We consider the role of the diffeomorphism constraint in the quantization of lattice formulations of diffeomorphism invariant theories of connections. It has been argued that in working with abstract lattices, one automatically takes care of the diffeomorphism constraint in the quantum theory. We use two systems in order to show that imposing the diffeomorphism constraint is imperative to obtain a physically acceptable quantum theory. First, we consider 2 + 1 gravity where an exact lattice formulation is available. Next, general theories of connections for compact gauge groups are treated, where the quantum theories are known –for both the continuum and the lattice– and can be compared.

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I. INTRODUCTION

Within the canonical quantization of diffeomorphism invariant theories of connections, at some stage one is forced to deal with the diffeomorphism constraint. Examples of such theories are given by gravity in 3 and 4 dimensions, when formulated as theories of connections [1,2]. As first noticed by Rovelli and Smolin [3], when the basic observables are the (traces of) holonomies of the connection \( W_\gamma = \text{tr}(h_\gamma) \) along loops \( \gamma \) in the manifold, finite diffeomorphisms act simply by shifting the loops as prescribed by the mapping \( \phi \): 
\[
U_\phi \circ W_\gamma = W_{\phi(\gamma)}.
\]
Further developments of this idea rigorously solve the diffeomorphism constraint in the continuum. For a complete discussion of the quantization of diffeomorphism invariant theories see [4].

In theories that have discretized space from the outset, the issue of the diffeomorphism constraint has a special character. In some of these discretized theories, space is replaced with an abstract lattice; that is, a lattice that does not “reside” in a manifold [5,6]. Hence the diffeomorphism symmetry is lost in this approach. However, an abstract lattice \( L \) can be viewed as an equivalence class of embedded lattices \( \{L\} \) under the equivalence relation defined by the diffeomorphism group, in the sense used in knot theory to define knot classes. It has been argued that this picture implies that the theory defined on an abstract lattice is manifestly diffeomorphism invariant, and therefore, there is no need to further impose the diffeomorphism constraint [6]. A different approach considers that even after discretization, the phase space associated to the abstract lattice still has a reminiscent symmetry related to the diffeomorphism constraint. This strategy has been successful in treating theories without local degrees of freedom like 2 + 1 gravity and \( B \wedge F \) theories [5,7]. A third strategy on the lattice, embedded in a manifold, is to require invariance under a discrete symmetry such that diffeomorphism invariance is recovered in the continuum limit [8,9]. We will focus our attention on the first two strategies for discrete theories.

The physically meaningful aspects of a lattice theory are the ones regarding its continuum limit. This makes comparison between different lattice approaches a delicate issue. To illustrate the discrepancy between the two lattice approaches described above, we shall consider 2 + 1 dimensional gravity. In this framework the role of the continuum limit is not very significant because it is a theory with only global degrees of freedom that are fully contained in the lattice theory already. We show that by assuming that a theory based on abstract lattices is manifestly diffeomorphism invariant one fails to ‘subtract’ the appropriate degrees of freedom from the unconstrained theory. Therefore, the resulting theory is not the one we wanted to quantize; it has too many degrees of freedom.

In the second part of the article, we consider diffeomorphism invariant theories for compact gauge groups. For this class of theories, there is a well defined formalism to treat the quantization both in the continuum and in the lattice.
For completeness, we first review this formalism and then show that, independently of the details of the continuum limit one might choose to take, there is still a residual symmetry associated to diffeomorphisms.

The article is organized as follows. Section II reviews the classical description of 2 + 1 gravity as a theory of connections, both in the continuum and in the lattice formulation. Section III is devoted to discussing the quantum theory for general theories of connections with compact gauge groups. We review first the quantization in the continuum and then in the lattice. It is shown that one must impose all constraints, including the ones associated to diffeomorphism, in order to recover the physical theory. We conclude in Sec. IV with a discussion.

II. 2 + 1 GRAVITY IN THE CONNECTION-DYNAMICS FORMULATION

A. 2 + 1 in the continuum

In this part we give a brief review of the continuum description following the notation of [11]. For a detailed exposition we refer the reader to the original papers [2,10]. Let $M$ be a 3-dimensional manifold of the form $\Sigma \times R$, where $\Sigma$ is a compact, oriented 2-manifold. In the Hamiltonian formulation of the theory, the phase space $\Gamma$ can be coordinatized by pairs $(A^I_a, \tilde{E}^b_I)$ on $\Sigma$. Here, $A^I_a$ is the pull-back to $\Sigma$ of the 3-dimensional Lorentz connection. As a field on the hyper-surface it is a one-form with values in the Lorentz Lie-algebra so(2,1). The canonical momenta is given by the vector densities $\tilde{E}^b_I$. The 1-forms $e_{af} = g_{ab} \tilde{E}^b_I$ are the pull-back to $\Sigma$ of the co-triad $\tilde{e}^a_I$ on $M$.

(Throughout we will raise and lower the internal index $I$ with the Minkowski metric $\eta_{IJ} = \text{diag}(-,+,+)$). The canonical pair of variables satisfy the usual Poisson bracket relations,

$$\{ A^I_a(x), \tilde{E}^b_J(y) \} = \delta^b_J \delta^I_a \delta^{(3)}(x,y).$$  \hspace{1cm} (2.1)

From this variables we can recover the 2-dimensional metric $g_{ab}$ on $\Sigma$ of the usual geometrodynamical formulation as follows:

$$g_{ab} = \eta_{IJ} e^I_a e^J_b.$$ \hspace{1cm} (2.2)

Since we want to consider positive definite metrics $g_{ab}$, we should impose the restriction on the phase space coordinates that $\tilde{E}$ be non-degenerate.

There are two constraints on the phase space, given by

$$D_a \tilde{E}^a_I = 0 \quad \text{and} \quad F^I_{ab} = 0,$$  \hspace{1cm} (2.3)

where $D$ denotes the generalized covariant derivative defined by $A^I_a$, namely $D_a v_b J := \partial_a v_b J + \Gamma^c_{ab} v_c J + \varepsilon_{JLM} A^L_a v_M K$. $F^I_{ab}$ is the curvature of the connection: $F^I_{ab} = 2\partial_a [A^I_b] + [A_a, A_b]^I$. The first constraint is known as the Gauss law. In order to analyze what the constraints generate, let us define the smeared functions on $\Gamma$,

$$G[v] := \int_{\Sigma} v^I D_a \tilde{E}^a_I,$$

$$F[\alpha] := \int_{\Sigma} \alpha^{ab} F^I_{ab}.$$  \hspace{1cm} (2.4)

The infinitesimal canonical transformation generated by the constraint functions will depend, of course, on the smearing fields $(v, \alpha)$.

The Gauss constraint $G[v]$ generates gauge transformations on the connection $A^I_a$ and `rotations' in the momenta $\tilde{E}^a_I$. The constraint function $F[\alpha]$ leaves the connection invariant and `shifts' the conjugate momenta. We know that this constrained system is equivalent to the geometrodynamical description [11]. Therefore, an appropriate combination of the constraint functions should generate the induced action of diffeomorphisms on $\Sigma$. Let us start by considering the connection $A$. Since the curvature constraint does not depend on the momenta $E$, it leaves the connection unchanged. That means that the only way one can generate diffeomorphism on the connection is via the Gauss constraint, with an appropriate smearing field $v^I$.

Consider the vector field $V^a$ on $\Sigma$, as generator of infinitesimal diffeomorphisms. The function,

$$\int_{\Sigma} (A^I_b V^b) D_a \tilde{E}^a_I,$$  \hspace{1cm} (2.5)
generates the required action on $A$ since,

$$\left\{ A^a_i, \int_\Sigma (A^b_j V^b) \mathcal{D}_a \mathcal{E}^j_b \right\} = -\mathcal{D}_a (A^b_j V^b),$$

$$= -V^b F^I_{ab} - \mathcal{L}_V A^b_a \equiv -\mathcal{L}_V A^b_a. \quad (2.6)$$

More generally, the generator of spatial diffeomorphisms is given by,

$$D[V] = \int_\Sigma V^b (A^b_j \mathcal{D}_a \mathcal{E}^j_b - \mathcal{E}^j_b F^I_{ab}). \quad (2.7)$$

Note that the second term, where we have chosen $\tilde{\alpha}^I_{ab} = V^b \mathcal{E}^j_b$, can also be interpreted as the projection of the curvature constraint along the vector density $\mathcal{E}^j_b$. One way to impose the curvature constraint is to require that the projections of the curvature in every internal direction vanish. In fact, after imposing (2.7) we need only one more projection of the curvature constraint.

$$H[N] := \int_\Sigma \mathcal{N} \varepsilon^I_{JK} \tilde{\mathcal{E}}^a_j \tilde{\mathcal{E}}^b_k F^I_{ab}, \quad (2.8)$$

where $\mathcal{N}$ is a scalar density of weight $-1$. More precisely, provided that the co-triad $e^I_a$ is non-degenerate, the Witten constraints (2.4) and the Ashtekar constraints (below) are equivalent [11].

$$\mathcal{D}_a \tilde{\mathcal{E}}^a_j = 0, \quad \tilde{\mathcal{E}}^b F^I_{ab} = 0, \quad \varepsilon^I_{JK} \tilde{\mathcal{E}}^a_j \tilde{\mathcal{E}}^b_k F^I_{ab} = 0. \quad (2.9)$$

The Witten constraints can be seen as ‘covariant’ since there is no explicit decomposition of the 3-dimensional diffeomorphisms into a ‘spatial’ and ‘orthogonal’ components. The Ashtekar constraints on the other hand, do exhibit this decomposition. They are more closely related to the constraints one encounters in the 3+1-dimensional theory. When one allows for degenerate triads (and therefore singular induced metrics), the constraints (2.4) do not imply the vanishing of the curvature $F^I_{ab}$. The structure of such extended phase space has been studied in [12].

**B. 2+1 gravity in the Lattice Formulation**

In this part, we recall the lattice formulation of 2 + 1 gravity due to Waehbroeck [3]. For the convenience of the reader the roles of the dual lattice and the lattice itself are reversed, with respect to [1]. The starting point is a two dimensional abstract lattice of arbitrary valence; the $N_0$ vertices of the lattice are labeled by greek indices $(\alpha), (\beta), \ldots$, the $N_1$ links are labeled as pairs of vertices $(\alpha\beta), (\epsilon\gamma), \ldots$ and the $N_2$ faces are labeled as $f_1, f_2, \ldots$. To each vertex $(\alpha)$ of the lattice we assign a three dimensional Minkowski frame (a copy of the Lie algebra so(2,1)); the role of the connection is played by 3-dimensional Lorentz matrices $M(\alpha\beta)$ that define parallel transport from vertex $(\beta)$ to vertex $(\alpha)$. In this way the configuration space of the lattice theory is $SO(2,1)^{N_1}$. The momenta $E(\alpha\beta)$, represented by vectors of the Lie algebra, label the left and right-invariant vector fields of $SO(2,1)$ as can be seen in the Poisson algebra

$$\{ E(\alpha\beta)^I, E(\alpha\beta)^J \} = \varepsilon^{IJ}_{JK} E(\alpha\beta)^K, \quad (2.10)$$

$$\{ E(\alpha\beta)^I, M(\alpha\beta)^J_{\ K} \} = \varepsilon^{IJ}_{LK} M(\alpha\beta)^L_{\ K}, \quad (2.11)$$

$$\{ E(\alpha\beta)^I, M(\beta\alpha)^J_{\ K} \} = \varepsilon^{IJ}_{LK} M(\beta\alpha)^L_{\ K}. \quad (2.12)$$

The rest of the non-vanishing brackets follow from the identities

$$M(\alpha\beta)^I_{\ J} M(\alpha\beta)^J_{\ K} = \delta^K_I, \quad (2.13)$$

$$M(\alpha\beta)^I_{\ J} M(\alpha\beta)^J_{\ K} = \eta^K_I, \quad (2.14)$$

$$M(\alpha\beta)^I_{\ J} E(\beta\alpha)^J = -E(\alpha\beta)^I. \quad (2.15)$$

3
Identity \((2.13)\) plays a very important role; first, it relates the Lie algebra vectors that label the left and right-invariant vector fields to give three independent momentum coordinates per link of the lattice. Second, thanks to \((2.15)\) a geometric interpretation of the variables is possible. The links of the dual lattice are placed in the Minkowski frames of the vertices of the original lattice (dual faces) according to the variables \(E_{(a\beta)}\). And after Gauss’s law is satisfied

\[ J(\alpha)^I := E_{(a\beta)}^I + E_{(a\gamma)}^I + \ldots = 0, \quad (2.16) \]

the faces of the dual lattice close. To make this lattice gauge theory describe vacuum \(2+1\) gravity we have to require that the curvature vanishes; in other words, the parallel transport around every face of the lattice (around every vertex of the dual lattice) must be the identity. That is,

\[ W(f_1)^I_\alpha := (M_{a\beta}M_{b\mu} \ldots M_{\rho\alpha})^I_\alpha = \delta^I_\alpha, \quad (2.17) \]

where the boundary of face \(f_1\) is composed by the links \((a\beta), (b\mu), \ldots, (\rho\alpha)\). Therefore the holonomy around every contractible loop of the lattice is restricted to be the identity. The curvature constraint can be replaced by \(P(f_1)^I := \frac{1}{2} \varepsilon^{I} J K W(f_1)^K_\alpha = 0\). The constraints are first class, and generate Lorentz transformations at the frame \((\alpha)\) and translations of vertex \((f_1)\) of the dual lattice respectively.

\[ \begin{align*}
\{ J(\alpha)^I, E_{(a\beta)}^J \} &= \varepsilon^{I} J K E_{(a\beta)}^K, \\
\{ J(\alpha)^I, M_{(a\beta)}^J \} &= \varepsilon^{I} J L M_{(a\beta)}^L, \\
\{ \varepsilon^{I} P(f_1)^I, E_{(a\beta)}^J \} &\approx \xi^J. 
\end{align*} \quad (2.18, 2.19, 2.20) \]

(The weak equality indicates that we are restricting ourselves to the constraint surface). Waelbroeck’s constraints \((2.16), (2.17)\) are the precise analog of the Witten constraints \((2.4)\) in the continuum.

When the \(E\)’s are non-degenerate the constraints can be written in an equivalent Ashtekar form. More precisely, when at every vertex \((\alpha)\) the vector space generated by \(E_{(a\beta)}, E_{(a\gamma)}, \ldots\) is at least two dimensional, the projected curvature constraints

\[ \begin{align*}
H_{(a\beta)} &:= B(\alpha)^I E_{(a\beta)}^I = 0, \\
H(\alpha) &:= \varepsilon^{I} J K B(\alpha)^I (E_{(a\beta)}^J E_{(a\gamma)}^K + E_{(a\gamma)}^J E_{(a\delta)}^K + \ldots) = 0, \quad (2.21, 2.22) \]
\]

imply \(B(\alpha)^I := \sum_{\alpha \in f_1} P(f_1)^I = 0\). Hence, the constraints \((2.16), (2.21)\) and \((2.22)\), that are of the Ashtekar type, form a first-class system.

The question of whether the projected constraints \((2.21), (2.22)\) are equivalent to the covariant constraint \((2.17)\) is studied in the appendix. We consider the case of a square \(M \times N\) lattice with periodic boundary conditions, i.e. \(T^2\) represented by rectangular grid; in this case we prove that the two sets of constraints are equivalent if and only if \(M\) and \(N\) are odd. Because of its interest in the case of a possible relation with Regge calculus, we consider the case of three-valent lattices. These are lattices whose duals are triangular lattices. Again there are cases of lattices where the covariant and projected versions of the constraints are not equivalent. However, given one of this lattices \(L\) where the two sets of constraints are not equivalent we construct a new tri-valent lattice \(L’\) where the the projected constraints \((2.21), (2.22)\) are equivalent to the covariant constraints \((2.17)\). We know that any surface can be represented by a three-valent lattice (because any surface can be triangulated). Therefore we can represent any two dimensional space by a lattice where the covariant constraints are equivalent to the projected constraints, in the sector where the \(E\)’s are non-degenerate.

The constraint \((2.22)\) is the lattice version of the Hamiltonian constraint (it is linear in the curvature and quadratic in the momenta), while \((2.21)\) plays the role of the diffeomorphism constraint (it is linear in the curvature and in the momenta). We see that algebraically \((2.21)\) takes care of the residual diffeomorphism symmetry. Geometrically the constraint is the generator of phase space symmetries that do correspond to translations of the vertices (dual faces) of the lattice.

\[ \text{The sum is over the curvature vectors of all the faces that contain vertex } (\alpha); \text{ the orientation of the faces is the one induced by the orientation of the lattice.} \]
In $2+1$ gravity the lattice theory is exact in the sense that it contains all the true degrees of freedom of the continuum theory. The number of variables of the phase space is $6N_1$, while the number of first-class constraints is $3N_0 + 3N_2$. Hence the dimension of the reduced phase space is

$$6N_1 - 2(3N_0 + 3N_2) = 12g - 12,$$

where $g$ is the genus of space. In this lattice approach two different lattices with the same topology give rise to two descriptions of the same reduced phase space.

In this subsection we have recalled Waelbroeck’s exact lattice formulation of $2+1$ gravity, and we have rewritten the constraints in a projected form. By the counting of degrees of freedom given above, we see that in a quantum theory based on an abstract lattice one needs to impose a diffeomorphism constraint to reproduce a classical theory with the correct number of degrees of freedom. This is the first remark of this paper.

### III. QUANTIZATION

In this section, we will obtain the second result of the article. The section is divided into three parts. In the first one, we give some heuristic arguments that suggest the existence of a residual diffeomorphism symmetry for theories defined on an abstract lattice. In the second part, we make a small detour and recall, for the continuum, the quantization of theories of connections that have a compact gauge group, such as gravity in $3+1$ dimensions and Euclidean $2+1$ gravity. In the last part, we consider the quantization of the lattice formulation and compare it to the continuum.

#### A. Heuristics

The issue of diffeomorphism invariance in the quantization of the lattice theory is a subtle one, since it is not apriori clear how to achieve a continuum limit which is compatible with diffeomorphism invariance. The proposal that by working on an abstract lattice one automatically solves the diffeomorphism constraint is an attractive one. The idea is that by quantizing the gauge theory on an abstract lattice, one gets a family of states that should be mapped to some diffeomorphism invariant states after the continuum limit is taken.

In order to test the full physical viability of the theory one should wait for the construction of a consistent continuum limit and then compare its predictions with the ‘known physics’. However, there are general arguments that one can give, that do not depend on the details of the particular continuum limit that one selects. That is the aim of the second half of the paper. We want to argue that, roughly speaking, there are states defined over the lattice that should be identified in any continuum limit that recovers diffeomorphism invariance, and that are clearly distinct from the viewpoint of the abstract lattice. Therefore, there is a residual diffeomorphism symmetry even when working in the abstract lattice.

Let us consider, for instance, the lattice of Fig.1, together with two loops $\gamma_1$ and $\gamma_2$. As loops defined on the lattice, they are clearly different. However, when seen as embedded in a manifold, there is a diffeomorphism that maps $\gamma_1$ to $\gamma_2$. One could expect that in a limit that recovers spatial diffeomorphisms, states defined over $\gamma_1$ and $\gamma_2$ are identified.
Fig. 1 The loops $\gamma_1, \gamma_2$ are given by $\gamma_1 = e_1 \circ e_2 \circ e_3 \circ e_4$ and $\gamma_2 = e_1 \circ e_2 \circ e_5 \circ e_6 \circ e_7 \circ e_4$ where $e_1, \ldots, e_7 \in L$. The loops are diffeomorphic but the states defined on them are different.

To make these intuitive arguments conclusive, we shall go in to the details of the quantization on the continuum, and later compare it to the quantization of the abstract lattice.

B. Quantization in the Continuum

In the quantization of the continuum theory, we can follow the procedure introduced in [1] for 3+1 dimensions. The main steps are the introduction of a kinematical Hilbert space where the quantum constraints are to be defined, and the construction of the Hilbert space of solutions to the Gauss and diffeomorphism constraint. One basic assumption in this program is that the holonomies of the connection $A_i$ are well defined operators in the kinematical Hilbert space. This assumption leads to $\mathcal{A}$ that ‘completes’ the space of smooth connections $\mathcal{A}$, to form the quantum configuration space of ‘generalized connections’.

Here we review the characterization of $\mathcal{A}$ as a limit of a family of configuration spaces living on finite ‘floating’ lattices $\{\Sigma|_{L}\}$. To each lattice $L$ embedded in $\Sigma$ we assign a configuration space $\mathcal{A}_L = G^{N_1}$. A point in $\mathcal{A}_L$ is represented by $(g_1, g_2, \ldots, g_{N_1})$, each group element $g_i$ is to be thought of as the path-ordered exponential of the connection along the link $e_i$ $(i = 1, \ldots, N_1)$. The collection of all these configuration spaces corresponding to every lattice gives an over-complete description of $\mathcal{A}$. It is possible to keep track of all the repetition by means of a projective structure. We say that lattice $L$ is a refinement of lattice $L'$ ($L \leq L'$) if every link $e \in L'$ either $e = e_1 \circ e_2$ or $e = e_1 \circ \ldots \circ e_n$ for some $e_1, \ldots, e_n \in L$. For any pair of lattices related by refinement $L \geq L'$ there is a projection $p_{L'L} : \mathcal{A}_L \rightarrow \mathcal{A}_{L'}$,

$$\left(g_1, g_2, g_3, \ldots, g_{N_1}\right) p_{L'L} (g_1' = g_2 g_1, g_2', \ldots, g_{N_1}'), \quad (3.1)$$

where $e = e_1 \circ e_2$, $e \in L'$, $e_1, e_2 \in L$.

The projection map and the refinement relation have two properties that will allow us to define $\mathcal{A}$ as ‘the configuration space of the finest lattice’. First, we can check that $p_{L'L'} \circ p_{L'L''} = p_{L'L''}$. Second, equipped with the refinement relation ‘$\geq$’, the set of embedded lattices $\mathcal{L}$ is a partially ordered, directed set; i.e. for all $L, L'$ and $L''$ in $\mathcal{L}$ we have:

$$L \geq L \quad ; \quad L \geq L' \quad \text{and} \quad L' \geq L \Rightarrow L = L' \quad ; \quad L \geq L' \quad \text{and} \quad L' \geq L'' \Rightarrow L \geq L'' ; \quad (3.2)$$

and, given any $L', L'' \in \mathcal{L}$, there exists $L \in \mathcal{L}$ such that

$$L \geq L' \quad \text{and} \quad L \geq L'' . \quad (3.3)$$

The space $\mathcal{A}$ is the projective limit of the projective family, defined as follows:

$$\mathcal{A} := \{(A_L)_{L \in \mathcal{L}} \in \times_{L \in \mathcal{L}} \mathcal{A}_L : L' \geq L \Rightarrow p_{L'L} A_{L'} = A_L\} . \quad (3.4)$$

That is, the projective limit is contained in the Cartesian product of all possible configuration spaces $\mathcal{A}_L$, subject to the consistency conditions stated above. There is a canonical projection $p_L$ from the space $\mathcal{A}$ to the spaces $\mathcal{A}_L$ given by,

$$p_L : \mathcal{A} \rightarrow \mathcal{A}_L, \quad p_L((A_L)_{L \in \mathcal{L}}) := A_L . \quad (3.5)$$

Given any function $f_L$ defined on the space $\mathcal{A}_L$ one can define a function on $\mathcal{A}$ via the pull-back $p_L^* : C^0(\mathcal{A}_L) \rightarrow \text{Fun}(\mathcal{A})$. Such functions are called cylindrical, and the space of such functions is denoted by $\text{Cyl}(\mathcal{A})$. In a similar fashion, we can define Hilbert spaces for each space $\mathcal{A}_L$ and ‘pull them back’ to the projective limit. That is, we have a projective family of Hilbert spaces $\mathcal{H}_L$.

The final picture is that the kinematical Hilbert space is the space of square integrable functions on $\mathcal{A}$ with respect to a measure $\mu$, that is,

$$\mathcal{H}_{\text{kin}} := L^2(\mathcal{A}, d\mu) . \quad (3.6)$$

In the case of compact gauge groups $G$, the uniform Haar measure $\mu_0$ on each copy of the group corresponding to the links, endows the Hilbert space $\mathcal{H}_{\text{kin}}$ with a probability measure that is diffeomorphism invariant (also denoted by $\mu_0$).

Cylindrical functions define a dense subset on the Hilbert space. It is important to point out that even though the states are associated to the lattice, the natural labels for them are graphs $\gamma$ on $L$. A (closed) graph $\gamma$ is just
a collection of links of the lattice \( L \). We shall therefore denote the states by \( \Psi_\gamma \), without explicitly writing down the lattice \( L \) where \( \gamma \) lies. Note that in this representation the connection is ‘diagonal’. More precisely, given a function \( \Psi_\gamma(A) \) defined on a lattice \( L \), for example the trace of the holonomy \( T_\eta \) along a loop \( \eta \) contained in \( L \), the corresponding operator will act by multiplication:

\[
(\hat{T}_\eta \cdot \Psi_\gamma)(A) := T_\eta(A)\Psi_\gamma(A). \tag{3.7}
\]

The next step involves defining the constraints on this Hilbert space \( \mathcal{H}_{\text{kin}} \); in practice we do it by defining the action of the constraints on \( \text{Cyl}(A) \) and extend it to the whole \( \mathcal{H}_{\text{kin}} \). The first constraint we will consider is the Gauss constraint, that generates gauge transformations. A finite gauge transformation takes the holonomy \( g_1 \) to \( g(\alpha)g_1g(\beta)^{-1} \) (where edge \( e_1 \) goes from vertex \( \alpha \) to vertex \( \beta \)). The quantum Gauss constraint should impose gauge invariance on the wave functions; therefore, on a cylindrical function, \( \Psi_\gamma(A) = \psi(g_1, \ldots, g_{N_i}) \), the Gauss constraint takes the form, for each vertex \( \alpha \) of the lattice \( L \):

\[
\sum_{e_{\alpha}} X_{e_{\alpha}}^I \cdot \phi_\gamma = 0, \tag{3.8}
\]

where \( X_{e_{\alpha}}^I \) is a right or left invariant vector field on the internal direction \( I \) depending on whether the link \( e_{\alpha} \) is incoming or outgoing, respectively. Gauge invariant functions on \( \text{Cyl}(A) \) can be seen as functions on the quotient space of \( \mathcal{A} \) by the space of generalized gauge transformation \( \mathcal{G} \). Thus, \( L^2(\mathcal{A}/\mathcal{G}, d\mu) \) is the Hilbert space of solutions to the Gauss constraint.

It turns out that \( L^2(\mathcal{A}/\mathcal{G}, d\mu) \) is not well suited to define the generator of diffeomorphisms (see [4]); instead, we represent finite diffeomorphisms as unitary operators on the Hilbert space. Let us denote by \( \varphi \) a finite diffeomorphism on \( \Sigma \). We want to define \( \hat{U}_{\varphi} : L^2(\mathcal{A}/\mathcal{G}, d\mu) \rightarrow L^2(\mathcal{A}/\mathcal{G}, d\mu) \), to be a unitary operator. For any diffeomorphism invariant measure \( \mu \) we can define such representation, it will be given by,

\[
\hat{U}_{\varphi} \cdot \Psi_\gamma(A) := \Psi_{\varphi^{-1}(\gamma)}(A). \tag{3.9}
\]

That is, the induced action of \( \varphi \in \text{Diff}(\Sigma) \) is to ‘move’ the graph \( \gamma \) by the inverse diffeomorphism. Solutions to the diffeomorphism constraint will be those states that are left invariant by the unitary operators \( \hat{U}_{\varphi} \). These states will not belong to the kinematical Hilbert space \( \mathcal{H}_{\text{kin}} \). This is a general feature of the Dirac quantization method for constrained systems, whenever the ‘gauge group’ generated by the constraint to be imposed is non-compact.

Solutions to the diffeomorphism constraint are distributions, living on the dual space \( \Phi' \) to the dense sub-space \( \Phi = \text{Cyl}(\mathcal{A}/\mathcal{G}) \) of gauge invariant, cylindrical functions. An element \( \hat{\phi} \) of \( \Phi' \) is said to be a solution to the diffeomorphism constraint if

\[
\hat{\phi}(\hat{U}_{\varphi} \circ \psi) = \hat{\phi}(\psi) \quad \forall \ \varphi \in \text{Diff}(\Sigma) \quad \text{and} \quad \psi \in \Phi, \tag{3.10}
\]

where ‘\( [\cdot] \)’ denotes the dual action, \( \hat{\phi}(\psi) = \int_{\mathcal{A}/\mathcal{G}} d\mu \hat{\phi}\psi \).

One can construct such distributions by ‘group averaging’ over the group \( \text{Diff}(\Sigma) \). The infinite size of \( \text{Diff}(\Sigma) \) makes a precise definition of the group average procedure very subtle; here we follow the procedure given in [4]. An inner product for the space of solutions is given by the same formula that defines the group averaging; therefore, a summation over all the elements of \( \text{Diff}(\Sigma) \) would yield states with infinite norm. In this sense, prescribing an adequate definition for the averaging over the group \( \text{Diff}(\Sigma) \) involves some ‘renormalization.’ We shall give the details below.

An appropriate definition of the group averaging procedure follows from two observations. First, the inner product between two states induced by the cylindrical functions \( f_\gamma, g_\delta \in \Phi \) must be zero unless there is a diffeomorphism \( \varphi_0 \in \text{Diff}(\Sigma) \) that connects the two graphs \( \gamma \) and \( \delta \) \( \gamma = \varphi_0\delta \). Second, the construction of generalized connections assigns group elements to un-parametrized edges. Therefore, two diffeomorphisms that restricted to a graph \( \gamma \) are equal except for a reparametrization of the edges of \( \gamma \) should be counted only once in the construction of group averaging of states based on graph \( \gamma \). Thus, given a cylindrical function \( f_\gamma \in \Phi \), we define \( F_\gamma \in \Phi' \) by

\[
F_\gamma[g_\delta] := \delta_{[\gamma][\delta]} \sum_{[\varphi] \in \text{GS}(\gamma)} \langle \hat{U}_{\varphi^{-1}} \circ f_\gamma \rangle \tag{3.11}
\]

where \( \delta_{[\gamma][\delta]} \) is non vanishing only if there is a diffeomorphism \( \varphi_0 \in \text{Diff}(\Sigma) \) that maps \( \gamma \) to \( \delta \); and \( \varphi \in \text{Diff}(\Sigma) \) is any element in the class of \( [\varphi] \) in \( \text{GS}(\gamma) \). The discrete group \( \text{GS}(\gamma) \) is the group of symmetries of \( \gamma \); i.e. elements of \( \text{GS}(\gamma) \) are maps between the edges of \( \gamma \). The group can be constructed from subgroups of \( \text{Diff}(\Sigma) \) as follows:
The matrices of parallel transport along the cylindrical functions are naturally labeled by graphs rather than by lattices; therefore a distribution by elements of the Lie algebra $g$ that does not map a whole lattice to itself can identify states that are different at the non-diffeomorphism invariant Hilbert space $H$. On the other hand, in the lattice construction, the Hilbert spaces of two different classes of lattices have a non-trivial intersection (due to the subtlety mentioned before), so the sum is not direct. Thus, the imposition of the Gauss constraint follows very closely that of the continuum, the condition of the wave functions being

$$-i \sum_{e_\alpha} X^I_{e_\alpha} \cdot \Psi = 0,$$

where $X^I_{e_\alpha}$ is a left or right invariant vector field depending on our choice of $E^I_{(\alpha\beta)}$ or $E^I_{(\beta\alpha)}$ to denote the momenta associated to the link $(\alpha\beta)$. Mathematically, this representation is identical to the Hilbert space $H_L$ of the continuum for any embedding $L$ of the lattice $L_0$ into $\Sigma$. One can prove that there is an isomorphism $F_{L_0} : H_{L_0} \rightarrow H_{[L]}$ from the Hilbert space constructed from the abstract lattice to the Hilbert space of diffeomorphism invariant states $H_{[L]}$.
But comparison of the two descriptions should be made between the Hilbert space \( \mathcal{H}_{\text{diff}} \) and the (‘continuum’) limit of the spaces \( L^2(\mathcal{A} / \mathcal{G}_{L_0}, d\mu) \) in which the abstract lattice \( L_0 \) fills all space. The fact that a definite procedure to take the continuum limit has not been completed does not prevent us from drawing some qualitative results. We now make the argument that we presented in part A precise, concluding that there is a residual diffeomorphism symmetry in the discretized theory:

- There are states \( p_L^* \phi_{\gamma_1}, p_L^* \phi_{\gamma_2} \in L^2(\mathcal{A} / \mathcal{G}, d\mu) \) that are identified by \( \text{Diff}(\Sigma) \) and such that the corresponding states in \( \mathcal{H}_{L_0} \) are different \( F_{L_0}^{-1}(\phi_{\gamma_1}) \neq F_{L_0}^{-1}(\phi_{\gamma_2}) \) (see Fig. 1). This corresponds to the case discussed in Sec. III.B, where two states are related by a diffeomorphism and therefore, identified in the continuum theory, but the lattices on which they are defined are not related by a diffeomorphism. The construction works for any pair of lattices \( L_0, L \) such that \( \gamma_1, \gamma_2 \in L \), where \( |L| = L_0 \), and gives the same answer. This demonstrates that the continuum limit of the Hilbert spaces \( \mathcal{H}_{L_0} \) would be too big to be physically correct.

This discussion allows us to reach our conclusion: when working with a theory defined on an abstract lattice, we still have to get rid of the residual diffeomorphism symmetry. This is the second result of this paper.

IV. DISCUSSION

Let us briefly summarize our results. We have discussed the issue of diffeomorphisms on lattice formulations of theories of connections. The case of 2+1 gravity is tailored for this comparison. It is a theory with only global degrees of freedom that are fully incorporated in the lattice theory even before taking the continuum limit. The imposition of the continuum limit is in this case almost trivial allowing us to draw general qualitative conclusions without the difficulties that one encounters in 3+1 lattice gravity. In the case of the Waelbroeck-Witten formulation, after imposing the curvature constraint (2.17) the resulting quantum theory, just as in the case of the continuum, will be given by gauge invariant functions having support on flat connections. For the Ashtekar constraints, one expects the quantum theory to be closely related to the flat case. This expectation is based on the fact that for non-degenerate \( E \)'s (geometrodynamic sector) the two sets of constraints are equivalent. In the classical theory only if both the ‘diffeomorphism’ (2.22) and the ‘Hamiltonian’ (2.21) constraints are imposed, the theory describes flat connections and has the correct 6 topological degrees of freedom. This example shows that there is a residual diffeomorphism invariance and it has to be taken care of by means of more constraints. If we only consider the ‘Hamiltonian constraint’ (2.21), and do not impose the quantum ‘translation constraints’ (2.22), we will end up with a theory very far from the one describing flat connections; it will have local degrees of freedom.

We have reviewed, in certain detail, the quantization of the continuum theory in order to show that, mathematically, the quantization on the abstract lattice is equivalent to a certain subspace of the Hilbert space of the continuum. Using this framework, we could give examples of states that are identified in the continuum theory, but that are not automatically identified in the quantization of a theory based on an abstract lattice. Thus, this leads us to conclude that the quantum lattice theory has a residual diffeomorphism symmetry that should be taken care of.

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APPENDIX

Consider a square lattice with periodic boundary conditions, i.e. a grid of \( N \) rows by \( M \) columns on a planar representation of \( T^2 \). If \( N \) and \( M \) are odd, then \( B(v_{ij}) = 0 \) for every vertex \( v_{ij} \) of the lattice implies \( P(f_{ij}) = 0 \) for every face \( f_{ij} \) of the lattice. That follows after proving that the system of equations defined by \( B(v_{ij}) := \sum_{v_{ij} \in f_{kl}} P(f_{kl}) \) can be inverted. The formula for \( P = P(B) \) is

\[
P(f_{kl}) = \frac{1}{4} \sum_{ij} C_{kl}(ij) B(v_{ij})
\]  

(4.1)
where \( C_{kl}(ij) = \pm 1 \) and the sign is given in figure (2a).

On the other hand, if \( N \) or \( M \) are even, the system is not invertible and \( B(v_i)I = 0 \) does not imply \( P(f_i)I = 0 \). Now we see why invertibility requires an odd number of rows. In formula (L.1) we would have for \( P(f_{11}) \) that the coefficients \( C_{11}(ij) \) with \( i = 1,2,\ldots,M \) and \( j = 1,2,\ldots,N \) are given by: \( C_{1N}, C_{MN}, C_{M1}, C_{11} = -C_{M1} + 4 - C_{1N} - C_{MN}, C_{M2}, C_{12} = -C_{M2} - (4 - C_{1N} - C_{MN}), \ldots \), \( C_{1,N-1} = -C_{M,N-1} + (-1)^N(4 - C_{1N} - C_{MN}) \). With the consistency condition \( C_{1N} + C_{MN} + (-1)^N(4 - C_{1N} - C_{MN}) = 0 \) that forces \( N \) to be an odd number. The same argument shows that the number of columns should be odd.

![Fig. 2a) Signs of \( C_{kl}(ij) = \pm 1 \). The labels of the faces are the ones given in the figure, and the vertices are labeled by the face in their immediate down-right. Every face, except for face \( kl \), is surrounded by two plus signs and two minus signs. b) The lattice \( L' \) is generated from \( L \) by replacing vertex \( v_N \) with vertices \( v_{N'}, v_{N+1}, v_{N+2} \). The lattice dual to \( L' \) corresponds to the dual of \( L \) after face \( v_N \) has been refined by adding a vertex and three new edges in its interior. c) The free parameters of \( V(L) = \{ P(f_i)I|B(v_i)I = 0 \} \) are \( a^t = P(f_1(v_i))I, b^t = P(f_2(v_i))I, c^t = P(f_3(v_i))I = -a^t - b^t \), which may be subject to further conditions.]

Now we study the case of a lattice with vertices of valence three. Again there are cases of lattices where \( B(v_i)I = 0 \) does not imply \( P(f_j)I = 0 \), this occurs in general for lattices with many symmetries. Our result is the following: Consider any finite three-valent lattice \( L \) where there is a vertex \( v_N \) (dual face) such that \( B(v_N)I = 0 \) but \( P(f_i(v_N))I \neq 0 \) for some of the three faces \( f_1(v_N), f_2(v_N), f_3(v_N) \) that contain vertex \( v \). The three-valent lattice \( L' \) constructed replacing vertex \( v_N \) by three vertices \( v_{N'}, v_{N+1}, v_{N+2} \) (see figure (2b)) is such that \( B(v_i)I = 0 \) implies \( P(f_j)I = 0 \).

First we prove that \( \dim(V(L)) \leq 2 \times \dim(su(2)) = 6 \), where \( V(L) = \{ P(f_j)I|B(v_i)I = 0 \} \) is the vector space of the curvature vectors of all the faces \( f_j \) of the lattice restricted to the condition \( B(v_i)I = 0 \) for every vertex \( v_i \). Since \( L \) is finite, we can number its vertices leaving \( v_N \) at the end \( v_1, v_2, \ldots, v_N \), and in such a way that vertex \( v_R \) for \( 1 < R \leq N \) is joined by an edge of \( L \) to \( v_S \) with \( 1 \leq S < R \) (if the lattice is not connected, we can prove our result independently for its connected components). Denote the curvature vectors of the faces containing the first vertex by \( a^t = P(f_1(v_i))I, b^t = P(f_2(v_i))I, c^t = P(f_3(v_i))I \). \( B(v_1)I = 0 \) implies \( a^t + b^t + c^t = 0 \) and \( B(v_2)I = 0 \) implies that \( P(f(v_2))I = a^t \); then we use \( B(v_3)I = 0 \) to see \( P(f(v_3))I = c^t \), etc (see figure (2c)). In this way we see that all the free parameters in \( V(L) \) are \( a^t, b^t, c^t = -a^t - b^t \), which proves \( \dim(V(L)) \leq 2 \times \dim(su(2)) = 6 \).

Now we see that \( \dim(V(L')) = 0 \). In constructing \( L' \) from \( L \) we generated only one new face (see figure(2b)). Therefore, the construction given above parameterizes the curvature vectors of all the faces of \( L' \) except for one. And the conditions \( B(v_N)I = 0, B(v_{N+1})I = 0, B(v_{N+2})I = 0 \) demand \( a^t = b^t = c^t = 0 \), which concludes our proof.

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