The finiteness theorem for invariants of a finite group

by
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The following is an entirely elementary finiteness proof—using only the theory of symmetric functions—for the invariants of a finite group, which at once supplies an actual statement of a complete system of invariants while the usual proof using the Hilbert basis theorem (Ann. 36) is only an existence proof.

Let the finite group \( H \) consist of \( h \) linear transformations (with non-vanishing determinant) \( A_1 \cdots A_h \) where \( A_k \) represents the linear transformation

\[
x^{(k)}_1 = \sum_{\nu=1}^{n} a^{(k)}_{1\nu} x_\nu, \ldots, x^{(k)}_n = \sum_{\nu=1}^{n} a^{(k)}_{n\nu} x_\nu
\]

or more briefly: \( (x^{(k)}) = A_k(x) \). So the group \( H \) takes the series \( (x) \) with elements \( x_1 \cdots x_n \) over into the series \( (x^{(k)}) \) with elements \( x^{(k)}_1 \cdots x^{(k)}_n \). Since the identity is among the \( A_1 \cdots A_h \) so \( (x) \) is among the series \( (x^{(k)}) \). An integral rational (absolute) invariant of the group is understood to be an integral rational function of \( x_1 \cdots x_n \) which is unaltered by application of the \( A_1 \cdots A_h \); so that when \( f(x) \) is such an invariant:

\[
f(x) = f(x^{(1)}) = \cdots = f(x^{(k)}) = \frac{1}{h} \sum_{k=1}^{h} f(x^{(k)})
\]

1. Formula (1) says \( f(x) \) is an integral, rational, symmetric function of the series of quantities \( (x^{(1)}) \cdots (x^{(h)}) \) and in fact, because each summand \( f(x^{(k)}) \) contains only one series of quantities \( (x^{(k)}) \) it is the simplest case, customarily called uniform. By the well known theorem on symmetric functions of series of quantities, then \( f(x) \) has an integral rational presentation by the elementary symmetric functions of these series, i.e. by the coefficients \( G_{\alpha_1 \cdots \alpha_n}(x) \) of the “Galois resolvent”:

\[
\Phi(z, u) = \prod_{k=1}^{h} (z + u_1 x^{(k)}_1 + \cdots + u_n x^{(k)}_n) \\
= z^h + \sum_{\alpha_1 \cdots \alpha_n} G_{\alpha_1 \cdots \alpha_n}(x) z^{\alpha_1} u_1^{\alpha_1} \cdots u_n^{\alpha_n} \left( \alpha_1 + \cdots + \alpha_n = h \right)
\]

where the \( G_{\alpha_1 \cdots \alpha_n}(x) \) are invariants of degree \( \alpha_1 + \cdots + \alpha_n \) in \( x \). So is proven:

The coefficients \( G_{\alpha_1 \cdots \alpha_n}(x) \) of the Galois resolvent make up a complete system of invariants of the group so every invariant has an integral rational presentation in terms of these finitely many invariants.

2. One can also use the following even more elementary considerations based on (1), which at once yield a second complete system. Let

\[
f(x) = a + bx_1^{\mu_1} \cdots x_n^{\mu_n} + \cdots + cx_1^{\nu_1} \cdots x_n^{\nu_n}
\]

\[\text{(*) See for example Weber, Lehrbuch der Algebra (2. Aufl.) 2. Band, \$57.}\]
\[\text{†) See the footnote to section 2.}\]
\[\text{‡) This results from a proof of the theorem on symmetric functions of series of quantities in the above mentioned uniform case.}\]
where \( a, b, \ldots, c \) refer to constants. Then according to (1):

\[
h \cdot f(x) = h \cdot a + b \cdot \sum_{k=1}^{h} x_1^{(k)} \cdots x_n^{(k)} + \cdots + c \cdot \sum_{k=1}^{h} x_1^{(k)} \cdots x_n^{(k)}
\]

\[
= h \cdot a + b \cdot J_{\mu_1 \cdots \mu_n} + \cdots + c \cdot J_{\nu_1 \cdots \nu_n}
\]

So every invariant is a integral linear combination of the special:

\[
J_{\mu_1 \cdots \mu_n} = \sum_{k=1}^{h} x_1^{(k)} \cdots x_n^{(k)}
\]

and it suffices to give the proof for these. But, apart from a numerical factor, \( J_{\mu_1 \cdots \mu_n} \) is the coefficient of \( u_{\mu_1} \cdots u_{\mu_n} \) in the expression

\[
S_{\mu} = \sum_{k=1}^{h} (u_1 x_1^{(k)} + \cdots + u_n x_n^{(k)})^\mu
\]

where \( \mu = \mu_1 + \cdots + \mu_n \), which represents the \( \mu \)-th power sum of the \( h \) linear forms

\[
\xi_1 = u_1 x_1^{(1)} + \cdots + u_n x_n^{(1)}, \quad \xi_h = u_1 x_1^{(h)} + \cdots + u_n x_n^{(h)}
\]

Now it is known that the infinitely many power sums \( S_{\mu} \) are all integral rational combinations of \( S_1, \cdots, S_h \)

whose coefficients are given by the invariants

\[
J_{\mu_1 \cdots \mu_n} \quad \mu_1 + \cdots + \mu_n \leq h
\]

so a second complete system is given:

A full system of invariants for the group is given by the invariants \( J_{\mu_1 \cdots \mu_n} \) where \( \mu_1 + \cdots + \mu_n \leq h \) and \( h \) is the order of the group.

The connection with 1. is shown by the remark that when the power sums \( S_1 \cdots S_h \) are given in terms of the elementary symmetric functions of \( \xi_1 \cdots \xi_h \) this leads to to the given complete system \( G_{\alpha \alpha_1 \cdots \alpha_n}(x) \). Both results show all invariants have an integral rational expression in terms of those whose degree in \( x \) does not exceed the order \( h \) of the group.

3. These results have consequences for rational representations. Every rational absolute invariant can be represented—as shown when both the numerator and denominator are expanded by all conjugates of the denominator under the group action—as a quotient of two integral rational absolute invariants which need not be free of common factors. It follows that every rational absolute invariant can be expressed rationally by the coefficients \( G_{\alpha \alpha_1 \cdots \alpha_n}(x) \) of the Galois resolvent \( \Phi(z, u) \).

This theorem can easily be proved another way without going through the finiteness theorem; as already in Weber II §58.\(^1\)

\(^1\)That proof is incorrect. It only shows the function \( \Psi(t) \) in formula (7) has invariant coefficients without showing these invariants can be rationally expressed by coefficients of \( \Phi(t) \). This gap is avoided by using a well known differentiation process instead of the Lagrange interpolation formula to represent the \( x_i^{(k)} \) by the \( \xi_i \). Differentiating the identity \( \Phi(-\xi_k, u) = 0 \) at each \( u \) gives a relation:

\[
\left[ \frac{\partial \Phi}{\partial u_i} - x_i^{(k)} \cdot \frac{\partial \Phi}{\partial z} \right]_{z = -\xi_k} = 0
\]
4. In closing let it be said that the results proved here are implicit in my paper “Körper und Systeme rationaler Funktionen”\(^3\); specifically the complete system of 1. in Theorems VI and VII, and a proof for rational invariants independently of the finiteness theorem in Theorem III\(^4\). The proof methods and results are much simplified in the special case here compared to the general theory. I came to apply the general investigation to invariants of finite groups through conversations with Herr E. Fischer who is also the source for the substance of the proof in 2—the complete system there is not rationally definable in the general case—and for the note to 3.

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\[^{\text{And so, instead of formulas (7) and (8) of Weber for each rational function } \omega(x), \text{ the following occurs:}}\]

\[
\omega(x_1^{(k)} \cdots x_n^{(k)}) = \omega \left( \frac{\partial \Phi}{\partial u_1} \cdots \frac{\partial \Phi}{\partial u_n} \right) \bigg|_{z = -\xi_k}
\]

which involves only coefficients of \(\Phi(z, u)\) and whose summation over all \(k\) gives the desired representations of invariants \(\omega(x)\).

\[^{\text{\(\Phi\) Math. Ann. 76, p. 161 (1915).}}\]

\[^{\text{\(\|\) Theorems VIII and IX contain a finiteness proof for relative invariants which rests on different grounds than the usual but which is still only an existence proof; this is because relative invariants do not form a field.}}\]