General Non-Static Spherically Symmetric Solutions of Einstein Vacuum Field Equations with $\Lambda$

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ABSTRACT

1– It is shown that the upper bound for $\alpha$ in the general solutions of spherically symmetric vacuum field equations (gr-qc/9812081, $\Lambda = 0$) is nearly $10^3$. This has been obtained by comparing the theoretical prediction for bending of light and precession of perihelia with observation. For a significant range of possible values of $\alpha (\alpha > 2)$ the metric is free of coordinate singularity. 2– It is checked that the singularity in the non-static spherically symmetric solution of Einstein field equations with $\Lambda$ (JHEP 04 (1999) 011, $\alpha = 0$) at the origin is intrinsic. 3– Using the techniques of these two works, a general class of non-static solutions is presented. They are smooth and finite everywhere and have an extension larger than the Schwarzschild metric. 4– The geodesic equations of a freely falling material particle for the general case are solved which reveal a Schwarzschild-de Sitter type potential field.

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I. INTRODUCTION

According to Birkhoff’s theorem the metric for a vacuum spherically symmetric gravitational field with $\Lambda = 0$, has a unique Schwarzschild form

$$ds^2 = (1 - \frac{2M}{r})dt^2 - (1 - \frac{2M}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$  

(1)

and with $\Lambda \neq 0$ it has a unique Schwarzschild-de Sitter form

$$ds^2 = (1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2)dt^2 - (1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2)^{-1}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$  

(2)

Eq. (1) has a coordinate type singularity at $r = 2M$ and an intrinsic singularity at $r = 0$ while Eq. (2) has two coordinate singularities at $r \approx 2M$ and $r \approx (\frac{3}{\Lambda})^{\frac{1}{2}}$, and an intrinsic singularity at $r = 0$. The intrinsic singularity is irremovable and this is indicated by diverging the Riemann tensor scalar invariant

$$R^a_{\ bcd}R_a^{bcd} = \frac{48M^2}{r^6}$$  

(3)

For case $\Lambda = 0$, a general class of solutions has been obtained which has the form

$$ds^2 = (1 - \frac{2M}{r + \alpha M})dt^2 - (1 - \frac{2M}{r + \alpha M})^{-1}dr^2 - (r + \alpha M)^2(d\theta^2 + \sin^2 \theta d\phi^2)$$  

(4)

where $\alpha$ is an arbitrary constant. In Sec.II we primarily discuss the apparent objection that they are subspaces of the Schwarzschild metric. Though it seems only a linear change of variables, but $\alpha$ does affect the curvature of space-time. Then it is shown that staticness is not a necessary initial condition if $g_{\theta\theta}$ is independent of time. Subsequently by deriving the equation of precession of perihelia and bending of light in a gravitational field and comparing them with observational measurements, we conclude that $\alpha$ may take small as well as large values up to $10^3$. Even though this may solve the singularity problem, but recent observations of type Ia supernovae indicate the existence of a positive cosmological constant. From other side, it has been shown that in the presence of cosmological constant, using static reference coordinate, is not suitable.

For case $\Lambda \neq 0$, a non-static solution for this system has been proposed which has the following form

$$ds^2 = (\sqrt{\left(1 - \frac{2M}{\rho} - \frac{\Lambda}{3}\rho^2\right)^2 + \frac{4\Lambda}{3}\rho^2} + (1 - \frac{2M}{\rho} - \frac{\Lambda}{3}\rho^2)\frac{2}{2})dt^2$$

$$- e^{\frac{2}{\sqrt{\frac{\Lambda}{3}}}} e^{-\frac{1}{2}\sqrt{\frac{\Lambda}{3}t}} \left[\left(\sqrt{\left(1 - \frac{2M}{\rho} - \frac{\Lambda}{3}\rho^2\right)^2 + \frac{4\Lambda}{3}\rho^2} + (1 - \frac{2M}{\rho} - \frac{\Lambda}{3}\rho^2)\right)\frac{2}{2}ight]$$

$$+ r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$  

(5)

where $\rho = re^{\sqrt{\frac{\Lambda}{3}t}}$. The result which evidently is free of any singularity for $r \neq 0$, is singular at $r = 0$. In Sec.III we show that it is indeed an intrinsic singularity. In Sec.IV, by making
use of the presented techniques in [25], a general class of non-static solutions will be obtained which has the form

\[
    ds^2 = \left(\frac{\sqrt{(1 - \frac{2M}{\rho} - \frac{\Lambda}{3}\rho^2)^2 + 4\frac{\Lambda}{3}\rho^2} + (1 - \frac{2M}{\rho} - \frac{\Lambda}{3}\rho^2)} + (1 - \frac{2M}{\rho} - \frac{\Lambda}{3}\rho^2)^2}{2}\right)dt^2
    - e^{2\sqrt{\frac{\Lambda}{3}}t} \left[\left(\frac{\sqrt{(1 - \frac{2M}{\rho} - \frac{\Lambda}{3}\rho^2)^2 + 4\frac{\Lambda}{3}\rho^2} + (1 - \frac{2M}{\rho} - \frac{\Lambda}{3}\rho^2)} + (1 - \frac{2M}{\rho} - \frac{\Lambda}{3}\rho^2)^2}{2}\right)^{-1}dr^2
    + (r + \alpha M)^2(d\theta^2 + \sin^2\theta d\phi^2)\right]
\]

(6)

where here \(\rho = (r + \alpha M)e^{\sqrt{\frac{\Lambda}{3}}t}\). These solutions are smooth and finite everywhere even at \(r = 0\). We show that they have an extension larger than the Schwarzschild metric. Obviously they should be checked for completeness before we may call them non-singular. Sec.V deals with solving the geodesic equations for a freely falling material particle in the general case and results a potential field which though is very large at \(r \approx 0\) but it is finite. Finally, derivations of Eq.(22), Eq.(26) and Eq.(29) which being used in Sec.II, are presented in Appendixes A and B.

II. CASE \(\Lambda = 0 \ , \alpha \neq 0\)

Since Eq. (11) transforms to Eq. (1) by simply replacing \(r' = r + \alpha M\) with the range of \(r' \geq \alpha M\), this may cause a confusion that (11) is a subspace of (1). The proof of completeness usually for a pseudo-Riemannian manifold is not an easy task. We will show the flaw in this argument by a Riemann counter-example. Taking \(R^2\) as a two-space of all points with coordinate \(r, \theta\) such that the metric is

\[
    ds^2 = dr^2 + r^2 d\theta^2
\]

(7)

where \(\theta = 0\) is identified with \(\theta = 2\pi\) and the point \(r = 0\) is included. This plane is complete and non-singular. Also \(R^2\) is a two-space of all points with coordinate \(r, \theta\) such that the metric is

\[
    ds^2 = dr^2 + (r + a)^2 d\theta^2
\]

(8)

where the range of \(r\) and \(\theta\) is the same as \(R^2\). If we transform \(r' = r + a\) Eq. (8) gives

\[
    ds^2 = dr'^2 + r'^2 d\theta^2 , \quad \text{and} \quad a \leq r'
\]

(9)

which apparently this means \(R'^2 \subset R^2\). We show that indeed this is not valid. Let’s consider a subspace of \(R^2\) and \(R'^2\) by restricting \(r < \rho\). The surface area of \(R^2(r < \rho)\) is \(\pi \rho^2\) while the surface area of \(R'^2(r < \rho)\) is \(\pi(\rho^2 + 2a\rho)\). This means for finite \(\rho\) we always have \(R^2(r < \rho) \subset R'^2(r < \rho)\). If we take the limit \(\rho \to \infty\) then we get \(R^2(r < \rho) \to R^2\) and \(R'^2(r < \rho) \to R'^2\). Since \(R^2\) is complete and from other side \(R^2(r < \rho) \subset R'^2(r < \rho)\), there is no way except to conclude that \(R^2 = R'^2\). This counter-example shows that how the conclusion that the general solutions are subspaces of the Schwarzschild metric may be
impulsory. Indeed the Schwarzschild solution Eq. (1), is a special case of the general solutions Eq. (4) for the case \( \alpha = 0 \). We must notice that both the Schwarzschild and the general solutions are in the same Schwarzschild coordinate, which manifestly have different forms. Any transformation to the new radial coordinate \( r' = r + \alpha M \) requires the Schwarzschild metric be written in this new coordinate too, which means we should replace \( r \) by \( r' - \alpha M \). Thus in this new coordinate they will have different forms too. As we know the Schwarzschild metric is an exterior solution which is only valid for \( r > 2M \) and incomplete. This is true for the general solutions but \( r > (2 - \alpha)M \) gives an extension larger than the Schwarzschild solution and even with \( \alpha > 2 \) the range of validity is \( r > 0 \). In the following we try to show how the spaces of general solutions for \( \alpha > 2 \) are complete.

A manifold endowed with an affine or metric geometry is said to be geodesically complete if all geodesics emanating from any point can be extended to infinite values of the affine parameters in both directions. In the case of a manifold with a positive definite metric it can be shown that geodesic completeness and metric completeness are equivalent. Focusing on \( \alpha > 2 \) which is the most likely values of it, the line element Eq. (4) is a vacuum solution abstracted away from any source for all values of \( r \). We also realize that

\[
g^{tt} = \left(1 - \frac{2M}{r + \alpha M}\right)^{-1}
\]

is always positive for all \( r > 0 \), so "\( t \)" is the invariant world time. The hypersurfaces \( t = \text{const.} \) are spacelike with a positive definite metric

\[
d\sigma^2 = \left(1 - \frac{2M}{r + \alpha M}\right)^{-1}dr^2 + (r + \alpha M)^2(d\theta^2 + \sin^2 \theta d\phi^2)
\]

which is a distance function. Since every Cauchy sequence with respect to this distance function converges to a point in the manifold, it is metrically complete.

Next we show that staticness is not a necessary initial condition if we merely take \( g_{\theta\theta} \) independent of time. Let’s assume that the line element has the following desired form

\[
ds^2 = B(r, t)\,dt^2 - A(r, t)dr^2 - D(r)(d\theta^2 + \sin^2 \theta d\phi^2)
\]

(11)

Taking \( ds^2 = -g_{\mu\nu}dx^\mu dx^\nu \), the nonvanishing components of the metric connection are

\[
\Gamma^t_{tt} = \frac{\dot{B}}{2B}, \quad \Gamma^t_{tr} = \frac{B'}{2B}, \quad \Gamma^t_{rr} = \frac{\dot{A}}{2B}, \quad \Gamma^r_{tt} = \frac{B'}{2A}, \quad \Gamma^r_{tr} = \frac{A'}{2A}, \quad \Gamma^r_{rr} = \frac{D'}{2A}, \quad \Gamma^\theta_{\phi\phi} = \frac{D'}{2A}\sin^2 \theta, \quad \Gamma^\theta_{r\theta} = \frac{D'}{2D}, \quad \Gamma^\phi_{\phi\phi} = -\sin \theta \cos \theta, \quad \Gamma^\phi_{r\phi} = \frac{D'}{2D}, \quad \Gamma_{r\phi} = \cot \theta.
\]

(12)

where (‘) and (˙) denote derivatives with respect to \( r \) and \( t \) respectively. The nonvanishing components of the Ricci tensor are
The Einstein field equations with $\Lambda = 0$ for vacuum give $\mathcal{R}_{\mu\nu} = 0$, so from setting $\mathcal{R}_{tr} = 0$, we have

$$\dot{A} = 0$$  

and with a reparametrization in the line element we may also have

$$\dot{B} = 0$$  

This means that with the choice Eq. (11) as line element, the solution is necessarily static.

By considering the bending of light, we try to find an upper bound for $\alpha$. The deflection of light in a gravitational field predicted by the general theory of relativity witnessed experimental verification in 1919. Since then, there has been much studies on the gravitational deflection of light by the Sun and gravitational lensing (GL). Under the great vision of Zwicky [7], observation of a QSO showed the first example of the GL phenomena [8], and thereafter it has become the most important tool for probing the universe. It is believed that GL can give valuable important information on important questions, such as masses of galaxies and clusters of galaxies, the existence of massive exotic objects, determination of cosmological parameters and can be also used to test the alternative theories of gravitation [9]. The gravitational deflection of light has now been measured more accurately at radio wavelengths with using Very-Long-Baseline Interferometry (VLBI), than at visible wavelengths with available optical techniques.

Appealing to the spherically symmetric nature of the metric, we consider the geodesics, without lose of generality, on the equatorial plane ($\theta = \frac{\pi}{2}$). Following Weinberg [10], we get the equation for the photon trajectories as

$$\phi(r) - \phi_{\infty} = \int_{r}^{\infty} \left[ \frac{A(r)}{D(r)} \right]^{1/2} \left[ \frac{D(r) B(r) - 1}{D(r) B(r) - 1} \right]^{-1/2} dr$$  

The Einstein deflection angle is given by

$$\Delta \phi = 2|\phi(r_o) - \phi_{\infty}| - \pi$$  

In Eq. (4) we have

$$B = \left(1 - \frac{2M}{r + \alpha M}\right), \quad A = \left(1 - \frac{2M}{r + \alpha M}\right)^{-1}, \quad D = (r + \alpha M)^2$$  

$$\mathcal{R}_{tt} = -\frac{B''}{2A} + \frac{B'}{4A} \left( \frac{A'}{A} + \frac{B'}{B} \right) - \frac{B'D'}{2AD} + \frac{\ddot{A}}{2A} - \frac{\dot{A}}{4A} \left( \frac{\ddot{A}}{A} + \frac{\dot{B}}{B} \right)$$  

$$\mathcal{R}_{tr} = -\frac{AD'}{2AD}$$  

$$\mathcal{R}_{rr} = \frac{B''}{2B} + \frac{D''}{D} - \frac{D'}{2D} \left( \frac{D'}{D} + \frac{A'}{A} \right) - \frac{B'}{4B} \left( \frac{A'}{A} + \frac{B'}{B} \right) - \frac{\ddot{A}}{2B} + \frac{\dot{A}}{4B} \left( \frac{\ddot{A}}{A} + \frac{\dot{B}}{B} \right)$$  

$$\mathcal{R}_{\theta\theta} = -1 + \frac{D'}{4A} \left( \frac{B'}{B} - \frac{A'}{A} \right) + \frac{D''}{2A}$$  

$$\mathcal{R}_{\phi\phi} = \sin^2 \theta \mathcal{R}_{\theta\theta}$$
which make the above integral well defined for \( r_o > (3 - \alpha)M \). Since the radial coordinate is always non-negative, \( \alpha \) merely take positive values. Different values of \( r_o \) and \( \alpha \), yield different expansions for \( B \) and \( D \), so we get different expressions for deflection angle. Our investigation is on very small and sufficiently large values of \( \alpha \), since the rest of the interval brings no more to us. We give the details of calculating in Appendix A and here use the results. For \( \alpha < 1 \), the Einstein deflection angle (up to the second order) is as Eq. (A10)

\[
\Delta \phi = 4x + 4x^2 \left[ \frac{15\pi}{16} - (1 + \alpha) \right] + \cdots
\] (22)

where

\[
x = \frac{MG}{r_o}
\]

and the restriction imposed by the integral singularity is

\[
0 < x < \frac{1}{3 - \alpha}
\]

thus the closest approach is \( r_o \approx (3 - \alpha)MG \). Switching off \( \alpha \) in this solution one recovers the well-known Schwarzschild solution, which has extensively examined by many authors (see [11] and references therein). However small values is qualitatively similar (but quantitatively different) to the Schwarzschild case. Further calculation shows that Eq. (22) is also valid for \( \alpha > 1 \) (see Eq. (A23)), but it may not contain the closest approach. Therefore, for the weak field limit (when \( r_o \) is much larger than \( MG \)) and all possible values of \( \alpha \), we rewrite the equation as

\[
\Delta \phi = \Delta \phi_{fo} \left[ 1 + \frac{MG}{r_o} \left( \frac{15\pi}{16} - (1 + \alpha) \right) \right] + \cdots
\] (23)

where

\[
\Delta \phi_{fo} = \frac{4MG}{r_o}
\]

is the first order deflection angle. The results of VLBI observations of extragalactic radio sources, show radio-wave deflection by the Sun [12] as

\[
\Delta \phi \approx \Delta \phi_{fo}(0.9998 \pm 0.0008)
\] (24)

Since the order of magnitude of \( MG/r_o \) for the Sun is \( 10^{-6} \), Eq. (23) and Eq. (24) make an upper bound for \( \alpha \) as

\[
0 < \alpha < 10^3
\] (25)

For sufficiently large values of \( \alpha \) (actually \( \alpha > 3 \)) we may obtain another expression for deflection angle (up to the second order) in the following form (see Eq. (A23))

\[
\Delta \phi = 8y + 4y^2 \left[ \frac{15\pi}{16} - 5(\alpha - 1) \right] + \cdots
\] (26)
where
\[ y = \frac{r_o}{\alpha(\alpha - 2)MG} \]
and the range of validity is
\[ 0 < y < \frac{1}{\alpha} \]
thus the closest approach is \( r_o \approx 0 \). Eq. (22) and Eq. (26) which contain the closest approach, can be used for testing the general theory of relativity in a strong gravitational field. Although, no test for the theory is known in this region, but there is an open room for such investigations. Several possible observational candidates have been proposed to test the Einstein theory of relativity in the vicinity of a compact massive object. One of the current topics is the study of point source lensing in the strong gravitational field region when the deflection angle can be very large [13]. Our calculation confirms that deflection angle may take any small as well as large values depending on \( \alpha \) and it would provide a good key for the gravitational lensing studies.

Consequently bending of light phenomena, both in the weak field and strong field limits, restrict all non-negative values of \( \alpha \) as Eq. (25) up to \( 10^3 \).

Now we would like to use observational data of the precession of perihelia measurements, in order to find a better bound for \( \alpha \). Following Weinberg [14], for a test particle moves on a timelike geodesic in the \( \theta = \pi/2 \) plane, the angle swept is given by
\[
\phi(r) - \phi(r_-) = \int_{r_-}^r dr \left[ \frac{A^{1/2}(r)}{D(r)} \right] \left[ \frac{D_- (B^{-1}(r) - B^{-1}_-) - D_+ (B^{-1}(r) - B^{-1}_+)}{D_+ D_-(B^{-1}_+ - B^{-1}_-)} - \frac{1}{D(r)} \right]^{-1/2}
\]
where
\[
D_\pm = D(r_\pm) \quad \& \quad B_\pm = B(r_\pm)
\]
The orbit precesses in each revolution by an angle
\[
\Delta \phi = 2|\phi(r_+) - \phi(r_-)| - 2\pi
\]
By using the metric components presented in Eq. (21), we have gotten the expression for the precession per revolution (up to the second order) which the details are given in Appendix B. So we may write Eq. (B14) in here as
\[
\Delta \phi = \Delta \phi_{fo} \left[ 1 + \frac{MG}{L} \left( \frac{19}{6} + \frac{e^2}{4} - 2\alpha(1 + e^2) \right) \right] + \cdots
\]
where “\( L \)” and “\( e \)” are the semilatus rectum and eccentricity respectively, and \( \Delta \phi_{fo} \) is the well-known first order approximation
\[
\Delta \phi_{fo} = \frac{6\pi MG}{L}
\]
Fortunately by development of Long-Baseline radio Interferometry and analysis of Radar Ranging Data, there are accurate measurements of precession which typically show

\[ \Delta \phi \approx \Delta \phi_f (1.003 \pm 0.005) \]  

(30)

Thus matching the theory with observation, with using typical values of \( \frac{M G}{L} \) and \( e \), we get an upper bound for \( \alpha \) as

\[ 0 < \alpha < 10^5 \]  

(31)

We conclude that measurements of these two tests of the general theory of relativity in the weak field limit according to Eq. (25) and Eq. (31), restrict the allowed values of \( \alpha \) to \( 10^3 \). Though the observation of the GL phenomena is a difficult task, it would support our presented metric components role in a strong gravitational field and would also give an accurate bound for \( \alpha \).

By calculating the Riemann tensor scalar invariant, we receive useful information about the existence of singularities. For the line element Eq. (4) in which \( \alpha \neq 0 \), it is equal to

\[ R_{abcd} R^{abcd} = \frac{48 M^2}{D^3} = \frac{48 M^2}{(r + \alpha M)^6} \]  

(32)

As it is evident, the presence of \( \alpha \) makes the scalar finite in the whole range of \( r \), meaning that the solutions are free of any intrinsic singularity. Meanwhile there may be a coordinate singularity at \( r = (2 - \alpha) M \) according to the restriction imposed by Eq. (25).

We would like to mention two points about this work. One is that, sometimes in literatures for discussing this problem, coordinate \( r \) is defined so that the area of the surfaces \( r = \text{const.} \) to be \( 4 \pi r^2 \). This generally is not the case, since before fixing the metric there is no possibility of speaking the distance, then in the same way, there is no possibility of speaking the area. We take \( r = (x^2 + y^2 + z^2)^{1/2} \) where \( (x, y, z) \) are usual cartesian space coordinates.

The other point which should be taken with caution is, while at first \( r \) was taken as a space coordinate with the range \( (0, \infty) \) and the particle was located at \( r = 0 \), at end we come to the conclusion that \( r \) is merely a space coordinate in the interval \( ((2 - \alpha) M, \infty) \). For the rest of the interval \( (0, (2 - \alpha) M) \), it is standing as a time coordinate. This contradiction or at least ambiguity raises the question that While the location of the particle is not well-defined, how may we speak of the value of \( D \) at this point? This ambiguity should be found a satisfactory explanation.

III. CASE \( \Lambda \neq 0, \alpha = 0 \)

Recently for vacuum spherically symmetric space, a non-static solution of Einstein field equations with cosmological constant in a proper FRW type coordinate system has been considered. The result shows a singularity at the origin which the intrinsic nature of
it may be checked by calculating the invariant Gaussian curvature or the Riemann tensor scalar invariant. In this section we choose tensor analysis to find the answer. So we start with the following form of the line element

\[ ds^2 = B(r, t)dt^2 - R^2(t) \left[ A(r, t)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right] \]  

(33)

where \( R(t) \) is the scale factor and satisfies the Friedmann equation

\[ \frac{\dot{R}^2}{R^2} = \frac{\ddot{R}}{R} = \frac{\Lambda}{3} \]  

(34)

and \( \Lambda \) is the cosmological constant. The cosmological time \( t \), which determines the evolution of the universe is measured by the proper time of clocks fixed on the geodesically moving galaxies comprising the universe, synchronized such that the time \( t = 0 \) corresponds to the beginning of the universe at the big bang.

According to Ref. [5] the nonvanishing components of the metric connection are

\[
\begin{align*}
\Gamma^t_{tt} &= \frac{\dot{B}}{B} , \\
\Gamma^t_{tr} &= \frac{B'}{2B} , \\
\Gamma^t_{rr} &= \frac{\dot{R}RA}{B} + \frac{R^2\dot{A}}{2B} , \\
\Gamma^r_{\theta\theta} &= \frac{\dot{R}r}{A} , \\
\Gamma^r_{\phi\phi} &= \frac{\dot{R}r^2}{A} \sin^2 \theta , \\
\Gamma^\theta_{t\theta} &= \frac{\dot{R}}{R} , \\
\Gamma^\phi_{t\phi} &= \frac{\dot{R}}{R} , \\
\Gamma^\theta_{r\theta} &= \frac{1}{r} , \\
\Gamma^\phi_{r\phi} &= \frac{1}{r} , \\
\Gamma^\theta_{\phi\phi} &= \cot \theta .
\end{align*}
\]  

(35)

The nonvanishing components of the curvature tensor then become

\[
\begin{align*}
\mathcal{R}^t_{trr} &= \frac{AR^2}{B} + \frac{\ddot{A}}{2B} - \frac{B''}{2B} + \frac{3\dot{A}R}{B} , \\
\mathcal{R}^t_{ttr} &= \frac{B'}{2AR^2} - \frac{\dot{A}}{2A} - \frac{3\dot{A}R}{2AR} - \frac{(\dot{R})^2}{R^2} , \\
\mathcal{R}^t_{\theta\theta} &= \frac{A'r}{2A^2} + \frac{r^2\dot{R}}{2A} \left( \frac{\dot{A}}{2A} + \frac{\dot{R}}{R} \right) , \\
\mathcal{R}^t_{\phi\phi} &= \mathcal{R}^t_{\theta\theta} \sin^2 \theta , \\
\mathcal{R}^\theta_{t\theta} &= \frac{\dot{B}}{2rAR} + \frac{\dot{B}}{2B} - \frac{\dot{R}}{R} , \\
\mathcal{R}^\phi_{r\phi} &= \mathcal{R}^\theta_{\theta\theta} \sin^2 \theta , \\
\mathcal{R}^\theta_{r\theta} &= \frac{\dot{A}'}{2rA} + \frac{\dot{A}}{R} \left( \frac{AR^3}{2B} + \frac{R^2\dot{A}}{2B} \right) , \\
\mathcal{R}^\phi_{\theta\phi} &= 1 - \frac{1}{A} + \frac{r^2\dot{R}^2}{B} .
\end{align*}
\]  

(36)

Since the metric components have no cross terms and furthermore the curvature tensors have some algebraic properties, the Riemann tensor scalar invariant takes a simpler form as

\[
\mathcal{R}^a_{\ bcd} \mathcal{R}_a^{\ bcd} = 2 \sum_{\mu \neq \nu} (g^{\mu\nu} \mathcal{R}^\nu_{\ \mu\nu})^2
\]  

(37)
Using Eq. (36) gives

\[
\mathcal{R}^a_{\ bcd}\mathcal{R}_a^{\ bcd} = 4\left(\frac{B''}{2ABR^2} - \frac{\ddot{A}}{2AB} - \frac{3\dot{A}\dot{R}}{2ABR} - \frac{\ddot{R}^2}{BR^2}\right)^2 + 16\left(\frac{A'}{2rA^2R^2} + \frac{\dot{A}\dot{R}}{2ABR} + \frac{\ddot{R}^2}{BR^2}\right)^2 + 4\left(\frac{1}{r^2R^2} - \frac{1}{r^2AR^2} + \frac{\ddot{R}^2}{BR^2}\right)^2 \tag{38}
\]

The \(rr\) component of the field equation gives

\[
\mathcal{R}_{rr} + \Lambda g_{rr} = 0
\]

and makes the equal terms for the first parentheses as

\[
\frac{B''}{2ABR^2} - \frac{\ddot{A}}{2AB} - \frac{3\dot{A}\dot{R}}{2ABR} - \frac{\ddot{R}^2}{BR^2} = -\left(\frac{1}{r^2R^2} - \frac{1}{r^2AR^2} + \frac{\ddot{R}^2}{BR^2}\right) \tag{39}
\]

The \(\theta\theta\) component gives

\[
\mathcal{R}_{\theta\theta} + \Lambda g_{\theta\theta} = 0
\]

which this leads to

\[
\frac{A'}{2rA^2R^2} + \frac{\dot{A}\dot{R}}{2ABR} + \frac{\ddot{R}^2}{BR^2} = -\frac{1}{2}\left(\frac{1}{r^2R^2} - \frac{1}{r^2AR^2} + \frac{\ddot{R}^2}{BR^2} - \frac{3\ddot{R}^2}{R^2}\right) \tag{40}
\]

By solving field equations we also have [3]

\[
1 - \frac{1}{A} + \frac{\Lambda}{3}r^2R^2(A - 1) = \frac{2M}{r\dot{R}} \tag{41}
\]

or

\[
B = A^{-1} = \frac{1}{2}\left[\sqrt{(1 - \frac{2M}{r\dot{R}} - \frac{\Lambda}{3}r^2R^2)^2 + \frac{4\Lambda}{3}r^2R^2} + (1 - \frac{2M}{r\dot{R}} - \frac{\Lambda}{3}r^2R^2)\right] \tag{42}
\]

Inserting (39), (40) and (41) in (38), gives

\[
\mathcal{R}^a_{\ bcd}\mathcal{R}_a^{\ bcd} = 48\frac{M^2}{r^6\dot{R}^6(t)} + \frac{24}{9}\Lambda^2 \tag{43}
\]

This evidently exhibits the existence of an intrinsic singularity at the origin. Removing this deficiency, leads us to the most general form of the solutions which comes next. As it is expected, Eq. (43) with \(R(t) = 1\) gives the result of the static case [17].

We end this section with emphasizing on some aspects of Eq. (42) and Eq. (43). Firstly, the existence of a nonzero cosmological constant regardless of its actual value, is sufficient to prevent from Schwarzschild singularity at \(r \approx 2M\). Recent observations of type Ia supernovae indicating a universal expansion, put forward the possible existence of a small
positive cosmological constant \([\Lambda]\). These evidences persuade us that in a \(\Lambda\)-dominated universe, we would have no troubles in describing the whole space.

Secondly, since there is no singularity for \(r > 0\), then there is no ambiguity in defining coordinate \(r\), which mentioned at the end of Sec.II. “\(r\)” is a space coordinate in the whole interval \((0, \infty)\) and we may speak of the value of \(D\) at \(r = 0\) without any problem.

Thirdly, a coordinate transformation \([\mathcal{C}]\), transforms the above non-static metric to the well-known Schwarzschild-de Sitter static metric Eq. (41). This metric has some deficiencies which we would like to discuss briefly. Despite of an intrinsic singularity at the origin which is similar to our case, it has a coordinate type singularity at \(r \approx \sqrt{\frac{A}{\Lambda}}\) in a \(\Lambda\)-dominated universe. Though the presence of cosmological constant removes the Schwarzschild singularity from our metric, but the transformed form of it, \(i.e.\) Schwarzschild-de Sitter metric, has again the problem of exchanging the meaning of space and time. On the other hand when \(M = 0\), the assumed FRW background could not be revisited but the attached homogenity and isotropy would resist. And more importantly, this metric shows a redshift-magnitude relation which contradicts the observational data \([4]\).

Therefore it is adequate to discard the Schwarzschild-de Sitter metric, in favor of our presented metric, as a proper frame of reference in the presence of \(\Lambda\), since all of the mentioned deficiencies do not exist in it. The metric asymptotically approaches to the non-static de Sitter metric which is appropriate for a \(\Lambda\)-dominated universe. Furthermore as we show next, the presented metric has the suitability even to remove the intrinsic singularity at the origin.

**IV. CASE \(\Lambda \neq 0, \alpha \neq 0\)**

Since it turns out that there is an intrinsic singularity with the choice \(\alpha = 0\), we would like to solve the problem by using the most general form of the line element. In this section we work out expressions for metric coefficients in the FRW universe and obtain an analytic metric everywhere. Therefore we choose the metric for the universe in terms of the coordinates \((r, t)\) to be

\[
d s^2 = B(r, t) dt^2 - R^2(t) \left[ A(r, t) dr^2 + D(r) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right]
\]  

(44)

The nonvanishing components of the metric and metric connection respectively are

\[
g_{tt} = -B, \quad g_{rr} = AR^2, \quad g_{\theta\theta} = DR^2, \quad g_{\phi\phi} = DR^2 \sin^2 \theta
\]

(45)

\[
\Gamma^t_{tt} = \frac{\dot{B}}{2B}, \quad \Gamma^t_{tr} = \frac{B'}{2B}, \quad \Gamma^r_{rr} = \frac{R \dot{A}}{2B} + \frac{R^2 \ddot{A}}{2B}, \quad \Gamma^t_{\theta\theta} = \frac{R \ddot{D}}{B}, \quad \Gamma^t_{\phi\phi} = \frac{R \ddot{D}}{B} \sin^2 \theta,
\]

\[
\Gamma^r_{rr} = \frac{B'}{2AR^2}, \quad \Gamma^r_{tr} = \frac{\dot{R}}{R} + \frac{\dot{A}}{2A}, \quad \Gamma^r_{\theta\theta} = \frac{A'}{2A}, \quad \Gamma^r_{\phi\phi} = -\frac{D'}{2A}, \quad \Gamma^\theta_{\phi\phi} = -\frac{D'}{2A} \sin^2 \theta,
\]

\[
\Gamma^\theta_{t\theta} = \frac{\dot{R}}{R}, \quad \Gamma^\theta_{r\theta} = \frac{D'}{2D}, \quad \Gamma^\phi_{\phi\phi} = -\sin \theta \cos \theta,
\]

\[
\Gamma^\phi_{t\phi} = \frac{\dot{R}}{R}, \quad \Gamma^\phi_{r\phi} = \frac{D'}{2D}, \quad \Gamma^\theta_{\phi\phi} = \cot \theta.
\]

(46)
where (') and (·) denote derivatives with respect to \( r \) and \( t \) respectively. The nonvanishing components of the Ricci tensor then become

\[
\mathcal{R}_{tt} = \frac{B}{AR^2} \left[ \frac{B''}{2B} + \frac{B'}{AB} \left( \frac{A'}{A} + \frac{B'}{B} \right) - \frac{B'D'}{2BD} \right] + \frac{\ddot{A}}{2A} - \frac{\dot{A}}{4A} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) + \frac{3\ddot{R}}{R} - \frac{\dot{R}}{R} \left( \frac{3\dot{B}}{2B} - \frac{\dot{A}}{A} \right)
\]

\[
\mathcal{R}_{tr} = -\frac{\dot{A}D'}{2AD} - \frac{B'R}{BR}
\]

\[
\mathcal{R}_{rr} = \frac{B''}{2B} + \frac{D''}{D} - \frac{D'}{2D} \left( \frac{D'}{D} + \frac{A'}{A} \right) - \frac{B'}{4B} \left( \frac{A'}{A} + \frac{B'}{B} \right) - \frac{R^2}{B} \left[ \frac{\ddot{R}}{R} + \frac{\dot{R}}{R} \left( \frac{2\dot{A}}{A} - \frac{\dot{B}}{2B} \right) + \frac{\ddot{A}}{2A} - \frac{\dot{A}}{4A} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) \right]
\]

\[
\mathcal{R}_{\theta\theta} = -1 + \frac{D'}{4A} \left( \frac{B'}{B} - \frac{A'}{A} \right) + \frac{D''}{2A} - \frac{R^2}{B} \left[ \frac{\ddot{R}}{R} + \frac{\dot{R}}{R} \left( \frac{2\dot{R}}{2B} + \frac{\dot{A}}{2A} - \frac{\dot{B}}{2B} \right) \right]
\]

\[
\mathcal{R}_{\phi\phi} = \sin^2 \theta \mathcal{R}_{\theta\theta}
\]

To solve the vacuum field equations \( \mathcal{R}_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \), we first begin with \( \mathcal{R}_{tr} = 0 \) and introduce a new variable \( \rho \) to be

\[
\rho = R(t)D^{1/2}(r)
\]

From this equation we obtain

\[
\frac{1}{2} \ddot{R}D'D^{-1/2} \left( \frac{A'}{A} + \frac{B'}{B} \right) = 0
\]

which (*) means differentiation with respect to \( \rho \). Since \( \dot{R} \) and \( D' \neq 0 \), we should have

\[
\frac{A'}{A} + \frac{B'}{B} = 0
\]

Integration with respect to \( \rho \) and imposing flat boundary condition at large distances yields

\[
AB = 1
\]

in agreement with the FRW background. In the next step, let’s consider

\[
\frac{\mathcal{R}_{tt}}{B} + \frac{\mathcal{R}_{rr}}{AR^2} = 0
\]

which we get

\[
-\frac{D'^2}{2AD^2R^2} + \frac{D''}{ADR^2} = 0
\]

or we may write

\[
(D'D^{-1/2})' = 0
\]
Integration with respect to $r$ gives

$$D^{1/2} = r + \alpha M \quad \text{or} \quad D = (r + \alpha M)^2 \quad (57)$$

As before $\alpha$ is a positive constant in the range $0 < \alpha < 10^3$.

Finally from the $\theta\theta$ component of the field equation, $\mathcal{R}_{\theta\theta} + \Lambda g_{\theta\theta} = 0$, we obtain the functional form of $A(r,t)$ as follows

$$-1 + \frac{1}{A} - \rho \frac{A^*}{A^2} - \Lambda \rho^2 (A - 1) - \frac{\Lambda}{3} \rho^3 A^* = 0$$

or

$$\frac{d}{d\rho}\left[ \rho(1 - \frac{1}{A}) \right] + \frac{\Lambda}{3} \frac{d}{d\rho}\left[ \rho^3 (A - 1) \right] = 0 \quad (58)$$

Integration then yields

$$\rho(1 - \frac{1}{A}) + \frac{\Lambda}{3} \rho^3 (A - 1) = c \quad (59)$$

where $c$ is constant of integration. Comparing our result with post-Newtonian limit gives $c = 2M$. Finally the result is

$$A = B^{-1} = \left( \frac{2\Lambda}{3} \rho^2 \right)^{-1} \left[ \sqrt{(1 - \frac{2M}{\rho} - \frac{\Lambda}{3} \rho^2)^2 + \frac{4\Lambda}{3} \rho^2} - (1 - \frac{2M}{\rho} - \frac{\Lambda}{3} \rho^2) \right]$$

or

$$B = A^{-1} = \frac{1}{2} \left[ \sqrt{(1 - \frac{2M}{\rho} - \frac{\Lambda}{3} \rho^2)^2 + \frac{4\Lambda}{3} \rho^2} + (1 - \frac{2M}{\rho} - \frac{\Lambda}{3} \rho^2) \right] \quad (60)$$

As it is evident from the functional form of $A$, $B$ and $D$, this metric has no apparent singularity and a straightforward calculation gives the Riemann tensor scalar invariant in the form

$$\mathcal{R}^a_{\ bcd} \mathcal{R}_a^{bcd} = \frac{48M^2}{D^3 R^c(t)} + \frac{24}{9} \Lambda^2 \quad (61)$$

If we set $\Lambda = 0$, $R(t) = 1$ in (61), Eq. (32) will be obtained, and furthermore with $D = r^2$, Eq. (3) would regain.

It is remarkable that having $\alpha \neq 0$, we would have a well-defined metric in the whole space which asymptotically approaches to the non-static de Sitter metric; i.e. the appropriate metric for a $\Lambda$-dominated universe.
Our next task is to obtain and solve the geodesic equations of a freely falling material particle in a proper FRW type coordinate system. In the static case, the basic equations which determine the geodesic structure of the manifold have been fully developed \[17,18\]. We solve the problem in the non-static case by using the equations of free fall

\[\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{ds} \frac{dx^\nu}{ds} = 0\] (62)

Using the nonvanishing components of affine connection, given by Eq. (46) we have

\[\frac{d^2t}{ds^2} + \frac{\dot{B}}{2B} \left(\frac{dt}{ds}\right)^2 + \frac{B'}{B} \frac{dt}{ds} + \frac{R\dot{R}A}{B} + \frac{R^2\ddot{A}}{2B} \left(\frac{dr}{ds}\right)^2 + \frac{R\ddot{D}}{B} \left(\frac{d\theta}{ds}\right)^2 \]

\[+ \frac{R\dddot{D}}{B} \sin^2 \theta \left(\frac{d\phi}{ds}\right)^2 = 0\] (63)

\[\frac{d^2r}{ds^2} + \frac{B'}{2AR^2} \left(\frac{dt}{ds}\right)^2 + 2 \left(\frac{\ddot{R}}{R} + \frac{\dot{A}'}{2A} \right) \frac{dt}{ds} + \frac{A'}{2A} \left(\frac{dr}{ds}\right)^2 - \frac{D'}{2A} \left(\frac{d\theta}{ds}\right)^2 \]

\[\frac{D'}{2A} \sin^2 \theta \left(\frac{d\phi}{ds}\right)^2 = 0\] (64)

\[\frac{d^2\theta}{ds^2} + \frac{2\ddot{R}}{R} \frac{dt}{ds} + \frac{D'}{D} \frac{dr}{ds} + \frac{D'}{D} \frac{d\theta}{ds} \sin \theta \cos \theta \left(\frac{d\phi}{ds}\right)^2 = 0\] (65)

\[\frac{d^2\phi}{ds^2} + 2 \frac{\dot{R}}{A} \frac{dt}{ds} + \frac{D'}{D} \frac{dr}{ds} + \frac{D'}{D} \frac{d\phi}{ds} + 2 \cot \theta \frac{d\theta}{ds} \frac{d\phi}{ds} = 0\] (66)

Since the field is isotropic, we may consider the orbit of our particle to be confined to the equatorial plane, that is \(\theta = \frac{\pi}{2}\). Then Eq. (65) immediately is satisfied and we may forget about \(\theta\) as a dynamical variable. Introducing the previous variable \(\rho\) and using the Friedmann equation, make the above equations as

\[\frac{d^2t}{ds^2} + \frac{\ddot{A}^*}{A^*} \frac{dt}{ds} \frac{dr}{ds} - \left(\frac{\ddot{A}^*}{A^*} + 2 \frac{\dot{A}^*}{A^*} + \frac{1}{2} \rho A A^* \right) \left(\frac{d\rho}{ds}\right)^2 \]

\[+ \frac{\dot{A}^*}{A^*} \frac{d\phi}{ds} \left(\frac{d\phi}{ds}\right)^2 + \sqrt{\frac{\ddot{A}^*}{A^*}} \left(\frac{d\phi}{ds}\right)^2 = 0\] (67)

\[\frac{d^2\rho}{ds^2} - \frac{\ddot{A}^*}{A^*} \frac{dt}{ds} - \left(\frac{\ddot{A}^*}{A^*} + 2 \frac{\dot{A}^*}{A^*} + \frac{1}{2} \rho A A^* \right) \left(\frac{d\rho}{ds}\right)^2 + \frac{\ddot{A}^*}{2A} \left(\frac{d\rho}{ds}\right)^2 - \frac{\rho}{A} \left(\frac{d\phi}{ds}\right)^2 = 0\] (68)

\[\frac{d^2\phi}{ds^2} + 2 \frac{\dot{A}^*}{A^*} \frac{dt}{ds} + \frac{D'}{D} \frac{dr}{ds} + \frac{D'}{D} \frac{d\phi}{ds} + 2 \cot \theta \frac{d\theta}{ds} \frac{d\phi}{ds} = 0\] (69)

The last equation gives

\[\frac{d}{ds} \left(\rho \frac{d\phi}{ds}\right) = 0\] (70)

Integrating (70) with respect to \(s\) we get

\[\frac{d\phi}{ds} = \frac{J}{\rho^2}\] (71)
where $J$ is constant of integration.

In order to derive the exact solution, first rewrite Eq. (67) as

$$
\frac{d}{ds} \left[ \left( \frac{1}{A} - \frac{\Lambda}{3} \rho^2 A \right) \frac{dt}{ds} \right] + \sqrt{\frac{\Lambda}{3} \rho A} \left[ \frac{h}{\Lambda} \frac{d^2 t}{ds^2} + \frac{A^*}{dt} + \frac{\Lambda}{3} \rho + \frac{\Lambda}{6} \rho^2 A^* \right] (\frac{dt}{ds})^2 + \frac{J^2}{A^2} \\
+ \sqrt{\frac{\Lambda}{3} (A + \frac{\rho}{2} A^*)} \left( \frac{d\rho}{ds} \right)^2 = 0 \tag{72}
$$

Substituting Eq. (68) into Eq. (72) yields

$$
\frac{d}{ds} \left[ \left( \frac{1}{A} - \frac{\Lambda}{3} \rho^2 A \right) \frac{dt}{ds} \right] + \sqrt{\frac{\Lambda}{3} \rho A} \left[ \frac{d\rho}{ds} + (A + \rho A^*) (\frac{d\rho}{ds})^2 \right] = 0
$$
or

$$
\frac{d}{ds} \left[ \left( \frac{1}{A} - \frac{\Lambda}{3} \rho^2 A \right) \frac{dt}{ds} + \sqrt{\frac{\Lambda}{3} \rho A} \left( \frac{d\rho}{ds} \right) \right] = 0 \tag{73}
$$

which can be integrated to get

$$
\left( \frac{1}{A} - \frac{\Lambda}{3} \rho^2 A \right) \frac{dt}{ds} + \sqrt{\frac{\Lambda}{3} \rho A} \frac{d\rho}{ds} = c_1 \tag{74}
$$

where $c_1$ is constant of integration. Derivative of $\rho$ with respect to $s$ gives

$$
\frac{d\rho}{ds} = \sqrt{\frac{\Lambda}{3} \rho A} \frac{dt}{ds} + R \frac{dr}{ds} \tag{75}
$$

if $r \to \infty$ or $\rho \to \infty$, then (75) reduces to

$$
\frac{d\rho}{ds} \to \sqrt{\frac{\Lambda}{3} \rho}
$$

because for free fall at infinity we have

$$
\frac{dr}{ds} = 0 \quad \& \quad \frac{dt}{ds} = 1
$$

Therefore $c_1$ would be fixed by asymptotic behavior as $c_1 = 1$. Eq. (74) then becomes

$$
\frac{dt}{ds} = \frac{1 - \sqrt{\frac{\Lambda}{3} \rho A} \frac{d\rho}{ds}}{\frac{1}{A} - \frac{\Lambda}{3} \rho^2 A} \tag{76}
$$

We have yet another equation, the line element equation, which in terms of $\rho$ variable has the form

$$
\left( \frac{1}{A} - \frac{\Lambda}{3} \rho^2 A \right) (\frac{dt}{ds})^2 + 2 \sqrt{\frac{\Lambda}{3} \rho A} \frac{dt}{ds} \frac{d\rho}{ds} - A (\frac{d\rho}{ds})^2 - \rho^2 (\frac{d\phi}{ds})^2 = 1 \tag{77}
$$
Rewriting Eq. (67) as

\[
\frac{d^2 t}{ds^2} + \sqrt{\frac{\Lambda}{3}} \rho \frac{A^*}{2A} \left( \frac{dt}{ds} \right)^2 - \frac{A^*}{A} \frac{dt}{ds} \frac{d\rho}{ds} = 0
\]

and using Eq. (77), we come to

\[
\frac{d^2 t}{ds^2} + \sqrt{\frac{\Lambda}{3}} (1 + \rho \frac{A^*}{A}) \left( \frac{dt}{ds} \right)^2 - \frac{A^*}{A} \frac{dt}{ds} \frac{d\rho}{ds} = 0 \quad (79)
\]

multiplying the above equation by \( \sqrt{\frac{\Lambda}{3}} \rho \) and inserting the result in Eq. (68) we find

\[
\frac{d^2 \rho}{ds^2} - \frac{A^*}{2A^2} \left[ (\frac{1}{A} - \frac{\Lambda}{3} \rho^2 A) \left( \frac{dt}{ds} \right)^2 + 2 \sqrt{\frac{\Lambda}{3}} \rho A \frac{dt}{ds} \frac{d\rho}{ds} - \sqrt{\frac{\Lambda}{3}} (A + \rho A^* + \frac{J^2}{2\rho} A^*) \right] = 0 \quad (81)
\]

From Eq. (77), we notice that the expression in the bracket is just equal to \( 1 + \frac{J^2}{\rho^2} \), so we obtain a second order differential equation for \( \rho \) as

\[
\frac{d^2 \rho}{ds^2} - \frac{A^*}{2A^2} (1 + \frac{J^2}{\rho^2}) - \frac{J^2}{\rho^3 A} - \frac{\Lambda}{3} \rho (A + \frac{\rho}{2} A^* + \frac{J^2}{2\rho} A^*) = 0 \quad (81)
\]

which can be written

\[
\frac{d^2 \rho}{ds^2} + \frac{1}{2} \frac{d \rho}{ds} \left[ (\frac{1}{A} - \frac{\Lambda}{3} \rho^2 A)(1 + \frac{J^2}{\rho^2}) \right] = 0 \quad (82)
\]

or

\[
\frac{d \rho}{ds} \left[ (\frac{d \rho}{ds})^2 + (\frac{1}{A} - \frac{\Lambda}{3} \rho^2 A)(1 + \frac{J^2}{\rho^2}) \right] = 0 \quad (83)
\]

Integration then yields

\[
(\frac{d \rho}{ds})^2 + (\frac{1}{A} - \frac{\Lambda}{3} \rho^2 A)(1 + \frac{J^2}{\rho^2}) = c_2 \quad (84)
\]

Asymptotic behavior shows that \( c_2 = 1 \), and thereby

\[
(\frac{d \rho}{ds})^2 = 1 - (\frac{1}{A} - \frac{\Lambda}{3} \rho^2 A)(1 + \frac{J^2}{\rho^2}) \quad (85)
\]

Eq. (59) together with \( J = 0 \) makes Eq. (82) as
\[
\frac{d^2 \rho}{ds^2} + \frac{d}{d\rho} \left( \frac{-M}{\rho} - \frac{\Lambda \rho^2}{6} \right) = 0 \quad (86)
\]

Since the potential field \(\Phi\) is defined as
\[
\frac{d^2 \rho}{ds^2} + \frac{d\Phi}{d\rho} = 0
\]
then
\[
\Phi = \frac{-M}{\rho} - \frac{\Lambda \rho^2}{6} \quad (87)
\]
which is equivalent to the Schwarzschild-de Sitter potential [19]. Although the potential at \(r = 0\) i.e. \(\rho = \alpha M R(t)\) is very large, but it is finite. It seems likely that the potential field of massive stars show this behavior and help us to find a physical mechanism for fixing \(\alpha\). It will be a great success if observing extra high energy phenomena in AGN’s and cosmic rays, provide a lower limit for \(\alpha\).

**REMARKS**

A space-time is said to be spherically symmetric, if it admits the group SO(3) as a group of isometries, with the group orbits spacelike two surfaces. A coordinate transformation like \(r \rightarrow r' = r + \alpha M\) which translates the center of symmetry and thereby breaks down spherical symmetry does not belong to SO(3). We have shown that the solutions of spherically symmetric vacuum Einstein field equations are necessarily neither static nor incomplete.

**APPENDIX A: DEFLECTION OF LIGHT**

We would like to derive the deflection of light formula which introduce in Sec.II. The orbit is described by Eq. (19), that is
\[
\phi(r) - \phi_\infty = \int_r^\infty \left[ \frac{A(r)}{D(r)} \right]^{1/2} \left[ \frac{D(r)}{D(r_o)} \frac{B(r_o)}{B(r)} - 1 \right]^{-1/2} dr \quad (A1)
\]
Using for \(A(r)\), \(B(r)\) and \(D(r)\) the Eq. (21), we notice that the integral is defined for \(r_o > (3 - \alpha)M\). The first term of the second square root is written as
\[
\frac{D(r)}{D(r_o)} \frac{B(r_o)}{B(r)} = \left[ \frac{r}{r_o} \right]^2 \frac{1 + \alpha M/r}{1 + \alpha M/r_o} \frac{1 - (2 - \alpha)M/r}{1 - (2 - \alpha)M/r_o} \quad (A2)
\]
We want to derive deflection angle up to the second order but the expansion explicitly depends on \(\alpha\) and \(M/r_o\). Since there is no common valid range of expansion for the involved parameters, we must treat the necessary and sufficient cases separately; i.e. \(\alpha < 1\) and \(\alpha > 1\).
We consider first \( \alpha < 1 \). Because of the integral singularity condition, we always have \( r_o > (2 - \alpha)M \) and therefore with a simple expansion we get

\[
\frac{D(r) B(r_o)}{D(r_o) B(r)} = \left[ \frac{r}{r_o} \right]^2 \left[ 1 + \frac{1}{1 - \frac{1}{r} - \frac{1}{r_o}} \right] + \frac{(2 - \alpha)^2 M^2}{r} \left( \frac{1}{r} - \frac{1}{r_o} \right) + 6\alpha M^2 \left( \frac{1}{r} - \frac{1}{r_o} \right)^2 + \cdots \tag{A3}
\]

The argument of the second square root in Eq. (A1) is then

\[
\frac{D(r) B(r_o)}{D(r_o) B(r)} - 1 = \left[ \left( \frac{r}{r_o} \right)^2 - 1 \right] \left[ 1 - \frac{2(1 + \alpha)M - 3\alpha^2 M^2/r}{2r_o(r + r_o)} \right] \left[ \frac{(2 - \alpha)^2 M^2}{r} \left( \frac{1}{r} - \frac{1}{r_o} \right) + \frac{6\alpha M^2(r - r_o)}{r_o(r + r_o)} + \cdots \right] \tag{A4}
\]

so it gives

\[
\left( \frac{D(r) B(r_o)}{D(r_o) B(r)} - 1 \right)^{-1/2} = \left[ \left( \frac{r}{r_o} \right)^2 - 1 \right]^{-1/2} \times \left[ 1 - \frac{2(1 + \alpha)M - 3\alpha^2 M^2/r}{2r_o(r + r_o)} \right] \left[ \frac{(2 - \alpha)^2 M^2}{r} \left( \frac{1}{r} - \frac{1}{r_o} \right) + \frac{6\alpha M^2(r - r_o)}{r_o(r + r_o)} + \cdots \right] \tag{A5}
\]

The argument of the first square root in Eq. (A1) and its expansion is

\[
\left[ \frac{A(r)}{D(r)} \right]^{1/2} = \frac{1}{r} \left[ (1 + \alpha M/r)(1 - (2 - \alpha)M/r) \right]^{-1/2} = \frac{1}{r} \left[ 1 + \frac{(1 - \alpha)M}{r} + \frac{(2\alpha^2 - 4\alpha + 3)M^2}{2r^2} + \cdots \right] \tag{A6}\]

Inserting these two equations into Eq. (A1) gives

\[
\phi(r_o) - \phi_\infty = \int_{r_o}^\infty \frac{dr}{r(\left( \frac{r}{r_o} \right)^2 - 1)^{1/2}} \left[ 1 + \frac{(1 - \alpha)M}{r} + \frac{(2\alpha^2 - 4\alpha + 3)M^2}{2r^2} + \cdots \right] \times \left[ 1 + \frac{2(1 + \alpha)M - 3\alpha^2 M^2/r_o}{2r_o(r + r_o)} \right] \frac{(2 - \alpha)^2 M^2}{2r_o(r + r_o)} \left( \frac{1}{r} - \frac{1}{r_o} \right) + \frac{6\alpha M^2(r - r_o)}{r_o(r + r_o)} + \cdots \right] \tag{A7}
\]

The integral is elementary, and gives

\[
\phi(r_o) - \phi_\infty = \frac{\pi}{2} + \frac{2M}{r_o} + \frac{2M^2}{r_o^2} \left( \frac{15\pi}{16} - (1 + \alpha) \right) + \cdots \tag{A8}
\]

The deflection of the orbit from a straight line is
\[ \Delta \phi = 2|\phi(r_o) - \phi_\infty| - \pi \]

Hence to second order in \( M/r_o \), the deflection angle is as follows

\[ \Delta \phi = \frac{4M}{r_o} + \frac{4M^2}{r_o^2} \left[ \frac{15\pi}{16} - (1 + \alpha) \right] + \cdots \]  

\[ (A9) \]

Next we consider \( \alpha > 1 \). Since in this case we may also have \( r_o < |2 - \alpha|M \) (e.g. for \( \alpha > 3 \) the integral singularity condition reduces to \( r_o > 0 \)), so in order to have a valid expansion we define the following parameters

\[ a \equiv \frac{\alpha M}{1 + \alpha M/r_o} \quad \& \quad b \equiv \frac{(2 - \alpha)M}{1 - (2 - \alpha)M/r_o} \]

and with \( \alpha > 1 \), we always have

\[ a/r_o < 1 \quad \& \quad b/r_o < 1 \]

So we rewrite Eq. \((A2)\) and expand it as

\[ \frac{D(r) B(r_o)}{D(r_o) B(r)} - 1 = \left[ \left( \frac{r}{r_o} \right)^2 - 1 \right] \left[ 1 - \frac{(3a + b)r}{r_o(r + r_o)} + \frac{(3a^2 + 3ab + b^2)(r - r_o)}{r_o^2(r + r_o)} + \cdots \right] \]

\[ (A12) \]

The argument of the second square root in Eq. \((A1)\) is then

\[ \frac{D(r) B(r_o)}{D(r_o) B(r)} - 1 = \left[ \left( \frac{r}{r_o} \right)^2 - 1 \right] \left[ 1 + \frac{(3a + b)r}{2r_o(r + r_o)} + \frac{3(3a + b)^2r^2}{8r_o^3(r + r_o)^2} - \frac{(3a^2 + 3ab + b^2)(r - r_o)}{2r_o^2(r + r_o)} + \cdots \right] \]

\[ (A13) \]

so it gives

\[ \left[ \frac{D(r) B(r_o)}{D(r_o) B(r)} - 1 \right]^{-1/2} = \left[ \left( \frac{r}{r_o} \right)^2 - 1 \right]^{-1/2} \left[ 1 + \frac{(3a + b)r}{2r_o(r + r_o)} + \frac{3(3a + b)^2r^2}{8r_o^3(r + r_o)^2} - \frac{(3a^2 + 3ab + b^2)(r - r_o)}{2r_o^2(r + r_o)} + \cdots \right] \]

\[ (A14) \]

The argument of the first square root in Eq. \((A1)\) is

\[ \left[ \frac{A(r)}{D(r)} \right]^{1/2} = \frac{1}{r} \left[ (1 + \alpha M/r)(1 - (2 - \alpha)M/r) \right]^{-1/2} \]

\[ (A15) \]

In terms of these new parameters, we have the following equalities

\[ 1 + \alpha M/r \equiv (1 - a/r_o)^{-1} \left[ 1 + a \left( \frac{1}{r} - \frac{1}{r_o} \right) \right] \]

\[ (A16) \]
\[ 1 - (2 - \alpha)M/r \equiv (1 + b/r_o)^{-1} \left[ 1 - b \left( \frac{1}{r} - \frac{1}{r_o} \right) \right] \quad (A17) \]

Inserting these results into Eq. (A15) and expanding it up to second order yields
\[
\left[ \frac{A(r)}{D(r)} \right]^{1/2} = \frac{1}{r} \left[ (1 - a/r_o)(1 + b/r_o) \right]^{1/2} \\
\times \left[ 1 - \frac{(a - b)}{2} \left( \frac{1}{r} - \frac{1}{r_o} \right) + \frac{(3a^2 - 2ab + 3b^2)}{8} \left( \frac{1}{r} - \frac{1}{r_o} \right)^2 + \ldots \right] \\
\times \left[ 1 + \frac{(3a + b)r}{2r_o(r + r_o)} + \frac{3(3a + b)^2 r^2}{8r_o^2(r + r_o)^2} - \frac{(3a^2 + 3ab + b^2)(r - r_o)}{2r_o^2(r + r_o)} + \ldots \right] \\
\quad (A18) \]

Substituting Eq. (A14) and Eq. (A18) into Eq. (A1) gives an elementary integral as
\[
\phi(r_o) - \phi_\infty = \int_{r_o}^\infty \frac{dr}{r[\left( \frac{1}{r_o} \right)^2 - 1]^{1/2}} \left[ (1 - a/r_o)(1 + b/r_o) \right]^{1/2} \\
\times \left[ 1 - \frac{(a - b)}{2} \left( \frac{1}{r} - \frac{1}{r_o} \right) + \frac{(3a^2 - 2ab + 3b^2)}{8} \left( \frac{1}{r} - \frac{1}{r_o} \right)^2 + \ldots \right] \\
\times \left[ 1 + \frac{(3a + b)r}{2r_o(r + r_o)} + \frac{3(3a + b)^2 r^2}{8r_o^2(r + r_o)^2} - \frac{(3a^2 + 3ab + b^2)(r - r_o)}{2r_o^2(r + r_o)} + \ldots \right] \\
\quad (A19) \]

which can be easily integrated
\[
\phi(r_o) - \phi_\infty = \frac{\pi}{2} + \frac{(a + b)}{r_o} - \frac{(a + b)(a + 3b)}{2r_o^2} + \frac{15\pi(a + b)^2}{32r_o^2} + \ldots \\
\quad (A20) \]

This gives the deflection angle in terms of the defined parameters as
\[
\Delta \phi = \frac{2(a + b)}{r_o} - \frac{(a + b)(a + 3b)}{2r_o^2} + \frac{15\pi(a + b)^2}{16r_o^2} + \ldots \\
\quad (A21) \]

For expanding the parameters, we must notice the region in which the expansion is valid. For \( r_o > \alpha M \) (and consequently \( r_o > |2 - \alpha|M \) the expansions to second order in \( M/r_o \) are
\[
a/r_o = \frac{\alpha M}{r_o} - \frac{\alpha^2 M^2}{r_o^2} + \ldots \\
b/r_o = \frac{(2 - \alpha)M}{r_o} + \frac{(2 - \alpha)^2 M^2}{r_o^2} + \ldots \\
\quad (A22) \]

Hence the Einstein deflection angle up to the second order in this region becomes
\[
\Delta \phi = \frac{4M}{r_o} + \frac{4M^2}{r_o^2} \left[ \frac{15\pi}{16} - (1 + \alpha) \right] + \ldots \\
\quad (A23) \]

The other region we are interested in, is \( r_o < |2 - \alpha|M \) (Actually this region exists if \( \alpha > 2.5 \)) which gives the second order expansions as
\[
a/r_o = 1 - \frac{r_o}{\alpha M} + \frac{r_o^2}{\alpha^2 M^2} + \cdots
\]

\[
b/r_o = -1 + \frac{r_o}{(\alpha - 2)M} - \frac{r_o^2}{(\alpha - 2)^2 M^2} + \cdots
\]  

(A24)

Therefore the Einstein deflection angle up to the second order in this region is

\[
\Delta \phi = \frac{8r_o}{\alpha(\alpha - 2)M} + \frac{4r_o^2}{\alpha^2(\alpha - 2)^2 M^2} \left[ \frac{15\pi}{16} - 5(\alpha - 1) \right] + \cdots
\]  

(A25)

**APPENDIX B: PRECESSION OF PERIHELIA**

In order to derive the Precession of Perihelia expression which is used in Sec.II, we consider a test particle bound in an orbit around a massive object. The angle swept is given by Eq. (27), that is

\[
\phi(r) - \phi(r_-) = \int_{r_-}^{r} \left[ \frac{D(r_-) (B^{-1}(r) - B^{-1}(r_-)) - D(r_+) (B^{-1}(r) - B^{-1}(r_+))}{D(r_+)D(r_-) (B^{-1}(r_+) - B^{-1}(r_-))} - \frac{1}{D(r)} \right]^{-1/2}
\]

\[
\times \left[ \frac{A^{1/2}(r)}{D(r)} \right] dr
\]  

(B1)

and the orbit precesses in each revolution by an angle

\[
\Delta \phi = 2|\phi(r_+) - \phi(r_-)| - 2\pi
\]  

(B2)

Following Weinberg, we make the argument of the first square root in Eq. (B1) a quadratic function of \(1/D(r)\) which vanishes at \(D(r_+)\), so

\[
\frac{D(r_-) (B^{-1}(r) - B^{-1}(r_-)) - D(r_+) (B^{-1}(r) - B^{-1}(r_+))}{D(r_+)D(r_-) (B^{-1}(r_+) - B^{-1}(r_-))} - \frac{1}{D(r)} = C \left( \frac{1}{D^{1/2}(r_-)} - \frac{1}{D^{1/2}(r)} \right) \left( \frac{1}{D^{1/2}(r)} - \frac{1}{D^{1/2}(r_+)} \right)
\]  

(B3)

The constant \(C\) can be determined by letting \(r \to \infty\)

\[
C = \frac{D(r_+) (1 - B^{-1}(r_+)) - D(r_-) (1 - B^{-1}(r_-))}{D^{1/2}(r_+)D^{1/2}(r_-) (B^{-1}(r_+) - B^{-1}(r_-))}
\]  

(B4)

and its expansion up to second order becomes

\[
C = 1 - 2M \left( \frac{1}{r_+} + \frac{1}{r_-} \right) + 2\alpha M^2 \left( \frac{1}{r_+^2} + \frac{1}{r_-^2} \right) + \cdots
\]  

(B5)
Using Eq. (B3) in Eq. (B1) gives then

\[ \phi(r) - \phi(r_-) = C^{-1/2} \int_{r_-}^{r} dr \left[ \frac{A^{1/2}(r)}{D(r)} \right] \left( \frac{1}{D^{1/2}(r_-)} - \frac{1}{D^{1/2}(r)} \right) \left( \frac{1}{D^{1/2}(r)} - \frac{1}{D^{1/2}(r_+)} \right)^{-1/2} \]  \hspace{1cm} (B6)

Introducing a new variable \( \psi \)

\[ \frac{1}{D^{1/2}} = \frac{1}{2} \left( \frac{1}{D^{1/2}(r_+)} + \frac{1}{D^{1/2}(r_-)} \right) + \frac{1}{2} \left( \frac{1}{D^{1/2}(r_+)} - \frac{1}{D^{1/2}(r_-)} \right) \sin \psi \]  \hspace{1cm} (B7)

makes the integral as

\[ \phi(r) - \phi(r_-) = C^{-1/2} \int_{-\pi/2}^{\psi} d\psi \left[ 1 - M \left( \frac{1}{D^{1/2}(r_+)} + \frac{1}{D^{1/2}(r_-)} \right) - M \left( \frac{1}{D^{1/2}(r_+)} - \frac{1}{D^{1/2}(r_-)} \right) \sin \psi \right]^{-1/2} \]  \hspace{1cm} (B8)

Expanding the square root to second order in \( M/D^{1/2}(r) \), and integrating yields

\[ \phi(r_+) - \phi(r_-) = C^{-1/2} \left[ \pi + \frac{\pi M}{2} \left( \frac{1}{D^{1/2}(r_+)} + \frac{1}{D^{1/2}(r_-)} \right) \right. 
\[ + \frac{3\pi M^2}{8} \left( \frac{1}{D^{1/2}(r_+)} + \frac{1}{D^{1/2}(r_-)} \right)^2 
\[ + \frac{3\pi M^2}{16} \left( \frac{1}{D^{1/2}(r_+)} - \frac{1}{D^{1/2}(r_-)} \right)^2 + \cdots \]  \hspace{1cm} (B9)

Using Eq. (B5) in the above equation and expanding it gives

\[ \phi(r_+) - \phi(r_-) = \left[ 1 + M \left( \frac{1}{r_+} + \frac{1}{r_-} \right) + \frac{3M^2}{2} \left( \frac{1}{r_+} + \frac{1}{r_-} \right)^2 - \alpha M^2 \left( \frac{1}{r_+^2} + \frac{1}{r_-^2} \right) + \cdots \right] 
\[ \times \left[ \pi + \frac{\pi M}{2} \left( \frac{1}{r_+} + \frac{1}{r_-} \right) - \frac{\pi \alpha M^2}{2} \left( \frac{1}{r_+^2} + \frac{1}{r_-^2} \right) \right. 
\[ + \frac{3\pi M^2}{8} \left( \frac{1}{r_+} + \frac{1}{r_-} \right)^2 + \frac{3\pi M^2}{16} \left( \frac{1}{r_+} - \frac{1}{r_-} \right)^2 + \cdots \]  \hspace{1cm} (B10)

So with Eq. (B2), we get the precession per revolution as

\[ \Delta \phi = 3\pi M \left( \frac{1}{r_+} + \frac{1}{r_-} \right) + \frac{19\pi M^2}{4} \left( \frac{1}{r_+} + \frac{1}{r_-} \right)^2 + \frac{3\pi M^2}{8} \left( \frac{1}{r_+} - \frac{1}{r_-} \right)^2 
\[ -3\pi \alpha M^2 \left( \frac{1}{r_+^2} + \frac{1}{r_-^2} \right) + \cdots \]  \hspace{1cm} (B11)

The elements of planetary orbits are \( a \) and \( e \) which defined by
\[ r_\pm = (1 \pm e)a \]
\[ L = (1 - e^2)a \]  
\[ \frac{1}{L} = \frac{1}{2} \left( \frac{1}{r_+} + \frac{1}{r_-} \right) \]  
(B12)

where

These result the desired form of precession per revolution up to the second order as

\[ \Delta \phi = \frac{6\pi MG}{L} \left[ 1 + \frac{MG}{L} \left( \frac{19}{6} + \frac{e^2}{4} - 2\alpha(1 + e^2) \right) \right] + \cdots \]  
(B14)

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