Integral structures in automorphic line bundles on the $p$-adic upper half plane

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Abstract

Given an automorphic line bundle $O_X(k)$ of weight $k$ on the Drinfel’d upper half plane $X$ over a local field $K$, we construct a $GL_2(K)$-equivariant integral lattice $O_{\hat{X}}(k)$ in $O_X(k) \otimes_K \hat{K}$, as a coherent sheaf on the formal model $\hat{X}$ underlying $X \otimes_K \hat{K}$. Here $\hat{K}/K$ is ramified of degree 2. This generalizes a construction of Teitelbaum from the case of even weight $k$ to arbitrary integer weight $k$.

We compute $H^*(\hat{X}, O_{\hat{X}}(k))$ and obtain applications to the de Rham cohomology $H^1_{dR}(\Gamma \setminus X, \text{Sym}^k_{\hat{K}}(\text{St}))$ with coefficients in the $k$-th symmetric power of the standard representation of $SL_2(K)$ (where $k \geq 0$) of projective curves $\Gamma \setminus X$ uniformized by $X$: namely, we prove the degeneration of a certain reduced Hodge spectral sequence computing $H^1_{dR}(\Gamma \setminus X, \text{Sym}^k_{\hat{K}}(\text{St}))$, we re-prove the Hodge decomposition of $H^1_{dR}(\Gamma \setminus X, \text{Sym}^k_{\hat{K}}(\text{St}))$ and show that the monodromy operator on $H^1_{dR}(\Gamma \setminus X, \text{Sym}^k_{\hat{K}}(\text{St}))$ respects integral de Rham structures and is induced by a “universal” monodromy operator defined on $\hat{X}$, i.e. before passing to the $\Gamma$-quotient.

Introduction

Let $K$ be a local field and let $X$ be the Drinfel’d upper half plane over $K$; that is, the projective line over $K$ with its $K$-rational points removed. $G = GL_2(K)$ acts on $X$. Let $O_X(k)$ be the structure sheaf on the rigid space $X$, endowed with the automorphic
action by $G$ of weight $k \in \mathbb{Z}$. For $k \geq 0$ and even, Teitelbaum [8] constructed a $G$-invariant integral lattice in $\mathcal{O}_X(k)$, as a line bundle on the natural formal $\mathcal{O}_K$-scheme $\mathfrak{X}$ underlying $X$. He then reduced this bundle modulo the maximal ideal of $\mathcal{O}_K$ and determined explicitly its global sections, as a representation of $G$ on an infinite dimensional vector space over the residue field $\mathbb{F}$ of $K$. The first aim of this paper is to extend his results to any weight $k \in \mathbb{Z}$. Now it is not hard to see that for odd $k$ there is no $G$-equivariant $\mathcal{O}_X$-line bundle lattice in $\mathcal{O}_X(k)$. Let $\hat{K}$ be a ramified extension of $K$ of degree 2, let $\hat{\mathfrak{X}} = \mathfrak{X} \otimes_{\mathcal{O}_K} \mathcal{O}_{\hat{K}}$ be the base extended formal $\mathcal{O}_{\hat{K}}$-scheme, let $\hat{\mathfrak{X}} = \mathfrak{X} \otimes_{\mathcal{O}_K} \mathbb{F} = \hat{\mathfrak{X}} \otimes_{\mathcal{O}_{\hat{K}}} \mathbb{F}$. We show that for any $k \in \mathbb{Z}$, if we twist the automorphic action on $\mathcal{O}_X(k)$ by a suitable character, there is a $G$-equivariant $\mathcal{O}_{\hat{\mathfrak{X}}}$-module $\mathcal{O}_{\hat{\mathfrak{X}}}(k)$ which is a lattice inside $\mathcal{O}_X(k) \otimes_K \hat{K}$.

If $k$ is even it is a line bundle, if $k$ is odd it is not: around the singular points of $\mathfrak{X}$ it needs two generators. We show that $H^0(\hat{\mathfrak{X}}, \mathcal{O}_{\hat{\mathfrak{X}}}(k))$ for $k \geq 0, k \neq 1$ and $H^1(\hat{\mathfrak{X}}, \mathcal{O}_{\hat{\mathfrak{X}}}(k))$ for $k \leq -1$ are precisely those cohomology groups which do not vanish. We prove that they are $\mathcal{O}_{\hat{K}}$-flat and that their formation commutes with base change to the special fibre $\hat{X}$. Finally, if $\text{char}(K) = 0$, we demonstrate that integral structures are a strong tool for studying the ”reduced” de Rham complex

$$R_X^\bullet = [\mathcal{O}_X(-k) \xrightarrow{dz} \mathcal{O}_X(k+2)]$$

on $X$ considered in [5], [6], for $k \geq 0$ (here $z$ is a global variable on $X \subset \mathbb{P}^1_K$). It computes the de Rham cohomology $H^\bullet(X, \Omega^\bullet_X \otimes \text{Sym}^k_X(\text{St}))$ of $X$ with coefficients in the $k$-th symmetric power $\text{Sym}^k_X(\text{St})$ of the standard representation of $\text{SL}_2(K)$. Its differential respects our integral structures, hence a complex

$$R_{\hat{\mathfrak{X}}}^\bullet = [\mathcal{O}_{\hat{\mathfrak{X}}}(-k) \xrightarrow{dz} \mathcal{O}_{\hat{\mathfrak{X}}}(k+2)]$$

on $\hat{\mathfrak{X}}$. We show that for $k > 0$ we have $H^j(\hat{\mathfrak{X}}, R_{\hat{\mathfrak{X}}}^\bullet) = 0$ for $j \neq 1$, while $H^1(\hat{\mathfrak{X}}, R_{\hat{\mathfrak{X}}}^\bullet)$ decomposes as

$$H^1(\hat{\mathfrak{X}}, R_{\hat{\mathfrak{X}}}^\bullet) \cong H^1(\hat{\mathfrak{X}}, \mathcal{O}_{\hat{\mathfrak{X}}}(-k)) \oplus H^0(\hat{\mathfrak{X}}, \mathcal{O}_{\hat{\mathfrak{X}}}(k+2))$$

(*).

As an application, we show that structural features of the cohomology of varieties uniformized by $X$ can be deduced from (*), thus show up already on $X$ (or rather $\hat{\mathfrak{X}}$) itself.
Namely we get the well known Hodge decomposition (first obtained by de Shalit [7], see also [5])

\[ H^1_{dR}(\Gamma \setminus X, \text{Sym}_K^k(\text{St})) = H^1(\Gamma, \text{Sym}_K^k(\text{St})) \oplus H^0(\pi, \mathcal{O}(k+2)) \]

of \( H^1_{dR}(\pi, \text{Sym}_K^k(\text{St})) = H^1(\Gamma \setminus X, (\Omega_X^n \otimes K \text{Sym}_K^k(\text{St}))^\Gamma) = H^1(\Gamma \setminus X, (\mathcal{R}_X^n)^\Gamma) \) simply by taking \( \Gamma \)-invariants for a cocompact discrete (torsionfree) subgroup \( \Gamma < \text{SL}_2(K) \); no higher \( \Gamma \)-group cohomology is needed. Again, while earlier proofs were truly analytic we reduce everything to algebraic geometry on the irreducible components of \( \tilde{X} \) (these are all isomorphic to \( \mathbb{P}^1_k \)).

As a bonus of our method we obtain the degeneration of the "reduced" Hodge spectral sequence computing \( H^1_{dR}(\Gamma \setminus X, \text{Sym}_K^k(\text{St})) \), as conjectured by Schneider [5], and a complete description (in particular their dimensions) of the cohomology spaces \( H^j(\Gamma \setminus X, \mathcal{O}(r)) \) (any \( j, r \)). Moreover, for \( k > 0 \), we describe a monodromy operator on \( H^1(\tilde{X}, \mathcal{R}_X^n) \) as an isomorphism \( H^0(\tilde{X}, \mathcal{O}_\tilde{X}(k+2)) \cong H^1(\tilde{X}, \mathcal{O}_\tilde{X}(-k)) \). It induces the monodromy operator on \( H^1_{dR}(\Gamma \setminus X, \text{Sym}_K^k(\text{St})) \) predicted by \( p \)-adic Hodge theory, so in particular we see that the latter respects integral de Rham structures (which in \( p \)-adic Hodge theory can not be expected in general) and that its monodromy filtration splits the Hodge filtration.

We mention that the integral structures in \( \mathcal{O}(k) \) and in the "reduced" de Rham complex considered in this paper play an important role in the recent work of Breuil [1].

Notations: \( K \) denotes a non-archimedean locally compact field and \( K_a \) its algebraic closure, \( \mathcal{O}_K \) its ring of integers, \( \pi \in \mathcal{O}_K \) a fixed prime element and \( \mathbb{F} \) the residue field with \( q \) elements, \( q \in \mathbb{P}^\mathbb{N} \). We choose \( \widehat{\pi} \in K_a \) such that \( \widehat{\pi}^2 = \pi \). Then \( \widehat{K} = K(\widehat{\pi}) \) is a ramified extension of \( K \) of degree 2 with ring of integers \( \mathcal{O}_{\widehat{K}} \). We let \( \omega : K_a^\times \rightarrow \mathbb{Q} \) be the extension of the discrete valuation \( \omega : K^\times \rightarrow \mathbb{Z} \) normalized by \( \omega(\pi) = 1 \). For formal \( \mathcal{O}_K \)-schemes resp. \( K \)-rigid spaces we denote by a superscript \( \widehat{\cdot} \) the formal \( \mathcal{O}_{\widehat{K}} \)-schemes resp. \( \widehat{K} \)-rigid space obtained by the base change \( \mathcal{O}_K \rightarrow \mathcal{O}_{\widehat{K}} \) resp. \( K \rightarrow \widehat{K} \). For \( E = K \) or \( E = \widehat{K} \) and a formal (admissible) \( \mathcal{O}_E \)-scheme \( \mathcal{M} \) we let \( \mathcal{M}_E \) be its generic fibre, as a \( E \)-rigid space. We need the characters \( \chi : G \rightarrow \widehat{K}^\times, \chi(\gamma) = \pi^{\omega(\gamma)}, \) and \( \varepsilon : G \rightarrow \mathcal{O}_K^\times, \varepsilon(\gamma) = \pi^{-\omega(\gamma)} \text{det } \gamma, \) of \( G = \text{GL}_2(K) \) and denote the Bruhat-Tits tree of \( G \) by \( \mathcal{BT} \). For \( r \in \mathbb{R} \) we define \( \lfloor r \rfloor, \lceil r \rceil \in \mathbb{Z} \) by requiring \( \lfloor r \rfloor \leq r < \lfloor r \rfloor + 1 \) and \( \lceil r \rceil - 1 < r \leq \lceil r \rceil \).

1 Integral structures in automorphic line bundles

Let \( X = \Omega^{(2)}_K \) be Drinfel’d’s symmetric space of dimension 1 over \( K \). This is the \( K \)-rigid space obtained by removing all \( K \)-rational points from the projective line \( \mathbb{P}^1_K \) over \( K \). We
choose a coordinate $z$ and define an action of $G$ on $X$ (on the left) by

$$
\gamma z = \frac{-b + az}{d - cz} \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G
$$

($\mathfrak{S}$ takes the other left action). Fix $k \in \mathbb{Z}$. For $f \in \mathcal{O}_{\tilde{X}}$ set

$$(1) \quad f|_{\gamma}(z) = \chi_k(\gamma)(a + cz)^{-k}f\left(\frac{b + dz}{a + cz}\right) \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.
$$

Denote by $\mathcal{O}_{\tilde{X}}(k)$ the structure sheaf of the $\tilde{K}$-rigid space $\tilde{X}$ endowed with the $G$-action on the left defined by (1). (This is a left action; in [8] a right action is considered.)

As explained in [8], the $K$-rigid space $X$ is the generic fibre of a certain $\pi$-adic strictly semistable formal $\mathcal{O}_K$-scheme $\mathfrak{X}$: the set $F^0$ of irreducible components of the reduction $\tilde{\mathfrak{X}}$ of $\mathfrak{X}$ is in natural bijection with the set of vertices of $BT$. Let $F^1$ be the set of subsets $\{Z_1, Z_2\} \subset F^0$ with $Z_1 \cap Z_2 \neq \emptyset$ and $Z_1 \neq Z_2$; it corresponds to the set of edges of $BT$.

Each $Z \in F^0$ is isomorphic to $\mathbb{P}^1_{\mathcal{O}_K}$. The action of $G$ on $X$ extends to $\tilde{\mathfrak{X}}$. The admissible open subset

$$
U = \{ P \in \mathbb{P}^1; \quad \omega(z(P)) > -1 \quad \text{and} \quad \omega(z(P) - x) < 1 \quad \text{for all} \quad x \in \mathcal{O}_K \}
$$

of $X$ is the tube (=preimage under the specialization map $X \to \mathfrak{X}$) of the central (with respect to $z$) irreducible component $Z_{\gamma_0}$ of $\tilde{\mathfrak{X}}$. For $\gamma \in G$ define the irreducible component $Z_{\gamma}$ of $\tilde{\mathfrak{X}}$ as $Z_{\gamma} = \gamma.Z_{\gamma_0}$. For $n \in \mathbb{Z}$ let

$$
\gamma_n = \begin{pmatrix} 1 & 0 \\ 0 & \pi^n \end{pmatrix} \in G.
$$

For a subset $E \subset F^0$ let $\tilde{\mathcal{U}}_E$ be the maximal open subscheme of $\tilde{\mathfrak{X}}$ contained in $\cup_{Z \in E} Z$; in other words, the complement in $\tilde{\mathfrak{X}}$ of the union of all irreducible components not in $E$. Let $\mathcal{U}_E$ be the open formal subscheme of $\mathfrak{X}$ lifting $\tilde{\mathcal{U}}_E$. Letting

$$
\mathfrak{Y} = \mathcal{U}_{\{Z_{\gamma_n}; n \in \mathbb{Z}\}}
$$

we have the open covering

$$
\mathfrak{X} = \bigcup_{g \in \text{SL}_2(K)} g.\mathfrak{Y}
$$

($\text{SL}_2(K)$ acts transitively on $F^1$). Let $f_{n, n} \in \mathcal{O}_{\mathfrak{Y}}(\mathcal{U}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}})$ (resp. $f_{n, n+1} \in \mathcal{O}_{\mathfrak{Y}}(\mathcal{U}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}})$) be an equation for the closed subscheme $Z_{\gamma_n} \cap \tilde{\mathcal{U}}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}}$ (resp. $Z_{\gamma_{n+1}} \cap \tilde{\mathcal{U}}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}}$) of $\mathcal{U}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}}$. In local coordinates, there is an open embedding $\mathcal{U}_{\{Z_{\gamma_n}, Z_{\gamma_{n+1}}\}} \to \text{Spf}(\mathcal{O}_K <
$X_1, X_2 > / (X_1 X_2 - \pi)$ such that $f_{n,n} = X_1$ and $f_{n,n+1} = X_2$. Viewing $f_{n,n}$ and $f_{n,n+1}$ as sections of $\mathcal{O}_{\mathfrak{H}}(\mathfrak{U}_f(z_n, z_{n+1}))$ we define

$$\mathcal{O}_{\mathfrak{U}_f(z_n, z_{n+1})}(k) = \mathcal{O}_{\mathfrak{U}_f(z_n, z_{n+1})}. f_{n,n}^{[k_n]} f_{n,n+1}^{[k_{n+1}]} + \mathcal{O}_{\mathfrak{U}_f(z_n, z_{n+1})}. \hat{\pi} f_{n,n}^{[k_n]} f_{n,n+1}^{[k_{n+1}]};$$

i.e. the $\mathcal{O}_{\mathfrak{U}_f(z_n, z_{n+1})}$-submodule of $\mathcal{O}_{\mathfrak{U}_f(z_n, z_{n+1})} \otimes \hat{K}$ generated by the two elements $f_{n,n}^{[k_n]} f_{n,n+1}^{[k_{n+1}]}$ and $\hat{\pi} f_{n,n}^{[k_n]} f_{n,n+1}^{[k_{n+1}]}$. If $k$ is even this is just the line bundle generated by the element $z^{-k}$. If $k$ is odd this is not a line bundle; an explicit pair of generators is $\hat{\pi}^{n+1} z^{-(k-1)/2}, \hat{\pi}^{-n} z^{-(k+1)/2}$.

The $\mathcal{O}_{\mathfrak{U}_f(z_n, z_{n+1})}(k)$ glue into an $\mathcal{O}_{\mathfrak{H}}$-submodule $\mathcal{O}_{\mathfrak{H}}(k)$ of $\mathcal{O}_{\mathfrak{H}} \otimes \hat{K}$. Note that

$$(2) \quad \mathcal{O}_{\mathfrak{H}}(k)|_{\mathfrak{U}_f(z_n)} = \hat{\pi}^{kn} \mathcal{O}_{\mathfrak{U}_f(z_n)} \quad \text{inside} \quad \mathcal{O}_{\mathfrak{U}_f(z_n)} \otimes \hat{K}.$$ 

As we remarked, if $k$ is even, $\mathcal{O}_{\mathfrak{H}}(k)$ is the line bundle generated by the element $z^{-k} \in H^0(\mathfrak{H}, \mathcal{O}_{\mathfrak{H}} \otimes \hat{K})$. For any $k$ again we have a canonical identification of sheaves $sp_* \mathcal{O}_{\hat{X}}(k) = \mathcal{O}_{\hat{X}} \otimes \hat{K}$ where $sp: \hat{X} \to \hat{K}$ is the specialization map; we write $sp_* \mathcal{O}_{\hat{X}}(k)$ when we refer to the $G$-equivariant structure on $\mathcal{O}_{\hat{X}} \otimes \hat{K}$ induced by that on $\mathcal{O}_{\hat{X}}(k)$.

**Proposition 1.1.** Let $\hat{\mathfrak{W}}, \hat{\mathfrak{W}}'$ be open formal subschemes of $\hat{\mathfrak{H}}$, let $\gamma \in G$ such that $\gamma \hat{\mathfrak{W}} = \hat{\mathfrak{W}}'$. Then the isomorphism

$$\gamma : sp_* \mathcal{O}_{\hat{X}}(k)|_{\hat{\mathfrak{W}}} \cong sp_* \mathcal{O}_{\hat{X}}(k)|_{\hat{\mathfrak{W}}'}$$

induces an isomorphism of subsheaves

$$\gamma : \mathcal{O}_{\mathfrak{H}}(k)|_{\hat{\mathfrak{W}}} \cong \mathcal{O}_{\mathfrak{H}}(k)|_{\hat{\mathfrak{W}}'}.$$ 

**PROOF:** (a) First we assume $\hat{\mathfrak{W}} \subset \hat{\mathfrak{U}}(z_n)$ for some $n$; then also $\hat{\mathfrak{W}}' \subset \hat{\mathfrak{U}}(z_{n'})$ for some $n'$ and (2) applies to $\hat{\mathfrak{W}}$ and $\hat{\mathfrak{W}}'$. In that situation we must show

$$(3) \quad 2\omega((a + cz(P))^{-k}) + k \omega(ad - bc) = k(n' - n) \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for each point $P$ in the generic fibre $\hat{\mathfrak{W}}_K^{\mathfrak{W}}$ of $\hat{\mathfrak{W}}'$. Note that $\gamma Z_{z_n} = Z_{z_{n'}}$, and thus $\gamma_{n'}^{-1} \gamma_{n}$ stabilizes $Z_{z_n}$, hence is an element of $K^\times, GL_2(\mathcal{O}_K)$; in other words, $\gamma = \gamma_0 \delta \gamma_{n}^{-1}$ for some $\delta \in K^\times, GL_2(\mathcal{O}_K)$. Therefore it suffices to check (3) in the cases

(i) $\gamma = \gamma_m$ and $n' = n + m$ for some $m \in \mathbb{Z}$;
(ii) $b = c = 0 = n = n'$ and $a = d$;
(iii) $n = n' = 0$ and $\gamma \in GL_2(\mathcal{O}_K)$.

In either case (3) is immediate; for the case (iii) note that $\omega(z(P) - \beta) = 0$ for any $\beta \in \mathcal{O}_K$. 

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(b) Now let \(\hat{\mathfrak{W}}, \hat{\mathfrak{W}}'\) be arbitrary. By construction, both \(L_1 = \gamma_*(O_{\hat{\mathfrak{W}}}(k)|_{\hat{\mathfrak{W}}})\) and \(L_2 = O_{\hat{\mathfrak{W}}}(k)|_{\hat{\mathfrak{W}}'}\) are \(O_{\hat{\mathfrak{W}}'}\)-modules contained in \(O_{\hat{\mathfrak{W}}'} \otimes_{\hat{\mathfrak{K}}} \hat{\mathfrak{K}}\) as lattices, i.e. \(L_i \otimes_{O_{\hat{\mathfrak{K}}} \hat{\mathfrak{K}}} \hat{\mathfrak{K}} = O_{\hat{\mathfrak{W}}'} \otimes_{O_{\hat{\mathfrak{K}}} \hat{\mathfrak{K}}} \hat{\mathfrak{K}}\). By (a) we have \(L_1|_{\hat{\mathfrak{W}}} = L_2|_{\hat{\mathfrak{W}}}\) for an open formal subscheme \(\hat{\mathfrak{W}}\) of \(\hat{\mathfrak{W}}'\) whose reduction is dense in the reduction of \(\hat{\mathfrak{W}}'\). All this implies \(L_1 = L_2\), using the following fact: for open formal subschemes \(\hat{\mathfrak{W}}_1 \subset \hat{\mathfrak{W}}_2\) of \(\hat{\mathfrak{W}}\) with \(\hat{\mathfrak{W}}_1\) dense in \(\hat{\mathfrak{W}}_2\), and for \(f \in (O_{\hat{\mathfrak{W}}_1}(k) \otimes_{O_{\hat{\mathfrak{K}}} \hat{\mathfrak{K}}} \hat{\mathfrak{K}})(\hat{\mathfrak{W}}_2)\) we have \(f \in O_{\hat{\mathfrak{W}}_2}(k)(\hat{\mathfrak{W}}_2)\) if and only \(f \in O_{\hat{\mathfrak{W}}_1}(k)(\hat{\mathfrak{W}}_1)\). To see this fact it suffices to show that for \(g \in (O_{\hat{\mathfrak{W}}_1}(k)/(\hat{\pi}))(\hat{\mathfrak{W}}_2)\) we have \(g = 0\) if and only \(g|_{\hat{\mathfrak{W}}_1} = 0\) in \((O_{\hat{\mathfrak{W}}_1}(k)/(\hat{\pi}))(\hat{\mathfrak{W}}_1)\). This is immediate from the local analysis in section 2 below. □

Thanks to \[1.1\] we can now move around \(O_{\hat{\mathfrak{W}}}(k)\) by means of the \(G\)-action on \(\hat{x}\) and obtain a \(G\)-equivariant coherent \(O_{\hat{x}}\)-module lattice \(O_{\hat{x}}(k)\) inside \(sp_*O_{\hat{x}}(k)\).

For \(k_1, k_2 \in \mathbb{Z}\) we have a \(G\)-equivariant surjective map (not needed in the sequel)

\[
O_{\hat{x}}(k_1) \otimes_{O_{\hat{x}}} O_{\hat{x}}(k_2) \to O_{\hat{x}}(k_1 + k_2)
\]

which is multiplication of functions. This follows from equation \[2\] and the argument in part (b) of the proof of \[1.1\]. It is an isomorphism if at least one of \(k_1\) or \(k_2\) is even, for in that case we are tensoring with a line bundle. On the other hand, it cannot be an isomorphism if both \(k_1\) and \(k_2\) are odd, because then the fibres of both \(O_{\hat{x}}(k_j)\) at singular points of \(\hat{x}\) are 2-dimensional, whereas \(O_{\hat{x}}(k_1 + k_2)\) is a line bundle (in this case).

## 2 Cohomology

For divisors \(D\) on \(\mathbb{P}_\mathbb{F}^1\) let \(L(D)\) be the corresponding line bundle on \(\mathbb{P}_\mathbb{F}^1\). By the usual convention, \(L(-D) \subset O_{\mathbb{P}_\mathbb{F}^1}\) if \(D\) is an effective divisor. Fix a system \(R\) of representatives for \(\mathbb{F}\) in \(O_{\mathfrak{K}}\). For \(a \in R\) and \(n \in \mathbb{Z}\) let

\[
\gamma_{a,n} = \begin{pmatrix} 1 & \pi^{-n}a \\ 0 & 1 \end{pmatrix}.
\]

An easy consideration on \(BT\) shows that

\[
\{Z_{\gamma_{a,n+1}}\} \cup \{Z_{\gamma_{a,n},\gamma_{n-1}}; a \in R\}
\]

is the set of the \(q + 1\) many irreducible components of \(\hat{x}\) meeting \(Z_{\gamma_n}\). (The function \(\pi^{n-1}z + \pi^{-1}a\) is a coordinate on \(\hat{U}_{\{Z_{\gamma_{a,n},\gamma_{n-1}}\}}\) in the sense that \(\omega(\pi^{n-1}z(P) + \pi^{-1}a) = 0\) for any \(P \in (\hat{U}_{\{Z_{\gamma_{a,n},\gamma_{n-1}}\}})\)). Since \(\gamma_{a,n}\) acts on \(sp_*O_{\hat{x}}(k)\) with trivial automorphy factor it induces an isomorphism

\[
\gamma_{a,n} : \hat{\pi}^{k(n-1)}O_{\hat{x}}(k)|_{\hat{U}_{\{Z_{\gamma_{a,n},\gamma_{n-1}}\}}} \cong \hat{\pi}^{k(n-1)}O_{\hat{x}}(k)|_{\hat{U}_{\{Z_{\gamma_{a,n},\gamma_{n-1}},\gamma_n\}}}.
\]
Using this we can now give a local description of the $G$-equivariant coherent $\mathcal{O}_X$-module $\mathcal{O}_\hat{X}(k)/(\tilde{\pi})$ which we denote by $\mathcal{O}_\hat{X}(k)$.

(a) First assume that $k$ is even. Let $h_a \in \mathcal{O}_{Z_{\gamma}}$ be a local equation for $Z_{\gamma} \cap Z_{\gamma,n} \cap Z_{\gamma,n+1}$ in $Z_{\gamma}$, let $h_\infty \in \mathcal{O}_{Z_{\gamma}}$ be a local equation for $Z_{\gamma} \cap Z_{\gamma,n+1}$ in $Z_{\gamma}$. Then $\mathcal{O}_\hat{X}(k) \otimes_{\mathcal{O}_X} \mathcal{O}_{Z_{\gamma}}$ is isomorphic to the following $\mathcal{O}_{Z_{\gamma}}$-submodule of the constant ”rational function field” sheaf on $Z_{\gamma} \cong \mathbb{P}_F^1$: locally around $Z_{\gamma} \cap Z_{\gamma,n} \cap Z_{\gamma,n+1}$ it is generated by $h_a^{k(n+1)-\frac{k(n+1)}{2}}$, locally around $Z_{\gamma,n} \cap Z_{\gamma,n+1}$ it is generated by $h_\infty^{\frac{k(n+1)}{2}}$, and locally around other points it coincides with $\mathcal{O}_{Z_{\gamma}}$. Thus

\begin{equation}
\mathcal{O}_\hat{X}(k) \otimes_{\mathcal{O}_X} \mathcal{O}_Z \cong \mathcal{L}(\frac{-k}{2} \infty + \sum \frac{k}{2} b)
\end{equation}

for $Z = Z_{\gamma}$. By equivariance we get \(5\) for any $Z \in F^0$. In particular, $\mathcal{O}_\hat{X}(k) \otimes_{\mathcal{O}_X} \mathcal{O}_Z$ is of degree $\frac{(q-1)k}{2}$.

(b) Now assume that $k$ is odd. For $Z \in F^0$ let

\begin{equation}
(\mathcal{O}_\hat{X}(k) \otimes_{\mathcal{O}_X} \mathcal{O}_Z)^c = \frac{\mathcal{O}_\hat{X}(k) \otimes_{\mathcal{O}_X} \mathcal{O}_Z}{(\mathcal{O}_\hat{X}(k) \otimes_{\mathcal{O}_X} \mathcal{O}_Z)_{torsion}}.
\end{equation}

We then have

\begin{equation}
\mathcal{O}_\hat{X}(k)|_{\tilde{\mathcal{U}}(z_{\gamma,n},z_{\gamma,n+1}+1)} = (\mathcal{O}_\hat{X}(k) \otimes_{\mathcal{O}_X} \mathcal{O}_{Z_{\gamma,n}})^c|_{\tilde{\mathcal{U}}(z_{\gamma,n},z_{\gamma,n+1}+1)} \oplus (\mathcal{O}_\hat{X}(k) \otimes_{\mathcal{O}_X} \mathcal{O}_{Z_{\gamma,n+1}})^c|_{\tilde{\mathcal{U}}(z_{\gamma,n},z_{\gamma,n+1}+1)}.
\end{equation}

Explicitly, $(\mathcal{O}_\hat{X}(k) \otimes_{\mathcal{O}_X} \mathcal{O}_{Z_{\gamma,n}})^c|_{\tilde{\mathcal{U}}(z_{\gamma,n},z_{\gamma,n+1}+1)}$ is generated by $f_n, f_{n,n+1}$ if $n$ is even, and by $\tilde{\pi} f_{n,n}^{k(n+1)-\frac{k(n+1)}{2}} f_{n+2,n}^{\frac{k(n+1)}{2}}$ if $n$ is odd. $(\mathcal{O}_\hat{X}(k) \otimes_{\mathcal{O}_X} \mathcal{O}_{Z_{\gamma,n+1}})^c|_{\tilde{\mathcal{U}}(z_{\gamma,n},z_{\gamma,n+1}+1)}$ is generated by $\tilde{\pi} f_{n,n}^{k(n+1)-\frac{k(n+1)}{2}} f_{n+2,n}^{\frac{k(n+1)}{2}}$ if $n$ is even, and by $f_{n,n}^{\frac{k(n+1)}{2}} f_{n+2,n}^{\frac{k(n+1)}{2}}$ if $n$ is odd. Now we proceed as in (a). By what we just saw, $(\mathcal{O}_\hat{X}(k) \otimes_{\mathcal{O}_X} \mathcal{O}_{Z_{\gamma,n}})^c$ is generated around $Z_{\gamma,n} \cap Z_{\gamma}$ by $h_\infty^{\frac{k(n+1)}{2}} - \frac{k(n+1)}{2}$, and around $Z_{\gamma,n} \cap Z_{\gamma,n+1}$ by $h_a^{\frac{k(n+1)}{2}} - \frac{k(n+1)}{2}$ (by equivariance, it suffices to check the latter for $a = 0$). Thus

\begin{equation}
\mathcal{O}_\hat{X}(k) = \prod_{Z \in F^0} (\mathcal{O}_\hat{X}(k) \otimes_{\mathcal{O}_X} \mathcal{O}_Z)^c,
\end{equation}

\begin{equation}
(\mathcal{O}_\hat{X}(k) \otimes_{\mathcal{O}_X} \mathcal{O}_Z)^c \cong \mathcal{L}(\frac{-k-1}{2} \infty + \sum \frac{k-1}{2} b)
\end{equation}

for $Z = Z_{\gamma}$. By equivariance we get \(6\) for any $Z \in F^0$. In particular, $(\mathcal{O}_\hat{X}(k) \otimes_{\mathcal{O}_X} \mathcal{O}_Z)^c$ is of degree $\frac{(q-1)(k-1)}{2} - 1$.  

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Theorem 2.1. (a) $H^*(\tilde{X}, \mathcal{O}_{\tilde{X}}(k))$ is $\mathcal{O}_R$-flat and

$$H^*(\tilde{X}, \mathcal{O}_{\tilde{X}}(k)) = H^*(\tilde{X}, \mathcal{O}_{\tilde{X}}(k))/\pi.$$ 

(b) For $k \leq -1$ and also for $k = 1$ we have $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(k)) = 0$.

c) For $k \geq 0$ we have $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(k)) = 0$.

PROOF: (i) First assume $k$ is even. To prove (c) it is enough to prove

$$\mathbb{R}^1 \lim_{\tilde{t}} H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(k)/(\tilde{\pi}^t)) = 0$$

and also for

$$\mathbb{R}^1 \lim_{\tilde{t}} H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(k)/(\tilde{\pi}^t)) = 0.$$ 

For (7) it suffices to show surjectivity of all transition maps $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(k)/(\tilde{\pi}^{t+1})) \to H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(k)/(\tilde{\pi}^t))$. Using the long exact cohomology sequence associated with

$$0 \to \mathcal{O}_{\tilde{X}}(k) \to \mathcal{O}_{\tilde{X}}(k)/(\tilde{\pi}^t) \to 0$$

this will be implied by

$$H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(k)) = 0.$$ 

Also (8) is reduced to (10) using (9), so let us prove (10). We have an exact sequence

$$0 \to \mathcal{O}_{\tilde{X}}(k) \to \prod_{Z \in F^0} \mathcal{O}_{\tilde{X}}(k) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_Z \to \prod_{\{Z_1, Z_2\} \in F^1} \mathcal{O}_{\tilde{X}}(k) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_{Z_1 \cap Z_2} \to 0$$

and a corresponding long exact sequence in cohomology. We know

$$H^1(\tilde{X}, \prod_{Z \in F^0} \mathcal{O}_{\tilde{X}}(k) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_Z) = \prod_{Z \in F^0} H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(k) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_Z) = 0$$

because $\mathcal{O}_{\tilde{X}}(k) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_Z$ is isomorphic to a line bundle on $\mathbb{P}^1 \cong Z$ of non-negative degree as we saw above (since $k \geq 0$). On the other hand

$$H^0(\tilde{X}, \prod_{Z \in F^0} \mathcal{O}_{\tilde{X}}(k) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_Z) \to H^0(\tilde{X}, \prod_{\{Z_1, Z_2\} \in F^1} \mathcal{O}_{\tilde{X}}(k) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_{Z_1 \cap Z_2})$$

is surjective: This follows from the contractibility of $\mathcal{B}T$ and again the fact that each $\mathcal{O}_{\tilde{X}}(k) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_Z$ has non-negative degree, which implies that

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(k) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_{Z_1}) \to H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(k) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_{Z_1 \cap Z_2})$$

for any $\{Z_1, Z_2\} \in F^1$ is surjective. To prove (b), since $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(k)) = \lim_{\tilde{t}} H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(k)/(\pi^t))$ we can reduce, using the long exact cohomology sequence associated with (9), to the statement

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(k)) = 0.$$
But this follows immediately from the injectivity of
\[ H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(k)) \rightarrow H^0(\tilde{X}, \prod_{Z \in P^0} \mathcal{O}_{\tilde{X}}(k) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_Z) \]
and the fact that \( \mathcal{O}_{\tilde{X}}(k) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_Z \) for each \( Z \in P^0 \) is isomorphic to a line bundle on \( \mathbb{P}^1 \cong Z \) of negative degree as we saw above. To see the \( \mathcal{O}_K \)-flatness of \( H^*(\tilde{X}, \mathcal{O}_{\tilde{X}}(k)) \) in (a) we need to show injectivity of multiplication with \( \pi \). This follows from (the proof of) (b) and (c) and the long exact cohomology sequence associated with
\[ 0 \rightarrow \mathcal{O}_{\tilde{X}}(k) \xrightarrow{\pi} \mathcal{O}_{\tilde{X}}(k) \rightarrow \mathcal{O}_{\tilde{X}}(k) \rightarrow 0. \]
The base change statement follows similarly.

(ii) For odd \( k \) the proofs are similar but easier in view of the decomposition \( [5] \). \( \square \)

The important vanishing \( H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(k)) = 0 \) was asserted for even \( k \geq 0 \) in \([6] \) Cor.24. However, the comparison with \( H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(k)) \) invoked there does not seem to be justified.

Let \( \Gamma < \text{SL}_2(K) \) be a cocompact discrete subgroup which for simplicity we assume to be torsion free (in general it contains a torsion free subgroup of finite index). Let \( X_\Gamma = \Gamma \backslash X, \tilde{X}_\Gamma = \Gamma \backslash \tilde{X}, \tilde{X}_\Gamma = \Gamma \backslash \tilde{\tilde{X}} \) and \( \tilde{X}_\Gamma = \Gamma \backslash \tilde{\tilde{X}} \) be the quotients for the free action by \( \Gamma \); they all algebraize to projective schemes.

**Corollary 2.2.** (a) For \( k > 0 \) we have
\[ H^0(\tilde{X}_\Gamma, \mathcal{O}_{\tilde{X}}(-k)^\Gamma) = H^1(\tilde{X}_\Gamma, \mathcal{O}_{\tilde{X}}(k + 2)^\Gamma). \]
In particular, \( H^0(X_\Gamma, \mathcal{O}_X(-k)^\Gamma) = H^1(X_\Gamma, \mathcal{O}_X(k + 2)^\Gamma) \).
(b) \( H^0(\tilde{X}_\Gamma, \mathcal{O}_{\tilde{X}}(k + 2)^\Gamma) \) and \( H^1(\tilde{X}_\Gamma, \mathcal{O}_{\tilde{X}}(-k)^\Gamma) \) are \( \mathcal{O}_K \)-flat and
\[ H^0(\tilde{X}_\Gamma, \mathcal{O}_{\tilde{X}}(k + 2)^\Gamma) \otimes_{\mathcal{O}_K} \mathbb{F} = H^0(\tilde{X}_\Gamma, \mathcal{O}_{\tilde{X}}(k + 2)^\Gamma) \]
\[ H^1(\tilde{X}_\Gamma, \mathcal{O}_{\tilde{X}}(-k)^\Gamma) \otimes_{\mathcal{O}_K} \mathbb{F} = H^1(\tilde{X}_\Gamma, \mathcal{O}_{\tilde{X}}(-k)^\Gamma). \]
(c) Serre duality identifies \( H^1(X_\Gamma, \mathcal{O}_X(-k)^\Gamma) \) with the dual of \( H^0(X_\Gamma, \mathcal{O}_X(k + 2)^\Gamma) \).
(d) \( H^j(\tilde{X}_\Gamma, \mathcal{O}_{\tilde{X}}(1)^\Gamma) = 0 \)
for any \( j \). In particular, \( H^j(X_\Gamma, \mathcal{O}_X(1)^\Gamma) = 0 \).

**Proof:** (a) For odd \( k \) literally the same proof as in \([2.1] \) applies, because in that case we have the decomposition \( [5] \) which allows us to reduce to problems on each irreducible component — these are the same for \( \tilde{X} \) and \( \tilde{X}_\Gamma \). Now let \( k \) be even. From \([2.1] \) we
get $H^0(\tilde{X}, \mathcal{O}_X(-k)) = 0$ and $H^0(\tilde{X}, \mathcal{O}_X(-k)) = 0$. In particular $H^0(\tilde{X}_\Gamma, \mathcal{O}_X(-k)\Gamma) = H^0(\tilde{X}, \mathcal{O}_X(-k))\Gamma = 0$ and $H^0(\tilde{X}_\Gamma, \mathcal{O}_X(-k)\Gamma) = H^0(\tilde{X}, \mathcal{O}_X(-k))\Gamma = 0$. Now we have a $SL_2(K)$-equivariant isomorphism $\mathcal{O}_X(2) \cong \Omega^1_{\tilde{X}}$ on $\tilde{X}$, where $\Omega^1_{\tilde{X}}$ is the sheaf of relative logarithmic differentials for the log smooth formal $\text{Spf}(\mathcal{O}_K)$-scheme $\tilde{X}$ (with respect to the pull back log structures from the canonical log structures on $X$ and $\text{Spf}(\mathcal{O}_K)$). Thus $\mathcal{O}_X(2)\Gamma$ can be identified with the sheaf of relative logarithmic differentials for the log smooth projective $\text{Spec}(\mathcal{O}_K)$-scheme $\tilde{X}_\Gamma$. This is a dualizing sheaf by [3] Ch.I, sect.2, where it is called the sheaf of regular differentials (the generalization to general projective log schemes is [9] Theorem 2.21). Since $\mathcal{O}_X(k + 2)\Gamma = (\mathcal{O}_X(-k)\Gamma)^{\otimes (-1)} \otimes \mathcal{O}_X(2)\Gamma$ (note that since $k$ is even we are dealing with line bundles here) we get $H^1(\tilde{X}_\Gamma, \mathcal{O}_X(k + 2)\Gamma) = 0$ by Serre duality. The same argument works for the sheaves $\mathcal{O}_X(.)$. For (b) we may now proceed as in [2.11]. For (c) note that $\mathcal{O}_X(2)$ is $SL_2(K)$-equivariantly isomorphic with the sheaf $\Omega^1_X$ of differentials on $X$, hence $\mathcal{O}_X(k + 2)\Gamma \cong (\mathcal{O}_X(-k)\Gamma)^{\otimes (-1)} \otimes \Omega^1_X\Gamma$ (for even $k$ we just saw the integral version in (a)). The statements in (d) follow immediately from [2.11]

The fact $H^0(X_\Gamma, \mathcal{O}_X(1)\Gamma) = 0$ ("there are no non zero automorphic forms for $\Gamma$ of weight one") was proven by analytic methods in [6] Cor.13. For the $K$-vector space dimensions of $H^1(X_\Gamma, \mathcal{O}_X(-k)\Gamma)$ and of $H^0(X_\Gamma, \mathcal{O}_X(k + 2)\Gamma)$ see [5,3] below.

### 3 Modular representations

Denote by $\mathcal{I} \subset \mathcal{O}_X$ the ideal sheaf of functions vanishing at the singular points of $\tilde{X}$. For $k \in \mathbb{Z}$ and $i \geq 0$ let

$$\mathcal{O}_X(k)(i) = \mathcal{O}_X(k) \otimes_{\mathcal{O}_X} \mathcal{T}^i.$$  

Let $Z \in F^0$. If $k$ is odd we let

$$(\mathcal{O}_X(k)(i) \otimes_{\mathcal{O}_X} \mathcal{O}_Z)^c = (\mathcal{O}_X(k) \otimes_{\mathcal{O}_X} \mathcal{O}_Z)^c \otimes_{\mathcal{O}_X} \mathcal{T}^i.$$  

To unify notations, if $k$ is even we let $(\mathcal{O}_X(k)(i) \otimes_{\mathcal{O}_X} \mathcal{O}_Z)^c = \mathcal{O}_X(k) \otimes_{\mathcal{O}_X} \mathcal{O}_Z$ and

$$(\mathcal{O}_X(k)(i) \otimes_{\mathcal{O}_X} \mathcal{O}_Z)^c = \mathcal{O}_X(k) \otimes_{\mathcal{O}_X} \mathcal{O}_Z \otimes_{\mathcal{O}_X} \mathcal{T}^i,$$

i.e. for even $k$ the outer $(.)^c$ is redundant. We have

$$\mathcal{O}_X(k)(i) = \prod_{Z \in F^0} (\mathcal{O}_X(k)(i) \otimes_{\mathcal{O}_X} \mathcal{O}_Z)^c. \tag{11}$$

if $k$ is odd and $i \geq 0$ arbitrary, and also if $k$ is even and $i > 0$. In particular, for such $(k, i)$ we have for any $Z \in F^0$ the natural injection

$$\iota_Z : (\mathcal{O}_X(k)(i) \otimes_{\mathcal{O}_X} \mathcal{O}_Z)^c \longrightarrow \mathcal{O}_X(k)(i)$$  

and the canonical projection map

\[ \rho_Z : \mathcal{O}_{\tilde{x}}(k)(i) \to (\mathcal{O}_{\tilde{x}}(k)(i) \otimes_{\mathcal{O}_{\tilde{x}}} \mathcal{O}_Z)^\gamma. \]

We denote maps induced by \( \iota_Z \) resp. \( \rho_Z \) in cohomology again by \( \iota_Z \) resp. \( \rho_Z \).

**Lemma 3.1.** Suppose \( i \geq 0 \) if \( k \) is odd, or \( i > 0 \) if \( k \) is even. Then we have a canonical \( G \)-equivariant isomorphism

\[ H^*(\tilde{x}, \mathcal{O}_{\tilde{x}}(k)(i)) \cong \text{Ind}^{\mathbb{Q}}_{K^\times \text{GL}_2(O_K)}H^*(\tilde{x}, (\mathcal{O}_{\tilde{x}}(k)(i) \otimes_{\mathcal{O}_{\tilde{x}}} \mathcal{O}_{Z_{\gamma_0}})^\gamma) \]

**Proof:** By definition, \( \text{Ind}^{\mathbb{Q}}_{K^\times \text{GL}_2(O_K)}H^*(\tilde{x}, (\mathcal{O}_{\tilde{x}}(k)(i) \otimes_{\mathcal{O}_{\tilde{x}}} \mathcal{O}_{Z_{\gamma_0}})^\gamma) \) is the space of locally constant functions \( u : G \to H^*(\tilde{x}, (\mathcal{O}_{\tilde{x}}(k)(i) \otimes_{\mathcal{O}_{\tilde{x}}} \mathcal{O}_{Z_{\gamma_0}})^\gamma) \) which satisfy \( u(\gamma) = \eta(u(\gamma)) \) for \( \eta \in K^\times \text{GL}_2(O_K) \), \( \gamma \in G \). The action of \( G \) is by \( (\gamma, u)(\gamma') = u(\gamma' \gamma) \). Note that \( K^\times \text{GL}_2(O_K) \) is the stabilizer of \( Z_{\gamma_0} \) in \( G \). Let \( S \subset G \) be a subset such that \( \gamma \mapsto Z_\gamma \) is a bijection between \( S \) and \( F^0 \). The desired map is

\[ f \mapsto [u : G \to H^*(\tilde{x}, (\mathcal{O}_{\tilde{x}}(k)(i) \otimes_{\mathcal{O}_{\tilde{x}}} \mathcal{O}_{Z_{\gamma_0}})^\gamma), \gamma \mapsto \rho_{Z_{\gamma_0}}(\gamma, f)]. \]

Its inverse is

\[ [u : G \to H^*(\tilde{x}, (\mathcal{O}_{\tilde{x}}(k)(i) \otimes_{\mathcal{O}_{\tilde{x}}} \mathcal{O}_{Z_{\gamma_0}})^\gamma)] \mapsto \sum_{\gamma \in S} \gamma(\iota_{Z_{\gamma_0}}(u(\gamma^{-1}))). \]

Note that \( \gamma(\iota_{Z_{\gamma_0}}(u(\gamma^{-1}))) \) is supported only on \( Z_\gamma \).

For a commutative ring \( A \) and integers \( n, s \) with \( n \geq 0 \) let us denote by \( \text{Sym}^n_A(\text{St})[s] \) the free \( A \)-module of homogeneous polynomials \( F(X, Y) \) of degree \( n \) in the variables \( X, Y \) with coefficients in \( A \), together with its \( \text{GL}_2(A) \)-action

\[ \gamma.F(X, Y) = (ad - bc)^s(F(dX + bY, cX + aY)) \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(A). \]

Now consider for \( k \in \mathbb{Z} \) the action of \( \text{GL}_2(\mathbb{F}) \) on \( \mathbb{F}(z) \) given by

\[ f|_{\gamma}(z) = \left( \frac{1}{a + cz} \right)^k f\left( \frac{b + dz}{a + cz} \right) \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \]

We view \( \mathbb{F}(z) \) as the function field of \( \mathbb{P}^1_{\mathbb{F}} = \text{Spec}(\mathbb{F}[z]) \cup \{ \infty \} \) and will consider line bundles on \( \mathbb{P}^1_{\mathbb{F}} \) stable for \( (12) \). Let \( i \geq 0 \).

**Lemma 3.2.** (a) Suppose \( k \) is even and \( t = \frac{(q - 1)k}{2} - i(q + 1) \geq 0 \). Then, as \( \text{GL}_2(\mathbb{F}) \)-representations,

\[ \text{Sym}^t_{\mathbb{F}}(\text{St})[i - \frac{k}{2}] \cong H^0(\mathbb{P}^1_{\mathbb{F}}, \mathcal{L}(\sum_{l \in \mathbb{F}} (\frac{k}{2} - i)b - (\frac{k}{2} + i).\infty)). \]
(b) Suppose $k$ is odd and $t = \frac{(q-1)k-(q+1)}{2} - i(q+1) \geq 0$. Then, as $GL_2(\mathbb{F})$-representations,

$$\text{Sym}_\mathbb{F}^t(\text{St})[i - \frac{k-1}{2}] \cong H^0(\mathbb{P}^1_{\mathbb{F}}, L(\sum_{b \in \mathbb{F}} \frac{k-1}{2} - i) \cdot b - \frac{k+1}{2} + i) \cdot \infty).$$

**Proof:** In (a) the map sends $X^r Y^{t-r}$ to $z^r(z - z')^{t-r}$ for $0 \leq r \leq t$. In (b) it sends $X^r Y^{t-r}$ to $z^r(z - z')^{t-r}$ for $0 \leq r \leq t$. We view $\text{Sym}^n(\text{St})[s]$ as a $GL_2(\mathcal{O}_K)$-representation via the canonical map $GL_2(\mathcal{O}_K) \to GL_2(\mathbb{F})$, and we then extend the action further to an action by $K \times GL_2(\mathcal{O}_K)$ by sending $\pi \in K^\times$ (i.e. the diagonal matrix with both entries equal to $\pi = \hat{\pi}^2$) to the identity.

**Theorem 3.3.** (a) Suppose $k$ is even, $i > 0$ and $t = \frac{(q-1)k}{2} - i(q+1) \geq 0$. Then we have a canonical $G$-equivariant isomorphism

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(k)(i)) \cong \text{Ind}_{K^\times GL_2(\mathcal{O}_K)}^G \text{Sym}_\mathbb{F}^t(\text{St})[i - \frac{k}{2}].$$

(b) Suppose $k$ is odd, $i \geq 0$ and $t = \frac{(q-1)(k-1)}{2} - 1 - i(q+1) \geq 0$. Then we have a canonical $G$-equivariant isomorphism

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(k)(i)) \cong \text{Ind}_{K^\times GL_2(\mathcal{O}_K)}^G \text{Sym}_\mathbb{F}^t(\text{St})[i - \frac{k-1}{2}].$$

**Proof:** We lift the $GL_2(\mathbb{F})$-action on $H^0(\mathbb{P}^1_{\mathbb{F}}, L(\sum_{b \in \mathbb{F}} \frac{k-1}{2} - i) \cdot b - \frac{k+1}{2} + i) \cdot \infty))$ if $k$ is even, resp. on $H^0(\mathbb{P}^1_{\mathbb{F}}, L(\sum_{b \in \mathbb{F}} \frac{k-1}{2} - i) \cdot b - \frac{k+1}{2} + i) \cdot \infty))$ if $k$ is odd, to an action by $K \times GL_2(\mathcal{O}_K)$ in the same way as explained for $\text{Sym}^n(\text{St})[s]$. Identifying the reduction of the global variable $z$ with our projective coordinate $z$ on $Z_{\gamma_0} \cong \mathbb{P}^1_{\mathbb{F}}$ we use [3] and [4] to get $K \times GL_2(\mathcal{O}_K)$-equivariant isomorphisms

$$H^0(\mathbb{P}^1_{\mathbb{F}}, L(\sum_{b \in \mathbb{F}} \frac{k-1}{2} - i) \cdot b - \frac{k+1}{2} + i) \cdot \infty)) = H^0(\tilde{X}, (\mathcal{O}_{\tilde{X}}(k)(i) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_{Z_{\gamma_0}}) \cdot c)$$

if $k$ is even, resp.

$$H^0(\mathbb{P}^1_{\mathbb{F}}, L(\sum_{b \in \mathbb{F}} \frac{k-1}{2} - i) \cdot b - \frac{k+1}{2} + i) \cdot \infty)) = H^0(\tilde{X}, (\mathcal{O}_{\tilde{X}}(k)(i) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_{Z_{\gamma_0}}) \cdot c)$$

if $k$ is odd, thus we conclude by 3.1 and 3.2.

We can now filter the representation $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(k))$ and determine its subquotients. For $k odd, i \geq 0$ and $t = \frac{(q-1)(k-1)}{2} - 1 - i(q+1) \geq q+1$ we have $H^1(\tilde{X}, (\mathcal{O}_{\tilde{X}}(k)(i+1) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_{Z_{\gamma_0}}) \cdot c) = 0$ (use [6]), hence

$$\frac{\text{Sym}_\mathbb{F}^t(\text{St})[i - \frac{k-1}{2}]}{\text{Sym}_\mathbb{F}^{t-(q+1)}(\text{St})[i + 1 - \frac{k-1}{2}]} \cong \frac{H^0(\tilde{X}, (\mathcal{O}_{\tilde{X}}(k)(i) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_{Z_{\gamma_0}}) \cdot c)}{H^0(\tilde{X}, (\mathcal{O}_{\tilde{X}}(k)(i+1) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_{Z_{\gamma_0}}) \cdot c)}$$
is a representation of $\text{GL}_2(\mathbb{F})$ on the $(q + 1)$-dimensional $\mathbb{F}$-vector space with basis the $\mathbb{F}$-rational points of $\mathbb{P}^1_{\mathbb{F}}$. Explicitly, this is the quotient

$$\frac{\text{Sym}_F^t(\text{St})[i - \frac{k-1}{2}]}{< XjY^{t-j} - Xq+j-1Y^{t-q-j+1}; 1 \leq j \leq t-q >_F}$$

One might ask for its composition series. For example, if $q = 2$, $k = 9$, $i = 0$, $t = 3$, then the class of $X^3 + Y^3 + X^2Y$ (i.e. the class of $X^3 + Y^3 + XY^2$) in this quotient spans a $\text{GL}_2(\mathbb{F})$-stable line. The results for even $k$ are similar, with $i > 0$ and $t = \frac{(q-1)k}{2} - i(q + 1) \geq q + 1$, see also [8]. For the last $i$, the one for which $q \geq t \geq 0$, we get $\text{Sym}_F^t(\text{St})[i - \frac{k-1}{2}]$ (if $k$ is odd), resp. $\text{Sym}_F^t(\text{St})[i - \frac{k}{2}]$ (if $k$ is even). To complete the picture it remains to observe that for even $k \geq 4$ we have

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(k)) \cong \text{Ind}_N^G 1$$

where $N \subset G$ denotes the stabilizer of an (arbitrary) non-oriented edge $\{Z_1, Z_2\} \in F^1$ and $1$ its trivial representation: use $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(k)(1)) = 0$.

As an application, if $q$ is odd, Teitelbaum [8] constructs modular forms mod $\pi$ of weight $q+1$ (in fact, elements of $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(q+1)(\frac{q+1}{2} - 1))$ for the entire group $\text{SL}_2(K)$. Here we will do the same if $q$ is even. The action of $\text{SL}_2(K)$ on the set $F^0$ has two orbits: the orbit $F^0_{\text{even}}$ of $Z_{\gamma_0} \in F^0$ and the orbit $F^0_{\text{odd}}$ of $Z_{\gamma_1} \in F^0$. Choose subsets $S_{\text{even}}$ and $S_{\text{odd}}$ of $\text{SL}_2(K)$ such that $\gamma \mapsto Z_{\gamma}$ defines bijections $S_{\text{even}} \cong F^0_{\text{even}}$ and $S_{\text{odd}} \cong F^0_{\text{odd}}$. Recall that we fixed a coordinate $z$ on $X$. For any $\gamma \in \text{SL}_2(K)$ we get another function $z \circ \gamma$ on $X$.

**Theorem 3.4.** The $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(q+1))$-elements

$$b^+_{q+1} = \sum_{\gamma \in S_{\text{even}}} (\iota_{Z_{\gamma}} \circ \rho_{Z_{\gamma}})((z \circ \gamma^{-1} - (z \circ \gamma^{-1})^q)^{-1})$$

$$b^-_{q+1} = \sum_{\gamma \in S_{\text{odd}}} (\iota_{Z_{\gamma}} \circ \rho_{Z_{\gamma}})((z \circ \gamma^{-1} - (z \circ \gamma^{-1})^q)^{-1})$$

are invariant for $\text{SL}_2(K)$, and interchanged by $\begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$.

Now let us look at the modular representations $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(k))$ for $k < 0$. If $k$ is even we get from

$$0 \rightarrow \mathcal{O}_{\tilde{X}}(k) \rightarrow \prod_{Z \in F^0} \mathcal{O}_{\tilde{X}}(k) \otimes \mathcal{O}_Z \rightarrow \prod_{\{Z_1, Z_2\} \in F^1} \mathcal{O}_{\tilde{X}}(k) \otimes \mathcal{O}_{\tilde{X}} \mathcal{O}_{Z_1 \cap Z_2} \rightarrow 0$$
the exact sequence

\[ 0 \rightarrow H^0(\tilde{\mathcal{X}}, \prod_{\{Z_1, Z_2\} \in \mathcal{F}^1} \mathcal{O}_{\tilde{\mathcal{X}}}(k) \otimes \mathcal{O}_{Z_1 \cap Z_2}) \rightarrow \]

\[ H^1(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{\mathcal{X}}}(k)) \rightarrow H^1(\tilde{\mathcal{X}}, \prod_{Z \in \mathcal{F}^0} \mathcal{O}_{\tilde{\mathcal{X}}}(k) \otimes \mathcal{O}_Z) \rightarrow 0. \]

Here \( H^0(\tilde{\mathcal{X}}, \prod_{\{Z_1, Z_2\} \in \mathcal{F}^1} \mathcal{O}_{\tilde{\mathcal{X}}}(k) \otimes \mathcal{O}_{Z_1 \cap Z_2}) \) is as in (13). If \( k \) is odd things are easier because then we have

\[ H^1(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{\mathcal{X}}}(k)) \cong H^1(\tilde{\mathcal{X}}, \prod_{Z \in \mathcal{F}^0} (\mathcal{O}_{\tilde{\mathcal{X}}}(k) \otimes \mathcal{O}_Z)^c). \]

Thus for any \( k \), even or odd, we need to understand \( H^1(\tilde{\mathcal{X}}, \prod_{Z \in \mathcal{F}^0} (\mathcal{O}_{\tilde{\mathcal{X}}}(k) \otimes \mathcal{O}_Z)^c) \) as a \( G \)-representation; by (the proof of) 3.1 this means understanding \( H^1(\tilde{\mathcal{X}}, (\mathcal{O}_{\tilde{\mathcal{X}}}(k) \otimes \mathcal{O}_{Z_0})^c) \) as a \( \text{GL}_2(\mathbb{F}) \)-representation. By an explicit computation on \( \mathbb{P}^1_{\mathbb{F}} \), using the formulas (4) and (6), we see that Serre duality yields a \( \text{GL}_2(\mathbb{F}) \)-equivariant isomorphism

\[ H^1(\tilde{\mathcal{X}}, (\mathcal{O}_{\tilde{\mathcal{X}}}(k) \otimes \mathcal{O}_{Z_0})^c) \cong \text{Hom}_F(H^0(\tilde{\mathcal{X}}, (\mathcal{O}_{\tilde{\mathcal{X}}}(-k + 2)(1) \otimes \mathcal{O}_{Z_0})^c), F) \]

if \( k \) is even, resp.

\[ H^1(\tilde{\mathcal{X}}, (\mathcal{O}_{\tilde{\mathcal{X}}}(k) \otimes \mathcal{O}_{Z_0})^c) \cong \text{Hom}_F(H^0(\tilde{\mathcal{X}}, (\mathcal{O}_{\tilde{\mathcal{X}}}(-k + 2) \otimes \mathcal{O}_{Z_0})^c), F) \]

if \( k \) is odd. The duals of these representations have been determined above. For example, for odd \( k < 0 \), setting \( t = \frac{(g-1)(k-1)}{2} - 1 \) we get a canonical \( G \)-equivariant isomorphism

\[ H^1(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{\mathcal{X}}}(k)) \cong \text{Ind}_K^{G} \times_{\text{GL}_2(\mathcal{O}_K)} \text{Hom}_F(\text{Sym}_{\mathbb{F}}^t(\mathcal{S}t)[\frac{k-1}{2}], F). \]

On the other hand, in section 31 below we will obtain for any \( k < 0 \), even or odd, \( G \)-equivariant isomorphisms

\[ H^1(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{\mathcal{X}}}(k)) \otimes \varepsilon^{-k-1} \cong H^0(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{\mathcal{X}}}(2 - k)). \]

4 Harmonic cochains

Fix \( k \geq 0 \). On \( \text{Hom}_{\tilde{\mathcal{X}}}(\text{Sym}_K^k(\mathcal{S}t)[1] \otimes \chi^{-k-2}, \tilde{\mathcal{K}}) \) the \( G \)-action is given by \( (\gamma, h)(x) = h(\gamma^{-1} \cdot x) \) for \( \gamma \in G \), \( x \in \text{Sym}_K^k(\mathcal{S}t)[1] \otimes \chi^{-k-2} \) and \( h \in \text{Hom}_{\tilde{\mathcal{X}}}(\text{Sym}_K^k(\mathcal{S}t)[1] \otimes \chi^{-k-2}, \tilde{\mathcal{K}}) \). — (In everything here and below we could replace \( \text{Hom}_{\tilde{\mathcal{X}}}(\text{Sym}_K^k(\mathcal{S}t)[1] \otimes \chi^{-k-2}, \tilde{\mathcal{K}}) \) by the isomorphic \( G \)-representation \( \text{Sym}_K^k(\mathcal{S}t)[-k-1] \otimes \chi^k+2 \); the isomorphism sends \( h_j \) (as defined below) to \( X^{k-j}Y^j \).) — We set

\[ C^1(k + 2) = \prod_{\{Z_1, Z_2\} \in \mathcal{F}^1} \text{Hom}_{\tilde{\mathcal{X}}}(\text{Sym}_K^k(\mathcal{S}t)[1] \otimes \chi^{-k-2}, \tilde{\mathcal{K}}). \]
\[ C^0(k + 2) = \prod_{Z \in F^0} \text{Hom}_{\hat{\mathbb{K}}}((\text{Sym}_k\hat{\mathbb{K}}(\text{St})[1] \otimes \chi^{-k-2}, \hat{\mathbb{K}}) \]

(products of copies of \(\text{Hom}_{\hat{\mathbb{K}}}((\text{Sym}_k\hat{\mathbb{K}}(\text{St})[1] \otimes \chi^{-k-2}, \hat{\mathbb{K}})\), indexed by \(F^1\) resp. \(F^0\)). On \(C^1(k + 2)\) we define a \(G\)-action by

\[ (\gamma.f)(z_1, z_2) = \gamma(f_{\gamma^{-1}}(z_1, z_2)) \]

for \(\gamma \in G\) and \((f_{\{z_1, z_2\}}(z_1, z_2) \in C^1(k + 2)\). For \(Z \in F^0\) let \(\text{sg}(Z) = 1\) if \(Z \in F^0_{\text{even}}\) and \(\text{sg}(Z) = -1\) if \(Z \in F^0_{\text{odd}}\). Moreover let

\[ * (Z) = \{ Z' \in F^0; \{ Z, Z' \} \in F^1 \}. \]

Then we have the operator

\[ C^1(k + 2) \xrightarrow{\Delta} C^0(k + 2), \quad (f_{\{z_1, z_2\}}(z_1, z_2) \mapsto (\text{sg}(Z) \sum_{Z' \in * (Z)} f_{\{z, z'\}})z \]

and we define \(C^1_{\text{har}}(k + 2)\) by the exact sequence

\[ 0 \longrightarrow C^1_{\text{har}}(k + 2) \longrightarrow C^1(k + 2) \xrightarrow{\Delta} C^0(k + 2). \]

This is the variant with non-trivial coefficients of the space \(C^1_{\text{har}}(\hat{\mathbb{K}})\) of \(\hat{\mathbb{K}}\)-valued harmonic cochains on \(\mathcal{B}T\) which is defined by the exact sequence

\[ 0 \longrightarrow C^1_{\text{har}}(\hat{\mathbb{K}}) \longrightarrow \prod_{\{Z_1, Z_2\} \in F^1} \hat{\mathbb{K}} \xrightarrow{\Delta} \prod_{Z \in F^0} \hat{\mathbb{K}}. \]

Let \(\Omega^1_{\hat{\mathbb{K}}}\) denote the sheaf of logarithmic differential forms for the morphism of log schemes \(\hat{\mathcal{X}} \rightarrow \text{Spf}(\mathcal{O}_{\hat{\mathbb{K}}})\) (with log structures defined by the respective special fibres). Define

\[ \text{res} : \Gamma(\hat{\mathcal{X}}, \Omega^1_{\hat{\mathbb{K}}}) \rightarrow C^1_{\text{har}}(\hat{\mathbb{K}}) \]

to be the unique \(G\)-equivariant morphism of \(\mathcal{O}_{\hat{\mathbb{K}}}-\text{modules with} \]

\[ \text{res}(\eta)_{\{Z_{\gamma_0}, Z_{\gamma-1}\}} = a_{-1} \]

for \(\eta \in \Gamma(\hat{\mathcal{X}}, \Omega^1_{\hat{\mathbb{K}}})\), where

\[ \eta(z) = \sum_{j \in \mathbb{Z}} a_j z^j dz \]

is the Laurent expansion of \(\eta\) on the annulus \(|Z_{\gamma_0} \cap Z_{\gamma-1}| = sp^{-1}(Z_{\gamma_0} \cap Z_{\gamma-1}) \subset \hat{\mathcal{X}}\) reducing to \(Z_{\gamma_0} \cap Z_{\gamma-1}\). (That \(\text{res}(\eta)\) indeed lies in \(C^1_{\text{har}}(\hat{\mathbb{K}})\) follows from the residue theorem on \(\mathbb{P}^1\).) This map also has a version with non-trivial coefficients, as follows. Consider the \(G\)-equivariant map

\[ \Gamma(\hat{\mathcal{X}}, \mathcal{O}_{\hat{\mathbb{K}}}(k + 2)) \longrightarrow \text{Hom}_{\hat{\mathbb{K}}}((\text{Sym}_k\hat{\mathbb{K}}(\text{St})[1] \otimes \chi^{-k-2}, \Gamma(\hat{\mathcal{X}}, \Omega^1_{\hat{\mathbb{K}}})) \]

\(g \mapsto \Phi_g\)
where $\Phi_g$ is defined by
\[
\Phi_g(X^iY^{k-i}) = g(z)z^idz, \quad 0 \leq i \leq k.
\]
We use it to define the $G$-equivariant map
\[
\text{Res}^0 : \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X}(k+2)) \to \text{Hom}(\text{Sym}^k_{\hat{K}}(\text{St})[1] \otimes \chi^{-k-2}, C^1_{\text{har}}(\hat{K})) = C^1_{\text{har}}(k+2)
\]
\[
g \mapsto \text{res} \circ \phi_g.
\]
We will work with the following more explicit description of $\text{Res}^0$: it is the unique $G$-equivariant morphism of $\mathcal{O}_\hat{K}$-modules with
\[
(\text{Res}^0(g)(z_{\gamma_0},z_{\gamma_{-1}})))(X^iY^{k-i}) = a_{-i-1}
\]
for $g \in \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X}(k+2))$ and $0 \leq i \leq k$, where
\[
g(z) = \sum_{j \in \mathbb{Z}} a_j z^j
\]
is the Laurent expansion of $g$ on the annulus $]Z_{\gamma_0} \cap Z_{\gamma_{-1}}[ = sp^{-1}(Z_{\gamma_0} \cap Z_{\gamma_{-1}}) \subset \mathcal{X}$ reducing to $Z_{\gamma_0} \cap Z_{\gamma_{-1}}$. Equivalently, $(\text{Res}^0(g)(z_1,z_2))(X^iY^{k-i})$ for arbitrary $\{Z_1, Z_2\} \in F^1$ can be described follows. Choose a $\gamma \in G$ such that $\gamma \cdot \{Z_1, Z_2\} = \{Z_{\gamma_0}, Z_{\gamma_{-1}}\}$. Let $\sum_{j \in \mathbb{Z}} a_j z^j$ be the Laurent expansion of $\gamma \cdot g$ on $]Z_{\gamma_0} \cap Z_{\gamma_{-1}}[ \cap \mathcal{X}$ and write
\[
\gamma \cdot (X^iY^{k-i}) = \sum_{s=0}^k c_s X^s Y^{k-s}
\]
in $\text{Sym}^k_{\hat{K}}(\text{St})[1] \otimes \chi^{-k-2}$. Then $(\text{Res}^0(g)(z_1,z_2))(X^iY^{k-i}) = \sum_{s=0}^k a_{s-1} c_s$. This is independent of the choice of $\gamma$.

We want to show that $\text{Res}^0$ is injective and to describe its image. For $Z \in F^0$ choose $\gamma \in G$ with $Z = Z_{\gamma}$ and define
\[
L_Z = \gamma \cdot \text{Hom}_{\mathcal{O}_\hat{K}}(\text{Sym}^k_{\mathcal{O}_\hat{K}}(\text{St})[1], \mathcal{O}_\hat{K}) \subset \text{Hom}_{\hat{K}}(\text{Sym}^k_{\hat{K}}(\text{St})[1] \otimes \chi^{-k-2}, \hat{K}).
\]
In this definition we consider $\text{Hom}_{\mathcal{O}_\hat{K}}(\text{Sym}^k_{\mathcal{O}_\hat{K}}(\text{St})[1], \mathcal{O}_\hat{K})$ not as a $\text{GL}_2(\mathcal{O}_K)$-representation but only as a $\mathcal{O}_\hat{K}$-submodule of the $\hat{K}$-vector space underlying the $G$-representation $\text{Hom}_{\hat{K}}(\text{Sym}^k_{\hat{K}}(\text{St})[1] \otimes \chi^{-k-2}, \hat{K})$. For $\{Z_1, Z_2\} \in F^1$ we write $L_{\{Z_1, Z_2\}} = L_{Z_1} \cap L_{Z_2}$ and then let
\[
Z^1(k+2) = \prod_{\{Z_1, Z_2\} \in F^1} L_{\{Z_1, Z_2\}},
\]
\[
Z^0(k+2) = \prod_{Z \in F^0} L_Z,
\]
subspaces of $C^1(k+2)$ resp. of $C^0(k+2)$. We define $Z^1_{\text{har}}(k+2)$ by the exact sequence
\[
0 \to Z^1_{\text{har}}(k+2) \to Z^1(k+2) \xrightarrow{\Delta} \prod_{Z \in F^0} L_Z.
\]
Lemma 4.1. The image of $\text{Res}^0$ lies in $Z^1_{\text{har}}(k + 2)$.

**Proof:** By $G$-equivariance it suffices to check $\text{Res}^0(g)(Z_{\gamma_0}, Z_{\gamma_1}) \in L_{Z_{\gamma_0}} \cap L_{Z_{\gamma_1}}$ for all $g \in \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}(k + 2))$. Let $g(z) = \sum_{j \in \mathbb{Z}} a_j z^j$ be the Laurent expansion of $g$ on $]Z_{\gamma_0} \cap Z_{\gamma_1}[$. From (2) we deduce

(16) \[ \omega(g(P)) \geq 0 \quad \text{for all closed points } P \in ]\tilde{U}_{Z_{\gamma_0}}[ \]

(17) \[ \omega(g(P)) \geq \frac{-k - 2}{2} \quad \text{for all closed points } P \in ]\tilde{U}_{Z_{\gamma_1}}[. \]

From (16) we get $\omega(a_j) \geq 0$ for all $j$ (with a point $P \in ]Z_{\gamma_0} \cap Z_{\gamma_1}[$ approach $]\tilde{U}_{Z_{\gamma_0}}[$], hence $\text{Res}^0(g)(Z_{\gamma_0}, Z_{\gamma_1}) \in \text{Hom}_{\hat{k}}(\text{Sym}^k_{\hat{k}}(\text{St})[1], \mathcal{O}_{\hat{K}}) = L_{Z_{\gamma_0}}$. From (17) we get $\omega(a_j) \geq \frac{-k - 2}{2}$ for all $j$ (with a point $P \in ]Z_{\gamma_0} \cap Z_{\gamma_1}[$ approach $]\tilde{U}_{Z_{\gamma_1}}[$). Now in $\text{Sym}^k_{\hat{k}}(\text{St})[1] \otimes \chi^{-k-2}$ we have $\gamma_{-1}(X^i Y^{k-i}) = \pi^{k-2i} X^i Y^{k-i}$. Thus $\text{Res}^0(g)(Z_{\gamma_0}, Z_{\gamma_1})(\gamma_{-1}(X^i Y^{k-i})) = \pi^{k-2i} \text{Res}^0(g)(Z_{\gamma_0}, Z_{\gamma_1})(X^i Y^{k-i}) = \pi^{k-2i} a_{-i-1}$ lies in $\mathcal{O}_{\hat{K}}$, thus $\gamma_{1}\text{Res}^0(g)(Z_{\gamma_0}, Z_{\gamma_1})$ lies in $\text{Hom}_{\hat{k}}(\text{Sym}^k_{\hat{k}}(\text{St})[1], \mathcal{O}_{\hat{K}})$, thus $\text{Res}^0(g)(Z_{\gamma_0}, Z_{\gamma_1})$ lies in $L_{Z_{\gamma_1}}$.

**Theorem 4.2.**

$\text{Res}^0 : \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}(k + 2)) \longrightarrow Z^1_{\text{har}}(k + 2)$ is an isomorphism.

**Proof:** (i) First we claim that the sequence (15) is also exact on the right. Let $\tilde{Z}^1(k + 2) = Z^1(k + 2)/\hat{\pi}$ and for $Z \in F^0$ let $\tilde{L}_Z = L_Z/\hat{\pi}$. Then it is enough to show that the map

$$\tilde{Z}^1(k + 2) \xrightarrow{\Delta} \prod_{Z \in F^0} \tilde{L}_Z$$

induced by $\Delta$ is surjective. For $\{Z_1, Z_2\} \in F^1$ let

$$D_{(Z_1, Z_2)}^\ast = \text{Im}(L_{(Z_1, Z_2)} \to \tilde{L}_{Z_1})$$

$$E_{(Z_1, Z_2)} = \text{Im}(L_{(Z_1, Z_2)} \to (L_{Z_1} + L_{Z_2})/\hat{\pi})$$

(images under the natural maps). Note that $\dim_{\pi}(D_{(Z_1, Z_2)}^\ast) = \frac{k+2}{2}$ and $\dim_{\pi}(E_{(Z_1, Z_2)}) = 1$ if $k$ is even, and $\dim_{\pi}(D_{(Z_1, Z_2)}^\ast) = \frac{k+1}{2}$ and $E_{(Z_1, Z_2)} = 0$ if $k$ is odd (for explicit descriptions see below). For $Z \in F^0$ let

$$\tilde{Z}^1(k + 2)_Z = \prod_{Z' \in \ast(Z)} D_{(Z, Z')}^\ast.$$ 

Then $\Delta$ factors as

$$\tilde{Z}^1(k + 2) \xrightarrow{\beta} \prod_{Z \in F^0} \tilde{Z}^1(k + 2)_Z \xrightarrow{\delta = \prod_{Z \in F^0} \delta_Z} \prod_{Z \in F^0} \tilde{L}_Z$$
where $\beta$ is the product of the natural projection maps. We have an exact sequence

$$0 \longrightarrow \tilde{Z}_i^1(k+2) \xrightarrow{\beta} \prod_{Z \in F^0} \tilde{Z}_i^1(k+2)_Z \xrightarrow{\alpha} \prod_{\{Z_1, Z_2\}} E_{\{Z_1, Z_2\}}$$

where $\alpha$ is defined as

$$\alpha(((gz, z')z'_{\ast}(z))_{Z \in F^0})_{\{Z_1, Z_2\}} = sg(Z_1)gz_1z_2 + sg(Z_2)gz_2z_1$$

for $\{Z_1, Z_2\} \in F^1$. For $Z \in F^0$ we define $\tilde{Z}_i^1(k+2)_Z$ by the exact sequence

$$0 \longrightarrow \tilde{Z}_i^1(k+2)_Z \xrightarrow{\nu_Z} \tilde{Z}_i^1(k+2)_Z \xrightarrow{\delta_Z} \tilde{L}_Z.$$ 

Now it is enough to prove that each $\delta_Z$ (and hence $\delta$) is surjective, and that

$$\prod_{Z \in F^0} \tilde{Z}_i^1(k+2)_Z \xrightarrow{\mu_{Z_1, Z_2}} \prod_{\{Z_1, Z_2\}} E_{\{Z_1, Z_2\}}$$

is surjective. The surjectivity of $\alpha \circ (\prod Z \nu_Z)$, an empty statement if $k$ is odd, will be implied by the surjectivity of its factors

$$\tilde{Z}_i^1(k+2)_{Z_1} \xrightarrow{\mu_{Z_1, Z_2}} E_{\{Z_1, Z_2\}}.$$ 

Let us make the objects explicit. By equivariance we may assume $Z = Z_{\gamma_0}$, resp. $\{Z_1, Z_2\} = \{Z_{\gamma_0}, Z_{\gamma-1}\}$. For $0 \leq j \leq k$ define $h_j \in \text{Hom}_K(K_k^j(\text{Sym}_K^k(\text{St})[1] \otimes \chi^{-k-2}, K))$ by

$$h_j(X^iY^{k-i}) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$ 

Then one finds

$$L_{\gamma_0} = \bigoplus_{j=0}^k O_K \cdot h_j, \quad L_{\gamma_1} = \bigoplus_{j=0}^k (\hat{\pi}^{k-2j}) \cdot h_j, \quad L_{\gamma-1} = \bigoplus_{j=0}^k (\hat{\pi}^{2j-k}) \cdot h_j,$$

$$D_{\{Z_{\gamma-1}, Z_{\gamma_0}\}}^{Z_{\gamma_0}} = \bigoplus_{j=0}^k \mathbb{F} \cdot h_j, \quad D_{\{Z_{\gamma_1}, Z_{\gamma_0}\}}^{Z_{\gamma_0}} = \bigoplus_{j=0}^k \mathbb{F} \cdot h_j$$

and if $k$ is even also $E_{\{Z_{\gamma-1}, Z_{\gamma_0}\}} = \mathbb{F} \cdot h_\frac{k}{2}$. The surjectivity of $\delta_{\gamma_0}$ follows from $\tilde{L}_{\gamma_0} = D_{\{Z_{\gamma-1}, Z_{\gamma_0}\}}^{Z_{\gamma_0}} + D_{\{Z_{\gamma_1}, Z_{\gamma_0}\}}^{Z_{\gamma_0}}$. For the surjectivity of $\mu_{Z_{\gamma_0}, Z_{\gamma-1}}$ (if $k$ is even): the element $h_\frac{k}{2} \in E_{\{Z_{\gamma-1}, Z_{\gamma_0}\}}$ is the image of the $\tilde{Z}_i^1(k+2)_{Z_{\gamma_0}}$-element with entry $h_\frac{k}{2}$ in the $\{Z_{\gamma-1}, Z_{\gamma_0}\}$-component, with entry $-h_\frac{k}{2}$ in the $\{Z_{\gamma_1}, Z_{\gamma_0}\}$-component, and with entry 0 at all other components.

(ii) Let $\tilde{Z}_i^1(k+2) = Z_i^1(k+2)/(\hat{\pi})$. To prove the theorem, since $\Gamma(\hat{\mathcal{X}}, O_{\hat{K}}(k+2))$ and $Z_i^1(k+2)$ are $\hat{\pi}$-adically complete and separated, and since $Z_i^1(k+2)$ is $O_{\hat{K}}$-flat, it is enough to prove that the induced map

$$\widetilde{\text{Res}}^0 : \Gamma(\hat{\mathcal{X}}, O_{\hat{K}}(k+2))/(\hat{\pi}) \rightarrow \tilde{Z}_i^1(k+2)$$

is surjective.
is an isomorphism. Since \( \prod_{Z \in F^0} L_Z \) is \( \mathcal{O}_\delta \)-flat it follows from (i) that (4.1) reduces modulo \((\tilde{\pi})\) to an exact sequence

\[
0 \longrightarrow \bar{Z}^1_{\text{har}}(k + 2) \longrightarrow \bar{Z}^1(k + 2) \overset{\bar{\delta}}{\longrightarrow} \prod_{Z \in F^0} \bar{L}_Z.
\]

We then also obtain from (i) for any \( Z \in F^0 \) exact sequences

\[
0 \to \bar{Z}^1_{\text{har}}(k + 2) \to \prod_{Z \in F^0} \bar{Z}^1_{\text{har}}(k + 2)_Z \to \prod_{\{Z_1, Z_2\} \in F^1} E_{\{Z_1, Z_2\}} \to 0
\]

and (by surjectivity of \( \delta_Z \)) the estimates

\[
\dim_F(\bar{Z}^1_{\text{har}}(k + 2)_Z) = \begin{cases} \frac{(q-1)(k+1)}{2} & : k \text{ odd} \\ \frac{(q-1)(k+2)}{2} + 1 & : k \text{ even} \end{cases}
\]

Now let us look at the source of \( \tilde{\text{Res}}^0 \). By [2.1] we know that this is \( H^0(\bar{\mathfrak{X}}, \mathcal{O}_{\bar{\mathfrak{X}}}(k + 2)) \). Our discussion in section 2 implies that the natural restriction maps induce an exact sequence (note \( H^1(\bar{\mathfrak{X}}, \mathcal{O}_{\bar{\mathfrak{X}}}(k + 2)) = 0 \))

\[
0 \to H^0(\bar{\mathfrak{X}}, \mathcal{O}_{\bar{\mathfrak{X}}}(k + 2)) \to \prod_{Z \in F^0} H^0(\bar{\mathfrak{X}}, (\mathcal{O}_{\bar{\mathfrak{X}}}(k + 2) \otimes_{\mathcal{O}_{\bar{\mathfrak{X}}}} \mathcal{O}_Z)^c) \to \prod_{\{Z_1, Z_2\} \in F^1} J_{\{Z_1, Z_2\}} \to 0
\]

where \( \dim_F(J_{\{Z_1, Z_2\}}) = 1 \) if \( k \) is even, and \( J_{\{Z_1, Z_2\}} = 0 \) if \( k \) is odd. Since \( \tilde{\text{Res}}^0 \) induces isomorphisms \( J_{\{Z_1, Z_2\}} \simeq E_{\{Z_1, Z_2\}} \) it now suffices to see that the map

\[
H^0(\bar{\mathfrak{X}}, (\mathcal{O}_{\bar{\mathfrak{X}}}(k + 2) \otimes_{\mathcal{O}_{\bar{\mathfrak{X}}}} \mathcal{O}_Z)^c) \to \bar{Z}^1_{\text{har}}(k + 2)_Z
\]

induced by \( \tilde{\text{Res}}^0 \) is an isomorphism for any \( Z \in F^0 \), or, by equivariance, for \( Z = Z_{\gamma_0} \). Recall that identifying \( \mathbb{P}_F^1 \cong Z_{\gamma_0} \) as before we have

\[
(\mathcal{O}_{\bar{\mathfrak{X}}}(k + 2) \otimes_{\mathcal{O}_{\bar{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_0}})^c \simeq \begin{cases} \mathcal{L}(\frac{k-3}{2}, \infty + \sum_{b \in \mathbb{F}} \frac{k+1}{2}b) & : k \text{ odd} \\ \mathcal{L}(\frac{k-2}{2}, \infty + \sum_{b \in \mathbb{F}} \frac{k+2}{2}b) & : k \text{ even} \end{cases}
\]

For \( k \) odd, if \( g \in H^0(\bar{\mathfrak{X}}, (\mathcal{O}_{\bar{\mathfrak{X}}}(k + 2) \otimes_{\mathcal{O}_{\bar{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_0}})^c) = H^0(\mathbb{P}_F^1, \mathcal{L}(\frac{k-3}{2}, \infty + \sum_{b \in \mathbb{F}} \frac{k+1}{2}b)) \) lies in the kernel of \( \tilde{\text{Res}}^0 \) then it is an element even of \( H^0(\mathbb{P}_F^1, \mathcal{L}(\frac{k-3}{2}, \infty)) \) and therefore it vanishes. Thus \( \tilde{\text{Res}}^0 \) is injective. Similarly for even \( k \). On the other hand by our above computation we find \( \dim_F(H^0(\bar{\mathfrak{X}}, (\mathcal{O}_{\bar{\mathfrak{X}}}(k + 2) \otimes_{\mathcal{O}_{\bar{\mathfrak{X}}}} \mathcal{O}_{Z_{\gamma_0}})^c)) = \dim_F(\bar{Z}^1_{\text{har}}(k + 2)_Z) \), thus \( \tilde{\text{Res}}^0 \) is also surjective and the proof is complete. \( \square \)

The \( p \)-adic Shimura isomorphism [3] p.98 is an immediate consequence of [4.2].
5 The reduced de Rham complex

In this section \( \mathrm{char}(K) = 0 \). Fix \( k \geq 0 \) and for our fixed coordinate \( z \) let \( \partial = \frac{d}{dz} \). Let \( (\Omega^\bullet_X \otimes_K \text{Sym}^k_K(\text{St}), \partial \otimes \text{id}) \) be the de Rham complex on \( X \) with coefficients in \( \text{Sym}^k_K(\text{St}) \). By \([6]\) p.97 this complex is \( \text{SL}_2(K) \)-equivariantly quasi-isomorphic with the "reduced de Rham complex"

\[
\mathcal{R}^\bullet_X = [\mathcal{O}_X(-k) \xrightarrow{\partial^{k+1}} \mathcal{O}_X(k+2)]
\]

on \( X \). (The genesis of this "theta operator" \( \partial^{k+1} \) from \( (\Omega^\bullet_X \otimes_K \text{Sym}^k_K(\text{St}), \partial \otimes \text{id}) \) is completely parallel to that of the theta operator on classical modular forms, cf. \([2]\)).

We change bases \( K \rightarrow \hat{K} \). Since \( \omega(z(P)) = -n \) for any \( n \) and any point \( P \in (\hat{U}_{(z_m)})_{\hat{K}} \) the operator \( \partial \) on \( \mathcal{O}_{\hat{U}_{(z_m)}} \otimes_{\hat{K}} \hat{K} = sp \mathcal{O}_{\hat{U}_{(z_m)}} \) restricts to a map \( \partial : \mathcal{O}_{\hat{U}_{(z_m)}} \rightarrow \pi^n \mathcal{O}_{\hat{U}_{(z_m)}} \). Iterating we get a map \( \partial^{k+1} : \mathcal{O}_X(-k)|_{\hat{U}_{(z_m)}} \rightarrow \mathcal{O}_X(k+2)|_{\hat{U}_{(z_m)}} \). By equivariance we see that \( \partial^{k+1} \) induces a map \( \partial^{k+1} : \mathcal{O}_X(-k)|_{\hat{U}(z)} \rightarrow \mathcal{O}_X(k+2)|_{\hat{U}(z)} \) for any \( Z \in F^0 \). By an argument similar to that at the end of the proof of \([4]\) it follows that \( \partial^{k+1} \) respects these integral structures also above the singular points of \( \hat{X} \), hence a complex

\[
\mathcal{R}^\bullet_{\hat{X}} = [\mathcal{O}_X(-k) \xrightarrow{\partial^{k+1}} \mathcal{O}_X(k+2)].
\]

We denote by \( \mathcal{H}^i(\mathcal{R}^\bullet_{\hat{X}}) \) for \( i = 0 \) and \( i = 1 \) the cohomology sheaves.

**Theorem 5.1.** For any \( i, j \) we have canonical isomorphisms

\[
H^i(\hat{X}, \mathcal{H}^j(\mathcal{R}^\bullet_{\hat{X}})) \cong H^j(\hat{X}, \mathcal{R}^i_{\hat{X}}).
\]

**Proof:** For \( i = 0 \) the map is induced by the canonical injection \( \mathcal{H}^0(\mathcal{R}^\bullet_{\hat{X}}) \rightarrow \mathcal{R}^0_{\hat{X}} = \mathcal{O}_X(-k) \), for \( i = 1 \) it is induced by the canonical surjection \( \mathcal{R}^1_{\hat{X}} \rightarrow \mathcal{H}^1(\mathcal{R}^\bullet_{\hat{X}}) \). Once we know the claim for \( i = 0 \) it follows that \( H^*(\hat{X}, \mathcal{B}) = 0 \) for \( \mathcal{B} = \text{Im}(\mathcal{R}^0_{\hat{X}} \rightarrow \mathcal{R}^1_{\hat{X}}) = \text{Ker}(\mathcal{R}^1_{\hat{X}} \rightarrow \mathcal{H}^1(\mathcal{R}^\bullet_{\hat{X}})) \), hence the claim for \( i = 1 \). Thus we concentrate on the case \( i = 0 \). Denote by \( (\cdot)_m \) reduction modulo \( \pi^m \). Since \( \mathcal{H}^0(\mathcal{R}^\bullet_{\hat{X}}) = \lim_m (\mathcal{H}^0(\mathcal{R}^\bullet_{\hat{X}}))_m \) and \( \mathcal{R}^0_{\hat{X}} = \lim_m (\mathcal{R}^0_{\hat{X}})_m \), the spectral sequence for the composition of derived functors \( \mathcal{R} \lim_m \mathcal{R}\Gamma(\hat{X}, \cdot) \) shows that it suffices to show

\[
H^i(\hat{X}, (\mathcal{H}^0(\mathcal{R}^\bullet_{\hat{X}}))_m) \cong H^i(\hat{X}, (\mathcal{R}^0_{\hat{X}})_m)
\]

for any \( m \). Now \( \mathcal{R}^0_{\hat{X}} \) and hence also its subsheaf \( \mathcal{H}^0(\mathcal{R}^\bullet_{\hat{X}}) \) is \( \mathcal{O}_{\hat{K}} \)-flat. Therefore one gets exact sequences of sheaves

\[
0 \rightarrow \mathcal{F}_{m-1} \xrightarrow{\hat{\pi}^{m-1}} \mathcal{F}_m \rightarrow \mathcal{F}_1 \rightarrow 0
\]

for \( \mathcal{F} = \mathcal{R}^0_{\hat{X}} \) and \( \mathcal{F} = \mathcal{H}^0(\mathcal{R}^\bullet_{\hat{X}}) \). Using the associated long exact cohomology sequences we reduce our task to proving the isomorphism just stated in the case \( m = 1 \). Now observe
that \( \mathcal{H}^0(\mathcal{R}_X^* \otimes_K \widehat{K}) \) is precisely the locally constant sheaf generated by the \( \widehat{K} \)-vector space of polynomials in the variable \( z \) of degree at most \( k \). Thus \( \mathcal{H}^0(\mathcal{R}_X^*) \) consists of such polynomials subject to growth conditions. Namely, since \( \mathcal{R}_X^0|_{U(z_m)} = \mathcal{O}_X(-k)|_{U(z_m)} = \pi^{-kn}\mathcal{O}_X|_{U(z_m)} \) and \( \omega(z(P)) = -n \) for any \( n \) and any point \( P \in (U(z_m)) \), we have

\[
\mathcal{H}^0(\mathcal{R}_X^*)_Z(z_m) = \{ \sum_{0 \leq t \leq k} d_t z^t \mid d_t \in \widehat{K}, \omega(d_t) \geq tn - \frac{kn}{2} \},
\]

\[
\mathcal{H}^0(\mathcal{R}_X^*)_{U(z_m, z_{m-1})} = \mathcal{H}^0(\mathcal{R}_X^*)_{U(z_m)} \cap \mathcal{H}^0(\mathcal{R}_X^*)_{U(z_{m-1})}
\]

\[
= \{ \sum_{0 \leq t \leq k} d_t z^t \mid d_t \in \widehat{K}, \omega(d_t) \geq \begin{cases} tn - \frac{kn}{2} & : t \geq \frac{k}{2} \\ t(n - 1) - \frac{k(n-1)}{2} & : t \leq \frac{k}{2} \end{cases} \}
\]

(any \( k \), even or odd). For \( Z \in F^0 \) let \( (\mathcal{H}^0(\mathcal{R}_X^*))^Z \) be the image of the composition

\[
\mathcal{H}^0(\mathcal{R}_X^*) \rightarrow \mathcal{R}_X^0 = \mathcal{O}_X(-k) \rightarrow (\mathcal{O}_X(-k) \otimes \mathcal{O}_Z)^c.
\]

Then the above shows

\[
(\mathcal{H}^0(\mathcal{R}_X^*))^Z_1(z_m, U(z_m)) = \{ \sum_{0 \leq t \leq k} d_t z^t \mid d_t \in \left( \frac{z^{(2t-k)n}}{z^{(2t-k)n+1}} \right) \}
\]

\[
(\mathcal{H}^0(\mathcal{R}_X^*))^Z_{U(z_m, z_{m-1})} = \{ \sum_{\frac{k}{2} \leq t \leq k} d_t z^t \mid d_t \in \left( \frac{z^{(2t-k)n}}{z^{(2t-k)n+1}} \right) \}
\]

\[
(\mathcal{H}^0(\mathcal{R}_X^*))^Z_{U(z_m, z_{m+1})} = \{ \sum_{0 \leq t \leq k} d_t z^t \mid d_t \in \left( \frac{z^{(2t-k)n}}{z^{(2t-k)n+1}} \right) \}
\]

Similar descriptions hold at other \( Z \in F^0 \), resp. \( \{Z_1, Z_2\} \in F^1 \), by equivariance. We find

\[
(\mathcal{H}^0(\mathcal{R}_X^*))_1 = \prod_{Z \in F^0} (\mathcal{H}^0(\mathcal{R}_X^*))^Z_1
\]

if \( k \) is odd (because then there are no summands \( d_t z^t \) to consider). If \( k \) is even we find an exact sequence

\[
0 \rightarrow (\mathcal{H}^0(\mathcal{R}_X^*))_1 \rightarrow \prod_{Z \in F^0} (\mathcal{H}^0(\mathcal{R}_X^*))^Z_1 \rightarrow \prod_{\{Z_1, Z_2\} \in F^1} (\mathcal{H}^0(\mathcal{R}_X^*))_1^{Z_1, Z_2} \rightarrow 0
\]

where \( (\mathcal{H}^0(\mathcal{R}_X^*))_1^{Z_1, Z_2} \) for \( \{Z_1, Z_2\} \in F^1 \) is a sheaf with \( (\mathcal{H}^0(\mathcal{R}_X^*))_1^{Z_1, Z_2}(U) \cong \mathbb{F} \) if \( U \cap Z_1 \cap Z_2 \neq \emptyset \), and 0 for other open \( U \subset \widehat{X} \). On the other hand we have

\[
(\mathcal{R}_X^0)_1 = \mathcal{O}_X(-k) = \prod_{Z \in F^0} (\mathcal{O}_X(-k) \otimes \mathcal{O}_Z)^c
\]
if \(k\) is odd, and an exact sequence

\[
0 \to (\mathcal{R}_x^0)_{1} \to \prod_{Z \in F^0} (\mathcal{O}_x(-k) \otimes \mathcal{O}_Z) \to \prod_{\{Z_1, Z_2\} \in F^1} \mathcal{O}_x(-k) \otimes \mathcal{O}_Z \cap \mathcal{O}_{Z_1 \cap Z_2} \to 0
\]

if \(k\) is even. For \(Z \in F^0\) let

\[
\alpha_Z : (\mathcal{H}^0(\mathcal{R}_x^*)_1)^Z \to (\mathcal{O}_x(-k) \otimes \mathcal{O}_Z)^c
\]

be the inclusion. If \(k\) is even then for \(\{Z_1, Z_2\} \in F^1\) the maps \(\alpha_{Z_1}\) and \(\alpha_{Z_2}\) commute with obvious isomorphisms

\[
\alpha_{Z_1, Z_2} : (\mathcal{H}^0(\mathcal{R}_x^*)_1)^{Z_1, Z_2} \to \mathcal{O}_x(-k) \otimes \mathcal{O}_{Z_1 \cap Z_2}.
\]

Since the \(\alpha_z\) also commute with our map \((\mathcal{H}^0(\mathcal{R}_x^*)_1) \to (\mathcal{R}_x^0)_{1}\) in question, it remains to prove that the \(\alpha_z\) induce isomorphisms in cohomology. By equivariance it is enough to do this for \(Z = Z_{\gamma_0}\). We identify \(\text{Spec}(\mathbb{F}[z]) \cup \{\infty\} = \mathbb{P}^1_{\mathbb{F}} \cong Z_{\gamma_0}\) such that this \(z\) on \(\mathbb{P}^1_{\mathbb{F}}\) is induced by the global variable \(z\) on \(X\). In particular, \(\infty \in \mathbb{P}^1_{\mathbb{F}}\) corresponds to \(Z_{\gamma_0} \cap Z_{\gamma_1}\), and \(0 \in \mathbb{P}^1_{\mathbb{F}}\) corresponds to \(Z_{\gamma_0} \cap Z_{\gamma_{-1}}\). Let \(\iota : \mathbb{P}^1_{\mathbb{F}} \cong Z_{\gamma_0} \to \tilde{x}\) be the closed immersion. Since we have

\[
H^*(\tilde{x}, \mathcal{F}) = H^*(\mathbb{P}^1_{\mathbb{F}}, \iota^{-1}\mathcal{F})
\]

for both \(\mathcal{F} = (\mathcal{H}^0(\mathcal{R}_x^*)_1)^{Z_{\gamma_0}}\) and \(\mathcal{F} = (\mathcal{O}_x(-k) \otimes \mathcal{O}_{Z_{\gamma_0}})^c\), we must show that

\[
H^*(\mathbb{P}^1_{\mathbb{F}}, \iota^{-1}(\mathcal{H}^0(\mathcal{R}_x^*)_1)^{Z_{\gamma_0}}) \to H^*(\mathbb{P}^1_{\mathbb{F}}, \iota^{-1}(\mathcal{O}_x(-k) \otimes \mathcal{O}_{Z_{\gamma_0}})^c)
\]

is an isomorphism. If on \(\mathbb{P}^1_{\mathbb{F}}\) we define the divisor

\[
D = \left\{ \begin{array}{ll}
\frac{k}{2}, \infty - \sum_{b \in \mathbb{F}_{k/2}} b & : k \text{ even} \\
\frac{k+1}{2}, \infty - \sum_{b \in \mathbb{F}_{k/2}} b - \frac{1}{2}, b & : k \text{ odd}
\end{array} \right.
\]

then we have a natural identification

\[
\mathcal{L}(D) = \iota^{-1}(\mathcal{O}_x(-k) \otimes \mathcal{O}_{Z_{\gamma_0}})^c.
\]

In this way we may view \(\iota^{-1}(\mathcal{O}_x(-k) \otimes \mathcal{O}_{Z_{\gamma_0}})^c\) as a subsheaf of \(\mathcal{L}(k, \infty)\). On the other hand we may view \(\iota^{-1}(\mathcal{H}^0(\mathcal{R}_x^*)_1)^{Z_{\gamma_0}}\) as a subsheaf of the constant \(\mathbb{F}\)-vector space sheaf \(\mathcal{H}\) on \(\mathbb{P}^1_{\mathbb{F}}\) with value \(\bigoplus_{i=0}^k \mathbb{F}.z^i\) (as a sub \(\mathbb{F}\)-vector space of the function field \(\mathbb{F}(z)\)). The inclusion \(\beta : \mathcal{H} \to \mathcal{L}(k, \infty)\) induces our map \(\iota^{-1}(\mathcal{H}^0(\mathcal{R}_x^*)_1)^{Z_{\gamma_0}} \to \mathcal{L}(D)\) in question. It also induces an isomorphism between the respective cokernel (skyscraper) sheaves

\[
\mathcal{H} \quad \frac{\mathcal{L}(k, \infty)}{\mathcal{L}(D)} \approx \frac{\mathcal{L}(k, \infty)}{\mathcal{L}(D)}
\]

(try the above local description of \(\iota^{-1}(\mathcal{H}^0(\mathcal{R}_x^*)_1)^{Z_{\gamma_0}}\). Since clearly \(\beta\) induces isomorphisms

\[
H^*(\mathbb{P}^1_{\mathbb{F}}, \mathcal{H}) \cong H^*(\mathbb{P}^1_{\mathbb{F}}, \mathcal{L}(k, \infty))
\]

we are done. \(\square\)
Corollary 5.2. We have the Hodge decomposition

\[(18) \quad H^1(\hat{X}, R^*_X) \cong H^0(\hat{X}, \mathcal{O}_\hat{X}(k+2)) \oplus H^1(\hat{X}, \mathcal{O}_\hat{X}(-k)).\]

**Proof:** Consider the canonical maps of sheaf complexes

\[\mathcal{H}^0(\mathcal{R}^*_X) \to \mathcal{R}^1_X \to \mathcal{R}^*_X\]

on \(\hat{X}\). By 5.1 both of them induce isomorphisms in cohomology; together we thus obtain the isomorphism

\[\mathbb{R}\Gamma(\hat{X}, R^*_X) \cong \mathbb{R}\Gamma(\hat{X}, [\mathcal{R}^0_X \to \mathcal{R}^1_X]).\]

We derive the stated Hodge decomposition. \(\square\)

Let again \(\Gamma < \text{SL}_2(K)\) be a cocompact discrete torsion free subgroup.

**Theorem 5.3.** (a) The reduced Hodge spectral sequence

\[E_1^{r,s} = H^s(X_\Gamma, (\mathcal{R}^r_X)\Gamma) \Rightarrow H^{r+s}(x_\Gamma, (\mathcal{R}^*_X)\Gamma) = H^{r+s}(X_\Gamma, (\Omega^r_X \otimes_K \text{Sym}_K^k(\text{St}))\Gamma)\]

degenerates in \(E_1\).

(b) \(H^1(X_\Gamma, (\Omega^*_X \otimes_K \text{Sym}_K^k(\text{St}))\Gamma) = H^1(X_\Gamma, (\mathcal{R}^*_X)\Gamma)\) decomposes naturally as

\[H^1(X_\Gamma, (\Omega^*_X \otimes_K \text{Sym}_K^k(\text{St}))\Gamma) = H^1(\Gamma, \text{Sym}_K^k(\text{St})) \oplus H^0(X_\Gamma, \mathcal{O}_X(k+2)\Gamma).\]

(c) If \(\Gamma\) is the free group on \(g\) generators and if \(k > 0\), then

\[\dim_K(H^1(X_\Gamma, \mathcal{O}_X(-k)\Gamma)) = \dim_K(H^0(X_\Gamma, \mathcal{O}_X(k+2)\Gamma)) = (g-1)(k+1).\]

**Proof:** We may of course change bases from \(K\) to \(\hat{K}\). Statement (a) is a consequence of 2.2 (if \(k > 0\)) but we can also argue as follows. It is enough to show that the inclusion of sheaf complexes

\[\mathcal{H}^0(\mathcal{R}^*_X)\Gamma \to (\mathcal{R}^1_X)\Gamma \hookrightarrow (\mathcal{R}^*_X)\Gamma\]

on \(\hat{X}_\Gamma\) induces isomorphisms

\[H^*(\hat{X}_\Gamma, [\mathcal{H}^0(\mathcal{R}^*_X)\Gamma \to (\mathcal{R}^1_X)\Gamma]) \cong H^*(\hat{X}_\Gamma, (\mathcal{R}^*_X)\Gamma).\]

For this it suffices to show that \(\mathcal{H}^0(\mathcal{R}^*_X)\Gamma \to (\mathcal{R}^0_X)\Gamma\) induces isomorphisms in cohomology. Now \(\hat{X}_\Gamma\) is quasi-compact, hence \(H^*(\hat{X}_\Gamma, .)\) commutes with \(\otimes_{\hat{K}}\). Therefore it suffices to show that the morphism of sheaves \(\mathcal{H}^0(\mathcal{R}^*_X)\Gamma \to (\mathcal{R}^0_X)\Gamma\) on \(\hat{X}_\Gamma\) induces isomorphisms

\[(19) \quad H^*(\hat{X}_\Gamma, \mathcal{H}^0(\mathcal{R}^*_X)\Gamma) \cong H^*(\hat{X}_\Gamma, (\mathcal{R}^0_X)\Gamma).\]
Using the covering spectral sequences
\[ E_2^{s,t} = H^t(\Gamma, H^s(\tilde{X}, \mathcal{F})) \Rightarrow H^{t+s}(\tilde{X}_\Gamma, \mathcal{F}^\Gamma) \]
for \( \mathcal{F} = \mathcal{H}^0(\mathcal{R}_\tilde{X}^\bullet) \) and \( \mathcal{F} = \mathcal{R}_\tilde{X}^0 \) we see that it is enough to prove that the maps
\[ H^t(\Gamma, H^s(\tilde{X}, \mathcal{H}^0(\mathcal{R}_\tilde{X}^\bullet))) \rightarrow H^t(\Gamma, H^s(\tilde{X}, \mathcal{R}_\tilde{X}^0)) \]
are isomorphisms. But they are, as follows from [5, 1]. We turn to (b). We have
\[
H^1(\tilde{X}_\Gamma, (\mathcal{R}_\tilde{X}^\bullet)^\Gamma) = H^1(\tilde{X}_\Gamma, (\mathcal{R}_\tilde{X}^\bullet)^\Gamma) \otimes_{\mathcal{O}_\tilde{X}} \hat{K} \\
= H^1(\tilde{X}_\Gamma, \mathcal{H}^0(\mathcal{R}_\tilde{X}^\bullet)^\Gamma) \otimes_{\mathcal{O}_\tilde{X}} \hat{K} + H^0(\tilde{X}_\Gamma, (\mathcal{R}_\tilde{X}^1)^\Gamma) \otimes_{\mathcal{O}_\tilde{X}} \hat{K} \\
= H^1(\tilde{X}_\Gamma, \mathcal{H}^0(\mathcal{R}_\tilde{X}^\bullet)^\Gamma) \oplus H^0(\tilde{X}_\Gamma, (\mathcal{R}_\tilde{X}^1)^\Gamma)
\]
where the first and the third equality follow again from the quasi-compactness of \( \tilde{X}_\Gamma \), and the second equality from [19]. Now \( \mathcal{R}_\tilde{X}^\bullet = \mathcal{O}_X(k+2) \), and on the other hand
\[
H^1(X_\Gamma, \mathcal{H}^0(\mathcal{R}_X^\bullet)^\Gamma) = H^1(\Gamma, \mathcal{R}(X, \mathcal{H}^0(\mathcal{R}_X^\bullet))).
\]
But \( H^0(X, \mathcal{H}^0(\mathcal{R}_X^\bullet)) = H^0(X, \mathcal{R}_X^\bullet) = \text{Sym}_K^k(St) \) and \( H^j(X, \mathcal{H}^0(\mathcal{R}_X^\bullet)) = 0 \) for \( j \neq 0 \) because \( \mathcal{H}^0(\mathcal{R}_X^\bullet) \) is the locally constant sheaf on \( X \) generated by \( H^0(X, \mathcal{R}_X^\bullet) = H^0(X, \mathcal{O}_X^\bullet \otimes K \text{Sym}_K^k(St)) = \text{Sym}_K^k(St) \). In (c) for the equality \( \dim_K(H^0(X_\Gamma, \mathcal{O}_X(k+2)^\Gamma)) = (g-1)(k+1) \) see [6] p.98. The equality \( \dim_K(H^1(X_\Gamma, \mathcal{O}_X(-k)^\Gamma)) = (g-1)(k+1) \) follows from statement (a) together with [5] p.628 and [6] p.98.

The decomposition in (b) is not new. It was established for the first time in [7] and later again in [5]. Both these (mutually different) proofs use sophisticated analytic methods (e.g. Coleman integration in [7]). The degeneration of the spectral sequence in (a) however, conjectured in [5], seemed to be unknown before (cf. [5] p.649). Note that the spectral sequence for the non-reduced de Rham complex does not degenerate in general at \( E_1 \) (cf. loc. cit.).

Corollary 5.4. The intersection of \( H^0(\tilde{X}, \mathcal{O}_\tilde{X}(k+2)) \) and of
\[
\text{Im}[H^0(\tilde{X}, \mathcal{O}_\tilde{X}(-k)) \otimes_k H^0(\tilde{X}, \mathcal{O}_\tilde{X}(k+2))] \]
inside \( H^0(\tilde{X}, \mathcal{O}_\tilde{X}(k+2)) = H^0(\tilde{X}, \mathcal{O}_\tilde{X}(k+2)) \) is zero. In particular, \( H^0(\tilde{X}, \mathcal{O}_\tilde{X}(k+2)) \) can be viewed as a submodule of \( H^1(\tilde{X}, \mathcal{R}_X^\bullet) = H^1(\tilde{X}, \mathcal{O}_X^\bullet \otimes K \text{Sym}_K^k(St)) \).

Proof: This follows immediately from the injectivity of the map \( R_s^0 \) in [4, 2].

Theorem 5.5. For \( k > 0 \) there is a natural \( G \)-equivariant isomorphism
\[
\theta : Z^1_{\text{har}}(k+2) \otimes k^{k+1} \cong H^1(\tilde{X}, \mathcal{H}^0(\mathcal{R}_X^\bullet)).
\]
PROOF: Observe $\varepsilon = \det \cdot \chi^{-2}$, which relates the twisting here to that in Section 4. We have a $G$-equivariant isomorphism

$$\text{Hom}_K(\text{Sym}_K^k(\text{St})[-k] \otimes \chi^k, \hat{K}) \xrightarrow{\sigma} H^0(\hat{\mathfrak{X}}, \mathcal{H}^0(\mathcal{R}_\hat{X}^*)),$$  \quad h_j \mapsto z^{k-j}$$

with $h_j \in \text{Hom}_K(\text{Sym}_K^k(\text{St})[-k] \otimes \chi^k, \hat{K})$ as in the proof of [4,2] i.e. $h_j(X^iY^{k-j}) = 1$ and $h_j(X^iY^{k-j}) = 0$ for $i \neq j$. For $Z \in F^0$ and $\{Z_1, Z_2\} \in F^1$ we define sheaves $\mathcal{G}_Z$ and $\mathcal{G}_{\{Z_1, Z_2\}}$ on $\hat{\mathfrak{X}}$: for open $U \subset \hat{\mathfrak{X}}$ we let

$$\mathcal{G}_Z(U) = \begin{cases} H^0(\mathcal{R}_\hat{X}^*)(\hat{\mathfrak{X}}(Z)) & : U \cap Z \neq \emptyset \\ 0 & : U \cap Z = \emptyset \end{cases}$$

$$\mathcal{G}_{\{Z_1, Z_2\}}(U) = \begin{cases} \mathcal{G}_{Z_1}(U) + \mathcal{G}_{Z_2}(U) & : U \cap Z_1 \cap Z_2 \neq \emptyset \\ 0 & : U \cap Z_1 \cap Z_2 = \emptyset \end{cases}.$$

Then we have an exact sequence

$$0 \rightarrow H^0(\mathcal{R}_\hat{X}^*) \rightarrow \prod_{Z \in F^0} \mathcal{G}_Z \xrightarrow{\delta} \prod_{\{Z_1, Z_2\} \in F^1} \mathcal{G}_{\{Z_1, Z_2\}} \rightarrow 0 \quad (20)$$

where $\delta$ is the product of all maps $sg(Z_1)\cdot \text{id} : \mathcal{G}_{Z_1} \rightarrow \mathcal{G}_{\{Z_1, Z_2\}}$. In cohomology we get

$$\frac{H^0(\hat{\mathfrak{X}}, \prod_{\{Z_1, Z_2\} \in F^1} \mathcal{G}_{\{Z_1, Z_2\}})}{H^0(\hat{\mathfrak{X}}, \prod_{Z \in F^0} \mathcal{G}_Z)} \cong H^1(\hat{\mathfrak{X}}, \mathcal{H}^0(\mathcal{R}_\hat{X}^*)].$$

We claim that

$$Z_{\text{har}}^1(k+2) \otimes \varepsilon^{k+1} \rightarrow H^0(\hat{\mathfrak{X}}, \prod_{\{Z_1, Z_2\} \in F^1} \mathcal{G}_{\{Z_1, Z_2\}})$$

$$(f(z_1, z_2))_{\{Z_1, Z_2\} \in F^1} \mapsto \prod_{\{Z_1, Z_2\} \in F^1} \sigma(f(z_1, z_2))$$

induces an isomorphism $\theta : Z_{\text{har}}^1(k+2) \otimes \varepsilon^{k+1} \rightarrow H^1(\hat{\mathfrak{X}}, \mathcal{H}^0(\mathcal{R}_\hat{X}^*))$. Since $H^1(\hat{\mathfrak{X}}, \mathcal{H}^0(\mathcal{R}_\hat{X}^*)) = H^1(\hat{\mathfrak{X}}, \mathcal{O}_\hat{X}(-k))$ is flat it suffices to show that the induced map

$$\tilde{\theta} = \theta/(\tilde{\pi}) : \tilde{Z}_{\text{har}}^1(k+2) \rightarrow H^1(\hat{\mathfrak{X}}, (\mathcal{H}^0(\mathcal{R}_\hat{X}^*))_1)$$

is an isomorphism (with notations from the proof of [3,1]). Let us first assume $k > 0$ is even. Consider the submodule

$$\tilde{Z}_{\text{har}}^1(k+2)(1) = \left\{ f = (f(z_1, z_2))_{\{Z_1, Z_2\} \in F^1} \in \tilde{Z}_{\text{har}}^1(k+2); \right. \left. (\gamma.f)(Z_{\gamma_1}, Z_{\gamma_2}) (X^{\frac{\gamma_1}{2}}Y^{\frac{\gamma_2}{2}}) = 0 \text{ for all } \gamma \in G \right\}$$

of $\tilde{Z}_{\text{har}}^1(k+2)$ (this is nothing but the image of $H^0(\hat{\mathfrak{X}}, \mathcal{O}_\hat{X}(k+2)(1))$ under $\text{Res}^0\cdot \tilde{\pi}$). If for $Z \in F^0$ we let $\tilde{Z}_{\text{har}}^1(k+2)(1)_Z$ be the image of $\tilde{Z}_{\text{har}}^1(k+2)(1) \rightarrow \tilde{Z}_{\text{har}}^1(k+2) \rightarrow \tilde{Z}_{\text{har}}^1(k+2)_Z$, then

$$\tilde{Z}_{\text{har}}^1(k+2)(1) = \prod_{Z \in F^0} \tilde{Z}_{\text{har}}^1(k+2)(1)_Z.$$
In particular we have natural injections \( \iota_Z : \tilde{Z}_{\text{har}}(k + 2)(1)_{\tilde{X}} \to \tilde{Z}_{\text{har}}(k + 2)(1) \). We claim that for each \( Z \in F^0 \) the composition

\[
\tilde{Z}_{\text{har}}(k + 2)(1)_{\tilde{X}} \xrightarrow{\iota_Z} \tilde{Z}_{\text{har}}(k + 2)(1) \xrightarrow{\tilde{\theta}} H^1(\tilde{X}, (\mathcal{H}^0(\mathcal{R}^\bullet_X))_1) \to H^1(\tilde{X}, (\mathcal{H}^0(\mathcal{R}^\bullet_X))_1^Z),
\]

which we denote by \( \beta_Z \), is an isomorphism. To see this we may assume \( Z = Z_{\gamma} \). From the proof of [5,11] we infer an exact sequence

\[
0 \to H^0(\mathbb{P}^1, \mathcal{H}) \to H^0(\mathbb{P}^1, \frac{\mathcal{H}}{l^{-1}(\mathcal{H}^0(\mathcal{R}^\bullet_X))_1^Z}) \to H^1(\tilde{X}, (\mathcal{H}^0(\mathcal{R}^\bullet_X))_1^Z) \to 0.
\]

Here \( \iota : \mathbb{P}^1 \cong Z_{\gamma} \to \tilde{X} \) is the natural embedding, \( \mathcal{H} \) is the constant sheaf with value \( \bigoplus_{i=0}^{k} \mathbb{F}.z^i \), and the quotient \( \mathcal{H}/l^{-1}(\mathcal{H}^0(\mathcal{R}^\bullet_X))_1^Z \) is a skyscraper sheaf whose only stalks are \( \frac{k}{2} \)-dimensional \( \mathcal{F} \)-vector spaces at the \( \mathcal{F} \)-rational points of \( \mathbb{P}^1 \). Namely, in notations from section [2] the \( \mathcal{F} \)-rational points of \( \mathbb{P}^1 \cong Z_{\gamma} \) are just the intersections \( Z_{\gamma} \cap Z_{\gamma_0, \gamma_1} \) with \( a \in R \), and \( Z_{\gamma} \cap Z_{\gamma_1} \). The stalk of \( \mathcal{H}/l^{-1}(\mathcal{H}^0(\mathcal{R}^\bullet_X))_1^Z \) at \( Z_{\gamma} \cap Z_{\gamma_0, \gamma_1} \) is (canonically identified with) \( \bigoplus_{i=0}^{k} \mathbb{F}.(z - \overline{a})^i \) (with \( \overline{a} \in \mathbb{F} \) the image of \( a \in R \)), and the stalk at \( Z_{\gamma} \cap Z_{\gamma_1} \) is (canonically identified with) \( \bigoplus_{i=0}^{k} \mathbb{F}.z^i \). From the proof of [1,2] we get the exact sequence

\[
0 \to \tilde{Z}_{\text{har}}(k + 2)(1)_{Z_{\gamma}} \to \bigoplus_{j=0}^{\frac{k}{2}} \mathbb{F}.h_j \times \prod_{a \in R} \bigoplus_{j=\frac{k}{2}+1}^{k} \mathbb{F}.(\gamma_a, 0)h_j \to \sum_{j=0}^{k} \bigoplus_{j=0}^{k} \mathbb{F}.h_j
\]

(the first factor in the middle term is the \( \{Z_{\gamma_0}, Z_{\gamma_1}\} \)-component). Now \( \sigma \) maps \( \gamma_a, 0 \) to \( \gamma_{a, 0}.z^{-k-j} = (z - \overline{a})^{k-j} \), hence defines a map

\[
\tilde{Z}_{\text{har}}(k + 2)(1)_{Z_{\gamma}} \to H^0(\mathbb{P}^1, \frac{\mathcal{H}}{l^{-1}(\mathcal{H}^0(\mathcal{R}^\bullet_X))_1^Z})
\]

whose composition with the projection to \( H^1(\tilde{X}, (\mathcal{H}^0(\mathcal{R}^\bullet_X))_1^Z) \) is an isomorphism: this isomorphism is our \( \beta_Z \). We have shown that \( \tilde{\theta} |_{\tilde{Z}_{\text{har}}(k+2)(1)} \) is injective and that its image \( \text{Im}(\tilde{\theta} |_{\tilde{Z}_{\text{har}}(k+2)(1)}) \subset H^1(\tilde{X}, (\mathcal{H}^0(\mathcal{R}^\bullet_X))_1) \) maps isomorphically to \( H^1(\tilde{X}, \prod_{Z \in F^0}(\mathcal{H}^0(\mathcal{R}^\bullet_X))_1^Z) \). From the exact sequence

\[
0 \to H^0(\tilde{X}, \prod_{\{Z_1, Z_2\} \in F^1} (\mathcal{H}^0(\mathcal{R}^\bullet_X))_1^{Z_1, Z_2}) \to H^1(\tilde{X}, (\mathcal{H}^0(\mathcal{R}^\bullet_X))_1) \to H^1(\tilde{X}, \prod_{Z \in F^0}(\mathcal{H}^0(\mathcal{R}^\bullet_X))_1^Z) \to 0
\]

we therefore get

\[
\frac{H^1(\tilde{X}, (\mathcal{H}^0(\mathcal{R}^\bullet_X))_1)}{\text{Im}(\tilde{\theta} |_{\tilde{Z}_{\text{har}}(k+2)(1)})} \cong H^0(\tilde{X}, \prod_{\{Z_1, Z_2\} \in F^1} (\mathcal{H}^0(\mathcal{R}^\bullet_X))_1^{Z_1, Z_2}).
\]

In particular we get a map

\[
\frac{\tilde{Z}_{\text{har}}(k + 2)(1)}{\tilde{Z}_{\text{har}}(k + 2)(1)} \to H^0(\tilde{X}, \prod_{\{Z_1, Z_2\} \in F^1} (\mathcal{H}^0(\mathcal{R}^\bullet_X))_1^{Z_1, Z_2})
\]
induced by $\tilde{\theta}$ and it remains to show that this map is bijective. But this is clear, as both sides can be identified with $\text{ker}(\mathcal{N})$. If $k > 0$ is odd things are easier since there are no terms $(\mathcal{H}^0(\mathcal{R}^\bullet_\hat{X}))_{1}^{Z_1, Z_2}$ and we only need to show bijectivity of the maps

$$\tilde{Z}_\text{har}^{1}(k+2)_Z \overset{\text{res}}{\rightarrow} \tilde{Z}_\text{har}^{1}(k+2) \overset{\tilde{\theta}}{\rightarrow} H^1(\tilde{\mathcal{X}}, (\mathcal{H}^0(\mathcal{R}^\bullet_\hat{X})))_{1} \rightarrow H^1(\tilde{\mathcal{X}}, (\mathcal{H}^0(\mathcal{R}^\bullet_\hat{X}))_{1}^{Z}).$$

We can proceed just as before, now the sums in our local analysis run from 0 to $k$, resp. from $k+\frac{1}{2}$ to $k$.

At this point we see that by considering integral structures in our automorphic line bundles $\mathcal{O}_X(k)$ on $X$ we obtain genuinely new structures in cohomology. Namely, whereas Theorem 4.2 does have a non-integral counterpart — the isomorphism

$$\text{Res} : \frac{\Gamma(X, \mathcal{O}_X(k+2))}{\text{im}[\Gamma(X, \mathcal{O}_X(-k)) \overset{\partial^{k+1}}{\rightarrow} \Gamma(X, \mathcal{O}_X(k+2))]} \cong C^1_{\text{har}}(K)$$

from [3] p.97 —, Theorem 5.5 has no non-integral counterpart (in fact $H^1(X, \mathcal{H}^0(\mathcal{R}^\bullet_\hat{X})) = 0$). As an application of Theorem 5.5 we get a global version of the monodromy operator, as follows. From [4.2], [5.1] and [5.5] we obtain $G$-equivariant isomorphisms (if $k > 0$)

$$H^0(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{X}}(k+2)) \cong Z^1_{\text{har}}(k+2) \cong H^1(\tilde{\mathcal{X}}, \mathcal{H}^0(\mathcal{R}^\bullet_\hat{X})) \otimes \varepsilon^{-k-1} \cong H^1(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{X}}(-k)) \otimes \varepsilon^{-k-1}$$

whose composition we denote by $\nu$.

**Definition:** The monodromy operator $N : H^1(\tilde{\mathcal{X}}, \mathcal{R}^\bullet_\hat{X}) \rightarrow H^1(\tilde{\mathcal{X}}, \mathcal{R}^\bullet_\hat{X})$ is the composition

$$H^1(\tilde{\mathcal{X}}, \mathcal{R}^\bullet_\hat{X}) \overset{\text{pr}}{\rightarrow} H^0(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{X}}(k+2)) \overset{\nu}{\rightarrow} H^1(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{X}}(-k)) \overset{i}{\rightarrow} H^1(\tilde{\mathcal{X}}, \mathcal{R}^\bullet_\hat{X})$$

where $\text{pr}$ resp. $i$ is the natural projection resp. inclusion in (18).

Thus $N$ is $G$-equivariant when viewed as a map $H^1(\tilde{\mathcal{X}}, \mathcal{R}^\bullet_\hat{X}) \rightarrow H^1(\tilde{\mathcal{X}}, \mathcal{R}^\bullet_\hat{X}) \otimes \varepsilon^{-k-1}$. Its monodromy filtration $\text{ker}(N) = \text{im}(N) = H^1(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{X}}(-k))$ splits the Hodge filtration $H^0(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{X}}(k+2))$ of $H^1(\tilde{\mathcal{X}}, \mathcal{R}^\bullet_\hat{X})$. Now we restrict our attention to the action by $\text{SL}_2(K)$. If $\Gamma < \text{SL}_2(K)$ is a cocompact discrete torsion free subgroup, we only need to take $\Gamma$-invariants and invert $p$ in (18) to obtain the Hodge decomposition

$$H^1(\tilde{\mathcal{X}}_{\Gamma}, (\mathcal{O}_{\tilde{X}}^\bullet \otimes_{\hat{K}} \text{Sym}^k_{\hat{K}}(\text{St}))^\Gamma) = H^0(\tilde{\mathcal{X}}_{\Gamma}, \mathcal{O}_{\tilde{X}}(k+2)^\Gamma) \oplus H^1(\Gamma, \text{Sym}^k_{\hat{K}}(\text{St}))$$

from [5.3] (we saw $H^1(\Gamma, \text{Sym}^k_{\hat{K}}(\text{St})) = H^1(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{X}}(-k))^\Gamma \otimes \mathbb{Q}$ in [5.3]): no higher $\Gamma$-group cohomology is needed for this passage. It is not hard to see that the monodromy operator we thus obtain on $H^1(\tilde{\mathcal{X}}_{\Gamma}, (\mathcal{O}_{\tilde{X}}^\bullet \otimes_{\hat{K}} \text{Sym}^k_{\hat{K}}(\text{St}))^\Gamma)$ is the one predicted by $p$-adic Hodge theory, using the description of the latter given in [4]. In particular this shows that $N$ respects the integral de Rham structures (as opposed to integral Hyodo-Kato cohomology
structures) in $H^1(\hat{X}_\Gamma, (\Omega^*_\hat{X} \otimes_K \text{Sym}^k_R(\text{St}))^\Gamma)$, a fact which the general $p$-adic Hodge theory does not seem to suggest. We so obtain an infinite rank filtered monodromy module over $\mathcal{O}_\hat{K}$ which comprises all the filtered monodromy modules $H^1(\hat{X}_\Gamma, (\Omega^*_\hat{X} \otimes_K \text{Sym}^k_R(\text{St}))^\Gamma)$ for the various $\Gamma$.

For $k = 0$ we still can define $N$ de Rham integrally as the composition

$$H^1(\hat{X}_\Gamma, (\mathcal{R}^*_\hat{X})^\Gamma) \xrightarrow{pr} H^0(\hat{X}_\Gamma, \mathcal{O}_{\hat{X}}(2)^\Gamma) \xrightarrow{Res^0} Z^1_{\text{par}}(2)^\Gamma,$$

$\xi \mapsto H^0(\hat{X}, \prod_{(Z_1, Z_2)} \mathcal{G}_{(Z_1, Z_2)}(\delta) \xrightarrow{i} H^1(\hat{X}_\Gamma, (\mathcal{R}^*_\hat{X})^\Gamma)$.

Here sheaves $\mathcal{G}_Z$ and $\mathcal{G}_{(Z_1, Z_2)}$ and a map $Z^1_{\text{par}}(2) \to H^0(\hat{X}, \prod_{(Z_1, Z_2)} \mathcal{G}_{(Z_1, Z_2)})$ are defined just as in the proof of 5.5 and $\xi$ is the restricted map on $\Gamma$-invariants. The map $\delta$ is the connecting homomorphism in group cohomology (observe that for $k = 0$ application of $H^0(\hat{X}, \cdot)$ to the sequence (20) preserves its exactness). Inverting $p$ in the above composition gives the correct $N$ on $H^1(\hat{X}_\Gamma, \Omega^*_\hat{X})$ (at least up to sign, see [4]).

6 Complements

(A) Let $\mathcal{R}^*_\hat{X} = \mathcal{R}^*_\hat{X}/(\hat{\pi})$. One can prove the analogs of 5.1 and 5.3 for $\mathcal{R}^*_\hat{X}$, namely:

$$H^1(\hat{X}, \mathcal{R}^*_\hat{X}) = H^1(\hat{X}, \mathcal{H}^0(\mathcal{R}^*_\hat{X})) \oplus H^0(\hat{X}, \mathcal{O}_{\hat{X}}(k + 2))$$

$$H^1(\hat{X}_\Gamma, (\mathcal{R}^*_\hat{X})^\Gamma) = H^1(\hat{X}_\Gamma, (\mathcal{H}^0(\mathcal{R}^*_\hat{X}))^\Gamma) \oplus H^0(\hat{X}_\Gamma, \mathcal{O}_{\hat{X}}(k + 2)^\Gamma).$$

Note that this is not obvious from the proof of 5.1 there we did not consider $\mathcal{H}^0(\mathcal{R}^*_\hat{X})$.

(B) Let $k \in \mathbb{Z}$ be even and let $\omega^\frac{k}{2}_{\hat{X}/\mathcal{O}_R}$ be the logarithmic differential module of the log smooth morphism $\hat{X} \to \text{Spf}(\mathcal{O}_K)$: an invertible $\text{PGL}_2(K)$-equivariant line bundle on $\hat{X}$.

We have an $\text{SL}_2(K)$-equivariant isomorphism

$$\mathcal{O}_{\hat{X}}(k) \cong \omega^\frac{k}{2}_{\hat{X}/\mathcal{O}_R}, \quad f \mapsto fdz^\frac{k}{2}.$$ 

Now $dz^\frac{k}{2}$ is not a generator of $\omega^\frac{k}{2}_{\hat{X}/\mathcal{O}_R}$, not even a global section of $\omega^\frac{k}{2}_{\hat{X}/\mathcal{O}_R}$ if $k < 0$. Let $k > 0$ and even. For $a \in \mathcal{O}_K$ the local section $d\log(z - a)$ is a generator of $\omega^\frac{k}{2}_{\hat{X}/\mathcal{O}_R}$ on an appropriate open formal subscheme of $\hat{X}$. There, the complex $\mathcal{R}^*_\hat{X}$ becomes isomorphic to

$$\omega^\frac{k}{2}_{\hat{X}/\mathcal{O}_R} \xrightarrow{\sim} \omega^\frac{k+2}{2}_{\hat{X}/\mathcal{O}_R},$$

$$f d\log(z - a) \xrightarrow{\text{def}} (D_a \prod_{j=1}^{k} (D_a^2 - j^2) f) d\log(z - a) \omega^\frac{k+2}{2}_{\hat{X}/\mathcal{O}_R}.$$
where $D_a = (z-a)\partial = \frac{(z-a)d}{d(z-a)}$. For the proof you need to show $(z-a)^{k+1} (z-a)^{\frac{1}{2}} = D_a \prod_{j=1}^{k} (D_a - j^2)$. For this show by induction on $n$, departing from $D_a = \partial (z-a) - 1$ that $(z-a)^n \partial^n = D_a (D_a-1) \cdots (D_a-n+1)$ and $\partial^n (z-a)^n = (D_a+n)(D_a+n-1) \cdots (D_a+1)$. Also note $-D_a = (z-a)^{-1} \frac{d}{d(z-a)}$.

(C) For even weights $k \in \mathbb{Z}$ the $\mathcal{O}_X$-modules $\mathcal{O}_X(k)$ are in fact line bundles, and the base extension $K \to \hat{K}$ is unnecessary, i.e. everything we did here descends from $\hat{K}$ to $K$. The automorphic action of even weight $k$ in [5] is the one we get by replacing the factor $\chi^k(\gamma)$ with the factor $\det(\gamma)^{\frac{k}{2}}$ in equation (11). All our results carry over to this situation (and in 5.5 no $\varepsilon^{k+1}$-twist is needed). But also if the weight $k$ is odd, if one is willing to restrict the automorphic action on $\mathcal{O}_X(k)$ to a smaller group, the base extension $K \to \hat{K}$ can be avoided and one has equivariant integral structures which are even line bundles. Let $G^{even} = \{\gamma \in G; \omega(\det(\gamma)) \text{ even}\}$. Note that the restriction to $G^{even}$ of the automorphic action (defined in equation (1)) only depends on the choice of $\pi$, not of $\hat{\pi}$. In notations from section 1 define the following $\mathcal{O}_{\hat{U}(Z_{\gamma n},Z_{\gamma n+1})}$-submodule of $\mathcal{O}_{\hat{U}(Z_{\gamma n},Z_{\gamma n+1})} \otimes \mathcal{O}_K K$:

$$\mathcal{O}_{\hat{U}(Z_{\gamma n},Z_{\gamma n+1})}(k) = \mathcal{O}_{\hat{U}(Z_{\gamma n},Z_{\gamma n+1})} \cdot f_{n,n}^{\left[\frac{k}{2}\right]} f_{n,n+1}^{\left[\frac{k+1}{2}\right]}.$$ 

The $\mathcal{O}_{\hat{U}(Z_{\gamma n},Z_{\gamma n+1})}(k)$ glue into an invertible $\mathcal{O}_{\hat{Y}}$-submodule $\mathcal{O}_{\hat{Y}}(k)$ of $\mathcal{O}_{\hat{Y}} \otimes \mathcal{O}_K K$. Observe

$$\mathcal{O}_{\hat{Y}}(k)|_{\hat{U}(Z_{\gamma n})} = \pi^{\left[\frac{k}{2}\right]} \mathcal{O}_{\hat{U}(Z_{\gamma n})} \text{ inside } \mathcal{O}_{\hat{U}(Z_{\gamma n})} \otimes \mathcal{O}_K K.$$ 

As in [11] one sees that $\mathcal{O}_{\hat{Y}}(k)$ globalizes to a $G^{even}$-equivariant line bundle $\mathcal{O}_X(k)$ on $X$, an integral structure in $\mathcal{O}_X(k)$. Our entire analysis of $\mathcal{O}_X(k)$ can be repeated with $\mathcal{O}_X(k)$, with essentially the same results (e.g. those from section 2 however, the case $k = 1$ is slightly harder in this context). One additional feature is that one has to study $\mathcal{O}_X(k) \otimes \mathcal{O}_Z$ for $Z \in F_{even}$ and for $Z \in F_{odd}$ separately (the two orbits of $G^{even}$ acting on $F^0$) and the shapes of these two types are indeed different if $k$ is odd.

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