Topological Field Theory
and
Second-Quantized Five-Branes

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Abstract

We construct the six-dimensional topological field theory appropriate to describe the ground-state configurations of D5-branes. A close examination on the degenerations of D5-branes gives us the physical observables which can be regarded as the Poincaré duals of the cycles of the moduli space. These observables are identified with the creation operators of the bound states of D5-branes and lead to the second quantization of five-branes. This identification of the bound states with the cycles also provides their topological stability and suggests that the bound states of five-branes have internal structures. The partition function of the second-quantized five-branes is also discussed.

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1 Introduction

The recent discovery of the Dirichlet branes (D-branes) [1] gives us a route to the quantization of solitonic objects in string theory. The quantum fluctuations of these solitonic objects are described by open strings with one or both of their boundaries constrained on them, for which they are named Dirichlet branes.

Their low energy effective worldvolume theory turns out [2], [3] to be essentially a dimensionally reduced supersymmetric gauge theory. For the case of the Dirichlet five branes (D5-branes) in Type I string theory, which are identified [3] with the dual of the small instanton limit of the gauge five branes [4] in the $SO(32)$ heterotic string, the effective worldvolume theory is a six-dimensional supersymmetric $Sp(1)$ gauge theory. It is also argued in [3] that the $Sp(1)$ gauge symmetry can be enhanced to $Sp(k)$ group for $k$ coincident D5-branes and that the $D$-flat condition gives us the ADHM equation [5] of $k$ instantons of four-dimensional $SO(32)$ gauge theory.

In addition to these low energy analysis, due to their coupling with open string, the D-branes are identified with the BPS states which have the Ramond-Ramond charges. These BPS states are allowed to have the bound states which are marginally stable. The existence of these marginally stable bound states is argued by several authors [6], [7], [8]. It is also pointed out [8] the possibility that the BPS states can be identified with the cohomology elements of instanton moduli spaces.

In this article topological aspects of the BPS states are studied from the worldvolume theoretical viewpoint. In particular, based on the effective worldvolume gauge theory, we examine the geometrical interpretation of the bound states. For this purpose we consider the five branes in the Type IIB theory with open strings having the $U(1)$ Chan-Paton factors. In section 2 we consider the case of the $U(n)$ Chan-Paton factors. The $D$-flat condition of the effective worldvolume theory leads to the ADHM equation of four-dimensional $U(n)$ gauge theory. We construct the six-dimensional topological $U(k)$ gauge theory of which physical moduli space consists of the solutions of the ADHM equation of the $k$ instantons. In section 3 we concentrate on the case of the $U(1)$ Chan-Paton factors. We begin by comparing the physical moduli space with the configuration space of D5-
branes. The degeneration of D5-branes are studied in detail. This close examination brings us to introduce the physical observable $O_k$ \((3.18)\) which can be regarded as the Poincaré dual of the cycle corresponding to $k$ coincident D5-branes. These physical observables $O_k$ turn out to give a field theoretical realization of the correspondences investigated by Nakajima \([9], [10]\) and are naturally identified with creation operators of the bound states of D5-branes. This identification of the bound states with the cycles provides their topological stability and suggests that the bound states of 5-branes have internal structures. The introduction of the creation operators also leads to the second quantization of D5-branes. The partition function of second-quantized D5-branes gives the generating function of the Poincaré polynomials of the physical moduli spaces. In section 4 we discuss further on the second quantization of D5-branes from the four-dimensional supersymmetric gauge theoretical point of view.

2 Topological Field Theory on D5-Brane Worldvolume

2.1 Effective Worldvolume Theory of D5-brane

Let us consider type IIB theory with the $U(n)$ open strings\(^4\). When there is a D5-brane an open string can have either free (Neumann) boundary with the (anti-)fundamental representation of $U(n)$, or Dirichlet boundary located on the D5-brane. The Dirichlet boundary has the index of the fundamental representation of $U(k)$ when there are $k$ coincident D5-branes\(^5\). The combinations of these boundary conditions correspond to three distinct open string sectors: Neumann-Neumann (NN), Dirichlet-Dirichlet (DD) and Dirichlet-Neumann (DN). The quantization of DD and DN strings leads to the (first) quantization of 5-brane\([1], [2], [3]\).

Suppose that there are $k$ coincident D5-branes at $x^6 = \cdots = x^9 = 0$ for definiteness. Their low energy effective worldvolume theory can be described by massless modes of

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\(^4\) This is equivalent to consider the type IIB theory under the background of $n$ coincident D9-branes\([1]\).
the DD and DN strings. It is a six-dimensional supersymmetric $U(k)$ gauge theory with global $U(n)$ symmetry\cite{2,3}.

There are two kinds of massless bosonic modes $A_\mu$ and $X^i$ in the DD sector. $A_\mu (\mu = 0, \cdots, 5)$ give a $U(k)$ gauge field. $X^i (i = 6, \cdots, 9)$ are scalar fields which belong to the adjoint representations of $U(k)$. Their $U(k)$ gauge transformations are

\begin{align}
A_\mu (x) & \to g(x)A_\mu (x)g^{-1}(x) - i(\partial_\mu g(x))g^{-1}(x), \\
X^i (x) & \to g(x)X^i (x)g^{-1}(x), \quad g(x) \in U(k).
\end{align}

Note that $A_\mu$ and $X^i$ are represented by $k \times k$ hermitian matrices. These fields are inert under the global $U(n)$ since the DD string has no Neumann boundary. The vacuum expectation values of $X^i (6 \leq i \leq 9)$ become the collective coordinates of D5-branes.

The $SO(4) \simeq SU(2) \times SU(2)_R$ which originates from the rotations in the four-dimensions $(x^6, \cdots, x^9)$ is a global symmetry group of the worldvolume theory. $SU(2)_R$ is identified with the $SU(2)$ R-symmetry. In order to make it clear, it is convenient to rewrite $X^i$ as

\begin{equation}
X_{A\dot{A}} = X^i \sigma^i_{A\dot{A}}, \quad \sigma^i_{A\dot{A}} = (i\tau^1, i\tau^2, i\tau^3, 1_2),
\end{equation}

where $\tau^{1,2,3}$ are the Pauli matrices and $1_2$ is a $2 \times 2$ identity matrix. $\dot{A}(= \dot{1}, \dot{2})$ is the $SU(2)_R$ index and $A(= 1, 2)$ is the $SU(2)$ index. In the DN sector, there is a $SU(2)_R$ doublet complex scalar $H_{\dot{A}}$. Since the DN string has both Neumann and Dirichlet boundaries, each $H_{\dot{A}}$ transforms as $(k, \bar{n})$ representation under the action of $U(k) \times U(n)$.

\begin{equation}
H_{\dot{A}}(x) \to g(x)H_{\dot{A}}(x)h^{-1}, \quad (g(x), h) \in U(k) \times U(n).
\end{equation}

Notice that $H_{\dot{A}}$ is represented by a $k \times n$ complex matrix.

The bosonic part of the effective action will be given by

\begin{equation}
S_{\text{boson}} = \int d^6x \ Tr \left[ \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} D^\mu X^{A\dot{A}} D_\mu X_{A\dot{A}} + D^\mu H_{\dot{A}} D_\mu H^{\dot{A}} - \frac{1}{2} D^A D_{AB} + D^{\dot{A}\dot{B}} (X_{A\dot{A}} X^{A\dot{B}} + H_{\dot{A}} H_{\dot{B}}) \right],
\end{equation}

where $D_\mu$ is the $U(k)$ gauge covariant derivative and

\begin{equation}
\bar{H}_{\dot{A}} = \epsilon_{\dot{A}\dot{B}} (H_B)^\dagger.
\end{equation}
The degenerate vacua (moduli space) are determined by the D-flat condition

\[ X_{A(\bar{A})}X_{B(\bar{B})}^A + H_{(\bar{A})\bar{B}} = 0, \quad (2.7) \]

which coincides with the ADHM equation [3] of \( U(n) \) instantons in \( \mathbb{R}^4 \).

So far we have implicitly assumed that the space-time is flat \( \mathbb{R}^{10} \), but we may consider more general cases. For the case that the four-dimensional space \( (x^6, \cdots, x^9) \) is an ALE space \( K \), ADHM equation (2.7) will be modified [11] to

\[ X_{A(\bar{A})}X_{B(\bar{B})}^A + H_{(\bar{A})\bar{B}} = \zeta_{\bar{A}\bar{B}}1_k, \quad (2.8) \]

where \( 1_k \) is a \( k \times k \) unit matrix and \( \zeta_{\bar{A}\bar{B}}(= \zeta_{\bar{B}\bar{A}}) \) are the constants related to the three Kähler forms \( \omega_{\bar{A}\bar{B}} \) of \( K : \zeta_{\bar{A}\bar{B}} = \int_{C_2} \omega_{\bar{A}\bar{B}} \). \( C_2 \in H_2(K) \). This modification of the ADHM equation can be also understood from the D-brane viewpoint as the result of the Fayet-Iliopoulos D-terms induced in a non-trivial gravitational background [12] :

\[ S_{FI} = -\zeta_{\bar{A}\bar{B}} \int d^6x \text{ Tr}D^{\bar{A}\bar{B}}. \quad (2.9) \]

Notice that, by using the \( SU(2)_R \) rotations, we may set these three constants \( \zeta_{\bar{A}\bar{B}} \) equal to zero except the only one component which we call \( \eta(>0) \). This can be done when we fix a complex structure and the constant \( \eta \) is related to the corresponding Kähler form \( \omega^R \) by \( \eta = \int_{C_2} \omega^R \).

As the first step to taking account of the gravitational effect (due to string) it may be enough to add Fayet-Iliopoulos D-terms (2.9) to the effective action. (The corresponding \( \eta \) is a positive constant.) Therefore the moduli space of the effective worldvolume \( U(k) \) gauge theory will be given as

\[ \mathcal{M}_{U(n)}^k = \left\{ (X_{\bar{A}A}, H_{\bar{A}}) \mid X_{A(\bar{A})}X_{B(\bar{B})}^A + H_{(\bar{A})\bar{B}} = \zeta_{\bar{A}\bar{B}}1_k, \quad \partial_\mu X_{\bar{A}A} = \partial_\mu H_{\bar{A}} = 0 \right\} \bigg/ U(k), \quad (2.10) \]

where the subscript \( U(n) \) denotes the Chan-Paton factors of the open strings. The quotient by \( U(k) \) in the R.H.S. of (2.10) must be taken due to the global part of the \( U(k) \) gauge symmetry. As regards the dimensionality of the moduli space, by simply counting up the degrees of freedom, it turns out to be

\[ \dim_{\mathbb{R}} \mathcal{M}_{U(n)}^k = 4kn. \quad (2.11) \]
Notice that the moduli space $\mathcal{M}_{U(n)}^k$ also describes the grand-state configurations of D5-branes in Type IIB theory with the $U(n)$ open strings.

2.2 Topological Field Theory

Let us construct a six-dimensional topological gauge theory appropriate to describe the cohomology theory on moduli space $\mathcal{M}_{U(n)}^k$ (2.10). To obtain a topological field theory [13] it may be convenient to use the standard BRST formulation [14]. To apply this formalism we shall begin by considering the following constraints on the field variables:

\begin{align}
F_{\mu\nu} &\equiv \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] = 0, \\
D_\mu X_{A\bar{A}} &\equiv \partial_\mu X_{A\bar{A}} - i[A_\mu, X_{A\bar{A}}] = 0, \\
D_\mu H_{\bar{A}} &\equiv \partial_\mu H_{\bar{A}} - iA_\mu H_{\bar{A}} = 0, \\
X_{A(A)}X_{\bar{B}}^{\bar{B}} + H_{(A)}H_{\bar{B}} &\equiv \zeta_{\bar{A}\bar{B}}1_k.
\end{align}

Since the worldvolume is $\mathbb{R}^6$ one can solve the first constraint $F_{\mu\nu} = 0$ by simply setting $A_\mu$ a pure gauge. Constraints (2.13) and (2.14) now reduce to

\begin{align}
\partial_\mu X_{A\bar{A}} = \partial_\mu H_{\bar{A}} = 0,
\end{align}

which means that $X_{A\bar{A}}$ and $H_{\bar{A}}$ are constant. Therefore the last constraint becomes equivalent to ADHM equation (2.8).

As the second step we define the (pre-)BRST transformation $\tilde{\delta}$ by introducing the corresponding (fermionic) ghost fields $\psi_\mu, \psi_{A\bar{A}}$ and $\psi_{\bar{A}}$:

\begin{align}
\tilde{\delta}A_\mu &= \psi_\mu, \quad \tilde{\delta}\psi_\mu = 0, \\
\tilde{\delta}X_{A\bar{A}} &= \psi_{A\bar{A}}, \quad \tilde{\delta}\psi_{A\bar{A}} = 0, \\
\tilde{\delta}H_{\bar{A}} &= \psi_{\bar{A}}, \quad \tilde{\delta}\psi_{\bar{A}} = 0.
\end{align}
To construct the action of topological field theory one may need (bosonic) auxiliary fields and (fermionic) anti-ghost fields. Let us introduce $D_{\dot{A}, \dot{B}}, \chi_{\mu, \nu}, \chi_{\mu, \dot{A}}$, and $\chi_{\mu, \dot{A}}$ as the auxiliary fields. The anti-ghost fields are denoted by $\eta_{\dot{A}, \dot{B}}, \eta_{\mu, \nu}, \eta_{\mu, \dot{A}}$, and $\eta_{\mu, \dot{A}}$. The (pre-)BRST transforms of these additional fields are given by

\[
\delta \eta_{\dot{A}, \dot{B}} = iD_{\dot{A}, \dot{B}}, \quad \delta D_{\dot{A}, \dot{B}} = 0,
\]

\[
\delta \eta_{\mu, \nu} = i\chi_{\mu, \nu}, \quad \delta \chi_{\mu, \nu} = 0,
\]

\[
\delta \eta_{\mu, \dot{A}} = i\chi_{\mu, \dot{A}}, \quad \delta \chi_{\mu, \dot{A}} = 0,
\]

\[
\delta \eta_{\mu, \dot{A}} = i\chi_{\mu, \dot{A}}, \quad \delta \chi_{\mu, \dot{A}} = 0.
\]

(2.18)

Notice that these (anti-)ghost and auxiliary fields are $k \times k$ or $k \times n$ matrix-valued with the definite values of ghost number. For instance the fermionic ghost $\psi_{\mu, \nu}$ is $k \times k$ hermitian matrix-valued and possesses the ghost number equal to one. Matrix sizes and ghost numbers of other fields are summarized in Table 1.

Now the $\tilde{\delta}$-invariant action will follow if we apply the standard BRST formalism with taking constraints (2.12)-(2.15) as the gauge-fixing functionals:

\[
\tilde{S} = -i \int d^6x \tilde{\delta} \text{Tr} \left[ \frac{1}{2} \eta^{\mu \nu} F_{\mu \nu} + \eta^{\mu \dot{A}} D_{\mu} X_{\dot{A}, \dot{A}} + \eta^{\mu \dot{A}} D_{\mu} \bar{H}^\dot{A} + (D_{\mu} H^\dot{A}) \bar{\eta}^{\mu \dot{A}} + \eta^{\dot{A} \dot{B}} (X_{\dot{A}, \dot{A}} X_{\dot{B}, \dot{B}} + H^\dot{A} \bar{H}^\dot{B} - \zeta_{\dot{A}, \dot{B}} 1_k) \right],
\]

(2.19)

which is evaluated into the form:

\[
\tilde{S} = \int d^6x \text{Tr} \left[ \frac{1}{2} \chi^{\mu \nu} F_{\mu \nu} + \chi^{\mu \dot{A}} D_{\mu} X_{\dot{A}, \dot{A}} + \chi^{\mu \dot{A}} D_{\mu} \bar{H}^\dot{A} + (D_{\mu} H^\dot{A}) \bar{\chi}^{\mu \dot{A}} + D^{\dot{A} \dot{B}} (X_{\dot{A}, \dot{A}} X_{\dot{B}, \dot{B}} + H^\dot{A} \bar{H}^\dot{B} - \zeta_{\dot{A}, \dot{B}} 1_k) - i \eta^{\mu \nu} D_{\mu} \bar{\psi}_\nu + i \eta^{\mu \dot{A}} D_{\mu} \bar{\psi}_{\dot{A}} + i \eta^{\mu \dot{A}} D_{\mu} \bar{\bar{\psi}}^\dot{A} - i (D_{\mu} \bar{\psi}) \bar{\eta}^{\mu \dot{A}} + i \eta^{\dot{A} \dot{B}} (\psi_{\dot{A}, \dot{A}} X_{\dot{B}, \dot{B}} + X_{\dot{A}, \dot{A}} \bar{\psi}_{\dot{B}} + \psi_{\dot{A}} \bar{H}_{\dot{B}} + H^\dot{A} \bar{\psi}_B) \right].
\]

(2.20)

Note that the equations of motion for $A_{\mu}, X_{\dot{A}, \dot{A}}$ and $H^\dot{A}$ fields are exactly equations (2.12)-(2.15) and that those for the ghost fields have the forms:

\[
D_{[\mu} \bar{\psi}_{\nu]} = 0,
\]

(2.21)

\[
D_{\mu} \bar{\psi}_{\dot{A}} = 0,
\]

(2.22)

\[
D_{\mu} \bar{\psi}_{\dot{A}} = 0,
\]

(2.23)

\[
\psi_{\dot{A}} X^\dot{A} + X_{A} \bar{\psi}^A + \psi_{\dot{A}} \bar{H}_{\dot{B}} + H^\dot{A} \bar{\psi}_B = 0.
\]

(2.24)
Since the worldvolume is $R^6$ one can also solve equations (2.21)-(2.23) by simply gauging away the nonzero modes of the ghosts (due to the triviality of $A_\mu$). Last equation (2.24) can be regarded as the equations of motion for the zero modes. Since it describes the (infinitesimal) deformation of the ADHM equation, the ghost zero modes can be thought of as the (co-)tangent vectors of moduli space $M_{U(n)}^k$ (2.10).

The above construction of topological field theory, however, is not complete yet. Some modifications are needed due to the existence of two residual local symmetries in action (2.20). One is the “super” gauge symmetry [13]. The transforms of $\psi_\mu$ and $\chi_{\mu\nu}$ under this fermionic symmetry have the forms:

$$\delta \psi_\mu = D_\mu \theta, \quad \delta \chi_{\mu\nu} = [\eta_{\mu\nu}, \theta],$$

(2.25)

where $\theta$ is the fermionic gauge parameter. The another residual symmetry is rather peculiar, which transforms the auxiliary and anti-ghost fields by anti-symmetric tensorial parameters $\Lambda_{\mu\nu\rho}$, for instance,

$$\delta \chi^{\mu\nu} = \frac{1}{3!} \epsilon^{\mu\nu\rho_1\rho_2\rho_3\rho_4} D_{\rho_1} \Lambda_{\rho_2\rho_3\rho_4},$$

(2.26)

where $\Lambda_{\mu\nu\rho}$ is the rank three anti-symmetric tensorial parameter. This peculiar symmetry originates in the properties of equations (2.12)-(2.15). For example symmetry (2.26) is due to the Bianchi identity, $D_{[\mu} F_{\nu]} = 0$.

The prescription for “super” gauge symmetry (2.25) is as follows. Let us introduce the (bosonic) ghost $\phi$. The corresponding (bosonic) anti-ghost and (fermionic) auxiliary fields are denoted by $\lambda$ and $\eta$ respectively. We will fix fermionic symmetry (2.25) by imposing the condition:

$$D_\mu \psi^\mu \equiv \partial_\mu \psi^\mu - i[A_\mu, \psi^\mu] = 0.$$  

(2.27)

(Pre-)BRST transforms (2.18) must be modified so that they are consistent with the above prescription. This requirement leads to the following definition of the BRST transfor-
Note that the transformation $\delta_b$ is nilpotent up to the $U(k)$ gauge transformation parametrized by the bosonic ghost $\phi$: $\delta_b^2 = \delta_{U(k)}(\phi)$.

As regards the residual symmetry of the anti-ghost and auxiliary fields it turns out that one can fix it by simply introducing non-zero gauge parameters in the action. Taking into account these two modifications we finally obtain the following form of the action:

$$S = -i \int d^6 x \delta_b \text{Tr} \left[ \frac{1}{2} \eta^{\mu \nu} F_{\mu \nu} + \lambda D^\mu \psi_\mu + \eta^{A \dot{A}} D_\mu X_{A \dot{A}} + \eta^{\mu A} D_\mu \bar{H}^A + (D_\mu H_A) \bar{\eta}^{\mu A} ight.$$

$$\left. + \eta^{A \dot{B}} (X_{A \dot{A}} X_{B \dot{B}} + H_A \bar{H}_{B} - \zeta_{A \dot{B}} I_k) - \frac{\alpha_1}{4} \eta^{\mu \nu} \chi_{\mu \nu} - \frac{\alpha_2}{2} \eta^{A \bar{A}} \chi_{\mu A \dot{A}} - \frac{\alpha_3}{2} (\eta_{\mu A} \bar{\eta}^{\mu A} + \chi_{\mu A} \chi^{\mu A}) - \frac{\alpha_4}{2} \eta^{A \dot{B}} D_{A \dot{B}} \right]$$

$$= \int d^6 x \text{Tr} \left[ \frac{1}{2} \chi^{\mu \nu} F_{\mu \nu} - \frac{\alpha_1}{4} \chi^{\mu \nu} \chi_{\mu \nu} + \chi^{A \bar{A}} D_\mu X_{A \dot{A}} - \frac{\alpha_2}{2} \chi^{A \bar{A}} \chi_{\mu A \dot{A}} ight.$$

$$\left. + D^{A \dot{B}} (X_{A \dot{A}} X_{B \dot{B}} + H_A \bar{H}_{B} - \zeta_{A \dot{B}} I_k) - \frac{\alpha_4}{2} D^{A \dot{B}} D_{A \dot{B}} \right.$$  

$$+ i \eta^{\mu \nu} D_\mu \psi_\nu + i \eta^{\mu A} \bar{D}_\mu \psi_{\mu A} + i \eta^{\mu \bar{A}} \bar{D}_\mu \bar{\psi}_{\bar{A}} - i (D_\mu \psi_\mu) \bar{\eta}^{\mu A} 
+ i \eta^{A \dot{B}} (\psi_{A \dot{A}} X_{B \dot{B}} + X_{A \dot{A}} \psi_{B \dot{B}} + \psi_{A \dot{A}} \bar{H}_{B} + H_A \bar{\psi}_{B}) 
+ \eta^{A \bar{A}} \psi_\mu, X_{A \dot{A}} + \eta^{\mu \bar{A}} \psi_\mu, \bar{H}^\dot{A} - [\psi_\mu, H_A] \bar{\eta}^{\mu A} 
- i \frac{\alpha_1}{4} \eta^{\mu \nu} [\phi, \eta_{\mu \nu}] - i \frac{\alpha_2}{2} \eta^{A \bar{A}} [\phi, \eta_{A \bar{A}}] - i \alpha_3 \eta_{A \dot{A}} [\phi, \bar{\eta}^{A \dot{A}}] \right]$$
\[-i \frac{\alpha_4}{2} \eta^{AB} [\phi, \eta_{AB}] - i \eta D^\mu \psi_\mu + i D^\mu \lambda D_\mu \phi - \lambda [\psi^\mu, \psi_\mu] \]

(2.29)

where \(\alpha_1, \ldots, \alpha_4\) are the gauge parameters which fix the residual symmetry of the anti-ghost and auxiliary fields.

Let us describe the relation between the action of the topological theory and that of the effective D5-brane worldvolume theory. Taking the Feynmann gauge \(\alpha_1 = \cdots = \alpha_4 = 1\) and then integrating out the auxiliary fields \(\chi\) in (2.29), we can find out that the action has the form:

\[
S = \int d^6 x \ Tr \left[ \frac{1}{4} F^{\mu \nu} F_{\mu \nu} + \frac{1}{2} D^\mu X^{A \bar{A}} D_\mu X_{A \bar{A}} + D^\mu H_A D_\mu \bar{H}^{\bar{A}} \right.
\]

\[
- \frac{1}{2} D^{\hat{A} \hat{B}} D_{\hat{A} \hat{B}} + D^{A \bar{B}} (X_{A \bar{A}} X_{B \bar{B}} + H_A \bar{H}_{B} - \zeta_{A \bar{B}} I_k) + \cdots \Bigg] , \quad (2.30)
\]

where \(\cdots\) denotes the terms including fermions. This is nothing but the bosonic part of action (2.23) of the effective D5-brane worldvolume theory with Fayet-Iliopoulos D-terms (2.9). Therefore our topological field theory can be regarded as the one describing the ground-state configurations of D5-branes.

In the next section, concentrating on the simple case that the open strings have the \(U(1)\) Chan-Paton factors, we will study the moduli space \(\mathcal{M}^k \equiv \mathcal{M}^k_{U(1)}\) from the D5-brane viewpoint and then, returning to the topological field theory, we will construct physical observables. These observables will be interpreted in terms of D5-branes.

3 Second Quantization of D5-Branes

3.1 Degenerations of D5-Branes

In this section we study D5-branes in Type IIB theory with the \(U(1)\) open strings. We will begin by describing the relation between the moduli space \(\mathcal{M}^k \equiv \mathcal{M}^k_{U(1)}\) and the configuration space of D5-branes. The configuration space of \(k\) D5-branes is the \(k\)-th symmetric product of \(\mathbb{R}^4\), \(S^k(\mathbb{R}^4)\). In fact, when the \(j\)-th D5-brane is located on \(P_j \in \mathbb{R}^4(1 \leq j \leq k)\), the corresponding configuration is represented as \(P_1 + \cdots + P_k\).

\(^6\)We assume that quantum statistics of D5-branes are bosonic.
$P_k \in S^k(\mathbb{R}^4)$. Let us consider the $U(k)$-invariant combinations of $(X_{\dot{A}\dot{A}}, H_{\dot{A}})$. The characteristic polynomial of $X_{\dot{A}\dot{A}}$ suffices:

$$\det(\lambda - X_{\dot{A}\dot{A}}) = \prod_{j=1}^{k}(\lambda - x_{j,\dot{A}\dot{A}}).$$

(3.1)

Each eigenvalue $x_{j,\dot{A}\dot{A}}$ can be thought of as the coordinates of $P_j \in \mathbb{R}^4$ on which the $j$-th D5-brane is located. Therefore this set of eigenvalues of $X_{\dot{A}\dot{A}}$, counting their multiplicity if they degenerate, defines the projection $\pi$ from the moduli space $\mathcal{M}^k$ to the configuration space of D5-branes.

$$\pi : \mathcal{M}^k \to S^k(\mathbb{R}^4) \quad (X_{\dot{A}\dot{A}}, H_{\dot{A}}) \mapsto P_1 + \cdots + P_k$$

(3.2)

When the positions of $k$ D5-branes are generic, $x_{j,\dot{A}\dot{A}}$ ($1 \leq j \leq k$) can be regarded as the coordinates of moduli space $\mathcal{M}^k$ (2.10).

We will consider the degeneration of D5-branes in moduli space $\mathcal{M}^k$ (2.10). In particular we will study the case when all the D5-branes are overlapping at a point. For this purpose it is convenient to use the center-of-mass and relative coordinates of $k$ D5-branes.

The center of $k$ D5-branes is described by $x_{\dot{A}\dot{A}}^{(0)}$:

$$x_{\dot{A}\dot{A}}^{(0)} = \frac{1}{k} \text{Tr}X_{\dot{A}\dot{A}},$$

(3.3)

and the relative motions are measured by $x_{\dot{A}\dot{A}}^{(j)}$:

$$x_{\dot{A}\dot{A}}^{(j)} = x_{j,\dot{A}\dot{A}} - x_{\dot{A}\dot{A}}^{(0)} \quad (1 \leq j \leq k - 1).$$

(3.4)

We denote the relative distances of $k$ D5-branes by $\Delta^{(j)}$:

$$\Delta^{(j)} = \det x_{\dot{A}\dot{A}}^{(j)} \quad (1 \leq j \leq k - 1).$$

(3.5)

It also turns out useful to introduce the following complex structure in the moduli space $\mathcal{M}^k$:

$$z_1^{(j)} = x_{21}^{(j)}, \quad z_2^{(j)} = x_{11}^{(j)} \quad (0 \leq j \leq k - 1),$$

(3.6)

which is equivalent to use the following complex matrices at the level of field variables:

$$N = X_{21}, \quad M = X_{11}.$$
In terms of these variables the ADHM equation has the form:

\[
\begin{align*}
[ N, M ] + H_1 H_2^\dagger &= 0, \\
[ N, N^\dagger ] + [ M, M^\dagger ] + H_1 H_1^\dagger - H_2 H_2^\dagger &= \eta 1_k .
\end{align*}
\] (3.8)

(i) \( k = 2 \) case

We firstly consider the degeneration of two D5-branes. Their center is located at \( z^{(0)} = (z_1^{(0)}, z_2^{(0)}) \) and the relative coordinate is given by \( z^{(1)} = (z_1^{(1)}, z_2^{(1)}) \). The configuration of these two D5-branes being separate from each other gives the following solution of ADHM equation (3.8) up to the \( U(2) \) action:

\[
\begin{align*}
N &= \begin{pmatrix}
z_1^{(0)} + z_1^{(1)}/2 & \eta u \sqrt{\eta(2 - \frac{u^2}{2})} z_1^{(1)}/\Delta^{(1)} \\
0 & z_1^{(0)} - z_1^{(1)}/2
\end{pmatrix}, \\
M &= \begin{pmatrix}
z_2^{(0)} + z_2^{(1)}/2 & \eta u \sqrt{\eta(2 - \frac{u^2}{2})} z_2^{(1)}/\Delta^{(1)} \\
0 & z_2^{(0)} - z_2^{(1)}/2
\end{pmatrix}, \\
H_1 &= \begin{pmatrix}
\eta/u \\
\sqrt{\eta(2 - \frac{u^2}{2})}
\end{pmatrix}, \\
H_2 &= \begin{pmatrix}
0 \\
0
\end{pmatrix},
\end{align*}
\] (3.9)

where

\[
u^2 = \frac{\eta}{2} \left\{ 1 + \frac{2\eta}{\Delta^{(1)} \Delta^{(1)}} + \sqrt{1 + \left( \frac{2\eta}{\Delta^{(1)} \Delta^{(1)}} \right)} \right\}.
\] (3.10)

The collision of two D5-branes may be described by studying the behavior of solution (3.9) in the region \( \Delta^{(1)} \ll \eta^{1/2} \). It turns out to be

\[
\begin{align*}
N &= \begin{pmatrix}
z_1^{(0)} & \sqrt{\eta} z_1^{(1)}/\Delta^{(1)} \\
0 & z_1^{(0)}
\end{pmatrix} + \begin{pmatrix}
z_1^{(1)}/2 & 0 \\
0 & -z_1^{(1)}/2
\end{pmatrix} + O(\Delta^{(1)2}/\eta), \\
M &= \begin{pmatrix}
z_2^{(0)} & \sqrt{\eta} z_2^{(1)}/\Delta^{(1)} \\
0 & z_2^{(0)}
\end{pmatrix} + \begin{pmatrix}
z_2^{(1)}/2 & 0 \\
0 & -z_2^{(1)}/2
\end{pmatrix} + O(\Delta^{(1)2}/\eta), \\
H_1 &= \begin{pmatrix}
0 \\
\sqrt{2\eta}
\end{pmatrix} + \begin{pmatrix}
\Delta^{(1)}/\sqrt{2} \\
0
\end{pmatrix} + O(\Delta^{(1)2}/\eta).
\end{align*}
\] (3.11)

It tells that, under the limit of two D5-branes colliding with each other, solution (3.9) does depend on how they approach. It really depends on the way that \( z^{(1)} = (z_1^{(1)}, z_2^{(1)}) \)
goes to (0, 0). It is parametrized by \( \lambda^{(1)} = (\lambda_1^{(1)}, \lambda_2^{(1)}) : \)

\[
(\lambda_1^{(1)}, \lambda_2^{(1)}) = \lim_{z^{(1)} \to (0,0)} \left( \frac{z_1^{(1)}}{\Delta^{(1)}}, \frac{z_2^{(1)}}{\Delta^{(1)}} \right). \tag{3.12}
\]

Notice that \( \lambda^{(1)} \in S^3 \). With this parametrization of their collision the solution of ADHM equation acquires the following form when two D5-branes overlap :

\[
N^{(0)} = \begin{pmatrix}
    z_1^{(0)} & \sqrt{\eta} \lambda_1^{(1)} \\
    0 & z_1^{(0)}
\end{pmatrix},
\]

\[
M^{(0)} = \begin{pmatrix}
    z_2^{(0)} & \sqrt{\eta} \lambda_2^{(1)} \\
    0 & z_2^{(0)}
\end{pmatrix},
\]

\[
H^{(0)}_1 = \begin{pmatrix}
    0 \\
    \sqrt{2\eta}
\end{pmatrix}, \quad H^{(0)}_2 = \begin{pmatrix}
    0
\end{pmatrix}. \tag{3.13}
\]

Although solution (3.13) has the collision parameter \( \lambda^{(1)} \) of two D5-branes besides the position \( z^{(0)} \) where they overlap, different collisions of two D5-branes at \( z^{(0)} \) do not necessarily correspond to different points in the moduli space \( \mathcal{M}^{k=2} \) due to the symmetry enhancement. In fact, let us consider the action of the \( U(1) \) subgroup, \( \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix} \in U(2) \right\} \).

Solution (3.13) transforms to :

\[
N^{(0)} g = \begin{pmatrix}
    z_1^{(0)} & \sqrt{\eta} \lambda_1^{(1)} e^{i\theta} \\
    0 & z_1^{(0)}
\end{pmatrix},
\]

\[
M^{(0)} g = \begin{pmatrix}
    z_2^{(0)} & \sqrt{\eta} \lambda_2^{(1)} e^{i\theta} \\
    0 & z_2^{(0)}
\end{pmatrix}, \tag{3.14}
\]

with \( H^{(0)}_A \) unchanged, \( H^{(0)}_A g = H^{(0)}_A \). Only the collision parameter \( \lambda^{(1)} = (\lambda_1^{(1)}, \lambda_2^{(1)}) \) is multiplied by \( e^{i\theta} \). This multiplication by the \( U(1) \) phase factor means that we can distinguish overlapping two D5-branes by its collision parameter \( \lambda^{(1)} \) only up to the \( U(1) \) action. Taking it in reverse, in the projection map (3.2), the inverse image of the configuration of two D5-branes degenerate at \( P \in \mathbb{R}^4 \) is equivalent to \( S^2 \) which is obtainable by modding out \( S^3 \) by \( U(1) : \pi^{-1}(2P) \cong S^2 \). By looking at solution (3.13)
we find out that the area of this $S^2$ is proportional to $\eta(>0)$ and, as we shall see later, this cycle can be regarded as one of cousins of the bound states of D5-branes.

At this stage it may be convenient to reinterpret these phenomena from the world-volume $U(2)$ gauge theoretical point of view. Notice that $U(2)$ gauge symmetry is completely broken by Higgs mechanism when the positions of two D5-branes are generic $(z^{(1)} \neq (0,0))$. At the point where two D5-branes overlap each other only the $U(1)$ gauge symmetry is restored and other gauge symmetries are still left broken due to $\eta(>0)$. By taking the parametrization

$$(z_1^{(i)}, z_2^{(i)}) = (\Delta^{(i)} \cos \beta e^{i\sigma_1^{(i)}}, \Delta^{(i)} \sin \beta e^{i\sigma_2^{(i)}}),$$

we can find that $\sigma^{(i)}_{U(1)} = \frac{1}{2}(\sigma_1^{(i)} + \sigma_2^{(i)})$ is the corresponding $U(1)$ Nambu-Goldstone boson.

(ii) $k \geq 3$ case

Let us consider the situation that $k$ D5-branes overlap at $P \in \mathbb{R}^4$. It is very similar to the case of $k = 2$. The $U(k)$ gauge symmetry on the worldvolume theory is completely broken by Higgs mechanism when these $k$ D5-branes are in generic positions. At the point where they degenerate occurs the restoration of $U(1) \times \cdots \times U(1)$ gauge symmetry and other gauge symmetries are still left broken due to $\eta(>0)$. To see explicitly, let $k$ D5-branes collide at $P$ with keeping their generic positions. As in the case of $k = 2$ we shall introduce the collision parameters $\lambda^{(j)} = (\lambda_1^{(j)}, \lambda_2^{(j)}) \in S^3$ from the phases of relative coordinates $z^{(j)} = (z_1^{(j)}, z_2^{(j)}) (1 \leq j \leq k-1)$. These collision parameters will appear in the solution of ADHM equation which is obtainable as the limit of the $k$-ply degeneration of D5-branes. Even though they survive in the solution we can not necessarily distinguish all the different collisions at $P$ due to the enhanced symmetry group $U(1) \times \cdots \times U(1)$. In particular the net degrees of freedom of $\lambda^{(j)}$s are $3(k-1) - (k-1) = 2(k-1)$. It means that in projection map $(3.2)$, $\pi^{-1}(kP)$, the inverse image of the configuration of $k$ D5-branes overlapping at $P$, is the cycle of $2(k-1)$ dimensions.

To give an exact description of this cycle, it is necessary to study the $k$-ply degenerations of $k$ D5-branes over all the patterns of their collisions. These patterns may be
parametrized by Young tableaux with length $k$. Namely, by letting $Y = [a_1, \cdots, a_l]$ be a Young tableau with length $k$ ($a_1 \geq \cdots \geq a_l \geq 1$, $|Y| \equiv a_1 + \cdots + a_l = k$), the corresponding pattern of collisions is that $a_j$ ($1 \leq j \leq l$) of $k$ D5-branes degenerate first and then they collide at $P$ in order. From this observation one may say that the stratification of the cycle $\pi^{-1}(kP)$ can be labelled by these Young tableaux: $\pi^{-1}(kP) = \bigsqcup_{|Y| = k} \pi^{-1}(kP)_Y$. The situation we have mentioned above includes the maximal stratum $\pi^{-1}(kP)_{Y = [k]}$.

### 3.2 Construction of Observables

Let us construct physical observables in our theory. Due to BRST symmetry (2.28) the physical content of the theory will become equivalent to the cohomology theory of the moduli space $\mathcal{M}^k$. Especially the BRST charge $Q_{\text{BRST}}$ can be regarded as the differential operator $d_{\mathcal{M}^k}$ on the moduli space:

$$Q_{\text{BRST}} = d_{\mathcal{M}^k} : \Omega^*(\mathcal{M}^k) \rightarrow \Omega^*(\mathcal{M}^k).$$ (3.16)

(i) $k = 1$ case

Notice that $\mathcal{M}^{k=1} = \mathbb{R}^4$. $X_{AA}$ gives the global coordinates of the moduli space. There is an unique physical observable $[e] \in H^0(\mathbb{R}^4)$. We shall denote it by $O_1$:

$$O_1 = [e].$$ (3.17)

(ii) $k \geq 2$ case

For the case of $k \geq 2$ our topological theory will be effectively described as the sum of two topological subsystems. One is the system of $x_{AA}^{(0)}$, which leads to the topological quantum mechanics on $\mathbb{R}^4$. The BRST charge $Q_{\text{BRST}}$ acts as the differential operator $d_{\mathbb{R}^4}$ on $\mathbb{R}^4$. So there is an unique physical observable $[e] \in H^0(\mathbb{R}^4)$ in this subsystem. The another is the system of relative coordinates $x_{AA}^{(j)}$ ($1 \leq j \leq k - 1$), which gives us the nontrivial physical observables.

Let us consider the following combinations of field variables:

$$O_k \equiv [e] \prod_{j=1}^{k-1} \frac{2}{\sqrt{\pi \rho}} e^{-\frac{\Delta_j(j)^{2}}{\rho}} \psi^{(j)}_\Delta \psi^{(j)}_{U(1)};$$ (3.18)
where $\rho$ is a positive constant and $[e]$ is the physical observable of the subsystem of $x^{(0)}_{AA}$, $\psi_{\Delta}^{(j)}$ is the BRST partner of the $j$-th relative distance $\Delta^{(j)}$

$$\delta_b \Delta^{(j)} = \psi_{\Delta}^{(j)} \quad (1 \leq j \leq k - 1),$$

(3.19)

and $\psi_{U(1)}^{(j)}$ is the BRST partner of the $j$-th Nambu-Goldstone boson $\sigma_{U(1)}^{(j)}$

$$\delta_b \sigma_{U(1)}^{(j)} = \psi_{U(1)}^{(j)} \quad (1 \leq j \leq k - 1).$$

(3.20)

$O_k$ is the physical observable with ghost number equal to $(1 + 1) \times (k - 1) = 2(k - 1)$. Let us investigate its geometrical meaning. Due to the exponential factors in (3.18) the support of $O_k$ in the moduli space $\mathcal{M}^k$ is on a tubular neighborhood (with it’s radius $\sim \sqrt{\rho}$) of the cycle $\mathcal{M}^{(k)}_k \equiv \{ \pi^{-1}kP : P \in \mathbb{R}^4 \}$. The dimensions of this cycle are $2(k - 1) + 4 = 2(k + 1)$ (or the co-dimensions equal $2(k - 1)$). Ghost fields $\psi_{\Delta}^{(j)}$ and $\psi_{U(1)}^{(j)}$ appearing in (3.18) can be regarded as the differential forms normal to this $2(k + 1)$-dimensional cycle. Therefore $O_k$ can be interpreted as the Poincaré dual of the cycle $\mathcal{M}^k_{(k)}$.

Besides the above understanding of $O_k$ based on the geometry of moduli space $\mathcal{M}^k$ it is quite surprising that $O_k$ itself has its own meaning in any topological system based on the worldvolume gauge theory with gauge group $U(k')$ ($k' \geq k$). In fact, suppose that there exist $k'$ D5-branes and $k$ pieces of them are colliding at a point. As regards these degenerate $k$ D5-branes we can apply the preceding discussions and then we will obtain $O_k$ as the physical observable. Due to this universality $O_k$ will be regarded as a field theoretical realization of the “correspondences” given in [9], [10].

Although one may ask the possibility that there exist physical observables other than $O_k$, we will see in the next subsection that these $O_k$ generate all the observables.

### 3.3 Second-Quantized D5-Branes

In order to study the implication of this universality of physical observables let us consider the quantization of the theory by means of operator formalism.

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7This kind of universality is familiar in the non-perturbative formulation of two-dimensional quantum gravity. Namely the gravitational descendents can be defined non-perturbatively and also have their geometrical expressions on the moduli space of Riemann surface with fixed genus.
Since the physical observables $O_k (k \geq 1)$ have the universality, the vacuum state should be characterized in terms of D5-branes, not by gauge theories. It may be very plausible to introduce the vacuum as the state which simply corresponds to the configuration of no D5-branes. Let us denote it by $|0\rangle$. We shall introduce the physical operator $\alpha_{-k}$ which corresponds to $O_k$. From the look of the explicit form of $O_k$ it is the bosonic operator with ghost number equal to $2(k - 1)$ and will be interpreted as the creation operator of the configuration of $k$ D5-branes degenerating at a point. One may also say that $\alpha_{-k}$ is the creation operator of the bound state of $k$ D5-branes. Notice that, because $O_k$ can be regarded as the Poincaré dual of an appropriate cycle in each moduli space $\mathcal{M}^{k'} (k' \geq k)$, this identification of $\alpha_{-k}$ with the creation operator of the bound state also gives the geometrical interpretation of the bound states of D5-branes. Stability of these bound states will be ensured topologically by their identification with the cycles of the moduli spaces. Since the creations of D5-branes should not depend on their order these operators will commute one another: $[\alpha_{-k_1}, \alpha_{-k_2}] = 0$. Being given the creation operators it is also conceivable to consider the annihilation of these bound states. The annihilation operator $\alpha_k (= \alpha_{-k}^\dagger)$ of the bound state of $k$ D5-branes will be the bosonic operator of ghost number equal to $-2(k - 1)$ satisfying the commutation relations,

$$[\alpha_m, \alpha_n] = m\delta_{m+n, 0} \quad (m, n \in \mathbb{Z}\{0\}). \quad (3.21)$$

Being restricted to the $U(k)$ gauge theory the basis of the physical Hilbert space $\mathcal{H}^k$ will have the form,

$$\alpha_{-k_1} \cdots \alpha_{-k_l} |0\rangle \quad (\forall k_i \geq 1, \ k_1 + \cdots + k_l = k), \quad (3.22)$$

which is the bosonic state with ghost number equal to $2(k - l)$. Because of this restriction on the allowed combinations of the creation operators it becomes quite reasonable to introduce the total Hilbert space $\mathcal{H}$:

$$\mathcal{H} \equiv \bigoplus_{k=0}^{\infty} \mathcal{H}^k, \quad (3.23)$$

and it is very tempting to interpret this total Hilbert space $\mathcal{H}$ as the physical Hilbert space of the second-quantized D5-branes (or string-solitons). Let us investigate this possibility.

---

8The correspondences are given in [9, 10].
furthur. Since, by taking mass of a D5-brane as an unit, the bound state energy of $k$ D5-branes is $k$, we shall introduce the Hamiltonian operator $\hat{H}$ of the second-quantized D5-branes as follows:

$$\hat{H} = \sum_{k \geq 1} \alpha_{-k} \alpha_k.$$  \hfill (3.24)

We also denote the ghost number operator by $\hat{N}$. Note that $\hat{N}$ measures degrees of the Poincaré duals of the cycles which geometrically realize the bound states of D5-branes. One may regard $\text{Tr}_{\hat{H}} t^N q^\hat{H}$ as an analogue of the partition function of these second-quantized D5-branes (or string solitons). It has the form

$$\text{Tr}_{\hat{H}} t^N q^\hat{H} = \frac{1}{\prod_{l \geq 1} (1 - t^{2(l-1)} q^l)},$$

which was shown to be equal to $\sum_{k \geq 0} q^k P_t(M^k)$ \cite{9}. ($P_t(M^k)$ is the Poincaré polynomial of $M^k$.) This coincidence of the partition function with the generating function of the Poincaré polynomials also shows that $\{O_k\}_{k \geq 1}$ generate all the physical observables of our topological field theory.

\subsection*{3.4 Generalization}

In the preceding discussions our study was limited to the case when D5-branes are on $\mathbf{R}^4$. Let us generalize it to the case that they are on a curved background $X$. The configuration space of $k$ D5-branes is the $k$-th symmetric product of $X$, $S^k(X)$. We may need to introduce the moduli space $\mathcal{M}^k(X)$, that is, an analogue of $\mathcal{M}^k(\equiv \mathcal{M}^k(\mathbf{R}^4))$. The moduli space $\mathcal{M}^k(X)$ will give a resolution of $S^k(X)$ as $\mathcal{M}^k(\mathbf{R}^4)$ does give a resolution of $S^k(\mathbf{R}^4)$. In particular those cycles which appear by the resolution will lead to the geometrical realization of the bound states of D5-branes on $X$. Since the singularities of $S^k(X)$ are those configurations that some of D5-branes are overlapping at points of $X$, their resolution will be prescribed by the local properties of $X$ in the neighborhood of these points, and therefore will be quite similar to the case of $\mathbf{R}^4$ or $\mathbf{C}^2$. For the case that $X$ is an algebraic surface the moduli space $\mathcal{M}^k(X)$ is known \cite{9} to be given by $X^{[k]}$, the Hilbert scheme of $k$ points of $X$.
Due to this locality our framework of the second quantization of D5-branes will practically work even in this case. Let us consider the same theory as before. The effect of the curved background will be taken into account by modifying the subsystem of the center of D5-branes to the topological quantum mechanics on \( X \). Although the division of the system into two parts, that is, two subsystems of the center and the relative motions of D5-branes seems to be meaningless in the curved space \( X \), it will work at least for the construction of physical observables. This is because, as we have seen, \( O_k \) is essentially zero except for the configurations that D5-branes are overlapping at a point.

Since the topological quantum mechanics on \( X \) is equivalent to the cohomology theory of \( X \), by applying the same construction as that of \( O_k([\varepsilon]) \equiv O_k \), each element \([\omega]\) of \( H^*(X) \) gives us the physical observable \( O_k([\omega]) \). It is bosonic (fermionic) when \( d_\omega \), the degrees of \( \omega \), is even (odd). It has the ghost number \( 2(k-1)+d_\omega \). Let us also introduce the creation operator \( \alpha_{-k}([\omega]) \) corresponding to the physical observable \( O_k([\omega]) \) and then we will identify it with the creation operator of the bound state of \( k \) D5-branes topologically constrained on the cycle \( C \) of \( X \). ([\omega] is the Poincaré dual of \( C \) in \( X \).) These creation operators of the bound states constrained on the even (odd) dimensional cycles of \( X \) constitute (anti-)commuting bosonic (fermionic) operators. The total physical Hilbert space \( \mathcal{H} \) can be defined as the Fock space of these creation operators. Therefore \( \text{Tr}_{\mathcal{H}}t^\hat{N}q^\hat{H} \) has the form

\[
\text{Tr}_{\mathcal{H}}t^\hat{N}q^\hat{H} = \frac{\prod_{l \geq 1}(1 + t^{2l-1}q^l)b_1(X)(1 + t^{2l+1}q^l)b_3(X)}{\prod_{l \geq 1}(1 - t^{2l-1}q^l)b_0(X)(1 - t^{2l+1}q^l)b_2(X)(1 - t^{2l}q^l)b_4(X)},
\]

where \( b_i(X) \) is the \( i \)-th Betti number of \( X \). For the case that \( X \) is a projective surface, it coincides \cite{9} with \( \sum_{k \geq 0} q^k P_t(X^{[k]}) \), the generating function of the Poincaré polynomials of the Hilbert scheme parametrizing points in \( X \) \cite{15}. In particular, by setting \( t = -1 \), \( \text{Tr}_{\mathcal{H}}(-)^\hat{N}q^\hat{H} (3.26) \) reduces to the generating function of the Euler numbers and coincides with the partition function of \( N = 4 \) supersymmetric Yang-Mills theory proposed by Vafa-Witten \cite{16}.

4 Discussion
CFT of second quantized D5-branes

The second quantization of D5-branes described in this article may be considered as the quantization in terms of two-dimensional conformal field theory\(^9\). This unexpected appearance of 2D CFT may be the origin of the integrability of four-dimensional \(N = 4, 2\) supersymmetric Yang-Mills theories. The Seiberg-Witten solutions\(^{[17]}\), that is, the exact low energy effective actions of these supersymmetric theories are prescribed by introducing the moduli of the curves characteristic of these theories. It is also pointed out\(^{[18]}\) that these exact solutions can be regarded as the semi-classical solutions of two dimensional integrable systems such as the Toda lattice\(^{[19]}\). Notice that these integrable system can be also realized\(^{[20]}\) by using 2D CFT. Since these exact solutions of 4D supersymmetric Yang-Mills theories should be ultimately obtainable from string theory these two appearances of 2D CFT will not be accidental.

An attempt to understand the Seiberg-Witten exact solutions from the 2D CFT viewpoint has been made in\(^{[21]}\), where, by emphasizing the similarity with the non-perturbative formulation of two-dimensional gravity\(^{10}\) isomonodromic deformation problems are addressed so that their semi-classical analysis precisely gives us the Seiberg-Witten solutions. One may say that our framework of the second quantization of D5-branes is on the line suggested by the authors of\(^{[21]}\) because the quantization follows from the universality of physical observables which, as we mentioned in the text, is also characteristic in the nonperturbative formulation of two-dimensional gravity. In order to step further it might be necessary to give a more precise treatment of the second quantization of D5-branes (presumably without using any gauge theories). The construction of matrix models which realize the correspondences given in\(^{[9],[10]}\) should be investigated.

\(^9\)The creation and annihilation operators \(\alpha_n\) of the bound states of D5-branes, with an addition of the zero-mode, constitute a free boson \(\partial \phi(z) \equiv \sum_n \alpha_n z^{-n-1}\).

\(^{10}\)The string equation, \([P, Q] = 1\), of 2D gravity defines an isomonodromy problem\(^{[22]}\).
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Table 1: The field contents of the topological field theory. \( k \times k \) (\( k \times n \)) in the above means that these fields are \( k \times k \) hermitian (\( k \times n \) complex) matrix-valued.