Application of Homotopy Perturbation Method to an Eco-epidemic Model

P. K. Bera(1), S. Sarwardi(2)* and Md. A. Khan(3)

(1) Department of Physics, Dumkal College, Basantapur, Dumkal Murshidabad - 742 303, West Bengal, India

(2) Department of Mathematics, Aliah University, Salt Lake City, Sector-V, Kolkata - 700 091, West Bengal, India.
email: s.sarwardi@gmail.com

(3) Department of Physics, Aliah University, Salt Lake City, Sector-V, Kolkata - 700 091, West Bengal, India.

Abstract

In this article, we apply Homotopy Perturbation Method (HPM) for solving three coupled non-linear equations which play an important role in biosystems. To illustrate the capability and reliability of this method. Numerical example is given which confirms our analytical findings.

Keywords: Homotopy Perturbation Method; Eco-epidemic model; Application

1 Introduction

There exist a wide class of literature dealing with the problem of approximate solutions to nonlinear equations with various different methodologies, called the perturbation methods. But almost all perturbation methods are based on small parameters so that the approximate solutions can be expanded in series of small parameters. Its basic idea is to transform by means of small parameters, a nonlinear problem of an infinite number of linear subproblems into an infinite number of simpler ones. The small parameter determines not only the accuracy of the perturbation approximations but also the validity of the perturbation method.

There exists some analytical approaches, such as the harmonic balance method [1], the Krylov-Bogolyubov-Mitropolsky method [2], weighted linearization method [3], perturbation procedure for limit cycle analysis [4], modified Lindstedt-Poincare method [5], artificial parameter method [6] and so on.

*Author to whom all correspondence should be addressed
In science and engineering, there exists many nonlinear problems, which do not contain any small parameters, especially those with strong nonlinearity. He [7, 8] developed the Homotopy Perturbation Method (HPM) for solving linear, nonlinear, initial and boundary value problems by merging the standard homotopy and the perturbation. The HPM was formulated by taking full advantage of the standard homotopy and perturbation methods. In this method the solution is given in an infinite series usually converging to an accurate solution.

Inspired and motivated by the ongoing research in the area of bioscience involving mainly ecological and eco-epidemiological systems, we apply HPM for solving three coupled nonlinear equations representing a prey-predator model system with disease in prey species only.

This paper is organized as follows: In Section 2, HPM has been illustrated. Based on the HPM, the approximate solutions of three coupled nonlinear equations are obtained in section 3. Finally, we have drawn the conclusion in section 4.

2 Analysis of the Homotopy Perturbation Method (HPM)

To illustrate the basic ideas of HPM for solving nonlinear differential equations, He[7, 8] considered the following nonlinear differential equation:

\[ A(u) - f(r) = 0, \ r \in \Omega, \]  \hspace{1cm} (2.1)

with the boundary conditions

\[ B\left(u, \frac{\partial}{\partial n}\right) = 0, \ r \in \Gamma, \]  \hspace{1cm} (2.2)

where \( A \) is a general differential operator, \( B \) is a boundary operator, \( f(r) \) is known anaclitic function, \( \Gamma \) is the boundary of the domain \( \Omega \) and \( \frac{\partial}{\partial n} \) denotes differentiation along the normal vector drawn outwards from \( \Omega \). The operator \( A \) can generally be divided into two parts \( L \) and \( N \), where \( L \) is linear and \( N \) is nonlinear. Therefore, Eq. (2.1) can be written as

\[ L(u) + N(u) - f(r) = 0, \ r \in \Omega. \]  \hspace{1cm} (2.3)

He [7, 8] constructed a homotopy as follows:

\[ H(v, p) = (1 - p)(L(v) - L(u_0)) + p(A(v) - f(r)) = 0 \]  \hspace{1cm} (2.4)

or,

\[ H_i(v, p) = L_i(v, p) - L_i(v_0, p) + p\left(L_i(v_0, p) + N_i(v, p)\right) = 0. \]  \hspace{1cm} (2.5)

where \( v(r, p) : \Omega \times [0, 1] \rightarrow R \). In Eq. (2.4), \( p \in [0, 1] \) is an embedding parameter and \( u_0 \) is the first approximation that satisfies the boundary condition. The changing
process of $p$ from zero to unity is just that of $H(v, p)$ from $L(v) - L(u_0)$ to $A(v) - f(r)$. In topology, this is called deformation. The terms $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopy. According to the homotopy perturbation method, the parameter $p$ is used as a small parameter and the solution of Eq. (2.4) can be expressed as a series in $p$ in the form

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \cdots \cdots \cdots \cdots$$  \hspace{1cm} (2.6)

when $p \to 1$, Eq. (2.4) corresponds to the original one, Eq. (2.5) the approximate solution of Eq. (2.1), i.e.

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots \cdots \cdots \cdots$$  \hspace{1cm} (2.7)

The convergence of the series in Eq. (2.6) has been discussed by He[7, 8].

3 Homotopy perturbation method for three coupled system

We consider three coupled nonlinear equations which describe a prey-predator model, consisting with two prey and a predator species. We also consider an infectious disease, which is transmissible among the prey species only to give the model realism, interested readers are referred to [9-11]. Let us assume $S(t)$ denotes susceptible prey population, $I(t)$ denotes infected prey population, and $P(t)$ denotes predator population at any time. The model under consideration is given by the following system of ordinary nonlinear differential equations

$$\frac{dS}{dt} = rS\left(1 - \frac{S + I}{K}\right) - c_1SP - \delta SI,$$  \hspace{1cm} (3.1)

$$\frac{dI}{dt} = \delta SI - c_2IP - d_1I,$$  \hspace{1cm} (3.2)

$$\frac{dP}{dt} = e(c_1S + c_2I)P - d_2P,$$  \hspace{1cm} (3.3)

where $S(0) > 0, I(0) > 0, P(0) > 0$ and $\dot{S}_0 = 0, \dot{I}_0 = 0$ and $\dot{P}_0 = 0$. Here $r$ is the growth rate of the prey population, $c_1$ and $c_2$ are the searching efficiency of the predators for the susceptible prey and infected prey respectively, similarly $\epsilon c_1$ and $\epsilon c_2$ are the conversion factors for the susceptible prey and infected prey respectively consumed by the predators. $K$ is the carrying capacity of the environment for the total (susceptible + infected) prey population, the disease spreads horizontally with mass action incidence rate $\delta SI$. $d_1$ is the mortality rate of infected prey population including disease related death, $d_2$ is the mortality rate of the predator population. All the parameters are non negative. The predators eat both susceptible and infected prey at different rates, since the susceptible prey more likely escapes from an attack, thus $c_1 < c_2$. It is to be noted that the value
of the system parameter ‘e’ is a proper fraction for most of the realistic prey-predator interactions. For application of HPM, now we write Eqs. (3.1)-(3.3) as

\[ H_i(S, I, P, p) = L_i(S, I, P, p) - L_i(S_0, I_0, P_0, p) + p \left( L_i(S, I, P, p) + N_i(S, I, P, p) \right) \] (3.4)

where \( i = 1, 2, 3 \) and we also consider

\[ S = S_0 + pS_1 + p^2S_2 + p^3S_3 + \cdots \] (3.5)

\[ I = I_0 + pI_1 + p^2I_2 + p^3I_3 + \cdots \] (3.6)

\[ P = P_0 + pP_1 + p^2P_2 + p^3P_3 + \cdots \] (3.7)

when \( p \to 1 \), Eqs. (3.5)-(3.7) become the approximate solution of Eqs. (3.8)-(3.10), i.e.,

\[ S_{approx} = \lim_{p \to 1} S = S_0 + S_1 + S_2 + S_3 + \cdots \] (3.8)

\[ I_{approx} = \lim_{p \to 1} I = I_0 + I_1 + I_2 + I_3 + \cdots \] (3.9)

\[ P_{approx} = \lim_{p \to 1} P = P_0 + P_1 + P_2 + P_3 + \cdots \] (3.10)

Here, boundary conditions are \( S_0 > 0, I_0 > 0, P_0 > 0 \) and \( \dot{S}_0 = \dot{I}_0 = \dot{P}_0 = 0 \).

For without perturbation, the Eqs. (3.1)-(3.3) can be written as

\[ \frac{dS_0(t)}{dt} = rS_0 \] (3.11)

\[ \frac{dI_0}{dt} = -d_1I_0 \] (3.12)

\[ \frac{dP_0}{dt} = -d_2P_0 \] (3.13)

whose solutions are \( S_0(t) = S(0)e^{rt}, I_0(t) = I(0)e^{-d_1t}, P_0(t) = P(0)e^{-d_2t} \).

With the help of Eq. (3.4), one can write the Eqs. (3.1)-(3.3) as follows:

\[ \frac{dS}{dt} - rS = p \left( -rS \left( \frac{S + I}{K} \right) - c_1SP - \delta SI \right), \] (3.14)

\[ \frac{dI}{dt} + d_1I = p \left( \delta SI - c_2IP \right), \] (3.15)

\[ \frac{dP}{dt} + d_2P = p \left( ec_1S + ec_2IP \right). \] (3.16)
Substituting the values of \( S(t), I(t) \) and \( P(t) \) from Eqs. (3.5)-(3.7) and equating the coefficients of embedding parameter \( p \), we get coefficient of \( p^0 \) as

\[
\frac{dS_0(t)}{dt} - rS_0 = 0, \tag{3.17}
\]
\[
\frac{dI_0}{dt} + d_1 I_0 = 0, \tag{3.18}
\]
\[
\frac{dP_0}{dt} + d_2 P_0 = 0. \tag{3.19}
\]

Equating the coefficient of \( p \), we have

\[
\frac{dS_1}{dt} - rS_1 = -\frac{r}{K}S_0^2 - \frac{r}{k}S_0 I_0 - c_1 S_0 P_0 - \delta S_0 I_0, \tag{3.20}
\]
\[
\frac{dI_1}{dt} + d_1 I_1 = \delta S_0 I_0 - c_2 I_0 P_0, \tag{3.21}
\]
\[
\frac{dP_1}{dt} + d_2 P_1 = ec_1 S_0 P_0 + ec_2 I_0) P_0, \tag{3.22}
\]

and coefficient of \( p^2 \) as

\[
\frac{dS_2}{dt} - rS_2 = -\frac{2r}{K}S_0 S_1 - \frac{r}{k}(I_0 S_1 + S_0 I_1) - c_1(P_0 S_1 + S_0 P_1) - \delta (S_1 I_0 + S_0 I_1), \tag{3.23}
\]
\[
\frac{dI_2}{dt} + d_1 I_2 = \delta(S_1 I_0 + S_0 I_1) - c_2(I_1 P_0 + I_0 P_1), \tag{3.24}
\]
\[
\frac{dP_2}{dt} + d_2 P_2 = ec_1(S_0 P_0 + S_0 P_1) + ec_2(I_1 P_0 + I_0 P_1), \tag{3.25}
\]

etc. Eqs. (3.1)-(3.3) can easily determine the components \( S_k, I_k \) and \( P_k \) and \( k \geq 0 \). So, it is possible to calculate more components in the decomposition series to enhance the approximation. Consequently, one can recursively determine every term of the series \( \Sigma_{k=0}^{\infty} S_k(t), \Sigma_{k=0}^{\infty} I_k(t) \) and \( \Sigma_{k=0}^{\infty} P_k(t) \) and hence the solutions \( S(t), I(t) \) and \( P(t) \) is readily obtained in the form of a series like

\[
S_{\text{approx.}} = S_0 e^{rt} + A_1 e^{2rt} + A_2 e^{(r-d_1)t} + A_3 e^{(r-d_2)t} + A_4 e^{3rt} + A_5 e^{(r-d_1-d_2)t} + A_6 e^{(r-2d_1)t} + A_7 e^{(r-2d_2)t} + A_8 e^{(2r-d_1)t} + A_9 e^{(2r-d_2)t}, \tag{3.26}
\]
\[
I_{\text{approx.}} = I_0 e^{-d_1t} + B_1 e^{(r-d_1)t} + B_2 e^{-(d_1+d_2)t} + B_3 e^{(-d_1-d_2)t} + B_4 e^{(r-2d_1)t} + B_5 e^{(2r-d_1)d_1) + B_6 e^{-(2d_1+d_2)t} + B_7 e^{-(d_1+2d_2)t}, \tag{3.27}
\]
\[
P_{\text{approx.}} = P_0 e^{-d_2t} + C_1 e^{(r-d_2)t} + C_2 e^{-(d_1+d_2)t} + C_3 e^{(-d_1-d_2)t} + C_4 e^{(r-2d_2)t} + C_5 e^{(2r-d_2)t} + C_6 e^{-(2d_1+d_1)t} + C_7 e^{-(d_2+2d_1)t}, \tag{3.28}
\]
where the constants \( A_i, \ i = 1 \) to \( 9 \) are given by

\[
A_1 = -\frac{S^2(0)}{K}, \quad A_2 = \frac{1}{d_1}\left(\frac{r}{K} + \delta\right)S(0)I(0), \quad A_3 = \frac{c_1}{d_2}S(0)P(0),
\]
\[
A_4 = \frac{S^3(0)}{K^2}, \quad A_5 = \frac{1}{d_1 + d_2}\left(\frac{r}{K} + \delta\right)\left(\frac{2c_2}{d_1} + \frac{c_1}{d_2}\right)S(0)I(0)P(0),
\]
\[
A_6 = \frac{rC_1}{2Kd_1d_2}I(0)S(0)P(0) + \frac{\delta}{2d_1^2}\left(\frac{r}{K} + \delta\right)I^2(0)S(0),
\]
\[
A_7 = \frac{c_1}{d_2}S(0)P^2(0), \quad A_8 = -\frac{1}{r - d_1}\left(\frac{\delta}{r} + \frac{1}{K}\right)\left(\frac{2r^2}{Kd_1} - \frac{r}{K} + \delta\right)S^2(0)I(0),
\]
\[
A_9 = -\frac{1}{r - d_2}\left(\frac{2rc_1}{Kd_2} - \frac{ec_1}{r} - \frac{c_1}{K}\right)S^2(0)P(0),
\]

the constants \( B_i, \ i = 1 \) to \( 7 \) are given by

\[
B_1 = \frac{\delta}{r}S(0)I(0), \quad B_2 = \frac{c_2}{d_1}I(0)P(0), \quad B_3 = \frac{1}{r - d_2}\left(\frac{\delta}{r} + \frac{1}{K}\right)\left(\frac{c_1}{d_2} + \frac{c_2}{d_1}\right)S(0)I(0)P(0),
\]
\[
B_4 = \frac{\delta c_1}{d_2(r - d_1)}S(0)I(0)P(0), \quad B_5 = \frac{\delta}{2r}\left(\frac{\delta}{r} - \frac{1}{K}\right)S^2(0)I(0),
\]
\[
B_6 = -\frac{ec_2}{d_2(d_1 + d_2)}I^2(0)P(0), \quad B_7 = \frac{ec_2}{2d_2d_1}P^2(0)I(0),
\]

and the constants \( C_i, \ i = 1 \) to \( 7 \) are given by

\[
C_1 = \frac{ec_1}{r}S(0)P(0), \quad C_2 = -\frac{ec_2}{d_2}I(0)P(0),
\]
\[
C_3 = \frac{1}{r - d_1}\left(\frac{ec_1c_2}{r} + \frac{ec_2d}{r} - \frac{ec_1d}{d_2}\right)S(0)I(0)P(0) + \frac{ec_2}{d_2}S(0)P^2(0),
\]
\[
C_4 = \frac{ec_2}{d_2(r - d_2)}S(0)P^2(0), \quad C_5 = \frac{ec_1}{2r}\left(\frac{ec_1}{r} - \frac{1}{K}\right)S^2(0)P(0),
\]
\[
C_6 = -\frac{ec_2}{d_2(d_1 + d_2)}P^2(0)I(0), \quad C_7 = \frac{ec_2}{2d_2d_1}I^2(0)P(0).
\]

Putting the different values of parameters \( r, k, c_1, c_2, \delta, e, d_1, d_2 \), we obtained the approximate solutions of Eqs. (3.1)-(3.3). To explain our results, we have drawn figures of \( S(t), I(t) \) and \( P(t) \) versus time for different parameters and initial condition values. From these figures, we see that the exact numerical results and the results obtained using HPM are nearly same.
4 Conclusion

This system of three coupled differential equations (3.1)–(3.3) plays an important role in biosystems. The basic goal of this paper is to study this model using HPM. The goal has been achieved by deriving solutions using few iterations only. The qualitative results of the present studies have been compared with the results obtained by numerical computation using $r = 0.1, k = 0.3, c_1 = 0.1, c_2 = 0.2, \delta = 0.1, e = 0.1, d_1 = 0.2, d_2 = 0.2$ as evident from the Fig. 1 and reveal that HPM is very effective and convenient for solving non-linear differential equations. Hope that with the help of these solutions, one can study qualitative and quantitative behaviors of realistic prey-predator interactions. The HPM introduces a significant improvement in this field. This makes the proposed scheme more powerful and gives a wider applicability.

![Figure 1: The $S(t), I(t), P(t)$ versus time for the different values of parameters $r = 0.1, k = 0.3, c_1 = 0.1, c_2 = 0.2, \delta = 0.1, e = 0.1, d_1 = 0.2, d_2 = 0.2$ and initial conditions $S_0(0) = 0.01, I_0(0) = 0.01$ and $P_0(0) = 0.01$. B-, D-, F-line represent the numerical solutions $S(t), I(t)$ and $P(t)$ respectively and C-, E-, G-line represent the approximate solutions of $S(t), I(t)$ and $P(t)$ respectively which have been obtained using HPM.](image-url)
Acknowledgement: This work is supported by University Grants Commission (UGC), Government of India (Project No. PSW-63/12-13(ERO)). Dr. S. Sarwardi is thankful to the Department of Mathematics, Aliah University for extending opportunities to perform the present work.

References

[1] A. H. Nayfeh and D. T. Mook, Nonlinear Oscillations, John Willey and Sons., New York, 1979.

[2] N. N. Bogolyubov and IU. A. Mitropolsky, Asymptotic methods in theory of nonlinear Vibrations, Moskva, 1974.

[3] V. P. Agrwal and H. Denman, Weighted linearrization technique for period approximation technique for period approximation in large amplitude Nonlinear Oscillations, J. Sound Vib. 57, 463–473, 1985.

[4] S. H. Chen, Y. K. Cheung, S. L. Lau, On Perturbation Procedure for limit cycle analysis, Int. J. Nonl. Mech., 26, 125–133, 1991.

[5] Y. K. Cheung, S. H. Chen, and S. L. Lau, A modified Lindstedt-Pioncare method for certain strong non-linear oscillations, Int. J. Nonl. Mech. 26, 367–378, 1991.

[6] G. L. Lau, New research direction in singular perturbation theory, artificial parameter approach and inverse-perturbation technique, National Conf. on 7th Modern Mathematics and Mechanics, 47–53, Shanghai 1997.

[7] J. H. He, Homotopy perturbation technique, Comp. Meth. Appl. Mech. Eng. 178, 257–262, 1999.

[8] J. H. He, A coupling method of a homotopy technique and a perturbation technique for nonlinear problems, Int. J. Nonl. Mech. 35, 37–43, 2000.

[9] M. Haque, J. Zhen, and E. Venturino, An ecoepidemiological predator-prey model with standard disease incidence, Math. Meth. Appl. Sci., 32(7), 875–898, 2009.

[10] O. Arino, J. Mikram, J. Chattopadhyay, Infection in prey population may act as a biological control in ratio-dependent predator-prey models. Nonlinearity, 17, 1101–1116, 2004.

[11] S. Sarwardi, M. Haque, and E. Venturino, A Leslie-Gower Holling-type II ecoepidemic model, J. Appl. Math. Comput., 35, 263–280, 2009.