Non-relativistic bound states: the long way back from the Bethe–Salpeter to the Schrödinger equation

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I review, in a personal perspective, the history of the theory of non-relativistic bound states in QED and QCD from the Bethe–Salpeter equation to the construction of effective field theories.

1.1. Introduction

The study of bound states and, in particular, of non-relativistic bound states has accompanied the quantum theory from its beginning through all its subsequent turning points up to what is now the Standard Model of particle physics. At the beginning it was the description of the hydrogen atom that led to the foundation of quantum mechanics, later the Lamb shift contributed to the development of relativistic field theories and renormalization, which eventually led to the foundation of Quantum Electrodynamics (QED); similarly, in the seventies, quarkonium played a special role in the foundation of Quantum Chromodynamics (QCD). The special role of non-relativistic bound states in particle physics is due to the striking experimental signatures that they provide and the fact that analytical (perturbative) methods are able to describe the relevant features of these signatures.

Despite this, it has proven very difficult to carry out theoretical analyses of a precision comparable with the data, in part due to the high quality of the data, but largely owing to the difficulties in performing bound-state calculations. These may be traced back to the presence of different energy

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1 Contribution to “Fundamental Interactions—A Memorial Volume for Wolfgang Kummer”, D. Grumiller, A. Rebhan, D.V. Vassilevich (eds.)
scales that make it a challenge to maintain a consistent book-keeping in the calculations.

Let us consider a non-relativistic particle of mass $m$ that propagates in a potential $V$ (in the case of a Coulomb potential: $V = -\alpha/r$). If the momentum of the particle is non-relativistic, then $p \sim mv, v \ll 1$ being the velocity of the particle. In the threshold region, the velocity is such that $mv^2 \sim V$. The balance between kinetic energy and potential creates the bound state: the particle propagator $G$ cannot be computed order by order in $V$, but comes from resumming all potential insertions in the free propagator $G_0 = 1/(E - p^2/2m)$:

$$G = G_0 + G_0 V G. \tag{1.1}$$

The function $G = 1/(E - p^2/2m - V)$ exhibits poles in correspondence of the bound-state energies $E_n \sim mv^2$ ($= -m\alpha^2/2n^2$ in the Coulombic case, which implies $v \sim \alpha$ and $1/r \sim m\alpha$), the residues at the poles, $\phi_n^* \phi_n$, satisfy the equation:

$$E_n \phi_n = \left( \frac{p^2}{2m} + V \right) \phi_n, \tag{1.2}$$

which is the Schrödinger equation for a non-relativistic bound state whose wave function is $\phi_n$.

Hence, the non-relativistic dynamics of a particle close to threshold is characterized by a hierarchy of energy scales: $m \gg mv \gg mv^2$. The scale of the mass is sometimes called “hard”, the scale of the typical momentum transfer, or inverse size of the system, $mv$, is called “soft” and the scale $mv^2$ is called “ultrasoft”.

At the level of non-relativistic quantum mechanics, $m$ does not play any dynamical role, for the kinetic energy is $p^2/2m$ rather than $\sqrt{p^2 + m^2}$. The contributions from the other scales are accounted for by the Schrödinger equations (1.1) or (1.2). The solutions of the Schrödinger equation are non-relativistic bound states of typical energy of order $mv^2$ and typical momentum (or inverse size) of order $mv$.

One may expect that a more complicated picture will emerge in a relativistic field theory, although the leading dynamics should be still described by a Schrödinger equation. In a relativistic field theory description of the bound state, we will have, besides the bound state, other degrees of freedom, for instance photons (in QED) and gluons (in QCD) emitted and exchanged by the bound state; for each of them, modes associated to each of the energy scales, $m$, $mv$ and $mv^2$ will appear. We shall discuss bound states in relativistic field theories in the next section.
1.2. The Bethe–Salpeter equation

Let us consider a particle and an antiparticle (e.g. an electron and a positron or a quark and an antiquark) that interact near threshold. In the centre-of-mass frame, their momenta $p$ and energies $E$ are small compared to their masses $m$: $p/m \sim v \ll 1$. We assume that we may express the interaction perturbatively in terms of Feynman diagrams. This is always the case in QED, but does not need to be so in QCD where, at the typical hadronic scale $\Lambda_{\text{QCD}}$, perturbation theory breaks down. Non-relativistic bound states in QCD are made by heavy quarks: this means that at least the quark mass is larger than $\Lambda_{\text{QCD}}$. A bound state of a heavy quark and a heavy antiquark is called quarkonium (examples are charmonium, a charm-anticharm bound state, and bottomonium, a bottom-antibottom bound state; a top-antitop bound state, which would be toponium, has no time to form due to the rapid top quark weak decay, however, near threshold, the bound-state enhancement should be visible in the top-antitop production cross section). A perturbative treatment of quarkonium, which requires $mv, mv^2 \gg \Lambda_{\text{QCD}}$, is justified only for top-antitop pairs near threshold and possibly for the ground state of bottomonium.

$$\alpha_s (1 + \alpha_s/v + \ldots)$$

Fig. 1.1. Resummed propagator near threshold.

How does the bound state emerge in a near threshold interaction? For certain sets of graphs, like those in Fig. 1.1, the perturbative expansion breaks down when $\alpha_s \sim v$ (for definiteness, we will consider here and in the following figures the QCD case: continuous lines stand for quarks and antiquarks, and the curly lines for gluons; the strong coupling constant is $\alpha_s$). The summation of all $\alpha_s/v$ contributions leads to the appearance of a bound-state pole of order $mv^2 \sim m\alpha_s^2$ in the resummed propagator. Indeed, in the leading non-relativistic limit, when the quark/antiquark propagators can be approximated by $\frac{i}{\pm p^0 + E/2 - \mathbf{p}^2/2m + i\epsilon}$ and the gluon exchange by $\frac{i}{q^2}$ (close to threshold we may expand in $|q^0|/|q| \ll 1$; $\gamma^0$ is a
Dirac matrix) the Green’s function shown in Fig. 1.1 satisfies Eq. (1.1).

Beyond the leading non-relativistic limit, diagrams will be much more complicated to calculate and contributions from the different energy scales will get entangled. This happens for any diagram, but the annihilation diagram shown in Fig. 1.2 provides a rather immediate way to see it. Assuming that the incoming quarks are near threshold, the different gluons entering the diagram are characterized by different scales: the annihilation gluons have a typical energy of order $m$; binding gluons carry the momentum of the incoming quarks, which is of order $mv$; and ultrasoft gluons, sensitive to the intermediate bound state, have energies of the order of the binding energy, i.e. $mv^2$.

\[ G = G_0 + G_0 K G, \]  

where $G$ is the two-particle Green’s function, $G_0$ the product of the free propagators of the two particles and the kernel $K$ is the sum of all amputated irreducible two-particle diagrams. Equation (1.3) does not represent an expansion, because, like (1.1), a bound state emerges only from the sum of all interactions, at least those shown in Fig. 1.1. However, unlike (1.1), the Bethe–Salpeter equation is not homogeneous in the momentum scale and an exact solution is unknown. To make the Bethe–Salpeter equation useful, the strategy has been to isolate from $K$ a kernel $K_c$ containing the leading contribution responsible for the formation of the bound state, i.e.
the Coulomb potential, and expand around it (see, for instance, Ref. 2). The most refined approach in this strategy can be found in Ref. 3 (see also Ref. 4): $K_c$ is chosen in such a way that the corresponding Bethe–Salpeter equation, $G_c = G_0 + G_0 K_c G_c$, may be solved in an analytically closed form, and the full Green’s function expanded around the exact solution:

$$G = G_c + G_c \delta K G,$$

(1.4)

where $\delta K = K - K_c$. Since, in the non-relativistic limit, $K_c$ becomes the Coulomb potential, $G_c$ is nothing else than a relativistic modification of the solution of Eq. (1.1) for $V$ equal to the Coulomb potential, which is known since long time. The difference between Eq. (1.3) and Eq. (1.4) is that the latter is a perturbative expansion in the kernel while the former is not.

The Bethe–Salpeter equation was the only systematic tool to treat bound states in field theory until the end of the eighties. However, around that time, it became increasingly clear that perturbative calculations for QED bound states, to which the Bethe–Salpeter equation had been mostly applied, could not be push beyond the reached limit if not at the cost of a formidable amount of work. It shows the difficulty of the approach the fact that going from the calculation of the $m \alpha^5$ correction in the hyperfine splitting of the positronium ground state to the $m \alpha^6 \ln \alpha$ term took twenty-five years! The main problem was the lack of an efficient way of disentangling the contributions coming from the different energy scales and organize them in a perturbative expansion (techniques for asymptotic expansions of Feynman integrals near threshold would be developed later):

In the late seventies and eighties, systematic calculations of quarkonium observables started (for a recent review see Ref. 14). The complicated dynamics of QCD made it more apparent that a treatment based on the Bethe–Salpeter equation was inadequate to perform high-precision quarkonium calculations. First, not all of the quarkonium scales are in general perturbative, lower ones may not be, so that a separation of scales is necessary to achieve factorization. Second, even if a perturbative treatment would be possible (like for the bottomonium ground state and for $t\bar{t}$ threshold production), the number and topology of diagrams makes the calculation prohibitive. It was felt that somehow going back to the Schrödinger equa-
tion and identifying a quarkonium potential would lead to a more treatable problem. In Refs. 15–19, a quarkonium potential was derived from the quark-antiquark scattering amplitude. In the same years, focusing in particular on toponium and $t\bar{t}$ threshold physics, a similar program was carried out by W. Kummer and collaborators$^{20–24}$ (see also the Ph.D. thesis in Ref. 25). In this case, the starting point was the Bethe–Salpeter equation and the generalization to QCD of the solution of the Bethe–Salpeter equation for positronium found in Ref. 3. Still, the goal was not the solution of the Bethe–Salpeter equation itself, but the derivation of a potential, facing, in the process, some of the problems that, in a few years, would have led to (and found a solution with) the construction of effective field theories for non-relativistic bound states. Among the problems mentioned or addressed at that time were the infrared sensitivity of the potential, the inclusion of a finite decay width (in Refs. 22,23, one can find addressed, for the first time in a formal way, how to include the top-quark instability beyond leading order), gauge invariance. The infrared sensitivity of the potential will be discussed in Sec. 1.5.

1.3. NRQED/NRQCD

In QED and QCD, one may take advantage of the hierarchy of scales that characterizes non-relativistic bound states by expanding Green’s functions in the ratios of low energy scales over large energy scales. Working out these expansions, however, turns out to be cumbersome and does not lead to a straightforward and easy way to organize the calculation. If such an expansion is instead implemented at the Lagrangian level, it leads to the construction of an effective field theory (EFT). In the effective field theory the large scale is integrated out from the beginning and does not appear anymore in the Green’s function. The terms in the EFT Lagrangian are organized as an expansion in powers of the inverse of the large scale that has been integrated out leading to a straightforward power counting.

The first EFT introduced for non-relativistic bound states in QED and QCD has been non-relativistic QED/QCD (NRQED/NRQCD).$^{26}$ The large scale that is integrated out in NRQED/NRQCD is the mass $m$ of the bound-state constituents. The degrees of freedom of NRQED/NRQCD are non-relativistic fermions and antifermions, and photons/gluons of energy and momentum smaller than $m$; they build up the operators $O_n$ of the La-
The Lagrangian is organized as an expansion in $1/m$:

$$\mathcal{L}_{\text{NRQED/NRQCD}} = \sum_n c_n(\alpha_s(m), \mu) \times \frac{O_n(\mu)}{m^n}. \quad (1.5)$$

Since, once $O_n$ has been run down to energies lower than $m$, the expectation value of $O_n$ scales like $mv$ or smaller scales, Eq. (1.5) provides, for any physical observable, a perturbative expansion in the ratio of the scale $mv$ or smaller scales over $m$. The Wilson coefficients $c_n$ are non-analytical in the scale $m$ and function of the factorization scale $\mu$. They are calculated by equating, “matching”, amplitudes in QED/QCD with amplitudes in NRQED/NRQCD order by order in $1/m$ and in the coupling constant since in both theories we have that $\alpha, \alpha_s(m) \ll 1$. The matching may be performed on scattering amplitudes, hence in a manner completely independent of the bound state. This is not surprising: the formation of the bound state takes place at a scale, $mv$, which is much smaller than $m$.

The diagram in Fig. 1.2 corresponds, via the optical theorem, to the imaginary part of the diagram shown on the left in Fig. 1.3. The same process would be described in NRQCD by the diagram shown on the right in Fig. 1.3 i.e. by a diagram where the two hard gluons coming from the annihilation are replaced by a contact interaction. The difference between the two diagrams is compensated by the Wilson coefficient $c \sim \alpha_s(m)^2$.

As our example may suggest, NRQCD is particularly well suited to describe heavy quarkonium decay and production\textsuperscript{27,28}. It is in the theory of quarkonium production that NRQCD has perhaps achieved its major success by explaining, in the nineties, the quarkonium production data at the Tevatron by a new mechanism allowed by the symmetries of NRQCD, the octet mechanism, but missed by previous approaches (see Ref. 14 and references therein).

Applications of NRQED have started in the nineties and with time have led to many new results (for some early works, see Refs. 29–31). However, the progress in high precision calculations in NRQED/NRQCD has been...
slowed down by two major shortcomings: first, the fact that soft and ultra-
soft degrees of freedom still remain entangled in NRQED/NRQCD, second,
the use in early NRQED/NRQCD calculations of a cut-off regularization
scheme. The first difficulty led to a power counting that was non homo-
geneous and to perturbative calculations that still involved two scales. To
overcome this difficulty, lower energy EFTs were developed; we shall dis-
cuss some of them in Sec. 1.5. The second difficulty, on one hand, pushed
the development of lattice NRQCD (see Ref. 33 for recent results on the
bottomonium spectrum), on the other hand, addressed analytical studies
towards a consistent formulation of NRQCD in dimensional regulariza-

1.4. The bound state in dimensional regularization

Surprisingly, it was only few years after NRQCD had been introduced that
an EFT for mesons made of a single heavy quark, the heavy quark effec-
tive theory (HQET), was formulated.\textsuperscript{34} In the two-fermion sector, the La-
grangian of HQET contains the same operators as the NRQCD Lagrangian.
However, HQET is a quite different theory from NRQCD: HQET contains
only a single dynamical scale, $\Lambda_{\text{QCD}}$, which governs its power counting. As
a consequence, the kinetic energy, which is of order $\Lambda_{\text{QCD}}^2/m$, is suppressed
with respect to the binding energy, which is of order $\Lambda_{\text{QCD}}$, while, in a
non-relativistic bound state, the two are of the same order.

It is precisely because, in HQET, propagators are expanded in the ki-
netic energy that we may use dimensional regularization in loop calcula-
tions. This has led to a rapid, vast and very successful use of the HQET in
precision studies of $D$ and $B$ mesons.\textsuperscript{35} Instead, keeping the kinetic energy
in the denominators of the propagators, as the power counting of NRQCD
seems to suggest, turns out to be disastrous and leads to the break down
of the power counting. The reason is that, in dimensional regularization,
integrals are not cut-off at high momenta and hard scale poles are going to
contribute if present in the denominators. Once, this had been realized in
Ref. 36, it became also clear that the way out was to compute the matching
to NRQCD in the same way as the matching to the HQET, i.e. order by
order in $1/m$. Since both in NRQCD and in the HQET the matching con-
ditions are computed in the same way, the two Lagrangians are the same:
not only the operators of the two theories coincide in the two-fermion sec-
tor, but also their matching coefficients do. Obviously, in order to compute
observables with the NRQCD Lagrangian, the usual non-relativistic power
counting rules, different from the HQET ones, should be used.
Having understood how to treat the bound state in dimensional regularization, opened, finally, the doors to analytical high-precision calculations also for non-relativistic bound states in NRQED/NRQCD.

1.5. pNRQED/pNRQCD

The problem of disentangling the soft from the ultrasoft scale in NRQED/NRQCD was addressed immediately after dimensional regularization was established as an useful tool for non-relativistic bound state calculations also. The history and details of the developments that have ultimately led to the construction of EFTs for the ultrasoft degrees of freedom of NRQED/NRQCD have been recollected in Ref. 37 and we refer the interested reader to it. Here, we would like just to stress the importance that the process of $tt\bar{t}$ production near threshold (see Ref. 14 and references therein) has played in these developments, providing the only near threshold, heavy quark-antiquark system in nature entirely accessible in perturbation theory. As it was mentioned before, this very special feature of the $tt\bar{t}$ system near threshold had already been appreciated by the groups working on the subject at the beginning of the nineties and, in particular, by the Vienna group.

In the following, in order to illustrate some general features, we will concentrate on the EFTs for ultrasoft degrees of freedom of NRQED/NRQCD known as potential NRQED\(^{38,39}\) and potential NRQCD\(^{40,41}\) (for an alternative formulation see Ref. 42 and the review in Ref. 43). The large scale that is integrated out in pNRQED/pNRQCD is the typical momentum transfer of the bound state, which, in coordinate space, is associated with the inverse of the typical distance $r$ between the two heavy particles. The degrees of freedom of pNRQED/pNRQCD are non-relativistic fermions and antifermions, and photons/gluons of energy and momentum smaller than $mv$. They build up the operators $O_{k,n}$ of the Lagrangian; the operators may be also chosen to be explicitly gauge invariant. The Lagrangian is organized as an expansion in $1/m$, inherited from NRQED/NRQCD, and an expansion in $r$ (multipole expansion), which is characteristic of the new EFT:

$$L_{pNRQED/pNRQCD} = \sum_{k,n} \frac{1}{m^k} \times c_k(\alpha_s(m), \mu) \times V_n(r, \mu, \mu') \times r^n O_{k,n}(\mu').$$

(1.6)

Since, once $O_{k,n}$ has been run down to the lowest energy $mv^2$, the expectation value of $O_{k,n}$ scales like $mv^2$, Eq. (1.6) provides, for any physical
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observable, a perturbative expansion in the ratio of \( mv^2 \) over \( mv \) or \( m \).

The Wilson coefficients \( c_k \) are those inherited from NRQED/NRQCD, the

Wilson coefficients \( V_n \) are the new ones of pNRQED/pNRQCD. They are

non analytical in the scale \( r \) and function of the new factorization scale \( \mu' \). They are calculated by matching, order by order in \( r \), Green’s function

in NRQED/NRQCD with Green’s function in pNRQED/pNRQCD. In pNRQED, the matching may be also done order by order in \( \alpha_s \). In pNRQCD, \( \alpha_s(mv) \ll 1 \) holds only for tightly bound states (short-range quarkonia, e.g. the bottomonium ground state or \( t\bar{t} \) near threshold), while, in general, higher excited quarkonium states (long-range quarkonia) are not accessible by perturbation theory. This means that we can rely on an expansion in \( \alpha_s(mv) \) only for the former states, while for the latter states the matching has to be done in a non-perturbative fashion.

\[
\text{NRQCD} \quad + \quad \text{pNRQCD} \quad + \ldots
\]

\[
\frac{1}{E - p^2/m - V(\mu, \mu')} \quad + \ldots
\]

Fig. 1.4. Matching to pNRQCD.

Let us consider the NRQCD diagram shown in Fig. 1.3 and the part of it where soft gluons are exchanged between the quark and antiquark. The sum of all soft-gluon exchanges would be described in pNRQCD by the diagram shown on the right in Fig. 1.4 where the single line stands for a quark-antiquark propagator in a color-singlet configuration, \( 1/(E - p^2/m - V) \), the double line for a quark-antiquark pair in a color-octet configuration, the curly line for ultrasoft gluons and the circle with a cross for a chromoelectric dipole interaction \( \sim \phi^\dagger \mathbf{r} \cdot \mathbf{E} \phi \) that comes from multipole expanding the gluon fields in the NRQCD Lagrangian. The non-analytical behaviour in \( r \) of the NRQCD diagram is reproduced in pNRQCD by the Wilson coefficient \( V \). Since \( V \), together with \( p^2/m \), makes up the pole of the quark-antiquark
propagator, the interpretation of $V$ is obvious: $V$ is the potential describing the interaction in the heavy quark-antiquark pair. At leading order in the multipole expansion, when we neglect diagrams involving ultrasoft gluons, the equation of motion of a non-relativistic fermion-antifermion pair is nothing else than the Schrödinger equation (1.2).

The Schrödinger equation is the equation governing non-relativistic bound states in quantum mechanics. The full relativistic description provided by field theory, which is richer and much more complex, is given by the Bethe–Salpeter equation. This complexity arises from the entanglement of different energy scales. Once the contributions of all these scales have been separated/factorized, we are left with an EFT of the ultrasoft degrees of freedom. The Schrödinger equation naturally emerges as the equation of motion of these ultrasoft degrees of freedom. But, because the EFT contains all the richness and complexity of the field theory, although unfolded in a systematic and organized way, the Schrödinger equation, which we have gotten from the EFT, is much more than the Schrödinger equation of quantum mechanics we have started with. First, the EFT provides a proper, field theoretically founded, definition of the potential: the potential is the Wilson coefficient of the dimension six operator of the EFT, containing two fermion and two antifermion fields, that encodes all contributions coming from modes whose energies and momenta are larger than the binding energy. It undergoes renormalization, develops scale dependence and satisfies renormalization group equations, which, in perturbation theory, allow to resum potentially large logarithms. Moreover, the EFT accounts also for effects that cannot be cast in a Schrödinger equation and that are due to the coupling of the fermion-antifermion pair with the other ultrasoft degrees of freedom.

Fig. 1.5. QCD diagrams responsible for the infrared sensitivity of the static potential.

In QCD, ultrasoft effects affect also the static potential. As first observed in Ref. 44, they come from the “non-Abelian Lamb shift” diagrams displayed in Fig. 1.5. At fixed order in perturbation theory, the diagrams
are infrared divergent; at order $\alpha_s^4$, the leading logarithmic correction is 

$$\delta V(r, \mu') = -\frac{3}{r} \frac{\alpha_s(\mu')}{\pi} \alpha_s^3 \frac{1}{r} \ln(r\mu').$$  \hspace{1cm} (1.7)$$

The result shows clearly the non-physical nature of the potential, which
depends on the renormalization scale $\mu'$. The potentially large logarithms, 
$\ln(r\mu')$, have been resummed by means of renormalization group equations 
in Ref. 46; subleading corrections have been calculated in Ref. 47. In 
physical observables, like the static energy or the quarkonium mass, the 
scale dependence of Eq. (1.7) cancels against ultrasoft contributions coming 
from the second diagram in the pNRQCD part of Fig. 1.4.

Higher-order terms in the relativistic expansion may be computed system-
atically in the EFT. Again, the full complexity and symmetries of the 
underlying field theory are not lost in the expansion. So, for instance, 
relativistic invariance imposes specific constraints on the Wilson coeffi-
cients/potentials of the EFT,\textsuperscript{48,49} which can be tested on the lattice.\textsuperscript{50,51}

Applications of pNRQCD and, more in general, of EFTs for the ultrasoft 
degrees of freedom of NRQCD have led to a plethora of new results in 
quarkonium physics (see Refs. 14,37,52–55 for some recent reviews) and, 
in particular, in $t\bar{t}$ threshold production (see Refs. 56–59 for the present 
status of the art). Also QED calculations have remarkably benefitted from 
the EFT approach and corrections of very high order in perturbation theory 
have been calculated in the last years for many observables after decades of 
very slow or no progress. As an example, we mention that for the hyperfine 
splitting of the positronium ground state the terms of order $\alpha^6$, $\alpha^7 \ln^2 \alpha$ and 
$\alpha^7 \ln \alpha$ have been calculated (for recent reviews on positronium precision 
studies and further references we refer to Refs. 60,61).

In Fig. 1.6 we summarize the hierarchy of EFTs for bound states in 
QED and, for heavy quarks, in QCD.

\textbf{1.6. Outlook}

The history of non-relativistic bound states in the quantum theory had in 
the last century a peculiar spiral behaviour. It started with the Schrödinger 
equation of the hydrogen atom and seemed to have written its ultimate 
chapter with the Bethe–Salpeter equation in the fifties. However, in face of 
the enormous difficulties in treating bound states in field theory by means 
of the Bethe–Salpeter equation, a long journey started in the seventies 
that took us back to the Schrödinger equation. This coming back, how-
ever, was not like closing a circle, it was more like building up a spiral. The
Schrödinger equation we have come back to, encompasses all the complexity of the Bethe–Salpeter equation, all the richness of field theory, in the elegant and systematic setting of non-relativistic effective field theories. The counting rules and structure of the EFTs have allowed us to perform calculation with unprecedented precision, where higher-order perturbative calculations were possible, and to systematically factorize short from long range contributions where observables were sensitive to the non-perturbative, infrared dynamics of QCD.

Non-relativistic EFTs have become nowadays the standard tool to treat non-relativistic bound states. Besides QED bound states and quarkonium, these include hadronic atoms like pionium,\textsuperscript{62} nucleon-nucleon systems,\textsuperscript{63,64} non-relativistic bound states at finite temperature\textsuperscript{65–67} and many others. The modern history of non-relativistic bound states is far from being finished and still needs to be told in its full extent.

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