Sketching for Two-Stage Least Squares Estimation

SOKBAE LEE*  SERENA NG†

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Abstract

When there is so much data that they become a computation burden, it is not uncommon to compute quantities of interest using a sketch of data of size $m$ instead of the full sample of size $n$. This paper investigates the implications for two-stage least squares (2SLS) estimation when the sketches are obtained by a computationally efficient method known as CountSketch. We obtain three results. First, we establish conditions under which given the full sample, a sketched 2SLS estimate can be arbitrarily close to the full-sample 2SLS estimate with high probability. Second, we give conditions under which the sketched 2SLS estimator converges in probability to the true parameter at a rate of $m^{-1/2}$ and is asymptotically normal. Third, we show that the asymptotic variance can be consistently estimated using the sketched sample and suggest methods for determining an inference-conscious sketch size $m$. The sketched 2SLS estimator is used to estimate returns to education.

Keywords: sketching, two-stage least squares, countsketch, asymptotic normality

1 Introduction

Big data sets in which a large number of observations are collected for each variable are now routinely used in scientific research. However, even in the modern computing environment, analyzing large datasets is time consuming and the demand for efficient computing is ever-present. One approach around the data bottlenecks is to work with a randomly chosen subset, or a sketch, of the data. The early works of Sarlos (2006), Drineas et al. (2006) and Drineas et al. (2011) consider sketching of the least squares estimator from an algorithmic

*Department of Economics, Columbia University and Institute for Fiscal Studies
†Department of Economics, Columbia University and NBER
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perspective. Subsequent work extends the analysis to ridge regression (e.g., Wang et al., 2017), and logistic regression (e.g., Wang, 2019) under the assumption that the covariates are exogenous. See, e.g., Drineas and Mahoney (2018) and Woodruff (2014) for a review.

However, recent work due to Ma et al. (2015), Raskutti and Mahoney (2016) and Ma et al. (2020) show that an optimal worse-case error may not yield an optimal mean-squared error. This has led to an increased interest to better understand the statistical implications of sketching. For example, Geppert et al. (2017) considers Bayesian estimation while Ahfock et al. (2019) provides the distribution theory for the sketched least squares estimator. Lee and Ng (2020) considers inference and highlights the tension between a large sample size required to maximize power in hypothesis testing, and computation efficiency which necessarily favors a smaller sample size.

This paper also aims to provide a better understanding of the statistical implications of sketching but our focus is two-stage least squares (2SLS) estimation. Given \( n \) observations for \( \{(y_i, X_i) : i = 1, \ldots, n\} \), we consider a linear regression model:

\[
y_i = X_i^T \beta_0 + e_i, \quad i = 1, \ldots, n, \tag{1}
\]

where \( y_i \) is the scalar dependent variable, \( X_i \) is a \( p \times 1 \) vector of regressors, \( e_i \) is the scalar regression error, \( \beta_0 \) is a \( p \times 1 \) vector of unknown parameters. If \( e_i \) and \( X_i \) are uncorrelated, \( \beta_0 \) can be consistently estimated by the least-squares estimator. However, unlike least squares estimation, we assume \( \mathbb{E}[X_i e_i] \neq 0 \). Such a problem can arise because (i) \( e_i \) includes an omitted random variable that is correlated with \( X_i \), (ii) \( X_i \) is measured with errors, or (iii) \( y_i \) and \( X_i \) are determined simultaneously, as in a demand and supply model in economics. When \( \mathbb{E}[X_i e_i] \neq 0 \), the least squares estimator is inconsistent.

To identify \( \beta_0 \), 2SLS estimation posits that there exists a \( q \times 1 \) vector of instrumental variables \( Z_i \) that is correlated with \( X_i \) and yet uncorrelated with \( e_i \), where \( q \geq p \). Formally,

\[
(i) \mathbb{E}(Z_i e_i) = 0, \quad (ii) \mathbb{E}(Z_i X_i^T) \text{ has full rank } p. \tag{2}
\]

These two conditions, known respectively as instrument exogeneity and instrument relevance, imply

\[
\mathbb{E}(Z_i y_i) = \mathbb{E}(Z_i X_i^T) \beta_0.
\]

In the ‘exactly identified’ case when \( p = q \) and \( \mathbb{E}(Z_i X_i^T) \) is non-singular, \( \beta_0 \) is identified by

\[
\beta_0 = \left[\mathbb{E}(Z_i X_i^T)\right]^{-1} \mathbb{E}(Z_i y_i)
\]
and can be estimated by its sample analog. A generalized version with \( q \geq p \) is given in the next section. The 2SLS estimator was developed in economics and is widely used in many disciplines to estimate causal effects from observational data. It has also received attention in the machine learning community (e.g., [Chen et al. 2018; Guo and Small 2016]).

In this paper, we are interested in 2SLS estimation using sketches of data of size \( m \). We focus on a type of random-projection sketching method called CountSketch ([Clarkson and Woodruff 2017]) that can be computed efficiently by streaming. We establish three properties of the sketched 2SLS estimator. First, given the full sample, we establish the conditions under which a sketched 2SLS estimate can be arbitrarily close to the full-sample 2SLS estimate with high probability. Second, we give conditions under which the sketched 2SLS estimator converges in probability to the true parameter at a rate of \( m^{-1/2} \) and is asymptotically normal. This asymptotic normality result explores a degenerate \( U \)-statistic structure of the leading term of the sketched 2SLS estimator. In general, the asymptotic distribution of degenerate \( U \)-statistics can be either a weighted average of independent, centered chi-square random variables with a complex form of weights or simply a centered normal distribution (Hall 1984). We establish that the Countsketch yields a sketched estimator that is asymptotically normally distributed which is convenient for statistical inference. Third, we make precise how the asymptotic variance can be consistently estimated using the sketched sample and propose to determine the sketch size using an inference-conscious rule that will give a statistical test with a target power.

The paper is organized as follows. Section 2 describes a sketched 2SLS estimator and Section 3 provides its algorithmic properties. In Section 4, we concentrate on CountSketch and show that it produces a computationally efficient high-quality solution to the 2SLS problem. In Section 5 we obtain statistical properties of the sketched 2SLS estimator. We focus on CountSketch and establish asymptotic normality of the sketched estimator. Section 6 gives practical guides for choosing the subsample size in sketching. Section 7 reports the results of a Monte Carlo experiment that shows the finite-sample performance of the sketched 2SLS estimator. In Section 8, we illustrate sketching with a well-known dataset from [Angrist and Keuueger 1991]. Section 9 concludes. Appendix A contains the proofs of theorems.

### 1.1 Notation

For any vector \( a \), let \( \| a \| \) denote the Euclidean norm of \( a \). For a real \((n \times d)\) matrix \( A \) with rank \( d \), the singular value decomposition (SVD) of \( A \) is \( A = U_A \Sigma_A V_A^T \), where \( U_A \in \mathbb{R}^{n \times d} \) is the matrix of the left singular vectors, \( \Sigma_A \in \mathbb{R}^{d \times d} \) is the diagonal matrix of singular values, and \( V_A \in \mathbb{R}^{d \times d} \) is the matrix of the right singular vectors. Here, \( U_A^T U_A = I_d \) and \( V_A^T V_A = I_d \).
where $I_d$ is the $d$-dimensional identity matrix. Let $\|A\|_2$ denote its spectral norm, and $\sigma_k(A)$ a singular value of $A$, where $k = 1, \ldots, d$. Let $\sigma_{\text{max}}(A)$ and $\sigma_{\text{min}}(A)$ denote the largest and smallest singular values of $A$. That is, $\|A\|_2 = \sigma_{\text{max}}(A)$. The superscript $T$ denotes the transpose of a matrix. For an integer $n \geq 1$, $[n]$ denotes the set of positive integers from 1 to $n$. Let $A_{ij}$ or $[A]_{ij}$ denote the $(i, j)$ element of a matrix $A$. Let $\rightarrow_p$ and $\rightarrow_d$, respectively, denote convergence in probability and in distribution.

## 2 Two-Stage Least Squares Estimators

In this section, we define the 2SLS estimator and describe how to conduct inference. It is simpler to work with matrix notation. Let $y$ be an $n \times 1$ vector of the dependent variable whose $i$-th row is $y_i$, $X$ an $n \times p$ matrix of regressors whose $i$-th row is $X_i^T$, and $Z$ an $n \times q$ matrix of instruments whose $i$-th row is $Z_i^T$. The matrices $X$ and $Z$ may have common columns; for example, conventionally the first columns of $X$ and $Z$ are a vector of ones.

The 2SLS estimator is defined as

$$\hat{\beta} := (X^T Z (Z^T Z)^{-1} Z^T X)^{-1} X^T Z (Z^T Z)^{-1} Z^T y.$$  \hspace{1cm} (3)

The estimator can be understood as a two step procedure in which $X$ is first projected on $Z$ to purge the variations in $X$ correlated with $e$, in the second step replace $X$ by the outcome of the projection $P_Z X$, where $P_Z := Z(Z^T Z)^{-1} Z^T$ is the projection matrix. The estimator exists if (i) $Z^T Z$ and (ii) $X^T P_Z X$ are non-singular. These conditions hold if the population moments non-singular.

**Assumption 1.**  (i) The data $D_n := \{(y_i, X_i, Z_i) \in \mathbb{R}^{1+p+q} : i = 1, \ldots, n\}$ are independent and identically distributed, where $y_i = X_i^T \beta_0 + e_i$ with $p \leq q$. (ii) $\mathbb{E}(e_i^2) > 0$, $\mathbb{E}(y_i^4) < \infty$, $\mathbb{E}(\|X_i\|^4) < \infty$, and $\mathbb{E}(\|Z_i\|^4) < \infty$. (iii) $\mathbb{E}(Z_i e_i) = 0$, $\mathbb{E}(Z_i Z_i^T)$ is positive definite, and $\mathbb{E}(Z_i X_i^T)$ has full rank $p$. (iv) $\mathbb{E}(e_i^2 Z_i Z_i^T) = \mathbb{E}(e_i^2) \mathbb{E}(Z_i Z_i^T)$.

Assumption (iv) is not needed for identification and consistent estimation of $\beta_0$; however, the 2SLS estimator is efficient under Assumption (iv). Under Assumption (i) as $n \to \infty$, it is well known that

$$\sqrt{n} (\hat{\beta} - \beta_0) \rightarrow_d N(0, V_0),$$  \hspace{1cm} (4)
where

\[ V_0 := \mathbb{E}(e_i^2) \left[ \mathbb{E}(X_iZ_i^T) \left[ \mathbb{E}(Z_iZ_i^T) \right]^{-1} \mathbb{E}(Z_iX_i^T) \right]^{-1}. \]  

(5)

The unknown \( V_0 \) can be consistently estimated by

\[ \hat{V} := \hat{e}^T \hat{e} \left( X^T Z (Z^T Z)^{-1} Z^T X \right)^{-1}, \]  

(6)

where \( \hat{e} := y - X \hat{\beta} \), the residuals evaluated at the 2SLS estimator (and not the residuals from the second step regression with \( P_Z X \) as regressor). In many applications, the parameter of interest can be written as \( c^T \beta_0 \) for a known vector \( c \). The standard error of \( c^T \hat{\beta} \) is

\[ \text{se}(c^T \hat{\beta}) := \left( c^T \hat{V}_c / n \right)^{1/2}. \]

3 Algorithmic Properties of Sketched 2SLS Estimators

A sketch of the data \((y, X, Z)\) is \((\tilde{y}, \tilde{X}, \tilde{Z})\), where \( \tilde{y} = \Pi y \), \( \tilde{X} = \Pi X \) and \( \tilde{Z} = \Pi Z \) where \( \Pi \) is a \( m \times n \) random matrix. Assuming \( \tilde{Z}^T \tilde{Z} \) and \( \tilde{X}^T \tilde{Z} (\tilde{Z}^T \tilde{Z})^{-1} \tilde{Z}^T \tilde{X} \) are both non-singular, a sketched version of the 2SLS estimator is

\[ \hat{\beta} := \left( \tilde{X}^T \tilde{Z} (\tilde{Z}^T \tilde{Z})^{-1} \tilde{Z}^T \tilde{X} \right)^{-1} \tilde{X}^T \tilde{Z} (\tilde{Z}^T \tilde{Z})^{-1} \tilde{Z}^T \tilde{y}. \]  

(7)

We assume the following subspace embedding results for \( \Pi \). Sufficient conditions will be given for a particular form of \( \Pi \) later. To state assumptions using the singular value decomposition of \( X \) and \( Z \), write \( X = U_X \Sigma_X V_X^T \) and \( Z = U_Z \Sigma_Z V_Z^T \). Also, recall that \( \hat{e} = y - X \hat{\beta} \). Throughout in this section, we let the data \( D_n \) be fixed. All statements regarding probability refers to randomized sketching \( \Pi \).

Assumption 2. Let the data \( D_n \) be fixed, \( Z^T Z \) and \( X^T P_Z X \) are non-singular. For given constants \( \varepsilon_1, \varepsilon_2, \varepsilon_3, \delta \in (0, 1/2) \), the following holds jointly with probability at least \( 1 - \delta \):

\begin{align*}
(i) & \quad \| U_Z^T \Pi^T U_Z - I_q \|_2 \leq \varepsilon_1, \\
(ii) & \quad \| U_Z^T \Pi^T U_X - U_Z^T U_X \|_2 \leq \varepsilon_2, \\
(iii) & \quad \| U_Z^T \Pi \hat{e} - U_Z^T \hat{e} \| \leq \varepsilon_3 \| \hat{e} \|. 
\end{align*}

Assumption 2(i) is equivalent to the statement that the all eigenvalues of \( U_Z^T \Pi^T U_Z \) are bounded between \([1-\varepsilon_1, 1+\varepsilon_1]\). Therefore, Assumption 2(i) ensures that \( \tilde{Z}^T \tilde{Z} \) is non-singular with probability at least \( 1 - \delta \).
Define
\[ f_1(\varepsilon_1, \varepsilon_2) := \frac{\varepsilon_1 + \varepsilon_2(\varepsilon_2 + 2)}{(1 - \varepsilon_1)}, \quad (8) \]
\[ f_2(\varepsilon_1, \varepsilon_2) := \frac{\varepsilon_2 + \varepsilon_1}{(1 - \varepsilon_1)} + \frac{\varepsilon_2 \varepsilon_1}{(1 - \varepsilon_1)}. \quad (9) \]

**Assumption 3.** For \( f_1(\varepsilon_1, \varepsilon_2) \) defined in equation (8), assume that
\[ \sigma_{\text{min}}^2(U_Z^T U_X) \geq 2 f_1(\varepsilon_1, \varepsilon_2). \]

This assumption strengthens non-singularity of \( X^T P_Z X \) to require that \( \sigma_{\text{min}}^2(U_Z^T U_X) \) is strictly positive and bounded below by the constant \( 2 f_1(\varepsilon_1, \varepsilon_2) \).

**Theorem 1.** Under Assumptions 2 and 3, the following holds with probability at least \( 1 - \delta \):
\[ \| \hat{\beta} - \hat{\beta} \| \leq \frac{f_2(\varepsilon_1, \varepsilon_2) + \varepsilon_3 \| \hat{\varepsilon} \| \left[ 1 + f_2(\varepsilon_1, \varepsilon_2) \right]}{\sigma_{\text{min}}(X) \sigma_{\text{min}}^2(U_Z^T U_X)} \left[ 1 + \frac{2 f_1(\varepsilon_1, \varepsilon_2)}{\sigma_{\text{min}}^2(U_Z^T U_X)} \right]. \]

Theorem 1 implies that the sketched 2SLS estimator is a more accurate approximation to the full-sample 2SLS estimator if (i) \( \varepsilon_j, j = 1, 2, 3 \), is smaller (the error of approximating matrix multiplication is smaller); (ii) \( \| \hat{\varepsilon} \| \) is smaller (the estimated residual from the full-sample 2SLS estimator is less variable); (iii) \( \sigma_{\text{min}}(X) \) is larger (the signal from \( X \) is stronger); (iv) \( \sigma_{\text{min}}(U_Z^T U_X) \) is larger (the instrument \( Z \) is more relevant for \( X \)).

### 4 CountSketch

We focus on the randomized algorithm called CountSketch (Clarkson and Woodruff, 2013).

A countsketch of sketching dimension \( m \) is a data-oblivious random linear map \( \Pi = PD : \mathbb{R}^n \to \mathbb{R}^m \), where \( D \) is an \( n \times n \) random diagonal matrix with entries chosen independently to be +1 or −1 with equal probability. Furthermore, \( P \in \{0, 1\} \) is an \( m \times n \) binary matrix such that \( P_{h(i), i} = 1 \) and \( P_{j, i} = 0 \) for all \( j \neq h(i) \), and \( h : [n] \to [m] \) is a random map such that for each \( i \in [n] \), \( h(i) = m' \) for \( m' \in [m] \) with probability \( \frac{1}{m} \).

The main appeal of the countsketch is that the run time needed to compute \( \Pi A \) can be reduced to \( O(\text{nnz}(A)) \), where \( \text{nnz}(A) \) denotes the number of non-zero entries of \( A \). The efficiency gain is due to extreme sparsity of a countsketch \( \Pi \) which only has one non-zero element per column. Furthermore, it is possible to compute the sketch by streaming without constructing \( \Pi \). Specifically, the streaming algorithm proceeds by initializing \( \tilde{A} \) to an \( m \times n \)
matrix of zeros. Each row $A(i)$ of $A$ is updated as

$$\tilde{A}_{h(i)} = \tilde{A}_{h(i)} + g(i)A(i),$$

where $h(i)$ sampled uniformly at random from $[m]$ and $g(i)$ sampled from $\{+1, -1\}$ are independent. Computation can be done one row at a time. See [Clarkson and Woodruff (2017)] and [Woodruff (2014)] for details.

We summarize the key properties of CountSketch in the following assumption.

**Assumption 4 (CountSketch).** Let $\Pi \in \mathbb{R}^{m \times n}$ be a random matrix such that (i) its $(j,i)$ element is $\Pi_{ji} = \delta_{ji}\sigma_{ji}$, where $\sigma_{ji}$’s are $\pm 1$ random variables and $\delta_{ji}$ is an indicator random variable for the event $\Pi_{ji} \neq 0$; (ii) $\sum_{j=1}^{m} \delta_{ji} = 1$ for each $i = 1, \ldots, n$; (iii) $\mathbb{E}(\sigma_{ji}) = 0$ and $\mathbb{E}(\delta_{ji}) = m^{-1}$; (iv) the columns of $\Pi$ are i.i.d.; (v) $\Pi$ is independent of the data $D_n$.

We now specialize Theorem 1 for CountSketch.

**Theorem 2.** Let the data $D_n$ be fixed, $Z^TZ$ and $X^TP_ZX$ are non-singular. Let $\Pi \in \mathbb{R}^{m \times n}$ satisfy Assumption 4 with $m \geq \max\{q(q+1), 2pq\}/(\varepsilon^2\delta)$ for some $\varepsilon \in (0, 1/3]$. Suppose that

$$\sigma^2_{\min}(U^T_Z U_X) \geq \frac{16\varepsilon(1 + \varepsilon)}{1 - \varepsilon}.$$

Then, the following holds with probability at least $1 - \delta$:

$$\left\| \tilde{\beta} - \hat{\beta} \right\| \leq \frac{4\varepsilon}{1 - \varepsilon} \left[ 2 + \frac{3\|\tilde{e}\|}{\sqrt{p}} \right] \left[ \sigma_{\min}(X)\sigma^2_{\min}(U^T_Z U_X) \right]^{-1}.$$

Theorem 2 establishes that the difference between a sketched 2SLS estimate and the full-sample 2SLS estimate can be arbitrarily small with high probability, provided that the subsample size $m$ is sufficiently large, $X$ is linearly independent, and the instrument $Z$ is sufficiently relevant for $X$. The constants appearing in the statement of the theorem are explicit; however, we did not attempt to find the best possible constants.

## 5 Statistical Properties of Sketched 2SLS Estimators

The previous section provides algorithmic properties of the sketched 2SLS estimators with CountSketch. Theorem 2 provides a worst-case bound for the difference between the sketched and full-sample 2SLS estimators, while the data $D_n$ being fixed.

In this section, we establish statistical properties of the sketched 2SLS estimator when the full sample size $n$ goes to infinity. We now assume that both $\Pi$ and $D_n$ are random but
independent of each other and the subsample size \( m = m_n \) is a sequence of \( n \) that diverges to infinity at a slower rate than \( n \). We focus on CountSketch and obtain the following asymptotic normality for the difference between sketched and full-sample 2SLS estimators, multiplied by the square root of \( m \).

**Theorem 3.** Let Assumptions 1(i)-(iii) and 4 hold. Suppose that \( m = m_n \to \infty \) but \( m_n/n \to 0 \) as \( n \to \infty \). Then, as \( n \to \infty \) and \( V_0 \) is defined in (5),

\[
m^{1/2}(\tilde{\beta} - \hat{\beta}) \to_d N(0, V_0),
\]

Theorem 3 states that \( \tilde{\beta} \) is asymptotically normally distributed with mean \( \hat{\beta} \) and variance \( V_0/m \). Thus, it provides a sort of average bound for the difference between the sketched and full-sample 2SLS estimators, thereby complementing the algorithmic properties obtained in Theorem 2. Interestingly, the asymptotic variance \( V_0 \) is the same as that in (4). However, the full sample result in (4) explicitly assumes homoskedasticity \( \mathbb{E}(e_i^2 Z_i Z_i^T) = \mathbb{E}(e_i^2) \mathbb{E}(Z_i Z_i^T) \), but in Theorem 3, Assumption 4(iv) is not assumed.

In Theorem 3, both \( \Pi \) and \( D_n \) are treated random. A crucial step to prove Theorem 3 is to recognize a \( U \)-statistic structure of the sample moments for the sketched 2SLS estimator when CountSketch is employed. To convey the main ideas, for \( u, v \in \mathbb{R}^n \), consider

\[
u^T \Pi^T \Pi v = \sum_{i=1}^n \sum_{j=i}^n \sum_{k=1}^m u_i \Pi_k_i \Pi_k_j v_j,
\]

where \( u_i \) and \( v_j \), respectively, are the \( i \)-th and \( j \)-th elements of \( u \) and \( v \), and \( \Pi_k_i \) is the \((k,i)\) element of \( \Pi \). We now invoke the condition that \( \sum_{k=1}^m \Pi_k_i = \sum_{k=1}^m \delta_{ki} = 1 \) for each \( i \in [n] \) in Assumption 4 to write

\[
u^T \Pi^T \Pi v - u^T v = \sum_{i=1}^n \sum_{j \neq i}^n \sum_{k=1}^m u_i \Pi_k_i \Pi_k_j v_j = \sum_{1 \leq i < j \leq n} H_{ij},
\]

where

\[
H_{ij} = (u_i v_j + u_j v_i) \sum_{k=1}^m \Pi_k_i \Pi_k_j.
\]

This indicates that \( u^T \Pi^T \Pi v - u^T v \) is a degenerate \( U \)-statistic, provided that \( \Pi \) is independent of \( u \) and \( v \). Sample moments appearing in the definition of \( \tilde{\beta} \) (e.g., elements of \( \tilde{X}^T \tilde{Z} \) in (7)) can be expressed as \( u^T \Pi^T \Pi v \) with suitable choices of \( u \) and \( v \). In particular, we make use of the central limit theorem for degenerate \( U \)-statistics of Hall (1984).

A recent working paper by Ahfock et al. (2019) established asymptotic normality for
sketched ordinary-least-squares (OLS) estimators that include CountSketch. Their approach is to condition on data $D_n$, and apply a central limit theorem for a triangular array of random variables. In contrast, we do not condition on $D_n$ and we treat data $D_n$ random. Another recent working paper by [Ma et al., 2020] considered random sampling algorithms for the OLS estimators and derived the asymptotic distributions under general sampling probabilities. Our work is distinct from theirs in that (i) we focus on the 2SLS and not the OLS estimator, and more importantly, (ii) CountSketch is a random projection method that is different from random sampling algorithms. These differences lead to different proof strategies. [Ma et al., 2020] exploited the fact that the leading stochastic term of the random sampling OLS estimators can be written as a weighted single sum of random variables. The proof strategy used [Ma et al., 2020] hinges crucially on the assumption that $\Pi^T\Pi$ is a diagonal matrix. An application of Hajek-Sidak central limit theorem is then used to establish the properties of least-squares estimators constructed from random sampling. In our case of the CountSketch which is a form of random projections, $\Pi^T\Pi$ is not a diagonal matrix as shown above. Our proof strategy is to explore a degenerate $U$-statistic structure of the leading term of the sketched 2SLS estimator under CountSketch. In general, the asymptotic distribution of degenerate $U$-statistics can be either a weighted average of independent, centered chi-square random variables with a complex form of weights, or simply a centered normal distribution ([Hall, 1984]). We establish that a Countsketch yields a centered normal distribution for $\tilde{\beta}$ which is convenient for inference.

Our proof method can be easily modified to study the ordinary least squares (OLS) estimator which is a special case when $Z = X$. We state this as a corollary.

**Corollary 1.** Let Assumptions [4](i)-(iii) and [4](iv) hold with $Z_i = X_i$. Suppose that $m = m_n \to \infty$ but $m_n/n \to 0$ as $n \to \infty$. Then, as $n \to \infty$,

$$m^{1/2}(\tilde{\beta}_{OLS} - \hat{\beta}_{OLS}) \to_d N(0, V_{OLS}),$$

$$V_{OLS} := \mathbb{E}(e_i^2) \left[ \mathbb{E}(X_iX_i^T) \right]^{-1}.$$

**5.1 Inference on $\beta_0$**

In applications, researchers would like to carry out inference on $\beta_0$ using the sketched 2SLS estimator. To use Theorem 3 for this purpose, note that

$$m^{1/2}(\tilde{\beta} - \hat{\beta}) = m^{1/2}(\tilde{\beta} - \beta_0) - (m/n)^{1/2} n^{1/2}(\tilde{\beta} - \beta_0)$$

$$= m^{1/2}(\tilde{\beta} - \beta_0) + o_p(1),$$
because \( m/n \to 0 \) and \( n^{1/2}(\tilde{\beta} - \beta_0) = O_p(1) \). To estimate \( V_0 \) in (5) using the sketched sample, define \( \tilde{e} := \tilde{y} - \tilde{X}\tilde{\beta} \) and

\[
\tilde{V} := \tilde{e}^T \tilde{e} \left( \tilde{X}^T \tilde{Z} (\tilde{Z}^T \tilde{Z})^{-1} \tilde{Z}^T \tilde{X} \right)^{-1}.
\] (10)

We obtain the following corollary directly from Theorem 3 by showing that \( \tilde{V} \to_p V_0 \).

**Corollary 2.** Let Assumptions in Theorem 3 hold. Then, as \( n \to \infty \),

\[
m^{1/2}\tilde{V}^{-1/2}(\tilde{\beta} - \beta_0) \to_d N(0, I_p),
\]

where \( I_p \) is the \( p \)-dimensional identity matrix.

Corollary 2 provides a method of carrying out asymptotic inference. As an example, suppose that we are interested in constructing a confidence interval for \( c^T \beta_0 \) for some known vector \( c \). In view of Corollary 2, an asymptotically valid \((1 - \alpha)\) confidence interval for \( c^T \beta_0 \) is constructed by

\[
\left[ c^T \tilde{\beta} - z(1 - \alpha/2)(c^T \tilde{V} c/m)^{1/2}, \ c^T \tilde{\beta} + z(1 - \alpha/2)(c^T \tilde{V} c/m)^{1/2} \right],
\]

where \( z(u) := \Phi(u) \) for each \( u \), where \( \Phi(\cdot) \) is the cumulative distribution function of the standard normal random variable.

### 6 The Choice of \( m \)

As shown in Theorem 2 we require that

\[
m \geq \max\{q(q + 1), 2pq\}/(\varepsilon^2 \delta),
\]

which implies that the desired \( m \) is data-oblivious. Since \( q \) needs to be as large as \( p \), \( m \) is of order \( q^2 \), assuming that \( \varepsilon \) and \( \delta \) are fixed. Then, as a simple rule of thumb, we may take

\[
m_1 = C_m q^2,
\] (11)

where \( C_m \) is a constant that needs to be chosen by a researcher. However, statistical analysis often cares about the variability of the estimates in repeated sampling, thereby implying that a larger \( m \) is always desirable in terms of efficiency, as demonstrated in Corollary 2. The question arises as to whether \( m \) can be designed to take both algorithmic and statistical
considerations into account. We follow our previous work (Lee and Ng, 2020) and suggest an inference-conscious guide.

To obtain a rule that is statistically motivated, let \( c^T \tilde{\beta} \) denote a linear combination of estimators, where \( c \) is a \( p \times 1 \) vector of constants. For example, it can be a particular element of \( \tilde{\beta} \) that a researcher is mainly interested in. Recall that \( \beta_0 \) denotes the true (unknown) value of \( \beta \). Define

\[
\xi_0(m) = \frac{c^T(\tilde{\beta} - \beta_0)}{\sigma(c^T \tilde{\beta})},
\]

where \( \sigma(c^T \tilde{\beta}) \) is the square-root of a consistent estimator of the asymptotic variance of \( \sqrt{mc^T (\tilde{\beta} - \beta_0)} \). That is, \( \sigma(c^T \tilde{\beta}) = (c^T \tilde{V} c/m)^{1/2} \).

Consider the usual \( t \)-statistic \( \tau_0(m) = \sqrt{m} \xi_0(m) \). In view of asymptotic normality of the sketched 2SLS estimator in Corollary 2, suppose that \( m \) is large enough such that \( P_{\beta_0}(\tau_0 > z) \approx \Phi(-z) \) for each \( z \). Now consider a one-sided test \( \tau_1 \) based on \( \tilde{\beta} \) against an alternative, say, \( \beta^0 \). The test \( \tau_1 \) is related to \( \tau_0 \) by

\[
\xi_1(m) = \frac{c^T(\tilde{\beta} - \beta_0)}{\sigma(c^T \tilde{\beta})} - \frac{c^T(\beta^0 - \beta_0)}{\sigma(c^T \tilde{\beta})} = \xi_0(m) - \xi_2(m).
\]

The power of \( \tau_1 \) at nominal size \( \alpha \) is then

\[
P_{\beta_0} \left[ \sqrt{m} (\xi_0 - \xi_2) > \Phi^{-1}_{(1-\alpha)} \right] \approx \Phi \left( -\Phi^{-1}_{(1-\alpha)} - \sqrt{m} \xi_2 \right) \equiv \gamma.
\]

Define

\[
S(\alpha, \gamma) = \Phi^{-1}_{\gamma} + \Phi^{-1}_{1-\alpha}.
\]

Solving for \( \sqrt{m} \xi_2 \) and taking the square of it gives

\[
m \xi_2^2(m) = S^2(\alpha, \gamma).
\]

In practice, we do not know the value of \( \xi_2^2(m) \). Instead, we approximate it by

\[
\xi_2^2(m) \approx \frac{m_1 \xi_2^2(m_1)}{m_1} = \frac{\tau_2^2(m_1)}{m_1} = \frac{1}{m_1} \left[ \frac{c^T(\beta^0 - \beta_0)}{\text{SE}(c^T \tilde{\beta})} \right]^2,
\]

where \( \text{SE}(c^T \tilde{\beta}) = \sigma(c^T \tilde{\beta})/\sqrt{m_1} = (c^T \tilde{V} c/m_1)^{1/2} \), which can be estimated using the initial sketch with \( m_1 \). Alternatively, setting \( m_1 \) to \( n \) gives \( m_2(n) = n S^2(\bar{\alpha}, \gamma)/\tau_2^2(n) \). This leads to the following two data-dependent inference-conscious guides for sketch size \( m \).
Algorithm 1: Inference-conscious sketch size

**Input:** $m_1$, $\bar{\alpha}$, $\bar{\gamma}$

**Output:** $m_2$ or $m_3$

1. Choose the nominal size $\bar{\alpha}$ and target power $\bar{\gamma}$ of a one-sided $t$-test.
2. Obtain the standard error, $\text{SE}(c^T \tilde{\beta})$, of $c^T \tilde{\beta}$ using an initial sketch of size $m_1 = C_m q^2$ for some constant $C_m$, say $C_m = 10$.
3. A data-dependent *inference-conscious* sketch size for pre-specified $\beta_0 - \beta_0$ is

$$m_2(m_1) = m_1 S^2(\bar{\alpha}, \bar{\gamma}) \left[ \frac{\text{SE}(c^T \tilde{\beta})}{c^T (\beta_0 - \beta_0)} \right]^2.$$  \hspace{1cm} (12)

4. A data-oblivious *inference-conscious* sketch size for a pre-specified $\tau_2(\infty)$ is

$$m_3 = n \frac{S^2(\bar{\alpha}, \bar{\gamma})}{\tau_2^2(\infty)},$$ \hspace{1cm} (13)

Note that $m_3$ only requires the choice of $\bar{\alpha}$, $\bar{\gamma}$, and $\tau_2(\infty)$ which, unlike $m_2$, can be computed without a preliminary sketch.

7 **A Monte Carlo Experiment**

In this section, we report the results of a Monte Carlo experiment. We focus on the asymptotic normality result and examine the finite sample performance of the sketched 2SLS estimators.

The instruments $Z_i = (1, Z_{1,i}, \ldots, Z_{q-1,i})^T$ consist of a constant term and a $(q - 1)$-dimensional random vector $(Z_{1,i}, \ldots, Z_{q-1,i})^T$ generated from a multivariate normal distribution with mean zero vector and the variance covariance matrix $\Sigma$, whose $(i,j)$ component is $\Sigma_{ij} = \rho^{|i-j|}$ with $\rho = 0.5$. A scalar right-hand side endogenous variable $X_{1i}$ is generated by

$$X_{1i} = \sum_{j=1}^{q-1} Z_{j,i} + V_i,$$

where $V_i \sim N(0,1)$. A $(p - 2)$-dimensional vector of right-hand side exogenous variables is $X_{2i} = (Z_{1,i}, \ldots, Z_{p-2,i})^T$; thus, only $(Z_{p-1,i}, \ldots, Z_{q,3})^T$ are excluded from the outcome equation:

$$y_i = X_i^T \beta_0 + e_i,$$

where $X_i = (1, X_{1i}, X_{2i})^T$, $\beta_0 = (0, 1, \ldots, 1)^T$, and $e_i = V_i + \eta_i$, where $\eta_i$ is generated
from the distribution in the Pearson system with mean 0, standard deviation 1, skewness 1, and kurtosis 5. The results for $\eta_i \sim N(0,1)$ are similar and not reported. Here, $V_i$, $\eta_i$ and $Z_i$ are mutually independent. The source of endogeneity comes from $V_i$, so that $E(X_{1i}e_i) = E(V_i^2) = 1 \neq 0$. Throughout the Monte Carlo experiment, we set $p = 6$ and $q = 21$.

Figure 1: Results of a Monte Carlo Experiment

Notes. The top panel shows the histogram of $\frac{\hat{\beta}_2 - \beta_{02}}{\text{SE}(\hat{\beta}_2)}$ with a normal distribution fit and also reports the mean and standard deviation in 10,000 replications. The second and third panels depict the histogram when the sketched samples are obtained by uniform sampling without replacement and the CountSketch, respectively. The bottom panel puts three subfigures together.

The parameter of interest is $\beta_{02} = c^T \beta_0$ with $c^T = (0, 1, 0, \ldots, 0)$, namely the coefficient for the endogenous variable $X_{1i}$. The full-sample and sketch sizes are $n = 10,000$ and $m = 500$, respectively. For each Monte Carlo repetition, we generated $n$ independent and identical observations of $(y_i, X_i, Z_i)$ and construct a sketch of size $m$ to compute both the full and sketched 2SLS estimates. For comparison, we also constructed a random subsample via
uniform subsampling without replacement and computed the subsampled 2SLS estimates. There were 10,000 repetitions in the Monte Carlo experiment.

Figure 1 summarizes the results of the experiment. The top panel shows the histogram of the full sample $(\hat{\beta}_2 - \beta_{02})/\text{se}(\hat{\beta}_2)$ with a normal distribution fit. The second and third panels depict the histograms for uniform sampling without replacement and the CountSketch, respectively. The bottom panel puts three subfigures together. Each of the subfigures is well approximated by a normal distribution. There is some bias for both uniform sampling and CountSketch; however, the overall finite-sample performance is encouraging.

8 An Empirical Illustration

An exemplary case of the 2SLS in economics is to estimate the return to education. Suppose that $y_i$ is the wages for worker $i$ (typically in logs) and $X_i$ contains educational attainment $\text{edu}_i$ (say, years of schooling completed). Here, the unobserved random variable $e_i$ includes worker $i$’s unobserved ability among other things. Then, $\text{edu}_i$ will be correlated with $e_i$ if workers with higher ability tends to attain higher levels of education. Therefore, the least-squares estimator may not be a consistent estimator of the return to schooling—that is, the causal effect of education—although it can estimate correlation between wages and schooling. To overcome this problem, economists use an instrumental variable that is uncorrelated with $e_i$ but correlated with $\text{edu}_i$. For example, Angrist and Keueger (1991) used quarter of birth as instrument for schooling. Their idea was that season of birth is unlikely to be correlated with workers’ ability but can affect educational attainment because of compulsory schooling laws.

In this section, we illustrate the sketched 2SLS estimator based on CountSketch using the well-known dataset of Angrist and Keueger (1991). Specifically, we look at the 2SLS estimate of the return to education in column (2) of Table IV in their paper. The dependent variable $y$ is the log weekly wages, the covariates $X$ include years of education, the intercept term and 9 year-of-birth dummies ($p = 11$), and the instruments $Z$ are a full set of quarter-of-birth times year-of-birth interactions ($q = 40$). The full-sample 2SLS estimate with $n = 247,199$ is 0.0769 with the standard error of 0.0150.

Table 1 gives the choices of $m$. In Panel A, we present $m_1 = C_mq^2$ with different values of $C_m$. The last column shows the percentage of the subsample size $m_1$ relative to the full sample size $n$. In Panel B, we compute $m_2(m_1)$ with $c^T \beta_0$ being the return to education. The average standard error of the subsample estimates of the return to education was 0.03 out of 100 randomly sketched subsamples using $m_1 = 10q^2$. A different value of $\tau_2(m_1)$ correspond to a different effect size: that is, $c^T(\beta^0 - \beta_0) \in \{0.03, 0.05, 0.07, 0.09\}$. In Panel C, we present
Table 1: Choices of $m$ in the empirical illustration

Panel A. $m_1 = C_m q^2$ for a given constant $C_m$

| $n$     | $p$ | $q$ | $C_m$ | $m_1$ | $100m_1/n$ |
|---------|-----|-----|-------|-------|------------|
| 247199  | 11  | 40  | 5     | 8000  | 3.24       |
|         | 10  |     |       | 16000 | 6.47       |
|         | 15  |     |       | 24000 | 9.71       |
|         | 20  |     |       | 32000 | 12.95      |

Panel B. $m_2(m_1) = m_1 S^2(\bar{\alpha}, \bar{\gamma})/\tau_2^2(m_1)$ as a function of $\tau_2(m_1)$ with $m_1 = 10q^2$

| $n$     | $\alpha$ | $\gamma$ | $\tau_2(m_1)$ | $m_2$ | $100m_2/n$ |
|---------|-----------|-----------|----------------|-------|------------|
| 247199  | 0.05      | 0.8       | 1.00           | 98883 | 40.00      |
|         |           |           | 1.67           | 35598 | 14.40      |
|         |           |           | 2.33           | 18162 | 7.35       |
|         |           |           | 3.00           | 10987 | 4.44       |

Panel C. $m_3 = n S^2(\bar{\alpha}, \bar{\gamma})/\tau_2^2(\infty)$ as a function of $\tau_2(\infty)$

| $n$     | $\alpha$ | $\gamma$ | $\tau_2(\infty)$ | $m_3$ | $100m_3/n$ |
|---------|-----------|-----------|-----------------|-------|------------|
| 247199  | 0.05      | 0.8       | 3               | 169749| 68.67      |
|         |           |           | 5               | 61110 | 24.72      |
|         |           |           | 7               | 31178 | 12.61      |
|         |           |           | 9               | 18861 | 7.63       |
$m_3$ as a function of $\tau_2(\infty)$.

We took three subsample sizes $m_1 = 16,000$ from Panel A, $m_2 = 35,598$ from Panel B, and $m_3 = 61110$ from Panel C. They range from 6.5% to 25% of the full sample size. To evaluate how data sketching by CountSketch works, we generated 1,000 subsamples independently and computed 2SLS subsample estimates 1,000 times. Table 2 and Figure 2 report the empirical distributions of sketched 2SLS estimates across three different values of $m$. All three cases center around the full-sample estimate of 0.0769 and the variability decreases as $m$ gets larger.

| $m$     | mean | median | standard deviation | 1%    | 99%    |
|---------|------|--------|--------------------|-------|-------|
| 16000   | 0.080| 0.080  | 0.030              | 0.009 | 0.152 |
| 35598   | 0.078| 0.079  | 0.027              | 0.017 | 0.141 |
| 61110   | 0.080| 0.079  | 0.024              | 0.025 | 0.133 |

Figure 2: Histograms of Sketched Estimates

9 Extensions and Future Research

We have introduced a random sketch for the 2SLS estimator and have analyzed its algorithmic and statistical properties, focusing on CountSketch. We have recommended selecting
the size of random sketch based on power calculation.

Our method for proving asymptotic normality result may be used to analyze sampling schemes other than the Countsketch. For a general random \( \Pi \), we have that

\[ n^{-1}(u^T \Pi^T \Pi v - u^T v) = n^{-1} \sum_{i=1}^{n} \psi_i u_i v_j + n^{-1} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} u_i \varphi_{ij} v_j, \tag{14} \]

where

\[ \psi_i := \sum_{k=1}^{m} \Pi^2_{ki} - 1 \quad \text{and} \quad \varphi_{ij} := \sum_{k=1}^{m} \Pi_{ki} \Pi_{kj}. \]

Consider the first term on the right-hand side of (14). For CountSketch, \( \psi_i \equiv 0 \) for each \( i \); however, in order to obtain asymptotic normality, it suffices to assume that

\[ n^{-1} \sum_{i=1}^{n} \psi_i u_i v_i = o_p \left( m^{-1/2} \right), \]

which will be true for a variety of algorithmic sampling methods.

We now turn to the second term on the right-hand side of (14). As before, we write

\[ n^{-1} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} u_i \varphi_{ij} v_j = \sum_{1 \leq i < j \leq n} H_{ij}, \]

where

\[ H_{ij} = (u_i v_j + u_j v_i) \varphi_{ij}. \]

Let \( W_i = (u_i, v_i, \Pi_{1i}, \ldots, \Pi_{mi}) \) denote all random variables associated with \( i \). Write \( H_{ij} = H_n(W_i, W_j) \) because \( W_i \) depends on \( m \) and therefore \( n \). Assume that \( \{W_1, \ldots, W_n\} \) are i.i.d. random vectors. Suppose that \( \mathbb{E}[H_n(W_1, W_2)|W_1] = 0 \) almost surely and \( \mathbb{E}[H_n^2(W_1, W_2)] < \infty \) for each \( n \). Let \( G_n(w_1, w_2) := \mathbb{E}[H_n(W_1, w_1)H_n(W_1, w_2)]. \) The crucial sufficient condition for asymptotic normality in Hall (1984, Theorem 1) is that

\[ \frac{\mathbb{E}[G_n^2(W_1, W_2)]}{\{\mathbb{E}[H_n^2(W_1, W_2)]\}^2} + n^{-1} \mathbb{E}[H_n^4(W_1, W_2)] \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \tag{15} \]

We verify (15) for CountSketch and expect it will hold for other random projection methods.

Our analysis can also be used to study other sketched estimators that are linear combinations of the data. Consider the two-sample (TS) 2SLS estimator analyzed in Angrist and Krueger (1992 [1995]) and Inoue and Solon (2010). Let \( (y_1, Z_1) \) be an \( n_1 \times (1 + q) \) matrix.
of the outcome variable and instruments in the first sample, and \((X_2, Z_2)\) an \(n_2 \times (p + q)\) matrix of regressors and instruments in the second sample. Then, the TS2SLS estimator is defined by

\[
\hat{\beta}_{TS2SLS} := (X_2^T Z_2 (Z_2^T Z_2)^{-1} Z_1^T Z_1 (Z_2^T Z_2)^{-1} Z_2^T X_2)^{-1} X_2^T Z_2 (Z_2^T Z_2)^{-1} Z_1^T y_1.
\]

Let \(\Pi_1\) and \(\Pi_2\), respectively, denote \(m_1 \times n_1\) and \(m_2 \times n_2\) random matrices. Here, \(m_1\) and \(m_2\) can be set to be equal or different. A sketch of the first sample is \((\tilde{y}_1, \tilde{Z}_1)\), where \(\tilde{y}_1 = \Pi_1 y_1\) and \(\tilde{Z}_1 = \Pi_1 Z_1\). Likewise, a sketch of the second sample is \((\tilde{X}_2, \tilde{Z}_2)\), where \(\tilde{X}_2 = \Pi_2 X_2\) and \(\tilde{Z}_2 = \Pi_2 Z_2\). Then, a sketched version of TS2SLS is defined by

\[
\hat{\beta}_{TS2SLS} := (\tilde{X}_2^T \tilde{Z}_2 (Z_2^T \tilde{Z}_2)^{-1} \tilde{Z}_1^T \tilde{Z}_1 (Z_2^T \tilde{Z}_2)^{-1} \tilde{Z}_2^T \tilde{X}_2)^{-1} \tilde{X}_2^T \tilde{Z}_2 (Z_2^T \tilde{Z}_2)^{-1} \tilde{Z}_1^T \tilde{y}_1.
\]

Assuming that \(\Pi_1\) and \(\Pi_2\) are independent, it would be straightforward to obtain theoretical results similar to those obtained in this paper. Moreover, our rules of choosing inference-conscious sketch size would be easily adopted.

In general, it would be conceptually the same if we consider estimators based on matrix multiplication. It would require a possibly different approach if we consider nonlinear estimators, e.g., the generalized method of moments, logit, quantile regression, maximum score estimators among others. These are topics for future research.

## A  Proofs

### A.1  Proof of Theorem 1

Recall that using the singular value decomposition of \(X\) and \(Z\), we write \(X = U_X \Sigma_X V_X^T\) and \(Z = U_Z \Sigma_Z V_Z^T\). Define

\[
\hat{\theta} := (U_X^T U_Z U_Z^T U_X)^{-1} U_X^T U_Z y
\]  

and

\[
\tilde{\theta} := (U_X^T \Pi^T \Pi U_Z (U_Z^T \Pi^T \Pi U_Z)^{-1} U_Z^T \Pi^T \Pi U_X)^{-1} U_X^T \Pi^T \Pi U_Z (U_Z^T \Pi^T \Pi U_Z)^{-1} U_Z^T \Pi^T \Pi y.
\]  

It would be convenient to work with \(U_X \hat{\theta}\) and \(U_X \tilde{\theta}\) in order to analyze algorithmic properties of sketched 2SLS estimators because \(U_X\) is an orthonormal matrix. The following lemma establishes the equivalence between \(X_\hat{\beta}\) and \(U_X \hat{\theta}\).
Lemma 1. Let Assumption 2 hold. Then, \( \hat{X} = U_X \tilde{\theta} \).

Proof. By the singular value decomposition of \( X \) and \( Z \), we have that

\[
Z^T Z = V_Z \Sigma_Z^2 V_Z^T, \\
Z(Z^T Z)^{-1} Z^T = U_Z U_Z^T, \\
X^T Z(Z^T Z)^{-1} Z^T X = V_X \Sigma_X U_X^T U_Z U_Z^T U_X \Sigma_X V_X^T, \\
(X^T Z(Z^T Z)^{-1} Z^T X)^{-1} = V_X \Sigma_X^{-1} (U_X^T U_Z U_Z^T U_X)^{-1} \Sigma_X^{-1} V_X^T, \\
X^T Z(Z^T Z)^{-1} Z^T \beta = V_X \Sigma_X U_X^T U_Z U_Z^T \beta.
\]

Therefore,

\[
\tilde{\beta} = V_X \Sigma_X^{-1} (U_X^T U_Z U_Z^T U_X)^{-1} U_X^T U_Z U_Z^T \beta, \\
X \tilde{\beta} = U_X (U_X^T U_Z U_Z^T U_X)^{-1} U_X^T U_Z U_Z^T \beta,
\]

which in turn implies the conclusion in view of the definition of \( \tilde{\theta} \) in (16). □

As in Lemma 1, the equivalence between \( X \tilde{\beta} \) and \( U_X \tilde{\theta} \) holds.

Lemma 2. Assume that (i) \( \tilde{Z}^T \tilde{Z} \) is non-singular and (ii) \( \tilde{X}^T \tilde{Z}(\tilde{Z}^T \tilde{Z})^{-1} \tilde{Z}^T \tilde{X} \) is non-singular. Then, \( \tilde{X} \tilde{\beta} = U_X \tilde{\theta} \).

Proof. As in the proof of Lemma 1 we have that

\[
\tilde{Z}^T \tilde{Z} = V_Z \Sigma_Z U_Z^T \Pi \Pi U_Z \Sigma V_Z^T, \\
(\tilde{Z}^T \tilde{Z})^{-1} = V_Z \Sigma_Z^{-1} (U_Z^T \Pi \Pi U_Z)^{-1} \Sigma_Z^{-1} V_Z^T, \\
\tilde{Z}(\tilde{Z}^T \tilde{Z})^{-1} \tilde{Z}^T \tilde{X} = V_X \Sigma_X U_X^T \Pi \Pi U_Z (U_Z^T \Pi \Pi U_Z)^{-1} U_Z^T \Pi \Pi U_X \Sigma_X V_X^T, \\
(\tilde{X}^T \tilde{Z}(\tilde{Z}^T \tilde{Z})^{-1} \tilde{Z}^T \tilde{X})^{-1} = V_X \Sigma_X^{-1} (U_X^T \Pi \Pi U_Z (U_Z^T \Pi \Pi U_Z)^{-1} U_Z^T \Pi \Pi U_X)^{-1} \Sigma_X^{-1} V_X^T, \\
\tilde{X}^T \tilde{Z}(\tilde{Z}^T \tilde{Z})^{-1} \tilde{Z}^T \tilde{Y} = V_X \Sigma_X U_X^T \Pi \Pi U_Z (U_Z^T \Pi \Pi U_Z)^{-1} U_Z^T \Pi \Pi U_Y.
\]

Therefore,

\[
\tilde{\beta} = V_X \Sigma_X^{-1} (U_X^T \Pi \Pi U_Z (U_Z^T \Pi \Pi U_Z)^{-1} U_Z^T \Pi \Pi U_X)^{-1} U_X^T \Pi \Pi U_Z (U_Z^T \Pi \Pi U_Z)^{-1} U_Z^T \Pi \Pi y, \\
X \tilde{\beta} = U_X (U_X^T \Pi \Pi U_Z (U_Z^T \Pi \Pi U_Z)^{-1} U_Z^T \Pi \Pi U_X)^{-1} U_X^T \Pi \Pi U_Z (U_Z^T \Pi \Pi U_Z)^{-1} U_Z^T \Pi \Pi Y,
\]

which again implies the conclusion in view of the definition of \( \tilde{\theta} \) in (17). □
Define
\[ \tilde{A} := U_X^T \Pi^T \Pi U_Z (U_Z^T \Pi^T \Pi U_Z)^{-1} U_Z^T \Pi^T \Pi U_X, \]
\[ \hat{A} := U_X^T U_Z U_Z^T U_X, \]
\[ \tilde{B} := U_X^T \Pi^T \Pi U_Z (U_Z^T \Pi^T \Pi U_Z)^{-1} U_Z^T \Pi^T \hat{c}, \]
\[ \hat{B} := U_X^T U_Z U_Z^T \hat{c}. \]

**Lemma 3.** Let Assumption 2 hold. Then, \( \hat{A}^{-1} \hat{B} = 0. \)

**Proof.** Note that
\[
\hat{A}^{-1} \hat{B} = (U_X^T U_Z U_Z^T U_X)^{-1} U_X^T U_Z U_Z^T \hat{c} \\
= (U_X^T U_Z U_Z^T U_X)^{-1} U_X^T U_Z U_Z^T y - (U_X^T U_Z U_Z^T U_X)^{-1} U_X^T U_Z U_Z^T X \hat{\beta} \\
= 0,
\]
since \( X \hat{\beta} = U_X (U_X^T U_Z U_Z^T U_X)^{-1} U_X^T U_Z U_Z^T y. \)

Under Assumption 2 we first obtain the following lemma.

**Lemma 4.** Let Assumption 2 hold. Then, the following holds jointly with probability at least \( 1 - \delta : \)
\[
\| \tilde{A} - \hat{A} \|_2 \leq f_1(\varepsilon_1, \varepsilon_2), \\
\| \tilde{B} - \hat{B} \|_2 \leq \varepsilon_3 \| \hat{c} \| + f_2(\varepsilon_1, \varepsilon_2) [1 + \varepsilon_3 \| \hat{c} \|].
\]

**Proof.** Let \( \tilde{A}_1 := U_Z^T \Pi^T \Pi U_X, \tilde{A}_2 := (U_Z^T \Pi^T \Pi U_Z)^{-1}, \hat{A}_1 := U_Z^T U_X, \) and \( \hat{A}_2 := I. \) Then we have that
\[
\tilde{A} - \hat{A} \\
= \tilde{A}_1^T \tilde{A}_2 \tilde{A}_1 - \hat{A}_1^T \hat{A}_2 \hat{A}_1 \\
= (\tilde{A}_1 - \hat{A}_1)^T \tilde{A}_2 (\tilde{A}_1 - \hat{A}_1) + \hat{A}_1^T \hat{A}_2 (\tilde{A}_1 - \hat{A}_1) + (\tilde{A}_1 - \hat{A}_1)^T \tilde{A}_2 \hat{A}_1 + \hat{A}_1^T (\tilde{A}_2 - \hat{A}_2) \hat{A}_1 \\
= (\tilde{A}_1 - \hat{A}_1)^T \tilde{A}_2 (\tilde{A}_1 - \hat{A}_1) + (\tilde{A}_1 - \hat{A}_1)^T (\tilde{A}_2 - \hat{A}_2) (\tilde{A}_1 - \hat{A}_1) \\
+ \hat{A}_1^T \tilde{A}_2 (\tilde{A}_1 - \hat{A}_1) + \hat{A}_1^T (\tilde{A}_2 - \hat{A}_2) (\tilde{A}_1 - \hat{A}_1) \\
+ (\tilde{A}_1 - \hat{A}_1)^T \tilde{A}_2 \hat{A}_1 + (\tilde{A}_1 - \hat{A}_1)^T (\tilde{A}_2 - \hat{A}_2) \hat{A}_1 \\
+ \hat{A}_1^T (\tilde{A}_2 - \hat{A}_2) \hat{A}_1.
\]
It is straightforward to show that $\left\| \hat{A}_2 - \hat{A}_2 \right\|_2 \leq \varepsilon_1/(1 - \varepsilon_1)$ using Assumption 2(i). Since $\left\| \hat{A}_1 \right\|_2 \leq \left\| U_Z \right\|_2 \left\| U_X \right\|_2 = 1$ and $\left\| \hat{A}_2 \right\|_2 = 1$, we have that

$$\left\| \tilde{A} - \hat{A} \right\|_2 \leq \varepsilon_2^2 + \varepsilon_2^2\varepsilon_1/(1 - \varepsilon_1) + 2\varepsilon_2 + 2\varepsilon_2\varepsilon_1/(1 - \varepsilon_1) + \varepsilon_1/(1 - \varepsilon_1)$$

$$= \frac{\varepsilon_1 + \varepsilon_2(\varepsilon_2 + 2)}{1 - \varepsilon_1} = f_1(\varepsilon_1, \varepsilon_2),$$

using Assumption 2. This proves the first desired result.

Now let $\tilde{B}_1 := U_Z^T \Pi^T \hat{\Pi} e$ and $\hat{B}_1 := U_T^T \hat{\Xi}$. Consider

$$\tilde{B} - \hat{B} = U_T^T \Pi^T \Pi U_Z (U_T^T \Pi^T \Pi U_Z)^{-1} U_Z^T \Pi^T \hat{\Xi} - U_T^T U_Z \hat{\Xi}$$

$$= A_1^T \hat{A}_2 B_1 - \hat{A}_1^T \hat{A}_2 B_1$$

$$= (\tilde{A}_1 - \hat{A}_1)^T \hat{A}_2 (\tilde{B}_1 - \hat{B}_1) + \hat{A}_1^T \hat{A}_2 (\tilde{B}_1 - \hat{B}_1) + (\tilde{A}_1 - \hat{A}_1)^T \hat{A}_2 \hat{B}_1 + \hat{A}_1^T (\tilde{A}_2 - \hat{A}_2) \hat{B}_1$$

$$= (\tilde{A}_1 - \hat{A}_1)^T \hat{A}_2 \tilde{B}_1 + (\tilde{A}_1 - \hat{A}_1)^T (\tilde{A}_2 - \hat{A}_2) (\tilde{B}_1 - \hat{B}_1)$$

$$+ \hat{A}_1^T \hat{A}_2 (\tilde{B}_1 - \hat{B}_1) + \hat{A}_1^T (\tilde{A}_2 - \hat{A}_2) (\tilde{B}_1 - \hat{B}_1)$$

$$+ (\tilde{A}_1 - \hat{A}_1)^T \hat{A}_2 \tilde{B}_1 + (\tilde{A}_1 - \hat{A}_1)^T (\tilde{A}_2 - \hat{A}_2) \tilde{B}_1$$

$$+ \hat{A}_1^T (\tilde{A}_2 - \hat{A}_2) \tilde{B}_1.$$

Since $\left\| \hat{B}_1 \right\|_2 = \left\| U_T^T \hat{\Xi} e \right\|_2 \leq \left\| U_Z \right\|_2 \left\| \hat{\Xi} e \right\| \leq \left\| \hat{\Xi} e \right\|$, we have that

$$\left\| \tilde{B} - \hat{B} \right\|_2 \leq \varepsilon_2 \left\| \hat{\Xi} e \right\| + \left[ \varepsilon_2 + \varepsilon_1/(1 - \varepsilon_1) + \varepsilon_2\varepsilon_1/(1 - \varepsilon_1) \right] \left\| 1 + \varepsilon_3 \left\| \hat{\Xi} e \right\| \right\|$$

$$= \varepsilon_3 \left\| \hat{\Xi} e \right\| + f_2(\varepsilon_1, \varepsilon_2) \left[ 1 + \varepsilon_3 \left\| \hat{\Xi} e \right\| \right],$$

again using Assumption 2. This proves the second desired result.

Lemma 5. Let Assumptions 2 and 3 hold. Then, the following holds with probability at least $1 - \delta$:

$$\sigma_{\min}(\hat{A}) \geq \frac{1}{2} \sigma_{\min}^2(U_Z^T U_X).$$

Proof. Use the fact that for real matrices $C$ and $D$,

$$\sigma_{\min}(C + D) \geq \sigma_{\min}(C) - \sigma_{\max}(D)$$
to obtain

\[ \sigma_{\min}(\tilde{A}) \geq \sigma_{\min}(\hat{A}) - \sigma_{\max}(\hat{A} - A). \]

Then the desired result follows from the first conclusion of Lemma 4, since

\[ \sigma_{\min}(\hat{A}) = \sigma_{\min}(U_X^T U_Z U_Z^T U_X) = \sigma_{\min}^2(U_Z^T U_X) \quad \text{and} \quad \sigma_{\max}(\tilde{A} - \hat{A}) \leq \| \tilde{A} - \hat{A} \|_2. \]

Lemma 5 implies that \( \tilde{A}^{-1} \) is well defined with probability at least \( 1 - \delta \).

**Lemma 6.** Let Assumptions 2 and 3 hold. Then, the following holds with probability at least \( 1 - \delta \):

\[ \| A^{-1} - \tilde{A}^{-1} \|_2 \leq 2 f_1(\varepsilon_1, \varepsilon_2) \sigma_{\min}^2(U_Z^T U_X). \]

**Proof.** Write

\[ \tilde{A}^{-1} - \hat{A}^{-1} = \hat{A}^{-1} (\hat{A} - \tilde{A}) \tilde{A}^{-1}. \]

Thus,

\[
\begin{align*}
\| A^{-1} - \tilde{A}^{-1} \|_2 & \leq \| \tilde{A}^{-1} \|_2 \| \hat{A} - \tilde{A} \|_2 \| \tilde{A}^{-1} \|_2 \\
& \leq \frac{2 f_1(\varepsilon_1, \varepsilon_2)}{\sigma_{\min}^2(U_Z^T U_X)}
\end{align*}
\]

since \( \| \tilde{A}^{-1} \|_2 = [\sigma_{\min}^2(U_Z^T U_X)]^{-1} \), by Lemma 4\( \| \hat{A} - \tilde{A} \|_2 \leq f_1(\varepsilon_1, \varepsilon_2) \) and, by Lemma 5, \( \| \tilde{A}^{-1} \|_2 \leq 2 [\sigma_{\min}^2(U_Z^T U_X)]^{-1} \) with probability at least \( 1 - \delta \).

**Proof of Theorem 7** By Lemmas 1 and 2,

\[ X(\tilde{\beta} - \hat{\beta}) = U_X(\tilde{\theta} - \hat{\theta}), \]

so that

\[ \sigma_{\min}(X) \| \tilde{\beta} - \hat{\beta} \| \leq \| \tilde{\theta} - \hat{\theta} \|. \]
Thus, it suffices to bound \( \|\tilde{\theta} - \hat{\theta}\| \). To do so, write

\[
\tilde{y} = \Pi \left( X \hat{\beta} + \hat{\epsilon} \right) = X \hat{\beta} + \hat{\epsilon} = \Pi U \hat{\theta} + \tilde{\epsilon},
\]

where \( \tilde{\epsilon} = \Pi \hat{\epsilon} \). Plugging (18) into (17) yields

\[
\tilde{\theta} - \hat{\theta} = \tilde{A}^{-1} \tilde{B}.
\]

Then, by Lemma 3, we have that \( \tilde{\theta} - \hat{\theta} = \tilde{A}^{-1} \tilde{B} = \tilde{A}^{-1} \hat{B} - \tilde{A}^{-1} \hat{B} \). Further, write

\[
\tilde{\theta} - \hat{\theta} = (\tilde{A}^{-1} - \hat{A}^{-1}) \hat{B} + \hat{A}^{-1} (\hat{B} - \tilde{B}) + (\tilde{A}^{-1} - \hat{A}^{-1}) (\tilde{B} - \hat{B}).
\]

Thus,

\[
\|\tilde{\theta} - \hat{\theta}\| = \|\tilde{A}^{-1} \tilde{B} - \hat{A}^{-1} \hat{B}\|_2
\]

\[
\leq \|\tilde{A}^{-1} - \hat{A}^{-1}\|_2 \|\tilde{B}\|_2 + \|\tilde{A}^{-1}\|_2 \|\hat{B} - \tilde{B}\|_2 + \|\tilde{A}^{-1} - \hat{A}^{-1}\|_2 \|\tilde{B} - \hat{B}\|_2
\]

\[
\leq \frac{2f_1(\varepsilon_1, \varepsilon_2)}{\sigma_{\min}(U_T^T U_X)} \|\tilde{\epsilon}\| + \frac{\varepsilon_3 \|\tilde{\epsilon}\| + f_2(\varepsilon_1, \varepsilon_2) [1 + \varepsilon_3 \|\tilde{\epsilon}\|]}{\sigma_{\min}^2(U_T^T U_X)}
\]

\[
+ \frac{2f_1(\varepsilon_1, \varepsilon_2)}{\sigma_{\min}^4(U_T^T U_X)} \left\{ \varepsilon_3 \|\tilde{\epsilon}\| + f_2(\varepsilon_1, \varepsilon_2) [1 + \varepsilon_3 \|\tilde{\epsilon}\|] \right\}
\]

\[
= f_2(\varepsilon_1, \varepsilon_2) + \varepsilon_3 \|\tilde{\epsilon}\| \left[ 1 + f_2(\varepsilon_1, \varepsilon_2) \right] \left[ 1 + \frac{2f_1(\varepsilon_1, \varepsilon_2)}{\sigma_{\min}^2(U_T^T U_X)} \right],
\]

where the last inequality follows from Assumption 3.

\[\square\]

### A.2 Proof of Theorem 2

To establish Lemma 9, given below, we first state the following known results in the literature.

**Lemma 7** (Theorem 6.2 of Kane and Nelson (2014)). Distribution \( \mathcal{D} \) over \( \mathbb{R}^{m \times n} \) is defined to have \((\varepsilon, \delta, 2)\)-JL (Johnson-Lindenstrauss) moments if for any \( x \in \mathbb{R}^n \) such that \( \|x\| = 1 \),

\[
\mathbb{E}_{\Pi \sim \mathcal{D}} \left[ \|\Pi x\|^2 - 1 \right]^2 \leq \varepsilon^2 \delta.
\]

Given \( \varepsilon, \delta \in (0, 1/2) \), let \( \mathcal{D} \) be any distribution over matrices with \( n \) columns with the \((\varepsilon, \delta, 2)\)-JL moment property. Then, for any \( A \) and \( B \) real matrices each with \( n \) rows,

\[
\mathbb{P}_{\Pi \sim \mathcal{D}} \left( \|A^T \Pi^T \Pi B - A^T B\|_F > 3\varepsilon \|A\|_F \|B\|_F \right) < \delta.
\]
Lemma 8 (Theorem 2.9 of Woodruff (2014)). Let \( \Pi \in \mathbb{R}^{m \times n} \) be CountSketch with \( m \geq 2/(\varepsilon^2 \delta) \). Then, \( \Pi \) satisfies the \((\varepsilon, \delta, 2)\)-JL moment property.

Lemma 9. Let \( \Pi \in \mathbb{R}^{m \times n} \) be CountSketch with \( m \geq \max\{q(q+1), 2pq\}/(\varepsilon^2 \delta) \) for some \( \varepsilon \in (0, 1/2) \). Then, Assumption 2 holds with \( \varepsilon_1 = \varepsilon, \varepsilon_2 = 3\varepsilon, \varepsilon_3 = 3\varepsilon p^{-1/2} \).

Proof. As shown in the proof of Theorem 2 of Nelson and Nguyen (2013),

\[
\mathbb{P}_{\Pi \sim \mathcal{D}} \left( \|U_Z^T \Pi^T \Pi U_Z - I_q\|_2 > \varepsilon \right) < \delta,
\]

provided that \( m \geq q(q+1)/(\varepsilon^2 \delta) \). This verifies the first condition of Assumption 2.

Now to verify conditions (ii) and (iii) of Assumption 2, note that since CountSketch with \( m \geq 2/(\varepsilon^2 \delta) \) satisfies the \((\varepsilon, \delta, 2)\)-JL moment property, we have, for any any \( A \) and \( B \) real matrices each with \( n \) rows,

\[
\mathbb{P}_{\Pi \sim \mathcal{D}} \left( \|A^T \Pi^T \Pi B - A^T B\|_2 > 3\varepsilon \|A\|_F \|B\|_F \right) \\
\leq \mathbb{P}_{\Pi \sim \mathcal{D}} \left( \|A^T \Pi^T \Pi B - A^T B\|_F > 3\varepsilon \|A\|_F \|B\|_F \right) < \delta.
\]

Since \( \|U_X\|_F^2 = p, \|U_Z\|_F^2 = q \) and \( \|\tilde{e}\|_F = \|\varepsilon\| \), we have that

\[
\mathbb{P}_{\Pi \sim \mathcal{D}} \left( \|U_Z^T \Pi^T \Pi U_X - U_Z^T U_X\|_2 > 3\varepsilon \sqrt{pq} \right) < \delta,
\]

\[
\mathbb{P}_{\Pi \sim \mathcal{D}} \left( \|U_Z^T \Pi^T \Pi \tilde{e} - U_Z^T \tilde{e}\| > 3\varepsilon \sqrt{q\|\varepsilon\|} \right) < \delta,
\]

provided that \( m \geq 2/(\varepsilon^2 \delta) \). Replacing \( \varepsilon \) with \( \varepsilon/\sqrt{pq} \) yields that

\[
\mathbb{P}_{\Pi \sim \mathcal{D}} \left( \|U_Z^T \Pi^T \Pi U_X - U_Z^T U_X\|_2 > 3\varepsilon \right) < \delta,
\]

\[
\mathbb{P}_{\Pi \sim \mathcal{D}} \left( \|U_Z^T \Pi^T \Pi \tilde{e} - U_Z^T \tilde{e}\| > 3\varepsilon p^{-1/2} \|\varepsilon\| \right) < \delta,
\]

provided that \( m \geq 2pq/(\varepsilon^2 \delta) \). Thus, we have proved Lemma 9.

Proof of Theorem 2. In view of Lemma 9, this theorem follows directly from applying Theorem 1 to the case when \( \Pi \) is a countsketch.

A.3 Proof of Theorem 3

Proof of Theorem 3. It follows from \((\tilde{X}^T \tilde{Z})^{-1}(\tilde{X}^T \tilde{Z})^{-1}\tilde{X}^T \tilde{Z})(\tilde{X} \beta_0 + \tilde{e}) \]

\[
= \beta_0 + \left\{ (\tilde{X}^T \tilde{Z}/n)(\tilde{Z}^T \tilde{Z}/n)^{-1}(\tilde{Z}^T \tilde{X}/n) \right\}^{-1}(\tilde{X}^T \tilde{Z}/n)(\tilde{Z}^T \tilde{Z}/n)^{-1}(\tilde{Z}^T \tilde{e}/n).
\]
By (3),
\[ \hat{\beta} = \beta_0 + [(X^T Z)(Z^T Z)^{-1} Z^T X/n]^{-1} (X^T Z)(Z^T Z)^{-1} Z^T e/n. \]

Write
\[ \tilde{\beta} - \hat{\beta} = (\bar{\Xi}_n - \Xi)\Upsilon_n + \Xi(\bar{\Upsilon}_n - \Upsilon_n) + (\bar{\Xi}_n - \Xi)(\bar{\Upsilon}_n - \Upsilon_n), \]
where \( \bar{\Upsilon}_n := \bar{Z}^T \bar{e}/n, \Upsilon_n := Z^T e/n, \)
\[ \bar{\Xi}_n := \left\{ (\bar{X}^T \bar{Z}/n)(\bar{Z}^T \bar{Z}/n)^{-1}(\bar{Z}^T \bar{X}/n) \right\}^{-1} (\bar{X}^T \bar{Z}/n)(\bar{Z}^T \bar{Z}/n)^{-1}, \]
\[ \Xi := \left[ \mathbb{E}(X_i Z_i^T) \left[ \mathbb{E}(Z_i Z_i^T)^{-1} \mathbb{E}(Z_i X_i^T) \right] \right]^{-1} \mathbb{E}(X_i Z_i^T). \]

It will be shown below that \( \bar{\Xi}_n - \Xi = O_p(1) \) by Lemma 14 and the continuous mapping theorem, and \( \bar{\Upsilon}_n - \Upsilon_n = O_p(m^{-1/2}) \) by Lemma 11. By the central limit theorem, \( \Upsilon_n = O_p(n^{-1/2}) \). Hence,
\[ \tilde{\beta} - \hat{\beta} = \Xi(\bar{\Upsilon}_n - \Upsilon_n) + o_p \left( n^{-1/2} + m^{-1/2} \right). \]

Moreover, it will be shown in Lemma 11 that
\[ m^{1/2}(\bar{\Upsilon}_n - \Upsilon_n) \rightarrow_d N(0, \Psi). \]

Combining all the arguments above yields
\[ m^{1/2}(\tilde{\beta} - \hat{\beta}) \rightarrow_d N(0, \Xi \Psi \Xi^T), \]
which gives the conclusion of the theorem. \( \square \)

We use the central limit theorem for degenerate U-statistics of Hall (1984). For the sake of easy referencing, we reproduce it below.

**Lemma 10** (Theorem 1 of Hall (1984)). Assume that \( \{W_1, \ldots, W_n\} \) are independent and identically distributed random vectors. Define
\[ U_n := \sum_{1 \leq i < j \leq n} H_n(W_i, W_j). \]
Assume \( H_n \) is symmetric, \( \mathbb{E}[H_n(W_1, W_2)|W_1] = 0 \) almost surely and \( \mathbb{E}[H_n^2(W_1, W_2)] < \infty \) for
each $n$. Let $G_n(w_1, w_2) := \mathbb{E}[H_n(W_1, w_1)H_n(W_1, w_2)]$. If
\[
\frac{\mathbb{E}[G_n^2(W_1, W_2)] + n^{-1}\mathbb{E}[H_n(W_1, W_2)]}{\mathbb{E}[H_n^2(W_1, W_2)]^2} \to 0
\]
as $n \to \infty$, then
\[
V_n^{-1/2} U_n \to_d N(0, 1),
\]
where $V_n := \frac{1}{2}\mathbb{E}[H_n^2(W_1, W_2)]$.

**Lemma 11.** Let Assumptions in Theorem 3 hold. Then, as $n \to \infty$,
\[
m^{1/2} \left( \frac{\bar{Z}^T \tilde{e}}{n} - \frac{Z^T e}{n} \right) \to_d N \left( 0, \mathbb{E}(\epsilon_i^2)\mathbb{E}(Z_i Z_i^T) \right).
\]

**Proof.** In view of the Cramer-Wold device, it suffices to show that for any nonzero constant vector $c \in \mathbb{R}^q$,
\[
m_n^{1/2} \left[ c^T \mathbb{E}(\epsilon_i^2)\mathbb{E}(Z_i Z_i^T)c \right]^{-1/2} c^T (\tilde{\Upsilon}_n - \Upsilon_n) \to_d N(0, 1).
\]
Let $\tilde{Z}_i(c) := \sum_u c_u Z_{iu}$, which is a weighted sum of elements of the $i$-th row of $Z$. Note that
\[
c^T Z^T \Pi^T \Pi e = \sum_i \sum_j \sum_k \tilde{Z}_j(c) \delta_{kj} \sigma_{kj} \delta_{ki} \sigma_{ki} e_i
\]
\[
= \sum_i \tilde{Z}_i(c) e_i + \sum_j \sum_n \sum_m \tilde{Z}_j(c) \delta_{kj} \sigma_{kj} \delta_{ki} \sigma_{ki} e_i
\]
using the facts that $\delta_{ki}^2 = \delta_{ki}$, $\sigma_{ki}^2 = 1$ and $\sum_{k=1}^m \delta_{ki} = 1$. Thus,
\[
c^T (Z^T \Pi^T \Pi e - Z^T e) = \sum_{i,j} \sum_{k=1}^m \tilde{Z}_j(c) \delta_{kj} \sigma_{kj} \delta_{ki} \sigma_{ki} e_i. \tag{19}
\]
Let $W_i = (Y_i, X_i^T, Z_i^T, \Pi_{i1}, \ldots, \Pi_{im})^T \in \mathbb{R}^{1+p+q+m}$, where $\Pi_{ki}$ is the $(k, i)$ element of $\Pi$. Since columns of $\Pi$ are i.i.d., we have that $\{W_i : i = 1, \ldots, n\}$ are i.i.d.

As a nonrandom index, let $w = (y, x^T, z^T, \pi_1, \ldots, \pi_m)^T$, where the lower case letters represent non-nonrandom elements. For each nonzero $c \in \mathbb{R}^q$, define $\tilde{H}_c(w_1, w_2) := \sum_{k=1}^m \tilde{z}_1(c) \pi_{k1} \pi_{k2} e_2$.
and \( H_c(w_1, w_2) := \tilde{H}_c(w_1, w_2) + \tilde{H}_c(w_2, w_1) \). Then,

\[
c^T(Z^T\Pi^T\Pi e - Z^T e) = \sum_{1 \leq i < j \leq n} H_c(W_i, W_j).
\]

Note that \( H_c(w_1, w_2) = H_c(w_2, w_1) \) and \( \mathbb{E}(H_c(W_1, W_2)|W_1) = \mathbb{E}(H_c(W_1, W_2)|W_2) = 0 \). Thus, \( c^T(Z^T\Pi^T\Pi e - Z^T e) \) is a degenerate \( U \)-statistic.

**Lemma 12.** Let Assumptions in Theorem 3 hold. Let \( G_c(w_1, w_2) := \mathbb{E}[H_c(W_i, w_1)H_c(W_i, w_2)] \). Then,

\[
\mathbb{E}[G_c^2(W_i, W_j)] = m^{-3}\{\mathbb{E}[\tilde{Z}_i(c)^2]\}^2\{\mathbb{E}(e_i^2)\}^2 = O(m^{-3}).
\]

**Proof.** First, write

\[
G_c(w_1, w_2) = \mathbb{E}\left\{ \left[ \tilde{H}_c(W_i, w_1) + \tilde{H}_c(w_1, W_i) \right] \left[ \tilde{H}_c(W_i, w_2) + \tilde{H}_c(w_2, W_i) \right] \right\}.
\]

Because \( \Pi_{ki,\ell \ell_i} = 0 \) whenever \( k \neq \ell \) for each \( i \), we have that

\[
\tilde{H}_c(W_i, w_1)\tilde{H}_c(W_i, w_2) = \sum_{k=1}^{m} Z_i(c)^2 e_1 e_2 \Pi_{ki,\pi k_1 \pi k_2},
\]

\[
\tilde{H}_c(w_1, W_i)\tilde{H}_c(w_2, W_i) = \sum_{k=1}^{m} \tilde{z}_1(c)\tilde{z}_2(c) e_1^2 \Pi_{ki,\pi k_1 \pi k_2},
\]

\[
\tilde{H}_c(W_i, w_1)\tilde{H}_c(w_2, W_i) = \sum_{k=1}^{m} \tilde{Z}_i(c)\tilde{Z}_2(c) e_1 e_2 \Pi_{ki,\pi k_1 \pi k_2}.
\]

Then, by simple algebra,

\[
G_c(W_i, W_j) = m^{-1} \sum_{k=1}^{m} \left\{ \mathbb{E}[\tilde{Z}_i(c)^2] e_i e_j + \tilde{Z}_i(c)\tilde{Z}_j(c)\mathbb{E}(e_i^2) \right\} \Pi_{ki,\pi kj}.
\]

Again, using the fact that \( \Pi_{ki,\ell \ell_i} = 0 \) whenever \( k \neq \ell \) for each \( i \), write

\[
G_c^2(W_i, W_j) = m^{-2} \sum_{k=1}^{m} \left\{ \mathbb{E}[Z_i(c)^2] e_i e_j + Z_i(c)Z_j(c)\mathbb{E}(e_i^2) \right\}^2 \Pi_{ki,\pi kj}^2,
\]

which immediately implies the desired result. \( \square \)
Lemma 13. Let Assumptions in Theorem \[ \Box \] hold. Then,

\[ \mathbb{E}[H^2_c(W_i, W_j)] = 2m^{-1}\mathbb{E}[Z_i(c)^2]\mathbb{E}(e_i^2) = O(m^{-1}). \]

Furthermore,

\[ \mathbb{E}[H^4_c(W_i, W_j)] = 2m^{-1}\mathbb{E}[Z_i(c)^4]\mathbb{E}[e_i^4] + 3m^{-1}\{\mathbb{E}[Z_i(c)^2e_i^2]\}^2 = O(m^{-1}). \]

**Proof.** For \( i \neq j \), write

\[ \mathbb{E}\{[H_c(W_i, W_j)]^2\} = \mathbb{E}\left\{ \left[ \tilde{H}_c(W_i, W_j) + \tilde{H}_c(W_j, W_i) \right] \left[ \tilde{H}_c(W_i, W_j) + \tilde{H}_c(W_j, W_i) \right] \right\}. \]

As in the previous lemma, using the fact that \( \Pi_{ki}\Pi_{\ell i} = 0 \) whenever \( k \neq \ell \) for each \( i \),

\[ T_{ijc1} := \tilde{H}_c(W_i, W_j)\tilde{H}_c(W_i, W_j) = \sum_{k=1}^{m} \tilde{Z}_i(c)^2\epsilon_j^2\Pi_{ki}\Pi_{kj}, \]

\[ T_{ijc2} := \tilde{H}_c(W_j, W_i)\tilde{H}_c(W_j, W_i) = \sum_{k=1}^{m} \tilde{Z}_j(c)^2\epsilon_i^2\Pi_{ki}\Pi_{kj}, \]

\[ T_{ijc3} := \tilde{H}_c(W_i, W_j)\tilde{H}_c(W_j, W_i) = \sum_{k=1}^{m} \tilde{Z}_i(c)\tilde{Z}_j(c)\epsilon_i\epsilon_j\Pi_{ki}\Pi_{kj}. \]

Thus,

\[ \{H_c(W_i, W_j)^2\} = T_{ijc1} + T_{ijc2} + 2T_{ijc3}. \]

Then, the first desired result follows simply by taking expectations on both sides of the equation above.

For the second desired result, write

\[ \{H_c(W_i, W_j)^4\} \]

\[ = (T_{ijc1} + T_{ijc2} + 2T_{ijc3})(T_{ijc1} + T_{ijc2} + 2T_{ijc3}) \]

\[ = \{\tilde{Z}_i(c)^4\epsilon_j^4 + \tilde{Z}_j(c)^4\epsilon_i^4 + 3\tilde{Z}_i(c)^2\tilde{Z}_j(c)^2\epsilon_i^2\epsilon_j^2 + \tilde{Z}_i(c)\tilde{Z}_j(c)\epsilon_i\epsilon_j^3 + \tilde{Z}_i(c)\tilde{Z}_j(c)^3\epsilon_i^3\epsilon_j \} \sum_{k=1}^{m} \Pi_{ki}\Pi_{kj}, \]

where the last equality again follows from the fact that \( \Pi_{ki}\Pi_{\ell i} = 0 \) whenever \( k \neq \ell \) for each \( i \) and that \( \Pi_{ki}^{\dagger} = \Pi_{ki}^2 = \delta_{ki} \). Then, the desired result follows by taking the expectations and use the assumption that \( \mathbb{E}(\tilde{Z}_i(c)^4) < \infty \) and \( \mathbb{E}(e_i^4) < \infty \). \( \square \)
We now return to the proof of Lemma 11. By Lemmas 12 and 13, we have that
\[
\frac{\mathbb{E}[G_c^2(W_1, W_2)] + n^{-1}\mathbb{E}[H_c^4(W_1, W_2)]}{\{\mathbb{E}[H_c^2(W_1, W_2)]\}^2} = O(m^{-1} + n^{-1}m) = o(1).
\]
Then, the conclusion of Lemma 11 follows directly by applying Lemma 10 along with Lemma 13.

**Lemma 14.** Let Assumptions in Theorem 3 hold. Then, as \(n \to \infty\),
\[
\frac{\tilde{Z}^T \tilde{Z}}{n} \to_p \mathbb{E}(Z_i Z_i^T) \quad \text{and} \quad \frac{\tilde{X}^T \tilde{Z}}{n} \to_p \mathbb{E}(X_i Z_i^T).
\]

**Proof.** Write
\[
\frac{\tilde{Z}^T \tilde{Z}}{n} - \mathbb{E}(Z_i Z_i^T) = \left\{ \frac{\tilde{Z}^T \tilde{Z}}{n} - Z^T Z/n \right\} + \left\{ Z^T Z/n - \mathbb{E}(Z_i Z_i^T) \right\}. \tag{20}
\]
Since the second term on the right-hand side of (20) converges in probability to zero by the law of large numbers, it suffices to show that the first term is \(o_p(1)\).

Let \(A_{ij}\) or \([A]_{ij}\) denote the \((i, j)\) element of a matrix \(A\). For any \(u, v \in [q]\), write
\[
[\tilde{Z}^T \tilde{Z} - Z^T Z]_{uv} = [Z^T \Pi^T \Pi Z - Z^T Z]_{uv} = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{m} Z_{ju} \Pi_{kj} \Pi_{ki} Z_{iv}.
\]
Note that the form of the equation above is similar to that of (19) in the proof of Lemma 11. Repeating the same arguments before, we have that as \(n \to \infty\),
\[
m^{1/2} \left[ n^{-1} \tilde{Z}^T \tilde{Z} - n^{-1} Z^T Z \right]_{uv} \to_d N \left[ 0, \mathbb{E}(Z_{iu}^2) \mathbb{E}(Z_{iv}^2) \right],
\]
which implies that
\[
\frac{\tilde{Z}^T \tilde{Z}}{n} - Z^T Z/n = o_p(1).
\]
Therefore, we have proved the first conclusion. Analogously, the second result can be proved and so details are omitted. \(\square\)
A.4 Proof of Corollary 1

Proof of Corollary 1. The proof is omitted since it can be proved using arguments similar to those used in the proof of Theorem 3.

A.5 Proof of Corollary 2

Proof of Corollary 2. As stated in the main text, we have that

\[ m^{1/2}(\tilde{\beta} - \hat{\beta}) = m^{1/2}(\tilde{\beta} - \beta_0) + o_p(1), \]

because \( m/n \to 0 \) and \( n^{1/2}(\hat{\beta} - \beta_0) = O_p(1) \). Thus, it suffices to show that \( \tilde{V} \to_p V_0 \).

Note that

\[ \tilde{e}^T \tilde{e} = e^T e - e^T U_X U_X^T e, \]
\[ \tilde{e}^T \hat{e} = e^T \Pi^T \Pi e - e^T \Pi^T \Pi U_X (U_X^T \Pi^T \Pi U_X)^{-1} U_X^T \Pi^T \Pi e. \]

Write

\[ |\tilde{e}^T \tilde{e} - \hat{e}^T \hat{e}| \leq |e^T \Pi^T \Pi e - e^T e| + \left| e^T \Pi^T \Pi U_X (U_X^T \Pi^T \Pi U_X)^{-1} U_X^T \Pi^T \Pi e - e^T U_X U_X^T e \right|. \]

As in the proofs of Lemmas 11 and 14 we can show that

\[ (e^T \Pi^T \Pi e - e^T e)/n = o_p(1), \]
\[ (U_X^T \Pi^T \Pi e - U_X^T e)/n = o_p(1), \]
\[ (U_X^T \Pi^T \Pi U_X - U_X^T U_X)/n = o_p(1), \]

which implies that

\[ \tilde{e}^T \tilde{e}/n = \tilde{e}^T \hat{e}/n + o_p(1) = E(e_i^2) + o_p(1), \]

where the last equality follows from the law of large numbers. We have already shown in the proof of Theorem 3 that

\[ \left\{ (\tilde{X}^T \tilde{Z}/n)(\tilde{Z}^T \tilde{Z}/n)^{-1}(\tilde{Z}^T \tilde{X}/n) \right\}^{-1} \to_p \left[ E(X_i Z_i^T) \left[ E(Z_i Z_i^T) \right]^{-1} E(Z_i X_i^T) \right]^{-1}. \]

Therefore, we have verified that \( \tilde{V} \to_p V_0 \).
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