Integrability of weak distributions on Banach manifolds

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Abstract

This paper concerns the problem of integrability of non closed distributions on Banach manifolds. We introduce the notion of weak distribution and we look for conditions under which these distributions admit weak integral submanifolds. We give some applications to Banach Lie algebroids and Banach Poisson manifold. The main results of this paper generalize the works presented in [ChSt], [Nu] and [Gl].

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1 Introduction

In differential geometry, a distribution on a smooth manifold $M$ is an assignment $D : x \mapsto D_x \subset T_x M$ on $M$, where $D_x$ is a subspace of $T_x M$. The distribution is integrable if, for any $x \in M$ there exists an immersed submanifold $f : L \rightarrow M$ such that $x$ belongs to $f(L)$ and for any $z \in L$ we have $Tf(T_z L) = D_f(z)$. On the other hand, $D$ is called involutive if, for any vector fields $X$ and $Y$ on $M$ which are tangent to $D$, the Lie bracket $[X, Y]$ is also tangent to $D$. The distribution is invariant if for any vector field $X$ tangent to $D$, the flow $\phi^X_t$ leaves $D$ invariant (see 2.1).

On finite dimensional manifold, when $D$ is a subbundle of $TM$, the classical Frobenius Theorem gives an equivalence between integrability and involutivity. In the other case, the distribution is "singular" and even under assumptions of smoothness on $D$, in general, the involutivity is not a sufficient condition for integrability (we need some more additional local conditions). These problems were clarified and resolved essentially in [Su], [St] and [Ba].

In the context of Banach manifolds, the Frobenius Theorem is again true, for distributions which are complemented subbundles in the tangent bundle. For singular distributions, some papers ([ChSt], [Nu] for instance) show that, when the distribution is closed and complemented (i.e. $D_x$ is a complemented Banach subspace of $T_x M$), we have equivalence between integrability and invariance. Under sufficient conditions about local involutivity we also get a result of integrability. A more recent work ([Gl]) proves analog results without the assumption that the distribution is complemented.

According to the notion of "weak submanifolds" in a Banach manifold introduced in ([El], [Pe]), in this paper, we consider "weak distributions": $D_x$ can be not closed in $T_x M$ but $D_x$ is endowed with its own Banach structure, so that the inclusion $D_x \rightarrow T_x M$ is continuous. Such a category of distributions takes naturally place in the framework of Banach Lie algebroids (morphisms from a Banach bundle over a Banach manifold into the tangent bundle of this manifold). Under conditions
of "local triviality”, our result can be seen like generalization as well of results of [Su] and [St], than of results of [Nu] and [Gl]. Note that, our proofs take in account Remarks of Balan in [Ba], about results of [Su] and [St] (see Observations 3.4).

The first section contains the most important definitions and properties about weak distributions. It contains also the first result of equivalence between integrability and invariance (Theorem 1), under local lower triviality assumption. This last property is, in fact, a generalization of the classical notion of "smoothness" for distributions (see Observations 2.7). In the second section, we adapt the arguments used in [ChSt] to our context: under condition of "Lie invariance”, we give a generalization of their results about the integrability of distributions (Theorem 2). In the second part, under the assumption of "upper triviality" (which is a general context for anchored bundles), we give some conditions of "local involutivity" which gives rise to an integrability property (Theorem 4). In the last section, we give some applications of these results in the context of Banach Lie algebroids and Banach Poisson manifold (cf [OdRa1] and [OdRa2]).

2 Integrability and invariance

2.1 Preliminaries and context

Let $M$ be a smooth connected Banach manifold modelled on a Banach space $E$. We denote by $C^\infty(M)$ the ring of smooth functions on $M$ and by $X(M)$ the Lie algebra of smooth vector fields on $M$. A local vector field $X$ on $M$ is a smooth section of the tangent bundle $TM$ defined on an open set of $M$ (denoted by Dom$(X)$). Let be $X_L(M)$ the set of all local vector fields on $M$. Such a vector field $X \in X_L(M)$ has a flow $\phi^X_t$ which is defined on a maximal open set $\Omega_X$ of $M \times \mathbb{R}$.

Using the terminology introduced in [El] or [Pe], a weak submanifold of $M$ is a pair $(N,f)$ where $N$ is Banach manifold modelled on a Banach space $F$ and $f : N \to M$ a smooth map such that:

1. there exists an injective continuous linear map $i : F \to E$ between these two Banach spaces;
2. $f$ is injective and the tangent map $T_nf : T_xN \to T_{f(x)}M$ is injective for all $x \in N$.

Remark 2.1

Given a weak submanifold $f : N \to M$, on the subset $f(N)$ in $M$ we have two topologies:

1. the induced topology from $M$
2. the topology for which $f$ is a homeomorphism from $N$ to $f(N)$.

With this last topology, via $f$, we get a structure of Banach manifold modelled on $F$. Moreover, the inclusion from $f(N)$ into $M$ is continuous as map from the Banach manifold $f(N)$ to $M$. In particular, if $U$ is open in $M$, then, $f(N) \cap U$ is an open set for the topology of the Banach manifold on $f(N)$.

Note that in [El] and [Pe] the definition of a "weak manifold" imposes that these two topologies are identical. Our definition is somewhat different and is motivated by the notion of "weak immersion" introduced in [OdRa1] and [OdRa2].

In this work, a weak distribution on $M$ is a map $D : x \to D_x$ which, for every $x \in M$, associates a vector subspace $D_x$ in $T_xM$ (not necessarily closed) endowed with a norm $\| \cdot \|_x$ so that $(D_x,\| \cdot \|_x)$ is a Banach space (denoted by $\tilde{D}_x$) and such that the natural inclusion $i_x : \tilde{D}_x \to T_xM$ is continuous.

Remark 2.2

When $D_x$ is closed, via any chart, we get a norm on $T_xM$ which induces a Banach structure on $D_x$. So if $D_x$ is closed for all $x \in M$, the previous assumption on the Banach structure $D_x$ is
always satisfied, and we get the usual definition of a distribution on $M$ (compare with [Gl], [ChSt], [Ny]). In this last situation we always endow $\mathcal{D}_x$ with this induced Banach structure and we say that $\mathcal{D}$ is closed.

Examples 2.3

1. Let $l^p$ (resp. $l^\infty$) be the Banach space of real sequences $(x_k)$ such that $\sum_{k=1}^{\infty} |x_k|^p < \infty$ (resp. absolutely bounded) and denote by $I_p$ the natural inclusion of $l^1$ in $l^p$, $p > 1$ or $p = \infty$. On the Banach space $l^p$, $x \mapsto \mathcal{D}_x = x + I_p(l^1)$ is a weak distribution which is not closed.

2. Let $E$ and $F$ be two Banach spaces and $T : F \to E$ a continuous operator. Denote by $\hat{T} : F/\ker T \to E$ the canonical quotient bijection associated to $T$ that is

\[
\begin{array}{ccc}
F & \xrightarrow{\hat{T}} & F/\ker T \\
\downarrow & & \downarrow \\
T(F) & \xrightarrow{T} & E
\end{array}
\]

We can endow $T(F)$ with the structure of Banach space such that $\hat{T}$ is an isometry. On $E$, the assignment $x \mapsto \mathcal{D}_x = x + T(F)$ is a weak distribution. This distribution is closed if and only if $T(F)$ is closed in $E$.

3. Let $L(F, E)$ be the set of continuous operators between the Banach spaces $F$ and $E$. Given a smooth map $\Psi : E \to L(F, E)$, we denote by $\Psi_x$ the continuous operator associated to $x \in E$. As in (1) denote by $\hat{T}_x$ the canonical bijection associated $\Psi_x$ and we endow $\mathcal{D}_x = \Psi_x(F)$ with the Banach structure such that $\hat{T}_x$ is an isometry. Then, $x \mapsto \mathcal{D}_x$ is a weak distribution on $E$ which is closed if and only if $\mathcal{D}_x$ is closed and $\mathcal{D}_x$ is a weak distribution for any $x \in E$.

A vector field $Z \in \mathcal{X}(M)$ is tangent to $\mathcal{D}$, if for all $x \in \text{Dom}(Z)$, $Z(x)$ belongs to $\mathcal{D}_x$. The set of local vector fields tangent to $\mathcal{D}$ will be denoted by $\mathcal{X}_\mathcal{D}$.

We say that $\mathcal{D}$ is generated by a subset $\mathcal{X} \subset \mathcal{X}_L(M)$ if, for every $x \in M$, the vector space $\mathcal{D}_x$ is the linear hull of the set $\{Y(x) : Y \in \mathcal{X}, x \in \text{Dom}(Y)\}$.

For a weak distribution $\mathcal{D}$ on $M$, we have the following classical properties:

- an integral manifold of $\mathcal{D}$ through $x$ is a weak submanifold $f : N \to M$ such that $f(\tilde{x}) = x$ for some $\tilde{x} \in N$ and and $T_{\tilde{y}}f(T_{\tilde{y}}N) = \mathcal{D}_{f(\tilde{y})}$ for all $\tilde{y} \in N$.

- $\mathcal{D}$ is called integrable if for any $x \in M$ there exists an integral manifold $N$ of $\mathcal{D}$ through $x$.

- if $\mathcal{D}$ is generated by $\mathcal{X} \subset \mathcal{X}_L(M)$, then $\mathcal{D}$ is called $\mathcal{X}$-invariant if for any $X \in \mathcal{X}$, the tangent map $T_x\phi^X_N$ sends $\mathcal{D}_x$ onto $\mathcal{D}_{\phi^X_N(x)}$ for all $(x, t) \in \Omega_X$. If $\mathcal{D}$ is $\mathcal{X}_\mathcal{D}$-invariant we simply say that $\mathcal{D}$ is invariant.

Now, we introduce two properties of "local triviality" which will play an essential role in the whole paper:

- $\mathcal{D}$ is (locally) lower trivial (lower trivial for short) if, for each $x \in M$, there exists an open neighbourhood $V$ of $x$, a smooth map $\Theta : \mathcal{D}_x \times V \to TM$ (called lower trivialization) such that:

(i) $\Theta(\mathcal{D}_x \times \{y\}) \subset \mathcal{D}_y$ for each $y \in V$

(ii) for each $y \in V$, $\Theta_y \equiv \Theta(\cdot, y) : \mathcal{D}_x \to T_yM$ is a continuous operator and $\Theta_x : \mathcal{D}_x \to T_xM$ is the natural inclusion $i_x$. 

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(iii) there exists a continuous operator \( \tilde{\Theta}_{y} : \tilde{D}_{x} \to \tilde{D}_{y} \) such that \( i_{y} \circ \tilde{\Theta}_{y} = \Theta_{y} \), \( \tilde{\Theta}_{y} \) is an isomorphism from \( \tilde{D}_{x} \) onto \( \Theta_{y}(\tilde{D}_{x}) \) and \( \tilde{\Theta}_{x} \) is the identity of \( \tilde{D}_{x} \).

\( \bullet \) \( D \) is called (locally) upper trivial (upper trivial for short) if, for each \( x \in M \), there exists an open neighbourhood \( V \) of \( x \), a Banach space \( F \) and a smooth map \( \Psi : F \times V \to TM \) (called upper trivialization) such that:

(i) for each \( y \in V \), \( \Psi_{y} \equiv \Psi(\cdot, y) : F \to T_{y}M \) is a continuous operator with \( \Psi_{y}(F) = D_{y} \);
(ii) \( \ker \Psi_{x} \) complemented in \( F \);
(iii) if \( F = \ker \Psi_{x} \oplus S \), the restriction \( \theta_{y} \) of \( \Psi_{y} \) to \( S \) is injective for any \( y \in V \);
(iv) \( \Theta(u, y) = (\theta_{y} \circ \theta_{x}^{-1})^{-1}(u, y) \) is a lower trivialization of \( D \).

In this case the map \( \Theta \) is called the associated lower trivialization.

**Examples 2.4**

1. The distribution \( D_{x} = x + T(F) \) (where \( T : F \to E \) is a bounded operator as in Example 2.3) on \( E \) is lower trivial. This distribution is upper trivial if and only if \( \ker T \) is complemented in \( E \).

2. Let be \( \Sigma \) a closed topological subset of a Banach space \( E \) and again \( T : F \to E \) a bounded operator with \( T(F) \neq E \). We consider the distributions \( D \) and \( D' \) on \( E \) defined in the following way:
   \( D_{x} = x + T(F) \) if \( x \in \Sigma \) and \( D_{x} = x + E \) otherwise;
   \( D'_{x} = x + E \) if \( x \in \Sigma \) and \( D'_{x} = x + T(F) \) otherwise.
   It is easy to see that \( D \) and \( D' \) are weak distributions on \( E \). Then \( D \) is lower trivial but not upper trivial and \( D' \) is neither lower trivial, nor upper trivial.

For an illustration of the property of lower local triviality and upper local triviality in a more large context, we give the following result, which is a generalization of Example 2.3.

**Proposition 2.5**

Let \( D : x \to D_{x} \subset T_{x}M \) be a field on \( M \) of normed subspaces. Suppose that for each \( x \in M \), there exists an open neighbourhood \( V \) of \( x \), a Banach space \( F \) and a smooth map \( \Psi : F \times V \to TM \) such that:

1. for each \( y \in V \), \( \Psi_{y} \equiv \Psi(\cdot, y) : F \to T_{y}M \) is a continuous operator such that \( \Psi_{y}(F) = D_{y} \).
2. Assume that \( \ker \Psi_{x} \) is complemented (i.e. \( F = \ker \Psi_{x} \oplus S \)). Then there exists an open neighbourhood \( W \) of \( x \) such that the restriction \( \theta_{y} \) of \( \Psi_{y} \) to \( S \) is injective for any \( y \in W \), and then \( \Theta(u, y) = (\theta_{y} \circ \theta_{x}^{-1})^{-1}(u, y) \) is a lower trivialization of \( D \). So, \( D \) is upper trivial, and \( \Psi \) is an upper trivialization of \( D \).

The context of Proposition 2.5 can be found in the framework of Banach Poisson manifold \( (M, \{ \cdot, \cdot \}) \) where \( \Psi : T^{*}M \to TM \subset T^{**}M \) is the canonical morphism associated to the Poisson structure (see for instance \([\text{OdRa1}]\) and \([\text{OdRa2}]\)).
Corollary 2.6
Let $\pi: F \to M$ be a Banach fiber bundle over $M$ with typical fiber $F$ and $\Psi : F \to TM$ a morphism of bundle. If the kernel of $\Psi$ is complemented in each fiber, then $D$ is an upper trivial weak distribution.

Observations 2.7

1. Recall the definition of "differentiability" (resp. "smoothness") of a closed and complemented distribution $D$, introduced in [ChSt] (resp. [Nu]): there exists some neighbourhood $V$ of $x \in M$ on which $TM$ is trivializable and there exists on $V$ a smooth field $y \to \Theta_y$ of isomorphisms from $T_x M$ to $T_y M$ so that $\Theta_x = Id_{T_x M}$ and $\Theta_y[D_x] \subset D_y$. For a closed distribution, we have $\tilde{D} \equiv D$ and so $\Theta(u,y) = \Theta_y(u) (u,y) \in D_x \times V$ is a lower trivialization and so, the property of lower triviality is nothing but a generalization of "differentiability". Note that, the property of "smoothness" for a closed distributions, introduced in [Gl] and [Nu], is also a generalization of the property of "differentiability" given in [ChSt] in the same way.

2. Consider an anchored bundle on $M$, that is a Banach bundle $\pi: F \to M$ over $M$ so that we have a bundle morphism $\Psi : F \to TM$. In this situation, we are going to see that the distribution $D = \Psi(F)$ is "continuous local lower trivial" in some sense, which is quite different of our previous definition. At first, recall that, from [Ma], if $p: F \to F/K$ is the canonical projection of a Banach space $F$ onto the Banach quotient $F/K$, there always exists a continuous selection $\sigma: F/K \to F$ such that $\sigma(\lambda u) = \lambda \sigma(u)$ for any $\lambda \in \mathbb{R}$ and $u \in F/K$. However, $\sigma$ is not linear. Of course, there exists such a linear selection $\sigma$, if and only if $K$ is complemented in $F$. Come back to the situation of an anchored bundle $\Psi: F \to TM$. We fix some $x \in M$, and choose a continuous selection $\theta_x: F_x/\ker\Psi_x$. We can identify $D_x = \Psi(F_x)$ with $F_x/\ker\Psi_x$. Thus, any $u \in D_x$ can be written $\Psi \circ \theta_x(u)$. Choose any neighbourhood $U$ of $x$ such that $F_x|_U$ is equivalent to $F_x \times U$. Via the previous equivalence we consider the map $\theta: D_x \times U \to TM$ defined by:

$$\theta(u,y) = \Psi(\theta_x(u), y)$$

Then $\theta$ is continuous and in general not linear in the first variable, and is smooth in the second variable. So, we can consider $\theta$ as a kind of local lower trivialization which, in general, is only continuous and not linear on the fiber $D_x$. On the other hand, this construction can be considered as a kind of "smoothness" of $D$: via such a (local) map $\theta$, each $u \in D_x$ can be extended to a smooth vector field which is tangent to $D$ (compare with [OdRa1] p 35).

Of course if $\theta_x$ is linear, we get a lower trivialization associated to $\Psi$ (see the proof of Proposition 2.5).

However, in this context, our criteria of integrability works only when we have a lower trivialization associated to an upper trivialization. For this reason, we have introduced the property of upper triviality.

3. In finite dimension, the definition of "differentiability" given in [St] is the same definition as that given in [ChSt]. On the other hand, in [St], a distribution is called smooth, if there exists a subset $X$ which generates $D$. In this context, for any $x \in M$ there exists a neighbourhood $V$ of $x$ and vector fields $X_1, \cdots, X_p$ which are defined and linearly independent on $V$ such that $\{X_1(x), \cdots, X_p(x)\}$ is a basis of $D_x$ so $D$ is lower trivial. On the other hand, for any $x \in M$, denote by $X_{D_x}$ the module of germs vector fields at $x$ which are tangent to $D$. If $X_{D_x}$ is finitely generated, then $D$ is upper trivial.

We end this section with the proof of Proposition 2.5 and its Corollary. For this proof, we need the following Lemma which will be also used later:
Lemma 2.8

1. Consider two Banach spaces $E_1$ and $E_2$ and $i : E_1 \to E_2$ an injective continuous operator. Let $y \to \Theta_y$ be a smooth field of continuous operators of $L(E_1, E_2)$ on an open neighbourhood $V$ of $x \in E_1$ such that $\Theta_x = i$. Then there exists a neighbourhood $W$ of $x$ in $V$ such that for each $y \in W$, $\Theta_y$ is an injective operator.

2. Let $f : U \to V$ be a $C^1$ map from two open sets $U$ and $V$ in Banach spaces $E_1$ and $E_2$ respectively, such that $T_u f$ is injective at $u \in U$. Then there exists an open neighbourhood $W$ of $u$ in $U$ such that the restriction of $f$ to $W$ is injective.

Proof of Lemma 2.8

There exists an open ball $B(x, r)$ included in $V$ such that $|\Theta_y - \Theta_x| \leq K|y - x|$ for any $y \in B(x, r)$. We can suppose that $r < 1$. According to Hahn-Banach theorem, there exists an open ball $B(x, r/n)$ included in $x$ such that $\Theta_x(h_n) = 0$. We have of course:

$$< \alpha, \Theta_x(h_n) > = 0$$

for all $\alpha \in E_2^*$.

It follows that we have:

$$| < \alpha, \Theta_x(h_n) > | = | < \alpha, (\Theta_x - \Theta_x)(h_n) > | \leq \frac{K||\alpha||}{n}$$

On the other hand, $\Theta_x = i$ is a continuous bijective operator from the Banach space $E_1$ onto the normed subspace $F = i(E_1)$ in $E_2$. So, the transpose operator $i^* \in L(F^*, E_1^*)$ is a monomorphism with a dense range (see [Ha, HaMb]). According to Hahn-Banach Theorem, there exists $\beta_n \in E_1^*$ such that $< \beta_n, h_n > = 1$ with $||\beta_n|| = 1$. From the density of $i^*(F^*)$, there exists $\alpha_n \in F^*$ such that $||\beta_n - i^*(\alpha_n)|| < \frac{1}{4}$, i.e. such that $\frac{3}{4} \leq ||i^*(\alpha_n)|| \leq \frac{5}{4}$. From these inequalities we get:

- $| < i^*(\alpha_n), h_n > - 1 | = | < i^*(\alpha_n) - \beta_n, h_n > | \leq \frac{1}{4}$

so we have $| < i^*(\alpha_n), h_n > | \geq \frac{3}{4}$

- as $||i^*(\alpha_n)|| \leq \frac{5}{4}$ and as $i^*$ is a monomorphism, we have $||i^*(\alpha_n)|| \geq k||\alpha_n||$ for some $k > 0$

and finally we get $||\alpha_n|| \leq \frac{5k}{4}$. On the other hand we can write:

$$| < i^*(\alpha_n), h_n > | = | < \alpha_n, i(h_n) > | \geq \frac{3}{4}$$

for any $n$.

From Hahn-Banach Theorem, we obtain the same relation with $\alpha_n \in E_2^*$. But from 3 we get:

$$| < \alpha_n, \Theta_x(h_n) > | \leq \frac{K||\alpha||}{n}$$

and so $| < \alpha_n, i(h_n) > | \leq \frac{K}{n} \frac{5k}{4k}$

for any $n$.

So we get a contradiction with 4 for $n$ large enough. So we have completed the proof of the part (1).

Let $f : U \to V$ be a map of class $C^1$. As the problem is local, without loss of generality, we can suppose that $U$ is an open ball of center $0 \in E_1$. As $f$ is of class $C^1$, there exists an open ball $B(0, r)$ such that

$$||T_u f - T_v f|| \leq K||u - v||$$

for $u, v \in B(0, r)$

\(^1\) an operator $T$ between two Banach space $E$ and $F$ is a monomorphism if we have $\inf(||T(u)||_F; ||u||_E = 1) \geq k > 0$
Moreover, we can choose $r$ so that $r < 1$.

Suppose that $f$ is not locally injective around 0. Given any pair $(u, v) \in [B(0, r)]^2$ such that $u \neq v$ but $f(u) = f(v)$, we set $h = v - u$. For any $\alpha \in E_2^*$ we consider the smooth curve $c_\alpha : [0, 1] \to \mathbb{R}$ defined by:

$$c_\alpha(t) = \alpha, f(u + th) - f(u) >$$

Of course we have $\dot{c}_\alpha(t) = \alpha, T_{u + th} f(h) >$.

Denote by $|u, v|$ the set of points $\{w = u + th, t \in [0, 1]\}$. As we have $c_\alpha(0) = c_\alpha(1) = 0$, from Rolle’s Theorem, there exists $u_n \in [u, v]$ such that

$$< \alpha, T_{u_n} f(h) > = 0$$

Replacing $h$ by $\frac{h}{||h||}$, we can suppose in (6) that $||h|| = 1$.

From our assumption it follows that, for each $n \in \mathbb{N}^*$, there exists $u_n$ and $v_n$ in $B(x, r/n)$ so that $u_n \neq v_n$ but $f(u_n) = f(v_n)$. So from the previous argument, for any $\alpha \in E_2^*$, we have

$$< \alpha, T_{u_n, n} f(h_n) > = 0$$

for some $u_{\alpha, n} \in [u_n, v_n]$ and with $h_n = \frac{v_n - u_n}{||v_n - u_n||}$

From (5) and (7), we get

$$| < \alpha, T_0 f(h) > | = | < \alpha, [T_0 f - T_{u_n, n} f](h_n) > | \leq ||\alpha|| \frac{K_r}{n} < ||\alpha|| \frac{K}{n}.$$  (8)

for any $\alpha \in E_2^*$.

Now, we can use the same argument as in part (1) and we get again a contradiction.

$\triangle$

**Proof of Proposition 2.3**

At first, for any $x \in M$, we have a natural Banach structure on $\mathcal{D}_x$ (again denoted by $\hat{\mathcal{D}}_x$) such that the natural morphism $\Psi_x : F/ \ker \Psi_x \to \hat{\mathcal{D}}_x$ is an isometry. On the other hand, take a local trivialization of $TM$ on a neighbourhood $W$ of $x$; so we have $TM \equiv E \times W$. In this context, on $W$, $\Psi$ can be identified with a smooth field of continuous operators $\Psi_y : F \to E$ such that $\mathcal{D}_y = \Psi_y(F) \times \{y\} \subset E \times \{y\} \equiv T_y M$. Let us consider the following commutative diagram:

\[
\begin{array}{ccc}
F & \xrightarrow{\theta} & F/ \ker \Psi_y \\
\downarrow & & \downarrow \\
\mathcal{D}_y & \xrightarrow{\Psi_y} & \Psi_y(F)
\end{array}
\]

where $q$ is the natural projection and $\hat{\Psi}_y$ is the natural bijection induced by $\Psi_y$. So, if we consider the Banach structure $\hat{\Psi}_y$, we get a continuous operator $\hat{\psi}_y = \hat{\Psi}_y \circ q : F \to \hat{\mathcal{D}}_y$ so that $\Psi_y = i_y \circ \hat{\psi}_y$.

Assume that $F = \ker \Psi_x \oplus S$, for some Banach space $S \subset F$. Let $\theta_y$ be the restriction to $S$ of $\Psi_y$ for any $y \in W$. Clearly, $\theta(u, y) = (\theta_y(u), y)$ defines a smooth map from $S \times W$ into $E \times V \equiv TM$ and $\theta_y : S \times \{x\} \to E \times \{x\} \equiv T_x M$ is a continuous operator whose image is contained in $\mathcal{D}_y$.

On the other hand, let $\hat{\theta}_y$ be the restriction of $\hat{\psi}_y$ to $S$, then, $\hat{\theta}_y$ is a continuous operator from $S$ to $\hat{\mathcal{D}}_y$ so that $\theta_y = i_y \circ \hat{\theta}_y$ for any $y \in W$. Of course, $\hat{\theta}_y : S \to \hat{\mathcal{D}}_y$ is an isometry and, in particular, it is an isomorphism. As, $\theta_x$ is injective, according to Lemma 2.3, without loss of
generality, we can suppose that \( \theta_y \) is injective for any \( y \in W \). It follows that \( \tilde{\theta}_y \) is a continuous injective operator from \( S \) into \( \tilde{D}_y \). As \( \theta_y \) is injective, we have \( \ker \Psi_y \cap S = \{0\} \). It follows that \( q_1 = q_S \) is an isomorphism onto \( q(S) \subset F/\ker \Psi_y \). Of course the restriction \( q_2 \) of the isomorphism \( \Psi_y : F/\ker \Psi_y \to q(S) \) is an isomorphism onto \( \tilde{\theta}_y(S) \) such that \( \tilde{\theta}_y = q_2 \circ q_1 \). So \( \tilde{\theta}_y \) is an isomorphism of \( S \) onto \( \tilde{\theta}_y(S) \).

Finally, the map
\[
\Theta : \tilde{D}_x \times W \to E \times W \equiv TM
\]
defined by \( \Theta(u, y) = (\theta_y \circ [\theta_x]^{-1}(u), y) \) is clearly a lower trivialization of \( D \).

Proof of Corollary 2.6

Given \( x \in M \) there exists a local trivialization of \( F \) on an open set \( V \) around \( x \). So we can identify \( F \) with \( F \times V \) on \( V \). In this context, in restriction to \( V \), the morphism \( \Psi \) can be identified, as a map \( \Psi : F \times V \to TM \) which satisfies assumption (i) and (ii) of Proposition 2.5.

2.2 Results

Let \( D \) be a lower trivial distribution on \( M \). For any \( x \in M \) and any lower trivialization \( \Theta : \tilde{D}_x \times V \to TM \) and any \( u \in \tilde{D}_x \), we consider
\[
X(z) = \Theta(u, z)
\]
Of course we also have \( X(z) = i_z \circ \tilde{\Theta}(u, z) \) where \( \tilde{\Theta}(u, z) = \tilde{\Theta}_z(u) \) and \( X \) is a local vector field on \( M \) tangent to \( D \) whose domain is \( V \). Moreover, the set of all such local vector fields spans \( D \).

A lower (local) section of a lower trivial weak distribution \( D \) is a map of type (9) for any lower trivialization \( \Theta \) any \( u \in \tilde{D}_x \) and any \( x \in M \). Note that the domain of all lower section defined by such a lower trivialization \( \Theta : \tilde{D}_x \times V \to TM \) is the open set \( V \). The set of such lower sections will be denoted by \( X_D \).

The following Proposition gives a relation between integral manifolds and \( X_D \)-invariant weak distributions:

**Proposition 2.9**

If a lower trivial weak distribution \( D \) (resp. lower trivial closed distribution) is integrable, then \( D \) is \( X_D \)-invariant (resp. \( X_D \)-invariant).

In this context, we obtain the following version of Stefan-Sussmann Theorem:

**Theorem 1** Let \( D \) be a lower trivial weak distribution on a Banach manifold \( M \).

1. \( D \) is integrable if and only if it is \( X_D \)-invariant.

2. if \( D \) is integrable, on \( M \), consider the binary relation
\[
x R y \text{ iff there exists an integral manifold } (N, f) \text{ of } D \text{ such that } x, y \in f(N).
\]

Then \( R \) is an equivalence relation and the equivalence class \( L(x) \) of \( x \) has a natural structure of connected Banach manifold modelled on \( \tilde{D}_x \).

Moreover \( (L(x), i_L(x)) \), is a maximal integral manifold of \( D \) in the following sense: for any integral manifold \( (N, f) \) of \( D \), such that \( f(N) \cap L(x) \) is not empty then \( f(N) \subset L(x) \).
Taking into account Remark 2.10, the property of lower triviality of a weak distribution corresponds to the usual assumptions on the distribution that we find in [CBSt], [Nu], [Gl]. When \( \mathcal{D} \) is closed (resp. complemented) in \( T_xM \) the following Corollary of Theorem 1 gives exactly the main result of integrability of distributions we can find in [Gl] (resp. CBSt, Nu):

**Corollary 2.10**

For a lower trivial closed distribution the following propositions are equivalent:

(i) \( \mathcal{D} \) is integrable;

(ii) \( \mathcal{D} \) is \( \mathcal{X}_{\mathcal{D}} \)-invariant;

(iii) \( \mathcal{D} \) is \( \mathcal{X}_{\tilde{\mathcal{D}}} \)-invariant.

We end this section with the proof of Proposition 2.10:

Consider a lower section \( X(y) = \Theta(u, y) \), associated to a lower trivialization \( \Theta : \tilde{\mathcal{D}}_x \times V \to TM \). So \( \text{Dom}(X) = V \). Fix such a lower section \( X \) and the associated lower trivialization \( \Theta \). Denote by \( \triangle \) (resp. \( \tilde{\triangle} \)) the subspace \( \mathcal{D}_x \) of \( T_xM \equiv E \) (resp. the Banach space \( \tilde{\mathcal{D}}_x \)). Let be \( \triangle_y = \Theta_y(\triangle) \) and \( \tilde{\triangle}_y \) the natural Banach structure induced by \( \mathcal{D}_y \).

Given any \( z \in V \), the map \( \Theta_y' = \Theta_y \circ [\tilde{\Theta}_2]^{-1} \) is a smooth field of continuous operators from \( \tilde{\triangle}_z \) into \( T_yM \equiv E \times \{ y \} \) and moreover, \( \Theta_{y}' = \Theta_y \circ [\tilde{\Theta}_2]^{-1} \) is an isomorphism between \( \triangle_z \) and \( \tilde{\triangle}_y \).

Of course, if \( v = \tilde{\Theta}_2(u) \), we have \( X(y) = \Theta_y'(v) \).

Let \( f : N \to M \) be an integral manifold of \( \mathcal{D} \) passing through some \( z \in V \). Then, \( N \) is a Banach manifold modelled on the Banach space \( \tilde{\mathcal{G}} = \tilde{\triangle}_z \). For any open neighbourhood \( U \) of \( z \) the set \( \tilde{U} = f^{-1}(U) \) is an open neighbourhood of \( \tilde{z} = f^{-1}(z) \). According to Remark 2.1 without loss of generality, we may assume that \( N \) is an open set in \( \tilde{\mathcal{G}} \) and \( M \) is an open set in \( E \). In these identifications, \( f \) is the natural inclusion \( i_N \) of \( N \) in \( M \), that is the restriction to \( N \) of the natural inclusion \( i : \tilde{\mathcal{G}} \to E \). In this context, on \( i(N) \subset M \), \( y \to \Theta_y' \) is a smooth field of continuous linear operators from \( \tilde{\triangle}_z \subset \tilde{\mathcal{G}} \) into \( \mathcal{D}_y \equiv E \times \{ y \} \). Moreover, \( \Theta_y' \) is an isomorphism between \( \tilde{\triangle}_z \subset \tilde{\mathcal{G}} \) and \( \triangle_y \subset \tilde{\mathcal{G}} \times \{ y \} \) for any \( y \in i(N) \).

**Lemma 2.11**

With the previous notations, the map \( y \mapsto \Theta_y' \) from \( N \) to \( L(\tilde{\triangle}_z, \tilde{\mathcal{G}}) \) is smooth (for the topology induced by \( \tilde{\triangle}_z \) on \( N \)).

From Lemma 2.11, \( \tilde{Y} = \Theta_y'(v) \) is a smooth vector field on the Banach manifold \( N \), and, moreover, and we have \( X(i(y)) = T_y[i\tilde{Y}](y) = (i_*Y)(y) \) on \( i(N) \). So the flow \( \phi_t^X \) satisfies the relation

\[
\phi_t^X \circ i = i \circ \phi_t^{\tilde{Y}}
\]

on a small neighbourhood \( W \) of \( z \) and for all \( t \) such that \( \phi_t^{\tilde{Y}} \) is defined on \( N \). Of course for any \( y \in W \) and \( t \) such that \( \phi_t^{\tilde{Y}}(y) \) is defined, we have

\[
T_y \phi_t^X(D_y) = T_y \phi_t^{\tilde{Y}}(i[T_yN]) = i \circ T_y \phi_t^{\tilde{Y}}(T_yN) = i[T_{\phi_t^{\tilde{Y}}(y)}N] = i[D_{\phi_t^{\tilde{Y}}(y)}] = D_{\phi_t^{\tilde{Y}}(y)}.
\]

Now, consider any \( (z, t) \in \Omega_X \). Denote by \( [\alpha_z, \beta_z] \) the maximal interval on which \( \Phi_t^X(z) \) is defined. Given any \( \tau \in [\alpha_z, \beta_z] \), consider the integral curve \( \gamma(t) = \Phi_t^X(z) \) for \( t \in [0, \tau] \). By compactness of \( [0, \tau] \) there exists a finite number of integral manifolds \( (N_1, f_1), \ldots, (N_r, f_r) \) so that \( \gamma([0, \tau]) \) is contains in \( \bigcup_{i=1}^r f_i(N_i) \). Using the previous argument, by induction, we obtain:

\[
T_z \phi_t^X(\mathcal{D}_z) = D_{\phi_t^{\tilde{Y}}(z)}.
\]
We deduce that integrability implies $\mathcal{X}_G$–invariance.

Now, if moreover $\mathcal{D}$ is closed, given an integral manifold $f : N \to M$ and any local section $X$ of $\mathcal{D}$ whose domain intersects $f(N)$, then $X$ induces, by restriction on $f(N)$, a smooth vector fields on $N$. So the same arguments used last part of in the previous proof works too. ( see [Gl]).

**Proof of Lemma 2.11**

From convenient analysis (see [KrMl]), recall that for a map $f$ from an open set $U$ in a Banach space $E_1$ to a Banach space $E_2$ we have the equivalent following properties

(i) $f$ is smooth;

(ii) for any smooth curve $c : \mathbb{R} \to U$ the map $t \mapsto f \circ c(t)$ is smooth;

(iii) the map $t \mapsto \langle \alpha, f \circ c(t) \rangle$ is smooth for any $\alpha \in E_2^*$;

(iv) for any smooth curve $c : \mathbb{R}^2 \to U$, all partial derivatives of $f \circ c$ exist and are locally bounded.

Fix some $v \in \tilde{\Delta}_z$. Note that, for any $\alpha \in \tilde{G}^*$ we have

$$\langle \alpha, \Theta'_y(v) \rangle = \langle \tilde{\Theta}'_y]^* (\alpha), v \rangle \quad (10)$$

If $i : \tilde{G} \to G$ is the natural inclusion, we have $[\Theta'_y]^* = [\tilde{\Theta}'_y]^* \circ i^*$.

For $y \in i(N) \subset \tilde{\Delta}_z$ and $\alpha \in G^*$ fixed, we consider the map

$h(y) = [\Theta'_y]^* (\alpha) = [\tilde{\Theta}'_y]^* (i^* \alpha))$

Clearly, $h$ is a smooth map from the open $i(N)$ in the normed space $\Delta_z \subset E$ to the Banach $[\tilde{\Delta}_z]^*$. Take any smooth curve $c : \mathbb{R} \to N \subset \tilde{\Delta}_z$. As the inclusion of $\Delta_z$ into $\tilde{\Delta}_z$ is linear continuous, $c$ is also a smooth map from $\mathbb{R}$ to $N \subset \Delta_z$, the map $h \circ c$ is a smooth map from $\mathbb{R}$ to $[\tilde{\Delta}_z]^*$. We conclude that $h$ is a smooth map from $N \subset \Delta_z$ to $[\tilde{\Delta}_z]^*$.

So from (10), we see that the map $y \mapsto \langle \tilde{i}^* \alpha, \tilde{\Theta}'_y(v) \rangle$ is a smooth from $N \subset \tilde{\Delta}_z$ to $\mathbb{R}$, for any $\alpha \in G^*$. As $\tilde{i}^*(G^*)$ is dense in $\tilde{G}^*$, given any $\beta \in \tilde{G}^*$ there exists a sequence $\alpha_n \in G^*$ so that $\tilde{i}^*(\alpha_n)$ converges to $\beta$ in $\tilde{G}^*$. For simplicity, we set $g(y) = \tilde{\Theta}'_y(v)$. Consider any smooth curve $c : \mathbb{R} \to N \subset \Delta_z$.

Now on any compact $K \subset \mathbb{R}$, and for any $p \in \mathbb{N}$ we have:

$$| \langle \beta, (g \circ c)^{(p)}(t) \rangle - \langle \tilde{i}^*(\alpha_n), (g \circ c)^{(p)}(t) \rangle | \leq ||\beta - \tilde{i}^*(\alpha_n)|| \sup_{t \in K} \|(g \circ c)^{(p)}(t)\|$$

So the map $\tilde{i}^* \alpha_n, (g \circ c)^{(p)}$ converges uniformly to $\langle \beta, (g \circ c)^{(p)} \rangle$ on $K$. It follows that $\langle \beta, g \circ c \rangle$ is a smooth map for any $\beta \in \tilde{G}^*$. On one hand, we have proved that the map $y \mapsto \tilde{\Theta}'_y(v)$ is smooth for any $v \in \Delta_z$. On the other hand, we know that $\tilde{\Theta}'_y$ is a continuous operator from $\tilde{\Delta}_z$ to $\tilde{G}$. It follows from (iv) that the map $y \mapsto \tilde{\Theta}'_y$ is a smooth map from $N \subset \tilde{\Delta}_z$ into $L(\tilde{\Delta}_z, \tilde{G})$.

**2.3 Proof of Theorem 1**

**Proof of Part (1)**

At first, according to the Proposition 2.9 integrability implies $\mathcal{X}_G$–invariance. So we have to prove the converse. In fact, this proof is an adaption of arguments of Chillingworth and Stefan used in [ChSt].

△
Given \( x \in M \), we may assume that \( M \) is an open set of \( E \) and \( TM = E \times M \). We denote by \( \triangle \) (resp. \( \tilde{\triangle} \)) the normed space (resp. the Banach space) \( \mathcal{D}_x \) (resp. \( \tilde{\mathcal{D}}_x \)). From the property of lower triviality, after restricting this open if necessary, we have a smooth fields \( y \to \Theta_y \) of continuous operators from \( \tilde{\triangle} \) to \( E \). Consider the family \( \{X_u(y) = \Theta_y(u), u \in \tilde{\triangle}\} \) of smooth vector fields on \( M \). By standard argument (see [CHSt] proof of Corollary 4.2), we can choose an open ball \( B(0, r) \subset D \) so that the flow \( \phi_t^{X_u} \) is defined on an open neighbourhood \( W \) of \( x \) for all \( |t| \leq 1 \). We set \( \Phi(t, y, u) = \phi_t^{X_u}(y), t \in [0, 1], y \in W \) and \( u \in B \equiv B(0, r) \subset \tilde{\triangle} \).

**Lemma 2.12**

For any smooth map \( \Phi : \mathbb{R} \times W \times B \to E \) we denote by \( D_t \Phi(t, y, u) \) (resp. \( D_y \Phi(t, y, u) \), resp. \( D_u \Phi(t, y, u) \)) the partial derivative of \( \Phi \) according to the first (resp. the second (resp. the third)) variable, at point \((t, y, u) \in \mathbb{R} \times W \times B \). With these notations, \( u \to \Phi(t, y, u) \) is smooth. Moreover assume that \( T_x \phi_t^{X_u} [\mathcal{D}_x] = \mathcal{D}_{\phi_t^{X_u}(x)} \) for all \( t \) such that \((x, t) \in \Omega_{X_u} \) and all \( u \in B \), then we have:

\[
D_u \phi(t, x, u)(\triangle) \subset \mathcal{D}_{x(t)}
\]

where \( x(t) = \phi(t, y, u) \).

**Proof**

At first, we fix \( y \in W \) and \( u \in B \), and we set:

- \( g(t) = \phi(t, y, u) \) (the integral curve of \( X_u \) though \( y \));
- \( X(t, y, u) = X_u(y(t, u)) \);
- \( A(t) = D_y X(t, y, u) \);
- \( B(t) = D_u X(t, y, u) \).

Of course, \( A \) (resp. \( B \)) is a smooth field of operators in \( L(E, E) \) (resp. \( L(\tilde{\triangle}, E) \)). In fact, we have \( B(t) = \Theta_{y(t)} \). So, in the Banach space \( L(\tilde{\triangle}, E) \), the linear differential equation

\[
\Sigma = A \circ \Sigma + B
\]

as an unique solution with initial condition \( U(0) = 0 \) given by

\[
\Sigma - t(u) = \Gamma_t \int_0^t (\Sigma(s))^{-1} \circ \Theta_{y(s)} ds
\]

where \( \Gamma_s \) is the solution of the differential equation

\[
\dot{\Gamma} = A \circ \Gamma
\]

with initial condition \( \Gamma_0 = Id_E \).

From (10.7.3) and (10.7.4) of [Di], we obtain that \( \phi \) is smooth in the third variable and we have

\[
D_u \phi(t, y, u) = \Sigma - t(u).
\]

We now look for the integral curve \( x(t) \) through \( x \). In this case, \( \Gamma_s \) is in fact the \( t \to \phi_t^{X_u}(x) \) (see [Di] (10.8.5)), taking in account our assumption of invariance by \( \phi_t^{X_u}(x) \), we have:

\[
\Gamma_s(\mathcal{D}_x) = \mathcal{D}_{x(s)}
\]

On the other hand, from the assumption of lower triviality, we have \( \Theta_{x(s)}(\mathcal{D}_x) \subset \mathcal{D}_{x(s)} \). So, we get

\[
(\Gamma_s)^{-1} \circ \Theta_{x(s)}(\mathcal{D}_x) \subset \mathcal{D}_x
\]

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and moreover by integration we also have
\[ \int_0^t (\Gamma_s)^{-1} \circ \Theta_{x(s)}(D_x) \subset D_x \]

Finally, using (14) and (12), we obtain the announced result.

We are now in situation to give a sufficient condition of the existence of an integral manifold through \( x \in M \):

**Proposition 2.13**

Consider the map:
\[ \Phi : B \rightarrow M \text{ defined by } \Phi(u) = \Phi(1,x,u) = \phi_t^{X_u}(x), \text{ for } u \in B \equiv B(0,r) \subset \tilde{\triangle}. \]  

(15)

There exists \( \delta > 0 \) such that \((B(0,\delta), \Phi)\) is a weak submanifold of \( M \). Moreover, if we have \( T_x \phi_t^{X_u}[D_x] = D_{\phi_t^{X_u}(x)} \) for all \( t \) such that \((x,t) \in \Omega_{X_u}\) and all \( u \in B \), then, for \( \delta > 0 \) small enough, \((B(0,\delta), \Phi)\) is an integral manifold of \( D \) through \( x \)

It is clear that Proposition 2.13 _ends the proof of part (1) of Theorem 1._

We now end this subsection with the proof of the previous Proposition.

**Proof**

According to Lemma 2.8 it follows that, for \( \delta > 0 \) small enough, \((B(0,\delta), \Phi)\) is a weak submanifold of \( M \).

Assume now that \( T_x \phi_t^{X_u}[D_x] = D_{\phi_t^{X_u}(x)} \) for all \( t \) such that \((x,t) \in \Omega_{X_u}\) and all \( u \in B \). From Lemma 2.12 for any \( u \in B \subset \tilde{\triangle} \), we have:

\[ D_u \Phi(\tilde{\triangle}) \subset D_{\Phi(u)}. \]

So, it follows that
\[ T_x \Phi(\tilde{\triangle}) \subset D_{\Phi(u)} \]

for all \( u \in B \).

Now, according to the assumption of invariance, we have
\[ [T_x \phi_t^{X_u}]^{-1} \circ T_x \Phi(\tilde{\triangle}) \subset [T_x \phi_t^{X_u}]^{-1}(D_{\Phi(u)}) = D_x \equiv F \]  

(16)

We set \( \Lambda_u = [T_x \phi_t^{X_u}]^{-1} \circ T_x \Phi \) for \( u \in B \). In particular, \( \Lambda_u \) is a continuous operator from the Banach space \( \tilde{\triangle} \) to the normed space \( \triangle \). The part (1) will be a consequence of the following Lemma:

**Lemma 2.14**

Let be \( E_1 \) (resp. \( E_2 \)) a Banach space (resp. a normed space). Suppose that the set \( L_s(E_1, E_2) \) of surjective operators in \( L(E_1, E_2) \) is non empty. Then, \( L_s(E_1, E_2) \) is an open set.

**Proof**

The first part of this proof is an adaptation of an argument which can be found in [QuZu].
Recall that an operator, $T \in L(E_1, E_2)$ is almost open, if for any open ball $B(0, r)$ in $E_2$, there exists an open ball $\tilde{B}(0, \rho) \subset E_1$ such that:

$$B(0, r) \subset \overline{\tilde{B}(0, \rho)}$$

Given $\alpha \in [0, 1[$, there exists $\rho > 0$ such that, for any $y \in B(0, 1)$ we can find $x_1 \in \tilde{B}(0, \rho)$ such that $||y - T(x_1)|| \leq \alpha$. So, $\frac{1}{\alpha}||y - T(x_1)|| \leq 1$, and then, there exists $x_2 \in \tilde{B}(0, \rho)$ such that

$$||\frac{1}{\alpha}(y - T(x_1)) - T(x_2)|| \leq \alpha \text{ i.e. } ||y - T(x_1) - \alpha T(x_2)|| \leq \alpha^2.$$

By induction, we can build a sequence $(x_n)$ such that $x_n \in \tilde{B}(0, \rho)$ and also

$$||y - T(x_1 + ax_2 + \cdots + \alpha^{n-1}x_n)|| \leq \alpha^n.$$ 

In the Banach space $E_1$, the series of general term $||\alpha^{n-1}x_n||$ converges. So, there exists $z \in E_1$ such that $z = \sum_{n=1}^{\infty} \alpha^{n-1}x_n$, with $||z|| \leq \frac{\rho}{1 - \alpha}$ and, of course, $y = T(z)$. It follows that $T$ must be surjective. On the other hand, the set of almost open operator in $L(E_1, E_2)$ is an open set (see [Hus, HaMB]), so the Lemma is proved.

△

Coming back to the proof of part (1), the map $T_0\Phi$ is the inclusion map of $\tilde{F}$ in $F$ and $[T_x\phi_1^{X_n}] = Id_E$ so $\Lambda_0$ is surjective. From Lemma 2.13 for $\delta > 0$ small enough, $\Lambda_u$ is surjective for all $u \in B(0, \delta)$; in particular, we get an equality

$$[T_x\phi_1^{X_n}]^{-1} o T_u \Phi(\tilde{F}) = [T_x\phi_1^{X_n}]^{-1}(\mathcal{D}(\Phi(u)))$$

in (10) which ends the proof of Proposition 2.13.

△

Proof of Part (2)

In this subsection, we will use the notations introduced in the previous one. In particular, for any $x \in M$, we associate an integral manifold $(B(0, \delta), \Phi)$ build in Proposition 2.13. Such an integral manifold will be called a slice through $x$.

At first, we must prove that the relation $\mathcal{R}$ is transitive. This fact is a direct consequence of the following Lemma:

Lemma 2.15

1. Given any integral manifold $(N, f)$ of $\mathcal{D}$ through $x \in M$, there exists a slice $(B(0, \delta), \Phi)$ such that $\Phi(0) = x$ and $f^{-1}[\Phi(B(0, \Delta))]$ is an open set in $N$.

2. For any two integral manifolds $(N, f)$ and $(N', f')$ through $x \in M$, then $f^{-1}[f(N) \cap f'(N')]$ (resp. $f'^{-1}[f(N) \cap f'(N')]$) is open in $N$ (resp. $N'$). Moreover, $L = f(N) \cup f'(N') \subset M$ has a natural structure of Banach manifold modelled on $\tilde{D}_x$ and $(L,i_L)$ is an integral manifold of $\mathcal{D}$ through $x$, where $i_L$ is the natural inclusion of $L$ in $M$.

Proof

We fix any $x \in f(N)$. Note that $N$ is a connected Banach manifold modelled on $\tilde{\Delta} \equiv \tilde{D}_x$. As the problem is local, according to Remark 2.1 we can assume that $N$ is an open subset of $\tilde{\Delta}$, $M$ is an open subset of $E \equiv T_x M$ and $f$ is the natural inclusion $i$ of $\tilde{\Delta}$ into $E$ (restricted to $N$). Consider a lower trivialization $\Theta : \tilde{\Delta} \times V \rightarrow M$ around $x$. Given any $u \in \tilde{\Delta}$, according to the arguments used in the proof of Proposition 2.10 (with $\Theta$ instead of $\Theta'$), we get that the restriction of $X_u = \Theta(u, )$ to $i(N)$ induces a vector field $Y_u$ on $i(N)$ relative to its natural Banach manifold structure. It follows that the integral curve $t \rightarrow \Phi_1^{X_u}(x)$ of $X_u$ through $x$ lies in $i(N)$. So, for $\delta$ small enough, $\Phi[B(0, \delta)]$
Let be \((N,f)\) and \((N',f')\) integral manifolds through \(x \in M\). Note that \(N\) and \(N'\) are connected Banach manifold modelled on \(\tilde{\Delta} \equiv \tilde{D}_x\). Applying part (1) for any \(z \in f(N) \cap f'(N')\) to the integral manifold \((N,f)\) (resp. \((N',f')\) we obtain that \(f^{-1}[f(N) \cap f'(N')\) (resp. \(f'^{-1}[f(N) \cap f'(N')]\) is open in \(N\) (resp. \(N'\)).

Consider \(L = f(N) \cup f'(N') \subset M\). It is clear that \(L\) is connected. From part(1), For any \(z \in L\) there exists a slice \((B(0,\delta),\Phi)\) such that \(\Phi(0) = z\) so we get a covering of \(L\) by slices. On the other hand, if we have two slices \((B(0,\delta),\Phi)\) and \((B(0,\delta'),\Phi')\) so that \(\Phi(B(0,\delta)) \cap \Phi'(B(0,\delta'))\) is not empty, then in keeping with part (1), the restriction of \(\Phi^{-1} \circ \Phi'\) to \(\Phi^{-1}[\Phi(B(0,\delta)) \cap \Phi'(B(0,\delta'))]\) is a diffeomorphism on \(\Phi^{-1}[\Phi(B(0,\delta)) \cap \Phi'(B(0,\delta'))]\). So we get a structure of connected Banach manifold on \(L\) modelled on \(\tilde{\Delta}\). Moreover, by construction, the natural inclusion \(i_L : L \rightarrow M\) is injective. As each slice is an integral manifold of \(D\) modelled on \(\tilde{\Delta}\), it follows that, that \((L,i_L)\) is an integral manifold of \(D\).

\[\triangle\]

It remains to show that any equivalent class \(L(x)\) of \(x \in M\) carries a natural structure of connected Banach manifold modelled on \(\tilde{D}_x\). Note that \(L(x)\) is the union of all the subset \(f(N)\) where \((N,f)\) any integral manifold through \(x\). So \(L(x)\) is connected. Moreover, as in the proof of part(2) Lemma 2.13 we can cover \(L(x)\) by slices and this gives rise to a natural structure of connected Banach manifold on \(L(x)\). Again, \((L(x),i_{L(x)})\) is an integral manifold of \(D\) through \(x\), which is maximal by construction.

3 Integrrability and Lie invariance

3.1 Case of lower trivial weak distribution

In this section we shall adopt the material and arguments used in [St] and [ChSt].

Let be \(X\) and \(Y\) be smooth vector fields on an open set \(U\) in a Banach space \(E\). Given any integral curve \(\gamma : I = [\alpha, \beta] \rightarrow U\) of \(X\) it is well known that the Lie bracket at some \(\gamma(t)\) is given by

\[\begin{align*}
[X,Y]\big|_{\gamma(t)} &= \frac{d}{dt} Y(\gamma(t)) - DX(Y(\gamma(t)) \quad (17)
\end{align*}\]

where \(DX\) is the differential of \(X\). Note that, for any diffeomorphism \(\phi : U \rightarrow V\), according to the "chain rule" in differentiation, the same type of formula for \(X = \phi_* X, Y = \phi_* Y\) and the integral curve \(\bar{\gamma} = \phi \circ \gamma\) of \(\bar{X}\) is compatible with as \(D\phi|[X,Y]|(\gamma(t)) = [D\phi \circ X, D\phi \circ Y]|(\phi \circ \gamma(t))\). On the other hand, (17) depends only of the map of \(Y \circ \gamma\). It follows that the following definition does not depend of the choice of the chart:

**Definition 3.1** Let be \(X\) an vector field on \(M\) and \(\gamma : I = [\alpha, \beta] \rightarrow U \subset M\) an integral curve of \(X\) whose range is contained in a chart domain \(U\) of a chart \((U,\phi)\). Given any vector field \(Y\) along \(\gamma\) (i.e. \(Y : I \rightarrow TM\) such that \(Y(t) \in T_{\gamma(t)}M\)) the Lie bracket \([X,Y]\) along \(\gamma\) is the vector field characterized by

\[\phi_*[X,Y]\big|_{\gamma(t)} = \frac{d}{dt}\phi_* Y(\gamma(t)) - D\phi_* X(Y(\gamma(t))\]

Note that the previous definition do not depends of the choice of the chart. So given any \(X \in \mathcal{X}(M)\) and any integral curve \(\gamma : I \rightarrow \text{Dom}(X)\), the Lie bracket \([X,Y]\) is well defined along \(\gamma\), for any vector field \(Y\) along \(\gamma\).
On the other hand, given any smooth curve \( \gamma : I = [\alpha, \beta] \to M \), we denote by \( T_\gamma M \) the restriction of \( TM \) to \( \gamma(I) \). For any Banach space \( G \), we denote by \( L_\gamma(G, TM) \) the bundle, over \( \gamma \), of morphisms between the trivial bundle \( G \times I \) and \( T_\gamma M \).

Let \( D \) be a lower trivial weak distribution on \( M \). Consider a local vector field \( X \) and an integral curve \( \gamma : [\alpha, \beta] \to M \) of \( X \).

**Definition 3.2**

1. An upper trivialization of \( D \) over \( \gamma \) is a smooth map \( \psi : [\alpha, \beta] \to L_\gamma(G, TM) \) (where \( G \) is some Banach space) such that, for each \( t \in [\alpha, \beta] \), the corresponding morphism \( \psi_t \in L(G, T_\gamma(t)M) \) satisfies \( \psi_t(G) = D_{\gamma(t)} \).

2. Given an upper trivialization \( \psi \) as in part 1, for each \( v \in G \), denote by \( \psi[v] \) the vector field along \( \gamma \) defined \( \psi[v](t) = \psi_t[v] \). The Lie derivative of \( \psi \) by \( X \) along \( \gamma \) is defined by:

\[
(L_X \psi)_t(v) = [X, \psi_t(v)](\gamma(t)) \tag{18}
\]

Remark that, with the previous notations, the map \( v \to (L_X \psi)_t(v) \) is linear so we get a smooth map \( L_X \psi : I \to L_\gamma(G, TM) \) so that for \( t \in I \), \( (L_X \psi)_t \in L(G, T_\gamma(t)M) \).

**Definition 3.3**

Let \( D \) be a lower trivial weak distribution.

1. Let \( \Theta : \tilde{D}_x \times V \to TM \), be a lower trivialization around \( x \), and \( X_u = \Theta(u, ) \) a lower section on \( V \).

   We say that the weak distribution \( D \) is **Lie invariant by** \( X_u \), if, for any \( y \in V \), we can find \( \varepsilon > 0 \), so that, for all \( 0 < \tau < \varepsilon \), there exists:

   - an upper trivialization \( \psi : [-\tau, \tau] \to L_\gamma(G, TM) \) of \( D \) over \( \gamma(t) = \phi_t^{X_u}(y) \) for \( t \in [-\tau, \tau] \),

   - a smooth field of operator \( \Lambda : [-\tau, \tau] \to L(G, G) \) which satisfy

\[
L_{X_u} \psi = \psi \circ \Lambda \tag{19}
\]

2. The weak distribution \( D \) is called **Lie invariant** if for any \( x \in M \) there exists a lower trivialization \( \Theta : \tilde{D}_x \times V \to TM \) such that, for any \( u \in D_x \), the weak distribution \( D \) is Lie invariant by \( X_u = \Theta(u, ) \).

As in [ChSt], we have the following Theorem but without the assumption of closeness and existence of a complement for all subspaces \( D_x \).

**Theorem 2**

Let \( D \) be a lower trivial weak distribution. The following properties are equivalent:

1. \( D \) is integrable;
2. \( D \) is Lie invariant;
3. \( D \) is \( X_{\tilde{D}_x} \)-invariant.

**Observations 3.4**

1. On finite dimensional manifold \( M \) consider a "smooth" or "differential" distribution \( D \) (see Observations 2.7 part 3). In a local context the following version of Theorem 4.7 of [Ba] (which is its central result) can be reformulated in the following way (Ballan’s proof leads exactly to this result).
2. In the same context of finite dimensional manifolds, recall the remark of Balan about the following conditions:

(a) For any \( x \in \mathbb{R}^n \) and \( 1 \leq i, j \leq p \) such that

\[ X_i(X_j(x)) = \sum_{j=1}^{p} \lambda_{ij}(t) X_j(x) \]

(b) for any \( x \in \mathbb{R}^n \) and \( 1 \leq j \leq p \), \( [X_i, X_j](x) = 0 \) for any \( i \neq j \)

(c) \( \mathcal{D} \) is generated by \( X_1(x), \ldots, X_p(x) \)

We will show that, under the previous assumptions (a), (b), (c), the condition 2 of Theorem 3.5 is fulfilled and so, we get that \( \mathcal{D} \) is integrable.

Note that using Proposition 2.3, the property of lower triviality and the stability by Lie bracket on an integral manifold we obtain easily the inverse implication.

Indeed, assume that previous assumptions (a), (b), (c), are fulfilled. At first consider a lower trivialization \( \Theta : \mathcal{D} \times V \rightarrow TM \) associated to a family of linear independent vector fields \( \{Z_1, \ldots, Z_n\} \) on a neighbourhood \( V \) of \( x \) in \( M \) (see Observations 2.7 part 3). Fix some lower section \( X_u = \sum_{j=1}^{n} u_j Z_j \) for some \( u = \sum_{j=1}^{n} u_j Z_j(x) \in \mathcal{D}_x \) and consider the integral curve

\[ \gamma(t) = \phi_t^{X_u}(x) \]

\( \gamma(t) \) is generated by \( X_u \). Fix some \( 0 < \tau < \varepsilon \). On \( [-\tau, \tau] \), we consider the smooth field of linear operators : \( t \rightarrow \psi_t, \) from \( G \) to \( T_{\gamma(t)}M \), defined by:

\[ \psi_t(v) = \sum_{j=1}^{p} v_j X_j(\gamma(t)) \]

From assumption (c) and Observations 2.7 part 3, we get an upper trivialization of \( \mathcal{D} \) along \( \gamma \).

From assumption (b), we can write

\[ [X_i, X_j](\gamma(t)) = \sum_{j=1}^{p} \lambda_{ij}(t) X_j(\gamma(t)) \]

Consider the field \( t \rightarrow \Lambda_t \) of endomorphisms of \( G \) defined by

\[ \Lambda_t(v) = \sum_{i,j=1}^{p} \lambda_{ij}(t) v_i e_j \] for any \( v = \sum_{i=1}^{p} v_i e_i \).

But we have

\[ (L_{X_u} \psi_t)(v) = [X_u, \psi_t(v)](\gamma(t)) = \sum_{i,j=1}^{p} \lambda_{ij}(t) v_i X_j(\gamma(t)) \]

So, we get the Lie invariance of \( \mathcal{D} \) and we can apply Theorem 3.5.

\( \square \)

2. In the same context of finite dimensional manifolds, recall the remark of Balan about the proof of Theorem 4.1 of [Su]. Consider the two following conditions:

(a) For any \( x \in M \) there exists vector fields \( X_1, \ldots, X_p \) defined on some neighbourhood of \( V \) of \( x \) such that

\[ (1) \mathcal{D}_x \text{ is generated by } X_1(x), \ldots, X_p(x) \]
Choose any $0 < \tau < \epsilon$

According to Theorem 1, we have only to prove the equivalence $(2) \iff (3)$.

Assume that $D$ is $X^\perp$-invariant. Let $x \in M$ be a fixed point and choose a lower trivialization $\Theta : D_x \times V \to TM$. Consider a lower section $X_u = \Theta(u, \cdot)$ and $y \in V$. Note that there exists $\epsilon > 0$ such that the integral curve $t \mapsto \phi^X_t(y)$ of $X_u$ through $y$ is defined for all $t \in [-\epsilon, \epsilon]$.

Choose any $0 < \tau < \epsilon$ and set $\gamma(t) = \phi^X_t(y)$ for $t \in [-\tau, \tau]$. From our assumption, we have $T_y \phi^X_t(D_y) = D_{\gamma(t)}$. If $i_y : D_y \to D_y$ is the natural inclusion, denote by $\psi_t = T_y \phi^X_t \circ i_y$. Set
\[ \Gamma_t = T_y \phi^X_{\gamma}. \] It is clear that \( \psi \) is an upper trivialization of \( D \) over \( \gamma \) with \( G = \tilde{D}_y \). On the other hand, we have:

\[ (L_{X_y} \psi)_t = [\dot{\Gamma}_t - DX_u(\gamma(t)) \circ \Gamma_t] \circ \psi \]

But, we have \( \dot{\Gamma_t} = DX_u(\gamma(t)) \circ \Gamma_t \) (see proof of Lemma 2424). So we obtain \( L_{X_y} \psi = 0 \) on \([-\tau, \tau]\).

Taking \( \Lambda = 0 \) in (19) we get Lie invariance for \( X_u \).

For the converse, as in [SI, ChSI] and [NM], we need the following result whose proof is somewhat different (each space \( D_x \) can be not closed here)

**Lemma 3.6**

Let \( X \) be a local vector field and \( \psi \) an upper trivialization of \( D \) defined over an integral curve \( \gamma : [-\epsilon, \epsilon] \rightarrow V = \text{dom}(X) \). Moreover we assume that, for any \( 0 < \tau < \epsilon \) there exists a smooth field \( \Lambda : [-\tau, \tau] \rightarrow L(G, G) \) such that

\[ L_X \psi = \psi \circ \Lambda. \]

Then, we have \( T_y \phi^X_\tau[D_y] = D_{\phi^X_\tau(y)} \) for all \( \tau < |t| < \epsilon \).

Now, assume that \( D \) is Lie invariant by \( X_u \); let us fix some \( y \in V = \text{Dom}(X_u) \), and take a maximal integral curve \( \gamma : [\alpha_y, \beta_y] \rightarrow V \) of \( X_u \). Consider the set \( I = \{ t \in [\alpha_y, \beta_y] : T_y \phi^X_t[D_y] = D_{\phi^X_t(y)} \} \). This set is clearly open according to Lemma 3.6. Take a sequence \((t_n)\) in \( I \) which converges to some \( t \in [\alpha_y, \beta_y] \). From the assumption (2) of the Theorem, and Lemma 3.6 applied to the point \( \phi^X_t(y) \), we have \( \phi^X_s(D_{\phi^X_t(y)} = D_{\phi^X_s(y)} \) for \( s \) in some neighbourhood \( |t - \eta, t + \eta| \) of \( t \). As we have some \( t_n \) which belongs to \( |t - \eta, t + \eta| \), we get that \( I \) is closed. So \( I = [\alpha_y, \beta_y] \) and finally we deduce that \( D \) is invariant by \( X_u \). \( \triangle \)

**Proof of Lemma 3.6**

Let \( \psi : [-\tau, \tau] \rightarrow L_\gamma(G, TM) \) be an upper trivialization of \( D \) over an integral curve \( \gamma \) of \( X \) such that \( \gamma(0) = y \in V \). Consider any smooth field of operators \( \sigma : [-\tau, \tau] \rightarrow L(G, G) \) and set \( \psi = \psi \circ \sigma \). On a chart domain, we have

\[ L_X \hat{\psi} = \hat{\psi} - DX \circ \phi^X_{\gamma} = \phi^X_{\gamma} \circ \psi \circ \phi^X_{\gamma} \circ \sigma = L_X \psi \circ \sigma + \psi \circ \sigma \]

Assume that \( L_X \psi = \psi \circ \Lambda \) for some smooth field of operators \( \Lambda : [-\tau, \tau] \rightarrow L(G, G) \). Then we have:

\[ L_X \hat{\psi} = \psi \circ \Lambda \circ \sigma + \psi \circ \sigma = \psi \circ [\Lambda \circ \sigma + \sigma] \]

Consider the solution (again denoted by \( \sigma \)) of the linear equation \( \dot{\sigma} = (\sigma \circ \sigma) \) with initial condition \( \sigma_0 = I \circ \sigma \). So \( \sigma \) is a smooth field of isomorphisms of \( G \) and in particular, for this choice of \( \sigma \), we have \( \hat{\psi}(t)[G] = D_{\gamma(t)} \) for all \( t \in [-\tau, \tau] \). Moreover, using (21), we have \( L_X \hat{\psi} = 0 \). In fact the relation (21) do not depends of the choice of the chart, so we can get the same result even if \( \gamma([-\tau, \tau]) \) is not contained in a chart domain.

So we can assume that there exists an upper trivialization \( \psi : [-\tau, \tau] \rightarrow L(G, TM) \) such that \( L_X \psi = 0 \) on \( \gamma \). Again we set \( \Gamma_t = T_y \phi^X_{\gamma} \). Then \( \Sigma_t = [\Gamma_t]^{-1} \circ \psi \) is a smooth field of continuous operators from \( G \) to \( E = T_x M \) defined on \([-\tau, \tau] \).

On a chart domain we have

\[ \hat{\psi} = \hat{\Gamma} \circ \Sigma + \Gamma \circ \hat{\Sigma} = DX_u(\gamma) \circ \Gamma \circ \Gamma^{-1} \circ \psi + \Gamma \circ \hat{\Sigma} = DX_u \circ \psi + \Gamma \circ \hat{\Sigma} \]

According to (17) and (18), on \([-\tau, \tau] \) we have

\[ L_X \psi = \Gamma \circ \hat{\Sigma} \]
Now, (22) do not depend of the choice of the chart. As $\Gamma$ and $\tilde{\Sigma}$ are intrinsically defined, so even if $\gamma([-\tau, \tau])$ is not contained in a chart domain, we can obtain the same relation.

Now from our assumption $L_X \psi = 0$, as $\Gamma_t$ is an isomorphism, we must have $\Sigma_t = \Sigma_0 = \psi_0$. We conclude that, for any $t \in [-\tau, \tau]$, we have $[\Gamma_t]^{-1} \circ \psi_t(G) = \psi_0(G) = D_y$ and finally

$$T_y \phi_t^X [D_y] = \psi_t(G) = D_{\gamma(t)}. \quad (23)$$

Now, according to the assumption in this Lemma, there exists $\varepsilon > 0$ such that, we are in the previous situation for any interval $[-\tau, \tau]$ with $0 < \tau < \varepsilon$. So (23) is true for any $0 < |t| < \varepsilon$. \hfill \triangle$

### 3.2 Case of upper trivial weak distribution

Let $D$ be an upper trivial weak distribution on $M$ (see subsection 2.1). By analogy with lower sections (see subsection 2.2), for any upper trivialization $\Psi : F \times V \to TM$ such that the associated lower trivialization $\Theta : ˜D_x \times V \to TM$, an upper section is a local vector field on $M$ defined by

$$Z(y) = \Psi(u, y) \text{ for any } u \in F \quad (24)$$

**Remark 3.7**

1. The set $X^+_D$ of upper sections generates $D$.

2. Given any upper trivialization $\Psi : F \times V \to TM$ at $x$, consider the module $X_D(V)$ of vector fields $X \in X(M)$ whose domain contains $V$ and which are tangent to $D$ on $V$. The set $X^+_D(V) = \{Z_v = \Psi(v, \cdot), v \in F \text{ is contained in } X_D(V) \text{ and of course } \{Z_v(y), v \in F\} \}$ generates $D_y$ for all $y \in V$. Moreover, if $F$ has a Schauder basis $\{e_\alpha, \alpha \in A\}$, then the convex hull of $\{Z_{e_\alpha}(y), \alpha \in A\}$ is dense in $D_y$.

3. If $\Theta : ˜D_x \times TM$ is the lower trivialization associated to the upper trivialization $\Psi$ then each lower section $X_u = \Theta(u, \cdot)$ can be written $X_u = \Theta(\Psi(u, x), \cdot)$ and so the set $X^+_D(V)$ of such lower sections is contained in $X^+_D(V)$.

Let $D$ be a upper trivial weak distribution on $M$. Let $V$ be the domain of a chart around $x \in M$. Consider a upper trivialization $\Psi : F \times V \to TM$ and $\Theta : ˜D_x \times V \to TM$ the associated lower section. Given any smooth function $\sigma : V \to F$, let $Z_\sigma = \Psi(\sigma, \cdot)$ be the associated vector field on $V$. Consider $\gamma : [-\tau, \tau] \to V$ an integral curve of $Z_\sigma$, then, $\Psi_\gamma$, defined by $\Psi_\gamma(t)[v] = \Psi(v, \gamma(t))$, is an upper trivialization of $D$ along $\gamma$, according to Definition 3.2. So the Lie derivative of $\Psi$ by $Z_\sigma$ along $\gamma$ is $L_{Z_\sigma} \Psi_\gamma$ which we simply denoted by $L_{Z_\sigma} \Psi$. We can also define directly $L_{Z_\sigma} \Psi$ by:

$$L_{Z_\sigma} \Psi_\gamma(t) = [Z_\sigma, Z_v](\gamma(t)) \text{ for any } Z_v = \Psi(v, \cdot) \quad (25)$$

**Definition 3.8**

An upper trivial weak distribution $D$ is called Lie bracket invariant if, for any $x \in M$, there exists an upper trivialization $\Psi : F \times V \to TM$ such that for any $u \in F$, there exists $\varepsilon > 0$, such that, for all $0 < \tau < \varepsilon$, there exists a smooth field of operators $\Lambda : [-\tau, \tau] \to L(F, F)$ with the following property

$$L_{X_u} \Psi = \Psi \circ \Lambda \quad (26)$$

along the integral curve $t \mapsto \phi^X_t(x)$ on $[-\tau, \tau]$ of any lower section $X_u = \Theta(\Psi(u, x), \cdot)$. \hfill \hfill

**Remark 3.9** According to (25), the property (26) is equivalent to

$$[X_u, Z_v](\gamma(t)) = \Psi(\Lambda_t(v), \gamma(t)) \text{ for any } Z_v = \Psi(v, \cdot) \quad (27)$$

along $\gamma(t) = \phi^X_t(x)$.

(27) justifies the term "Lie bracket invariant" in Definition 3.8.
With these definitions we have:

**Theorem 3**

Let $D$ be an upper trivial weak distribution. The following propositions are equivalent:

1. $D$ is integrable;
2. $D$ is Lie bracket invariant;
3. $D$ is $X_D$-invariant.

**Remark 3.10**

1. The assumption "the kernel of $\Psi$ is complemented in each fiber" is automatically satisfied if the kernel of $\Psi$ is finite dimensional or finite codimensional in each fiber, or in the context of Hilbert manifold.
2. Recall that when $M$ is a finite dimensional manifold, if, for any $x \in M$, each module $X_{D_x}$ of germs of vector fields is finitely generated then $D$ is upper trivial (see Observations 2.7 part 3.) So, the formulation Theorem 4.7 of [Ba] can be given in its original way:

   if each $X_{D_x}$ is finitely generated for any $x \in M$, then $D$ is integrable if and only if $D$ satisfies the properties (a), (b) and (c) in Theorem 3.5.

Of course, this result is a direct consequence of this last Theorem, but, we can easily see that this Theorem 4.7 can be directly deduced from Theorem 3. This proof is left to the reader.

Coming back to the context of Corollary 2.6 let $\Pi : F \to M$ be a Banach fiber bundle over $M$ with typical fiber $F$, $\Psi : F \to TM$ a morphism of bundle whose kernel is complemented in each fiber. We denote by $S(F)$ the set of local sections of $\Pi : F \to M$, that is smooth maps $\sigma : U \subset M \to F$ such that $\Pi \circ \sigma = Id_U$ where $U$ is an open set of $M$. The maximal such open set is called the domain of $\sigma$ and denoted $\text{Dom}(\sigma)$.

A subset $S$ of $S(F)$ is called a generating upper set of $D$ if for any $x \in M$, the set $X_S = \{\Psi \circ \sigma, \sigma \in S\}$ contains $X_{D_x}$. Of course $S(F)$ is a maximal generating upper set. We introduce some condition on $X_S$ which will be used in the next theorem:

$X_S$ satisfies the condition (LB) if:

for any local section $\sigma \in S$, there exists an open set $V \subset \text{Dom}(\sigma)$ on which $F$ is trivializable and for any $x \in V$ we have the following property:

given any integral curve $\gamma : ]-\epsilon,\epsilon[ \to V$ of $X = \Psi \circ \sigma$ with $\gamma(0) = x$, there exists a smooth field $\Lambda : ]-\epsilon,\epsilon[ \to \mathcal{L}(F_x,S_x)$ such that

$$[\Psi \circ \sigma, (\Psi(u, \cdot))(\gamma(t))] = \Psi(\Lambda_t(u), \gamma(t)) \text{ for any } t \in ]-\epsilon,\epsilon[ \text{ for any } u \in F_x$$

(28)

Then, using Theorem 3 we get the following Theorem

**Theorem 4**

Let $\Pi : F \to M$ be a Banach fiber bundle over $M$ with typical fiber $F$ and $\Psi : F \to TM$ a morphism of bundles such that the kernel of $\Psi$ is complemented in each fiber and denote $D = \text{Im} \Psi$.

1. Then $D$ is an integrable distribution if and only there exists a generating upper set $S$ such that $X_S$ satisfies the condition (LB) Moreover, when (LB) is satisfied, if $S_x$ fulfills $F_x = \ker \Psi_x \oplus S_x$, there exists $\Lambda : ]-\epsilon,\epsilon[ \to \mathcal{L}(F_x,S_x)$ which satisfies (28).

2. If $D$ is a closed distribution, then this distribution is integrable if and only if (LB) is satisfied where (28) can be replaced by

$$[\Psi \circ \sigma, (\Psi(u, \cdot))(\gamma(t))] \in \Psi(\gamma(t))S_x \text{ for any } t \in ]-\epsilon,\epsilon[ \text{ for any } u \in F_x$$

(29)
3.3 Proofs of Theorem 3 and Theorem 4

Proof of Theorem 3

According to Theorem 1, we have only to prove (1) \iff (2).

From Lemma 3.11 property 2 of Theorem 3 implies that for any \( x \in M \), we have \( T_x \phi_t^{\ast} (D_x) = D_{\phi_t^{\ast} (x)} \) for all \( t \) such that \((x, t) \in \Omega_{X_u}\). From Proposition 2.12 \((B(0, \delta), \Phi)\) is an integral manifold through \( x \). So \( (2) \Rightarrow (1) \).

For the converse, we will use the following Lemma:

Lemma 3.11

Assume that \( D \) is integrable. Let \( \Psi : F \times V \to TM \) be a upper trivialization, and \( \sigma : V \to F \) a smooth map and let \( X = \Psi(\sigma, \cdot) \) be the associated vector field on \( V \). Consider an integral curve \( \gamma : [-\varepsilon, \varepsilon] \rightarrow V \) of \( X \) such that \( \gamma(0) = x \). Then there exists a smooth field \( \Lambda : [-\varepsilon, \varepsilon] \rightarrow L(F, F) \) such that:

\[
L_X \Psi(v, \gamma(t)) = \Psi(\Lambda_t(v), \gamma(t))
\]

So, for \( \sigma(y) = (u, y) \), with \( u \in S \), the vector field \( Z_\sigma \) is the lower section \( X_u \) for \( u \in S \) et clearly Lemma 3.11 proves \((1) \Rightarrow (2)\).

\( \square \)

Proof of Lemma 3.11

Recall that we have assumed that \( D \) is integrable. Fix some \( x \in M \). Take an upper trivialization \( \Psi : F \times V \to TM \) around \( x \) and let \( \Theta : \tilde{D}_x \times V \to TM \) be the associated lower trivialization. We can choose \( V \) such that \( TM|_V \equiv E \times V \). Recall that we have a decomposition \( F = \ker \Psi_x \oplus S \), and \( \Theta = (\theta_y \circ \theta_x^{-1}, \cdot) \) where \( \theta_y \) is the restriction to \( S \) of \( \Psi_y \) (see the proof of Proposition 2.15). At first note that any lower section \( X_u \) can be written \( X_u = \Theta(\Psi(u, x), \cdot) \) for some \( u \in F \) and also \( X_u = \theta(u, \cdot) \) but with \( u \in S \). On the other hand, according to Lemma 2.11 \((B(0, \delta), \Phi)\) is an integral manifold of \( D \) through \( x \) (associated to \( \Theta \)). In particular, \( \theta_y \) is an isomorphism from \( S \) to \( \tilde{D}_y \). Given \( u \in F \), there exists an unique \( v \in S \) such that \( \Psi_y(u) = \theta_y(v) \) so \( u \in \ker \Psi_y \oplus S \). It follows that \( F = \ker \Psi_y \oplus S \).

Set \( N = \Phi(B(0, \delta)) \subset M \) endowed with its Banach manifold structure. Without loss of generality, we can identify \( \Psi_y \) with \( \tilde{\theta}_y (S) = \tilde{D}_y \) and so we can consider \( N \) as an open set of \( \tilde{D}_y \). Denote by \( i : \tilde{D}_x \to T_x M \equiv E \) the canonical inclusion. We have \( T_y N \equiv S \times \{ y \} \). On \( N \), each \( \Psi_y \) can be considered as an element of \( L(F, S) \). By arguments similar to those used in the proof of Lemma 2.11 we can show that \( y \mapsto \Psi_y \) is a smooth field of operators in \( L(F, S) \). So, \( y \mapsto \tilde{\theta}_y \) is a smooth field of isomorphisms of \( S \). We get a smooth field \( \pi_y = [\tilde{\theta}_y]^{-1} \circ \Psi_y \) of operators in \( L(F, S) \) with the following properties:

\[
\Psi_y = \theta_y \circ \pi_y
\]

\[
\ker \pi_y = \ker \Psi_y = \ker \Psi_y
\]

\[
\pi_y(u) = u \text{ for all } u \in S
\]

Take any smooth map \( \sigma : V \to F \). Then \( Z_\sigma(y) = \Psi_y \circ \sigma(y) \) for \( y \in V \) (resp. \( \tilde{Z}_\sigma(y) = \Psi_y \circ \sigma(y) \) for \( y \in N \)) is a smooth vector field on \( V \) (resp. on \( N \)) and we have the relations:

\[
\Psi(\sigma(y), y) = Z_\sigma(i(y)) = i[\tilde{Z}_\sigma(y)] = i \circ \tilde{\theta}_{i(y)} \circ \pi_y(\sigma(y)) = \theta_{i(y)} \circ \pi_y(\sigma(y)) = \theta(\pi_y(\sigma(y)), y)
\]

Consider the integral curves \( \gamma(t) = \phi_t^{Z_\sigma}(x) \) and \( \tilde{\gamma}(t) = \phi_t^{\tilde{Z}_\sigma}(x) \) for \( t \in [-\varepsilon, \varepsilon] \). Of course we have \( \gamma(0) = x \). For simplicity, we set:

\[
\sigma(\gamma(t)) = \sigma(t) \text{ and } \sigma(\tilde{\gamma}(t)) = \tilde{\sigma}(t)
\]
\[ P(t) = \pi_{\tilde{\gamma}(t)}. \]

Note that, using \((\ref{eq:26})\) we have
\[ \Psi(v, \gamma(t)) = \theta(P(t)[v], \gamma(t)) \] (34)

Now, in keeping with \((\ref{eq:25})\), for any \(v \in S\), we have:
\[ L_{\hat{Z}_v} \Psi(v, \gamma(t)) = [Z_\sigma, X_v](\gamma(t)) = L_{\hat{Z}_v} \theta(v, \gamma(t)). \] (35)

As we have \(Z_\sigma = i_* \tilde{Z}_\sigma\) and \(X_v = i_* \tilde{X}_v\), on the Banach manifold \(N\), we get
\[ [\tilde{Z}_\sigma, \tilde{X}_v](\tilde{\gamma}(t)) \in T_{\tilde{\gamma}(t)}N \equiv S \times \{ \tilde{\gamma}(t) \}. \]

Note that we have \([Z_\sigma, X_v] = i_* [\tilde{Z}_\sigma, \tilde{X}_v]\). It follows that we have:
\[ L_{\hat{Z}_v} \theta(v, \gamma(t)) = i_* [\tilde{Z}_\sigma, \tilde{X}_v](\tilde{\gamma}(t)) \] (36)

Without loss of generality, we can choose \(\delta > 0\) small enough such that on \(N = \Phi(B(0, \delta))\) we have:
\[ ||\dot{\theta}(. , y)|| \leq K \text{ and } ||D_2 \dot{\theta}(., y)|| \leq K \text{ for all } y \in N. \] (37)

On the other hand, we have:
\[ [\tilde{Z}_\sigma, \tilde{X}_v](\tilde{\gamma}(t)) = D_2 \dot{\theta}(v, \tilde{\gamma}(t))[\dot{\theta}(P(t)[\tilde{\sigma}(t)], \tilde{\gamma}(t))] - D_2 \dot{\theta}(\tilde{\sigma}(t), \tilde{\gamma}(t))[\tilde{\gamma}(t)]. \] \(\text{From } (37), \text{ we have:}\)
\[ ||D_2 \dot{\theta}(v, \tilde{\gamma}(t))[\dot{\theta}(P(t)[\tilde{\sigma}(t)], \tilde{\gamma}(t))]|| \leq K||v|| ||P(t)|| ||\tilde{\sigma}(t)|| \leq K^2 ||v|| ||P(t)|| ||\tilde{\sigma}(t)|| \leq K^2 ||v|| ||P|| ||\tilde{\sigma}||. \] \(\text{In the second member of } (38), \text{ the same majoration is true for}\)
\[ ||D_2 \dot{\theta}(\tilde{\sigma}(t), \tilde{\gamma}(t))[\dot{\theta}(v, \tilde{\gamma}(t))]||. \]

So we get
\[ ||[\tilde{Z}_\sigma, \tilde{X}_v](\tilde{\gamma}(t))|| \leq 2K^2 ||P|| ||\tilde{\sigma}|| ||v|| \]

It follows that, for each \(t \in [\epsilon, \epsilon]\), the map \(v \to [\tilde{Z}_\sigma, \tilde{X}_v](\tilde{\gamma}(t))\) is linear continuous map \(\tilde{\Lambda}(t)\) from \(S\) to \(S \times \{ \tilde{\gamma}(t) \}. \) We set
\[ \Lambda(t) = [\dot{\theta}_{\tilde{\gamma}(t)}]^{-1} \circ \tilde{\Lambda}(t) \]

Clearly, \(t \mapsto \Lambda(t)\) is a smooth field of endomorphisms of \(S\) and taking in account \((\ref{eq:30})\) we get
\[ L_{\hat{Z}_v} \theta(v, \gamma(t)) = \theta(\Lambda(t)[v], \gamma(t)) \] (40)

Now from \((\ref{eq:31})\), with the same argument \((\ref{eq:20})\) used in the proof of Lemma \(3.6\) we get:
\[ L_{\hat{Z}_v} \Psi(\gamma) = L_{\hat{Z}_v} \{ \theta(\gamma) \circ P \} + \theta(\gamma) \circ \dot{P} = \theta(\gamma) \circ P \circ \dot{\Lambda} + \theta(\gamma) \circ \dot{P} \] (41)

But, according to the definition of \(P\) and \((\ref{eq:32})\), we have \(P \circ \dot{P} = \dot{P}\). So using \((\ref{eq:11})\), we get:
\[ L_{\hat{Z}_v} \Psi = \Psi(\dot{\Lambda} + \dot{P}). \]

which ends the proof of Lemma \(3.11\) by setting \(\Lambda = \dot{\Lambda} + \dot{P}. \)

\(\Box\)

\textit{Proof of Theorem 4}
(1) According to the context of the proof of Corollary 2.6, for any given \( x \in M \) we consider a local trivialization of \( \mathcal{F} \) on an open set \( V \) around \( x \), so that the morphism \( \Psi \) can be identified, as a map \( \Psi: \mathcal{F}_x \times V \to TM \) and \( \mathcal{F}_x = \ker \Psi_x \oplus S_x \) and let \( \Theta: S_x \times V \to TM \) be the associated lower trivialization. In this context, taking any \( \sigma(y) = (u,x) \), for any \( u \in S_x \) in (LB) for any \( x \in M \), we get property (2) of Theorem 3 so (LB) is a sufficient condition for integrability of \( \mathcal{D} \).

Assume now that \( \mathcal{D} \) is integrable and consider an upper generating set \( \mathcal{S} \) of \( \mathcal{D} \) and any section \( \sigma \in \mathcal{S} \) defined on an open set \( U \). Fix any \( x \in U \). From Lemma 3.11 we get (28) with \( \Lambda: [\varepsilon,\varepsilon] \to L(F_x, S_x) \) and \( \mathcal{F} \) is integrable and by Lemma 3.11 we can find \( \Lambda': [\varepsilon,\varepsilon] \to L(F_x, S_x) \) which satisfies (28).

(2) If now \( \mathcal{D} \) is a closed distribution, then in each fiber \( \mathcal{F}_x = \ker \Psi_x \oplus S_x \), \( \theta_x \) is a continuous bijective morphism between both Banach space \( S_x \) and \( \mathcal{D}_x \) so \( \theta_x \) is an isomorphism. In particular, \( \mathcal{D}_x \) and \( \mathcal{D}_x \) are equivalent as Banach spaces. Coming back to the previous local context of the upper trivialization \( \Psi: \mathcal{F}_x \times V \to TM \), the map \( y \to \theta_y \) is a smooth field of isomorphisms from \( S_x \) to \( \Psi_{\gamma(t)}(S_x) \).

If \( \mathcal{D} \) is integrable, from (28) and the properties of \( \Lambda \) we obtain (29). For the converse, it is sufficient to set

\[
\Lambda_t(u) = [\theta_{\gamma(t)}]^{-1}[Z_{\sigma}, Z_u](\gamma(t))
\]

to get (28).

\[\triangle\]

4 Applications

4.1 Banach Lie Algebroid

The concept of Lie algebroid was first introduced by J. Pradines in relation with Lie groupoids (cf [Pr]). The theory of algebroids was developed by A. Weinstein ([We]) and, independently, by M. Karasev ([Ka]), in view of the symplectization of Poisson manifolds. This theory also has an important role as models in mechanics and mathematical physics (for a survey, see [Li] and [Kos2] for instance). On the other hand, this concept of Lie algebroid can be extend in the infinite dimensional case: in [KisLe] the authors build variational Lie algebroids of the infinite jet bundles over a vector bundle over a finite dimensional manifolds. This construction can be situated in the Frechet manifold framework. In fact, this context is very special: infinite jet bundles over a vector bundle over finite dimensional manifolds are projective limit of finite dimensional Banach spaces so we get a set of coordinate on such a space. The existence of these coordinates is crucial in this construction of the variational Lie algebroid. In this paper, we look for the Banach manifold context and in this framework we do not have (local) coordinates in general. According to the classical definition of a Lie algebroid in finite dimension we introduce:

**Definition 4.1**: A Banach Lie algebroid structure on a Banach bundle \( \Pi: \mathcal{A} \to M \) is a quadruple \((\mathcal{A}, \Psi, M, \{,\})\) such that

1. a bracket \(\{,\}\), i.e. is a composition law \((\sigma_1, \sigma_2) \mapsto \{\sigma_1, \sigma_2\}\) on the set of global sections \(\mathcal{S}(\mathcal{A})\) of \(\Pi: \mathcal{A} \to M\), such that, \((\mathcal{S}(\mathcal{A}), \{,\})\) has a Lie algebra structure;

2. \(\Psi: \mathcal{A} \to TM\) is a smooth vector bundle morphism;

3. the Leibniz property is satisfied: for any smooth function \(f\) defined on \(M\) and any sections \(\sigma_1, \sigma_2 \in \mathcal{S}(\mathcal{A})\) we have:

\[
\{\sigma_1, f\sigma_2\} = f\{\sigma_1, \sigma_2\} + df(Z_{\sigma_1})\sigma_2
\]

where \(Z_{\sigma_1} = \Psi \circ \sigma_1\) is the vector field associated to \(\sigma_1\).
The quadruplet \((\mathcal{A}, \Psi, M, \{,\})\) is called a Banach Lie algebroid and \(\{,\}\) (resp. \(\Psi\)) is called the Lie bracket on \(\mathcal{A}\), (resp. the anchor morphism).

As in finite dimension, the Jacobi identity and the Leibniz property implies that \(\Psi\) gives rise to a Lie algebra morphism from \(\mathcal{S}(\mathcal{A})\) into \(\mathcal{X}(M)\) i.e.

\[
[\Psi \circ \sigma_1, \Psi \circ \sigma_2] = \Psi \circ \{\sigma_1, \sigma_2\}
\]

(42)

Given some open set \(U\) in \(M\), we denote by \(\mathcal{A}_U\) the restriction of the Banach bundle \(\Pi : \mathcal{A} \to M\) to the Banach manifold \(U\): \(\mathcal{A}_U = \Pi^{-1}(U)\); the set of sections of \(\mathcal{A}_U\) will be denote by \(\mathcal{S}(\mathcal{A}_U)\).

In finite dimension, it is classical that a bracket \(\{,\}\) on a Lie algebroid \((\mathcal{A}, \Psi, M, \{,\})\) is compatible the sheaf of sections of \(\Pi : \mathcal{A} \to M\) or, for short, is localizable (see for instance [Ma]). By this property, we mean the following:

(i) for any open set \(U\) of \(M\), there exists a unique bracket \(\{,\}_U\) on the space of sections \(\mathcal{S}(\mathcal{A}_U)\) such that, for any \(s_1\) and \(s_2\) in \(\mathcal{S}(\mathcal{A})\), we have:

\[
\{(s_1|_U, s_2|_U) = (\{s_1, s_2\})|_U
\]

(ii) (compatibility with restriction) if \(V \subset U\) are open sets, then, \(\{,\}_U\) induces a unique bracket \(\{,\}_V\) on \(\mathcal{S}(\mathcal{A}_V)\) which coincides with \(\{,\}_V\) (induced by \(\{,\}_U\)).

Using the same arguments as in finite dimension, when \(M\) is smooth regular, we can prove that, for any Lie algebroid \((\mathcal{A}, \Psi, M, \{,\})\), its bracket is localizable (see [CaPe]). But, if \(M\) is not smooth regular, we can no more used this argument. Unfortunately, we have no example of Lie algebroid for which the Lie bracket is not localizable. Note that, according to [KrMi] sections 32.1, 32.4, 33.2 and 35.1, this problem is similar to the problem of localization (in an obvious sense) of global derivations of the module of smooth functions on \(M\) or the module of differential forms on \(M\). In [KrMi] and, to our known, more generally in the literature, there exists no example of such derivations which are not localizable. On the other hand, even if \(M\) is not regular, the classical Lie bracket of vector fields on \(M\) is localizable. So, there always exists an anchored bundle \(\mathcal{A} = TM\) and a Lie bracket algebroid \((TM, \text{Id}, M, [; , ])\) for which its Lie bracket is localizable. Moreover, in Examples 4.3 we do not assume that \(M\) is regular but, nevertheless, these Lie brackets are also localizable.

Thus, in the Definition 4.1 we moreover impose that, if \(M\) is not regular, then the Lie bracket of the Lie algebroid is localizable even if this assumption could be, in fact, unnecessary when \(M\) is not regular.

Remark 4.2 : 

In finite dimension we have many equivalent definitions of a Lie algebroid: a Lie algebroid structure on a vector bundle \(\mathcal{A} \to M\) may be characterized by:

- a Lie bracket on an anchored bundle \((\mathcal{A}, \rho)\);
- a linear Poisson structure on \(\mathcal{A}^*\);
- a linear Schouten structure on the exterior algebra \(\Lambda^*\mathcal{A}^*\);
- a differential operator \(d\) on the module of sections \(\mathcal{S}(\Lambda^*\mathcal{A}^*)\) with \(d \circ d = 0\).

This last approach can be interpreted in the context of supermanifolds (see [Kos1]). It is precisely this last aspect which is used in [KisLe] for the construction of variational Lie algebroids. However, in the context of Banach manifolds, we have many obstructions in the generalization of the previous equivalent definitions. For instance, when the typical fiber of \(\mathcal{A}\) is an infinite dimensional Banach space, in general, such a differential operator \(d\) could be not localizable (see [KrMi] section 35.1). Moreover, according to the fact that the bidual \(E^{**}\) of a Banach space \(E\) may contains strictly \(E\), we must impose complementary conditions on \(d\), to get a Lie algebroid structure on \(\mathcal{A}\) by this way. On the other hand, the set of sections of the graded algebra \(\Lambda^*\mathcal{A}\) is
not generated by elements of degree 0 and 1. So, we cannot extend the Lie algebroid bracket to a unique linear Schouten bracket on \( \Sigma(A^*A) \), such that \( \{s,f\} = df(\rho(s)) \) for any \( s \in \Sigma(A) \) and any smooth function \( f \) on \( M \). Such problems are studied in [CaPu].

Now, in the general situation of a Lie Banach algebroid \((A,\Psi,\{,\})\), if the kernel of \( \Psi \) is complemented in each fiber \( A_x \), then the distribution \( D = \Psi(A) \) is upper trivial. So, from Theorem 4 we then get:

Theorem 5
Let \((A,\Psi,\{,\})\) be a Banach Lie algebroid. If the kernel of \( \Psi \) is complemented in each fiber, then \( D = \Psi(A) \) is an integrable weak distribution.

Example 4.3

1. Let \( \Pi : A \to M \) be a weak subbundle of \( TM \to M \) i.e. \( A \subset TM \) , \( \Pi \) is the restriction to \( A \) of the canonical projection of \( TM \) onto \( M \), and the canonical inclusion \( i : A \to TM \) is a morphism bundle. Any section of \( \Pi : A \to M \) induces a vector field on \( M \). Assume that the set of sections \( S(A) \) is stable by Lie bracket of vector fields, which means that the associated weak distribution \( D = i(A) \) is involutive. Then, \((A,i,M,\{,\},S(A))\) is a Banach Lie algebroid. So it follows from Theorem 5 that \( D \) is an integrable distribution. Thus, we get a version of Frobenius Theorem, as we can find in [Gl], when \( \Pi : A \to M \) is a (closed) subbundle of \( TM \to M \). In the previous general situation, we can also consider this result as an appropriate version of Frobenius Theorem.

2. Let \( \Pi : A \to M \) be a Banach bundle and \( \Psi : A \to TM \) an injective morphism bundle. If \( D = \Psi(A) \) satisfies the condition (LB) of Theorem 4, then \( D \) is integrable. From the injectivity of \( \Psi \), it follows that we can define a Lie algebra structure on the sections \( S(A) \), by:
\[
\{s_1,s_2\} = \Psi^{-1}(\{\Psi(s_1),\Psi(s_2)\})
\]
So, we get a Banach Lie algebroid structure on \( A \).

3. Consider a smooth right action \( \psi : M \times G \to M \) of a connected Banach Lie group \( G \) over a Banach manifold \( M \). Denote by \( G \) the Lie algebra of \( G \). We have a natural morphism \( \xi \) of Lie algebras from \( G \) to \( X(M) \) which is defined by
\[
\xi_X(x) = T_{(x,e)}\psi(0,X)
\]
For any \( X \) and \( Y \) in \( G \), we have:
\[
\xi(X,Y) = [\xi_X,\xi_Y]
\]
where \( \{,\} \) denote the Lie algebra bracket on \( G \) (see for instance [KrMi] chap. VIII, 36.12 or [Bi]). On the trivial bundle \( M \times G \), each section can be identified with a map \( \sigma : M \to G \); we define a Lie bracket on the set of sections by
\[
\{\{\sigma,\sigma'\}\}(x) = \{\sigma(x),\sigma'(x)\} + d\sigma(\xi_{\sigma(x)}) - d\sigma'(\xi_{\sigma(x)})
\]
According to the triviality of \( M \times G \), we get a localizable Lie bracket. An anchor morphism \( \Psi : M \times G \to TM \) is defined by setting \( \Psi(x,X) = \xi_X(x) \).
It follows that \((M \times G,\Psi,\{,\})\) has a Banach Lie algebroid structure on \( M \).
Denote by \( G_x \) the closed subgroup of isotropy of a point \( x \in M \) and \( G_x \subset G \) its Lie subalgebra. Of course, we have ker \( \Psi_x = G_x \). According to Theorem 4, if \( G_x \) is complemented in \( G \) for any \( x \in M \), the weak distribution \( D = \Psi(M \times G) \) is integrable. In fact the leaf through \( x \) is its orbit \( \psi(x,G) \).
Proof of Theorem 5

We will show that the property (LB) of Theorem 4 is satisfied in our context. Of course the set \( S(\mathcal{A}) \) of local sections of \( \mathcal{A} \) is a generating upper set. Suppose that we have a structure of Banach Lie algebroid on \( \mathcal{A} \). As (LB) is a local property, we may assume that \( M \) is an open set of \( E \) and \( \mathcal{A} \equiv F \times M \) if \( F \) is the typical fiber of \( \mathcal{A} \). So we adopt the (local) notation used in the proof of Theorem 4.

Consider any section \( \sigma \in S(\mathcal{A}) \) and fix some \( x \in V \). Again, we set \( Z_\sigma = \Psi \circ \sigma \) and \( Z_u = \Psi(u, ) \) an upper section. Given an integral curve \( \gamma(t) = \phi^\varepsilon_t(x) \) on \([\varepsilon, \varepsilon]\), from [OdRa2], we have
\[
[Z_\sigma, Z_u](\gamma(t)) = \Psi([\sigma, s_u](\gamma(t))) \text{ where } s_u(x) = (u, x).
\]
But, using the same arguments as the ones used in the proof of Lemma 3.11 we can show that the map
\[
t \mapsto \{\sigma, s_u\}(\gamma(t))
\]
is a smooth field of endomorphisms of \( F \). It follows that \( \mathcal{D} \) satisfies (LB), and then, \( \mathcal{D} \) is integrable. \( \triangle \)

4.2 Banach Poisson manifold

We first recall the context of Banach Poisson manifold studied these last years (see for example [OdRa2]). In particular, we will prove in a large context the existence of weak symplectic leaves.

A Lie bracket on \( C^\infty(M) \) is \( \mathbb{R} \)-bilinear antisymmetric pairing \( \{ , \} \) on \( C^\infty(M) \) which satisfies the Leibniz rule: \( \{fg, h\} = f\{g, h\} + g\{f, h\} \) and the Jacobi identity:
\[
\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0 \text{ for any } f, g, h \in C^\infty(M).
\]

A Poisson anchor on \( M \) is a bundle morphism \( \Psi : T^*M \to TM \) which is antisymmetric (i.e. such that \( <\alpha, \Psi\beta> = -<\beta, \Psi\alpha> \) for any \( \alpha, \beta \in T^*M \)).

We can associate to a such morphism an \( \mathbb{R} \)-bilinear antisymmetric pairing \( \{ , \} \) on the set \( \mathcal{A}^1(M) \) of 1-form on \( M \) defined by:
\[
\{\alpha, \beta\}_\Psi = <\beta, \Psi\alpha>
\]
Moreover, for any \( f \in C^\infty(M) \) we have:
\[
\{f\alpha, \beta\}_\Psi = <\beta, f\Psi\alpha> = f\{\alpha, \beta\}
\]
So, we get a bracket \( \{ , \}_\Psi \) on \( C^\infty(M) \) defined by
\[
\{f, g\}_\Psi = \{df, dg\}_\Psi
\]
As in finite dimension, \( \{ , \}_\Psi \) satisfies the Jacobi identity if and only if the the Schouten-Nijenhuis bracket \([P, P]\) of \( P \) is identically zero (see for instance [MaRa]). In this case, \( C^\infty(M) \) has a structure of Lie algebra and \( (M, \{ , \}_\Psi) \) is called a Banach Poisson manifold (see for instance [OdRa1] or [OdRa2]).

From now, the Poisson anchor \( \Psi \) is fixed and for simplicity we denote \( \{ , \} \) the Lie bracket associated to \( \Psi \).

Given such a Banach Poisson manifold, the distribution \( \mathcal{D} = \Psi(T^*M) \) is called the characteristic distribution. Of course, in general \( \mathcal{D} \) is not a closed distribution but it is a weak distribution.

Associated to \( \{ , \} \), on \( T^*M \), we have a natural skew-symmetric bilinear form \( \omega \) defined as follows:
for any \( \alpha \) and \( \beta \) in \( T^*_xM \), we have \( \omega(\alpha, \beta) = \{f, g\} \) if \( f \) and \( g \) are smooth functions defined on a neighbourhood of \( x \) and such that \( df(x) = \alpha \) and \( dg(x) = \beta \) (this definition is independent of the
choice of \(f\) and \(g\).

For each \(x\), on the quotient \(T^*_xM/\ker\Psi_x\) we get a skew-symmetric bilinear form \(\tilde{\omega}_x\). On the other hand, let \(\check{\Psi}_x : T^*_xM/\ker\Psi_x \to \mathcal{D}_x\) be the canonical isomorphism associated to \(\Psi_x\) between Banach spaces. Finally we get a skew-symmetric bilinear form \(\tilde{\omega}_x\) on \(\mathcal{D}_x\) such that:

\[
[\check{\Psi}_x]^*\tilde{\omega}_x = \tilde{\omega}_x
\]

According to [OdRa2], a **symplectic leaf** of \(\mathcal{D}\) is a weak submanifold \((\mathcal{L}, i)\) where \(\mathcal{L} \subseteq M\) and \(i : \mathcal{L} \to M\) is the natural inclusion with the following properties:

(i) \((\mathcal{L}, i)\) is a maximal integral manifold of \(\mathcal{D}\) (in the sense of Theorem 1 part (2));

(ii) on \(\mathcal{L}\) we have a weak symplectic form \(\omega_{\mathcal{L}}\) such that \((\omega_{\mathcal{L}})_x = \tilde{\omega}_x\) for all \(x \in \mathcal{L}\)

**Remark 4.4**

As in the context of Lie Banach algebroids, we will say that a Lie bracket \{ , \} on \(C^\infty(M)\) is **localizable** if \{ , \} is compatible with the sheaf of germs of functions on \(M\). From our definition of Banach Poisson manifold, the Lie bracket associated to a Poisson anchor \(\Psi\) is always localizable.

On the other hand, given any Lie bracket \{ , \} on \(C^\infty(M)\), when \(M\) is regular, we can prove that \{ , \} is localizable, and then we have a morphism \(\Psi : T^*M \to T^*M\) naturally associated (see [CaPe]). If moreover, \(\Psi(T^*M) \subseteq TM\), then we get the previous definition of Banach Poisson manifold (see for instance [OdRa1] or [OdRa2]).

**Theorem 6**

Let be \(\Psi : T^*M \to TM\) a Poisson anchor. If the kernel of \(\Psi\) is complemented in each fiber, then the associated characteristic distribution \(\mathcal{D}\) is integrable. Moreover, each maximal integral manifold has a natural structure of weak symplectic leaf.

For an illustration of this result, the reader will find many examples of Banach Poisson manifolds in [OdRa1] and [OdRa2].

**Proof of Theorem 6**

At first, we can observe that the set

\[
\mathcal{S} = \{ \Psi(df) : f \in C^\infty(U), U \text{ any open set in } M \}
\]

is an upper generating set for \(\mathcal{D}\): given any \(x \in M\), modulo any local chart around \(x\), we can suppose that \(M\) is an open subset of \(E\) and \(T^*M \equiv E^* \times M\); for any \(\alpha \in E^*\) the function \(f_\alpha(x) = \langle \alpha, x \rangle\) is a smooth map on \(M\) such that \(df_\alpha(y) = \alpha\) for any \(y \in M\); so \(Z_\alpha = \Psi(\alpha, y) = \Psi(df_\alpha(y))\) is an upper section.

For any smooth local function \(f : U \to \mathbb{R}\), we set \(Z_f = \Psi(df, )\). From the Jacobi identity in \(C^\infty(M)\) we have

\[
[Z_f, Z_g] = \Psi(dfg), \text{ for any } f, g \in C^\infty(M) \quad (43)
\]

According to Theorem 4, to prove the integrability of \(\mathcal{D}\), we have only to prove (LB) for the generating upper set \(\mathcal{S}\). As (LB) is a local property, again we assume that \(M\) is an open set in \(E\). So fix some smooth function \(f : M \to \mathbb{R}\) and consider an integral curve \(\gamma(t) = \phi_t^f(x)\) through \(x \in M\) defined on \([-\varepsilon, \varepsilon]\). For any \(\alpha \in E^*\), using (43), we have:

\[
[Z_f, Z_\alpha](\gamma(t)) = \Psi(df, \Psi(df_\alpha))(\gamma(t)) = \Psi(df, f_\alpha)(\gamma(t))
\]

But, using the same arguments as the ones used in the proof of Lemma 3.11, we can show that the map

\[
y \mapsto [\alpha \mapsto df \circ f_\alpha(y)]
\]

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is a smooth field of continuous operators from $E^*$ to $E^*$. It follows that $D$ satisfies (LB), and then, $D$ is integrable.

Assume now that $D$ is integrable and choose any maximal leaf $L$. As $T_xL = \tilde{D}_x$, on $T_xL$ we have the skew-symmetric bilinear form $\tilde{\omega}_x$ previously defined. We will show that $\tilde{\omega}_x$ defines a closed 2-form $\omega_L$ on $L$, which is a weak symplectic form.

Fix $x \in L$. We have $T_x' = \ker \Psi_x \oplus S_x$. So $L$ is a Banach manifold modelled on $S_x$. From the definition of $\tilde{\omega}_x$, we have

$$\tilde{\omega}_x(\tilde{\theta}_x(\alpha), \tilde{\theta}_x(\beta)) = \langle \alpha, \tilde{\theta}_x(\beta) \rangle.$$  \hspace{1cm} (44)

As we know that $\tilde{\theta}_x$ is an isomorphism from $S_x$ to $T_xL$ it follows that $\tilde{\omega}_x$ is a weak symplectic 2-form on the Banach space $T_xL$. On one hand, locally, in keeping to Lemma 2.11 it follows that $\omega_L$ defined by $(\omega_L)_x = \tilde{\omega}_x$ is a smooth differential 2-form on $L$. On the other hand, for any smooth function $f$ defined on an open set $U \subset M$, we set $\tilde{f} = f \circ \tilde{i}$. So for any smooth functions $f$, $g$ and $h$ defined on $U$, the Jacobi identity is satisfied for $\tilde{f}$, $\tilde{g}$ and $\tilde{h}$ on the open set $\tilde{i}^{-1}(U) \subset \mathcal{L}$. So, by classical arguments of Poisson bracket (see for instance [LiMa], [OdRa1], [OdRa2]), we get:

$$d\omega_L(\tilde{i}^*Z_f, \tilde{i}^*Z_g, \tilde{i}^*Z_h) = 0$$

for any choice of functions $f$, $g$ and $h$. So $\omega_L$ is closed and the proof of Theorem 6 is complete.

\[ \triangle \]

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