A mathematical solution of the optimum takeoff angle in long jump

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Abstract

We successfully derive the optimum takeoff angle in long jump based on the maximization problem of the flight distance. The takeoff model proposed here includes three parameters: the horizontal speed of the centre of mass, the takeoff speed and the takeoff angle. The optimum takeoff angle is determined explicitly as a function of these parameters. The estimation of the optimum angle shows good agreement with the measured data of Mike Powell and Carl Lewis, which demonstrate the effectiveness of the present model and its solution.

Keywords: long jump; takeoff; optimum angle; mechanical model; three degrees of freedom; maximization problem; analytical solution; projectile; flat aerial path

1. Introduction

Long jump consists of four phases; namely, run-up, takeoff, aerial and landing phases. The three phases except the run-up affect the total distance of the jump. In other words, the total distance \( L \) is divided into three parts of \( L_1 \), \( L_2 \) and \( L_3 \) [1]. The first part \( L_1 \) comes from the takeoff phase, and it is the horizontal distance between the front edge of a takeoff board and the centre of mass of a jumper. The second part \( L_2 \) is the horizontal distance of the aerial phase in which the jumper’s centre of mass moves in the air. The landing phase generates the last part \( L_3 \) that is the horizontal distance between the jumper’s centre of mass and the landing position.

The distances \( L_1 \) and \( L_3 \) are basically determined from the physical size of a jumper, and the sum of the distances is approximately 10% of the total distance \( L \). In contrast, the distance \( L_2 \) occupies 90% of the total distance [1]. This means that the success of long jump depends on the aerial phase and therefore this phase is the most significant factor for improving the record of long jump. For simplicity, we call the distance \( L_2 \) the flight distance hereafter.

During the aerial phase, forces acting on a jumper are gravity and air resistance. Then, the motion of a jumper in this phase can be described by the motion equation of a point particle and the jumper’s centre of mass is moving.
along the trajectory of a projectile [2]. This fact suggests that there exists the optimum angle of launching velocity. The takeoff with a launching velocity of this angle brings the longest flight distance in the same launching speed.

It is known well that the optimum angle of a projectile is 45° in a vacuum. However, it is impossible for long jumpers to take off at such a large angle. Long jumpers utilize the kinetic energy of run-up in order to take off with large launching velocity. The process of takeoff converts the horizontal momentum in run-up into the momentum in launching direction. This conversion determines the launching angle.

In fact, most jumpers use the angles of 20°–30° at their takeoff. A crude estimation with horizontal and vertical speeds qualitatively explains the reason why the launching angle in long jump lies in this range [3]. However, this estimation cannot lead to any quantitative results such as the flight distance and the optimum launching angle.

In order to clarify the optimum launching angle in long jump, several attempts have been achieved. For example, the energy conservation law at takeoff was utilized and a solution of the optimum angle was derived theoretically [4]. The result, however, showed rather large values of the launching angle.

As another approach, the launching speed was obtained as a function of the launching angle based on practical measurements [5]. It is true indeed this approach provided good results on the optimum launching angle, but it needs measured data and cannot clarify the mechanics of the takeoff process.

In the present paper, we attempt to mathematically derive a solution of the optimum launching angle in long jump based on the takeoff model that includes three parameters of horizontal and takeoff speeds and takeoff angle. We can regard this model as a link mechanism of human body with the least degrees of freedom. Based on this model, a maximization problem of the flight distance is formulated and the optimum takeoff angle is obtained in an analytical form. Also, the optimum angle is estimated from measured data in several cases.

2. Trajectory of aerial phase

In the aerial phase, the motion of the jumper’s center of mass is described by the motion equation of a particle,

\[ u = -\alpha u_0 \sqrt{u^2 + v^2}, \quad \dot{v} = -\alpha v_0 \sqrt{u^2 + v^2} - g, \]  

where \( u \) and \( v \) mean the horizontal and vertical velocity components of the center of mass, and \( \alpha \) indicates the drag coefficient per unit mass.

The exact solution of eq. (1) has not been obtained [6] and then we start our modeling with an approximated trajectory of eq. (1),

\[ y(x) = x \left( \frac{v}{u} + \frac{g}{2\alpha u} \right) - \frac{g}{4\alpha u} (e^{2\alpha x} - 1), \]

where \((u, v)\) denotes the launching velocity, and the origin of this expression is set to the position of the center of mass at takeoff.

In deriving eq. (2) from eq. (1), it is assumed that \(|v|\) is sufficiently smaller than \(|u|\) and therefore eq. (2) is called flat aerial path [7]. In order to obtain a more usef ul expression, we introduce another assumption that the drag coefficient \( \alpha \) is also sufficiently small. As shown later, the second assumption is really valid.

Under these assumptions, we can obtain a perturbed solution of the trajectory in the following form,

\[ y(x) = y_i(x) + \alpha x y_i(x) = x \left( \frac{v}{u} - \frac{g}{2u} x \right) - \alpha \frac{g}{3u^3} x^3. \]  

When the vertical displacement of the center of mass at landing is \( h (> 0) \), the equation \( y(X) = -h \) of the flight distance \( X \) is obtained and we have the following solution;

\[ X = X_0(1 + \alpha X_0), \quad X_0 = \frac{u}{g} \left( v + \sqrt{v^2 + 2gh} \right), \quad X_1 = -\frac{gX_0^2}{3u\sqrt{v^2 + 2gh}}. \]  

In order to confirm the accuracy of eq. (4), we estimate the flight distance based on some measured data shown in Table 1. In this table, WS means the data used by Ward-Smith in his paper [2]. Also, MP and CL denote the data measured in practical performance by Mike Powell and Carl Lewis in 1991 [8–9]. Some data with the symbol of * mean corrected values: the distances \( L* \) are modified from the actual values of \( L \) according to the wind effect [9]
while the vertical distances $h$ are estimated by the author from the measured data of the maximum height in aerial phase and the height at landing [10].

Table 1 Measured data of long jump for reference

|     | $L$ [m] | $L'$ [m] | $u_i$ [m/s] | $v_i$ [m/s] | $\alpha$ [1/m] | $h$ [m] |
|-----|--------|----------|-------------|-------------|----------------|--------|
| WS  | 8.03   | 8.03     | 10.10       | 2.87        | 0.003          | 0.50   |
| MP  | 8.95   | 8.92     | 9.27        | 4.26        | 0.012          | 0.61   |
| CL  | 8.91   | 8.62     | 9.11        | 3.37        | 0.011          | 0.78   |

The estimated results are summarized in Table 2. The distance $L_2$ is obtained from 90% of $L'$ in Table 1 and $X_{RK}$ indicates numerical evaluations with the Runge-Kutta method of eq. (1). In the case of WS, the agreement between $L_2$ and $X$ is excellent. In contrast, the comparison in the cases of MP and CL shows a difference in tens of centimeters in each case. However, the relative errors of the difference are 6% in MP and 2% in CL respectively, and it seems to be sufficiently small. Thus, the approximated distance $X$ of eq. (4) works well in estimating the flight distance in long jump.

Table 2 Estimation of the flight distance based on the data in Table 1

|     | $L_2$ [m] | $X_0$ [m] | $X$ [m] | $X_{RK}$ [m] |
|-----|----------|-----------|--------|--------------|
| WS  | 7.23     | 7.34      | 7.24   | 7.24         |
| MP  | 8.03     | 9.22      | 8.62   | 8.64         |
| CL  | 7.76     | 7.93      | 7.55   | 7.56         |

3. The optimum takeoff angle

Figure 1 shows the takeoff model of the present study and it is characterized by three parameters $V$, $w$ and $\psi$. The parameter $V$ means the horizontal speed of the centre of mass at takeoff and therefore represents the speed of the trunk of a jumper. The speed $w$ is named the takeoff speed and a push by a jumper’s leg generates it. The takeoff angle $\psi$ is defined here as the angle between $V$ and $w$. Thus, this takeoff model has three degrees of freedom and provides a link mechanism of human body with the least degrees of freedom. These parameters give the following expression of the launching velocity ($u_i$, $v_i$):

$$u_i = V + w \cos \psi, \quad v_i = w \sin \psi,$$

and we use the symbol $\theta$ for representing the angle of the launching velocity given as

$$\tan \theta = \frac{v_i}{u_i} = \frac{w \sin \psi}{V + w \cos \psi}.$$

Substituting the expression of eq. (5) into eq. (4) of the flight distance, we obtain the distance $X$ as a function of the takeoff angle $\psi$ instead of the launching velocity ($u_i$, $v_i$). This defines the following maximization problem of the flight distance:

$$\text{maximize } X(\psi) \quad \text{under the condition of } 0^\circ \leq \psi \leq 180^\circ \text{ for a given } w \text{ and } V.$$

The maximization problem (7) is equivalent to the following requirement

$$\frac{dX}{d\psi} = \frac{dX_0}{d\psi} = 0,$$
where the effect of air resistance to the angle is neglected here because it is of the order of 0.01° [10].

Substituting the distance \( X(\psi) \) into eq. (8) and after some manipulations, the requirement (8) reduces to the following cubic equation of \( \cos \psi \);

\[
2wV \cos^3 \psi + (2w^2 + V^2 + 2gh) \cos^2 \psi - (w^2 + 2gh) = 0. \tag{9}
\]

In order to investigate the solutions of eq. (9), we define the cubic function \( F(\cos \psi) \) that equals to the left hand side of eq. (9). Then, it is obvious that \( F(-\infty) = -\infty \) and \( F(+\infty) = +\infty \). Furthermore, the function \( F \) has the following properties;

\[
F(-\infty) = -(w^2 + 2gh) < 0 \quad F(+\infty) = (V + w)^2 > 0.
\]

These properties lead to the conclusion that eq. (9) has three solutions in real number, one of which has a positive value and the others have negative values. Also, since one of the negative solutions is smaller than \(-1\), it is obviously invalid as the present solution. Consequently, we have two solutions of eq. (9) in the range from \(-1\) to \(1\), and the positive one gives the optimum solution of the maximization problem (7).

In particular, it is noted that eq. (9) includes the special solution in \( V = 0 \). In this case, we have

\[
\cos \psi_{\text{opt}} = \frac{w^2 + 2gh}{2(w^2 + gh)}, \tag{10}
\]

which follows the well-known solution in a projectile;

\[
\tan \psi_{\text{opt}} = \frac{w}{w^2 + 2gh}. \tag{11}
\]

According to the formula of cubic equation [12], we can obtain the positive solution of eq. (9) as follows;

\[
\cos \psi_{\text{opt}} = \frac{1}{3} \left( \frac{V}{w} \right) \left( \frac{1}{2} \right)^{1/3} \left( \frac{w}{V} \right)^{2/3} \left[ \frac{gh}{V^2} + \cos \varphi \right]^{-1/3}, \tag{12}
\]

where

\[
\cos \varphi = -1 + \frac{27}{2} \left( \frac{w}{V} \right)^2 \left[ \frac{1}{2} \left( \frac{w}{V} \right)^2 + \frac{gh}{V^2} \right] \left( \frac{1}{2} \left( \frac{w}{V} \right)^2 + \frac{gh}{V^2} \right)^{1/3}. \tag{13}
\]

The present solution implies that two non-dimensional parameters \( w/V \) and \( gh/V^2 \) determine the optimum angle. The former is the ratio of the takeoff speed to the horizontal speed at takeoff while the latter means the ratio of the potential energy obtained in aerial phase to the kinetic energy in run-up. The optimum angle \( \psi_{\text{opt}} \) is shown in Fig. 2 as the function of these parameters. The angle \( \psi_{\text{opt}} \) monotonously decreases with increasing the two parameters.

4. Discussion

As shown in Fig. 1, the present takeoff model includes the parameters \( w \) and \( V \). The estimation of \( w \) and \( V \) is, therefore, inevitable in order to determine the optimum takeoff angle \( \psi_{\text{opt}} \). In evaluating the horizontal speed \( V \), we employ the fact that a run-up speed decelerates to approximately 80–90% of the full speed at just the moment of takeoff [1]. For example, world class athletes are able to run 100 m in 10 seconds, which leads to the estimation of \( V \approx 8.0–9.0 \) m/s.

On the other hand, no data on the takeoff speed \( w \) have been found, and thereby we introduce an assumption to this parameter. Eliminating the angle \( \psi \) in eq. (5), we have the following relation;

\[
w = \sqrt{(u_i - V)^2 + v_{i,0}^2}, \tag{14}
\]
which implies that the takeoff speed $w$ is evaluated from the launching velocity and the horizontal speed $V$.

Here, employing the expression (14), we show that the angle defined by eq. (12) is actually the optimum value. When measured data of the launching velocity $(u_i, v_i)$ and a value of the horizontal speed $V$ are given, eq. (14) provides the corresponding value of the takeoff speed $w$. Then, a value of $\psi$ in eq. (5) yields another launching velocity instead of the used launching velocity. This launching velocity and the vertical displacement at landing predict the flight distance $X$ according to eq. (4).

The results of $V=7.0$ m/s and $9.5$ m/s are shown in Fig. 3, in which the data of WS in Table 1 are used in the measured launching velocity and the vertical displacement at landing. Equation (12) gives the optimum angles of $61.6^\circ$ and $68.6^\circ$ to $V=7.0$ m/s and $9.5$ m/s, respectively. We can confirm in Fig. 3 that the maximum flight distance actually appears at the value in each case.

Now, we summarize the procedure for evaluating the optimum takeoff angle. As mentioned above, a takeoff speed $w$ is estimated from eq. (14) with values of the horizontal speed $V$ and the measured launching velocity $(u_i, v_i)$. Then, it is possible to obtain the optimum takeoff angle $\psi_{opt}$ in eqs. (12) and (13) with a value of the vertical displacement $h$. The values of $V$, $w$ and $\psi_{opt}$ determine the optimum launching velocity $(u_i, v_i)_{opt}$ which disagrees with the original launching velocity in general. Then, the optimum launching angle $\theta_{opt}$ is obtained from eq. (5), and consequently eq. (4) yields the optimum distance $X_{opt}$ in this case.

According to this procedure, we attempt to estimate the takeoff angle from the data of WS in Table 1. In order to clarify the feature of eq. (12), the horizontal speed $V$ is varied in the range from 0 to $u_i$. Also, eq. (5) defines the actual takeoff angle $\psi_{pr}$:

$$\tan \psi_{pr} = \frac{v_i}{u_i - V}, \quad (15)$$

which gives the measured launching velocity in a given value of $V$.

The result is shown in Fig. 4, in which the variation of the actual takeoff angle is also shown with a broken line. Since the velocity $u_i$ is equal to $w \cos \psi$ in the case of $V = 0$, the takeoff angle $\psi_{pr}$ is consistent with the launching angle $\theta$ and the value becomes $15.87^\circ$. Then, the optimum takeoff angle $\psi_{opt}$ has the value of $43.96^\circ$. In contrast, when $V$ agrees with $u_i$, $w$ is identical with $v_i$, and thus $\psi_{pr}$ becomes $90^\circ$. The value of $\psi_{opt}$ in this limit becomes $69.6^\circ$.

The curves of $\psi_{pr}$ and $\psi_{opt}$ increase monotonously with increasing the value of $V$, and the range of the former is larger than that of the latter: for example, $\psi_{pr}$ varies from $15.87^\circ$ to $90^\circ$ whereas $\psi_{opt}$ from $43.96^\circ$ to $69.6^\circ$. As the result, these curves has the intersection at $V = 8.9$ m/s ($\equiv V_c$). This means that if the horizontal velocity at takeoff was equal to this value in the practice, the jump was performed with the optimum takeoff angle.

Furthermore, the results of the optimum takeoff angle obtained from the data of MP and CL in Table 1 are summarized in Table 3. Here, we assume that $V$ varies from 7.0 m/s to 9.0 m/s. In this range of $V$, the optimum launching angles of M. Powell and C. Lewis are in the range from $19.6^\circ$ to $24.0^\circ$ and $16.3^\circ$ to $20.9^\circ$, respectively.
The measured angles of 24.6° and 20.3° lie on the upper limits of the range and this result supports the validity of the present optimum angle. The comparisons of the optimum launching angle $\theta_{opt}$ and the actual takeoff angle $\psi_{pr}$ show that the results of M. Powell and C. Lewis are different in each case. However, the range of the optimum takeoff angle $\psi_{opt}$ in both the results agrees with each other. Although the reason has not been evident yet, it is new finding and an interesting result of the present study.

Table 3 Estimated results on the optimum takeoff angle of M. Powell and C. Lewis based on the measured data in 1991.

|       | $\psi_{pr}$ [°] | $\psi_{opt}$ [°] | $\theta_{opt}$ [°] | $\theta$ [°] | $V_c$ [m/s] |
|-------|-----------------|------------------|-------------------|-------------|-------------|
| MP    | 61.9-86.4       | 60.0-64.5        | 24.0-19.6         | 24.6        | 6.8         |
| CL    | 57.9-88.1       | 59.9-64.6        | 20.9-16.3         | 20.3        | 7.2         |

5. Conclusion

In order to analytically obtain the optimum takeoff angle in long jump, a perturbed solution of the trajectory is derived from the flat aerial path that is an approximate solution of the motion equation with the quadratic resistance law. The approximate solution of the trajectory includes the effect of air resistance in the linear form. Since the effect of air resistance is sufficiently small in long jump, the comparison of this solution with measured data shows good agreement.

Based on the approximate trajectory, we formulate a maximization problem of the flight distance with a takeoff model. The present takeoff model includes three parameters: the horizontal speed of the center of mass ($V$), the takeoff speed (w) and the takeoff angle ($\psi$). As the solution of the maximization problem, the optimum takeoff angle is obtained analytically and given explicitly by a function of $w/V$ and $gh/V^2$.

The flight distance and the takeoff angle are evaluated using the measured data of M. Powell and C. Lewis. The evaluated distances are in sufficient agreement with the measured values. Also, the present solution of the optimum takeoff angle provides well-known values of the launching angle.

The comparisons of the evaluated results with the measured values demonstrate the effectiveness of the present approach and its solution. In particular, the present study suggests that the optimum range of the takeoff angle ($\psi_{opt}$) becomes the same in both the famous world-class jumpers, although the launching angles are different in their practical performance. It is new finding and an interesting result of the present study.

Finally, the following point should be noted. In the case that we are interested in the flight distance of long jump, the launching velocity at takeoff can provide the sufficient estimation. However, in the estimation of the optimum takeoff angle, further information on the takeoff phase (the horizontal and takeoff speeds in the present model) is necessary. This is the conclusion of the present study and the measurement of these speeds will make the estimation of the optimum takeoff angle more accurate.

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