The limiting spectral distribution of the generalized Wigner matrix

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Abstract

The properties of eigenvalues of large dimensional random matrices have received considerable attention. One important achievement is the existence and identification of the limiting spectral distribution of the empirical spectral distribution of eigenvalues of Wigner matrix. In the present paper, we explore the limiting spectral distribution for more general random matrices, and, furthermore, give an application to the energy of general random graphs, which generalizes the result of Nikiforov.

Keywords: eigenvalues, random matrix, Wigner matrix, empirical spectral distribution, limiting spectral distribution, moment approach, Stieltjes transform, graph energy.

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1 Introduction

In quantum mechanics, the energy levels of quanta can be characterized by the eigenvalues of a matrix. The empirical spectral distribution (ESD) of a matrix, however, is rather complicated when the order of the matrix is high. Wigner [8, 9] considered the limiting spectral distribution (LSD) for large dimensional random matrices, and obtained the famous semi-circle law. We recall a generalization here due to Bai [1]. To be precise, Wigner investigated the LSD for a random matrix, so-called Wigner matrix,

\[ \mathbf{X}_n := (x_{ij}), \ 1 \leq i, j \leq n, \]

which satisfies the following properties:

- \( x_{ij} \)'s are independent random variables with \( x_{ij} = x_{ji} \);

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the \( x_i \)'s have the same distribution \( F_1 \), while the \( x_{ij} \)'s are to possess the same distribution \( F_2 \);
\[ \text{Var}(x_{ij}) = \sigma^2 < \infty \text{ for all } 1 \leq i < j \leq n. \]

Set
\[ Y_n = \frac{1}{2\sqrt{n}}X_n. \]

We denote the eigenvalues of \( Y_n \) by \( \lambda_{1,n}, \lambda_{2,n}, \ldots, \lambda_{n,n} \), and their ESD by \( \Phi_n(x) = N_n(x)/n \) where
\[ N_n(x) := \#\{\lambda_{k,n} \mid \lambda_{k,n} \leq x, \ k = 1, 2, \ldots, n\}. \]

It is readily seen that for any given real number \( x \), the ESD \( \Phi_n(x) \) is a random variable on the space \( X_n \) consisting of Wigner matrices, while for any given matrix \( X_n \) in \( X_n \), \( \Phi_n(x) \) can be regarded as a distribution function of the eigenvalues of \( X_n \).

**Theorem 1.1** (Wigner [8, 9]). Let \( F_1 \) and \( F_2 \) be distribution functions mentioned above. Then
\[ \lim_{n \to \infty} \Phi_n(x) = \Phi(x) \text{ a.s.,} \]
where \( \Phi(x) \) is the limiting spectral distribution with density
\[ \phi(x) = \begin{cases} \frac{2}{\pi \sigma_2} \sqrt{\sigma_2^2 - x^2} & \text{if } |x| \leq \sigma_2, \\ 0 & \text{if } |x| > \sigma_2. \end{cases} \]

In the present paper, we explore the LSD for more general random matrices. Let \( m := m(n) \geq 2 \) be an integer, and let \( V_1, \ldots, V_m \) be a partition of \([n] := \{1, \ldots, n\}\) such that \( |V_k| = n\nu_k \), where \( \nu_k \) might be the function of \( n \) and \( k = 1, \ldots, m \). We consider the random matrix \( A_n(\nu_1, \ldots, \nu_m) \) (or \( A_n \) for short) satisfying the following properties:

- \( a_{ij} \)'s are independent random variables with \( a_{ij} = a_{ji} \);
- the \( a_{ij} \)'s have the same distribution \( F_1 \) if \( i \) and \( j \) \( \in V_k \), while the \( a_{ij} \)'s are to possess the same distribution \( F_2 \) if \( i \in V_k \) and \( j \in [n] \setminus V_k \), where \( k \) is an integer with \( 1 \leq k \leq m \);
- \( |a_{ij}| \leq K \).

Set
\[ B_n = \frac{1}{2\sqrt{n}}A_n. \]

Let \( \Psi_n(x) \) be the ESD of \( B_n \). In section 2, we establish the LSD of \( B_n \) for special random matrices \( A_n \).

**Theorem 1.2.** Let \( F_1 \) and \( F_2 \) be distribution functions mentioned above.
(i) If
\[
\lim_{n \to \infty} \max\{\nu_1(n), \ldots, \nu_m(n)\} > 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{\nu_i(n)}{\nu_j(n)} = 1 \quad \text{for all} \quad 1 \leq i, j \leq m,
\] (1)
then
\[
\lim_{n \to \infty} \Psi_n(x) = \Psi(x) \quad \text{a.s.},
\]
where \(\Psi(x)\) is the limiting spectral distribution with density
\[
\psi(x) = \begin{cases} 
\frac{2m}{\pi(\sigma_1^2 + (m-1)\sigma_2^2)} \sqrt{\frac{\sigma_1^2 + (m-1)\sigma_2^2}{m}} - x^2 & \text{if} \quad |x| \leq \sqrt{\frac{\sigma_1^2 + (m-1)\sigma_2^2}{m}} \\
0 & \text{if} \quad |x| > \sqrt{\frac{\sigma_1^2 + (m-1)\sigma_2^2}{m}}.
\end{cases}
\]

(ii) If
\[
\lim_{n \to \infty} \max\{\nu_1(n), \ldots, \nu_m(n)\} = 0,
\] (2)
then \(\lim_{n \to \infty} \Psi_n(x) = \Phi(x) \quad \text{a.s.}

Remark. We require \(|a_{ij}| \leq K\) here for some fixed integer \(K\). In fact, one can readily obtain the same LSD for more general distributions \(F_1\) and \(F_2\) satisfying that \(\sigma_1^2 < \infty\) and \(\sigma_2^2 < \infty\) by employing the classical truncation method (see [1] for instance).

We then show that for general case, the random matrix \(A_n\) has no such LSD in section 3, and, besides, propose a conjecture concerning the LSD of \(A_n\). Finally, we give an application about the energy of a simple graph.

2 Proof of Theorem 1.2

The main goal of this section is to show Theorem 1.2. Since we can centralize the general distribution functions \(F_1\) and \(F_2\), we first prove Theorem 1.2 on condition that the expectations \(\mu_1\) and \(\mu_2\) are equal to zero, and then prove the theorem for general distributions in subsection 2.2.

2.1 LSD for centralized distributions

In this part, we employ the moment approach to prove Theorem 1.2 supposing that \(\mu_1 = \mu_2 = 0\).

Above all, we deal with the first part of Theorem 1.2. It is turned out that we need to prove that the moments \(M_{k,n} = \int_{-\infty}^{\infty} \lambda^k d\Psi_n \quad (k = 1, 2, \ldots)\) satisfies almost surely (a.s.) the following condition:
\[
\lim_{n \to \infty} M_{k,n} = \gamma_k = \begin{cases}
0, & \text{if} \quad k \text{ is odd,} \\
\frac{k!}{2^{(k/2)((k/2)+1)}} f(m, \sigma_1, \sigma_2)^k, & \text{if} \quad k \text{ is even,}
\end{cases}
\] (3)
where \( f(m, \sigma_1, \sigma_2) = \sqrt{\frac{\sigma_1^2 + (m-1)\sigma_2^2}{m}} \). Let \( X_{\Psi} \) be a random variable with the distribution \( \Psi(x) \). Then, by the linearity of expectation, we have

\[
\mathbb{E}(e^{itX_{\Psi}}) = \sum_{k \geq 0} \frac{(it)^k}{k!} \mathbb{E}(X_{\Psi}^k).
\]

On the other hand

\[
\sum_{k \geq 0} \frac{(it)^k}{k!} \gamma_k = \sum_{j \geq 0} \frac{(-1)^j}{j! (j+1)!} \left( \frac{t}{2} f(m, \sigma_1, \sigma_2) \right)^{2j} = \frac{2}{t \cdot f(m, \sigma_1, \sigma_2)} J_1(t \cdot f(m, \sigma_1, \sigma_2))
\]

where \( J_1 \) denotes the Bessel function of order 1 of the first kind. Therefore,

\[
\lim_{n \to \infty} M_{k,n} = \gamma_k = \mathbb{E}(X_{\Psi}^k) \text{ a.s., } k = 1, 2, \ldots,
\]

and thus \( \Psi_n(x) \to \Psi(x) \) a.s. \( (n \to \infty) \) according to the Moment Convergence Theorem (\[\text{[1]}, \text{pp. 613}\]).

In order to show that \( M_{k,n} \to \gamma_k \) a.s. \( (n \to \infty) \), we first prove that

\[
\lim_{n \to \infty} \mathbb{E}(M_{k,n}) = \gamma_k,
\]

and then prove that

\[
\lim_{n \to \infty} \left( M_{k,n} - \mathbb{E}(M_{k,n}) \right) = 0 \text{ a.s.}
\]

We now proceed with the calculation of \( M_{k,n} \). It is not difficult to see that

\[
M_{k,n} = \int_{-\infty}^{\infty} \lambda^k d\Psi_n = n^{-1} \sum_{j=1}^{n} \lambda_{j,n}^k = n^{-1} \text{tr}(B_n^k)
\]

\[
= 2^{-k} n^{-1-k/2} \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}.
\]

For \( 1 \leq v \leq k \), denote by \( S_{v,k,n} \) the sum of \( 2^{-k} n^{-1-k/2} \mathbb{E}(a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}) \) over all sequences \( i_1, \ldots, i_k \) where \( v : = \# \{ i_1, \ldots, i_k \} \) (not counting multiplicities) is the order of a sequence. Since the expectation of \( a_{ij} \) equals zero, if some \( a_{ij} \) in the product \( a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1} \) has multiplicity one then the expectation of the product is zero. According to the pigeon hole principle, if \( v > k/2 + 1 \) then \( S_{v,k,n} = 0 \), and thus

\[
\mathbb{E}(M_{k,n}) = \sum_{v=1}^{k/2+1} S_{v,k,n}.
\]

Notice that a product \( a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1} \) corresponds to an unique closed walk

\[
(i_1, i_2)(i_2, i_3) \cdots (i_k, i_1)
\]
of length \(k\) in the complete graph \(K_n\) on the set \([n]\) \((K_n\) can contain loops here). A closed walk \((i_1, i_2)(i_2, i_3)\ldots(i_k, i_1)\) is said to be good if \(\mathbb{E}(a_{i_1i_2}a_{i_2i_3}\ldots a_{i_ki_1}) \neq 0\). Hence the estimation of \(S_{v,k,n}\) relies on a bound on the number of good walks. Let \(W_{v,k,n}\) be the number of good walks in \(K_n\) of length \(k\) and order \(v\). Clearly, there are \(n(n-1)\ldots(n-v+1)\) ways to fix an ordered good walk of \(v\) distinct vertices in \(K_n\). Moreover, for a fixed order, \(W_{v,k,n}\) is a function \(g(v, k)\) of variables \(v\) and \(k\), and thus

\[
W_{v,k,n} = n(n-1)\ldots(n-v+1) \cdot g(v, k).
\]

For odd \(k\), since \((i_1, i_2)(i_2, i_3)\ldots(i_k, i_1)\) is a closed walk and \(\mathbb{E}(a_{ij}) = 0, 1 \leq i \leq j \leq n\), it is easily seen that \(v \leq (k-1)/2\) if \((i_1, i_2)(i_2, i_3)\ldots(i_k, i_1)\) is a good walk. Therefore, for \(v = 1, \ldots, (k-1)/2\),

\[
S_{v,k,n} = 2^{-k}n^{1-k/2} \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \mathbb{E}(a_{i_1i_2}a_{i_2i_3}\ldots a_{i_ki_1})
\leq 2^{-k}n^{-1-k/2}n^v g(v, k)K^k
\leq 2^{-k}n^{-1}g(v, k)K^k \to 0 \; (n \to \infty).
\]

For even \(k\), since \(v \leq k/2 + 1\), we again find, by a similar way, that \(S_{v,k,n} \to 0\) as \(n \to \infty\) when \(v < k/2 + 1\). For the case that \(v = k/2 + 1\), set

\[
T_{k/2} = g(k/2 + 1, k),
\]

\(i.e., T_k\) denotes the number of good walks \(W\) in \(K_n\) of length \(2k\) and order \(k + 1\) such that the order of the vertices appearing in \(W\) is fixed. We use \(T'_k\) to denote the number of good walks \(W = (i_1, i_2)(i_2, i_3)\ldots(i_{2k}, i_1)\) which contain no \(i_1\) except the first and the last member. It is easy to see that

\[
T'_k = T_{k-1}, \quad T'_1 = T_0 = 1,
\]

and

\[
T_k = \sum_{j=1}^{k} T'_j T_{k-j} = \sum_{j=1}^{k} T_{j-1} T_{k-j} = \sum_{i=0}^{k-1} T_i T_{k-1-i}, \quad k = 1, 2, \ldots. \quad (4)
\]

The generating function of \(T_k\) is defined below

\[
T(x) = \sum_{k \geq 0} T_k x^k.
\]

The recursive formula \(4\) then gives

\[
T(x) = 1 + xT(x)^2.
\]

It follows that

\[
T(x) = (2x)^{-1}(1 \pm (1-4x)^{1/2}).
\]

Since \(T_0 = 1\), we have \(T(x) = (2x)^{-1}(1 - (1-4x)^{1/2})\). Thus,

\[
T_k = \frac{1}{2} \left( \frac{1}{k + 1} \right)(-4)^{k+1} = \frac{(2k)!}{k!(k+1)!}.
\]
To calculate $S_{k/2+1,k,n}$, we need to estimate the quantity that
\[
\sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \mathbb{E} \left( a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_k i_1} \right),
\]
where $a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_k i_1}$ corresponds to a closed walk of order $k/2 + 1$. In order to avoid the tedious analysis, we further assume that
\[
\lim_{n \to \infty} \max\{\nu_1(n), \ldots, \nu_m(n)\} > 0 \quad \text{and} \quad \nu_i = \nu_j \quad \text{for all} \quad 1 \leq i, j \leq m.
\]
Indeed, one can readily obtain the same estimation for (5) on condition (1) by the trick we employ below.

Obviously, to estimate (5), the crucial step is to estimate the sum of expectations of good walks of order $k/2 + 1$ and length $k$. In this case, it is readily seen that each edge appears exactly twice in a good walk. For a fixed order of vertices appearing in a good walk $(i_1, i_2)(i_2, i_3) \ldots (i_k, i_1)$ of order $k/2 + 1$, we get a term
\[
(s_1^2 r)(s_2^{2k/2 - r}T_{k/2},
\]
where $r$ is an integer with $0 \leq r \leq k/2$. An edge $(i_j, i_{j+1})$ in a good walk of order $k/2 + 1$ and length $k$ is said to be secondary if $i_j, i_{j+1} \in V_l$ for some part $V_l$ of the partition $V_1 \cup \cdots \cup V_m = [n]$, otherwise, the edge is chief. We then pick up vertices according to the fixed positions of chief and secondary edges when $n$ is large enough. By the condition (6), we have the term, for large enough $n$,
\[
n(\sigma_1^2 r) \left( \frac{n}{m} \right)^r (\sigma_2^{2k/2 - r}) \left( \frac{(m-1)n}{m} \right)^{k/2-r} \cdot T_{k/2} = n^{1+k/2}T_{k/2} \left( \frac{\sigma_1^2}{m} \right)^r \left( \frac{(m-1)\sigma_2^2}{m} \right)^{k/2-r}.
\]
For any fixed value of $r$, we next choose the possible positions for chief and secondary edges, and thus get the term
\[
n^{1+k/2}T_{k/2} \left( \frac{k/2}{r} \right) \left( \frac{\sigma_1^2}{m} \right)^r \left( \frac{(m-1)\sigma_2^2}{m} \right)^{k/2-r}.
\]
According to the condition (6), $r$ may take any value from $\{0, 1, \ldots, k/2\}$ when $n$ is large enough. Hence, we finally obtain the estimation of (5) that
\[
n^{1+k/2}T_{k/2} \sum_{r=0}^{k/2} \left( \frac{k/2}{r} \right) \left( \frac{\sigma_1^2}{m} \right)^r \left( \frac{(m-1)\sigma_2^2}{m} \right)^{k/2-r} = n^{1+k/2}T_{k/2} \left( \frac{\sigma_1^2}{m} + \frac{(m-1)\sigma_2^2}{m} \right)^{k/2}.
\]
Therefore, for large enough $n$, we have
\[
S_{k/2+1,k,n} = 2^{-k}n^{-1-k/2} \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \mathbb{E} \left( a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_k i_1} \right)
\]
\[
= 2^{-k}n^{-1-k/2} \cdot n^{1+k/2}T_{k/2} \left( \frac{\sigma_1^2 + (m-1)\sigma_2^2}{m} \right)^{k/2}
\]
\[
= 2^{-k} \frac{k!}{(k/2)!((k/2)+1)!} \left( \frac{\sigma_1^2 + (m-1)\sigma_2^2}{m} \right)^{k/2}
\]
\[
= \frac{2^k(k/2)!(k/2+1)!}{f(m, \sigma_1, \sigma_2)^k} = \gamma_k.
\]
Consequently,
\[ \mathbb{E}(M_{k,n}) = S_{k/2 + 1, k,n} \rightarrow \gamma_k \quad (n \rightarrow \infty). \]

We now estimate the difference \( M_{k,n} - \mathbb{E}(M_{k,n}) \). Using Markov’s inequality, we have
\[ \mathbb{P}[|M_{k,n} - \mathbb{E}(M_{k,n})| > \epsilon] \leq \mathbb{E}[(M_{k,n} - \mathbb{E}(M_{k,n}))^2]/\epsilon^2. \]
Hence, to prove that \( \mathbb{P}[\lim_{n \rightarrow \infty} (M_{k,n} - \mathbb{E}(M_{k,n})) = 0] = 1 \), it suffices to show that
\[ \sum_{n=1}^{\infty} \mathbb{E}[(M_{k,n} - \mathbb{E}(M_{k,n}))^2] < \infty, \text{ for any given } k. \quad (7) \]

One can readily see that
\[
\begin{align*}
\mathbb{E}[(M_{k,n} - \mathbb{E}(M_{k,n}))^2] &= \mathbb{E}[(M_{k,n})^2] - (\mathbb{E}[M_{k,n}])^2 \\
&= 2^{-2k}n^{-2k} \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} \left[ \mathbb{E}(a_{i_1i_2 \ldots i_ki_1} a_{j_1j_2 \ldots j_kj_1}) - \mathbb{E}(a_{i_1i_2 \ldots i_ki_1}) \mathbb{E}(a_{j_1j_2 \ldots j_kj_1}) \right].
\end{align*}
\]

We use \( a_i \) and \( a_j \) to denote, respectively, the sequences \( a_{i_1i_2 \ldots i_ki_1} \) and \( a_{j_1j_2 \ldots j_kj_1} \).

Obviously, to prove (7), it suffices to show that if \( \mathbb{E}(a_i) \mathbb{E}(a_j) = 0 \), then \( |V(W_i) \cup V(W_j)| \leq k \), where \( W_i \) and \( W_j \) denote the two closed walks \( W_i := (i_1, i_2) \ldots (i_k, i_1) \) and \( W_j := (j_1, j_2) \ldots (j_k, j_1) \), respectively.

One can easily see that if \( a_i \) and \( a_j \) are independent or \( \mathbb{E}(a_i \cdot a_j) = 0 \) then \( \mathbb{E}(a_i \cdot a_j) = \mathbb{E}(a_i) \mathbb{E}(a_j) = 0 \). Thus it is sufficient to consider the case that \( a_i \) and \( a_j \) are not independent and \( \mathbb{E}(a_i \cdot a_j) \neq 0 \).

**Claim 1.** If \( a_i \) and \( a_j \) are not independent and \( \mathbb{E}(a_i \cdot a_j) \neq 0 \), then \( |V(W_i) \cup V(W_j)| \leq k \).

Clearly, if \( a_i \) and \( a_j \) are not independent then \( V(W_i) \cap V(W_j) \neq \emptyset \). Then \( W_i \cup W_j \) is a closed walk of length \( 2k \) since \( W_i \) and \( W_j \) are two closed walks of length \( k \), respectively. If \( \mathbb{E}(a_i \cdot a_j) \neq 0 \) then the order of \( W_i \cup W_j \) is not more than \( k + 1 \) by pigeon hole principle. Furthermore, if \( |V(W_i) \cup V(W_j)| = k + 1 \) then \( a_i \) and \( a_j \) are independent. In fact, each edge in \( W_i \cup W_j \) appears exactly twice when \( |V(W_i) \cup V(W_j)| = k + 1 \). Thus, those edges induce (not counting multiplicities) a tree in \( K_n \) of order \( k + 1 \) since \( W_i \cup W_j \) is connected.

We further assert that \( E(W_i) \cap E(W_j) = \emptyset \). Suppose, for a contradiction, that there exists one element \( a_{i_1}a_{i_{k+1}} \) appears only once in \( W_i \). Then the subgraph graph induced by \( W_i \) should contain a cycle since \( W_i \) is a closed walk, which contradicts to the fact that the subgraph induced by \( W_i \cup W_j \) is a tree. Therefore, \( E(W_i) \cap E(W_j) \) is empty, and thus \( a_i \) and \( a_j \) are independent. Hence, our claim follows.

We thus have
\[ \mathbb{E}[(M_{k,n} - \mathbb{E}(M_{k,n}))^2] \leq n^{-2}, \]
and then (7) holds. Therefore, (3) follows, and this completes our proof of the first part of Theorem 1.2 on condition that \( \mu_1 = \mu_2 = 0 \).

We next show the second part of Theorem 1.2 on condition that \( \mu_1 = \mu_2 = 0 \). We can hold the desire by applying the moment approach again. In fact, by the approach,
it is sufficient to show that the moments $M_{k,n}$ ($k = 1, 2, \ldots$) satisfies a.s. the following condition:

$$
\lim_{n \to \infty} M_{k,n} = \gamma_k = \begin{cases} 
0, & \text{if } k \text{ is odd}, \\
\frac{k^k}{2^k \Gamma(k/2+1)!} \sigma_2^k, & \text{if } k \text{ is even}.
\end{cases}
$$

(8)

It is similar to the proof of the first part that the crucial step is to estimate the quantity when $k$ is even. Evidently, in this case, each edge appears exactly twice in a good walk. Let $W'_{k/2+1,k,n}$ be the set of good walks in $K_n$ of order $k/2 + 1$ and length $k$ in which each walk contains at least one secondary edge (not counting multiplicities). Set

$$W'_{k/2+1,k,n} = |W'_{k/2+1,k,n}| \text{ and } W''_{k/2+1,k,n} = W_{k/2+1,k,n} - W'_{k/2+1,k,n}.$$ 

It is not hard to see that

$$W'_{k/2+1,k,n} \leq \frac{k}{2} \cdot n^{k/2} o(n) T_{k/2} = o(n^{k/2+1}).$$ 

Thus,

$$\lim_{n \to \infty} \frac{W'_{k/2+1,k,n}}{n^{k/2+1}} = \lim_{n \to \infty} \frac{W_{k/2+1,k,n} + W''_{k/2+1,k,n}}{n^{k/2+1}} = \lim_{n \to \infty} \frac{W''_{k/2+1,k,n}}{n^{k/2+1}},$$

and

$$\sum_{W_i \in W'_{k/2+1,k,n}} \mathbb{E}(W_i) \leq o(n^{k/2+1}) K^k.$$ 

On the other hand,

$$W_{k/2+1,k,n} = n(n-1) \cdots (n-k/2) \cdot T_{k/2} = n(n-1) \cdots (n-k/2) \frac{(2k)!}{k!(k+1)!}.$$ 

Hence, for large enough $n$, we have

$$S_{k/2+1,k,n} = 2^{-k} n^{-1-k/2} \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \mathbb{E}(a_{i_1,i_2} a_{i_2,i_3} \cdots a_{i_k,i_1})$$

$$= 2^{-k} n^{-1-k/2} \cdot W''_{k/2+1,k,n} (\sigma_2^2)^{k/2}$$

$$= 2^{-k} n^{-1-k/2} \cdot W_{k/2+1,k,n} (\sigma_2^2)^{k/2}$$

$$= \frac{k!}{2^k (k/2)! (k/2+1)!} \sigma_2^k = \gamma_k.$$ 

Therefore,

$$\lim_{n \to \infty} \mathbb{E}(M_{k,n}) = \gamma_k.$$ 

Using a similar way in the proof of the first part, one can also prove that

$$M_{k,n} \to \mathbb{E}(M_{k,n}) \text{ a.s. (} n \to \infty \text{).}$$ 

Thus (8) holds and the second part of Theorem 1.2 follows when $\mu_1 = \mu_2 = 0$. 

8
2.2 LSD for general distributions

In this subsection, we show that Theorem 1.2 holds for general distribution functions $F_1$ and $F_2$ by two distinct tools.

In the following, the norm $||f||$ of a real function $f$ is always defined as follows:

$$||f|| = \sup_x |f(x)|.$$

**Lemma 2.1 (Rank Inequality [1]).** Let $U$ and $V$ be two Hermitian matrices of order $n$, and let $\Psi_U(x)$ and $\Psi_V(x)$ be the ESD of $U$ and $V$, respectively. Then

$$||\Psi_U(x) - \Psi_V(x)|| \leq \frac{1}{n} \text{rank}(U - V).$$

The Stieltjes transform $S(z)$ of a function $F(x)$ is defined below

$$S(z) = \int_{-\infty}^{\infty} (x - z)^{-1} dF(x), \text{ Im}(z) > 0.$$  

One can readily see that for the ESD $\Psi_n(x)$ of $B_n$, we have

$$\int_{-\infty}^{\infty} (x - z)^{-1} d\Psi_n(x) = n^{-1} \text{tr}(B_n - zI)^{-1}.$$  

Here, we need two facts about this transform, and refer the readers to [1] for details.

**Lemma 2.2.**

(i) $F(x)$ is uniquely determined by $S(z)$.

(ii) For probability distribution, $F_n(x) \to F(x)$ if and only if $S_n(z) \to S(z)$ pointwise.

Let $A_n$ be a symmetric matrix, and let $D_n$ be a symmetric quasi-diagonal matrix. We use $\Psi_n(x)$ to denote the ESD of $A_n + D_n$. Then we have the following result.

**Lemma 2.3.** Let $S_n(z)$ and $\overline{S}_n(z)$ be the Stieltjes transforms of $\Psi_n(x)$ and $\overline{\Psi}_n(x)$, respectively. Then

$$|S_n(z) - \overline{S}_n(z)| \leq \text{Im}(z)^{-2} ||D_n||_1,$$

where $||D_n||_1 := \max_{j \in [n]} \{\sum_{i=1}^{n} D_n(ij)\}$ is the 1-normal number of $D_n$.

Denote by $\lambda(M)$ the spectral radius for some real symmetric matrix $M$ of order $n$. Clearly, $\frac{1}{n}|\text{tr}(M)| \leq \lambda(M) \leq ||M||_1$. As is well known, the eigenvalues of $M$ are real. Then $\lambda(M - zI) \geq |\text{Im}(z)|$. By these observations, we show lemma 2.3 as follows.

**Proof.** Clearly, $(A_n - zI)^{-1} - (A_n + D_n - zI)^{-1} = (A_n + D_n - zI)^{-1}D_n(A_n - zI)^{-1}$. Then

$$|S_n(z) - \overline{S}_n(z)| \leq n^{-1} |\text{tr}((A_n + D_n - zI)^{-1}D_n(A_n - zI)^{-1})|$$

$$\leq \lambda((A_n + D_n - zI)^{-1}D_n(A_n - zI)^{-1})$$

$$\leq (\text{Im}(z))^{-2} ||D_n||_1.$$  

$\square$
We assume, without loss of generality, that $A_n$ is a random matrix with the partition $V_1, \ldots, V_m$ such that $n \nu_i \to \infty$, $i = 1, 2, \ldots, l$, and $n \nu_i < \infty$, $i = l + 1, \ldots, m$, as $n \to \infty$. Let $H_n$ be a quasi-diagonal matrix of order $n$ such that
\[
h_{ij} = \begin{cases} 1, & \text{if } i \text{ and } j \in V_k(1 \leq k \leq m), \\ 0, & \text{otherwise}, \end{cases}
\]
and let $H_n'$ be a matrix such that
\[
h'_{ij} = \begin{cases} 1, & \text{if } i \text{ and } j \in V_k(k \leq l), \\ 0, & \text{otherwise}. \end{cases}
\]
Set
\[
H''_n = H_n - H_n',
\]
\[
C'_n = \frac{1}{2\sqrt{n}}(A_n - (\mu_1 - \mu_2)H_n' - \mu_2J_n),
\]
and
\[
C''_n = C'_n - \frac{1}{2\sqrt{n}}((\mu_1 - \mu_2)H''_n),
\]
where $J_n$ is the matrix in which all elements equal 1.

Since $E(C''_n(ij)) = 0$, the LSD of $C''_n$ is $\Psi(x)$ if \([1]\) holds (or $\Phi(x)$ if \([2]\) holds), as shown above. Let $\Psi''_n$ and $\Psi'_n$ be the ESD of $C''_n$ and $C'_n$, respectively. Then, for their corresponding Stieltjes transforms $S''_n(z)$ and $S'_n(z)$, we have from Lemma 2.3 that
\[
|S''_n(z) - S'_n(z)| \leq (\text{Im}(z))^{-2} \frac{1}{2\sqrt{n}}\|(\mu_1 - \mu_2)H''_n\|_1.
\]
Since each block matrix on the diagonal of $H''_n$ is of finite order, we have
\[
\frac{1}{2\sqrt{n}}\|(\mu_1 - \mu_2)H''_n\|_1 \to 0 \ (n \to \infty).
\]
Then, we can get that $\lim_{n \to \infty} S''_n(z) = \lim_{n \to \infty} S'_n(z)$ for any $z$ such that $\text{Im}z > 0$. Because $\Psi(x)$ (or $\Phi(x)$) is the LSD of $C''_n$, from Lemma 2.2(ii), we have that $\lim_{n \to \infty} S''_n(z) = S^*(z)$, where $S^*(z)$ is the Stieltjes transform of $\Psi(x)$ (or $\Phi(x)$). Therefore, $\lim_{n \to \infty} S'_n(z) = S^*(z)$, and thus the ESD $\Psi'_n(x)$ of $C'_n$ converges to $\Psi(x)$ (or $\Phi(x)$). So, $C'_n$ has the same LSD as $C''_n$.

Furthermore, since each of the block matrices on the diagonal of $H'_n$ is of infinite order, there are $o(n)$ such block matrices. Then we have $\text{rank}((\mu_1 - \mu_2)H'_n + \mu_2J_n) = o(n)$. By employing Lemma 2.1 for $B_n$ and $C'_n$, $\Psi_n(x)$ of $B_n$ converges to the LSD $\Psi(x)$ (or $\Phi(x)$).

Therefore, Theorem 1.2 holds for general distribution functions $F_1$ and $F_2$.

3 The LSD for more general random matrices

In this section, we shall show that there is no LSD for general random matrix $A_n(\nu_1, \ldots, \nu_m)$. Actually, we shall prove that $A_n(\nu_1, \ldots, \nu_m)$ has no LSD for some special cases. To be precise, we shall show that if $F_1 \equiv 0$ then the LSD $A_n(\nu_1, \nu_2)$ exists if and only if
Conjecture 1. Let $A_n$ be a random matrix with partition $V_1, \ldots, V_m$ such that $\lim_{n \to \infty} \max\{\nu_1(n), \ldots, \nu_m(n)\} > 0$. If the LSD of $A_n$ exists, then $\lim_{n \to \infty} \frac{\nu_i(n)}{\nu_j(n)} = 1$ for all $i, j \in [m]$.

Remark. According to Theorem 12 if $\lim_{n \to \infty} \frac{\nu_1(n)}{\nu_2(n)} = 1$ for all $i, j \in [m]$ then the LSD of $A_n$ exists. Thus, if Conjecture 1 is true, then the condition that $\lim_{n \to \infty} \frac{\nu_2(n)}{\nu_1(n)} = 1$ for all $i, j \in [m]$ is necessary and sufficient for the existence of the LSD of $A_n$.

Firstly, we investigate the LSD for $A_n(\nu_1, \nu_2)$ when $F_1 \equiv 0$ and $0 < \nu_1 < 1$. A function $f(t)$ is defined to be nonnegative if it satisfies that

$$\sum_{k=1}^l \sum_{j=1}^l f(t_k - t_j) r_k t_j \geq 0,$$

for any positive integer $l$, and any real numbers $t_1, \ldots, t_l$ and complex numbers $r_1, \ldots, r_l$. In order to avoid the tedious analysis, we further assume that $\nu_1$ and $\nu_2$ are real numbers. Indeed, one can readily obtain the same result under the condition $\lim_{n \to \infty} \frac{\nu_1(n)}{\nu_2(n)} = 1$ by the method we employ below.

Theorem 3.1. Let $A_n(\nu_1, \nu_2)$ be a random matrix with $F_1 \equiv 0$ and $0 < \nu_1 < 1$. If $\nu_1 \neq \nu_2$, then $A_n(\nu_1, \nu_2)$ has no LSD.

Proof. We prove the assertion by a contradiction. Since we can centralize the general distribution $F_2$, we further assume that $E(a_{ij}) = 0$ in what follows. We suppose that $\nu_1 \neq \nu_2$ and there exists a function $\Psi(x)$ such that the ESD $\Psi_n(x)$ of $B_n = A_n(\nu_1, \nu_2)/(2\sqrt{n})$ converges to it almost surely as $n \to \infty$. Then $M_{k,n} = \int_{-\infty}^{\infty} x^k d\Psi_n$ converges to $\gamma_k = \int_{-\infty}^{\infty} x^k d\Psi$ almost surely as $n \to \infty$ ($k = 1, 2, \ldots$). So we can get the estimation of $\gamma_k$ by calculating the moment $M_{k,n}$.

We first estimate $E(M_{k,n}) = \sum_{v=1}^{k/2+1} S_{v,k,n}$, as we did in subsection 2.1. For the similar reason, one can readily see that we merely need to compute $E(M_{k,n})$ for even $k = 2j$. Moreover, if $v < j + 1$ then $S_{v,2j,n} \to 0$ as $n \to \infty$. Thus, to get the estimation of $E(M_{k,n})$, it suffices to focus on $S_{j+1,2j,n}$ which indeed satisfies that

$$\lim_{n \to \infty} S_{j+1,2j,n} = \begin{cases} 2^{-2j} \cdot (\nu_1 \nu_2)^{2j} \cdot T_j & \text{if } j \text{ is even}, \\ 2^{-2j} \cdot (\nu_1 \nu_2)^{2j} \cdot T_j & \text{if } j \text{ is odd}. \end{cases}$$

One then can prove by an analogous way in Subsection 2.1 that

$$\lim_{n \to \infty} \left( M_{k,n} - E(M_{k,n}) \right) = 0 \text{ a.s.}$$

Set $\tilde{\nu} = (\nu_1 \nu_2)^{1/4}$. Apparently, $0 < \tilde{\nu} < \sqrt{1/2}$ since $\nu_1 \neq \nu_2$. Then, the following equality holds a.s.

$$\lim_{n \to \infty} M_{k,n} = \begin{cases} 0 & k \equiv 1, 3 \mod 4, \\ \frac{2k! \tilde{\nu}^k \sigma_2^k}{2^k(k/2)! \Gamma((k/2 + 1)!)} & k \equiv 0 \mod 4, \\ \frac{k! \tilde{\nu}^k \sigma_2^k}{\tilde{\nu}^2 2^k(k/2)! \Gamma((k/2 + 1)!)} & k \equiv 2 \mod 4. \end{cases}$$
Let $X_{\Psi}$ be a random variable with the distribution $\Psi$, and let $f(t) := \mathbb{E}(e^{itX_{\Psi}}) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} (it)^k$ be the character function of $X_{\Psi}$. Since $M_{k,n} \to \gamma_k$ a.s. ($n \to \infty$), $k = 1, 2, \ldots$, the following equality holds a.s.

\[
\mathbb{E}(e^{itX_{\Psi}}) = \sum_{j \text{ is even}}^{\infty} \frac{2 \cdot (-1)^j}{j!(j+1)!} \left( \frac{t}{2} \right)^{2j} \tilde{\nu}^{2j} \sigma_1^{2j} + \sum_{j \text{ is odd}}^{\infty} \frac{(-1)^j}{j!(j+1)!} \left( \frac{t}{2} \right)^{2j} \tilde{\nu}^{2j} \sigma_1^{2j} = 2 \cdot \frac{J_1(\tilde{\nu} \sigma_F t) + I_1(\tilde{\nu} \sigma_F t)}{\tilde{\nu} \sigma_F t} + \frac{1}{\tilde{\nu}^2} \cdot \frac{J_1(\tilde{\nu} \sigma_F t) - I_1(\tilde{\nu} \sigma_F t)}{\tilde{\nu} \sigma_F t} = (2 + \frac{1}{\tilde{\nu}^2}) \cdot \frac{J_1(\tilde{\nu} \sigma_F t)}{\tilde{\nu} \sigma_F t} + (2 - \frac{1}{\tilde{\nu}^2}) \cdot \frac{I_1(\tilde{\nu} \sigma_F t)}{\tilde{\nu} \sigma_F t} = (2 + \frac{1}{\tilde{\nu}^2}) \cdot \frac{1}{\pi} \int_{-1}^{1} e^{i\tilde{\nu} \sigma_F t x} \sqrt{1-x^2} dx + (2 - \frac{1}{\tilde{\nu}^2}) \cdot \frac{1}{\pi} \int_{-1}^{1} e^{-\tilde{\nu} \sigma_F t x} \sqrt{1-x^2} dx
\]

($J_1$ denotes the Bessel function of order 1 of the first kind and $I_1$ denotes the modified Bessel function of order 1 of the first kind, such that $I_1(t) = -iJ_1(it)$). We further assume that $l = 2, r_1 = 1, r_2 = 1$ and $t_2 = 0$. Since $0 < \tilde{\nu} < \sqrt{\frac{1}{2}}$, if $t_1$ is large enough then

\[
\sum_{k=1}^{l} \sum_{j=1}^{l} f(t_k - t_j) r_k \tilde{r}_j = \int_{-1}^{1} \sum_{k=1}^{2} \sum_{j=1}^{2} \left[ (2 + \frac{1}{\tilde{\nu}^2}) e^{i\tilde{\nu} \sigma_2 (t_k - t_j)x} \cdot \frac{1}{\pi} \sqrt{1-x^2} + (2 - \frac{1}{\tilde{\nu}^2}) e^{-i\tilde{\nu} \sigma_2 (t_k - t_j)x} \cdot \frac{1}{\pi} \sqrt{1-x^2} \right] dx \
\leq \int_{-1}^{1} (2 + \frac{1}{\tilde{\nu}^2}) \cdot \frac{1}{\pi} \sqrt{1-x^2} \sum_{k=1}^{2} e^{i\tilde{\nu} \sigma_2 t_k x} \left| 2 \right| dx + \int_{-1}^{1} (2 - \frac{1}{\tilde{\nu}^2}) e^{-i\tilde{\nu} \sigma_2 t_k x} \cdot \frac{1}{\pi} \sqrt{1-x^2} dx \
\leq 2(2 + \frac{1}{\tilde{\nu}^2}) + (2 - \frac{1}{\tilde{\nu}^2}) \int_{1/4}^{1/2} \frac{1}{\pi} \cdot e^{i\tilde{\nu} \sigma_2 t_1/4} \cdot \frac{\sqrt{3}}{2} dx \
\leq 2(2 + \frac{1}{\tilde{\nu}^2}) + (2 - \frac{1}{\tilde{\nu}^2}) \frac{1}{\pi} \cdot e^{i\tilde{\nu} \sigma_2 t_1/4} \cdot \frac{\sqrt{3}}{8} < 0,
\]

which contradicts to the fact that the character function should be nonnegative. Hence, the necessity follows.

For general random matrix $A_n(\nu_1, \ldots, \nu_m), m \geq 3$, we fail to obtain the result in the same way as the case for $m = 2$. Indeed, if we estimate the moment $\gamma_k$ by the same method we used above, the step to count the number of good walks for $S_{j+1,2j,n}$ is much complicated. Worse still, the character function is also harder to get. However, we can still verify Conjecture for some special cases.

**Proposition 1.** Suppose $F_1 \equiv 0$, $\sigma_2 = 1$, $\nu_1 > 3/4$ and $\nu_2 = \cdots = \nu_m$, where $m \geq 3$. Then, there exists no LSD for $A_n(\nu_1, \ldots, \nu_m)$ a.s.

**Proof.** Since we can centralize the general distribution $F_2$, we further assume $\mu_2 = 0$ in what follows. For a contradiction, we assume $\Psi(x)$ is the LSD of $B_n$ such that $\lim_{n \to \infty} \Psi_n(x) = \Psi(x)$ a.s. Then, $M_{k,n} = \int x^k d\Psi_n(x) \to \int x^k d\Psi(x) = \gamma_k$ a.s. ($n \to \infty$).
It is readily seen that for any \( k + 1 \) real numbers \( t_0, t_1, \ldots, t_k \),
\[
\sum_{p,q=0}^{k} \gamma_{p+q} t_p t_q = \int_{-\infty}^{\infty} (t_0 + t_1 x + \cdots + t_k x^k)^2 d\Psi(x) \geq 0. \tag{9}
\]

Obviously, \( \gamma_0 = 1 \). Set
\[
\Delta_k = \begin{pmatrix}
\gamma_0 & \gamma_1 & \cdots & \gamma_k \\
\gamma_1 & \gamma_2 & \cdots & \gamma_{k+1} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k}
\end{pmatrix}.
\]

Then due to (9), we have
\[
(t_0, \ldots, t_k) \Delta_k (t_0, \ldots, t_k)^T \geq 0.
\]

Thus, the symmetric matrix \( \Delta_k \) is non-negative definite for any \( k \). Therefore, \( |\Delta_k| \geq 0 \), \( k = 0, 1, 2, \ldots \).

Then, applying the same method as in Section 2.1, we can compute the moment \( \gamma_k \).

At first, to estimate \( E(M_{k,n}) \), we just need to focus on the case for \( k = 2j \), and, moreover, it suffices to calculate \( S_{j+1,2j,n} \).

Let \( j = 1 \). We have
\[
S_{2,2,n} = 2^{-2} n^{-2} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} E(a_{i_1 i_2}^2).
\]

If \( i_1, i_2 \) are in the same part \( V_i \) (\( 1 \leq i \leq m \)), \( E(a_{i_1 i_2}^2) = 0 \) since \( F_i \equiv 0 \). Thus, only for \( i_1, i_2 \) that do not belong to the same part \( V_i \), the expectation \( E(a_{i_1 i_2}^2) \) contributes 1 to the value \( S_{2,2,n} \). Combining the fact \( (m-1)\nu_2 + \nu_1 = 1 \), we have
\[
S_{2,2,n} \rightarrow 2^{-2} \cdot (\nu_1(m-1)\nu_2 + (m-1)\nu_2(1-\nu_2)) \text{ as } n \to \infty.
\]

One then can prove by an analogous way in Subsection 2.1 that
\[
\lim_{n \to \infty} (M_{k,n} - E(M_{k,n})) = 0 \text{ a.s.}
\]

Therefore, the value \( \gamma_2 = 2^{-2} \cdot (\nu_1(m-1)\nu_2 + (m-1)\nu_2(1-\nu_2)) \text{ a.s.} \)

By this means, we have
\[
\begin{align*}
\gamma_2 &= 2^{-2} \cdot (\nu_1(m-1)\nu_2 + (m-1)\nu_2(1-\nu_2)), \\
\gamma_4 &= 2 \cdot 2^{-4} \cdot (\nu_1(m-1)\nu_2(1-\nu_2) + (m-1)\nu_2\nu_1(m-1)\nu_2 \\
&\quad + (m-1)\nu_2(m-2)\nu_2(1-\nu_2)), \\
\gamma_6 &= 5 \cdot 2^{-6} \cdot (\nu_1(m-1)\nu_2\nu_1(m-1)\nu_2 + \nu_1(m-1)\nu_2(m-2)\nu_2(1-\nu_2) \\
&\quad + (m-1)\nu_2\nu_1(m-1)\nu_2(1-\nu_2) + (m-1)\nu_2(m-2)\nu_2\nu_1(m-1)\nu_2 \\
&\quad + (m-1)\nu_2(m-2)\nu_2(m-2)\nu_2(1-\nu_2)).
\end{align*}
\]

and \( \gamma_1, \gamma_3, \gamma_5 = 0 \).

But then \( |\Delta_3| < 0 \), which contradicts to the fact \( |\Delta_k| \geq 0 \). Thus, the proposition follows. \( \Box \)
4 Application to the energy of random graphs

In this section, we shall compute the energy of a random graph by the results established in the previous sections. Our notions and terminology are standard, and we refer the readers to [2] for the conceptions not defined here. Let $G$ be a simple graph of order $n$. The eigenvalues $\lambda_1, \ldots, \lambda_n$ of the adjacent matrix of $G$ are said to be the eigenvalues of $G$. In chemistry, there is a correspondence between the graph eigenvalues and the molecular orbital energy levels of $\pi$-electrons in conjugated hydrocarbons. For the H"{u}ckel molecular orbital (HMO) approximation, the total $\pi$-electron energy $\mathcal{E}(G)$ in conjugated hydrocarbons is given by the sum of absolute values of the eigenvalues corresponding to the molecular graph $G$. In 1970s, Gutman [5] extended the conception of energy to all simple graphs who defined

$$\mathcal{E}(G) = \sum_{i=0}^{n} |\lambda_i|,$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $G$. Recently, this graph invariant has attracted a lot of attention, and the readers are referred to [6] for further details.

Let $G_n(p)$ be a random graph of $G_n(p)$. It is easy to see that if $F_1 \equiv 0$ and $F_2$ is a Bernoulli distribution with mean $p$, then the random matrix $X_n$ is the adjacent matrix of $G_n(p)$. According to Theorem 1.1, almost every (a.e.) random graph $G_n(p)$ enjoys the equation below:

$$\mathcal{E}(G_n(p)) = 2\sqrt{n} \cdot n \left( \frac{2}{\pi \sigma_2^2} \left| \int_{-\sigma_2}^{\sigma_2} |x| \sqrt{\sigma_2^2-x^2} \, dx + o(1) \right| \right)$$

$$= \frac{n^{3/2}}{\pi} \left( \frac{8}{3\pi} \sigma_2 + o(1) \right).$$

Note that for $p = \frac{1}{2}$, Nikiforov in [7] got the above formula. Here, our result is for any probability $p$, which could be seen as a generalization of his result. Next we will get the energy for random $m$-partite graphs.

We use $K_{n;\nu_1,\ldots,\nu_m}$ to denote the complete $m$-partite graph of order $n$ whose parts $V_1, \ldots, V_m$ are such that $|V_i| = n\nu_i$, $i = 1, \ldots, m$, where $m = m(n) \geq 2$ is an integer. Let $G_{n;\nu_1,\ldots,\nu_m}(p)$ be the set of random graphs in which the edges are chosen independently with probability $p$ from $K_{n;\nu_1,\ldots,\nu_m}$. Especially, we denote by $K[n; m]$ and $G_{n,m}(p)$, respectively, the complete $m$-partite graph and the set of $m$-partite random graphs satisfying

$$\lim_{n \to \infty} \max \{\nu_1(n), \ldots, \nu_m(n)\} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{\nu_i(n)}{\nu_j(n)} = 1.$$

One can readily see that if a random matrix $A_n$ and the complete $m$-partite graph $K[n; m]$ have the same partition, and $F_1 \equiv 0$ and $F_2$ is a Bernoulli distribution with mean $p$, then $A_n$ is the adjacent matrix of $G_{n,m}(p) \in G_{n,m}(p)$. Employing the first part of Theorem 1.2.
a.e. random graph $G_{n,m}(p)$ enjoys the following equation:

$$
\varepsilon(G_{n,m}(p)) = 2\sqrt{n} \cdot n \left( \frac{2m}{\pi(m-1)\sigma_2^2} \int_{-\sqrt{\frac{m-1}{m}\sigma_2^2}}^{\sqrt{\frac{m-1}{m}\sigma_2^2}} |x| \sqrt{\frac{(m-1)\sigma_2^2}{m} - x^2} \, dx + o(1) \right)
$$

\[= n^{3/2} \left( \frac{8}{3\pi} \sqrt{\frac{m-1}{m}\sigma_2 + o(1)} \right) \]

Furthermore, we can get the energy $\varepsilon$ of a random graph $G_{n;\nu_1,\ldots,\nu_m}(p) \in G_{n;\nu_1,\ldots,\nu_m}(p)$ if $\lim_{n \to \infty} \max\{\nu_1(n), \ldots, \nu_m(n)\} = 0$ by Theorem 1.2 (ii). In fact, note that if $A_n$ and $K_{n;\nu_1,\ldots,\nu_m}$ have the same partition, $F_1 \equiv 0$ and $F_2$ is a Bernoulli distribution with mean $p$, then $A_n$ is the adjacent matrix of $G_{n;\nu_1,\ldots,\nu_m}(p)$. Thus, by Theorem 1.2 (ii), a.e. random graph $G_{n;\nu_1,\ldots,\nu_m}(p)$ enjoys the following equation:

$$
\varepsilon(G_{n;\nu_1,\ldots,\nu_m}(p)) = n^{3/2} \left( \frac{8}{3\pi} \sqrt{p(1-p) + o(1)} \right).
$$

For $m$-partite random graphs $G_{n;\nu_1,\ldots,\nu_m}(p)$ such that

$$
\lim_{n \to \infty} \max\{\nu_1(n), \ldots, \nu_m(n)\} > 0 \quad \text{and there exist } \nu_i \text{ and } \nu_j \text{ such that } \lim_{n \to \infty} \frac{\nu_i(n)}{\nu_j(n)} < 1,
$$

we can establish lower and upper bounds for its energy. For the purpose, we first introduce the following an auxiliary assertion due to [3].

**Lemma 4.1.** Let $X, Y, Z$ be square matrices of order $n$ such that $X + Y = Z$, then

$$
\sum_{i=1}^{n} s_i(X) + \sum_{i=1}^{n} s_i(Y) \geq \sum_{i=1}^{n} s_i(Z)
$$

where $s_i (i = 1, \ldots, n)$ is the singular values of a matrix.

Similarly, suppose $A_n$ and $G_{n;\nu_1,\ldots,\nu_m}(p)$ have the same partition $V_1, \ldots, V_m (|V_i| = \nu_i n)$. Then, $A_n$ is the adjacent matrix of $G_{n;\nu_1,\ldots,\nu_m}(p)$ providing $F_1 \equiv 0$ and $F_2$ is a Bernoulli distribution $B(p)$. Without loss of generality, we assume, for some $r \geq 1$, $|V_1|, \ldots, |V_r|$ are of order $O(n)$ while $|V_{r+1}|, \ldots, |V_m|$ of order $o(n)$. Let $X'_n$ be a random symmetric matrix such that

$$
X'_n(ij) = \begin{cases} 
A_n(ij) & \text{if } i \text{ or } j \notin V_s(1 \leq s \leq r), \\
B(p) & \text{if } i, j \in V_s(1 \leq s \leq r) \text{ and } i > j, \\
0 & \text{if } i, j \in V_s(1 \leq s \leq r) \text{ and } i = j.
\end{cases}
$$

From Theorem 1.2 (ii), $X'_n$ has the same LSD as $X_n$ on condition that $F_1 \equiv 0$ and $F_2 = B(p)$. Set

$$
D_n = X'_n - A_n = \begin{pmatrix} K_1 \\ K_2 \\ \vdots \\ K_r \\ O \end{pmatrix}_{n \times n}
$$

(10)
Let \( M \) be a matrix. We use \( \mathcal{E}(M) \) to denote the sum of singular values of \( M \). Evidently, if \( M \) is the adjacent matrix of a simple graph \( G \) then \( \mathcal{E}(G) = \mathcal{E}(M) \). One can readily see that a.e. matrix \( K_i \) \((i = 1, \ldots, r)\) enjoys the following:

\[
\mathcal{E}(K_i) = \left( \frac{8}{3\pi} \sqrt{p(1-p)} + o(1) \right) (\nu_i n)^{3/2},
\]

and then a.e. matrix \( D_n \) satisfies the following:

\[
\mathcal{E}(D_n) = \left( \frac{8}{3\pi} \sqrt{p(1-p)} + o(1) \right) (\nu_1^{3/2} + \cdots + \nu_r^{3/2}) n^{3/2}.
\]

By (10), we have \( A_n + D_n = X'_n \) and \( X'_n + (-D_n) = A_n \). Employing Lemma 4.1 we deduce

\[
\mathcal{E}(X'_n) - \mathcal{E}(D_n) \leq \mathcal{E}(A_n) \leq \mathcal{E}(X'_n) + \mathcal{E}(D_n).
\]

Therefore, we establish the following result.

**Theorem 4.2.** Let \( G_{n;\nu_1\ldots\nu_m}(p) \) be a random graph of \( G_{n;\nu_1\ldots\nu_m}(p) \). Then a.e. random graph \( G_{n;\nu_1\ldots\nu_m}(p) \) satisfies the following inequality

\[
\left( 1 - \sum_{i=1}^r \nu_i^{3/2} \right) n^{3/2} \leq \mathcal{E}(G_{n;\nu_1\ldots\nu_m}(p)) \left( \frac{8}{3\pi} \sqrt{p(1-p)} + o(1) \right)^{-1} \leq \left( 1 + \sum_{i=1}^r \nu_i^{3/2} \right) n^{3/2}.
\]

**Remark.** Since \( \nu_1, \ldots, \nu_r \) are positive real numbers with \( \sum_{i=1}^r \nu_i \leq 1 \), we have \( \sum_{i=1}^r \nu_i (1 - \nu_i^{1/2}) > 0 \). Therefore, \( \sum_{i=1}^r \nu_i > \sum_{i=1}^r \nu_i^{3/2} \), and thus \( 1 > \sum_{i=1}^r \nu_i^{3/2} \). Hence, we can deduce, by the theorem above, that a.e. random graph \( G_{n;\nu_1\ldots\nu_m}(p) \) enjoys the following

\[
\mathcal{E}(G_{n;\nu_1\ldots\nu_m}(p)) = O(n^{3/2}).
\]

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