Conditional Expectations of Correspondences

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Abstract

We characterize the properties of convexity, compactness and preservation of upper hemicontinuity for conditional expectations of correspondences. These results are then applied to obtain a necessary and sufficient condition for the existence of pure strategy equilibria in Bayesian games.

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1 Introduction

For a correspondence $F$ from an atomless probability space $(T, \mathcal{T}, \lambda)$ to the Euclidean space $\mathbb{R}^n$, let $\mathcal{I}_F^{(T, \mathcal{G})}$ be the set of all $E(f|\mathcal{G})$ such that $f$ is an integrable selection of $F$, where $E(f|\mathcal{G})$ is the conditional expectation of $f$ on some given sub-$\sigma$-algebra $\mathcal{G}$ of $T$. By the classical Lyapunov Theorem, the atomless property of $(T, \mathcal{T}, \lambda)$ implies the convexity of $\mathcal{I}_F^{(T, \mathcal{G})}$ if $\mathcal{G}$ is the trivial $\sigma$-algebra $\{T, \emptyset\}$. However, it is easy to see that such a convexity result fails when we work with a general sub-$\sigma$-algebra $\mathcal{G}$ of $T$. Similarly, some other common regularity properties, such as compactness and preservation of upper hemicontinuity, also fail to hold in the general case. The purpose of this paper is to characterize the properties of convexity, compactness and preservation of upper hemicontinuity for conditional expectations of correspondences.

The key condition we will work with is that $T$ has no $\mathcal{G}$-atom, which means that $T$ does not coincide with $\mathcal{G}$ when they are restricted on any non-trivial set in $T$. Based on this condition, Dynkin and Evstigneev (1976) established the equivalence of $\mathcal{I}_F^{(T, \mathcal{G})}$ and $\mathcal{I}_{\text{co}(F)}^{(T, \mathcal{G})}$ for any measurable, integrably bounded and closed valued correspondence $F$, where $\text{co}(F)(t)$ is the convex hull of $F(t)$ for each $t \in T$. We show that $\mathcal{I}_F^{(T, \mathcal{G})}$ is convex for any correspondence $F$ if and only if $T$ has no $\mathcal{G}$-atom. We also prove that this condition is necessary and sufficient for the weak/weak$^*$ compactness of $\mathcal{I}_F^{(T, \mathcal{G})}$ for any integrably bounded and closed valued correspondence $F$. A similar necessity and sufficiency result holds for the property on preservation of upper hemicontinuity. Thus, we not only generalize the classical results on integration of correspondences$^2$ to the case of conditional expectation, but also demonstrate the optimality of the relevant condition.

To illustrate the application of the main results, Bayesian games with finite actions will be considered. We formulate the notion of “inter-player information” to describe the influence of player $i$’s private information in other players’ payoffs. The condition of “coarser inter-player information” is proposed below and we show that this condition is not only sufficient but also necessary for the existence of pure strategy equilibrium. In particular, interdependent payoffs and correlated types are allowed in our setting. In addition, we prove the purification results for any behavioral strategy profile.

The rest of the paper is organized as follows. Some basic definitions are given

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$^1$Suppose that $F(t) = \{0, 1\}$ for all $t \in T$, which is a measurable, integral bounded and compact valued correspondence. Given $\mathcal{G} = T$, then the conditional expectation of $F$ conditional on $\mathcal{G}$ is the set of all integrable selections, which is not convex.

$^2$See, for example, Aumann (1965) and Hildenbrand (1974).

$^3$There is a substantial literature studying the existence of pure strategy equilibrium in Bayesian games with finite actions, see Radner and Rosenthal (1982), Milgrom and Weber (1985), Khan, Rath and Sun (2006) and Barelli and Duggan (2013).
in Section 2. We state the results on conditional expectations of correspondences in Section 3. An application to Bayesian games is presented in Section 4. The proofs are left in Section 5.

2 Basic Definitions

Suppose that \((T, \mathcal{T}, \lambda)\) is a complete probability space endowed with a countably additive probability measure \(\lambda\). A correspondence \(F\) from \(T\) to \(\mathbb{R}^n\) is a mapping from \(T\) to the family of nonempty subsets of \(\mathbb{R}^n\). It is said to be measurable if for all open sets \(O \subseteq \mathbb{R}^n\), we have \(\{t \in T : F(t) \cap O \neq \emptyset\} \in \mathcal{T}\). A measurable function \(f\) from \(T\) to \(\mathbb{R}^n\) is called a selection of \(F\) if \(f(t) \in F(t)\) for \(\lambda\)-almost all \(t \in T\).

A correspondence \(F\) from \(T\) to \(\mathbb{R}^n\) is said to be convex (resp. closed, compact) valued if \(F(t)\) is convex (resp. closed, compact) for \(\lambda\)-almost all \(t \in T\).

A correspondence \(F\) from a topological space \(Y\) to another topological space \(Z\) is said to be upper hemicontinuous at \(y_0 \in Y\) if for any open set \(O \subseteq Z\) that contains \(F(y_0)\), there exists an open neighborhood \(O \subseteq Y\) of \(y_0\) such that \(\forall y \in O\), \(F(y) \subseteq O\). \(F\) is upper hemicontinuous if it is upper hemicontinuous at every point \(y \in Y\).

Let \(L^\mathcal{G}_p(T, \mathbb{R}^n)\) and \(L^\mathcal{G}_\infty(T, \mathbb{R}^n)\) be the set of all \(\mathcal{G}\)-measurable mappings from \(T\) to \(\mathbb{R}^n\) with the usual norm. That is,

\[
L^\mathcal{G}_p(T, \mathbb{R}^n) = \left\{ f : f \text{ is } \mathcal{G}\text{-measurable and } \left( \int_T \|f\|^p \, d\lambda \right)^{\frac{1}{p}} < \infty \right\},
\]

\[
L^\mathcal{G}_\infty(T, \mathbb{R}^n) = \left\{ f : f \text{ is } \mathcal{G}\text{-measurable and essentially bounded under } \lambda \right\},
\]

where \(\|\cdot\|\) is the usual norm in \(\mathbb{R}^n\) and \(1 \leq p < \infty\). By the Riesz Representation Theorem (see Theorem 13.26-28 of Aliprantis and Border (2006)), \(L^\mathcal{G}_q(T, \mathbb{R}^n)\) can be viewed as the dual space of \(L^\mathcal{G}_p(T, \mathbb{R}^n)\), where \(\frac{1}{p} + \frac{1}{q} = 1\). Similarly, one can define \(L^\mathcal{T}_p(T, \mathbb{R}^n)\) and \(L^\mathcal{T}_\infty(T, \mathbb{R}^n)\).

If there is a real valued function \(h \in L^\mathcal{T}_p(T, \mathbb{R})\) such that \(\sup\{\|x\| : x \in F(t)\} \leq h(t)\) for \(\lambda\)-almost all \(t \in T\), then the correspondence \(F\) is said to be \(p\)-integrably bounded, \(1 \leq p \leq \infty\) (it is said to be integrably bounded if \(p = 1\)).

For any nonnegligible subset \(D \in \mathcal{T}\), the restricted probability space \((D, \mathcal{G}^D, \lambda^D)\) is defined as follows: \(\mathcal{G}^D\) is the \(\sigma\)-algebra \(\{D \cap D' : D' \in \mathcal{G}\}\) and \(\lambda^D\) the probability
measure re-scaled from the restriction of $\lambda$ on $\mathcal{G}^D$. Furthermore, $(D, \mathcal{T}^D, \lambda^D)$ can be defined similarly. A subset $D \in \mathcal{T}$ is said to be a $\mathcal{G}$-atom if $\lambda(D) > 0$ and given any $D_0 \in \mathcal{T}^D$, there exists a set $D_1 \in \mathcal{G}^D$ such that $\lambda(D_0 \Delta D_1) = 0$.

3 The Main Results

In this section, we will show that the condition that $\mathcal{T}$ has no $\mathcal{G}$-atom is sufficient and necessary for the validity of several regularity properties for conditional expectations of correspondences (convexity, compactness and upper hemicontinuity).

The sufficiency part of the following theorem is due to Dynkin and Evstigneev (1976, Theorem 1.2), while the necessity part is from He and Sun (2013, Proposition 1).

**Theorem 1.** $\mathcal{I}_F^{(\mathcal{T}, \mathcal{G})} = \mathcal{I}_F^{(\mathcal{T}, \mathcal{G})}_{\text{co}(F)}$ for any $\mathcal{T}$-measurable, integrably bounded and closed valued correspondence $F$ if and only if $\mathcal{T}$ has no $\mathcal{G}$-atom.

By the Kuratowski-Ryll-Nardzewski Selection Theorem (see Aliprantis and Border (2006, Theorem 18.13)), $\mathcal{I}_F^{(\mathcal{T}, \mathcal{G})}$ is nonempty for any $\mathcal{T}$-measurable, integrably bounded and closed valued correspondence $F$.

If $\mathcal{T}$ has no $\mathcal{G}$-atom, then the convexity of $\mathcal{I}_F^{(\mathcal{T}, \mathcal{G})}$ for any correspondence $F$ is a simple corollary of Theorem 1. It can be also shown that this condition is necessary for such convexity property.

**Corollary 1.** The set $\mathcal{I}_F^{(\mathcal{T}, \mathcal{G})}$ is convex for any correspondence $F$ if and only if $\mathcal{T}$ has no $\mathcal{G}$-atom.

Next, we consider the weak/weak∗ compactness of $\mathcal{I}_F^{(\mathcal{T}, \mathcal{G})}$ for a correspondence $F$.

**Theorem 2.** The set $\mathcal{I}_F^{(\mathcal{T}, \mathcal{G})}$ is weakly compact (resp. weak∗ compact) in $L^G(T, \mathbb{R}^n)$ when $1 \leq p < \infty$ (resp. $p = \infty$, and $\mathcal{G}$ is countably generated) for any $p$-integrably bounded and closed valued correspondence $F$ if and only if $\mathcal{T}$ has no $\mathcal{G}$-atom.\(^4\)

The last property is the preservation of weak/weak∗ upper hemicontinuity for conditional expectations of correspondences.

**Theorem 3.** The following conditions are equivalent.

\(^4\)The equivalence of compactness and sequential compactness in the weak topology of a Banach space is important in the proofs of Theorems 2 and 3 below. Such an equivalence still holds for the weak∗ topology of $L^G_\infty(T, \mathbb{R}^n)$ when $\mathcal{G}$ is countably generated.
1. For any closed valued correspondence $F$ from $T \times Y \to \mathbb{R}^n$ ($Y$ is a metric space) such that there is a $p$-integrably bounded and compact valued correspondence $G$ from $T$ to $\mathbb{R}^n$ and
   
   a. $F(t,y) \subseteq G(t)$ for $\lambda$-almost all $t \in T$ and all $y \in Y$;
   b. $F(\cdot, y)$ is $T$-measurable for all $y \in Y$;
   c. $F(t, \cdot)$ is upper hemicontinuous for $\lambda$-almost all $t \in T$;
   
   $H(y) = I^{(T,\mathcal{G})}_{F_y}$ is weakly (resp. weak*) upper hemicontinuous in $L_p^\mathcal{G}(T, \mathbb{R}^n)$ when $1 \leq p < \infty$ (resp. $p = \infty$, and $\mathcal{G}$ is countably generated).

2. $T$ has no $\mathcal{G}$-atom.

   Note that if $1 < p < \infty$, then $L_p^\mathcal{G}(T, \mathbb{R}^n)$ is reflexive. Thus, the weak compactness (resp. weak upper hemicontinuity) and the weak* compactness (resp. weak* upper hemicontinuity) are equivalent in $L_p^\mathcal{G}(T, \mathbb{R}^n)$ for $1 < p < \infty$.

**Remark 1.** He and Sun (2013) proved the existence of stationary Markov perfect equilibria in discounted stochastic games with coarser transition kernels by using Theorem 1.2 of Dynkin and Evstigneev (1976). Recall that $P$ is an equilibrium payoff correspondence from $T \times V$ to $\mathbb{R}^n$ such that $P(t, \cdot)$ is upper hemicontinuous and $P(\cdot, v)$ is $T$-measurable. Let $R(v)$ be the set of all selections of $P_v$ for each $v \in V$. The classical Fan-Glicksberg Fixed Point Theorem is applied to the correspondence $\text{co}(R)$, which is convex valued and upper hemicontinuous, to obtain a selection $v'$ of $\text{co}(R)$. Then $v'$ is an equilibrium payoff correspondence from $T \times V$ to $\mathbb{R}^n$ such that $P(t, \cdot)$ is upper hemicontinuous and $P(\cdot, v)$ is $T$-measurable. Let $R(v)$ be the set of all selections of $P_v$ for each $v \in V$. Then the correspondence $\text{co}(R)$ is convex, compact valued and upper hemicontinuous. Then the existence result can be also proved by applying the Fan-Glicksberg Fixed Point Theorem to the correspondence $\text{co}(R)$.

### 4 Bayesian Games with Inter-player Information

In this section, we shall propose the condition of “coarser inter-player information”, and show that this condition is not only sufficient but also necessary for the existence of pure strategy equilibria in Bayesian games with finite actions. Purification results of behavioral strategy profiles will be also considered.

#### 4.1 Model

A Bayesian game $\Gamma$ can be described as follows:
• The set of players: \( I = \{1, 2, \ldots, n\} \).

• The (private) information space for each player: \( \{T_i\}_{i \in I} \). Each \( T_i \) is endowed with a countably generated \( \sigma \)-algebra \( \mathcal{T}_i \). Let \( T = \times_{i=1}^n T_i \) and \( \mathcal{T} = \otimes_{i=1}^n \mathcal{T}_i \).

• For each player \( i \in I \), \( X_i \) is a finite set of actions. Let \( X = \prod_{1 \leq i \leq n} X_i \).

• The information structure: \( \lambda \), a probability measure on the measurable space \( (T, \mathcal{T}) \). For each \( i \in I \), \( \lambda_i \) is the marginal probability of \( \lambda \) on \( T_i \) and \( (T_i, \mathcal{T}_i, \lambda_i) \) is atomless. \( \lambda \) is absolutely continuous with respect to \( \otimes_{1 \leq i \leq n} \lambda_i \) and \( q(t_1, \ldots, t_n) \) is the Radon-Nikodym derivative.\footnote{This assumption is standard in the literature, see Milgrom and Weber (1985).}

• The payoff functions: \( \{u_i\}_{i \in I} \). Each \( u_i \) is an integrably bounded mapping from \( X \times T \) to \( \mathbb{R} \) such that \( u_i(x, \cdot) \) is \( \mathcal{T}_i \)-measurable for each \( x \in X \).

Hereafter, the notation \(-i\) denotes the set of all players except player \( i \). Let \( \lambda_{-i} = \otimes_{j \neq i} \lambda_j \). Without loss of generality, we can assume that the mixture of actions of player \( i \) is the simplex \( \mathcal{M}(X_i) \), and the pure actions in \( X_i \) correspond to vertices of \( \mathcal{M}(X_i) \). For each player \( i \in I \), a behavioral strategy (resp. pure strategy) is a measurable function from \( T_i \) to \( \mathcal{M}(X_i) \) (resp. \( X_i \)), and \( L^T_i \) is the set of all behavioral strategies. \( L^T = \times_{i \in I} L^T_i \).

Given a strategy profile \( f = (f_1, \ldots, f_n) \), player \( i \)'s expected payoff is
\[
U_i(f) = \int_T \int_X u_i(x, t) \prod_{j \in I} f_j(t_j, dx_j) \lambda(dt).
\]

A behavioral (resp. pure) strategy equilibrium is a behavioral (resp. pure) strategy profile \( f^* = (f^*_1, f^*_2, \ldots, f^*_n) \) such that \( f^*_i \) maximizes \( U_i(f_i, f^*_{-i}) \) for each \( i \in I \).

Consider the density weighted payoff of player \( i \): \( w_i(x, t) = u_i(x, t) \cdot q(t) \) for each \( x \in X \) and \( t \in T \). Let \( G_i \) be the \( \sigma \)-algebra generated by the collection of mappings
\[
\{w_j(x, \cdot, t_{-i}): x \in X, t_{-i} \in T_{-i}, \forall j \neq i\}.
\]

Then \( G_i \subseteq T_i \) denotes player \( i \)'s inter-player information. That is, \( G_i \) is player \( i \)'s information flow to all other players, which describes the influence of player \( i \)'s private information in other players' payoffs.

### 4.2 Existence of pure strategy equilibria

In this section, we will prove the existence of the pure strategy equilibrium in Bayesian games under an appropriate condition called "coarser inter-player
information”. More importantly, we will show that this condition is necessary for the existence result.

**Definition 1.** Player $i$ is said to have **coarser inter-player information** if $T_i$ has no $G_i$-atom under $\lambda_i$.

A Bayesian game is said to have coarser inter-player information if each player has coarser inter-player information.

**Theorem 4.** Every Bayesian game with coarser inter-player information has a pure strategy equilibrium.

**Remark 2.** For Bayesian games with coarser inter-player information, players’ payoffs might be interdependent and types could be correlated. In particular, it is inessential whether types are independent or correlated, since the derivative $q$ can be absorbed into the density weighted payoff.

If $G_i$ is the trivial $\sigma$-algebra $\{\emptyset, T_i\}$ for each player $i \in I$, then players have independent priors and private values, and the condition of “coarser inter-player information” is automatically satisfied since $(T_i, T_i, \lambda_i)$ is atomless.

In Theorem 4, we show that the condition of “coarser inter-player information” is sufficient for the existence of pure strategy equilibrium. The next theorem demonstrates that this condition is also necessary.

Given any $n \geq 2$ and the player space $I = \{1, 2, \ldots, n\}$, player $i$ has private information space $(T_i, T_i, \lambda_i)$ and inter-player information $G_i$ such that $(T_i, G_i, \lambda_i)$ is atomless for each $1 \leq i \leq n$. Let $H_n$ be the collection of all Bayesian games with the player space $I$ and the above private information spaces $\{(T_i, T_i/G_i, \lambda_i)\}_{i \in I}$.

**Theorem 5.** Given the player space $I = \{1, \ldots, n\}$ for $n \geq 2$ and the private information space $(T_i, T_i/G_i, \lambda_i)$ for each $i \in I$, every player $i$ has coarser inter-player information if either of the following conditions holds:

1. every Bayesian game in $H_n$ with type-irrelevant payoffs has a pure strategy equilibrium;
2. every Bayesian game in $H_n$ with independent types has a pure strategy equilibrium.

### 4.3 Purification

In this subsection, we will consider the purification of behavioral strategy profiles in Bayesian games with finite actions.

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\[(\text{A Bayesian game is said to have type-irrelevant payoffs if the payoff function of each player does not depend on the type } t \in T).\]
Definition 2. Let \( f = (f_1, f_2, \ldots, f_n) \) and \( g = (g_1, g_2, \ldots, g_n) \) be two behavioral strategy profiles.

1. The strategy profiles \( f \) and \( g \) are said to be payoff equivalent if for each player \( i \in I \), \( U_i(f) = U_i(g) \).

2. The strategy profiles \( f \) and \( g \) are said to be strongly payoff equivalent if
   \( (a) \) they are payoff equivalent;
   \( (b) \) for each player \( i \in I \) and any given behavioral strategy \( h_i \), the two strategy profiles \( (h_i, f_{-i}) \) and \( (h_i, g_{-i}) \) are payoff equivalent.

3. The strategy profiles \( f \) and \( g \) are said to be distribution equivalent if for each player \( i \in I \),
   \( \int_{T_i} f_i(t, \cdot) \, d\lambda_i(t) = \int_{T_i} g_i(t, \cdot) \, d\lambda_i(t) \).

4. Suppose that \( f \) is a pure strategy profile. For player \( i \), \( f_i \) is said to be belief consistent with \( g_i \) if \( f_i(t_i) \in \text{supp} g_i(t_i) \) for \( \lambda_i \)-almost all \( t_i \in T_i \). Moreover, \( f \) is said to be belief consistent with \( g \) if they are belief consistent for each player \( i \in I \).

Now we are ready to give the definitions of purification.

Definition 3. Suppose that \( g \) is a pure strategy profile and \( f \) is a behavioral strategy profile. Then \( g \) is said to be a strong purification of \( f \) if they are strongly payoff equivalent, distribution equivalent, and belief consistent.

Proposition 1. In a Bayesian game with coarser inter-player information, every behavioral strategy profile \( f \) possesses a strong purification \( g \).

5 Appendix

5.1 Proofs in Section 3

For a sequence of sets \( \{A_m\}_{m \in \mathbb{N}} \) in a topological space \( X \), let \( \text{Ls}(A_m) \) be the set of all \( x \) such that for any neighborhood \( O_x \) of \( x \) there are infinitely many \( m \) with \( O_x \cap A_m \neq \emptyset \). The following lemma will be needed in the proofs of the main results.

Lemma 1. Denote \( \{\phi_m\}_{m \in \mathbb{N}} \) as a sequence of measurable and \( p \)-integrably bounded mappings from an atomless probability space \( (T, \mathcal{T}, \lambda) \) to \( \mathbb{R}^n \), \( 1 \leq p < \infty \). Let \( h_m = E(\phi_m | \mathcal{G}) \) for each \( m \in \mathbb{N} \), where \( \mathcal{G} \) is a sub-\( \sigma \)-algebra of \( \mathcal{T} \). Assume that \( h_m \) weakly converges to some \( h_0 \in L_p^G(T, \mathbb{R}^n) \) as \( m \to \infty \). If \( \mathcal{T} \) has no \( \mathcal{G} \)-atom, then there exists a \( \mathcal{T} \)-measurable mapping \( \phi_0 \) such that

1. \( \phi_0(t) \in \text{Ls}(\phi_m(t)) \) for \( \lambda \)-almost all \( t \in T \),
2. \( E(\phi_0 | \mathcal{G}) = h_0 \).
Proof. Since the sequence $\{\phi_m\}_{m \in \mathbb{N}}$ is $p$-integrably bounded in $L_p^r(T, \mathbb{R}^n)$, it has a weakly convergent subsequence by the Riesz/Dunford-Pettis Weak Compactness Theorem in Royden and Fitzpatrick (2010, p.408/p.412). Without loss of generality, we assume that $\phi_m$ weakly converges to some $\phi \in L_p^r(T, \mathbb{R}^n)$. Given any $g \in L_q^r(T, \mathbb{R}^n)$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$
\int_T h_m g \, d\lambda = \int_T E(\phi_m | G) g \, d\lambda = \int_T E(\phi | G) g \, d\lambda
$$

Thus, $h_m$ weakly converges to $E(\phi | G)$ in $L_q^r(T, \mathbb{R}^n)$, which implies that $h_0 = E(\phi | G)$. In addition, $\{\phi_k, \phi_{k+1}, \ldots\}$ also weakly converges to $\phi$ for each $k \in \mathbb{N}$. By Theorem 29 of Royden and Fitzpatrick (2010, p.293), there is a sequence of convex combination of $\{\phi_k, \phi_{k+1}, \ldots\}$ that converges to $\phi$ in $L_p$ norm. For each $k \in \mathbb{N}$, assume that $\varphi_k$ is the convex combination $\{\phi_k, \phi_{k+1}, \ldots\}$ such that $\|\varphi_k - \phi\|_p \leq \frac{1}{k}$. Thus, there is a subsequence of $\{\varphi_k\}$, say itself, which converges to $\phi$ $\lambda$-almost everywhere.

Fix $t \in T$ such that $\varphi_k(t)$ converges to $\phi(t)$. By Carathéodary’s convexity theorem (see Aliprantis and Border (2006, Theorem 5.32)), $\varphi_k(t) = \sum_{j=0}^{n} \alpha_j k \gamma_j(t)$, where

1. for each $k \in \mathbb{N}$, $\alpha_{jk} \geq 0$ for any $j$ and $\sum_{j=0}^{n} \alpha_{jk} = 1$;
2. for each $k \in \mathbb{N}$, $\gamma_0(t), \ldots, \gamma_n(t) \in \{\phi_k(t), \phi_{k+1}(t), \ldots\}$.

Without loss of generality, assume that for each $0 \leq j \leq n$, $\alpha_{jk} \rightarrow \alpha_j$ and $\gamma_{jk}(t) \rightarrow \gamma_j(t)$. Then $\alpha_1, \ldots, \alpha_n \geq 0$ and $\sum_{j=0}^{n} \alpha_j = 1$. Moreover, $\gamma_j(t) \in Ls(\phi_m(t))$. Let $G(t) = Ls(\phi_m(t))$. Then $\phi(t) \in co(G(t))$.

Since $T$ has no $G$-atom and $G$ is measurable, integrably bounded and closed valued, Theorem 1 implies that $\mathcal{I}_G^{(T, G)} = \mathcal{I}_{co(G)}^{(T, G)}$. Thus, there exists a $T$-measurable selection $\phi_0$ of $G$ such that $E(\phi_0 | G) = E(\phi | G) = h_0$, which completes the proof. \qed

Proof of Corollary 1. First, we assume that $T$ has no $G$-atom. Pick two measurable selections $\phi_1$ and $\phi_2$ of $F$. Let $G(t) = \{\phi_1(t), \phi_2(t)\}$. Then $G$ is a $T$-measurable, integrably bounded, closed valued correspondence. By Theorem 1, we have $\mathcal{I}_G^{(T, G)} = \mathcal{I}_{co(G)}^{(T, G)}$, which implies that $\mathcal{I}_G^{(T, G)}$ is convex. For any $\alpha \in [0, 1]$, there exists a $T$-measurable selection $\phi_0$ of $G$ such that $E(\phi_0 | G) = \alpha E(\phi_1 | G) + (1 - \alpha)E(\phi_2 | G)$. Since $\phi_0$ is also a selection of $F$, $\mathcal{I}_F^{(T, G)}$ is convex.

Conversely, suppose that $T$ has a $G$-atom $D$ with $\lambda(D) > 0$. Define a
correspondence

\[
F(t) = \begin{cases} 
\{0, 1\} & t \in D; \\
\{0\} & t \notin D.
\end{cases}
\]

It is shown in Proposition 1 of He and Sun (2013) that \( \mathcal{I}_F^{(T,G)} \) is not convex. \( \square \)

Below we prove Theorem 2.

**Proof of Theorem 2.** Suppose that \( T \) has no \( G \)-atom and \( 1 \leq p < \infty \). It is sufficient to show that \( \mathcal{I}_F^{(T,G)} \) is weakly sequentially compact in \( L_p^G(T, \mathbb{R}^n) \). Fix an arbitrary sequence of \( T \)-measurable selections \( \{\phi_m\}_{m \in \mathbb{N}} \) of \( F \). Let \( h_m = E(\phi_m|G) \) for each \( m \in \mathbb{N} \). We need to show that there is a subsequence of \( \{h_m\}_{m \in \mathbb{N}} \) which weakly converges in \( L_p^G(T, \mathbb{R}^n) \) to some point in \( \mathcal{I}_F^{(T,G)} \). Since the sequence \( \{\phi_m\}_{m \in \mathbb{N}} \) is \( p \)-integrably bounded, it has a weakly convergent subsequence due to the Riesz/Dunford-Pettis Weak Compactness Theorem in Royden and Fitzpatrick (2010, p.408/p.412). Without loss of generality, assume that \( \phi_m \) weakly converges to some \( \phi \in L_p^T(T, \mathbb{R}^n) \). As shown in the proof of Lemma 1, \( h_m \) also weakly converges to \( E(\phi|G) \) in \( L_p^G(T, \mathbb{R}^n) \). By Lemma 1, there exists a \( T \)-measurable selection \( \phi_0 \) of \( \text{Ls}(\phi_m) \) such that \( E(\phi_0|G) = E(\phi|G) \). Since \( F \) is compact valued, \( \text{Ls}(\phi_m(t)) \subseteq F(t) \) for \( \lambda \)-almost all \( t \in T \). Thus, \( \phi_0 \) is a selection of \( F \), and we are done.

Next, we consider the case \( p = \infty \) and \( G \) is countably generated. Since \( F \) is essentially bounded by some positive constant \( C \), \( \mathcal{I}_F^{(T,G)} \) is also norm bounded by \( C \). By Alaoglu’s Theorem (see Theorem 6.21 of Aliprantis and Border (2006)), the closed ball with radius \( C \) (the \( C \)-ball) is weak* compact in \( L_p^G(T, \mathbb{R}^n) \). We only need to show that \( \mathcal{I}_F^{(T,G)} \) is weak* closed in the \( C \)-ball. Since \( G \) is countably generated, \( L_p^G(T, \mathbb{R}^n) \) is separable, which implies that the \( C \)-ball is metrizable in the weak* topology (see Theorem 6.30 of Aliprantis and Border (2006)). Suppose that \( \{\phi_m\} \) is a sequence of \( T \)-measurable selections of \( F \) and \( h_m \) weak* converges to \( h_0 \in L_p^G(T, \mathbb{R}^n) \) as \( m \to \infty \), where \( h_m = E(\phi_m|G) \) for each \( m \). Then \( h_m \) also weakly converges to \( h_0 \) in \( L_p^G(T, \mathbb{R}^n) \). Moreover, the condition that \( F \) is integrably bounded (i.e., essentially bounded) implies that \( F \) is integrably bounded. By Lemma 1, there exists a \( T \)-measurable selection \( \phi_0 \) of \( \text{Ls}(\phi_m) \) such that \( h_0 = E(\phi_0|G) \). Since \( F \) is compact valued, \( \phi_0(t) \in \text{Ls}(\phi_m(t)) \subseteq F(t) \) for \( \lambda \)-almost all \( t \in T \). That is, \( \phi_0 \) is a \( T \)-measurable selection of \( F \) and \( h_0 \in \mathcal{I}_F^{(T,G)} \). Therefore, \( \mathcal{I}_F^{(T,G)} \) is weak* closed in the \( C \)-ball.

\( ^7 \)For simplicity, the target space of the correspondence is \( \mathbb{R} \). One can easily define a new correspondence on \( \mathbb{R}^n \) such that each of other \( n - 1 \) dimensions only contains 0.
Conversely, suppose that $\mathcal{T}$ has a $\mathcal{G}$-atom $D$ with $\lambda(D) > 0$. Consider the correspondence $F$ as defined in the proof of Corollary 1. Pick an orthonormal subset \{\varphi_m\}_{m \in \mathbb{N}} of $L^2_{\mathcal{T}}(D, \mathbb{R})$ on the atomless probability space $(D, \mathcal{T}^D, \lambda^D)$ such that $\varphi_m$ takes value in \{-1, 1\} and $\int_D \varphi_m \, d\lambda^D = 0$ for each $m \in \mathbb{N}$. Let

$$
\phi_m(t) = \begin{cases} \frac{\varphi_m(t)+1}{2} & t \in D; \\ 0 & t \notin D. \end{cases}
$$

Then $\phi_m$ is a $\mathcal{T}$-measurable selection of $F$ for each $m \in \mathbb{N}$.

Pick a set $E \in \mathcal{T}^D$. By Bessel’s inequality (see (Royden and Fitzpatrick, 2010, p.316)), $\int_D 1_E \varphi_m \, d\lambda^D \to 0$ as $m \to \infty$, where $1_E$ is the indicator function of the set $E$. Thus, for any $E_1 \in \mathcal{T}$,

$$
\int_T 1_{E_1} \phi_m \, d\lambda = \frac{1}{2} \int_T 1_{E_1 \cap D} \phi_m \, d\lambda + \frac{1}{2} \lambda(E_1 \cap D) \to \frac{1}{2} \lambda(E_1 \cap D). \quad (1)
$$

Given any nonnegative function $\psi \in L^1_T(T, \mathbb{R})$, $\psi$ will be the increasing limit of a sequence of simple functions \{\psi_k\}_{k \in \mathbb{N}} (finite linear combination of measurable indicator functions). Fix any $\epsilon > 0$. By the dominated convergence theorem, there exists a positive integer $K_0 > 0$ such that for each $k \geq K_0$, $\int_T |\psi - \psi_k| \, d\lambda < \epsilon$.

Then we have

$$
\begin{align*}
&\left| \int_T \psi \phi_m \, d\lambda - \frac{1}{2} \int_T \psi \, d\lambda \right| \\
&\quad + \left| \int_T \psi \phi_{K_0} \, d\lambda - \frac{1}{2} \int_T \psi \, d\lambda \right| \\
&\quad + \left| \int_T \psi \phi_{K_0} \, d\lambda - \frac{1}{2} \int_T \psi \phi_m \, d\lambda \right| \\
&\quad + \left| \int_T \psi \phi_{K_0} \, d\lambda - \frac{1}{2} \int_T \psi \phi_{K_0} \, d\lambda \right| \\
&\leq \int_T |\psi - \psi_{K_0}| \, d\lambda + \frac{1}{2} \int_T \psi \phi_{K_0} \, d\lambda - \frac{1}{2} \int_T \psi \phi_{K_0} \, d\lambda + \int_T |\psi_{K_0} - \psi| \, d\lambda.
\end{align*}
$$

The first and the third terms are less than $\epsilon$. By Equation (1) and the fact that $\psi_{K_0}$ is a simple function, the second term goes to 0 as $m \to \infty$. Hence, $\int_T \psi \phi_m \, d\lambda \to \frac{1}{2} \int_T \psi \, d\lambda$ as $m \to \infty$. Given any $\psi \in L^1_T(T, \mathbb{R})$, we can obtain $\int_T \psi \phi_m \, d\lambda \to \frac{1}{2} \int_T \psi \, d\lambda$ as $m \to \infty$ by writing $\psi$ as the sum of its positive and negative parts.

Therefore, $\phi_m$ weak* converges to $\phi = \frac{1}{2} \mathbf{1}_D$ in $L^p_T(T, \mathbb{R})$. Thus, $E(\phi_m | G) \in \mathcal{I}_F^{(T, \mathcal{G})}$ weak* converges to $\frac{1}{2} E(1_D | G)$ in $L^p_{\infty}(T, \mathbb{R})$ as shown in the proof of Lemma 1. It is shown in He and Sun (2013) that $\frac{1}{2} E(1_D | G) \notin \mathcal{I}_F^{(T, \mathcal{G})}$, which implies that $\mathcal{I}_F^{(T, \mathcal{G})}$ is not weak* compact in $L^p_{\infty}(T, \mathbb{R})$.

For $1 \leq p < \infty$, just note that $F$ is also $p$-integrably bounded, and $\phi_m$ weakly converges to $\phi = \frac{1}{2} \mathbf{1}_D$ in $L^p_T(T, \mathbb{R})$. \hfill \Box

**Proof of Theorem 3.** Suppose that $\mathcal{T}$ has no $\mathcal{G}$-atom and $1 \leq p < \infty$. By Theorem 2, we know that $\mathcal{I}_G^{(T, \mathcal{G})}$ is weakly compact, and hence weakly sequentially
compact. Pick \( \{y_m\}_{m=0}^{\infty} \subseteq Y \) and \( \{\phi_m\}_{m \in \mathbb{N}} \) such that \( \phi_m \) is a \( \mathcal{T} \)-measurable selection of \( F_{y_m} \). Let \( h_m = E(\phi_m|\mathcal{G}) \) for each \( m \in \mathbb{N} \). Suppose that \( h_m \) weakly converges to some \( h_0 \in L^p_G(T, \mathbb{R}^n) \) and \( y_m \) converges to some \( y_0 \in Y \). By Lemma 1, there exists a \( \mathcal{T} \)-measurable selection \( \phi_0 \) of \( \mathcal{L}(\phi_m) \) such that \( h_0 = E(\phi_0|\mathcal{G}) \).

Since \( F_t(\cdot) \) is upper hemicontinuous for \( \lambda \)-almost all \( t \in T \), \( \phi_0(t) \in \mathcal{L}(\phi_m(t)) \subseteq \mathcal{L}(F_{y_0}(t)) \subseteq F_{y_0}(t) \) for \( \lambda \)-almost all \( t \in T \). That is, \( \phi_0 \) is a \( \mathcal{T} \)-measurable selection of \( F_{y_0} \) and \( h_0 \in H(y_0) \). Therefore, \( H \) is weakly upper hemicontinuous. The case that \( p = \infty \) and \( \mathcal{G} \) is countably generated follows from a similar argument by noting that any closed ball in \( L^G_\infty(T, \mathbb{R}^n) \) is metrizable.

Conversely, suppose that \( \mathcal{T} \) has a \( \mathcal{G} \)-atom \( D \) with \( \lambda(D) > 0 \). Let \( G \) be the correspondence as in Corollary 1

\[
G(t) = \begin{cases} 
\{0, 1\} & t \in D; \\
\{0\} & t \notin D.
\end{cases}
\]

Let \( Y = \{\frac{1}{m}\}_{m \geq 1} \cup \{0\} \) endowed with the usual metric, \( F(t, 0) = G(t) \) and \( F(t, \frac{1}{m}) = \{\phi_m(t)\} \) for all \( t \in T \) and \( m \geq 1 \), where \( \phi_m \) is the same as in the converse part of the proof of Theorem 2. Then \( G \) is compact valued and bounded, and \( F(t, \cdot) \) is upper hemicontinuous for all \( t \in T \).

Consider the correspondence \( G \). For \( 1 \leq p < \infty \), since \( I_G^{(T, \mathcal{G})} \) is \( p \)-integrably bounded, it is relatively weakly sequentially compact in \( L^p_G(T, \mathbb{R}) \) due to the Riesz/Dunford-Pettis Weak Compactness Theorem in Royden and Fitzpatrick (2010, p.408/p.412), and hence relatively weakly compact. For \( p = \infty \), \( I_G^{(T, \mathcal{G})} \) is relatively weak* compact in \( L^G_\infty(T, \mathbb{R}^n) \) due to Alaoglu’s Theorem. Thus, \( H(y) \) is a subset of a fixed weakly (resp. weak*) compact set for all \( y \in Y \) when \( 1 \leq p < \infty \) (resp. \( p = \infty \)).

For the sequence \( \{\frac{1}{m}\}, \frac{1}{m} \to 0 \) and \( \phi_m \) is a selection of \( F_{\frac{1}{m}} \). As shown in the proof above, \( E(\phi_m|\mathcal{G}) \) weakly (resp. weak*) converges to \( \frac{1}{p}E(1_D|\mathcal{G}) \) in \( L^p_G(T, \mathbb{R}) \) for \( 1 \leq p < \infty \) (resp. \( p = \infty \)), but there is no \( \mathcal{T} \)-measurable selection \( \phi_0 \) of \( \mathcal{G} \) such that \( E(\phi_0|\mathcal{G}) = \frac{1}{p}E(1_D|\mathcal{G}) \). Therefore, \( \frac{1}{p}E(1_D|\mathcal{G}) \notin H(0) = I_{K_0}^{(T, \mathcal{G})} \), which implies that \( H(y) \) is neither weakly upper hemicontinuous in \( L^G_\infty(T, \mathbb{R}) \) for \( 1 \leq p < \infty \) nor weak* upper hemicontinuous in \( L^G_\infty(T, \mathbb{R}) \).

5.2 Proofs in Section 4

5.2.1 Proofs in Section 4.2

**Proof of Theorem 4.** Suppose that \( m_i \) is the cardinality of \( X_i \) for each \( i \in I \). Let \( \Delta_i = \{(a_1^i, \ldots, a_{m_i}^i) : \sum_{1 \leq k \leq m_i} a_k^i = 1, a_k^i \geq 0 \text{ for } 1 \leq k \leq m_i \} \). The mixture of
actions of player $i$ in $\mathcal{M}(X_i)$ can be regarded as elements in the simplex $\triangle_i$, and the pure actions in $X_i$ correspond to the extreme points of $\triangle_i$. Denote $L^G_i$ as the set of all $G_i$-measurable functions from $T_i$ to $\mathcal{M}(X_i)$. Without loss of generality, it can be viewed as $L^G_i(T_i, \triangle_i)$, and embedded in $L^G_i(T_i, \mathbb{R}^{m_i})$ endowed with the weak* topology.

By the Riesz representation theorem (see Theorem 13.28 of Aliprantis and Border (2006)), $L^G_i(T_i, \mathbb{R}^{m_i})$ can be viewed as the dual space of $L^G_i(T_i, \mathbb{R}^{m_i})$. Then $L^G_i(T_i, \mathbb{R}^{m_i})$ is a locally convex, Hausdorff topological vector space under the weak* topology. By Alaoglu’s Theorem (see Theorem 6.21 of Aliprantis and Border (2006)), the closed ball with radius $C \geq 1$ (the $C$-ball) is weak* compact in $L^G_i(T_i, \mathbb{R}^{m_i})$. Since $G_i$ is countable generated, $L^G_i(T_i, \mathbb{R}^{m_i})$ is separable, which implies that the $C$-ball is metrizable in the weak* topology (see Theorem 6.30 of Aliprantis and Border (2006)).

It is obvious that $L^G_i$ is included in the $C$-ball, we need to show that $L^G_i$ is weak* closed. That is, for any sequence $\{g_k\}_{k \in \mathbb{N}} \in L^G_i$ such that $g_k$ weak* converges to some $g_0 \in L^G_i$, we need to show $g_0 \in L^G_i$. Since $g_k$ weak* converges to $g_0 \in L^G_i$, it also weakly converges to $g_0$ in $L^G_i(T_i, \mathbb{R}^{m_i})$. Following an analogous argument in the proof of Lemma 1, one can show that $g_0(t_i) \in \text{co}(\{ls(g_k(t_i))\})$ for $\lambda_i$-almost all $t_i \in T_i$. Since $\triangle_i$ is closed and convex, $\text{co}(\{ls(g_k(t_i))\}) \subseteq \triangle_i$ for $\lambda_i$-almost all $t_i \in T_i$. Thus, $g_0(t_i) \in \triangle_i$ for $\lambda_i$-almost all $t_i \in T_i$, and $g_0 \in L^G_i$. Therefore, $L^G_i$ is nonempty, convex, and compact under the weak* topology. Let $L^G = \times_{i \in I} L^G_i$ endowed with the product topology.

Given a pure strategy profile $h$, let $\overline{h}_i = E^{\lambda_i}(h_i|G_i) \in L^G_i$, and $h^k_i$ denote the $k$-th dimension of $h_i$ for each player $i \in I$ and $1 \leq k \leq m_i$. For any distinct $i, j \in I$, $x_{-j} \in X_{-j}$, $t_{-j} \in T_{-j}$, and $D \in G_j$,

$$
\int_{T_j} 1_D(t_j)w_i(x_{-j}, h_j(t_j), t_{-j}, t_j)q(t_j, t_{-j})\lambda_j(dt_j)
= \int_{T_j} 1_D(t_j)w_i(x_{-j}, h_j(t_j), t_{-j}, t_j)\lambda_j(dt_j)
= \int_{T_j} 1_D(t_j) \sum_{k=1}^{m_j} \left(w_i(x_{-j}, a^k_i, t_{-j}, t_j) \cdot h^k_j(t_j) \right) \lambda_j(dt_j)
= \int_{T_j} E^{\lambda_j} \left( 1_D(t_j) \sum_{k=1}^{m_j} \left(w_i(x_{-j}, a^k_i, t_{-j}, t_j) \cdot h^k_j(t_j) \right) | G_j \right) \lambda_j(dt_j)
= \int_{T_j} 1_D(t_j) \sum_{k=1}^{m_j} E^{\lambda_j} \left(w_i(x_{-j}, a^k_i, t_{-j}, t_j) \cdot h^k_j(t_j) | G_j \right) \lambda_j(dt_j)
= \int_{T_j} 1_D(t_j) \sum_{k=1}^{m_j} w_i(x_{-j}, a^k_i, t_{-j}, t_j) \cdot E^{\lambda_j} \left(h^k_j(t_j) | G_j \right) \lambda_j(dt_j)
$$

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integration. Thus, for the first three equalities are obvious. The fourth and fifth equalities hold since \( \mathbf{1}_D \) and \( w_i(x, t_{-\cdot}) \) are \( \mathcal{G}_j \)-measurable for any \( x \in X \) and \( t_{-j} \in T_{-j} \). The sixth equality is due to the definition of \( \overline{\mu}_j \), and the last one is just rewriting the summation as integration. Thus, for \( \lambda_j \)-almost all \( t_j \in T_j \),

\[
E^{\lambda_j} \left( w_i(x_{-j}, h_j(t_j), t_{-j}, t_j) | \mathcal{G}_j \right) = \int_{X_j} w_i(x_{-j}, x_j, t_{-j}, t_j) \overline{\mu}_j(t_j, dx_j). \tag{2}
\]

Fix player 1. For any \( t_1 \in T_1 \) and \( x_1 \in X_1 \), we have

\[
\int_{T_{-1}} u_1(x_1, h_{-1}(t_{-1}), t_1, t_{-1}) q(t_1, t_{-1}) \lambda_{-1}(dt_{-1})
= \int_{T_{-1}} w_1(x_1, h_{-1}(t_{-1}), t_1, t_{-1}) \lambda_{-1}(dt_{-1})
= \int_{T_{-(1,2)}} \int_{T_2} w_1(x_1, h_2(t_2), h_{-(1,2)}(t_{-(1,2)}), t_{-2}, t_2) \lambda_2(dt_2) \lambda_{-(1,2)}(dt_{-(1,2)})
= \int_{T_{-(1,2)}} \int_{T_2} E^{\lambda_2} \left( w_1(x_1, h_2(t_2), h_{-(1,2)}(t_{-(1,2)}), t_{-2}, t_2) | \mathcal{G}_2 \right) \lambda_2(dt_2) \lambda_{-(1,2)}(dt_{-(1,2)})
= \int_{T_{-(1,2)}} \int_{T_2} \int_{X_2} w_1(x_1, x_2, h_{-(1,2)}(t_{-(1,2)}), t_{-2}, t_2) \overline{\mu}_2(t_2, dx_2) \lambda_2(dt_2) \lambda_{-(1,2)}(dt_{-(1,2)})
= \cdots
= \int_{T_{-1}} \int_{X_{-1}} w_1(x_1, x_{-1}, t_1, t_{-1}) \overline{\mu}_{-1}(t_{-1}, dx_{-1}) \lambda_{-1}(dt_{-1}),
\]

where the subscript \( -(1,2) \) denotes the set of all players except players 1 and 2. The first equality is due to the definition of density weighted payoff. The second equality is due to the Fubini property. The third equality holds by taking the conditional expectation. The fourth equality is implied by Equation (2). Then the previous four equalities are repeated for \( n - 2 \) times (from \( T_3 \) to \( T_n \)). This procedure is omitted in the fifth equality, and finally leads to the last equality. One can repeat the argument and show that for any \( i \in I, x_i \in X_i \) and \( t_i \in T_i \)

\[
\int_{T_{-i}} u_i(x_i, h_{-i}(t_{-i}), t_i, t_{-i}) q(t_i, t_{-i}) \lambda_{-i}(dt_{-i})
= \int_{T_{-i}} \int_{X_{-i}} w_i(x_i, x_{-i}, t_i, t_{-i}) \overline{\mu}_{-i}(t_{-i}, dx_{-i}) \lambda_{-i}(dt_{-i}). \tag{3}
\]

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For each $i \in I$, let $F_i$ be a mapping from $T_i \times X_i \times L^G$ to $\mathbb{R}$ defined as follows:

$$F_i(t_i, x_i, g_1, \ldots, g_n) = \int_{T_{-i}} \int_{X_{-i}} w_i(x_i, x_{-i}, t_i, t_{-i}) g_{-i}(t_{-i}, dx_{-i}) \lambda_{-i}(dt_{-i}).$$

It is clear that $F_i$ is $T_i$-measurable on $T_i$ and continuous on $L^G$, where $L^G$ is endowed with the weak* topology. For each $i \in I$, the best response correspondence $G_i$ from $T_i \times L^G$ to $X_i$ is given by

$$G_i(t_i, g_1, \ldots, g_n) = \text{argmax}_{x_i \in X_i} F_i(t_i, x_i, g_1, \ldots, g_n).$$

For each $t_i$, Berge’s maximal theorem implies that $G_i$ is nonempty, compact-valued, and upper-hemicontinuous on $L^G$. For any $x_i$ and $(g_1, \ldots, g_n)$, $F_i$ is $T_i$-measurable. Then $G_i(\cdot, g_1, \ldots, g_n)$ admits a measurable selection. Thus, $E^{\lambda_i}(G_i(\cdot, g_1, \ldots, g_n)|G_i)$ is nonempty. Since $T_i$ has no $G_i$-atom, by Corollary 1 and Theorems 2 and 3, it is convex, weak* compact-valued, and weak* upper-hemicontinuous on $L^G$.

Consider a correspondence from $L^G$ to itself:

$$\psi(g_1, \ldots, g_n) = \times_{i=1}^n E^{\lambda_i}(G_i(\cdot, g_1, \ldots, g_n)|G_i).$$

It is clear that $\psi$ is nonempty, convex, weak* compact-valued, and weak* upper-hemicontinuous on $L^G$. By Fan-Glicksberg’s fixed-point theorem, there exists a fixed point $(g_1^*, \ldots, g_n^*)$ of $\psi$. Thus for each $i$, there exists some $T_i$-measurable selection $f_i^*$ of $G_i(\cdot, g_1^*, \ldots, g_n^*)$ such that $g_i^* = E^{\lambda_i}(f_i^*|G_i)$.

With the strategy profile $(f_1^*, \ldots, f_n^*)$, the payoff of player $i$ is

$$U_i(f^*) = \int_T w_i(f_i^*(t_i), f_{-i}^*(t_{-i}), t_i, t_{-i}) \lambda(dt)$$

$$= \int_{T_i} \int_{T_{-i}} w_i(f_i^*(t_i), f_{-i}^*(t_{-i}), t_i, t_{-i}) \lambda_{-i}(dt_{-i}) \lambda_i(dt_i)$$

$$= \int_{T_i} \int_{T_{-i}} \int_{X_{-i}} w_i(f_i^*(t_i), x_{-i}, t_i, t_{-i}) g_{-i}^*(t_{-i}, dx_{-i}) \lambda_{-i}(dt_{-i}) \lambda_i(dt_i).$$

The first equality holds due to the definition of $U_i$. The second equality holds based on the Fubini property, and the third equality relies on Equation (3). By the choice of $(g_1^*, \ldots, g_n^*)$, we have that $(f_1^*, \ldots, f_n^*)$ is a pure strategy equilibrium.

To prove Theorem 5, we first consider an auxiliary game.

**Example 1.** Consider an $m \times m$ zero-sum “matching pennies” game $\Gamma$ with asymmetric information. There are two players, and the action space for both players is $A_1 = A_2 = \{a_1, a_2, \ldots, a_m\}$, $m \geq 2$. The payoff matrix for player 1 is
given below.

Player 1

|   | $a_1$ | $a_2$ | $a_3$ | $\cdots$ | $a_m$ |
|---|---|---|---|---|---|
| $a_1$ | 1 | -1 | 0 | $\cdots$ | 0 |
| $a_2$ | 0 | 1 | -1 | $\cdots$ | 0 |
| $a_3$ | 0 | 0 | 1 | $\cdots$ | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $a_m$ | -1 | 0 | $\cdots$ | 0 | 1 |

Player $i$ has a private information space $L_i = [0, 1]$ and $(l_1, l_2)$ follows the uniform distribution $\tau$ on the triangle of the unit square $0 \leq l_1 \leq l_2 \leq 1$. Then it is obvious that the Radon-Nikodym derivative of $\tau$ with respect to the Lebesgue measure on the unit square is

$$\rho(l_1, l_2) = \begin{cases} 2, & 0 \leq l_1 \leq l_2 \leq 1; \\ 0, & \text{otherwise}. \end{cases}$$

Let $\tau_i$ be the marginal distribution of $\tau$ on $L_i$ for $i = 1, 2$. Then the Lebesgue measure $\eta$ is absolutely continuous with respect to $\tau_1$ on $[0, 1]$ with the Radon-Nikodym derivative $\beta_1(l_1) = \frac{1}{2(1-l_1)}$ if $0 < l_1 < 1$ and 0 otherwise, and $\eta$ is absolutely continuous with respect to $\tau_2$ with the Radon-Nikodym derivative $\beta_2(l_2) = \frac{1}{2l_2}$ if $0 < l_2 < 1$ and 0 otherwise. The two probability measures $\tau_1$ and $\tau_2$ are both atomless. Let $\rho'$ be the corresponding Radon-Nikodym derivative of $\tau$ with respect to $\tau_1 \otimes \tau_2$:

$$\rho'(l_1, l_2) = \rho(l_1, l_2) \cdot \beta_1(l_1) \cdot \beta_2(l_2) = \begin{cases} \frac{1}{2(1-l_1)l_2}, & 0 < l_1 \leq l_2 < 1; \\ 0, & \text{otherwise}. \end{cases}$$

As is well known, there exists a measure preserving mapping $h_i$ from $(T_i, G_i, \lambda_i)$ to $([0, 1], \mathcal{B}, \tau_i)$ such that for any $E \in G_i$, there exists a set $E' \in \mathcal{B}$ such that $\lambda_i(E \Delta h_i^{-1}(E')) = 0$. For $i = 1, 2$, let $\pi_i$ be a probability measure on $(T_i, \mathcal{T}_i)$ which is absolutely continuous with respect to $\lambda_i$ with the Radon-Nikodym derivative $\beta_i(h_i(t_i))$. Since $\beta_i(h_i(t_i))$ is positive for $\lambda_i$-almost all $t_i$, $\lambda_i$ is also absolutely continuous with respect to $\pi_i$.

**Proof of Theorem 5.**

(1) First we consider the following 2-player game $\Gamma'$, and then extend it to an $n$-player game. Player 1 and 2’s action spaces and payoffs are the same as in the game $\Gamma$. The private information space for player $i$ is $(T_i, \mathcal{T}_i, \lambda_i)$, $q(t_1, t_2) =$
$\rho'(h_1(t_1), h_2(t_2))$, and the common prior $\lambda$ has the Radon-Nikodym derivative $q$ with respect to $\lambda_1 \otimes \lambda_2$. It can be easily checked that $\lambda_i$ is the marginal probability measure of $\lambda$ for $i = 1, 2$.

Suppose that $\Gamma_1$ has a pure strategy equilibrium $(f_1, f_2)$. Let $E^1_j = \{t_1 \in T_1 : f_1(t_1) = a_j\}$ and $E^2_j = \{t_2 \in T_2 : f_2(t_2) = a_j\}$ for $1 \leq j \leq m$. Then we shall show that for $\lambda_2$-almost all $t_2 \in T_2$

$$\int_{E^1_i} q(t_1, t_2) \lambda_1(dt_1) = \ldots = \int_{E^m_i} q(t_1, t_2) \lambda_1(dt_1), \quad (4)$$

and for $\lambda_1$-almost all $t_1 \in T_1$,

$$\int_{E^1_i} q(t_1, t_2) \lambda_2(dt_2) = \ldots = \int_{E^m_i} q(t_1, t_2) \lambda_2(dt_2). \quad \text{(5)}$$

Suppose that $\alpha$ and $-\alpha$ are the equilibrium payoffs of player 1 and player 2, respectively. Denote $a_{m+1} = a_1$, $E^1_{m+1} = E^1_1$ and $E^2_{m+1} = E^2_1$. For $j = 1, \ldots, m$, let

$$C^j_1 = \left\{ t_1 \in T_1 : \int_{E^j_1} q(t_1, t_2) \lambda_2(dt_2) > \int_{E^j_{j+1}} q(t_1, t_2) \lambda_2(dt_2) \right\}$$

and

$$C^j_2 = \left\{ t_2 \in T_2 : \int_{E^j_1} q(t_1, t_2) \lambda_1(dt_1) > \int_{E^j_{j+1}} q(t_1, t_2) \lambda_1(dt_1) \right\}.$$

Now we define a new strategy for players 1 and 2 as follows:

$$f'_1(t_1) = \begin{cases} a_j & t_1 \in C^j_1 \setminus \left( \cup_{1 \leq k < j} C^j_k \right), \\ a_1 & \text{otherwise}; \end{cases}$$

and

$$f'_2(t_2) = \begin{cases} a_{j+1} & t_2 \in C^j_2 \setminus \left( \cup_{1 \leq k < j} C^j_k \right), \\ a_1 & \text{otherwise}. \end{cases}$$

We claim that player 2 can choose the strategy $f'_2$ and get a nonnegative payoff.

If player 2 takes action $a_{j+1}$ at state $t_2$, then his interim expected payoff is

$$\int_{T_1} u_2(f_1(t_1), a_{j+1}) q(t_1, t_2) \lambda_1(t_1) = \sum_{k=1}^{m} \int_{E^k_i} u_2(a_k, a_{j+1}) q(t_1, t_2) \lambda_1(t_1) = \int_{E^j_1} q(t_1, t_2) \lambda_1(t_1) - \int_{E^j_{j+1}} q(t_1, t_2) \lambda_1(t_1).$$

1. if $t_2 \in C^j_2$, then choosing the action $a_{j+1}$ gives player 2 a strictly positive payoff.
2. if \( t_2 \in T_2 \setminus \left( \bigcup_{1 \leq j \leq m} C_j^2 \right) \), then player 2 is indifferent between any action and gets a payoff 0.

Thus, player 2 can choose the strategy \( f'_2 \) and guarantee himself a nonnegative payoff, which implies that \( \alpha \leq 0 \). Similarly, one can analyze the payoff of player 1 and show that \( \alpha \geq 0 \). As a result, \( \alpha = 0 \), which implies that \( \lambda_1 \left( \bigcup_{1 \leq j \leq m} C_j^1 \right) = 0 \) and \( \lambda_2 \left( \bigcup_{1 \leq j \leq m} C_j^2 \right) = 0 \). As a result, Equation (4) holds for \( \lambda_2 \)-almost all \( t_2 \in T_2 \), and Equation (5) holds for \( \lambda_1 \)-almost all \( t_1 \in T_1 \).

For \( \lambda_2 \)-almost all \( t_2 \in T_2 \), \( \int_{T_1} q(t_1, t_2) \lambda_1(dt_1) = 1 \), we have

\[
\int_{E_j^1} q(t_1, t_2) \lambda_1(dt_1) = \frac{1}{m}
\]

for \( 1 \leq j \leq m \). For each \( E_j^1 \), there exists a set \( D_j \subseteq I_1 \) such that \( \lambda_1(E_j^1 \Delta h_1^{-1}(D_j)) = 0 \), implying that

\[
\int_{D_j} \rho'(l_1, h_2(t_2)) \tau_1(dl_1) = \int_{D_j} \rho(l_1, h_2(t_2)) \beta_2(h_2(t_2)) \eta(dl_1) = \frac{1}{m},
\]

and hence \( \int_{D_j} \rho(l_1, h_2(t_2)) \eta(dl_1) = \frac{2(h_2(t_2))}{m} \) for \( \lambda_2 \)-almost all \( t_2 \in T_2 \). That is, for \( \lambda_2 \)-almost all \( t_2 \in T_2 \), \( \eta(D_j \cap [0, h_2(t_2)]) = \frac{h_2(t_2)}{m} \).

As a result,

\[
\pi_1(E_j^1 \cap h_1^{-1} ([0, h_2(t_2)])) = \int_{E_j^1 \cap h_1^{-1} ([0, h_2(t_2)])} \beta_1(h_1(t_1)) \lambda_1(dt_1)
\]

\[
= \int_{D_j \cap [0, h_2(t_2)]} \beta_1(l_1) \tau_1(dl_1)
\]

\[
= \eta(D_j \cap [0, h_2(t_2)])
\]

\[
= \frac{h_2(t_2)}{m}.
\]

Thus, \( \pi_1(E_j^1) = \frac{1}{m} \). In addition, \( \pi_1(h_1^{-1} ([0, h_2(t_2)])) = \eta ([0, h_2(t_2)]) = h_2(t_2) \) for \( \lambda_2 \)-almost all \( t_2 \in T_2 \). Therefore, \( \pi_1(E_j^1 \cap h_1^{-1} ([0, h_2(t_2)])) = \pi_1(E_j^1) \cdot \pi_1(h_1^{-1} ([0, h_2(t_2)])) \) for \( \lambda_2 \)-almost all \( t_2 \in T_2 \). Since \( \{ [0, h_2(t_2)] \}_{t_2 \in T_2} \) generates the Borel \( \sigma \)-algebra on \([0, 1]\) modulo null sets, \( \{ h_1^{-1} ([0, h_2(t_2)]) \}_{t_2 \in T_2} \) generates \( \mathcal{G}_1 \) on \( T_1 \) modulo null sets, which implies that \( E_j^1 \) is independent of \( \mathcal{G}_1 \) under \( \pi_1 \). As \( m \) is arbitrary, we have proved that for any natural number \( m \geq 2 \), there exist \( m \) disjoint subsets \( \{ E_j^1 \}_{1 \leq j \leq m} \) which are of measure \( \frac{1}{m} \) and independent of \( \mathcal{G}_1 \) under \( \pi_1 \). Thus, \( T_1 \) has no \( \mathcal{G}_1 \)-atom under \( \pi_1 \). Since \( \pi_1 \) and \( \lambda_1 \) are absolutely continuous with respect to each other, \( T_1 \) has no \( \mathcal{G}_1 \)-atom under \( \lambda_1 \). Similarly, one can show that \( T_2 \) has no \( \mathcal{G}_2 \)-atom under \( \lambda_2 \).
We extend the game $\Gamma'$ to an $n$-player game $\Gamma_2$. Players 1 and 2 in $\Gamma_2$ share the same payoffs, action sets and private information spaces with those in the game $\Gamma'$. Other players in $\Gamma_2$ are dummy in the sense that player $k$ has private information space $(T_k, T_k, \lambda_k)$, and only one action set $X_k = \{a\}$ for $3 \leq k \leq n$. The common prior $\lambda$ is absolutely continuous with respect to $\otimes_{1 \leq i \leq n} \lambda_i$ with the Radon-Nikodym derivative $q(t_1, t_2)$. Hence, the payoffs of all players in the Bayesian game $\Gamma_2$ are type-irrelevant. If $\Gamma_2$ has a pure strategy equilibrium, then the analysis above shows that players 1 and 2 have coarser inter-player information.

For any $3 \leq j \leq n$, one can construct a new $n$-player game $\Gamma_j$ in which players 1 and $j$ are active while all other players are dummy. The payoff functions, action sets and private information spaces of players 1 and $j$ are defined similarly as those of players 1 and 2 in the game $\Gamma_2$. Adopting the above argument, it can be shown that players 1 and $j$ have coarser inter-player information. Therefore, all players have coarser inter-player information.

(2) Now we construct a new game $\Gamma''$ based on the game $\Gamma'$ above. Suppose that players 1 and 2’s action spaces and private information spaces are the same, while the payoff of player $i$ in the game $\Gamma''$ is given by $v_i(x, t) = u_i(x) \cdot q(t)$ for each $x \in X$ and $t \in T$, where $u_i$ is the payoff function of player $i$ and $q$ is the Radon-Nikodym derivative in the game $\Gamma'$. Players have independent types and the common prior $\lambda = \lambda_1 \otimes \lambda_2$. It is obvious that the game $\Gamma''$ is essentially the same compared with the game $\Gamma'$ if one considers the density weighted payoff. Extending the game $\Gamma''$ to an $n$-player game with 2 active players and $n - 2$ dummy players, then one can follow the proof in (1) and show that the two active players have coarser inter-player information if the game $\Gamma''$ has a pure strategy equilibrium. Since those two active players are arbitrarily chosen, all players have coarser inter-player information.

5.2.2 Proofs in Section 4.3

Proof of Proposition 1. Given any behavioral strategy profile $f$, $x_i \in X_i$ and $t_i \in T_i$, let

$$V_i^f(x_i, t_i) = \int_{T_{-i}} \int_{X_{-i}} u_i(x_i, x_{-i}, t_i, t_{-i})q(t_i, t_{-i}) \prod_{j \neq i} f_j(t_j, dx_j) \lambda_{-i}(dt_{-i}).$$

For any $\mu_i \in \mathcal{M}(X_i)$, define

$$W_i^{(\mu_i, f)}(t_i) = \int_{X_i} V_i^f(x_i, t_i) \mu_i(dx_i).$$
Let \( c_i(t_i, x_i) = 1_{\text{supp } f_i(t_i)}(x_i) \) for each \( t_i \in T_i, x_i \in X_i \) and \( i \in I \). Denote \( c_i^{\mu_i}(t_i) = \int_{X_i} c_i(t_i, x_i)\mu_i(dx_i) \) for any \( \mu_i \in \mathcal{M}(X_i) \). Then given any behavioral strategy \( h_i \), we slightly abuse the notation by letting \( W_i^{(h_i, f_i)}(t_i) = \int_{X_i} V_i^{f_i}(x_i, t_i)h_i(t_i, dx_i) \) and \( c_i^{h_i}(t_i) = \int_{X_i} c_i(t_i, x_i)h_i(t_i, dx_i) \).

Define a correspondence

\[
H_i^f(t_i) = \{ (x_i, V_i^{f_i}(x_i, t_i), c_i(t_i, x_i)) : x_i \in X_i \}.
\]

We have

\[
\text{co}(H_i^f(t_i)) = \{ (\mu_i, W_i^{(\mu_i, f_i)}(t_i), c_i^{\mu_i}(t_i)) : \mu_i \in \mathcal{M}(X_i) \}.
\]

By Theorem 1, we have \( E^{\lambda_i}(H_i^f|G_i) = E^{\lambda_i}(\text{co}(H_i^f)|G_i) \).

For each \( i \in I \), \( (f_i, W_i^{(f_i, f_i)}, c_i^{f_i}) \) is a measurable selection of \( \text{co}(H_i^f) \). Thus, there is a \( T_i \)-measurable mapping \( g_i \) from \( T_i \) to \( X_i \) such that \( E^{\lambda_i}(g_i|G_i) = E^{\lambda_i}(f_i|G_i), E^{\lambda_i}(W_i^{(g_i, f_i)}|G_i) = E^{\lambda_i}(W_i^{(f_i, f_i)}|G_i) \) and \( E^{\lambda_i}(c_i^{g_i}|G_i) = E^{\lambda_i}(c_i^{f_i}|G_i) \). Then \( E^{\lambda_i}(g_i|G_i) = E^{\lambda_i}(f_i|G_i) \) for each \( i \) implies that \( f \) and \( g \) are distribution equivalent.

Given any \( t_i \in T_i \) and \( x_i \in X_i \),

\[
V_i^g(x_i, t_i) = \int_{T_{-i}} \int_{X_{-i}} w_i(x_i, x_{-i}, t_i, t_{-i})g(t_i, t_{-i}) \prod_{j \neq i} g_j(t_j, dx_j)\lambda_{-i}(dt_{-i})
\]

\[
= \int_{T_{-i}} \int_{X_{-i}} w_i(x_i, x_{-i}, t_i, t_{-i}) \prod_{j \neq i} g_j(t_j, dx_j)\lambda_{-i}(dt_{-i})
\]

\[
= \int_{T_{-i}} \int_{X_{-i}} w_i(x_i, x_{-i}, t_i, t_{-i}) \prod_{j \neq i} E^{\lambda_j}(g_j|G_j)(t_j, dx_j)\lambda_{-i}(dt_{-i})
\]

\[
= \int_{T_{-i}} \int_{X_{-i}} w_i(x_i, x_{-i}, t_i, t_{-i}) \prod_{j \neq i} E^{\lambda_j}(f_j|G_j)(t_j, dx_j)\lambda_{-i}(dt_{-i})
\]

\[
= \int_{T_{-i}} w_i(x_i, f_{-i}(t_{-i}), t_i, t_{-i})\lambda_{-i}(dt_{-i})
\]

\[
= V_i^f(x_i, t_i).
\]

The third and fifth equalities are due to Equation (3), and the fourth equality holds since \( E^{\lambda_i}(g_i|G_i) = E^{\lambda_i}(f_i|G_i) \) for each \( i \in I \). Thus, \( W_i^{(h_i, g_i)}(t_i) = W_i^{(h_i, f_i)}(t_i) \) for any \( h_i \) and \( t_i \in T_i \).

We have

\[
U_i(g) = \int_{T_i} W_i^{(g_i, g_i)}(t_i)\lambda_i(dt_i) = \int_{T_i} W_i^{(g_i, f_i)}(t_i)\lambda_i(dt_i)
\]

\[
= \int_{T_i} E^{\lambda_i}(W_i^{(g_i, f_i)}|G_i)\lambda_i(dt_i) = \int_{T_i} E^{\lambda_i}(W_i^{(f_i, f_i)}|G_i)\lambda_i(dt_i)
\]

\[
= \int_{T_i} W_i^{(f_i, f_i)}(t_i)\lambda_i(dt_i) = U_i(f),
\]

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and

\[ U_i(h_i, g_{-i}) = \int_{T_i} W_i^{(h_i, g)}(t_i) \lambda_i(dt_i) = \int_{T_i} W_i^{(h_i, f)}(t_i) \lambda_i(dt_i) = U_i(h_i, f_{-i}). \]

Thus, \( f \) and \( g \) are strongly payoff equivalent.

Finally, since \( E^{\lambda_i}(c^g_i \mid G_i) = E^{\lambda_i}(c^f_i \mid G_i) \), we have

\[ \int_{T_i} c^g_i(t_i) \lambda_i(dt_i) = \int_{T_i} c^f_i(t_i) \lambda_i(dt_i) = \int_{T_i} \int_{X_i} c_i(t_i, x_i) f_i(t_i, dx_i) \lambda_i(dt_i) = 1 \]

which implies that \( c(t_i, g_i(t_i)) = c^g_i(t_i) = 1 \) for \( \lambda_i \)-almost all \( t_i \in T_i \). That is, \( g_i(t_i) \in \text{supp } f_i(t_i) \) for \( \lambda_i \)-almost all \( t_i \in T_i \), \( f_i \) and \( g_i \) are belief consistent. Since \( i \) is arbitrarily chosen, \( f \) and \( g \) are belief consistent.

Therefore, \( g \) is a strong purification of \( f \). \( \square \)

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