Effective magnetic field induced by inhomogeneous Fermi velocity in strained honeycomb structures

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In addition to the known pseudomagnetic field, nonuniform strains independently induce a position-dependent Fermi velocity (PDFV) in graphene. Here we demonstrate that, due to the presence of a PDFV, the Dirac fermions on a nonuniform (strained) honeycomb lattice may experiment a sort of magnetic effect, which is linearly proportional to the momentum of the quasiparticle. As a consequence, the quasiparticles have a sublinear dispersion relation. Moreover, we analyze the general consequence of a PDFV on the Klein tunneling of electrons through pseudomagnetic barriers. In particular, we report an anomalous (Klein) tunneling for an electron passing across velocity barriers with magnetic features. Our findings about the effects induced by a PDFV on Dirac fermions in (2D) strained honeycomb lattice could be extended to (3D) Dirac and Weyl semimetals and/or its analogous artificial systems.

I. INTRODUCTION

One of the key features of graphene is its conical electronic band-structure at the so-called Dirac cones. As a consequence, in this two-dimensional conductor, low-energy electrons behave as massless chiral Dirac fermions [1, 2]. This special electronic behavior has suggested the possibility of observing Klein tunneling [3–5], originally predicted in the context of particle physics.

Another relevant feature of graphene lies in the unusual influence that the elastic lattice deformations have on its electronic properties [6–8]. Nonuniform strains in particular constitute a useful tool to implement the concept of strain engineering in graphene because they may induce a pseudomagnetic field [9, 10]. Many scanning tunneling microscopy studies in graphene have reported pseudo-Landau levels as signatures of a strain-induced pseudomagnetic field [11–15], as was also confirmed by means of angle-resolved photoemission [16]. The effects of such an elastic gauge field on the electronic transport properties of graphene are actively being investigated, particularly those related to the control of the valley degree of freedom: graphene valleytronics [17–20]. In general, nonuniform strained graphene opens new opportunities to research exotic and in some cases unique behaviors, such as a metal-insulator transition [21, 22], a fractal spectrum [23, 24], superconducting states [25, 26], and the quantum Hall effect [27, 28].

In addition to the mentioned strain-induced pseudomagnetic field, another known effect that arises from nonuniform strains is a position-dependent Fermi velocity (PDFV) [29], which might induce confinement effects [30–34]. Evidence of the PDFV effect in strained graphene has been detected through scanning tunneling spectroscopy (STS) [35, 36]. These experiments are based on the fact that the slope of V-shaped STS spectra shows variations at different positions of the sample if the Fermi velocity is spatially varying. Alternatively, Landau-level spectroscopy measurements can be used to confirm PDFV effects [37, 38]. Fingerprints of a PDFV on the Landau level spectrum have been theoretically addressed [38–40]. For example, from tight-binding and Dirac approaches, it was demonstrated in terms of a PDFV that nonuniform uniaxial strains (ripples) produce position-dependent local density-of-states peaks of graphene under an external uniform magnetic field [38]. Otherwise, Landau levels in rippled graphene are shifted towards lower energies proportionally to the average deformation [40].

More recently, Lantagne et al. [41] drew attention to the possibility of achieving spatially separated valley currents and a valley analog to the chiral anomaly in graphene nanoribbons under uniaxial nonuniform strain. Their findings are based on the following remarkable feature: In the presence of certain uniform strain-induced pseudomagnetic fields, the resulting pseudo-Landau levels in graphene are not flat but disperse linearly for small wave-vectors away from the Dirac point, with an opposite slope between the two valleys. As previously reported in the literature [42, 43], such dispersive behavior of the pseudo-Landau levels was explained by Lantagne and colleagues [41] as a consequence of a PDFV, confirming that its presence has important physical consequences. However, they do not discuss the possible magnetic effect of a PDFV. The present paper is devoted to demonstrating that, even in the absence of a strain-induced pseudomagnetic field, a PDFV may induce by itself a sort of magnetic field (and therefore a Lorentz-like force) on the charge carriers in strained graphene.

For the sake of generality, we present this work in the broader context of strained honeycomb lattices, i.e., beyond strained graphene. Nowadays, synthetic systems with honeycomb lattices are artificially created to mimic the behavior of Dirac quasiparticles, such as molecular graphene [44], ultracold atoms [45], photonic lattices

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polartonic systems [48, 49], and acoustic structures [50, 51]. These artificial graphene-like systems offer the advantage of tuning, in a controlled and independent manner, the hopping parameter between different lattice sites. As a consequence, they open a door to explore the physical effects as well as applications of gauge fields [44–51]. For instance, for photonic lattices, it has been envisioned that such pseudoelectromagnetic fields will be useful for applications, such as chip-scale nonlinear optics and coupling to quantum emitters, where strong enhancement of light-matter interaction is achieved from the high density-of-states associated with flat bands [47].

To a large degree, applications of synthetic gauge fields concern phenomena related to the physics of topological quantum matter [52, 53]. Moreover, it is worth pointing out that such gauge fields also offer the opportunity to investigate novel effects that come from other fields of research, such as high-energy physics [52].

The manuscript is organized as follows. Sec. II presents the low-energy theory of Dirac fermions on a nonuniform (strained) honeycomb lattice. In Sec. III, we derive and discuss magnetic effects due to a PDFV on charge carriers. Further, Sec. IV illustrates the consequences of such PDFV-induced effects on the (Klein) tunneling of electrons through pseudomorphic barriers. Finally, in Sec. V, our conclusions are presented.

II. MODEL AND THEORY

Homogeneous Anisotropy.—Within a nearest-neighbor tight-binding model for a homogeneous anisotropic honeycomb lattice, e.g. uniformly strained graphene, with different hoppings $t_1$, $t_2$, and $t_3$ between nearest sites but independent of the position (see Fig. 1(a)), the Hamiltonian in momentum space can be expressed by a $(2 \times 2)$ matrix of the form [54]

$$H(k) = -\sum_{\mathbf{a}_n} t_n \begin{pmatrix} 0 & e^{-ik\cdot\mathbf{a}_n} \\ e^{ik\cdot\mathbf{a}_n} & 0 \end{pmatrix},$$

where $\mathbf{a}_n$ are the nearest-neighbor vectors. This Hamiltonian leads to the dispersion relation of two bands,

$$E(k) = \pm |t_1 e^{ik\cdot\mathbf{a}_1} + t_2 e^{ik\cdot\mathbf{a}_2} + t_3 e^{ik\cdot\mathbf{a}_3}|.$$

As is well known, for the isotropic (unstrained) case, $t_{1,2,3} = t$, the positions $(\mathbf{K}^D)$ of the Dirac cones are located at the corners $(\mathbf{K})$ of the first Brillouin zone. Nevertheless, for an anisotropic honeycomb lattice, $\mathbf{K}^D$ do not coincide with $\mathbf{K}$, as illustrated in Fig. 1(b). In consequence, to obtain the appropriate effective Dirac Hamiltonian, one can no longer expand the Hamiltonian (1) around $\mathbf{K}$, but around $\mathbf{K}^D$ [30, 55–57].

Writing the hoppings as

$$t_n = t(1 + \delta_n),$$

and the nearest-neighbor vectors as

$$\mathbf{a}_1 = \frac{a}{2}(\sqrt{3}, 1), \quad \mathbf{a}_2 = \frac{a}{2}(-\sqrt{3}, 1), \quad \mathbf{a}_3 = a(0, -1),$$

$a$ being the distance between nearest sites, the shift of the Dirac cones (up to first order in the parameters $\delta_n$) can be expressed as [58]

$$\mathbf{K}^D \approx \mathbf{K} + \mathbf{A},$$

where

$$A_x = \frac{\tau}{3a}(2\delta_1 - \delta_1 - \delta_2), \quad A_y = \frac{\tau}{\sqrt{3}a}(\delta_1 - \delta_2),$$

and $\tau$ is the valley index of $\mathbf{K}$ which can take the values $\pm 1$ [59]. Then, by expanding the tight-binding Hamiltonian in momentum space around $\mathbf{K}^D$, with $\mathbf{k} = \mathbf{K}^D + \mathbf{q}$, the effective low-energy Hamiltonian of a uniform anisotropic honeycomb lattice becomes [58]

$$\mathcal{H} = \hbar v_F \sum_{i,j} \sigma_i(I_{ij} + \Delta_{ij})q_j,$$

where $v_F = 3at/2\hbar$, $I_{ij}$ is the $2 \times 2$ identity matrix, $\sigma = (\sigma_x, \sigma_y)$ is a vector of Pauli matrices and $\Delta_{ij}$ is the symmetric matrix with the following entries

$$\Delta_{ij} = \begin{pmatrix} 1/3(2\delta_1 + 2\delta_2 - \delta_3) & 1/\sqrt{3}(\delta_1 - \delta_2) \\ 1/\sqrt{3}(\delta_1 - \delta_2) & \delta_3 \end{pmatrix}.$$ 

Therefore, from Eq. (7) one can recognize an anisotropic Fermi velocity tensor expressed by

$$v_{ij} = v_F(I_{ij} + \Delta_{ij}),$$

which is directly related to the presence of the deformed Dirac cones with an elliptical cross-section in the relation dispersion due to the hopping anisotropy.

The general expression (7) enables us to obtain straightforwardly the effective Dirac Hamiltonian once the explicit form of the hopping parameters is given. For example, for uniform strained graphene, the hopping parameters $t_n$ can be approximated up to first order in the
strain tensor $\epsilon_{ij}$ by $t_n \approx t[1 - (\beta/a^2) \sum_{i,j} a^T_{ij} \epsilon_{ij} a^i_j]$, where $t = 2.7 eV$ and $\beta \sim 3$ [60, 61]. Then, for graphene under uniform strain, $\delta_n = -(\beta/a^2) \sum_{ij} a^T_{ij} \epsilon_{ij} a^i_n$, and hence Eq. (8) results in $\Delta_{ij} = -\beta \epsilon_{ij}$.

Inhomogeneous Anisotropy.—Now, let us to consider that the hoppings are functions of the coordinates, i.e. $t_n(r) = t(1 + \delta_n(r))$, varying slowly on the lattice scale. For this general case, the effective low-energy Hamiltonian of the nonuniform anisotropic honeycomb lattice can be obtained from Hamiltonian (7) as similar to previous considerations for graphene [29, 30, 56, 57, 62] or Dirac and Weyl semimetals [63, 64] under nonuniform strain. Note that with the simple replacement $\delta_n \rightarrow \delta_n(r)$ in the Hamiltonian (7), the terms of the form $\delta_n(r) q_i$ break the hermiticity of the resulting Hamiltonian. Therefore, to assure hermiticity, the usual procedure consists of taking Eq. (7) and doing the symmetric substitution

$$\delta_n q_i \rightarrow \delta_n(r) \left( -i \partial_i - K_i^T(r) \right) - \frac{i}{2} \partial_i \delta_n(r).$$

Using this rule, and considering up to first order in the parameters $\delta_n(r)$, the effective low-energy Hamiltonian of the nonuniform anisotropic honeycomb lattice can be written as

$$\mathcal{H} = -i\hbar \sum_{ij} \sigma_i v_{ij} \partial_j - \hbar v_F \sum_i \sigma_i A_i - \hbar v_F \sum_i \sigma_i \Gamma_i,$$

where the Fermi velocity tensor $v_{ij}(r)$ and the gauge field $A(r)$ are respectively given by Eqs. (9) and (6), but now both are functions of the position vector $r = (x, y)$ due to the spatial dependence of the parameters $\delta_n(r)$. The presence of a PDFV is accompanied by the purely imaginary vector field $\Gamma(r)$ given by

$$\Gamma_i = \frac{i}{2v_F} \sum_j \partial_j v_{ij}(r) = \frac{i}{2} \sum_j \partial_j \Delta_{ij}(r),$$

whose specific form guarantees the hermiticity of the Hamiltonian (11). Unlike the well-known gauge field $A$, $\Gamma$ does not give rise to pseudo Landau levels; however, it has physical significance [62].

If the effect of the hopping variation on the Fermi velocity is not included, Eq. (11) reduces to the simpler expression $\hbar v_F \sum_i \sigma_i (i \partial_i - A_i)$. This last Hamiltonian is typically used in the literature to evaluate and/or interpret the strain effects on electronic properties in terms of a pseudomagnetic field perpendicular to the lattice plane and of strength $B_{ps} = \hbar (\partial_x A_y - \partial_y A_x)/e$, with $e$ being the quasiparticle charge.

Moreover, it is important to mention that Hamiltonian (11) resembles the one for the Dirac fermions moving in a curved background. Within a quantum field theory description, the Fermi velocity tensor represents the vielbein (tetrad), which encoded the metric of the effective curved space [29, 39, 56, 57]. As a consequence, the effects of a PDFV that we discuss in the next section can be interpreted in terms of the emergent vielbein field. For a detailed correspondence between the tight-binding approach followed in this paper and a quantum field theory approach, see for example Refs. [29, 56].

III. MAGNETIC EFFECTS INDUCED BY PDFV

We now consider two illustrative cases of hopping variation in order to evaluate the corresponding magnetic-like effects due to a PDFV on the quasiparticle dynamics from Eq. (11).

A. Dispersive pseudo-Landau levels

It is well known that certain variations of the hopping parameters lead to a uniform pseudomagnetic field (see Ref. [52] and references therein). For example, this happens for

$$\delta_1(y) = \delta_2(y) = 3c_0 y/4 \quad \text{and} \quad \delta_3 = 0,$$

since $A = (-\tau c_0 y/2a, 0)$, and thus the pseudomagnetic field turns out to be $B_{ps} = \tau h c_0/(2ea)$. In addition to the uniform pseudomagnetic field, the hopping variation (13) induces also a PDFV tensor given by

$$v_{ij} = v_F \left( 1 + c y \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right),$$

such that the Fermi velocity in the $x$-direction depends on the $y$-coordinate. Moreover, note that $\Gamma$ is zero.

So, for this case, the effective Dirac Hamiltonian (11) reads as

$$\mathcal{H} = -i\hbar v_F (1 + c y) \sigma_x \partial_x - i\hbar v_F \sigma_y \partial_y + v_F e B_{ps} \sigma_x,$$

which has translational symmetry along the $x$-direction. As a consequence, the eigenfunctions can be represented as $\Psi(r) = e^{iq_x x} \Phi(y)$ leading to the following eigenvalues problem

$$[(q_x + eB y/h) - \partial_y] \phi_2 = \frac{E}{hv_F} \phi_1,$$

$$[(q_x + eB y/h) + \partial_y] \phi_1 = \frac{E}{hv_F} \phi_2,$$

where $B = B_{ps} + h q_x c_0/e = B_{ps}(1 + \tau 2q_x a)$ and $E$ is the energy. By simple inspection, one can see that Eq. (16) is analogous to the Landau level problem for massless Dirac fermions in the presence of a uniform magnetic field of strength $B$, whose spectrum is $E_n = \pm \sqrt{2h v_F^2 |B| n}$, with $n$ an integer including zero [65]. Therefore, the eigenvalues of Eq. (16) can be expressed as

$$E_{n, q_x} = \pm h v_F \sqrt{|c_0/a| n \sqrt{1 + \tau 2q_x a}},$$

which can be understood as a result of the combined action of two fields, i.e. the standard pseudomagnetic field $B_{ps} = \tau h c_0/(2ea)$ and an effective magnetic field...
Then, for the considered case (18), the effective Dirac Hamiltonian (11) can be written as

$$\mathcal{H} = -i\hbar v_F \sigma_y \partial_x - i\hbar v_F (1 + c_0 x) \sigma_y \partial_y,$$

(20)

with translational symmetry along the \( y \)-direction. Therefore, the eigenfunctions can be casted as \( \Psi(r) = e^{i q_y y} \Phi(x) \), which leads to the following eigenvalues problem

$$\begin{align*}
[\partial_x + (q_y + e B_x e^2/\hbar)] \phi_1 &= \frac{i E}{\hbar v_F} \phi_1, \\
[\partial_x - (q_y + e B_x e^2/\hbar)] \phi_2 &= \frac{i E}{\hbar v_F} \phi_2,
\end{align*}$$

(21)

where \( e B_c = \hbar q_y c_0 \) is due to the PDFV (19). As in the previous example, one can recognize that problem (21) is analogous to the Landau level problem for massless Dirac fermions in the presence of a uniform magnetic field of strength \( B_c \). Therefore, the eigenvalues of Eq. (21) are given by

$$E_{n,q_y} = \pm \hbar v_F \sqrt{2|q_y c_0| n}.$$  

(22)

In short, in a honeycomb lattice with a PDFV as given by Eq. (19), and in the absence of the standard pseudomagnetic field, the dynamics of the Dirac quasiparticles can be understood as if they feel an effective magnetic field of strength \( B_c = \hbar q_y c_0 / e \). As a consequence, the energy no longer depends linearly on the momentum \( q_y \), but on the square-root \( E \sim \sqrt{|q_y|} \).

It is worthwhile to remark that, as mentioned at the end of Sec. II, the results here obtained as a consequence of a PDFV could be derived from the equivalent vielbein field. In that case, the spectrum (22) would have been called torsional Landau levels [66, 67], since the strength of the vielbein field is referred to as the torsion or torsional magnetic field [68, 69].

**B. A sublinear dispersion relation**

Now let us consider an inhomogeneous honeycomb lattice characterized by the following variation of the hopping parameters

$$4 \delta_1(x) = 4 \delta_2(x) = \delta_3(x) = c_0 x,$$

(18)

whose resulting gauge field is \( A = (c_0 x / 2a, 0) \) and, thus, the corresponding pseudomagnetic field \( B_{ps} \) is zero. Otherwise, the hopping variation (18) induces a PDFV tensor given by

$$v_{ij} = v_F \begin{pmatrix} 1 & 0 \\ 0 & 1 + c_0 x \end{pmatrix},$$

(19)

such that the Fermi velocity in the \( y \)-direction depends on the \( x \)-coordinate and, besides \( \Gamma = 0 \).

**IV. KLEIN TUNNELING UNDER PSEUDOMAGNETIC BARRIERS**

Because of Klein tunneling, Dirac electrons can tunnel through electrostatic potential barriers without reflection, particularly for normal incidence [3–5]. This fact makes it difficult to confine Dirac electrons, turning the electronic switching into a big challenge for graphene-based nanoelectronics. In contrast to electrostatic potential barriers, magnetic barriers (as well as magnetic quantum dots) are able to confine Dirac electrons [70–73].

Consider a square-well magnetic barrier in the region \( |x| \leq d \), such that the vector potential is given by

$$A_x = 0 \quad \text{and} \quad A_y(x) = B_0 \begin{cases} -d, & x < -d \\ x, & |x| \leq d \\ d, & x > d \end{cases},$$

(23)
with $B_0$ being the strength of the external magnetic field within the strip (see Fig. 3(a)). Proceeding as in Refs. [70, 71], the electron scattering entering from the left side can be characterized as follows. For $x < -d$, the wave function can be written as

$$\Psi_1(x) = \left( \begin{array}{c} e^{iq_x x} + re^{-iq_x x} \\ e^{iq_x x + i\phi} - re^{-iq_x x - i\phi} \end{array} \right), \quad (24)$$

where $\mathbf{q} = (q_x, q_y)$ is the incoming wave vector, $r$ is the reflection amplitude, and $\phi$ is just the kinematic incidence angle $\phi_i$, i.e. $\phi = \phi_i$, because the gauge-invariant velocity is $\mathbf{v} = v_F(\cos \phi, \sin \phi)$. Also, note that the wave vector is given by $q_x = \varepsilon \cos \phi$ and $q_y = \varepsilon \sin \phi + d/l_B^2$, with $\varepsilon = E/\hbar v_F$ and $l_B = \sqrt{\hbar/eB_0}$ the magnetic length.

Within the region $|x| \leq d$, the wave function can be expressed as a linear combination of parabolic cylinder functions $D_u$ (Weber functions),

$$\Psi_II(x) = \left( \begin{array}{c} c_1 D_{\eta-1}(X) + c_2 D_{\eta-1}(-X) \\ i\sqrt{2}/(\varepsilon l_B) [c_1 D_{\eta}(X) - c_2 D_{\eta}(-X)] \end{array} \right), \quad (25)$$

where $c_{1,2}$ are complex constants, $X = \sqrt{2}(x/l_B + q_y l_B)$ and $\eta = (\varepsilon l_B^2)/2$.

Otherwise, for $x > d$ the transmitted wave function is

$$\Psi_III(x) = t \sqrt{\frac{q_x}{q'_x}} e^{iq'_x x}, \quad (26)$$

with $t$ the transmission amplitude and $\phi' (\mathbf{q}')$ the exit angle (wave vector). From the conservation of $q_y$, it results that $\sin \phi' = 2d/(\varepsilon l_B^2) + \sin \phi$, which implies that for certain incidence angles $\phi$, that fulfill the condition

$$-1 \leq \sin \phi \leq 1 - 2d/(\varepsilon l_B^2), \quad (27)$$

transmission is possible. However, when $\varepsilon l_B \leq d/l_B$, any incoming electron is reflected, regardless of the incident angle $\phi$. Then, matching the wave functions and the flux at the interfaces $x = \pm d$, one can obtain in closed form that the transmission amplitude $t$ is given by [70],

$$t = \frac{2i\varepsilon l_B \sqrt{2q_x'/q_x} \cos \phi}{e^{i(q_x' + q_x)} Q} (u_2^+ v_2^- + v_2^+ u_2^-), \quad (28)$$

where

$$Q = (i\sqrt{2}v_1^+ - \varepsilon l_B e^{i\phi} u_2^+) (-i\sqrt{2}v_1^- + \varepsilon l_B e^{-i\phi} u_1^-) + (i\sqrt{2}v_2^+ + \varepsilon l_B e^{i\phi} u_2^+)(i\sqrt{2}v_2^- + \varepsilon l_B e^{-i\phi} u_1^+),$$

$$u_1^\pm = D_{\eta-1}[\pm\sqrt{2}(d/l_B + q_y l_B)],$$

$$v_2^\pm = D_{\eta}[\pm\sqrt{2}(d/l_B + q_y l_B)],$$

whereas $u_1^\pm$ and $v_1^\pm$ follow by letting $d \rightarrow d$ in the last two expressions, respectively. Finally, the transmission probability is obtained as $T(\phi) = |t|^2$, which is ultimately a function of the quantities $\phi, \varepsilon l_B$ and $d/l_B$.

Figure 3(a) shows the $\phi$-dependent transmission probability $T(\phi)$ of Dirac electrons (in graphene) for different energies through a magnetic barrier with $B_0 = 10 T$ and $d = 50 a \approx 7.1$ nm, with $a$ being the carbon-carbon distance. For this barrier, electrons with energies less than $\hbar v_F d/l_B^2 \approx 62$ meV are reflected, whereas for electrons with energies of 70 meV, 100 meV and 200 meV, the tunneling is possible for incident angles from $-90^\circ$ to the limit angles $-50.6^\circ$, $-13.9^\circ$ and $22.3^\circ$, respectively (see Fig. 3(a)). If we consider the magnetic field to be $B_0 = -10 T$, the resulting transmission probability can be obtained from the symmetry relation $T(-B_0, \phi) = T(B_0, -\phi)$. On the other hand, Fig. 3(b) depicts the same scenario of electron tunneling (presented

![Fig. 3. (a) On left-side, scattering geometry of Dirac electrons through a magnetic barrier along the y-axis with uniform strength $B_0$ and width $2d$. On right-side, the respective polar graphs of the transmission probability $T(\phi)$ for $B_0 = 10 T$, $d \approx 7.1$ nm and different energies. (b) Same as panel (a), but with the magnetic barrier along the x-axis.](image-url)
Let us now suppose that the hopping parameters vary as

$$\delta_1(y) = \delta_2(y) = 3c_0/4 \begin{cases} -d, & y < -d \\ y, & |y| \leq d \\ d, & y > d \end{cases},$$

and $\delta_3 = 0$. In the strip $|y| < d$, this variation of the hopping parameters induces a standard pseudomagnetic field $B_{ps} = \tau \hbar c_0/(2ea)$ and a Fermi velocity in the $x$-direction depends on the $y$-coordinate. As discussed in Sec. III A, both effects can be understood as if electrons experience an effective $q_y$-dependent magnetic field given by $B = B_{ps}(1 + \tau 2q_ya)$. As a consequence, the previous results about the transmission probability $T$ through a magnetic barrier (along the $x$-axis) can be extended to the present case by replacing $B_0$ with $B$, and thus the magnetic longitude should be redefined as $l_B = \sqrt{\hbar/eB}$. Following this procedure, $T$ is obtained as a function of the angle $\phi$. Nevertheless, it is important to note that for this case, in the incidence region $y < -d$, the gauge-invariant velocity is $v = v_F(1 - c_0 d) \sin \phi, \cos \phi$. Hence, $\phi$ and the incident angle $\phi_i$ no longer coincide. They are related by the expression

$$\tan \phi_i = (1 - c_0 d) \tan \phi,$$

which can be approximated by $\phi_i \approx \phi - (1 - c_0 d) \sin 2\phi$ (in radians) for $c_0 d < 1$. At last, Eq. (30) allows to parametrically express $T$ as a function of $\phi_i$.

Figure 4(a) illustrates the transmission probability $T(\phi)$ for electrons (with valley index $\tau = +1$) tunneling through the pseudomagnetic barrier induced by the hopping variation (29). We assumed $d = 50a$, and the value of $c_0$ was chosen such that $B_{ps} = 10T$. Note that if the position dependence of Fermi velocity is disregarded, and only the standard pseudomagnetic field $B_{ps}$ is taken into account, then the transmission probability would be as discussed above for a magnetic barrier with 10T, which is represented by the blue dashed line in Fig. 4 for reference. Instead, the solid lines in Fig. 4 show the resulting $T(\phi)$ when the position dependence of Fermi velocity is taken into account, i.e., by considering that electrons feel the effective $q_y$-dependent magnetic field $B$. Comparing these results, it can be observed that, in general, the higher the energy of the incident particle through the pseudomagnetic barrier (29), the larger is the effect of the PDFV on the transmission probability $T$. In detail, for a given energy (see Fig. 4(b)), at grazing incidence angles (close to 90°) the resulting $T(\phi)$ is slightly lower because $B > B_{ps}$ since $q_y > 0$ for incidence angles. Otherwise, as illustrated in Fig. 4(c), at incident angles close to the limit angle of tunneling, $T(\phi)$ turns out to be higher due to $B < B_{ps}$, which also leads to a shift of the limit angle itself. In other words, the consideration of the position dependence of Fermi velocity yields a broader spectrum of tunneling angles.

**B. A velocity barrier with magnetic features**

Finally, let us consider the variation of the hopping parameters,

$$4\delta_1(x) = 4\delta_2(x) = 4\delta_3(x) = c_0 \begin{cases} -d, & x < -d \\ x, & |x| \leq d \\ d, & x > d \end{cases}.$$
kind of magnetic velocity barrier with incidence angle \( \phi_i > 0 \) (\( \phi_i < 0 \)) feels an effective magnetic field \( B_v > 0 \) (\( B_v < 0 \)), while for normal incidence \( B_v = 0 \). Therefore, it is expected that the transmission probability fulfills \( T(\phi_i) = T(-\phi_i) \) and \( T(0) = 0 \) (see Fig. 5(a)).

Again the transmission probability \( T(\phi) \) can be obtained by using the results for a real magnetic barrier, but now defining the magnetic longitude as \( l_{B_v} = \sqrt{\hbar c/|B_v|} = \sqrt{|T|/|q_B v_o|} \). Also for this case, it is worth noting that \( \phi \) and the incident angle \( \phi_i \) do not coincide because, in the incidence region \( x < -d \), the gauge-invariant velocity is \( v = v_F (\cos \phi_i (1 - c_0 d) \sin \phi) \). Hence, both angles are related by expression (30).

Figure 5(a) shows \( T(\phi_i) \) for electrons with different energies tunneling through the magnetic velocity barrier produced by the hopping variation (31). As in the previous example, here we assumed that \( d = 50a \) and \( c_0d \approx 0.03 \). The reported transmission probability spectrum is notably different from those of electrostatic potential barriers [3–5], magnetic barriers [70, 71] and velocity barriers [74, 75]. Its distinguishing feature is that, irrespective of the electron energy, the transmission is almost perfect (\( T \approx 1 \)) for the incidence angles in the approximate range \( |\phi| \lesssim \phi_L \), whereas it is completely inhibited (\( T = 0 \)) for \( |\phi| \geq \phi_L \) (see Fig. 5(a)-(b)). The limit angle \( \phi_L \) is only determined by the barrier parameter \( c_0d \). By adapting Eq. (27) to the present case, we obtain that \( \phi_L \) is given by

\[
\phi_L(c_0d) = \arctan \left[ (1 - c_0d) \tan \left[ \arcsin \left( \frac{1 - c_0d}{1 + c_0d} \right) \right] \right],
\]

which is plotted (in degrees) in Fig. 5(c). For instance, \( \phi_L(0.03) \approx 69.8^\circ \), \( \phi_L(0.1) \approx 52.0^\circ \) and \( \phi_L(0.2) \approx 35.6^\circ \). Thus, varying \( c_0d \) we could tune the collimation of an electron flux that passes through this magnetic velocity barrier.

V. CONCLUSIONS

In closing, we reported new effects due to a PDFV on Dirac fermions in a nonuniform honeycomb lattice, such as, for example, strained graphene or/and artificial graphene-like systems. It was shown that, for certain spatial dependencies of Fermi velocity, the electrons feel an effective magnetic field. Unlike the known strain-induced pseudomagnetic field, this new sort of magnetic field depends on the electron momentum but not on the valley index. Moreover, we studied fingerprints of a PDFV on the transmission probability of electrons through pseudomagnetic barriers and velocity barriers. For the last ones, we reported anomalous tunneling with magnetic features, which suggests the possibility of using these types of velocity barriers to achieving electron collimation as an alternative to other routes [76–78]. In general, our results confirmed that the presence of a PDFV can not be obviated because it has important physical consequences, expanding the concept of strain-engineering. Finally, the reported magnetic effects due to a PDFV could be extended to (3D) Dirac and Weyl semimetals, including its artificial versions, where a strain-induced PDFV has also drawn attention [64, 79, 80].

ACKNOWLEDGMENTS

This work has been partially supported by CONACyT-Mexico under Grant No. 254414. MOL acknowledges a postdoctoral fellowship from CONACyT–Mexico. JEBV acknowledges funding from PAIP Facultad de Química, UNAM (grant 5000-9173).

[1] K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, M. I. Katsnelson, I. V. Grigorieva, S. V. Dubonos, and A. A. Firsov, “Two-dimensional gas of massless Dirac fermions in graphene,” Nature 438, 197–200 (2005).

[2] Y. Zhang, Y.-W. Tan, H. L. Stormer, and P. Kim, “Experimental observation of the quantum Hall effect and Berry’s phase in graphene,” Nature 438, 201204 (2005).
[3] M. I. Katsnelson, K. S. Novoselov, and A. K. Geim, “Chiral tunnelling and the Klein paradox in graphene,” Nature Physics 2, 620–625 (2006).

[4] A. F. Young and P. Kim, “Quantum interference and Klein tunnelling in graphene heterojunctions,” Nat. Phys. 5, 222–226 (2009).

[5] N. Stander, B. Huard, and D. Goldhaber-Gordon, “Evidence for Klein tunneling in graphene $p-n$ junctions,” Phys. Rev. Lett. 102, 026807 (2009).

[6] B. Amorim, A. Cortijo, F. de Juan, A. G. Grushin, F. Guinea, A. Gutiérrez-Rubio, H. Ochoa, V. Parente, R. Roldán, P. San-Jose, J. Schiefele, M. Sturla, and M. A. H. Vozmediano, “Novel effects of strains in graphene and other two dimensional materials,” Physics Reports 617, 1–54 (2016).

[7] G. G. Naumis, S. Barraza-Lopez, M. Oliva-Leyva, and H. Terrones, “Electronic and optical properties of strained graphene and other strained 2D materials: a review,” Rep. Prog. Phys. 80, 096501 (2017).

[8] D. Zhai and N. Sandler, “Electron dynamics in strained graphene,” Modern Physics Letters B 33, 1930001 (2019).

[9] F. Guinea, M. I. Katsnelson, and A. K. Geim, “Energy gaps and a zero-field quantum Hall effect in graphene by strain engineering,” Nat Phys 6, 30–33 (2010).

[10] M. A. H. Vozmediano, M. I. Katsnelson, and F. Guinea, “Gauge fields in graphene,” Phys. Rep. 496, 109 (2010).

[11] N. Levy, S. A. Burke, K. L. Meaker, M. Panlasigui, A. Zettl, F. Guinea, A. H. Castro Neto, and M. F. Crommie, “Strain-induced pseudo–magnetic fields greater than 300 tesla in graphene nanobubbles,” Science 329, 544–547 (2010).

[12] N.-C. Yeh, M.-L. Teague, S. Yeom, B. L. Standley, R. T.-P. Wu, D. A. Boyd, and M. W. Bockrath, “Strain-induced pseudo-magnetic fields and charging effects on cvd-grown graphene,” Surface Science 605, 1649 – 1656 (2011).

[13] J. Lu, A. H. Castro Neto, and K. P. Loh, “Transforming Moiré blisters into geometric graphene nano-bubbles,” Nature Communications 3, 823 (2012).

[14] Y. Liu et al., “Tailoring sample-wide pseudo-magnetic fields on a graphene–black phosphorus heterostructure,” Nature Nanotechnology 13, 828–834 (2018).

[15] R. Banerjee, V.-H. Nguyen, T. Granzient-Nakajima, L. Pabbi, A. Lherbier, A. R. Binion, J.-C. Charlier, M. Terrones, and E. W. Hudson, “Strain modulated superlattices in graphene,” Nano Letters 20, 3113–3121 (2020).

[16] P. Nigge et al., “Room temperature strain-induced Landau levels in graphene on a wafer-scale platform,” Science Advances 5, eaaw5593 (2019).

[17] M. Settnes, S. R. Power, M. Brandbyge, and A.-P. Jauho, “Graphene nanobubbles as valley filters and beam splitters,” Phys. Rev. Lett. 117, 276801 (2016).

[18] T. Stegmann and N. Szpak, “Current splitting and valley polarization in elastically deformed graphene,” 2D Materials 6, 015024 (2018).

[19] D. Faria, C. León, L. R. F. Lima, A. Latgé, and N. Sandler, “Valley polarization braiding in strained graphene,” Phys. Rev. B 101, 081410 (2020).

[20] S.-Y. Li, Y. Su, Y.-N. Ren, and L. He, “Valley polarization and inversion in strained graphene via pseudo-Landau levels, valley splitting of real Landau levels, and confined states,” Phys. Rev. Lett. 124, 106802 (2020).

[21] H.-K. Tang, E. Laksono, J. N. B. Rodrigues, P. Senagupta, F. F. Assaad, and S. Adam, “Interaction-driven metal-insulator transition in strained graphene,” Phys. Rev. Lett. 115, 186602 (2015).

[22] S. Sorella, K. Seki, O. O. Brovko, T. Shirakawa, S Miyakoshi, S. Yunoki, and E. Tosatti, “Correlation-driven dimerization and topological gap opening in isotropically strained graphene,” Phys. Rev. Lett. 121, 066402 (2018).

[23] G. G. Naumis and P. Roman-Taboada, “Mapping of strained graphene into one-dimensional hamiltonians: Quasicrystals and modulated crystals,” Phys. Rev. B 89, 241404 (2014).

[24] P. Roman-Taboada and G. G. Naumis, “Spectral butterfly and electronic localization in rippled-graphene nanoribbons: Mapping onto effective one-dimensional chains,” Phys. Rev. B 92, 035406 (2015).

[25] B. Uchoa and Y. Barlas, “Superconducting states in pseudo-Landau-levels of strained graphene,” Phys. Rev. Lett. 111, 046604 (2013).

[26] V. J. Kauppiä, F. Aikebaier, and T. T. Heikkilä, “Flat-band superconductivity in strained dirac materials,” Phys. Rev. B 93, 214505 (2016).

[27] G. Wagner, F. de Juan, and D. X. Nguyen, “Quantum Hall effect in curved space realized in strained graphene,” (2019), arXiv:1911.02028 [cond-mat.str-el].

[28] E. Sela, Y. Bloch, F. von Oppen, and M. B. Shalom, “Quantum Hall response to time-dependent strain gradients in graphene,” Phys. Rev. Lett. 124, 026602 (2020).

[29] F. de Juan, M. Sturla, and M. A. H. Vozmediano, “Space dependent Fermi velocity in strained graphene,” Phys. Rev. Lett. 108, 227205 (2012).

[30] M. Oliva-Leyva and G. G. Naumis, “Generalizing the Fermi velocity of strained graphene from uniform to nonuniform strain,” Phys. Lett. A 379, 2645 (2015).

[31] C. A. Downing and M. E. Portnoi, “Localization of massless dirac particles via spatial modulations of the fermi velocity,” Journal of Physics: Condensed Matter 29, 153501 (2017).

[32] K. Flouris, M. Mendoza Jimenez, J.-D. Debus, and H. J. Herrmann, “Confining massless dirac particles in two-dimensional curved space,” Phys. Rev. B 98, 155419 (2018).

[33] A. Contreras-Astorga, V. Jakubsy, and A. Raya, “On the propagation of Dirac fermions in graphene with strain-induced inhomogeneous Fermi velocity,” Journal of Physics: Condensed Matter 32, 295301 (2020).

[34] A.-L Phan, D.-N Le, V.-H Le, and P. Roy, “Electronic spectrum in 2D Dirac materials under strain,” Physica E: Low-dimensional Systems and Nanostructures 121, 114084 (2020).

[35] H. Yan, Z.-D. Chu, W. Yan, M. Liu, L. Meng, M. Yang, Y. Fan, J. Wang, R.-F. Dou, Y. Zhang, Z. Liu, J.-C. Nie, and L. He, “Superlattice dirac points and space-dependent fermi velocity in a corrugated graphene monolayer,” Phys. Rev. B 87, 075405 (2013).

[36] W.-J. Jang et al., “Observation of spatially-varying fermi velocity in strained-graphene directly grown on hexagonal boron nitride,” Carbon 74, 139 (2014).

[37] O. Storz, A. Cortijo, S. Wilfert, K. A. Kohk, O. E. Tereshchenko, M. A. H. Vozmediano, M. Bode, F. Guinea, and P. Sessi, “Mapping the effect of defect-induced strain disorder on the Dirac states of topological insulators,” Phys. Rev. B 94, 121301 (2016).
Z. V. Khaidukov and M. A. Zubkov, “Landau levels in graphene in the presence of emergent gravity,” The European Physical Journal B 89, 213 (2016).

J.-D. Debus, M. Mendoza, and H. J. Herrmann, “Shifted Landau levels in curved graphene sheets,” Journal of Physics: Condensed Matter 30, 415503 (2018).

É. Lantagne-Hurtubise, X.-X. Zhang, and M. Franz, “Dispersive Landau levels and valley currents in strained graphene nanoribbons,” Phys. Rev. B 101, 085423 (2020).

G. Salerno, T. Ozawa, H. M. Price, and I. Carusotto, “How to directly observe Landau levels in driven-dissipative strained honeycomb lattices,” 2D Materials 2, 034015 (2015).

G. Salerno, T. Ozawa, H. M. Price, and I. Carusotto, “Propagating edge states in strained honeycomb lattices,” Phys. Rev. B 95, 245418 (2017).

K. K. Gomes, W. Mar, W. Ko, F. Guinea, and H. C. Manoharan, “Designer Dirac fermions and topological phases in molecular graphene,” Nature 483, 306–310 (2012).

L. Tarruell, T. Greif, D.and Uehlinger, G. Jotzu, and T. Esslinger, “Creating, moving and merging Dirac points with a Fermi gas in a tunable honeycomb lattice,” Nature 483, 302305 (2012).

M. C. Rechtsman, J. M. Zeuner, A. Tünnermann, T. C. H. Liew, M. C. Rechtsman, D. Schuster, J. Simon, O. Zilberberg, and T. Esslinger, “Exciton-polariton topological insulator,” Nature 527, 453–458 (2015).

J. Guglielmon, M. C. Rechtsman, and M. I. Weinstein, “Landau levels in strained two-dimensional photonic crystals,” (2020), arXiv:2003.06690 [physics.optics].

S. Klembt, T. H. Harder, O. A. Egorov, K. Winkler, R. Ge, M. A. Bandres, M. Emmerling, L. Worschech, T. C. H. Liew, M. Segev, C. Schneider, and S. Höfling, “Exciton-polariton topological insulator,” Nature 562, 552–556 (2018).

C.-R. Mann, S. A. R. Horsley, and E. Mariani, “Tunable pseudo-magnetic fields for polaritons in strained meta-surfaces,” (2020), arXiv:2001.11931 [physics.optics].

Z. Yang, F. Gao, Y. Yang, and B. Zhang, “Strain-induced gauge field and Landau levels in dielectric structures,” Nature Photonics 7, 153–158 (2013).

M. Aidelsburger, S. Nascimbene, and N. Goldman, “Artificial gauge fields in materials and engineered systems,” Comptes Rendus Physique 19, 394 – 432 (2018).

T. Ozawa, H. M. Price, A. Amo, N. Goldman, M. Hafezi, L. Lu, M. C. Rechtsman, D. Schuster, J. Simon, O. Zilberberg, and I. Carusotto, “Topological photonics,” Rev. Mod. Phys. 91, 015006 (2019).

Y. Hasegawa, R. Konno, H. Nakano, and M. Kohmoto, “Zero modes of tight-binding electrons on the honeycomb lattice,” Phys. Rev. B 74, 033413 (2006).

M. Oliva-Leyva and G. G. Naumis, “Understanding electron behavior in strained graphene as a reciprocal space distortion,” Phys. Rev. B 88, 085430 (2013).

G. E. Volovik and M. A. Zubkov, “Emergent Horava gravity in graphene,” Ann. Phys. 340, 352 (2014).

G. E. Volovik and M. A. Zubkov, “Emergent geometry experienced by fermions in graphene in the presence of dislocations,” Annals of Physics 356, 255 – 268 (2015).

M. Oliva-Leyva and G. G. Naumis, “Effective Dirac hamiltonian for anisotropic honeycomb lattices: Optical properties,” Phys. Rev. B 93, 035439 (2016).

C. Bena and G. Montambaux, “Remarks on the tight-binding model of graphene,” New Journal of Physics 11, 095003 (2009).

V. M. Pereira, A. H. Castro Neto, and N. M. R. Peres, “Tight-binding approach to uniaxial strain in graphene,” Phys. Rev. B 80, 045401 (2009).

A. R. Botello-Méndez, J. C. Obeso-Jureidini, and G. G. Naumis, “Toward an accurate tight-binding model of graphene’s electronic properties under strain,” The Journal of Physical Chemistry C 122, 15753–15760 (2018).

F. de Juan, J. L. Mänes, and M. A. H. Vozmediano, “Gauge fields from strain in graphene,” Phys. Rev. B 87, 165131 (2013).

M. A. Zubkov, “Emergent gravity and chiral anomaly in Dirac semimetals in the presence of dislocations,” Annals of Physics 360, 655 – 678 (2015).

A. Cortijo and M. A. Zubkov, “Emergent gravity in the cubic tight-binding model of Weyl semimetal in the presence of elastic deformations,” Annals of Physics 366, 45 – 56 (2016).

J. W. McClure, “Diamagnetism of graphite,” Phys. Rev. 104, 666–671 (1956).

Z.-M. Huang, B. Han, and M. Stone, “Nieh-Yan anomaly: Torional Landau levels, central charge, and anomalous thermal Hall effect,” Phys. Rev. B 101, 125201 (2020).

L. Liang and T. Ojanen, “Topological magnetotorsional effect in Weyl semimetals,” Phys. Rev. Research 2, 022016 (2020).

A. Shitade, “Heat transport as torsional responses and Keldysh formalism in a curved spacetime,” Progress of Theoretical and Experimental Physics 2014, 123I01 (2014).

O. Parrilkar, T. L. Hughes, and R. G. Leigh, “Torsion, parity-odd response, and anomalies in topological states,” Phys. Rev. D 90, 105004 (2014).

A. De Martino, L. Dell’Anna, and R. Egger, “Magnetic confinement of massless Dirac fermions in graphene,” Phys. Rev. Lett. 98, 066802 (2007).

M. Ramezani Masir, P. Vasilopoulos, A. Matulis, and F. M. Peeters, “Direction-dependent tunneling through nanostructured magnetic barriers in graphene,” Phys. Rev. B 77, 235443 (2008).

P. Roy, T. K. Ghosh, and K. Bhattacharya, “Localization of Dirac-like excitations in graphene in the presence of smooth inhomogeneous magnetic fields,” Journal of Physics: Condensed Matter 24, 055301 (2012).

J. A. Downing and M. E. Portnoi, “Massless Dirac fermions in two dimensions: Confinement in nonuniform magnetic fields,” Phys. Rev. B 94, 165407 (2016).

A. Raoux, M. Polini, R. Asgari, A. R. Hamilton, R. Fazio, and A. H. MacDonald, “Velocity-modulation control of electron-wave propagation in graphene,” Phys. Rev. B 81, 073407 (2010).

A. Concha and Z. Tesanovic, “Effect of a velocity barrier on the ballistic transport of Dirac fermions,” Phys. Rev.
[76] H. van Houten, B. J. van Wees, J. E. Mooij, C. W. J. Beenakker, J. G. Williamson, and C. T. Foxon, “Coherent electron focusing in a two-dimensional electron gas,” EPL (Europhysics Letters) 5, 721 (1988).

[77] C.-H. Park, Y.-W. Son, L. Yang, M. L. Cohen, and S. G. Louie, “Electron beam supercollimation in graphene superlattices,” Nano Letters 8, 2920–2924 (2008).

[78] Z. Wang and F. Liu, “Manipulation of electron beam propagation by hetero-dimensional graphene junctions,” ACS Nano 4, 2459–2465 (2010).

[79] S. A. Yang, H. Pan, and F. Zhang, “Chirality-dependent Hall effect in Weyl semimetals,” Phys. Rev. Lett. 115, 156603 (2015).

[80] V. Arjona and M. A. H. Vozmediano, “Rotational strain in Weyl semimetals: A continuum approach,” Phys. Rev. B 97, 201404 (2018).