WHAT DOES A VECTOR FIELD KNOW ABOUT VOLUME?

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Abstract. This note provides an affirmative answer to a question of Viterbo concerning the existence of nondiffeomorphic contact forms that share the same Reeb vector field. Starting from an observation by Croke–Kleiner and Abbondandolo that such contact forms define the same total volume, we discuss various related issues for the wider class of geodesible vector fields. In particular, we define an Euler class of a geodesible vector field in the associated basic cohomology and give a topological characterisation of vector fields with vanishing Euler class. We prove the theorems of Gauß–Bonnet and Poincaré–Hopf for closed, oriented 2-dimensional orbifolds using global surfaces of section and the volume determined by a geodesible vector field. This volume is computed for Seifert fibred 3-manifolds and for some transversely holomorphic flows.

1. Introduction

This paper is concerned with a question about Reeb flows posed to me by Claude Viterbo: are there nondiffeomorphic contact forms with the same Reeb vector field? Viterbo’s question was prompted by Alberto Abbondandolo’s discovery of a miraculous identity on differential forms.

Lemma 1.1 (Abbondandolo). Given two differential 1-forms $\alpha, \beta$ on the same manifold, the identity

\[(1) \quad \alpha \wedge (d\alpha)^n - \beta \wedge (d\beta)^n = (\alpha - \beta) \wedge \sum_{j=0}^{n}(d\alpha)^j \wedge (d\beta)^{n-j} + d(\alpha \wedge \beta \wedge \sum_{j=1}^{n-1}(d\alpha)^j \wedge (d\beta)^{n-1-j})\]

holds for any $n \in \mathbb{N}_0$.

Identity (1), whose verification is straightforward, has the following striking consequence, which — as we learned in the meantime — has been observed earlier by Croke and Kleiner [10, Lemma 2.1]. They do not state identity (1), but give a quite similar proof.

Proposition 1.2 (Croke–Kleiner). Let $X$ be a nonsingular vector field on a closed, oriented manifold $M$ of dimension $2n + 1$. Let $\alpha, \beta$ be 1-forms on $M$ that are invariant under the flow of $X$ and satisfy

\[\alpha(X) = \beta(X) = 1.\]

Then

\[\int_M \alpha \wedge (d\alpha)^n = \int_M \beta \wedge (d\beta)^n.\]
Proof. Given (2), the invariance condition $L_X\alpha = L_X\beta = 0$ is equivalent to

\begin{equation}
 i_Xd\alpha = i_Xd\beta = 0
\end{equation}

by the Cartan formula. Then (3) is immediate from (1) and Stokes’s theorem. □

In particular, this proposition says that any two contact forms on a closed, oriented manifold that share the same Reeb vector field give rise to volume forms that integrate to the same total volume. In other words, this total volume is determined by the Reeb vector field alone. Abbondandolo has raised the question whether one can compute this volume from a given Reeb vector field, not knowing a contact form it is associated with.

Remark 1.3. Croke and Kleiner used this proposition to conclude that two compact Riemannian manifolds with $C^1$-conjugate geodesic flows have the same volume [10, Proposition 1.2]. This follows by considering the canonical contact form on the unit cotangent bundle, whose Reeb vector field generates the cogeodesic flow [11, Theorem 1.5.2].

As we shall see, the existence of a 1-form $\alpha$ as in Proposition 1.2 is equivalent to the vector field $X$ being geodesible (Definition 3.1, Proposition 3.3).

Definition 1.4. We write $\text{vol}_X$ for the real number defined by (3) and call it the \textit{volume} of $X$, even though $\alpha \wedge (d\alpha)^n$ is not, in general, a volume form.

Much of this paper is a rumination on the consequences and ramifications of Proposition 1.2, leading us ultimately towards an affirmative answer to Viterbo’s question (Theorem 10.1), which shows that Proposition 1.2 is indeed a nontrivial statement, even within the class of Reeb vector fields. We pay special attention to the cases where the geodesible vector field $X$ generates an $S^1$-action, or where the flow of $X$ admits a global surface of section. In these cases, one can compute $\text{vol}_X$ and give it a geometric interpretation.

Along the way, we introduce the Euler class $e_X$ of a geodesible vector field $X$ in the basic cohomology of the foliation it determines, and we argue that Proposition 1.2 ought to be interpreted as a statement in basic cohomology (Proposition 5.6). These considerations will allow us to establish a criterion for the vanishing of $e_X$ in terms of the existence of a transverse invariant foliation (Theorem 5.7). Geodesible vector fields $X$ with $e_X = 0$ exist precisely on manifolds that fibre over $S^1$ (Corollary 5.8).

In Section 6 we compute $\text{vol}_X$ for vector fields that define a Seifert fibration on a 3-manifold. This computation involves the use of global surfaces of section. With similar arguments we prove the theorems of Gauß–Bonnet and Poincaré–Hopf for closed, oriented 2-dimensional orbifolds in Section 7.

For certain geodesible vector fields $X$ whose flow admits a transverse holomorphic structure, we can relate $\text{vol}_X$ to the Bott invariant of that structure. This is the content of Section 8.

In Section 9 we derive a formula for $\text{vol}_X$ when $X$ admits a global surface of section. After presenting the answer to Viterbo’s question in Section 10, we end the paper in Section 11 with a brief discussion of orbit equivalent geodesible vector fields.
2. Dimension three

In dimension three, the answer to Viterbo’s question is negative.

**Proposition 2.1.** Let \( \alpha_0, \alpha_1 \) be two contact forms on a closed 3-manifold \( M \) sharing the same Reeb vector field \( R \). Then \( \alpha_0 \) and \( \alpha_1 \) define the same orientation of \( M \). Furthermore, there is an isotopy \( (\psi_t)_{t \in [0,1]} \) of \( M \), starting at \( \psi_0 = \text{id}_M \), such that \( \psi_t^* \alpha_1 = \alpha_0 \) and \( (\psi_t^*)^{-1} \alpha_0 \) is a contact form with Reeb vector field \( R \) for all \( t \in [0,1] \).

**Proof.** The fact that \( \alpha_0 \) and \( \alpha_1 \) define the same orientation of \( M \) follows from Proposition 1.2, since by (3) the two volume forms \( \alpha_i \wedge d\alpha_i \) must have the same sign.

Set \( \alpha_t := (1-t)\alpha_0 + t\alpha_1 \), \( t \in [0,1] \). Since \( d\alpha_0 \) and \( d\alpha_1 \) restrict to nondegenerate 2-forms defining the same orientation on any tangent 2-plane field transverse to \( R \), so does \( d\alpha_t \). It follows that \( \alpha_t \) is likewise a contact form with Reeb vector field \( R \). Now apply the Moser trick [11, p. 60] to the equation

\[ \psi_t^* \alpha_t = \alpha_0, \]

where we would like the isotopy \( (\psi_t) \) to be the flow of a time-dependent vector field \( X_t \in \ker \alpha_t \). Under this last assumption, by differentiating (5) we find

\[ \alpha_1 - \alpha_0 + i_{X_t} d\alpha_t = 0, \]

which has a unique solution \( X_t \in \ker \alpha_t \). \( \square \)

Nonetheless, the question how to compute \( \text{vol}_R \) for the Reeb vector field \( R \) on a closed contact 3-manifold \( (M, \alpha) \) is extremely interesting. Cristofaro-Gardiner, Hutchings and Ramos [9, Theorem 1.2] have established a deep connection between \( \text{vol}_R \) and embedded contact homology (ECH). For a contact 3-manifold \( (M, \alpha) \) with nonzero contact ECH invariant and finite ECH capacities \( c_k(M, \alpha) \), \( k \in \mathbb{N}_0 \), the volume of \( R \) can be computed as

\[ \text{vol}_R = \lim_{k \to \infty} \frac{c_k(M, \alpha)^2}{2k}. \]

Through this asymptotic formula, \( \text{vol}_R \) is determined in a subtle way by the periodic Reeb orbits and their actions.

3. Geodesible vector fields and taut foliations

As shown by Wadsley [31], for a nonsingular vector field \( X \) the existence of a 1-form \( \alpha \) satisfying conditions (2) and (4) is equivalent to \( X \) being geodesible. Here we briefly recall the proof of this result, since it is essential to our discussion; see also [16, 28]. Notice that \( \text{vol}_X \) is only defined for vector fields \( X \) on closed manifolds of odd dimension, but all the considerations about geodesible vector fields in this and the following two sections make sense, unless stated otherwise, for manifolds of arbitrary dimension.

**Definition 3.1.** (a) A nonsingular vector field \( X \) on a manifold \( M \) is called geodesible if there exists a Riemannian metric on \( M \) with respect to which \( X \) has unit length and the flow lines of \( X \) are geodesics.

(b) A 1-dimensional foliation \( \mathcal{F} \) on a manifold \( M \) is called taut if there exists a Riemannian metric on \( M \) for which the leaves of \( \mathcal{F} \) (suitably parametrised) are geodesics.
Lemma 3.2. Let \((M, \langle ., . \rangle)\) be a Riemannian manifold with Levi-Civita connection \(\nabla\). Let \(X\) be a vector field of unit length, and set \(\alpha = \langle X, . \rangle\). Then
\[
L_X\alpha = \langle \nabla_X X, . \rangle.
\]

Proof. The claimed identity is a pointwise statement. Locally one can always extend a tangent vector \(Y_p \in T_pM\) to an \(X\)-invariant vector field \(Y\), i.e. a vector field satisfying \([X, Y] = 0\). Therefore, it suffices to verify the identity
\[
(L_X\alpha)(Y) = \langle \nabla_X X, Y \rangle
\]
for such \(X\)-invariant vector fields \(Y\). Notice that \(\nabla\) being torsion-free then translates into \(\nabla_X Y = \nabla_Y X\). Using the fact that the Lie derivative commutes with contraction, we compute
\[
(L_X\alpha)(Y) = L_X(\alpha(Y)) - \alpha(L_X Y) = L_X(\alpha(Y)) = X\langle Y, \nabla_X X \rangle
\]
\[
= \langle \nabla_X X, Y \rangle + \langle X, \nabla_Y X \rangle
\]
\[
= \langle \nabla_X X, Y \rangle + \frac{1}{2} Y\langle X, X \rangle
\]
\[
= \langle \nabla_X X, Y \rangle.
\]

Example 3.4. The Reeb vector field of a contact form or a stable Hamiltonian structure [7] is geodesible.

Proposition 3.3. Let \(X\) be a nonsingular vector field on a manifold \(M\). Then the following are equivalent:

(i) \(X\) is geodesible;
(ii) there exists a 1-form \(\alpha\) on \(M\) with \(\alpha(X) = 1\) and \(L_X \alpha = 0\);
(iii) there exists a 1-form \(\alpha\) on \(M\) with \(\alpha(X) = 1\) and \(i_X d\alpha = 0\);
(iv) there is a hyperplane field \(\eta\) transverse to \(X\) and invariant under the flow of \(X\).

Proof. The equivalence of (ii) and (iii) is clear from the Cartan formula. We first prove the equivalence of (i) and (ii).

Assuming (i), we take \((., .)\) to be the metric for which the flow lines of \(X\) are geodesics parametrised by arc length and set \(\alpha = \langle X, . \rangle\). Then \(\nabla_X X = 0\), and (ii) follows from the lemma.

Conversely, given \(\alpha\) as in (ii) we choose a metric \((., .)\) on \(M\) with \(\langle X, X \rangle = 1\) and \(X \perp \ker \alpha\). Then \(\alpha = \langle X, . \rangle\), and the vanishing of \(L_X\alpha\) implies, by the lemma, that \(\nabla_X X = 0\).

Next we show the equivalence of (ii) and (iv). Given (ii), the hyperplane field \(\eta := \ker \alpha\) satisfies (iv). Conversely, given \(\eta\) as in (iv), define a 1-form \(\alpha\) on \(M\) by the conditions \(\alpha(X) = 1\) and \(\ker \alpha = \eta\). Then \(i_X d\alpha = L_X \alpha\), and the latter equals \(f \alpha\) for some \(f \in C^\infty(M)\) by the invariance of \(\eta\). Thus, \(i_X d\alpha\) vanishes on \(\eta\). Since \(TM = \eta \oplus \langle X \rangle\), the Lie derivative \(L_X\alpha = i_X d\alpha\) vanishes identically.

Example 3.4. The Reeb vector field of a contact form or a stable Hamiltonian structure [7] is geodesible.
The following characterisation of oriented taut 1-dimensional foliations, first observed in [28], is then immediate. We write $F = \langle X \rangle$ with any nonsingular vector field $X$ whose flow lines are the leaves of $F$.

**Proposition 3.5.** The oriented 1-dimensional foliation $F = \langle X \rangle$ is taut if and only if there is a 1-form $\alpha$ on $M$ with $\alpha(X) > 0$ and $i_X d\alpha = 0$.

**Proof.** If $F = \langle X \rangle$ is taut, rescale $X$ to a vector field of length 1 with respect to the metric that makes the leaves of $F$ geodesics. Then the existence of the desired 1-form $\alpha$ follows from the equivalence of (i) and (iii) in Proposition 3.3.

Conversely, given $\alpha$, the rescaled vector field $X/\alpha(X)$, which likewise spans $F$, satisfies (iii) in Proposition 3.3. □

**Remark 3.6.** (1) Alternatively, one can derive the equivalence of (i) and (iii) in Proposition 3.3 from the identity

$$i_X d\alpha = \langle \nabla_X X, . \rangle - d(\langle X, . \rangle/2),$$

where again $\alpha = \langle X, . \rangle$; this identity holds for any vector field $X$, see [7, Section 2.3].

(2) The main point of Sullivan’s article [28] is a characterisation of taut foliations in terms of the absence of “tangent homologies”. I refer to [16] for a beautiful discussion of Sullivan’s theorem; there one can find examples of 1-dimensional oriented foliations that are not taut.

### 4. Basic cohomology

Here are the elementary notions of basic differential forms and basic cohomology associated with a foliation. I restrict attention to oriented 1-dimensional foliations $F = \langle X \rangle$; for a more comprehensive treatment see [30, Chapter 4].

**Definition 4.1.** A differential form $\omega$ on $(M, F)$ is called basic if

$$i_X \omega = 0 \quad \text{and} \quad i_X d\omega = 0.$$

Notice that this definition does not depend on the choice of vector field $X$ spanning $F$. We write $\Omega^k_B(F)$ for the vector space of basic $k$-forms on $(M, F)$. The usual exterior differential $d$ restricts to

$$d_B: \Omega^k_B \to \Omega^{k+1}_B,$$

and the basic cohomology groups $H^k_B(F)$ are defined as the cohomology groups of the complex $(\Omega^\cdot_B(F), d_B)$. The cohomology class of a $k$-form $\omega \in \ker d_B$ is written as $[\omega]_B \in H^k_B(F)$.

The following definitions are motivated by Propositions [3.3 and 3.5]. The notation $C_X, C_F$ is chosen because the 1-form $\alpha = \langle X, . \rangle$ (with $X$ of unit length) is the characteristic form of $F$ [30, p. 69] with respect to the metric $(.,.)$. We adapt this definition to the case of geodesible vector fields, where it is reasonable to consider only those metrics for which the flow lines of $X$ are geodesics.

**Definition 4.2.** Let $X$ be a geodesible vector field. Any 1-form $\alpha$ with $\alpha(X) = 1$ and $i_X d\alpha = 0$ is called a characteristic 1-form of $X$. We write $\text{char}_X$ for the space of these characteristic forms.
Definition 4.3. (a) Let $X$ be a geodesible vector field on a manifold $M$. Set
\[ \Omega^1_X := \{ \alpha \in \Omega^1(M) : \alpha(X) = c \text{ for some } c \in \mathbb{R}^+, \ i_X \alpha = 0 \} \]
and
\[ C_X := \Omega^1_X / \sim, \]
where
\[ \alpha \sim \beta :\iff \alpha(X) = \beta(X). \]
The equivalence class of $\alpha \in \Omega^1_X$ is written as $[\alpha]_X \in C_X$. Obviously there is a canonical identification of $C_X$ with $\mathbb{R}^+$.\]
(b) Let $F = \langle X \rangle$ be an oriented taut 1-dimensional foliation on $M$. Set
\[ \Omega^1_F := \{ \alpha \in \Omega^1(M) : \alpha(X) > 0, \ i_X \alpha = 0 \} \]
and
\[ C_F := \Omega^1_F / \sim, \]
where the equivalence relation $\sim$ is defined as in (a). The equivalence class of $\alpha \in \Omega^1_F$ is written as $[\alpha]_F$. Notice that these definitions do not depend on the choice of $X$.\]
The assumptions on geodesibility and tautness, respectively, guarantee that we are not talking about empty sets.\]
The spaces $\Omega^1_X$, $\text{char}_X$ and $\Omega^1_F$ are obviously convex. The proof of Proposition 3.3 shows that, for a geodesible vector field $X$, the map
\[ \text{Met}_X \longrightarrow \text{char}_X \]
\[ \langle \cdot, \cdot \rangle \longmapsto \alpha = \langle X, \cdot \rangle \]
from the space $\text{Met}_X$ of metrics for which $X$ has unit length and geodesic flow lines is a Serre fibration with fibre the space of metrics on a hyperplane field transverse to $X$, which can be seen as follows. Given a family $\alpha_{q,t} \in \text{char}_X$, where $t \in [0, 1]$ and $q$ varies in some parameter space, and a family of metrics $\langle \cdot, \cdot \rangle_{q,0}$ with $\langle X, \cdot \rangle_{q,0} = \alpha_{q,0}$, one simply defines $\langle \cdot, \cdot \rangle_{q,t}$ by the following requirements:
(i) $\langle X, X \rangle_{q,t} = 1$;
(ii) $\ker \alpha_{q,t} \perp X$ with respect to $\langle \cdot, \cdot \rangle_{q,t}$;
(iii) $\langle \cdot, \cdot \rangle_{q,t}|_{\ker \alpha_{q,t}} = \langle \cdot, \cdot \rangle_{q,0}|_{\ker \alpha_{q,0}}$ under the identification of $\ker \alpha_{q,t}$ with $\ker \alpha_{q,0}$ given by projection along $X$.\]
Of course, this Serre fibration property is not terribly useful, since all spaces in question are contractible.\]

Proposition 4.4. Let $M$ be a closed, oriented manifold of dimension $m$.\]
(a) Let $X$ be a geodesible vector field on $M$. Set $F = \langle X \rangle$. Then the map
\[ C_X \times H^{-1}_B(F) \longrightarrow \mathbb{R} \]
\[ ([\alpha]_X, [\sigma]|_B) \longmapsto [\alpha]_X \cdot [\sigma]|_B := \int_M \alpha \wedge \sigma \]
is well defined.\]
(b) Let $F$ be an oriented taut 1-dimensional foliation on $M$. Then the map
\[ C_F \times H^{-1}_B(F) \longrightarrow \mathbb{R} \]
\[ ([\alpha]_F, [\sigma]|_B) \longmapsto [\alpha]_F \cdot [\sigma]|_B := \int_M \alpha \wedge \sigma \]
is well defined.
Proof. We prove (b); the proof of (a) is completely analogous. Write $F = \langle X \rangle$ with some nonsingular vector field $X$ spanning $F$.

(i) We have $i_X \sigma = 0$, since $\sigma \in \Omega^{m-1}_B(F)$. Suppose $[\alpha]_F = [\alpha']_F$, which means that the function $\alpha(X) - \alpha'(X)$ is identically zero. It follows that the $m$-form $(\alpha - \alpha') \wedge \sigma$ vanishes identically, since its interior product with the nonsingular vector field $X$ vanishes.

(ii) Suppose $[\sigma]_B = [\sigma']_B \in H^{m-1}_B(F)$, that is, $\sigma - \sigma' = d\tau$ for some $\tau \in \Omega^{m-2}_B(F)$.

Then
\[
\int_M \alpha \wedge (\sigma - \sigma') = \int_M \alpha \wedge d\tau = -\int_M d(\alpha \wedge \tau) + \int_M d\alpha \wedge \tau.
\]

The first summand vanishes by Stokes’s theorem; the integrand of the second summand vanishes identically, since $i_X (d\alpha \wedge \tau) = 0$. \hfill \Box

Observe that the maps defined in this proposition are positively homogeneous of degree 1 on the first factor, and linear in the second factor.

5. The Euler class of a geodesible vector field

Let $X$ be a geodesible vector field on a manifold $M$ and set $F = \langle X \rangle$. Choose a characteristic $1$-form $\alpha$ for $X$.

Lemma 5.1. The basic cohomology class $e_X := -[d\alpha]_B \in H^2_B(F)$ is determined by $X$.

Proof. Let $\beta$ be a further characteristic $1$-form. Then $\gamma := \alpha - \beta \in \Omega^1_B(F)$, and $d\alpha - d\beta = d\gamma = d_B \gamma$, hence $[d\alpha]_B = [d\beta]_B$. \hfill \Box

I do not know whether the following definition has been made before, but it is certainly a very natural one.

Definition 5.2. The class $e_X \in H^2_B(F)$ is called the Euler class of the geodesible vector field $X$.

Example 5.3. (1) If the flow of $X$ generates a principal $S^1$-action, where we think of $S^1$ as $\mathbb{R}/\mathbb{Z}$, then $e_X$ can be naturally identified with the real Euler class $e \otimes \mathbb{R} \in H^2(M/S^1; \mathbb{R})$ of the $S^1$-bundle $M \to M/S^1$. Our definition accords with the usual sign convention, cf. [24, Section 6.2], [11, Section 7.2].

(2) If the flow of $X$ generates a locally free $S^1$-action, then $H^*_B(F)$ may be thought of as the orbifold cohomology of the orbifold $M/S^1$, and $e_X$ as the real Euler class of the $S^1$-orbibundle $M \to M/S^1$. We discuss examples of this kind in detail in Sections 6 and 7. For more information on $S^1$-orbibundles in the general sense see [18].

We shall meet further examples in Section 9 where we discuss surfaces of section for the flow of $X$.

The next lemma is the generalisation of a result for connection $1$-forms of principal $S^1$-bundles.

Lemma 5.4. Let $X$ be a geodesible vector field and $\omega \in \Omega^2_B(F)$ a basic $2$-form with $-[\omega]_B = e_X$. Then there is a characteristic $1$-form $\beta$ with $d\beta = \omega$. 
Proof. Since $[\omega]_B = [d\alpha]_B$, we find a basic 1-form $\gamma \in \Omega^1_B(\mathcal{F})$ with $\omega = d\alpha + d\gamma$. Then $\beta := \alpha + \gamma$ is the desired characteristic form. \qed

This lemma implies the following proposition.

**Proposition 5.5.** A geodesible vector field $X$ on a manifold $M$ of dimension $2n+1$ is the Reeb vector field of a contact form if and only if the Euler class $e_X$ has an odd-symplectic representative, i.e. if there is a closed basic 2-form $\omega \in \Omega^2_B(M)$ with $-[\omega]_B = e_X$ and $\omega^n \neq 0$. \qed

The following expression of the volume $\text{vol}_X$ in terms of the Euler class is immediate from the definitions. This is the promised cohomological interpretation and generalisation of Proposition 1.2.

**Proposition 5.6.** Let $X$ be a geodesible vector field on a closed, oriented manifold $M$ of dimension $2n+1$, and $\alpha$ a characteristic form for $X$. Then$$\text{vol}_X = (-1)^n [\alpha]_X \cdot e^n_X.$$If $X$ generates a free $S^1$-action, we have — with $e \in H^2(B;\mathbb{Z})$ denoting the Euler class of the fibration $M \to M/S^1 =: B$ —$$\text{vol}_X = (-1)^n \langle e^n, [B] \rangle,$$where $[B]$ denotes the fundamental class of $B$ and $\langle \cdot, \cdot \rangle$ the Kronecker pairing. \qed

Here is a useful vanishing criterion for the Euler class. For the flow of $X$ to be globally defined, we assume $M$ to be closed.

**Theorem 5.7.** The Euler class $e_X \in H^2_B(\mathcal{F})$ of a geodesible vector field $X$ on a closed manifold $M$ vanishes if and only if $X$ admits a transverse foliation $\mathcal{T}$ invariant under the flow of $X$.

**Proof.** Suppose that $e_X = 0$. As before we write $\mathcal{F} = \langle X \rangle$. Choose a 1-form $\alpha$ with $\alpha(X) = 1$ and $i_X d\alpha = 0$. Then $[d\alpha]_B = -e_X = 0$, so there is a basic 1-form $\gamma \in \Omega^1_B(\mathcal{F})$ with $d\gamma = d\alpha$. Then $\beta := \alpha - \gamma$ is a closed 1-form with $\beta(X) = 1$. In particular, $\ker \beta$ defines a foliation $\mathcal{T}$ transverse to $X$, and $\mathcal{T}$ is invariant under the flow of $X$ since $L_X \beta = d(\beta(X)) + i_X d\beta = 0$.

Conversely, let $\mathcal{T}$ be a transverse invariant foliation. Define a 1-form $\alpha$ by $\alpha(X) = 1$ and $i_Y \alpha = L_Y \alpha$, where $L_Y$ denotes the distribution of tangent spaces to $\mathcal{T}$. Then $i_X d\alpha = L_X \alpha$, and the latter equals $f \alpha$ for some $f \in C^\infty(M)$ by the invariance of $\mathcal{T}$. This implies $d\alpha(X, Y) = 0$ for $Y \in \Gamma(T\mathcal{T})$.

Given two (local) vector fields $Y_1, Y_2 \in \Gamma(T\mathcal{T})$, we compute$$d\alpha(Y_1, Y_2) = Y_1 \alpha(Y_2) - Y_2 \alpha(Y_1) - \alpha([Y_1, Y_2]) = 0.$$Thus, we conclude that $d\alpha = 0$, and hence $e_X = 0$. \qed

**Corollary 5.8.** A closed manifold $M$ admits a geodesible vector field $X$ with $e_X = 0$ if and only if $M$ fibres over $S^1$.

**Proof.** If $M$ admits a geodesible vector field with $e_X = 0$, the fact that $M$ fibres over $S^1$ follows from the existence of a closed, nonsingular 1-form on $M$, established in the foregoing proof, and a result of Tischler [29], cf. [8, Section 9.3].

Conversely, a manifold $M$ that fibres over $S^1$ always admits a geodesible vector field [16]. Such a manifold $M$ can be written as $[0,1] \times F/(1, x) \sim (0, \psi(x))$, where $F$ denotes the fibre and $\psi$ the monodromy of the bundle. Let $g_\theta, \theta \in [0,1]$ be any
smooth family of metrics on $F$ with $\psi^*g_0 = g_1$. Then $d\theta^2 + g_\theta$ defines a metric on $[0, 1] \times F$ for which the segments $[0, 1] \times \{x\}$ are geodesics, and this metric descends to $M$.

Alternatively, let $\alpha$ be the pull-back of the 1-form $d\theta$ under the bundle projection $M \to S^1$. Then $d\alpha = 0$, so any vector field $X$ on $M$ with $\alpha(X) = 1$, i.e. any lift of $\partial_\theta$, is geodesible, and clearly $e_X = 0$. □

**Example 5.9.** For Seifert fibred 3-manifolds (see the next section), the statement of Corollary 5.8 can be found in [26, Theorem 5.4].

### 6. Seifert fibred 3-manifolds

In this section we take $M \to B$ to be a Seifert fibration of a closed, oriented 3-manifold $M$ over a closed, oriented 2-dimensional orbifold $B$. Let $X$ be the vector field whose flow defines an $S^1$-action on $M$ with orbits equal to the Seifert fibres, where the minimal period of the regular fibres is assumed to be equal to 1. I refer to [15] and [17] for the basic terminology of Seifert fibrations.

Suppose the Seifert invariants of $M \to B$ are $(g; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$, where $g \in \mathbb{N}_0$ is the genus of $B$, and the $(\alpha_i, \beta_i)$, $i = 1, \ldots, n$, are pairs of coprime integers with $\alpha_i \neq 0$. Here the $\alpha_i$ give the multiplicities of the singular fibres; the pairs with $\alpha_i = 1$ do not correspond to singular fibres, but contribute to the Euler class of the fibration.

Concretely, $M$ is recovered from these Seifert invariants as follows. Let $B$ be the closed, oriented surface of genus $g$, and remove $n$ disjoint discs to obtain $B_0 = B \setminus \text{Int}(D^2_1 \sqcup \ldots \sqcup D^2_n)$.

Over this surface with boundary, we take the trivial $S^1$-bundle $M_0 = B_0 \times S^1 \to B_0$. Write the boundary $\partial B_0$ with the opposite of its natural orientation as $-\partial B_0 = S^1_1 \sqcup \ldots \sqcup S^1_n$.

We write the fibre class of this trivial fibration as $h = \{\ast\} \times S^1$, and on $\partial M_0$ we consider the curves $q_i = S^1_i \times \{0\}$, $i = 1, \ldots, n$; recall that we think of the fibre $S^1$ as $\mathbb{R}/\mathbb{Z}$. The labels $h, q_1, \ldots, q_n$ should be read as isotopy classes of curves on $\partial M_0$.

Choose integers $\alpha_i', \beta_i'$, $i = 1, \ldots, n$, such that

$$\begin{vmatrix} \alpha_i & \alpha_i' \\ \beta_i & \beta_i' \end{vmatrix} = 1.$$

Further, take $n$ copies $V_i = D^2 \times S^1$ of a solid torus, where $D^2$ is the unit disc in $\mathbb{R}^2$, with respective meridian and longitude $\mu_i = \partial D^2 \times \{0\}$, $\lambda_i = \{1\} \times S^1 \subset \partial V_i$.

Then glue the $V_i$ to $M_0$ along the boundary, where $\partial V_i$ is identified with the component $S^1_i \times S^1$ of $\partial M_0$ via

$$h = -\alpha_i'\mu_i + \alpha_i\lambda_i, \quad q_i = \beta_i'\mu_i - \beta_i\lambda_i.$$
Notice that the fibration of $M_0$ given by the fibre class $h$ extends to a fibration of $V_i$ with the central fibre of multiplicity $\alpha_i$. This is the description of $M \to B$ with the given Seifert invariants.

The Euler number $e$ of the Seifert fibration with the given Seifert invariants, defined as the obstruction to the existence of a section (in the Seifert sense) [17 Section 3], is

$$e = - \sum_{i=1}^{n} \frac{\beta_i}{\alpha_i}.$$  

We now want to use a global surface of section (in a slightly generalised sense) to derive this formula.

**Proposition 6.1.** Let $X$ be a vector field on a closed, oriented $3$-manifold $M$ defining a Seifert fibration of regular period $1$ with invariants

$$(g; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)).$$

Then

$$\langle e_X, [B] \rangle = - \sum_{i=1}^{n} \frac{\beta_i}{\alpha_i},$$

where $\langle \cdot, \cdot \rangle$ denotes the Kronecker pairing between $H_2^B(F)$ and $H_2(B)$.

Recall that a global surface of section (s.o.s.) for the flow of $X$ is an embedded compact surface $\Sigma \subset M$ whose boundary consists of orbits of $X$, whose interior $\text{Int}(\Sigma)$ is transverse to $X$, and such that the flow line of $X$ through any point not on $\partial \Sigma$ hits $\text{Int}(\Sigma)$ in forward and backward time. We now describe such an s.o.s. for the situation at hand.

**Proof of Proposition 6.1.** In $M_0$ we can take $B_0 \times \{0\}$ as section. The boundary of this section consists of the curves $-q_i$, $i = 1, \ldots, n$, which are identified with $-\beta_i \mu_i + \beta_i \lambda_i$ on $\partial V_i$. Thus, by isotoping these respective curves radially towards the spine $\sigma_i = \{0\} \times S^1$ of $V_i$, we sweep out a surface $\Sigma$ that is not quite an s.o.s. in the sense of the definition above, but which has the following properties:

- the inclusion $\Sigma \subset M$ is an embedding on $\text{Int}(\Sigma)$;
- the boundary of $\Sigma$ is made up of the curves $\sigma_i$, each covered $\beta_i$ times;
- the interior $\text{Int}(\Sigma)$ is intersected positively in a single point by each $X$-orbit different from the $\sigma_i$.

We now choose a specific connection $1$-form $\alpha$ on $M \to B$, i.e. a characteristic $1$-form for $X$. On $V_i = D^2 \times S^1$ we write $(r, 2\pi \phi)$ for polar coordinates on the $D^2$-factor, and $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$. It follows from (6) that on $V_i$ we may assume $X$ to be given by $-\alpha_i \partial_\phi + \alpha_i \partial_\theta$. So we choose the $1$-form $\alpha$ equal to $\alpha = d\theta/\alpha_i$ near the spine of $V_i$, and then extend arbitrarily as a connection form over $M$ (using a partition of unity).

We then compute

$$\langle e_X, [B] \rangle = - \int_B d\alpha = - \int_{\Sigma} d\alpha = - \int_{\partial \Sigma} \alpha = - \sum_{i=1}^{n} \frac{\beta_i}{\alpha_i}.$$

Notice that the integral $\int_B d\alpha$ is well defined, since $d\alpha$ is a basic form. \qed
Remark 6.2. As in Proposition 5.5 one argues that if the Euler number $e$ of the Seifert fibration is nonzero, one can choose $\alpha$ as a contact form (defining the correct orientation of $M$ if $e < 0$, the opposite one if $e > 0$).

Corollary 6.3. The volume $\text{vol}_X$ of a vector field $X$ defining a Seifert fibration on a closed, orientable 3-manifold, with the regular fibres having minimal period 1, equals minus the Euler number of that Seifert fibration. In particular, with $m$ denoting the least common multiple of the multiplicities $\alpha_1, \ldots, \alpha_n$, we have that $m \cdot \text{vol}_X$ is an integer.

Proof. The value of the integral of $\alpha \wedge d\alpha$ over $M$ does not change when we remove the singular fibres of the Seifert fibration. But then the integral equals
\[
\int_{\text{Int}(\Sigma) \times S^1} \alpha \wedge d\alpha = \int_{\Sigma} d\alpha = -e. \quad \square
\]

Remark 6.4. The integrality statement has been observed in greater generality by Weinstein [32].

Example 6.5. The positive Hopf fibration
\[
\mathbb{C}^2 \supset S^3 \longrightarrow \mathbb{C}\mathbb{P}^1 = S^2
\]
\[
(z_1, z_2) \mapsto [z_1 : z_2]
\]
is given by the vector field $X = 2\pi(\partial_{\varphi_1} + \partial_{\varphi_2})$ of period 1, where $\varphi_1, \varphi_2 \in \mathbb{R}/2\pi\mathbb{Z}$. The corresponding connection 1-form is
\[
\alpha = \frac{1}{2\pi} (r_1^2 \, d\varphi_1 + r_2^2 \, d\varphi_2).
\]

With $r^2 = r_1^2 + r_2^2$ one computes
\[
rd\varphi \wedge \alpha \wedge d\alpha = \frac{1}{2\pi^2} (r_1^2 + r_2^2) \cdot (r_1 \, dr_1 \wedge d\varphi_1 \wedge r_2 \, dr_2 \wedge d\varphi_2).
\]

So along the unit sphere $S^3 = \{ r = 1 \}$, the 3-form $\alpha \wedge d\alpha$ restricts to the standard volume form up to a factor $1/2\pi^2$, hence
\[
\int_{S^3} \alpha \wedge d\alpha = \frac{1}{2\pi^2} \text{vol}(S^3) = 1.
\]

A section of the Hopf fibration over $\mathbb{C} \cong \mathbb{C}\mathbb{P}^1 \setminus \{[0 : 1]\}$ is defined by
\[
re^{i\varphi} \longmapsto [1 : e^{i\varphi}] \longmapsto \left( \frac{1}{\sqrt{1 + r^2}}, \frac{re^{i\varphi}}{\sqrt{1 + r^2}} \right).
\]

Under this map, $d\alpha$ pulls back to
\[
\frac{1}{\pi} \cdot \frac{r}{(1 + r^2)^2} \, dr \wedge d\varphi.
\]

This yields
\[
\int_{S^2} d\alpha = \frac{1}{\pi} \int_0^{2\pi} \int_0^\infty \frac{r}{(1 + r^2)^2} \, dr \, d\varphi = 1.
\]

Thus, both computations confirm that the positive Hopf fibration has Euler number $e = -1$, see also [5 Lemma 2.2].
7. The theorems of Gauss–Bonnet and Poincaré–Hopf

In this section we formulate and prove the theorems of Gauß–Bonnet and Poincaré–Hopf for oriented 2-dimensional orbifolds, using an s.o.s. argument as in the preceding section. Versions of these theorems for higher-dimensional orbifolds (including those with boundary) can be found in [25] and [27]. In order to avoid confusion with formulas found elsewhere, in this section we follow the usual convention that the regular fibres in the unit tangent bundle of a 2-dimensional orbifold have length $2\pi$.

Thus, let $B$ be a closed, oriented 2-dimensional Riemannian orbifold with underlying surface of genus $g$ and $n$ cone points of multiplicities $\alpha_1, \ldots, \alpha_n$. There is a well-defined unit tangent bundle $STB$, cf. [13], which is a Seifert manifold with invariants 

$$(g, (1, 2g - 2), (\alpha_1, \alpha_1 - 1), \ldots, (\alpha_n, \alpha_n - 1)).$$

The orbifold Euler characteristic $\chi_{orb}(B)$ is the Euler number of the Seifert fibration $\pi: STB \to B$, so by Proposition 6.1 we have

$$\chi_{orb}(B) = 2 - 2g - n + \sum_{i=1}^{n} \frac{1}{\alpha_i}.$$ 

This formula can also be derived combinatorially, using the Riemann–Hurwitz formula for coverings, see [26, p. 427].

Just like the unit tangent bundle of a smooth surface, the unit tangent bundle $STB$ of an orbifold admits a pair of Liouville–Cartan forms $\lambda_1, \lambda_2$ and a connection 1-form $\tilde{\alpha}$ satisfying the structure equations

$$d\lambda_1 = -\lambda_2 \wedge \tilde{\alpha}, \quad d\lambda_2 = -\tilde{\alpha} \wedge \lambda_1, \quad d\tilde{\alpha} = -(\pi^* K) \lambda_1 \wedge \lambda_2,$$

where $K$ is the Gauß curvature of the Riemannian metric on $B$. See [3, Section 2.1] for the surface case, and [12, Section 7] for a discussion of Liouville–Cartan forms for orbifolds.

**Theorem 7.1 (Gauß–Bonnet).** The total curvature of a closed, oriented 2-dimensional Riemannian orbifold $B$ equals

$$\int_B K \, dA = 2\pi \chi_{orb}(B).$$

**Proof.** The characteristic 1-form $\alpha$ for the vector field $X$ that makes the regular fibres of $STB$ of length 1 is $\alpha = \tilde{\alpha}/2\pi$. Therefore, with $e = \chi_{orb}(B)$ we obtain

$$\int_B K \, dA = - \int_{\Sigma} d\tilde{\alpha} = -2\pi \int_{\Sigma} d\alpha = 2\pi \chi_{orb}(B),$$

where $\Sigma$ is as in the proof of Proposition 6.1. \qed

Now let $Y$ be a vector field with isolated zeros on the orbifold $B$. In order to formulate the Poincaré–Hopf theorem we need to give a definition of the index $\text{ind}_p Y$ in an orbifold singularity $p \in B$, cf. [25, Section 3.2]. First, let $p \in B$ be a smooth point where $Y$ has a zero. Choose a small disc $D_c(p) \subset B$ not containing other zeros of $Y$ (and hence, as we shall see, in particular no orbifold points of $B$).
Choose a trivialisation $TD_z(p) \cong D_z(p) \times \mathbb{R}^2$. Then $\text{ind}_pY$ is the degree of the map $\partial D_z(p) \to S^1, x \mapsto Y(x)/|Y(x)|$.

When the zero of $Y$ happens to be an orbifold singularity $p_i \in B$ of order $\alpha_i$, we consider a local description $\pi_{\alpha_i} : D^2 \to D^2/\mathbb{Z}_{\alpha_i} \cong D^2$ of the singularity, where the cyclic group $\mathbb{Z}_{\alpha_i}$ is generated by the rotation about $0 \in \mathbb{R}^2$ through an angle $2\pi/\alpha_i$.

We drop the index $i$ for the time being; there should be little grounds for confusing $\alpha$ in the following discussion with the connection 1-form.

The fibre of $STB$ over the singular point $p \in B$ has Seifert invariants $(\alpha, \beta = \alpha - 1)$, so we may take $\alpha' = \beta' = 1$. Then, cf. [15], the local description of the fibration $STB \to B$ near the orbifold point $p$ is given by

$$\pi : D^2 \times S^1 \to D^2 \quad (re^{i\varphi}, e^{i\theta}) \mapsto re^{i(\alpha\varphi + \theta)}.$$}

where we identify $p$ with $0 \in D^2$. Notice that the fibres of $\pi$ are described by $\alpha\varphi + \theta = \text{const.}$, or in parametric form as

$$t \mapsto (\varphi(t), \theta(t)) = (\varphi_0 - t, \theta_0 + \alpha t).$$

This accords with (6).

Now consider the following commutative diagram,

$$\begin{array}{ccc}
D^2 \times S^1 & \rightrightarrows & (re^{i\varphi}, e^{i\theta}) \\
\pi \downarrow & & \downarrow \pi \\
D^2 & \rightrightarrows & (re^{i(\alpha\varphi + \theta)}, e^{i\theta}) \\
\tilde{\pi} \downarrow & & \downarrow \pi \\
\tilde{\pi}_\alpha : (\tilde{\varphi}, \tilde{\theta}) \rightrightarrows (\varphi, \theta) = (\tilde{\varphi}, \alpha \tilde{\theta}).
\end{array}$$

where the quotient map $\pi_{\alpha}$ under the $\mathbb{Z}_{\alpha}$-action is given by

$$\pi_{\alpha}(re^{it}) = re^{i\alpha t},$$

its lift $\tilde{\pi}_\alpha$ to the unit tangent bundle by

$$\tilde{\pi}_\alpha : (\tilde{\varphi}, \tilde{\theta}) \rightrightarrows (\varphi, \theta) = (\tilde{\varphi}, \alpha \tilde{\theta}).$$

Up to homotopy, the section $Y/|Y|$ of $\pi$ over $\partial D^2$ is of the form

$$\begin{equation}
(\varphi(t), \theta(t)) = (2\pi kt, 2\pi(1 - ka)t), \quad t \in [0, 1],
\end{equation}$$

for some $k \in \mathbb{Z}$; notice that $\alpha\varphi(t) + \theta(t)$ goes from 0 to $2\pi$ as $t$ goes from 0 to 1.

The lift of the $\alpha$-fold traversal of this curve under the map $\tilde{\pi}_\alpha$ is described by

$$\begin{equation}
(\tilde{\varphi}(t), \tilde{\theta}(t)) = (2\pi kt, 2\pi\frac{1 - ka}{\alpha}t), \quad t \in [0, \alpha];
\end{equation}$$

here $\tilde{\varphi}(t) + \tilde{\theta}(t) = \frac{2\pi}{\alpha}t$ goes from 0 to $2\pi$ as $t$ goes from 0 to $\alpha$.

The fibres of $\tilde{\pi}$ are described by $\tilde{\varphi} + \tilde{\theta} = \text{const.}$, and a single right-handed Dehn twist along a meridional twist along a meridional disc of $D^2 \times S^1$,

$$(\tilde{\varphi}', \tilde{\theta}') := (\tilde{\varphi} + \tilde{\theta}, \tilde{\theta}),$$

will bring these fibres into the form $\tilde{\varphi}' = \tilde{\varphi}'_0$, and the curve (5) becomes

$$\begin{equation}
(\tilde{\varphi}'(t), \tilde{\theta}'(t)) = \left(\frac{2\pi}{\alpha}t, 2\pi\frac{1 - ka}{\alpha}t\right), \quad t \in [0, \alpha].
\end{equation}$$
The index $\text{ind}_pY$ at an orbifold point of multiplicity $\alpha$ is defined as $\text{ind}_p\tilde{Y}/\alpha$, where $\tilde{p} = \pi^{-1}_\alpha(p)$ and $\tilde{Y}$ is the lifted vector field. Our considerations show that, in dependence on $k \in \mathbb{Z}$, this index is
\[
\text{ind}_pY = \frac{1}{\alpha} - k.
\]
Notice that $k = 0$ corresponds to a rotationally symmetric source or sink of $Y$, which lifts to an identically looking zero of $\tilde{Y}$. Also, for $\alpha > 1$ the term $1 - k\alpha$ in the $\theta$-component of (7) never equals zero, no matter what $k \in \mathbb{Z}$, which means that orbifold points always must be zeros of $Y$.

**Theorem 7.2** (Poincaré–Hopf). Let $Y$ be a vector field with isolated zeros on an orbifold $B$. Then
\[
\sum_{p \in B} \text{ind}_pY = \chi_{\text{orb}}(B).
\]

**Proof.** The idea is simply to compute the Euler number $e = \chi_{\text{orb}}(B)$ of the Seifert fibration $STB \to B$ with the help of an s.o.s. $\Sigma^Y$ adapted to $Y$.

Outside small disc neighbourhoods of the zeros of $Y$ we may normalise the vector field and regard it as a section $\Sigma^Y_0$ of $STB \to B$ outside this set of discs in $B$. This surface $\Sigma^Y_0$ extends to an s.o.s. $\Sigma^Y$ of $STB$, with boundary components certain multiple covers of the fibres over the zeros $p$ of $Y$, as in the proof of Proposition 6.1. The multiplicity of the covering is determined by the number of full turns the boundary component makes in the $\theta$-direction. Notice that the orientation of the collection of circles $\pi(\partial\Sigma^Y_0)$ is the opposite of the orientation as boundaries of the removed discs. Thus, the multiplicity is $-\text{ind}_pY$ at a smooth point and, by (8), equal to $-(1 - k_\alpha \alpha_i) = -\alpha_i \text{ind}_pY$ at an orbifold point of order $\alpha_i$, where $k_i \in \mathbb{Z}$ is the integer describing that particular zero of $Y$, see also Remark 7.3.

With a connection 1-form $\alpha$ corresponding to regular fibres having length 1 as in the proof of Proposition 6.1, that is, equal to $d\theta/2\pi$ near the fibres over smooth zeros of $Y$, and equal to $d\theta/2\pi \alpha_i$ over an orbifold point of order $\alpha_i$, we have
\[
\chi_{\text{orb}}(B) = e = -\int_{\Sigma^Y} d\alpha = -\int_{\partial\Sigma^Y} \alpha = \sum_{p \in B} \text{ind}_pY. \quad \Box
\]

**Remark 7.3.** It may be helpful to reformulate the first part of the proof in terms of meridians and longitudes, similar to the discussion of the topology of surfaces of section in [5].

First, consider a zero of $Y$ at a smooth point $p \in B$. Let $V$ be a tubular neighbourhood of the fibre $ST_pB$. Let $\mu$ be the meridian on $\partial V$, and $\lambda$ the longitude determined by the parallel fibres. We orient $\lambda$ as the fibres, and $\mu$ in such a way that $(\mu, \lambda)$ gives the positive orientation of $\partial V$. The component of $\partial\Sigma^Y_0$ on $\partial V$ is $(-1, -\text{ind}_pY)$ in terms of the $(\mu, \lambda)$-basis. So this component is isotopic to $-\text{ind}_pY$ times the spine of $V$. Also, notice that the intersection number of the fibre $(0, 1)$ with $(-1, -\text{ind}_pY)$ is +1, which is consistent with $\Sigma^Y_0$ being a section.

For a zero of $Y$ at a singular point of order $\alpha_i$, we take a neighbourhood $V_i$ of $ST_pB$ with $\mu_i, \lambda_i$ as in Section 6. Now the component of $\partial\Sigma^Y_0$ on $\partial V_i$ is $(-k_i, -1 + k_i \alpha_i)$ by (7), which is isotopic to $-1 + k_i \alpha_i$ times the spine. Again, the intersection of the fibre $(-1, \alpha)$, see (9), with $(-k_i, -1 + k_i \alpha_i)$ is +1.
8. Transversely holomorphic foliations and the Bott invariant

In [14] with Jesús Gonzalo we proved a generalised Gauß–Bonnet theorem for transversely holomorphic 1-dimensional foliations on 3-manifolds. In certain situations, which I am going to describe now, this can be interpreted as a statement about \( \text{vol}_X \) for a vector field \( X \) whose flow defines such a foliation.

The following definition is from [12].

**Definition 8.1.** A pair of contact forms \((\omega_1, \omega_2)\) on a closed, oriented 3-manifold \( M \) is called a Cartan structure if

\[
\omega_1 \wedge d\omega_1 = \omega_2 \wedge d\omega_2 \neq 0
\]

\[
\omega_1 \wedge d\omega_2 = \omega_2 \wedge d\omega_1 = 0.
\]

Such structures exist in abundance, see [12, Theorem 1.2]. They are special cases of what we christened taut contact circles in that paper: any linear combination \( \lambda_1 \omega_1 + \lambda_2 \omega_2 \) with \((\lambda_1, \lambda_2) \in S^1 \subset \mathbb{R}^2 \) is again a contact form defining the same volume form. The defining equations for a Cartan structure can be rephrased as saying that there is a uniquely defined nowhere vanishing 1-form \( \alpha \) such that

\[
d\omega_1 = \omega_2 \wedge \alpha,
\]

\[
d\omega_2 = \alpha \wedge \omega_1.
\]

In terms of the complex-valued 1-form \( \omega_c := \omega_1 + i\omega_2 \), these equations can be rewritten as

\[
d\omega_c = i\alpha \wedge \omega_c.
\]

Observe that

\[
0 \neq \omega_1 \wedge d\omega_1 = \omega_1 \wedge \omega_2 \wedge \alpha,
\]

so \( \alpha \) is nonzero on the common kernel of \( \omega_1 \) and \( \omega_2 \).

**Lemma 8.2.** Let \( X \) be the vector field defined by \( X \in \ker \omega_1 \cap \ker \omega_2 \) and \( \alpha(X) = 1 \). Then \( i_X d\alpha = 0 \). Hence, by Proposition 7.3, \( X \) is geodesible.

**Proof.** By taking the exterior derivative of the defining equations for \( \alpha \) we find

\[
0 = d^2 \omega_1 = d\omega_2 \wedge \alpha - \omega_2 \wedge d\alpha = -\omega_2 \wedge d\alpha,
\]

and similarly

\[
0 = d\alpha \wedge \omega_1.
\]

This implies that \( i_X d\alpha \) must be a multiple both of \( \omega_1 \) and \( \omega_2 \), but these forms are pointwise linearly independent. \( \square \)

The 1-form \( \omega_c \) is formally integrable in the sense that \( \omega_c \wedge d\omega_c = 0 \). In [14] it is shown that this is equivalent to saying that \( \omega_c \) defines a transverse holomorphic structure for the 1-dimensional foliation defined by the flow of \( X \).

In general, the formal integrability of a complex-valued 1-form \( \omega_c \) only implies the existence of a (not uniquely defined) complex-valued 1-form \( \alpha_c \) such that

\[
d\omega_c = \alpha_c \wedge \omega_c.
\]

A Godbillon–Vey type argument shows that the complex number

\[
\int_M \alpha_c \wedge d\alpha_c,
\]

called the *Bott invariant*, is an invariant of the transversely holomorphic foliation that does not depend on the choice of \( \omega_c \) and \( \alpha_c \).
The generalised Gauß–Bonnet theorem \[14\] Theorem 3.3] says that for transversely holomorphic foliations coming from a Cartan structure, this Bott invariant depends only on the 1-dimensional foliation defined by the common kernel flow, not on the specific transverse holomorphic structure. As observed before, for \(\omega_c\) coming from a Cartan structure we can take \(\alpha_c = i\alpha\). Thus, the generalised Gauß–Bonnet theorem from \[14\] can be rephrased as follows.

**Theorem 8.3.** If the vector field \(X\) derives from a Cartan structure as described, then \(\text{vol}_X\) equals the negative of the Bott invariant of any transversely holomorphic structure on the foliation \(\langle X\rangle\).

The paper \[14\] contains examples which show this to be a nontrivial statement. There are instances of the generalised Gauß–Bonnet theorem where the transverse holomorphic structure is indeed not unique. In \[14\] one can also find a complete classification of the transversely holomorphic foliations on \(S^3\), originally due (for all 3-manifolds) to Brunella and Ghys, and a computation of their Bott invariant.

### 9. Global surfaces of section

We now want to compute \(\text{vol}_X\) under the assumption that the geodesible vector field \(X\) admits a global surface of section \(\Sigma \subset M\). For simplicity, we assume that \(M\) is a closed, oriented manifold of dimension 3, although our considerations extend in an obvious manner to global hypersurfaces of section in manifolds of higher odd dimension for an appropriate definition of that concept.

Given such an s.o.s., we can associate with each point \(p \in \text{Int}(\Sigma)\) its return time \(\tau(p) \in \mathbb{R}^+\), i.e. the smallest positive real number with \(\phi_{\tau(p)}(p) \in \text{Int}(\Sigma)\), were \(\phi_t\) denotes the flow of \(X\).

**Proposition 9.1.** Let \(\sigma\) be a basic 2-form on \(M\) that represents the Euler class \(e_X\). Then

\[
\text{vol}_X = -\int_{\text{Int}(\Sigma)} \tau \sigma,
\]

where we interpret \(\sigma\) as a 2-form on the transversal \(\text{Int}(\Sigma)\) for the flow of \(X\).

**Proof.** Let \(\alpha\) be a characteristic 1-form of \(X\). By Proposition 4.4 we have

\[
\text{vol}_X = \int_M \alpha \wedge d\alpha = -\int_M \alpha \wedge \sigma.
\]

To compute the integral on the right, we consider the injective immersion

\[
\Phi: \quad [0, 1) \times \text{Int}(\Sigma) \rightarrow M
\]

\[
(t, p) \mapsto \phi_{\tau(p)}(p).
\]

Since \(T\Phi(\partial_t)\) is a multiple of \(X\), and \(\sigma\) a basic differential form, we can compute

\[
\int_M \alpha \wedge \sigma = \int_{M \setminus \partial\Sigma} \alpha \wedge \sigma
\]

\[
= \int_{[0, 1) \times \text{Int}(\Sigma)} \Phi^* (\alpha \wedge \sigma)
\]

\[
= \int_{\text{Int}(\Sigma)} \left( \int_0^1 (\Phi^* \alpha)_{(t,p)}(\partial_t) \, dt \right) \sigma
\]

\[
= \int_{\text{Int}(\Sigma)} \tau \sigma.
\]
In the last line we used that
\[(\Phi^*\alpha)_{(t,p)}(\partial_t) = \alpha_{\Phi(t,p)}(T\Phi(\partial_t)) = \alpha_{\Phi(t,p)}(\tau(p)X) = \tau(p).\]
Hence \(\text{vol}_X = -\int_{\text{Int}(\Sigma)} \tau\sigma\), as claimed. \(\square\)

**Example 9.2.** On \(D^2\) with polar coordinates \((r,\varphi)\) we write \(\lambda = r^2 \frac{d\varphi}{2}\) for the primitive 1-form of the standard area form \(\omega = d\lambda = r \, dr \wedge d\varphi\). On \(\mathbb{R}/\mathbb{Z} \times D^2\) we consider the 1-form
\[\alpha = H \, d\theta + \lambda,\]
where \(H\) is a smooth function of \(r^2\). In the sequel it will always be understood that \(H\) or its derivative \(H'\) is evaluated at \(r^2\). Then
\[d\alpha = 2r H' \, dr \wedge d\theta + \omega\]
and
\[\alpha \wedge d\alpha = (H - r^2 H') \, d\theta \wedge \omega.\]
We assume that \(H - r^2 H' > 0\); then \(\alpha\) is a contact form. As discussed in [1], this 1-form descends to a contact form (still denoted \(\alpha\)) on \(S^3\), obtained from \(S^1 \times D^2\) by collapsing the circle action on the boundary \(S^1 \times \partial D^2\) generated by
\[\partial_\theta - 2H(1)\partial_\varphi \in \ker \alpha|_{T(S^1 \times \partial D^2)}.\]
The Reeb vector field of \(\alpha\) (on \(S^1 \times D^2\)) is
\[X = \frac{\partial_\theta - 2H'\partial_\varphi}{H - r^2 H'}.\]
Thus,
\[\text{vol}_X = \int_{S^3} \alpha \wedge d\alpha = \int_{S^1 \times D^2} \alpha \wedge d\alpha = \int_{D^2} (H - r^2 H')\omega.\]
On the other hand, the disc \(\{0\} \times D^2\) descends to an s.o.s. for the Reeb flow on \(S^3\), and by [3] the return time is
\[\tau = H - r^2 H'.\]
So we see that \(\text{vol}_X\) can likewise be computed as
\[\text{vol}_X = \int_{D^2} \tau\sigma\]
with \(\sigma = d\alpha|_{T D^2}\) or any other 2-form that differs from \(d\alpha\) by the differential of a basic 1-form for \(X\) on \(S^3\) (not just on \(S^1 \times D^2\)).

**Remark 9.3.** For an expression of \(\text{vol}_X\) in the preceding example in terms of the Calabi invariant of the return map on the s.o.s. see [2].

10. **Contact forms with the same Reeb vector field**

In this section we present examples of nondiffeomorphic contact forms with the same Reeb vector field.

**Theorem 10.1.** In any odd dimension \(\geq 9\) there is a closed manifold admitting a countably infinite family of contact forms that are pairwise nondiffeomorphic but share the same Reeb vector field.
Proof. We construct these manifolds as Boothby–Wang bundles \cite{6}, \cite{11} Section 7.2 over integral symplectic manifolds. Starting point for our construction are examples of symplectic manifolds, in any even dimension $\geq 8$, with cohomologous but nondiffeomorphic symplectic forms, devised by McDuff \cite{22}. In dimension eight, one begins with the manifold $S^2 \times T^2 \times S^2 \times S^2$ with the standard split symplectic form. We think of $T^2$ as $(\mathbb{R}/\mathbb{Z})^2$. One then twists this symplectic form by a diffeomorphism $(p_1; s_2, t_2; p_3; p_4) \mapsto (p_1; s_2, t_2, \psi_k(p_1, t_2)(p_3); p_4)$, where $\psi_k(p_1, t_2): S^2 \to S^2$ is the rotation of $S^2$ about the axis determined by $\pm p_1$ through an angle $2\pi kt_2$. Finally, one takes the symplectic blow-up of these forms along $S^2 \times T^2 \times \{(p_3, p_4)\}$ with the same blow-up parameter (giving the 'size' of the blow-up) for all $k \in \mathbb{N}_0$.

The resulting symplectic forms $\omega_k$ on the blown-up manifold $W$ are cohomologous and homotopic through (noncohomologous) symplectic forms, but they are pairwise nondiffeomorphic. By taking products with copies of $S^2$, one obtains similar examples in higher dimensions.

The cohomology class of the symplectic form on a manifold obtained as a blow-up has been computed in \cite{21}, and from there one sees that the blow-up can be chosen in such a way that this cohomology class is rational. Hence, after a constant rescaling we may assume the symplectic forms $\omega_k$ to be integral, i.e. their de Rham cohomology class $[\omega_k]$ lies in the image of the inclusion $H^2(W; \mathbb{Z}) \subset H^2(W; \mathbb{R}) = H_{dR}^2(W)$.

Now choose a class $e \in H^2(W; \mathbb{Z})$ with $e \otimes \mathbb{R} = -[\omega_k]$, and let $\pi: M \to W$ be the $S^1$-bundle over $W$ of Euler class $e$. One then finds, for each $k \in \mathbb{N}_0$, a connection 1-form $\alpha_k$ on $M$ with curvature form $\omega_k$, that is, $d\alpha_k = \pi^* \omega_k$, see \cite{11} Section 7.2. (This normalisation corresponds to thinking of $S^1$ as $\mathbb{R}/\mathbb{Z}$.) Hence, the $\alpha_k$ are contact forms with Reeb vector field given by the unit tangent vector field along the fibres.

The $\alpha_k$, $k \in \mathbb{N}_0$, are pairwise nondiffeomorphic, because any diffeomorphism between $\alpha_k$ and $\alpha_\ell$ would preserve the Reeb vector field and hence descend to a diffeomorphism between $\omega_k$ and $\omega_\ell$. \hfill $\Box$

Remark 10.2. (1) I do not know whether the contact structures $\text{ker } \alpha_k$ are diffeomorphic. They all have the same underlying almost contact structure.

(2) I hedge my bets concerning dimensions 5 and 7.

(3) Contact forms with all Reeb orbits closed and of the same minimal period are also called Zoll contact forms \cite{2} \cite{1}.

11. Orbit equivalence

A slightly weaker question than the one asked by Viterbo is the following: are there examples of contact forms with the same Reeb vector field up to scaling by a function? Or, put differently, one asks for nondiffeomorphic contact forms whose Reeb flows are smoothly orbit equivalent. For the more general class of geodesible vector fields, this problem is best phrased as follows: on a manifold $M$, is there a geodesible vector field $X$ and a function $f \in C^\infty(M, \mathbb{R}^+)$ such that $fX$ is likewise geodesible? Of course, one should exclude the trivial case of $f$ being constant, where one simply rescales the metric by the inverse constant.
This is related, but not equivalent to the question about nontrivially geodesically equivalent metrics, where two Riemannian metrics share the same geodesics up to reparametrisation (so the geodesic flows are orbit equivalent), but one metric is not a constant multiple of the other.

Matveev [19] has shown that among closed, connected 3-manifolds, examples of nontrivially geodesically equivalent metrics exist only on lens spaces and Seifert manifolds with Euler number zero. See also [20] for a discussion of this phenomenon in the context of general relativity.

Our question asks about the nontrivial equivalence of two foliations by geodesible vector fields. In some sense, this is a weaker question; on the other hand, a nontrivial equivalence between two Riemannian metrics may well become trivial when restricted to any geodesic foliation.

**Example 11.1.** On the 2-torus $T^2 = (\mathbb{R}/\mathbb{Z})^2$ we consider the standard flat metric $g_1 = dx_1^2 + dx_2^2$ and a second flat metric $g_2 = dx_1^2 + \alpha dx_2^2$ with $\alpha \in \mathbb{R}^+ \setminus \{1\}$. Then $g_2$ is not a constant multiple of $g_1$, but the two metrics are geodesically equivalent: the geodesics in both cases are the images of straight lines in $\mathbb{R}^2$ under the projection to $T^2$. A geodesic foliation is given by the straight lines of some constant slope, and along those parallel lines the unit vector fields for the two metrics differ by a constant.

Using an idea going back to Beltrami and explained in [19], we can exhibit a simple example of geodesically equivalent metrics on $S^3$ that give rise to a geodesible vector field admitting nontrivial rescalings into likewise geodesible vector fields. Here, by construction, the vector fields are diffeomorphic. As I shall explain, rescalings of geodesible vector fields that define an $S^1$-fibration will always be diffeomorphic.

**Example 11.2.** For $a_1, a_2 \in \mathbb{R}^+$, consider the linear map $A = A_{a_1,a_2} : (z_1, z_2) \mapsto (a_1 z_1, a_2 z_2)$ on $\mathbb{C}^2 = \mathbb{R}^4$. Then define $\phi = \phi_{a_1,a_2} : S^3 \to S^3$ by $\phi(p) = A(p)/|A(p)|$. Let $g_{a_1,a_2} = \phi^* g_0$ be the pull-back of the round metric $g_0$ on $S^3$. Since $\phi$ takes great circles to great circles, the metric $\phi^* g_0$ is geodesically equivalent to $g_0$, nontrivially so unless $a_1 = a_2$.

A straightforward computation yields the following expression for $g_{a_1,a_2}$:

$$g_{a_1,a_2} = \frac{a_1^2}{\Delta} (dx_1^2 + dy_1^2) + \frac{a_2^2}{\Delta} (dx_2^2 + dy_2^2)$$

$$- \frac{a_1^4}{\Delta^2} (x_1 dx_1 + y_1 dy_1)^2 - \frac{a_2^4}{\Delta^2} (x_2 dx_2 + y_2 dy_2)^2$$

$$- \frac{2a_1 a_2}{\Delta^2} (x_1 dx_1 + y_1 dy_1) (x_2 dx_2 + y_2 dy_2),$$

where we write

$$\Delta = \Delta_{a_1,a_2}(r_1, r_2) = a_1^2 r_1^2 + a_2^2 r_2^2.$$  

Recall that in terms of polar coordinates we have $dx_1^2 + dy_1^2 = dr_1^2 + r_1^2 d\phi_1^2$ and $x_1 dx_1 + y_1 dy_1 = r_1 dr_1$.

The positive Hopf fibration is generated by $X_0 = \partial_{x_1} + \partial_{x_2}$. This vector field has constant length 1 with respect to all the metrics $g_{a_1,a_2}$, so from the viewpoint of geodesic foliations this yields nothing new. Also, the corresponding contact forms

$$\alpha_{a_1,a_2} = g_{a_1,a_2}(X_0, \cdot) = \frac{a_1^2 r_1^2 d\phi_1 + a_2^2 r_2^2 d\phi_2}{\Delta}.$$
all have $X_0$ as Reeb vector field, and so they are just diffeomorphic deformations of the standard contact form $\alpha_{1,1}$ by Proposition 2.1.

A more interesting choice is to take the great circle foliation generated by

$$X_1 = x_1 \partial x_2 - x_2 \partial x_1 + y_1 \partial y_2 - y_2 \partial y_1.$$ 

We write $L = L_{a_1, a_2} = (g_{a_1, a_2}(X_1, X_1))^{1/2}$ for the length of $X_1$ with respect to $g_{a_1, a_2}$. One computes

$$L^2 = \frac{a_1^2 y_2^2 + a_2^2 x_1^2}{\Delta} - \frac{(a_1^2 - a_2^2)^2}{\Delta^2} (x_1 x_2 + y_1 y_2)^2.$$

Thus, we have found the nontrivial family of geodesible vector fields $X_1/L_{a_1, a_2}$, with corresponding metric $g_{a_1, a_2}$, all generating the same foliation of $S^3$ by great circles. The corresponding 1-form

$$\alpha = \alpha_{a_1, a_2} = g_{a_1, a_2}(X_1/L_{a_1, a_2}, \cdot)$$

can be computed explicitly as

$$L \alpha = -\frac{a_1^2}{\Delta} (x_2 \, dx_1 + y_2 \, dy_1) + \frac{a_2^2}{\Delta} (x_1 \, dx_2 + y_1 \, dy_2) + \frac{a_1^2 - a_2^2}{\Delta^2} (x_1 x_2 + y_1 y_2) (x_2 \, dx_1 + y_1 \, dy_1) - \frac{a_2^4 - a_1^2 a_2^2}{\Delta^2} (x_1 x_2 + y_1 y_2) (x_2 \, dx_2 + y_2 \, dy_2).$$

I did not check whether these are contact forms for all $a_1, a_2 \in \mathbb{R}^+$, but by the openness of the contact condition they certainly are for $a_1, a_2$ close to 1. Then $X_1/L_{a_1, a_2}$ will be the Reeb vector field of $\alpha_{a_1, a_2}$.

The following proposition gives a more systematic statement about rescalings of geodesible vector fields that define an $S^1$-fibration. This is essentially due to Wadsley [31] (in greater generality); for the case at hand it can be retraced to the work of Boothby and Wang [6].

**Proposition 11.3.** Let $X$ be a geodesible vector field on a closed manifold $M$ such that the flow lines of $X$ are the fibres of a principal $S^1$-bundle $M \to M/S^1$. Then, after a constant rescaling of $X$ all orbits have (minimal) period 1, so that the flow of $X$ defines the $S^1$-action. A rescaling $fX$ of $X$ is likewise geodesible if and only if all orbits have the same period. When this period is 1, the vector fields $X$ and $fX$ are diffeomorphic by a diffeomorphism that sends each fibre to itself and is isotopic to the identity via such diffeomorphisms.

**Proof.** If $X$ is geodesible, we find a 1-form $\alpha$ with $\alpha(X) = 1$ and $i_X d\alpha = 0$ by Proposition 2.3. Then [11] Lemmas 7.2.6 and 7.2.7, which fill a gap in [6], show that the orbits of $X$ all have the same period. Notice that Lemma 7.2.7 in [11] is formulated for Reeb vector fields, but the proof only uses the property $i_X d\alpha = 0$, not the nondegeneracy of $d\alpha|_{\ker \alpha}$.

If $fX$ is geodesible, the same argument applies. Conversely, if the orbits of $fX$ all have the same period, then the flow of $X$ defines an $S^1$-bundle structure, and any connection 1-form for this bundle is a characteristic 1-form for $fX$, which makes $fX$ geodesible.

Now suppose the period of $fX$ equals 1. Given a local section $U \cong D^2$ of $X$, the flow of $X$ defines a trivialisation $U \times S^1$ of the bundle, and $f$ gives rise to a family of
1-periodic velocity functions \( v_u : [0, 1] \to \mathbb{R}^+ \) with \( \int_0^1 v_u(t) \, dt = 1 \) for every \( u \in U \).

(I refrain from writing \( v_u \) as a function on \( S^1 \), since the time parameter \( t \) should not be confused with the fibre parameter defined by the flow of \( X \).) Let \( \psi : D^2 \to [0, 1] \) be a bump function equal to 1 on a disc of radius \( 1/2 \), say, and supported in the interior of \( D^2 \). Then

\[
\mu \left( \psi(u)v_u + 1 - \psi(u) \right) + 1 - \mu
\]

defines for each \( u \in U \) and \( \mu \in [0, 1] \) a 1-periodic velocity function \([0, 1] \to \mathbb{R}^+\) of integral 1. This gives rise to an isotopy along fibres whose time-1 map sends \( X \) to \( fX \) on fibres where \( \psi(u) = 1 \), and which is stationary on fibres along which \( f = 1 \). This allows us to patch together such local isotopies to obtain the desired result.

The next corollary also applies to Example 11.2.

**Corollary 11.4.** If \( R \) and \( fR \) are Reeb vector fields on a closed 3-manifold with all orbits periodic of the same period 1, then any corresponding contact forms are related via a fibre-preserving isotopy.

**Proof.** This follows immediately by combining the last statement of Proposition 11.3 with Proposition 2.1.

Alternatively, one can give a direct proof, using a refinement of the proof of Proposition 2.1. Let \( \alpha_0 \) be a contact form with Reeb vector field \( R \), and \( \alpha_1 \) a contact form for \( fR \). Set \( \alpha_t := (1-t)\alpha_0 + t\alpha_1 \). We would like to find an isotopy \((\psi_t)_{t \in [0,1]}\) satisfying (5) as in the proof of Proposition 2.1.

The \( \alpha_t \) are contact forms with Reeb vector field \( R_t \) proportional to \( R \). We try to find an isotopy \((\psi_t)\) generated by a vector field \( X_t \) of the form

\[
X_t = h_t R_t + Y_t
\]

with \( Y_t \in \ker \alpha_t \). Differentiating (5) we find

\[
\alpha_1 - \alpha_0 + dh_t + i_{Y_t} d\alpha_t = 0.
\]

When we plug \( R \) into this equation, we find

\[
f^{-1} - 1 + dh_t(R) = 0.
\]

The condition that the period of \( fR \) be 1 translates into \( f^{-1} \) integrating to 1 along any fibre of the \( S^1 \)-bundle. This allows us to define a family of functions \( h_t \) satisfying (11), and then there is a unique vector field \( Y_t \in \ker \alpha_t \) satisfying (10). Both \( \alpha_1 - \alpha_0 + dh_t \) and \( d\alpha_t \) are lifts of differential forms on the quotient surface \( M/S^1 \), hence the flow of \( Y_t \) preserves fibres.

**Remark 11.5.** If \( R \) is the Reeb vector field of a contact form \( \alpha \) (on a connected manifold \( M \)), then the rescaled vector field \( f^{-1}R \) is never the Reeb vector field of \( f\alpha \), unless the function \( f \) is constant, for the identity

\[
0 = i_R d(f\alpha) = i_R (df \wedge \alpha) = df(R)\alpha - df
\]

implies that \( df \) vanishes on all vectors tangent to the contact structure \( \ker \alpha \), and by [11, Theorem 3.3.1] any two points in \( M \) can be joined by a curve tangent to the contact structure.
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