THE ARAKELOV-ZHANG PAIRING AND JULIA SETS

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Abstract. The Arakelov-Zhang pairing \( \langle \psi, \phi \rangle \) is a measure of the “dynamical distance” between two rational maps \( \psi \) and \( \phi \) defined over a number field \( K \). It is defined in terms of local integrals on Berkovich space at each completion of \( K \). We obtain a simple expression for the important case of the pairing with a power map, written in terms of integrals over Julia sets. Under certain disjointness conditions on Julia sets, our expression simplifies to a single canonical height term; in general, this term is a lower bound. As applications of our method, we give bounds on the difference between the canonical height \( h_\psi \) and the standard Weil height \( h \), and we prove a rigidity statement about polynomials that satisfy a strong form of good reduction.

1. Introduction

Let \( K \) be a number field with fixed algebraic closure \( \overline{K} \). For \( z \in \mathbb{P}^1(\overline{K}) \), we use \( h(z) \) to denote the standard logarithmic Weil height of \( z \) and \( h_\phi(z) \) to denote the Call-Silverman canonical height of \( z \) with respect to \( \phi \in K(x) \). We recall background on these heights in Section 2.

Let \( \psi, \phi : \mathbb{P}^1 \to \mathbb{P}^1 \) be rational maps defined over \( K \) (equivalently, rational functions in \( K(x) \)) each of degree \( \geq 2 \). In [20] (see also [22]), Petsche, Szpiro, and Tucker introduced the Arakelov-Zhang pairing \( \langle \psi, \phi \rangle \), a symmetric, non-negative, real-valued pairing on the space of rational maps. In Section 2 we present their definition of the pairing as a sum of integrals over Berkovich space at each place of \( K \). Using [20, Theorem 11] and standard results on equidistribution of preimages, we also give a more intuitive equivalent definition as the limiting average of \( h_\psi \) evaluated at the preimages under \( \phi \) of any non-exceptional point \( \beta \in \mathbb{P}^1(\overline{K}) \) (here “exceptional” means that \( \beta \) has finite backward orbit under \( \psi \)):

\[
\langle \psi, \phi \rangle = \lim_{n \to \infty} \frac{1}{(\deg \phi)^n} \sum_{\phi^n(x) = \beta} h_\psi(x).
\]

The pairing can be understood as a “dynamical distance” between \( \psi \) and \( \phi \). For example, by [20, Theorem 3], \( \langle \psi, \phi \rangle \) vanishes precisely when the canonical height functions \( h_\psi \) and \( h_\phi \) agree; this in turn holds if and only if the sets of preperiodic points of \( \psi \) and \( \phi \) coincide. Thus the specific pairing \( \langle x^2, \phi \rangle \) may be interpreted as a measure of the dynamical complexity of \( \phi \), as the height function \( h_{x^2} \) equals the standard height \( h \). As above, we have

\[
\langle x^2, \phi \rangle = \lim_{n \to \infty} \frac{1}{(\deg \phi)^n} \sum_{\phi^n(x) = \beta} h(x).
\]

for non-exceptional \( \beta \). We note that \( \langle x^d, \phi \rangle = \langle x^2, \phi \rangle \) for any \( d \in \mathbb{Z} \setminus \{-1, 0, 1\} \).

In this paper, we study the relationship between the Arakelov-Zhang pairing and Julia sets, both in the classical and the non-archimedean setting. We produce a formula for the pairing \( \langle x^2, \phi \rangle \), which can be computed exactly under certain disjointness conditions on Julia sets.

Let \( M_K \) be the set of places of \( K \). For \( \nu \in M_K \), let \( r_\nu = [K_\nu : \mathbb{Q}_\nu] / [K : \mathbb{Q}] \), and let \( \mu_{\phi, \nu} \) be the canonical \( \phi \)-invariant probability measure on the Berkovich projective line \( \mathbb{P}^1 \) over \( \mathbb{C}_\nu \) (see Section 2 for definitions). Our main theorem is as follows:

Theorem 1.1. Let \( \phi \in K(x) \). Then

\[
\langle x^2, \phi \rangle = h_\phi(0) - \sum_{\nu \in M_K} r_\nu \int_{|\alpha| < 1} \log |\alpha| d\mu_{\phi, \nu}.
\]

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Theorem 1.6. They show the following theorem:

bound on the difference between the canonical height of $\phi$ in $\mathbb{P}^1$, we may interpret it as an integral over the Julia set. The integral evaluates to 0 if $\phi$ has good reduction at the place $\nu$, so the sum over all $\nu \in M_K$ is actually a finite sum.

We give two proofs of Theorem 1.1. The first proof directly uses the formula for the Arakelov-Zhang pairing in Equation [12] as the limiting average height of preimages, and avoids the extensive local analytic machinery used to define the pairing in [20]. However, due to measure-theoretic difficulties, this proof does not work when 0 is in the Julia set of $\phi$ at some place. The second proof is less elementary, but it does not require an assumption on the Julia set.

We give some consequences of Theorem 1.1 which are easier to state. Corollary 1.2 is a fundamental inequality between the pairing $\langle x^2, \phi \rangle$ and the canonical height $h_\phi(0)$, which are equal under a disjointness condition on Julia sets. See Section 3 for details on when this condition is satisfied.

Corollary 1.2. For any $\phi \in K(x)$,

$$\langle x^2, \phi \rangle \geq h_\phi(0),$$

with equality if and only the Julia set of $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ is disjoint from the open unit disk in $\mathbb{P}^1$ at every completion of $K$.

Corollary 1.3 is the purely archimedean version of our main theorem, which becomes simpler (and requires no reference to Berkovich space) in the case that $\phi$ is a monic polynomial with integer coefficients. Here $\mu_\phi$ is the invariant measure on $\mathbb{P}^1(\mathbb{C})$, the integral is over the complex unit disk, and the canonical height $h_\phi$ may be interpreted over $\mathbb{Q}$ (or any number field).

Corollary 1.3. Let $\phi \in \mathbb{Z}[x]$ be monic. Then

$$\langle x^2, \phi \rangle = h_\phi(0) - \int_{|z|<1} \log |z| d\mu_\phi.$$

Remark 1.4. A similar statement to Corollary 1.3 holds if $\phi$ has algebraic integer coefficients, but the integral must be replaced with a sum of integrals corresponding to every possible embedding into $\mathbb{C}$, and the statement is essentially that of Theorem 1.1 (though again with no reference to Berkovich space). See Section 3. The restriction to algebraic integers is interesting from the dynamical point of view because of its connection to families of post-critically finite mappings. For example, in the family of complex quadratic polynomials $\phi_c(z) = z^2 + c$, the mappings $\phi_c$ for which the critical point 0 is preperiodic arise from certain algebraic integer parameters $c$ (the roots of the famous Gleason and Misiurewicz polynomials, see, e.g., [7]).

Our method also allows us to prove Theorem 1.5, a rigidity statement about polynomials with certain reduction conditions, by combining our work with a result of Kawaguchi-Silverman on maps with equal canonical height functions [10].

Theorem 1.5. Let $\phi(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in K[x]$ with $\beta$ a preperiodic point of $\phi$. Suppose that $\phi$ has good reduction at every non-archimedean place. Further suppose that the Julia set of $\phi$ at every archimedean place $\nu$ does not intersect the open unit disc centered at $\beta$ in $\mathbb{P}^1$ over $\mathbb{C}_\nu$. Then $\phi(x) = (x - \beta)^d + $.

Petsche, Szpiro, and Tucker also show that the pairing $\langle x^2, \phi \rangle$ can be used to give an upper bound on the difference between the canonical height of $\phi$ and the standard height. More precisely, they show the following theorem:

Theorem 1.6. [21] Theorem 15] Let $\phi$ be a rational function of degree at least 2 defined over a number field $K$. Then for any $z \in \mathbb{P}^1(K)$,

$$h_\phi(z) - h(z) \leq \langle x^2, \phi \rangle + h_\phi(\infty) + \log 2.$$

The explicit nature of Theorem 1.1 allows us to compute the pairing with some rational functions where the canonical measure is known explicitly (such as Chebyshev polynomials). For these examples, we then apply Theorem 1.6 to bound the difference between the Weil height and the canonical height.

The paper is organized as follows. In Section 2, we recall some relevant background and define the Arakelov-Zhang pairing in terms of local integrals. We also prove Theorem 1.1 and
its corollaries. In Section 3, we recall some facts about Julia sets and prove Theorem 1.5. In Section 4, we compute some explicit examples.

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2. Background

We sketch the background needed to properly define the Arakelov-Zhang pairing, starting with a brief overview of height functions. See [17] for background and basic properties of height functions, and see [9] for background on the canonical height function.

Let \( K \) be a number field. The logarithmic height of \( x \in K \) is defined by

\[
h(x) = \sum_{\nu \in M_K} r_\nu \log \max\{|x|_\nu, 1\},
\]

where \( r_\nu = [K_\nu : \mathbb{Q}_\nu]/[K : \mathbb{Q}] \) as in the introduction. This definition immediately extends in a compatible way to any finite extension \( K' \) of \( K \) by the local-global degree formula, and so \( h \) is a function \( h : \overline{K} \to \mathbb{R} \). We extend \( h \) to a function \( h : \mathbb{P}^1(K) \to \mathbb{R} \) by setting \( h(\infty) = 0 \). See [5] or [15] for an alternate but equivalent way of defining the height function.

Fix a rational function \( \phi \in K(x) \) with \( d = \deg \phi \geq 2 \). We use \( \phi^n \) to mean the \( n \)-fold composition of \( \phi \) with itself. The Call-Silverman canonical height relative to \( \phi \) is defined by

\[
h_\phi(x) = \lim_{n \to \infty} \frac{h(\phi^n(x))}{d^n}
\]

for all \( x \in \mathbb{P}^1(\overline{K}) \). In [9] it is shown that this limit exists, and its basic properties are established. Importantly, for all \( x \in \mathbb{P}^1(\overline{K}) \),

\[
h_\phi(\phi(x)) = dh_\phi(x), \quad \text{and} \quad |h(x) - h_\phi(x)| < C_\phi
\]

for an absolute constant \( C_\phi \) (in fact, the canonical height is uniquely characterized by these two properties). The other property of the canonical height that we will use is that \( h_\phi(x) = 0 \) if and only if \( x \) is preperiodic for \( \phi \), i.e., if \( \phi^n(x) = \phi^m(x) \) for some \( n > m \geq 0 \). In the setting of number fields, this fact is a simple consequence of Northcott’s theorem [21, Theorem 3.22]. It also holds if \( K \) is a function field and \( \phi \) is not isotrivial, due to work of Benedetto [3].

We say that a map \( \phi \in K(x) \) has good reduction at a non-archimedean place \( \nu \) if the degree of \( \phi \) is unchanged after reducing the coefficients to the residue field \( k_\nu \). To be precise, we must first write \( \phi \) as a map \( \mathbb{P}^1 \to \mathbb{P}^1 \) in homogeneous coordinates and choose a normalized form. See [19] or [21] Theorem 2.18 for details. For a polynomial

\[
\phi(x) = a_dx^d + a_{d-1}x^{d-1} + \cdots + a_1x + a_0,
\]

good reduction has the simple interpretation that \( \nu(a_d) = 0 \) and \( \nu(a_i) \geq 0 \) for \( 0 \leq i \leq d-1 \). There are evidently only finitely many non-archimedean places \( \nu \in M_K \) for which \( \phi \) has bad reduction (i.e., does not have good reduction).

We recall some notation and terminology from [20]. Let \( \mathbb{K} \) be either the complex numbers \( \mathbb{C} \) or the field \( \mathbb{C}_v \) which is the completion of the algebraic closure of \( K_v \), and let \( |\cdot| \) be the standard absolute value on \( \mathbb{K} \). Let \( \mathbb{P}^1 \) denote the Berkovich projective line over \( \mathbb{K} \); for \( \mathbb{K} = \mathbb{C} \), \( \mathbb{P}^1 \) is simply \( \mathbb{P}^1(\mathbb{C}) \). See [11] for background on Berkovich space in the context of dynamics.

Let \( \phi : \mathbb{P}^1 \to \mathbb{P}^1 \) be a morphism of degree \( d \) defined over \( \mathbb{K} \). A polarization \( \epsilon \) of \( \phi \) is an isomorphism \( \epsilon : \mathcal{O}(d) \to \mathcal{O}(1) \), where by \( \mathcal{O}(d) \) we mean \( \mathcal{O}_{\mathbb{P}^1}(d) \). More concretely, a choice of a polarization is equivalent to a choice of a homogeneous lift of \( \phi \) to a polynomial endomorphism \( \Phi : \mathbb{A}^2 \to \mathbb{A}^2 \) (see [20] Section 2.1).

Recall that a metric on a line bundle \( \mathcal{L} \) is a non-negative, real-valued function on \( \mathcal{L} \) such that the restriction to each fiber \( \mathcal{L}_x \) is a norm on \( \mathbb{K} \)-vector space. The standard metric \( || \cdot ||_{st} \) on \( \mathcal{O}(1) \) is characterized by the identity

\[
||s(x)||_{st} = \frac{|s(x_0, x_1)|}{\max\{|x_0|, |x_1|\}}
\]
Remark 2.2. The Arakelov-Zhang pairing may be understood as a dynamical distance, but it is clear from the notation, the local pairing does not depend on the choice of polarization $\epsilon$.

The standard measure $\mu_{st}$ on the Berkovich projective line $\mathbb{P}^1$ over $\mathbb{K}$ is the Haar measure on the unit circle for $\mathbb{K} = \mathbb{C}$, and the Dirac point mass at the Gauss point $\zeta_0,1$ for $\mathbb{K}$ non-archimedean. The canonical invariant probability measure $\mu_{\phi}$ is defined as the weak limit as $k \to \infty$ of the sequence of measures given by

$$
\mu_{\phi,0} = \mu_{st},
$$

$$
\mu_{\phi,k+1} = \frac{1}{d} \phi^* \mu_{\phi,k}
$$

for all $k$. The measure satisfies $\phi_* \mu_{\phi} = \mu_{\phi}$ and $\phi^* \mu_{\phi} = d \cdot \mu_{\phi}$.

The Julia set of $\phi$ can be defined in many equivalent ways. The most classical definition is that the Fatou set of $\phi$ is the locus on which the family of iterates $\{\phi^n\}_{n=1}^{\infty}$ is equicontinuous, and the Julia set is the complement of the Fatou set. The support of the invariant measure $\mu_{\phi}$ is precisely the Julia set of $\phi$ - this was proved in [11] for $\mathbb{C}$ and by Rivera-Letelier in the non-archimedean setting (see [1] Theorem 10.56) for a writeup of the proof).

For the moment, let $\psi, \phi : \mathbb{P}^1 \to \mathbb{P}^1$ be defined over $\mathbb{K}$. Fix a polarization $\epsilon$ of $\phi$. Let $s, t \in \Gamma(\mathbb{P}^1, \mathcal{O}(1))$ be sections with $\text{div}(s) \neq \text{div}(t)$. The local Arakelov-Zhang pairing is defined by

$$
\langle \psi, \phi \rangle_{s,t} = \log \|s(\text{div}(t))\|_{\phi,\epsilon} - \int \log \|s(x)\|_{\phi,\epsilon} d\mu_{\psi}(x). \quad (2.1)
$$

Now let $\psi, \phi$ be defined over the number field $K$. For $\nu \in M_K$, the local pairing of $\psi$ and $\phi$ for $\mathbb{K} = \mathbb{C}_\nu$ is denoted $\langle \psi, \phi \rangle_{s,t,\nu}$. The global Arakelov-Zhang pairing is then defined by

$$
\langle \psi, \phi \rangle = \sum_{\nu \in M_K} r_{\nu} \langle \psi, \phi \rangle_{s,t,\nu} + h_{\psi}(\text{div}(s)) + h_{\phi}(\text{div}(t)). \quad (2.2)
$$

As is clear from the notation, the local pairing does not depend on the choice of polarization $\epsilon$, and the global pairing does not depend on the choice of sections $s$ and $t$ (see [20]).

**Remark 2.2.** The Arakelov-Zhang pairing may be understood as a dynamical distance, but it is not a metric on the space of rational maps. However, as observed by Fili, it coincides with the square of a metric of mutual energy defined on a space of adelic measures [13]. These measures have associated canonical height functions, which in the dynamical setting agree with the Call-Silverman canonical height. This connection has been fruitful in studying “unlikely intersection” problems of some relation to the questions studied in this paper, e.g., the recent work in [14] on uniform Manin-Mumford.
As mentioned in the introduction, the formula given for $\langle \psi, \phi \rangle$ in Equation 1.1 follows from the main results of [20] combined with a result on equidistribution of preimages of a non-exceptional point. We prove this as Proposition 2.4. First we recall a theorem of Petsche-Szpiro-Tucker:

**Theorem 2.3.** [20] Theorem 1] Let $\psi$ and $\phi$ be rational functions defined over a number field $K$. Let $\{x_n\} \in \mathbb{P}^1(\overline{K})$ be a sequence of distinct points such that $h_\psi(x_n) \to 0$. Then $h_\phi(x_n) \to \langle \psi, \phi \rangle$.

Recall that an exceptional point $\beta \in \mathbb{P}^1(\overline{K})$ of a rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ is a point such the set of all $x \in \mathbb{P}^1(\overline{K})$ such that $\phi^n(x) = \beta$ for some $n \geq 1$ (the backward orbit of $\beta$) is a finite set. It is not hard to show that, if $\beta$ is exceptional for $\phi$, then up to conjugacy by Möbius transformations, either $\phi$ is a polynomial and $\beta = \infty$ or $\phi(x) = x^d$ for $d \in \mathbb{Z}$ and $\beta \in \{0, \infty\}$.

**Proposition 2.4.** Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d \geq 2$ defined over a number field $K$. Suppose $\beta \in \mathbb{P}^1(\overline{K})$ is not exceptional for $\phi$. Then

$$\langle \psi, \phi \rangle = \lim_{n \to \infty} \frac{1}{d^n} \sum_{\phi^n(x) = \beta} h_\psi(x),$$

where the summation is counted with multiplicity.

**Proof of Proposition 2.4.** If, for all $n$, there are precisely $d^n$ points $x$ with $\phi^n(x) = \beta$, then the claim follows directly from Theorem 2.3. In order to prove the Proposition in general, we need to show that the points in the multiset $\{\phi^n(x) = \beta\}$ cannot occur with too large a multiplicity as $n \to \infty$. Fix an embedding $\overline{K} \hookrightarrow \mathbb{C}$. In particular, we may view $\phi$ as a rational function with complex coefficients under this embedding. Since $\beta$ is a non-exceptional point, the $n$th preimages of $\beta$ equidistribute along the complex Julia set of $\phi$ under the canonical measure $\mu_\phi$ [14].

It follows from [15] Theorem 4] that $\mu_\phi$ has no point masses (the existence of point masses would violate the “balanced measure” condition). Thus the number of times any $\alpha \in \mathbb{P}^1(\mathbb{C})$ occurs in the multiset of $d^n$ preimages of $\beta$ is $o(d^n)$ as $n \to \infty$.

For any $\varepsilon > 0$, there is $n$ such that there are only finitely many $\alpha$ with $h_\phi(\alpha) < h_\phi(\beta)d^{-n}$ and $|h_\psi(\alpha) - \langle x^2, \phi(x) \rangle| > \varepsilon$, since otherwise we could find a sequence of distinct points $\{x_n\}$ such that $h_\phi(x_n) \to 0$ but $h_\psi(x_n)$ does not tend to $\langle \psi, \phi \rangle$, which would violate Theorem 2.3. Fix such an $n$, and call these points $\alpha_1, \alpha_2, \ldots, \alpha_k$. For $n$ sufficiently large, these $\alpha_i$ will occur at most $d^n \varepsilon / \max\{|h_\psi(\alpha_i) - \langle x^2, \phi(x) \rangle|\}$ times as $n$th preimages of $\beta$. For such $n$, we see that

$$\frac{1}{d^n} \sum_{\phi^n(\alpha) = \beta} h_\psi(\alpha) - \langle x^2, \phi(x) \rangle < 2\varepsilon.$$

The claim follows. \(\square\)

We now proceed to proofs of the main theorems in the introduction. The first proof we give assumes that 0 does not lie in the Julia set of $\phi$ at any place of $K$. This proof uses the simple formula given in Proposition 2.4. The second proof does not make any assumption on $\phi$, but it uses the more complicated definition of the Arakelov-Zhang pairing given in Equation 2.2.

**First Proof of Theorem 1.1.** Choose $\beta \in K$ a non-exceptional point such that $\beta$ is not in the forward orbit of 0, that is, $\phi^n(0) \neq \beta$ for any $n \geq 1$. Let $| \cdot |_\nu$ be an absolute value on $K$. Let $L$
denote the splitting field of \( \phi^n(x) - \beta \). Recall that \( r_\nu = \frac{[K_\nu : Q_\nu]}{[K : K_\nu]} \). We compute

\[
\sum_{\omega \in M_\nu, |\nu|} \left[ \frac{L_\omega : Q_\nu}{L : Q} \right] \sum_{\phi^n(\alpha) = \beta} \log \max\{|\alpha|_\omega, 1\}
\]

\[
= \sum_{\omega \in M_\nu, |\nu|} \left[ \frac{L_\omega : Q_\nu}{L : Q} \right] \left( \sum_{\phi^n(\alpha) = \beta} \log |\alpha|_\omega - \sum_{\phi^n(\alpha) = \beta, |\alpha|_\omega < 1} \log |\alpha|_\omega \right)
\]

\[
= \sum_{\omega \in M_\nu, |\nu|} \left[ \frac{L_\omega : Q_\nu}{L : Q} \right] \left( \log |\phi^n(0) - \beta|_\nu - \sum_{\phi^n(\alpha) = \beta, |\alpha|_\omega < 1} \log |\alpha|_\omega \right)
\]

\[
r_\nu \log |\phi^n(0) - \beta|_\nu - \sum_{\omega \in M_\nu, |\nu|} \left[ \frac{L_\omega : Q_\nu}{L : Q} \right] \sum_{\phi^n(\alpha) = \beta, |\alpha|_\omega < 1} \log |\alpha|_\omega,
\]

where we use that for any \( \alpha \in K \), the extension formula implies that

\[
\prod_{\omega \in M_\nu, |\nu|} |\alpha|_{\nu, \omega} = |r_\nu|_{\nu, \nu, \omega}.
\]

Fix a completion of the algebraic closure of \( K_\nu, C_\nu \), with an absolute value that extends \(| \cdot |_\nu\). There is a distinguished place \( \omega \) on \( L \) extending \( \nu \) such that \( \omega \) agrees with the valuation on \( C_\nu \). Since \( G = \text{Gal}(L/K) \) acts transitively on the set of valuations above \( \nu \), every valuation \( \omega' \) on \( L \) above \( \nu \) is of the form \(| \cdot |_{\omega'} = |\sigma(\cdot)|_\omega\) for some \( \sigma \in G \). Let \( D_\omega \) denote the decomposition group of \( \omega \), i.e. the stabilizer of \( \omega \) in \( G \). Then the set of valuations above \( \nu \) is isomorphic as a \( G \)-set to \( G/D_\omega \). Note that

\[
|D_\omega| = \frac{|G|}{[L_\omega : K_\nu]} = \frac{|L : K|}{[L_\omega : K_\nu]}.
\]

Thus we may write

\[
\sum_{\omega \in M_\nu, |\nu|} \left[ \frac{L_\omega : Q_\nu}{L : Q} \right] \sum_{\phi^n(\alpha) = \beta, |\alpha|_\omega < 1} \log |\alpha|_\omega = \frac{[L_\omega : Q_\nu]}{[L : Q]} \sum_{g \in G/D_\omega} \sum_{\phi^n(\alpha) = \beta, |\alpha|_{g\omega} < 1} \log |\alpha|_{g\omega},
\]

where as usual we identify \( G/D_\omega \) with a given choice of a left transversal. For each \( \alpha \) with \(|\alpha|_\omega < 1\), the term \( \log |\alpha|_\omega \) appears once in the inner sum for each \( g \in G/D_\omega \). Therefore

\[
\sum_{\omega \in M_\nu, |\nu|} \left[ \frac{L_\omega : Q_\nu}{L : Q} \right] \sum_{\phi^n(\alpha) = \beta, |\alpha|_\omega < 1} \log |\alpha|_\omega = \frac{[K_\nu : Q_\nu]}{[K : Q]} \sum_{\phi^n(\alpha) = \beta, |\alpha|_\omega < 1} \log |\alpha|_\omega.
\]

If the Julia set of \( \phi \) at a completion \( \nu \) of \( K \) does not include 0, then the function \( \alpha \mapsto \log |\alpha|_\nu \) is a bounded continuous function on the support of the canonical invariant measure \( \mu_{\phi, \nu} \). Therefore, by the weak convergence of the average of point masses \( \frac{1}{n} \sum_{\phi^n(\alpha) = \beta} \delta_\alpha \) to the canonical measure, we have that

\[
\lim_{n \to \infty} \frac{1}{d^n} \sum_{\phi^n(\alpha) = \beta, |\alpha|_\omega < 1} \log |\alpha|_\nu = \int_{|\alpha| < 1} \log |\alpha|_\nu d\mu_{\phi, \nu}.
\]

Using the same notation as previously, we compute:

\[
\frac{1}{d^n} \sum_{\phi^n(\alpha) = \beta} h(\alpha) = \frac{1}{d^n} \sum_{\nu \in M_K} \sum_{\omega \in M_\nu, |\nu|} \left[ \frac{L_\omega : Q_\nu}{L : Q} \right] \sum_{\phi^n(\alpha) = \beta} \log \max\{|\alpha|_\omega, 1\}
\]

\[
= \sum_{\nu \in M_K} \left( r_\nu \log |\phi^n(0) - \beta|_\nu - r_\nu \sum_{\phi^n(\alpha) = \beta, |\alpha|_\omega < 1} \log |\alpha|_\omega \right)
\]

\[
= \frac{h(\phi^n(0) - \beta)}{d^n} - \sum_{\nu \in M_K} r_\nu \sum_{\phi^n(\alpha) = \beta, |\alpha|_\omega < 1} \log |\alpha|_\omega.
\]

Taking the limit as \( n \to \infty \), the first term approaches \( h(0) \) by basic properties of heights. Since the average height of the preimages of a non-exceptional point tends to the pairing \( \langle x^2, \phi \rangle \) by Proposition 2.4, this implies the result. \( \square \)
We now give a second proof of Theorem 1.1. The proof is somewhat similar to the proof of Proposition 16 in [20].

Second Proof of Theorem 1.1. Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be defined over $K$. Fix homogeneous coordinates $x_0, x_1$ on $\mathbb{P}^1$, and set $x = [x_0 : x_1]$. Consider the map $\mathbb{P}^1 \to \mathbb{P}^1$ given by $x^2$ (i.e., $[x_0 : x_1] \mapsto [x_0^2 : x_1^2]$), and choose the polarization $\epsilon$ that yields its homogeneous lift $\Phi(x_0, x_1) = (x_0^2, x_1^2)$.

The squaring map has good reduction at all non-archimedean places of $K$, so by [20 Proposition 6] we have

$$\|s(x)\|_{x^2, \epsilon, \nu} = \|s(x)\|_{s,t,\nu}$$

for each non-archimedean place $\nu$. This equality also holds at each archimedean place – simply observe that the identity in Equation 13 of [20, Proposition 6] is clearly true for $\nu$ archimedean, and the rest of the argument goes through verbatim.

To compute the local pairing, we choose sections $s, t \in \Gamma(\mathbb{P}^1(\mathbb{C}_\nu), \mathcal{O}(1))$. Let $s(x_0, x_1) = x_0$ and $t(x_0, x_1) = x_0 + x_1$. We compute $\text{div}(s) = [0 : 1]$ and $\text{div}(t) = [1 : -1]$. Identify $\mathbb{P}^1$ with $\mathbb{K} \cup \{1 : 0\}$ in the usual way, with $\alpha \in \mathbb{K}$ corresponding to $[\alpha : 1]$. Then for any place $\nu \in M_K$,

$$\|s(a)\|_{x^2, \epsilon, \nu} = \|s(a)\|_{s,t,\nu} = \frac{|a|_\nu}{\max\{|a|_\nu, 1\}} = \min\{|a|_\nu, 1\}.$$ 

Therefore,

$$\int \log \|s(x)\|_{x^2, \epsilon, \nu} d\mu_{\phi, \nu}(x) = \int \log |\alpha|_\nu d\mu_{\phi, \nu}(\alpha).$$

We compute $\|s(\text{div}(t))\|_{x^2, \epsilon} = \|s(\text{div}(t))\|_{s,t,\epsilon} = 1$. So for any place $\nu$,

$$\langle x^2, \phi \rangle_{s,t,\nu} = -\int_{|\alpha|_\nu < 1} \log |\alpha| d\mu_{\phi, \nu}(\alpha)$$

by the definition of the local pairing in Equation 2.1.

Now observe that $h_\phi(\text{div}(s)) = h_\phi(0)$ and $h_\phi(\text{div}(t)) = 0$, as $[1 : -1]$ is preperiodic under the map $x^2$. Using Equation 2.2 we compute

$$\langle x^2, \phi \rangle = \sum_{\nu \in M_K} r_\nu \langle x^2, \phi \rangle_{s,t,\nu} + h_\phi(\text{div}(t)) + h_\phi(\text{div}(s))$$

$$= h_\phi(0) - \sum_{\nu \in M_K} r_\nu \int_{|\alpha|_\nu < 1} \log |\alpha| d\mu_{\phi, \nu}(\alpha),$$

as claimed. \qed

Proof of Corollary 1.2. The Julia set is the support of the canonical measure. Thus if the Julia set is disjoint from the open unit disk for a valuation $\nu$, then

$$\int_{|\alpha|_\nu < 1} \log |\alpha| d\mu_{\phi, \nu} = 0,$$

so $\langle x^2, \phi \rangle = h_\phi(0)$.

Suppose the Julia set of $\phi$ at some valuation $\nu$ intersects the open unit disk. Then the Julia set intersects the open ball of radius $r$ for some $r < 1$. The measure of this ball is then positive, and we have that $\log |\alpha| < -\varepsilon < 0$ for some $\varepsilon$ on this ball. Thus

$$\int_{|\alpha|_\nu < 1} \log |\alpha| d\mu_{\phi, \nu} < 0,$$

implying the result. \qed

Proof of Corollary 1.3. Let $\nu \in M_K$ be a non-archimedean place. If $\phi$ has good reduction at $\nu$, then the Julia set in Berkovich space is the Gauss point, hence is disjoint from the open unit disk. A monic polynomial with integer coefficients has good reduction at every such $\nu$, and we are done. \qed
3. Applications to Julia sets

First, we give two conditions that guarantee that
\[ \int_{|a|<1} \log|\alpha|d\mu_{\phi,\nu} = 0. \]

The first condition is potentially good reduction. A rational function is said to have potentially good reduction if it can be conjugated by a M"obius transform to a rational function with good reduction. If \( \phi \) is a rational function with good reduction at \( \nu \), then the Julia set of \( \phi \) at \( \nu \) is the Gauss point \( \zeta_{a,1} \) [1 Chapter 10.5]. Since \( \zeta_{a,1} \) is not contained in the open unit disk for any \( a \), the integral vanishes when we have potentially good reduction. Note that, as rational functions have good reduction away from finitely many places, this implies that only finitely many of the terms in the sum in Theorem 1.3 are nonzero.

Our second condition is condition * in the following lemma:

**Lemma 3.1.** Let \( f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \) be a polynomial with coefficients in a number field \( K \), and let \( \nu \) be a non-archimedean valuation which satisfies the condition
\[ \nu(a_0) \leq 0 \text{ and } \nu(a_i) < \nu(a_1) \text{ for every } i. \]

Then the Julia set in the \( \nu \)-adic Berkovich space \( \mathbb{P}^1 \) does not intersect the open unit disk.

**Proof.** Let \( g(x) = x^d + b_{d-1}x^{d-1} + \cdots + b_0 \), with \( \nu(b_i) < \nu(b_1) \) for each \( i \) and with \( \nu(b_0) \leq 0 \). We want to show that \( f(g(x)) \) also has this property. Then
\[ f(g(0)) = \sum_{i=0}^{n} b_ia_i^b. \]

Note that \( \nu(b_ia_i^b) = \nu(b_i) + i\nu(a_0) \geq \nu(a_0) \), with equality if and only if \( i = n \). Thus \( \nu(f(g(0))) = \nu(\nu(a_0)) \). Every other coefficient of \( f(g(x)) \) is a sum of multinomial coefficients multiplied by \( b_ia_i^b \) for some \( i, j \leq n, k < n \). As multinomial coefficients are integral and thus have non-negative valuation at \( \nu \), each coefficient of \( f(g(x)) \) has valuation strictly greater than \( \nu(a_0) \). It follows by induction that \( \mathbb{P} \) is preserved by iteration.

If \( 0 \) is exceptional, then \( g(x) = x^d \) because \( g \) is a polynomial and the result is obvious. Therefore we may assume that \( 0 \) is not exceptional and \( g \) has at least two non-zero coefficients. Then the computation in the previous paragraph implies that for any \( k \), all segments of the Newton polygon of \( f^k \) have non-negative slopes. By the theory of Newton polygons, all roots of \( f^k \) have absolute value \( |\cdot|_\nu \) greater than or equal to 1, and hence lie outside of the open unit disk. As the \( \nu \)-adic Julia set is the set of accumulation points of the backwards orbit of \( 0 \) [1 Theorem 10.22], this implies that the Julia set is contained in the (closed) complement of the open unit disk. \( \square \)

Let \( \phi \in \mathbb{K}[x] \) be a monic polynomial such that, for every non-archimedean place \( \nu \), either \( \phi \) has good reduction at \( \nu \) or \( \phi \) satisfies the hypothesis of Lemma 3.1. For example, this criterion applies to every \( \phi \) in the quadratic family \( x^2 + c \). Lemma 3.1 implies that
\[ \langle x^2, \phi \rangle = h_\phi(0) - \sum_{\nu \text{ archimedean}} r_\nu \int_{|\alpha|<1} \log|\alpha|d\mu_{\phi,\nu}. \]

Equation 3.1 is a slight generalization of Theorem 1.3. Note that the hypothesis holds in the case that \( \phi \) has coefficients in the ring of integers \( \mathcal{O}_K \), as mentioned in Remark 1.3.

We now prove Theorem 3.3 which will immediately imply Theorem 1.5 as a special case. First, we recall Proposition 3.2, a result of Kawaguchi-Silverman on polynomials with equal canonical height functions [10].

**Proposition 3.2.** Let \( \psi, \phi \in \mathbb{K}[x] \) for \( \mathbb{K} \) a number field. Suppose \( h_\psi = h_\phi \). Then after a simultaneous conjugation and up to multiplication by appropriate roots of unity, one of the following is true:
- \( \psi \) and \( \phi \) are both powers of \( x \).
- \( \psi \) and \( \phi \) are both Chebyshev polynomials.
- \( \psi \) and \( \phi \) are both iterates of a common polynomial.
If $f$ is a Möbius transformation and $\phi$ is a rational function, let $\phi^f = f^{-1} \circ \phi \circ f$.

**Theorem 3.3.** Let $\phi$ be a polynomial of degree $d \geq 2$ defined over a number field $K$, and let $\beta \in K$ be preperiodic under $\phi$. Suppose that at every non-archimedean place $\nu$, either

1. $\phi$ has potentially good reduction at $\nu$, or
2. $\phi(x - \beta) + \beta$ satisfies the hypothesis of Lemma 3.1 at $\nu$.

Suppose further that, at each archimedean place $\nu$, the Julia set of $\phi$ in $\mathbb{P}_K^1$ over $\mathbb{C}_\nu$ does not intersect the open unit disk around $\beta$. Then $\operatorname{ord}_{x_0}(x - \beta)/\nu$ for some root of unity $\eta$.

**Proof.** The Arakelov-Zhang pairing is defined independently of coordinates on $\mathbb{P}_K^1$, and hence is invariant under simultaneous conjugation of the two rational maps. Let $f(x) = x + \beta$. By Theorem 1.1,

$$\langle x^2, \phi^f \rangle = h_{\phi^f}(0) - \sum_{\nu \in M_K} r_{\nu} \int_{|\alpha| < 1} \log |\alpha| d\mu_{\phi^f, \nu} = h_{\phi}(\beta) - \sum_{\nu \in M_K} r_{\nu} \int_{|\alpha - \beta| < 1} \log |\alpha - \beta| d\mu_{\phi, \nu}$$

We have $\beta$ preperiodic for $\phi$ and so $h_{\phi}(\beta) = 0$. Also, our hypotheses imply that the support of $\mu_{\phi, \nu}$ is disjoint from the open unit disk centered at $\beta$ (appealing to Lemma 3.1 if necessary). Therefore $\langle x^2, \phi^f \rangle = 0$. By [20, Theorem 3], the canonical heights $h_{x^2}$ and $h_{\phi^f}$ are equal. It follows from Proposition 3.2 that $\phi^f = \eta x^d$ for some root of unity $\eta$. □

Now Theorem 3.3 follows from Theorem 3.1. In fact, the stronger statement of the following Corollary is true, where we may replace the condition of good reduction with the hypothesis of Lemma 3.1.

**Corollary 3.4.** Let $\phi(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in K[x]$ with $0$ a preperiodic point of $\phi$. Suppose that for every non-archimedean place $\nu$, either $\phi$ has good reduction at $\nu$, or $\nu(a_0) \leq 0$ and $\nu(a_i) < \nu(a_0)$ for every $i$. Further suppose that the Julia set of $\phi$ at every archimedean place $\nu$ does not intersect the open unit disc in $\mathbb{P}_K^1$ over $\mathbb{C}_\nu$. Then $\phi(x) = x^d$.

**Remark 3.5.** Observe that Corollary 3.4 applies to quadratic polynomials of the form $x^2 + c$, as the hypothesis of Lemma 3.1 is trivially satisfied. It follows that, for example, if $c$ is a preperiodic parameter in the Mandelbrot set (which is necessarily an algebraic integer), then the Julia set of $\phi_c(x) = x^2 + c'$ must intersect the open unit disk in $\mathbb{C}$ for some Galois conjugate $c'$ of $c$.

4. **Explicit computations**

Because of the explicit nature of Theorem 3.1, we are able to compute the Arakelov-Zhang pairing in several situations, extending the computations in [20].

First, we compute the Arakelov-Zhang pairing between $x^2$ and the Chebyshev polynomials $T_n(x)$, an important family of polynomials in dynamics. The polynomial $T_n$ is characterized by the equation $T_n(x + x^{-1}) = x^n + x^{-n}$ (see e.g. [21] for basic facts about Chebyshev polynomials).

**Proposition 4.1.** Let $T_n(x)$ be a Chebyshev polynomial for $n \geq 2$. Then

$$\langle x^2, T_n \rangle = \frac{1}{2\pi} \int_{-1}^{1} \frac{\log |x|}{\sqrt{1 - x^2}/4} dx = \frac{3\sqrt{3}}{4\pi} \log(2, \chi) \approx 0.3231,$$

where $L(s, \chi)$ is the Dirichlet $L$-function associated to the nontrivial character with modulus 3.

**Proof.** The Chebyshev polynomials are monic polynomials with integer coefficients, so Corollary 3.1 applies. We note that the complex Julia set of the $T_n(x)$ for $n \geq 2$ is the interval $[-2, 2]$, and $0$ is preperiodic. It is known (see e.g. [3] Example 2.6) and can be easily verified that the canonical measure on the complex Julia set of $T_n$ is given by

$$d\mu_{T_n} = \frac{1}{2\pi} \frac{1}{\sqrt{1 - x^2}/4} dx,$$
where $dx$ is the standard Lebesgue measure. Then Corollary 1.3 gives

$$
\langle x^2, T_n \rangle = 0 - \int_{|x| \leq 1} \log |x| d\mu_{T_n} = -\frac{1}{2\pi} \int_1^{\pi/4} \log |x| dx.
$$

We claim that

$$
-\frac{1}{2\pi} \int_{-1}^{1} \frac{\log |x|}{\sqrt{1-x^2/4}} dx = \frac{1}{\pi} \int_{-\pi/3}^{\pi/3} \log |2 + 2 \sin t| dt = \frac{3\sqrt{3}}{4\pi} L(2, \chi).
$$

The later equality is shown by Smyth in an appendix to [4].

The substitution $x = 2 \sin(t/2)$ gives

$$
-\frac{1}{2\pi} \int_{-1}^{1} \frac{\log |x|}{\sqrt{1-x^2/4}} dx = -\frac{1}{2\pi} \int_{-\pi/3}^{\pi/3} \log |2 \sin(t/2)| dt = -\frac{1}{2\pi} \int_{0}^{\pi/3} \log(4 \sin^2(t/2)) dt.
$$

Using the Pythagorean identity, this is equal to

$$
-\frac{1}{2\pi} \int_{0}^{\pi/3} \log(4 \sin^2(t/2)) = -\frac{1}{2\pi} \int_{0}^{\pi/3} (\log(2 - 2 \cos(2t/3)) + \log(2 + 2 \cos(2t/3))) dt
$$

$$
= -\frac{1}{\pi} \int_{0}^{\pi/6} (\log(2 - 2 \cos(t)) + \log(2 + 2 \cos(t))) dt
$$

$$
= -\frac{1}{\pi} \int_{\pi/3}^{\pi/2} (\log(2 - 2 \sin(t)) + \log(2 + 2 \sin(t))) dt.
$$

Substituting $t \to -t$ on the interval from $-\pi/3$ to $0$, we see that

$$
\frac{1}{\pi} \int_{-\pi/3}^{\pi/3} \log |2 \sin t| dt = \int_{0}^{\pi/3} (\log(2 + 2 \sin(t)) + \log(2 - 2 \sin(t))) dt.
$$

Then the result follows as

$$
\int_{0}^{\pi/2} (\log(2 + 2 \sin(t)) + \log(2 - 2 \sin(t))) dt = \int_{0}^{\pi/2} \log(2 \cos(t)) dt = 0,
$$

where the last equality is well-known. \hfill \Box

We are also able to extend the some of the explicit computations done in [20]. For example, Petsche-Szpiro-Tucker prove Proposition 4.2.

**Proposition 4.2.** [20 Proposition 19] Let $\phi(x) = x^2 + c$ for $c \in K$. Then

$$
(1/2) h(c) - \log 3 \leq \langle x^2, \phi \rangle \leq (1/2) h(c) + \log 2.
$$

We show the following:

**Proposition 4.3.** Let $\phi(x) = x^2 + c$ for $c \in K$. Then

$$
\langle x^2, \phi \rangle \geq h_\phi(0).
$$

Suppose that for all archimedean place $\nu$ of $K$ we have that $|c|_\nu \geq 2 + \sqrt{2}$. Then

$$
\langle x^2, \phi \rangle = h_\phi(0).
$$

**Proof.** The first statement follows immediately from Corollary 1.2. As mentioned directly above Equation 5.1 polynomials of the form $x^2 + c$ satisfy the hypothesis of Lemma 5.1. We show that if $|c|_\nu \geq 2 + \sqrt{2}$, then the Julia set of $\phi$ at $\nu$ is disjoint from the open unit disk. Indeed, if $|c|_\nu \geq 2 + \sqrt{2}$, then every point in the open unit disk lies in the basin of attraction of infinity at $\nu$. If $|x|_\nu > \frac{1 + \sqrt{1 + 4|c|_\nu}}{2}$, then $|\phi(x)|_\nu > |x|_\nu$ and hence $|\phi^n(x)|_\nu$ grows geometrically. If $|c|_\nu \geq 2 + \sqrt{2}$ and $|a|_\nu < 1$, then

$$
|a^2 + c|_\nu > |c|_\nu - 1 \geq \frac{1 + \sqrt{1 + 4|c|_\nu}}{2}.
$$

Thus if $|c|_\nu \geq 2 + \sqrt{2}$ for all archimedean places $\nu$, then the Julia set of $\phi$ at every place of $K$ is disjoint from the open unit disk, so the proposition follows from Corollary 1.2. \hfill \Box
We can combine Propositions 4.2 and 4.3 to derive a bound on the difference between \( h_\phi(0) \) and \((1/2)h(\phi)\). Since
\[
h_\phi(0) = \lim_{n \to \infty} \frac{h(\phi^n(0))}{2^n},
\]
one may expect that \((1/2)h(\phi(0))\) is a reasonable approximation for \(h_\phi(0)\). We show that this is a good approximation uniformly in \(\phi\):

**Proposition 4.4.** Let \(\phi(x) = x^2 + c\). Then we have that \(h_\phi(0) \leq (1/2)h(\phi(0)) + \log 2\).

**Proof.** By Proposition 4.3
\[
\langle x^2, \phi \rangle \geq h_\phi(0).
\]
By Proposition 4.2 this implies that \(h_\phi(0) \leq (1/2)h(\phi(0)) + \log 2\). \(\square\)

Petsche-Szpiro-Tucker also consider the pairing between the squaring map \(x^2\) and the map \(\alpha - (\alpha - x)^2\) which is a conjugate of the squaring map by translation by \(\alpha\). Let \(\sigma(x) = x^2\), so that, for a Möbius transformation \(f\), we have \(\sigma^f(x) = f^{-1}(f(x)^2)\).

**Proposition 4.5.** [20] Proposition 18 For \(t \geq 0\), let \(I(t) = -\int_0^1 \log \min \{1, |t + e^{2\pi i \theta}| \} d\theta\). Suppose \(f(x) = \alpha - x\) is defined over a number field \(K\). Then
\[
\langle x^2, \sigma^f \rangle = h(\alpha) + \sum_{\nu | \infty} r_\nu I(|\alpha|_\nu).
\]

We simplify the proof of this this and extend it to the case of any Möbius transformation, which allows us to bound the difference between the standard height and canonical height in these cases.

**Proposition 4.6.** Let \(f\) be a Möbius transform defined over a number field. Then
\[
\langle x^2, \sigma^f \rangle = h(f(0)) + \sum_{\nu | \infty} r_\nu I(|f^{-1}(0)|_\nu).
\]

Note that \(\sum_{\nu | \infty} r_\nu = 1\) and \(I(t)\) attains its maximal value at 1 (see [20] Lemma 17]). Petsche-Szpiro-Tucker compute that
\[
I(1) = \frac{3\sqrt{3}}{4\pi} L(2, \chi),
\]
where as in Proposition 4.4 \(L(s, \chi)\) is the Dirichlet \(L\)-function associated to the nontrivial character with modulus 3. Thus
\[
\langle x^2, \sigma^f \rangle \leq h(f(0)) + \frac{3\sqrt{3}}{4\pi} L(2, \chi).
\]

**Proof.** Note that \(h_{\sigma^f}(x) = h(f(x))\), so \(h_{\sigma^f}(0) = h(f(0))\). Then the result will follow by Theorem 1.11 once we show that
\[
\int_{|\alpha| < 1} \log |\alpha| d\mu_{\sigma^f, \nu} = \int_0^1 \log \min \{1, |f^{-1}(0)|_\nu + e^{2\pi i \theta}| \} d\theta.
\]
Recall that the canonical measure of \(\sigma(x) = x^2\) at a place \(\nu\) (viewed as an embedding of \(K\) into \(\mathbb{C}\)) is the uniform measure on the unit circle. Since the canonical measure of \(\sigma^f\) is the pushforward of the canonical measure of \(\sigma\) under \(f\), the canonical measure of \(\sigma^f\) is the uniform measure on the circle of radius 1 centered at \(f^{-1}(0))\). Since \(e^{2\pi i \theta} + t\) parameterizes the unit circle centered at \(t\) at constant speed at \(\theta\) goes from 0 to 1, this implies the result. \(\square\)

We now apply these explicit computations and Theorem 1.6 to bound the height difference between the canonical height and the standard height.

**Proposition 4.7.** Let \(T_n(x)\) denote the \(n\)th Chebyshev polynomial. Let \(\mu\) be a Möbius transform defined over a number field \(K\). Let \(\sigma^\mu(x) = \mu^{-1}(\mu(x)^2)\). Let \(c = \frac{3\sqrt{3}}{4\pi} L(2, \chi) + \log 2\). Then for any \(n \geq 2\) and any \(x \in \mathbb{P}^1(K)\),
\[
h_{T_n}(x) - h(x) \leq c,
\]
\[
h_{\sigma^\mu}(x) - h(x) \leq c + h(\mu(0)) + h(\alpha^\infty).
\]
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