Dynamics of the Cosmological Constant
in Two-Dimensional Universe

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ABSTRACT

We consider a two-dimensional model of gravity with the cosmological constant as a dynamical variable. The effective cosmological constant is derived when the universe has no initial boundary. It turns out to be extremely small if the universe is sufficiently large.
1. Introduction

It is a profound mystery about the Universe that the observational bounds for
the cosmological constant are incredibly small. This has motivated various ideas\cite{1} on the subject of gravitation and cosmology.

Recent interest in two-dimensional gravity might be largely rooted in stringy
approach to unified theory. However, we hope that two-dimensional theories of
gravity may also serve as toy models for investigating qualitative features of realistic
gravity in four dimensions.

In this paper, we consider a simple model of two-dimensional gravity where
the cosmological constant appears as a constant of integration. Namely, the cos-
mological constant is determined by an initial condition for a dynamical variable,
whose expectation value will be computed when the state of the universe is of
the Hartle-Hawking type.\cite{2} The effective cosmological constant turns out to be
extremely small if the universe is sufficiently large.

2. The Model

Let us consider the following model Lagrangian in two dimensions:

\[ \mathcal{L} = \mathcal{L}_c(g_{\mu \nu}) + \frac{1}{2\pi} (\rho \sqrt{\gamma} + \lambda \sqrt{\gamma} + \lambda \epsilon^{\mu \nu} \partial_{\mu} A_{\nu}), \]  

(1)

where \( \mathcal{L}_c(g_{\mu \nu}) \) denotes the effective Lagrangian for conformal matter with central
charge \( c \) coupled to gravity,\cite{3,4} \( \rho \) is a renormalized cosmological constant, \( \sqrt{\gamma} \) repre-
sents the invariant volume density,\cite{5} in terms of the metric tensor \( g_{\mu \nu} \), \( \lambda \) is a scalar
field which contributes to the effective cosmological constant, \( A_\mu \) is an abelian
 gauge field, and \( \epsilon^{\mu \nu} \) denotes the Levi-Civita tensor.

The model (1) may be regarded as a two-dimensional analogue of the covariant
form of unimodular gravity in four dimensions given by Henneaux and Teitelboim.\cite{6}
Classically its physical contents are almost the same as those of the conventional
gravity. A major difference results from equations of motion \( \partial_\mu \lambda = 0 \), which indi-
cate that the effective cosmological constant appears as a constant of integration.
Thus it is determined by an initial condition for the universe.
In the following sections, we will calculate the expectation value of the observable $\lambda$ in the case of no-boundary universe, which involves an initial condition we need. That is, we adopt as our universe a hemisphere with $w$ wormholes attached. Then the desired expectation value will be obtained as a one-point function $\langle \lambda \rangle$ on a closed Riemann surface $M$ with $h = 2w$ handles.

3. Partition Function

In this section, we estimate the partition function for the model $\lambda$ exposed in the previous section:

$$Z = \int Dg D\lambda D\phi e^{-S},$$  \hspace{1cm} (2)

where

$$S = \int_M d^2x \mathcal{L}.$$  \hspace{1cm} (3)

We note that integration over the multiplier field $\lambda$ should be performed along the direction of the imaginary axis $\text{[7]}$ so as to put the theory $\lambda$ properly in the Euclidean path integral (2).

Let us first define the zero mode $\lambda_0$ of the field $\lambda$ as follows:

$$\lambda = \lambda_0 + \lambda_1, \quad \partial_\mu \lambda_0 = 0,$$  \hspace{1cm} (4)

where $\lambda_1$ satisfies a condition

$$\partial_\mu \lambda_1 = 0 \iff \lambda_1 = 0.$$  \hspace{1cm} (5)

Then the action (3) is written as

$$S = S_c + \lambda_0 \left( \frac{1}{2\pi} \int_M d^2x \sqrt{-T} \right) + \frac{1}{2\pi} \int_M d^2x (\lambda_1 \sqrt{+} + \lambda_1 \epsilon_{\mu\nu} \partial_\mu A_\nu).$$  \hspace{1cm} (6)
Here we have introduced
\[ S_c = \int_M d^2x (L_c + \frac{1}{2\pi} \rho \sqrt{\cdot}), \quad T = -\frac{1}{2\pi} \int_M d^2x \epsilon^{\mu\nu} \partial_\mu A_\nu, \] (7)
where \( T \) comes out to be a number which is independent of fluctuation in the field \( A_\mu \).

The form (6) of the action \( S \) allows us to perform successive path integration in (2) over the fields \( A_\mu \) and \( \lambda_1 \) to obtain
\[ Z = \int \mathcal{D}g \mathcal{D}\lambda_0 e^{-S'}, \] (8)
where
\[ S' = S_c + \lambda_0 \left( \frac{1}{2\pi} \int_M d^2x \sqrt{-T} \right). \] (9)

Further integration over the variable \( \lambda_0 \) results in the expression
\[ Z = \int \mathcal{D}g e^{-S_c} \left( \frac{1}{2\pi} \int_M d^2x \sqrt{-T} \right), \] (10)
which implies that \( T \) characterizes the size of the universe.

This expression of the partition function \( Z \) makes its \( T \) dependence apparent through scaling behavior:
\[ Z \sim T^X e^{-\rho T}, \] (11)
where we have defined a constant
\[ X = \frac{1}{12} (h - 1) (25 - c + \sqrt{(25 - c)(1 - c)}) - 1. \] (12)

Note that the form (11) is universal in the sense that it appears independent of the detailed content of matter-gravity action \( S_c \) when the volume \( T \) is large.
4. Cosmological Constant

Now we proceed to compute the desired one-point function \( \langle \lambda \rangle \). With the aid of the equations (8) and (9), we see

\[
Z^{-1} \frac{\partial}{\partial T} Z = \langle \lambda_0 \rangle = \langle \lambda \rangle, \tag{13}
\]

where the last equality follows from the definition (4)-(5). Thus, by means of (11), we obtain

\[
\langle \lambda \rangle = \frac{X}{T} - \rho. \tag{14}
\]

Quantum fluctuation \( \tilde{\lambda} \) is defined by

\[
\lambda = \langle \lambda \rangle + \tilde{\lambda}, \tag{15}
\]

which satisfies \( \langle \tilde{\lambda} \rangle = 0 \). Substituting the above expressions into the Lagrangian (1), we immediately get

\[
\mathcal{L} = \mathcal{L}_c + \frac{1}{2\pi} (\Lambda \sqrt{+} \tilde{\lambda} \sqrt{+} \tilde{\lambda} \epsilon^{\mu\nu} \partial_\mu A_\nu + \langle \lambda \rangle \epsilon^{\mu\nu} \partial_\mu A_\nu), \tag{16}
\]

where we have written

\[
\Lambda = \frac{X}{T}. \tag{17}
\]

As a conceivable interpretation, these results imply that the effective cosmological constant, which directly affects the motion of the metric \( g_{\mu\nu} \), is given by \( \Lambda \) with the fluctuation \( \tilde{\lambda} \) contributing to it no more. In view of (17), we conclude that the effective cosmological constant is expected to be extremely small when the universe is sufficiently large.

5. Discussion
We have computed the effective cosmological constant (17) in the theory (1) of two-dimensional gravity when the state of the universe is of the Hartle-Hawking type. Two remarks are in order:

(i) Although the value $\Lambda$ is tiny for large $T$, it turned out to be non-zero. Observational cosmology suggests that this feature might be realized in the Universe.

(ii) The effective cosmological constant $\Lambda$ is not necessarily small when the size of the universe $T$ is not so large. This might be adequate for inflationary scenarios which need dominance of the cosmological-constant effect in an early epoch of the Universe.

It seems interesting to ask whether these features will be attained in realistic four-dimensional quantum gravity yet to come.

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