An involution for symmetry of hook length and part length of pointed partitions

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Abstract

A pointed partition of $n$ is a pair $(\lambda, v)$ where $\lambda \vdash n$ and $v$ is a cell in its Ferrers diagram. We construct an involution on pointed partitions of $n$ exchanging “hook length” and “part length”. This gives a bijective proof of a recent result of Bessenrodt and Han.

1 Introduction

A partition $\lambda$ is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$. The integers $\lambda_1, \ldots, \lambda_\ell$ are called the parts of $\lambda$, the number $\ell$ of parts is denoted by $\ell(\lambda)$ and called the length of $\lambda$. The sum of parts is denoted by $|\lambda|$. If $|\lambda| = n$, we say that $\lambda$ is a partition of $n$ and write $\lambda \vdash n$. Each partition can be represented by its Ferrers diagram, we shall identify a partition with its Ferrers diagram. We refer the reader to Andrews’ book [And98] for general reference on partitions.

A pointed partition of $n$ is a pair $(\lambda, v)$ where $\lambda \vdash n$ and $v$ is a cell in its Ferrers diagram. Let $F_n$ be the set of pointed partitions of $n$. For each pointed partition $(\lambda, v)$, we define the arm length $a_v := a_v(\lambda)$ (resp. leg length $l_v$, coarm length $m_v$, coleg length $g_v$) to be the number of cells lying in the same row as $v$ and to the right of $v$ (resp. in the same column as $v$ and above $v$, in the same row as $v$ and to the left of $v$, in the same column as $v$ and under $v$), see Figure 1. The hook length $h_v$ and part length $p_v$ of $v$ in $\lambda$ are defined by $h_v = l_v + a_v + 1$ and $p_v = m_v + a_v + 1$, respectively. The joint distribution of $(h_v, p_v)$ on $F_4$ is given in Figure 2, where $(h_v, p_v)$ is written in $(\lambda, v)$.

Figure 1: The arm, leg, coarm and coleg length of $(\lambda, v)$
In a recent paper, generalizing the result of Bessenrodt [Bess98], Bessenrodt and Han [BH09] showed that the bivariate joint distribution \((h_v, p_v)\) over \(F_n\) is symmetric, that is,
\[
\sum_{(\lambda, v) \in F_n} x^{h_v} y^{p_v} = \sum_{(\lambda, v) \in F_n} x^{p_v} y^{h_v}.
\]

Since the proof in [BH09] uses a generating function argument, this raises the natural question of finding a bijective proof of their result. The aim of this paper is to construct an involution \(\Phi\) on \(F_n\) exchanging hook length and part length and give bijective proofs of the main results in [BH09].

2 Hook and rim hook

Let \(\lambda\) be a partition. Denote its conjugate by \(\lambda' = (\lambda'_1, \lambda'_2, \ldots)\), where \(\lambda'_i\) is the number of parts of \(\lambda\) that are \(\geq i\). For a cell \(v \in \lambda\), the hook \(H_v := H_v(\lambda)\) of the cell \(v\) in \(\lambda\) is the set of all cells of \(\lambda\) lying in the same column above \(v\) or in the same row to the right of \(v\), including \(v\) itself. Obviously, the number of all cells in a hook \(H_v\) equals the hook length \(h_v\) for any \(v\) in \(\lambda\). A cell \(v\) in \(\lambda\) is called boundary cell if the upright corner of \(v\) is in boundaries of \(\lambda\). A border strip is a sequence \(x_0, x_1, \ldots, x_m\) of boundary cells in \(\lambda\) such that \(x_{j-1}\) and \(x_j\) have a common side for \(1 \leq j \leq m\). The rim hook \(R_v := R_v(\lambda)\) of \((\lambda, v)\) is a border strip \(x_0, x_1, \ldots, x_m\) of cells in \(\lambda\) such that \(x_0\) (resp. \(x_m\)) is the uppermost (resp. rightmost) cell of the hook \(H_v\). The rim hook length \(r_v\) of \(v\) in \(\lambda\) is defined to be the number of all cells in the rim hook \(R_v\). It is easy to see that the hook \(H_v\) and the rim hook \(R_v\) have the same length, same height (or number of rows), and same width (or number of columns). See Figure 3.
For a given nonnegative integers \( a \) and \( m \), let \( \mathcal{A} \) be the set of partitions whose largest part is bounded by \( m \), \( \tilde{\mathcal{A}} \) the set of partitions whose largest part is bounded by \( m \) and parts are at most \( a \), and \( \mathcal{R} \) the set of nondecreasing sequences \( (r_1, \ldots, r_t) \), \( t \geq 0 \), with
\[
a + 1 \leq r_1 \leq r_2 \leq \cdots \leq r_t \leq a + m.
\]

We first describe an important algorithm, called Pealing Algorithm, which will be used in the construction of the involution \( \Phi \). From the leftmost top cell of the diagram of \( A \in \mathcal{A} \), remove a rim hook of height \( a + 1 \), if any, in such a way that what remains is a diagram of a partition, and continue removing rim hooks of height \( a + 1 \) in this way as long as possible. Denote the remained partition by \( \tilde{\mathcal{A}} \) as shown in Figure 4. Clearly, the length of \( \tilde{\mathcal{A}} \) is less than or equal to \( a \) and \( \tilde{\mathcal{A}} \in \mathcal{A} \).

Each removed rim hook in the above transformation corresponds to some cell in the first column of \( A \). Let \( v_1, \ldots, v_t \) be the cells from top to bottom corresponding to removed rim hooks. If \( r_1, \ldots, r_t \) are the lengths of removed rim hooks \( R_1, \ldots, R_t \), then
\[
r_i = a_{v_i} + a + 1 \quad \text{for all } 1 \leq i \leq t.
\]

Since \( A \) is a partition with largest part bounded by \( m \), we have
\[
0 \leq a_{v_1} \leq a_{v_2} \leq \cdots \leq a_{v_t} \leq m - 1
\]
and \( (r_1, \ldots, r_t) \in \mathcal{R} \).

**Lemma 1** (Pealing Algorithm). The mapping \( A \mapsto (\tilde{A}; r_1, \ldots, r_t) \) is a bijection from \( \mathcal{A} \) to \( \tilde{\mathcal{A}} \times \mathcal{R} \).

**Proof.** It is sufficient to construct the inverse of pealing algorithm.

Starting from \( (A; r_1, \ldots, r_t) \in \tilde{\mathcal{A}} \times \mathcal{R} \), let \( A_t := \tilde{A} = (\lambda_1, \ldots, \lambda_\ell) \) with \( \ell \leq a \). If there exists a cell whose arm length \( < r_t - a - 1 \), let \( v_t := (1, \alpha) \) be the lowest such cell. Otherwise, let \( v_t := (1, \alpha) \) with \( \alpha = \ell(A_t) + 1 \). Since the length of \( A_t \) is less than or equal to \( a \), we have \( l_{v_t}(A_t) < a \). Define the partition
\[
A_{t-1} := (\lambda_1, \ldots, \lambda_{\alpha-1}, r_t - a, \lambda_\alpha + 1, \ldots, \lambda_{\alpha+a-1} + 1),
\]
Remark. The pealing algorithm gives a bijective proof of the formula

\[
\lambda_a = t = a
\]

where \(\lambda_i := 0\) for \(i > \ell\). Clearly

\[
l_{v_i}(A_{t-1}) = a \quad \text{and} \quad a_{v_i}(A_{t-1}) = r_t - a - 1.
\]  

(1)

Next, we proceed by induction on \(i\) from \(t - 1\) to 1. Suppose that we have found \(A_{t-1}, A_{t-2}, \ldots, A_i = (\lambda_1, \ldots, \lambda_{\ell})\) with

\[
l_{v_{i+1}}(A_i) = a \quad \text{and} \quad a_{v_{i+1}}(A_i) = r_{i+1} - a - 1.
\]

(2)

If there exists a cell whose arm length \(< r_i - a - 1\), let \(v_i := (1, \alpha)\) be the lowest such cell. Otherwise, let \(v_i := (1, \alpha)\) with \(\alpha = \ell(A_i) + 1\). Since we have \(a_{v_{i+1}}(A_i) = r_{i+1} - a - 1 \geq r_i - a - 1\) by (2), the cell \(v_i\) is above \(v_{i+1}\). Also, since \(l_{v_{i+1}}(A_i) = a\) by (2), we have \(l_{v_1}(A_i) < a\). Let

\[
A_{i-1} := (\lambda_1, \ldots, \lambda_{\alpha-1}, r_i - a, \lambda_\alpha + 1, \ldots, \lambda_{\alpha+a-1} + 1),
\]

where \(\lambda_i := 0\) for \(i > \ell\). Clearly,

\[
l_{v_i}(A_{i-1}) = a \quad \text{and} \quad a_{v_i}(A_{i-1}) = r_i - a - 1.
\]

(3)

Finally, define \(A := A_0 \in \mathcal{A}\). Applying pealing algorithm to such a partition \(A\) induced from \((\tilde{A}; r_1, \ldots, r_t)\), by (1) and (3), we get \((\tilde{A}; r_1, \ldots, r_t)\) again.

\[\square\]

For nonnegative integers \(m\) and \(n\), the \(q\)-ascending factorial is defined by \((a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})\) and the \(q\)-binomial coefficient is defined by

\[
\binom{n}{m}_q = \frac{(q; q)_n}{(q; q)_m(q; q)_{n-m}} \quad \text{for} \quad 0 \leq m \leq n.
\]

Remark. The pealing algorithm gives a bijective proof of the formula

\[
\frac{1}{(q; q)_m} = \left[ \begin{array}{c} m + a \\ a \end{array} \right]_q \times \frac{1}{(q^{a+1}; q)_m} = \left[ \begin{array}{c} m + a \\ a \end{array} \right]_q \times \sum_{t=0}^{\infty} q^{(a+1)t} \left[ \begin{array}{c} m - 1 + t \\ t \end{array} \right]_q,
\]

where \(t\) means the number of removed rim hooks. See [And98, Eq. (3.3.7)].

3 Main result

Let \(\mathcal{F}_n(a, l, m)\) be the set of pointed partitions \((\lambda, v)\) of \(n\) such that \(a_v = a\), \(l_v = l\) and \(m_v = m\). We shall divide the construction of the involution \(\Phi\) in three basic steps.

**Step 1: Bessenrodt-Han decomposition** \(\varphi_{a,l,m} : (\lambda, v) \mapsto (A, B, C, D, E)\)

Let \(\mathcal{Q}_n(a, l, m)\) be the set of quintuples \((A, B, C, D, E)\) such that \(A\) is a partition whose largest part is bounded by \(m\), \(B\) is a partition whose diagram fits inside an \(l \times a\) rectangle, \(C\) is a partition whose all parts are greater than or equal to \(m+a+1\), \(D\) is a partition whose diagram is an \((l+1) \times (m+1)\) rectangle, \(E\) is a partition whose diagram is an \(1 \times a\) rectangle, and

\[
|A| + |B| + |C| + |D| + |E| = n.
\]
Suppose that \((\lambda, v) \in \mathcal{F}_n(a, l, m)\). First of all, we decompose the diagram of \(\lambda\) into five regions \((A, B, C, D, E)\) as shown in Figure 5, modified slightly from the decomposition made in [BH09], in which \(E\) is implicit. Denote this decomposition by \(\varphi_{a,l,m}\).

Merging the other partitions \(B, C, D,\) and \(E\) in bottom of \(A\), we recover a pointed partition \((\lambda, v)\). So \(\varphi_{a,l,m}\) is a bijection from \(\mathcal{F}_n(a, l, m)\) to \(\mathcal{Q}_n(a, l, m)\).

Example. For the pointed partition \((\lambda, v) \in \mathcal{F}_{101}(3, 3, 5)\) with 
\[
\lambda = (12, 10, 10, 9, 8, 7, 5, 4, 3, 2, 2, 1, 1), \quad v = (6, 5),
\]
the decomposition is given by \((A, B, C, D, E)\) with 
\[
A = (5, 3, 1), \quad B = (2, 1, 1), \quad C = (12, 10, 10, 9), \quad D = (6, 6, 6, 6), \quad E = (3).
\]

Step 2: Transformation \(\psi_{a,l,m} : (A, B, C, D, E) \mapsto (\tilde{A}, B, \tilde{C}, D, E)\)

Let \(\tilde{\mathcal{Q}}_n(a, l, m)\) be the set of quintuples \((\tilde{A}, B, \tilde{C}, D, E)\) such that \(\tilde{A}\) is a partition whose diagram fits inside an \(a \times m\) rectangle, \(B\) is a partition whose diagram fits inside an \(l \times a\) rectangle, \(\tilde{C}\) is a partition whose all parts are greater than or equal to \(a + 1\), \(D\) is a partition whose diagram is an \((l + 1) \times (m + 1)\) rectangle, \(E\) is a partition whose diagram is an \(1 \times a\) rectangle, and 
\[
|\tilde{A}| + |B| + |\tilde{C}| + |D| + |E| = n.
\]
Note that the partitions \(\tilde{A}, B, \tilde{C},\) and \(E\) could be empty. The mapping \(\psi_{a,l,m}\) from \(\mathcal{Q}_n(a, l, m)\) to \(\tilde{\mathcal{Q}}_n(a, l, m)\) is defined by \(\psi_{a,l,m}(A, B, C, D, E) = (\tilde{A}, B, \tilde{C}, D, E)\) as follows:

- Applying the pealing algorithm to \(A\), we have \((\tilde{A}; r_1, \ldots, r_t)\). In the above example, we have \(\tilde{A} = (5, 3, 1)\) and \((r_1, r_2, r_3) = (5, 7, 8)\) with \(t = 3\).

- Gluing \(r_1, \ldots, r_t\) to \(C\), we get the partition \(\tilde{C} = (C, r_t, r_{t-1}, \ldots, r_1)\) obtained by adding \(r_i\)'s to the partition \(C\). Clearly, all parts of \(\tilde{C}\) are greater than or equal to \(a + 1\).
We can construct the inverse of the mapping $\psi_{a,l,m}$ as follows: Move out all parts $r_i$, $i = 1, \ldots, t$, of size at most $a + m$ from a partition $\tilde{C}$ and denote the remained partition by $C$. By Lemma 1, $A$ can be recovered from $(\tilde{A}; r_1, \ldots, r_t)$.

**Example.** Continuing the previous example, we have $(r_1, r_2, r_3) = (5, 7, 8), (\tilde{A}, B, \tilde{C}, D, E) \in \mathcal{Q}_{101}(3, 3, 5)$ with $\tilde{A} = (5, 3, 1), B = (2, 1, 1), \tilde{C} = (12, 10, 10, 9, 8, 7, 5), D = (6, 6, 6, 6)$ and $E = (3)$.

So every pointed partition $(\lambda, v) \in \mathcal{F}_n(a, l, m)$ can be transformed into five partitions $(\tilde{A}, B, \tilde{C}, D, E) \in \mathcal{Q}_n(a, l, m)$ by $\psi_{a,l,m} \circ \varphi_{a,l,m}$, as shown in Figure 6. Let $\tilde{\mathcal{Q}}_n$ be the disjoint union of the sets $\tilde{\mathcal{Q}}_n(a, l, m)$ for all $a, l, m \geq 0$. Define the bijection $\Psi$ from $\mathcal{F}_n$ to $\tilde{\mathcal{Q}}_n$ by

$$
\Psi(\lambda, v) = \psi_{a,l,m} \circ \varphi_{a,l,m}(\lambda, v) \quad \text{if} \quad (\lambda, v) \in \mathcal{F}_n(a, l, m).
$$

**Step 3: The involution** $\rho : (\tilde{A}, B, \tilde{C}, D, E) \mapsto (B', \tilde{A}', \tilde{C}, D', E)$

Define the involution $\rho$ on $\tilde{\mathcal{Q}}_n$ by

$$
\rho(\tilde{A}, B, \tilde{C}, D, E) = (B', \tilde{A}', \tilde{C}, D', E),
$$

where $X'$ is the conjugate of the partition $X$.

**Example.** Continuing the previous example, we have $\rho(\tilde{A}, B, \tilde{C}, D, E) = (B', \tilde{A}', \tilde{C}, D', E)$, where $B' = (3, 1), \tilde{A}' = (3, 2, 2, 1, 1), D' = (4, 4, 4, 4, 4)$.

**Theorem 2.** For all $n \geq 0$, the mapping $\Phi = \Psi^{-1} \circ \rho \circ \Psi$ is an involution on $\mathcal{F}_n$ such that if $\Phi : (\lambda, v) \mapsto (\mu, u)$ then

$$
(a_v, l_v, m_v)(\lambda) = (a_u, m_u, l_u)(\mu).
$$

(4)
In particular, the mapping $\Phi$ also satisfies

$$(h_v, p_v)(\lambda) = (p_u, h_u)(\mu).$$

(5)

Proof. By definition, the restriction of $\rho$ on $\tilde{Q}_n(a, l, m)$ is a bijection from $\tilde{Q}_n(a, l, m)$ to $\tilde{Q}_n(a, m, l)$. Hence, for any $(\lambda, v) \in F_n(a, l, m)$, we have

$$\Phi(\lambda, v) = \Psi^{-1} \circ \rho \circ \Psi(\lambda, v) = \varphi_{a,m,l}^{-1} \circ \psi_{a,m,l}^{-1} \circ \rho \circ \psi_{a,l,m} \circ \varphi_{a,l,m}(\lambda, v) \in F_n(a, m, l).$$

Clearly the mapping $\Phi$ is an involution on $F_n$ satisfying (4) and (5).

Remark. In other words, we have the following diagram:

![Diagram](image)

Example. Continuing the above example, we have $\Phi(\lambda, v) = (\mu, u) \in F_{101}(3, 5, 3)$ with

$$\mu = (12, 10, 10, 9, 8, 7, 7, 6, 6, 5, 5, 3, 2, 2, 1, 1)$$

and $u = (4, 7)$.

To end this section we draw a graph on $F_4$ to illustrate the bijection $\Phi$ on $F_4$ by connecting each $(\lambda, v)$ to $\Phi(\lambda, v)$ (see Figure 2) as follows:

![Graph](image)

4 Some consequences

We derive immediately the following result of Bessenrodt and Han [BH09, Theorem 3].

Corollary 3. The triple statistic $(a_v, l_v, m_v)$ has the same distribution as $(a_v, m_v, l_v)$ over $F_n$. In other words, the polynomial

$$Q_n(x, y, z) = \sum_{(\lambda, v) \in F_n} x^{a_v} y^{l_v} z^{m_v}$$

is symmetric in $y$ and $z$. 
Let \( f_n(a,l,m) \) be the cardinality of \( \mathcal{F}_n(a,l,m) \), that is, \( f_n(a,l,m) \) is the coefficient of \( x^ay^lz^m \) in \( Q_n(x,y,z) \). We can apply the bijection \( \Phi \) to give a different proof of Bessenrodt and Han’s formula [BH09, Theorem 2] for \( \sum_{n \geq 0} f_n(a,l,m)q^n \).

**Corollary 4.** The generating function of \( f_n(a,l,m) \) is given by the following formula

\[
\sum_{n \geq 0} f_n(a,l,m)q^n = \frac{1}{(q^{a+1};q)_\infty} \left[ \frac{m+a}{a} \right]_q \left[ \frac{l+a}{a} \right]_q q^{(m+1)(l+1)+a},
\]

where \( (a;q)_\infty := \prod_{i=0}^{\infty} (1 - aq^i) \).

**Proof.** By the bijection \( \psi \) the generating function \( \sum_\lambda a[\lambda] \), where \( (\lambda, \nu) \in \mathcal{F}_n(a,l,m) \), is equal to the product of the corresponding generating functions of the five partitions \( A, B, C, D \) and \( E \). In view of the basic facts about partitions (see [And98, Chapter 3]), it is easy to see that \( D(q) = q^{(m+1)(l+1)}, E(q) = q^a \), and

\[
A(q) = \left[ \frac{m+a}{a} \right]_q, \quad B(q) = \left[ \frac{l+a}{a} \right]_q, \quad C(q) = \frac{1}{(q^{a+1};q)_\infty}.
\]

Multiplying the five generating functions yields the result. \( \square \)

A polynomial \( P(x,y) \) in two variables \( x \) and \( y \) is super-symmetric if

\[
[x^\alpha y^\beta]P(x,y) = [x^{\alpha'} y^{\beta'}]P(x,y)
\]

when \( \alpha + \beta = \alpha' + \beta' \). Clearly, any super-symmetric polynomial is also symmetric.

It is known (see [Bes98, BM02, BH09]) that the generating function for the pointed partitions of \( \mathcal{F}_n \) by the two joint statistics arm length and co-arm length (resp. leg length) is super-symmetric. In other words, the polynomial

\[
\sum_{\lambda,v} x^{a_\nu}y^{m_\nu} \quad \text{(resp.} \sum_{\lambda,v} x^{a_\nu}y^{l_\nu} \text{)}
\]

is super-symmetric. Note that the above two polynomials are actually equal due to Corollary 2.

Let \( \mathcal{F}_n(a,*,*) \) (resp. \( \mathcal{F}_n(a,l,*) \)) be the set of pointed partitions \( (\lambda, \nu) \) of \( n \) such that \( a_\nu = a \) and \( m_\nu = m \) (resp. \( a_\nu = a \) and \( l_\nu = l \)).

It is easy to give a combinatorial proof of the super-symmetry of the first polynomial \( \sum_{(\lambda,v) \in \mathcal{F}_n} x^{a_\nu}y^{m_\nu} \). Indeed, if \( \alpha + \beta = \alpha' + \beta' \) and \( (\lambda, \nu) \in \mathcal{F}_n(\alpha,*,\beta) \), let \( u \) be the unique cell on the same row as \( v \) satisfying \( a_\nu(\lambda) = \alpha' \) and \( m_\nu(\lambda) = \beta' \), then \( \tau_{\alpha,\beta,\alpha',\beta'} : (\lambda, \nu) \mapsto (\lambda, u) \), is a bijection from \( \mathcal{F}_n(\alpha,*,\beta) \) to \( \mathcal{F}_n(\alpha',*,\beta') \).

Combining the bijection \( \tau_{\alpha,\beta,\alpha',\beta'} \) with the involution \( \Phi \) we can prove bijectively the super-symmetry of the polynomial \( \sum_{(\lambda,v) \in \mathcal{F}_n} x^{a_\nu}y^{l_\nu} \). More precisely we have the following result.

**Theorem 5.** If \( \alpha + \beta = \alpha' + \beta' \), the mapping \( \zeta_{\alpha,\beta,\alpha',\beta'} = \Phi \circ \tau_{\alpha,\beta,\alpha',\beta'} \circ \Phi \) is a bijection from \( \mathcal{F}_n(\alpha,\beta,*) \) to \( \mathcal{F}_n(\alpha',\beta',*) \).

**Proof.** Fix nonnegative integers \( \alpha, \beta, \alpha' \), and \( \beta' \) satisfying \( \alpha + \beta = \alpha' + \beta' \). By Theorem 2 the mapping \( \Phi \) is a bijection from \( \mathcal{F}(\alpha,\beta,*) \) to \( \mathcal{F}_n(\alpha,*,\beta) \) and also a bijection from \( \mathcal{F}(\alpha',*,\beta') \) to \( \mathcal{F}_n(\alpha',\beta',*) \). Since the mapping \( \tau_{\alpha,\beta,\alpha',\beta'} \) is a bijection from \( \mathcal{F}(\alpha,*,\beta) \) to \( \mathcal{F}_n(\alpha',*,\beta') \), it
is obvious that the mapping \( \zeta_{\alpha,\beta,\alpha',\beta'} = \Phi \circ \tau_{\alpha,\beta,\alpha',\beta'} \circ \Phi \) is a bijection from \( F_n(\alpha, \beta, *) \) to \( F(\alpha', \beta', *) \). The bijection \( \zeta_{\alpha,\beta,\alpha',\beta'} \) is illustrated as follows:

\[
\begin{array}{c}
F_n(\alpha, \beta, * ) \xrightarrow{\zeta_{\alpha,\beta,\alpha',\beta'}} F_n(\alpha', \beta', *) \\
\Phi \downarrow \quad \Phi \\
F_n(\alpha, *, \beta ) \xrightarrow{\tau_{\alpha,\beta,\alpha',\beta'}} F_n(\alpha', *, \beta')
\end{array}
\]

We are done.

Theorem 4 yields that the generating function of \( F_n \) by the bivariate joint distribution of arm length and leg length is super-symmetric.

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