REITERATION THEOREM FOR $\mathcal{R}$ AND $\mathcal{L}$-SPACES WITH THE SAME PARAMETER.

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Abstract. Let $E, F, E_0, E_1$ be rearrangement invariant spaces; let $a, b, b_0, b_1$ be slowly varying functions and $0 < \theta_0, \theta_1 < 1$. We characterize the interpolation spaces

\[
\left( X_{\theta_0, a_0, E_0, a, F}, X^c_{\theta_1, b_1, E_1, a, F} \right)_{\eta, b, E}, \quad 0 \leq \eta \leq 1,
\]

when the parameters $\theta_0$ and $\theta_1$ are equal (under appropriate conditions on $b_i(t)$, $i = 0, 1$). This completes the study started in [9, 19], which only considered the case $\theta_0 < \theta_1$. As an application we recover and generalize interpolation identities for grand and small Lebesgue spaces given by [23].

1. Introduction

This paper continues the project started in [9, 10, 17, 18, 19] where we have proved reiteration theorems for couples formed by arbitrary combinations of the spaces

\[
X_{\theta, b, E}, \quad X^R_{\theta, b, E, a, F}, \quad X^c_{\theta, b, E, a, F}
\]

when $0 \leq \theta \leq 1$, $a$ and $b$ are slowly varying functions and $E, F$ are rearrangement invariant (r.i.) Banach function spaces or Lebesgue spaces $L_q$, $0 < q \leq \infty$; see §4 for precise definitions. In particular, we have identified the spaces

\[
\left( X^R_{\theta_0, a_0, a_0, F_0}, X^c_{\theta_1, b_1, a_1, F_1} \right)_{\eta, b, E}, \quad 0 \leq \eta \leq 1,
\]

under the condition $\theta_0 \neq \theta_1$.

Our goal in this paper is to identify (1.1) in the special case $\theta_0 = \theta_1$, provided that $a_0 = a_1$, $F_0 = F_1$ and for suitable $b_0$ and $b_1$; that is the space

\[
\left( X^R_{\theta_0, b_0, a_0, F_0}, X^c_{\theta_1, b_1, E_1, a_1, F_1} \right)_{\eta, b, E}, \quad 0 \leq \eta \leq 1.
\]

The resulting spaces in (1.2) are very different from those obtained in (1.1). While in the case $\theta_0 \neq \theta_1$ the spaces belong to the classical scale $X_{\eta, b, E}$ for all $0 < \eta < 1$, now (1.2) will always be an $\mathcal{R}$ or $\mathcal{L}$-space, depending on the values of $\eta$; see Theorem 5.3 for the precise statements. Moreover, in the cases $\eta = 0$ or $\eta = 1$ we will need intersections of $\mathcal{R}$ or $\mathcal{L}$-space with the extreme scales of ($\mathcal{R}, \mathcal{L}$)-spaces or ($\mathcal{L}, \mathcal{R}$)-spaces, respectively. These extremal constructions were introduced in [17].

As in [9, 17, 18, 19], the present work finds its motivation in applications to the interpolation of grand and small Lebesgue spaces $L^{p, \alpha}$, $L^{p, \alpha}$, $1 < p < \infty$, $\alpha > 0$ (see
Definition [6,4] below). These spaces are limiting interpolation spaces for the couples $(L_1, L_p)$ and $(L_p, L_\infty)$, respectively, and they can also be identified as $\mathcal{R}$ and $\mathcal{L}$-spaces in the following way

$$L^{p,\alpha} = (L_1, L_p)^\mathcal{R}_{\alpha} = (L_1, L_\infty)^\mathcal{R}_{1-\frac{1}{p}}$$

and

$$L^{p,\alpha} = (L_p, L_\infty)^\mathcal{L}_{\alpha}$$

where $\ell(t) = 1 + |\log(t)|$, $t \in (0, 1)$. Then, from our results concerning (1.2) we will obtain identities for the interpolation spaces

$$(L^{p,\alpha}, L^{p,\beta})_{\eta,E}, \quad 0 \leq \eta \leq 1.$$  

These recover (and extend) recent results of Fiorenza et al. [23, Theorem 6.2], which they obtained in the case $\alpha = \beta = 1$, $\eta \in (0, 1)$, $b \equiv 1$ and $E = L_q$. See Corollary 6.5 for the precise statements.

The paper has a very technical profile. Some of these details are worth to mention. For example, an important feature of the rearrangement invariant spaces, extension indices of functions and slowly varying functions, such as $b(t) = 1 + |\log(t)|$, $t > 0$, which have the same behavior near 0 and near $\infty$, but they are true working for example with broken logarithms

$$\ell^{(\alpha,\beta)}(t) = \begin{cases} 
\ell^{\alpha}(t), & 0 < t \leq 1 \\
\ell^{\beta}(t), & t > 1 
\end{cases} \quad (\alpha, \beta) \in \mathbb{R}^2,$$

provided $\alpha$ and $\beta$ have different signs (see [12, Lemma 4.2]). In this sense, we follow the ideas developed in [15], that is, we work with slowly varying functions satisfying that $b(t) \sim b(t^2)$ and we define the associated functions $B_0$ and $B_\infty$ by

$$B_0(u) = b(e^{-\frac{1}{u}}) \quad \text{and} \quad B_\infty(u) = b(e^{\frac{u}{u-1}}), \quad \text{for} \quad 0 < u \leq 1.$$  

Under appropriate conditions in the extension indices of $B_0$ and $B_\infty$ the limiting Hardy type inequalities in the whole line $(0, \infty)$ are true.

Another ingredient that we will use is a new identity between the $\mathcal{R}$-spaces and the $(\mathcal{R}, \mathcal{L})$-spaces (or $\mathcal{L}$-spaces with $(\mathcal{L}, \mathcal{R})$-spaces, respectively); see Corollary 4.5. These are based on new equivalences for norm functionals involving suitable slowly varying functions; see Lemma 3.6. It must be said that the proof of Lemma 3.6 is very technical and needs the discretization of the norm of the spaces involved as well as weighted Hardy type inequalities for r.i. sequence spaces (see Lemma 3.4).

The organization of the paper is the following: In Section 2 we review basic concepts about rearrangement invariant spaces, extension indices of functions and slowly varying functions. In Section 3 we restrict to slowly varying functions which satisfy that $b(t) \sim b(t^2)$, collect some essential lemmas (equivalence lemma, limiting estimates and limiting Hardy type inequalities) and prove the technical lemmas that we shall need for the
identification of the spaces in [1,2]. The description of the interpolation methods we shall work with, namely \(X_{\theta,b,E}\), the \(R, L, (R, L)\) and \((L, R)\) constructions can be found in Section 4. Generalized Holmstedt type formula, the change of variables and the main reiteration theorem appear in Section 5. Finally, Section 6 is devoted to applications to the interpolation of grand and small Lebesgue spaces.

2. Preliminaries

We refer to the monographs [2, 3, 5, 28, 33] for the main definitions and properties concerning r.i. spaces and interpolation theory. Recall that a Banach function space \(E\) (on the semiaxis \(\Omega = (0, \infty)\) with the Lebesgue measure) is called rearrangement invariant (r.i.) if, for any two measurable functions \(f, g\),

\[
g \in E \quad \text{and} \quad f^* \leq g^* \implies f \in E \quad \text{and} \quad \|f\|_E \leq \|g\|_E,
\]

where \(f^*\) and \(g^*\) stand for the non-increasing rearrangements of \(f\) and \(g\). We shall denote the r.i. space by \(e\), when \(\Omega = \mathbb{Z}\) with the counting measure. Moreover, following [2], we always assume that every Banach function space \(E\) enjoys the Fatou property. Under this assumption, every r.i. space \(E\) is an exact interpolation space with respect to the Banach couple \((L_1, L_\infty)\), that is

\[
E = (L_1, L_\infty)^K_D
\]

for some suitable choice of parameter \(D\).

Let us recall more explicitly what this notation means. For any compatible (quasi-) Banach couple \(\mathcal{X} = (X_0, X_1)\), the Peetre \(K\)-functional \(K(t, f; X_0, X_1) \equiv K(t, f)\) is defined for \(f \in X_0 + X_1\) and \(t > 0\) by

\[
K(t, f) = \inf \left\{ \|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1, \ f_i \in X_i, \ i = 0, 1 \right\}.
\]

A Banach lattice \(D\) of measurable functions on \((0, \infty)\) is called a parameter if it contains the function \(\min(1, t)\). For each such \(D\), the real interpolation space \(X^K_D\) consists of all \(f \in X_0 + X_1\), for which

\[
\|f\|_{X^K_D} = \|K(t, f)\|_D < \infty.
\]

We shall work with three different measures on \((0, \infty)\): the usual Lebesgue measure \(dt\), the homogeneous measure \(dt/t\) and the measure \(dt/t\ell(t)\) where \(\ell(t) := (1 + |\log t|), t > 0\). As in [15], we use letters with a tilde for spaces with the measure \(dt/t\) and with a hat for the measure \(dt/t\ell(t)\). For example, the spaces \(\tilde{L}_1\) and \(\hat{L}_1\) are defined by the norms

\[
\|f\|_{\tilde{L}_1} = \int_0^\infty |f(t)| \frac{dt}{t} \quad \text{and} \quad \|f\|_{\hat{L}_1} = \int_0^\infty |f(t)| \frac{dt}{t\ell(t)},
\]

respectively, while \(\tilde{L}_\infty\) and \(\hat{L}_\infty\) coincide with \(L_\infty\). Similarly, each r.i. space \(E\) on \(((0, \infty), dt)\) generates two more versions of \(E\) on the other measure spaces. Namely, if \(D\) is a parameter such that \(E = (L_1, L_\infty)^K_D\), then we set

\[
\tilde{E} = (\tilde{L}_1, L_\infty)^K_D \quad \text{and} \quad \hat{E} = (\hat{L}_1, L_\infty)^K_D.
\]

It should be pointed out that the definition of the spaces \(\tilde{E}\) and \(\hat{E}\) do not depend on any particular choice of parameter \(D\). In fact, there are simple formulae which directly
connect the norms of the spaces $E, \tilde{E}$ and $\hat{E}$, without any reference to the parameter $D$. For all measurable function $f : (0, \infty) \rightarrow (0, \infty)$ we have

$$\|f\|_{E(0,1)} = \|f(e^{-u})\|_E \quad \text{and} \quad \|f\|_{E(1,\infty)} = \|f(e^u)\|_E,$$

while

$$\|f\|_{\tilde{E}(0,1)} = \|f(e^{1-e^{-u}})\|_E \quad \text{and} \quad \|f\|_{\tilde{E}(1,\infty)} = \|f(e^{e^{-u}-1})\|_E.$$

Throughout the paper, given two (quasi-) Banach spaces $X$ and $Y$, we will write $X = Y$ if $X \hookrightarrow Y$ and $Y \hookrightarrow X$, where the latter means that $Y \subset X$ and the natural embedding is continuous. Similarly, $f \sim g$ means that $f \lesssim g$ and $g \lesssim f$, where $f \lesssim g$ is the abbreviation of $f(t) \leq Cg(t)$, $t > 0$, for some positive constant $C$ independent of $f$ and $g$. Moreover, we do not distinguish the notions “increasing” and “non-decreasing” as well as “decreasing” and “non-increasing” and we say a function $f$ is almost increasing (decreasing) if it is equivalent to an increasing (decreasing) function.

2.1. Extension indices. Given a non-negative everywhere finite function $\varphi$ on $(0, a)$, $0 < a \leq \infty$, we consider its dilation function

$$m_\varphi(t) = \sup_{0 < s < \min(a, \frac{t}{\varphi})} \frac{\varphi(ts)}{\varphi(s)}, \quad 0 < t < \infty.$$  

If $m_\varphi(t)$ is finite everywhere, then there exist the lower and upper extension indices of $\varphi$ given by

$$\pi_\varphi = \lim_{t \to 0} \frac{\ln m_\varphi(t)}{\ln t}, \quad \rho_\varphi = \lim_{t \to \infty} \frac{\ln m_\varphi(t)}{\ln t}.$$  

In general, $-\infty < \pi_\varphi \leq \rho_\varphi < \infty$, but $0 \leq \pi_\varphi \leq \rho_\varphi < \infty$ if the function $\varphi$ is increasing and $0 \leq \pi_\varphi \leq \rho_\varphi \leq 1$ if $\varphi$ is quasi-concave. Moreover, if $0 < \pi_\varphi \leq \rho_\varphi < \infty$, then $\varphi$ is equivalent to the increasing function $\int_0^t \varphi(s) \frac{ds}{s}$ while if $-\infty < \pi_\varphi \leq \rho_\varphi < 0$ then the function $\varphi$ is equivalent to the decreasing function $\int_t^\infty \varphi(s) \frac{ds}{s}$ (see [28] pg. 57). As an example, $\varphi(t) = t^\alpha \ell(t)^\beta$, $\alpha, \beta \in \mathbb{R}$, has $\pi_\varphi = \rho_\varphi = \alpha$.

The following properties of extension indices will be useful in the sequel and can be easily proved:

(i) Let $\sigma \in \mathbb{R}$ and $\varphi(t) = t^\sigma \psi(t)$, then $\pi_\varphi = \sigma + \pi_\psi$ and $\rho_\varphi = \sigma + \rho_\psi$.
(ii) If $\varphi(t) = \alpha(t)\psi(t)$ then $\pi_\varphi \geq \pi_\alpha + \pi_\psi$ and $\rho_\varphi \leq \rho_\alpha + \rho_\psi$.
(iii) If $\varphi(t) = \psi(t^{-1})$ then $\pi_\varphi = -\rho_\psi$ and $\rho_\varphi = -\pi_\psi$.
(iv) If $\varphi(t) = 1/\psi(t)$ then $\pi_\varphi = -\rho_\psi$ and $\rho_\varphi = -\pi_\psi$.
(v) If $\varphi(t) = \theta(\psi(t))$ and $\theta$ is almost increasing then $\pi_\varphi \geq \pi_\theta \pi_\psi$ and $\rho_\varphi \leq \rho_\theta \rho_\psi$.
(vi) If $\varphi \sim \psi$ then $\pi_\varphi = \pi_\psi$ and $\rho_\varphi = \rho_\psi$.

From (i) it follows that the ratio $\varphi(t)/t^\sigma$ is almost increasing for any $\sigma < \pi_\varphi$ and almost decreasing for any $\sigma > \rho_\varphi$.

Every r.i. space $E$ has its fundamental function $\varphi_E(\lambda) = \|\chi_{(0,\lambda)}\|_E$, which is continuous and quasi-concave. Moreover, the space $E$ always admits an equivalent renorming such that $\varphi_E$ becomes concave and the derivative $\varphi'_E$ exists a.e. and is decreasing. In particular, $0 \leq \pi_{\varphi_E} \leq \rho_{\varphi_E} \leq 1$. 
2.2. Slowly varying functions. In this subsection we recall the definition and basic properties of slowly varying functions. See [4, 24, 29].

**Definition 2.1.** A positive Lebesgue measurable function $b$, $0 \neq b \neq \infty$, is said to be slowly varying on $(0, \infty)$ (notation $b \in SV$) if, for each $\epsilon > 0$, the function $t \sim t^\epsilon b(t)$ is almost increasing on $(0, \infty)$ and $t \sim t^{-\epsilon} b(t)$ is almost decreasing on $(0, \infty)$.

The class $SV(0, 1)$ can be defined similarly, changing the interval $(0, \infty)$ by $(0, 1)$. Examples of $SV$-functions include powers of logarithms, $\ell^\alpha(t) = (1 + |\log t|)^\alpha$, reiterated logarithms $(\ell \circ \ldots \circ \ell)^\alpha(t)$, $\alpha \in \mathbb{R}$, $t > 0$, “broken” logarithmic functions $\ell^{\alpha, \beta}(t)$ (see (1.3)) and also the family of functions as $t \sim \exp(|\log t|^{\alpha})$, $\alpha \in (0, 1)$, $t > 0$.

Some basic properties of slowly varying functions are summarized in the following lemmas.

**Lemma 2.2.** Let $b, b_1, b_2 \in SV$ and let $\mu$ be a non-negative measurable function on $(0, \infty)$.

(i) Then $b_1 b_2 \in SV$, $b(1/t) \in SV$ and $b^\epsilon \in SV$ for all $r \in \mathbb{R}$.

(ii) If for some $\delta > 0$, the functions $\mu$ and $t/\mu^\delta(t)$ are almost increasing on $(0, \infty)$, then $b \circ \mu \in SV$.

(iii) If $\epsilon, s > 0$ then there are positive constants $c_\epsilon$ and $C_\epsilon$ such that

$$c_\epsilon \min\{s^\epsilon, s^{-\epsilon}\} b(t) \leq b(st) \leq C_\epsilon \max\{s^\epsilon, s^{-\epsilon}\} b(t) \quad \text{for every } t > 0.$$

In particular, $\pi_b = \rho_b = 0$.

(iv) $b \circ f \sim b \circ g$ if $f$ and $g$ are positive finite equivalent functions on $(0, \infty)$.

(v) If $b_1$ is almost monotone, then $b \circ b_1 \in SV$.

**Lemma 2.3.** Let $E$ be an r.i. space on $(0, \infty)$ and $b \in SV$.

(i) If $\alpha > 0$, then, for all $t > 0$,

$$\|s^\alpha b(s)\|_{E(0,t)} \sim t^\alpha b(t) \quad \text{and} \quad \|s^{-\alpha} b(s)\|_{E(t,\infty)} \sim t^{-\alpha} b(t).$$

(ii) If $\alpha \in \mathbb{R}$, then, for all $t > 0$,

$$\|s^\alpha b(s)\|_{E(t,2t)} \sim t^\alpha b(t).$$

(iii) The following functions belong to $SV$

$$t \sim \|b\|_{E(0,t)} \quad \text{and} \quad t \sim \|b\|_{E(t,\infty)}, \quad t > 0.$$

(iv) For all $t > 0$,

$$b(t) \lesssim \|b\|_{E(0,t)} \quad \text{and} \quad b(t) \lesssim \|b\|_{E(t,\infty)}.$$

We refer to [24, 13, 16] for the proof of Lemma 2.2 and 2.3 respectively.

**Remark 2.4.** The property (iii) of Lemma 2.2 implies that if $b \in SV$ is such that $b(t_0) = 0$ ($b(t_0) = \infty$) for some $t_0 > 0$, then $b \equiv 0$ ($b \equiv \infty$). Thus, by Lemma 2.3 (ii), if $\|b\|_{E(0,1)} < \infty$ then $\|b\|_{E(t,\infty)} < \infty$ for all $t > 0$, and if $\|b\|_{E(1,\infty)} < \infty$ then $\|b\|_{E(t,\infty)} < \infty$ for all $t > 0$. 

In this paper we are concerned with those slowly varying functions \( b \) such that \( b(t^2) \sim b(t) \). All previous examples, except \( b(t) = \exp(|\log t|^\alpha) \), \( \alpha \in (0,1) \), satisfy this condition. For every \( b \in SV \) with this property, we define new functions \( B_0 \) and \( B_\infty \) by

\[
B_0(u) = b(e^{u-1}) \quad \text{and} \quad B_\infty(u) = b(e^{u-1}), \quad \text{for} \quad 0 < u \leq 1.
\]

For example, if \( b(t) = t^{(\alpha,\beta)}(t) \) then \( B_0(u) = 1/u^\alpha \) and \( B_\infty(u) = 1/u^\beta \), \( u \in (0,1] \). The condition \( b(t^2) \sim b(t) \) implies that \( B_0 \) and \( B_\infty \) satisfy the \( \Delta_2 \) condition, that is \( B_0(u) \sim B_0(2u) \) and \( B_\infty(u) \sim B_\infty(2u) \). Hence the extension indices of \( B_0 \) and \( B_\infty \) exist and are both finite.

In what follows we need to consider different functions with the above properties, so we shall denote these functions by \( a, b, \phi, \) etc, and their associated functions by \( A_0, A_\infty, B_0, B_\infty, \Phi_0, \Phi_\infty, \) etc (with capital letters).

Next, we collect some useful lemmas about this class of slowly varying functions from [15]. The first lemma gives the counterparts of Lemma 2.3 i) in the limiting case \( \alpha = 0 \). The third one contains limiting Hardy-type inequalities.

**Lemma 3.1.** [15] Lemma 3.2] Let \( E \) be an r.i. space and let \( b \in SV \) such that \( b(t^2) \sim b(t) \), with associated functions \( B_0 \) and \( B_\infty \) defined in (3.1).

i) If \( \rho_{B_\infty} < \pi_{\varphi_E} \leq \rho_{\varphi_E} < \pi_{B_0} \), then

\[
\|b\|_{E(0,t)} \sim b(t)\varphi_E(\ell(t)), \quad t > 0.
\]

ii) If \( \rho_{B_0} < \pi_{\varphi_E} \leq \rho_{\varphi_E} < \pi_{B_\infty} \), then the equivalence is

\[
\|b\|_{E(t,\infty)} \sim b(t)\varphi_E(\ell(t)), \quad t > 0.
\]

**Lemma 3.2.** [15] Lemma 3.5] Let \( E \) be an r.i. space and let \( b \in SV \) such that \( b(t^2) \sim b(t) \), with associated functions \( B_0 \) and \( B_\infty \) defined in (3.1).

i) If \( 0 < \pi_{B_0} \) and \( \varphi \) is an almost decreasing function, then for all \( t > 0 \)

\[
\|b(s)\varphi(s)\|_{E(0,t)} \lesssim \int_0^t b(s)\varphi(s)\varphi_E(\ell(s)) \frac{ds}{s\ell(s)}.
\]

ii) If \( 0 < \pi_{B_\infty} \) and \( \varphi \) is an almost increasing function, then for all \( t > 0 \)

\[
\|b(s)\varphi(s)\|_{E(t,\infty)} \lesssim \int_t^\infty b(s)\varphi(s)\varphi_E(\ell(s)) \frac{ds}{s\ell(s)}.
\]

**Lemma 3.3.** [15] Lemma 3.6] Let \( E \) be an r.i. space and let \( b \in SV \) such that \( b(t^2) \sim b(t) \), with associated functions \( B_0 \) and \( B_\infty \) defined in (3.1).

i) If \( \rho_{B_0} < 0 < \pi_{B_\infty} \), then

\[
\|b(t)\int_0^t f(s) \, ds\|_{E(0,t)} \lesssim \|\ell b(t)f(t)\ell(t)\|_E
\]

for any non-negative measurable function \( f \) on \((0,\infty)\).
ii) If $\rho_{B_e} < 0 < \pi_{B_0}$, then
\[
\left\| b(t) \int_t^\infty f(s) \, ds \right\|_{\hat{E}} \lesssim \left\| tb(t)f(t)\ell(t) \right\|_{\hat{E}}
\]
for any non-negative measurable function $f$ on $(0, \infty)$.

3.1. Technical lemmas. This subsection contains the most technical lemmas that we shall need in the proof of the main result of this paper.

The first one is a weighted Hardy type inequality, which is well known for Lebesgue sequence spaces $e = \ell_q$, $0 < q \leq \infty$ (see [24, 26]). In our case, we shall need it for any r.i. sequence space $e$, that is, spaces of the form $e = (\ell_1, \ell_\infty)^D$ for some suitable choice of the parameter $D$.

Lemma 3.4. Let $(\sigma_k)_{k \in \mathbb{Z}}$ be a non-negative sequence and let $e$ be an r.i. sequence space.

i) If $\sup_{k \in \mathbb{Z}} \frac{\sigma_{k+1}}{\sigma_k} < 1$, then
\[
\left\| \left( \frac{\sigma_k \sum_{m \leq k} x_m}{k \in \mathbb{Z}} \right) \right\|_e \sim \left\| (\sigma_k x_k)_{k \in \mathbb{Z}} \right\|_e
\]
for any non-negative sequence $(x_k)_{k \in \mathbb{Z}}$.

ii) If $\inf_{k \in \mathbb{Z}} \frac{\sigma_{k+1}}{\sigma_k} > 1$, then
\[
\left\| \left( \frac{\sigma_k \sum_{m \geq k} x_m}{k \in \mathbb{Z}} \right) \right\|_e \sim \left\| (\sigma_k x_k)_{k \in \mathbb{Z}} \right\|_e
\]
for any non-negative sequence $(x_k)_{k \in \mathbb{Z}}$.

Proof. Let $\sigma := \sup_{k \in \mathbb{Z}} \frac{\sigma_{k+1}}{\sigma_k} < 1$. The estimate “$\gtrsim$” in (3.2) is clearly true, so only the estimate “$\lesssim$” has to be proved. We consider the operator $R : e \to e$ defined by
\[
R((y_k)_{k \in \mathbb{Z}}) = \left( \frac{\sigma_k \sum_{m \leq k} y_m}{\sigma_m} \right)_{k \in \mathbb{Z}}.
\]
We claim that $R$ is bounded in $\ell_1$ and in $\ell_\infty$. Actually, let $(y_k)_{k \in \mathbb{Z}} \in \ell_1$, then
\[
\|R((y_k)_{k \in \mathbb{Z}})\|_{\ell_1} = \sum_{k \in \mathbb{Z}} \left| \frac{\sigma_k \sum_{m \leq k} y_m}{\sigma_m} \right| \leq \sum_{k \in \mathbb{Z}} \sigma_k \sum_{m \leq k} \frac{|y_m|}{\sigma_m} = \sum_{m \in \mathbb{Z}} \frac{|y_m|}{\sigma_m} \sum_{k \geq m} \sigma_k \\
\leq \sum_{m \in \mathbb{Z}} |y_m|(1 + \sigma + \sigma^2 + \ldots) = \frac{1}{1 - \sigma} \| (y_m)_{m \in \mathbb{Z}} \|_{\ell_1}.
\]
Take now $(y_k)_{k \in \mathbb{Z}} \in \ell_\infty$, then
\[
\|R((y_k)_{k \in \mathbb{Z}})\|_{\ell_\infty} = \sup_{k \in \mathbb{Z}} \left| \frac{\sigma_k \sum_{m \leq k} y_m}{\sigma_m} \right| \leq \| (y_m)_{m \in \mathbb{Z}} \|_{\ell_\infty} \sup_{k \in \mathbb{Z}} \left\{ \sigma_k \sum_{m \leq k} \frac{1}{\sigma_m} \right\} \\
\leq \| (y_m)_{m \in \mathbb{Z}} \|_{\ell_\infty} (1 + \sigma + \sigma^2 + \ldots) = \frac{1}{1 - \sigma} \| (y_m)_{m \in \mathbb{Z}} \|_{\ell_\infty}.
\]
Since $e$ is an interpolation space for the couple $(\ell_1, \ell_\infty)$ we have that the operator $R$ is bounded in $e$. To complete the proof of i) it suffices to take $y_k = \sigma_k x_k$, $k \in \mathbb{Z}$. The proof of ii) can be done in a similar way. □
In next lemmas we shall consider the following sequence

\[ \lambda_k = \begin{cases} 
  e^{1-e^{-k}} & \text{if } -k \in \mathbb{N}, \\
  e^{k-1} & \text{if } k \in \mathbb{N} \cup \{0\}.
\end{cases} \tag{3.4} \]

Observe that the special slowly varying functions we are working with, \( b \in SV \) such that \( b(t^2) \sim b(t) \), satisfy that

\[ b(\lambda_{k-1}) \sim b(t) \sim b(\lambda_k) \text{ for } t \in [\lambda_{k-1}, \lambda_k], \tag{3.5} \]

with the equivalent constants being independent of \( k \in \mathbb{Z} \).

**Lemma 3.5.** Let \((\lambda_k)_{k \in \mathbb{Z}}\) be the sequence defined by (3.4) and let \( b \in SV \) such that \( b(t^2) \sim b(t) \), with associated functions \( B_0 \) and \( B_\infty \) defined in (3.1).

i) If \( \rho_{B_0} < 0 < \pi_{B_\infty} \), then there exist a measurable function \( \Phi \) equivalent to \( b \) such that \( \sup_{k \in \mathbb{Z}} \frac{\Phi(\lambda_{k+1})}{\Phi(\lambda_k)} < 1 \).

ii) If \( \rho_{B_\infty} < 0 < \pi_{B_0} \), then there exist a measurable function \( \Psi \) equivalent to \( b \) such that \( \inf_{k \in \mathbb{Z}} \frac{\Psi(\lambda_{k+1})}{\Psi(\lambda_k)} > 1 \).

**Proof.** An straightforward computation shows that

\[ b(\lambda_k) = \begin{cases} 
  B_0(e^k) & \text{if } -k \in \mathbb{N}, \\
  B_\infty(e^{-k}) & \text{if } k \in \mathbb{N} \cup \{0\}.
\end{cases} \]

Assume now that \( \rho_{B_0} < 0 < \pi_{B_\infty} \). Then, there exist \( \alpha, \beta \in \mathbb{R} \) such that \( \rho_{B_0} < \alpha < 0 < \beta < \pi_{B_\infty} \). By properties of extension indices (see, e.g., [28, Section II.1.2]), the function \( t^{-\alpha}B_0(t) \) is almost decreasing and \( t^{-\beta}B_\infty(t) \) is almost increasing. This implies the existence of two functions \( \Phi_0 \sim B_0 \) and \( \Phi_\infty \sim B_\infty \) such that \( t^{-\alpha}\Phi_0(t) \) is strictly decreasing while \( t^{-\beta}\Phi_\infty(t) \) is strictly increasing. So, it follows that the function

\[ \phi(\lambda_k) = \begin{cases} 
  \Phi_0(e^k) & \text{if } -k \in \mathbb{N}, \\
  \Phi_\infty(e^{-k}) & \text{if } k \in \mathbb{N} \cup \{0\},
\end{cases} \]

satisfies that \( \sup_{k \in \mathbb{Z}} \frac{\phi(\lambda_{k+1})}{\phi(\lambda_k)} < max\{e^\alpha, e^{-\beta}\} < 1 \). The argument to prove ii) is analogous. \( \square \)

In the next two lemmas we shall need to work with the discretization of \( \hat{E} \) that we denote by \( d\hat{E} \). Let us explain more explicitly the definition of \( d\hat{E} \). Consider the intervals

\[ I_k = \begin{cases} 
  [\lambda_{k-1}, \lambda_k] & \text{if } -k \in \mathbb{N}, \\
  (\lambda_{-1}, \lambda_0) & \text{if } k = 0, \\
  [\lambda_{k-1}, \lambda_k) & \text{if } k \in \mathbb{N}
\end{cases} \]

where \((\lambda_k)_{k \in \mathbb{Z}}\) is the sequence defined by (3.4). Hence, \( \bigcup_{k \in \mathbb{Z}} I_k \) is a decomposition of \((0, \infty)\) into disjoint intervals \( I_k \) such that

\[ \int_{I_k} \frac{dt}{t \ell(t)} = 1, \quad \forall k \in \mathbb{Z}. \]
It is easy to prove that the operator
\[ Tf = \left( \int_{I_k} f(t) \frac{dt}{t^2} \right)_{k \in \mathbb{Z}} \]
is bounded from $\hat{L}_1$ into $\ell_1(\mathbb{Z})$ and from $L_\infty$ into $\ell_\infty(\mathbb{Z})$. By interpolation, $T$ is bounded from $\hat{E} = (\hat{L}_1, L_\infty)^K$ into $d\hat{E} := (\ell_1(\mathbb{Z}), \ell_\infty(\mathbb{Z}))^K_D$ for the suitable choice of parameter $D$. Similarly, the operator
\[ S((x_k)_{k \in \mathbb{Z}}) = \sum_{k \in \mathbb{Z}} x_k \chi_{I_k} \]
is bounded from $\ell_1(\mathbb{Z})$ into $\hat{L}_1$ and from $\ell_\infty(\mathbb{Z})$ into $L_\infty$. Then, using interpolation $S$ is bounded from $d\hat{E} = (\ell_1(\mathbb{Z}), \ell_\infty(\mathbb{Z}))^K$ into $\hat{E} = (\hat{L}_1, L_\infty)^K_D$. Moreover, $T S x = x$ for all $x \in d\hat{E}$ and $S T f = f$ for all $f \in \hat{E}_c$, where $\hat{E}_c$ denotes the collection of all functions on $\hat{E}$ constant on each interval $I_k$. Hence we can identify $\hat{E}_c$ with the space $d\hat{E}$ of all sequences $x = (x_k)_{k \in \mathbb{Z}}$ such that
\[ \|x\|_{d\hat{E}} = \left\| \sum_{k \in \mathbb{Z}} x_k \chi_{I_k} \right\|_{\hat{E}} < \infty. \]
The space $d\hat{E}$ is an r.i. sequence space, that we shall called the discretization of $\hat{E}$.

**Lemma 3.6.** Let $E$, $F$, $G$ be r.i. spaces and let $a$, $b \in SV$ such that $a(t^2) \sim a(t)$, $b(t^2) \sim b(t)$, with associated functions $A_0$, $A_\infty$, $B_0$ and $B_\infty$ defined in (3.1).

i) If $\rho_{A_\infty} < \pi_{\varphi_F} \leq \rho_{\varphi_F} < \pi_{A_0}$ and $\rho_{B_0} < 0 < \pi_{B_\infty}$, then
\[ \left\| b(u) \| f \|_{\hat{G}(0,u)} \right\|_{\hat{E}} \sim \left\| \frac{b(u)}{a(t)} \right\|_{\hat{E}} \left\| a(t) \| f \|_{\hat{G}(t,u)} \right\|_{\hat{E}} \]
for any measurable function $f$ on $(0, \infty)$.

ii) If $\rho_{A_0} < \pi_{\varphi_F} \leq \rho_{\varphi_F} < \pi_{A_\infty}$ and $\rho_{B_\infty} < 0 < \pi_{B_0}$, then
\[ \left\| b(u) \| f \|_{\hat{G}(u,\infty)} \right\|_{\hat{E}} \sim \left\| \frac{b(u)}{a(t)} \right\|_{\hat{E}} \left\| a(t) \| f \|_{\hat{G}(u,t)} \right\|_{\hat{E}} \]
for any measurable function $f$ on $(0, \infty)$.

**Proof.** We observe that the inequality “$\lesssim$” always holds, so in both cases only the inequality “$\gtrsim$” has to be proved. Let us begin with the proof of i). Using (3.1) and the discretization of the norm in $\hat{E}$ we derive
\[ (3.6) \]
\[ \left\| b(u) \| f \|_{\hat{G}(0,u)} \right\|_{\hat{E}} = \left\| \sum_{k \in \mathbb{Z}} b(u_k) \| f \|_{\hat{G}(0,u_k)} \chi_{I_k}(u) \right\|_{\hat{E}} \]
\[ \gtrsim \left\| \sum_{k \in \mathbb{Z}} b(\lambda_k) \| f \|_{\hat{G}(0,\lambda_k)} \chi_{I_k}(u) \right\|_{\hat{E}} \]
\[ \gtrsim \left\| \sum_{k \in \mathbb{Z}} b(\lambda_k) \left( \sum_{j \leq k} \| f \|_{\hat{G}(\lambda_k,\lambda_j)} \right) \chi_{I_k}(u) \right\|_{\hat{E}} \]
\[ = \left\| \left( b(\lambda_k) \sum_{j \leq k} \| f \|_{\hat{G}(\lambda_k,\lambda_j)} \right) \chi_{I_k}(u) \right\|_{\hat{E}}. \]
Now by Lemma 3.3 i) \(\rho_{B_0} < \pi_{B_{\infty}}\), there exists an equivalent function of \(b\), that we denote in the same way, such that \(\sup_{k \in \mathbb{Z}} \frac{b(\lambda_{k+1})}{b(\lambda_k)} < 1\). Then, Lemma 3.1 ii) implies that
\[
\left\| \left( b(\lambda_k) \sum_{j \leq k} \| f \| \tilde{G}(\lambda_j, \lambda_j) \right)_{k \in \mathbb{Z}} \right\|_{dE} \sim \left\| \left( b(\lambda_k) \| f \| \tilde{G}(\lambda_{k-1}, \lambda_k) \right)_{k \in \mathbb{Z}} \right\|_{dE}.
\]
Therefore, using again the discretization of the norm in \(\hat{E}\), we obtain
\[
\left\| b(u) \| f \| \tilde{G}(0, u) \right\|_{\hat{E}} \lesssim \left\| \left( b(\lambda_k) \| f \| \tilde{G}(\lambda_{k-1}, \lambda_k) \right)_{k \in \mathbb{Z}} \right\|_{dE}
= \sum_{k \in \mathbb{Z}} b(\lambda_k) \| f \| \tilde{G}(\lambda_{k-1}, \lambda_k) \chi_{I_{k+1}}(u) \right\|_{\hat{E}}
\sim \sum_{k \in \mathbb{Z}} \frac{b(u)}{a} \| f \| \tilde{G}(\lambda_{k-1}, \lambda_k) \chi_{I_{k+1}}(u) \right\|_{\hat{E}}.
\]
Additionally, Lemma 3.1 i) gives that \(\| a \| \tilde{F}(0, u) \approx a(u) \varphi_F(\ell(u))\). Hence, \(\| a \| \tilde{F}(0, u)\) is a slowly varying function such that \(\| a \| \tilde{F}(0, u) \sim \| a \| \tilde{F}(0, u)\). Thus, for any \(u \in I_{k+1}\) we have that \(\| a \| \tilde{F}(0, u) \sim \| a \| \tilde{F}(0, \lambda_{k-1})\). Moreover,
\[
\| a \| \tilde{F}(0, \lambda_{k-1}) \| f \| \tilde{G}(\lambda_{k-1}, \lambda_k) \| \tilde{F}(0, \lambda_{k-1}) \lesssim \| a(t) \| f \| \tilde{G}(t, \lambda_k) \| \tilde{F}(0, \lambda_{k-1})
\leq \| a(t) \| f \| \tilde{G}(t, \lambda_k) \| \tilde{F}(0, \lambda_{k-1}).
\]
Summing up,
\[
\left\| b(u) \| f \| \tilde{G}(0, u) \right\|_{\hat{E}} \lesssim \sum_{k \in \mathbb{Z}} \frac{b(u)}{a} \| a(t) \| f \| \tilde{G}(t, \lambda_k) \| \tilde{F}(0, \lambda_{k-1}) \right\|_{\hat{E}}
\leq \sum_{k \in \mathbb{Z}} \frac{b(u)}{a} \| a(t) \| f \| \tilde{G}(t, \lambda_k) \| \tilde{F}(0, \lambda_{k-1}) \right\|_{\hat{E}}
= \left\| \frac{b(u)}{a} \| a(t) \| f \| \tilde{G}(t, \lambda_k) \| \tilde{F}(0, \lambda_{k-1}) \right\|_{\hat{E}}.
\]
This concludes the proof of i).
The proof of ii) can be done in a similar vein. Indeed, arguing as in (3.6) we have
\[
\left\| b(u) \| f \| \tilde{G}(u, \infty) \right\|_{\hat{E}} = \left\| \sum_{k \in \mathbb{Z}} b(u) \| f \| \tilde{G}(u, \infty) \chi_{I_{k+1}}(u) \right\|_{\hat{E}}
\leq \sum_{k \in \mathbb{Z}} b(\lambda_k) \| f \| \tilde{G}(\lambda_{k-1}, \lambda_k) \chi_{I_{k+1}}(u) \right\|_{\hat{E}}
= \left\| \left( b(\lambda_k) \sum_{j \geq k} \| f \| \tilde{G}(\lambda_j, \lambda_j) \right)_{k \in \mathbb{Z}} \right\|_{dE}.
\]
This time, \( \rho_{B_{\infty}} < 0 < \pi_{B_0} \). Then, by Lemma 3.5 ii), there exists an equivalence function of \( b \), that again we denote in the same way, such that \( \inf_{k \in \mathbb{Z}} \frac{b(\lambda_{k+1})}{b(\lambda_k)} > 1 \) and using Lemma 3.4 ii) we obtain the relation

\[
\left\| \left( b(\lambda_k) \sum_{j \geq k} \| f \|_{\tilde{G}(\lambda_j, \lambda_{j+1})} \right)_{k \in \mathbb{Z}} \right\|_{d\tilde{E}} \sim \left\| \left( b(\lambda_k) \| f \|_{\tilde{G}(\lambda_k, \lambda_{k+1})} \right)_{k \in \mathbb{Z}} \right\|_{d\tilde{E}}.
\]

Hence,

\[
\left\| b(u) \| f \|_{\tilde{G}(u, \infty)} \right\|_{\tilde{E}} \lesssim \left\| \left( b(\lambda_k) \| f \|_{\tilde{G}(\lambda_k, \lambda_{k+1})} \right)_{k \in \mathbb{Z}} \right\|_{d\tilde{E}} = \left\| \sum_{k \in \mathbb{Z}} b(\lambda_k) \| f \|_{\tilde{G}(\lambda_k, \lambda_{k+1})} \chi_{I_k}(u) \right\|_{\tilde{E}} \sim \left\| \sum_{k \in \mathbb{Z}} \frac{b(u)}{a(\tilde{F}(u, \infty))} \| a \|_{\tilde{F}(u, \infty)} \| f \|_{\tilde{G}(\lambda_k, \lambda_{k+1})} \chi_{I_k}(u) \right\|_{\tilde{E}}.
\]

By Lemma 3.4 ii) the function \( u \sim \| a \|_{\tilde{F}(u, \infty)} \) is slowly varying with \( \| a \|_{\tilde{F}(u^2, \infty)} \). Then, for any \( u \in I_k \) we have that \( \| a \|_{\tilde{F}(u, \infty)} \sim \| a \|_{\tilde{F}(\lambda_{k+1}, \infty)} \), \( k \in \mathbb{Z} \), and

\[
\left\| a \|_{\tilde{F}(\lambda_{k+1}, \infty)} \| f \|_{\tilde{G}(\lambda_k, \lambda_{k+1})} \right\|_{\tilde{F}(\lambda_{k+1}, \infty)} = \left\| a(t) \| f \|_{\tilde{G}(\lambda_k, \lambda_{k+1})} \right\|_{\tilde{F}(\lambda_{k+1}, \infty)} \lesssim \left\| a(t) \| f \|_{\tilde{G}(\lambda_k, \lambda_{k+1})} \right\|_{\tilde{F}(\lambda_{k+1}, \infty)} \lesssim \left\| a(t) \| f \|_{\tilde{G}(\lambda_k, \lambda_{k+1})} \right\|_{\tilde{F}(\lambda_{k+1}, \infty)}.
\]

Therefore,

\[
\left\| b(u) \| f \|_{\tilde{G}(u, \infty)} \right\|_{\tilde{E}} \lesssim \left\| \sum_{k \in \mathbb{Z}} \frac{b(u)}{a(\tilde{F}(u, \infty))} \| a \|_{\tilde{F}(u, \infty)} \| f \|_{\tilde{G}(\lambda_k, \lambda_{k+1})} \chi_{I_k}(u) \right\|_{\tilde{E}} \lesssim \left\| \sum_{k \in \mathbb{Z}} \frac{b(u)}{a(\tilde{F}(u, \infty))} \| a \|_{\tilde{F}(u, \infty)} \| f \|_{\tilde{G}(u, \infty)} \chi_{I_k}(u) \right\|_{\tilde{E}} = \left\| \frac{b(u)}{\| a \|_{\tilde{F}(u, \infty)}} \| a \|_{\tilde{F}(u, \infty)} \chi_{I_k}(u) \right\|_{\tilde{E}}.
\]

The proof of the lemma is complete.

\[\square\]

**Lemma 3.7.** Let \( E, F, G \) be r.i. spaces and let \( d\tilde{E} \) be the discretization of \( \tilde{E} \). Let \( a, b \in SV \) such that \( a(t^2) \sim a(t) \), \( b(t^2) \sim b(t) \) and let \( A_0, A_{\infty}, B_0, B_{\infty} \) be their associated functions defined in (3.7). The following statements holds:

i) If \( \rho_{A_{\infty}} < \pi_{\varphi_F} \leq \rho_{\varphi_F} < \pi_{A_0} \) and \( \rho_{B_0} < 0 < \pi_{B_{\infty}} \), then

\[
\left\| b(u) \| a(t) \| f \|_{\tilde{G}(u, \infty)} \right\|_{\tilde{E}} \sim \left\| \left( b(\lambda_k) \| a \|_{\tilde{F}(0, \lambda_k)} \right)_{k \in \mathbb{Z}} \right\|_{d\tilde{E}}
\]

for any measurable function \( f \) on \( (0, \infty) \).

ii) If \( \rho_{A_0} < \pi_{\varphi_F} \leq \rho_{\varphi_F} < \pi_{A_{\infty}} \) and \( \rho_{B_\infty} < 0 < \pi_{B_0} \), then

\[
\left\| b(u) \| a(t) \| f \|_{\tilde{G}(u, \infty)} \right\|_{\tilde{E}} \lesssim \left\| \left( b(\lambda_k) \| a \|_{\tilde{F}(\lambda_k, \infty)} \right)_{k \in \mathbb{Z}} \right\|_{d\tilde{E}}
\]

for any measurable function \( f \) on \( (0, \infty) \).
Proof. We start with (i). Arguing as in (3.6) it follows that
\[
I := \left\| b(u) \| a(t) \| f \| \overline{G}(t,u) \| F(0,u) \right\|_{\tilde{E}} \sim \left\| \sum_{k \in \mathbb{Z}} b(\lambda_k) \| a(t) \| f \| \overline{G}(t,u) \| F(0,u) \chi_k(u) \right\|_{\tilde{E}} \\
\leq \left\| \sum_{k \in \mathbb{Z}} b(\lambda_k) \| a(t) \| f \| \overline{G}(t,u) \| F(0,u) \chi_k(u) \right\|_{\tilde{E}} \\
= \left\| \left( b(\lambda_k) \| a(t) \| f \| \overline{G}(t,u) \| F(0,u) \right)_{k \in \mathbb{Z}} \right\|_{d\tilde{E}}.
\]
Moreover, for the factor \( \| a(t) \| f \| \overline{G}(t,u) \| F(0,u) \) the following estimate holds
\[
(3.7) \quad \left\| a(t) \| f \| \overline{G}(t,u) \| F(0,u) \right\| \leq \sum_{m \leq k} \left\| a(t) \| f \| \overline{G}(t,u) \| F(0,u) \right\| F(\lambda_{m-1}, \lambda_m) \\
\leq \sum_{m \leq k} \left\| a \| F(\lambda_{m-1}, \lambda_m) \| f \| \overline{G}(\lambda_{m-1}, \lambda_m) \right\| \\
\leq \sum_{m \leq k} \left\| a \| F(\lambda_{m-1}, \lambda_m) \| f \| \overline{G}(\lambda_{j-1}, \lambda_j) \right\|.
\]
Hence,
\[
I \lesssim \left\| \left( b(\lambda_k) \sum_{m \leq k} \| a \| F(\lambda_{m-1}, \lambda_m) \sum_{j=m}^k \| f \| \overline{G}(\lambda_{j-1}, \lambda_j) \right)_{k \in \mathbb{Z}} \right\|_{d\tilde{E}}.
\]
Moreover, due to Lemma 3.3 (i) \( \rho_{A_{\infty}} < \pi_{\varphi_F} \leq \rho_{\varphi_F} < \pi_{A_0} \) we have that
\[
(3.8) \quad \left\| a \right\| F(\lambda_{m-1}, \lambda_m) \lesssim \left\| a \right\| F(0, \lambda_m) \sim a(\lambda_m) \varphi_F(\ell(\lambda_m)).
\]
Let us denote by \( \tilde{a} \) the slowly varying function \( u \sim a(u) \varphi_F(\ell(u)), \) \( u > 0. \) This function satisfies that \( \tilde{a}(u^2) \sim \tilde{a}(u) \) and its associated functions are \( \tilde{A}_0(u) = A_0(u) \varphi_F(1/u), \) \( \tilde{A}_\infty(u) = A_\infty(u) \varphi_F(1/u), \) \( u \in (0, 1]. \) Using the properties of the extension indices it is clear that
\[
\rho_{\tilde{A}_\infty} \leq \rho_{A_\infty} - \pi_{\varphi_F} < 0 < \pi_{A_0} - \rho_{\varphi_F} \leq \pi_{\tilde{A}_0}.
\]
Hence, by Lemma 3.3 (ii), there exists an equivalent function to \( \tilde{a} \), that we denote in the same way, such that \( \inf_{k \in \mathbb{Z}} \frac{\tilde{a}(k+1)}{\tilde{a}(k)} > 1. \) Applying Lemma 3.3 (ii), with \( e = \ell_1, \) we derive
\[
\sum_{m \leq k} \tilde{a}(\lambda_m) \sum_{j=m}^k \| f \| \overline{G}(\lambda_{j-1}, \lambda_j) \lesssim \sum_{m \leq k} \tilde{a}(\lambda_m) \| f \| \overline{G}(\lambda_{m-1}, \lambda_m).
\]
Therefore, by (3.8), we obtain
\[
I \lesssim \left\| \left( b(\lambda_k) \sum_{m \leq k} \| a \| F(0, \lambda_m) \| f \| \overline{G}(\lambda_{m-1}, \lambda_m) \right)_{k \in \mathbb{Z}} \right\|_{d\tilde{E}}.
\]
Since \( \rho_{B_0} < 0 < \pi_{B_\infty} , \) we can apply again Lemmas 3.3 and 3.4 (i) with \( e = d\tilde{E} \) to deduce that
\[
I \lesssim \left\| \left( b(\lambda_k) \| a \| F(0, \lambda_k) \| f \| \overline{G}(\lambda_{k-1}, \lambda_k) \right)_{k \in \mathbb{Z}} \right\|_{d\tilde{E}}.
\]
In order to prove the reverse inequality \( \gtrsim \) we proceed as follows. An argument similar to that of (3.6) yields
\[
I \sim \left\| \sum_{k \in \mathbb{Z}} b(\lambda_k) \| a(t) \| f(t, u) \| \tilde{G}(0, u) \chi_{I_{k+1}}(u) \right\|_{E}
\]
\[
\gtrsim \left\| \sum_{k \in \mathbb{Z}} b(\lambda_k) \| a(t) \| f(t, \lambda_k) \| \tilde{F}(0, \lambda_k) \chi_{I_{k+1}}(u) \right\|_{E}
\]
\[
= \left\| \left( b(\lambda_k) \| a(t) \| f(t, \lambda_k) \right) \right\|_{E}.
\]
Moreover
\[
\| a(t) \| f(t, \lambda_k) \| \tilde{F}(0, \lambda_k) \gtrsim \| a(t) \| f(t, \lambda_k) \| \tilde{F}(0, \lambda_{k-1}) \gtrsim \| a(t) \| f(t, \lambda_{k-1}, \lambda_k) \| \tilde{F}(0, \lambda_{k-1})
\]
\[
\sim \| a \| \tilde{F}(0, \lambda_k) \| f \| \tilde{G}(\lambda_{k-1}, \lambda_k).
\]
Hence
\[
I \gtrsim \left\| \left( b(\lambda_k) \| a \| \tilde{F}(0, \lambda_k) \| f \| \tilde{G}(\lambda_{k-1}, \lambda_k) \right) \right\|_{E}
\]
and the proof of i) is complete.

Now we proceed with the proof of ii). Similar arguments to the ones given in (3.6) allow to obtain that
\[
II : = \left\| \left( b(\lambda_k) \| a(t) \| f(t, u) \| \tilde{G}(u, \infty) \right) \right\|_{E}
\]
\[
\lesssim \left\| \left( b(\lambda_k) \| a(t) \| f \| \tilde{G}(\lambda_{k-1}, u) \right) \right\|_{E} \sum_{k \in \mathbb{Z}} \left\| \tilde{G}(\lambda_{k-1}, u) \right\|_{E}
\]
And arguing as in (3.7)
\[
\| a(t) \| f \| \tilde{G}(\lambda_{k-1}, u) \| \tilde{F}(\lambda_{k-1}, \infty) \leq \sum_{m \geq k} \| a(t) \| f \| \tilde{G}(\lambda_{k-1}, \lambda_m) \| \tilde{F}(\lambda_{m-1}, \lambda_m)
\]
\[
\leq \sum_{m \geq k} \| a \| \tilde{F}(\lambda_{m-1}, \lambda_m) \| f \| \tilde{G}(\lambda_{k-1}, \lambda_m)
\]
\[
\leq \sum_{m \geq k} \| a \| \tilde{F}(\lambda_{m-1}, \lambda_m) \sum_{j=k}^{m} \| f \| \tilde{G}(\lambda_j, \lambda_j)
\]

Lemma 3.1 \((\rho_{A_0} < \pi_{\varphi_F} \leq \rho_{\varphi_F} \leq \pi_{A_\infty})\) establishes that
\[
\| a \| \tilde{F}(\lambda_{m-1}, \lambda_m) \leq \| a \| \tilde{F}(\lambda_{m-1}, \infty) \sim a(\lambda_m) \varphi_F(\ell(\lambda_m)).
\]
We denote again by \( \tilde{a} \) the slowly varying function \( u \sim a(u) \varphi_F(\ell(u)), u > 0 \). Its associated functions are \( \tilde{A}_0(u) = A_0(u) \varphi_F(1/u), \tilde{A}_\infty(u) = A_\infty(u) \varphi_F(1/u) \) and this time
\[
\rho_{\tilde{A}_0} < \rho_{A_0} - \pi_{\varphi_F} < 0 < \pi_{A_\infty} - \rho_{\varphi_F} \leq \pi_{\tilde{A}_\infty}.
\]
Then, there exist an equivalent function, that we denote in the same way, such that \( \sup_{k \in \mathbb{Z}} \frac{\tilde{a}(\lambda_{k+1})}{\tilde{a}(\lambda_k)} < 1 \). Using Lemma 3.4(i) with \( e = \ell_1 \), we have
\[
\sum_{m \geq k} \tilde{a}(\lambda_m) \sum_{j=k}^{m} \| f \| \tilde{G}(\lambda_{j-1}, \lambda_j) \lesssim \sum_{m \geq k} \tilde{a}(\lambda_m) \| f \| \tilde{G}(\lambda_{m-1}, \lambda_m).
\]
All together,
\[ II \lesssim \left\| \left( b(\lambda_k) \sum_{m \geq k} a(\lambda_m) \| f \|_{\tilde{G}(\lambda_{m-1},\lambda_m)} \right)_{k \in \mathbb{Z}} \right\|_d E. \]

Finally, by Lemmas 3.3 y 3.4 with \( e = d \tilde{E} (\rho_{B_0} < 0 < \pi_{B_0}) \) we obtain the desired inequality
\[ II \lesssim \left\| \left( b(\lambda_k) \| a \|_{\tilde{F}(\lambda_k,\infty)} \| f \|_{\tilde{G}(\lambda_{k-1},\lambda_k)} \right)_{k \in \mathbb{Z}} \right\|_d E. \]

4. Interpolation Methods

In the sequel, \( \overline{X} = (X_0, X_1) \) will be a compatible (quasi-) Banach couple. Next, we collect, following [13], the necessary definitions and statements dealing with the real Banach space \( \theta \), provided that any of the following conditions holds, the space is trivial, that is \( \| b \|_{\tilde{E}(0,1)} < \theta \), and \( \| b \|_{\tilde{E}(1,\infty)} < \infty \).

**Definition 4.1.** Let \( E \) be an r.i. space, \( b \in SV \) and \( 0 \leq \theta \leq 1 \). The real interpolation space \( \overline{X}_{\theta,b,E} \equiv (X_0, X_1)_{\theta,b,E} \) consists of all \( f \) in \( X_0 + X_1 \) for which
\[ \| f \|_{\theta,b,E} := \left\| t^{-\theta} b(t) K(t,f) \right\|_{\tilde{E}} < \infty. \]

It is a well-known fact that \( \overline{X}_{\theta,b,E} \) is a (quasi-) Banach space, and it is intermediate for the couple \( \overline{X} \), that is,
\[ X_0 \cap X_1 \hookrightarrow \overline{X}_{\theta,b,E} \hookrightarrow X_0 + X_1, \]
provided that \( 0 < \theta < 1 \), \( \theta = 0 \) and \( \| b \|_{\tilde{E}(1,\infty)} < \infty \) or \( \theta = 1 \) and \( \| b \|_{\tilde{E}(0,1)} < \infty \). If none of these conditions holds, the space is trivial, that is \( \overline{X}_{\theta,b,E} = \{0\} \).

When \( E = L_q \) and \( b \equiv 1 \), then \( \overline{X}_{\theta,b,E} \) coincides with the classical real interpolation space \( \overline{X}_{\theta,q} \).

The reiteration spaces
\[ (\overline{X}_{\theta_0,b_0,E_0}, \overline{X}_{\theta_1,b_1,E_1})_{\theta,b,E}, \]
with \( \theta_0 < \theta_1 \), have been studied in detail in [13] [14] [15] for general r.i. spaces \( E \), and in [1] [24] for \( E = L_q \), \( 0 < q \leq \infty \). For other special cases see e.g. [6] [8] [11] [17] [32]. When \( \theta = 0,1 \) the resulting reiteration spaces do not belong to the same scale and the \( \mathcal{R} \) and \( \mathcal{L} \) constructions are needed to describe them.

**Definition 4.2.** Let \( E, F \) be two r.i. spaces, \( a, b \in SV \) and \( 0 \leq \theta \leq 1 \). The (quasi-) Banach space \( \overline{X}_{\theta,b,E,a,F} \equiv (X_0, X_1)_{\theta,b,E,a,F} \) consists of all \( f \in X_0 + X_1 \) for which
\[ \| f \|_{\mathcal{R}_{\theta,b,E,a,F}} := \left\| b(t) s^{-\theta} a(s) K(s,f) \right\|_{\tilde{F}(t,\infty)} \|_{\tilde{E}} < \infty. \]

The space \( \mathcal{R} \) is intermediate for the couple \( \overline{X} \), that is,
\[ X_0 \cap X_1 \hookrightarrow \overline{X}_{\theta,b,E,a,F} \hookrightarrow X_0 + X_1 \]
provided that any of the following conditions holds:
1. \( 0 < \theta < 1 \) and \( \| b \|_{\tilde{E}(0,1)} < \infty \),
2. \( \theta = 0 \), \( \| b \|_{\tilde{E}(0,1)} < \infty \) and \( \| b(t) \|_{\tilde{F}(t,\infty)} \|_{\tilde{E}(1,\infty)} < \infty \) or
3. \( \theta = 1, \|b\|_{\tilde{E}(0,1)} < \infty, \|b(t)\|_{\tilde{F}(t,1)} < \infty \) and \( \|ab\|_{\tilde{E}(0,1)} < \infty \).

Otherwise, \( \Xi_{\theta,b,E,a,F}^R = \{0\} \).

**Definition 4.3.** Let \( E, F \) be two r.i. spaces, \( a, b \in SV \) and \( 0 \leq \theta \leq 1 \). The space \( \Xi_{\theta,b,E,a,F}^L \equiv (X_0, X_1)_{\theta,c,E,a,F}^L \) consists of all \( f \in X_0 + X_1 \) for which

\[
\|f\|_{L,\theta,b,E,a,F} = \|b(t)\|_{\tilde{F}(t,1)} < \infty.
\]

This is a (quasi-) Banach space. Moreover, it is intermediate for the couple \( \Xi^L \),

\[
X_0 \cap X_1 \hookrightarrow \Xi_{\theta,b,E,a,F}^L \hookrightarrow X_0 + X_1,
\]

provided that:

1. \( 0 < \theta < 1 \) and \( \|b\|_{E(1,\infty)} < \infty \),
2. \( \theta = 0, \|b(t)\|_{F(0,t)} < \infty \) or
3. \( \theta = 1, \|b\|_{E(\infty)} < \infty \) and \( \|b(t)\|_{F(0,t)} < \infty \).

If none of these conditions holds, then \( \Xi_{\theta,b,E,a,F}^L \) is the trivial space.

The spaces \( \Xi_{\theta,b,E,a,F}^R \) and \( \Xi_{\theta,b,E,a,F}^L \) can be defined analogously replacing \( \tilde{E} \) by \( \hat{E} \) in previous definitions.

We refer to the recent papers \([9, 10, 17, 18, 19]\) for reiteration theorems for couples formed by arbitrary combinations of the previous spaces under the condition that the parameters \( \theta_0 \) and \( \theta_1 \) are not equal. Again, in the extremal cases \( \theta = 0, 1 \) the resulting reiteration spaces belong to some extremal constructions introduced in \([10, 17, 18]\). Let us recall the definition of the interpolation methods \((R, \mathcal{L})\) and \((\mathcal{L}, R)\).

**Definition 4.4.** Let \( E, F, G \) be r.i. spaces, \( a, b, c \in SV \) and \( 0 < \theta < 1 \). The space \( \Xi_{\theta,c,E,b,F,a,G}^{R,\mathcal{L}} \equiv (X_0, X_1)_{\theta,c,E,b,F,a,G}^{R,\mathcal{L}} \) is the set of all \( f \in X_0 + X_1 \) for which

\[
\|f\|_{R,\mathcal{L};\theta,c,E,b,F,a,G} := \|c(u)\|_{\tilde{F}(t,u)} < \infty.
\]

The space \( \Xi_{\theta,c,E,b,F,a,G}^L \equiv (X_0, X_1)_{\theta,c,E,b,F,a,G}^L \) is the set of all \( f \in X_0 + X_1 \) such that

\[
\|f\|_{\mathcal{L};R;\theta,c,E,b,F,a,G} := \|c(u)\|_{\tilde{F}(t,u)} < \infty.
\]

Lemma \([3,6]\) allows us to obtain the following inclusions:

**Corollary 4.5.** Let \( E, F, G \) be r.i. spaces, let \( a, b, c \in SV \) such that \( b(t^2) \sim b(t) \), \( c(t^2) \sim c(t) \) and let \( B_0, B_\infty, C_0 \), and \( C_\infty \) be their respective associated functions defined by \([2,1]\). The following statements holds:

i) If \( \rho_{B_\infty} < \pi_{\varphi_F} \leq \rho_{\varphi_F} < \pi_{B_0} \) and \( \rho_{C_0} < 0 < \pi_{C_\infty} \), then

\[
\Xi_{\theta,d,E,b,F,a,G}^{R,\mathcal{L}} = \Xi_{\theta,c,E,a,G}^L
\]

where \( d(t) = c(t)/\|b\|_{\tilde{F}(0,u)}, u > 0 \).
ii) If $\rho_{B_0} < \pi_{\varphi_F} \leq \rho_{\varphi_F} < \pi_{B_\infty}$ and $\rho_{C_\infty} < 0 < \pi_{C_0}$, then
\[
X_{\theta,d,\tilde{E},h,F,a,G}^R = X_{\theta,c,\tilde{E},a,G}^R
\]
where $d(t) = c(t)/\|b\|_{\tilde{F}(u,\infty)}$, $u > 0$.

5. Reiteration theorem

In this section we shall prove the main result of this paper, the characterization of the interpolation space
\[
(\tilde{X}_{\theta,b_0,E_0,a,F}, \tilde{X}_{\theta,b_1,E_1,a,F})_{\eta,b,E}
\]
for all possible values of $\eta \in [0, 1]$. To this end we will additionally need a generalized Holsmtedt type formula and a change of variables.

**Theorem 5.1.** Let $0 < \theta < 1$. Let $E_0$, $E_1$, $F$ be r.i. spaces; $a$, $b_0$, $b_1 \in SV$ with $\|b_0\|_{E_0(0,1)} < \infty$ and $\|b_1\|_{E_1(1,\infty)} < \infty$. Then, for every $f \in \tilde{X}_{\theta,b_0,E_0,a,F} + \tilde{X}_{\theta,b_1,E_1,a,F}$ and $u > 0$
\[
K(\phi(u), f; \tilde{X}_{\theta,b_0,E_0,a,F}, \tilde{X}_{\theta,b_1,E_1,a,F}) \sim \|b_0(t)\|_{E_0(0,u)} \|s^{-\theta}a(s)K(s,f)\|_{\tilde{F}(t,u)}
+ \phi(u)\|b_1(t)\|_{E_1(1,\infty)} \|s^{-\theta}a(s)K(s,f)\|_{\tilde{F}(u,t)}
\]
where
\[
\phi(u) = \frac{\|b_0\|_{E_0(0,u)}}{\|b_1\|_{E_1(1,\infty)}}, \quad u > 0.
\]

**Proof.** Given $f \in X_0 + X_1$ and $u > 0$, we consider the following (quasi-) norms
\[
(P_0f)(u) = \|b_0(t)\|_{E_0(0,u)} \|s^{-\theta}a(s)K(s,f)\|_{\tilde{F}(t,u)},
\]
\[
(Q_1f)(u) = \|b_1(t)\|_{E_1(1,\infty)} \|s^{-\theta}a(s)K(s,f)\|_{\tilde{F}(u,t)}.
\]
Denote $Y_0 = \tilde{X}_{\theta,b_0,E_0,a,F}$ and $Y_1 = \tilde{X}_{\theta,b_1,E_1,a,F}$. The proof of the estimate
\[
K(\phi(u), f; Y_0, Y_1) \lesssim (P_0f)(u) + \phi(u)(Q_1f)(u)
\]
can be done exactly as we did in Theorem 3.3 from [19]. Therefore, we only have to prove the converse estimate, that is,
\[
(P_0f)(u) + \phi(u)(Q_1f)(u) \lesssim K(\phi(u), f; Y_0, Y_1)
\]
for all $f \in Y_0 + Y_1$ and $u > 0$.

Fix $u > 0$ and let $f = f_0 + f_1$ be any decomposition of $f$ where $f_0 \in Y_0$ and $f_1 \in Y_1$. By the (quasi)-subadditivity of the $K$-functional and the definition of the norms in $Y_0$ and $Y_1$, we derive
\[
(P_0f)(u) \lesssim (P_0f_0)(u) + (P_0f_1)(u) \leq \|f_0\|_{Y_0} + (P_0f_1)(u),
\]
\[
(Q_1f)(u) \lesssim (Q_1f_0)(u) + (Q_1f_1)(u) \leq (Q_1f_0)(u) + \|f_1\|_{Y_1}.
\]
Thus, we have to study the boundedness of $(P_0f_1)(u) + \phi(u)(Q_0f_1)(u)$ by $\|f_0\|_Y + \phi(u)\|f_1\|_{Y_1}$. Let us start with $(P_0f_1)(u)$. It is easy to observe that

$$
(P_0f_1)(u) \leq \|b_0\|_{\tilde{E}_0(0,u)}\|s^{-\theta}a(s)K(s,f_1)\|_{\tilde{F}(0,u)}
$$

$$
= \phi(u)\|b_1\|_{\tilde{E}_1(u,\infty)}\|s^{-\theta}a(s)K(s,f_1)\|_{\tilde{F}(0,u)}
$$

$$
\leq \phi(u)\|b_1(t)\|_{\tilde{E}_1(t,\infty)}\|s^{-\theta}a(s)K(s,f_1)\|_{\tilde{F}(0,t)}
$$

$$
\leq \phi(u)\|b_1(t)\|_{\tilde{E}_1(t,\infty)}\|s^{-\theta}a(s)K(s,f_1)\|_{\tilde{F}(0,t)}
$$

$$
\leq \phi(u)\|b_1(t)\|_{\tilde{E}_1(t,\infty)}\|s^{-\theta}a(s)K(s,f_1)\|_{\tilde{F}(0,t)}
$$

Similarly,

$$
\phi(u)(Q_0f_1)(u) \leq \|b_0\|_{\tilde{E}_0(0,u)}\|s^{-\theta}a(s)K(s,f_0)\|_{\tilde{F}(u,\infty)}
$$

$$
= \phi(u)\|b_0(t)\|_{\tilde{E}_0(t,\infty)}\|s^{-\theta}a(s)K(s,f_0)\|_{\tilde{F}(0,u)}
$$

$$
\leq \phi(u)\|b_0(t)\|_{\tilde{E}_0(t,\infty)}\|s^{-\theta}a(s)K(s,f_0)\|_{\tilde{F}(0,u)}
$$

$$
\leq \phi(u)\|b_0(t)\|_{\tilde{E}_0(t,\infty)}\|s^{-\theta}a(s)K(s,f_0)\|_{\tilde{F}(0,u)}
$$

$$
\leq \phi(u)\|b_0(t)\|_{\tilde{E}_0(t,\infty)}\|s^{-\theta}a(s)K(s,f_0)\|_{\tilde{F}(0,u)}
$$

Putting together the previous estimates we establish the inequality

$$(P_0f)(u) + \phi(u)(Q_1f)(u) \lesssim \|f_0\|_Y + \phi(u)\|f_1\|_{Y_1}.$$
Similarly, since the indices of the function $\Phi_\infty(u) = \Phi_\infty(1/u)$, $1 \leq u < \infty$, are strictly positive and finite, there exist a smooth function $\Psi_\infty \sim \Phi_\infty$ such that $u\Psi_\infty(u) \sim \Psi_\infty(u)$, $1 \leq u < \infty$, $\Psi(1) = \Phi(1)$ and $\lim_{u \to \infty} \Psi(u) = \infty$. Consequently, changing variables, $t = \Psi(u)$, it follows

$$\int_1^\infty |H(t)| \frac{dt}{t} = \int_1^\infty |H(\Psi(\psi(u)))| \frac{\psi'(u)}{\psi(u)} du \sim \int_1^\infty |H(\Psi(u))| \frac{du}{u}.$$  

Using that $H \circ \Psi_\infty \sim H \circ \Phi_\infty$, the change of variables $u = \ell(t)$ and the fact that $\Phi_\infty(\ell(t)) = \phi(t)$, $1 \leq t < \infty$, we have

$$(5.3) \quad \int_{\Phi_\infty(1)}^\infty \frac{|H(s)|}{s} \frac{ds}{s} \sim \int_1^\infty |H(\Phi_\infty(u))| \frac{du}{u} = \int_1^\infty |H(\phi(t))| \frac{dt}{t\ell(t)}.$$  

The equivalences (5.2) and (5.3) give

$$\|H\|_{L_1} \sim \|H \circ \phi\|_{L_1}$$  

(observe that $\Phi_0(1) = \Phi_\infty(1)$). The same is true for $E = L_\infty$, that is $\|H\|_{L_\infty} \sim \|H \circ \phi\|_{L_\infty}$. Thus, by the interpolation properties of the space $E$, we obtain the inequality

$$\|H \circ \phi\|_E \lesssim \|H\|_E.$$

The reverse inequality can be proved applying the same techniques with inverse functions.

Now, we are in position to establish the main interpolation theorem of our paper.

**Theorem 5.3.** Let $0 < \theta < 1$. Let $E$, $E_0$, $E_1$, $F$ be $r.i.$ spaces and let $a$, $b$, $b_0$, $b_1 \in SV$ be such that $b_0(t) \sim b_0(t^2)$ and $b_1(t) \sim b_1(t^2)$. Assume that $E_0$, $E_1$ and the associated functions $B_0, B_0, B_1, B_1$ of $b_0$ and $b_1$, respectively, defined in (3.1) satisfy that

$$\rho_{B_0,0} < \pi_{\varphi_{E_0}} < \pi_{\varphi_{E_0}} \quad \text{and} \quad \rho_{B_1,0} < \pi_{\varphi_{E_1}} < \pi_{B_1,\infty}.$$  

Let $0 \leq \eta \leq 1$ be a parameter and define

$$B_\eta(u) = (b_0(u)\varphi_{E_0}(\ell(u)))^{1-\eta} (b_1(u)\varphi_{E_1}(\ell(u)))^{\eta} \begin{pmatrix} b_0(u) & \varphi_{E_0}(\ell(u)) \\ b_1(u) & \varphi_{E_1}(\ell(u)) \end{pmatrix}, \quad u > 0.$$

a) If $0 < \eta < M_1 := \max\left\{ \left(1 - \frac{\pi_{B_1,0} - \pi_{\varphi_{E_1}}}{\pi_{B_0,0} - \pi_{\varphi_{E_0}}} \right)^{-1}, \left(1 - \frac{\rho_{B_1,\infty} - \pi_{\varphi_{E_1}}}{\rho_{B_0,\infty} - \pi_{\varphi_{E_0}}} \right)^{-1} \right\}$, then

$$\begin{pmatrix} \mathcal{X}_{\theta,b_0,E_0,a,F}^R, \mathcal{X}_{\theta,b_1,E_1,a,F}^C \end{pmatrix}_{\eta,b,E} = \mathcal{X}_{\theta,B_0,E,a,F}^R, \quad \mathcal{X}_{\theta,b_0,E_0,a,F}^C, \mathcal{X}_{\theta,b_1,E_1,a,F}^C \end{pmatrix}_{\eta,b,E} = \mathcal{X}_{\theta,B_1,E,a,F}^C.$$

b) If $M_2 := \max\left\{ \left(1 - \frac{\rho_{B_1,0} - \pi_{\varphi_{E_1}}}{\rho_{B_0,0} - \pi_{\varphi_{E_0}}} \right)^{-1}, \left(1 - \frac{\pi_{B_1,\infty} - \pi_{\varphi_{E_1}}}{\pi_{B_0,\infty} - \pi_{\varphi_{E_0}}} \right)^{-1} \right\} < \eta < 1$, then

$$\begin{pmatrix} \mathcal{X}_{\theta,b_0,E_0,a,F}^R, \mathcal{X}_{\theta,b_1,E_1,a,F}^C \end{pmatrix}_{\eta,b,E} = \mathcal{X}_{\theta,B_0,E,a,F}^R, \quad \mathcal{X}_{\theta,b_0,E_0,a,F}^C, \mathcal{X}_{\theta,b_1,E_1,a,F}^C \end{pmatrix}_{\eta,b,E} = \mathcal{X}_{\theta,B_1,E,a,F}^C.$$

c) If $M_1 \leq \eta \leq M_2$, then

$$\begin{pmatrix} \mathcal{X}_{\theta,b_0,E_0,a,F}^R, \mathcal{X}_{\theta,b_1,E_1,a,F}^C \end{pmatrix}_{\eta,b,E} = \mathcal{X}_{\theta,B_0,E,a,F}^R, \quad \mathcal{X}_{\theta,b_0,E_0,a,F}^C, \mathcal{X}_{\theta,b_1,E_1,a,F}^C \end{pmatrix}_{\eta,b,E} = \mathcal{X}_{\theta,B_1,E,a,F}^R$$

where $B_\eta^\#(u) = \frac{B_\eta(u)}{b_0(u)\varphi_{E_0}(\ell(u))}$ and $a^\#(u) = a(u)b_0(u)\varphi_{E_0}(\ell(u))$, $u > 0$. 


d) If $\|b\|_{\overline{E}(1,\infty)} < \infty$, then
\[
\left( X^{R}_{b,b_{0},E_{0},a,F}, X^{L}_{b,b_{1},E_{1},a,F} \right)_{0,b,E} = X^{R}_{b,b_{0},\overline{E},a,F} \cap X^{L}_{b,b_{0},\overline{E},b_{1},E_{1},a,F},
\]
where $\phi(u) = \frac{b_{0}(u)\varphi_{E_{0}}(\ell(u))}{b_{1}(u)\varphi_{E_{1}}(\ell(u))}$, $u > 0$.

e) If $\|b\|_{\overline{E}(0,1)} < \infty$, then
\[
\left( X^{R}_{b,b_{0},E_{0},a,F}, X^{L}_{b,b_{1},E_{1},a,F} \right)_{1,b,E} = X^{L}_{b_{1},\overline{E},a,F} \cap X^{R}_{b_{0},\overline{E},b_{1},E_{1},a,F},
\]
where $\phi$ is defined above.

Proof. Throughout the proof we use the notation
\[
Y_{0} = X^{R}_{b,b_{0},E_{0},a,F}, \quad Y_{1} = X^{L}_{b,b_{1},E_{1},a,F},
\]
\[
K(u,f) = K(u,f;Y_{0},Y_{1}), \quad u > 0,
\]
and
\[
\phi(u) = \frac{\|b_{0}\|_{E_{0}(0,u)}}{\|b_{1}\|_{E_{1}(u,\infty)}}, \quad u > 0.
\]
It is clear that $\phi$ is an increasing slowly varying function. Moreover, from Lemma \[5.1\] we have
\[
\phi(u) \sim \phi(u^{2}) \quad \text{and its associated functions, in the sense of } (3.1), \quad \text{are}
\]
\[
\Phi_{0}(v) = \frac{B_{0,0}(v)\varphi_{E_{0}}(t(v))}{B_{1,0}(v)\varphi_{E_{1}}(t(v))}, \quad \Phi_{\infty}(v) = \frac{B_{0,\infty}(v)\varphi_{E_{0}}(t(v))}{B_{1,\infty}(v)\varphi_{E_{1}}(t(v))}, \quad 0 < v \leq 1.
\]
By the properties of the extension indices, it holds
\[
\rho_{\Phi_{\infty}} \leq \rho_{B_{0,0}} - \pi_{\varphi_{E_{0}}} - \pi_{B_{1,\infty}} + \rho_{\varphi_{E_{1}}} < 0 < \pi_{B_{0,0}} - \rho_{\varphi_{E_{0}}} - \rho_{B_{1,0}} + \pi_{\varphi_{E_{1}}} \leq \pi_{\Phi_{0}}.
\]
Then, Lemma \[5.2\] establishes the equivalence
\[
\|f\|_{\overline{Y}_{\eta,b,E}} = \|u^{-\eta}b(u)K(u,f)\|_{\overline{E}} \sim \|\phi(u)^{-\eta}b(\phi(u))K(\phi(u),f)\|_{\overline{E}}.
\]
Applying generalized Holmstedt type formula, Theorem \[5.1\] and the triangular inequality we obtain that
\[
\max(I_{1},I_{2}) \leq \|f\|_{\overline{Y}_{\eta,b,E}} \leq I_{1} + I_{2} \quad \text{for all } 0 \leq \eta \leq 1.
\]
where
\[
I_{1} := \|\phi(u)^{-\eta}b(\phi(u))\|_{0,b_{0}}\|s^{\eta}a(s)K(s,f)\|_{\overline{F}(t(u))}\|\overline{E}(0,u)}_{\overline{Y}_{\eta,b,E}}
\]
and
\[
I_{2} := \|\phi(u)^{1-\eta}b(\phi(u))\|_{b_{1}}\|s^{\eta}a(s)K(s,f)\|_{\overline{F}(t(u))}\|\overline{E}(u,\infty)}_{\overline{Y}_{\eta,b,E}}.
\]
Therefore, in order to identify the space $\overline{Y}_{\eta,b,E}$ we have to estimate $I_{1}$ and $I_{2}$. First, we proceed with the estimates from above of both quantities. It is clear by \[5.4\] and the definition of $B_{\eta}$ that
\[
I_{1} \leq \|\phi(u)^{-\eta}b(\phi(u))\|_{0,b_{0}}\|s^{\eta}a(s)K(s,f)\|_{\overline{F}(t(u))}\|\overline{E}(0,u)}_{\overline{Y}_{\eta,b_{0},\overline{E},a,F}}
\]
\[
\sim \|B_{\eta}(u)\|s^{\eta}a(s)K(s,f)\|_{\overline{F}(t(u))}\|\overline{E}(0,u)}_{\overline{Y}_{\eta,b_{0},\overline{E},a,F}}.
\]
and

\[ I_2 \leq \left\| \phi(u)^{1-\eta}b(\phi(u))b_1 \right\|_{E_1(u, \infty)} \left\| s^{-\theta}a(s)K(s, f) \right\|_{F(u, \infty)} \]

\[ \sim \left\| B_\eta(u) \right\|_{F(u, \infty)} \left\| s^{-\theta}a(s)K(s, f) \right\|_{F(u, \infty)} = \left\| f \right\|_{X_{E, B, E, a, F}^\infty}. \]

Hence

\[ I_1 + I_2 \lesssim \left\| f \right\|_{X_{E, B, E, a, F}^\infty} + \left\| f \right\|_{X_{E, B, E, a, F}^\infty} \quad \text{for all} \quad 0 \leq \eta \leq 1. \]

Next, we will prove that

\[ I_1 + I_2 \lesssim \min(\left\| f \right\|_{X_{E, B, E, a, F}^\infty}, \left\| f \right\|_{X_{E, B, E, a, F}^\infty}) \quad \text{when} \quad 0 < \eta < 1. \]

The slowly varying function

\[ \psi(u) := \phi(u)^{-\eta}b(\phi(u)), \quad u > 0, \]

satisfies that \( \psi(u) \sim \psi(u^2) \) and has as associated functions, in the sense of (3.1),

\[ \Psi_0(v) = (\Phi_0(v))^{-\eta}b(\Phi_0(v)) \quad \text{and} \quad \Psi_\infty(v) = (\Phi_\infty(v))^{-\eta}b(\Phi_\infty(v)), \]

for \( 0 < v \leq 1 \). Using that \( b(\Phi_0(v)) \) and \( b(\Phi_\infty(v)) \) are slowly varying in \((0, 1)\) and the properties of the extension indices, it is clear that

\[ \rho_{\Psi_0} \leq -\eta \pi_{\Phi_0} < 0 < -\eta \rho_{\Phi_\infty} \leq \pi_{\Psi_\infty}. \]

Hence, Lemma (3.2) (\( \pi_{B_0,0} > 0 \)) and limiting Hardy type inequality i) (\( \rho_{\Psi_0} < 0 < \pi_{\Psi_\infty} \)) give

\[ I_1 \lesssim \left\| \psi(u) \right\|_{\inf} \left( \int_0^u b_0(t) \left\| s^{-\theta}a(s)K(s, f) \right\|_{F(t, u)} \frac{dt}{t(E(t))} \right) \]

\[ \leq \left\| \psi(u) \right\|_{\inf} \left( \int_0^u b_0(t) \left\| s^{-\theta}a(s)K(s, f) \right\|_{F(t, \infty)} \frac{dt}{t(E(t))} \right) \]

\[ \leq \left\| \psi(u) \right\|_{\inf} b_0(u) \left\| s^{-\theta}a(s)K(s, f) \right\|_{F(u, \infty)} \]

\[ \sim \left\| B_\eta(u) \right\|_{F(u, \infty)} \left\| s^{-\theta}a(s)K(s, f) \right\|_{F(u, \infty)} = \left\| f \right\|_{X_{E, B, E, a, F}^\infty}. \]

Similarly, take the slowly varying function \( \gamma(u) := \phi(u)^{1-\eta}b(\phi(u)), \quad u > 0 \), which satisfies that \( \gamma(u) \sim \gamma(u^2) \) and has as associated functions

\[ \Gamma_0(u) = (\Phi_0(u))^{1-\eta}b(\Phi_0(u)) \quad \text{and} \quad \Gamma_\infty(u) = (\Phi_\infty(u))^{1-\eta}b(\Phi_\infty(u)), \]

for \( 0 < u \leq 1 \). The indices of these functions satisfy that

\[ \rho_{\Gamma_\infty} \leq (1 - \eta) \rho_{\Phi_\infty} < 0 < (1 - \eta) \pi_{\Phi_0} \leq \pi_{\Gamma_0}. \]
Then, Lemma 3.2 ii) \((\pi_{B_{1,\infty}} > 0)\) and limiting Hardy type inequality ii) \((\rho_{I_{\infty}} < 0 < \pi_{I_{0}})\) can be applied to obtain that
\[
I_2 \lesssim \left\| \gamma(u) \int_{u}^{\infty} b_1(t) \|s^{-\theta}a(s)K(s, f)\| \mathcal{F}(u, t) \mathcal{F}(t) \right\|_{E} \\
\leq \left\| \gamma(u) \int_{u}^{\infty} b_1(t) \|s^{-\theta}a(s)K(s, f)\| \mathcal{F}(u, t) \mathcal{F}(t) \right\|_{E} \\
\lesssim \|B_\eta(u)\|s^{-\theta}a(s)K(s, f)\| \mathcal{F}(u, t) \|_{E} = \|f\|_{\mathcal{X}_{\theta, B_\eta, E,a,F}^\infty}.
\]
Therefore
\[
(5.8) \quad I_1 + I_2 \lesssim \min(\|f\|_{\mathcal{X}_{\theta, B_\eta, E,a,F}^\infty}, \|f\|_{\mathcal{X}_{\theta, B_\eta, E,a,F}^\infty}) \quad \text{for all} \quad 0 < \eta < 1.
\]

Secondly, we shall give separate arguments for the proof of the lower estimates of \(I_1\) and \(I_2\) for each of the cases. In case a) we are assuming that \(\eta\) belongs to the interval \((0, M_1)\). By the definition of \(M_1\), we have
\[
(1 - \eta)[\rho_{B_{0,\infty}} - \pi_{E_0}] + \eta[\rho_{B_{1,\infty}} - \pi_{E_1}] < 0 < (1 - \eta)[\pi_{B_{0,\infty}} - \rho_{E_0}] + \eta[\pi_{B_{1,\infty}} - \rho_{E_1}]
\]
and hence, the indices of the associated functions of the \(B_\eta\) satisfy that \(\rho_{B_{1,\infty}} < 0 < \pi_{B_{0,\infty}}\). Applying Lemma 3.6 ii) \((\rho_{B_{1,\infty}} < \rho_{E_0} < \pi_{B_{1,\infty}})\) we obtain that
\[
(5.9) \quad \|f\|_{\mathcal{X}_{\theta, B_\eta, E,a,F}^\infty} = \left\| B_\eta(u) \|s^{-\theta}a(s)K(s, f)\| \mathcal{F}(u, \infty) \right\|_{E} \\
\lesssim \left\| \frac{B_\eta(u)}{b_1(t)} \right\| \|b_1(t)\|s^{-\theta}a(s)K(s, f)\| \mathcal{F}(u, \infty) \|_{E} \sim I_2.
\]
Summing up this estimate with (5.3) and (5.8) we complete the proof of a).

In case b) we are assuming the condition \(M_2 < \eta < 1\). By definition of \(M_2\), we have
\[
(1 - \eta)[\rho_{B_{0,\infty}} - \pi_{E_0}] + \eta[\rho_{B_{1,\infty}} - \pi_{E_1}] < 0 < (1 - \eta)[\pi_{B_{0,\infty}} - \rho_{E_0}] + \eta[\pi_{B_{1,\infty}} - \rho_{E_1}]
\]
and then the function \(B_\eta\) satisfies that \(\rho_{B_{1,\infty}} < 0 < \pi_{B_{0,\infty}}\). Applying Lemma 3.6 i) \((\rho_{B_{1,\infty}} < \rho_{E_0} < \pi_{B_{0,\infty}})\) we deduce
\[
(5.10) \quad \|f\|_{\mathcal{X}_{\theta, B_\eta, E,a,F}^\infty} = \left\| B_\eta(u) \|s^{-\theta}a(s)K(s, f)\| \mathcal{F}(u, \infty) \right\|_{E} \\
\lesssim \left\| \frac{B_\eta(u)}{b_0(t)} \right\| \|b_0(t)\|s^{-\theta}a(s)K(s, f)\| \mathcal{F}(t, u) \|_{E} \sim I_1.
\]
Therefore, using (5.5) and (5.8) we show that \(\Upsilon_{\eta,b,E} = \mathcal{X}_{\theta, B_\eta, E,a,F}^\infty\) if \(\eta \in (M_2, 1)\).

Next we proceed with the proof of c). Lemma 3.7 yields that
\[
I_1 \sim \left\| \left( B_\eta(\lambda_k) \right) \|s^{-\theta}a(s)K(s, f)\| \mathcal{F}(\lambda_{k-1}, \lambda_k) \right\|_{k \in \mathbb{Z}} \|d\mathcal{E} \]
and
\[
I_2 \lesssim \left\| \left( B_\eta(\lambda_k) \right) \|s^{-\theta}a(s)K(s, f)\| \mathcal{F}(\lambda_{k-1}, \lambda_k) \right\|_{k \in \mathbb{Z}} \|d\mathcal{E} \|
\]
where $d\hat{E}$ is the discretization of $E$. Then, by (6.1), it follows that
\[ \|f\|_{\mathcal{X}_{\eta,b,E}} \sim \left\| \left( B_\eta(\lambda_k) \|s^{-\theta}a(s)K(s,f)\|_{\tilde{F}(\lambda_{k-1},\lambda_k)} \right)_{k \in \mathbb{Z}} \right\|_{d\hat{E}}.\]

Since $B_\eta(\lambda_k) \sim \psi(\lambda_k)\|b_0\|_{\tilde{E}_0(0,\lambda_k)}$, we deduce
\[ \|f\|_{\mathcal{X}_{\eta,b,E}} \sim \left\| \left( \psi(\lambda_k)\|b_0\|_{\tilde{E}_0(0,\lambda_k)} \|s^{-\theta}a(s)K(s,f)\|_{\tilde{F}(\lambda_{k-1},\lambda_k)} \right)_{k \in \mathbb{Z}} \right\|_{d\hat{E}}.\]

We note that
\[ \|b_0\|_{\tilde{E}_0(0,\lambda_k)} \|s^{-\theta}a(s)K(s,f)\|_{\tilde{F}(\lambda_{k-1},\lambda_k)} \sim \|s^{-\theta}a(s)\|_{\tilde{E}_0(0,s)} \|K(s,f)\|_{\tilde{F}(\lambda_{k-1},\lambda_k)}.\]

Therefore, using Lemmas 3.3 and 3.5 ($\rho_{\psi_0} < 0 < \pi_{\psi_\infty}$) we obtain
\[ \|f\|_{\mathcal{X}_{\eta,b,E}} \sim \left\| \left( \psi(\lambda_k)\|s^{-\theta}a(s)\|_{\tilde{E}_0(0,s)} \|K(s,f)\|_{\tilde{F}(\lambda_{k-1},\lambda_k)} \right)_{k \in \mathbb{Z}} \right\|_{d\hat{E}}.\]

This completes the proof of c).

Finally, we take care of the case $\eta = 0$ and $\eta = 1$. If $\eta = 0$, it is clear that due to (5.7) and (5.9), $I_1 = \|f\|_{\mathcal{X}_{\eta,b,E}}$ while $I_2 \sim \|f\|_{\mathcal{X}_{\eta,b,E}}$. We have to observe that the arguments gave in (5.9) are also true if $\eta = 0$.

If $\eta = 1$, by (5.6) and (5.10) we have that $I_1 \sim \|f\|_{\mathcal{X}_{\eta,b,E}}$. Again we note that (5.10) is true in the case $\eta = 1$. On the other hand, it is a fact that $I_2 = \|f\|_{\mathcal{X}_{\eta,b,E}}$. The proof of the theorem is finished.

\[ \square \]

\section{6. Applications}

For simplicity, we apply our results to ordered (quasi)-Banach couples $\mathcal{X} = (X_0, X_1)$, in the sense that $X_1 \hookrightarrow X_0$. One of the most classical examples of an ordered couple is $(L_1(\Omega, \mu), L_\infty(\Omega, \mu))$ when $\Omega$ is a finite measure space.

\subsection*{6.1. Ordered couples.}

We briefly review how our definitions adapt to this simpler setting of ordered couples. When $X_1 \hookrightarrow X_0$, the real interpolation $\mathcal{X}_{\theta,b,E}$ can be equivalently defined as the space of all $f \in X_0$ such that
\[ \|f\|_{\mathcal{X}_{\theta,b,E}} = \|t^{-\theta}b(t)K(t,f)\|_{\tilde{E}(0,1)} < \infty,\]
where $0 \leq \theta \leq 1$, $E$ is an r.i. space on $(0,1)$ and $b \in SV(0,1)$ (assuming the condition $\|b\|_{\tilde{E}(0,1)} < \infty$ if $\theta = 1$).

Similarly, given a real parameter $0 \leq \theta \leq 1$, $a, b, c \in SV(0,1)$ and r.i. spaces $E, F, G$ on $(0,1)$, the spaces $\mathcal{X}^c_{\theta,b,E,a,F}$, $\mathcal{X}^c_{\theta,b,E,a,F}$ and $\mathcal{X}^c_{\theta,c,E,b,F,a,G}$ are defined just as in
Lemma 6.1. The first embedding is an easy consequence of the fact that \( \tilde{E}(0, \infty) \) must be replaced by \( \tilde{E}(0, 1) \) and \( \tilde{E}(0, \infty) \) by \( \tilde{E}(0, 1) \). Likewise, the spaces \( \tilde{X}_{\theta,b,E,a,F}^{\mathcal{R}} \) and \( \tilde{X}_{\theta,c,E,b,F,a,G}^{\mathcal{L},\mathcal{R}} \) are defined as

\[
\tilde{X}_{\theta,b,E,a,F}^{\mathcal{R}} = \left\{ f \in X_0 : \| b(t) \| s^{-\theta} a(s) K(s,f) \|_{\tilde{F}(t,1)} \|_{\tilde{E}(0,1)} < \infty \right\}
\]

and

\[
\tilde{X}_{\theta,c,E,b,F,a,G}^{\mathcal{L},\mathcal{R}} = \left\{ f \in X_0 : \| c(u) \| b(t) \| s^{-\theta} a(s) K(s,f) \|_{\tilde{G}(u,t)} \|_{\tilde{F}(u,t)} \|_{\tilde{E}(0,1)} < \infty \right\}.
\]

The space \( \tilde{X}_{\theta,b,E,a,F}^{\mathcal{R}} \) can be defined analogously replacing \( \tilde{E} \) by \( \tilde{F} \).

If \( \tilde{X} \) is an ordered couple, then \( (\tilde{X}_{\theta,b_0,E_0,a,F_0}, \tilde{X}_{\theta,b_1,E_1,a,F_1}) \) is also ordered as we proved in the following lemma.

Lemma 6.1. Let \( \tilde{X} \) be an ordered (quasi-) Banach couple, \( E_0, E_1, F \) r.i. spaces on \((0,1)\) and \( a, b_0, b_1 \in SV(0,1) \) such that \( \| b_0 \|_{\tilde{E}_0(0,1)} < \infty \). If \( 0 \leq \theta < 1 \) or \( \theta = 1 \) and \( \| a \|_{\tilde{F}(0,1)} < \infty \), then

\[
\tilde{X}_{\theta,b_1,E_1,a,F} \hookrightarrow \tilde{X}_{\theta,a,F} \hookrightarrow \tilde{X}_{\theta,b_0,E_0,a,F}.
\]

Proof. The first embedding is an easy consequence of the fact that interval \((0,1)\) in (6.1) can be reduce to \((0,1/2)\). Indeed,

\[
\| f \|_{\tilde{X}_{\theta,b_1,E_1,a,F}} \geq \| b_1(t) \| \cdot s^{-\theta} a(s) K(s,f) \|_{\tilde{F}(0,t)} \|_{\tilde{E}_1(1/2,1)} \geq \| b_1 \|_{\tilde{E}_1(1/2,1)} \cdot \| s^{-\theta} a(s) K(s,f) \|_{\tilde{F}(0,1/2)} \sim \| f \|_{\tilde{X}_{\theta,a,F}}.
\]

The second one follows directly using the definition of the norm in the \( \mathcal{R} \)-space.

Of course, the results of the previous sections remain true if we work with slowly varying functions on \((0,1)\), r.i. spaces on \((0,1)\) and ordered couples. In these cases all assumptions concerning the interval \((1, \infty)\) must be omitted. For example, Lemma 3.1 reads as follows:

Lemma 6.2. Let \( E \) be an r.i. space on \((0,1)\) and let \( b \in SV(0,1) \) such that \( b(t^2) \sim b(t) \), with associated function \( B_0 \) defined in (3.1).

i) If \( \rho_{\varphi_E} < \pi_{B_0} \), then

\[
\| b \|_{\tilde{E}(0,t)} \sim b(t) \varphi_E(\ell(t)), \quad t \in (0,1).
\]

ii) If \( \rho_{B_0} < \pi_{\varphi_E} \), then the equivalence is

\[
\| b \|_{\tilde{E}(t,1)} \sim b(t) \varphi_E(\ell(t)), \quad t \in (0,1/2).
\]

That’s reduction to the interval \((0,1/2)\) is not a problem since if \( a \) and \( b \) are two slowly varying functions such that \( b(t) \sim a(t) \) for all \( t \in (0,1/2) \) then \( \tilde{X}_{\theta,b,E} = \tilde{X}_{\theta,a,E} \). A similar identity holds for \( \mathcal{R}, \mathcal{L} \) spaces and for the extreme constructions \( \mathcal{R}, \mathcal{L} \) and \( \mathcal{L}, \mathcal{R} \).

Moreover, if the couple \( \tilde{X} \) is ordered Theorem 5.3 reads as follows:
Theorem 6.3. Let $0 < \theta < 1$. Let $E$, $E_0$, $E_1$, $F$ be r.i. spaces and let $a$, $b$, $b_0$, $b_1 \in SV(0,1)$ be such that $b_0(t) \sim b_0(t^2)$ and $b_1(t) \sim b_1(t^2)$. Assume that $E_0$, $E_1$ and the associated functions $B_{0,0}$, $B_{1,0}$ of $b_0$ and $b_1$, respectively, satisfy that
\[
\rho_{\varphi_{E_0}} < \pi_{B_{0,0}} \quad \text{and} \quad \rho_{B_{1,0}} < \pi_{\varphi_{E_1}}.
\]
Let $0 \leq \eta \leq 1$ be a parameter and define
\[
B_\eta(u) = \left(b_0(u)\varphi_{E_0}(\ell(u))\right)^{1-\eta} \left(b_1(u)\varphi_{E_1}(\ell(u))\right)^\eta \frac{b_0(u)\varphi_{E_0}(\ell(u))}{b_1(u)\varphi_{E_1}(\ell(u))}, \quad u \in (0,1).
\]

a) If $0 \leq \eta < M_1 := \left(1 - \frac{\rho_{B_{1,0}} - \rho_{\varphi_{E_1}}}{\rho_{B_{0,0}} - \rho_{\varphi_{E_0}}}\right)^{-1}$, then
\[
\left(\mathcal{X}_{\theta,b_0,E_0,a,F}, \mathcal{X}_{\theta,b_1,E_1,a,F}\right)_{\eta,b,E} = \mathcal{X}_{\theta,B_\eta,E,a,F}^R.
\]

b) If $M_2 := \left(1 - \frac{\rho_{B_{1,0}} - \rho_{\varphi_{E_1}}}{\rho_{B_{0,0}} - \rho_{\varphi_{E_0}}}\right)^{-1} < \eta \leq 1$, then
\[
\left(\mathcal{X}_{\theta,b_0,E_0,a,F}, \mathcal{X}_{\theta,b_1,E_1,a,F}\right)_{\eta,b,E} = \mathcal{X}_{\theta,B_\eta,E,a,F}^C.
\]

c) If $M_1 \leq \eta \leq M_2$, then
\[
\left(\mathcal{X}_{\theta,b_0,E_0,a,F}, \mathcal{X}_{\theta,b_1,E_1,a,F}\right)_{\eta,b,E} = \mathcal{X}_{\theta,B_\eta,E,a,F}^C
\]
where $B_\eta^c(u) = \frac{B_\eta(u)}{b_0(u)\varphi_{E_0}(\ell(u))}$ and $a^c(u) = a(u)b_0(u)\varphi_{E_0}(\ell(u))$, $u \in (0,1)$.

d) If $\eta = 0$, then
\[
\left(\mathcal{X}_{\theta,b_0,E_0,a,F}, \mathcal{X}_{\theta,b_1,E_1,a,F}\right)_{0,b,E} = \mathcal{X}_{\theta,B_0,E,a,F}^R \cap \mathcal{X}_{\theta,b_0,\phi,E,b_0,E_0,a,F}^R.
\]

where $\phi(u) = \frac{b_0(u)\varphi_{E_0}(\ell(u))}{b_1(u)\varphi_{E_1}(\ell(u))}$, $u \in (0,1)$.

e) If $\eta = 1$ and $\|b\|_{E(0,1)} < \infty$, then
\[
\left(\mathcal{X}_{\theta,b_0,E_0,a,F}, \mathcal{X}_{\theta,b_1,E_1,a,F}\right)_{1,b,E} = \mathcal{X}_{\theta,B_1,E,a,F}^C \cap \mathcal{X}_{\theta,b_0,\phi,E,b_1,E_1,a,F}^R.
\]

where $\phi$ is defined in d).

6.2. Interpolation between grand and small Lebesgue spaces. Here, we present interpolation identities for the grand and small Lebesgue spaces as application of our reiteration theorem.

The identity
\[
(L_1, L_\infty)_1^{1 - \frac{1}{p}, b,E} = L_{p,b,E}
\]
for $1 < p \leq \infty$, follows from Peetre’s formulal for the $K$-functional
\[
K(t, f; L_1, L_\infty) = \int_0^t f^*(s) \, ds = tf^{**}(t), \quad t > 0,
\]
and the equivalence $\|t^{1/p}b(t)f^{**}(t)\|_E \sim \|t^{1/p}b(t)f^*(t)\|_E$, $1 < p \leq \infty$, $b \in SV$ (see, e.g. [7] Lemma 2.16). Analogously, it can be proved that
\[
(L_1, L_\infty)^R_1^{1 - \frac{1}{p}, b,E,a,F} = L_{p,b,E,a,F}^R \quad \text{and} \quad (L_1, L_\infty)^C_1^{1 - \frac{1}{p}, b,E,a,F} = L_{p,b,E,a,F}^C
\]
where
\[ L^R_{p,b,E,a,F} := \left\{ f \in \mathcal{M}(\Omega, \mu) : \left\| b(t) \|s^{1/p}a(s)f^*(s)\| \bar{F}(t,1) \right\|_{E(0,1)} < \infty \right\} \]
and
\[ L^L_{p,b,E,a,F} := \left\{ f \in \mathcal{M}(\Omega, \mu) : \left\| b(t) \|s^{1/p}a(s)f^*(s)\| \bar{F}(t,0) \right\|_{E(0,1)} < \infty \right\} \]
for \( E, F \) r.i. spaces, \( a, b \in SV \) and \( 1 < p \leq \infty \). In a similar vein
\[ (L_1, L_\infty)_{1-\frac{1}{p}, E, b, F, a, G} = L^R_{p,c,E,b,F,a,G} \quad \text{and} \quad (L_1, L_\infty)_{\frac{1}{p}, E, b, F, a, G} = L^L_{p,c,E,b,F,a,G} \]
where the spaces \( L^R_{p,c,E,b,F,a,G} \) and \( L^L_{p,c,E,b,F,a,G} \) are defined as the set of all \( f \in \mathcal{M}(\Omega, \mu) \) for which (1.1)-(1.2) are satisfied after the change of \( \omega a(s)K(s, f) \) for \( s^{1/p}a(s)f^*(s), s > 0 \), respectively.

We follow the paper by Fiorenza and Karadzhov [21] in order to give the next definition:

**Definition 6.4.** Let \( (\Omega, \mu) \) be a finite measure space such that \( \mu(\Omega) = 1 \), let \( 1 < p < \infty \) and \( \alpha > 0 \). The grand Lebesgue space \( L^{p,\alpha} \) is the set of all \( f \in \mathcal{M}(\Omega, \mu) \) such that
\[ \|f\|_{p,\alpha} = \left\| \ell^{\frac{-\alpha}{p}}(t) \|s^{1/p}f^*(s)\| L_p(t,1) \right\|_{L_\infty(0,1)} < \infty. \]
The small Lebesgue space \( L^{p,\alpha}(\Omega) \) is the set of all \( f \in \mathcal{M}(\Omega, \mu) \) such that
\[ \|f\|_{p,\alpha} = \left\| \ell^{\frac{-\alpha}{p}}(t) \|s^{1/p}f^*(s)\| L_p(t,1) \right\|_{L_1(0,1)} < \infty \]
where \( \frac{1}{p} + \frac{1}{p'} = 1 \).

The classical grand Lebesgue space \( L^p(\Omega) := L^{p,1}(\Omega) \) was introduced by Iwaniec and Sbordone in [27] while the classical small Lebesgue space \( L^p(\Omega) := L^{p,1}(\Omega) \) was characterized by Fiorenza in [20] as its associate; that is \( (L^p)' = L^p \). For more information about these spaces and their generalizations see the recent paper [22].

As observed in [21] [30], these spaces can be characterized as \( \mathcal{R} \) and \( \mathcal{L} \)-spaces in the following way
\[ L^{p,\alpha} = L^R_{p,\tilde{\alpha}/p,1,L_p} \quad \text{and} \quad L^{p,\alpha} = L^L_{p,\tilde{\alpha},p-1,u,L_1,L_p}. \]
Thus, we can apply the main theorem of this paper in order to identify the interpolation space \( (L^{p,\alpha}, L^{p,\beta})_{p,b,E} \). Next result recovers Theorem 6.2 form [23], and additionally completes it with the case \( \alpha \neq \beta \) and with the extreme cases \( \eta = 0, 1 \).

**Corollary 6.5.** Let \( E \) be an r.i. space on \((0,1), b \in SV(0,1), 1 < p < \infty \) and \( \alpha, \beta > 0 \).
Let \( 0 \leq \eta \leq 1 \) a parameter and define \( B_\eta(u) = \ell^{-\frac{\alpha(1-\eta)}{p} + \frac{2\eta}{p'}}(u) b(\ell^{\frac{\beta-\alpha}{p'}(u)}, u \in (0,1). \)
The following statements hold:

a) If \( 0 < \eta < \frac{\alpha}{\alpha - \beta + p\beta} \), then
\[ (L^{p,\alpha}, L^{p,\beta})_{\eta,b,E} = L^R_{p,B_\eta,E,1,L_p}. \]
b) If \( \frac{\alpha}{\alpha - \beta + p \delta} < \eta < 1 \), then
\[
(L^p, \alpha, L^{(p, \beta)})_{\eta, b, E} = L^\mathcal{L}_{p, B_\eta^\# L_p, E, 1, L_p}.
\]

c) If \( \eta = \frac{\alpha}{\alpha - \beta + p \delta} \), then
\[
(L^p, \alpha, L^{(p, \beta)})_{\eta, b, E} = L^\mathcal{L}_{p, B_\eta^\# L_p, E, \ell^{-p/\delta - \beta}(u), L_p}
\]
where \( B_\eta^\#(u) = \ell^{\eta \frac{\alpha - \beta}{p} + \beta}(u) \) \( b(\ell^{\frac{\alpha - \beta}{p} - \beta}(u)) \), \( u \in (0, 1) \).

d) If \( \eta = 0 \), then
\[
(L^p, \alpha, L^{(p, \beta)})_{0, b, E} = L^\mathcal{R}_{p, B_0^\# E, 1, L_p} \cap L^\mathcal{L}_{p, b_0 \phi, E, \ell^{-p/\delta - \beta}(u), L_p}
\]
where \( \phi(u) = \ell^{\frac{-\alpha}{p} - \beta}(u) \), \( u \in (0, 1) \).

e) If \( \eta = 1 \) and \( \|b\|_{\ell(0, 1)} < \infty \), then
\[
(L^p, \alpha, L^{(p, \beta)})_{1, b, E} = L^\mathcal{L}_{p, B_1^\# E, 1, L_p} \cap L^\mathcal{L}_{p, b_1 \phi, E, \ell^{p/\delta - 1}(u), L_1, 1, L_p}
\]
where \( \phi \) is defined in d).

Proof. Denote \( b_0(u) = \ell^{-p/\delta}(u) \), \( b_1(u) = \ell^{p/\delta - 1}(u) \), \( u \in (0, 1) \), \( E_0 = L_\infty \) and \( E_1 = L_1 \).
Then, it is clear that \( \pi_{B_0, 0} = \rho_{B_0, 0} = \frac{-\alpha}{p} \), \( \pi_{B_1, 0} = \rho_{B_1, 0} = \frac{\beta}{p} - 1 \), \( \pi_{\phi, E_0} = \rho_{\phi, E_0} = 0 \) and that \( \pi_{\phi, E_0} = \rho_{\phi, E_0} = 1 \). Hence, \( M_1 = M_2 = \frac{\alpha}{\alpha - \beta + p \delta} \). Moreover,
\[
b_0(u) \phi_{E_0}(u) = \ell^{-p/\delta}(u) \quad \text{and} \quad b_1(u) \phi_{E_1}(u) = \ell^{p/\delta}(u), \quad u \in (0, 1).
\]
Thus, the result follows by direct application of Theorem 6.3 \( \square \)

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