Special Ergodic Theorem for hyperbolic maps.

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Abstract

Let $f : M \to M$ be a smooth transformation of a compact manifold $M$, admitting an observable SRB measure $\mu$. For a continuous test function $\varphi : M \to \mathbb{R}$ and a constant $\alpha > 0$, consider the set $X_{\varphi, \alpha}$ of the initial points for which the Birkhoff time averages of the function $\varphi$ differ from its $\mu$–space average by at least $\alpha$. As the measure $\mu$ is an SRB one, the intersection of $X_{\varphi, \alpha}$ with the basin of attraction of $\mu$ should have zero Lebesgue measure.

The special ergodic theorem, whenever it holds, claims that, moreover, this intersection has the Hausdorff dimension less than the dimension of $M$. We prove that for $C^1$-local diffeomorphisms, the special ergodic theorem follows from the dynamical large deviations principle. Applying a theorem of L. S. Young, we conclude that in the case of $C^2$-diffeomorphisms, the special ergodic theorem holds for hyperbolic attractors as well as for Anosov maps.

Introduction

Let $f : M \to M$ be a smooth map of a smooth compact manifold $M$ to itself. The famous Birkhoff ergodic theorem states that if $\mu$ is an ergodic invariant measure for $f$, then for any continuous function $\varphi \in C(M)$ and for $\mu$-almost every $x \in M$, at the point $x$ the time averages $\varphi_n$ of the function $\varphi$ converge to its space average $\bar{\varphi}$:

$$
\lim_{n \to \infty} \varphi_n(x) = \varphi, \quad \text{where} \quad \varphi_n(x) := \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ f^k(x), \quad \bar{\varphi} := \int_X \varphi \, d\mu. \quad (1)
$$

The conclusion of this theorem has one large drawback. It is usually much more natural (in particular, from the “physical” point of view) to take an initial point $x$, which is generic in the sense of the Lebesgue measure, and if the measure $\mu$ is supported on some “thin” set (for instance, on a point), one cannot deduce any conclusions for initial points outside this set.

This leads to another famous notion, the one of Sinai-Ruelle-Bowen measure (or SRB measure for short):

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Definition 1. A measure $\mu$ is an (observable) SRB measure for $f : M \to M$ with a basin of attraction $U \subset M$ if for Lebesgue-almost every $x \in U$ one has

$$\varphi_n(x) \to \varphi := \int_M \varphi \, d\mu.$$  

(2)

Not all the systems possess SRB measures. The first example of a system with no SRB measures, the heteroclinic attractor, was given by Bowen himself. However, under certain assumptions, e.g. hyperbolicity of the system, one can guarantee its existence; we refer the reader to [11] for the survey of this subject, as well as to [2, Sec. 14].

But even the conclusion in the definition of an SRB measure sometimes turns out to be insufficient. In some situations (cf. [3], [4], [5]), it is necessary not only to conclude that the set of the “bad” initial conditions has zero Lebesgue measure, but also to state the same for its images under Hölder (but not necessarily absolutely continuous!) conjugacy between two dynamical systems.

One of the ways of handling this difficulty bases on the analysis of Hausdorff dimension of this set. It leads to the concept of special ergodic theorem, proposed by Yu. Ilyashenko, to which this paper is devoted. We will now introduce some auxiliary notations, and present this concept in Def. 2.

For any $\varphi \in C(M)$, $\alpha > 0$, define the set

$$X_{\alpha, \varphi} := \left\{ x \in X : \lim_{n \to \infty} |\varphi_n(x) - \varphi| > \alpha \right\}.$$  

In other words, $X_{\alpha, \varphi}$ is the set of points for which the statement “time averages converge to the space one” is violated “in an essential way”.

Let now $f : M \to M$ be a $C^1$-local diffeomorphism, and $U \subset M$ a trapping domain, $f(U) \subset U$. Assume that $\mu$ is an SRB-measure supported in $U$ and hence in the maximal attractor $A := \cap_{n} f^n(U)$.

Definition 2. Say that for $(f, \mu, U)$ holds the special ergodic theorem, if for any continuous function $\varphi \in C(M)$ and any $\alpha > 0$ the Hausdorff dimension of the set $X_{\alpha, \varphi} \cap U$ is strictly less than the dimension of the phase space:

$$\forall \varphi \in C(M), \alpha > 0 \quad \dim_H(X_{\alpha, \varphi} \cap U) < \dim M.$$  

(3)

Remark 1. Definition 2 as well as the definitions and statements below, can be applied in case $U = M$: we do not require the strict contraction $U \neq f(U)$.

Remark 2. Formally speaking, to state the property (3) one doesn’t need to claim the existence of SRB measure. Though, the conclusion of the definition immediately implies that $\mu$ is an SRB measure. Indeed, the set $X_{0, \varphi} \cap U$ of points in $U$ for which the equality (2) between time and space averages does not hold is exhausted by a countable number of sets $X_{1/n, \varphi} \cap U$ of zero Lebesgue measure.

Having introduced this notion, Yu. Ilyashenko [3] proved that the special ergodic theorem holds for the circle doubling map; later, P. Saltykov [8] proved it also for the linear Anosov map on the 2-torus. Analogous questions were also considered in the work of Gurevich and Tempelman (cf. [11]).
The aim of this note is to prove the special ergodic theorem for a larger class of maps — in particular, for all the hyperbolic ones. Namely, we notice and prove, that the special ergodic theorem can be obtained as a corollary of another property of the system, the dynamical large deviations principle:

**Definition 3.** Let a $C^1$-local diffeomorphism $f : M \to M$, a trapping domain $U \subset M$ and an invariant measure $\mu$ supported in $U$ be given. Say, that for $(f, \mu, U)$ holds the dynamical large deviations principle if for any $\varphi \in C(M), \alpha > 0$ there exist constants $C = C(\alpha, \varphi), h = h(\alpha, \varphi) > 0$ such that
\[
\forall n \in \mathbb{N} \quad \text{Leb}(X_{\alpha,\varphi,n} \cap U) \leq Ce^{-nh},
\]
where
\[
X_{\alpha,\varphi,n} := \{ x \in M : |\varphi_n(x) - \bar{\varphi}| \geq \alpha \}.
\]

**Remark 3.** The measure $\mu$ in Definition 3 is automatically an SRB measure (attracting Lebesgue-a.e. point from $U$) due to the Borel–Cantelli Lemma type arguments.

**Theorem 1** (Main result). If for a $C^1$-local diffeomorphism $f : M \to M$, a trapping domain $U$ and an invariant measure $\mu$ holds the dynamical large deviations principle, then for this system also holds the special ergodic theorem.

As the reader will see in the next section, the proof of Theorem 1 is indeed rather simple (moreover, one needs only Lipschitz condition on the map $f$). However, the conclusion itself seems very interesting to us — especially, due to the possibilities to apply it to the study of the skew products.

The dynamical large deviations principle is a dynamical counterpart of the large deviations principle of the probability theory. The latter states that under some assumptions on the distribution of i.i.d. random variables $\xi_j$, for any $\alpha > 0$ the probability of a large deviation $P(|(\xi_1 + \cdots + \xi_n)/n - E\xi| > \alpha)$ decreases exponentially in $n$ as it goes to infinity.

The dynamical large deviations principle is well-studied, and was proved for all hyperbolic attractors, that is, maximal attractors that are hyperbolic sets, by L.-S. Young [10], see Thm. 2. (In particular, this theorem also covers the case of Anosov diffeomorphisms.) There are also large deviations results for non-uniformly hyperbolic systems, see [6] and [7], but in these works the property (4) is proven for the invariant measure $\mu$ instead of the Lebesgue measure. Thus, in the latter case, one can apply Theorem 1 only if the measure $\mu$ is absolutely continuous with density bounded away from zero.

**Theorem 2** (L.-S. Young, [10], Thm. 2(2)). If $f : M \to M$ is a $C^2$-diffeomorphism, $U$ is a trapping domain, and the maximal attractor $A := \cap_n f^n(U)$ is hyperbolic, then on $A$ is supported an SRB measure $\mu$ such that the dynamical large deviations principle holds for $(f, \mu, U)$.

Thus, “plugging” this theorem into our main result, we immediately obtain

**Corollary 1.** The special ergodic theorem holds for all hyperbolic attractors of $C^2$-diffeomorphisms, in particular, for all $C^2$ Anosov diffeomorphisms.

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1For a fixed function $\varphi$, the function $h = h(\alpha)$ is called the "rate function" or "Cramér function"; see, for example, [7, 9].
Proof of the main theorem

Let $f: M \to M$ be a $C^1$-local diffeomorphism on a $d$-dimensional compact manifold $M$ with a trapping region $U$ and SRB measure $\mu$ supported on it, for which the dynamical large deviations principle holds. Choose and fix $\varphi \in C(M)$, $\alpha > 0$, and denote $L := \max(\text{Lip}(f), 2)$. Note, that as $f$ is a Lipschitz map, and $X_{\alpha, \varphi}$ is $f$-invariant, it suffices to estimate $\dim_H (X_{\alpha, \varphi} \cap f(U))$ instead of $\dim_H (X_{\alpha, \varphi} \cap U)$.

We are going to prove that

$$\dim_H (X_{\alpha, \varphi} \cap f(U)) \leq d_0 := d - \frac{h(\alpha/2, \varphi)}{\ln L} < d.$$ 

That is, for each $d' > d_0$ we want to find a cover of $X_{\alpha, \varphi} \cap f(U)$ with arbitrarily small $d'$-dimensional volume.

To construct such a cover, we first choose (and fix) $\delta > 0$ such that

$$\forall x, y \in X, \text{dist}(x, y) < \delta \quad |\varphi(x) - \varphi(y)| < \alpha/2.$$ 

Then we have the following lemma:

**Lemma 1.** If $x \in X_{\alpha, \varphi, n}$, then $B(x, r) \subset X_{\alpha/2, \varphi, n}$, where $r = r(n) := L^{-n} \delta$ and $B(x, r)$ is a ball centered at $x$ with radius $r$.

**Proof.** Let $y \in B(x, r)$. Then $\text{dist}(x, y) < L^{-n} \delta$ and consequently

$$\text{dist}(f^k(x), f^k(y)) < \delta \quad \text{for } k = 0, 1, \ldots n.$$ 

Therefore, by the choice of $\delta$, we have

$$|\varphi(f^k(x)) - \varphi(f^k(y))| < \alpha/2 \quad \text{for } k = 0, 1, \ldots n,$$ 

what implies (by averaging over $k$) that $|\varphi_n(x) - \varphi_n(y)| < \alpha/2$. Therefore

$$|\varphi_n(y) - \varphi| \geq |\varphi - \varphi_n(x)| - |\varphi_n(x) - \varphi_n(y)| \geq \alpha - \alpha/2 = \alpha/2.$$ 

\[\square\]

Now, for any $n$, choose a maximal $\frac{r(n)}{2}$-separated set $S_n$ in $f(U)$. The union of $\frac{r(n)}{2}$-neighborhoods of elements of $S_n$ is a cover of $f(U)$, and in particular, of $X_{\alpha, \varphi, n} \cap f(U)$. Remove from this cover the neighborhoods that do not intersect $X_{\alpha, \varphi, n} \cap f(U)$: let

$$S'_n := \{ x \in S_n : B_{r(n)/2}(x) \cap (X_{\alpha, \varphi, n} \cap f(U)) \neq \emptyset \}$$

and

$$C_n := \{ B_{r(n)/2}(x) \mid x \in S'_n \}.$$ 

By construction, $C_n$ is still a cover of $X_{\alpha, \varphi, n} \cap f(U)$. Also, noticing that any point of $X_{\alpha, \varphi} \cap f(U)$ belongs to an infinite number of $X_{\alpha, \varphi, n} \cap f(U)$, we see that for any $N \in \mathbb{N}$ the union of $C_n$ for $n \geq N$ is a cover of $X_{\alpha, \varphi}$. 

4
Let us estimate the $d'$-dimensional volumes of these covers. First, estimate the cardinality of $C_n$: note that radius $r(n)/4$ balls centered at the elements of $S_n$ are pairwise disjoint. Hence, the volume of the union

$$V_n := \bigcup_{x \in S'_n} B_{r(n)/4}(x)$$

is at least $\text{card}(C_n) \cdot c \cdot r(n)^d$, where the constant $c > 0$ depends only on the geometry of the manifold $M$.

On the other hand, for any $n$ such that the $r(n)/4$-neighborhood of $f(U)$ is included in $U$, due to Lemma 1 one has $V_n \subset X_{\alpha/2, \varphi, n} \cap U$. Hence, due to the assumed large deviations principle, $\text{Leb}(V_n) \leq C e^{-h(\alpha/2, \varphi)n}$. Thus,

$$\text{card}(C_n) \cdot c \cdot r(n)^d \leq C e^{-h(\alpha/2, \varphi)n}$$

and hence

$$\text{card}(C_n) \leq C' L^{nd} e^{-h(\alpha/2, \varphi)n} = C' L^{nd_0}.$$ 

Taking any sufficiently big $N$, we see that the $d'$-dimensional volume of the union of covers $V_n$ for $n \geq N$ can be estimated as

$$\sum_{n=N}^{\infty} \text{card}(C_n) \cdot r(n)^d \leq \text{const} \cdot \sum_{n=N}^{\infty} L^{-n(d'-d_0)}$$

This series converges, as $d' > d_0$ by the choice of $d'$. Thus, choosing $N$ sufficiently large, we obtain covers of $X_{\alpha, \varphi} \cap f(U)$ with arbitrarily small $d'$-dimensional volumes, and hence $\dim_H X_{\alpha, \varphi} \cap f(U) \leq d'$ for any $d' > d_0$. This implies that

$$\dim_H X_{\alpha, \varphi} \cap f(U) \leq d_0,$$

what concludes the proof of the theorem.

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Special ergodic theorems
and dynamical large deviations

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Abstract

Let $f : M \to M$ be a self-map of a compact manifold $M$, admitting an global
SRB measure $\mu$. For a continuous test function $\varphi : M \to \mathbb{R}$ and a constant $\alpha > 0$,
consider the set $K_{\varphi,\alpha}$ of the initial points for which the Birkhoff time averages of
the function $\varphi$ differ from its $\mu$–space average by at least $\alpha$. As the measure $\mu$ is
an SRB one, the intersection of $K_{\varphi,\alpha}$ with the basin of attraction of $\mu$ should have
zero Lebesgue measure.

The special ergodic theorem, whenever it holds, claims that, moreover, this
intersection has the Hausdorff dimension less than the dimension of $M$. We prove
that for Lipschitz maps, the special ergodic theorem follows from the dynamical
large deviations principle.

Applying theorems of L. S. Young and of V. Araújo and M. J. Pacífico, we
conclude that the special ergodic theorem holds for transitive hyperbolic attractors
of $C^2$-diffeomorphisms, as well as for some other known classes of maps (including
the one of partially hyperbolic non-uniformly expanding maps).

1 Introduction

1.1 Basic concepts

Let $f : M \to M$ be a self-map of a compact Riemannian manifold $M$ to itself. The
famous Birkhoff ergodic theorem states that if $\mu$ is an ergodic invariant measure for $f$,
then for any continuous function $\varphi \in C(M)$ and for $\mu$-almost every $x \in M$ the time
averages $\varphi_n$ of the function $\varphi$ at the point $x$ converge to the space average $\bar{\varphi}$:

$$\lim_{n \to \infty} \varphi_n(x) = \bar{\varphi}, \quad \text{where} \quad \varphi_n(x) := \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ f^k(x), \quad \bar{\varphi} := \int_M \varphi \, d\mu. \quad (1)$$

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The conclusion of this theorem has one large drawback. It is usually much more natural (in particular, from the “physical” point of view) to take an initial point \( x \), which is generic in the sense of the Lebesgue measure. But if the measure \( \mu \) is supported on some “thin” set (for instance, on a point), one cannot deduce anything for initial points outside this set.

This leads to another famous notion, the one of Sinai-Ruelle-Bowen measure (or SRB measure for short):

**Definition 1.** An invariant measure \( \mu \) is called an (observable) SRB measure for \( f : M \to M \) if there is a set \( U \supset M \) of positive Lebesgue measure such that for every \( x \in U \) and for every continuous function \( \varphi \) on \( M \) one has

\[
\lim_{n \to \infty} \varphi_n(x) \to \bar{\varphi}.
\]  

(2)

An SRB measure is called *global* if one can take \( U = M \).

Not all the systems possess SRB measures. The first example of a system with no SRB measures, the heteroclinic attractor (sf. [12]), was given by Bowen himself. However, under certain assumptions, e.g. hyperbolicity of the system, one can guarantee its existence; we refer the reader to [15] and to [4, Sec. 14] for the survey of this subject.

But even the existence of SRB measure sometimes turns out to be insufficient. In some situations (cf. [5], [6], [7]), it is necessary not only to conclude that the set of the “bad” initial conditions has zero Lebesgue measure, but also to state the same for its images under H"older (but not necessarily absolutely continuous!) conjugacy between two dynamical systems.

One of the ways of handling this difficulty bases on the analysis of Hausdorff dimension of this set. It leads to the concept of the *special ergodic theorem*, proposed by Yu. Ilyashenko, to which this paper is devoted. We will now introduce some auxiliary notations, and present this concept in Def. 2.

Taking a continuous test-function \( \varphi \in C(M) \) and any \( \alpha \geq 0 \), we define the set of \((\varphi, \alpha)\)-nontypical points as

\[
K_{\varphi, \alpha} := \{ x \in X : \lim_{n \to \infty} |\varphi_n(x) - \bar{\varphi}| > \alpha \}.
\]

In other words, \( K_{\varphi, \alpha} \) is the set of points for which the statement “time averages converge to the space one” is violated “in an essential way”.

By definition, if \( \mu \) is a global SRB measure, then \( \text{Leb}(K_{\varphi, 0}) = 0 \). Thereinafter we assume that \( \mu \) actually is a global SRB measure. Otherwise we can restrict all our considerations to its (invariant) basin of attraction \( U \subset M \), that is, the set of all points for which (2) holds.

**Definition 2.** Say that for \((f, \mu)\) the special ergodic theorem holds, if for any continuous function \( \varphi \in C(M) \) and any \( \alpha > 0 \) the Hausdorff dimension of the set \( K_{\varphi, \alpha} \) is strictly less than the dimension of the phase space:

\[
\forall \varphi \in C(M), \alpha > 0 \quad \dim_H K_{\varphi, \alpha} < \dim M.
\]  

(3)
Remark 1. Formally speaking, to state the property (3) one doesn’t need to claim the existence of SRB measure. Though, this property itself immediately implies that $\mu$ actually is an SRB measure. Indeed, the set $K_{\varphi,0}$ of points for which the equality (2) between time and space averages does not hold is exhausted by a countable number of sets $K_{\varphi,1/n}$ of zero Lebesgue measure.

Having introduced this notion, Yu. Ilyashenko [5] proved that the special ergodic theorem holds for the circle doubling map; later, P. Saltykov [11] proved it also for the linear Anosov map on the 2-torus. Analogous questions were also considered in the work of Gurevich and Tempelman [3], where the Hausdorff dimensions of level sets of Birkhoff averages for finite spin lattice systems were evaluated.

The aim of this note is to prove the special ergodic theorem for a larger class of maps. Namely, we notice and prove, that the special ergodic theorem can be obtained as a corollary of another property of the system, the dynamical large deviations principle. This implies, that the special ergodic theorem holds, for instance, for all the hyperbolic transitive $C^2$–diffeomorphisms, for partially hyperbolic non-uniformly expanding $C^1$-diffeomorphisms, and for some other examples.

1.2 Large deviations principles

The dynamical large deviations principle is a dynamical counterpart of the large deviations principle in the probability theory. The latter states that under some assumptions on the distribution of i.i.d. random variables $\xi_j$, for any $\alpha > 0$ the probability of a large deviation $P(|(\xi_1 + \cdots + \xi_n)/n - E\xi| > \alpha)$ decreases exponentially in $n$ as $n$ goes to infinity, see, for example, [13].

Definition 3. Suppose that $\mu$ is an invariant measure for a self-map $f : M \to M$, and $m$ be an arbitrary measure on $M$. Say, that for $(f, \mu)$ the dynamical large deviations principle for reference measure $m$ holds, if for any $\varphi \in C(M), \alpha > 0$ there exist constants $C = C(\alpha, \varphi), h = h(\alpha, \varphi) > 0$ such that

$$\forall n \in \mathbb{N} \quad m(K_{\varphi,\alpha,n}) \leq C(\alpha, \varphi)e^{-nh(\alpha, \varphi)},$$

where

$$K_{\varphi,\alpha,n} := \{x \in M : |\varphi_n(x) - \bar{\varphi}| \geq \alpha\}.$$  

Usually, for $m$ in (4) stands either Lebesgue measure or the invariant measure $\mu$ itself. In this paper we will use the case $m = \text{Leb}$ (otherwise our arguments wouldn’t work). Under this assumption, $\mu$ in Definition 3 is a global SRB measure due to the Borel–Cantelli Lemma type arguments.

The dynamical large deviations principle with respect to the invariant measure was widely studied (see, for example, [8], [11]). Unfortunately, we cannot use directly any of these numerous results, unless if the invariant measure is absolutely continuous with respect to the Lebesgue one.

1for a fixed function $\varphi$, the function $h = h(\alpha)$ is called the “rate function” or “Cramér function”
1.3 Theorem and applications

**Theorem 1** (Main result). Let $f : M \to M$ be a Lipschitz self-map of compact Riemannian manifold $M$ and $\mu$ be a global SRB measure. If the dynamical large deviations principle with Lebesgue reference measure holds for $(f, \mu)$, then for this map the special ergodic theorem also holds.

As will be shown in the next section, the proof of Theorem 1 is indeed rather simple. However, the conclusion itself seems very interesting to us — especially, due to the possibilities to apply it to the study of the skew products.

There are numerous applications of the theorem.

**Definition 4.** A $C^1$-map $f : M \to M$ is **non-uniformly expanding**, if

$$\limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n} \log ||Df(f^j(x))^{-1}|| < 0$$

for Lebesgue almost all $x$. A $C^1$-map $f : M \to M$ is **partially hyperbolic non-uniformly expanding**, if the tangent bundle of $M$ decomposes into direct sum of two continuous sub-bundles such that one of them is uniformly contracting, and the other is non-uniformly expanding.

**Corollary 1.** The special ergodic theorem holds:

(i) for transitive hyperbolic attractors of $C^2$-diffeomorphisms; in particular, for transitive $C^2$ Anosov diffeomorphisms;

(ii) for the stochastic maps from real quadratic family $f(x) = x^2 + c$;

(iii) for typical maps from general $C^2$ unimodal families;

(iv) for partially hyperbolic non-uniformly expanding $C^1$-diffeomorphisms;

(v) for Lorenz-like maps.

All the statements follow from the large deviations principles for these maps. The principle for the first item was proven in [14], see Thm. 2 (2) and (ii) in the discussion after it. For other items, see [2], section 2.

Note that the existence of a global SRB measure for a smooth map does not imply the special ergodic theorem. In [10] there is presented such an example of $C^\infty$-smooth diffeomorphism of a compact manifold that admits a global SRB measure but for which the special ergodic theorem doesn’t hold.

1.4 Flows

All the above concepts and statements can be defined for flows as well as for maps. Namely, let $\{f_t\}$ be a flow on a compact Riemannian manifold $M$ with an invariant
measure $\mu$. For any continuous function $\varphi \in C(M)$ one defines its $T$-time average $\varphi_T$ over the time $T > 0$ as

$$\varphi_T(x) := \frac{1}{T} \int_{t=0}^{T} \varphi \circ f_t(x) dt. \quad (5)$$

In the same way as for the maps, given $\alpha \geq 0$, we define the set of $(\varphi, \alpha)$-nontypical points:

$$K_{\varphi,\alpha} := \left\{ x \in X : \lim_{T \to \infty} |\varphi_T(x) - \bar{\varphi}| > \alpha \right\},$$

where, as above, $\bar{\varphi} = \int_M \varphi \, d\mu$ denotes a space average of the function $\varphi$. We will also use the notation

$$K_{\varphi,\alpha,T} := \left\{ x \in M : |\varphi_T(x) - \bar{\varphi}| \geq \alpha \right\}.$$

**Definition 5.** Say that for the flow $F$ with an invariant measure $\mu$ the special ergodic theorem holds, if for any continuous function $\varphi \in C(M)$ and any $\alpha > 0$ the Hausdorff dimension of the set $K_{\varphi,\alpha}$ is strictly less than the dimension of the phase space:

$$\forall \varphi \in C(M), \alpha > 0 \quad \dim_H K_{\varphi,\alpha} < \dim M. \quad (6)$$

We also say that the dynamical large deviations principle with respect to the reference measure $m$ holds for $F$, if there exist constants $C = C(\alpha, \varphi)$, $h = h(\alpha, \varphi) > 0$ such that

$$\forall T > 0 \quad m(K_{\varphi,\alpha,T}) \leq C(\alpha, \varphi) e^{-nh(\alpha, \varphi)} \quad (7)$$

In the same way as for the maps, we have the following

**Theorem 2.** Let $\mu$ be an invariant measure for a flow (a continuous action of $\mathbb{R}$) $\{f_t\}$ on a compact Riemannian manifold $M$. Suppose that $f_1 : M \to M$ is a Lipschitz self-map of $M$. If the dynamical large deviations principle with the Lebesgue reference measure holds for $\{f_1, \mu\}$, then the special ergodic theorem also holds for this flow.

**Corollary 2.** The special ergodic theorem holds for Anosov flows, as well as for the Lorenz flow and, moreover, for geometric Lorenz flows.

The statements follow from the large deviations principles for these flows. For Anosov flows, this principle was proven in [14] (see Thm. 2 (2) and (ii) in the discussion after it). For precise definitions of (geometric) Lorenz flows and for large deviations principles for these flows see [1].

## 2 Proofs

### 2.1 Proof of the main theorem

Let $f : M \to M$ be Lipschitz map on a $d$-dimensional compact manifold $M$ with a global SRB measure $\mu$ supported on it, for which the dynamical large deviations principle holds. Choose and fix $\varphi \in C(M)$, $\alpha > 0$, and denote $L := \max(\text{Lip}(f), 2)$. 

We are going to prove that
\[
\dim_H K_{\varphi,\alpha} \leq d_0 := d - \frac{h(\alpha/2, \varphi)}{\ln L} < d.
\]
That is, for each \(d' > d_0\) we want to find a cover of \(K_{\varphi,\alpha}\) with arbitrarily small \(d'\)-dimensional volume. Thus we deduce that \(d'\)-dimensional volume of \(K_{\varphi,\alpha}\) is zero, and hence \(\dim_H K_{\varphi,\alpha} \leq d'\) for all \(d' > d_0\).

To construct such a cover, we first choose (and fix) \(\delta > 0\) such that
\[
\text{dist}(x, y) < \delta \implies |\varphi(x) - \varphi(y)| < \alpha/2.
\]

Then we have the following lemma:

**Lemma 1.** If \(x \in K_{\varphi,\alpha,n}\), then \(B(x, r) \subset K_{\varphi,\alpha/2,n}\), where
\[
r = r(n) := L^{-n} \delta
\]
and \(B(x, r)\) is a ball centered at \(x\) with radius \(r\).

**Proof.** Let \(y \in B(x, r)\). Then \(\text{dist}(x, y) < L^{-n} \delta\) and consequently
\[
\text{dist}(f^k(x), f^k(y)) < \delta \quad \text{for} \quad k = 0, 1, \ldots, n.
\]
Therefore, by the choice of \(\delta\), we have
\[
|\varphi(f^k(x)) - \varphi(f^k(y))| < \alpha/2 \quad \text{for} \quad k = 0, 1, \ldots, n,
\]
which implies (by averaging over \(k\)) that \(|\varphi_n(x) - \varphi_n(y)| < \alpha/2\). Hence
\[
|\varphi_n(y) - \varphi| \geq |\varphi - \varphi_n(x)| - |\varphi_n(x) - \varphi_n(y)| \geq \alpha - \alpha/2 = \alpha/2.
\]

Now, for any \(n\), let \(r(n)\) be the same as in (8). Choose a maximal \(\frac{r(n)}{2}\)-separated set \(S_n\) in \(M\). The union of \(\frac{r(n)}{2}\)-neighborhoods of elements of \(S_n\) is a cover of \(M\), and in particular, of \(K_{\varphi,\alpha,n}\). Remove from this cover the neighborhoods that do not intersect \(K_{\varphi,\alpha,n}\): let
\[
S'_n := \{x \in S_n : B_{r(n)/2}(x) \cap K_{\varphi,\alpha,n} \neq \emptyset\}
\]
and
\[
C_n := \left\{B(x, \frac{r(n)}{2}) \mid x \in S'_n\right\}.
\]

By construction, \(C_n\) is still a cover of \(K_{\varphi,\alpha,n}\). Also, noticing that any point of \(K_{\varphi,\alpha}\) belongs to an infinite number of \(K_{\varphi,\alpha,n}\), we see that for any \(N \in \mathbb{N}\) the union of \(C_n\) for \(n \geq N\) is a cover of \(K_{\varphi,\alpha}\).

Let us estimate the \(d'\)-dimensional volumes of these covers. First, estimate the cardinality of \(C_n\): note that radius \(r(n)/4\) balls centered at the elements of \(S_n\) are pairwise disjoint. Hence, the volume of the union
\[
V_n := \bigcup_{x \in S'_n} B_{r(n)/4}(x)
\]
is at least \( \text{card}(C_n) \cdot c \cdot r(n)^d \), where the constant \( c > 0 \) depends only on the geometry of the manifold \( M \).

On the other hand, by Lemma 1, \( V_n \subset K_{\varphi,\alpha/2,n} \). Hence, due to the assumed large deviations principle, \( \text{Leb}(V_n) \leq Ce^{-h(\alpha/2,\varphi)n} \). Thus,

\[
\text{card}(C_n) \cdot c \cdot r(n)^d \leq Ce^{-h(\alpha/2,\varphi)n}
\]

and hence

\[
\text{card}(C_n) \leq C'L^{nd}e^{-h(\alpha/2,\varphi)n} = C'L^{nd_0}.
\]

Therefore we see that the \( d' \)-dimensional volume of the union of covers \( V_n \) for \( n \geq N \) can be estimated as

\[
\sum_{n=N}^{\infty} \text{card}(C_n) \cdot r(n)^{d'} \leq \text{const} \cdot \sum_{n=N}^{\infty} L^{-n(d'-d_0)}
\]

This series converges, as \( d' > d_0 \) by the choice of \( d' \). Thus, choosing \( N \) sufficiently large, we obtain covers of \( K_{\varphi,\alpha} \) with arbitrarily small \( d' \)-dimensional volumes, and hence \( \dim_H K_{\varphi,\alpha} \leq d' \) for any \( d' > d_0 \). This implies that

\[
\dim_H K_{\varphi,\alpha} \leq d_0,
\]

which concludes the proof of Theorem 1.

### 2.2 Proof of Theorem 2

Suppose \( x \in K_{\varphi,\alpha} \) for the flow \( F \). It means that there exists a sequence \( t_n \) growing to \( \infty \) such that

\[
|\varphi_{t_n}(x) - \bar{\varphi}| > \alpha.
\]

Note that

\[
|\varphi_{t_n}(x) - \varphi_{[t_n]}(x)| < \frac{\max_{y \in M} |\varphi(y)|}{[t_n]},
\]

where \([\cdot]\) stands for an integer part of a value. Then for all sufficiently large \( n \)

\[
|\varphi_{[t_n]}(x) - \bar{\varphi}| > \alpha/2.
\]

Hence \( x \in K_{\varphi,\alpha/2} \) for the time 1 map \( f_1 \) of a flow \( F \). This set has a Hausdorff dimension less than the dimension of \( M \) due to Theorem 1.

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