deg_G(f_1) = a + c \leq \Delta^*\) and \(\deg_G(f_2) = b + d \leq \Delta^*\). Let, w.l.o.g., \(b \geq a\). Then \(a + d \leq \Delta^*\) and there is, in \(G\), a separating cycle \(C_n\) of length at most \(\Delta^*\), namely \(C_n = C_{a+d}\).

\[\]

4. Two Lemmas

**Lemma 1.** If \(G\) is a 2-connected plane graph with exactly three faces and maximum face degree \(\Delta^*(G)\), then

\[\chi_c(G) \leq \left\lfloor \frac{3}{2} \Delta^*(G) \right\rfloor.\]

**Proof.** The graph \(G\) consists of three edge disjoint paths \(P_a, P_b,\) and \(P_c\) joining two vertices \(u\) and \(v\). Because for any two vertices there is, in \(G\), a common face incident with both of them, we have \(\chi_c(G) = a + b + c - 1\). Observe that \(a + b \leq \Delta^*(G)\), \(a + c \leq \Delta^*(G)\), and \(c + b \leq \Delta^*(G)\). This implies

\[2(a + b + c - 1) + 2 = 2\chi_c(G) + 2 \leq 3\Delta^*(G)\]

which immediately gives the statement of the lemma. □

The construction of the following lemma first appears in [[3]].

**Lemma 2.** For every \(t \geq 0\) there exists a 2-connected plane graph \(H\) of maximum face degree \(\Delta^*(H)\) with the reduction \(R(H)\) being simple 3-connected plane graph, \(t = t(H)\), and

\[\chi_c(H) = \Delta^*(H) + t + 2 = \left\lfloor \frac{3}{2} \Delta^*(H) \right\rfloor.\]

**Proof.** The triangular prism \(D_3\) with three edges joining the two triangles replaced by disjoint paths of equal length \(t + 1\), \(t \geq 0\), gives a graph \(H\) with \(\chi_c(H) = \Delta^*(H) + t + 2 = \left\lfloor \frac{3}{2} \Delta^*(H) \right\rfloor\). □

5. Reduction of Conjecture [[1]]

To prove Conjecture [[1]] it is enough to prove the following:

**Conjecture 3.** If \(G\) is subdivision of a simple 3-connected plane graph with maximum face degree \(\Delta^*(G)\), then

\[\chi_c(G) \leq \left\lfloor \frac{3}{2} \Delta^*(G) \right\rfloor.\]

**Theorem 4.** Conjecture [[1]] holds if and only if Conjecture [[3]] holds.

**Proof.** It is enough to prove that Conjecture [[1]] follows from Conjecture [[3]]. Suppose it is not true.

Let \(H\) be a counterexample with minimum number of faces and then with the minimum number of vertices. It is easy to see that \(H\) is 2-connected, has at least four faces (because Lemma [[1]]), and is not a subdivision of any simple 3-connected plane graph. Then, by Theorem [[3]] \(H\) contains a separating cycle \(C_n\), with \(n \leq \Delta^*(H)\). The graphs \(H_1 = C_n \cup \text{int}_G(C_n)\) and \(H_2 = C_n \cup \text{ext}_G(C_n)\), are smaller than \(H\), therefore, for \(i \in \{1, 2\}\) we have

\[\chi_c(H_i) \leq \left\lfloor \frac{3}{2} \Delta^*(H_i) \right\rfloor.\]

Since the graphs \(H_1\) and \(H_2\) have only vertices of the cycle \(C_n\) in common, \(n \leq \Delta^*(H)\), and each face of \(H\) is also a face in \(H_1\) or in \(H_2\), there is \(\Delta^*(H_i) \leq \Delta^*(H)\). So we can combine the cyclic colorings of \(H_1\) and \(H_2\) to obtain a cyclic coloring of \(H\) using at most \(\frac{3}{2} \Delta^*(H)\) colors. A contradiction. □
6. Cyclic colorings of subdivisions of simple 3-connected plane graphs

As in any simple 3-connected plane graph every two distinct faces have at most one edge in common, for any subdivision $G$ of a simple 3-connected plane graph we have $k^* = t + 2, t = t(G)$. As a result, from Theorem 4 we have:

**Theorem 5.** If $G$ is a subdivision of a simple 3-connected plane graph with $\Delta^*(G) \geq 5$, then

$$\chi_c(G) \leq \max\{\Delta^*(G) + 3t + 8, \Delta^*(G) + 14\}.$$ 

**Corollary 1.** If $G$ is a subdivision of a simple 3-connected plane graph with $\Delta^*(G) \geq \max\{6t + 16, 28\}$, then

$$\chi_c(G) \leq \left\lceil \frac{3}{2} \Delta^*(G) \right\rceil.$$ 

Let $G$ be a 2-connected plane graph with the reduction $R = R(G)$ being a simple 3-connected plane graph. We associate with the graph $G$ the following plane multigraph $S = S(G)$, called the subdivision multigraph of $G$, whose vertex set $V(S) = F(G)$, the face set of $G$, and edge set $E(S) = V_2(G)$, the set of vertices of degree 2 of $G$. The edge $v$ joins the vertices $f_a$ and $f_b$ in $S$ if and only if the 2-vertex $v$ is incident, in $G$, with both faces $f_a$ and $f_b$. Observe that the multigraph $S$ has the maximum vertex degree $\Delta(S) = \max_{f \in E(G)} \{\deg_G(f) - \deg_R(f')\}$ where $f'$ is the face of $R$ corresponding to the face $f$ of $G$. Let $\chi'(S)$ denote the chromatic index of the subdivision multigraph of $S$.

**Theorem 6.** Let $G$ be a subdivision of a simple 3-connected plane graph $R$ (which is the reduction $R = R(G)$) and let $S = S(G)$ be the subdivision multigraph of $G$. Then

$$\chi_c(G) \leq \chi'(S) + \chi_c(R).$$

Moreover, the bound is tight.

**Proof.** Let $\phi : V(R) \to \{1, \ldots, \chi_c(R)\}$ be a cyclic coloring of $R$.

Let $\psi : V_2(G) = E(S) \to \{\chi_c(R) + 1, \ldots, \chi_c(R) + \chi'(S)\}$ be a proper edge-coloring of $S$ with $\chi'(S)$ colors. The edge-coloring $\psi$ induces a vertex-coloring of vertices of $V_2(G)$ in which every two distinct vertices $v_1$ and $v_2$ of degree 2, that are adjacent with the same face $f$, receive different colors. It is easy to see that the colorings $\phi$ and $\psi$ together give a cyclic coloring of $G$.

The bound is tight for any 2-connected plane graph in which any two of its vertices are cyclically adjacent. For an example of such a graph $H$ consider the triangular prism $D_3$ with three edges joining two triangles replaced by three paths of lengths $a + 1$, $b + 1$, and $c + 1$. It is easy to see that

$$\chi_c(H) = (a + b + c) + 6 = \chi'(S(H)) + \chi_c(R(H)).$$

**Theorem 7.** If $G$ is a subdivision of a 3-connected plane graph $R$, then

$$\chi_c(G) \leq \left\lceil \frac{3}{2} \max_{f \in E(G)} \{\deg_G(f) - \deg_R(f')\} \right\rceil + \chi_c(R),$$

where $f'$ is the face of $R$ corresponding to the face $f$ of $G$. Moreover, the bound is tight.

**Proof.** By theorem of Shannon [26] it holds that $\chi'(S) \leq \left\lceil \frac{3}{2} \Delta(S) \right\rceil$. To see the tightness, consider the graph $H$ of the proof of Theorem 6 with $a = b = c \geq 2$.

Next corollary provides three other wide families of graphs for which Borodin’s conjecture holds.


Corollary 2. If $G$ is a subdivision of a plane triangulation, a simple 3-connected plane quadrangulation, or a simple 3-connected plane pentagulation (all faces are of degree 5) with even maximum face degree, then

$$\chi_c(G) \leq \frac{3}{2} \Delta^*(G).$$

Proof. We apply Theorem 7. If $G$ is a triangulation, then $\chi_c(G) \leq \frac{3}{2}(\Delta^*(G) - 3)] + 4 = \frac{3}{2} \Delta^*(G) - 9 + 8 \leq \frac{3}{2} \Delta^*(G)$.

If $G$ is a simple 3-connected plane quadrangulation, then $\chi_c(G) \leq \frac{3}{2}(\Delta^*(G) - 4)] + 6 = \frac{3}{2} \Delta^*(G) - 12 + 12 = \frac{3}{2} \Delta^*(G)$.

If $G$ is a subdivision of a simple 3-connected plane pentagulation of even maximum face degree, then $\chi_c(G) \leq \frac{3}{2}(\Delta^*(G) - 5)] + 8 = \frac{3}{2} \Delta^*(G) - 15 + 16 = \frac{3}{2} \Delta^*(G)$.

\qed

7. One more upper bound on cyclic chromatic number

Theorem 8. If $G$ is a subdivision of a simple 3-connected plane graph $R$, then

$$\chi_c(G) \leq \max_{f \in F(G)} \{\deg_G(f) - \deg_R(f')\} + t(G) + \chi_c(R),$$

where $f'$ is the face of $R$ corresponding to the face $f$ of $G$. Moreover, the bound is tight.

Proof. We use for $\chi'(S)$ in the proof of Theorem 7 instead of the Shannon bound, the bound of Vizing and Gupta (see [27], p. 275), saying that $\chi'(S) \leq \Delta(S) + \mu(S)$ where $\mu(S)$ denotes the maximum edge multiplicity of $S$, and the fact that $\mu(S) = t(G)$. To see the tightness, consider the graph $H$ from the proof of Theorem 6 with $a = b = c = t(H) \geq 2$.

\qed

Corollary 3. If $G$ is a subdivision of a simple 3-connected plane graph $R$ with $\Delta^*(G) \geq 2\chi_c(R) + 2t(G) - 6$, then

$$\chi_c(G) \leq \frac{3}{2} \Delta^*(G).$$

Note that if $\Delta^*(R) \leq 2t(G) + 6$ for $t \geq 2$ or $\Delta^*(R) \leq 12 - t$ for $t \leq 1$, then the bound of Theorem 8 is better than that of Theorem 6 and, hence, of Theorem 1.

The graph $G$ is a regular subdivision of a graph $R$ if it is obtained from $R$ by replacing each edge of $R$ with a path of length $k + 1$ for some constant $k \geq 0$.

Corollary 4. If $G$ is a regular subdivision of a simple 3-connected plane graph $R$, then

$$\chi_c(G) \leq \Delta^*(G) + t(G) + r,$$

where $r$ is a constant depending on the properties of $R$, see Theorem 2.

Proof. It is easy to see that for any face $f$ of $G$ there is $\deg_G(f) = \deg_R(f')(k + 1)$ for some integer $k \geq 0$, where $f'$ is the face of $R$ corresponding to the face $f$. Then $t(G) = k$ and, by Theorem 6, $\chi_c(G) \leq \max_{f \in F(G)} \{\deg_G(f) - \deg_R(f')\} + t(G) + \Delta^*(R) + r = \max_{f \in F(G)} \{\deg_R(f')\} + t(G) + \Delta^*(R) + r = \Delta^*(R)(k + 1) + t(G) + r = \Delta^*(G) + t(G) + r$.

\qed

Corollary 5. If $G$ with $t(G) \geq 1$ is a regular subdivision of a simple 3-connected plane graph $R$, then

$$\chi_c(G) \leq \frac{3}{2} \Delta^*(G).$$

Proof. Let $G$ be a regular subdivision of the graph $R$ with $t(G) = k \geq 1$. From $\Delta^*(G) \geq 2 + \frac{8}{k + 1}$ and Corollary 4 we have $\chi_c(G) \leq \Delta^*(R)(k + 1) + k + 5 \leq \frac{3}{2} \Delta^*(R)(k + 1) = \frac{3}{2} \Delta^*(G)$.

\qed
8. One new conjecture

We believe that the following conjecture, which involves both, the Plummer and Toft conjecture and the corresponding part of Conjecture 2, holds:

**Conjecture 4.** If $G$ is a subdivision of a simple 3-connected plane graph, then

$$\chi_c(G) \leq \Delta^*(G) + t(G) + 2.$$ 

Observe, that Conjecture 4 holds for regular subdivisions of simple 3-connected plane graphs $R$ with $\Delta^*(R) \geq 16$, or $\Delta^*(R) \leq 4$, or $\delta(R) = 4$ and $\Delta^*(R) \geq 6$, or $\delta(R) = 5$, or $R$ being locally connected (see Corollary 4 and Theorem 2, cases $r = 1$ and $r = 2$), for subdivisions of plane triangulations, and for subdivisions of simple 3-connected plane quadrangulations.

In the case of triangulations we have from Theorem 8:

$$\chi_c(G) \leq \Delta^*(G) - 3 + t(G) + 3 + 1 = \Delta^*(G) + t(G) + 1.$$ 

In the case of quadrangulations Theorem 8 provides:

$$\chi_c(G) \leq \Delta^*(G) - 4 + t(G) + 4 + 2 = \Delta^*(G) + t(G) + 2.$$ 

From Lemma 2 we know that the upper bound on $\chi_c(G)$ in our Conjecture 4 cannot be improved.

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