Flat topology on prime, maximal and minimal prime spectra of quantales

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Abstract

Several topologies can be defined on the prime, the maximal and the minimal prime spectra of a commutative ring; among them, we mention the Zariski topology, the patch topology and the flat topology. By using these topologies, Tarizadeh and Aghajani obtained recently new characterizations of various classes of rings: Gelfand rings, clean rings, absolutely flat rings, mp-rings, etc. The aim of this paper is to generalize some of their results to quantales, structures that constitute a good abstractization for lattices of ideals, filters and congruences. We shall study the flat and the patch topologies on the prime, the maximal and the minimal prime spectra of a coherent quantale. By using these two topologies one obtains new characterization theorems for hyperarchimedean quantales, normal quantales, B-normal quantales, mp-quantales and PF-quantales. The general results can be applied to several concrete algebras: commutative rings, bounded distributive lattices, MV-algebras, BL-algebras, residuated lattices, commutative unital l-groups, etc.

Keywords: flat topology, coherent quantale, reticulation, hyperarchimedean quantales, normal and B-normal quantales, mp-quantales.

1 Introduction

The flat topology on the prime spectrum of a ring was introduced by Hochster in [22] under the name of inverse topology. It was rediscovered by Doobs et al. in [14], where the terminology of "flat topology" appeared. The flat topology is strongly related to other two topologies defined on the prime spectrum of a commutative ring: spectral topology and patch topology [13, 25, 43]. These three topologies have a deep impact on some important themes in ring theory (see [22, 13, 11]). Recently, Tarizadeh proposed in [43] purely algebraic definitions for flat and patch topologies on the prime spectrum $Spec(R)$ of a commutative ring $R$. For examples, the closed sets in the flat topology on $Spec(R)$ are defined
as the images $\text{Im}(f^*)$, where $f^* : \text{Spec}(A) \to \text{Spec}(R)$ is the map induced by a flat morphism $f : R \to A$. Further, the flat and the patch topologies were used to obtain new properties and new characterizations of Gelfand rings and clean rings, as well as of new classes of rings: mp-rings and purified rings (see [1], [14], [15]). A natural problem is to use the flat and the patch topology for obtaining new results on the spectra of other types of algebras. On the other hand, the quantales constitute a good abstraction of the lattices of ideals, filters and congruences in various algebraic structures. A rich literature was dedicated to the spectra of quantales (see [15], [17], [35], [36], [38]). Besides the quantales, other types of multiplicative lattices were proposed to abstractize the lattices of ideals, filters and congruences [19], [30], [40], [41]. In this paper we shall study the flat topology on prime, maximal and minimal prime spectra of a coherent quantale in connection to important classes of quantales: hyperarchimedean, normal, B-normal, mp - quantales, etc. We shall obtain new characterizations of these types of quantales in terms of some algebraic and topological properties of spectra. Our results extend some theorems proven in [1], [14], [15], [3], [4], etc. for the spectra of commutative rings. The proofs of some results will use the reticularation of a quantale, a construction that assigns to each coherent quantale $A$ a unique bounded distributive lattice $L(A)$, whose prime spectrum $\text{Spec}_{Id}(L(A))$ is homeomorphic to the prime spectrum of $A$.

Now we shall describe the content of this paper. In Section 2 we present some notions and basic results on the prime and the maximal spectra of a quantale, their spectral topologies and the radical elements (cf. [38], [30], [15], [36]).

Section 3 contains the axiomatic definition of reticularation of a coherent quantale $A$ and the connection between the $m$-prime elements of $A$ and the prime ideals of the reticularation $L(A)$. The homeomorphism between the prime spectrum $\text{Spec}(A)$ of $A$ and the space $\text{Spec}_{Id}(L(A))$ of the prime ideals in $L(A)$ (with the Stone topology) induces on $\text{Spec}(A)$ a spectral topology (this spectral space is denoted by $\text{Spec}_{Z}(A)$, because it generalizes the Zariski topology).

Section 4 deals with the Boolean center $B(A)$ of a quantale $A$. We present a short proof that $B(A)$ is isomorphic to the Boolean algebra $B(L(A))$ of complemented elements in $L(A)$. Section 5 concerns the patch and the flat topology on $\text{Spec}(A)$, associated with the spectral space $\text{Spec}_{Z}(A)$; these two topological spaces will be denoted by $\text{Spec}_{P}(A)$, resp. $\text{Spec}_{F}(A)$. We also introduce the Pierce spectrum $Sp(A)$ of the quantale $A$. In Section 6 we continue the study of the hyperarchimedean quantales, initiated in [9]. The main result of the section presents new characterizations of these objects in terms of the flat and the patch topologies.

Section 7 contains results about the flat topology on the maximal spectrum $\text{Max}(A)$ of the coherent quantale $A$. The new topological space $\text{Max}_{F}(A)$ is Hausdorff and zero - dimensional. We use the flat topology on $\text{Max}(A)$ in order to obtain new properties that characterize the normal and the B-normal quantales [9], [20], [35], [11]. The normal quantales constitute an abstractization of the lattices of ideals in Gelfand rings [25], [27], [32] and in normal lattices [11], [37], [19], while the B-normal quantales generalize the lattices of ideals in clean rings [34], [23] and in B-normal lattices [10], [8]. Thus our theorems on normal and
B-normal quantales can be applied to various types of structures: Gelfand rings and normal lattices, normal and $B$-normal lattices, normal and $B$-normal lattices, Gelfand residuated lattices and residuated lattices with Boolean lifting property [18], clean unital $l$-groups [21], etc.

In Section 8 we study two topologies on the set $\text{Min}(A)$ of the minimal $m$-prime elements of a coherent quantale $A$. Thus we obtain two topological spaces $\text{Min}_Z(A)$ and $\text{Min}_F(A)$: the first one is a subspace of $\text{Spec}_Z(A)$ and the second one is a subspace of $\text{Spec}_F(A)$. By using the reticulation we prove that $\text{Min}_Z(A)$ is a zero-dimensional Hausdorff space and $\text{Min}_F(A)$ is a compact $T_1$ space. The main results of this section focus on the $mp$-quantales, a notion that generalizes the $mc$-rings of [1]. We present several conditions that characterize $mp$-quantales. For example, we prove that a coherent quantale $A$ is an $mc$-quantale iff the reticulation $L(A)$ is a conormal lattice [40] iff $\text{Spec}_F(A)$ is a normal space. We introduce the $PF$-quantales as a generalization of the $PF$-rings [4] and we prove that a coherent quantale $A$ is a $PF$-quantale if and only if it is a semiprime $mp$-quantale. Further we obtain a characterization theorem for $PF$-quantales.

2 Preliminaries

Let $(A, \lor, \land, \cdot, 0, 1)$ be a quantale and $K(A)$ the set of its compact elements. $A$ is said to be integral if $(A, \cdot, 1)$ is a monoid and commutative, if the multiplication $\cdot$ is commutative. A frame is a quantale in which the multiplication coincides with the meet [25]. The quantale $A$ is algebraic if any $a \in A$ has the form $a = \bigvee X$ for some subset $X$ of $K(A)$. An algebraic quantale $A$ is coherent if $1 \in K(A)$ and $K(A)$ is closed under the multiplication. Throughout this paper, the quantales are assumed to be integral and commutative. Often we shall write $ab$ instead of $a \cdot b$. We fix a quantale $A$.

Lemma 2.1 [7] For all elements $a, b, c$ of the quantale $A$ the following hold:

1. If $a \lor b = 1$ then $a \cdot b = a \land b$;
2. If $a \lor b = 1$ then $a^n \lor b^n = 1$ for all integer numbers $n \geq 1$;
3. If $a \lor b = a \lor c = 1$ then $a \lor (b \cdot c) = a \lor (b \land c) = 1$;
4. If $a \lor b = 1$ and $a \leq c$ then $a \lor (b \cdot c) = c$.

One can define on the quantale $A$ a residuation operation $a \rightarrow b = \bigvee \{x | ax \leq b\}$ and a negation operation $a^\perp = a \rightarrow 0 = \bigvee \{x | ax = 0\}$. Thus $(A, \lor, \land, \cdot, \rightarrow, 0, 1)$ is a residuation lattice [16], [26]. In this paper we shall use without mention the basic arithmetical properties of a residuated lattice.

An element $p < 1$ of $A$ is $m$-prime if for all $a, b \in A$, $ab \leq p$ implies $a \leq b$ or $b \leq p$. If $A$ is an algebraic quantale, then $p < 1$ is $m$-prime if and only if for all $c, d \in K(A)$, $cd \leq p$ implies $c \leq p$ or $d \leq p$. Let us introduce the following notations: $\text{Spec}(A)$ is the set of $m$-prime elements and $\text{Max}(A)$ is the
set of maximal elements of $A$. If $1 \in K(A)$ then for any $a < 1$ there exists $m \in Mar(A)$ such that $a \leq m$. The same hypothesis $1 \in K(A)$ implies that $Max(A) \subseteq Spec(A)$.

Let $R$ be a (unital) commutative ring and $L$ a bounded distributive lattice. Let us denote by $Id(R)$ the quantale of ideals in $R$ and by $Id(L)$ the frame of ideals in $L$. Thus the set $Spec(R)$ of prime ideals in $R$ is the prime spectrum of the quantale $Id(R)$ and the set of prime ideals in $L$ is the prime spectrum of the frame $Id(L)$.

The radical $\rho(a) = \rho_A(a)$ of an element $a \in A$ is defined by $\rho_A(a) = \bigwedge\{p \in Spec(A)|a \leq p\}$; if $a = \rho(a)$ then $a$ is a radical element. We shall denote by $R(A)$ the set of radical elements of $A$. The quantale is *semiprime* if $\rho(0) = 0$.

**Lemma 2.2** [38] For all elements $a, b \in A$ the following hold:

1. $a \leq \rho(a)$;
2. $\rho(a \wedge b) = \rho(ab) = \rho(a) \wedge \rho(b)$;
3. $\rho(a) = 1$ iff $a = 1$;
4. $\rho(a \vee b) = \rho(a) \vee \rho(b)$;
5. $\rho(\rho(a)) = \rho(a)$;
6. $\rho(a) \vee \rho(b) = 1$ iff $a \vee b = 1$;
7. $\rho(a^n) = \rho(a)$, for all integer $n \geq 1$.

For an arbitrary family $(a_i)_{i \in I} \subseteq A$, the following equality holds: $\rho(\bigvee_{i \in I} a_i) = \rho(\bigvee_{i \in I} \rho(a_i))$. If $(a_i)_{i \in I} \subseteq R(A)$ then we denote $\bigvee_{i \in I} a_i = \rho(\bigvee_{i \in I} a_i)$. Thus it easy to prove that $(R(A), \bigvee, \wedge, \rho, 1)$ is a frame [38].

**Lemma 2.3** [9] If $1 \in K(A)$ then $Spec(A) = Spec(R(A))$ and $Max(A) = Max(R(A))$.

**Lemma 2.4** [30] Let $A$ be a coherent quantale and $a \in A$. Then

1. $\rho(a) = \bigvee\{c \in K(A)|c^k \leq a \text{ for some integer } k \geq 1\}$;
2. For any $c \in K(A), c \leq \rho(a)$ iff $c^k \leq a$ for some $k \geq 1$.

**Lemma 2.5** [9] If $A$ is a coherent quantale then $K(R(A)) = \rho(K(A))$ and $R(A)$ is a coherent frame.

For any element $a$ of a coherent quantale $A$ let us consider the interval $[a]_A = \{x \in A|a \leq x\}$ and for all $x, y \in [a]_A$ denote $x \cdot_a y = xy \vee a$. Thus $[a]_A$ is closed under the multiplication $\cdot_a$ and $([a]_A, \vee, \wedge, \cdot_a, 0, 1)$ is a coherent quantale.
Lemma 2.6 [9] The quantale \(((\rho(a))_A, \vee, \wedge, \cdot, 0, 1)\) is semiprime and \(\text{Spec}(A) = \text{Spec}((\rho(a))_A), \text{Max}(A) = \text{Max}((\rho(a))_A)\).

Let \(A, B\) be two quantales. A function \(f : A \to B\) is a morphism of quantales if it preserves the arbitrary joins and the multiplication; \(f\) is an integral morphism if \(f(1) = 1\).

Lemma 2.7 [9] Let \(A\) be a coherent quantale and \(a \in A\).

1. The function \(u^A_a : A \to [a]_A\), defined by \(u^A_a(x) = x \lor a\), for all \(x \in A\), is an integral quantale morphism;

2. If \(c \in K(A)\) then \(u^A_a(c) \in K([a])\).

Let \(A\) be a quantale such that \(1 \in K(A)\). For any \(a \in A\), denote \(D(a) = \{p \in \text{Spec}(A) | a \nleq p\}\) and \(V(a) = \{p \in \text{Spec}(A) | a \leq p\}\). Then \(\text{Spec}(A)\) is endowed with a topology whose closed sets are \((V(a))_{a \in A}\). If the quantale \(A\) is algebraic then the family \((D(c))_{c \in K(A)}\) is a basis of open sets for this topology. The topology introduced here generalizes the Zariski topology (defined on the prime spectrum \(\text{Spec}(R)\) of a commutative ring \(R\) [2]) and the Stone topology (defined on the prime spectrum \(\text{Spec}_{ld}(L)\) of a bounded distributive lattice \(L\) [5]).

Thus we denote by \(\text{Spec}_{Z}(A)\) the prime spectrum \(\text{Spec}(A)\) endowed with the above defined topology: \(\text{Max}_{Z}(A)\) will denote the maximal spectrum \(\text{Max}(A)\) considered as a subspace of \(\text{Spec}_{Z}(A)\).

Let \(L\) be a bounded distributive lattice. For any \(x \in L\), denote \(D_{ld}(x) = \{P \in \text{Spec}_{ld}(L) | x \notin P\}\) and \(V_{ld}(x) = \{P \in \text{Spec}_{ld,Z}(L) | x \in P\}\). The family \((D_{ld}(x))_{x \in L}\) is a basis of open sets for the Stone topology on \(\text{Spec}_{ld}(L)\); this topological space will be denoted by \(\text{Spec}_{ld,Z}(L)\). Let \(\text{Max}_{ld}(L)\) be the set of maximal ideals of \(L\). Thus \(\text{Max}_{ld}(L) \subseteq \text{Spec}_{ld}(L)\) and \(\text{Max}_{ld}(L)\) becomes a subspace of \(\text{Spec}_{ld}(L)\), denoted \(\text{Max}_{ld,Z}(L)\).

3 Reticulation of a coherent quantale

In this section we shall recall from [9,17] the axiomatic definition of the reticulation of the coherent quantale and some of its basic properties. Let \(A\) be a coherent quantale and \(K(A)\) the set of its compact elements.

Definition 3.1 [9] A reticulation of the quantale \(A\) is a bounded distributive lattice \(L\) together a surjective function \(\lambda : K(A) \to L\) such that for all \(a, b \in K(A)\) the following properties hold

1. \(\lambda(a \lor b) \leq \lambda(a) \lor \lambda(b)\);
2. \(\lambda(ab) = \lambda(a) \land \lambda(b)\);
3. \(\lambda(a) \leq \lambda(b)\) iff \(a^n \leq b\), for some integer \(n \geq 1\).
In [9,17] there were proven the existence and the unicity of the reticulation for each coherent quantale $A$; this unique reticulation will be denoted by $(L(A), \lambda_A : K(A) \to L(A))$ or shortly $L(A)$. The reticulation $L(R)$ of a commutative ring $R$ there was introduced by many authors, but the main references on this topic remain [40], [25]. We remark that $L(R)$ is isomorphic to the reticulation $L(Id(R))$ of the quantale $Id(R)$.

**Lemma 3.2** [9] For all elements $a, b \in K(A)$ the following properties hold:

1. $a \leq b$ implies $\lambda_A(a) \leq \lambda_A(b)$;
2. $\lambda_A(a \lor b) = \lambda_A(a) \lor \lambda_A(b)$;
3. $\lambda_A(a) = 1$ iff $a = 1$;
4. $\lambda_A(0) = 0$;
5. $\lambda_A(a) = 0$ iff $a^n = 0$, for some integer $n \geq 1$;
6. $\lambda_A(a^n) = \lambda_A(a)$, for all integer $n \geq 1$;
7. $\rho(a) = \rho(b)$ iff $\lambda_A(a) = \lambda_A(b)$;
8. $\lambda_A(a) = 0$ iff $a \leq \rho(0)$;
9. If $A$ is semiprime then $\lambda_A(a) = 0$ implies $a = 0$.

For any $a \in A$ and $I \in Id(L(A))$ let us denote $a^* = \{\lambda_A(c) | c \in K(A), c \leq a\}$ and $I_* = \forall\{c \in K(A)|\lambda_A(c) \in I\}$.

**Lemma 3.3** [9] The following assertions hold

1. If $a \in A$ then $a^*$ is an ideal of $L(A)$ and $a \leq (a^*)_*$;
2. If $I \in Id(L(A))$ then $(I_*)^* = I$;
3. If $p \in Spec(A)$ then $(p^*)_* = p$ and $p^* \in Spec(Id(L(A)))$;
4. If $P \in Spec(Id((L(A)))$ then $P_* \in Spec(A)$;
5. If $p \in K(A)$ then $c^* = (\lambda_A(c))$.

**Lemma 3.4** [9] If $a \in A$ and $I \in Id(L(A))$ then $\rho(a) = (a^*)_*$, $a^* = (\rho(a))^*$ and $\rho(I_*) = I_*$.

**Lemma 3.5** If $c \in K(A)$ and $I \in Id(L(A))$ then $c \leq I_*$ iff $\lambda_A(c) \in I$.

**Proof.** If $c \leq \forall\{d \in K(A)|\lambda_A(d) \in I\}$ then there exists $d \in K(A)$ such that $\lambda_A(c) \in I$ and $c \leq d$. Thus $\lambda_A(c) \leq \lambda_A(d)$, so $\lambda_A(c) \in I$. The converse implication is obvious.
Lemma 3.6 Assume that $c \in K(A)$ and $p \in \text{Spec}(A)$. Then $c \leq p$ iff $\lambda_A(c) \in p^*$.  

According to Lemma 3.3, one can consider the following order-preserving functions: $u : \text{Spec}(A) \to \text{Spec}_{Id}(L(A))$ and $v : \text{Spec}_{Id}(L(A)) \to \text{Spec}(A)$, defined by $u(p) = p^*$ and $v(P) = P^*$, for all $p \in \text{Spec}(A)$ and $P \in \text{Spec}_{Id}(L(A))$. Sometimes the previous functions $u$ and $v$ will be denoted by $u_A$ and $v_A$.

Lemma 3.7 [9] The functions $u$ and $v$ are homeomorphisms, inverse to one another.

Corollary 3.8 $\text{Max}_{Z}(A)$ and $\text{Max}_{Id,Z}(L(A))$ are homeomorphic.

Proposition 3.9 [9] The functions $\Phi : R(A) \to \text{Id}(L(A))$ and $\Psi : \text{Id}(L(A)) \to R(A)$ defined by $\Phi(a) = a^*$ and $\Psi(I) = I^*$, for all $a \in R(A)$ and $I \in \text{Id}(L(A))$, are frame isomorphisms, inverse to one another.

Corollary 3.10 If $I, J$ are ideals of $L(A)$ then $(I \lor J)_* = \rho(I_* \lor J_*)$.

Proof. The equality $(I \lor J)_* = \rho(I_* \lor J_*)$ follows from the fact that the frame isomorphism $\Psi$ preserves the finite joins.

4 Boolean center of a quantale and the reticulation

The Boolean center of a quantale $A$ is the Boolean algebra $B(A)$ of complemented elements of $A$ (cf. [7],[24]). In this section we shall study some basic properties of the Boolean center of a coherent quantale versus the reticulation.

Lemma 4.1 [7],[27] Let $A$ be a quantale and $a, b \in A$, $e \in B(A)$. Then the following properties hold:

1. $a \in B(A)$ iff $a \lor a^\perp = 1$;
2. $a \land b = ae$;
3. $e \to a = e^\perp \lor a$;
4. If $a \lor b = 1$ and $ab = 0$, then $a, b \in B(A)$;
5. $(a \land b) \lor e = (a \lor e) \land (b \land e)$;
6. For any integer $n \geq 1$, $a \lor b = 1$ and $a^n b^n = 0$ implies $a^n, b^n \in B(A)$.

Proof. The properties (1)-(5) are taken from [7],[24] and (6) follows by (4) and Lemma 2.1(ii).
Lemma 4.2 \cite{7} If $1 \in K(A)$ then $B(A) \subseteq K(A)$.

For a bounded distributive lattice $L$ we shall denote by $B(L)$ the Boolean algebra of the complemented elements of $L$. It is well-known that $B(L)$ is isomorphic to the Boolean center $B(Id(L))$ of the frame $Id(L)$ (see \cite{7, 25, 8}).

Let us fix a coherent quantale $A$.

Lemma 4.3 Assume $c \in K(A)$. Then $\lambda_A(c) \in B(L(A))$ if and only if $c^n \in B(A)$, for some integer $n \geq 1$.

Proof. Assume $\lambda_A(c) \in B(L(A))$, hence $\lambda_A(c) \vee \lambda_A(d) = 1$ and $\lambda_A(c) \wedge \lambda_A(d) = 0$, for some $d \in K(A)$. Then $\lambda_A(c \vee d) = 1$ and $\lambda_A(cd) = 0$, hence, by Lemma 3.2,(2) and (5), it follows that $c \vee d = 1$ and $c^n d^n = 0$, for some integer $n \geq 1$. Therefore by Lemma 4.1,(6) one gets $c^n, d^n \in B(A)$. Conversely, if $c^n \in B(A)$ then $\lambda_A(c) = \lambda_A(c^n)$ is an element of $B(L(A))$.

Corollary 4.4 \cite{7} The function $\lambda_A|_{B(A)} : B(A) \rightarrow B(L(A))$ is a Boolean isomorphism.

Proof. It is easy to see that the function $\lambda_A|_{B(A)} : B(A) \rightarrow B(L(A))$ is an injective Boolean morphism. The surjectivity follows by using Lemma 4.3. \qed

If $L$ is bounded distributive lattice and $I \in Id(L)$ then the annihilator of $I$ is the ideal $Ann(I) = \{x \in L \mid x \wedge y = 0$, for all $y \in L\}$.

The next two propositions concern the behaviour of reticulation w.r.t. the annihilators.

Proposition 4.5 If $a$ is an element of a coherent quantale then $Ann(a^*) = (a \rightarrow \rho(0))^*$; if $A$ is semiprime then $Ann(a^*) = (a^*)^*$.

Proof. Assume $x \in Ann(a^*)$, so $x = \lambda_A(c)$ for some $c \in K(A)$ with the property that for all $d \in K(A)$, $d \leq a$ implies $\lambda_A(cd) = \lambda_A(c) \wedge \lambda_A(d) = 0$. By Lemma 3.2(8) one gets $cd \leq \rho(0)$, so $c \leq d \rightarrow \rho(0)$. Thus the following hold: $c \leq \bigwedge\{d \rightarrow \rho(0) \mid d \in K(A), d \leq a\} = (\bigvee\{d \in K(A) \mid d \leq a\}) \rightarrow \rho(0) = a \rightarrow \rho(0)$, hence $x = \lambda_A(c) \in (a \rightarrow \rho(0))^*$. We conclude that $Ann(a^*) \subseteq (a \rightarrow \rho(0))^*$.

In order to prove that $(a \rightarrow \rho(0))^* \subseteq Ann(a^*)$ assume that $x \in (a \rightarrow \rho(0))^*$, so $x = \lambda_A(c)$ for some $c \in K(A)$ such that $c \leq a \rightarrow \rho(0)$. For all $d \in K(A)$ with $d \leq a$ we have $c \leq a \rightarrow \rho(0) \leq d \rightarrow \rho(0)$. By Lemma 3.2,(8) one gets $\lambda_A(c) \wedge \lambda(d) = \lambda_A(cd) = 0$, so $x = \lambda_A(c) \in Ann(a^*)$.

\qed

Proposition 4.6 Assume that $A$ is a coherent quantale. If $I$ is an ideal of $L(A)$ then $(Ann(I))_* = I_* \rightarrow \rho(0)$; if $A$ is semiprime then $(Ann(I))_* = (I_*)^\perp$.

Proof. In order to verify that $I_* \rightarrow \rho(0) \leq (Ann(I))_*$, it suffices to show that for all $c \in K(A)$, $c \leq I_* \rightarrow \rho(0)$ implies $c \leq (Ann(I))_*$. If $c \leq I_* \rightarrow \rho(0)$ then
\( \forall \{cd|d \in K(A), \lambda_A(d) \in I\} = c(\forall \{d \in K(A)|\lambda_A(d) \in I\}) = cI_\ast \leq \rho(0). \)

Thus for all \( d \in K(A) \) with \( \lambda_A(d) \in I \) we have \( cd \leq \rho(0) \) so \( c^n d^n = 0 \) for some integer \( n \geq 1 \) (cf. Lemma 2.4 (ii)). It follows that \( \lambda_A(c) \wedge \lambda_A(d) = \lambda_A(c^n d^n) = 0 \), hence \( \lambda_A(c) \in \text{Ann}(I) \), i.e. \( c \leq (\text{Ann}(I))_\ast \).

Assume now that \( c \in K(A) \) and \( c \leq (\text{Ann}(I))_\ast \), hence by Lemma 3.5, \( \lambda_A(c) \in \text{Ann}(I) \). For any \( c \in K(A) \) with \( \lambda_A(d) \in I \) we have \( \lambda_A(cd) = \lambda_A(c) \wedge \lambda_A(d) = 0 \), hence by Lemma 3.2 (8) one gets \( cd \leq \rho(0) \). Therefore we have \( cI_\ast = \forall \{cd|d \in K(A), \lambda_A(d) \in I\} \leq \rho(0) \), i.e. \( c \leq I_\ast \to \rho(0) \). Then the inequality \( (\text{Ann}(I))_\ast \leq I_\ast \to \rho(0) \) is proven, so the equality \( (\text{Ann}(I))_\ast = I_\ast \to \rho(0) \) follows.

An element \( a \) of an arbitrary quantale \( A \) is said to be pure (or virginal, in the terminology of [20]) if for all \( c \in K(A) \), \( c \leq a \) implies \( a \lor c^\perp = 1 \). The pure elements in a quantale extend the pure ideals of a ring [27], [41] and the \( \sigma \) -ideals of a bounded distributive lattice [12], [19]. More precisely, an ideal \( I \) of bounded distributive lattice \( L \) is a \( \sigma \) - ideal if for all \( x \in I \), we have \( I \lor \text{Ann}(x) = L \).

**Lemma 4.7** If an element \( a \) of coherent quantale \( A \) is pure then \( a^\ast \) is a \( \sigma \) - ideal of the reticulation \( L(A) \). If moreover \( A \) is semiprime then for each \( \sigma \) - ideal \( J \) of \( L(A) \), \( J_\ast \) is a pure element of \( A \).

**Proof.** Assume \( x \in a^\ast \), so \( x = \lambda_A(c) \) for some \( c \in K(A) \) with \( c \leq a \). Since \( a \) is pure, \( c \leq a \) implies \( a \lor c^\perp = 1 \), so \( d \lor e = 1 \) for some \( d,e \in K(A) \) with the properties \( d \leq a \) and \( e \leq c^\perp \). Thus \( \lambda_A(d) \in a^\ast \) and \( \lambda_A(c) \wedge \lambda_A(e) = \lambda_A(ce) = \lambda(0) = 0 \), i.e. \( \lambda_A(e) \in \text{Ann}(\lambda_A(c)) \). We observe that \( \lambda_A(d) \lor \lambda_A(e) = \lambda_A(d \lor e) = 1 \), so \( a^\ast \lor \text{Ann}(\lambda_A(c)) = L(A) \). Thus \( a^\ast \) is a \( \sigma \) - ideal.

Now we assume that \( A \) is semiprime and \( J \) is \( \sigma \) - ideal of \( L(A) \). In order to prove that \( J_\ast \) is a pure element of \( A \) let us consider a compact element \( c \) of \( A \) such that \( c \leq J_\ast \). By Lemma 3.5 we have \( \lambda_A(c) \in J \), hence \( J \lor \text{Ann}(\lambda_A(c)) = L(A) \), so there exist two compact elements \( d \) and \( e \) of \( A \) such that \( \lambda_A(d) \in J \), \( \lambda_A(e) \in \text{Ann}(\lambda_A(c)) \) and \( \lambda_A(d \lor e) = \lambda_A(d) \lor \lambda_A(e) = 1 \). According to Lemmas 3.5 and 3.2, (3) we get \( d \leq J_\ast \) and \( d \lor e = 1 \). From \( \lambda_A(e) \in \text{Ann}(\lambda_A(c)) \) we infer \( \lambda_A(ce) = \lambda_A(c) \wedge \lambda_A(e) = 0 \), hence \( ce = 0 \) (because \( A \) is semiprime). Thus \( e \leq c^\perp \), therefore \( 1 = d \lor e \leq J_\ast \lor c^\perp \). It follows that \( J_\ast \lor c^\perp = 1 \), hence \( J \) is a \( \sigma \) - ideal of \( L(A) \).

5 Three topological structures on the prime spectrum

In this section we shall discuss some basic properties concerning three topologies defined on the prime spectrum \( \text{Spec}(A) \) of a coherent quantale \( A \): spectral topology, flat topology and patch topology. A topological space \((X, \Omega)\) is said to be spectral [22] (or coherent in the terminology of [25]) if it is sober and the
family $K(\Omega)$ of compact open sets of $X$ is closed under finite intersections, and forms a basis for the topology. The standard examples of spectral spaces are the prime spectrum $\text{Spec}(A)$ of a commutative ring $R$ (with the Zariski topology) and the prime spectrum $\text{Spec}_I(L)$ of bounded distributive lattice $L$ (with the Stone topology). If $(X, \Omega)$ is a spectral space then in a standard way (see [13],[25]) one can define on $X$ the following two topologies:

- the patch topology, having as basis the family of sets $U \cup V$, where $U$ is a compact open set in $X$ and $V$ is the complement of a compact open set (this topological space is denoted by $X_P$);
- the flat topology, having as basis the family of the complements of compact open sets in $X$ (this topological space is denoted by $X_F$).

**Lemma 5.1** [13],[27] $X_P$ is a Boolean space and $X_F$ is a spectral space.

**Remark 5.2** If $L$ is a bounded distributive lattice and $X$ is the spectral space $\text{Spec}_I(L(A))$, then the family $(D_L(x))_{x,y \in L}$ is a basis of open sets for $X_P$ and the family $(V_L(y))_{y \in L}$ is a basis of open sets for $X_F$.

Let $A$ be a coherent quantale. By Proposition 3.7, $\text{Spec}_Z(A)$ is homeomorphic with the spectral space $\text{Spec}_I(L(A))$, hence it is a spectral space. The family of open sets in $\text{Spec}_Z(A)$ will be denoted by $\mathcal{Z} = \mathcal{Z}_A$. For any subset $S$ of $\text{Spec}(A)$, $cl_Z(S) = V(\bigcap S)$ is the closure of $S$ in $\text{Spec}_Z(A)$; for all $p \in \text{Spec}(A)$, we have $cl_Z(\{p\}) = V(p)$. Now we can consider the two topologies associated with the spectral space $X = \text{Spec}_Z(A)$: the patch topology and the flat topology. We will denote $\text{Spec}_P(A) = X_P$ and $\text{Spec}_F(A) = X_F$; $\mathcal{P} = \mathcal{P}_A$ will be the family of open sets in $\text{Spec}_P(A)$ and $\mathcal{F} = \mathcal{F}_A$ the family of open sets in $\text{Spec}_F(A)$.

**Remark 5.3** (i) The family $\{D(c) \cap V(d) | c, d \in K(A)\}$ is a basis of open sets for $\text{Spec}_P(A)$;

(ii) The family $\{V(c) | c \in K(A)\}$ is a basis of open sets for $\text{Spec}_F(A)$.

**Remark 5.4** (i) The patch topology on $\text{Spec}(A)$ is finer than the spectral and the flat topologies on $\text{Spec}(A)$ (i.e. $\mathcal{Z} \subseteq \mathcal{P}$ and $\mathcal{F} \subseteq \mathcal{P}$);

(ii) The inclusions $\mathcal{Z} \subseteq \mathcal{P}$ and $\mathcal{F} \subseteq \mathcal{P}$ show that the identity functions id : $\text{Spec}_P(A)$ → $\text{Spec}_Z(A)$ and id : $\text{Spec}_F(A)$ → $\text{Spec}_Z(A)$ are continuous.

**Proposition 5.5** The two inverse functions $u : \text{Spec}(A) \rightarrow \text{Spec}_I(L(A))$ and $v : \text{Spec}_I(L(A)) \rightarrow \text{Spec}(A)$ from Proposition 3.7 are homeomorphisms w.r.t the patch and the flat topologies.

**Proof.** Applying Lemma 3.6 it is easy to prove that for all $c \in K(A)$, the following equalities $u^{-1}(V_L(\lambda_A(c))) = V(c)$ and $v^{-1}(D_L(\lambda_A(c))) = D(c)$ hold. Therefore, by Remarks 5.2 and 5.3, $u$ is patch and flat continuous.

For any $p \in \text{Spec}(A)$, let us denote $A(p) = \{q \in \text{Spec}(A) | q \leq p\}$. 


**Proposition 5.6** For any $p \in \text{Spec}(A)$, the flat closure of the set $\{p\}$ is $\text{cl}_F \{p\} = \Lambda(p)$.

**Proof.** According to the definition of the closure $\text{cl}_F(p) = \text{cl}_F \{p\}$, the following equalities hold:

$$\text{cl}_F(p) = \{q \in \text{Spec}(A) | \forall c \in K(A) (q \in V(c) \Rightarrow V(c) \cap \{p\} \neq \emptyset)\}$$

$$= \{q \in \text{Spec}(A) | \forall c \in K(A) (c \leq q \Rightarrow c \leq p)\}.$$

In order to prove that $\text{cl}_F(p) \subseteq \Lambda(p)$, let us consider $q \in \text{cl}_F(p)$ and $c \in K(A)$. Then $c \leq q$ implies $c \leq q$, therefore

$$q = \bigvee \{c \in K(A) | c \leq q\} \leq \bigvee \{c \in K(A) | c \leq p\} = p.$$  

Conversely, assume that $q \in \Lambda(p)$, so $q \leq p$. Thus for any $c \in K(A)$, $c \leq q$ implies $c \leq p$, hence $q \in \text{cl}_F(p)$.

**Proposition 5.7** If $S \subseteq \text{Spec}(A)$ is compact in $\text{Spec}_Z(A)$ then its flat closure is $\text{cl}_F(S) = \bigcup_{p \in S} \Lambda(p)$.

**Proof.** Applying Proposition 5.6, it follows that for any $p \in S$ we have $\Lambda(p) = \text{cl}_F(p) \subseteq \text{cl}_F(S)$, so $\bigcup_{p \in S} \Lambda(p) \subseteq \text{cl}_F(S)$. Let us prove the converse inclusion $\text{cl}_F(S) \subseteq \bigcup_{p \in S} \Lambda(p)$. Assume by absurdum that there exists $q \in \text{cl}_F(S) - \bigcup_{p \in S} \Lambda(p)$, so $q \not\leq p$ for all $p \in S$. Then for all $p \in S$ there exists $c_p \in K(A)$ such that $c_p \not\leq p$ and $c_p \leq q$. This means that $S \subseteq \bigcup_{p \in S} D(c_p)$, so $S \subseteq \bigcup_{i=1}^n D(c_{p_i})$ for some $p_1, \ldots, p_n \in S$. Denote $c = \bigvee_{i=1}^n c_{p_i}$, so $c \in K(A)$ and $S \subseteq D(c)$. One remarks that $c \leq q$, so $q \in V(c)$. Since $q \in \text{cl}_F(S)$ and $V(c)$ is an open neighbourhood of $q$ in the flat topology, it follows that $S \cap V(c) \neq \emptyset$. This contradicts $S \subseteq D(c)$, hence $\text{cl}_F(S) \subseteq \bigcup_{p \in S} \Lambda(p)$. We conclude that $\text{cl}_F(S) = \bigcup_{p \in S} \Lambda(p)$.

An element $a \in A$ is regular if it is a join of complemented elements. A maximal element in the set of proper regular elements is called max-regular. The set $\text{Sp}(A)$ of max-regular elements of $A$ is called the Pierce spectrum of the quantale $A$. For any proper regular element $a$ there exists $p \in \text{Sp}(A)$ such that $a \leq p$. If $e \in B(A)$ then we denote $U(e) = \{p \in \text{Sp}(A) | e \not\leq a\}$. Thus it is easy to prove that the family $(U(e))_{e \in B(A)}$ is a basis of open sets for a topology on $\text{Sp}(A)$.

For any $p \in \text{Spec}(A)$ we define $s_A(p) = \bigvee \{e \in B(A) | e \leq p\}; s_A(p)$ is regular and $s_A(p) \leq p < \Lambda(p)$.

**Lemma 5.8** $s_A(p)$ is a max-regular element of $A$.  

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Proof. In order to prove that \( s_A(p) \in Sp(A) \) is suffices to have: \( e \in B(A) \) and \( e \not\leq p \) implies \( s_A(p) \lor e = 1 \). Assume by absurdum that there exists \( e \in B(A) \) such that \( e \not\leq p \) and \( s_A(p) \lor e < 1 \). The element \( s_A(p) \lor e \) is regular so there exists a max - regular element \( q \) such that \( s_A(p) \lor e \leq q \). Since \( e \not\leq p \) and \( p \in Spec(A) \) we have \( \neg e \leq p \), so \( 1 = e \lor \neg \leq p \). This contradiction shows that \( s_A(p) \in Sp(A) \).

According to the previous lemma, for each \( p \in Spec(A) \), \( s_A(p) \) is a max - regular element of \( A \), so one obtains a function \( s_A : Spec(A) \rightarrow Sp(A) \).

**Proposition 5.9** \( Sp(A) \) is a Boolean space and \( s_A : Spec(A) \rightarrow Sp(A) \) is surjective and continuous w.r.t. both flat and spectral topologies on \( Spec(A) \).

**Proof.** Assume that \( q \in Sp(A) \) then \( q \leq p \) for some \( p \in Spec(A) \), hence \( q \leq s_A(p) \). Since \( q \) and \( s_A(p) \) are max - regular we have \( q = s_A(p) \), so \( s_A \) is surjective. It is easy to see that \( Sp(A) \) is a Hausdorff space and \( (U(e))_{e \in B(A)} \) is basis of clopen set for \( Sp(A) \). For all \( e \in B(A) \) we have \( s_A^{-1}(U(e)) = D(e) = V(\neg e) \), hence the functions \( s_A : Spec_2(A) \rightarrow Sp(A) \) and \( s_A : Spec_F(A) \rightarrow Sp(A) \) are continuous. Therefore the topological space \( Sp(A) = Im(s_A) \) is compact. We conclude that \( Sp(A) \) is a Boolean space. 

6 Hyperarchimedean quantales

The hyperarchimedean quantales were introduced in [9], where a characterization theorem of these objects was proven. This section contains new algebraic and topological characterizations of the hyperarchimedean quantales. Let \( A \) be a coherent quantale. By [9], \( A \) is said to be hyperarchimedean if for any \( c \in K(A) \) there exists an integer \( n > 1 \) such that \( c^n \in B(A) \). Applying Lemma 4.2, it follows that a coherent frame \( A \) is hyperarchimedean if and only if \( B(A) = K(A) \) (see [31]).

**Remark 6.1** Recall from [6] that a frame \( L \) is zero - dimensional if any element \( a \in A \) is a joint of complemented elements. On the other hand, by Remark 2.1,(i) of [24], an algebraic frame is zero - dimensional if and only if each compact element is complemented. Thus a coherent frame is hyperarchimedean if and only if it is zero - dimensional.

**Proposition 6.2** If \( A \) is a coherent quantale the the following are equivalent:

1. \( A \) is hyperarchimedean;
2. \( L(A) \) is a Boolean algebra;
3. \( Spec(A) = Max(A) \);
4. The quantale \( \rho(0)_A \) is hyperarchimedean;
(5) \( R(A) \) is a hyperarchimedean frame;
(6) \( R(A) \) is a zero-dimensional frame;

**Proof.** The equivalence of (1), (2), (3) and (6) was proven in [9], but for sake of completeness we shall present a short proof of the proposition.

(1) \( \iff \) (2) In accordance to Lemma 4.3, the following assertions are equivalent:
- \( L(A) \) is a Boolean algebra;
- for all \( c \in K(A) \), \( \lambda_A(c) \in B(L(A)) \);
- for all \( c \in K(A) \), there exists an integer \( n > 0 \) such that \( c^n \in B(A) \);
- \( A \) is hyperarchimedean.

(2) \( \iff \) (3) By the Nachbin theorem [5] and Proposition 3.7, \( L(A) \) is a Boolean algebra iff \( \text{Spec} \text{Id}(L(A)) = \text{Max} \text{Id}(L(A)) \) iff \( \text{Spec}(A) = \text{Max}(A) \).

(3) \( \iff \) (4) This equivalence follows by using Lemma 2.6.

(1) \( \iff \) (5) According to Lemma 6 of [9], \( \text{Spec}(A) = \text{Spec}(R(A)) \) and \( \text{Max}(A) = \text{Max}(R(A)) \), therefore this equivalence follows by using that (1) and (3) are equivalent.

(5) \( \iff \) (6) By Remark 6.1.

**Lemma 6.3** If \( L \) is a bounded distributive lattice then the following are equivalent:

(1) \( L \) is a Boolean algebra;

(2) \( \text{SpecId}(L) = \text{MaxId}(L) \);

(3) For all distinct prime ideals \( M \) and \( N \) of \( L \) there exist \( x \not\in M \) and \( y \not\in N \) such that \( x \land y = 0 \).

**Proof.** (1) \( \iff \) (2) By the Nachbin theorem.

(1) \( \Rightarrow \) (3) Assume that \( L \) is a Boolean algebra and \( M, N \) are distinct prime ideals of \( L \). Thus \( M, N \) are distinct maximal ideals of \( L \) so there exists an element \( x \in M - N \). Denoting \( y = \neg x \) one gets \( x \not\in N \), \( y \not\in M \) and \( x \land y = 0 \).

(3) \( \Rightarrow \) (1) Assume by absurdum that there exist two ideals \( M \) and \( N \) of \( L \) such that \( M \not\subseteq N \). By hypothesis there exist two elements \( x \) and \( y \) of \( L \) such that \( x \not\in M \), \( y \not\in N \) and \( x \land y = 0 \). Since \( x \not\in M \) and \( x \land y = 0 \) implies \( y \in M \), it follows a contradiction, so \( M = N \). It follows that \( \text{SpecId}(L) = \text{MaxId}(L) \).

**Proposition 6.4** Assume that \( A \) is a coherent quantale. The following following properties are equivalent:

(1) For all distinct \( p, q \in \text{Spec}(L(A)) \) there exist \( c, d \in K(A) \) such that \( c \not\leq p, d \not\leq q \) and \( cd = 0 \);

(2) For all distinct prime ideals \( I, J \) of \( L(A) \) there exist two elements \( x, y \) of \( L(A) \) such that \( x \not\in I \), \( y \not\in J \) and \( x \land y = 0 \).
Proof. Assume that $I, J$ are two distinct prime ideals of $L(A)$. In accordance to Proposition 3.7 there exist two $p, q \in \text{Spec}(A)$ such that $I = p^*$, $J = q^*$ and $p \neq q$. By hypothesis there exist $c, d \in K(A)$ such that $c \nleq p$, $d \nleq q$ and $cd = 0$. Applying Lemmas 3.2 and 3.6 one gets $\lambda_A(c) \land \lambda_A(d) = \lambda_A(cd) = \lambda_A(0) = 0$ and $\lambda_A(c) \notin p^*$, $\lambda_A(d) \notin q^*$.

Assume now that $p, q \in \text{Spec}(A)$, with $p \neq q$ so $p^*$, $q^*$ are distinct prime ideals of $L(A)$. Thus there exist $c, d \in K(A)$ such that $\lambda_A(c) \notin p^*$, $\lambda_A(d) \notin q^*$ and $\lambda_A(cd) = \lambda_A(c) \land \lambda_A(d) = 0$. Applying Lemma 3.6 one obtains $c \nleq p$ and $d \nleq q$. Since $A$ is semiprime, $\lambda_A(cd) = 0$ implies $cd = 0$ (by Lemma 3.2,(9)), so there exist an integer $n \geq 1$ such that $c^nd^n = 0$. Let us denote $u = c^n$ and $v = d^n$. Thus $u$ and $v$ are two compact elements of $A$ such that $u \nleq p$, $v \nleq q$ (because $p, q$ are $m$-prime elements) and $uv = 0$.

Proposition 6.5 Assume that $A$ is a coherent quantale. The following properties are equivalent:

(1) $A$ is hyperarchimedean;

(2) $L(A)$ is a Boolean algebra;

(3) For all distinct prime ideals $I, J$ of $L(A)$ there exist two elements $x, y$ of $L(A)$ such that $x \nleq I$, $y \nleq J$ and $x \land y = 0$;

(4) For all distinct $p, q \in \text{Spec}(A)$ there exist $c, d \in K(A)$ such that $c \nleq p$, $d \nleq q$ and $cd = 0$;

Proof. (1) $\leftrightarrow$ (2) By Proposition 6.2.

(2) $\leftrightarrow$ (3) By Lemma 6.3.

(3) $\leftrightarrow$ (1) By Proposition 6.4.

Corollary 6.6 If $A$ is a coherent quantale then it is hyperarchimedean if and only if for all distinct $p, q \in \text{Spec}(A)$ there exist $c, d \in K(A)$ such that $c \nleq p$, $d \nleq q$ and $cd = \rho(0)$.

Lemma 6.7 Assume that $X$ and $Y$ are topological spaces, $X$ is compact and $Y$ is Hausdorff. Any continuous function $f : X \to Y$ is a closed map. Moreover, if $f$ is bijective, then it is a homeomorphism.

Theorem 6.8 If $A$ is a coherent quantale then the following are equivalent:

(1) $A$ is hyperarchimedean;

(2) For all distinct $p, q \in \text{Spec}(A)$ there exist $c, d \in K(A)$ such that $c \nleq p$, $d \nleq q$ and $cd = 0$;
(3) \( \text{Spec}_2(A) \) is Hausdorff;

(4) \( \text{Spec}_2(A) \) is a Boolean space;

(5) \( Z = \mathcal{P} \);

(6) \( \text{Spec}_F(A) \) is Hausdorff;

(7) \( \text{Spec}_F(A) \) is Boolean space;

(8) \( Z = \mathcal{F} \).

Proof.

The equivalence (1) \( \Leftrightarrow \) (2) follows from Proposition 6.5 and the equivalences (3) \( \Leftrightarrow \) (4), (6) \( \Leftrightarrow \) (7) are well-known from the general topology.

(2) \( \Rightarrow \) (3) Let \( p, q \) be two distinct elements of \( \text{Spec}(A) \). Then there exist \( c, d \in K(A) \) such that \( c \not\leq p, d \not\leq q \) and \( cd = 0 \). It follows that \( p \in D(c), q \in D(d) \) and \( D(c) \cap D(d) = D(cd) = \emptyset \). Thus \( \text{Spec}_2(A) \) is a Hausdorff space.

(3) \( \Rightarrow \) (5) Assume that \( \text{Spec}_2(A) \) is a Hausdorff space. Recall from Remark 5.4(ii) that the identity function \( id : \text{Spec}_F(A) \rightarrow \text{Spec}_2(A) \) is continuous. Since \( \text{Spec}_F(A) \) is compact (cf. Lemma 5.1) and \( \text{Spec}_2(A) \) is Hausdorff, by Lemma 6.7 it follows that the map \( id : \text{Spec}_F(A) \rightarrow \text{Spec}_2(A) \) is a homeomorphism, hence \( Z = \mathcal{P} \).

(5) \( \Rightarrow \) (1) Assume that \( Z = \mathcal{P} \) and \( p \in \text{Spec}(A) \). We want to show that \( p \in \text{Max}(A) \). According to Remark 5.4(i) and the hypothesis (5) we have \( \mathcal{F} \subseteq \mathcal{P} = Z \). From \( \mathcal{F} \subseteq \mathcal{P} \) it follows that any closed set in \( \text{Spec}_F(A) \) is closed in \( \text{Spec}_2(A) \), so \( \text{cl}_Z(\{p\}) \subseteq \text{cl}_F(\{p\}) \). Thus by applying Proposition 5.6 one gets \( V(p) \subseteq \Lambda(p) \). Thus \( V(p) = \{p\} \), so \( p \in \text{Max}(A) \). It follows that \( \text{Spec}(A) = \text{Max}(A) \), hence, by Proposition 6.2 we conclude that \( A \) is hyperarchimedean.

(1) \( \Rightarrow \) (6) Assume that \( A \) is hyperarchimedean and \( p, q \) are distinct elements of \( \text{Spec}(A) \), hence, by Proposition 6.2 we have \( p, q \in \text{Max}(A) \), therefore \( p \lor q = 1 \). Since \( 1 \in K(A) \) there exist \( p, q \in K(A) \) such that \( c \leq p, d \leq q \) and \( c \lor d = 1 \). Then \( p \in V(c), q \in V(d) \) and \( V(c), V(d) \) are open sets of \( \text{Spec}_F(A) \) such that \( V(c) \cap V(d) = V(c \lor d) = V(1) = \emptyset \). It follows that \( \text{Spec}_F(A) \) is a Hausdorff space.

(6) \( \Rightarrow \) (1) Assume that \( \text{Spec}_F(A) \) is a Hausdorff space. Let \( p \in \text{Spec}(A) \) and \( q \in \text{Max}(A) \) such that \( p \leq q \). Since \( \text{Spec}_F(A) \) is Hausdorff we have \( \text{cl}_F(\{q\}) = \{q\} \). According to Proposition 5.6 we have \( p \in \Lambda(q) = \text{cl}_F(\{q\}) = \{q\} \), hence \( p = q \). Thus \( \text{Spec}(A) = \text{Max}(A) \), so \( A \) is hyperarchimedean.

(6) \( \Rightarrow \) (8) Assume that \( \text{Spec}_F(A) \) is a Hausdorff space. The identity function \( id : \text{Spec}_F(A) \rightarrow \text{Spec}_F(A) \) is continuous, \( \text{Spec}_F(A) \) is compact and \( \text{Spec}_F(A) \) is Hausdorff. Hence by Lemma 6.7 it follows that \( id : \text{Spec}_F(A) \rightarrow \text{Spec}_F(A) \) is a homeomorphism, so \( \mathcal{F} = \mathcal{P} \). According to the previous proof of the implications (6) \( \Rightarrow \) (1), and (1) \( \Rightarrow \) (5) we have \( Z = \mathcal{P} \), therefore \( Z = \mathcal{F} \).
(3) ⇒ (5) Assume that \( Z = \mathcal{F} \). If \( p \in \text{Spec}(A) \) then \( \Lambda(p) = \text{cl}_F(\{p\}) = \{p\} = V(p) \), hence \( p \in \text{Max}(A) \). Thus \( \text{Spec}(A) = \text{Max}(A) \), so the quantale \( A \) is hyperarchimedean. \( \blacksquare \)

**Remark 6.9** If we apply the previous theorem to the quantale \( \text{Id}(R) \) of the ideals of a commutative ring \( R \) then we obtain the main part of Theorem 3.3 from [\( \square \)].

7 Flat topology on the maximal spectrum

In this section we shall study the flat topology on the maximal spectrum of coherent quantales in order to obtain new results on the normal and \( B \)-normal quantales.

We fix a coherent quantale \( A \). Recall that \( \text{Max}_F(\mathcal{A}) \) is the maximal spectrum \( \text{Max}(\mathcal{A}) \) of \( \mathcal{A} \) endowed with the restriction of the flat topology of \( \text{Spec}(\mathcal{A}) \).

**Lemma 7.1** If \( c \in K(\mathcal{A}) \) then \( V(c) \cap \text{Max}(\mathcal{A}) \) is a clopen set of \( \text{Max}_F(\mathcal{A}) \).

**Proof.** Assume that \( c \) is a compact element of \( \mathcal{A} \). According to Remark 5.3,(ii), \( V(c) \cap \text{Max}(\mathcal{A}) \) is an open set of \( \text{Max}_F(\mathcal{A}) \). It remains to prove that the set \( D(c) \cap \text{Max}(\mathcal{A}) \) is open in \( \text{Max}_F(\mathcal{A}) \). Let \( p \in D(c) \cap \text{Max}(\mathcal{A}) \), hence \( c \not\leq p \) and \( p \in \text{Max}(\mathcal{A}) \). If \( c \lor p < 1 \) then \( p < c \lor p \leq q \) for some \( q \in \text{Max}(\mathcal{A}) \), contradicting the maximality of \( p \). Thus \( c \lor p = 1 \), hence there exists \( d \in K(\mathcal{A}) \) such that \( d \leq p \) and \( c \lor d = 1 \). One obtains \( V(c) \cap V(d) = V(c \lor d) = V(1) = \emptyset \), hence \( V(d) \subset D(c) \). It follows that \( p \in V(d) \cap \text{Max}(\mathcal{A}) \subseteq D(c) \cap \text{Max}(\mathcal{A}) \), so \( D(c) \cap \text{Max}(\mathcal{A}) \) is an open subset of \( \text{Max}_F(\mathcal{A}) \). \( \blacksquare \)

**Proposition 7.2** The topological space \( \text{Max}_F(\mathcal{A}) \) is Hausdorff and zero-dimensional.

**Proof.** Let \( p, q \) be two distinct maximal elements of \( \mathcal{A} \), hence \( p \lor q = 1 \). Thus there exist \( c, d \in K(\mathcal{A}) \) such that \( c \leq p, d \leq q \) and \( c \lor d = 1 \), therefore \( p \in V(c), d \in V(d) \) and \( V(c) \cap V(d) = V(c \lor d) = V(1) = \emptyset \). It results that \( \text{Max}_F(\mathcal{A}) \) is a Hausdorff space. In accordance to Lemma 7.1, the family \( (V(c) \cap \text{Max}(\mathcal{A}))_{c \in K(\mathcal{A})} \) is a basis of clopen sets for \( \text{Max}_F(\mathcal{A}) \), so this topological space is zero-dimensional. \( \blacksquare \)

Following [\( \square \)] we shall denote \( r(\mathcal{A}) = \bigwedge \text{Max}(\mathcal{A}) \). One remarks that \( r(\mathcal{A}) \) extends the notion of Jacobson radical of a commutative ring. It is obvious that \( \rho(0) \leq r(\mathcal{A}) \).

**Theorem 7.3** \( \text{Max}_F(\mathcal{A}) \) is compact if and only if \( [r(\mathcal{A})]_\mathcal{A} \) is a hyperarchimedean quantale.
Proof.  

(⇒) Assume that $\text{Max}_F(A)$ is compact. First we shall prove that $L(A)/(r(A))^*$ is a Boolean algebra. Let $c$ be a compact element of $A$ such that $\lambda_A(c)/(r(A))^* \neq 0/(r(A))^*$, so $\lambda_A(c) \not\leq (r(A))^*$. By Lemma 3.5 we have $c \not\leq r(A)$, so there exists $m_c \in \text{Max}(A)$ such that $c \not\leq m_c$. For any $m \in \text{Max}(A)$ we have $c \leq m$ or $c \not\leq m$; if $c \not\leq m$ then there exists $d_m \in K(A)$ such that $d_m \leq m$ and $c \lor d_m = 1$. Since $c \not\leq m_c$, the family $\{m \in \text{Max}(A) | c \not\leq m\}$ is non-empty. One remarks that $\text{Max}(A) = \{m \in \text{Max}(A) | c \leq m\} \cup \{m \in \text{Max}(A) | c \not\leq m\} \subseteq V(c) \cup \bigcup_{c \not\leq m} V(d_m)$.

By hypothesis $\text{Max}_F(A)$ is compact so there exist $m_1, \ldots, m_n \in \text{Max}(A)$ such that $c \not\leq m_i$ for $i = 1, \ldots, n$ and $\text{Max}(A) \subseteq V(c) \cup \bigcup_{i=1}^n V(d_m)$. Let us denote $d_i = d_{m_i}$ for $i = 1, \ldots, n$ and $d = d_1d_2 \ldots d_m$. Therefore $\text{Max}(A) \subseteq V(c) \cup V(d) = V(c)$, so $cd \leq r(A)$. By Lemma 3.5, $cd \leq r(A)$ implies $\lambda_A(cd) \in (r(A))^*$.

According to Lemma 2.1 (i), from $c \lor d_i = 1$, $i = 1, \ldots, n$ one gets $c \lor d = 1$. Since $\lambda_A(c) \lor \lambda_A(d) = \lambda_A(c \lor d) = \lambda_A(1) = 1$ and $\lambda_A(c) \land \lambda_A(d) = \lambda_A(cd)$, the following equalities hold:

\[
\lambda_A(c)/(r(A))^* \lor \lambda_A(d)/(r(A))^* = 1/(r(A))^*; \\
\lambda_A(c)/(r(A))^* \land \lambda_A(d)/(r(A))^* = \lambda_A(cd)/(r(A))^* = 0/(r(A))^*.
\]

It follows that $L(A)/(r(A))^*$ is a Boolean algebra. By Proposition 6 of [9], the lattices $L([r(A)]_A)$ and $L([r(A)]_A)$ are isomorphic, so the reticulation $L([r(A)]_A)$ of the quantale $[r(A)]_A$ is a Boolean algebra. Applying Proposition 6.1, it follows that $[r(A)]_A$ is a hyperarchimedean quantale.

(⇐) Assume that the quantale $[r(A)]_A$ is hyperarchimedean. By Proposition 6.2 we have $\text{Max}_F([r(A)]_A) = \text{Spec}_F([r(A)]_A)$, so $\text{Max}_F([r(A)]_A)$ is compact (cf. Lemma 5.1). It is easy to see that $\text{Max}_F(A) = M_F([r(A)]_A)$, so $\text{Max}_F(A)$ is compact.

\[\blacksquare\]

**Proposition 7.4** The topology of $\text{Max}_F(A)$ is finer than the topology of $\text{Max}_Z(A)$.

**Proof.** A basic open subset of $\text{Max}_Z(A)$ has the form $U = \text{Max}(A) \cap D(c)$, for some $c \in K(A)$. Let us consider an element $m \in U$ so $m \in \text{Max}(A)$ and $c \not\leq m$, hence $c \lor m = 1$. Thus we have $c \lor d = 1$ for some $d \in K(A)$ with $d \leq m$. Therefore $V(c) \cap V(d) = V(c \lor d) = V(1) = \emptyset$, so $m \in \text{Max}(A) \cap V(d)$, and $\text{Max}(A) \cap V(d)$ is included in $U$. It follows that $U$ is an open subset of $\text{Max}_F(A)$.

\[\blacksquare\]

The following proposition characterizes the quantales $A$ for which $\text{Max}_Z(A)$ and $\text{Max}_F(A)$ coincide.

**Proposition 7.5** If $A$ is a coherent quantale then the following are equivalent:

1. $\text{Max}_F(A)$ is compact;
(2) The topological spaces $Max_Z(A)$ and $Max_F(A)$ coincide;

(3) $[r(A)]_A$ is a hyperarchimedean quantale.

**Proof.** The equivalence of (1) and (3) follows by Theorem 7.3. We remark that the following equalities hold: $Max_F(A) = Max_F([r(A)]_A)$ and $Max_Z([r(A)]_A) = Max_Z(A)$. According to Theorem 6.8 the assertions (2) and (3) are equivalent. 

Following [25], p.199 we say that a commutative ring $R$ is a Gelfand ring if each prime ideal of $R$ is contained in a unique maximal ideal. Recall from [40], [25] that a bounded distributive lattice $L$ is called normal if for all elements $x, y \in L$ such that $x \vee y = 1$ there exist $u, v \in L$ such that $x \vee u = y \vee v = 1$ and $uv = 0$. We know from [25], p.68 that a bounded distributive lattice $L$ is normal if and only if each prime ideal of $L$ is contained in a unique maximal ideal. The normal quantales were introduced in [35] as an abstractization of the lattices of ideals of Gelfand rings and normal lattices.

According to [35], a quantale $A$ is said to be normal if for all $a, b \in A$ such that $a \vee b = 1$ there exist $e, f \in A$ such that $a \vee e = b \vee f = 1$ and $ef = 0$. If $1 \in K(A)$ then $A$ is normal if and only if for all $c, d \in K(A)$ such that $c \vee d = 1$ there exist $e, f \in K(A)$ such that $c \vee e = d \vee f = 1$ and $ef = 0$ (cf. Lemma 20 of [9]). One observes that a commutative ring $R$ is a Gelfand ring iff $Id(R)$ is a normal quantale and a bounded distributive lattice $L$ is normal iff $Id(L)$ is a normal frame.

The normal quantales offer an abstract framework in order to unify some algebraic and topological properties of commutative Gelfand rings [25], [23], [32], [29], [39], normal lattices [25], [19], [37], [40], commutative unital $l$-groups [7], $F$-rings [41], [20], MV-algebras and BL-algebras [16], [28], Gelfand residuated lattices [18], etc.

Let us fix a coherent quantale $A$.

**Proposition 7.6** [9] The quantale $A$ is normal if and only if the reticulation $L(A)$ is a normal lattice (in the sense of [40], [25]).

**Proposition 7.7** [35], [20], [41] If $A$ is a coherent quantale then the following are equivalent:

1. $A$ is a normal quantale;
2. For all distinct $m, n \in Max(A)$ there exist $c_1, c_2 \in K(A)$ such that $c_1 \nleq m, c_2 \nleq n$ and $c_1c_2 = 0$;
3. The inclusion $Max(A) \subseteq Spec(A)$ is a Hausdorff embedding (i.e. any distinct points in $Max(A)$ have disjoint neighbourhoods in $Spec_Z(A)$);
4. For any $p \in Spec(A)$ there exists a unique $m \in Max(A)$ such that $p \leq m$;
5. $Spec_Z(A)$ is a normal space;
(6) The inclusion $\text{Max}_\mathbb{Z}(A) \subseteq \text{Spec}_\mathbb{Z}(A)$ has a continuous retraction $\gamma : \text{Spec}_\mathbb{Z}(A) \to \text{Max}_\mathbb{Z}(A)$;

(7) If $m \in \text{Max}(A)$ then $\Lambda(m)$ is a closed subset of $\text{Spec}_\mathbb{Z}(A)$.

**Remark 7.8** A proof of the previous proposition can be obtained by using Proposition 7.6 and some characterizations of normal lattices given in [19], [23], [37], [40].

**Theorem 7.9** Assume that $A$ is a normal quantale. Then the retraction map $\gamma : \text{Spec}(A) \to \text{Max}(A)$ is flat continuous if and only if $\text{Max}_F(A)$ is a compact space.

**Proof.** Assume that the retraction map $\gamma : \text{Spec}_F(A) \to \text{Max}_F(A)$ is continuous. Since $\text{Spec}_F(A)$ is compact it follows that $\text{Max}_F(A)$ is also compact.

Conversely, assume that $\text{Max}_F(A)$ is compact, hence by Proposition 7.2 it is a Boolean space. By Proposition 7.3 it results that $[(r(A))_A]$ is a hyperarchimedean quantale. Applying the condition (4) of Theorem 6.7 one gets $\text{Max}_\mathbb{Z}((r(A))_A) = \text{Max}_F((r(A))_A)$. We remark that $\text{Max}_\mathbb{Z}(A)$ and $\text{Max}_F((r(A))_A)$ are homeomorphic, thus by Theorem 6.7(4) it follows that $\text{Max}_\mathbb{Z}(A)$ is a Boolean space. Thus $(\text{Max}(A) \cap D(c))_{c \in K(A)}$ is a basis of clopen sets in $\text{Max}_\mathbb{Z}(A)$.

Let us consider an element $c \in K(A)$; in accordance to the continuity of $\gamma : \text{Spec}_F(A) \to \text{Max}_F(A)$, it follows that $\gamma^{-1}(\text{Max}(A) \cap D(c))$ is a clopen set in $\text{Spec}_\mathbb{Z}(A)$. Applying Lemma 2.4 of [3] we find an element $e \in B(A)$ such that $\gamma^{-1}(\text{Max}(A) \cap D(c)) = V(e)$. Therefore $\gamma^{-1}(\text{Max}(A) \cap D(c))$ is a clopen subset of $\text{Max}_F(A)$ (cf. Lemma 7.1), hence the map $\gamma : \text{Spec}_F(A) \to \text{Max}_F(A)$ is continuous.

**Proposition 7.10** If $\text{Max}_\mathbb{Z}(A)$ is Hausdorff and $\rho(0) = r(A)$ then the quantale $A$ is normal.

**Proof.** Assume by absurdum that the quantale $A$ is not normal, so by Proposition 7.7.(4) there exist $p \in \text{Spec}(A)$ and $q, r \in \text{Max}(A)$ such that $q \neq r$, $p \leq q$ and $p \leq r$. Since $\text{Max}_\mathbb{Z}(A)$ is Hausdorff there exist $c, d \in K(A)$ such that $q \in D(c)$, $r \in D(d)$ and $D(cd) \cap \text{Max}(A) = D(c) \cap D(d) \cap \text{Max}(A) = \emptyset$.

If $cd \not\leq \rho(0)$ then $cd \not\leq r(A)$, so $cd \not\leq m$ for some $m \in \text{Max}(A)$. It results that $m \in D(cd) \cap \text{Max}(A)$, contradicting $D(cd) \cap \text{Max}(A) = \emptyset$. Thus $cd \leq \rho(0)$, hence one gets $c \leq p$ or $d \leq p$. If $c \leq p$ then $c \leq q$, contradicting $q \in D(c)$; similarly, $c \leq p$ contradicts $r \in D(d)$. We conclude that $A$ is normal.

**Corollary 7.11** $\text{Max}_\mathbb{Z}(A)$ is a Hausdorff space if and only if $[r(a)_A]$ is a normal quantale.
Proof. We observe that the quantale $C = [r(a)]_A$ verifies the conditions $\rho_C(0) = r(C)$ and $\text{Max}_Z(A) = \text{Max}_C(A)$. Applying Proposition 7.10 to the quantale $C$ the following equivalences hold: $\text{Max}_Z(A)$ is Hausdorff iff $\text{Max}_Z(C)$ is Hausdorff iff $C$ is a normal quantale.

Following [9] we say that a quantale $A$ is said to be $B$-normal if for all $c, d \in K(A)$ there exist $e, f \in B(A)$ such that $c \vee e = d \vee f = 1$ and $ce = 0$. If the $B$-normal quantale $A$ is a frame then we shall say that $A$ is a $B$-normal frame. The $B$-normal quantales constitute an abstract setting in which we can generalize various results on the clean commutative rings [23], [34], the $B$-normal (bounded distributive) lattices [10], the clean unital $l$-groups [21], the quasi-local BL-algebras [28], the quasi-local residuated lattices [33], etc.

Lemma 7.12 If $A$ is normal quantale then $(D(e) \cap \text{Max}(A))_{e \in B(A)}$ is the family of the clopen subsets of $\text{Max}_Z(A)$.

Proof. Let $K$ be a clopen subset of $\text{Max}_Z(A)$. If $\gamma : \text{Spec}_Z(A) \rightarrow \text{Max}_Z(A)$ is the continuous retract of the inclusion $\text{Max}_Z(A) \subseteq \text{Spec}_Z(A)$, then $L = \gamma^{-1}(K)$ is a clopen set in $\text{Spec}_Z(A)$. By Lemma 24 of [9] there exists an element $e \in B(A)$ such that $L = D(e)$. Thus $K = \gamma(D(e)) = \{ \gamma(p) \mid p \in \text{Spec}(A), e \not\leq p \}$. It is easy to see that for all $p \in \text{Spec}(A)$ we have $e \not\leq p$ iff $e \not\leq \gamma(p)$, hence $K = \{ \gamma(p) \mid p \in \text{Spec}(A), e \not\leq \gamma(p) \} = D(e) \cap \text{Max}(A).

The following theorem contains some conditions that characterize the $B$-normal algebras.

Theorem 7.13 If $A$ is a coherent quantale then the following are equivalent:

1. $A$ is $B$-normal;
2. $R(A)$ is a $B$-normal frame;
3. The reticulation $L(A)$ is a $B$-normal lattice;
4. For all distinct $p, q \in \text{Max}(A)$ there exists $e \in B(A)$ such that $e \leq p$ and $\neg e \leq q$;
5. $A$ is a normal quantale and $\text{Max}_Z(A)$ is a zero-dimensional space;
6. $A$ is a normal quantale and $\text{Max}_Z(A)$ is a Boolean space;
7. The family $(D(e) \cap \text{Max}(A))_{e \in B(A)}$ is a basis of open sets for $\text{Max}_Z(A)$;
8. The function $s_A|_{\text{Max}(A)} : \text{Max}_Z(A) \rightarrow \text{Sp}(A)$ is a homeomorphism.

Proof. The equivalence of the properties (1), (2), (3), (5) and (6) was established in [9], hence it remains to prove the equivalence of the other conditions.

(1) $\Rightarrow$ (4) Let $p, q$ be two distinct maximal elements of $A$, hence $p \vee q = 1$. Since $A$ is $B$-normal, there exist $e, f \in B(A)$ such that $p \vee f = q \vee e = 1$ and
$ef = 0$. From $p \vee f = q \vee e = 1$ one gets $f \not\leq p$, $e \not\leq q$, hence $-f \leq p$ and $-e \leq q$. The equality $ef = 0$ implies $e \leq -f \leq p$.

(6) $\Rightarrow$ (7) Since $Max_Z(A)$ is a Boolean space, the family of its clopen subsets is a basis of open sets. By Lemma 7.12, the family $(D(e) \cap Max(A))_{e \in B(A)}$ is exactly this basis of open sets for $Max_Z(A)$.

(7) $\Rightarrow$ (4) Let $p, q$ be two distinct maximal elements of $A$. We observe that $U = Spec(A) - \{q\} = Spec(A) - V(q) = D(q)$ is open in $Spec_Z(A)$, so $U \cap Max(A)$ is an open subset of $Max_Z(A)$ that contains $p$. In accordance to the hypothesis, there exists $e \in B(A)$ such that $p \in D(e) \cap Max(A) \subseteq U \cap Max(A)$. It follows that $e \not\leq p$ and $e \leq q$, so $-e \leq p$ and $e \leq q$.

(1) $\Rightarrow$ (8) Assume that $A$ is $B$-normal. In accordance to Proposition 5.9, $s_A|_{Max(A)} : Spec_Z(A) \to Sp(A)$ is a surjective continuous map. We shall prove that the restriction of $s_A$ to $Max(A)$ is injective. Let $m, n \in Max(A)$ such that $m \neq n$. We know that the conditions (1) and (4) are equivalent, so there exists $e \in B(A)$ such that $e \leq m$, $-e \leq n$, hence $e \leq s_A(m)$ and $e \not\leq s_A(n)$. It follows that $s_A(m) \neq s_A(n)$, so $s_A$ is injective.

In order to prove that $s_A|_{Max(A)} : Max_Z(A) \to Sp(A)$ is surjective assume that $q \in Sp(A)$, hence $q = s_A(p)$, for some $p \in Spec(A)$. Let $\gamma(p)$ be the unique maximal element of $A$ such that $p \leq \gamma(p)$. Thus $q = s_A(p) \leq s_A(\gamma(p))$, so $q = \gamma(p)$, because $q$ and $\gamma(p)$ are max-regular elements. We know already that (1) and (6) are equivalent, so $Max(A)$ is a Boolean space. By Proposition 5.9, $Sp(A)$ is also a Boolean space. Therefore, by applying Lemma 6.8 it follows that $s_A|_{Max(A)} : Max_Z(A) \to Sp(A)$ is a homeomorphism.

(8) $\Rightarrow$ (6) Taking into account the hypothesis (8), it follows that the function $(s_A|_{Max(A)})^{-1} \circ s_A : Spec_Z(A) \to Max_Z(A)$ is a continuous retraction of the inclusion $Max_Z(A) \subset Spec_Z(A)$, so $A$ is a normal quantale. Moreover, by (8) and Proposition 5.9 it follows that $Max_Z(A)$ is a Boolean space.

\[\blacksquare\]

**Corollary 7.14** Let $A$ be a normal quantale. If $[r(A)]_A$ is a hyperarchimedean quantale then $A$ is $B$-normal.

**Proof.** Assume that $[r(A)]_A$ is hyperarchimedean, so by Proposition 4.5 we have $Max_F(A) = Max_Z(A)$. Therefore by using Proposition 7.2 it follows that $Max_Z(A)$ is zero-dimensional. Applying Theorem 7.13,(5) one gets that $A$ is a $B$-normal quantale.

\[\blacksquare\]

### 8 Flat topology on the minimal prime spectrum

If $A$ is a quantale then we denote by $Min(A)$ the set of minimal $m$-prime elements of $A$; $Min(A)$ is called the minimal prime spectrum of $A$. If $1 \in K(A)$ then for any $p \in Spec(A)$ there exists $q \in Min(A)$ such that $q \leq p$. For any bounded distributive lattice $L$ we denote by $Min_{Id}(L)$ the set of minimal prime ideals in $L$; $Min_{Id}(L)$ is the minimal prime spectrum of the frame $Id(L)$.
We will obtain a description of the minimal \( m \)-prime elements of a coherent quantale \( A \) by using the reticulation. First we remember from [40] the following result.

**Lemma 8.1** A prime ideal \( P \) of a bounded distributive lattice \( L \) is minimal prime if and only if for all \( x \in P \) we have \( \text{Ann}(x) \not\subseteq P \).

Let us fix a coherent quantale \( A \).

**Lemma 8.2** If \( c \in K(A) \) and \( p \in \text{Spec}(A) \) then \( \text{Ann}(\lambda_A(c)) \subseteq p^* \) if and only if \( c \to \rho(0) \leq p \).

**Proof.** If \( \text{Ann}(\lambda_A(c)) \subseteq p^* \), then by using Lemma 3.4 and Proposition 4.5, one gets \( c \to \rho(0) \leq (c \to \rho(0))^\ast = (\text{Ann}(\lambda_A(c)))^\ast \leq (p^\ast)^\ast = p \). Conversely, if \( c \to \rho(0) \leq p \), then by using Proposition 4.5 we have \( \text{Ann}(\lambda_A(c)) = (c \to \rho(0))^\ast \subseteq p^\ast \).

**Proposition 8.3** If \( p \in \text{Spec}(A) \) then the following are equivalent:

1. \( p \in \text{Min}(A) \);
2. \( p^* \in \text{Min}_{Id}(L(A)) \);
3. For all \( c \in K(A) \), \( \lambda_A(p) \in p^* \) implies \( \text{Ann}(\lambda_A(p)) \not\subseteq p^* \);
4. For all \( c \in K(A) \), \( c \leq p \) if and only if \( c \to \rho(0) \not\leq p \).

**Proof.** (1) \( \Leftrightarrow \) (2) Let us consider the order-preserving map \( u : \text{Spec}(A) \to \text{Spec}_{Id}(L(A)) \) defined by \( u(p) = p^* \), for all \( p \in \text{Spec}(A) \). According to Proposition 3.7, \( u \) is an order-isomorphism, hence the conditions (1) and (2) are equivalent.

(2) \( \Leftrightarrow \) (3) By Lemma 3.3.(3), \( p^* \) is a prime ideal of the lattice \( L(A) \). Therefore, by using Lemma 8.1 it follows that the properties (2) and (3) are equivalent.

(3) \( \Leftrightarrow \) (4) By Lemmas 3.6 and 8.2.

**Corollary 8.4** If \( A \) is semiprime and \( p \in \text{Spec}(A) \) then \( p \in \text{Min}(A) \) if and only if for all \( c \in K(A) \), \( c \leq p \) implies \( c^\perp \not\leq p \).

Let us denote by \( \text{Min}_{Z}(A) \) (resp. \( \text{Min}_{F}(A) \)) the topological space obtained by restricting the topology of \( \text{Spec}_{Z}(A) \) (resp. \( \text{Spec}_{F}(A) \)) to \( \text{Min}(A) \). Similarly, for a bounded distributive lattice \( L \) we denote by \( \text{Min}_{Id,Z}(L) \) (resp. \( \text{Min}_{Id,F}(L) \)) the space obtained by restricting to \( \text{Min}_{Id}(L) \) the Stone topology (resp. the flat topology) of \( \text{Spec}_{Id}(L) \).

**Lemma 8.5** The topological spaces \( \text{Min}_{Z}(A) \) and \( \text{Min}_{Id,Z}(L) \) (resp. \( \text{Min}_{F}(A) \) and \( \text{Min}_{Id,F}(L) \)) are homeomorphic.
Corollary 8.6 \( \text{Min}_Z(A) \) is a zero-dimensional Hausdorff space and \( \text{Min}_F(A) \) is a compact \( T_1 \) space.

Proof. By [42], \( \text{Min}_{Id,Z}(L) \) is a zero-dimensional Hausdorff space and \( \text{Min}_{Id,F}(L) \) is a compact \( T_1 \) space. \( \blacksquare \)

Proposition 8.7 \([42]\) If \( L \) is a bounded distributive lattice then the following are equivalent:

1. \( \text{Min}_{Id,Z}(L) = \text{Min}_{Id,F}(L) \);
2. \( \text{Min}_{Id,Z}(L) \) is a compact space;
3. \( \text{Min}_{Id,Z}(L) \) is a Boolean space;
4. For any \( s \in L \) there exists \( y \in L \) such that \( x \land y = 0 \) and \( \text{Ann}(x \lor y) = \{0\} \).

Theorem 8.8 If \( A \) is a semiprime quantale then the following are equivalent:

1. \( \text{Min}_Z(A) = \text{Min}_F(A) \);
2. \( \text{Min}_Z(A) \) is a compact space;
3. \( \text{Min}_Z(A) \) is a Boolean space;
4. For any \( c \in K(A) \) there exists \( d \in K(A) \) such that \( cd = 0 \) and \( (c \lor d)^\perp = 0 \).

Proof.

(1) \( \Leftrightarrow \) (2) \( \Leftrightarrow \) (3) These equivalences follow by applying Lemma 8.5 and taking into account the equivalence of the assertions (i), (ii) and (iii) from Proposition 8.7.

(1) \( \Rightarrow \) (2) Assume that \( c \in K(A) \). By Lemma 8.5 we have \( \text{Min}_{Id,Z}(L(A)) = \text{Min}_{Id,F}(L(A)) \), therefore by applying Proposition 8.7 to the lattice \( L(A) \) there exists \( d \in K(A) \) such that \( \lambda_A(cd) = \lambda_A(c) \land \lambda_A(d) = 0 \) and \( \text{Ann}(\lambda_A(c \lor d)) = \text{Ann}(\lambda_A(c) \lor \lambda_A(d)) = \{0\} \). The quantale \( A \) is semiprime, hence by using Lemma 3.2,(9) and Proposition 4.5, one obtains \( cd = 0 \) and \( ((c \lor d)^\perp)^* = \{0\} \). If \( z \in K(A) \) and \( z \leq (c \lor d)^\perp \) then \( \lambda_A(z) \in ((c \lor d)^\perp)^* \), so \( \lambda_A(z) = 0 \). Since \( A \) is semiprime it follows that \( z = 0 \) (cf. Lemma 3.2,(9)). We conclude that \( (c \lor d)^\perp = 0 \).

(4) \( \Rightarrow \) (1) Assume that \( x \in L(A) \) hence \( x = \lambda_A(c) \) for some \( c \in K(A) \). Then there exists \( d \in K(A) \) such that \( cd = 0 \) and \( (c \lor d)^\perp = 0 \). Denoting \( y = \lambda_A(d) \) we obtain \( x \land y = \lambda_A(cd) = 0 \) and \( \text{Ann}(x \lor y) = \text{Ann}(\lambda_A(c \lor d)) = ((c \lor d)^\perp)^* = 0 \). By Proposition 8.7 we have \( \text{Min}_{Id,Z}(L(A)) = \text{Min}_{Id,F}(L(A)) \), hence \( \text{Min}_Z(A) = \text{Min}_F(A) \).
Recall from [1] that an \( mp \)-ring is a commutative ring \( R \) with the property that each prime ideal of \( R \) contains a unique minimal prime ideal. Let us extend this notion to quantales: a quantale \( A \) is an \( mp \)-quantale if for any \( p \in \text{Spec}(A) \) there exist a unique \( q \in \text{Min}(A) \) such that \( q \leq p \). An \( mp \)-frame is an \( mp \)-quantale which is a frame. We remark that a ring \( R \) is an \( mp \)-ring if and only if the quantale \( \text{Id}(R) \) of ideals of \( R \) is an \( mp \)-quantale.

The \( mp \)-quantales can be related to the conormal lattices, introduced by Cornish in [11] under the name of "normal lattices". According to [40], [25], a conormal lattice is a bounded distributive lattice \( L \) such that for all \( x, y \in L \) with \( x \wedge y = 0 \) there exist \( u, v \in L \) having the properties \( x \wedge u = y \wedge v = 0 \) and \( u \vee v = 1 \). In [11] Cornish obtained several characterizations of the conormal lattices.

**Proposition 8.9** [11] A bounded distributive lattice \( L \) is conormal if and only if any prime ideal of \( L \) contains a unique minimal prime ideal.

**Corollary 8.10** A coherent quantale \( A \) is an \( mp \)-quantale if and only if the reticulation \( L(A) \) is a conormal lattice.

**Proof.** Recall that the two functions \( u_A : \text{Spec}(A) \to \text{Spec}_{\text{Id}}(L(A)) \) and \( v_A : \text{Spec}_{\text{Id}}(L(A)) \to \text{Spec}(A) \) from Proposition 3.7 are order-isomorphisms (the order is the inclusion). Thus the corollary follows by using Proposition 8.9.

By [22] for any bounded distributive lattice \( L \) there exists a commutative ring \( R \) such that the lattices \( L \) and \( L(A) \) are isomorphic (see also the discussion from Section 3.13 of [25]). Thus for any coherent quantale \( A \) there exists a commutative ring \( R \) such that the lattices \( L(A) \) and \( L(R) \) are isomorphic (we shall identify these isomorphic lattices). Let us fix this ring \( R \) associated with the quantale \( A \).

In accordance with Proposition 3.7 we have the following homeomorphisms:

\[
\begin{align*}
\text{Spec}_{\text{Id}}(A) & \xrightarrow{u_A} \text{Spec}_{\text{Id}}(L(A)) \xrightarrow{v_A} \text{Spec}(A) \\
\text{Spec}(R) & \xrightarrow{u_R} \text{Spec}_{\text{Id}}(L(A)) \xrightarrow{v_A} \text{Spec}(A)
\end{align*}
\]

By restricting these four maps to minimal prime spectra one gets the following homeomorphisms:

\[
\begin{align*}
\text{Min}_{\text{Id}}(A) & \xrightarrow{u_A} \text{Min}_{\text{Id}}(L(A)) \xrightarrow{v_A} \text{Min}(R) \\
\text{Min}(R) & \xrightarrow{u_R} \text{Min}_{\text{Id}}(L(A)) \xrightarrow{v_A} \text{Spec}(A)
\end{align*}
\]

(we denote the restrictions by the same symbols).

**Remark 8.11** Taking into account Proposition 5.5 it is easy to prove that the following maps: \( \text{Spec}_{F}(A) \xrightarrow{u_A} \text{Spec}_{\text{Id}}(L(A)) \), \( \text{Spec}_{\text{Id}}(L(A)) \xrightarrow{v_A} \text{Spec}_{F}(A) \), \( \text{Spec}_{F}(R) \xrightarrow{u_R} \text{Spec}_{\text{Id}}(L(A)) \) and \( \text{Spec}_{\text{Id}}(L(A)) \xrightarrow{v_R} \text{Spec}_{F}(R) \) are homeomorphisms.

**Lemma 8.12** The coherent quantale \( A \) is an \( mp \)-quantale if and only if \( R \) is an \( mp \)-ring.
Proof. We apply Proposition 8.9 to the isomorphic reticulations $L(A)$ and $L(R)$ of the quantale $A$ and the ring $R$. □

Proposition 8.13 For a coherent quantale $A$ the following are equivalent:

(1) The inclusion $\text{Min}_F(A) \subseteq \text{Spec}_F(A)$ has a flat continuous retraction;

(2) The inclusion $\text{Min}_F(R) \subseteq \text{Spec}_F(R)$ has a flat continuous retraction.

Proof. According to Remark 8.11, each of these two conditions is equivalent to the following property: the inclusion $\text{Min}_{\text{Id},F}(L(A)) \subseteq \text{Spec}_{\text{Id},F}(L(A))$ has a flat continuous retraction.

Theorem 8.14 If $A$ is a coherent quantale then the following are equivalent:

(1) $A$ is an mp - quantale;

(2) For any distinct elements $p, q \in \text{Min}(A)$ we have $p \vee q = 1$;

(3) $R(A)$ is an mp - frame;

(4) $[\rho(0)]_A$ is an mp - quantale;

(5) The inclusion $\text{Min}_F(A) \subseteq \text{Spec}_F(A)$ has a flat continuous retraction;

(6) $\text{Spec}_F(A)$ is a normal space;

(7) If $p \in \text{Min}(A)$ then $V(p)$ is a closed subset of $\text{Spec}_F(A)$.

Proof.

(1) $\Rightarrow$ (2) Suppose that $p, q$ are distinct elements of $\text{Min}(A)$. If $p \vee q < m$ then $p \vee q \leq m$ for some $m \in \text{Max}(A)$. Then there exist two distinct $p, q \in \text{Min}(A)$ such that $p \leq m$ and $q \leq m$, contradicting that $A$ is an mp - quantale. It follows that $p \vee q = 1$.

(2) $\Rightarrow$ (1) Assume by absurdum that there exist $p \in \text{Spec}(A)$ and two distinct $q, r \in \text{Min}(A)$ such that $q \leq p$ and $r \leq p$. Thus $q \vee r = 1$, hence $p = 1$, contradicting that $p \in \text{Spec}(A)$.

(1) $\Leftrightarrow$ (3) By Lemma 6 of [9] we have $\text{Spec}(A) = \text{Spec}(R(A))$, hence $\text{Min}(A) = \text{Min}(R(A))$, so the equivalence of (1) and (3) is immediate.

(1) $\Leftrightarrow$ (4) This equivalence follows from $\text{Spec}(A) = \text{Spec}([\rho(0)]_A)$ and $\text{Min}(A) = \text{Min}([\rho(0)]_A)$.

(1) $\Leftrightarrow$ (5) By using Lemma 8.12, Proposition 8.13 and the equivalence of the conditions (i), (v) from Theorem 6.2 of [11] it results that the properties (1) and (5) are equivalent.

(1) $\Leftrightarrow$ (6) By Remark 8.11, Lemma 8.12 and Theorem 6.2 of [11], it follows that $A$ is an mp - quantale iff $R$ is an mp - ring iff $\text{Spec}_F(R)$ is normal iff $\text{Spec}_F(A)$ is normal.

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(2) \( \Rightarrow \) (7) Assume that \( p \in \text{Min}(A) \) and \( q \in D(p) \). Consider an element \( r \in \text{Min}(A) \) such that \( r \leq q \). Since \( p \nleq q \) we have \( p \neq r \), hence \( p \lor r = 1 \) by the hypothesis (2), so there exist \( c, d \in K(A) \) such that \( c \leq p \), \( d \leq r \) and \( c \lor d = 1 \). Thus \( V(c) \cap V(d) = V(c \lor d) = V(1) = \emptyset \), so \( V(d) \subseteq D(c) \subseteq D(p) \). From \( d \leq r \leq q \) one gets \( q \in V(d) \). Since \( q \in V(d) \subseteq D(p) \) and \( V(d) \) is a basic open set of \( \text{Spec}_F(A) \) it follows that \( D(p) \) is open in \( \text{Spec}_F(A) \). We conclude that \( V(p) \) is closed in \( \text{Spec}_F(A) \).

(7) \( \Rightarrow \) (2) Assume by absurdum that there exist two distinct \( p, q \in \text{Min}(A) \) such that \( p \lor q < 1 \), so \( p \lor q \leq m \) for some \( m \in \text{Max}(A) \). Therefore \( m \in V(p) \) and \( m \in V(q) \), hence \( V(p) \cap V(q) \neq \emptyset \). Since \( V(q) \) is an open neighborhood of \( p \) in \( \text{Spec}_F(A) \) and \( V(p) \) is flat closed, one gets \( q \in V(p) \). Thus \( q \leq p \) so \( q = p \) because \( p \) and \( q \) are minimal \( m \)-prime elements. This contradiction shows that for all distinct minimal \( m \)-prime elements \( p, q \) we have \( p \lor q = 1 \).

\[ \Box \]

**Remark 8.15** If \( R \) is a commutative ring and \( A \) is the quantale \( \text{Id}(R) \) of ideals of \( R \), then applying the previous result one obtains some of the characterizations of \( \text{mp} \)-rings, contained in Theorem 6.2 of [1].

Recall from [3] that a commutative ring \( R \) is said to be an \( \text{PF} \)-ring if the annihilator of each element of \( R \) is a pure ideal. We shall generalize this notion to quantales. Then a quantale \( A \) is a \( \text{PF} \)-quantale if for each \( c \in K(A) \), \( c^\perp \) is a pure element. For any commutative ring \( R \), \( \text{Id}(R) \) is a \( \text{PF} \)-quantale if and only if \( R \) is a \( \text{PF} \)-ring.

**Proposition 8.16** [3] If \( A \) is a bounded distributive lattice \( L \) then the following are equivalent

1. \( L \) is conormal;
2. For all \( x \in L \), \( \text{Ann}(x) \) is a \( \sigma \)-ideal;
3. Any minimal prime ideal of \( L \) is a \( \sigma \)-ideal.

In other words, a bounded distributive lattice \( L \) is conormal if and only if \( \text{Id}(L) \) is a \( \text{PF} \)-frame.

In what follows we shall establish a relationship between \( \text{PF} \)-quantales and \( \text{mp} \)-quantales. We fix a coherent quantale \( A \).

**Lemma 8.17** Any \( \text{PF} \)-quantale \( A \) is semiprime.

**Proof.** Let \( c \) be a compact element of \( A \) such that \( c^n = 0 \) for some integer \( n \geq 1 \). Then \( c^{n-1} \leq (c^{n-1})^\perp \), hence \( (c^{n-1})^\perp = (c^{n-1})^\perp \lor (c^{n-1})^\perp = 1 \), because \( (c^{n-1})^\perp \) is pure. Thus \( c^{n-1} \leq (c^{n-1})^\perp = 0 \), so \( c^{n-1} = 0 \). By using many times this argument one gets \( c = 0 \). According to Lemma 2.4,(2) it follows that \( A \) is semiprime.

\[ \Box \]
Proposition 8.18 If $A$ is a $PF$-quantale then the reticulation $L(A)$ is a conormal lattice.

Proof. By Lemma 8.17, the $PF$-quantale $A$ is semiprime. Assume that $x \in L(A)$ so $x = \lambda_A(c)$, for some $c \in K(A)$. Applying Proposition 4.5 one obtains $Ann(x) = Ann(\lambda_A(c)) = Ann(c^*) = (c^\perp)^*$. By hypothesis, $c^\perp$ is a pure element of $A$, therefore by Lemma 4.7 it follows that $Ann(x) = (c^\perp)^*$ is a $\sigma$-ideal of the lattice $L(A)$. Applying Proposition 8.16 it follows that $L(A)$ is a conormal lattice.

Theorem 8.19 For a coherent quantale $A$ the following are equivalent:

1. $A$ is a $PF$-quantale;
2. $A$ is a semiprime $mp$-quantale.

Proof.

(1) $\Rightarrow$ (2) By Proposition 8.18 and Corollary 8.10.

(2) $\Rightarrow$ (1) In order to prove that $A$ is a $PF$-quantale let us assume that $c \in K(A)$. We shall prove that $c^\perp$ is a pure element of $A$. Let $d$ be a compact element of $A$ such that $d \leq c^\perp$, hence $\lambda_A(c) \land \lambda_A(d) = \lambda_A(ad) = \lambda_A(0) = 0$, i.e. $\lambda_A(d) \in Ann(c^*) = Ann(\lambda_A(c))$. By Corollary 8.10 $L(A)$ is a conormal lattice, hence $Ann(\lambda_A(c))$ is $\sigma$-ideal of $L(A)$ (cf. Proposition 8.16). It follows that $Ann(c^*) \lor Ann(d^*) = Ann(\lambda_A(c)) \lor Ann(\lambda_A(d)) = L(A)$.

According to Lemma 3.4, Proposition 4.5 and Corollary 3.10, the following equalities hold:

$$
\rho(\rho(c^\perp) \lor d^\perp) = \rho(((c^\perp)^* \lor ((c^\perp)^*)^*)_*) = \rho((Ann(c^*))_* \lor (Ann(d^*))_*) = (Ann(\lambda_A(c)) \lor Ann(\lambda_A(d))_*) = (L(A))_* = 1.
$$

By Lemma 2.2,(3) and (6) one gets $c^\perp \lor d^\perp = 1$, hence $c^\perp$ is pure.

Theorem 8.20 For a coherent quantale $A$ consider the following conditions:

1. Any minimal $m$-prime element of $A$ is pure;
2. $A$ is an $mp$-quantale.

Then (1) implies (2). If the quantale $A$ is semiprime then the converse implication holds.

Proof. First we shall prove that (1) implies (2). According to Corollary 8.10 it suffices to check that the reticulation $L(A)$ is a conormal lattice. Let $P$ be a minimal prime ideal of $L(A)$, hence $P = p^*$ for some $p \in Min(A)$. By taking into account the hypothesis, it results that $p$ is a pure element of $A$. In accordance to Lemma 4.7, $P = p^*$ is a $\sigma$-ideal of $L(A)$, so any minimal prime ideal of $L(A)$ is a $\sigma$-ideal. Applying Proposition 8.16 it follows that the lattice $L(A)$ is conormal.
Assume now that $A$ is a semiprime $mp$-quantale and $p \in \text{Min}(A)$, so $p^*$ is a minimal prime ideal of $L(A)$. By Corollary 8.10, $L(A)$ is a conormal lattice, thus any minimal prime ideal of $L(A)$ is a $\sigma$-ideal. Therefore $p^*$ is a $\sigma$-ideal of $L(A)$. Since $A$ is semiprime, by applying Proposition 4.7 it follows that $p = (p^*)_*$ is a pure element of $A$. ■

**Corollary 8.21** Let $A$ be a semiprime quantale. Then $A$ is a PF-quantale if and only if any minimal $m$-prime element of $A$ is pure.

**Proof.** We apply Theorems 8.19 and 8.20. ■

**Theorem 8.22** For a coherent quantale $A$ the following are equivalent:

1. $A$ is a PF-quantale;
2. $A$ is a semiprime $mp$-quantale;
3. If $c, d \in K(A)$ then $cd = 0$ implies $c^\perp \lor d^\perp = 1$;
4. If $c, d \in K(A)$ then $(cd)^\perp = c^\perp \lor d^\perp$;
5. For each $c \in K(A)$, $c^\perp$ is a pure element.

**Proof.** The equivalence of (1) and (2) follows from Theorem 8.19 and that the conditions (3) and (5) are equivalent is obvious.

(2) $\Rightarrow$ (3) Assume by absurdum that there exist $c, d \in K(A)$ such that $cd = 0$ and $c^\perp \lor d^\perp < 1$, hence there exists a minimal $m$-prime element $p$ such that $p \leq m$. Since $cd = 0$ and $p \in \text{Spec}(A)$ we have $c \leq p$ or $d \leq p$. Assume that $c \leq p$, so $p \lor c^\perp = 1$ (by Theorem 8.20, the minimal $m$-prime element $p$ is pure). This contradicts $p \lor c^\perp \leq m$, so the implication is proven.

(3) $\Rightarrow$ (2) First we prove that $A$ is semiprime. Let $c \in K(A)$ such that $c^n = 0$ for some integer $n \geq 1$. Assuming $n > 1$, from $c^{n-1}c = 0$ one gets $(c^{n-1})^\perp = c^\perp \lor (c^{n-1})^\perp = 1$, hence $c^{n-1} = 0$. By using many times this argument one obtains $c = 0$, so $A$ is semiprime.

Now we shall prove that $A$ is an $mc$-quantale. According to Theorem 8.14 it suffices to show that for any distinct minimal $m$-prime elements $p, q$ we have $p \lor q = 1$. Assume that $p, q \in \text{Min}(A)$, $p \neq q$, so there exists $d \in K(A)$ with $d \leq p$ and $d \not\leq q$. By Corollary 8.4 we have $d^\perp \not\leq p$, so there exists $c \in K(A)$ such that $c \leq d^\perp$ and $c \not\leq p$. Thus $cd = 0$ implies $c^\perp \lor d^\perp = 1$, so there exist $u, v \in K(A)$ such that $u \leq c^\perp$, $v \leq d^\perp$ and $u \lor v = 1$. Since $p$ is $m$-prime, from $uc = 0$ and $c \not\leq p$ it results that $u \leq p$. Similarly, one obtains $v \leq q$, so in the both cases we have $p \lor q = 1$.

(3) $\Rightarrow$ (4) In order to prove that $(cd)^\perp \leq c^\perp \lor d^\perp$ let us consider a compact element $e$ of $A$ such that $e \leq (cd)^\perp$. Then $ecd = 0$, so by hypothesis one gets $(ec)^\perp \lor (d)^\perp = 1$, therefore there exist $u, v \in K(A)$ such that $u \leq (ec)^\perp$, $v \leq (d)^\perp$.
and \( u \lor v = 1 \). It follows that \( eu \leq c^\perp \), \( ev \leq d^\perp \) and \( e = e(u \lor v) = eu \lor ev \), hence \( e \leq c^\perp \lor d^\perp \). Then the inequality \( (cd)^\perp \leq c^\perp \lor d^\perp \) was proven. The converse inequality is clear, so \( (cd)^\perp = c^\perp \lor d^\perp \).

(4) \( \Rightarrow \) (3) It is easy to see that this implication holds.

\[\square\]

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