VIRTUALLY FREE GROUPS ARE STABLE IN PERMUTATIONS

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ABSTRACT. We prove that finitely generated virtually free groups are stable in permutations. As an application, we show that almost-periodic almost-automorphisms of labelled graphs are close to periodic automorphisms.

1. INTRODUCTION

A finitely generated group $G$ is called stable in permutations (in short P-stable) if every almost action of $G$ on a finite set is close to an honest action (see §2 for definitions). For the ubiquitous class of sofic groups, the property of P-stability can be seen as a stronger form of residual finiteness [1]. Our main result is:

**Theorem A.** Every finitely generated virtually free group is P-stable.

It is trivially true that free groups are P-stable. But while residual finiteness is preserved under passing to finite index subgroups (or rather to any subgroup), this fact is not clear in general for P-stability.

To the best of our knowledge Theorem A gives the first examples of P-stable groups which are not free products of P-stable amenable groups. Note that while fundamental groups of closed orientable surfaces are known to be flexibly P-stable [3], it is not clear if these groups are P-stable in the strict sense.

As a special case of Theorem A we answer the following question of Lubotzky.

**Corollary 1.1.** The modular group $\text{SL}_2(\mathbb{Z})$ is P-stable.

Interestingly P-stability is not, generally speaking, preserved under direct products — for example, the groups $F_2 \times \mathbb{Z}$ are not P-stable [2]. This phenomenon is to be contrasted with the fact that the product groups $F_2 \times (\mathbb{Z}/n\mathbb{Z})$ are P-stable for all $n \in \mathbb{N}$, as follows from Theorem A. As a consequence of the P-stability of these groups we are able to deduce:

**Corollary B.** Fix some $d, n \in \mathbb{N}$. Let $F_d$ be the free group of rank $d$ and $\mathcal{G}_d$ be the family of finite labelled Schreier graphs of $F_d$. Then for every graph $\Gamma \in \mathcal{G}_d$ and every $\delta$-almost automorphism $\alpha$ of $\Gamma$ of $\delta$-almost order $n$, there is a graph $\Gamma' \in \mathcal{G}_d$ on the same vertex set as $\Gamma$ and $O(\delta)$-close to $\Gamma$ and an automorphism $\alpha'$ of $\Gamma'$ which is $O(\delta)$-close to $\alpha$ and has order $n$.

More details and a precise statement of Corollary B can be found in §8 below.

**Stable epimorphisms.** Stallings theorem on ends of groups [5,6] implies that a finitely generated group $G$ is virtually free if and only if $G$ is isomorphic to the fundamental group $\pi_1(G,T)$ of a finite graph of groups $\mathcal{G}$ with finite vertex groups with respect to some maximal spanning tree $T$ (see §3 for the definition of $\pi_1(G,T)$).
Naturally associated to the graph of groups $\mathcal{G}$ and the maximal spanning tree $T$ there is another group $\pi_1(\mathcal{G}, T)$ admitting a quotient map $\pi_1(\mathcal{G}, T) \to \pi_1(\mathcal{G}, T)$. This group is isomorphic to the free product of the vertex groups of $\mathcal{G}$ with the topological fundamental group of the underlying graph of $\mathcal{G}$. As finite groups are P-stable it follows immediately that the group $\pi_1(\mathcal{G}, T)$ is P-stable.

Motivated by this, we introduce a relative notion of P-stable epimorphisms, see Definition 2.1. In particular, a finitely generated group $G$ is P-stable in the usual sense if and only if the natural epimorphism from the free group in the generators $G$ onto the group $G$ is P-stable. Theorem 1.2 is thereby reduced to the following statement, to which the major part of this work is dedicated.

**Theorem 1.2.** The epimorphism $\pi_1(\mathcal{G}, T) \to \pi_1(\mathcal{G}, T)$ is P-stable.

A detailed outline of the proof of Theorem 1.2 can be found in §2 below, after the necessary definitions and notations are set in place.

2. **P-stable epimorphisms**

Let $X$ be a finite set. Consider the normalized Hamming distance $d_X$ on the symmetric group $\text{Sym}(X)$ given by

$$d_X(\sigma_1, \sigma_2) = \frac{1}{|X|}|\{x \in X : \sigma_1(x) \neq \sigma_2(x)\}|$$

for all pairs $\sigma_1, \sigma_2 \in \text{Sym}(X)$. Note that the metric $d_X$ is bi-invariant.

Let $\mathcal{G}$ be a group with finite generating set $S$. Define a metric $d_{X,S}$ on the set $\text{Hom}(\mathcal{G}, \text{Sym}(X))$ of all group homomorphisms $\rho : \mathcal{G} \to \text{Sym}(X)$ by

$$d_{X,S}(\rho, \rho') = \sum_{s \in S} d_X(\rho(s), \rho(s'))$$

for each pair $\rho, \rho' \in \text{Hom}(\mathcal{G}, \text{Sym}(X))$.

Let $N \triangleleft \mathcal{G}$ be a normal subgroup normally generated by some finite subset $R \subseteq \mathcal{G}$. Denote $G = \mathcal{G}/N$. We say that an action $\rho : \mathcal{G} \to \text{Sym}(X)$ is a $\delta$-almost $G$-action if

$$\sum_{r \in R} d_X(\rho(r), \text{id}) < \delta.$$ 

This terminology is justified by the observation that $\rho$ is an honest $G$-action if and only if it is a $\delta$-almost $G$-action with respect to $\delta = 0$. Note that strictly speaking this notion depends on fixing the normal generating set $R$.

**Definition 2.1.** The epimorphism $\phi : \mathcal{G} \to G$ is P-stable if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $\delta$-almost $G$-action $\rho : \mathcal{G} \to \text{Sym}(X)$ there is a $G$-action $\rho' : G \to \text{Sym}(X)$ with $d_{X,S}(\rho, \rho' \circ \phi) < \varepsilon$.

**Lemma 2.2.** The P-stability of the epimorphism $\phi : \mathcal{G} \to G$ is a well-defined notion (i.e. it is independent of the choices of the finite sets $S$ and $R$).

**Proof.** It is easy to see that if $S_1$ and $S_2$ are two finite generating sets for the group $G$ then the resulting metrics $d_{X,S_1}$ and $d_{X,S_2}$ on the set $\text{Hom}(\mathcal{G}, \text{Sym}(X))$ are bi-Lipschitz equivalent. A similar argument, taking into account the bi-invariance of the normalized Hamming metric $d_X$, shows that if $R_1$ and $R_2$ are two finite normal generating sets for the subgroup $N \triangleleft \mathcal{G}$ then there is a constant $C = C(R_1, C_2) > 1$ such that

$$C^{-1} \sum_{r \in R_2} d_X(\rho(r), \text{id}) \leq \sum_{r \in R_1} d_X(\rho(r), \text{id}) \leq C \sum_{r \in R_2} d_X(\rho(r), \text{id}).$$
The conclusion follows from these observations.

Let $H$ be any group admitting a finite generating set $S$ and $F(S)$ be the free group in the generators $S$. Observe that the natural homomorphism $F(S) \to H$ is $P$-stable if and only if the group $H$ is $P$-stable in the usual sense.

**Remark 2.3.** Every split epimorphism is $P$-stable.

The following follows immediately from Definition 2.1.

**Lemma 2.4.** Let $G \xrightarrow{\phi} G \xrightarrow{\psi} G$ be a sequence of epimorphisms with normally finitely generated kernels. If $\phi$ and $\psi$ are $P$-stable then $\psi \circ \phi$ is $P$-stable.

We have occasion to use Lemma 2.4 only in the following special form: if the group $G$ is $P$-stable and $\phi : G \xrightarrow{} G$ is a $P$-stable epimorphism then the group $G$ is $P$-stable.

**Remark 2.5.** It seems an interesting problem to look for other non-trivial instances of $P$-stable epimorphisms.

### 3. The fundamental group of a graphs of groups

We recall the definition of the fundamental group of a graph of groups and in particular list its defining relations. This is followed by a detailed sketch of proof for our Theorem A as well as for the “relative” Theorem 1.2. Lastly we introduce some useful asymptotic notations.

**Graphs of groups.** We use Serre’s notation for graphs [4]. In this notation, a graph $\Gamma$ consists of a set of vertices $V(\Gamma)$ and a set of edges $E(\Gamma)$. Each edge $e \in E(\Gamma)$ has an origin $o(e) \in V(\Gamma)$ and a terminus $t(e) \in V(\Gamma)$. Moreover each edge $e \in E(\Gamma)$ admits a distinct opposite edge $\bar{e} \in E(\Gamma)$ that satisfies $\bar{\bar{e}} = e$, $o(\bar{e}) = t(e)$ and $t(\bar{e}) = o(e)$. Every pair of “oriented” edges $\{e, \bar{e}\} \subset E(\Gamma)$ represents a single “geometric” edge. An orientation of the graph $\Gamma$ is a subset $\vec{E}(\Gamma) \subset E(\Gamma)$ containing exactly a single edge from each pair $\{e, \bar{e}\}$.

**Definition 3.1.** A graph of groups $\mathcal{G}$ is

$$\mathcal{G} = (\Gamma, \{G_v\}_{v \in V(\Gamma)}, \{G_e\}_{e \in E(\Gamma)}, \{i_e : G_e \to G_{t(e)}\}_{e \in E(\Gamma)})$$

where $\Gamma$ is a connected graph, $G_v$ is a vertex group for all $v \in V(\Gamma)$, $G_e$ is an edge group for all edges $e \in E(\Gamma)$ with $G_e = G_{\bar{e}}$ and $i_e : G_e \to G_{t(e)}$ are injective homomorphisms.

Let $\mathcal{G}$ be a graph of groups. Fix an orientation $\tilde{E}(\Gamma)$ and a maximal spanning tree $T \subset \Gamma$. Consider the group $\pi_1(\mathcal{G}, T)$ defined as the free product

$$\pi_1(\mathcal{G}, T) = \ast_{v \in V(\Gamma)} G_v \ast F(\{s_e\}_{e \in \tilde{E}(\Gamma)})$$

where $F(\cdot)$ denotes the free group over the given basis. It will be convenient to consider the following generating set

$$S_{\mathcal{G}} = \bigcup_{v \in V(\Gamma)} G_v \cup \{s_e\}_{e \in \tilde{E}(\Gamma)}.$$
Definition 3.2. The fundamental group \(\pi_1(G, T)\) of the graph of groups \(\mathcal{G}\) with respect to the subtree \(T\) is the quotient of the free product \(\pi_1(G, T)\) by the normal subgroup generated by the relations

\[
R_G = \begin{cases} 
    s_e = 1 & \forall e \in E(T), \\
    s_e^{-1}i_e(g_e)s_e = i_e(g_e) & \forall e \in E(T), g_e \in G_e.
\end{cases}
\]

Remark 3.3. The fundamental group \(\pi_1(G, T)\) as well as the group \(\pi_1(G, T)\) are independent of the choice of maximal spanning tree \(T\) up to isomorphism [4 I.§5].

For the remainder of the paper we will assume that \(\mathcal{G}\) is a finite graph of groups, with finite vertex groups. In particular \(S_G\) is a finite generating set for the group \(\pi_1(G, T)\).

Outline of the proof and of the paper. Note that the group \(\pi_1(G, T)\) is a free product of finite groups and of a free group. As such \(\pi_1(G, T)\) is easily seen to be P-stable [1]. In light of Lemma 2.4 and the remarks following it, our main result Theorem A follows immediately from Theorem 1.2 of the introduction. In other words it suffices to show that the epimorphism \(\pi_1(G, T) \to \pi_1(G, T)\) is P-stable.

Towards this goal consider some \(\delta\)-almost \(\pi_1(G, T)\)-action \(\rho : \pi_1(G, T) \to \text{Sym}(X)\). In particular \(\rho\) restricts to actions of the finite vertex groups \(G_v\). For \(\rho\) to factorize through the fundamental group \(\pi_1(G, T)\) it is necessary that for every edge \(e \in E(T)\) the two actions \(\rho \circ i_e\) and \(\rho \circ i_e\) of the edge group \(G_e\) are isomorphic.

It is clear that the isomorphism type of an action of a finite group on a finite set is characterized by the number of occurrences of each of its finitely many transitive action types. In [4] we show how to represent this data using a vector in some canonical \(\mathbb{Z}\)-module associated to the group. The restriction maps \(\rho|_{G_v} \to (\rho \circ i_e)|_{G_e}\) define a \(\mathbb{Z}\)-linear map \(d_G\) between the respective \(\mathbb{Z}\)-modules. The above mentioned condition (that the two actions \(\rho \circ i_e\) and \(\rho \circ i_e\) of the edge group \(G_e\) are isomorphic) can be described as the kernel of this \(\mathbb{Z}\)-linear map \(d_G\). Lastly, the fact that \(\rho\) is a \(\delta\)-almost action of \(\pi_1(G, T)\) translates to having a small image under the map \(d_G\).

In [4] we show that “\(\mathbb{Z}\)-linear maps are stable” in the following sense: an exact \(\mathbb{Z}\)-solution to a linear system of equations and inequalities can be found nearby a \(\delta\)-almost solution. This is applied to the linear map \(d_G\). An exact \(\mathbb{Z}\)-solution represents an isomorphism type of an action of \(\pi_1(G, T)\) on a finite set of the same size as \(X\), which is statistically close to \(\rho\) and admits refinements to well-defined isomorphism types of actions of the edge groups.

Finally, in [6] we show how given a \(\delta\)-almost action \(\rho\), and a nearby exact \(\mathbb{Z}\)-solution to the corresponding linear system of equations, one can find a nearby action \(\rho'\) factoring via \(\pi_1(G, T)\).

We will make repeated use of the finiteness of vertex and edge groups via:

Observation 3.4. Let \(G\) be a finite group. If \(G\) acts on a finite set \(X\) and \(Y \subseteq X\) then there exists a \(G\)-invariant subset \(Y' \subseteq Y\) such that \(|X - Y'| \leq |G||X - Y|\).

Notations. We will need to consider inequalities involving quantities depending on the graph of groups \(\mathcal{G}\) in question (such as the number of vertices or edges, the sizes of the vertex groups \(G_v\), etc.). To avoid cumbersome formulas it would be convenient to introduce the following asymptotic notation.

We write \(A \lesssim B\) if there exists a constant \(c = c(\bullet)\) such that \(A \leq cB\). We omit the subscript when it is clear from the context.
4. Set of actions on finite sets

Let \( G \) be any group. Let \( \text{Acts}(G) \) denote the set of all actions of the group \( G \) on finite sets considered up to isomorphism. Similarly let \( \text{Trans}(G) \) be the set of transitive actions of \( G \) on finite sets considered up to isomorphism.

Every action \( \rho : G \to \text{Sym}(X) \) on some finite set \( X \) can be decomposed into a disjoint union of its finitely many orbits \( O_1, \ldots, O_n \subseteq X \). The restriction \( \rho \rvert_{O_i} : G \to \text{Sym}(O_i) \) of \( \rho \) to each orbit \( O_i \) is transitive for all \( i = 1, \ldots, n \). The isomorphism class of the action \( \rho \) is determined by counting the isomorphism classes of its restricted actions \( \rho \rvert_{O_i} \) with multiplicity.

This observation enables us to identify the set of actions \( \text{Acts}(G) \) with a non-negative cone in the free \( \mathbb{Z} \)-module \( \Lambda_G \) with basis \( \text{Trans}(G) \), namely

\[
\Lambda_G = \bigoplus_{\rho \in \text{Trans}(G)} \mathbb{Z}\rho.
\]

More precisely, given an action \( \rho \in \text{Acts}(G) \) we define

\[
\rho^\dagger = \sum_{O \subseteq G \setminus X} \rho \rvert_{O} \Lambda_G.
\]

The correspondence \( \rho \mapsto \rho^\dagger \) is injective and its image \( \text{Acts}(G)\dagger \) in \( \Lambda_G \) is the non-negative cone

\[
\Lambda^+_G := \{ (\lambda_\rho)_{\rho \in \text{Trans}(G)} : \lambda_\rho \geq 0 \}.
\]

We observe that the correspondence \( \rho \mapsto \rho^\dagger \) is additive in the following sense: any two actions \( \rho_1, \rho_2 \in \text{Acts}(G) \) with \( \rho_i : G \to \text{Sym}(X_i) \) for \( i \in \{1, 2\} \) satisfy

\[
(\rho_1 \bigsqcup \rho_2)^\dagger = \rho_1^\dagger + \rho_2^\dagger
\]

where \( \rho_1 \bigsqcup \rho_2 : G \to \text{Sym}(X_1 \bigsqcup X_2) \) is the disjoint union of \( \rho_1 \) and \( \rho_2 \).

We find it convenient to introduce a norm \( \| \cdot \|_G \) on the \( \mathbb{Z} \)-module \( \Lambda_G \) by

\[
\| \lambda \|_G = \sum_{\rho \in \text{Trans}(G) \mid \rho \rvert_{\text{Sym}(X_\rho)}} |\lambda_\rho| \cdot |X_\rho| \quad \forall \lambda = (\lambda_\rho) \in \Lambda_G.
\]

This norm is chosen in such a way that every action \( \rho \in \text{Acts}(G) \) with \( \rho : G \to \text{Sym}(X) \) satisfies \( \| \rho^\dagger \|_G = |X| \).

**Pullback on set of actions.** Let \( H \) be any group admitting a homomorphism \( i : H \to G \). There is a pullback map \( i^* \) on the corresponding sets of isomorphism classes of actions on finite sets is given by

\[
i^* : \text{Acts}(G) \to \text{Acts}(H), \quad i^* \rho = \rho \circ i \quad \forall \rho \in \text{Acts}(G).
\]

Allowing for a slight abuse of notation, we also let \( i^* \) denote the resulting \( \mathbb{Z} \)-linear map \( i^* : \Lambda_G \to \Lambda_H \) defined in terms of the basis by

\[
i^*(\rho^\dagger) = (\rho \circ i)^\dagger \quad \rho \in \text{Trans}(G).
\]

**Observation 4.1.** Let \( \phi : H \to \text{Sym}(X) \) be a group action such that \( \phi^\dagger = i^*(\lambda) \) for some \( \lambda \in \Lambda_G \). Then there exists a group action \( \rho : G \to \text{Sym}(X) \) satisfying \( \rho^\dagger = \lambda \) and \( \rho \circ i = \phi \).
Set of actions for a graph of groups. We extend notions introduced above to the setting of graphs of groups. Recall that
\[ \mathcal{G} = (\Gamma, \{G_v\}_{v \in V(\Gamma)}, \{G_e\}_{e \in E(\Gamma)}, \{i_e : G_e \to G_{i(e)}\}_{e \in E(\Gamma)}) \]
is a finite graph of groups with finite vertex groups. We define the \( \mathbb{Z} \)-modules
\[ \Lambda_V = \bigoplus_{v \in V(\Gamma)} \Lambda_{G_v} \quad \text{and} \quad \Lambda_E = \bigoplus_{e \in E(\Gamma)} \Lambda_{G_e} \]
and the respective positive cones
\[ \Lambda_V^+ = \bigoplus_{v \in V(\Gamma)} \Lambda_{G_v}^+ \quad \text{and} \quad \Lambda_E^+ = \bigoplus_{e \in E(\Gamma)} \Lambda_{G_e}^+. \]

It will be convenient to consider the \( \mathbb{Z} \)-modules with the norms
\[ \| \cdot \|_V = \frac{1}{|V(\Gamma)|} \sum_{v \in V(\Gamma)} \| \cdot \|_{G_v}, \quad \| \cdot \|_E = \frac{1}{|E(\Gamma)|} \sum_{e \in E(\Gamma)} \| \cdot \|_{G_e} \]
where \( \| \cdot \|_{G_v} \) and \( \| \cdot \|_{G_e} \) are the norms defined on the \( \mathbb{Z} \)-modules \( \Lambda_{G_v} \) and \( \Lambda_{G_e} \) as above.

Let \( d_\mathcal{G} : \Lambda_V \to \Lambda_E \) be the \( \mathbb{Z} \)-linear map defined on each direct summand \( \Lambda_{G_v} \) of the \( \mathbb{Z} \)-module \( \Lambda_V \) by
\[ (d_\mathcal{G}|_{\Lambda_{G_v}}) = \sum_{e \in \{v\} \to v} i_e^* - \sum_{e \in \{v\} \to v} i_{\overline{v}}^*. \]
In other words, the image of the vector \( \lambda = (\lambda_v)_v \in \Lambda_V \) under the \( \mathbb{Z} \)-linear map \( d_\mathcal{G} \) in each coordinate \( e \in E(\Gamma) \) is given by
\[ (d_\mathcal{G}(\lambda))_e = i_e^*(\lambda_{i(e)}) - i_{\overline{e}}^*(\lambda_{\overline{v}(e)}). \]

Actions of the group \( \pi_1(\mathcal{G}, T) \) on finite sets. Let \( X \) be a fixed finite set. Given an action \( \rho : \pi_1(\mathcal{G}, T) \to \text{Sym}(X) \) we denote (abusing our previous notations)
\[ \rho^I \in \Lambda_X, \quad (\rho^I)_v = (\rho|_{G_v})^I \ \forall v \in V. \]
Note that \( \|\rho^I\|_V = |X| \). Moreover the vector \( \rho^I \) depends only on the restrictions of the action \( \rho \) to the vertex groups \( G_v \)'s but not to the free factor \( F(\{s_e\}) \).

**Proposition 4.2.** If the action \( \rho : \pi_1(\mathcal{G}, T) \to \text{Sym}(X) \) factors through \( \pi_1(\mathcal{G}, T) \) then \( \rho^I \in \ker d_\mathcal{G} \).

**Proof.** The \( \Lambda_{G_v} \)-coordinate of the image of the vector \( \rho^I \) under the \( \mathbb{Z} \)-linear map \( d_\mathcal{G} \) for any fixed oriented edge \( e \in \widetilde{E}(\Gamma) \) is given by
\[ (d_\mathcal{G}(\rho^I))_e = i_e^*((\rho|_{G_{i(e)}})^I) - i_{\overline{e}}^*((\rho|_{G_{\overline{v}(e)}})^I) = (\rho \circ i_e)^I - (\rho \circ i_{\overline{e}})^I \in \Lambda_{G_v}. \]
The two actions \( \rho \circ i_e \) and \( \rho \circ i_{\overline{e}} \) of the edge group \( G_v \) on the finite set \( X \) are conjugate via the permutation \( \rho(s_e) \). Therefore \( (\rho \circ i_e)^I = (\rho \circ i_{\overline{e}})^I \) so that the \( \Lambda_{G_v} \)-coordinate in question vanishes. This concludes the proof as the oriented edge \( e \in E(\Gamma) \) was arbitrary. \( \square \)

We remark that the converse of Proposition 4.2 is also true, in the sense that if a vector \( \lambda \in \Lambda^+_V \) is in \( \ker d_\mathcal{G} \) then there exists a finite set \( Y \) with \( \|\lambda\|_V = |Y| \) and some action \( \rho : \pi_1(\mathcal{G}, T) \to \text{Sym}(Y) \) such that \( \rho^I = \lambda \). We will need a much sharper version of this fact proved in Proposition 4.3 below.

**Proposition 4.3.** Let \( \rho : \pi_1(\mathcal{G}, T) \to \text{Sym}(X) \) be an action. If \( \rho \) is a \( \delta \)-almost \( \pi_1(\mathcal{G}, T) \)-action then
\[ \|d_\mathcal{G}(\rho^I)\|_E < \delta \|\rho^I\|_V. \]
Proof: Fix an oriented edge \( e \in \tilde{E}(\Gamma) \) with \( t(e) = u \) and \( o(e) = v \). For each group element \( g \in G_e \) consider the subset

\[
X_g = \{ x \in X : \rho(s_e^{-1}i_e(g)s_e)(x) = \rho(i_e(g))(x) \}.
\]

The assumption that \( \rho \) is a \( \delta \)-almost \( \pi_1(\mathcal{G}, T) \)-action implies that \( |X_g| \geq (1-\delta)|X| \). Denote \( X_e = \cap_{g \in G_e} X_g \) so that \( |X_e| \geq (1-\delta|G_e|)|X| \) and

\[
\rho(s_e^{-1}i_e(g)s_e)(x) = \rho(i_e(g))(x) \quad \forall x \in X_e, g \in G_e.
\]

According to Observation 3.4 there is some \( i_e(G_e) \)-invariant subset \( Y_e \subset X_e \) satisfying \( |Y_e| \geq (1-\delta|G_e|)|X| \). Note that the set \( \rho(s_e)(Y_e) \) is \( i_e(G_e) \)-invariant. Moreover, the two actions \( (\rho \circ i_e) |_{\rho(s_e)(Y_e)} \) and \( (\rho \circ i_e) |_{Y_e} \) of the group \( G_e \) are isomorphic (via conjugation by the permutation \( \rho(s_e) \)).

To simplify our notations let \( \rho_u = \rho|G_u \) and \( \rho_v = \rho|G_v \) for the remainder of this proof. The previous paragraph implies that

\[
((\rho_u \circ i_e) |_{\rho(s_e)(Y_e)})^I = ((\rho_v \circ i_e) |_{Y_e})^I.
\]

The norm of the coordinate of the vector \( d^I_G(\rho^I) \) corresponding to the edge \( e \) is given by

\[
\|d^I_G(\rho^I)|_{G_e} = \|i^U_e(\rho_u^I) - i^U_e(\rho_v^I)\|_{G_e}
= \|(X \circ \rho_u |_{\rho(s_e)(Y_e)^I})^I + (X \circ \rho_v |_{\rho(s_e)(Y_e)^I})^I
- (X \circ \rho_v |_{Y_e^I})^I - (X \circ \rho_u |_{Y_e^I})^I\|_{G_e}
\leq \|i^U_e(\rho_u |_{\rho(s_e)(Y_e)^I})^I\|_{G_e} + \|i^U_e(\rho_v |_{\rho(s_e)(Y_e)^I})^I\|_{G_e}
= |X - s_e(Y_e)| + |X - Y_e| \leq 2\delta|G_e|^2|X|.
\]

Averaging the above estimate over all oriented edges \( e \in \tilde{E}(\Gamma) \) gives

\[
\|d^I_G(\rho^I)\|_E \leq c\delta|X| = c\delta\|\rho^I\|_V
\]

with respect to the constant \( c = 2\max_{e \in \tilde{E}(\Gamma)}|G_e|^2 \). This constant depends only on the graph of groups \( \mathcal{G} \).

\[
\square
\]

5. LINEAR ALGEBRA AND CONES

This section is, formally speaking, independent of the rest of the paper. Its goal is to show that “\( \mathbb{Z} \)-linear maps are stable”, in the sense that an approximate solution to a system of linear equations and inequalities must be close to an exact \( \mathbb{Z} \)-solution (see Lemma 5.3 below for a precise statement).

Let \( \Lambda_1 \) and \( \Lambda_2 \) be a pair of finitely generated free \( \mathbb{Z} \)-modules. Let \( d : \Lambda_1 \to \Lambda_2 \) be a \( \mathbb{Z} \)-linear map.

Let \( V_i = \Lambda_i \otimes \mathbb{R} \) be the \( \mathbb{R} \)-vector spaces obtained by extending scalars from \( \Lambda_i \) and \( \| \cdot \|_i \) be norms on \( V_i \) for \( i = 1, 2 \). By abuse of notation, we continue using \( d : V_1 \to V_2 \) to denote the \( \mathbb{R} \)-linear extension of \( d : \Lambda_1 \to \Lambda_2 \). We will make essential use of the fact that \( d : V_1 \to V_2 \) is defined over \( \mathbb{Q} \). Denote \( K = \ker d \) so that \( K \) is a \( \mathbb{Q} \)-subspace of the \( \mathbb{R} \)-vector space \( V_1 \).

Assume that \( C \subset V_1 \) is a closed positive cone defined by finitely many inequalities over \( \mathbb{Q} \) and satisfying \( \text{Span}(C) = V_1 \). Denote \( \Lambda_1^+ = C \cap \Lambda_1 \) so that the subset \( \Lambda_1^+ \) is closed under addition.

**Lemma 5.1.** For all \( v \in C \) there exists \( v'' \in C \cap K \) such that \( \|v - v''\|_1 \leq \|dv\|_2 \).
We point out that the intersection $C \cap K$ is non-empty for it contains the zero vector $0 \in V_1$. Lemma 5.1 does not require the assumption that the subspace $K$, the linear map $d$ and the positive cone $C$ are all defined over $\mathbb{Q}$. We do need however the assumption that $C$ is defined by finitely many inequalities.

**Proof of Lemma 5.1.** We argue by induction on the $\mathbb{R}$-dimension of $V_1$. The base case where $\dim_{\mathbb{R}} V_1 = 0$ is trivial.

Consider the $\mathbb{R}$-subspace $d(V_1)$ of the $\mathbb{R}$-vector space equipped with two different norms, namely the restriction of norm $\| \cdot \|_2$ coming from $V_2$, and the quotient norm $\| \cdot \|_1$ defined by

$$[d v]_1 := \inf_{w \in K} \| v - w \|_1 \quad \forall v \in V_1.$$  

Since any two norms on a finite dimensional $\mathbb{R}$-vector space are bi-Lipschitz equivalent, there is some constant $c > 0$ such that $[d v]_1' \leq c [d v]_2$ for all $v \in V_1$.

Fix some vector $v \in C$. The infimum appearing in the definition of the quotient norm $[d v]_1$ is attained at some vector $w \in K$, hence

$$[d v]_1' = \| v - w \|_1 \leq c [d v]_2.$$  

If $w \in C$, then we are done by choosing $v' = w \in C \cap K$. Otherwise, let $u$ be the closest point to $w$ along the closed segment $[v, w] \subseteq V_1$ and belonging to the closed cone $C$. Then clearly

$$\| v - u \|_1 \leq \| v - w \|_1 = [d v]_1' \leq c [d u]_2 \quad \text{and} \quad [d u]_2 \leq [d v]_2.$$  

Since the point $u$ lies on the boundary of the positive cone $C$, it belongs to some proper face $D \subset C$ spanning a $\mathbb{Q}$-subspace $U_1 = \text{Span}_{\mathbb{Q}}(D) \subseteq V_1$ of strictly lower dimension. By the induction hypothesis there exists some constant $c_D$, such that for $u \in D$ there exists a point $v' \in D \cap K$ with $\| u - v' \|_1 \leq c_D [d u]_2$. Hence,

$$\| v - v' \|_1 \leq \| v - u \|_1 + \| u - v' \|_1 \leq (c + c_D) [d v]_2 \leq C_1 [d v]_2$$  

where $C_1 = c + \max_{D \subset C} c_D$ and the maximum is taken over the finite set of proper faces of the positive cone $C$. \qed

Recall that $K$ denotes the kernel of the linear map $d$ regarded as a $\mathbb{Q}$-subspace of the $\mathbb{R}$-vector space $V_1$.

**Lemma 5.2.** There are constants $c_1, A > 0$ such that if $v \in C$ then there is a vector $\lambda \in \Lambda^+ \cap K$ satisfying $\| v - \lambda \|_1 \leq c_1 [d v]_2 + A$.

**Proof.** Let $v \in C$ be any vector. By Lemma 5.1 there exists a vector $v'' \in C \cap K$ such that $\| v - v'' \|_1 \leq c_1 [d v]_2$ for some constant $c_1 > 0$ independent of $v$.

Note that $C \cap K$ is a positive cone defined over $\mathbb{Q}$. Let $U = \text{Span}_{\mathbb{Q}}(C \cap K) \subseteq V_1$ so that $C \cap U$ is a closed cone with a non-empty interior in the vector subspace $U$. Denote $B_A = \{ w \in V_1 : \| w \|_1 \leq A \}$. Therefore $C \cap U \cap B_A$ contains in its interior a ball in $U$ of an arbitrary large radius, provided the radius $A > 0$ is sufficiently large. Since $\Lambda \cap U$ is a lattice in the $\mathbb{R}$-vector space $U$, the set $C \cap U \cap B_A$ surjects onto $U/(U \cap \Lambda)$ for all $A > 0$ sufficiently large. Fix any sufficiently large such $A > 0$.

Since $v'' \in C \cap K$ the translated set $v'' + C \cap U \cap B_A \subseteq C \cap K$ also surjects onto $U/(U \cap \Lambda)$. In particular this set admits a point $\lambda \in \Lambda \cap C \cap K = \Lambda^+ \cap K$. We conclude that $\| v - \lambda \|_1 \leq \| v - v'' \|_1 + \| v'' - \lambda \|_1 \leq c_1 [d v]_2 + A$ as required. \qed

**Lemma 5.3.** For any vector $\lambda \in \Lambda^+$ there is another vector $\lambda' \in \Lambda^+ \cap K$ satisfying $\| \lambda - \lambda' \|_1 < [d \lambda]_2$ and $\| \lambda' \|_1 \leq \| \lambda \|_1$.  

Proof. Let the vector \( \lambda \in \Lambda^* \) be fixed. If \( \lambda \in K = \ker d \) then there is nothing to prove, for we may simply take \( \lambda' = \lambda \in \Lambda^* \cap K \). Assume therefore that \( \lambda \notin K \).

Since the linear map \( d \) is defined over \( \mathbb{Q} \) there is a constant \( M > 0 \) such that
\[
\|d\lambda\|_2 \geq M \quad \text{for every vector } \lambda \in \Lambda \setminus K.
\]
Denote
\[
\theta = \frac{c_1 \|d\lambda\|_2 + A}{\|\lambda\|_1}
\]
where the constants \( c_1 \) and \( A \) are as in Lemma [5.2].

If \( \theta \geq 1 \) then we may take \( \lambda' = \lambda \). This vector \( \lambda' \) satisfies \( 0 = \|\lambda'\|_1 \leq \|\lambda\|_1 \) and
\[
\|\lambda - \lambda'\|_1 \leq \theta \|\lambda\|_1 = c_1 \|d\lambda\|_2 + A \leq (c_1 + \frac{A}{M}) \|d\lambda\|_2
\]
as desired (the constants \( c_1, A \) and \( M \) are all independent of \( \lambda \)).

Finally assume that \( 0 < \theta < 1 \). Apply Lemma [5.2] to the vector \( v = (1 - \theta)\lambda \). This gives a new vector \( \lambda' \in \Lambda^* \cap K \) with
\[
\|v - \lambda'\|_1 \leq c_1 \|dv\|_2 + A \leq c_1 \|d\lambda\|_2 + A.
\]
This verifies the second condition. As for the first condition, we have
\[
\|\lambda - \lambda'\|_1 \leq c_1 \|v\|_2 + c_1 \|v'\|_2 + A \leq 2c_1 \|d\lambda\|_2 + 2A \leq 2(c_1 + A/M) \|d\lambda\|_2.
\]
This concludes the proof, noting as above that the constants \( c_1, A \) and \( M \) are all independent of the vector \( \lambda \).

\[ \square \]

6. From linear algebra back to actions

We show that any \( \delta \)-almost \( \pi_1(G,T) \)-action \( \rho \) whose isomorphism type \( \rho^\dagger \) is compatible with some \( \pi_1(G,T) \)-action, can be corrected to such an action. More precisely, we establish the following result, using without further mention all the notations introduced in [2] [3] and [4].

**Proposition 6.1.** Let \( \rho : \pi_1(G,T) \to \Sym(X) \) be a \( \delta \)-almost \( \pi_1(G,T) \)-action with \( \lambda = \rho^\dagger \). Let \( \lambda' \in \Lambda^*_V \) be any vector with \( \|\lambda'\|_V = \|\lambda\|_V \). If

(a) \( \lambda' \in \ker d_\mathfrak{g} \) and
(b) \( \|\lambda - \lambda'\|_V \leq \delta \|\lambda\|_V \)

then there is a group action \( \rho' : \pi_1(G,T) \to \Sym(X) \) satisfying

(i) \( \lambda' = (\rho')^\dagger \) and
(ii) \( d_{X,S_0}(\rho_i, \rho'_i) < G \delta \).

We precede the proof of Proposition 6.1 with an analogous statement in the simpler context of a single group homomorphism.

**Lemma 6.2.** Let \( i : H \to G \) be a homomorphism of finite groups. Let \( \phi : H \to \Sym(X) \) and \( \rho : G \to \Sym(X) \) be a pair of group actions. Denote \( \lambda = \rho^\dagger \). If \( \lambda' \in \Lambda^*_G \) and \( \delta > 0 \) are such that

(a) \( d_{X,H}(\rho \circ i, \phi) \leq \delta \),
(b) \( i(\lambda') = \phi^\dagger \), and
(c) \( \|\lambda - \lambda'\|_G \leq \delta \|\lambda\|_G \)

then there exists a group action \( \rho' : G \to \Sym(X) \) satisfying

(i) \( \rho' \circ i = \phi \),
(ii) \( \lambda' = (\rho')^I \), and
(iii) \( d_{X,G}(\rho,\rho') < \delta_o \).

Note that the “small” action \( \phi \) of the group \( H \) is not being changed, rather the “large” action \( \rho \) is being replaced with a new action \( \rho' \) compatible with \( \phi \).

**Proof of Lemma 6.2.** We combine Assumption (a) and Observation 3.4 applied with respect to the finite group \( H \) in order to find a \( \phi(H) \)-invariant subset \( X_0 \subset X \) satisfying \( \phi \upharpoonright X_0 = (\rho \circ i) \upharpoonright X_0 \) and \( |X_0| \geq (1 - \delta[H]|G|)|X| \). By applying Observation 3.3 a second time with respect to the finite group \( G \), we find a \( \rho(G) \)-invariant subset \( X_1 \subset X_0 \) satisfying \( |X_1| \geq (1 - \delta[H]|G|)|X| \).

Consider the vector \( \lambda_1 = (\rho \upharpoonright X_1)^I \in \Lambda^+_G \). Let \( \mu_1 \in \Lambda^+_G \) be the component-wise minimum of the two vectors \( \lambda' \) and \( \lambda_1 \), i.e \( \mu_1 \) is the vector given by

\[
(\mu_1)_\chi = \min\{(\lambda')_\chi, (\lambda_1)_\chi\} \quad \forall \chi \in \text{Trans}(G).
\]

The previous paragraph implies that \( \|\lambda - \lambda_1\|_G \leq \delta[H]|G|X| \). Therefore Assumption (c) gives

\[
\max\{\|\lambda' - \mu_1\|_G, \|\lambda_1 - \mu_1\|_G\} \leq \|\lambda' - \lambda_1\|_G \leq \|\lambda' - \lambda\|_G + \|\lambda - \lambda_1\|_G \\
\leq \delta[X] + \delta[H]|G|X| = \delta(1 + |H||G|)|X|.
\]

Let \( Y_1 \subset X_1 \) be any \( \rho(G) \)-invariant subset satisfying \( \mu_1 = (\rho \upharpoonright Y_1)^I \). Write \( \mu_2 = \lambda' - \mu_1 \in \Lambda^+_G \) and \( Y_2 = X \setminus Y_1 \) so that \( \lambda' = \mu_1 + \mu_2 \) and \( X = Y_1 \bigsqcup Y_2 \). It will not be the case in general that \( (\rho \upharpoonright Y_2)^I \) coincides with \( \mu_2 \). However \( |Y_2| = \|\mu_2\|_G \) and in particular the size of the subset \( Y_2 \) is bounded by

\[
|Y_2| = \|\mu_2\|_G \leq \delta(1 + |H||G|)|X|.
\]

We define a new action \( \rho' : G \to \text{Sym}(X) \) as follows. To begin with, the restriction of \( \rho' \) to the \( \rho(G) \)-invariant subset \( Y_1 \) coincides with \( \rho \), namely \( \rho' \upharpoonright Y_1 = \rho \upharpoonright Y_1 \). As \( i^*(\lambda') = \phi^I \) by Assumption (b) and as \( i^*(\mu_1) = (\phi \upharpoonright Y_1)^I \) by the choice of the subset \( Y_1 \) we have \( i^*(\mu_2) = (\phi \upharpoonright Y_2)^I \). It remains to define the action \( \rho' \) on the \( \rho(G) \)-invariant complement \( Y_2 = X \setminus Y_1 \). Taking into account Observation 4.1 we let \( \rho' \upharpoonright Y_2 \) be an arbitrary action satisfying \( (\rho' \circ i) \upharpoonright Y_2 = \phi \upharpoonright Y_2 \) and \( (\rho' \upharpoonright Y_2)^I = \mu_2 \). This completes the definition of the new action \( \rho' \).

Statements (i) and (ii) of the Lemma hold true since \( \rho' \circ i = \phi \) and

\[
\rho'^I = (\rho' \upharpoonright Y_1)^I + (\rho' \upharpoonright Y_2)^I = \mu_1 + \mu_2 = \lambda'.
\]

To verify Statement (iii) we compute

\[
d_X(\rho(g),\rho'(g)) = \frac{|Y_1|}{|X|} d_{Y_1}(\rho(g) \upharpoonright Y_1, \rho'(g) \upharpoonright Y_1) + \frac{|Y_2|}{|X|} d_{Y_2}(\rho(g) \upharpoonright Y_2, \rho'(g) \upharpoonright Y_2)
\leq \frac{|Y_2|}{|X|} \leq (1 + |H||G|)\delta \leq 2|H||G|\delta
\]

for all elements \( g \in G \). Therefore \( d_{X,G}(\rho,\rho') \leq 2|H||G|\delta \) as required. \( \square \)

We are now in a position to prove the main result of Section 6.

**Proof of Proposition 6.7.** We define the new action \( \rho' : \pi_1(G,T) \to \text{Sym}(X) \) of the fundamental group \( \pi_1(G,T) \) by specifying it on the finite generating set \( S_G \). This is done in two steps: first we define \( \rho' \) on the vertex groups \( G_v \) and then on the generators of the form \( s_v \).
Step 1. Defining \( \rho' \) on \( G_v \) for all \( v \in V(\Gamma) \). Fix an arbitrary base vertex \( v_0 \) in \( V(\Gamma) \). We define the vertex group actions \( \rho'_v \) by induction on the distance in the spanning tree \( T \) of the vertex \( v \) from the base vertex \( v_0 \) such that:

1. \( \rho'_v \) is trivial and the vector \( \lambda'_v \in \Lambda^v_0 \) is the coordinate \( \lambda'_v \) of the vertex group \( \rho'_v \).
2. \( \rho'_v \) is the edge group \( \rho'_e \) of the edge \( e \) such that \( t(e) = v \) and \( o(e) \) implies \( \rho'_v \).
3. \( \rho'_v \) is the action \( \phi \) of the group \( G_v \) on \( G_v \) such that \( \lambda'_v \) is the coordinate \( \lambda'_v \) of the vertex group \( \rho'_v \).

For all \( v_0 \) in the tree \( T \) and \( e \in E(T) \) be the unique edge such that \( t(e) = v \) and \( o(e) \) is distance \( n-1 \) from the base vertex \( v_0 \). Denote \( u = o(e) \). By the induction hypothesis the vertex group action \( \rho' \) has been defined and satisfies \( \rho'_v \).

We apply Lemma 0.1 with the following data: the group \( G \) is the vertex group \( G_v \), the group \( H \) and the homomorphism \( i : H \to G \) are trivial and the vector \( \lambda'_v \in \Lambda^v_0 \) is the coordinate \( \lambda'_v \). This results in a new action \( \rho'_v \) of the vertex group \( G_v \) satisfying \( d_X(G_v) < \delta \) and \( \rho'_v \).

Induction step. Let \( v \in V(\Gamma) \) be a vertex of distance \( n \in \mathbb{N} \) from the base vertex \( v_0 \) in the tree \( T \) and \( e \in E(T) \) be the unique edge such that \( t(e) = v \) and \( o(e) \) is distance \( n-1 \) from the base vertex \( v_0 \). Denote \( u = o(e) \). By the induction hypothesis the vertex group action \( \rho'_v \) has been defined and satisfies \( \rho'_v \).

We apply Lemma 0.1 with the following data: the group \( G \) is the vertex group \( G_v \), the group \( H \) and the homomorphism \( i : H \to G \) are trivial and the vector \( \lambda'_v \in \Lambda^v_0 \) is the coordinate \( \lambda'_v \). This results in a new action \( \rho'_v \) of the vertex group \( G_v \) satisfying \( d_X(G_v) < \delta \) and \( \rho'_v \).

We proceed to verify the assumptions of Lemma 0.1. The induction hypothesis combined with the assumption that \( \lambda'_v \) implies

\[
i^*(\lambda'_v) = i^*_e(\lambda'_e) = i^*_e(\lambda'_e) = i^*_e((\rho'_v)^t) = (\rho'_{i_e})^t = \phi^t.
\]

By the triangle inequality the two actions \( \rho \circ i \) and \( \phi \) of the edge group \( G_e \) satisfy

\[
d_X(G_e)(\rho \circ i, \phi) = d_X(G_e)(\rho \circ i, \rho' \circ i) \\
\leq d_X(G_e)(\rho \circ i, \rho(s_e)) + d_X(G_e)(\rho(s_e) \circ (\rho \circ i), (\rho \circ i)) \\
+ d_X(G_e)(\rho(s_e) \circ (\rho \circ i), (\rho(s_e) \circ (\rho(s_e)^{-1}) \\
+ d_X(G_e)(\rho \circ i, \rho \circ i) + d_X(G_e)(\rho \circ i, \rho' \circ i).
\]

The normalized Hamming metric \( d_X \) is bi-invariant so that the first and second summands are both less than \( d_X(G_v)(\rho(s_e), \text{id}) < \delta \). The third summand is also less than \( \delta \) as \( \rho(s) \) is a \( \delta \)-almost \( \pi_1(G,T) \)-action and taking into account the corresponding relation in \( R_G \). Lastly, the fourth summand satisfies \( d_X(G_v)(\rho \circ i, \rho' \circ i) < \delta \) by the induction hypothesis. We conclude that

\[
d_X(G_v)(\rho \circ i, \phi) < \delta.
\]

Having verified all of the assumptions for Lemma 0.1 we get a new action \( \rho'_{G_v} \) of the vertex group \( G_v \) such that \( \rho' \circ i_e = \rho' \circ i_e \) on the edge group \( G_e \), \( d_X(G_e)(\rho'_{G_e}, \rho'_{G_e}) < \delta \) and \( (\rho'_e)^t = \lambda'_e \). This completes the step of the induction.

Proceed with the induction until the new action \( \rho' \) is defined on all vertex groups.

Step 2. Defining \( \rho' \) on the generators \( s_e \) for all \( e \in \tilde{E}(\Gamma) \). Let \( e \in \tilde{E}(\Gamma) \) be a directed edge with \( o(e) = u \) and \( t(e) = v \).
Assume that $e \in E(T)$. Define $\rho'(s_e) = \text{id}$. Recall that the action $\rho'$ of the edge group $G_e$ satisfies $\rho' \circ i_e = \rho' \circ i_e$ by Step 1. Therefore

$$\rho'((i_e(g_e))s_e)(x) = \rho'(s_ei_e(g_e))(x)$$

for all points $x \in X$ and all elements $g_e \in G_e$. Moreover, since $\rho$ is a $\delta$-almost $\pi_1(G, T)$-action we have $d_X(\rho(s_e), \rho'(s_e)) \leq \delta$.

Assume that $e \in E(\Gamma) \setminus E(T)$. According to Observation 3.3 there exists a $\rho'|_{i_\cdot(G_e)}$-invariant subset $X_e \subseteq X$, such that $|X - X_e| < \delta |X|$ and the following conditions are satisfied for all points $x \in X_e$ and all elements $g_e \in G_e$

$$\rho(i_e(g_e)s_e)(x) = \rho(s_ei_e(g_e))(x),$$
$$\rho(i_e(g_e))(x) = \rho'(i_e(g_e))(x),$$
$$\rho(i_e(g_e))(\rho(s_e)x) = \rho'(i_e(g_e))(\rho(s_e)x).$$

Define the restriction $\rho'(s_e) \upharpoonright X_e$ of the new action to be the same as $\rho(s_e) \upharpoonright X_e$. The above conditions imply that the permutation $\rho'(s_e)$ satisfies

$$\rho'((i_e(g_e))s_e)(x) = \rho'(s_ei_e(g_e))(x)$$

for all points $x \in X_e$ and all edge group elements $g_e \in G_e$.

It remains to define the permutation $\rho'(s_e)$ on the complement $X - X_e$ and verify the above relation for all points $x \in X - X_e$. The two actions $\rho' \circ i_e$ and $\rho' \circ i_e$ of the edge group $G$ are conjugate as $X' \in \ker d_{\mathcal{G}}$ and $\rho' = \lambda'$. Since the permutation $\rho(s_e)$ conjugates $(\rho' \circ i_e) \upharpoonright X_e$ to $(\rho' \circ i_e) \upharpoonright \rho(s_e) \upharpoonright X_e$, we know that their complements must be conjugate as well. Define the restriction $\rho'(s_e) \upharpoonright X - X_e$ to be an arbitrary bijection from $X \setminus X_e$ to $X \setminus \rho(s_e)X_e$ implementing this isomorphism of actions. Note that $d_X(\rho(s_e), \rho'(s_e)) < \delta$. This concludes the definition of the permutation $\rho'(s_e)$ for this particular oriented edge $e$.

A bound on $d_{X, S_{\mathcal{G}}}(\rho, \rho')$. The $\pi_1(G, T)$-action $\rho'$ has been constructed in Steps 1 and 2. It was specified in terms of the finite generating set $S_{\mathcal{G}}$ while making sure that the defining relations $R_{\mathcal{G}}$ of the fundamental group $\pi_1(G, T)$ hold true. It follows from the construction that $\rho' = \lambda'$. To conclude the proof it remains to bound the normalized Hamming distance $d_{X, S_{\mathcal{G}}}(\rho, \rho')$. Namely

$$d_{X, S_{\mathcal{G}}}(\rho, \rho') = \sum_{\sigma \in S_{\mathcal{G}}} d_X(\rho(\sigma), \rho'(\sigma))$$

$$\leq \sum_{v \in V(G) \setminus G_e} d_X(\rho(g), \rho'(g))$$

$$+ \sum_{e \in E(T)} d_X(\rho(s_e), \rho'(s_e))$$

$$+ \sum_{e \in E(\Gamma) \setminus E(T)} d_X(\rho(s_e), \rho'(s_e))$$

$$< \delta$$

as required. \hfill \Box

7. Proof of the main theorem

We are ready to show that the epimorphism $\pi_1(G, T) \to \pi_1(G, T)$ is P-stable.
Proof of Theorem 1.2. Let X be a finite set admitting a $\delta$-almost $\pi_1(G,T)$-action $\rho: \pi_1(G,T) \to \Sym(X)$. Denote $\lambda = \rho^d$. We know by Proposition 4.3 that

(7.1) $\|d_G(\lambda)\|_E \approx_G \delta \|\lambda\|_V$

Lemma 5.3 allows us to find a vector $\lambda'' \in \Lambda_{\rho}^+ \cap \ker d_G$ such that

$$|\lambda'' - \lambda|_V < \delta |\lambda|_V \quad \text{and} \quad |\lambda''|_V \leq |\lambda|_V.$$

We will make an auxiliary use of the action of the fundamental group $\pi_1(G,T)$ on a singleton. Denote this action by $s$. By Proposition 4.2 we know that $s^d \in \ker d_G$. Moreover $\|s^d\|_V = 1$. Let

$$\lambda' = \lambda' + (\|\lambda\|_V - \|\lambda''\|_V)s^d.$$

It is clear that $\lambda' \in \ker d_G$, $\|\lambda'|_V = \|\lambda\|_V = \|\rho^d\|_V = |X|$ and

$$|\lambda' - \lambda|_V \leq |\lambda - \lambda''|_V + |\lambda' - \lambda''|_V < \delta |X|.$$

To conclude the proof we apply Proposition 6.1 and obtain the desired action $\rho': \pi_1(G,T) \to \Sym(X)$ satisfying $(\rho')^d = \lambda'$ and $d_X.s_\delta(\rho,\rho') < \delta$. \hfill $\Box$

Remark 7.1. It follows from the proof that one can take $\delta < \epsilon$ for the P-stability of $\pi_1(G,T)$.

The derivation of Theorem A from the above Theorem 1.2 is immediate and has been discussed in §3.

8. Graph automorphisms of finite order

Fix some $d \in \mathbb{N}$ and let $F_d = F(s_1, \ldots, s_d)$ be the free group of rank $d$.

A finite Schreier graph $A$ of the group $F_d$ is a finite directed graph edge-labelled by the generators $s_1, \ldots, s_d$ such that every vertex has exactly one incoming and one outgoing edge of each label. We indicate the labelling of the directed edges $E(A)$ using a function $c = c_A: E(A) \to \{s_1, \ldots, s_d\}$.

A weak $\delta$-almost-automorphism $\alpha$ of the Schreier graph $A$ is a pair of bijections $\alpha: V(A) \to V(A)$ and $\alpha: E(A) \to E(A)$ (we use the same letter for both by abuse of notation) such that for all directed edges $e \in \tilde{E}(A)$ except for a subset of size $\delta|E(A)|$ we have

$$c(\alpha(e)) = c(e), \quad o(\alpha(e)) = o(e) \quad \text{and} \quad t(\alpha(e)) = t(e).$$

A $\delta$-almost-automorphism $\alpha$ is a weak $\delta$-almost-automorphism that moreover satisfies the first two conditions, namely $c(\alpha(e)) = c(e)$ and $o(\alpha(e)) = o(e)$, for all directed edges $e \in E(A)$.

Observation 8.1. Let $\alpha$ be a weak $\delta$-almost-automorphism of the finite Schreier graph $A$. Up to changing $\alpha$ on at most $O(\delta|E(A)|)$ edges we can make $\alpha$ into a $\delta$-almost-automorphism.

Fix some integer $n \in \mathbb{N}$.

Definition 8.2. A (weak) $\delta$-almost-automorphism $\alpha$ has $\delta$-almost order $n$ if the condition $\alpha^n(v) = v$ holds true for all $v \in V(A)$ except for a subset of size $\delta|V(A)|$.
Given an action $\rho : F_d \ast \langle a \rangle \to \text{Sym}(X)$ on some finite set $X$ we denote by $A_\rho$ the Schreier graph of the restricted action $\rho : F_d \to \text{Sym}(X)$. Let $\alpha_\rho$ denote the bijection on the vertices of the Schreier graph $A_\rho$ defined by $\alpha_\rho = \rho(a)$. Moreover, by abuse of notation, let $\alpha_\rho$ denote the bijection of the directed edges of $A_\rho$ defined for every $e \in E(A)$ by $\alpha_\rho(e) = e'$ where $e'$ is the unique edge satisfying $c(e) = c(e')$ and $o(e') = o(\alpha_\rho(e))$.

**Observation 8.3.** If $\rho : F_d \ast \langle a \rangle \to \text{Sym}(X)$ is a $\delta$-almost $(F_d \times \mathbb{Z})$-action (resp. $\delta$-almost-order $n$)-action on some finite set $X$ then $\alpha_\rho$ is a $\delta$-almost-automorphism of the Schreier graph $A_\rho$ (resp. of $\delta$-almost-order $n$).

Vice versa, if $A$ is a finite Schreier graph of the group $F_d$ and $\alpha$ is $\delta$-almost-automorphism (resp. of $\delta$-almost-order $n$) of the free group $F_d$ then there exists a $\delta$-almost $(F_d \times \mathbb{Z})$-action (resp. $\delta$-almost $(F_d \times (\mathbb{Z}/n\mathbb{Z}))$-action) $\rho : F_d \ast \langle a \rangle \to \text{Sym}(X)$ such that $A = A_\rho$ and $\alpha = \alpha_\rho$.

Theorem [A] applied to the virtually free group $F_d \times (\mathbb{Z}/n\mathbb{Z})$ and combined with the above observations immediately gives the following corollary.

**Corollary 8.4.** Let $A$ be a finite Schreier graph of the free group $F_d$ and let $\alpha$ be a weak $\delta$-almost automorphism of $\delta$-almost order $n$. Then there exists a Schreier graph $A'$ of the group $F_d$ with $V(A) = V(A')$, and an automorphism $\alpha'$ of $A'$ of order $n$ such that the graphs $A$ and $A'$ differ on at most $O(\delta|E|)$ edges, and the automorphisms $\alpha$ and $\alpha'$ differ on at most $O(\delta|V|)$ vertices. □

Note that Corollary 8.4 is false without requiring that $\alpha$ has $\delta$-almost order $n$ since $F_d \times \mathbb{Z}$ is non P-stable by [2].

We end this paper with the following related question.

**Question 8.5.** Is the conclusion of Corollary 8.4 true in the setting of general $d$-regular graphs and graph automorphisms (rather than Schreier graphs of $F_d$)?

**References**

[1] Lev Glebsky and Luis Manuel Rivera. Almost solutions of equations in permutations. *Taiwanese Journal of Mathematics*, 13(2A):493–500, 2009.

[2] Adrian Ioana. Stability for product groups and property $(\tau)$. *Journal of Functional Analysis*, 279(9):108729, 2020.

[3] Nir Lazarovich, Arie Levit, and Yair Minsky. Surface groups are flexibly stable. *arXiv preprint arXiv:1901.07182*, 2019.

[4] Jean-Pierre Serre. *Arbres, Amalgames, SL$_2$*. 1977.

[5] John Stallings. On torsion-free groups with infinitely many ends. *Annals of Mathematics*, pages 312–334, 1968.

[6] John Stallings. *Group theory and three-dimensional manifolds*. Yale University Press, 1972.