Convergence of adaptive algorithms for weakly convex constrained optimization

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Abstract

We analyze the adaptive first order algorithm AMSGrad, for solving a constrained stochastic optimization problem with a weakly convex objective. We prove the $\tilde{O}(t^{-1/4})$ rate of convergence for the norm of the gradient of Moreau envelope, which is the standard stationarity measure for this class of problems. It matches the known rates that adaptive algorithms enjoy for the specific case of unconstrained smooth stochastic optimization. Our analysis works with mini-batch size of 1, constant first and second order moment parameters, and possibly unbounded optimization domains. Finally, we illustrate the applications and extensions of our results to specific problems and algorithms.

1 Introduction

Adaptive first order methods have become a mainstay of neural network training in recent years. Most of these methods build on the AdaGrad framework [12], which is a modification of online gradient descent by incorporating the sum of the squared gradients in the step size rule. Based on the practical shortcomings of AdaGrad for training neural networks, RMSprop [26] and Adam [17] proposed to use exponential moving averages for gradients and squared gradients with parameters $\beta_1$ and $\beta_2$, respectively. These methods have seen a huge practical success.

The recent work [24] identified a technical issue that affects Adam and RMSprop and proposed a new Adam-variant called AMSGrad that does not suffer from the same problem. Theoretical properties of AMSGrad, AdaGrad and their variants for nonconvex optimization problems are studied in a number of recent papers [3, 4, 9, 20, 28, 31]. These works focus on unconstrained smooth stochastic optimization, where the standard analysis framework of the stochastic gradient descent (SGD) [14] can be used. Convergence of adaptive methods for the more general setting of constrained and/or nonsmooth stochastic nonconvex optimization has remained open, while these settings have broad practical applications [6, 11, 16, 21, 23, 27].

In this work, we take a step towards this direction and establish the convergence of AMSGrad for solving the problem

$$\min_{x \in X} \{ f(x) = \mathbb{E}_\xi [f(x; \xi)] \},$$

where $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is $\rho$-weakly convex, $X \subset \mathbb{R}^d$ is a closed convex set, and $\xi$ is a random variable following a fixed unknown distribution. This template captures the setting of previous analyses when $f$ is $L$-smooth, as this implies $L$-weak convexity, and $X = \mathbb{R}^d$. On the other hand, there exist many applications when $X \neq \mathbb{R}^d$ [16, 21, 23, 27] or when $f$ is not $L$-smooth [6, Section 2.1],[11, 13].

It is well known that constrained stochastic optimization with nonconvex functions presents challenges not met in the convex setting [5, 15]. In particular, until the recent work of Davis and Drusvyatskiy [6], even for SGD, increasing mini-batch sizes were required for convergence in constrained optimization. To study the behavior of AMSGrad for solving (1), we build on the analysis framework of [6].
Contributions. We show that AMSGrad achieves $O(\log(T)/\sqrt{T})$ rate for near-stationarity, see (8), for solving (1). Key specifications for this result are the following:

- We can use a mini-batch size of 1.
- We can use constant moment parameters $\beta_1, \beta_2$ which are used in practice [1, 4, 17, 24].
- We do not assume boundedness of the domain $\mathcal{X}$.

We present particular cases of our results for constrained optimization with $L$-smooth objectives and for a variant of RMSprop. We also extend our analysis for the scalar version of AdaGrad with first order momentum. For easy reference, we compare our results with state-of-the-art in Table 1.

1.1 Related work

Adaptive algorithms based on AdaGrad [12] and Adam [1, 17, 24] are classically analyzed in online optimization framework with convex objective functions. Recent works studied the behavior of these methods for nonconvex optimization [2, 3, 4, 9, 20, 28, 30, 31]. The common characteristic of these results is that they are based on the well established proof templates of SGD [14] that only works in the simplest case of unconstrained smooth stochastic minimization. Moreover, as mentioned in [1], unconstrained optimization makes it easier to use a constant $\beta_1$ parameter in Adam-type methods. In particular, many results for constrained optimization require a fast diminishing schedule for $\beta_1$ parameter, while a constant parameter is used in practice [5, 17, 24].

The specific case of (1) with $L$-smooth $f$ is studied by Chen et al. [5], where the authors proposed a zeroth order variant of AMSGrad. This result applies for the specific case of $\beta_1 = 0$ which corresponds to a variant of RMSprop [24, 26]. More importantly, since its analysis follows the one of Ghadimi et al. [15], increasing mini-batch sizes are required [5, Theorem 2].

As also mentioned in [5, 6], analysis of SGD for constrained problems introduces specific difficulties that are not observed in the convex case. Due to this, classical works analyzing SGD for nonconvex constrained optimization used large mini-batches to ensure convergence [15]. Showing convergence for SGD for constrained optimization with a single sample had been an open question until Davis and Drusvyatskiy [6] gave a positive answer in the framework of weakly convex stochastic optimization, which includes constrained smooth stochastic optimization as a special case.

Weakly convex optimization is well studied with SGD based methods [6, 7, 13]. A recent work by Mai and Johansson [22], considers momentum SGD for problem (1). However, this algorithm (i) does not use momentum with $\beta_2$ and (ii) uses a scalar stepsize with $\hat{v}_t = 1$ in the notation of Algorithm 1 and $\alpha_t = \alpha/\sqrt{T}$. These make the algorithm less practical, while simpler for analysis.

Another promising direction of research concerns nonsmooth nonconvex problems under more general assumptions. For instance, tameness and Hadamard semi-differentiability are used in [8] and [29], respectively, where convergence guarantees are established for SGD-based methods. Because of the generality of the problem class in these works, the algorithms studied there are simpler than the Adam-type algorithms considered in this paper, and the stationarity measures are less standard [29].

1.2 Notation

We adopt the convention of using the standard operations $ab$, $a^2$, $a/b$, $a^{1/2}$, $1/a$, $\max\{a, b\}$ as element-wise, given two vectors $a, b \in \mathbb{R}^d$. To denote $i$th element of the vector $a_t \in \mathbb{R}^d$, we use
We now present the assumptions of our analysis.

\[ f(X) \text{ weighted projection operator onto } \mathcal{X} \]

A standard property of the weighted projection is that \( \forall \|\cdot\\) expectation, it directly follows that \( E(x) \text{ for weakly convex functions is that, } \forall x \in \mathcal{X} \) and \( I_x(x) = +\infty \) otherwise.

Given the elements \( v_i > 0, i = 1, \ldots, d \), we define a weighted norm \( \|x\|^2 := \langle x, \text{diag}(v)x \rangle \). The weighted projection operator onto \( \mathcal{X} \) is defined as

\[
P_{\mathcal{X}}(x) = \arg\min_{y \in \mathcal{X}} \|y - x\|^2.
\]

A standard property of the weighted projection is that \( \forall x, y \in \mathbb{R}^d, P_{\mathcal{X}}(x) \) is nonexpansive:

\[
\|P_{\mathcal{X}}(y) - P_{\mathcal{X}}(x)\| \leq \|y - x\|.
\]

Due to nonconvexity, we cannot use standard definition of subgradients to form a global under-estimator. \textit{Regular subdifferential}, denoted as \( \partial f \), for nonconvex functions [25, Ch. 8] is defined as the set of vectors \( q \in \mathbb{R}^d \) such that, \( \forall x, y \in \mathbb{R}^d, q \in \partial f(x) \) if

\[
f(y) \geq f(x) + \langle y - x, q \rangle + o(\|y - x\|), \quad \text{as } y \to x.
\]

When \( f \) is convex, this reduces to standard definition of a subdifferential and when \( f \) is differentiable, this set coincides with \( \{ \nabla f(x) \} \).

We say that \( f \) is \( \rho \)-weakly convex w.r.t. \( \|\cdot\| \), if \( f(x) + \frac{\rho}{2} \|x\|^2 \) is convex. An equivalent representation for weakly convex functions is that, \( \forall x, y \in \mathbb{R}^d \), where \( q \in \partial f(x) \) [6, Lemma 2.1],

\[
f(y) \geq f(x) + \langle y - x, q \rangle - \frac{\rho}{2}\|y - x\|^2.
\]

Moreover, we say \( f \) is \( L \)-smooth, if it holds that, \( \forall x, y \in \mathbb{R}^d \)

\[
\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|.
\]

Given random iterates \( x_1, \ldots, x_t \), we denote the filtration generated by these realizations as \( \mathcal{F}_t = \sigma(x_1, \ldots, x_t) \), and the corresponding conditional expectation as \( \mathbb{E}_t[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_t] \). By the law of total expectation, it directly follows that \( \mathbb{E}[\mathbb{E}_t[\cdot]] = \mathbb{E}[\cdot] \).

We now present the assumptions of our analysis.

**Assumption 1.**
- \( f : \mathbb{R}^d \to \mathbb{R} \) is \( \rho \)-weakly convex with respect to norm \( \|\cdot\| \).
- The set \( \mathcal{X} \subset \mathbb{R}^d \) is convex and closed.
- There exists \( g_t \in \partial f(x_t, \xi_t) \) such that \( \|g_t\|_\infty \leq G, \forall t \).
- \( f \) is lower bounded: \( f^* \leq f(x), \forall x \in \mathcal{X} \).

**Remark 1.** We note that when \( f \) is \( \rho \)-weakly convex w.r.t. \( \|\cdot\| \), then it is \( \frac{\rho}{\sqrt{\delta}} \)-weakly convex w.r.t. \( \|\cdot\|_{1/2} \), \( \forall t \), since \( \psi_{t,i} \geq \delta \) (see Algorithm 1). We denote \( \hat{\rho} = \frac{\rho}{\sqrt{\delta}} \).

It is easy to verify this remark by noticing that \( x \mapsto f(x) + \frac{\hat{\rho}}{2}\|x\|^2 \) is convex and \( \frac{\hat{\rho}}{2}\|x\|^2_{1/2} \geq \frac{\rho}{2}\|x\|^2_{1/2} \geq \frac{\rho}{2}\|x\|^2 \).

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**Algorithm 1 AMSGrad [24]**

**Input:** \( x_1 \in \mathcal{X}, \alpha_t = \frac{\alpha}{\sqrt{t}}, \text{ for } t \geq 1, \alpha > 0, \beta_1 < 1, \beta_2 < 1, \)

\( m_0 = v_0 = 0, \delta_0 = \delta I, 1 \geq \delta > 0. \)

**for** \( t = 1, 2, \ldots T \) **do**

\( g_t \in \partial f(x_t, \xi_t) \)

\( m_t = \beta_1 m_{t-1} + (1 - \beta_1)g_t \)

\( v_t = \beta_2 v_{t-1} + (1 - \beta_2)g_t^2 \)

\( \hat{v}_t = \max(v_{t-1}, v_t) \)

\( x_{t+1} = P_{\mathcal{X}}(\frac{\hat{v}_t^{1/2}}{v_t^{1/2}} (x_t - \alpha_t \hat{v}_t^{-1/2} m_t) \)

**end for**

**Output:** \( x_{t*}, \text{ where } t^* \text{ is selected uniformly at random from } \{1, \ldots, T\}. \)
A few remarks are in order for Assumption 1. First, we do not require boundedness of the domain $X$. Second, weak convexity assumption is weaker than smoothness assumption on $f$ and the assumption of bounded gradients is standard [3, 4, 9]. In principle, it is possible to relax the bounded gradient assumption to the weaker requirement $E\|g_t\|^2 \leq G$ as in [31, Remark 6. (ii)] with a slightly worse and complicated convergence rate. Thus, for clarity, we stick with Assumption 1.

2 Algorithm and preliminaries

We analyze the algorithm AMSGrad proposed in [24]. On top of Adam [17], it includes a step to ensure monotonicity of the exponential average of squared gradients. It is standard in stochastic nonconvex optimization to output a randomly selected iterate [6, 14, 15], which we also adopt.

We next define the composite objective
\[
\varphi(x) = f(x) + I_X(x).
\]

For nonsmooth problems, the standard stationarity measures such as the norm of subgradients are no longer applicable, see [6, 22] and [11, Section 4]. This motivates the following definitions that, as we show below, relate to a relaxed form of stationarity. Based on $\varphi$ and a parameter $\bar{\rho} > 0$, we define the proximal point of $x_t$ and the Moreau envelope
\[
\hat{x}_t = \text{prox}_{\varphi/\bar{\rho}}(x_t) = \text{argmin}_y \left\{ \varphi(y) + \frac{\bar{\rho}}{2}\|y - x_t\|^2_{\hat{v}_t/2} \right\},
\]
\[
\varphi_{1/\bar{\rho}}(x_t) = \min_y \left\{ \varphi(y) + \frac{\bar{\rho}}{2}\|y - x_t\|^2_{\hat{v}_t/2} \right\}.
\]

We compare the definitions with that of Davis and Drusvyatskiy [6]. Due to the use of variable metric $v_t$ in adaptive methods, we have a time dependent Moreau envelope, where the corresponding vector $\hat{v}_t$ is used for defining the norm. Important considerations for these quantities are the uniqueness of $\hat{x}_t$ and the smoothness of $\varphi_{1/\bar{\rho}}$. As we shall see now, choice of $\bar{\rho}$ is critical for ensuring these.

In light of Remark 1, selecting $\bar{\rho} > \hat{\rho} = \frac{G}{\sqrt{3}}$, and by using similar arguments as in [6, Lemma 2.2], it follows that $\hat{x}_t$ is unique and $\varphi_{1/\bar{\rho}}$ is smooth with the gradient
\[
\nabla \varphi_{1/\bar{\rho}}(x_t) = \bar{\rho} \hat{v}_t/2 (x_t - \hat{x}_t).
\]

Near stationarity. Near-stationarity conditions follow from the optimality condition of $\hat{x}_t$: $0 \in \partial \varphi(\hat{x}_t) + \bar{\rho} \hat{v}_t/2 (\hat{x}_t - x_t)$, where we have used $\hat{v}_{t,i} \leq G^2$:
\[
\left\{ \begin{array}{l}
\|x_t - \hat{x}_t\|^2_{\hat{v}_t/2} = \frac{1}{\bar{\rho}} \|\nabla \varphi_{1/\bar{\rho}}(x_t)\|^2_{\hat{v}_t/2} \\
\text{dist}^2(0, \partial \varphi(\hat{x}_t)) \leq G \|\nabla \varphi_{1/\bar{\rho}}(x_t)\|^2_{\hat{v}_t/2} \\
\varphi(\hat{x}_t)
\end{array} \right.
\]
(8)

Consistent with previous literature [6, 22], we will state the convergence guarantees in terms of the norm of the gradient of Moreau envelope. Given (8), this means that the iterate $x_t$ is close to its proximal point $\hat{x}_t$ and $\hat{x}_t$ is an approximate stationary point.

3 Convergence analysis

3.1 Preliminary results

We start with a result showing that under Assumption 1, the quantity $\|x_t - \hat{x}_t\|$ from (8) stays bounded. This is the main reason why we do not need to assume boundedness of $X$. The proof of this lemma given in Appendix A combines the definition of $\hat{x}_t$ with weak convexity to reach the result.

Lemma 1. Let Assumption 1 hold. Let $\bar{\rho} > \hat{\rho}$, and $\hat{v}_t \geq \delta > 0$ (see Algorithm 1). It follows that
\[
\|x_t - \hat{x}_t\|^2 \leq \hat{D}^2 := \frac{4\delta G^2}{\delta (\rho - \hat{\rho})^2},
\]
A key challenge in the analysis of adaptive algorithms is the dependence of \( \hat{v}_t \) and \( g_t \) that couples \( \hat{x}_t \) and \( g_t \) (see (7)), preventing taking expectation of \( \langle x_t - \hat{x}_t, g_t \rangle \) that we use for obtaining the stationarity measure in the proof. Since this was not the case in [6], we need a more refined analysis.

**Lemma 2.** Let Assumption 1 hold. Let \( q_t = E_t[g_t] \in \partial f(x_t) \), then it follows that

\[
\alpha_t E_t(x_t - \hat{x}_t, g_t) \geq \alpha_t (\bar{\rho} - \hat{\rho}) E_t ||x_t - \hat{x}_t||_{\hat{v}_{t-1}}^2 \left( \alpha_{t-1} - \alpha_t \right) \sqrt{\bar{a}DG} - \frac{\bar{\rho} - \hat{\rho}}{4\hat{\rho}} E_t ||\hat{x}_t - \hat{x}_{t-1}||_{\hat{v}_{t-1}}^2 \right.
- \frac{\alpha_{t-1}}{2} E_t ||m_{t-1}||_{\hat{v}_{t-1}}^2 - \left( \frac{1}{2} + \frac{\bar{\rho}}{\rho - \hat{\rho}} \right) \alpha_{t-1} E_t ||g_t||^2.
\]

We review the terms in this bound to gain some intuition. The first term in the RHS is the stationarity measure (see (8)), second term will sum to a constant, fourth and fifth terms will sum to \( \log(T) \) by Lemma 4. Handling the third term in RHS is not as obvious, but we can show that we can cancel it using the contribution from another part of the analysis that we detail in the full proof (see (12)).

As alluded earlier, one critical issue for Adam-type algorithms is to obtain results with constant \( \beta_1 \) parameter. A recent paper [1] studied this problem for constrained convex problems. The following lemma from [1] also plays an important role in our analysis.

**Lemma 3.** [1, Lemma 1] Let \( m_t = \beta_1 m_{t-1} + (1 - \beta_1)g_t \). Then for any vectors \( A_{t-1}, A_t \), we have

\[
\langle A_t, g_t \rangle = \frac{1}{1 - \beta_1} \left( \langle A_t, m_t \rangle - \langle A_{t-1}, m_{t-1} \rangle \right) + \langle A_{t-1}, m_{t-1} \rangle + \frac{\beta_1}{1 - \beta_1} \langle A_{t-1} - A_t, m_{t-1} \rangle.
\]

This lemma derives a decomposition for handling \( \beta_1 \) parameter in the beginning of the analysis. As explained in [1, Section 3.1], using a decomposition for \( m_t \) later in the analysis results in a requirement of decreasing \( \beta_1 \), especially for constrained problems, which we would like to avoid.

Next lemma is a standard estimation used for the analysis of Adam-based methods, dating back to [17]. For easy reference we point out to [1] where this bound is included as a separate lemma with tighter estimations than previous works, due to using a constant \( \beta_1 \). It bounds the sum of the norms of first moment vectors multiplied by the step size sequence.

**Lemma 4.** Let \( \beta_1 < 1, \beta_2 < 1, \gamma = \frac{\beta_1^2}{\beta_2} < 1 \), then it holds that

\[
\sum_{t=1}^{T} \alpha_t^2 ||m_t||_{\hat{v}_{t-1}}^2 \leq \frac{1 - \beta_1}{\sqrt{(1 - \beta_2)(1 - \gamma)}} dG(1 + \log T).
\]

### 3.2 Main result

Equipped with the preliminary results from the previous section, we proceed to our main theorem that shows that the norm of the gradient of Moreau envelope converges to 0 at the claimed rate, resulting in near-stationarity of \( x_t \), as in (8).

**Theorem 1.** Let Assumption 1 hold. Let \( \beta_1 < 1, \beta_2 < 1, \gamma = \frac{\beta_1^2}{\beta_2} < 1, \bar{\rho} = 2\hat{\rho} \). Then, for iterate \( x_t \), generated by Algorithm 1, it follows that

\[
E ||\nabla f_{1/\rho}^t(x_t^*)||_{\hat{v}_{t-1}}^2 \leq \frac{2}{\alpha \sqrt{T}} \left[ C_1 + (1 + \log T)C_2 + C_3 \right],
\]

where \( C_1 = \frac{4\rho \beta_1}{\sqrt{\rho(1 - \beta_1)}} \sqrt{\bar{a}DG} + \varphi_{1/\rho}^t(x_1) - f^* \),

\[
C_2 = \frac{5\rho}{6} dG^2 + \frac{2\rho}{\sqrt{T}} \left( 1 + \frac{\alpha}{\sqrt{\rho}} + \frac{\beta_1}{1 - \beta_1} + \frac{2\beta_1^2}{(1 - \beta_1)^2} \right) \frac{1 - \beta_1}{\sqrt{(1 - \beta_2)(1 - \gamma)}} dG,
\]

\[
C_3 = \frac{2\rho}{\sqrt{T}} \sum_{t=1}^{d} E \hat{\delta}_{t+1}^{1/2}, \text{ and } \hat{\delta} := \frac{2\sqrt{T}G}{\rho}.
\]

We delay the discussion about the result to Section 3.3 and continue with the proof sketch of the theorem, which is a careful combination of the preliminary results mentioned in the previous section. The sketch includes the necessary bounds, but omits the tedious estimations required in some steps. The full proof with the details is given in Appendix A.
Proof sketch. We sum the result of Lemma 3 and use $A_1 = A_0$, with $m_0 = 0$. We note that we have $A_t = \tilde{\rho} \alpha_t (x_t - \hat{x}_t)$, for $t \geq 1$.
\[
\sum_{t=1}^{T} \langle A_t, g_t \rangle = \frac{\beta_1}{1 - \beta_1} \langle A_T, m_T \rangle + \sum_{t=1}^{T} \langle A_t, m_t \rangle + \frac{\beta_1}{1 - \beta_1} \sum_{t=1}^{T-1} \langle A_t - A_{t+1}, m_t \rangle.
\] (9)

After plugging in the value of $A_t$, (9) becomes
\[
\sum_{t=1}^{T} \tilde{\rho} \alpha_t \langle x_t - \hat{x}_t, g_t \rangle \leq \frac{\beta_1 \tilde{\rho} \alpha_T}{1 - \beta_1} \langle x_T - \hat{x}_T, m_T \rangle + \sum_{t=1}^{T} \tilde{\rho} \alpha_t \langle x_t - \hat{x}_t, m_t \rangle
\]
\[
+ \frac{\beta_1 \tilde{\rho}}{1 - \beta_1} \sum_{t=1}^{T-1} \langle \alpha_t (x_t - \hat{x}_t) - \alpha_{t+1} (x_{t+1} - \hat{x}_{t+1}), m_t \rangle.
\] (10)

LHS of this bound is suitable for applying Lemma 2 to obtain the stationarity measure. We have to estimate the three terms on the RHS. It is easy to bound the first term using Cauchy-Schwarz inequality and Lemma 1. Other two terms require longer estimations which we sketch below.

- **Bound for** $\frac{\beta_1}{1 - \beta_1} \sum_{t=1}^{T-1} \langle \alpha_t (x_t - \hat{x}_t) - \alpha_{t+1} (x_{t+1} - \hat{x}_{t+1}), m_t \rangle$ in (10).

Decomposing this term gives
\[
\langle \alpha_t (x_t - \hat{x}_t) - \alpha_{t+1} (x_{t+1} - \hat{x}_{t+1}), m_t \rangle = \langle \alpha_t - \alpha_{t+1} \rangle \langle x_t - \hat{x}_{t+1}, m_t \rangle + \alpha_t \langle x_t - x_{t+1}, m_t \rangle + \alpha_t \langle \hat{x}_{t+1} - \hat{x}_t, m_t \rangle.
\]

For the first term, we use that $\alpha_t \geq \alpha_{t+1}$ and Cauchy-Schwarz inequality
\[
\sum_{t=1}^{T-1} \langle \alpha_t - \alpha_{t+1} \rangle \langle x_t - \hat{x}_{t+1}, m_t \rangle \leq \sum_{t=1}^{T-1} (\alpha_t - \alpha_{t+1}) \bar{D} \sqrt{\bar{d}} G \leq \alpha_1 \bar{D} \sqrt{\bar{d}} G.
\]

For the second term we deduce by Cauchy-Schwarz inequality and nonexpansiveness of the projection $\alpha_t \langle x_t - x_{t+1}, m_t \rangle \leq \alpha_t^2 \|m_t\|_{v_{t+1}}^2$.

For the third term, we use Young’s inequality to obtain the bound
\[
\sum_{t=1}^{T-1} \frac{\beta_1 \tilde{\rho}}{1 - \beta_1} \langle \alpha_t (x_t - \hat{x}_t) - \alpha_{t+1} (x_{t+1} - \hat{x}_{t+1}), m_t \rangle \leq \frac{\beta_1 \tilde{\rho}}{1 - \beta_1} \alpha_1 \bar{D} \sqrt{\bar{d}} G + \sum_{t=1}^{T} \frac{\beta_1 \tilde{\rho} \alpha_t^2}{1 - \beta_1} \|m_t\|_{v_{t+1}}^2
\]
\[
+ \sum_{t=1}^{T} \frac{\tilde{\rho} - \tilde{\rho}}{4} \|x_t - \hat{x}_{t+1}\|_{v_{t+1}}^2 + \frac{\tilde{\rho}^2}{(\tilde{\rho} - \tilde{\rho}) (1 - \beta_1)^2} \sum_{t=1}^{T} \alpha_t^2 \|m_t\|_{v_{t+1}}^2,
\] (11)

- **Bound for** $\sum_{t=1}^{T} \tilde{\rho} \alpha_t \langle x_t - \hat{x}_t, m_t \rangle$ in (10).

We proceed similar to [6], with a tighter estimation (resulting in the negative term on RHS) to obtain
\[
\varphi_{1/\tilde{\rho}}^{t+1}(x_{t+1}) \leq \varphi_{1/\tilde{\rho}}^t(x_t) + \tilde{\rho} \alpha_t \langle \hat{x}_t - x_t, m_t \rangle + \frac{\tilde{\rho}}{2} \alpha_t^2 \|m_t\|_{v_{t+1}}^2
\]
\[
+ \frac{\tilde{\rho}}{2} \|x_t - x_{t+1}\|_{v_{t+1}}^2 - \frac{\tilde{\rho} - \tilde{\rho}}{2} \|\hat{x}_t - \hat{x}_{t+1}\|_{v_{t+1}}^2.
\] (12)

Then we manipulate the fourth term on RHS with standard $\|a - b\|^2 \leq 2 \|a\|^2 + 2 \|b\|^2$, and Lemma 1,
\[
\frac{\tilde{\rho}}{2} \|x_t - x_{t+1}\|_{v_{t+1}}^2 - \frac{\tilde{\rho} - \tilde{\rho}}{2} \|\hat{x}_t - \hat{x}_{t+1}\|_{v_{t+1}}^2 \leq \tilde{\rho} \|\hat{x}_t - x_t\|_{v_{t+1}}^2 + \frac{G \tilde{\rho}}{\sqrt{\delta}} \|x_t - x_{t+1}\|_{v_{t+1}}^2
\]
\[
\leq \tilde{\rho} \bar{D}^2 \sum_{i=1}^{d} \langle \hat{v}_{t+1, i}^{1/2} - \hat{v}_{t, i}^{1/2} \rangle + \frac{G \tilde{\rho}}{\sqrt{\delta}} \alpha_t^2 \|m_t\|_{v_{t+1}}^2.
\] (13)

We use this estimation in (12) and sum to get
\[
\tilde{\rho} \alpha_t \sum_{t=1}^{T} \langle x_t - \hat{x}_t, m_t \rangle \leq \varphi_{1/\tilde{\rho}}^1(x_1) - \varphi_{1/\tilde{\rho}}^{T+1}(x_T) + \sum_{t=1}^{T} \left( \frac{1}{2} + \frac{G \tilde{\rho}}{\sqrt{\delta}} \right) \tilde{\rho} \alpha_t^2 \|m_t\|_{v_{t+1}}^2
\]
\[
+ \tilde{\rho} \bar{D}^2 \sum_{i=1}^{d} \hat{v}_{T+1, i}^{1/2} - \sum_{t=1}^{T} \frac{\tilde{\rho} - \tilde{\rho}}{2} \|\hat{x}_t - \hat{x}_{t+1}\|_{v_{t+1}}^2.
\] (14)
We collect (11) and (14) in (10). Finally, we have to obtain the stationarity criterion on the LHS of (10) by taking conditional expectation. This is not immediate due to coupling of  \( \hat{x}_t, \hat{v}_t, \) and \( g_t \). We use Lemma 2 to handle this issue and the negative term in (14) is utilized to cancel the third term in the RHS of the result of Lemma 2. Then, we use (8), plug in Lemma 4 and \( \hat{p} = 2\hat{p} \) to conclude.

### 3.3 Discussion

In the context of near-stationarity (8), Theorem 1 states that to have \( x_{t^*} \) in Algorithm 1 such that \( \|\nabla \varphi_{1/\rho}^*(x_{t^*})\|_{\hat{v}_{t^*}^{-1/2}} \leq \epsilon \), we require \( \tilde{O}(\epsilon^4) \) iterations. This matches the known complexities for adaptive methods in unconstrained smooth stochastic optimization [1, 3, 4, 9, 20, 28, 30, 31], and SGD-type methods in weakly convex optimization [6, 22].

Our first remark is about the metric of the norm used for the gradient of the Moreau envelope in Theorem 1. We then continue to discuss the dependence of our bound w.r.t. important quantities.

**Remark 2.** By (8), one has \( \|\nabla \varphi_{1/\rho}^*(x_t)\|_{\hat{v}_{t}^{-1/2}}^2 = \tilde{\rho}^2\|x_t - \hat{x}_t\|_{\hat{v}_{t}^{-1/2}}^2 \). We note that \( \|x_t - \hat{x}_t\|_{\hat{v}_{t}^{-1/2}} \geq \sqrt{\delta}\|x_t - \hat{x}_t\|_{\hat{v}_{t}^{-1/2}} \) as \( \hat{v}_{t,i} \geq \delta \). It also holds that \( \hat{v}_{t,i} \leq G^2 \). Therefore, one can convert our guarantees to \( \|x_{t^*} - \hat{x}_{t^*}\|_{\hat{v}_{t^*}^{-1/2}} \) or \( \|\nabla \varphi_{1/\rho}^*(x_{t^*})\| \) by multiplying the right hand side by appropriate quantities depending on \( \delta \) or \( G \). We leave the result with the metric however, as \( \delta \) and \( G \) are the worst case bounds.

**Dependence of \( \beta_1 \).** Comparing with the previous work, the scaling of our bound in terms of \( \beta_1 \) is \((1 - \beta_1)^{-1}\) matching the state-of-the-art dependence for the unconstrained setting [1, 9].

**Dependence of \( d \).** Standard dependence of \( d \) in the convergence rates for Adam-type algorithms for unconstrained case is \( d/\sqrt{T} \) [1, 9].

Even though in Theorem 1, the constant \( C_3 \) has worst case dependence \( d^2 \), this is merely due to assumptions. The main reason is that we do not assume boundedness of the sequence \( x_t \), instead we prove the necessary result for the analysis in Lemma 1. However, this result gives a bound for \( \|x_t - \hat{x}_t\| \), which is naturally dimension dependent. We used this bound in (13), where we need to use \( \|x_t - \hat{x}_t\|_\infty \). If we had assumed a bound for \( \|x_t - \hat{x}_t\|_\infty \), then in (13) we could have used it instead of Lemma 1 to have standard \( d/\sqrt{T} \) in \( C_3 \). We note that boundedness assumption also would remove a factor of \( \frac{1}{\sqrt{\delta}} \) in the bound, as those appear in the steps where we avoid boundedness assumption.

**Dependence of \( \delta \).** Our bound has a polynomial dependence of \( 1/\delta \) similar to [1, 3, 4]. In [9], a more refined technique from [28] is used to have a logarithmic dependence of \( 1/\delta \). This technique, used on the case of smooth unconstrained problems in these works, did not seem to apply to our setting.

### 4 Applications and extensions

#### 4.1 Applications

**RMSprop.** The counterexamples presented in [24] show that RMSprop, similar to Adam might diverge in simple problems. Setting \( \beta_1 = 0 \) in AMSGrad [24] results in an algorithm similar to RMSprop, with the difference of having \( \hat{v}_t \) as the output of the max step. Therefore, our analysis also applies to this version of RMSprop with similar guarantees.

**Corollary 1.** Let \( \beta_1 = 0 \). Then, for a variant of RMSprop [24], obtained by setting \( \beta_1 \) in Algorithm 1, Theorem 1 applies with \( \beta_1 = 0 \).

It is easy to see that \( \beta_1 = 0 \) gives a better bound in Theorem 1. This is in fact common for the bounds of Adam-type algorithms even in the convex case [24]. Setting nonzero momentum parameters \( \beta_1, \beta_2 \) do not predict improvement, however, in practice they are routinely observed to improve performance.

**SGD with momentum.** When \( \hat{v}_t = 1, \forall t \), then AMSGrad reduces to an algorithm similar to SGD with momentum. Lack of diagonal step sizes in this case simplifies the analysis as weighted

\footnote{We note that in [3] better dependence is obtained by using step sizes in the order of \( \frac{1}{\sqrt{\delta}} \), which we do not consider, as this choice forces small step sizes.}
projections are not used in the algorithm. This specific case is studied in the recent work [22], with a slightly different way to set $m_t$. Our analysis can be seen as an alternative derivation of convergence for a method similar to [22].

**Constrained smooth optimization.** A special case of (1) is when $f$ is $L$-smooth. In this case, the standard convergence measure is the gradient mapping [15], which is used in [5]

$$G(x) = \frac{\hat{v}_t^{1/4}}{\lambda} \left( x - P_X^{\hat{v}_t^{1/2}}(x - \lambda\hat{v}_t^{-1/2}\nabla f(x)) \right). \quad (15)$$

It is instructive to observe that when $X = \mathbb{R}^d$, then $\|G(x)\| = \|\nabla f(x)\|_{\hat{v}_t^{-1/2}} \geq \frac{1}{C} \|\nabla f(x)\|$ which is the stationarity measure for smooth unconstrained problems. In the cases when $X \neq \mathbb{R}^d$, gradient mapping is used as a standard stationarity measure [6, 15, 22].

As illustrated in [6], for the specific case of constrained smooth minimization, norm of the Moreau envelope is of the same order as the norm of the gradient mapping, therefore, the results can be converted to guarantees on gradient mapping norms. Using similar ideas as in [10, Theorem 3.5], [6], one can show that $\|G_{1/\rho}(x_t)\| \leq C_{g,m}\|\nabla \varphi_{1/\rho}^{\beta}(x_t)\|_{\hat{v}_t^{-1/2}}$, for a constant $C_{g,m}$ (see Appendix B).

### 4.2 An extension: Scalar AdaGrad with momentum

An alternative adaptive algorithm is AdaGrad [12] and its variants with first order momentum are referred to as AdamNC [24] or AdaFOM [4]. In unconstrained smooth stochastic optimization, it has been observed that the same proof structure applies to AMSGrad and AdaGrad-based methods simultaneously [4, 9]. However, in our setting, the analysis we developed for AMSGrad does not directly apply to AdaGrad-based methods.

The main reason is that $v_t$ in the case of AdaGrad does not admit a lower bound separated from 0, unlike AMSGrad where $0 < \delta \leq \hat{v}_t$. The uniform lower bound is necessary for converting regular weak convexity assumption w.r.t. norm $\|\cdot\|$ to the one w.r.t. the weighted norm $\|\cdot\|_{v_t^{1/2}}$ in the sense of Remark 1. Naively assuming the existence of $\hat{\rho}$ in Remark 1 is not consistent, since $v_t$ is not separated from zero due to $\hat{v}_t \geq \frac{\rho}{\sqrt{t}}$ in AdaGrad, and hence, the norm $\|\cdot\|_{v_t^{1/2}}$ is not well-defined.

In this section, we provide partial results on this direction. In particular, we show that the scalar version of AdaGrad, that is used for example in [18, 19, 20, 28], along with its variant with first order momentum estimation also has the same convergence rate. In the framework of Algorithm 1, scalar (non-diagonal) version of these methods iterate as, for $g_t \in \partial f(x_t, \xi_t),$

$$\begin{cases}
m_t = \beta_1 m_{t-1} + (1 - \beta_1) g_t \\
v_t = \frac{1}{\delta + \frac{1}{2} \sum_{j=1}^t \|g_j\|^2} \\
x_{t+1} = P_X(x_t - \frac{\alpha_t}{\sqrt{v_t}} m_t),
\end{cases} \quad (16)$$

where $P_X$ denotes a standard Euclidean projection (without any metric). The factor of $1/d$ in front of gradient norms is to normalize the step size, as $\ell_2$-norm is dimension dependent. This factor only affects the dimension dependence of the bound.

In this case, one does not need the time-dependent definitions for Moreau envelope and proximal point. Thus, one can define $\hat{x}_t = \text{prox}_{1/\hat{\rho}}(x_t)$ and $\varphi_{1/\hat{\rho}}(x) = \min_{y \in X} f(y) + \frac{\rho}{2} \|y - x\|^2$, due to lack of weighted projection in the algorithm since $v_t$ is now a scalar. The proof then is similar to [6] with AdaGrad step sizes. The difficulties arising due to adaptive step sizes and existence of $\beta_1$, are handled using the results in Lemma 1, Lemma 3, and Lemma 4.

**Theorem 2.** Let Assumption 1 hold. Then, for the method sketched in (16), with $\beta_1 < 1, \alpha_t = \frac{\alpha}{\sqrt{t}}$ it holds

$$\mathbb{E}\|\nabla \varphi_{1/2\rho}(x_t)\|^2 \leq \frac{2G}{\alpha \sqrt{T}} \left[ C_1 + \left( 1 + \log \left( \frac{T \gamma^2}{\delta} + 1 \right) \right) C_2 \right],$$

where $C_1 = \varphi_{1/2\rho}(x_1) - f^* + 2 \rho \left( \frac{2\beta_1}{1 - \beta_1} + 1 \right) \omega \frac{\sqrt{G}}{\sqrt{8}}, C_2 = 2 \rho \alpha d \left( \frac{1}{2} + \frac{\beta_1}{1 - \beta_1} + \frac{\beta_1^2}{(1 - \beta_1)\gamma} \right),$ and $\omega = \frac{2\gamma G}{\sqrt{\rho}}.$

We leave it as an open question to derive similar results for AdaGrad-based methods with diagonal step sizes.
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A Proofs

Lemma 1. Let Assumption 1 hold. Let $\hat{\rho} > \check{\rho}$, and $\hat{\nu}_t \geq \delta > 0$ (see Algorithm 1). It follows that
\[
\|x_t - \hat{x}_t\|^2 \leq \hat{D}^2 := \frac{4dG^2}{\delta(\hat{\rho} - \check{\rho})^2}.
\]

Proof. By the definition of $\hat{x}_t$ in (7), it follows that
\[
\varphi(\hat{x}_t) + \frac{\hat{\rho}}{2}\|x_t - \hat{x}_t\|^2_{\hat{\nu}_t} \leq \varphi(x_t) + \frac{\hat{\rho}}{2}\|x_t - x_t\|^2_{\hat{\nu}_t} = \varphi(x_t).
\]
Next, we use $\check{\rho}$-weak convexity of $\varphi$ with respect to norm $\|\cdot\|_{\hat{\nu}_t}$ from Remark 1, and the fact that $x_t, \hat{x}_t \in X$ to get for any vector $q_t$ such that $q_t \in \partial f(x_t)$,
\[
\varphi(x_t) - \varphi(\hat{x}_t) \leq \langle x_t - \hat{x}_t, q_t \rangle + \frac{\check{\rho}}{2}\|x_t - \hat{x}_t\|^2_{\hat{\nu}_t}.
\]
We sum two inequalities and apply Cauchy-Schwarz inequality
\[
\frac{\hat{\rho} - \check{\rho}}{2}\|x_t - \hat{x}_t\|^2_{\hat{\nu}_t} \leq \langle x_t - \hat{x}_t, q_t \rangle \leq \|q_t\|_{\hat{\nu}_t}^2 \|x_t - \hat{x}_t\|_{\hat{\nu}_t},
\]
which yields
\[
\frac{\hat{\rho} - \check{\rho}}{2}\|x_t - \hat{x}_t\|^2_{\hat{\nu}_t} \leq \|q_t\|_{\hat{\nu}_t}^2.
\]
As $\hat{\nu}_t \geq \delta$ and for $q_t$ such that $E q_t = q_t$, $\|q_t\|^2 = \|E q_t\|^2 \leq \|E q_t\|^2 \leq dG^2$ by Assumption 1, we have
\[
\|q_t\|^2_{\hat{\nu}_t} \leq \frac{dG^2}{\sqrt{\delta}}
\]
and the final bound follows immediately. \(\square\)

Lemma 2. Let Assumption 1 hold. Let $q_t = E_t[1] \in \partial f(x_t)$, then it follows that
\[
\alpha_t E_t(x_t - \hat{x}_t, g_t) \geq \alpha_t(\hat{\rho} - \check{\rho})E_t\|x_t - \hat{x}_t\|^2_{\check{\nu}_t} - (\alpha_{t-1} - \alpha_t)\sqrt{\hat{d}}G - \frac{\hat{\rho} - \check{\rho}}{4\check{\rho}}E_t\|\hat{x}_t - \hat{x}_{t-1}\|^2_{\check{\nu}_t} - \frac{\alpha_{t-1}}{2}E_t\|m_{t-1}\|^2_{\check{\nu}_t} - \left(\frac{1}{2} + \frac{\hat{\rho} - \check{\rho}}{\check{\rho}}\right)\frac{\alpha_{t-1}}{\sqrt{\delta}}E_t\|g_t\|^2.
\]

Proof. We first decompose the LHS
\[
\alpha_t(x_t - \hat{x}_t, g_t) = \alpha_t(x_t - \hat{x}_t, q_t) + \alpha_t(x_t - \hat{x}_t, q_t - q_t)
\]
\[
= \alpha_t(x_t - \hat{x}_t, q_t) + \alpha_t(x_t - \hat{x}_t) - \alpha_{t-1}(x_{t-1} - \hat{x}_{t-1}) - (\alpha_{t-1} - \alpha_t)\langle x_{t-1} - \hat{x}_{t-1}, \check{\nu}_t \rangle + \langle \alpha_{t-1}(x_{t-1} - \hat{x}_{t-1}), g_t - q_t \rangle
\]
(17)
In this bound, the last term will be 0 after taking conditional expectation $E_t$ as $\hat{x}_{t-1}$ depends on $\check{\nu}_t$, which, in turn, depends only on $g_1, \ldots, g_{t-1}$, thus, independent of $g_t$.

For the first term in (17), we recall that $\hat{x}_t \in X, x_t \in X, q_t \in \partial f(x_t)$. Then we use $\check{\rho}$-weak convexity of $f$ with respect to $\|\cdot\|_{\check{\nu}_t}$,
\[
\langle x_t - \hat{x}_t, q_t \rangle \geq f(x_t) - f(\hat{x}_t) + \frac{\check{\rho}}{2}\|x_t - \hat{x}_t\|^2_{\check{\nu}_t}
\]
\[
= \left(f(x_t) + \frac{\hat{\rho}}{2}\|x_t - \hat{x}_t\|^2_{\hat{\nu}_t}\right) - \left(f(\hat{x}_t) + \frac{\hat{\rho}}{2}\|x_t - \hat{x}_t\|^2_{\hat{\nu}_t}\right) + \frac{\hat{\rho} - \check{\rho}}{2}\|x_t - \hat{x}_t\|^2_{\hat{\nu}_t}
\]
\[
\geq (\hat{\rho} - \check{\rho})\|x_t - \hat{x}_t\|^2_{\hat{\nu}_t},
\]
(18)
where the last step is due to $x \mapsto f(x) + I_X(x) + \frac{\check{\rho}}{2}\|x - x_t\|^2_{\check{\nu}_t}$ being $\check{\rho} - \hat{\rho}$ strongly convex w.r.t. $\|\cdot\|_{\check{\nu}_t}$, with the minimizer $\hat{x}_t$, and $x_t, \hat{x}_t \in X$.

Next, we need to lower bound the second term in (17), for which we upper bound the term given by
\((\alpha_{t-1}(x_{t-1} - \tilde{x}_{t-1}) - \alpha_t(x_t - \tilde{x}_t), g_t - q_t) = (\alpha_{t-1} - \alpha_t)(x_t - \tilde{x}_t, g_t - q_t) + \alpha_{t-1}(x_{t-1} - x_t, g_t - q_t) + \alpha_{t-1}(\tilde{x}_t - \tilde{x}_{t-1}, g_t - q_t). \quad (19)\)

We proceed with bounding the first term in the RHS of (19), using \(\alpha_t \leq \alpha_{t-1},\)

\[
\mathbb{E}_t(\alpha_{t-1} - \alpha_t)\langle x_t - \tilde{x}_t, g_t - q_t \rangle \leq (\alpha_{t-1} - \alpha_t)\mathbb{E}_t\|g_t - q_t\| \\
\leq (\alpha_{t-1} - \alpha_t)\hat{D}\mathbb{E}_t\|g_t\|^2 \\
\leq (\alpha_{t-1} - \alpha_t)\hat{D}\sqrt{\mathbb{E}_t}\|g_t\|^2,
\]

where the second inequality follows from Lemma 1 and third inequality follows from Jensen’s inequality and \(\mathbb{E}_t\|g_t - \mathbb{E}_tg_t\|^2 \leq \mathbb{E}_t\|g_t\|^2\).

For the second term in the RHS of (19) we use Cauchy-Schwarz and Young’s inequalities and nonexpansiveness of weighted projection to get

\[
\mathbb{E}_t\alpha_{t-1}(x_{t-1} - x_t, g_t - q_t) \leq \frac{1}{2}\mathbb{E}_t\|x_t - x_{t-1}\|_{\hat{\epsilon}_{t-1}}^2 + \frac{\alpha_{t-1}^2}{2}\mathbb{E}_t\|g_t - q_t\|_{\hat{\epsilon}_{t-1}}^2 \\
\leq \frac{\alpha_{t-1}^2}{2}\mathbb{E}_t\|m_{t-1}\|_{\hat{\epsilon}_{t-1}}^2 + \frac{\alpha_{t-1}^2}{2}\hat{\rho}\mathbb{E}_t\|g_t - q_t\|^2 \\
\leq \frac{\alpha_{t-1}^2}{2}\mathbb{E}_t\|m_{t-1}\|_{\hat{\epsilon}_{t-1}}^2 + \frac{\alpha_{t-1}^2}{2}\hat{\rho}\mathbb{E}_t\|g_t\|^2.
\]

Similarly, we estimate the third term in the RHS of (19)

\[
\mathbb{E}_t\alpha_{t-1}(\tilde{x}_t - \tilde{x}_{t-1}, g_t - q_t) \leq \frac{\hat{\rho} - \hat{\rho}}{4\hat{\rho}}\|\tilde{x}_t - \tilde{x}_{t-1}\|_{\hat{\epsilon}_{t-1}}^2 + \frac{\alpha_{t-1}^2}{2}\hat{\rho}\mathbb{E}_t\|g_t - q_t\|^2 \\
\leq \frac{\hat{\rho} - \hat{\rho}}{4\hat{\rho}}\|\tilde{x}_t - \tilde{x}_{t-1}\|_{\hat{\epsilon}_{t-1}}^2 + \frac{\alpha_{t-1}^2}{2}\hat{\rho}\mathbb{E}_t\|g_t\|^2.
\]

Combining all the bounds gives the result. \(\square\)

**Lemma 4.** Let \(\beta_1 < 1, \beta_2 < 1, \gamma = \frac{\beta_1^2}{\beta_2} < 1,\) then it holds that

\[
\sum_{t=1}^T \alpha_t^2\|m_t\|_{\hat{\epsilon}_{t-1/2}}^2 \leq \frac{1 - \beta_1}{\sqrt{(1 - \beta_2)(1 - \gamma)}}\int G(1 + \log T).
\]

**Proof.** We start with the result of [1, Lemma 3]

\[
\|m_t\|_{\hat{\epsilon}_{t-1/2}}^2 \leq \frac{(1 - \beta_1)^2}{\sqrt{(1 - \beta_2)(1 - \gamma)}} \sum_{t=1}^{T} \sum_{j=1}^{d} \beta_{t-j} \|g_{j,i}\|.
\]

We will proceed similar to [1, Lemma 4] with the only change of having \(\alpha_t^2\) instead of \(\alpha_t\)

\[
\sum_{t=1}^T \alpha_t^2\|m_t\|_{\hat{\epsilon}_{t-1/2}}^2 \leq \frac{(1 - \beta_1)^2}{\sqrt{(1 - \beta_2)(1 - \gamma)}} \sum_{t=1}^{T} \sum_{j=1}^{d} \alpha_t \sum_{t-j}^{T} \beta_{t-j} \|g_{j,i}\| \\
= \frac{(1 - \beta_1)^2}{\sqrt{(1 - \beta_2)(1 - \gamma)}} \sum_{t=1}^{T} \sum_{j=1}^{d} \sum_{t-j}^{T} \alpha_t^2 \beta_{t-j} \|g_{j,i}\| \\
\leq \frac{1 - \beta_1}{\sqrt{(1 - \beta_2)(1 - \gamma)}} \sum_{t=1}^{T} \sum_{j=1}^{d} \alpha_t^2 \|g_{j,i}\| \\
\leq \frac{1 - \beta_1}{\sqrt{(1 - \beta_2)(1 - \gamma)}} \int G(1 + \log T). \quad \square
\]
Theorem 1. Let Assumption 1 hold. Let $\beta_1 < 1, \beta_2 < 1, \gamma = \frac{\beta_1^2}{\beta_2^2} < 1, \bar{\rho} = 2\bar{\rho}$. Then, for iterate $x_t$, generated by Algorithm 1, it follows that

$$E\|\nabla f(x_t)^{\bar{\rho}}(x_t)\|_2^{2/\bar{\rho} - 1/2} \leq \frac{2}{\alpha\sqrt{T}} \left[ C_1 + (1 + \log T)C_2 + C_3 \right],$$

where

$$C_1 = \frac{4n\bar{\rho}}{\sqrt{T}d(1-\beta_1)} \sqrt{dD} + \varphi_{1/\bar{\rho}}(x_1) - f^*,$$

$$C_2 = \frac{5\rho}{d} + \frac{2\rho}{d} \left( 1 + \frac{C_0}{\sqrt{t}} + \frac{\beta_1}{1-\beta_1} + \frac{2\beta_1^2}{(1-\beta_1)^2} \right) \frac{1-\beta_1}{\sqrt{(1-\beta_2)(1-\gamma)}} dG,$$

$$C_3 = \bar{\rho} \hat{D}^{2/\bar{\rho}} \sum_{i=1}^{d} \mathbb{E}_{t_i^{1/\bar{\rho}}}^{1/\bar{\rho}} \text{ and } \hat{D} := 2\sqrt{dG}/\rho.$$

Proof. We sum the result of Lemma 3 and use $A_1 = A_0$. with $m_0 = 0$. We note that we have $A_t = \bar{\rho}\alpha_t(x_t - \hat{x}_t)$, for $t \geq 1$.

$$\sum_{t=1}^{T} \langle A_t, g_t \rangle = \frac{\beta_1}{1-\beta_1} \langle A_T, m_T \rangle + \sum_{t=1}^{T} \langle A_t, m_t \rangle + \frac{\beta_1}{1-\beta_1} \sum_{t=1}^{T-1} \langle A_t - A_{t+1}, m_t \rangle. \tag{20}$$

After plugging in the value of $A_t$, (20) becomes

$$\sum_{t=1}^{T} \bar{\rho} \alpha_t \langle x_t - \hat{x}_t, g_t \rangle \leq \frac{\beta_1}{1-\beta_1} \bar{\rho} \alpha_T \langle x_T - \hat{x}_T, m_T \rangle + \sum_{t=1}^{T} \bar{\rho} \alpha_t \langle x_t - \hat{x}_t, m_t \rangle$$

$$+ \frac{\beta_1\bar{\rho}}{1-\beta_1} \sum_{t=1}^{T-1} \langle \alpha_t(x_t - \hat{x}_t) - \alpha_{t+1}(x_{t+1} - \hat{x}_{t+1}), m_t \rangle. \tag{21}$$

LHS of this bound is suitable for applying Lemma 2 to obtain the stationarity measure. We have to estimate the three terms on the RHS.

- **Bound for $\frac{\beta_1}{1-\beta_1} \langle x_T - \hat{x}_T, m_T \rangle$ in (21).**

Applying Cauchy-Schwarz inequality and using Lemma 1 is enough to bound this term, with $\|m_t\|_{\infty} \leq G$:

$$\langle x_T - \hat{x}_T, m_T \rangle \leq \|x_T - \hat{x}_T\| \|m_T\| \leq \hat{D} \sqrt{dG}. \tag{22}$$

- **Bound for $\frac{\beta_1\bar{\rho}}{1-\beta_1} \sum_{t=1}^{T-1} \langle \alpha_t(x_t - \hat{x}_t) - \alpha_{t+1}(x_{t+1} - \hat{x}_{t+1}), m_t \rangle$ in (21).**

We have

$$\langle \alpha_t(x_t - \hat{x}_t) - \alpha_{t+1}(x_{t+1} - \hat{x}_{t+1}), m_t \rangle = (\alpha_t - \alpha_{t+1}) \langle x_{t+1} - \hat{x}_{t+1}, m_t \rangle + \alpha_t \langle x_t - x_{t+1}, m_t \rangle + \alpha_t \langle \hat{x}_{t+1} - \hat{x}_t, m_t \rangle. \tag{23}$$

For the first term in (23), we use that $\alpha_t \geq \alpha_{t+1}$, Lemma 1, Cauchy-Schwarz inequality and $\|m_t\|_{\infty} \leq G$ to obtain

$$\sum_{t=1}^{T-1} (\alpha_t - \alpha_{t+1}) \langle x_{t+1} - \hat{x}_{t+1}, m_t \rangle \leq \sum_{t=1}^{T-1} (\alpha_t - \alpha_{t+1}) \hat{D} \sqrt{dG} \leq \alpha_1 \hat{D} \sqrt{dG}.$$

For the second term of (23), using nonexpansiveness of weighted projection, we deduce

$$\alpha_t \langle x_t - x_{t+1}, m_t \rangle \leq \alpha_t \|x_t - x_{t+1}\|_{\nu_t^{1/2}} \|m_t\|_{\nu_t^{1-1/2}}$$

$$= \alpha_t \|x_t - P_{\mathcal{X}}^{\nu_t^{1/2}} (x_t - \alpha_t \nu_t^{-1/2} m_t)\|_{\nu_t^{1/2}} \|m_t\|_{\nu_t^{1-1/2}} \leq \alpha_t^2 \|m_t\|_{\nu_t^{1-1/2}}^2.$$
We insert this estimate into (25) and use the definition of

$$\varphi(t) = \frac{1}{2} \left( x_t - \dot{x}_t \right)^2$$

We will manipulate the second to last term, by using

$$x = f(x) + I(x) + \frac{\rho}{2} \left\| x - x_{t+1} \right\|_{\mathcal{F}_t}^2$$

and Lemma 1

$$\varphi(t) \leq \varphi(t) + \rho \alpha \left( x_t - x_{t+1} \right) + \frac{\rho^2}{2} \left\| x_t - x_{t+1} \right\|_{\mathcal{F}_t}^2$$

We use this estimate in (26) and sum the inequality to get

$$\sum_{t=1}^{T-1} \varphi(t) \leq \sum_{t=1}^{T-1} \left( \varphi(t) + \rho \alpha \left( x_t - x_{t+1} \right) + \frac{\rho^2}{2} \left\| x_t - x_{t+1} \right\|_{\mathcal{F}_t}^2 \right) + \frac{\rho^2}{2} \left\| x_T - x_{T+1} \right\|_{\mathcal{F}_t}^2$$

where we used Young’s inequality in the last step.

**Bound for \( \sum_{t=1}^{T} \rho \alpha \left( x_t - \dot{x}_t, m_t \right) \) in (21).**

We proceed as in eq. (3.8) in [6], but with a tighter bound in the beginning, where we use

$$x \mapsto f(x) + I(x) + \frac{\rho}{2} \left\| x - x_{t+1} \right\|_{\mathcal{F}_t}^2$$

being \( \rho \) strongly convex w.r.t. \( \left\| \cdot \right\|_{\mathcal{F}_t}^2 \), with the minimizer \( \dot{x}_{t+1} \)

$$\dot{x}_{t+1} = \arg\min_{x} \left\{ f(x) + I(x) + \frac{\rho}{2} \left\| x - x_{t+1} \right\|_{\mathcal{F}_t}^2 \right\}$$

and Lemma 1

$$\varphi(t) \leq \varphi(t) + \rho \alpha \left( x_t - x_{t+1} \right) + \frac{\rho^2}{2} \left\| x_t - x_{t+1} \right\|_{\mathcal{F}_t}^2$$

We estimate the second term in the RHS of (25) by the definition of \( x_{t+1} \), then using \( \dot{x}_t \in \mathcal{X} \) and nonexpansiveness of the weighted projection in the weighted norm

$$\varphi(t+1) \leq \varphi(t) + \rho \alpha \left( x_t - x_{t+1} \right) + \frac{\rho^2}{2} \left\| x_t - x_{t+1} \right\|_{\mathcal{F}_t}^2$$

We insert this estimate into (25) and use the definition of \( \varphi(t+1) \) to obtain

$$\varphi(t+1) \leq \varphi(t) + \rho \alpha \left( x_t - x_{t+1} \right) + \frac{\rho^2}{2} \left\| x_t - x_{t+1} \right\|_{\mathcal{F}_t}^2$$

We will manipulate the second to last term, by using

$$\left\| a + b \right\|^2 \leq 2 \left\| a \right\|^2 + 2 \left\| b \right\|^2, \ \dot{v}_{t,i} \geq \hat{v}_{t,i},$$

and Lemma 1

$$\varphi(t+1) \leq \varphi(t) + \rho \alpha \left( x_t - x_{t+1} \right) + \frac{\rho^2}{2} \left\| x_t - x_{t+1} \right\|_{\mathcal{F}_t}^2$$

We use this estimate in (26) and sum the inequality to get

$$\sum_{t=1}^{T} \varphi(t) \leq \sum_{t=1}^{T} \left( \varphi(t) + \rho \alpha \left( x_t - x_{t+1} \right) + \frac{\rho^2}{2} \left\| x_t - x_{t+1} \right\|_{\mathcal{F}_t}^2 \right) + \frac{\rho^2}{2} \left\| x_T - x_{T+1} \right\|_{\mathcal{F}_t}^2$$

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Combining estimates into (21). We now plug in (22), (24), (27) into (21) and use \( \alpha_T \leq \alpha \), \( \hat{v}_{i+1}^{1/2} \geq \hat{v}_{i}^{1/2} \) to get

\[
\sum_{t=1}^{T} \bar{\rho} \alpha_t \langle x_t - \hat{x}_t, g_t \rangle \leq \frac{2\beta_1 \bar{\rho} \alpha}{(1 - \beta_1)} \hat{D} \sqrt{dG} + \varphi_1^{1/\rho}(x_1) - \varphi_{1/\rho}^T(x_{T+1}) + \bar{\rho} \hat{D}^2 \sum_{t=1}^{d} \hat{v}_{t+1,i}^{1/2} \\
+ \sum_{t=1}^{T} \left( \frac{1}{2} + \frac{G}{\sqrt{\delta}} + \frac{\beta_1}{1 - \beta_1} + \frac{\bar{\rho}}{\bar{\rho} - \rho} \frac{\beta_1^2}{(1 - \beta_1)^2} \right) \bar{\rho} \alpha_t^2 \| m_t \|_{\hat{v}_{i-1}^{1/2}}^2 - \sum_{t=1}^{T} \frac{\bar{\rho} - \rho}{4} \| \hat{x}_t - \hat{x}_{t+1} \|_{\hat{v}_{i+1}^{1/2}}^2.
\]

(28)

At this point, due to the coupling between \( \hat{x}_t, \hat{v}_t \), and \( g_t \), we cannot directly take expectations, so we will use the estimations of Lemma 2. First we sum the result of Lemma 2 which gives

\[
\sum_{t=1}^{T} \mathbb{E}_t [\alpha_t \langle x_t - \hat{x}_t, g_t \rangle] \geq \sum_{t=1}^{T} \mathbb{E}_t (\bar{\rho} - \rho) \alpha_t \| x_t - \hat{x}_t \|_{\hat{v}_{i}^{1/2}}^2 - (\alpha_0) \sqrt{\hat{D} G}
\]

\[- \sum_{t=1}^{T} \frac{\bar{\rho} - \rho}{4} \mathbb{E}_t (\| x_t - \hat{x}_{t-1} \|_{\hat{v}_{i}^{1/2}}^2) - \sum_{t=1}^{T} \frac{\alpha_t - 1}{2} \mathbb{E}_t (\| m_{t-1} \|_{\hat{v}_{i}^{1/2}}^2) - \sum_{t=1}^{T} \left( \frac{1}{2} + \frac{\bar{\rho}}{\bar{\rho} - \rho} \right) \frac{\alpha_t^2}{\sqrt{\delta}} \mathbb{E}_t \| g_t \|^2.
\]

We use here the assignments used for convenience: \( \alpha_0 = 0 \) and \( \hat{x}_0 = \hat{x}_1 \) and recall that \( m_0 = 0 \). We plug this estimation after taking full expectation into (28) and use \( \hat{v}_{i}^{1/2} \leq \hat{v}_{i+1}^{1/2} \) to obtain

\[
\bar{\rho} (\bar{\rho} - \rho) \sum_{t=1}^{T} \alpha_t \mathbb{E}_t \| x_t - \hat{x}_t \|_{\hat{v}_{i}^{1/2}}^2 \leq \frac{2\beta_1 \bar{\rho} \alpha}{(1 - \beta_1)} \hat{D} \sqrt{dG} + \varphi_1^{1/\rho}(x_1) - \varphi_{1/\rho}^T(x_{T+1}) + \bar{\rho} \hat{D}^2 \sum_{t=1}^{d} \hat{v}_{t+1,i}^{1/2} \\
+ \sum_{t=1}^{T} \left( \frac{1}{2} + \frac{G}{\sqrt{\delta}} + \frac{\beta_1}{1 - \beta_1} + \frac{\bar{\rho}}{\bar{\rho} - \rho} \frac{\beta_1^2}{(1 - \beta_1)^2} \right) \bar{\rho} \alpha_t^2 \mathbb{E}_t \| m_t \|_{\hat{v}_{i-1}^{1/2}}^2 \\
+ \sum_{t=1}^{T} \left( \frac{1}{2} + \frac{\bar{\rho}}{\bar{\rho} - \rho} \right) \frac{\alpha_t^2}{\sqrt{\delta}} \mathbb{E}_t \| g_t \|^2.
\]

The only quantities left to estimate are \( \sum_{t=1}^{T} \alpha_{t-1}^2 \| g_t \|^2 \) and \( \sum_{t=1}^{T} \alpha_{t}^2 \| m_t \|_{\hat{v}_{i}^{1/2}}^2 \). Using Lemma 4 and \( \alpha_0 = 0 \) shows that both these quantities are bounded by \( O(\log T) \):

\[
\sum_{t=1}^{T} \alpha_{t}^2 \| m_t \|_{\hat{v}_{i}^{1/2}}^2 \leq \frac{1 - \beta_1}{\sqrt{(1 - \beta_2)(1 - \gamma)}} dG (1 + \log T),
\]

\[
\sum_{t=1}^{T} \alpha_{t-1}^2 \| g_t \|^2 \leq \sum_{t=2}^{T} \alpha_{t-1}^2 \| g_t \|^2 \leq dG^2 (1 + \log T).
\]

The proof then follows by using (8), \( f^* \leq f(x), \forall x \in X \), picking \( \bar{\rho} = 2\rho \), using \( \alpha_t \geq \alpha_T \), and in the end dividing both sides by \( T \alpha_T \).

Before, moving onto the proof of Theorem 2, we need a lemma analogous to Lemma 4. This lemma can be seen as a simplified version of the similar results, for example in [1, 24].

**Lemma 5.** Let Assumption 1 hold. Let \( \beta_1 < 1 \) and \( \alpha_t, v_t \) are set as in (16). Then, we have

\[
\sum_{t=1}^{T} \frac{\alpha_t^2}{v_t} \| m_t \|^2 \leq \alpha d \left( 1 + \log \left( \frac{T G^2}{\delta} + 1 \right) \right).
\]

**Proof.** We note that \( \frac{\alpha_t^2}{v_t} = \frac{\alpha}{\delta + \frac{1}{2} \sum_{j=1}^{T} \| g_j \|^2} \). We proceed as [1, Lemma 5, 6] with the difference of not having diagonal \( v_t \):

\[
\sum_{t=1}^{T} \frac{\alpha_t^2}{v_t} \| m_t \|^2 = \sum_{t=1}^{T} \frac{\alpha_t^2}{v_t} \sum_{i=1}^{d} (m_{t,i})^2 = \sum_{t=1}^{T} \frac{\alpha_t^2}{v_t} \sum_{i=1}^{d} \left( \sum_{j=1}^{t} (1 - \beta_1) \beta_1^{t-j} g_{j,i} \right)^2.
\]
We now let \( A \) recall the definitions from [6] for standard SGD.

This proof will be midway between the proof we have presented for Theorem 1 and the proof to conclude.

\[
\leq (1 - \beta_1)^2 \sum_{t=1}^{T} \frac{\alpha_t}{\bar{v}_t} \sum_{i=1}^{d} \left( \sum_{j=1}^{t-1} \beta_1^{t-j} g_{j,i}^2 \right) \sum_{j=1}^{t} \beta_1^{t-j} g_{j,i}^2 \tag{29}
\]

\[
\leq (1 - \beta_1) \alpha \sum_{i=1}^{d} \sum_{j=1}^{t} \frac{\beta_1^{t-j} g_{j,i}^2}{\bar{v}_t} \tag{30}
\]

\[
\leq (1 - \beta_1) \alpha \sum_{i=1}^{d} \sum_{j=1}^{t} \frac{\beta_1^{t-j} g_{j,i}^2}{\bar{v}_t} \sum_{j=1}^{t} \beta_1^{t-j} g_{j,i}^2 \tag{31}
\]

\[
= (1 - \beta_1) \alpha \sum_{i=1}^{d} \sum_{j=1}^{t} \frac{\beta_1^{t-j} g_{j,i}^2}{\bar{v}_t} \sum_{j=1}^{t} \beta_1^{t-j} g_{j,i}^2 \tag{32}
\]

\[
\leq \alpha \sum_{i=1}^{d} \sum_{j=1}^{t} \frac{\beta_1^{t-j} g_{j,i}^2}{\bar{v}_t} \sum_{j=1}^{t} \beta_1^{t-j} g_{j,i}^2 \tag{33}
\]

where (29) is by Cauchy-Schwarz inequality, (30) is by summing a geometric series, (31) is by \( j \leq t \), (32) is by changing the order of summation, (33) is by summing a geometric series and the last step is by changing the order of summation.

Now we can apply a standard inequality, for nonnegative numbers \( a_i, \forall i \) and \( \delta > 0 \) [19, Lemma A.3]

\[
\sum_{j=1}^{T} \frac{a_j}{\delta + \sum_{k=1}^{l} a_j} \leq 1 + \log \left( \frac{\sum_{j=1}^{T} a_j}{\delta} + 1 \right)
\]

to conclude.

**Theorem 2.** Let Assumption 1 hold. Then, for the method sketched in (16), with \( \beta_1 < 1 \), \( \alpha_t = \frac{\alpha}{\sqrt{t}} \) it holds

\[
\mathbb{E} \| \nabla \varphi_{1/2\rho}(x_t) \|^2 \leq \frac{2G}{\alpha \sqrt{T}} \left[ C_1 + \left( 1 + \log \left( \frac{T G^2}{\delta} + 1 \right) \right) C_2 \right],
\]

where \( C_1 = \varphi_{1/2\rho}(x_1) - f^* + 2\rho \left( \frac{\alpha_1}{1 - \beta_1} + 1 \right) \frac{\alpha \beta_1 \rho G}{\sqrt{\delta}} \), \( C_2 = 2\rho \delta d \left( \frac{1}{2} + \frac{\alpha_1}{1 - \beta_1} + \frac{2\beta_1^2}{1 - \beta_1 \tau} \right) \), and \( \hat{D} = \frac{2G}{\sqrt{\rho}} \).

**Proof.** This proof will be midway between the proof we have presented for Theorem 1 and the proof from [6] for standard SGD.

We recall the definitions

\[
\hat{x}_t = \arg\min_{x \in \mathcal{X}} f(x) + \frac{\rho}{2} \| x - x_t \|^2,
\]

\[
\varphi_{1/\rho}(x_t) = \min_{x \in \mathcal{X}} f(x) + \frac{\rho}{2} \| x - x_t \|^2,
\]

\[
\alpha_t = \frac{\alpha}{\sqrt{t}} \sqrt{\delta + \frac{1}{\alpha} \sum_{j=1}^{l} \| g_j \|^2}
\]

Same as Theorem 1, we sum the result of Lemma 3 to get

\[
\sum_{t=1}^{T} \langle A_t, g_t \rangle = \frac{\beta_1}{1 - \beta_1} \langle A_T, m_T \rangle + \sum_{t=1}^{T-1} \langle A_t, m_t \rangle + \frac{\beta_1}{1 - \beta_1} \sum_{t=1}^{T-1} \langle A_t - A_{t+1}, m_t \rangle. \tag{34}
\]

We now let \( A_t = \hat{\rho} \frac{\alpha_t}{\sqrt{t}} (x_t - \hat{x}_t) \) and (34) becomes

\[
\sum_{t=1}^{T} \hat{\rho} \frac{\alpha_t}{\sqrt{t}} (x_t - \hat{x}_t, g_t) \leq \frac{\beta_1 \rho \alpha_T}{\sqrt{T} (1 - \beta_1)} (x_T - \hat{x}_T, m_T) + \sum_{t=1}^{T} \hat{\rho} \frac{\alpha_t}{\sqrt{t}} (x_t - \hat{x}_t, m_t)
\]

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\[ + \frac{\beta_1 \hat{\rho}}{1 - \beta_1} \sum_{t=1}^{T-1} \left( \frac{\alpha_t}{\sqrt{v_t}} (x_t - \hat{x}_t) - \frac{\alpha_{t+1}}{\sqrt{v_{t+1}}} (x_{t+1} - \hat{x}_{t+1}, m_t) \right). \] (35)

- Bound for \( \frac{\beta_1 \hat{\rho}}{1 - \beta_1} \sum_{t=1}^{T} (\frac{\alpha_t}{\sqrt{v_t}} (x_t - \hat{x}_t) - \frac{\alpha_{t+1}}{\sqrt{v_{t+1}}} (x_{t+1} - \hat{x}_{t+1}, m_t) \) in (35)

We deduce similar to (23)

\[ \left\langle \frac{\alpha_t}{\sqrt{v_t}} (x_t - \hat{x}_t) - \frac{\alpha_{t+1}}{\sqrt{v_{t+1}}} (x_{t+1} - \hat{x}_{t+1}, m_t) \right\rangle = \left( \frac{\alpha_t}{\sqrt{v_t}} - \frac{\alpha_{t+1}}{\sqrt{v_{t+1}}} \right) \left\langle x_{t+1} - \hat{x}_{t+1}, m_t \right\rangle \]

\[ + \frac{\alpha_t}{\sqrt{v_t}} \langle x_t - x_{t+1}, m_t \rangle + \frac{\alpha_t}{\sqrt{v_t}} \langle \hat{x}_{t+1} - \hat{x}_t, m_t \rangle. \]

We note that since \( \frac{\alpha_t}{\sqrt{v_t}} \) is decreasing,

\[ \sum_{t=1}^{T-1} \left( \frac{\alpha_t}{\sqrt{v_t}} - \frac{\alpha_{t+1}}{\sqrt{v_{t+1}}} \right) \left\langle x_{t+1} - \hat{x}_{t+1}, m_t \right\rangle \leq \sum_{t=1}^{T-1} \left( \frac{\alpha_t}{\sqrt{v_t}} - \frac{\alpha_{t+1}}{\sqrt{v_{t+1}}} \right) \tilde{D} \sqrt{dG} \]

\[ \leq \frac{\alpha_1}{\sqrt{v_1}} \tilde{D} \sqrt{dG}. \]

Next, we use Cauchy-Schwarz inequality, definition of \( x_{t+1} \) and nonexpansiveness

\[ \frac{\alpha_t}{\sqrt{v_t}} \langle x_t - x_{t+1}, m_t \rangle \leq \frac{\alpha_t}{\sqrt{v_t}} \| x_t - \mathcal{P}_X (x_t - \frac{\alpha_t}{\sqrt{v_t}} m_t) \| m_t \| \leq \frac{\alpha_t^2}{v_t} \| m_t \|^2. \]

We use Young’s inequality to get

\[ \frac{\alpha_t}{\sqrt{v_t}} \langle \hat{x}_{t+1} - \hat{x}_t, m_t \rangle \leq \frac{(\tilde{\rho} - \rho)(1-\beta_1)}{4\tilde{\rho}_1} \| \hat{x}_{t+1} - \hat{x}_t \|^2 + \frac{\alpha_t^2 \tilde{\rho} \beta_1}{v_t (\tilde{\rho} - \rho)(1-\beta_1)} \| m_t \|^2. \]

Collecting all the bounds in this part gives

\[ \frac{\beta_1 \hat{\rho}}{1 - \beta_1} \sum_{t=1}^{T-1} \left( \frac{\alpha_t}{\sqrt{v_t}} (x_t - \hat{x}_t) - \frac{\alpha_{t+1}}{\sqrt{v_{t+1}}} (x_{t+1} - \hat{x}_{t+1}, m_t) \right) \leq \frac{\tilde{\rho} \beta_1 \alpha_1}{(1-\beta_1)\sqrt{v_1}} \tilde{D} \sqrt{dG} \]

\[ + \frac{\beta - \rho}{4} \sum_{t=1}^{T} \| \hat{x}_{t+1} - \hat{x}_t \|^2 + \sum_{t=1}^{T} \left( \tilde{\rho} \beta_1 \frac{\alpha_1}{1-\beta_1} + \frac{\tilde{\rho}^2 \beta_1^2}{(\tilde{\rho} - \rho)(1-\beta_1)^2} \right) \frac{\alpha_t^2}{v_t} \| m_t \|^2. \] (36)

- Bound for \( \sum_{t=1}^{T} \frac{\rho \alpha_t}{\sqrt{v_t}} (x_t - \hat{x}_t, m_t) \) in (35)

Since \( x \rightarrow f(x) + I_X(x) + \frac{\beta}{2} \| x - x_{t+1} \|^2 \) is \( (\tilde{\rho} - \rho) \)-strongly convex with the minimizer, \( \hat{x}_{t+1} \)

\[ \varphi_{1/\tilde{\rho}}(x_{t+1}) \leq f(\hat{x}_t) + \frac{\tilde{\rho}}{2} \| \hat{x}_t - x_{t+1} \|^2 - \frac{\beta - \rho}{2} \| \hat{x}_t - \hat{x}_{t+1} \|^2. \] (37)

By using \( \hat{x}_t \in X \)

\[ \frac{\rho}{2} \| x_{t+1} - \hat{x}_t \|^2 = \frac{\rho}{2} \| \mathcal{P}_X (x_t - \frac{\alpha_t}{\sqrt{v_t}} m_t) - \mathcal{P}_X (\hat{x}_t) \|^2 \leq \frac{\rho}{2} \| x_t - \frac{\alpha_t}{\sqrt{v_t}} m_t - \hat{x}_t \|^2 \]

\[ = \frac{\rho}{2} \| x_t - \hat{x}_t \|^2 - \frac{\rho \alpha_t}{\sqrt{v_t}} \langle x_t - \hat{x}_t, m_t \rangle + \frac{\rho \alpha_t^2}{2 v_t} \| m_t \|^2. \]

Then, (37) becomes

\[ \varphi_{1/\tilde{\rho}}(x_{t+1}) \leq \varphi_{1/\tilde{\rho}}(x_t) - \frac{\rho \alpha_t}{\sqrt{v_t}} \langle x_t - \hat{x}_t, m_t \rangle + \frac{\rho \alpha_t^2}{2 v_t} \| m_t \|^2 - \frac{\beta - \rho}{2} \| \hat{x}_{t+1} - \hat{x}_t \|^2. \]

Summing this inequality gives

\[ \sum_{t=1}^{T} \frac{\rho \alpha_t}{\sqrt{v_t}} \langle x_t - \hat{x}_t, m_t \rangle \leq \varphi_{1/\tilde{\rho}}(x_1) - \varphi_{1/\tilde{\rho}}(x_{T+1}) + \tilde{\rho} \sum_{t=1}^{T} \frac{\alpha_t^2}{2 v_t} \| m_t \|^2 - \frac{\beta - \rho}{2} \sum_{t=1}^{T} \| \hat{x}_{t+1} - \hat{x}_t \|^2. \] (38)
We now note (36) and (38) into (35)

\[
\sum_{t=1}^{T} \frac{\hat{\rho}}{v_{t}} (x_{t} - \hat{x}_{t}, g_{t}) \leq \frac{\hat{\rho} \beta_{1} \alpha_{T}}{\sqrt{v_{T}(1 - \beta_{1})}} \hat{\Delta} \sqrt{dG} + \varphi_{1/\hat{\rho}}(x_{1}) - \varphi_{1/\hat{\rho}}(x_{T+1}) + \hat{\rho} \sum_{t=1}^{T} \left( \frac{1}{2} + \frac{\beta_{1}}{1 - \beta_{1}} \right) \frac{\alpha_{t}^{2}}{v_{t}} \|m_{t}\|^{2} + \frac{\hat{\rho} \beta_{1} \alpha_{1}}{\sqrt{v_{1}(1 - \beta_{1})}} \hat{\Delta} \sqrt{dG}. \tag{39}
\]

Due to coupling of \(v_{t}\) and \(g_{t}\), we estimate LHS as

\[
\mathbb{E}_{t} \hat{\rho} \frac{\alpha_{t}}{v_{t}} (x_{t} - \hat{x}_{t}, g_{t}) = \hat{\rho} \frac{\alpha_{t-1}}{v_{t-1}} (x_{t} - \hat{x}_{t}, \mathbb{E}_{t} g_{t}) + \hat{\rho} \mathbb{E}_{t} \left( \frac{\alpha_{t}}{v_{t}} - \frac{\alpha_{t-1}}{v_{t-1}} \right) (x_{t} - \hat{x}_{t}, g_{t}) \geq \hat{\rho} \frac{\alpha_{t-1}}{v_{t-1}} \rho \mathbb{E}_{t} \|x_{t} - \hat{x}_{t}\|^{2} - \hat{\rho} \mathbb{E}_{t} \left( \frac{\alpha_{t}}{v_{t}} - \frac{\alpha_{t-1}}{v_{t-1}} \right) \hat{\Delta} \sqrt{dG},
\]

where in the last inequality holds for any \(t > 1\); in order to have it for \(t = 1\), we need to have \(\alpha_{0}\), which we can choose arbitrarily. For convenience, we set \(\frac{\alpha_{0}}{\sqrt{v_{0}}} = \frac{\alpha_{1}}{\sqrt{v_{1}}}\). Then we take expectation of (39), use \(\frac{\alpha_{0}}{\sqrt{v_{0}}} \leq \frac{\alpha_{1}}{\sqrt{v_{1}}} \leq \frac{\beta}{\sqrt{G}}\) and plug in the last inequality to get

\[
\sum_{t=1}^{T} \rho \left( \frac{\alpha_{t}}{v_{t}} \right) \mathbb{E}_{t} \|x_{t} - \hat{x}_{t}\|^{2} \leq \varphi_{1/\hat{\rho}}(x_{1}) - \mathbb{E}_{t} \varphi_{1/\hat{\rho}}(x_{T+1}) + \rho \frac{\alpha}{\sqrt{\beta}} \mathbb{E}_{t} \|x_{t} - \hat{x}_{t}\|^{2} + \rho \frac{\alpha}{\sqrt{\beta}} \hat{\Delta} \sqrt{dG}.
\]

We now note \(\frac{\alpha_{t-1}}{v_{t-1}} \geq \frac{\alpha_{t}}{v_{t}} \geq \frac{\beta}{\sqrt{G}}\). We also use \(\rho \|x_{t} - \hat{x}_{t}\| \leq \|\nabla \varphi_{1/\hat{\rho}}(x)\|\).

For \(\hat{\Delta}\), we use Lemma 1 without the metric to obtain

\[
\|\hat{x}_{t} - x_{t}\| \leq \hat{\Delta}^{2} = \frac{4dG^{2}}{\rho - \hat{\rho}}.
\]

We select \(\hat{\rho} = 2\rho\) and collect the bounds to complete the proof. \(\square\)

**B Relation between gradient mapping and Moreau envelope**

We show how to determine the constant for the inequality \(\|G_{1/\hat{\rho}}(x_{t})\| \leq C_{g,m} \|\nabla \varphi_{1/\hat{\rho}}(x_{t})\|_{v_{t}^{-1/2}}\), by following arguments similar to [10, Theorem 3.5].

We start with the definitions

\[
\varphi(x) = f(x) + r(x) := f(x) + I_{X}(x)
\]

\[
G_{\lambda}(x) = \lambda^{1/4} \left( x - \lambda^{\beta/2} (x - \lambda^{\beta/2} \nabla f(x)) \right)
\]

\[
\hat{x}_{t} = \text{prox}_{\hat{\beta}/\hat{\rho} \varphi} \left( x_{t} \right) = \argmin_{y} \left\{ \varphi(y) + \frac{\hat{\rho}}{2} \|y - x_{t}\|_{v_{t}^{-1/2}}^{2} \right\}
\]

Let us use the notation \(z := \nabla \varphi_{1/\hat{\rho}}(x_{t}) = \hat{\beta} v_{t}^{1/2} (x_{t} - \hat{x}_{t})\) and \(\alpha := \hat{\rho}^{-1} v_{t}^{-1/2}\). As \(\hat{x}_{t} = (I + \hat{\rho}^{-1} v_{t}^{-1/2} \partial \varphi)^{-1}(x_{t})\), we have

\[
z = \hat{\beta} v_{t}^{1/2} (x_{t} - \hat{x}_{t}) \iff \alpha z = x_{t} - (I + \alpha \partial \varphi)^{-1}(x_{t}) \iff x_{t} \in (I + \alpha \partial \varphi)(x_{t} - \alpha z) \iff x_{t} \in (I + \alpha \partial \rho)(x_{t} - \alpha z) + \alpha \nabla f(x_{t} - \alpha z) + \alpha \nabla f(x_{t}) - \alpha \nabla f(x_{t}).
\]

Let \(w = \alpha \nabla f(x_{t} - \alpha z) - \alpha \nabla f(x_{t}).\) Then

\[
x_{t} - \alpha \nabla f(x_{t}) - w \in (I + \alpha \partial \rho)(x_{t} - \alpha z) \iff x_{t} - (I + \alpha \partial \rho)^{-1}(x_{t} - \alpha \nabla f(x_{t}) - w) = \alpha z.
\]
We now plug in the value of $\alpha = \bar{\rho}^{-1} \hat{v}_t^{-1/2}$

$$ \| \bar{\rho}^{1/4} t (x_t - \text{pro}_{\bar{\rho}} (x_t - \bar{\rho}^{-1} \hat{v}_t^{-1/2} \nabla f (x_t) - w)) \| = \| \hat{v}_t^{-1/2} z \|. \quad (40) $$

By the triangle inequality and nonexpansiveness, we have that

$$ \text{LHS} \geq \| \bar{\rho}^{1/4} t (x_t - \text{pro}_{\bar{\rho}} (x_t - \bar{\rho}^{-1} \hat{v}_t^{-1/2} \nabla f (x_t))) \|
- \| \bar{\rho}^{1/4} t (\text{pro}_{\bar{\rho}} (x_t - \bar{\rho}^{-1} \hat{v}_t^{-1/2} \nabla f (x_t) - w) - \text{pro}_{\bar{\rho}} (x_t - \bar{\rho}^{-1} \hat{v}_t^{-1/2} \nabla f (x_t)))) \|
\geq \| \mathcal{G}_{1/\bar{\rho}} (x_t) \| - \| \bar{\rho} w \|_{\hat{v}_t^{1/2}}. $$

Thus, we deduce from (40) that

$$ \| \mathcal{G}_{1/\bar{\rho}} (x_t) \| \leq \| \nabla \phi_{1/\hat{\rho}} (x_t) \|_{\hat{v}_t^{1/2}} + \| \bar{\rho} w \|_{\hat{v}_t^{1/2}}. $$

We lastly estimate $\| w \|_{\hat{v}_t^{1/2}}$ using $L$-smoothness of $f$. Let us denote by $\hat{L}$ the smoothness constant of $f$ w.r.t. norm $\| \cdot \|_{\hat{v}_t^{1/2}}$:

$$ \| \nabla f (x) - \nabla f (y) \|_{\hat{v}_t^{1/2}} \leq \hat{L} \| x - y \|_{\hat{v}_t^{1/2}}. $$

Then

$$ \bar{\rho} \| w \|_{\hat{v}_t^{1/2}} = \| \nabla f (x_t - \bar{\rho}^{-1} \hat{v}_t^{-1/2} z) - \nabla f (x_t) \|_{\hat{v}_t^{1/2}} \leq \hat{L} \bar{\rho}^{-1} \| \hat{v}_t^{-1/2} z \|_{\hat{v}_t^{1/2}}
= \hat{L} \bar{\rho}^{-1} \| z \|_{\hat{v}_t^{1/2}} = \hat{L} \bar{\rho}^{-1} \| \nabla \phi_{1/\hat{\rho}} (x_t) \|_{\hat{v}_t^{1/2}}. $$

Recall that in our main theorem we have chosen $\bar{\rho} = 2 \hat{\rho}$ where $\hat{\rho}$ was the weak convexity constant of $f$ w.r.t. norm $\| \cdot \|_{\hat{v}_t^{1/2}}$. Similarly, here we have a constant depending on $\hat{\rho}^{-1} \hat{L}$, where $\hat{L}$ is the Lipschitz constant of $f$ on the weighted norm.