THE ANOMALY FLOW AND THE FU-YAU EQUATION

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Abstract

The Anomaly flow is shown to converge on toric fibrations with the Fu-Yau ansatz, for both positive and negative values of the slope parameter $\alpha'$. This implies both results of Fu and Yau on the existence of solutions for Hull-Strominger systems, which they proved using different methods depending on the sign of $\alpha'$. It is also the first case where the Anomaly flow can even be shown to exist for all time. This is in itself remarkable from the point of view of the theory of fully nonlinear partial differential equations, as the elliptic terms in the flow are not concave.

1 Introduction

The Hull-Strominger system [25, 36] is a system of equations for supersymmetric compactifications of the heterotic string, which is less restrictive than the Ricci-flat and Kähler conditions originally proposed by Candelas, Horowitz, Strominger, and Witten [6]. More specifically, let $Y$ be a compact 3-fold with a nowhere vanishing holomorphic $(3,0)$-form $\Omega_Y$ and a vector bundle $E \to Y$. Then the Hull-Strominger system is the following system of equations for a Hermitian metric $\chi$ on $Y$ and a Hermitian metric $H$ on $E$,

$$F^{2,0} = F^{0,2} = 0, \quad \chi^2 \wedge F^{1,1} = 0 \quad (1.1)$$

$$i\partial \bar{\partial} \chi - \frac{\alpha'}{4} \text{Tr}(Rm \wedge Rm - F \wedge F) = 0 \quad (1.2)$$

$$d(\|\Omega_Y\|_{\chi}^2) = 0. \quad (1.3)$$

Here $\alpha'$ is a constant parameter called the slope, $\|\Omega_Y\|_{\chi}$ is the norm of $\Omega_Y$ with respect to $\chi$ defined by $\|\Omega_Y\|_{\chi}^2 = i\Omega_Y \wedge \Omega_Y \chi^{-3}$, and $Rm$ and $F$ are respectively the curvatures of the Chern unitary connections of $\chi$ and $H$, viewed as $(1,1)$-forms valued in the bundles of endomorphisms of $T^{1,0}(Y)$ and $E^2$. For fixed $\chi$, the first equation (1.1) is just the well-known Hermitian-Yang-Mills equation. The essential novelty in the Hull-Strominger system, from the point of view of both non-Kähler geometry and non-linear partial differential equations, resides rather in the last two equations (1.2) and (1.3). The equation (1.3) says that the

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2The equation (1.3) was originally written in [25, 36] as $d^\chi = i(\partial - \bar{\partial}) \log \|\Omega\|_{\chi}$. That (1.3) is an equivalent formulation is an important insight due to J. Li and S.T. Yau [29].
metric $\chi$ is not required to be Kähler, but conformally balanced, and the equation (1.2) defines a new curvature condition, which differs markedly from more familiar conditions such as Einstein since it is quadratic in the curvature tensor. By now, many special solutions of the Hull-Strominger system have been found, both in the physics (see e.g. [3, 5, 11]) and the mathematics literature (see e.g. [1, 2, 12, 14, 15, 16, 17, 18, 19, 22, 29, 30, 42]). The main goal in the present paper is rather to develop PDE techniques towards an eventual general solution.

A first major difficulty in the Hull-Strominger system, symptomatic of non-Kähler geometry, is to implement the conformally balanced condition (1.3) in the absence of an analogue of the $\partial \bar{\partial}$-lemma. While balanced metrics can be produced by many Ansätze, none seems more natural than the others, and all lead to unwieldy expressions for the equation (1.2). This is why it was proposed by the authors in [31] to bypass completely the choice of an Ansatz by considering instead the flow

$$H^{-1} \partial_t H = -\Lambda_\chi F(H)$$

$$\partial_t (\parallel Y \parallel_{\chi^2}) = i \partial \bar{\partial} \chi - \frac{\alpha'}{4} \text{Tr}(Rm(\chi) \wedge Rm(\chi) - F(H) \wedge F(H))$$

starting from a metric $H(0)$ on $E$, and a metric $\chi(0)$ on $Y$ which is conformally balanced. Here $\Lambda_\chi \psi = \chi^2 \wedge \psi \chi^{-3}$ is the Hodge operator on $(1, 1)$-forms $\psi$. The point of this flow is that, by Chern-Weil theory, the right hand side of (1.5) is a closed $(2, 2)$-form, so if the initial metric $\chi(0)$ is conformally balanced, then the metric $\chi(t)$ will remain conformally balanced for all $t$. It suffices then to determine whether the flow exists for all time and converges. Flows of the form (1.4) have been called Anomaly flows in [31], in recognition of the fact that the equation (1.2) originates from the famous Green-Schwarz anomaly cancellation mechanism [23] required for the consistency of superstring theories.

The flow (1.4) has been shown in [31] to be weakly parabolic when $|\alpha' Rm(\chi)|$ is small, which implies its short-time existence. However, for any given elliptic system, there are many parabolic flows with it as stationary point, so the true test of whether a particular parabolic flow is the right one is its long-time existence and convergence. In the general theory of fully non-linear PDE’s, it is customary to select the parabolic flow by some desirable properties, such as, in the case of scalar equations, concavity in the second derivatives [8, 24, 28]. In the present case, there is no further flexibility, as the particular flow (1.4) provides the only known way of implementing the conformally balanced condition (1.3) without appealing to any particular Ansatz. In effect, the geometric constraint of the metric being conformally balanced has excluded the customary desirable analytic properties for flows, and it is vital at this stage to develop some new analytic tools. For this, we shall consider the Anomaly flow on the model case of Calabi-Eckmann fibrations. This case retains enough features of the general case to provide a valuable guide in the future (see §2.4 below). It is also the case where J.X. Fu and S.T. Yau [18, 19] found, by a particularly difficult and delicate analysis, the first non-perturbative, non-Kähler solution
of the Hull-Strominger system. The precise $C^0$ estimate that they obtained has a great influence on the present work. On the other hand, we shall see that the Anomaly flow requires very different $C^1$, $C^2$, and $C^{2,\alpha}$ estimates, which actually sharpen the elliptic estimates in many ways. In particular, they define a new region, narrower than the space of positive Hermitian forms, which is preserved by the flow and where the unknown metric ultimately belongs. We now describe more precisely our results.

Let $(X, \hat{\omega})$ be a Calabi-Yau surface, equipped with a nowhere vanishing holomorphic $(2, 0)$-form $\Omega$ normalized to satisfy $\|\Omega\|_{\hat{\omega}} = 1$ and two harmonic forms $\omega_1, \omega_2 \in H^{1,1}(X, \mathbb{Z})$. Building on an earlier construction of Calabi and Eckmann [4], Goldstein and Prokushkin [21] have shown how to associate to this data a toric fibration $\pi : Y \to X$, equipped with a $(1, 0)$-form $\theta$ with the property that $\Omega_Y = \Omega \wedge \theta$ is holomorphic and non-vanishing on $Y$, and $\omega_u = \pi^*(e^u \hat{\omega}) + i\theta \wedge \bar{\theta}$ is a conformally balanced metric on $Y$ for all $u \in C^\infty(X)$ (see §2.1 for more details). We shall prove

**Theorem 1** Let $\pi : Y \to X$ be a Goldstein-Prokushkin fibration over a Calabi-Yau surface $(X, \hat{\omega})$ with Ricci-flat metric $\hat{\omega}$ as above, and $E_X \to (X, \hat{\omega})$ a stable vector bundle with zero slope and Hermitian-Yang-Mills metric $H_X$. Assume that the cohomological condition $\int_X \mu = 0$ is satisfied, where $\mu$ is defined in (2.18) below. Set $E = \pi^*(E_X)$ and consider the Anomaly flow (1.4, 1.5) on $(Y, E)$, with initial data $H(0) = \pi^*(H_X)$, and $\chi(0) = \pi^*(M \hat{\omega}) + i\theta \wedge \bar{\theta}$, where $M$ is a positive constant. Then there exists $M_0 > 1$ such that, for all $M \geq M_0$, the flow exists for all time, and converges smoothly to a solution $(\chi_\infty, \pi^*(H_X))$ of the Hull-Strominger system for $(Y, E)$.

In particular, the theorem recaptures at one stroke the results of Fu and Yau in [18, 19], where they proved the existence of solutions of the Hull-Strominger system on Goldstein-Prokushkin fibrations satisfying the cohomological condition $\int_X \mu = 0$, when $\alpha' > 0$ and $\alpha' < 0$ respectively.

Restricted to a Goldstein-Prokushkin fibration, the Anomaly flow becomes equivalent to the following flow for a metric $\omega = ig_{k\bar{j}}dz^j \wedge d\bar{z}^k$ on a Calabi-Yau surface $X$, equipped with a nowhere vanishing holomorphic $(2, 0)$-form $\Omega$,

$$
\partial_t \omega = -\frac{1}{2\|\Omega\|_{\hat{\omega}}} \left( \frac{R}{2} - |T|^2 - \frac{\alpha'}{4} \sigma_2(i\text{Ric}_\omega) + 2\alpha' i\partial \bar{\partial}(\|\Omega\|_{\hat{\omega}}^4) - 2 \frac{\mu}{\omega^2} \right) \omega \tag{1.6}
$$

where $\sigma_2(\Phi) = \Phi \wedge \Phi \omega^{-2}$ is the usual determinant of a real $(1, 1)$-form $\Phi$, relative to the metric $\omega$. The expression $|T|^2$ is the norm of the torsion of $\omega$ defined in (2.33) below. Thus the theorem that we shall actually prove is the following

**Theorem 2** Let $(X, \hat{\omega})$ be a Calabi-Yau surface, equipped with a Ricci-flat metric $\hat{\omega}$ and a nowhere vanishing holomorphic $(2, 0)$-form $\Omega$ normalized by $\|\Omega\|_{\hat{\omega}} = 1$. Let $\alpha'$ be a non-zero real number, and let $\rho$ and $\mu$ be smooth real $(1, 1)$ and $(2, 2)$-forms respectively, with
\( \mu \) satisfying the integrability condition

\[ \int_X \mu = 0. \] (1.7)

Consider the flow (1.6), with an initial metric given by \( \omega(0) = M \hat{\omega} \), where \( M \) is a constant. Then there exists \( M_0 \) large enough so that, for all \( M \geq M_0 \), the flow (1.6) exists for all time, and converges exponentially fast to a metric \( \omega_\infty \) satisfying the Fu-Yau equation

\[ i\partial \bar{\partial}(\omega_\infty - \alpha'\|\Omega\|\omega_\infty \rho) - \frac{\alpha'}{8} \text{Ric}_{\omega_\infty} \wedge \text{Ric}_{\omega_\infty} + \mu = 0, \] (1.8)

and the normalization \( \int_X \|\Omega\|\omega_\infty \frac{\omega_\infty^2}{2} = M \).

Rewriting the flow as a flow of the conformal factor \( u \), we can discuss now more precisely its features in the context of the general theory of non-linear parabolic PDE’s. The flow (1.6) can be expressed as

\[ \partial_t u = \frac{1}{2} \left( \Delta u + \alpha' e^{-u} \hat{\sigma}_2 (i\partial \bar{\partial} u) - 2\alpha' e^{-u} \frac{i\partial \bar{\partial} (e^{-u} \rho)}{\hat{\omega}^2} + |Du|^2 + e^{-u} \tilde{\mu} \right) \] (1.9)

where \( \tilde{\mu} = 2\mu \hat{\omega}^{-2} \) is a time-independent scalar function, and both the Laplacian \( \Delta \) and the determinant \( \hat{\sigma}_2 \) are written with respect to the fixed metric \( \hat{\omega} \). Setting the right hand side to 0 gives the equation solved by Fu and Yau [18, 19], so the Anomaly flow is indeed a parabolic version of the Fu-Yau equation. Moreover, the equation (1.9) can be rewritten in the form

\[ 2\alpha' e^u \partial_t u = \frac{\left( e^u \omega + e^{-u} \rho + \alpha' i\partial \bar{\partial} u \right)^2}{\omega^2} + w(\mu, \rho, u, Du), \] (1.10)

and it is a parabolic complex Monge-Ampère type equation. However, unlike the Kähler-Ricci flow where the elliptic term is \( \log \det (g_{ij} + u_{ij}) \), the equation (1.10) has none of the desirable concavity properties of elliptic and parabolic equations. In particular, none of the techniques used in [18, 19] for the elliptic case (as well as the ones for the more general equations in [33, 34]) can be adapted here besides the Moser iteration technique for the \( C^0 \) estimate. We shall see below that the proof of Theorem 2 relies instead, in an essential manner, on the geometric formulation of (1.6), and that the estimates are obtained using the metric which evolves with the flow.

Various geometric flows have been studied in non-Kähler complex geometry (see e.g. [9, 20, 37, 38, 39, 40, 41] and references therein). The main difficulty in studying (1.6) is that it is quadratic in the Ricci curvature. This creates substantial problems in applying known techniques to try and obtain estimates on the torsion, curvature, and derivatives of curvature. To overcome these issues, we first start the flow with a metric with vanishing
Ricci curvature and torsion, and the objective is to show that for suitably large normalization of the initial metric, we can prevent the terms which are nonlinear in curvature from growing too large and dominating the behavior of the flow. Proposition 4 in §7 shows exactly how the estimates on Ricci curvature and torsion depend on the normalization of the initial metric. More precisely, the metric $\omega(t)$ will be proved to belong at all times in the following set

$$C^{-1} M \hat{\omega} \leq \omega(t) \leq CM \hat{\omega}, \quad |T(\omega(t))| \leq CM^{-1/2}, \quad |\alpha' Ric_{\omega(t)}| \leq CM^{-1/2}$$

(1.11)

for a constant $C$ depending only on the geometric data defining the Goldstein-Prokushkin fibration, the Hermitian-Yang-Mills metric $H_X$, and the slope $\alpha'$, but independent of the normalization constant $M$. In particular, it is a set much narrower and more specific than the cone of metrics on $X$.

The normalization of the initial metric actually sets a scale for the problem, and Proposition 4 is an example of estimates with scales. The key underlying strategy of the present paper can be viewed as the use of estimates with scales to tame higher powers of the curvature tensor. This strategy appears both flexible and powerful: since the initial posting online in 2016 of the original version of the present paper, we have applied it successfully in our paper “Fu-Yau Hessian equations”, arXiv:1801.09842, in fact with the same test functions used in the proof of Proposition 4, to solve the Fu-Yau equation and its Hessian generalizations in all dimensions, for both signs of the parameter $\alpha'$. See also the paper “The Fu-Yau equation in higher dimensions” by J. Chu, L. Huang, and X.H. Zhu, arXiv:1801.09351.

The paper is organized as follows. In Section §2, we provide the background on Goldstein-Prokushkin fibrations, and show how to reduce the Anomaly flow in this case to the flow (1.6) for metrics on a Calabi-Yau surface. Sections §3-§6 are devoted to successive estimates: the uniform boundedness of the metrics $\omega$ in §3, the estimates for the torsion in Section §4, the estimates for the curvature, and higher order derivatives of both the torsion and the curvature in Sections §5-§6. Finally, long time existence is shown in Section §7 and the convergence of the flow is proved in Section §8.

### 2 Anomaly flows on Goldstein-Prokushkin fibrations

#### 2.1 The Goldstein-Prokushkin fibration

We would like to restrict the Anomaly flow from a general 3-fold $Y$ to the special case of a Goldstein-Prokushkin fibration $\pi : Y \to X$. We begin by recalling the basic properties of Goldstein-Prokushkin fibrations that we need.
Let \((X, \hat{\omega})\) be a compact Calabi-Yau manifold of dimension 2, with \(\hat{\omega}\) a Ricci-flat Kähler metric, and \(\Omega\) a nowhere vanishing holomorphic \((2, 0)\)-form, normalized so that
\[
1 = \|\Omega\|_\hat{\omega}^2 = \Omega \wedge \overline{\Omega} \hat{\omega}^{-2}. \tag{2.1}
\]
Let \(\omega_1, \omega_2 \in 2\pi H^2(X, \mathbb{Z})\) be two \((1, 1)\)-forms such that \(\omega_1 \wedge \hat{\omega} = \omega_2 \wedge \hat{\omega} = 0\). From this data, Goldstein and Prokushkin [21] construct a compact 3-fold \(Y\) which is a toric fibration \(\pi: Y \to X\) over \(X\) equipped with a \((1, 0)\) form \(\theta\) on \(Y\) satisfying
\[
\overline{\partial} \theta = \pi^*(\omega_1 + i\omega_2) \quad \partial \theta = 0. \tag{2.2}
\]
Furthermore, the \((3, 0)\)-form
\[
\Omega_Y = \sqrt{3} \Omega \wedge \theta \tag{2.3}
\]
is holomorphic and nowhere vanishing, and the \((1, 1)\)-form
\[
\chi_0 = \pi^*(\hat{\omega}) + i\theta \wedge \bar{\theta} \tag{2.4}
\]
is positive-definite on \(Y\). Observe that
\[
i\Omega_Y \wedge \overline{\Omega_Y} = 3\Omega \wedge \overline{\Omega} \wedge i\theta \wedge \bar{\theta} = \|\Omega\|_\omega^2(3\hat{\omega}^2 \wedge i\theta \wedge \bar{\theta}) = \|\Omega\|_\omega^2 \chi_0^3. \tag{2.5}
\]
Thus, defining the norm \(\|\Omega_Y\|_\chi\) of the holomorphic form \(\Omega_Y\) on \(Y\) with respect to a metric \(\chi\) as \(\|\Omega_Y\|_\chi^2 = i\Omega_Y \wedge \overline{\Omega_Y} \wedge \chi^{-3}\), we have \(\|\Omega_Y\|_{\chi_0} = \|\Omega\|_\omega = 1\). Consequently,
\[
\|\Omega_Y\|_{\chi_0} \chi_0^2 = \hat{\omega}^2 + 2i\hat{\omega} \wedge \theta \wedge \bar{\theta}. \tag{2.6}
\]
This implies that \(d(\|\Omega_Y\|_{\chi_0} \chi_0^2) = 0\) by (2.2) and the fact that \(\omega_1\) and \(\omega_2\) wedged with \(\hat{\omega}\) gives zero. Thus \(\chi_0\) is a conformally balanced metric on \(Y\).

More generally, for any smooth function \(u\) on \(Y\), introduce the following metrics \(\omega_u\) and \(\chi_u\) on the manifolds \(X\) and \(Y\) respectively,
\[
\omega_u = e^u \hat{\omega}, \quad \chi_u = \pi^*(e^u \hat{\omega}) + i\theta \wedge \bar{\theta}. \tag{2.7}
\]
Then the same arguments that we just used show that
\[
\|\Omega_Y\|_{\chi_u} = \|\Omega\|_{\omega_u} = e^{-u}, \tag{2.8}
\]
and furthermore,
\[
\|\Omega_Y\|_{\chi_u} \chi_u^2 = \|\Omega\|_{\omega_u} \omega_u^2 + 2i\hat{\omega} \wedge \theta \wedge \bar{\theta}. \tag{2.9}
\]
This shows that \(d(\|\Omega_Y\|_{\chi_u} \chi_u^2) = 0\) since \(d(\|\Omega\|_{\omega_u} \omega_u^2)\) is the pull-back of a differential form of rank 5 defined on the 4-dimensional manifold \(X\), and \(2id(\hat{\omega} \wedge \theta \wedge \bar{\theta})\) is zero as before in view of (2.2). It follows that the metric \(\chi_u\) is a conformally balanced metric on \(Y\) for any choice of \(u\).
2.2 The Fu-Yau Ansatz

In [18, 19], Fu and Yau obtain a solution of the Hull-Strominger system in the following manner. Let $\pi : Y \to X$ be a Goldstein-Prokushkin fibration, constructed as described above from a Calabi-Yau surface $(X, \hat{\omega})$, equipped with two integer-valued harmonic $(1, 1)$-forms $\omega_1/(2\pi)$ and $\omega_2/(2\pi)$.

Let $E_X \to X$ be a stable holomorphic vector bundle over $X$, with slope $\int_X c_1(E_X) \wedge \hat{\omega} = 0$. Then by the Donaldson-Uhlenbeck-Yau [10, 43] theorem, $E_X$ admits a metric $H_X$ with respect to $\hat{\omega}$ satisfying the Hermitian-Yang-Mills equation $\hat{\omega} \wedge F(H_X) = 0$. Let $E = \pi^*(E_X) \to Y$ be the pull-back bundle over $Y$, and let $H = \pi^*(H_X)$. Since

$$\chi_u^2 \wedge F(H) = \pi^*(e^u \hat{\omega} \wedge F(H_X)) \wedge (e^u \hat{\omega} + 2i\theta \wedge \bar{\theta}) = 0,$$

it follows that $H$ is Hermitian-Yang-Mills with respect to $\chi_u$, for any $u$. Now recall that $\chi_u$ is conformally balanced for any $u$. This means that, if we look for a solution of the Strominger system under the form $(Y, E)$, equipped with the metrics $\chi_u$ and $H$, then the only equation which is left to solve is the Green-Schwarz anomaly cancellation equation (1.2), with the scalar function $u$ defining the metric $\chi_u$ as the unknown.

The key property that allows this approach to work is that, for metrics of the form $\chi_u$ in a Goldstein-Prokushkin fibration, the equation on $Y$

$$i\bar{\partial}\partial \chi_u - \frac{\alpha'}{4} \text{Tr}(Rm(\chi_u) \wedge Rm(\chi_u) - F \wedge F) = 0$$

(descends to an equation on the base $X$). This was established by Fu and Yau [18] and a summary of their results is as follows.

First, the term $i\bar{\partial}\partial \chi_u$ is readily worked out, using the properties (2.2) of the form $\theta$,

$$i\partial\bar{\partial} \chi_u = i\partial\bar{\partial} \omega_u - \partial \theta \wedge \partial \bar{\theta}.$$

Next, the quadratic term in the curvature tensor can be worked out to be (Proposition 8 in [18])

$$\text{Tr}(Rm(\chi_u) \wedge Rm(\chi_u)) = \text{Tr}(Rm(\omega_u) \wedge Rm(\omega_u)) + \partial\bar{\partial}(\|\omega_u \text{Tr}(\partial B \wedge \partial B^* \cdot \hat{\omega}^-))^2.$$

Here $B$ is a $(1, 0)$-form depending on the data $(\hat{\omega}, \omega_1, \omega_2)$, which is only locally defined. However the full expression $\text{Tr}(\partial B \wedge \partial B^* \cdot \hat{\omega}^-)$ is not only globally well-defined on $Y$, but it is the pull-back of a globally defined real $(1, 1)$-form $\rho$ on $X$,

$$\frac{1}{4} \text{Tr}(\partial B \wedge \partial B^* \cdot \hat{\omega}^-) = \pi^*(\rho).$$

On the other hand, from $\omega_u = e^u \hat{\omega}$, it follows that

$$Rm(\omega_u) = -\partial\bar{\partial} u \otimes I + Rm(\hat{\omega})$$

(2.15)
and hence, in view of the fact that the metric $\hat{\omega}$ is Ricci-flat,

$$\text{Tr}(Rm(\omega_u) \wedge Rm(\omega_u)) = \text{Tr}(Rm(\hat{\omega}) \wedge Rm(\hat{\omega})) + 2\partial\bar{\partial}u \wedge \partial\bar{\partial}u. \quad (2.16)$$

Altogether, the Green-Schwarz anomaly cancellation equation can be written as the following equation on the base manifold $X$,

$$0 = i\partial\bar{\partial}(\omega_u - \alpha'\|\Omega\|\omega_u^\rho) - \frac{\alpha'}{2}(\partial\bar{\partial}u) \wedge (\partial\bar{\partial}u) + \mu \quad (2.17)$$

where we have set

$$\mu = -\bar{\theta} \wedge \partial\bar{\theta} - \frac{\alpha'}{4}\text{Tr}(Rm(\hat{\omega}) \wedge Rm(\hat{\omega})) + \frac{\alpha'}{4}\text{Tr}(F(H_X) \wedge F(H_X)). \quad (2.18)$$

which is a well-defined $(2,2)$-form on $X$. The equation (2.17) is the Fu-Yau equation. Clearly, a necessary condition for the existence of solutions is

$$\int_X \mu = 0. \quad (2.19)$$

This condition was shown to be sufficient by Fu and Yau in [18] for $\alpha' > 0$ and in [19] for $\alpha' < 0$. Examples of fibrations $\pi : Y \rightarrow X$ and vector bundles $E \rightarrow X$ satisfying $\int_X \mu = 0$ are exhibited in [18, 19].

### 2.3 Reduction of the Anomaly flow by the Fu-Yau Ansatz

We consider now the Anomaly flow (1.5) on a Goldstein-Prokushkin fibration $\pi : Y \rightarrow X$, equipped with the holomorphic $(3,0)$-form $\Omega_Y$ and restricted to metrics of the form $\chi = \chi_u$. Recall that $F = F(\pi^*(H_X))$ is fixed, and that $\Lambda_\chi F = 0$. Thus the metric $H_X$ remains fixed along the flow, and we need only concentrate on the flow for the metric $\chi_u$.

We work out both sides of the flow (1.5) in this setting. Recall that we have set $\omega_u = e^{au}\hat{\omega}$ which is a metric on $X$. From the earlier equation (2.9) and the fact that the term $\hat{\omega} \wedge \theta \wedge \bar{\theta}$ is time-independent, it follows at once that

$$\partial_t(||\Omega_Y||_{\chi_u}^2) = \partial_t(||\Omega||_{\omega_u}^2). \quad (2.20)$$

On the other hand, the same formulas derived by Fu and Yau [18] for their reduction of the anomaly equation on $Y$ to an equation on $X$ and which we described in the previous section give

$$i\partial\bar{\partial}\chi_u = -\frac{\alpha'}{4}\text{Tr}(Rm(\chi_u) \wedge Rm(\chi_u) - F \wedge F)$$

$$= i\partial\bar{\partial}(\omega_u - \alpha'\|\Omega\|\omega_u^\rho) - \frac{\alpha'}{2}\partial\bar{\partial}u \wedge \partial\bar{\partial}u + \mu \quad (2.21)$$
with $\mu$ the $(2, 2)$-form defined by (2.18). Thus the Anomaly flow for Goldstein-Prokushkin fibrations is equivalent to the flow for metrics on $X$ given by
\[
\partial_t(\|\Omega\|_\omega \omega^2) = i\partial\bar{\partial}(\omega - \alpha'\|\Omega\|_\omega \rho) - \frac{\alpha'}{2}\partial\bar{\partial}u \wedge \partial\bar{\partial}u + \mu.
\] (2.22)

Since we wish to apply techniques of geometric flows, it is useful to re-express the flow entirely in terms of curvature. If we denote by $\text{Ric}_{\omega}$ the Chern-Ricci tensor of $\omega$, we have
\[
\text{Ric}_{\omega} = -2 \partial\bar{\partial}u
\] (2.23)
since the metric $\hat{\omega}$ is Ricci-flat. Thus the Anomaly flow can be rewritten as the following flow of metrics on $X$,
\[
\partial_t(\|\Omega\|_\omega \omega^2) = i\partial\bar{\partial}(\omega - \alpha'\|\Omega\|_\omega \rho) - \frac{\alpha'}{8}\text{Ric}_{\omega} \wedge \text{Ric}_{\omega} + \mu
\] (2.24)
which we can take now as our starting point. Here we have suppressed the subindex $u$ in $\omega_u$.

A technical issue in Anomaly flows is that they are formulated in terms of flows for $\|\Omega\|_\omega \omega^2$, and not of $\omega$ itself. This issue was addressed in all generality in [32] in dimension 3. For the above Anomaly flow on the surface $X$ arising from the Goldstein-Prokushkin fibration, the metric $\omega$ is already characterized by its volume form, and we can proceed more directly as follows.

First, using the two-dimensional identity $2\partial_t\omega \wedge \omega = (\partial_t \log \omega^2)\omega^2$ and the fact that $\omega^2 = \|\Omega\|^{-2}\hat{\omega}^2$, we can rewrite the left hand side as
\[
\partial_t(\|\Omega\|_\omega \omega^2) = \|\Omega\|_\omega (\partial_t \log \|\Omega\|_\omega \omega^2 + 2\partial_t \omega \wedge \omega) = -\|\Omega\|_\omega (\partial_t \log \|\Omega\|_\omega)\omega^2.
\] (2.25)

Next, we work out the right hand side more explicitly. Following [32], we define the torsion $T(\omega) = \frac{1}{2}T_{kpq} \, dz^p \wedge dz^q \wedge dz^k$ of a Hermitian metric $\omega$ by
\[
T = i\partial\omega, \quad \bar{T} = -i\bar{\partial}\omega,
\] (2.26)
and we also introduce the $(1, 0)$-form $T_m$ and the $(0, 1)$-form $\bar{T}_m$ by
\[
T_m = g^{jk}T_{kjm}, \quad \bar{T}_m = g^{jk}\bar{T}_{kjm}.
\] (2.27)

Then
\[
(i\partial\bar{\partial}\omega)_{kj\ell} = R_{kj\ell} - R_{k\ell m}f_j + R_{\ell km}f_j - R_{\ell kjm} + g^{sk}T_{mj}\bar{T}_{sk\ell}.
\] (2.28)

In general, there are several notions of Ricci curvature for Hermitian metrics, given by
\[
R_{kj} = R_{kj}^p p, \quad \bar{R}_{kj} = R_{kj}^p \bar{p}, \quad R_{kj}' = R_{kj}^p p, \quad R_{kj}'' = R_{kj}^p \bar{p}.
\] (2.29)
For metrics of the form $\omega = e^u \omega_0$, where $\omega_0$ is Kähler and Ricci-flat, the following important relations between torsion and curvature hold,

$$ T_q(\omega) = \partial_q \log \|\Omega\|_\omega, \quad \bar{T}_q(\omega) = \partial_{\bar{q}} \log \|\Omega\|_\omega $$

and

$$ R_{kj}(\omega) = 2\nabla_k T_j(\omega) = 2\nabla_j \bar{T}_k(\omega), \\
R'_{kj}(\omega) = R''_{kj}(\omega) = \frac{1}{2} R_{kj}(\omega). \quad (2.31) $$

Also, because $\omega_0$ is Kähler, we have

$$ T(\omega) = i \partial u \wedge \omega $$

so that the $(1,0)$-form $T_m$ actually determines in our case the full $(2,1)$ torsion tensor $T(\omega)$. Henceforth, unless explicitly indicated otherwise, we shall designate by $T$ the $(1,0)$-form $T_m dz^m$ rather than the $(2,1)$-form $i \partial \omega$. For example, the norm $|T|^2$ will designate the expression

$$ |T|^2 = g^{m\bar{\ell}} T_m \bar{T}_{\bar{\ell}} $$

rather than $|i \partial \omega|^2$ (which can be verified to be equal to $2|T|^2$).

Using these relations, and the fact that we are in dimension 2, we find

$$ i \partial \bar{\partial} \omega = \frac{1}{2} (-R + 2|T|^2) \omega^2. \quad (2.34) $$

Substituting this equation and (2.25) in the flow (2.24), we obtain

$$ \partial_t \log \|\Omega\|_\omega = \frac{1}{\|\Omega\|_\omega} \left( \frac{R}{2} - |T|^2 + 2\alpha' i \bar{\partial} (\|\Omega\|_\omega^\rho) \omega^2 + \frac{\alpha'}{4} \sigma_2(i \text{Ric}_\omega) - \|\Omega\|_\omega^2 \mu \right), \quad (2.35) $$

where we have introduced the time-independent, scalar function $\tilde{\mu}$ by $\mu = \tilde{\mu} \omega^2$, and the $\sigma_2$ operator with respect to the evolving metric

$$ 2\sigma_2(i \text{Ric}_\omega) \omega^2 = i \text{Ric}_\omega \wedge i \text{Ric}_\omega. \quad (2.36) $$

Since the metric $\omega = e^u \omega_0$ is entirely determined by the conformal factor $e^u$, this flow for the volume form is equivalent to the flow of metrics (1.6) quoted in the Introduction. The flow in terms of the conformal factor $u$ is easily worked out to be given by the equation (1.9).
2.4 Comparisons between the 3-dimensional Anomaly flow and its 2-dimensional reduction

It may be noteworthy that the flow (2.24) retains many of the features of the original Anomaly flow in 3-dimensions. Indeed, as shown in [32], the conformally balanced condition (1.3) in dimension 3 implies that the Hermitian metric $\chi$ on the 3-fold $Y$ satisfies exactly the same relations (2.30) and (2.3) between torsion and curvature as the metric $\omega = e^u \hat{\omega}$ on the surface $X$,

$$T_q(\chi) = \partial_q \log \|\Omega_Y\|_{\chi}, \quad \bar{T}_\bar{q}(\chi) = \partial_{\bar{q}} \log \|\Omega_Y\|_{\chi}$$

(2.37)

and

$$R_{kj}(\chi) = 2\nabla_k T_j(\chi) = 2\nabla_j \bar{T}_k(\chi),$$

$$R'_{kj}(\chi) = R''_{kj}(\chi) = \frac{1}{2} R_{kj}(\chi).$$

(2.38)

This suggests that the flow (1.6) is interesting not just as a special case of the general Anomaly flow, but also as a good model for developing general methods for studying the flow.

2.5 Starting the Flow

In [31] general conditions were given for the short-time existence of the Anomaly flow, using the Nash-Moser implicit function theorem. However, the short-time existence of the flow can be seen more directly from the parabolicity of the flow, which holds when the form

$$\omega' = e^u \hat{\omega} + \alpha' e^{-u} \rho + \alpha' i\partial\bar{\partial}u > 0,$$

(2.39)

is positive definite. This can be seen from the scalar equation (1.9). We will always assume that we start the flow from a large constant multiple of the background metric

$$u(x, 0) = \log M \gg 1, \quad \omega(0) = e^{u(0)} \hat{\omega} = M \hat{\omega}.$$  

(2.40)

Recall that $\mu$ is defined in (2.18). In all that follows, we will assume that the cohomological condition

$$\int_X \mu = 0,$$

(2.41)

is satisfied. Integrating (2.22) and using the fact that $\|\Omega\|_{\omega^2} = e^u \hat{\omega}^2$ gives the following conservation law

$$\frac{\partial}{\partial t} \int_X e^u \hat{\omega}^2 = 0.$$

(2.42)

Hence

$$\int_X e^u \hat{\omega}^2 = M,$$

(2.43)

along the flow.
3 The $C^0$ estimate of the conformal factor

In this section, we will work with equation (2.22), since it will be easier to work with differential forms to obtain integral estimates. We let $\hat{\omega}$ denote the fixed background Kähler form of $X$. We can rescale $\hat{\omega}$ such that $\int_X \hat{\omega}^2 = 1$. We will omit the background volume form $\hat{\omega}^2$ when integrating scalar functions. All norms in the current section will be taken with respect to the background metric $\hat{\omega}$. The starting point for the Moser iteration argument is to compute the quantity

$$\int_X i\partial\bar{\partial}(e^{-ku}) \wedge \omega',$$

in two different ways. Recall that $\omega'$ is defined in (2.39). On one hand, by the definition of $\omega'$ and Stokes’ theorem, we have

$$\int_X i\partial\bar{\partial}(e^{-ku}) \wedge \omega' = \int_X \{e^u \hat{\omega} + \alpha' e^{-u} \rho\} \wedge i\partial\bar{\partial}(e^{-ku}).$$

Expanding

$$\int_X i\partial\bar{\partial}(e^{-ku}) \wedge \omega' = k^2 \int_X e^{-ku} \{e^u \hat{\omega} + \alpha' e^{-u} \rho\} \wedge i\partial\bar{\partial}(e^{-ku}) - k \int_X e^{-ku} \{e^u \hat{\omega} + \alpha' e^{-u} \rho\} \wedge i\partial\bar{\partial}u. \quad (3.3)$$

On the other hand, without using Stokes’ theorem, we obtain

$$\int_X i\partial\bar{\partial}(e^{-ku}) \wedge \omega' = k^2 \int_X e^{-ku} \{e^u \hat{\omega} + \alpha' e^{-u} \rho\} \wedge i\partial\bar{\partial}(e^{-ku}) - \alpha' k \int_X e^{-ku} i\partial\bar{\partial}u \wedge i\partial\bar{\partial}u. \quad (3.4)$$

We equate (3.3) and (3.4)

$$0 = -k^2 \int_X e^{-ku} i\partial\bar{\partial}u \wedge \partial u \wedge \omega' + k^2 \int_X e^{-ku} \{e^u \hat{\omega} + \alpha' e^{-u} \rho\} \wedge i\partial\bar{\partial}u \wedge i\partial\bar{\partial}u.$$

Using equation (2.22) and that $\|\Omega\|_{\omega^2} = e^u \hat{\omega}^2$,

$$0 = -k^2 \int_X e^{-ku} i\partial\bar{\partial}u \wedge \partial u \wedge \omega' + k^2 \int_X e^{-ku} \{e^u \hat{\omega} + \alpha' e^{-u} \rho\} \wedge i\partial\bar{\partial}u \wedge \bar{\partial}u - 2k \int_X e^{-ku} \mu.$$
$$- \int_X e^{-(k-1)u} i \partial \bar{\partial} u \wedge \omega - \int_X e^{-(k-1)u} i \partial \bar{\partial} u \wedge \omega - \alpha' \int_X e^{-(k+1)u} i \partial \bar{\partial} u \wedge \rho$$

$$+ \alpha' \int_X e^{-(k+1)u} i \partial \bar{\partial} u \wedge \rho + \alpha' \int_X e^{-(k+1)u} i \partial \bar{\partial} \rho$$

$$- 2\alpha' \text{Re} \int_X e^{-(k+1)u} i \partial \bar{\partial} u \wedge \bar{\partial} \rho + 2 \int_X e^{-(k-1)u} \partial_t \bar{\partial} u \wedge 2 \hat{\omega}^2. \quad (3.7)$$

Integration by parts gives

$$0 = -\frac{k}{2} \int_X e^{-ku} i \partial \bar{\partial} u \wedge \bar{\partial} u \wedge \omega' - \frac{k}{2} \int_X e^{-ku} \{ e^{u \omega} + \alpha' \rho \} \wedge i \partial \bar{\partial} u \wedge \omega' - \int_X e^{-ku} \mu + \alpha' \int_X e^{-(k+1)u} i \partial \bar{\partial} \rho - \alpha' \text{Re} \int_X e^{-(k+1)u} i \partial \bar{\partial} u \wedge \bar{\partial} u$$

$$+ 2 \int_X e^{-(k-1)u} \partial_t \bar{\partial} u \wedge 2 \hat{\omega}^2. \quad (3.8)$$

One more integration by parts yields the following identity:

$$\frac{k}{2} \int_X e^{-ku} \{ e^{u \omega} + \alpha' \rho \} \wedge i \partial \bar{\partial} u \wedge \bar{\partial} u + \frac{\partial}{\partial t} \frac{2}{k-1} \int_X e^{-(k-1)u} \hat{\omega}^2 2! \quad (3.9)$$

$$= -\frac{k}{2} \int_X e^{-ku} i \partial \bar{\partial} u \wedge \bar{\partial} u \wedge \omega' - \int_X e^{-ku} \mu + (\alpha' - \frac{k}{k+1}) \int_X e^{-(k+1)u} i \partial \bar{\partial} \rho. \quad (3.10)$$

The identity (3.9) will be useful later to control the infimum of $u$, but to control the supremum of $u$, we replace $k$ with $-k$ in (3.9). Then, for $k \neq 1$,

$$\frac{k}{2} \int_X e^{(k+1)u} \{ \omega + \alpha' e^{-2u} \partial \rho \} \wedge i \partial \bar{\partial} u \wedge \bar{\partial} u + \frac{\partial}{\partial t} \frac{2}{k+1} \int_X e^{(k+1)u} \hat{\omega}^2 2! \quad (3.11)$$

$$= -\frac{k}{2} \int_X e^{ku} i \partial \bar{\partial} u \wedge \bar{\partial} u \wedge \omega' + \int_X e^{ku} \mu - (\alpha' - \frac{k}{1-k}) \int_X e^{(k+1)u} i \partial \bar{\partial} \rho. \quad (3.12)$$

### 3.1 Estimating the supremum

**Proposition 1** Start the flow with initial data $e^{u(x,0)} = M$. Suppose the flow exists for $t \in [0,T)$ with $T > 0$, and that $\inf_X e^u \geq 1$ and $\alpha' e^{-2u} \rho \geq -\frac{1}{2} \hat{\omega}$ for all time $t \in [0,T)$. Then

$$\sup_{X \times [0,T]} e^u \leq C_1 M, \quad (3.11)$$

where $C_1$ only depends on $(X, \hat{\omega}), \rho, \mu, \alpha'$.  

**Proof:** As long as the flow exists, we have

$$i \partial \bar{\partial} u \wedge \omega' \geq 0. \quad (3.12)$$
Let \( \beta = \frac{n}{n - 1} = 2 \). We can use (3.12), (3.10), and \( \alpha' e^{-2u} \rho \geq -\frac{1}{2} \dot{\rho} \) to derive the following estimate for any \( k \geq \beta \)

\[
\frac{k}{4} \int_X e^{(k+1)u} |Du|^2 + \frac{\partial}{\partial t} \frac{2}{k+1} \int_X e^{(k+1)u} \leq (\|\mu\|_{L^\infty} + 2|\alpha'|\|\rho\|_{C^2}) \left( \int_X e^{ku} + \int_X e^{(k-1)u} \right). 
\]

(3.13)

Here we omit the background volume form \( \frac{\sqrt{g}}{2} \) when integrating scalars.

We now consider two cases: the case of small time and the case of large time. We must consider both these cases carefully because the objective is more than just to control \( e^u \) uniformly in time; rather, we need to establish that \( e^u \) stays comparable to the scale \( M \) for all times.

We begin with the estimate for large time. Suppose \( T \in [n, n+1] \) for an integer \( n \geq 1 \). Let \( n - 1 < \tau < \tau' < T \). Let \( \zeta(t) \geq 0 \) be a monotone function which is zero for \( t \leq \tau \), identically 1 for \( t \geq \tau' \), and \( |\zeta'| \leq 2(\tau' - \tau)^{-1} \). Multiplying inequality (3.13) by \( \zeta \) gives, for any \( k \geq \beta \),

\[
\frac{k \zeta}{4} \int_X e^{(k+1)u} |Du|^2 + \frac{\partial}{\partial t} \frac{2 \zeta}{k+1} \int_X e^{(k+1)u} \leq (\|\mu\|_{L^\infty} + 2|\alpha'|\|\rho\|_{C^2}) \left\{ \zeta \int_X e^{(k-1)u} + \zeta \int_X e^{ku} \right\} + \frac{2 \zeta'}{k+1} \int_X e^{(k+1)u}. 
\]

(3.14)

Let \( \tau' < s \leq T \). Integrating from \( \tau \) to \( s \) yields

\[
\frac{k}{4} \int_{\tau'}^s \int_X e^{(k+1)u} |Du|^2 + \frac{2}{k+1} \int_X e^{(k+1)u}(s) \leq C \left\{ \int_{\tau}^T \int_X e^{(k-1)u} + \int_{\tau}^T \int_X e^{ku} + \frac{1}{\tau' - \tau} \int_{\tau}^T \int_X e^{(k+1)u} \right\},
\]

(3.15)

(3.16)

for any \( k \geq \beta \), where \( C \) only depends on \( \alpha', \rho, \mu \). We rearrange this inequality to obtain, for \( k \geq \beta + 1 \),

\[
\frac{(k - 1)}{k} \int_{\tau'}^s \int_X |De^{ku}|^2 + \int_X e^{ku}(s) \leq C_k \left\{ \int_{\tau}^T \int_X e^{(k-2)u} + \int_{\tau}^T \int_X e^{(k-1)u} + \frac{1}{\tau' - \tau} \int_{\tau}^T \int_X e^{ku} \right\}. 
\]

(3.17)

Using \( e^{-u} \leq 1 \),

\[
\int_{\tau'}^s \int_X |De^{ku}|^2 + \int_X e^{ku}(s) \leq C_k \left\{ 1 + \frac{1}{\tau' - \tau} \right\} \left\{ \int_{\tau}^T \int_X e^{ku} \right\}. 
\]

(3.18)

The Sobolev inequality gives us

\[
\left( \int_X e^{k\beta u} \right)^{\frac{1}{\beta}} \leq C_X' \left( \int_X |e^{\frac{k}{2}u}|^2 + \int_X |De^{\frac{k}{2}u}|^2 \right),
\]

(3.19)
where $C'_X$ is the Sobolev constant on manifold $(X, \hat{\omega})$. Let $\beta^*$ be such that $\frac{1}{\beta} + \frac{1}{\beta^*} = 1$.

By Hölder’s inequality and the Sobolev inequality,

$$
\int_{\tau}^{T} \int_X e^{ku} e^{\frac{k}{\beta^*}u} \leq \int_{\tau}^{T} \left( \int_X e^{k\beta u} \right)^{1/\beta} \left( \int_X e^{ku} \right)^{1/\beta^*} \leq C'_X \sup_{t \in [\tau', T]} \left( \int_X e^{ku} \right)^{1/\beta} \int_{\tau'}^{T} \left\{ \int_X e^{ku} + \int_X |D e^{\frac{k}{2}u}|^2 \right\}. \tag{3.20}
$$

Using estimate (3.18), and defining $\gamma = 1 + \frac{1}{\beta^*} = 1 + \frac{1}{2}$, we have for $k \geq 1 + \beta$,

$$
\left( \int_{\tau}^{T} \int_X e^{\gamma ku} \right)^{1/\gamma} \leq Ck \left\{ 1 + \frac{1}{\tau' - \tau} \right\} \int_{\tau}^{T} \int_X e^{ku}. \tag{3.21}
$$

We will iterate with $\tau_k = (n - 1) + \theta_1 - \gamma^{-k}(\theta_1 - \theta_2)$, for fixed $0 < \theta_2 < \theta_1 \leq 1$.

$$
\left( \int_{\tau_{k+1}}^{T} \int_X e^{\gamma^{k+1} u} \right)^{1/\gamma^{k+1}} \leq \left\{ C\gamma^k + (\theta_1 - \theta_2)^{-1} C\gamma^{2k} \frac{\gamma^k}{1 - \gamma^{-1}} \right\} \left\{ \int_{\tau_k}^{T} \int_X e^{ku} \right\}^{1/\gamma^k}. \tag{3.22}
$$

Iterating, and using $\sum_i \gamma^{-i} = 3$, we see that for $p = \gamma \eta^0 \geq 1 + \beta$, there holds

$$
\sup_{X \times [n-1+\theta_1, T]} e^u \leq \frac{C}{(\theta_1 - \theta_2)^3} \| e^u \|_{L^p(X \times [n-1+\theta_2, T])}. \tag{3.23}
$$

where $C$ only depends on $(X, \hat{\omega}), \rho, \mu,$ and $\alpha'$. A standard argument can be used to relate the $L^p$ norm of $e^u$ to $\int_X e^u = M$. Indeed, by Young’s inequality,

$$
\sup_{X \times [n-1+\theta_1, T]} e^u \leq C(\theta_1 - \theta_2)^{-3} \left( \sup_{X \times [n-1+\theta_2, T]} e^{(1-1/p)u} \right) \left( \int_{X \times [n-1+\theta_2, T]} e^u \right)^{1/p} \leq \frac{1}{2} \sup_{X \times [n-1+\theta_2, T]} e^u + C(\theta_1 - \theta_2)^{-3} \int_{X \times [n-1, T]} e^u. \tag{3.24}
$$

for all $0 < \theta_2 < \theta_1 \leq 1$. We iterate this inequality with $\theta_0 = 1$ and $\theta_{i+1} = \theta_i - \frac{1}{2}(1 - \eta)\eta^{i+1}$, where $1/2 < \eta^{3p} < 1$. Then for each $k > 1$,

$$
\sup_{X \times [n, T]} e^u \leq \frac{1}{2k} \left( \sup_{X \times [n-1+\theta_k, T]} e^u \right) + \frac{2^{3p} CM \eta^{3p}}{(1 - \eta)^{3p} e^{3p}} \sum_{i=0}^{k-1} \left( \frac{1}{2\eta^{3p}} \right)^i. \tag{3.25}
$$

Taking the limit as $k \to \infty$, we obtain a constant $C$ depending only on $(X, \hat{\omega}), \rho, \mu, \alpha'$ such that

$$
\sup_{X \times [n, T]} e^u \leq CM, \tag{3.26}
$$

for any $T \in [n, n + 1]$ and integer $n \geq 1$.  

15
Next, we adapt the previous estimate to the small time region $[0, T] \subseteq [0, 1]$. The argument is similar in essence, and we provide all details for completeness. Integrating (3.13) from 0 to $0 < s \leq T$ yields

$$\frac{k}{4} \int_0^s \int_X e^{(k+1)u} |Du|^2 + \frac{2}{k+1} \int_X e^{(k+1)u}(s) \leq C \left\{ \int_0^T \int_X e^{(k-1)u} + \int_0^T \int_X e^{ku} + M^{k+1} \right\},$$

for any $k \geq \beta$, where $C$ only depends on $\alpha', \rho, \mu$. We rearrange this inequality to obtain, for $k \geq \beta + 1$,

$$\frac{(k-1)}{k} \int_0^s \int_X |De^{\frac{1}{k}u}|^2 + \int_X e^{ku}(s) \leq Ck \left\{ \int_0^T \int_X e^{(k-2)u} + \int_0^T \int_X e^{(k-1)u} + M^k \right\}. \quad (3.27)$$

Using $e^{-u} \leq 1$, we obtain the following estimate, which holds uniformly for all $0 < s \leq T$.

$$\int_0^s \int_X |De^{\frac{1}{k}u}|^2 + \int_X e^{ku}(s) \leq Ck \left\{ \int_0^T \int_X e^{ku} + M^k \right\}. \quad (3.28)$$

As estimate in (3.20), by the Hölder and Sobolev inequalities there holds

$$\int_0^T \int_X e^{ku} e^{\frac{1}{k}u} \leq C_X \sup_{s \in [0, T]} \left( \int_X e^{ku} \right)^{1/\beta^*} \left\{ \int_0^T \int_X e^{ku} + \int_X |De^{\frac{1}{k}u}|^2 \right\}. \quad (3.29)$$

Recall that $\gamma = 1 + \frac{1}{\beta^*}$. Thus for $k \geq 1 + \beta$,

$$\int_0^T \int_X e^{k\gamma u} \leq (Ck)^\gamma \left( \int_0^T \int_X e^{ku} + M^k \right)^\gamma. \quad (3.30)$$

Therefore

$$\left( \int_0^T \int_X e^{k\gamma u} + M^{k\gamma} \right)^{1/\gamma} \leq \left( (Ck)^\gamma \left( \int_0^T \int_X e^{ku} + M^k \right)^\gamma + M^{k\gamma} \right)^{1/\gamma}, \quad (3.31)$$

and hence

$$\left( \int_0^T \int_X e^{k\gamma u} + M^{k\gamma} \right)^{1/\gamma} \leq Ck \left\{ \int_0^T \int_X e^{ku} + M^k \right\}. \quad (3.32)$$

It follows that for all $\gamma^k \geq 1 + \beta$,

$$\left( \int_0^T \int_X e^{\gamma^k u} + M^{\gamma^k} \right)^{1/\gamma^k} \leq \left\{ C\gamma^k \right\}^{1/\gamma^k} \left\{ \int_0^T \int_X e^{\gamma^ku} + M^{\gamma^k} \right\}^{1/\gamma^k}. \quad (3.33)$$

Iterating, we see that for all $k$ such that $\gamma^k \geq \gamma^{\alpha_0} \geq 1 + \beta$,

$$\left( \int_0^T \int_X e^{\gamma^{k+1} u} \right)^{1/\gamma^{k+1}} \leq \left\{ \prod_{i=\alpha_0}^k \left( C\gamma^i \right)^{1/\gamma^i} \right\} \left\{ \int_0^T \int_X e^{\gamma^{\alpha_0 u}} + M^{\gamma^{\alpha_0}} \right\}^{1/\gamma^{\alpha_0}}. \quad (3.34)$$
Sending $k \to \infty$, we obtain for $p = \gamma^\alpha$,

$$\sup_{X \times [0,T]} e^u \leq C(\|e^u\|_{L^p(X \times [0,T])} + M),$$

(3.35)

where $C$ only depends on $(X, \hat{\omega}), \rho, \mu,$ and $\alpha'$. Lastly, we relate the $L^p$ norm of $e^u$ to $\int_X e^u = M$. By the previous estimate

$$\sup_{X \times [0,T]} e^u \leq C\left(\sup_{X \times [0,T]} e^{(p-1)u}\right)^{1/p} \left(\int_{X \times [0,T]} e^u\right)^{1/p} + CM.$$  

(3.36)

We absorb the supremum term on the right-hand side using Young’s inequality. Therefore,

$$\sup_{X \times [0,T]} e^u \leq CTM + CM \leq CM,$$  

(3.37)

for any $0 < T \leq 1$, and $C$ only depends on $(X, \hat{\omega}), \rho, \mu,$ and $\alpha'$.

By combining (3.26) and (3.37), we conclude the proof of Proposition 1. Q.E.D.

### 3.2 Estimating the infimum

We introduce the constant

$$\theta = \frac{1}{2C_1 - 1}.$$  

(3.38)

Note that since $C_1 \geq 1$, we must have $0 < \theta \leq 1$. Fix a small constant $0 < \delta < 1$ such that

$$\delta < \frac{\theta}{4C_X(|\alpha'|\|\rho\|_{C^2} + \|\mu\|_{C^0})}, \quad \text{and} \quad \alpha' \delta^2 \rho \geq -\frac{1}{2} \hat{\omega},$$  

(3.39)

where $C_X$ is the Poincaré constant for the reference Kähler manifold $(X, \hat{\omega})$. Define

$$S_\delta := \{t \in [0,T) : \sup_X e^{-u} \leq \delta\}.$$  

(3.40)

Recall that we start the flow at $u_0 = \log M$. It follows that if $M > \delta^{-1}$, then the flow starts in the region $S_\delta$. At any time $\hat{t} \in S_\delta$, we consider $U = \{z \in X : e^{-u} \leq \frac{\delta}{M}\}$. Then by Proposition 1,

$$M = \int_U e^u + \int_{X \setminus U} e^u \leq |U| \sup_X e^u + (1 - |U|) \frac{M}{2} \leq C_1 M |U| + (1 - |U|) \frac{M}{2}.$$  

(3.41)

It follows that at any $\hat{t}$,

$$|U| > \theta > 0.$$  

(3.42)

We will also need the constant $C_0 > 1$ defined by

$$C_0 = \frac{1}{1 - \frac{\theta}{4}} \left(1 + \frac{2}{\theta} \right) \left(\frac{2}{\theta^2}\right).$$  

(3.43)
3.2.1 Integral estimate

**Proposition 2** Start the flow at \( u_0 = \log M \), where \( M \) is large enough such that the flow starts in the region \( S_\delta \). Suppose \([0, T] \subseteq S_\delta \). Then on \([0, T]\), there holds

\[
\int_X e^{-u} \leq \frac{2C_0}{M}. \tag{3.44}
\]

**Proof:** At \( t = 0 \), we have \( \int_X e^{-u} = \frac{1}{M} < \frac{2C_0}{M} \). Suppose \( \hat{t} \in S_\delta \) is the first time when we reach \( \int_X e^{-u} = \frac{2C_0}{M} \). Then we must have

\[
\frac{\partial}{\partial t} \big|_{t=\hat{t}} \int_X e^{-u} \geq 0. \tag{3.45}
\]

Setting \( k = 2 \) in (3.9) and dropping the negative term involving \( \omega' \geq 0 \), we have

\[
\int_X e^{-u} \{ \omega + \alpha' e^{-2u} \rho \} \wedge i \partial u \wedge i \partial \bar{u} + 2 \frac{\partial}{\partial t} \int_X e^{-u} \leq \left( |\alpha'||\rho||c^2 \int_X e^{-3u} + ||\mu||c^0 \int_X e^{-2u} \right). \]

Since \( \frac{\partial}{\partial t} \big|_{t=\hat{t}} \int_X e^{-u} \geq 0 \), and \( e^{-u} \leq \delta < 1 \), there holds at \( \hat{t} \),

\[
\int_X |De^{-\frac{\hat{\pi}}{2}}|^2 \leq (|\alpha'||\rho||c^2 + ||\mu||c^0)\delta \int_X e^{-u}. \tag{3.46}
\]

By the Poincaré inequality

\[
\int_X e^{-u} - \left( \int_X e^{-\frac{\hat{\pi}}{2}} \right)^2 = \int_X \left| e^{-\frac{\hat{\pi}}{2}} - \int_X e^{-\frac{\hat{\pi}}{2}} \right|^2 \leq C_X \int_X |De^{-\frac{\hat{\pi}}{2}}|^2. \tag{3.47}
\]

By (3.39), we have

\[
\int_X e^{-u} - \left( \int_X e^{-\frac{\hat{\pi}}{2}} \right)^2 \leq \frac{\theta}{4} \int_X e^{-u}, \tag{3.48}
\]

and it implies

\[
\int_X e^{-u} \leq \frac{1}{1 - \frac{\theta}{4}} \left( \int_X e^{-\frac{\hat{\pi}}{2}} \right)^2. \tag{3.49}
\]

We may use the measure estimate and (3.49) to obtain

\[
\left( \int_X e^{-\frac{\hat{\pi}}{2}} \right)^2 \leq \left( 1 + \frac{2}{\theta} \right) \left( \int_U e^{-\frac{\hat{\pi}}{2}} \right)^2 + (1 + \frac{\theta}{2}) \left( \int_X \chi_U e^{-\frac{\hat{\pi}}{2}} \right)^2 \\
\leq \left( 1 + \frac{2}{\theta} \right)|U| \int_U e^{-u} + (1 + \frac{\theta}{2})(1 - |U|) \int_X \chi_U e^{-u} \\
\leq \left( 1 + \frac{2}{\theta} \right) \frac{2}{M} + (1 + \frac{\theta}{2})(1 - \theta) \frac{1}{1 - \frac{\theta}{4}} \left( \int_X e^{-\frac{\hat{\pi}}{2}} \right)^2. \tag{3.50}
\]

Thus

\[
\left( \int_X e^{-\frac{\hat{\pi}}{2}} \right)^2 \leq \left( 1 + \frac{2}{\theta} \right) \frac{2}{M} \left( \frac{1}{1 - (1 + \frac{\theta}{2})(1 - \theta)(1 - \frac{\theta}{4})} \right). \tag{3.51}
\]
For any $\theta \geq 0$, we have the elementary estimate
\[(1 + \frac{\theta}{2})(1 - \theta)(1 - \frac{\theta}{4})^{-1} \leq 1 - \theta^2.\] (3.52)

Using this and (3.49),
\[\int_X e^{-u} \leq \frac{1}{1 - \frac{\theta}{4}}(1 + \frac{2}{\theta})(\frac{2}{\theta^2}) \frac{1}{M} = \frac{C_0}{M}.\] (3.53)

This contradicts that $\int_X e^{-u} = \frac{2C_0}{M}$ at $\hat{t}$. It follows that $\int_X e^{-u}$ stays less than $\frac{2C_0}{M}$ for all time $t \in S_\delta$.

### 3.2.2 Iteration

**Proposition 3** Start the flow with initial data $e^{u(x,0)} = M$. Suppose the flow exists for $t \in [0,T)$ with $T > 0$, and $[0,T) \subseteq S_\delta$. Then
\[\sup_{X \times [0,T)} e^{-u} \leq \frac{C_2}{M},\] (3.54)

where $C_2$ only depends on $(X,\hat{\omega})$, $\rho$, $\mu$, $\alpha'$.

**Proof:** We can drop the negative terms involving $\omega' \geq 0$ and use $\alpha' e^{-2u}\rho \geq -\frac{1}{2} \hat{\omega}$ in (3.9) to obtain the estimate, for $k \geq 2$,
\[\frac{k}{4} \int_X e^{-(k-1)u} |Du|^2 + \frac{\partial}{\partial t} \frac{2}{k-1} \int_X e^{-(k-1)u} \leq C \left( \int_X e^{-(k+1)u} + \int_X e^{-ku} \right).\] (3.55)

As in the upper bound on $e^u$, we split the argument into the cases of large time and small time, and first consider the case of large time.

Suppose $T \in [n, n+1]$ for an integer $n \geq 1$. Let $n - 1 < \tau < \tau' < T$. Let $\zeta(t) \geq 0$ be a monotone function which is zero for $t \leq \tau$ and identically 1 for $t \geq \tau'$. Multiplying (3.55) by $\zeta$ gives
\[\frac{k \zeta}{4} \int_X e^{-(k-1)u} |Du|^2 + \frac{\partial}{\partial t} \frac{2 \zeta}{k-1} \int_X e^{-(k-1)u} \leq C \left\{ \zeta \int_X e^{-(k+1)u} + \zeta \int_X e^{-ku} + \zeta' \int_X e^{-(k-1)u} \right\}.\] (3.56)

Let $\tau' < s \leq T$. Integrating from $\tau$ to $s$
\[\frac{k}{4} \int_{\tau}^{s} \int_X e^{-(k-1)u} |Du|^2 + \frac{2}{k-1} \int_X e^{-(k-1)u}(s) \leq C \left\{ \int_{\tau}^{T} \int_X e^{-(k+1)u} + \int_{\tau}^{T} \int_X e^{-ku} + \frac{1}{\tau' - \tau} \int_{\tau}^{T} \int_X e^{-(k-1)u} \right\}.\] (3.57)
We rearrange this inequality to obtain, for \( k \geq 1, \)
\[
\int_{\tau'}^s \int_X |D e^{-\frac{k}{2} u}|^2 + 2 \int_X e^{-ku}(s) \leq C k \left\{ \int_{\tau'}^T \int_X e^{-(k+2)u} + \int_{\tau'}^T \int_X e^{-(k+1)u} + \frac{1}{\tau' - \tau} \int_{\tau'}^T \int_X e^{-ku} \right\}.
\]

Since \( e^{-u} \leq \delta < 1, \) we have
\[
\int_{\tau'}^s \int_X |D e^{-\frac{k}{2} u}|^2 + 2 \int_X e^{-ku}(s) \leq C k \left\{ 1 + \frac{1}{\tau' - \tau} \right\} \left\{ \int_{\tau}^T \int_X e^{-ku} \right\}. \tag{3.58}
\]

Recall that we denote \( \beta = \frac{n}{n-1} = 2, \beta^* \) such that \( \frac{1}{\beta} + \frac{1}{\beta^*} = 1, \) and \( \gamma = 1 + \frac{1}{\beta^*}. \) By the Sobolev inequality
\[
\int_{\tau'}^T \int_X e\gamma ku \leq \int_{\tau'}^T \left( \int_X e^{k\beta u} \right)^{1/\beta} \left( \int_X e^{-ku} \right)^{1/\gamma^*} \leq C \sup_{t \in [\tau', T]} \left( \int_X e^{-ku} \right)^{1/\beta^*} \int_{\tau'}^T \left\{ \int_X e^{-ku} + \int_X |D e^{-\frac{k}{2} u}|^2 \right\}. \tag{3.59}
\]

Using estimate (3.58), we arrive at
\[
\left( \int_{\tau'}^T \int_X e\gamma ku \right)^{1/\gamma} \leq C k \left\{ 1 + \frac{1}{\tau' - \tau} \right\} \left\{ \int_{\tau}^T \int_X e^{-ku} \right\}. \tag{3.60}
\]

Iterating with \( \tau_k = (1 - \gamma^{-(k+1)}) + (n-1), \)
\[
\left( \int_{\tau_{k+1}}^T \int_X e\gamma^{k+1} u \right)^{1/\gamma^{k+1}} \leq \left\{ C \gamma^k + \frac{C \gamma^{2k}}{1 - \gamma^{-1}} \right\}^{1/\gamma^k} \left\{ \int_{\tau_k}^T \int_X e^{\gamma^k u} \right\}. \tag{3.61}
\]

Note \( \tau_k \geq n - \frac{2}{3}. \) Sending \( k \to \infty, \) we have the \( C^0 \) estimate
\[
\sup_{X \times [n,T]} e^{-u} \leq C \| e^{-u} \|_{L^1(X \times [n - \frac{2}{3}, T])}. \tag{3.62}
\]

By Proposition 2, for \( n \leq T \leq n + 1 \) and \( n \geq 1, \) we obtain
\[
\sup_{X \times [n,T]} e^{-u} \leq \frac{C}{M}. \tag{3.63}
\]

Next, we consider the small time region \([0,T] \subseteq [0,1]. \) Integrating (3.55) from 0 to \( 0 < s < T, \) we obtain
\[
\frac{k}{4} \int_0^s \int_X e^{-(k-1)u} |Du|^2 + \frac{2}{k-1} \int_X e^{-(k-1)u}(s) \leq C \left\{ \int_0^T \int_X e^{-(k+1)u} + \int_0^T \int_X e^{-ku} + \frac{M^{-(k-1)}}{k-1} \right\}.
\]

We rearrange this inequality to obtain, for \( k \geq 1, \)
\[
\int_0^s \int_X |D e^{-\frac{k}{2} u}|^2 + 2 \int_X e^{-ku}(s) \leq C k \left\{ \int_0^T \int_X e^{-(k+2)u} + \int_0^T \int_X e^{-(k+1)u} + M^{-k} \right\}. \tag{3.64}
\]

20
Since \( e^{-u} \leq \delta < 1 \), we have
\[
\int_0^s \int_X |D e^{-\frac{k}{2} u}|^2 + 2 \int_X e^{-ku}(s) \leq C k \left\{ \int_0^T \int_X e^{-ku} + M^{-k} \right\}. \tag{3.65}
\]

As before, by the Sobolev inequality
\[
\int_0^T \int_X e^{-ku} e^{-\frac{k}{2} u} \leq C \sup_{s \in [0,T]} \left( \int_X e^{-ku} \right)^{1/\beta^*} \int_0^T \left\{ \int_X e^{-ku} + \int_X |D e^{-\frac{k}{2} u}|^2 \right\}. \tag{3.66}
\]

Combining this with (3.65) yields
\[
\left( \int_0^T \int_X e^{-\gamma ku} + M^{-\gamma k} \right)^{1/\gamma} \leq C k \left\{ \int_0^T \int_X e^{-ku} + M^{-k} \right\}. \tag{3.67}
\]

Iterating, we obtain the \( C^0 \) estimate
\[
\sup_{X \times [0,T]} e^{-u} \leq C \|e^{-u}\|_{L^1(X \times [0,T])} + CM^{-1}. \tag{3.68}
\]

By Proposition 2, for \( 0 < T \leq 1 \) we obtain
\[
\sup_{X \times [0,T]} e^{-u} \leq C TM^{-1} + CM^{-1} \leq \frac{C}{M}. \tag{3.69}
\]

By combining (3.63) and (3.69), we conclude the proof of Proposition 3. Q.E.D.

**Theorem 3** Suppose the flow exists for \( t \in [0,T) \), and initially starts with \( u_0 = \log M \). There exists \( M_0 \gg 1 \) such that for all \( M \geq M_0 \), there holds
\[
\sup_{X \times [0,T]} e^u \leq C_1 M, \quad \sup_{X \times [0,T]} e^{-u} \leq \frac{C_2}{M}, \tag{3.70}
\]

where \( C_1, C_2 \) only depend on \((X, \hat{\omega}), \rho, \mu, \alpha'\).

**Proof:** By Proposition 1 and Proposition 3, the estimates hold as long as we stay in \( S_\delta \). Choose \( M_0 \) such that
\[
\frac{C_2}{M_0} < \frac{\delta}{2}, \tag{3.71}
\]

where recall \( \delta \) is defined in (3.39). Then at \( t = 0 \), we have \( e^{-u_0} < \delta \), and the estimate is preserved on \([0,T)\). The theorem follows. Q.E.D.
4 Evolution of the torsion

Before proceeding, we clearly state the conventions and notation that will be used for the maximum principle estimates of Sections 4-6. All norms from this point on will be with respect to the evolving metric $\omega = e^u \hat{\omega}$, unless denoted otherwise. We will write $\omega = ig_{kj}dz^j \wedge dz^k$. We will use the Chern connection of $\omega$ to differentiate

$$\nabla_k V^\alpha = \partial_k V^\alpha, \quad \nabla_k V^\alpha = g^{\alpha \beta} \partial_k (g_{\beta \gamma} V^\gamma). \tag{4.1}$$

The curvature of the metric $\omega$ is

$$R_{kj}{}^\alpha{}_{\beta} = -\partial_k (g^{\alpha \gamma} \partial_j g_{\gamma \beta}) = \hat{R}_{kj}{}^\alpha{}_{\beta} - u_{kj} \delta^\alpha{}_{\beta}. \tag{4.2}$$

The torsion tensor of the metric $\omega$ is $T_{kmj} = \partial_m g_{kj} - \partial_j g_{km}$, and since $\hat{\omega}$ has zero torsion, we may compute

$$T^\lambda{}_{mj} = g^{\lambda k} T_{kmj} = u_m \delta^\lambda{}_{j} - u_j \delta^\lambda{}_{m}. \tag{4.3}$$

We note the following formulas for the torsion and Chern-Ricci curvature of the evolving metric

$$R_{kj} = R_{kj}{}^\alpha{}_{\alpha} = -2u_{kj}, \quad T_j = T^\lambda{}_{\lambda j} = -\partial_j u. \tag{4.4}$$

Recall that $|T|^2$ refers to the norm of $T_j$, as noted in (2.33). We will often use the following commutation formulas to exchange covariant derivatives

$$[\nabla_j, \nabla_k] V_i = -R_{kj}{}^p{}_{i} V_p, \quad [\nabla_j, \nabla_k] V_i = -T^\lambda{}_{jk} \nabla^\lambda V_i. \tag{4.5}$$

To handle the differentiation of the equation, we will rewrite the terms involving $\rho$ in the flow (1.6). Compute

$$-\alpha' i \partial \bar{\partial} (e^{-u} \rho) = -\alpha' e^{-u} i \partial \bar{\partial} \rho + 2\alpha' \text{Re}\{e^{-u} i \partial \rho \wedge \bar{\partial} \rho\} + \alpha' e^{-u} i \partial \bar{\partial} \rho \wedge \rho - \alpha' i e^{-u} \partial \rho \wedge \bar{\partial} \rho \wedge \rho. \tag{4.6}$$

We introduce the notation

$$-\alpha' i \partial \bar{\partial} (e^{-u} \rho) = \left( -\alpha' e^{-u} \psi \rho + \alpha' e^{-u} \text{Re}\{b^i_{\rho} u_i\} + \alpha' e^{-u} \bar{\rho} k_{\bar{u}} \rho_{\bar{u}} - \alpha' e^{-u} \bar{\rho} q_{\bar{u}} \rho_{\bar{u}} \right) \frac{\dot{\omega}^2}{2}, \tag{4.7}$$

where $\psi_{\rho}(z), b^i_{\rho}(z), \bar{\rho} k_{\bar{u}}(z)$ are defined one by one corresponding to the previous expression. We note that $\psi_{\rho}, b^i_{\rho}, \bar{\rho} k_{\bar{u}}$ are bounded in $C^\infty$ by constants depending only on the form $\rho$ and the background metric $\hat{\omega}$. We also note that $\bar{\rho} k_{\bar{u}}$ is Hermitian since $\rho$ is real. We may rewrite this expression as

$$-\alpha' i \partial \bar{\partial} (e^{-u} \rho) = \left( -\alpha' e^{-3u} \psi \rho - \alpha' e^{-3u} \text{Re}\{b^i_{\rho} T_i\} - \frac{\alpha'}{2} e^{-3u} \bar{\rho} k_{\bar{u}} R_{kj} - \alpha' e^{-3u} \bar{\rho} q_{\bar{u}} \bar{T}_{\bar{u}} \bar{T}_{\bar{u}} \right) \frac{\dot{\omega}^2}{2}. \tag{4.8}$$
With all the introduced notation, we can write the flow (1.6) in the following way.

\[ \partial_t g_{kj} = \frac{1}{2\|\Omega\|} \left( -\frac{R}{2} - \frac{\alpha'}{2}\|\Omega\|^3 \bar{\rho}^{pq} R_{qp} + \frac{\alpha'}{4} \sigma_2(i \text{Ric}_\omega) + |T|^2 + \|\Omega\|^2 \nu \right) g_{kj}, \quad (4.9) \]

where

\[ \nu = -\alpha'\|\Omega\|\psi_\rho - \alpha'\|\Omega\|\text{Re}\{b_i^j T_i\} - \alpha'\|\Omega\|\bar{\rho}^{pq} T_p \bar{T}_q + \bar{\mu}. \quad (4.10) \]

In the following, we will use \( \|\Omega\| \) to replace \( \|\Omega\|_\omega \) for simplicity, if there is no confusing of the notation.

### 4.1 Torsion tensor

Using \( \|\Omega\| = e^{-u} \) and \( g_{kj} = e^u \hat{g}_{kj}, \) (4.9) implies the following evolution of \( \|\Omega\|, \)

\[ \partial_t \log \|\Omega\| = \frac{1}{2\|\Omega\|} \left( \frac{R}{2} + \frac{\alpha'}{2}\|\Omega\|^3 \bar{\rho}^{pq} R_{qp} - |T|^2 - \frac{\alpha'}{4} \sigma_2(i \text{Ric}_\omega) - \|\Omega\|^2 \nu \right). \quad (4.11) \]

Using (2.30) and (4.11), we evolve

\[ \partial_t T_j = \partial_j \partial_t \log \|\Omega\| = \nabla_j \left\{ \frac{1}{2\|\Omega\|} \left( \frac{R}{2} + \frac{\alpha'}{2}\|\Omega\|^3 \bar{\rho}^{pq} R_{qp} - |T|^2 - \frac{\alpha'}{4} \sigma_2(i \text{Ric}_\omega) \right) \right\}. \quad (4.12) \]

Using \( \partial_j \|\Omega\| = \|\Omega\| T_j \) and the definition of \( \nu \) (4.10), a straightforward computation gives

\[ \partial_t T_j = \frac{1}{2\|\Omega\|} \left\{ -\frac{1}{2} T_j R + T_j |T|^2 + \frac{\alpha'}{4} T_j \sigma_2(i \text{Ric}_\omega) \right. \\
+ \frac{1}{2} \nabla_j R + \frac{\alpha'}{2}\|\Omega\|^3 \bar{\rho}^{pq} \nabla_j R_{qp} - \nabla_j |T|^2 - \frac{\alpha'}{4} \nabla_j \sigma_2(i \text{Ric}_\omega) + E_j \left\}, \quad (4.13) \right. \]

where

\[ E_j = 2\alpha' \|\Omega\|^3 \psi_\rho T_j + 2\alpha' \|\Omega\|^3 \text{Re}\{b_j^i T_i\} T_j + \alpha' \|\Omega\|^3 \bar{\rho}^{pq} R_{qp} T_j \\
+ 2\alpha' \|\Omega\|^3 (\bar{\rho}^{pq} T_p \bar{T}_q) T_j - \|\Omega\|^2 \bar{\mu} T_j + \alpha' \|\Omega\|^3 \nabla_j \psi_\rho \\
+ \alpha' \|\Omega\|^3 \text{Re}\{\nabla_j b_j^i T_i\} + \alpha' \|\Omega\|^3 \text{Re}\{b_j^i \nabla_j T_i\} + \frac{\alpha'}{2} \|\Omega\|^3 (\nabla_j \bar{\rho}^{pq}) R_{qp} \\
+ \alpha' \|\Omega\|^3 (\nabla_j \bar{\rho}^{pq}) T_p \bar{T}_q + \alpha' \|\Omega\|^3 \bar{\rho}^{pq} \nabla_j T_p \bar{T}_q + \frac{\alpha'}{2} \|\Omega\|^3 \bar{\rho}^{pq} T_p \bar{T}_q \\
- \|\Omega\|^2 \nabla_j \bar{\mu}. \quad (4.14) \]

Our reason for treating \( E_j \) as an error term is that the \( C^0 \) estimate tells us that \( \|\Omega\| = e^{-u} \ll 1 \) if we start the flow from a large enough constant \( \log M \). As we will see, the terms appearing in \( E_j \) will only slightly perturb the coefficients of the leading terms in the proof of Theorem 4.
We need to express the highest order terms in (4.13) as the linearized operator acting on torsion. First, we write the Ricci curvature in terms of the conformal factor
\[
\nabla_j R_{\bar{q}p} = -2\nabla_j \nabla_p \nabla_q u.
\]
(4.15)

Exchanging covariant derivatives
\[
-2\nabla_j \nabla_p \nabla_q u = -2\nabla_p \nabla_q \nabla_j u - 2T^\lambda_{\ p\ j} \nabla_\lambda \nabla_q u.
\]
(4.16)

It follows from (4.4) that
\[
\nabla_j R_{\bar{q}p} = 2\nabla_p \nabla_q T_j + T^\lambda_{\ p\ j} R_{\bar{q}\lambda}.
\]
(4.17)

Hence
\[
\nabla_j R - \frac{\alpha'}{2} \nabla_j \sigma_2(i\text{Ric}_\omega) + \alpha'\|\Omega\|^3 \bar{\rho}_{\bar{p}q} \nabla_j R_{\bar{q}p}
\]
\[
= g^{p\bar{q}} \nabla_j R_{\bar{p}j} + \alpha'\|\Omega\|^3 \bar{\rho}_{\bar{p}q} \nabla_j R_{\bar{q}p} - \frac{\alpha'}{2} \sigma_2^{p\bar{q}} \nabla_j R_{\bar{q}p}
\]
\[
= 2F^{p\bar{q}} \nabla_p \nabla_q T_j + F^{p\bar{q}} T^\lambda_{\ p\ j} R_{\bar{q}\lambda},
\]
(4.18)

where we introduced the notation
\[
\sigma_2^{p\bar{q}} = R g^{p\bar{q}} - R_{\bar{p}q},
\]
(4.19)

and
\[
F^{p\bar{q}} = g^{p\bar{q}} + \alpha'\|\Omega\|^3 \bar{\rho}_{\bar{p}q} - \frac{\alpha'}{2} (R g^{p\bar{q}} - R_{\bar{p}q}).
\]
(4.20)

The tensor $F^{p\bar{q}}$ is Hermitian, and in Section §5 we will show that $F^{p\bar{q}}$ stays close to $g^{p\bar{q}}$ along the flow. Substituting (4.18) into (4.13)
\[
\partial_t T_j = \frac{1}{2\|\Omega\|} \left\{ F^{\bar{p}q} \nabla_p \nabla_q T_j - \nabla_j |T|^2 - \frac{1}{2} T_j R + \frac{\alpha'}{4} T_j \sigma_2(i\text{Ric}_\omega) \right. \\
+ \left. \frac{\alpha'}{2} F^{p\bar{q}} T^\lambda_{\ p\ j} R_{\bar{q}\lambda} + T_j |T|^2 + E_j \right\}.
\]
(4.21)

Before proceeding, let us discuss $\sigma_2^{\bar{p}q}$ and $F^{p\bar{q}}$ using convenient coordinates. Suppose we work at a point where the evolving metric $g_{ij} = \delta_{ij}$ and $R_{\bar{k}j}$ is diagonal. Let $A_{ij} = g^{\bar{k}j} R_{\bar{k}j}$. The function $\sigma_2(A_{ij})$ maps a Hermitian endomorphism to the second elementary symmetric polynomial of its eigenvalues. We are working in dimension $n = 2$, so $\sigma_2(A_{ij})$ is the product of the two eigenvalues of $A$. Our operator $\sigma_2(i\text{Ric}_\omega)$ defined in (2.36) is with respect to the evolving metric $\omega$, so denoting $A_{ij} = g^{\bar{k}j} R_{\bar{k}j}$, we have $\sigma_2(i\text{Ric}_\omega) = \sigma_2(A)$. We define $\sigma_2^{\bar{p}q} = \frac{\partial \sigma_2}{\partial A^{\bar{p}q}} g^{\bar{k}j}$. It is well-known that $\frac{\partial \sigma_2}{\partial A_{11}} = A^2_{12}$, $\frac{\partial \sigma_2}{\partial A_{12}} = A^1_{11}$, and $\frac{\partial \sigma_2}{\partial A_{22}} = 0$ if $A$ is diagonal. Then in our case,
\[
\sigma_2^{1\bar{1}} = R_{\bar{2}2}, \quad \sigma_2^{2\bar{2}} = R_{\bar{1}1}, \quad \sigma_2^{1\bar{2}} = \sigma_2^{2\bar{1}} = 0.
\]
(4.22)

We obtain
\[
F^{1\bar{1}} = 1 + \alpha'\|\Omega\|^3 \bar{\rho}^{1\bar{1}} - \frac{\alpha'}{2} R_{\bar{2}2}, \quad F^{2\bar{2}} = 1 + \alpha'\|\Omega\|^3 \bar{\rho}^{2\bar{2}} - \frac{\alpha'}{2} R_{\bar{1}1},
\]
\[
F^{1\bar{2}} = \alpha'\|\Omega\|^3 \bar{\rho}^{1\bar{2}}, \quad F^{2\bar{1}} = \alpha'\|\Omega\|^3 \bar{\rho}^{2\bar{1}}.
\]
(4.23)
4.2 Norm of the torsion

We will compute
\[
\partial_t |T|^2 = \partial_t \{g^{ij} T_i T_j \}. \tag{4.24}
\]

We have
\[
\partial_t g^{ij} = -g^{i\lambda} g^{\gamma j} \partial_t g_{\lambda\gamma} = \frac{1}{2\|\Omega\|} \left( \frac{R}{2} + \frac{\alpha'}{\|\Omega\|} \alpha' \bar{R}^{pq} R_{qp} - \frac{\alpha'}{4} \sigma_2(i \text{Ric}_\omega) - |T|^2 - \|\Omega\|^2 \nu \right) g^{ij}. \tag{4.25}
\]

Hence
\[
\partial_t |T|^2 = \text{Re} \langle \partial_t T, T \rangle + \frac{|T|^2}{2\|\Omega\|} \left( \frac{R}{4} + \frac{\alpha'}{\|\Omega\|} \alpha' \bar{R}^{pq} R_{qp} - \frac{\alpha'}{4} \sigma_2(i \text{Ric}_\omega) - |T|^2 - \|\Omega\|^2 \nu \right) \tag{4.26}
\]

Next, using the notation \( |W|^2_{Fg} = F^{pq} g^{ij} W_p \bar{W}_q \)
\[
F^{pq} \nabla_p \nabla_q |T|^2 = \frac{F^{pq} g^{ij} \nabla_p \nabla_q T_i \bar{T}_j + F^{pq} g^{ij} T_i \nabla_p \nabla_q \bar{T}_j + |\nabla T|^2_{Fg} + |\nabla T|^2}{2}\]
\[
+ |\nabla T|^2_{Fg} + |\nabla T|^2 \tag{4.27}
\]

We introduce the notation \( \Delta_F = F^{pq} \nabla_p \nabla_q - \text{Ric}_\omega \). We have shown
\[
\Delta_F |T|^2 = 2\text{Re} \langle \Delta_F T, T \rangle + |\nabla T|^2_{Fg} + |\nabla T|^2 + F^{pq} g^{ij} T_i R_{qpj} \lambda \bar{T}_\lambda. \tag{4.28}
\]

Combining (4.21), (4.26), and (4.28), we obtain
\[
\partial_t |T|^2 = \frac{1}{2\|\Omega\|} \left\{ \Delta_F |T|^2 - |\nabla T|^2_{Fg} - |\nabla T|^2_{Fg} - 2\text{Re} \{g^{ij} \nabla_i |T|^2 T_j \}
\right. \\
- \frac{1}{2} R |T|^2 + \frac{\alpha'}{4} \sigma_2(i \text{Ric}_\omega) |T|^2 + \text{Re} \{F^{pq} g^{ij} T_i R_{qpj} \lambda \bar{T}_\lambda \}
\left. \\
- F^{pq} g^{ij} T_i R_{qpj} \lambda \bar{T}_\lambda + |T|^4 + \frac{\alpha'}{2} \|\Omega\|^3 \bar{R}^{pq} R_{qp} |T|^2 \\
- \|\Omega\|^2 |T|^2 \nu + 2\text{Re} \langle E, T \rangle \}. \tag{4.29}
\]

4.3 Estimating the torsion

**Theorem 4** There exists \( M_0 \gg 1 \) such that all \( M \geq M_0 \) have the following property. Start the flow with a constant function \( u_0 = \log M \). If
\[
|\alpha' \text{Ric}_\omega| \leq 10^{-6} \tag{4.30}
\]
along the flow, then there exists \( C_3 > 0 \) depending only on \( (X, \omega), \rho, \bar{\mu} \) and \( \alpha' \), such that
\[
|T|^2 \leq \frac{C_3}{M} \ll 1. \tag{4.31}
\]
Denote $\Lambda = 1 + \frac{1}{8}$. We will study the test function
\[
G = \log |T|^2 - \Lambda \log \|\Omega\|.
\] (4.32)

Taking the time derivative gives us
\[
\partial_t G = \frac{\partial_t |T|^2}{|T|^2} - \Lambda \partial_t \log \|\Omega\|.
\] (4.33)

Computing using (2.30) and (4.20),
\[
\Delta F \log \|\Omega\| = F^{pq} \nabla_i \bar{T}_q = \frac{1}{2} F^{pq} R_{qp}
\] = \frac{1}{2} R - \frac{\alpha'}{4} \sigma_2^{pq} R_{qp} + \frac{\alpha'}{2} \|\Omega\|^3 \bar{\rho}^{pq} R_{qp}
\] = \frac{1}{2} R - \frac{\alpha'}{2} \sigma_2(i\Ric) + \frac{\alpha'}{2} \|\Omega\|^3 \bar{\rho}^{pq} R_{qp}. \quad (4.34)

Therefore by (4.11)
\[
\partial_t \log \|\Omega\| = \frac{1}{2\|\Omega\|} \left\{ \Delta F \log \|\Omega\| - |T|^2 + \frac{\alpha'}{4} \sigma_2(i\Ric) - \|\Omega\|^2 \nu \right\}. \quad (4.35)
\]

Substituting (4.29) and (4.35) into (4.33), we have
\[
\partial_t G = \frac{1}{2\|\Omega\|} \left\{ \Delta F G + \frac{\|\nabla|T|^2\|^2}{|T|^4} - \frac{|\nabla T|^2}{|T|^2} - \frac{|\nabla T|^2}{|T|^2} - \frac{2}{|T|^2} \Re \{ g^{ij} \nabla_i |T|^2 \bar{T}_j \} \right\}
\] = \frac{1}{2} R + \frac{\alpha'}{4} \sigma_2(i\Ric) + \frac{1}{|T|^2} \Re \{ F^{pq} g^{ij} T_i R_{pqj} \bar{T}_j \}
\] = \frac{1}{|T|^2} F^{pq} g^{ij} T_i R_{pqj} \bar{T}_j + |T|^2 + \frac{\alpha'}{2} \|\Omega\|^3 \bar{\rho}^{pq} R_{qp} - \|\Omega\|^2 \nu
\] + \frac{2}{|T|^2} \Re \{ E, T \} + \Lambda |T|^2 - \frac{\alpha'}{4} \Lambda \sigma_2(i\Ric) + \Lambda \|\Omega\|^2 \nu \}. \quad (4.36)

Let $(p, t_0)$ be the point in $X \times [0, T]$ where $G$ attains its maximum. Since we start the flow at $t = 0$ with a constant function $u_0 = \log M$, the torsion is zero at the initial time. It follows that $t_0 > 0$. The following computation will be done at this point $(p, t_0)$, and we note that $|T|^2 > 0$ at $(p, t_0)$. The critical equation $\nabla G = 0$ gives
\[
0 = \frac{\nabla_i |T|^2}{|T|^2} - \Lambda T_i. \quad (4.37)
\]

Using (2.38), this can be rewritten in the following way
\[
\frac{\langle \nabla_i T, T \rangle}{|T|^2} = \Lambda T_i - \frac{\langle T, \nabla_i T \rangle}{|T|^2} = \Lambda T_i - \frac{1}{2|T|^2} g^{jk} T_j R_{ki}. \quad (4.38)
\]
Therefore, by Cauchy-Schwarz and the critical equation,
\[
-\frac{|\nabla T|^2_{F^2}}{|T|^2} \leq -\frac{(\nabla T, T)}{|T|^2} = -\Lambda T_i - \frac{1}{2|T|^2}g^{jk}T_jR_{ki}^2_{F^2} \\
= -\Lambda^2|T|_F^2 - \frac{1}{4|T|^4}|g^{jk}T_jR_{ki}^2_{F^2} + \frac{\Lambda}{|T|^2}\text{Re}\{F^{pq}g^{jk}T_jR_{kp}\bar{T}_q}\}. \quad (4.39)
\]
Here we used the notation \(|V|_F^2 = F^{pq}V_p\bar{V}_q\). We may also expand the following term using the definition of \(F^{pq}\),
\[
4|\nabla T|^2_{F^2} = F^{pq}g^{ij}R_{qi}R_{jp} = |\text{Ric}_w|^2 - \frac{\alpha'}{2}g^{ij}\sigma_2^{pq}R_{qi}R_{jp} + \alpha'\|\Omega\|^3g^{ij}\rho^{pq}R_{qi}R_{jp}. \quad (4.40)
\]
Set \(\varepsilon = 1/100\). Using (4.39) and (4.40), and the critical equation (4.37) once more on the first and last term, we obtain
\[
\frac{|\nabla T|^2_{F^2}}{|T|^4} - (1 - \varepsilon)\frac{|\nabla T|^2_{F^2}}{|T|^2} - \frac{2}{|T|^2}\text{Re}\{g^{ij}\nabla_i|T|^2\bar{T}_j\} \\
\leq \Lambda^2|T|_F^2 - (1 - \varepsilon)\Lambda^2|T|_F^2 - (1 - \varepsilon)\frac{1}{4|T|^4}|g^{jk}T_jR_{ki}|_F^2 \\
+ (1 - \varepsilon)\frac{\Lambda}{|T|^2}\text{Re}\{F^{pq}g^{jk}T_jR_{kp}\bar{T}_q\} - \frac{1}{4|T|^2}|\text{Ric}_w|^2 + \frac{\alpha'}{8|T|^2}g^{ij}\sigma_2^{pq}R_{qi}R_{jp} \\
- \frac{\alpha'}{4|T|^2}\|\Omega\|^3g^{ij}\rho^{pq}R_{qi}R_{jp} - 2\Lambda|T|^2. \quad (4.41)
\]
Substituting this inequality into (4.36), our main inequality becomes
\[
\partial_t G \leq \frac{1}{2\|\Omega\|^2} \left\{ \frac{\Delta_F G - \varepsilon|\nabla T|^2_{F^2}}{|T|^2} - \frac{1}{4}\frac{|\text{Ric}_w|^2}{|T|^2} - (\Lambda - 1)|T|_F^2 + \varepsilon\Lambda^2|T|_F^2 - \frac{1}{2}R \\
- \frac{\alpha'}{4}(\Lambda - 1)\sigma_2(\text{i}\text{Ric}_w) + \frac{\alpha'}{8|T|^2}g^{ij}\sigma_2^{pq}R_{qi}R_{jp} + (1 - \varepsilon)\frac{\Lambda}{|T|^2}\text{Re}\{F^{pq}g^{jk}T_jR_{kp}\bar{T}_q\} \\
- \frac{1}{|T|^2}F^{pq}g^{ij}T_iR_{pqj}\hat{T}_j^\lambda + \frac{1}{|T|^2}\text{Re}\{F^{pq}g^{ij}\hat{T}_i\bar{T}_j\bar{T}_q\} \\
- \frac{(1 - \varepsilon)}{4|T|^4}|g^{jk}T_jR_{ki}|_F^2 + \frac{\alpha'}{4|T|^2}\|\Omega\|^3g^{ij}\rho^{pq}R_{qi}R_{jp} + \frac{\alpha'}{2}\|\Omega\|^3\rho^{pq}R_{qp} \\
+ (\Lambda - 1)\|\Omega\|^2\nu + \frac{2}{|T|^2}\text{Re}\{E, T\} \right\}, \quad (4.42)
\]
which holds at \((p, t_0)\). Next, we use (4.2) to write the evolving curvature as
\[
R_{qpj}^\lambda = \hat{\nabla}_{q\rho}^\lambda + \frac{1}{2}R_{q\rho}\delta_j^\lambda. \quad (4.43)
\]
This identity allows us to write

\[-\frac{1}{|T|^2} F^{pq} g^{ij} T_i R_{qpl} \hat{T}_j = -\frac{1}{|T|^2} F^{pq} g^{ij} T_i \hat{R}_{qpl} \hat{T}_j - \frac{1}{2} F^{pq} R_{qp}. \tag{4.44}\]

Next, by (4.3), the torsion can be written as

\[T^\lambda_{\rho i} = T_i \delta^\lambda_p - T_p \delta^\lambda_i, \tag{4.45}\]

so we may rewrite

\[\frac{1}{|T|^2} \text{Re} \{ F^{pq} g^{ij} T^\lambda_{\rho p} R_{q\lambda T} \} = F^{pq} R_{qp} - \frac{1}{|T|^2} \text{Re} \{ F^{pq} g^{ij} R_{qij} \hat{T}_j T_p \}. \tag{4.46}\]

Together, we have

\[-\frac{1}{|T|^2} F^{pq} g^{ij} T_i R_{qpl} \hat{T}_j + \frac{1}{|T|^2} \text{Re} \{ F^{pq} g^{ij} T^\lambda_{\rho p} R_{q\lambda T} \} = -\frac{1}{|T|^2} F^{pq} g^{ij} T_i \hat{R}_{qpl} \hat{T}_j + \frac{1}{2} F^{pq} R_{qp} - \frac{1}{|T|^2} \text{Re} \{ F^{pq} g^{ij} R_{qij} \hat{T}_j T_p \} \]

\[= -\frac{1}{|T|^2} F^{pq} g^{ij} T_i \hat{R}_{qpl} \hat{T}_j + \frac{1}{2} R - \frac{1}{2} \alpha' \sigma_2(i\text{Ric}_\omega) + \frac{\alpha'}{2} \|\Omega\|^2 \hat{\rho}^{pq} R_{qp} - \frac{1}{|T|^2} \text{Re} \{ F^{pq} g^{ij} R_{qij} \hat{T}_j T_p \}. \tag{4.47}\]

We also compute

\[|T|^2_F = |T|^2 + \alpha'\|\Omega\|^3 \hat{\rho}^{pq} T_p T_q - \frac{\alpha'}{2} \sigma_2 R^{pq} T_p T_q. \tag{4.48}\]

Substituting (4.47) and (4.48) in the main inequality (4.42), we see that the terms of order \(R\) have cancelled.

\[\partial_t G \leq \frac{1}{2\|\Omega\|} \{ \Delta_F G - \varepsilon \frac{\|\nabla T\|^2_F}{|T|^2} - \frac{1}{4} \frac{|\text{Ric}_\omega|^2}{|T|^2} - (\Lambda - 1 - \varepsilon \Lambda^2)|T|^2 - \varepsilon \Lambda^2 \frac{\alpha'}{2} \sigma^2 T_p T_q - (1 + \Lambda) \frac{\alpha'}{4} \sigma_2(i\text{Ric}_\omega) + \frac{\alpha'}{8 |T|^2} g^{ij} \sigma_2^{pq} R_{qi} R_{jp} \]

\[+ (\Lambda - \alpha - 1) \frac{1}{|T|^2} \text{Re} \{ F^{pq} g^{jk} T_j R_{kp} T_q \} - \frac{(1 - \varepsilon)}{4 |T|^4} \|g^{jk} T_j R_{kp}\|^2_F \]

\[-\frac{1}{|T|^2} F^{pq} g^{ij} T_i \hat{R}_{qpl} \hat{T}_j - \frac{\alpha'}{4 |T|^2} \|\Omega\|^3 \hat{g}^{ij} \hat{\rho}^{pq} R_{qi} R_{jp} + \varepsilon \Lambda^2 \alpha' \|\Omega\|^3 \hat{\rho}^{pq} T_q T_p \]

\[+ \alpha' \|\Omega\|^3 \hat{\rho}^{pq} R_{qp} + (\Lambda - 1) \|\Omega\|^2 \nu + \frac{2}{|T|^2} \text{Re} \langle E, T \rangle \}. \tag{4.49}\]

We now substitute \(\Lambda = 1 + \frac{4}{8}\) and \(\varepsilon = \frac{1}{100}\). Then

\[\partial_t G \leq \frac{1}{2\|\Omega\|} \{ \Delta_F G - \frac{1}{100} \frac{\|\nabla T\|^2_F}{|T|^2} - \frac{1}{4} \frac{|\text{Ric}_\omega|^2}{|T|^2} - \frac{1}{9} |T|^2 - \left( \frac{9}{8} \right)^2 \frac{1}{100} \frac{\alpha'}{2} \sigma^2 T_p T_q \]
\[-\frac{17}{16} \alpha' \sigma_2(i\text{Ric}_\omega) + \frac{\alpha'}{8|T|^2} g^{ij} \sigma_2^{pq} R_{qj} R_{jp} + \left( \frac{1}{8} - \frac{9}{800} \right) \frac{1}{|T|^2} \text{Re}\{F^{pq} g^{jk} T_j R_{kp} \hat{T}_q \} - \frac{99}{400} \frac{1}{|T|^4} g^{jk} T_j R_{ki} \right)^2 - \frac{1}{|T|^2} F^{pq} g^{ij} T_i \hat{R}_{qj} \lambda T_\lambda - \frac{\alpha'}{4|T|^2} \Omega^3 g^{ij} \bar{\rho}^{pq} R_{qj} R_{jp} \]
\[+ \alpha' \|\Omega\|^3 \bar{\rho}^{pq} R_{qp} + \frac{1}{100} \left( \frac{9}{8} \right)^2 \alpha' \|\Omega\|^3 \bar{\rho}^{pq} T_p \hat{T}_q + \frac{1}{8 \|\Omega\|^2} \nu + \frac{2}{|T|^2} \text{Re}\{E(T)\} \right) \quad (4.50)\]

We are assuming in the hypothesis of Theorem 4 that $|\alpha' \text{Ric}_\omega| < 10^{-6}$. By Theorem 3, we know that $\|\Omega\| \leq \frac{C}{M} \ll 1$, so for $M$ large enough we can assume

\[(1 - 10^{-6}) g^{ij} \leq F^{ij} \leq (1 + 10^{-6}) g^{ij}. \quad (4.51)\]

One way to see this inequality is by writing $F^{ij}$ in coordinates (4.23). Using (4.51), we can estimate

\[-\frac{17}{16} \alpha' \sigma_2(i\text{Ric}_\omega) - \frac{\alpha'}{8|T|^2} g^{ij} \sigma_2^{pq} R_{qj} R_{jp} + \left( \frac{1}{8} - \frac{9}{800} \right) \frac{1}{|T|^2} \text{Re}\{F^{pq} g^{jk} T_j R_{kp} \hat{T}_q \}
\leq \frac{17}{16} \frac{1}{|T|^2} |\text{Ric}_\omega||\text{Ric}_\omega| + \frac{1}{8} \frac{1}{|T|^2} \|\text{Ric}_\omega\|^2 + \frac{1}{7} |\text{Ric}_\omega|
\leq \frac{1}{(2)(3)^2} |T|^2 + \left( \frac{1}{100} + \frac{1}{(2)(2)^2} \right) |\text{Ric}_\omega|^2. \quad (4.52)\]

We also notice

\[-\left( \frac{9}{8} \right)^2 \frac{1}{100} \frac{\alpha'}{2} \sigma_2^{pq} T_p \hat{T}_q \leq |\alpha' \text{Ric}| |T|^2 \leq \frac{1}{10^6} |T|^2, \quad (4.53)\]

and

\[\frac{1}{|T|^2} F^{pq} g^{ij} T_i \hat{R}_{qj} \lambda T_\lambda \leq C e^{-u} = C \|\Omega\|. \quad (4.54)\]

Substituting these estimates into (4.50) gives

\[
\partial_t G \leq \frac{1}{2 \|\Omega\|^2} \left\{ \Delta_F G - \frac{1}{200} \frac{|\nabla T|^2}{|T|^2} - \frac{1}{100} \frac{|\text{Ric}_\omega|^2}{|T|^2} - \frac{1}{100} |T|^2 \right.
\[+ C \|\Omega\| + \frac{\alpha'}{4|T|^2} \|\Omega\|^3 g^{ij} \bar{\rho}^{pq} R_{qj} R_{jp} + \alpha' \|\Omega\|^3 \bar{\rho}^{pq} R_{qp} \]
\[\left. + \frac{1}{100} \left( \frac{9}{8} \right)^2 \alpha' \|\Omega\|^3 \bar{\rho}^{pq} T_p \hat{T}_q + \frac{1}{8 \|\Omega\|^2} \nu + \frac{2}{|T|^2} \text{Re}\{E(T)\} \right\}. \quad (4.55)\]

By the definition of $E$ (4.14) and $\nu$ (4.10), the terms on the last two lines can only slightly perturb the coefficients of the first line since $\|\Omega\| = e^{-u} \leq \frac{C}{M} \ll 1$ for $M \gg 1$ large enough. We recall that $\bar{\rho}^{pq}$ and $b_i^p \rho$ are bounded in $C^\infty$ in terms of the background metric $\hat{g}$, so for example,

\[\|\Omega\| g^{pq} \leq C e^{-u} \hat{g}^{pq} = C g^{pq}, \quad \|\Omega\|^{1/2} |b_i^p T_i| \leq C |T|. \quad (4.56)\]

29
This allows us to bound certain terms such as
\[ \alpha' \|\Omega\|^3 \bar{\rho}^q R_{qp} \leq C \|\Omega\|^2 |\text{Ric}_\omega| \leq \frac{C}{2} \|\Omega\|^2 \frac{|\text{Ric}_\omega|^2}{|\mathcal{R}|^2} + \frac{C}{2} \|\Omega\|^2 |\mathcal{R}|^2, \]  
(4.57)
and
\[ \alpha' \|\Omega\|^3 \text{Re}\{b_i^p T_i\} \leq C \|\Omega\|^2 |\mathcal{R}| \leq C \|\Omega\|^2 \frac{|\mathcal{R}|^2}{2} + C \|\Omega\|^2. \]  
(4.58)
Covariant derivatives with respect to the evolving metric act like \( \nabla_i = \partial_i - T_i \), so we can bound terms such as
\[ \frac{2}{|\mathcal{R}|^2} \|\Omega\|^3 g^{jk} (\nabla_j b_i^q) R_{qp} \bar{T}_k \leq C \|\Omega\|^2 \frac{|\text{Ric}_\omega|}{|\mathcal{R}|} + C \|\Omega\|^2 |\text{Ric}_\omega| |\mathcal{R}|. \]  
(4.59)
The inequality \( 2ab \leq a^2 + b^2 \) can be used to absorb terms into the first line. We also bound terms
\[ -\frac{2}{|\mathcal{R}|^2} \|\Omega\|^2 g^{jk} \nabla_j \tilde{\mu} \bar{T}_k \leq C \|\Omega\|^2 \frac{\|\Omega\|^1/2}{|\mathcal{R}|}. \]  
(4.60)
Using these estimates, it is possible to show that at the maximum point \((p, t_0)\) of \( G \), for \( \|\Omega\| \leq \frac{C}{M} \ll 1 \), there holds
\[ 0 \leq \frac{1}{2\|\Omega\|} \left\{ \Delta F G - \frac{1}{200} |\mathcal{R}|^2 + C \|\Omega\| \left( 1 + \frac{\|\Omega\|^1/2}{|\mathcal{R}|} \right) \right\}. \]  
(4.61)
By (4.51), \( \Delta F G \leq 0 \) at the maximum \((p, t_0)\) of \( G \), hence
\[ |\mathcal{R}|^2 \leq C \|\Omega\| \leq \frac{C}{M}. \]  
(4.62)
Therefore
\[ G \leq G(p, t_0) \leq \log \frac{C}{M} + \Lambda u(p). \]  
(4.63)
By Theorem 3,
\[ |\mathcal{R}|^2 \leq \frac{C}{M} \exp \{ \Lambda (u(p) - u) \} \]
\[ \leq \frac{C}{M} \left( \sup_{X \times [0, T]} e^u \right)^\Lambda \left( \sup_{X \times [0, T]} e^{-u} \right)^\Lambda \]
\[ \leq \frac{C}{M} (C_2 C_4)^\Lambda \ll 1. \]  
(4.64)
This proves Theorem 4.
5 Evolution of the curvature

5.1 Ricci curvature

In this subsection, we flow the Ricci curvature of the evolving Hermitian metric $e^u \tilde{g}$. We will use the well-known general formula for the evolution of the curvature tensor

$$\partial_t R_{k\bar{j}}^{\alpha \beta} = -\nabla_k \nabla_j (\tilde{g}^\alpha \partial_t \tilde{g}_{\beta \bar{\gamma}}).$$

(5.1)

Recall that we defined $R_{k\bar{j}} = R_{k\bar{j}}^{\alpha \alpha}$, hence substituting (4.9) yields

$$\partial_t R_{k\bar{j}} = -\nabla_k \nabla_j \left\{ \frac{1}{2 \| \Omega \|} \left( -R - \alpha' \| \Omega \| \tilde{\rho}^q R_{q\bar{p}} + \frac{\alpha'}{2} \sigma_2 (i \text{Ric}_\omega) + 2 |T|^2 + 2 \| \Omega \|^2 \nu \right) \right\}. \quad (5.2)$$

Expanding out terms gives

$$\partial_t R_{k\bar{j}} = \frac{1}{2 \| \Omega \|^2} \left\{ \nabla_k \nabla_j R + \alpha' \| \Omega \| \tilde{\rho}^q \nabla_k \nabla_j R_{q\bar{p}} - \nabla_k \nabla_j \frac{\alpha'}{2} \sigma_2 (i \text{Ric}_\omega) - 2 \nabla_k \nabla_j |T|^2 \right. \right.$$  

$$+ \alpha' \nabla_k (\| \Omega \| \tilde{\rho}^q) \nabla_j R_{q\bar{p}} + \alpha' \nabla_j (\| \Omega \| \tilde{\rho}^q) \nabla_k R_{q\bar{p}}$$  

$$+ \alpha' \nabla_k \nabla_j (\| \Omega \| \tilde{\rho}^q) R_{q\bar{p}} - \nabla_k \nabla_j 2 \| \Omega \|^2 \nu \} \left\{ R + \alpha' \| \Omega \| \tilde{\rho}^q R_{q\bar{p}} \right. \right.$$  

$$- \frac{\nabla_j \| \Omega \|}{2 \| \Omega \|^2} \nabla_k \left\{ R + \alpha' (\| \Omega \| \tilde{\rho}^q R_{q\bar{p}}) - \frac{\alpha'}{2} \sigma_2 (i \text{Ric}_\omega) - 2 |T|^2 - 2 \| \Omega \|^2 \nu \right\}$$  

$$- \frac{\nabla_k \| \Omega \|}{2 \| \Omega \|^2} \nabla_j \left\{ R + \alpha' (\| \Omega \| \tilde{\rho}^q R_{q\bar{p}}) - \frac{\alpha'}{2} \sigma_2 (i \text{Ric}_\omega) - 2 |T|^2 - 2 \| \Omega \|^2 \nu \right\}$$  

$$\left. + \left\{ \frac{2 \nabla_k \| \Omega \| \nabla_j \| \Omega \|}{2 \| \Omega \|^3} \right\} \left\{ R + \alpha' \| \Omega \| \tilde{\rho}^q R_{q\bar{p}} \right. \right.$$  

$$- \frac{\alpha'}{2} \sigma_2 (i \text{Ric}_\omega) - 2 |T|^2 - 2 \| \Omega \|^2 \nu \right\}. \quad (5.3)$$

Using $\nabla_j \| \Omega \| = \| \Omega \||T|_j$,

$$\partial_t R_{k\bar{j}} = \frac{1}{2 \| \Omega \|^2} \left\{ \nabla_k \nabla_j R + \alpha' \| \Omega \| \tilde{\rho}^q \nabla_k \nabla_j R_{q\bar{p}} - \nabla_k \nabla_j \frac{2 \alpha'}{2} \sigma_2 (i \text{Ric}_\omega)$$  

$$- 2 \nabla_k \nabla_j |T|^2 + \alpha' \nabla_k (\| \Omega \| \tilde{\rho}^q) \nabla_j R_{q\bar{p}} + \alpha' \nabla_j (\| \Omega \| \tilde{\rho}^q) \nabla_k R_{q\bar{p}}$$  

$$+ \alpha' \nabla_k \nabla_j (\| \Omega \| \tilde{\rho}^q) R_{q\bar{p}} - 2 \nabla_k \nabla_j (\| \Omega \|^2 \nu) - T_j \nabla_k R$$  

$$- \alpha' T_j \nabla_k (\| \Omega \| \tilde{\rho}^q R_{q\bar{p}}) + 2 T_j \nabla_k |T|^2 + \frac{\alpha'}{2} T_j \nabla_k (\sigma_2 (i \text{Ric}_\omega)) + 2 T_j \nabla_k \left\{ \| \Omega \|^2 \nu \right\}$$  

$$- T_k \nabla_j R - \alpha' T_k \nabla_j (\| \Omega \| \tilde{\rho}^q R_{q\bar{p}}) + 2 T_k \nabla_j |T|^2 + \frac{\alpha'}{2} T_k \nabla_j (\sigma_2 (i \text{Ric}_\omega))$$  

$$+ 2 T_k \nabla_j \left\{ \| \Omega \|^2 \nu \right\} + R T_j T_k + \alpha' T_j T_k (\| \Omega \| \tilde{\rho}^q R_{q\bar{p}}) - 2 |T|^2 T_j T_k$$  

$$- \frac{\alpha'}{2} \sigma_2 (i \text{Ric}_\omega) T_j T_k - 2 T_j T_k \left\{ \| \Omega \|^2 \nu \right\} - R \nabla_k T_j - \alpha' \nabla_k T_j (\| \Omega \| \tilde{\rho}^q R_{q\bar{p}})$$  

$$+ 2 |T|^2 \nabla_k T_j + \frac{\alpha'}{2} \sigma_2 (i \text{Ric}_\omega) \nabla_k T_j + 2 \nabla_k T_j \left\{ \| \Omega \|^2 \nu \right\}. \quad (5.4)$$
We now study the highest order terms, namely
\[ \nabla_k \nabla_j R_{qp} = -2 \nabla_k \nabla_j \nabla_p \nabla_q u. \quad (5.5) \]
We will use the following commutation formula for covariant derivatives in Hermitian geometry
\[ \nabla_k \nabla_j \nabla_p \nabla_q u = \nabla_p \nabla_q \nabla_j \nabla_k u + T^\lambda_{pj} \nabla_q \nabla_\lambda \nabla_k u + \bar{T}^\lambda_{qk} \nabla_p \nabla_j \nabla_\lambda u \\
+ R_{kj} \nabla_p \nabla_q u - R_{qpk} \nabla_\lambda u + \bar{T}^\lambda_{qk} T^\gamma_{pj} \nabla_q u. \quad (5.6) \]
Using \( R_{qp} = -2u_{qp} \), we obtain
\[ \nabla_k \nabla_j R_{qp} = \nabla_p \nabla_q R_{kj} + T^\lambda_{pj} \nabla_q R_{k\lambda} + \bar{T}^\lambda_{qk} \nabla_p R_{k\lambda} \\
+ R_{kj} \nabla_p \nabla_q u - R_{qpk} \nabla_\lambda u + \bar{T}^\lambda_{qk} T^\gamma_{pj} \nabla_q u. \quad (5.7) \]
Hence
\[ \nabla_k \nabla_j R = g^{pq} \nabla_p \nabla_q R_{kj} + g^{pq} T^\lambda_{pj} \nabla_q R_{k\lambda} + g^{pq} \bar{T}^\lambda_{qk} \nabla_p R_{k\lambda} \\
+ R_{kj} \nabla_p \nabla_q u - R_{qpk} \nabla_\lambda u + g^{pq} \bar{T}^\gamma_{pj} \nabla_q u. \quad (5.8) \]
Differentiating \( \sigma_2^{pq} \) (4.19) leads to the following definition
\[ \sigma_2^{pq,rs} = g^{pq} g^{rs} - g^{pr} g^{qs}. \quad (5.9) \]
With this notation, we now differentiate \( \sigma_2(i\text{Ric}_\omega) \) twice.
\[ \nabla_k \nabla_j \sigma_2(i\text{Ric}_\omega) = \nabla_k (\sigma_2^{pq} \nabla_j R_{qp}) \\
= \sigma_2^{pq} \nabla_k \nabla_j R_{qp} + \sigma_2^{pq,rs} \nabla_k R_{sr} \nabla_j R_{qp} \\
= \sigma_2^{pq} \nabla_p \nabla_q R_{kj} + \sigma_2^{pq,rs} \nabla_k R_{sr} \nabla_j R_{qp} + \sigma_2^{pq} T^\lambda_{pj} \nabla_q R_{k\lambda} \\
+ \sigma_2^{pq} \bar{T}^\lambda_{qk} \nabla_p R_{k\lambda} + \sigma_2^{pq} R_{kj} \nabla_p \nabla_q u - \sigma_2^{pq} R_{qpk} \nabla_\lambda u \\
+ \sigma_2^{pq} \bar{T}^\gamma_{pj} \nabla_q u. \quad (5.10) \]
By (5.8) and (5.10), and proceeding similarly for the \( \rho \) term, we obtain
\[ \nabla_k \nabla_j R + \alpha' ||\Omega||^3 \rho^{pq} \nabla_k \nabla_j R_{qp} - \frac{\alpha'}{2} \nabla_k \nabla_j \sigma_2(i\text{Ric}_\omega) \\
= F^{pq} \nabla_p \nabla_q R_{kj} - \frac{\alpha'}{2} \sigma_2^{pq,rs} \nabla_k R_{sr} \nabla_j R_{qp} + F^{pq} T^\lambda_{pj} \nabla_q R_{k\lambda} + F^{pq} \bar{T}^\lambda_{qk} \nabla_p R_{k\lambda} \\
+ F^{pq} R_{kj} \nabla_p \nabla_q u - F^{pq} R_{qpk} \nabla_\lambda u + F^{pq} \bar{T}^\gamma_{pj} \nabla_q u, \quad (5.11) \]
where the definition of \( F^{pq} \) was given in (4.20).
Using (2.38), we may convert derivatives of torsion $\nabla T$ into curvature terms, but terms $\nabla T$ are of different type and must be treated separately. For example

$$-2 \nabla_k \nabla_j |T|^2 = -2g^{pq} \nabla_k \nabla_j T_p T_q - 2g^{pq} \nabla_j T_p \nabla_k T_q - \frac{1}{2}g^{pq} R_{kp} R_{qj} - g^{pq} T_p \nabla_k R_{qj}$$

$$= -g^{pq} \nabla_j R_{kp} T_q - 2g^{pq} R_{kj} ^\lambda \, _p T_q T_j - 2g^{pq} \nabla_j T_p \nabla_k T_q - \frac{1}{2}g^{pq} R_{kp} R_{qj} - g^{pq} T_p \nabla_k R_{qj}.$$ \hspace{1cm} (5.12)

Substituting (5.11) and (5.12) into (5.4),

$$\partial_q R_{kj} = \frac{1}{2 \| \Omega \|} \left\{ F^{\rho q} \nabla_p \nabla_q R_{kj} - \frac{\alpha'}{2} \sigma_2 ^{p q, r s} \nabla_k R_{sr} \nabla_j R_{qp} \right.$$  

$$+ 2\alpha' \| \Omega \| ^3 \rho^{pq} \nabla_j T_p \nabla_k T_q - 2g^{pq} \nabla_j T_p \nabla_k T_q + Y_{kj} \right\}. \hspace{1cm} (5.13)$$

where $Y_{kj}$ contains various combinations of torsion and curvature terms, but is linear in first derivatives of curvature and torsion and does not contain higher order derivatives of curvature and torsion. Explicitly,

$$Y_{kj} = F^{\rho q} T^\lambda _{pj} \nabla_q R_{k\lambda} + F^{\rho q} - T^\lambda _{qj} \nabla_k R_{p\lambda} + F^{\rho q} R_{kj} ^\lambda \, _p R_{q\lambda} - F^{\rho q} R_{qk} \tilde{\lambda} R_{kj}$$

$$+ F^{\rho q} T^\lambda _{qj} T_p R_{k\gamma} + g^{pq} \nabla_j R_{kp} T_q - 2g^{pq} R_{kj} ^\lambda \, _p T_q T_j - \frac{1}{2}g^{pq} R_{kp} R_{qj}$$

$$- g^{pq} T_p \nabla_k R_{qj} + \alpha' \nabla_j (\| \Omega \| ^3 \rho^{pq}) \nabla_j R_{qp} + \alpha' \nabla_j (\| \Omega \| ^3 \rho^{pq}) \nabla_k R_{qp}$$

$$+ \alpha' (\nabla_k \nabla_j (\| \Omega \| ^3 \rho^{pq}) R_{qp} - T_j F^{\rho q} \nabla_k R_{qp} - \alpha' T_j \nabla_k (\| \Omega \| ^3 \rho^{pq}) R_{qp} + g^{pq} T_j R_{kp} T_q$$

$$+ 2g^{pq} T_j T_p \nabla_k T_q - 2 \left\{ - \alpha' \nabla_k \nabla_j (\| \Omega \| ^3 \rho^{pq}) - \frac{\alpha'}{2} \Re \{ \| \Omega \| ^3 \phi_j \nabla_j R_{ki} \} \right.$$  

$$- \alpha' \Re \{ \| \Omega \| ^3 \phi_j \nabla_j T_i \} - \alpha' \Re \{ \nabla_k (\| \Omega \| ^3 \phi_j) \nabla_j T_i \}$$

$$- \alpha' \Re \{ \nabla_j (\| \Omega \| ^3 \phi_j) \nabla_k T_i \} - \alpha' \Re \{ \nabla_k \nabla_j (\| \Omega \| ^3 \phi_j) T_i \} + \nabla_k \nabla_j (\| \Omega \| ^3 \phi_j) \}$$

$$+ \left\{ 2\alpha' (\nabla_k \nabla_j (\| \Omega \| ^3 \rho^{pq}) T_p T_q + 2\alpha' \nabla_k (\| \Omega \| ^3 \rho^{pq}) \nabla_j (T_p T_q) \right.$$  

$$+ 2\alpha' \nabla_j (\| \Omega \| ^3 \rho^{pq}) \nabla_j (T_p T_q) + \alpha' \| \Omega \| ^3 \rho^{pq} \nabla_j R_{kp} T_q + 2\alpha' \| \Omega \| ^3 \rho^{pq} R_{kj} ^\lambda \, _p T_q T_j$$

$$+ \alpha' \| \Omega \| ^3 \rho^{pq} T_p \nabla_k R_{qj} + \frac{\alpha'}{2} \| \Omega \| ^3 \rho^{pq} R_{kp} R_{qj} \right\}$$

$$+ 2T_k \nabla_j \left\{ - \alpha' \| \Omega \| ^3 \phi_j \nabla_j (\| \Omega \| ^3 \rho^{pq}) T_p T_q + 2T_k F^{\rho q} \nabla_j R_{kp} \nabla_j (\| \Omega \| ^3 \rho^{pq}) R_{qp} + g^{pq} T_k \nabla_j R_{qp} + g^{pq} T_k T_p R_{qj} \right.$$  

$$+ 2T_k \nabla_j \left\{ - \alpha' \| \Omega \| ^3 \phi_j \nabla_j (\| \Omega \| ^3 \rho^{pq}) R_{qp} - \alpha' \| \Omega \| ^3 \rho^{pq} \nabla_j T_i \} - \alpha' \| \Omega \| ^3 \rho^{pq} T_p T_q + \| \Omega \| ^3 \phi_j \}$$

$$+ RT_j T_k + \alpha' T_j T_k (\| \Omega \| ^3 \rho^{pq} R_{qp}) - 2T_j T_k T_k - \frac{\alpha'}{2} \sigma_2 (i \text{Ric}_j) T_j T_k \right\}.$$  

33
\[-2T\gamma_k \left\{ -\alpha'\|\Omega\|^3\psi_p - \alpha'\|\Omega\|^3\text{Re}\{b^k_p T_k\} - \alpha'\|\Omega\|^3\bar{\rho}^{pq} T_p \bar{T}_q + \|\Omega\|^2 \bar{\mu} \right\} \]

\[-\frac{1}{2} \alpha' \rho_T k_j (\|\Omega\|^3 \bar{\rho}^{pq} R_{qp} + |T|^2 R_{k\bar{j}} + \frac{\alpha'}{4} \sigma_2 (i\text{Ric}_\omega) R_{k\bar{j}} \]

\[+ R_{k\bar{j}} \left\{ -\alpha'\|\Omega\|^3\psi_p - \alpha'\|\Omega\|^3\text{Re}\{b^k_p T_k\} + \alpha'\|\Omega\|^3\bar{\rho}^{pq} T_p \bar{T}_q + \|\Omega\|^2 \bar{\mu} \right\}. \tag{5.14} \]

The terms in brackets indicate terms which come from substituting the definition of \(\nu\) (4.10).

### 5.2 Evolving the norm of the curvature

We will compute

\[\partial_t |\text{Ric}_\omega|^2 = \partial_t \{g^{k\bar{i}} g^{i\bar{j}} R_{\bar{k}i} R_{k\bar{j}}\}. \tag{5.15} \]

We have

\[\partial_t g^{i\bar{j}} = -g^{i\bar{k}} g^{j\bar{\gamma}} \partial_t g_{\bar{k}\gamma} = \frac{1}{2} \frac{\|\Omega\|}{\|\Omega\|^3} \left( R + \frac{\alpha'}{2} \|\Omega\|^3 \bar{\rho}^{pq} R_{qp} - \frac{\alpha'}{2} \sigma_2 (i\text{Ric}_\omega) - |T|^2 - \|\Omega\|^2 \nu \right) g^{i\bar{j}}. \tag{5.16} \]

Hence

\[\partial_t |\text{Ric}_\omega|^2 = 2 \text{Re} \langle \partial_t \text{Ric}_\omega, \text{Ric}_\omega \rangle + \|\text{Ric}_\omega\|^2 \left( R + \alpha'\|\Omega\|^3 \bar{\rho}^{pq} R_{qp} - \frac{\alpha'}{2} \sigma_2 (i\text{Ric}_\omega) - 2 |T|^2 - 2 \|\Omega\|^2 \nu \right). \tag{5.17} \]

Next,

\[Q^{pq} \nabla_p \nabla_q |\text{Ric}_\omega|^2 = g^{k\bar{i}} g^{i\bar{j}} Q^{pq} \nabla_p \nabla_q R_{\bar{k}i} R_{k\bar{j}} + g^{k\bar{i}} g^{i\bar{j}} R_{\bar{k}i} Q^{pq} \nabla_p \nabla_q R_{k\bar{j}} + \|\nabla \text{Ric}_\omega\|^2_{Fgg} + \|\nabla \text{Ric}_\omega\|^2_{Fg} \]

\[= g^{k\bar{i}} g^{i\bar{j}} Q^{pq} \nabla_p \nabla_q R_{\bar{k}i} R_{k\bar{j}} + g^{k\bar{i}} g^{i\bar{j}} R_{\bar{k}i} Q^{pq} \nabla_p \nabla_q R_{k\bar{j}} - g^{k\bar{i}} g^{i\bar{j}} R_{\bar{k}i} Q^{pq} R_{q\bar{\lambda}} R_{\bar{j}\lambda} + g^{k\bar{i}} g^{i\bar{j}} R_{\bar{k}i} Q^{pq} R_{q\bar{\lambda}} R_{\bar{j}\lambda} + \|\nabla \text{Ric}_\omega\|^2_{Fgg} + \|\nabla \text{Ric}_\omega\|^2_{Fg}. \tag{5.18} \]

We have shown

\[\Delta_F |\text{Ric}_\omega|^2 = 2 \text{Re} \langle \Delta_F \text{Ric}_\omega, \text{Ric}_\omega \rangle + \|\nabla \text{Ric}_\omega\|^2_{Fgg} + \|\nabla \text{Ric}_\omega\|^2_{Fg} - g^{k\bar{i}} g^{i\bar{j}} R_{\bar{k}i} Q^{pq} R_{q\bar{\lambda}} R_{\bar{j}\lambda} + g^{k\bar{i}} g^{i\bar{j}} R_{\bar{k}i} Q^{pq} R_{q\bar{\lambda}} R_{\bar{j}\lambda}. \tag{5.19} \]

Substituting (5.13) into (5.17) gives

\[\partial_t |\text{Ric}_\omega|^2 = \frac{1}{2} \frac{\|\Omega\|}{\|\Omega\|^3} \left( \Delta_F |\text{Ric}_\omega|^2 - \|\nabla \text{Ric}_\omega\|^2_{Fgg} - \|\nabla \text{Ric}_\omega\|^2_{Fg} \right). \tag{5.20} \]
Lemma 1 Let \( 0 < \delta, \epsilon < \frac{1}{2} \) be such that \(-\frac{1}{4} g^{\bar{p}\bar{q}} < \alpha' \delta^2 \| \Omega \| \rho^{\bar{p}\bar{q}} < \frac{1}{4} g^{\bar{p}\bar{q}}\), and

\[
\| \Omega \|^2 \leq \delta, \quad |T|^2 \leq \delta, \quad |\alpha' \text{Ric}_\omega| \leq \epsilon,
\]

at a point \((p, t_0)\). Let \( \Lambda > 1 \) be any constant. Then at \((p, t_0)\) there holds

\[
\partial_t (|\alpha' \text{Ric}_\omega|^2 + \Lambda |T|^2) \\
\leq \frac{1}{2 \| \Omega \|^2} \left\{ \Delta_F (|\alpha' \text{Ric}_\omega|^2 + \Lambda |T|^2) - \left( \frac{1}{2} - 2 \epsilon \right) |\alpha' \nabla \text{Ric}_\omega|^2 \right\} \\
- \left( \frac{\Lambda}{4} - (5 + C \delta^2) \epsilon |\alpha'|^{-1} \right) |\nabla T|^2 - \frac{\Lambda}{8} |\text{Ric}_\omega|^2 + C(1 + \Lambda) \epsilon \delta + C \epsilon^2 + C \Lambda \delta,
\]

for some constant \( C \) only depending on \( \tilde{\mu}, \rho, \alpha' \), and the background manifold \((X, \tilde{\omega})\).

Proof: Since \( \epsilon \) and \( \delta \) are assumed to be small, we have

\[
F^{\bar{p}\bar{q}} = g^{\bar{p}\bar{q}} + \alpha' \| \Omega \|^2 \rho^{\bar{p}\bar{q}} - \frac{\alpha'}{2} \sigma_2^{\bar{p}\bar{q}} - \frac{1}{2} g^{\bar{p}\bar{q}} < F^{\bar{p}\bar{q}} < \frac{3}{2} g^{\bar{p}\bar{q}}.
\]

We note the following estimate

\[
-\alpha' \text{Re} \{ g^{\bar{j}\bar{k}} g^{m\bar{n}} \sigma_2^{\bar{p}\bar{q},\bar{r}\bar{s}} R_{\bar{i}\bar{m}} \nabla_{\bar{k}} R_{\bar{s}\bar{r}} \nabla_{\bar{j}} R_{\bar{q}\bar{p}} \} \leq |\alpha' \text{Ric}_\omega| \| \nabla \text{Ric}_\omega \|^2.
\]

We will estimate and group terms in (5.20) and (5.14). We will convert \( F^{\bar{p}\bar{q}} \) into the metric \( g^{\bar{p}\bar{q}} \), and handle \( \rho^{\bar{p}\bar{q}} \) and \( b^i \) as in (4.56). We will also use that the norm of the full torsion \( T(\omega) = i \partial \omega \) is \( 2 |T|^2 \), \( \nabla_i \| \Omega \| = \| \Omega \| T_i \), \( \nabla_k \nabla_i \| \Omega \| = \| \Omega \| T_i \nabla_k + 2^{-1} \| \Omega \| R_{kij} \), and \( \| \Omega \| \leq 1 \).

\[
\partial_t |\text{Ric}_\omega|^2 \leq \frac{1}{2 \| \Omega \|^2} \left\{ \Delta_F |\text{Ric}_\omega|^2 - \frac{1}{2} |\nabla \text{Ric}_\omega|^2 - \frac{1}{2} |\nabla \text{Ric}_\omega|^2 \right\} \\
+ |\alpha' \text{Ric}_\omega| |\nabla \text{Ric}_\omega|^2 + (4 + C \| \Omega \|^2) |\text{Ric}_\omega| |\nabla T|^2
\]

5.3 Estimating Ricci curvature
\[ + \frac{C}{2\|\Omega\|} \left\{ |T||\text{Ric}_\omega||\nabla\text{Ric}_\omega| + \|\Omega\|^2(1 + |T|)|\text{Ric}_\omega||\nabla\text{Ric}_\omega| \\
+ (|\text{Ric}_\omega| + |\text{Ric}_\omega|^2)|T|^2|\nabla T| + |Rm||\text{Ric}_\omega|^2 + |Rm||\text{Ric}_\omega||T|^2 \\
+ |\text{Ric}_\omega|^2|T|^2 + |\text{Ric}_\omega||T|^4 + |\text{Ric}_\omega|^3(|T| + 1)^2 + |\text{Ric}_\omega|^4 \\
+ \|\Omega\|^2|\text{Ric}_\omega|(|T| + 1)^4(|\text{Ric}_\omega| + |Rm| + |\nabla T| + 1) \right\}. \]

First, we estimate
\[ C(|\text{Ric}_\omega| + |\text{Ric}_\omega|^2)|T|^2|\nabla T| \leq |\text{Ric}_\omega||\nabla T|^2 + \frac{C^2}{2}|\text{Ric}_\omega|(1 + |\text{Ric}_\omega|)|T|^4. \] (5.26)
\[ C|T||\text{Ric}_\omega||\nabla\text{Ric}_\omega| \leq \frac{1}{2}|\alpha'|\text{Ric}_\omega||\nabla\text{Ric}_\omega|^2 + \frac{C^2}{2|\alpha'|}|\text{Ric}_\omega||T|^2. \] (5.27)

We may estimate, using \(|T| \leq 1|
\[ C\|\Omega\|^2|\text{Ric}_\omega|(|T| + 1)^4|\nabla T| \leq \|\Omega\|^2|\text{Ric}_\omega||\nabla T|^2 + \frac{C^2}{4}(2)^8\|\Omega\|^2|\text{Ric}_\omega|, \] (5.28)
\[ C\|\Omega\|^2(1 + |T|)|\text{Ric}_\omega||\nabla\text{Ric}_\omega| \leq \frac{1}{2}|\alpha'|\text{Ric}_\omega||\nabla\text{Ric}_\omega|^2 + \frac{1}{2|\alpha'|}(2C\|\Omega\|^2)|\text{Ric}_\omega|. \] (5.29)

Recall that
\[ R_{kj}^\alpha \beta = \hat{R}_{kj}^\alpha \beta + \frac{1}{2}R_{kj} \delta^\alpha \beta. \] (5.30)

Hence, using \(|\Omega| \leq 1|, \|T\| \leq 1| and \|\alpha'|\text{Ric}_\omega| \leq 1| on lower order terms, from (5.25) and the above estimates, we get
\[ \partial_t|\text{Ric}_\omega|^2 \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F|\text{Ric}_\omega|^2 - \left( \frac{1}{2} - 2\|\text{Ric}_\omega\| \right)|\nabla\text{Ric}_\omega|^2 + (5 + C\|\Omega\|^2)|\text{Ric}_\omega||\nabla T|^2 \right\} \\
+ \frac{C}{2\|\Omega\|} \left\{ |\text{Ric}_\omega||T|^2 + |\text{Ric}_\omega|^2 + \|\Omega\|^2|\text{Ric}_\omega| \right\}. \] (5.31)

In terms of \(0 < \epsilon, \delta < 1\), we have
\[ \partial_t|\alpha'|\text{Ric}_\omega|^2 \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F|\alpha'|\text{Ric}_\omega|^2 - \left( \frac{1}{2} - 2\epsilon \right)|\alpha'|\nabla\text{Ric}_\omega|^2 \\
+ (5 + C\delta^2)\epsilon|\alpha'|^{-1}|\nabla T|^2 + C\delta \epsilon + C\epsilon^2 \right\}. \] (5.32)

Using the evolution of the torsion (4.29)
\[ \partial_t|T|^2 = \frac{1}{2\|\Omega\|} \left\{ \Delta_F|T|^2 - |\nabla T|^2_{FG} - \frac{1}{4}|\text{Ric}_\omega|^2_{FG} - 2\text{Re}\{g^{ij}g^{pq}\nabla_iT_p\bar{T}_q\bar{T}_j\} \\
- \text{Re}\{g^{ij}g^{pq}T_iR_{q\bar{i}\bar{T}_j}\} - \frac{1}{2}R|T|^2 + \frac{\alpha'}{4}\sigma_2(i\text{Ric}_\omega)|T|^2 \\
+ \text{Re}\{F_{pq}g^{ij}T_{\lambda}^{\lambda}T_i\bar{T}_j\bar{T}_\lambda + R_{q\bar{i}\bar{T}_j}\bar{T}_\lambda + |T|^4 \\
+ \frac{\alpha'}{2}\|\Omega\|^2\bar{\rho}^{pq}R_{q\bar{i}}|T|^2 - |T|^2\|\Omega\|^2\nu + 2\text{Re}\langle E, T \rangle \right\}. \] (5.33)
Estimating by replacing $F^q\tilde{p}$ by the evolving metric $g^p\tilde{q}$,

$$
\partial_t|T|^2 \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F|T|^2 - \frac{1}{2} |\nabla T|^2 - \frac{1}{8} |\text{Ric}_\omega|^2 + 2|\nabla T||T|^2 + C|\text{Ric}_\omega||T|^2 \\
+ |\mathcal{R}|T|^2 + \frac{\alpha'}{4} |\text{Ric}_\omega|^2 T|^2 + \|\Omega\||\hat{\text{R}} \omega|T|^2 + |T|^4 \\
+ C\|\Omega\|^2(|T|^4 + |T|^3 + |T|^2 + |T|)(1 + |\text{Ric}_\omega| + |\nabla T|) \right\}. \tag{5.34}
$$

Estimate

$$
2|\nabla T||T|^2 \leq \frac{1}{8}|\nabla T|^2 + 8|T|^4, \tag{5.35}
$$

and

$$
C\|\Omega\|^2(|T|^4 + |T|^3 + |T|^2 + |T|)|\nabla T| \leq \frac{1}{8}|\nabla T|^2 + 2C^2\|\Omega\|^4(4)^2. \tag{5.36}
$$

Using $0 < \delta, \epsilon < 1$,

$$
\partial_t|T|^2 \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F|T|^2 - \frac{1}{4} |\nabla T|^2 - \frac{1}{8} |\text{Ric}_\omega|^2 + C\epsilon\delta + C\delta \right\}. \tag{5.37}
$$

Combining (5.32) and (5.37), we obtain the desired estimate.

**Theorem 5** Start the flow with a constant function $u_0 = \log M$. There exists $M_0 \gg 1$ such that for every $M \geq M_0$, if

$$
\|\Omega\|^2 \leq \frac{C_2^2}{M^2}, \quad |T|^2 \leq \frac{C_3}{M}, \tag{5.38}
$$

along the flow, then

$$
|\alpha'\text{Ric}_\omega| \leq \frac{C_5}{M^{1/2}}, \tag{5.39}
$$

where $C_5$ only depends on $(X, \hat{\omega}), \rho, \tilde{\mu}$ and $\alpha'$. Here, $C_2$ and $C_3$ are the constants given in Theorems 3 and 4 respectively.

**Proof:** Denote

$$
\epsilon = \frac{1}{M^{1/2}}, \quad \delta = \frac{C_3}{M}. \tag{5.40}
$$

Let $C_4$ denote the constant $C$ on the right-hand side of (5.22), which only depends on $(X, \hat{\omega}), \rho, \tilde{\mu}$ and $\alpha'$. For $M_0$ large enough, we can simultaneously satisfy the hypothesis of Lemma 1, and the inequalities $2\epsilon < \frac{1}{2}$ and $(5 + C_4\delta^2)\epsilon \leq 1$. We will study the evolution equation of

$$
|\alpha'\text{Ric}_\omega|^2 + \Lambda|T|^2, \tag{5.41}
$$

where $\Lambda$ is a constant given by

$$
\Lambda = \max\{ 4|\alpha'|^{-1}, 8|\alpha'|^2(C_4 + 1) \}. \tag{5.42}
$$
With this choice of $\Lambda$ and $M_0$, we have

$$
\left( \frac{1}{2} - 2\epsilon \right) \geq 0, \quad \left( \frac{\Lambda}{4} - (5 + C_4\delta^2)\epsilon|\alpha'|^{-1} \right) \geq 0.
$$

At $t = 0$, $u_0 = \log M$ and it follows that

$$
\alpha'^2|Ric_\omega|^2 + \Lambda|T|^2 = 0.
$$

(5.44)

Suppose that along the flow, we reach

$$
\alpha'^2|Ric_\omega|^2 + \Lambda|T|^2 = (2\Lambda C_3 + 1)\epsilon^2,
$$

at some point $p \in X$ at a first time $t_0 > 0$. By Lemma 1,

$$
\partial_t (|\alpha' Ric_\omega|^2 + \Lambda|T|^2) \leq \frac{1}{2\|\Omega\|} \left\{ -\frac{\Lambda}{8}|Ric_\omega|^2 + C_4(1 + \Lambda)\epsilon\delta + C_4\epsilon^2 + C_4\Lambda\delta \right\}. \quad (5.46)
$$

At $(p, t_0)$, we have

$$
|\alpha' Ric_\omega|^2 = (2\Lambda C_3 + 1)\epsilon^2 - \Lambda|T|^2 \geq (2\Lambda C_3 + 1)\epsilon^2 - \Lambda\delta.
$$

(5.47)

Thus

$$
\partial_t (|\alpha' Ric_\omega|^2 + \Lambda|T|^2) \leq \frac{1}{2\|\Omega\|} \left\{ -\frac{\Lambda}{8}|\alpha'|^2 \epsilon^2 + C_4\epsilon^2 - \frac{\Lambda^2}{8|\alpha'|^2}(2C_3\epsilon^2 - \delta) + C_4\Lambda\delta + C_4(1 + \Lambda)\epsilon\delta \right\}.
$$

(5.48)

After substituting the definition of $\epsilon$ and $\delta$, we obtain

$$
\partial_t (|\alpha' Ric_\omega|^2 + \Lambda|T|^2) \leq \frac{1}{2\|\Omega\|} \left\{ -\left( \frac{\Lambda}{8|\alpha'|^2} - C_4 \right) \frac{1}{M} - \left( \frac{\Lambda}{8|\alpha'|^2} - C_4 \right) \frac{C_3\Lambda}{M} + C_3C_4(1 + \Lambda) \frac{1}{M^{1/2}} \right\}.
$$

(5.49)

By our choice of $\Lambda$ (5.42), for $M_0 \gg 1$ depending only on $(X, \hat{\omega})$, $\alpha'$, $\tilde{\mu}$, $\rho$, for all $M \geq M_0$ we have at $(p, t_0)$

$$
\partial_t (|\alpha' Ric_\omega|^2 + \Lambda|T|^2) \leq 0.
$$

(5.50)

Hence along the flow, there holds

$$
|\alpha' Ric_\omega|^2 + \Lambda|T|^2 \leq (2\Lambda C_3 + 1)\epsilon^2.
$$

(5.51)

It follows that

$$
|\alpha' Ric_\omega| \leq (2\Lambda C_3 + 1)^{1/2}\epsilon
$$

is preserved along the flow.
6 Higher order estimates

6.1 The evolution of derivatives of torsion

6.1.1 Covariant derivative of torsion

Since $\nabla_k T_j = \frac{1}{2} R_{kj}$, we only need to look at $\nabla_k T_j$. We will compute

$$\partial_t \nabla_i T_j = \nabla_i \partial_t T_j - \partial_t \Gamma^\lambda_{ij \lambda}.$$  \hspace{1cm} (6.1)

First, using the standard formula for the evolution of the Christoffel symbols and (1.6), we compute

$$\partial_t \Gamma^\lambda_{ij} = g^{\mu i} \nabla_i \partial_t g_{\mu j}$$

$$= \nabla_i \left\{ \frac{1}{2\|\Omega\|} \left( -\frac{R}{2} - \frac{\alpha'}{2} \|\Omega\|^3 \bar{\rho}^{pq} R_{qp} + |T|^2 + \frac{\alpha'}{4} \sigma_2 (i \text{Ric}_\omega) + \alpha' \|\Omega\|^2 \nu \right) \right\} \delta^\lambda_j$$

$$= \frac{1}{2\|\Omega\|} \left\{ -\nabla_i R - \frac{\alpha'}{2} \|\Omega\|^3 \bar{\rho}^{pq} \nabla_i R_{qp} + \frac{\alpha'}{4} \sigma_2^{pq} \nabla_i R_{qp} + g^{pq} \nabla_i T_p T_q$$

$$+ \frac{1}{2} g^{pq} T_p R_{qi} + \frac{R}{2} T_i - |T|^2 T_i - \frac{\alpha'}{4} \sigma_2 (i \text{Ric}_\omega) T_i - E_i \right\} \delta^\lambda_j.$$  \hspace{1cm} (6.2)

We recall that the definition of $E_i$ is given in (4.14). Using (4.21)

$$\partial_t \nabla_i T_j = \frac{1}{2\|\Omega\|} \nabla_i \left\{ F^{pq} \nabla_p \nabla_q T_j - \nabla_j |T|^2 - \frac{1}{2} T_j R + \frac{\alpha'}{4} T_j \sigma_2 (i \text{Ric}_\omega) + \frac{1}{2} F^{pq} T^\lambda_{pj} R_{q\lambda}$$

$$+ T_j |T|^2 + E_j \right\} + \nabla_i \left\{ \frac{1}{2\|\Omega\|} \left\{ F^{pq} \nabla_p \nabla_q T_j - g^{pq} \nabla_j T_p T_q - \frac{1}{2} g^{pq} T_p R_{qi}$$

$$- \frac{1}{2} T_j R + \frac{\alpha'}{4} T_j \sigma_2 (i \text{Ric}_\omega) + \frac{1}{2} F^{pq} T^\lambda_{pj} R_{q\lambda} + T_j |T|^2 + E_j \right\}$$

$$- \frac{1}{2\|\Omega\|} \left\{ - F^{pq} \nabla_p \nabla_q T_i + g^{pq} \nabla_i T_p T_q + \frac{1}{2} g^{pq} T_p R_{qi}$$

$$+ \frac{R}{2} T_i - |T|^2 T_i - \frac{\alpha'}{4} \sigma_2 (i \text{Ric}_\omega) T_i - E_i \right\} T_j.$$  \hspace{1cm} (6.3)

First, we may rewrite

$$F^{pq} \nabla_p \nabla_q T_j = \frac{1}{2} F^{pq} \nabla_p R_{qj}.$$  \hspace{1cm} (6.4)

Next,

$$\nabla_i \left\{ F^{pq} \nabla_p \nabla_q T_j \right\} = F^{pq} \nabla_i \nabla_p \nabla_q T_j + \nabla_i \left( \alpha' \|\Omega\|^2 \bar{\rho}^{pq} - \frac{\alpha'}{2} \sigma_2^{pq} \right) \nabla_p \nabla_q T_j$$

$$= F^{pq} \nabla_{\nu} \nabla_q T_j + F^{pq} T^\lambda_{pj} \nabla_{\lambda} \nabla_q T_j + \alpha' \nabla_i (\|\Omega\|^3 \bar{\rho}^{pq}) \nabla_p \nabla_q T_j$$

$$- \frac{\alpha'}{2} \sigma_2^{pq}_{\nu,\tau} \nabla_i R_{\nu \tau} \nabla_p \nabla_q T_j$$

$$= F^{pq} \nabla_p \nabla_q T_j - F^{pq} \nabla_p (R_{qi} \sigma_{i \lambda} T_j) + F^{pq} T^\lambda_{pj} \nabla_{\lambda} R_{qi}$$

$$+ \frac{\alpha'}{2} \nabla_i (\|\Omega\|^3 \bar{\rho}^{pq}) \nabla_p R_{qj} - \frac{\alpha'}{4} \sigma_2^{pq}_{\nu,\tau} \nabla_i R_{\nu \tau} \nabla_p R_{qj}. \hspace{1cm} (6.5)$$
We also compute
\[ \nabla_i \nabla_j |T|^2 = g^{pq} \nabla_i \nabla_j T_p \bar{T}q + g^{pq} \nabla_i T_p \nabla_j \bar{T}q + g^{pq} \nabla_i \bar{T}_p \nabla_j \bar{T}q = g^{pq} \nabla_i \nabla_j T_p \bar{T}q + \frac{1}{2} g^{pq} \nabla_i T_p R_{qi} + \frac{1}{2} g^{pq} \nabla_i \bar{T}_p R_{qi} + \frac{1}{2} g^{pq} T_p \nabla_i \bar{R}_{qj}. \] (6.6)

We introduce the notation \( \mathcal{E} \), which denotes any combination of terms involving only \( Rm \), \( T \), \( g \), \( \|\Omega\| \), \( \alpha' \), \( \rho \) and \( \mu \), as well as any derivatives of \( \rho \) and \( \mu \). Note that \( F^{pq} \) is an element of \( \mathcal{E} \). The notation \( * \) refers to a contraction using the evolving metric \( g \). The notation \( D \mathcal{E} \) denotes any term which is a covariant derivative of a term in \( \mathcal{E} \). For example, the group \( D \mathcal{E} \) contains terms involving \( \nabla_T \), \( \nabla \bar{T} \), and \( \nabla \text{Ric}_\omega \). Substituting (6.4), (6.5), (6.6) gives
\[ \partial_t \nabla_i T_j = \frac{1}{2\|\Omega\|} \left\{ \Delta_F \nabla_i T_j - \frac{\alpha'}{4} \sigma_2^{pq,rs} \nabla_i R_{sr} \nabla_p \bar{R}_{qj} + \nabla \nabla T * \mathcal{E} + D \mathcal{E} * \mathcal{E} + \mathcal{E} \right\}. \] (6.7)

Here we also used that \( \nabla_i E_j = \nabla \nabla T * \mathcal{E} + D \mathcal{E} * \mathcal{E} + \mathcal{E} \) which can be verified from the definition of \( E_j \) given in (4.14)

### 6.1.2 Norm of covariant derivative of torsion

We will compute
\[ \partial_t |\nabla T|^2 = \partial_t \{ g^{ij} g^{kl} \nabla_i T_k \nabla_j \bar{T}_l \}. \] (6.8)

As in (4.26), we have
\[ \partial_t |\nabla T|^2 = 2 \text{Re}(\partial_t \nabla T, \nabla T) + 2\frac{|\nabla T|^2}{2\|\Omega\|^2} \left( \frac{R}{2} + \frac{\alpha'}{2} \|\Omega\|^3 p^{pq} R_{qp} - \frac{\alpha'}{4} \nu_2 (i \text{Ric}_\omega) - |T|^2 - \|\Omega\|^2 \nu \right). \] (6.9)

Next,
\[ \Delta_F |\nabla T|^2 = \text{Re}(\Delta_F \nabla T, \nabla T) + g^{ij} g^{kl} \nabla_i \nabla_j T_k \nabla_j \bar{T}_l + \|\nabla T\|^2_{F_{gg}} + \text{Ric}_\omega \nabla \bar{T}_l. \]

The last term can be written as a norm of \( \nabla \text{Ric}_\omega \) plus commutator terms. Explicitly,
\[ F^{pq} g^{ij} g^{kl} \nabla_q \nabla_T T_k \nabla_p \nabla_j \bar{T}l = F^{pq} g^{ij} g^{kl} \nabla_q \nabla_T T_k \nabla_p \nabla_j \bar{T}l + F^{pq} g^{ij} g^{kl} \nabla_q T_k \nabla_p \nabla_j \bar{T}l = F^{pq} g^{ij} g^{kl} \nabla_q \nabla_T T_k \nabla_p \nabla_j \bar{T}l + F^{pq} g^{ij} g^{kl} \nabla_q \nabla_T T_k \nabla_p \nabla_j \bar{T}l + F^{pq} g^{ij} g^{kl} \nabla_q T_k \nabla_p \nabla_j \bar{T}l + F^{pq} g^{ij} g^{kl} \nabla_q \nabla_T T_k \nabla_p \nabla_j \bar{T}l \]
\[ = \frac{1}{4} F^{pq} g^{ij} g^{kl} \nabla_q R_{kp} \nabla_j \bar{T}l + \frac{1}{2} F^{pq} g^{ij} g^{kl} \nabla_q \nabla_T R_{kp} \nabla_j \bar{T}l + \frac{1}{2} F^{pq} g^{ij} g^{kl} \nabla_q \nabla_T R_{kp} \nabla_j \bar{T}l = \left(6.10\right) \]
Hence
\[
\Delta_F |\nabla T|^2 = 2 \text{Re} \langle \Delta_F \nabla T, \nabla T \rangle + |\nabla \nabla T|^2_{Fg} + \frac{1}{4}|\nabla \text{Ric}_{\omega}|^2_{Fg} \\
+ g^{ij} g^{kl} \nabla_i T_k F^{pq} R_{ipq} \lambda^\gamma \nabla_j \bar{T}^\gamma + g^{ij} g^{kl} \nabla_i T_k F^{pq} R_{ipq} \lambda^\gamma \nabla_j T_\lambda \\
+ \frac{1}{2} F^{pq} g^{ij} g^{kl} \nabla_i R_{jk} R_{\bar{p}q\bar{l}} \lambda^\gamma \nabla_j T_\lambda + \frac{1}{2} F^{pq} g^{ij} g^{kl} R_{\bar{q}_i \bar{k}} T_j R_{\bar{j}_{p\bar{l}}} \\
+ F^{pq} g^{ij} g^{kl} R_{\bar{q}_i \bar{k}} T_j R_{\bar{j}_{p\bar{l}}} \lambda^\gamma \nabla_j \bar{T}^\gamma. \tag{6.11}
\]

Therefore, by (6.7), (6.9) and (6.11),
\[
\partial_t |\nabla T|^2 = \frac{1}{2} \left\{ \Delta_F |\nabla T|^2 - |\nabla \nabla T|^2_{Fg} - \frac{1}{4}|\nabla \text{Ric}_{\omega}|^2_{Fg} \\
- \frac{\alpha'}{2} \text{Re} \{ g^{ij} g^{kl} \sigma_2^{pq,rs} \nabla_i R_{jk} \nabla_p R_{qk} \lambda^\gamma \nabla_j \bar{T}^\gamma \} + \nabla \nabla T \ast \nabla T \ast \mathcal{E} \\
+ D \mathcal{E} \ast D \mathcal{E} \ast \mathcal{E} + D \mathcal{E} \ast \mathcal{E} + \mathcal{E} \right\}. \tag{6.12}
\]

### 6.2 The evolution of derivatives of curvature

#### 6.2.1 Derivative of Ricci curvature

We will compute
\[
\partial_t \nabla_i R_{kj} = \nabla_i \partial_t R_{kj} - \partial_t \Gamma_{ij}{}^\lambda R_{k\lambda}. \tag{6.13}
\]

Using (5.13) and (6.2), we obtain
\[
\partial_t \nabla_i R_{kj} = \left\{ \nabla_i \left( F^{pq} \nabla_p \nabla_q R_{kj} \right) - \frac{\alpha'}{2} \nabla_i \left( \sigma_2^{pq,rs} \nabla_k R_{sr} \nabla_j R_{qp} \right) \\
+ \left( 2 g^{pq} + 2 \alpha' \|\Omega\| \|\bar{\rho}^{pq}\| \right) \ast \nabla \nabla T \ast \nabla T + 2 D \mathcal{E} \ast \mathcal{E} \\
+ D \mathcal{E} \ast D \mathcal{E} \ast \mathcal{E} + D \mathcal{E} \ast \mathcal{E} + \mathcal{E} \right\}. \tag{6.14}
\]

Here, we used that $\nabla \nabla \bar{T} = \nabla \text{Ric}_\omega + \text{Rm} \ast \bar{T}$. Compute
\[
\nabla_i (F^{pq} \nabla_p \nabla_q R_{kj}) = F^{pq} \nabla_i \nabla_p \nabla_q R_{kj} + \alpha' \nabla_i (\|\Omega\| \|\bar{\rho}^{pq}\|) \nabla_p \nabla_q R_{kj} - \frac{\alpha'}{2} \nabla_i (\sigma_2^{pq}) \nabla_p \nabla_q R_{kj} \\
= F^{pq} \nabla_i \nabla_p \nabla_q R_{kj} + F^{pq} T^{\lambda}{}_{pq} \nabla_\lambda \nabla_q R_{kj} \\
+ \alpha' \nabla_i (\|\Omega\| \|\bar{\rho}^{pq}\|) \nabla_p \nabla_q R_{kj} - \frac{\alpha'}{2} \sigma_2^{pq,rs} \nabla_i R_{sr} \nabla_q R_{kj} \\
= F^{pq} \nabla_p \nabla_q R_{kj} + F^{pq} \nabla_p (R_{\bar{q}_k \bar{i}} R_{\bar{i} \bar{j}} - R_{\bar{q}_i \bar{j}} \lambda^\gamma R_{k\lambda}) \\
+ F^{pq} T^{\lambda}{}_{pq} \nabla_\lambda \nabla_q R_{kj} + \alpha' \nabla_i (\|\Omega\| \|\bar{\rho}^{pq}\|) \nabla_p \nabla_q R_{kj} \\
- \frac{\alpha'}{2} \sigma_2^{pq,rs} \nabla_i R_{sr} \nabla_p \nabla_q R_{kj}. \tag{6.15}
\]
Hence, using that $\nabla_i \sigma^p_q r^s = 0$ (5.9), we obtain

\[
\partial_t \nabla_i R_{kj} = \frac{1}{2\|\Omega\|} \left\{ \Delta_F \nabla_i R_{kj} - \frac{\alpha'}{2} \sigma^p_q r^s \nabla_i \nabla_k R_{sr} \nabla_j R_{qp} \\
- \frac{\alpha'}{2} \sigma^p_q r^s \nabla_k R_{sr} \nabla_i \nabla_j R_{qp} - \frac{\alpha'}{2} \sigma^p_q r^s \nabla_i R_{sr} \nabla_p \nabla_q R_{kj} \\
+ (2g^{pq} + 2\alpha'\|\Omega\|^3 \beta^{pq}) \nabla \nabla T \ast \nabla T \\
+ DDE \ast E + DE \ast E + D\nabla E \ast \nabla E \ast E \ast \nabla E \ast E \ast E \right\}.
\] (6.16)

**6.2.2 Norm of derivative of Ricci curvature**

We will compute

\[
\partial_t |\nabla \text{Ric}_\omega|^2 = \partial_t \{ g^{ia} g^{bj} \nabla_i R_{kj} \nabla_a R_{bc} \}.
\] (6.17)

As in (4.26), we have

\[
\partial_t |\nabla \text{Ric}_\omega|^2 = 2\text{Re}(\partial_t \nabla \text{Ric}_\omega, \nabla \text{Ric}_\omega) + 3 |\nabla \text{Ric}_\omega|^2 F_{ggg} + 3 |\nabla \text{Ric}_\omega|^2 F_{ggg} + |\nabla \nabla \text{Ric}_\omega|^2 F_{ggg} + |\nabla \nabla \text{Ric}_\omega|^2 F_{ggg} + \nabla \nabla E \ast E + E.
\] (6.18)

Commuting covariant derivatives

\[
|\nabla \nabla \text{Ric}_\omega|^2 F_{ggg} = |\nabla \nabla \text{Ric}_\omega|^2 F_{ggg} + \nabla \nabla E \ast E + E.
\] (6.19)

Hence

\[
\partial_t |\nabla \text{Ric}_\omega|^2 = \frac{1}{2\|\Omega\|} \left\{ \Delta_F |\nabla \text{Ric}_\omega|^2 - |\nabla \nabla \text{Ric}_\omega|^2 F_{ggg} - |\nabla \nabla \text{Ric}_\omega|^2 F_{ggg} \\
+ \frac{1}{2\|\Omega\|} 2\text{Re} \left\{ - \alpha' g^{ia} g^{jk} g^{\ell a} \sigma^p_q r^s \nabla_i \nabla_k R_{sr} \nabla_j R_{qp} \nabla_a R_{bc} \\
- \frac{\alpha'}{2} g^{ia} g^{jk} g^{\ell a} \sigma^p_q r^s \nabla_i R_{sr} \nabla_k \nabla_j R_{qp} \nabla_a R_{bc} \\
- \frac{\alpha'}{2} g^{ia} g^{jk} g^{\ell a} \sigma^p_q r^s \nabla_i R_{sr} \nabla_p \nabla_q R_{kj} \nabla_a R_{bc} \\
+ (2g^{pq} + 2\alpha'\|\Omega\|^3 \beta^{pq}) \nabla \nabla T \ast \nabla T \ast \nabla \text{Ric}_\omega \\
+ DDE \ast DE \ast E + DDE \ast E + DE \ast DE \ast E \ast E + DE \ast DE \ast E + DE \ast E \right\}.
\] (6.20)
Lemma 2  Suppose $|\alpha'\text{Ric}_\omega| \leq \frac{1}{4}$ and $-\frac{1}{8}g^{pq} < \alpha'\|\Omega\|^3 \bar{\rho}^{pq} < \frac{1}{8}g^{pq}$. Then

$$\partial_t|\nabla\text{Ric}_\omega|^2 \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F|\nabla\text{Ric}_\omega|^2 - \frac{1}{2} |\nabla\nabla\text{Ric}_\omega|^2 - \frac{1}{2} |\nabla\nabla\text{Ric}_\omega|^2 \right\}$$

$$+ \frac{1}{2\|\Omega\|} \left\{ 9\alpha'^2|\nabla\text{Ric}_\omega|^4 + 5|\nabla\nabla T||\nabla T||\nabla\text{Ric}_\omega| \right.$$  

$$+ DD\mathcal{E} \ast D\mathcal{E} \ast \mathcal{E} + (D\mathcal{E} + \mathcal{E})^3 \right\}. \quad (6.21)$$

Proof: By assumption, we may use

$$|\nabla\nabla\text{Ric}_\omega|^2_{F^{ggg}} + |\nabla\nabla\text{Ric}_\omega|^2_{F^{ggg}} \geq \frac{3}{4}(|\nabla\nabla\text{Ric}_\omega|^2 + |\nabla\nabla\text{Ric}_\omega|^2). \quad (6.22)$$

In coordinates where the evolving metric $g$ is the identity, we have $\sigma^p_{s^q} = \pm 1$. Using $2ab \leq a^2 + b^2$, estimate (6.21) follows from (6.20).

6.3 Higher order estimates

Theorem 6  There exists $0 < \delta_1, \delta_2$ with the following property. Suppose

$$-\frac{1}{8}g^{pq} < \alpha'\|\Omega\|^3 \bar{\rho}^{pq} < \frac{1}{8}g^{pq}, \quad \|\Omega\| \leq 1, \quad (6.23)$$

$$|\alpha'\text{Ric}_\omega| \leq \delta_1, \quad (6.24)$$

and

$$|T|^2 \leq \delta_2, \quad (6.25)$$

along the flow. Then

$$|\nabla\text{Ric}_\omega| \leq C, \quad |\nabla T| \leq C, \quad (6.26)$$

where $C$ depends only on $\delta_1, \delta_2, \alpha', \rho, \bar{\mu}$, and $(\mathcal{X}, \hat{\omega})$.

Proof: Let us assume that $\delta_1 < \frac{1}{4}$. This will allow us to use the estimate

$$\frac{3}{4}g_{kj} \leq F^{jk} \leq 2g_{kj}. \quad (6.27)$$

This follows from the definition of $F^{jk}$, see (4.23). From (5.20), with assumptions (6.24) and (6.27) we may estimate

$$\partial_t|\text{Ric}_\omega|^2 \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F|\text{Ric}_\omega|^2 - \frac{1}{2} |\nabla\text{Ric}_\omega|^2 \right\}$$

$$+ \frac{1}{2\|\Omega\|} \text{Re} \left\{ D\mathcal{E} \ast \mathcal{E} + 5\nabla T \ast \nabla T \ast \text{Ric} + \mathcal{E} \right\}. \quad (6.28)$$
Here we used
\[ -\alpha' \text{Re} \{ g^{\bar{t}}_t g^{\bar{m}k} \sigma_2^{p,q,r,s} R_{\bar{t}m} \nabla_k R_{\bar{s}t} \nabla_j R_{\bar{q}p} \} \leq \delta_1 |\nabla \text{Ric}_\omega|^2, \quad (6.29) \]
to absorb this term into the \(- |\nabla \text{Ric}_\omega|^2\) term. We will compute the evolution of
\[ G = (|\alpha' \text{Ric}_\omega|^2 + \tau_1)|\nabla \text{Ric}_\omega|^2 + (|T|^2 + \tau_2)|\nabla T|^2, \quad (6.30) \]
where \(\tau_1\) and \(\tau_2\) are constants to be determined. First, we compute
\[ \partial_t \{ (|\alpha' \text{Ric}_\omega|^2 + \tau_1)|\nabla \text{Ric}_\omega|^2 \} = \alpha^2 \partial_t |\text{Ric}_\omega|^2 |\nabla \text{Ric}_\omega|^2 + (|\alpha' \text{Ric}_\omega|^2 + \tau_1) \partial_t |\nabla \text{Ric}_\omega|^2. \quad (6.31) \]
By (6.21) and (6.28)
\begin{align*}
\partial_t \{ (|\alpha' \text{Ric}_\omega|^2 + \tau_1)|\nabla \text{Ric}_\omega|^2 \} & \\
& \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F (|\alpha' \text{Ric}_\omega|^2)|\nabla \text{Ric}_\omega|^2 - \frac{\alpha^2}{2} |\nabla \text{Ric}_\omega|^4 \right\} \\
& + \frac{1}{2\|\Omega\|} \text{Re} \left\{ D\mathcal{E} \ast \mathcal{E} + 5 \nabla T \ast \nabla T \ast \text{Ric} + \mathcal{E} \right\} \alpha^2 |\nabla \text{Ric}_\omega|^2 \\
& + \frac{(|\alpha' \text{Ric}_\omega|^2 + \tau_1)}{2\|\Omega\|} \left\{ \Delta_F |\nabla \text{Ric}_\omega|^2 - \frac{1}{2} |\nabla \nabla \text{Ric}_\omega|^2 - \frac{1}{2} |\nabla \nabla \nabla \text{Ric}_\omega|^2 \right\} \\
& + \frac{(|\alpha' \text{Ric}_\omega|^2 + \tau_1)}{2\|\Omega\|} \left\{ 9\alpha^2 |\nabla \text{Ric}_\omega|^4 + 5 |\nabla \nabla T||\nabla T||\nabla \text{Ric}_\omega| \right. \\
& \left. + |\nabla \nabla \mathcal{E} \ast \mathcal{E} + |\nabla \nabla \mathcal{E} \ast \mathcal{E} | \ast \mathcal{E} + (D\mathcal{E} + \mathcal{E})^3 \right\}. \quad (6.32) 
\end{align*}
Hence
\begin{align*}
\partial_t \{ (|\alpha' \text{Ric}_\omega|^2 + \tau_1)|\nabla \text{Ric}_\omega|^2 \} & \\
& \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F \{ (|\alpha' \text{Ric}_\omega|^2 + \tau_1)|\nabla \text{Ric}_\omega|^2 \} - \left( \frac{1}{2} - 9|\alpha' \text{Ric}_\omega|^2 - 9\tau_1 \right) \alpha^2 |\nabla \text{Ric}_\omega|^4 \\
& - \frac{1}{2} |\nabla \nabla \text{Ric}_\omega|^2 (|\alpha' \text{Ric}_\omega|^2 + \tau_1) - \frac{1}{2} |\nabla \nabla \nabla \text{Ric}_\omega|^2 (|\alpha' \text{Ric}_\omega|^2 + \tau_1) \\
& - 2\text{Re} \left\{ F_j^i \nabla_i |\alpha' \text{Ric}_\omega|^2 \nabla_j |\nabla \text{Ric}_\omega|^2 \right\} + 6(\delta_1^2 + \tau)|\nabla \nabla T||\nabla T||\nabla \text{Ric}_\omega| \right\} \\
& + \frac{\alpha^2 |\nabla \text{Ric}_\omega|^2}{2\|\Omega\|} \text{Re} \left\{ 5 \nabla T \ast \nabla T \ast \text{Ric} + D\mathcal{E} \ast \mathcal{E} + \mathcal{E} \right\} \\
& + \frac{(|\alpha' \text{Ric}_\omega|^2 + \tau_1)}{2\|\Omega\|} \left\{ |\nabla \nabla \mathcal{E} \ast \mathcal{E} + |\nabla \nabla \mathcal{E} \ast \mathcal{E} | \ast \mathcal{E} + (D\mathcal{E} + \mathcal{E})^3 \right\}. \quad (6.33) 
\end{align*}
We estimate
\begin{align*}
-2\text{Re} \left\{ F_j^i \nabla_i |\alpha' \text{Ric}_\omega|^2 \nabla_j |\nabla \text{Ric}_\omega|^2 \right\} & \\
& \leq 8|\alpha'| \delta_1 |\nabla \text{Ric}_\omega|^2 (|\nabla \nabla \text{Ric}_\omega| + |\nabla \nabla \text{Ric}_\omega| + \mathcal{E}) \\
& \leq \frac{\alpha^2}{2\tau} |\nabla \text{Ric}_\omega|^4 + 2^8 \delta_1^2 (|\nabla \nabla \text{Ric}_\omega|^2 + |\nabla \nabla \text{Ric}_\omega|^2) + C |\nabla \text{Ric}_\omega|^2, \quad (6.34) 
\end{align*}
By (6.12), we have
\[
\|\nabla T\|^2 + 2^1 3^2 (\delta_1^2 + \tau_1) |\nabla T|^2 |\nabla \text{Ric}_\omega|^2
\]
\[
\leq \frac{1}{2} (\delta_1^2 + \tau_1) |\nabla T|^2 + \frac{\alpha^2}{24} |\nabla \text{Ric}_\omega|^4 + 2^4 3^4 \alpha^{-2} (\delta_1^2 + \tau_1)^2 |\nabla T|^4, \quad (6.35)
\]
\[
\frac{\alpha^2 |\nabla \text{Ric}_\omega|^2}{2 \|\Omega\|} \Re \left\{ 5 |\nabla T* \nabla T* \text{Ric} + \nabla \mathcal{E} * \mathcal{E} + \mathcal{E} \right\}
\]
\[
\leq \frac{1}{2 \|\Omega\|} \left\{ \frac{\alpha^2}{2}\|\nabla \text{Ric}_\omega\|^4 + 2^2 5^2 |\nabla T|^4 + C|\nabla \text{Ric}_\omega|^3 + C|\nabla T|^3 + C \right\}. \quad (6.36)
\]
\[
\frac{(|\alpha' \text{Ric}_\omega|^2 + \tau_1)}{2 \|\Omega\|} \left\{ |\nabla \nabla \mathcal{E} * D \mathcal{E} * \mathcal{E} + \nabla \nabla \mathcal{E} * D \mathcal{E} * \mathcal{E} + (D \mathcal{E} + \mathcal{E})^3 \right\}
\]
\[
\leq \frac{1}{2 \|\Omega\|} \left\{ \frac{1}{4} |\nabla \nabla \text{Ric}_\omega|^2 (|\alpha' \text{Ric}_\omega|^2 + \tau_1) + \frac{1}{4} |\nabla \nabla \text{Ric}_\omega|^2 (|\alpha' \text{Ric}_\omega|^2 + \tau_1) + \left(\frac{2^4 3^4 \alpha^{-2} (\delta_1^2 + \tau_1)^2 + 2^5 5^2 \delta_1^2}{2^4}\right) |\nabla T|^4 + C_{\alpha', \tau, \delta} |\nabla \text{Ric}_\omega|^3 + C_{\alpha', \tau, \delta} |\nabla T|^3 + C_{\alpha', \tau, \delta} \right\}. \quad (6.37)
\]
Therefore
\[
\partial_t \{(|\alpha' \text{Ric}_\omega|^2 + \tau_1)|\nabla \text{Ric}_\omega|^2 \}
\]
\[
\leq \frac{1}{2 \|\Omega\|} \left\{ \Delta_F \{(|\alpha' \text{Ric}_\omega|^2 + \tau_1)|\nabla \text{Ric}_\omega|^2 \} - \left(\frac{1}{4} - 9 \delta_1^2 - 9 \tau_1 \right) \alpha^2 |\nabla \text{Ric}_\omega|^4 
\right.
\]
\[
- \left( |\nabla \nabla \text{Ric}_\omega|^2 + |\nabla \nabla \text{Ric}_\omega|^2 \right) \left(\frac{T_1}{4} - 2^8 \delta_1^2 \right) + (\delta_1^2 + \tau_1) |\nabla T|^2 
\]
\[
+ \left(2^4 3^4 \alpha^{-2} (\delta_1^2 + \tau_1)^2 + 2^5 5^2 \delta_1^2 \right) |\nabla T|^4 + C_{\alpha', \tau, \delta} |\nabla \text{Ric}_\omega|^3 + C_{\alpha', \tau, \delta} |\nabla T|^3 + C_{\alpha', \tau, \delta} \right\}. \quad (6.38)
\]
Next, we compute
\[
\partial_t \{(|T|^2 + \tau_2)|\nabla T|^2 \} = \partial_t |T|^2 |\nabla T|^2 + (|T|^2 + \tau_2) \partial_t |\nabla T|^2. \quad (6.39)
\]
By (4.29), we have
\[
\partial_t |T|^2 \leq \frac{1}{2 \|\Omega\|} \left\{ \Delta_F |T|^2 - |\nabla T|^2 \right\} + C |\nabla T| + C. \quad (6.40)
\]
By (6.12), we have
\[
\partial_t |\nabla T|^2 \leq \frac{1}{2 \|\Omega\|} \left\{ \Delta_F |\nabla T|^2 - |\nabla \nabla T|^2 + |\alpha' ||\nabla T||\nabla \text{Ric}_\omega|^2 
\right.
\]
\[
+ C |\nabla \nabla T||\nabla T| + C |\nabla T|^2 + C |\nabla \text{Ric}_\omega|^2 + C \right\}. \quad (6.41)
\]
45
By our assumption $|\alpha'\text{Ric}_\omega| \leq \frac{1}{4}$, we have $|\nabla\nabla T|_{F_{gg}}^2 \geq \frac{1}{2}|\nabla\nabla T|^2$ and $|\nabla T|^2_{F_g} \geq \frac{1}{2}|\nabla T|^2$. Therefore
\[
\partial_t\{(|T|^2 + \tau_2)|\nabla T|^2\} 
\leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F\{(|T|^2 + \tau_2)|\nabla T|^2\} - 2\text{Re}\{F_{ij}\nabla_i|T|^2\nabla_j|\nabla T|^2\} \right. 
\left. - \frac{1}{4}|\nabla T|^4 - (|T|^2 + \tau_2)\frac{1}{4}|\nabla\nabla T|^2 + C|\nabla\text{Ric}_\omega|^3 + C|\nabla T|^3 + C \right\}. 
\tag{6.42}
\]

Here we used Young’s inequality $|\nabla T||\nabla\text{Ric}_\omega|^2 \leq \frac{1}{3}|\nabla T|^3 + \frac{2}{3}|\nabla\text{Ric}_\omega|^3$. In the following, we will use that $\nabla T$ can be expressed as Ricci curvature. We estimate
\[
-2\text{Re}\{F_{ij}\nabla_i|T|^2\nabla_j|\nabla T|^2\} 
\leq 4|T||\nabla T|(|\nabla T| + |\nabla T|)(|\nabla\nabla T| + |\nabla\nabla T|) 
\leq 4|T||\nabla T|^2|\nabla\nabla T| + 4|T||\nabla T|^2|\nabla\text{Ric}_\omega| + 4|T||\nabla T||\nabla\text{Ric}_\omega||\nabla T| 
+ 4|T||\nabla T||\nabla\text{Ric}_\omega||\nabla\text{Ric}_\omega| + 4|T||\nabla T|(|\nabla T| + |\nabla T|)|R \ast T|. 
\tag{6.43}
\]

We may estimate the first term in the following way
\[
4|T||\nabla T|^2|\nabla\nabla T| \leq 4|\nabla T|^2(\delta_2)^{1/2}|\nabla T| \leq \frac{1}{23}|\nabla T|^4 + 2^5\delta_2|\nabla\nabla T|^2. 
\tag{6.44}
\]

The other terms may be estimated using Young’s inequality, and we can derive
\[
-2\text{Re}\{F_{ij}\nabla_i|T|^2\nabla_j|\nabla T|^2\} \leq \frac{1}{23}|\nabla T|^4 + 2^6\delta_2|\nabla\nabla T|^2 + C|\nabla T|^3 + C|\nabla\text{Ric}_\omega|^3 + C. 
\]

Hence
\[
\partial_t\{(|T|^2 + \tau_2)|\nabla T|^2\} \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F\{(|T|^2 + \tau_2)|\nabla T|^2\} - \frac{1}{8}|\nabla T|^4 
- (\frac{\tau_2}{4} - 2^6\delta_2)|\nabla\nabla T|^2 + C|\nabla\text{Ric}_\omega|^3 + C|\nabla T|^3 + C \right\}. 
\tag{6.45}
\]

Combining (6.38) and (6.45) gives
\[
\partial_tG \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F G - \left(\frac{1}{4} - 9\delta_1^2 - 9\tau_1\right)\alpha^2|\nabla\text{Ric}_\omega|^4 
- \left(\frac{\tau_1}{4} - 2^8\delta_1^2\right)|\nabla\nabla\text{Ric}_\omega|^2 + |\nabla\nabla\text{Ric}_\omega|^2\right) - \left(\frac{\tau_2}{4} - 2^6\delta_2 - \delta_1^2 - \tau_1\right)|\nabla\nabla T|^2 
- \left(\frac{1}{8} - 2^43\alpha'\delta_2(\delta_1^2 + \tau_1)^2 - 2^5\delta_1^2\right)|\nabla T|^4 
+ C\alpha',\tau,\delta|\nabla\text{Ric}_\omega|^3 + C\alpha',\tau,\delta|\nabla T|^3 + C\alpha',\tau,\delta \right\}. 
\tag{6.46}
\]

We may choose $\tau_1 = \min\{2^{-7}, 2^{-5}3^{-2}|\alpha'|\}$ and $\tau_2 = 1$. Then for any $\delta_1, \delta_2 > 0$ such that
\[
\delta_1, \delta_2 \leq 2^{-6}\tau_1 \ll \tau_2 = 1, 
\tag{6.47}
\]
we have the estimate
\[ \partial_t G \leq \frac{1}{2\|\Omega\|} \left\{ \Delta_F G - \frac{1}{8} \alpha^2 |\nabla \text{Ric}_\omega|^4 - \frac{1}{16} |\nabla T|^4 + C_{\alpha', \tau, \delta} \right\}. \] (6.48)

Now, suppose \( G \) attains its maximum at a point \((z,t)\) where \( t > 0 \). From the above estimate, at this point we have
\[ \frac{1}{8} \alpha^2 |\nabla \text{Ric}_\omega|^4 + \frac{1}{16} |\nabla T|^4 \leq C_{\alpha', \tau, \delta}. \] (6.49)

It follows that \( G \) is uniformly bounded along the flow, and hence
\[ |\nabla \text{Ric}_\omega| \leq C, \quad |\nabla T| \leq C, \] (6.50)
along the flow.

**Corollary 1** There exists \( 0 < \delta_1, \delta_2 \) with the following property. Suppose
\[ -\frac{1}{8} g^{pq} < \alpha' \|\Omega\|^3 \rho^{pq} < \frac{1}{8} g^{pq}, \] (6.51)
\[ |\alpha' \text{Ric}_\omega| \leq \delta_1, \] (6.52)
and
\[ |T|^2 \leq \delta_2, \] (6.53)
along the flow. If there exists \( \delta_0 > 0 \) such that \( 0 < \delta_0 \leq \|\Omega\| \leq 1 \) along the flow, then
\[ |D^k \text{Ric}_\omega| \leq C, \quad |D^k T| \leq C, \] (6.54)
where \( C \) depends only on \( \delta_0, \delta_1, \delta_2, \alpha', \rho, \tilde{\mu}, \) and \((X, \hat{\omega})\).

**Proof:** Since \( \|\Omega\| = e^{-u} \), we are assuming that \( |u| \) stays bounded, and that the metrics \( \hat{g} \) and \( g = e^u \hat{g} \) are equivalent. We are also assuming that \( e^{-u} |Du|^2_{\hat{g}} \ll 1 \) and \( e^{-u} |\alpha' u_{kj}| \hat{g} \ll 1 \). By Theorem 6, there exists \( \delta_1 \) and \( \delta_2 \) such that \( |\nabla \nabla u| \) and \( |\nabla \nabla \nabla u| \) stay bounded along the flow. We will estimate partial derivatives in coordinate charts. Since
\[ \partial_i \partial_j \partial_k u = \nabla_i \nabla_j \nabla_k u + \Gamma^\lambda_{ik} u_{j\lambda}, \quad \partial_i \partial_j u = \nabla_i \nabla_j u + \Gamma^\lambda_{ij} u_\lambda, \] (6.55)
and the Christoffel symbol
\[ \Gamma^\lambda_{ik} = e^{-u} \hat{g}^{\lambda \gamma} \partial_i (e^u \hat{g}_{\gamma k}) = u_i \delta^\lambda_k + \hat{\Gamma}^\lambda_{ik} \] (6.56)
stays bounded, we have that
\[ |u|, |\partial u|, |\partial \partial u|, |\partial \partial \partial u| \leq C. \] (6.57)
The scalar equation is

\[ \partial_t u = \Delta u + \alpha' e^{-2u} \bar{\rho}^p q_q u^{pq} + \alpha' e^{-u} \bar{\sigma}_2 (i \bar{\partial} \partial u) + |Du|^2 + e^{-u} \nu. \]  

(6.58)

where \( \nu(x, u, Du) \). Differentiating once gives

\[ \partial_t Du = \hat{F}^p q_q Du^{pq} + \alpha' D(e^{-2u} \bar{\rho}^p q_q) u^{pq} + \alpha' e^{-u} \bar{\sigma}_2 (i \bar{\partial} \partial u) Du + D(e^{-u} \nu), \]  

(6.59)

where

\[ \hat{F}^p q_q = \hat{g}^p q_q + \alpha' e^{-2u} \bar{\rho}^p q_q + \alpha' e^{-u} \bar{\sigma}_2 p_q. \]  

(6.60)

We note that \( \hat{F}^{jk} \) only differs from \( F^{jk} \) (4.20) by a factor of \( e^u \). From our assumptions on \( |\alpha' \text{Ric}_\omega| = e^{-u} |\alpha' \partial \bar{\partial} u|_\hat{g} \) and \( \| \Omega \| = e^{-u} \), we have uniform ellipticity of \( \hat{F}^{jk} \). Differentiating twice yields

\[ \partial_t u_{k}^j = \hat{F}^p q_q \partial_p \partial_u u_{k}^j + \Psi(x, u, \partial u, \partial \partial u, \partial \bar{\partial} u, \partial \partial u), \]  

(6.61)

where \( \Psi \) is uniformly bounded along the flow. By the Krylov-Safonov theorem [27], we have that \( u_{k}^j \) is bounded in the \( C^{\alpha/2, \alpha} \) norm. The function \( u \) and the spacial gradient \( Du \) are also bounded in the \( C^{\alpha/2, \alpha} \) norm since the right-hand sides of (6.58) and (6.59) are bounded. We may now apply parabolic Schauder theory (for example, in [26]) to the linearized equation (6.59). Standard theory and a bootstrap argument give higher order estimates of \( u \), and hence we obtain estimates on derivatives of the curvature and torsion of \( g = e^u \hat{g} \).

7 Long time existence

**Proposition 4** Let \( C_1, C_2, C_5 \) be the named constants as before, which only depend on \((X, \hat{g}), \mu, \rho, \) and \( \alpha' \). There exists \( M_0 \gg 1 \) such that for all \( M \geq M_0 \), the following statement holds. If the flow exists on \([0, t_0)\), and initially starts with \( u_0 = \log M \), then along the flow

\[ \frac{1}{C_1 M} \leq e^{-u} \leq \frac{C_2}{M}, \quad |T|^2 \leq \frac{C_3}{M}, \quad |\alpha' \text{Ric}_\omega| \leq \frac{C_5}{M^{1/2}}, \]  

(7.1)

and

\[ |D^k u|^2 \leq \bar{C}_k, \quad \frac{1}{2} \hat{g}^{ij} \leq \hat{F}^{ij} \leq 2 \hat{g}^{ij}, \]  

(7.2)

where \( \bar{C}_k \) only depends on \((X, \hat{g}), \mu, \rho, \alpha', M \).

**Proof:** Let \( \delta_1 \) and \( \delta_2 \) be the constants from Corollary 1, and choose a smaller \( \delta_1 \) if necessary to ensure \( \delta_1 < 10^{-6} \). Recall that from Theorem 3,

\[ \frac{1}{C_1 M} \leq \| \Omega \| = e^{-u} \leq \frac{C_2}{M} \]  

(7.3)
along the flow for $M$ large enough. Consider the set

$$I = \{ t \in [0,t_0) \text{ such that } |\alpha'\text{Ric}| \leq \delta_1, \ |T|^2 \leq \delta_2 \text{ holds on } [0,t] \}. \quad (7.4)$$

Since at $t = 0$ we have $|\alpha'\text{Ric}| = |T|^2 = 0$, we know that $I$ is non-empty. By definition, $I$ is relatively closed. We now show that $I$ is open. Suppose $\hat{t} \in I$. By definition of $I$, the hypothesis of Theorem 4 is satisfied, hence $|T|^2 \leq \frac{C}{M} < \delta_2$ at $\hat{t}$ as long as $M$ is large enough. It follows that the hypothesis of Theorem 5 is satisfied as long as $M$ is large enough, hence $|\alpha'\text{Ric}| \leq \frac{C_5}{M^{1/2}} < \delta_1$ at $\hat{t}$. We can conclude the existence of $\varepsilon > 0$ such that $[\hat{t} + \varepsilon) \subset I$, and hence $I$ is open.

It follows that $I = [0,t_0)$. We know that $-C\tilde{g}^{pq} \leq \tilde{\rho}^{pq} \leq C\tilde{g}^{pq}$ since $\tilde{\rho}$ can be bounded using the background metric. For $M$ large enough, we can conclude

$$-\frac{1}{8} e^{-u} \tilde{g}^{pq} < \alpha' e^{-3u} \tilde{\rho}^{pq} < \frac{1}{8} e^{-u} \tilde{g}^{pq}, \quad (7.5)$$

and we can apply Corollary 1 to obtain higher order estimates of $u$. Uniform ellipticity follows from the definition of $\tilde{F}^{jk}$ (6.60) and the estimates on $|\alpha'\text{Ric}| = e^{-u}|\alpha'\bar{\partial}\partial u|_g$ and $\|\Omega\|$. Q.E.D.

Theorem 7 There exists $M_0 \gg 1$ such that for all $M \geq M_0$, if the flow initially starts with $u_0 = \log M$, then the flow exists on $[0, \infty)$.

Proof: By short-time existence [31], we know the flow exists for some maximal time interval $[0,T)$. If $T < \infty$, we may apply the previous proposition to extend the flow to $[0, T + \varepsilon)$, which is a contradiction. Q.E.D.

8 Convergence of the flow

We may apply Theorem 7 to construct solutions to the Fu-Yau equation.

Theorem 8 There exists $M_0 \gg 1$ such that for all $M \geq M_0$, if the flow initially starts with $u_0 = \log M$, then the flow exists on $[0, \infty)$ and converges smoothly to a function $u_\infty$, where $u_\infty$ solves

$$0 = i\bar{\partial}(e^{u_\infty} \omega - \alpha' e^{-u_\infty} \rho) + \frac{\alpha'}{2} i\bar{\partial}u_\infty \wedge i\bar{\partial}u_\infty + \mu, \quad \int_X e^{u_\infty} = M. \quad (8.1)$$

Proof: Since we will work with the scalar equation, all norms in this section will be with respect to the background metric $\omega$. Let $v = \partial_t e^u$. Recall that

$$\int_X v = 0, \quad (8.2)$$

49
along the flow. Differentiating equation (2.22) with respect to time gives

\[
2\partial_t \hat{\omega}^2 = i\partial\bar{\partial}(v\hat{\omega} + \alpha'e^{-2u}\nu\rho) + \alpha'i\partial\bar{\partial}u \wedge i\partial\bar{\partial}(e^{-u}v). \quad (8.3)
\]

Consider the functional

\[
J(t) = \int_X v^2 \hat{\omega}^2. \quad (8.4)
\]

Compute

\[
\frac{dJ}{dt} = \int_X vi\partial\bar{\partial}(v\hat{\omega} + \alpha'e^{-2u}\nu\rho) + \alpha'\int_X vi\partial\bar{\partial}u \wedge i\partial\bar{\partial}(e^{-u}v)
\]

\[
= -\int_X |\nabla v|^2 - \alpha'\int_X e^{-2u}vi\partial\nu \wedge \bar{\partial}v \wedge \rho + 2\alpha'\int_X e^{-2u}vi\partial\nu \wedge \bar{\partial}u \wedge \rho
\]

\[-\alpha'\int_X e^{-2u}vi\partial\nu \wedge \bar{\partial}\rho - \alpha'\int_X e^{-u}i\partial\bar{\partial}u \wedge i\partial\nu \wedge \bar{\partial}v + \alpha'\int_X e^{-u}vi\partial\bar{\partial}u \wedge i\partial\nu \wedge \bar{\partial}v. \quad (8.5)
\]

We may estimate

\[
\frac{dJ}{dt} \leq -\int_X |\nabla v|^2 + \alpha'\|\nu\|\int_X e^{-2u}|\nabla v|^2 + 2\alpha'\|\nu\||\nabla u||\int_X e^{-2u}|v| |\nabla v|
\]

\[+\alpha'\|\bar{\partial}\nu\|\int_X e^{-2u}|v| |\nabla v| + \||\alpha'e^{-u}i\partial\bar{\partial}u\||\int_X |\nabla v|^2
\]

\[+\|\nabla u\||\alpha'e^{-u}i\partial\bar{\partial}u||\int_X |v| |\nabla v|. \quad (8.6)
\]

By Proposition 4, we know that on \([0, \infty)\) we have the estimates

\[
e^{-u} \leq \frac{C_2}{M} \ll 1, \quad |\nabla u|^2_\hat{g} \leq C_3C_1, \quad |\alpha'e^{-u}u_{k\bar{j}}|_\hat{g} \leq \frac{C_5}{M^{1/2}}. \quad (8.7)
\]

Hence for any \(\varepsilon > 0\), we can choose \(M\) large enough such that

\[
\frac{dJ}{dt} \leq -\frac{1}{2} \int_X |\nabla v|^2 + \varepsilon \int_X |v| |\nabla v| \leq -\left(\frac{1}{2} - \frac{\varepsilon}{2}\right) \int_X |\nabla v|^2 + \frac{\varepsilon}{2} \int_X |v|^2. \quad (8.8)
\]

Since \(f_X v = 0\), we may use the Poincaré inequality to obtain, for \(\varepsilon > 0\) small enough,

\[
\frac{dJ}{dt} \leq -\eta \int_X v^2 = -\eta J, \quad (8.9)
\]

with \(\eta > 0\). This implies that

\[
J(t) \leq Ce^{-\eta t}, \quad (8.10)
\]

that is,

\[
\int_X v^2 \leq Ce^{-\eta t}. \quad (8.11)
\]
From this estimate, we see that for any sequence $v(t_j)$ converging to $v_\infty$, we have $v_\infty = 0$.

We can now show convergence of the flow. Following the argument given in Proposition 2.2 in [7], we have

$$
\int_X |e^u(x, s') - e^u(x, s)| \leq \int_X \int_s^{s'} |\partial_t e^u(x, t)| = \int_s^{s'} \int_X |v(x, t)| \\
\leq \int_s^{s'} \left( \int_X v^2 \right)^{\frac{1}{2}} dt \leq \int_s^{+\infty} \left( \int_X v^2 \right)^{\frac{1}{2}} dt \\
\leq C \int_s^{+\infty} e^{-\frac{\eta}{2} t} dt \tag{8.12}
$$

Recall that we normalized the background metric such that $\int_X \tilde{\omega}^2 = 1$. This estimate shows that, as $t \to +\infty$, $e^u(x, t)$ are Cauchy in $L^1$ norm. Thus $e^u(x, t)$ converges in the $L^1$ norm to some function $e^u_\infty(x)$ as $t \to \infty$.

By our uniform estimates, $e^{u_\infty}$ is bounded in $C^\infty$, and a standard argument shows that $e^u$ converges in $C^\infty$. Indeed, if there exist a sequence of times such that $\|e^{-u(x, t_j)} - e^{-u_\infty(x)}\|_{C^k} \geq \epsilon$, then by our estimates a subsequence converges in $C^k$ to $e^{-u'_\infty}$. Then $\|e^{-u'_\infty(x)} - e^{-u_\infty(x)}\|_{L^1} = 0$ but $\|e^{-u'_\infty(x)} - e^{-u_\infty(x)}\|_{C^k} \geq \epsilon$, a contradiction.

It follows from (8.11) that $e^{u_\infty}$ satisfies the Fu-Yau equation (8.1).

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