THE BLOW UP METHOD FOR 
BRAKKE FLOWS: NETWORKS NEAR TRIPLE JUNCTIONS 

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Abstract. We introduce a parabolic blow-up method to study the asymptotic behavior of an integral Brakke flow of planar networks (i.e. a 1-dimensional integral Brakke flow in a two-dimensional region) weakly close in a space-time region to a static multiplicity 1 triple junction $J$. We show that such a network flow is regular in a smaller space-time region, in the sense that it consists of three curves coming smoothly together at a single point at 120 degree angles, staying smoothly close to $J$ and moving smoothly. Using this result and White’s stratification theorem, we deduce that whenever an integral Brakke flow of networks in a space-time region $\mathcal{R}$ has no static tangent flow with density $\geq 2$, there exists a closed subset $\Sigma \subset \mathcal{R}$ of parabolic Hausdorff dimension at most 1 such that the flow is classical in $\mathcal{R} \setminus \Sigma$, i.e. near every point in $\mathcal{R} \setminus \Sigma$, the flow, if non-empty, consists of either an embedded curve moving smoothly or three embedded curves meeting smoothly at a single point at 120 degree angles and moving smoothly. In particular, such a flow is classical at all times except for a closed set of times of ordinary Hausdorff dimension at most $\frac{1}{2}$.

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1. Introduction

Our goal in this paper is to introduce a general framework—a parabolic blow up method—to study the asymptotic nature of a multiplicity 1 Brakke flow near certain generic singularities of the flow. The theorems we prove here using this framework, which we shall describe shortly, concern the simplest non-trivial situation, namely, the asymptotic behavior near a static triple junction of a Brakke flow of planar networks, i.e., a 1-parameter family of 1-dimensional sets (corresponding to integral 1-varifolds) moving by generalized curvature in a domain $U \subset \mathbb{R}^2$ over a time interval.

In what might be called the classical setting, such a 1-dimensional flow consists of a locally finite union of smoothly embedded open curves moving smoothly with velocity equal to the curvature vector at each point and time, and such that at each of their boundary points (in $U$), three curves meet smoothly at 120 degree angles. Since globally in space the flow decreases the total length of the curves in time, the curvature flow of networks models the motion of grain boundaries driven by interfacial surface tension [12]. The 120 degree angle condition in this context is called the Herring condition.

While such classical solutions may stay classical time-globally in some special cases [11, 14, 19, 24, 25], in general, various singularities may occur in finite time. For instance, physically, in the motion of grain boundaries, one observes that two or more triple junctions collide with each other and small grains are eliminated. This is a process of grain coarsening which should be an integral part of the mathematical modelling of the motion. Motivated partly by such phenomena, in the pioneering work [3], Brakke introduced a generalized notion of mean curvature flow (abbreviated MCF hereafter) using the notion of varifolds in Geometric Measure Theory, and studied existence and regularity of surfaces of any dimension and codimension moving by mean curvature. Brakke’s MCF naturally accommodates flows with singularities, but allows the possibility of sudden loss of measure and non-uniqueness.

While the study of regularity of various classes of stationary varifolds (generalized minimal submanifolds), which are the equilibrium solutions of the Brakke flow, has seen several advances in the past 50 years or so, much less has been known concerning Brakke flows apart from Brakke’s own original work [3]. Recently, the work of Kasai and the first author [18] and of the first author [33] gave a new, streamlined proof of a generalization of Brakke’s local regularity theorem (3) which establishes a.e. smoothness in time and space under the hypothesis that the moving surfaces have multiplicity 1 a.e. The main result in [18] in case of equilibrium reduces to (the most general version of) Allard’s regularity theorem [1] for $k$-dimensional varifolds with mean curvature locally in $L^p$ for $p > k$, and it shows roughly speaking that a nearly flat part of a unit density Brakke flow is necessarily a smooth MCF. Neither [18] nor [3] however gives any structural information about the flow in the vicinity of singularities including triple junctions. We note that there have been important results on Brakke’s regularity theorem when one is interested in special Brakke flows such as those arising as weak limits of smooth MCFs (see for instance [8, 9, 38]) or those produced by Ilmanen’s elliptic regularization method ([15]).
For both minimal submanifolds and mean curvature flows, as well as for numerous other problems in geometric analysis and non-linear PDE, describing the asymptotic behavior of the objects in question on approach to their singular sets, and understanding the structure of the singular sets themselves, remain largely open major challenges. For multiplicity 1 classes of minimal submanifolds, the seminal work of Simon [28, 29, 30] established asymptotics near certain singularities, and also the structure results for the singular sets in the full generality of varying tangent cone types and when there is no topological obstruction to perturbing singularities away. Earlier work of Allard–Almgren [2], Taylor [32] and White [34] proved similar results in situations where the tangent cones satisfy more restrictive conditions in addition to the multiplicity 1 condition. Recent work of the second author [40, 41] and of Krummel and the second author [21, 22] establish regularity results and asymptotics near singularities (branch points) for certain classes of minimal and related submanifolds for which the multiplicity 1 condition either fails or is not assumed a priori. The work [32, 34, 29, 30, 41, 22] establish, for various classes of minimal submanifolds, fine properties of the singular sets themselves, such as smoothness or rectifiability.

Among the known results in this direction for MCF is the recent deep work of Colding–Minicozzi [6, 7] (aided also by the work of Colding–Ilmanen–Minicozzi [5]) which proves, for any flow of hypersurfaces, uniqueness of the tangent flow whenever a multiplicity 1 shrinking cylinder occurs as one tangent flow, and provides strong structural information on the singular sets of hypersurface flows whose tangent flows at singularities are all shrinking multiplicity 1 cylinders. In particular, these results apply to MCFs of mean-convex hypersurfaces, for which the earlier work of White [36, 37, 39] had established that all tangent flows at singularities are shrinking multiplicity 1 cylinders. The work of Schulze [26] established such asymptotics near compact singularities of multiplicity 1 flows.

Simon’s work [28, 29, 30] mentioned above developed two far reaching methods—one based on an infinite dimensional Lojasiewicz inequality and the other based on the so called blow-up method—for studying the singular sets of minimal submanifolds. The work of Colding–Minicozzi [6, 7] and of Schulze [26] referred to above study singularities of MCFs by establishing appropriate Lojasiewicz inequalities. In the present paper we introduce a parabolic version of the blow-up method for studying fine properties of Brakke flow singularities, and implement it fully in the simplest non-trivial case with moving singularities, namely, for 1-dimensional Brakke flows in the vicinity of a static multiplicity 1 triple junction.

Our main result here (Theorem 2.1 below) is a precise version of the following:

**Theorem 1.** If a 1-dimensional integral Brakke flow in a planar region is weakly close in a space-time neighborhood to a static multiplicity 1 triple junction $J$, then in a smaller space-time neighborhood, it is regular in the sense that it consists of three curves coming smoothly together at a single point, staying smoothly close to $J$ and moving by curvature.

Here, by weakly close we mean that the flow has small space-time $L^2$ distance from $J$ and satisfies suitable mass hypotheses at the initial and final times. (See the statement of Theorem 2.1.) This is a natural, easily verifiable criterion in the analysis of singularities. For example, if a 1-dimensional integral Brakke flow at a space-time singular point has a tangent flow equal to a static multiplicity 1 triple junction, then our theorem is applicable near that point, and implies uniqueness of the tangent flow, and moreover, that in a space-time neighborhood of the point, the flow itself is a regular triple junction moving by curvature. Thus, using the above result and White’s stratification theorem [35], we deduce the following partial regularity result (Theorem 2.2) for 1-dimensional Brakke flows in a planar region:
**Theorem 2.** If a 1-dimensional integral Brakke flow in a planar region has no static tangent flow consisting of more than 3 half-lines meeting at the origin (or, equivalently, if the density of every static tangent flow is \(< 2\)), then the flow is a classical network flow away from a closed singular set of parabolic Hausdorff dimension at most 1; in particular, such a flow is a classical network flow at each instance of time except for a closed set of times of ordinary Hausdorff dimension at most 1/2.

The hypothesis concerning tangent flows in Theorem 2 is motivated by the physics of motion of grain boundaries where the triple junction seems to be the unique stable junction; other types of junctions may form but seem to disappear instantly. (See more discussion after the statement of Theorem 2.2.)

The above results are formulated and proved here for a class of 1-dimensional flows more general than Brakke flows, in which the “velocity of motion” is given by the curvature vector plus any given space-time dependent vector field satisfying an optimal integrability condition.

The simplicity of the spatial 1-dimensionality of the problem considered here allows us to essentially isolate the difficulties arising from the presence of the time variable. Although some of our arguments here take advantage of the spatial 1-dimensionality, the overall method introduced here appears to hold promise for much further development. Indeed, many of the estimates developed here either directly extend to or can easily be modified to work for Brakke flows of general dimension and codimension weakly close to certain types of multiplicity 1 tangent flows, including higher dimensional static triple junctions. However, there are also a few ingredients for which the arguments needed in higher dimensions seem to be much more complicated. We shall address such generalizations elsewhere.

One may also naturally wonder what could go wrong if the triple junction \(J\) is replaced by a 1-dimensional multiplicity 1 stationary junction \(J_N\) with \(N(>3)\) half lines meeting at the origin. In this case, the direct analogue of the conclusion of Theorem 1, namely that in the space-time interior the flow consists of \(N\) embedded curves coming together smoothly at a single point, is false. For instance in case \(N = 4\), consider the static junction \(J_4\) consisting of two intersecting lines at the origin with a 120 degree angle between them. We may construct a static configuration arbitrarily close to \(J_4\) with precisely two triple junction singularities by splitting apart \(J_4\) at the origin into two pairs of half-lines each making a 120 degree angle and connecting their vertices by a short line segment, and imagine non static flows that remain close to this configuration. From the point of view of our method here (see below for an outline), general uniform regularity estimates fail (as they must in view of the example just mentioned) without further hypotheses in case \(N \geq 4\) because the flow need not have the property that the moving curve at time \(t\) has a singular point of density \(\geq N/2\) for a.e. \(t\). A further complicating issue in this case is that even when the curve does have singularities with the right density, its tangent cones may contain higher multiplicity lines or half-lines. Neither of these issues arises in the case \(N = 3\).

Smoothly embedded 1-dimensional flows on the other hand cannot stay close to a singular static junction for too long. In higher dimensions, an analogue is the question of what one can say about minimal surfaces weakly close to a pair of transverse planes (say, in \(\mathbb{R}^3\)). In that case, the difficulties are illustrated by Scherk’s surfaces which show that no uniform estimates can hold without further hypotheses.

**An outline of the proof of Theorem 1:** Without loss of generality, let \(B_2 \times [0,4]\) be the space-time region in Theorem 1, where \(B_2\) is the open ball in \(\mathbb{R}^2\) (space) with radius 2
and center at the origin. Let \( V_t, t \in [0, 4] \) denote the moving 1-varifold at time \( t \), and let \( J \) denote a fixed stationary triple junction with vertex at the origin. Thus \( J \) consists of three half-lines meeting at 120 degree angles at the origin. By assumption, the flow is weakly close to \( J \), which in particular means that the space-time \( L^2 \) distance (height excess) \( \mu \) of the flow \( \{V_t\}_{t \in [0,4]} \) relative to \( J \), defined by

\[
\mu = \left( \int_0^4 \int_{B_2} \text{dist}^2(x, J) \, d\|V_t\|(x) \, dt \right)^{1/2},
\]

is small.

As mentioned before, our proof of Theorem 1 is based on a parabolic version of the blow-up method. We first use the full strength of [13] to obtain (in Proposition 5.1) a graphical representation of the varifolds \( V_t \), with an appropriate estimate, away from the center of \( J \). We use this graphical representation to establish various a priori space-time and time uniform \( L^2 \)-estimates that control the behavior of the flow in the region near the center of \( J \). In particular, a key step is to show that \( \mu \) does not concentrate near the center of \( J \).

Our approach to establishing this a priori non-concentration estimate is inspired by the basic strategy developed by Simon [29] for minimal submanifolds. A key ingredient in Simon’s method is the monotonicity formula for minimal submanifolds, whose role here is played by a certain local estimate inspired by the Huisken monotonicity formula. In the present parabolic setting, there are several interesting new aspects also. These stem from firstly the fact that all we have at our disposal is Brakke’s inequality defining the flow—which a priori only tells us something about the rate of change of mass (length) and not much about the velocity of motion—and secondly the fact that we need a number of nontrivial preliminary estimates involving curvature, which in the case of minimal submanifolds are not needed (regardless of the dimension). A key such estimate (established in Proposition 4.5) gives an interior space-time \( L^2 \) bound for the generalized curvature \( h = h(V_t, x) \) of \( V_t \) (where \( x \in \text{spt} \|V_t\| \)) in terms of \( \mu \) whenever \( \mu \) is sufficiently small; said more precisely,

\[
\int_1^3 \int_{B_{3/2}} |h|^2 \, d\|V_t\| \, dt \leq c \mu^2
\]

provided \( \mu \) is sufficiently small, where \( c \) is a fixed constant independent of the flow. We use this estimate and computations similar to those used in the derivation of the Huisken monotonicity formula [13] (see also [16]) to establish (in Proposition 5.1), whenever \( \mu \) is sufficiently small, that

\[
\int_{5/4}^s \int_{B_1} \left| h + \frac{x^+}{2(s-t)} \right|^2 \rho_{(0,s)}(x, t) \, d\|V_t\|(x) \, dt \leq c \mu^2
\]

for any \( s \in [3/2, 3] \) such that \( h(V_s, \cdot) \in L^2_{\text{loc}}(\|V_s\|) \) and \( \Theta(\|V_s\|, 0) \geq \Theta(\|J\|, 0) = 3/2 \), where \( c \) is a fixed constant independent of the flow, \( \rho_{(0,s)}(x, t) = (4\pi(s-t))^{-1/2} e^{-\frac{s-t}{4}} \) \((-\infty < t < s < \infty)\) is the backwards heat kernel with pole at \((0, s)\) and \( \Theta(\|V\|, Z) \) denotes the density of \( V \) at \( Z \). This bound is then used (in Proposition 5.2) to obtain, for any \( s \in [3/2, 3] \) as above and any \( \kappa \in (0, 1) \), the crucial estimate

\[
\sup_{t \in [5/4, s]} |s-t|^{-\kappa} \int_{B_{3/4}} \rho_{(0,s)}(\cdot, t) \, \text{dist}^2(\cdot, J) \, d\|V_t\|(x) \leq c_0 \mu^2,
\]

again provided \( \mu \) is sufficiently small, where \( c_0 \) depends only on \( \kappa \). This says that the \( L^2 \) distance of \( V_t \) from the triple junction \( J \) weighted by the backwards heat kernel decays
quickly in time; in particular, this estimate implies that the contribution to $\mu^2$ coming from a small spatial neighborhood of the origin and a slightly smaller time interval is a small proportion of $\mu^2$.

Using these estimates, we carry out a careful blow-up analysis in Sec. 4. We emphasize that the term $(s - t)^{-\kappa}$ appearing in the preceding estimate, though not needed for the non-concentration conclusion just pointed out, plays an important role in the blow up analysis. Once the appropriate asymptotic decay for the blow-ups are established, we obtain a space-time excess improvement lemma (Lemma 7.13) for the flow, the iteration of which leads to Theorem 1 in a fairly standard way.

**Organization of the paper:** In Sec. 2 we fix notation and state our main results. In Sec. 3 we use results of [13] to give a graph representation of the moving curves away from the center of the triple junction. The main result in Sec. 4 is Proposition 4.3, which gives a time-uniform estimate on the difference of length between the moving curve and the triple junction in terms of the space-time $L^2$ distance of the flow to the triple junction. The same estimate gives an $L^2$ curvature estimate in terms of the $L^2$ distance. Sec. 5 contains the main non-concentration estimate, Proposition 5.2, which shows that the $L^2$ distance does not concentrate around the junction point. This is used in Sec. 6 to estimate the location of and the Hölder norm (in time) for the junction points in term of the $L^2$ distance. All of these estimates are used to carry out a blow-up argument in Sec. 7 on each of the three rays of the triple junction and to show that the three pieces of the blow-up come together at a single point in a regular fashion. Sec. 8 describes the iteration procedure giving a Hölder estimate of the gradient up to the (moving) junction points, proving the main local regularity theorem (Theorem 2.1). Sec. 9 contains the proof of the partial regularity theorem (Theorem 2.2). Sec. 10 contains a further result concerning the nature of the tangent flows at singular points of a flow satisfying the hypotheses of Theorem 2.2.

### 2. Notation, background and the main theorems

#### 2.1. Basic notation

Let $\mathbb{N}$ be the the set of natural numbers and let $\mathbb{R}^+ := \{x \geq 0\}$. For $r \in (0, \infty)$ and $a \in \mathbb{R}^2$, define $B_r(a) := \{x \in \mathbb{R}^2 : |x - a| < a\}$ and when $a = 0$, define $B_r := B_r(0)$. We write $L^1$ for the Lebesgue measure on $\mathbb{R}$ and $\mathcal{H}^1$ for the 1-dimensional Hausdorff measure on $\mathbb{R}^2$. The restriction of $\mathcal{H}^1$ to a set $A$ is denoted by $\mathcal{H}^1|_A$. For an open set $U \subseteq \mathbb{R}^2$ let $C_c(U)$ be the set of all compactly supported continuous functions on $U$ and let $C_c(U; \mathbb{R}^2)$ be the the set of all compactly supported continuous vector fields. The index $k$ of $C^k$ indicates continuous $k$-th order differentiability. $\nabla$ always indicates differentiation with respect to the space variable. For $g \in C^1_c(U; \mathbb{R}^2)$, $\nabla g(x)$ is a $2 \times 2$ matrix-valued function.

For any Radon measure $\lambda$ on $\mathbb{R}^2$ and $\phi \in C_c(\mathbb{R}^2)$, we shall often write $\lambda(\phi)$ for $\int_{\mathbb{R}^2} \phi d\lambda$. We let $\text{spt} \lambda$ be the support of $\lambda$, and $\Theta(\lambda, x)$ be the 1-dimensional density of $\lambda$ at $x$, i.e., $\Theta(\lambda, x) = \lim_{r \to 0^+} \lambda(B_r(x))/(2r)$, when the limit exists. For a $\lambda$ measurable function $f$, $f \in L^2(\lambda)$ means $\int_{\mathbb{R}^2} |f|^2 d\lambda < \infty$ and $f \in L^2_{\text{loc}}(\lambda)$ means $\int_K |f|^2 d\lambda < \infty$ for each compact set $K \subseteq \mathbb{R}^2$.

For $-\infty < t < s < \infty$ and $x, y \in \mathbb{R}^2$, define the 1-dimensional backwards heat kernel $\rho_{(y,s)}(x, t)$ by

$$
(2.1) \quad \rho_{(y,s)}(x, t) = \frac{1}{\sqrt{4\pi(s-t)}} \exp\left(-\frac{|x-y|^2}{4(s-t)}\right).
$$
Let \( G(2,1) \) be the space of 1-dimensional subspaces of \( \mathbb{R}^2 \). For \( S \in G(2,1) \), we identify \( S \) with orthogonal projection (and the \( 2 \times 2 \) matrix associated with orthogonal projection) of \( \mathbb{R}^2 \) onto \( S \) and we let \( S^\perp \in G(2,1) \) be the orthogonal complement of \( S \) in \( \mathbb{R}^2 \). For \( A, B \in \text{Hom}(\mathbb{R}^2; \mathbb{R}^2) \) let \( A \cdot B = \text{trace}(A^* \circ B) \), where \( \circ \) denotes composition and \( A^* \) is the transpose of \( A \). Let \( u \otimes v \in \text{Hom}(\mathbb{R}^2; \mathbb{R}^2) \) be the tensor product of \( u, v \in \mathbb{R}^2 \).

2.2. Varifolds. We next recall the notion of varifolds and some related definitions. For a detailed discussion on varifolds, see \[1, 27\]. For an open set \( U \subseteq \mathbb{R}^2 \), define \( G_1(U) = U \times G(2,1) \). A 1-varifold in \( U \) is a Radon measure on \( G_1(U) \). The set of 1-varifolds in \( U \) is denoted by \( \mathbf{V}_1(U) \). Varifold convergence is the usual measure convergence on \( G_1(U) \). In this paper we are only concerned with 1-varifolds and shall often just refer to them as varifolds subsequently. For a varifold \( V \in \mathbf{V}_1(U) \) let \( \|V\| \) denote the weight measure associated to \( V \), defined by \( \|V\|(\phi) = \int_{G_1(U)} \phi(x) dV(x,S) \) for \( \phi \in C_c(U) \). Given a \( \mathcal{H}^1 \) measurable countably 1-rectifiable set \( M \subseteq U \) with locally finite \( \mathcal{H}^1 \) measure, there is a natural varifold denoted by \( |M| \) and defined by

\[
|M|(\phi) = \int_M \phi(x, \text{Tan}_x M) d\mathcal{H}^1(x), \quad \forall \phi \in C_c(G_1(U)),
\]

where \( \text{Tan}_x M \in G(2,1) \) is the approximate tangent space of \( M \) at \( x \) which exists \( \mathcal{H}^1 \) a.e. on \( M \). \( V \in \mathbf{V}_1(U) \) is called integral if

\[
V(\phi) = \int_M \phi(x, \text{Tan}_x M) \theta(x) d\mathcal{H}^1(x), \quad \forall \phi \in C_c(G_1(U)),
\]

for some \( \mathcal{H}^1 \) measurable countably 1-rectifiable set \( M \subseteq U \) and \( \mathcal{H}^1 \) a.e. integer-valued locally \( \mathcal{H}^1 \) integrable function \( \theta \) defined on \( M \). The function \( \theta \) is called the multiplicity of \( V \). Set of all integral 1-varifolds in \( U \) is denoted by \( \mathbf{IV}_1(U) \). \( V \) is called a unit density varifold if \( V \) is integral with \( \theta = 1 \) a.e., that is, if \( V = |M| \) with \( M \) as above. For \( V \in \mathbf{IV}_1(U) \) and a given \( \|V\| \)-measurable vector field \( g \) on \( U \), we shall often write \( \int_U (g(x))^\perp d\|V\|(x) \) for \( \int_{G_1(U)} S^\perp(g(x)) dV(x,S) \).

For \( V \in \mathbf{V}_1(U) \), let \( \delta V \) denote the first variation of \( V \), defined by

\[
\delta V(g) = \int_{G_1(U)} \nabla g(x) \cdot S dV(x,S), \quad \forall g \in C^1_c(U; \mathbb{R}^2).
\]

Let \( \|\delta V\| \) be the total variation measure of \( \delta V \) when it exists (which is the case precisely when \( \delta V \) is locally bounded). If \( \|\delta V\| \) is absolutely continuous with respect to \( \|V\| \), we have for some \( \|V\| \) measurable vector field \( h(V, \cdot) \),

\[
(2.2) \quad \delta V(g) = -\int_U g(x) \cdot h(V,x) d\|V\|(x), \quad \forall g \in C^1_c(U; \mathbb{R}^2).
\]

By \( h(V, \cdot) \), we always mean the vector field satisfying (2.2). We simply call \( h(V, \cdot) \) the (generalized) curvature of \( V \). If \( V = |M| \) and \( M \) is a \( C^2 \) curve, then \( h(V, \cdot) \) is the curvature of \( M \) times the unit normal vector.

For any \( V \in \mathbf{IV}_1(U) \) with locally bounded first variation, we note that Brakke’s perpendicularity theorem \[3, \text{Ch. 5}\] says that

\[
(2.3) \quad \int_U (g(x))^\perp \cdot h(V,x) d\|V\|(x) = \int_U g(x) \cdot h(V,x) d\|V\|(x), \quad \forall g \in C_c(U; \mathbb{R}^2).
\]
2.3. The right-hand side of the curvature flow equation. For \( V \in \mathbf{V}_1(U) \), a vector field \( u \in L^2_{loc}(\|V\|) \) and \( \phi \in C^1_c(U; \mathbb{R}^+) \), define \( \mathcal{B}(V, u, \phi) \) as follows:

\[
\mathcal{B}(V, u, \phi) = \int_U (-\phi(x)h(V, x) + \nabla \phi(x)) \cdot (h(V, x) + (u(x))^\perp) \, d\|V\|(x)
\]

if \( V \in \mathbf{IV}_1(U) \), \( \|\delta V\| \) exists and is absolutely continuous with respect to \( \|V\| \) and \( h(V, \cdot) \in L^2_{loc}(\|V\|) \); otherwise \( \mathcal{B}(V, u, \phi) = -\infty \). If \( \{M_t\} \) is a family of smoothly embedded curves moving with normal velocity \( v = h + u^\perp \), where \( u \) is a given smooth ambient vector field, then one can prove that

\[
\frac{d}{dt} \mathcal{H}^1 \mathcal{L}_{M_t}(\phi) = \mathcal{B}(|M_t|, u, \phi) \quad \forall \phi \in C^1_c(U; \mathbb{R}^+).
\]

Conversely, having the property (2.5) with \( \leq \) in place of equality for smooth \( M_t \) implies that it holds with equality and that \( v = h + u^\perp \) on \( M_t \), so we may use (2.5) with \( \leq \) in place of equality as a weak formulation of the condition \( v = h + u^\perp \). (See Hypothesis (A4) below.)

2.4. The triple junction \( J \), some test functions and norms. Let \( J \subset \mathbb{R}^2 \) be defined by

\[
J = \{(s, 0) : s \geq 0\} \cup \left\{(-s, \sqrt{3}s) : s \geq 0\right\} \cup \left\{(-s, -\sqrt{3}s) : s \geq 0\right\}.
\]

For \( \theta \in \mathbb{R} \) denote by \( R_\theta : \mathbb{R}^2 \to \mathbb{R}^2 \) the map corresponding to the counterclockwise orthogonal rotation by angle \( \theta \). Define a similarity class of \( J \) by

\[
\mathcal{J} = \left\{ R_\theta(J) + \xi : \theta \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right) \text{ and } \xi \in \mathbb{R}^2 \right\},
\]

and for \( R \in (0, \infty) \), define \( d_R : \mathcal{J} \times \mathcal{J} \to \mathbb{R} \) by

\[
d_R(J_1, J_2) = \max \left\{ R^{-1}|\xi_1 - \xi_2|, |\theta_1 - \theta_2| \right\}
\]

for \( J_1 = R_{\theta_1}(J) + \xi_1 \) and \( J_2 = R_{\theta_2}(J) + \xi_2 \). It is clear that \( d_R \) defines a metric on \( \mathcal{J} \). We set

\[
d(J_1, J_2) = d_1(J_1, J_2).
\]

Let \( \hat{\phi} : \mathbb{R}^2 \to \mathbb{R} \) be a smooth radially symmetric non-negative function such that \( \hat{\phi}(x) = 1 \) for \( |x| \leq \frac{1}{4} \), \( \hat{\phi}(x) = 0 \) for \( |x| \geq \frac{1}{2} \), \( 0 \leq \hat{\phi} \leq 1 \) and \( |\nabla \hat{\phi}| \leq 8 \). Set

\[
c_1 = \int_\mathbb{R} \hat{\phi}(s, 0) \, ds.
\]

Define

\[
\phi_j(x) = \hat{\phi} \left( R_{\theta_j}(\frac{2\pi}{3}x) - (1, 0) \right)
\]

for \( j = 1, 2, 3 \) and \( x \in \mathbb{R}^2 \). Note that \( \phi_1 \) is just \( \hat{\phi} \) composed with translation by \((1, 0)\), and \( \phi_2 \) and \( \phi_3 \) are \( \phi_1 \) composed with orthogonal rotation by \( \frac{2\pi}{3} \) and \( \frac{4\pi}{3} \) respectively. We have \( \int_{\mathcal{J}} \phi_j \, d\mathcal{H}^1 = c_1 \) for each \( j = 1, 2, 3 \). For \( J' = R_{\theta}(J) + \xi \in \mathcal{J} \), \( R \in (0, \infty) \) and \( j = 1, 2, 3 \), define

\[
\phi_{j, J, R}(x) = \phi_j(R_{-\theta}(R^{-1}(x - \xi))), \quad \phi_{j, J}(x) = \phi_{j, J, 1}(x)
\]

for \( x \in \mathbb{R}^2 \), where \( \phi_j \) is as in (2.11). Note that \( \phi_{j, J} = \phi_j \).
For $R > 0$, let $Q_R = \{(s, t) \in (-R, R) \times (-R^2, R^2)\}$. We shall use the following norm for functions $f : Q_R \to \mathbb{R}$:

$$
\|f\|_{C^{1, \xi}(Q_R)} = \sup_{(s, t) \in Q_R} \left( R^{-1} |f(s, t)| + |\nabla f(s, t)| \right) \\
+ \sup_{(s_1, t_1), (s_2, t_2) \in Q_R, (s_1, t_1) \neq (s_2, t_2)} \frac{R^\xi |\nabla f(s_1, t_1) - \nabla f(s_2, t_2)|}{\max\{|s_1 - s_2|, |t_1 - t_2|^{\frac{2}{\xi}}\}} \\
+ \sup_{(s, t_1), (s, t_2) \in Q_R, t_1 \neq t_2} \frac{R^\xi |f(s, t_1) - f(s, t_2)|}{|t_1 - t_2|^{\frac{2}{\xi}}}.
$$

Note that this norm is invariant under the parabolic change of variables in the sense that if $f(\hat{s}, \hat{t}) = R^{-1} f(s, t)$ where $\hat{s} = R^{-1} s$ and $\hat{t} = R^{-2} t$, then $\|f\|_{C^{1, \xi}(Q_1)} = \|f\|_{C^{1, \xi}(Q_R)}$. We shall denote by $C^{1, \xi}(Q_R)$ the space of functions $f : Q_R \to \mathbb{R}$ with $\|f\|_{C^{1, \xi}(Q_R)} < \infty$.

2.5. Hypotheses and the main theorems. Let $U \subseteq \mathbb{R}^2$ be open and let $I \subseteq \mathbb{R}$ be an interval (i.e. a connected subset of $\mathbb{R}$). Assume

$$(A0) \quad p \in [2, \infty) \text{ and } q \in (2, \infty) \text{ are fixed numbers such that}$$

$$
\xi \equiv 1 - \frac{1}{p} - \frac{2}{q} > 0.
$$

For each $t \in I$, let $V_t$ be a 1-varifold in $U$ and $u(\cdot, t) : U \to \mathbb{R}^2$ a $\|V_t\|$ measurable vector field such that:

(A1) $V_t \in IV_1(U)$ for a.e. $t \in I$;

(A2) there exists $E_1 \in [1, \infty)$ such that for each $B_r(x) \subset U$ and each $t \in I$,

$$
\|V_t\|(B_r(x)) \leq 2rE_1;
$$

(A3) $u$ satisfies

$$
\left( \int_I \left( \int_U |u(x, t)|^p d\|V_t\|(x) \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}} < \infty;
$$

(A4) for each $\phi \in C^1(U \times I; \mathbb{R}^+)$ with $\phi(\cdot, t) \in C^1_c(U)$ for $t \in I$, and for any $t_1, t_2 \in I$ with $t_1 < t_2$,

$$
\|V_t\|(\phi(\cdot, t))_{t=t_1}^{t_2} \leq \int_{t_1}^{t_2} B(V_t, u(\cdot, t), \phi(\cdot, t)) dt + \int_{t_1}^{t_2} \int_U \frac{\partial \phi}{\partial t}(\cdot, t) d\|V_t\| dt.
$$

Remark: If $u = 0$, Huisken’s monotonicity formula which can be derived from (A4) (more precisely, the corresponding “monotonicity inequality” for Brakke flows, see e.g. [10]) shows that (A2) is locally satisfied for some $E_1$. Thus (A2) may be dropped in the case of classical Brakke flows along with (A3) (since $u = 0$). (A4) is a weak formulation of the condition $v = h + u^\perp$, allowing time-dependent test functions. Thus, in case $u = 0$, (A1)-(A4) are simply the definition of Brakke flow in integrated form (see [3 3.5]).

The following is our $\varepsilon$-regularity theorem, whose proof takes up a major part of the paper:

**Theorem 2.1.** Corresponding to $p, q$ as in (A0), $E_1 \in [1, \infty)$ and $\nu \in (0, 1)$, there exist $\varepsilon_1 \in (0, 1)$ and $c_2 \in (1, \infty)$ such that the following holds: For $R \in (0, \infty)$ and $U = B_{4R}$, let $\{V_t\}_{t \in [-2R^2, 2R^2]}$ and $\{u(\cdot, t)\}_{t \in [-2R^2, 2R^2]}$ satisfy (A1)-(A4) with $I = [-2R^2, 2R^2]$. Suppose

$$
\mu \equiv \left( R^{-5} \int_{-2R^2}^{2R^2} \int_{B_{4R}} \text{dist}(\cdot, J)^2 d\|V_t\| dt \right)^{\frac{1}{2}} < \varepsilon_1;
$$


there exist \( j_1, j_2 \in \{1, 2, 3\} \) such that

\[
R^{-1}||V_{-2R}\|((\phi_{j_1,J,R}) \leq (2 - \nu)c_1, \quad R^{-1}||V_{2R}\|((\phi_{j_2,J,R}) \geq \nu c_1
\]

where \( J, c_1 \) and \( \phi_{j_2,J,R} \) are as defined in (2.6), (2.10) and (2.12) respectively, and

\[
\|u\| \leq R^c \left( \int_{-2R}^{2R} \left( \int_{B_{4R}} |u|^p d|V_1| \right)^{\frac{2}{p}} \right)^{\frac{1}{2}} < c_1.
\]

Then there exists \( \hat{a} \in C^{1+\xi}((-2R^2, 2R^2); B_R) \) and, letting

\[
l_j(t) = \text{first coordinate of } R \left( \frac{2R}{3} \right) \cap \{l_j(t), R\} \cap \{l_j(t), R\} = \{x, f_j(x, t) : x \in [l_j(t), R]\}
\]

there exist functions \( f_j \in C^{1,\xi}(D_j), j = 1, 2, 3, \) such that for all \( t \in [-2R^2, 2R^2] \), we have

\[
\frac{\partial f_1}{\partial x}(l_1(t), t) = \frac{\partial f_2}{\partial x}(l_2(t), t) = \frac{\partial f_3}{\partial x}(l_3(t), t).
\]

Furthermore, we have

\[
\|\hat{a}\|_{C^{1+\xi}((-2R^2, 2R^2); B_R)} + \sum_{j=1}^{3} \|f_j\|_{C^{1,\xi}(D_j)} \leq c_2 \max\{\mu, \|u\|\}.
\]

Remark: In case \( u \in C^\beta(B_{4R} \times [-2R^2, 2R^2]; R^2) \) (where Hölder continuity is in the usual parabolic sense), the result of [33] shows that \( f_j \) are \( C^{2,\beta} \) away from the junction point and that the flow satisfies \( v = h + u^\perp \) in the classical sense. In fact in this case, up-to-the-junction-point \( C^{2,\beta} \) regularity of \( f_j \) as well as \( C^{1,\frac{\xi}{2}} \) regularity of \( \hat{a} \) also hold, and can be proved by the well-known reflection technique of [20] combined with the regularity theory of linear parabolic systems [31]. If \( u \) is smooth, or zero in particular, then \( \hat{a} \) is smooth, and \( f_j \) are smooth up to the junction point. Note that for the reflection technique of [20], having \( C^{1,\xi} \) regularity given by our theorem provides the crucial starting hypothesis.

We note that all quantities are scale invariant under parabolic change of variables so we may and we shall, without loss of generality, set \( R = 1 \) in the proof of Theorem 2.1. The inequality (2.17) provides a closeness to \( J \) of \( \|V_t\| \) in the \( L^2 \) distance, and (2.18) requires some closeness to \( J \) in terms of measure at the initial and final times. The latter also prevents complete loss of measure \( \|V_t\| \) during this time interval. Assumption (2.19) ensures smallness of the perturbation from the \( u = 0 \) case, and obviously is not needed if \( u = 0 \). The conclusion is that each time slice of the flow in a smaller space-time domain consists of three embedded curves meeting precisely at one common junction point, that this junction point \( \hat{a}(t) \) at time \( t \) is a Hölder continuous function of \( t \), and that the three curves are represented as \( C^{1,\xi} \) graphs up to the junction point as in (2.20) and (2.21), satisfying the estimate (2.23). Moreover, at each time, the three curves meet at 120 degree angles, as expressed by (2.22). If more regularity is assumed on \( u \), then we have, as stated in the Remark above, better regularity up to the junction point.

By combining Theorem 2.1 with a stratification theorem of White [35], we obtain the following partial regularity theorem. We make an assumption on the density of static tangent flows, which is equivalent to assuming that any static tangent flow is either a unit density line or a unit density triple junction. For the precise definition of tangent flow, see
Sec.9. Tangent flows are analogous to tangent cones for minimal submanifolds: at each point in space-time, at least one tangent flow is obtained by passing to a subsequential varifold limit of parabolic rescalings of the flow, and tangent flows enjoy a nice homogeneity property called backwards-cone-like (see Sec.9 (b)).

**Theorem 2.2.** In addition to the assumptions (A1)-(A4), assume that

\[(A5)\] at each point in space-time, whenever a tangent flow to \(\{V_t\}_{t \in I}\) is static, the density at the origin of the tangent flow is strictly less than 2.

Then there exists a closed set \(\Sigma_1 \subset U \times I\) with the parabolic Hausdorff dimension at most 1 such that the flow in \(U \times I \setminus \Sigma_1\) is classical in the sense that for any \((x, t) \in U \times I \setminus \Sigma_1\), there exists a space-time neighborhood \(U_{x,t}\) containing \((x, t)\) such that \(U_{x,t} \cap \cup_{t'}(\text{spt } [V_t'] \times \{t'\})\) is either empty, a \(C^{1,\zeta}\) graph over a line segment or a \(C^{1,\zeta}\) triple junction as described in Theorem 2.1.

**Remarks:**

(1) Under hypotheses (A1)-(A4), we may completely classify all non-trivial static tangent flows. They are time-independent stationary integral varifolds whose supports are unions of half-lines emanating from the origin. Thus hypothesis \((A5)\) requires that any static tangent flow is a single line or a triple junction, either one with unit density, and nothing else. This hypothesis is motivated by the fact that any static tangent flow with density greater than or equal to 2 should be unstable for various physical models. For the motion of grain boundaries, one observes that junctions with more than 3 edges appear and break up instantaneously. Mathematically, any junction (including lines) with multiplicity strictly greater than 1 is not mass minimizing in the sense that one can always set the multiplicity equal to 1 and reduce the mass. Any unit density junction with more than 3 edges may be mapped by a suitable Lipschitz function so that the image of the map has locally less \(H^1\) measure as a set. (Note that the usual varifold push-forward counts multiplicities of the image and the mapping here is different from it.) It is called *reduced mass model* according to Brakke [3, p.57].

(2) Other than the singularities coming from collisions of triple junctions, we may also have some curve disappearing suddenly. Such singularities are included in the closed set \(\Sigma_1\).

(3) The parabolic Hausdorff dimension counts the time variable as 2. Thus, \(\Sigma_1\) having parabolic dimension at most 1 implies that the times at which singularities can occur form a closed subset of \(I\) of usual Hausdorff dimension at most \(1/2\).

(4) The short-time existence of classical network flows (i.e. those consisting of curves meeting smoothly and only at a locally finite number of triple junctions) was established by Bronsard–Reitich [4] when the initial network itself is classical, and has recently been extended to more general initial networks satisfying certain regularity and non-degeneracy assumptions by Ilmanen–Neves–Schulze [17]. (The work [17] also gives a result that says that a flow weakly close to a triple junction \(J\) is \(C^{1,\alpha}\) close to \(J\) in the interior (see [17], Theorem 1.3 and remark (iii)) in the special case when the flow is a priori assumed to be regular, using methods limited to such an priori regularity hypothesis.) It remains an interesting open problem to prove a general existence theorem for curvature flows satisfying (A1)-(A5).

(5) See Sec.10 for a more detailed characterization of \(\Sigma_1\) in terms of tangent flows.
3. A graph representation away from the singularity of $J$

We apply results from [18] to show that the supports of the moving varifolds, in the region outside a small neighborhood of the singularity of $J$, are represented as a $C^{1,\xi}$ graphs, with the $C^{1,\xi}$ norm bounded in terms of the $L^2$ distance of the flow to $J$. This result will be used frequently in the rest of the paper.

Proposition 3.1. Corresponding to $\tau \in (0, \frac{1}{4})$, $\nu \in (0, 1)$, $E_1 \in [1, \infty)$ and $p \in [2, \infty)$, $q \in (2, \infty)$ satisfying (2.13), there exist $\varepsilon_2 \in (0, 1)$ and $c_2 \in (1, \infty)$ such that the following holds: Suppose that $\{V_t\}_{t \in [0,4]}$ and $\{u(\cdot, t)\}_{t \in [0,4]}$ satisfy (A1)-(A4) with $U = B_2$ and $I = [0,4]$, and that

\[ \mu \equiv \left( \int_0^4 \int_{B_2} \text{dist}(\cdot, J)^2 d\|V_t\| dt \right)^{\frac{1}{2}} \leq \varepsilon_2, \]

(3.1)

\[ \|u\| \equiv \left( \int_0^4 \left( \int_{B_2} |u|^p d\|V_t\| \right) \frac{1}{2} dt \right)^{\frac{1}{2}} \leq \varepsilon_2, \]

(3.2)

\[ \|V_0\|(\phi_{j_1}) \leq (2 - \nu)c_1 \text{ for some } j_1 \in \{1, 2, 3\} \text{ and}, \]

(3.3)

\[ \|V_0\|(\phi_{j_2}) \geq \nu c_1 \text{ for some } j_2 \in \{1, 2, 3\}. \]

Then

\[ B_2 - \frac{\tau}{5} \cap \left\{ \text{dist}(\cdot, J) > \frac{\tau}{3} \right\} \cap \text{spt} \|V_t\| = \emptyset \text{ for each } t \in [\tau, 4], \]

(3.5)

and there exist $f_j : [\tau, 2 - \tau] \times [\tau, 4 - \tau] \to \mathbb{R}$, $j = 1, 2, 3$, such that $f_1, f_2, f_3$ are $C^{1,\xi}$ in space and $C^{\frac{1}{2}}$ in time with

\[ \|f_j\|_{C^{1,\xi}(Q)} \leq c_3 \max\{\mu, \|u\|\} \]

where $Q = [\tau, 2 - \tau] \times [\tau, 4 - \tau]$, and

\[ ([\tau, 2 - \tau] \times [-\tau, \tau]) \cap \left( \mathbb{R} \times \text{supp}(\text{spt} \|V_t\|) \right) = \left\{ (s, f_j(s, t)) : s \in [\tau, 2 - \tau] \right\}, \]

(3.7)

for $j = 1, 2, 3$ and for all $t \in [\tau, 4 - \tau]$. Furthermore, for a.e. $t \in [\tau, 4 - \tau]$, we have that

\[ \Theta(\|V_t\|, x) \in \{1, 3/2\} \text{ for each } x \in B_{2-\tau} \cap \text{spt} \|V_t\|. \]

(3.8)

Proof. The claim (3.5) may be deduced by applying [18] Prop. 6.4 & Cor. 6.3. We also see that for fixed $\tau \in (0, 1/2)$, $\text{spt} \|V_t\| \cap B_{2-\tau}$ approaches $J$ as $\mu, \|u\| \to 0$ for all $t \in [\tau, 4]$.

To prove the existence of a graph representation and the $C^{1,\xi}$ estimate as asserted, assume for fixed $E_1$, $\nu$, $\tau$ that the claim is false. Then for each $m \in \mathbb{N}$ there exist $\{V_t^{(m)}\}_{t \in [0,4]}$ and $\{u^{(m)}(\cdot, t)\}_{t \in [0,4]}$ satisfying (A1)-(A4) and (3.1)-(3.4) with $U = B_2$, $I = [0,4]$, $\varepsilon_2 = m^{-1}$ and with $V_t^{(m)}$, $u^{(m)}$ in place of $V_t$, $u$, but there are no functions $f_1$, $f_2$, $f_3$ with the stated regularity satisfying (3.1) and (3.4) with $c_3 = m$, $u^{(m)}$ in place of $u$ and

\[ \mu^{(m)} = \left( \int_0^4 \int_{B_2} \text{dist}^2(\cdot, J) d\|V_t^{(m)}\| dt \right)^{\frac{1}{2}} \text{ in place of } \mu. \]

To obtain a contradiction, we will use [18] Th. 8.7 which shows the existence of such a graph representation as in the asserted conclusion under a set of hypotheses. In order to check that the hypotheses of [18] Th. 8.7, with $V_t^{(m)}$, $u^{(m)}$ in place of $V_t$, $u$, are satisfied for sufficiently large $m$, we first prove that

\[ \|V_t^{(m)}\| \to H^1 \mathcal{L}_J \]

(3.9)
as $m \to \infty$ on $B_2$ for all $t \in (0, 4)$. To see this, take any $\varphi \in C^2_\infty(B_2; \mathbb{R}^+)$ and use (A4) to obtain for any $t_1, t_2 \in [0, 4]$ with $t_1 < t_2$ that
\begin{equation}
\|V_{t_2}^{(m)}(\varphi) - V_{t_1}^{(m)}(\varphi)\| \leq \int_{t_1}^{t_2} \int -\frac{|h^{(m)}|^2}{2} \varphi + \frac{\nabla \varphi}{2\varphi} + |u^{(m)}|^2 \varphi + |\nabla \varphi|u^{(m)}|d\|V_{t}^{(m)}\|dt
\end{equation}
where $h^{(m)} = h(V_{t}^{(m)}, x)$. Define $\Phi^{(m)}(t) = \int_0^t \int -\frac{|\nabla \varphi|^2}{2\varphi} + |u^{(m)}|^2 \varphi + |\nabla \varphi|u^{(m)}|d\|V_{t}^{(m)}\|dt$. Since $\sup |\nabla \varphi|^2/\varphi \leq \sup 2|\nabla^2 \varphi|$, we see by Hölder’s inequality combined with the fact that $\|u^{(m)}\| \leq m^{-1}$ and $\|V_{t}^{(m)}\|(\text{spt} \varphi) \leq 4E_1$, that $\{\Phi^{(m)}\}_{m\in \mathbb{N}}$ is bounded uniformly in the $\frac{2-2}{q}$-Hölder norm on $[0, 4]$. In particular, we may choose a uniformly convergent subsequence of $\{\Phi^{(m)}\}$. We also see from (3.10) that $\|V_{t}^{(m)}(\varphi) - \Phi^{(m)}(t)\|$ is monotone decreasing in $t$. Because of this, we may extract a subsequence $ \{m_j\}_{j\in \mathbb{N}}$ such that, for each $t$ except for a countable number of $t$, $\{\|V_{t}^{(m_j)}(\varphi)\|_{j\in \mathbb{N}}$ is a Cauchy sequence. Next choose a countable set $\{\varphi_i\}_{i\in \mathbb{N}} \subset C^2_\infty(B_2, \mathbb{R}^+)$ such that it is dense with respect to the $C^0$ topology in $C^2_\infty(B_2, \mathbb{R}^+)$. By a diagonal argument, we may choose a subsequence so that $\{\|V_{t}^{(m_j)}(\varphi_i)\|_{j\in \mathbb{N}}$ is a Cauchy sequence except for a countably many $t$ and for all $l \in \mathbb{N}$. Since $\{\|V_{t}^{(m_j)}\|\}_{j\in \mathbb{N}}$ is a set of uniformly bounded measures, this shows that $\|V_{t}^{(m_j)}\|$ converges to a Radon measure, say, $\lambda_t$ for all $t \in [0, 4]$ except for countably many $t$. By the weak compactness of Radon measures, we may extend $\lambda_t$ to all $t \in [0, 4]$ as Radon measures such that passing to a further subsequence, $\|V_{t}^{(m_j)}\|$ converges to $\lambda_t$ for all $t \in [0, 4]$. By the first part of the proof, we know that $\text{spt} \lambda_t \subset J$ for all $t \in [0, 4].$ By (3.3) and (3.4), we also have that
\begin{equation}
\lambda_0(\phi_{j_1}) \leq (2 - \nu)c_1, \; \lambda_4(\phi_{j_2}) \geq \nu c_1,
\end{equation}
for some $j_1, j_2 \in \{1, 2, 3\}$. Next note that by (3.10), for any fixed $\varphi \in C^2_\infty(B_2, \mathbb{R}^+)$, we have
\begin{equation}
\int_0^4 \int h(V_{t}^{(m_j)}, \cdot) \frac{|\nabla \varphi|^2}{2}\varphi d\|V_{t}^{(m_j)}\|dt \leq \|V_{t}^{(m_j)}(\varphi) + \Phi^{(m)}(4)
\end{equation}
where the right-hand side is uniformly bounded. By Fatou’s lemma applied to (3.12), for a.e. $t \in [0, 4]$, there exists a (time-dependent) subsequence such that
\begin{equation}
\int h(V_{t}^{(m_{j_k})}, \varphi)\varphi d\|V_{t}^{(m_{j_k})}\| \leq C(t, \varphi, E_1)
\end{equation}
for all $k \in \mathbb{N}$. By (A1), for a.e. $t \in [0, 4]$, $V_{t}^{(m_{j_k})}$ is integral. By the compactness theorem for integral varifolds $\mathbb{V}$, there exists a further subsequence which converges to a limit varifold $V_t$ which is integral with locally $L^2$ generalized curvature. Since $\text{spt} \|V_t\| \subset J$ and $h(V_t, \cdot) \in L^2(\|V_t\|)$, the density function of $\|V_t\|$ must be constant on each line segment of $J \cap B_2$. In fact, the density must be constant on all of $J \cap B_2$. Since $\|V_t\| = \lambda_t$, we conclude that for a.e. $t \in [0, 4]$, $\lambda_t = \theta(t)\mathcal{H}^1 L_{m\cap B_2}$ for some $\theta(t) \in \{0\} \cup \mathbb{N}$. On the other hand, using $\varphi = \phi_j$ in (3.10) and taking a limit, we obtain for any $0 \leq t_1 < t_2 \leq 4$
\begin{equation}
\lambda_{t_2}(\phi_j) - \lambda_{t_1}(\phi_j) \leq (t_2 - t_1)E_1 \sup \frac{|\nabla \phi_j|^2}{2\phi_j}
\end{equation}
In general, we have $\lambda_t(\phi_j) = \theta(t)c_1$, but by (3.11) and (3.14), we see that $\lambda_t(\phi_j) = c_1$ for a.e. $t \in [0, 4]$. Thus we have shown that $\theta(t) = 1$ for a.e. $t \in [0, 4]$. Since (3.14) holds also for any function in $C^2_\infty(B_2, \mathbb{R}^+)$ in place of $\phi_j$, we see that $\lambda_t = \mathcal{H}^1 L_{J \cap B_2}$ for all $t \in (0, 4)$. Since
the limit measure is uniquely determined, the whole sequence converges, and this establishes (3.9).

We are now ready to use [18 Th. 8.7]. For any \((x_0, t_0) \in J \cap (B_2 \setminus B_\tau) \times (\tau, 4 - \tau)\), consider a small domain containing \((x_0, t_0)\) which is a positive distance away from the origin. In view of (3.9), we have (8.85) and (8.86) (with \(\nu = 1/2\), for example) of [18 Th. 8.7] satisfied for all sufficiently large \(m\), and so are the other assumptions (8.83) and (8.84). Note that [18 Th. 8.7] assumes the varifold has unit density a.e. but one can indeed prove using a variant of Huisken’s monotonicity formula (see [18 Prop. 6.2]) and (3.9) that there cannot be a point \((x, t) \in B_{2-\tau} \times (\tau, 4 - \tau)\) with \(\Theta(||V^t||, x) \geq 2\) for all sufficiently large \(m\). Thus, near \((x_0, t_0)\), \(\text{spt} ||V^t||\) is represented as a graph a function of the desired regularity satisfying the estimate (3.6) on this domain. By covering \(Q\) with a finite number of such small domains, we obtain (3.6) with a suitable constant \(c_3\) which depends only on \(\tau, \nu, E_1, p, q\).

To see (3.8), first note that we have \(h(V_t, \cdot) \in L^2_{\text{loc}}(||V_t||)\) and that \(V_t\) is integral for a.e. \(t\); thus, for such \(t\), \(\Theta(||V_t||, x)\) exists and is greater than or equal to 1 for all \(x \in \text{spt} ||V_t|| \cap B_2\). Moreover, at each \(x \in \text{spt} ||V_t|| \cap B_2\), there exists a tangent cone which is a stationary 1-dimensional integral varifold, and thus we may conclude that \(\Theta(||V_t||, x)\) is an integer multiple of 1/2. Again using a variant of Huisken’s monotonicity formula and (3.9), we conclude that \(\Theta(||V_t||, x) \leq 3/2\) for \(x \in B_{2-\tau}\) for sufficiently small \(\varepsilon_2\). This proves (3.8). \(\Box\)

4. A priori estimates I: the space-time \(L^2\)-curvature estimate

The estimates in this section are analogous to those in [18 Sec. 5] where, roughly speaking, one obtains a time-uniform estimate for the difference of the \(k\)-dimensional area of the moving varifolds and that of a flat plane. There are a few subtle differences however. First, unlike in [18 Prop. 4.6], we do not have a useful Lipschitz graph approximation for the varifolds near the triple junction. Here, in Proposition 4.2 we rely on the fact that \(L^2\) control of curvature gives good \(C^{1,1}\) norm control since the varifolds are 1-dimensional, with the end result being similar to [18 Prop. 5.2]. Lemma 4.4 is similar in spirit to [18 Lem. 5.5], although since we do not have, unlike in [18 Sec. 6], an \(L^\infty\) estimate for \(\text{dist} (\cdot, J)\) on \(\text{spt} ||V||\) in terms of the \(L^2\)-distance, we need to develop an different estimate in Lemma 4.3. Once this lemma is established, we obtain (4.36) and (4.37) just as we obtain the corresponding estimates in [18 Th. 5.7].

**Definition 4.1.** Let \(\phi_{\text{rad}} : \mathbb{R}^2 \to \mathbb{R}^+\) be a non-negative radially symmetric function such that \(\phi_{\text{rad}} = 1\) on \(B_1\), \(|\nabla \phi_{\text{rad}}| \leq 4\) and \(\phi_{\text{rad}} \in C^\infty_c(B_{1/2})\). Define

\[
(4.1) \quad c = \int_j \phi^2_{\text{rad}} d\mathcal{H}^1.
\]

Proposition 4.2 and Corollary 4.3 below do not involve the time variable.

**Proposition 4.2.** Corresponding to \(E_1 \in [1, \infty)\) there exist constants \(c_4 \in (1, \infty), \alpha_1, \beta_1 \in (0, 1)\) such that the following holds. For \(V \in IV_1(B_2)\) with \(h(V, \cdot) \in L^2(||V||)\), define

\[
(4.2) \quad \hat{\alpha} = \left( \int_{B_2} |h(V, x)|^2 \phi_{\text{rad}}(x)^2 d||V|| (x) \right)^{1/2},
\]

and

\[
(4.3) \quad \hat{\mu} = \left( \int_{B_2} \text{dist} (x, J)^2 d||V|| (x) \right)^{1/2}.
\]
Assume the following (4.4)-(4.9):

(4.4) \[ \|V\| (B_2) \leq 4E; \]

(4.5) \[ \Theta(\|V\|, x) \in \left\{ 1, \frac{3}{2} \right\} \text{ for each } x \in \text{spt } \|V\|; \]

(4.6) \[ \left( \left[ \frac{1}{2}, \frac{3}{2} \right] \times \left[ -\frac{1}{2}, \frac{1}{2} \right] \right) \cap \left( R_{-\frac{1}{2}}(\text{spt } \|V\|) \right) = \left\{ (s, f_j(s)) : |s - 1| \leq \frac{1}{2} \right\} \]

for \( j = 1, 2, 3 \) and for some \( f_j \in C^1(\{ s \in \mathbb{R} : |s - 1| \leq \frac{1}{2} \}) \) with

(4.7) \[ \hat{\beta} \equiv \max_{j \in \{1, 2, 3\}} \| f_j \|_{C^1(\{ |s - 1| \leq \frac{1}{2} \})} \leq \beta_1; \]

(4.8) \[ \text{spt } \|V\| \cap B_{\frac{2}{10}} \cap \left\{ x \in \mathbb{R}^2 : \text{dist} (x, J) \geq \frac{1}{10} \right\} = \emptyset \]

and

(4.9) \[ \tilde{\alpha} \leq \alpha_1. \]

Then there exist three \( C^{1, \frac{1}{2}} \) curves \( l_1, l_2, l_3 \) having one common end point near the origin in \( B_1 \) meeting at 120 degree angles such that \( \text{spt } \|V\| \cap B_1 = \cup_{j=1}^{3} l_j \cap B_1 \). Moreover, we have

(4.10) \[ |\mathcal{H}^1(\text{spt } \|V\| \cap B_1) - 3| \leq c_4(\hat{\alpha} \hat{\mu} + \hat{\beta}^2) \]

and

(4.11) \[ |\|V\| (\phi_{rad}^2 - c)| \leq c_4(\hat{\alpha} \hat{\mu} + \hat{\beta}^2). \]

**Proof.** If \( \Theta(\|V\|, x) = 1 \), the Allard regularity theorem [1] combined with the fact that \( h(V, \cdot) \in L^2(\|V\|) \) shows that \( \text{spt } \|V\| \) is an embedded \( C^{1, \frac{1}{2}} \) curve in some neighborhood of \( x \). A standard argument shows that the set of points with \( \Theta(\|V\|, x) = \frac{3}{2} \) is discrete. Specifically, assume for a contradiction that this set has an accumulation point \( a \in B_2 \) and let \( a_1, a_2, \ldots \) be such that \( a_j \neq a, \Theta(\|V\|, a_j) = \frac{3}{2} \) for each \( j = 1, 2, \ldots \) and \( a_j \to a \) as \( j \to \infty \). By (4.5), \( \Theta(\|V\|, a) = \frac{3}{2} \). Consider a subsequential limit \( C \) of rescalings of \( V \) about \( a \) by the scale factors \( |a_i - a|^{-1} \). This limit \( C \) is a stationary integral cone which has density \( \frac{3}{2} \) at the origin and at \( b = \lim_{s \to \infty} \frac{a_i - a}{|a_i - a|} \in S^1 \), hence along the whole ray determined by \( b \), producing a positive measure portion of \( \text{spt } \|C\| \) on which the density is equal to \( \frac{3}{2} \), a contradiction to integrality of \( C \). In particular this shows that there are only finitely many points \( a_1, \ldots, a_N \) in \( B_1 \) with \( \Theta(\|V\|, a_j) = \frac{3}{2} \). Away from these points, one may parametrize the curves by the arc length parameters. One can prove that the weak second derivative is precisely \( h \) a.e., thus the \( \frac{3}{2} \)-Hölder norms of the unit tangent vectors along the curves can be estimated by \( L^2 \) norm of \( h \). Because of this fact, at each \( a_1, \ldots, a_N \), \( \text{spt } \|V\| \) consists of three emanating \( C^{1, \frac{1}{2}} \) curves, and due to the stationarity of tangent cone, the meeting angles of these three curves are all 120 degrees. Let us call these points ‘junction point’. On \( B_1 \setminus \{ a_1, \ldots, a_N \} \), \( \text{spt } \|V\| \) consists of \( C^{1, \frac{1}{2}} \) curves, with the end points on \( \partial B_1 \) or at junction point or without any end points (i.e. closed curve). Since any closed curve \( l \subset B_1 \) have \( \int |h| \geq 2\pi \), Hölder inequality and (4.4) shows \( \hat{\alpha} \geq 2\pi/\sqrt{4E_1} \). Thus for small \( \alpha_1 \), there cannot be any closed curve in \( B_1 \). By (4.6) and (4.8), there are at least three curves \( l_1, l_2, l_3 \) which are the extensions of curves represented as graphs of \( f_j, f_2, f_3 \), respectively. As noted already, small \( \alpha_1 \) implies small change of unit tangent vector along
the curves. Thus choosing sufficiently small $\alpha_1$ and $\beta_1$, we may assume that $l_1, l_2, l_3$ are very close to straight lines and close to $J$ in $C^1$ norm. They hit one of junction point, or otherwise \((4.8)\) would be violated. We note that there cannot be more than one triple junction. Suppose otherwise. Suppose one follows the parametrization of $l_1$ from the right and one hits the first junction point $a_1$. Then one follows the curve emanating from $a_1$ by turning 60 degrees ‘to the right’. Suppose that one hits another junction point $a_2$ along this curve. Then one follows the curve just like before by turning 60 degrees. By choosing $\beta_1$ and $\alpha_1$ small, we can make sure that the latter curve has the tangent vector whose direction is only at most, say, 1 degree different from that of \((1, \sqrt{3})\). Note that $l_1$ is very close to $x$-axis, so after turning 60 degrees twice, the curve should be almost parallel to such vector. Unless one hits the next junction point $a_3$ along this curve, we would have a contradiction to \((4.8)\). In this manner, one can argue that there would have to be infinitely many junction points in $B_1$, noting that the sum of total variations of tangent vector for each curve can be made arbitrarily small. This contradicts the finiteness of the number of triple junctions. Thus we may conclude that $l_1, l_2, l_3$ meet at a unique junction point close to the origin under appropriate restrictions on $\alpha_1$ and $\beta_1$. This proves the first part of the claim. For the rest of the proof, continue to denote the graph representations of $l_1, R_{-\frac{\pi}{3}}l_2, R_{-\frac{2\pi}{3}}l_3$ by (for $j = 1, 2, 3$, respectively)

\[
\{(s, f_j(s)) \in \mathbb{R}^2 : s \in [s_j, 1]\},
\]

where we note that

\[
\begin{align*}
(s_1, f_1(s_1)) = R_{-\frac{2\pi}{3}}(s_2, f_2(s_2)) = R_{-\frac{2\pi}{3}}(s_3, f_3(s_3))
\end{align*}
\]

is the meeting point of triple junction of three curves. It then follows from \((4.13)\) and $R_0 + R_{-\frac{2\pi}{3}} + R_{-\frac{4\pi}{3}} = 0$ that

\[
\begin{align*}
s_1 + s_2 + s_3 &= 0 \\
f_1(s_1) + f_2(s_2) + f_3(s_3) &= 0.
\end{align*}
\]

The fact that the curves meet at 120 degrees implies that

\[
\begin{align*}
f'_1(s_1) = f'_2(s_2) = f'_3(s_3),
\end{align*}
\]

where $f'_j$ here means the right derivative of $f_j$. Recalling $|h| = |f''|/(1 + |f'|^2)^{\frac{3}{2}}$, \((4.12)\) and \((4.7)\), we have

\[
\begin{align*}
\sup_{s \in [s_j, 1]} |f'_j(s)| &\leq |f'_j(1)| + \int_{s_j}^1 |f''_j(s)| \, ds \\
&\leq \tilde{\beta} + 2\tilde{\alpha}
\end{align*}
\]

for all sufficiently small $\alpha_1$ and $\beta_1$. For each $j = 1, 2, 3$, we next compute $\mathcal{H}^1(l_j \cap B_1)$.

\[
\begin{align*}
|\mathcal{H}^1(l_j \cap B_1) - (1 - s_j)| &\leq \tilde{\beta}^2 + \int_{s_j}^1 (\sqrt{1 + |f'_j(s)|^2} - 1) \, ds \\
&\leq \tilde{\beta}^2 + \frac{1}{2} \int_{s_j}^1 |f'_j(s)|^2 \, ds,
\end{align*}
\]

where the first inequality is due to the error estimate using \((4.7)\) near $\partial B_1$ and the second inequality is by $\sqrt{1 + t^2} - 1 \leq t^2/2$. Summing over $j$ and using \((4.14)\), we obtain

\[
\begin{align*}
\sum_{j=1}^3 \mathcal{H}^1(l_j \cap B_1) - 3 &\leq 3\tilde{\beta}^2 + \frac{1}{2} \sum_{j=1}^3 \int_{s_j}^1 |f'_j(s)|^2 \, ds.
\end{align*}
\]
By integration by parts, and using (4.15), (4.16) and (4.7), we have
\[
(4.20) \sum_{j=1}^{3} \int_{s_j}^{1} |f_j'|^2 = 3 \sum_{j=1}^{3} (f_j(1)f_j'(1) - \int_{s_j}^{1} f_jf_j'') \leq 3\beta^2 + 2\alpha \left( \sum_{j=1}^{3} \int_{s_j}^{1} |f_j|^2 \right)^{\frac{1}{2}}.
\]
For each $j$, we note that $|f_j(s)| = \text{dist} ((s, f_j(s)), J)$ away from the origin and close to $J$. More precisely, the equality holds when $(s, f_j(s))$ and positive $x$-axis has angle $\leq \frac{\pi}{3}$. When $(s, f_j(s))$ and negative $x$-axis has angle $\leq \frac{\pi}{3}$, then we have $|f_j(s)| \leq \text{dist} ((s, f_j(s)), J)$. For $s \in (s_1, s_2)$, suppose $(s, f_j(s))$ lies in the sector $\{tv : t > 0, |v| = 1, \cos \frac{\pi}{3} > v \cdot (1, 0) > \cos \frac{5\pi}{6}\}$, where we have $|f_j(s)| > \text{dist} ((s, f_j(s)), J)$. Denote $\hat{\delta} := \hat{s}_2 - \hat{s}_1$ and note that $\hat{\delta}$ is small when $\alpha_1$ and $\beta_1$ are small. Considering the geometry of graph, for sufficiently small $\sup |f_j'|$, we have
\[
(4.21) \hat{\delta} \sup_{s \in (\hat{s}_1, \hat{s}_2)} |f_j(s)|^2 \leq 2\hat{\delta} \inf_{s \in (\hat{s}_2, \hat{s}_2 + \hat{\delta})} |f_j(s)|^2 \leq 2 \int_{J_1 \cap B_1} \text{dist} (x, J^2) d\|V\|(x).
\]
Outside of $(\hat{s}_1, \hat{s}_2)$, as stated, we have $|f_j(s)| \leq \text{dist} ((s, f_j(s)), J)$, thus we have by (4.21)
\[
(4.22) \int_{s_j}^{1} |f_j|^2 \leq 3 \int_{J_1 \cap B_1} \text{dist} (x, J^2) d\|V\|(x).
\]
Combining (4.19), (4.20) and (4.22), we obtain (4.11). To obtain (4.11), note that $\phi_{\text{rad}} = 1$ on $B_1$ and thus we need to be concerned with region of integration over $B_2 \setminus B_1$ of $\|V\|$. But we have (4.7), thus the difference of integrations in this region can be estimated by $c\beta^2$. In the estimate one uses the radial symmetry of $\phi_{\text{rad}}$ to obtain the quadratic estimate. Thus we obtain (4.11) with some suitable constant $c_4$. \hfill $\Box$

**Corollary 4.3.** For a given $E_1 \in [1, \infty)$, let $c_4, \alpha_1, \beta_1$ be the corresponding constants obtained in Proposition 4.2. For $V \in IV_1(B_2)$ with $h(V, \cdot) \in L^2(\|V\|)$, define $\hat{\alpha}, \hat{\beta}, \hat{\mu}$ as (4.2), (4.7) and (1.3). Assume (4.4)-(4.8). Define
\[
(4.23) \hat{E} = \|V\|(\phi_{\text{rad}}^2) - c,
\]
and assume that
\[
(4.24) 3c_4\beta^2 \leq |\hat{E}|.
\]
Then we have
\[
(4.25) \hat{\alpha}^2 \geq \min\{\alpha_1^2, (2c_4\hat{\mu})^{-2}|\hat{E}|^2\}.
\]

**Proof.** If $\hat{\alpha} \geq \alpha_1$ holds, then (4.25) holds and there is nothing further to prove. Thus consider the case $\hat{\alpha} < \alpha_1$. Since (4.4)-(4.8) are assumed, we fulfill all the assumptions of Proposition 4.2 thus we have (4.11). Using the notation of (4.23), this implies that we have either $|\hat{E}| \leq 2c_4\hat{\alpha}\hat{\mu}$, or $|\hat{E}| \leq 2c_4\beta^2$. The last possibility is excluded by (4.24). Thus we have $\hat{\alpha}^2 \geq (2c_4\hat{\mu})^{-2}|\hat{E}|^2$. Thus we have either $\hat{\alpha}^2 \geq \alpha_1^2$ or the last possibility. This proves (4.25). \hfill $\Box$

The next ODE lemma connects Corollary 4.3 to (A4).

**Lemma 4.4.** Corresponding to $P, T \in (0, \infty)$ there exist $c_5, c_6 \in (0, \infty)$ such that the following holds: Given a non-negative function $g \in L^2([0, T])$ and a monotone decreasing function $\Phi : [0, T] \to \mathbb{R}$, define $f : [0, T] \to \mathbb{R}^+$ by
\[
(4.26) f(t) = P \min \{1, g(t)^{-2}|\Phi(t)|^2\}
\]
when \( g(t) > 0 \) and \( f(t) = P \) when \( g(t) = 0 \), and suppose that
\[
(4.27) \quad \Phi(t_2) - \Phi(t_1) \leq - \int_{t_1}^{t_2} f(t) \, dt, \quad 0 \leq t_1 < t_2 \leq T.
\]
Then
\[
(1) \text{ if } \Phi(0) \leq c_5, \text{ then } \Phi(T) \leq c_6 \|g\|_{L^2([0,T])}^2.
\]
\[
(2) \text{ if } \Phi(T) \geq -c_5, \text{ then } \Phi(0) \geq -c_6 \|g\|_{L^2([0,T])}^2.
\]

Proof. We prove (1) first. Set
\[
(4.30) \quad c_5 = \frac{PT}{8}, \quad c_6 = \frac{8}{PT^2}.
\]
We may assume \( \Phi(t) > 0 \) for all \( t \in [0,T] \) since \( \Phi(T) \leq 0 \) otherwise and (4.28) is trivially true. Assume for a contradiction that (4.28) were false. Set
\[
(4.31) \quad c = \|g\|_{L^2} \sqrt{2/T}
\]
and define \( A_1 = \{ t \in [0,T] : g(t) \geq c \} \) and \( A_2 = [0,T] \setminus A_1 \). It is easy to check that \( \mathcal{L}^1(A_1) \leq T/2 \), and thus
\[
(4.32) \quad \mathcal{L}^1(A_2) \geq T/2.
\]
We next define \( A_{2,a} = \{ t \in A_2 : \|g(t)\|_{L^2} \geq 1 \} \) and \( A_{2,b} = A_2 \setminus A_{2,a} \). By (4.32), we have either \( \mathcal{L}^1(A_{2,a}) \geq T/4 \) or \( \mathcal{L}^1(A_{2,b}) \geq T/4 \). In the first case, since \( f(t) = P \) on \( A_{2,a} \), we have
\[
(4.33) \quad \Phi(T) - \Phi(0) \leq - \int_{A_{2,a}} f(t) \, dt = -P \mathcal{L}^1(A_{2,a}) \leq -\frac{PT}{4},
\]
We have \( \Phi(T) - \Phi(0) \geq -c_5 \) and (4.33) gives a contradiction to (4.30). In the second case, using \( f(t) = P g(t)^{-2} \Phi(t)^2 \) on \( A_{2,b} \) and integrating \( (\Phi(t)^{-1})' \leq -P g(t)^{-2} \) we have
\[
(4.34) \quad -\Phi(T)^{-1} + \Phi(0)^{-1} \leq -P \int_{A_{2,b}} \frac{dt}{g(t)^2} \leq -P c_2 \mathcal{L}^1(A_{2,b}) \leq -\frac{PT^2}{8} \|g\|_{L^2}^{-2}.
\]
We used \( g(t) \leq c \) on \( A_{2,b} \subset A_2 \) and (4.31). Since we are assuming (4.28) is false, by (4.30), we have
\[
(4.35) \quad -\Phi(T)^{-1} > -c_6^{-1} \|g\|_{L^2}^{-2} = -\frac{PT^2}{8} \|g\|_{L^2}^{-2}.
\]
Two inequalities (4.34) and (4.35) give a contradiction. This proves (1). For (2), replace \( \Phi(\cdot) \) by \(-\Phi(T - \cdot) \) and \( f(\cdot) \) by \( f(T - \cdot) \), and then apply the previous argument. If \(-\Phi(T) \leq c_5 \), then one concludes that \(-\Phi(0) \leq c_6 \|g\|_{L^2}^{-2} \). Thus we obtain the result of (2). \( \square \)

**Proposition 4.5.** Corresponding to \( \nu, E_1, p, q \) there exist \( \varepsilon_3 \in (0,1) \) and \( c_7 \in (1,\infty) \) with the following property. Suppose \{ \( V_1 \) \( t \in [0,4] \) and \{ \( u(\cdot,t) \) \( t \in [0,4] \) satisfy (A1)-(A4) on \( B_2 \times [0,4] \). Assume (3.3) and (3.2) with \( \varepsilon_2 \) replaced by \( \varepsilon_3 \), (3.3) and (3.4). Then we have
\[
(4.36) \quad \sup_{t \in [1,3]} \|V_t\|_{(\phi_{rad})^2} - c \leq c_7 \max \{ \mu, \|u\| \}^2
\]
and
\[
(4.37) \quad \int_1^3 \int_{B_2} |h(V_t, \cdot)|^2 \phi_{rad}^2 \, dV_t \, dt \leq c_7 \max \{ \mu, \|u\| \}^2.
\]
Proof. We first use Proposition 3.1 with \( \tau = \frac{1}{4} \) to obtain \( \varepsilon_2 \) and \( c_3 \) so that we have (3.5)–(3.8) with \( \tau = \frac{1}{4} \) there. We also use Proposition 4.2 to obtain \( c_4, \alpha_1, \beta_1 \) corresponding to \( E_1 \). Then, for a.e. \( t \in [\frac{1}{2}, \frac{7}{2}] \), we have conditions (4.4)–(4.6) and (4.8) satisfied for \( V = V_t \) there. If we further assume that

\[
(4.38) \\
c_3 \max\{\mu, \|u\|\} \leq \beta_1,
\]

then (4.7) is also satisfied due to (3.5). We restrict \( \varepsilon_3 \) so that (4.38) holds by the following:

\[
(4.39) \\
\varepsilon_3 \leq \min\{\varepsilon_2, \beta_1^{-1}\}.
\]

Next fix \( P \) and \( T \) as

\[
(4.40) \\
P = \frac{1}{16} \min\{\alpha_1^2, (2c_4)^{-2}\}, \quad T = \frac{1}{2},
\]

With these choices of \( P \) and \( T \), we obtain \( c_5 \) and \( c_6 \) by Lemma 4.4. With \( c_5 \) fixed, we choose a small \( \tau \) and then restrict \( \varepsilon_3 \) so that, by using Proposition 3.1, we have

\[
(4.41) \\
\|V_2\|(\phi^2_{rad}) \leq c + \frac{c_5}{2}
\]

and

\[
(4.42) \\
\|V_2\|(\phi^2_{rad}) \geq c - \frac{c_5}{2}.
\]

We will fix \( c_7 \) later. We also set

\[
(4.43) \\
\beta_* = \max_{j=1,2,3} \sup_{t \in [\frac{1}{4}, \frac{7}{2}]} \|f_j(\cdot, t)\|_{C^1(|t| \leq \frac{1}{2})} \leq c_3 \max\{\mu, \|u\|\} \leq \beta_1,
\]

and define \( C(u) \) and estimate it by Hölder’s inequality as follows:

\[
(4.44) \\
C(u) = \int_0^1 \int_{B_2} |u|^2 d\|V_t\| dt \leq c_8(p, q, E_1)\|u\|^2.
\]

Define for \( t \in [\frac{1}{4}, \frac{7}{2}] \)

\[
(4.45) \\
E(t) = \|V_t\|(\phi^2_{rad}) - c - \int_{\frac{1}{2}}^t \int_{B_2} |u|^2 \phi^2_{rad} d\|V_s\| ds - c_9 \beta_*^2 (t - \frac{1}{2}),
\]

where \( c_9 \) will be fixed later. We first prove that

\[
(4.46) \\
E(t_2) - E(t_1) \leq -\frac{1}{4} \int_{t_1}^{t_2} \int_{B_2} |h(V_t, \cdot)|^2 \phi^2_{rad} d\|V_t\| dt, \quad \frac{1}{2} \leq \forall t_1 < \forall t_2 \leq \frac{7}{2}.
\]

By (A1), (A2) and (A4), for a.e. \( t \), we have \( V_t \in IV_1(B_2) \), \( h(V_t, \cdot) \in L^2(\|V_t\|) \), \( u(\cdot, t) \in L^2(\|V_t\|) \). At such time \( t \), using the perpendicularity of mean curvature (2.3), (omitting \( t \) dependence for simplicity)

\[
(4.47) \\
B(V, u, \phi^2_{rad}) \leq \int_{B_2} -|h|^2 \phi^2_{rad} + \phi^2_{rad} d\|h\|_{u} + |u^\perp \cdot \nabla \phi^2_{rad}| + (\nabla \phi^2_{rad})^\perp \cdot d\|V\|.
\]

The last term of (4.47) may be computed as

\[
(4.48) \\
\int_{G_1(B_2)} 2 \phi^2_{rad} S^\perp(\nabla \phi_{rad}) \cdot dV \leq \frac{1}{4} \int_{B_2} |h|^2 \phi^2_{rad} d\|V\| + 4 \int_{G_1(B_2)} |S^\perp(\nabla \phi_{rad})|^2 dV.
\]

Note that \( \nabla \phi_{rad} \) is 0 outside of \( B_\frac{3}{2} \setminus B_1 \), and \( \text{spt} \|V\| \) is represented as the union of three graphs of \( C^1 \) functions by (3.5) and (3.7) in \( B_\frac{3}{2} \setminus B_1 \). Consider the neighborhood of \((1, 0)\) in which \( \text{spt} \|V\| \) is represented by \( f_1 \). Since \( \phi_{rad} \) is radially symmetric function, \( \nabla \phi_{rad} \) at
Thus by (4.48) and (4.50), the last term of (4.47) may be estimated by

\[ \left( \frac{f'_1}{f'_1} - \frac{f_1}{1} \right) \]

which is obtained by computing \( I - \hat{v} \otimes \hat{v} \) with \( \hat{v} = (1 + (f'_1)^2)^{1/2} (1, f'_1) \), \( I \) being the identity \( 2 \times 2 \) matrix. Thus, we have

\[ |S^\perp (\nabla \phi_{\text{rad}})| \leq \frac{4}{\sqrt{s^2 + f'_1(s)^2}} |S^\perp \left( \begin{array}{c} s \\ f'_1(s) \end{array} \right) | \leq c \sqrt{(f'_1)^2 + f_1^2} \leq c \beta_* \]

by (4.43), where \( c \) is an absolute constant. We have similar computations for \( f_2 \) and \( f_3 \). Thus by (4.48) and (4.50), the last term of (4.47) may be estimated by

\[ \int_{B_2} (\nabla \phi_{\text{rad}}^2)^{1/2} \cdot h \, d\|V\| \leq \frac{1}{4} \int_{B_2} |h| \phi_{\text{rad}}^2 \, d\|V\| + c \beta_*^2. \]

The same computations show that the third term of (4.47) may be estimated by

\[ \int_{B_2} |u^\perp \cdot \nabla \phi_{\text{rad}}^2| \, d\|V\| \leq \frac{1}{2} \int_{B_2} |u|^2 \phi_{\text{rad}}^2 \, d\|V\| + c \beta_*^2. \]

The second term of (4.47) may be estimated by

\[ \int_{B_2} \phi_{\text{rad}}^2 |h| |u| \, d\|V\| \leq \frac{1}{2} \int_{B_2} \phi_{\text{rad}}^2 |h|^2 \, d\|V\| + \frac{1}{2} \int_{B_2} \phi_{\text{rad}}^2 |u|^2 \, d\|V\|. \]

Combining (4.47), (4.51)-(4.53), we obtain (by recovering the notation for \( t \) dependence)

\[ B(V_t, u(\cdot, t), \phi_{\text{rad}}^2) \leq -\frac{1}{4} \int_{B_2} |h(V_t, \cdot)|^2 \phi_{\text{rad}}^2 \, d\|V_t\| + \int_{B_2} |u(\cdot, t)|^2 \phi_{\text{rad}}^2 \, d\|V_t\| + c \beta_*^2, \]

where \( c \) is an absolute constant, and this holds for a.e. \( t \in [\frac{1}{2}, \frac{7}{2}] \). Due to (A4) and (4.54), now it is clear that the inequality (4.46) holds if we define \( \bar{E}(t) \) as in (4.43). We restrict \( \beta_3 \) further by

\[ \varepsilon_3^2 \leq \min \left\{ \frac{c_9}{4c_8}, \frac{c_5}{16c_9} \beta_*^2 \right\}. \]

We proceed to prove (4.36). For a.e. \( t \in [\frac{1}{2}, \frac{7}{2}] \), we have (4.43)-(4.8). Denoting

\[ \hat{E}(t) = \|V_t\| (\phi_{\text{rad}}^2) - c, \]

(4.56)

\[ \alpha(t) = \left( \int_{B_2} |h(V_t, \cdot)|^2 \phi_{\text{rad}}^2 \, d\|V_t\| \right)^{1/4}, \quad \mu(t) = \left( \int_{B_2} \text{dist} (\cdot, J)^2 \, d\|V_t\| \right)^{1/4}, \]

Corollary 4.3 and the definition of \( \beta_* \) in (4.43) show that

\[ 3c_4 \beta_*^2 \leq |\hat{E}(t)| \implies \alpha(t)^2 \geq \min \{ \alpha_1^2, (2c_4 \mu(t))^{-2} |\hat{E}(t)|^2 \}. \]

Fix any \( \hat{s} \in [1, 3] \) and we first prove the following upper bound,

\[ \hat{E}(\hat{s}) \leq c_6 \mu^2 + (3c_4 + 4c_9) \beta_*^2 + 2C(u). \]

Proof of (4.59).

(i) Suppose that there exists some \( t_0 \in [\frac{1}{2}, 1] \) such that

\[ \hat{E}(t_0) < (3c_4 + c_9) \beta_*^2 + C(u). \]

(4.60)
By the monotone decreasing property of $E(\cdot)$, \((4.45)\) and \((4.60)\), we then have
\[(4.61) \quad E(\hat{s}) \leq E(t_0) \leq \hat{E}(t_0) < (3c_4 + c_9)\beta_*^2 + C(u).\]
But then again by \((4.45)\), \((4.61)\) and \((4.43)\), we have
\[(4.62) \quad \hat{E}(\hat{s}) \leq E(\hat{s}) + C(u) + 3c_9\beta_*^2 < (3c_4 + 4c_9)\beta_*^2 + 2C(u).\]
With \((4.62)\) we proved \((4.59)\) under the assumption of (i). Now consider the complementary situation.

(ii) Suppose that for all $t \in [\frac{1}{2}, 1]$, we have
\[(4.63) \quad \hat{E}(t) \geq (3c_4 + c_9)\beta_*^2 + C(u).\]
This in particular means $|\hat{E}(t)| \geq 3c_4\beta_*^2$, thus \((4.63)\) and \((4.58)\) show
\[(4.64) \quad \alpha(t)^2 \geq \min\{\alpha_t^2, (2c_4\mu(t))^{-2}|\hat{E}(t)|^2\}\]
for a.e. $t \in [\frac{1}{2}, 1]$. By \((4.45)\) and \((4.63)\),
\[(4.65) \quad \hat{E}(t) \geq E(t) \geq \hat{E}(t) - c_9\beta_*^2 - C(u) \geq 0.\]
\((4.65)\) shows $|\hat{E}(t)| \geq |E(t)|$ in particular and by \((4.64)\) and \((4.40)\) we have
\[(4.66) \quad \alpha(t)^2 \geq 4P \min\{1, \mu(t)^{-2}|E(t)|^2\}\]
for a.e. $t \in [\frac{1}{2}, 1]$. Now we are in the position to apply Lemma 4.4. Define
\[(4.67) \quad \Phi(t) = E(t + \frac{1}{2}), \quad f(t) = P \min\{1, \mu(t + \frac{1}{2})^{-2}\Phi(t)|^2\}, \quad t \in [0, \frac{1}{2}].\]
By \((4.46)\), \((4.66)\) and \((4.67)\), we have for $0 \leq \forall t_1 < \forall t_2 \leq \frac{1}{2}$
\[(4.68) \quad \Phi(t_2) - \Phi(t_1) \leq -\frac{1}{4} \int_{t_1 + \frac{1}{2}}^{t_2 + \frac{1}{2}} \alpha(t)^2 dt - \int_{t_1}^{t_2} f(t) dt.\]
We also have
\[(4.69) \quad \Phi(0) = E(\frac{1}{2}) = \|V_\frac{1}{2}\|\langle \phi_{rad}^2 \rangle - c \leq \frac{c_8}{2}\]
by \((4.45)\) and \((4.41)\). Hence the assumptions of Lemma 4.4 are all satisfied with $g(t) = \mu(t + \frac{1}{2})$, and noticing that $\|g\|_{L^2}^2 \leq \mu^2$, we conclude
\[(4.70) \quad (E(1) =) \Phi(\frac{1}{2}) \leq c_6\mu^2.\]
For any $\hat{s} \in [1, 3]$, by \((4.45)\), \((4.70)\) and the monotone decreasing property of $E$, we have
\[(4.71) \quad \hat{E}(\hat{s}) \leq E(\hat{s}) + C(u) + 3c_9\beta_*^2 \leq c_6\mu^2 + C(u) + 3c_9\beta_*^2.\]
Thus \((4.71)\) shows that \((4.59)\) holds under the assumption of (ii). This concludes the proof of \((1.59)\).
Fix any $\hat{s} \in [1, 3]$ and we next prove the following lower bound,
\[(4.72) \quad -c_6\mu^2 - (3c_4 + 8c_9)\beta_*^2 - 2C(u) \leq \hat{E}(\hat{s}).\]
The idea is similar to the upper bound estimate with a few differences, but we present the proof for the completeness.

Proof of \((4.72)\).

(i) Suppose that there exists some $t_0 \in [3, \frac{7}{2}]$ such that
\[(4.73) \quad \hat{E}(t_0) > -(3c_4 + 4c_9)\beta_*^2 - C(u).\]
By (4.45) and (4.42),
\[(4.74)\quad E(t_0) \geq \hat{E}(t_0) - C(u) - 4c_9\beta^2_* > -(3c_4 + 8c_9)\beta^2_* - 2C(u).
\]
By the monotone decreasing property of $E$, we have $E(\hat{s}) \geq E(t_0)$ while $\hat{E}(\hat{s}) \geq E(\hat{s})$ by (4.35). Thus (4.74) proves (4.72) in case of (i).

(ii) Suppose that for all $t \in [3, \frac{7}{2}]$, we have
\[(4.75)\quad \hat{E}(t) \leq -(3c_4 + 4c_9)\beta^2_* - C(u).
\]
This means $|\hat{E}(t)| \geq 3c_4\beta^2_*$, thus by (4.58), we have (4.64) for a.e. $t \in [3, \frac{7}{2}]$. We need to change $\hat{E}$ in (4.64) to $E$. To do so, observe that
\[(4.76)\quad |E(t)| \leq |\hat{E}(t)| + C(u) + 4c_9\beta^2_* \leq 2|\hat{E}(t)|,
\]
the last inequality of (4.76) coming from (4.75). Thus, (4.64) with (4.76) (as well as recalling (4.40)) shows (4.66) for a.e. $t \in [3, \frac{7}{2}]$. Again we apply Lemma 4.4. Set
\[(4.77)\quad \Phi(t) = E(t + 3), \quad f(t) = P \min\{1, \mu(t + 3)^{-2}|\Phi(t)|^2\}, \quad t \in [0, \frac{1}{2}].
\]
By having (4.66), we have (4.68) and
\[(4.78)\quad \Phi(\frac{1}{2}) = E(\frac{7}{2}) \geq \|V_2\|\|(\phi^2_{rad}) - c - C(u) - 4c_9\beta^2_* \geq -\frac{c_5}{2} - c_8\zeta^2 - 4c_9\zeta^2 \geq -c_5
\]
by (4.77), (4.45), (4.42), (4.44), (4.43) and the last inequality due to (4.55). The assumptions of Lemma 4.4 (for case (2)) are thus satisfied, and we obtain
\[(4.79)\quad -c_6\mu^2 \leq \Phi(0)(= E(3)).
\]
Since $E$ is decreasing, for any $\hat{s} \in [1, 3]$, we have
\[(4.80)\quad \hat{E}(\hat{s}) \geq E(\hat{s}) \geq E(3).
\]
Hence under the assumption of (ii), (4.79) and (4.80) show (4.72).

Since $\hat{s} \in [1, 3]$ is arbitrary, (4.59) and (4.72) combined with (4.43) and (4.44) prove the first claim (4.36) with a suitable constant $c_7$.

To prove (4.37), observe that (4.46) with $t_2 = 3$ and $t_1 = 1$ shows (recalling (4.45))
\[(4.81)\quad \int_1^3 \int_{B_2} |h|^2 \phi^2_{rad} d\|V_i\| dt \leq 4(E(1) - E(3)) \leq 4(\hat{E}(1)) + |\hat{E}(3)| + C(u) + 2c_9\beta^2_*.
\]
Then using (4.36) to (4.81), we obtain (4.37), again with a suitable choice of $c_7$. \qed

In the next two sections we derive further a priori estimates for the flow $\{V_i\}$ whenever it is weakly close, in space-time at scale one, to the static triple junction $J$. These estimates provide enough control of the behavior of the moving curves near the singularity of $J$ for us to establish (in Section 7) decay, by a fixed factor at a fixed smaller scale, of the space-time $L^2$ distance of the flow to $J$, and consequently (by iterating this decay result) Theorem 2.1. These estimates are in the spirit of those proved first by L. Simon (29), for a similar purpose, for the case of multiplicity 1 minimal submanifolds weakly close to certain cylindrical minimal cones (in arbitrary dimension and codimension). However, in the present parabolic setting, their statements are often different and proofs require new ideas.
5. A PRIORI ESTIMATES II: NON-CONCENTRATION OF THE $L^2$-DISTANCE NEAR THE SINGULARITY OF $J$

The main result in this section is the estimate (5.20) of Proposition 5.2. This estimate in full strength plays an important role in Section 7 where we establish asymptotics for the blow-ups of sequences of flows converging weakly to the static triple junction $J$. It also implies that the space-time $L^2$ distance $\mu$ of the flow from $J$ does not concentrate near the singularity of $J$, a fact that is indispensable in the proof of the key decay result (Proposition 7.13) for $\mu$.

An essential ingredient in the proof of Proposition 5.2 is Proposition 5.1 below, which is based on the results of Section 4 and (the main idea behind) Huisken’s monotonicity formula.

**Proposition 5.1.** Corresponding to $\nu \in (0, 1)$, $E_1 \in [1, \infty)$ and $p, q$ as in (A0), there exist $\varepsilon_4 \in (0, \varepsilon_3]$ (where $\varepsilon_3 = \varepsilon_3(\nu, E_1, p, q)$ is as in Proposition 4.3) and $c_{10} \in (1, \infty)$ with the following property: If $\{V_t^i\}_{t \in [0,4]}$ and $\{u(\cdot, t)\}_{t \in [0,4]}$ satisfy (A1)-(A4) with $U = B_2$ and $I = [0, 4]$, if (3.1), (3.2), (3.3), (3.4) hold with $\varepsilon_4$ in place of $\varepsilon_3$ and if

\[
(5.1) \quad V_{t_0} \in IV_1(B_2), \quad h(V_{t_0}, \cdot) \in L^2(\|V_{t_0}\|) \quad \text{and} \quad \Theta(\|V_{t_0}\|, 0) = \frac{3}{2}
\]

for some $t_0 \in \left[\frac{3}{4}, 3\right]$, then

\[
(5.2) \quad \int_{5/4}^{t_0} \int_{B_1} \left[ h + \frac{x^1}{2(t_0 - t)} \right]^2 \rho(0, t_0)(x, t) \, d\|V_t\| \, dt \leq c_{10} \max\{\mu, \|u\|\}^2.
\]

**Proof.** By (5.1), there exists a tangent cone to $V_{t_0}$ at $x = 0$ which is $|R_\theta(J)|$ for some $\theta \in [0, 2\pi)$. From this fact, it follows that

\[
(5.3) \quad \lim_{\varepsilon \searrow 0} \int_{B_2} \phi_{rad}^2(x) \rho(0, t_0 + \varepsilon)(x, t_0) \, d\|V_{t_0}\|(x) = \frac{3}{2}.
\]

In the following, we fix $\varepsilon > 0$ arbitrarily close to 0. We choose $t_1 \in \left[1, \frac{5}{4}\right]$ so that

\[
(5.4) \quad \int_{B_2} |h(V_1, \cdot)|^2 \phi_{rad}^2 d\|V_1\| \leq c_7 \max\{\mu, \|u\|\}^2
\]

where $c_7 = c_7(\nu, E_1, p, q)$ is as in Proposition 4.3. Arguing as in Proposition 4.2 for $\varepsilon_4$ suitably small (so that $8c_7 \max\{\mu, \|u\|\}^2 \leq \alpha_1^2$ where $\alpha_1 = \alpha_1(E_1)$ is as in Proposition 4.2), we may conclude that $\text{spt} \|V_1\| \cap B_1$ consists of three $C^{1, \frac{1}{2}}$ curves $l_1, l_2, l_3$ meeting at a common point $p$ near the origin, with associated numbers $s_1, s_2, s_3 \in (-1/2, 1/2)$ and functions $f_j \in C^{1,1/2}([s_j, 1])$, $j = 1, 2, 3$, such that

\[
(5.5) \quad R_{2\alpha_1 - 1} l_j = \{(s, f_j(s)) \in \mathbb{R}^2 : s \in [s_j, 1]\}
\]

for $j = 1, 2, 3$; furthermore, using the estimate (4.17), we see that sup $\{\text{dist} (x, J) : x \in l_j\}$ for $j = 1, 2, 3$, and hence also $|p|$, are all $\leq 2\beta_s + 2\sqrt{8c_7} \max\{\mu, \|u\|\}$, where $\beta_s$ is as in (4.3). These estimates and radial symmetry of $\phi_{rad}$ and $\rho(0, t_0)(\cdot, t_1)$ imply, for a suitable choice of $c_{10}$ depending only on $p, q, \nu, E_1$, that

\[
(5.6) \quad \int_{J} \phi_{rad}^2 \rho(0, t_0)(\cdot, t_1) \, dH^1 - \int_{B_2} \phi_{rad}^2 \rho(0, t_0)(\cdot, t_1) \, d\|V_{t_1}\| \leq c_{10} \max\{\mu, \|u\|\}^2.
\]

Here it is important that $t_1 \in [1, \frac{5}{4}]$ so that $t_0 - t_1 \geq \frac{1}{4}$, allowing the choice of $c_{10}$ to be independent of $t_0$ and $t_1$. 

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We next use \( \rho_{(0,t_0+\epsilon)}(\cdot,t)\phi_{\text{rad}}^2 \) as a test function in (2.14) with \( t \in [t_1,t_0] \). For simplicity of notation write \( \rho \) for \( \rho_{(0,t_0+\epsilon)} \) and define \( \hat{\rho}(x,t) = \rho_{(0,t_0+\epsilon)}(x,t)\phi_{\text{rad}}(x)^2 \). By direct computation,

\[
(5.7) \quad \mathcal{B}(V_t,u(\cdot,t),\hat{\rho}(\cdot,t)) = \int_{B_2} \left( -h\hat{\rho} + \nabla \hat{\rho} \cdot (h + u^+) \right) d\|V_t\|
\]

\[
= \int_{B_2} |-h|^2 \hat{\rho} + 2\nabla \hat{\rho} \cdot h + u^+ \cdot (-h\hat{\rho} + \nabla \hat{\rho}) - \nabla \hat{\rho} \cdot h \ d\|V_t\|
\]

\[
= \int_{B_2} -\hat{\rho} |h - (\nabla \hat{\rho})^\perp| + \left( \frac{|(\nabla \hat{\rho})^\perp|^2}{\hat{\rho}} \right) + u \cdot (-h\hat{\rho} + (\nabla \hat{\rho})^\perp) + S \cdot \nabla^2 \hat{\rho} d\|V_t\|
\]

\[
\leq \int_{B_2} -\frac{\hat{\rho}}{2} |h - (\nabla \hat{\rho})^\perp|^2 + \left( \frac{|(\nabla \hat{\rho})^\perp|^2}{\hat{\rho}} \right) + \frac{|u|^2 \hat{\rho}}{2} + S \cdot \nabla^2 \hat{\rho} d\|V_t\|.
\]

where we have used the fact that by (2.3), for \( \|V_t\| \) a.e., \( h \cdot \nabla \hat{\rho} = h \cdot (\nabla \hat{\rho})^\perp \). We now need to carefully evaluate the terms involving \( \nabla \phi_{\text{rad}}^2 \) in the above. For the second term on the right hand side of (5.7), we have

\[
(5.8) \quad \frac{|(\nabla \hat{\rho})^\perp|^2}{\hat{\rho}} = \phi_{\text{rad}}^2 \frac{|(\nabla \rho)^\perp|^2}{\rho} + 2(\nabla \hat{\rho})^\perp \cdot (\nabla \phi_{\text{rad}}^2) + \rho \frac{|(\nabla \phi_{\text{rad}}^2)^\perp|^2}{\phi_{\text{rad}}^2} \leq \phi_{\text{rad}}^2 \frac{|(\nabla \rho)^\perp|^2}{\rho} + c\beta^2
\]

where the last inequality follows from the estimate (4.50) which holds also with \( \rho \) in place of \( \phi_{\text{rad}} \). The constant \( c \) is an absolute constant which may differ from line to line. By combining (5.7) and (5.8), we obtain

\[
(5.9) \quad \mathcal{B}(V_t,u(\cdot,t),\hat{\rho}(\cdot,t)) \leq \int_{B_2} -\frac{\hat{\rho}}{2} |h - (\nabla \hat{\rho})^\perp|^2 + \phi_{\text{rad}}^2 \frac{|(\nabla \rho)^\perp|^2}{\rho} + S \cdot \nabla^2 \hat{\rho}
\]

\[
\leq \frac{|u|^2 \hat{\rho}}{2} + c\beta^2 + 2S \cdot (\nabla \rho \times \nabla \phi_{\text{rad}}^2) + \rho S \cdot \nabla^2 \phi_{\text{rad}}^2 \ d\|V_t\|.
\]

By (5.9), (A4) and the identity

\[
(5.10) \quad \frac{\partial \rho}{\partial t} + S \cdot \nabla^2 \rho + \frac{|S^\perp(\nabla \rho)|^2}{\rho} = 0,
\]

we conclude that

\[
(5.11) \quad \int_{B_2} \hat{\rho}(\cdot,t) \ d\|V_t\|\bigg|_{t=t_1}^{t_0} \leq \int_{t_1}^{t_0} \int_{B_2} \left[ -\frac{\hat{\rho}}{2} |h - (\nabla \hat{\rho})^\perp|^2 + \frac{|u|^2 \hat{\rho}}{2} + c\beta^2
\]

\[
+ 2S \cdot (\nabla \rho \times \nabla \phi_{\text{rad}}^2) + \rho S \cdot \nabla^2 \phi_{\text{rad}}^2 \ d\|V_t\| \ dt.
\]

We next proceed to estimate the last two terms on the right hand side of (5.11). Note that the integrands in these two terms are zero outside \( B_{\frac{3}{2}} \setminus B_{\frac{1}{2}} \). (So in particular the derivatives of \( \rho \) appearing there are bounded uniformly.) Since by Proposition 3.11 integration with respect to \( \|V_t\| \) in \( B_{\frac{3}{2}} \setminus B_{\frac{1}{2}} \) is along three \( C^{1,\frac{1}{2}} \) curves \( l_1^{(t)}, l_2^{(t)}, l_3^{(t)} \) represented as graphs of functions \( f_1(\cdot,t), f_2(\cdot,t), f_3(\cdot,t) \) respectively as in (3.7), one can compute them explicitly. For instance for the curve \( l_1^{(t)} \), by explicit calculation (suppressing the t dependence of the
functions involved),

\[
\int_{t_1}^{3} 2S \cdot (\nabla \rho \otimes \nabla \phi_{rad}^2) + \rho S \cdot \nabla^2 \phi_{rad}^2 \\
= \int_{t_1}^{3} \left( 1 + (f_1')^2 \right)^{-\frac{1}{2}} \left( \frac{1}{f_1'} \frac{f_1''}{f_1'^2} \right) \cdot \left\{ 2r^{-2}(x \otimes x) \frac{d\rho}{dt} \frac{d\phi_{rad}}{dt} \right. \\
\quad \left. + \rho r^{-2}(x \otimes x) \frac{d^2\phi_{rad}}{dr^2} - r^{-1} \frac{d}{dr} \phi_{rad}^2 + r^{-1} \rho I \frac{d\phi_{rad}^2}{dr} \right\} ds,
\]

(5.12)

where \( f_1 = f_1(s, t), r = \sqrt{s^2 + (f_1')^2}, x = (s, f_1(s, t)) \), \( I \) is the identity \( 2 \times 2 \) matrix and \( d/dr \) is the differentiation with respect to the radial direction. Since \( |f_1|, |f_1'| \leq \beta_s \) by (14.43), we may estimate terms on the right hand side of (5.12) up to errors of order \( \beta_s^2 \) to obtain

\[
\int_{t_1}^{3} 2S \cdot (\nabla \rho \otimes \nabla \phi_{rad}^2) + \rho S \cdot \nabla^2 \phi_{rad}^2 \\
\leq \int_{t_1}^{3} \left( \frac{2}{\rho} \frac{d\rho}{dt} \frac{d\phi_{rad}^2}{dt} + \rho \left( \frac{d^2\phi_{rad}^2}{dr^2} - s^{-1} \frac{d}{dr} \phi_{rad}^2 + s^{-1} \rho \frac{d\phi_{rad}^2}{dr} \right) \right) ds + c\beta_s^2
\]

(5.13)

\[
= \int_{t_1}^{3} \left( \frac{2}{\rho} \frac{d\rho}{dt} \frac{d\phi_{rad}^2}{dt} + \rho \frac{d^2\phi_{rad}^2}{dr^2} \right) ds + c\beta_s^2.
\]

Since the functions appearing in the integrand on the right hand side of the above are radially symmetric, their values at \( (s, f_1(s, t)) \) and those at \( (s, 0) \) differ by at most \( c\beta_s^2 \). Thus we have

\[
\int_{t_1}^{3} 2S \cdot (\nabla \rho \otimes \nabla \phi_{rad}^2) + \rho S \cdot \nabla^2 \phi_{rad}^2 \\
\leq \int_{t_1}^{3} \left( \frac{2}{\rho} \frac{d\rho}{dt} \frac{d\phi_{rad}^2}{dt} + \rho \frac{d^2\phi_{rad}^2}{dr^2} \right) (s, 0) ds + c\beta_s^2 = \int_{t_1}^{3} \frac{\partial \rho}{\partial x} \frac{\partial \phi_{rad}^2}{\partial x} (s, 0) ds + c\beta_s^2
\]

(5.14)

where we integrated by parts and used the property that \( \frac{\partial \phi_{rad}^2}{\partial x} (s, 0) = 0 \) at \( s = 1 \pm \frac{1}{2} \). The same computation holds after rotation for the other two curves \( t_2^{(1)}, t_3^{(1)} \), so by (5.11) we deduce

\[
\int_{B_2} \hat{\rho}(\cdot, t) d\|V_1\|_t^{t_0} \leq \int_{t_1}^{t_0} \int_{B_2} -\hat{\rho} \frac{1}{2} |h - \left( \frac{\nabla \rho}{\rho} \right) |^2 + \frac{|u|^2 \hat{\rho}}{2} d\|V_1\| dt + 3 \int_{t_1}^{t_0} \int_{\{(x, 0) \in \mathbb{R}^2 : |x - 1| < \frac{1}{2}\}} \frac{\partial \rho}{\partial x} \frac{\partial \phi_{rad}^2}{\partial x} dH^1 dt + c\beta_s^2.
\]

(5.15)

We note that

\[
\int_{t_1}^{t_0} \frac{\partial \hat{\rho}}{\partial t} (-\hat{\rho}^{\frac{1}{2}}) dt dH^1 = 3 \int_{t_1}^{t_0} \int_{\{(x, 0) \in \mathbb{R}^2 : x \geq 0\}} \frac{\partial \rho}{\partial x} \frac{\partial \phi_{rad}^2}{\partial x} dH^1 dt
\]

by radial symmetry of \( \hat{\rho} \), and thus, since \( \frac{\partial \hat{\rho}}{\partial t} = -\hat{\rho} \frac{\partial \rho}{\partial x} \) on the \( x \)-axis,

\[
\int_{t_1}^{t_0} \frac{\partial \hat{\rho}}{\partial t} dH^1 = 3 \int_{t_1}^{t_0} \int_{\{(x, 0) \in \mathbb{R}^2 : x \geq 0\}} \frac{\partial \rho}{\partial x} \frac{\partial \phi_{rad}^2}{\partial x} dH^1 dt
\]

(5.16)

\[
= 3 \int_{t_1}^{t_0} \int_{\{(x, 0) \in \mathbb{R}^2 : |x - 1| < \frac{1}{2}\}} \frac{\partial \rho}{\partial x} \frac{\partial \phi_{rad}^2}{\partial x} dH^1 dt
\]
where we used integration by parts and the fact that $\frac{\partial \phi}{\partial x} = 0$ at $x = 0$ and $\phi_{rad}^2 = 0$ at $x = \frac{3}{2}$.

Substituting (5.16) into (5.15), we obtain

(5.17) \[
\left( \int_{B_2} \hat{\rho}(\cdot, t) \| V_i \| - \int_{f_j} \hat{\rho}(\cdot, t) \mathcal{H}^1 \right) \left|_{t=t_1}^{t_0} \right. \leq \int_{t_1}^{t_0} \int_{B_2} -\frac{\hat{\rho}}{2} |h - \frac{(\nabla \hat{\rho})}{\hat{\rho}}|^2 + \hat{\rho} \frac{|u|^2}{2} \| V_i \| dt + c \beta^2.
\]

We now let $\epsilon \to 0$ in (5.17). Since $\int_f \rho(x, t) \mathcal{H}^1 \to \frac{3}{2}$, in view of (5.3) and (5.6), we obtain from (5.17) (using also the fact that $\phi_{rad} = 1$ on $B_1$ and $= 0$ on $\mathbb{R}^2 \setminus B_2$) that

(5.18) \[
\int_{t_1}^{t_0} \int_{B_2} \rho |h - \frac{(\nabla \rho)}{\rho}|^2 \| V_i \| dt \leq \int_{t_1}^{t_0} \int_{B_2} \rho \frac{|u|^2}{2} \| V_i \| dt + c \beta^2 + c_{10} \max\{\mu, \| u \|\}^2
\]

where $\rho = \rho_{(0,t_0)}(x,t)$. Lastly, the term above involving $u$ may be estimated as in [18 (6.7)-(6.8)] to get

(5.19) \[
\int_{t_1}^{t_0} \int_{B_{\frac{3}{2}}} \rho |u|^2 \| V_i \| dt \leq c(p,q)E_1^{1-\frac{2}{p}} \| u \|^2.
\]

Since $\frac{\sum \rho}{\rho} = -\frac{x}{2(t_0-t)}$, the desired estimate follows after redefining $c_{10}$ depending only on $p, q, \nu, E_1$. \(\square\)

**Proposition 5.2.** Fix $\kappa \in [0,1)$. Under the same assumptions as in Proposition 5.1, we have

(5.20) \[
\sup_{t \in [\frac{4}{5}, t_0]} (t_0 - t)^{-\kappa} \int_{B_{\frac{3}{2}}} \rho_{(0,t_0)}(\cdot, t) \text{dist} (\cdot, J)^2 \| V_i \| \leq c_{11} \max\{\mu, \| u \|\}^2
\]

where $c_{11}$ depends only on $\kappa, p, q, \nu, E_1$.

**Proof.** Define $\tilde{d} : \mathbb{R}^2 \to \mathbb{R}$ such that $\tilde{d}$ is positively homogeneous of degree one (i.e. $\tilde{d}(\lambda x) = \lambda \tilde{d}(x)$ $\forall \lambda \geq 0$ and $\forall x \in \mathbb{R}^2$), smooth away from $J$, and

(5.21) \[
\begin{align*}
\tilde{d}(x) &= \text{dist} (x, J) \quad \forall x \text{ with } \text{dist} (x, J) < \frac{|x|}{5}, \\
\frac{1}{2} \text{dist} (x, J) &\leq \tilde{d}(x) \leq 2 \text{dist} (x, J) \quad \forall x \in \mathbb{R}^2, \\
|\nabla \tilde{d}(x)| &\leq 1 \quad \forall x \notin J.
\end{align*}
\]

By homogeneity, we have $x \cdot \nabla (\tilde{d}^2/|x|^2) = 0$, which gives after a little computation that

(5.22) \[
x \cdot \nabla \tilde{d}^2 = 2 \tilde{d}^2.
\]

Let $0 \leq \eta \leq 1$ be a non-negative smooth radially symmetric function such that

(5.23) \[
\eta = 0 \quad \forall x \notin B_1 \quad \text{and} \quad \eta = 1 \quad \forall x \in B_{\frac{3}{4}}.
\]

Since $\text{spt} \nabla \eta \subset B_1 \setminus B_{\frac{3}{4}}$, we may assume that $\text{spt} |\nabla \eta| \cap \| V_i \|$ is contained in $\cup_{j=1}^3 \text{graph} f_j(\cdot, t)$ for all $t \in [1,3]$, where $f_j$ are as in Proposition 3.1. Fix $t_1 \in [1, \frac{5}{2}]$ such that

(5.24) \[
\int_{B_2} \text{dist} (\cdot, J)^2 \| V_i \| \leq 4\mu^2.
\]

Such $t_1$ exists by (3.1), the definition of $\mu$. We next use $(t_0 - t)^{-\kappa} g(t) \rho_{(0,t_0)}(x,t)\tilde{d}(x)^2 \eta(x)$ as a test function in (2.10) over the time interval $[t_1, t_2]$ with arbitrary $t_2 \in [\frac{5}{4}, t_0]$, where $g$ is a fixed smooth non-negative function with

(5.25) \[
0 < g(t) \leq 1
\]
which will be chosen later. Denoting, for notational convenience, \( \rho(0,t_0) \) by \( \rho \) and 
\((t_0-t)^{-\kappa}g(t)\rho(0,t_0)(x,t) \) by \( \hat{\rho} \) respectively, we obtain from (2.16) that

\[
(5.26) \quad \int_{B_1} \eta^2 \hat{\rho} d\|V_t\| \mid_{t=t_1}^{t_2} \leq \int_{t_1}^{t_2} \int_{B_1} \left( -h\hat{\rho}\eta d^2 + \nabla(\hat{\rho}\eta d^2) \right) \cdot (h + u^\perp) + \eta^2 \frac{\partial \hat{\rho}}{\partial t} d\|V_t\| dt.
\]

Since by direct calculation and estimation

\[
\left( -h\hat{\rho}\eta d^2 + \nabla(\hat{\rho}\eta d^2) \right) \cdot (h + u^\perp)
\]

\[
= (-|h|^2\hat{\rho} + (\nabla \hat{\rho} \cdot h)) \eta d^2 + \hat{\rho}\nabla(\eta d^2) \cdot h + \eta d^2 (-h\hat{\rho} + \nabla \hat{\rho}) \cdot u^\perp + \hat{\rho}\nabla(\eta d^2) \cdot u^\perp
\]

\[
\leq -\hat{\rho}|h| - \frac{(\nabla \hat{\rho})}{\hat{\rho}} \eta d^2 - (\nabla \hat{\rho} \cdot h) \eta d^2 + \frac{|(\nabla \hat{\rho})|^2}{\hat{\rho}} \eta d^2 + \hat{\rho}\nabla(\eta d^2) \cdot h
\]

\[
+ \frac{1}{2} \hat{\rho}|h| - \frac{(\nabla \hat{\rho})}{\hat{\rho}} |\eta d^2| + \frac{1}{2} \hat{\rho} d^2 |u|^2 + \hat{\rho}\nabla(\eta d^2) \cdot u^\perp,
\]

it follows from (5.26) that

\[
(5.27) \quad \int_{B_1} \eta^2 \hat{\rho} d\|V_t\| \mid_{t=t_1}^{t_2} \leq \int_{t_1}^{t_2} \int_{B_1} - (\nabla \hat{\rho} \cdot h) \eta d^2 + \frac{|(\nabla \hat{\rho})|^2}{\hat{\rho}} \eta d^2 + \hat{\rho}\nabla(\eta d^2) \cdot h
\]

\[
+ \frac{1}{2} \hat{\rho} d^2 |u|^2 + \hat{\rho}\nabla(\eta d^2) \cdot u^\perp + \eta^2 \frac{\partial \hat{\rho}}{\partial t} d\|V_t\| dt.
\]

Next note that by (2.2),

\[
(5.28) \quad \int_{B_1} - (\nabla \hat{\rho} \cdot h) \eta d^2 d\|V_t\| = \int_{B_1} \int_{S} \left( \eta^2 \nabla^2 \hat{\rho} + \nabla \hat{\rho} \otimes \nabla(\eta d^2) \right) dV_t(\cdot, S) \quad \text{for a.e. } t.
\]

Using (5.28) in (5.27) and using (5.10) (keeping in mind that \( \nabla \hat{\rho} = (t_0 - t)^{-\kappa}g(t) \nabla \rho, \nabla^2 \hat{\rho} = (t_0 - t)^{-\kappa}g(t) \nabla^2 \rho \)), we obtain

\[
(5.29) \quad \int_{B_1} \eta^2 \hat{\rho} d\|V_t\| \mid_{t=t_1}^{t_2} \leq \int_{t_1}^{t_2} \int_{B_1} S \cdot \left( \nabla \hat{\rho} \otimes \nabla(\eta d^2) \right) + \hat{\rho}\nabla(\eta d^2) \cdot h
\]

\[
+ \frac{1}{2} \hat{\rho} d^2 |u|^2 + \hat{\rho}\nabla(\eta d^2) \cdot u^\perp + \eta^2 \frac{\partial \hat{\rho}}{\partial t} \left( (t_0 - t)^{-\kappa}g \right) dV_t(\cdot, S) dt
\]

\[
= I_1 + I_2 + I_3 + I_4 + I_5
\]

where \( I_1, \ldots, I_5 \) denote the five integrals corresponding to the five summands, in the order listed, in the integrand on the right hand side of the above. We analyse these integrals as follows:

**Estimation of \( I_1 + I_2 \).**
Since \( S \cdot (v_1 \otimes v_2) = v_1 \cdot v_2 - v_1^\perp \cdot v_2 \) for any two vectors \( v_1, v_2 \), we have for a.e. \( t \) (recalling that \( v^\perp = v - S(v) \)),

\[
S \cdot (\nabla \hat{\rho} \otimes \nabla(\eta d^2)) + \hat{\rho}\nabla(\eta d^2) \cdot h = \nabla \hat{\rho} \cdot \nabla(\eta d^2) - (\nabla \hat{\rho})^\perp \cdot \nabla(\eta d^2) + \hat{\rho}\nabla(\eta d^2) \cdot h
\]

\[
= \nabla \hat{\rho} \cdot \nabla(\eta d^2) + \hat{\rho}\nabla(\eta d^2) \left( h - \frac{(\nabla \hat{\rho})^\perp}{\hat{\rho}} \right)
\]

\[
\leq -\frac{\hat{\rho} x}{2(t_0 - t)} \cdot \nabla(\eta d^2) + \frac{\hat{\rho}}{2} \left| h + \frac{x^\perp}{2(t_0 - t)} \right|^2 + \frac{1}{2} (t_0 - t)^{-\kappa} |\nabla(\eta d^2)|^2 \hat{\rho}.
\]
Here we also used (5.25). The terms involving $\nabla \eta$ is non-zero only on $B_1 \setminus B_{\frac{3}{4}}$ where $|\tilde{d}| \leq c\beta_*$ by (1.43) and $(t_0 - t)^{-1} \tilde{\rho}$ is uniformly bounded. Thus, using also (5.22) and (5.21), we obtain from (5.30) that

\[(5.31) \quad I_1 + I_2 \leq \int_{t_1}^{t_2} \int_{B_1} -\frac{\rho \eta \tilde{d}^2}{t_0 - t} + \frac{\rho}{2} |h + \frac{x}{2(t_0 - t)}|^2 + 4(t_0 - t)^{-\kappa} \rho \eta \tilde{d}^2 \|V_t\| dt + c\beta_*^2.\]

**Estimation of $I_3$.**

We separate integration with respect to the spatial variable $x$ into the region $A_1 = \{|x| \leq (t_0 - t)^{\frac{5}{4}}\} \cap B_1$ and its complement $A_2 = B_1 \setminus A_1$. On $A_1$, $\tilde{d}(x) \leq 2 \operatorname{dist} (x, J) \leq 2(t_0 - t)^{\frac{5}{4}}$ by (5.21) so recalling that $\tilde{\rho} = (t_0 - t)^{-\kappa} \rho$ and (5.25), we see that

\[(5.32) \quad \int_{t_1}^{t_2} \int_{A_1} \tilde{\rho} \eta \tilde{d}^2 |u|^2 d\|V_t\| dt \leq \int_{t_1}^{t_2} \int_{B_1} 4 \rho \eta |u|^2 d\|V_t\| dt \leq c(p, q, E_1) \|u\|^2\]

where the last inequality follows from (5.19). On $A_2$, we have

\[(5.33) \quad \tilde{\rho} \leq (4\pi)^{-\frac{1}{2}} (t_0 - t)^{-\kappa - \frac{1}{2}} \exp \left( -\frac{1}{4(t_0 - t)^{1-\kappa}} \right)\]

and hence $\tilde{\rho}$ is uniformly bounded for $t \in [1, t_0)$. Thus

\[\int_{t_1}^{t_2} \int_{A_2} \tilde{\rho} \eta \tilde{d}^2 |u|^2 d\|V_t\| dt \leq c(\kappa, p, q, E_1) \|u\|^2\]

and we conclude that

\[(5.34) \quad I_3 \leq c(\kappa, p, q, E_1) \|u\|^2.\]

**Estimation of $I_4$.**

Since

\[(5.35) \quad \int_{t_1}^{t_2} \int_{B_1} \tilde{\rho} |\nabla (\eta \tilde{d}^2)| |u| d\|V_t\| dt \leq \frac{1}{2} \int_{t_1}^{t_2} \int_{B_1} (t_0 - t)^{-\kappa} \tilde{\rho} |\nabla (\eta \tilde{d}^2)|^2 + \rho |u|^2 d\|V_t\| dt,\]

and the term involving $\nabla \eta$ in (5.35) can be estimated by $c\beta_*^2$, and the last term may be estimated as in (5.19), we obtain

\[(5.36) \quad I_4 \leq \int_{t_1}^{t_2} \int_{B_1} 4(t_0 - t)^{-\kappa} \tilde{\rho} \eta \tilde{d}^2 d\|V_t\| dt + c\beta_*^2 + c(p, q, E_1) \|u\|^2.\]

**Computation of $I_5$.**

Now we make the explicit choice of $g$ given by

\[(5.37) \quad g(t) = \exp \left( -8 \int_{t_1}^{t} (t_0 - s)^{-\kappa} ds \right) \quad \forall t \in [t_1, t_0),\]

and note that $g(t) \leq 1$ and also, since $\kappa < 1$, that

\[(5.38) \quad \inf_{t \in [t_1, t_0)} g(t) \geq c(\kappa) > 0.\]

We emphasize that $c(\kappa)$ here may be chosen independently of $t_1 \in [1, \frac{5}{4})$ and $t_0 \in [\frac{5}{4}, 3]$. With this choice of $g$,

\[(5.39) \quad \rho \frac{d}{dt} ((t_0 - t)^{-\kappa} g) = \frac{\kappa \tilde{\rho}}{t_0 - t} - 8(t_0 - t)^{-\kappa} \tilde{\rho}\]
so we have
\begin{equation}
I_5 = \int_{t_1}^{t_2} \int_{B_1} \frac{\kappa \hat{\rho} \hat{d}^2}{t_0 - t} - 8(t_0 - t)^{-\kappa} \hat{\rho} \hat{d}^2 \|V_t\| dt.
\end{equation}

Conclusion.
By combining (5.31), (5.34), (5.36) and (5.40), and using the estimates (5.2), (5.24), (5.38) and (5.21), we deduce the desired estimate (5.20). 

\section{A priori estimates III: Distance and approximate continuity estimates for the junction point}

Fix $p, q$ as in (A0), $\nu \in (0, 1), E_1 \in [1, \infty)$ and $\kappa \in [0, 1)$. Let $c_\gamma$ be as in Proposition 4.3, $\varepsilon_4, c_{10}$ be as in Proposition 5.1 and $c_{11}$ be as in Proposition 5.2. With $U = B_3$ and $I = [0, 4]$, suppose $\{V_t\}_{t \in [0, 4]}$ and $\{u(\cdot, t)\}_{t \in [0, 4]}$ satisfy (A1)-(A4). Assume further, with the following new definitions of $\mu$ and $\|u\|$ (in which spatial integration is over $B_3$ rather than over $B_2$), that
\begin{equation}
\mu = \left( \int_0^4 \int_{B_3} \text{dist} \left( \cdot, J \right)^2 d\|V_t\| dt \right)^{\frac{1}{2}} \leq \frac{\varepsilon_4}{2},
\end{equation}
\begin{equation}
\|u\| = \left( \int_0^4 \left( \int_{B_3} |u|^p d\|V_t\| \right)^{\frac{2}{p}} dt \right)^{\frac{1}{2}} \leq \frac{\varepsilon_4}{2},
\end{equation}
\begin{equation}
\|V_0\|_\infty (\phi_{j_1}) \leq \frac{3 - \nu}{2} c_1, \quad \|V_4\|_\infty (\phi_{j_2}) \geq \frac{1 + \nu}{2} c_1 \quad \text{for some } j_1, j_2 \in \{1, 2, 3\}
\end{equation}
where $\phi_j$ are as in (2.11). Note that (6.1)-(6.3) are more restrictive conditions than hypotheses (3.1)-(3.4) of Proposition 5.1. For $\xi \in \mathbb{R}^2$ with $|\xi| < 1$, let $V_t^{(\xi)}$ be the translation of $V_t$ by $-\xi$; thus,
\begin{equation}
V_t^{(\xi)}(\phi) = \int_{G_1(B_3)} \phi(x - \xi, S) dV_t(x, S) \quad \text{for each } \phi \in C_c(G_1(B_2)).
\end{equation}
Note that if $\text{spt}\|V_t\|$ has a junction point at $\xi$, then $\text{spt}\|V_t^{(\xi)}\|$ has a junction point at the origin. Now, depending only on $\nu, E_1, \varepsilon_4$ (hence ultimately only on $p, \nu, E_1$), there exists a small $\delta_1 \in (0, 1)$ such that, if $|\xi| \leq \delta_1$, then $\{V_t^{(\xi)}\}_{t \in [0, 4]}$ and $\{u(\cdot + \xi, t)\}_{t \in [0, 4]}$ satisfy (3.1)-(3.4) with $\varepsilon_4$ in place of $\varepsilon_3$. In particular, if $|\xi| \leq \delta_1$, $\{V_t^{(\xi)}\}$ and $\{u(\cdot + \xi, t)\}$ satisfy the hypotheses of Proposition 5.1 on $B_2 \times [0, 4]$. Let us fix such $\delta_1$.

\begin{lemma}
For $\xi \in \mathbb{R}^2 \setminus \{0\}$ define $J_\xi = \{x \in \mathbb{R}^2 : x - \xi \in J\}$. Then on one of the three connected components of $\{x \in \mathbb{R}^2 : \text{dist} (x, R_\xi(J)) > |\xi|\}$, we have
\begin{equation}
\frac{\sqrt{3}}{2} |\xi| \leq \text{dist} (x, J) + \text{dist} (x, J_\xi).
\end{equation}
\end{lemma}

\begin{proof}
First note that $R_\xi(J) \setminus \{0\}$ is the set of points $x$ such that the closest point to $J$ from $x$ is not unique. Given $\xi \in \mathbb{R}^2 \setminus \{0\}$, let $A = \{x \in \mathbb{R}^2 : \text{dist} (x, R_\xi(J)) > |\xi|\}$. Since $x \in A$ is away from $R_\xi(J)$ by at least $|\xi|$, the closest points in $J$ and $J_\xi$ to $x$ are both unique. Let $x, J, J_\xi$ be the closest point to $x$ in $J$. One checks easily that the closest point to $J_\xi$ from $x$ is $x + \xi^\perp$, where $\xi^\perp$ is $(T_x J)^\perp(\xi)$. This implies $\text{dist} (x, J) = |x - x_J|$ and $\text{dist} (x, J_\xi) = |x - x_J - \xi^\perp|$. Then the triangle inequality gives $|\xi| \leq \text{dist} (x, J) + \text{dist} (x, J_\xi)$.
\end{proof}
For \( \xi \), there is at least one component of \( A \) on which \(|\xi^j| \geq \frac{\sqrt{3}}{2} |\xi| \) holds. On this component, we have \((6.5)\). \( \square \)

**Proposition 6.2.** There exist \( \varepsilon_5 \in (0, \frac{\varepsilon_2}{2}) \) and \( c_{12} \in (1, \infty) \) depending only on \( p, q, \nu, E \) such that if \( \{V_t\}_{t \in [0,4]} \) satisfy (A1)-(A4) with \( U = B_3 \), \( I = [0,4] \) and \((6.1), (6.2), (6.3)\) hold with \( \varepsilon_5 \) in place of \( \frac{\varepsilon_2}{2} \) then for any \( \xi \in B_1 \) and \( t_0 \in [\frac{5}{4}, 3] \) with \( h(V_t, \cdot) \in L^2(|V_t|) \) and \( \Theta(\|V_t\|, \xi) = \frac{3}{2} \), we have

\[
|\xi| \leq c_{12} \max\{\mu, \|u\|\}.
\]

In addition, given \( \kappa \in [0,1) \), there exists \( c_{13} \in (1, \infty) \) depending only on \( \kappa, p, q, \nu, E \) such that

\[
\sup_{t \in [\frac{5}{4}, t_0]} (t_0 - t)^{-\kappa} \int_{B_{\frac{3}{4}}(\xi)} \rho(\xi, t_0)(\cdot, t) \text{dist}(\cdot, J_\xi)^2 \|V_t\| \leq c_{13} \max\{\mu, \|u\|\}^2.
\]

**Proof.** Recall \( \delta_1 \) which is fixed before Lemma 6.1. Corresponding to \( \kappa = 1/2 \), let \( c_{11} \) be chosen using Proposition 5.2. Fix \( r_0 \in (0, \frac{1}{4}) \) by

\[
r_0 = \min\left\{1, \frac{1}{4}, \frac{1}{2}, \left(\frac{16}{3} \frac{2c_{12} E}{16}\right)^{-1}\right\}.
\]

Corresponding to \( \tau = \min\{\delta_1/2, r_0/8\} \), fix \( \varepsilon_2 \) using Proposition 3.1. We will choose \( \varepsilon_5 \in (0, \varepsilon_2] \) by restricting further in the following. For any \( \xi \in B_1 \) and \( t_0 \) satisfying the assumptions, due to the choice of \( \varepsilon_2 \), the claim of Proposition 3.1 shows that we have \( |\xi| \leq 2\tau = \delta_1 \). Then due to the choice of \( \delta_1 \) (see the discussion before Lemma 5.1), \( \{V_t^{(\xi)}\}_{t \in [0,4]} \) and \( \{u(\cdot + \xi, t)\}_{t \in [0,4]} \) satisfy assumptions of Proposition 5.1 and 5.2. Thus we have

\[
\sup_{t \in [\frac{5}{4}, t_0]} (t_0 - t)^{-1/2} \int_{B_{\frac{3}{4}}(\xi)} \rho(\xi, t_0)(\cdot, t) \text{dist}(\cdot, J_\xi)^2 \|V_t^{(\xi)}\| \leq c_{11} \max\{\mu, \|u(\cdot + \xi)\|\}^2,
\]

where \( \mu_\xi \) is the corresponding quantities for \( V_t^{(\xi)} \) with integration over \( B_2 \) and \( \|u(\cdot + \xi)\| \) is integration over \( B_2 \). By the definition of \( V_t^{(\xi)} \) and dist \((x, J_\xi)^2 \leq 2 \text{dist}(x, J)^2 + 2|\xi|^2\), we have

\[
\mu_\xi^2 \leq 2\mu^2 + 32E_1|\xi|^2
\]

and

\[
\|u(\cdot + \xi)\| \leq \|u\|,
\]

where \( \|u\| \) is integration over \( B_3 \). In the time interval \([t_0 - 2r_0^2, t_0 - r_0^2] \subset [\frac{5}{4}, t_0] \), we choose \( t_1 \) so that

\[
\int_{B_3} \text{dist}(\cdot, J)^2 \|V_t\| \leq \frac{3}{r_0^2} \mu^2,
\]

\[
\int_{B_1} \|h(V_t, \cdot)^2\| \|V_t\| \leq \frac{3c_7}{r_0^2} \max\{\mu, \|u\|\}^2.
\]

Such \( t_1 \) exists by the definition of \( \mu \) and by \((4.37)\). Using \( t_1 \) in \((6.9)\) as well as \((6.10), (6.11)\) and recalling the definition of \( V_t^{(\xi)} \), we then obtain

\[
\int_{B_{\frac{3}{4}}(\xi)} \rho(\xi, t_0)(\cdot, t_1) \text{dist}(\cdot, J_\xi)^2 \|V_t\| \leq c_{11} (t_0 - t_1)^{1/2} \max\{2\mu^2 + 32E_1|\xi|^2, \|u\|^2\}.
\]
On $B_{r_0}$, we have (since $|\xi| \leq 2\tau \leq \frac{3\tau}{4}$ and $2r_0^2 \geq t_0 - t_1 \geq r_0^2$)

\begin{equation}
\rho(\xi, t_0)(x, t_1) \geq \exp \left( \frac{\|x|^2 + |\xi|^2}{2(t_0 - t_1)} \right) \geq \frac{e^{-1}}{\sqrt{8\pi r_0}}.
\end{equation}

Using $t_0 - t_1 \leq 2r_0^2$ and noting that $r_0 \leq \frac{\tau}{4}$ and $|\xi| \leq \frac{\tau_0}{4}$ (implying $B_{r_0} \subset B_{\frac{\tau}{4}}(\xi)$), we obtain from (6.14) and (6.15)

\begin{equation}
\frac{1}{r_0} \int_{B_{r_0}} \text{dist}(\cdot, J_\xi)^2 d\|V_r\| \leq e^{\sqrt{16\pi c_1 r_0}} \max\{2\mu^2 + 32E_1 |\xi|^2, \|u\|^2\}.
\end{equation}

Next, we consider the set $\{x \in \mathbb{R}^2 : \text{dist}(x, R^J_{\hat{r}}(J)) > \frac{\tau}{4}\}$. Denote the three connected components of this set by $W_1, W_2, W_3$. By the argument in the proof of Proposition 4.2 by choosing $\varepsilon_5$ depending only on $r_0$ (as in (6.12)) and $c_7$ (as in (6.13)) (which ultimately depend only on $p, q, \nu, E_1$), we can ensure that

\begin{equation}
\|V_{t_j}\|(W_j \cap B_{r_0}) \geq \frac{r_0}{2}, \quad j = 1, 2, 3.
\end{equation}

By Lemma 6.1 and the fact that $|\xi| \leq r_0/4$, on one of the components, we have (6.5). Without loss of generality, let this component be $W_1$. Then (6.17) implies

\begin{equation}
|\xi|^2 \leq \frac{2}{r_0} \int_{W_1 \cap B_{r_0}} |\xi|^2 d\|V_{t_1}\| \leq \frac{16}{3r_0} \int_{W_1 \cap B_{r_0}} \left( \text{dist}(\cdot, J)^2 + \text{dist}(\cdot, J_\xi)^2 \right) d\|V_{t_1}\|.
\end{equation}

The first term of the right-hand side of (6.18) may be estimated by an appropriate constant times $\mu^2$ due to (6.12). For the second term, we use (6.16) and (6.8) to deduce

\begin{equation}
\frac{16}{3r_0} \int_{B_{r_0}} \text{dist}(\cdot, J_\xi)^2 d\|V_{t_1}\| \leq \frac{16}{3} 2e^{\sqrt{16\pi c_1 r_0}} \max\{\mu, \|u\|\}^2 + \frac{|\xi|^2}{2}.
\end{equation}

By relegating the last term of (6.19) to the left-hand side in (6.18) and setting an appropriate $c_{12}$, we obtain the desired estimate (6.6). By applying Proposition 5.2 to $V_{t_1}(\xi)$ and using (6.10) and (6.3), we obtain (6.7).

\begin{proposition}
Corresponding to $\gamma \in (0, \frac{1}{2})$, $\kappa \in (0, 1)$, $p, q, \nu, E_1, \gamma$ and $c_{14} \in (1, \infty)$ depending only on $p, q, \nu, E_1, \gamma$ and $c_{14} \in (1, \infty)$ depending only on $p, q, \nu, E_1, \gamma, \kappa$ such that the following holds: Suppose $\{V_{t_j}\}_{t \in [0,4]}$ and $\{u(\cdot, t)\}_{t \in [0,4]}$ satisfy (A1)-(A4) with $U = B_3$ and $I = [0, 4]$. Assume that we have (6.1) and (6.2) with $\varepsilon_6$ in place of $\frac{\tau_0}{2}$ and (6.3). Suppose that we have two points $\xi_1, \xi_2 \in B_1$ and two times $t_1, t_2 \in [\frac{3}{2}, 3]$ such that $h(V_{t_j}, \cdot) \in L^2(\|V_{t_j}\|)$ and $\Theta(\|V_{t_j}\|, \xi_j) = \frac{3}{2}$ for $j = 1, 2$. Then we have

\begin{equation}
|\xi_1 - \xi_2| \leq c_{14} \left( \max\{\mu, \|u\|\}^\gamma + \sqrt{|t_1 - t_2|} \right) \max\{\mu, \|u\|\}.
\end{equation}

\end{proposition}

\textbf{Proof.} Assume $t_1 \leq t_2$ without loss of generality. Write for simplicity

\begin{equation}
\hat{\mu} = \max\{\mu, \|u\|\}, \quad \hat{s} = \hat{\mu}^\gamma + \sqrt{|t_2 - t_1|}, \quad \hat{\xi} = \xi_2 - \xi_1.
\end{equation}

From Proposition 6.2, we already know that $|\hat{\xi}| \leq 2c_{12} \hat{\mu}$. We choose $\hat{t} \in [t_1 - 2\hat{s}^2, t_1 - \hat{s}^2]$ so that

\begin{equation}
\int_{B_{\hat{t}}} |h(V_{t_1}, \cdot)|^2 d\|V_{\hat{t}}\| \leq \frac{2c_{17}}{s^2} \hat{\mu}^2.
\end{equation}
This is possible by (4.37). Since $\hat{s} \geq \hat{\mu}^\gamma$ with $\gamma < \frac{1}{2}$, $\hat{s}$ is relatively larger than $\hat{\mu}$ for all sufficiently small $\hat{\mu}$. We utilize this in the following. We restrict $\hat{\mu}$ depending only on $c_{12}$ and $\gamma$ so that

\[ (6.23) \quad 2c_{12}\hat{\mu} \leq \frac{\hat{s}}{10} \]

E.g., $\hat{\mu} \leq (20c_{12})^{-\frac{1}{1-\gamma}}$ is sufficient. Consider the set $B_3 \cap \{x \in \mathbb{R}^2 : \text{dist}(x, R_{\frac{1}{2}}(J)) > 2c_{12}\hat{\mu}\}$. Due to (6.23), this set consists of three non-empty connected components, denoted by $W_1, W_2, W_3$. We have

\[ (6.24) \quad \frac{\hat{\mu}^2}{s^2} \leq \hat{\mu}^{2-2\gamma} = \hat{\mu}^{2-4\gamma} \cdot \hat{\mu}^{2\gamma} \]

with $\hat{\mu}^{2-4\gamma}$ chosen as small as one likes (note $\gamma < 1/2$). Thus, restricting $\hat{\mu}$ depending only on $\gamma$ and $c_{12}$, we can make sure using (6.22) and (6.24) that $\text{spt} \|V_i\|$ lies $O(\hat{\mu}^{-\gamma})$-neighborhood of $J$ (using the argument in Proposition 4.2) in $B_1$. The same can be said about $\text{spt} \|V_i^{(\xi)}\|$, since $|\xi| \leq c_{12}\hat{\mu}$. In particular, since $\hat{s} \geq \hat{\mu}^\gamma$, we have

\[ (6.25) \quad \|V_i^{(\xi)}\|(W_j) \geq \frac{\hat{s}}{4} \]

for $j = 1, 2, 3$ under this restriction on $\hat{\mu}$. Since $|\hat{\xi}| \leq 2c_{12}\hat{\mu}$, by Lemma 6.1 on one of $W_j$, say on $W_1$, we have (6.5) with $\hat{\xi}$ in place of $\xi$. Thus by (6.25) and (6.5) we obtain

\[ (6.26) \]

\[ |\hat{\xi}|^2 \leq \frac{4}{\hat{s}} \int_{W_1} |\hat{\xi}|^2 d\|V_i^{(\xi)}\| \leq \frac{32}{3\hat{s}} \int_{W_1} \text{dist}(\cdot, J)^2 + \text{dist}(\cdot, J_\xi)^2 d\|V_i^{(\xi)}\| \]

\[ \leq \frac{32}{3\hat{s}} \sum_{j=1}^2 \int_{B_{2\hat{s}}(\xi_j)} \text{dist}(\cdot, J_\xi)^2 d\|V_i\|. \]

For each $j = 1, 2$ and $x \in B_{2\hat{s}}(\xi_j)$, we have

\[ (6.27) \]

\[ \rho(\xi_j, t_j)(x, \hat{t}) = \exp\left(\frac{-(x-x_{\xi_j})^2}{4(t_j-\hat{t})}\right) \geq \exp(-1) \sqrt{12\pi \hat{s}}, \]

where we used $\hat{s}^2 \leq t_j - \hat{t} \leq 3\hat{s}^2$ which follows easily from the definition of $\hat{s}$ and $\hat{t}$. By (6.20) and (6.27), we obtain

\[ (6.28) \]

\[ |\hat{\xi}|^2 \leq \frac{32\sqrt{12}\pi}{3} \sum_{j=1}^2 \int_{B_{2\hat{s}}(\xi_j)} \rho(\xi_j, t_j)(\cdot, \hat{t})\text{dist}(\cdot, J_\xi)^2 d\|V_i\|. \]

For $\kappa \in [0, 1)$, by Proposition 6.2 each of the last two integrals is bounded by $c_{13}(t_j - \hat{t})^\kappa \hat{\mu}^2$. Since $t_j - \hat{t} \leq 3\hat{s}^2$, by defining $c_{14}$ appropriately, we obtain the desired estimate (6.20). \qed

7. Blow-up analysis and improvement of the space-time $L^2$-distance

Throughout this section, we prove a sequence of propositions under the following assumptions. Suppose that $\{V_i^{(m)}\}_{t \in [0, 4]}$ and $\{u^{(m)}(\cdot, t)\}_{t \in [0, 4]}$ ($m \in \mathbb{N}$) are arbitrary sequences satisfying (A1)-(A4) and (6.3) with $U = B_3$, $I = [0, 4]$ and with $V_i^{(m)}$, $u^{(m)}$ in place of $V_i$, $u$.
Proposition 7.1. In the following, we denote subsequences by the same index.

(7.1) \[
\left( \int_0^4 \int_{B_3} \text{dist}(\cdot, J)^2 d\|V_t^{(m)}\| dt \right)^{\frac{1}{2}} \leq \mu^{(m)},
\]

(7.2) \[
\left( \int_0^4 \left( \int_{B_3} |u^{(m)}|^p d\|V_t^{(m)}\| \right)^{\frac{p}{2}} dt \right)^{\frac{1}{p}} \leq \|u^{(m)}\|
\]

and

(7.3) \[
\lim_{m \to \infty} \left( \mu^{(m)} \right)^{-1} \|u^{(m)}\| = 0.
\]

Fix a decreasing sequence \( \{\tau_m\}_{m \in \mathbb{N}} \subset (0, \frac{1}{2}) \) with \( \lim_{m \to \infty} \tau_m = 0 \) and use Proposition 3.1 with \( \tau = \tau_m \) to obtain \( \varepsilon_2(m) \) and \( c_3(m) \) corresponding to \( \tau_m \). We choose a subsequence so that, after relabelling, \( \max \{\mu^{(m)}, \|u^{(m)}\|\} \leq \varepsilon_2(m) \) for all \( m \in \mathbb{N} \). Then we can apply Proposition 3.1 to \( \{V_t^{(m)}\}_{t \in [0,4]} \) and \( \{u^{(m)}(\cdot, t)\}_{t \in [0,4]} \) with \( \tau = \tau_m \). Let \( f_j^{(m)} : [\tau_m, 2 - \tau_m] \times [\tau_m, 4 - \tau_m] \to \mathbb{R} \), \( j = 1, 2, 3 \), be the resulting functions, satisfying (3.6) and (3.7). For each fixed \( \tau \in (0, \frac{1}{2}) \) and for all \( m \in \mathbb{N} \) with \( \tau_m \leq \tau \), note that \( f_j^{(m)} \) satisfies (3.6) with \( Q = Q_\tau = [\tau, 2 - \tau] \times [\tau, 4 - \tau] \) and \( c_3 = c_3(\tau) \), i.e.,

(7.4) \[
\|f_j^{(m)}\|_{C^{1,\xi}(Q_\tau)} \leq c_3(\tau) \max \{\mu^{(m)}, \|u^{(m)}\|\}.
\]

For each \( m \in \mathbb{N} \) and \( j = 1, 2, 3 \), define

(7.5) \[
\tilde{f}_j^{(m)} = (\mu^{(m)})^{-1} f_j^{(m)}.
\]

By (7.4), (7.5), (7.3) and the Ascoli-Arzelà compactness theorem, \( \{\tilde{f}_j^{(m)}\}_{m \in \mathbb{N}} \) has a subsequence which converges locally uniformly on \((0, 2) \times (0, 4)\) to some limit function \( \tilde{f}_j \), \( j = 1, 2, 3 \). We also have the estimate

(7.6) \[
\|\tilde{f}_j\|_{C^{1,\xi}(Q_\tau)} \leq c_3(\tau).
\]

In the following, we denote subsequences by the same index.

Proposition 7.1. The function \( \tilde{f}_j \) belongs to \( C^\infty((0, 1) \times (1, 3)) \) and satisfies the heat equation \( \frac{\partial \tilde{f}_j}{\partial t} = \frac{\partial^2 \tilde{f}_j}{\partial x^2} \) on \((0, 1) \times (1, 3)\) for \( j = 1, 2, 3 \).

Proof. It is enough to prove the claim for \( \tilde{f}_1 \) since the proof for the other two is similarly carried out after suitable rotations. Fix \( \phi \in C^\infty_c((0, 1) \times (1, 3); \mathbb{R}^+) \), and fix \( \tau \in (0, \frac{1}{2}) \) so that \( \text{spt} \phi \subset Q_\tau \). For all sufficiently large \( m \), we have \( \tau_m < \tau \) and we only consider such \( m \). Let \( c_3 = c_3(\tau) \) be a constant to be fixed depending only on \( \tau \). We take in (2.16)

\( \phi^{(m)}(x, t) \equiv (x_2 + 2c_3\mu^{(m)})(x_1, t)\eta^{(m)}(x_2) \) as a test function, where, \( x = (x_1, x_2) \) and \( \eta^{(m)} \) is a \( C^\infty \) function such that \( \eta = 1 \) for \( x_2 \in [-c_3\mu^{(m)}, c_3\mu^{(m)}] \), \( \eta^{(m)} = 0 \) for \( x_2 \notin [-2c_3\mu^{(m)}, 2c_3\mu^{(m)}] \) and \( 0 \leq \eta^{(m)} \leq 1 \). Note that \( \phi^{(m)}(\cdot, t) \) has compact support in \( B_3 \) and is non-negative, so is a valid choice as a test function. Since \( x_2 = f^{(m)}_1(x_1, t) \) for \( (x_1, x_2) \in \text{spt} \|V_t^{(m)}\| \), and since \( |f^{(m)}| \leq c_3 \mu^{(m)} \) by (3.6) and (7.3), for all sufficiently large \( m \), we have \( \eta^{(m)} = 1 \) on \( \text{spt} \|V_t^{(m)}\| \). Thus in the following computation, even though we need \( \eta^{(m)} \) for \( \phi^{(m)} \) to have non-negativity, we ignore \( \eta^{(m)} \). We then have

(7.7) \[
0 \leq \int_1^3 \int_{B_2} (-h(V_t^{(m)}, \cdot) \phi^{(m)} + \nabla \phi^{(m)} \cdot (h(V_t^{(m)}, \cdot) + (u^{(m)})^\perp) + \frac{\partial \phi^{(m)}}{\partial t} \|V_t^{(m)}\|) dt.
\]
By the Cauchy-Schwarz inequality and by dropping a negative $|h|^2$ term, we obtain from (7.8)
\begin{equation}
0 \leq \int_1^3 \int_{B_2} |u^{(m)}|^2 \phi^{(m)} + |u^{(m)}| |\nabla \phi^{(m)}| + \frac{\partial \phi^{(m)}}{\partial t} + \nabla \phi^{(m)} \cdot h(V_t^{(m)}, \cdot) \, dV_t^{(m)} \, dt
\end{equation}
\begin{equation}
= I_1^{(m)} + I_2^{(m)} + I_3^{(m)} + I_4^{(m)}, \quad \text{say.}
\end{equation}

We next estimate $\lim_{m \to \infty} (\mu^{(m)})^{-1} I_j^{(m)}$. By the Hölder inequality, we have
\begin{equation}
\lim_{m \to \infty} (\mu^{(m)})^{-1} I_j^{(m)} \leq \lim_{m \to \infty} c(p, q)(\mu^{(m)})^{-1} \|u^{(m)}\|^2 = 0,
\end{equation}
where we used (7.3). Similarly, since $|\nabla \phi^{(m)}| \leq c(\phi)$ on spt $\|V_t^{(m)}\|$ and by (7.3),
\begin{equation}
\lim_{m \to \infty} (\mu^{(m)})^{-1} I_2^{(m)} \leq \lim_{m \to \infty} c(\phi)(\mu^{(m)})^{-1} \|u^{(m)}\| = 0.
\end{equation}

For $I_3^{(m)}$, we have
\begin{equation}
\lim_{m \to \infty} (\mu^{(m)})^{-1} I_3^{(m)} = \lim_{m \to \infty} \int_1^3 \int_0^1 \int_0^1 (\tilde{f}_1^{(m)} + 2c_3) \frac{\partial \phi}{\partial t} \sqrt{1 + |\partial x_1 f_1^{(m)}|^2} \, dx_1 \, dt
\end{equation}
\begin{equation}
= \int_1^3 \int_0^1 \int_0^1 (\tilde{f}_1 + 2c_3) \frac{\partial \phi}{\partial t} \, dx_1 \, dt,
\end{equation}
where we used the uniform convergence $\tilde{f}_1^{(m)} \to \tilde{f}_1$ and (7.4). For $I_4^{(m)}$, since $\nabla \phi^{(m)} = (0, 1) \phi + (x_2 + 2c_3 \mu^{(m)}) \nabla \phi$, writing $h(V_t^{(m)}, x) = h = (h_1, h_2)$, we have
\begin{equation}
(\mu^{(m)})^{-1} I_4^{(m)} = (\mu^{(m)})^{-1} \int_1^3 \int_{B_2} \{ \phi h_2 + (x_2 + 2c_3 \mu^{(m)}) \nabla \phi \cdot h \} \, dV_t^{(m)} \, dt.
\end{equation}

Since $|x_2 + 2c_3 \mu^{(m)}| \leq 3c_3 \mu^{(m)}$ and using the estimate (4.37) which is valid here, the second term of the integral converges to 0. For the first term, by the first variation formula,
\begin{equation}
\int_{B_2} \phi h_2 \, dV_t^{(m)} = - \int_{B_2} S \cdot (\nabla \phi \otimes (0, 1)) \, dV_t^{(m)}(\cdot, S).
\end{equation}

Since $S = (1 + |\partial x_1 f_1^{(m)}|^2)^{-1} (1, \partial x_1 f_1^{(m)}) \otimes (1, \partial x_1 f_1^{(m)})$ and $\nabla \phi = (\partial x_1 \phi, 0)$, we have
\begin{equation}
- \int_{B_2} S \cdot (\nabla \phi \otimes (0, 1)) \, dV_t^{(m)}(\cdot, S) = - \int_0^1 (1 + |\partial x_1 f_1^{(m)}|^2)^{-1} \partial x_1 f_1^{(m)} \partial x_1 \phi \, dx_1.
\end{equation}

Since $\nabla \tilde{f}_1^{(m)} \to \nabla \tilde{f}_1$ uniformly, (7.12)-(7.14) show that
\begin{equation}
\lim_{m \to \infty} (\mu^{(m)})^{-1} I_4^{(m)} = - \int_1^3 \int_0^1 \partial x_1 \tilde{f}_1 \partial x_1 \phi \, dx_1.
\end{equation}

Combining (7.8)-(7.11) and (7.15), we obtain (writing $x_1$ as $x$
\begin{equation}
0 \leq \int_1^3 \int_0^1 (\tilde{f}_1 + 2c_3) \frac{\partial \phi}{\partial t} - \partial t \tilde{f}_1 \frac{\partial \phi}{\partial x} \, dx \, dt.
\end{equation}

We carry out the same argument with $\phi^{(m)} := (2c_3 \mu^{(m)} - x_2) \phi(x_1, t) \eta^{(m)}(x_2)$, which is again non-negative with compact support. The limit in this case produces
\begin{equation}
0 \leq \int_1^3 \int_0^1 (2c_3 - \tilde{f}_1) \frac{\partial \phi}{\partial t} + \partial t \tilde{f}_1 \frac{\partial \phi}{\partial x} \, dx \, dt.
\end{equation}
Since $\phi$ has a compact support in $(0,1) \times (1,3)$, the term involving $c_3$ is 0. Thus (7.16) and (7.17) give
\[
0 = \int_1^3 \int_0^1 f_1 \frac{\partial \phi}{\partial t} - \frac{\partial f_1}{\partial x} \frac{\partial \phi}{\partial x} \, dx dt.
\]
We have proved that (7.18) holds for arbitrary $\phi \in C_\infty^1((0,1) \times (1,3); \mathbb{R}^+)$. One can then prove that (7.18) holds for $\phi \in C_\infty^1((0,1) \times (1,3))$ which is not necessarily non-negative. By the standard regularity theory of parabolic equation, $\tilde{f}_1$ is smooth and satisfies the heat equation. 

For the sequence $\{\tilde{V}_t^{(m)}\}_{t \in [0,4]}$ ($m \in \mathbb{N}$) under consideration, we define the following.

**Definition 7.2.**
\[
T_g = \cap_{m \in \mathbb{N}} \left\{ t \in \left[ \frac{3}{2}, 3 \right] : \tilde{V}_t^{(m)} \text{ is a unit density 1-varifold, } h(V_t^{(m)}, \cdot) \in L^2_{\text{loc}}(\|V_t^{(m)}\|) \text{ and } \Theta(\|V_t^{(m)}\|, x) = 1 \text{ or } \frac{3}{2}, \quad \forall x \in \text{spt} \|V_t^{(m)}\| \right\},
\]
where all the conditions are required to be satisfied in $B_3$.

Since above conditions are satisfied for a.e. $t \in [\frac{1}{2}, 4]$ for each $\{\tilde{V}_t^{(m)}\}_{t \in [0,4]}$, $T_g$ is a full measure set in $[\frac{3}{2}, 3]$, i.e., $\mathcal{L}^1([\frac{3}{2}, 3] \setminus T_g) = 0$. We next define the following sets.

**Definition 7.3.** For $m \in \mathbb{N}$ and $t \in T_g$, define
\[
\xi^{(m)}(t) = \left\{ x \in B_1 : \Theta(\|V_t^{(m)}\|, x) = \frac{3}{2} \right\}, \quad \tilde{\xi}^{(m)}(t) = \left\{ \frac{x}{\mu^{(m)}} : x \in \xi^{(m)}(t) \right\}.
\]

Since spt $\|V_t^{(m)}\|$ away from the origin consists of three $C^{1,\zeta}$ curves, there has to be at least one point $x$ in $B_1$ with $\Theta(\|V_t^{(m)}\|, x) = \frac{3}{2}$. Otherwise $\Theta(\|V_t^{(m)}\|, x) = 1 \quad \forall x \in \text{spt} \|V_t^{(m)}\| \cap B_1$ and spt $\|V_t^{(m)}\|$ has to be a union of regular embedded curves, a contradiction. Thus $\xi^{(m)}(t) = \emptyset$ for all $t \in T_g$. We now apply Proposition 6.2 and 6.3. Fix $\gamma \in (0, \frac{1}{2})$ and $\kappa \in (0, 1)$. Then for all sufficiently large $m$, (6.6) and (6.20) combined with (7.3) show that for any $a \in \tilde{\xi}^{(m)}(t_1)$, (7.21)
\[
|a| \leq c_{12},
\]
and for any $a \in \tilde{\xi}^{(m)}(t_1)$ and $b \in \tilde{\xi}^{(m)}(t_2)$,
\[
|a - b| \leq c_{14}(\mu^{(m)} \gamma + \sqrt{|t_1 - t_2|^\kappa}).
\]

**Proposition 7.4.** There exists a $\frac{\gamma}{2}$-Hölder continuous function $\tilde{\xi} : [\frac{3}{2}, 3] \to \mathbb{R}^2$ such that
\[
\sup_{t \in [\frac{3}{2}, 3]} |\tilde{\xi}(t)| \leq c_{12},
\]
\[
\sup_{t_1, t_2 \in [\frac{3}{2}, 3], t_1 \neq t_2} \frac{|\tilde{\xi}(t_1) - \tilde{\xi}(t_2)|}{|t_1 - t_2|^\frac{\gamma}{2}} \leq c_{14}
\]
and $\tilde{\xi}^{(m)}(t)$ converges uniformly on $T_g$ to $\tilde{\xi}(t)$ in the Hausdorff distance.

**Proof.** Choose a countable dense set $\{t_i\}_{i \in \mathbb{N}} \subset T_g$. For all sufficiently large $m$, $\tilde{\xi}^{(m)}(t_i)$ is bounded uniformly by (7.21). Also by (7.22), one notes that the diameter of $\tilde{\xi}^{(m)}(t_i)$ is $\leq c_{14}(\mu^{(m)} \gamma$ and converges to 0 as $m \to \infty$. Thus for each fixed $i \in \mathbb{N}$, $\{\tilde{\xi}^{(m)}(t_i)\}_{m \in \mathbb{N}}$ has a converging subsequence whose limit is a single point. By the diagonal argument, we can
extract a subsequence (denoted by the same index) so that \( \{ \tilde{\xi}^{(m)}(t_i) \}_{m \in \mathbb{N}} \) converges to a limit point denoted by \( \xi(t_i) \). By (7.22), \( \xi \) is H"{o}lder continuous on this countable set, and one can extend the definition of \( \xi \) uniquely to the whole \([\frac{3}{2}, 3]\) with the same H"{o}lder constant.

For \( t \in T_y \setminus \{ t_1 \}_{t \in \mathbb{N}} \), by using (7.22), one can prove that \( \{ \tilde{\xi}^{(m)}(t) \}_{m \in \mathbb{N}} \) also converges to \( \tilde{\xi}(t) \) and that the convergence is uniform.

**Definition 7.5.** For each \( t \in \left[ \frac{3}{2}, 3 \right] \) and \( j = 1, 2, 3 \), let \( \tilde{\xi}_j(t) \in \mathbb{R} \) be obtained as follows. For \( j = 1 \), set \( \tilde{\xi}_1(t) \) to be the second coordinate of \( \tilde{\xi}(t) \). For \( j = 2, 3 \), rotate \( \tilde{\xi}(t) \) by \( \frac{2\pi(j-1)}{3} \) clockwise, and take its second coordinate to be \( \tilde{\xi}_j(t) \).

Since \( R_0 + R_{-\frac{2\pi}{3}} + R_{-\frac{4\pi}{3}} = 0 \), we have

\[
(7.25) \quad \tilde{\xi}_1(t) + \tilde{\xi}_2(t) + \tilde{\xi}_3(t) = 0
\]

for all \( t \in \left[ \frac{3}{2}, 3 \right] \).

**Proposition 7.6.** For any \( t_0 \in \left[ \frac{3}{2}, 3 \right] \), we have

\[
(7.26) \quad \sup_{t \in [t_0, t_0]} \left( t_0 - t \right)^{-\frac{\kappa}{2}} \int_0^2 e^{-\frac{t_0 - t}{2}} \sum_{j=1}^{3} \left| \tilde{f}_j(x, t) - \tilde{\xi}_j(t_0) \right|^2 dx \leq 4\pi c_{13}.
\]

**Proof.** If we prove

\[
(7.27) \quad \left( t_0 - t \right)^{-\frac{\kappa}{2}} \int_0^2 e^{-\frac{t_0 - t}{2}} \sum_{j=1}^{3} \left| \tilde{f}_j(x, t) - \tilde{\xi}_j(t_0) \right|^2 dx \leq 4\pi c_{13},
\]

for arbitrary \( t_0 \in T_y, t \in \left[ \frac{3}{2}, t_0 \right) \) and \( \tau \in (0, \frac{3}{2}) \), then by the continuity of \( \tilde{\xi}_j \), (7.27) is true for all \( t_0 \in \left[ \frac{3}{2}, 3 \right] \) and we will end the proof of (7.26). Thus we fix arbitrary such \( t_0, t, \tau \).

By (7.19), there exists a sequence of non-empty sets \( \{ \xi^{(m)}(t_0) \}_{m \in \mathbb{N}} \) as in (7.20). From each \( \xi^{(m)}(t_0) \), choose one point \( \xi^{(m)}_j(t_0) \in \xi^{(m)}(t_0) \). Now, for all sufficiently large \( m \), we may apply Proposition 7.2 with \( \xi \) replaced by \( \xi^{(m)}_j(t_0) \). Thus for all sufficiently large \( m \), we have

\[
(7.28) \quad \left( t_0 - t \right)^{-\kappa} \int_{\mathbb{R}^3} \rho_{\xi^{(m)}_j(t_0)}(\cdot, t) \text{dist}(\cdot, J_{\xi^{(m)}_j(t_0)})^2 d\|V^{(m)}_t\| \leq c_{13}(\mu^{(m)})^2.
\]

As we have seen already, we may represent \( \|V^{(m)}_t\| \) by \( f^{(m)}_j \) in the relevant domain after suitable rotations. At a point in \( x \in \text{spt} \|V^{(m)}_t\| \) represented by \( (s, f^{(m)}_j(s, t)) \) after a rotation, (7.29)

\[
\text{dist}(x, J_{\xi^{(m)}_j(t_0)}) = \|f^{(m)}_j(s, t) - \xi^{(m)}_j(t_0)\|,
\]

where \( \xi^{(m)}_j(t_0) := \text{second coordinate of } R_{-\frac{2\pi}{3}}(\xi^{(m)}_j(t_0)) \). Since \( \mu^{(m)}(s, f^{(m)}_j(s, t)) \rightarrow \tilde{\xi}(t_0) \), we have \( \mu^{(m)}(s, f^{(m)}_j(s, t)) \rightarrow \tilde{\xi}_j(t_0) \). Since \( \mu^{(m)}(s, f^{(m)}_j(s, t)) \rightarrow \tilde{\xi}_j(t_0) \) uniformly away from the origin, with (7.29), we have

\[
\lim_{m \rightarrow \infty} \left( \mu^{(m)} \right)^{-2} \int_{\mathbb{R}^3} \rho_{\xi^{(m)}_j(t_0)} \text{dist}(\cdot, J_{\xi^{(m)}_j(t_0)})^2 d\|V^{(m)}_t\| = \sum_{j=1}^{3} \int_{\mathbb{R}^3} \rho_{\xi^{(m)}_j(t_0)} \|\tilde{f}_j - \tilde{\xi}_j(t_0)\|^2 dx.
\]

Recalling the definition of \( \rho_{\xi^{(m)}_j(t_0)} \), (7.28) and (7.30) prove (7.27) and we end the proof.
Lemma 7.7. There exists $c_{15} \in (1, \infty)$ depending only on $\kappa, p, q, \nu, E_1$ with the property that

\[
\sup_{x \in [\frac{\kappa}{2}, \frac{\kappa}{2}]} \left( \int_0^t |f_j(x, t)|^2 dx \right)^{\frac{1}{2}} \leq c_{15}
\]

for $j = 1, 2, 3$.

Proof. We simply choose $t_0 = 3$ in Proposition 7.6. Then, for any $t \in \left[ \frac{\kappa}{2}, \frac{\kappa}{2} \right]$ and $x \in (0, \frac{1}{2})$, $t_0 - t \leq \frac{t}{2}$ and $\frac{x^2}{4(t_0 - t)} \leq \frac{1}{8}$. Moreover, $|\xi_j^\perp(3)| \leq c_{12}$ by (7.23). Combining these facts and with a suitable constant depending only on $c_{12}$ and $c_{13}$, and thus ultimately depending only on $\kappa, p, q, \nu, E_1$, we obtain (7.31) from (7.26). \qed

Definition 7.8. We define $\bar{f}_{AV} \in C^\infty((0, 1) \times (\frac{5}{4}, \frac{5}{2}))$ by

\[
\bar{f}_{AV}(x, t) = \frac{1}{3}(\bar{f}_1(x, t) + \bar{f}_2(x, t) + \bar{f}_3(x, t)).
\]

Proposition 7.9. The odd extension of $\bar{f}_{AV}$ with respect to $x$ satisfies the heat equation on $(-1, 1) \times (\frac{5}{4}, \frac{5}{2})$ and is $C^\infty$ there. In particular, $\bar{f}_{AV}(0, t) = 0$ for $t \in (\frac{5}{4}, \frac{5}{2})$.

Proof. Fix $\tau \in (0, \frac{1}{4})$. In (7.20), we use $t \in (\frac{5}{4}, \frac{5}{2})$ and $t_0 = \frac{\tau}{4} + t$. Then we obtain

\[
\int_0^\tau e^{-\frac{t}{4}} \left| \int_{\tau}^3 |f_j(x, t)|^2 dx \right| \leq \sqrt{4\pi} c_{13} \left( \frac{\tau^2}{4} \right)^{\kappa + \frac{1}{2}}.
\]

Using (7.25), (7.32) and (7.33), we obtain

\[
\int_0^\tau |\bar{f}_{AV}(x, t)|^2 dx \leq e^{\sqrt{4\pi} 4^{-\kappa + \frac{1}{2}} c_{13} \tau^{2\kappa + 1}}.
\]

We continue to denote the odd extension of $\bar{f}_{AV}(x, t)$ for $x \in (-1, 0)$ by the same notation. For $\phi \in C_{c}^\infty((-1, 1) \times (\frac{5}{4}, \frac{5}{2}))$, we need to prove

\[
\int_Q \bar{f}_{AV} \frac{\partial \phi}{\partial t} + \bar{f}_{AV} \frac{\partial^2 \phi}{\partial x^2} dx dt = 0
\]

with $Q := (-1, 1) \times (\frac{5}{4}, \frac{5}{2})$. Since $\bar{f}_{AV}$ is odd with respect to $x$, we only need to prove (7.35) for odd $\phi$. Let $\eta_\tau \in C_{c}^\infty(\mathbb{R})$ be a function such that $\eta_\tau(x) = 1$ for $|x| \geq \tau$, $\eta_\tau(x) = 0$ for $|x| \leq \frac{\tau}{2}$, $0 \leq \eta_\tau \leq 1$ and $|\partial^j \eta_\tau| \leq 4^{j-1}$ and $|\partial^j \eta_\tau| \leq 16 \tau^{-2}$. By integration by parts and the fact that $\bar{f}_{AV}$ satisfies the heat equation away from $\{x = 0\}$, we have

\[
\int_Q \bar{f}_{AV} \frac{\partial \phi}{\partial t} + \bar{f}_{AV} \frac{\partial^2 \phi}{\partial x^2} dx dt = \lim_{\tau \to 0^+} \int_Q \bar{f}_{AV} \eta_\tau \frac{\partial \phi}{\partial t} + \bar{f}_{AV} \eta_\tau \frac{\partial^2 \phi}{\partial x^2} dx dt
\]

\[
= - \lim_{\tau \to 0^+} \int_Q \eta_\tau \frac{\partial \phi}{\partial x} + \eta_\tau \frac{\partial \phi}{\partial x} \bar{f}_{AV} dx dt.
\]

Since $|\phi(x, t)| \leq c(|\nabla \phi||x|)$ by the oddness of $\phi$, we have $|\partial^j \eta_\tau| \leq c \tau$. Then by using (7.34), we may prove that (7.36) is $= 0$, which proves (7.35). The standard regularity theory shows that $\bar{f}_{AV}$ is $C^\infty$ on $Q$. \qed

Proposition 7.10. For $j, j' \in \{1, 2, 3\}$, $t \in (0, \frac{1}{4})$ and $x \in (-1, 0)$, define $(\tilde{f}_j - \tilde{f}_{j'})(x, t) = (\tilde{f}_j - \tilde{f}_{j'})(-x, t)$. Then $\tilde{f}_j - \tilde{f}_{j'}$ satisfies the heat equation on $(-1, 1) \times (\frac{5}{4}, \frac{5}{2})$ and is $C^\infty$ there. In particular, we have $\frac{\partial (\tilde{f}_j - \tilde{f}_{j'})}{\partial x}(0, t) = 0$ for $t \in (\frac{5}{4}, \frac{5}{2})$. 

Proof. We consider \( \tilde{f}_1 - \tilde{f}_2 \) since others can be similarly proved. Given \( \tau \in (0, \frac{1}{4}) \), fix an arbitrary \( \tau' \in (0, \tau) \). With respect to \( \tau' \), we obtain \( c_3(\tau') \) by Proposition 3.1. For all sufficiently large \( m \), define

\[
T_g^{(m)} = \left\{ t \in T_g \cap \left[ \frac{5}{4}, \frac{5}{2} \right] : \int_{B_2} |h(V_t^{(m)}, \cdot)|^2 \phi_{rad}^2 \, d\|V_t^{(m)}\| \leq \alpha_1^2 \right\},
\]

where \( \alpha_1 \) is from Proposition 4.2. By (4.37), note that we have

\[
(7.37) \quad \mathcal{L}^1 \left( \left[ \frac{5}{4}, \frac{5}{2} \right] \setminus T_g^{(m)} \right) \leq \alpha_1^{-2}(\mu^{(m)})^2 c_7.
\]

For any \( t \in T_g^{(m)} \), by Proposition 4.2 spt \( \|V_t^{(m)}\| \) is represented by \( f_j^{(m)} \) \((j = 1, 2, 3)\) inside \( B_1 \), and there is only one junction point where three curves are joined with angle \( \frac{2\pi}{3} \). Using the similar notation in the proof of Proposition 4.2, each curves are represented as in (4.12) with \( f_j^{(m)}(\cdot, t) \) and \( s_j^{(m)} \) in place of \( f_j \) and \( s_j \) there. We have (4.13)-(4.16) satisfied similarly. Suppose without loss of generality that \( s_2^{(m)} \leq s_3^{(m)} \). For all sufficiently large \( m \), we have \( |s_2^{(m)}| < \tau' \) since \( |s_j^{(m)}| \leq c_{12} \mu^{(m)} \) by (6.6). For each \( s \in (\tau', \tau) \), we have (omitting \( t \) dependence)

\[
|\frac{\partial f_1^{(m)} - f_2^{(m)}}{\partial x}(s)| \leq \left| \frac{\partial f_1^{(m)} - f_2^{(m)}}{\partial x}(s_2^{(m)}) \right| + \sum_{j=1,2} \left| \frac{\partial^2 f_j^{(m)}}{\partial x^2} \right| \, dx \\
\leq \left| \frac{\partial f_1^{(m)}(s_2^{(m)}) - \partial f_2^{(m)}(s_2^{(m)})}{\partial x} \right| + 2 \int_{B_{2\tau}} |h(V_t^{(m)}, \cdot)| \, d\|V_t^{(m)}\|,
\]

where we have used \( \frac{\partial f_j^{(m)}(s_1^{(m)})}{\partial x} = \frac{\partial f_j^{(m)}(s_2^{(m)})}{\partial x} \) which follows from (4.16). The first term of (7.38) may be bounded by the second term, so we obtain from (7.38)

\[
(7.39) \quad \sup_{s \in (\tau', \tau)} \left| \frac{\partial f_1^{(m)} - f_2^{(m)}}{\partial x}(s, t) \right| \leq 4 \int_{B_{2\tau}} |h(V_t^{(m)}, \cdot)| \, d\|V_t^{(m)}\|.
\]

For any \( t \in \left[ \frac{5}{4}, \frac{5}{2} \right] \setminus T_g^{(m)} \), we have

\[
(7.40) \quad \sup_{s \in (\tau', \tau)} \left| \frac{\partial f_1^{(m)} - f_2^{(m)}}{\partial x}(s, t) \right| \leq 2c_3(\tau') \mu^{(m)}.
\]

Combining (7.37), (7.39) and (7.40), we obtain

\[
(7.41) \quad \int_{\frac{\tau}{4}}^{\frac{5}{4}} \sup_{s \in (\tau', \tau)} \left| \frac{\partial f_1^{(m)} - f_2^{(m)}}{\partial x}(s, t) \right| \, dt \leq 4 \int_{\frac{\tau}{4}}^{\frac{5}{4}} \int_{B_{2\tau}} |h(V_t^{(m)}, \cdot)| \, d\|V_t^{(m)}\| + 2c_3 \alpha_1^{-2} c_7 (\mu^{(m)})^3.
\]

We may estimate

\[
(7.42) \quad \int_{\frac{\tau}{4}}^{\frac{5}{4}} \int_{B_{2\tau}} |h(V_t^{(m)}, \cdot)| \, d\|V_t^{(m)}\| \leq (5\tau E_1)^\frac{1}{2} \left( \int_{\frac{\tau}{4}}^{\frac{5}{4}} \int_{B_{2}} |h(V_t^{(m)}, \cdot)|^2 \phi_{rad}^2 \, d\|V_t^{(m)}\| \right)^{\frac{1}{2}} \\
\leq (5\tau E_1)^\frac{1}{2} \frac{1}{2} \mu^{(m)},
\]
Recall the definition of $\xi(t)$ (in Definition 7.5). For each $m \in \mathbb{N}$, define $J^{(m)}(t)$ to be the set obtained by first rotating $J$ counterclockwise by $\arctan(\mu^{(m)} \frac{\partial \tilde{f}}{\partial x}(0,2))$ and then translating by $\mu^{(m)} \tilde{f}(2)$. In the similarity class of $J$, $J^{(m)}$ is the element characterized by the properties that it has junction point at $\mu^{(m)} \tilde{f}(2)$, and the slope of its ray close to the positive $x$-axis is equal to $\mu^{(m)} \frac{\partial \tilde{f}}{\partial x}(0,2)$. Denote the junction point of $J^{(m)}$ by $a^{(m)}$ (thus $a^{(m)} = \mu^{(m)} \tilde{f}(2)$).

To clarify the property of $J^{(m)}$ concerning the slope of its ray close the the $x$-axis, we recall that the second coordinate of $R_{\frac{-\pi}{32\pi 2}}(\xi(t))$ has been denoted by $\xi^{(1)}_+(t)$ and is equal

$$
\phi \quad \text{time, it is sufficient to use even test functions}
$$

$R$ recall that the second coordinate of $\tau$ with respect to $x$ of differentiation. Since $c$ we may write $\tilde{f}_1 - \tilde{f}_2$ of $\tilde{f}$ may be smoothly extended for $\{x \leq 0\}$. More precisely, for $x \in (-1,0)$ and $t \in \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$, define

$$
\tilde{f}_1(x,t) = \left\{ - \tilde{f}_1 + \frac{1}{3} (\tilde{f}_1 - \tilde{f}_2) + \frac{1}{3} (\tilde{f}_1 - \tilde{f}_3) \right\}_{(-x,t)} = \frac{1}{3} (\tilde{f}_1 - 2\tilde{f}_2 - 2\tilde{f}_3)(-x,t).
$$

Then $\tilde{f}_1$ is in $C^\infty((-1,1) \times (\frac{\pi}{2}, \frac{\pi}{2}))$ and satisfies the heat equation on its domain. Moreover, by (7.31) and (7.45), we have

$$
\sup_{t \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]} \left( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\tilde{f}_1(x,t)|^2 \, dx \right) \leq \frac{8}{3} c_{15},
$$

and by (7.23), $\tilde{f}_1(0,t) = \tilde{c}_1(t)$ holds for $t \in \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$. Since $\tilde{f}_1$ satisfies the heat equation with the estimate (7.46), the standard regularity theory (23) shows that any partial derivatives of $\tilde{f}_1$ on $(-\frac{\pi}{4}, \frac{\pi}{4}) \times (\frac{\pi}{2}, \frac{\pi}{2})$ can be bounded by a constant depending only on $c_{15}$ and the order of differentiation. Since $c_{15}$ depends only on $\kappa, p, q, \nu, E_1$, we have (7.44) with a suitable constant $c_{16}$. 

**Definition 7.12.** Recall the definition of $\tilde{\xi}(t)$ (in Definition 7.5). For each $m \in \mathbb{N}$, define $J^{(m)} \subset \mathbb{R}^2$ to be the set obtained by first rotating $J$ counterclockwise by $\arctan(\mu^{(m)} \frac{\partial \tilde{f}}{\partial x}(0,2))$ and then translating by $\mu^{(m)} \tilde{f}(2)$. In the similarity class of $J$, $J^{(m)}$ is the element characterized by the properties that it has junction point at $\mu^{(m)} \tilde{f}(2)$, and the slope of its ray close to the positive $x$-axis is equal to $\mu^{(m)} \frac{\partial \tilde{f}}{\partial x}(0,2)$. Denote the junction point of $J^{(m)}$ by $a^{(m)}$ (thus $a^{(m)} = \mu^{(m)} \tilde{f}(2)$).
to $\tilde{f}_j(0,t)$ for $j = 1, 2, 3$. The ray of $\tilde{J}^{(m)}$ close to the $x$-axis can be expressed as
\begin{equation}
(7.47) \quad \left\{ (x, \mu^{(m)} f_j(0,2) + \mu^{(m)}(x - \mu^{(m)} \tilde{c}_1(2)) \frac{\partial f_j}{\partial x}(0,2)) \in \mathbb{R}^2 : x \in (\mu^{(m)} \tilde{c}_1(2), \infty) \right\}.
\end{equation}
More generally, for $j = 1, 2, 3$, the half line of $\mathbf{R}_{-\mu^{(m)}x}(\tilde{J}^{(m)})$ close to the $x$-axis is
\begin{equation}
(7.48) \quad \left\{ (x, \mu^{(m)} f_j(0,2) + \mu^{(m)}(x - \mu^{(m)} v_j) \frac{\partial f_j}{\partial x}(0,2)) \in \mathbb{R}^2 : x \in (\mu^{(m)} v_j, \infty) \right\},
\end{equation}
where $v_j$ is the first coordinate of $\mathbf{R}_{-\mu^{(m)}x}(\tilde{J}^{(m)})$. It is important to note that we used
\[ \frac{\partial f_j}{\partial x}(0,2) = \frac{\partial f_j}{\partial x}(0,2) (j = 2, 3) \]
which follows from Proposition 7.10 and 7.11.

**Proposition 7.13.** There exists $c_{17}$ depending only on $p, q, \nu, E_1$ such that, for all $\theta \in (0, \frac{1}{4})$, we have
\begin{equation}
(7.49) \quad \lim_{m \to \infty} \sup \frac{1}{(\mu^{(m)})^2 \theta^2 \int_{2-\theta^2}^{2+\theta^2} t^{1/2} \int_{B_\theta(a^{(m)})} \text{dist}(\cdot, \tilde{J}^{(m)})^2 d\|V_t^{(m)}\| dt \leq c_{17}\theta^2
\end{equation}
and
\begin{equation}
(7.50) \quad d(\tilde{J}^{(m)}, J) \leq c_{17}\mu^{(m)}.
\end{equation}

**Proof.** Fix $\theta \in (0, \frac{1}{4})$ and $\tau \in (0, \theta)$. For any $t \in (2 - \theta^2, 2 + \theta^2)$ with $t + \tau^2 \in T_y$, choose a point $\tilde{\xi}^{(m)} \in \xi^{(m)}(t + \tau^2)$ (recall (7.20)). Then by Proposition 6.2 with $\kappa = \frac{1}{2}$ fixed, for all sufficiently large $m$, we have
\begin{equation}
(7.51) \quad \tau^{-2k} \int_{B_{\frac{\kappa}{4} \tilde{\xi}^{(m)}}} \rho(\tilde{\xi}^{(m)} \cdot t + \tau) \text{dist}(\cdot, J_{\tilde{\xi}^{(m)}})^2 d\|V_t^{(m)}\| \leq c_{13}(\mu^{(m)})^2.
\end{equation}
Since $\rho(\tilde{\xi}^{(m)} \cdot t + \tau)(x,t) \geq (4\pi \tau^2)^{-\frac{1}{2}} e^{-1}$ for $|x - \tilde{\xi}^{(m)}| = 2\tau$, we have from (7.51)
\begin{equation}
(7.52) \quad \int_{B_{\tau}} \text{dist}(\cdot, J_{\tilde{\xi}^{(m)}})^2 d\|V_t^{(m)}\| \leq c_{13}(4\pi)^{\frac{1}{2}} \tau e^{1+2k} (\mu^{(m)})^2.
\end{equation}
Here, the integration should be over $B_{2\tau}(\tilde{\xi}^{(m)})$, but since $\tilde{\xi}^{(m)} \to 0$ (uniformly in $t$) as $m \to \infty$, we have $B_{\tau} \subset B_{2\tau}(\tilde{\xi}^{(m)})$ for sufficiently large $m$ and we obtain (7.52). We next wish to replace $J_{\tilde{\xi}^{(m)}}$ by $\tilde{J}^{(m)}$ in (7.52). The Hausdorff distance between $J_{\tilde{\xi}^{(m)}}$ and $J_{\mu^{(m)}\tilde{c}_2}$ in $B_1$ may be estimated by $3c_{12}\mu^{(m)}$ for all sufficiently large $m$ due to $|(\mu^{(m)})^{-1}\tilde{\xi}^{(m)} - \tilde{\xi}(t + \tau^2)| \leq o(1)$ and $|\tilde{\xi}(t + \tau^2) - \tilde{\xi}(2)| \leq 2c_{12}$ by Proposition 7.4. The distance between $J_{\mu^{(m)}\tilde{c}_2}$ and $\tilde{J}^{(m)}$ is bounded by $c_{16}\mu^{(m)}$ due to the estimate (7.44) (with $k = 1$ and $l = 0$) for the angle of rotation. Thus we have for any $x \in B_{\tau}$
\begin{equation}
(7.53) \quad \text{dist}(x, \tilde{J}^{(m)})^2 \leq 2 \text{dist}(x, J_{\tilde{\xi}^{(m)}})^2 + 2(\mu^{(m)})^2(3c_{12} + c_{16})^2.
\end{equation}
Now combining (7.52) and (7.53), we obtain
\begin{equation}
(7.54) \quad \lim_{m \to \infty} \sup \frac{1}{(\mu^{(m)})^2} \int_{B_{\tau}} \text{dist}(\cdot, \tilde{J}^{(m)})^2 d\|V_t^{(m)}\| \leq 2c_{13}(4\pi)^{\frac{1}{2}} \tau e^{1+2k} + 6\tau (3c_{12} + c_{16})^2.
\end{equation}
We next estimate the integration over $B_{2\theta} \setminus B_{\tau}$. Fix any $t \in (2 - \theta^2, 2 + \theta^2)$. For all sufficiently large $m$ depending on $\tau$, $\text{spt}\|V_t^{(m)}\| \cap B_{2\theta} \setminus B_{\tau}$ is represented as a union of
graphs using \( \hat{f}_j^{(m)}(\cdot, t) = \mu^{(m)} f_j^{(m)}(\cdot, t) \). Recalling (7.48), we have (denoting \( \frac{\partial}{\partial x} \) by sub-index \( x \) for simplicity)

\[
\frac{1}{(\mu^{(m)})^2} \int_{B_{2\theta} \setminus B_{\theta}} \text{dist}(\cdot, \hat{f}^{(m)})^2 \, d\|V^{(m)}\| \leq \sum_{j=1}^{3} \int_{\frac{\theta}{2}}^{2\theta} |\hat{f}_j^{(m)}(x, t) - \hat{f}_j(0, 2) - (x - \mu^{(m)} u_j)(\hat{f}_j)_x(0, 2)|^2 \sqrt{1 + (\mu^{(m)})^2 (\hat{f}_j^{(m)})_x^2} \, dx
\]

(7.55)

\[
\leq \sum_{j=1}^{3} \int_{\frac{\theta}{2}}^{2\theta} 2 |\hat{f}_j^{(m)}(x, t) - \hat{f}_j(0, 2) - x(\hat{f}_j)_x(0, 2)|^2 \, dx + c(c(\tau), c_16)(\mu^{(m)})^2.
\]

We know already that \( \hat{f}_j^{(m)} \) converges to \( \hat{f}_j \) on \( [\frac{\theta}{2}, 2\theta] \), and

\[
|\hat{f}_j(x, t) - \hat{f}_j(0, 2) - x(\hat{f}_j)_x(0, 2)| \leq c_{16}(|x|^2 + |t - 2|)
\]

by Taylor’s theorem and (7.44). Since \( |x| \leq 2\theta \) and \( |t - 2| \leq \theta^2 \), (7.55) and (7.56) prove

\[
\limsup_{m \to \infty} \frac{1}{(\mu^{(m)})^2} \int_{B_{2\theta} \setminus B_{\theta}} \text{dist}(\cdot, \hat{f}^{(m)})^2 \, d\|V^{(m)}\| \leq 24c_{16}^2 \theta^5.
\]

Since \( \tau \) is arbitrary, combining (7.54) and (7.57) and setting \( c_{17} \geq 48c_{16}^2 \), we obtain the desired estimate (7.49), also by observing \( B_{\theta}(a^{(m)}) \subset B_{2\theta} \) for all sufficiently large \( m \). By the definition of \( J^{(m)} \), (7.50) follows as well with a suitable choice of constant. \( \Box \)

8. Pointwise estimates: Proof of Theorem 2.1

With Proposition 7.13 established, a standard iteration argument establishes the desired estimates as well as the expected geometry of the flow as a regular triple junction moving by curvature. For completeness, we present the detailed argument.

**Proposition 8.1.** Corresponding to \( p, q, \nu, E_1 \), there exist \( \varepsilon_7 \in (0, 1), \theta_* \in (0, \frac{1}{4}) \) and \( c_{18} \in (1, \infty) \) such that the following holds: For \( \theta \in (0, \infty) \) and \( U = B_{4\theta} \), suppose \( \{V_t\}_{t \in [-2R^2, 2R^2]} \) and \( \{u(\cdot, t)\}_{t \in [-2R^2, 2R^2]} \) satisfy (A1)-(A4). Assume

\[
\mu = \left(R^{-5} \int_{-2R^2}^{2R^2} \int_{B_{4\theta}} \text{dist}(\cdot, J)^2 \, d\|V_t\| \, dt \right)^{\frac{1}{2}} < \varepsilon_7,
\]

(8.1)

\[
\exists j_1, j_2 \in \{1, 2, 3\}: R^{-1}\|V_{-2R^2}\|(|\phi_{j_1, J|R}| \leq \frac{3 - \nu}{2} c_1, \ R^{-1}\|V_{2R^2}\|(|\phi_{j_2, J|R}| \geq \frac{1 + \nu}{2} c_1,
\]

(8.2)

and denote

\[
\|u\| = R^{\frac{1}{2}} \left( \int_{-2R^2}^{2R^2} \left( \int_{B_{4\theta}} |u|^p \, d\|V_t\| \right)^{\frac{2}{p}} \right)^{\frac{1}{2}}.
\]

(8.3)

Then there exists \( J' = R_0(J) + \xi \in J \) such that

\[
d_R(J', J) \leq c_{18} \mu \quad \text{and}
\]

(8.4)

\[
\left(\theta_* R\right)^{-5} \int_{-2(\theta_* R)^2}^{2(\theta_* R)^2} \int_{B_{4\theta_* R}^{(\xi)}} \text{dist}(\cdot, J')^2 \, d\|V_t\| \, dt \leq \theta_*^2 \max\{\mu, c_{18}\|u\|\}.
\]

(8.5)
Moreover, if we additionally assume that \( \|u\| \leq \varepsilon_7 \), then we have
\[
(\theta_* R)^{-1}\|V_{2(\theta_* R)^2}(\phi_{j,\theta_* R})\| \leq \frac{3 - \nu}{2} c_1, \quad (\theta_* R)^{-1}\|V_{2(\theta_* R)^2}(\phi_{j,\theta_* R})\| \geq \frac{1 + \nu}{2} c_1, \quad j = 1, 2, 3.
\]

Proof. We may assume that \( R = 1 \) after a parabolic change of variables. We prove the claim by contradiction. For all \( m \in \mathbb{N} \), consider a set of sequences \( \{V_{(m)}(\cdot, t)\}_{1 \leq t \leq 2} \) satisfying (A1)-(A4) with \( U = B_4 \) such that (8.1) and (8.2) are satisfied with \( V_{1(m)} \), that is, for all \( m \in \mathbb{N} \),
\[
(\theta_* R)^{-1}\|V_{2(\theta_* R)^2}(\phi_{j,\theta_* R})\| \leq \frac{3 - \nu}{2} c_1, \quad (\theta_* R)^{-1}\|V_{2(\theta_* R)^2}(\phi_{j,\theta_* R})\| \geq \frac{1 + \nu}{2} c_1, \quad j = 1, 2, 3.
\]
Define \( \|u_{(m)}\| \) as in (8.3) with \( R = 1 \) and \( u_{(m)} \) in place of \( u \). The negation then implies that for any \( J' = \mathbb{R}_\theta(J) + \xi \in J \) with
\[
d(J', J) \leq m\mu_{(m)},
\]
we have
\[
(\theta_* R)^{-5}\|\{\theta_* R\} \int_{-2}^2 \int_{B_{\theta_* R}(\xi)} \text{dist} (\cdot, J')^2 d\|V_{1(m)}\| dt \|^{\frac{1}{2}} > \theta_*^\kappa \max\{\mu_{(m)}, m\|u_{(m)}\|\}.
\]
Here \( \theta_* \in (0, \frac{1}{4}) \) will be chosen depending only on \( p, q, E_1 \).

We next proceed to use the argument in the previous section. First, use \( J' = J \) in (8.10). Then we have
\[
\theta_*^\kappa \max\{\mu_{(m)}, m\|u_{(m)}\|\} < (\theta_* R)^{-5}\|\{\theta_* R\} \int_{-2}^2 \int_{B_{\theta_* R}(\xi)} \text{dist} (\cdot, J')^2 d\|V_{1(m)}\| dt \|^{\frac{1}{2}} \leq \theta_*^{\frac{1}{2}} \mu_{(m)}.
\]
Thus, (8.11) shows (7.3) is satisfied. We also have \( \lim_{m \to \infty} \mu_{(m)} = 0 \) by (8.7). We shift \( t \) by \(-2 \) so that the time interval will be \([0, 4]\) from \([-2, 2]\). We have (7.1), (7.2) and (6.3) satisfied thus all the assumptions in the previous section are satisfied. In the argument, we may fix \( \kappa = \frac{1}{2} \) from the beginning. The conclusion of Proposition 7.13 shows that for all \( m > c_{17}, \bar{J}^{(m)} \) satisfies (8.9) due to (7.50) while we have (7.49). On the other hand, (8.10) shows
\[
\theta_*^{2\kappa} \leq \lim_{m \to \infty} \sup_{\theta_* R} \frac{1}{(\mu_{(m)})^2} \theta_* R^5 \|\{\theta_* R\} \int_{-2}^2 \int_{B_{\theta_* R}(\xi)} \text{dist} (\cdot, \bar{J}^{(m)})^2 d\|V_{1(m)}\| dt \|.
\]
If we let \( \theta_* = 4\theta_* \) in (7.49) and compare (8.12), we obtain
\[
\theta_*^{2\kappa} \leq 4^7 c_{17} \theta_*^{2\kappa}.
\]
Note that \( c_{17} \) and \( \zeta \) depend only on \( p, q, E_1 \). Since \( \zeta \in (0, 1) \), we obtain a contradiction for suitably small \( \theta_* \) depending only on \( c_{17}, \zeta \) and thus ultimately only on \( p, q, E_1 \). Once \( \theta_* \) is fixed, then we may use Proposition 5.3 with \( \tau = \frac{\theta_*}{2} \). For suitably small \( \varepsilon_7 \), we can make sure that \( \text{spt}\|V_{1}\| \) on \( B_{\theta_*}(\xi) \setminus B_{\frac{\theta_*}{2}}(\xi) \) is close to \( J \) (and thus to \( J' \)) in \( C^1\zeta \) and (8.6) can be guaranteed. \( \square \)
Proposition 8.2. Corresponding to $\nu$, $E_1,p,q$ there exist $\varepsilon_8 \in (0,1)$ and $c_{19} \in (1,\infty)$ with the following property. Under the assumptions of Proposition 8.1, where $\varepsilon_7$ is replaced by $\varepsilon_8$ and with $\|u\| \leq \varepsilon_8$,

(1) there exists $J_0 \in \mathcal{J}$ with the junction point at $\dot{a}$ such that

\[ d_R(J_0, J) \leq c_{19} \max\{\mu, c_{18}\|u\|\}, \]

(2) for $0 < s < R$, there exists $J_s \in \mathcal{J}$ with the junction point at $a_s$ such that

\[ d_s(J_s, J_0) + \left( s^{-5} \int_{-2s^2}^{2s^2} \int_{B_{4s}(a_s)} \text{dist} (\cdot, J_s)^2 d\|V\| dt \right)^{\frac{1}{2}} \leq c_{19} (s/R)^{\zeta} \max\{\mu, c_{18}\|u\|\}, \]

(3) $\text{spt}\|V_0\| \cap B_{2R}$ consists of three curves meeting at $\dot{a}$ with 120 degrees.

Proof. We may assume $R = 1$ after a change of variables. We choose $\varepsilon_8 \in (0,1)$ and $c_{19} \in (1,\infty)$ so that

\[ c_{18}\varepsilon_8 < \varepsilon_7, \]

\[ (1 + \theta_*^{-1})c_{18} \sum_{j=0}^{\infty} \theta_*^{(j-1}\zeta) \leq c_{19}, \]

\[ 2\theta_*^{\frac{2}{2} - \zeta} \leq c_{19}. \]

Set $J^{(0)} = J$ and we inductively prove that for $m = 1, 2, \cdots$, we have $J^{(m)} = R_{e(m)}(J) + a^{(m)} \in \mathcal{J}$ such that

\[ d_{\theta_*^{m}}(J^{(m)}, J^{(m-1)}) \leq c_{18} \theta_*^{(m-1)\zeta} \max\{\mu, c_{18}\|u\|\}, \]

\[ \mu^{(m)} = \left( \theta_*^{-5m} \int_{-2\theta_*^{2m}}^{2\theta_*^{2m}} \int_{B_{4\theta_*^{2m}}(a^{(m)})} \text{dist} (\cdot, J^{(m)})^2 d\|V\| dt \right)^{\frac{1}{2}} \leq \theta_*^{m\zeta} \max\{\mu, c_{18}\|u\|\}. \]

\[ \theta_*^{-m}\|V - 2\theta_*^{2m}\| (\phi_{j, J^{(m)}, \theta_*^{m}}) \leq \frac{3 - \nu}{2} c_1, \quad \theta_*^{-m}\|V - 2\theta_*^{2m}\| (\phi_{j, J^{(m)}, \theta_*^{m}}) \geq \frac{1 + \nu}{2} c_1, \quad j = 1, 2, 3. \]

Since $\varepsilon_8 < \varepsilon_7$ by (8.16), Proposition 8.1 gives the proof for the existence of $J^{(1)} := J'$ satisfying (8.19)-(8.21) for $m = 1$ case. Under the inductive assumption up to $m$, we have

\[ \mu^{(m)} \leq \theta_*^{m\zeta} \max\{\mu, c_{18}\|u\|\} < \varepsilon_7 \]

by (8.1), $\|u\| \leq \varepsilon_8$ and (8.16). Then with (8.21) and $J$ replaced by $J^{(m)}$ and $R = \theta_*^{m}$, we have the assumptions (8.1) and (8.2) satisfied. Hence we have a $J^{(m+1)} \in \mathcal{J}$ such that

\[ d_{\theta_*^{m}}(J^{(m+1)}, J^{(m)}) \leq c_{18} \mu^{(m)} \leq c_{18} \theta_*^{m\zeta} \max\{\mu, c_{18}\|u\|\} \]

by (8.4) and (8.20) and

\[ \mu^{(m+1)} \leq \theta_* \max\{\mu^{(m)}, c_{18} \theta_*^{m\zeta}\|u\|\} \leq \theta_*^{(m+1)\zeta} \max\{\mu, c_{18}\|u\|\} \]

by (8.5) and (8.20), (8.21) for $m + 1$ is also satisfied due to (8.6). Thus (8.19)-(8.21) for $m + 1$ in place of $m$ are verified. We next check that $J^{(m)}$ converges. By (8.19) and (8.17), $\{\tau^{(m)}\}$ is a Cauchy sequence and

\[ |\tau^{(m)}| \leq \sum_{j=1}^{m} d_{\theta_*^{j-1}}(J^{(j)}, J^{(j-1)}) \leq \frac{c_{19}}{2} \max\{\mu, c_{18}\|u\|\}. \]
and \( \tau^{(m)} \) converges to some \( \hat{\tau} \). We also have

\[
|a^{(m)}| \leq \sum_{j=1}^{m} |a^{(j)} - a^{(j-1)}| \leq \sum_{j=1}^{m} \theta_s^{-1} d_{\theta_s^{-1}}(J^{(j)}, J^{(j-1)}) \leq \frac{C_1\max\{\mu, c_{18}\|u\|}{2}
\]

and \( a^{(m)} \) converges to some \( \hat{a} \). Setting \( J_0 := R_+ (J) + \hat{a} \), we prove (8.14) by (8.25) and (8.26). Next, for \( \theta_s^{n+1} \leq s < \theta_s^n \), let \( J_s := J^{(m)} \). Then,

\[
d_s(J_s, J_0) = \sum_{j=m+1}^{\infty} (s^{-1} |a^{(j)} - a^{(j-1)}| + |\tau^{(j)} - \tau^{(j-1)}|) \leq \sum_{j=m+1}^{\infty} (s^{-1} \theta_s^{-1} + 1) d_{\theta_s^{-1}}(J^{(j)}, J^{(j-1)})
\]

\[
\leq \sum_{j=m+1}^{\infty} (\theta_s^{-1} + 1) c_{18} \theta_s^{(j-1)\xi} \max\{\mu, c_{18}\|u\|\} \leq \frac{s^\xi}{2} c_{19} \max\{\mu, c_{18}\|u\|\}
\]

by (8.19) and (8.17),

\[
(8.28) \quad \left( s^{-5} \int_{-2s^2}^{2s^2} \int_{B_2(a^{(m)})} \text{dist}(\cdot, J_s)^2 d\|V_t\| dt \right)^{\frac{1}{2}} \leq \theta_s^{-\frac{1}{2}} \mu^{(m)} \leq \frac{s^\xi}{2} c_{19} \max\{\mu, c_{18}\|u\|\}
\]

by (8.20) and (8.18). Thus (8.27) and (8.28) prove (8.15). Now we see that the junction point \( a_s \) of \( J_s \) satisfies \( |\hat{a} - a_s| = O(s^{1+\xi}) \), and the decay of (8.28) via the graphical representation shows that the dilation of \( \text{spt} \|V_0\| \) centered at \( \hat{a} \) consists of three \( C^{1,\xi} \) curves approaching to \( J_0 \). Also there cannot be any other junction point, thus the claim of (3) follows.

**Proof of Theorem 2.7.** We may assume that \( R = 1 \). For each \( s \in [-1, 1] \), we use Proposition 8.2 with \( R = 1/2 \) and the time variable shifted by \( s \). For this purpose, choose \( \varepsilon_1 \) so that \( 2\pi \varepsilon_1 < \varepsilon_8 \). Then for all \( s \in [-1, 1] \), we have

\[
(8.29) \quad \left( (1/2)^{-5} \int_{s-1/2}^{s+1/2} \int_{B_2} \text{dist}(\cdot, J)^2 d\|V_t\| dt \right)^{\frac{1}{4}} < \varepsilon_8
\]

by (2.17). Since \( \varepsilon_1 < \varepsilon_8 \), we also have

\[
(8.30) \quad (1/2)^\xi \left( \int_{s-1/2}^{s+1/2} \left( \int_{B_2} |u|^n d\|V_t\| \right)^{\frac{2}{n}} dt \right)^{\frac{1}{n}} < \varepsilon_8
\]

by (2.19). By Proposition 3.1 and restricting \( \varepsilon_1 \) appropriately, we may guarantee that

\[
(8.31) \quad 2\|V_{s-1/2}\|(\phi_{J,j,1/2}) \leq \frac{3 - \nu}{2} c_1, \quad 2\|V_{s+1/2}\|(\phi_{J,j,1/2}) \geq \frac{1 + \nu}{2} c_1, \quad j = 1, 2, 3
\]

for all \( s \in [-1, 1] \). Since the assumptions of Proposition 8.2 are satisfied by (8.29)-(8.31), there exist \( J_0(s) \in \mathcal{J} \) with the junction point at \( \hat{a}(s) \) and satisfying (8.14), and for each \( \lambda \in (0, 1/2) \), \( J_s(s) \in \mathcal{J} \) with the junction point at \( \hat{a}_s(s) \) satisfying (8.15). The point \( \hat{a}(s) \) is the unique junction point of \( \text{spt} \|V_0\| \) in \( B_1 \). We next prove \( \hat{a} \in C^{1,\xi} \) and the corresponding norm estimate. Since (8.14) gives \( \sup_{s \in [-1, 1]} |\hat{a}(s)| \) bound (with \( 2\pi c_{19} \) in place of \( c_{19} \)), we only need to check the \( 1+\xi \)-Hölder norm bound of \( \hat{a} \). For any \(-1 \leq s_1 < s_2 \leq 1\), set
\[ \lambda := \min \left\{ \frac{1}{4}, \sqrt{s_2 - s_1} \right\}. \]

Then by (8.15), we have
\begin{equation}
(8.32) \quad d_\lambda(J_\lambda(s_1), J_0(s_1)) + \left( \lambda^{-5} \int_{s_1 - 2\lambda^2}^{s_1 + 2\lambda^2} \text{dist}(\cdot, J_\lambda(s_1))^2 \, d\|V_t\| \, dt \right)^{\frac{1}{2}} \leq 2\varepsilon_c \lambda^\kappa \max \{\mu, c_{18}\|u\|\}.
\end{equation}

Since \( [s_2 - \frac{1}{2}\lambda^2, s_2 + \frac{1}{2}\lambda^2] \subset [s_1 - 2\lambda^2, s_1 + 2\lambda^2] \), by restricting \( \varepsilon \) so that \( 2\varepsilon_c \lambda^\kappa \max \{\mu, c_{18}\|u\|\} < \varepsilon \) holds, we may use Proposition 8.2 again with \( R = \frac{\lambda}{2} \) and \( J \) replaced by \( J_\lambda(s_1) \). Since we know that \( J_0(s_2) \) is the unique triple junction of spt\( \|V_s\| \), (8.14) gives
\begin{equation}
(8.33) \quad d_\lambda(J_0(s_2), J_\lambda(s_1)) \leq c_{18} \lambda^\kappa \max \{\mu, c_{18}\|u\|\} \cdot C_{18}(\frac{\lambda}{2})^\kappa \|u\|.
\end{equation}

Now (8.32) and (8.33) give the desired \( C^{1,\kappa} \) estimate for \( \tilde{a} \) with an appropriate choice of \( c_2 \). From the argument up to this point, it is clear that we have (2.21) and (2.22). The estimates (8.32) and (8.33) also prove
\begin{equation}
(8.34) \quad \left\| \frac{\partial f_j}{\partial x}(\lambda(s), \cdot) \right\|_{C^0([-1,1])} \leq c_2 \max \{\mu, \|u\|\}.
\end{equation}

With a similar argument, one can prove (using \( \lambda = x - l_j(s) \)) and Proposition 8.2
\begin{equation}
(8.35) \quad \left| \frac{\partial f_j}{\partial x}(x, s) - \frac{\partial f_j}{\partial x}(l_j(s), s) \right| \leq c_2 (x - l_j(s))^\kappa \max \{\mu, \|u\|\}.
\end{equation}

To obtain \( C^{1,\kappa} \) estimate for \( f_j \), fix \(-1 \leq s_1 < s_2 \leq 1 \) and \( l_j(s_i) \leq x_i \leq 1 \) \((i = 1, 2)\) and set \( \lambda := \min \{1/4, \max\{|x_1 - x_2|, \sqrt{s_2 - s_1}\}\} \). If \( \max\{|x_1 - l_j(s_1), x_2 - l_j(s_2)| > \lambda \), one can check that
\begin{equation}
(8.36) \quad \left| \frac{\partial f_j}{\partial x}(x_1, s_1) - \frac{\partial f_j}{\partial x}(x_2, s_2) \right| \leq c_2 \lambda^\kappa \max \{\mu, \|u\|\}
\end{equation}
by using (8.34), (8.35) and the triangle inequalities. The similar estimate for \( f_j \) (with \( \lambda^{1+\kappa} \) in place of \( \lambda^\kappa \) on the right-hand side) can be obtained. If \( \max\{|x_1 - l_j(s_1), x_2 - l_j(s_2)| > \lambda \), and assuming \( x_1 - l_j(s_1) > \lambda \) without loss of generality, we may use Proposition 8.2 with \( \lambda = x_1 - l_j(s_1) \) and Proposition 3.1 to obtain a \( C^{1,\kappa} \) estimate. With an appropriate choice of \( c_2 \), we may finish the proof of (2.23). \( \square \)

9. Partial regularity: Proof of Theorem 2.2

In this section, we combine Theorem 2.1 with a stratification theorem of singular sets. The general idea of stratification using tangent cone goes back to Federer [10] and it has been adapted in a number of variational problems. Here we use a non-trivial adaptation to Brakke flows due to White [35]. For the proof of Theorem 2.2 we first recall some definitions and results from [16, 35].

(a) Existence of tangent flow. For any fixed \((y, s) \in U \times (0, \Lambda)\) and \( \lambda > 0 \), define
\begin{equation}
(9.1) \quad V_t^{(y, s), \lambda}(\phi) = \lambda^{-1} \int_{G_1(\lambda^{-1}(U-y))} \phi(y + \lambda x, S) \, dS \, dV_s \, dV_t(x, S)
\end{equation}
for \( \phi \in C_c(G_1(\lambda^{-1}(U-y))) \) and \( t \in (-\lambda^{-2}s, \lambda^{-2}(\Lambda - s)) \). \( V_t^{(y, s), \lambda} \) is a parabolically rescaled flow at \((y, s)\). For any positive sequence \( \{\lambda_j\}_{j \in \mathbb{N}} \) converging to 0, there exist a subsequence \( \{\lambda_j\}_{j \in \mathbb{N}} \) and a family of varifolds \( \hat{V}_t \in \hat{IV}_1(\mathbb{R}^2) \) for a.e. \( t \in \mathbb{R} \), \( \hat{V}_t \) satisfies (2.16) with \( u = 0 \) on \( \mathbb{R}^2 \times \mathbb{R} \), \( \hat{V}_t = V_t^{(y, s), \lambda} \) for all \( t < 0 \) and \( \lambda, \mu > 0 \), and \( \lim_{t \to \infty} \|V_t^{(y, s), \lambda_j} - \|\hat{V}_t\| \) for all \( t \in \mathbb{R} \). The proof of the existence of such flow
is in [16, Lem. 8] and also see [35, Sec. 7]. It is for \( u = 0 \), but the proof goes through even with non-zero \( u \) since Huisken’s monotonicity formula [13] holds with a minor error term due to \( u \) (see [18, Sec. 6] for the detail) and it vanishes as \( \lambda \to 0 \). We call \( \{\tilde{V}_t\}_{t \in \mathbb{R}} \) a tangent flow at \((y, s)\). Note that \( \{\tilde{V}_t\}_{t \in \mathbb{R}} \) inherits the property of (A2) with the same constant \( E_1 \).

(b) backwards-cone-like functions. For any tangent flow \( \{\tilde{V}_t\}_{t \in \mathbb{R}} \) at \((y, s) \in U \times (0, \Lambda) \) and for \((x, t) \in \mathbb{R}^2 \times \mathbb{R} \), define

\[
g(x, t) = \lim_{\tau \to 0^+} \int_{\mathbb{R}^2} \rho_{(x,t)}(x', t - \tau) \, d\|\tilde{V}_{t-\tau}\|(x').
\]

By Huisken’s monotonicity formula, the limit in (9.2) always exists. The set of all such \( g \) obtained from a tangent flow at \((y, s)\) is denoted by \( \mathcal{G}(y, s) \). The function \( g \) has the following property which is called backwards-cone-like [35, Sec. 8]:

\[
g(x, t) \leq g(0, 0) \quad \forall (x, t) \in \mathbb{R}^2 \times \mathbb{R},
\]

\[
g(x, t) = g(0, 0) \implies g(x + x', t + t') = g(x + \lambda x', t + \lambda^2 t') \quad \forall t', \forall x' \in \mathbb{R}^2, \forall \lambda > 0.
\]

Define

\[
\mathcal{V}(g) = \{ x \in \mathbb{R}^2 : g(x, 0) = g(0, 0) \}, \quad \mathcal{S}(g) = \{ (x, t) \in \mathbb{R}^2 \times \mathbb{R} : g(x, t) = g(0, 0) \}.
\]

We note that \( \mathcal{V}(g) \) and \( \mathcal{S}(g) \) are denoted by \( V(g) \) and \( S(g) \) in [35], respectively, but we changed the notation here to avoid possible confusion. Then [35, Th. 8.1] proves that

\[
g(x, t) = g(x + x', t) \quad \forall x \in \mathbb{R}^2, \forall x' \in \mathcal{V}(g), \forall t \leq 0,
\]

\( \mathcal{V}(g) \) is a vector subspace of \( \mathbb{R}^2 \), and \( \mathcal{S}(g) \) is either \( \mathcal{V}(g) \times \{0\} \) or \( \mathcal{V}(g) \times (-\infty, a] \) for some \( a \in [0, \infty] \). In the latter case, \( g \) is time-independent up to time \( t = a \): that is, \( g(x, t) = g(x, t') \) for all \( t \leq t' < a \) and \( x \in \mathbb{R}^2 \). Depending on whether \( \mathcal{S}(g) \) is equal to \( \mathcal{V}(g) \times \mathbb{R} \), \( \mathcal{V}(g) \times (-\infty, a] \) for some \( a \in [0, \infty] \) or \( \mathcal{V}(g) \times \{0\} \), \( g \) is called static, quasi-static or shrinking, respectively. \( \mathcal{V}(g) \) is called the spatial spine of \( g \). In the present situation of \( \mathbb{R}^2 \), dimension of \( \mathcal{V}(g) \) denoted by \( \dim \mathcal{V}(g) \) can be either 2, 1 or 0.

(c) Stratification. Define for \( g \in \mathcal{G}(y, s) \)

\[
\mathcal{D}(g) = \begin{cases} 
2 + \dim \mathcal{V}(g) & \text{if } g \text{ is static}, \\
\dim \mathcal{V}(g) & \text{if } g \text{ is quasi-static or shrinking},
\end{cases}
\]

and define for \( k \in \{0\} \cup \mathbb{N} \)

\[
\Sigma_k = \{(y, s) \in U \times (0, \Lambda) : \mathcal{D}(g) \leq k \quad \forall g \in \mathcal{G}(y, s)\}.
\]

Then [35, Th. 8.2] proves that \( \dim \Sigma_k \leq k \) and \( \Sigma_0 \) is a discrete set. Here, \( \dim \) is the Hausdorff dimension with respect to the parabolic metric.

**Proof of Theorem 2.2** Define \( \Sigma_1 \) as in (9.8) whose parabolic Hausdorff dimension is at most 1. Let \((y, s) \in U \times (0, \Lambda) \setminus \Sigma_1 \). By the definition of (9.8), there exists \( g \in \mathcal{G}(y, s) \) with \( \mathcal{D}(g) \geq 2 \). In the following, fix such \( g \).

Claim 1. There exists a constant \( g_0 > 0 \) depending only on \( E_1 \) such that either \( g(x, t) \geq g_0 \) or \( g(x, t) = 0 \). In the latter case, there exists a space-time neighborhood \( U_{x,t} \) of \((x, t)\) with \( U_{x,t} \cap \{t' \in \mathbb{R} : \|\tilde{V}_{t'}\| = 0\} = \emptyset \).

**Proof of claim 1.** This is a well-known fact but we include the proof for the convenience of the reader. By Brakke’s clearing out lemma [3] or [18, Cor. 6.3], there exist constants \( \tilde{g}_0 > 0 \) and \( L > 1 \) depending on \( E_1 \) such that for any \( \tau > 0 \), \( \|\tilde{V}_{t-\tau}\|(B_{\sqrt{\tau}L}(x)) < \tilde{g}_0 \sqrt{\tau} \) implies
\[\|V_t\|(B_{\sqrt{\tau}}(x)) = 0 \text{ for all } t' \in [t - \tau, t + \tau].\] Assume \(g(x, t) > 0\). By the monotone property of (9.2), for sufficiently small \(\tau\), we have
\[2g(x, t) > \int_{B_{\sqrt{\tau}}(x)} \rho(x, t)(x', t - 2\tau) \, d\|V_t\|((x') \geq e^{-\frac{t^2}{8\pi \tau}} \|V_{t-2\tau}\|(B_{\sqrt{\tau}L}(x)).\]

Thus, for sufficiently small \(g_0\) depending on \(g_0\) and \(L\), if \(g(x, t) < g_0\), we have \(\|V_t\|(B_{\sqrt{\tau}}(x)) = 0\) as above and this implies \(g(x, t) = 0\), leading to a contradiction. Thus we have \(g(x, t) \geq g_0\) if \(g(x, t) > 0\). If \(g(x, t) = 0\), then the same argument shows the last statement, concluding the proof of claim 1.

Claim 2. \(\dim \mathcal{V}(g) = 2\) implies that there exists a space-time neighborhood \(U_{y,s} \subset U \times (0, \Lambda)\) such that \(U_{y,s} \cap \cup_{t}(\text{spt } V_t \times \{t\}) = \emptyset\).

Proof of claim 2. By (9.3), \(g(\cdot, 0)\) is a constant function on \(\mathbb{R}^2\). Suppose \(g(0, 0) \geq g_0\). By the monotone property of (9.2) and using (A2), we have for any \(x \in \mathbb{R}^2\), \(t > 0\) and \(R > 0\)
\[g_0 \leq g(x, 0) \leq \int_{B_{\sqrt{\tau}R}(x)} \rho(x, 0)(x', -\tau) \, d\|V_t\|((x') + E_1(1)(9.9)\]
where \(o(1)\) here means \(\lim_{R \to \infty} o(1) = 0\). Hence, fixing large \(R\) depending only on \(E_1\) so the last term is less than \(\frac{2\pi}{\tau}\), and then set \(\delta = \frac{\pi}{2\sqrt{4\pi}}\). With this choice and (9.9) show
\[\delta\sqrt{\tau} \leq \|V_t\|(B_{\sqrt{\tau}R}(x))\]
for all \(\tau > 0\) and \(x \in \mathbb{R}^2\). But then, since \(B_R\) may contain \(O(\tau^{-1})\) number of disjoint balls of radius \(\sqrt{\tau}R\), we may prove that \(\|V_t\|(B_R) \geq C\delta\sqrt{\tau}\) which goes to infinity as \(\tau \to 0\). This is a contradiction to (A2). Thus \(g(0, 0) = 0\), and by (9.3), \(g\) is identically 0 on \(\mathbb{R}^2 \times \mathbb{R}\). Then claim 1 shows \(\|V_t\| = 0\) for all \(t \in \mathbb{R}\). Let \(\lambda_j\) be a sequence such that \(\|V_t^{(g,s),\lambda_j}\| \to \|\tilde{V_t}\|(= 0)\) as was described in (a). Then one has \(\lim_{j \to \infty} \|V_t^{(g,s),\lambda_j}\|(B_{\tau R}) = 0\) for \(t' \in [-1, 1]\), \(L\) as in claim 1. Then by [18 Cor.6.3], for sufficiently large \(j\), we have \(\|V_t^{(g,s),\lambda_j}\|(B_{\tau 1}) = 0\) for all \(t' \in [-1, 1]\). This shows that there exists a space-time neighborhood of \((y, s)\) on which \(\|V_t\|\) has measure zero, completing the proof of claim 2.

Claim 3. \(\dim \mathcal{V}(g) = 1\) implies that there exists a space-time neighborhood \(U_{y,s} \subset U \times (0, \Lambda)\) such that \(U_{y,s} \cap \cup_{t}(\text{spt } V_t \times \{t\})\) is represented as a \(C^{1,\lambda}\) graph.

Proof of claim 3. Since \(D(g) \geq 2\), (9.7) shows that \(g\) is static, i.e., \(\mathcal{S}(g) = \mathcal{V}(g) \times \mathbb{R}\). We have \(g(0, 0) \geq g_0\), or else, \(g\) is identically 0 and \(\dim \mathcal{V}(g) = 2\). As described in (b), \(g\) is independent of \(t\), and by (6.9), invariant in \(\mathcal{V}(g)\) direction. Thus, if \(g(x, t)(= g(x, 0)) > 0\) for some \(x \in \mathbb{R}^2 \setminus \mathcal{V}(g)\), then \(g(\lambda x + x', t) = g(x, t)\) for all \(x' \in \mathcal{V}(g)\) and \(\lambda > 0\), letting \(g\) have a positive constant on the half-space of \(\mathbb{R}^2\) with the boundary \(\mathcal{V}(g)\). This leads to a contradiction by the similar argument in the proof of claim 2. Thus we have \(g = 0\) outside of \(\mathcal{S}(g)\) and positive constant on \(\mathcal{S}(g)\). Similarly, we may also prove that \(\|V_t\|(\mathbb{R}^2 \setminus \mathcal{V}(g)) = 0\) for all \(t \in \mathbb{R}\). For a.e. \(t \in \mathbb{R}\), we have \(\tilde{V_t} \in L^1_{\text{loc}}(\mathbb{R}^2)\), thus \(\tilde{V_t} = \theta(t, x)\mathcal{V}(g)\) for some \(H^1\) a.e. integer-valued function \(\theta(\cdot, t)\). Since \(h(\tilde{V_t}, \cdot) \in L^2_{\text{loc}}\) for a.e. \(t\), going back to the definition of the first variation, one can check that \(\theta(\cdot, t)\) has to be a constant function. Since \(g\) is constant on \(\mathcal{S}(g)\), one easily sees that \(\theta\) is independent of \(t\). Thus with some integer \(\theta_0\), \(\tilde{V_t} = \theta_0\|\mathcal{V}(g)\|\) for all \(t\). Now by (A5), we necessarily have \(\theta_0 = 1\). Let \(V_t^{(g,s),\lambda_j}\) be a sequence converging to \(\tilde{V_t}\). Then by [18 Th. 8.7 or Prop. 9.1], for sufficiently large \(j\), we may conclude that \(\cup_{t}(\text{spt } V_t^{(g,s),\lambda_j} \times \{t\})\) is represented as a \(C^{1,\lambda}\) graph in \(B_1 \times (-1, 1)\). This concludes the proof of claim 3.

Claim 4. \(\dim \mathcal{V}(g) = 0\) implies that there exists a space-time neighborhood \(U_{y,s} \subset U \times (0, \Lambda)\) such that \(U_{y,s} \cap \cup_{t}(\text{spt } V_t \times \{t\})\) is represented as a \(C^{1,\lambda}\) triple junction as in Theorem 2.1.
Proof of claim 4. Since $D(g) \geq 2$, by (9.7), $g$ is static. $g$ is independent of $t$ and $g(x, t) = g(x, 0)$ for all $x \in \mathbb{R}^2$ and $\lambda > 0$. We shall write $g(x)$ instead of $g(x, t)$. Define $W = \{x \in \mathbb{R}^2 : |x| = 1, g(x) \geq g_0\}$. Then following the similar argument as in the proof of claim 2, we may prove that the number of element of $W$ is finite. More precisely, if we pick $W' \subset W$ consisted of $N$ elements, we may choose $O(N/\sqrt{r})$ number of disjoint balls of radius $\sqrt{r}R$ centered at $\cup_{\lambda > 0} \lambda W'$ inside of $B_R$, each satisfying (9.11). Then we would have $\| \tilde{V}_{-r}(B_R) \| \geq O(N)$, thus $N$ cannot go to $\infty$. Thus $\{g \geq g_0\}$ consists of a finite number of half rays denoted by $\{l_j \subset \mathbb{R}^2 : j = 1, \ldots, N\}$ emanating from $0$, and $g$ is constant on each half ray. As in the proof of claim 3, one can argue that there exist some positive integers $\theta_j$ such that $\tilde{V}_t = \sum_{j=1}^N \theta_j |l_j|$. Then again using (A5) which says $\Theta(\|V_s\|, y) < 2$ and arguing as before, we have $N \leq 3$ and $\sum_{j=1}^N \theta_j \leq 3$. The conditions $h(\tilde{V}_t, \cdot) \in L^2_{loc}$ and $\dim V(g) = 0$ limit the possibility to $\tilde{V}_t = |R_\theta(J)|$ with some $\theta \in [0, 2\pi)$. We are now ready to apply Theorem 2.1 to $V_{(y,s),\lambda}$. Note that (after a rotation by $\theta$) that (2.17)-(2.19) are satisfied for all sufficiently large $j$. Thus this concludes the proof of claim 4. Since $U_{y,s}$ in all three cases do not intersect with $\Sigma_1$, $\Sigma_1$ is a closed set and this ends the proof of Theorem 2.2.

10. The top dimensional part of the genuine singular set

Under the hypotheses (A1)-(A5) of Theorem 2.2 let $\Sigma_1$ and $\Sigma_0$ be defined as in (9.8). We know that $\Sigma_1$ is closed (by Theorem 2.2), and that $\dim \Sigma_1 \leq 1$ where dim is the parabolic Hausdorff dimension. Moreover, $\Sigma_0$ is discrete by [35] Th. 8.2.

We may further characterize the “top dimensional part” of the singular set, i.e. $\Sigma_1 \setminus \Sigma_0$, in terms of tangent flows as follows:

Theorem 10.1. For any $(y, s) \in \Sigma_1 \setminus \Sigma_0$, there exists a quasi-static tangent flow $\{\tilde{V}_t\}_{t \in \mathbb{R}}$ such that, $\text{spt} |\tilde{V}_t| = 0$ or $\text{spt} |\tilde{V}_t| = S$ for some $S \in G(2,1)$. Moreover, there exists a set of integers $\theta_1 > \cdots > \theta_N \geq 0 \ (N \geq 2)$ and real numbers $0 \leq a_1 < \cdots < a_{N-1}$ such that $\tilde{V}_t = \theta_j |S|$ for $t \in (a_{j-1}, a_j)$, $j = 1, \ldots, N$, where $a_0 = -\infty$ and $a_N = \infty$. If $\theta_1 = 1$, then $a_1 = 0$ and $\theta_2 = 0$.

A heuristic meaning of the above is that $\Sigma_1 \setminus \Sigma_0$ is the set of points where some curve instantaneously disappears.

Proof. Take any $(y, s) \in \Sigma_1 \setminus \Sigma_0$. Then there exists $g \in G(y, s)$ such that $D(g) = 1$. By (9.7), it has to be quasi-static or shrinking, and $\dim V(g) = 1$. By (9.6), $g$ is invariant in $V(g)$ direction while having the backwards-cone-like property (9.4) with respect to $(0,0)$. The set $\{x \in \mathbb{R}^2 : g(x, -1) > 0\}$ then consists of a finite number of lines parallel to $V(g)$ and by the argument in the proof of Theorem 2.2, one can prove that $\tilde{V}_{-1}$ is a sum of varifolds with integer multiples supported on such lines. Due to the backwards-cone-like property and also the fact that $\tilde{V}_t$ is a curvature flow, one can prove that $\tilde{V}_t = \theta_j |V(g)|$ for $t < 0$ (otherwise it has to move at non-zero speed even if it is a line). This shows that $g$ has to be quasi-static. By [35], we know that $S(g) = V(g) \times (\infty, a]$ for some $a \in [0, \infty)$. This in particular shows $\tilde{V}_t = \theta_j |V(g)|$ for $t \in (\infty, a)$. One can then prove, for example using the clearing-out lemma [35], that $\text{spt} |\tilde{V}_t| \subset V(g)$ for all $t > 0$ and by $h(\tilde{V}_t, \cdot) \in L^2_{loc}$, that $\tilde{V}_t = \theta(t) |V(g)|$ for some $\theta(t) \in \mathbb{N}$. The fact that they are time-discretely decreasing can be easily seen from the curvature flow inequality. If $\theta_1 = 1$, and if $a_1 > 0$, then this would mean that $\tilde{V}_t = |V(g)|$ in a neighborhood of $(0, 0)$. Since $V_{(y,s),\lambda}$ is approaching to $\tilde{V}_t$, for sufficiently large $j$, we may apply [18] Th. 8.7 to $V_{(y,s),\lambda}$ in some small neighborhood of...
(0, 0) and conclude that \((y, s)\) is a \(C^{1,\infty}\) regular point of \(V_t\) (see the definition in [18]). But then the tangent flow at \((y, s)\) should be static, a contradiction. Thus if \(\theta_1 = 1\), then \(a_1 = 0\). This completes the proof. \(\square\)

Remark: If we assume further that there exists no quasi-static tangent flow with \(\dim V(g) = 1\), Theorem 10.1 shows that \(\Sigma_1 \setminus \Sigma_0 = \emptyset\). If this is satisfied, the picture is akin to that of the motion of grain boundaries where networks of curves joined by triple junctions move continuously with occasional collisions of junctions only at discrete points in \(\Sigma_0\).

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