THE RATE OF INCREASE FOR RECURRENCE 
WITH QUADRATIC NON-LINEARITY

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\[ \begin{align*}
D(n + 1) &= a + b \cdot D^2(n), \quad a, b = \text{const} > 0, \quad n = 0, 1, 2, \ldots, \quad D(n) = D_{a,b}(n). \\
\end{align*} \tag{1.1} \]

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Abstract.

We investigate in this short report the rate of increase of positive numerical recursion with quadratic non-linearity. More exactly, we intent to calculate the logarithmic index of its increasing.

We present also the possible application in the theory of the Navier-Stokes equations.

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1 Notations. Statement of problem. Conditions. Possible applications.

Let us consider the following numerical recurrence relation (dynamical system)

\[ \begin{align*}
D(n + 1) &= a + b \cdot D^2(n), \quad a, b = \text{const} > 0, \quad n = 0, 1, 2, \ldots, \quad D(n) = D_{a,b}(n). \\
\end{align*} \tag{1.1} \]

We can and will suppose without loss of generality \( D(0) = 1 \) (initial condition).

We impose in the sequel on the parameters \( a, b \) the following conditions:

\[ \begin{align*}
b &> 0; \quad a \cdot b \geq 1/4. \\
\end{align*} \tag{1.2} \]

This conditions guarantee the monotonic increasing of the sequence \( D(n) : D(n + 1) \geq D(n) \).

Something similar occurred in the theory of elliptical curves and following in the coding theory [16]. Another applications and investigations of these equations are discussed in the book [15].
The authors are faced with this kind of equation by the investigation of numerical method for Navier-Stokes equation, see [13], [14]. We describe in greater detail.

The mild solution 

\[ u_{n+1}(x, t) = u_0(x, t) + G[u_n, u_n](x, t), n = 0, 1, 2, \ldots, \]

where \( u_0(x, t) \) is the solution of heat equation with correspondent initial value and right-hand side and \( G[u, v] \) is bilinear unbounded pseudo-differential operator, [6], [7]. See also the articles [1], [2], [3], [4], [5], [8], [9], [10], [11] etc. The second iteration is investigated in [12].

Recall that the function 

\[ u = u(x, t) \]

and hence the functions 

\[ u_n(x, t), n = 0, 1, 2, \ldots \]

are vector functions:

\[ u(x, t) = \vec{u}(x, t) = \{ u(i)(x, t) \}, \quad i = 1, 2, \ldots, d; \]

therefore the functional 

\[ G[u, v] = G[\vec{u}, \vec{v}] = \vec{G}[\vec{u}, \vec{v}] \]

may be interpreted as a tensor:

\[ \vec{G} = \{ g_{i,j}^m \}; \quad \vec{G}[\vec{u}, \vec{v}]_m = \sum_{i=1}^{d} \sum_{j=1}^{d} g_{i,j}^m u(i)v(j), \quad m = 1, 2, \ldots, d. \]

We denote by \( D(n) \) the amount of independent summands in the expression for the \( n^{th} \) iteration:

\[ u_{n}^{(i)} = \sum_{s = 1}^{D(n)} \omega_{n,s1}^{(i)}, \quad v_{n}^{(j)} = \sum_{s = 1}^{D(n)} \kappa_{n,s2}^{(j)}, \quad \omega_{n,s1}^{(i)}, \kappa_{n,s2}^{(j)} = \omega_{n,s1}(x, t), \kappa_{n,s2}(x, t), \]

then

\[ \vec{G}[\vec{u}_n, \vec{v}_n]_m = \sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{s = 1}^{D(n)} \sum_{s = 1}^{D(n)} \omega_{n,s1}^{(i)} \kappa_{n,s2}^{(j)} u_{n}^{(i)} v_{n}^{(j)}, \quad k = 1, 2, \ldots, d. \]

The last expression contains exactly in general case \( d^2 \cdot D^2(n) \) independent summands.

Obviously for all the values \( k D(0) = 1 \) and

\[ D(n + 1) = 1 + d^2 \cdot D^2(n), \]

i.e. in this case \( a = 1, \ b = d \). Since \( d \geq 1 \), the conditions (1.2) are satisfied.

Our claim in this report is investigation of recurrence equation (1.1) under condition (1.2): obtaining of upper and lower bounds and calculating the asymptotic for the solution.
2 Main results: bilateral bounds and asymptotic behavior for solution.

Theorem.

\[ \forall k, l = 1, 2, \ldots \Rightarrow 1 \leq \frac{b}{[bD(l)]^{2k}} \leq \left[ 1 + \frac{a}{bD^2(l)} \right]^{2k-1}; \quad (2.1) \]

\[ \forall k \geq 1 \Rightarrow \lim_{l \to \infty} \frac{b}{[bD(l)]^{2k}} = 1. \quad (2.2) \]

**Proof. Lower bound:**

\[ D(k + 1) \geq bD^2(k), \; k = l, l + 1, l + 2, \ldots; \; l = \text{const} = 0, 1, \ldots \]

We deduce:

\[ D(l + 1) \geq bD^2(l), \; D(l + 2) \geq b^3D^4(l), \; D(l + 3) \geq b^7D^8(l), \ldots \]

By induction:

\[ D(l + k) \geq b^{2k-1}D^{2k}(l). \quad (2.3) \]

**Upper bound.** We deduce denoting

\[ Q(l) = 1 + \frac{a}{bD^2(l)} \]

and taking into account the monotonicity of the sequence \( D_{a,b}(n) \): if \( k \geq l \) then

\[ D(k + 1) = a + bD^2(k) = bD^2(k) \left( 1 + \frac{a}{bD^2(k)} \right) \leq b^{2}(k)Q(l), \]

and we find analogously

\[ D(k + l) \leq b^{2k-1}D^{2k}(l)[Q(l)]^{2k-1}. \quad (2.4) \]

The assertion (2.2) follows immediately from the bilateral estimates (2.1).

3 Examples.

We intent to illustrate by building of some numerical examples the huge growth rate \( D(n) \) to infinity.

1. As regards to the Navier-Stokes equation in real case.
Here \(d = 3\); i.e. \(D(n) = D_{1,9}(n)\); \(D(0) = 1\), \(D(n + 1) = 1 + 9D^2(n)\):

\[
D(0) = 1, \ D(1) = 10, \ D(2) = 901, \ D(3) = 811\ 802, \ D(4) = 659\ 022\ 487\ 205,
\]

\[
D(5) = 434\ 310\ 638\ 641\ 864\ 388\ 712\ 026, \ D(6) \approx 1.886257308 \cdot 10^{47}, \ D(7) \approx 3.5579666 \cdot 10^{84}.
\]

2. Let now \(a = b = 1\); i.e. \(D(n) = D_{1,1}(n)\); then

\[
D(0) = 1, \ D(1) = 2, \ D(2) = 5, \ D(3) = 26, \ D(4) = 677, \ D(5) = 458\ 330,
\]

\[
D(6) = 210\ 066\ 388\ 901, \ D(7) = 44\ 127\ 887\ 745\ 906\ 175\ 987\ 802.
\]

3. For comparison:

\[
D(n) \geq \tilde{D}(n) := 2^{(2^n-1)};
\]

\[
\tilde{D}(0) = 1, \ \tilde{D}(1) = 2, \ \tilde{D}(2) = 4, \ \tilde{D}(3) = 16, \ \tilde{D}(4) = 256, \ \tilde{D}(5) = 65\ 536,
\]

\[
\tilde{D}(6) = 4\ 294\ 967\ 296, \ \tilde{D}(7) = 18\ 446\ 744\ 073\ 709\ 551\ 616.
\]

The great difference between \(D_{1,1}(n)\) and \(\tilde{D}(n)\) show us the influence of free member "a" in the source equation (1.1).

4 Concluding remarks.

A. At the same method may be used by investigation of the non-linear recursion

\[
D(n + 1) = F(n, D(n))
\]

with monotonic increasing power of non-linearity such that

\[
C_1z^{1+\Delta} \leq F(n, z) \leq C_2z^{1+\Delta}, \ z \geq 1, \ \Delta = \text{const} > 0.
\]

B. The vector analog of the equation (1.1) has a form

\[
D(n + 1) = \vec{a} + D^2(n)\vec{b},
\]
where $D(n)$ is the square matrix $m \times m$ and $\dim \vec{a} = \dim \vec{b} = m$, $m = 2, 3, \ldots$.

C. Obviously, if in addition both the numbers $a$ and $b$ are integer, then quite sequence $\{D(n)\}$ is integer. Therefore

$$
\forall k, l = 1, 2, \ldots \Rightarrow b^{2k-1} D(l)^{2k} \leq D(k + l) \leq
$$

$$
\text{Ent}\left\{ 1 + \frac{a}{bD^2(l)} \right\}^{2k-1} \cdot b^{2k-1} \cdot D(l)^{2k} \right\},
$$

where $\text{Ent}(z)$ denotes the integer part of the real number $z$, since $D(n)$ is integer sequence.

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