For $-\pi \leq \beta_1 < \beta_2 \leq \pi$, denote by $\Phi_{\beta_1, \beta_2}(Q)$ the amount of algebraic numbers of degree $2m$, elliptic height at most $Q$, and arguments in $[\beta_1, \beta_2]$, lying on the unit circle. It is proved that

$$\Phi_{\beta_1, \beta_2}(Q) = Q^{m+1} \int_{\beta_1}^{\beta_2} p(t) \, dt + O(Q^m \log Q), \quad Q \to \infty,$$

where $p(t)$ coincides up to a constant factor with density of the roots of a random trigonometrical polynomial. This density is calculated explicitly using the Edelman–Kostlan formula. Bibliography: 15 titles.

1. Introduction

The problem of distribution of real and complex algebraic numbers has been studied intensively in the last two decades and turned out to be closely related with distribution of roots of random polynomials. Before describing the problem considered in this paper, we first give a brief review of recent results in this area.

Let $\mathbb{A}$ denote the field of (all) algebraic numbers over $\mathbb{Q}$, and let $\mathbb{A}_n$ denote the set of algebraic numbers of degree $n \in \mathbb{N}$. Note that the set $\mathbb{A}_n$ is countable and any open subset of $\mathbb{R}$ or $\mathbb{C}$ contains infinitely many algebraic numbers. In order to study the distribution of these algebraic numbers, we need to choose finite (ordered) subsets of $\mathbb{A}_n$. To this end, we consider a height function $h : \mathbb{A} \to \mathbb{R}_+$ such that for any $n \in \mathbb{N}$ and $Q > 0$, there are only finitely many algebraic numbers $\alpha$ of degree $n$ with $h(\alpha) \leq Q$. Note that one usually requires (and we always assume) that $h(\alpha') = h(\alpha)$ for all conjugates of $\alpha$.

A natural question is to determine the asymptotics of the number of all $\alpha \in \mathbb{A}_n$ lying in a given subset of $\mathbb{R}$ or $\mathbb{C}$ and such that $h(\alpha) \leq Q$ and the degree $n$ is fixed as $Q \to \infty$. This problem has been studied by Masser and Vaaler in [11] with height function being the Mahler measure. Later, Kaliada determined in [9] the asymptotic number of real algebraic numbers ordered by the naive height; the same result for complex algebraic numbers was obtained in [7]. A generalization of the naive height; the weighted $l_p$-norm (which also generalizes the length, the Euclidean norm, and the Bombieri $p$-norm), was considered in [8]. Another interesting example of heights, the house of algebraic number, was studied in [2], where the distribution of the Perron numbers is considered.

In the present note we would like to study the distribution of algebraic numbers on unit circle. Although our methods work for any weighted $l_p$-norm (including the naive height), we consider the weighted Euclidean (or elliptic) height only: this case corresponds to the simplest asymptotic distribution formula.
The paper is organized as follows. In the next section, we formulate our main result Theorem 2.1 and give some corollaries. The proof of Theorem 2.1 is given in Sec. 3. Section 4 contains the proof of some auxiliary statements, and the corollaries are proved in Sec. 5.

2. Basic notation and main result

Given a polynomial \( P(t) := a_n t^n + \cdots + a_1 t + a_0 \) and a vector of positive weights \( \lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n) \), we define the elliptic height of \( P \) as

\[
h_\lambda(P) := \left( \sum_{k=0}^{n} \frac{a_k^2}{\lambda_k^2} \right)^{1/2}.
\]

Let \( \mathcal{P}(Q) \) denote the class of integral polynomials (that is, with integral coefficients) of degree \( n \) and with height \( h_\lambda \) at most \( Q \):

\[
\mathcal{P}(Q) := \{ P \in \mathbb{Z}[t] : \deg P = n, \ h_\lambda(P) \leq Q \}.
\]

We say that an integral polynomial is prime if it is irreducible over \( \mathbb{Q} \), primitive (with pairwise coprime coefficients), and its leading coefficient is positive. Denote by \( \mathcal{P}^*(Q) \) the class of prime polynomials from \( \mathcal{P}(Q) \),

\[
\mathcal{P}^*(Q) := \{ P \in \mathcal{P}(Q) : P \text{ is prime} \}.
\]

The minimal polynomial of an algebraic number \( \alpha \) is a uniquely defined prime polynomial \( P \in \mathbb{Z}[t] \) satisfying the equation \( P(\alpha) = 0 \). Then the elliptic height of \( \alpha \) is defined as the elliptic height of its minimal polynomial,

\[
h_\lambda(\alpha) := h_\lambda(P).
\]

The other roots of \( P \) are called algebraic conjugates of \( \alpha \).

We aim to investigate the asymptotic behavior of algebraic numbers lying on the unit circle \( \mathbb{T} \subset \mathbb{C} \),

\[
\mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \}.
\]

We start with a simple observation that any algebraic number on \( \mathbb{T} \) has even degree and the coefficients of its minimal polynomial posses some symmetry property. Indeed, consider an algebraic number \( \alpha \in \mathbb{T} \) with minimal polynomial

\[
P(t) = a_n t^n + \cdots + a_1 t + a_0.
\]

Since the coefficients of \( P \) are real, \( \bar{\alpha} \) is also a root of \( P \). Moreover, \( \alpha \in \mathbb{T} \) implies \( \bar{\alpha} = 1/\alpha \). Thus,

\[
P(\alpha) = P \left( \frac{1}{\alpha} \right) = 0,
\]

which means that \( \alpha \) is a root of the polynomial

\[
\tilde{P}(t) := t^n P(t^{-1}) = a_0 t^n + \cdots + a_{n-1} t + a_n.
\]

Hence, \( P \) is a multiple of \( \tilde{P} \) by definition of minimal polynomial. This leaves us with two possibilities only: \( P \equiv -\tilde{P} \) or \( P \equiv \tilde{P} \). The former one would imply that 1 is a root of \( P \), which is impossible due to its irreducibility. Therefore, \( P \equiv \tilde{P} \) which means that

\[
a_k = a_{n-k}, \quad 0 \leq k \leq n.
\]

For odd \( n \), this condition implies that \(-1\) is a root of \( P \) which, again, contradicts its irreducibility. Thus from now on we consider only algebraic numbers of even degree

\[
n = 2m, \quad m \geq 1.
\]

The number \( m \) will stay throughout the paper.
Now we are ready to formulate our main result. For
\[-\pi \leq \beta_1 < \beta_2 \leq \pi,\]
we denote by $\Phi_{\beta_1, \beta_2}(Q)$ the number of algebraic numbers of degree $n = 2m$ on the unit circle with arguments in $[\beta_1, \beta_2]$ and elliptic height at most $Q$:
\[
\Phi_{\beta_1, \beta_2}(Q) := \# \left\{ \theta \in [\beta_1, \beta_2] : e^{i\theta} \in B_{2m}, h_\lambda(e^{i\theta}) \leq Q \right\}.
\]
Denote by $\zeta(\cdot)$ the Riemann zeta function and by $B^{m+1}$ the $(m+1)$-dimensional unit ball of volume $\text{Vol}(B^{m+1})$.

**Theorem 2.1.** For any fixed symmetric vector of positive weights $\lambda = (\lambda_0, \ldots, \lambda_m, \ldots, \lambda_0)$, we have
\[
\Phi_{\beta_1, \beta_2}(Q) = \frac{\text{Vol}(B^{m+1})}{2^{m/2+1} \pi \zeta(m+1)} \lambda_0 \cdots \lambda_m Q^{m+1} \int_{\beta_1}^{\beta_2} p_\lambda(t) \, dt + O(Q^m \log Q), \quad Q \to \infty,
\]
where
\[
p_\lambda(t) := \left[ \frac{\partial^2}{\partial x \partial y} \log \left( \frac{\lambda_m^2}{2} + \sum_{k=1}^{m} \lambda_{m-k} \cos(kx) \cos(ky) \right) \bigg|_{x=y=t} \right]^{1/2}. \tag{2}
\]
The implicit constant depends on $\lambda$ and $m$ only and the log $Q$ factor can be omitted for $m \geq 3$.

The limit density $p_\lambda$ is difficult to analyze in this general form. However, there are some examples of $\lambda$, where (2) can be simplified. The first one is the vector of the Bombieri 2-norm weights.

**Corollary 2.2.** For $\lambda_k = \sqrt{\binom{2m}{k}}, k = 0, \ldots, 2m$, we have
\[
p_\lambda(t) = \sqrt{\frac{m}{2}} \cdot \frac{|\sin t| \sqrt{\sum_{k=0}^{2m-2} (\cos t)^{2k} + (2m - 1)(\cos t)^{2m-2}}}{(\cos t)^{2m} + 1}.
\]
The second example is the Euclidean height.

**Corollary 2.3.** For $\lambda = (1, \ldots, 1)$, we have
\[
p_\lambda(t) = \left( b_m + \frac{\sin(b_m t)}{\sin t} \right)^{-1} \cdot \left( \frac{b_m \sin(b_m t)}{2(\sin t)^3} - \frac{b_m^2 \cos(b_m t) \cos t}{2(\sin t)^2} + \frac{(\sin(b_m t))^2}{4(\sin t)^4} - \frac{b_m^3 + 2b_m \sin(b_m t)}{6} \sin t \right) - \frac{b_m^2}{4(\sin t)^2} + \frac{(m^2 + m)b_m^2}{3})^{1/2},
\]
where $b_m = 2m + 1$.

As was mentioned above, the results about distribution of the roots of random algebraic polynomials were used in [7, 8] to study the asymptotic behavior of algebraic numbers in a domain of the complex plane. In the present paper, we show that the distribution of algebraic numbers on the unit circle can be described in terms of random *trigonometrical* polynomials. Namely, one can easily see from the proof of Theorem 2.1 that
\[
\mathbb{E} \mu_{\lambda}(\beta) = \frac{1}{\pi} \int_{\beta_1}^{\beta_2} \rho_\lambda(t) \, dt,
\]
where
\[ F(\theta) := \frac{\lambda_m}{2} \eta_m + \frac{1}{\sqrt{2}} \sum_{k=1}^{m} \lambda_{m-k} \eta_{m-k} \cos(k\theta) \]
is a random trigonometrical polynomial with coefficients \( \eta_0, \ldots, \eta_m \) being i.i.d. real-valued standard Gaussian random variables and \( \mu_F([\beta_1, \beta_2]) \) denotes the number of the real roots of \( F \) lying in the interval \([\beta_1, \beta_2]\). Thus, up to the factor \( \pi \), the limit density \( p_\lambda \) coincides with density of the roots of \( F \).

3. PROOF OF THEOREM 2.1

As argued in Sec. 2 we can restrict our attention to the following subclass of symmetric polynomials of even degree \( n = 2m \):

\[ \mathcal{P}_{\text{sym}}(Q) := \left\{ P \in \mathcal{P}(Q) : P(t) = \sum_{k=0}^{2m} a_k t^k, a_k = a_{2m-k} \right\}. \]

Moreover, let \( \mathcal{P}_{\text{sym}}^*(Q) \) denote the subclass of prime polynomials from \( \mathcal{P}_{\text{sym}}(Q) \):

\[ \mathcal{P}_{\text{sym}}^*(Q) := \mathcal{P}_{\text{sym}}(Q) \cap \mathcal{P}^*(Q). \]

Given a function \( g : \mathbb{C} \to \mathbb{C} \) and a subset \( B \subset \mathbb{C} \), denote by \( \mu_g(B) \) the number of the roots of \( g \) lying in \( B \). For \( -\pi \leq \beta_1 < \beta_2 \leq \pi \), we denote by \( T_{\beta_1, \beta_2} \) the arc

\[ T_{\beta_1, \beta_2} := \{ z \in T : \arg z \in [\beta_1, \beta_2] \}. \]
Since $\mathcal{P}_{\text{sym}}^*(Q)$ is the set of all minimal polynomials for algebraic numbers $\alpha \in \mathbb{T}$ with $\deg \alpha = 2m$ and $h_\lambda(\alpha) \leq Q$, we have

$$
\Phi_{\beta_1, \beta_2}(Q) = \sum_{P \in \mathcal{P}_{\text{sym}}^*(Q)} \mu_P(\mathbb{T}_{\beta_1, \beta_2})
$$

and since $\mu_P(\mathbb{T}_{\beta_1, \beta_2}) \leq 2m$, we can write

$$
\Phi_{\beta_1, \beta_2}(Q) = \sum_{l=0}^{2m} l \# \{ P \in \mathcal{P}_{\text{sym}}^*(Q) : \mu_P(\mathbb{T}_{\beta_1, \beta_2}) = l \}. \tag{3}
$$

Thus, our aim is to estimate the number of prime symmetric polynomials having a prescribed number of roots in a given set. Since these polynomials can be identified with the vectors of their coefficients, our first step is to find a way of counting integral points in multidimensional regions.

For a Borel set $A \subset \mathbb{R}^d$, we denote by $\gamma(A)$ (respectively, $\gamma^*(A)$) the number of points in $A$ with integral (respectively, coprime integral) coordinates. For a real number $r$, we define the dilated set

$$
rA := \{ r\mathbf{x} : \mathbf{x} \in A \}.
$$

For our purposes we would like to know the asymptotic behavior of the quantity $\gamma^*(QA)$ as $Q \to \infty$. In order to get a good estimate, one should impose some regularity conditions on the boundary $\partial A$ of $A$. According to [15, Definition 2.2], we say that $\partial A \subset \mathbb{R}^d$ is of Lipschitz class $(M, L)$ if there exist $M$ maps $\phi_1, \ldots, \phi_M : [0, 1]^{d-1} \to \mathbb{R}^d$ satisfying the Lipschitz condition

$$
|\phi_i(x) - \phi_i(y)| \leq L|x - y| \quad \text{for} \ x, y \in [0, 1]^{d-1}, \quad i = 1, \ldots, M,
$$

and such that $\partial A$ is covered by the images of the maps $\phi_i$.

The following lemma provides the asymptotics of $\gamma^*(QA)$ as $Q \to \infty$.
Lemma 3.1. Let $A \subset \mathbb{R}^d$, $d \geq 2$, be a bounded region such that the boundary $\partial A$ of $A$ is of Lipschitz class $(M, L)$. Then $A$ is Lebesgue measurable and

$$\left| \frac{\chi^*(Q A)}{Q^d} - \frac{\text{Vol}(A)}{\zeta(d)} \right| \leq C \frac{\log^\chi(d) Q}{Q^d},$$

where $C$ depends on $d, L, M$ only and

$$\chi(d) := \begin{cases} 1 & d = 2, \\ 0 & \text{otherwise}. \end{cases}$$  \hspace{1cm} (4)$$

Proof. See, e.g., [8, Lemma 6.3].

For $l = 0, 1, \ldots, 2m$, we denote by $A_l \subset \mathbb{R}^{m+1}$ the set of points $(a_0, \ldots, a_m)$ such that the polynomial $P(t) = a_0 t^{2m} + \ldots + a_m t^m + \ldots + a_0$ is such that $\mu_P(T_{\beta_1, \beta_2}) = l$ and $h_\lambda(P) \leq 1$. The latter condition is equivalent to $(a_0, \ldots, a_m) \in E_\lambda$, where $E_\lambda \subset \mathbb{R}^{m+1}$ is the ellipsoid defined by

$$E_\lambda := \left\{(a_0, \ldots, a_m) \in \mathbb{R}^{m+1} : a_0^2 \frac{\lambda_0}{\lambda_m} + 2 \sum_{k=0}^{m-1} \frac{a_k^2}{\lambda_k^2} \leq 1 \right\}.$$ We note that

$$\text{Vol}(E_\lambda) = 2^{-m/2} \lambda_0 \ldots \lambda_m \text{Vol}(\mathbb{R}^{m+1}).$$

Then by definition of a primitive polynomial, we have

$$\gamma^*(Q A_l) = \# \{ P \in \mathcal{P}_{\text{sym}}(Q) : P \text{ is primitive}, \mu_P(T_{\beta_1, \beta_2}) = l \},$$

which implies

$$\left| \frac{1}{2} \gamma^*(Q A_l) - \# \{ P \in \mathcal{P}_{\text{sym}}^*(Q) : \mu_P(T_{\beta_1, \beta_2}) = l \} \right| \leq r(Q).$$ \hspace{1cm} (5)$$

Here, $r(Q)$ denotes the number of the polynomials reducible in $\mathcal{P}_{\text{sym}}(Q)$ (i.e., can be represented as a product of two symmetric integral polynomials of positive degree). The factor $1/2$ in (5) comes from the positiveness of the leading coefficient of a prime polynomial.

Thus, our next step is to estimate $\gamma^*(Q A_l)$ and $r(Q)$. The latter is established in the following lemma.

Lemma 3.2. For some $C > 0$ depending on $m$ and $\lambda$ only, we have

$$r(Q) \leq C \cdot Q^m \log^\chi(m) Q,$$

where $\chi(\cdot)$ is defined in (4).

The proof of Lemma 3.2 is postponed to Sec. 4. In order to estimate $\gamma^*(Q A_l)$, we are going to apply Lemma 3.1. Before we do this, we need to verify that the boundary of $A_l$ is of Lipschitz class.

Lemma 3.3. For each $0 \leq l \leq m$, there exist constants $M, L$ depending on $m, \lambda$ only and such that the boundary $\partial A_l$ is of Lipschitz class $(M, L)$.

This lemma is a slightly modified and simplified version of [8, Lemma 6.4]. For the reader’s convenience, we give a detailed proof of Lemma 3.3 in Sec. 4.

Now taking Lemma 3.3 into account, we can apply Lemma 3.1 to the set $A_l$. This together with (5) and Lemma 3.2, gives

$$\# \{ P \in \mathcal{P}_{\text{sym}}^*(Q) : \mu_P(T_{\beta_1, \beta_2}) = l \} = \frac{\text{Vol}(A_l)}{2 \zeta(m+1)} Q^{m+1} + O \left( Q^m \log^\chi(m) Q \right).$$
In view of (3), we conclude that
\[ \Phi_{\beta_1, \beta_2}(Q) = \frac{Q^{m+1}}{2\zeta(m+1)} \sum_{l=0}^{2m} l \, \text{Vol}(A_l) + O\left( Q^m \log^\chi(m) \right). \tag{6} \]

To compute the sum on the right-hand side of (6), we consider the random polynomial

\[ G(z) := \sum_{k=0}^{m-1} \xi_k (z^k + z^{2m-k}) + \xi_m z^m, \]

where the random vector
\[ \left( \frac{\xi_0}{\lambda_0/\sqrt{2}}, \ldots, \frac{\xi_{m-1}}{\lambda_{m-1}/\sqrt{2}}, \frac{\xi_m}{\lambda_m} \right) \tag{7} \]
is uniformly distributed over the \((m+1)\)-dimensional unit ball \( \mathbb{B}^{m+1} \). Then, the definition of \( A_l \) and the fact that the semi-axes of \( \mathcal{E}_\lambda \) are \( \lambda_0/\sqrt{2}, \ldots, \lambda_{m-1}/\sqrt{2}, \lambda_m \), imply that
\[ \mathbb{P} [ \mu_G (\mathbb{T}_{\beta_1, \beta_2}) = l] = \frac{\text{Vol}(A_l)}{\text{Vol} (\mathcal{E}_\lambda)}. \tag{8} \]

Taking \( z = e^{i\theta} \in \mathbb{T} \) leads to
\[ G(z) = \sum_{k=0}^{m-1} \xi_k (e^{ik\theta} + e^{i(2m-k)\theta}) + \xi_m e^{im\theta} = 2e^{im\theta} \left( \sum_{k=0}^{m-1} \xi_k \frac{e^{-i(m-k)\theta} + e^{i(m-k)\theta}}{2} + \frac{\xi_m}{2} \right) \]
\[ = 2e^{im\theta} \left( \sum_{k=1}^{m} \xi_{m-k} \cos (k\theta) + \frac{\xi_m}{2} \right) =: 2e^{im\theta} \widehat{F}(\theta). \]

Therefore,
\[ \mathbb{P} [ \mu_G (\mathbb{T}_{\beta_1, \beta_2}) = l] = \mathbb{P} [ \mu_\widehat{F} ([\beta_1, \beta_2]) = l]. \tag{9} \]

The latter probability is still difficult to calculate, because of the dependency of the coefficients of \( \widehat{F} \). However, by proper normalization, which does not affect the roots, we can achieve independence.

Namely, let \( \eta_0, \ldots, \eta_m \) be i.i.d. real-valued standard Gaussian random variables, and let \( Z \) be a standard exponential random variable. It is known (see, e.g., [6, Chap. 2]) that the random vector
\[ \left( \frac{\eta_0, \eta_1, \ldots, \eta_m}{\left( \sum_{k=0}^{m} \eta_k^2 + Z \right)^{1/2}} \right) \]
is uniformly distributed in the unit ball \( \mathbb{B}^{m+1} \), and hence has the same distribution as (7). Thus,
\[ \left( \frac{\lambda_0 \eta_0}{\sqrt{2}}, \ldots, \frac{\lambda_{m-1} \eta_{m-1}}{\sqrt{2}}, \frac{\lambda_m \eta_m}{\sqrt{2}} \right) \sim (\xi_0, \ldots, \xi_m). \]

Using the fact that dividing a polynomial by a nonzero constant does not affect its roots, we conclude that the polynomials \( \widehat{F}(\theta) \) and
\[ F(\theta) := \frac{\lambda_m}{2} \eta_m + \frac{1}{\sqrt{2}} \sum_{k=1}^{m} \lambda_{m-k} \eta_{m-k} \cos (k\theta) \]
have the same distribution of the roots and
\[ \mathbb{P} [ \mu_{\widehat{F}} ([\beta_1, \beta_2]) = l] = \mathbb{P} [ \mu_F ([\beta_1, \beta_2]) = l]. \]
Combining this with (8) and (9), we obtain
\[
\sum_{l=0}^{2m} l \text{Vol}(A_l) = \text{Vol}(E_{\lambda}) \sum_{l=0}^{2m} l P[\mu_F ([\beta_1, \beta_2]) = l] = \text{Vol}(E_{\lambda}) E[\mu_F ([\beta_1, \beta_2])].
\] (10)

Finally, applying the Edelman–Kostlan formula (see [5, Theorem 3.1]) to the random function \(F\) leads to
\[
E[\mu_F ([\beta_1, \beta_2])] = \frac{1}{\pi} \int_{\beta_1}^{\beta_2} p_\lambda(t) \, dt,
\]
where
\[
p_\lambda(t) = \left. \frac{\partial^2}{\partial x \partial y} \log \left( \frac{\lambda^2}{2} + \sum_{k=1}^{m} \lambda_{m-k}^2 \cos(kx) \cos(ky) \right) \right|_{x=y=t}^{1/2}.
\]

Together with (6) and (10), this completes the proof.

4. Proofs of Lemma 3.2 and Lemma 3.3

4.1. Proof of Lemma 3.2. The proof follows a scheme described in [13].

For nonnegative functions \(f, g\) we write \(f \ll g\) if there exists a nonnegative constant \(C\) depending on \(m, \lambda\) only and such that \(f \leq C g\). Given a polynomial \(P(t) = a_n t^n + \cdots + a_1 t + a_0\), denote by \(H(P)\) its naive height: \(H(P) := \max_{0 \leq i \leq n} |a_i|\). Our first step is to prove the lemma for the naive height instead of \(h_\lambda\).

Let \(\tilde{r}(Q)\) be the number of symmetric integral polynomials of degree \(2m\) and naive height at most \(Q\), which can be represented as the product of two symmetric polynomials of positive degrees. Denote by \(s(Q)\) the number of pairs \((P_1, P_2)\) of symmetric integral polynomials such that \(\deg P_1 + \deg P_2 = 2m\) and
\[
H(P_1)H(P_2) \leq Q.
\]
It is easy to see that the number of symmetric integral polynomials of degree \(2k\) and naive height \(Q\) is of order \(O(Q^k)\). Hence,
\[
s(Q) \ll \sum_{k=1}^{m-1} \sum_{x, y \in \mathbb{Z}, x, y \geq 1, xy \leq Q} x^k y^{m-k} \ll Q^m \log \chi(m) Q.
\]
The proof of the second inequality is given in [13, eq. (3.2)].

It is known (see, e.g., [14, Theorem 4.2.2]) that if \(P_1\) and \(P_2\) are integral polynomials of degrees \(n_1\) and \(n_2\), respectively, then
\[
(2^{n_1+n_2-2} \sqrt{n_1+n_2+1})^{-1} H(P_1)H(P_2) \leq H(P_1 P_2),
\]
which implies that
\[
\tilde{r}(Q) \leq \sum_{n_1+n_2=2m} s \left(2^{2m-2} (2m+1)^{1/2} Q\right) \ll Q^m \log \chi(m) Q.
\]
Now to complete the proof, it suffices to note that
\[
H(P) \ll h_\lambda(P) \ll H(P).
\]
4.2. Proof of Lemma 3.3. Recall that $A_l \subset \mathbb{R}^{m+1}$ is a set of points $(a_0, \ldots, a_m) \in E_\lambda$ such that the polynomial

$$P(z) = a_0 z^2 + \cdots + a_m z^m + \cdots + a_0$$

satisfies $\mu_P(T_{\beta_1, \beta_2}) = l$. For $z = e^{i\theta}$, we have

$$P(z) = 2e^{im\theta} \left( \sum_{k=1}^{m} a_{m-k} \cos(k\theta) + \frac{a_m}{2} \right) = 2e^{im\theta} \tilde{P}(\theta).$$

Hence, $A_l$ is a set of points $(a_0, \ldots, a_m) \in E_\lambda$ such that $\mu_{\tilde{P}}([\beta_1, \beta_2]) = l$.

It is easy to see that the boundary of $A_l$ is contained in the union of the following three sets:

1. the boundary of $E_\lambda$;
2. the set
   $$A' = \left\{ (a_0, \ldots, a_m) \in E_\lambda : \tilde{P}(\beta_1) = 0 \text{ or } \tilde{P}(\beta_2) = 0 \right\};$$
3. the set $A''$ consisting of the points $(a_0, \ldots, a_m)$ such that the trigonometrical polynomial $\tilde{P}$ has double real roots in $[\beta_1, \beta_2]$.

Thus it suffices to show that each of these sets is of Lipschitz class.

(i) The boundary of $E_\lambda$. Since $E_\lambda$ is a convex body, its boundary belongs to the Lipschitz class in accordance with [15, Theorem 2.6].

(ii) The set $A'$. Without loss of generality we may assume that $P(\beta_1) = 0$, which is equivalent to

$$a_m = -2 \sum_{k=1}^{m} a_{m-k} \cos(k\beta_1).$$

Since $(a_0, \ldots, a_m) \in E_\lambda$, we have $a_0, \ldots, a_{m-1} \leq C := \max_i \lambda_i$. Consider a Lipschitz map $\phi = (\phi_0, \ldots, \phi_m) : [0,1]^m \to \mathbb{R}^{m+1}$ defined as follows:

$$\phi_i(t_0, \ldots, t_{m-1}) := Ct_i, \quad i = 0, \ldots, m-1,$$

and

$$\phi_m(t_0, \ldots, t_{m-1}) := -2C \sum_{k=1}^{m} t_{m-k} \cos(k\beta_1).$$

We obviously have

$$a_i = \phi_i(a_0/C, \ldots, a_{m-1}/C), \quad i = 0, \ldots, m-1,$$

which implies $A' \subset \phi([0,1]^m)$. Therefore, $A'$ is of Lipschitz class.

(iii) The set $A''$. Assume that $(a_0, \ldots, a_m) \in A''$. Then

$$\tilde{P}(\theta) = \sum_{k=1}^{m} a_{m-k} \cos(k\theta) + \frac{a_m}{2}$$

has a multiple real root, say $\beta_0$, which implies that

$$\tilde{P}(\beta_0) = 0, \quad \tilde{P}'(\beta_0) = 0,$$

or, excluding the trivial case $\beta_0 = 0$, equivalently,

$$a_{m-1} = -\sum_{k=2}^{m} k a_{m-k} \frac{\sin(k\beta_0)}{\sin \beta_0},$$

$$a_m = -2 \sum_{k=2}^{m} a_{m-k} \cos(k\beta_0) + 2 \cos \beta_0 \sum_{k=2}^{m} k a_{m-k} \frac{\sin(k\beta_0)}{\sin \beta_0}.$$
Again, $a_0, \ldots, a_{m-1} \leq C := \max \lambda_i$. Moreover, we have $|\beta_0| \leq \pi$ (see (1)). Consider a map $\phi = (\phi_0, \ldots, \phi_m) : [0,1]^m \to \mathbb{R}^{m+1}$ defined as follows:

\[
\phi_i(t, t_0, \ldots, t_{m-2}) := Ct_i, \quad i = 0, \ldots, m-2, \\
\phi_{m-1}(t, t_0, \ldots, t_{m-2}) := -C \sum_{k=2}^{m} k t_{m-k} \sin(k \pi t) / \sin(\pi t),
\]

and

\[
\phi_m(t, t_0, \ldots, t_{m-2}) := -2C \sum_{k=2}^{m} t_{m-k} \cos(k \pi t) + 2C \cos(\pi t) \sum_{k=2}^{m} k t_{m-k} \sin(k \pi t) / \sin(\pi t).
\]

Since $\phi$ is continuously differentiable in a compact, it satisfies the Lipschitz condition. We obviously have

\[
a_i = \phi_i(\beta_0/\pi, a_0/C, \ldots, a_{m-2}/C), \quad i = 0, \ldots, m,
\]

which implies $A'' \subset \phi([0,1]^m)$. Therefore, $A''$ is of Lipschitz class.

5. PROOFS OF COROLLARIES

Consider the kernel

\[
K_\lambda(x, y) = \frac{\lambda^2_m}{2} + \sum_{k=1}^{m} \lambda^2_{m-k} \cos(kx) \cos(ky).
\]

We obviously have

\[
p_\lambda(t) = \left[ \frac{\partial^2}{\partial x \partial y} \log K_\lambda(x, y) \right]_{x=y=t}^{1/2}
\]

\[
= \left[ K_\lambda(t, t) \cdot \frac{\partial^2}{\partial x \partial y} K_\lambda(x, y) \bigg|_{x=y=t} - \frac{\partial}{\partial x} K_\lambda(x, t) \bigg|_{x=t} \cdot \frac{\partial}{\partial y} K_\lambda(t, y) \bigg|_{y=t} \right]^{1/2}. \tag{12}
\]

5.1. Proof of Corollary 2.2. In this case the kernel looks as follows:

\[
K_\lambda(x, y) = \frac{1}{2} \left( \begin{array}{c} 2m \\ m \end{array} \right) + \sum_{k=1}^{m} \left( \begin{array}{c} 2m \\ m-k \end{array} \right) \cos(kx) \cos(ky)
\]

\[
= \frac{1}{2} \left( \begin{array}{c} 2m \\ m \end{array} \right) + \sum_{k=1}^{m} \left( \begin{array}{c} 2m \\ m-k \end{array} \right) \frac{e^{-ikx} + e^{ikx}}{2} \cdot \frac{e^{-iky} + e^{iky}}{2}
\]

\[
= e^{-im(x+y)} / 4 \left( \sum_{k=0}^{2m} \left( \begin{array}{c} 2m \\ k \end{array} \right) e^{ik(x+y)} + \sum_{k=0}^{2m} \left( \begin{array}{c} 2m \\ k \end{array} \right) e^{iky} e^{i(2m-k)x} \right)
\]

\[
= e^{-im(x+y)} / 4 \left( (1 + e^{i(x+y)})^{2m} + (e^{iy} + e^{ix})^{2m} \right)
\]

\[
= \frac{e^{-im(x+y)}}{4} \tilde{K}_\lambda(x, y).
\]
Since the factor $e^{-im(x+y)}/4$ does not affect the result, we obtain

$$p_\lambda(t) = \left[ \frac{\partial^2}{\partial x \partial y} \log K_\lambda(x, y) \right]_{x=y=t}^{1/2} = \left[ \frac{\partial^2}{\partial x \partial y} \log \tilde{K}_\lambda(x, y) \right]_{x=y=t}^{1/2}$$

$$= \left[ \tilde{K}_\lambda(t, t) \cdot \frac{\partial^2}{\partial x \partial y} \tilde{K}_\lambda(x, y) \bigg|_{x=y=t} - \frac{\partial}{\partial x} \tilde{K}_\lambda(x, t) \bigg|_{x=t} - \frac{\partial}{\partial y} \tilde{K}_\lambda(t, y) \bigg|_{y=t} \right]^{1/2} \cdot \frac{\partial \tilde{K}_\lambda(x, y)}{\partial K^2(t, t)}.$$

The task now is to find the partial derivatives of $\tilde{K}_\lambda(x, y)$ at $x = y = t$. We have

$$\tilde{K}_\lambda(t, t) = (1 + e^{2it})^{2m} + 2m e^{2imt} = 2^m e^{2imt} ((\cos t)^{2m} + 1),$$

$$\left. \frac{\partial}{\partial x} \tilde{K}_\lambda(x, t) \right|_{x=t} = \left. \frac{\partial}{\partial y} \tilde{K}_\lambda(t, y) \right|_{y=t} = 2im e^{2it} (1 + e^{2it})^{2m-1} + 2m 2^{m-1} e^{2imt},$$

and

$$\left. \frac{\partial^2}{\partial x \partial y} \tilde{K}_\lambda(x, y) \right|_{x=y=t} = -2m e^{2it} (1 + e^{2it})^{2m-1} - 2m(2m - 1)e^{4it} (1 + e^{2it})^{2m-2} - 2m(2m - 1)2^{2m-2} e^{2imt}$$

$$= -2m^2 2^{2m-2} e^{2imt} \left( 2e^{it} (\cos t)^{2m-1} + (2m - 1)e^{2it} (\cos t)^{2m-2} + (2m - 1) \right).$$

Therefore,

$$p_\lambda(t) = \sqrt{\frac{m}{2}} \left( (\cos t)^{2m} + 1 \right)^{-1} \left( e^{2it} (\cos t)^{4m-2} - 2e^{it} (\cos t)^{4m-1} - (2m - 1)(\cos t)^{2m} - (2m - 1) e^{2it} (\cos t)^{2m-2} + (2m - 1) e^{it} (\cos t)^{2m-1} + 1 \right)^{1/2}.$$  

Using the identities

$$e^{it} = \cos t + i \sin t,$$

$$e^{2it} = \cos(2t) + i \sin(2t) = 2(\cos t)^2 - 1 + 2i \sin t \cos t,$$

we conclude with

$$p_\lambda(t) = \sqrt{\frac{m}{2}} \left( 1 - (\cos^2 t)^{2m-1} + (2m - 1)(\cos t)^{2m-2} (1 - \cos^2 t) \right)^{1/2}$$

$$= \sqrt{\frac{m}{2}} \cdot \sqrt{1 - \cos^2 t} \left( \sum_{k=0}^{2m-2} (\cos t)^{2k} + (2m - 1)(\cos t)^{2m-2} \right)^{1/2}$$

$$= \sqrt{\frac{m}{2}} \cdot \sin t \left( \sum_{k=0}^{2m-2} (\cos t)^{2k} + (2m - 1)(\cos t)^{2m-2} \right)^{1/2}.$$
5.2. Proof of Corollary 2.3. First, we recall some trigonometrical formulas to be used in our calculations:

\[
\sin \left( (N + \frac{1}{2})(x - y) \right) = 1 + 2 \sum_{k=1}^{N} \cos(k(x - y));
\]
\[
\sin \left( \frac{N(x - y)}{2} \right) = \sum_{k \text{ odd}} (-1)^{(k-1)/2} \binom{N}{k} \left( \cos \frac{x - y}{2} \right)^{N-k} \left( \sin \frac{x - y}{2} \right)^{k};
\]
\[
\cos \left( \frac{N(x - y)}{2} \right) = \sum_{k \text{ even}} (-1)^{k/2} \binom{N}{k} \left( \cos \frac{x - y}{2} \right)^{N-k} \left( \sin \frac{x - y}{2} \right)^{k}.
\]

Consider the function \( p_\lambda(t) \) with weights \( \lambda = (1, \ldots, 1) \). In this case the kernel has the form

\[
K_\lambda(x, y) = \frac{1}{2} + \sum_{k=1}^{m} \cos(kx) \cos(ky) = \frac{1}{2} + \frac{1}{2} \sum_{k=1}^{m} \cos k(x + y) + \frac{1}{2} \sum_{k=1}^{m} \cos k(x - y)
\]

\[
= \sin \left( (m + \frac{1}{2})(x + y) \right) + \sin \left( (m + \frac{1}{2})(x - y) \right). \quad \frac{4 \sin \frac{x+y}{2}}{2} + \frac{4 \sin \frac{x-y}{2}}{2}
\]

In order to compute \( p_\lambda(t) \), we find the partial derivatives of the kernel \( K_{\lambda,m}(x, y) \) at \( x = y = t \).

Recall that \( b_m = 2m + 1 \). Thus we have

\[
K_{\lambda}(t, t) = \frac{\sin(b_m t)}{4 \sin t} + \frac{\sin \left( (m + \frac{1}{2})(x - y) \right)}{4 \sin \frac{x-y}{2}} \Big|_{x=y=t} = \frac{b_m}{4} + \frac{\sin(b_m t)}{4 \sin t},
\]

\[
\frac{\partial}{\partial x} K_{\lambda,m}(x, t) \Big|_{x=t} = \frac{\partial}{\partial y} K_{\lambda,m}(t, y) \Big|_{y=t} = \left( \frac{m \cos \frac{b_m(x-y)}{2}}{4 \sin \frac{x-y}{2}} - \frac{\sin \frac{2m(x-y)}{2}}{8 \sin^2 \frac{x-y}{2}} \right) \Big|_{x=y=t}
\]

\[+\frac{m \cos(b_m t)}{4 \sin t} - \frac{\sin(2mt)}{8 \sin^2 t} = \frac{m}{4 \sin \frac{x-y}{2}} \Big|_{x=y=t} - \frac{m}{4 \sin \frac{x-y}{2}} \big|_{x=y=t}
\]

\[+\frac{m \cos(b_m t)}{4 \sin t} - \frac{\sin(2mt)}{8 \sin^2 t} = \frac{m \cos(b_m t)}{4 \sin t} - \frac{\sin(2mt)}{8 \sin^2 t},
\]

and

\[
\frac{\partial^2}{\partial x \partial y} K_{\lambda,m}(x, y) \big|_{x=y=t} = \left( \frac{m^2 \sin \frac{b_m(x-y)}{2}}{4 \sin \frac{x-y}{2}} + \frac{m \cos \frac{2m(x-y)}{2}}{4 \sin^2 \frac{x-y}{2}} - \frac{\sin \frac{2m(x-y)}{2}}{8 \sin^3 \frac{x-y}{2}} \right) \big|_{x=y=t}
\]

\[= \frac{-m^2 \sin(b_m t)}{4 \sin t} - \frac{-m \cos(2mt)}{4 \sin^2 t} - \frac{\cos t \sin(2mt)}{8 \sin^3 t}
\]

\[= \frac{m^2 b_m}{4} + \frac{m}{4 \sin^2 \frac{x-y}{2}} \big|_{x=y=t} - \frac{m^2(2m-1)}{4} - \frac{m}{4 \sin^2 \frac{x-y}{2}} \big|_{x=y=t}
\]

\[+ \frac{m(2m-1)(m-1)}{2} - \frac{m \cos(b_m t)}{4 \sin t} - \frac{m \cos(2mt)}{4 \sin^2 t} + \frac{\cos t \sin(2mt)}{8 \sin^3 t}
\]

\[= \frac{m(2m-1)(m-1)}{8 \sin^3 t} - \frac{m^2 \sin(b_m t)}{4 \sin t} - \frac{m \cos(2mt)}{4 \sin^2 t} + \frac{m^2 + m b_m}{12}.
\]

Substituting this into (12) completes the proof.

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