Log Canonical Thresholds and Generalized Eckardt Points

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Abstract. Let $X$ be a smooth hypersurface of degree $n \geq 3$ in $\mathbb{P}^n$. We prove that the log canonical threshold of $H \in |-K_X|$ is at least $\frac{n-1}{n}$. Under the assumption of the Log minimal model program, we also prove that a hyperplane section $H$ of $X$ is a cone in $\mathbb{P}^{n-1}$ over a smooth hypersurface of degree $n$ in $\mathbb{P}^{n-2}$ if and only if the log canonical threshold of $H$ is $\frac{n-1}{n}$.

Bibliography: 20 titles.

1. Introduction

All varieties are assumed to be defined over $\mathbb{C}$, unless otherwise stated. Main definitions and notations appear in [3] and [20].

To measure how far from log canonicity log pairs are, V. Shokurov introduced log canonical thresholds in [20]. It is known that they have many amazing properties.

Definition 1.1. Let $(X, B)$ be a log canonical pair and let $Z$ be a closed subvariety of $X$. Suppose that $D$ is a $\mathbb{Q}$-Cartier divisor on $X$. The log canonical threshold of $D$ along $Z$ with respect to $K_X + B$ is the number:

$$lct_Z(D; X, B) = \sup \{c : K_X + B + cD \text{ is log canonical along } Z \}.$$ 

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It is easy to check $0 \leq \text{lct}_Z(D; X, B) \leq 1$. If $B = 0$, we use $\text{lct}_Z(D; X)$ instead of $\text{lct}_Z(D; X, 0)$. For the case $Z = X$ we use the notation $\text{lct}(D; X, B)$ instead of $\text{lct}_X(D; X, B)$.

We may find log canonical thresholds in several other branches of mathematics in various disguises. For example, we consider the log canonical threshold $\text{lct}_0(D; C^n)$ of $D = (f = 0)$ at the origin with respect to $(C^n, 0)$, where $f$ is a nonconstant holomorphic function near the origin. Then we can see ([11]) that this number is the same as the following number:

$$\sup \{c : |f|^c \text{ is locally } L^2 \text{ near the origin}\}.$$

Bernstein-Sato polynomials provide another example. Bernstein-Sato polynomials appear in differential operator theory—in particular, $\mathcal{D}$-module theory ([8]). Let us briefly explain what Bernstein-Sato polynomials are. It is known that for any convergent power series $f \in \mathbb{C}\{z_1, \cdots, z_n\}$, there is a nonzero polynomial $b(s) \in \mathbb{C}[s]$ and a linear differential operator $Q = \sum_{I,j} f_{I,j} s_j \frac{\partial}{\partial z_I}$ such that

$$b(s)f^s = Qf^{s+1},$$

where each $f_{I,j}$ is a convergent power series. For a fixed $f$, such $b(s)$’s form an ideal of $\mathbb{C}[s]$. The monic generator of this ideal is called the Bernstein-Sato polynomial of $f$. We can see that $\text{lct}_0(D; C^n)$ is the absolute value of the largest root of the Bernstein-Sato polynomial of $f$ ([11]).

Recently, M. Mustaţă investigated log canonical thresholds via jet schemes ([13]). He obtained

**Theorem 1.2.** Let $X$ be a smooth variety and $D$ an integral effective divisor on $X$. Then the log canonical threshold of $D$ with respect to $K_X$ is given by

$$\text{lct}(D; X) = \dim X - \sup_{m \geq 0} \frac{\dim D_m}{m + 1},$$

where $D_m$ is the $m$-th jet scheme of $D$.

The $m$-th jet scheme $X_m$ of a variety $X$ is a scheme whose closed points over $x \in X$ are morphisms $O_{X,x} \longrightarrow k[t]/(t^{m+1})$. If $X$ is a smooth variety, then $X_m$ is an affine bundle over $X$ of dimension $(m + 1)\dim X$.

To understand a given variety, it is important to investigate linear systems related to the canonical divisor. One such investigation is to find “extreme” elements in the linear
systems. We have two kinds of extreme elements in the linear systems. One is a “good” element, and the other is a “bad” element. We need to explain what “good” elements are and what “bad” elements are. It is natural that singularities should distinguish between the “good” and the “bad”. Since we always consider linear systems related to canonical divisors (or log canonical divisors), these concepts should involve canonical divisors.

For a “good” element, M. Reid considered a general elephant. Following him, V. Shokurov introduced more general concepts.

**Definition 1.3.** Let $X$ be a normal variety and let $D = S + B$ be a subboundary on $X$ such that $S$ and $B$ have no common components, $S$ is a reduced divisor, and $|B| \leq 0$. We say that $K_X + D$ is $n$-complementary if there is a divisor $D^+$ on $X$ satisfying the following conditions:

1. $nD^+$ is integral and $n(K_X + D^+)$ is linearly trivial.
2. $nD^+ \geq nS + \lfloor (n + 1)B \rfloor$.
3. $K_X + D^+$ is log canonical.

The divisor $K_X + D^+$ is called an $n$-complement of $K_X + D$.

We now introduce a counterpart of these “good” elements. It was originally introduced by S. Keel and J. McKernan ([10]). Strictly speaking, it is a counterpart of M. Reid’s general elephant.

**Definition 1.4.** Let $X$ be a normal variety. Let $B$ be an effective $\mathbb{Q}$-divisor on $X$. A special tiger for $K_X + B$ is an effective $\mathbb{Q}$-divisor $D$ such that $K_X + B + D$ is numerically trivial, but not Kawamata log terminal.

Suppose that a log pair $(X, B)$ is log canonical. Then, using log canonical thresholds, we can compare special tigers for $K_X + B$. To this end, we introduce

**Definition 1.5.** Let $(X, B)$ be a log canonical pair with nonempty $-(K_X + B)|$. The total log canonical threshold of $(X, B)$ is the real number

$$\text{totalct}(X, B) = \sup\{r : K_X + B + rD \text{ is log canonical for any } D \in -(K_X + B)|\}.$$
Note that $0 \leq \text{totallct}(X, B) \leq 1$. If $B = 0$, the total log canonical threshold of $(X, B)$ will be denoted by $\text{totallct}(X)$ instead of $\text{totallct}(X, B)$. The total log canonical threshold of $(X, B)$ measures how bad elements of $|-(K_X + B)|$ can be. It is worthwhile to pay attention to special tigers realizing the total log canonical threshold.

**Definition 1.6.** Let $X$ be a normal variety. Let $B$ be an effective $\mathbb{Q}$-divisor on $X$. A divisor $D$ on $X$ is called a wild tiger of $K_X + B$ if the following conditions are satisfied:

1. $K_X + B + D$ is linearly trivial.
2. $\text{lct}(D; X, B) = \text{totallct}(X, B)$.

By a wild tiger for $X$, we mean a wild tiger for $K_X$.

Comparing with the definition of a special tiger, we note that numerical triviality is replaced by linear triviality in the definition of a wild tiger. The concept of a wild tiger is a sort of counterpart of $1$-complement of $K_X + B$. We can find beautiful interactions between $1$-complements and wild tigers in [14]. In the present paper, we are mainly interested in finding wild tigers for smooth hypersurfaces of degree $n \geq 3$ in $\mathbb{P}^n$.

**2. Eckardt points**

An Eckardt point is a point on a smooth cubic surface $\Sigma$ at which three lines on $\Sigma$ intersect each other. In other words, it is a point $p$ on $\Sigma$ such that there is an element in $|-(K_\Sigma)|$ which is a cone with vertex $p$ and base consisting of three different points.

Now, we investigate smooth del Pezzo surfaces (in particular cubic surfaces) to find their wild tigers. During this investigation, we will observe the special feature of Eckardt points. When hunting wild tigers, the first step is to calculate total log canonical thresholds. But, due to the concrete geometric description of smooth del Pezzo surfaces, we may find wild tigers of smooth del Pezzo surfaces by investigating cubic curves and points in general position on $\mathbb{P}^2$.

**Proposition 2.1.** Let $S$ be a smooth del Pezzo surface of degree $d$. Then we have the following table:
| $d$ | Remarks | total $lct(S)$ | Wild tiger |
|-----|---------|----------------|------------|
| 9   | Fano index 3 | $\frac{1}{3}$ | Triple line |
| 8   | Fano index 1 | $\frac{1}{3}$ | 3(0) 2(-1) |
| 8   | Fano index 2 | $\frac{1}{2}$ | 2(0) 2(0) |
| 7   |          | $\frac{1}{3}$ | 2(-1) 3(-1) 2(-1) |
| 6   |          | $\frac{1}{2}$ | 1(-1) 2(0) 1(-1) |
| 5   |          | $\frac{1}{2}$ | 1(-1) 2(-1) |
| 4   |          | $\frac{2}{3}$ | 1(0) 1(-1) |
| 3   | $S$ has an Eckardt point. | $\frac{2}{3}$ | |
| 3   | Generic case | $\frac{3}{4}$ | A line and a conic intersecting tangentially with intersection number 2 |
| 2   | $S$ has an effective anticanonical divisor with a tacnode. | $\frac{3}{4}$ | Two lines intersecting tangentially with intersection number 2 |
| 2   | Generic case | $\frac{5}{6}$ | Cuspidal cubic |
| 1   |          | $\frac{5}{6}$ | Cuspidal cubic |

where $\bullet$'s denote smooth rational curves, numbers are multiplicities, and the numbers in parentheses are self-intersection numbers.
Proof. The proof is straightforward. Refer to [14] for some direction. Q.E.D.

We now pay attention to smooth cubic surfaces. We observe that in this case, there are two different wild tigers. If a cubic surface has an Eckardt point, then its wild tiger consists of three lines intersecting at a single point which is an Eckardt point. If not, then its wild tiger consists of a line and a conic intersecting tangentially with intersection number 2. Note that having an Eckardt point is a codimension one condition.

It is expected that Eckardt points indicate special features to del Pezzo fibrations of degree 3. The evidence can be found in [6] and [14]. For this reason, it is necessary to generalize Eckardt points to the case of smooth hypersurfaces of degree $n \geq 4$ in $\mathbb{P}^n$.

Definition 2.2. Let $X$ be a smooth hypersurface of degree $n \geq 3$ in $\mathbb{P}^n$. A point $p$ on $X$ is called an (generalized) Eckardt point if there is an element $S$ in $|-K_X|$ which is a cone in $\mathbb{P}^{n-1}$ over a smooth hypersurface of degree $n$ in $\mathbb{P}^{n-2}$ with vertex $p$.

It is clear that a generalized Eckardt point coincides with the classical one when $n = 3$.

As we noted earlier, if a smooth cubic surface has an Eckardt point, then its wild tiger is a cone over three different points. This fact can be generalized as follows:

Theorem 2.3. Let $X$ be a smooth hypersurface of degree $n \geq 3$ in $\mathbb{P}^n$. If $X$ has an Eckardt point $p$, then $S \in |-K_X|$ as used in Definition 2.2 is a wild tiger for $X$. In particular, $\text{lct}(S; X) = \text{total lct}(X) = \frac{n-1}{n}$ and the locus of log canonical singularities $\text{LCS}(X, \frac{n-1}{n}S)$ of $(X, \frac{n-1}{n}S)$ is one point $\{p\}$.

We will prove the theorem in the next section.

We see that if a smooth cubic surface has total log canonical threshold $\frac{2}{3}$, then it has an Eckardt point. It is natural to conjecture

Conjecture 2.4. Let $X$ be a smooth hypersurface of degree $n \geq 3$ in $\mathbb{P}^n$. If the total log canonical threshold of $X$ is $\frac{n-1}{n}$, then a wild tiger $S$ for $X$ is a cone in $\mathbb{P}^{n-1}$ over a smooth hypersurface of degree $n$ in $\mathbb{P}^{n-2}$ with vertex $p$. Moreover, $\text{LCS}(X, \frac{n-1}{n}S) = \{p\}$, where $p$ is an Eckardt point of $X$.

Of course, the conjecture holds for $n = 3$. In the fourth section, we will prove the conjecture under the assumption of the Log minimal model program. Since the Log minimal model program holds up to dimension 3, the conjecture is true up to $n = 4$. 
Before proceeding, we should mention one more problem related to total log canonical thresholds and wild tigers of smooth Fano varieties. We see that a generic cubic surface has total log canonical threshold 3/4 and that its wild tiger consists of a line and a conic intersecting tangentially with intersection number 2. A generic del Pezzo surface of degree 2 has total log canonical threshold 5/6, and its tiger is a cuspidal cubic. This seems to be quite a common phenomenon. So, we ask what the total log canonical threshold of a generic hypersurface of degree $n$ in $\mathbb{P}^n$ is and what its wild tiger is.

3. A Lower Bound for Total Log Canonical Thresholds

Let $W$ be a smooth hypersurface of degree $m$ in $\mathbb{P}^n$ and $H$ be a hyperplane section of $W$, where $n \geq 4$. It follows from the Lefschetz theorem that the Picard group of $W$ is a free abelian group generated by a hyperplane section $H$, i.e., $\text{Pic}(W) = \mathbb{Z}H$. Therefore, a hyperplane section $H$ of $W$ is irreducible and reduced.

**Lemma 3.1.** For any curve $C$ on $H$, $\text{mult}_CH = 1$.

**Proof.** Let $p$ be a general point in $\mathbb{P}^n \setminus W$. We consider a cone $P_p$ with the vertex $p$ and the base $C$. Then we have

$$P_p \cap W = C \cup R_p,$$

where $R_p$ is the residual curve of degree $(m - 1)\deg(C)$. The curves $C$ and $R_p$ intersect at $(m - 1)\deg(C)$ different points (see [17]). Since $H$ is a hyperplane section of $W \subset \mathbb{P}^n$, we have

$$H \cdot R_p = \deg(R_p) = (m - 1)\deg(C).$$

On the other hand,

$$H \cdot R_p \geq \deg(R_p)\text{mult}_CH = (m - 1)\deg(C)\text{mult}_CH.$$

Q.E.D.

**Corollary 3.2.** A hyperplane section $H$ has only isolated singularities. In particular, it is normal.
**Proof.** The first statement immediately follows from Lemma 3.1. Since $H$ is a smooth in codimension 1 hypersurface of a smooth variety, it is normal. Q.E.D.

**Theorem 3.3.** The log canonical threshold of $H$ in $W$ is at least $\lambda = \min\{\frac{n-1}{m}, 1\}$.

**Proof.** Let $0 < \alpha < \lambda$. We may consider the log pair $(\mathbb{P}^{n-1}, \alpha H)$ instead of $(W, \alpha H)$. Suppose that $K_{\mathbb{P}^{n-1}} + \alpha H$ is not Kawamata log terminal. Then the log canonical singularity subscheme $L = LCS(\mathbb{P}^{n-1}, \alpha H)$ associated to $(\mathbb{P}^{n-1}, \alpha H)$ is a zero-dimensional subscheme. Now, we consider a Cartier divisor $D$ which is numerically equivalent to $K_{\mathbb{P}^{n-1}} + \alpha H + (\lambda - \alpha)H'$, where $H'$ is a generic element in $|H|$. Note that

$$O_{\mathbb{P}^{n-1}}(D) = \begin{cases} O_{\mathbb{P}^{n-1}}(-1) & \text{if } m - n \geq -1, \\ O_{\mathbb{P}^{n-1}}(m - n) & \text{otherwise} \end{cases}$$

By Shokurov’s vanishing theorem (see [1]), we have an exact sequence

$$H^0(\mathbb{P}^{n-1}, O_{\mathbb{P}^{n-1}}(D)) \rightarrow H^0(L, O_L(D)) \rightarrow 0.$$  

But the first term is zero even though the second term is not. This is a contradiction. Q.E.D.

**Corollary 3.4.** Suppose that $n = m$. Then the total log canonical threshold of $W$ is at least $\frac{n-1}{n}$.

**Proof.** This immediately follows from Theorem 3.3. Q.E.D.

**Proof of Theorem 2.3.** It is obvious that the log canonical threshold of $S$ with respect to $K_X$ is $\frac{n-1}{n}$. The theorem then follows from Corollary 3.4. Q.E.D.

**Proposition 3.5.** If the log canonical threshold $\alpha$ of $H$ in $W$ is not 1, then the locus of log canonical singularities $LCS(W, \alpha H)$ of $(W, \alpha H)$ consists of a single point.

**Proof.** The proof is similar to that of Theorem 3.3. Q.E.D.
Example 3.6. Suppose that a hypersurface $W$ in $\mathbb{P}^n$ is given by equation

$$x_0^m - x_1^m + \sum_{i=2}^{n} x_i^m = 0$$

and a hyperplane section $H$ is given by $x_0 - x_1 = 0$. Then the log canonical threshold of $H$ is $\lambda$. Thus, our $\lambda$ is the sharp lower bound for log canonical thresholds of hyperplane sections of smooth hypersurfaces of degree $m$ in $\mathbb{P}^n$.

4. Proof of the Conjecture via the Log Minimal Model Program

Let $X$ be a smooth hypersurface of degree $n \geq 4$ in $\mathbb{P}^n$. Let $S$ be a hyperplane section of $X$. The goal of this section is to prove Conjecture 2.4 for $n = 4$. Specifically, under the assumption of the Log minimal model program in dimension $\leq n - 1$, we will show that if a log pair $(X, n-1 \cdot S)$ is not Kawamata log terminal, then $S$ is a cone in $\mathbb{P}^{n-1}$ over a smooth hypersurface of degree $n$ in $\mathbb{P}^{n-2}$.

We suppose that $(X, n-1 \cdot S)$ is not Kawamata log terminal. Then, $LCS(X, n-1 \cdot S)$ is nonempty and consists of only finite number of points (in fact, only one point by Proposition 3.5). So, we may forget about the hypersurface $X$ and deal only with the log pair $(\mathbb{P}^{n-1}, n-1 \cdot S)$.

From now on, we assume that the Log minimal model program holds for dimension $\leq n - 1$.

Lemma 4.1. There exists a birational morphism $f : V \to \mathbb{P}^{n-1}$ satisfying the following:

- $f$ is an isomorphism outside of $LCS(\mathbb{P}^{n-1}, n-1 \cdot S) = \{p\}$,

- $V$ has $\mathbb{Q}$-factorial terminal singularities, and

- there is an effective $f$-exceptional $\mathbb{Q}$-divisor $E$ on $V$ such that the support of $E$ coincides with that of the $f$-exceptional locus, $|E| \neq 0$, and $K_V + \frac{n-1}{n} f_s^{-1}S + E = f^*(K_{\mathbb{P}^{n-1}} + \frac{n-1}{n} S)$.

Proof. Let $g : V' \to \mathbb{P}^{n-1}$ be a log terminal blow-up of $(\mathbb{P}^{n-1}, n-1 \cdot S)$. Since $V'$ has $\mathbb{Q}$-factorial Kawamata log terminal singularities, we may take a terminal blow-up $h : V \to V'$ with respect to $V'$ (see [13]). Then the birational morphism $f = g \circ h : V \to \mathbb{P}^{n-1}$ will satisfy the conditions.

Q.E.D.
We fix such a birational morphism $f : V \to \mathbb{P}^{n-1}$. Let $\tilde{S} = f^{-1}(S)$. Since the log pair $(\mathbb{P}^{n-1}, \frac{1}{n} S)$ is log canonical, the log pair $(V, \frac{1}{n} \tilde{S} + E)$ is also log canonical. We see that $K_V + \frac{1}{n} \tilde{S} + E$ is not nef and that $-(K_V + \frac{1}{n} \tilde{S} + E)$ is not ample. Therefore, there is an extremal contraction $g : V \to W$ such that $-(K_V + \frac{1}{n} \tilde{S} + E)$ is $g$-ample. Because $-(K_V + \frac{1}{n} \tilde{S} + E)$ is $g$-ample and $f$-numerically trivial, no curve contracted by $g$ is contained in the fibers of $f$. In particular, $W$ is not a point.

**Lemma 4.2.** Suppose that the extremal contraction $g$ contracts a subvariety $F$ of $V$ to a subvariety $Z$ of $W$. Then $\dim F - \dim Z = 1$.

**Proof.** Suppose that $\dim F - \dim Z > 1$. Let $G$ be a fiber of $g$ over $Z$. Since $G \cap E \neq \emptyset$ and $V$ is $\mathbb{Q}$-factorial, there is a curve on $G \cap E$ which is contracted by both $f$ and $g$. But this is impossible. Q.E.D.

**Proposition 4.3.** If the extremal contraction $g : V \to W$ is a Mori fiber space, then $g$ is a conic bundle.

**Proof.** This follows from Lemma 4.2. Q.E.D.

The following lemma is due to V. Shokurov’s paper ([19]), which is in preparation. It generalizes X. Benveniste and S. Mori’s results ([2] and [12]) under the assumption of the existence of flips.

**Lemma 4.4.** Suppose that $Y$ has at worst $\mathbb{Q}$-factorial terminal singularities. Let $h : Y \to Z$ be a birational contraction. If a curve $C$ on $Y$ is an irreducible component of the exceptional locus of $h$, then $K_Y \cdot C > -1$.

**Proof.** It is enough to consider the statement over an analytic neighborhood of $h(C) = q \in Z$. We choose a divisor $H$ on $Y$ with $H \cdot C = 1$. In addition, we may assume that the exceptional locus of $h$ is the curve $C$. Suppose that $K_Y \cdot C \leq -1$. Then $(K_Y + H) \cdot C = 0$ (see [19]). We consider an $K_Y$-flip of $h$;

$$
\begin{array}{c}
Y \xrightarrow{\phi} Y^+ \\
h \downarrow \quad Z \quad \downarrow h^+ \\
\end{array}
$$
Let $H^+$ be the birational transform of $H$ to $Y^+$ via $\phi$. Since we have $\dim Ex(h) + \dim Ex(h^+) \geq \dim Y - 1$ (see 3), we get $\dim Ex(h^+) = \dim Y - 2$.

Note that the numerical $h$-triviality of $K_Y + H$ implies $H^+ \cdot C' < 0$ for any curve $C'$ on $Ex(h^+)$. Therefore, $Ex(h^+) \subset H^+$.

Let $E$ be the exceptional divisor of blow-up centered at a component of $Ex(h^+)$. We may assume that the center of $E$ on $Y$ is not contained in $H$. Then we have the following inequality:

$$a(E; Y, H) \leq a(E; Y^+, H^+) \leq 0,$$

where $a(E; Y, H)$ (resp. $a(E; Y^+, H^+)$) is the discrepancy of $E$ with respect to $K_Y + H$ (resp. $K_{Y^+} + H^+$). This is a contradiction because $Y$ is terminal. Q.E.D.

**Lemma 4.5.** The extremal contraction $g$ is not a small contraction whose exceptional locus has a curve as an irreducible component.

**Proof.** Suppose that $g$ is a small contraction whose exceptional locus has a curve $C$ as an irreducible component. We have $K_V \cdot C > -1$ by Lemma 4.4. On the other hand,

$$(K_V + \frac{n-1}{n}\tilde{S}) \cdot C = (f^*O_{\mathbb{P}^{n-1}}(-1) - E) \cdot C = -\deg (f_*C) - E \cdot C \leq -1.$$ 

Thus, $\tilde{S} \cdot C < 0$ and $C \subset \tilde{S}$, and hence

$$(K_V + \tilde{S}) \cdot C < (K_V + \frac{n-1}{n}\tilde{S}) \cdot C \leq -1.$$ 

Let $\nu : \tilde{S} \rightarrow \check{S}$ be a normalization of $\tilde{S}$. By adjunction, we have

$$K_{\check{S}} + \text{Diff}_{\check{S}}(0) = \nu^*((K_V + \tilde{S})|_{\check{S}}).$$

Since $\check{S}$ is smooth at a generic point of the curve $C$, the curve $\nu^{-1}_*C$ cannot be contained in $\text{Diff}_{\check{S}}(0)$, and hence $K_{\check{S}} \cdot \nu^{-1}_*C < -1$. On the other hand, $K_{\check{S}} \cdot \nu^{-1}_*C \geq -1$ since the curve $\nu^{-1}_*C$ is contractible. Q.E.D.

So far, we proved that the extremal contraction $g$ is a contraction of a subvariety $F$ of $V$ to a subvariety $Z$ of $W$ with $\dim F - \dim Z = 1$ and $\dim Z \geq 1$. Let $C$ be a general fiber of the morphism $g$ over $Z$. Then we have

$$(K_V + \frac{n-1}{n}\tilde{S}) \cdot C = (f^*O_{\mathbb{P}^{n-1}}(-1) - E) \cdot C = -\deg (f_*C) - E \cdot C < -\deg (f_*C),$$
because the curve $C$ should meet the exceptional locus of $f$.

**Lemma 4.6.** If the extremal contraction $g : V \to W$ is not a conic bundle, then $F = \tilde{S}$.

**Proof.** Let $C$ be a general enough fiber of the morphism $g$. The inequality

$$-1 \leq K_V \cdot C = (K_V + \frac{n-1}{n} \tilde{S}) \cdot C - \frac{n-1}{n} \tilde{S} \cdot C < -\deg(f_* C) - \frac{n-1}{n} \tilde{S} \cdot C$$

implies that $F \subset \tilde{S}$. If the codimension of the subvariety $F$ of $V$ is greater than 1, then we get

$$-1 \geq -\deg(f_* C) > (K_V + \frac{n-1}{n} \tilde{S}) \cdot C = (K_V + \tilde{S}) \cdot C - \frac{1}{n} \tilde{S} \cdot C > -1 - \frac{1}{n} \tilde{S} \cdot C > -1,$$

where the second to the last inequality is implied by Lemma 4.4. This is absurd. Hence, the subvariety $F$ has codimension 1. Consequently, $F = \tilde{S}$. Q.E.D.

Now we know that the extremal contraction $g$ is either a conic bundle or a contraction of the divisor $\tilde{S}$ of $V$ to a subvariety $Z$ of $W$ with $\dim Z = n - 3$.

**Theorem 4.7.** If the Log minimal model program holds for dimension $\leq n - 1$, then Conjecture 2.4 holds for $n$.

**Proof.** Suppose that $g$ is a conic bundle. Then $[E] \cap C \neq \emptyset$ since no component of the divisor $E$ lies in the fibers of $g$. Therefore,

$$\deg(f_* C) = -(K_V + \frac{n-1}{n} \tilde{S} + E) \cdot C = 2 - \frac{n-1}{n} \tilde{S} \cdot C - E \cdot C < 2.$$ 

Consequently, $f_* C$ is a line on $S$. This implies that $S$ is a cone in $\mathbb{P}^{n-1}$.

Now, we suppose that the morphism $g$ is not a conic bundle. Then it is a contraction of the divisor $\tilde{S}$ of $V$ to a subvariety $Z$ of $W$ with $\dim Z = n - 3$. Therefore, we have

$$-\deg(f_* C) > (K_V + \frac{n-1}{n} \tilde{S}) \cdot C = (K_V + \tilde{S}) \cdot C - \frac{1}{n} \tilde{S} \cdot C = -2 - \frac{1}{n} \tilde{S} \cdot C > -2.$$ 

Thus, $f_* C$ is a line on $S$. Consequently, $S$ is a cone in $\mathbb{P}^{n-1}$. Q.E.D.

**Corollary 4.8.** Conjecture 2.4 holds for $n = 4$. 
Proof. The Log minimal model program has been proven for dimension 3. Theorem 4.7 implies the statement. Q.E.D.

Corollary 4.9. The total log canonical threshold of a smooth quartic $X$ in $\mathbb{P}^4$ is $\frac{3}{4}$ if and only if the quartic $X$ has an Eckardt point.

Proof. This immediately follows from Corollary 4.8. Q.E.D.

5. Application

Let $O$ be a discrete valuation ring with quotient field $K$. We assume that the residue field is of characteristic zero. For a scheme $\pi: X \to \text{Spec} O$, we denote its scheme-theoretic fiber $\pi^*(o)$ by $S_X$, where $o$ is the closed point of $\text{Spec} O$.

Theorem 5.1. Let $X$ and $Y$ be smooth Fano fibrations over $\text{Spec} O$ such that their generic fibers are each isomorphic to a smooth hypersurface of degree $n$ in $\mathbb{P}^n_K$, where $n \geq 3$. Then any birational map of $X$ into $Y$ over $\text{Spec} O$ which is biregular on the generic fiber is biregular.

Proof. Note that our birational map cannot be an isomorphism in codimension 1 (see [5]). Anticanonical divisors of $S_X$ and $S_Y$ are very ample. Moreover, their total log canonical thresholds are strictly larger than $\frac{1}{2}$. Therefore, the same method as in [14] works. Q.E.D.

Remark. Birational rigidity has been proven for any smooth hypersurface of degree $n$ in $\mathbb{P}^n$ and any generic hypersurface of degree $m$ of $\mathbb{P}^m$, where $4 \leq n \leq 8$ and $m \geq 9$. For details, refer to [1], [7], [8], [16], and [18]. Using birational rigidity, we can easily prove Theorem 5.1 in the case of $n \leq 8$. Also, we can obtain a weaker statement than Theorem 5.1 for $n \geq 9$.

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