Faber polynomial coefficient estimation of subclass of bi-subordinate univalent functions

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Abstract — In this paper, a comprehensive subclass of bi-univalent functions class are introduced and investigated. Using the Faber polynomials, estimation of the coefficients $|a_n|$ and certain Fekete-Szegő inequality of Maclaurin expansion of functions in this subclass are concluded. Finally, some earlier results are pointed out and improved.

Keywords and phrases: Analytic function; Univalent function; Bi-univalent function; Faber polynomial; Fekete Szegő inequalities; Bounded functions.

2010 Mathematics Subject Classification. 30C45 Secondary: 30C50, 30C55.

1. Introduction

Let $A$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ defined in the open unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

Let $S$ be the subclass of $A$ consisting of all functions of the form (1) which are univalent in $U$. Let $\varphi$ be an analytic univalent function in $U$ with positive real part and $\varphi(U)$ be symmetric with respect to the real axis, starlike with respect to $\varphi(0) = 1$ and $\varphi'(0) > 0$. Ma and Minda [17] gave a unified presentation of various subclasses of starlike and convex functions by introducing the classes $S^*(\varphi)$ and $K(\varphi) \cup \{1\}$ of functions $f \in S$ satisfying $(z f'(z)/f(z)) < \varphi(z)$ and $1 + (z f''(z)/f'(z)) < \varphi(z)$ respectively, which includes several well-known classes as special case. For example, when $\varphi(z) = (1 + Az)/(1 + Bz)$ with a condition $(-1 < B < A \leq 1)$, the classes $S^*(\varphi)$ and $K(\varphi) \cup \{1\}$ converted to the class $S^*[A,B]$ and $K[A,B]$, respectively, introduced by Janowski [15]. Although, for a special choose of the value of $A = 1 - 2\beta$, $B = -1 (0 \leq \beta < 1)$, the classes $S^*[A,B]$ and $K[A,B]$ reduced to the classes $S^*(\beta)$ and $K(\beta)$, respectively, which are the class of starlike and convex functions of order $\beta$. For another choose of

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the function \( \varphi(z) = ((1 + z)/(1 - z))^a \), we obtain the classes \( S_\alpha \) and \( K_\alpha \), which are the class of strongly starlike and strongly convex functions of order \( \alpha \) \( (0 < \alpha \leq 1) \).

The Koebe one quarter theorem [8] ensures that the image of \( U \) under every univalent function \( f \in S \) contains a disk of radius \( \frac{1}{4} \). Thus every univalent function \( f \) has an inverse \( f^{-1} \) satisfying

\[
f^{-1}(f(z)) = z, \quad (z \in U) \text{ and } f(f^{-1}(w)) = w \quad (|w| < r_0(f), r_0(f) \leq \frac{1}{4}).
\]

A function \( f \in S \) is said to be bi-univalent in \( U \) if both \( f \) and \( f^{-1} \) are univalent in \( U \). Let \( \Sigma \) denote the class of all bi-univalent functions defined in the unit disk \( U \). Since \( f \in \Sigma \) has the Maclaurin series expansion given by (1), a simple calculation shows that its inverse \( g = f^{-1} \) has the series expansion

\[
g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - ....
\]

Examples of functions in the class \( \Sigma \) are

\[
\frac{z}{1 - z}, \quad -\log(1 - z) \quad \text{and} \quad \frac{1}{2} \log\left(\frac{1 + z}{1 - z}\right),
\]

and so on. However, the familiar Koebe function is not a member of \( \Sigma \). Other common examples of functions in \( S \) such as

\[
z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1 - z^2}
\]

are also not members of \( \Sigma \) (see [20]).

Many papers concerning bi-univalent functions have been published recently (for mentioned but a few, [5, 6, 9, 11]). A function \( f \in \Sigma \) is in the class \( S_\beta \) of bi-starlike function of order \( \beta \) \( (0 \leq \beta < 1) \), or \( K_\beta \) of bi-convex function of order \( \beta \) if both \( f \) and \( f^{-1} \) are respectively starlike or convex functions of order \( \beta \). For \( 0 < \alpha \leq 1 \), the function \( f \in \Sigma \) is strongly bi-starlike function of order \( \alpha \) if both the functions \( f \) and \( f^{-1} \) are strongly starlike functions of order \( \alpha \). The class of all such functions is denoted by \( S_{\alpha, \alpha} \). These classes were introduced by Brannan and Taha [5]. They obtained estimates on the initial coefficients \( |a_2| \) and \( |a_3| \) for functions in these classes. The research into \( \Sigma \) was started by Lewin [16]. He focused on problems connected with coefficients and showed that \( |a_2| < 1.51 \). Subsequently, Brannan and Clunie [4] conjectured that \( |a_2| < \sqrt{\alpha} \). Netanyahu [19] concluded that \( \max |a_2| = \frac{\alpha}{4} \).

The coefficient estimate problem for each of the following Taylor Maclaurin coefficients \( |a_n|, \ n \in \{2, 3, \ldots\} \) is presumably still an open problem. This is because the bi-univalency requirement makes the behavior of the coefficients of the function \( f \) and \( f^{-1} \) unpredictable. The Faber polynomials play an important role in various areas of mathematical sciences, especially in geometric function theory. The recent publications [12, 13] applying the Faber polynomial expansions to meromorphic bi-univalent functions motivated us to apply this technique to classes of analytic bi-univalent functions. In the literature, there are only a few works determining the general coefficient bounds \( |a_n| \) for the analytic bi-univalent functions given by (1) using Faber polynomial expansions.

In this present work, we use the Faber polynomials in obtaining bounds of Maclaurin coefficients \( |a_n|, \ n \in \mathbb{N} \) and bounds for the Fekete-Szegő functional \( |a_3 - 2a_2^2| \) of a new defined subclass of \( \Sigma \) to generalize some earlier results.
2. Construction of the subclass $\mathcal{H}_\Sigma(\tau, \lambda, \delta; \varphi)$

Throughout this section, let us assume that $\varphi$ be an analytic function with positive real part in the unit disc $U$ satisfying $\varphi(0) = 1$, $\varphi'(0) > 0$ and $\varphi(U)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form
\[
\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1 > 0).
\]
where $B_n \in \mathbb{R}$, for all $n = 2, 3, \ldots$.

Using the Faber polynomial $[1,2]$ expansion of the functions $f \in \Sigma$ of the form (1), the inverse function $g = f^{-1}$ may be expressed as
\[
g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n,
\]
where
\[
A_n = \frac{1}{n} \mathcal{K}^{-n}_{n-1}(a_2, a_3, \ldots, a_n).
\]

Now, for any $p \in \mathbb{Z} := \{0, \pm 1, \pm 2, \cdots\}$, the expansion of $\mathcal{K}_n^p$ is given by
\[
\mathcal{K}_n^p = p a_n + \frac{p!}{(p - 2)!} D_n^2 + \frac{p!}{(p - 3)!} D_n^3 + \cdots + \frac{p!}{(p - n)!} D_n^n,
\]
where
\[
D_n^m = D_n^m(a_1, a_2, \ldots, a_n),
\]
\[
= \sum_{n=2}^{\infty} \frac{m!}{\mu_1! \mu_2! \cdots \mu_n!} a_1^{\mu_1} a_2^{\mu_2} a_3^{\mu_3} \cdots a_n^{\mu_n},
\]
while $a_1 = 1$ and the sum is taken over all non-negative integers $\mu_1, \mu_2, \mu_3, \ldots, \mu_n$ satisfying
\[
\mu_1 + \mu_2 + \mu_3 + \cdots + \mu_n = m, \quad \mu_1 + 2\mu_2 + \cdots + n\mu_n = n.
\]

It is observed that
\[
D_n^n(a_1, a_3, \ldots, a_n) = a_1^n.
\]

Thus, from equation (5) together with (6) we get an expression of $\mathcal{K}^{-n}_{n-1}$ as
\[
\mathcal{K}^{-n}_{n-1}(a_2, a_3, \ldots, a_n) = \frac{(-n)!}{(-2n+1)(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))(n-3)!} a_2^{n-3} a_3 \\
+ \frac{(-n)!}{(-2n+3)! (n-4)!} a_2^{n-4} a_4 \\
+ \frac{(-n)!}{(2(-n+2))(n-5)!} a_2^{n-5} (a_5 + (-n+2)a_3^2) \\
+ \frac{(-n)!}{(-2n+5)! (n-6)!} a_2^{n-6} (a_6 + (-2n+5)a_3 a_4) \\
+ \sum_{j \geq 7} a_2^{n-j} V_j,
\]
where such expressions as \((-n)\)! are to be interpreted by
\[\text{\((-n)!: = \Gamma \left(1 - n\right) = (-n)(-n-1)(-n-2) \cdots (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} ),}\]
and \(V_j (7 \leq j \leq n)\) is a homogeneous polynomial in the variables \(a_2, a_3, \ldots, a_n\). In particular, in case of \(n = 2, 3, 4\) the expression of \(\mathcal{K}_{n-1}^<\) is reduced to
\[
\begin{align*}
\mathcal{K}_{1}^2 &= -2a_2, \\
\mathcal{K}_{2}^3 &= 3(2a_2^2 - a_3), \\
\mathcal{K}_{3}^4 &= -4(5a_2^3 - 5a_2a_3 + a_4).
\end{align*}
\]

**Definition 2.1.** Let \(\lambda \geq 1, \tau \in \mathbb{C}^* = \mathbb{C} - \{0\}, 0 < \delta \leq 1\) and \(f, g \in \Sigma\) given by (1) and (3) respectively, then \(f\) is said to be in the class \(\mathcal{H}_\Sigma(\tau, \lambda, \delta; \varphi)\) if
\[
1 + \frac{1}{\tau} \left(1 - \lambda - \frac{f(z)}{z} + \lambda f'(z) + \delta f''(z) - 1\right) < \varphi(z),
\]
and
\[
1 + \frac{1}{\tau} \left(1 - \lambda - \frac{g(w)}{w} + \lambda g'(w) + \delta g''(w) - 1\right) < \varphi(w),
\]
where \(z, w \in U\) and \(\varphi(z)\) is given by (2).

**Remark 1.** For special choices of the parameters \(\lambda, \tau, \delta\) and the function \(\varphi(z)\), the class \(\mathcal{H}_\Sigma(\tau, \lambda, \delta; \varphi)\) reduced to the following subclasses:

1. \(\mathcal{H}_\Sigma(\tau, 1, \gamma; \varphi) = \Sigma(\tau, \gamma, \varphi)\) which introduced by A.E. Tudor [23] and recently studied by H.M. Srivastava and Deepak Bansal [22].
2. \(\mathcal{H}_\Sigma(1, 1, 0; \varphi) = \mathcal{H}_\sigma(\varphi)\) which defined and studied by Rosihan M. Ali et al. [3].
3. \(\mathcal{H}_\Sigma(1, 1, \beta; \left(\frac{1 + z}{1 - z}\right)^n) = \mathcal{H}_\Sigma(\alpha, \beta)\) which introduced by B.A. Frasin [11].
4. \(\mathcal{H}_\Sigma(1, 1, 0; \left(\frac{1 + z}{1 - z}\right)^n) = \mathcal{H}_\Sigma^a\) which introduced by H.M. Srivastava et al. [20].
5. \(\mathcal{H}_\Sigma(1, \lambda, 0; \left(\frac{1 + z}{1 - z}\right)^n) = \mathcal{B}_\Sigma(\alpha, \lambda)\) which is introduced by B.A. Frasin and M.K. Aouf [10], and recently studied by H.M. Srivastava et al. [21].
6. \(\mathcal{H}_\Sigma(1 - \gamma, 1, \beta; \frac{1 + z}{1 - z}) = \mathcal{H}_\Sigma(\gamma, \beta)\) which introduced by B.A. Frasin [11].
7. \(\mathcal{H}_\Sigma(1 - \alpha, \lambda, \delta; \frac{1 + z}{1 - z}) = \mathcal{N}_\Sigma(\alpha, \lambda, \delta)\) which introduced by S. Bulut [6].
8. \(\mathcal{H}_\Sigma(1 - \beta, 1, 0; \frac{1 + z}{1 - z}) = \mathcal{H}_\Sigma(\beta)\) which introduced by H.M. Srivastava et al. [20].
9. \(\mathcal{H}_\Sigma(1 - \beta, \lambda, 0; \frac{1 + z}{1 - z}) = \mathcal{B}_\Sigma(\beta, \lambda)\) which introduced by B.A. Frasin and M.A. Aouf [10] and recently studied by J.M. Jahangiri and S.G. Hamidi [14].
10. \(\mathcal{H}_\Sigma(\tau, 1, \gamma; \frac{1 + z}{1 + \delta z}) = \mathcal{R}_{\gamma, \tau}^\alpha(A, B)\) which introduced by A.E. Tudor [23].

**Lemma 2.2.** [18] Let \(u(z)\) be analytic function in the unit disc \(U\) with \(u(0) = 0\) and \(|u(z)| < 1\) for all \(z \in U\) with the power series expansion
\[u(z) = \sum_{n=1}^{\infty} c_n z^n,\]
then \(|c_n| \leq 1\) for all \(n = 1, 2, 3, \ldots\). Furthermore, \(|c_n| = 1\) for some \(n = 1, 2, 3, \ldots\) if and only if
\[u(z) = e^{i\theta} z^n,\quad \theta \in \mathbb{R}.\]
Lemma 2.3. [7] Let the function \( p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \) be so that \( \Re(p(z)) > 0 \) for \( z \in U \). Then for \(-\infty < \alpha < \infty\),

\[
|p_2 - \alpha p_1^2| \leq \begin{cases} 
2 - \alpha |p_1|^2 & ; \alpha < \frac{1}{2} \\
2 - (1 - \alpha)|p_1|^2 & ; \alpha \geq \frac{1}{2}
\end{cases}.
\] (9)

Let \( \varphi(z) = \sum_{n=1}^{\infty} a_n z^n \) be a Schwarz function so that \( |\varphi(z)| < 1 \), \( z \in U \). Set \( p(z) = \frac{1 + \varphi(z)}{1 - \varphi(z)} \) where \( p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \) is so that \( \Re(p(z)) > 0 \) for \( z \in U \). Comparing the corresponding coefficients of powers of \( z \) yields \( p_1 = 2\varphi_1 \) and \( p_2 = 2(\varphi_2 + \varphi_1^2) \). Now, substituting for \( p_1 \) and \( p_2 \) and letting \( \eta = 1 - 2\alpha \) in (9), we obtain

\[
|\varphi_2 + \eta \varphi_1^2| \leq \begin{cases} 
1 - (1 - \eta)|\varphi_1|^2 & ; \eta > 0 \\
1 - (1 + \eta)|\varphi_1|^2 & ; \eta < 0
\end{cases}.
\] (10)

2.1. Coefficient bounds of members of \( \mathcal{H}_\Sigma(\tau, \lambda; \varphi) \)

Unless otherwise mentioned, let us assume in the reminder of this section that \( z \in U \), \( \lambda \geq 1 \), \( 0 \leq \delta \leq 1 \) and \( \tau \in \mathbb{C} - \{0\} \).

Theorem 2.4. Let \( f \) defined by (1) belong to the class \( \mathcal{H}_\Sigma(\tau, \lambda; \varphi) \) and \( a_k = 0 \) \( (2 \leq k \leq n - 1) \), then

\[
|a_n| \leq \frac{B_1|\tau|}{1 + (n - 1)(\lambda + n\delta)} \quad (n \geq 4).
\] (11)

Proof. Since \( f \in \mathcal{H}_\Sigma(\tau, \lambda; \varphi) \), then we have

\[
1 + \frac{1}{\tau} \left((1 - \lambda)\frac{f(z)}{z} + \lambda f'(z) + \delta z f''(z) - 1\right) = 1 + \sum_{n=2}^{\infty} a_n z^{n-1},
\] (12)

and since the inverse map \( g = f^{-1} \) represented by (3) also belonging to the same subclass, then

\[
1 + \frac{1}{\tau} \left((1 - \lambda)\frac{g(w)}{w} + \lambda g'(w) + \delta z g''(w) - 1\right) = 1 + \sum_{n=2}^{\infty} A_n w^{n-1}.
\] (13)

Now, Since \( f, g \in \mathcal{H}_\Sigma(\tau, \lambda; \varphi) \), by the definition 2.1, there exist two Schwarz functions \( u(z) = \sum_{n=1}^{\infty} c_n z^n \) and \( v(w) = \sum_{n=1}^{\infty} d_n w^n \) such that

\[
1 + \frac{1}{\tau} \left((1 - \lambda)\frac{f(z)}{z} + \lambda f'(z) + \delta z f''(z) - 1\right) = \varphi(u(z)),
\] (14)

\[
1 + \frac{1}{\tau} \left((1 - \lambda)\frac{g(w)}{w} + \lambda g'(w) + \delta z g''(w) - 1\right) = \varphi(v(w)),
\] (15)

such that

\[
\varphi(u(z)) = 1 - \sum_{n=2}^{\infty} B_1 \mathcal{K}_{n-1}^{-1}(c_1, ..., c_{n-1}; B_1, ..., B_{n-1}) z^{n-1},
\] (16)

\[
\varphi(v(w)) = 1 - \sum_{n=2}^{\infty} B_1 \mathcal{K}_{n-1}^{-1}(d_1, ..., d_{n-1}; B_1, ..., B_{n-1}) w^{n-1},
\] (17)
where in general $\mathcal{K}_n^p = \mathcal{K}_n^p(\rho_1, ..., \rho_n, B_1, ..., B_n)$ are defined by

$$
\mathcal{K}_n^p = \frac{p!}{(p-n)!(n)!} \rho_1^n B_n \frac{B_n}{B_1} + \frac{p!}{(p-n+1)!(n-1)!(n-2)!} \rho_1^{n-2} \rho_2 B_{n-1} \frac{B_{n-1}}{B_1} \\
+ \frac{p!}{(p-n+2)!(n-3)!} \rho_1^{n-3} \rho_3 B_{n-2} \frac{B_{n-2}}{B_1} \\
+ \frac{p!}{(p-n+3)!(n-4)!} \rho_1^{n-4} \left( \frac{B_{n-3}}{B_1} + \frac{p-n+3}{2} \rho_2 \frac{B_{n-2}}{B_1} \right) \\
+ \frac{p!}{(p-n+4)!(n-5)!} \rho_1^{n-5} \left( \frac{B_{n-4}}{B_1} + (p-n+4)\rho_2 \rho_3 \frac{B_{n-3}}{B_1} \right) \\
+ \sum_{j \geq 6} \rho_1^{n-j} X_j,
$$

(18)

where $X_j$ is a homogeneous polynomial of degree $j$ in the variables $\rho_1, \rho_2, ..., \rho_n$.

Now, comparing the coefficients in both sides of equations (14) and (15) after substituting about $\varphi(u(z))$ and $\varphi(v(w))$ from equations (16) and (17), we have

$$
\frac{1 + (n-1)(\lambda + n\delta)}{\tau} a_n = -B_1 \mathcal{K}_{n-1}^{-1}(c_1, ..., c_{n-1}; B_1, ..., B_{n-1}),
$$

(19)

$$
\frac{1 + (n-1)(\lambda + n\delta)}{\tau} A_n = -B_1 \mathcal{K}_{n-1}^{-1}(d_1, ..., d_{n-1}; B_1, ..., B_{n-1}).
$$

(20)

Since $a_k = 0 \ (2 \leq k \leq n-1)$, then from equation (4) it is easy to conclude

$$
A_n = -a_n.
$$

(21)

Therefore, equations (19) and (20) reduced to

$$
\frac{1 + (n-1)(\lambda + n\delta)}{\tau} a_n = B_1 c_{n-1},
$$

(22)

$$
\frac{1 + (n-1)(\lambda + n\delta)}{\tau} a_n = B_1 d_{n-1}.
$$

(23)

By subtracting equation (23) from equation (22) obtained

$$
a_n = \frac{B_1 \tau (c_{n-1} - d_{n-1})}{2(1 + (n-1)(\lambda + n\delta))}.
$$

(24)

Applying Lemma 2.2 for the coefficients $c_{n-1}$ and $d_{n-1}$ in equation (24) which reduced to the desired estimation. The proof is completed.

By putting $\tau = 1 - \alpha (0 \leq \alpha < 1)$ and $\varphi(z) = \frac{1 + \alpha z}{1 - z} \ (B_1 = 2)$ in Theorem 2.4, we conclude

**Corollary 2.5.** [6, Theorem 2] Let $f \in \mathcal{N}_\Sigma(\alpha, \lambda, \delta)$ and $a_k = 0 \ (2 \leq k \leq n-1)$, then

$$
|a_n| \leq \frac{2(1 - \alpha)}{1 + (n-1)(\lambda + n\delta)} \ \ (n \geq 4).
$$

Let us put $\lambda = 1$ in Corollary 2.6, we have
Theorem 2.8. Let \( f \in \mathcal{N}_\Sigma^{(a, \lambda)} \) and \( a_k = 0 \) \((2 \leq k \leq n - 1)\), then

\[
|a_n| \leq \frac{2(1 - \alpha)}{n(1 + \delta(n - 1))} \quad (n \geq 4).
\]

Let us put \( \delta = 0 \) in Corollary 2.6, we obtain

Corollary 2.7. [14, Theorem 1] If \( f \in \mathcal{D}(a, \lambda) \) and \( a_k = 0 \) \((2 \leq k \leq n - 1)\), then

\[
|a_n| \leq \frac{2(1 - \alpha)}{1 + \lambda(n - 1)} \quad (n \geq 4).
\]

Theorem 2.8. Let \( f \in \mathcal{H}_\Sigma(t, \lambda, \delta; p) \) and \( B_1 \geq |B_2| \), then

\[
|a_2| \leq \begin{cases} 
\frac{B_1 \sqrt{p|t|}}{B_1^2|t|(1 + 2\lambda + 6\delta) + (B_1 + B_2)(1 + \lambda + 2\delta)^2} & \text{if } B_2 < 0, B_1 + B_2 \leq 0 \\
\frac{B_1 \sqrt{p|t|}}{B_1^2|t|(1 + 2\lambda + 6\delta) + (B_1 - B_2)(1 + \lambda + 2\delta)^2} & \text{if } B_2 > 0, B_1 - B_2 \leq 0
\end{cases},
\]

(25)

\[
|a_3| \leq \begin{cases} 
\frac{|B_2|\sqrt{p|t|}}{1 + 2\lambda + 6\delta} & ; \ B_1 > |B_2| \\
\frac{|B_2|\sqrt{p|t|}}{1 + 2\lambda + 6\delta} & ; \ B_1 < |B_2|
\end{cases},
\]

(26)

and

\[
|a_3 - 2a_2^2| \leq \begin{cases} 
\frac{|B_2|\sqrt{p|t|}}{1 + 2\lambda + 6\delta} & ; \ B_1 > |B_2| \\
\frac{|B_2|\sqrt{p|t|}}{1 + 2\lambda + 6\delta} & ; \ B_1 < |B_2|
\end{cases}.
\]

(27)

Proof. Let us set \( n = 2, n = 3 \) in the equations (19) and (20), we deduce

\[
\frac{1 + \lambda + 2\delta}{\tau}a_2 = B_1c_1,
\]

(28)

\[
-\frac{1 + \lambda + 2\delta}{\tau}a_2 = B_1d_1,
\]

(29)

\[
\frac{1 + 2\lambda + 6\delta}{\tau}a_3 = B_1c_2 + B_2c_1^2,
\]

(30)

and

\[
\frac{1 + 2\lambda + 6\delta}{\tau}(2a_2^2 - a_3) = B_1d_2 + B_2d_1^2.
\]

(31)

From equations (28) and (29), we deduce

\[
c_1 = -d_1,
\]

(32)

and

\[
a_2 = \frac{B_1c_1\tau}{1 + \lambda + 2\delta}.
\]

(33)

Now, adding equation (30) to (31) obtains

\[
a_2^2 = \tau \left( \frac{(B_1(c_2 + d_2) + B_2(c_1^2 + d_1^2))}{2(1 + 2\lambda + 6\delta)} \right).
\]

(34)
\[ a_2^2 = \frac{B_1 \tau}{2(1 + 2\lambda + 6\delta)} \left[ (c_2 + \frac{B_2}{B_1} c_1^2) + (d_2 + \frac{B_2}{B_1} d_1^2) \right]. \] (35)

Firstly, let \( B_2 < 0 (\eta = \frac{B_2}{B_1} < 0, B_1 + B_2 \geq 0) \) and applying Lemma 2.3 with using equation (32), we obtain
\[ |a_2|^2 \leq \frac{B_1 \tau}{1 + 2\lambda + 6\delta} \left[ 1 - \left( \frac{B_1 + B_2}{B_1} \right) |c_1|^2 \right]. \] (36)

By substituting \( c_1 \) from equation (33), we conclude
\[ |a_2|^2 \leq \frac{|\tau|^2 B_1^3}{B_1^2 |\tau| (1 + 2\lambda + 6\delta) + (B_1 + B_2)(1 + \lambda + 2\delta)^2}. \] (37)

Taking the square root of the both side of inequality (37), we have
\[ |a_2| \leq \frac{|\tau| B_1 \sqrt{B_1}}{\sqrt{B_1^2 |\tau| (1 + 2\lambda + 6\delta) + (B_1 + B_2)(1 + \lambda + 2\delta)^2}}. \] (38)

Second, let \( B_2 > 0 (\eta = \frac{B_2}{B_1} > 0, B_1 - B_2 \geq 0) \) and applying Lemma 2.3 with using equation (32), then
\[ a_2^2 \leq \frac{B_1 \tau}{1 + 2\lambda + 6\delta} \left[ 1 - \left( \frac{B_1 - B_2}{B_1} \right) |c_1|^2 \right]. \] (39)

By substituting \( c_1 \) from equation (33), we conclude
\[ |a_2|^2 \leq \frac{|\tau|^2 B_1^3}{B_1^2 |\tau| (1 + 2\lambda + 6\delta) + (B_1 - B_2)(1 + \lambda + 2\delta)^2}. \] (40)

Taking the square root of the both side of inequality (41), we have
\[ |a_2| \leq \frac{|\tau| B_1 \sqrt{B_1}}{B_1^2 |\tau| (1 + 2\lambda + 6\delta) + (B_1 - B_2)(1 + \lambda + 2\delta)^2}. \] (41)

Combining the last inequality with inequality (38), we obtain the desired estimate on the coefficient \( |a_2| \) which given by (25).

In order to deduce the estimation of \( |a_3| \), subtracting equation (31) from (30) with using equation (33), obtains
\[ a_3 = a_2^2 + \frac{B_1 \tau (c_2 - d_2)}{2(1 + 2\lambda + 6\delta)}. \] (42)

By substituting \( a_2^2 \) from equation (34) into (42), we conclude
\[ a_3 = \frac{\tau (B_1 c_2 + B_2 c_1^2)}{1 + 2\lambda + 6\delta}. \] (43)

Taking the modulus of both sides of equation (43), we get
\[ |a_3| \leq \frac{B_1 |\tau|}{1 + 2\lambda + 6\delta} \left| c_2 + \frac{B_2}{B_1} c_1^2 \right|. \] (44)

By applying Lemma 2.3, let first \( B_2 < 0 (\eta = \frac{B_2}{B_1} < 0) \), then
\[ |a_3| \leq \frac{B_1 |\tau|}{1 + 2\lambda + 6\delta} \left[ 1 - \frac{B_1 - B_2}{B_1} |c_1|^2 \right]. \] (45)
If $B_1 - B_2 > 0$, then we must put $|c_1|$ by its least value $|c_1| = 0$. Thus

$$|a_3| \leq \frac{B_1|\tau|}{1 + 2\lambda + 6\delta}. \quad (46)$$

If $B_1 - B_2 < 0$, then we must put $|c_1|$ by its maximum value $|c_1| = 1$ (using Lemma 2.3). Thus

$$|a_3| \leq \frac{B_2|\tau|}{1 + 2\lambda + 6\delta}. \quad (47)$$

Second, let us put $B_2 > 0(\eta = \frac{B_2}{B_1} > 0)$, then

$$|a_3| \leq \frac{B_1|\tau|}{1 + 2\lambda + 6\delta} \left[ 1 - \frac{B_1 + B_2}{B_1} |c_1|^2 \right]. \quad (48)$$

If $B_1 + B_2 > 0$, then we must put $|c_1|$ by its least value $|c_1| = 0$. Thus

$$|a_3| \leq \frac{B_1|\tau|}{1 + 2\lambda + 6\delta}. \quad (49)$$

If $B_1 + B_2 < 0$, then we must put $|c_1|$ by its maximum value $|c_1| = 1$ (using Lemma 2.2). Thus

$$|a_3| \leq \frac{-B_2|\tau|}{1 + 2\lambda + 6\delta}. \quad (50)$$

By comparing the estimates of $|a_3|$ in relations from (46) to (49) which obtain the desired estimate given by (26). Finally, using equation (31), gives

$$a_3 - 2a_2^2 = \frac{-\tau(B_1d_2 + B_2d_2^2)}{1 + 2\lambda + 6\delta}. \quad (51)$$

Using the same technique in proving the estimate of $|a_3|$, we get the desired estimate given by (27), then we prefer to omit it.

In case of $\lambda = 1$, Theorem 2.8 becomes

**Corollary 2.9.** [22, Theorem 1] Let $f \in \Sigma(\tau, \delta, \varphi)$, then

$$|a_2| \leq \begin{cases} \frac{B_1 \sqrt{B_1|\tau|}}{\sqrt{3B_1^2|\tau|(1 + 2\delta) + 4(B_1 + B_2)(1 + \delta)^2}} & B_2 < 0 \text{ and } B_1 + B_2 \geq 0 \\ \frac{B_1 \sqrt{B_1|\tau|}}{\sqrt{3B_1^2|\tau|(1 + 2\delta) + 4(B_1 - B_2)(1 + \delta)^2}} & B_2 > 0 \text{ and } B_1 - B_2 \geq 0 \\ \frac{B_1 |\tau|}{3(1 + 2\delta)} & B_1 > |B_2| \\ \frac{|B_2| |\tau|}{3(1 + 2\delta)} & B_1 < |B_2| \end{cases}.$$ 

Let us put $\varphi(z) = \left(\frac{1 + z}{1 - z}\right)^\alpha$, $B_1 = 2\alpha$ and $B_2 = 2\alpha^2$, and $\tau = 1$ in Corollary 2.9 we have

**Corollary 2.10.** [11, Theorem 2.2] Let $f \in \mathcal{H}_2(\alpha, \delta)$, then

$$|a_2| \leq \frac{2\alpha}{\sqrt{2(2 + \alpha) + 4\delta(\alpha + \delta - \alpha\delta + 2)}}.$$ 

$$|a_3| \leq \frac{2\alpha}{3(1 + 2\delta)}.$$
By putting $\tau = 1 - \gamma$ and $\varphi(z) = \frac{1 + z}{1 - z}$, $B_1 = B_2 = 2$, in Corollary 2.9, we obtain

**Corollary 2.11.** [11, Theorem 3.2] Let $f \in \mathcal{H}_\Sigma(\gamma, \delta)$, then

$$|a_2| \leq \frac{2(1 - \gamma)}{3(1 + 2\delta)},$$

$$|a_3| \leq \frac{2(1 - \gamma)}{3(1 + 2\delta)}.$$

In case of $\tau = 1$, $\delta = 0$ and $\varphi(z) = \frac{1 + z}{1 - z}$, $B_1 = B_2 = 2$, in Theorem 2.8, we have

**Corollary 2.12.** [10, Theorem 2.2] Let $f \in \mathcal{B}_\Sigma(\alpha, \lambda)$, then

$$|a_2| \leq \frac{2\alpha}{\sqrt{(1 + \lambda)^2 + \alpha(1 + 2\lambda - \lambda^2)}},$$

$$|a_3| \leq \frac{2\alpha}{1 + 2\lambda}.$$

Let us put $\tau = 1 - \gamma$ and $\varphi(z) = \frac{1 + z}{1 - z}$, $B_1 = B_2 = 2$, in Theorem 2.8, we obtain

**Corollary 2.13.** [6, Theorem 5] Let $0 \leq \alpha < 1$ and $f \in \mathcal{N}_\Sigma(\gamma, \lambda, \delta)$, then

$$|a_2| \leq \sqrt{\frac{2(1 - \gamma)}{1 + 2\lambda + 6\delta}},$$

$$|a_3| \leq \frac{2(1 - \gamma)}{1 + 2\lambda + 6\delta},$$

and

$$|a_3 - 2a_2^2| \leq \frac{2(1 - \gamma)}{1 + 2\lambda + 6\delta}.$$

By putting $\delta = 0$ in Corollary 2.13, gets

**Corollary 2.14.** [10, Theorem 3.2] If $f$ belong to $\mathcal{B}_\Sigma(\gamma, \lambda)$ and $0 \leq \gamma < 1$, then

$$|a_2| \leq \sqrt{\frac{2(1 - \gamma)}{1 + 2\lambda}},$$

$$|a_3| \leq \frac{2(1 - \gamma)}{1 + 2\lambda}.$$

**Remark 2.** Some results investigated in Corollaries from 2.9 to 2.14 represented an improvement of the estimate of $|a_3|$ of the earlier corresponding results.

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