Plane Symmetric Solutions of Gravitational Field Equations in Five Dimensions

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Abstract

We present the effective field equations obtained from a generalized gravity action with Euler-Poincaré term and a cosmological constant in a \(D\) dimensional bulk space-time. A class of plane-symmetric solutions that describe a 3-brane world embedded in a \(D = 5\) dimensional bulk space-time are given.

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1 Introduction

Brane-world theories that receive a lot of interest recently are strictly motivated by string models [1]. They were mainly proposed to provide new solutions to the hierarchy problem and compactification of extra dimensions [2],[3]. The main content of the brane-world idea is that we live in a four dimensional world embedded in a higher dimensional bulk space-time. According to the brane-world scenarios, the gauge fields, fermions and scalar fields of the Standard Model should be localised on a 3-brane, while gravity may freely propagate into the higher dimensional bulk.

In our previous work [4] we derived covariant gravitational field equations on a 3-brane embedded in a five-dimensional bulk space-time with $\mathbb{Z}_2$ symmetry in a generalization that included a dilaton scalar as well as the second order Euler-Poincaré density in the action. We introduced a general ADM-type coordinate setting to show that the effective gravitational field equations on the 3-brane remain unchanged, however, the evolution equations off the brane are significantly modified due to the acceleration of normals to the brane surface in the non-geodesic, ADM slicing of space-time.

In the second part of this paper, using the language of differential forms, we present the field equations of a generalized gravity model with a dilaton 0-form and an axion 3-form in Einstein frame from an action that includes the second order Euler-Poincaré term and a cosmological constant in a D-dimensional bulk space-time. In the third part, we present some plane-symmetric solutions that generalize the well-known domain-wall solution [5].

2 Model

We consider a $D$-dimensional bulk space-time manifold $M$ equipped with a metric $g$ and a torsion-free, metric compatible connection $\nabla$. We determine our gravitational field equations by a variational principle from a $D$-dimensional action that includes the second order Euler-Poincaré term and a cosmological constant

$$I[e, \omega, \phi, H] = \int_M \mathcal{L}$$ (2.1)
where in the Einstein frame the Lagrangian density D-form [6]

\[
\mathcal{L} = \frac{1}{2} R^{ab} \wedge (e_a \wedge e_b) - \frac{\alpha}{2} d\phi \wedge *d\phi + \frac{\beta}{2} e^{-\beta_2 \phi} H \wedge *H + \Lambda e^{-\beta_1 \phi} * 1 \\
+ \eta R^{ab} \wedge R^{cd} \wedge * (e_a \wedge e_b \wedge e_c \wedge e_d) \\
+ (de^a + \omega^a_{\ b} \wedge e^b) \wedge \lambda_a + (dH - \frac{\varepsilon}{2} R_{ab} \wedge R^{ab}) \wedge \mu .
\]

(2.2)

Here \( \lambda_a \) and \( \mu \) are Lagrange multiplier forms that upon variation impose the zero-torsion and anomaly-freedom constraints.

The final form of the variational field equations to be solved are the Einstein field equations

\[
\frac{1}{2} R^{ab} \wedge * (e_a \wedge e_b \wedge e_c) = -\frac{\alpha}{2} \tau_c[\phi] + \frac{\beta}{2} e^{-\beta_2 \phi} \tau_c[H] - \Lambda e^{-\beta_1 \phi} e^c \\
- \frac{\eta}{4} R^{ab} \wedge R^{dg} \wedge * (e_a \wedge e_b \wedge e_d \wedge e_g \wedge e_c) \\
- 2\varepsilon \beta D(e^{-\beta_2 \phi} \tau_b(R^b_\ c \wedge *H)) - \frac{\varepsilon}{2} \beta \ e^c \wedge D(e^{-\beta_2 \phi} \tau_s \tau_l(R^s \wedge *H)),
\]

(2.3)

where the dilaton stress-energy forms

\[
\tau_a[\phi] = \iota_a d\phi \wedge *d\phi + \iota_a \wedge d\phi
\]

and the axion stress-energy forms

\[
\tau_a[H] = \iota_a H \wedge *H + H \wedge \iota_a \wedge H,
\]

the dilaton scalar field equation

\[
ad(*d\phi) = \frac{\beta_2 \beta}{2} e^{-\beta_2 \phi} H \wedge *H + \Lambda \beta_1 e^{-\beta_1 \phi} \wedge 1,
\]

(2.4)

and the axion field equations

\[
dH = \frac{\varepsilon}{2} R_{ab} \wedge R^{ab}, \quad d(e^{-\beta_2 \phi} \wedge H) = 0.
\]

(2.5)

3 Plane symmetric solutions in \( D = 5 \)

We investigate below a class of plane symmetric solutions in 5-dimensions. We consider the metric

\[
g = -f^2(t, \omega) dt^2 + u^2(t, \omega) d\omega^2 + g^2(t, \omega) \left( \frac{dx^2 + dy^2 + dz^2}{(1 + \frac{k r^2}{4})^2} \right),
\]

(3.1)
the dilaton scalar field
\[ \phi = \phi(t, \omega) \] (3.2)

and 3-form gauge field
\[ H = h(t, \omega) \frac{dx \wedge dy \wedge dz}{(1 + \frac{k r^2}{4})^3} \] (3.3)
in terms of local coordinates
\[ x^M : \{ x^0 = t, x^5 = \omega, x^1 = x, x^2 = y, x^3 = z \}. \]

We choose our co-frame 1-forms as
\[ e^0 = f(t, \omega) dt, \quad e^5 = u(t, \omega) d\omega, \quad e^i = g(t, \omega) \frac{dx^i}{(1 + \frac{k r^2}{4})}, \quad i = 1, 2, 3. \] (3.4)

Then we calculate the Levi-Civita connection 1-forms
\[ \omega^0_i = \frac{gt}{fg} e^i, \quad \omega^i_j = \frac{k}{2g} (x^i e^j - x^j e^i), \] (3.5)
\[ \omega^0_5 = \frac{ut}{fu} e^5 + \frac{fu}{fu} e^0, \quad \omega^i_5 = \frac{g\omega}{ug} e^i. \] (3.6)

and the corresponding curvature 2-forms
\[ R^{ij} = \frac{1}{g^2} \left\{ k + \left( \frac{gt}{f} \right)^2 - \left( \frac{g\omega}{u} \right)^2 \right\} e^i \wedge e^j, \] (3.7)
\[ R^{05} = \frac{1}{fu} \left\{ \left( \frac{fu}{f} \right)_t - \left( \frac{u}{f} \right)_t \right\} e^5 \wedge e^0, \] (3.8)
\[ R^{0t} = \frac{1}{fg} \left\{ \left( \frac{gt}{f} \right)_t - \frac{f\omega g\omega}{u^2} \right\} e^0 \wedge e^t + \frac{1}{ug} \left\{ \left( \frac{gt}{f} \right)_\omega - \frac{u_t g\omega}{fu} \right\} e^5 \wedge e^t, \] (3.9)
\[ R^{5i} = \frac{1}{fg} \left\{ \frac{f\omega g\omega}{fu} - \left( \frac{g\omega}{u} \right)_t \right\} e^i \wedge e^0 + \frac{1}{ug} \left\{ \left( \frac{g\omega}{u} \right)_\omega - \frac{g\omega u_t}{f^2} \right\} e^5 \wedge e^i. \] (3.10)

From these expressions we note that \( R_{ab} \wedge R^{ab} = 0 \). Therefore \( dH = 0 \) implying that
\[ H = \frac{Q}{g^3} e^1 \wedge e^2 \wedge e^3. \] (3.11)
where $Q$ may be identified as a magnetic charge. Now, for simplicity, we let $k = 0$ and take the functions $g$, $f$ and $u$ independent of time. Then we obtain the following system of coupled ordinary differential equations ($'$ denotes derivative with respect to $\omega$):

\[
2G - 2C - B - A = -\eta(2CG - AB) - \frac{\alpha}{2} \left( \frac{\phi'}{u} \right)^2 - \frac{\beta}{2} \frac{Q^2}{g^6} e^{-\beta_1 \phi} + \Lambda e^{-\beta_2 \phi},
\]

(3.12)

\[
3A - 3G = 3\eta GA + \frac{\alpha}{2} \left( \frac{\phi'}{u} \right)^2 - \frac{\beta}{2} \frac{Q^2}{g^6} e^{-\beta_1 \phi} - \Lambda e^{-\beta_1 \phi},
\]

(3.13)

\[
3C + 3A = -3\eta CA - \frac{\alpha}{2} \left( \frac{\phi'}{u} \right)^2 - \frac{\beta}{2} \frac{Q^2}{g^6} e^{-\beta_2 \phi} - \Lambda e^{-\beta_1 \phi},
\]

(3.14)

\[
\alpha \left( \frac{\phi' f g^3}{u} \right)' \frac{1}{g^3 f^u} = \frac{\beta_2 \beta}{2} \frac{Q^2}{g^6} e^{-\beta_2 \phi} + \Lambda \beta_1 e^{-\beta_1 \phi}.
\]

(3.15)

where

\[
A = - \left( \frac{g'}{g} \right)^2 \frac{1}{u^2}
\]

\[
B = - \left( \frac{f'}{u} \right)' \frac{1}{f u},
\]

(3.16)

\[
C = - \frac{f' g'}{u^2 f g},
\]

\[
G = \left( \frac{g'}{u} \right)' \frac{1}{u g}.
\]

(3.17)

We will give below some special classes of solutions:

**Case:** $\phi = constant$, $H = 0$ and $\eta = 0$.

Here the Euler-Poincaré term is absent, $H = 0$ and the dilaton scalar is constant. We obtain the AdS solution in 5-dimensions that is also known as Randall-Sundrum model [3]:

\[
g = d\omega^2 + e^{\mp 2p\omega}(-dt^2 + dx^2 + dy^2 + dz^2).
\]

(3.18)

where $p^2 = \frac{\Lambda}{6}$.

**Case:** $\phi = constant$, $H = 0$.

Here $H = 0$ and the dilaton scalar is constant. Solutions are given by the metric

\[
g = d\omega^2 + e^{\mp 2p\omega}(-dt^2 + dx^2 + dy^2 + dz^2)
\]

(3.19)
where
\[ s^2 = \frac{1 + \sqrt{1 - \frac{\eta \Lambda}{3}}}{\eta} \] (3.20)
provided that \( \Lambda \eta \leq 3 \). When \( \eta \Lambda = 3 \), the solution may alternatively be given in AdS form as
\[ g = -4 \cosh^2(l \omega) dt^2 + d\omega^2 + 4 \sinh^2(l \omega)(dx^2 + dy^2 + dz^2) \] (3.21)
where \( l^2 = \frac{1}{\eta} \).

**Case: \( \eta = 0, H = 0 \).**

Here the Euler-Poincaré term is absent and \( H = 0 \). We obtain the following solution:
\[ g = e^{\frac{4\alpha}{\beta_1}} \phi(\omega) d\omega^2 + e^{\frac{4\alpha}{3\beta_1}} \phi(\omega)(-dt^2 + dx^2 + dy^2 + dz^2) \] (3.22)
with
\[ \phi(\omega) = \frac{1}{\left(\frac{\beta_1}{2} - \frac{8\alpha}{3\beta_1}\right)} \ln \left| \frac{2\beta_1 \Lambda}{\left(\frac{16\alpha}{3\beta_1} - \beta_1\right) \alpha \left(\frac{\beta_1}{2} - \frac{8\alpha}{3\beta_1}\right) \omega + C_0} \right| \] (3.23)
where \( C_0 \) is an integration constant. When \( \beta_1 = 2 \), it reduces to a supersymmetric domain wall solution presented in [5].

**Case: \( \eta = 0 \).**

In this case the solution possesses a magnetic charge. It is given by
\[ g = e^{\frac{4\alpha}{\beta_1} \phi(\omega)} d\omega^2 + e^{\frac{4\alpha}{3\beta_1} \phi(\omega)}(-dt^2 + dx^2 + dy^2 + dz^2) \] (3.24)
with
\[ \phi(\omega) = \frac{6}{4\beta_2 - \beta_1} \ln \left| \left(4\beta_2 - \beta_1\right) \sqrt{\frac{6}{(\beta_1 - 4\beta_2)\alpha}} \omega + C \right| \] (3.25)
provided that the constants satisfy
\[ (\beta_1 - \beta_2) \left(\frac{\beta_2 Q^2 + \beta_1 \Lambda}{2} + \beta_1 \Lambda\right) = \left(\frac{\beta Q^2}{2} + 4\Lambda\right) \alpha. \] (3.26)
\( C \) is an integration constant. \( H \) is given by

\[
H = Qe^{\frac{(\beta_2 - \beta_1)}{2} \phi(\omega)} e^1 \wedge e^2 \wedge e^3
\]  

(3.27)

We note that when \( Q = 0 \) and the constants \( \beta_1 \) and \( \beta_2 \) satisfy \( \beta_1 - \beta_2 = \frac{4\alpha}{\beta_1} \), the solutions reduce to (3.22) and (3.23).

We also note that an electric dual of solutions (3.24) and (3.25) may be given by defining a 2-form field

\[
F = e^{\beta_2 \phi} \ast H.
\]  

(3.28)

Then the solutions are identified as electrically charged solutions.

4 Conclusion

We have given a class of solutions to the variational field equations of a generalized theory of gravity in a \( D \) dimensional bulk space-time derived from an action that includes the second-order Euler-Poincaré term and a cosmological constant. The theory describes a heterotic type first order effective string model in \( D \) dimensions in the Einstein frame. The special class of plane-symmetric solutions of this model in 5-dimensions we gave refer to a 3-brane world also called a domain wall solution in the literature [5].
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