Theory space of one unitary matrix model and its critical behavior associated with Argyres-Douglas theory

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Abstract

The lowest critical point of one unitary matrix model with cosine plus logarithmic potential is known to correspond with the \((A_1, A_3)\) Argyres-Douglas (AD) theory and its double scaling limit derives the Painlevé II equation with parameter. Here, we consider the critical points associated with all cosine potentials and determine the scaling operators, their vevs and their scaling dimensions from perturbed string equations at planar level. These dimensions agree with those of \((A_1, A_{4k-1})\) AD theory.

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1 Introduction

Matrix models are useful in the analysis of four dimensional supersymmetric gauge theory. The instanton partition function [1, 2] of the four dimensional $SU(2)$ linear quiver superconformal gauge theories can be identified with the conformal block of regular vertex operators in two dimensional conformal field theory by the AGT correspondence [3]. The integral representation of the conformal block [4] is the multi-Penner type $\beta$-deformed matrix model whose potential has logarithmic terms.

The simplest example is given by the four dimensional $\mathcal{N} = 2$ $SU(2)$ supersymmetric gauge theory with four hypermultiplets. The instanton partition function corresponds to a four point conformal block by the regular vertex operators whose integral representation is the three-Penner type matrix model [5, 6]. The instanton partition functions for the cases with less than four flavors which are asymptotically free are obtained by taking a degeneration limit of the regular vertex operators [7, 8]. As a result, an irregular conformal block is formed. The emergent matrix model contains rational terms [9, 10]. A similar limit can be taken for the four dimensional $\mathcal{N} = 2$ $SU(2)$ linear quiver superconformal gauge theories and we can obtain corresponding asymptotically free theories. The potentials of these matrix models contain rational terms with higher powers [9, 11]. For reviews, see [12, 13, 14].

Connection with integrable systems is understood better by considering a discrete Fourier transform of the instanton partition function at $\beta = 1$ [15] with regard to the filling fraction, which is the Coulomb moduli\(^1\). Connection between the tau function of Painlevé equations and the generating function of the instanton partition function of four dimensional $\mathcal{N} = 2$ $SU(2)$ gauge theory with $N_f = 0, \ldots, 4$ and that of Argyres-Douglas (AD) type theories [17] obtained in [18, 19] \(^2\) has been pointed out in [23, 24, 25, 26, 27]. Some of these cases can be derived by using matrix models [28, 29, 30]. In [28, 29, 31], it was shown that in the case of two flavors can be represented by the unitary matrix model of potential $U + U^\dagger + \log U$ type. This is a Gross-Witten-Wadia (GWW) model [32, 33, 34] with a logarithmic potential. The string equations, which are a set of difference equations arising from the recursion relations among orthogonal polynomials, have been shown to be the alternate discrete Painlevé II equations (alt-dPII)\(^3\). We have also shown that the partition function of this model is the

\(^1\)The generating function of the $q$-deformed matrix model has been considered in [16].
\(^2\)References in the higher rank case include [20, 21, 22].
\(^3\)It is also called the discrete Painlevé equation d-P($(2A_1)^{(1)}/D_6^{(1)}$) in [35].
tau function of alt-dPII equation which is closely related to the Painlevé III$_1$ equation. Using the partition function of this model, we have constructed the tau function of Painlevé III$_1$ equation. By taking the double scaling limit, alt-dPII equation turns into Painlevé II equation with accessory parameter$^4$.

Extension of GWW model plus a logarithmic potential is given by a potential with higher powers of $U$ and those of $U^\dagger$. It was considered in [37] without a logarithmic potential. (For the recent development about the phase structures of a generalized GWW model, see [38].) It is natural to identify the generating functional of this extended unitary matrix model with that of $\hat{A}_{2m,2n}$ [39, 40] asymptotically free theory. In this viewpoint, the multicritical points are AD points of various type. In fact, in [41], we have taken the double scaling limit at the next to the lowest order critical point and derived the system of differential equations associated with the $(A_1, A_7)$ AD theory.

In this paper we develop a further extension of this type. We will examine the planar critical behavior at various critical points. It is known that the critical points of the symmetric unitary matrix model are labeled by an integer $k$ [42, 43]. The analysis of critical behavior and the construction of the even type scaling operators which perturb these critical points are given there.$^5$ We take the viewpoint that the Coulomb branch operators of some AD theory corresponds to these scaling operators. Note that not all of the Coulomb branch operators are reproduced by [42, 43]. We will construct scaling operators of odd type and logarithmic type in addition to even type and derive their vevs, and determine their scaling dimensions from these. By comparing their dimensions with those of the Coulomb branch operator from AD theory, we give further evidence for the correspondence between the $k$-th multicritical point of unitary matrix model and the $(A_1, A_{4k-1})$ AD point.

The paper is organized as follows. In section 2, we first review the method of orthogonal polynomial for a unitary matrix model and explain how planar string equations are given. Then we derive an explicit form of these equations at the $k$-th multicritical point. We determine the vevs and their dimensions of scaling operators of various type. In the last section, we give a dictionary between the scaling operators at the $k$-th multicritical point and the Coulomb branch operators in the $(A_1, A_{4k-1})$ AD theory by comparing their scaling dimensions.

$^4$For an alternative approach based on the genus expansion of two-cut Hermitian cubic model, see [36].

$^5$See [44] for the hermitian matrix case.
2 One unitary matrix model

The partition function of the one unitary matrix model is defined by

\[ Z_U = \frac{1}{N!} \left( \prod_{i=1}^{N} \oint \frac{dz_i}{2\pi i z_i} \right) \Delta(z) \Delta(z^{-1}) \exp \left( \sum_{i=1}^{N} W(z_i) \right), \]  

(2.1)

where \( \Delta(z) \) is the Vandermonde determinant \( \Delta(z) = \prod_{i<j} (z_i - z_j) \) and \( W(z) \) is the potential

\[ W(z) = -\frac{1}{2g_s} \left[ \sum_{p=1}^{\infty} g^+_p \left( z^p + \frac{1}{z^p} \right) + g^-_p \left( z^p - \frac{1}{z^p} \right) \right]. \]  

(2.2)

When all \( g^-_p \) vanish, the model is called symmetric. We can perform the large \( N \) expansion for the free energy of (2.1):

\[ F \equiv \log Z_U = \sum_{g=0}^{\infty} N^{2-2g} F_g(\tilde{S}). \]  

(2.3)

The coefficients in the expansion are the function of the parameter \( \{g^+_p, g^-_p\} \) and the t’ Hooft coupling \( \tilde{S} \equiv g_s N \) which is fixed in the large \( N \) limit.

A way to evaluate the free energy is to resort to the method of orthogonal polynomials [42, 43, 45, 46, 47, 48, 49]. For a review in the hermitian case, see [50] and other approaches, in particular, based on Virasoro constraints, see [51, 52, 53, 54]. Let us introduce the set of monic orthogonal polynomials \( \{p_n(z), \tilde{p}_n(1/z)\} \). Their orthogonality condition with regard to the measure is

\[ \oint d\mu(z)p_n(z)\tilde{p}_m(1/z) = h_n \delta_{n,m}, \quad d\mu(z) \equiv \frac{dz}{2\pi i z} e^{W(z)}. \]  

(2.4)

Here, \( p_n(\tilde{p}_n) \) is the polynomial in \( z(z^{-1}) \) of degree \( n \)

\[ p_n(z) = z^n + \cdots + A_n, \quad \tilde{p}_n(1/z) = z^{-n} + \cdots + B_n. \]  

(2.5)

We have denoted the constant terms by \( A_n \equiv p_n(0), B_n \equiv \tilde{p}_n(0) \). They are related to \( h_n \) by

\[ \frac{h_n}{h_{n-1}} = 1 - A_n B_n. \]  

(2.6)

From the orthogonality condition (2.4), one can show that these polynomials obey the following recursion relations

\[zp_n(z) = p_{n+1}(z) - \sum_{k=0}^{n} h_n \frac{h_k}{h_{k+1}} A_{n+1} B_k \tilde{p}_k(z), \]  

(2.7)

\[ z^{-1} \tilde{p}_n(1/z) = \tilde{p}_{n+1}(1/z) - \sum_{k=0}^{n} h_n \frac{h_k}{h_{k+1}} A_k B_{n+1} \tilde{p}_k(1/z). \]  

(2.8)
Rewriting the Vandermonde determinant in (2.1) as
\[ \Delta(z) = \det(p_{j-1}(z_i))_{1 \leq i,j \leq N}, \quad \Delta(1/z) = \det(\tilde{p}_{j-1}(1/z_i))_{1 \leq i,j \leq N}, \]
and using the orthogonality, we have
\[ Z_U = \prod_{k=0}^{N-1} h_k = h_0^N \prod_{j=1}^{N-1} (1 - A_j B_j)^{N-j}. \]
Then, in terms of the coefficients in orthogonal polynomials, the free energy is written as
\[ F = N \log h_0 + \sum_{j=1}^{N-1} (N - j) \log(1 - A_j B_j). \]
In particular, the planar free energy \( F_0 \) can be evaluated as
\[ F_0 \sim \int_0^1 dx (1 - x) \log(1 - A(x)B(x)), \quad \frac{n}{N} \to x, \quad A_n \to A(x), \quad B_n \to B(x). \]
Here \( \sim \) means dropping the higher contributions in \( 1/N \).

### 2.1 string equation and \( k \)-th multicritical points

It is known that the parameter space of the symmetric unitary matrix model has a set of critical points which are labeled by an integer \( k \). At such points, the coefficients in (2.3) behave as
\[ F_g(S) \sim (\bar{S}_c^{(k)} - S)^{(2-\gamma)(2-2\varphi)/2}, \]
where \( \bar{S}_c^{(k)} \) is the \( k \)-th critical value of the t’ Hooft coupling and \( \gamma = -1/k \) is susceptibility. Therefore, around this point, we can also expand the free energy by sending the parameters to their critical values together with the large \( N \) limit
\[ F = \sum_{g=0}^{\infty} \kappa^{2g-2} f_g(c), \quad 1 - \bar{S}/\bar{S}_c^{(k)} = a^2 c, \quad \kappa^{-1} \equiv N a^{2-\gamma}. \]
Here \( a \) is the auxiliary parameter, and \( \kappa \) is kept fixed under \( a \to 0 \).

The free energy depends on the coefficients \( A_n \) and \( B_n \), which are controlled by the recursion relations called string equations. They are given by a set of identities
\[ 0 = \oint dz \frac{\partial}{\partial z} \left\{ \frac{z^k}{2\pi i} e^{W(z)} p_l(z) \tilde{p}_m(1/z) \right\}. \]
Therefore, the critical behavior of the free energy can be evaluated by solving a set of string equations at the critical point. In particular, the cases \((k, \ell, m) = (-1, n, n-1)\) and \((k, \ell, m) = (0, n, n)\) are important:

\[
\oint d\mu(z)W'(z)p_n(z)p_{n-1}(1/z) = n(h_n - h_{n-1}), \tag{2.16}
\]

\[
\oint d\mu(z)zW'(z)p_n(z)p_n(1/z) = 0. \tag{2.17}
\]

For later convenience, we introduce the new variables \(H_n, R_n\) and \(G_n\) by

\[
H_n = \sqrt{\frac{h_n}{h_{n-1}}}, \quad A_n = R_nD_n, \quad B_n = \frac{R_n}{D_n}, \quad \frac{D_{n+1}}{D_n} = 1 + G_n. \tag{2.18}
\]

In terms of these variables, (2.6) reads

\[
H_n^2 = 1 - R_n^2. \tag{2.19}
\]

Appropriate bases for (2.16) and (2.17) are respectively

\[
\mathcal{F}_p^\pm(H_m, G_n) \equiv \frac{1}{h_n} \oint d\mu(z) \left( z^{p-1} \mp \frac{1}{z^{p+1}} \right) p_n(z)p_{n-1}(1/z), \tag{2.20}
\]

\[
\mathcal{G}_p^\pm(H_m, G_n) \equiv \frac{1}{h_n} \oint d\mu(z) \left( z^p \mp \frac{1}{z^p} \right) p_n(z)p_n(1/z). \tag{2.21}
\]

Using these equations, (2.16) and (2.17) can be written as

\[
\sum_{n} n \mathcal{J}_n \equiv \frac{1 - R_n^2}{2R_n^2} \sum_{p=1}^{\infty} \left( g_p^+ \mathcal{F}_p^+(R, G) + g_p^- \mathcal{F}_p^-(R, G) \right), \tag{2.22}
\]

\[
0 \equiv \frac{1 - R_n^2}{2R_n^2} \sum_{p=1}^{\infty} \left( g_p^+ \mathcal{G}_p^+(R, G) + g_p^- \mathcal{G}_p^-(R, G) \right). \tag{2.23}
\]

Here we have used (2.19) to write the functions \(\mathcal{F}, \mathcal{G}\) in terms of \(R_n, G_n\).

To derive the explicit form of (2.20) and (2.21), we follow [43] with small modification. Let us define the normalized orthogonal polynomial

\[
P_n(z) \equiv \frac{1}{\sqrt{h_n}} p_n(z), \quad \tilde{P}_n(1/z) \equiv \frac{1}{\sqrt{h_n}} \tilde{p}(1/z). \tag{2.24}
\]

They are orthonormal with respect to \(d\mu(z)\)

\[
\oint d\mu(z)P_n(z)\tilde{P}_m(1/z) = \delta_{n,m}. \tag{2.25}
\]
Let us introduce the operators \(\hat{\ell}\) and \(\hat{u}\) by
\[
\hat{\ell} P_n(z) =nP_n(z), \quad \hat{u} P_n(z) = P_{n+1}(z).
\] (2.26)

Let us write \(z\) and \(1/z\) in terms of \(\hat{\ell}\) and \(\hat{u}\). From (2.7) and (2.8), we obtain
\[
\oint d\mu P_m(z) \tilde{P}_n(1/z) z
= \begin{cases} 
H_{m+1} & n = m + 1 \\
-R_{m+1} R_m (1 + G_m) & n = m \\
-R_{m+1} H_m (1 + G_m) \cdots H_{n+1} (1 + G_{n+1}) R_n (1 + G_n) & n < m 
\end{cases},
\] (2.27)
\[
\oint d\mu P_m(z) \tilde{P}_n(1/z) z^{-1}
= \begin{cases} 
H_m & n = m - 1 \\
-R_{m+1} R_m (1 + G_m)^{-1} & n = m \\
-R_m (1 + G_m)^{-1} H_{m+1} (1 + G_{m+1})^{-1} \cdots H_n (1 + G_n)^{-1} R_{n+1} & n > m 
\end{cases}.
\] (2.28)

Therefore, denoting \(H(n) = H_n\), ... etc, we have
\[
z = H(\hat{\ell}) \hat{u} - R(\hat{\ell}) (1 + G(\hat{\ell})) \hat{u}^{-1} \frac{1}{1 - H(\hat{\ell}) (1 + G(\hat{\ell})) \hat{u}^{-1} R(\hat{\ell})} \hat{u},
\] (2.29)
\[
\frac{1}{z} = \hat{u}^{-1} H(\hat{u}) - \hat{u}^{-1} R(\hat{\ell}) \hat{u} \frac{1}{1 - H(\hat{\ell}) (1 + G(\hat{\ell}))^{-1} \hat{u}^{-1} 1 + G(\hat{\ell})}.
\] (2.30)

We are interested in the planar limit where the coefficients \(H_n, R_n\) and \(G_n\) turn into the continuous function \(H_n \rightarrow H(x), R_n \rightarrow R(x), G_n \rightarrow G(x), n/N \rightarrow x\). The operators \(z\) and \(1/z\) in this limit become respectively
\[
z = -\frac{1 + G(x) - H(x) u}{1 - H(x) (1 + G(x)) u^{-1}}, \quad \frac{1}{z} = -\frac{(1 + G(x))^{-1} - H(x) u^{-1}}{1 - H(x) (1 + G(x))^{-1} u}.
\] (2.31)

Substitute these representations into (2.20) and (2.21), we obtain the planar form of \(\mathcal{F}_p^\pm (R, G)\).
and \( G_p^\pm (R, G) \) by extracting appropriate coefficient in \( u \):

\[
\mathcal{F}_p^+(H, G) = \frac{1}{H} \mathcal{F}_p(H, G) = \frac{1}{H} \int d\mu(z) \left( z^{p-1} - \frac{1}{z^{p+1}} \right) P_n(z) \bar{P}_{n-1}(1/z) \\
- \frac{1}{H} \left\{ \left( \frac{1 + G - Hu}{1 - H(1+G)u^{-1}} \right)^{p-1} - \left( \frac{(1 + G)^{-1} - Hu^{-1}}{1 - H(1+G)^{-1}u} \right)^{p+1} \right\}_{u^{-1}} \\
= (1 - H^2) \left( (1 + G)^p \sum_{n=0}^{p-1} (-1)^{p+n-1} \frac{\Gamma(p + n + 1)}{p \Gamma(n+1) \Gamma(n+2) \Gamma(p-n-1)} H^{2n} \\
+ \frac{1}{(1+G)^p} \sum_{n=0}^{p-1} (-1)^{p+n-1} \frac{\Gamma(p + n + 2)}{p \Gamma(n+1) \Gamma(n+2) \Gamma(p-n)} H^{2n} \right), \quad (2.32)
\]

\[
\mathcal{F}_p^-(H, G) = (1 - H^2) \left( (1 + G)^p \sum_{n=0}^{p-1} (-1)^{p+n-1} \frac{\Gamma(p + n + 1)}{p \Gamma(n+1) \Gamma(n+2) \Gamma(p-n-1)} H^{2n} \\
- \frac{1}{(1+G)^p} \sum_{n=0}^{p-1} (-1)^{p+n-1} \frac{\Gamma(p + n + 2)}{p \Gamma(n+1) \Gamma(n+2) \Gamma(p-n)} H^{2n} \right), \quad (2.33)
\]

and

\[
G_p^\pm (H, G) = \left\{ (1 + G)^p \mp \frac{1}{(1+G)^p} \right\} (1 - H^2) \sum_{n=0}^{p-1} (-1)^{p+n} \frac{\Gamma(p + n + 1)}{p \Gamma(n+1) \Gamma(n+2) \Gamma(p-n)} H^{2n}. \quad (2.34)
\]

For later convenience, let us write the planar string equations as

\[
\vec{s}(x) = \left( \frac{1 - R^2}{2R^2} \left( \sum_{p=1}^{\infty} \sum_{i=+,-} g_p^i \mathcal{F}_p^i(R, G) \right), \frac{1 - R^2}{2R^2} \left( \sum_{p=1}^{\infty} \sum_{i=+,-} g_p^i \mathcal{G}_p^i(R, G) \right) \right)^t, \quad (2.35)
\]

where \( \vec{s}(x) = (\vec{S} x, 0)^t \).

Let us move on to study critical points of this model. While we do not give the derivation [43], the \( k \)-th multicritical potential is given by

\[
W^{(k)}(z) = -\frac{1}{2g} \sum_{p=1}^{k} t_p^{(k)} \left( z^{p} + \frac{1}{z^{p}} \right), \quad (2.36)
\]

where the critical values of the couplings at this critical point are

\[
g_p^+ \big|_{\text{crit}} = t_p^{(k)} = (-1)^{p+1} \frac{\Gamma(k) \Gamma(k+2)}{\Gamma(k-p+1) \Gamma(k+p+1)}, \quad 1 \leq p \leq k \quad (2.37)
\]
and zero otherwise. \(^6\)

The set of string equations can be expanded around \(R = 0\) and \(G = 0\) as

\[
\vec{s}(x) = \sum_{n=0}^{\infty} \sum_{r+s=n} \frac{R^n G^s}{r!s!} \times \partial_R^r \partial_G^s \left( \frac{1 - R^2}{2R^2} \left( \sum_{p=1}^{\infty} \sum_{i=+,-} g_p^i \mathcal{F}_p^i(R, G) \right), \frac{1 - R^2}{2R^2} \left( \sum_{p=1}^{\infty} \sum_{i=+,-} g_p^i \mathcal{G}_p^i(R, G) \right) \right)^t .
\]

By a tedious but straightforward calculation, (2.38) is resummed to take the following form:

\[
\vec{s}(x) = \vec{s}_{c}^{(k)} + a^{2-1/k} \sum_{n=1}^{k} \alpha_{n}^{(k)} r^{2(k-n)} g^{2n-1} + a^2 \sum_{n=0}^{k} \beta_{n}^{(k)} r^{2(k-n)} g^{2n} + \mathcal{O}(a^{2+1/k}).
\] (2.39)

Here we have set \(R = a^{1/k} r, G = a^{1/k} g\) and introduced

\[
\vec{s}_{c}^{(k)} = \begin{pmatrix} \vec{S}_{c}^{(k)} \\ 0 \end{pmatrix} = \begin{pmatrix} k+1 \\ 2k \end{pmatrix}
\]

as well as

\[
\alpha_n^{(k)} = \frac{(-1)^n}{2} \frac{\Gamma(k)\Gamma(k+2)}{\Gamma(2n+1)\Gamma(k-n+1)\Gamma(k-n+2)} \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\] (2.41)

and

\[
\beta_n^{(k)} = \frac{(-1)^{n+1}}{2} \frac{\Gamma(k)\Gamma(k+2)}{\Gamma(2n+1)\Gamma(k-n+1)\Gamma(k-n+2)} \begin{pmatrix} (k-n+1)+n(2n-1) \\ n(2n-1) \end{pmatrix} .
\] (2.42)

Eq.(2.39) gets further simplified by multiplying by

\[
T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},
\]

as \(T\vec{s}(x) = \vec{s}(x)\) and \(T\vec{s}_{c} = \vec{s}_{c}^{(k)} :\)

\[
T\alpha_n^{(k)} = \frac{(-1)^n}{2} \frac{\Gamma(k)\Gamma(k+2)}{\Gamma(2n+1)\Gamma(k-n+1)\Gamma(k-n+2)} \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\] (2.44)

\[
T\beta_n^{(k)} = \frac{(-1)^{n+1}}{2} \frac{\Gamma(k)\Gamma(k+2)}{\Gamma(2n+1)\Gamma(k-n+1)\Gamma(k-n+1)^2} \begin{pmatrix} 1 \\ n(2n-1) \end{pmatrix} \frac{1}{k-n+1} .
\] (2.45)

\(^6\)Here, we normalize the critical coupling \(t_1^{(k)}\) to be 1. This normalization is different from [43].
The first component of (2.39) in this form is \( \mathcal{O}(a^2) \) and the second component is \( \mathcal{O}(a^{2-1/k}) \). Defining
\[
1 - \frac{\tilde{S}x}{\tilde{S}^{(k)}} = a^2t, \quad 1 - \frac{\tilde{S}/\tilde{S}^{(k)}}{c} = a^2c,
\]
we obtain a set of planar string equations by the scaling variables \( t, r \) and \( g \)
\[
t = \sum_{n=0}^{k} (-1)^n \frac{\Gamma(k + 1)^2}{\Gamma(2n + 1) \Gamma(k - n + 1)^2} r^{2(k-n)} g^{2n},
\]
\[
0 = \sum_{n=1}^{k} (-1)^n \frac{\Gamma(k + 1)^2}{\Gamma(2n) \Gamma(k - n + 1) \Gamma(k - n + 2)} r^{2(k-n)} g^{2n-1}.
\]
Note that \( g = 0 \) is always a solution to (2.48), where (2.47) reduces to
\[
t = r^{2k}.
\]

We can now compute the \( k \)-th critical behavior of the planar free energy\(^7\) from (2.12)
\[
F_0 \sim a^{4+2/k} \int_0^c dt \left\{ (t - c) t^{1/k} + \mathcal{O}(a^2) \right\}
\]
\[
\sim - a^{4+2/k} \frac{k^2}{(2k + 1)(k + 1)} c^{2+1/k}.
\]
Then defining \( \kappa^{-1} = N a^{2+1/k} \) which is kept fixed under the limit \( N \to \infty, a \to 0 \), we obtain the planar free energy at the \( k \)-th multicritical point
\[
\mathcal{F} = -\kappa^{-2} \frac{k^2}{(2k + 1)(k + 1)} c^{2+1/k} + \ldots,
\]
namely
\[
f_0 = - \frac{k^2}{(2k + 1)(k + 1)} c^{2+1/k}.
\]

### 2.2 perturbation for critical points

In this section, we will construct the scaling operators which perturb the \( k \)-th critical planar string equations (2.47) and (2.48). Let us first review how to construct the scaling operators which preserve the \( z \to 1/z \) invariance [43]. We will call them even type scaling operators. We also construct the odd and logarithmic type scaling operators which are odd under \( z \to 1/z \).

\(^7\)Note that our definition of the free energy in (2.3) omits the minus sign. This explains a few minus signs in what follows.
2.2.1 even type perturbation

Let us consider the following perturbation of the \( k \)-th critical potential

\[
W^{(k)}(z) \to W^{(k)}(z) + m^{+B}_{\ell} \sigma^{+B}_{\ell}(z), \quad \sigma^{+B}_{\ell}(z) = W^{(\ell)}(z),
\]

where \( m^{+B}_{\ell} \) is “bare” coupling and \( \sigma^{+B}_{\ell}(z) \) is “bare” scaling operator. It is clear that perturbed potential is invariant under \( z \to 1/z \). This perturbation changes the string equation (2.39) into

\[
\vec{s}(x) = s_{c}^{(k)} + a^{2-1/k} \sum_{n=1}^{k} T \vec{a}_{n}^{(k)} r^{2(k-n)} g^{2n-1} + a^{2} \sum_{n=0}^{k} T \vec{b}_{n}^{(k)} r^{2(k-n)} g^{2n} + m^{+B}_{\ell} \left( s_{c}^{(\ell)} + a^{(2\ell-1)/k} \sum_{n=1}^{\ell} T \vec{a}_{n}^{(\ell)} r^{2(\ell-n)} g^{2n-1} + a^{2\ell/k} \sum_{n=0}^{\ell} T \vec{b}_{n}^{(\ell)} r^{2(\ell-n)} g^{2n} \right),
\]

(2.54)

Since the critical value of \( \tilde{S}_{c}^{(k)} \) shifts by

\[
\tilde{S}_{c}^{(k)} \to \tilde{S}_{c}^{(k)'} = \tilde{S}_{c}^{(k)} + m^{+B}_{\ell} \tilde{S}_{c}^{(\ell)},
\]

(2.55)

we redefine the scaling variable for \( \tilde{S} \) by

\[
1 - \tilde{S}x/\tilde{S}_{c}^{(k)'} = a^{2}t'.
\]

(2.56)

In order for the first component of (2.54) to be order \( a^{2} \), we should scale \( m^{+B}_{\ell} \) by

\[
m^{+B}_{\ell} = a^{2(1-\ell/k)} \tilde{S}_{c}^{(k)} \tilde{S}_{c}^{(\ell)} m^{+}_{\ell}.
\]

(2.57)

For \( \ell = 1, \cdots, k-1 \), the critical value \( \tilde{S}_{c}^{(k)'} \) goes to \( \tilde{S}_{c}^{(k)} \) in the \( a \to 0 \) limit. It means that the critical value of \( \tilde{S}_{c}^{(k)} \) does not change at the \( k \)-th multicritical point. For \( \ell = k \), the perturbation changes only the critical values of \( \tilde{S}^{(k)} \). For \( \ell > k \), we must set \( m^{+}_{\ell} = 0 \) to make \( \tilde{S}_{c}^{(k)'} \) into finite. We should therefore consider the \( \ell = 1, \cdots, k-1 \) cases, where the
perturbed string equations read

\[ t = \left( \sum_{n=0}^{k} (-1)^n \frac{\Gamma(k+1)^2}{\Gamma(2n+1)\Gamma(k-n+1)^2} r^{2(k-n)} g^{2n} \right) + m_\ell^+ \left( \sum_{n=0}^{\ell} (-1)^n \frac{\Gamma(\ell+1)^2}{\Gamma(2n+1)\Gamma(\ell-n+1)^2} r^{2(\ell-n)} g^{2n} \right), \]  

(2.58)

\[ 0 = \left( \sum_{n=1}^{k} (-1)^n \frac{\Gamma(k+1)^2}{\Gamma(2n)\Gamma(k-n+1)\Gamma(k-n+2)^2} r^{2(k-n)} g^{2n-1} \right) + m_\ell^+ \left( \sum_{n=1}^{\ell} (-1)^n \frac{\Gamma(\ell+1)^2}{\Gamma(2n)\Gamma(\ell-n+1)\Gamma(\ell-n+2)^2} r^{2(\ell-n)} g^{2n-1} \right). \]  

(2.59)

Note that (2.59) has a solution \( g = 0 \) for \( \ell = 1, \ldots, k - 1 \) and (2.58) becomes

\[ t = m_\ell^+ r^{2\ell} + r^{2k}. \]  

(2.60)

### 2.2.2 odd type perturbation

Let us now consider the following perturbation

\[ W^{(k)}(z) \to W^{(k)}(z) + m_\ell^{-B} \sigma_\ell^{-B}(z), \]  

(2.61)

where \( 1 \leq \ell \leq k - 1 \) and

\[ \sigma_\ell^{-B}(z) = -\frac{1}{2g} \sum_{p=1}^{\ell} \tilde{t}_p^{(\ell)} \left( z^p - \frac{1}{z^p} \right), \quad \tilde{t}_p^{(\ell)} = p t_p^{(\ell)}. \]  

(2.62)

Here \( t_p^{(\ell)} \) is given by (2.37). The perturbed string equation is

\[ \tilde{s}(x) = \left( \frac{1 - R^2}{2R^2} \left( \sum_{p=1}^{k} t_p^{(k)} F_p^+(R, G) \right), \frac{1 - R^2}{2R^2} \left( \sum_{p=1}^{k} t_p^{(k)} G_p^+(R, G) \right) \right)^t + m_\ell^{-B} \left( \frac{1 - R^2}{2R^2} \left( \sum_{p=1}^{\ell} t_p^{(\ell)} F_p^-(R, G) \right), \frac{1 - R^2}{2R^2} \left( \sum_{p=1}^{\ell} t_p^{(\ell)} G_p^-(R, G) \right) \right)^t. \]  

(2.63)

The first line has been already computed in (2.39). The second line can be resummed to give

\[ m_\ell^{-B} a^{(2\ell-2)/k} \sum_{n=1}^{\ell} c_n r^{2(\ell-n)} g^{2n-2} + m_\ell^{-B} a^{(2\ell-1)/k} \sum_{n=1}^{\ell} d_n r^{2(\ell-n)} g^{2n-1} + \mathcal{O}(a^{2\ell/k}), \]  

(2.64)
where
\[
\vec{c}_n = \frac{(-1)^n}{2} \frac{\Gamma(\ell)\Gamma(\ell + 2)}{\Gamma(2n - 1)\Gamma(\ell - n + 1)\Gamma(\ell - n + 2)} \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\]
\[
\vec{d}_n = \frac{(-1)^{n+1}}{2} \frac{\Gamma(\ell)\Gamma(\ell + 2)}{\Gamma(2n)\Gamma(\ell - n + 1)\Gamma(\ell - n + 2)} \begin{pmatrix} 1 + \ell - n + (n - 1)(2n - 1) \\ (n - 1)(2n - 1) \end{pmatrix}.
\]

These can be also transformed into a simpler form by the multiplication by \(T\):
\[
T\vec{c}_n = \frac{(-1)^n}{2} \frac{\Gamma(\ell)\Gamma(\ell + 2)}{\Gamma(2n - 1)\Gamma(\ell - n + 1)\Gamma(\ell - n + 2)} \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]
\[
T\vec{d}_n = \frac{(-1)^n}{2} \frac{\Gamma(\ell)\Gamma(\ell + 2)}{\Gamma(2n)\Gamma(\ell - n + 1)^2} \begin{pmatrix} 1 \\ \frac{\ell - n + 1}{(n - 1)(2n - 1)} \end{pmatrix}.
\]

Thus, this perturbation shifts the first string equation at \(O(a^{2(\ell - 1)/k})\) and the second string equation at \(O(a^{2(\ell - 2)/k})\). In order to make the first string equation \(O(a^2)\), we should rescale the coupling \(m_\ell^{-B}\) as
\[
m_\ell^{-B} = a^{2-(2\ell - 1)/k} \frac{S_c^{(k)}}{S_c^{(\ell)}} m_\ell^{-},
\]
\[
(2.69)
\]

The resulting perturbed string equations are
\[
t = \sum_{n=0}^{k} (-1)^n \frac{\Gamma(k + 1)^2}{\Gamma(2n + 1)\Gamma(k - n + 1)^2} r^{2(k-n)} g^{2n}
\]
\[
+ m_\ell^{-} \sum_{n=1}^{\ell} (-1)^{n+1} \frac{\Gamma(\ell + 1)^2}{\Gamma(2n)\Gamma(\ell - n + 1)^2} r^{2(\ell-n)} g^{2n-1},
\]
\[
0 = \sum_{n=1}^{k} (-1)^n \frac{\Gamma(k + 1)^2}{\Gamma(2n)\Gamma(k - n + 1)\Gamma(k - n + 2)} r^{2(k-n)} g^{2n-1}
\]
\[
+ m_\ell^{-} \sum_{n=1}^{\ell} (-1)^n \frac{\Gamma(\ell + 1)^2}{\Gamma(2n - 1)\Gamma(\ell - n + 1)\Gamma(\ell - n + 2)} r^{2(\ell-n)} g^{2n-2}.
\]
\[
(2.70)
\]

2.2.3 logarithmic type perturbation

Let us finally consider the following perturbation
\[
W^{(k)}(z) \to W^{(k)}(z) + m_\ell^{B} \sigma_0^{B}(z), \quad \sigma_0^{B}(z) = -\frac{1}{2g_s} \log(z).
\]
\[
(2.72)
\]
Because
\[ \frac{1}{\hbar_n} \oint d\mu(z) \frac{1}{z} p_n(z) p_{n-1}(1/z) = \frac{1}{\hbar_n} \oint d\mu(z) p_n(z) p_n(1/z) = 1, \]
the net effect of this perturbation is just to shift r.h.s. of (2.39) by
\[ \frac{1 - R^2}{R^2} (m^B_{\log}, m^B_{\log})^t. \]

The \( k \)-th critical string equations perturbed by (2.72) are
\[ \vec{s}(x) = \vec{s}_c + (a^{-2/k} r^{-2} - 1) (0, m^B_0)^t + a^{2-1/k} \sum_{n=1}^k T \vec{a}_n r^{2(k-n)} g^{2n-1} + a^2 \sum_{n=0}^k T \vec{b}_n r^{2(k-n)} g^{2n}. \]

This perturbation affects the second string equation only. For the second component of (2.75) to be \( O(a^{2-1/k}) \), we should scale \( m^B_0 \) as
\[ m^B_{\log} = a^{2+1/k} m_{\log}. \]

The perturbed string equations of the logarithmic type are
\[ t = \sum_{n=0}^k (-1)^n \frac{\Gamma(k+1)^2}{\Gamma(2n+1) \Gamma(k-n+1)^2} r^{2(k-n)} g^{2n}, \]
\[ 0 = m_{\log} + \sum_{n=1}^k (-1)^n \frac{\Gamma(k+1)^2}{\Gamma(2n) \Gamma(k-n+1) \Gamma(k-n+2)} r^{2(k-n+1)} g^{2n-1} \]

2.3 scaling behavior and the scaling dimension

2.3.1 scaling operators

In section 2.2, we derived the perturbed string equations obtained by the various perturbations. The vevs of the scaling operators at the \( k \)-th multicritical point are given by the derivative of free energy with respect to the scaled perturbation parameter:
\[ \langle \sigma^\pm_\ell \rangle = \frac{\partial F}{\partial m^\pm_\ell}, \quad \langle \sigma_{\log} \rangle = \frac{\partial F}{\partial m_{\log}}. \]

Since they can be also expanded in \( \kappa \) as is the free energy, their leading parts are written as
\[ \langle \sigma^\pm_{\ell,0} \rangle = \frac{\partial f_0}{\partial m^\pm_\ell}, \quad \langle \sigma_{\log,0} \rangle = \frac{\partial f_0}{\partial m_{\log}}. \]
We can evaluate \( \langle \sigma^\pm_{\ell,0} \rangle \) and \( \langle \sigma_{\log,0} \rangle \) by solving the perturbed string equations.

Let us consider the even type scaling operators which are studied in [43]:

\[
\langle \sigma^+_{\ell,0} \rangle = \frac{\partial f_0}{\partial m_\ell^+} \sim \int^c dt \left( t - c \right) \frac{\partial r(t; \{ m_\ell^+ \})^2}{\partial m_\ell^+},
\]

(2.81)

where \( r(t; \{ m_\ell^+ \}) \) is the solution of

\[
t = \sum_{\ell=1}^{k-1} m_\ell^+ r^{2\ell} + r^{2k},
\]

(2.82)

which can be obtained in the same way as in [55]:

\[
r(t; \{ m_\ell^+ \})^2 = \frac{t^{1/k}}{1 + \sum_{p=1}^{\infty} \frac{(-1)^p}{k! p!} \left( \frac{\partial}{\partial t} \right)^{p-1} \left\{ \left( \sum_{\ell=1}^{k-1} m_\ell^+ t^\ell/k \right)^p \right\} t^{1+1/k}}.
\]

(2.83)

We obtain

\[
\langle \sigma^+_{\ell,0} \rangle \sim \frac{k}{(k + \ell + 1)(\ell + 1)} c^{1+(\ell+1)/k} + O(m_\ell^+).
\]

(2.84)

To evaluate the odd type scaling operators, we solve the set of equations (2.70) and (2.71). We assume that \( m^-_\ell \) is small and \( g \) can be expanded in \( m^-_\ell \) as

\[
g(r; m^-_\ell) = \sum_{n=1}^{\infty} g_n(r)(m^-_\ell)^{2n-1}.
\]

(2.85)

We choose the solution of (2.71) to satisfy \( g(r; m^-_\ell = 0) = 0 \) so that it reduces to (2.49) when there is no deformation. Similarly, we assume that \( r \) has the following expansion

\[
r(t; m^-_\ell)^2 = \frac{t^{1/k}}{1 + \sum_{n=1}^{\infty} r_n(t)(m^-_\ell)^{2n}}
\]

(2.86)

such that \( r = t^{1/(2k)} \) at \( m^-_\ell = 0 \). Substituting (2.85) into (2.70), and solving order by order, we obtain

\[
r(t; m^-_\ell)^2 = t^{1/k} + \frac{\ell^2}{2k^2} (k + 2\ell) t^{-2+2\ell/k} (m^-_\ell)^2 + O((m^-_\ell)^4),
\]

(2.87)

and

\[
\langle \sigma^-_{\ell,0} \rangle = \frac{\partial f_0}{\partial m_\ell^-} = \frac{\ell^2}{k^2} (k + 2\ell) m^-_\ell \int^c_{a-2+c} dt (t - c) t^{-2+2\ell/k} + O((m^-_\ell)^3)
\]

\[
= \frac{\ell(k+2\ell)}{2(k-2\ell)} c^{2\ell/k} m^-_\ell + O((m^-_\ell)^3).
\]

(2.88)
Similarly, for the logarithmic type scaling operator, we obtain from the solution of (2.77) and (2.78)

\[ r(m_{\log})^2 = t^{1/k} + \frac{1}{2t^2}(m_{\log})^2 + \mathcal{O}((m_{\log})^4), \]  

and

\[ \langle \sigma_{\log,0} \rangle = \frac{\partial f_0}{\partial m_{\log}} \int_{a-2+c}^c dt(t-c)t^{-2} + \mathcal{O}((m_{\log})^3) \]

\[ \sim m_{\log} (\log c + 1) + \mathcal{O}((m_{\log})^3). \]  

(2.89)

### 2.3.2 scaling dimensions

We search for the possibility that the free energy at the \(k\)-th multicritical point is that of the AD theory of some type. Then, we define the scaling dimension of the sphere contribution of the free energy \(f_0\) to be \([f_0] = 2\), since it should be identified the Seiberg-Witten prepotential of corresponding AD theory. The scaling dimensions of the other variables are easily determined from the critical behavior of the free energy (2.52) and the vevs of the scaling operators (2.84), (2.88) and (2.90):

\[ [c] = \frac{2k}{2k+1}, \]

\[ [\sigma_\ell^+] = \frac{2k + 2\ell + 2}{2k + 1}, \quad [\sigma_\ell^-] = \frac{2k + 2\ell + 1}{2k + 1}, \quad [\sigma_{\log,0}] = 1, \quad \ell = 1, \ldots, k - 1, \]  

(2.91)

and

\[ [m_\ell^+] = \frac{2k - 2\ell}{2k + 1}, \quad [m_\ell^-] = \frac{2k - 2\ell + 1}{2k + 1}, \quad [m_{\log}] = 1, \quad \ell = 1, \ldots, k - 1. \]

(2.92)

### 3 Comparison with Argyres-Douglas theory

Let us compare the scaling operators which we constructed in section 2.3 with those of the Coulomb branch operators in the AD theory. In this way, we can easily see the relationship between the \(k\)-th multicritical point and the AD theory. In fact, in the \((A_1, A_{4k-1})\) theory, the scaling dimension for the Coulomb branch operators \(u_{2k+1+i}\) and their coupling constant \(m_i \equiv u_{2k+1-i}\) are respectively\(^8\)

\[ [u_{2k+1+i}] = \frac{2k + 1 + i}{2k + 1}, \quad [m_i] = \frac{2k + 1 - i}{2k + 1}, \quad i = 1, \ldots, 2k - 1. \]  

(3.1)

\(^8\)Here, the Seiberg-Witten curve of the \((A_1, A_{4k-1})\) AD theory is given by \[x^2 = z^{4k} + u_2z^{4k-2} + \cdots + u_{4k}.\]
We obtain a dictionary by setting $i = 2\ell$ and $i = 2\ell + 1$

\[
\begin{align*}
[s_{\ell,0}^-] &= [u_{2k+1+2\ell}], & [m_\ell^+] &= [m_{2\ell}], & \ell &= 1, \ldots, k-1, \\
[s_{\ell,0}^+] &= [u_{2k+2+2\ell}], & [m_\ell^+] &= [m_{2\ell+1}], & \ell &= 0, \ldots, k-1,
\end{align*}
\]

and

\[
[m_{\log}] = [u_{2k+1}].
\]

Here we introduced

\[
\sigma_{0,0}^+ \equiv \frac{\partial f_0}{\partial c}, \quad m_{0}^+ \equiv c.
\]

The perturbations from the $k$-th multicritical point can capture all Coulomb branch operators which are contained in the $(A_1, A_{4k-1})$ theory. We conclude that the $k$-th multicritical point at the even potential of the one unitary matrix model corresponds to the $(A_1, A_{4k-1})$ theory.

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