Super WZNW with Reductions to Supersymmetric and Fermionic Integrable Models

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Abstract

A systematic construction for an action describing a class of supersymmetric integrable models as well as for pure fermionic theories is discussed in terms of the gauged WZNW model associated to twisted affine Kac-Moody algebras. Explicit examples of the $N = 1, 2$ super sinh(sine)-Gordon models are discussed in detail. Pure fermionic theories arises for cosets $sl(p, 1)/sl(p) \otimes u(1)$ when a maximal kernel condition is fulfilled. The integrability condition for such models is discussed and it is shown that the simplest example when $p = 2$ leads to the constrained Bukhvostov-Lipatov, Thirring, scalar massive and pseudo-scalar massless Gross-Neveu models.

Keywords: Classical Super-Integrable Field Theory, Super-Toda Models, Twisted Affine superalgebras, Fermionic Integrable Models.

1 Introduction

The construction and classification of integrable models is known to be underlined by an affine Lie algebraic structure. Moreover, a systematic construction of soliton solutions may be understood in terms of representation theory of affine Lie algebras. The action of several relativistic integrable models can be derived from reductions of the Wess-Zumino-Novikov-Witten (WZNW) model representing a 2D field theory in a group manifold. For the finite dimensional Lie algebra, the formulation of the Hamiltonian reduction involving Kac-Moody currents (one-loop) was proposed in [1]. In ref. [2] a generalization of WZNW...
model with currents satisfying a two-loop Kac-Moody algebra was proposed and its reduction lead to the construction of conformal affine Toda models. A gauged two-loop WZNW version was considered in [3] in order to derive a class of dyonic \((U(1))\) integrable models and was further generalized for non abelian internal structure in [4].

The construction of supersymmetric integrable models of the sinh-Gordon (mKdV)-type was proposed in [5] based upon the Riemann-Hilbert (RH) factorization problem. The object of this paper is to propose an alternative and complementary construction based upon the gauged two-loop WZNW action for affine super Lie algebras. The key ingredient is the decomposition of a twisted affine structure into integer and semi-integer subspaces followed by a Gauss-type parametrization which, in turn, defines the bosons and the fermions of the theory. Our construction uncovers the group theoretic origin of the physical fields within which the fermions parametrize grade \(\pm 1/2\) subspace, \(W_{\pm 1/2} \in g_{\pm 1/2}\) whilst the bosons parametrize the zero grade subspace \(g_0\). A local supersymmetry condition found in ref. [5] naturally arises within the formalism and is responsible for the truncation of the potential. Explicit examples for the twisted affine super algebras \(\hat{sl}(2,1)\) and \(\hat{sl}(2,2)\) yields the \(N = 1\) and \(N = 2\) super sinh-Gordon models. The implementation of the supersymmetry condition naturally leads to constructing the model associated to grade two of the higher grading generalization of the Toda systems coupled to matter proposed in [6], [7] in terms of zero curvature representation.

By a suitable decomposition of the \(\hat{sl}(p,1)\) affine twisted super algebra, a structure with maximal kernel within the bosonic zero grade sector can be constructed. The whole bosonic subalgebra, namely \(sl(p) \otimes U(1)\) may be gauged away by constructing a subsequent gauged WZNW action similarly to one of ref. [4] resulting in a pure fermionic model. We develop explicitly the \(sl(2,1)\) case and discuss its integrability. Fermionic integrable models like constrained Bukhvostov-Lipatov, Thirring, scalar massive and pseudo-scalar massless Gross-Neveu models may be generated from this formalism.

This paper is organized as follows. Section 2 is devoted to review the construction of the Lax pair of the Leznov-Saveliev (LS) equations by solving the RH factorization problem by means of the dressing technique. In Section 3 we discuss the Hamiltonian reduction of a two-loop WZNW. The main ingredients is the introduction of auxiliary gauge fields and a Gauss-type decomposition of the super group element. Invariance of the gauged WZNW action allows the reduction of the degrees of freedom leading to effective coset elements within the \(0, \pm 1/2\) graded subspaces, \(g_0\) and \(g_{\pm 1/2}\), parametrized by the bosons and fermions of the theory respectively. The effective action simplifies considerably after imposing a subsequent constraint \(Q^{(2)}_{0\pm} = 0\) already considered in ref. [5] within the locality of supersymmetry transformation.

Section 4 is devoted to the explicit construction of examples involving the supersymmetric \(N = 1\) and \(N = 2\) sinh (sine)-Gordon models. In Section 5 we discuss the extreme cases where the bosonic Kernel is maximal. The basic prototype is given by the coset \(sl(p,1)/sl(p) \otimes U(1)\) which generates pure...
fermionic theories. The integrability of these theories is discussed and the simplest example when $p = 2$ is considered in detail. Models like the constrained Bukhvostov-Lipatov, Thirring, scalar massive and pseudo-scalar massless Gross-Neveu models are shown to belong to such class related to $sl(2,1)/sl(2) \otimes U(1)$.

2 The extended Riemann-Hilbert factorization problem and the Dressing formalism.

Here we briefly review the algebraic approach given in [8] where the dressing formalism was used to unify symmetry flows (Isospectral and Non-abelian) of Integrable Hierarchies related to $\mathbb{Z}$-graded Affine Lie algebras. These ideas were applied to Twisted Affine Lie superalgebras in [5] with a $\mathbb{Z}/2$ gradation were the Lax operators for several supersymmetric integrable Hierarchies were derived by solving an extended Riemann-Hilbert factorization problem.

2.1 Symmetry flows and Isospectral times.

The Riemann-Hilbert factorization problem allows us to define an Integrable structure and a related hierarchy of non-linear partial differential equations. This also provides the explicit form of the Lax operators from which the equations of motion can be written as a zero curvature representation.

Consider a Twisted Loop superalgebra decomposed by a grading operator $Q$ into $\mathbb{Z}/2$-graded spaces satisfying $[Q, \hat{g}_i] = i\hat{g}_i$, $i \in \mathbb{Z}/2$ such that $\hat{g} = \bigoplus_{i \in \mathbb{Z}/2} \hat{g}_i$. Let $E^{(n)}$ be a constant Bosonic semisimple element of degree $n$ which induces the following algebra decomposition $\hat{g} = K \bigoplus M$, where the spaces $K$ and $M$ are the kernel and the image of the adjoint operator $adE^{(n)}(*) \equiv [E^{(n)}, *]$ respectively.

The integrable structure treated here is derived from an extended Riemann-Hilbert factorization problem involving positive and negative time flows

$$\exp \left[ - \sum_{n=1}^{\infty} E^{(n)} t_n \right] g_0 \exp \left[ \sum_{n=1}^{\infty} E^{(-n)} t_{-n} \right] = \Theta^{-1}(t)\Pi(t), \quad (1)$$

where $g_0$ is a constant element in $\tilde{G}$ and the dressing matrices are given by the following exponentials

$$\Theta(t) = \exp \left( \sum_{i \in \mathbb{Z}/2=1/2}^{+\infty} W_{-i}(t) \right), \quad \Pi(t) = B(t)M(t)$$

$$M(t) = \exp \left( \sum_{i \in \mathbb{Z}/2=1/2}^{+\infty} W_{+i}(t) \right), \quad B(t) = \exp (\hat{g}_0).$$
From (1) we get the flow equations
\[ \frac{\partial}{\partial t_n} \Theta(t) = \left( \Theta E^{(n)} \Theta^{-1} \right)_- \Theta(t), \quad \frac{\partial}{\partial t_n} \Pi(t) = - \left( \Theta E^{(n)} \Theta^{-1} \right)_+ \Pi(t) \] (2)
\[ \frac{\partial}{\partial t_{-n}} \Theta(t) = - \left( \Pi E^{(-n)} \Pi^{-1} \right)_- \Theta(t), \quad \frac{\partial}{\partial t_{-n}} \Pi(t) = \left( \Pi E^{(-n)} \Pi^{-1} \right)_+ \Pi(t), \] (3)
where \((\ast)_\pm\) denotes projection on the \(\geq 0\) and \(< 0\) grades respectively. Taking \(n = 1\) for the first equation in (2) we get
\[ \frac{\partial}{\partial t_1} \Theta(t) = \Theta(t) E_+ - \left( E_+ + A_0 + A_{1/2} + Q^{(2)}_{0+} \right) \Theta(t), \]
where \(A_0 \equiv [W_{-1}, E_+] \in \mathcal{M}, A_{1/2} \equiv [W_{-1/2}, E_+] \in \mathcal{M}\) and \(Q^{(2)}_{0+} \equiv \frac{1}{2} [W_{-1/2}, [W_{-1/2}, E_+]] \in \mathcal{K} \oplus \mathcal{M}\). From this equation we find the dressing relation
\[ L_1 = \Theta L^V_1 \Theta^{-1} \]
\[ L_1 = \frac{\partial}{\partial t_1} + E_+ + A_0 + A_{1/2} + Q^{(2)}_{0+} \] (4)
\[ L^V_1 = \frac{\partial}{\partial t_1} + E_+, \]
which relates the Lax operator \(L_1\) and the vacuum Lax operator \(L^V_1\) by means of a Dressing gauge transformation. Similarly, by considering the negative flow \(n = 1\) for the first equation in (3) we get
\[ \frac{\partial}{\partial t_{-1}} \Theta(t) = \Theta(t) E_- - \left( E_- + A_0 - A_{1/2} + Q^{(2)}_{0+} \right) \Theta(t), \]
where \(J_{-1/2} \equiv - [W_{+1/2}, E_-] \in \mathcal{M}\).

The additional term \(Q^{(2)}_{0+}\) appearing in the Lax pair comes entirely from the \(-1/2\) grade component of the dressing matrix \(\Theta\) and when different from zero gives rise to Integrable models with non-local supersymmetry c.f [5]. Local supersymmetry transformations are obtained when
\[ Q^{(2)}_{0+} = \frac{1}{2} \text{ad}^2 W_{-1/2} (E_+) = \frac{1}{2} [W_{-1/2}, [W_{-1/2}, E_+]] = 0. \] (6)
The algebra of symmetries of the model is identified with a centralizer of the generator of isospectral deformations, namely \(E_+\). In this setting, we will associate to all positive grade elements \(K^+_i \subset \mathcal{K}\), the following transformation equations
\[ \delta_{K_i} \Theta(t) = (\Theta K_i \Theta^{-1})_- \Theta(t) \]
\[ \delta_{K_i} \Pi(t) = -(\Theta K_i \Theta^{-1})_+ \Pi(t). \]
The map defined by $K_i \rightarrow \delta_{K_i}$ defines the Homomorphism $[\delta_{K_i}, \delta_{K_j}] \Theta = \delta_{[K_i,K_j]} \Theta$ and due to the fact that $K_i^+ \in \mathcal{K}$ of the operator $adE_\pm(*) \equiv [E_\pm, *]$ they commute with the isospectral deformations and are symmetries of the Integrable model. The existence of a non-trivial Fermionic kernel for the operators $adE_\pm(*)$ will lead to the existence of half integer flows and therefore to supersymmetry transformations. Given a grade one-half element, $D^{1/2} \in \mathcal{K}$, define a Fermionic flow

$$\frac{\partial}{\partial t_{1/2}} \Theta(t) \equiv \delta_{D^{1/2}} \Theta(t) = \left( \Theta D^{1/2} \Theta^{-1} \right) \Theta(t) \quad (7)$$

from which we get the dressing expression

$$L_{+1/2} = \Theta L_{+1/2}^{V} \Theta^{-1}$$

$$L_{-1/2} = \left( \frac{\partial}{\partial t_{1/2}} + D^{1/2} + D^{0} \right)$$

$$L_{+1/2}^{V} = \left( \frac{\partial}{\partial t_{1/2}} + D^{1/2} \right)$$

where $D^{0} \equiv [W_{-1/2}, D^{1/2}]$. This flow commutes with the ones generated by $\frac{\partial}{\partial t_{1}}$ and $\frac{\partial}{\partial t_{-1}}$ and the relations

$$[L_{+1/2}, L_{-1}] = 0$$

$$[L_{+1/2}, L_{+1}] = 0$$

$$[L_{+1}, L_{-1}] = 0 \quad (8)$$

follows as compatibility equations which guarantees the invariance under supersymmetry transformations of the equations of motion written as zero curvature representation, i.e., $[L_{+1}, L_{-1}] = 0$.

The grade -1 components of the first and third equations (8) allows to write

$$A_0 = -\frac{\partial}{\partial t_{1}} BB^{-1}, \quad D^{0} = -\frac{\partial}{\partial t_{1/2}} BB^{-1}.$$ 

from which we extract the supersymmetry transformations among the matrix fields

$$\partial_{1/2} J_{-1/2} = \left[ E_{-}, B^{-1} D^{1/2} B \right] \quad (9)$$

$$\partial_{-1} \left( \partial_{1/2} BB^{-1} \right) = \left[ BJ_{-1/2} B^{-1}, D^{1/2} \right]$$

$$[E_{+}, \partial_{1/2} BB^{-1}] = \left[ A_{1/2}, D^{1/2} \right]$$

$$\partial_{1/2} A_{1/2} = \left[ \partial_{1/2} BB^{-1}, A_{1/2} \right] - \left[ \partial_{+1} BB^{-1}, D^{1/2} \right].$$

From this analysis we obtain the Super-Integrable Leznov-Saveliev equations...
\[ \partial_{-1} A_{1/2} = \left[ E_+, BJ_{-1/2} B^{-1} \right] \]
\[ \partial_{-1} (\partial_{+1} BB^{-1}) = \left[ BE_ - B^{-1}, E_+ \right] + \left[ BJ_{-1/2} B^{-1}, A_{1/2} \right] \]
\[ \partial_{+1} J_{-1/2} = \left[ E_-, B^{-1} A_{1/2} B \right]. \]

Together its Lax pair

\[ L_{+1} = \partial_{+1} + \partial_{+1} BB^{-1} + E_+ + A_{1/2} \]
\[ L_{-1} = \partial_{-1} + B (E_- + J_{-1/2}) B^{-1}. \]

The relativistic hierarchy \((t_{-1}-\text{flow}, t_{-1} = z, t_{1} = \bar{z})\) given by equations (10) can also be formulated as Hamiltonian reduction procedure of a 2-Loop WZNW model by imposing an infinite number of first class constraints, as we will now show.

3 Supersymmetric Affine Toda Field Theories s-\(\text{ATFT} \).

We now show how the action governing the equations of motion of the Supersymmetric Affine Toda models can be derived from a gauged 2-Loop WZNW action where the fundamental matrix fields takes values on a Twisted Affine Lie superalgebra.

3.1 Hamiltonian reduction of the 2-Loop WZNW model.

We assume the existence of a Twisted Affine Lie Superalgebra (see for instance [9]) \( \mathfrak{g} \) endowed with a Half-integer gradation according to a grading operator \( Q \). We also identify the \( Z_2 \) statistics of the fields with the \( Z_2 \) structure of the classical superalgebra \( \hat{\mathfrak{g}} \). This means that Bosonic-Fermionic fields will parametrize respectively the even-odd part of the superalgebra with \( \Phi_B \in \mathcal{I} \otimes \mathfrak{g}_B \) and \( \Psi_F \in \mathcal{I} \otimes \mathfrak{g}_F \) given by

\[ \Phi_B \equiv \phi_i \otimes \mathfrak{g}_B^i, \quad \Psi_F \equiv \psi_k \otimes \mathfrak{g}_F^k, \]

where \( \mathcal{I} \) is the field space depending on \( t_{\pm 1} \).

The structure of the twisted affine superalgebra \( \hat{\mathfrak{g}} \) is then given by

\[ \hat{\mathfrak{g}} = \bigoplus_{i \in \mathbb{Z}/2 = -\infty}^{+\infty} \hat{\mathfrak{g}}_i, \quad [Q, \hat{\mathfrak{g}}_i] = i \hat{\mathfrak{g}}_i. \]

Let us consider a group element expressed in a generalized Gauss-type form as follows

\[ g = K_< \Gamma K_> \quad \Gamma = \Phi B \Psi, \]
where
\[ K_\le \in \exp(\hat{\mathfrak{g}}\le_{-1}), \quad K_\ge \in \exp(\hat{\mathfrak{g}}\ge_{1}), \] (16)
\[ \Phi = \exp(W_{-1/2}), \quad B \in \exp(\hat{\mathfrak{g}}_0), \quad \Psi = \exp(W_{1/2}) \] (17)
and \( W_{\pm 1/2} \in \hat{\mathfrak{g}}_{\pm 1/2} \). Propose the gauged WZNW action
\[
S[g, A_\pm] = \left. S_{WZNW}[g] \right| - \frac{k}{2\pi} \int \langle A_- (\partial_+ g g^{-1} - E_+) + A_+ (g^{-1} \partial_- g - E_-) + A_- g A_+ g^{-1} \rangle,
\] (18)
which is invariant under the local gauge transformation
\[
g \rightarrow g' = \alpha g \beta, \quad \alpha \in \exp(\hat{\mathfrak{g}}\le_{-1}) \quad \text{and} \quad \beta \in \exp(\hat{\mathfrak{g}}\ge_{1})
\]
\[
A'_+ = \beta^{-1} A_+ \beta + \partial_+ \beta^{-1} \beta
\]
\[
A'_- = \alpha A_- \alpha^{-1} + \alpha \partial_- \alpha^{-1},
\] (19)
where \( A_+ \in \hat{\mathfrak{g}}_{\ge 1}, \quad A_- \in \hat{\mathfrak{g}}_{\le 1} \) and \( \langle \ast \rangle \) is the generalized supertrace, i.e take the supertrace followed by projection using the orthogonality condition \( \langle \hat{\mathfrak{g}}, \hat{\mathfrak{g}} \rangle = 0 \) \[ \delta_{\ell i +} \delta_{\ell j 0}. \] Since the action (18) is invariant under (19) we may choose \( \alpha = K_\le^{-1} \) and \( \beta = K_\ge^{-1} \) which in practice amounts to the elimination of an infinite number of fields (first class constraints) ending up with
\[
S[\Gamma, A'_\pm] = \left. S_{WZNW}[\Gamma] \right| - \frac{k}{2\pi} \int \langle A'_- (\partial_+ \Gamma g^{-1} - E_+) + A'_+ (\Gamma^{-1} \partial_- \Gamma - E_-) + A'_- \Gamma A'_+ \Gamma^{-1} \rangle.
\]
For the action \( S_{WZNW}[\Gamma] \) above, we use the Polyakov-Wiegmann identity\( ^1 \) to decompose it as
\[
S_{WZNW}[\Gamma] = S_{WZNW}[B] - \frac{k}{2\pi} \int \langle (\Phi^{-1} \partial_- \Phi) B (\partial_+ \Phi \Psi^{-1}) B^{-1} \rangle
\] (21)
and for the second term we seek for the non-zero contributions of the inner product
\[
I = \langle A_- (\partial_+ \Gamma g^{-1} - E_+) + A_+ (\Gamma^{-1} \partial_- \Gamma - E_-) + A_- \Gamma A_+ \Gamma^{-1} \rangle.
\]
\[ ^1 \text{This is given by:}
\]
\[
S_{WZNW}[ABC] = \left. S_{WZNW}[A] + S_{WZNW}[B] + S_{WZNW}[C] \right| - \frac{k}{2\pi} \int \langle (A^{-1} \partial_+ A) (\partial_+ BB^{-1}) + (B^{-1} \partial_- B) (\partial_+ CC^{-1}) \rangle
\]
\[
- \frac{k}{2\pi} \int \langle (A^{-1} \partial_+ A) B (\partial_+ CC^{-1}) B^{-1} \rangle,
\] (20)
where
\[
S_{WZNW}[g] = -\frac{k}{4\pi} \int \langle (g^{-1} \partial_+ g) (g^{-1} \partial_- g) \rangle + \frac{k}{12\pi} \int \epsilon^{IJK} \langle (g^{-1} \partial_+ g) (g^{-1} \partial_+ J) (g^{-1} \partial_+ K) g \rangle
\]
Analyzing term by term we have
\[
\langle A_- ((\partial_+ \Gamma^{-1})|_{\geq 1} - E_+) \rangle = \langle A_- (\Phi B \partial_+ \Psi \Psi^{-1} B^{-1} \Phi^{-1} - E_+) \rangle \\
\langle A_+ ((\Gamma^{-1} \partial_- \Gamma)|_{\leq -1} - E_-) \rangle = \langle A_+ (\Psi^{-1} B^{-1} \Phi^{-1} \partial_- \Phi B \Psi - E_-) \rangle.
\]
Solving now the equations of motion for the gauge fields,
\[
A_+ = \Gamma^{-1} E_+ \Gamma - \Psi^{-1} \partial_+ \Psi \\
A_- = \Gamma E_- \Gamma^{-1} - \partial_- \Phi \Phi^{-1}
\]
and plugging them back in (3.1) yields the effective action,
\[
I_{eff} = -\langle (\Phi^{-1} E_+ \Phi) B (\Psi E_- \Psi^{-1}) B^{-1} \rangle - \langle (\Phi^{-1} \partial_- \Phi) B (\partial_+ \Psi \Psi^{-1}) B^{-1} \rangle \\
+ \langle (\Phi^{-1} E_+ \Phi) (\Phi^{-1} \partial_- \Phi) + (\Psi E_- \Psi^{-1}) (\partial_+ \Psi \Psi^{-1}) \rangle.
\]
(22)

The term involving the two derivatives in (21) and (22) cancel each other leaving only linear terms in the derivatives which are the correct kinetic terms expected for Fermionic fields. Finally, we have
\[
S_{eff}[\Phi, B, \Psi] = S_{WZNW}[B] + \frac{k}{2\pi} \int \langle (\Phi^{-1} E_+ \Phi) B (\Psi E_- \Psi^{-1}) B^{-1} \rangle \\
- \frac{k}{2\pi} \int \langle (\Phi^{-1} E_+ \Phi) (\Phi^{-1} \partial_- \Phi) + (\Psi E_- \Psi^{-1}) (\partial_+ \Psi \Psi^{-1}) \rangle.
\]
(23)
The effective action (23) possess, in principle, an infinite number of terms due to the Baker-Haussdorff expansion in the potential. Although these terms may eventually truncate due to either, the Grassmaniann character of the fields or to the Nilpotency of the subspaces involved, other cases of interest may be obtained by reduction as we shall now discuss.

3.2 The case where \( Q_{0\pm} = 0 \) and the Supersymmetric Affine Toda Field Theories

Now, guided by (6) and the symmetric structure of (23), we propose the following subsidiary conditions,
\[
Q_{0+}^{(2)} = (\Phi^{-1} E_+ \Phi)|_0 = \frac{1}{2} [W_{-1/2}, [W_{-1/2}, E_+]] = 0, \\
Q_{0-}^{(2)} = (\Psi E_- \Psi^{-1})|_0 = \frac{1}{2} [W_{1/2}, [W_{1/2}, E_-]] = 0,
\]
(24)
which ensures finite number of terms coming from the Baker-Haussdorff expansions of \((\Phi^{-1} E_+ \Phi)\) and \((\Psi E_- \Psi^{-1})\) appearing in the effective action (23).
Assuming \((24)\) we write the action \((23)\) in terms of fields given in \((17)\) as,

\[
S_{\text{Toda}}[B, W_{\pm 1/2}] = S_{WZNW} - \frac{k}{2\pi} \int \left\langle \left( \left[ W_{-1/2}, E_+ \right] B \left[ W_{+1/2}, E_- \right] B^{-1} \right) \right\rangle - \frac{k}{4\pi} \int \left\langle \partial_- W_{-1/2} \left[ W_{-1/2}, E_+ \right] - \partial_+ W_{+1/2} \left[ W_{+1/2}, E_- \right] \right\rangle + \frac{k}{2\pi} \int \left\langle \left( E_+ B E_- B^{-1} \right) \right\rangle
\]

(25)

An arbitrary variation of this action, i.e.,

\[
\frac{2\pi}{k} \delta S_{\text{Toda}} = \int \left\langle \left( \delta B B^{-1} \right) \left\{ \partial_- \left( \partial_+ B B^{-1} \right) - \left[ E_+, B E_- B^{-1} \right] - \left[ A_{1/2}, B J_{-1/2} B^{-1} \right] \right\} \right\rangle
\]

+ \int \left\langle \delta W_{+1/2} \left\{ \partial_+ J_{-1/2} - \left[ E_-, B^{-1} A_{1/2} B \right] \right\} \right\rangle

+ \int \left\langle \delta W_{-1/2} \left\{ \partial_- A_{1/2} + \left[ E_+, B J_{-1/2} B^{-1} \right] \right\} \right\rangle,
\]

yields the following equations of motion

\[
\partial_- A_{1/2} = \left[ B J_{-1/2} B^{-1}, E_+ \right],
\]

\[
\partial_- \left( \partial_+ B B^{-1} \right) = \left[ E_+, B E_- B^{-1} \right] + \left[ A_{1/2}, B J_{-1/2} B^{-1} \right]
\]

\[
\partial_+ J_{-1/2} = \left[ E_-, B^{-1} A_{1/2} B \right].
\]

(26)

where

\[
A_{1/2} \equiv \left[ W_{-1/2}, E_+ \right], \quad J_{-1/2} \equiv - \left[ W_{+1/2}, E_- \right].
\]

(27)

Equations (26) coincide precisely with the LS found before (see (10)) if we identify the flows \(\frac{\partial}{\partial \pm 1} = \partial_{\pm 1} \rightarrow \pm \partial_\pm\).

The integrability of the model can be shown from the Lax operators

\[
L_{+1} = \partial_+ - \partial_+ B B^{-1} + E_+ + A_{1/2}, \quad L_{-1} = \partial_- - B(E_- + J_{-1/2}) B^{-1}
\]

(28)

where the equations of motion (26) are written in the zero curvature representation

\[
[L_{+1}, L_{-1}] = 0
\]

(29)

A natural question that arises at this point is whether Supersymmetry transformations (9) keeps the action (25) unchanged. Such proof is given in appendix A where it is shown that (25) is in fact invariant under the supersymmetry transformation

\[
\partial_{1/2} J_{-1/2} \equiv \left[ E_-, B^{-1} D^{1/2} B \right]
\]

\[
\partial_{1/2} B B^{-1} \equiv \left[ D^{1/2}, W_{-1/2} \right]
\]

\[
\partial_{1/2} A_{1/2} \equiv \left[ D^{1/2}, W_{-1/2} \right] - \left[ \partial_+ B B^{-1}, D^{1/2} \right].
\]

(30)
Notice that equations (26) are of the same form as those introduced in [6] (eqns. (2.12) - (2.14)) within the construction of $M = 2$ higher grading Toda field theories coupled to matter, namely,

\[
\begin{align*}
\partial_- (g_0 \partial_+ W_1^+ g_0^{-1}) &= [E_+, \partial_- W_1^-], \\
\partial_+ (g_0^{-1} \partial_- g_0) &= \left[ E_-; g_0^{-1} E_+ g_0 \right] + [g_0^{-1} \partial_- W_1^- g_0, \partial_+ W_1^+], \\
\partial_+ g_0^{-1} \partial_- W_1^- g_0 &= \left[ E_-, \partial_+ W_1^+ \right].
\end{align*}
\] (32)

After the change of variables ( and $\partial_- \rightarrow - \partial_-$)

\[
\begin{align*}
A_{1/2} &= [W_{-1/2}, E_+] \rightarrow g_0 \partial_+ W_1^+ g_0^{-1}, \\
J_{-1/2} &= -[W_{1/2}, E_-] \rightarrow g_0^{-1} \partial_- W_1^- g_0
\end{align*}
\] (33)

we find that eqns. (32) coincide precisely with our eqns. (26) if we identify $B \rightarrow g_0$. Although their fields $W_\pm^\pm$ were Bosonic and $Z$-gradations were considered, their action coincide precisely with (25) with the difference that no analogue for the subsidiary conditions (24) was considered. Here the matter fields are Fermionic and are introduced within a group theoretic background by $W_{-1/2}$. We also emphasize on the locality of the action (25) in contrast to what has been argued in the literature [6], [7], in the sense that the connection between the action and the equations of motion involves a non-local field transformation as in (33).

4 Examples Super sinh-Gordon Type Models

4.1 The $N = 1 \hat{sl}(2,1)^{(2)}$-Model.

Let us first consider the grading operator $Q = 2d + \frac{1}{2}h_1$ and $E_\pm$ given by

\[ E_\pm = h_1^{(1/2)} + 2h_2^{(1/2)} - \left( E_{\alpha_1}^{(0)} + E_{-\alpha_1}^{(1)} \right) \] (34)

which yields the following (finite dimensional Lie super algebra) decomposition

\[
\begin{align*}
\mathcal{K}_B &= \{ K_1 = \lambda_2 H, \quad K_2 = E_{\alpha_1} + E_{-\alpha_1} \} \\
\mathcal{K}_F &= \left\{ f_1 = E_{\alpha_2 + \alpha_1}, \quad f_2 = f_1^\dagger \right\} \\
\mathcal{M}_B &= \{ M_1 = h_1, \quad M_2 = E_{\alpha_1} - E_{-\alpha_1} \} \\
\mathcal{M}_F &= \left\{ g_1 = E_{\alpha_2 - \alpha_1 + \alpha_2}, \quad g_2 = g_1^\dagger \right\}.
\end{align*}
\] (36)

Notice that the Cartan subalgebra contribution, $h_1 + 2h_2$, in (34) is crucial in order to generate a nontrivial fermionic kernel. In extending the Lie super algebra $sl(2,1)$ to a twisted affine structure we assign to each generator a loop index $n$ which may be either, integer $n \in Z$ or semi-integer $n \in Z + 1/2$. In particular this yields two generators of grade $1/2$ within the Kernel, $K_{1/2} = \ldots$
\{ F_1^{(1/2)} = (f_1^{(1/2)} + f_2^{(1/2)}), \quad F_2^{(1/2)} = f_1^{(1/2)} - f_2^{(1/2)} \} \text{ and two other generators within the Image, } \mathcal{M}_{1/2} = \{ G_1^{(1/2)} = g_1^{(1/2)} + g_2^{(1/2)}, \quad G_2^{(1/2)} = g_1^{(1/2)} - g_2^{(1/2)} \}.

Since $W_{\pm 1/2} \in g_{\pm 1/2}$, they can be parametrized as

$$W_{-1/2} = \bar{\psi}_1 G_1^{(-1/2)} + \bar{\psi}_2 G_2^{(-1/2)}, \quad W_{1/2} = \psi_1 G_1^{(1/2)} + \psi_2 G_2^{(1/2)}$$

(37)

so that

$$Q_{0,+}^{(2)} = \frac{1}{2} [E^+, W_{-1/2}], W_{-1/2} = 4 \bar{\psi}_1 \bar{\psi}_2 (K_2^{(0)} - K_1^{(0)}), \quad Q_{0,-}^{(2)} = \frac{1}{2} [E^-, W_{1/2}], W_{1/2} = 4 \psi_1 \psi_2 (K_2^{(0)} - K_1^{(0)})$$

(38)

In order to satisfy the subsidiary conditions $Q_{0,\pm}^{(2)} = 0$ we take

$$\psi_2 = \bar{\psi}_1 = 0$$

which yields (after renaming $\psi_1 \equiv \psi$ and $\bar{\psi}_2 \equiv \bar{\psi}$)

$$A_{1/2} = -2 \psi G_1^{(1/2)}, \quad J_{-1/2} = 2 \bar{\psi} G_2^{(-1/2)}, \quad B = \exp \phi M_1^{(0)}$$

(39)

The action (25) gives the following Lagrangian density

$$L_{N=1} = -\frac{k}{2\pi} \{ \partial_+ \psi \partial_- \phi - \mp \gamma^\mu \partial_\mu \Psi - V_{N=1} \}$$

(40)

and the equations of motion correspond to the $N=1$ super sinh-Gordon equations:

$$\partial_+ \partial_- \phi = 2 \sinh [2\phi] + \mp \Psi \sinh [\phi]$$

$$0 = (\gamma^\mu \partial_\mu + 2 \cosh [\phi]) \Psi,$$

which are invariant under the supersymmetry transformation

$$\partial_{1/2} \phi = \epsilon \bar{\psi}, \quad \partial_{1/2} \psi = \epsilon \partial_x \phi.$$

---

2In 2-dimensions, the Majorana-Weyl basis for the gamma matrices is:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. $$

In the light cone we have $\gamma^\pm = \frac{1}{2} (\gamma^0 \pm \gamma^1)$ and the projectors $\mathcal{P}^\pm = \frac{1}{2} (1 \pm \gamma^5)$. Define

$$\Psi = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}, \quad \bar{\Psi} = \Psi^T \gamma_0.$$

3The Bosonic limit:

$$\mathcal{L} = -\frac{k}{2\pi} \{ \partial_+ \phi \partial_- \phi - 2 \mu^2 \cosh [2\phi] \}$$

gives the Polnhmeyer reduction of a Classical Bosonic String in an $AdS_2 \times S^1$ background.
Remark 1 Notice that in order to satisfy the subsidiary conditions \( \text{[35]} \), we have constrained a pair of Fermi fields, i.e., \( \psi_2 = \overline{\psi}_1 = 0 \). This, effectively is equivalent in choosing the subalgebra \( \hat{\text{sl}}(2,1)_{(2)} \) (see appendix B) with only one generator within the subspaces \( \mathfrak{g}_{\pm 1/2} \).

Following the same line of thought we now consider the second twisted affine subalgebra \( \hat{\text{sl}}(2,1)_{(2)} \) (see appendix B) with parametrization

\[
B = \exp[\phi M_2^0], \quad W_{+1/2} = \psi G_1^{1/2}, \quad W_{-1/2} = \overline{\psi} G_2^{-1/2},
\]

yielding the Lagrangian density \( \mathcal{L}_{N=1} \)

\[
\mathcal{L}_{N=1} = \frac{k}{2\pi} \left[ \partial_+ \phi \partial_- \phi + \overline{\Psi} \gamma^\mu \partial_\mu \Psi + V_{N=1} \right]
\]

\[
V_{N=1} = 2 \left( \cos[2\phi] + \overline{\Psi} \Psi \cos[\phi] \right),
\]

and equations of motion given by the \( N = 1 \) super sine-Gordon equations:

\[
\partial_+ \partial_- \phi = 2 \sin[2\phi] + \overline{\Psi} \Psi \sin[\phi]
\]

\[
0 = (\gamma^\mu \partial_\mu + 2 \cos[\phi]) \Psi.
\]

invariant under \( (\partial_+ \rightarrow \partial_-) \)

\[
\partial_{1/2} \phi = -\epsilon \overline{\psi}, \quad \partial_{1/2} \overline{\psi} = \epsilon \partial_+ \phi.
\]

Remark 2 We see that the twisted superalgebra \( \hat{\text{sl}}(2,1)^{(2)} \) supports the two models, note also that one is the field analytic continuation of the other but only at the action level and not in their Lax connections which is where the models are truly specified \( \text{[12]} \). We are using a two-valued spectral parameter \( \lambda^{\pm 1/2} \) due to the gradation used in contrast to the single-valued Lax pairs usually considered so this \( \lambda^{\pm 1/2} \) plays a role. The Lax connections are different and have cuts, this opens the question of having a formal Zakharov-Shabat like construction involving a two-valued spectral parameter.

Remark 3 The untwisted Affine superalgebra \( \text{sl}(2,1)^{(1)} = \hat{\text{sl}}(2,1) \) does not admit a purely Fermionic simple root system (the longest root being Bosonic). This means that the usual criteria \( \text{[14]}, \text{[13]} \) that only Contragredient Lie superalgebras with a purely Fermionic simple root system admit supersymmetric Integrable extensions does not apply when \( \mathbb{Z}/2 \)-gradations are taken into account. The \( \text{sl}(2,1)^{(2)} \) is an counter-example of this statement, from it we could construct the \( N = 1 \) super Toda model. We expect this construction help us to have a better understanding of the intricate relation between supersymmetric Toda models and Lie superalgebras.

\[4\]The Bosonic limit:

\[
\mathcal{L} = \frac{k}{2\pi} \left[ \partial_+ \phi \partial_- \phi + 2\mu^2 \cos[2\phi] \right],
\]

gives the Polhymeyer reduction of a Classical Bosonic String in an \( R_t \times S^2 \) background \( \text{[15]} \).
4.2 The $N = 2 \hat{sl}(2, 2)_{(2)}$-Model.

Consider the grading operator $Q = d + \frac{1}{2} (h_1 + h_3)$ and

$$E_+ = \left( E^{(0)}_{\alpha_1} + E^{(2)}_{-\alpha_1} \right) + \left( E^{(2)}_{\alpha_3} + E^{(0)}_{-\alpha_3} \right) + I^{(1)} \quad (42)$$

In the Appendix B we have found that there are four sub-superalgebras of $\hat{sl}(2, 2)_{(2)}$ namely [98]-[101] which solves $Q_{\pm}^2 = 0$. We will write only the ones giving inequivalent solutions.

For $psl(2, 2)_{[1]}$ in (98), we take the parametrization

$$B = \exp[\phi_1 M_1^0 + \phi_3 M_3^0]$$

$$W_{+1/2} = \psi_1 G_1^{1/2} + \psi_3 G_3^{1/2}$$

$$W_{-1/2} = \overline{\psi}_2 G_2^{-1/2} + \overline{\psi}_4 G_4^{-1/2},$$

to obtain the Lagrangian density

$$-\frac{2\pi}{k} \mathcal{L}_{N=2} = \partial_+ \phi_1 \partial_- \phi_1 - \partial_+ \phi_3 \partial_- \phi_3$$

$$+ \psi_2 \partial_+ \psi_1 + \overline{\psi}_2 \partial_- \overline{\psi}_1 - \psi_3 \partial_+ \psi_3 - \overline{\psi}_4 \partial_- \overline{\psi}_4 - V_{N=2} \quad (43)$$

where

$$V_{N=2} = 2 \cosh[2\phi_1] - 2 \cosh[2\phi_3] - 4 \left( \psi_1 \overline{\psi}_2 - \psi_3 \overline{\psi}_4 \right) \cosh[\phi_1] \cosh[\phi_3]$$

$$- 4 \left( \psi_1 \overline{\psi}_4 - \psi_3 \overline{\psi}_2 \right) \sinh[\phi_1] \sinh[\phi_3] \quad (44)$$

leading to the $N = 2$ super sinh-Gordon model (see for instance [18]).

For $psl(2, 2)_{[2]}$ in (99), we have the parametrization

$$B = \exp[\phi_1 M_1^0 + \phi_3 M_3^0]$$

$$W_{+1/2} = \psi_1 G_1^{1/2} + \psi_3 G_3^{1/2}$$

$$W_{-1/2} = \overline{\psi}_2 G_2^{-1/2} + \overline{\psi}_4 G_4^{-1/2},$$

and the Lagrangian density

$$-\frac{2\pi}{k} \mathcal{L}_{N=2} = -\partial_+ \phi_1 \partial_- \phi_1 + \partial_+ \phi_3 \partial_- \phi_3$$

$$+ \psi_1 \partial_+ \psi_1 + \overline{\psi}_2 \partial_- \overline{\psi}_2 - \psi_3 \partial_+ \psi_3 - \overline{\psi}_4 \partial_- \overline{\psi}_4 - V_{N=2} \quad (45)$$

where

$$V_{N=2} = 2 \cos[2\phi_1] - 2 \cos[2\phi_3] - 4 \left( \psi_1 \overline{\psi}_2 - \psi_3 \overline{\psi}_4 \right) \cos[\phi_1] \cos[\phi_3]$$

$$+ 4 \left( \psi_1 \overline{\psi}_4 - \psi_3 \overline{\psi}_2 \right) \sin[\phi_1] \sin[\phi_3]$$

yields the $N = 2$ super sine-Gordon model.
For $\text{psl}(2, 2)_{[3]}^{(2)}$, we have the parametrization

$$
B = \exp[\phi_1 M_0^0 + \phi_3 M_4^0],
$$
$$
W_{+1/2} = \psi_1 G_1^{1/2} + \psi_3 G_4^{1/2},
$$
$$
W_{-1/2} = \bar{\psi}_2 G_2^{-1/2} + \bar{\psi}_4 G_3^{-1/2},
$$

and the Lagrangian density

$$
- \frac{2\pi}{k} \mathcal{L}_{N=2} = \partial_+ \phi_1 \partial_- \phi_1 + \partial_+ \phi_3 \partial_- \phi_3 + \psi_1 \partial_+ \psi_1 + \bar{\psi}_2 \partial_- \bar{\psi}_2 - \psi_3 \psi_4 \partial_+ \bar{\psi}_4 - V_{N=2} \quad (46)
$$

where

$$
V_{N=2} = 2 \cosh[2\phi_1] - 2 \cos[2\phi_3] - 4 \left( \psi_1 \bar{\psi}_2 - \psi_3 \bar{\psi}_4 \right) \cosh[\phi_1] \cos[\phi_3] + 4 \left( \psi_1 \bar{\psi}_4 + \psi_3 \bar{\psi}_2 \right) \sinh[\phi_1] \sin[\phi_3]
$$

which mixes the $N = 1$ sinh-Gordon and sine-Gordon models.

For $\text{psl}(2, 2)_{[4]}^{(2)}$, we have essentially the same model as the one obtained with $\text{psl}(2, 2)_{[3]}^{(2)}$ with the replacements $\sinh \rightarrow \sin$, $\cosh \rightarrow \cos$ and some sign changes.

Another interesting example is the pure bosonic composed sinh-sine Gordon model obtained from

$$
B = \exp[\phi_1 M_0^0 + \phi_2 M_2^0],
$$

and Lagrangian density:

$$
\mathcal{L} = - \frac{k}{2\pi} \left[ \partial_+ \phi_1 \partial_- \phi_1 - \partial_+ \phi_2 \partial_- \phi_2 - V_{\ominus} \right] \quad (47)
$$

$$
V_{\ominus} = 4 \sinh^2 \left( \sqrt{\frac{\phi_1^2}{\phi_2^2}} \right),
$$

which leads to the sinh-Gordon when $\phi_2 \rightarrow 0$ and to the sine-Gordon when $\phi_1 \rightarrow 0$.

### 4.3 Solving $Q_{0,\pm}^{(2)} = 0$

Let us assume that the space $\mathcal{M}_F$ has the following decomposition:

$$
\mathcal{M}_F = \left\{ I_+^N \bigoplus I_-^N \mid [E, I_\pm^N] \subset \pm I_\pm^N, \quad N = 1/2 \dim \mathcal{M}_F = 1, 2, 4, \ldots \right\}. \quad (48)
$$

This supersymmetric Lagrangian gives the Polhmeyer reduction of a Classical superstring in an $\text{AdS}_2 \times S^2$ background and its Bosonic limit:

$$
\frac{2\pi}{k} \mathcal{L} = \partial_+ \phi_1 \partial_- \phi_1 + \partial_+ \phi_3 \partial_- \phi_3 - \mu^2 (2 \cosh[2\phi_1] - 2 \cos[2\phi_3])
$$

gives the reduction for a Classical Bosonic String in an $\text{AdS}_2 \times S^2$ background [13].
This means we have two \( \text{ad}E(*) \) -- Invariant eigenspaces \( I_\pm \) of dimension \( N \) and eigenvalues \( \pm 1 \). Furthermore, we assume that the following result also holds:

\[
\text{ad}^2 E(*) = \Lambda I(*) \quad \Lambda \in \mathbb{R}^+.
\] (49)

Now we show a situation when \( Q_{0\pm}^{(2)} = 0 \) happens. We have the following

**Proposition 4** The conditions \( Q_{0\pm}^{(2)} = 0 \) are satisfied if the adjoint representation matrix of the constant element \( E \) on \( \mathcal{M}_F \), i.e., \( [E, g_i] = (\text{Ad}E)_{ij} g_i \) decomposes in a block diagonal matrix of the form \( [\text{Ad}E] = \text{diag}(\Sigma, -\Sigma) \), with \( \Sigma \) an \( N \times N \) symmetric matrix commuting with the structure constants \( M, N \) and \( R \) defined by \( \{g_i^+, g_j^-\} = M_{ij} X_0^0 \), \( \{g_i^-, g_j^-\} = N_{ij} Y_0^0 \) and \( \{g_i^+, g_j^-\} = R_{ij} (Z_v^0) \), i.e., \( [\Sigma, M^0] = [\Sigma, N^0] = [\Sigma, R^0] = 0 \), \( i, j = 1, \ldots, N \) where \( g_1^+ \in I_\pm \).

**Proof.** The proof is made by direct computation. We write a general element \( W = \psi_1 g_1^+ + \psi_2 g_2^- \) where \( g^\pm \in I_\pm \) are such that \( [E, g^\pm] = \pm S_{ij} g^\pm \). Now we see that \( Q_0^2 = \frac{1}{2} [W, [W, E]] \) can be written as

\[
4Q_0 = \left( \begin{pmatrix} \psi_1^T & [\Sigma, M^0] \end{pmatrix} \mathbf{x}_0^0 + \left( \begin{pmatrix} \psi_1^T & [\Sigma, N^0] \end{pmatrix} \mathbf{\bar{\psi}}_0^0 \right) \mathbf{y}_0^0 + 2 \left( \begin{pmatrix} \psi_1^T & [\Sigma, R^0] \end{pmatrix} \mathbf{\bar{\psi}}_0^0 \right) \mathbf{z}_0^0 \right),
\]

from which the statement follows. \( \blacksquare \)

As an illustration, let us consider the \( \text{sl}(2,1) \) case where \( N = 1, g_1^+ = g_1 = E_{a2} + E_{a1+a2} \) and \( g_2^- = g_2 = E_{-a2} + E_{-a1-a2} \). The commutation relations for the odd generators defined in \( (50) \) can be evaluated using the matrix representation given in the appendix B. We therefore find,

\[
[E_+, g_1^+] = 2g_1^+, \quad [E_+, g_1^-] = -2g_1^-,
\] (50)

and \( \Sigma \) is diagonal which commutes with any other matrices.

Another example is \( \text{sl}(2,2) \) where \( N = 2 \). The adjoint representation of \( E \) is defined by

\[
[E_+, g_1^+] = 2g_1^+, \quad [E_+, g_1^-] = -2g_1^-, \quad i = 1, 2
\] (51)

where \( g_1^+ = g_2 + g_1, \quad g_2^+ = g_2 + g_3, \quad g_1^- = g_1 - g_3, \quad g_2^- = g_2 - g_3 \) are defined in \( (55) \). It therefore follows from the algebra given in the appendix of re. \( [17] \) that again \( \Sigma \) is proportional to the unit matrix and commutes with all other matrices.

## 5 \( Q_{0\pm}^{(2)} \neq 0 \) and Pure Fermionic Models

We now discuss the construction of pure fermionic theories by considering the maximal Kernel condition i.e., \( \mathcal{M}_B = \emptyset \). The bosonic fields in this cases lie in \( \mathcal{K}_B \) and may be gauged away by considering the coset \( G/\mathcal{K}_B \). As a general prototype, consider the super algebra \( \text{sl}(p, 1) \) with homogeneous gradation \( Q = d \) and \( E^+ = l_p \cdot H^{(1)} \) (where \( l_p \) is the \( p-th \) fundamental weight) such that

\[
\mathcal{K} \cap E^+ = \mathcal{K}_B = \text{sl}(p) \otimes u(1), \quad \mathcal{M}_B = \emptyset
\] (52)
and fermionic subspace generated by \( \{ E_{\alpha_1, \ldots, \alpha_i}, E_{-(\alpha_1, \ldots, \alpha_i)}, i = 1, 2, \ldots, p \} \).

From (52), we now construct the gauged WZNW action

\[
S[W_{\pm 1/2}, B, A_\pm]_{G/K_B} = S_{WZNW}[B] + \frac{k}{2\pi} \int \langle \Phi^{-1} E_+ \Phi \Psi E_0 \Psi^{-1} B^{-1} \rangle
\]

\[
- \frac{k}{4\pi} \int \langle D_- W_{1/2} [W_{-1/2}, E_+] - D_+ W_{1/2} [W_{1/2}, E_-] \rangle + \]

\[
- \frac{k}{2\pi} \int \langle A_- (\partial_+ B B^{-1}) + A_+ (B^{-1} \partial_- B) + A_- B A_+ B^{-1} + A_0^0 A_0^0 \rangle,
\]

(53)

where

\[
D_- W_{1/2} = \partial_+ W_{-1/2} + [A_-, W_{1/2}] = \partial_+ W_{1/2} - [A_+, W_{1/2}]
\]

\( A_\pm = A_0^0 + \tilde{A}_0^0 \in K_B \), where \( A_0^0 \) lies in the Cartan subalgebra of \( sl(p) \otimes u(1) \) and \( \tilde{A}_0^0 \) in its orthogonal complement, are the auxiliary gauge fields and \( D_\pm \) the covariant derivatives. The action (53) is invariant under the following transformation

\[
A_0^0 = A_0^0 - \gamma_0^{-1} \partial_+ \gamma_0
\]

\[
A_0^0 = A_0^0 - \gamma_0^{-1} \partial_- \gamma_0
\]

\[
A'_- = \Gamma_- A_- \Gamma_-^{-1} - \partial_- \Gamma_- \Gamma_-^{-1}
\]

\[
A'_+ = \Gamma_+ A_+ \Gamma_+^{-1} - \partial_+ \Gamma_+ \Gamma_+^{-1}
\]

(54)

and

\[
B' = \Gamma_- B \Gamma_+
\]

\[
W'_{-1/2} = \Gamma_- W_{-1/2} \Gamma_-^{-1}
\]

\[
W'_{+1/2} = \Gamma_+ W_{+1/2} \Gamma_+^{-1}
\]

(55)

(56)

where \( \Gamma_- = \gamma_0 \gamma_- \) and \( \Gamma_+ = \gamma_+ \gamma_0 \) are both in \( K_B \). Here \( \gamma_0, \gamma_\pm \) are exponentials of Cartan subalgebra and positive/negative step operators of \( \mathfrak{sl}(p) \otimes u(1) \) respectively. The two terms involving covariant derivatives in (53) are clearly invariant. The last term was introduced in (4) within the coset structure \( \mathfrak{sl}(3)/\mathfrak{sl}(2) \otimes u(1) \) and its invariance under the \( \mathfrak{g}_0 = \mathfrak{sl}(2) \otimes u(1) \) was shown to lead to Isospin conservation laws.

The invariance of the action (53) under (55) allows us to choose a gauge in which \( B' = I \). Under such circumstances, we find

\[
S'[W'_{\pm 1/2}, B = I, A'_\pm, \tilde{A}'_\pm]_{G/K_B} = \frac{k}{2\pi} \int \langle \Phi^{-1} E_+ \Phi \Psi E_0 \Psi^{-1} \rangle
\]

\[
- \frac{k}{4\pi} \int \langle \partial_- W'_{-1/2} [W'_{1/2}, E_+] - \partial_+ W'_{1/2} [W'_{1/2}, E_-] \rangle + \]

\[
- \frac{k}{2\pi} \int \langle A'_- \tilde{Q}'_0^{(2)} + A'_+ \tilde{Q}'_0^{(2)} + A'_- A'_+ + A_0^0 A_0^0 \rangle
\]

(57)
Since the action is quadratic in the auxiliary fields and $\langle A_+^0 A_-^0 \rangle = 0$ we can use Gaussian integration to obtain the effective action

$$S[\psi_{\pm 1/2}]_{\text{effective}} = \frac{k}{2\pi} \int \langle \Phi^{-1} E_+ \Phi \Psi E_- \Psi^{-1} + \frac{3}{2} Q_0^{(2)} Q_{-0}^{(2)} \rangle - \frac{k}{4\pi} \int \langle \partial_+ W_{-1/2} [W_{-1/2}, E_] - \partial_+ W_{+1/2} [W_{+1/2}, E_-] \rangle$$

(58)

As an illustration consider the simplest case where $g = sl(2,1)$. Let us parametrize

$$W_{1/2} = \psi_1 E_{\alpha_1}^{(1/2)} + \psi_2 E_{\alpha_1 + \alpha_2}^{(1/2)} + \psi_3 E_{-\alpha_1}^{(1/2)} + \psi_4 E_{-\alpha_1 - \alpha_2}^{(1/2)}$$

$$W_{-1/2} = \bar{\psi}_1 E_{\alpha_1}^{(-1/2)} + \bar{\psi}_2 E_{\alpha_1 + \alpha_2}^{(-1/2)} + \bar{\psi}_3 E_{-\alpha_1}^{(-1/2)} + \bar{\psi}_4 E_{-\alpha_1 - \alpha_2}^{(-1/2)}$$

(59)

from where we evaluate

$$Q_{0-}^{(2)} = \langle \psi_2 \psi_4 \rangle h_1 + \langle \psi_1 \psi_3 + \psi_2 \psi_4 \rangle h_2 + \langle \psi_2 \psi_4 \rangle E_{\alpha_1} + \langle \psi_1 \psi_4 \rangle E_{-\alpha_1},$$

$$Q_{(1/2)-}^{(3)} = \frac{1}{3} W_{1/2, Q_{0-}^{(2)}} = \frac{2}{3} \langle \psi_1 \psi_2 \psi_4 \rangle E_{\alpha_1}^{(1/2)}$$

$$- \frac{2}{3} \langle \psi_1 \psi_2 \psi_3 \rangle E_{\alpha_1 + \alpha_2}^{(1/2)} + \frac{2}{3} \langle \psi_2 \psi_3 \psi_4 \rangle E_{-\alpha_2}^{(1/2)},$$

$$Q_{(1)-}^{(4)} = \frac{1}{4} W_{1/2, Q_{(3)}^{(1/2)-}} = \frac{1}{6} \langle \psi_1 \psi_2 \psi_3 \psi_4 \rangle (h_1^{(1)} + h_2^{(1)})$$

(60)

and similar for $Q_{0+}^{(2)}, Q_{(-1/2)+}^{(3)}$ and $Q_{(-1)+}^{(4)}$ in terms $\bar{\psi}$ fields. The potential term in the Lagrangian decomposes according to the number of fermions involving the following individual contributions,

$$\langle Q_{0+}^{(2)}, Q_{0-}^{(2)} \rangle = - \langle \bar{\psi}_2 \psi_4 \rangle \langle \psi_1 \psi_3 \rangle - \langle \bar{\psi}_1 \psi_3 \rangle \langle \psi_2 \psi_4 \rangle$$

$$+ \langle \bar{\psi}_2 \psi_3 \rangle \langle \psi_1 \psi_4 \rangle + \langle \bar{\psi}_1 \psi_4 \rangle \langle \psi_2 \psi_3 \rangle$$

$$\langle Q_{(-1/2)+}^{(3)}, Q_{(1/2)-}^{(3)} \rangle = - \frac{4}{9} \langle \bar{\psi}_2 \psi_3 \psi_4 \rangle \langle \psi_1 \psi_3 \psi_4 \rangle - \langle \bar{\psi}_2 \psi_3 \psi_4 \rangle \langle \psi_1 \psi_2 \psi_4 \rangle$$

$$\langle Q_{(-1)+}^{(4)}, Q_{(1)-}^{(4)} \rangle = \frac{1}{9} \langle \bar{\psi}_1 \psi_3 \bar{\psi}_4 \psi_1 \psi_2 \psi_4 \rangle$$

(61)

The Lagrangian density with normalized coupling constants\footnote{Coupling constants $g$ and $\mu$ may be introduced by $W_{1/2} \rightarrow g W_{1/2}$ and $E_{\pm} \rightarrow \mu E_{\pm}$} then becomes

$$- \frac{2\pi}{k} L_{p=2} \psi_1 \partial_+ \psi_3 - \bar{\psi}_1 \partial_- \bar{\psi}_3 + \psi_2 \partial_+ \psi_4 - \bar{\psi}_2 \partial_- \bar{\psi}_4$$

$$- \bar{\psi}_1 \psi_3 + \bar{\psi}_3 \psi_1 - \bar{\psi}_2 \psi_4 + \bar{\psi}_4 \psi_2 + \bar{\psi}_2 \psi_4 \psi_1 \psi_3$$

$$+ \psi_1 \psi_3 \psi_2 \psi_4 - \bar{\psi}_2 \psi_3 \bar{\psi}_4 \psi_1 \psi_3$$

$$- \frac{4}{9} \bar{\psi}_1 \psi_3 \psi_2 \psi_4 \psi_1 \psi_2 \psi_4 + \frac{4}{9} \bar{\psi}_2 \psi_3 \bar{\psi}_4 \psi_1 \psi_2 \psi_4$$

$$- \frac{1}{9} \bar{\psi}_1 \psi_3 \bar{\psi}_4 \psi_1 \psi_2 \psi_4$$

(62)
which can be put in the Dirac form

\[ L = \bar{\Psi}_D (i\gamma^\mu \partial_\mu - 1) \Psi_D + \bar{\Phi}_D (i\gamma^\mu \partial_\mu - 1) \Phi_D \]
\[ + \frac{5}{4} (\bar{\Psi}_D \gamma^\mu \Psi_D) (\bar{\Phi}_D \gamma_\mu \Phi_D) + \frac{5}{4} (\bar{\Psi}_D \Psi_D) (\bar{\Phi}_D \Phi_D) \]
\[ - \frac{5}{4} (\bar{\Psi}_D \gamma^5 \Psi_D) (\bar{\Phi}_D \gamma^5 \Phi_D) - \frac{1}{9} (\bar{\Phi}_D \gamma_\mu \Phi_D) (\bar{\Phi}_D \phi_\mu \Phi_D) (\bar{\Psi}_D \Psi_D) \]
\[ + \frac{1}{144} (\bar{\Phi}_D \gamma^\mu \Phi_D) (\bar{\Phi}_D \gamma_\mu \Phi_D) (\bar{\Psi}_D \gamma^\nu \Psi_D) \]

where the complex Dirac spinor components are $\Psi_{D1} = i\psi_3$, $\Psi_{D2} = -\bar{\psi}_3$, $\Psi_{D1}^+ = -\bar{\psi}_1$, $\Psi_{D2}^- = i\bar{\psi}_1$ and $\Phi_{D1} = i\psi_4$, $\Phi_{D2} = -\bar{\psi}_4$, $\Phi_{D1}^+ = -\bar{\psi}_2$, $\Phi_{D2}^- = i\bar{\psi}_2$.

5.1 Zero Curvature Representation and Integrability Conditions

We now discuss the integrability conditions of the class of models constructed so far. Inspired by the Lax form when $Q^{(2)}_{0\pm} = 0$ in (28), propose the following structure

\[ \tilde{L}_{+1} = \partial_+ + E_+ + A_{1/2} + XQ^{(2)}_{0\pm} \]
\[ \tilde{L}_{-1} = \partial_- - E_- - J_{-1/2} + YQ^{(2)}_{0\pm} \]

where $A_{1/2}$, $J_{-1/2}$, $Q^{(2)}_{0\pm}$ are defined in terms of $W_{\pm 1/2}$ as in (27) and (68), $X$ and $Y$ are parameters to be adjusted. The zero curvature condition (29) yields the following equations,

\[ \partial_- A_{1/2} = [E_+, J_{-1/2}] + Y[A_{1/2}, Q^{(2)}_{0\pm}] \]
\[ \partial_+ J_{-1/2} = -[A_{1/2}, E_-] - X[Q^{(2)}_{0\pm}, J_{-1/2}] \]

\[ X\partial_+ Q^{(2)}_{0\pm} + Y\partial_- Q^{(2)}_{0\pm} = -[A_{1/2}, J_{-1/2}] + XY[Q^{(2)}_{0\pm}, Q^{(2)}_{0\pm}] \]

Consider now the equations of motion derived from the Lagrangian (68) when $Q^{(3)}_{(\pm 1/2)\pm} = Q^{(4)}_{(\pm 1)} = 0$,

\[ \partial_- A_{1/2} = -[E_+, J_{-1/2}] + [A_{1/2}, Q^{(2)}_{0\pm}] \]
\[ \partial_+ J_{-1/2} = -[A_{1/2}, E_-] + [Q^{(2)}_{0\pm}, J_{-1/2}] \]

from where we derive

\[ \partial_- W_{-1/2} = J_{-1/2} + [W_{-1/2}, Q^{(2)}_{0\pm}] \]
\[ \partial_+ W_{1/2} = A_{1/2} - [W_{1/2}, Q^{(2)}_{0\pm}] \]

Using the the definition of $Q^{(2)}_{0\pm}$, we find after employing eqns. (67) and (68) and Jacobi identities,

\[ Y\partial_+ Q^{(2)}_{0\pm} + X\partial_- Q^{(2)}_{0\pm} = (X + Y) ([A_{1/2}, J_{-1/2}] + 2[Q^{(2)}_{0\pm}, Q^{(2)}_{0\pm}]) \]
where we have used the fact that, for \(g = sl(p,1), [J_{-1/2}, W_{-1/2}] \in \mathcal{K}\) and 
\([A_{1/2}, W_{1/2}] \in \mathcal{K}\) and therefore 
\([E_+, [J_{-1/2}, W_{-1/2}]] = [E_-, [A_{1/2}, W_{1/2}]] = 0\).
Comparing (69) with the l.h.s. of (66) we find compatibility between zero curvature representation and equations of motion if
\[x + y = 1\] and
\[Q_{0+} = Q_{0-} = 0.\] (70)
Notice that the Lax (63), \(\tilde{L}_{\pm 1}\) coincide with \(L_{\pm 1}\) of eqns. (4) and (5) when \(B = I, A_0 = 0\) for \(X = 1, Y = 0\). Other solutions of \(X + Y = 1\) correspond to different models related by non local gauge transformations involving \(Q_{0+}^{(2)} \in \mathcal{K}\).

5.2 Examples
We now consider some explicit examples where \(Q_{(\mp 1/2)}^{(3)} = Q_{(\mp 1)}^{(4)} = 0\) constructed by imposing the integrability conditions (70) to the basic model (63).
In order to fulfill the integrability conditions (70) let us consider the following different cases by taking for instance \(X = Y = 1/2\):

5.2.1 Constrained Bukhvostov-Lipatov Model
Take the constraint
\[\psi_2 \psi_3 = \psi_1 \psi_4 = 0, \quad \bar{\psi}_2 \bar{\psi}_3 = \bar{\psi}_1 \bar{\psi}_4 = 0\] (71)
Under such conditions we find from eqns. (60) and (61),
\[Q_{0-}^{(2)} = \psi_2 \psi_3 h_1 + (\psi_1 \psi_3 + \psi_2 \psi_4) h_2\]
\[Q_{0+}^{(2)} = \bar{\psi}_2 \bar{\psi}_3 h_1 + (\bar{\psi}_1 \bar{\psi}_3 + \bar{\psi}_2 \bar{\psi}_4) h_2\] (72)
and
\[\langle Q_{0+}^{(2)}, Q_{0-}^{(2)} \rangle = -\bar{\psi}_2 \bar{\psi}_3 \psi_1 \psi_3 - \bar{\psi}_1 \bar{\psi}_3 \psi_2 \psi_4.\] (73)
The Lagrangian density becomes [16],
\[L = \bar{\Psi}_D (i \gamma^\mu \partial_\mu - 1) \Psi_D + \bar{\Phi}_D (i \gamma^\mu \partial_\mu - 1) \Phi_D\]
\[+ \frac{5}{4} (\bar{\Psi}_D \gamma^\mu \Psi_D)(\bar{\Phi}_D \gamma_\mu \Phi_D)\] (74)
with fields satisfying constraint (71).

5.2.2 Thirring Model
Take now a solution of (71), namely,
\[\psi_1 = -\psi_4, \quad \psi_2 = -\psi_3, \quad \bar{\psi}_1 = \bar{\psi}_4, \quad \bar{\psi}_2 = \bar{\psi}_3\] (75)
\[ Q^{(2)}_{0-} = \bar{\psi}_2 \psi_4 \ h_1, \quad Q^{(2)}_{0+} = \bar{\psi}_2 \bar{\psi}_4 \ h_1 \]  

yielding the Lagrangian density

\[ L = \bar{\Psi}_D (i\gamma^\mu \partial_\mu - 1) \Psi_D + (\bar{\Psi}_D \gamma^\mu \Psi_D)(\bar{\Psi}_D \gamma^\mu \Psi_D) \]  

which correspond to the Thirring model.

**5.2.3 Pseudo scalar, massless Gross-Neveu model**

Consider the constraint

\[ \bar{\psi}_2 = \bar{\psi}_4, \quad \bar{\psi}_1 = \bar{\psi}_3, \quad \psi_2 = \psi_4, \quad \psi_1 = \psi_3 \]  

(78)

giving

\[ Q^{(2)}_{0-} = -\psi_1 \psi_2 \ h_1, \quad Q^{(2)}_{0+} = -\bar{\psi}_1 \bar{\psi}_2 \ h_1 \]  

(79)

and therefore yielding the Gross-Neveu model,

\[ L = \bar{\Psi}_D (i\gamma^\mu \partial_\mu) \Psi_D + \bar{\Phi}_D (i\gamma^\mu \partial_\mu) \Phi_D - (\bar{\Psi}_D \gamma^5 \Psi_D)(\bar{\Phi}_D \gamma^5 \Phi_D) \]  

(80)

**5.2.4 Scalar, Massive Gross-Neveu model**

Consider

\[ \bar{\psi}_2 = \bar{\psi}_4, \quad \bar{\psi}_1 = \bar{\psi}_3, \quad \psi_2 = -\psi_4, \quad \psi_1 = -\psi_3 \]  

(81)

with

\[ Q^{(2)}_{0-} = \psi_1 \psi_2 \ (E_{\alpha_1} - E_{-\alpha_1}), \quad Q^{(2)}_{0+} = \bar{\psi}_1 \bar{\psi}_2 \ (E_{\alpha_1} - E_{-\alpha_1}) \]  

(82)

yielding the Gross-Neveu model,

\[ L = \bar{\Psi}_D (i\gamma^\mu \partial_\mu - 1) \Psi_D + \bar{\Phi}_D (i\gamma^\mu \partial_\mu - 1) \Phi_D - (\bar{\Psi}_D \Psi_D)(\bar{\Phi}_D \Phi_D) \]  

(83)

**6 Concluding remarks**

Based on the gauged WZNW model, we have established a general framework for constructing systematically the action for a class of \( N = 1, 2 \) supersymmetric relativistic integrable models of sinh(sine)-Gordon type. It is important to stress that the field content of the theory is established by the group theoretic structure of a coset \( G/K \) and the latter by the decomposition of an twisted affine super Kac-Moody algebra. Within this context, it would be interesting to uncover the algebraic structure of affine supersymmetric integrable models with higher supersymmetries, \( N > 2 \) as well as higher Toda models.

Moreover, the higher grading generalization of the Toda systems proposed by Gervais and Saveliev and its connection to matter fields arises naturally from the coset and the gauged WZNW structures.
Another important achievement of our formalism is the construction of pure fermionic theories by considering the coset \( sl(p, 1)/sl(p) \otimes U(1) \) where all bosonic fields lie in the maximal kernel subalgebra \( K = sl(p) \otimes U(1) \in sl(p, 1) \). General integrability conditions were discussed and explicit examples for \( p = 2 \) were constructed. The generalization for \( p > 2 \) would be interesting and would lead to other fermionic integrable theories.

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7 Appendix A: Invariance of the Toda action under the SUSY Flow.

Here we prove that the action (25) is invariant under the 1/2-grade flow (9) generated by the element \( D^{1/2} \in K \), namely

\[
\begin{align*}
\partial_{1/2}J_{-1/2} &= \left[ E_-, B^{-1} D^{1/2} B \right] \\
\partial_-(\partial_{1/2}BB^{-1}) &= \left[ D^{1/2}, BJ_{-1/2}B^{-1} \right] \\
\left[ E_+, \partial_{1/2}BB^{-1} \right] &= \left[ A_{1/2}, D^{1/2} \right] \\
\partial_{1/2}A_{1/2} &= \left[ \partial_{1/2}BB^{-1}, A_{1/2} \right] - \left[ \partial_+ BB^{-1}, D^{1/2} \right],
\end{align*}
\]

i.e., \( \partial_{1/2}S_{\text{Toda}} = 0 \). We will assume that (39) holds. This allows to rewrite (84) as \( \partial_{1/2}BB^{-1} = [D^{1/2}, W_{-1/2}] \) which is \( \Lambda \)-independent. The relevant transformations are then given by

\[
\begin{align*}
\partial_{1/2}J_{-1/2} &= \left[ E_-, B^{-1} D^{1/2} B \right] \\
\partial_{1/2}BB^{-1} &= \left[ D^{1/2}, W_{-1/2} \right] \\
\partial_{1/2}A_{1/2} &= \left[ \left[ D^{1/2}, W_{-1/2} \right], A_{1/2} \right] - \left[ \partial_+ BB^{-1}, D^{1/2} \right].
\end{align*}
\]

An arbitrary variation of the action (25) is given by

\[
\frac{2\pi}{k} \delta S_{\text{Toda}} = \int \left\{ \left( \delta BB^{-1} \right) \left\{ \partial_- \left( \partial_+ BB^{-1} \right) - E_+ \right\} \right. - \left[ E_+, BE_- B^{-1} \right] - \left[ A_{1/2}, BJ_{-1/2}B^{-1} \right] \right\} + \\
+ \left( \delta J_{-1/2} \left\{ \partial_+ W_{-1/2} - B^{-1} A_{1/2} B \right\} \right) + \left( \delta A_{1/2} \left\{ - \partial_- W_{-1/2} + BJ_{-1/2}B^{-1} \right\} \right),
\]

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taking $\delta \to \partial_{1/2}$ and using (86) we have
\[
\frac{2\pi}{R} \partial_{1/2} S_{\text{Toda}} = \int \left\langle \left\{ D^{1/2}, W_{-1/2} \right\} \left\{ \partial_+ (\partial_+ BB^{-1}) - [E_+, BE_B^{-1}] - [A_{1/2}, BJ_{-1/2}B^{-1}] \right\} \right\rangle + \\
+ \left\langle \left\{ E_- B^{-1} D^{1/2} B \right\} \left\{ \partial_+ W_{1/2} - B^{-1} A_{1/2} B \right\} \right\rangle + \\
+ \left\langle \left\{ \left[D^{1/2}, W_{-1/2} \right], A_{1/2} \right\} - \left\{ \partial_+ BB^{-1}, D^{1/2} \right\} \right\rangle \left\{ - \partial_+ W_{-1/2} + BJ_{-1/2} B^{-1} \right\} \right\rangle.
\]

Analising term by term, we find using the identity $\langle a [b, c] \rangle = \langle [a, b] c \rangle$ we obtain a total derivative
\[
\left\langle \left[D^{1/2}, W_{-1/2} \right] \partial_- (\partial_+ BB^{-1}) + \left[D^{1/2}, \partial_+ BB^{-1} \right] \partial_- W_{-1/2} \right\rangle = \left\langle \partial_- (\partial_+ BB^{-1}) \left[D^{1/2}, W_{-1/2} \right] \right\rangle + \left\langle (\partial_+ BB^{-1}) \left[D^{1/2}, \partial_- W_{-1/2} \right] \right\rangle = \partial_- \left\langle (\partial_+ BB^{-1}) \left[D^{1/2}, W_{-1/2} \right] \right\rangle.
\]

The term
\[
- \left\langle \left[D^{1/2}, W_{-1/2} \right] A_{1/2} \partial_- W_{-1/2} \right\rangle = - \left\langle \left[D^{1/2}, W_{-1/2} \right] A_{1/2} \partial_- W_{-1/2} \right\rangle = 0,
\]

since $[A_{1/2}, \partial_- W_{-1/2}] \in \mathcal{K}_B$, $[D^{1/2}, W_{-1/2}] \in \mathcal{M}_B$ and the orthogonality $\langle \mathcal{K}_B, \mathcal{M}_B \rangle = 0$

The term
\[
- \left\langle \left[D^{1/2}, W_{-1/2} \right] A_{1/2}, BJ_{-1/2} B^{-1} \right\rangle + \left\langle \left[D^{1/2}, W_{-1/2} \right] A_{1/2}, BJ_{-1/2} B^{-1} \right\rangle = 0
\]
is also zero because of the identity $\langle a [b, c] \rangle = \langle [a, b] c \rangle$.

For the term
\[
\left\langle - \left[D^{1/2}, W_{-1/2} \right] E_+, BE_B^{-1} \right\rangle - \left[D^{1/2}, E_- B^{-1} \right] B^{-1} A_{1/2} B \right\rangle = -2 \left\langle \left[D^{1/2}, W_{-1/2} \right] E_+, BE_B^{-1} \right\rangle = -2 \left\langle E_- B^{-1} \left[D^{1/2}, A_{1/2} \right] B \right\rangle
\]
we use the Jacobi identity $[E_+, [BE_B^{-1}, D^{1/2}]] = - [D^{1/2}, [E_+, BE_B^{-1}]]$.

We also have
\[
\left\langle \left[D^{1/2}, W_{-1/2} \right] E_+, BE_B^{-1} \right\rangle - \left[D^{1/2}, \partial_+ BB^{-1} \right] BJ_{-1/2} B^{-1} \right\rangle = - \partial_+ \left\langle \left(B^{-1} D^{1/2} B \right) J_{-1/2} \right\rangle + 2 \left\langle BJ_{-1/2} B^{-1} \left[D^{1/2}, \partial_+ BB^{-1} \right] \right\rangle.
\]

Then, we have that the remaining sum
\[
- \partial_+ \left\langle \left(B^{-1} D^{1/2} B \right) J_{-1/2} \right\rangle + 2 \left\langle BJ_{-1/2} B^{-1} \right\rangle = 2 \left\langle D^{1/2} B \partial_+ J_{-1/2} B^{-1} \right\rangle + 2 \left\langle D^{1/2} \partial_+ BB^{-1} \right\rangle BJ_{-1/2} B^{-1} \right\rangle
\]
is total derivative because of the equations of motion for the field $J_{-1/2}$ and the identity $\langle a [b, c] \rangle = \langle [a, b] c \rangle$. The result then follows.
8 Appendix B: Usefull quantities.

Here we present the the structure of some of the Lie superalgebras used above. The simple root system can be constructed from the following basis \((e_i, \zeta_k)\) of a real pseudo-euclidean \((m + n)\) -dimensional space:

\[
e_i e_j = \delta_{ij}, \quad \zeta_k \zeta_l = -\delta_{kl}, \quad e_i \zeta_k = 0,
\]

The non-degenerate Cartan matrix is given by \(K_{ij} = (\alpha_i, \alpha_j)\). The commutation relations are

\[
[h_i, E_{\pm \alpha_j}] = \pm K_{ij} E_{\pm \alpha_j},
\]

\[
[E_{\alpha_i}, E_{-\alpha_j}] = \delta_{ij} h_i,
\]

\[
[h_i, h_j] = 0 .
\]

We will write explicitly the representation matrices only for \(sl(2,1)\). For the other ones a similar obvious construction holds.

8.1 The \(sl(2,1)\) superalgebra.

This algebra can be represented by \(3 \times 3\) matrices with 4 Bosonic elements:

\[
h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

\[
E_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{-\alpha_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

and 4 the Fermionic ones:

\[
E_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{-\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\]

\[
E_{\alpha_1+\alpha_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{-\alpha_1-\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}
\]

The simple roots are

\[
\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - \zeta_1
\]

In extending the Lie super algebra \(sl(2,1)\) to a twisted affine structure we assign to each generator a loop index \(n\) which may be either, integer \(n \in Z\) or semi-integer \(n \in Z + 1/2\). Let \(Q = 2d + \frac{1}{2}h_1\) be the grading operator and consider

\[
E_\pm = h_1^{(1/2)} + 2h_2^{(1/2)} - (E_{\alpha_1} + E_{-\alpha_1})
\]
When considering the affine structure of the $sl(2,1)$ super Kac-Moody algebra we find that there are two affine twisted subalgebras solving the locality conditions (24), namely,

- $sl(2,1)^{(2)}_{[1]}$

\[
\begin{align*}
K_{\text{Bose}} &= \{ K_1^{(2n+1)} = -(E_{\alpha_1}^{(n)} + E_{-\alpha_1}^{(n+1)}), \\ K_2^{(2n+1)} = l_2 \cdot H^{(n+1/2)} \}, \\
M_{\text{Bose}} &= \{ M_1^{(2n+1)} = -E_{\alpha_1}^{(n)} + E_{-\alpha_1}^{(n+1)}, \\ M_2^{(2n)} = h_1^{(n)} \},
\end{align*}
\]

and

\[
\begin{align*}
K_{\text{Fermi}} &= \{ F_1^{(2n+3/2)} = (E_{\alpha_1+\alpha_2}^{(n+1/2)} - E_{\alpha_2}^{(n+1)}), \\ F_2^{(2n+1/2)} = -(E_{\alpha_1+\alpha_2}^{(n)} - E_{\alpha_2}^{(n+1/2)}) + (E_{-\alpha_1-\alpha_2}^{(n+1/2)} - E_{\alpha_2}^{(n)}), \\
M_{\text{Fermi}} &= \{ G_1^{(2n+1/2)} = (E_{\alpha_1+\alpha_2}^{(n)} + E_{\alpha_2}^{(n+1/2)}) + (E_{-\alpha_1-\alpha_2}^{(n+1/2)} + E_{\alpha_2}^{(n)}), \\ G_2^{(2n+3/2)} = -(E_{\alpha_1+\alpha_2}^{(n+1/2)} + E_{\alpha_2}^{(n+1)}) + (E_{-\alpha_1-\alpha_2}^{(n+1)} + E_{\alpha_2}^{(n+1/2)}) \},
\end{align*}
\]

- $sl(2,1)^{(2)}_{[2]}$

\[
\begin{align*}
K_{\text{Bose}} &= \{ K_1^{(2n+1)} = -(E_{\alpha_1}^{(n)} + E_{-\alpha_1}^{(n+1)}), \\ K_2^{(2n+1)} = l_2 \cdot H^{(n+1/2)} \}, \\
M_{\text{Bose}} &= \{ M_1^{(2n)} = -E_{\alpha_1}^{(n+1/2)} + E_{-\alpha_1}^{(n+1/2)}, \\ M_2^{(2n+1)} = h_1^{(n+1/2)} \},
\end{align*}
\]

and

\[
\begin{align*}
K_{\text{Fermi}} &= \{ F_1^{(2n+1/2)} = (E_{\alpha_1+\alpha_2}^{(n)} - E_{\alpha_2}^{(n+1/2)}) + (E_{-\alpha_1-\alpha_2}^{(n+1/2)} - E_{\alpha_2}^{(n)}), \\ F_2^{(2n+3/2)} = -(E_{\alpha_1+\alpha_2}^{(n+1/2)} - E_{\alpha_2}^{(n+1)}), \\
M_{\text{Fermi}} &= \{ G_1^{(2n+1/2)} = (E_{\alpha_1+\alpha_2}^{(n+1/2)} + E_{\alpha_2}^{(n+1)}) + (E_{-\alpha_1-\alpha_2}^{(n+1/2)} + E_{\alpha_2}^{(n+1)}), \\ G_2^{(2n+3/2)} = -(E_{\alpha_1+\alpha_2}^{(n+1/2)} + E_{\alpha_2}^{(n+1)}) + (E_{-\alpha_1-\alpha_2}^{(n+1)} + E_{\alpha_2}^{(n+1/2)}) \},
\end{align*}
\]

The algebra of such operators was given in the appendix A of ref [17].

### 8.2 The $sl(2,2)$ superalgebra.

This algebra can be represented by 8 Bosonic, \{ $h_1$, $h_2$, $h_3$, $E_{\pm \alpha_1}$, $E_{\pm \alpha_3}$ \} and 8 Fermionic, \{ $E_{\pm \alpha_2}$, $E_{\pm (\alpha_1+\alpha_2)}$, $E_{\pm (\alpha_2+\alpha_3)}$, $E_{\pm (\alpha_1+\alpha_2+\alpha_3)}$ \} generators. The simple roots are

\[ \alpha_1 = \epsilon_1 - \epsilon_2, \quad \alpha_2 = \epsilon_2 - \zeta_1, \quad \alpha_3 = \zeta_1 - \zeta_2. \]
The principal grading for the $\widehat{sl}(2|2)$ algebra is defined in terms of the operator

\[ Q = d + \frac{1}{2}(h_1 + h_3) \]  

(91)

and the grade one semisimple element $E$ is chosen as

\[ E_+ = (E^{(0)}_{\alpha_1} + E^{(2)}_{-\alpha_1}) + (E^{(2)}_{\alpha_3} + E^{(0)}_{-\alpha_3}) + I^{(1)} \]  

(92)

where is the identity, $I = h_1 + 2h_2 + h_3$. The odd (fermionic) part of the kernel of $E_+$ of grade $n + 1/2$ consists of $K_{Fermi} = \{ f_i^{(n+\frac{1}{2})}, i = 1, \ldots, 4, \ n \in \mathbb{Z} \}$, with

\[
\begin{align*}
  f_1^{(n+\frac{1}{2})} &= (E^{(n-\frac{1}{2})}_{\alpha_1+\alpha_2} + E^{(n+\frac{1}{2})}_{-\alpha_1-\alpha_2}) + (E^{(n+\frac{1}{2})}_{\alpha_3} + E^{(n-\frac{1}{2})}_{-\alpha_3}), \\
  f_2^{(n+\frac{1}{2})} &= (-E^{(n-\frac{1}{2})}_{\alpha_1+\alpha_2} + E^{(n+\frac{1}{2})}_{\alpha_1-\alpha_2}) + (-E^{(n+\frac{1}{2})}_{\alpha_3} + E^{(n-\frac{1}{2})}_{-\alpha_3}), \\
  f_3^{(n+\frac{1}{2})} &= (E^{(n+\frac{1}{2})}_{\alpha_1+\alpha_2+\alpha_3} + E^{(n-\frac{1}{2})}_{\alpha_1-\alpha_2-\alpha_3}) + (E^{(n+\frac{1}{2})}_{\alpha_2} + E^{(n-\frac{1}{2})}_{-\alpha_2}), \\
  f_4^{(n+\frac{1}{2})} &= (-E^{(n+\frac{1}{2})}_{\alpha_1+\alpha_2+\alpha_3} + E^{(n-\frac{1}{2})}_{\alpha_1-\alpha_2-\alpha_3}) + (-E^{(n+\frac{1}{2})}_{\alpha_2} + E^{(n-\frac{1}{2})}_{-\alpha_2}),
\end{align*}
\]

(93)

while the bosonic part $K_{Base} = \{ K_i^{(n)}, i = 1, \ldots, 3, \ n \in \mathbb{Z} \}$ of the kernel $K$ of grade $n$ contains

\[ K_1^{(n)} = E^{(n-1)}_{\alpha_1} + E^{(n+1)}_{-\alpha_1}, \quad K_2^{(n)} = E^{(n+1)}_{\alpha_3} + E^{(n-1)}_{-\alpha_3}, \quad K_3^{(n)} = l^n I. \]

The fermionic part of the image $\mathcal{M}_{Fermi}$ of $E_+$ consists of

\[ \mathcal{M}_{Fermi} = \{ g_i^{(n+\frac{1}{2})}, i = 1, \ldots, 4, \ n \in \mathbb{Z} \} \]  

(94)

where

\[
\begin{align*}
  g_1^{(n+\frac{1}{2})} &= (E^{(n-\frac{1}{2})}_{\alpha_1+\alpha_2} + E^{(n+\frac{1}{2})}_{\alpha_1-\alpha_2}) - (E^{(n+\frac{1}{2})}_{\alpha_3} + E^{(n-\frac{1}{2})}_{-\alpha_3}), \\
  g_2^{(n+\frac{1}{2})} &= (-E^{(n-\frac{1}{2})}_{\alpha_1+\alpha_2} + E^{(n+\frac{1}{2})}_{\alpha_1-\alpha_2}) + (E^{(n+\frac{1}{2})}_{\alpha_3} - E^{(n-\frac{1}{2})}_{-\alpha_3}), \\
  g_3^{(n+\frac{1}{2})} &= (E^{(n+\frac{1}{2})}_{\alpha_1+\alpha_2+\alpha_3} + E^{(n-\frac{1}{2})}_{\alpha_1-\alpha_2-\alpha_3}) - (E^{(n+\frac{1}{2})}_{\alpha_2} + E^{(n-\frac{1}{2})}_{-\alpha_2}), \\
  g_4^{(n+\frac{1}{2})} &= (-E^{(n+\frac{1}{2})}_{\alpha_1+\alpha_2+\alpha_3} + E^{(n-\frac{1}{2})}_{\alpha_1-\alpha_2-\alpha_3}) + (E^{(n+\frac{1}{2})}_{\alpha_2} - E^{(n-\frac{1}{2})}_{-\alpha_2}),
\end{align*}
\]

(95)

There are four bosonic generators

\[
\begin{align*}
  M_1^{(n)} &= h_1^{(n)}, \quad M_2^{(n)} = -E^{(n-1)}_{\alpha_1} + E^{(n+1)}_{-\alpha_1}, \\
  M_3^{(n)} &= h_3^{(n)}, \quad M_4^{(n)} = -E^{(n+1)}_{\alpha_3} + E^{(n-1)}_{-\alpha_3},
\end{align*}
\]

in the image $\mathcal{M}$ of $E_+$. Note that $M_1^{(n)}$ and $M_3^{(n)}$ are in the Cartan subalgebra. This algebra can be represented by $4 \times 4$ block matrices and their structure is
given in [17]. Define

\[
F_1^{(n+1/2)} = \frac{1}{\sqrt{2}} (f_1^{(n+1/2)} + f_3^{(n+1/2)}), \quad F_2^{(n+1/2)} = \frac{1}{\sqrt{2}} (f_2^{(n+1/2)} + f_4^{(n+1/2)})
\]

\[
F_3^{(n+1/2)} = \frac{1}{\sqrt{2}} (f_1^{(n+1/2)} - f_3^{(n+1/2)}), \quad F_4^{(n+1/2)} = \frac{1}{\sqrt{2}} (f_2^{(n+1/2)} - f_4^{(n+1/2)})
\]

(96)

\[
G_1^{(n+1/2)} = \frac{1}{\sqrt{2}} (g_1^{(n+1/2)} + g_3^{(n+1/2)}), \quad G_2^{(n+1/2)} = \frac{1}{\sqrt{2}} (g_2^{(n+1/2)} + g_4^{(n+1/2)})
\]

\[
G_3^{(n+1/2)} = \frac{1}{\sqrt{2}} (g_1^{(n+1/2)} - g_3^{(n+1/2)}), \quad G_4^{(n+1/2)} = \frac{1}{\sqrt{2}} (g_2^{(n+1/2)} - g_4^{(n+1/2)})
\]

(97)

The four subalgebras which solve the locality conditions are defined by:

\[
sl(2, 2)^{(2)}_{[1]} = \begin{cases} 
K_1^{2n+1}, K_2^{2n+1}, K_3^{2n+1}, M_1^{2n}, \\
M_2^{2n+1}, M_3^{2n+1}, M_4^{2n+1}, F_1^{2n+3/2}, \\
F_2^{2n+1/2}, F_3^{2n+3/2}, F_4^{2n+1/2}, G_1^{2n+1/2}, \\
G_2^{2n+3/2}, G_3^{2n+1/2}, G_4^{2n+3/2}
\end{cases}
\]

(98)

\[
sl(2, 2)^{(2)}_{[2]} = \begin{cases} 
K_1^{2n+1}, K_2^{2n+1}, K_3^{2n+1}, M_1^{2n+1}, \\
M_2^{2n+1}, M_3^{2n+1}, M_4^{2n+1}, F_1^{2n+1/2}, \\
F_2^{2n+3/2}, F_3^{2n+1/2}, F_4^{2n+3/2}, G_1^{2n+1/2}, \\
G_2^{2n+3/2}, G_3^{2n+1/2}, G_4^{2n+3/2}
\end{cases}
\]

(99)

\[
sl(2, 2)^{(2)}_{[3]} = \begin{cases} 
K_1^{2n+1}, K_2^{2n+1}, K_3^{2n+1}, M_1^{2n}, \\
M_2^{2n+1}, M_3^{2n+1}, M_4^{2n+1}, F_1^{2n+1/2}, \\
F_2^{2n+3/2}, F_3^{2n+3/2}, F_4^{2n+1/2}, G_1^{2n+1/2}, \\
G_2^{2n+3/2}, G_3^{2n+3/2}, G_4^{2n+1/2}
\end{cases}
\]

(100)

\[
sl(2, 2)^{(2)}_{[4]} = \begin{cases} 
K_1^{2n+1}, K_2^{2n+1}, K_3^{2n+1}, M_1^{2n}, \\
M_2^{2n}, M_3^{2n}, M_4^{2n}, F_1^{2n+3/2}, \\
F_2^{2n+1/2}, F_3^{2n+1/2}, F_4^{2n+3/2}, G_1^{2n+1/2}, \\
G_2^{2n+3/2}, G_3^{2n+3/2}, G_4^{2n+1/2}
\end{cases}
\]

(101)

9 Appendix C: Vector, Scalar and Pseudo Scalar Currents

Here we give explicit relations of vector, scalar and pseudo scalar currents in terms of Fermi fields components parametrizing $W_{\pm 1/2}$.

\[
\Psi_D \Psi_D = \psi_1 \tilde{\psi}_3 + \tilde{\psi}_1 \psi_3
\]

\[
\Phi_D \Phi_D = \psi_2 \tilde{\psi}_4 + \tilde{\psi}_2 \psi_4
\]

(102)
\[ \Psi_D \gamma^5 \Psi_D = -\psi_1 \bar{\psi}_3 + \bar{\psi}_1 \psi_3 \]
\[ \Phi_D \gamma^5 \Phi_D = -\psi_2 \bar{\psi}_4 + \bar{\psi}_2 \psi_4 \]  
(103)

\[ \Psi_D \gamma^0 \Psi_D = -i\psi_1 \bar{\psi}_3 + i\bar{\psi}_1 \bar{\psi}_3 \]
\[ \Phi_D \gamma^0 \Phi_D = -i\psi_2 \bar{\psi}_4 + i\bar{\psi}_2 \bar{\psi}_4 \]
\[ \Psi_D \gamma^1 \Psi_D = i\psi_1 \bar{\psi}_3 - i\bar{\psi}_1 \bar{\psi}_3 \]
\[ \Phi_D \gamma^1 \Phi_D = -i\psi_2 \bar{\psi}_4 + i\bar{\psi}_2 \bar{\psi}_4 \]  
(104)

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