Linear Complexity of Geometric Sequences Defined by Cyclotomic Classes and Balanced Binary Sequences Constructed by the Geometric Sequences *

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Abstract

Pseudorandom number generators are required to generate pseudorandom numbers which have good statistical properties as well as unpredictability in cryptography. An m-sequence is a linear feedback shift register sequence with maximal period over a finite field. M-sequences have good statistical properties, however we must nonlinearize m-sequences for cryptographic purposes. A geometric sequence is a sequence given by applying a nonlinear feedforward function to an m-sequence. Nogami, Tada and Uehara proposed a geometric sequence whose nonlinear feedforward function is given by the Legendre symbol, and showed the period, periodic autocorrelation and linear complexity of the sequence. Furthermore, Nogami et al. proposed a generalization of the sequence, and showed the period and periodic autocorrelation. In this paper, we first investigate linear complexity of the geometric sequences. In the case that the Chan–Games formula which describes linear complexity of geometric sequences does not hold, we show the new formula by considering the sequence of complement numbers, Hasse derivative and cyclotomic classes. Under some conditions, we can ensure that the geometric sequences have a large linear complexity from the results on linear complexity of Sidelnikov sequences. The geometric sequences have a long period and large linear complexity under some conditions, however they do not have the balance property. In order to construct sequences that have the balance property, we propose interleaved sequences of the geometric sequence and its complement. Furthermore, we show the periodic autocorrelation and linear complexity of the proposed sequences. The proposed sequences have the balance property, and have a large linear complexity if the geometric sequences have a large one.

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1 Introduction

Pseudorandom number generators play an important role to ensure that cryptography is safely used. Therefore, pseudorandom number generators are required to generate pseudorandom numbers which have good statistical properties as well as unpredictability in cryptography.

An m-sequence [10, 15] is well-distributed sequence with long period. However, we must nonlinearize this sequence for cryptographic purposes. A geometric sequence [6] is a sequence given by applying a nonlinear feedforward function to an m-sequence, such as GMW sequences [14, 29] and cascaded GMW sequences [19, 12].

Nogami, Tada and Uehara [26] proposed a geometric sequence whose nonlinear feedforward function is given by the Legendre symbol (this sequence is referred to as the NTU sequence). They showed the period, periodic autocorrelation and linear complexity of the NTU sequence. The studies on generalization of the sequences have made progress in recent years [4, 28, 27, 25, 20, 3, 17, 23, 32]. In particular, Nogami et al. [27] proposed a generalization of the NTU sequence (this sequence is referred to as the generalized NTU sequence), and showed the period and periodic autocorrelation.

In this paper, we first investigate the linear complexity of the generalized NTU sequences. Chan and Games [6] obtained a formula which describes the linear complexity of geometric sequences by that of sequences with a shorter period. However, under certain conditions, we cannot apply the formula to the generalized NTU sequences. In these cases, we show a similar formula. This formula induces some conditions to have a large linear complexity from the results on linear complexity of Sidelnikov sequences.

Generalized NTU sequences have a long period and large linear complexity under some conditions, however they do not have the balance property. In order to construct sequences that have the balance property, we propose interleaved sequences of the generalized NTU sequence and its complement as in the case of NTU sequences [32]. Furthermore, we show the periodic autocorrelation and linear complexity of the interleaved sequences. The interleaved sequences have the balance property by the interleaved structure.

This paper is organized as follows: Sect. 2 describes some preliminaries. Sect. 3 introduces the generalized NTU sequences. Sect. 4 considers the linear complexity of the generalized NTU sequences. Sect. 5 proposes the interleaved sequences of the generalized NTU sequence and its complement and shows the period, periodic autocorrelation and linear complexity of the proposed sequences. Finally, Sect. 6 concludes the paper.

We give some notations. For a prime number $p$ and an integer $a$, $(a/p)$ denotes the Legendre symbol. For a prime power $q$, $\mathbb{F}_q$ denotes the finite field with $q$ elements. For the extension field $\mathbb{F}_{q^m}$ of degree $m$ of $\mathbb{F}_q$, $\text{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q} : \mathbb{F}_{q^m} \to \mathbb{F}_q$ denotes the trace map, namely, $\text{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\alpha) = \alpha + \alpha^q + \cdots + \alpha^{q^{m-1}} \in \mathbb{F}_q$ for $\alpha \in \mathbb{F}_{q^m}$. Let $D$ be a subset of $\mathbb{F}_q$. For $c \in \mathbb{F}_q$, we define $D + c = \{d + c \mid d \in D\}$ and $cD = \{cd \mid d \in D\}$. Let $f(x) \in \mathbb{F}_q[x]$ be a polynomial over $\mathbb{F}_q$. $\deg f$ denotes the degree of $f(x)$. For $\xi \in \overline{\mathbb{F}_q}$, $\text{ml}_\xi(f(x))$ denotes the multiplicity of $\xi$ as zero of $f(x)$, where $\overline{\mathbb{F}_q}$ is the algebraic closure of $\mathbb{F}_q$. For an integer $n > 1$ and an integer $a$, $a \mod n$ (resp. $a \mod n$) denotes the integer $u \in \{0, 1, \ldots, n-1\}$ (resp. $u \in \{1, 2, \ldots, n\}$) such that $u \equiv a \mod n$. 
2 Preliminaries

In this section, we introduce some definitions and known results used throughout the paper.

2.1 Linear Complexity

Let $\ell$ be a prime number. Let $S = (S_n)_{n \geq 0}$ be a sequence over $\mathbb{F}_\ell$. The linear complexity $L(S)$ of $S$ is the length $L$ of the shortest linear recurrence relation

$$S_{n+L} = a_{L-1}S_{n+L-1} + \cdots + a_0S_n, \quad n \geq 0$$

for some $a_0, \ldots, a_{L-1} \in \mathbb{F}_\ell$, and the minimal polynomial $m_S(x)$ of $S$ is the polynomial $x^L - a_{L-1}x^{L-1} - \cdots - a_0 \in \mathbb{F}_\ell[x]$. Assume that $S$ is a periodic sequence of period $N$. $S(x)$ denotes the polynomial $S_0 + S_1x + \cdots + S_{N-1}x^{N-1} \in \mathbb{F}_\ell[x]$. Then $m_S(x) = (x^N-1)/\gcd(x^N-1, S(x))$ and $L(S) = N - \deg \gcd(x^N-1, S(x))$ (cf. [8, Lemma 8.2.1]).

2.2 Periodic Cross-Correlation and Periodic Autocorrelation

Let $S^{(1)} = (S_n^{(1)})_{n \geq 0}$ and $S^{(2)} = (S_n^{(2)})_{n \geq 0}$ be periodic sequences of period $N$ over $\mathbb{F}_2$. For $\tau \in \{0, 1, \ldots, N-1\}$, $R_{S^{(1)}, S^{(2)}}(\tau)$ is defined as

$$R_{S^{(1)}, S^{(2)}}(\tau) = \sum_{i=0}^{N-1} (-1)^{S_i^{(1)} + S_{i+\tau}^{(2)}}.$$

Note that $S_i^{(1)} + S_{i+\tau}^{(2)}$ means an addition in $\mathbb{F}_2$ and $i + \tau$ is performed modulo $N$. For an integer $\tau \notin \{0, 1, \ldots, N-1\}$, $R_{S^{(1)}, S^{(2)}}(\tau)$ means $R_{S^{(1)}, S^{(2)}}(\tau \mod N)$ in this paper. If $S^{(1)} \neq S^{(2)}$, then $R_{S^{(1)}, S^{(2)}}(\tau)$ is called a periodic cross-correlation of $S^{(1)}$ and $S^{(2)}$. If $S := S^{(1)} = S^{(2)}$, then $R_S(\tau) := R_{S,S}(\tau)$ is called a periodic autocorrelation of $S$. By the definition, $R_S(0) = N$.

2.3 Geometric Sequences

Let $q$ be an odd prime power, $m > 1$ an integer, $\omega$ a primitive element in $\mathbb{F}_{q^m}$ and $R = (R_n)_{n \geq 0}$ an m-sequence defined as $R_n = \text{Tr}_{q^m/q}(\omega^n)$, $n \geq 0$. Put $\nu = (q^m - 1)/(q - 1)$ and $g = \omega^\nu \in \mathbb{F}_q$. Let $\ell$ be a prime divisor of $q - 1$ and $\rho : \mathbb{F}_q \to \mathbb{F}_\ell$ a (nonlinear) map. Let $S = (S_n)_{n \geq 0}$ be a sequence defined as $S_n = \rho(R_n)$, $n \geq 0$, and $s = (s_n)_{n \geq 0}$ a sequence defined as $s_n = \rho(g^n)$, $n \geq 0$. $S$ is called a geometric sequence. Chan and Games [6] showed that the linear complexity of $S$ is described by that of $s$ under certain condition as follows:

Lemma 1 ([6] Theorem). Assume that $\rho(0) = 0$. Then

$$L(S) = \nu L(s).$$

Chan and Games showed (1) in the case of $\ell = 2$. In fact, (1) holds for any divisor $\ell$ of $q - 1$. 

3
2.4 Cyclotomic Classes

We recall the definition of cyclotomic classes and difference parameters. For details, see [8, 13.3]. Let $p$ be a prime number and $d$ a divisor of $p - 1$. Let $g$ be a primitive element in $\mathbb{F}_p$. The sets

$$D_0 = \left\{ g^{id} \mid 0 \leq i < \frac{p - 1}{d} \right\}, \quad D_j = g^j D_0, \quad 1 \leq j \leq d - 1$$

are called cyclotomic classes of order $d$. For $a \in \mathbb{F}_p$, the difference parameter is defined as $d(i, j; a) = \#(D_i \cap (D_j - a))$, $i, j \in \{0, 1, \ldots, d - 1\}$. If $a = 1$, then $d(i, j; 1)$ is the cyclotomic number $(i, j)_d$ of order $d$.

Assume that $d = 2$. Then $D_0$ and $D_1$ are the sets of non-zero quadratic residues modulo $p$ and of quadratic non-residues modulo $p$, respectively. For $a \in \mathbb{F}_p$, $d(i, j; a)$ is given as in Table 1.

| $(i, j)$       | $(a/p) = 1$ | $(a/p) = -1$ |
|---------------|-------------|--------------|
| $(0, 0)$      | $(p - 5)/4$ | $(p - 1)/4$  |
| $(0, 1)$      | $(p - 1)/4$ | $(p - 1)/4$  |
| $(1, 0)$      | $(p - 1)/4$ | $(p - 3)/4$  |
| $(1, 1)$      | $(p - 5)/4$ | $(p - 3)/4$  |

2.5 Hasse Derivative

Let $\ell$ be a prime number. For $S(x) = \sum_{i=0}^{N-1} a_i x^i \in \mathbb{F}_\ell[x]$ and $k \geq 1 \in \mathbb{Z}$,

$$S^{(k)}(x) = \sum_{i=k}^{N-1} \binom{i}{k} a_i x^{i-k}$$

is called the $k$th Hasse derivative of $S(x)$. Let $\xi \in \overline{\mathbb{F}}_\ell$ be a root of $S(x)$, where $\overline{\mathbb{F}}_\ell$ is the algebraic closure of $\mathbb{F}_\ell$. Then $\text{ml}_\ell S(x) = v$ if and only if $S(\xi) = S^{(1)}(\xi) = \cdots = S^{(v-1)}(\xi) = 0$ and $S^{(v)}(\xi) \neq 0$ (cf. [23, Lemma 6.51]).

2.6 Interleaved Sequences and Left Cyclic Shift Sequences

For a family $S = \{S^{(i)} = (S_n^{(i)})_{n \geq 0} \mid 0 \leq i \leq T - 1\}$ of periodic sequences of period $N$, the sequence $U = (U_n)_{n \geq 0}$ defined as

$$U_n = S_j^{(i)} \text{ if } n = j \times T + i, \quad 0 \leq i \leq T - 1, \quad 0 \leq j$$

is called an interleaved sequence of $S$. 
For an periodic sequence \( S = (S_n)_{n \geq 0} \) of period \( N \) and \( \epsilon \in \{0, 1, \ldots, N-1\} \), the sequence \( L^\epsilon(S) = (L^\epsilon(S)_n)_{n \geq 0} \) defined as

\[
L^\epsilon(S)_n = S_{n+\epsilon}, \quad n \geq 0
\]
is called a left cyclic shift sequence.

3 Generalized NTU Sequences

In this section, we survey properties of the generalized NTU sequences. For details, see [26, 25, 27].

Let \( p \) be an odd prime number, \( m \geq 1 \) an integer, \( \omega \) a primitive element in \( \mathbb{F}_{p^m} \) and \( R = (R_n)_{n \geq 0} \) an \( m \)-sequence defined as \( R_n = \text{Tr}_{p^m/p}(\omega^n), \ n \geq 0 \). Put \( \nu = (p^m - 1)/(p-1) \) and \( g = \omega^\nu \in \mathbb{F}_p \). Let \( \ell \) be a prime divisor of \( p-1 \) and \( D_j, 0 \leq j \leq \ell - 1 \) cyclotomic classes of order \( \ell \). Let \( A \in \mathbb{F}_p \). We define a map \( \rho^\text{NTU}_A : \mathbb{F}_p \to \mathbb{F}_\ell \) as

\[
\rho^\text{NTU}_A(x) = \begin{cases} 
0 & \text{if } x + A = 0 \\
1 & \text{if } x + A \in D_k, \ 0 \leq k \leq \ell - 1.
\end{cases}
\]  

Let \( T^\text{NTU}_A = (T^\text{NTU}_{A,n})_{n \geq 0} \) be a sequence defined as

\[
T^\text{NTU}_{A,n} = \rho^\text{NTU}_A(R_n), \quad n \geq 0,
\]  

and \( t^\text{NTU}_{A,n} = (t^\text{NTU}_{A,n})_{n \geq 0} \) a sequence defined as

\[
t^\text{NTU}_{A,n} = \rho^\text{NTU}_A(g^n), \quad n \geq 0.
\]

\( T^\text{NTU}_A \) is called an \( \ell \)-ary generalized NTU sequence.

If \( A = 0 \), then \( T^\text{NTU}_0 \) is an \( \ell \)-ary NTU sequence [26, 25]. \( T^\text{NTU}_0 \) is a periodic sequence of period \( N_0 := \ell(p^m - 1)/(p-1) \). The autocorrelation distribution of \( T^\text{NTU}_0 \) have been explicitly obtained in [26, Property 6] and [25, Property 5]. The linear complexity of \( T^\text{NTU}_0 \) is \( L(T^\text{NTU}_0) = 2(p^m - 1)/(p-1) \). In particular, if \( \ell = 2 \), then \( L(T^\text{NTU}_0) \) attains the maximal value \( N_0 \).

Assume that \( A \neq 0 \). Then \( T^\text{NTU}_A \) is a periodic sequence of period \( N := p^m-1 \). The autocorrelation distribution of \( T^\text{NTU}_A \) have been explicitly obtained in [27, 3.3]. In particular, if \( \ell = 2 \), then the autocorrelation distribution of \( T^\text{NTU}_A \) is

\[
R_{T^\text{NTU}_A}(\tau) = \begin{cases} 
N & \text{if } \tau = 0 \\
N^{(j)} & \text{if } \tau = j\nu, \ 1 \leq j \leq p - 2 \\
N_2 & \text{otherwise}
\end{cases}
\]

for \( \tau \in \{0, 1, \ldots, N-1\} \). Here \( N^{(j)} \) and \( N_2 \) are defined as

\[
N^{(j)} = p^{m-1} \left\{ (-1)^{j} g^\nu A + (-1)^{j+1} A \right\} - 1, \quad (5)
\]
\[
N_2 = p^{m-2} - 1,
\]
respectively.

Remark 2. If \( A = 1 \), then \( t^\text{NTU}_1 \) is an \( \ell \)-ary Sidelnikov sequence [30, 22].
Lemma 3. Let $A_1, A_2 \in \mathbb{F}_p \setminus \{0\}$. If $A_1$ and $A_2$ belong to the same cyclotomic class, then there exists $i \in \{0, 1, \ldots, p^m - 2\}$ such that $T_{A_2,n}^{\text{NTU}} = T_{A_1,n+i}^{\text{NTU}}$, $n \geq 0$.

Proof. The statement follows from the shift-and-add property of m-sequences.

4 Linear Complexity of Generalized NTU Sequences

In this section, we consider linear complexity of generalized NTU sequences.

Let $p, m, \omega, R = (R_n)_{n \geq 0}, \nu, g, \ell$ and $D_j, 0 \leq j \leq \ell - 1$ be as in Sect. 3. Let $A \in \mathbb{F}_p \setminus \{0\}$. Let $\rho_A^{\text{NTU}} : \mathbb{F}_p \to \mathbb{F}_\ell$ be a map defined as (2), $T_A^{\text{NTU}} = (T_{A,n}^{\text{NTU}})_{n \geq 0}$ a sequence defined as (3) and $t_A^{\text{NTU}} = (t_{A,n}^{\text{NTU}})_{n \geq 0}$ a sequence defined as (4).

4.1 The Case of $A \in D_0$

Assume that $A \in D_0$. Since $\rho_A^{\text{NTU}}(0) = 0$, we can apply (1) to $L(T_A^{\text{NTU}})$. By Lemma 3, we have

$$L(T_A^{\text{NTU}}) = \nu L(t_1^{\text{NTU}}).$$

Since $t_1^{\text{NTU}}$ is a Sidelnikov sequence (see [30, 22] and Remark 2), $L(T_A^{\text{NTU}})$ is obtained from some known results [18, 21, 24, 1, 36, 5] on $L(t_1^{\text{NTU}})$.

4.2 The Case of $A \in D_k$, $k \neq 0$

Assume that $A \in D_k$, $k \neq 0$. Since $\rho_A^{\text{NTU}}(0) \neq 0$, we can not apply (1) to $L(T_A^{\text{NTU}})$. In order to apply (1), we introduce a sequence whose any term is added by a complement of $k$.

We define a map $\overline{\rho}_A^{\text{NTU}} : \mathbb{F}_p \to \mathbb{F}_\ell$ as

$$\overline{\rho}_A^{\text{NTU}}(x) = \rho_A^{\text{NTU}}(x) - k, \quad x \in \mathbb{F}_p.$$ (7)

Let $T_{A,n}^{\text{NTU}} = (T_{A,n}^{\text{NTU}})_{n \geq 0}$ be a sequence defined as

$$T_{A,n}^{\text{NTU}} = \rho_A^{\text{NTU}}(R_n), \quad n \geq 0,$$ (8)

and $t_{A,n}^{\text{NTU}} = (t_{A,n}^{\text{NTU}})_{n \geq 0}$ a sequence defined as

$$t_{A,n}^{\text{NTU}} = \rho_A^{\text{NTU}}(g^n), \quad n \geq 0.$$ (9)

Since $\rho_A^{\text{NTU}}(0) = \rho_A^{\text{NTU}}(0) - k = 0$, we can apply (1) to $L(T_{A,n}^{\text{NTU}})$.

Lemma 4 ([23] Theorem 8.62). Let $T = (T_n)_{n \geq 0}$ be a periodic sequence of period $N$ over $\mathbb{F}_\ell$. For $a \in \mathbb{F}_\ell \setminus \{0\}$, let $\overline{T} = (\overline{T}_n)_{n \geq 0}$ be a sequence defined by $\overline{T}_n = T_n + a$, $n \geq 0$.

1. If $\text{ml}_1(x^N - 1) \leq \text{ml}_1(T(x))$, then $L(\overline{T}) = L(T) + 1$.

2. If $\text{ml}_1(x^N - 1) = \text{ml}_1(T(x)) + 1$, then $L(\overline{T}) = L(T) - 1$.
3. If \( ml_1 (x^N - 1) > ml_1 (T(x)) + 1 \), then \( L(T) = L(T) \).

Proposition 5. Assume that \( A \in D_k, k \neq 0 \). If \( \ell = 2 \) and \( p \equiv 1 \mod 4 \) or \( \ell \geq 3 \), then

\[
L(T_A^{\text{NTU}}) = \nu L(t_A^{\text{NTU}}).
\]

Proof. Since \( p \equiv 1 \mod \ell \), \( p^m - 1 \equiv p - 1 \equiv 0 \mod \ell \). Hence \( x^{p^m - 1} - 1 = (x^{p^{m-1}/\ell - 1})^\ell \) and \( x^{p^{m-1} - 1} = (x^{(p-1)/\ell - 1})^\ell \). Therefore, \( ml_1 (x^{p^{m-1} - 1}) \geq \ell \geq 2 \) and \( ml_1 (x^{p^{m-1} - 1}) \geq \ell \geq 2 \). On the other hand,

\[
T_A^{\text{NTU}}(1) = \sum_{i=0}^{p^m-2} \frac{p^m-1}{2} - k
\]

and

\[
t_A^{\text{NTU}}(1) = \sum_{i=0}^{p^m-2} \frac{p^m-1}{2} - k
\]

Hence the statement follows from Lemma 1 and Lemma 4.

Example 6. Let \( p = 29, m = 3 \) and \( \ell = 2 \). Assume that \( (A/p) = -1 \). Then

\[
L(T_A^{\text{NTU}}) = L(T_A^{\text{NTU}}) = 24388,
\]

\[
L(t_A^{\text{NTU}}) = L(t_A^{\text{NTU}}) = 28,
\]

\[
\nu = \frac{p^m - 1}{p - 1} = 871.
\]

If \( \ell = 2 \), \( (A/p) = -1 \) and \( p \equiv 3 \mod 4 \), then \( ml_1 (T_A^{\text{NTU}} (x)) \geq 1 \) and \( ml_1 (t_A^{\text{NTU}} (x)) \geq 1 \). Therefore, we evaluate the first Hasse derivative of \( T_A^{\text{NTU}} (x) \) at 1.

Lemma 7. Assume that \( \ell = 2, (A/p) = -1, p \equiv 3 \mod 4 \) and \( m \equiv 1 \mod 2 \). Then

\[
T_A^{\text{NTU}(1)}(1) = \begin{cases} 
0 & \text{if } p \equiv 3 \mod 8 \\
1 & \text{if } p \equiv 7 \mod 8. 
\end{cases}
\]

Proof. Since

\[
T_A^{\text{NTU}(1)}(x) = \sum_{i=1}^{p^m-2} \binom{i}{1} T_A^{\text{NTU}} x^{i-1} = \sum_{i=0}^{(p^m-3)/2} T_A^{\text{NTU}} x^{2i+1},
\]

we have

\[
T_A^{\text{NTU}(1)}(1) = \sum_{i=0}^{(p^m-3)/2} T_A^{\text{NTU}} x^{2i+1}.
\]
For $0 \leq j \leq \nu - 1$, put $R_j = (R_{w+j})_{0 \leq i \leq p-2} \in \mathbb{F}_p^p$.

Assume that $R_j = 0$. Since all the terms of $(T_{A,w+j}^{NTU})_{0 \leq i \leq p-2} \in \mathbb{F}_2^{p-1}$ are 1, the number of terms satisfying $T_{A,w+j}^{NTU} = 1$ and $i \nu + j \equiv 1 \mod 2$ is $(p-1)/2$.

Assume that $R_j \neq 0$. Then

$$
\{ R_{2i+w+j} \mid 0 \leq i \leq \frac{p-3}{2} \} = D_k \quad \text{and} \quad \{ R_{(2i+1)w+j} \mid 0 \leq i \leq \frac{p-3}{2} \} = D_{1-k},
$$

where $k = 0$ or 1. Hence the number of terms of $(T_{A,w+j}^{NTU})_{0 \leq i \leq p-2} \in \mathbb{F}_2^{p-1}$ such that $T_{A,w+j}^{NTU} = 1$ and $i \nu + j \equiv 1 \mod 2$ is $d(0,1;A)$ or $d(1,1;A)$, that is $(p-3)/4$ by Table 1.

On the other hand, the number of terms of $(R_j)_{0 \leq j \leq \nu-1} \in \mathbb{F}_p$ such that $R_j = 0$ is $(p^{\nu-1} - 1)/(p-1)$ (cf. [6, Lemma 3]). Hence

$$
\sum_{i=0}^{(p^{\nu-1})/2} T_{A,2i+1}^{NTU} = \frac{p^{\nu-1} - 1}{p-1} \times \frac{p-1}{2} + \left\{ \nu - \frac{p^{\nu-1} - 1}{p-1} \right\} \times \frac{p-3}{4}
$$

$$
= \frac{p^{\nu-1} - 2}{4}
$$

$$
= \begin{cases} 0 & \text{if } p \equiv 3 \mod 8 \\ 1 & \text{if } p \equiv 7 \mod 8. \end{cases}
$$

\[ \square \]

**Theorem 8.** Assume that $\ell = 2$, $(A/p) = -1, p \equiv 3 \mod 4$ and $m \equiv 1 \mod 2$.

Then

$$
L(T_{A}^{NTU}) = \begin{cases} \nu L(t_{A}^{NTU})^{-1} + 1 & \text{if } p \equiv 3 \mod 8 \\ \nu L(t_{A}^{NTU})^{-1} + 1 & \text{if } p \equiv 7 \mod 8. \end{cases}
$$

**Proof.** Since $p \equiv 3 \mod 4$ and $m \equiv 1 \mod 2$, $ml_1(x^{p^{\nu-1}} - 1) = 2$. If $p \equiv 3 \mod 8$ (resp. if $p \equiv 7 \mod 8$), then $ml_1(T_{A}^{NTU}(x)) \geq 2$ (resp. $ml_1(T_{A}^{NTU}(x)) = 1$) by Lemma 7. Hence, we have

$$
L(T_{A}^{NTU}) = \begin{cases} \nu L(t_{A}^{NTU})^{-1} - 1 & \text{if } p \equiv 3 \mod 8 \\ \nu L(t_{A}^{NTU}) + 1 & \text{if } p \equiv 7 \mod 8. \end{cases}
$$

by Lemma 1 and Lemma 4. On the other hand, since $p \equiv 3 \mod 4$, $ml_1(x^{p^{\nu-1}} - 1) = 2$. By Table 1,

$$
n_{A}^{NTU(1)}(1) = \sum_{i=0}^{(p-3)/2} n_{A,2i+1}^{NTU} = d(1,1;A) = \frac{p-3}{4}
$$

$$
= \begin{cases} 0 & \text{if } p \equiv 3 \mod 8 \\ 1 & \text{if } p \equiv 7 \mod 8. \end{cases}
$$

Hence, if $p \equiv 3 \mod 8$ (resp. if $p \equiv 7 \mod 8$), then $ml_1(t_{A}^{NTU}(x)) \geq 2$ (resp. $ml_1(t_{A}^{NTU}(x)) = 1$). Therefore the statement follows from Lemma 4. \[ \square \]

**Example 9.** Let $p = 43, m = 3$ and $\ell = 2$. Assume that $(A/p) = -1$. Then

$$
L(T_{A}^{NTU}) = 77612, \quad L(t_{A}^{NTU}) = 77613,
$$

$$
L(t_{A}^{NTU}) = 40, \quad L(t_{A}^{NTU}) = 41,
$$

$$
\nu = \frac{p^p - 1}{p - 1} = 1893.
$$
Lemma 12. Let \( p = 47, m = 3 \) and \( \ell = 2 \). Assume that \( (A/p) = -1 \). Then
\[
L(T_A^{\text{NTU}}) = 99309, \quad L(\overline{T_A^{\text{NTU}}}) = 99308,
\]
\[
L(t_A^{\text{NTU}}) = 45, \quad L(\overline{T_A^{\text{NTU}}}) = 44,
\]
\[
\nu = \frac{p^m - 1}{p - 1} = 2257.
\]

Remark 11. If \( m \equiv 0 \mod 2 \), we need to evaluate higher Hasse derivative of \( T_A^{\text{NTU}}(x) \) at \( 1 \). In [17], a formula was conjectured by numerical experiments as follows: Assume that \( \ell = 2, (A/p) = -1, p \equiv 3 \mod 4 \) and \( m \equiv 0 \mod 2 \). Then
\[
L(T_A^{\text{NTU}}) = \begin{cases} 
\nu (L(t_A^{\text{NTU}}) + 1) & \text{if } p \equiv 3 \mod 8 \\
\nu (L(t_A^{\text{NTU}}) - 1) + 1 & \text{if } p \equiv 7 \mod 8.
\end{cases}
\]

4.3 Conditions to Have a Large Linear Complexity

In order to show conditions to have a large linear complexity, we refer to the result shown by Meidl and Winterhof [24] and Brandstätter and Meidl [5].

Lemma 12 ([24] Proposition 3, [5] Proposition 1). Let \( r \neq \ell \) be a prime divisor of \( p - 1 \). If \( \ell \) is a primitive root in \( \mathbb{F}_p \), and \( r \geq \sqrt{p} + 1 \), then for each \( r \)th root of unity \( \beta \neq 1 \) we have \( t_A^{\text{NTU}}(\beta) \neq 0 \).

Lemma 12 was shown in the case of \( A = 1 \). However, Lemma 12 holds for any \( A \in \mathbb{F}_p \setminus \{0\} \). Therefore, we obtain conditions to have a large linear complexity in the case of \( \ell = 2 \) as follows:

Corollary 13. Assume that \( \ell = 2 \). Let \( p \) be a prime number of the form \( p = 2^r + 1 \), where \( r \) is an odd prime number such that \( 2 \) is a primitive root in \( \mathbb{F}_r \) and \( r \geq \sqrt{p} + 1 \).

1. If \( A = 1 \) and \( s = 1 \) or \( (A/p) = -1 \) and \( s \geq 2 \), then the minimal polynomials \( m_{e_A^{\text{NTU}}}(x) \) and \( m_{f_A^{\text{NTU}}}(x) \) of \( t_A^{\text{NTU}} \) and \( T_A^{\text{NTU}} \) are given as
\[
m_{e_A^{\text{NTU}}}(x) = x^{p - 1} + 1, \quad m_{f_A^{\text{NTU}}}(x) = x^{p^{m - 1}} + 1,
\]
respectively. Therefore the linear complexity \( L(t_A^{\text{NTU}}) \) and \( L(T_A^{\text{NTU}}) \) of \( t_A^{\text{NTU}} \) and \( T_A^{\text{NTU}} \) are given as
\[
L(t_A^{\text{NTU}}) = p - 1, \quad L(T_A^{\text{NTU}}) = p^{m} - 1,
\]
respectively.

2. If \( (A/p) = -1, m \equiv 1 \mod 2 \) and \( p \equiv 7 \mod 8 \), then the minimal polynomials \( m_{e_A^{\text{NTU}}}(x) \) and \( m_{f_A^{\text{NTU}}}(x) \) of \( t_A^{\text{NTU}} \) and \( T_A^{\text{NTU}} \) are given as
\[
m_{e_A^{\text{NTU}}}(x) = \frac{x^{p - 1} + 1}{x + 1}, \quad m_{f_A^{\text{NTU}}}(x) = \frac{(x + 1)(x^{p^{m - 1}} + 1)}{x^{2} + 1},
\]
respectively. Therefore the linear complexity \( L(t_A^{\text{NTU}}) \) and \( L(T_A^{\text{NTU}}) \) of \( t_A^{\text{NTU}} \) and \( T_A^{\text{NTU}} \) are given as
\[
L(t_A^{\text{NTU}}) = p - 2, \quad L(T_A^{\text{NTU}}) = \frac{p^{m+1} - 3p^m + 2}{p - 1},
\]
respectively.
Proof. We only show the case that \((A/p) = -1, m \equiv 1 \mod 2\) and \(p \equiv 7 \mod 8\) (the proofs for the other cases are trivial). By the argument in the previous subsection,
\[
m_{\iota}^{NTU}(x) = (x + 1)m_{\iota}^{NTU}(x), \quad m_{\iota}^{NTU}(x) = (x + 1)m_{\iota}^{NTU}(x).
\]
Noting that \(m_{\iota}^{NTU}(x) = m_{\iota}^{NTU}(x')\) (cf. [6, p.551, Theorem]), we have
\[
m_{\iota}^{NTU}(x) = \frac{x + 1}{x^{2^{\nu} + 1}} m_{\iota}^{NTU}(x') = \frac{(x + 1)(x^{p^m - 1} + 1)}{x^{2^{\nu} + 1}}.
\]
\[
\square
\]
Remark 14. For the case of \(\ell = 3\), we can obtain some conditions to have a large linear complexity by [5, Theorem 1].

5 Interleaved Sequences of Generalized NTU Sequences

Generalized NTU sequences have good properties for linear complexity as discussed in the previous section, however they do not have the balance property. In order to construct sequences that have the balance property, we introduce interleaved sequences as in the case of NTU sequences [32]. Gong [11] introduced the concept of interleaved sequences that were obtained by merging sequences, and sequence interleaving was widely used to improve the balance property of existing sequences such as in [31, 16, 13, 9, 7, 35, 34].

In this section, we propose interleaved sequences of a generalized NTU sequence and its complement. By interleaving them, the interleaved sequence has the balance property by the interleaved structure. Furthermore, we show periodic autocorrelation and linear complexity of the interleaved sequences.

Let \(p, m, \omega, R = (R_n)_{n \geq 0}, \nu, g, \ell, 0 \leq j \leq \ell - 1\) be as in Sect. 3. Let \(A \in \mathbb{F}_p\). Let \(\rho_{A}^{NTU} : \mathbb{F}_p \to \mathbb{F}_\ell\) be a map defined as (2) and \(T_{A}^{NTU} = (T_{A,n}^{NTU})_{n \geq 0}\) a sequence defined as (3). Let \(\rho_{A}^{NTU} : \mathbb{F}_p \to \mathbb{F}_\ell\) be a map defined as (7) and \(T_{A}^{NTU} = (T_{A,n}^{NTU})_{n \geq 0}\) a sequence defined as (8). Throughout this section, assume that \(\ell = 2\). Put \(N = p^m - 1\).

In the case of \(A = 0\), interleaved sequences of NTU sequences were studied in [32]. We apply a similar argument to the case of \(A \neq 0\).

5.1 Interleaved Sequences of Generalized NTU Sequences

For \(e \in \{0, 1, \ldots, N - 1\}\), we define a sequence \(S^e = (S^e_n)_{n \geq 0}\) as the interleaved sequence of \(\left\{T_{A}^{NTU}, L_e \left(\overline{T_{A}^{NTU}}\right)\right\}\).

Example 15. Let \(p = 3\) and \(m = 2\), so that \(N = 8\). Let \(\omega\) be a root of the primitive polynomial \(x^2 + 2x + 2\). Then \(T_{A}^{NTU}\) is given as
\[
T_{A}^{NTU} = (0, 1, 0, 1, 1, 0, 0, \ldots).
\]
Put \(e = 2\). Then \(S^e = S^2\) is given as
\[
S^2 = (0, 1, 1, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 0, 0, \ldots).
\]
5.2 Periodic Autocorrelation of the Proposed Sequences

We show some lemmas to obtain the periodic autocorrelation of the proposed sequences.

Lemma 16. For $\tau \in \{0, 1, \ldots, N-1\}$,

$$R_{\overline{T_A, T_A \overline{NTU}}} (\tau) = -R_{\overline{T_A \overline{NTU}}} (\tau).$$

Proof. It follows that

$$R_{\overline{T_A, T_A \overline{NTU}}} (\tau) = \sum_{i=0}^{N-1} (-1)^{T_{A,i} + T_{A,i+\tau}} = \sum_{i=0}^{N-1} (-1)^{T_{A,i} + T_{A,i+\tau} + 1}$$

$$= \# \{ i \mid T_{A,i} = T_{A,i+\tau} + 1, 0 \leq i \leq N-1 \}$$

$$- \# \{ i \mid T_{A,i} \neq T_{A,i+\tau} + 1, 0 \leq i \leq N-1 \}$$

$$= \# \{ i \mid T_{A,i} \neq T_{A,i+\tau}, 0 \leq i \leq N-1 \}$$

$$- \# \{ i \mid T_{A,i} = T_{A,i+\tau}, 0 \leq i \leq N-1 \}$$

$$= -R_{\overline{T_A \overline{NTU}}} (\tau).$$

Lemma 17. If $\tau = 2\tau_0, \tau_0 \in \{0, 1, \ldots, N-1\}$, then

$$R_{S^1, S^2} (\tau) = R_{\overline{T_A \overline{NTU}}} (\tau_0) + R_{\overline{T_A \overline{NTU}}} (e_2 - e_1 + \tau_0).$$

If $\tau = 2\tau_0 + 1, \tau_0 \in \{0, 1, \ldots, N-1\}$, then

$$R_{S^1, S^2} (\tau) = -R_{\overline{T_A \overline{NTU}}} (e_2 + \tau_0) - R_{\overline{T_A \overline{NTU}}} (e_1 - \tau_0 - 1).$$

Proof. Assume that $\tau = 2\tau_0, \tau_0 \in \{0, 1, \ldots, N-1\}$. Then

$$R_{S^1, S^2} (\tau) = R_{\overline{T_A \overline{NTU}}} (\tau_0) + R_{\overline{T_A \overline{NTU}}} (e_2 - e_1 + \tau_0)$$

$$= R_{\overline{T_A \overline{NTU}}} (\tau_0) + R_{\overline{T_A \overline{NTU}}} (e_2 - e_1 + \tau_0).$$

Assume that $\tau = 2\tau_0 + 1, \tau_0 \in \{0, 1, \ldots, N-1\}$. By Lemma 16,

$$R_{S^1, S^2} (\tau) = R_{\overline{T_A \overline{NTU}}} (e_2 + \tau_0) + R_{\overline{T_A \overline{NTU}}} (e_1 - \tau_0 - 1)$$

$$= R_{\overline{T_A \overline{NTU}}} (e_2 + \tau_0) + R_{\overline{T_A \overline{NTU}}} (e_1 - \tau_0 - 1).$$

The periodic autocorrelation of the proposed sequence is obtained as follows:
Theorem 18. For \( e \in \{0, 1, \ldots, N - 1\} \), let \( S^e \) be the interleaved sequence of \( \{T_A^{\text{NTU}}, L^e \} \). For \( \tau = 2\tau_0, \tau_0 \in \{0, 1, \ldots, N - 1\} \), the periodic autocorrelation \( R_S^e(\tau) \) of \( S^e \) is given as

\[
R_S^e(\tau) = \begin{cases} 
2N & \text{if } \tau_0 = 0 \\
2N^{(j)} & \text{if } \tau_0 = j\nu, 1 \leq j \leq p - 2 \\
2N_2 & \text{otherwise.}
\end{cases}
\] (10)

For \( \tau = 2\tau_0 + 1, \tau_0 \in \{0, 1, \ldots, N - 1\} \), the periodic autocorrelation \( R_S^e(\tau) \) of \( S^e \) is given as

\[
R_S^e(\tau) = \begin{cases} 
-N - N_2 & \text{if } \tau_0 \equiv -e, e - 1 \text{ mod } N \\
-N^{(j)} - N_2 & \text{if } \tau_0 \equiv -e + j\nu, e - 1 - j\nu \text{ mod } N, 1 \leq j \leq p - 2 \\
-2N_2 & \text{otherwise}
\end{cases}
\] (11)

if \( 2e \not\equiv 1 + j\nu \text{ mod } N \) for any \( j \in \{1, \ldots, p - 2\} \), and

\[
R_S^e(\tau) = \begin{cases} 
-N - N^{(j_0)} & \text{if } \tau_0 \equiv -e, e - 1 \text{ mod } N \\
-N^{(j)} - N^{(j_0-j \text{ mod } p-1)} & \text{if } \tau_0 \equiv -e + j\nu, e - 1 - j\nu \text{ mod } N, 1 \leq j \leq p - 2, j \neq j_0 \\
-2N_2 & \text{otherwise}
\end{cases}
\] (12)

if there exists a \( j_0 \in \{1, \ldots, p - 2\} \) such that \( 2e \equiv 1 + j_0\nu \text{ mod } N \). Here \( N^{(j)}_1 \) and \( N_2 \) are defined as (5) and (6), respectively.

Proof. Assume that \( \tau = 2\tau_0, \tau_0 \in \{0, 1, \ldots, N - 1\} \). Since \( R_S^e(\tau) = 2R_{T_A^{\text{NTU}}}(\tau_0) \) by Lemma 17, \( R_S^e(\tau) \) is given as (10).

Assume that \( \tau = 2\tau_0 + 1, \tau_0 \in \{0, 1, \ldots, N - 1\} \). Then we have \( R_S^e(\tau) = -R_{T_A^{\text{NTU}}}(\epsilon + \tau_0) - R_{T_A^{\text{NTU}}}(\epsilon - \tau_0 - 1) \) by Lemma 17. On the other hand,

\[
R_{T_A^{\text{NTU}}}(\epsilon + \tau_0) = \begin{cases} 
N & \text{if } \tau_0 \equiv -e \text{ mod } N \\
N^{(j)}_1 & \text{if } \tau_0 \equiv -e + j\nu \text{ mod } N, 1 \leq j \leq p - 2 \\
N_2 & \text{otherwise}
\end{cases}
\]

and

\[
R_{T_A^{\text{NTU}}}(\epsilon - \tau_0 - 1) = \begin{cases} 
N & \text{if } \tau_0 \equiv e - 1 \text{ mod } N \\
N^{(j)}_1 & \text{if } \tau_0 \equiv e - 1 - j\nu \text{ mod } N, 1 \leq j \leq p - 2 \\
N_2 & \text{otherwise}
\end{cases}
\]

If \( 2e \not\equiv 1 + j\nu \text{ mod } N \) for any \( j \in \{1, \ldots, p - 2\} \), then \(-e, -e + j\nu, e - 1, e - 1 - j\nu \) are incongruent modulo \( N \). Hence \( R_S^e(\tau) \) is given as (11). If there exists a \( j_0 \in \{1, \ldots, p - 2\} \) such that \( 2e \equiv 1 + j_0\nu \text{ mod } N \), then \(-e + j_0\nu \equiv e - 1 \text{ mod } N \) and \(-e \equiv e - 1 - j_0\nu \text{ mod } N \). Hence \( R_S^e(\tau) \) is given as (12).

Corollary 19. \( S^e \) has period \( 2N = 2p^m - 1 \).

By the interleaved structure, \( S^e \) has the balance property, namely the number of zeros and ones in one period of \( S^e \) are \( N \).

Remark 20. If \( m \) is even, then \( 2e \not\equiv 1 + j\nu \text{ mod } N \) for any \( j \in \{1, \ldots, p - 2\} \).
Figure 1: The graph of the periodic autocorrelation of $S^2$ in the case of $p = 7, m = 2$.

Figure 2: The graph of the periodic autocorrelation of $S^6$ in the case of $p = 5, m = 3$.

**Example 21.** Let $p = 7$ and $m = 2$, so that $N = 48$. Figure 1 describes the graph of the periodic autocorrelation of $S^2$.

**Example 22.** Let $p = 5$ and $m = 3$, so that $N = 124$. Figure 2 and Fig. 3 describe the graphs of the periodic autocorrelation of $S^6$ and $S^{16}$, respectively. Note that $\nu = 31$ and $2 \cdot 16 = 1 + 1 \cdot 31 = 32$.

### 5.3 Linear Complexity of the Proposed Sequences

The minimal polynomial and linear complexity of the proposed sequence are obtained as follows:

**Theorem 23.** For $e \in \{0, 1, \ldots, N - 1\}$, let $S^e$ be the interleaved sequence of $\left\{ T_A^{\text{NTU}}, L^e \left( T_A^{\text{NTU}} \right) \right\}$. Assume that $p$ is a prime number of the form $p = 2^s r + 1$, where $r$ is an odd prime number such that $2$ is a primitive root in $\mathbb{F}_r$ and $r \geq \sqrt{p} + 1$. 

---
Figure 3: The graph of the periodic autocorrelation of $S^{16}$ in the case of $p = 5, m = 3$.

1. If $A = 1$ and $s = 1$ or $(A/p) = -1$ and $s \geq 2$, then the minimal polynomial $m_{S^e}(x)$ of $S^e$ is given as

$$m_{S^e}(x) = \frac{x^{2N} + 1}{x^{G(N,e)} + 1},$$

where $G(N,e) = \gcd(-2e + 1 \mod N/2^{\nu_2(N)}, N/2^{\nu_2(N)})$ and $\nu_2(N)$ is the exponent of the largest power of 2 which divides $N$. Therefore the linear complexity $L(S^e)$ of $S^e$ is given as

$$L(S^e) = 2N - G(N,e) = 2L(T_A^{\text{NTU}}) - G(N,e).$$

2. If $(A/p) = -1$, $m \equiv 1 \mod 2$ and $p \equiv 7 \mod 8$, then the minimal polynomial $m_{S^e}(x)$ of $S^e$ is given as

$$m_{S^e}(x) = \frac{(x^{2N} + 1)(x^2 + 1)(x^{H_0(N,e)} + 1)}{(x^{4\nu} + 1)(x^{H_1(N,e)} + 1)},$$

where $H_0(N,e) = \gcd(-2e + 1 \mod \nu, \nu)$ and $H_1(N,e) = \gcd(-2e + 1 \mod N/2, N/2)$. Therefore the linear complexity $L(S^e)$ of $S^e$ is given as

$$L(S^e) = 2N + 2 + 4\nu + H_0(N,e) - H_1(N,e) = 2L(T_A^{\text{NTU}}) + H_0(N,e) - H_1(N,e).$$

Proof. Since

$$L^e(T_A^{\text{NTU}})(x) \equiv x^{N-e}T_A^{\text{NTU}}(x) \mod x^N + 1$$

$$\equiv x^{N-e}T_A^{\text{NTU}}(x) + x^{N-e} \frac{x^N + 1}{x + 1} \mod x^N + 1,$$
Therefore we have
\[ S^\circ(x) = T_A^{NTU}(x^2) + xL_e \left( T_A^{NTU}(x^2) \right) \]
\[ \equiv T_A^{NTU}(x^2) + x \left( x^{2N-2e}T_A^{NTU}(x^2) + x^{2N-2e} \cdot \frac{x^{2N} + 1}{x^2 + 1} \right) \mod x^{2N} + 1 \]
\[ \equiv \left( x^{2N-2e+1} + 1 \right) T_A^{NTU}(x^2) + x^{2N-2e+1} \cdot \frac{x^{2N} + 1}{x^2 + 1} \mod x^{2N} + 1. \]

Assume that \( A = 1 \) and \( s = 1 \) or \( (A/p) = -1 \) and \( s \geq 2 \). By Corollary 13, \( L(T_A^{NTU}) = N \). Since \( 2N - 2e + 1 \equiv 1 \mod 2 \), \( x^2 + 1 \) does not divide \( x^{2N-2e+1} + 1 \). Hence it follows that
\[ \gcd (x^{2N} + 1, S^\circ(x)) = \gcd (x^{2N} + 1, x^{2N-2e+1} + 1) \]
\[ = x^{\gcd(2N,2N-2e+1)} + 1 \]
\[ = x^{G(N,e)} + 1. \]

Therefore we have
\[ m_S^\circ(x) = \frac{x^{2N} + 1}{\gcd (x^{2N} + 1, S^\circ(x))} = \frac{x^{2N} + 1}{x^{G(N,e)} + 1}. \]

and
\[ L(S^\circ) = 2N - \deg \gcd (x^{2N} + 1, S^\circ(x)) \]
\[ = 2N - \deg \left( x^{G(N,e)} + 1 \right) \]
\[ = 2N - G(N,e). \]

Assume that \( (A/p) = -1, m \equiv 1 \mod 2 \) and \( p \equiv 7 \mod 8 \). Put \( \varphi(x) = \gcd (x^N + 1, T_A^{NTU}(x)) = (x^{2\nu} + 1)/(x + 1) \) and \( \psi(x) = T_A^{NTU}(x)/\varphi(x) \). By Corollary 13, \( m_{\nu} (\varphi(x)) = m_{\nu} (m_{T_A^{NTU}}(x)) = 1 \), \( m_{\nu} (\psi(x)) = 0 \) and \( \gcd (m_{T_A^{NTU}}(x), \psi(x)) = 1 \). Then
\[ S^\circ(x) \equiv \varphi(x^2) \left\{ (x^{2N-2e+1} + 1)\psi(x^2) + x^{2N-2e+1} \cdot \frac{m_{T_A^{NTU}}(x^2)}{x^2 + 1} \right\} \mod x^{2N} + 1 \]
\[ \equiv \varphi(x^2) \left\{ (x^{2N-2e+1} + 1)\psi(x^2) + x^{2N-2e+1} \cdot \frac{x^{2N} + 1}{x^{2\nu} + 1} \right\} \mod x^{2N} + 1. \]

Hence it follows that
\[ \gcd (x^{2N} + 1, S^\circ(x)) = \varphi(x^2) \cdot \frac{x^{\gcd(2N-2e+1,2N)} + 1}{x^{\gcd(2N-2e+1,2\nu)} + 1} \]
\[ = \varphi(x^2) \cdot \frac{x^H_{1,N,e} + 1}{x^H_{\nu,e} + 1} \]
\[ = \frac{(x^{4\nu} + 1)(x^H_{1,N,e} + 1)}{(x^2 + 1)(x^H_{\nu,e} + 1)}. \]

Therefore we have
\[ m_S^\circ(x) = \frac{x^{2N} + 1}{\gcd (x^{2N} + 1, S^\circ(x))} = \frac{(x^{2N} + 1)(x^2 + 1)(x^H_{\nu,e} + 1)}{(x^{4\nu} + 1)(x^H_{1,N,e} + 1)}, \]
Table 2: The values of $H_0(\nu,e)$ and $H_1(N,e)$ in the case of $e \equiv 5 \mod 9$.  

| $e$                   | $H_0(\nu,e)$ | $H_1(N,e)$ |
|-----------------------|--------------|------------|
| $e \equiv 86 \mod 171$| 57           | 171        |
| $e \equiv 5 \mod 9, e \not\equiv 86 \mod 171$ | 3           | 9          |

and

$$L(S^e) = 2N - \deg \gcd(x^{2N} + 1, S^e(x)) = 2N + 2 - 4\nu + H_0(\nu,e) - H_1(N,e) = 2L(T_N^{NTU}) + H_0(\nu,e) - H_1(N,e).$$

\[\square\]

**Corollary 24.** Assume that $S^e$ satisfies the conditions in the first case of Theorem 23, that is $L(T_N^{NTU}) = N$. Then the upper and lower bounds on the linear complexity $L(S^e)$ are obtained as follows:

1. $L(S^e) \leq 2N - 1$. The equality holds if and only if $G(N,e) = 1$.
2. $L(S^e) \geq 2N - N/2^{\nu^2(N)}$. The equality holds if and only if $-2\nu + 1 \equiv 0 \mod N/2^{\nu^2(N)}$.

**Example 25.** Let $p = 17$ and $m = 2$, so that $N = 288 = 2^5 \cdot 3^2$. Assume that $(A/p) = -1$. Then the linear complexity $L(S^e)$ of $S^e$ is given as

$$L(S^e) = \begin{cases} 
567 & \text{if } e \equiv 5 \mod 9 \\
573 & \text{if } e \equiv 2 \mod 9 \\
575 & \text{otherwise.}
\end{cases}$$

**Example 26.** Let $p = 7$ and $m = 3$, so that $N = 342 = 2 \cdot 3^2 \cdot 19$. Assume that $(A/p) = -1$. Then the linear complexity $L(S^e)$ of $S^e$ is given as

$$L(S^e) = \begin{cases} 
344 & \text{if } e \equiv 86 \mod 171 \\
452 & \text{if } e \equiv 5 \mod 9, e \not\equiv 86 \mod 171 \\
458 & \text{otherwise.}
\end{cases}$$

Note that $H_0(\nu,e)$ and $H_1(N,e)$ are given as in Table 2 if $e \equiv 5 \mod 9$.

**6 Conclusions**

In this paper, we first investigate linear complexity of generalized NTU sequences, which are geometric sequences defined by cyclotomic classes. In the case that the Chan–Games formula does not hold, we show the new formula by considering the sequence of complement numbers, Hasse derivative and cyclotomic classes. Under some conditions, we can ensure that the generalized NTU sequences have a large linear complexity from the results on linear complexity of Sidel’nikov sequences.
Next, we propose interleaved sequences of a binary generalized NTU sequence and its complement. Furthermore, we show the periodic autocorrelation and linear complexity of the proposed sequences. The proposed sequences have the balance property and double the period of the generalized NTU sequences by the interleaved structure. The autocorrelation distributions of the proposed sequences have a few peaks and troughs. The question is whether a security is affected by these distributions. We obtain the linear complexity of the proposed sequences in the case that the generalized NTU sequences have a large linear complexity. In particular, if the linear complexity of the generalized NTU sequence attains the maximal value, the formula induces the upper and lower bounds on the linear complexity, and the conditions to attain the upper and lower bounds, respectively. Therefore, although the linear complexity do not attain the maximal value, one can choose a shift width with which a sequence has a large linear complexity, especially such as the period minus one. Thus the proposed sequences have good cryptographic properties for linear complexity.

In this paper, we only consider the frequency for a single symbol, however the proposed sequences do not have the balance property for block of length larger than one. Applying a linear transformation, one can expect to obtain well balanced sequence for block of a certain length. However, the correlation properties are not kept. As a future work, we should give a transformation, by which transformed sequences have the balance property and keep some good properties of the proposed sequences.

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