Solvable Models of Supercooled Liquids at the Avoided Mode-Coupling-Theory Transition

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Mode-Coupling Theory (MCT) provides an accurate quantitative description of many supercooled liquids models in the early stages of dynamical slowing down. In realistic systems the description becomes incorrect approaching the MCT singularity that is eventually avoided and replaced by a crossover. The main open problem is to amend MCT and extend it quantitatively to the crossover region and beyond. We consider a family of models in three dimensions obtained by the application of the M-layer construction to the Fredrickson-Andersen Kinetically-Constrained-Model. We argue that deviations from mean-field MCT predictions in these models are described by the Stochastic-Beta-Relaxation (SBR) equations and compute independently the free parameters of the theory. We show that SBR predictions, obtained by numerical solution of the equations, yield an accurate description of the dynamics as obtained from Monte-Carlo simulations, thus providing the first instance of a model that can be described quantitatively at the glass crossover without any fitting parameter.

I. INTRODUCTION

The ubiquity of glass in nature and technology has driven research in this area for decades but there is still no agreement on the basic mechanism by which a supercooled liquid forms a glass [1,3]. An essential difficulty for assessing the validity of any of the competing theories is the fact that often they make only qualitative statements. For instance, the classic thermodynamics vs. dynamics controversy resolves around the putative divergences of susceptibilities and correlation lengths but, while the debate has produced many conceptual developments [1,2], there are no precise quantitative predictions to be matched with experiments and numerical simulations. In this respect Mode-Coupling-Theory (MCT) [3] is an exception and this is the main reason why its relevance is largely agreed upon.

MCT captures many qualitative features of the physics of liquids upon supercooling, notably two-step relaxation and stretched exponential decay. Furthermore it agrees quantitatively with numerical simulations although the agreement is not complete as one has to replace the values of some MCT parameters with values extracted from data fits [4-7]. The essential problem however is that it predicts a dynamical arrest transition, characterized by power-law divergences, that is not observed in experiments and simulations. In spite of this serious drawback many believe that the MCT transition is still relevant and, albeit avoided, determines a crossover from power-law to exponential increase of the relaxation time that is widely observed. Further support to the relevance of the spurious MCT transition comes from the fact that simple liquid models in the limit of infinite physical dimension d display a sharp transition qualitatively similar to the one of MCT, although MCT itself is quantitatively wrong in this limit [8]. In that case the sharp transition is considered a mean-field artifact due to the d → ∞ limit and it should become a crossover as soon as the dimension is finite. The fundamental problem of fixing somehow MCT and develop a theory of supercooled liquids valid down to the crossover temperature and below is thus open. It is generically believed that the spurious transition predicted by MCT is destroyed by fluctuations and the technical problem is how to take into account them.

There have been many unsatisfactory attempts to incorporate fluctuations at the microscopic level but recently progress has been made starting from the assumption that relevant fluctuations occur on large length-scales. As a consequence one can argue that the microscopic details of the original systems are not essential and that large-distance fluctuations are described by an effective coarse-grained theory depending on few (five) parameters. More precisely the assumption should be valid in the β time regime where the dynamical correlation stays close to a plateau. Once this assumption is made the choice of the effective theory is dictated by the symmetries of the problem in much the same way as one derives the Landau effective theory for ferromagnetism, in that case one starts with the Ising model and ends up with the φ⁴ field theory. The dynamical effective theory corresponding to MCT can indeed be identified [9] building on the connection between MCT and spin-glass with one step of Parisi Replica-Symmetry-Breaking (1RSB) discovered more that thirty years ago [10].

Studying effective theories beyond the mean-field approximation is typically very difficult but it turns out that the theory associated to MCT is equivalent to an
intuitive set of stochastic dynamical equations called Stochastic-Beta-Relaxation (SBR) that can be solved numerically. Conceptually the model is simple and intuitive, it is basically MCT supplemented with random spatial fluctuations of the temperature that are quenched on the scale of the $\beta$ regime. These results are promising because SBR seems to cure the drawbacks of MCT, displaying in particular the power-law to exponential crossover and dynamical heterogeneities without spoiling its successes, in particular two-step relaxation and stretched exponential decay.

In order to determine if an actual supercooled liquid can be described by SBR one should assess the validity of the assumption that fluctuations in the $\beta$ regime can be described by an effective theory. Generically one expects the assumption to be approximately valid if the dynamical correlation length is sufficiently large. It is widely reported that molecular dynamics data display a rather large dynamical correlation length [14–17] but putting the condition in a precise quantitative form is a complex open problem and will not be considered here. In the following instead we will study models where the validity of an effective theory description is granted by construction with the aim of i) demonstrating the claim that, in this case, the avoided transition is actually described by SBR and ii) present a practical application that can be useful to better understand a few subtleties of the SBR equations.

These models have a tunable parameter $M$: when the parameter is set to infinity the system is effectively a mean-field system with a sharp MCT transition, instead for any finite, albeit large, value of $M$ the model is effectively finite dimensional and the MCT transition is avoided. Standard arguments suggest that for large but finite values of $M$ an effective theory description, and then SBR, should become accurate. Indeed we will compute independently the five SBR parameters and solve numerically the dynamical stochastic equations to show that SBR provides an accurate qualitative and quantitative parameter-free dynamical description of these models as obtained from Monte-Carlo numerical simulations.

We note that this is the first instance of a three dimensional supercooled liquid model that is solvable (i.e. its dynamics can be described quantitatively) in the region where the MCT transition is avoided, i.e. beyond mean-field theory.

A classic strategy to design models that interpolate between mean-field and finite dimension, is the one of Kac [13] for the Ising model: the original model with nearest-neighbor interactions is generalized in such a way that each spin interacts weakly with all spins at a distance comparable with some length $\gamma$. In the limit $\gamma \to \infty$ the model is equivalent to the solvable fully-connected Curie-Weiss model that is characterized by a ferromagnetic transition with mean-field critical exponents but for any finite, no matter how large, value of $\gamma$ the critical exponents are non-mean-field. In this case deviations from mean-field at large but finite $\gamma$ are precisely described by an effective theory that is easily identified with the $\phi^4$ field theory. We mention that the Kac construction in the context of spin-glass models has been explored in [19–21], while alternative strategies have been studied in [22–24] for supercooled liquid models and in [25] for Kinetically-Constrained-Models (KCM). In particular the setting of Ref. [25] is closely related to ours, and it is likely that quantitative SBR predictions could be also made in that context. In the following we will consider instead the $M$-layer construction discussed in [26] and apply it to the Fredrickson-Andersen model, a classic lattice model with facilitated dynamics.

The paper is organized as follows: in the next section present the model and recall the definition of the SBR equations, we then compare Monte-Carlo data in three dimensions and numerical solutions of the SBR equations and finally we discuss a time-dependent Ginzburg criterion to explain why an effective theory naturally holds on the $M$-layer lattice. In section [11] we will further discuss and compare data corresponding to the zero-dimensional case that is relevant for finite-size effects in mean-field models. In section [IV] we will discuss the computation of the SBR parameters from the connection between the Fredrickson-Andersen model and bootstrap percolation. In section [V] we give our conclusions. Several technical results are presented in the appendix.
II. KINETICALLY CONSTRAINED MODELS ON THE M-LAYER LATTICE

A. The model and Stochastic-Beta-Relaxation equations

We have considered the Fredrickson-Andersen (FA) kinetically constrained model (KCM) on the three-dimensional (random) lattice obtained by the application of the $M$-layer construction of 20 to the diamond lattice (that has connectivity $c = 4$). The key property of such a random lattice is that it is locally tree-like at short distances but is finite dimensional at large distances. To obtain an instance of the random lattice we start from a finite-dimensional lattice of connectivity $c$, e.g. the square lattice in $d = 2$, we replicate it $M$ times and then we rewire through a random permutation the $M$ links corresponding to the same link on the original ($M = 1$) lattice, repeating the procedure for each link of the original lattice as shown in Fig. 1.

For large values of $M$ short loops in the lattice are rare and in the limit $M \to \infty$ the lattice becomes identical to a Bethe lattice with connectivity $c$. As a consequence for $M$ large but finite the Bethe approximation is accurate except close to the Bethe lattice critical point where deviations from mean-field behavior must occur. In this region, for sufficiently large but finite values of $M$, deviations from the mean-field Bethe solution are expected to be accurately described by an effective theory, as will be further discussed in subsection III C.

The FA model is defined by putting Ising spins on the sites of the lattice. The spins are independent, the Hamiltonian being $H = \sum_i s_i$, but obey a kinetically constrained dynamics: a spin can flip only if it has at least $m = 2$ of its $c$ nearest neighbors in the excited (up) state. An equilibrium configuration is thus easily generated and there is a simple relationship between the inverse temperature $\beta$ and the fraction $p = (1 + e^{-\beta})^{-1}$ of negative spins. The ($c = 4, m = 2$) model considered here is typically studied on the two-dimensional regular lattice but we choose the diamond lattice to work in three dimensions. Starting from an equilibrium initial condition the basic observable we consider is the persistence. More precisely we define the local persistence $\phi_i(t)$ as equal to one if the spin was negative at all times smaller than $t$ and zero otherwise, thus the averaged persistence is the number of negative sites that have never flipped at time $t$ divided by the total number of spins.

The FA model on the Bethe lattice is known to exhibit a dynamical arrest transition 20, 35. At the critical temperature (corresponding to a probability $p = p_c$) the persistence remains blocked to a plateau value $\phi_{plat}$ that is approached in a power-law fashion. The FA dynamical transition is intimately related to bootstrap percolation (BP) and both $p_c$ and $\phi_{plat}$ can be computed from its solution on the Bethe lattice as discussed in appendix C.

In particular the average persistence $\phi(t)$ obeys:

$$g(t) \equiv \phi(t) - \phi_{plat},$$

with

$$g(t) \approx \frac{1}{(t/t_0)}^p, \quad p = p_c, \quad t \gg 1,$$

$$p_c = \frac{8}{9}, \quad \phi_{plat} = \frac{21}{32}.$$

Unfortunately, at present, no analytic expressions of $t_0$ and $a$ are available but they can be estimated from numerical simulations on the Bethe lattice, see appendix B

$$a \approx 0.352, \quad t_0 \approx 2.30.$$

In the $M$-layer construction we have $M$ spins $s_i^a, a = 1, \ldots, M$ for each site $i = 1, \ldots, N$ of the original lattice ($M = 1$) and the total number of sites is $N_{tot} = M \times N$. The natural local order parameter is the average over the layers of the local persistence minus the plateau value

$$g_i^a(t) \equiv \phi_i^a(t) - \phi_{plat}$$

$$g(x, t) \equiv \frac{1}{M} \sum_{a=1}^M g_i^a(t),$$

where $x$ is the spatial coordinate of site $i$. For $M = \infty$ the system displays the Bethe lattice behavior while for large but finite $M$ we argue that deviations are described by an effective theory, more precisely, following the arguments and computations of 9, 11, we expect that the generic $K$-point average obeys for $1 \ll M < \infty$: $$(g(x_1, t_1) \ldots g(x_K, t_K)) \approx [\hat{g}(x_1, t_1) \ldots \hat{g}(x_K, t_K)].$$

In the LHS the angle brackets means average with respect to: i) different instances of the random lattice generated by the $M$-layer construction, ii) different initial equilibrium configurations and iii) different thermal histories.

In the RHS $\hat{g}(x, t)$ is the solution of the SBR equations:

$$\sigma + s(x) = -\alpha \nabla^2 \hat{g}(x, t) - \lambda \hat{g}^2(x, t) + \frac{d}{dt} \int_0^t \hat{g}(x, t-s) \hat{g}(x, s) ds.$$  

The separation parameter $\sigma$ measures the distance from the critical point and vanishes at $p = p_c$. The square brackets mean average with respect to the field $s(x)$ that is a time-independent random fluctuation of $\sigma$, Gaussian and delta-correlated in space:

$$[s(x)] = 0, \quad [s(x)s(y)] = \Delta \sigma^2 \delta(x-y).$$

The equations have to be solved with the small-time condition

$$\lim_{t \to 0} \hat{g}(x, t)(t/t_0)^a = 1.$$
where \( \lambda \) and \( a \) are related by the MCT relationship
\[
\lambda = \frac{\Gamma^2(1-a)}{\Gamma(1-2a)}.
\] (11)

Note that on the \( M \)-layer lattice \( g(x,t) \) is non-zero only on the points of the lattice while the SBR field \( \hat{g}(x,t) \) is defined on the continuum therefore equation (7) is intended to hold only on lattice points. The fact that a lattice theory becomes equivalent to a continuum theory is due to the fact that, as we will see in the following, eq. (7) holds on length scales considerably larger than the lattice spacing and the actual microscopic structure of the lattice become irrelevant.

Within SBR\(_L\) mean-field theory is recovered setting \( \Delta \sigma^2 = 0 \), in this case \( \hat{g}(x,t) \) is constant in space, the gradient term plays no role and one recovers the critical MCT equation, in particular \( p = p_c \) corresponds to \( \sigma = 0 \) and \( \phi(t) \) will approach the plateau value. The \( M \)-layer construction allows to have a finite but small \( \Delta \sigma \) so that the MCT transition is avoided and \( \phi(t) \) crosses the plateau at a finite time for all values of \( \sigma \), including \( \sigma = 0 \).

Equation (7) embodies the power of the effective theory approach: on the LHS we have a model with a complex microscopic dynamics for which no analytic treatment of dynamics is available (not even on the Bethe lattice), on the RHS we have a (numerically) solvable set of equations depending on few parameters \( a, t_0, \alpha, \Delta \sigma \) and \( \sigma \) that were derived in [9, 11] starting solely from symmetry considerations and without reference to any microscopic model.

The microscopic details determines the actual values of the SBR parameters and can be computed independently of the numerical data on the \( M \)-layer. We have already seen that \( \lambda = 1/8 \) and \( D_{NBW} = 1 \) as can be obtained from a general formula derived in appendix D. Although we have chosen the three-dimensional diamond lattice the previous relationships are valid for a generic lattice of connectivity four and can be used to repeat the numerical analysis on many other lattices, including notably the regular two-dimensional lattice where \( \rho = 1 \) and \( D_{NBW} = 1/2 \). Formulas for lattices with generic connectivity and topology can be obtained from appendix D and E.

In appendix A we will show that the solution of SBR corresponding to generic values of \( \alpha, \Delta \sigma \) and \( t_0 \) can be expressed in terms of the solution with \( \alpha = \Delta \sigma = t_0 = 1 \) through appropriate rescalings of the correlators \( \phi = b \phi, g \), of distances \( x \to b_x x \) and of times \( t \to b_t t \).

From the corresponding formulas we get that the \( M \)-dependence of the rescaling factors is the following:
\[
\Delta \sigma^2 = O(M^{-1}) \to \begin{cases} 
\phi & = O(M^{-\frac{1}{\rho-1}}) \\
b_x & = O(M^{\frac{1}{\rho-1}}) \\
b_t & = O(M^{\frac{\rho}{\rho-1}}) 
\end{cases}
\]
(15)
for generic dimension \( d \) smaller than the critical dimension \( d_c = 8 \). The above relationships allows to understand the phenomenology we expect to observe on the \( M \)-layer lattice at the avoided MCT transition. They tell us that as \( p = p_c \) deviations from the Bethe lattice behavior occur at times that increase with \( M \) as \( O(M^{\frac{\rho}{\rho-1}}) \). Thus when the system starts to deviate it should be well described by the large-time limit of the Bethe solution corresponding to eq. (2) and this must match smoothly the small-time behavior on the time-scale of deviations thus clarifying the origin of the initial condition eq. (10) of the SBR equations. Furthermore at times \( O(M^{\frac{\rho}{\rho-1}}) \) the order parameter, i.e. the deviations of the persistence from the plateau value, is small \( O(M^{\frac{\rho}{\rho-1}}) \) and fluctuates over a large length-scale \( O(M^{\frac{\rho}{\rho-1}}) \), thus justifying \textit{a posteriori} the validity of an effective theory description and the expectation that SBR predictions becomes increasingly accurate for larger values of \( M \). In subsection II C we will give an \textit{a priori} justification, starting from the mean-field expression of fluctuations in the \( M \)-layer and applying a time-dependent Ginzburg criterion that will also lead to the above scaling eq. (15).

In the following we will discuss numerical simulations at \( p = p_c \) corresponding to \( \sigma = 0 \) but we want to stress that SBR holds also for \( p \neq p_c \). More precisely all the features of the crossover predicted by SBR as the separation parameter goes from \(-\infty\) to \( +\infty \) should be observed in a region close to \( p_c \) that shrinks with increasing \( M \) as
\[
|p - p_c| = O(M^{\frac{1}{\rho-1}}). 
\]
(16)

These include notably the non-monotonous behavior of the dynamical correlation length and the qualitative change in the nature of fluctuations with the appearance of strong dynamical heterogeneities [13].
B. Numerical Results and Theoretical Predictions

We now turn to the comparison of numerical data and SBR predictions. All simulations were performed at $p = p_c$ on random lattices obtained by the application of the $M$-layer construction to the diamond cubic lattice with periodic boundary conditions. The diamond lattice is generated by repeating in the three directions a basic unit cell of length $L = 4$. In each unit cell there are eight lattice points that can be divided into two groups: blue lattice points have coordinates $(0, 0, 0), (2, 0, 2), (0, 2, 2)$ and $(2, 2, 0)$, red lattice points have coordinates $(3, 3, 3), (3, 1, 1)$, $(1, 3, 1)$ and $(1, 1, 3)$. Each red (blue) lattice point is connected with its four blue (red) nearest neighbors, see fig. (1). All sites in the same group are updated in parallel with a Metropolis rule, see appendix B for further details. In fig. (2) we plot the time decay of the persistence for a system with $M = 3000$ and for one with $M = 200000$. At initial times the data follows the mean-field (MF) curve corresponding to $M = \infty$ (obtained from numerical simulations on the Bethe lattice (appendix B) and deviates from it at large times reaching the plateau value at a finite time. As expected, deviations from MF occur at times increasing with $M$. The dashed line are the prediction of SBR solved numerically with the above values of the five parameters.

We start observing that SBR predictions are expected to be accurate only in the non-trivial region where the data start to deviate from the MF curve. In particular it is obviously not accurate at very small times when the data follow the MF curve but the MF curve itself is still in the microscopic non universal regime and does not yet display the large-time asymptotic limit $(t/t_0)^\alpha$ that matches the small-time tail of SBR. Another way to understand this is to note that according to eqs. $\chi_4(t)$ describes the $\beta$ regime, i.e. the region of times where the data are slightly above or below the plateau value, more precisely at a distance $O(M^{-\frac{1}{2-\alpha}})$. Thus it is not expected to be accurate both at small times (the microscopic regime) and at large times where the persistence finally decays to zero (the $\alpha$ regime). For convenience we considered and compare the data only until they cross the plateau value. From the figure we see that for $M = 3000$ SBR describes the bulk of the deviations and the error is due to the fact that the system departs from the MF curve at times where the small time $1/t^{2-\alpha}$ corrections to the asymptotic behavior $1/t^\alpha$ are still significant (see discussion in appendix B). The effect is expected to disappear increasing $M$ and indeed the agreement between numerical simulations and theory is very good for $M = 200000$, we recall that there is no fitting parameter in the SBR predictions.

We want to stress that, according to eq. (7), SBR is a complete theory and provides a description not only of the average dynamical order parameter but also of all its fluctuations. To demonstrate this we consider the persistence density

$$g(t) \equiv \frac{1}{ML^3} \rho \sum_{i,\alpha} g^\alpha_i(t)$$

whose fluctuations define the $\chi_4(t)$ function:

$$\chi_4(t) \equiv ML^3 \rho \left( \langle g^2(t) \rangle - \langle g(t) \rangle^2 \right).$$

In the MF regime corresponding to $1 \ll t \ll O(M^{-\frac{1}{4(1-\alpha)}})$ the total susceptibility is the same that one measures on the Bethe lattice (see section IV.A) and di-
In order to compare predictions for dynamical fluctuations in space (i.e. dynamical heterogeneities) we define

$$\Gamma(x - y, t) = \langle g(x, t) g(y, t) \rangle - \langle g(x, t) \rangle \langle g(y, t) \rangle \ .$$  

In fig. (4) we plot $\Gamma(0, t)$ (i.e. the local order parameter squared) and $\Gamma_{av}$ defined as:

$$\Gamma_{av}(t) \equiv \frac{1}{V} \int \Gamma(x, t) \, dx \ .$$  

Note that $\Gamma_{av}(t)$ is equal to $\chi_4(t)/(ML^3 \rho)$. We see that the SBR predictions for both quantities are in good agreement with the numerical data in the regime where deviations from MF are observed.

Within SBR $\Gamma(x, t)$ takes is maximum value at $x = 0$ and decreases to zero at $x = \infty$, therefore $\Gamma(0, t)$ is always strictly larger than $\Gamma_{av}$. In the thermodynamic limit $\Gamma_{av}$ actually should be proportional to the inverse of the volume, while $\Gamma(0, t)$ remains finite. Note instead that in fig. (4) the two quantities are comparable at large times (small $g(t)$). This happens because the correlation length is comparable with the system size which is $L = 8$ for the data with $M = 3000$ shown in the previous figures. To be consistent we had to solve the SBR equations in a cubic box of the same size $L = 8$. This demonstrates that SBR is also able to capture quantitatively finite-size effects when they are expected, indeed the key assumption leading to SBR is that the correlation length is larger than the microscopic length-scale (in this case the lattice spacing) and this can be true even if it is comparable with the system size.

Note that, as we will see in the next subsection, in the MF regime valid at initial times the correlation length grows with time as $t^{d/2}$ while the time where SBR replaces MF theory grows with $M$ as $O(M^{1/(8-d)})$. Thus if the correlation length becomes comparable with the system size before deviations from MF occur, $\Gamma(x, t)$ is a constant for all $x$ also in the SBR regime. In this regime the solution of the SBR equations do not have significant spatial variation and can be obtained considering the equations without the gradient term. These are effectively zero-dimensional equations in the sense that we have to set $d = 0$ in the scaling formulas [15].

This situation will always occur if we increase $M$ keeping the size $L$ of the original $(M = 1)$ lattice fixed. This can be also understood noticing that according to eqs. [15] if we go from $M$ to a larger value $M'$ we should also increase the system size $L$ of the original lattice by a factor $(M'/M)^{1/(8-d)}$, if we do not do that we are systematically reducing the effective system size until we reach a length-scale over which spatial fluctuations are negligible due to the gradient term. The data at $M = 200000$ in fig. 2 correspond to $L = 4$ and are indeed in this regime, correspondingly the SBR prediction displayed in the figure was computed with the zero-dimensional equations. In section 11 we will discuss numerical data at $L = 4$ for values of $M$ corresponding to the zero-dimensional regime. The data for $M = 3000$ and $L = 8$ are instead strictly three-dimensional because, although we are not in the thermodynamic limit, we had to consider the full three-dimensional SBR equation that depends on the value of $D_{NBW}$ on the lattice and on the system size.

The reader may naturally ask what is the value of $M$ above which the SBR description is accurate for a system of size $L$. The practical answer is to solve the SBR equations for given values of $M$ and $L$ and compare the solution with the MF curve. As seen in fig. (3) for all values of $M$ and $L$ the SBR solution follows the asym-
totic curve $(t/t_0)^{-a}$ at small times and deviates from it at larger times. On the other hand the MF curve is described by $(t/t_0)^{-a}$ at large times but not at initial times for obvious reasons. If for the given value of $M$ and $L$ the SBR solution deviates too early, that is at times when the MF curve is still significantly different from its long-time limit $(t/t_0)^{-a}$, then the description is not expected to be quantitatively accurate.

C. Time-dependent Ginzburg Criterion

In this subsection we will explain why it is natural to expect that on the $M$-layer lattice deviations from mean-field behavior (i.e. the fact that the MCT transition occurring on the Bethe lattice is avoided) are described by an effective theory. We will not discuss instead the arguments leading to the choice of the effective theory and the computation leading from the effective theory to SBR for which we refer to [9][11].

As we said already, in the limit $M \to \infty$, the lattice is locally tree-like, the original three-dimensional structure becomes irrelevant and every observable takes the Bethe lattice value: in particular at $p = p_c$, dynamics displays the spurious MCT mean-field dynamical arrest transition, $t(t)$ never becomes negative and approaches zero with a $1/t^a$ power law. For $M$ large but finite the system recovers its finite-dimensional character: it deviates from mean-field theory and the dynamical arrest transition is avoided. Let us discuss the nature of the deviations from MF theory. One can see that on the $M$-layer the two-point correlation of the order parameter at any distance are small for $M$ sufficiently large at any finite time. More precisely in section [IV A] we will show that (neglecting constants that are irrelevant for the present discussion) the leading $O(1/M)$ expression of fluctuations in Fourier space at large times obey:

$$\langle g(k,t)g(k',t) \rangle - \langle g(k,t) \rangle \langle g(k',t) \rangle =$$

$$= \frac{t^{2a}}{M} \delta(k + k') \left( \frac{1}{1 + (k \xi)^2} \right)^2, \; \xi \propto t^{a/2}, \; t \gg 1 \quad (24)$$

Thus at the MF level the total susceptibility diverges with time as $t^{2a}$ and the correlation length diverges as $t^{a/2}$. Both these behaviors will change when the system starts to deviate from MF: the correlation length will not increase indefinitely and $g(x,t)$ will become negative at a finite time. In this regime corrections of all orders in $1/M$ become equally relevant and one must abandon MF theory, technically using SBR in place of the MF expressions eq. [2] and eq. [24] amounts to include corrections at all orders in powers of $1/M$. We note en passant that MF fluctuations do not have the Ornstein-Zernicke form but rather the square of it. Thus it should be noted that, while the use of the OZ form to fit numerical data of dynamical fluctuations is widespread in the literature [13][17], it has no theoretical justification. Both MF theory and SBR certainly do not support OZ in the β regime, while the α regime predictions are at present not available, not even in MF theory. In real space the above expression leads to:

$$\langle g(x,t)g(y,t) \rangle - \langle g(x,t) \rangle \langle g(y,t) \rangle = \frac{t^{2a(2-d/2)}}{M} f \left( \frac{x - y}{\xi} \right). \quad (25)$$

Fluctuations as $O(1/M)$ but since the prefactor diverges at large times we can apply a Ginzburg criterion and identify the time $t_G$ where deviations from MF theory must occur with the time where MF fluctuations of $g(x,t)$ on the scale of the correlation length become comparable with its MF average squared, this leads to:

$$\frac{t^{2a(2-d/2)}}{M} \approx \frac{1}{t^{2a}} \to t_G \approx M^{-\frac{1}{4\alpha(2-\delta)\xi}}. \quad (26)$$

Thus the system starts to deviate from MF at large times $O(M^{-1/4\alpha(2-\delta)\xi})$, so that according to eq. [2] $g(x,t)$ has already decreased considerably and is very small ($O(M^{-1/4\alpha(2-\delta)\xi})$) while according to eq. [24] the correlation length has increased considerably and is rather large ($O(M^{-1/4\alpha(2-\delta)\xi})$). The key observation to be made now is that small order parameter and large (but not necessarily diverging) correlation length define exactly the regime in which an effective theory is expected to be appropriate. Let us recall why: in general the probability distribution of the order parameter $g(x,t)$ has a complicated functional expression $P[g] \propto e^{A[g]}$, but if $g(x,t)$ is small we can safely retain only the lowest order terms in the Taylor expansion of $A[g]$ in powers of $g$. Furthermore, if the correlation length is large only the lowest-order space derivatives of the order parameter matter. Thus one concludes that $A[g]$ can be approximated by an effective theory i.e. a polynomial in $g$ plus a term involving the gradient squared. Typically the precise form of the effective theory can be guessed from symmetry considerations while the microscopic details of the original models determines only the actual value of the coupling constants.

In [9][11] it is argued, starting from an integral representation of dynamics, that the effective theory of MCT is the so-called Glassy-Critical-Theory (GCT) whose form is suggested by the replica-dynamics analogy in the context of the Random-First-Order-Theory [2]. However, identifying the correct effective theory does not at all solve the problem: we claim that deviations are described by an effective theory but we still have to compute them. For example, one thing is to say that deviations from MF in a ferromagnetic model can be described by the $\phi^4$ theory, another thing is to extract the critical exponents from it. In the case of the GCT the problem is greatly simplified by the fact that the GCT can be shown rigorously to be equivalent to the SBR equations [9]. The great technical advantage is that the unfeasible computation of the integral of the action (which is in most cases can only be computed perturbatively through Feynman
diagrams) is replaced by the average over instances of the random fields $s(x)$ that can be done numerically.

We should note that the use of the GCT and SBR to describe the FA models on $M$-layer lattices cannot at present be completely justified because we lack an analytic treatment of the dynamics starting from the microscopic. It is not even possible to show analytically that the FA model has a MCT transition on the Bethe lattice. On the other hand there are strong numerical evidences that is a Gaussian random fluctuation of the separation parameter:

$$\langle g^K(t) \rangle \approx \langle \hat{g}^K(t) \rangle,$$  

where in the RHS $g(t)$ is the solution of the zero-dimensional SBR equations \[9, 11, 12\]:

$$\sigma + s = -\lambda \hat{g}^2(t) + \frac{d}{dt} \int_0^t \hat{g}(t-s) \hat{g}(s) ds.$$  

The square brackets mean average with respect to the field $s$ that is a time-independent Gaussian random fluctuation of the separation parameter:

$$[s] = 0, \quad [s^2] = \Delta \sigma^2.$$  

The equations have to be solved with the small-time condition

$$\lim_{t \to 0} \hat{g}(t) (t/t_0)^a = 1.$$  

In the following we will consider simulations for $L = 4$ and values of $M$ that are large enough to grant, according to the discussion of the previous section, the SBR equations to be used are the zero dimensional ones. This setting is particularly important in applications also outside the structural glass problem because is the one relevant to describe finite-size effects in mean-field models on fully connected or sparse random graphs with one-step of Parisi Replica-Symmetry-Breaking at the so-called dynamical temperature $T_d$ \[38\]. Considering the shifted total persistence

$$g(t) = \frac{1}{ML^3 \rho} \sum_{i,\alpha} g_{i,\alpha}(t)$$  

we have for a generic cumulant of order $K$:

$$\langle g^K(t) \rangle \approx \langle \hat{g}^K(t) \rangle,$$  

where in the RHS $g(t)$ is the solution of of the zero-dimensional SBR equations \[9, 11, 12\]:

$$\sigma + s = -\lambda \hat{g}^2(t) + \frac{d}{dt} \int_0^t \hat{g}(t-s) \hat{g}(s) ds.$$  

The dotted lines in the fig. (6) represents respectively $N_{\text{tot}} = M L^3 \rho$ of sites of the lattice that controls $\Delta \sigma^2$ through:

$$\Delta \sigma^2 = \frac{1}{N_{\text{tot}}} 2187 \frac{1}{8192} (1-\lambda)^2,$$  

that follows from expression (12). The above expression does not depend on how the lattice is actually generated, as we said before it describes generically finite-size corrections in mean-field models, and in particular it is also correct for a random-regular-graph of size $N_{\text{tot}}$ \[33\]. In the present setting $N_{\text{tot}} = M L^3 \rho = 8 M$.

In fig. (6) we show the relaxation for $L = 4$ and $M = 5 \times 10^4$, $M = 10^5$ and $M = 2 \times 10^5$ averaged over respectively 10840, 3840 and 3840 samples. As expected the data follows the mean-field curve (corresponding to $M = \infty$) at small times and deviate from it at times that increase with $M$. The dotted lines represent the predictions of SBR corresponding to the different values of $M$. In fig. (6) we show the same data rescaled horizontally and vertically with the appropriate powers of $M$ that lead to collapse on a unique SBR curve. They are obtained setting $d = 0$ in expressions (15) and this is the technical reason why this is called the zero-dimensional regime. The rescaling factors are $M^{1/4}$ for the shifted persistence and $M^{-1/(4a)}$ for time.

The dotted lines in the fig. (6) represents respectively the mean-field asymptotic expression $(t/t_0)^{-a}$ (top) and the SBR prediction. We see that the curves tend indeed to collapse onto the SBR prediction. Note that the agreement is not yet perfect and this can be tracked to the fact that at the values of $M$ considered the curves start to deviate from the mean-field curve at times where the $M = \infty$ mean-field curve itself has still relevant small-
In the MF regime corresponding to $1 \ll t \ll O(M^{1/2})$ the total susceptibility is the same that one measures on the Bethe lattice (see section IV.A) and diverges at large times as:

$$\chi_4(t) \approx \frac{2187}{8192} (1 - \lambda)^2 A_2^2 (t/t_0)^{2a}$$

while deviations are observed for times of order $M^{1/4}$. A consequence in fig. (6) the data for small rescaled times $t/t_0$ of order 5 – 20 display an approximate 20% deviation from the asymptotic expression $(t/t_0)^{-a}$ that should describe the data for small rescaled times at large values of $M$ but agreement improves considering larger values of $M$. On the other hand small-time corrections gets smaller at larger rescaled times $\approx 50$ and consequently the data agree much better with the SBR curve. We recall that in order to fully appreciate the agreement between theory and data it is important to remember that the SBR curves in fig. (5) and (6) were computed independently of the data. We note that the rescaling factor $M^{1/4}$ that leads to the collapse of the curves could not be obtained from the data shown in fig. (2) because the data for $M = 3000$ and $L = 8$ are genuinely three-dimensional while the data for $M = 200000$ and $L = 4$ are effectively zero-dimensional.

We now turn to the analysis of fluctuations defined as before:

$$\chi_4(t) = \chi_{tot}(g^2(t) - \langle g(t) \rangle^2).$$

In fig. (7) we plot parametrically the susceptibility vs. rescaled persistence for $L = 4$ and $M = 5 \times 10^4$, $M = 10^5$ and $M = 2 \times 10^5$ (Dots,bottom to top). The data collapse on the SBR prediction (solid). The SBR prediction for $\chi_4(t)$ converges at small times to the MF asymptote (dashed).

FIG. 6: Rescaled shifted persistence for $L = 4$ and $M = 5 \times 10^4$, $M = 10^5$ and $M = 2 \times 10^5$. The dotted lines are the mean-field asymptotic expression $(t/t_0)^{-a}$ (top) and the SBR prediction (bottom). The data for different $M$ display a good collapse on the SBR curve at large rescaled times.

FIG. 7: Parametric plot of the fluctuations vs. shifted persistence for $L = 4$ and $M = 5 \times 10^4$, $M = 10^5$ and $M = 2 \times 10^5$ (bottom to top). The data approaches the MF asymptote (dashed) at large times and deviates from it at times that increase with $M$, corresponding to smaller values of the persistence.

FIG. 8: Parametric plot of the rescaled fluctuations vs. rescaled persistence for $L = 4$ and $M = 5 \times 10^4$, $M = 10^5$ and $M = 2 \times 10^5$.
We can then write
\[ \langle (g(x,t)g(y,t)) - \langle g(x,t) \rangle \langle g(y,t) \rangle = \frac{1}{M} \sum_{L=0}^{\infty} N_L(x,y)c_L^{\text{Bethe}}(t) \]
(35)

where \( c_L^{\text{Bethe}}(t) \) is the correlation of the shifted persistence between two sites \( i \) and \( j \) at distance \( L \) on the Bethe lattice:
\[ c_L^{\text{Bethe}}(t) \equiv \langle g_i(t)g_j(t) \rangle - \langle g_i(t) \rangle \langle g_j(t) \rangle \]
(36)

and \( N_L(x,y) \) is the number of non-backtracking paths between point \( x \) and point \( y \) on the original lattice (corresponding to \( M = 1 \)).

As we have discussed before, for \( M \) finite but large, in the regime where SBR provides a quantitatively accurate description, the length-scale of the fluctuations of \( g(x,t) \) is large and thus we are interested in the regime where \( x-y \) (and thus \( L \)) in \( N_L(x,y) \) is also large. In this regime \( N_L(x,y) \) tends to a Gaussian with a \( O(L) \) variance:
\[ N_L(x,y) \approx \frac{c(c-1)^L}{\rho} G(x-y). \]
(37)
The above relationship is valid only if \( x \) and \( y \) correspond to coordinates of points of the lattice and is zero otherwise. This explains the prefactor, indeed since as we will see the variance of the Gaussian is much large than the lattice spacing we can replace the sum over lattice points as an integral on the continuum \( \sum_{j} \rightarrow \int \rho d^2x \). The integral on the other hand must be equal to the total number of paths of length \( L \) originating from a point, i.e. \( c(c-1)^L \). In terms of the unitary Fourier transform we can then write
\[ N_L(k,k') \approx \delta(k+k') \frac{c}{\rho(c-1)} (c-1)^L \exp[-L \sigma_{NBW} k^2] \]
(38)

Where \( \sigma_{NBW} \) is by definition the diffusion coefficient of non-backtracking random walks \( \{x_1, \ldots, x_t\} \) on the original lattice with \( M = 1 \):
\[ \sigma_{NBW} = \lim_{t \rightarrow \infty} \frac{\langle ||x_t||^2 \rangle}{2 dt}. \]
(39)

In appendix C we will provide a simple expression for \( \sigma_{NBW} \) in terms of the dimension of the lattice and of its connectivity validity for a huge class of lattices.

If we compute quantities at the level of the MF approximation we will get the spurious dynamical arrest transition. In particular, the FA model on the Bethe lattice below the critical temperature (corresponding to \( p < p_c = 8/9 \)) displays a glassy phase. In the glassy phase it is convenient to study the persistence and its fluctuations in the infinite-time limit. Furthermore in this limit the critical properties of the blocked (negative) sites are exactly the same of the \( k \)-core of bootstrap percolation. In particular we have
\[ \lim_{t \rightarrow \infty} \langle (g(x,t)g(y,t)) - \langle g(x,t) \rangle \langle g(y,t) \rangle \rangle = \frac{27}{16\sqrt{2}} \delta p^{1/2} \]
(40)

and
\[ \lim_{t \rightarrow \infty} c_L^{\text{Bethe}}(t) = c_L^{\text{Bethe}}(p) \approx c_1 L \mu_L(p) \]
(41)

where \( \mu(p) \) is the \( k \)-core density and \( \delta p \) is the correlation coefficient. The above formula for \( c_L^{\text{Bethe}}(p) \) (first appeared in [41]) is derived in the supplemental material of Ref. [36] (see also appendix C) where it is also shown that \( \mu(p) \) tends to the critical value \((c-1)^{-1}\) at large times with a correction of order \( \sqrt{\delta p} \)
\[ \mu(p) \approx \frac{1}{c-1} (1 - c_2 \delta p^{1/2}) \]
(42)

and the numerical constants \( c_1 \) and \( c_2 \) read for \( c = 4 \) and \( m = 2 \) [39]:
\[ c_1 = \frac{81}{512}, \quad c_2 = \frac{3}{\sqrt{2}}. \]
(43)

In appendix C we report the general formulas from which \( c_1 \) and \( c_2 \) were obtained. Putting eqs. (35), (38) and (41) together and performing the summation over \( L \) we can write for the unitary Fourier transform of the fluctuations:
\[ \lim_{t \rightarrow \infty} \langle (g(k,t)g(k',t)) - \langle g(k,t) \rangle \langle g(k',t) \rangle \rangle = \]
\[ = \frac{1}{M} \delta(k+k') \frac{c_1 c}{\rho(c-1)} \left( \frac{1}{c_2 \delta p^{1/2} + \sigma_{NBW} k^2} \right)^2. \]
(44)

We stress once again the difference from a simple Ornstein-Zernicke form due to the square that appears because of the \( O(L) \) prefactor in (41).

The above expression has to be compared with the long-time limit of the mean-field approximation to SBR in the glassy phase \( \sigma > 0 \). The MF approximation to SBR corresponds to assume \( \Delta \sigma^2 \) is negligible and leads to:
\[ \lim_{t \rightarrow \infty} [g(x,t)] = \left( \frac{\sigma}{1-\lambda} \right)^{1/2}. \]
(45)
The MF approximation to fluctuations can be computed treating the fields \( h(x) \) as small compared to \( \sigma \) and is given by:

\[
\lim_{t \to \infty} \left( \langle \hat{g}(k, t) \hat{g}(k', t) \rangle - \langle \hat{g}(k, t) \rangle \langle \hat{g}(k', t) \rangle \right) =
\]

\[
= \Delta \sigma^2 \delta(k + k') \left( \frac{1}{2\sqrt{(1-\lambda)\sigma + \alpha k^2}} \right)^2
\]

(46)

By equating expressions (45) and (40) we obtain, as quoted before,

\[
\sigma = (1 - \lambda) \frac{729}{512} (p - p_c)
\]

(47)

By equating expressions (46) and (44) we obtain:

\[
\Delta \sigma^2 = \frac{1}{M \rho} \frac{2187}{8192} (1 - \lambda)^2
\]

(48)

\[
\alpha = (1 - \lambda) \frac{9}{8} D_{NBW}
\]

(49)

Note that the MF approximations we have discussed are utterly wrong because neither the FA model on the M-layer nor the SBR equation actually display a dynamic arrest transition and there is no static limit. Nevertheless we are making in both cases the same (wrong) approximation and it is thus correct to compare the outcome.

### V. CONCLUSIONS

We have considered a supercooled liquid model in finite dimension where deviations from mean-field MCT predictions are expected to be described by an effective theory and we have demonstrated that the corresponding theory is SBR by comparing Monte-Carlo data and the numerical solution of the SBR equations. All the parameters of SBR could be independently computed from the Bethe lattice version of the model, in part analytically (through the connection with bootstrap percolation) and in part numerically. We thus provide the first instance of a model that can be described both qualitatively and quantitatively in the region where the MCT transition is avoided.

As we mentioned in the introduction, even in the region where MCT offers a good description of molecular dynamics data one needs to adjust by hand some of its parameters (i.e. the critical temperature \( T_c \) while there is no fitting parameter at all in the theoretical curves used here. It is tempting to attribute the aforementioned discrepancies in MCT fits to non-mean field effects that could possibly be described by SBR but a careful analysis is left for future work. This is a particularly pressing issue given that there is widespread belief that numerical simulations of a variety of models are already or will shortly attain the MCT critical temperature. These claims should be taken with some caution given that estimates of the critical temperature are obtained from fits that are by definition bound to fail close to it, be as it may they further underline the need for a quantitative description of crossover effects.
Most quantitative theoretical first-principles approaches to supercooled liquids are plagued by the presence of the spurious MCT transition, including notably simple liquids in the infinite-dimensional limit. We stress that none of the techniques used in those approaches could have led to a description of the numerical simulations presented here. This is because one has to overcome two essential difficulties. The first is going beyond mean-field theory given that these are genuinely three-dimensional models and display a dynamic arrest transition if treated at the MF level. The second is that one must work in a purely dynamical framework, because the glassy phase do not exist beyond MF and as a consequence many useful tools (e.g. the replica method) cannot be used.

The results presented here illustrate and consolidate the claim of that if one assumes that deviations from MCT in a given system are described by an effective theory, then that theory is SBR. The chain of arguments leading to the claim is made of an additional assumption and an exact computation. The first assumption is that deviations from MCT are described by an effective theory, the second assumption (motivated by symmetry arguments) is that the effective theory is the so-called Glassy-Critical-Theory (GCT), and finally the rigorous computation shows that the GCT is equivalent to SBR.

As we have discussed before, the M-layer construction, much as the Kac construction, is specifically designed to satisfy the first assumption by having native small deviations and large correlation lengths, but these assumptions could as well be satisfied in actual supercooled liquids. Probably these assumptions hold at smaller values of $M$ as well but to have an accurate quantitative description one should choose different values for the five SBR parameters. In practice SBR integrates long-wavelength fluctuations but its parameters depend on short wavelength fluctuations that can be integrated by a mean-field theory. Here we computed them using the Bethe approximation that completely neglects the presence of loops in the lattice and is certainly too crude at intermediate values of $M$ where some other approximation should be used. Thus the reason why we had to consider large values of $M$ to demonstrate the agreement between theory and numerical experiments is because we wanted the Bethe approximation to be accurate at short distances and short times, while the essential condition that the correlation length is larger than the lattice spacing likely holds already at smaller values of $M$.

The analysis presented here can be repeated for FA models on lattices in other dimensions but it can be also applied to any model that exhibits a MCT-like transition on the Bethe lattice, including notably 1RSB spin-glasses. First we should identify a lattice in finite dimension that has the same connectivity and then we could apply the $M$-layer construction to it. The case of spin-glass models is particularly interesting because in that case the replica (and cavity) method allows to compute in a static context also the parameter exponent $\lambda$ and thus all the four mesoscopic parameters $\Delta \sigma$, $\alpha$, $\sigma$ and $\lambda$ of SBR could be computed analytically.

As we mentioned in the introduction SBR has been obtained building on the connection between MCT and spin-glass with one step of Parisi Replica-Symmetry-Breaking discovered more that thirty years ago. The most famous outcome of this analogy is the Random-First-Order-Theory and the case of 1RSB Spin-Glass models on the $M$-layer lattice allows to clarify the connection between SBR and RFOT. The finite dimensional version of these models (as realized for instance by means of the $M$-layer construction) are the classic example of a system for which RFOT should hold.

Within RFOT the Kauzmann transition is always preceded at the mean-field level by an MCT transition where the phase space breaks into many components with a finite configurational entropy. RFOT describes the mechanism by which ergodicity is restored at temperatures between the avoided MCT singularity and the genuine Kauzmann singularity. The theory shares strong analogies with the decay of a metastable phase by nucleation, in particular the divergence of the radius of the critical droplet at the coexistence point is analogous to the divergence of the point-to-set correlation length at the Kauzmann temperature.

A feature that is often overlooked is that metastability at the spinodal of the metastable phase, that in the analogy picture corresponds to the MCT critical point, is qualitatively different from what happens at the coexistence point. Analogously in the $M$-layer framework SBR replaces mean-field theory in the crossover region that corresponds to values of the temperature in a neighborhood of $T_{MCT}$ of size $O(M^{-4/(8-d)})$. Decreasing the temperature outside the critical region the standard RFOT nucleation mechanism should sets in for 1RSB spin-glasses and this should match the behavior of the SBR equations at large values of $\sigma$. Thus SBR does not exclude the occurrence of a genuine thermodynamic transition at lower temperatures but it may also describe models, like the one considered here, that have no thermodynamics transition.

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Appendix A: SBR in general form

The equations of SBR are:

\[ \sigma + s(x) = -\alpha \nabla^2 \hat{g}(x,t) - \lambda \hat{g}^2(x,t) + \frac{d}{dt} \int_0^t \hat{g}(x,t-s)\hat{g}(x,s)ds \]  

(A1)

where the field \( s(x) \) is a time-independent random fluctuation of the separation parameter, Gaussian and delta-correlated in space:

\[ [s(x)] = 0, \quad [s(x)s(y)] = \Delta \sigma^2 \delta(x-y) \]  

(A2)

and they have to be solved with the condition

\[ \lim_{t \to 0} \hat{g}(x,t)(t/t_0)^a = 1. \]  

(A3)

Thus the equations depend on five coupling constants: \( \lambda \), \( t_0 \), \( \alpha \), \( \sigma \) and \( \Delta \sigma \). However one can fix \( c_0 = \alpha = \Delta \sigma = 1 \) and then the general solution can be obtained by rescalings. This means that in practice the SBR equations at fixed \( \lambda \) need only to be solved for varying values of \( \sigma \). More precisely one can easily verify that for a generic \( K \)-point function we have

\[ \hat{g}(x_1,t_1)\ldots\hat{g}(x_K,t_K)\big|_{\alpha,\Delta \sigma,t_0,\sigma,\lambda} = \]

\[ \left[ b^K_\phi \hat{g}(x_1/bx,t_1/bt)\ldots\hat{g}(x_K/bx,t_K/bt)\right]_{1,1,1,\sigma/b_\sigma,\lambda} \]

where the notation \([\ldots]_{\alpha,\Delta \sigma,t_0,\sigma,\lambda}\) means that the SBR equations above are to be solved with the corresponding values of the five parameters. The rescaling parameters read:

\[ b_\phi = \Delta \sigma^{\frac{1}{4-a}} \alpha^{-\frac{1}{8-a}} \]  

(A4)

\[ b_x = (\alpha/b_\phi)^{1/2} \]  

(A5)

\[ b_\sigma = b_\phi^2 \]  

(A6)

\[ b_t = t_0 b_\phi^{-1/a}. \]  

(A7)

Appendix B: Mean-Field Behavior

In figure (9) we plot \( g(t) = \phi(t) - \phi_{flat} \) for \( L = 4 \) and \( M = 1.6 \times 10^6 \). In the range of times shown these data have converged on the Bethe lattice solution corresponding to \( M \to \infty \). In fig. (10) we plot parametrically the (discrete) logarithmic derivative of \( g(t) \) vs. \( g(t) \). For a \( g(t) \) decaying at large times as \( 1/t^a \), the logarithmic derivative should converge to \(-a\) in the limit of \( g(t) \to 0 \). The data display linear behavior in the small \( g \) region corresponding to a correction of order \( 1/t^{2a} \). This is precisely the small-time correction that one would expect within MCT: the leading term is given by a quadratic equation while various subleading cubic terms induce a \( 1/t^{2a} \) correction. The following behavior

\[ g(t) = \frac{1}{(t/t_0)^a} + \frac{\delta_1}{(t/t_0)^{2a}} + \ldots \]  

(B1)

leads to

\[ \frac{\Delta \ln g}{\Delta \ln t} = -a - \delta_1 g + \ldots \]  

(B2)

and thus from a linear fit we can extract \( a \) and \( \delta_1 \). The constant \( t_0 \) can then be estimated fitting the numerical data with the asymptotic form (B1). In fig. (9) we show both the \( 1/(t/t_0)^a \) term and the corrected expression. We note that in the range of times accessed numerically the \( 1/t^{2a} \) correction is significant and a linear fit of the data in fig. (9) would give an incorrect smaller exponent \( a \). Instead the analysis of the second derivative is much safer and inconsistent with the value \( 0.28 - 0.3 \) reported in previous studies.

We note that within the replica treatment the parameter exponent (and thus the exponent \( a \)) is available from the statics (41, 42) and only the coefficient \( t_0 \) must be extracted numerically. Unfortunately a static replica
treatment of the problem is not available at present for these problems.

1. Comparison between different dynamics

The numerical simulations were performed in a Chessboard/Metropolis setting; we have divided the lattice in red and blue sites in such a way that blue sites are connected only to red sites and vice versa. Then we perform a sequential update of all the red spins followed by a sequential update of all the blue spins. The single update is made with a Metropolis move: a negative mobile spin is flipped with probability $e^{-\beta}$ and a positive mobile spin is flipped with probability one. The standard dynamics of the FA model is instead a Random-site/Metropolis one in which the site to be updated is chosen at random. The choice of the Chessboard setting is more convenient because each Monte-Carlo step (MCS) requires less CPU time, besides in MCS unit the relaxation is faster as can be seen in fig. (11) where we display the relaxation for a system with $L = 4$ and $M = 4 \times 10^5$ with different dynamics, including Chessboard with heat-bath (Glauber) update. The key point is that at large enough times the different curves differ only by a constant shift in time: this can be seen more clearly considering the parametric plot of the logarithmic derivative of $g(t)$ vs. $g$. In fig. (12) we see that at small values of $g$ the parametric curves collapse onto a single curve independently of the dynamics. Note that the collapsing curve deviates from the linear fit describing the data at large $M$.

is not invertible, furthermore irreducibly of the Markov chain is not granted in general in sequential update with Metropolis moves, but these features, as usual, appear to be harmless in the interesting region of long-time critical behavior.

Appendix C: Bootstrap Percolation on the Bethe lattice

The dynamics transition of FA models on the Bethe lattice of connectivity $c$ is intimately related with $k$-core, or bootstrap, percolation [30]. Let us recall the definition of BP: first the sites of a given lattice are populated with probability $p$, then each site with less than $k$ neighbors is removed and the process is repeated until each site has at least $k$ neighbors. When the process is completed the remaining occupied sites, if any, form the so-called the $k$-core.

One can argue that the cluster of blocked negative sites on the FA model is exactly the $k$-core with $k$ given by

$$k = c - m + 1 \quad (C1)$$

therefore $k = 3$ for the $(c = 4, m = 2)$ FA models considered here. The relationship between temperature in the FA model and $p$ in BP is easily found:

$$p = \frac{e^\beta}{1 + e^\beta} \quad (C2)$$

With the definition $P_{s,t} \equiv \sum_{i=s}^t \binom{t}{i} P^i (1 - P)^{t-i}$ the Bethe solution of BP is determined by the following equations:

$$P = pP_{k-1,c-1}, \quad P_{\text{site}} = pP_{k,c} \quad (C3)$$
Where $P_{\text{site}}$ is the density of the $k$-core and $p$. For all $k > 2$ a solution with non-zero $P$ is found at large values of $p$; the solution disappears at a critical value $p = p_c$ with a square-root singularity thus exhibiting the celebrated mixed first-order/second-order character.

As shown in the Supplemental Material for [36] the connected probability that two points at large distances $L$ on the Bethe lattice are both on the $k$-core is given:

$$c_L(p) \approx \frac{p^2 \Delta}{\mu} L \mu^L + O(\lambda^k)$$  
where

$$\mu = p \left( \frac{c-2}{k-2} \right) P^{k-2}(1 - P)^{c-k}$$

Note that in [36] the parameter $\mu$ is called $\lambda$ but we changed notation to avoid confusion with the MCT parameter exponent.

$$\Delta \equiv \frac{1 - \mu}{\mu} P(1 - P)$$

In the case $(c = 4, k = 3)$ we have

$$P = \frac{3}{4}, \quad \mu = \frac{1}{3}, \quad P_{\text{site}} = \frac{21}{32} \text{ for } p = p_c = \frac{8}{9}$$

From which the expressions for $c_1$ and $c_2$ given in section [IV] can be obtained.

**Appendix D: Diffusion Coefficient of Non-Backtracking random walks on generic lattices**

We consider lattices with connectivity $c$ such that we can classify sites in two classes, say red and blue, such that a blue site is connected to the red sites located in the $c$ directions $v_{\mu}$. Similarly a red site is connected to the blue sites located in the $c$ directions $-v_{\mu}$. We will not require the condition $\{v_{\mu}\} = -\{v_{\mu}\}$, meaning that the set of directions need not be invariant under inversion of the coordinate axes. Thus we extend previous results [20] to include e.g. the honeycomb lattice in $d = 2$ and the diamond lattice in $d = 3$. We assume instead that

$$\sum_{\mu} v_{\mu} = 0, \quad \|v_{\mu}\| = v.$$  

A non-backtracking random walk can be described as a sequence of steps at consecutive times $s$ in the directions $\mu(s)$:

$$x_t = \sum_{s=0}^{t-1} (-1)^s v_{\mu(s)}$$

where the minus sign comes from the fact that the set of possible directions changes as the walker moves from a blue and to a red site. The average can then be written in terms of the joint probability $P_{\mu,\mu'}^{\mu(s,s')}$:

$$\langle \|x_t\|^2 \rangle = \sum_{s,s'=0}^{t-1} (-1)^{s+s'} v_{\mu} \cdot v_{\mu'} P_{\mu,\mu'}^{\mu(s,s')}$$

Note that we use Einstein’s convention of implicit summation over repeated indexes $\mu$. Since the directions are uniformly distributed at zero time we have exactly

$$P_{\mu,\mu'}^{(s,s')} = P_{\mu,\mu'}^{(0,s-s')}$$

and we can write for large $t$

$$\langle \|x_t\|^2 \rangle \approx t v^2 \left( \sum_{s=1}^{\infty} (-1)^s v_{\mu} \cdot v_{\mu'} P_{\mu,\mu'}^{(0,s)} + 1 \right)$$

The probability can be computed recursively

$$P_{\mu,\mu'}^{(0,s+1)} = T_{\mu,\mu'}^{\mu,\mu''} P_{\mu,\mu''}^{(0,s)}$$

where $T_{\mu,\mu''}^{\mu,\mu''}$ is the $c \times c$ matrix with zero diagonal elements and off-diagonal elements equal to $1/(c - 1)$. We can thus write

$$T = P_s - \frac{1}{c-1} Q_s$$

where $P_s$ is the projector on the vector with all equal components and $Q_s = I - P_s$ is the orthogonal projector. It follows that

$$P^{(0,0)} = \frac{1}{c} I, \quad P^{(0,s)} = \frac{1}{c} T^s = \frac{1}{c} P_s + \frac{1}{c} \left( \frac{-1}{c-1} \right)^s Q_s$$

since $P_s$ is a matrix with all equal elements equal it gives zero contribution when summed over the directions because of the condition $\sum_{\mu} v_{\mu} = 0$ and we have

$$\hat{v}_{\mu} \cdot \hat{v}_{\mu'} P_{\mu,\mu'}^{(0,s)} = \left( \frac{-1}{c-1} \right)^s v^2$$

replacing the above expression in the large time expression we obtain:

$$\langle \|x_t\|^2 \rangle \approx t \frac{c v^2}{c-2}$$

and therefore (taking into account that each vector has $d$ components)

$$D_{\text{NBW}} \equiv \lim_{t \to \infty} \frac{\langle \|x_t\|^2 \rangle}{2 d t} = \frac{c v^2}{2 d (c-2)}.$$  

On regular lattice $c = 2d$ and we recover the result $D_{\text{NBW}} = 1/(2d - 2)$. On the diamond cubic lattice studied in the paper we have $c = 4$ and $v^2 = 3$ leading to $D_{\text{NBW}} = 1$. The honeycomb lattice can be realized repeating a $3 \times \sqrt{3}$ unit cell with two blue points at coordinates $(0,0), (\sqrt{3}/2, 3/2)$ and two red points with coordinates $(0,1), (\sqrt{3}/2, 5/2)$. This leads to $\rho = 4.3^{-3/2}$ and $D_{\text{NBW}} = 3/4$. 


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