Asymptotic soliton-like solutions to the singularly perturbed Benjamin–Bona–Mahony equation with variable coefficients

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Abstract. The paper deals with a problem of asymptotic soliton-like solutions to the Benjamin–Bona–Mahony (BBM) equation with a small parameter at the highest derivative and variable coefficients depending on the variables $x, t$ as well as a small parameter. An algorithm for constructing solutions to the BBM equation has been proposed and theorems on accuracy of such solutions have been proved.

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1. Introduction In the modern mathematics and theoretical physics, much attention is paid to the Korteweg-de Vries (KdV) equation

$$u_t + uu_x - u_{xxx} = 0$$

as well as its different generalizations.

After the discovery of particular solutions to the KdV equation describing waves with special features, which took place in 1965, the equation has become an object for comprehensive study by many researchers. These waves interact among themselves in a special way, namely, they don’t change their shapes after their collision. Such particular solutions, called solitons [1], have been studied mostly by physicists because soliton solutions describe propagation of long waves in shallow water as well as in various media. Later, similar soliton solutions have been also found for a variety of nonlinear dynamical systems, in particular for the modified KdV equation [2], the nonlinear Shrodinger equation [3] and the derivative Shrodinger equation [4], the Kadomtsev-Petviashvili equation [5], the Kaup system [6], and many others. At present, the soliton
theory is widely applied in hydrodynamics, plasma physics, nonlinear optic, quantum field theory, solid physics, biology, etc.

A number of monographs and numerous papers are devoted to different aspects of the soliton theory. Numerous works investigated different analytical and qualitative properties of solutions to the KdV equation and its diverse generalizations using various methods and approaches [1], [7] – [18], including numerical methods, that are usually used for researching complex nonlinear models.

Studies of the KdV equation stimulated efforts to find new nonlinear systems possessing similar properties, namely the propagation of long waves and the existence of soliton solutions. In 1966, Peregrine D.H. [19] proposed the following equation

\[ u_t + u_x + uu_x - u_{xxx} = 0 \]  

(2)
as an alternative to the KdV equation. Equation (2) describes the propagation of long waves in nonlinear dispersive media and has soliton solutions. It was thoroughly studied by Benjamin T.B., Bona J.L., and Mahony J.J. in [20]. These researchers have demonstrated the existence of classical solutions to (2) and the uniqueness of solution to the initial-value problem for it. In [20], it was also proved that solutions depend continuously on the initial values as well as on the forcing functions added to the right-hand side of equation (2). In other words, the initial-value problem for equation (2) is confirmed to be classically well posed in the Hadamard sense.

Equation (2) was originally called the regularized long wave equation [20]. At present, equation (2) is also called the Benjamin–Bona–Mahony equation or the BBM equation.

After the initial publication [20], equation (2) has been discovered to describe a variety of physical phenomena and processes, in particular, the propagation of long waves in liquids, acoustic waves in anharmonic crystals, acoustic-gravity waves in compressible fluids and others [21]. Subsequent studies showed that equation (2) possesses numerous interesting properties. It was stated in [22, 23] that equation (2) has, in some sense, "much nicer mathematical properties than the KdV equation".

Indeed, the KdV equation and the BBM equation have several different mathematical properties. First of all, it should be noted the BBM equation is more convenient for studying through numerical technique than the KdV equation because the numerical schemes [19], [22, 23] applied for (2) are stable for greater time steps [12] compared to numerical schemes proposed for (1). That is why the BBM equation was first intensively studied through different
numerical methods. As a result, many improbable properties of equation (2) have been found. For example, in the case of the BBM equation, the numerical study of the two-soliton collision and three-soliton collision appointed "collisional stability" or elastic interaction [22, 23]. In other words, it was numerically demonstrated that after the nonlinear interaction, the solitary waves again have their initial amplitudes.

On the other hand, the inelastic collision of two solitary waves moving in opposite directions was also found [24]. Moreover, the BBM equation has only three conservation laws while the KdV equation has an infinite conservation law hierarchy. As a consequence, the equation (2) is not integrable.

Because of the numerical results on the existence of both the two- and three-soliton solutions to the BBM equation, it is natural to clarify whether the equation has an analytical $N$-soliton solution as the KdV equation. Many authors studied the problem through various techniques but without success [24] – [27]. At present it is known that the BBM equation has neither exact two-soliton solution nor exact multi-soliton solution [12, p. 649].

At the same time, a number of exact and numerical solutions and new qualitative properties to the BBM equation have been discovered. For example, different exact solutions to (2) were found in [21, 28, 29] using the sech – tanh-method and the cos-function method. In particular, it was found through the sech – tanh-method that the BBM equation has a soliton of the following form [29]

$$u(x, t) = 3(a - 1) \cosh^{-2} \left( \frac{1}{2} \sqrt{\frac{a - 1}{a}} (x - at) + C \right), \quad (3)$$

where $a, C$ are some real constants.

Moreover, shock solutions to equation (2) where found in [30], asymptotic stability of its solitary waves was studied in [31], the existence of the global attractor for it was proved in [32], and in [33], the existence of the global solutions to the BBM equation was established.

Taking into account that equation (2) has one-soliton solution mentioned above, it's naturally to consider the problem of constructing asymptotic soliton-like solutions to the BBM equation with variable coefficients and a small parameter at the highest derivative analogously to the KdV equation [34] – [36].

Thus, the present paper deals with the singularly perturbed equation of the following form

$$a(x, t, \varepsilon) u_t + b(x, t, \varepsilon) u_x + c(x, t, \varepsilon) u u_x - \varepsilon^n u_{xxx} = 0, \quad (x, t) \in \mathbb{R} \times [0; T], \quad (4)$$
where \( \varepsilon > 0 \) is a small parameter, \( n \) is natural and functions \( a(x, t, \varepsilon), b(x, t, \varepsilon), c(x, t, \varepsilon) \) are generally infinitely differentiable with respect to all variables \( (x, t, \varepsilon) \in \mathbb{R} \times [0; T] \times [0; \varepsilon_0) \) for some \( T > 0, \varepsilon_0 > 0 \).

The functions \( a(x, t, \varepsilon), b(x, t, \varepsilon), c(x, t, \varepsilon) \) are supposed to have asymptotic expansions

\[
\begin{align*}
a(x, t, \varepsilon) &= \sum_{k=0}^{N} \varepsilon^k a_k(x, t) + O(\varepsilon^{N+1}), \\
b(x, t, \varepsilon) &= \sum_{k=0}^{N} \varepsilon^k b_k(x, t) + O(\varepsilon^{N+1}), \\
c(x, t, \varepsilon) &= \sum_{k=0}^{N} \varepsilon^k c_k(x, t) + O(\varepsilon^{N+1}).
\end{align*}
\]

In addition we suppose \( a_0(x, t) b_0(x, t) c_0(x, t) \neq 0 \) for all \( (x, t) \in \mathbb{R} \times [0; T] \).

We search functions represented as asymptotic expansions in a small parameter that satisfy equation (4) with certain accuracy. These functions are chosen so that in the case of constant coefficients, they coincide with the exact soliton solutions of equation (4). Therefore, the searched functions are called asymptotic soliton-like solutions to the given equation.

To find such functions, general ideas and methods of asymptotic analysis can be applied. We need to define the form of these functions, to propose an recurrent algorithm determining all of the members of the corresponding asymptotic solution, and, in addition, to evaluate the accuracy with which the asymptotic approximations satisfy the equation.

The problem of the existence of solutions to a equation is not usually studied in asymptotic analysis [17], [37], [38], [39], [40], [41], [42], since this question relates to problems of another type, which are usually complex and require completely different methods and approaches for their analysis.

For example, for studying the problem of existence of solutions to the generalized KdV equation with variable coefficients in the Schwartz space, methods of parabolic regularization and a priori estimates [43] are used.

Below, we present an algorithm for constructing asymptotic soliton-like solutions to equation (4) and find the accuracy with which constructed asymptotic solutions satisfy equation (4). The algorithm is analogous to the one developed for constructing asymptotic one-, two- and multi-phase soliton-like solutions to the KdV equation with variable coefficients [34], [35], [36]. It is based on the nonlinear WKB technique. It should be noted that asymptotic soliton-like solutions to the integrable type equations were firstly constructed by Maslov V.P. and his coauthors [40]. They applied the nonlinear WKB technique
developed for constructing quasi-periodic solutions to the singularly perturbed KdV equation with constant coefficient \[44\] in 1974.

The present paper is organized as follows. In Section 2, we give preliminary notes and formulate auxiliary definitions. In Section 3, an algorithm for constructing an asymptotic soliton-like solution to the BBM equation is proposed and described in details. We discuss procedures of finding terms of asymptotic expansions and, in particular, solvability of differential equations for regular and singular parts of the asymptotic solutions. In Section 4, theorems on justification of proposed algorithm are proved.

2. Preliminary notes and definitions

Let \( S = S(\mathbb{R}) \) be a space of quickly decreasing functions, i.e. the space of infinitely differentiable on \( \mathbb{R} \) functions such that for any integers \( m, n \geq 0 \) the condition
\[
\sup_{x \in \mathbb{R}} |x^m D^n u(x)| < +\infty
\]
is satisfied.

Let \( G_1 = G_1(\mathbb{R} \times [0; T] \times \mathbb{R}) \) be a space of infinitely differentiable functions \( f = f(x, t, \tau), (x, t, \tau) \in \mathbb{R} \times [0; T] \times \mathbb{R} \) such that there are fulfilled the following conditions \[40\]:

1°. the relation
\[
\lim_{\tau \to +\infty} \tau^n \frac{\partial^p}{\partial x^p} \frac{\partial^q}{\partial t^q} \frac{\partial^r}{\partial \tau^r} f(x, t, \tau) = 0, \quad (x, t) \in K,
\]
takes place;

2°. there exists such a differentiable function \( f^-(x, t) \) that on any compact set \( K \subset \mathbb{R} \times [0; T] \) condition
\[
\lim_{\tau \to -\infty} \tau^n \frac{\partial^p}{\partial x^p} \frac{\partial^q}{\partial t^q} \frac{\partial^r}{\partial \tau^r} (f(x, t, \tau) - f^-(x, t)) = 0, \quad (x, t) \in K,
\]
is satisfied for any non-negative integers \( n, p, q, r \) uniformly with respect to \( (x, t) \in K \).

Let \( G_1^0 = G_1^0(\mathbb{R} \times [0; T] \times \mathbb{R}) \subset G_1 \) be a space of functions \( f = f(x, t, \tau) \in G_1, (x, t, \tau) \in \mathbb{R} \times [0; T] \times \mathbb{R} \) such that uniformly with respect to variables \( (x, t) \) on any compact \( K \subset \mathbb{R} \times [0; T] \) the following condition
\[
\lim_{\tau \to -\infty} f(x, t, \tau) = 0
\]
takes place.

We use the following definition of an asymptotic soliton-like function.
Definition. A function \( u = u(x, t, \varepsilon) \), where \( \varepsilon \) is a small parameter, is called an asymptotic one phase soliton-like function if for any integer \( N \geq 0 \) it can be represented in the form of

\[
u(x, t, \varepsilon) = \sum_{j=0}^{N} \varepsilon^{j} [u_{j}(x, t) + V_{j}(x, t, \tau)] + O(\varepsilon^{N+1}), \quad \tau = \frac{x - \varphi(t)}{\varepsilon},
\]

where \( \varphi(t) \in C^\infty([0; T]) \) is a scalar real-valued function; \( u_{j}(x, t) \in C^\infty(\mathbb{R} \times [0; T]) \), \( j = 0, N \); \( V_{0}(x, t, \tau) \in G_{1}^{0} \); \( V_{j}(x, t, \tau) \in G_{1}^{1} \), \( j = 1, N \).

The function \( x - \varphi(t) \) is called a phase of the one-phase soliton-like function \( u(x, t, \varepsilon) \). A curve determined by equation \( x - \varphi(t) = 0 \) is called a curve of discontinuity for function (6).

It should be noted that the definition 1 was formulated in monograph [40] concerning problem of constructing asymptotic soliton-like solution to the KdV equation with small parameter at the highest derivative for the case when power of small parameter is equal to 2. In general, an asymptotic soliton-like solution to the singularly perturbed KdV equation has more complicated structure [34] depending on the power of a small parameter at the highest derivative.

In the sequel, we use the notation of asymptotic analysis of the following form \( \Psi(x, t, \varepsilon) = O(\varepsilon^{N}), \varepsilon \to 0. \) It means that there exist such values \( \varepsilon_{0} > 0, C > 0 \) that \( |\Psi(x, t, \varepsilon)| \leq C \varepsilon^{N} \) for all \( \varepsilon \in (0; \varepsilon_{0}), (x, t) \in K, \) where compact set \( K \subset \mathbb{R} \times [0; T] \) and value \( C \) is only depending on the number \( N \) and the set \( K \subset \mathbb{R} \times [0; T] \).

3. Algorithm for constructing asymptotic solutions. Let us consider a problem of constructing asymptotic one-phase soliton-like solutions to equation (4). Analogously to the case of the KdV equation [34, 35, 36], the form of the asymptotic solutions depends on the power \( n \) of a small parameter at the highest derivative in (4). So, asymptotic solutions to equation (4) are written in the form

\[
u(x, t, \varepsilon) = \sum_{j=0}^{N} \varepsilon^{j} u_{j}(x, t) + \sum_{j=0}^{N} \varepsilon^{j} V_{j}(x, t, \tau) + O(\varepsilon^{N+1}), \quad \tau = \frac{x - \varphi(t)}{\varepsilon^{n/2}},
\]

when \( n \) is an even number, and

\[
u(x, t, \varepsilon) = \sum_{j=0}^{k} \varepsilon^{j} u_{j}(x, t) + \sum_{j=0}^{k} \varepsilon^{j} V_{j}(x, t, \tau) + \varepsilon^{k} \sum_{j=1}^{2N-2k} \varepsilon^{j/2} u_{j}(x, t) +
\]

\[+ \varepsilon^{k} \sum_{j=1}^{2N-2k} \varepsilon^{j/2} V_{j}(x, t, \tau) + O(\varepsilon^{N+1/2}), \quad \tau = \frac{x - \varphi(t)}{\sqrt{\varepsilon} \varepsilon^{k}},
\]

when \( n \) is such an odd number that \( n = 2k + 1, k \in \mathbb{N} \cup \{0\} \).
Further, we consider the case $n = 2$ allowing us to demonstrate all details of algorithm on searching asymptotic solutions to equation (4). Thus, the asymptotic solutions are constructed as follows
\[ u(x, t, \varepsilon) = Y_N(x, t, \tau, \varepsilon) + O(\varepsilon^{N+1}), \]  
(9)
where
\[ Y_N(x, t, \tau, \varepsilon) = \sum_{j=0}^{N} \varepsilon^j [u_j(x, t) + V_j(x, t, \tau)], \quad \tau = \frac{x - \varphi(t)}{\varepsilon}. \]

The function
\[ U_N(x, t, \varepsilon) = \sum_{j=0}^{N} \varepsilon^j u_j(x, t) \]  
(10)
is called a regular part of asymptotic solution (9), and the function
\[ V_N(x, t, \tau, \varepsilon) = \sum_{j=0}^{N} \varepsilon^j V_j(x, t, \tau) \]  
(11)
is called a singular part of asymptotic solution (9).

It’s clear $Y_N(x, t, \tau, \varepsilon) = U_N(x, t, \varepsilon) + V_N(x, t, \tau)$.

In order to obtain differential equations for terms of the regular and the singular parts of asymptotic series (9), we apply basic ideas of asymptotic analysis [37], [45], [46]. In more details, we put series (9) into equation (4), take into account property $V_j(x, t, \tau) \in G_1, \ j = 0, N$, and equate coefficients at the same powers of a small parameter in left and right sides of the relation after the substitution.

The terms of the regular part (10) satisfy the following system of the first order partial differential equations
\[ a_0(x, t) \frac{\partial u_0}{\partial t} + b_0(x, t) \frac{\partial u_0}{\partial x} + c_0(x, t) u_0 \frac{\partial u_0}{\partial x} = 0, \]  
(12)
\[ a_0(x, t) \frac{\partial u_j}{\partial t} + b_0(x, t) \frac{\partial u_j}{\partial x} + c_0(x, t) \left( u_j \frac{\partial u_0}{\partial x} + u_0 \frac{\partial u_j}{\partial x} \right) = f_j(x, t, u_0, u_1, \ldots, u_{j-1}), \quad j = 1, N, \]  
(13)
where functions $f_j(x, t, u_0, u_1, \ldots, u_{j-1}), j = 1, N$, are recurrently determined.

System (12), (13) contains the only quasi-linear equation while the others are linear ones. The functions $u_j(x, t), j = 0, N$, can be easy found recurrently through integrating equations (12), (13), for example, by means of the method of characteristics [47]. Regarding it the terms of regular part (10) of asymptotic are supposed to be known.
3.1. Singular part of asymptotic. The terms of singular part are defined as solutions to system of the third order partial differential equations

\[ \varphi'(t) \frac{\partial^3 V_0}{\partial \tau^3} + (b_0(x,t) - \varphi'(t) a_0(x,t)) \frac{\partial V_0}{\partial \tau} + c_0(x,t) (u_0 + V_0) \frac{\partial V_0}{\partial \tau} = 0, \quad (14) \]

\[ \varphi'(t) \frac{\partial^3 V_j}{\partial \tau^3} + (b_0(x,t) - \varphi'(t) a_0(x,t)) \frac{\partial V_j}{\partial \tau} + c_0(x,t) \left( u_0 \frac{\partial V_j}{\partial \tau} + \frac{\partial}{\partial \tau} (V_0 V_j) \right) = F_j(x,t,\tau), \quad (15) \]

where functions \( F_j(x,t,\tau) = F_j(t,V_0(x,t,\tau),\ldots,V_{j-1}(x,t,\tau), u_0(x,t), \ldots, u_j(x,t)) \), are defined recurrently after determining the functions \( u_0(x,t), \ldots, u_1(x,t), \ldots, u_j(x,t), V_0(x,t,\tau), V_1(x,t,\tau), \ldots, V_{j-1}(x,t,\tau), j = 1, N \).

Recall that the solutions to equations (14), (15) must belong to the spaces \( G_1^0, G_1 \) correspondingly. Besides, under the searching the functions \( V_j(x,t,\tau), j = 0, N \), we have also to find a function \( \varphi = \varphi(t) \) defining a discontinuity curve \( \Gamma = \{(x,t) \in \mathbb{R} \times [0;T] : x = \varphi(t)\} \).

Taking the remarks into attention, we may study system (14), (15) as follows. Firstly, we assume the function \( \varphi = \varphi(t) \) is known. Then equations (14), (15) are considered on the discontinuity curve \( \Gamma \) and value \( t \) in the equations is supposed to be a parameter. In this connection the function \( v_0(t,\tau) = V_0(x,t,\tau) \bigg|_{x=\varphi(t)} \) can be found in explicit form.

Secondly, we prove \( v_0(t,\tau) \) to be a quickly decreasing function with respect to variable \( \tau \), i.e. \( v_0(t,\tau) \in G_0^1 \). Then using property \( V_1(x,t,\tau) \in G_1 \), we find the solution \( v_1(t,\tau) = V_1(x,t,\tau) \bigg|_{x=\varphi(t)} \) in explicit form too. Moreover, we receive necessary and sufficient condition on existence the solution in the space of quickly decreasing functions with respect to the variable \( \tau \) as \( \tau \to +\infty \). Later, the condition is used for obtaining nonlinear ordinary differential equation for function \( \varphi = \varphi(t) \).

We now proceed with the description of the algorithm in details. Denote \( v_j = v_j(t,\tau) = V_j(x,t,\tau) \bigg|_{x=\varphi(t)}, j = 0, N \). From (4), (14) it follows that functions \( v_j, j = 0, N \), satisfy differential equations:

\[ \varphi'(t) \frac{\partial^3 v_0}{\partial \tau^3} + (b_0(\varphi, t) - \varphi'(t) a_0(\varphi, t) + c_0(\varphi, t) u_0(\varphi, t)) \frac{\partial v_0}{\partial \tau} + c_0(\varphi, t) v_0 \frac{\partial v_0}{\partial \tau} = 0; \quad (16) \]

\[ \varphi'(t) \frac{\partial^3 v_j}{\partial \tau^3} + (b_0(\varphi, t) - \varphi'(t) a_0(\varphi, t) + c_0(\varphi, t) u_0(\varphi, t)) \frac{\partial v_j}{\partial \tau} + \]

\[ + c_0(\varphi, t) \frac{\partial}{\partial \tau} (v_0 v_j) = F_j(t,\tau), \quad (17) \]
where \( \mathcal{F}_j(t, \tau) = F_j(t, V_0(x, t, \tau), \ldots, V_{j-1}(x, t, \tau), u_0(x, t), \ldots, u_j(x, t)) \bigg|_{x=\varphi(t)} \), are easy defined recurrently after finding functions \( u_0(x, t), u_1(x, t), \ldots, u_j(x, t), V_0(x, t, \tau), V_1(x, t, \tau), \ldots, V_{j-1}(x, t, \tau), j = 1, \mathcal{N} \). Here and below \( \varphi = \varphi(t) \).

In (17), for example,

\[
\mathcal{F}_1(t, \tau) = -a_0(\varphi, t) \frac{\partial v_0}{\partial t} - c_0(\varphi, t) u_0(x, \varphi, t) v_0 - \\
- (c_0x(\varphi, t) u_0(\varphi, t) + c_0(\varphi, t) u_0x(\varphi, t) - \varphi'(t) a_0x(\varphi, t) + b_0x(\varphi, t)) \frac{\tau}{\partial \tau} - \\
- (c_0x(\varphi, t) \tau + c_1(\varphi, t)) v_0 \frac{\partial v_0}{\partial \tau} - \\
- (c_0(\varphi, t) u_1(\varphi, t) + c_1(\varphi, t) u_0(\varphi, t) - \varphi'(t) a_1(\varphi, t) + b_1(\varphi, t)) \frac{\partial v_0}{\partial \tau} + \frac{\partial^3 v_0}{\partial \tau^2 \partial t}.
\]

Let us consider equation (16). We can find its solution in the following way. By integrating it with respect to \( \tau \), we obtain

\[
\varphi'(t) \frac{d^2 v_0}{d \tau^2} = A(\varphi, t) v_0(t, \tau) - \frac{1}{2} c_0(\varphi, t) v_0^2(t, \tau) + C_1(t),
\]

where

\[
A(\varphi, t) = \varphi'(t) a_0(\varphi, t) - b_0(\varphi, t) - c_0(\varphi, t) u_0(\varphi, t).
\]

Since \( v_0(t, \tau) \in G_1^{0} \) we can put \( C_1(t) \equiv 0 \).

Multiplying both sides of equation (19) by \( dv_0 / d\tau \) gives us

\[
\frac{1}{2} \varphi'(t) \frac{d}{d\tau} \left( \frac{dv_0}{d\tau} \right)^2 = \frac{1}{2} A(\varphi, t) \frac{dv_0^2}{d\tau} - \frac{1}{6} c_0(\varphi, t) \frac{dv_0^3}{d\tau}.
\]

By further integration of the relation in \( \tau \) we get

\[
\frac{1}{2} \varphi'(t) \left( \frac{dv_0}{d\tau} \right)^2 = \frac{1}{2} A(\varphi, t) v_0^2 - \frac{1}{6} c_0(\varphi, t) v_0^3 + C_2(t).
\]

Property \( v_0(t, \tau) \in G_1^{0} \) allows us to take \( C_2(t) \equiv 0 \).

Finally, we have solution to equation (16) in the following form

\[
v_0(t, \tau) = \frac{3A(\varphi, t)}{c_0(\varphi, t)} ch^{-2} \left( \frac{1}{2} \sqrt{\frac{A(\varphi, t)}{\varphi'(t)}(\tau + C_0(t))} \right).
\]

Function (22) is evidently quickly decreasing with respect to variable \( \tau \) for all \( t \in [0; T] \).

Thus the next lemma is true.
Lemma 1. If inequality
\[ \varphi'(t)A(\varphi(t), t) > 0 \]  \text{(23)}
is fulfilled for all \( t \in [0; T] \), then function \text{(22)} is a solution to equation \text{(16)} and belongs to the space \( G_1^0 \).

3.2. Orthogonality condition. Let us discuss now the problem on existence of solution to equation \text{(17)} in the space \( G_1 \).

The following lemma ensures necessary and sufficient conditions of solvability of the problem.

Lemma 2. Let inequality \text{(23)} be fulfilled for all \( t \in [0; T] \) and \( F_j(t, \tau) \in G_1^0, \ j = 1, N \). Then equation \text{(17)} has a solution \( v_j(t, \tau) \in G_1, \ j = 1, N \), if and only if
\[ \int_{-\infty}^{+\infty} F_j(t, \tau)v_0(t, \tau)d\tau = 0, \ j = 1, N. \] \text{(24)}

Proof. Firstly, we show that a solution to equation \text{(17)} can be represented as follows
\[ v_j(t, \tau) = \nu_j(t)\eta_j(t, \tau) + \psi_j(t, \tau), \ j = 1, N, \] \text{(25)}
where \( \eta_j(t, \tau) \in G_1 \) and \( \lim_{\tau \to -\infty} \eta_j(t, \tau) = 1; \ \psi_j(t, \tau) \in G_1^0, \)
\[ \nu_j(t) = [-\varphi'(a_0(\varphi, t) + b_0(\varphi, t) + c_0(\varphi, t)u_0(\varphi, t)]^{-1} \lim_{\tau \to -\infty} \Phi_j(t, \tau), \] \text{(26)}
\[ \Phi_j(t, \tau) = \int_{-\infty}^{\tau} F_j(t, \xi)d\xi + E_j(t). \] \text{(27)}
Here value \( E_j(t) \) no depends on variable \( \tau \) and it can be found from condition
\[ \lim_{\tau \to +\infty} \Phi_j(t, \tau) = 0. \]

To state representation \text{(25)} we integrate equation \text{(17)} in \( \tau \) in limits from \( -\infty \) to \( \tau \). So, we obtain differential equation
\[ Lv_j = \Phi_j(t, \tau), \] \text{(28)}
where operator \( L \) is defined with formula
\[ L = \varphi'(t) \frac{d^2}{dT^2} - \varphi'(t) a_0(\varphi, t) + b_0(\varphi, t) + c_0(\varphi, t)u_0(\varphi, t) + c_0(\varphi, t)v_0(t, \tau). \] \text{(29)}

By virtue of formulae \text{(25)}, \text{(28)} the function \( \psi_j(t, \tau), \ j = 1, N, \) has to satisfy inhomogeneous equation
\[ L\psi_j = \Phi_j - \nu_j L\eta. \] \text{(30)}
Owing to property \( \text{ker } L^* = \{v_0\} \) and theorem on existence of a solution to the inhomogeneous equation with the one-dimensional Schrodinger operator in the space of quickly decreasing functions \( [48] \), we obtain the following statement: equation (30) has a solution in the space \( G^0_1 \) iff the following orthogonality condition

\[
\int_{-\infty}^{+\infty} (\Phi_j - \nu_j L\eta) v_0 \tau d\tau = 0, \quad j = 1, N,
\]

(31)

takes place.

Finally, from (31), (27), (28), we deduce condition (24). Lemma 2 is proved.

Relation (24) is called orthogonality condition. In case \( j = 1 \) it may be used for deducing differential equation for function \( \varphi = \varphi(t) \). The condition can be also used for determining an interval \([0; T]\) where asymptotic one-phase soliton-like solution has to be considered. The problem is studied in details below.

The following lemma describes more exact properties of solution to equation (17).

**Lemma 3.** Let conditions of lemma 2 and relation (24) be satisfied. Then \( v_j(t, \tau) \in G^0_1, \quad j = 1, N \), if and only if the condition

\[
\lim_{\tau \to -\infty} \Phi_j(t, \tau) = 0, \quad j = 1, N,
\]

(32)
is fulfilled.

The proof is trivial and it follows from representation (25).

In particular case as \( j = 1 \) condition (32) has the following form

\[
a_0(\varphi, t) \frac{d}{dt} \sqrt{A(\varphi, t) \varphi'(t)} + \sqrt{A(\varphi, t) \varphi'(t)} \frac{\varphi'(t)}{c_0(\varphi, t)} \times
\]

\[
\times \left[ a_{0x}(\varphi, t) \varphi'(t) - b_{0x}(\varphi, t) - c_{0x}(\varphi, t) u_0(\varphi, t) - \frac{A(\varphi, t)}{c_0(\varphi, t)} \right] = 0.
\]

(33)

**3.3. Differential equation for discontinuity curve.** Using (24) as \( j = 1 \) and (18) by means of tedious but not complicated calculations we find the second order ordinary differential equation for function \( \varphi = \varphi(t) \) in the following form

\[
[A_1 \varphi''^2 + A_2 \varphi' + A_3] \varphi'' + A_4 \varphi'^4 + A_5 \varphi'^3 + A_6 \varphi'^2 + A_7 \varphi' = 0,
\]

(34)

where coefficients \( A_k = A_k(\varphi, t), \quad k = 1, 7 \), are given as follows

\[
A_1 = 24 a_0^2 c_0, \quad A_2 = -8 a_0 c_0 \alpha, \quad A_3 = -c_0 \alpha^2, \quad A_4 = -40 c_{0x} a_0^2 + 30 a_0 a_{0x} c_0, \quad A_5 = -120 a_0 c_{0x} a_0^2 + 120 c_{0x} a_0 a_{0x} c_0, \quad A_6 = -120 c_{0x} a_0^2 + 120 a_0 a_{0x} c_0, \quad A_7 = 24 a_0^2 c_{0x},
\]
\[ A_5 = 60 a_0 c_{0x} \alpha + 20 a_0 a_{0t} c_0 - 24 a_0^2 c_{0t} - 30 a_0 c_0 \alpha_x - 15 a_{0x} c_0 \alpha + 20 a_0 c_0^2 u_{0x}, \]
\[ A_6 = -20 a_0 c_0 \alpha_t - 5 a_0 c_0 \alpha + 15 c_0 \alpha_x + 28 a_0 c_{0t} \alpha - 20 c_0^2 u_{0x} \alpha - 20 c_{0x} \alpha^2, \]
\[ A_7 = 5 c_0 \alpha \alpha_t - 20 c_{0t} \alpha^2, \]

where \( \alpha = b_0 + c_0 u_0, \) \( a_0 = a_0(\varphi, t), \) \( b_0 = b_0(\varphi, t), \) \( c_0 = c_0(\varphi, t), \) \( u_0 = u_0(\varphi, t). \)

Differential equation (34) is nonlinear and has smooth coefficients. A problem of existence of its solution must be studied in every case. In general, the equation possesses a solution on only finite time interval. Therefore, the supposed interval of existing its solution is denoted by \([0; T].\)

### 3.4. Exact solutions to system of differential equations (16), (17).

The general solution to equation (16) in space \( G_1^0 \) is given by formula (22). Let us proceed to searching a general solution to inhomogeneous equation (17) in exact form through lemma 2 and lemma 3 providing us with necessary and sufficient conditions of existing solutions to equation (17) in space \( G_1. \)

It is possible to find solution to (17) after getting a solution to equation (28). The last one can be solved using the method of variation of parameters because it’s a linear inhomogeneous ordinary differential equation. According to described above procedure of constructing function \( v_0(t, \tau) \) in (22), function \( w_1(t, \tau) = v_0 \tau \) is a non-trivial solution to homogeneous equation \( Lw = 0. \) The other linearly independent solution is given through Abel’s formula [49]

\[
 w_2(t, \tau) = w_1(t, \tau) \int_{\tau_0}^{\tau} w_1^{-2}(t, \tau) d\tau = v_0 \tau(t, \tau) \int_{\tau_0}^{\tau} v_0^{-2}(t, \tau) d\tau.
\]

Thus, general solution to inhomogeneous equation (17) is written in the form

\[
 v_j(t, \tau) = \left( \int_{\tau_0}^{\tau} \Phi_j(t, \tau_1) v_{0\tau}(t, \tau_1) d\tau_1 + C_3 \right) v_{0\tau}(t, \tau) \int_{\tau_0}^{\tau} v_0^{-2}(t, \tau_1) d\tau_1 -
\]

\[
 - \left( \int_{\tau_0}^{\tau} \Phi_j(t, \tau_1) v_{0\tau}(t, \tau_1) \int_{\tau_0}^{\tau_1} v_0^{-2}(t, \xi) d\xi d\tau_1 + C_4 \right) v_{0\tau}(t, \tau), \quad (35)
\]

where values \( C_3, C_4 \) are some real constants.

### 3.5. Constructing terms of the singular part of asymptotic.

At last functions \( V_j(x, t, \tau), \) \( j = 0, N, \) can be determined outside of the discontinuity curve \( \Gamma. \) Taking into consideration formulae (22), (35) providing us with their values on the curve \( \Gamma \) we define functions \( V_j(x, t, \tau), \) \( j = 0, N, \) through extension \( v_j(t, \tau), \) \( j = 0, N, \) from the curve \( \Gamma \) to its neighborhood.

Since \( v_0(t, \tau) \in G_1^0 \) we can put

\[
 V_0(x, t, \tau) = v_0(t, \tau). \quad (36)
\]
While extending $V_j(x, t, \tau), j = 1, N$, two cases should be considered. Firstly, we suppose condition (32) takes place, i.e. $v_j(t, \tau) \in G^0_1$. The case is similar to the one for $v_0(t, \tau)$. It means that extension of function $v_j(t, \tau), j = 1, N$, from $\Gamma$ to its neighborhood can be written as

$$V_j(x, t, \tau) = v_j(t, \tau). \quad (37)$$

In the opposite case when there isn’t satisfied condition (32) we make use of representation (25). Thus extension of the function is realized as follows

$$V_j(x, t, \tau) = u_j^-(x, t)\eta(t, \tau) + \psi_j(t, \tau), \quad (38)$$

where functions $\eta_j(t, \tau), \psi_j(t, \tau), j = 1, N$, are defined under describing formulae (25), (26) while function $u_j^-(x, t), j = 1, N$, satisfies differential equation

$$\Lambda u_j^-(x, t) = f_j^-(x, t), \quad j = 1, N, \quad (39)$$

$$\Lambda = a_0(x, t)\frac{\partial}{\partial t} + b_0(x, t)\frac{\partial}{\partial x} + c_0(x, t)u_0(x, t)\frac{\partial}{\partial x} + c_0(x, t)u_0x(x, t). \quad (40)$$

In particular, here

$$f_1^-(x, t) = 0, \quad f_2^-(x, t) = -a_1(x, t)\frac{\partial u_1^-}{\partial t} - b_1(x, t)\frac{\partial u_1^-}{\partial x} - c_1(x, t)u_1^- \frac{\partial u_0}{\partial x} -$$

$$-c_0(x, t)u_1^- \frac{\partial u_1^-}{\partial x} - c_0(x, t)u_1^- \frac{\partial u_1^-}{\partial x} - c_0(x, t)u_1^- \frac{\partial u_1^-}{\partial x} - c_1(x, t)u_0 \frac{\partial u_1^-}{\partial x}. \quad (41)$$

In addition, function $u_j^-(x, t), j = 1, N$, is clear to satisfy condition

$$u_j^-(x, t)\big|_\Gamma = v_j(t), \quad j = 1, N, \quad (42)$$

following from (25).

Thus, $u_j^-(x, t), j = 1, N$, is a solution to the Cauchy problem (39), (42).

In general, the Cauchy problem is well posed because the curve $\Gamma$ is transversal to characteristics of the operator $\Lambda$. It follows that the problem has a solution in some neighborhood $\Omega_e(\Gamma)$ of the curve $\Gamma$.

Therefore, the problem of constructing the singular part of asymptotic (9) is completely solved.

Summarizing the results given above, the following should be noted. We have found the form of asymptotic solutions to the singularly perturbed BBM equation with variable coefficients and have described in details the algorithm for constructing such solutions. Thus, we solved the first main problem of asymptotic analysis, methods of which were successfully applied to obtain approximate solutions of special form to the BBM equation (4).
Remark 1. The constructed asymptotic soliton-like solution to equation (1) doesn’t belong to the Schwartz space in the general case. It is a sum of regular part \( U_N(x, t, \varepsilon) \) and singular part \( V_N(x, t, \tau, \varepsilon) \). The regular part \( U_N(x, t, \varepsilon) \) is only enough smooth function and it doesn’t belong to the Schwartz space in general.

If \( U_N(x, t, \varepsilon) \equiv 0 \) then the constructed asymptotic solution may belong to the Schwartz space only in particular case, because all terms of the singular part except the main term belong to the space \( G_1 \not\subset G_0 \). The function \( V_N(x, t, \tau, \varepsilon) \) belongs to the Schwartz space only under the condition (32) of lemma 3 in our paper. In general, the constructed asymptotic solutions don’t approximate the quickly decreasing functions.

Remark 2. Through the proposed technique we can find asymptotic solutions coinciding with exact soliton solutions in the case of constant coefficients as well as we can construct the other type of asymptotic solutions among them there are asymptotic step-like solutions [51] and asymptotic \( \Sigma \)-solutions [52,53]. So, the set of constructed asymptotic solutions is wider than the set of quickly decreasing solutions to the BBM equation (4).

At present we turn to another important problem of asymptotic analysis relating to justification of the constructed asymptotic solutions.

4. Precision of asymptotic solution (9). Discussing the problem of asymptotic estimates for approximate solutions built for equation (4), we need to analyze the accuracy with which the solutions satisfy this equation.

The asymptotic solutions for equation (4) are represented by formula (9) where their singular part is written in two ways depending on condition (32). However, both forms of the asymptotic solutions satisfy the equation with the same precision as confirmed by the following relevant statements.

Theorem 1. Let the following conditions be supposed:
1. functions \( a_k(x, t), b_k(x, t), c_k(x, t) \in C^\infty(\mathbb{R} \times [0; T]), k = 0, N; \)
2. inequality (23) is fulfilled;
3. orthogonality conditions (24) are satisfied;
4. conditions (32) are realized.

Then asymptotic one-phase soliton-like solution to equation (4) is written as
\[
U_N(x, t, \varepsilon) = \begin{cases}
Y_N^-(x, t, \varepsilon), & (x, t) \in D^- \setminus \Omega_\varepsilon(\Gamma), \\
Y_N(x, t, \varepsilon), & (x, t) \in \Omega_\varepsilon(\Gamma), \\
Y_N^+(x, t, \varepsilon), & (x, t) \in D^+ \setminus \Omega_\varepsilon(\Gamma),
\end{cases}
\] (43)
where

\[ Y^-(x, t, \varepsilon) = \sum_{j=0}^{N} \varepsilon^j u_j(x, t), \quad (x, t) \in D^- = \{(x, t) \in \mathbb{R} \times [0; T] : x - \varphi(t) < 0\}, \]

(44)

\[ Y^+(x, t, \varepsilon) = \sum_{j=0}^{N} \varepsilon^j [u_j(x, t) + V_j(t, \tau)], \quad \tau = \frac{x - \varphi(t)}{\varepsilon}, \quad (x, t) \in \Omega^\varepsilon(\Gamma), \]

(45)

\[ Y^+ (x, t, \varepsilon) = \sum_{j=0}^{N} \varepsilon^j u_j(x, t), \quad (x, t) \in D^+ = \{(x, t) \in \mathbb{R} \times [0; T] : x - \varphi(t) > 0\}. \]

(46)

In addition, function (43) satisfies equation (4) on the set \( \mathbb{R} \times [0; T] \) with accuracy \( O(\varepsilon^N) \). As \( \tau \to \pm \infty \) the function satisfies (4) with asymptotic estimate \( O(\varepsilon^{N+1}) \).

**Proof of theorem 1.** Proving the theorem is not complicated and it is done in the standard way similarly to the proof of the analogous statement for the singularly perturbed KdV equation with variable coefficients that was described in details in [36]. That is why we avoid demonstration of tedious calculations in full here and describe only the basic idea.

We have to obtain an asymptotic estimate for discrepancy of the approximate solution that is given by formula (43).

From the structure of function (43), we can see that in domains \( D^+ \setminus \Omega^\varepsilon(\Gamma) \), \( D^- \setminus \Omega^\varepsilon(\Gamma) \) asymptotic solution contains only regular part of the asymptotic and in them it satisfies the equation with accuracy \( O(\varepsilon^N) \) through its determining.

Thus, it remains only to consider domain \( \Omega^\varepsilon(\Gamma) \). Function (43) is substituted into equation (4). Further, equations (12), (13) for regular part of asymptotic (43) are taken into account.

The next step is considering functions \( a_j(x, t), \ b_j(x, t), \ c_j(x, t), \ u_j(x, t), \ j = 0, N \), in neighborhood of the discontinuity curve \( \Gamma \) and their representation as Taylor polynomials with the required accuracy.

Finally, equations (16), (17) for singular parts of asymptotic (43) as well as property of functions \( V_j(t, \tau) \in C^0_1, \ j = 0, N \), are used for obtaining asymptotic estimation of discrepancy for the approximate solution (43). This completes the proof of theorem 1.

**Theorem 2.** Let the following conditions be satisfied:
1. functions \( a_k(x, t), \ b_k(x, t), \ c_k(x, t) \in C^\infty(\mathbb{R} \times [0; T]) \), \( k = 0, N \);
2. inequality (23) is fulfilled;
3. orthogonality conditions (24) are satisfied;
4. the Cauchy problem (39), (42) has a solution on the set \( \{(x, t) \in \mathbb{R} \times [0; T] : x - \varphi(t) \leq 0\} \).
Then the asymptotic one-phase soliton-like solution to equation (4) can be written as

\[
  u_N(x, t, \varepsilon) = \begin{cases} 
  Y_N^-(x, t, \varepsilon), & (x, t) \in D^- \setminus \Omega_\varepsilon(\Gamma), \\
  Y_N(x, t, \varepsilon), & (x, t) \in \Omega_\varepsilon(\Gamma), \\
  Y_N^+(x, t, \varepsilon), & (x, t) \in D^+ \setminus \Omega_\varepsilon(\Gamma), 
\end{cases}
\]  

(47)

where

\[
  Y_N^-(x, t, \varepsilon) = \sum_{j=0}^N \varepsilon^j u_j(x, t) + \sum_{j=1}^N \varepsilon^j u_{-j}^-(x, t), 
\]

(48)

\[
  Y_N(x, t, \varepsilon) = \sum_{j=0}^N \varepsilon^j [u_j(x, t) + V_j(x, t, \tau)], 
\]

\[
  \tau = \frac{x - \varphi(t)}{\varepsilon}, 
\]

(49)

\[
  Y_N^+(x, t, \varepsilon) = \sum_{j=0}^N \varepsilon^j u_j(x, t), 
\]

(50)

In addition, function (47) satisfies equation (4) with accuracy \(O(\varepsilon^N)\) on the set \(R \times [0; T]\). As \(\tau \to \pm \infty\) the solution satisfies (4) with asymptotic estimate \(O(\varepsilon^{N+1})\).

Proof of theorem 2. Function (47) gives another form of asymptotic one-phase soliton-like solution to equation (4). However, the proof is similar to proof of theorem 1. In other words, we have also to obtain asymptotic estimate for discrepancy of the approximate solution that is given by formula (47).

In domains \(\Omega_\varepsilon(\Gamma), D^+ \setminus \Omega_\varepsilon(\Gamma)\) function (47) is the same as (43) through composition, in fact. So, the proof of theorem 2 is analogous to proof of theorem 1 for the case of these domains.

Finally, we consider set \(D^- \setminus \Omega_\varepsilon(\Gamma)\). Here function (47) differs from (43) in expression \(\sum_{j=1}^N \varepsilon^j u_j^-(x, t)\), \(j = 1, N\). The last equates relation (4) with asymptotic precise \(O(\varepsilon^{N+1}), \varepsilon \to 0\).

Thus, on the set \(R \times [0; T]\) function (47) satisfies equation (4) with accuracy \(O(\varepsilon^N)\). Due to properties of functions forming the singular part of the asymptotic solution, it satisfies equation (4) with precision \(O(\varepsilon^{N+1})\) as \(\tau \to \pm \infty\).

Theorem 2 is proved.

5. Discussions and conclusions.

This paper deals with one of the main problems of asymptotic analysis that concerns the development of an algorithm for constructing approximate (asymptotic) solutions to a partial differential equation with singular perturbation. The solutions satisfy the perturbed equation with a certain (asymptotic) accuracy.
We have found the form of the asymptotic solutions to the equation and proposed and described in details an algorithm for constructing asymptotic solutions of a special form for the singularly perturbed Benjamin–Bona–Mahony (BBM) equation with variable coefficients. Statements on justification of the algorithm are proved.

The asymptotic solutions consist of regular and singular parts. The singular part gives a solution that describes the soliton wave of the BBM equation in the case of constant coefficients [29]. In the case of variable coefficients for the BBM equation, a singular part does not necessarily belong to a space of functions that are quickly decreasing with respect to a phase variable. But in some cases, such a property is fulfilled. Therefore, similar solutions can be considered as a generalization of soliton ones and they can be called soliton-like [40]. The regular part creates a background on which soliton-like waves move.

It should be also mentioned that the properties of solutions of initial problems for evolution equations depend essentially on the initial functions, especially in the case of nonlinear partial differential equations. In this connection, we can mention the Cauchy problem for the Korteweg-de Vries equation with smooth initial function [50]. It has singular solutions that are destroyed at finite time "roughly" or accordingly to scenario of gradient catastrophe. The problem of existence of different type of solutions to the Cauchy problem for the BBM equation is also interesting, but it is not discussed in the paper. This challenging problem requires additional in-depth research.

The proposed algorithm can be applied to constructing asymptotic soliton-like solutions of different nonlinear partial differential equations of integrable type with variable coefficients because the algorithm allows us to obtain exact soliton solutions to the equations in the case of constant coefficients.

Moreover, the proposed technique allows us to construct not only asymptotic soliton-like solutions coinciding with exact soliton solutions in case of constant coefficients but also the other types of approximate solutions, including asymptotic step-like [51] and asymptotic Σ – solutions [52], [53].

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