UNIQUENESS OF SCATTERER IN INVERSE ACOUSTIC OBSTACLE SCATTERING WITH A SINGLE INCIDENT PLANE WAVE

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Abstract. In this paper, we give a positive answer to a challenging open problem for recovering unknown obstacle (which is usually referred to as a scatterer) by acoustic wave probe associated to the Helmholtz equation. We show that the acoustic scattering amplitude $A(\beta, \alpha_0, k_0)$, known for all $\beta \in S^2$, where $S^2$ is the unit sphere in $R^3$, $\alpha_0 \in S^2$ is fixed, $k_0 > 0$ is fixed, determines the obstacle $D$ and the boundary condition on $\partial D$ uniquely (The boundary condition on $\partial D$ is either the Dirichlet, or Neumann, or the impedance one).

1. Introduction

Throughout this paper, $D$ is assumed to be a bounded domain with boundary $\partial D$ of class $C^2$ and with the connected complement $R^3 \setminus \bar{D}$. Consider the acoustic scattering problem:

\begin{align}
\Delta u + k^2 u &= 0 \quad \text{in } R^3 \setminus \bar{D}, \\
u(x) &= e^{ik\alpha \cdot x} + u^s(x), \\
B u &= 0 \quad \text{on } \partial D, \\
\lim_{r \to \infty} r \left( \frac{\partial u^s}{\partial r} - iku^s \right) &= 0,
\end{align}

where $u^i(x) := e^{ik\alpha \cdot x}$ is the incident plane wave with the wave number $k > 0$ and the direction $\alpha \in S^2$ of the propagation, which is an entire solution to the Helmholtz equation $\Delta u + k^2 u = 0$, (1.4) is the Sommerfeld radiation condition which guarantees that the scattered wave is outgoing and $r = |x|$. The boundary condition $Bu = 0$ can be assumed to be either the Dirichlet boundary condition (i.e., sound-soft obstacle) $B_1 u := u|_{\partial D} = 0$, or the Neumann boundary condition (i.e., sound-hard obstacle) $B_2 u := \frac{\partial u}{\partial \nu}|_{\partial D} = 0$ or the impedance boundary condition $B_3 u := \left( \frac{\partial u}{\partial \nu} + hu \right)|_{\partial D} = 0$, where $h$ is a constant with $\text{Im} h \geq 0$, $\nu$ is the unit normal to $\partial D$ pointing out of $D$. We refer to [27], [20] or Theorem 1.3 on p. 179 of [40] for the unique existence of an $H^2_{loc}(R^3 \setminus \bar{D})$ solution to the problem (1.1)–(1.4). It is known that $u^s$ has the following asymptotic expansion (see [8] or [35]),

\begin{equation}
\lim_{|x| \to \infty} \frac{u^s(x)}{|x|} = \frac{e^{ik|x|}}{|x|} A(\beta, \alpha, k) + O\left( \frac{1}{|x|^2} \right), \quad |x| \to \infty, \quad \frac{x}{|x|} = \beta,
\end{equation}

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where $A(\beta, \alpha, k)$ is known as the scattering amplitude (or far field pattern) of the scattered acoustic wave, $\beta \in S^2$ is the direction of the scattered wave (or the observation angle). The scattering amplitude $A(\beta, \alpha, k)$ is very important physics data in scattering theory which can be measured experimentally. The interest in scattering amplitudes lies in the fact that the basic inverse problem in scattering theory is to determine $D$ from a knowledge of $A(\beta, \alpha, k)$ for $\beta$ and $\alpha$ on the unit sphere.

The study of uniqueness of the inverse scattering problem for acoustic wave is of fundamental important to many areas of science and technology, such as radar and sonar, geophysical exploration, non-destructive testing, and medical imaging. The first result for uniqueness of the inverse scattering problem was due to Schiffer (see Lax and Phillips [21]) who showed that the scattering amplitude $A(\beta, \alpha, k)$ for all $\beta, \alpha \in S^2$ and $k$ fixed uniquely determines the sound-soft scattering obstacle $D$. The same uniqueness conclusion still holds by $A(\beta, \alpha, k)$ for all $\beta$ and $k$ along with a fixed $\alpha$ (see [13] and [8]). The unique recovery of purely a sound-hard $D$ by knowledge of $A(\beta, \alpha, k)$ for all $\beta$ and $\alpha$ along with a fixed $k$ is due to Isakov's singular source method [14], [17]. The uniqueness of recovering purely a sound-soft or a sound-hard $D$ by knowledge of $A(\beta, \alpha, k)$ with all $\beta$ and finitely many $k$ and $\alpha$ were discussed in [11], [11]. Karp [16] showed that if $A(\beta, \alpha, k) = A(Q\beta, Q\alpha, k)$ for all rotations $Q$ and all $\beta, \alpha \in S^2$ then $D$ is a ball centered at the origin. A great progress was made by Colton and Sleeman [10] who proved that the sound-soft scatterer is uniquely determined by the scattering amplitude of a finite number of incident plane waves provided a priori information on the size of the obstacle is available. Stefanov and Uhlmann [39] showed that a sound-soft scatterer $D$ is also uniquely determined if $D$ is closed to a given obstacle. Liu in [23] proved that a sound-soft ball is uniquely determined by the scattering amplitude for one incident plane wave. For the polyhedron scatterer $D$, a single incident plane wave uniquely determines $D$ as established by Cheng and Yamamoto [4], by Alessandrini and Rondi [1] and by Liu and Zhou [29].

It has been a challenging open problem (see Problem 6.3-6.4 on p. 162 of [13], or p. 123 of [8], or [28]) that for a fixed wave number $k$ and a fixed incident direction $\alpha$, whether the scattering amplitude determines the scatterer $D$ and its boundary condition uniquely? Also a remark was made in 1998 by Isakov (see [13]) for the uniqueness in inverse acoustic obstacle scattering problem and it reads as follows: “This is a well-known question that supposedly can be solved by elementary means. However, it has been open for thirty to forty years.” There is already a vast literature on inverse acoustic scattering problems using the scattering amplitude $A(\beta, \alpha, k)$, and many progresses have been made in this problem. We refer the readers to [7], [8], [21], [34], [6], [10], [39], [23], [26], [4], [1], [11], [29], [28], [14], [5], [38], [9] [17], [18], [2], [12], [3] and [19] and the references therein for the studies on uniqueness of this inverse problem.

In this paper, we give a positive answer to the longstanding open problem mentioned above. Our main result is the following:

**Theorem 1.1.** Assume that $D_1$ and $D_2$ are two scatterers with boundary conditions $B^{D_1}$ and $B^{D_2}$ such that for a fixed wave number $k_0$ and a fixed incident direction $\alpha_0$ the scattering amplitude of both scatterers coincide (i.e., $A_1(\beta, \alpha_0, k_0) = A_2(\beta, \alpha_0, k_0)$ for all $\beta$ in an open subset of $S^2$). Then $D_1 = D_2$ and $B^{D_1} = B^{D_2}$.

Let us remark that in the proof of Theorem 1.1 we will use a novel idea and an elementary means by discussing all possible positions of two scatterers and applying the spectral theory of the Laplacian. In particular, we subtly apply three basic tools: the property of the
Laplacian eigenfunction in a bounded domain, the interior analyticity of the solutions for the Helmholtz equation, and the asymptotic property of the scattering solution as $|x| \to \infty$.

This paper is organized as follows. In Section 2 we present some known results and prove a useful lemma (Lemma 2.7) which shows that the scattering amplitude determines the total scattering wave in the unbounded connected component of $\mathbb{R}^3 \setminus (D_1 \cup D_2)$. Section 3 is devoted to the proof of the main result.

2. Preliminaries

Let $g(x)$ be a real-valued function defined in an open set $\Omega$ in $\mathbb{R}^n$. For $y \in \Omega$ we call $g$ real analytic at $y$ if there exist $\gamma \in \mathbb{R}^n$ and a neighborhood $U$ of $y$ (all depending on $y$) such that

$$g(x) = \sum_{|\gamma|} a_\gamma (x - y)^\gamma$$

for all $x \in U$, where $\gamma = (\gamma_1, \cdots, \gamma_n)$ is a multi-index (a set of non-negative integers), $|\gamma| = \sum_{j=1}^n \gamma_j$, and $(x - y)^\gamma = (x_1 - y_1)^{\gamma_1} \cdots (x_n - y_n)^{\gamma_n}$. We say $g$ is real analytic in $\Omega$, if $g$ is real analytic at each $y \in \Omega$.

Lemma 2.1 (Unique continuation of real analytic function, see, for example, p. 65 of [15]). Let $\Omega$ be a connected open set in $\mathbb{R}^n$, and let $g$ be real analytic in $\Omega$. Then $g$ is determined uniquely in $\Omega$ by its values in any nonempty open subset of $\Omega$.

Lemma 2.2 (The interior real analyticity of the solutions for real analytic elliptic equations, see [30], [31], [32] or [33]). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let $L$ be a strongly elliptic linear differential operator of order $2m$

$$Lu = \sum_{|\gamma| \leq 2m} a_\gamma(x) D^\gamma u(x).$$

If the coefficients $a_\gamma(x), |\gamma| \leq 2m$, and the right-hand side $f(x)$ of the equation $Lu = f$ are real analytic with respect to $x = (x_1, \cdots, x_n)$ in the domain $\Omega$, then any solution $u$ of this equation is also real analytic in $\Omega$.

Lemma 2.3 (see Remark 3.4 of [8] or p.21 of [8]) Let $D$ be a bounded domain with boundary $\partial D$ of class $C^2$ and with the connected complement $\mathbb{R}^3 \setminus \bar{D}$. Assume that $u \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ is a solution to the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \bar{D}$$

satisfying the Sommerfeld radiation condition

$$\lim_{r \to \infty} r \left( \frac{\partial u}{\partial r} - iku \right) = 0$$

where $r = |x|$ and the limit is assumed to hold uniformly in all directions $\frac{x}{|x|}$. Then $u$ automatically satisfies

$$u(x) = O\left( \frac{1}{|x|} \right), \quad |x| \to \infty,$$

uniformly for all directions $\frac{x}{|x|}$. 
Lemma 2.4 (see p. 21 of [8]). Let \( u \) be a solution to the Helmholtz equation in \( \mathbb{R}^3 \) satisfying the Sommerfeld radiation condition. Then \( u \) must vanish identically in \( \mathbb{R}^3 \).

Lemma 2.5 (Holmgren’s theorem for the acoustic scattering equation, see Theorem 2 on p. 42 in [77], or Theorem 2.3 on p. 19 in [8]). Let \( D \) be a bounded domain with boundary \( \partial D \) of class \( C^2 \) and with the connected complement \( \mathbb{R}^3 \setminus \bar{D} \), and let \( \Gamma \subset \partial D \) be an open subset with \( \Gamma \cap (\mathbb{R}^3 \setminus D) \neq \emptyset \). Assume that \( u \) is a solution of the scattering problem for the Helmholtz equation

\[
\begin{align*}
\Delta u + k^2 u &= 0, & u(x) = e^{ik\alpha \cdot x} + u^s(x) & \text{in } \mathbb{R}^3 \setminus \bar{D}, \\
\lim_{|x| \to \infty} |x| \left( \frac{\partial u^s(x)}{\partial |x|} - iku^s(x) \right) &= 0,
\end{align*}
\]

such that

(2.2) \[ u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma. \]

Then \( u \equiv 0 \) in \( \mathbb{R}^3 \setminus \bar{D} \).

The following Lemma will be needed in the proof of Lemma 2.7.

Lemma 2.6 (Rellich’s lemma, see p. 33 of [8] or p. 178 of [40]). Assume the bounded set \( D \) is the open complement of an unbounded domain and let \( v \in C^2(\mathbb{R}^3 \setminus \bar{D}) \) be a solution to the Helmholtz equation \((\Delta + k^2)v = 0\) satisfying \( \int_{\partial B_r(0)} |v|^2 ds \to 0 \) as \( r \to \infty \), where \( \partial B_r(0) \) is the sphere \( \{ x \in \mathbb{R}^3 | |x| = r \} \). Then \( v(x) = 0 \) for all \( x \in \mathbb{R}^3 \setminus \bar{D} \).

Let \( D_j \) be a bounded domain in \( \mathbb{R}^3 \) with a connected boundary \( \partial D_j \) of class \( C^2 \) \((j = 1, 2)\). Let \( u_j(x, \alpha, k) \) be the solution of the scattering problem in \( \mathbb{R}^3 \setminus \bar{D}_j \), i.e., \( u_j(x, \alpha, k) := e^{ik\alpha \cdot x} + u_j^s(x, \alpha, k), \ j = 1, 2, \) satisfies the Helmholtz equation

(2.3) \[
\begin{align*}
\Delta u_j + k^2 u_j &= 0, & u_j(x, \alpha, k) &= e^{ik\alpha \cdot x} + u_j^s(x, \alpha, k) & \text{in } \mathbb{R}^3 \setminus \bar{D}_j \\
B_l u_j &= 0 & \text{on } \partial D_j, & l = 1, 2, 3
\end{align*}
\]

and

\[
\frac{\partial u_j^s}{\partial r} - iku_j^s = O\left(\frac{1}{r}\right) \quad \text{as } r := |x| \to \infty
\]

uniformly for all direction \( \frac{x}{|x|} \). As pointed out in Section 1, we can write

(2.4) \[
u_j(x, \alpha, k) = e^{ik\alpha \cdot x} + e^{ik|x|} |x|^{-1} A_j(\beta, \alpha, k) + O\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty, \quad \beta = \frac{x}{|x|},
\]

where \( A_j(\beta, \alpha, k) \) are the scattering amplitude for the exterior domains \( \mathbb{R}^3 \setminus \bar{D}_j, \ j = 1, 2. \)

Now, we have the following useful lemma:

Lemma 2.7. Let \( D_j \) be a bounded domain with boundary \( \partial D_j \) of class \( C^2 \) and with the connected complement \( \mathbb{R}^3 \setminus \bar{D}_j \) \((j = 1, 2)\). Let \( u_j(x, \alpha_0, k_0) \) be the solution of scattering problem for the Helmholtz equation in \( \mathbb{R}^3 \setminus \bar{D}_j \). If \( A_1(\beta, \alpha_0, k_0) = A_2(\beta, \alpha_0, k_0) \) for all \( \beta \in \mathbb{S}^2 \), a fixed \( \alpha_0 \in \mathbb{S}^2 \) and a fixed \( k_0 \in \mathbb{R}^1 \), then

(2.5) \[ u_1(x, \alpha_0, k_0) = u_2(x, \alpha_0, k_0) \quad \text{for all } x \in D_{12}, \]

where \( D_{12} \) is the unbounded connected component of \( \mathbb{R}^3 \setminus (\bar{D}_1 \cup \bar{D}_2) \).
Proof of theorem 1.1. For convenience, we assume below the obstacle has the Dirichlet boundary condition, but our proof is valid for the Neumann or the impedance boundary condition as well. It is well-known that for a $C^2$-smooth bounded obstacle the scattering amplitude $A(\beta, \alpha, k)$ is an analytic function of $\beta$ and $\alpha$ on the non-compact analytic variety $K := \{z \in \mathbb{C}^3 | z \cdot z = 1\}$, where $z \cdot z := \sum_{m=1}^{\infty} z_m^2$. The unit sphere $S^2$ is a subset of $K$. It is clear that if $A_1(\beta, \alpha_0, k_0) = A_2(\beta, \alpha_0, k_0)$ for all $\beta$ in an open subset of $S^2$, then the same is true for all $\beta \in S^2$ by analyticity. It is also an obvious fact that if two bounded domains $D_1$ and $D_2$ of class $C^2$ satisfying $D_1 \neq D_2$, then either $D_1 \neq D_2$ and $D_1 \cap D_2 = \emptyset$, or $D_1 \neq D_2$ and $D_1 \cap D_2 \neq \emptyset$. We will show that the above two cases can never occur.

Case 1. Suppose by contradiction that $D_1 \neq D_2$ and $D_1 \cap D_1 = \emptyset$. As pointed out above, we have that $A_1(\alpha_0, \beta, k_0) = A_2(\alpha_0, \beta, k_0)$ for all $\beta \in S^2$. From Lemma 2.7 we get that $u_1(x, \alpha_0, k_0) = u_2(x, \alpha_0, k_0)$ for all $x \in \mathbb{R}^{12}$, where $u_j(x, \alpha_0, k_0)$ is the solution of scattering problem for the Helmholtz equation in $\mathbb{R}^{12} \setminus D_j$ ($j = 1, 2$), and $D_{12}$ is the unbounded connected component of $\mathbb{R}^{12} \setminus (D_1 \cup D_2)$. Note that the real part and imaginary part of $u_j$ are both real analytic in $\mathbb{R}^{12} \setminus D_j$ ($j = 1, 2$) (see Lemma 2.2 or Theorem 2.2 of [5]). Since $u_1(x, \alpha_0, k_0)$ is defined in $D_2$ and satisfies there the Helmholtz equation (1.1), the unique continuation property implies that $u_2(x, \alpha_0, k_0)$ can be defined in $D_2$ and satisfies there the Helmholtz equation. Consequently, $u_2(x, \alpha_0, k_0)$ is defined in $\mathbb{R}^{12}$, it is a smooth function that satisfies the Helmholtz equation in $\mathbb{R}^{12}$, and the same is true for $u_1(x, \alpha_0, k_0)$. Therefore the scattered parts $u_1^s$ and $u_2^s$ of the scattering solutions $u_1$ and $u_2$ satisfy the Helmholtz equation $\Delta u + k^2 u = 0$ in $\mathbb{R}^{12}$ and have the Sommerfeld radiation condition. It follows from Lemma 2.4 that $u_1^s = u_2^s = 0$ in $\mathbb{R}^{12}$ and hence $u_1 = u_2 = e^{i k_0 \alpha_0 \cdot \mathbf{x}}$ in $\mathbb{R}^{12}$. This is impossible since $u_j = 0$ on $\partial D_j$, $j = 1, 2$, while $e^{i k_0 \alpha_0 \cdot \mathbf{x}} \neq 0$ for any $x \in \partial D_j$. Thus, we must have $D_1 = D_2$. 

Proof. Without loss of the generality, we only consider the scattering problem with Dirichlet boundary condition. By (2.4) we have

$$u_2(x, \alpha_0, k_0) - u_1(x, \alpha_0, k_0) = \frac{e^{ik_0 \cdot x}}{|x|} \left[ A_2(\beta, \alpha_0, k_0) - A_1(\beta, \alpha_0, k_0) \right] + O\left( \frac{1}{|x|^2} \right) \quad \text{as} \quad |x| \to \infty, \quad \beta = \frac{x}{|x|}.$$ 

In view of

$$A_1(\beta, \alpha_0, k_0) = A_2(\beta, \alpha_0, k_0) \quad \text{for all} \quad \beta \in S^2,$$

we obtain

$$u_1(x, \alpha_0, k_0) - u_2(x, \alpha_0, k_0) = O\left( \frac{1}{|x|^2} \right) \quad \text{as} \quad |x| \to \infty, \quad \beta = \frac{x}{|x|},$$

Obviously, $u_1 - u_2$ still satisfies the Helmholtz equation in $D_{12}$, i.e.,

$$\Delta (u_1(x, \alpha_0, k_0) - u_2(x, \alpha_0, k_0)) + k_0^2 (u_1(x, \alpha_0, k_0) - u_2(x, \alpha_0, k_0)) = 0 \quad \text{in} \quad D_{12}.$$ 

It follows from (2.7) and Lemma 2.6 (Rellich's lemma) that

$$u_1(x, \alpha_0, k_0) = u_2(x, \alpha_0, k_0) \quad \text{for all} \quad x \in D_{12}.$$ 

□

3. PROOF OF MAIN THEOREM
Case 2. Suppose by contradiction that $D_2 \neq D_2$ and $D_1 \cap D_2 \neq \emptyset$. Then either $(\mathbb{R}^3 \setminus \bar{D}_1) \cap (\mathbb{R}^3 \setminus \bar{D}_2)$ or $(\mathbb{R}^3 \setminus \bar{D}_1) \cap (\mathbb{R}^3 \setminus \bar{D}_2)$ has only finitely many connected components, and each of them adjoins the unbounded domain $D_1$ by sharing a common $C^2$-smooth surface, where $D_1$ is the unbounded connected component of $\mathbb{R}^3 \setminus (\bar{D}_1 \cup \bar{D}_2)$. Let us assume that $\Omega$ be any one of the above connected components. Clearly, $\Omega$ is a bounded domain with piecewise $C^2$-smooth boundary. Without loss of generality, we let $\Omega \subset \mathbb{R}^3 \setminus \bar{D}_1$. Since $A_1(\alpha_0, \beta, k_0) = A_2(\alpha_0, \beta, k_0)$ for all $\beta \in S^2$ by analyticity, applying Lemma 2.7 once more we find that

$$u_1(x, \alpha_0, k_0) = u_2(x, \alpha_0, k_0)$$

for all $x \in \partial D_1$, where $u_j$ is the solution of (1.1)–(1.4) in $\mathbb{R}^3 \setminus \bar{D}_j$ $(j = 1, 2)$. Note that $u_j|_{\partial D_j} = 0$ $(j = 1, 2)$ and $u_1|_{\partial D_{12}} = u_2|_{\partial D_{12}} = 0$. It is easy to see from this and the definition of $\Omega$ that the restriction of $u_1$ to $\Omega$ satisfies

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases}$$

i.e., the restriction of $u_1$ to $\Omega$ is a Dirichlet eigenfunction of the Laplacian corresponding to the Dirichlet eigenvalue $k^2$. Since $u_1 \in C^2(\mathbb{R}^3 \setminus \bar{D}_1) \cap C(\mathbb{R}^3 \setminus \bar{D}_1)$ and since the coefficients of the Helmholtz operator $\Delta + k^2$ is real analytic in $\mathbb{R}^3$, we find by Lemma 2.2 (see also Theorem 2.2 of [8]) that $\text{Re } u_1$ and $\text{Im } u_1$ are both real analytic in $\mathbb{R}^3 \setminus \bar{D}_1$, where $\text{Re } u_1$ and $\text{Im } u_1$ are the real part and imaginary part of the solution $u_1$, i.e., $u_1 = \text{Re } u_1 + i\text{Im } u_1$. By the definition of $u_1$, we have that for all $x \in \mathbb{R}^3 \setminus \bar{D}_1$,

$$u_1(x, \alpha_0, k_0) = e^{ik_0 \alpha_0 \cdot x} + u_1^s(x, \alpha_0, k_0)$$

$$= \cos(k_0 \alpha_0 \cdot x) + i \sin(k_0 \alpha_0 \cdot x) + \text{Re } u_1^s(x, \alpha_0, k_0) + i \text{Im } u_1^s(x, \alpha_0, k_0)$$

$$= (\cos(k_0 \alpha_0 \cdot x) + \text{Re } u_1^s(x, \alpha_0, k_0)) + i(\sin(k_0 \alpha_0 \cdot x) + \text{Im } u_1^s(x, \alpha_0, k_0)).$$

It follows from the spectral theory for the Laplacian that the Dirichlet eigenfunction $u_1$ must be a real-valued function in $\Omega$. From this and (3.3), we get that $\text{sin}(k_0 \alpha_0 \cdot x) + \text{Im } u_1^s(x, \alpha_0, k_0)$ must vanish identically for all $x \in \Omega$, i.e.,

$$\text{Im } u_1^s(x, \alpha_0, k_0) = -\sin(k_0 \alpha_0 \cdot x)$$

for all $x \in \Omega$. From Lemma 2.1, we know that the real analytic function $\text{Im } u_1^s$ is uniquely determined in $(\Omega \cup D_{12} \cup ((\partial \Omega) \cap (\partial D_{12})))^o$ by its values in the subset domain $\Omega$, where $(\Omega \cup D_{12} \cup ((\partial \Omega) \cap (\partial D_{12})))^o$ is the interior of $\Omega \cup D_{12} \cup ((\partial \Omega) \cap (\partial D_{12}))$. Let us remark that $(\Omega \cup D_{12} \cup ((\partial \Omega) \cap (\partial D_{12})))^o$ is still an unbounded connected component (i.e., an unbounded domain in $\mathbb{R}^3$). Note also that the real analytic function $-\sin(k_0 \alpha_0 \cdot x)$ defined for $x \in \Omega$ has just a unique real analytic extension to $(\Omega \cup D_{12} \cup ((\partial \Omega) \cap (\partial D_{12})))^o$, that is,

$$-\sin(k_0 \alpha_0 \cdot x)$$

for $x \in (\Omega \cup D_{12} \cup ((\partial \Omega) \cap (\partial D_{12})))^o$.

Thus, we have that for all $x \in (\Omega \cup D_{12} \cup ((\partial \Omega) \cap (\partial D_{12})))^o$,

$$\text{Im } u_1^s(x, \alpha_0, k_0) = -\sin(k_0 \alpha_0 \cdot x).$$

Since $u_1^s(x, \alpha_0, k_0)$ is the scattering solution of the Helmholtz equation in $\mathbb{R}^3 \setminus \bar{D}_1$ satisfying the Sommerfeld radiation condition, by (2.6) of Lemma 2.3 we get $\lim_{|x| \to \infty} |u_1^s(x, \alpha_0, k_0)| = 0$. On the other hand, from (3.6) we see that

$$|u_1^s(x, \alpha_0, k_0)|^2 \geq \left| \left( \text{Re } u_1^s(x, \alpha_0, k_0) \right)^2 + \left( \text{Im } u_1^s(x, \alpha_0, k_0) \right)^2 \right|^{1/2} \geq \left| \sin(k_0 \alpha_0 \cdot x) \right| \text{ for all } x \in (\Omega \cup D_{12} \cup ((\partial \Omega) \cap (\partial D_{12})))^o,$$
and so \(|u_1^\ast(x, \alpha_0, k_0)|\) cannot tend to zero (uniformly for all directions $\frac{x}{|x|}$) as $|x| \to \infty$. This is a contradiction, which implies that any domain $\Omega$ mentioned above can never appear. Therefore we must have $D_1 = D_2$.

Finally, denoting $D = D_1 = D_2$, $u = u_1 = u_2$, we assume that we have different boundary condition $B^{D_1}u \neq B^{D_2}u$. For the sake of generality, consider the case where we have impedance boundary conditions with two different continuous impedance functions $h_1 \neq h_2$. Then, from $\frac{\partial u}{\partial n} + h_j u = 0$ on $\partial D$ for $j = 1, 2$ we observe that $(h_1 - h_2)u = 0$ on $\partial D$. Therefore for the open set $\Gamma := \{x \in \partial D | h_1(x) \neq h_2(x)\}$ we have that $u = 0$ on $\Gamma$. Consequently, we obtain $\frac{\partial u}{\partial n} = 0$ on $\Gamma$ by the given boundary condition. Hence, by Holmgren’s uniqueness theorem for the acoustic scattering (Helmholtz) equation (see Lemma 2.5), we obtain that $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$, which contradicts the fact that $|u(x)| \to 1$ as $|x| \to \infty$. Hence $h_1 = h_2$ on $\partial D$. The case where one of the boundary conditions is the Dirichlet or Neumann boundary condition can be treated analogously. \(\square\)

**Remark 3.1.** Using a method analogous to $n = 3$, we can also show that the same conclusion is still true for any $n$-dimensional case ($n \geq 2$).

**Remark 3.2.** With the similar ideas, by using some new techniques we also give positive answer to two more difficult open problems on the electromagnetic obstacle scattering and elastic obstacle scattering (see [24] and [25]).

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