Classification of GHZ-type, W-type and GHZ-W-type multiqubit entanglements

Lin Chen and Yi-Xin Chen
Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou 310027, China

We propose the concept of SLOCC-equivalent basis (SEB) in the multiqubit space. In particular, two special SEBs, the GHZ-type and the W-type basis are introduced. They can make up a more general family of multiqubit states, the GHZ-W-type states, which is a useful kind of entanglement for quantum teleportation and error correction. We completely characterize the property of this type of states, and mainly classify the GHZ-type states and the W-type states in a regular way, which is related to the enumerative combinatorics. Many concrete examples are given to exhibit how our method is used for the classification of these entangled states.

I. INTRODUCTION

The recent development of quantum information theory (QIT) shows the importance of entanglement. As a kind of quantum nonlocal correlation, entanglement has been used in many new tasks which are unpractical or even impossible in the classical scenarios \[1, 2\]. The study of entanglement is thus greatly inspired by these novel phenomena, and much progress has been made in quantifying and classifying entanglement in both bipartite and multipartite system \[3, 4\]. In spite of lots of effort, a better understanding of multiple entanglement is still required, which contains a much richer configuration than the bipartite setting \[5\].

In recent years, a subject of most interest in QIT is to characterize the entanglement properties of multiqubit states \[2, 3, 4\]. The main motivation is that the multiqubit entangled states play the role of quantum correlations in a quantum computer, since any unitary operation could be decomposed into a series of two-level unitary operations \[2, 10\], each of which act on one spin-1/2 particle. On the other hand, the latest experimental progress shows the multiqubit entanglement is also a kind of practically physical resource, such as the decoherence-free quantum information processing \[11\]. In particular, the characterization of two important kinds of 3-qubit states, the Greenberger-Horne-Zeilinger (GHZ) state, \[\ket{GHZ} = (\ket{000} + \ket{111})/\sqrt{2}\], and W state, \[\ket{W} = (\ket{001} + \ket{010} + \ket{100})/\sqrt{3}\] has been realized by using quantum state tomography \[12, 13, 14\]. By virtue of these facts, one can expect to generally construct and manipulate macroscopic quantum states. In this sense, the multiqubit states are considered as the most fundamental quantum resource.

The case becomes complicated when trying to classify the multiqubit entanglement under a specific constraint, e.g., local operations and classical communications (LOCC), because of the exponentially increasing number of parameters \[3\]. Under the LOCC criterion, two pure states \[\ket{\psi}\] and \[\ket{\phi}\] are interconvertible with certainty if and only if (iff) they are related by local unitary operations \[3\]. The meaning of classification is that the states belonging to the same family can be used to finish the same quantum-information task, possibly with a different probability. Although the LOCC criterion can explicitly judge the equivalence of two states, it is exceedingly inconvenient when classifying multiple entanglement \[4, 15\]. For simplicity, a more useful criterion requires that two states are regarded as equivalent by LOCC in a stochastic manner (SLOCC), i.e., related by an invertible local operator (ILO) only with a nonvanishing probability \[14, 16, 17\]. As far as the SLOCC criterion is concerned, GHZ state and W state are the only two classes of states of the genuine 3-qubit system \[14\]. These two types of states (including the generalized GHZ and W state of multipartite system) have been applied to many quantum-information process, such as the quantum secret sharing \[18\], quantum key distribution \[19\] and quantum teleportation \[20, 21, 22, 23\]. Noticeably, the scheme of many-party controlled teleportation in \[20\] employed a “quasi” GHZ state like this,

\[
\psi = \frac{1}{\sqrt{2}} \left( \ket{000} \otimes \ket{000} + \ket{111} \otimes \ket{111} \right),
\]

which can be viewed as a combination of the EPR pair of different bipartite system. This type of state more frequently appears in the error correction, e.g., the four-qubit code \[24\], which corrects one erasure by the encoding

\[
\alpha \ket{0} + \beta \ket{1} \rightarrow \frac{\alpha}{2} (\ket{00} + \ket{11}) \otimes (\ket{00} + \ket{11}) + \frac{\beta}{2} (\ket{00} - \ket{11}) \otimes (\ket{00} - \ket{11}).
\]

Due to these facts, the states composed of the GHZ states (and W states) can be regarded as an important kind of quantum resource. On the other hand, such a state has a simple configuration compared to other multiqubit states, so it is likely to give a complete investigation of it.

In the following we will characterize this kind of entanglement. Different from the existing results such as the 4-qubit entanglement \[25\], our work is devoted to the classification of a general family of pure multiqubit states under SLOCC criterion (we shall use \sim to denote the SLOCC equivalence), by constructing the so-called...
SLOCC-equivalent basis. Given the number of parties in the system, this family can be seen as a subset of corresponding multiqubit entangled classes. In particular, the family of states will be regularly classified into infinitely many subclasses, each of which contains certain parameters. It is shown that the general classification of the family requires the technique of enumerative combinatorics. Furthermore, we give out the relative ILO’s making one class into another, and one can efficiently remove the parameters to obtain a brief expression of every class. So our classification is a more explicit scheme than that in [25]. This helps completely classify the general multiqubit states. Besides, it is easy to convert our results into the explicit classification under LOCC.

In section II we recall the range criterion [27], based on which the SLOCC equivalent basis (SEB) is proposed. We explain the meaning of SEB and particularly construct two general kinds of multiqubit SEBs, the GHZ-type basis and W-type basis, which is two special cases of the rank-2 basis. We prove that the GHZ state is the only type consisting of the GHZ-type basis in each two-qubit subspace, while the W state is the only one consisting of the W-type basis in each two-qubit subspace. Subsequently, we exhibit a new family of pure multiqubit states, i.e., the GHZ-W-type state, which is composed of the GHZ-type basis and W-type basis, which is two special cases of the type consisting of the GHZ-type basis in each two-qubit subspace.

In particular, the correspondence relation greatly simplifies the traditional procedure of entanglement detection. Besides, we also give a practical criterion to distinguish whether a general multiqubit state with unknown coefficient could be fully entangled. Second, we explore the simplest form of the GHZ-W-type state by using of the GHZ-criterion and W-criterion newly established. Moreover, we calculate the ways of partitions into which an N-qubit GHZ-W-type state can be divided by the theory of combinatorics. By virtue of these techniques, we first classify the GHZ-type state under the SLOCC criterion in section III, where some useful notions are proposed, such as the relative ILO and column. These tricks can be applied to catalog the W-type state, while the general GHZ-W-type state can be viewed as the mixture of the GHZ-type and the W-type state, as shown in section IV. We describe how to explicitly write out the inequivalent classes of states by many typical examples. The conclusion is given in section V.

II. GHZ-TYPE BASIS, W-TYPE BASIS AND GHZ-W-TYPE MULTIQUBIT ENTANGLEMENT

Several leading theories of entangled bases have been established in QIT. For example, there is an intimate correlation between the unextendible product bases (UPBs) and the bound entanglement [28], while the mutually unbiased bases (MUBs) [29] has also been particularly investigated since they can be used for the the discrete Wigner function [30] and mean king problem [31]. In the present work, we apply the entangled bases to the classification of true multiqubit entangled states under SLOCC, by virtue of the range criterion [27]. Let \( \rho \) be an operation on space \( \mathcal{H} \), then the range of \( \rho \) is defined by \( R(\rho) = \rho | \Phi \rangle \), for some \( | \Phi \rangle \in \mathcal{H} \). For generality, a density operator \( \rho \) can be written as \( \rho = \sum_{i=0}^{n-1} p_i | \psi_i \rangle \langle \psi_i | \), where \( \{ | \psi_i \rangle, i = 0, 1, ..., n - 1 \} \) are a set of linearly independent vectors. It is thus easy to see that

\[
R(\rho) = \sum_{i=0}^{n-1} p_i | \psi_i \rangle \langle \psi_i | \Phi \rangle.
\]

In this case, \( R(\rho) \) can be regarded as a vector space spanned by a set of basis \( \{ | \psi_i \rangle, i = 0, 1, ..., n - 1 \} \), which is a set of entangled bases. Define the reduced density operator \( \rho_{\Phi} = tr_{A_{k+1},A_{k+2},...,A_{1N}}(\rho_{\Phi}, i_1, i_2, ..., i_k \in \{ 1, 2, ..., N \}, k \leq N - 1 \). It has been shown that the range of the reduced density operator of a multipartite pure state \( | \Psi \rangle_{A_1,A_2,...,A_N} \) essentially determines the sort of this state.

Range Criterion. Consider two multiple states \( | \Psi \rangle_{A_1,A_2,...,A_N} \) and \( | \Phi \rangle_{A_1,A_2,...,A_N} \) with the identical local rank of each party. Then there exist certain ILO’s \( V_i, i = 1, ..., N \) making \( | \Psi \rangle_{A_1,A_2,...,A_N} = V_1 \otimes V_2 \otimes \cdots \otimes V_N \otimes | \Phi \rangle_{A_1,A_2,...,A_N} \) if \( R(\rho_{\Phi}^{A_1,A_2,...,A_{N-1}}) = V_1 \otimes V_2 \otimes \cdots \otimes V_{N-1} R(\rho_{\Phi}^{A_{k+1},A_{k+2},...,A_{1N}}) \). It is easy to see that \( V_i \) exists if and only if \( \rho_{\Phi}^{A_1,A_2,...,A_{N-1}} \) is nonsingular.

The range criterion has been used for the classification of several families of true multipartite states, such as the \( 2 \times 3 \times N \) and \( 2 \times 4 \times 4 \) entanglement containing finite and infinite kinds of states respectively [27]. A prominent feature of these existing results is that most of them have distinct local ranks, which greatly helps distinguish the correspondence relation of parties since the ILO’s cannot alter the local rank [14]. For example, if two \( 2 \times 3 \times 4 \) states \( | \Psi_0 \rangle_{A,B,C} \) and \( | \Psi_1 \rangle_{A',B',C'} \) are equivalent under SLOCC, then the correspondence relation must be \( A \rightarrow A', B \rightarrow B' \) and \( C \rightarrow C' \). In other word, we only need to consider the structure of the ranges, e.g., the number of product states therein. While in the case of multiqubit system, all parties have the same local rank two. It thus becomes difficult to distinguish the correspondence relation when classifying this crucial kind of entanglement. However, we will show that the entangled bases can help explore the structure of multiqubit states.

Suppose \( \rho_1 \) and \( \rho_2 \) are two reduced density operators of states \( | \Psi_1 \rangle \) and \( | \Psi_2 \rangle \) respectively. If they have the same local ranks, then the equivalence of them under SLOCC implies \( R(\rho_1) \sim R(\rho_2) \) due to the range criterion. We thus can say that \( R(\rho_1) \) and \( R(\rho_2) \) are equivalent spaces under SLOCC, i.e., they are spanned by the SLOCC-equivalent basis (SEB). Concretely, let \( S_0 = \{ | \psi_i \rangle_{A_1,A_2,...,A_N}, i = 0, 1, ..., n - 1 \} \) be a set of bases, then its SEB must have the form \( S_1 = \{ | \phi_i \rangle_{A_1,A_2,...,A_N} = V_1 \otimes V_2 \otimes \cdots \otimes V_N \sum_{j=0}^{n-1} a_{ij} | \psi_j \rangle_{A_1,A_2,...,A_N}, i = 0, 1, ..., n - 1 \} \), the coefficient matrix \( A^{n \times n} = [a_{ij}] \) is nonsingular and
we can catalog the multipartite entanglement by finding a unique fully product state and contains one and two product states respectively. Due to the above argument, the set of GHZ class can be expressed as

\[ |\Psi_{GHZ}\rangle_{ABC} = V_A \otimes V_B (a_{00} |00\rangle + a_{01} |11\rangle)_{AB} |0\rangle_C + V_A \otimes V_B (a_{10} |00\rangle + a_{11} |11\rangle)_{AB} |1\rangle_C, \]

where \( a_{00}a_{11} - a_{01}a_{10} \neq 0 \), \( V_A \) and \( V_B \) are nonsingular. Clearly, any state SLOCC-equivalent to the GHZ state has the above form. In the same vein one can write out the expression of the W class \( |\Psi_W\rangle_{ABC} \).

The SEBs \( R_{20} \) and \( R_{21} \) are two cases of the rank-2 basis of the multiqubit space, i.e., either of them only contains two independent vectors (they are also the only two SEBs in the true 2-qubit space). It is natural to refer to the rank-k basis as those including \( k \) independent vectors. In general, any state can be seen as a synthesis consisting of the rank-k basis in each subspace of several parties, e.g., \( |GHZ\rangle_{ABC} \) is composed of \( R_{21} \) spanning \( AB, AC \) and \( BC \) spaces. Generalize the SEBs \( R_{20} \) and \( R_{21} \) to the multipartite case, i.e., define the W-type basis \( R_{20} \equiv \{ |00\rangle_N, |W\rangle_N \} \) and the GHZ-type basis \( R_{21} \equiv \{ |00\rangle_N, |11\rangle_N \} \), where

\[ |0\rangle_N \equiv |0,0,\ldots,0\rangle, \quad |1\rangle_N \equiv |1,1,\ldots,1\rangle, \]

\[ |W\rangle_N \equiv |0,0,\ldots,0\rangle + |0,1,\ldots,0\rangle + \cdots + |0,0,\ldots,1\rangle, \quad N \geq 2. \]

Evidently, \( |W\rangle_N \) is the N-partite W state and another well-known N-partite state has the form \( |GHZ\rangle_N \equiv |00\rangle_N + |11\rangle_N \). Because the space spanned by \( R_{20} \) contains a unique fully product state and \( R_{21} \) has two, they are inequivalent under SLOCC. The states containing the two SEBs can be respectively written as

\[ |\Omega_{N0}\rangle \equiv |W\rangle_N |\psi_0\rangle + |0\rangle_N |\phi_0\rangle, \]

\[ |\Omega_{N1}\rangle \equiv |00\rangle_N |\psi_1\rangle + |11\rangle_N |\phi_1\rangle, \]

where each pair of \( |\psi_i\rangle, |\phi_i\rangle \), \( i = 0, 1 \), which are linearly independent, denote the states belonging to the Hilbert space of extra parties. As both of \( R_{20} \) and \( R_{21} \) are symmetric under particle exchange, we always have

\[ |\Omega_{N0}\rangle = \langle (|00\rangle + |10\rangle)_{AB} |0\rangle_{N-2} + |00\rangle_{AB} |W\rangle_{N-2} |\psi_0\rangle + |00\rangle_{AB} |0\rangle_{N-2} |\phi_0\rangle = \langle (|01\rangle + |10\rangle)_{AB} |\psi_1\rangle + |00\rangle_{AB} |\phi_1\rangle \sim |\Omega_{N0}\rangle, \]

where \( A, B \) are randomly chosen from the first \( N \) parties. Similarly, \( |\Omega_{N1}\rangle \) is a special case of \( |\Omega_{N2}\rangle \). Notice that both of the states \( |W\rangle_N \) and \( |GHZ\rangle_N \) consist of the rank-2 basis \( (R_{20} \text{ and } R_{21} \text{ respectively}) \) in each two-qubit subspace. We consider the converse of this observation. That is, do there exist other multiqubit states composed of \( R_{20} \) or \( R_{21} \) in each two-qubit subspace, besides the \( |GHZ\rangle_N \) and \( |W\rangle_N \)? The answer is phrased as follows.

**Theorem 1.** Suppose the multiqubit state \( |\Psi\rangle_{A_1A_2\ldots A_N} \) consists of \( R_{20} \) or \( R_{21} \) in each two-qubit subspace, then \( |\Psi\rangle_{A_1A_2\ldots A_N} \) is equivalent to either \( |GHZ\rangle_N \) or \( |W\rangle_N \) under SLOCC.

**Proof.** Two existing ILO’s are useful fortheco. They are defined by \( O_1^2((|\phi\rangle, |\alpha\rangle) : |\phi\rangle_A \rightarrow |\alpha|\phi\rangle_A \text{ and } O_2^2((|\phi\rangle, |\psi\rangle)) : |\phi\rangle_A \rightarrow |\phi\rangle_A + |\psi\rangle_A \text{ respectively}. Besides, we will omit the specific form of ILO’s in the deduction if unnecessary. First consider the separable state, which is the direct product of at least two states of different systems, i.e., \( |\Psi\rangle_{A_1A_2\ldots A_N} = \bigotimes \prod \{ |\Psi_i\rangle \} \), every \( |\Psi_i\rangle \) truly entangled. Choose the first two states and we can always write \( |\Psi\rangle_{AB}\bigotimes |\Psi_1\rangle_{CD} = (|0\rangle_0 |\psi_0\rangle + |1\rangle_1 |\psi_1\rangle)_{AB} (|0\rangle_0 |\phi_0\rangle + |1\rangle_1 |\phi_1\rangle)_{CD} \), where \( (B \text{ and } D) \) can be composite system. If \( |\psi_0\rangle \text{ and } |\psi_1\rangle \text{, } |\phi_0\rangle \text{ and } |\phi_1\rangle \text{ are linearly independent respectively, then } R(\rho_{ABC}) \text{ is spanned by the rank-4 SEB. For the case of linear dependence of one pair, e.g., } |\psi_0\rangle \text{ and } |\psi_1\rangle \text{, we have} \)

\[ R(\rho_{ABC}) \sim a |00\rangle + b |01\rangle. \text{ Hence this separable state is not fully spanned by } R_{20} \text{ or } R_{21}. \text{ Next, let us move to the case of true entanglement, which has two subcases.} \]

(i) The states containing \( R_{20} \), which must be in the form \( |\Omega_{N0}\rangle = (|01\rangle + |10\rangle) |\psi_0\rangle + |00\rangle |\phi_0\rangle \). We show that it must also be \( |\Omega_{N0}\rangle \sim (|W\rangle_N |\psi_0\rangle + |00\rangle |\phi_0\rangle \) by induction. Evidently, \( |\Omega_{N0}\rangle \) is the case of \( N = 2 \) and we can write

\[ |\Omega_{N0}\rangle = |W\rangle_N (|0\rangle_B |\psi_0\rangle + |1\rangle_B |\psi_0\rangle) + (|0\rangle_N |00\rangle_B |\psi_0\rangle + |00\rangle_N |\psi_1\rangle) \]

\[ = (|00\rangle_{AB} (|W\rangle_{N-1} |\psi_0\rangle + |00\rangle_{N-1} |\psi_0\rangle) + (|01\rangle_{AB} (|W\rangle_{N-1} |\psi_1\rangle + |00\rangle_{N-1} |\psi_1\rangle) + \gamma |0\rangle_{N-1} |\psi_0\rangle + \delta |0\rangle_{N-1} |\psi_1\rangle, \]

where \( A \) is one of the first \( N \) particles. To keep \( AB \) space is spanned by \( R_{20} \text{ or } R_{21}, \) there must be

\[ \alpha (|W\rangle_{N-1} |\psi_0\rangle + |0\rangle_{N-1} |\psi_0\rangle) + \beta (|W\rangle_{N-1} |\psi_0\rangle + |0\rangle_{N-1} |\psi_1\rangle) + \gamma |0\rangle_{N-1} |\psi_0\rangle + \delta |0\rangle_{N-1} |\psi_1\rangle = 0, \]

or more explicitly

\[ \alpha |\psi_0\rangle + \beta |\psi_1\rangle = 0, \]

and

\[ \alpha |\psi_0\rangle + \beta |\psi_1\rangle + \gamma |\psi_0\rangle + \delta |\psi_1\rangle = 0. \]

Observe the two equations. If \( |\psi_0\rangle \text{ and } |\psi_1\rangle \) are linearly independent, then it has to be \( \alpha = \beta = \gamma = \delta = 0, \)
means $AB$ space is spanned by rank-4 basis. It follows that $|\psi_0\rangle$ and $|\psi_1\rangle$ must be linearly dependent. By some ILO’s $O_B^1$ and $O_B^2$, we obtain

$$\Omega_{N0} \sim |W\rangle_N |0\rangle_B |\psi_0\rangle + |0\rangle_N |0\rangle_B |\psi_0\rangle + |1\rangle_B |\psi_1\rangle \rangle.$$  

(13)

Analyzing this equation in the same vein gives rise to $|\psi_1\rangle = k |\psi_0\rangle$, where $k$ is some universal factor. Afterwards, We perform an operation $O_B^B(1, 1/k)$ and have

$$\Omega_{N0} \sim |W\rangle_N |0\rangle_B |\psi_0\rangle + |0\rangle_N |0\rangle_B |\psi_0\rangle + |1\rangle_B |\psi_0\rangle = |W\rangle_{N+1} |\psi_0\rangle + |0\rangle_{N+1} |\psi_1\rangle = \Omega_{N+1, 0}.$$  

(14)

Therefore, the states containing $R_N0$ must also be in the form of $\Omega_{N+1, 0}$. Continue the above procedure and at last we arrive at the case in which $|\psi_0\rangle$ is local. By some ILO we obtain $\Omega_{20} \sim |W\rangle_{N-1} |0\rangle + |0\rangle_{N-1} |1\rangle = |W\rangle_N$. This result asserts that the N-partite W state is the unique one composed of $R_{20}$ in each two-qubit space of N particle system.

(ii) The states containing $R_{21}$, which must be in the form of $\Omega_{21} = |00\rangle |\psi_0\rangle + |11\rangle |\psi_1\rangle$. One can do it following the technique in (i). Write out

$$\Omega_{N1} = |0\rangle_N |0\rangle_B |\psi_0\rangle + |1\rangle_B |\psi_0\rangle + |1\rangle_B |\psi_1\rangle,$$

$$= |00\rangle_{AB} |0\rangle_{N-1} |\psi_0\rangle + |01\rangle_{AB} |0\rangle_{N-1} |\psi_0\rangle + |10\rangle_{AB} |1\rangle_N |\psi_0\rangle + |11\rangle_{AB} |1\rangle_N |\psi_1\rangle.$$  

and thus

$$\alpha |\psi_0\rangle + \beta |\psi_1\rangle = 0, \quad \gamma |\psi_1\rangle + \delta |\psi_1\rangle = 0.$$  

(16) (17)

By the reason similar to that in (i), $|\psi_1\rangle$ disappears and thus $|\psi_0\rangle, |\psi_1\rangle$ are linearly dependent. After an operation $O_B^2$, one finds that

$$\Omega_{21} \sim |0\rangle_{N+1} |\psi_0\rangle + |1\rangle_{N+1} |\psi_1\rangle.$$  

(18)

Again we iterate this procedure and finally obtain $\Omega_{21} \sim |GHZ\rangle_N$, so the N-partite GHZ state is the one unique composed of $R_{20}$ in each two-qubit subspace of N particle system. This completes the proof.

Theorem 1 says that for the N-partite system, $|GHZ\rangle_N$ and $|W\rangle_N$ are the only two types consisting of rank-2 basis in each two-qubit subspace under SLOCC criterion. Conversely, this also implies that a multiquit state containing both $R_{20}$ and $R_{21}$ must contain higher rank basis. For example, consider the following state,

$$|\Psi\rangle = |000\rangle_{ABC} |01\rangle_{DE} + |11\rangle_{ABC} |00\rangle_{DE}. \quad (19)$$

Here, $R^A_{\psi}^{ABC}$ and $R^D_{\psi}^{ABC}$ are spanned by $R_{20}$ and $R_{21}$ respectively. One can briefly check that $R^C_{\psi}^{CD}$ is spanned by the rank-3 basis, so $|\Psi\rangle$ equals neither the GHZ state nor the W state under SLOCC. Nonetheless, it remains a simple type of multiquit entanglement for it consists of the rank-2 basis. As mentioned above, a general multiquit state can be seen as the combination of the rank-k basis in each subspace of several parties. Intuitively, the states consisting of the rank-2 basis have a simpler structure compared to those composed of the rank-3 basis, etc, and it is always a wise decision to first treat the easier cases of a difficult problem. As the GHZ-type and W-type basis are two special kinds of rank-2 SEBs, we will particularly characterize a general family of multiquit states consisting of these two types of SEBs. Differing from other existing results such as the 3-qubit and 4-qubit system, our classification describes a subclass belonging to any multiquit entanglement, so it helps get insight into the structure of a general multiquit state. Let this family of states be $\Omega_{GHZ-W^{k,n-k}, 0 \leq k \leq n}$, which is defined in the following form

$$|0\rangle_{N0} |0\rangle_N |1\rangle_{N-1} |0\rangle_{N-2} |0\rangle_{N-3} \cdots |0\rangle_{N-k-2} |0\rangle_{N-k-1} + |\psi_0\rangle \rangle,$$

$$|a_0 |0\rangle_{Nk} |0\rangle_{Nk+1} |\cdots |0\rangle_{Nk+1} |0\rangle_{N-k} + |a_1 |0\rangle_{Nk} |0\rangle_{N-k-1} |0\rangle_{N-k} + |a_2 |0\rangle_{Nk} |0\rangle_{N-k-2} |0\rangle_{N-k} + \cdots + |a_{2^{n-k-1}} |0\rangle_{Nk} |0\rangle_{N-k} |0\rangle_{N-k} |0\rangle_{N-k-1} + |0\rangle_{N0} |0\rangle_{N1} |0\rangle_{N2} \cdots |0\rangle_{Nk} |1\rangle_{N-k} + |a_{2^{n-k-1}} |0\rangle_{Nk} |0\rangle_{N-k+1} |0\rangle_{N-k} + |a_{2^{n-k-1}} |0\rangle_{Nk} |0\rangle_{N-k+1} |0\rangle_{N-k} + \cdots + |a_{2^{n-k-1}} |0\rangle_{Nk} |0\rangle_{N-k} |0\rangle_{N-k} |0\rangle_{N-k-1} + |a_{2^{n-k-1}} |0\rangle_{Nk} |0\rangle_{N-k} |0\rangle_{N-k} |0\rangle_{N-k-1} + \cdots.$$  

(20)

That is, $|\Omega_{GHZ-W^{k,n-k}}\rangle_{k,n-k}$ represents the most universal form composed of the GHZ-type basis and the W-type basis, so we call it the GHZ-W-type state. In particular, when $k = n$ the state is completely composed of the GHZ-type basis, and define $\Omega_{GHZ} = |\Omega_{GHZ-W^{k,n}}\rangle_{0,n}$, which we call the GHZ-type state. Similarly, the W-type state is defined as $|W\rangle_n = |\Omega_{GHZ-W^{k,n}}\rangle_{0,n}$. Although the GHZ-type and W-type states are merely two special cases of $|\Omega_{GHZ-W^{k,n-k}}\rangle_{k,n-k}$, we can characterize many entanglement properties of $|\Omega_{GHZ-W^{k,n-k}}\rangle_{k,n-k}$ by investigating.
these two representatives. In what follows we analyze the property of these three types of states, including detecting the full entanglement and the condition for the simplest form of the GHZ-W-type state. Our interest mainly focuses on the classification of this multiqubit entanglement. In particular, we will classify the GHZ-type and W-type state in a regular way, which proves to be related to the theory of enumerative combinatorics [20]. We thus have converted the issue of classification of this family of multiqubit state into a combinatorial problem.

A. Detecting the entanglement of the GHZ-W-type state

Let us first find out the condition on which the state $|\Omega_{GHZ-W}\rangle_{k,n-k}$ is fully entangled, since the separable state can always be expressed as a direct product of several truly entangled states. It is easy to see that the structure of $|\Omega_{GHZ-W}\rangle_{k,n-k}$ resembles that of the general pure multiqubit state,

$$|\Omega\rangle_n \equiv b_0 |0,0,...,0,0\rangle + b_1 |0,0,...,1,0\rangle + b_2 |0,0,...,1,1\rangle + \cdots + b_{2^{n-1}} |1,1,...,1,1\rangle.$$

However, giving an exclusive condition on which one can judge whether $|\Omega\rangle_n$ is fully entangled is difficult. Theoretically, one can do it by checking the rank of each k-partite reduced matrix operator, $1 \leq k \leq [n/2]$; that is, $|\Omega\rangle_n$ is fully entangled iff each rank is larger than 1. In this case, one has to check $2^{n-1} - 1$ matrices in all. It actually concerns the separability problem [37] and remains a puzzling aspect of QIT. Furthermore, usually the state $|\Omega_{GHZ-W}\rangle_{k,n-k}$ is not the locally symmetric extension (LSE) of some $|\Omega\rangle_n$ [37], for both of them are pure. Nevertheless, this "pseudo" parallelism still helps detect the entanglement of the GHZ-W-type state, an interesting relation can be constructed.

**Theorem 2.** Construct the corresponding state $|\Omega\rangle_n$ of $|\Omega_{GHZ-W}\rangle_{k,n-k}$, by "concentrating" the $i'th$ group consisting of $N_i$ parties in $|\Omega_{GHZ-W}\rangle_{k,n-k}$ to the $i'th$ party in $|\Omega\rangle_n$, i.e., $|0\rangle_{N_i} \rightarrow |0_i\rangle$ or $|1\rangle_{N_i} \rightarrow |1_i\rangle$. Then $|\Omega_{GHZ-W}\rangle_{k,n-k}$ is fully entangled iff $|\Omega\rangle_n$ is fully entangled.

**Proof.** The necessity could be lightly obtained by the parallelism between $|\Omega_{GHZ-W}\rangle_{k,n-k}$ and $|\Omega\rangle_n$. For the case of sufficiency, we can prove the assertion by reduction to absurdity. Regard $|\Omega_{GHZ-W}\rangle_{k,n-k}$ as a bipartite state by random partition, and the local rank of this bipartite state must always be larger than 1 iff it is fully entangled. Suppose the bipartition happens in the $i'th$

$$|\Omega_{GHZ-W}\rangle_{k,n-k} = (|0\rangle\langle 0| + |1\rangle\langle 1|)_{N_i}$$

$$\otimes (|0\rangle\langle 0|_{N_{n-m} - l} + |1\rangle\langle 1|_{N_{n-m} - l})$$

$$= (|0\rangle\langle 0|_{N_{n-m} - l} |\phi_0\rangle + |1\rangle\langle 1|_{N_{n-m} - l} |\phi_1\rangle)$$

$$= |0\rangle\langle 0|_{N_{n-m} - l} |\phi_0\rangle + |1\rangle\langle 1|_{N_{n-m} - l} |\phi_1\rangle,$$

where $0 < m < N_i$.

(22) If $|\psi_0\rangle$ and $|\psi_1\rangle$ are linearly dependent, the space of $i'th$ group is spanned by the basis $\{ |\alpha\rangle_{m} + |\beta\rangle_{m} \}$, which evidently cannot be $R_{N_i,0}$ or $R_{N_i,1}$, so they are mutually independent. Similarly, $|\phi_0\rangle$ and $|\phi_1\rangle$ are also linearly independent and thus the local rank of $i'th$ group is 4, which is inconsistent with the assumption that the corresponding state $|\Omega\rangle_n$ is a truly entangled multiqubit state. For the case in which bipartition happens in one group of $R_{N_0}$, the above argument entirely applies. Consequently, $|\Omega_{GHZ-W}\rangle_{k,n-k}$ is fully entangled iff its corresponding state $|\Omega\rangle_n$ is fully entangled.

Thus we can judge whether $|\Omega_{GHZ-W}\rangle_{k,n-k}$ is fully entangled by only analyzing its corresponding state $|\Omega\rangle_n$. For example, the following state is always fully entangled $|\Omega\rangle_n = |000\rangle |000\rangle + |001\rangle |000\rangle + |010\rangle |000\rangle + |100\rangle |000\rangle$, whose corresponding state is $|001\rangle + |010\rangle + |100\rangle$.

In this sense, theorem 2 efficiently reduces the calculation required in entanglement detection. As mentioned above, one has to calculate $2^{n-1} - 1$ matrices to judge whether a pure $(\sum_{i=0}^{n-1} N_i)$-qubit state is fully entangled. However, if it can be structurally adjusted to fit the form of GHZ-W-type state, then it suffices to calculate $2^{n-1} - 1$ matrices in all. To make for a better effect in the following text, we propose an important property of pure multiqubit states. Though detecting the multiqubit entanglement is generally difficult, it is shown to be much easier to determine whether a multiqubit state with undermined coefficients could be fully entangled by choosing proper coefficients. This helps characterize the multiqubit entanglement.

**Theorem 3.** Suppose $\{i_0,i_1,...,i_{n-1}\}$, $i_k = 0,1,k = 0,1,...,n-1$ is the set of basis vectors on the space $H_{p_0,p_1,...,p_{n-1}}$ and choose $g \geq 2$ vectors $|\phi_j\rangle = |\phi_{i_0},i_1,...,i_{n-1}\rangle$, $j = 0,1,...,k-1$ from this set. Then there exist infinitely many sets of coefficients $\alpha_0,\alpha_1,...,\alpha_{k-1}$, such that $\sum_{i=0}^{k-1} \alpha_i |\phi_i\rangle$ is fully entangled iff there exist at least two inequivalent states in any set $\{ |i_s\rangle, j = 0,1,...,k-1, s \in [0,n-1]. \}$ That is, $| \phi_j\rangle's$, $j = 0,1,...,k-1$ cannot simultaneously be $|0\rangle$ or $|1\rangle$.

**Proof.** We only need to verify the sufficiency. Let $|\Psi\rangle_{p_0,p_1,...,p_{n-1}} = \sum_{i=0}^{k-1} \alpha_i |\phi_i\rangle + x |\phi_{k-1}\rangle$, $x = \alpha_{k-1}$. Assume that $x,\alpha_i > 0, i = 0,1,...,k-2$, since it suffices to
prove the statement in this case. For the first step, consider the 1-party reduced matrix operator of \( P_0 \) (possibly after performing a permutation matrix),

\[
\rho P_0 = (b|0\rangle + c|1\rangle)(b|0\rangle + c|1\rangle) + \sum_i (c_i|0\rangle + d_i|1\rangle)(c_i|0\rangle + d_i|1\rangle),
\]

and let \( A = \sum_i c^2_i, B = \sum_i c_id_i, C = \sum_i d^2_i \), then

\[
\rho P_0 = \left( A + b^2B + bx + bx^2 \right), \det \rho P_0 = Ax^2 - 2Bbx + AC + b^2C - B^2.
\]

Clearly, \( P_0 \) is separate from the other parties iff \( \det \rho P_0 = 0 \). Let us consider the condition making \( \det \rho P_0 = 0 \), i.e., no matter what \( x \) equals. First, it requires \( A = 0 \), which means \( c_i = 0 \) for each \( i \). Thus, \( B = 0 \) and \( b > 0 \) for \( x > 0 \) exist at least one vector \( |0\rangle P_0 \), which also implies \( C > 0 \). In this case, \( \det \rho P_0 = b^2C > 0 \). Consequently, it is not possible to realize \( \rho P_0 = 0 \), so there is only a finite number of \( x \)'s (at most two here) making \( \det \rho P_0 = 0 \). Construct a set \( S_0 \), such that \( x \in S_0 \) making \( \det \rho P_0 = 0 \). Similarly, we can construct the sets \( S_1, S_2, \ldots, S_{n-1} \) for the parties \( P_1, P_2, \ldots, P_{n-1} \) respectively. Subsequently for the second step, let us move to the case of 2-party reduced matrix operator of \( P_0 P_1 \),

\[
\rho P_0 P_1 = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix},
\]

which could be rewritten as

\[
\rho P_0 P_1 = \begin{pmatrix} a^2 & ab & ac \\ ba & b^2 & bc \\ ca & cb & c^2 \end{pmatrix} + \begin{pmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{pmatrix},
\]

Consider the principal submatrices of \( \rho P_0 P_1 \),

\[
M_0 = \begin{pmatrix} A_{00} & a^2 \\ A_{30} & ax \end{pmatrix}, \quad M_1 = \begin{pmatrix} A_{11} & b^2 \\ A_{31} & bx \end{pmatrix}, \quad M_2 = \begin{pmatrix} A_{22} & c^2 \\ A_{32} & cx \end{pmatrix}.
\]

Notice that \( P_0, P_1 \) are always separate from the other parties iff \( \text{rank}[\rho P_0 P_1] = 1 \), which requires \( \det M_i = 0, i = 0, 1, 2 \). Applying the result in the first step to these matrices leads to \( A_{00} = A_{11} = A_{22} = 0 \), which implies there is at least one nonzero number in \( a, b, c \), as well as \( A_{33} > 0 \) based on the reason similar to that in the first step. Without loss of generality, choose \( a > 0 \) and \( \det M_0 = a^2A_{33} > 0 \). So it is also impossible to make \( \text{rank}[\rho P_0 P_1] = 1 \). Again we construct a finite set \( S_{n0} \), such that \( x \in S_{n0} \) making \( \rho P_0 P_1 \) = 1. Similarly, we can construct the sets \( S_{n2}, S_{12}, S_{13}, \ldots, S_{n-2,n-1} \) for the subgroups \( P_0 P_2, P_1 P_2, P_1 P_3, \ldots, P_{n-2}P_{n-1} \) respectively. Continue until all reduced matrix operators have been checked and exhibit the set

\[
S = S_0 \sqcup S_1 \sqcup \cdots \sqcup S_{n-1} \sqcup S_{n0} \sqcup S_{n2} \sqcup S_{12} \sqcup \cdots \sqcup S_{n-2,n-1} \sqcup S_{n0}, \cdots, n-3, n-2 \sqcup S_{0,1,2,\ldots,n-3,2} \sqcup S_{0,1,2,\ldots,n-3,1} \sqcup \cdots \sqcup S_{1,2,3,\ldots,n-3,2,n-1}.
\]

which contains a finite amount of (complex) numbers. Choose \( x \notin S \), then the set of numbers \( \{a_0, a_1, \ldots, a_{k-1}\} \) makes \( \sum_{i=0}^{k-1} a_i |\phi_i\rangle \) fully entangled. Since we can randomly choose \( a_0, a_1, \ldots, a_{k-2} > 0 \), there exist infinitely many sets of numbers satisfying the statement. This completes the proof.

This result gives a useful criterion for determining whether a multiqubit state can be fully entangled, especially in the cases containing a great number of qubits. By virtue of theorem 3, we say that a general multiqubit state can always be fully entangled and we have to classify it, if no range of its single reduced matrix operator merely contains the same computational basis \( |0\rangle \) or \( |1\rangle \). Without loss of generality, we suppose that the states in the following text always satisfies this condition and are supposed to be fully entangled, e.g., \(|\Psi\rangle = \alpha |00\rangle (|01\rangle + |10\rangle) + \beta |11\rangle |00\rangle \).

**B. Simplest state of GHZ-W-type state**

In this subsection, we investigate the condition under which the GHZ-W-type state will be of the simplest condition defined below. Like the condition of full entanglement in section A, the simplest form is also necessary when classifying the GHZ-W-type multiqubit entanglement. Let

\[
|\Omega_{GHZ-W}\rangle_{k,n-k} = \sum_i c_i |a_{i,0}\rangle_{N_0} |a_{i,1}\rangle_{N_1} \cdots |a_{i,k-1}\rangle_{N_{k-1}} |a_{i,k}\rangle_{N_k} |a_{i,k+1}\rangle_{N_{k+1}} \cdots |a_{i,n-1}\rangle_{N_{n-1}},
\]

where \( a_{i,j} = 0,1 \) for \( j = 0, \ldots, k-1 \) and \( a_{i,j} = 0, W \) for \( j = k, \ldots, n-1 \). Due to the definition of GHZ-W-type state, the space of the \( i \)th group consisting of \( N_i \) particles is spanned by \( R_{N_i,0} \) or \( R_{N_i,1} \). Interestingly, there may exist several groups containing \( N_i \) particles respectively, which will be absorbed into a larger group containing \( \sum_i N_i \) particles and the space of this new group is spanned by \( R_{\sum_i N_i,0} \) or \( R_{\sum_i N_i,1} \). So we can collect these groups respectively to get a more brief expression of the state \(|\Omega_{GHZ-W}\rangle_{k,n-k}\). Continue this procedure and finally we can get

\[
|\Omega_{GHZ-W}\rangle_{k',n'-k'} = \sum_{i'} c_{i'} |a_{i',0}\rangle_{N_0} |a_{i',1}\rangle_{N_1} \cdots |a_{i',k'-1}\rangle_{N_{k'}} |a_{i',k'+1}\rangle_{N_{k'+1}} \cdots |a_{i',n'-1}\rangle_{N_{n'-1}},
\]

which is the simplest form of the GHZ-W-type state.
which is a succinct expression of $|\Omega_{GHZ-W}\rangle_{k',n'-k'}$ containing $n' \leq n$ groups. In this state, the space of the $i$'th group consisting of $N'_i$ particles is spanned by $R_{N_0}$ or $R_{N_1}$, and no two groups will be absorbed into a larger group whose space is still spanned by the GHZ-type or W-type basis. We then call the above expression the simplest form of $|\Omega_{GHZ-W}\rangle_{k,n-k}$ (or this expression is simplest). For instance, the following state is in the simplest form,

$$|\Psi\rangle = |0\rangle_{N_0} |0\rangle_{N_1} + |1\rangle_{N_0} |W\rangle_{N_1},$$ (29)

where the two groups here cannot form a larger group whose space is spanned by the GHZ-type or W-type basis. However, it is not easy to check whether a general GHZ-W-type state is under the simplest form, especially for those consisting of many groups. Consider a 3-group W-type state

$$|\Psi\rangle = |0\rangle_{N_0} |0\rangle_{N_1} |0\rangle_{N_2} + \alpha |W\rangle_{N_0} |W\rangle_{N_1} |0\rangle_{N_2} + \beta |0\rangle_{N_0} |W\rangle_{N_1} |W\rangle_{N_2}, \alpha, \beta \neq 0.$$ (30)

By theorem 2 and 3, it is fully entangled. Notice the coefficients $\alpha$ and $\beta$ can be moved away by the ILO's $\prod_{i=0}^{N_0-1} O_i(1, 1/\alpha)$ and $\prod_{i=0}^{N_2-1} O_i(1, 1/\beta)$ on the first $N_0$ and last $N_2$ parties respectively. Now, one can find out that the 0'th and 2'th groups can form a larger group whose space is spanned by the W-type basis,

$$|\Psi\rangle \sim |0\rangle_{N_1} |0\rangle_{N_0} |0\rangle_{N_2} + |W\rangle_{N_1} |W\rangle_{N_0} |0\rangle_{N_2} + |0\rangle_{N_0} |W\rangle_{N_1} |W\rangle_{N_0} + N_2.$$ (31)

Hence, this expression is actually the simplest form of $|\Psi\rangle$. Similar mergers exist in the case of GHZ-type state. For example, we prove theorem 2 by disproval where the bipartition makes a group of partners whose space is spanned by the GHZ-type basis into two groups, while either of them is in the space spanned by the GHZ-type basis.

To characterize the entangled family $|\Omega_{GHZ-W}\rangle_{k,n-k}$, we must find out the simplest form of any state of it. The reason derives from that the local rank is invariant under the ILO's. By the reset of the positions of the particles in the system, one can merge several groups into a new group consisting of more particles whose space is spanned by the same SEB, i.e., the local rank of the new group is identical to that of either of the subgroups in it. In this case, it is hard to decide the correspondence of the initial particles and the final ones in the system, just similar to that any local rank is 2 in the multiqubit state. Consequently, a theoretical depiction is required for finding out the simplest form of $|\Omega_{GHZ-W}\rangle_{k,n-k}$. Rewrite it as follows,

$$|\Omega_{GHZ-W}\rangle_{k,n-k} = |0\rangle_{N_0} |0\rangle_{N_1} |\psi_0\rangle + |0\rangle_{N_0} |1\rangle_{N_1} |\psi_1\rangle + |1\rangle_{N_0} |0\rangle_{N_1} |\psi_2\rangle + |1\rangle_{N_0} |1\rangle_{N_1} |\psi_3\rangle = |00\rangle_{AB} |0\rangle_{N_0-1} |0\rangle_{N_1-1} |\psi_0\rangle + |01\rangle_{AB} |0\rangle_{N_0-1} |1\rangle_{N_1-1} |\psi_1\rangle + |10\rangle_{AB} |1\rangle_{N_0-1} |0\rangle_{N_1-1} |\psi_2\rangle + |11\rangle_{AB} |1\rangle_{N_0-1} |1\rangle_{N_1-1} |\psi_3\rangle.$$ (32)

notice $N_0, N_1 \geq 2$ here. In the light of theorem 1, a space is spanned by $R_{N_0}$ or $R_{N_1}$ if any two-qubit subspace of it is spanned by $R_{20}$ or $R_{21}$. This implies that if the 0'th and 1'th group can be merged into one group, the $AB$ space must be spanned by $R_{21}$. However, the four nonvanishing vectors of system $C$ form a set of orthogonal bases. Thus, the two groups make up a new group iff $|\psi_0\rangle = |\psi_3\rangle = 0$, or $|\psi_1\rangle = |\psi_2\rangle = 0$.

On the other hand, the GHZ-W-type state can also be expressed in terms of the W-type basis,

$$|\Omega_{GHZ-W}\rangle_{k,n-k} = |0\rangle_{N_0} |0\rangle_{N_1} |\psi_0\rangle + |0\rangle_{N_0} |W\rangle_{N_1} |\psi_1\rangle + |W\rangle_{N_0} |0\rangle_{N_1} |\psi_2\rangle + |W\rangle_{N_0} |W\rangle_{N_1} |\psi_3\rangle = |00\rangle_{AB} |0\rangle_{N_0-1} |0\rangle_{N_1-1} |\psi_0\rangle + |01\rangle_{AB} |0\rangle_{N_0-1} |1\rangle_{N_1-1} |\psi_1\rangle + |10\rangle_{AB} |1\rangle_{N_0-1} |0\rangle_{N_1-1} |\psi_2\rangle + |11\rangle_{AB} |1\rangle_{N_0-1} |1\rangle_{N_1-1} |\psi_3\rangle.$$ (33)

Due to theorem 1, the $AB$ space should be spanned by $R_{20}$. If $|\psi_3\rangle \neq 0$, one can easily check that the four vectors of system $C$ are linearly independent. Since the rank of the $AB$ system should be equal to that of the $C$ system, there must be $|\psi_3\rangle = 0$, i.e.,

$$\alpha (|0\rangle_{N_0-1} |0\rangle_{N_1-1} |\psi_0\rangle + |0\rangle_{N_0-1} |W\rangle_{N_1-1} |\psi_1\rangle + |W\rangle_{N_0-1} |0\rangle_{N_1-1} |\psi_2\rangle) + \beta (|0\rangle_{N_0-1} |0\rangle_{N_1-1} |\psi_1\rangle + \gamma |0\rangle_{N_0-1} |0\rangle_{N_1-1} |\psi_2\rangle = 0,$$ (34)

which leads to $\alpha = 0, \beta |\psi_1\rangle = -\gamma |\psi_2\rangle \neq 0$. Combined with the result of GHZ-type state, we have
Corollary 1. Given a fully entangled state $|\Omega_{GHZ-W}^k\rangle_{k,n-k}$, it is in the simplest form if $|\psi_0\rangle = |\psi_1\rangle = |\psi_2\rangle = 0$ or $|\psi_1\rangle = |\psi_2\rangle = 0$ when $|\Omega_{GHZ-W}^k\rangle_{k,n-k}$ is the state

$$
|\Psi\rangle = a_0 |0\rangle_{N_0} |0\rangle_{N_1} |0\rangle_{N_2} |0\rangle_{N_3} |0\rangle_{N_4} |0\rangle_{N_5} + a_1 |0\rangle_{N_0} |0\rangle_{N_1} |1\rangle_{N_2} |W_{N_1}\rangle |0\rangle_{N_4} |0\rangle_{N_5} + a_2 |0\rangle_{N_0} |0\rangle_{N_1} |0\rangle_{N_2} |W_{N_2}\rangle |0\rangle_{N_4} |N_5\rangle + a_3 |1\rangle_{N_0} |0\rangle_{N_1} |0\rangle_{N_2} |0\rangle_{N_3} |W_{N_3}\rangle |N_4\rangle |W_{N_5}\rangle,
$$

(35)

which is fully entangled by theorem 3. Notice the state is partly symmetric, i.e., unchanged under the exchange of groups 0 and 1, 4 and 5 simultaneously, etc. We analyze the part of GHZ-type basis and W-type basis respectively. Since there always exist three kind of bases in each group's space, i.e., $|0\rangle_{N_i}, |0\rangle_{N_i} |1\rangle_{N_i}, |1\rangle_{N_i} |0\rangle_{N_i}$, so the part of GHZ-type basis is under the simplest form. While for the case of another part, observe that, e.g., the states $|0\rangle_{N_2} |N_2\rangle |0\rangle_{N_3} |0\rangle_{N_4} |0\rangle_{N_5}$ and $|1\rangle_{N_2} |0\rangle_{N_3} |0\rangle_{N_4} |0\rangle_{N_5}$ are orthogonal, which means the part of W-type basis is also under the simplest form. Conclusively, the state $|\Psi\rangle$ is under the simplest form.

In general, an arbitrary GHZ-W-type state can always be turned into its simplest form, which is an effective constitution at our disposal. We then catalog $|\Omega_{GHZ-W}^k\rangle_{k,n-k}$ under its simplest form. First, two states $|\Omega_{GHZ-W}^A\rangle_{k_0,n_0-k_0}$ and $|\Omega_{GHZ-W}^B\rangle_{k_1,n_1-k_1}$ can be classified into the same sort if $n_0 = n_1 = n$, $k_0 = k_1$, $N_i^A = N_i^B, i = 0, 1, \ldots, n - 1$, up to some permutation of the subgroups in the state. This is a necessary step in the process of classification, and we will concretely describe how to write out the inequivalent classes of entanglement under SLOCC with definite $k_0, n_0$ in the next section. Second, from the above assertion, it is natural to ask how many ways of partition in which an N-qubit GHZ-W-type state can be divided. To exemplify it, a 6-qubit GHZ-type state can be in the following bipartite forms,

$$
|\Omega_{GHZ}^A\rangle_6 = \psi_0 |0\rangle_{N_0} |0\rangle_{N_1} |0\rangle_{N_2} |1\rangle_{N_3} |0\rangle_{N_4} |1\rangle_{N_5} + \psi_1 |0\rangle_{N_0} |1\rangle_{N_1} |0\rangle_{N_2} |1\rangle_{N_3} |0\rangle_{N_4} |1\rangle_{N_5},
$$

(36)

where $6 = 2 + 4 = 3 + 3$. However, one may notice there exists the third way of partition for $6 = 2 + 2 + 2$, i.e.,

$$
|\Omega_{GHZ}^C\rangle_6 = c_0 |0\rangle_{N_0} |0\rangle_{N_1} |0\rangle_{N_2} |0\rangle_{N_3} |1\rangle_{N_4} |1\rangle_{N_5},
$$

(37)

The three states are in the simplest form by GHZ-criterion. So there are three main classes of states in $|\Omega_{GHZ}\rangle_6$, though either of which still requires a more detailed analysis. For a natural number $N \geq 4$, first we divide it into two parts $M_{GHZ}$ and $M_W$, representing the numbers of GHZ-type and W-type basis in the state respectively. Mathematically, it is a problem of ordered partition and the result is $N - 3$ (notice $N_i \geq 2$ for every group in $|\Omega_{GHZ-W}^k\rangle_{k,n-k}$). Furthermore, one can consider the ways of partition in $M_{GHZ}$ or $M_W$. This is a problem of unordered partition and can be solved by the Ferrers diagram. Suppose the generating function is

$$
G(y) = (\sum_{i=0}^{\infty} y^i)(\sum_{i=0}^{\infty} y^{2i}) \cdots (\sum_{i=0}^{\infty} y^{Ni}),
$$

(38)

and the coefficient of $y^n$ is $C(G(y),N)_n$ Then the results are $C(G(y),M_{GHZ}) = C(G(y),M_{GHZ-1})M_{GHZ-1}$ for the partition of GHZ-type basis and $C(G(y),M_W) = C(G(y),M_{W-1})M_{W-1}$ for that of W-type basis, where $M_{GHZ} = N - M_W \in \{0\} \cup \{2, N - 2\} \cup \{N\}$. Taking one with another, a general GHZ-W-type state can be divided into $L$ kinds of partitions, where

$$
L = \sum_{k=2}^{N-2} (C(G(y),k) - C(G(y),k-1))
\times (C(G(y),N-k) - C(G(y),N-k-1))
+ 2C(G(y),N) - C(G(y),N-1).
$$

(39)

Consequently, by virtue of the theory of unordered partition one can concretely write out all distinct ways in which $N$ can be divided, and ulteriorly find out all main classes of states $|\Omega_{GHZ-W}^k\rangle_{k,n-k}$. Since the permutations between the subgroups have been considered by unordered partition, so our deduction will not lead to the repetition of classification.

III. CLASSIFYING THE GHZ-TYPE STATE

So far we managed to roughly characterize the structure of a general GHZ-W-type state. In what follows,
clarify the form of the allowed ILO’s operated by the
of enumerative combinatorics. These tricks can be gen-
In this section we character the GHZ-type state. In par-
the general argument, it is a wise choice to analyze the
on every class of state particularly. Before moving to
divided into many main classes of entanglement in terms
of the theory of unordered partition, and we now work
on every class of state particularly. Before moving to
the general argument, it is a wise choice to analyze the
GHZ-type and W-type state respectively, for they can be
seen as two fundamental parts of a GHZ-W-type state.
In this section we character the GHZ-type state. In par-
icular, some important concepts are exhibited such as
the relative ILO’s and columns, as well as the technique
of enumerative combinatorics. These tricks can be gen-
eralized to the case of the W-type state, etc.
As for the SLOCC-classification, it is necessary to
clarify the form of the allowed ILO’s operated by the
partners in the system. Consider the form of ILO
\(V_0,1,\ldots,N-1 = \bigotimes_{i=0}^{N-1} V_i\) making \(R_{N1} \to R_{N1}\) on the
space \(\mathcal{H}_{P_0,P_1,\ldots,P_{N-1}}\), where
\[
V_i = \left( \begin{array}{cc}
a_i & b_i \\
c_i & d_i
\end{array} \right)_{P_i}, i = 0, 1, \ldots, N-1,
\]
and we have
\[
V_{0,1,\ldots,N-1} = \left( \alpha_0 |0,0,1,\ldots,0_{N-1}\rangle + \alpha_1 |1,0,1,\ldots,1_{N-1}\rangle \right)
+ \beta_0 |0,0,1,\ldots,0_{N-1}\rangle + \beta_1 |1,0,1,\ldots,1_{N-1}\rangle ,
\]
\(\alpha_0, \alpha_1, \beta_0, \beta_1 \in \mathbb{C}\. \)
Hence, there must be \(b_i = c_i = 0\) or \(c_i = d_i = 0\, i = 0, 1, \ldots, N-1\). In other words, the relative ILO can be
expressed as
\[
V_{0,1,\ldots,N-1} = \left( \begin{array}{cc}
a_i & 0 \\
n_i & 1
\end{array} \right)_{P_i}, \forall a_i \neq 0, \text{ or}
\]
\[
V_{0,1,\ldots,N-1} = \left( \begin{array}{cc}
0 & b_i \\
m_i & 1
\end{array} \right)_{P_i}, \forall b_i \neq 0.
\]
The total effect of this operator on the system is
\[
V_{0,1,\ldots,N-1} |0\rangle_N = |0\rangle_N, \quad V_{0,1,\ldots,N-1} |1\rangle_N = |1\rangle_N,
\]
or
\[
V_{0,1,\ldots,N-1} |0\rangle_N = |1\rangle_N, \quad V_{0,1,\ldots,N-1} |1\rangle_N = |0\rangle_N.
\]
In both cases, the universal factors have been omit-
ted. Therefore, the relative ILO only can real-
ize the exchange of “0” and “1”, or they remain
invariant. The essential change occurs in the as-
pect of coefficients, by the optionally nonvanishing set
\(\{a_0, a_1, \ldots, a_{N-1}\}\) and \(\{b_0, b_1, \ldots, b_{N-1}\}\). Regard each product
\(\bigotimes_{i=0}^{N-1} |x_i\rangle_{N_i}, x_i = 0 \text{ or } 1\), as a term in the state
\(\Omega_{\text{GHZ}}\), then the ILO \(V_{0,1,\ldots,N-1}\) does not change the
number of terms. That is,

**Corollary 2.** Suppose two GHZ-type states are equivalent
under SLOCC, then they have the same number of terms.

By virtue of this corollary, we can describe how to write
out the concrete form of the GHZ-type state. For conve-
nience, let \(\bigotimes_{i=0}^{N-1} |x_i\rangle_{N_i} = |x_0, x_1, \ldots, x_{N-1}\rangle\). That is,
we firstly find out all sorts of entanglement under the
assumption that the position of each group is “equivalent”,
or under an average division \(N_i = N_j, i, j = 0, 1, \ldots, n-1\).
This means that we can arbitrarily exchange the groups
in the state. Subsequently, one can permute the true
sequence of every sort. \(N_0, N_1, \ldots, N_{n-1}\) for the groups
in the state to generate a series of states. By comparing
these states and eliminating the repeating cases, all resi-
dual states are indeed inequivalent. Finally, we should also
use the ILO’s to adjust the coefficient of every term. Of
all above, we will mostly emphasize the first step for the
posterior two steps are more trivial. Some examples will
be given to illustrate them.

Write out the general form of the GHZ-type state,
\[
\Omega_{\text{GHZ}} = a_0 |0,0,0\rangle + a_1 |0,0,1\rangle + \cdots
+ a_2^n |1,1,1\rangle.
\]
For the first step, let us start by an identification of the following
state,
\[
\Psi = a_0 |1,0,0,0\rangle + a_1 |0,1,0,0\rangle + a_2 |0,0,1,0\rangle + a_3 |0,0,0,1\rangle,
\]
which is under the simplest form in terms of the
GHZ-criterion. If we observe this state in a “vertical” manner,
it can be referred to as a combination of four “GHZ-
column”’s (or column for short in this section) of which
each column represents a vector consisting of the
components of the GHZ-type basis describing the system
of one group. For example, the first column of such “GHZ-
column”’s is
\[
\begin{pmatrix}
|1\rangle \\
|0\rangle \\
|0\rangle
\end{pmatrix}.
\]
Generally, a column contains \(p |1\rangle\)’s and \(q |0\rangle\)’s. In
the light of the relative ILO’s, we can always make that
\(p \leq q\). In this case, two columns are different if they
contain different numbers of \(|1\rangle\)’s (the sequence of \(|1\rangle\)’s
and \(|0\rangle\)’s makes no difference). Define the direct sum
of two columns by \(\oplus\), e.g.,
\[
\Delta = \left( \begin{array}{ccc}
|1\rangle & |1\rangle & |0\rangle \\
|0\rangle & |0\rangle & |0\rangle
\end{array} \right) = a_0 |1,0\rangle + a_1 |0,1\rangle + a_2 |0,0\rangle + a_3 |0,0\rangle.
\]
That is, \(\oplus\) represents all ways of combinations in which
two groups will not make up one new group in the
GHZ-type basis under the GHZ-criterion. One may notice
there exists another way of combination like this,
\[
\Delta = a_0 |1,1\rangle + a_1 |0,0\rangle + a_2 |0,0\rangle + a_3 |0,0\rangle.
\]

[48]
which is just the GHZ state and it does not belong to the outcome of the operation $\oplus$. In this sense, the state (45) can be written as

$$|\Psi\rangle = \left( |1\rangle \right) \oplus \left( |1\rangle \right) \oplus \left( |1\rangle \right) \oplus \left( |1\rangle \right).$$

(49)

Notice that $|\Psi\rangle$ is composed of the same kind of columns. In fact, the most universal form of this combination is

$$|\Omega_{GHZ}^{p,q}\rangle_n = \oplus_{i=0}^{n-1} (p|1\rangle, q|0\rangle)_i, 1 \leq p \leq q,$$

(50)

where $(p|1\rangle, q|0\rangle)_i$ represents the $i$'th column containing $p|1\rangle$'s and $q|0\rangle$'s. By permuting the groups in the state, identical columns are collected together and hence $|\Omega_{GHZ}^{p,q}\rangle_n$ can be decomposed into the direct sum of many $|\Omega_{GHZ}^{p,q}\rangle_{n'}$'s,

$$|\Omega_{GHZ}^{p,q}\rangle_n = \oplus_{i=0}^{k-1} (p_{ni}|1\rangle, q_{ni}|0\rangle)_i, p_i \leq q_i \sum_{i=0}^{k-1} n_i = n.$$

(51)

In this sense, two GHZ-type states $|\psi\rangle$ and $|\phi\rangle$ belong to the same sort iff $k^p = k^\phi, n_i^\psi = n_i^\phi, p_i^\psi = p_i^\phi, q_i^\psi = q_i^\phi$ by corollary 2, $i = 0, 1, ..., k - 1$, up to some permutation of the groups. In addition, it is easy to see that two groups in two distinct columns never make up a new group whose space is in some GHZ-type basis. In the light of these facts, it is wise to firstly analyze the structure of $|\Omega_{GHZ}^{p,q}\rangle_n$. We can exemplify it as follows. Consider the case of $|\Omega_{GHZ}^{2,q}\rangle_n$.

$$\left( \begin{array}{c} |1\rangle_0 \\ |0\rangle_0 \\ \vdots \\ |0\rangle_q \end{array} \right) \oplus \left( \begin{array}{c} |1\rangle_1 \\ |0\rangle_1 \\ \vdots \\ |0\rangle_q \end{array} \right) \oplus \cdots \oplus \left( \begin{array}{c} |1\rangle_n \\ |0\rangle_n \\ \vdots \\ |0\rangle_q \end{array} \right).$$

(52)

According to the definition of direct sum $\oplus$, there is a unique kind of combination of the same columns $(1|1\rangle, q|0\rangle)$,

$$|\Omega_{GHZ}^{2,q}\rangle_n = \left( \begin{array}{c} |1,0, ..., 0,0\rangle_0 \\ |0,1, ..., 0,0\rangle_1 \\ \vdots \\ |0,0, ..., 1,0\rangle_{n-2} \\ |0,0, ..., 0,1\rangle_{n-1} \\ |0,0, ..., 0,0\rangle_n \end{array} \right), q \geq n - 1.$$  

(53)

As far as the family $|\Omega_{GHZ}^{2,q}\rangle_n$ is concerned, there exist two inequivalent sorts of entanglement deriving from this expression, $|\Omega_{GHZ}^{p,q}\rangle_n$ and $|\Omega_{GHZ}^{2,q}\rangle_n$. However, it is generally necessary to retain the identical terms $|0,0, ..., 0,0\rangle_k, k = n, ..., q$ as a result of the direct sum. The reason lies in the decomposition of the general GHZ-type state, e.g., consider the following state,

$$|\Psi\rangle = [1,0,0]\rangle_{12} [0,0]\rangle_{34} + [0,1,0]\rangle_{12} [0,0]\rangle_{34} + [0,0,1]\rangle_{12} [1,1]\rangle_{34} + [0,0,0]\rangle_{12} [1,1]\rangle_{34} + [0,0,0]\rangle_{12} [0,1]\rangle_{34} + [0,0,0]\rangle_{12} [1,0]\rangle_{34}.$$  

(54)

This state contains two kinds of columns $(1|1\rangle, 5|0\rangle)$ and $(3|1\rangle, 3|0\rangle)$, and it is in the simplest form in terms of the GHZ-criterion. Clearly, the “redundant” $|0,0,0\rangle_{12}$’s is a necessary part of the terms. This state is also a typical example of the general GHZ-type state by the combination of different columns.

Subsequently, let us move to the case of $|\Omega_{GHZ}^{2,q}\rangle_n$, which is much more sophisticated than the former one. Write out the direct sum of the columns $(2|1\rangle, q|0\rangle)$,

$$\left( \begin{array}{c} |1\rangle_0 \\ |1\rangle_1 \\ |0\rangle_2 \\ \vdots \\ |0\rangle_{q+1} \end{array} \right) \oplus \left( \begin{array}{c} |1\rangle_0 \\ |1\rangle_1 \\ |0\rangle_2 \\ \vdots \\ |0\rangle_{q+1} \end{array} \right) \oplus \cdots \oplus \left( \begin{array}{c} |1\rangle_0 \\ |1\rangle_1 \\ |0\rangle_2 \\ \vdots \\ |0\rangle_{q+1} \end{array} \right) .$$

(55)

Again we need to know the possibly valued region of $q$. It is obvious that $q_{sup} = \infty$. However, it is not easy to find out $q_{inf}$, which is defined by that the state $|\Omega_{GHZ}^{2,q}\rangle_n$ always holds the simplest form if the value of $q$ is larger than it. Consider the case of $n = 4$. A possible form of the state is

$$|\Psi_0\rangle = \left( \begin{array}{c} |1,1,1,1\rangle \\ |1,0,0,0\rangle \\ |0,1,0,0\rangle \\ |0,0,1,0\rangle \end{array} \right),$$

(56)

i.e., $q = 3$, and it keeps the simplest form. We advance another question, whether $|\Psi_0\rangle$ is the form in which the bases $|1\rangle$’s distributes in the smallest “vertical” region in all columns? Interestingly, there exists another better arrangement,

$$|\Psi_1\rangle = \left( \begin{array}{c} |1,1,1,0\rangle \\ |1,0,0,1\rangle \\ |0,1,0,1\rangle \\ |0,0,1,0\rangle \\ |0,0,0,0\rangle \end{array} \right),$$

(57)

in which the bases $|1\rangle$’s only takes up 4 terms. Imagine a column to be a combination of certain $|0\rangle$’s and $|1\rangle$’s in the vertical region, i.e. the positions of these components make the difference. These identical columns constructing the state then must be different combinations from
each other. Clearly, if \( q \) is less, the total number of combinations will be fewer. For the state \( |\Omega_{\text{GHZ}}^{p,q}\rangle_n \), let

\[
(p+q)_p = n,
\]

whose least positive solution of \( q \) is \( q_{\text{min}} \). Choose the least integer \( q_{\text{inf}} \geq q_{\text{min}} \), so when \( q \geq q_{\text{inf}} \) the state can always be simplest. For the above example where \( p = 2, n = 4 \), we have \( q_{\text{inf}} = 2 \). Here, the case of \( q_{\text{inf}} < p \) is also allowed, since the redundant terms \( |0,0,\ldots,0\rangle\)’s are freely added similar to that of \( |\Omega_{\text{GHZ}}^{p,q}\rangle_n \). In this sense, we modify the definition of \( |\Omega_{\text{GHZ}}^{p,q}\rangle_n \) and let \( |\omega_{\text{GHZ}}^{p,q}\rangle_n \) be the direct sum of the columns containing \( |1\rangle \)'s and \( q = q(p,n) \in [q_{\text{inf}}, \infty] \) is a secondary parameter. \( p + q \) indeed describes the distribution of \( |1\rangle \)'s in the vertical direction. Due to corollary 2, every value of \( p + q \) leads to an essential kind of state.

Return to the above example, where \( |\Psi_1\rangle \) has proven to be the state reaching the lowest value of \( q \). One may ask whether there exist other such states. In this expression, there are 8 \( |1\rangle \)'s vertically distributed in the manner \( 8 = 3 + 2 + 2 + 1 \), i.e., again we meet the problem of unordered partition, which is similar to that in the last section. Write out other ways of partitions. For the case of \( 8 = 4+2+1+1 \), the state cannot be simplest. However, for \( 8 = 2 + 2 + 2 + 2 \), we can write out the second form of \( |\omega_{\text{GHZ}}\rangle^2 \).

\[
|\Psi_2\rangle = \begin{pmatrix} |1,1,0,0\rangle \\ |1,0,1,0\rangle \\ |0,1,0,1\rangle \\ |0,0,1,1\rangle \\ |0,0,0,0\rangle \end{pmatrix},
\]

and there is no the third subclass of state in the sort of \( |\omega_{\text{GHZ}}^2\rangle^2 \) (the relative ILO exchanging \( |0\rangle \) and \( |1\rangle \) does not generate new class). Generally, to write out all kinds of \( |\omega_{\text{GHZ}}^{p,q}\rangle_n \) where \( p \) and \( n \) are given and \( q \in [q_{\text{inf}}, \infty] \), it suffices to analyze the terms containing more than one \( |1\rangle \), for those containing a unique \( |1\rangle \) are automatically placed. Moreover, analyzing the number of columns related to the terms containing more than one \( |1\rangle \) is also necessary. For example,

\[
|\omega_{\text{GHZ}}^{2,2n-3}\rangle_n = \begin{pmatrix} |1,1,...,0,0\rangle_0 \\ |1,0,...,0,0\rangle_1 \\ |0,1,...,0,0\rangle_2 \\ |0,0,1,...,0\rangle_3 \\ \vdots \\ |0,0,...,0,0\rangle_{2n-2} \\ |0,0,...,0,1\rangle_{2n-1} \\ |0,0,...,0,0\rangle \end{pmatrix},
\]

and

\[
|\omega_{\text{GHZ}}^{2,2n-3}\rangle_n = \begin{pmatrix} |1,1,...,0,0\rangle_0 \\ |1,0,...,0,0\rangle_1 \\ |0,1,...,0,0\rangle_2 \\ \vdots \\ |0,0,...,0,1\rangle_{2n-3} \\ |0,0,...,0,0\rangle_{2n-2} \end{pmatrix}.
\]

Although they are two trivial cases, the latter of which has implies that what really functions is the first term \( |1,1,...,0,0\rangle_{10} \), while other terms does not work. A complicated case from \( |\omega_{\text{GHZ}}^{2,2n-5}\rangle_n \) completely exhibits the above statement.

\[
|\psi_{00}\rangle = \begin{pmatrix} |1,1,1,1,0,\ldots,0\rangle_{(58)} \\ \vdots \\ |1,1,0,0,0,\ldots,0\rangle_{(64)} \end{pmatrix},
\]

and

\[
|\psi_{00}\rangle = \begin{pmatrix} |1,1,1,0,0,\ldots,0\rangle_{(63)} \\ \vdots \\ |1,1,0,1,0,\ldots,0\rangle_{(67)} \end{pmatrix},
\]

\[
|\psi_{20}\rangle = \begin{pmatrix} |1,1,...,0,0,0,0\rangle \\ |0,0,0,0,1,0,0\rangle \\ |0,0,0,0,1,1,0\rangle \\ |0,0,0,0,1,1,1\rangle \end{pmatrix}.
\]

The seven kinds of states are obtained firstly in terms of the different factorizations of 8, while the number of columns plays a key role in the second step of discrimination, e.g., the number of related columns is 6,5,4,3 for \( |\psi_{20}\rangle, |\psi_{21}\rangle, |\psi_{22}\rangle, |\psi_{23}\rangle \) respectively. As for the more
general case, writing out all entangled classes is indeed a complicated enumerative problem, whose prior rule has been set up here, including the full entanglement, the GHZ-criterion and the effective combination of different columns, since a general GHZ-type state $|\Omega_{\text{GHZ}}\rangle_n = \sum_{i=0}^{k-1} |\omega_{\text{GHZ}}^{pi,q_i}\rangle_n$, $q_i \in [q_{in},f,\infty]$, $\sum_{i=0}^{k-1} a_i = n$. Besides, the relative ILO can make the exchange of $|0\rangle$ and $|1\rangle$ in the columns with the same number of $|0\rangle$ and $|1\rangle$, which is also an interesting discipline of this problem. For all, the technique provided in the present paper has primarily described a feasible method for this difficult problem.

Hitherto, we have said a lot about the first step. Here we briefly exemplify how to distill the essential sorts of entanglement from a set of states generated by the full permutation of partitions $N_0, N_1, ..., N_{n-1}$. Consider the states $|\Psi_0\rangle$ and $|\Psi_1\rangle$,

$$
\begin{pmatrix}
1, 1, 1, 1 \\
1, 0, 0, 0 \\
0, 1, 0, 0 \\
0, 0, 1, 0 \\
0, 0, 0, 1
\end{pmatrix}, \text{ and }
\begin{pmatrix}
1, 1, 1, 0 \\
1, 0, 0, 1 \\
0, 1, 0, 1 \\
0, 0, 1, 0 \\
0, 0, 0, 0
\end{pmatrix}.
$$

(69)

Observe the position of each column. Evidently, every position of the column in $|\Psi_0\rangle$ is equivalent and thus there still exists only one class of entanglement for the state $|\Psi_0\rangle$, since any state generated by some sequence of partition leads to the same result up to a permutation of the groups. On the other hand, the scenario differs a lot when analyzing $|\Psi_1\rangle$, in which only the positions of column 1 and 2 are equivalent. By some simple fact of the combinatorics we know there are at most 12 kinds of subclasses of $|\Psi_1\rangle$, that is,

$$(N_0, N_1, N_2, N_3), (N_0, N_1, N_3, N_2), (N_0, N_2, N_1, N_3), (N_0, N_2, N_3, N_1), (N_1, N_2, N_0, N_3), (N_1, N_2, N_3, N_0), (N_0, N_3, N_1, N_2), (N_0, N_3, N_2, N_1), (N_1, N_3, N_0, N_2), (N_1, N_3, N_2, N_0), (N_2, N_3, N_0, N_1), (N_2, N_3, N_1, N_0).$$

(70)

Practically, there may be some identical numbers in the sequence and the repeating classes will be eliminated. Finally, by some relative ILO’s one can move away the coefficients of the state if unnecessary.

To summarize, we have described the method of classifying the GHZ-type state, which is actually related to the theory of combinatorics. A significant feature of the technique is that we can previously find out the concrete forms of relative ILO’s for this kind of entanglement by virtue of the range criterion. This is mainly decided by the symmetry of the GHZ-type and W-type basis, which also simplifies the calculation of entanglement of an arbitrary pair of particles. The relative ILO’s actually restrict the evolvement of the GHZ-type state under the SLOCC, and there exist the analogous rule for the W-type and GHZ-W-type state.

IV. CLASSIFYING THE W-TYPE STATE AND THE GHZ-W-TYPE STATE

This section is devoted to characterizing the W-type state by virtue of the techniques similar to that of the last section, and some details will be omitted if unnecessary. Subsequently, we will primarily analyze the universal GHZ-W-type state by some examples. Anyhow, the general classification of the GHZ-W-type state always concerns the theory of enumerative combinatorics.

Let us start by considering the form of the relative ILO. Write out $V_{0,1,...,N-1} = \bigotimes \prod_{i=0}^{N-1} V_i$ making $R_{N0} \rightarrow R_{N0}$ on the space $\mathcal{H}_{P_0,P_1,...,P_{N-1}}$, where

$$V_i = \begin{pmatrix}
a_1 & b_i \\
c_i & d_i
\end{pmatrix}, i = 0, 1, ..., N - 1,$$

(71)

and hence

$$V_{0,1,...,N-1}(\alpha_0 |0_0, 0_1, ..., 0_{N-1} + \alpha_1 |W_N\rangle \rangle = \beta_0 |0_0, 0_1, ..., 0_{N-1} + \beta_1 |W_N\rangle \rangle , \alpha_0, \alpha_1, \beta_0, \beta_1 \in C.$$ (72)

More explicitly,

$$\bigotimes_{i=0}^{N-1} \left( \begin{array}{c}
a_i \\
c_i
\end{array} \right)_{P_i} = \beta_0 |0_0, 0_1, ..., 0_{N-1} + \beta_1 |W_N\rangle \rangle .$$ (73)

(73)

and

$$\begin{pmatrix}
b_0 \\
d_0
\end{pmatrix}_{P_0} \bigotimes \prod_{i=1}^{N-1} \begin{pmatrix}
a_i \\
c_i
\end{pmatrix}_{P_i} + \begin{pmatrix}
a_0 \\
c_0
\end{pmatrix}_{P_0} \bigotimes \begin{pmatrix}
b_1 \\
d_1
\end{pmatrix}_{P_1} \bigotimes \prod_{i=2}^{N-1} \begin{pmatrix}
a_i \\
c_i
\end{pmatrix}_{P_i} \nonumber \bigotimes \prod_{i=0}^{N-1} \begin{pmatrix}
b_0 \\
d_0
\end{pmatrix}_{P_{N-1}} = \beta_0 |0_0, 0_1, ..., 0_{N-1} + \beta_1 |W_N\rangle \rangle .$$ (74)

(74)

By the first expression, $c_j c_i = 0, i, j = 0, 1, ..., N - 1, i \neq j$, so every $c_i = 0$. Without loss of generality, let $a_i = 1$, $i = 0, 1, ..., N - 1$. Then by the second expression, $d_i = d_j$, $i, j = 0, 1, ..., N - 1$. In other words, the relative ILO can be expressed as

$$V_{0,1,...,N-1} = \bigotimes \prod_{i=0}^{N-1} \begin{pmatrix}
1 & b_i \\
0 & x
\end{pmatrix}_{P_i} , x \neq 0,$$

so the total effect on the system is

$$V_{0,1,...,N-1} = |0_N\rangle \rangle = |0_N\rangle \rangle ,$$ (75)

$$V_{0,1,...,N-1} = x |W_N\rangle \rangle + \sum_{i=0}^{N-1} b_i |0_i\rangle \rangle .$$ (76)

(76)

Therefore for the W-type state, the essential change by the relative ILO is $|W\rangle \rightarrow |W\rangle + \alpha |0\rangle$, where $\alpha$ is an arbitrarily determined parameter in advance. Besides, the base $|0\rangle$ remains invariant, and the coefficients in the state can be efficiently reduced by the universal factor $x$ in the expression of $V_{0,1,...,N-1}$. Regard each product form $\bigotimes \prod_{i=0}^{N-1} |0_i\rangle \rangle , \bigotimes \prod_{i=k}^{N-1} |W_i\rangle \rangle$, as a term.
in the state \(|\Omega_W\rangle_n\), then \( V_{0,1,...,N-1} \) does not change the number of the terms containing the most \(|W\rangle\)’s. That is,  

**Corollary 3.** Suppose two W-type states are equivalent under SLOCC, then they have the same number of the terms containing the most \(|W\rangle\)’s. We call it the highest-term.

For instance, consider the two-group W-type state,

\[
|\Omega_W\rangle_2 = |W\rangle_{N_0} |W\rangle_{N_1} + a_1 |W\rangle_{N_0} |0\rangle_{N_1} + a_2 |0\rangle_{N_0} |W\rangle_{N_1} + a_3 |0\rangle_{N_0} |0\rangle_{N_1},
\]

By using of two ILO’s making \(|W\rangle_{N_0} \rightarrow |W\rangle_{N_0} - a_2 |0\rangle_{N_0}\) and \(|W\rangle_{N_1} \rightarrow |W\rangle_{N_1} - a_1 |0\rangle_{N_1}\), respectively, we obtain the unique class of the two-group W-type state

\[
|\Omega_W\rangle_2 = |W\rangle_{N_0} |W\rangle_{N_1} + |0\rangle_{N_0} |0\rangle_{N_1},
\]

where the coefficient has been moved away by some ILO \(O_{i,j}^{N_0}\). Evidently, the highest-term \(|W\rangle_{N_0} |W\rangle_{N_1}\) does not disappear under the ILO’s. Moreover, the W-type state has no decomposition form similar to that of the GHZ-type state. Fortunately, the procedure of classification thereof still serves. Let \( \prod_{i=0}^{n-1} |x_i\rangle_{N_i} = |x_0, x_1, ..., x_{n-1}\rangle, x_i = 0, W, \) to firstly find out all sorts of entanglement under an average division \(N_i = N_j, i, j = 0, 1, ..., n - 1\), and then get all states by a full permutation of the partitions \(N_0, N_1, ..., N_{n-1}\). After eliminating the repeating cases and moving away the coefficients as many as possible, the essential classes of states are obtained. Here, we only describe the first step and omit the trivial parts, which one can deal with by following the techniques in the last section. Write out the general form of the W-type state,

\[
|\Omega_W\rangle_n = a_0 |W, W, ..., W\rangle + \sum_{i=1}^{C_n^1} a_{1,i} P_i(|W, W, ..., W, 0\rangle)
\]

\[
+ \sum_{i=1}^{C_n^2} a_{2,i} P_i(|W, W, ..., W, 0, 0\rangle) + \cdots + \sum_{i=1}^{C_n^{n-1}} a_{n-1,i} P_i(|W, 0, ..., 0, 0\rangle) + a_n |0, 0, ..., 0, 0\rangle,
\]

where \(\{P_i\}\) is the set of all \(C_n^m\) distinct permutations of the groups. Due to corollary 3, we can catalog this state by the changing number of highest-terms. Define \(\Omega^{0,n}_W\) as the W-type state containing \(p\) highest-terms \(P_i(|W, W, ..., W, 0, ..., 0\rangle), 1 \leq p \leq C_n^1, 2 \leq q \leq n, \) e.g., there is one term \(|W, W, ..., W\rangle\), \(n\) terms \(P_i(|W, W, ..., W, 0\rangle)\), etc. Totally, there exist \(M = 1 + \binom{n}{q} + \binom{n}{2} + \cdots + \binom{n}{n-2} = 2^n - n - 1\) main classes of W-type entanglement. However, even for a class with a certain number of highest-terms, there may still exist some subclasses of states by permutation similar to that in last section. Observe the following two states on which some ILO’s has been operated,

\[
\Omega^{0,2,2,4}_W = |W, W, 0, 0\rangle + |0, 0, W, W\rangle + \alpha |0, 0, 0, 0\rangle, \\
\Omega^{2,2,4}_W = |W, W, 0, 0\rangle + |W, 0, W, 0\rangle + |0, 0, 0, W\rangle + |0, 0, W, 0\rangle.
\]

Notice the parameter \(\alpha = 0, 1\) is necessary here, since each of them leads to an essential sort of W-type state, which can be easily checked by the relative ILO’s. The main point we emphasize is the inequivalence of \(\Omega^{2,2,4}_W\) and \(\Omega^{2,2,4}_W\). In the light of the W-criterion, both of them are in the simplest form. Similar to the case of the GHZ-type state, the components \(|W\rangle\)’s distribute in all four columns in \(\Omega^{2,2,4}_W\), while in only three columns in \(\Omega^{2,2,4}_W\), so they are inequivalent under SLOCC. This distinction derives from the different arrangement of the components \(|W\rangle\)’s. However, the intrinsic change happens in the lower-terms, i.e., those containing less \(|W\rangle\)’s. As the relative ILO’s can make \(\forall \alpha, |W\rangle \rightarrow |W\rangle + \alpha |0\rangle\), the number of the lower-terms generally can not be the evidence of inequivalent states, and finding out a regular arrangement of \(|W\rangle\)’s remains a difficult problem. We analyze some situations to illustrate it.

(i) The state \(\Omega^{1,n}_W\). By performing the ILO’s \(|W\rangle_{N_i} \rightarrow |W\rangle_{N_i} + \alpha |0\rangle\), \(i = 0, 1, ..., n - 1\), it is in a more succinct form,

\[
\Omega^{1,n}_W = |W, W, ..., W\rangle + \sum_{i=1}^{\binom{n}{2}} a_{2,i} P_i(|W, W, ..., W, 0, 0\rangle)
\]

\[
+ \cdots + \sum_{i=1}^{\binom{n}{n-1}} a_{n-1,i} P_i(|W, 0, ..., 0, 0\rangle) + a_n |0, 0, ..., 0, 0\rangle.
\]

The above state has an important character, i.e., the relative ILO’s on this state must make \(|W\rangle \rightarrow |W\rangle\) and \(|0\rangle \rightarrow |0\rangle\), otherwise some term \(P_i(|W, W, ..., W, W, 0\rangle)\) will always appear. In this case, once more the different numbers of the lower-terms lead to inequivalent classes of W-type states, which implies the existing techniques can be applied to this scenario.

(ii) The state \(\Omega^{p,n-1}_W\). The part of the highest-terms can be written in terms of columns,

\[
\begin{pmatrix}
|W, W, W, ..., W, 0\rangle_0 \\
|W, W, W, ..., W, 0\rangle_1 \\
|W, W, W, ..., W, 0\rangle_2 \\
\vdots \\
|W, W, W, ..., W_0, 0\rangle_{p-1}
\end{pmatrix},
\]

which is the unique form of the state \(\Omega^{p,n-1}_W\). Interestingly, the backward main diagonal of the right \(p\) columns is composed of identical \(|0\rangle\)’s. Besides, only the state
$|\Omega_{W}^{n-1}\rangle_{n}$ could be not in the simplest form by the W-criterion. However, it is difficult to decide the form of the relative ILO’s, for any ILO making $|W\rangle \rightarrow |W\rangle + \alpha |0\rangle$ will affect all the lower-terms, while the unknown coefficients are a lot. This awful fact appears in all other cases of $|\Omega_{W}^{n}\rangle_{n}$. Nonetheless, we can still deal with many classes by the existing techniques, e.g., $|\Omega_{W}^{2,3}_{A}\rangle$ and $|\Omega_{W}^{2,3}_{B}\rangle$ are the only two types of states of $|\Omega_{W}^{2}\rangle_{4}$. A more involved case is the state $|\Omega_{W}^{3,2}_{A}\rangle$. We demonstrate the three distinct forms of the highest terms,

$$
\begin{pmatrix}
6=3+1+1+1 \\
|W, W, 0, 0\rangle_{0} \\
|W, 0, W, 0\rangle_{1} \\
|W, 0, 0, W\rangle_{2}
\end{pmatrix}, \quad \begin{pmatrix}
6=2+2+1+1 \\
|W, W, 0, 0\rangle_{0} \\
|W, 0, W, 0\rangle_{1} \\
|W, 0, 0, W\rangle_{2}
\end{pmatrix},
$$

and

$$
\begin{pmatrix}
6=2+2+2 \\
|W, W, 0, 0\rangle_{0} \\
|0, W, W, 0\rangle_{1} \\
|W, 0, 0, W\rangle_{2}
\end{pmatrix}.
$$

By some simple ILO’s, the three subclasses of $|\Omega_{W}^{3,2}_{A}\rangle$ can be briefly written as

$$
|\Omega_{W}^{3,2}_{A}\rangle_{4} = |W, W, 0, 0\rangle + |W, 0, W, 0\rangle + |W, 0, 0, W\rangle + a_{0}|0, 0, 0, 0\rangle + a_{1}|0, 0, 0, W\rangle + a_{2}|0, 0, W, 0\rangle,
$$

$$
|\Omega_{W}^{3,2}_{B}\rangle_{4} = |W, W, 0, 0\rangle + |W, 0, W, 0\rangle + |0, W, 0, W\rangle + b_{0}|0, 0, 0, W\rangle + b_{1}|0, 0, W, 0\rangle + b_{2}|0, W, 0, 0\rangle,
$$

$$
|\Omega_{W}^{3,2}_{C}\rangle_{4} = |W, W, 0, 0\rangle + |0, W, W, 0\rangle + |W, 0, 0, W\rangle + c_{0}|0, 0, 0, W\rangle + c_{2}|0, 0, W, 0\rangle,
$$

in each of them the coefficients characterize the structure of this state. Moreover, we can simplify the third expression by the ILO’s $|W\rangle_{N_{i}} \rightarrow |W\rangle_{N_{i}} + x_{i}|0\rangle_{N_{i}}$, $i = 0, 1, 2, 3$ such that

$$
|\Omega_{W}^{3,2}_{A}\rangle_{4} = |W, W, 0, 0\rangle + |0, W, W, 0\rangle + |W, 0, 0, W\rangle + \sum_{i=0}^{3} x_{i}|0, 0, 0, W\rangle + (x_{0} + x_{1} + c_{0})|0, 0, 0, W\rangle + |0, 0, 0, W\rangle,
$$

and supposing $x_{1} + x_{2} = x_{0} + x_{2} = x_{0} + x_{1} + c_{0} = 0$ leads to

$$
|\Omega_{W}^{3,2}_{C}\rangle_{4} = |W, W, 0, 0\rangle + |0, W, W, 0\rangle + |W, 0, 0, W\rangle + |0, 0, 0, W\rangle,
$$

which is the most succinct form of $|\Omega_{W}^{3,2}_{C}\rangle_{4}$. One can simplify other expressions above in the similar way, despite some tedious algebra.

Of all above, we have analyzed two typical families of the GHZ-W-type state, the GHZ-type state and the W-type state. Generally, the state $|\Omega_{GHZ-W}\rangle_{k,n-k}$ can be obtained by the direct sum of the state $|\Omega_{GHZ}\rangle_{k}$ and $|\Omega_{W}\rangle_{n-k}$,

$$
|\Omega_{GHZ-W}\rangle_{k,n-k} = |\Omega_{GHZ}\rangle_{k} \otimes |\Omega_{W}\rangle_{n-k}.
$$

For convenience, we can regard the two states as two universal kinds of “basis”, i.e., the universal GHZ-type and universal W-type basis respectively. In this sense, the operation $\otimes$ means all matches of every pairwise components in the two kinds of basis respectively. Besides, the relative ILO’s for the two parts are known and either of the distinct classes of the two basis will lead to the essential classes of the GHZ-W-type state. Furthermore, the rules for determining whether the state is simplest still serve here. Due to the operation of the direct sum, the part of the W-type basis becomes more unrestricted. For instance, the state $|\Omega_{W}^{n}\rangle_{n}$ always cannot be in the simplest form since

$$
\begin{pmatrix}
|0, 0, 0, \ldots, 0, 0, W\rangle_{0} \\
|0, 0, 0, \ldots, W, 0\rangle_{1} \\
\vdots \\
|0, 0, 0, \ldots, 0, 0\rangle_{p-1} \\
|0, 0, 0, \ldots, 0, 0\rangle_{p-1}
\end{pmatrix} \sim \sum_{i=0}^{n-1} N_{i} |W\rangle_{i}^{n-1} + \sum_{i=n-p}^{n-1} N_{i} |W\rangle_{i-p}^{n-1}.
$$

However, the following state remains simplest,

$$
|\Psi\rangle = |\omega_{GHZ}^{1,2}\rangle_{3} \otimes |\Omega_{W}^{3,2}_{C}\rangle_{2} = |1, 0, 0\rangle(a_{0}|W, 0\rangle + a_{1}|0, W\rangle + a_{2}|0, 0\rangle) + |0, 1, 0\rangle(b_{0}|W, 0\rangle + b_{1}|0, W\rangle + b_{2}|0, 0\rangle) + |0, 0, 1\rangle(c_{0}|W, 0\rangle + c_{1}|0, W\rangle + c_{2}|0, 0\rangle) + |0, 0, 0\rangle(d_{0}|W, 0\rangle + d_{1}|0, W\rangle + d_{2}|0, 0\rangle),
$$

when the coefficients are appropriately given such that it satisfies the W-criterion. We are interested in the classification of this universal type of multiqubit state. Since the position of the columns in the part of the GHZ-basis is identical, there exists some repeating cases when setting the coefficients. By corollary 3, the number of the highest terms does not change under the relative ILO’s. In this case, the numbers of $P_{i}(|W, 0\rangle)$’s in each bracket are fixed. Denote these numbers in the prior three brackets by $(p_{0}, p_{1}, p_{2})$, and the number “0” means the highest term is $|0, 0\rangle$. Then there are nine main classes of the state $|\Psi\rangle$, whose numbers are

$$
(2, 2, 2), (2, 2, 1), (2, 2, 0), (2, 1, 1), (2, 1, 0),
$$

$$
(2, 0, 0), (1, 1, 1), (1, 1, 0), (1, 0, 0),
$$

up to some permutation of the groups (the combination $(0, 0, 0)$ does not lead to the simplest form). More explicitly, there are two subclasses in the states with numbers $(2, 1, 1), (1, 1, 1)$, and $(1, 1, 0)$. By some simple ILO’s we
can write out the succinct forms as follows,

\[
|\Psi^A_{(2,1,1)}\rangle = |1,0,0\rangle (|W,0\rangle + |0,W\rangle) + |1,0,0\rangle (|W,0\rangle + b_2 |0,0\rangle) + |0,0,1\rangle |W,0\rangle
+ |0,0,0\rangle (d_0 |W,0\rangle + d_1 |0,W\rangle + d_2 |0,0\rangle),
\]

\[
|\Psi^B_{(2,1,1)}\rangle = |1,0,0\rangle (|W,0\rangle + |0,W\rangle) + |1,0,0\rangle (|W,0\rangle + b_2 |0,0\rangle) + |0,0,1\rangle |W,0\rangle
+ |0,0,0\rangle (d_0 |W,0\rangle + d_1 |0,W\rangle + d_2 |0,0\rangle),
\]

\[
|\Psi^A_{(1,1,1)}\rangle = |1,0,0\rangle (|W,0\rangle + a_2 |0,0\rangle) + |0,1,0\rangle (|W,0\rangle + |0,0,1\rangle |0,W\rangle + |0,0,0\rangle (d_0 |W,0\rangle + d_1 |0,W\rangle + d_2 |0,0\rangle),
\]

\[
|\Psi^B_{(1,1,1)}\rangle = |1,0,0\rangle (|W,0\rangle + |0,0,1\rangle |0,W\rangle) + |0,0,0\rangle (d_0 |W,0\rangle + d_1 |0,W\rangle + d_2 |0,0\rangle). \tag{95}
\]

One can also list other states containing only one sub-classes by the similar techniques. Predictably, the general GHZ-W-type state can be both qualitatively and quantitatively characterized by the techniques developed by us.

V. CONCLUSION

In this paper, first we introduced the concept of the SLOCC-equivalent basis (SEB), and proposed two general SEBs, the GHZ-type and the W-type basis in the multiqubit space. By virtue of the two SEBs, we proved that the GHZ state and the W state are the only two types consisting of the basis \(R_{20}\) and \(R_{21}\) in any two-qubit subspace respectively. Subsequently, a universal type of pure multiqubit state consisting of the GHZ-type and W-type basis, i.e., the GHZ-W-type state was carefully investigated. We proposed the condition on which this state can be fully entangled and be in the simplest form. The main purpose of this paper is to classify the GHZ-W-type multiqubit state, which can be realized by characterizing the GHZ-type and W-type state respectively. Our result has shown that the full understanding of this type of multiqubit state tightly relates to the theory of combinatorics.

There are several aspects for the future work. First, more algebraic effort is required, especially for the case of the W-type state. The classification of the GHZ-W-type state proves to be a difficult problem in the enumerative combinatorics, and it is expected that a more explicit method can be given. Although we have classified the entanglement under the SLOCC criterion, it is easy to modify our argument so that it becomes the LOCC criterion, since most of the relative ILO’s are diagonal or anti-diagonal (one can set the entries \(b_i = 0, \forall i\) in the ILO’s for the W-type state). Therefore, the classification becomes utterly precise and most of our conclusions still serve. Second, recently a few types of multiqubit entanglement have been demonstrated by the spontaneous parametric down-conversion (SPDC), such as the 4-qubit GHZ state and 5-qubit GHZ state [36]. In particular, Wei further et al. [37] has realized a superposition of a four photon GHZ state and a product of two EPR pairs by using the type-II down-conversion [38].

\[
|\Psi^{(3)}\rangle = \sqrt{\frac{2}{3}} |\text{GHZ}_{aa'bb'}\rangle - \sqrt{\frac{1}{3}} |\text{EPR}_{aa'}\rangle |\text{EPR}_{bb'}\rangle. \tag{98}
\]

Here, the EPR state is the Bell state \((|01\rangle + |10\rangle)/\sqrt{2}\) with \(x = a, b\). Evidently, this state has a more complicated configuration than the GHZ-W-type state since it consists of the rank-3 basis in each bipartite subspace. It is thus possible to realize the 4-qubit GHZ-W-type state by the similar experimental setup, for it can be seen as the superposition of a product of two EPR pairs and another replaceable term, which can be \(|0000\rangle, |0000\rangle + |0011\rangle, etc. An alternative method of producing it can be obtained by the joint measurement of the state of the composite system, e.g., the state \(|\Psi_{\text{sub}}\rangle\) in [39] is indeed the 4-qubit GHZ-type state. Finally, the character of the GHZ-W-type state is also an interesting topic, such as the multiqubit entanglement measures [40] and the robustness of it [41].

The work was partly supported by the NNSF of China Grant No.90503009 and 973 Program Grant No.2005CB724508.

[1] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. 70, 1895 (1993).
[2] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, England, 2000).
[3] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A 53, 2046 (1996).
[4] N. Linden and S. Popescu, Fortsch. Phys. 46, 567 (1998).
[5] G. Vidal, J. Mod. Opt. 47, 355 (2000); M. Nielsen, Phys. Rev. Lett. 83, 436 (1999); G. Vidal, ibid. 83, 1046 (1999).
[6] J. Eisert and D. Gross, e-print quant-ph/0505149.
[7] W. Dür and J. I. Cirac, Phys. Rev. A 61, 042314 (2000).
[8] W. K. Wootters, e-print quant-ph/0001114; W. Dür, Phys. Rev. A 63, 020303(R) (2000); K. M. O’Connor and W. K. Wootters, Phys. Rev. A 63, 052302 (2001).
[9] J. K. Stockton, J. M. Geremia, A. C. Doherty, and H. Mabuchi, Phys. Rev. A 67, 022112 (2003).
[10] M. Möttönen, J. J. Vartiainen, V. Bergholm, and M. M. Salomaa, Phys. Rev. Lett. 93, 130502 (2004).
[11] M. Bourennane, M. Eibl, S. Gaertner, C. Kurtsiefer, A. Cabello, and H. Weinfurter, Phys. Rev. Lett. 92, 107901 (2004).
[12] K. J. Resch, P. Walther, and A. Zeilinger, Phys. Rev. Lett. 94, 070402 (2005).
[13] H. Mikami, Y. Li, K. Fukuoka, and T. Kobayashi, Phys. Rev. Lett. 95, 150404 (2005).
[14] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A 62, 062314 (2000).
[15] A. Acín, A. Andrianov, L. Costa, E. Jané, J. I. Latorre, and R. Tarrach, Phys. Rev. A 61, 042314 (2000).
[16] C. H. Bennett, S. Popescu, D. Rohrlich, J. A. Smolin and A. V. Thapliyal, Phys. Rev. A 63, 012307 (2000).
[17] F. Verstraete, J. Dehaene, and B. De Moor, Phys. Rev. A 64, 010101(R) (2001).
[18] M. Hillery, V. Bužek, and A. Berthiaume, Phys. Rev. A 59, 1829 (1999).
[19] J. Kempe, Phys. Rev. A 60, 910 (1999).
[20] C. P. Yang, S. I. Chu, and S. Han, Phys. Rev. A 70, 022329 (2004).
[21] G. Rigolin, Phys. Rev. A 71, 032303 (2005).
[22] F. G. Deng, C. Y. Li, Y. S. Li, H. Y. Zhou, and Y. Wang, Phys. Rev. A 72, 022308 (2005).
[23] Y. Yeo and W. K. Chan, Phys. Rev. Lett. 96, 060502 (2006).
[24] L. Vaidman, L. Goldenberg, and S. Wiesner, Phys. Rev. A 54, R1745 (1996).
[25] F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde, Phys. Rev. A 65, 052112 (2002).
[26] R. P. Stanley, Enumerative Combinatorics, Volume 1,2 (Cambridge University Press, Cambridge, England, 1999).
[27] L. Chen and Y. X. Chen, Phys. Rev. A 73, 052310 (2006); L. Chen, Y. X. Chen, and Yu-Xue Mei, e-print quant-ph/0604184 (2006).
[28] C. H. Bennett, D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, Phys. Rev. Lett. 82, 5385 (1999).
[29] J. L. Romero, G. Björk, A. B. Klímov, and L. L. Sánchez-Soto, Phys. Rev. A 72, 062310 (2005).
[30] K. S. Gibbons, M. J. Hoffman, and W. K. Wootters, Phys. Rev. A 70, 062101 (2004).
[31] A. Hayashi, M. Horibe, and T. Hashimoto, Phys. Rev. A 71, 052331 (2005).
[32] R. Horn and C. Johnson, Matrix Analysis (Cambridge University Press, Cambridge, England, 1985).
[33] A. Sampera, R. Tarrach, and G. Vidal, Phys. Rev. A 58, 826 (1998).
[34] In general, it is hard to seek the generic form of all rank-2 SEBs in the multiqubit space, because too many parameters appear as the number of parties increases. We will apply the theory of entangled bases to the characterization of 5-qubit entanglement in a posterior work.
[35] A. C. Doherty, P. A. Parrilo, and F. M. Spedalieri, Phys. Rev. A 71, 032333 (2005).
[36] J. W. Pan, M. Daniell, S. Gasparoni, G. Weihs, and A. Zeilinger, Phys. Rev. Lett. 86, 4435 (2001); Z. Zhao, Y. A. Chen, A. N. Zhang, T. Yang, H. J. Briegel, and J. W. Pan, Nature (London), 430, 54 (2004).
[37] H. Weinfurter and M. Zukowski, Phys. Rev. A 64, 010102(R) (2001); M. Eibl, S. Gaertner, M. Bourennane, C. Kurtsiefer, M. Zukowski, and H. Weinfurter, Phys. Rev. Lett. 90, 200403 (2003).
[38] P. G. Kwiat et al, Phys. Rev. Lett. 75, 4337 (1995).
[39] F. G. Deng, X. H. Li, C. Y. Li, P. Zhou, and H. Y. Zhou, Phys. Rev. A 72, 044301 (2005).
[40] A. Wong and N. Christensen, Phys. Rev. A 63, 044301 (2001).
[41] G. Vidal and R. Tarrach, Phys. Rev. A 59, 141 (1998).