COHOMOLOGY OF TWISTED SYMMETRIC POWERS OF COTANGENT BUNDLES OF SMOOTH COMPLETE INTERSECTIONS

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Abstract. In this paper, we provide two different resolutions of structural sheaves of projectivized tangent bundles of smooth complete intersections. These resolutions allow in particular to obtain convenient (and completely explicit) descriptions of cohomology of twisted symmetric powers of cotangent bundles of complete intersections, which are easily implemented on computers. We then provide several applications. First, we recover the known vanishing theorems on the subject, and show that they are optimal via some non-vanishing theorems. Then, we study the symmetric algebra $\bigoplus_{m\in\mathbb{N}}H^0(X, S^m\Omega_X(m))$ of a smooth complete intersection of codimension $c < \frac{N}{2}$, improving the known results in the literature. We also study partial ampleness of cotangent bundles of general hypersurfaces. Finally, we illustrate how the explicit descriptions of cohomology can be implemented on computer. In particular, this allows to exhibit new and simple examples of family of surfaces along which the canonically twisted pluri-genera do not remain constant.

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The main goal of this paper is to provide a convenient and explicit description of cohomology groups of (negatively) twisted symmetric powers of cotangent bundles of complete intersections, and provide several applications. Some of them are, to our knowledge, new and some others consist of alternative proofs of known results.

The study of this subject is not new, but there seems to be, as far as we know, surprisingly few references. In [Brj71], the author proves a vanishing theorem for global sections of tensor powers of cotangent bundles of hypersurfaces. In [Sak79], the author proves that a smooth complete intersection in $\mathbb{P}^N$ of codimension $c < \frac{N}{2}$ has no global symmetric differentials. (Note that in both previous references, twists by the Serre line bundle are not considered). In [BR90], the authors consider more generally Schur powers of cotangent bundles of complete intersections, and prove strong vanishing results (see [BR90][Theorem 4 (i), (ii), (iii)]). In our first application (see Section 4.1), we provide new proofs of these vanishing theorems, and show that they are optimal via non-vanishing theorems. These non-vanishing results are, to our knowledge, new, and allowed us to generalize results of Bogomolov–De Oliveira in [BDO08]: see Theorem 0.0.10 below (or Theorem 4.1.3).

Let us continue our overview. In [Sch92], the author proves that vanishing theorems [BR90][Theorem 4 (i), (ii) for symmetric powers] are actually true for any smooth subvarieties (within the codimension range allowed by the hypothesis of the theorem). Schneider’s result was then generalized by Brotbek in [Bro11], allowing to handle the case of a general Schur power. It remains, to our knowledge, unknown whether [BR90][Theorem 4 (iii) for symmetric powers] remains valid for any smooth subvariety in $\mathbb{P}^N$ (of codimension $c < \frac{N}{2}$). Note that, by Hartshorne’s conjecture [Laz04][Conjecture 3.2.8], this should be the case at least when $c < \frac{N}{3}$. In [BDO08], the authors studied more specifically symmetric powers of $\Omega_X(1)$, where $X \subset \mathbb{P}^N$ is a smooth subvariety. Amongst other things, they recovered [BR90][Theorem (iii) for symmetric powers] in the case of hypersurfaces (see Theorem B in loc.cit), and they showed that a very particular case of [BR90][Theorem (iii) for symmetric powers] holds for smooth subvarieties of codimension 2: see Section 4.1.1 for more details. Later, in [DOL19], the study of the algebra $\bigoplus_{m \in \mathbb{N}} H^0(X, S^m \Omega_X(m))$ was pushed further. In loc.cit, the authors proved in particular that, for a smooth complete intersection $X$ of codimension $c < \frac{(N+2)}{3}$, there is an isomorphism of algebra

$$\bigoplus_{m \in \mathbb{N}} H^0(X, S^m \mathcal{O}_X(m)) \cong \bigoplus_{m \in \mathbb{N}} S^m(H^0(X, \mathcal{I}_X(2))).$$
where $\mathcal{I}_X$ is the ideal sheaf of the smooth complete intersection $X$. We improved their result by showing that the statement holds more generally in codimension $c < \frac{N}{2}$ (via a completely different method): see Theorem 0.0.11 below or Theorem 4.2.1.

In [Deb05], the author studied cohomology of twisted symmetric powers of cotangent bundles of complete intersections in abelian varieties. Most notably, he showed that a sufficiently general and sufficiently ample complete intersection in an abelian variety, with codimension at least as large as its dimension, has ample cotangent bundle. By analogy, he suggested that the same statement should hold for projective spaces: for a bit more than a decade, this was known as Debarre’s conjecture on ampleness. This conjecture attracted a lot of attention, and the strategy to tackle it was initiated by Brotbek in [Bro15]. In loc.cit, the author provides an explicit description of cohomology of (negatively) twisted symmetric powers of cotangent bundles of smooth complete intersections (see [Bro15][Theorem A], and Section 2.5 in loc.cit for details). The description provided is rather technical. It is probably worth specifying that the descriptions we provide in Theorem 3.2.3 or Theorem 3.4.1 are different, and, at least it seems to us, more concise and less technical. Using this description, Brotbek was able to prove particular cases of Debarre’s conjecture (see [Bro15][Theorem D]). A bit later, Brotbek’s strategy was further developed by Brotbek himself and Darondeau in [BD17], and they proved in loc.cit Debarre’s conjecture in full generality. Around the same period, with a similar but more technical approach, Xie provided a different proof of Debarre’s conjecture in [Xie18].

After this introductory overview on the literature on symmetric powers of cotangent bundles, let us now detail the content of the present paper.

In Section 1, we recall the study of twisted symmetric powers of cotangent bundles (and tangent bundles) of projective spaces, which allows to compute their cohomology. Whereas this is very classical, we provide here a very explicit description: it may very well be possible that our approach has already been described elsewhere, but if so, we were not aware of it.

The starting point is the following generalization of the so-called Euler exact sequence (see Section 1.1):

**Theorem 0.0.1 (Generalized Euler exact sequence).** Let $N \in \mathbb{N}_{\geq 1}$. For any $m \in \mathbb{N}_{\geq 1}$, and any $n \in \mathbb{Z}$, one has the following short exact sequence:

$$0 \longrightarrow S^m \Omega_{\mathbb{P}^N} (m+n) \longrightarrow \mathbb{C}[Y]_m \otimes \mathcal{O}_{\mathbb{P}^N} (n) \overset{\delta}{\longrightarrow} \mathbb{C}[Y]_{m-1} \otimes \mathcal{O}_{\mathbb{P}^N} (n+1) \longrightarrow 0,$$

where $\delta := \sum_{i=0}^N X_i \frac{\partial}{\partial Y_i}$.

This result allows us to interpret local twisted symmetric differentials of the projective space $\mathbb{P}^N$ (i.e. local sections of twisted symmetric powers of $\Omega_{\mathbb{P}^N}$) as solutions in $\mathbb{C}(X)[Y]$ of the partial differential equation

$$\delta := \sum_{i=0}^N X_i \frac{\partial}{\partial Y_i},$$

where $X := (X_0, \ldots, X_N)$ and $Y := (Y_0, \ldots, Y_N)$. This is the point of view that we are going to adopt throughout the paper.

The geometric interpretation is quite straightforward once one interprets the projectivization\(^1\) tangent bundle $\mathbb{P}(TP^N)$ as the flag variety $\text{Flag}_{(1,2)} \mathbb{C}^{N+1}$ of lines included in planes in $\mathbb{C}^{N+1}$: see Section 1.2.

\(^1\)Throughout the paper, the projectivization $\mathbb{P}(E)$ of a vector bundle $E$ on a variety means the projectivization of lines.
The generalized Euler exact sequence allows to compute the cohomology of $\text{Ker} \, \delta$ (or equivalently, twisted symmetric powers of $\Omega_{\mathbb{P}^n}$). The key fact in the computations (at least for the $H^0$ and the $H^1$) is that the partial differential equation $\delta$ is equivariant with respect to the natural action of $\text{GL}_{N+1} \mathbb{C}$ on $\mathbb{C}[Y, X]$ (which allows to use representation theory). Using rather the geometric interpretation $\mathbb{P}(\mathcal{T}\mathbb{P}^n) \cong \text{Flag}(1, 2) \mathbb{C}^{N+1}$, the cohomology of $\text{Ker} \, \delta$ follows immediately from Bott’s formulas (see Appendix A). These two approaches are detailed in Section 1.3.

By dualizing the generalized Euler exact sequence, i.e. by applying the functor $\text{Hom}(\cdot, \mathcal{O}_{\mathbb{P}^n})$, one also gets a description of twisted symmetric powers of the tangent bundle:

$$
\begin{array}{ccc}
0 & \longrightarrow & \mathbb{C}[Y]_{m-1} \otimes \mathcal{O}_{\mathbb{P}^n}(n-1) \\
& \overset{\delta^*}{\longleftarrow} & \mathbb{C}[Y]_{m} \otimes \mathcal{O}_{\mathbb{P}^n}(n) \\
& \overset{\longrightarrow}{\longrightarrow} & S^m \mathcal{T}\mathbb{P}^n(n-m) \\
& \longrightarrow & 0.
\end{array}
$$

Here, $m$ is a positive natural number, $n$ an integer, and one uses the natural isomorphism $\text{Hom}(\mathcal{O}_{\mathbb{P}^n}(i), \mathcal{O}_{\mathbb{P}^n}) \cong \mathcal{O}_{\mathbb{P}^n}(-i)$. It turns out that, doing so, the dual map $\delta^*$ is quite unaesthetic. In order to obtain a convenient description, one has to renormalize the vector space $\mathbb{C}[Y]$ by the following map:

$$
u: \left( \frac{\mathbb{C}[Y]}{Y^\alpha} \rightarrow \frac{\mathbb{C}[Y]}{Y^\alpha} \right).$$

After this renormalization, the modified map (denoted $\delta_*$) becomes simply the multiplication by the quadratic polynomial

$$q := \sum_{i=0}^{N} X_i Y_i.$$

We refer to Section 1.4 for more details. We chose to insist on this renormalization map in the introduction because it will be used repeatedly throughout the paper. This might seem somewhat anecdotal, but it will turn out to be crucial to obtain neat descriptions (of cohomology of twisted symmetric powers of cotangent bundles of complete intersections).

In Section 1.5, we set some natural notations, important for the whole paper.

We conclude Section 1 with Section 1.6, in which we study partial differential equations of the form

$$
\sum_{i=0}^{N} P_i \frac{\partial}{\partial Y_i},
$$

where $P_0, \ldots, P_N$ are homogeneous polynomials of same degree $d \geq 1$, sharing only the origin as common zero. More precisely, we want to understand the set of bi-homogeneous polynomials satisfying such a partial differential equation. The main result of this section is that, in order to understand this set, it is enough to:

1. understand the set of polynomials satisfying the differential equation $\delta$;
2. understand the push-forward sheaf $f_* \mathcal{O}_{\mathbb{P}^n}$, where

$$f := [P_0 : \cdots : P_N]$$

is the finite map induced by the polynomials $P_0, \ldots, P_N$.

The first item is dealt with (as an outcome of the computation of the cohomology of $\text{Ker} \, \delta$), and the real difficulty lies in the second item. It is known that $f_* \mathcal{O}_{\mathbb{P}^n}$ splits as a direct sum of line bundles (it follows from a result of Horrocks, see e.g. [Bea00] Section 1 for details), but the question of identifying the factors appearing in the decomposition seems very difficult in general. Let us state the main result of Section 1.6:
Theorem 0.0.2. The partial differential equation

\[ \sum_{i=0}^{N} P_i \frac{\partial}{\partial Y_i} \]

admits a bi-homogeneous polynomial \( A \in \mathbb{C}[Y, X]_{m,n} \) as a (non-trivial) solution if and only if \( d m \leq n \).

In Section 2, we provide two different locally free resolutions of the structural sheaf \( \mathcal{O}_{P(TX)} \) of the projectivized tangent bundle \( P(TX) \to X \) of a smooth complete intersection \( X \subset P^N \), and we describe their push-forward on the base \( X \) (more precisely, we describe the push-forward of various twists of these resolutions).

For both resolutions, we start with the case of smooth hypersurfaces (see Sections 2.1 and 2.3), and then move onto the general case of smooth complete intersections (see Sections 2.2 and 2.4). The reason for this presentation is the following. The first (resp. the second) resolution is given by a Koszul complex associated to sections of rank 1 (resp. rank 2) vector bundles on \( P(TP^N|_X) \) (resp. on \( P(TP^N) \)). Therefore, we take the time in Sections 2.1 and 2.3 to do properly the construction for a single section, which is easy in the first case, and less straightforward in the second case.

The push-forward on the base of the previous twisted resolutions provide locally free resolutions of twisted symmetric powers of the cotangent bundle \( \Omega_X \) of the complete intersection \( X \). It is thus important to obtain an explicit description. For the first resolution provided, it turns out to be easy (see also Theorem 2.2.3):

Theorem 0.0.3. Let \( X := \{P_1 = \cdots = P_c = 0\} \subset P^N \) be a smooth complete intersection of codimension \( c \) and multi-degree \( d = (d_1, \ldots, d_c) \). Denote by \( \alpha(P_i) \) the multiplication map by the bi-homogeneous polynomial \( \frac{1}{d_i} (d_P) X(Y) \in \mathbb{C}[Y, X] \). The Koszul complex

\[ \mathcal{K}(\alpha(P_1), \ldots, \alpha(P_c)) \]

induced by the multiplication maps \( (\alpha(P_i))_{1 \leq i \leq c} \) on \( \ker \delta|_X \) provides a locally free bi-graded\(^2\) resolution of

\[ \bigoplus_{m \geq c, n \in \mathbb{Z}} S^m \Omega_X(m + n). \]

However, for the second resolution, the interpretation of the push-forward twisted resolution is much more complicated, and we choose to provide an explicit description only in codimension 1 and 2 (see also Theorem 2.3.7 and Theorem 2.4.4):

Theorem 0.0.4. Let \( H := \{P = 0\} \subset P^N \) be a smooth hypersurface of degree \( d \geq 1 \). Denote by \( \beta(P) \) the following map

\[ \beta(P)(\cdot) := \cdot \times P - \alpha(P) \circ \delta(\cdot), \]

where one recalls that \( \alpha(P) \) is the multiplication map by \( \frac{1}{d} (d_P) X(Y) \).

The complex

\[ 0 \to \ker \delta[-1, -2d + 1] \xrightarrow{\alpha(P)} \ker \delta^2[0, -d] \xrightarrow{\beta(P)} \ker \delta \]

provides a locally free bi-graded resolution of

\[ \bigoplus_{m \geq 1, n \in \mathbb{Z}} S^m \Omega_H(m + n). \]

\(^2\)The bi-graduations on \( \bigoplus_{m \geq c, n \in \mathbb{Z}} S^m \Omega_X(m + n) \) and on \( \ker \delta \) are the natural ones.

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Theorem 0.0.5. Let $X := \{P_1 = P_2 = 0\} \subset \mathbb{P}^N$ be a smooth complete intersection of codimension 2 and multi-degree $d = (d_1, d_2)$. The complex

$$
\begin{array}{cccccc}
0 & \to & \text{Ker} \delta[-2, -2|d| + 2] & \to & (f_{1i})_{1 \leq i \leq 2} \\
\downarrow & & \downarrow & & \\
& \text{Ker} \delta^2[-1, -2|d| + d_2 + 1] & \oplus & \text{Ker} \delta^2[-1, -2|d| + d_1 + 1] & \to & (f_{ai})_{1 \leq i \leq 4} \\
& \downarrow & \downarrow & \downarrow & & \\
& (\oplus \text{Ker} \delta[-1, -2d_i + 1]) & \oplus & \text{Ker} \delta^3[0, -|d|] & \oplus & \text{Ker}[-1, -|d| + 1] \\
& \downarrow & & \downarrow & & \\
& (f_{ai})_{1 \leq i \leq 2} & \to & \text{Ker} \delta^2[0, -d_1] & \oplus & \text{Ker} \delta^2[0, -d_2] & \to & f_{41} & \to \text{Ker} \delta,
\end{array}
$$

where

$$
\begin{align*}
f_{1i} &= \alpha(P_i), 1 \leq i \leq 2; \\
f_{21}(A, B) &= \beta(P_2)(B) \text{ and } f_{22}(A, B) = \beta(P_1)(A); \\
f_{23}(A, B) &= \alpha(P_2)(A) - \alpha(P_1)(B); \\
f_{24}(A, B) &= \beta(P_2)(A) + \beta(P_1)(B); \\
f_{31}(A, B, C, D) &= \alpha(P_1)(A) + \frac{1}{2}(\beta(P_2)(C) + P_2 C + \alpha(P_2)(D)); \\
f_{32}(A, B, C, D) &= \alpha(P_2)(B) - \frac{1}{2}(\beta(P_1)(C) + P_1 C + \alpha(P_1)(D)); \\
f_{41}(A, B) &= \beta(P_1)(A) + \beta(P_2)(B).
\end{align*}
$$

provides a locally free bi-graded resolution of

$$
\bigoplus_{m \geq 2, n \in \mathbb{Z}} S^m \Omega_X(m + n).
$$

At this point, one may wonder why bother using such a complicated expression, whereas the complex given in Theorem 0.0.3 (in the codimension 2 case) seems much easier to apprehend. We must admit that we were not able yet to take advantage of it (it is not used in the applications, only the resolution upstairs from which this resolution comes from is used). However, we do believe that this resolution is better than the one in Theorem 0.0.3. We discuss this further below, but in a few words the reason is roughly the following. In the complex of Theorem 0.0.3, both the objects and the arrows depend on the complete intersection, whereas in the complex of Theorem 0.0.4 or Theorem 0.0.5, the dependency on $X$ only lies in the arrows. We believe that this should somehow make the study more tractable.

In Section 3, we use the resolutions provided in Theorem 0.0.3, 0.0.4 and 0.0.5 to exhibit complexes computing the cohomology of (negatively) twisted symmetric powers of cotangent bundles of smooth complete intersections: see respectively Section 3.2, 3.3 and 3.4.

The basic observation is that suitable bi-graded parts of the above complexes have their cohomology supported in maximal degree. Therefore, by a simple exercise in cohomological algebra, one can compute the cohomology by applying the functor $H^{top}$ to the (suitable graded parts of the) complexes. However, for computational purposes, it is not practical to work with $H^{top}$. Therefore, we use Serre duality to provide a reformulation of the complex. In order to obtain a neat description, it is crucial to use the renormalization map defined earlier in the introduction: see Section 3.1 for details (the ideas are the same in the subsequent Sections 3.2, 3.3 and 3.4).
Using the resolution given in Theorem 0.0.3, we obtain the following (see Theorem 3.2.3):

**Theorem 0.0.6.** Let \(X := \{P_1 = \cdots = P_c = 0\} \subset \mathbb{P}^N\) be a smooth complete intersection of codimension \(c\) and multi-degree \(d = (d_1, \ldots, d_c)\). Denote by \(\alpha^*(P_i)\) the following partial differential equation:

\[
\alpha^*(P_i)(\cdot) := \frac{1}{d_i} \left( \sum_{i=0}^{N} \frac{\partial P_i}{\partial X_i} \frac{\partial}{\partial Y_i}(\cdot) \right),
\]

which induces a map of bi-graded algebra \(\mathbb{C}[Y,X]_{(P_1,\ldots,P_c,q)}[-1,-d_i+1] \rightarrow \mathbb{C}[Y,X]_{(P_1,\ldots,P_c,q)}\).

Then, the cohomology of \(\bigoplus_{m \geq c,n>1} S^m \Omega_X(m-n)\) can be computed via the Koszul complex on \(\mathbb{C}[Y,X]_{(P_1,\ldots,P_c,q)}\):

\[
K(\alpha^*(P_1), \ldots, \alpha^*(P_c))[-c, -(N+1) + 2|d| - c].
\]

Namely, the \(i\)th cohomology group of one graded component of \(\bigoplus_{m \geq c,n>1} S^m \Omega_X(m-n)\) is isomorphic to the \((N - c - i)\)th cohomology group of the corresponding graded part of the Koszul complex.

Using the resolutions given in Theorem 0.0.4 and 0.0.5, we obtain respectively (see Theorem 3.3.3 and 3.4.1):

**Theorem 0.0.7.** Let \(H := \{P = 0\} \subset \mathbb{P}^N\) be a smooth hypersurface of degree \(d \geq 1\). Denote by \(\beta^*(P)\) the following map

\[
\beta^*(P)(\cdot) := \cdot \times P - q \times \alpha^*(P)(\cdot),
\]

where one recalls that \(\alpha^*(P)\) is the partial differential equation \(\frac{1}{d} \left( \sum_{i=0}^{N} \frac{\partial P_i}{\partial X_i} \frac{\partial}{\partial Y_i}(\cdot) \right)\).

Then, the cohomology of \(\bigoplus_{m \geq 1,n>1} S^m \Omega_H(m-n)\) can be computed via the graded complex

\[
S_{(q)}[0,-(N+1)] \xrightarrow{\beta^*(P)} S_{(q^2)}[0,-(N+1)+d] \xrightarrow{\alpha^*(P)} S_{(q^3)}[-1,-(N+1)+2d-1],
\]

where \(S := \mathbb{C}[Y,X]\).

Namely, the \(i\)th cohomology group of one graded component of \(\bigoplus_{m \geq 1,n>1} S^m \Omega_H(m-n)\) is isomorphic to the \((N - i)\)th cohomology group of the corresponding graded part of the graded complex.

**Theorem 0.0.8.** Let \(X := \{P_1 = P_2 = 0\} \subset \mathbb{P}^N\) be a smooth complete intersection of codimension 2 and multi-degree \(d = (d_1,d_2)\). The cohomology of

\[
\bigoplus_{m \geq 2,n>1} S^m \Omega_X(m-n)
\]
can be computed via the following bi-graded complex

\[
\begin{array}{c}
\left( \frac{S}{(q^d)} \right)_{i,j} \xrightarrow{(g_{11})_{i,j} \leq i \leq j \leq 2} \frac{S}{(q^d)}[0, d_1] \oplus \frac{S}{(q^d)}[0, d_2] \xrightarrow{(g_{21})_{i,j} \leq i \leq j \leq 4} \\
\xrightarrow{(g_{31})_{i,j} \leq i \leq j \leq 2} \frac{S}{(q^d)}[-1, 2d_i - 1] \oplus \frac{S}{(q^d)}[0, -|d|] \oplus \frac{S}{(q^d)}[-1, |d| - 1] \\
\xrightarrow{(g_{41})_{i,j} \leq i \leq j \leq 2} \frac{S}{(q^d)}[-1, 2|d| - d_2 - 1] \oplus \frac{S}{(q^d)}[-1, 2|d| + d_1 - 1] \\
\xrightarrow{g_i} \frac{S}{(q^d)}[-2, 2|d| - 2] \oplus \frac{S}{(q^d)}[-2, 2|d| - 2]
\end{array}
\]

where \( S := \mathbb{C}[Y, X] \), and where one has:

\[
\begin{align*}
  g_{11} &= \beta^*(P_1) \text{ and } g_{12} = \beta^*(P_2) \\
  g_{21}(A, B) &= \alpha^*(P_1)(A) \text{ and } g_{22}(A, B) = \alpha^*(P_2)(B) \\
  g_{31}(A, B, C) &= \beta^*(P_1)(B) + \alpha^*(P_2)(C) + \beta^*(P_2)(D) \\
  g_{32}(A, B, C, D) &= \beta^*(P_1)(B) + \alpha^*(P_2)(C) + \beta^*(P_2)(D) \\
  g_{41}(A, B) &= \alpha^*(P_1)(A) + \alpha^*(P_2)(B).
\end{align*}
\]

Namely, the \( i \)th cohomology group of one graded component of \( \bigoplus_{m \geq 2, n \geq 1} S^m \Omega_X(m - n) \) is isomorphic to the \( (N - i) \)th cohomology group of the corresponding graded part of the above complex.

The main issue with the complex given in Theorem 0.0.6 is that we have to deal with quotients. Consider the simplest case of a hypersurface: in order to study the cohomology of the complex, one has to understand the kernel of the following map

\[
\alpha^*(P) : \frac{S}{(P, q)}[-1, -d] \to \frac{S}{(P, q)}.
\]

Said otherwise, one has to study the solutions of the partial differential equation

\[
\sum_{i=0}^N \frac{\partial^i P}{\partial x_i} \frac{\partial^i}{\partial t^i} \text{ modulo the ideal } (P, q).
\]

We managed to do so when the polynomial \( P \) has a particular shape: see Section 4.3. The rough idea is that, in this case, one can reduce the problem to a situation where there is no quotient, so that one can use Theorem 0.0.2.

Whereas there are still quotients in the complexes in Theorems 0.0.7 and 0.0.8, the upshot is that they are of relatively mild form, namely of the following form

\[
\frac{S}{(q^i)}
\]

where \( i \in \mathbb{N}_{\geq 1} \). Note that the quadratic form \( q = \sum_{i=0}^N X_i Y_i \) is left unchanged under the following action of \( \text{GL}_{N+1}(\mathbb{C}) \)

\[
A \in \text{GL}_{N+1}(\mathbb{C}), \quad A \cdot (Y, X) := ((A^{-1})^T Y, AX).
\]

Therefore, representation theory of the general linear group \( \text{GL}_{N+1}(\mathbb{C}) \) applies. In theory, this should allow to get rid of the quotient via a section of the quotient map

\[
S \to \frac{S}{(q^i)}.
\]

The real difficulty is to understand how the arrows in the complex behaves...
with respect to these sections: this is the object of a current work.

Finally, in Section 4, we provide several applications. In Section 4.1, we use the Koszul complex on Flag_{1,2} C^{N+1} \simeq P(TP^N), whose construction is detailed in Section 2.4, in order to recover all known vanishing theorems on twisted symmetric powers of cotangent bundles of complete intersection (namely, [BR90][Theorem (i), (ii), (iii)]). Then, we prove non-vanishing theorems, which show in particular that the above vanishing theorems are optimal: these non-vanishing statements are, to our knowledge, new. In particular, we prove the following (see also Theorem 4.1.2):

Theorem 0.0.9. Let n \in \mathbb{N}, d = (d_1, \ldots, d_c), d := \min\{d_i \mid 1 \leq i \leq c\}, and suppose furthermore that d \geq 2. A smooth complete intersection X \subset P^N of codimension c < \frac{N+1}{2}, and multi-degree d satisfies the following non-vanishing statements:

- for d \geq 3, H^0(X, S^{d-1}\Omega_X(2d-3)) \neq (0);
- for d = 2, H^0(X, S^2\Omega_X(2)) \neq (0).

The combination of some of these vanishing and non-vanishing theorems allows us to prove the following theorem (see also Theorem 4.1.3):

Theorem 0.0.10. Let d \in \mathbb{N}_{\geq 2} be a natural number, and let X \subset P^N be a non-degenerate smooth complete intersection of codimension c < \frac{N}{2}. The complete intersection X satisfies

\[ \bigoplus_{m \geq \max(d-2,1)} H^0(X, S^m\Omega_X(m + \max(d - 3, 0))) = (0) \]

if and only if X is not included in an hypersurface of degree 2 \leq i \leq d.

This result can be seen as a generalization of results obtained in [BDO08][Theorems B and D]. In view of Hartshorne’s conjecture on smooth projective varieties that are complete intersections (see [Laz04][Conjecture 3.2.8], and Section 4.1.1), a natural problem would be to give a proof of Theorem 0.0.10 that works for any smooth subvariety of codimension c < \frac{N}{2}.

In Section 4.2, we study the algebra

\[ \bigoplus_{m \in \mathbb{N}} H^0(X, S^m\Omega_X(m)) \]

of a smooth complete intersection X \subset P^N of codimension c < \frac{N}{2}, using the explicit resolution of O_{P(TX)} detailed in Section 2.4. We proved the following (see also Theorem 4.2.1):

Theorem 0.0.11. Let

\[ X := \{q_1 = 0\} \cap \cdots \cap \{q_k = 0\} \cap \{P_{k+1} = 0\} \cap \cdots \cap \{P_c = 0\} \subset P^N \]

be a smooth complete intersection of codimension c < \frac{N}{2}, where q_1, \ldots, q_k are homogeneous quadratic polynomials, and where P_{k+1}, \ldots, P_c are homogeneous of degree > 2.

There is a natural graded isomorphism of \mathbb{C}-algebras

\[ \bigoplus_{m \in \mathbb{N}} H^0(X, S^m\Omega_X(m)) \simeq \mathbb{C}[q_1, \ldots, q_k] \subset \mathbb{C}[X]. \]

\[ \text{This means that the complete intersection is not included in an hyperplane.} \]
This theorem generalizes a result in [DOL19]: in loc.cit, under the more restrictive hypothesis on the codimension $c < \frac{N+2}{3}$, the authors provided a completely different proof of the same statement. Let us note that in the case where $X = \{q_1 = 0\} \cap \{q_2 = 0\} \subset \mathbb{P}^4$ (which is not covered by the above theorem), we showed that the conclusion of Theorem 0.0.11 still holds: see Remark 4.2.2. This provides an alternative proof of the fact that the tangent bundle of $X$ is not big, which was proved independently in [HLS22] and [Mal21].

In Section 4.3, we use the complex given in Theorem 0.0.6 (or the one given in Theorem 3.1.6) in order to prove a vanishing theorem for cohomology of negatively twisted symmetric powers of cotangent bundles of special kinds of hypersurfaces (see also Theorem 4.3.1):

**Theorem 0.0.12.** Let $H$ be a smooth hypersurface of degree $d$, whose defining equation $P$ is of the following form

$$P := X_N^d - F(X_0, \ldots, X_{N-1}),$$

where $F$ is a smooth\(^4\) polynomial, homogeneous of degree $d$. Let $m \in \mathbb{N}$, $n \in \mathbb{Z}$, and suppose that $d \geq 3$. As soon as $m(d-3) > n$, one has the vanishing:

$$H^{N-1}(H, S^m \Omega_H(d-n-(N+1))) = (0).$$

In particular, as soon as $d \geq 4$, this shows that $\Omega_H$ is uniformly q-ample (with uniform bound $\lambda = \frac{1}{d-3}$ if $d \geq N+1$), and hence so is a general hypersurface of degree $d \geq 4$.

For the notion of (uniform) intermediate ampleness, we refer to Section 4.3. A stronger version of this vanishing theorem was actually proved recently in [HLS22], in a different context and via a different method. In loc.cit, the authors prove that for any smooth hypersurface $H \subset \mathbb{P}^N, N \geq 3$, of degree $d \geq 3$, and for any $m \in \mathbb{N}, n \in \mathbb{Z}$ satisfying the broad inequality

$$m(d-3) \geq n,$$

one has the vanishing

$$H^0(H, S^mTX(n)) = (0).$$

(One recovers our statement via Serre duality).

In Section 4.4, we use the explicit description of cohomology detailed in Section 3. Once implemented on computers (see Appendix B), it allows to compute completely the cohomology for small twists and small symmetric powers. In particular, we showed from a purely computational point of view the existence of a (unique) global section of $S^6\Omega_H(8)$ on a quartic Fermat hypersurface in $\mathbb{P}^4$. From a theoretical point of view, this fact is well-known as such a section defines the exceptional divisor of bi-tangent lines in $\mathbb{P}(TH)$; see Section 4.4.1 for more details.

We also provide a simple example of a family of surfaces in $\mathbb{P}^4$ along which the canonically twisted symmetric pluri-genera do not remain constant (see also Theorem 4.4.1)

**Theorem 0.0.13.** The 1-parameter family

$$X_t := \{X_0^4 + X_1^4 + \cdots + X_4^4 - tX_0^2X_1^2 = 0\} \cap \{-2X_0^4 - X_1^4 + X_3^4 + 2X_4^4 = 0\} \subset \mathbb{P}^4$$

satisfies $h^0(S^6\Omega_{X_0}(K_{X_0})) = 1$, whereas $h^0(S^6\Omega_{X_0}(K_{X_t})) = 0$ for a general $t \in \mathbb{C}$.

It was asked by Paun if such a phenomenon occurs, and it was first answered negatively in [BDO08], via a counter-example more sophisticated than the one provided here.

Finally, in Section 4.5, we discuss possible strategies to provide explicit examples of complete intersection surfaces in $\mathbb{P}^4$ whose cotangent bundle is ample. There is nothing

\(^4\)By definition, this means that the associated hypersurface (in $\mathbb{P}^{N-1}$ here) is smooth.
particularly new in this section, but we felt that it was worth mentioning, particularly because this is one of the questions that originally motivated this paper.

**Conventions.** The terminology *Bott’s formulas* is written at many places in the text. It encodes a general statement, that is often too general for the purposes at hand. Therefore, from time to time, it happens that what we are referring to as *Bott’s formulas* is not originally due to Bott. However, we found it convenient to englobe in one expression a general statement that can be declined in various ways. For that reason, we wrote an appendix on these formulas (see Appendix A). This appendix is also the occasion to recall some notations regarding Schur functors: we therefore invite the unfamiliar reader to briefly read it.

Throughout the paper, the projective space $\mathbb{P}^N$ is fixed, and we suppose that $N \geq 2$.

**Acknowledgements.** I would like to particularly thank Andreas Höring, as the starting point of this work was the following question he asked me: on a surface $S \subset \mathbb{P}^3$ (say, a Fermat surface for simplicity), can we construct (via "Wronskian methods") global sections of $S^m\Omega_S(n)$ with ratio $\frac{n}{m} < \frac{1}{2}$? As I found it hopeless to try constructing them blind, I sought for a way to detect where one could hope to find such global sections. This is what eventually lead to this paper, as along the way many other questions popped off. The original question is still to be answered, but at least we have a better idea where to find these sections!

I also would like to thank Lionel Darondeau and Damian Brokhe for initiating me to the general subject of cohomology of symmetric powers of cotangent bundles of complete intersections. During my thesis, they offered me to work with them in order to construct explicitly a surface in $\mathbb{P}^4$ with ample cotangent bundle, following the description and strategy detailed in their work. I implemented on computer the strategy, but it quickly became evident that it would require too much computations to be able to conclude. This is this very problem that gave me the motivation to write explicitly the (complicated) complex detailed in Theorem 0.0.5 (even if I was not able to get anything out of it!).

Finally, I would like to thank Stéphane Drul, who indicated me the following result: for a finite map $f : \mathbb{P}^N \to \mathbb{P}^N$, the push-forward vector bundle $f_*\mathcal{O}_{\mathbb{P}^N}$ splits.

1. **Cohomology of twisted symmetric powers of cotangent bundles of projective spaces.**

1.1. **Generalized Euler exact sequence.** Recall that on the projective space $\mathbb{P}^N$, there exists canonical (negatively twisted) vector fields, usually called *Euler vector fields*, defined as follows. Denote $p : \mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{P}^N$ the canonical projection onto the projective space of dimension $N$, and denote $(e_i)_{0 \leq i \leq N}$ the canonical basis of $\mathbb{C}^{N+1}$.

Define then for $0 \leq i \leq N$ $
\gamma_i : \begin{pmatrix} \mathbb{P}^N \\
[x] \end{pmatrix} \to TP^N(-1) \begin{pmatrix} [x], (d_p)_x(e_i) \end{pmatrix}.
$

The negative twist (necessary to make the maps $\gamma_i$’s well-defined) comes from the fact that for any $\lambda \in \mathbb{C}^*$, one has the equality

$$(d_p)_x = \lambda (d_p)_x.$$

Using the Euler vector fields, one constructs the usual *Euler exact sequence*

\begin{equation}
0 \longrightarrow \Omega_{\mathbb{P}^N}(1) \longrightarrow \bigoplus_{i=0}^{N+1} \mathcal{O}_{\mathbb{P}^N} \longrightarrow \mathcal{O}_{\mathbb{P}^N}(1) \longrightarrow 0,
\end{equation}
where the first arrow is given by
\[( [x], \omega ) \mapsto ( [x], (\omega(\gamma_0), \ldots, \omega(\gamma_N)) ) \]
and the second arrow by
\[( [x], (v_0, \ldots, v_N) ) \mapsto ( [x], \sum_{i=0}^{N} X_i(x)v_i ) , \]
where \( X_0, \ldots, X_N \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \) are the usual homogeneous coordinates on the projective space \( \mathbb{P}^N \).

In order to generalize the previous exact sequence to higher symmetric powers of the cotangent bundle of \( \mathbb{P}^N \), note that the exact sequence (1) can be rewritten as follows:
\[
0 \longrightarrow \Omega_{\mathbb{P}^N}(1) \longrightarrow \mathbb{C}[Y]_1 \otimes \mathcal{O}_{\mathbb{P}^N} \overset{\delta}{\longrightarrow} \mathcal{O}_{\mathbb{P}^N}(1) \longrightarrow 0 ,
\]
where \( \delta \) is the (differential) operator
\[
\delta := \sum_{i=0}^{N} X_i \frac{\partial}{\partial Y_i} .
\]
Here, the sections \( X_0, \ldots, X_N \) are seen as global sections of the line bundle \( \mathcal{O}_{\mathbb{P}^N}(1) \) on the base space, and \( Y := (Y_0, \ldots, Y_N) \) are seen as the variables of the graded vector space of polynomials in \( N + 1 \) variables
\[
\mathbb{C}[Y] = \bigoplus_{i=0}^{\infty} \mathbb{C}[Y]_i ,
\]
where \( \mathbb{C}[Y]_i \) is the set of homogeneous polynomials of degree \( i \in \mathbb{N} \).

We are then naturally lead to consider, for any \( m \in \mathbb{N}_{\geq 1} \), the exact sequence of vector bundles
\[
0 \longrightarrow \text{Ker}(\delta) \longrightarrow \mathbb{C}[Y]_m \otimes \mathcal{O}_{\mathbb{P}^N} \overset{\delta}{\longrightarrow} \mathbb{C}[Y]_{m-1} \otimes \mathcal{O}_{\mathbb{P}^N}(1) .
\]
The complex (2) is actually exact on the right, i.e. the map \( \delta \) is surjective. We record it in a lemma:

**Lemma 1.1.1.** In the above complex (2), the map \( \delta \) is surjective.

**Proof.** Let \( x = [x_0 : \cdots : x_N] \in \mathbb{P}^N \), and suppose without loss of generality that \( x_0 \neq 0 \). Consider the local sections around \( x \)
\[
\begin{cases}
L_0 := \frac{Y_i}{X_0} , \\
L_i := Y_i - \frac{X_i}{X_0} Y_0 & \text{for } 1 \leq i \leq N .
\end{cases}
\]
Note that \( \delta(L_0) = 1 \), whereas \( \delta(L_i) = 0 \) for \( 1 \leq i \leq N \). By the Leibnitz rule, one has therefore the following equality for any \( \alpha = (\alpha_0, \ldots, \alpha_N) \in \mathbb{N}^{N+1} \):
\[
\delta(L_0^{\alpha_0+1}L_1^{\alpha_1} \cdots L_N^{\alpha_N}) = (\alpha_0 + 1)L_0^{\alpha_0} \cdots L_N^{\alpha_N} .
\]
The surjectivity of \( \delta \) now follows from the equality
\[
\mathcal{O}_{\mathbb{P}^N,x}[Y_0, \ldots, Y_N] = \mathcal{O}_{\mathbb{P}^N,x}[L_0, \ldots, L_N] .
\]

The locally free sheaf \( \text{Ker}(\delta) \) in the the complex (2) can naturally be identified with the \( m \)th symmetric power of \( \Omega_{\mathbb{P}^N}(1) \):

**Lemma 1.1.2.** There is a natural isomorphism
\[
\text{Ker}(\delta) \simeq S^m \Omega_{\mathbb{P}^N}(m) .
\]
Proof. Consider the natural injective map

\[
i: \left( S^m \Omega_{P^N}(m) \rightarrow \mathbb{C}[Y]_m \otimes \mathcal{O}_{P^N}\right),
\]

where one recalls that \( \gamma_i \) denotes the \( i \)th Euler vector field, with \( 0 \leq i \leq N \). It is a straightforward computation to see that the image actually lies in \( \text{Ker}(\delta) \):

\[
\delta\left( \sum_{i_1, \ldots, i_m} \omega(\gamma_{i_1} \cdots \gamma_{i_m})Y_{i_1} \cdots Y_{i_m} \right)
= \sum_{k=1}^{m} \sum_{i_k-1, s_{k+1}}^{N} \left( \sum_{i=0}^{N} X_i \omega(\gamma_{i_1} \cdots \gamma_{i_k-1} \gamma_{i_{k+1}} \cdots \gamma_{i_m-1} \cdot \gamma_i)Y_{i_1} \cdots Y_{i_{k-1}} Y_{i_{k+1}} \cdots Y_{i_m-1} \right)
= 0.
\]

The fact that it is identically zero follows from the observation that, for \( i_1 \cdots i_{m-1} \) fixed, \( \omega(\gamma_{i_1} \cdots \gamma_{i_m}) \) is a (local) section of \( \Omega_{P^N}(1) \).

Reciprocally, let\(^5\)

\[
P = \sum_{|\alpha|=m} P_\alpha Y^\alpha
\]

be a local section of \( \text{Ker}(\delta) \), where the \( P_\alpha \)'s are local functions on \( P^N \). At every point \( x = [x_0 : \ldots : x_N] \), the fiber \( (S^m TP^N(-m))_x \) has the following description

\[
(S^m TP^N(-m))_x = \bigoplus_{|\alpha|=m} \mathbb{C} \cdot \gamma^\alpha,
\]

where by definition \( \gamma^\alpha := \gamma_1^{\alpha_1} \cdots \gamma_N^{\alpha_N} \in H^0(P^N, S^m TP^N(-m)) \). Now, for any \( x \) where \( P \) is defined and any \( \alpha \) of length \( |\alpha| = m \), define

\[
\omega_x(\gamma^\alpha) := \alpha! P_\alpha(x).
\]

As \( P \) is in \( \text{Ker}(\delta) \), this descends to the quotient defining \( (S^m TP^N(-m))_x \), so that \( \omega \) defines a local section of \( S^m \Omega_{P^N}(m) \).

One immediately checks that two maps described above are inverse to each other, so that the lemma is proved. \( \square \)

Putting together Lemma 1.1.1 and Lemma 1.1.2, we have thus proved:

**Theorem 1.1.3** (Generalized Euler exact sequence). For any \( m \in \mathbb{N}_{\geq 1} \), the \( m \)th symmetric power of \( \Omega_{P^N}(1) \) fits into the short exact sequence:

\[
\begin{array}{cccccc}
0 & \rightarrow & S^m \Omega_{P^N}(m) & \rightarrow & \mathbb{C}[Y]_m \otimes \mathcal{O}_{P^N} & \rightarrow & \mathbb{C}[Y]_{m-1} \otimes \mathcal{O}_{P^N}(1) & \rightarrow & 0,
\end{array}
\]

where \( \delta = \sum_{i=0}^{N} X_i \partial_{Y_i} \).

\(^5\)We adopt the usual multi-indexes notation: \( Y^\alpha := Y_0^{\alpha_0} \cdots Y_N^{\alpha_N} \).
1.2. Geometric interpretation. The generalized Euler exact sequence has a simple geometric interpretation. The projectivization $P(TP^N)$ of the tangent space $TP^N$ is naturally isomorphic to the flag variety $\text{Flag}_{(1,2)} \mathbb{C}^{N+1}$ via the following map:

$$
j : \left( \begin{array}{c}
P(TP^N) \\
([x], v)
\end{array} \right) \rightarrow \left( \begin{array}{c}
\text{Flag}_{(1,2)} \mathbb{C}^{N+1} \\
(C \cdot x \subset C \cdot x \oplus C \cdot v)
\end{array} \right).
$$

This is indeed well-defined, as the tangent space at $[x]$ may be described as follows

$$T_{[x]} P^N \simeq \frac{\mathbb{C}^{N+1}}{C \cdot x}$$

(indeed, $(d p)_x : \mathbb{C}^{N+1} \rightarrow T_{[x]} P^N$ is surjective, with kernel $C \cdot x$), and it is clear that $j$ is an isomorphism.

Recall that there exists two tautological vector bundles on $\text{Flag}_{(1,2)} \mathbb{C}^{N+1}$, defined as sub-vector bundles of the trivial bundle $\text{Flag}_{(1,2)} \mathbb{C}^{N+1} \times \mathbb{C}^{N+1}$ as follows:

- $U_1$ such that the fiber above $\xi := (C \cdot x \subset C \cdot x \oplus C \cdot v)$ is the line spanned by $x$;
- $U_2$ such that the fiber above $\xi := (C \cdot x \subset C \cdot x \oplus C \cdot v)$ is the plane spanned by $x$ and $v$.

It gives rise to the short exact sequence of vector bundles

$$0 \rightarrow U_1 \rightarrow U_2 \rightarrow U_2/U_1 \rightarrow 0,$$

where the two extremities are line bundles on $\text{Flag}_{(1,2)} \mathbb{C}^{N+1}$. Denote, for any $n, m \in \mathbb{Z}$,

$$L_{m,n} := \left( \frac{U_2}{U_1} \right)^m \otimes \left( \frac{U_1'}{U_1} \right)^n.$$

We have the following proposition:

**Proposition 1.2.1.** The pull-back line bundle $j^* L_{m,n}$ on $P(TP^N)$ is equal to

$$O_{P(TP^N)}(m) \otimes \pi^* O_{P^N}(m+n),$$

where $\pi : \text{Flag}_{(1,2)} \mathbb{C}^{N+1} \rightarrow P^N$ is the natural projection onto the space of lines in $\mathbb{C}^{N+1}$.

**Proof.** First, observe that one has tautologically the following equality of line bundles:

$$j^* U_1 = \pi^* O_{P^N}(-1).$$

Then, observe that there is a natural isomorphism

$$j^* \left( \frac{U_2}{U_1} \right) \simeq O_{P(TP^N)}(-1) \otimes \pi^* O_{P^N}(-1)$$

defined as follows:

$$\left( ([x], [v]), [\pi] \right) \mapsto \left( ([x], [v]), \sum_{i=0}^N v_i \gamma_i([x]) \right).$$

(Note that it is indeed well defined since $\sum_{i=0}^N x_i \gamma_i([x]) = 0$). The lemma follows immediately from the previous two observations. \hfill \Box

Recall Bott’s formulas (in the absolute setting, see Appendix A), that says in particular that for any $n \in \mathbb{Z}$ and any $m \in \mathbb{N}$, one has the following isomorphism:

$$H^0(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, L_{m,n}) \simeq S^{(n,m)} \mathbb{C}^{N+1}.$$

Here, $S^{(n,m)}$ denotes the Schur functor associated to the partition $(n, m)$ if $n \geq m$, and it denotes the zero functor otherwise, by convention. On the other hand, Bott’s formulas...
(in the relative setting, in a particular case) implies also that for any \( n \in \mathbb{Z} \) and any \( m \in \mathbb{N} \) one has the isomorphism
\[
H^0(\mathbb{P}(\mathbb{P}^N), \mathcal{O}_{\mathbb{P}(\mathbb{P}^N)}(m) \otimes \pi^* \mathcal{O}_{\mathbb{P}^N}(n + m)) \simeq H^0(\mathbb{P}^N, S^m \Omega_{\mathbb{P}^N}(m + n)).
\]
Therefore, one has the isomorphism for any \( n \in \mathbb{Z} \) and any \( m \in \mathbb{N} \)
\[
H^0(\mathbb{P}^N, S^m \Omega_{\mathbb{P}^N}(m + n)) \simeq S^{(n,m)} \mathbb{C}^{N+1}.
\]

In order to relate this isomorphism to the generalized Euler exact sequence, recall that the vector space \( S^{(n,m)} \mathbb{C}^{N+1} \) has the following interpretation as a sub-vector space of the set of bi-homogeneous polynomials of bi-degrees \( (m,n) \) relatively to the variables \( Y = (Y_0, \ldots, Y_N) \) and \( X = (X_0, \ldots, X_N) \)(see [Dem88][Sect.2] for more details):
\[
S^{(n,m)} \mathbb{C}^{N+1} \simeq \{ P \in \mathbb{C}[Y,X]_{m,n} \mid \forall t \in \mathbb{C}, P(Y + tX, X) = P(Y, X) \}.
\]
One then easily sees that the set of \((m,n)\) bi-homogeneous polynomials satisfying the above functional equation coincides with set of \((m,n)\) bi-homogeneous polynomials \( P \) satisfying the partial differential equation
\[
\sum_{i=0}^{N} X_i \frac{\partial}{\partial Y_i}(P) \equiv 0.
\]

We recover therefore the description given by taking the long exact sequence in cohomology associated to the generalized Euler exact sequence.

### 1.3. Cohomology of twisted symmetric powers of \( \Omega_{\mathbb{P}^N} \).

Via the geometric interpretation given in the previous paragraph, and more precisely using Proposition 1.2.1, the computation of cohomology of twisted symmetric powers of cotangent bundles of projective spaces is a straightforward application of Bott’s formulas (see Appendix A).

A different approach would be to rather use the generalized Euler exact sequence (3). We will describe the two approaches.

**Theorem 1.3.1** (Cohomology of twisted symmetric powers of \( \Omega_{\mathbb{P}^N} \)). Let \( n \in \mathbb{Z} \) be a natural number, and \( m \in \mathbb{N}_{\geq 0} \) be an integer.

- If \( n \geq m \geq 0 \), then
  \[
  \begin{cases}
  H^0(\mathbb{P}^N, S^m \Omega_{\mathbb{P}^N}(m + n)) \simeq S^{(n,m)} \mathbb{C}^{N+1} \\
  H^i(\mathbb{P}^N, S^m \Omega_{\mathbb{P}^N}(m + n)) = (0) \text{ for } i > 0.
  \end{cases}
  \]

- If \(-(N+1) < n < m\), then
  \[
  \begin{cases}
  H^1(\mathbb{P}^N, S^m \Omega_{\mathbb{P}^N}(m + n)) \simeq S^{(m-1,n+1)} \mathbb{C}^{N+1} \\
  H^i(\mathbb{P}^N, S^m \Omega_{\mathbb{P}^N}(m + n)) = (0) \text{ for } i \neq 1
  \end{cases}
  \]

- If \( n \leq -(N+1) \), then
  \[
  \begin{cases}
  H^N(\mathbb{P}^N, S^m \Omega_{\mathbb{P}^N}(m + n)) \simeq S^{\lambda(n,m)} \mathbb{C}^{N+1} \\
  H^i(\mathbb{P}^N, S^m \Omega_{\mathbb{P}^N}(m + n)) = (0) \text{ for } i \neq N
  \end{cases}
  \]
where \( \lambda(n,m) \) is the partition
\[
(m - (n + N + 1), -(n + N + 1), \ldots, -(n + N + 1)) \times (N-1)
\]

For the last item, the dimension of \( H^N(\mathbb{P}^N, S^m \Omega_{\mathbb{P}^N}(m + n)) \) is equal to
\[
\dim(S^m \mathbb{C}^{N+1} \otimes S^{-(N+1)-n} \mathbb{C}^{N+1}) - \dim(S^{m-1} \mathbb{C}^{N+1} \otimes S^{-(N+1)-n-1} \mathbb{C}^{N+1}).
\]
Proof using Bott’s formulas. Use the isomorphism $P(TP^N) \simeq \text{Flag}_{(1,2)} \mathbb{C}^{N+1}$ under which $O_{P(TP^N)}(1)$ corresponds to $L_{1-1}$ by Proposition 1.2.1. A first application of Bott’s formulas (in the relative setting) implies that for any $m \geq 0$ and any $n \in \mathbb{Z}$, one has the following equality:

$$H^*(P^N, S^m \Omega_{P^N}(m+n)) \simeq H^*(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, L_{m,n}).$$

Apply now a second time Bott’s formulas, this time in the absolute setting: see Corollary A.0.5, and notations therein. In the case at hand, consider

$$\alpha = (n,m,0,\ldots,0),$$

and distinguish several cases:

(i) If $n \geq m$, then $\alpha$ is already a partition and one is in situation (2) of Corollary A.0.5.

(ii) If $-(N+1) < n < m$ and $m-1 \geq n+1 \geq 0$, then for the permutation $\sigma = (12)$ one has:

$$\tilde{\sigma}(\alpha) = (m-1, n+1, 0, \ldots, 0),$$

so that one is in situation (2) of Corollary A.0.5

(iii) If $-(N+1) < n < m$ and $m-1 < n+1$, then one necessarily has $n = m-1$. Therefore, for the permutation $\sigma = (1,2)$, one has

$$\tilde{\sigma}(\alpha) = \alpha,$$

so that one is in situation (1) of Corollary A.0.5.

(iv) If $-(N+1) < n < m$ and $n < -1$, then for the permutation $\sigma = (1,n)$ one has

$$\tilde{\sigma}(\alpha) = \alpha$$

so that one is in situation (1) of Corollary A.0.5.

(v) If $n \leq -(N+1)$, then for the cyclic permutation $\sigma = (1,N+1,N,\ldots,2)$, one obtains:

$$\tilde{\sigma}(\alpha) = (m-1, -1, -1, \ldots, -1, n+N),$$

which is non-increasing, so that one is in the situation (2) of Corollary A.0.5.

Putting all these cases together, one obtains the description given in the statement. It remains to prove the last equality regarding the dimension of $H^N$: this is a simple application of general formulas to compute the dimension of Schur bundles: see e.g. [Ful97]. (This will also follow from the proof using the generalized Euler exact sequence).

Proof using the generalized Euler exact sequence. Let $m \in \mathbb{N}$, and $n \in \mathbb{Z}$. If $n < -1$, then the results follow immediately from the (twisted) generalized Euler exact sequence combined with the well-known cohomology of the Serre line bundles $O_{P^N}(i), i \in \mathbb{Z}$.

Suppose now that $n \geq -1$. Using the (twisted) generalized Euler exact sequence, one sees that, in order to compute the 0th and 1st cohomology groups of $S^m \Omega_{P^N}(m+n)$, one has to understand the kernel and cokernel of the following map:

$$\delta_{m,n} := \sum_{i=0}^N X_i \frac{\partial}{\partial Y_i} : \mathbb{C}[Y,X]_{m,n} \rightarrow \mathbb{C}[Y,X]_{m-1,n+1}.$$

Note that if $n = -1$, the result is obvious. Suppose accordingly that $n \geq 0$. The crucial observation is that the map $\delta$ is equivariant with respect to the following natural action of the general linear group $\text{GL}_{N+1}(\mathbb{C})$ on $\mathbb{C}[Y,X]$: $A \in \text{GL}_{N+1}(\mathbb{C}), A \cdot (Y,X) := (AY,AX)$. 

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Therefore, $\delta_{m,n}$ is a morphism of $GL_{N+1}(\mathbb{C})$-representations.

Suppose first that $n \geq m$. By Pieri’s formula (see e.g [Ful97]), one has the following decomposition of $\mathbb{C}[Y, X]_{m,n}$ into irreducible components:

$$\mathbb{C}[Y, X]_{m,n} \cong \bigoplus_{i=0}^{m} S^{(n+i,m-i)} \mathbb{C}^{N+1}.$$ 

On the other hand, for any $0 \leq i \leq m$, the vector

$$X_0^{m-n+i}Y_0^i (X_1Y_0 - X_0Y_1)^{m-i}$$

is a highest weight vector, which spans accordingly the irreducible representation isomorphic to $S^{(n+i,m-i)} \mathbb{C}^{N+1}$. One now easily sees that $\delta_{m,n}$ sends $S^{(n,m)} \mathbb{C}^{N+1}$ to zero, and realizes an isomorphism between $S^{(n+i,m-i)} \mathbb{C}^{N+1}$ and its image for any $1 \leq i \leq m$. Therefore, the kernel of $\delta_{m,n}$ is isomorphic to $S^{(m,n)} \mathbb{C}^{N+1}$, and using once again Pieri’s formula for $\mathbb{C}[Y, X]_{m-1,n+1}$, one sees that $\delta_{m,n}$ is surjective.

Suppose now that $0 \leq n < m$. Similarly, by Pieri’s formula, one has the decomposition:

$$\mathbb{C}[Y, X]_{m,n} \cong \bigoplus_{i=0}^{n} S^{(m+i,n-i)} \mathbb{C}^{N+1}.$$ 

The highest weight vectors spanning the irreducible components in the above decomposition are this time the following:

$$\left( X_0^{m-n+i}Y_0^i (X_1Y_0 - X_0Y_1)^{n-i} \right)_{0 \leq i \leq n}.$$ 

These vectors are never sent to zero by $\delta_{m,n}$, so that the map $\delta_{m,n}$ is injective. Using again Pieri’s formula for $\mathbb{C}[Y, X]_{m-1,n+1}$, one easily sees that the cokernel of $\delta_{m,n}$ is isomorphic to $S^{(m-1,n+1)} \mathbb{C}^{N+1}$, this finishes the proof. \qed

1.4. Dualized generalized Euler sequence. By Serre duality, one immediately deduces from Theorem 1.3.1 the cohomology of twisted symmetric powers of tangent bundles of projective spaces. However, it is always useful to have an explicit description of these cohomology groups. By dualizing (and twisting) the generalized Euler exact sequence (3), i.e. by applying to it the functor $\text{Hom}(\cdot, \mathcal{O}_{P^N})$, one deduces the following exact sequence for any $m \in \mathbb{N}$ and $n \in \mathbb{Z}$

$$0 \longrightarrow \mathbb{C}[Y]_{m-1} \otimes \mathcal{O}_{P^N}(n-1) \overset{\delta^*}{\longrightarrow} \mathbb{C}[Y]_m \otimes \mathcal{O}_{P^N}(n) \longrightarrow S^m TP^N(-m+n) \longrightarrow 0.$$ 

(Recall that $\mathcal{E}xt^1(\mathcal{O}_{P^N}(i), \mathcal{O}_{P^N}) = 0$ for any $i \in \mathbb{Z}$). The dual map $\delta^*$ can be computed explicitly, and it takes the following form:

$$\delta^* = \left( \sum_{i=0}^{N} X_i Y_i \right) \times \text{Id} + \sum_{i=0}^{N} X_i Y_i \frac{\partial}{\partial Y_i}.$$ 

It has quite an unaesthetic form, and indeed this is not the "right" description of $S^m TP^N(-m+n)$. Consider rather the renormalization map

$$u: \left( \mathbb{C}[Y] \longleftrightarrow \mathbb{C}[Y] \right),$$

where by definition $\alpha! := \alpha_0! \cdots \alpha_N!$. For any $m \in \mathbb{N}$ and any $n \in \mathbb{Z}$, the renormalization map induces an isomorphism of sheaves

$$u_{m,n}: \left( \mathbb{C}[Y]_m \otimes \mathcal{O}_{P^N}(n) \longrightarrow \mathbb{C}[Y]_m \otimes \mathcal{O}_{P^N}(n) \right).$$
where the map \( \delta^* \) is the multiplication by 
\( q := \sum_{i=0}^{N} X_i Y_i \). The diagram is commutative:

one indeed computes that for any \( \alpha, \|\alpha\| = m - 1 \), and any \( s \in \mathcal{O}_{P^N}(n - 1) \) local section:

\[
(u_{m,n} \circ \delta^*)(Y^\alpha \otimes s) = \sum_{i=0}^{N} sX_i \otimes (u(Y^{\alpha+e_i}) + \alpha_i u(Y^{\alpha+e_i}))
\]

\[
= \sum_{i=0}^{N} sX_i \otimes \frac{Y^\alpha}{\alpha!} Y_i
\]

\[
= (\delta_* \circ u_{m-1,n-1})(Y^\alpha \otimes s).
\]

Therefore, the above diagram (4) naturally induces an isomorphism between \( Q \) and \( S^m T P^N(-m + n) \), and this is this description that we will favor. We record this in the following theorem:

**Theorem 1.4.1** (Dualized generalized Euler exact sequence). For any \( m \in \mathbb{N}_{\geq 1} \), the \( m \)th symmetric power of \( T P^N(-1) \) fits into the short exact sequence:

\[
0 \longrightarrow \mathbb{C}[Y]_{m-1} \otimes \mathcal{O}_{P^N}(n - 1) \xrightarrow{\delta^*} \mathbb{C}[Y]_m \otimes \mathcal{O}_{P^N}(n) \longrightarrow S^m T P^N(-m + n) \longrightarrow 0,
\]

where \( \delta_* = \cdot \times (\sum_{i=0}^{N} X_i Y_i) \).

1.5. Some notations. In view of the previous sections, it is natural to consider the partial differential operator \( \delta \) as acting on the following (infinite dimensional) bi-graded vector bundle:

\[
\mathcal{S} := \bigoplus_{m,n} \mathbb{C}[Y]_m \otimes \mathcal{O}_{P^N}(n).
\]

The bi-graduation is the natural one, i.e. \( \mathcal{S}_{m,n} := \mathbb{C}[Y]_m \otimes \mathcal{O}_{P^N}(n) \), and \( \delta \) shifts the graduation by \((-1,1)\), i.e.

\( \delta(\mathcal{S}_{m,n}) \subset \mathcal{S}_{m-1,n+1} \).

Note that this is actually an equality. If we wish to restrict the operator \( \delta \) to a given graded part (as we implicitly did in the previous sections), we will denote:

\[
\delta_{m,n} := \delta|_{\mathcal{S}_{m,n}}.
\]

With these notations, the generalized Euler exact sequence Theorem 1.1.3 tells us exactly that:

\[
\ker \delta = \bigoplus_{m,n} \ker \delta_{m,n} \simeq \bigoplus_{m,n} S^m \Omega_{P^N}(m + n).
\]
Another equivalent but aesthetic way of writing the above equality is the following:

\[ \text{Ker} \delta \simeq \bigoplus_{m \in \mathbb{N}, n \in \mathbb{Z}} S^m(\text{Ker}(\delta_{1,0}))(n). \]

The same considerations hold for the operator \( \delta_* \), which this time shifts the graduation by \((1, 1)\). The dualized generalized Euler exact sequence provides this time the following equality:

\[ \text{Coker}(\delta_*) = \bigoplus_{m \in \mathbb{N}, n \in \mathbb{Z}} \text{Coker}(\delta_*)_{m,n} \simeq \bigoplus_{m \in \mathbb{N}, n \in \mathbb{Z}} S^m TP^N(m - n). \]

In the following sections, we will deal with other partial differential operators acting on \( S \), and shifting the bi-graduation. We therefore introduce the following terminology:

**Definition 1.5.1.** Let \( i, j \in \mathbb{Z} \) be natural numbers. A \((i, j)\)-bigraded operator on \( S \) is an endomorphism of the sheaf \( S \) that shifts the graduation by \((i, j)\).

**Example 1.5.2.** For any \( k \in \mathbb{N} \), \( \delta^k \) is a \((-k, k)\) operator on \( S \).

1.6. **Study of a natural class of endomorphisms of \( S \).** To any finite endomorphism \( f : \mathbf{P}^N \to \mathbf{P}^N \) is naturally associated an endomorphism of \( S \) as follows. Denote \( d := \deg(f) \), and recall that there exists (unique up to multiplication by a non-zero scalar) homogeneous polynomials \( P_0, \ldots, P_N \in \mathbb{C}[X,d] \) of degree \( d \) such that \( f \) writes on \( \mathbf{P}^N \)

\[ f([x]) = [P_0(x) : \ldots : P_N(x)]. \]

Consider then the endomorphism of \( S \) defined as follows

\[ \delta(f) := \sum_{i=0}^N P_i \frac{\partial A}{\partial Y_i}, \]

and note that it is a \((-1, d)\)-bigraded operator on \( S \). Note also that for \( f = \text{Id} \), one recovers the endomorphism \( \delta \). The goal of this section is to see how cohomology provides a natural way to understand the global morphism

\[ \delta(f)(\mathbf{P}^N) : S \to S, \ A(Y,X) \mapsto \sum_{i=0}^N P_i(X) \frac{\partial A}{\partial Y_i}(Y,X), \]

where by definition \( S := H^0(S) = \mathbb{C}[Y,X] \).

Observe that, by definition, the vector bundle \( \text{Ker} \delta(f)_{1,0} \) is the rank \( N \) vector bundle on \( \mathbf{P}^N \) whose fiber over \([x] \in \mathbf{P}^N\) is the hyperplane defined by the equation:

\[ P_0(x)Y_0 + \cdots + P_N(x)Y_N = 0. \]

Furthermore, the simple but key observation is that \( \text{Ker} \delta(f)_{1,0} \) is nothing but the pull-back by \( f \) of \( \text{Ker} \delta_{1,0} \), i.e.

\[ \text{Ker} \delta(f)_{1,0} = f^* \text{Ker} \delta_{1,0}. \]

In the spirit of what we did in the previous sections, we prove the following:

**Proposition 1.6.1.** For any \( m \in \mathbb{N} \), the \( m \)th symmetric power of \( \text{Ker} \delta(f)_{1,0} \) fits into the exact sequence

\[ 0 \to S^m \text{Ker} \delta(f)_{1,0} \to \mathbb{C}[Y]_m \otimes \mathcal{O}_{\mathbf{P}^N} \xrightarrow{\delta(f)} \mathbb{C}[Y]_{m-1} \otimes \mathcal{O}_{\mathbf{P}^N}(d) \to 0. \]

**Proof.** The injection of locally free sheaves \( S^m \text{Ker} \delta(f)_{1,0} \to \mathbb{C}[Y]_m \otimes \mathcal{O}_{\mathbf{P}^N} \) is given by the map

\[ ([x], v^1 \cdots v^m) \mapsto ([x], \sum_{0 \leq i_1, \ldots, i_m \leq N} v^1_{i_1} \cdots v^m_{i_m} Y_{i_1} \cdots Y_{i_m}). \]
Its image does indeed lie in the kernel of $\delta(f)$:

$$
(\delta(f) \circ i)(v^1 \cdots v^n) = \sum_{i_1, \ldots, i_m} \sum_{k=1}^m v^1_{i_1} \cdots v^m_{i_m} P_{i_k} Y_{i_1} \cdots Y_{i_m}
$$

$$
= \sum_{k=1}^m \sum_{i_1, \ldots, i_{k-1}, i_{k+1}, \ldots} v^1_{i_1} \cdots v^{k-1}_{i_{k-1}} Y_{i_{k-1}} (\sum_{i=0}^N P_i(x) v^k_{i_{k+1}} Y_{i_{k+1}}) \cdots v^m_{i_m} Y_{i_m}
$$

$$
= 0,
$$
since all the vectors $v^1, \ldots, v^m$ lie in $\text{Ker} \, \delta(f)_{1,0}(x)$.

The surjectivity of the map $\delta(f)$ is proved as in the proof of Lemma 1.1.1 (almost verbatim). The only difference is that, in order to construct an antecedent of a local section, one rather considers the covering of affine open sets $\{P_i \neq 0\} \subseteq \mathbb{N}$, and the following local sections of $\mathcal{C}[Y]_1 \otimes \mathcal{O}_\mathbb{P}^N$

$$
\left\{ \begin{array}{l}
L_i := \frac{Y_i}{P_i} \\
L_j := \frac{P_j}{P_i} Y_i \quad \text{for } j \neq i
\end{array} \right.
$$

As for the exactness in the middle, it follows from the equality of dimension:

$$
\dim \mathbb{C}[Y]_m = \dim \mathbb{C}[Y]_{m-1} + \text{rank}(S^m \text{ Ker} \, \delta(f)_{1,0}).
$$

\[\square\]

**Remark 1.6.2.** Note that for $f = \text{Id}$, one recovers the generalized Euler exact sequence.

In plain words, the previous statement says that for any $x \in \mathbb{P}^N$, the map

$$
\delta(f)_x := \sum_{i=0}^N P_i(x) \frac{\partial}{\partial Y_i}
$$

is the defining equation of $(\text{S}^m \text{ Ker} \, \delta(f)_{1,0})_x$ in $\text{S}^m \mathbb{C}^{N+1} \simeq \mathbb{C}[Y]_m$.

By taking the long exact sequence in cohomology associated to the short exact sequence (1.6.1) twisted by $\mathcal{O}_\mathbb{P}^N(n), n \in \mathbb{N}$, one finds that

$$
\left\{ \begin{array}{l}
\text{Ker} \left( \delta(f)_{m,n}(\mathbb{P}^N) \right) \simeq H^0(\mathbb{P}^N, (\text{S}^m \text{ Ker} \, \delta(f)_{1,0}) \otimes \mathcal{O}_\mathbb{P}^N(n)) \\
\text{Coker} \left( \delta(f)_{m,n}(\mathbb{P}^N) \right) \simeq H^1(\mathbb{P}^N, (\text{S}^m \text{ Ker} \, \delta(f)_{1,0}) \otimes \mathcal{O}_\mathbb{P}^N(n))
\end{array} \right.
$$

Since one still has the following equality for any $m \in \mathbb{N}_{\geq 1}$

$$
f^* \text{S}^m \text{ Ker} \, \delta_{1,0} = \text{S}^m \text{ Ker} \, \delta(f)_{1,0},
$$

the projection formula implies the following:

$$
H^*(\mathbb{P}^N, \text{S}^m \text{ Ker} \, \delta(f)_{1,0} \otimes \mathcal{O}_\mathbb{P}^N(n)) \simeq H^*(\mathbb{P}^N, \text{S}^m \text{ Ker} \, \delta_{1,0} \otimes f_* \mathcal{O}_\mathbb{P}^N(n))_{\simeq \text{S}^m \text{ Ker} \, \delta_{1,0} \otimes f_* \mathcal{O}_\mathbb{P}^N(n)}.
$$

Accordingly, in order to understand the kernel and cokernel of the global map

$$
\delta(f)(\mathbb{P}^N) : \mathbb{C}[Y, X] \longrightarrow \mathbb{C}[Y, X],
$$

it is enough to understand the direct images $f_* (\mathcal{O}_\mathbb{P}^N(n))$ for any $n \in \mathbb{Z}$. It turns out that such direct images are particularly well-understood, and one has:

**Lemma 1.6.3.** Let $0 \leq r < d$ be an integer. The vector bundle $f_* (\mathcal{O}_\mathbb{P}^N(r))$ splits as a direct sum of line bundles, all of whom have degree less or equal than zero.
Lemma 2.1.1. The map defined pointwise on $A$ first Koszul complex: the case of hypersurfaces. 2.1. Let $H$ space to the base space the canonical projection from the restricted projectivized tangent bundle of the projective space writes as a direct sum of pieces of the following form $k$.

Proof. Since $f$ is a finite map, the push-forward sheaf $f_*(\mathcal{O}_{\mathbb{P}^N}(r))$ is locally free. Furthermore, the projection formula implies that for any $j \in \mathbb{Z}$, one has:

$$H^*(\mathbb{P}^N, f^*(\mathcal{O}_{\mathbb{P}^N}(j)) \otimes \mathcal{O}_{\mathbb{P}^N}(r)) \simeq H^*(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(j) \otimes f_* \mathcal{O}_{\mathbb{P}^N}(r))$$

In particular, this implies that

$$H^i(\mathbb{P}^N, f_*(\mathcal{O}_{\mathbb{P}^N}(r)) \otimes \mathcal{O}_{\mathbb{P}^N}(j)) = (0)$$

for any $1 \leq i \leq N - 1$ and any $j \in \mathbb{Z}$. By a result of Horrocks (see [Bea00][Corollary 1.3]), this implies that $f_*(\mathcal{O}_{\mathbb{P}^N}(r))$ splits as a direct sum of line bundles.  

Moreover, this immediately implies that all the line bundles appearing in the splitting of $f_*(\mathcal{O}_{\mathbb{P}^N}(r))$ have degree less or equal than zero. \hfill $\square$ 

Knowing what degrees appear in the splitting of the line bundles $f_* \mathcal{O}_{\mathbb{P}^N}(r)$, $0 \leq r < d$, seems however to be complicated in full generality.

For later use, we record the following result, which is interesting in itself:

Theorem 1.6.4. Let $m \in \mathbb{N}$, and $n \in \mathbb{N}$. As soon as $dm > n$, the linear map

$$\delta(f)_{m,n}(\mathbb{P}^N): S_{m,n} \to S_{m-1,n+d}$$

is injective, and the bound is sharp.

Proof. By the above (and the projection formula), the kernel of $\delta(f)_{m,n}(\mathbb{P}^N)$ is isomorphic to

$$H^0(\mathbb{P}^N, S^m \mathcal{O}_{\mathbb{P}^N}(m + q) \otimes \gamma_s(\mathcal{O}_{\mathbb{P}^N}(r))),$$

where $n = qd + r$ is the euclidean division of $n$ by $d$. By Lemma 1.6.3, the above vector space writes as a direct sum of pieces of the following form

$$H^0(\mathbb{P}^N, S^m \mathcal{O}_{\mathbb{P}^N}(m + k)),$$

where $k \leq q = \lceil \frac{m}{d} \rceil$. By Theorem 1.3.1, these cohomology groups are zero if and only if $k < m$: this finishes the proof. \hfill $\square$

2. Koszul complexes resolving structural sheaves of projectivized tangent bundles of smooth complete intersections.

2.1. A first Koszul complex: the case of hypersurfaces. Let us fix $H := \{P = 0\} \subset \mathbb{P}^N$ a smooth hypersurface of degree $d \geq 1$, and let us denote by

$$\pi_H: \mathbb{P}(\mathit{TP}_{|H}^N) \to H$$

the canonical projection from the restricted projectivized tangent bundle of the projective space to the base space $H$.

2.1.1. Construction of the Koszul complex. A simple but crucial observation is that the differential $d P$ allows to define a global section of $\mathcal{O}_{\mathbb{P}(\mathit{TP}_{|H}^N)(1)} \otimes \pi_H^* \mathcal{O}_H(d)$:

Lemma 2.1.1. The map defined pointwise on $\xi = ([x], [\mathcal{N}]) \in \mathbb{P}(\mathit{TP}_{|H}^N(-1))$ by $^6$

$$s(P)(\xi) := (\xi, v \mapsto \frac{1}{d}(d P)_x(v))$$

The multiplicative factor $\frac{1}{d}$ is motivated by the identity:

$$\delta(\frac{1}{d}(d P)_x(Y)) = P.$$
glues to a global section of $\mathcal{O}_{\mathbf{P}(T \mathbf{P}^N|_H)}(1) \otimes \pi_H^* \mathcal{O}_H(d)$.

Proof. Note first that, in the statement, one implicitly identifies $(T \mathbf{P}^N(-1))[x]$ with its natural description via the Euler vector fields, i.e.

$$(T \mathbf{P}^N(-1))[x] \simeq \frac{\mathbb{C} \cdot \gamma_0 + \cdots + \mathbb{C} \cdot \gamma_N}{\mathbb{C} \cdot (x_0 \gamma_0 + \cdots + x_N \gamma_N)}.$$ 

Therefore, a vector $\boldsymbol{v} \in (T \mathbf{P}^N(-1))[x]$ is seen as an equivalence class of a vector $v = (v_0, \ldots, v_N)$ in $\mathbb{C}^{N+1}$.

Let then $\lambda, \mu \in \mathbb{C}^*$. One has the following two simple identities:

1. $(d \mathbf{P})_{\lambda x}(\mu v) = \lambda^{d-1} \mu \cdot (d \mathbf{P})_x(v)$;
2. $(d \mathbf{P})_x(v + \lambda x) = (d \mathbf{P})_x(v) + \lambda^{d-1} P(x)$.

Then observe that the zero locus of the global section $\gamma : \mathbf{P}(T \mathbf{P}^N(-1)) \to H$ is the canonical projection.

The lemma now follows from the natural identification\(^7\) between

$$\mathcal{O}_{\mathbf{P}(T \mathbf{P}^N|_H)(-1))}(1) \otimes \tilde{\pi}_H^* \mathcal{O}_H(d - 1)$$

and

$$\mathcal{O}_{\mathbf{P}(T \mathbf{P}^N|_H)}(1) \otimes \pi_H^* \mathcal{O}_H(d).$$

Now, observe that the zero locus of the global section $s(P)$ defines set-theoretically the projectivized tangent bundle $\mathbf{P}(T \mathbf{H})$ inside $\mathbf{P}((T \mathbf{P}^N|_H))$, by very definition of the tangent bundle. Furthermore, by smoothness of $H$, the ideal sheaf spanned by $s(P)$ is reduced. Therefore, we deduce the following exact sequence:

$$(\mathbf{P}(T \mathbf{P}^N|_H))(-1) \otimes \pi_H^* \mathcal{O}_H(-d) \xrightarrow{\times s(P)} \mathcal{O}_{\mathbf{P}(T \mathbf{P}^N|_H)},$$

which provides a locally free resolution of $j_* \mathcal{O}_{\mathbf{P}(T \mathbf{H})}$, where $j : \mathbf{P}(T \mathbf{H}) \to \mathbf{P}(T \mathbf{P}^N|_H)$ is the closed immersion of $\mathbf{P}(T \mathbf{H})$ inside $\mathbf{P}(T \mathbf{P}^N|_H)$.

2.1.2. Interpretation of the (twisted) Koszul complex on the base. By twisting the exact sequence (7) by $\mathcal{O}_{\mathbf{P}(T \mathbf{P}^N|_H)}(m) \otimes \pi_H^* \mathcal{O}_H(m + n)$\(^8\), where $m \in \mathbb{N}_{\geq 1}$, and $n \in \mathbb{Z}$, one obtains the exact sequence:

$$(\mathbf{P}(T \mathbf{P}^N|_H))(-1) \otimes \pi_H^* \mathcal{O}_H(m + n - d) \xrightarrow{\times s(P)} \mathcal{O}_{\mathbf{P}(T \mathbf{P}^N|_H)}(m) \otimes \pi_H^* \mathcal{O}_H(m + n),$$

which provides a locally free resolution of $j_* \mathcal{O}_{\mathbf{P}(T \mathbf{H})}(m) \otimes \pi_H^* \mathcal{O}_H(m + n)$\(^9\).

Recall that, as soon as $r \geq 0$, Bott’s formulas (see Appendix A) imply in particular that the higher direct images

---

\(^7\)Keep in mind that, throughout the paper, we use projectivization of lines.

\(^8\)One may wonder why we twist by $\pi_H^* \mathcal{O}_H(m + n)$ and not by $\pi_H^* \mathcal{O}_H(n)$: we do it for the sake of coherence, as throughout the paper, it will be more natural to compute the cohomology of $S^m \Omega X(m + n)$, where $X$ is a complete intersection, rather than the cohomology of $S^m \Omega X(n)$.

\(^9\)Throughout the text, we will often omit to write these push-forward inclusions, as they are implicit (and often burdens the notations).
(1) \((R^i\pi)_* (\mathcal{O}_{\mathbb{P}(TP_{\mathbb{P}^N})}(r) \otimes \pi_H^* \mathcal{O}_H(s))\)

(2) \((R^i\pi)_* (j_* (\mathcal{O}_{\mathbb{P}(TH)})(r) \otimes \pi_H^* \mathcal{O}_H(s)) \simeq (R^i(j \circ \pi_H)_*) (\mathcal{O}_{\mathbb{P}(TH)}(r) \otimes (j \circ \pi_H)^* \mathcal{O}_H(s))\)

vanish for \(i > 0\) and any \(s \in \mathbb{Z}\). They also imply the following two equalities for any \(r \in \mathbb{N}_{\geq 0}\) and any \(s \in \mathbb{Z}\):

(1) \((\pi_H)_* (\mathcal{O}_{\mathbb{P}(TP_{\mathbb{P}^N})}(r) \otimes \pi_H^* \mathcal{O}_H(s)) \simeq S^r \mathcal{O}_{\mathbb{P}^N}(s)|_H;\)

(2) \((\pi_H)_* (j_* \mathcal{O}_{\mathbb{P}(TH)}(r) \otimes \pi_H^* \mathcal{O}_H(s)) \simeq S^r \mathcal{O}_H(s).\)

By pushing forward by \(\pi_H\) the exact sequence (8), one therefore keeps an exact sequence, which takes the following form

\[
0 \longrightarrow S^{m-1} \Omega_{\mathbb{P}^N}(m+n-d)|_H \xrightarrow{(\pi_H)_*(s(P))} S^m \Omega_{\mathbb{P}^N}(m+n)|_H.
\]

and which provides a locally free resolution of \(S^m \Omega_H(m+n)\). By very construction, one easily sees that the push-forward map \((\pi_H)_*(s(P))\) is nothing but the map induced by the multiplication by the \((1, d-1)\) bi-homogeneous polynomial \(\frac{1}{d}(d \cdot P)_X(Y)\):

\[
\begin{array}{c}
S^{m-1} \Omega_{\mathbb{P}^N}(m+n-d)|_H \\
\downarrow \\
C[Y]_{m-1} \otimes \mathcal{O}_H(n-d+1) \\
\downarrow \delta \\
\mathbb{C}[Y]_{m-2} \otimes \mathcal{O}_H(n-d+2) \\
\downarrow \\
0 \\
\end{array}
\quad \begin{array}{c}
\xrightarrow{\alpha(P)} \\
\downarrow \delta \\
\xrightarrow{\delta} \\
0 \\
\end{array}
\begin{array}{c}
S^m \Omega_{\mathbb{P}^N}(m+n)|_H \\
C[Y]_{m} \otimes \mathcal{O}_H(n) \\
\end{array}
\]

Here, by definition, one has denoted

\[\alpha(P) : = \cdot \times \frac{1}{d}(d \cdot P)_X(Y)\]

the multiplication map by \(\frac{1}{d}(d \cdot P)_X(Y)\). Note that the induced map is indeed well-defined, since one has the following identity:

\[\delta \circ \alpha(P) = \cdot \times (P + \alpha(P)) \circ \delta.\]

Using the notations of Section 1, we can summarize the results of this section in the following statement:

**Theorem 2.1.2.** The bi-graded (Koszul)complex

\[
K(\alpha(P)) : 0 \longrightarrow (\text{Ker} \, \delta)|_H[-1,-d+1] \xrightarrow{\alpha(P)} (\text{Ker} \, \delta)|_H.
\]

provides a locally free bi-graded resolution of

\[
\bigoplus_{m \geq 1, n \in \mathbb{Z}} S^m \Omega_H(m+n).
\]
2.1.3. A remark on the more general case of a smooth hypersurface of a smooth projective variety. Consider the following more general setting: take $X$ a smooth projective variety, fix $L$ a line bundle on $X$, and suppose that there exists $s \in H^0(X, L)$ whose zero locus $H := \{s = 0\}$ defines a smooth hypersurface. One easily sees that Lemma 2.1.1 generalizes as follows:

**Lemma 2.1.3.** There exists a global section of the line bundle 
$$\mathcal{O}_{P(TX|_H)}(1) \otimes \pi_H^*L$$
which is locally induced by the differential of $s$.

**Proof.** Let $(U_i)_{i \in I}$ be an open covering of $X$ trivializing both $TX$ and $L$, and denote by $(s_i)_{i \in I}$ the local sections on $U_i$ induced by $s$. It is straightforward to check that the transition maps of the local sections defined pointwise on $U_i$ by 
$$([x], [v]) \mapsto ([x], v \mapsto (d s_i)_x(v))$$
does glue to a global section of $\mathcal{O}_{P(TX|_H)}(1) \otimes \pi_H^*L$. \[\square\]

In this more general setting, Bott’s formulas still holds, which allows to show the following (in exactly the same fashion as in the previous Section 2.1.2):

**Proposition 2.1.4.** Let $m \in \mathbb{N}$. The twisted symmetric power $S^m \Omega_H$ fits into the short exact sequence
$$0 \longrightarrow S^{m-1} \Omega_X(-L) \longrightarrow S^m \Omega_X \longrightarrow S^m \Omega_H \longrightarrow 0.$$  

2.2. A first Koszul complex: the case of smooth complete intersections. Let us now fix $c$ hypersurfaces $(H_i = \{P_i = 0\})_{1 \leq i \leq c}$ of degrees $d_i \geq 1$ in such a way that their intersection
$$X := H_1 \cap \cdots \cap H_c \subset P^N$$
defines a smooth complete intersection. Let us also denote by 
$$\pi_X : P(TP^N|_X) \to X$$
the canonical projection from the restricted projectivized tangent bundle of the projective space to the base space $X$.

2.2.1. Construction of the Koszul complex. As in the previous section, for every $1 \leq i \leq c$, the differential $d P_i$ induces a global section 
$$s(P_i) \in H^0(P(TP^N|_{H_i}), \mathcal{O}_{P(TP^N|_{H_i})}(1) \otimes \pi_H^* \mathcal{O}_{H_i}(d_i)).$$
After restriction to $X$, such a global section $s(P_i)$ induces as well a global section of 
$$\mathcal{O}_{P(TP^N|_X)}(1) \otimes \pi_X^* \mathcal{O}_X(d_i).$$
We will denote $s_i$ this restricted section, and define $s := (s_1, \ldots, s_c)$.

Observe that the zero locus of the sections $s_1, \ldots, s_c$ defines set-theoretically the projectivized tangent bundle $P(TX)$ inside $P(TP^N|_X)$. Furthermore, the smoothness of $X$ implies that the ideal sheaf spanned by these sections is reduced (as one easily sees via the Jacobian criterion). By constructing the Koszul complex $\mathcal{K}(s)$ associated to the sections $s_1, \ldots, s_c$ (see e.g. [Laz04][Appendix B.2]), one therefore obtains a complex abutting to $\mathcal{O}_{P(TX)}$. What is more, the smoothness of $P(TP^N|_X)$ implies that this complex is exact, therefore providing a resolution of $\mathcal{O}_{P(TX)}$. 

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2.2. Interpretation of the (twisted) Koszul complex on the base. We can follow the same line of reasoning as in the case of hypersurfaces. Twisting by

\[ \mathcal{O}_{P(TP_X^N)}(m) \otimes \pi_X^* \mathcal{O}_X(m + n) \]

the Koszul complex \( \mathcal{K}(s) \), where \( m \in \mathbb{N}_{\geq c} \) and \( n \in \mathbb{Z} \), we obtain a resolution of \( \mathcal{O}_{P(TP_X^N)}(m) \otimes \pi_X^* \mathcal{O}_X(m + n) \). As in the case of hypersurfaces, we denote this resolution by \( \mathcal{K}(s)_{m,n} \).

Using Bott’s formulas, this resolution can be pushed-forward by \( \pi_X \) into a resolution of \( S^m \Omega_X(m + n) \). The pushed-forward maps \( (\pi_X)_* (s_i) \) are again induced by the multiplication by the \((1, d_i - 1)\)-bihomogeneous polynomial \( \frac{1}{d_i} (d P_i) \). Namely, this is nothing but the map of graded vector bundles:

\[ \alpha(P_i): (\text{Ker} \delta)_{|X}[-1, -d_i + 1] \rightarrow (\text{Ker} \delta)_{|X}. \]

The push-forward Koszul complex \( (\pi_X)_* (\mathcal{K}(s)_{m,n}) \) is then nothing but the suitable graded pieces of the Koszul complex

\[ \mathcal{K}(\alpha(P_1), \ldots, \alpha(P_c)) \]

on \( \text{Ker} \delta_{|X} \).

Example 2.2.2. For \( c = 2 \), the (push-forward) Koszul complex \( (\pi_X)_* (\mathcal{K}(s)_{m,n}) \) takes the following form:

\[ 0 \rightarrow (\text{Ker} \delta_{m-2,n-d_1-d_2+2})_{|X} \rightarrow (\text{Ker} \delta_{m-1,n-d_2+1})_{|X} \rightarrow (\text{Ker} \delta_{m-1,n-d_1+1})_{|X} \]

\[ \cdots \rightarrow (\text{Ker} \delta_{m,n})_{|X}, \]

and it provides a locally free resolution of \( S^m \Omega_X(m + n) \).

The main result of this section can thus be summarized in the following statement:

**Theorem 2.2.3.** The Koszul complex

\[ \mathcal{K}(\alpha(P_1), \ldots, \alpha(P_c)) \]

induced by the multiplication maps \( (\alpha(P_i))_{1 \leq i \leq c} \) on \( (\text{Ker} \delta)_{|X} \) provides a locally free bi-graded resolution of

\[ \bigoplus_{m \geq c, n \in \mathbb{Z}} S^m \Omega_X(m + n). \]

2.3. A second Koszul complex: the case of smooth hypersurfaces. Let \( H := (P = 0) \) be a smooth hypersurface of degree \( d \) in the projective space \( P^N \), with \( P \in \mathbb{C}[X] \) an homogeneous polynomial of degree \( d \) in the variables \( X := (X_0, \ldots, X_N) \). Following the same idea as in Section 1.2, we define a closed immersion of \( P(TH) \) inside \( \text{Flag}_{(1,2)} \mathbb{C}^{N+1} \simeq P(TP^N) \) as follows:

\[ j: \left( \begin{array}{c} P(TH) \\ ([x], v) \end{array} \right) \mapsto \text{Flag}_{(1,2)} \mathbb{C}^{N+1} \]

\[ (C \cdot x \subseteq C \cdot x \oplus C \cdot v) \]

Note that the exact same proof of Proposition 1.2.1 gives the following:
Proposition 2.3.1. The pull-back line bundle $j^*L_{m,n}$ on $\mathbf{P}(TH)$ is equal to $\mathcal{O}_{\mathbf{P}(TH)}(m) \otimes \pi_H^* \mathcal{O}_H(m + n)$, where $\pi_H : \mathbf{P}(TH) \to H$ is the canonical projection onto the hypersurface $H$.

The goal of this section is to construct a rank 2 vector bundle on $\text{Flag}_{(1,2)} \mathbb{C}^{N+1}$ admitting a global section defining the ideal sheaf of (the image of) $\mathbf{P}(TH)$ inside $\text{Flag}_{(1,2)} \mathbb{C}^{N+1}$. From such a data follows classically the construction of a Koszul complex on $\text{Flag}_{(1,2)} \mathbb{C}^{N+1}$ resolving the structural sheaf of $j_* \mathcal{O}_{\mathbf{P}(TH)}$. We then give the interpretation of the push-forward complex on the base space $H$.

### 2.3.1 Construction of the Koszul complex

We keep the notations of Section 1.2, and in particular still denote by $\pi : \text{Flag}_{(1,2)} \mathbb{C}^{N+1} \to \mathbf{P}^N$ the canonical projection. Consider the covering family of open sets of $\text{Flag}_{(1,2)} \mathbb{C}^{N+1}$

$$
\bigl(V_i := \pi^{-1}(V_i)\bigr)_{0 \leq i \leq N},
$$

where by definition $V_i := \{X_i \neq 0\}$. On each open set $\tilde{V}_i$, there exists a family of local sections $(g_{ji})_{0 \leq j \leq N} \in \mathcal{L}_{1,0}(V_i)$ defined as follows. Let $\xi = (\mathbb{C} \cdot x \subseteq \mathbb{C} \cdot x \oplus \mathbb{C} \cdot y)$ be a point in $\tilde{V}_i$, and define $g_{ji}(\xi, \lambda x + \mu y) = \frac{x_i \lambda - \mu x_j}{x_i}$. If we substitute $x$ with $\alpha x$, and $y$ with $\beta_1 y + \beta_2 x$ (for $\alpha, \beta_1 \in \mathbb{C}^*$ and $\beta_2 \in \mathbb{C}$), then the value of $g_{ji}$ is multiplied by $\beta_1$. In other words, $g_{ji}(\xi, \cdot)$ defines a linear functional on the fiber $(\frac{U_i}{U_j})_\xi$, so that $g_{ji}$ is indeed a local section of $\mathcal{L}_{1,0} = (\frac{U_i}{U_j})^\vee$ on $V_i$.

We then construct a rank 2 vector bundle on $\text{Flag}_{(1,2)} \mathbb{C}^{N+1}$ by glueing the rank 2 vector bundles

$$
(\mathcal{L}_{0,0} \oplus \mathcal{L}_{1,0})|_{\tilde{V}_i}
$$

with the transition maps from the $i$th chart to the $j$th chart

$$
\varphi_{ij} : \begin{pmatrix} (\mathcal{L}_{0,0} \oplus \mathcal{L}_{1,0})|_{V_i \cap V_j} & \mapsto & (\mathcal{L}_{0,0} \oplus \mathcal{L}_{1,0})|_{V_j \cap V_i} \\
(s_1, s_2) & \mapsto & (s_1, s_2) \begin{pmatrix} X_i & g_{ij} \\
0 & 1 \end{pmatrix}
\end{pmatrix}.
$$

Note that this indeed makes sense since

$$
\begin{pmatrix} X_i & g_{ij} \\
0 & 1 \end{pmatrix} \begin{pmatrix} X_j & g_{jk} + g_{ij} \\
0 & 1 \end{pmatrix}
= \begin{pmatrix} X_i X_j & g_{ik} \\
0 & 1 \end{pmatrix}.
$$

We record this construction in the following definition:

**Definition 2.3.2.** Denote by $\mathcal{E}$ the rank 2 vector bundle on $\text{Flag}_{(1,2)} \mathbb{C}^{N+1}$ obtained by the previous construction. For any $m, n \in \mathbb{Z}$, denote as well

$$
\mathcal{E}_{m,n} := \mathcal{E} \otimes \mathcal{L}_{m,n}.
$$

The vector bundle $\mathcal{E}$ has a simple structure, as it is a (non-trivial) extension of $\mathcal{L}_{1,0}$ by $\mathcal{L}_{0,1}$:

**Proposition 2.3.3.** The vector bundle $\mathcal{E}$ fits canonically into a short exact sequence

$$
0 \longrightarrow \mathcal{L}_{1,0} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}_{0,1} \longrightarrow 0.
$$
Lemma 2.3.4. The local sections $s_i$ glue to a global section defining $j$. It suffices to check that

It is then straightforward to check that the local sections $s^i$ glue to give a global section $s(P) \in \mathcal{E}_{0,d-1}$. We record it in the following lemma:

Lemma 2.3.4. The local sections

$$s^i = \left( \frac{P(X)}{X_i}, \frac{1}{d} (dP)_x(Y) - \frac{Y_i}{X_i} P(X) \right) \in (\mathcal{L}_{0,d-1} \oplus \mathcal{L}_{1,d-1})(\tilde{V}_i)$$

glue to a global section $s(P)$ of $\mathcal{E}_{0,d-1}$.

Proof. It suffices to check that

$$s^i \left( \frac{X_i}{Y_i}, \frac{g_{ij}}{1} \right) = s^j.$$

It is clear that the (image of the multiplication map by the) section

$$s := s(P)$$

defines the ideal sheaf associated to the closed immersion $j: \mathbb{P}(TH) \hookrightarrow \text{Flag}_{(1,2)} \mathbb{C}^{N+1}$, by its very construction. We can accordingly construct the Koszul complex associated to it

$$\mathcal{K}(s): 0 \longrightarrow \mathcal{L}_{-1,-2d+1} \longrightarrow \mathcal{E}_{-1,-d} \longrightarrow \mathcal{L}_{0,0},$$

which provides a locally free resolution of $j_* \mathcal{O}_{\mathbb{P}(TH)}$. The (twisted) third term in the complex comes from the isomorphism

$$\bigwedge^2 \mathcal{E}_{0,d-1} \cong \mathcal{L}_{1,2d-1}$$

obtained by twisting by $\mathcal{L}_{0,d-1}$ the exact sequence in Proposition 2.3.3, and taking its determinant. It is an exact complex, since $\text{Flag}_{(1,2)} \mathbb{C}^{N+1}$ is smooth, and codim $j(\mathbb{P}(TH)) = \text{codim}_{\mathbb{P}(TH)} \text{Flag}_{(1,2)} \mathbb{C}^{N+1}$.
rank(ℰ_{0,d-1}) = 2. We can always twist this exact complex by ℳ_{m,n} to obtain the following exact complex for any n, m ∈ ℤ:

\begin{equation}
\mathcal{K}(s)_{m,n}: 0 \longrightarrow \mathcal{L}_{m-1,n-2d+1} \longrightarrow \mathcal{E}_{m-1,n-d} \longrightarrow \mathcal{L}_{m,n},
\end{equation}

which provides a locally free resolution of j_*\mathcal{O}_{P(T)} \otimes \mathcal{L}_{m,n}.

2.3.2. Interpretation of the (twisted) Koszul complex on the base. As soon as m ≥ 0, Bott’s formulas (see Appendix A) imply that the higher direct image sheaves

\[ R^i\pi_*\mathcal{L}_{m,n} \]

vanish for i > 0 and any n ∈ ℤ, where we recall that π : Flag_{1,2} \mathbb{C}^{N+1} → \mathbb{P}^{N} is the canonical projection. Using the isomorphism of Proposition 1.2.1, Bott’s formulas also imply that

\[ \pi_*\mathcal{L}_{m,n} \simeq S^m\Omega_{\mathbb{P}^{N}}(m + n) \simeq \text{Ker}(\delta_{m,n}). \]

Using still Bott’s formulas, and the projection formula, one also gets that

\[ \pi_*(j_*\mathcal{O}_{P(T)} \otimes \mathcal{L}_{m,n}) \simeq i_*(S^m\Omega_H)(m + n), \]

where i : H → \mathbb{P}^{N} is the closed immersion of the hypersurface H inside the projective space \mathbb{P}^{N}. As soon as m ≥ 1, if one takes the push forward by π of the exact sequence (10), one keeps an exact sequence, which takes the following form

\begin{equation}
\pi_*(\mathcal{K}(s)_{m,n}): 0 \longrightarrow \text{Ker} \delta_{m-1,n-2d+1} \longrightarrow \pi_*(\mathcal{E}_{m-1,n-d}) \longrightarrow \text{Ker} \delta_{m,n},
\end{equation}

and which resolves i_*S^m\Omega_H(m + n).

In order to apprehend the above complex, we must understand the push-forward sheaf \pi_*((\mathcal{E}_{m-1,n-d}). It turns out that it has a simple interpretation:

**Proposition 2.3.5.** For any s ∈ ℤ and any r ∈ ℂ, the locally free sheaf \pi_*((\mathcal{E}_{r,s}) is canonically isomorphic to the kernel sheaf of the map:

\[ \mathbb{C}[Y]_{r+1} \otimes \mathcal{O}_{\mathbb{P}^{N}}(s) \xrightarrow{\delta \delta} \mathbb{C}[Y]_{r-1} \otimes \mathcal{O}_{\mathbb{P}^{N}}(s + 2), \]

i.e. \pi_*((\mathcal{E}_{r,s}) \simeq \text{Ker}\delta^2_{r+1,s}.

**Proof.** Recall that, by definition, the locally free sheaf \mathcal{E}_{s,r} is obtained by suitably glueing the sheaves

\[ (\mathcal{L}_{r,s} \oplus \mathcal{L}_{r+1,s})|_{\tilde{V}_{i}} \]

on the covering open sets \tilde{V}_{i} = \pi^{-1}({X_i \neq 0}) (see Subsection 2.3.1). Pushing everything forward by π, one sees that \pi_*((\mathcal{E}(s,r)) is therefore obtained by glueing

\[ (\text{Ker} \delta_{r,s} \oplus \text{Ker} \delta_{r+1,s})|_{V_{i}} \]

via the transition maps

\[ A_{ij} := \left(\begin{array}{cc} X_i & X_iY_j - X_jY_i \\ Y_i & 0 \end{array}\right). \]

There is then a natural map from \pi_*((\mathcal{E}(s,r)) to \text{Ker}\delta^2_{r+1,s} defined as follows. On each trivializing open set \tilde{V}_{i}, consider the map

\[ \Theta_{i} : (\text{Ker} \delta_{r,s}|_{\tilde{V}_{i}} \oplus (\text{Ker} \delta_{r+1,s})|_{\tilde{V}_{i}})_{(s_1,s_2)} \longrightarrow (\text{Ker} \delta^2_{r+1,s}|_{\tilde{V}_{i}})_{s_2 + \tilde{Y}_is_1}.\]

Note that the image does indeed lie in \text{Ker}\delta^2_{r+1,s}|_{\tilde{V}_{i}}, since (using the Leibniz rule for \delta) one has the equality

\[ (\delta \circ \delta)(s_2 + \tilde{Y}_is_1) = (\delta \circ \delta)(\tilde{Y}_i) \times s_1 = 0. \]
These local maps glue to a global map
\[ \Theta: \pi_\ast(\mathcal{E}(r,s)) \rightarrow \text{Ker} \delta^2_{r+1,s}, \]
which follows from the equality:
\[ \begin{pmatrix} X_i \\ X_i Y_i - X_i Y_i \\ 0 \\ X_i \\ 1 \end{pmatrix} \begin{pmatrix} Y_i \\ 1 \end{pmatrix} = \begin{pmatrix} Y_i \\ 1 \end{pmatrix}. \]

The injectivity of the map \( \Theta \) is straightforward to check: if one has locally on \( V_i \) the equality
\[ s_2 + Y_i s_1 = 0, \]
then by applying \( \delta \) one gets the equality \( \delta(Y_i)s_1 = 0 \Leftrightarrow X_is_1 = 0 \). This implies that \( s_1 \) is zero, and thus so is \( s_2 \). As for the surjectivity, observe that if \( s \in (\text{Ker} \delta^2_{r+1,s})|_{V_i} \), then by setting
\[ \begin{cases} s_1 := \frac{\delta(s)}{X_i} \\ s_2 := s - \frac{\delta(s)}{X_i} Y_i \end{cases} \]
one has the equality
\[ s = s_2 + Y_i s_1 = \Theta((s_1, s_2)), \]
with \( \delta(s_1) = \delta(s_2) = 0 \). This finishes the proof. \( \square \)

Note that in view of the above Proposition 2.3.5, the push-forward of the twisted exact sequence
\[ 0 \rightarrow \mathcal{L}_{s,r+1} \rightarrow \mathcal{E}_{r,s} \rightarrow \mathcal{L}_{s+1,r} \rightarrow 0 \]
of Proposition 2.3.3 writes as follows:
\[ 0 \rightarrow \text{Ker} \delta_{r+1,s} \rightarrow \text{Ker} \delta^2_{r+1,s} \rightarrow \text{Ker} \delta_{r,s+1} \rightarrow 0. \]

Furthermore, the resolution of \( i_\ast(S^m\Omega_H)(m + n) \), where \( m \geq 1 \), can be rewritten as follows:
\[ \begin{align*}
0 & \rightarrow \text{Ker} \delta_{m-1,n-2d+1} \rightarrow \text{Ker} \delta^2_{m,n-d} \rightarrow \text{Ker} \delta_{m,n}.
\end{align*} \]

(12)

It now remains to describe the different arrows in this complex. Let us justify that they take the following form:
\[ \begin{cases} 
\alpha(P) := \cdot \times \frac{1}{d}(dP)_X(Y) \\
\beta(P) := \cdot \times P - \alpha(P) \circ \delta
\end{cases} \]

On the trivializing open set \( \tilde{V}_i \), the section \( s(P) \) takes, by construction, the following form:
\[ s(P) \equiv \left( \frac{P(X)}{X_i}, \frac{1}{d}(dP)_X(Y) - \frac{Y_i}{X_i} P(X) \right). \]

Under the isomorphism between \( \pi_\ast(\mathcal{E}_{0,d-1}) \) and \( \text{Ker} \delta^2_{1,d-1} \), the section becomes
\[ \left( \frac{1}{d}(dP)_X(Y) - \frac{Y_i}{X_i} P(X) \right) + Y_i \times \left( \frac{P(X)}{X_i} \right) = \frac{1}{d}(dP)_X(Y). \]

Hence the shape of the first map. As for the second map, consider \( t = (t_1, t_2) \in \mathcal{E}_{0,d_1}(\tilde{V}_i) \) a local section. One computes that
\[ s(P) \wedge t \equiv \frac{P(X)}{X_i} t_2 - \left( \frac{1}{d}(dP)_X(Y) - \frac{Y_i}{X_i} P(X) \right) t_1 \]
\[ \equiv \frac{1}{X_i} \left( P(X) \times (t_2 + Y_i t_1) - \alpha(P)(X_i t_1) \right). \]
Now, observe that under the isomorphism between $\pi_*(\mathcal{E}_{0,d-1})$ and $\text{Ker} \delta^2_{1,-d-1}$, $t$ is identified with $t_2 + Y_1 t_1$, where $t_1, t_2 \in \text{Ker} \delta$, and compute accordingly that

$$\delta(t_2 + Y_1 t_1) = X_1 t_1.$$ 

Therefore, one has that, locally,

$$s(P) \wedge t \overset{\text{loc}}{=} \frac{1}{X_1} \beta(P)(t_2 + Y_1 t_1).$$

To conclude, observe that the factor $\frac{1}{X_1}$ comes from the (implicit in the above writing) trivialization of the determinant line bundle $\text{det}(\mathcal{E})^{10}$. 

**Remark 2.3.6.** Using the equality:

$$\delta \circ \alpha - \alpha \circ \delta = \cdot \times P,$$

one easily sees that that the chain of maps (12) is indeed a complex (and well-defined).

Note that the map $\alpha(P)$ is a $(1,d-1)$ operator on $S$, whereas the map $\beta(P)$ is a $(0,d)$-operator on $S$. The fundamental result that we have shown in this section can be summed up as follows:

**Theorem 2.3.7.** The complex

$$0 \longrightarrow \text{Ker} \delta[-1,-2d+1] \overset{\alpha(P)}{\longrightarrow} \text{Ker} \delta^2[0,-d] \overset{\beta(P)}{\longrightarrow} \text{Ker} \delta$$

provides a locally free bi-graded resolution of

$$\bigoplus_{m \geq 1, n \in \mathbb{Z}} S^m \Omega_H(m+n).$$

### 2.4. A second Koszul complex: the case of smooth complete intersections.

Let $X := (P_1 = 0) \cap \ldots \cap (P_c = 0)$ be a smooth complete intersection of multi-degree $d := (d_1, \ldots, d_c)$ in the projective space $\mathbb{P}^N$, with $P_i \in \mathbb{C}[X]$ an homogeneous polynomial of degree $d_i$ in the variables $X := (X_0, \ldots, X_N)$. As in the previous Section 2.3, we define a closed immersion of $\mathbb{P}(TX)$ inside $\text{Flag}_{(1,2)} \mathbb{C}^{N+1} \simeq \mathbb{P}(TP^N)$ as follows:

$$j: \left( \begin{array}{c} \mathbb{P}(TX) \\ ([x], v) \end{array} \right) \longrightarrow \text{Flag}_{(1,2)} \mathbb{C}^{N+1} \left( \begin{array}{c} ([x], (\mathbb{C} \cdot x \oplus \mathbb{C} \cdot v)) \end{array} \right).$$

Once again, the exact same proof of Proposition 1.2.1 gives the following:

**Proposition 2.4.1.** The pull-back line bundle $j^* \mathcal{L}_{m,n}$ on $\mathbb{P}(TX)$ is equal to

$$\mathcal{O}_{\mathbb{P}(TX)}(m) \otimes \pi_X^* \mathcal{O}_X(m+n),$$

where $\pi_X: \mathbb{P}(TX) \to X$ is the canonical projection onto the smooth complete intersection $X$.

This section is organized exactly as in the previous one: we provide the construction of a Koszul complex on $\text{Flag}_{(1,2)} \mathbb{C}^{N+1}$ resolving the structural sheaf $j_* \mathcal{O}_{\mathbb{P}(TX)}$, and give the interpretation of the push-forward complex on the base space $\mathbb{P}^N$ in the case of codimension 2 complete intersections (for sake of simplicity).

---

10Recall that $\mathcal{E}$ is obtained by suitably glueing $\mathcal{L}_{1,0} \mathbb{P}^N \oplus \mathcal{L}_{0,0} \mathbb{P}^N$ on the open sets $\tilde{V}_i$. 

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2.4.1. Construction of the Koszul complex. Following notations of the previous Section 2.3, consider, for \(1 \leq i \leq c\), the global sections \(s_i := s(P_i) \in H^0(\text{Flag}(1,2) \mathbb{C}^{N+1}, \mathcal{E}_{c-1,0})\). They allow to define a global section \(s := (s_1, \ldots, s_c)\) of \(E := E_0, d_1 - 1 \oplus \cdots \oplus E_0, d_c - 1\).

The zero set of the section \(s\) coincide with with \(\mathcal{I}(s) := j(\mathcal{P}(TX))\) since, by construction, it is given by the equations:

\[
\begin{align*}
P_1 &= \cdots = P_c = 0 \\
dP_1 &= \cdots = dP_c = 0
\end{align*}
\]

Furthermore, the smoothness of \(X\) implies that the ideal sheaf \(I(s)\) spanned by these equations is reduced (as an immediate application of, say, the Jacobian criterion). One has therefore the short exact sequence:

\[
0 \to I(s) \to \mathcal{O}_\text{Flag}(1,2) \mathbb{C}^{N+1} \xrightarrow{pr} j_* \mathcal{O}_{\mathcal{P}(TX)} \to 0.
\]

Constructing in the usual fashion the Koszul complex associated to the section \(s\) (see [Laz04][Appendix B]), one obtains accordingly the following complex:

\[
K(s) := 0 \to L_{c-2d+c} \xrightarrow{s^x} (E^d)_{c-2d+c} \xrightarrow{s^\wedge} \cdots \xrightarrow{s^\wedge} L_0, 0,
\]

which provides a locally free resolution of \(\mathcal{O}_{\mathcal{P}(TX)}\).

In order to understand explicitly this complex, we will push it forward by \(\pi : \text{Flag}(1,2) \mathbb{C}^{N+1} \to \mathbb{P}^N\).

However, whereas the push-forward sheaf \(\pi_* E\) has a simple description, as indicated by Proposition 2.3.5 of the previous Section 2.3.2, it is not as simple when it comes to computing the pushed-forward sheaves \(\pi_* E \otimes k\) for \(k \geq 2\) (which are the building blocks of \(\pi_* \Lambda E^d\)).

In the next section, we will give a full description in the case of smooth complete intersection of codimension 2. The path followed can of course be adapted to the general case, but it becomes much more complicated as far as notations, combinatorics, and determination of the push-forward maps are concerned. As we will not need this description in the applications (the Koszul complex upstairs on \(\text{Flag}(1,2) \mathbb{C}^{N+1}\) will be enough), we refrain ourselves of making such a general study.

2.4.2. Interpretation of the (twisted) Koszul complex on the base: the case of smooth complete intersections of codimension 2. In exactly the same fashion as in the beginning of Section 2.3, for any \(i \in \mathbb{N}_{\geq 1}\), and any \(r \in \mathbb{Z}, s \in \mathbb{N}\), we can push-forward the Koszul complex (14), and keep an exact complex.

In order to understand the push-forward on the base of the Koszul complex, we first give a satisfying description of the push-forward sheaf \(\pi_* ((E^\otimes k)_{r,s})\). We start with the following lemma, which generalizes the construction in Proposition 2.3.5:

**Lemma 2.4.2.** For any \(k \in \mathbb{N}_{\geq 1}\), for any \(s \in \mathbb{Z}\) and any \(r \in \mathbb{N}\), there is a natural map of locally free sheaves

\[
\pi_* ((E^\otimes k)_{r,s}) \xrightarrow{\Theta} \text{Ker} \delta^\otimes_{r+k+1}.
\]

**Proof.** Recall that, by definition, the locally free sheaf \(E\) is obtained by suitably glueing the free sheaves \((L_0, 0) \oplus (L_1, 0))_{V_i}\).
on the covering open sets $\tilde{V}_i = \pi^{-1}(\{X_i \neq 0\})$ (see Section 2.3). Note that there are canonical isomorphisms for any $0 \leq i \leq N$:

\[
(\mathcal{L}_{0,0} \oplus \mathcal{L}_{1,0})|_{\tilde{V}_i} \simeq \bigoplus_{\alpha \in \{0,1\}^k} (\mathcal{L}_{|\alpha|,0})|_{\tilde{V}_i},
\]

where $|\alpha|$ is the sum of the coordinates of $\alpha$ (i.e., the number of coordinates equal to one). Twisting by $\mathcal{L}_{s,r}$ and pushing forward by $\pi$, one gets the following canonical isomorphism on each open set $V_i$, $0 \leq i \leq N$:

\[
(\mathcal{E}^\otimes k)_{r,s}|_{V_i} \simeq \bigoplus_{\alpha \in \{0,1\}^k} (\text{Ker} \, \delta_{r+|\alpha|,s})|_{V_i}.
\]

For notational reasons, adopt the following formalism. Fix $e_0$ and $e_1$ two formal symbols, and denote for $\alpha \in \{0,1\}^k$:

\[
e_\alpha := e_{a_0} \otimes \cdots \otimes e_{a_k}.
\]

Rewrite then the above isomorphism (15) as follows:

\[
(\mathcal{E}^\otimes k)_{r,s}|_{V_i} \simeq \bigoplus_{\alpha \in \{0,1\}^k} (\text{Ker} \, \delta_{r+|\alpha|,s})|_{V_i} \cdot e_\alpha.
\]

Denote then $A^*_i := \begin{pmatrix} \frac{X_i}{X_jY_iY_j} & 0 \\ \frac{X_j}{X_iY_iY_j} & 1 \end{pmatrix}$, and interpret this matrix as an isomorphism of the $\mathbb{C}(X)[Y]$ vector space of dimension 2 with canonical basis

\[
\mathbb{C}(X)[Y] \cdot e_0 \oplus \mathbb{C}(X)[Y] \cdot e_1.
\]

With these notations, the transition maps between the above trivializations (16) are given by the matrix tensor product $(A^*_i)^{\otimes k}$. In other words, for any $s \in (\text{Ker} \, \delta_{r+|\alpha|,s})|_{V_i}$, the element

\[
s \cdot e_\alpha = s \cdot (e_{a_0} \otimes \cdots \otimes e_{a_k})
\]

becomes, after a change of trivialization from the $i$th chart to the $j$th chart,

\[
s \cdot (A^*_i e_{a_0} \otimes \cdots \otimes A^*_i e_{a_k}).
\]

There is then a natural map from $\pi_*(\mathcal{E}^\otimes k)_{r,s}$ to $\text{Ker} \, \delta_{r+k,s}$ defined as follows. On each trivializing open set $V_i$, consider the map

\[
\Theta_i: \bigoplus_{\alpha \in \{0,1\}^k} (\text{Ker} \, \delta_{r+|\alpha|,s})|_{V_i} \cdot e_\alpha \longrightarrow (\text{Ker} \, \delta_{r+k,s})|_{V_i} \cdot \sum_{\alpha \in \{0,1\}^k} e_{s \alpha}.
\]

One easily sees that the image of $\Theta_i$ does indeed lie in $\text{Ker} \, \delta_{r+k,s}|_{V_i}$. Furthermore, the maps $\Theta_i$ glue to a global map. Indeed, by linearity, it suffices to show the following equality for any $\alpha \in \{0,1\}^k$ and any $s \in (\text{Ker} \, \delta_{r+|\alpha|,s})|_{V_i \cap V_j}$:

\[
\Theta_j(s \cdot A^*_i e_{a_0} \otimes \cdots \otimes A^*_i e_{a_k}) = \Theta_i(s \cdot e_\alpha).
\]

Suppose without loss of generality that $\alpha = (0, \ldots, 0, 1, \ldots, 1)$. One has that

\[
A^*_i e_{a_0} \otimes \cdots \otimes A^*_i e_{a_k} = \left( \frac{X_i}{X_j} e_0 + \frac{X_i Y_i - X_j Y_j}{X_j} e_1 \right) \otimes \cdots \otimes e_1 \otimes \cdots \otimes e_1 \otimes \cdots \otimes e_1.
\]

\text{It will "dualize" the notations used so far, but it is convenient for the proof here.}
Then, one computes that:

\[
\Theta_j(s \cdot A_{ij}^* e_{\alpha_0} \otimes \cdots \otimes A_{ij}^* e_{\alpha_k}) = s \sum_{p=0}^{\ell} \binom{\ell}{p} Y_j^p \left( \frac{X_i}{X_j} \right)^p \left( \frac{X_j Y_i - X_i Y_j}{X_j} \right)^{\ell-p} = s Y_j^\ell = \Theta_j(s \cdot e_\alpha).
\]

This concludes the proof. □

We can now provide a convenient description of the various twists of the push-forward sheaf \(\pi_* (E \otimes k)\) for \(k = 2\):

**Proposition 2.4.3.** For any \(s \in \mathbb{Z}\) and any \(r \in \mathbb{N}\), the locally free sheaf \(\pi_* ((E \otimes 2)_{s,r})\) is isomorphic to

\[
\text{Ker} \, \delta_{r+2,s}^3 \oplus \text{Ker} \, \delta_{r+1,s+1}^3.
\]

**Proof.** Keep the notations introduced in the proof of Lemma 2.4.2, and consider the map

\[
\Theta : \pi_* ((E \otimes 2)_{s,r}) \to \text{Ker} \, \delta_{r+2,s}^3.
\]

Recall that, locally on the open sets \(V_i\), it is defined as follows:

\[
\text{Ker} \, \delta_{r,s} |_{V_i} \oplus \text{Ker} \, \delta_{r+1,s} |_{V_i} \oplus \text{Ker} \, \delta_{r+2,s} |_{V_i} \to \text{Ker} \, \delta_{r+2,s}^3 |_{V_i}
\]

\[
(s_0, s_1, s_2, s_3) \mapsto s_0 Y_i^2 + (s_1 + s_2) Y_i + s_3.
\]

Consider also the following map of locally free sheaves

\[
\Delta : \pi_* ((E \otimes 2)_{s,r}) \to \text{Ker} \, \delta_{r+1,s+1}^3
\]

defined on each open set \(V_i\) as follows:

\[
\text{Ker} \, \delta_{r,s} |_{V_i} \oplus \text{Ker} \, \delta_{r+1,s} |_{V_i} \oplus \text{Ker} \, \delta_{r+2,s} |_{V_i} \to \text{Ker} \, \delta_{r+1,s+1}^3 |_{V_i}
\]

\[
(s_0, s_1, s_2, s_3) \mapsto X_i(s_2 - s_1).
\]

Note that it is well defined: on the \(j\)th chart, the element \((0, s_1, s_2, 0)\) becomes

\[
(0, X_i / s_1, X_i / s_2, s),
\]

so that \(X_i(s_2 - s_1) = X_j(s_j^2 s_2 - X_i s_1)\).

It remains to justify that the morphism of locally free sheaves

\[
(\Theta, \Delta) : \pi_* ((E \otimes 2)_{s,r}) \to \text{Ker} \, \delta_{r+2,s}^3 \oplus \text{Ker} \, \delta_{r+1,s+1}^3
\]

is an isomorphism. For the injectivity, suppose that

\[
s = (s_0, s_1, s_2, s_3) \in \text{Ker} \, \delta_{r,s} |_{V_i} \oplus \text{Ker} \, \delta_{r+1,s} |_{V_i} \oplus \text{Ker} \, \delta_{r+2,s} |_{V_i}
\]

satisfies \(\Theta(s) = \Delta(s) = 0\), namely:

\[
\begin{cases}
  s_0 Y_i^2 + (s_1 + s_2) Y_i + s_3 = 0; \\
  X_i(s_2 - s_1) = 0.
\end{cases}
\]
The second equation implies $s_2 = s_1$. Applying successively $\delta^2$ and $\delta$ to the first equation implies successively the equalities $s_0 = s_1 = s_2 = s_3 = 0$, which proves the injectivity. As for the surjectivity, let $(u,v) \in (\ker \delta_{r+2,s} | V_i) \oplus (\ker \delta_{r+1,s+1} | V_i)$. Consider

$$
\begin{align*}
  s_0 &= \frac{\delta^2(u)}{X_i^2}, \\
  s_1 &= \frac{\delta(u)}{X_i} - \frac{\delta^2(u)Y_i}{X_i^2} - v, \\
  s_2 &= \frac{\delta(u)}{X_i} - \frac{\delta^2(u)Y_i}{X_i^2} + v, \\
  s_3 &= u - (s_2 + s_1)Y_i - s_0Y_i^2
\end{align*}
$$

and denote $s = (s_0, s_1, s_2, s_3)$. By construction, one has that $(\Theta, \Delta)(s) = (u, v)$. One then checks immediately that $\delta(s_0) = \delta(s_1) = \delta(s_2) = 0$. Furthermore, compute that

$$
\delta(s_3) = \delta(u) - (s_1 + s_2)X_i - 2s_0Y_iX_i = \delta(u) - \delta(u) - \delta^2(u)\frac{Y_i}{X_i} - \delta^2(u)\frac{Y_i}{X_i} = 0.
$$

This proves the surjectivity, and finishes the proof. \(\square\)

Let us now go back to our Koszul complex (14) (where $d = (d_1, d_2)$ as we work in codimension 2 here):

$$
L_{-2,-2|d|+2} \xrightarrow{s_{X}} (E^d)_{-2,-2|d|+2} \xrightarrow{s_{Y}} (\bigwedge^2 E^d)_{-2,-2|d|+2} \xrightarrow{s_{Y}} (\bigwedge^3 E^d)_{-2,-2|d|+2} \xrightarrow{s_{Y}} L_{0,0}.
$$

From elementary multi-linear algebra, one has the isomorphisms

$$
\begin{align*}
\bigwedge^2 E^d &\simeq L_{1,2d_1-1} \oplus L_{1,2d_2-1} \oplus (E^{\otimes 2})_{0,|d|-2}, \\
\bigwedge^3 E^d &\simeq E_{1,d_1+2d_2-2} \oplus E_{1,d_1+2d_1-2}.
\end{align*}
$$

Let $m \in \mathbb{N}_{\geq 2}$, $n \in \mathbb{Z}$, and twist the above complex (17) by $L_{m,n}$. Using the isomorphisms given in Propositions 2.3.5 and 2.4.3, we see that the (twisted) push-forward Koszul complex $\pi_*(K(s)_{m,n})$ takes the following form:

$$
\begin{align*}
\ker \delta_{m-2,n-2|d|+2} &\xrightarrow{(f_{11}, f_{22})} \ker \delta^2_{m-1,n-d_1-2d_2+1} \oplus \ker \delta^2_{m-1,n-d_2-2d_1+1} \xrightarrow{(f_{31})_{1\leq 4}} \\
\ker \delta_{m-1,n-2d_1+1} &\oplus \ker \delta^3_{m,n-|d|} \oplus \ker \delta_{m-1,n-|d|+1} \\
\ker \delta_{m,n-|d|+1} &\xrightarrow{(f_{11}, f_{22})} \ker \delta^2_{m,n-d_1} \oplus \ker \delta^2_{m,n-d_2} \xrightarrow{f_{i}} \ker \delta_{m,n},
\end{align*}
$$

and it provides a locally free resolution of $S^m \Omega_X (m + n)$.

In order to have a complete picture of the push-forward Koszul complex, it remains to describe the arrows. Recall first that for any polynomial $P \in \mathbb{C}[X]$, we denoted:

$$
\begin{align*}
\alpha(P) &= \cdot \times (dP)_X(Y), \\
\beta(P) &= \cdot \times P - \alpha(P) \circ \delta.
\end{align*}
$$
It is a (tedious) verification, that we leave to the reader\(^{12}\), to check that the maps in the
complex (18) are the following:

\[
\begin{cases}
  f_{1i} = \alpha(P_i), 1 \leq i \leq 2; \\
  f_{21}(A, B) = \beta(P_2)(B) \text{ and } f_{22}(A, B) = \beta(P_1)(A); \\
  f_{23}(A, B) = \alpha(P_2)(A) - \alpha(P_1)(B); \\
  f_{24}(A, B) = \beta(P_2)(A) + \beta(P_1)(B); \\
  f_{31}(A, B, C, D) = \alpha(P_1)(A) + \frac{1}{4}(\beta(P_2)(C) + P_3C - \alpha(P_2)(D)); \\
  f_{32}(A, B, C, D) = \alpha(P_2)(B) - \frac{3}{2}(\beta(P_1)(C) + P_1C + \alpha(P_1)(D)); \\
  f_4(A, B) = \beta(P_1)(A) + \beta(P_2)(B).
\end{cases}
\]

We can thus summarize the main result of this section in the following statement:

**Theorem 2.4.4.** The complex

\[
0 \longrightarrow \text{Ker} \, \delta[-2, -2|d| + 2] \longrightarrow \\
\text{Ker} \, \delta^2[-1, -2|d| + d_2 + 1] \oplus \text{Ker} \, \delta^2[-1, -2|d| + d_1 + 1] \longrightarrow \\
\bigoplus_i (\oplus \text{Ker} \, \delta[-1, -2d_i + 1]) \oplus \text{Ker} \, \delta^3[0, -|d|] \oplus \text{Ker} \, \delta[-1, -|d| + 1] \longrightarrow \\
\text{Ker} \, \delta,
\]

whose arrows are given in (19), provides a locally free bi-graded resolution of

\[
\bigoplus_{m \geq 2, n \in \mathbb{Z}} S^m \Omega_X(m + n).
\]

3. **Cohomology of twisted symmetric powers of cotangent bundles of complete intersections.**

3.1. **A first complex computing cohomology: the case of smooth hypersurfaces.** For this whole part, we fix \( H := \{ P = 0 \} \subset \mathbb{P}^N \) a smooth hypersurface of degree \( d \geq 1 \).

3.1.1. **A complex computing the cohomology.** In order to compute cohomology of (negatively) twisted symmetric powers of cotangent bundles of hypersurfaces, note that as soon as the following inequality is satisfied

\[ (*) \quad n < -1, \]

the cohomology of the two terms in the Koszul complex \( K(\alpha(P)) \) (see Theorem 2.1.2) is supported in maximal degree, i.e. in degree \( N - 1 \). Indeed, this follows from the short exact sequence

\[
0 \longrightarrow \text{Ker} \, \delta_{r,s-d} \longrightarrow \text{Ker} \, \delta_{r,s} \longrightarrow (\text{Ker} \, \delta_{r,s})|_H \longrightarrow 0,
\]

combined with Theorem 1.3.1, once one has taken the long exact sequence in cohomology. Recall indeed that Theorem 1.3.1 implies that the cohomology of

\[
\text{Ker} \, \delta_{r,s} \simeq S^r \Omega_{\mathbb{P}^N}(r + s)
\]
is supported in degree \( N \) as soon as \( s < -1 \).

\(^{12}\)It is similar to the verification made in the case of hypersurfaces, but more involved.
For the rest of this section, we suppose that the condition $(\ast)$ is satisfied, so that the cohomology of the terms in the Koszul complex $K(\alpha(P))$ is supported in maximal degree. An easy exercise in cohomological algebra allows then to deduce that the cohomology of $S^m\Omega_H(m+n)$ is read off through the following complex:

\[(20) \quad H^{N-1}(H,(\text{Ker}\delta_{m-1,n-d+1}))(H) \xrightarrow{\alpha(P)} H^{N-1}(H,(\text{Ker}\delta_{m,n}))(H).\]

Namely, the kernel is isomorphic to $H^{N-2}(H,S^m\Omega_H(m+n))$, and the cokernel is isomorphic to $H^{N-1}(H,S^m\Omega_H(m+n))$.

3.1.2. Reformulation of the complex (20) using Serre duality. Let us recall that for any $i \geq N+1$, the $N$th cohomological group $H^N(P^N,\mathcal{O}_{P^N}(-i))$ has for natural basis the set of Laurent polynomials $(\frac{1}{X^\gamma})_\gamma$ with $\gamma \in \mathbb{N}^{N+1}$ satisfying $\gamma_j \geq 1$ and $|\gamma| = i$.

One sees that there is therefore a natural identification between $H^N(P^N,\mathcal{O}_{P^N}(-i))$ and $H^0(P^N,\mathcal{O}_{P^N}(i-(N+1)))$ (or rather its dual): this is the very first manifestation of Serre duality, and it takes the following form:

\[
\left( H^N(P^N,\mathcal{O}_{P^N}(-i)) \rightarrow H^0(P^N,\mathcal{O}_{P^N}(i-(N+1)))^\vee \right),
\]

where by definition, for any $\alpha \in \mathbb{N}^{N+1}$, the symbol $(X^\alpha)^*$ denotes the linear functional on $\mathbb{C}[X]$ taking the value 1 on $X^\alpha$ and zero elsewhere. We have then the following elementary lemma:

**Lemma 3.1.1.** Let $i \in \mathbb{N}$, and let $P$ be an homogenous polynomial of degree $d$ in $N+1$ variables. Under the identification given by Serre duality, the dual of the map in $N$th cohomology induced by the multiplication map

\[
\mathcal{O}_{P^N}(-i) \xrightarrow{\cdot X^\alpha} \mathcal{O}_{P^N}(-i+d)
\]

is simply the multiplication by $P$:

\[
\left( H^0(P^N,\mathcal{O}_{P^N}(i-d-(N+1))) \rightarrow H^0(P^N,\mathcal{O}_{P^N}(i-(N+1))) \right).
\]

**Proof.** By linearity, it suffices to check the statement for a monomial $X^\alpha$, $|\alpha| = d$. Let $\beta \in \mathbb{N}^{N+1}$, with $\beta_j \geq 1$ and $|\beta| = i$. Then one has that, at the $N$th cohomological level,

\[
X^\alpha : \frac{1}{X^\beta} = \begin{cases} 0 & \text{if } \beta_j \leq \alpha_j \text{ for some } j \\ \frac{1}{X^{\beta-\alpha}} & \text{otherwise} \end{cases}.
\]

Under the isomorphism given by Serre duality, this says that for $(X^\beta)^* \in H^0(P^N,\mathcal{O}_{P^N}(i-(N+1)))^\vee$, the action of the multiplication by the monomial $X^\alpha$ is the following

\[
X^\alpha \cdot (X^\beta)^* = \begin{cases} 0 & \text{if } \beta_j < \alpha_j \text{ for some } j \\ (X^{\beta-\alpha})^* & \text{otherwise} \end{cases}.
\]

Now, let $X^\gamma \in H^0(P^N,\mathcal{O}_{P^N}(i-d-(N+1)))$ and $(X^\beta)^* \in H^0(P^N,\mathcal{O}_{P^N}(i-(N+1)))^\vee$.

The dual map of the multiplication by $X^\alpha$ acts as follows:

\[
((\cdot \times X^\alpha)^*(X^\gamma))((X^\beta)^*) = (X^\alpha \cdot (X^\beta)^*)(X^\gamma) = \begin{cases} 1 & \text{if } \beta = \alpha + \gamma \\ 0 & \text{otherwise} \end{cases}.
\]

Therefore, one indeed has that $(\cdot \times X^\alpha)^*(X^\gamma) = X^{\alpha+\gamma}$, which proves the lemma. ∎

With this small reminder on Serre duality on projective spaces, we can now give a convenient description of the spaces appearing in the complex (20):
Lemma 3.1.2. Let \( r \in \mathbb{N} \), and \( s \in \mathbb{Z} \). There is a natural isomorphism:

\[
H^{N-1}(H, (\text{Ker} \delta_{r,s}))[H]^\vee \simeq \frac{S_{r,s}-(N+1)+d}{(P,q)},
\]

where one recalls that \( S := \mathbb{C}[Y, X] \), and \( q = \sum_{i=0}^{N} X_i Y_i \).

Proof. Via the short exact sequence

\[
0 \to \text{Ker} \delta_{r,s-d} \to \text{Ker} \delta_{r,s} \to (\text{Ker} \delta_{r,s})[H] \to 0,
\]

and Theorem 1.3.1, one deduces that \( H^{N-1}(H, (\text{Ker} \delta_{r,s}))[H] \) is isomorphic to

\[
\text{Ker} \left( H^N(P^N, \text{Ker} \delta_{r,s-d}) \to H^N(P^N, \text{Ker} \delta_{r,s}) \right).
\]

On the other hand, for any \( r \geq 0 \) and any \( s \in \mathbb{Z} \), the vector space \( H^N(P^N, \text{Ker} \delta_{r,s}) \) is isomorphic to

\[
\text{Ker} \left( \mathbb{C}[Y]_r \otimes H^N(P^N, \mathcal{O}_{P^N}(s)) \to \mathbb{C}[Y]_{r-1} \otimes H^N(P^N, \mathcal{O}_{P^N}(s+1)) \right).
\]

One can thus form the following commutative diagram

(21)

\[
\begin{array}{ccc}
0 & \to & H^{N-1}(H, (\text{Ker} \delta_{r,s}))[H] \\
\downarrow & & \downarrow \\
0 & \to & H^N(P^N, \text{Ker} \delta_{r,s-d}) \\
\downarrow & & \downarrow \\
0 & \to & H^N(P^N, \text{Ker} \delta_{r,s})
\end{array}
\]

where the columns and lines are exact.

Now, by dualizing everything, using Serre duality and Lemma 3.1.1, the above diagram becomes:

\[
\begin{array}{ccc}
0 & \to & H^{N-1}(H, (\text{Ker} \delta_{r,s}))[H]^\vee \\
\downarrow & & \downarrow \\
0 & \to & H^N(P^N, \text{Ker} \delta_{r,s-d})^\vee \\
\downarrow & & \downarrow \\
0 & \to & H^N(P^N, \text{Ker} \delta_{r,s})^\vee
\end{array}
\]

As in Section 1.4, use the renormalization map

\[
u: \left( \begin{array}{c} \mathbb{C}[Y] \\ Y^\beta \end{array} \right) \to \left( \begin{array}{c} \mathbb{C}[Y] \\ \frac{Y^\beta}{\partial P} \end{array} \right)
\]
to change the two columns in the above diagram, so that it becomes:

\[
\begin{array}{cc}
0 & 0 \\
S_{r-1,-s-1-(N+1)+d} & S_{r-1,-s-1-(N+1)} \\
\cdot \times q & \cdot \times q \\
S_{r,-s-(N+1)+d} & S_{r,-s-(N+1)} \\
0 & 0 \\
\end{array}
\]

\[\begin{array}{c}
H^{N-1}(H, (\text{Ker } \delta_{r,s})|_H)^\vee \\
H^N(P^N, \text{Ker } \delta_{r,s-d})^\vee \\
H^N(P^N, \text{Ker } \delta_{r,s})^\vee \\
0 \\
0
\end{array}\]

The lemma now follows immediately from the above diagram. \(\square\)

Let us now use dualize the complex (20). Using the above Lemma 3.1.2, it becomes:

\[(22)\]

\[\frac{S_{m,n-(N+1)+d}}{(P^q)} \xrightarrow{\alpha^*(P)} \frac{S_{m-1,n-(N+1)+2d-1}}{(P^q)}.\]

It remains to understand the induced map, that we denoted by \(\alpha^*(P)\):

**Lemma 3.1.3.** The map \(\alpha^*(P)\) is induced by the map (still denoted \(\alpha^*(P)\) by a slight abuse of notations)

\[\alpha^*(P): \left(\begin{array}{c}
S \\
A
\end{array}\right) \mapsto \frac{1}{d} \left(\sum_{i=0}^{N} \frac{\partial P_i}{\partial Y_i}(A)\right),\]

where one recalls that \(S := \mathbb{C}[Y, X].\)

**Remark 3.1.4.** In other words, up to a non-zero multiplicative factor, this is the map induced by \(\delta(\gamma(P))(P^N),\) where \(\gamma(P)\) is the Gauss map associated to the hypersurface \(H = \{ P = 0 \}\) (see Section 1.6 for notations).

**Proof.** Consider the commutative diagram (21) introduced in the previous Lemma 3.1.2, which provides a convenient description of \(H^{N-1}(H, (\text{Ker } \delta_{r,s})|_H).\) The map induced (at the cohomological level) by the multiplication by \(\frac{1}{d}(d P)|_X(Y)\)

\[H^{N-1}(H, (\text{Ker } \delta_{r,s})|_H) \xrightarrow{\alpha(P)} H^{N-1}(H, (\text{Ker } \delta_{r+1,s+d-1})|_H)\]

is induced by the map

\[\mathbb{C}[Y]_r \otimes H^N(P^N, \mathcal{O}_{P^N}(s-d)) \xrightarrow{\alpha(P)} \mathbb{C}[Y]_{r+1} \otimes H^N(P^N, \mathcal{O}_{P^N}(s-1)).\]

This map fits into the following commutative diagram:

\[
\begin{array}{ccc}
\mathbb{C}[Y]_r \otimes H^{N}(P^N, \mathcal{O}_{P^N}(s-d)) & \xrightarrow{\alpha(P)} & \mathbb{C}[Y]_{r+1} \otimes H^{N}(P^N, \mathcal{O}_{P^N}(s-1)) \\
\text{Serre duality} & & \text{Serre duality} \\
\mathbb{C}[Y]_r \otimes H^{0}(P^N, \mathcal{O}_{P^N}(-s-(N+1)+d))^\vee & \xrightarrow{u \text{ (renormalization map)}} & \mathbb{C}[Y]_{r+1} \otimes H^{0}(P^N, \mathcal{O}_{P^N}(-s-N))^\vee \\
\text{u (renormalization map)} & & \text{u (renormalization map)} \\
\mathbb{C}[Y]_r \otimes H^{0}(P^N, \mathcal{O}_{P^N}(-s-(N+1)+d))^\vee & \xrightarrow{m(P)} & \mathbb{C}[Y]_{r+1} \otimes H^{0}(P^N, \mathcal{O}_{P^N}(-s-N))^\vee.
\end{array}
\]

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It is thus enough to understand the dual of the map $m(P)$, and the goal is to show the following equality:

$$m(P)^* = \alpha^*(P).$$

Note that the correspondances $P \mapsto m(P)$ and $P \mapsto \alpha^*(P)$ are linear in $P$, so that it suffices to prove this equality for monomials in $\mathbb{C}[X]_d$. The proof now follows from Lemma 3.1.1, and the same computation as the one carried over in Section 1.4 on the dualized generalized Euler exact sequence.

\[\square\]

**Remark 3.1.5.** More generally, under the Serre duality isomorphism and the renormalization map, the action of the multiplication by a polynomial $Q(Y,X)$ on the top cohomology group corresponds to the partial differential equation $Q(\partial_{Y},X)$ on the $0$th cohomology group. This is the content of the proof of the previous Lemma 3.1.3.

We have therefore shown the following theorem:

**Theorem 3.1.6.** The cohomology of $\bigoplus_{m \geq 1, n > 1} S^m \Omega_H^{m-n}$ can be computed via the graded (Koszul) complex $K(\alpha^*(P))[-1, -(N+1) + 2d - 1]$:

$$S(P_q)[0, -(N+1) + d] \xrightarrow{\alpha^*(P)} S(P_q)[-1, -(N+1) + 2d - 1].$$

Namely, the $i$th cohomology group of one graded component of $\bigoplus_{m \geq 1, n > 1} S^m \Omega_H^{m-n}$ is isomorphic to the $(N-1-i)$th cohomology group of the corresponding graded part of the graded complex.

3.2. **A first complex computing cohomology: the case of smooth complete intersections.** The same line of reasoning works for smooth complete intersections. Keeping the notations of Section 2.2, consider the push-forward twisted Koszul complex

$$(\pi_X)_*(K(s))_{m,n},$$

which provides a locally free resolution of $S^m \Omega_X^{m+n}$ for $m \geq c$ and any $n \in \mathbb{Z}$. Observe that the pieces in the locally free resolution $(\pi_X)_*(K(s))_{m,n}$ all write as a direct sum of terms of the following form:

$$(\text{Ker} \delta_{r,s})|_X,$$

where $r \leq m$ and $s \leq n$.

Suppose that the condition $(*)$ holds, i.e. that the following inequality holds:

$$n < -1.$$

By twisting by $\text{Ker} \delta_{r,s}$ the Koszul complex

$$(\pi_X)_*(K(P_1, \ldots, P_c))$$

resolving $O_X$ (recall that $P_1, \ldots, P_c$ are the homogeneous polynomials defining the complete intersection $X$), one obtains a locally free resolution of $(\text{Ker} \delta_{r,s})|_X$. Under the condition $(*)$, for any $r \leq m$ and any $s \leq n$, the cohomology of the pieces in the locally free complex resolving $(\text{Ker} \delta_{r,s})|_X$

$$K(P_1, \ldots, P_c) \otimes \text{Ker} \delta_{r,s}$$

is supported in degree $N$. This allows to show that the cohomology of $(\text{Ker} \delta_{r,s})|_X$ is supported in degree $\geq N-c$. On the other hand, since $\dim(X) = N-c$, the cohomology of $(\text{Ker} \delta_{r,s})|_X$ is supported in degree less or equal than $N-c$.

Therefore, the cohomology of the pieces appearing in the resolution $(\pi_X)_*(K(s))_{m,n}$ is supported in maximal degree, i.e. in degree $N-c = \dim(X)$. Accordingly, we deduce
that the cohomology of $S^m\Omega_X(m + n)$ is read off through the complex obtained by applying the functor $H^{-c}(X, \cdot)$ to the locally free resolution $(\pi_X)_*(\mathcal{K}(s)_{m,n})$.

Using Serre duality as in the case of hypersurfaces, we can reformulate this complex as follows. The analogous of Lemma 3.1.2 takes the following form:

**Lemma 3.2.1.** Let $r \in \mathbb{N}$, and $s \in \mathbb{Z}$. There is a natural isomorphism:

$$H^{N-c}(X, (\text{Ker} \, \delta_{r,s})|_X)^\vee \simeq \frac{S_{r,-s-(N+1)+|d|}}{(P_1, \ldots, P_c, q)}$$

**Proof.** The proof is completely similar the one of Lemma 3.1.2. Namely, consider the resolution $K(P_1, \ldots, P_c) \otimes \text{Ker} \, \delta_{r,s}$ of $(\text{Ker} \, \delta_{r,s})|_X$, which allows to give a convenient description of $H^{N-c}(X, (\text{Ker} \, \delta_{r,s})|_X)$. By dualizing this description, using Serre duality and Lemma 3.1.1, the result follows. □

As in the case of hypersurfaces, the maps in the dualized complex

$$H^{N-c}(X, (\pi_X)_*(\mathcal{K}(s_1, \ldots, s_c)(m, n)))^\vee$$

are induced (under the isomorphism of Lemma 3.2.1) by the maps $\alpha^*(P_1), \ldots, \alpha^*(P_c)$.

**Example 3.2.2.** For $c = 2$, the complex $H^{N-c}(X, (\pi_X)_*(\mathcal{K}(s)_{m,n}))^\vee$ takes the following form:

$$\frac{S_{m-2,-n-(N+1)+2d_1+d_2-1}}{(P_1, P_2, q)} \oplus \frac{S_{m-1,-n-(N+1)+2d_1+d_2-1}}{(P_1, P_2, q)}$$

Recall that the cohomology of this complex computes the cohomology of $S^m\Omega_X(m + n)$ as soon as $n < -1$ and $m \geq c = 2$. Namely:

- the first cohomology group in the above complex is isomorphic to $H^{N-2}(X, S^m\Omega_X(m + n))$;
- the middle cohomology group in the above complex is isomorphic to $H^{N-3}(X, S^m\Omega_X(m + n))$;
- the last cohomology group in the above complex is isomorphic to $H^{N-4}(X, S^m\Omega_X(m + n))$,

and all the other cohomology groups of the vector bundle $S^m\Omega_X(m + n)$ are zero.

More generally, note that for any $1 \leq i \leq c$, the map $\alpha^*(P_i)$ induces a map of bigraded algebra:

$$\alpha^*(P_i) : \frac{S}{(P_1, \ldots, P_c, q)}[-1, -d_i + 1] \to \frac{S}{(P_1, \ldots, P_c, q)}.$$

We can therefore consider the Koszul complex

$$K(\alpha^*(P_1), \ldots, \alpha^*(P_c)),$$

which, by what we just saw, computes the cohomology of $S^m\Omega_X(m + n)$ for $m \geq c$, and $n < -1$:

**Theorem 3.2.3.** The cohomology of $\bigoplus_{m \geq c, n > 1} S^m\Omega_X(m - n)$ can be computed via the Koszul complex on $\frac{S[Y, X]}{(P_1, \ldots, P_c, q)}$:

$$K(\alpha^*(P_1), \ldots, \alpha^*(P_c))[-c, -(N + 1) + 2|d| - c].$$
Namely, the $i$th cohomology group of one graded component of $\bigoplus_{m \geq 1, n > 1} S^m\Omega_X(m - n)$ is isomorphic to the $(N - c - i)$th cohomology group of the corresponding graded part of the Koszul complex.

### 3.3. A second complex computing cohomology: the case of smooth hypersurfaces.

We consider the same setting as in Section 3.1: we let $H = \{ P = 0 \} \subset \mathbb{P}^N$ be a smooth hypersurface, and fix $m \in \mathbb{N}_{\geq 1}$ and $n \in \mathbb{Z}$ two natural numbers satisfying the following condition:

\[(*) \quad n < -1.\]

This time, instead of considering the first resolution given in Section 2.1, we use the second resolution given in Section 2.3. The reasoning is then completely similar to the one carried over in Section 3.1. Namely, we first apply the functor $H^N(\mathbb{P}^N, \cdot)$ to the resolution (11) of $S^m\Omega_X(m + n)$, which we recall here:

\[
\begin{array}{c}
\pi_s(K(s)_{m,n}) : 0 \to \text{Ker} \delta_{m-1,n-2d+1} \to \pi_s(\mathcal{E}_{m-1,n-d}) \to \text{Ker} \delta_{m,n}.
\end{array}
\]

Let us justify that, under the condition $(*)$, the complex $H^N(\mathbb{P}^N, \pi_s(K(s)_{m,n}))$ does indeed compute the cohomology. This will follow from the following elementary lemma:

**Lemma 3.3.1.** Let $m \in \mathbb{Z}, n \in \mathbb{Z}$ and $k \in \mathbb{N}$. The cohomology of $(\mathcal{E}^{\otimes k})_{m,n}$ is supported in degree 0, 1 and $N$. Furthermore, if one supposes that $n < -1 - k$, then the cohomology is supported in degree $N$.

**Proof.** Proceed by induction on $k$. For $k = 0$, all statements follow immediately from Bott’s formula (see Appendix A). Suppose that the results holds for $k - 1 \geq 0$. By Proposition 2.3.3, one deduces that the vector bundle $(\mathcal{E}^{\otimes k})_{m,n}$ fits into the short exact sequence:

\[(23) \quad 0 \to (\mathcal{E}^{\otimes (k-1)})_{m+1,n} \to (\mathcal{E}^{\otimes k})_{m,n} \to (\mathcal{E}^{\otimes (k-1)})_{m,n+1} \to 0.
\]

This immediately implies that the cohomology of $(\mathcal{E}^{\otimes k})_{m,n}$ remains supported in degree 0, 1 and $N$.

Suppose now that $n < -1 - k$. By induction hypothesis, the cohomology of the two extremities of the short exact sequence (23) is supported in degree $N$. Therefore, so does the cohomology of the middle term of the short exact sequence: this finishes the proof.

For instance, if we consider the term $\pi_s(\mathcal{E}_{m-1,n-d})$ in the complex $\pi_s(K(s)_{m,n})$, the above Lemma 3.3.1 (combined with Bott’s formula) implies that its cohomology is indeed supported in maximal degree, since by hypothesis one has $n - d \leq n - 1 < -2$. The two other terms in the complex are treated similarly.

Now, the analogue of Lemma 3.1.2 is the following:

**Lemma 3.3.2.** Let $r \in \mathbb{N}, s \in \mathbb{Z}$. Then one has the following isomorphism\textsuperscript{13}

\[H^N(\mathbb{P}^N, \text{Ker} \delta_{r,s}^k) \simeq \left( \frac{S}{(q^k)} \right)_{r-s-(N+1)}.
\]

**Proof.** Consider the short exact sequence

\[
0 \to \text{Ker} \delta_{r,s}^k \to C[Y]_r \otimes \mathcal{O}_{\mathbb{P}^N}(s) \to C[Y]_{r-k} \otimes \mathcal{O}_{\mathbb{P}^N}(s+k) \to 0.
\]

Taking the long exact sequence in cohomology, one sees that

\[
H^N(\mathbb{P}^N, \text{Ker} \delta_{r,s}^k) \simeq \text{Ker} \left( C[Y]_r \otimes H^N(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(s)) \to C[Y]_{r-k} \otimes H^N(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(s+k)) \right).
\]

\textsuperscript{13}Recall that throughout the paper (see Conventions), $N$ is supposed to be greater or equal than 2.
Using Serre duality, and the renormalization map \( u \) (see Section 1.4), one deduces the lemma.

With the previous lemma, the complex \( H^N(P^N, \mathcal{K}(s)(m,n))^\vee \) writes as follows\(^{14}\)

\[
\begin{array}{cccc}
0 & \longrightarrow & S_{m,-n-(N+1)}(q) & \longrightarrow & S_{m,-n-(N+1)+d}(q^2) & \longrightarrow & S_{m-1,-n-(N+1)+2d-1}(q).
\end{array}
\]

Following the same reasoning as in Lemma 3.1.3, one easily shows that the maps \( \beta^*(P) \) and \( \alpha^*(P) \) are the following:

\[
\begin{align*}
\alpha^*(P)(\cdot) &= \frac{1}{d} \sum \frac{\partial P}{\partial X_i} \frac{\partial}{\partial Y_i}(\cdot); \\
\beta^*(P)(\cdot) &= \cdot \times P - q \times \alpha^*(P)(\cdot).
\end{align*}
\]

The main result of this section can then be stated as follows:

**Theorem 3.3.3.** The cohomology of \( \bigoplus_{m \geq 2, n > 1} S^m \Omega_X(m - n) \) can be computed via the graded complex

\[
\begin{array}{cccc}
S((q)[0, -1, -(N + 1)]) & \beta^*(P) & S((q^2)[0, -(N + 1) + d]) & \alpha^*(P) & S_q[-1, -(N + 1) + 2d - 1].
\end{array}
\]

Namely, the \( i \)th cohomology group of one graded component of \( \bigoplus_{m \geq 2, n > 1} S^m \Omega_H(m - n) \) is isomorphic to the \( (N - i) \)th cohomology group of the corresponding graded part of the graded complex.

### 3.4. A second complex computing cohomology: the case of smooth complete intersections of codimension 2

We keep the notations of Section 2.4. At this point, it must be clear to the reader what complex we are going to obtain from the resolution (18):

- apply the functor \( H^N(P^N, \cdot) \) to (18); Lemma 3.3.1 allows to show that the complex formed in this way does compute the sought cohomology;
- use Serre duality and the renormalization map, combined with Lemma 3.3.2, in order to provide a reformulation of the complex;
- consider the "recipe" given in Remark 3.1.5, namely, after dualization and renormalization
  - any multiplication map \( \cdot \times A(Y, X) \) becomes a partial differential equation \( A(\frac{\partial}{\partial Y}, X) \) (with reversed arrow);
  - any partial differential equation \( A(\frac{\partial}{\partial Y}, X) \) becomes a multiplication map \( \cdot \times A(Y, X) \) (with reversed arrow, as we dualize);
- apply this recipe to the various maps appearing in the complex (18).

By applying all these steps, we obtain a graded complex computing the cohomology of

\[
\bigoplus_{m \geq 2, n > 1} S^m \Omega_X(m - n),
\]

yielding the following statement:

\(^{14}\)The injectivity of the map \( \beta^*(P) \) can be checked directly, but it follows from the general fact that the cohomology of \( S^m \Omega_H(m + n) \) is supported in degree less or equal than \( N - 1 = \dim(H) \).
Theorem 3.4.1. The cohomology of $\bigoplus_{m \geq 2, n \geq 1} S^m \Omega_X(m - n)$ can be computed via the following complex

\begin{equation}
\begin{array}{c}
\left( \frac{S}{(q)} \right) \xrightarrow{(g_n)_{1 \leq i \leq 2}} S_{(q^2)}[0, d_1] \oplus S_{(q^2)}[0, d_2] \xrightarrow{(g_n)_{1 \leq i \leq 4}} \\
\rightarrow (\oplus_{1} S_{(q^2)}[-1, 2d_1 - 1]) \oplus S_{(q^2)}[0, -|d|] \oplus S_{(q^2)}[-1, |d| - 1] \\
\rightarrow (g_n)_{1 \leq i \leq 2} S_{(q^2)}[-1, +2|d| - d_2 - 1] \oplus S_{(q^2)}[-1, 2|d| + d_1 - 1] \\
\rightarrow g^4 \rightarrow S_{(q)}[-2, +2|d| - 2)][0, -(N + 1)],
\end{array}
\end{equation}

where one has:

\begin{align*}
g_{11} &= \beta^*(P_1) \text{ and } g_{12} = \beta^*(P_2); \\
g_{21}(A, B) &= \alpha^*(P_1)(A) \text{ and } g_{22}(A, B) = \alpha^*(P_2)(B); \\
g_{23}(A, B) &= \frac{1}{2}(\beta^*(P_2)(A) + P_2 A - \beta^*(P_1)(B) - P_1 B); \\
g_{24} &= -\frac{1}{2}(\alpha^*(P_2)(A) + \alpha^*(P_1)(B)); \\
g_{31}(A, B, C, D) &= \beta^*(P_1)(B) + \alpha^*(P_2)(C) + \beta^*(P_2)(D); \\
g_{32}(A, B, C, D) &= \beta^*(P_2)(A) - \alpha^*(P_1)(C) - \beta^*(P_1)(D); \\
g_{4}(A, B) &= \alpha^*(P_1)(A) + \alpha^*(P_2)(B).\end{align*}

Namely, the $i$th cohomology group of one graded component of $\bigoplus_{m \geq 2, n \geq 1} S^m \Omega_X(m - n)$ is isomorphic to the $(N - i)$th cohomology group of the corresponding graded part of the complex (24).

Let us mention that, as we know that the cohomology of $\bigoplus_{m \geq 1, n \geq 1} S^m \Omega_X(m - n)$ is supported in degrees ranging from $N - 4$ to $N - 2$, we are only interested in a truncated part of the complex given in the above theorem (in either way, it is easy to check the exactness of the complex where it is supposed to be).

4. Applications.

4.1. First application: vanishing and non-vanishing theorems. In this section, we use the resolution given in Section 2.4 in order to study cohomology of symmetric powers of cotangent bundles of complete intersections. We first recover the known vanishing results of Bruckmann-Rackvitz [BR90]. Then, we prove a few non-vanishing results, which in particular show that the previous vanishing results are optimal (which, to our knowledge, is new). It also allows us to generalize results of Bogomolov–DeOliveira [BDO08]: see Theorem 4.1.3 below.

Keep accordingly the notations of Section 2.4, and recall that the line bundle $\mathcal{O}_{T_X}(m) \otimes \pi_X^* \mathcal{O}_X(m + n)$ admits the following resolution:

\begin{equation}
0 \longrightarrow \mathcal{L}_{m-c,n+c-2|d|} \longrightarrow \mathcal{E}^d \otimes \mathcal{L}_{m-c,n+c-2|d|} \longrightarrow \cdots \longrightarrow (\Lambda^{2c-1} \mathcal{E}^d) \otimes \mathcal{L}_{m-c,n+c-2|d|} \longrightarrow \mathcal{L}_{m,n}.
\end{equation}
Cutting this resolution into $2c-2$ short exact sequences, one gets:

(26)

\[
0 \rightarrow I_0 \xrightarrow{} L_{m,n} \xrightarrow{} \mathcal{O}_{\mathbb{P}(TX)}(m) \otimes \pi_X^* \mathcal{O}_X(m+n) \rightarrow 0
\]

\[
0 \rightarrow I_1 \xrightarrow{} (\bigwedge^{2c-1} \mathcal{E}^d)_{m-c,n+c-2|d|} \xrightarrow{} I_0 \rightarrow 0
\]

\[
0 \rightarrow I_2 \xrightarrow{} (\bigwedge^{2c-2} \mathcal{E}^d)_{m-c,n+c-2|d|} \xrightarrow{} I_1 \rightarrow 0
\]

\[
0 \rightarrow L_{m-c,n+c-2|d|} \xrightarrow{} (\mathcal{E}^d)_{m-c,n+c-2|d|} \xrightarrow{} I_{2c-2} \rightarrow 0.
\]

All the results in this Section 4.1 and the next one Section 4.2 will be obtained via diagram chasing, and the use of Lemma 3.3.1. Note that we have not indicated any index relatively to the dependance on $m$ and $n$ for the intermediate coherent sheaves $I_0, \ldots, I_{2c-2}$ in the cut-out resolution: within the context they are used, they will always be implicit, leading, we believe, to no confusion.

4.1.1. Vanishing and non-vanishing theorems for $H^0$, in codimension $c < \frac{N}{2}$. In this subsection, let us first give an alternative proof of the known vanishing theorem of Bruckmann-Rackvitz [BR90][Theorem 4 (iii)], in the case of symmetric powers:

**Theorem 4.1.1.** Let $X \subset \mathbb{P}^N$ be a smooth complete intersection of codimension $c < \frac{N}{2}$, and multi-degree $d = (d_1, \ldots, d_c)$. Denote $d := \min\{d_i \mid 1 \leq i \leq c\}$.

For any $m \in \mathbb{N}, n \in \mathbb{Z}$ satisfying the inequalities $m > n$ and $n < d - 2$, one has the following vanishing result:

\[ H^0(X, S^m \Omega_X(m+n)) = (0). \]

**Proof.** Use the cut-out resolution (26) of $\mathcal{O}_{\mathbb{P}(TX)}(m) \otimes \pi_X^* \mathcal{O}_X(m+n)$ given above. Observe that, since $n < m$, the line bundle line bundle $L_{m,n}$ has no global sections. Therefore, one deduces from the first short exact sequence an injection

(27) \[ 0 \rightarrow H^0(X, S^m \Omega_X(m+n)) \rightarrow H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, I_0). \]

To prove the result, it is thus enough to show that $H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, I_0) = (0)$.

Note that the locally free sheaf $(\bigwedge^{2c-1} \mathcal{E}^d)_{m-c,n+c-2|d|}$ writes as a direct sum of the locally free sheaves

(28) \[ (\mathcal{E}_{0,d_i-1} \otimes (\bigotimes_{j \neq i} L_{1,2d_j-1})) \otimes L_{m-c,n+c-2|d|} \simeq \mathcal{E}_{m-1,n-d_i} \]

for $1 \leq i \leq c$. By Lemma 3.3.1, the locally free sheaf (28) has its cohomology supported in maximal degree as soon as

\[ n - d_i < -2. \]

This holds by hypothesis, and one deduces in particular that

\[ H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, (\bigwedge^{2c-1} \mathcal{E}^d)_{m-c,n+c-2|d|}) = (0). \]
Now, one deduces successively from the short exact sequences of (26) and Lemma 3.3.1 the following chain of isomorphism:

\[ H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, I_0) \simeq H^2(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, I_1) \simeq \cdots \simeq H^{2c-1}(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, I_{2c-2}), \]

as well as an injection

\[ H^{2c-1}(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, I_{2c-2}) \hookrightarrow H^{2c}(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, \mathcal{L}_{m-c,n+c-2[d_1]}). \]

Since \( 2c < N \), one has the vanishing

\[ H^{2c}(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, \mathcal{L}_{m-c,n+c-2[d_1]}) = (0), \]

and with (27), this indeed shows that \( S^m \Omega_X(m + n) \) has no global sections. \( \square \)

Let us now show that Theorem 4.1.1 is optimal in the extremal case, namely if we pick \( n = d - 2 \), and \( m = d - 1 \) (for \( d \geq 3 \))\(^{15}\)

**Theorem 4.1.2.** Let \( n \in \mathbb{N} \), \( d = (d_1, \ldots, d_c) \), \( d := \min \{ d_i \mid 1 \leq i \leq c \} \), and suppose furthermore that \( d \geq 2 \). A smooth complete intersection \( X \subset \mathbb{P}^N \) of codimension \( c < \frac{N+1}{2} \) and multi-degree \( d \) satisfies the following non-vanishing statements:

- for \( d \geq 3 \), \( H^0(X, S^{d-1} \Omega_X(2d - 3)) \neq (0) \);
- for \( d = 2 \), \( H^0(X, S^2 \Omega_X(2)) \neq (0) \).

**Proof.** Suppose first that \( d \geq 3 \), and fix \( n = d - 2 \), as well as \( m = d - 1 > n \). One deduces from the reasoning carried over during the proof of Theorem 4.1.1 the following two exact sequences:

\[
0 \longrightarrow H^0(X, S^n \Omega_X(m + n)) \longrightarrow H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, I_0) \longrightarrow H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, \mathcal{L}_{m,n})
\]

and

\[
H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, I_0) \longrightarrow H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, (\bigwedge^{2c-1} \mathcal{E}^{d})_{m-c,n+c-2[d_1]}) \longrightarrow H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, I_0).
\]

Observe that the first cohomology group of \( I_1 \) vanishes if and only if

\[ H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, (\bigwedge^{2c-2} \mathcal{E}^{d})_{m-c,n+c-2[d_1]}) = (0). \]

(This follows from Lemma 3.3.1, the successive short exact sequences of (26) and the hypothesis on the codimension.) Note that the locally free sheaf \((\bigwedge^{2c-2} \mathcal{E}^{d})_{m-c,n+c-2[d_1]}\) writes as a direct sum of terms of the following form (where \( i \neq j \)):

\[ ((\mathcal{E}_{0,d_i-1} \otimes \mathcal{E}_{0,d_j-1}) \otimes (\bigotimes_{k \neq i,j} \mathcal{L}_{1,2^{dk-1}})) \otimes \mathcal{L}_{m-c,n+c-2[d_1]} \simeq (\mathcal{E}^{d}_{i,j})_{m-2,n-(d_i+d_j)}. \]

Therefore, by Lemma 3.3.1, the first cohomology group of \((\bigwedge^{2c-2} \mathcal{E}^{d})_{m-c,n+c-2[d_1]}\) vanishes as soon as

\[ n < 2d - 3, \]

which holds since \( n = d - 2 \) and \( d \geq 2 \). Accordingly, the exact sequence (30) becomes:

\[
0 \longrightarrow H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, (\bigwedge^{2c-1} \mathcal{E}^{d})_{m-c,n+c-2[d_1]}) \longrightarrow H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, I_0).
\]

\(^{15}\)Observe that the hypothesis on the codimension has been somewhat relaxed in the statement, as the upper bound has become \( \frac{d+1}{2} \) instead of \( \frac{d}{2} \).
Putting together (29) and (32), one therefore deduces that the space of global section
\( H^0(X, S^m \Omega_X (m + n)) \) is non-zero as soon as the kernel
\[
\text{Ker} \left( \bigoplus_{i=1}^c H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N^+1}, \mathcal{E}_{m-1,n-d_i}) \xrightarrow{\psi} H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N^+1}, \mathcal{L}_{m,n}) \right)
\]
is non-trivial (here, one has used the isomorphism (28)). Note that in the direct sum
\[
\bigoplus_{i=1}^c H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N^+1}, \mathcal{E}_{m-1,n-d_i}),
\]
the only non-zero spaces are the one corresponding to the indexes \( i \) such that \( d_i = d \).
Without loss of generality, suppose that \( d_1 = \cdots = d_k = d \), and that \( d_i > 2 \) for \( i > k \).
Recall that \( \mathcal{E}_{m-1,n-d} = \mathcal{E}_{m-1,-2} \) fits into the short exact sequence
\[
0 \longrightarrow \mathcal{L}_{m,-2} \longrightarrow \mathcal{E}_{m-1,-2} \longrightarrow \mathcal{L}_{m-1,-1} \longrightarrow 0.
\]
Therefore, using Theorem 1.3.1, one obtains that
\[
(33) \quad H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N^+1}, \mathcal{E}_{m-1,-2}) \simeq H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N^+1}, \mathcal{L}_{m-1,-1}).
\]
Using the above isomorphism (33), one sees that the map \( \psi \) becomes
\[
\tilde{\psi}: \bigoplus_{i=1}^k H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N^+1}, \mathcal{L}_{m-1,-1}) \longrightarrow H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N^+1}, \mathcal{L}_{m,d-2}).
\]
Now, recall that the explicit description of \( H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N^+1}, \mathcal{L}_{m-1,-1}) \) is given by
\[
H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N^+1}, \mathcal{L}_{m-1,-1}) = \text{Coker} \left( \mathbb{C}[Y,X]_{m-1,-1} \xrightarrow{\delta} \mathbb{C}[Y,X]_{m-2,0} \right)
\]
\[
= \mathbb{C}[Y]_{m-2}.
\]
Similarly, one has
\[
H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N^+1}, \mathcal{L}_{m,d-2}) \simeq \text{Coker} \left( \mathbb{C}[Y,X]_{m,d-2} \xrightarrow{\delta} \mathbb{C}[Y,X]_{m-1,d-1} \right).
\]
Coming back to the explicit description of the resolution (25), and following all the
isomorphisms described above, one sees that the map the map \( \tilde{\psi} \) is given explicitly as follows:
\[
\tilde{\psi}: \begin{pmatrix}
\mathbb{C}[Y]_{m-2} \\
(A_1, \ldots, A_k)
\end{pmatrix} \longrightarrow \text{Coker} \left( \mathbb{C}[Y,X]_{m,d-2} \xrightarrow{\delta} \mathbb{C}[Y,X]_{m-1,d-1} \right)
\]
\[
=(d P_1)_X(Y) A_1 + \cdots + (d P_k)_X(Y) A_k.
\]
One has to show that kernel of \( \tilde{\psi} \) is non-trivial, and to this end, it is enough to find a
non-zero element \( A \in \mathbb{C}[Y]_{m-2} = \mathbb{C}[Y]_{d-3} \) such that
\[
A \times (d P_1)_X(Y) \in \text{Im}(\delta).
\]
As a matter of fact, one shows that for any \( A \in \mathbb{C}[Y]_{d-3} \) one has
\[
A \times (d P_1)_X(Y) \in \text{Im}(\delta).
\]
Consider the following polynomial:
\[
B := \sum_{i=1}^{d-2} (-1)^i \delta^{d-2-i}(A) \delta^i(P_1(Y)).
\]
Using the Leibnitz rule, compute that
\[
\delta(B) = - \left( \delta^{d-3}(A) \delta^2(P_1(Y)) \right) + \left( \delta^{d-3}(A) \delta^2(P_1(Y)) + \delta^{d-4}(A) \delta^3(P_1(Y)) \right) + \cdots + (-1)^{d-2} \left( \delta(A) \delta^{d-2}(A) + A \delta^{d-1}(P_1(Y)) \right)
\]
\[
= (-1)^{d-2} A \times \delta^{d-1}(P_1(Y)).
\]
In order to conclude, one shows by induction on \(d \geq 1\) the following identity for any \(R \in \mathbb{C}[X]_d\):
\[
\delta^{d-1}(R(Y)) = (d-1)! \times (d R)_X(Y).
\]
For \(d = 1\), the result is straightforward. Let therefore \(d \in \mathbb{N}_{\geq 2}\). By linearity, it suffices to prove the identity for any \(R \in \mathbb{C}[X]_d\) polynomial of the form
\[
R = X_i R_1,
\]
where \(0 \leq i \leq N\). By the Leibnitz rule, one has the equality:
\[
\delta^{d-1}(R(Y)) = \binom{d-1}{1} X_i \delta^{d-2}(R_1(Y)) + \binom{d-1}{0} Y_i \delta(\delta^{d-2}(R_1(Y))).
\]
Then, by induction hypothesis, compute that:
\[
\delta^{d-1}(R(Y)) = (d-1)! \times X_i (d R_1)_X(Y) + (d-2)! \times Y_i \delta((d R_1)_X(Y))
\]
\[
= (d-1)!(X_i (d R_1)_X(Y) + Y_i R_1(X))
\]
\[
= (d-1)!(d R)_X(Y),
\]
hence the identity. This proves the theorem for \(d \geq 3\).

As for the case where \(d = 2\), fix \(m = d = 2\) (instead of \(m = d - 1\)), and keep \(n = d - 2 = 0\). The same line of reasoning works, and one has to show that kernel of the map (keeping the same notations as above):
\[
\tilde{\psi} : \left( \mathbb{C}[Y] \oplus \mathbb{C}[Y, X]_2 \times \cdots \times \mathbb{C}[Y, X]_k \right) \rightarrow \mathbb{C}[X, X]_{2,0} \times \cdots \times \mathbb{C}[X, X]_{1,1}
\]
is not trivial. This fact follows immediately from the identity:
\[
\delta(P_i(Y)) = (d P_i)_X(Y).
\]
This concludes the proof of the theorem. \(\Box\)

Using the previous vanishing and non-vanishing statements, we can now prove the following theorem, which can be seen as a generalization of results in [BDO08][Theorems B and D]:

**Theorem 4.1.3.** Let \(d \in \mathbb{N}_{\geq 2}\) be a natural number, and let \(X \subset \mathbb{P}^N\) be a non-degenerate\(^{16}\) smooth complete intersection of codimension \(c < \frac{N}{2}\). The complete intersection \(X\) satisfies
\[
\bigoplus_{m \geq \max(d - 2, 1)} H^0 \left( X, S^m \Omega_X(m + \max(d - 3, 0)) \right) = (0)
\]
if and only if \(X\) is not included in an hypersurface of degree \(2 \leq i \leq d\).

\(^{16}\)Recall that it means that the complete intersection is not included in an hyperplane.
Proof. One direction of the equivalence is clear by the vanishing result of Theorem 4.1.1. For the other direction, suppose that $X$ is included in an hypersurface of minimal possible degree $2 \leq i \leq d$. If $i = 2$, it follows from the non-vanishing result of Theorem 4.1.1 that

$$H^0(X, S^2 \Omega_X(2)) \neq (0).$$

This proves the wanted result in the case $i = 2$. Suppose now that $3 \leq i \leq d$. This time, it follows from the non-vanishing result of Theorem 4.1.1 that

$$H^0(X, S^{i-1} \Omega_X(2i - 3)) \neq (0).$$

Since $\Omega_X(2)$ is globally generated, this implies that

$$H^0(X, S^{d-2} \Omega_X(2i - 3 + 2(d - i - 1))) \neq (0).$$

This finishes the proof.

In [BDO08][Theorems B and D], the authors proved that a smooth subvariety $X \subset \mathbb{P}^N$ (not necessarily a complete intersection) of codimension $c \in \{1, 2\}$, with $c < \frac{N}{2}$, satisfies the following:

$$\bigoplus_{m \geq 1} H^0(X, S^m \Omega_X(m)) = (0) \iff X \text{ is not included in a quadric.}$$

In view of Hartshorne’s conjecture (see [Laz04][Conjecture 3.2.8]), such a smooth subvariety of codimension 2 should be a complete intersection as soon as $N \geq 7$.

As an evidence towards Hartshorne’s conjecture, a natural problem would be to generalize Theorem 4.1.3 to the case of an arbitrary smooth subvariety $X \subset \mathbb{P}^N$ of codimension $c \leq \frac{N}{3}$, the case $c = 2, d = 2$ being settled by [BDO08][Theorem D].

4.1.2. Vanishing and non-vanishing theorems for $H^1$, in codimension $c < \frac{N}{2} - 1$. In this subsection, let us first give an alternative proof of the known vanishing theorem of Bruckmann-Rackvitz [BR90][Theorem 4 (ii)], in the case of symmetric powers:

Theorem 4.1.4. Let $X \subset \mathbb{P}^N$ be a smooth complete intersection of codimension $c < \frac{N}{2} - 1$, and multi-degree $d = (d_1, \ldots, d_c)$. Denote $d := \min\{d_i \mid 1 \leq i \leq c\}$.

For any $m \in \mathbb{N}$ and any $n \in \mathbb{Z}$ satisfying the inequality $n < -1$, one has the following vanishing result:

$$H^1(X, S^m \Omega_X(m + n)) = (0).$$

Proof. Considering the first short exact sequence of (26), and the vanishing $H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, L_{m,n}) = (0)$ (which holds under the assumption $n < -1$, see Theorem 1.3.1), one sees that it is enough to show that

$$H^2(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, I_0) = (0).$$

This follows from Lemma 3.3.1, the successive short exact sequences of (26), and the fact that $c < \frac{N}{2} - 1$ (in a classical manner, that was already encountered in Theorem 4.1.1).

Let us now show that this vanishing theorem is optimal:

Theorem 4.1.5. Let $X \subset \mathbb{P}^N$ be a smooth complete intersection of codimension $c < \frac{N}{2} - 1$, and multi-degree $d = (d_1, \ldots, d_c)$. Denote $d := \min\{d_i \mid 1 \leq i \leq c\}$.

For any $m \in \mathbb{N}$, and any $n \in \mathbb{Z}$ satisfying the inequality $-1 < n < \min(d - 1, m - 2)$, one has the following non-vanishing result:

$$H^1(X, S^m \Omega_X(m + n)) \neq (0).$$
Proof. Consider the exact sequence coming from the first short exact sequence of (26):

\[ H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, I_0) \longrightarrow H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, L_{m,n}) \longrightarrow H^1(X, S^m \Omega_X(m + n)). \]

Observe that the hypothesis on \( n \) implies the non-vanishing \( h^1(L_{m,n}) \neq 0 \), by Theorem 1.3.1. Therefore, in order to prove the non-vanishing of \( H^1(X, S^m \Omega_X(m + n)) \), it is enough to show that

\[ h^1(I_0) = 0. \]

Using the second short exact sequence of (26), one sees that it is actually enough to show that

\[ h^1\left( \bigwedge^{2c-1} \mathcal{E}^d \right)_{m-c,n+c-2|d|} = 0. \]

Using the identity (28) and Lemma 3.3.1, one deduces the sought result. \[ \square \]

4.1.3. Vanishing theorems for \( H^i \), in codimension \( c < \frac{N}{2} - i \). Using the same reasoning as in the proof of Theorem 4.1.1 or 4.1.4, we can again give an alternative proof of the known vanishing theorem of Bruckmann-Rackvitz [BR90][Theorem 4 (i)], in the case of symmetric powers:

**Theorem 4.1.6.** Let \( i \in \mathbb{N} \) be an integer. Let \( X \subset \mathbb{P}^N \) be a smooth complete intersection of codimension \( c < \frac{N}{2} - i \), and multi-degree \( d = (d_1, \ldots, d_c) \). For any \( m \in \mathbb{N} \) and any \( n \in \mathbb{Z} \), one has the following vanishing result:

\[ H^i(X, S^m \Omega_X(m + n)) = (0). \]

We leave the (easy) details to the reader.

4.2. Second application: the algebra \( \bigoplus_{m \in \mathbb{N}} H^0(X, S^m \Omega_X(m)) \) for smooth complete intersections of codimension \( c < \frac{N}{2} \). In this section, we fix a smooth complete intersection of codimension \( c \):

\[ X = \{ q_1 = 0 \} \cap \cdots \cap \{ q_k = 0 \} \cap \{ P_{k+1} = 0 \} \cap \cdots \cap \{ P_{c} = 0 \} \subset \mathbb{P}^N, \]

where \( q_1, \ldots, q_c \) are quadratic polynomials, and where \( \deg(P_i) > 2 \) for \( i > k \). Denote \( d = (\underbrace{2, \ldots, 2}_{k}, d_{k+1}, \ldots, d_c) \) the multi-degree of \( X \).

Following the proof of the non-vanishing Theorem 4.1.2, we can prove the following result:

**Theorem 4.2.1.** Suppose that \( X \) is of codimension \( c < \frac{N}{2} \). Then there is a natural graded isomorphism of \( \mathbb{C} \)-algebras

\[ \bigoplus_{m \in \mathbb{N}} H^0(X, S^m \Omega_X(m)) \simeq \mathbb{C}[q_1, \ldots, q_k] \subset \mathbb{C}[X]. \]

**Proof.** Follow the same reasoning as in the proof of 4.1.2, but set instead \( m \in \mathbb{N}_{\geq 1} \) arbitrary, and fix \( n = 0 \) (so that \( m > n \), which was required during the reasoning). For the purposes of the proof of Theorem 4.1.2, it was enough to know that there is an injection

\[ (34) \quad 0 \longrightarrow H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, \bigwedge^{2c-1} \mathcal{E}^d)_{m-c,n+c-2|d|} \longrightarrow H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, I_0). \]

(For which the hypothesis \( c < \frac{N+1}{2} \) was enough). Here, one needs the above map to be an isomorphism. This follows from Lemma 3.3.1, the successive short exact sequences of (26), and the hypothesis on the codimension \( c < \frac{N}{2} \).
That being said, the reasoning of the proof of Theorem 4.1.2 shows that for any \( m \in \mathbb{N}_{\geq 1} \), the space of global sections \( H^0(X, S^m\Omega_X(m)) \) is isomorphic to the kernel of the map

\[
\phi_m : \left( \mathbb{C}[Y]_{m-2}^{\oplus k} \rightarrow \text{Coker} \left( \mathbb{C}[Y, X]_{m,0} \xrightarrow{\delta} \mathbb{C}[Y, X]_{m-1,1} \right) \right) \to \left( (d q_1)_X(Y) A_1 + \cdots + (d q_k)_X(Y) A_k \right).
\]

Start by roughly analyzing the constraints on the elements in the kernel. Let \( (A_1, \ldots, A_k) \) be in the kernel of \( \phi_m \). By definition, there exists a polynomial \( B \in \mathbb{C}[Y]_m \) such that

\[
(35) \quad \delta(B) = \delta(q_1(Y)) A_1 + \cdots + \delta(q_k(Y)) A_k.
\]

Consider the operator \( \delta^{rev} := \sum_{i=0}^N Y_i \frac{\partial}{\partial X_i} \), and compute that

\[
\delta^{rev} \circ \delta = \sum_{i=0}^N Y_i \frac{\partial}{\partial Y_i} + \sum_{0 \leq i, j \leq N} Y_i X_j \frac{\partial^2}{\partial X_j \partial Y_j}.
\]

Applying \( \delta^{rev} \) to both sides of the equality (35) (and using the Leibniz rule), one finds the equality:

\[
m B = 2 \sum_{i=1}^k q_i A_i.
\]

(A word of caution: from now on, when writing the symbol \( q_i \), one means the quadratic polynomial \( q_i \) in the variables \( Y \)). Reinjecting the above equality into the equality (35), one finds that the following equalities are satisfied for any \( 0 \leq j \leq N \):

\[
\frac{\partial}{\partial Y_j} \left( \sum_{i=1}^k q_i A_i \right) = \frac{m}{2} \sum_{i=1}^k \frac{\partial q_i}{\partial Y_j} A_i.
\]

Now, observe that this set of equalities is equivalent to the following equality

\[
(36) \quad d \left( \sum_{i=1}^k q_i A_i \right) = \frac{m}{2} \sum_{i=1}^k A_i \ d q_i,
\]

where \( d \) is the exterior differential. Reciprocally, one immediately checks that any \( k \)-uple of polynomials \( (A_1, \ldots, A_k) \in \mathbb{C}[Y]^{\oplus k} \) satisfying the equality (36) defines an element in the kernel of \( \phi_m \).

Now that one has given a characterization of the kernel of \( \phi_m \) in complex analytic terms, one can use tools from complex analysis to study it. Since \( X \) is a complete intersection, the Jacobian matrix formed with the partial derivatives of \( q_1, \ldots, q_k \) is of maximal rank on a non-empty open subset \( U \) of \( \{ q_1 = 0 \} \cap \ldots \cap \{ q_k = 0 \} \). Fix \( u \in U \), and consider accordingly system of holomorphic coordinates around \( u \) of the following form:

\[
Z := ( z_1 := q_1, \ldots, z_k := q_k, z_{k+1}, \ldots, z_{N+1} ).
\]

One is interested in the convergent power series \( (A_1, \ldots, A_k) \) in the variables \( Z \) satisfying the equality:

\[
(37) \quad d \left( \sum_{i=1}^k z_i A_i \right) = \frac{m}{2} \sum_{i=1}^k A_i \ d z_i.
\]
Note first that the equality (37) implies that the differential form \( \sum_{i=1}^{k} A_i \, d \, z_i \) is closed. Hence, by the Poincaré lemma, there exists a convergent power series \( A \), depending only the variables \((z_1, \ldots, z_k)\) and which can be supposed to vanish at \( u \), such that for any \( 1 \leq i \leq k \), the following holds
\[
\frac{\partial A}{\partial z_i} = A_i.
\]
Decompose the power series \( A \) into its homogeneous parts
\[
A = A^1 + A^2 \cdots,
\]
where \( A^i \) is a homogeneous polynomial of degree \( i \) in the variables \((z_1, \ldots, z_k)\). The equality (37) can then be rewritten as follows:
\[
d(A^1 + 2A^2 + \cdots) = \frac{m}{2} d(A^1 + A^2 + \cdots).
\]
This implies that \( A \) must be zero if \( m \) is odd, and that \( A \) must be a homogeneous polynomial of degree \( \frac{m}{2} \) if \( m \) is even.

In conclusion, the reasoning in the previous paragraphs shows the following:

- if \( m \) is odd, then the kernel of \( \phi_m \) is trivial;
- if \( m \) is even, then the kernel of \( \phi_m \) is equal to
\[
\left\{ \left( \frac{\partial P_1}{\partial q_1}, \ldots, \frac{\partial P_k}{\partial q_k} \right) \mid P(q_1, \ldots, q_k) \text{ homogeneous of degree } \frac{m}{2} (\text{in the variables } (q_1, \ldots, q_k)) \right\}.
\]

In the second case, the elements in the kernel are thus in natural bijection with \( \mathbb{C}[q_1, \ldots, q_k]_{\frac{m}{2}} \). All of this allows to define a natural bijective map
\[
\bigoplus_{m \in \mathbb{N}} H^0(X, S^m \Omega_X(m)) \rightarrow \mathbb{C}[q_1, \ldots, q_k],
\]
and it is a simple verification to show that it respects the structure of algebra. This finishes the proof of the theorem. \( \square \)

**Remark 4.2.2.** Let us note that there are still interesting features that can be said in the case where the codimension is equal to \( c = \frac{m}{2} \). In order to apply the previous proof, one has to show that the map (34) is an isomorphism for any \( m \in \mathbb{N} \). By the successive short exact sequences (26), one has the isomorphisms:
\[
H^2(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, I_1) \cong \cdots \cong H^{2c-1}(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, I_{2c-2}).
\]
Furthermore, with the last short exact sequence of (26), one has that the last space \( H^{2c-1}(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, I_{2c-2}) \) is equal to:
\[
\text{Ker} \left( H^N(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, \mathcal{E}_{m-c, c-2} | d |) \xrightarrow{\psi} \bigoplus_{i=1}^{c} H^N(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, \mathcal{E}_{m-c, c-2} | d | + d_i, -1) \right).
\]
Using the ideas and reformulation techniques described throughout Section 3, the map \( \psi \) can be described as follows:
\[
\bigoplus_{i=1}^{c} \mathbb{C}[Y, X]_{m-c+1, 2|d|+1-d_i-c-1-N+1} \xrightarrow{\psi} \mathbb{C}[Y, X]_{m-c+2|d|-c-1-N+1},
\]
where we recall that \( \alpha^*(P_j)(\cdot) := \sum_{i=0}^{N} \frac{\partial P}{\partial X_j} \frac{\partial}{\partial Y_j}(\cdot) \). Proving that (34) is an isomorphism is then equivalent to proving that the above map is surjective.
For example, in the case where $X = \{q_1 = 0\} \cap \{q_2 = 0\} \subset \mathbb{P}^4$ is a smooth complete intersection of two quadrics in the 4-dimensional projective space, the surjectivity is easily checked. Indeed, for any $m \in \mathbb{N}$, the map

$$\alpha^* (q_1): \mathbb{C}[Y, X]_{m,0} \longrightarrow \mathbb{C}[Y, X]_{m-1,1}$$

is already surjective: it can be checked directly, or one can notice that the cokernel of this map is actually isomorphic to $H^1(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, \mathcal{L}_{m,0})$, which is zero. This allows to recover the result that the tangent bundle $TX$ is not big, which was recently proved in [Mal21] and independently in [HLS22]. Indeed, one has the isomorphism $TX \simeq \Omega_X(1)$, and the description of

$$\bigoplus_{m \in \mathbb{N}} H^0(X, S^m \Omega_X(m)) \simeq \mathbb{C}[q_1, q_2]$$

readily implies that $TX$ is not big (using the description of bigness via the asymptotic growth of the dimension of the space of global section of larger and larger symmetric powers of the tangent bundle).

4.3. Third application: intermediate ampleness of cotangent bundles of (general) hypersurfaces. Recall that, on a projective variety $(X, \mathcal{O}_X(1))$ polarized by a very ample line bundle, a vector bundle $\mathcal{E}$ is said (uniformly) $q$-ample if there exists a constant $\lambda > 0$ such that for any $m, n \in \mathbb{N}$ satisfying the inequality $m \geq \lambda n$, and any $i > q$, the following holds:

$$H^i(X, S^m \mathcal{E}(-n)) = (0).$$

For $q = 0$, one recovers the usual ampleness for vector bundles, and note that these notions form a string of implication, namely

$$0\text{-ampleness} \Rightarrow 1\text{-ampleness} \Rightarrow \cdots \Rightarrow \dim(X)\text{-ampleness}. $$

Also, note that any vector bundle on $X$ is automatically $\dim(X)$-ample (since any coherent sheaf on a projective variety of dimension $n$ has its cohomology supported in degree less or equal than $n$ by Grothendieck’s vanishing theorem (see [Har77][III. Theorem 2.7])).

We refer to [Tot13] or the survey [GK15] for more details on partial ampleness.

If we consider the projective variety $(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$, one easily sees by Theorem 1.3.1 that the cotangent bundle $\Omega_{\mathbb{P}^N}$ has the least possible positivity property, namely it is merely $N$-ample. Recall Xie-Brotbek–Darondeau theorem (see [Xie18] and [BD17][N]), which asserts that a sufficiently general complete intersection $X$, of codimension $c \geq \frac{N}{2}$ and of multi-degree large enough, has ample cotangent bundle. In view of this result, it is natural to address the following question. Consider a smooth projective variety $X$, and suppose that $\Omega_X$ is $q$-ample. Does it hold that, if one cuts $X$ by a sufficiently ample and sufficiently general (smooth) hypersurface $H$, the cotangent bundle $\Omega_H$ is $(q-2)$-ample?\footnote{By convention, one sets the notion of $q$-ampleness to be equivalent to the notion of $0$-ampleness for any $q \leq 0$.}

This question has a positive answer in the case of hypersurfaces in projective spaces, and it turns out that, in this case:

- it is enough to suppose $\deg(H) \geq 4$;
- the generality assumption on the hypersurface can be dropped.

One then may wonder if the generality assumption could be dropped in the above question: we do not know, but it would definitely be striking if we could!

The result in the case of hypersurfaces is quite recent, as it was proved in [HLS22]. In loc.cit, the authors rather look at the space of twisted global symmetric vector fields on
an hypersurface $X \subset \mathbb{P}^N$, and prove a vanishing result implying the $(N - 2)$-ampleness of the cotangent bundle $\Omega_X$ (via Serre duality). Their proof relies on the isomorphism

$$TX \simeq \bigwedge^{N-2} \Omega_X \otimes K_X$$

and the vanishing theorem of Bruckmann-Rackvitz [BR90][Theorem 4 (iii)] for the composition of Schur functors $S^m \circ \bigwedge^{N-2}$ (which can be shown, via some Plethysm, to be the Schur functor associated to the partition $(m, \ldots, m)$, so that the vanishing theorem applies, see [HLS22]).

In this section, we use the description of cohomology given in Section 3.1 to prove the following vanishing theorem, slightly weaker than the one in [HLS22]:

**Theorem 4.3.1.** Let $H$ be a smooth hypersurface of degree $d$, whose defining equation $P$ is of the following form

$$P := X_N^d - F(X_0, \ldots, X_{N-1}),$$

where $F$ is a smooth polynomial, homogeneous of degree $d$. Let $m \in \mathbb{N}$, $n \in \mathbb{Z}$, and suppose that $d = \text{deg}(H) \geq 3$. As soon as $m(d - 3) > n$, one has the vanishing:

$$H^{N-1}(H, S^m \Omega_H(d - n - (N + 1))) = (0).$$

In particular, as soon as $d \geq 4$, this shows that $\Omega_H$ is uniformly $q$-ample (with uniform bound $\lambda = \frac{1}{d - 3}$ if $\text{deg}(H) \geq N + 1$), and hence so is a general hypersurface of degree $d \geq 4$.

**Remark 4.3.2.** The bound in the statement is the best possible. Indeed, consider the simplest case of a curve $H \subset \mathbb{P}^2$. By adjunction, one has the isomorphism

$$TH^{\otimes m} \otimes \mathcal{O}_H(n) \simeq \mathcal{O}_H(n - m(d - 3)),$$

and the space of global sections of such a line bundle is zero if and only if $m(d - 3) > n$.

The strategy to prove the result is simply the following: by Theorem 3.1.6, the vanishing result is equivalent to the injectivity of the map

$$\alpha^*(P)_{m,m+n} : S_{m,m+n}(P,q) \rightarrow S_{m-1,m+n+d-1}(P,q).$$

Denote by $I := (P,q)$ the ideal spanned by $P$ and $q$. For later use, record the following elementary lemma:

**Lemma 4.3.3.** The ideal $I$ is prime.

**Proof.** The Jacobian criterion allows to show that the variety defined by the vanishing of $P$ and $q$ is smooth. This immediately implies the primeness of $I$. \qed

Let us now proceed to the proof of Theorem 4.3.1:

**Proof of Theorem 4.3.1.** Let $A \in S_{m,m+n}$ be a polynomial such that $\alpha^*(P)_{m,m+n} \equiv 0 \mod I$. One must show that

$$A \equiv 0 \mod I.$$  

Note first that, up to adding an element of $I$ to $A$ (therefore leaving the class $A \mod I$ unchanged), one can always suppose that

$$\alpha^*(P)_{m,m+n} \equiv 0 \mod (q).$$

Indeed, by hypothesis, one can write that

$$\alpha^*(P)_{m,m+n} = PU + qV$$

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for some polynomials $U, V$. Then, observe that the following equality holds:

$$
\alpha^*(P)(A - qU) = q(V + \alpha^*(P)(U)).
$$

Now, deshomogenize the variable $X$ by dividing by $X^N$, and set $x_i := \frac{X_i}{X_N}$.

$$
\begin{align*}
q &\mapsto q^{in} = \sum_{i=0}^{N-1} x_i Y_i + Y_N \\
P &\mapsto P^{in}(x) = 1 + F(x) \\
I &\mapsto I^{in} := (P^{in}, q^{in}) \\
\alpha^*(P) &\mapsto \alpha^*(P)^{in} = \frac{1}{d} \left( \sum_{i=0}^{N-1} \frac{\partial F}{\partial x_i} \frac{\partial}{\partial Y_i} + \frac{\partial}{\partial Y_N} \right) \\
A &\mapsto A^{in}(x, Y)
\end{align*}
$$

Note that one still has the equalities:

$$
\begin{align*}
\alpha^*(P)^{in}(A^{in}) &\equiv 0 \mod q^{in} \\
\alpha^*(P)^{in}(q^{in}) &= P^{in}
\end{align*}
$$

As $q^{in}$ is now unitary in $Y_N$, one can write uniquely (by Euclidean division)

$$
A = \sum_{k=0}^{m} a_k(x, Y^N)(q^{in})^k,
$$

where $a_k(x, Y^N)$ does not depend of the variable $Y_N$. Beware that the degree in $x$ of the $a_k$s has increased, and one has the following inequality for $0 \leq k \leq m$:

$$
\deg_x(a_k) \leq n + 2m.
$$

The equations (38) imply that

$$
\alpha^*(P)^{in}(a_0) + P^{in}a_1 \equiv 0 \mod q^{in}.
$$

Since $\alpha^*(P)^{in}(a_0) + P^{in}a_1$ is independent of the variable $Y_N$, the above equality implies in turn the equality:

$$
\alpha^*(P)^{in}(a_0) + P^{in}a_1 = 0.
$$

Now, the key observation, which uses crucially the form of the equation $P$, is that one has the following equality:

$$
\alpha^*(P)^{in}(a_0) = \frac{1}{d} \left( \sum_{i=0}^{N-1} \frac{\partial F}{\partial x_i} \frac{\partial}{\partial Y_i} \right)(a_0) = \alpha^*(F)(a_0),
$$

where $F$ is homogeneous of degree $d$ in the variables $x = (x_0, \ldots, x_{N-1})$. Therefore, the equality (39) rewrites:

$$
\alpha^*(F)(a_0) + (1 + F)a_1 = 0.
$$

One now deduces from (40) the following lemma:

**Lemma 4.3.4.** The element $a_1$ is in the image of $\alpha^*(F)$.

**Proof.** For $a \in S' := \mathbb{C}[x, Y^N]$, denote by $a^{min}$ its minimal homogenous component. Since $\alpha(F)$ is a graded map (i.e. it shifts the grading, since $F$ is homogeneous), one deduces from (40) the following equality:

$$
\alpha^*(F)(a_0^{min}) + a_1^{min} = 0.
$$
Therefore, $a_{1}^{\text{min}} \in \text{Im} \alpha^{*}(F)$. One can therefore rewrite (40) as follows:

$$\alpha^{*}(F)(a_{0} + (1 + F)a_{0}^{\text{min}}) + (1 + F)(a_{1} - a_{1}^{\text{min}}) = 0.$$ 

One can now repeat the previous reasoning, so that one eventually shows that every homogeneous component of $a_{1}$ lies in the image of $\alpha^{*}(F)$, which shows the result. \qed

With the previous Lemma 4.3.4, one can thus write

$$a_{1} = \alpha^{*}(F)(b),$$

with $b \in \mathbb{C}[x, Y]$. Accordingly, the equation (40) rewrites:

$$\alpha^{*}(F)(a_{0} + P^{\text{lin}}b) = 0.$$ 

Note that $\deg_{x}(a_{0} + P^{\text{lin}}b) = m$ (it has remained homogeneous of degree $m$ in the variables $Y$), whereas $\deg_{x}(a_{0} + P^{\text{lin}}b) \leq n + 2m$. By hypothesis, one has that

$$(d - 3)m > n \implies (d - 1)m > n + 2m.$$ 

Therefore, Proposition 1.6.4 applies, and the map

$$\alpha^{*}(F) : S_{m,i} \to S_{m,i+d-1}$$

is injective for any $i \leq n + 2m$. Therefore, one deduces that

$$a_{0} = -P^{\text{lin}}b.$$ 

Now, rehomogenize everything by multiplying by a suitable power of $X_{N}$. One deduces that there exists $M \in \mathbb{N}_{\geq 1}$ such that

$$X_{N}^{M}A \in I = (P, q).$$

By Lemma 4.3.3, the ideal $I$ is prime, so that $A \in I$: the proof is complete. \qed

4.4. Fourth application: effective computations in low-dimensional cases. The goal of this section is to illustrate how the explicit description of cohomology given in Section 3 can be used to recover some known results, and provide new examples of families (of surfaces) that do not satisfy a given property.

4.4.1. Surface of bi-tangent lines to quartic surfaces. Consider $S \subset \mathbb{P}^{3}$ a quartic surface of Picard rank one. By [GO20][Corollary 4.2], it is known that the closure of the effective cone of $\mathbb{P}(TS) \xrightarrow{\pi} S$ is generated by $H := \pi^{*}\mathcal{O}_{S}(1) + 3\xi + 4H$, where $\xi := \mathcal{O}_{\mathbb{P}(TS)}(1)$. It is then natural to ask whether or not a multiple of $3\xi + 4H$ is effective. The answer is positive, as it is known since the works of [Tih80] and [Wel81] that the surface of bi-tangent lines to $S$ defines an hypersurface in the linear system $|6\xi + 8H|$.

Suppose for a moment that one does not know the existence of such a particular surface (which exists for any smooth quartic in $\mathbb{P}^{3}$). By Serre duality, since $K_{S} = \mathcal{O}_{S}$, one has the isomorphism

$$H^{2}(S, S^{6}\Omega_{S}(-8)) \simeq H^{0}(S, S^{6}TS(8)).$$

On the other hand, since $S$ is a K3 surface, one also has the isomorphism $TS \simeq \Omega_{S}$, so that one has

$$H^{0}(S, S^{6}\Omega_{S}(8)) \simeq H^{2}(S, S^{6}\Omega_{S}(-8)).$$

For sake of simplicity, pick $S := \{X_{0}^{4} + \cdots + X_{4}^{4} = 0\}$ a Fermat hypersurface. Using the description given in Section 3.1 (or alternatively in Section 3.3), one computes (see Appendix B) that there is indeed one (and only one) global section of $S^{6}\Omega_{S}(8)$. 

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Explicitly, the rank computation carried over in Appendix B shows that there exists a (unique modulo \((P,q)\)) polynomial \(W \in S_{6,14} = \mathbb{C}[Y, X]_{6,14}\) which

- is not zero modulo \((P,q)\) (recall that \(q = X_0 Y_0 + \cdots + X_3 Y_3\));
- satisfies the differential equation \(\sum_{i=0}^3 X_i \frac{\partial W}{\partial Y_i} \equiv 0 \mod (P,q)\).

With some powerful computing machine, one could even find explicitly a solution, i.e. an explicit equation for the surface of bi-tangent lines. If one is optimistic, it might even be possible to extrapolate a general formula for any smooth quartic.

4.4.2. Examples of non-invariance under deformation of (canonically twisted) symmetric pluri-genera. In the spirit of Brotek’s examples of family of surfaces in \(\mathbb{P}^4\) (see [Bro15]) along which the 2-symmetric pluri-genus (i.e. the dimension of the second symmetric power of the cotangent bundle) does not remain constant, one can consider the 1-parameter family of surfaces in \(\mathbb{P}^4\)

\[
X_t := \{X_0^5 + X_1^5 + \cdots + X_4^5 = 0\} \cap \{-2X_0^5 - X_1^5 + tX_0X_1X_2X_3X_4 + X_3^5 + 2X_4^5 = 0\}.
\]

In Appendix B, explicit computations show that \(h^0(S^2\Omega_{X_t}) = 1\), whereas \(h^0(S^2\Omega_{X_t}) = 0\) (which, by upper semi-continuity of cohomology, implies that a general member of the family has no global 2-symmetric differential).

It was asked by Paun whether or not the invariance under deformation holds if one considers instead symmetric powers twisted by the canonical bundle. It was answered negatively in [BDO08]. We propose to give an explicit and very simple example of a one parameter family \((X_t)_{t \in \mathbb{C}}\) of surfaces in \(\mathbb{P}^4\) along which the invariance of \(S^6\Omega_{X_t}(K_{X_t})\) does not hold:

**Theorem 4.4.1.** The 1-parameter family

\[
X_t := \{X_0^4 + X_1^4 + \cdots + X_4^4 - tX_0^2X_1^2 = 0\} \cap \{-2X_0^4 - X_1^4 + X_3^4 + 2X_4^4 = 0\} \subset \mathbb{P}^4
\]

satisfies \(h^0(S^6\Omega_{X_0}(K_{X_0})) = 1\), whereas \(h^0(S^6\Omega_{X_0}(K_{X_1})) = 0\) for a general \(t \in \mathbb{C}\).

**Proof.** By the adjunction formula, for any \(t \in \mathbb{C}\), one has the equality

\[
S^6\Omega_{X_t}(K_{X_t}) = S^6\Omega_{X_t}(3).
\]

In Appendix B, explicit computations show that \(h^0(S^6\Omega_{X_0}(3)) = 1\), whereas \(h^0(S^6\Omega_{X_1}(3)) = 0\). By semi-continuity of cohomology, this proves the statement. \(\square\)

4.5. Fifth application: a criteria ensuring ampleness of cotangent bundles of complete intersection surfaces in \(\mathbb{P}^4\). This section is not devoted to proving anything new (as the scheme of proof for the criteria given in Lemma 4.5.2 is classical), but rather to discuss strategies to provide concrete examples of surfaces with ample cotangent bundles (which is still an open question).

4.5.1. Discussion on Brotek’s work: explicit constructions of symmetric differentials. In his seminal paper [Bro15], Brotek provided an explicit way to describe the space of global sections of complete intersections of codimension at least as large as their dimension. For people very familiar with Brotek’s work, the description we give in Section 3.2 (as far as the space of global sections is concerned) is reminiscent of Brotek’s description.

The drawback of Brotek’s description, or the one presented in this paper, is that it only allows to tell whether or not there exists global (negatively twisted) symmetric
differentials\(^\text{18}\), but it gives no information on their form (contrarily to the case of global symmetric vector fields: see Section 4.4.1\(^\text{19}\)).

The crucial input in Brotbek’s work, first implemented in [Bro15] to prove particular cases of Debarre’s conjecture, then improved in [BD17] to prove Debarre’s conjecture in full generality, is the following. In the special case of Fermat-type complete intersections, Brotbek’s description of cohomology allows to tell that there exists global negatively twisted symmetric differentials. The difficult part consists in understanding the explicit form of these sections: this is exactly what Brotbek did in [Bro15][Lemma 4.5]. Of course, there is still a lot of work to be done from here to obtain Brotbek’s results, but we believe that they are mainly of technical nature.

It is important to mention that, in order to be able to control the base locus of the constructed global sections (which is necessary to obtain ampleness), it is fundamental to consider the whole situation in family (and this is where technicality arises). It is of course always possible to consider a fixed example, but in this case, there is almost no flexibility to work with: one is reduced to proving that the base locus of the set of sections is empty\(^\text{20}\). Theoretical reasons for which this could be true are yet to be found, but one could always try doing this on a computer. It amounts to computing a Groebner basis, which, for the situation under study, seems out of reach.

4.5.2. A cohomological criteria ensuring ampleness. The approach we propose here is transversal to Brotbek’s approach: instead of exhibiting enough global sections to derive ampleness, we would like to use a cohomological approach. We are going to give a simple criteria, but as in the previous case, it would most probably require too much computations to conclude.

Let us consider \(\text{Flag}_{(1,2)} \mathbb{C}^{N+1} \simeq \mathbb{P}(TP^N)\), and \(\xi \in \text{Flag}_{(1,2)} \mathbb{C}^{N+1}\). The ideal sheaf \(m_\xi\) of the closed point \(\xi\) admits a simple resolution:

\[
\text{Lemma 4.5.1.} \quad \text{The ideal sheaf } m_\xi \text{ admits a locally free resolution of the following form}
\]

\[
0 \to A_{2N-2} \to A_{2N-3} \to \cdots \to A_0 \to m_\xi \to 0,
\]

where for each \(0 \leq i \leq 2N-2\), the vector bundle \(A_i\) writes as a direct sum of line bundles of the following type

\[
\mathcal{L}_{-k,-i}
\]

for \(0 \leq k \leq i\).

\[\text{Proof.}\] Up to applying a suitable automorphism of \(\mathbb{P}^N\), one can always suppose that

\[
\xi = (Ce_0 \subset Ce_0 \oplus Ce_1),
\]

where \(e_0, \ldots, e_N\) is the canonical basis of \(\mathbb{C}^{N+1}\). Consider the \(N\) global sections of \(\mathcal{L}_{0,1}\)

\[
(s_i = X_i)_{1 \leq i \leq N},
\]

as well as the \(N-2\) global sections of \(\mathcal{L}_{1,1}\)

\[
(t_i = X_0 Y_i - X_i Y_0)_{2 \leq i \leq N}.
\]

Here, one identifies \(H^0(\text{Flag}_{(1,2)} \mathbb{C}^{N+1}, L_{1,1})\) with the set of \((1,1)\) bi-homogeneous polynomials \(P \in \mathbb{C}[Y,X]\) satisfying the functional equation (see Section 1.2)

\[
\forall t \in \mathbb{C}, P(Y + tX, X) = P(Y,X).
\]

\(^{18}\)Recall that knowing that a given complete intersection, say a smooth surface in \(\mathbb{P}^4\), has global symmetric differentials vanishing along an ample is particularly interesting if one wishes to study the positivity of the cotangent bundle.

\(^{19}\)The specificity in this case is that one has a natural isomorphism between the tangent and cotangent bundle.

\(^{20}\)zero dimensional is actually enough.
One easily sees that the zero locus of the sections $s_1, \ldots, s_N, t_2, \ldots, t_N$ defines the closed point $\xi$ without multiplicity. Therefore, the Koszul complex associated to these sections provide a locally free resolution of the ideal sheaf $m_\xi$, which takes the form given in the statement. \qed

With this Lemma 4.5.1 at hand, we can now prove the following criterion:

**Lemma 4.5.2.** Let $X \subset P^4$ be smooth surface. Suppose that there exists $m \geq 2$ such that

- $H^1(X, S^m\Omega_X(-1)) = (0)$;
- $H^2(X, S^m\Omega_X(-2)) = H^2(X, S^{m-1}\Omega_X(-3)) = (0)$.

Then the cotangent bundle $\Omega_X$ is ample.

**Proof.** Note first that it is enough to prove that there exists $m \geq 2$ such that $\mathcal{O}_{P(TX)}(m) \otimes \pi^*_X \mathcal{O}_X(-1)$ is globally generated. Indeed, in this case, $\mathcal{O}_{P(TX)}(m)$ writes as the sum of nef line bundle and a $\pi_X$-relatively ample line bundle, so that $\mathcal{O}_{P(TX)}(m)$, and thus $\mathcal{O}_{P(TX)}(1)$, is ample.

Consider $\xi \in P(TX)$ a closed point. By Lemma 4.5.1 and Proposition 1.2.1, the ideal sheaf $m_\xi$ admits a resolution of the following form:

$$0 \to A_3 \to A_2 \to A_1 \to A_0 \to m_\xi \to 0,$$

where $A_i$ is a direct summand of line bundles of the form $\mathcal{O}_{P(TX)}(-k) \otimes (\pi^*_X \mathcal{O}_X(-k-i))$, where $k$ is allowed to range between 0 and $i$.

Now, tensor the above complex by $\mathcal{O}_{P(TX)}(m) \otimes \pi^*_X \mathcal{O}_X(-1)$. By cutting it into 3 short exact sequences, taking long exact sequences in cohomology and using:

- the two vanishing hypothesis of the statement;
- the fact that, for any $0 \leq i \leq 3$, the cohomology of $A_i \otimes \mathcal{O}_{P(TX)}(m) \otimes \pi^*_X \mathcal{O}_X(-1)$ is supported in degrees less or equal than 2 (by Bott’s formulas, and the fact that the cohomology of a coherent sheaf on a surface is supported in degree less or equal than 2),

one sees that the following vanishing result holds:

$$H^1(P(TX), \mathcal{O}_{P(TX)}(m) \otimes \pi^*_X \mathcal{O}_X(-1) \otimes m_\xi) = (0).$$

In order to conclude, consider the short exact sequence induced by the evaluation map at $\xi$

$$0 \to m_\xi \to \mathcal{O}_{P(TX)} \to \mathcal{C}_\xi \to 0,$$

where $\mathcal{C}_\xi$ is the skyscraper sheaf supported on $\{\xi\}$. Twisting this short exact sequence by $\mathcal{O}_{P(TX)}(m) \otimes \pi^*_X \mathcal{O}_X(-1)$, taking the long exact sequence in cohomology and using the above vanishing result, one obtains a surjection:

$$H^0(P(TX), \mathcal{O}_{P(TX)}(m) \otimes \pi^*_X \mathcal{O}_X(-1)) \longrightarrow H^0(P(TX), \mathcal{O}_{P(TX)}(m) \otimes \pi^*_X \mathcal{O}_X(-1) \otimes \mathcal{C}_\xi).$$

This shows that there exists a global section of $\mathcal{O}_{P(TX)}(m) \otimes \pi^*_X \mathcal{O}_X(-1)$ that does not vanish when evaluated at $\xi$. As this holds for any closed point $\xi$, one deduces that the line bundle $\mathcal{O}_{P(TX)}(m) \otimes \pi^*_X \mathcal{O}_X(-1)$ is globally generated. With the remark at beginning, this concludes the proof. \qed

With this criterion, one only has to check a few vanishing of cohomology groups (three to be precise) in order to conclude that the cotangent bundle is ample. Since we have a completely explicit description of these cohomology groups for any complete intersection surfaces, one could hope to provide explicit examples by pure computations. The issue is that one would need a huge computing capacity in order to do so. This might be within reach provided the bound $m$ for which the statement of Lemma 4.5.2 holds is small,
say less than 30, and provided that the multi-degree \( d = (d_1, d_2) \) for which there exists complete intersection surfaces with ample cotangent bundle is also small, ideally \( d_1 \leq 6 \). And even in these specific cases, the amount of computation remains very important: being able to find a concrete example via this method might be a mirage.

There is still the purely theoretic approach, and this is mainly the reason why we described completely the complex computing cohomology of twisted symmetric powers of cotangent bundles of complete intersections of codimension 2 (see Section 3.4). This is a subject we are working on. As a first step, one would already need to settle the case of hypersurfaces in full generality: see Section 4.3 where we treat a particular case.

### APPENDIX A. REMINDER ON BOTT’S FORMULAS.

The reference for this part is [Wey03].

Let \( V \) be a \( \mathbb{C} \)-vector space of dimension \( n \). For a decreasing sequence of integers

\[
\mathbf{s} = (s_1 > s_2 > \cdots > s_t),
\]

with \( t \geq 1 \), and \( s_1 \leq n - 1 \), the partial flag variety \( \text{Flag}_s(V) \) of sequence \( \mathbf{s} \) is defined as

\[
\text{Flag}_s(V) := \{ V \supseteq F_1 \supseteq \cdots \supseteq F_t \supseteq \{0\} \}/ F_t \text{ is a } k\text{-vector subspace of dimension } s_1.
\]

On \( \text{Flag}_s(V) \), there is a filtration of the trivial bundle \( \text{Flag}_s(V) \times V \) by universal sub-bundles \( U_{s_1} \supseteq \cdots \supseteq U_{s_t} \), given over the point \( \eta = (F_1 \supseteq \cdots \supseteq F_t) \in \text{Flag}_s V \) by \( U_{s_i}(\eta) = F_i \).

**Example A.0.1.** If we take the sequence \( \mathbf{s} = (k) \), where \( k \geq 1 \), then \( \text{Flag}_s(V) \) is the Grassmannian of \( k \)-dimensional subspaces of \( V \), denoted \( \text{Grass}(k, V) \), together with the tautological subbundle \( U_k \).

A \( n \)-uple of integer \( \alpha \in \mathbb{Z}^n \) is called admissible with jump sequence \( \mathbf{s} \) if and only if it writes (uniquely) as follows:

\[
\alpha = (\alpha_1, \ldots, \alpha_{s_1}, \alpha_{s_2}, \ldots, \alpha_{s_2}, \ldots, \alpha_{s_t}, \alpha_t),
\]

where the \( \alpha_i \)'s are all distinct. To any such \( \alpha \), one associates the line bundle \( \mathcal{L}_\alpha(V) \) on \( \text{Flag}_s(V) \) by setting:

\[
\mathcal{L}_\alpha(V) := \bigotimes_{i=1}^t \det((U_{s_i}/U_{s_{i+1}})^\vee)^{\alpha_{s_i}},
\]

where by convention \( U_{s_{t+1}} = \{0\} \).

**Example A.0.2.** For \( \alpha = (m, \ldots, m) \in \mathbb{N}^n \), \( \mathcal{L}_\alpha(V) \) is the \( m \)-th tensor power of the Plücker line bundle \( \mathcal{O}(1) = \det(U_k^\vee) \) on \( \text{Grass}(k, V) \). This line bundle allows one to embed \( \text{Grass}(k, V) \) in the projective space \( \mathbb{P}(\Lambda^k V) \).

Let us now take \( E \) a vector bundle of rank \( n \) over a variety \( M \). Let us fix a sequence \( \mathbf{s} = (s_1 > s_2 > \cdots > s_t) \) and an admissible \( n \)-uple of integers \( \alpha \in \mathbb{Z}^n \) with jump sequence \( \mathbf{s} \). The previous considerations make sense fiberwise, and the constructions globalize. This allows one to define

- the projective bundle \( \pi: \text{Flag}_s(E) \rightarrow M \), such that the fiber over \( x \in M \) is \( \text{Flag}_s(E_x) \);
- the line bundle \( \mathcal{L}_\alpha(E) \) over \( \text{Flag}_s(E) \), such that \( (\mathcal{L}_\alpha(E))|_{\text{Flag}_s(E_x)} = \mathcal{L}_\alpha(E_x) \).

\footnote{Note that, in the body of the text, we used the abbreviated notation \( \mathcal{L}_\alpha \): this does not lead to any confusion as the underlying vector space remains the same everywhere (namely, \( \mathbb{C}^{n+1} \)).}
The object of Bott’s formulas is to compute the higher direct image functors
\[ R^i \pi_* L_\alpha(E) \]
for any admissible \( n \)-uple \( \alpha \in \mathbb{Z}^n \) with jump sequence \( \mathbf{s} \), and for any \( i \geq 0 \). In order to state the result properly, let us first make a couple of definitions. First, for sake of simplicity, let us define Schur functors in the following fashion (for more details on Schur functors, see e.g. [Wey03] or [Ful97]):

**Definition A.0.3.** Let \( \lambda \in \mathbb{Z}^n \) be an admissible \( n \)-uple with jump sequence \( \mathbf{s} \). Suppose furthermore that \( \lambda \) is a partition, namely that \( \lambda_1 \geq \cdots \geq \lambda_n \geq 0 \). The **Schur bundle** associated to the partition \( \lambda \) with jump sequence \( \mathbf{s} \) is the direct image
\[ S^\lambda(E) := \pi_* (L_\lambda(E^\vee)). \]

Next, let us denote \( \rho = (n-1, n-2, \ldots, 0) \in \mathbb{N}^n \).

For any permutation \( \sigma \in S_n \), define the following induced action on \( \mathbb{Z}^n \):
\[ ˜\sigma(\cdot) := \sigma(\cdot + \rho) - \rho. \]

We can now state Bott’s formulas\(^{22}\), where we keep the notations introduced above:

**Theorem A.0.4 (Bott).** Let \( \alpha \in \mathbb{Z}^n \) be an admissible \( n \)-uple with jump sequence \( \mathbf{s} \). One of the mutually exclusive possibilities occurs.

1. There exists \( \sigma \in S_n \setminus \{\text{Id}\} \) such that
   \[ ˜\sigma(\alpha) = \alpha. \]
   In this case, all higher direct images \( R^i \pi_* L_\alpha(E) \) vanish for \( i \geq 0 \).
2. There exists a unique \( \sigma \in S_n \) and a unique \( r \in \mathbb{Z} \) such that
   \[ ˜\sigma(\alpha) + (r, \ldots, r) = (\beta_1, \ldots, \beta_{n-1}, 0) \]
   is a partition. In this case, all higher direct images \( R^i \pi_* L_\alpha(E) \) vanish for \( i \neq \ell(\sigma) \)
   \[ ^{23} \]
   and one has
   \[ R^{\ell(\sigma)} \pi_* L_\alpha(E) \simeq S^\beta E^\vee \otimes \det(E^\vee)^{-r}. \]

Let us now mention two particular cases where these formulas apply.

First, let us consider the simplest where the flag variety is the variety of lines. In this situation, one deals with the projectivized bundle \( P(E) \to M \), and Bott’s formula allows to recover the description given in [Har77][III. Exercise 8.4]. In particular, this says that
\[ R^i \pi_* O_{P(E)}(m) = (0) \]
for any \( i > 0 \) and any \( m \in \mathbb{N} \). Second, let us consider the case where \( M \) is a point. In this case, a straightforward corollary of Bott’s formula is the following:

**Corollary A.0.5.** Let \( \alpha \in \mathbb{Z}^n \) be an admissible \( n \)-uple with jump sequence \( \mathbf{s} \). One of the mutually exclusive possibilities occurs.

1. There exists \( \sigma \in S_n \setminus \{\text{Id}\} \) such that
   \[ ˜\sigma(\alpha) = \alpha. \]
   In this case, all cohomology groups of the line bundle \( L_\alpha(\mathbb{C}^n) \to \text{Flag}_n \mathbb{C}^n \) are zero.

\(^{22}\) Almost in its full generality: see [Wey03], where a larger class of line bundles than the one introduced here is considered.

\(^{23}\) \( \ell(\sigma) \) is the cardinality of the support of the partition \( \sigma \).
(2) There exists a unique $\sigma \in S_n$ and a unique $r \in \mathbb{Z}$ such that
\[ \tilde{\sigma}(\alpha) + (r, \ldots, r) = (\beta_1, \ldots, \beta_{n-1}, 0) \]

is a partition. In this case, all cohomology groups of $L_{\alpha}(\mathbb{C}^n)$ vanish for $i \neq \ell(\sigma)$, and one has
\[ H^{\ell(\sigma)}\left( \text{Flag}_s \mathbb{C}^n, L_{\alpha}(\mathbb{C}^n) \right) \cong S^{\beta} \mathbb{C}^n. \]

Let us note that, if one takes $\alpha$ to be a partition, then the conclusion of the previous corollary regarding the space of global sections (i.e. the $H^0$) is due to Borei-Weil [Ser95].

**APPENDIX B. COMPUTATIONS ON SAGE.**

In this section, we provide the outcome of a few programs written on the language Sage. We used the description of the $H^0$ provided by Theorem 3.2.3 (for codimension $1$ and codimension $2$ complete intersections).
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