THE DETERMINED PROPERTY OF BAIRE IN REVERSE MATH

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Abstract. We define the notion of a determined Borel code in reverse math, and consider the principle DPB, which states that every determined Borel set has the property of Baire. We show that this principle is strictly weaker than ATR₀. Any ω-model of DPB must be closed under hyperarithmetic reduction, but DPB is not a theory of hyperarithmetic analysis. We show that whenever $M \subseteq 2^\omega$ is the second-order part of an ω-model of DPB, then for every $Z \in M$, there is a $G \in M$ such that $G$ is $\Delta^1_1$-generic relative to $Z$.

The program of reverse math aims to quantify the strength of the various axioms and theorems of ordinary mathematics by assuming only a weak base theory (RCA₀) and then determining which axioms and theorems can prove which others over that weak base. Five robust systems emerged, (in order of strength, RCA₀, WKL, ACA₀, ATR₀, Π¹₁-CA) with most theorems of ordinary mathematics being equivalent to one of these five (earning this group the moniker “the big five”). The standard reference is [Sim09]. In recent decades, most work in reverse math has focused on the theorems that do not belong to the big five but are in the vicinity of ACA₀. Here we discuss two principles which are outside of the big five and located in the general vicinity of ATR₀: the property of Baire for determined Borel sets (DPB) and the Borel dual Ramsey theorem for 3 partitions and ℓ colors (Borel-DRT³). Both principles involve Borel sets.

Our motivation is to make it possible to give a meaningful reverse math analysis of theorems whose statements involve Borel sets. The way that Borel sets are usually defined in reverse math forces many theorems that even mention a Borel set to imply ATR₀, in an unsatisfactory sense made precise in [DFSW17]. Here we propose another definition for a Borel set in reverse math, distinguished from the original by the terminology determined Borel set, and to put bounds on the strength of the statement

DPB : “Every determined Borel set has the property of Baire”

This statement should be compared with the usual “Every Borel set has the property of Baire”, which [DFSW17] showed is equivalent to ATR₀ for aforementioned empty reasons. In contrast, working with DPB requires working with hyperarithmetic generics, giving this theorem more thematic content. While we do not claim that DPB is the “right” formalization of the principle that Borel sets have the Baire property, it is a step in that direction.

We show that over RCA₀, DPB is implied by ATR₀ and implies $L_{\omega_1,\omega^-}$-CA. Our first main theorems say that both implications are strict.

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Figure 1. DPB, ATR$_0$, and some theories of hyperarithmetic analysis. The new results are those concerning DPB. A double arrow indicates a strict implication.

**Theorem 0.1.** There is an $\omega$-model of DPB in which ATR$_0$ fails.

**Theorem 0.2.** There is an $\omega$-model of $L_{\omega_1,\omega}$-CA in which DPB fails. In fact, HY P is such an $\omega$-model.

This establishes that DPB is located in the general vicinity of the theories of hyperarithmetic analysis, a mostly linearly ordered collection of logical principles which are strong enough to support hyperarithmetic reduction, but too weak to imply the existence of jump hierarchies. With the exception of Jullien’s indecomposability theorem [Mon06], no theorems of ordinary mathematics are known to exist in this space. And the only known statement of hyperarithmetic analysis that is not linearly ordered with the others is the arithmetic Bolzano-Weierstrass theorem (see [Fri75], [Con12]). Now, DPB is not a theory of hyperarithmetic analysis because it does not hold in HY P. But these theories of hyperarithmetic analysis are the closest principles to DPB that have already been studied.

To elaborate on the factors preventing DPB from being a theory of hyperarithmetic analysis, we prove the following generalization of Theorem 0.2 above, establishing that hyperarithmetic generics must appear in any $\omega$-model of DPB.

**Theorem 0.3.** If $M$ is an $\omega$-model of DPB, then for any $Z \in M$, there is a $G \in M$ that is $\Delta^1_1(Z)$-generic.

As an application, we use DPB to analyze the theorem Borel-DRT$_\ell^3$, whose statement contains no concept of mathematical logic apart from that of Borel sets. (The statement of this theorem can be found in Section 7.) We show that, under appropriate formalization, Borel-DRT$_\ell^3$ is strictly weaker than ATR$_0$ and shares some properties with the theories of hyperarithmetic analysis. It is left open whether Borel-DRT$_\ell^3$ is a statement of hyperarithmetic analysis.

**Theorem 0.4.** For any finite $\ell \geq 2$, the principle Borel-DRT$_\ell^3$ is strictly implied by ATR$_0$. Any $\omega$-model of Borel-DRT$_\ell^3$ is closed under hyperarithmetic reduction.
The first section gives the preliminaries. In Section 2 we give the definition of a determined Borel code and prove its basic properties. In Section 3 we construct an \( \omega \)-model to separate \( \text{DPB} \) from \( \text{ATR}_0 \). In Section 4 we develop the machinery of decorating trees which will be used in Sections 5 and 6. In Section 5, we prove that \( \text{DPB} \) does not hold in \( \text{HYP} \). In Section 6, we prove Theorem 1.3. This is a strictly stronger theorem than the one proved in Section 5, but also a bit longer to prove, so Section 5 could be regarded as a warm-up. In Section 7 we discuss the implications of all this for the Borel dual Ramsey theorem. Section 8 has the open questions.

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1. Preliminaries

1.1. Notation, Borel sets and Borel codes. We typically denote elements of \( \omega^{<\omega} \) by \( \sigma, \tau \) and elements of \( 2^{<\omega} \) by \( p, q \). We write \( p \preceq q \) to indicate that \( p \) is an initial segment of \( q \), with \( \prec \) if \( p \neq q \). The empty string is denoted by \( \lambda \). A string with a single component of value \( n \in \omega \) is denoted by \( \langle n \rangle \). String concatenation is denoted by \( \sigma \triangleright \tau \). Usually we write \( \sigma \triangleright n \) instead of the more technically correct but uglier \( \sigma \triangleright \langle n \rangle \).

If \( U \) is a set of strings (for example, a tree, or a coded open subset of \( 2^\omega \)), and \( \sigma \) is any string, we write \( \sigma \triangleright U \) to mean \( \{ \sigma \triangleright \tau \mid \tau \in U \} \). If \( T \) is a tree and \( \sigma \in T \), we write \( T_\sigma \) to mean \( \{ \tau \mid \sigma \triangleright \tau \in T \} \).

The Borel subsets of a topological space are the smallest collection which contains the open sets and is closed under complements and countable unions (and thus countable intersections).

A Borel code is a well-founded tree \( T \subseteq \omega^{<\omega} \) whose leaves are labeled by basic open sets or their complements, and whose inner nodes are labeled by \( \cup \) or \( \cap \). The Borel set associated to a Borel code is defined by induction, interpreting the labels in the obvious way. Any Borel set can be represented this way, by applying DeMorgan’s laws to push any complementation out to the leaves.

1.2. Ordinal notations, alternating and ranked trees. We assume the reader is familiar with ordinal notations and overflow arguments. A standard reference is [Sac90a]. Recall that there is a computable procedure \( p \) which, on input \( a \in \mathcal{O} \), gives an index for \( W_p(a) = \{ b \in \mathcal{O} : b <_\mathcal{O} a \} \). By overflow, \( W_p(a) \) also produces reasonable-looking stuff for many \( a \notin \mathcal{O} \).

**Definition 1.1.** For arbitrary \( a, b \in \mathbb{N} \), define \( b <_a a \) to mean \( b \in W_{p(a)} \). Let \( \mathcal{O}^* \subseteq \mathbb{N} \) be defined by

\[
\mathcal{O}^* = \{ a \in \mathbb{N} : 1 \in W_{p(a)}, W_{p(a)} \text{ is linearly ordered by } <_a \text{ and } \text{W}_{p(a)} \text{ has no hyperarithmetic } <_a\text{-descending sequences} \}.
\]

If \( a^* \in \mathcal{O}^* \setminus \mathcal{O} \), then \( W_{p(a^*)} \) has an initial segment \( \{ b \in \mathcal{O} : b <_a a^* \} \) of length \( \omega_1^{ck} \).

According to the conventions governing ordinal notations, if \( b \in \mathcal{O} \), the successor of \( b \) is \( 2^b \). This is cumbersome when adding a finite number to the ordinal represented by \( b \), so instead we write \( b + k \) to mean take the \( k \)th successor of \( b \). It does not make sense to ever add ordinals using the ordinary addition on \( \mathbb{N} \), so this...
should not create confusion. Also, sometimes we will take a fixed but unspecified number of successors of \( b \), and the result is denoted \( b + O(1) \).

The following definition of a ranking for a tree is looser than given by some authors. We only require that the notations decrease, rather than the strong requirement that \( \rho(\sigma) = \sup_n (\rho(\sigma^\frown n) + 1) \). Additionally, it is technically convenient for us to assume that leaves have the smallest possible rank, but nothing serious hinges on this.

**Definition 1.2.** If \( T \subseteq \omega^{<\omega} \) is any tree, and \( \rho : T \to \mathcal{O}_* \), we say that \( \rho \) ranks \( T \) if

1. for all \( \sigma \) and \( n \) such that \( \sigma^\frown n \in T \), we have \( \rho(\sigma^\frown n) < \rho(\sigma) \), and
2. for each leaf \( \sigma \in T \), \( \rho(\sigma) = 1 \).

If \( T \) is ranked by \( \rho \) and \( \rho(\lambda) = a \), we say that \( T \) is a\( a \)-ranked by \( \rho \).

If \( T \) is a ranked tree and the name of the ranking function is not explicitly given, then its name is \( \rho_T \).

Trees appear for us in two contexts: as codes for formulas of \( L_{\omega_1, \omega} \) and codes for Borel sets. In both cases, interior nodes are labeled with one of \( \{\cap, \cup\} \). The nicest codes alternate these.

**Definition 1.3.** If \( T \subseteq \omega^{<\omega} \) is a tree with a labeling function \( \ell \) then we say \((T, \ell)\) alternates if for every \( \sigma^\frown n \in T \), we have \( \ell(\sigma) \neq \ell(\sigma^\frown n) \).

The main point about alternating trees is that it is always safe to assume that we have them. If we start with a labeled, \( a \)-ranked tree, we can effectively transform it into an alternating \( a \)-ranked tree, with no effect on the logic of the tree (assuming that whatever model we are working in does not contain any paths, if the tree is truly ill-founded.)

### 1.3. Borel sets in reverse math

In reverse math, open subsets of \( 2^\omega \) are represented by sets of strings \( p \in 2^{<\omega} \). If \( U \) is such a code, we will abuse notation and write \( X \in U \) to mean that for some \( p \in U \), \( p \prec X \). This is in addition to also sometimes speaking of the strings \( p \in U \). Context will tell which usage is meant.

For arbitrary Borel sets, we will make a more careful distinction between code and object. We restrict attention to Borel subsets of \( 2^\omega \). A clopen subset \( C \) of \( 2^\omega \) is represented by an element of \( \omega \) which canonically codes a finite subset \( F \subseteq 2^{<\omega} \). As above, for \( X \in 2^\omega \), we say \( X \in C \) if and only if \( p \prec X \) for some \( p \in F \). A code for \( C \) as a clopen set gives more information about \( C \) than an open code for the same set, because the number of elements of \( F \) is computable from the code. Effectively in a standard code for a clopen set, one can find a standard code for its complement.

We take the following as the definition of a (labeled) Borel code in reverse math.

**Definition 1.4.** A labeled Borel code is a well-founded tree \( T \subseteq \omega^{<\omega} \), together with a function \( \ell \) whose domain is \( T \), such that if \( \sigma \) is an interior node, \( \ell(\sigma) \) is either \( \cup \) or \( \cap \), and if \( \sigma \) is a leaf, \( \ell(\sigma) \) is a standard code for a clopen subset of \( 2^\omega \).

We call this a labeled Borel code instead of a Borel code, because we have added a labeling function to the original definition to improve readability.\footnote{The original definition of a Borel code in reverse math \cite{Sim09} is a well-founded tree \( T \) such that for exactly one \( m \in \omega \), \( \langle m \rangle \in T \). Some conventions are then adopted: if \( \langle m \rangle \in T \) is a leaf, then \( T \) represents a clopen set coded by \( m \) according to a standard computable look-up; if \( \langle m \rangle \) is not a leaf, then \( T \) represents a union}
we may simply say “σ is a union node”, and similarly for ∩. We will also usually suppress mention of ℓ, in an abuse of notation.

If \( T \) is a labeled Borel code and \( X \in 2^\omega \), the existence of an evaluation map is used to determine whether \( X \) is in the set coded by \( T \).

**Definition 1.5.** If \( T \) is a labeled Borel code and \( X \in 2^\omega \), an evaluation map for \( X \in T \) is a function \( f : T \rightarrow \{0,1\} \) such that

- If \( \sigma \) is a leaf, \( f(\sigma) = 1 \) if and only if \( X \) is in the clopen set coded by \( \ell(\sigma) \).
- If \( \sigma \) is a union node, \( f(\sigma) = 1 \) if and only if \( f(\sigma^\wedge n) = 1 \) for some \( n \in \omega \).
- If \( \sigma \) is an intersection node, \( f(\sigma) = 1 \) if and only if \( f(\sigma^\wedge n) = 1 \) for all \( n \in \omega \).

We say that \( X \) is in the set coded by \( T \), denoted \( X \in |T| \), if there is an evaluation map \( f \) for \( X \) in \( T \) such that \( f(\lambda) = 1 \).

Note that \( X \in |T| \) is a \( \Sigma^1_1 \) statement. Because evaluation maps are naturally constructed by arithmetic transfinite recursion, \( \text{ATR}_0 \) proves that if \( T \) is a Borel code and \( X \in 2^\omega \), there is an evaluation map \( f \) for \( X \) in \( T \). Furthermore, \( \text{ACA}_0 \) proves that if an evaluation map exists, then it is unique.

Because we are considering these definitions in the context of reverse math, there will sometimes be an ill-founded \( T \) which a model thinks is well-founded. In these cases, the statement \( X \in |T| \) is meaningful inside the model, or in the context of a proof inside second order arithmetic, but is not meaningful outside a model. However, the criteria defining what it means to be an evaluation map are absolute, so we can and will construct evaluation maps on ill-founded but otherwise coherent labeled Borel codes. If \( T \) is ill-founded, we will never use the notation \(|T|\) outside of a model. But if \( T \) is well-founded, then every \( X \) has a unique evaluation map in \( T \). In that case we give the notation “\(|T|\)” the obvious meaning of

\[ \{X : \text{the unique evaluation map } f \text{ for } X \text{ in } T \text{ satisfies } f(\lambda) = 1\} \]

when we refer to it outside the context of a model.

Operations on Borel sets are carried out easily. Observe that the operation which corresponds to complementation on a labeled Borel code is primitive recursive: just swap all the ∪ and ∩ labels, and replace every clopen leaf label with its complementary label.

**Definition 1.6.** If \((T, \ell)\) is a labeled Borel code, let \((T, \ell^c)\) denote the labeled Borel code whose tree is the same, and whose labeling \(\ell^c\) is complementary to \(\ell\) as described above.

Continuing the abuse of notation, if \( T \) is used to refer to some \((T, \ell)\), then \( T^c \) will be shorthand for \((T, \ell^c)\). Observe that \( \text{RCA}_0 \) proves that if \( T \) is a labeled Borel code, then \( T^c \) is a labeled Borel code. Similarly, if \((T_n)_{n\in\omega}\) is a sequence of labeled Borel codes, in \( \text{RCA}_0 \) we can construct a code for the intersection or union of these sets in the obvious effective way, and \( \text{RCA}_0 \) will prove that the result is a labeled Borel code.

or intersection according to the parity of \( m \), and the sets to be thus combined are those coded by the subtrees \( T_n = \{\langle n \rangle \wedge \sigma : \langle m,n \rangle \wedge \sigma \in T\} \). Classically, one can translate easily between this definition and the definition of Borel code given above, but one direction of the translation requires \( \text{ACA}_0 \) because one cannot effectively determine when a node is a leaf. All the principles considered in this paper will imply \( \text{ACA}_0 \) over \( \text{RCA}_0 \), so nothing will be muddled, but for the sake of fastidious readers, we will always call these labeled Borel codes to acknowledge the distinction.
1.4. On the maxim that “Borel sets need ATR$_0$”. Because making meaning out of a standard (labeled) Borel code requires evaluation maps to be around, ATR$_0$ is typically taken as the base theory when evaluating theorems involving Borel sets. Even when ATR$_0$ is not taken as the base theory, theorems involving Borel sets tend to imply ATR$_0$. The probable reason for this was observed in [DFSW17].

Theorem 1.7 ([DFSW17]). In RCA$_0$, the statement “For every Borel code $T$, there exists $X$ such that $X \in |T|$ or $X \in |T^c|$” implies ATR$_0$.

The strength comes from the fact that this statement is asserting the existence of an evaluation map for $X$ in $T$. If $f$ is an evaluation map for $X$ in $T$, then $1 - f$ is an evaluation map for $X$ in $T^c$.

Restatement (of Theorem 1.7). The statement “For every Borel set, either it or its complement is nonempty” is equivalent to ATR$_0$ over RCA$_0$.

This can make the reverse math of some standard theorems about Borel sets feel rather empty. Here is an example. Recall that a set $A \subseteq 2^\omega$ has the property of Baire if it differs from an open set by a meager set. That is, there are open sets $U$ and $\{D_n\}_{n \in \omega}$ such that $U \cap V = \emptyset$ and the sets $U \cup V$ and $D_n$ are dense.

A basic proposition is that every Borel set has the property of Baire, but what is the strength of that proposition in reverse math? In [DFSW17], the relevant notions were formalized as follows.

Definition 1.8. A Baire code is a collection of open sets $U, V, \{D_n\}_{n \in \omega}$ such that $U \cap V = \emptyset$ and the sets $U \cup V$ and $D_n$ are dense.

The statement $BP$ below formalizes the proposition “Every Borel set has the property of Baire.”

Definition 1.9. If $T$ is a Borel code and $U, V, \{D_n\}$ is a Baire code, we say that $U, V, \{D_n\}$ is a Baire approximation to $T$ if for all $X \in \cap_n D_n$, $X \in U \Rightarrow X \in |T|$ and $X \in V \Rightarrow X \in |T^c|$.

Definition 1.10. Let $BP$ denote the statement “Every Borel code has a Baire approximation.”

Proposition 1.11. [DFSW17] In RCA$_0$, ATR$_0$ is equivalent to $BP$.

Proof. $(\Rightarrow)$ The standard proof uses arithmetic transfinite recursion.

$(\Leftarrow)$ If a set has the property of Baire, either it or its complement is nonempty. □

The reverse direction of this proof is highly unsatisfactory. The purpose of this paper is to propose a variant on the definition of a Borel set which avoids this and similar unsatisfactory reversals to ATR$_0$.

1.5. Some landmarks between ATR$_0$ and $\text{JI}$. We will end up placing a variant of $BP$ somewhere in a zoo which exists just below ATR$_0$. Much of what is known about this region concerns theories, such as $\Delta^1_1$-CA$_0$, whose $\omega$-models are closed under join, hyperarithmetic reduction, and not much more.

Definition 1.12. A statement of hyperarithmetic analysis is any statement $S$ such that

$\text{[DFSW17]}$ is for original Borel codes, but the proof of the theorem remains valid for labeled Borel codes.
(1) whenever $M$ is an $\omega$-model which satisfies $S$, its second-order part $M$ is closed under hyperarithmetic reduction.
(2) For every $Y$, $\text{HYP}(Y)$ is the second-order part of an $\omega$-model of $S$, where $\text{HYP}(Y) = \{ X : X \leq_h Y \}$.

A theory of hyperarithmetic analysis is any theory which satisfies the same requirements as above.

It would be tempting to hope that there would be some theory of hyperarithmetic analysis whose $\omega$-models are exactly the Turing ideals which are closed under hyperarithmetic reduction, in analogy to the theorems characterizing the $\omega$-models of $\text{RCA}_0$ as the Turing ideals, the $\omega$-models of $\text{WKL}_0$ as the Scott ideals, and the $\omega$-models of $\text{ACA}_0$ as the Turing ideals closed under arithmetic reduction. However, no such theory can exist.

Theorem 1.13. [VW77] For every theory $T$, all of whose $\omega$-models are closed under hyperarithmetic reduction, there is a strictly weaker theory $T'$, all of whose $\omega$-models are also closed under hyperarithmetic reduction, and which has more $\omega$-models than $T$.

Therefore, we are stuck with an infinitely descending zoo of statements/theorems of hyperarithmetic analysis.

One theory of hyperarithmetic analysis is most relevant to us. Recall that a formula of $L_{\omega_1,\omega}$ is a formula constructed from the usual building blocks of first-order logic, together with countably infinite conjunctions and disjunctions. In a language which contains no atomic formulas other than true and false, a formula of $L_{\omega_1,\omega}$ is just a well-founded tree whose interior nodes are labeled with either $\cup$ (infinite disjunction) or $\cap$ (infinite conjunction), and whose leaves are labeled with either true or false. An evaluation map for a formula of $L_{\omega_1,\omega}$ is defined the same as an evaluation map for an element $X$ in a Borel code $T$, except that the evaluation map must satisfy $f(\sigma) = 1$ if $\ell(\sigma) = \text{true}$ and $f(\sigma) = 0$ if $\ell(\sigma) = \text{false}$. A formula of $L_{\omega_1,\omega}$ is determined if it has an evaluation map. Classically, every formula of $L_{\omega_1,\omega}$ is determined, but in weaker theories the witnessing function could fail to exist. A formula is called true if it has a witnessing function which maps the formula itself to true.

The following definition and result essentially appear in [Mon06], where $L_{\omega_1,\omega}$-CA goes by the name CDG-CA, and is stated in terms of games. The name $L_{\omega_1,\omega}$-CA and the definition given here were introduced in [Mon09].

Definition 1.14 (similar to [Mon06]). The principle $L_{\omega_1,\omega}$-CA is this statement: If $\{ \phi_i : i \in \mathbb{N} \}$ is a sequence of determined $L_{\omega_1,\omega}$ formulas, then the set $X = \{ i : \phi_i \text{ is true} \}$ exists.

Theorem 1.15 (essentially [Mon06]). The principle $L_{\omega_1,\omega}$-CA is a statement of hyperarithmetic analysis.

2. Determined Borel codes

We propose the following variation on the definition of a Borel code. We shall see that when this variant is used, the unsatisfactory shortcut in Proposition 1.11 vanishes, and indeed the reversal no longer holds.
Definition 2.1. A labeled Borel code $T$ is called determined if every $X \in 2^\omega$ has an evaluation map in $T$. A determined Borel code is a labeled Borel code that is determined.

When we formalize statements in reverse math, in order to not conflict with existing convention, we will say determined Borel set to indicate when the formalized version of the statement should call for a determined Borel code.

The following facts are immediate.

Proposition 2.2. In $\text{RCA}_0$,

1. If $T$ is a determined Borel code, then $T^c$ is also a determined Borel code.
2. For every determined Borel set $A$ and $X \in 2^\omega$, either $X \in A$ or $X \not\in A$.

With only a slight amount of effort, we also have the following.

Proposition 2.3. In $\text{RCA}_0$, if $A$ is a determined Borel set and $h : 2^\omega \to 2^\omega$ is continuous, then $h^{-1}(A)$ is a determined Borel set.

Proof. Let $T$ be a determined Borel code and $h : 2^\omega \to 2^\omega$ a continuous function. Define $S$ by starting with $S = T$ and modifying each leaf $\sigma \in T$ as follows:

1. In $S$, $\sigma$ is a union.
2. For each $n$, $\sigma \wedge n \in S$ and is a leaf.
3. If $U$ is the clopen set attached to $\sigma$ in $T$, let $\sigma \wedge n$ be labeled with a code for the clopen subset of $h^{-1}(U)$ inferable from the first $n$ bits of partial information about $h$.

We claim that $S$ is determined and $X \in |S|$ if and only if $h(X) \in |T|$. Let $f$ be an evaluation map for $h(X)$ in $T$. We claim that $f$ can be extended to an evaluation map for $X$ in $S$ by adding $f(\sigma \wedge n) = 1$ if and only if $X$ is in the clopen set attached to $\sigma \wedge n$ in $S$. One only needs to check that the logic of the evaluation map is correct at each $\sigma$ which was a leaf in $T$. □

The fact that Borel sets are closed under countable union, which was trivial using the standard definition of a Borel set, has quite some power for determined Borel sets.

Proposition 2.4. In $\text{RCA}_0$, the statement “A countable union of determined Borel sets is a determined Borel set” is equivalent to $\text{L}_{\omega_1,\omega}-\text{CA}$.

Proof. If $\{T^k : k \in \mathbb{N}\}$ are determined Borel codes, and $T = \{\lambda\} \cup \{(k) \wedge \sigma : \sigma \in T^k\}$, we claim that, assuming $\text{L}_{\omega_1,\omega}-\text{CA}$, $T$ is determined. Fixing $X$, let $\phi_{k,\sigma}$ be the formula obtained by replacing each clopen set at each leaf of $T^k$ by true or false according to whether $X$ is in each clopen set. Any evaluation map for $X$ in $T^k$ can be restricted to an evaluation map for $X$ in $T^k$, which is an evaluation map for $\phi_{k,\sigma}$, so all these formulas are determined. One obtains an evaluation map for $X$ in $T$ by letting $f(\sigma) = 1$ if and only if $\phi_{k,\sigma}$ is true, and the non-uniformly filling in $f(\lambda)$ to its unique correct value.

Conversely, if $\{\phi_k : k \in \mathbb{N}\}$ are determined, these formulas can be modified at the leaves to become determined Borel codes $T^k$ for $\emptyset$ or $2^\omega$ according to whether they are true or false. Defining $T$ as above, any evaluation map $f$ for $T$ satisfies $f((k)) = 1$ if and only if $\phi_k$ is true. □

Now we consider the determined variant of $\text{BP}$. 
Definition 2.5. Let DPB be the statement “Every determined Borel set has the property of Baire.”

Our main question is: what is the reverse math strength of DPB?

Proposition 2.6. In RCA₀, DPB implies L_{ω₁, ω}-CA.

Proof. Any sequence \( \{φ_k : k ∈ N\} \) of determined formulas of \( L_{ω₁, ω} \) can be modified at the leaves to produce a sequence of determined Borel codes which code either \([0^k1]\) or \(∅\) depending on whether \(φ_k\) is true or false. The union of these remains determined because each \(X\) passes through at most one of these sets. Any Baire approximation to \(∪_{k : φ_k \text{ is true}} [0^k1]\) computes \(\{k : φ_k \text{ is true}\}\). □

This places DPB somewhere in the general area of ATR₀ and the theories of hyperarithmetic analysis. If DPB were equivalent to \(L_{ω₁, ω}-CA\), our variant would be subject to the same kinds of critique that we made of the original definition (all the strength of the theorem coming essentially from Proposition 2.4). However, it turns out DPB is equivalent to none of the principles mentioned so far.

When considering how to show that DPB is strictly weaker than ATR₀, it is informative to consider the usual proof that every Borel set has the property of Baire. This proof uses arithmetic transfinite recursion on the Borel code of the given set to construct not only a Baire code for given set, but along the way also it constructs, in a uniform way, Baire codes for all Borel sets used to build up the given one (these ancillary sets are coded by \(T_{σ} := \{τ : σ⌢τ ∈ T\} \) for each \(σ ∈ T\)). Below, we give the name Baire decomposition to this extended object that ATR₀ would have created. Superficially, DPB would seem weaker than the statement “every determined Borel set has a Baire decomposition”, and one might wonder whether the additional information in the Baire decomposition carries any extra strength. The purpose of the rest of this section is to show that it does not (Proposition 2.8), and to mention exactly how a Baire approximation is constructively obtained from a Baire decomposition (Proposition 2.9).

The point is that any model separating DPB from ATR₀ will need another method of producing an entire Baire decomposition, not just the Baire approximation.

Definition 2.7. Let \(T\) be a determined Borel code. A Baire decomposition for \(T\) is a collection of open sets \(U_σ\) and \(V_σ\) for \(σ ∈ T\) such that for each \(σ ∈ T\) and each \(p ∈ 2^{<ω}\),

1. \(U_σ ∪ V_σ\) is dense and \(U_σ ∩ V_σ = ∅\),
2. if \(σ\) is a leaf, then \(U_σ = \text{the clopen set coded by } σ\) and \(V_σ = \overline{U_σ}\),
3. if \(σ\) is a union node, then \(U_σ\) is dense in \(∪_n U_σ⌢n\) and \(∪_n U_σ⌢n\) is dense in \(U_σ\),
4. if \(σ\) is an intersection node, then \(V_σ\) is dense in \(∪_n V_σ⌢n\) and \(∪_n V_σ⌢n\) is dense in \(V_σ\).

Proposition 2.8 (ACA₀). DPB implies that every determined Borel set has a Baire decomposition.

Proof. Let \(T\) be a determined Borel code. Informally, we partition the space into countably many disjoint clopen pieces (plus one limit point) and put an isomorphic copy of the set coded by \(T_σ\) in the \(σ\)-th piece. Then we show that a Baire approximation to this disintegrated set can be translated back to a Baire decomposition for the original set coded by \(T\).
More formally, for any \( p \in 2^{<\omega} \), let \( T[p] \) denote the labeled Borel code for \( \{ p^\prec X : X \in |T| \} \). This is an effective operation on codes. Recall that each leaf codes a clopen set by a finite list \( F \subseteq 2^{<\omega} \). By replacing each such \( F \) with \( \{ p^\prec q : q \in F \} \), we achieve the desired effect.

For any \( \sigma \in \omega^{<\omega} \), let \([\sigma]\) be a natural number which codes \( \sigma \) in a canonical way. Define \( S \) to be the labeled Borel code

\[
S = \{ \lambda \} \cup \{ [\sigma]^\tau : \tau \in T_\sigma[0[^\sigma]1], \sigma \in T \}
\]

where \( \lambda \) is a \( \cup \) and all other labels are inherited from the \( T_\sigma[0[^\sigma]1] \). Then \( S \) is determined: for any \( X \), if \( X = 0^\omega \), then the identically zero map is an evaluation map for \( X \); if \( X = 0^n 1^\omega \), then if \( f \) is an evaluation map for \( Y \) in \( T \) and \( n = [\sigma] \), an evaluation map \( g \) for \( X \) in \( S \) can be defined by letting \( g([\sigma]^\tau) = f(\sigma^\tau) \) on

\[
[ [\sigma]^\tau : \tau \in T_\sigma[0[^\sigma]1] ]
\]

\( g(\lambda) = f(\sigma) \), and \( g \) identically zero elsewhere. Therefore, for all \( Y \) and \( \sigma \),

\[
0[^\sigma]1^\omega Y \in |S| \iff Y \in |T_\sigma|.
\]

Now suppose that \( (U, V, \{ D_k \}_{k \in \omega}) \) is a Baire approximation for \( S \). Then define \( U_\sigma = \{ q : 0[^\sigma]1^\omega q \in U \} \) and \( V_\sigma = \{ q : 0[^\sigma]1^\omega q \in V \} \). Properties (1) and (2) of a Baire decomposition are clear. For property (3), suppose that \( \cup_k U_{\sigma \wedge n} \) is not dense in \( [p] \). Let \( q \) extend \( p \) such that for all \( n \), \( U_{\sigma \wedge n} \cap |q| = \emptyset \). Then define \( Y \) so that \( q \prec Y \) and the following collection of comeager events occur:

- (i) For all \( n \), \( Y \in V_{\sigma \wedge n} \)
- (ii) For all \( n \), \( 0[^\sigma]1^\omega Y \in \cap_k D_k \)
- (iii) \( Y \in U_\sigma \cup V_\sigma \)
- (iv) \( 0[^\sigma]1^\omega Y \in \cap_k D_k \)

The first comeager event guarantees that \( Y \in V_{\sigma \wedge n} \) for all \( n \). Together with second comeager event this implies that \( 0[^\sigma]1^\omega Y \not\in |S| \), and therefore \( Y \not\in |T_{\sigma \wedge n}| \). Therefore, \( Y \not\in |T_\sigma| \). In the third dense event, if we had \( Y \in U_\sigma \), the fourth comeager event would imply that \( Y \in |T_\sigma| \); therefore it must be that \( Y \in V_\sigma \), and so \( U_\sigma \) is not dense in \( [p] \). On the other hand, if \( U_\sigma \) is not dense in \( [p] \), then assuming \( \cup_k U_{\sigma \wedge n} \) is dense in \( [p] \) leads to a contradiction, for we may similarly define \( Y \) to meet \( V_\sigma \cap |p| \) and \( \cup_k U_{\sigma \wedge n} \) to meet \( V_\sigma \cap |p| \), while also satisfying (iii) and (iv).

The proof of (4) is similar to the proof of (3).

Turning a Baire decomposition into a Baire approximation involves extracting the comeager set on which the approximation should hold. The following proposition gives a canonical sequence of dense open sets which suffices for this.

**Proposition 2.9 (ACA₀).** Let \( T \) be a determined Borel code and \( (U_\sigma, V_\sigma)_{\sigma \in T} \) be a Baire decomposition for \( T \). Let \( \{ D_n \}_{n \in \omega} \) consist of the following dense open sets:

1. \( U_\sigma \cup V_\sigma \) for \( \sigma \in T \),
2. \( V_\sigma \cup \bigcup_{n} U_{\sigma \wedge n} \) for \( \sigma \in T \) a union node, and
3. \( U_\sigma \cup \bigcup_{n} V_{\sigma \wedge n} \) for \( \sigma \in T \) an intersection node.

Then, \( (U_\lambda, V_\lambda, \{ D_n \}_{n \in \omega}) \) is a Baire approximation for \( T \).

**Proof.** The properties of a Baire decomposition suffice to ensure that \( (U_\lambda, V_\lambda, \{ D_n \}_{n \in \omega}) \) is a Baire code. We must show that if \( X \in \cap_n D_n \), then \( X \in U_\lambda \iff X \in |T| \) and \( X \in V_\lambda \iff X \not\in |T| \). Fix such \( X \). We prove by arithmetic transfinite induction
that for all $\sigma \in T$, if $X \in U_\sigma$ then $X \in |T_\sigma|$ and if $X \in V_\sigma$ then $X \notin |T_\sigma|$. This holds when $\sigma$ is a leaf.

If $\sigma$ is a union node, suppose $X \in U_\sigma$. Then $X \notin V_\sigma$, but $X \in V_\sigma \cup \bigcup_n U_{\sigma \cap n}$, so $X \in U_{\sigma \cap n}$ for some $n$. Then the induction hypothesis gives us $X \in |T_{\sigma \cap n}|$, so $X \in |T_\sigma|$ since $\sigma$ is a union.

On the other hand, if $X \in V_\sigma$, let $p \prec X$ with $p \in V_\sigma$. Then $U_\sigma \cap [p] = \emptyset$, so $\bigcup_n U_{\sigma \cap n} \cap [p] = \emptyset$. So for each $n$, $V_{\sigma \cap n}$ is dense in $[p]$. Therefore, $X$ meets each $V_{\sigma \cap n}$, so by induction $X \notin |T_{\sigma \cap n}|$ holds for all $n$. Therefore, $X \notin |T_\sigma|$.

The case where $\sigma$ is an intersection node is similar. \hfill $\Box$

3. DPB does not imply $\text{ATR}_0$

Our non-$\text{ATR}_0$ method of producing a Baire decomposition involves polling sufficiently generic $X$ to see whether they are in or out of a given set.

To say just how generic, we assume a general familiarity with hyperarithmetic theory, and refer the reader to [Sac90a] for definitions and details. For $G \in 2^\omega$, it is well-known that an element $X$ of $2^\omega$ is $\Delta_1^1(G)$ if and only if it is $\text{HYP}(G)$, if and only if there is some $b \in \mathcal{O}^G$ such that $X \leq_T H_b^G$.

Recall that if $\Gamma$ is a pointclass, $X \in 2^\omega$ is called $\Gamma$-generic if $X$ meets or avoids every open set $U$ with a code in $\Gamma$. We are interested in $\Delta_1^1$-generics $G$ with the additional property that $\omega_1^{ck} = \omega_1^G$. By [CM17], these are precisely the $\Sigma^1_1$-generics.

The following three propositions must be folklore, but we give their proofs here. Recall that $A$ and $B$ are relatively $\Gamma$-generic if $A$ is $\Gamma(B)$-generic and $B$ is $\Gamma(A)$-generic.

**Proposition 3.1.** For $G_0, G_1 \in 2^\omega$, we have $G_0 \oplus G_1$ is $\Sigma^1_1$-generic if and only if $G_0$ and $G_1$ are relatively $\Sigma^1_1$-generic.

**Proof.** Consider the argument in [DH10] Thm. 8.20.1 (originally due to [?]), where it is shown that $A \oplus B$ is $\alpha$-generic if and only if $A$ and $B$ are relatively $\alpha$-generic. Observe that at no point do they make use of the fact that $\alpha$ is finite, and the same argument goes through if $\alpha$ is replaced with any $a \in \mathcal{O}$. Therefore, if we define $a$-generic to mean $\Sigma^\alpha_0$-generic, the same argument shows that $A \oplus B$ is $a$-generic if and only if $A$ and $B$ are relatively $a$-generic. Observe that $A$ is $\Delta^1_1$-generic if and only if $A$ is $a$-generic for all $a \in \mathcal{O}$.

Now suppose that $G_0 \oplus G_1$ is $\Sigma^1_1$-generic. We will show that $G_0$ is $\Sigma^1_1(G_1)$-generic. We have $\omega_1^{G_0 \oplus G_1} = \omega_1^{G_1}$, so it suffices to show that $G_0$ is $\Delta^1_1(G_1)$-generic, or equivalently, that $G_0$ is $a$-generic relative to $G_1$ for all $a \in \mathcal{O}$ (here we use the fact that $\omega_1^{G_1} = \omega_1^{ck}$). This follows from the previous paragraph because $G_0 \oplus G_1$ is $a$-generic.

On the other hand, if $G_0$ and $G_1$ are relatively $\Sigma^1_1$-generic, then in particular each is $\Sigma^1_1$-generic, so $\omega_1^{G_0} = \omega_1^{G_1} = \omega_1^{ck}$, and by relative $\Sigma^1_1$-genericity, we also have $\omega_1^{G_0 \oplus G_1} = \omega_1^{ck}$. Therefore it suffices to show that $G_0 \oplus G_1$ is $\Delta^1_1$-generic, or equivalently, that it is $a$-generic for all $a \in \mathcal{O}$. This follows because $G_0$ and $G_1$ are relatively $a$-generic for all $a \in \mathcal{O}$.

**Proposition 3.2.** If $G_0 \oplus G_1$ is $\Sigma^1_1$-generic, then $\Delta^1_1(G_0) \cap \Delta^1_1(G_1) = \Delta^1_1$.\footnote{A $\Sigma^\alpha_0$ set or relation is one given by a computable $a$-ranked formula of $L_{\omega_1, \omega}$, where the leaves are labeled with computable formulas instead of just the symbols $\text{true}$ or $\text{false}$. An object belongs in such a set if the formula evaluates to $\text{true}$ when that object is given as input at all the leaves.}
Proof. If \( X \in \Delta^1_1(G_0) \cap \Delta^1_1(G_1) \), then since \( \omega_1^{G_0} = \omega_1^{G_1} = \omega_1^{c^k} \), there are \( a, b \in \mathcal{O} \) and indices \( e \) and \( f \) such that \( X = \phi_e(H_a^{G_0}) = \phi_f(H_b^{G_1}) \). Consider the set

\[
W = \{ Y \oplus Z : \phi_e(H_a^Y) = \phi_f(H_b^Z) \}.
\]

This set is \( \Delta^1_1 \), so it has the property of Baire, and in particular there is a \( \Delta^1_1 \) open set \( V \) such that every sufficiently generic \( Y \oplus Z \) is an element of \( V \) if and only if it is an element of \( W \). Here the amount of genericity needed is not full \( \Delta^1_1 \)-genericity, but rather \( c \)-genericity, where \( c \) is some element of \( \mathcal{O} \) that can be determined from \( a, b \) and the definition of \( W \). Since \( G_0 \oplus G_1 \) is \( \Delta^1_1 \)-generic and in \( W \), it is in \( V \). Let \( p, q \in 2^{<\omega} \) be such that \( p \prec G_0, q \prec G_1 \) and \( p \oplus q \in V \). Now let \( Y \) be any \( c \)-generic, hyperarithmetic real with \( p \prec Y \). Then since \( G_1 \) is \( \Delta^1_1 \)-generic, it is \( \Delta^1_1 \) generic relative to \( Y \), so in particular it is \( c \)-generic relative to \( Y \), so \( Y \oplus G_1 \) is \( \Delta^1_1 \)-generic, and meets \( V \). Therefore, \( Y \oplus G_1 \in W \), and we obtain a \( \Delta^1_1 \) formula for \( X \), that is, \( X = \phi_e(H_a^Y) \).

\[ \square \]

**Proposition 3.3.** Let \( G_0 \) be \( \Sigma^1_1 \)-generic and \( P \) a hyperarithmetic predicate. If there is a \( Y \in \Delta^1_1(G_0) \) such that \( P(Y) \) holds, then for all \( \Delta^1_1 \)-generic \( G_1 \), there is a \( Y \in \Delta^1_1(G_1) \) such that \( P(Y) \) holds.

**Proof.** Since \( \omega_1^{c^k} = \omega_1^{G_0} \), there is some \( a \in \mathcal{O} \) and an index \( e \) such that \( Y = \phi_e(H_a^{G_0}) \). Then \( R(X) := \exists \phi(H_a^X) \) is a hyperarithmetic predicate that holds of \( G_0 \), and holds of \( p \prec G_0 \) for any \( p \in 2^{<\omega} \). Therefore, for any \( \Delta^1_1 \)-generic \( G_1 \), \( R(G_1) \) holds.

\[ \square \]

Let \( G = \bigoplus_i G_i \) be a \( \Sigma^1_1 \) generic. Let \( \mathcal{M} = \bigcup_n \Delta^1_1(\bigoplus_{i<n} G_i) \). This is the model which will be used to separate DPB and ATR_0. But first, some lemmas.

**Lemma 3.4.** \( \mathcal{M} \models L_{\omega_1, \omega} \text{-CA} \). Furthermore, whenever \( F \subseteq \omega \) is finite and the determined sequence of formulas \( \{ \phi_k : k \in \mathbb{N} \} \) is in \( \Delta^1_1(\bigoplus_{i \in F} G_i) \), we also have

\[
\{ k : \phi_k \text{ is true in } M \} \in \Delta^1_1 \left( \bigoplus_{i \in F} G_i \right).
\]

**Proof.** We begin with three facts. First, applying Proposition 3.1 to the decomposition \( G = \bigoplus_{i \in F} G_i \oplus \bigoplus_{i \in F} G_i \), we conclude that \( \bigoplus_{i \notin F} G_i \) is \( \Sigma^1_1(\bigoplus_{i \in F} G_i) \)-generic.

Second, fix \( j \notin F \). Applying Proposition 3.1 to \( G = G_j \oplus \bigoplus_{i \neq j} G_i \), we have that \( G_j \) is \( \Sigma^1_1(\bigoplus_{i \neq j} G_i) \)-generic and hence \( G_j \) is \( \Sigma^1_1(\bigoplus_{i \in F} G_i) \)-generic.

Third, fix \( j_0, j_1 \notin F \) with \( j_0 \neq j_1 \). By the same argument, we have that \( G_{j_0} \) is \( \Sigma^1_1(\bigoplus_{i \neq j} G_i) \)-generic and that \( G_{j_1} \) is \( \Sigma^1_1(\bigoplus_{i \in F} G_i) \)-generic. By Proposition 3.2 relativized to \( \bigoplus_{i \in F} G_i \), it follows that \( \Delta^1_1(G_{j_0} \oplus \bigoplus_{i \notin F} G_i) \cap \Delta^1_1(G_{j_1} \oplus \bigoplus_{i \notin F} G_i) = \Delta^1_1(\bigoplus_{i \in F} G_i) \).

We now apply Proposition 3.3 relativized to \( \bigoplus_{i \in F} G_i \). Fix \( j \notin F \) and \( k \in \omega \). Since \( \bigoplus_{i \in F} G_i \) is \( \Sigma^1_1(\bigoplus_{i \in F} G_i) \)-generic, \( G_j \) is \( \Sigma^1_1(\bigoplus_{i \in F} G_i) \)-generic and there is a \( \Delta^1_1(G) \) evaluation map for \( \phi_k \), it follows that \( \phi_k \) is determined in \( \Delta^1_1(G \oplus \bigoplus_{i \in F} G_i) \). Because this holds for any \( j \notin F \), \( \phi_k \) is determined in \( \Delta^1_1(\bigoplus_{i \in F} G_i) \) by the third fact above. Since \( L_{\omega_1, \omega} \text{-CA} \) is a theory of hyperarithmetic analysis, the conclusion follows.

\[ \square \]

**Proposition 3.5.** \( \mathcal{M} \models \neg \text{ATR}_0 \).
Proof. Let \( a^* \in \mathcal{O}^* \). Then \( \mathcal{M} \) believes that \( a^* \) is an ordinal. For if there were a \( \Delta_1^1(G) \)-computable descending sequence in \( a^* \), then for some \( b \in \mathcal{O} \) (here we use the fact that \( \omega_1^{ck} = \omega_0^G \)) the statement \( R(X) : "H_X^b \) computes a descending sequence in \( a^*" \) is a hyperarithmetic predicate which holds of \( G \). As \( R \) holds of \( p \upharpoonright G \) for any \( p \in 2^\omega \), the set of \( X \) for which \( R \) holds is comeager. Furthermore, \( R(X) \) is of Borel class \( b + O(1) \), so \( R(X) \) holds for any \( X \) which is \( b + O(1) \)-generic. There is a hyperarithmetic such \( X \). But then \( H_X^b \) is also hyperarithmetic, contradicting that \( a^* \) has no hyperarithmetic descending sequence. So \( a^* \) is well-founded, according to \( \mathcal{M} \).

For contradiction, suppose there were a jump hierarchy on \( a^* \) in \( \Delta_1^1(G) \). Then for some \( b \in \mathcal{O} \), \( R(X) := "H_X^b \) computes a jump hierarchy on \( a^*" \) is again a hyperarithmetic predicate of Borel class \( b + O(1) \), where \( R \) holds of \( G \). (Recall that being a jump hierarchy on \( a^* \) is just a \( \Pi_2^0 \) property). Arguing as above, hyperarithmetically in any \( b + O(1) \)-generic \( X \), we would have a jump hierarchy on \( a^* \), which is impossible since \( a^* \) has no hyperarithmetic jump hierarchy. \( \square \)

Below, the way that \( \mathcal{M} \) can produce a Baire decomposition without resorting to arithmetic transfinite recursion is by polling a sufficiently generic element \( p \) that being a jump hierarchy on \( a \), which is impossible since \( a \) has no hyperarithmetic descending sequence. So \( a^* \) is well-founded, according to \( \mathcal{M} \).

Theorem 3.6. There is an \( \omega \)-model of \( DPB \) that does not satisfy \( ATR_0 \).

Proof. Let \( \mathcal{M} \) be as above. Let \( T \in \mathcal{M} \) be a labeled Borel code which is determined in \( \mathcal{M} \). We consider the case where \( T \in \Delta^1_1 \); the case where \( T \in \Delta^1_1(\bigoplus_{i \in \omega} G_i) \) follows by relativization. Since \( T \) is determined, for each \( G_i \) and each \( p \in 2^{\omega} \), the statements \( p \upharpoonright G_i \in |T_n| \) can be understood as a determined formulas of \( L_{\omega_1, \omega} \) (by replacing the leaves of \( T_n \) with 0 or 1 according to whether \( p \upharpoonright G_i \) is in those sets). These formulas are uniformly \( \Delta^1_1(G_i) \). Therefore, by Lemma 3.3 we have

\[
\{(\sigma, p) : p \upharpoonright G_i \in |T_n|\} \in \Delta^1_1(G_i)
\]

Therefore, for each \( i \), \( \Delta^1_1(G_i) \) contains the sequence \((U^i_{\sigma}, V^i_{\sigma})_{\sigma \in T} \) defined by

\[
U^i_{\sigma} = \{ p : \forall q \geq p, q \upharpoonright G_i \in |T_{\sigma}| \}, \quad V^i_{\sigma} = \{ p : \forall q \geq p, q \notin U^i_{\sigma} \}.
\]

We claim that for each \( i \neq j \) and for each \( \sigma \in T \), the collections

\[
(U^i_{\sigma \upharpoonright \tau}, V^i_{\sigma \upharpoonright \tau})_{\tau \in T_{\sigma}}, \quad (U^j_{\sigma \upharpoonright \tau}, V^j_{\sigma \upharpoonright \tau})_{\tau \in T_{\sigma}}
\]

are Baire decompositions for \( T_{\sigma} \), and are equal. The proof (for fixed \( i, j \)) is carried out inside of \( \mathcal{M} \) by arithmetic transfinite induction on the rank of \( \sigma \) in \( T \). Specifically, we claim that

1. If \( \sigma \) is a leaf, then \( U^i_{\sigma} = \text{the clopen set coded by } \sigma \) and \( V^i_{\sigma} = \overline{U^i_{\sigma}} \).
2. If \( \sigma \) is a union node, then for all \( p \in 2^{<\omega} \), \( p \in U^i_{\sigma} \) if and only if \( \bigcup_n U^i_{\sigma \upharpoonright n} \) is dense in \( [p] \).
3. If \( \sigma \) is an intersection node, then for all \( p \in 2^{<\omega} \), \( p \in V^i_{\sigma} \) if and only if \( \bigcap_n V^i_{\sigma \upharpoonright n} \) is dense in \([p] \).
4. \( U^j_{\sigma} = U^i_{\sigma} \) (and thus \( V^j_{\sigma} = V^i_{\sigma} \)).

Note that the definition of the \( V^i_{\sigma} \) in term of \( U^i_{\sigma} \) guarantees that \( U^i_{\sigma} \cup V^i_{\sigma} \) is dense and \( U^i_{\sigma} \cap V^i_{\sigma} = \emptyset \), and the remaining parts of the claim suffice to establish that we have a Baire decomposition.
When $\sigma$ is a leaf, it is clear that $U^i_1$ and $U^j_2$ consist of precisely those $p$ such that $[p]$ is contained in the clopen set coded by $\ell(\sigma)$.

Now fix an interior node $\sigma$. By induction, we can assume that for all $\tau \in T$ properly extending $\sigma$, condition (4) holds, so we drop the superscripts and denote these open sets by $U_\tau$ and $V_\tau$. Since Properties (1)-(3) hold for $\rho$ extending such $\tau$, we have that $(U_\rho, V_\rho)$, $\rho \in \tau$, are a Baire decomposition for $T_\tau$. We let $D_{m, \tau}$ denote the canonical sequence of dense open sets corresponding to this Baire decomposition. Since $(D_{m, \tau})_m \in \Delta^1_1(G_1) \cap \Delta^1_1(G_2)$, so by Proposition 3.2, $(D_{m, \tau})_m \in \Delta^1_1$. Therefore, for all $p \not\in 2^{\omega \omega}$, we have $p \supseteq \rho \cap \rho \cap \cap m_n D_{m, \tau}$. Therefore, if $p \supseteq G_1 \in U^i_1$, then $p \supseteq G_1 \in [T_\tau]$, and if $p \supseteq G_1 \in V^i_1$, then $p \supseteq G_1 \not\in [T_\tau]$, and the same holds for $G_j$.

Suppose that $\sigma$ is a union node. To prove ($\Rightarrow$) in (2), fix $q \in U^i_\sigma$. We need to show that $\{r \in 2^{\omega \omega} : q \supseteq r \in \bigcap_n U^i_{\sigma \cap n}\}$ is dense. For a contradiction, suppose $[q \supseteq r_0] \cap \bigcap_n U^i_{\sigma \cap n} = \emptyset$ for some fixed $r_0$. To obtain a contradiction, we will show that for all $n$, we have $q \supseteq r_0 \cap G_i \not\in [T_{\sigma \cap n}]$. Since $\sigma$ is a union node, it follows that $q \supseteq r_0 \cap G_i \not\in [T_\sigma]$ contradicting the fact that $q \in U^i_\sigma$.

Fix $n$ and let $\tau = \sigma \cap n$. Since $\tau$ properly extends $\sigma$, we have that $q \supseteq r_0 \cap G_i \in \bigcap_m D_{m, \tau}$ by the comments two paragraphs above. Since $U_{\tau} \cup V_{\tau}$ is dense, it follows that $V_{\tau}$ is dense in $[q \supseteq r_0]$ and therefore $q \supseteq r_0 \cap G_i \in V_{\tau}$. From $q \supseteq r_0 \cap G_i \in \bigcap_m D_{m, \tau}$ and $q \supseteq r_0 \cap G_i \in V_{\tau}$, it follows that $q \supseteq r_0 \cap G_i \not\in [T_\tau]$ as required to complete the contradiction.

To prove ($\Leftarrow$) in (2), assume that $\bigcup_n U^i_{\sigma \cap n}$ is dense in $[q]$. We need to show that $q \in U^i_\sigma$. Fix $r_0 \in 2^{\omega \omega}$. Since $\bigcup_n U^i_{\sigma \cap n}$ is dense in $[q \supseteq r_0]$, it is also dense in $[q \supseteq r_0]$. By the induction hypothesis and Proposition 3.2, $\bigcup_n U^i_{\sigma \cap n}$ is $\Delta^1_1$. Let $A = \{\tau : \exists n (q \supseteq r_0 \cap \tau \in U^i_{\sigma \cap n}\}$. $A$ is dense and is $\Delta^1_1$. Therefore, $G_i$ meets the set $A$. Fix $\tau \in A$ such that $\tau \supseteq G_i$ and fix $n$ such that $q \supseteq r_0 \cap \tau \in U^i_{\sigma \cap n}$. Then $q \supseteq r_0 \cap G_i \in U^i_{\sigma \cap n}$. So, as noted above, $q \supseteq r_0 \cap G_i \in \bigcap_m D_{m, \sigma \cap n}$ and $q \supseteq r_0 \cap G_i \in [T_{\sigma \cap n}]$. As $r_0$ was arbitrary, this shows that $q \in U^i_\sigma$.

The exact same argument shows that (2) is also satisfied when $i$ is replaced by $j$. Therefore, $U^i_2$ and $U^j_2$ are described by exactly the same condition, so they are equal.

Finally, let $\sigma$ be an intersection node. First, consider the direction ($\Leftarrow$) of (3): Suppose that $q \not\in V^j_1$ and fix $r_0$ such that $q \supseteq r_0 \in U^i_\sigma$. We will show that $q \supseteq r_0 \in U^i_{\sigma \cap n}$ for all $n$, so $\bigcup_n V^j_{\sigma \cap n}$ is not dense in $[q]$ (it is disjoint from $[q \supseteq r_0]$).

Fixing $n$, consider an arbitrary string $p$ extending $q \supseteq r_0$. Since $q \supseteq r_0 \in U^i_\sigma$, we know that $p \supseteq G_i \in T_\sigma$. Since $\sigma$ is an intersection node, it follows that $p \supseteq G_i \in T_{\sigma \cap n}$. Since $p$ was an arbitrary string extending $q \supseteq r_0$, this implies $q \supseteq r_0 \in U_{\sigma \cap n}$ as required to complete this direction of (3).

To prove ($\Rightarrow$) in (3), assume $\bigcup_n V_{\sigma \cap n}$ is not dense in $[q]$. We need to show that $q \not\in V^j_1$. Fix $r_0$ such that $[q \supseteq r_0] \cap \bigcup_n V^j_{\sigma \cap n} = \emptyset$. Therefore, for each $n$, $U^j_{\sigma \cap n}$ is dense in $[q \supseteq r_0]$.

Fix an arbitrary string $p$ extending $q \supseteq r_0$. We claim that for all $n$, we have $p \supseteq G_i \in U^j_{\sigma \cap n}$. First, note that $U^j_{\sigma \cap n}$ is dense in $[p]$ and that by the induction hypothesis and Proposition 3.2, $U^j_{\sigma \cap n}$ is $\Delta^1_1$. We shift $U^j_{\sigma \cap n}$ to a set $A = \{\tau : p \supseteq \tau \in U^i_{\sigma \cap n}\}$ which is dense and $\Delta^1_1$, so $G_i$ meets $A$. Let $\tau \in A$ such that $\tau \supseteq G_i$. Then, $p \supseteq \tau \in U_{\sigma \cap n}$ and so $p \supseteq G_i \in U^j_{\sigma \cap n}$. Furthermore, as noted above, since $p \supseteq G_i \in \bigcap_m D_{m, \sigma \cap n}$, it follows that $p \supseteq G_i \in [T_{\sigma \cap n}]$. Since this property holds for each $n$ and since $\sigma$ is an intersection node, it follows that $p \supseteq G_i \in [T_\sigma]$. The string $p$ extending $q \supseteq r_0$ was
arbitrary, so by the definition of $U^i_\sigma$, we have $q^\sim r_0 \in U^i_\sigma$, and therefore $q \not\in V^i_\sigma$ to complete the proof of (3).

We have actually proved a little more. Inspecting the argument for $(\Rightarrow)$ in (3), we see that whenever $[q] \cap \bigcup_n V_{\sigma \sim n} = \emptyset$, we have $q \in U^i_\sigma$; and inspecting the argument for $(\Leftarrow)$ in (3), we see that whenever $q \in U^i_\sigma$, we have $[q] \cap \bigcup_n V_{\sigma \sim n} = \emptyset$. This gives a definition of $U^i_\sigma$ that does not depend on $i$, and indeed the arguments above could be repeated exactly for $U^j_\sigma$. Therefore, $U^i_\sigma = U^j_\sigma$ in the case where $\sigma$ is an intersection as well.

We conclude that $(U_\sigma, V_\sigma)_{\sigma \in T}$ is a Baire decomposition for $T$, and so $T$ has a Baire approximation in $M$. Therefore $M$ satisfies DPB but not ATR$^0$. □

4. Decorating trees

In order to show that DPB is strictly stronger than L$_{\omega_1, \omega}$-CA, we need to make some techniques for building non-standard Borel codes in a way that ensures they are determined.

A non-standard Borel code is a code that is not actually well-founded, but which the model thinks is well-founded. These fake codes are essential for the strength of DPB. If a Borel code is truly well-founded, then it has a Baire code which is hyper-arithmetic in itself. Since any $\omega$-model of L$_{\omega_1, \omega}$-CA is closed under hyperarithmetic reduction, L$_{\omega_1, \omega}$-CA alone would be enough to guarantee the Baire code exists in the case when the Borel code is truly well-founded (at least in $\omega$-models). So now we are going to describe how to construct a non-standard Borel code which makes every effort to be determined.

If we make a Borel code $T$ which is not well-founded, the most likely scenario is that it is also not determined. This is because, in general, it might take a jump hierarchy the height of the rank of $T$ in order to produce an evaluation map. So in this section, we show how to add “decorations” to the tree, which shortcut the logic of the tree to make sure that for a small set of $X$, there is an evaluation map for $X$ in the decorated tree. In Section 5 “small” is countable, and in Section 6 “small” is meager. This comes at the cost of trashing any information about whether $X$ was in the original set but if that set had a Baire approximation, then its decorated version should have the same Baire approximation, since the set of $X$ whose membership facts were overwritten is small. We use this to show that if the model satisfies DPB, then the “small” set cannot be the entire second-order part of the model.

Suppose that we have a partial computable function $h$ which maps a number $b \in O^*$ to a pair of $b$-ranked labeled trees $(P_b, N_b)$. We do not mind if $h$ happens to also make some outputs for $b \not\in O^*$.

The intention is that when $b \in O$, any $X \in |P_b| \cup |N_b|$ will have an approximately $H^X_b$-computable evaluation map in the decorated tree, and $X$ will be in the decorated tree if $X \in |P_b|$ and out of the decorated tree if $X \in |N_b|$. (In practice we will always have $|P_b| \cap |N_b| = \emptyset$.) The computable operation Decorate is defined using the recursion theorem.
**Definition 4.1.** The operation \( \text{Decorate} \) is defined as follows. The inputs are an \( a \)-ranked labeled tree \( T \) and a partial computable function \( h \) as above.

\[
\text{Decorate}(T, h) = \{\lambda\} \cup \bigcup_{(n) \in T} (2n)^{\sim} \text{Decorate}(T_{(n)}, h) \\
\cup \bigcup_{b < \rho_T(\lambda)} (2b + 1)^{\sim} \text{Decorate}(Q_b, h)
\]

where \( Q_b = P_b \) if \( \lambda \) is a \( \cup \) in \( T \), and \( Q_b = N_b^c \) if \( \lambda \) is a \( \cap \) in \( T \).

The rank and label of \( \lambda \) in \( \text{Decorate}(T, h) \) are defined to coincide with the rank and label of \( \lambda \) in \( T \).

Since \( P_b \) and \( N_b \) are \( b \)-ranked, \( \text{Decorate}(T, h) \) satisfies the local requirements on a ranking. So if \( T \) is \( a \)-ranked, so is \( \text{Decorate}(T, h) \).

Similarly, if \( T \) and each \( P_b \) and \( N_b \) are alternating, and each \( P_b \) and \( N_b \) have an intersection or leaf at their root, then \( \text{Decorate}(T, h) \) will also be alternating. (Note that in this case, \( N_b^c \) has a union at its root).

The following is the essential feature of a decorated tree.

**Proposition 4.2.** If \( \sigma \in \text{Decorate}(T, h) \) has rank \( b \), then for all \( d <_a b \),

\[
\text{Decorate}(T, h)_{\sigma^\sim(2d+1)} = \text{Decorate}(Q_d, h),
\]

where \( Q_d = P_d \) or \( N_d^c \) as appropriate.

**Proof.** By induction on the length of \( \sigma \). \( \square \)

**Definition 4.3.** A nice decoration generator is a partial computable function which maps any \( b \in O^* \) to alternating, \( b \)-ranked trees \( (P_b, N_b) \), where each \( P_b \) and \( N_b \) have an intersection or a leaf at their root.

**Lemma 4.4.** Let \( h \) be a nice decoration generator. Suppose \( b \in O \), and suppose that \( X \notin |P_d| \cup |N_d| \) for any \( d <_a b \). Then for any \( b \)-ranked tree \( T, X \in |\text{Decorate}(T, h)| \) if and only if \( X \in [T] \).

**Proof.** By induction on \( b \). Since \( b \in O \), \( T \) is truly well-founded, so there is a unique evaluation map \( f \) for \( X \) in \( T \). Further, for each \( d <_a b \), there are unique evaluation maps \( g_{P,d}, g_{N,d} \) for \( X \) in \( \text{Decorate}(P_d, h) \) and \( \text{Decorate}(N_d^c, h) \). Consider the function \( g : \text{Decorate}(T, h) \to \{0, 1\} \) defined by

\[
g(\sigma) = \begin{cases}
    f(\sigma) & \text{if each component of } \sigma \text{ is even} \\
g_{Q,d}(\sigma_1) & \text{if } \sigma = \sigma_0^\sim (2d+1)^{\sim} \sigma_1 \text{ and each component of } \sigma_0 \text{ is even},
\end{cases}
\]

where the division \( \sigma/2 \) is taken componentwise, and where \( Q \) is either \( P \) or \( N \) depending on whether \( \sigma_0 \) is a union or intersection in \( \text{Decorate}(T, h) \).

Since \( g(\lambda) = f(\lambda) \), it is enough to show that \( g \) is an evaluation map for \( X \) in \( \text{Decorate}(T, h) \). Clearly \( g \) satisfies the logic of the tree at leaves and at nodes which have an odd component. Consider \( \sigma \in \text{Decorate}(T, h) \) where \( \sigma \) is a \( \cup \) and all components of \( \sigma \) are even. By induction, since \( P_d \) is a \( d \)-ranked tree, \( X \in |\text{Decorate}(P_d, h)| \) if and only if \( X \in |P_d| \). By hypothesis, \( X \notin |P_d| \), so \( g_{P,d}(\lambda) = 0 \), so by Proposition 1.2 \( g(\sigma^\sim(2d+1)) = 0 \). Therefore, the nodes of this form can be ignored: we have

\[
\exists m(g(\sigma^m) = 1) \iff \exists n(g(\sigma^\sim(2m)) = 1) \iff f(\sigma/2) = 1
\]
so \(g(\sigma)\) takes the correct value. The argument if \(\sigma\) is a \(\cap\) is similar, except that as \(X \not\in |N_b|\), we have \(X \in |N_b|\), and therefore \(g_{|N_b}(\lambda) = 1\), meaning that nodes of the form \(\sigma^\sim(2d + 1)\) can be safely ignored when taking an intersection. \(\square\)

**Lemma 4.5.** Let \(a \in \mathcal{O}^*\) and \(b \in \mathcal{O}\) with \(b <^* a\). Let \(T\) be an alternating, a-ranked tree and let \(h\) be a nice decoration generator. Suppose \(X \in |P_b| \cup |N_b|\). Then

1. \(X\) has a unique evaluation map in \(\text{Decorate}(T, h)\).
2. This evaluation map is \(H^{X \oplus T}_{b+O(1)}\)-computable.
3. If \(b\) is \(<^*_\ast\)-minimal such that \(X \in |P_b| \cup |N_b|\), and \(b <^*_\ast \rho_T(\langle n \rangle)\) for all \(\langle n \rangle \in T\), and \(g\) is the unique evaluation map for \(X\) in \(\text{Decorate}(T, h)\), then
   
   \[
   \begin{align*}
   & (a) \ X \in |P_b| \setminus |N_b| \Rightarrow g(\lambda) = 1 \\
   & (b) \ X \in |N_b| \setminus |P_b| \Rightarrow g(\lambda) = 0.
   \end{align*}
   \]

**Proof.** It suffices to show all three parts in the case when \(b\) is \(<^*_\ast\)-minimal such that \(X \in |P_b| \cup |N_b|\).

We prove (1) and (2) by showing that for each \(\sigma \in \text{Decorate}(T, h)\), there is only one possible value for \(g(\sigma)\) for any evaluation map \(g\) for \(X\) in \(\text{Decorate}(T, h)\) and that \(H^{X \oplus T}_{b+O(1)}\) suffices to compute this value. Since these unique values satisfy the internal logic of the tree (which the reader can verify from the description below), they constitute an evaluation function for \(X\) in \(\text{Decorate}(T, h)\), proving (1) and (2).

To show that there is only one possible value for \(g(\sigma)\), we break into cases depending on the rank and label of \(\sigma\) in \(\text{Decorate}(T, h)\) and on whether \(X \in |P_b|\) or \(X \in |N_b|\). Note that \(H^{X \oplus T}_{b}\) can uniformly determine the appropriate case for each \(\sigma\).

**Case 1.** Suppose \(\rho(\sigma) \leq^*_\ast h\). Since \(b \in \mathcal{O}\), \(\text{Decorate}(T, h)\) is truly well-founded. Therefore, there is a unique evaluation map \(f\) for \(X\) in \(\text{Decorate}(T, h)\), and we have \(g(\sigma) = f(\lambda)\). The map \(f\) is uniformly \(H^{X \oplus T}_{b+O(1)}\)-computable.

**Case 2.** Suppose \(b <^*_\ast \rho(\sigma)\), \(\sigma\) is a union node in \(\text{Decorate}(T, h)\) and \(X \in |P_b|\). In this case, we claim that \(g(\sigma) = 1\). By Proposition 4.2, all nodes extending \(\sigma^\sim(2b + 1)\) have rank \(b\) or less. Therefore, there is a unique evaluation map \(f\) on \(\text{Decorate}(T, h)\) of rank \(2b + 1\) and so \(g(\sigma^\sim(2b + 1)) = f(\lambda)\). By Lemma 4.4, \(X \in |P_b|\) implies \(f(\lambda) = 1\). Therefore, \(g(\sigma^\sim(2b + 1)) = 1\) and because \(\sigma\) is a union node, \(g(\sigma) = 1\).

**Case 3.** Suppose \(b <^*_\ast \rho(\sigma)\), \(\sigma\) is an intersection node in \(\text{Decorate}(T, h)\) and \(X \in |P_b|\). Since \(\text{Decorate}(T, h)\) is alternating, each node \(\sigma^\sim m\) is either a union node or a leaf. If \(\rho(\sigma^\sim m) \leq^*_\ast b\), then the value of \(g(\sigma^\sim m)\) is fixed as in Case 1. If \(b <^*_\ast \rho(\sigma^\sim m)\), then \(g(\sigma^\sim m) = 1\) as in Case 2. Together, these values determine \(g(\sigma)\) uniquely. \(H^{X \oplus T}_{b}\) suffices to compute the values of \(g(\sigma^\sim m)\) and it takes one extra jump to determine if \(g(\sigma^\sim m) = 1\) for all \(m\), and hence determine \(g(\sigma)\).

**Case 4.** Suppose \(b <^*_\ast \rho(\sigma)\), \(\sigma\) is an intersection node in \(\text{Decorate}(T, h)\) and \(X \in |N_b|\). An analogous argument to Case 2 shows that \(g(\sigma) = 0\).

**Case 5.** Suppose \(b <^*_\ast \rho(\sigma)\), \(\sigma\) is a union node in \(\text{Decorate}(T, h)\) and \(X \in |N_b|\). This case is analogous to Case 3 and the unique value of \(g(\sigma)\) can be determined with one extra jump.

These cases are exhaustive, but if \(P_b \cap N_b \neq \emptyset\), then more than one case can apply. However, if \(X \in P_b \cap N_b\), there is no conflict between the values given in the cases. In this degenerate case, we have that for any \(\sigma\) such that \(b <^*_\ast \rho(\sigma)\), \(g(\sigma) = 1\) if \(\sigma\) is a union node and \(g(\sigma) = 0\) if \(\sigma\) is an intersection node. This completes the proof of (1) and (2).
For (3), if $X \in |P_b| \setminus |N_b|$, and if $\lambda$ is $\cup$, then $g(\lambda) = 1$ just as above. But if $\lambda$ is $\cap$, then we claim that for each $m$, $g(\langle m \rangle) = 1$. If $m = 2n$ for some $(n) \in T$, or if $m = 2d + 1$ for some $d > b$, then because $b <_\ast \rho_T((n))$ for all $n$, and each $\langle m \rangle$ is a union, again we have $g(\langle m \rangle) = 1$ for such $m$. In the remaining case, when $m = 2d + 1$ with $d \leq b$, then since $b$ is minimal such that $X \in |P_b| \cup |N_b|$, and $X \not\in |N_b|$, we have $X \in |N^T_b|$. So by Lemma 4.4, $X \in |\text{Decorate}(N^T_b; h)|$, so $g(\langle 2d + 1 \rangle) = 1$. Since $g(\langle m \rangle) = 1$ for all $m$, we have $g(\lambda) = 1$ as well. A complementary argument establishes (3b). □

5. DPB does not hold in HYPP

We now show that DPB is not a theory of hyperarithmetic analysis by showing that DPB fails in the $\omega$-model HYPP. In brief, we let $E_a$ code a canonical universal $\Sigma^0_\alpha$ set. Using overflow, we make a computable code for the set

$$\bigcup_b |E_b| \cap \{X : b \text{ is least s.t. } X \leq_T H_b\}.$$ 

We decorate the code to give each $H_b$-computable set an $H_b$-computable evaluation map. Then we argue that the result is a code which HYPP thinks is well-founded and determined, but which can have no HYPP Baire code.

**Theorem 5.1.** DPB does not hold in HYPP.

**Proof.** There is a computable procedure which, on inputs $a \in \mathcal{O}$, $e \in \mathbb{N}$, $p \in 2^{<\omega}$, outputs an index for a $2^a$-ranked computable $L_{\omega_1, \omega}$ formula $F_{a, e, p}$, which holds true if and only if $p \in W^F_{a^e}$. Transform each formula $F_{a, e, p}$ into a Borel code by swapping $\textbf{false}$ for $\emptyset$, and $\textbf{true}$ for $[0^e \check{1}^p]$. Then take the union of all of these, obtaining a code $E_a$ of rank $a + O(1)$ such that for all $a \in \mathcal{O}$,

$$|E_a| = \bigcup_{e, p : p \in W^F_a} [0^e \check{1}^p].$$

By overflow, there is also a pseudo-ordinal $a^*$, for which $W_{p(a^*)}$ is not well-founded, but has no hyperarithmetical descending sequence, so that HYPP believes $W_{p(a^*)}$ is well-founded. Then HYPP also believes that $E_b$ is well-founded for any $b <_\ast a^*$. We may assume that $E_b$ are alternating and $(b + O(1))$-ranked for all $b \leq a^*$. For the sake of a later application of Lemma 4.5, note that we can also assume that the rank of $E_{a^*}$ is a successor, so of the form $2^x$ for some $x$, and that for each $\langle n \rangle \in E_{a^*}$, the rank of $\langle n \rangle$ in $E_{a^*}$ is $x$.

Similarly, there is a computable procedure which, for each $b \in \mathcal{O}$, outputs a $(b + O(1))$-ranked Borel code $S_b$ such that

$$|S_b| = \{X \in 2^a : X \leq_T H_b \text{ and for all } c <_\ast b, X \not\leq_T H_c\}.$$ 

We think of $S_b$ as coding a slice of HYPP. By overflow, for any $b < a^*$, HYPP thinks that $S_b$ is well-founded.

For each $b <_\ast a^*$, define $P^b$ and $N^b$ so that they are alternating, and

$$|P^b| = |S_b| \cap |E_b|, \quad |N^b| = |S_b| \cap |E_b^\alpha|.$$ 

Observe that $P^b$ and $N^b$ can be both $(b + k)$-ranked, where $k$ is some fixed finite ordinal. Let $h$ be the function which, on input $b$, outputs $P_b = P^{b-k}$ and $N_b = N^{b-k}$ if the operation $b - k$ can be performed, and outputs a degenerate $b$-ranked tree coding the empty set, if $b$ is less than $k$ successors from a limit ordinal.
We claim that \(\text{Decorate}(E_{a^*, h})\) is determined in \(HYP\). Observe that \(h\) is a nice decoration generator. Let \(X \in HYP\). Then there is some \(b \in O\) with \(b \prec a^*\) such that \(X \leq_T H_b\). Since \(a^*\) is nonstandard, \(b + O(1) \prec a^*\) is satisfied. By the choice of \(b\) we have \(X \in |S_b| = |P_b + k| \cup |N_{b + k}|\). Therefore, by Lemma 1.3, \(X\) has a \(HYP\) evaluation map. Therefore, \(\text{Decorate}(E_{a^*, h})\) is determined in \(HYP\).

Suppose for contradiction that \(\text{Decorate}(E_{a^*, h})\) had a \(HYP\) Baire code. Let \(b \in O\) with \(b \prec a^*\) and with the Baire code \((U, V, \{D_n\}_{n \in \omega}) \leq_T H_b\). By the recursion theorem, there is an index \(e\) such that

\[
W_e^{H_b} = \{p : 0^e \upharpoonright p \in V\}
\]

where \(H_b\) is used to compute \(V\). Choose \(p\) with \(0^e \upharpoonright p \in U \cup V\), this is possible as \(U \cup V\) is dense. Let \(X \in HYP\) be such that

1. \(0^e \upharpoonright p \prec X\)
2. \(X \leq_T H_b\) but \(X \not\leq_T H_c\) for any \(c \prec b\),
3. \(X \in D_n\) for all \(n\).

This is possible because the \(D_n\), and the dense sets which need to be met to avoid being computed by \(H_c\) for \(c \prec b\), are uniformly \(H_b\)-computable.

Now \(b + k\) is least such that \(X \in |P_b + k| \cup |N_{b + k}| = |S_b|\). By Lemma 1.3, \(X \in |\text{Decorate}(E_{a^*, h})|\) if and only if \(X \in |E_b|\). Because \(X\) meets each \(D_n\) and \(U \cup V\), by the definition of a Baire code, we have \(X \in |\text{Decorate}(E_{a^*, h})|\) if and only if \(X \in U\). To establish the contradiction, it suffices to show that \(X \in |E_b|\) if and only if \(X \in V\).

Observe \(X \in |E_b|\), if and only if, for some \(q\) extending \(p\), we have \(0^e \upharpoonright q \prec X\) and \(q \in W_e^{H_b}\). But this happens if and only if for some such \(q\), we have \(0^e \upharpoonright q \in V\). \(\square\)

6. \(DPB\) implies \(HYP\) generics exist in \(\omega\)-models

The next theorem shows that \(DPB\) implies the existence of hyperarithmetic generics in \(\omega\)-models. In short, if \(M\) has \(Z\) but no \(\Delta^1_1(Z)\)-generics, there is a pseudo-ordinal which \(M\) thinks is well-founded. This pseudo-ordinal can be used to construct a code for the following subset of \(M\):

\[
\bigcup_{b} E^Z_b \cap \{X : b \text{ is least s.t. } X \text{ is not generic relative to } H^Z_b\}
\]

After decorating this code, it becomes determined for every non-\(\Delta^1_1(Z)\)-generic. If this code has a Baire decomposition, meeting the associated dense sets creates a \(\Delta^1_1(Z)\)-generic.

**Theorem 6.1.** If \(M\) is an \(\omega\)-model which satisfies \(DPB\), then for every \(Z \in M\), there is a \(G \in M\) such that \(G\) is \(\Delta^1_1\)-generic relative to \(Z\).

**Proof.** Let \(M\) be the second-order part of an \(\omega\)-model which satisfies \(DPB\). Then by Proposition 2.4, whenever \(Z \in M\), we also have that \(H^Z_b \in M\) for every \(b \in O\).

Case 1: Suppose \(M\) is a \(\beta\)-model (that is, for every tree \(T \in M\), if \(M \models \text{"}T\text{ is well-founded}.\) then \(T\) is truly well-founded.) Let \(Z \in M\). Because \(\{G : G \text{ is } \Delta^1_1(Z)\text{-generic}\} \) is a \(\Sigma^1_1(Z)\) set, the \(Z\)-computable tree corresponding to the \(\Sigma^1_1(Z)\) statement “there is a \(\Delta^1_1(Z)\)-generic” has a path in \(M\), and that path computes a \(\Delta^1_1(Z)\)-generic \(G\). Therefore, the theorem holds when \(M\) is a \(\beta\)-model.

Case 2: Suppose that there exists some \(T \in M\) which \(M\) believes is well-founded, but in reality is ill-founded. Let \(Z \in M\), and without loss of generality assume that \(Z \geq_T T\) (without this assumption we find a \(\Delta^1_1(Z \oplus T)\)-generic \(G\), but such \(G\) is
also \(\Delta_1^1(Z)\)-generic. There is a \(Z\)-computable function which, given the index of a truly well-founded \(Z\)-computable tree, outputs an element of \(O^Z\) which bounds the rank of the tree. Applying that function to \(T\) produces a pseudo-notation \(a^*\) such that \(W^Z_{p[a^*]}\) is not truly well-founded, but it has no descending sequence in \(M\).

Relativize the definitions of \(<_s\), ranked trees, Decorate, and Lemmas 4.4 and 4.5 to \(Z\).

As in the previous theorem, there are \(Z\)-computable procedures which map any \(b \in O^Z\) to a code \(E_b\) of rank \(b + O(1)\) such that
\[
|E_b| = \bigcup_{c.r.r \in W_e^{H_b^Z}} [0^c 1^{\omega}],
\]
and map each \(b \in O^Z\) to a code \(S_b\) of rank \(b + O(1)\) such that
\[
|S_b| = \{X \in 2^\omega : X \text{ is not } 1\text{-generic relative to } H_b^Z, \quad \text{but for all } c <_Z^s b, X \text{ is } 1\text{-generic relative to } H_c^Z\},
\]
and alternating codes \(P_b\) and \(N_b\) of rank \(b + O(1)\) such that
\[
|P_{b+O(1)}| = |S_b| \cap |E_b|, \quad |N_{b+O(1)}| = |S_b| \cap |E_b^c|.
\]
By overflow, these functions are total for any \(b <^Z a^*\). And fixing a large enough finite number \(k\), we may assume the resulting trees are all alternating and \((b + k)\)-ranked relative to \(Z\).

Letting \(h\) be the name of the nice decorating function mapping \(b + k\) to \((P_b, N_b)\), consider the code \(T := \text{Decorate}^Z(E_{a^*}, h)\). Observe that since \(\lambda\) in \(E_{a^*}\) is a \(\cup\), we know that \(\lambda\) in \(T\) is a \(\cup\).

If \(T\) is not determined, let \(G \in M\) be such that \(G\) is not determined. We claim that \(G\) is \(\Delta_1^1(Z)\)-generic. If \(G\) is not \(\Delta_1^1(Z)\)-generic, then there is some least \(b \in O^Z\) with \(b <^Z a^*\) such that \(G\) is not \(1\)-generic relative to \(H_b^Z\). Then we would have \(G \in |S_b|\), and therefore by Lemma 4.5, \(G\) would be determined in \(T\).

If \(T\) is determined, then since \(M\) models DPB, let \((U_\sigma, V_\sigma)_{\sigma \in T} \in M\) be a Baire decomposition for \(T\). Let \(\{D_i\}_{i < \omega} \subseteq M\) be the associated sequence of dense sets as in Proposition 2.9. For any \(p \in 2^{<\omega}\), define \(D_{i, p} = \{q : p \upharpoonright q \in D_i\}\). We claim that any \(G \in \bigcap_{i,p} D_{i, p}\) is \(\Delta_1^1(Z)\)-generic. For this we argue that every dense open \(B \in \Delta_1^1(Z)\) actually contains \(D_{i, p}\) for some \(i, p\). Let \(b \in O^Z\) and \(e\) be such that \(B = W_e^{H_b^Z}\). Then \(T_{(2(b+k)+1)} = \text{Decorate}(P_{b+k}, h)\), where \(|P_{b+k}| = |S_b| \cap |E_b|\).

Therefore, there is some \(\sigma \in T\) such that \(T_{\sigma} = \text{Decorate}(E_b, h)\). Since \(E_b\) has a union at the root, this \(\sigma\) is a union. Let \(p = 0^e 1\). We claim that \(D_{i, p} \subseteq B\), where \(D_{i} = \bigcup_{n} U_{\sigma_n} \cup V_\sigma\). Let \(q\) be such that \(p \upharpoonright q \in D_i\). To finish the proof, we need to show that \(|q| \subseteq B\).

For the remainder of this proof, any \(X\) which meets the following conditions will be called sufficiently generic:

- \(X \in \bigcap_{i} D_i\), and
- \(X\) is \(1\)-generic relative to \(H_{b+k}^Z\)

Observe that for every \(r \in 2^\omega\), there is a sufficiently generic \(X \in M\) with \(r < X\). Also, observe that for all such \(X\) and all codes \(R\) which are \(c\)-ranked for some \(c \leq_s b + k\), the second condition implies that \(c, X\) and \(R\) satisfy the conditions of Lemma 4.3 and so \(X \in |\text{Decorate}(R, h)|\) if and only if \(X \in |R|\). Finally, by
Proposition 2.9 for all sufficiently generic $X$ and all $\tau \in T$, we have $X \in |T_\tau|$ if and only if $X \in U_\tau$.

If $X$ is sufficiently generic and $p^\tau q \prec X$, then $X \in p^\tau B$, and so $X \in |E_0|$, and so by Lemma 4.4, $X \in |\text{Decorate}(E_0, h)| = |T_\sigma|$. Therefore, it is impossible that $X \in V_\tau$, so we conclude $p^\tau q \notin U_{\sigma \prec m}$ for some $m$. Therefore, for sufficiently generic $X$ with $p^\tau q \prec X$, we have $X \in |T_{\sigma \prec m}|$.

If $m = 2c + 1$ for some $c \leq b + k$, then $T_{\sigma \prec m} = \text{Decorate}(P_c, h)$. But for any sufficiently generic $X$, we have $X \notin |P_c|$, so this case is impossible. Therefore, $m = 2n$ for some $\langle n \rangle \in E_b$. It follows from the definition of Decorate that $T_{\sigma \prec m} = \text{Decorate}(\langle E_b \rangle_{\langle n \rangle}, h)$. So for sufficiently generic $X$ with $p^\tau q \prec X$, we have $X \in |\langle E_b \rangle_{\langle n \rangle}|$.

Now we will use a property of the codes $E_b$ which follows from how they are defined at the beginning of the proof of Theorem 5.1. The code $E_b$ was obtained as the union of many codes $F_{b,e,r}$, at whose leaves the only options are $[0^e1^r]$ or $\emptyset$. The code $E_b$ was also post-processed so that it would be alternating, but while this process can break up the codes $F_{b,e,r}$, it can never combine them together. Therefore, for every $\langle n \rangle \in E_b$, there is an $r$ such that whenever $\langle n \rangle^\tau \in E_b$ is a leaf, its attached clopen set is either $[0^e1^r]$ or $\emptyset$. Fixing $r$ associated to $n = m/2$ for the $m$ found above, we observe that an evaluation map that works for one $Y \in [0^e1^r]$ works for all such $Y$, and we conclude that $|\langle E_b \rangle_{\langle n \rangle}|$ is equal to either $\emptyset$ or $[0^e1^r]$. It must be the latter because $X \in |\langle E_b \rangle_{\langle n \rangle}|$ for all sufficiently generic $X$ with $p^\tau q \prec X$. It follows that $[r] \subseteq B$. Furthermore, any sufficiently generic $X$ that does not extend $p^\tau r$ must be out of $|\langle E_b \rangle_{\langle n \rangle}|$, so it must be that $[q] \subseteq [r]$. Therefore, $[q] \subseteq B$, as desired. 

7. Application to the Borel dual Ramsey theorem

As an application of Theorem 3.6 we identify a natural formulation of the Borel dual Ramsey theorem for 3 partitions and $\ell$ colors (Borel-DRT$_\ell^3$) as a principle which lies strictly below ATR$_0$, but all of whose $\omega$-models are closed under hyperarithmetic reduction.

Theorem 7.1 (Borel dual Ramsey theorem, [CSS3]). For every Borel $\ell$-coloring of the set of partitions of $\omega$ into exactly $k$ pieces, there is an infinite partition $p$ of $\omega$ and a color $i < \ell$ such that every way of coarsening $p$ down to exactly $k$ pieces is given color $i$.

Since the set of partitions of $\omega$ into exactly $k$ pieces can be coded naturally as a Borel subset of $k^\omega$, a natural way to formulate the hypotheses of the above theorem is roughly “Whenever there are Borel codes $T_1, \ldots, T_\ell$ such that for every $X \in k^\omega$, we have $X \in |\cup_{i < \ell} T_i|$, ...” (See below for a precise formalization).

Therefore, the Borel dual Ramsey theorem has a natural formulation in terms of determined Borel sets. In [PV85, DFSW17], it was shown that a solution to Borel-DRT$_\ell^3$ can in general be obtained by a two-step process:

1. Use the fact that every Borel set has the property of Baire to come up with a Baire approximation to each color in the given coloring.

2. Apply a purely combinatorial principle CDRT$_\ell$ to a coloring of $(k - 1)^{<\omega}$ obtainable from the Baire approximation from (1).

If we represent the coloring in the natural way described below, then DPB can be used to carry out (1). It was known to Simpson (see [DFSW17]) that CDRT$_\ell^3$
follows from Hindman’s Theorem (HT), which follows from ACA\(^+_0\) by [BHS87]. Therefore, the following natural formalization of Borel-DRT\(^3_\ell\) follows from DPB + ACA\(^+_0\). We first give the formalization of the space of \(k\)-partitions of \(\omega\), and then the formalization of Borel-DRT\(^3\).

**Definition 7.2** (Partitions of \(\omega\). [DFSW17]). In RCA\(_0\), a partition of \(\omega\) into exactly \(k\) pieces is a function \(p \in k^\omega\) such that \(p\) is surjective, and for each \(i < k - 1\),

\[
\min\{ n : p(n) = i \} < \min\{ n : p(n) = i + 1 \}.
\]

A partition of \(\omega\) into infinitely many pieces is a surjective function \(p \in \omega^\omega\) which satisfies the above condition for each \(i \in \omega\).

The set of partitions described above is an open subset of \(k^\omega\) representable in RCA\(_0\) by a determined Borel code, as the reader can verify. (For the case \(k = 3\), the set in question is the union of the sets \(O_{a,b}\) introduced at the start of the proof of Theorem 7.3.) Let \(P_3\) denote this determined Borel code in the case \(k = 3\).

**Definition 7.3** (Formal Borel dual Ramsey theorem for \(3\) partitions and \(\ell\) colors). In RCA\(_0\), Borel-DRT\(^3_\ell\) is the principle which states: Whenever \(T_0, \ldots, T_{\ell - 1}\) are Borel codes such that for all \(X \in [P_3]\), we have \(X \in |\bigcup_{i<\ell} T_i|\), then there is an infinite partition \(p\) of \(\omega\) and a color \(i < \ell\) such that whenever \(X \in [P_3]\), \(X \circ p \in [T_i]\).

It follows from the hypotheses of this theorem that the codes \(\{T_i\}_{i<\ell}\) are all determined. Therefore, the discussion preceding the formal definitions proves that DPB + ACA\(^+_0\) \(\vdash\) Borel-DRT\(^3\) over RCA\(_0\).

The \(\omega\)-model which was constructed to prove Theorem 8.3 is closed under hyperarithmetic reduction, and therefore satisfies ACA\(^+_0\) as well as DPB. Therefore, Borel-DRT\(^3_\ell\) holds in this model, while ATR\(_0\) does not. This shows that the formulation of Borel-DRT\(^3_\ell\) discussed here is strictly weaker than ATR\(_0\).

On the other hand, we have the following, which essentially follows from a more detailed version of the analysis in Section 4 of [DFSW17].

**Theorem 7.4.** Let \(\ell \in \omega\) with \(\ell \geq 2\). Every \(\omega\)-model of Borel-DRT\(^3_\ell\) is closed under hyperarithmetic reduction.

**Proof.** It suffices to consider the case \(\ell = 2\). We will first define some important subsets of \(3^\omega\). For each \(a, b\) with \(0 < a < b\), let \(O_{a,b}\) be the clopen set given by the finite collection of strings

\[
O_{a,b} = \{ \sigma \in 3^{b+1} : a = \min\{ n : \sigma(n) = 1 \} \text{ and } b = \min\{ n : \sigma(n) = 2 \} \}
\]

Then the set of partitions of \(\omega\) into exactly \(3\) pieces is given by \(P_3 = \bigcup_{0 < a < b} O_{a,b}\).

Let \(M\) be the second-order part of an \(\omega\)-model \(\mathcal{M}\) of Borel-DRT\(^3_2\). We first show that \(\mathcal{M}\) satisfies ACA\(_0\). Let \(A \in M\). Let \(R\) be the following labeled Borel code\(^4\)

\[
R = \bigcup_{0 < a < b \in M} \bigcap_{s > b} C_{a,b,s} \text{ where } C_{a,b,s} = \begin{cases} O_{a,b} & \text{if } A'_s \upharpoonright a = A'_s \upharpoonright a \\ \emptyset & \text{otherwise.} \end{cases}
\]

Then \(R\) is determined. For any \(X \in 3^\omega\), there is at most one pair \(a, b\) such that \(X \in O_{a,b}\), so an evaluation map for \(X\) in \(R\) may safely put zeros at every node of

\(^4\)We use standard computability-theoretic notation: for any \(s \in \mathbb{N}\), let \(\phi^{A}_s\) denote \(\{ x < s : \phi^{A}_s(x) \downarrow \}\), and for any \(X\) let \(X \upharpoonright s\) denote the string of length \(s\) describing the characteristic function of \(X\) on \(\{0, \ldots, s - 1\}\).
R except for the root and the nodes of the distinguished subtree $\cap_{a>b}C_{a,b,s}$. The leaves of that subtree can be $X \oplus A$-computably filled out. Then the root of $R$ and the root of the subtree $\cap_{a>b}C_{a,b,s}$ may be non-uniformly supplied with their unique correct values.

Exactly as in the proof of [DFSW17, Theorem 4.5], for any infinite partition $p$ of $\omega$ which is homogeneous for the coloring defined by $|R|, |R^c|$, we have $p \geq_T A'$. (The coloring defined here is the same as the one defined in the proof of that theorem; the function needing to be dominated is the least modulus function for $A'$. Here we have just been more precise about the code $R$ in order to argue that it is determined.) Therefore, $\mathcal{M} \models \text{ACA}_0$.

Now suppose that $A \in \mathcal{M}$ and $3 \cdot 5^e \in \mathcal{O}^A$. Suppose that for all $d \leq_{\mathcal{O}} 3 \cdot 5^3$, we have $H^A_d \in \mathcal{M}$. Then we claim that $H^A_{3 \cdot 5^e} \in \mathcal{M}$. By a result of Jockusch [Joc68] discussed in more detail below, the hyperarithmetic sets are exactly those that can be computed from sufficiently fast-growing functions. As in [DFSW17, Theorem 4.7], we construct a Borel coloring which forces any solution to $\text{Borel-DRT}_3^2$ to compute a sufficiently fast-growing function. To prove associated Borel code is determined, we need a more detailed analysis than what was given in [DFSW17].

More specifically, Jockusch’s result has plenty of uniformity: there are computable functions $h$ and $k$ such that for all $d \in \mathcal{O}^A$, whenever $g : \omega \rightarrow \omega$ dominates the increasing function $f_d(n) := \phi_{h(d)}(n)$, we have

$$\phi_{k(d)}(A \oplus g) = H^A_d.$$  

(To get this from the proof of [Joc68, Theorem 6.8], apply [Rog87, Exercise 16-98] to conclude that the sets $H^A_d$ are in fact uniformly Turing equivalent to implicitly $\Pi^0_1(A)$-definable functions $f_d$.)

Uniformly in $d \in \mathcal{O}^A$ and $a, b, \in \omega$, and $A$, there are Borel codes $C_{a,b,d}$ of well-founded rank $d + O(1)$ such that

$$C_{a,b,d} = \begin{cases} O_{a,b} & \text{if } b \geq f_d(a) \\ \emptyset & \text{otherwise.} \end{cases}$$

The uniformity follows from the existence of $h$ above and the $A$-uniformity of producing a formula of $L_{\omega_1,\omega}$ to assess facts about $H^A_d$.

For each $n < \omega$, let $d_n = \phi_e(n)$. Now let $R$ be the labeled Borel code

$$R = \bigcup_{0 < a < b, i \leq a} C_{a,b,d_i}.$$  

For any $X \in \mathcal{O}^\omega$, there is at most one pair of $a, b$ such that $X \in O_{a,b}$, so as above, any evaluation map for $X$ in $R$ can safely fill in zeros everywhere except for the root of $R$ and the distinguished subtree rooted at $\cap_{i \leq a} C_{a,b,d_i}$. This subtree has well-founded rank $d_a + O(1)$, so the unique evaluation map on it is $H^A_d + O(1)$-computable. Because $H^A_d \in \mathcal{M}$ for all $d \leq_{\mathcal{O}} 3 \cdot 5^e$, this evaluation map exists in $\mathcal{M}$. Therefore, $R$ is determined in $\mathcal{M}$.

Now let $p \in \mathcal{M}$ be any infinite partition of $\omega$ which is a solution to $\text{Borel-DRT}_2^3$ for the coloring $|R|, |R^c|$. Define, for each $i$,

$$p_i = \min\{n : p(n) = i\}.$$
As in the proof of [DFSW17, Theorem 4.5], for every $0 < s < t$, there is a coarsening $X_{s,t}$ of $p$ to exactly three blocks such that $p_s = \min\{n : X_{s,t}(n) = 1\}$ and $p_t = \min\{n : X_{s,t}(n) = 2\}$. Since $t$ can be chosen arbitrarily large, for every $s$ there is a $t$ such that

$$X_{s,t} \in \bigcap_{i \leq p_s} C_{p_s,p_t,d_i}$$

and therefore $P_i \circ p$ is monochromatic for color $R_i$ and $s < t$ implies that for all $i \leq p_s$, we have $p_t \geq f_{d_i}(p_s)$. Therefore, $p$ computes a sequence of functions \{\(g_i : i \in \omega\)\} such that for all $i$ and $n$, $g_i(n) \geq f_{d_i}(n)$. (Given $i$ and $n$, let $s$ be large enough that $i, n \leq p_s$, and output $p_{s+1}$.) Therefore, $A \oplus p$ computes

$$\bigoplus_i \phi_{b(d_i)}(A \oplus g_i) = \bigoplus_i H_{d_i}^A = H_{3,5^s}^A,$$

as was needed. \hfill \square

We end this section with a question about robustness. The formalization of Borel-DRT\(_1^2\) given above is one we find quite natural. However, another possible way to state the hypothesis of this theorem would be “Whenever there are Borel codes $T_1, \ldots, T_\ell$ such that for every $X \in k^\omega$, there is an $i$ such that $X \in |T_i|$, ...”

The subtle difference lies in the fact that if $X \in \bigcup_{i < \ell} |T_i|$, the evaluation map for $X$ in that code must also prove that $X \in |T_i|$ or $X \in |T_i'|$ for each $i < \ell$. In the slight variant just mentioned, it is enough to know that for some $i$, $X \in T_i$ (and possibly have no information about $X$ in the codes $T_j$ for $j \neq i$.) This variant does not, at least on its face, lead to any conclusion about whether, or in what sense, any of the $T_i$ must be determined.

**Question 7.5.** How robust is the given formalization of Borel-DRT\(_1^2\)? In particular, is it equivalent to the variant described above?

## 8. Questions

Several directions of further questions immediately suggest themselves. Most results here concern $\omega$-models. It is not immediately clear how to formalize the statement “for every $Z$, there is a $\Delta^1_1(Z)$-generic” in reverse math. To our knowledge no one has even yet examined the reverse math strength of “$\Delta^1_1 = HY P$”. Once a reasonable reverse math way of formalizing these principles exists, it would be natural to ask how these principles are related to principles about (determined) Borel sets.

In the context of $\omega$-models, there are some gaps remaining. For example, we have seen that every $\omega$-model of DPB models $L_{\omega_1,\omega}$-CA and the existence of $\Delta^1_1$ generics.

**Question 8.1.** Suppose $M \subseteq 2^{\omega}$ is closed under join, satisfies $L_{\omega_1,\omega}$-CA, and for every $Z \in M$, there is a $G \in M$ that is $\Delta^1_1(Z)$-generic. Does it follow that $M \models$ DPB?

One way that the above question could have a negative answer would be if DPB implied some theory of hyperarithmetic analysis strictly stronger than $L_{\omega_1,\omega}$-CA.

**Question 8.2.** Which theorems of hyperarithmetic analysis are implied by DPB, and which are incomparable with it?

We built an $\omega$-model of DPB by adjoining many mutually $\Sigma^1_1$-generics.
**Question 8.3.** Does every $\omega$-model of DPB contain a $\Sigma^1_1$-generic?

Whether in $\omega$-models or full reverse math, many other theorems involving Borel sets may now have interesting reverse math content when considering their determined versions. A forthcoming paper will address the strength of “Every determined Borel set is measurable.” We mention that the statement “Every determined Borel set has the perfect set property” is equivalent to $\text{ATR}_0$, because “Every closed set has the perfect set property” already implies $\text{ATR}_0$ by [Sim09, V.5.5], so here the way of defining a Borel set does not add additional strength.

Turning now to Borel-$\text{DRT}_3^3$, we have seen that any $\omega$-model of it is closed under hyperarithmetic reduction.

**Question 8.4.** Is Borel-$\text{DRT}_3^3$ a theory of hyperarithmetic analysis?

Finally, there is the issue of robustness. There are some possible variations on what could be considered as an evaluation map. For example, a weaker version of an evaluation map would be a partial function $f : \subseteq T \to \{0, 1\}$ such that $f(\lambda)$ is defined; and whenever $\sigma \in T$ is a $\cup$, and $f(\sigma) = 1$, there is an $n$ such that $f(\sigma \cap n) = 1$; and whenever $\sigma \in T$ is a $\cap$ and $f(\sigma) = 1$, for all $n$, $\sigma \cap n \in T$ implies $f(\sigma \cap n) = 1$; and similarly for when $f(\sigma) = 0$. We did not investigate, but it would be interesting to know, the extent to which the results of this paper are robust under variations of this definition.

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