The Deformation Complex for DG Hopf Algebras

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Abstract. Let $H$ be a DG Hopf algebra over a field $k$. This paper gives an explicit construction of a triple cochain complex that defines the Hochschild-Cartier cohomology of $H$. A certain truncation of this complex is the appropriate setting for deforming $H$ as an $H(q)$-structure. The direct limit of all such truncations is the appropriate setting for deforming $H$ as a strongly homotopy associative structure. Sign complications are systematically controlled. The connection between rational perturbation theory and the deformation theory of certain free commutative differential graded algebras is clarified.

1. Introduction

The purpose of this paper is two-fold: (1) to give an explicit construction of the deformation complex for differential graded Hopf algebras and (2) to relate the rational perturbation theory of Felix [9] and Halperin-Stasheff [11] to the deformation theory of certain free commutative differential graded algebras. The untruncated deformation complex constructed here directs the deformation of a differential graded Hopf algebra $H$ as an $H(\infty)$-structure; appropriate truncations direct the deformation of $H$ as an $H(q)$-structure. The special case $q = 3$ is applied by Lazarev and Movshev in their paper Deformations of the de Rham Algebra [17], which follows as a sequel.

In [10], Gerstenhaber and Schack showed how to deform a biassociative Hopf algebra $H$ over a field $k$ relative to its algebraic cohomology. Following their cues, we define the algebraic cohomology of a connected biassociative differential graded Hopf algebra $H$ and give a brief exposition of the related deformation theory.

This exposition minimizes the sign complications that arise in a graded theory by adopting two strategies: (1) we work at the (coordinate free) operator level and (2) we base our constructions on $H$-free resolutions with differentials of internal degree zero. Thus, elements of $H$ never "move past" graded cochains and graded cochains are free to "move past" the resolution differentials without complicating signs. While the first strategy is evident in Gerstenhaber and Schack’s exposition [10], the second was used by Burghelea and Poirrier [3] to define the Hochschild and cyclic cohomologies of free commutative associative differential graded algebras.

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in characteristic zero. The recent work of Penkava and Schwarz demonstrates that careful attention to signs can be critical.

This paper is organized as follows: Section 2 establishes the necessary preliminaries and section 3 reviews the "classical" (co)bar resolution of a graded (co)algebra and its extension to a differential graded (co)algebra. These resolutions, which are not meant to model chains on some contractible space, have differentials of internal degree zero and avoid the dimension shifts of Adams and Eilenberg and Mac Lane. Section 4 dualizes and generalizes the notion of a differential graded bimodule over a differential graded algebra, which is implicit in, to analogous structures over differential graded coalgebras and Hopf algebras.

In section 5 we define the Hochschild cohomology of a connected associative differential graded algebra (d.g.a.) \( M \) with coefficients in a differential graded \( A \)-bimodule \( M \). The deformation complex for \( A \) is obtained by setting \( M = A \) and appropriately truncating the Hochschild cochain complex. A construction of this cohomology was given by Markl in but with two significant differences: (1) our underlying bar resolution does not use the Eilenberg-Mac Lane dimension shift and (2) we transfer the theory from the level of \( A \)-bimodules to the level of \( k \)-modules at which the cohomology and deformation theory are clearly linked.

We also define the Harrison cohomology of a commutative d.g.a. (c.d.g.a) with coefficients in a symmetric d.g. A-bimodule \( M \). The Harrison cohomology of free c.d.g.a.'s with trivial coefficients was defined earlier by Burghelea and Poirrier. We show how to interpret the rational perturbation theory of "big graded models" in terms of the "appropriately truncated" Harrison cohomology of the model with coefficients in itself. We observe that for free c.d.g.a.'s, all flexibility lies in the direction of the differential. The Lie algebra analogs of these constructions recently appeared in.

Next we dualize and obtain the Cartier cohomology of a connected coassociative differential graded coalgebra (d.g.c.) \( C \) with coefficients in a differential graded \( C \)-bicomodule \( N \); the deformation complex for \( C \) is obtained by setting \( N = C \) and appropriately truncating the Cartier cochain complex. Finally, we join these dual theories and obtain the Hochschild-Cartier cohomology of a connected coassociative differential graded Hopf algebra (d.g.h.a.) \( H \); the deformation complex for \( H \) is an appropriate truncation of the triple cochain complex for this cohomology. Section 6 concludes the discussion with a brief exposition of the deformation theory for d.g.h.a.'s.

2. Notation and Preliminaries

Let \( R \) be a commutative ring with identity \( 1_R \) and let \( M \) be a (non-negatively) graded \( R \)-module. \( M \) is connected if \( M^0 = R \). Unless indicated otherwise, all tensor products will be defined over \( R \). Let \( \{M_i\} \) be a sequence of graded \( R \)-modules; the subspace of \( \bigotimes_i M_i \) consisting of all elements homogeneous in degree \( p \) is denoted by \( (\bigotimes_i M_i)^p \). Let \( M^{*n} = M \otimes \cdots \otimes M \) with \( n > 0 \) factors and define \( M^{*0} = R \). Let \( T_{p,n} M = (M^{*n})^p \), then \( TM = \sum_{p,n \geq 0} T_{p,n} M \) is a bigraded space; a bihomogeneous element \( x \in TM \) has bidegree \((p,n)\) and is said to have internal degree \( p \) and external degree \( n \). The symbol \(|x|\) denotes the internal degree of \( x \).

A map \( f : M \to M \) has degree \( p \) if \(|f(x)| = |x| + p \) for each homogeneous \( x \in M \), in which case we write \(|f| = p \). The identity map \( 1 : M \to M \) and the canonical isomorphisms \( i_1 : M \to R \otimes M, j_1 : R \otimes M \to M, i_2 : M \to M \otimes R, \) and
$j_2 : M \otimes R \to M$ are maps of degree zero. Another such map is the permutation operator $\sigma : M^{\otimes n} \to M^{\otimes n}$, defined by $\sigma(x_1 \otimes \cdots \otimes x_n) = \pm x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)}$, where $\sigma \in S_n$ and the sign is given by the standard sign commutation rule with respect to internal degree: whenever two symbols $u$ and $v$ with internal degrees are interchanged, affix the sign $(-1)^{|u||v|}$ (see [23], p. 164). If $x, y \in M$ and $f, g : M \to M$, the sign commutation rule gives: $(f \otimes g)(x \otimes y) = (-1)^{|g||x|}f(x) \otimes g(y)$. An $R$-module map $d : M \to M$ of degree $\pm 1$ such that $d \circ d = 0$ is called a differential on $M$; the pair $(M, d)$ is a differential graded (d.g.) $R$-module.

Let $A$ be a graded symmetric $R$-module. A multiplication on $A$ is an $R$-module map $\mu : A \otimes A \to A$ of degree 0; the pair $(A, \mu)$ is a graded $R$-algebra. An $R$-algebra $(A, \mu)$ is associative if $\mu \circ (\mu \otimes 1) = \mu \circ (1 \otimes \mu)$; it is commutative if $\mu = \mu \circ (1, 2)$. It is unital if there exists an $R$-algebra map $\eta : R \to A$ of degree 0 such that $\mu \circ (\eta \otimes 1) \circ i_1 = \mu \circ (1 \otimes \eta) \circ i_2 = 1$, in which case $\eta$ is called the unit. The element $1_A = \eta(1_R)$ acts as a two-sided identity for $\mu$. When $A$ is connected, the unique algebra isomorphism $R \to A^0$ is a canonical unit. A derivation of $(A, \mu)$ is an $R$-module map $\theta : A \to A$ such that $\theta \circ \mu = \mu \circ (\theta \otimes 1 + 1 \otimes \theta)$. If a differential $d$ on $A$ is a derivation of $(A, \mu)$, then $d$ is an algebra differential and the triple $(A, d, \mu)$ is a differential graded $R$-algebra (d.g.a.).

Let $n \in \mathbb{N} \cup \{\infty\}$. An $(n)$-algebra is defined to be a graded $R$-module $A$ together with maps $\{\mu^{(i)} \in \text{Hom}_{R}^{2-\ell}(A^{\otimes \ell}, A)\}_{1 \leq \ell \leq n}$ such that for each $\ell \leq n$,

$$\sum_{0 \leq i < \ell; \ j+k=\ell+1} (-1)^{i+k+k+\ell}(\mu^{(j)} \circ (1^{\otimes i} \otimes \mu^{(k)} \otimes 1^{\otimes (j-i-1)}) = 0.$$  

The signs here agree with those in [23]; we use upper indices and reserve the lower for indexing coefficients in a deformation. An $(n)$-algebra is strict if $\mu^{(n)} = 0$. Every d.g.a. $(A, d, \mu)$ is a strict $(n)$-algebra for all $n \geq 3$ via $\mu^{(1)} = d$, $\mu^{(2)} = \mu$, and $\mu^{(i)} = 0$ for $3 \leq i \leq n$.

Let $C$ be a graded symmetric $R$-module. A comultiplication on $C$ is an $R$-module map $\Delta : C \to C \otimes C$ of degree 0; the pair $(C, \Delta)$ is a graded $R$-coalgebra. An $R$-coalgebra $(C, \Delta)$ is coassociative if $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$; it is counital if $\Delta = (1, 2) \circ \Delta$. It is counital if there exists an $R$-coalgebra map $\varepsilon : C \to R$ of degree 0 such that $j_1 \circ (\varepsilon \otimes 1) \circ \Delta = j_2 \circ (1 \otimes \varepsilon) \circ \Delta = 1$, in which case $\varepsilon$ is called the counit. When $A$ is connected, the unique coalgebra isomorphism $C_0 \to R$ extended to the zero map in positive degrees is a canonical counit. A coderivation of $(C, \Delta)$ is an $R$-module map $\omega : C \to C$ such that $\Delta \circ \omega = (\omega \otimes 1 + 1 \otimes \omega) \circ \Delta$. If a differential $d$ on $C$ is a coderivation of $(C, \Delta)$, then $d$ is a coalgebra differential and the triple $(C, d, \Delta)$ is a differential graded $R$-coalgebra (d.g.c.).

Let $m \in \mathbb{N} \cup \{\infty\}$. An $(m)$-coalgebra is defined to be a graded $R$-module $C$ together with maps $\{\Delta^{(\ell)} \in \text{Hom}_{R}^{2-\ell}(C, C^{\otimes \ell})\}_{1 \leq \ell \leq m}$ such that for each $\ell \leq m$,

$$\sum_{0 \leq i < \ell; \ j+k=\ell+1} (-1)^{i+k+k+\ell}(1^{\otimes i} \otimes \Delta^{(k)} \otimes 1^{\otimes (j-i-1)}) \circ \Delta^{(j)} = 0.$$  

An $(m)$-coalgebra is strict if $\Delta^{(m)} = 0$. Every d.g.c. $(C, d, \Delta)$ is a strict $(m)$-coalgebra for all $m \geq 3$.

Let $H$ be a graded symmetric $R$-module, and suppose that $H$ is equipped with a multiplication $\mu$, a unit $\eta$, a comultiplication $\Delta$, and a counit $\varepsilon$ such that $\eta$ and $\varepsilon$ are $R$-bialgebra maps and $\Delta \circ \mu = (\mu \otimes \mu) \circ (2, 3) \circ (\Delta \otimes \Delta)$; then $(H, \mu, \eta, \Delta, \varepsilon)$ is a graded $R$-bialgebra. This latter condition is equivalent to requiring that $\mu$
and $\Delta$ be, respectively, coalgebra and algebra maps. An antipode for a graded $R$-bialgebra $H$ is an $R$-antialgebra map $S : H \to H$ of degree 0 such that $\mu \circ (S \otimes 1) \circ \Delta = \mu \circ (1 \otimes S) \circ \Delta = \eta \circ \varepsilon$. A graded $R$-bialgebra $H$ is biaassociative if it is both associative and coassociative. When $H$ is connected and biassociative, there is a unique inductively defined antipode $S$ that acts as the identity in degree 0 and by $S(x) = -x - \sum x(1) S(x(2))$ in positive degrees, where $\Delta(x) = \sum x(1) \otimes x(2)$; see [20]. A graded $R$-bialgebra $(H, \mu, \eta, \Delta, \varepsilon)$ equipped with antipode $S$ is a graded $R$-Hopf algebra $(g.h.a.)$. Furthermore, if $(H, d, \mu)$ is a d.g.a., $(H, d, \Delta)$ is a d.g.c. and $(H, \mu, \eta, \Delta, \varepsilon, S)$ is a g.h.a., then $(H, d, \mu, \eta, \Delta, \varepsilon, S)$ is a differential graded $R$-Hopf algebra $(d,g.h.a.)$.

Henceforth, all objects are assumed to be graded; all $R$-modules are assumed to be connected; all $R$-algebras, $R$-coalgebras, and $R$-Hopf algebras are assumed to be associative, coassociative, and biassociative, respectively. An $R$-Hopf algebra will be unambiguously denoted by $(H, \mu, \Delta)$.

3. Two-sided Bar and Cobar Resolutions

3.1. The Bar Resolution. Let $k$ be a field and let $(A, \mu)$ be a $k$-algebra. For each $m \geq 0$, inductively define $k$-linear maps $\partial_{(m)} : A^\otimes(m+2) \to A^\otimes(m+1)$ by

$$\partial_{(0)} = \mu,$$

and

$$\partial_{(m)} = \mu \otimes 1^\otimes m - 1 \otimes \partial_{(m-1)}.$$

In more familiar form this is

$$\partial_{(m)} = \sum_{i=0}^{m} (-1)^i (1^\otimes i \otimes \mu \otimes 1^\otimes (m-i)),$$

but many of the facts we need flow more easily from the inductive form. Let $\partial = \sum_{m \geq 0} \partial_{(m)}$; using induction and the fact that $\mu$ is associative, it is a simple matter to show that $\partial \circ \partial = 0$. Hence $\partial$ is a differential with respect to external degree. The chain complex

$$A \xleftarrow{\partial_{(0)}} A \otimes A \xleftarrow{\partial_{(1)}} A \otimes A \otimes A \xleftarrow{\partial_{(2)}} A^\otimes 4 \xleftarrow{\partial_{(3)}} \cdots$$

is called the (classical) two-sided bar resolution of $A$ [18]. Furthermore, this resolution is acyclic via contracting homotopy $s = \sum_{m \geq -1} s_m$ where $s_m = [\eta \otimes 1] \circ i_{1+m} \otimes 1^\otimes (m+1)$. Note that maps $\partial$ and $s$ have degree zero with respect to the internal grading.

The bar resolution extends to a d.g.a. $(A, d, \mu)$ as follows. For each $m \geq -1$, inductively define $k$-linear maps $d_{(m)} : A^\otimes(m+2) \to A^\otimes(m+2)$ by

$$d_{(-1)} = d$$

and

$$d_{(m)} = d \otimes 1^\otimes (m+1) + 1 \otimes d_{(m-1)},$$

which in more familiar form is

$$d_{(m)} = \sum_{i=0}^{m+1} 1^\otimes i \otimes d \otimes 1^\otimes (m-i+1).$$

It is easy to check that $d_{(s)}$ is a differential with respect to internal degree and

$$\partial \circ d_{(s)} - d_{(s-1)} \circ \partial = 0.$$
We refer to the double complex \( \{ TA, d, \partial \} \) as the *two-sided bar resolution of \( A \); the differentials \( d \) and \( \partial \) have respective bidegree \((1,0)\) and \((0,-1)\) (see Figure 1). This resolution is acyclic with respect to \( \partial \) via the contracting homotopy \( s \) given above.

An isomorphic (and more familiar) construction appears in \cite{4} but with the Eilenberg-Mac Lane shift in dimension. This dimension shift introduces a set of signs that give rise to a standard double complex whose subdiagrams anticommute; in this case \( D = d + \partial \) is a differential. But short of that, we are better off without the dimension shift since the subsequent signs unnecessarily complicate the exposition and formulas. When total differentials are required, it is a simple matter to introduce artificial signs; this is the strategy we adopt.

\[
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\partial & \downarrow & & & & \\
k^\otimes 4 & \rightarrow & (A^\otimes 4)^1 & \rightarrow & (A^\otimes 4)^2 & \rightarrow & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & & \\
k^\otimes 3 & \rightarrow & (A^\otimes 3)^1 & \rightarrow & (A^\otimes 3)^2 & \rightarrow & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & & \\
k^\otimes 2 & \rightarrow & (A^\otimes 2)^1 & \rightarrow & (A^\otimes 2)^2 & \rightarrow & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & & \\
k & \rightarrow & A^1 & \rightarrow & A^2 & \rightarrow & \cdots \\
\end{array}
\]

The Bar Resolution

Figure 1.

3.2. The Cobar Resolution. Now consider a d.g.c. \((C, d, \Delta)\). The 2-sided cobar resolution of \( C \) is an \( C \)-free resolution dual to the 2-sided bar resolution.

As in \cite{2}, consider the differentials with respect to internal degree \( d_{(n)} : C^\otimes (n+2) \rightarrow C^\otimes (n+2) \) given by

\[
d_{(n)} = \sum_{i=0}^{n+1} 1^\otimes i \otimes d \otimes 1^\otimes (n+1-i),
\]

Inductively define maps \( \delta_{(n)} : C^\otimes (n+2) \rightarrow C^\otimes (n+3) \) by

\[
\delta_{(-1)} = \Delta
\]

and

\[
\delta_{(n)} = \Delta \otimes 1^\otimes (n+1) - 1 \otimes \delta_{(n-1)},
\]

which in more familiar form is

\[
\delta_{(n)} = \sum_{i=0}^{n+1} (-1)^i (1^\otimes i \otimes \Delta \otimes 1^\otimes (n+1-i)).
\]

Let \( \delta = \sum_{n\geq -1} \delta_{(n)} \); using induction and the fact that \( \Delta \) is coassociative, it is easy to check that \( \delta \) is a differential with respect to external degree and

\[
\delta \circ d_{(*)} - d_{(*)+1} \circ \delta = 0.
\]
We refer to the double complex \( TC, d_M, \delta \) as the two-sided cobar resolution of \( C \). This resolution is acyclic with respect to \( \delta \) via contracting homotopy \( \tau = \sum_{n \geq 0} \tau_n \) where \( \tau_n = [j_1 \circ (\varepsilon \otimes 1)] \otimes 1^{\otimes n} \).

4. Two-sided Differential Graded \( k \)-Modules

4.1. Differential Graded \( A \)-Bimodules. Let \( k \) be a field.

**Definition 1.** Let \((A, \mu)\) be a \( k \)-algebra and let \( M \) be a \( k \)-module for which there exist \( k \)-linear structure maps \( \lambda : A \otimes M \to M \) and \( \rho : M \otimes A \to M \) of degree zero such that

1. \( \lambda \circ (\mu \otimes 1) = \lambda \circ (1 \otimes \lambda) \),
2. \( \rho \circ (1 \otimes \mu) = \rho \circ (\rho \otimes 1) \), and
3. \( \lambda \circ (\eta \otimes 1) \circ i_1 = \rho \circ (1 \otimes \eta) \circ i_2 = 1 \).

Then \((M, \lambda, \rho)\) is a \( A \)-bimodule; it is symmetric if \( \rho = \lambda \circ (1, 2) \). If \((M, \lambda, \rho)\) and \((M', \lambda', \rho')\) are \( A \)-bimodules, a map \( f \in \text{Hom}_k^* (M, M') \) is a map of \( A \)-bimodules if \( f \circ \lambda = \lambda' \circ (1 \otimes f) \) and \( f \circ \rho = \rho' \circ (f \otimes 1) \). The category of \( A \)-bimodules and \( A \)-bimodule maps is denoted by \( A \text{-bimod} \).

**Example 1.** Let \( V \) be any \( k \)-module, let \( M = A \otimes V \otimes A \), and consider structure maps \( \lambda^x = \mu \otimes 1 \otimes 1 \) and \( \rho^x = 1 \otimes 1 \otimes \mu \). Then \((A \otimes V \otimes A, \lambda^x, \rho^x)\) is an exterior \( A \)-bimodule; \( \lambda^x \) and \( \rho^x \) are called exterior bimodule structure maps. This is not to be confused with the notion of an exterior algebra.

Let \( V \) be a \( k \)-module, let \((M', \lambda', \rho')\) be an \( A \)-bimodule, and consider the exterior \( A \)-bimodule \((A \otimes V \otimes A, \lambda^x, \rho^x)\). There is a \( k \)-linear isomorphism

\[
\Phi_V : \text{Hom}_k^* \text{-bimod}(A \otimes V \otimes A, M') \cong \text{Hom}_k^*(V, M')
\]

given by \( \Phi_V(f) = f \circ (\eta \otimes 1 \otimes \eta) \circ (i_1 \otimes 1) \circ i_2 \).

Let \( W \) be a \( k \)-module and consider the exterior \( A \)-bimodules \( A \otimes V \otimes A \) and \( A \otimes W \otimes A \). An \( A \)-bimodule map \( \Theta : A \otimes V \otimes A \to A \otimes W \otimes A \) of degree \( p \) induces a \( k \)-linear map

\[
\Theta^* : \text{Hom}_k^*(W, M') \to \text{Hom}_k^{*+p}(V, M')
\]

given by \( \Theta^* = \Phi_V \circ \text{Hom}_A((\Theta, M') \circ \Phi_W^{-1}) \), where \( \Phi_W^{-1}(g) = \lambda' \circ (1 \otimes \rho') \circ (1 \otimes g \otimes 1) \). A critical point here is the fact that \( \lambda' \circ (1 \otimes \rho') \circ (1 \otimes g \otimes 1) : A \otimes V \otimes A \to M' \) is an \( A \)-bimodule map; the reader may wish to supply the proof.

**Definition 2.** Let \((A, d, \mu)\) be a d.g.a. and let \((M, \lambda, \rho)\) be an \( A \)-bimodule equipped with a differential \( d_M \). Then \((M, d_M)\) is a differential graded (d.g.) \( A \)-bimodule provided that

1. \( d_M \circ \lambda = \lambda \circ (d \otimes 1 + 1 \otimes d_M) \) and
2. \( d_M \circ \rho = \rho \circ (d_M \otimes 1 + 1 \otimes d) \).

If \((M, d_M)\) and \((M', d_M')\) are d.g. \( A \)-bimodules, a map \( f \in \text{Hom}_A^* \text{-bimod}(M, M') \) is a map of d.g. \( A \)-bimodules if \((-1)^{|f|} f \circ d_M - d_M' \circ f = 0 \).

**Example 2.** Every d.g.a. \((A, d, \mu)\) is a d.g. \( A \)-bimodule with respect to structure maps \( \lambda = \mu = \rho \).
EXAMPLE 3. Let \((A, d, \mu)\) be a d.g.a., let \(m \geq 0\), and identify \(A \otimes k \otimes A\) with \(A \otimes A\). Consider the differentials \(d_{(m)} : A^{\otimes (m+2)} \to A^{\otimes (m+2)}\) and \(\partial_{(m)} : A^{\otimes (m+2)} \to A^{\otimes (m+1)}\) defined in (3.2) and (3.3), respectively. Then \((A^{\otimes (m+2)}, d_{(m)})\) is an exterior d.g. \(A\)-bimodule, and by (3.3), \(\partial_{(m)}\) is a map of exterior d.g. \(A\)-bimodules.

Given d.g. \(A\)-bimodules \((M, \lambda, \rho, d_M)\) and \((M', \lambda', \rho', d_{M'})\), define a map \(\overline{d} : \text{Hom}_{A-bimod}^s(M, M') \to \text{Hom}_{A-bimod}^{s+1}(M, M')\) by

\[
(4.3) \quad \overline{d}(f) = (-1)^{|f|} f \circ d_M - d_{M'} \circ f.
\]

The following fact will be useful in the construction that follows:

**PROPOSITION 1.** \(\overline{d}(f)\) is a map of d.g. \(A\)-bimodules.

**Proof:** We check the compatibility of \(\overline{d}(f)\) with the structure map \(\rho\); the compatibility with \(\lambda\) is similar.

\(\overline{d}(f) \circ \rho = (-1)^{|f|} f \circ d_M \circ \rho - d_{M'} \circ \rho \circ f \circ \rho = (-1)^{|f|} f \circ \rho \circ (d_M \otimes 1 + 1 \otimes d) - d_M' \circ \rho' \circ (f \otimes 1) \circ (1 \otimes d) - d_M' \circ \rho' \circ (f \otimes 1) = (-1)^{|f|} \rho' \circ (f \otimes 1) \circ (d_M' \otimes 1 + 1 \otimes d) - \rho' \circ (d_M' \otimes 1 + 1 \otimes d) \circ (f \otimes 1) = (-1)^{|f|} \rho' \circ (f \circ d_M \otimes 1) - \rho' \circ (d_M' \circ f \otimes 1) - \rho' \circ (\overline{d}(f) \otimes 1)\).

It is trivial to check that \(\overline{d}(\overline{d}(f)) = 0\); hence \(\overline{d}(f)\) respects differentials.

Let \((H, \mu, \Delta)\) be a \(k\)-Hopf algebra.

**DEFINITION 3.** Let \((M, \lambda, \rho)\) and \((M', \lambda', \rho')\) be \(H\)-bimodules. The internal (bimodule) tensor product of \((M, \lambda, \rho)\) with \((M', \lambda', \rho')\) is the so called interior \(H\)-bimodule \(M \boxtimes M' = (M \otimes M', \lambda \boxtimes \lambda', \rho \boxtimes \rho')\) with

\[
\lambda \boxtimes \lambda' = (\lambda \otimes \lambda') \circ (2, 3) \circ (\Delta \otimes 1 \otimes 1)
\]

and

\[
\rho \boxtimes \rho' = (\rho \otimes \rho') \circ (2, 3) \circ (1 \otimes 1 \otimes \Delta).
\]

Since \(\Delta\) is coassociative, \((\lambda \boxtimes \lambda') \boxtimes \lambda' = \lambda \boxtimes (\lambda \boxtimes \lambda')\) and \((\rho \boxtimes \rho') \boxtimes \rho'' = \rho \boxtimes (\rho' \boxtimes \rho'')\). Thus, the internal tensor product can be associatively applied to any finite family of \(H\)-bimodules.

**DEFINITION 4.** Let \((M, \lambda, \rho)\) be an \(H\)-bimodule. The structure maps \(\overline{\lambda} = \mu \boxtimes \lambda \mu\) and \(\overline{\rho} = \mu \rho\mu\) on the interior \(H\)-bimodule \(H \boxtimes M \boxtimes H\) are called the two-sided interior extensions of \(\lambda\) and \(\rho\) by \(\mu\), respectively.

**DEFINITION 5.** Let \((M, \lambda, \rho)\) be an \(H\)-bimodule. The interior \(H\)-bimodule \(M^{\otimes n} = (M^{\otimes n}, \lambda^n, \rho^n)\), with \(\lambda^n = \lambda \boxtimes \lambda \cdots \lambda = (\lambda \otimes \lambda \cdots \lambda) \circ (1 \otimes \Delta \otimes 1 \otimes \cdots \otimes 1)\) and \(\rho^n = \rho \circ \cdots \circ (n, n+1) \circ (1 \otimes \cdots \otimes \Delta)\), is called the \(n\)-fold interior (bimodule) tensor power of \(M\).

**EXAMPLE 4.** For each \(n \geq 1\), the \(n\)-fold interior (bimodule) tensor power of \(H\) is the interior \(H\)-bimodule \(H^{\otimes n} = (H^{\otimes n}, \lambda^n, \rho^n)\) with

\[
\lambda^n = \mu^{\otimes n} \circ (1 \ 3 \ 5 \ \cdots \ (2n - 1) \ 2 \ 4 \ 6 \ \cdots \ 2n) \circ \prod_{i=n}^{2n-2} (\Delta \otimes 1 \otimes (3n-i-2))
\]

and

\[
\rho^n = \mu^{\otimes n} \circ (1 \ 3 \ 5 \ \cdots \ (2n - 1) \ 2 \ 4 \ 6 \ \cdots \ 2n) \circ \prod_{i=n}^{2n-2} (1 \otimes (3n-i-2) \otimes \Delta).
\]
Example 5. Let \((H, d, \mu, \Delta)\) be a d.g.h.a., let \(n \geq 0\), and identify the interior \(H\)-bimodules \(H \otimes_k H\) and \(H \otimes H\). Then \((H^{(n+2)}, d_{(n)})\) is an interior d.g. \(H\)-bimodule and the differential \(\delta_{(n)} : H^{(n+2)} \to H^{(n+3)}\) defined in (3.4) is a d.g. \(H\)-bimodule map.

4.2. Differential Graded \(C\)-Bicomodules. Let \(k\) be a field.

Definition 6. Let \((C, \Delta)\) be a \(k\)-coalgebra and let \(N\) be a \(k\)-module for which there exist \(k\)-linear structure maps \(\lambda : N \to C \otimes N\) and \(\rho : N \to N \otimes C\) of degree zero such that

- \((\Delta \otimes 1) \circ \lambda = (1 \otimes \lambda) \circ \lambda\),
- \((1 \otimes \Delta) \circ \rho = (\rho \otimes 1) \circ \rho\), and
- \(j_1 \circ (\varepsilon \otimes 1) \circ \lambda = j_2 \circ (1 \otimes \varepsilon) \circ \rho = 1\).

Then the triple \((N, \lambda, \rho)\) is an \(C\)-bicomodule. If \((N, \lambda, \rho)\) and \((N', \lambda', \rho')\) are \(C\)-bicomodules, a map \(g \in \text{Hom}_k^* (N, N')\) is a map of \(C\)-bicomodules if \(\lambda' \circ g = (1 \otimes g) \circ \lambda\) and \(\rho' \circ g = (g \otimes 1) \circ \rho\). The category of \(C\)-bicomodules and \(C\)-bicomodule maps is denoted by \(\text{C-bicomod}\).

Example 6. Let \(V\) be any \(k\)-module, let \((N, \lambda, \rho)\) be any \(C\)-bicomodule, and consider structure maps \(\lambda : N \to C \otimes N\) and \(\rho : N \to N \otimes C\). Then \((C \otimes V \otimes C, \lambda_\Delta, \rho_\Delta)\) is an exterior \(C\)-bicomodule; \(\lambda_\Delta\) and \(\rho_\Delta\) are called exterior bicomodule structure maps.

Let \(V\) be any \(k\)-module, let \((N, \lambda, \rho)\) be any \(C\)-bicomodule, and consider the exterior \(C\)-bicomodule \((C \otimes V \otimes C, \lambda_\Delta, \rho_\Delta)\). There is a \(k\)-linear isomorphism

\[
\Psi_V : \text{Hom}_{C \otimes C \text{-bicomod}}^*(N, C \otimes V \otimes C) \approx \text{Hom}_k^*(N, V)
\]

given by \(\Psi_V(g) = j_1 \circ (1 \otimes j_2) \circ (\varepsilon \otimes 1 \otimes \varepsilon) \circ g\).

Let \(W\) be a \(k\)-module and consider the exterior \(C\)-bicomodules \(C \otimes V \otimes C\) and \(C \otimes W \otimes C\). An \(C\)-bicomodule map \(\Xi : C \otimes V \otimes C \to C \otimes W \otimes C\) of degree \(q\) induces a \(k\)-linear map

\[
\Xi_* : \text{Hom}_k^*(N, V) \to \text{Hom}_k^{*+q}(N, W)
\]

given by \(\Xi_* = \Psi_W \circ \text{Hom}_{C \otimes C \text{-bicomod}}(N, \Xi) \circ \Psi_V^{-1}\), where \(\Psi_V^{-1}(g) = (1 \otimes g \otimes 1) \circ (1 \otimes \rho) \circ \lambda\). Again, it is important to note that \((1 \otimes g \otimes 1) \circ (1 \otimes \rho) \circ \lambda : N \to C \otimes V \otimes C\) is an \(C\)-bicomodule map.

Definition 7. Let \((C, d, \Delta)\) be a d.g.c. and let \((N, \lambda, \rho)\) be an \(C\)-bicomodule equipped with a differential \(d_N\). Then \((N, d_N)\) is a d.g. \(C\)-bicomodule provided that

- \(\lambda \circ d_N = (d \otimes 1 + 1 \otimes d_N) \circ \lambda\), and
- \(\rho \circ d_N = (d_N \otimes 1 + 1 \otimes d) \circ \rho\).

If \((N, d_N)\) and \((N', d_{N'})\) are d.g. \(C\)-bicomodules, a map \(g \in \text{Hom}_{C \otimes C \text{-bicomod}}(N, N')\) is a map of d.g. \(C\)-bicomodules if \((-1)^{|g|} g \circ d_N - d_{N'} \circ g = 0\).

Example 7. Every d.g.c. \((C, d, \Delta)\) is a d.g. \(C\)-bicomodule with respect to structure maps \(\lambda = \rho = \Delta\).

Example 8. Let \((C, d, \Delta)\) be a d.g.c., let \(n \geq 0\), and identify \(C \otimes k \otimes C\) with \(C \otimes C\). Consider the differentials \(d_{(n)} : C^{\otimes (n+2)} \to C^{\otimes (n+2)}\) and \(\delta_{(n)} : C^{\otimes (n+2)} \to C^{\otimes (n+3)}\) defined in (3.4) and (3.5), respectively. Then \((C^{\otimes (n+2)}, d_{(n)})\) is an exterior d.g. \(C\)-bicomodule, and by (3.4), \(d_{(n)}\) is a map of exterior d.g. \(C\)-bimodules.
Given d.g. C-bicomodules \((N, \lambda, \rho, d_N)\) and \((N', \lambda', \rho', d_{N'})\), define a map \(\overline{d} : \text{Hom}_{C^\otimes bicomod}(N, N') \to \text{Hom}_{k^\otimes}^{+1}(N, N')\) as in (4.3). The reader can check that:

**Proposition 2.** \(\overline{d}(g)\) is a map of d.g. C-bicomodules.

Let \((H, \mu, \Delta)\) be a \(k\)-Hopf algebra.

**Definition 8.** Let \((N, \lambda, \rho)\) and \((N', \lambda', \rho')\) be \(H\)-bicomodules. The internal (bicomodule) tensor product of \((N, \lambda, \rho)\) with \((N', \lambda', \rho')\) is the so-called \(H\)-bicomodule \(N \otimes N' = (N \otimes N', \lambda \lambda', \rho \rho')\) with

\[
\lambda \lambda' = (\mu \otimes 1 \otimes 1) \circ (2, 3) \circ (\lambda \otimes \lambda')
\]

and

\[
\rho \rho' = (1 \otimes 1 \otimes \mu) \circ (2, 3) \circ (\rho \otimes \rho')
\]

Since \(\mu\) is associative, \((\lambda \lambda') \lambda'\lambda'' = \lambda (\lambda' \lambda'\lambda'')\) and \((\rho \rho') \rho'\rho'' = \rho (\rho' \rho'\rho'')\). Thus, the internal tensor product can be associatively applied to any finite family of \(H\)-bicomodules.

**Definition 9.** Let \((N, \lambda, \rho)\) be an \(H\)-bicomodule. The structure maps \(\overline{\lambda}_\Delta = \Delta \otimes \lambda \otimes \Delta\) and \(\overline{\rho}_\Delta = \Delta \otimes \rho \otimes \Delta\) on the interior \(H\)-bicomodule \(H \otimes N \otimes H\) are called the two-sided interior extensions of \(\lambda\) and \(\rho\) by \(\Delta\), respectively.

**Definition 10.** Let \((N, \lambda, \rho)\) be an \(H\)-bicomodule. The interior \(H\)-bicomodule \(N \otimes^m = (N \otimes^m, \lambda_m, \rho_m)\), with \(\lambda_m = \lambda \otimes \lambda_{m-1} = (\mu \otimes 1 \otimes 1) \circ (2, 3) \circ (\lambda \otimes \lambda_{m-1})\) and 
\[
\rho_m = \rho_m - 1 \otimes \rho = (1 \otimes m) \circ (m, m+1) \circ (\rho_m - 1 \otimes \rho),
\]

is called the \(m\)-fold internal (bicomodule) tensor power of \(N\).

**Example 9.** Consider the \(H\)-bicomodule \((H, \lambda, \rho)\) where \(\lambda = \rho = \Delta\). For each \(m \geq 1\), the \(m\)-fold internal (bicomodule) tensor power of \(H\) is the interior \(H\)-bicomodule \(H \otimes^m = (H \otimes^m, \lambda_m, \rho_m)\) with

\[
\lambda_m = \prod_{i=m}^{2m-2} (\mu \otimes 1 \otimes 1) \circ (1 3 5 \cdots (2m-1) 2 4 6 \cdots 2m)^{-1} \circ \Delta \otimes^m
\]

and

\[
\rho_m = \prod_{i=m}^{2m-2} (1 \otimes i \otimes 1) \circ (1 3 5 \cdots (2m-1) 2 4 6 \cdots 2m)^{-1} \circ \Delta \otimes^m.
\]

**Example 10.** Let \((H, d, \mu, \Delta)\) be a d.g.h.a., let \(m \geq 0\), and identify the interior \(H\)-bicomodules \(H \otimes^k H\) and \(H \otimes H\). Then \((H \otimes (m+2), d_{(m)})\) is an interior d.g. \(H\)-bicomodule. Furthermore, \(\partial_{(m)} : H \otimes (m+2) \to H \otimes (m+1)\) defined in (3.1) is a map of interior d.g. \(H\)-bicomodules.

### 4.3. Differential Graded \(H\)-bidimodules

Let \(k\) be a field and let \((H, \mu, \Delta)\) be a \(k\)-Hopf algebra.

**Definition 11.** Let \(E\) be a \(k\)-module such that \((E, \lambda, \rho)\) is an \(H\)-bimodule and \((E, \lambda, \rho)\) is an \(H\)-bicomodule. Then \((E, \lambda, \rho)\) is an \(H\)-bidimodule if its \(H\)-bimodule and \(H\)-bicomodule structures are compatible in the following sense:

1. \(\lambda^\# \in \text{Hom}_{H^\otimes bicomod}(H \otimes E, E)\),
2. \(\rho^\# \in \text{Hom}_{H^\otimes bicomod}(E \otimes H, E)\),
3. \(\lambda^\# \in \text{Hom}_{H^\otimes bimod}(E, H \otimes E)\), and
(4) $\rho_\# \in Hom_{H\text{-}bimod}(E, E \text{cl} H)$.

A map of $H$-bidimodules preserves bimodule and bicomodule structure. Denote the category of $H$-bidimodules and $H$-bidimodule maps by $H$-bidimod.

**Example 11.** Let $(M, \lambda, \rho)$ be any $H$-bimodule. Then $H \text{cl} M \text{cl} H = (H \otimes M \otimes H, \lambda^\Delta, \rho_\Delta)$ is an $H$-bidimodule, where $(H \otimes M \otimes H, \lambda^\mu, \rho^\delta)$ is an interior $H$-bimodule and $(H \otimes M \otimes H, \lambda_\Delta, \rho_\Delta)$ is an exterior $H$-bicomodule. Dually, if $(N, \lambda, \rho)$ is any $H$-bimodule, then $H \text{cl} N \otimes H = (H \otimes N \otimes H, \lambda^\mu, \rho^\delta, \lambda_\Delta, \rho_\Delta)$ is an $H$-bidimodule. In particular:

**Example 12.** $H^{(m+2)}$ and $H^{(n+2)}$ are $H$-bidimodules.

Isomorphisms $[4.3]$ and $[4.4]$ extend to the $H$-bidimodules in Example $11$ giving

$$\Phi_N : Hom^*_{H\text{-}bimod}(H \otimes N \otimes H, H \text{cl} M \otimes H) \approx Hom^*_{H\text{-}bicomod}(N, H \text{cl} M \otimes H)$$

and

$$\Psi_M : Hom^*_{H\text{-}bimod}(H \otimes N \otimes H, H \text{cl} M \otimes H) \approx Hom^*_{H\text{-}bimod}(H \otimes N \otimes H, M).$$

Thus,

$$\Phi_N \circ \Psi_M = \Psi_M \circ \Phi_N : Hom^*_{H\text{-}bimod}(H \otimes N \otimes H, H \text{cl} M \otimes H) \approx Hom^*_{H}(N, M).$$

In particular, for $m, n \geq 0$, $N = H^{m}$ and $M = H^{n}$ we have

$$\Phi_N \circ \Psi_M : Hom^*_{H \text{-} bimod}(H^{m+2}, H^{n+2}) \approx Hom^*_{H}(H^{m}, H^{n}).$$

Now consider the $H$-bidimodules $H \otimes N \otimes H, H \otimes N' \otimes H$ and $H \otimes M \otimes H$. If $\Theta : H \otimes N \otimes H \to H \otimes N' \otimes H$ is an $H$-bidimodule map of degree $p$, there is an induced map $\Theta^* : Hom^*_{H}(N', M) \to Hom^*_{H}(N, M)$ given by

$$\Theta^* = (\Phi_N \circ \Psi_M) \circ Hom^*_{H \text{-} bimod}(\Theta, H \text{cl} M \otimes H) \circ (\Phi_N \circ \Psi_M)^{-1}.$$

Dually, given $H$-bidimodules $H \text{cl} M \otimes H, H \text{cl} M' \otimes H$, and $H \otimes N \otimes H$, an $H$-bidimodule map $\Xi : H \text{cl} M \otimes H \to H \text{cl} M' \otimes H$ of degree $q$ induces a map $\Xi_* : Hom^*_{H}(N, M) \to Hom^*_{H}(N, M')$ via

$$\Xi_* = (\Psi_M \circ \Phi_N) \circ Hom^*_{H \text{-} bimod}(H \otimes N \otimes H, \Xi) \circ (\Psi_M \circ \Phi_N)^{-1}.$$

**Definition 12.** Let $(E, \lambda^#, \rho^#, \lambda^\#, \rho^\#)$ be an $H$-bidimodule equipped with a differential $d_E$. Then $(E, d_E)$ is a d.g. $H$-bidimodule if $(E, \lambda^#, \rho^#, d_E)$ is a d.g. $H$-bimodule and $(E, \lambda^\#, \rho^\#, d_E)$ is a d.g. $H$-bicomodule. If $(E, d_E)$ and $(E', d_{E'})$ are d.g. $H$-bidimodules, a map $h \in Hom^*_{H\text{-}bimod}(E, E')$ is a d.g. $H$-bidimodule map if $(-1)^{|h|} h \circ d_E = d_{E'} \circ h = 0$.

**Example 13.** For each $m, n \geq 0$, $(H^{(m+2)}, d_{(m)})$ and $(H^{(n+2)}, d_{(n)})$ are d.g. $H$-bidimodules; maps $\partial_{(m)}$ and $\delta_{(n)}$ are d.g. $H$-bidimodule maps.

Let $(E, \lambda^#, \rho^#, \lambda^\#, \rho^\#, d_E)$ and $(E', \lambda'^#, \rho'^#, \lambda'^\#, \rho'^\#, d_{E'})$ be d.g. $H$-bidimodules and define $\overline{d} : Hom^*_{H\text{-}bimod}(E, E') \to Hom^*_{H}(E, E')$ by $\overline{d}(h) = (-1)^{|h|} h \circ d_E - d_{E'} \circ h$.

**Proposition 3.** Then $\overline{d}(h)$ is a map of d.g. $H$-bidimodules.
5. The Deformation Complex for Differential Graded Structures

5.1. The Deformation Complex for Differential Graded Algebras. We begin by defining the Hochschild cohomology with coefficients in a d.g. A-bimodule. Let \((A, d, \mu)\) be a d.g.a. and let \((M, \lambda, \rho, dM)\) be a d.g. A-bimodule. For each \(m \geq 0\), consider the exterior d.g. A-bimodule \((A \otimes (m + 2), d_m)\), where \(d_m\) is defined as in (3.2). By Proposition 1 there is a map \(d\) linear isomorphism \(\Phi: \text{Hom}^1_{A-(\otimes m+2)}(M) \rightarrow \text{Hom}_{A-(\otimes m+2)}^1(M)\) given by

\[
d^p_m(f) = (-1)^p f \circ d_m - d_m \circ f.
\]

It is easy to check that \(d^*\) is a differential of bidegree \((1, 0)\). Consider the k-linear isomorphism \(\Phi: \text{Hom}^+_A(\otimes (m + 2), M) \rightarrow \text{Hom}^+_A(\otimes m, M)\) defined in (3.3). Then \(d^*\) induces a k-linear differential \(d^*_{\otimes} : \text{Hom}^+_A(\otimes m, M) \rightarrow \text{Hom}^+_A(\otimes (m + 2), M)\) of bidegree \((0, 1)\) via

\[
d^*_{\otimes} = \Phi \circ d^* \circ \Phi^{-1}.
\]

Furthermore, the A-bimodule map \(\partial_m: A \otimes (m + 2) \rightarrow A \otimes (m + 1)\) induces a map \(\partial^{m+2} = \text{Hom}^+_{A-(\otimes m+2)}(\partial_m, M)\) and subsequently, as in (3.2), a k-linear map \(\partial^*_B: \text{Hom}_B^+(\otimes m, M) \rightarrow \text{Hom}_B^+(\otimes (m + 1), M)\) of bidegree \((1, 0)\) defined by

\[
\partial^*_B = \Phi \circ \partial^{m+1} \circ \Phi^{-1}.
\]

It is easy to check that \(\partial_B\) is a differential; the fact that \(\partial_B \circ d_B - d_B \circ \partial_B = 0\) follows easily from (3.3). At a particular \(f \in \text{Hom}_B^+(\otimes m, M)\), the differentials \(d_B\) and \(\partial_B\) can be written as

\[
d^{p,m}_B(f) = (-1)^p f \circ d_{m-2} - d_m \circ f
\]

and

\[
\partial^{p,m}_B(f) = \lambda \circ (1 \otimes f) - f \circ \partial_{m-1} + (-1)^{m+1} \rho \circ (f \otimes 1),
\]

where \(d_{-2} = 0\) and \(\partial_{-1} = 0\).

Refer to \(B^p,A;M\) = \(\text{Hom}_B^+(\otimes m, M)\) as the space of Hochschild m-cochains on \(A\) of degree \(p\). The double complex \(\{B^{p,q}(A;M), d_B, \partial_B\}\) is called the Hochschild cochain complex on \(A\) with coefficients in the d.g. A-bimodule \(M\) (see Figure 2).

Define the space of total r-cochains by \(B^r(A;M) = \sum_{p \in \mathbb{Z}} B^p,A;M\) and define \(D_B\) on the component \(B^{p,r-p}(A;M)\) by

\[
D_B = d_B - (-1)^p \partial_B.
\]

Then \(D_B\) is a total differential; the sign \(-(-1)^p\) is introduced so that \(D_B^2 = 0\). Now define the Hochschild cohomology of \(A\) with coefficients in \(M\), denoted by \(H^*_{d.g.a}(A;M)\), to be the homology of the total complex \(\{B^*(A;M), D_B\}\).
The Hochschild cochain complex on $A$

Figure 2.

We say that an $m$-cochain $f \in B^{p;m}(A;M)$ is normalized if $f(a_1 \otimes \cdots \otimes a_m) = 0$ whenever $|a_i| = 0$ for some $i = 1, 2, \ldots, m$. The differential $\partial_B$ restricts to the subspace of normalized cochains and a standard theorem \[\text{[8]}\] assures that the subcomplex given by such a restriction is cochain homotopic to the full cochain complex. Consequently, we shall use normalized cochains; the symbol $B^{p;m}(A;M)$ will henceforth denote the space of normalized $m$-cochains of degree $p$.

The applications require a bitruncation of the Hochschild cochain complex. First, delete the bottom row in Figure 2 above; the subsequent theory is referred to as the restricted Hochschild cohomology of $A$. Denote the space of restricted total cochains by $\overline{B}^*(A;M)$ and the restricted cohomology by $\overline{H}^*(A;M)$. For $n \in \{3, 4, \ldots\} \cup \{\infty\}$, further restrict the Hochschild complex to those cochains in bidegree $(p,m)$ with $p \geq 3 - n$ and $m \geq 1$. Denote the bitruncated total $r$-cochains by $\overline{B}^r(A;M;n) = \sum_{r-1 \geq p \geq 3-n} B^{p;r-p}(A;M)$. The homology of the complex $\{\overline{B}^*(A;M;n), D_B\}$, denoted by $\overline{H}_{d.g.a.}^*(A;M;n)$, is called the restricted Hochschild cohomology of $A$ truncated at degree $3 - n$.

The deformation complex for $A$ as a strict $A(n)$-algebra is the cochain complex $\{\overline{B}^*(A;A;n), D_B\}$. The cohomology $\overline{H}_{d.g.a.}^*(A;A;n)$ directs the deformation theory in the following way: If $A_t = (A[t], \mu_t^1, \mu_t^2, \ldots)$ is a deformation of $A$ as a strict $A(n)$-algebra, we agree that for $1 \leq i \leq n$ the maps $\mu_t^i = \mu_0^i + t \mu_1^i + t^2 \mu_2^i + \cdots$ satisfy $\mu_1^i \in B^{2-i,1}(A;A)$ with $\mu_0^1 = d$, $\mu_0^2 = \mu$, and $\mu_0^i = 0$ for $i > 2$. Then $a = \sum_{i=1}^n \mu_t^i$ is a deformation $D_B(a)$ satisfies $D_B(a) = 0$. Furthermore, given a total cocycle $a = \sum_{i=1}^n \mu_t^i$ in $\overline{B}^2(A;A;n)$, the obstructions to extending the corresponding linear approximation $(d + t \mu_1^1, \mu + t \mu_1^2, \mu_1^3, \ldots, t \mu_1^i, \ldots)$ to a

| $\cdots$ | $\cdots$ | $\cdots$ |
|---------|---------|---------|
| $\partial_B \uparrow$ | $\uparrow$ | $\uparrow$ |
| $\cdots$ $\text{Hom}_k^{p-1}(A^3, M)$ $\rightarrow$ $\text{Hom}_k^p(A^3, M)$ $\rightarrow$ $\text{Hom}_k^{p+1}(A^3, M)$ $\cdots$ |
| $\partial_B \uparrow$ | $\uparrow$ | $\uparrow$ |
| $\cdots$ $\text{Hom}_k^{p-1}(A^2, M)$ $\rightarrow$ $\text{Hom}_k^p(A^2, M)$ $\rightarrow$ $\text{Hom}_k^{p+1}(A^2, M)$ $\cdots$ |
| $\partial_B \uparrow$ | $\uparrow$ | $\uparrow$ |
| $\cdots$ $\text{Hom}_k^{p-1}(A^1, M)$ $\rightarrow$ $\text{Hom}_k^p(A^1, M)$ $\rightarrow$ $\text{Hom}_k^{p+1}(A^1, M)$ $\cdots$ |
| $\partial_B \uparrow$ | $\uparrow$ | $\uparrow$ |
| $\cdots$ $\text{Hom}_k^{p-1}(k, M)$ $\rightarrow$ $\text{Hom}_k^p(k, M)$ $\rightarrow$ $\text{Hom}_k^{p+1}(k, M)$ $\cdots$ |

The Hochschild cochain complex on $A$
deformation appear as an inductively defined sequence of cocycles in $\tilde{B}^{3}(A;A;n)$. We note that the deformation theory of $A$ as an $A(\infty)$-algebra also appears in \cite{15} and \cite{21}. The case $n = 3$ is discussed in some detail in section 6 (as a special case) and subsequently by Lazarev and Movshev in the sequel \cite{17}. When $n = 3$, we adopt the standard notation $A_1 = (A[t], d_1, \mu_1)$ with $d_1 = d + td_1 + t^2d_2 + \cdots$ and $\mu_1 = \mu + tf_1 + t^2f_2 + \cdots$, in which case $A_1$ is an (associative) d.g.a. Now suppose that such an $A$ is commutative.

If $(A, d, \mu)$ is a commutative d.g.a. (c.d.g.a.), a general deformation $A_t = (A[[t]], d_t, \mu_t)$ fails to be commutative. In some applications, such as the classification of rational homotopy type for example, one desires only commutative deformations; in this case it is necessary to restrict the Hochschild cochain complex to the subcomplex of cochains with the potential to spawn commutative deformations. A discussion of this subcomplex, called the Harrison cochain complex on $A$, now follows.

Let $(A, d, \mu)$ be a c.d.g.a. and let $(M, \lambda, \rho, d_M)$ be a symmetric d.g. $A$-bimodule. The desired cochains $f \in B^{*,2}(A;M)$ are the symmetric functions, and in general, the desired cochains $f \in B^{*,m}(A;M)$ vanish on sums of certain shuffle permutations. Precisely, let $\sigma_{r,s} \in S_n$ denote a $(r,s)$-shuffle \cite{18}; denote its sign as a permutation by $(-1)^{\sigma_{r,s}}$. Then $f \in B^{p,m}(A;M)$ is a Harrison $m$-cochain on $A$ in degree $p$ if and only if $f$ vanishes on $\sum_{r,s,m}(-1)^{r,m-r,s,m-s,m-r} \sigma_{r,m-r,s,m-s,m-r}$ for each $r = 1, 2, \ldots, m - 1$.

The space of Harrison $m$-cochains of degree $p$ is denoted by $Ch^{p,m}(A;M)$; note that $Ch^{*,1}(A;M) = B^{*,1}(A;M)$. The differentials $d_B$ and $\partial_B$ restrict to $Ch^{*,*}(A;M)$, so let $Ch^{*}(A;M) = \sum_{p+m=r} Ch^{p,m}(A;M)$ and define the Harrison cohomology of $A$, $\text{Harr}_{c.d.g.a.}(A;M)$, to be the homology of the total complex $\{Ch^{*}(A;M), D_B\}$.

As in the general Hochschild case, the symbol $\tilde{Ch}^{*}(A;M;n)$ denotes the total bi-truncated Harrison cochains and $\tilde{\text{Harr}}_{c.d.g.a.}(A;M;n)$ denotes the corresponding cohomology. The complex $\{\tilde{Ch}^{*}(A;A;n), D_B\}$ is the deformation complex for $A$ as a ”balanced” $A(n)$-algebra; see \cite{14} and \cite{19}.

If $(A, \mu)$ is a c.g.a. sans differential, one can forget the internal grading and grade the Harrison cohomology externally (with respect to the number of tensor factors) as one does classically. Let $\text{Harr}^{*}(A;M)$ denote the Harrison cohomology graded in this way; one has the following result, which is a consequence of the Hochschild, Kostant, and Rosenberg Theorem \cite{12}.

**Theorem 1.** Let $k$ be a field of characteristic 0. If $A$ is a free commutative $k$-algebra and $(M, \lambda, \mu)$ is any symmetric $A$-bimodule, then $\text{Harr}^{n}(A;M) = 0$ whenever $n > 1$.

The requirement that $k$ have characteristic 0 is critical here; the result fails in characteristic $p > 0$ \cite{2}. Theorem \cite{2} allows us to view the rational perturbation theory of Felix and Halperin-Stasheff in terms of the deformation theory of free c.d.g.a.’s in characteristic zero. The discussion that follows makes the connection precise.

Throughout the remainder of this section, $k$ denotes a field of characteristic zero. Let $(A, d, \mu)$ be a free c.d.g.a. over $k$ and let $(M, \lambda, \rho, d_M)$ be a symmetric d.g. $A$-bimodule. By forgetting the differentials, Theorem \cite{2} implies that $\text{Harr}^{n}(A;M) = 0$ for $n > 1$ so that each column in the restricted Harrison complex is exact (see Figure 3).
\[
\begin{align*}
\cdots & \to Ch^{n,3}(A;M) & \to \cdots \\
\partial_n & \uparrow \\
\cdots & \to Ch^{n,2}(A;M) & \to \cdots \\
\partial_n & \uparrow \\
\cdots & \to Ch^{n,1}(A;M) & \to \cdots 
\end{align*}
\]

An Exact Column
Figure 3.

Let \( \{E, d'_E, d''_E\} \) be a standard double cochain complex with differentials of respective bidegree \((1,0)\) and \((0,1)\).

**Definition 13.** Let \( n \geq 1, n \geq k \geq 0, \) and let \( f = \sum_{0 \leq i \leq n-1} g_{i,n-i} \) be a total \( n \)-cochain for which \( \{f_{i,n-i} \in E^{i,n-i}\}_{0 \leq i \leq n-1} \). Then \( f \) is an \((n,k)\)-cochain if \( f_{i,n-i} = 0 \) for \( 0 \leq i \leq k-1 \). An \((n,n)\)-cochain is said to be concentrated in bidegree \((n-1,1)\).

**Lemma 1.** If the columns of \( \{E, d'_E, d''_E\} \) are exact, then every class \([x] \in H^n(E, d'_E + d''_E)\) can be represented by a total \( n \)-cochain \( h \) concentrated in bidegree \((n-1,1)\).

**Proof:** Let \( f = \sum_{0 \leq i \leq n-1} g_{i,n-i} \) be a total \( n \)-cochain and note that \( f \) is an \((n,k)\)-cochain for some \( 0 \leq k \leq n \). If \( k = n \) there is nothing to prove; so assume that \( k < n \). By exactness, there exists a cochain \( g_{k,n-k-1} \in E^{k,n-k-1} \) such that \( d''_E(g_{k,n-k-1}) = f_{k,n-k} \) and \( d'_E(f_{k+1,n-k-1} - d''_E(g_{k,n-k-1})) = 0 \). Hence \( h_{k+1} = \sum_{k+1 \leq i \leq n-1} f_{i,n-i} - d'_E(g_{k,n-k-1}) \) is an \((n,k+1)\)-cochain and \( f - h_{k+1} = d'_E(g_{k,n-k-1}) + f_{k,n-k} = (d'_E + d''_E)(g_{k,n-k-1}) \). Proceed inductively until \( f \) is totally cohomologous to some \((n,n)\)-cochain \( h_n \).

I should note that "staircase" arguments such as this are not new, having appeared as early as 1952 in a paper by Weil [25].

**Definition 14.** Let \( (M, \lambda, \rho) \) be an \( A \)-bimodule. A \( k \)-linear map \( \theta : A \to M \) is a derivation if \( \theta \circ \mu = \rho \circ (\theta \otimes 1) + \lambda \circ (1 \otimes \theta) \). The set of all derivations of degree \( p \) is denoted by \( \text{Der}^p(A, M) \).

**Corollary 1.** Let \( (A, d, \mu) \) be a free c.d.g.a. over \( k \) and let \( (M, \lambda, \rho, d_M) \) be a symmetric \( d.g. \) \( A \)-bimodule. If \( f = \sum_{0 \leq i \leq n-1} f_{i,n-i} \in Ch^n(A;M;3) \) is a total \( n \)-cochain, there exists \( g \in \text{Der}^{n-1}(A, M) \) such that \( f - g \) is totally cohomologous to zero.
Proof: By Lemma 1, there exists $n$-cocycle $g$ concentrated in bidegree $(n - 1, 1)$ such that $f - g$ is totally cohomologous to zero. But then $g \in \ker \partial_B$ and $\text{Der}^{n-1}(A, M) = \ker \partial_B$ by the definition of $\partial_B$.

**Theorem 2.** Let $(A, d, \mu)$ be a free c.d.g.a. over $k$ and let $(M, \lambda, \rho, d_M)$ be a symmetric d.g. $A$-bimodule. There is an isomorphism

$$
\Gamma : H^{* - 1}(\text{Der}(A, M), d_B) \xrightarrow{\sim} \text{Harr}^*_{\text{c.d.g.a.}}(A; M; 3).
$$

Proof: Consider a $d_B$-cocycle $\theta \in \text{Der}^{n-1}(A, M)$; $\theta$ is a normalized $(n - 1)$-cochain since $\theta(1_k) = 0$. On the other hand, $\theta$ is an $(n, n)$-cocycle in $\text{Ch}^n(A; M; 3)$. So define $\Gamma[\theta] = [\theta]$; surjectivity follows from Corollary 1.

Since every Harrison n-class $[x]$ can be represented by a total $n$-cocycle concentrated in bidegree $(n - 1, 1)$, the dimension shift in isomorphism (5.3) is superficial to the extent that it emphasizes one of two points-of-view. We can think of $[x]$ either in terms of a representative in $\text{Der}^{n-1}(A, M) \subset \text{Ch}^{n-1,1}(A; M)$ or in terms of a representative in $\text{Ch}^{n-1}(A; M; 3)$.

Now set $M = A$. The adjoint action of $d$ on $\text{Der}^*(A, A)$ as a Lie algebra of derivations is given by $\text{ad}(d)(\theta) = [d, \theta]$, where $[d, \theta] = d \circ \theta - (-1)^{\theta \circ d} \theta \circ d$. In this case, the differential in (5.3) is simply

$$
d_B = -ad(d).
$$

In particular, let $\Lambda(x_i)$ denote the free c.g.a. on generators $\{x_i\}$ over $k$ and let $\langle x_i \rangle$ denote the $k$-module with basis $\{x_i\}$. Since derivations of a free c.g.a. are determined by their action on generators and $k$-linear maps are determined by their action on a basis we have:

**Corollary 2.** If $d$ is an algebra differential on $A = \Lambda(x_i)$, there is an isomorphism

$$
\text{Harr}^*_{\text{c.d.g.a.}}(A; A; 3) \approx H(\text{Hom}^{* - 1}_k(\langle x_i \rangle, A), \text{ad}(d)).
$$

Note that the right-hand-side of isomorphism (5.4) does not depend upon the multiplication $\mu$. Thus a non-vanishing Harrison 2-class $[x] \in \text{Harr}^2_{\text{c.d.g.a.}}(A; A; 3)$ signals a potential change in the differential $d$—not in the multiplication $\mu$. But changing the differential is exactly the game Felix and Halperin-Stasheff play.

Let $X$ be a formal space with rational cohomology algebra $A = H^*(X; \mathbb{Q})$. A rational minimal model for $A$ is a free c.d.g.a. $(\Lambda, d)$ over $\mathbb{Q}$ such that $d(\Lambda) \subset \Lambda \cdot \Lambda$ and $H^*(\Lambda, d) \approx A$. For Felix, this isomorphism is additive; for Halperin and Stasheff it is multiplicative. A perturbation $p$ of the differential $d$ is a derivation $p \in \text{Der}^1(\Lambda, A)$ such that $(d + p)^2 = 0$ and $H^*(\Lambda, d + p) \approx A$. The linearization of a perturbation $p$ represents a 1-class on the right-hand-side of (5.4). Given a perturbation $p$, there exists a rational space $Y$ and an isomorphism of higher order structures $H^*(\Lambda, d + p) \cong H^*(Y; \mathbb{Q})$. Conversely, given a rational space $Y$ with $H^*(Y; \mathbb{Q}) \approx A$, there exists a perturbation $p_Y$ such that $H^*(\Lambda, d + p_Y) \cong H^*(Y; \mathbb{Q})$. Spaces $Y$ and $Y'$ have the same rational homotopy type if and only if the corresponding perturbations $p_Y$ and $p_{Y'}$ are equivalent as deformations (see section 6 below).

Felix and Halperin-Stasheff apply this theory in two somewhat different ways. One can obtain a minimal model for $A$ as the limit of a Tate-Josefiak resolution...
of $A$ \cite{13, 24}. Thought of this way, the minimal model is naturally bigraded with respect to internal and resolution degrees. In either approach, this "bigraded model" is perturbed to a "filtered model": the perturbations of Felix arbitrarily decrease filtration, while those of Halperin and Stasheff decrease filtration by at least two. The former effectively fixes the additive structure and varies the multiplicative structure on $A$, while the latter fixes the multiplicative structure and varies the higher order algebra structure on $A$. The set of perturbations that decrease filtration by at least two is a subcomplex of $\{\text{Der}^*(\Lambda, \Lambda), \text{ad}(d)\}$.

Finally, I should mention that a comparison between the Harrison cohomology of $(\Lambda, d)$ and the homology of $\{\text{Coder}(L^c\Lambda, A), d^c\}$, where $L^c\Lambda$ is the free $d.g.$ Lie coalgebra on $\Lambda$ and $d^c$ is the differential induced by $d$, was given by Schleissinger and Stasheff in \cite{22}.

### 5.2. The Deformation Complex for Differential Graded Coalgebras.

Let $(C, d, \Delta)$ be a $d.g.c.$ The deformation complex for $C$ is a double complex dual to the one discussed in 5.1; this discussion is included for notational purposes. Let $(N, \lambda, \rho, d_N)$ be a $d.g.$ $C$-bicomodule. For each $n \geq 0$, consider the exterior $C$-bicomodule $(C^\otimes(n+2), d(n))$ where $d(n)$ is defined as in \cite{13}. By proposition 2 and an easy calculation, there is a map $d^{q,n} : \text{Hom}^q_{C\text{-bicomod}}(N, C^\otimes(n+2)) \to \text{Hom}^q_{C\text{-bicomod}}(N, C^\otimes(n+2))$ given by

$$d^{q,n}(g) = (-1)^q g \circ d_N - d(n) \circ g.$$ 

This induces a $k$-linear differential $d^{q,n}_{\Omega_1} : \text{Hom}^q_k(N, C^\otimes n) \to \text{Hom}^{q+1}_k(N, C^\otimes n)$ of bidegree $(1, 0)$ via \cite{13}:

$$d^{*, *}_{\Omega_1} = \Psi \circ d^{*, *} \circ \Psi^{-1}.$$ 

Via \cite{13}, the $C$-bicomodule map $\delta(n) : C^\otimes(n+2) \to C^\otimes(n+3)$ induces a $k$-linear differential $\delta^{q,n}_{\Omega_1} : \text{Hom}^q_k(N, C^\otimes n) \to \text{Hom}^q_k(N, C^\otimes(n+1))$ of bidegree $(1, 0)$ given by

$$\delta^{*, *}_{\Omega_1} = \Psi \circ \delta^{*, *} \circ \Psi^{-1}.$$ 

It is easy to check that $d_\Omega \circ \delta_\Omega - \delta_\Omega \circ d_\Omega = 0$; at a particular $g \in \text{Hom}^q_k(N, C^\otimes n)$ we have

$$d^{q,n}_{\Omega_1}(g) = (-1)^q g \circ d_N - d_{(n-2)} \circ g$$

and

$$\delta^{q,n}_{\Omega_1}(g) = (1 \otimes g) \circ \lambda - \delta_{(n-2)} \circ g + (-1)^{n+1}(g \otimes 1) \circ \rho,$$

where $d_{(-2)} = 0$ and $\delta_{(-2)} = 0$.

Refer to $\Omega^{*,n}(C; N) = \text{Hom}^q_k(N, C^\otimes n)$ as the space of Cartier $n$-cochains on $C$ of degree $q$. The double complex $\{\Omega^{*,*}(C; N), d_\Omega, \delta_\Omega\}$ is called the Cartier cochain complex on $C$ with coefficients in the $d.g.$ $A$-bicomodule $N$. Define the space of total $s$-cochains by $\Omega^s(C; N) = \sum_{q \in \mathbb{Z}; \ s \geq q} \Omega^{s-q}(C; N)$ and define $D_\Omega$ on the component $\Omega^{s,n}(C; N)$ by

$$D_\Omega = \delta_\Omega - (-1)^n d_\Omega.$$ 

Then $D_\Omega$ is a total differential; the sign $-(-1)^n$ is introduced so that $D^2_\Omega = 0$. Now define the Cartier cohomology of $C$ with coefficients in the $d.g.$ $A$-bicomodule $N$, denoted by $H^{*,s}_{\Omega, d.g.}(C; N)$, to be the homology of the total complex $\{\Omega^{*,*}(C; N), D_\Omega\}$.

We say that an $n$-cochain $g \in \Omega^{*,n}(C; N)$ is normalized if $g(x) = a_1 \otimes \cdots \otimes a_n = 0$ whenever $|a_i| = 0$ for some $i = 1, 2, \ldots, n$. The differential $\delta_\Omega$ restricts to the
subspace of normalized cochains and a standard theorem \[18\] assures that the subcomplex given by such a restriction is cochain homotopic to the full cochain complex. Consequently, we shall use normalized cochains; the symbol \( \Omega^{q,n}(C;N) \) will henceforth denote the space of normalized \( n \)-cochains of degree \( q \).

As on the algebra side, obtain the restricted Cartier cohomology of \( C \) by deleting the bottom row of the Cartier cochain complex. Denote the space of restricted total cochains by \( \tilde{\Omega}^*(C;N) \) and the restricted cohomology by \( \tilde{H}^*_{d.g.c.}(C;N) \). For \( m \in \{3,4,\ldots\} \cup \{\infty\} \), further restrict the Cartier complex to those cochains in bidegree \((q,n)\) with \( q \geq 3-m \) and \( n \geq 1 \). Denote the truncated total \( n \)-cochains by \( \tilde{\Omega}^*(C;N;m) = \sum_{s-1 \geq q \geq 3-m} \Omega^{q,s-q}(C;N) \). The homology of the complex \( \{\tilde{\Omega}^*(C;N;m);D_\Omega\} \), denoted by \( \tilde{H}^*_{d.g.c.}(C;N;m) \), is called the restricted Cartier cohomology of \( C \) truncated at degree \( 3-m \).

The deformation complex for \( C \) as a strict \( C(m) \)-coalgebra is the cochain complex \( \{\tilde{\Omega}^*(C;C;m);D_\Omega\} \). The cohomology \( \tilde{H}^*_{d.g.c.}(C;C;m) \) directs the deformation theory in a way completely analogous to the algebra case. If \( C_t = (C[[t]], \Delta_t^{(1)}, \Delta_t^{(2)}, \ldots) \) is a deformation of \( C \) as a strict \( C(m) \)-coalgebra, we agree that for \( 1 \leq i \leq m \) the maps \( \Delta_t^{(i)} = \Delta_0^{(i)} + t\Delta_1^{(i)} + t^2\Delta_2^{(i)} + \cdots \) satisfy \( \Delta_0^{(i)} \in \Omega^{2-i,i}(C;C) \) with \( \Delta_0^{(1)} = d \), \( \Delta_0^{(2)} = \Delta \), and \( \Delta_0^{(i)} = 0 \) for \( i > 2 \). Then \( C = \sum_{i=1}^m \Delta_t^{(i)} \in \tilde{\Omega}^*(C;C;m) \) satisfies \( D_\Omega(c) = 0 \). Furthermore, given a total cocycle \( c = \sum_{i=1}^m \Delta_t^{(i)} \in \tilde{\Omega}^*(C;C;m) \), the obstructions to extending the corresponding linear approximation \( (d + t\Delta_1^{(1)}, \mu + t\Delta_2^{(2)}, \ldots, t\Delta_3^{(3)}, \ldots) \) to a deformation appear as an inductively defined sequence cochains in \( \tilde{\Omega}^*(C;C;m) \). In the special case \( m = 3 \) we adopt the standard notation \( C_t = (C[[t]], \Delta_t) \) with \( \Delta_t = d + td_1 + t^2d_2 + \cdots \) and \( \Delta_t = \Delta + td_1 + t^2d_2 + \cdots \), in which case \( C_t \) is a (coassociative) \( d.g.c. \). We refer further discussion of this latter case to section 6.

5.3. The Deformation Complex for Differential Graded Hopf Algebras. Let \( (H,d,\mu,\Delta) \) be a \( d.g.h.a. \). In Example\[3\] we observed that \( (H\underline{\otimes}(m+2),d_{(m)}) \) and \( (H\underline{\otimes}(n+2),d_{(n)}) \) are \( d.g. \) \( H \)-bimodules and that \( \partial_{(m)} : H\underline{\otimes}(m+2) \to H\underline{\otimes}(m+1) \) and \( \delta_{(n)} : H\underline{\otimes}(n+2) \to H\underline{\otimes}(n+3) \) are \( d.g. \) \( H \)-bimodule maps. By proposition \[8\] there is a map \( d^{p,m,n} : \text{Hom}^P_{H-bimod}(H\underline{\otimes}(m+2),H\underline{\otimes}(n+2)) \to \text{Hom}^P_{H-bimod}(H\underline{\otimes}(m+2),H\underline{\otimes}(n+2)) \) given by \( d^{p,m,n}(f) = (-1)^pf \circ d_{(m)} - d_{(n)} \circ f \).

In section 5.1 and 5.2 we observed that \( d^{*,*,*} \) commutes with the induced maps \( \partial^{p,m,n} = \text{Hom}^P_{H-bimod}(\partial_{(m)},H\underline{\otimes}(n+2)) \) and \( \delta^{p,m,n} = \text{Hom}^P_{H-bimod}(\partial_{(n)},H\underline{\otimes}(m+2)) \); consequently \( d^{*,*,*} \) commutes with \( \partial^{*,*,*} \) and \( \delta^{*,*,*} \) in the common subcategory of \( H \)-bimodules. But there is more. For each \( p \in \mathbb{Z} \) and each \( m,n \geq 0 \), the bar and cobar differentials \( \tilde{\partial}^{*,*,*} \) and \( \tilde{\delta}^{*,*,*} \) functorially induce the commutativity of \( \partial^{*,*,*} \) and \( \delta^{*,*,*} \) in this subcategory (see Figure 4). This remarkable structural compatibility gives rise to a triple complex \( \{\text{Hom}^P_{H-bimod}(H\underline{\otimes}(m+2),H\underline{\otimes}(n+2)),d^{*,*,*},\partial^{*,*,*},\delta^{*,*,*}\} \) at the level of \( k \)-modules is obtained via isomorphism \[4,\delta\] with differentials induced by \( \partial^{*,*,*} \), \( \partial^{*,*,*} \), and \( \delta^{*,*,*} \) as in \( (4,\delta) \) and \( (4,\delta) \). We have obtained our main result.

**Theorem 3.** Let \( (H,d,\mu,\Delta) \) be a \( d.g.h.a. \) and let \( C^{p,m,n}(H;H) = \text{Hom}^P_k(H\underline{\otimes}k,H\underline{\otimes}k) \). There is a triple complex \( \{C^{*,*,*}(H;H),d_{C},\partial_{C},\delta_{C}\} \) whose
differentials have respective tridegree \((1, 0, 0), (0, 1, 0),\) and \((0, 0, 1)\) and arise from \(d, \mu,\) and \(\Delta\) as follows:

\[
d^{\star, \star, \star}_C = (\Phi \circ \Psi) \circ d^{\star, \star, \star} \circ (\Phi \circ \Psi)^{-1},
\]

\[
\partial^{\star, \star, \star}_C = (\Phi \circ \Psi) \circ \partial^{\star, \star+1, \star} \circ (\Phi \circ \Psi)^{-1},\quad \text{and}
\]

\[
\delta^{\star, \star, \star}_C = (\Phi \circ \Psi) \circ \delta^{\star, \star, \star} \circ (\Phi \circ \Psi)^{-1}.
\]

At a particular \(f \in C^{p,m,n}(H; H)\) these expand to

\[
\partial^{p,m,n}_C(f) = (-1)^p f \circ d_{(m-2)} - d_{(n-2)} \circ f,
\]

\[
\partial^{p,m,n}_C(f) = \lambda^n \circ (1 \otimes f) - f \circ \partial_{(m-1)} + (-1)^{m+1} \rho^n \circ (f \otimes 1),\quad \text{and}
\]

\[
\partial^{p,m,n}_C(f) = (1 \otimes f) \circ \lambda_m - d_{(n-2)} \circ f + (-1)^{n+1}(f \otimes 1) \circ \rho_m.
\]

\[
\Hom^p_{H-\text{bidmod}}(H^m, H^\otimes n + 3) \quad \longrightarrow \quad \Hom^p_{H-\text{bidmod}}(H^m, H^\otimes n + 3)
\]

\[
\delta^{p,m,n} \quad \uparrow \quad \delta^{p,m+1,n}
\]

\[
\Hom^p_{H-\text{bidmod}}(H^m, H^\otimes n + 2) \quad \longrightarrow \quad \Hom^p_{H-\text{bidmod}}(H^m, H^\otimes n + 2)
\]

Figure 4.

The space \(C^{p,m,n}(H; H)\) of normalized cochains is referred to as the Hochschild-Cartier \((m,n)\)-cochains on \(H\) of degree \(p\); the triple complex \(\{C^{\star, \star, \star}(H; H), d_C, \partial_C, \delta_C\}\) is called the Hochschild-Cartier cochain complex on \(H\). Define the space of total \(r\)-cochains by \(C^r(H; H) = \sum_{p \in \mathbb{Z}, m,n \geq 0; p+m+n=r+1} C^{p,m,n}(H; H)\) and define \(D_C\) on the component \(C^{p,m,n}(H; H)\) by

\[
D_C = (-1)^{m(n+1)} d_C + (-1)^n(p+1) \partial_C + (-1)^p(m+1) \delta_C.
\]

Then \(D_C\) is a total differential; the sign adjustments are introduced so that \(D_C^2 = 0\) and determine those in \(\{\overline{1}, \overline{2}, \overline{3}\}\) and \(\{\overline{4}, \overline{6}\}\) made earlier—take coefficients in \(H\) and set either \(m = 1\) or \(n = 1\) as appropriate. Finally, define the Hochschild-Cartier cohomology of \(H\), denoted by \(H^*_\text{HC}(H; H)\); to be the homology of the complex \(\{C^*(H; H), D_C\}\).

Obtain the restricted Hochschild-Cartier cohomology of \(H\) by deleting the two "coordinate planes" \(m = 0\) and \(n = 0\) in the Hochschild-Cartier triple cochain complex. Denote the space of restricted cochains by \(\overline{C}^*(H; H)\); denote the corresponding restricted cohomology by \(\overline{H}^*_\text{HC}(H; H)\). For \(q \in \{3, 4, \ldots\} \cup \{\infty\}\), further restrict the Hochschild-Cartier complex to those cochains in tridegree \((p, m, n)\) with \(p \geq 3 - q\) and \(m, n \geq 1\). Denote the space of tritruncated total \(r\)-cochains by \(\overline{C}^r(H; H; q) = \sum_{m,n \geq 1; p \geq 3-q; p+m+n=r+1} C^{p,m,n}(H; H)\).

The deformation complex for \(H\) as a strict \((H; q)\)-structure is the complex \(\{\overline{C}^*(H; H; q), D_C\}\); its homology, which directs the deformation theory, is called the restricted Hochschild-Cartier cohomology of \(H\) truncated at degree \(3 - q\) and is denoted by \(\overline{H}^*_\text{HC}(H; H; q)\). When \(q = 3\) we obtain the deformation complex for \(H\) as a \(d.g.h.a\). If the Hopf algebra \(H\) is commutative as an algebra, we obtain the restricted Harrison cohomology of \(H\) truncated at degree \(3 - q\), denoted by
corresponding linear approximation \( (H; q), \) as the homology of \( \{ \overline{C}H^*(H; q), D_C \} \). We conclude our discussion with a brief exposition of the deformation theory of \( d.g.h.a.'s. \)

6. Deformation Theory of Differential Graded Hopf Algebras

Let \( H_0 = (H, d, \mu, \Delta) \) be a \( d.g.h.a. \) over a field \( k \). Let \( t \) be an indeterminant of degree 0 and let \( k[[t]] \) denote the commutative ring of formal power series in \( t \). Consider the (graded) \( k[[t]] \)-module \( H[[t]] \) of formal power series in \( t \) with coefficients in \( H \). We give \( H[[t]] \) the \( t \)-adic topology in which \( a \) and \( b \) are \( t^r \)-close if \( a \equiv b(\text{mod } t^r) \). In the \( t \)-adic topology, every formal power series is the limit of its sequence of partial sums.

Given a \( k \)-linear map \( f : H[[t]]^m \to H[[t]]^n \), extend \( f \) to a \( k[[t]] \)-linear map \( f : H[[t]]^m \to H[[t]]^n \), where we tensor over \( k[[t]] \), by defining \( f(\sum t^a_i \otimes \sum t^j b_j \otimes \cdots) = \sum t^{a_i+j} f(a_i \otimes b_j \otimes \cdots) \); this is the unique \( k[[t]] \)-linear extension of \( f \) to \( H[[t]]^m \). In particular, so extending \( d, \mu \) and \( \Delta \) gives a \( k[[t]] \)-\( d.g.h.a. \) \( H_0[[t]] = (H[[t]], d, \mu, \Delta) \), which possibly deforms to some \( k[[t]] \)-\( d.g.h.a. \) \( H_0 = (H[[t]], d_t, \mu_t, \Delta_t) \). Indeed, the deformation theories of algebras and coalgebras discussed above suggest the possibility of further deforming to some \( H_t \) in the category of \( H(q) \)-bialgebras”, but we limit our discussion to the deformation of \( H_0 \) as a \( d.g.h.a. \) here.

**Definition 15.** A deformation of \( H_0 \) is a \( k[[t]] \)-\( d.g.h.a. \) \( H_t = (H[[t]], d_t, \mu_t, \Delta_t) \) such that

1. \( d_t = d + t d_0 + t^2 d_2 + \cdots \),
2. \( \mu_t = \mu + t \mu_1 + t^2 \mu_2 + \cdots \), and
3. \( \Delta_t = \Delta + t \Delta_1 + t^2 \Delta_2 + \cdots \).

The fundamental problem is to classify all "non-trivial" deformations. Given a deformation \( H_t = (H[[t]], d_t, \mu_t, \Delta_t) \), consider the linear terms in the associativity condition \( \mu_t \circ (1 \otimes \mu_t) = \mu_t \circ (\mu_t \otimes 1) \). Equating coefficients gives \( \mu \circ (1 \otimes \mu) + \mu \circ (1 \otimes \mu) = \mu \circ (1 \otimes \mu) + \mu \circ (1 \otimes \mu) \). With \( \lambda^1 = \rho^1 = \mu \) we have \( \partial^0_{C,1} \mu_1 = \lambda^1 \circ (1 \otimes \mu_1) - \mu_1 \circ (1 \otimes \mu) + \mu_1 \circ (1 \otimes \mu) = 0 \). Similarly, with \( \lambda_t = \rho_t = \Delta_t \), the linear terms in the coassociativity condition \( (1 \otimes \Delta_t) \circ \Delta_t = (\Delta_t \otimes 1) \circ \Delta_t \) imply that \( \partial^0_{C,1} \Delta_1 = 0 \), and the linear terms in the differential condition \( d_t \circ d_t = 0 \) imply that \( \partial^0_{C,1}(d_t) = 0 \). Furthermore, the linear terms in the bialgebra condition

\[
\Delta_t \circ \mu_t = (\mu_t \otimes \mu_t) \circ (2,3, \Delta_t \otimes \Delta_t)
\]

and the linear terms in the coderivation condition\n
\[
\Delta_t \circ d_t = (d_t \otimes 1 + 1 \otimes d_t) \circ \Delta_t
\]

imply that \( \partial^0_{C,1} \Delta_1 = 0 \). Hence \( D_C \) is a restricted total Hochschild-Cartier 2-cocycle. Restricting these calculations to "planes" \( n = 1 \) and \( m = 1 \) establishes similar claims made earlier in the algebra and coalgebra settings, respectively.

Conversely, given a restricted total Hochschild-Cartier 2-cocycle \( d_t + \mu_1 + \Delta_1 \in C^{1,1,1}(H; H) \oplus C^{0,2,1}(H; H) \oplus C^{0,1,2}(H; H) \), there is an inductively defined sequence of cocycles in \( \overline{C}^2(H; H; 3) \) whose vanishing allows one to inductively extend the corresponding linear approximation \( (d + t d_1, \mu + t \mu_1, \Delta + t \Delta_1) \) to a deformation \( H_t \) in a standard way. For a detailed discussion of the bialgebra case see [10].

Recall that a map \( \phi_t \in Aut_k[H[[t]]] \) satisfies \( \phi_t \equiv 1(\text{mod } t) \); hence there exist \( k \)-linear maps \( \{ \phi_t \}_{t \geq 1} \) such that \( \phi_t = 1 + t \phi_1 + t^2 \phi_2 + \cdots \).
Definition 16. Two deformations \( H_t = (H[[t]], d_t, \mu_t, \Delta_t) \) and \( H'_t = (H[[t]], d'_t, \mu'_t, \Delta'_t) \) are equivalent if there exists some d.g.h.a. map \( \phi_t \in \text{Aut}_{k[[t]]}(H[[t]]) \) such that

1. \( d_t \circ \phi_t = \phi_t \circ d'_t \),
2. \( \mu_t \circ (\phi_t \otimes \phi_t) = \phi_t \circ \mu'_t \), and
3. \( \Delta_t \circ \phi_t = (\phi_t \otimes \phi_t) \circ \Delta'_t \).

When \( H_t \) and \( H'_t \) are equivalent via \( \phi_t \), we write \( \phi_t : H_t \sim H'_t \) and refer to \( \phi_t \) as an equivalence. Any deformation \( H_t \) equivalent to \( H_0[[t]] = (H[[t]], d, \mu, \Delta) \), thought of as a deformation with \( d_i = \mu_i = \Delta_i = 0 \) for all \( i \), is called a trivial deformation. If every deformation \( H_t \) is trivial we say that \( H_0 \) is rigid as a d.g.h.a.

Given an equivalence \( \phi_t : H_t \sim H'_t \), equate coefficients in the linear terms of the naturality condition to obtain \( d^{(1,1)}(\phi_t) = \phi_t \circ d - d \circ \phi_t = d_t - d'_t \). Similarly, \( -d^{(1,1)}(\phi_t) = -[\mu \circ (1 \otimes \phi_t) - \phi_t \circ \mu + \mu \circ (\phi_t \otimes 1)] = \mu_t - \mu'_t \) since \( \phi_t \) is an algebra map, and \( d^{(1,1)}(\phi_t) = (1 \otimes \phi_t) \circ \Delta - \Delta \circ \phi_t + (\phi_t \otimes 1) \circ \Delta = \Delta_t - \Delta'_t \) since \( \phi_t \) is a coalgebra map. Therefore \( D_C(\phi_t) = (d_t + \mu_t + \Delta_t) - (d'_t + \mu'_t + \Delta'_t) \) so that \( d_t + \mu_t + \Delta_t \) and \( d'_t + \mu'_t + \Delta'_t \) are totally cohomologous as restricted Hochschild-Cartier 2-cocycles.

If \( H^2_{d.g.h.a.}(H; H; 3) = 0 \) and \( H_t = (H[[t]], d_t, \mu_t, \Delta_t) \) is a deformation, choose \( \phi_t \in \overline{C}^1 (H; H; 3) \) such that \( d^{(1,1)}(\phi_t) = d_t + \mu_t + \Delta_t \) and consider \( \phi_t = (1 - t \phi_t) \in \text{Aut}_{k[[t]]}(H[[t]]) \). There is a deformation \( H^{(1)}_t = (H[[t]], d^{(1)}_t, \mu^{(1)}_t, \Delta^{(1)}_t) \), where \( d^{(1)}_t = \phi_t^{(1)} \circ d_t \circ [\phi_t^{(1)}]^{-1} \), \( \mu^{(1)}_t = \phi_t^{(1)} \circ \mu_t \circ [\phi_t^{(1)}]^{-1} \), and \( \Delta^{(1)}_t = [\phi_t^{(1)} \circ \phi_t^{(1)}] \circ \Delta_t \circ [\phi_t^{(1)}]^{-1} \), and an equivalence \( \phi^{(1)}_t : H^{(1)}_t \sim H_t \). An easy calculation gives \( d^{(1)}_t = \mu^{(1)}_t = \Delta^{(1)}_t = 0 \). Inductively, given a deformation \( H^{(n)}_t = (H[[t]], d^{(n)}_t, \mu^{(n)}_t, \Delta^{(n)}_t) \) with \( d^{(n)}_t = \mu^{(n)}_t = \Delta^{(n)}_t = 0 \) for \( 1 \leq i \leq n \), and an equivalence \( \phi^{(n)}_t = \prod_{i=1}^{n} (1 - t^i \phi_t) : H^{(n)}_t \sim H_t \), choose \( \phi_{n+1} \in C^1 (H; H; 3) \) such that \( d^{(n+1)}_{C_t} = d^{(n+1)}_t + \mu^{(n+1)}_t + \Delta^{(n+1)}_t \). There is a deformation \( H^{(n+1)}_t = (H[[t]], d^{(n+1)}_t, \mu^{(n+1)}_t, \Delta^{(n+1)}_t) \) with \( d^{(n+1)}_t = \mu^{(n+1)}_t = \Delta^{(n+1)}_t = 0 \) for \( 1 \leq i \leq n+1 \), and an equivalence \( \phi^{(n+1)}_t = \prod_{i=1}^{n+1} (1 - t^i \phi_t) : H^{(n+1)}_t \sim H_t \). There is a sequence \( \{ \phi^{(n)}_t : H^{(n)}_t \sim H_t \}_{n \geq 1} \) that converges \( t \)-adically to an equivalence \( \phi^{(\infty)}_t = \prod_{i=1}^{\infty} (1 - t^i \phi_t) : H_0[[t]] \sim H_t \) and we conclude that:

Theorem 4. If \( H^2_{d.g.h.a.}(H; H; 3) = 0 \) then \( (H, d, \mu, \Delta) \) is rigid as a d.g.h.a.

Furthermore, since the obstructions to extending some finite approximation to a deformation lie in \( H^3_{d.g.h.a.}(H; H; 3) \) we have:

Theorem 5. If \( H^3_{d.g.h.a.}(H; H; 3) = 0 \) then every linear approximation extends to a deformation.

Thus \( H^2_{d.g.h.a.}(H; H; 3) \) directs the deformation theory of \( H_0 \) as a d.g.h.a. in the direction of the "infinitesimals", i.e., 2-cocycles. In the sequel, to which we refer the reader, Lazarev and Movshev\[17\] apply this particular theory to analyze the deformations of the de Rham cochains on a Lie group.

Finally, we note that the appropriate setting for the deformtion theory of Hopf algebras as quasi-Hopf algebras \( \overline{H} \) is \( \{ C^{0,m,n}(H; H), \partial_C, \delta_C \}_{m \geq 1; n \geq 0} \), i.e., the subcomplex of cochains in the "semi-restricted coordinate plane" \( p = 0, m \geq 1 \).
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