ON QUASIMODULARITY OF SOME EQUIVARIANT
INTERSECTION NUMBERS ON THE HILBERT SCHEMES

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Abstract. We observe that certain equivariant intersection numbers of Chern
characters of tautological sheaves on Hilbert schemes for suitable circle actions
can be computed using the Bloch-Okounkov formula, hence they are related
to Gromov-Witten invariants of elliptic curves and its operator formalism in
terms of operators on the Fock space.

1. Introduction

Okounkov [15] proposed several intriguing conjectures relating intersection num-
bers of Chern classes on Hilbert schemes of points to multiple q-zeta values. For
earlier results that lead to this conjecture, see [3, 4, 5]. For recent results on this
conjecture, see e.g. [17]. For related works, see e.g. [19, 20].

In this paper we will compute some \( S^1 \)-equivariant intersection numbers on
Hilbert schemes of the affine plane, and show that their suitably normalized gener-
ating series are quasimodular forms. We will also explain how to reduce their com-
putations to the Okounkov-Bloch formula [2], hence a connection to the Gromov-
Witten theory of elliptic curves [16] can be observed. We will also compute the
torus-equivariant intersection numbers and show that they can be reduced to the
deformed Bloch-Okounkov \( n \)-point functions defined by Cheng and Wang [6].

For the \( S^1 \)-equivariant case, our idea is simply to combine the results Li-Qin-
Wang [13] with equivariant localization and the theory of quasimodular forms [10,
2, 21]. We reinterpret what has been computed in [13, Theorem 4.5] as the trace of
an operator on the Fock space as the generating series of equivariant intersection
numbers of equivariant Chern characters on the Hilbert schemes of points. By
localization, equivariant intersection numbers on Hilbert schemes of points on \( \mathbb{C}^2 \)
are given by summations over fixed points of torus actions on the Hilbert schemes,
which are indexed by partitions of integers. Hence generating series of equivariant
intersection numbers on \( \mathbb{C}^2[n] \) are given by summations over partitions. This is why
they are related to the Bloch-Okounkov formula and quasimodularity.

Quasimodularity was suggested by Dijkgraaf [7] in mirror symmetry of elliptic
curves. The mathematical theory was formulated and developed by Kaneko and
Zagier [10]. Bloch and Okounkov [2] established the quasimodularity of summations
of over partitions for a large class of functions on partition function called shifted
symmetric polynomials and derived an explicit formula. Zagier [21] developed a
new approach to the results of Bloch and Okounkov. Our note is a combination of
some ideas from [3, 13] with some ideas from [2, 21]. In an appendix we will present
some new way to derive Zagier’s Theorem 1 in [21]. Since the space of space of
quasimodular forms is contained in the space of multiple q-zeta values [15], we are
dealing with a special case of Okounkov’s conjecture.
For the torus-equivariant case, the idea is similar. Equivariant localization reduces the computation to a summation over partitions which we identify with deformed $n$-point functions defined by Cheng and Wang [6] using a vertex operator realization of the Macdonald operator [1, 9]. Unfortunately closed formulas are available in this case only for $n = 1$ and 2, so a discussion of quasimodularity has to be left to further investigations at present.

We arrange the rest of this note as follows. In §2 after introducing some notations on partitions, we recall the work of Bloch and Okounkov [2] on summation over partitions and quasi-modular forms (see also Zagier [21]). We also make use of [13, Lemma 3.1] and reinterpret [13, Theorem 4.5]. In §3 we recall the deformed $n$-point function and deformed Bloch-Okounkov formula for 1- and 2-point functions [6]. In §4 we explain how to reduce the computations of some equivariant intersection numbers of equivariant Chern characters of tautological sheaves on $\mathbb{C}^2$ to Bloch-Okounkov formula or deformed Bloch-Okounkov characters. In the Appendix we present a natural formalism that yields a formula that generalizes both Jacobi’s triple product identity and Zagier’s recursion relations for $q$-brackets of shifted symmetric polynomials.

2. Partitions, Bloch-Okounkov Formula, and Quasimodular Forms

In this section we first recall some notations on partitions and recall a formula of Bloch-Okounkov [2]. We also recall their work on quasimodularity.

2.1. Notations on partitions. A partition is a nonincreasing sequence $\lambda$ of non-negative integers $\lambda_1 \geq \lambda_2 \geq \cdots$, with only finitely many zero terms. One writes $|\lambda| := \sum_{i=1}^{\infty} \lambda_i$ and call it the weight of $\lambda$. The length of the partition is defined to be the number of nonzero terms in $\lambda$ and is denoted by $l(\mu)$. The partition with length equal to 0 is called the empty partition and is denoted by $\emptyset$.

A partition $\lambda$ can be graphically represented by its Young diagram. First one can assign the following numbers to a box $s = (i, j) \in \lambda$:

$$a_\lambda(s) = \lambda_i - j,$$
$$l_\lambda(s) = \lambda_j^t - i,$$
$$a'_\lambda(s) = j - 1,$$
$$l'_\lambda(s) = i - 1,$$

where $s$ is located at the $i$-row and the $j$-th column. Note that

$$a_{\lambda^t}(s^t) = l_{\lambda^t}(s),$$
$$a'_{\lambda^t}(s^t) = l'_{\lambda^t}(s),$$

where $\lambda^t$ is obtained from $\lambda$ by switching the roles of rows and columns, and $s^t$ is the box in $\lambda^t$ that corresponds to $s$ in $\lambda$. The content of a box $s \in \lambda$ is defined by

$$c(s) := j - i = a'_\lambda(s) - l'_\lambda(s).$$

2.2. Some functions on partitions. An advantage of the graphical representation by Young diagram is that one can naturally define some functions on the set of partitions.

Recall the following formula [13, Lemma 3.1]:

$$\sum_{s \in \lambda} e^{zc_\lambda(s)} = \frac{1}{\varsigma(z)} \left( \sum_{i=1}^{\infty} e^{z(\lambda_i - i + 1/2)} - \frac{1}{\varsigma(z)} \right),$$

(2)
where \( \varsigma(z) = e^{z/2} - e^{-z/2} \). The proof of the formula in [13] is nice and simple. For the convenience of the reader we recall their proof here. The contents of \( \lambda \) are:

\[-i + 1, -i + 2, \ldots, -i + \lambda_i, \quad i = 1, \ldots, l(\lambda),\]

therefore,

\[
\sum_{s \in \lambda} e^{z\varsigma(s)} = \sum_{i=1}^{l(\lambda)} \left( z^{-i+1} + z^{-i+2} + \ldots + z^{-i+\lambda_i} \right)
\]

\[
= \sum_{i=1}^{l(\lambda)} \frac{e^{z(\lambda_i - i + 1)} - e^{z(-i + 1)}}{e^z - 1}
\]

\[
= \sum_{i=1}^{l(\lambda)} \frac{e^{z(\lambda_i - i + 1/2)} - e^{z(-i + 1/2)}}{e^{z/2} - e^{-z/2}}
\]

\[
= \frac{1}{\varsigma(z)} \sum_{i=1}^{\infty} \left( e^{z(\lambda_i - i + 1/2)} - e^{z(-i + 1/2)} \right)
\]

\[
= \frac{1}{\varsigma(z)} \left( \sum_{i=1}^{\infty} e^{z(\lambda_i - i + 1/2)} - \sum_{i=1}^{\infty} e^{z(-i + 1/2)} \right)
\]

\[
= \frac{1}{\varsigma(z)} \left( \sum_{i=1}^{\infty} e^{z(\lambda_i - i + 1/2)} - \frac{1}{\varsigma(z)} \right).
\]

Here in the last equality the following identity has been used:

\[
(3) \quad \varsigma(z) = \sum_{i=1}^{\infty} e^{z(-i + 1/2)}.
\]

We will write the left-hand side of (2) as \( \text{ch}_z(\lambda) \), and call it the Chern character of \( \lambda \) for reasons to be manifest in the next section. It defines a function on the set \( \mathcal{P} \) of partitions. For the empty partition \( \emptyset \), our convention is that \( \text{ch}_z(\emptyset) = 1 \). We also write

\[
(4) \quad \text{ch}_k(\lambda) = \frac{1}{k!} \sum_{c \in \mu} c(\lambda)^k.
\]

From our convention for \( \text{ch}_z(\emptyset) \), we have

\[
(5) \quad \text{ch}_k(\emptyset) = \delta_{k,0}.
\]

Another advantage of using the Young diagram is that one can introduce the Frobenius notation \((a_1, \ldots, a_r | b_1, \ldots, b_r)\) graphically, where \( a_i \) (resp. \( b_i \)) are the numbers of the cells to the right of (resp. below) the \( i \)-th cells on the diagonal. An alternative way to understand these numbers is to use the Dirac sea. Consider the set \( X_\lambda := \{ \lambda_i - i + 1/2 | i = 1, 2, \ldots \} \). Denote by \( (\mathbb{Z} + \frac{1}{2})_\pm \) the positive (resp. negative) half integers, then one has (see e.g. [21 (21)]):

\[
X_\lambda \cap (\mathbb{Z} + \frac{1}{2})_+ = C_\lambda^+ := \{ a_r + \frac{1}{2}, \ldots, a_1 + \frac{1}{2} \},
\]

\[
(\mathbb{Z} + \frac{1}{2})_+ - X_\lambda = C_\lambda^- := \{ -b_1 - \frac{1}{2}, \ldots, -b_r - \frac{1}{2} \}.
\]
One can then define a sequence $P_k(\lambda)$ of functions of partitions for $k \geq 0$ as follows:

$$
P_k(\lambda) = \sum_{j=1}^{r} ((a_j + \frac{1}{2})^k - (-b_j - \frac{1}{2})^k).
$$

It is easy to see that

$$
P_0(\lambda) = 0.
$$

One can relate these functions to the Chern characters of partitions. From (2) we have

$$
\varsigma(z) \, \text{ch}_z(\lambda) = \sum_{k=1}^{\infty} z^k k! P_k(\lambda).
$$

Indeed, this follows from the following straightforward computations:

$$
\varsigma(z) \, \text{ch}_z(\lambda) = \varsigma(z) \sum_{s \in \lambda} e^{z \varsigma_s(s)} = \sum_{i=1}^{\infty} e^{z(\lambda_i - i + 1/2)} - \frac{1}{\varsigma(z)}
$$

$$
= \sum_{\lambda_i - i + 1/2 \in C_\lambda^+} e^{z(\lambda_i - i + 1/2)} - \sum_{\lambda_i - i + 1/2 \in C_\lambda^-} e^{z(\lambda_i - i + 1/2)}
$$

$$
= \sum_{j=1}^{r} (e^{z(a_j + \frac{1}{2})} - e^{-z(b_j + \frac{1}{2})})
$$

$$
= \sum_{j=1}^{r} \sum_{k=0}^{\infty} \frac{z^k}{k!} ((a_j + \frac{1}{2})^k - (-b_j - \frac{1}{2})^k)
$$

$$
= \sum_{k=0}^{\infty} \frac{z^k}{k!} P_k(\lambda).
$$

By comparing the coefficients of $z^k$, one then gets

$$
P_k(\lambda) = k! \sum_{2l+m=k} \frac{2^{1-2l}}{(2l)!} \text{ch}_m(\lambda).
$$

One also has

$$
\text{ch}_z(\lambda) = \frac{1}{e^{z/2} - e^{-z/2}} \sum_{k=1}^{\infty} \frac{z^k}{k!} P_k(\lambda).
$$

Write

$$
\frac{z/2}{e^{z/2} - e^{-z/2}} = \sum_{n=0}^{\infty} \beta_n z^n
$$

as in [21], where

$$
\beta_n = \frac{B_n}{n!} (\frac{1}{2^n} - \frac{1}{2}) = \frac{1}{2} \frac{B_n(1/2)}{n!}.
$$

Then one can get

$$
\text{ch}_n(\lambda) = \sum_{k+l=n+1} 2\beta_k \frac{P_l(\lambda)}{l!}.
$$
Zagier \cite{21} introduced another sequence of functions $Q_n$ of partitions:

\begin{equation}
\sum_{i=1}^{\infty} e^{z(\lambda_i - i + 1/2)} = \sum_{k=0}^{\infty} Q_k(\lambda)z^{k-1}.
\end{equation}

Since we have

\begin{align*}
\sum_{i=1}^{\infty} e^{z(\lambda_i - i + 1/2)} &= \sum_{k=0}^{\infty} \frac{z^k}{k!} P_k(\lambda) + \frac{1}{e^{z/2} - e^{-z/2}} \\
&= \sum_{k=0}^{\infty} \frac{z^k}{k!} P_k(\lambda) + 2 \sum_{k=0}^{\infty} \beta_k z^{k-1},
\end{align*}

the following equality holds:

\begin{equation}
Q_k(\lambda) = \frac{P_{k+1}(\lambda)}{(k+1)!} + 2\beta_k = \frac{P_{k+1}(\lambda)}{(k+1)!} + \frac{B_k(1/2)}{k!}.
\end{equation}

2.3. Bloch-Okounkov formula. Let $f(\lambda)$ be a function on partitions, Bloch and Okounkov \cite{2} define

\begin{equation}
\langle f \rangle_q := \frac{\sum_{\lambda} f(\lambda)q^{\lambda}}{\sum_{\lambda} q^{\lambda}}.
\end{equation}

For $n = 1, 2, 3, \ldots$, they also define

\begin{equation}
F(t_1, \ldots, t_n; q) := \left\langle \prod_{k=1}^{n} \left( \sum_{i=1}^{\infty} t_k^{\lambda_i - i + \frac{1}{2}} \right) \right\rangle_q,
\end{equation}

and prove the following famous formula:

\begin{equation}
F(t_1, \ldots, t_n) = \sum_{\sigma \in S_n} \det \left( \frac{\theta(\tau(j+1), t_{\sigma(1)} \cdots t_{\sigma(n-j)}; \tau(I-j+1)})}{\theta(t_{\sigma(1)}, \cdots, t_{\sigma(n); \tau(I+1)})} \right).
\end{equation}

Here $\theta(x)$ is the Jacobi theta function defined by

\begin{equation}
\theta(t) := \eta(t)^{-3} \sum_{n \in \mathbb{Z}} (-1)^n x^{n+1/2} \frac{q^{(n+1/2)^2}}{(q)_{\infty}^{-2}(t^{1/2} - t^{-1/2})(qt)_{\infty}(q/t)_{\infty}},
\end{equation}

and $\theta^{(p)}(t) = (t \frac{d}{dt})^p \theta(t)$. See \cite{10} §5 for the relationship of this formula to the stationary Gromov-Witten invariants of elliptic curves.

Combining formula \cite{2} with \cite{13}, one can easily compute the $n$-point function of Chern characters of partitions:

\begin{equation}
\langle \text{ch}_{z_1} \cdots \text{ch}_{z_n} \rangle_q = \left\langle \prod_{j=1}^{n} \frac{1}{\xi(z_j)} \left( \sum_{i=1}^{\infty} e^{z_j(\lambda_i - i + 1/2)} - \frac{1}{\xi(z_j)} \right) \right\rangle_q
\end{equation}

by expanding the right-hand side:

\begin{equation}
\langle \text{ch}_{z_1} \cdots \text{ch}_{z_n} \rangle_q = \frac{1}{\prod_{j=1}^{n} \xi(z_j)} \sum_{k=0}^{n} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{F(e^{z_{i_1}}, \ldots, e^{z_{i_k}})(-1)^{n-k}\xi(z_{i_1}) \cdots \xi(z_{i_k})}{\prod_{j=1}^{n} \xi(z_j)}.
\end{equation}

See \cite{13} Theorem 4.5] for the interpretation of this formula in terms of the trace of an operator on the Fock space.
2.4. Quasimodular forms. Quasimodularity was suggested by Dijkgraaf [7] in the context of mirror symmetry of elliptic curves. The mathematical theory of quasimodular forms was developed by Kaneko and Zagier [10]. The ring of quasimodular forms is $\mathbb{Q}[E_2, E_4, E_6]$. The work of Bloch and Okounkov [2] provides many quasimodular forms. They have shown that for any polynomial $f$ in $P_1, P_2, \ldots$, $(f)_q$ is a quasimodular form. For a different proof of this result, see Zagier [21]. As a consequence, $(\text{ch}_k \cdots \text{ch}_n)_q$ is a quasimodular form.

3. Deformed Bloch-Okounkov Formula

In this section we recall the work on deformed Bloch-Okounkov formula due to Cheng and Wang [6]. We follow their notations and presentation closely with only minor modifications.

3.1. Deformed vertex operator. Consider the Heisenberg algebra generated by $I$ and $a_n$, $n \in \mathbb{Z}$, with the commutation relations:

$$[a_m, a_n] = \kappa m \delta_{m,-n} I,$$

where $\kappa$ is a Planck constant. The bosonic Fock space $B$ has a basis $a_{-\lambda} := a_{-\lambda_1}a_{-\lambda_2} \cdots |0\rangle$, where $\lambda = (\lambda_1, \lambda_2, \ldots)$ runs over all partitions. After identifying it with the ring of symmetric function $\Lambda$ by identifying $a_{-\lambda}$ with the power-sum symmetric functions $p_\lambda$, one has ($n \geq 1$):

$$a_{-n} = p_n, \quad a_0 = 0, \quad a_n = \kappa n \frac{\partial}{\partial p_n}, \quad I = 1.$$

Denote by $\Lambda_{q,t}$ the ring of symmetric functions with coefficients in $\mathbb{Q}(q,t)$. Denote by $L_0$ the usual energy operator on $\Lambda_{q,t}$: $L_0 g = qg$ if $g$ is a symmetric function of degree $n$. For $f \in \text{End}(\Lambda_{q,t})$, the trace of $A$ is defined by:

$$\text{Tr}_v A := \text{Tr}(v^{L_0} A)|_{\Lambda_{q,t}}.$$

As in [2], for a function $f$ on the set $\mathcal{P}$ of partitions, define the $v$-bracket of $f$ by

$$(f)_v := (v)_\infty \sum_{\lambda \in \mathcal{P}} f(\lambda)v^{[\lambda]}.$$

Cheng and Wang introduced the following deformed vertex operator:

$$(25) \quad V(z; q_1, t_1, q_2, t_2) = \exp \left( \sum_{k \geq 1} (q_1^k - q_2^k) a_{-k} \frac{z^k}{k} \right) \cdot \exp \left( - \sum_{k \geq 1} (t_1^k - t_2^k) \alpha_k \frac{z^{-k}}{k} \right).$$

They proved the following formula [6] Theorem 16:

$$(26) \quad \left\langle \prod_{i=1}^n V(z_i; s_i, t_i, u_i, w_i) \right\rangle_v = \prod_{1 \leq i < j \leq n} \left[ \frac{(1 - t_i w_j z_i^{-1} z_j)(1 - s_i u_j z_i^{-1} z_j)}{(1 - t_i u_j z_i^{-1} z_j)(1 - s_i w_j z_i^{-1} z_j)} \right]^\kappa \cdot \prod_{i,j=1}^n \left[ \frac{(t_i w_j z_i^{-1} z_j v)^\infty (s_i u_j z_i^{-1} z_j v)^\infty}{(t_i u_j z_i^{-1} z_j v)^\infty (s_i w_j z_i^{-1} z_j v)^\infty} \right]^\kappa,$$

where $(a)_{\infty} := \prod_{i=0}^{\infty} (1 - av^i)$. For example, when $n = 1$,

$$(27) \quad \langle V(z_1; s_1, t_1, u_1, w_1) \rangle_v = \left[ \frac{(t_1 w_1 v)^\infty (s_1 u_1 v)^\infty}{(t_1 u_1 v)^\infty (s_1 w_1 v)^\infty} \right]^\kappa.$$
Write

\[
V(z; q_1, t_1, q_2, t_2) = \sum_{m \in \mathbb{Z}} V_m(q_1, q_2, t_1, t_2) z^m.
\]

The operator \( V_0 \) is called the zero mode of \( V \). By (27) one can get:

\[
\langle V_0(s_1, t_1, u_1, w_1) \rangle_v = \left[ \frac{(t_1 u_1 v)_\infty (s_1 u_1 v)_\infty}{(t_1 u_1 v)_\infty (s_1 w_1 v)_\infty} \right]^{\kappa}.
\]

This is [6, Theorem 13] proved by a different method.

3.2. **Deformed Bloch-Okounkov formula.** As pointed out by Cheng and Wang [6, Remark 12], when \( \kappa = 1, q_2 = t_2 = 1 \), and write \( q = q_1 \) and \( t = t_1 \), the operator \( V_0 \) provides a vertex operator realization for the Macdonald operator \( \hat{B}_{q,t} \):

\[
\hat{B}_{q,t} := \frac{1}{(1-q)(1-t)} \cdot V_0(q, 1, t, 1).
\]

This formula appears in a different form in the study of Macdonald polynomials by Garsia and Haiman [9, (73)]. It also appears in [1, (32)]. This operator has Macdonald functions \( P_\lambda(x; q, t) \) as eigenfunctions:

\[
\hat{B}_{q,t} P_\lambda(x; q, t) = \hat{B}_\lambda(q, t) \cdot P_\lambda(x; q, t),
\]

\[
\hat{B}_\lambda(q, t) := \frac{1}{1-q} \sum_{i \geq 1} t^{-i+1} q^{\lambda_i}.
\]

There is a related operator:

\[
\mathfrak{B}_{q,t} P_\lambda(x; q, t) = B_\lambda(q, t) \cdot P_\lambda(x; q, t),
\]

\[
B_\lambda(q, t) := \sum_{\square \in \lambda} q^{a(\square)} q^{l(\square)}.
\]

The \( n \)-point (correlation) functions are defined to be

\[
F(q_1, t_1; \ldots, q_n, t_n) := \text{Tr}_v (\mathfrak{B}_{q_1, t_1} \cdots \mathfrak{B}_{q_n, t_n}),
\]

\[
\hat{F}(q_1, t_1; \ldots, q_n, t_n) := \text{Tr}_v (\hat{B}_{q_1, t_1} \cdots \hat{B}_{q_n, t_n}).
\]

By [6 Lemma 1],

\[
B_\lambda(q, t) = \hat{B}_\emptyset(q, t) - \hat{B}_\lambda(q, t) = \frac{1}{(1-q)(1-t)},
\]

so one convert between the computations of \( F \) and that of \( \hat{F} \). By [6 Lemma 2],

\[
F(q_1, t_1; \ldots, q_n, t_n) = (v)_\infty^{-1} \left( \prod_{k=1}^{n} B_\lambda(q_k, t_k) \right)_v.
\]
The one-point function and the two-point function have been computed by Cheng and Wang [6, Theorem 5, Theorem 9]:

\[
\hat{F}(q, t) = \frac{(vqt)_\infty}{(q)_\infty(t)_\infty},
\]

\[
\hat{F}(q_1, t_1; q_2, t_2) = \frac{1}{(1 - q_1)(1 - q_2)(1 - t_1t_2)} \cdot \frac{(vq_1q_2t_1t_2)_\infty}{(vt_1t_2)_\infty(vq_2q_2)_\infty} \cdot \left[\frac{q_1q_2t_1t_2 - 1}{(1 - q_1t_1)(1 - q_2t_2)} + \frac{1}{1 - q_1t_1}\Phi(v, q_1t_1, vq_1q_2; vq_1, vq_1q_1t_2; v; t_2) + \frac{1}{1 - q_1t_1}\Phi(v, q_2t_2, vq_1q_2; vq_2, vq_1q_1t_2; v; t_1)\right],
\]

where \(\Phi\) is \(v\)-basic hypergeometric series:

\[
\Phi(a_1, a_2, a_3; b_1, b_2; z) := \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m(a_3)_m}{(b_1)_m(b_2)_m} \frac{z^m}{(v)_m},
\]

and \((a)_0 = 1, (a)_m = \prod_{i=0}^{m-1} (1 - av^i).

4. Equivariant Intersection Numbers on \(\mathbb{C}^n\) and Quasimodularity

See [14] for references on equivariant indices on Hilbert schemes of points on \(\mathbb{C}^2\). We have followed the notations in [19, 20].

4.1. Localizations on Hilbert schemes of the affine plane. By a theorem of Fogarty [3], the Hilbert scheme \((\mathbb{C}^2)^n\) is a nonsingular variety of dimension \(2n\). The torus action on \(\mathbb{C}^2\) given by

\[
(t_1, t_2) \cdot x = t_1x, \quad (t_1, t_2) \cdot y = t_2y
\]
on linear coordinates induces an action on \((\mathbb{C}^2)^n\). The fixed points are isolated and parameterized by partitions \(\lambda = (\lambda_1, \ldots, \lambda_l)\) of weight \(n\). They correspond to ideals

\[
I_\lambda = \langle y^{\lambda_1}, xy^{\lambda_2}, \ldots, x^{l-1}y^{\lambda_l}, x^l \rangle.
\]
The weight decomposition of the tangent bundle of \(T(\mathbb{C}^2)^n\) at a fixed point \(\lambda\) is given by [14]:

\[
\sum_{(i,j) \in \lambda} t_1^{(\lambda'_j - i)} t_2^{-(\lambda_i - j + 1)} + t_1^{-(\lambda'_j - i + 1)} t_2^{(\lambda_i - j)}
\]

\[
= \sum_{s \in \lambda} t_1^{(l(s))} t_2^{-(a(s)+1)} + t_1^{-l(s)+1} t_2^{a(s)},
\]

where \((t_1, t_2) \in \mathbb{C}^* \times \mathbb{C}^*\). It follows that the equivariant Euler class is given at \(I_\lambda\) by:

\[
e_T(T\mathbb{C}^n)|_{I_\lambda} = ((\lambda'_j - i)t_1 - (\lambda_i - j + 1)t_2) \cdot (-1^{l(s)+1} t_1) + (\lambda_i - j)t_2)
\]

\[
= \prod_{s \in \lambda} (l(s)t_1 - (a(s)+1)t_2)(-l(s)+1)t_1 + a(s)t_2).
\]
4.2. Tautological bundles. Let \( \mathcal{Z}_n \subset X \times X^{[n]} \) be the universal family of sub-schemes parameterized by \( X^{[n]} \). Denote by \( p_1 : \mathcal{Z}_n \to X \) and \( \pi : \mathcal{Z}_n \to X^{[n]} \) the projection onto the \( X \) and \( X^{[n]} \) respectively. For any locally free sheaf \( F \) on \( X \) let \( F^{[n]} = \pi_*(\mathcal{O}_{\mathcal{Z}_n} \otimes p_1^*F) \). With this notation we write \( \xi_n = \xi_n^X = \mathcal{O}_{X}^{[n]} \). The tautological bundle \( \xi_n \) on \( (\mathbb{C}^2)^{[n]} \) has its weight decomposition at a fixed point \( I_\lambda \) given by \([13]\):

\[
\xi_n|_{I_\lambda} = \sum_{(i,j) \in \lambda} t_1^{i-1} t_2^{j-1} = \sum_{s \in \lambda} t_1^{l'(s)} t_2^{a'(s)}.
\]

So we have

\[
\text{ch}(\xi_n)_{T}|_{I_\lambda} = \sum_{(i,j) \in \lambda} e^{(i-1)t_1 + (j-1)t_2} = \sum_{s \in \lambda} e^{l'(s)t_1 + a'(s)t_2}.
\]

In particular,

\[
\text{ch}_k(\xi_n)_{T}|_{I_\lambda} = \frac{1}{k!} \sum_{(i,j) \in \lambda} ((i-1)t_1 + (j-1)t_2)^k = \frac{1}{k!} \sum_{s \in \lambda} (l'(s)t_1 + a'(s)t_2)^k.
\]

For a vector \( A = (a, b) \in \mathbb{Z}^2 \), denote by \( \mathcal{O}_{\mathbb{C}^2}^{A} \) the \( T^2 \)-equivariant line bundle on \( \mathbb{C}^2 \) with weight \( A \). Recall the universal family \( \mathcal{Z}_n \) lies in \( \mathbb{C}^2 \times (\mathbb{C}^2)^{[n]} \), and denote by \( p_1 : \mathcal{Z}_n \to \mathbb{C}^2 \) the projection onto the first factor. Let \( \xi_n^A = \pi_*(\mathcal{O}_{\mathcal{Z}_n} \otimes p_1^*\mathcal{O}_{\mathbb{C}^2}^{A}) \). Then one has:

\[
\xi_n^A|_{I_\lambda} = \sum_{(i,j) \in \lambda} t_1^{i-1} t_2^{j-1} t_1^a t_2^b = \sum_{s \in \lambda} t_1^{l'(s)} t_2^{a'(s)} t_1^a t_2^b.
\]

So we have:

\[
\text{ch}(\xi_n^A)_{T}|_{I_\lambda} = \sum_{(i,j) \in \lambda} e^{(i-1+a)t_1 + (j-1+b)t_2} = \sum_{s \in \lambda} e^{l'(s)t_1 + (a'(s)+b)t_2}.
\]

In particular,

\[
\text{ch}_k(\xi_n^A)_{T}|_{I_\lambda} = \frac{1}{k!} \sum_{(i,j) \in \lambda} ((i-1+a)t_1 + (j-1+b)t_2)^k = \frac{1}{k!} \sum_{s \in \lambda} (l'(s) + a)(t_1 + (a'(s) + b)t_2)^k.
\]

Hence by localization formula, we have

\[
\sum_{n \geq 0} q^n \int_{(\mathbb{C}^2)^{[n]}} \text{ch}_k(\xi_n^{A_1})_{T} \cdots \text{ch}_k(\xi_n^{A_n})_{T} \cdot c_T(T\mathbb{C}^{[n]})
\]

\[
= \sum_{\lambda} q^{|\lambda|} \prod_{j=1}^{N} \frac{1}{k_j!} \sum_{s \in \lambda} ((l'(s) + a_j)t_1 + (a'(s) + b_j)t_2)^{k_j}.
\]
Consider the circle subgroup $S^1 \rightarrow T$, $e^{it} \mapsto (e^{-it}, e^{it})$. I.e., let $-t_1 = t_2 = t$.

$$
\sum_{n \geq 0} q^n \int_{(\mathbb{C}^2)^{|n|}} \operatorname{ch}_{k_1} (\xi_{n}^{A_1})_{S^1} \cdots \operatorname{ch}_{k_N} (\xi_{n}^{A_N})_{S^1} \cdot e_{S^1} (T \mathbb{C}^{|n|})
$$

$$
= \prod_{n=1}^{\infty} (1-q^n) \sum_{n \geq 0} q^n \int_{(\mathbb{C}^2)^{|n|}} \operatorname{ch}_{k_1} (\xi_{n}^{A_1})_{S^1} \cdots \operatorname{ch}_{k_N} (\xi_{n}^{A_N})_{S^1} \cdot e_{S^1} (T \mathbb{C}^{|n|})
$$

The right-hand side is a linear combination of terms of the form:

$$
\prod_{j=1}^{N} \sum_{\lambda \in \Lambda} q^{\lambda} \sum_{j \in \Lambda} c(s)^{\lambda j}
$$

Therefore, by the results of last section, \( \prod_{n=1}^{\infty} (1-q^n) \sum_{n \geq 0} q^n \int_{(\mathbb{C}^2)^{|n|}} \operatorname{ch}_{k_1} (\xi_{n}^{A_1})_{S^1} \cdots \operatorname{ch}_{k_N} (\xi_{n}^{A_N})_{S^1} \cdot e_{S^1} (T \mathbb{C}^{|n|}) \) is a quasimodular form. Therefore, we have proved:

**Theorem 4.1.** For integers \( k_1, \ldots, k_N \geq 0 \), \( A_1, \ldots, A_N \in \mathbb{Z}^2 \),

$$
\prod_{n=1}^{\infty} (1-q^n) \sum_{n \geq 0} q^n \int_{(\mathbb{C}^2)^{|n|}} \operatorname{ch}_{k_1} (\xi_{n}^{A_1})_{S^1} \cdots \operatorname{ch}_{k_N} (\xi_{n}^{A_N})_{S^1} \cdot e_{S^1} (T \mathbb{C}^{|n|})
$$

is a quasimodular form.

In a similar way, one has

$$
\prod_{n=1}^{\infty} (1-q^n) \sum_{n \geq 0} q^n \int_{(\mathbb{C}^2)^{|n|}} \operatorname{ch}_{k_1} (\xi_{n}^{A_1})_{S^1} \cdots \operatorname{ch}_{k_N} (\xi_{n}^{A_N})_{S^1} \cdot e_{S^1} (T \mathbb{C}^{|n|})
$$

$$
= \prod_{n=1}^{\infty} (1-q^n) \sum_{n \geq 0} q^n \int_{(\mathbb{C}^2)^{|n|}} \operatorname{ch}_{k_1} (\xi_{n}^{A_1})_{S^1} \cdots \operatorname{ch}_{k_N} (\xi_{n}^{A_N})_{S^1} \cdot e_{S^1} (T \mathbb{C}^{|n|})
$$

$$
= \prod_{n=1}^{\infty} (1-q^n) \sum_{n \geq 0} q^n \int_{(\mathbb{C}^2)^{|n|}} \operatorname{ch}_{k_1} (\xi_{n}^{A_1})_{S^1} \cdots \operatorname{ch}_{k_N} (\xi_{n}^{A_N})_{S^1} \cdot e_{S^1} (T \mathbb{C}^{|n|})
$$

$$
= \prod_{n=1}^{\infty} (1-q^n) \sum_{n \geq 0} q^n \int_{(\mathbb{C}^2)^{|n|}} \operatorname{ch}_{k_1} (\xi_{n}^{A_1})_{S^1} \cdots \operatorname{ch}_{k_N} (\xi_{n}^{A_N})_{S^1} \cdot e_{S^1} (T \mathbb{C}^{|n|})
$$

$$
= \prod_{n=1}^{\infty} (1-q^n) \sum_{n \geq 0} q^n \int_{(\mathbb{C}^2)^{|n|}} \operatorname{ch}_{k_1} (\xi_{n}^{A_1})_{S^1} \cdots \operatorname{ch}_{k_N} (\xi_{n}^{A_N})_{S^1} \cdot e_{S^1} (T \mathbb{C}^{|n|})
$$

Hence it is possible to reduce to the Bloch-Okounkov formula by [24]. In order to compute the series in the above Theorem, one can consider their generating series:
For the torus-equivariant intersection numbers, we consider:

\[
\sum_{n \geq 0} q^n \int_{(\mathbb{C}^2)^n} \prod_{j=1}^{N} \sum_{k_j \geq 0} z_{j}^{k_j} \cdot \left( e_{(A)}^{\lambda} \right)_T \cdot e_T(T\mathbb{C}^{[n]})
\]

\[
= \sum_{\alpha} q^{\left|\alpha\right|} \prod_{k=1}^{N} \sum_{(i,j) \in \lambda} e^{z_{k}[(i-1+a)_{1}+(j-1+b)_{2}]} 
\]

\[
= e^{\sum_{j=1}^{N} (a_{j}f_{1}+b_{j}f_{2})z_{j}} \cdot \sum_{\alpha} q^{\left|\alpha\right|} \prod_{k=1}^{N} \sum_{s \in \Delta} e^{z_{k}[(s)_{1}+(a'_{s})_{2}]}.
\]

By (54), its computation is reduced to the deformed n-point function

\[
\langle \mathfrak{B}_{\lambda}(e^{x_{1}t_{1}}, e^{x_{2}t_{2}}) \cdots \mathfrak{B}_{\lambda}(e^{x_{N}t_{N}}, e^{x_{1}t_{1}}) \rangle_{q}.
\]

This solves the problem of finding geometric interpretations of such n-point functions posed by Cheng and Wang in the end of their paper [6].

Appendix A. Spectral Flow, Dirac Flow, and Zagier Recursion Relations

In this Appendix we interpret Zagier’s approach in [21] in terms of spectral flow and Dirac flow, and derive a formula that generalizes both the Jacobi triple product identity and Zagier’s recursion relations [21, Theorem 1].

A.1. The spectral flow. For \( \alpha \in \mathbb{R} \), note

\[
D e^{2\pi i(n+\alpha)x} = (n + \alpha) \cdot e^{2\pi i(n+\alpha)x},
\]

where \( D := \frac{d}{2\pi i \cdot dx} \) can be understood as the Dirac operator on the line, acting on the spaces of quasi-periodic functions:

\[
f(x + 1) = e^{2\pi i \alpha} \cdot f(x).
\]

The functions \( e_{n+\alpha}(x) := e^{2\pi i(n+\alpha)x} \) are the eigenfunctions in this space, and one can use them to form the “Dirac sea” as follows.

Denote by \( (\mathbb{Z} + \alpha)_{\pm} \) the nonnegative (resp. negative) numbers in the set \( \mathbb{Z} + \alpha := \{n + \alpha \mid n \in \mathbb{Z}\} \). It is clear that when \( \alpha - \alpha' \in \mathbb{Z} \), then \( \mathbb{Z} + \alpha = \mathbb{Z} + \alpha' \), and so \( (\mathbb{Z} + \alpha)_{\pm} = (\mathbb{Z} + \alpha')_{\pm} \). It then suffices to consider \( \alpha \in [0,1] \). For a sequence \( a = \{a_1, a_2, \ldots \} \subset \mathbb{Z} + \alpha \) with \( a_1 > a_2 > \cdots \), define

\[
a_{+} := a \cap (\mathbb{Z} + \alpha)_{+}, \quad a_{-} := (\mathbb{Z} + \alpha)_{-} - a.
\]

When both \( a_{+} \) and \( a_{-} \) are finite sets, we say \( a \) is admissible. Denote by \( \mathcal{X}_{\alpha} \) the set of all admissible sequences in \( \mathbb{Z} + \alpha \). To each \( a \in \mathcal{X}_{\alpha} \), define

\[
e_{a} := e_{a_{1}} \wedge e_{a_{2}} \wedge \cdots,
\]

and define the (fermionic) Fock space \( \mathcal{F}_{\alpha} \) by

\[
\mathcal{F}_{\alpha} := \text{span}\{e_{a} \mid a \in \mathcal{X}_{\alpha}\}.
\]

Two special cases have been widely used in the mathematical literature: \( \mathcal{F}_{0} \) (the Ramond-Ramond sector) and \( \mathcal{F}_{1/2} \) (the Neveu-Schwarz sector). But the spectral flow in the physics literature [18, 12] indicates that it is interesting to relate these two sectors by continuously going through other sectors. In our case, this can be
easily done by introducing a shift operator. For \( \beta \in \mathbb{R} \), define \( U(\beta) : \mathbb{Z} + \alpha \to \mathbb{Z} + \alpha + \beta \) simply by

\[
n + \alpha \mapsto n + \alpha + \beta.
\]

The family \( \{ U(\beta) \}_{\beta \in \mathbb{R}} \) is called the spectral flow. First of all, if is a flow because:

\[
U(0) = \text{id}, \quad U(\beta_1)U(\beta_2) = U(\beta_1 + \beta_2);
\]

secondly, it flows the eigenfunctions and the eigenvalues. The operator \( U(\beta) \) induces a shift operator, also denoted by \( U(\beta) \), from \( X_\alpha \) to \( X_{\alpha + \beta} \):

\[
a \in X_\alpha \mapsto a + \beta \in X_{\alpha + \beta},
\]

where \( a + \beta = \{ a_1 + \beta, a_2 + \beta, \ldots \} \) when \( a = \{ a_1, a_2, \ldots \} \). One therefore also has an induced operator, also denoted by \( U(\beta) \), from \( \mathcal{F}_\alpha \) to \( \mathcal{F}_{\alpha + \beta} \), defined by \( e_a \mapsto e_{U(\beta)e_a} \).

The following is the starting point of the approach of Zagier [21]. Define the charge of \( e_a \) by \( |a| - |a^c| \). Inside \( \mathcal{F}_{1/2} \), the charge zero \( e_a \) are in one-to-one correspondence with the set of partitions. Given a partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \), define

\[
a_\lambda = (\lambda_1 - 1 + 1/2, \lambda_2 - 2 + 1/2, \ldots),
\]

then \( e_{a_\lambda} \) has charge zero in \( \mathcal{F}_{1/2} \), and all \( e_a \) in \( \mathcal{F}_{1/2} \) of charge zero has this form. Zagier noted an one-to-one correspondence \( (Z + 1/2) \times \mathcal{P} \to X_0 \) given by

\[
(\beta, \lambda) \mapsto U(\beta)a_\lambda,
\]

where \( \mathcal{P} \) denotes the set of all partitions. This can be generalized to an one-to-one correspondence:

\[
(Z + 1/2 + \alpha) \times \mathcal{P} \to X_\alpha.
\]

For those of whom do not feel comfortable working with infinite wedge products, \( \mathcal{F}_\alpha \) can also be treated in the following way. For \( a \in X_\alpha \), write \( a_+ = \{ a_1, \ldots, a_r \} \) and \( a_- = \{ -b_1, \ldots, -b_s \} \), where \( b_1 < \cdots < c_s \) are nonnegative. Define

\[
\hat{e}_a = e_{a_1} \wedge \cdots \wedge e_{a_r} \wedge e_{-b_1} \wedge \cdots \wedge e_{-b_s}.
\]

Let \( \hat{\mathcal{F}}_\alpha = \text{span}\{ \hat{e}_a \mid a \in X_\alpha \} \). Then \( \mathcal{F}_\alpha \cong \hat{\mathcal{F}}_\alpha \).

A.2. Dirac flow and its asymptotic expansion. We call the family of operators \( \{ e^{tD} \}_{t \in \mathbb{R}} \) the Dirac flow.

\[
e^{tD}e_{n+\alpha}(x) = e^{(n+\alpha)t} \cdot e_{n+\alpha}(x).
\]

We define the action of \( e^{tD} \) on \( \mathcal{F}_\alpha \) as follows. For \( a \in X_\alpha \),

\[
e^{tD}e_a = \sum_{i=1}^{\infty} e_{a_1} \wedge \cdots \wedge e^{tD}e_{a_i} \wedge \cdots.
\]

It is clear that

\[
e^{tD}e_a = \sum_{i=1}^{\infty} e^{ta_i} \cdot e_a.
\]
For \( \alpha \in [0, 1) \), we then have

\[
W_a(t) = \sum_{i=1}^{\infty} e^{t a_i} = \sum_{a_i \in a_+} e^{t a_i} - \sum_{a_i \in a_-} e^{t a_i} + \sum_{n=0}^{\infty} e^{-(n+\alpha)t}
\]

\[
= \sum_{a_i \in a_+} e^{t a_i} - \sum_{a_i \in a_-} e^{t a_i} + e^{-\alpha t} \frac{1}{1 - e^{-t}}
\]

\[
= \sum_{n=0}^{\infty} \frac{t^n}{n!} \left( \sum_{a_i \in b_+} a_i^n - \sum_{a_i \in b_-} a_i^n \right) + \frac{1}{t} + \sum_{k=1}^{\infty} \frac{B_k (1-\alpha)}{k!} t^{k-1}.
\]

So if one writes

\[
(61) \quad W_a(t) = \sum_{k=0}^{\infty} Q_k(a) t^{k-1},
\]

then

\[
(62) \quad Q_0(a) = 1,
\]

\[
(63) \quad Q_k(a) = \frac{1}{(k-1)!} \left( \sum_{a_i \in b_+} a_i^{k-1} - \sum_{a_i \in b_-} a_i^{k-1} \right) + \frac{B_k (1-\alpha)}{k!}.
\]

This gives us an asymptotic expansion of the Dirac flow on the Fock space \( F_\alpha \) as \( t \to 0 \). Recall the Hurwitz zeta function is defined by:

\[
(64) \quad \zeta(s, \theta) = \sum_{n=0}^{\infty} \frac{1}{(n+\theta)^s}.
\]

And its values at nonpositive integers are given by the following formula:

\[
(65) \quad \zeta(-n, \theta) = -\frac{B_{n+1}(\theta)}{n+1}.
\]

So we have

\[
(66) \quad (k-1)! Q_k(a) = \sum_{a_i \in b_+} a_i^{k-1} - \sum_{a_i \in b_-} a_i^{k-1} - \zeta(-k+1, 1-\alpha).
\]

We understand \( Q_k(a) \) as the eigenvalue some operator \( Q_k \) acting on \( e^a \). The operators \( Q_k \) will be called the modes of the Dirac flow.

**Remark 1.** Zagier [21] considered \( W_a \) and \( Q_k(a) \) for \( \alpha = 0 \) and \( \frac{1}{2} \).

**A.3. Character of the modes of the Dirac flow.** Define

\[
(67) \quad \chi_\alpha(q_0, q_1, \ldots) = \sum_{a \in X_\alpha} q_0^{Q_1(a)} q_1^{Q_2(a)} \cdots = \text{Tr} e^{t_0 Q_1 + t_1 Q_2 + \cdots} |_{F_\alpha},
\]

where \( q_n = e^{t_n} \). One can compute \( \chi_\alpha \) in two different ways. First of all, using the isomorphism \( F_\alpha \cong \hat{F}_\alpha \), one can convert the computation to that of a trace of some operator on the latter space, which is easily evaluated to give us:

\[
(68) \quad \chi_\alpha(q_1, q_2, \ldots) = \exp \left( -\sum_{n=0}^{\infty} \frac{t_n}{(n-1)!} \zeta(-n, 1-\alpha) \right)
\]

\[
\cdot \prod_{n \geq 1} \left( 1 + \exp \left( \sum_{j=0}^{\infty} \frac{t_j}{j!} (n-1+\alpha)^j \right) \right) (1 + \exp \left( -\sum_{j=0}^{\infty} \frac{t_j}{j!} (-n+\alpha)^j \right)).
\]
Secondly, one can use the one-to-one correspondence in [68] as follows:

\[ \chi_\alpha(q_1, q_2, \ldots) = \sum_{n \in \mathbb{Z}} \sum_{\lambda \in \mathcal{P}} q_1^{\alpha_\lambda + n + 1/2 + \alpha} Q_2^{\alpha_\lambda + n + 1/2 + \alpha}, \ldots \]

One can use (61) to compute \( Q_k(a_\lambda + n + 1/2 + \alpha) \) in the following way:

\[
\sum_{k=0}^{\infty} Q_k(a_\lambda + n + 1/2 + \alpha)t^{k-1} = W_{a_\lambda + n + 1/2 + \alpha}(t) = \sum_{i=1}^{\infty} e^{(\lambda_i - i + 1/2 + n + 1/2 + \alpha)i}t = \sum_{i=1}^{\infty} e^{(n + 1/2 + \alpha)i}t \cdot W_{a_\lambda}(t)
\]

\[
= \sum_{l \geq 0} \frac{t^l}{l!}(n + 1/2 + \alpha)^l \cdot \frac{1}{l} \sum_{m=0}^{\infty} Q_m(a_\lambda)t^m,
\]

it follows that

\[
Q_k(a_\lambda + n + 1/2 + \alpha) = \sum_{l+m=k} \frac{1}{l!} (n + 1/2 + \alpha)^l \cdot Q_m(a_\lambda).
\]

Now we have:

\[
\sum_{k=0}^{\infty} t_{k-1}Q_k(a_\lambda + n + 1/2 + \alpha) = \sum_{k=1}^{\infty} t_{k-1} \sum_{l+m=k} \frac{1}{l!} (n + 1/2 + \alpha)^l \cdot Q_m(a_\lambda)
\]

\[
= \sum_{k=1}^{\infty} \frac{t_{k-1}}{k!} (n + 1/2 + \alpha)^k + \sum_{m=1}^{\infty} Q_m(a_\lambda) \sum_{l=0}^{\infty} \frac{t_{l+m-1}}{l!} (n + 1/2 + \alpha)^l.
\]

Therefore, we get another formula for \( \chi_\alpha \):

\[
\chi_\alpha(q_1, q_2, \ldots)
\]

\[
= \sum_{n \in \mathbb{Z}} \sum_{\lambda \in \mathcal{P}} \exp \left( \sum_{k=1}^{\infty} \frac{t_{k-1}}{k!} (n + 1/2 + \alpha)^k + \sum_{m=1}^{\infty} Q_m(a_\lambda) \sum_{l=0}^{\infty} \frac{t_{l+m-1}}{l!} (n + 1/2 + \alpha)^l \right).
\]

Now by comparing (68) with (70), we get the following formula:

\[
\sum_{n \in \mathbb{Z}} \sum_{\lambda \in \mathcal{P}} \exp \left( \sum_{k=1}^{\infty} \frac{t_{k-1}}{k!} (n + 1/2 + \alpha)^k + \sum_{m=1}^{\infty} Q_m(a_\lambda) \sum_{l=0}^{\infty} \frac{t_{l+m-1}}{l!} (n + 1/2 + \alpha)^l \right)
\]

\[
= \exp \left( \sum_{n=0}^{\infty} \frac{t_n}{n!} \cdot (-n, 1 - \alpha) \right)
\]

\[
\cdot \prod_{n \geq 1} \left( 1 + \exp \left( \sum_{j=0}^{\infty} \frac{t_j}{j!} (n - 1 + \alpha)^j \right) \right) \left( 1 + \exp \left( \sum_{j=0}^{\infty} \frac{(-\alpha)^j}{j!} \right) \right).
\]
Let us look at some special case of this formula. First take $\alpha = \frac{1}{2}$ to get:

$$\sum_{n \in \mathbb{Z}} \sum_{\lambda \in \mathcal{P}} \exp\left(\sum_{k=1}^{\infty} \frac{t_{k-1}}{k!} n^k + \sum_{m=1}^{\infty} Q_m(a_\lambda) \sum_{l=0}^{\infty} \frac{t_{l+m-1}}{l!} n^l\right)$$

$$= \exp\left(-\sum_{n=0}^{\infty} \frac{t_n}{n!} \zeta(-n, \frac{1}{2})\right)$$

$$\cdot \prod_{n \geq 1} \left(1 + \exp\left(\sum_{j=0}^{\infty} \frac{t_j}{j!} (n - \frac{1}{2} j^2)\right)\right) \cdot \left(1 + \exp\left(-\sum_{j=0}^{\infty} \frac{t_j}{j!} (-n + \frac{1}{2} j^2)\right)\right).$$

Next we take $t_2 = t_3 = \cdots = 0$, and use the following identities [21, (3)]

$$Q_1(a_\lambda) = 0, \quad Q_2(a_\lambda) = |\lambda| - \frac{1}{24}$$

to get:

$$\sum_{n \in \mathbb{Z}} \sum_{\lambda \in \mathcal{P}} \exp\left(nt_0 + \frac{1}{2} n^2 t_1 + (|\lambda| - \frac{1}{24}) t_1\right)$$

$$= q_1^{-1/24} \cdot \prod_{n \geq 1} \left(1 + q_0 q_1^{n - \frac{1}{4}}\right) \cdot \left(1 + q_0^{-1} q_1^{-\frac{1}{2}}\right).$$

After using the Euler identity:

$$\sum_{\lambda \in \mathcal{P}} q_1^{n |\lambda|} = \frac{1}{\prod_{n=1}^{\infty} (1 - q_1^n)},$$

we recover the Jacobi triple product identity:

$$\sum_{n \in \mathbb{Z}} q_0^n q_1^{\frac{1}{4} n^2} = \prod_{n \geq 1} \left(1 - q_1^n\right) \cdot \left(1 + q_0 q_1^{n - \frac{1}{4}}\right) \cdot \left(1 + q_0^{-1} q_1^{-\frac{1}{2}}\right).$$

Next, we take $\alpha = 0$ in (71) to get:

$$\sum_{n \in \mathbb{Z}} \sum_{\lambda \in \mathcal{P}} \exp\left(\sum_{k=2}^{\infty} \frac{t_{k-1}}{k!} (n + \frac{1}{2} k^2) + \sum_{m=2}^{\infty} Q_m(a_\lambda) \sum_{l=0}^{\infty} \frac{t_{l+m-1}}{l!} (n + \frac{1}{2} l^2)\right)$$

$$= \exp\left(-\sum_{n=0}^{\infty} \frac{t_n}{n!} \zeta(-n)\right) \cdot (1 + \exp t_0)$$

$$\cdot \prod_{n \geq 1} \left(1 + \exp\left(\sum_{j=0}^{\infty} \frac{t_j}{j!} n^j\right)\right) \cdot \left(1 + \exp\left(-\sum_{j=0}^{\infty} \frac{t_j}{j!} (-n)^j\right)\right).$$

After taking $t_0 = \pi \sqrt{-1}$, we get:

$$\sum_{n \in \mathbb{Z}} (-1)^n \sum_{\lambda \in \mathcal{P}} \exp\left(\sum_{k=2}^{\infty} \frac{t_{k-1}}{k!} (n + \frac{1}{2} k^2) + \sum_{m=2}^{\infty} Q_m(a_\lambda) \sum_{l=0}^{\infty} \frac{t_{l+m-1}}{l!} (n + \frac{1}{2} l^2)\right)$$

$$= 0.$$
It can be further simplified if one uses Zagier’s operator \( \partial \) defined by:

\[
\partial = \sum_{k=1}^{\infty} Q_{k-1} \frac{\partial}{\partial Q_k}.
\]

Then the above recursion relation can be rewritten as

\[
\sum_{n \in \mathbb{Z}} (-1)^n \sum_{\lambda \in P} \exp \left[ \left( e^{(n+\frac{1}{2})\partial} \sum_{k=2}^{\infty} t_{k-1} Q_k \right)(a_{\lambda}) \right] = 0.
\]

This is just Zagier’s recursion relations [21, Theorem 1] in the form of a generating series. Therefore, (71) is a generalization of both Jacobi triple product identity and Zagier’s recursion relations for \( q \)-brackets of shifted symmetric polynomials.

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