Tau Function Approach to Theta Functions

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Abstract

We study theta functions of a Riemann surface of genus $g$ from the view point of $\tau$-function of a hierarchy of soliton equations. We study two kinds of series expansions. One is the Taylor expansion at any point of the theta divisor. We describe the initial term of the expansion by the Schur function corresponding to the partition determined by the gap sequence of a certain flat line bundle. The other is the expansion of the theta function and its certain derivatives in one of the variables on the Abel-Jacobi images of $k$ points on a Riemann surface with $k \leq g$. We determine the initial term of the expansion as certain derivatives of the theta function successively. As byproducts, firstly we obtain a refinement of Riemann’s singularity theorem. Secondly we determine normalization constants of higher genus sigma functions of a Riemann surface, defined by Korotkin and Shramchenko, such that they become modular invariant.

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1 Introduction

The notion of τ-function of an integrable hierarchy of soliton equations is considered as an extension of the notion of theta function. This is not only because the τ-function becomes a theta function in a special case but also because the structural similarities like addition formulae \[34\] which are equivalent to the the hierarchy itself \[36, 30\]. More concretely let us consider the KP-hierarchy which is considered to be a universal system of integrable differential equations in the sense that various soliton equations are derived as special cases of it. Its solution space is determined by Sato \[34, 33\] as a certain infinite dimensional Grassmann manifold called the universal Grassmann manifold (UGM). In other words there is a one to one correspondence between solutions of the KP-hierarchy(τ-functions) and points of UGM.

Sato’s theory on the KP-hierarchy was successfully applied to the study of Novikov’s conjecture \[28, 23, 31\]. The results show that theta functions of algebraic curves are characterized by the KP-hierarchy among those of principally polarized Abelian varieties (ppAv). It means that tau functions of the KP-hierarchy supplied by the general properties of theta functions of ppAv can produce all the properties of theta functions corresponding to Riemann surfaces. Therefore it is quite natural to study theta functions by way of τ-functions.

However it seems that a τ-function approach in the study of theta functions is not fully developed yet. In papers \[25, 27, 9, 2\] we have studied the higher genus sigma functions of various algebraic curves from the viewpoint of τ-functions. The aim of this paper is to develop these researches further and to add examples which are effectively studied by an approach of τ-functions.

We study two kinds of series expansions of the theta function of a Riemann surface. One is the series expansion at any point on the theta divisor. We determine the initial term, with respect to certain weight, of the expansion as the Schur function with the partition determined from the gap sequence of the flat line bundle corresponding to a point on the theta divisor. The other is the expansion of the theta function and certain derivatives of the theta function with respect to one of the variables on the Abel-Jacobi images of \(k\) points on a Riemann surface of genus \(g\) with \(k \leq g\). We determine the initial term of the expansion as a certain explicit derivative of the theta function.

As a consequence of the study on the expansions we get an extension and a refinement of Riemann’s singularity theorem. As another corollary we determine normalization constants of higher genus sigma functions of a Riemann surface introduced by Korotkin and Shramchenko \[18\] so that they are modular invariant. In their paper the modular invariance of sigma functions is proved up to multiplications of certain roots of unity. We propose apparently different normalization constants from theirs and prove the modular invariance. Let us explain our results in more detail.

Let \(X\) be a compact Riemann surface of genus \(g\), \(\{\alpha_i, \beta_i\}\) a canonical homology basis, \(\Omega\) the normalized period matrix and \(\theta(Z|\Omega)\) Riemann’s theta function. We consider the data \((X, \{\alpha_i, \beta_i\}, p_\infty, e)\) consisting of \(X, \{\alpha_i, \beta_i\}\) as above, a point \(p_\infty\) on \(X\) and a point \(e\) of the theta divisor. We associate a partition to such a data as follows.

To this end we need a notion of gaps of a line bundle. Let \(L\) be a holomorphic line
bundle on $X$ of degree zero, which we call a flat line bundle. A non-negative integer $n$ is called a gap of $L$ at $p_\infty$ if there does not exist a meromorphic section of $L$ which is holomorphic on $X\setminus\{p_\infty\}$ and has a pole of order $n$ at $p_\infty$. By Riemann-Roch it can be easily proved that there are exactly $g$ gaps in $\{0, 1, \ldots, 2g-1\}$ for any $(L, p_\infty)$. Let $\delta$ be Riemann’s constant, $L_{e+\delta}$ the flat line bundle corresponding to the point $e+\delta$ on the Jacobian and

$$b_1 < \cdots < b_g,$$

$$w_1 < \cdots < w_g,$$

the gaps of $L_{e+\delta}$ and $L_0$ at $p_\infty$ respectively, where $L_0$ is the trivial line bundle. We define the partition $\lambda = (\lambda_1, \ldots, \lambda_g)$ by

$$\lambda = (b_g, b_{g-1}, \ldots, b_1) - (g-1, g-2, \ldots, 0),$$

and consider the Schur function $s_\lambda(t)$, $t = (t_1, t_2, \ldots)$. The special property of $s_\lambda(t)$ is that it depends only on $t_{w_i}$, $1 \leq i \leq g$. This property is crucial when we connect it to the theta function.

To give a relation of $s_\lambda(t)$ with the theta function we need to make a change of variables which is given by a certain non-normalized period matrix. To define it we specify a local coordinate $z$ around $p_\infty$. Then there is a basis $du_{w_i}$, $1 \leq i \leq g$, of holomorphic one forms which has the expansion at $p_\infty$ of the form

$$du_{w_i} = (z^{w_i-1} + O(z^{w_i}))dz, \quad 1 \leq i \leq g.$$ 

It is not unique. We take any one of them. Let $2\omega_1$ be the $\{\alpha_i\}$ period matrix of $\{du_{w_i}\}$. Let $u = t(u_{w_1}, \ldots, u_{w_g})$. We assign the weight $i$ to variables $u_i$ and $t_i$. Then the Schur function $s_\lambda(t)$ becomes a weight-homogeneous polynomial with the weight $|\lambda| = \lambda_1 + \cdots + \lambda_l$ for $\lambda = (\lambda_1, \ldots, \lambda_l)$.

We prove that the theta function has the expansion of the form

$$C\theta((2\omega_1)^{-1}u + e|\Omega) = s_\lambda(t)|_{t_{w_i} = u_{w_i}} + \text{higher weight terms},$$

for some constant $C$ which is given explicitly by a theta constant (see Theorem $[10]$).

Next we study the expansion of the function $\theta(p_1 + \cdots + p_g + e|\Omega)$ in $z_g = z(p_g)$ where $p_i$ inside the theta function denotes the image of $p_i$ by the Abel-Jacobi map with the base point $p_\infty$. To describe the results we need some sequence of numbers which we call $a$-sequence.

Let $0 \leq b_1^* < b_2^* < \cdots$ be non-gaps of $L_{e+\delta}$ at $p_\infty$. For $0 \leq k \leq g-1$ define $m_k$ by

$$m_k = \#\{i \mid b_i^* < g - k\}$$

and $a_i^{(k)}$, $1 \leq i \leq m_k$, by

$$(a_1^{(k)}, \ldots, a_{m_k}^{(k)}) = (b_{g-k}, b_{g-k-1}, \ldots, b_{g-k-m_k+1}) - (b_1^*, \ldots, b_{m_k}^*).$$
Any \( a_i^{(k)} \) is proved in \( \{w_j\} \). In general for a non-empty subset \( I = \{i_1, ..., i_l\} \) we set
\[
\partial I = \partial u_{i_1} \cdots \partial u_{i_l}, \quad \partial u_i = \frac{\partial}{\partial u_i}.
\]
and set \( \partial I = 1 \) for \( I = \emptyset \). Let \( A_k = \{a_i^{(k)}\} \) for \( k \geq 1 \) and \( A_0 = \emptyset \). We show that the following expansion is valid for \( 1 \leq k \leq g \):
\[
\partial_{A_k} \theta(\sum_{i=1}^{k} p_i + e|\Omega) = \tilde{c}_k \partial_{A_{k-1}} \theta(\sum_{i=1}^{k-1} p_i + e|\Omega) z_k z_k^{\lambda_k} + O(z_k^{\lambda_k+1}),
\]
where \( z_k = z(p_k) \) and \( c_k = \pm 1 \) is explicitly given (Theorem 12). The non-vanishing of the left hand side follows from (1). This type of expansion was first pointed out in [29] in the case of hyperelliptic curves and \( e = -\delta \) and was applied to addition formulae of the fundamental sigma function. The results are extended to the case of \((n, s)\) curves and \( e = -\delta \) in [27, 22] and to that of telescopic curves and \( e = -\delta \) in [2]. Here we extend the results to the case of an arbitrary Riemann surface and an arbitrary point \( e \) on the theta divisor.

The results on the expansions of the theta function above and the way to prove it implies an interesting extension and a refinement of Riemann’s singularity theorem.

Riemann’s singularity theorem asserts that the multiplicity of \( \theta(Z|\Omega) \) at \( e \) is \( m_0 \). In other words it says that any derivative of \( \theta(Z|\Omega) \) of degree less than \( m_0 \) vanishes at \( e \) and some derivative of degree \( m_0 \) does not vanish at \( e \), where the degree signifies the degree as a differential operator. Here we should notice that the theorem tells nothing on which derivatives do not vanish in general.

We derive the following properties of the theta function from those of \( \tau \) functions and Schur functions:

(i) \( \partial_I \theta(e|\Omega) = 0 \) for any \( I = (i_1, ..., i_m) \) if \( i_1 + \cdots + i_m < |\lambda| \).

(ii) \( \partial_I \theta(e|\Omega) = 0 \) for any \( I = (i_1, ..., i_m) \) if \( m < m_0 \).

(iii) \( \partial_{A_0} \theta(e|\Omega) \neq 0 \).

The properties (ii) and (iii), in particular, implies Riemann’s singularity theorem. Moreover we see that the \( A_0 \)-derivative gives the non-vanishing derivative of degree \( m_0 \) explicitly. The vanishing property (i) is a new vanishing property which does not follow from Riemann’s singularity theorem. Therefore (i)-(iii) give an extension and a refinement of Riemann’s singularity theorem.

Finally this \( A_0 \)-derivative can be used to define an appropriate normalization in defining the sigma function such that the resulting function becomes modular invariant. In order to define sigma functions we need a certain bilinear meromorphic differential. The normalized bilinear differential \( \omega(p_1, p_2) = d_{p_1} d_{p_2} \log E(p_1, p_2) \), where \( E(p_1, p_2) \) is the prime form, plays a fundamental role in the theory of theta functions [12]. However \( \omega(p_1, p_2) \) depends on the choice of canonical homology basis. Klein modified \( \omega \) so that it does not depend on the choice of canonical homology basis [12, 14].
which we call Klein form. Let \( \hat{\omega}(p_1, p_2) \) be a bilinear differential which is obtained from the Klein form by adding \( \sum c_{ij} du_{w_i} du_{w_j} \), where \( \{c_{ij}\} \) are independent of the choice of canonical homology basis and satisfy \( c_{ij} = c_{ji} \). The sigma function associated with \( (X, \{\alpha_i, \beta_i\}, p_\infty, z, e, \{du_{w_i}\}, \hat{\omega}) \) is defined as follows.

Let us write \( e = \Omega \varepsilon^* + \varepsilon'' \) with \( \varepsilon', \varepsilon'' \in \mathbb{R}^g \) and set \( \varepsilon = t(\varepsilon', \varepsilon'') \). Using Riemann’s theta function with the characteristics \( \varepsilon \) we set

\[
C_e = \partial_{\Lambda_0} \theta[\varepsilon](0|\Omega),
\]

which does not vanish due to (iii) above. Then we define the sigma function with the characteristics \( \varepsilon \) by

\[
\sigma[\varepsilon](u) = C_e^{-1} \exp\left(\frac{1}{2} t u \eta_1 \omega_1^{-1} u \right) \theta[\varepsilon](2 \omega_1)^{-1} u | \Omega),
\]

where \( \eta_1 \) is the period of certain second kind differentials which is computed from the \( \alpha_i \)-integral of \( \hat{\omega} \). The part without \( C_e \) of the right hand side, which we call the main part, is already proposed in [4] without explicit construction of \( \eta_1 \). In [18] Korotkin and Shramchenko proposed to use Klein form to define \( \eta_1 \). They have shown that the main part, multiplied by a certain theta constant which is apparently different from \( C_e \), is invariant under the change of the canonical homology basis up to multiplication of \( 8N \)-th root of unity, where \( N \) is the number of non-singular even half periods. We show that the sigma function normalized by \( C_e \) is invariant under the action of \( \text{Sp}(2g, \mathbb{Z}) \) on canonical homology basis. We call this property the modular invariance of the sigma function.

There remain several fundamental problems to be solved. We have determined the initial term of the expansion of the theta function with respect to weight. In applications sometimes the initial term with respect to degree is necessary [26]. In [4] the minimal degree term is determined for a hyperelliptic curve with \( e \) being certain half periods as certain determinants. In this paper we have determined the minimal degree term in the minimal weight term for arbitrary \( (X, e) \). It is interesting to determine the full minimal degree term. To this end it is necessary to study higher weight terms in the expansion of \( \tau \)-function. The results in this direction can be applied to the study on inversions of hyperelliptic integrals [10].

The relation of Klein form with the bilinear meromorphic differentials of \( (n, s) \) curves, telescopic curves and others [6, 4, 11, 24, 16, 17], which are constructed algebraically, should be clarified (see [8] for some examples). It is also interesting to determine the explicit relation between the normalization constants given in this paper and those in [18]. In the case of genus one the relation is given by the celebrated Jacobi’s derivative formula.

In the case of an \( (n, s) \) curve the coefficients of the series expansion of the fundamental sigma function, which corresponds to \( e = -\delta \), are polynomials of the coefficients of the defining equation of the curve [24, 25]. It is known that to any algebraic curve there exists a certain normal form of defining equations [21]. It is expected that the coefficients of the series expansion of the fundamental sigma function of an algebraic
curve can be expressed by the coefficients of defining equations of the curve as in the case of an \((n, s)\) curve. To this end we need to construct the (modified) Klein form algebraically using the defining equation of the curve. This construction is an independent interesting problem.

The paper is organized as follows. In section 2 after the review on divisors and line bundles on a Riemann surface we define the partition corresponding to a geometric data using gaps of line bundles. The properties of the Schur functions corresponding to geometric data are studied in section 3. In section 4 the properties of the function which has a similar expansion to \(\tau\)-function of the KP-hierarchy is studied. Sato's theory of the KP-hierarchy is reviewed in section 5. In section 6 the point of UGM corresponding to an algebro-geometric solution of the KP-hierarchy is determined. It is shown that the solution corresponding to \((X, \{\alpha_i, \beta_i\}, p_\infty, z, e)\) is in the cell \(UGM^\lambda\) of UGM with the partition \(\lambda\) corresponding to this geometric data. This is the extension of the result in \cite{15} to the non-generic case. The series expansions of the theta function are studied in section 7. We give examples of partitions corresponding to geometric data here. In the case of the hyperelliptic curve defined by an odd degree polynomial, the partition \(\lambda\) corresponding to any data of the form \((X, \{\alpha_i, \beta_i\}, \infty, e)\) are determined explicitly.

In section 8 sigma functions with arbitrary real characteristics are defined and they are shown to be modular invariant.

2 Geometric Data

2.1 Preliminaries

Here we collect necessary facts on Riemann surfaces following mainly \cite{12}.

Let \(X\) be a compact Riemann surface of genus \(g\) and \(\{\alpha_i, \beta_i\}\) a canonical homology basis of \(X\). Then the normalized basis \(\{dv_i\}\) of holomorphic one forms and the normalized period matrix is determined:

\[
\int_{\alpha_i} dv_i = \delta_{ij}, \quad \Omega = \left( \int_{\beta_j} dv_i \right).
\]

Riemann’s theta function with characteristics \(\varepsilon = \iota(\varepsilon', \varepsilon'')\), \(\varepsilon', \varepsilon'' \in \mathbb{R}^g\) is defined by

\[
\theta[\varepsilon](z|\Omega) = \sum_{n \in \mathbb{Z}^g} \exp(2\pi \iota(n + \varepsilon')\Omega(n + \varepsilon') + 2\pi \iota(n + \varepsilon')(z + \varepsilon'')) , \quad z = \iota(z_1, ..., z_g).
\]

For \(\varepsilon = \iota(0, 0)\), \(\theta[\varepsilon](z|\Omega)\) is denoted by \(\theta(z|\Omega)\).

Let \(J(X) = \mathbb{C}^g / (\mathbb{Z}^g + \Omega \mathbb{Z}^g)\) be the Jacobian variety \(X\). By fixing a base point \(p_\infty\) we have the Abel-Jacobi map:

\[
I : X \to J(X), \quad I(p) = \int_{p_\infty}^p dv,
\]
where $dv$ is the vector of normalize holomorphic one forms, $dv = t(dv_1, ..., dv_g)$. The Jacobian $J(X)$ is isomorphic to the group of divisor classes of degree zero by the Abel-Jacobi map. We sometimes identify a divisor of degree zero with its Abel-Jacobi image in $J(X)$:

$$\sum (p_i - q_i) = \sum (I(p_i) - I(q_i)).$$

A choice $\{\alpha_i, \beta_i\}$ specifies the Riemann divisor $\Delta$. It is a divisor class of degree $g - 1$ and satisfies $2\Delta = K_X$, where $K_X$ is the canonical divisor class of $X$. Further if a point $p_\infty$ on $X$ is specified, Riemann’s constant $\delta$ is determined as an element of $J(X)$. It is related to $\Delta$ by

$$\Delta - (g - 1)p_\infty = \delta.$$

To each divisor $D$ is associated a holomorphic line bundle $L_D$ on $X$ (see [12] for example). If $D$ is of degree zero, then

$$\frac{\theta (p - p_\infty - D - f)}{\theta (p - p_\infty - f)}$$

is a meromorphic section of $L_D$, where $f$ is a generic point of $\mathbb{C}^g$ such that both denominator and numerator do not vanish identically. In terms of the transformation law, if $D$ is represented by $c = t(c_1, ..., c_g) \in \mathbb{C}^g$ in $J(X)$, a section $F$ of $L_D$ satisfies

$$F(p + \alpha_j) = F(p), \quad F(p + \beta_j) = e^{2\pi ic_j} F(p). \quad (2)$$

In this case $L_D$ is also denoted by $L_c$. A different choice of the representative $c$ of $D$ gives a holomorphically equivalent line bundle. The holomorphic line bundle corresponding to a divisor of degree zero is called a flat line bundle since it admits a holomorphic flat connection.

To each holomorphic line bundle there corresponds the sheaf of germs of holomorphic sections of it. We shall introduce some notation related with this sheaf.

For two positive divisors $A$, $B$ and a holomorphic line bundle $L$ we denote by $L(B - A)$ the sheaf of germs of meromorphic sections of $L$ whose poles are at most at $B$ and whose zeros are at least at $A$. Let $\mathcal{O}$ denote the sheaf corresponding to the trivial line bundle $L_0$. Then we have

$$L_D \simeq \mathcal{O}(D), \quad L_D(B - A) \simeq \mathcal{O}(D + B - A), \quad (3)$$

as sheaves of $\mathcal{O}$ modules.

We set

$$h^0(L(B - A)) = \dim H^0(X, L(B - A)).$$

By (3) we have

$$h^0(L_D(B - A)) = h^0(\mathcal{O}(D + B - A)).$$

We denote the right hand side of this equation by $h^0(D + B - A)$.

For a point $p$ of $X$ let $L(*p)$ denote the sheaf of germs of meromorphic sections of $L$ which have a pole of any order at $p$. 

6
2.2 Gaps

**Definition 1** A non-negative integer \( b \) is called a gap of a flat line bundle \( L \) at \( p_{\infty} \) if there does not exist a meromorphic section of \( L \) which is holomorphic outside \( p_{\infty} \) and has a pole of order \( b \) at \( p_{\infty} \). A non-negative integer which is not a gap is called a non-gap of \( L \) at \( p_{\infty} \).

**Lemma 1** There are exactly \( g \) gaps for any pair \((L, p_{\infty})\) of a flat line bundle \( L \) and a point \( p_{\infty} \) of \( X \).

**Proof.** Let \( L \) correspond to the divisor \( D \). By the Riemann-Roch theorem we have

\[
h^0(D + np_{\infty}) - h^0(K_X - D - np_{\infty}) = 1 - g + n. \tag{4}
\]

If \( n \geq 2g - 1 \),

\[
h^0(K_X - D - np_{\infty}) = 0,
\]

since the degree of \( K_X - D - n \) is negative. In particular

\[
h^0(D + (2g - 1)p_{\infty}) = g.
\]

Since we have \( 2g \) integers between 0 and \( 2g - 1 \), it means that there are \( g \) gaps in \( \{0, 1, \ldots, 2g - 1\} \). It is obvious that there are no other gaps by (4). \( \square \)

2.3 Partition associated with a geometric data

Let \( e \) be a zero of the theta function \( \theta(z \mid \Omega) \). By Riemann’s vanishing theorem \( e \) can be written as

\[
e = q_1 + \cdots + q_{g-1} - \Delta, \tag{5}
\]

for some \( q_1, \ldots, q_{g-1} \in X \).

We consider the divisor \( e + \delta \) of degree zero:

\[
e + \delta = q_1 + \cdots + q_{g-1} - (g - 1)p_{\infty}. \tag{6}
\]

The proof of Lemma 4 shows that there are exactly \( g \) gaps and \( g \) non-gaps of \((L, p_{\infty})\) in \( \{0, 1, \ldots, 2g - 1\} \).

We introduce two kinds of gaps and non-gaps simultaneously. Let

\[
b_1 < b_2 < \cdots < b_g,
\]

\[
b_1^* < b_2^* < \cdots
\]

be gaps and non-gaps of \( L_{e+\delta} \) at \( p_{\infty} \) and

\[
w_1 < w_2 < \cdots < w_g,
\]

\[
w_1^* < w_2^* < \cdots
\]
those of $L_0$ at $p_\infty$.

**Remark** The sequence $(w_1, ..., w_g)$ is the gap sequence of a Riemann surface $X$ at $p_\infty$ [11]. A point $p_\infty$ for which $(w_1, ..., w_g) \neq (1, ..., g)$ is called a Weierstrass point. We have $w_1 = 1$ and $w_1^* = 0$.

**Definition 2** We define the partition $\lambda = (\lambda_1, ..., \lambda_g)$ associated with $(X, \{\alpha_i, \beta_i\}, p_\infty, e)$ by

$$\lambda = (b_g, ..., b_1) - (g - 1, ..., 1, 0). \quad (7)$$

In the next section we study the properties of the Schur function corresponding to $\lambda$.

## 3 Schur function of a Riemann surface

### 3.1 Dependence on variables

The Schur function $s_\mu(t)$, $t = (t_1, t_2, t_3, ..)$, [20] corresponding to a partition $\mu = (\mu_1, ..., \mu_l)$ is defined by

$$s_\mu(t) = \det (p_{\mu_i-j}(t))_{1 \leq i,j \leq l},$$

where $p_n(t)$ is the polynomial defined by

$$\exp \left( \sum_{n=1}^{\infty} t_n k^n \right) = \sum_{n=0}^{\infty} p_n(t) k^n; \quad p_n(t) = 0 \text{ for } n < 0.$$

We identify a partition $\mu = (\mu_1, ..., \mu_l)$ with $(\mu_1, ..., \mu_l, 0^g)$ for any $i$. The Schur function $s_\mu(t)$ does not depend on the choice of $i$.

We define the weight and degree of the variable $t_i$ to be $i$ and 1 respectively:

$$\text{wt}(t_i) = i, \quad \text{deg}(t_i) = 1.$$

With respect to weight the Schur function $s_\mu(t)$ is a homogeneous polynomial with the weight $|\mu| = \mu_1 + \cdots + \mu_l$, while it is not homogeneous with respect to degree in general.

By the definition $s_\mu(t)$ is a polynomial of $t_1, ..., t_{\mu_1+t-1}$. However the Schur function corresponding to a geometric data depends on fewer number of variables.

**Proposition 1** Let $\lambda$ be the partition associated with $(X, \{\alpha_i, \beta_i\}, p_\infty, e)$. Then $s_\lambda(t)$ is a polynomial of $t_{w_1}, ..., t_{w_g}$.

**Proof.** Notice that the space $H^0(X, L_{e+\delta}(\ast p_\infty))$ is a $H^0(X, \mathcal{O}(\ast p_\infty))$-module. Therefore if we define $N_1 = \{w_i^* | i \geq 1\}$ and $N_2 = \{b_i^* | i \geq 1\}$, then $N_1$ acts on $N_2$ by addition. Namely for any $i, j$ we have

$$w_i^* + b_j^* \in N_2.$$

It follows that, if $b_j - w_i^* \geq 0$ then it is a gap of $L$ at $p_\infty$. In fact if it is not the case, $b_j - w_i^* \in N_2$. Then $b_j \in N_2 + w_i^* \subset N_2$ which is absurd. The proposition can be proved in a similar manner to Proposition 2 of [27] using this property. □
3.2 \textit{a-sequence}

**Definition 3** For an integer \( k \) such that \( 0 \leq k \leq g - 1 \) we define the integer \( m_k \) by

\[
m_k = h^0(q_1 + \cdots + q_{g-1} - kp_\infty).
\]

It is possible to describe \( m_k \) in terms of gaps or non-gaps of \( L_{e+\delta} \).

**Lemma 2** We have

\[
m_k = \#\{i \mid b_i^* < g - k\} = g - k - \#\{i \mid b_i < g - k\}.
\]

**Proof.** By the definition of \( L_{e+\delta} \) we have

\[
m_k = h^0(L_{e+\delta}((g - k - 1)p_\infty)),
\]

which proves the first equation of the lemma. Since gaps and non-gaps are complements to each other in \( \{0, 1, ..., g - k - 1\} \) the second equality follows. \( \square \)

In order to describe more detailed properties of \( s_\lambda(t) \) we need

**Definition 4** For \( 0 \leq k \leq g - 1 \) define the sequence \( A_k = (a_1^{(k)}, ..., a_{m_k}^{(k)}) \) by

\[
A_k = (b_{g-k}, b_{g-k-1}, ..., b_{g-k-m_k+1}) - (b_1^*, ..., b_{m_k}^*).
\]

The sequence \( A_k \) is referred to as \textit{a-sequence}.

For a partition \( \mu = (\mu_1, ..., \mu_l) \) and \( 0 \leq k \leq l - 1 \) we set

\[
N_{\mu,k} = \sum_{i=k+1}^{l} \mu_i.
\]

In the case \( k = 0 \), \( N_{\mu,0} = \sum_{i=1}^{l} \mu_i = |\mu| \) is the weight of \( \mu \).

**Lemma 3** Suppose that \( m_k > 0 \). Then

(i) \( a_1^{(k)} > \cdots > a_{m_k}^{(k)} \).

(ii) \( a_i^{(k)} \in \{w_j\} \) for any \( i \).

(iii) \( \sum_{i=1}^{m_k} a_i^{(k)} = N_{\lambda,k} \).

**Proof.** (i) is obvious from the definition of \( a_i^{(k)} \). Let us prove (ii). We first show that \( a_{m_k}^{(k)} > 0 \). By Lemma 2 we have \( b_1^* < \cdots < b_{m_k}^* < g - k \). Since \( \{b_i^*\} \) and \( \{b_i\} \) are complement in the set of nonnegative integers to each other,

\[
\{b_1^*, ..., b_{m_k}^*\} \cup \{b_1, ..., b_{g-k-m_k}\} = \{0, 1, ..., g - k - 1\}.
\]
Thus $b^*_i < g - k \leq b_{g - k - m_k + 1}$ and $a^{(k)}_m = b_{g - k - m_k + 1} - b^*_m > 0$. Now, suppose that $a^{(k)}_i = b_{g - k + 1 - i} - b^*_i \notin \{w_j\}$. Since $a^{(k)}_i > 0$, we have $a^{(k)}_i = w^*_j$ for some $j$. Then $b_{g - k + 1 - i} = b^*_i + w^*_j$ is a non-gap of $L_{e + \delta}$, which is impossible. Thus the assertion (ii) is proved.

(iii): We have

$$\sum_{i=1}^{m_k} a^{(k)}_i = \sum_{i=g-k-m_k+1}^{g-k} b_i - \sum_{i=1}^{m_k} b^*_i = \sum_{i=1}^{g-k} b_i - \sum_{i=1}^{g-k-1} i,$$

(9)

where we use (8). Since $\lambda_i = b_{g - k + 1 - i} - (g - i)$, the right hand side of (9) equals to $\lambda_{k+1} + \cdots + \lambda_g$. □

3.3 Vanishing and non-vanishing

We introduce the analogue of the Abel-Jacobi map for Schur function.

Definition 5 Define $[x]$ by

$$[x] = (x, \frac{x^2}{2}, \frac{x^3}{3}, \ldots).$$

Using the a-sequence we can describe the properties of derivatives of $s_{\lambda}(t)$.

For a sequence $I = (i_1, \ldots, i_r)$ of positive integers we set

$$\partial_t, I = \partial_{t_{i_1}} \cdots \partial_{t_{i_r}}, \quad \partial_{t_i} = \frac{\partial}{\partial t_i}.$$ 

In the following we sometimes use the expressions $\sum_{i=1}^{k} [x_i]$ and $s_{(\mu_1, \ldots, \mu_k)}(t)$ for $k = 0$. They should be understood as

$$\sum_{i=1}^{k} [x_i] = 0, \quad s_{(\mu_1, \ldots, \mu_k)}(t) = 1.$$

Theorem 1 Suppose $m_k > 0$. Let $\lambda$ be the partition defined by (7) and $\mu = (\mu_1, \ldots, \mu_l)$ a partition satisfying $\mu_i = \lambda_i$ for $i \geq k + 1$. Then

$$\partial_t, \lambda, s_{\mu}(\sum_{i=1}^{k} [x_i]) = c_k s_{(\mu_1, \ldots, \mu_k)}(\sum_{i=1}^{k} [x_i]),$$

where

$$c_k = \text{sgn} \begin{pmatrix} b^*_1 & \cdots & b^*_m & b_{g - k - m_k} & \cdots & b_1 \\ g - k - 1 & \cdots & . & . & \cdots & 1 \end{pmatrix}. \quad (10)$$
Proof. The theorem can be proved in a completely similar way to Theorem 1 in [27]. □

For two partitions \( \mu = (\mu_1, \ldots, \mu_l), \nu = (\nu_1, \ldots, \nu_l) \) we define \( \mu \leq \nu \) by the condition \( \mu_i \leq \nu_i \) for any \( i \).

Then we have the following vanishing theorem for Schur functions.

**Theorem 2** Let \( \lambda \) be the partition defined by \( \{7\} \). Then

(i) Let \( \mu \) be any partition. Then, for any sequence \( I = (i_1, \ldots, i_m) \), \( m \geq 1 \) satisfying \( \sum_{j=1}^{m} i_j \neq |\mu| \) we have

\[
\partial_{t,I} s_{\mu}(0) = 0,
\]

(ii) Let \( \mu \) be a partition satisfying \( \mu \geq \lambda \). If \( m < m_0 \) we have

\[
\partial_{t,I} s_{\mu}(0) = 0,
\]

for any and \( I = (i_1, \ldots, i_m) \).

(iii) \( \partial_{t,A_0} s_\lambda(0) = c_0 \), where \( c_0 = \pm 1 \) is given by \( \{10\} \).

Proof. Since \( s_{\mu}(t) \) is weight-homogeneous with the weight \( |\mu| \), (i) is obvious. (iii) is the case of \( k = 0 \) of Theorem \( \{1\} \). So let us prove (ii).

If \( \sum_{j=1}^{m} i_j \neq |\mu| \), the left hand side of \( \{12\} \) vanishes by (i). So we assume \( \sum_{j=1}^{m} i_j = |\mu| = N_\mu \).

Let \([i_1, \ldots, i_l]\) be the determinant of the \( l \times l \) matrix whose \( j \)-th row is given by\( (p_{i_j-1}(t), \ldots, p_{i_j-1}(t), p_{i_j}(t)) \).

We write \([i_1, \ldots, i_l](t)\) if it is necessary to indicate \( t \). Let us define the strictly decreasing sequence \( (b'_1, \ldots, b'_i) \) corresponding to \( \mu = (\mu_1, \ldots, \mu_l) \) by

\[
(b'_1, \ldots, b'_i) = (\mu_1, \ldots, \mu_l) + (l-1, \ldots, 1, 0).
\]

We take \( l \geq g \) by inserting several 0’s to the end of \( \mu \) if necessary. Then the condition \( \mu \geq \lambda \) is

\[
b'_i \geq b_{i-l+g} + l - g, \quad l - g + 1 \leq i \leq l.
\]

With this notation we have

\[
s_{\mu}(t) = [b'_1, \ldots, b'_l].
\]

Since \( \partial_{t,i} p_j(t) = p_{j-i}(t) \) and the derivative of the determinant by \( \partial_{t,i} \) is the sum of the determinant whose \( j \)-th row is differentiated, we have

\[
\partial_{t,i} s_{\mu}(t) = \sum_{j=1}^{l} [b'_1, \ldots, b'_j - i, \ldots, b'_l].
\]
Thus the left hand side of (12) is written as a sum of the determinants of the form

$$[b'_1 - r_l, ..., b'_1 - r_1](0).$$  \(13\)

If \(r_j > 0\) then it means that the \(j\)-th row is differentiated at least once.

We show that all terms (13) appearing in the left hand side of (12) vanish.

Suppose that there is a non-zero term (13). Since the number of the derivatives in the left side of (12) is \(m < m_0\), some row among \(m_0\) rows labeled by \(l - m_0 + 1, ..., l\) is not differentiated. We call this row the \(j\)-th row. By (8) we have

$$b_1 < \cdots < b_{g-m_0} < g \leq b_{g-m_0+1} < \cdots < b_g.$$  

Thus

$$b'_j \geq b_{j-l+g} + l - g \geq l,$$

since \(g - m_0 + 1 \leq j - l + g \leq g\) for \(l - m_0 + 1 \leq j \leq l\). By Lemma 2 (ii) in [27] such a term is zero, which contradicts the assumption. \(\square\)

**Remark** The assertions (ii) and (iii) of Theorem 2 is an analogue of Riemann’s singularity theorem for Schur functions.

### 3.4 Minimal degree term

Assertions (ii) and (iii) of Theorem 2 mean that the term \(t_{a_1^{(0)}} \cdots t_{a_{m_0}^{(0)}}\) is one of monomials with the minimal degree which appear in \(s_{\lambda}(t)\). In fact it is possible to determine the minimal degree term of \(s_{\lambda}(t)\).

Let \(\mu = (\mu_1, ..., \mu_l)\) be a partition and \(L_{\mu}(t)\) the minimal degree term of \(s_{\mu}(t)\):

$$s_{\mu}(t) = L_{\mu}(t) + \text{higher degree terms}.$$  

Define \(m_0(\mu)\) by

$$m_0(\mu) = l - \sharp \{ i \mid b'_i < l \}, \quad (b'_1, ..., b'_l) = (\mu_1, ..., \mu_l) + (l - 1, ..., 1, 0).$$

By Lemma 2 we have \(m_0(\lambda) = m_0\).

**Proposition 2** We have

$$L_{\mu}(t) = (-1)^{N_{\mu,m_0(\mu)}} \det(t_{\mu_i-i+j}^{(0)}|1 \leq i \leq m_0(\mu), j \neq l-b'_1, ..., l-b'_{l-m_0(\mu)}).$$

In particular the degree of \(L_{\mu}(t)\) is \(m_0(\mu)\).

**Proof.** Since

$$p_n(t) = t_n + \text{higher degree terms}$$
we have

\[ s_\mu(t) = \det(t_{\mu_i-i+j})_{1 \leq i, j \leq l} + \text{higher degree terms}, \]

where we set \( t_0 = 1 \) and \( t_i = 0 \) for \( i < 0 \). Consider the determinant in the right hand side of (14). Consider \( i \) with \( b'_i < l \). The \( i \)-th row from the bottom of the matrix \((t_{\mu_i-i+j})_{1 \leq i, j \leq l}\) is

\[(0, \ldots, 0, t_0, t_1, \ldots, t_{b'_i}).\]

We first expand the determinant in the \( l \)-th row, which corresponds to \( i = 1 \), and pick up the term containing \( t_0 \). It is the determinant of the matrix which is obtained by removing \( l \)-th row and \((l-b'_1)-\text{th column times } (-1)^{l+(l-b'_1)} \). We proceed similarly for \((l-1)\)-th row, \((l-2)\)-th row, \ldots, \((m_0(\mu)+1)\)-th row. Then we get

\[ s_\mu(t) = (-1)^{N_{\mu,m_0(\mu)}} \det(t_{\mu_i-i+j})_{1 \leq i \leq m_0(\mu), j \neq \ldots, l-b'_1, \ldots, l-b'_m(\mu)} + \text{higher degree terms}. \]

What we have to check is that the first term of the right hand side is not identically zero. Notice the monomial in \( t_i \)'s in the determinant which is obtained by taking the product of anti-diagonal components. This monomial is unique among the \( m_0(\mu)! \) terms in the expansion of the determinant and has \( \pm 1 \) as its coefficient. □

Example Consider a hyperelliptic curve of genus \( g \) defined by \( y^2 = f(x) \), where \( f(x) \) is a polynomial of degree \( 2g+1 \) without multiple zeros and take \( p_\infty = \infty, e = -\delta \). Then \( \lambda = (g, g-1, \ldots, 1) \). In this case \( m_0 = \lceil \frac{g+1}{2} \rceil \), \( N_{\lambda,m_0} = (1/2)(g-m_0)(g+1-m_0) \) and

\[ L_{\lambda}(t) = (-1)^{N_{\lambda,m_0}} \det(t_{2k+1-2i+2j})_{1 \leq i, j \leq k} \text{ for } g = 2k, \]

\[ L_{\lambda}(t) = (-1)^{N_{\lambda,m_0}} \det(t_{2k+1-2i+2j})_{1 \leq i, j \leq k+1} \text{ for } g = 2k+1. \]

If we introduce the variables with different indices by

\[(u_1, u_2, \ldots, u_g) = (t_{2g-1}, t_{2g-3}, \ldots, t_1),\]

then

\[ L_{\lambda}(t) = (-1)^{(1/2)g(g+1)+gm_0} \begin{vmatrix} u_1 & u_2 & \ldots & u_{m_0} \\ u_2 & u_3 & \ldots & u_{m_0+1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{m_0} & u_{m_0+1} & \ldots & u_{2m_0-1} \end{vmatrix}, \]

which is precisely the Hankel determinant formula for the minimal degree term of the series expansion of the hyperelliptic theta function derived in [4, 3].
4 Tau function

4.1 Expansion on Abel-Jacobi images

Let $\lambda$ be the partition (7) associated with $(X, \{\alpha_i, \beta_i\}, p_\infty, e)$. In this section we consider an arbitrary function of the form

$$\tau(t) = s_\lambda(t) + \sum_{\lambda < \mu} x_\mu s_\mu(t).$$

For $1 \leq k \leq g$ we set

$$\tau^{(k)}(t) = s_{(\lambda_1, \ldots, \lambda_k)}(t) + \sum_\mu x_\mu s_{(\mu_1, \ldots, \mu_k)}(t),$$

where the sum in the right hand side runs over partitions $\mu = (\mu_1, \ldots, \mu_g)$ satisfying the conditions $\lambda < \mu$, $\mu_i = \lambda_i$ for $k + 1 \leq i \leq g$. We set $\tau^{(0)}(t) = 1$.

**Theorem 3** Suppose that $m_k > 0$. Let $c_k$ be given by (10). Then

(i) $\partial_{t,A_k} \tau \left( \sum_{i=1}^k [x_i] \right) = c_k \tau^{(k)} \left( \sum_{i=1}^k [x_i] \right)$.

(ii) $\tau^{(k)} \left( \sum_{i=1}^k [x_i] \right) = \tau^{(k-1)} \left( \sum_{i=1}^{k-1} [x_i] \right) x_\lambda^k + O(x_\lambda^{k+1})$.

(iii) $\partial_{t,A_k} \tau \left( \sum_{i=1}^k [x_i] \right) = \frac{c_k}{c_{k-1}} \partial_{t,A_{k-1}} \tau \left( \sum_{i=1}^{k-1} [x_i] \right) x_\lambda^k + O(x_\lambda^{k+1})$.

**Proof.** The proof of the theorem is similar to that of Theorem 5 in [27]. For the sake of completeness we give a proof here.

Let $\mu = (\mu_1, \ldots, \mu_l)$ be a partition satisfying $\lambda \leq \mu$. Here $l$ is not necessarily the length of $\mu$. By (iii) of Lemma 3 we have

$$\sum_{i=1}^{m_k} a_i^{(k)} = \sum_{i=k+1}^g \lambda_i \leq \sum_{i=k+1}^l \mu_i = N_{\mu,k}.$$

In case the last inequality is a strict inequality

$$\partial_{t,A_k}^\ast \mu \left( \sum_{i=1}^k [x_i] \right) = 0$$

by Proposition 3 in [27]. Thus, if the left hand side of (15) is not zero we have

$$\sum_{i=k+1}^g \lambda_i = \sum_{i=k+1}^l \mu_i$$

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which implies

\[ \lambda_i = \mu_i, \quad k + 1 \leq i \leq g, \]

and \( \mu_i = 0 \) for \( i > g \), since \( \lambda \leq \mu \). Then we have, by Theorem 1

\[ \partial_{t,A_k}s_\mu \left( \sum_{i=1}^k [x_i] \right) = c_k s_{(\mu_1, \ldots, \mu_k)} \left( \sum_{i=1}^k [x_i] \right). \]

The assertion (i) follows from this. The assertion (ii) is already proved in (ii) of Theorem 5 in [27], since no specialty of the partition \( \lambda \) associated with an \( (n, s) \) curve is used there. (iii) follows from (i) and (ii). \( \square \)

### 4.2 Vanishing and non-vanishing

The following vanishing theorem for the function \( \tau \) is valid.

**Theorem 4**  
(i) For any \( I = (i_1, \ldots, i_m), m \geq 1, \) satisfying \( \sum_{j=1}^m i_j < |\lambda| \) we have

\[ \partial_{t,I}\tau(0) = 0. \]

(ii) If \( m < m_0 \)

\[ \partial_{t,I}\tau(0) = 0, \]

for any \( I = (i_1, \ldots, i_m) \).

(iii) \( \partial_{t,A_0}\tau(0) = c_0 \), where \( c_0 = \pm 1 \) is given by (10).

**Proof.** The assertions of the theorem follow from (ii), (iii) of Theorem 2 and the definition of the function \( \tau(t) \). \( \square \)

Notice that this theorem is an analogue of Riemann’s singularity theorem for the function \( \tau(t) \).

### 5 Sato’s theory on soliton equations

In this section we review Sato’s theory of the KP-hierarchy [34] which makes a one to one correspondence between solutions of the KP-hierarchy and points of an infinite dimensional Grassmann manifold, called the universal Grassmann manifold (UGM).

### 5.1 The KP-hierarchy

The KP-hierarchy is the infinite system of differential equations for \( \tau(t) \) given by

\[ \int \tau(t - s - [k^{-1}])\tau(t + s + [k^{-1}])e^{-2 \sum_{i=1}^{\infty} s_i k^i}dk = 0 \quad (16) \]
where \( t = t(t_1, t_2, \ldots) \) and \( s = t(s_1, s_2, \ldots) \) and the integral signifies taking residue at \( k = \infty \). Namely we formally expand the integrand in the series of \( k \) and \( y \) and equate the coefficient of \( k^{-1}s_1^\gamma_1 s_2^\gamma_2 \cdots \) to zero. Then we get an infinite number of differential equations for \( \tau(t) \). In particular taking the coefficient of \( k^{-1}s_3 \) we get the bilinear form of the Kadomtsev-Petviashvili (KP) equation:

\[
(D_1^4 + 3D_2^2 - 4D_1D_3)\tau(t) \cdot \tau(t) = 0,
\]

where \( D_i \) is the Hirota derivative defined by

\[
\tau(t+s)\tau(t-s) = \sum (D_1^\gamma_1 D_2^\gamma_2 \cdots) \tau(t) \cdot \tau(t) \frac{s_1^\gamma_1 s_2^\gamma_2 \cdots}{\gamma_1! \gamma_2! \cdots}.
\] (17)

The initial value problem of the KP-hierarchy is uniquely solvable and the set of the initial values forms the infinite dimensional Grassmann manifold UGM.

5.2 UGM

Let us give the definition of UGM [34]. To this end we consider the ring of microdifferential operators with the coefficients in the formal power series ring \( R = \mathbb{C}[[x]] \) in one variable \( x \). Namely \( \mathcal{E}_R \) consists of all expressions of the form

\[
a_n(x)\partial^n + a_{n-1}(x)\partial^{n-1} + \cdots, \quad n \in \mathbb{Z}, \quad a_i(x) \in R,
\]

where \( \partial = \partial/\partial x \). Using

\[
am\partial^n = \sum_{i=0}^{\infty} (-1)^i \binom{n}{i} \partial^{n-i} a^{(i)} \quad a^{(i)} = \frac{d^i a}{dx^i},
\]

elements of \( \mathcal{E}_R \) can equally be rewritten in the form

\[
\sum_{i \leq n} \partial^i b_i, \quad n \in \mathbb{Z}, \quad b_i \in R.
\] (18)

Next we introduce the left \( \mathcal{E}_R \)-module \( V \) by

\[
V = \mathcal{E}_R/\mathcal{E}_Rx.
\]

The expression (15) implies that \( V \) is isomorphic to the space of microdifferential operators with constant coefficients:

\[
V \simeq \mathbb{C}((\partial^{-1})).
\] (19)

Then we see that \( V \) has the decomposition of the form

\[
V = V_\emptyset \oplus V_0, \quad V_\emptyset = \mathbb{C}[\partial], \quad V_0 = \mathbb{C}[[\partial^{-1}]]\partial^{-1}.
\]
Let us define the element $e_i$ of $V$ by

$$e_i = \partial^{-i-1} \mod \mathcal{E}_R x, \quad i \in \mathbb{Z}.$$  

Then

$$V_\phi = \bigoplus_{i=-\infty}^{-1} \mathbb{C} e_i, \quad V_0 = \prod_{i=0}^\infty \mathbb{C} e_i.$$  

The action of $\mathcal{E}_R$ on $V$ is given by

$$\partial e_i = e_{i+1}, \quad xe_i = (i+1)e_{i+1}.$$  

UGM is defined as the set of subspaces of $V$ which are comparable with $V_\phi$. To be precise we need some notation. For a subspace $U$ of $V$ we denote $\pi_U$ the map $\pi_U : U \rightarrow V/V_0 \cong V_\phi$, which is obtained as the composition of the inclusion $U \hookrightarrow V$ and the natural projection $V \rightarrow V/V_0$.

**Definition 6** The universal Grassmann manifold UGM is the set of subspaces $U \subset V$ such that $\ker \pi_U$ and $\coker \pi_U$ are of finite dimension and satisfy

$$\dim(\ker \pi_U) - \dim(\coker \pi_U) = 0.$$  

A point $U$ of UGM is specified by giving a frame of $U$, which is an ordered basis of $U$.

To each point $U$ of UGM there exists the unique sequence of integers $\rho = (\rho(i))_{i<0}$ and the unique frame $\xi = (\xi_j)_{j<0}, \xi_j = \sum_{i \in \mathbb{Z}} \xi_{ij} e_i$, of $U$ such that

$$\rho(-1) > \rho(-2) > \cdots, \quad \rho(i) = i \text{ for } i << 0,$$

$$\xi_{ij} = \begin{cases} 0 & i < \rho(j) \text{ or } i = \rho(j') \text{ for some } j' > j \\ 1 & i = \rho(j). \end{cases}$$

(20)

This frame $\xi$ is called the normalized frame of $U$.

The dimensions of $\ker \pi_U$ and $\coker \pi_U$ are expressed by $\rho$:

$$\dim(\ker \pi_U) = \#\{i \mid \rho(i) \geq 0\}, \quad \dim(\coker \pi_U) = \#\{i < 0 \mid i \notin \{\rho(j)\}\}.$$  

In general for a sequence $\rho = (\rho(i))_{i<0}$ satisfying the condition (20) define the partition $\lambda_\rho$ by

$$\lambda_\rho = (\lambda_{\rho,1}, \lambda_{\rho,2}, ...) = (\rho(-1), \rho(-2), ...) + (1, 2, ...).$$

(21)

The partition $\lambda = (0, 0, ...)$ corresponding to $\rho = (-1, -2, ...)$ is denoted by $\phi$.

Conversely for any partition $\lambda = (\lambda_1, ..., \lambda_l)$ one can construct $\rho$ satisfying (20) using (21), where we set $\lambda_i = 0$ for $i > l$.

For a partition $\lambda$ let $UGM^\lambda$ be the set of points $U$ of UGM such that the partition associated with $U$ is $\lambda$. Then UGM has the decomposition

$$UGM = \sqcup_\lambda UGM^\lambda,$$

where $\lambda$ runs over all partitions.
5.3 Fundamental theorems of Sato’s theory

Let $U$ be a point of UGM, $\xi = (\xi_j)_{j<0}$ the normalized frame of $U$ and $\mu$ an arbitrary partition. Let us write $\mu = \lambda_\rho$ with $\rho$ satisfying (20). We define the Plücker coordinate $\xi_\mu$ of $U$ associated with $\mu$ by

$$\xi_\mu = \det \left( \xi_{\rho(i),j} \right)_{i,j<0}.$$  

If $U \in UGM^\lambda$ the Plücker coordinates satisfy

$$\xi_\mu = \begin{cases} 
0 & \text{unless } \mu \geq \lambda \\
1 & \mu = \lambda. 
\end{cases} \quad (22)$$

**Definition 7** Define the tau function corresponding to $U$ by

$$\tau(t; U) = \sum_\mu \xi_\mu s_\mu(t). \quad (23)$$

**Theorem 5** [34, 33] For any $U$, $\tau(t; U)$ is a solution of the KP-hierarchy. Conversely any formal power series solution $\tau(t)$ of the KP-hierarchy there exists a unique point $U$ of UGM such that $\tau(t) = c\tau(t; U)$ for some constant $c$.

Notice that, for a point $U$ of $UGM^\lambda$, we have

$$\tau(t; U) = s_\lambda(t) + \sum_{\lambda<\mu} \xi_\mu s_\mu(t) \quad (24)$$

due to (22).

Now we explain how to recover $U$ from $\tau(t)$.

Let $K = \mathbb{C}((x))$ be the field of formal Laurent series in $x$ and $E_K = K((\partial^{-1}))$ the ring of microdifferential operators with the coefficients in $K$.

**Definition 8** Let $\mathcal{W}$ be the set of $W$ in $E_K$ of the form

$$W = \sum_{i \leq 0} w_i \partial^i, \quad w_0 = 1,$$

which satisfies the condition

$$x^m W, \quad W^{-1} x^m \in E_R, \quad (25)$$

for some non-negative integer $m$.

Then

**Theorem 6** [34, 33] Let $W$ be an element of $\mathcal{W}$ and $m$ an integer as in (25). Then

$$\gamma(W) = W^{-1} x^m V^\phi$$

defines a bijection $\gamma : W \to UGM$. 

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We remark that the map \( \gamma \) does not depend on the choice of \( m \), since \( x : V_\phi \to V_\phi \) is a surjection.

Given a formal power series solution of the KP-hierarchy \( \tau(t) \neq 0 \) the wave function \( \bar{\Psi}(t; z) \) and the adjoint wave function \( \Psi(t, z) \) are defined by

\[
\bar{\Psi}(t; z) = \frac{\tau(t - [z])}{\tau(t)} \exp(\sum_{i=1}^{\infty} t_i z^{-i}).
\]

\[
\Psi(t; z) = \frac{\tau(t + [z])}{\tau(t)} \exp(-\sum_{i=1}^{\infty} t_i z^{-i}).
\]

Let

\[
\frac{\tau(t - [z])}{\tau(t)} = \sum_{i=0}^{\infty} w_i z^i, \quad W = \sum_{i=0}^{\infty} w_i \partial^{-i}.
\]

Then

\[
\bar{\Psi}(t, z) = W \exp\left(\sum_{i=1}^{\infty} t_i z^{-i}\right),
\]

where we set \( t_1 = x \). The equation (16) implies that \( \Psi \) can be written as

\[
\Psi(t, z) = (W^*)^{-1} \exp(-\sum_{i=1}^{\infty} t_i z^{-i}).
\]

where \( P^* = \sum (-\partial)^i a_i(x) \) is the formal adjoint of \( P = \sum a_i(x) \partial^i \). We have \((P^*)^{-1} = (P^{-1})^*\) for an invertible \( P \in \mathcal{E}_K \) [7].

If \( \tau(t) \) is not identically 0, \( \tau(x, 0, 0, \ldots) \) is not identically zero [35] (see Lemma 4 in [25]). Let \( m_0 \) be the order of zeros of \( \tau(x, 0, 0, \ldots) \) at \( x = 0 \) and \( m \geq m_0 \). Obviously we have

\[
x^m W(x, 0, \ldots), \quad W(x, 0, \ldots)^{-1} x^m \in \mathcal{E}_R
\]

which implies \( W \in \mathcal{W} \). Then

**Theorem 7** [32, 34] There is a constant \( C \) such that

\[
C\tau(t) = \tau(t; \gamma(W(x, 0, \ldots))).
\]

The image \( \gamma(W(x, 0, \ldots)) \) can be computed from \( \Psi(t, z) \) using the following proposition [25] (see also [32, 15]).

**Proposition 3** Let

\[
x^m \Psi(x, 0, \ldots; z) = \sum_{i=0}^{\infty} \Psi_i(z) \frac{x^i}{i!}.
\]

(26)
Then we have, for \( i \geq 0 \),
\[
W(x, 0, \ldots) e_{-1}^{-1} x^m e_{-1-i} = (-1)^i \Psi_i (\partial^{-1}) e_{-1}.
\]
In particular
\[
\gamma(W(x, 0, \ldots)) = \text{Span}_\mathbb{C} \{ \Psi_i (\partial^{-1}) e_{-1} | i \geq 0 \},
\]
where \( \text{Span}_\mathbb{C} \{ \cdots \} \) signifies the vector space spanned by \( \{ \cdots \} \).

6 Algebro-geometric solution

In this section we determine the point of UGM corresponding to a theta function solution of the KP-hierarchy. We assume that a data \((X, \{ \alpha_i, \beta_i \}, p_\infty, z)\) is given, where \((X, \{ \alpha_i, \beta_i \}, p_\infty)\) is as before and \(z\) is a local coordinate at \(p_\infty\).

6.1 Prime form

Let us first recall the prime form [12]. It is known that there exists a non-singular odd half period \( e' \in J(X) \). We write \( e' \) as
\[
e' = q'_1 + \cdots + q'_{g-1} - \Delta, \quad q'_i \in X.
\]
Let \( \varepsilon_0 = t'(\varepsilon'_0, \varepsilon''_0) \), \( \varepsilon'_0, \varepsilon''_0 \in \mathbb{R}^g \) be the characteristics of \( e' \):
\[
e' = \Omega \varepsilon'_0 + \varepsilon''_0.
\]
Then the zero divisor of the holomorphic one form
\[
\sum_{i=1}^{g} \frac{\partial \theta[\varepsilon_0]}{\partial z_i}(0) dv_i
\]
is \( 2 \sum_{i=1}^{g-1} q'_i \). Since \( e' \) is non-singular, there exists a unique, up to constant multiples, holomorphic section of \( L_{e'} \otimes L_\Delta \) which vanishes exactly at \( q'_1 + \cdots + q'_{g-1} \).

Let \( h_{\varepsilon_0} \) be a holomorphic section of \( L_{e'} \otimes L_\Delta \) such that
\[
h^2_{\varepsilon_0} = \sum_{i=1}^{g} \frac{\partial \theta[\varepsilon_0]}{\partial z_i}(0) dv_i.
\]
Then the prime form \( E(p_1, p_2) \) is defined by
\[
E(p_1, p_2) = \frac{\theta[\varepsilon_0](p_2) \int_{P_1} dv_{p_1}}{h_{\varepsilon_0}(p_1) h_{\varepsilon_0}(p_2)}.
\]
(27)
Let \( \pi_i \) be the projection of \( X \times X \) to the \( i \)-th component \( X \) and \( I_{21} \) the map from \( X \times X \) to \( J(X) \) defined by \( I_{21}(p_1, p_2) = I(p_2) - I(p_1) \). We denote by \( \Theta \) the holomorphic line bundle on \( J(X) \) of which \( \theta(z|\Omega) \) is a holomorphic section.
Then the prime form is a holomorphic section of \( \pi_1^* L_\Delta^{-1} \otimes \pi_2^* L_\Delta^{-1} \otimes I_2^* (\Theta) \). The prime form \( E(p_1, p_2) \) is skew symmetric in \((p_1, p_2)\) and vanishes at the diagonal \( \{(p, p) | p \in X\} \) to the first order. For other properties see \([12]\).

We need the object which is obtained from the prime form by restricting one of the variables to a point. Using the local coordinate \( z \) around \( p_\infty \) let us write

\[
E(p_1, p_2) = E(z_1, z_2) \sqrt{dz_1 dz_2},
\]

where \( z_i = z(p_i) \). Then we set

\[
E(p, p_\infty) = E(z, 0) \sqrt{dz}, \tag{28}
\]

which is a constant multiple of

\[
\frac{\theta[\varepsilon_o](I|\Omega)}{\sqrt{\hbar \varepsilon_o}}.
\]

The proportional constant is determined so that the expansion of \( E(p, p_\infty) \sqrt{dz} \) at \( p_\infty \) has the form \(-z + O(z^2)\). So it depends on the choice of \( z \). It is a section of the line bundle \( L_\Delta^{-1} \otimes I^* \Theta \). The holomorphic line bundle \( I^* \Theta \) on \( X \) is described by the transformation rule

\[
f(p + \alpha_j) = f(p),
\]

\[
f(p + \beta_j) = e^{-\pi i \Omega_j} e^{2\pi i \int_{p_\infty}^p dv_j} f(p). \tag{29}
\]

### 6.2 Theta function solution

Let

\[
\omega(p_1, p_2) = d_{p_1} d_{p_2} \log E(p_1, p_2),
\]

be the fundamental normalized differential of the second kind \([12]\). Using the local coordinate \( z \) we expand \( \omega(p_1, p_2) \) and \( dv \) as as

\[
\omega(p_1, p_2) = \left( \frac{1}{(z_1 - z_2)^2} + \sum_{i,j=1}^\infty q_{ij} z_1^{i-1} z_2^{j-1} \right) dz_1 dz_2, \tag{30}
\]

\[
dv_i = \left( \sum_{j=1}^\infty a_{ij} z^{j-1} \right) dz,
\]

where \( z_i = z(p_i) \). Set

\[
q(t) = \sum_{i,j=1}^\infty q_{ij} t_i t_j, \quad A = (a_{ij})_{1 \leq i \leq g, 1 \leq j}.
\]

It is well known that

\[
\tau(t) = e^{\frac{1}{2} q(t)} \theta(A t + e | \Omega) \tag{31}
\]

is a solution of the KP-hierarchy \([16]\) for any \( e = t(e_1, \ldots, e_g) \in \mathbb{C}^g \).
6.3 The point of UGM

Let us determine the point of UGM corresponding to (31). To this end we compute the adjoint wave function associated with \( \tau(t) \).

Let \( dr_n, n \geq 1 \) be the meromorphic one form with a pole only at \( p_\infty \) of order \( n + 1 \) which satisfies the following conditions:

\[
dr_n = d \left( \frac{1}{z^n} + O(z) \right) \quad \text{near } p_\infty, \]
\[
\int_{\alpha_i} dr_n = 0 \quad \text{for any } i. \tag{32}
\]

Then we have \([13, 15, 25]\)

\[
z^{-1}\Psi(t; z)\sqrt{dz} = \frac{1}{E(p, p_\infty)} \frac{\theta(I(p) + At + e)}{\theta(At + e)} \exp \left( -\sum_{n=1}^{\infty} t_n \int^p dr_n \right), \tag{33}
\]

where the integral is normalized as

\[
\lim_{p \to p_\infty} \left( \int^p dr_n - \frac{1}{z^n} \right) = 0. \]

Notice that

\[
\theta(I(p) + At + e) \exp \left( -\sum_{n=1}^{\infty} t_n \int^p dr_n \right) \quad \text{(33)}
\]

is a section of the holomorphic line bundle \( I^*(\Theta) \otimes L_{-e} \) on \( X \) and specified by the transformation rule \((34)\). Therefore \( z^{-1}\Psi(t; z)\sqrt{dz} \) can be considered a section of the line bundle \( L_\Delta \otimes L_{-e} \).

Now we define the map

\[
i : H^0(X, (L_\Delta \otimes L_{-e})(*p_\infty)) \longrightarrow V,
\]

in the following way.

First we specify the local trivialization of \( L_\Delta \otimes L_{-e} \) as in \((32)\). Namely a section of this bundle is written as \( E(p, p_\infty)^{-1} \) times a section of \( L_{-e} \otimes I^*\Theta \) and the latter is realized as a multiplicative functions on \( X \) which obeys the transformation rule given by

\[
\begin{align*}
f(p + \alpha_j) &= f(p), \\
f(p + \beta_j) &= e^{-\pi i \Omega_{ij}} e^{-2\pi i (f_{p_\infty}^p dv_j + e_j)} f(p). \tag{34}
\end{align*}
\]

Take an element \( \varphi(p) \) of \( H^0(X, (L_\Delta \otimes L_{-e})(*p_\infty)) \) and expand it around \( p_\infty \) in \( z \):

\[
\varphi(p) = \sum_{-\infty << n << \infty} c_n z^n \sqrt{dz}. \]

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Then we set
\[ \iota(\varphi) = \sum_{-\infty < n < \infty} c_n e_n \in V. \] (35)

Define the subspace \( U \) of \( V \) as the image of \( \iota \):
\[ U = \iota \left( H^0(X, (L_\Delta \otimes L_{-e})(*p_\infty)) \right). \] (36)

Let \( \lambda \) be the partition defined by (7) if \( \theta(e|\Omega) = 0 \) and \( e \) is given by (5). We set \( \lambda = \phi \) if \( \theta(e|\Omega) \neq 0 \). Then

**Theorem 8** The subspace \( U \) is a point of \( UGM^\lambda \).

**Proof.** For \( e \) satisfying \( \theta(e|\Omega) \neq 0 \) the theorem is proved in [15]. Let us prove the theorem in case \( \theta(e|\Omega) = 0 \).

Writing \( e \) as in (5) we have
\[
\begin{align*}
L_\Delta \otimes L_{-e} &\simeq K_X(-q_1 - \cdots - q_{g-1}), \\
L_\Delta \otimes L_e &\simeq O(q_1 + \cdots + q_{g-1}).
\end{align*}
\] (37) (38)

Then
\[
\begin{align*}
L_{-e+\delta} &= K_X(-q_1 - \cdots - q_{g-1} - (g-1)p_\infty), \\
L_{e+\delta} &= O(q_1 + \cdots + q_{g-1} - (g-1)p_\infty).
\end{align*}
\]

Therefore
\[
\begin{align*}
(L_\Delta \otimes L_{-e})(np_\infty) &\simeq L_{-e+\delta} ((g-1+n)p_\infty), \\
(L_\Delta \otimes L_e)(np_\infty) &\simeq L_{e+\delta} ((g-1+n)p_\infty).
\end{align*}
\] (39) (40)

Let
\[
\begin{align*}
0 &\leq b'_1 < \cdots < b'_g \leq 2g-1, \\
0 &\leq b^*_1 < b^*_2 < \cdots,
\end{align*}
\]
be gaps and non-gaps of \( L_{-e+\delta} \) at \( p_\infty \) respectively.

By (39) we have
\[
\dim (\text{Ker } \pi_U) = \sharp\{i \mid b^*_i - (g-1) \leq 0\}, \quad \dim (\text{Coker } \pi_U) = \sharp\{i \mid b'_i - (g-1) > 0\}.
\]

Since the number of gaps is \( g \) we have
\[
\dim (\text{Coker } \pi_U) = g - \sharp\{i \mid b'_i - (g-1) \leq 0\}.
\]

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Then

\[
\dim (\text{Ker } \pi_U) - \dim (\text{Coker } \pi_U) = \sharp \{i \mid b_i^* - (g - 1) \leq 0\} + \sharp \{i \mid b'_i - (g - 1) \leq 0\} - g = g - g = 0.
\]

Thus \( U \in UGM \).

By the definition of gaps of \( L_{-e+\delta} \) and \([39]\) the sequence \( \rho = (\rho(i))_{i<0} \) corresponding to \( U \) is given by

\[
\rho(-i) = -b_i^* + g - 1.
\]

(41)

In order to prove that \( U \) belongs to \( UGM^\lambda \) we need to rewrite this in terms of gaps of \( L_{e+\delta} \).

**Lemma 4** (i) \((b'_1, \ldots, b'_g) = (2g - 1 - b_g^*, \ldots, 2g - 1 - b_1^*)\).

(ii) \((b_1^*, \ldots, b_g^*) = (2g - 1 - b_g, \ldots, 2g - 1 - b_1)\).

**Proof.** For simplicity we denote \( L_{\pm e+\delta}(kp) \) by \( L_{\pm e+\delta}(k) \) respectively. By Riemann-Roch theorem, \([39], [40]\) we have

\[
h^0(L_{e+\delta}(g - 1 + n)) - h^0(L_{-e+\delta}(g - 1 - n)) = n.
\]

(42)

This equation implies that \( g - 1 + n \) is a gap of \( L_{e+\delta} \) if and only if \( g - n \) is a non-gap of \( L_{-e+\delta} \). In fact the condition that \( g - 1 + n \) is a gap of \( L_{e+\delta} \) is equivalent to the equation

\[
h^0(L_{e+\delta}(g - 1 + (n - 1))) = h^0(L_{e+\delta}(g - 1 + n)),
\]

which, by \([42]\), is equivalent to

\[
h^0(L_{e+\delta}(g - n)) = h^0(L_{-e+\delta}(g - n - 1)) + 1.
\]

The last equation is equivalent to the condition that \( g - n \) is a non-gap of \( L_{-e+\delta} \).

Let \( n \) be defined by \( g - 1 + n = b_i \). Since \( b_i \) is a gap of \( L_{e+\delta} \),

\[
g - (b_i - g + 1) = 2g - 1 - b_i
\]

is a non-gap of \( L_{-e+\delta} \). Similarly \( 2g - 1 - b_i^* \) is a gap of \( L_{-e+\delta} \) for any \( i \). Since

\[
0 \leq 2g - 1 - b_i; 2g - 1 - b_i^* \leq 2g - 1, \quad 1 \leq i \leq g,
\]

and these numbers are all distinct, they exhaust all gaps and non-gaps of \( L_{-e+\delta} \) contained in \( \{0, 1, \ldots, 2g - 1\} \). Thus the lemma is proved. □

Notice that \( b_i^* = i + g - 1 \) for \( i \geq g + 1 \). Then we have, by (2) of Lemma 4,

\[
\rho = (g - 1 - b_1^*, \ldots, g - 1 - b_g^*, -(g + 1), -(g + 2), \ldots)
\]

\[
= (b_g - g, \ldots, b_1 - g, -(g + 1), -(g + 2), \ldots).
\]

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Thus
\[ \lambda_{\rho} = \rho + (1, 2, 3, \ldots) = (b_\rho - (g - 1), \ldots, b_1, 0, 0, \ldots) = \lambda, \]
which completes the proof of Theorem 8. \(\square\)

Now we have

**Theorem 9** Let \(\tau(t)\) be the solution of the KP-hierarchy given by (31) and \(U\) the point of UGM given by (36). Then
\[ C\tau(t) = \tau(t; U) \] (43)
for some non-zero constant \(C\).

**Proof.** Notice that the factors of automorphy of \(f(p)\) in (34) do not depend on \(t\). Thus if we expand (33) in \(t = t_1, t_2, \ldots\), all the coefficients are sections of \(I^* (\Theta) \otimes L_{-e}\). It implies that if we expand \(z^{-1} \Psi(t; z) \sqrt{dz} \theta(At + e)\) in \(t\), any coefficient is an element of \(H^0(X, L_\Delta \otimes L_{-e}(p_\infty))\). Let \(m_0\) be the order of the zero of \(\tau(x, 0, \ldots)\) at \(x = 0\). We expand \(x^{m_0} \Psi(x, 0, \ldots; z)\) as in (26). Then
\[ z^{-1} \Psi_i(z) \sqrt{dz} \in H^0(X, L_\Delta \otimes L_{-e}(p_\infty)), \] (44)
for any \(i\). Let
\[ \Psi_i(z) = \sum_{-\infty < n < \infty} \Psi_{in} z^n. \]
Then
\[ \Psi_i(\partial^{-1}) e_{-1} = \sum_{-\infty < n < \infty} \Psi_{in} e_{n-1}. \] (45)
On the other hand we have, by the definition (35) of the map \(\iota\),
\[ \iota(z^{-1} \Psi_i(z) \sqrt{dz}) = \iota( \sum_{-\infty < n < \infty} \Psi_{in} z^{n-1} \sqrt{dz} ) = \sum_{-\infty < n < \infty} \Psi_{in} e_{n-1}. \] (46)
Let
\[ U' = \text{Span}_C \{ \Psi(\partial^{-1}) e_{-1} \mid i \geq 0 \}. \]
Then \(U'\) is the point of UGM corresponding to the solution (31) by Proposition 3 and Theorem 7. By (44), (45) and (46) we have \(U' \subset U\). Since a strict inclusion relation is impossible for two points of UGM we have \(U' = U\). Thus the theorem is proved. \(\square\)

As a corollary of Theorem 9 we have, by (24),

**Corollary 1** The \(\tau(t)\) given by (31) has the expansion of the form
\[ C\tau(t) = s_\lambda(t) + \sum_{\lambda < \mu} \xi_\mu s_\mu(t), \]
for some constant \(C\).

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7 Series expansion of the theta function

In this section we assume that a data \((X, \{\alpha_i, \beta_i\}, p_\infty, z, e)\) is given, where notation is as before.

7.1 Taylor expansion

In order to study the series expansion of the theta function we need to change the variables to appropriate ones.

Let \(\{du_{w_i}\}\) be a basis of holomorphic one forms such that \(du_{w_i}\) has the following expansion at \(p_\infty\):

\[
du_{w_i} = z^{w_i - 1}(1 + O(z))dz.
\]

The existence of such a basis is easily proved using Riemann-Roch theorem (c.f. [18]). However a basis which satisfies the condition (47) is not unique. For the moment we do not know what is the best choice. So we take any one of them.

By integrating this basis over the canonical homology basis we define the period matrices \(\omega_1, \omega_2\) by

\[
2\omega_1 = \left(\int_{\alpha_j} du_i\right), \quad 2\omega_2 = \left(\int_{\beta_j} du_i\right).
\]

The normalized period matrix \(\Omega\) is given by \(\Omega = \omega_1^{-1}\omega_2\).

We label the \(g\) variables of the theta function by gaps at \(p_\infty\) as

\[
u = \left(u_{w_1}, \ldots, u_{w_g}\right),
\]

and consider the function of the form

\[
\theta((2\omega_1)^{-1}u + e | \Omega).
\]

We set \(\partial_i = \partial/\partial u_i\) and \(\partial_I = \partial_{i_1} \cdots \partial_{i_r}\) for \(I = (i_1, \ldots, i_r)\). If we differentiate theta function \(d_j\) should be considered in \(\{w_1, \ldots, w_g\}\). We assign weight \(i\) to \(u_i\). Then the initial term, with respect to weight, of the Taylor series expansion of this function is given by the following theorem.

**Theorem 10** Suppose that a data \((X, \{\alpha_i, \beta_i\}, p_\infty, z, \{du_{w_i}\}, e)\) is given. Let \(\lambda\) be the partition associated with it. Then we have

\[
C\theta((2\omega_1)^{-1}u + e | \Omega) = s_\lambda(t)|_{w_i = u_{w_i}} + \text{higher weight terms},
\]

where \(C = c_0\partial_A_0\theta(e | \Omega)\) and \(c_0 = \pm 1\) is given by \((10)\).
Proof. Let us expand $du_{w_i}$ at $p_\infty$ in $z$ as

$$du_{w_i} = \sum_{j=1}^{\infty} b_{ij} z^{j-1} dz,$$

and define the matrix $B = (b_{ij})$. By (47) $B$ has the following triangular structure:

$$b_{ij} = \begin{cases} 0 & \text{if } j < w_i \\ 1 & \text{if } j = w_i \end{cases} \quad (48)$$

**Lemma 5** We have $B = 2\omega_1 A$.

Proof. Write

$$du_{w_i} = \sum_{j=1}^{g} c_{ij} dv_j. \quad (49)$$

Integrating this equation over $\alpha_j$ we get $(2\omega_1)_{ij} = c_{ij}$. Then the lemma follows by comparing the expansion coefficients of (49).

By the lemma the tau function (31) can be written as

$$\tau(t) = e^{\frac{1}{2} q(t)} (2\omega_1)^{-1} B t + e \mid \Omega). \quad (50)$$

Let $u = t(u_{w_1}, ..., u_{w_g}) = B t$ and set $t_j = 0$ for all $j$ except $t_{w_i}, 1 \leq i \leq g$. Then we can write

$$u = \tilde{B} \tilde{t} , \quad \tilde{t} = t(t_{w_1}, ..., t_{w_g}),$$

where the $g \times g$ matrix $\tilde{B} = (\tilde{b}_{ij})$ is the upper triangular matrix whose diagonal entries are all equal to 1 due to (48). Thus $\tilde{B}$ is invertible and $\tilde{B}^{-1}$ has a similar form to $B$. Then

$$\theta((2\omega_1)^{-1} u + e \mid \Omega) = e^{-\frac{1}{2} q(\tilde{u})} \tau(\tilde{u}), \quad (51)$$

where $\tilde{u}$ in $\tau(t)$ and $q(t)$ should be understood that $\tilde{u}_{w_i}$ sits on the $w_i$-th component of $t = t(t_1, t_2, ...)$ and other components of $t$ are zero. Since $\tilde{u}_{w_i} = u_{w_i}$ modulo higher weight terms, the theorem follows from Corollary 4 and Theorem 2 (iii). $\square$

### 7.2 Duality

There is a relation of the expansions of the theta function at $e$ and at $-e$.

**Theorem 11** Under the same assumption as in Theorem 10 we have

$$(-1)^{|\lambda|} C \theta((2\omega_1)^{-1} u - e \mid \Omega) = s^t \lambda(t)|_{t_{w_i}=u_{w_i}} + \text{higher weight terms},$$

where $^t \lambda$ is the conjugate partition of $\lambda$ and $C$ is the same as in Theorem 10.
Proof. Since \( \theta(Z | \Omega) \) is an even function of \( Z \)
\[
\theta((2\omega_1)^{-1}u - e | \Omega) = \theta((2\omega_1)^{-1}(-u) + e | \Omega).
\]
Then the theorem follows from the known relation for the Schur functions \[20\]
\[
 s_{\mu}(-t) = (-1)^{|\mu|} s_{\mu}(t),
\]
and Theorem \[10\]. □

**Corollary 2** Let \( \lambda \) be the partition associated with \((X, \{\alpha_i, \beta_i\}, p_{\infty}, e)\). Then the partition associated with \((X, \{\alpha_i, \beta_i\}, p_{\infty}, -e)\) is \( ^t \lambda \).

**Proof.** Since \( s_{\mu}(t) = s_{\nu}(t) \) implies \( \mu = \nu \) for two partitions \( \mu \) and \( \nu \), the assertion follows from Theorem \[10\] and Theorem \[11\]. □

### 7.3 Examples

We give examples of the partition \( \lambda \) given by \[7\] which, by Theorem \[10\], determines the initial term of the Taylor expansion of the theta function.

**Example 1** Let \( X \) be any compact Riemann surface of genus \( g \), \( p_{\infty} \) a non-Weierstrass point of \( X \) and \( e = -\delta \). In this case \( L_{e+\delta} \cong \mathcal{O} \) and we have
\[
(b_1, ..., b_g) = (w_1, ..., w_g) = (1, 2, ..., g).
\]
Thus \( \lambda = (1^g) \). In this case \( s_\lambda(t) = (-1)^g p_g(-t) \). For example, for \( 1 \leq g \leq 4 \) they are given by
\[
 s_{(1)}(t) = t_1, \quad s_{(1,1)}(t) = -t_2 + \frac{t_1^2}{2}, \quad s_{(1,1,1)}(t) = t_3 - t_1t_2 + \frac{t_1^3}{3!},
\]
\[
 s_{(1,1,1,1)}(t) = -t_4 + t_1t_3 + \frac{t_2^2}{2} - \frac{t_1^2t_2}{2} + \frac{t_1^4}{4!}.
\]
The \((3, 4, 5)\) curve studied in \[16\] is included in this Example.

**Example 2** Let \( X \) and \( p_{\infty} \) be the same as in Example 1. Take \( e = \delta \). Then, by duality, \( \lambda = (g) \). In this case \( s_\lambda(t) = p_g(t) \).

**Example 3** Let \( X \) be an \((n, s)\) curve \[5\] or a telescopic curve of genus \( g \) \[21\]. Take \( p_{\infty} = \infty \) and \( e = -\delta \). Then
\[
 \lambda = (w_g, ..., w_1) - (g - 1, ..., 1, 0).
\] (52)
In this case \( \delta = -\delta \) in \( J(X) \) and \( \lambda \) satisfies \( ^t \lambda = \lambda \) by Corollary \[2\]. These are cases which were studied in most of the literatures.
Example 4 In [17] sigma functions of (3,7,8) \((g = 4)\) and (6,13,14,15,16) \((g = 12)\) curves are studied. If we take \(e = -\delta\), then (52) holds. For (3,7,8)-curve, gaps are \((1,2,4,5)\) and \(\lambda = (2,2,1,1)\). For (6,13,14,15,16)-curve, gaps are \((1,2,3,4,5,7,8,9,10,11,17,23)\) and \(\lambda = (12,7,2,2,2,2,1,1,1,1)\). In the latter case the maximal gap 23 is equal to \(2g - 1\). It implies \(\delta = -\delta\) in \(J(X)\) and \(\lambda = \lambda\).

If we take \(e = \delta\), then we have \(\lambda = (4,2)\) for (3,7,8)-curve by duality.

7.4 The case of hyperelliptic curves

In the case of hyperelliptic curves it is possible to determine the partition \(\lambda\) corresponding to any point on the theta divisor explicitly.

Let \(X\) be the hyperelliptic curve of genus \(g\) given by
\[ y^2 = f(x), \quad f(x) = \prod_{i=1}^{2g+1} (x - e_i), \]
where \(e_i \neq e_i\) for any \(i \neq j\). We take a canonical homology basis \(\{\alpha_i, \beta_i\}\) and the base point \(p_\infty = \infty\). Let \(\phi\) be the hyperelliptic involution, \(\phi(x, y) = (x, -y)\). It is known that the points on the theta divisor with the multiplicity \(m_0 \geq 1\) are given by
\[ q_1 + \cdots + q_{g+1-2m_0} + (2m_0 - 2)\infty - \Delta, \tag{53} \]
where \(q_i\)'s are points on \(X\) satisfying \(q_i \neq \phi(q_j)\) for any \(i \neq j\) [12]. We denote this set of points by \(\Theta_{m_0}\).

We set
\[ \Theta_{m_0,o} = \{q_1 + \cdots + q_{g+1-2m_0} + (2m_0 - 2)\infty - \Delta \in \Theta_{m_0} \mid q_i \neq \infty \forall i\}, \]
\[ \Theta_{m_0,e} = \{q_1 + \cdots + q_{g-2m_0} + (2m_0 - 1)\infty - \Delta \in \Theta_{m_0} \mid q_i \neq \infty \forall i\}. \]
Then

**Proposition 4** The partition \(\lambda\) associated with \((X, \{\alpha_i, \beta_i\}, \infty, e)\) is given as follows.

(i) \(\lambda = (2m_0 - 1,2m_0 - 2,\ldots,1)\) for \(e \in \Theta_{m_0,o}\).

(ii) \(\lambda = (2m_0,2m_0 - 1,\ldots,1)\) for \(e \in \Theta_{m_0,e}\).

This proposition follows from

**Lemma 6** The gaps of \(L_{e+\delta}\) are given as follows.

(i) \(\{0,1,\ldots,g - 2m_0, g - 2m_0 + 2i \mid 1 \leq i \leq 2m_0 - 1\}\) for \(e \in \Theta_{m_0,o}\).

(ii) \(\{0,1,\ldots,g - 1 - 2m_0, g - 1 - 2m_0 + 2i \mid 1 \leq i \leq 2m_0\}\) for \(e \in \Theta_{m_0,e}\).
Proof. Let us prove (i). The proof of (ii) is similar. We have
\[ e + \delta = q_1 + \cdots + q_{g+1-2m_0} - (g + 1 - 2m_0)\infty. \]

Since
\[ L_{e+\delta}(n\infty) \simeq \mathcal{O}_X(q_1 + \cdots + q_{g+1-2m_0} + (n - (g + 1 - 2m_0))\infty), \]
\[ H^0(X, L_{e+\delta}(n\infty)) \] is identified with the space of meromorphic functions on \( X \) with at most a simple pole at \( q_i \) \((1 \leq i \leq g+1-2m_0)\) and a pole of order at most \( n-(g+1-2m_0) \) at \( \infty \) if \( n-(g+1-2m_0) \geq 0 \) or with at most a simple pole at \( q_i \) \((1 \leq i \leq g+1-2m_0)\) and a zero of order at least \(-n+(g+1-2m_0)\) at \( \infty \) if \( n-(g+1-2m_0) < 0 \).

Let \( q_i = (x_i, y_i) \). Since a meromorphic function on \( X \) which has a pole only at \( \infty \) is a polynomial of \( x \) and \( y \), any element of \( H^0(X, L_{e+\delta}(n\infty)) \) can be written in the form
\[ F(x, y) = \frac{A(x) + B(x)y}{(x-x_1) \cdots (x-x_{g+1-2m_0})}, \] (54)

\( A(x) \) and \( B(x) \) are some polynomials of \( x \). Since \( F \) has at most a simple pole at \( q_i = (x_i, y_i) \), \( A(x) \) and \( B(x) \) should satisfy
\[ A(x) - B(x) = 0, \quad 1 \leq i \leq g + 1 - 2m_0. \] (55)

First consider the case \( n < g + 1 - 2m_0 \). In this case \( F \) must have a zero at \( \infty \) of order at least \( g + 1 - 2m_0 - n \). Let us estimate the order of zero or pole at \( \infty \) of \( F \) given by (54).

If \( B(x) = 0 \), then by (55), \( A(x) \) must be divided by \( \prod_{i=1}^{g+1-2m_0} (x-x_i) \) and \( F \) becomes a polynomial of \( x \). Thus \( F \) can not have a zero at \( \infty \).

Suppose that \( B(x) \neq 0 \). Let \( k \) and \( l \) be degrees of \( A(x) \) and \( B(x) \) respectively. Since \( x \) and \( y \) has a pole of order 2 and \( 2g + 1 \) at \( \infty \), the order of a zero of \( F \) at \( \infty \), denoted by \( \text{ord}(F) \), is given by
\[ \text{ord}(F) = 2(g + 1 - 2m_0) - \max\{2k, 2l + 2g + 1\}. \]

Notice that \( \max\{2k, 2l + 2g + 1\} \geq 2g + 1 \). Then
\[ \text{ord}(F) \leq 2(g + 1 - 2m_0) - (2g + 1) = 1 - 4m_0 < 0. \]

Thus \( F \) can not have a zero at \( \infty \) in this case too. Consequently \( 0, 1, \ldots, g - 2m_0 \) are gaps of \( L_{e+\delta} \) at \( \infty \).

If \( n = g + 1 - 2m_0 \), the constant function 1 can be considered as a global section of \( L_{e+\delta}(n\infty) \). Thus \( n \) is a non-gap.

Suppose that \( n > g + 1 - 2m_0 \). In this case \( x^{2i} \), \( i \geq 1 \) gives a global section of \( L_{e+\delta}(n\infty) \) with \( n = 2i + (g + 1 - 2m_0) \). Let us prove that \( n = 2i - 1 + (g + 1 - 2m_0) \), \( 1 \leq i \leq 2m_0 - 1 \) is a gap. The order of a pole of \( F \) at \( \infty \), which is denoted by \(-\text{ord}(F)\), is given by
\[ -\text{ord}(F) = \max\{2k, 2l + 2g + 1\} - 2(g + 1 - 2m_0) \geq 2l - 1 + 4m_0. \]
Therefore, if $n = 2i - 1 + (g + 1 - 2m_0)$ is a non gap, it must satisfy

$$2i - 1 \geq 2l - 1 + 4m_0,$$

which implies $i \geq 2m_0 + l \geq 2m_0$. Thus $n = 2i - 1 + (g + 1 - 2m_0), 1 \leq i \leq 2m_0 - 1$ is a gap.

The number of gaps we obtained so far is $(g - 2m_0 + 1) + (2m_0 - 1) = g$. Thus those exhaust all gaps. Thereby the proof of the proposition is completed. □

**Remark** If $g$ is odd and $m_0 = \frac{g + 1}{2}$ then $e = -\delta \in \Theta_{m_0, o}$. In this case $\lambda = (g, g - 1, ..., 1)$. These results are the same as those given in Example 3 for $(2, 2g + 1)$ curves as it should be.

### 7.5 Expansion on Abel-Jacobi images

We denote by $l(\mu)$ the length of a partition $\mu$.

**Proposition 5** Let $0 \leq k \leq l(\lambda)$. Then, for any $I = (i_1, ..., i_m), m \geq 1$, satisfying $\sum_{j=1}^{m} i_j < N_{\lambda, k}$ we have

$$\partial_I \theta \left( \sum_{i=1}^{k} (p_i - p_\infty) + e \mid \Omega \right) = 0,$$

for any $p_i \in X, 1 \leq i \leq k$.

**Proof.** Differentiate (51) by $\partial_I$ and use Proposition 4 in [27], Corollary 1, to get the result. □

**Theorem 12** Suppose that $m_k > 0$. Let $c_k$ be given by (10) and $z_j = z(p_j)$ for $p_j \in X$. Then

1. $C \partial_{A_k} \theta \left( \sum_{j=1}^{k} (p_j - p_\infty) + e \mid \Omega \right) = c_k s_{(\lambda_1, ..., \lambda_k)} \left( \sum_{j=1}^{k} [z_j] \right) + \text{higher weight terms},$

   where $C$ is the constant in Theorem 10 and $c_k = \pm 1$ is given by (10).

2. $\partial_{A_k} \theta \left( \sum_{j=1}^{k} (p_j - p_\infty) + e \mid \Omega \right) = \frac{c_k}{c_{k-1}} \partial_{A_{k-1}} \theta \left( \sum_{j=1}^{k-1} (p_j - p_\infty) + e \mid \Omega \right) z_k^{\lambda_k} + O(z_k^{\lambda_k + 1}),$

   where $z_j = z(p_j)$.

**Proof.** The theorem easily follows from Corollary 1, Theorem 3 and (50) as in the proof of Corollary 4 in [27]. □
7.6 Refined Riemann’s singularity theorem

Corollary 3 We assume the same conditions as in Theorem 10. Then
(i) For any \( I = (i_1, \ldots, i_m), m \geq 1, \) satisfying \( \sum_{j=1}^{m} i_j < |\lambda| \) we have
\[ \partial_I \theta(e \mid \Omega) = 0. \]
(ii) If \( m < m_0 \) we have
\[ \partial_I \theta(e \mid \Omega) = 0, \]
for any \( I = (i_1, \ldots, i_m) \).
(iii) \( \partial_{A_0} \theta(e \mid \Omega) \neq 0. \)

Proof. (i) is the special case \( k = 0 \) of Proposition 5. (ii) follows from Theorem 4 and (51). (iii) is the special case \( k = 0 \) of Theorem 12 (i). □

Notice that Riemann’s singularity theorem is a part of this corollary. Moreover the assertion (iii) gives one non-vanishing derivative with degree \( m_0 \) explicitly and (i) gives a new vanishing property of the theta function which does not follow from Riemann’s singularity theorem. In this sense Corollary 3 is an extension and a refinement of Riemann’s singularity theorem. By Proposition 2 we can say more.

Corollary 4 In the series expansion of the theta function \( C\theta((2\omega_1)^{-1} u + e \mid \Omega) \) at \( u = 0 \) the terms with the minimal weight and the minimal degree are given by \( L_\lambda(t)|_{t_{w_i} = u_{w_i}} \), where \( C \) is the constant in Theorem 10.

8 Sigma function

Sigma functions of an arbitrary Riemann surface have been introduced by Korotkin and Shramchenko[18]. In this section we first review their construction in a more unified framework. Next we prove the modular invariance of sigma functions with characteristics by specifying a suitable normalization constant which is obtained as a result of Corollary 3 and is apparently different from that in [18].

In this section we assume that a data \( (X, \{\alpha_i, \beta_i\}, p_\infty, z, e, \{du_{w_i}\}) \) is given.

8.1 Bilinear meromorphic differential

The key ingredients in constructing sigma function are certain bilinear meromorphic differentials. So we first study general properties of such bilinear differentials.

Let \( \tilde{\omega}(p_1, p_2) \) be a symmetric bilinear meromorphic differential on \( X \times X \) such that it is holomorphic outside the diagonal \( \{(p, p) \mid p \in X\} \) where it has a double pole and at any \( p_0 \in X \) it has the expansion of the form
\[ \tilde{\omega}(p_1, p_2) = \left( \frac{1}{(w_1 - w_2)^2} + \text{holomorphic in } w_1, w_2 \right) dw_1 dw_2, \]
where \( w \) is a local coordinate at \( p_0 \) and \( w_i = w(p_i) \).

The fundamental normalized differential of the second kind \( \omega(p_1, p_2) \) is an example (see (30)). In general \( \tilde{\omega}(p_1, p_2) \) has the following structure.

**Proposition 6** For any \( \tilde{\omega}(p_1, p_2) \) there exist \( \Omega(p_1, p_2) \) and \( \{d\tilde{r}_i\}_{i=1}^{\infty} \) such that

\[
\tilde{\omega}(p_1, p_2) = \text{d} p_2 \tilde{\Omega}(p_1, p_2) + \sum_{i=1}^{g} du_{w_i}(p_1) d\tilde{r}_i(p_2).
\]

(56)

Here \( d\tilde{r}_i(p) \) is a locally exact meromorphic one form on \( X \) which has a pole only at \( p_\infty \) and \( \tilde{\Omega}(p_1, p_2) \) is a meromorphic one form on \( X \times X \) which satisfies the following conditions.

(i) It is a meromorphic one form in \( p_1 \) for a fixed \( p_2 \).
(ii) It is a meromorphic function in \( p_2 \) for a fixed \( p_1 \).
(iii) It is holomorphic except \( \{(p, p) \mid p \in X\} \cup \{(p_\infty, p) \mid p \in X\} \cup \{(p, p_\infty) \mid p \in X\} \).
(iv) It has a simple pole at the diagonal \( \{(p, p) \mid p \in X\} \).

Proof. By Lemma 7 of [24] \( \tilde{\omega} \) can be written in terms of \( \omega \) as

\[
\tilde{\omega}(p_1, p_2) = \omega(p_1, p_2) + \sum_{i,j=1}^{g} c_{ij} du_{w_i}(p_1) du_{w_j}(p_2),
\]

for some constants \( c_{ij} \) satisfying \( c_{ij} = c_{ji} \). Therefore it is sufficient to prove (56) for \( \omega \).

Let us consider the exact sequence of sheaves

\[
0 \rightarrow \mathbb{C} \rightarrow \mathcal{O}(\ast p_\infty) \rightarrow \text{d}\mathcal{O}(\ast p_\infty) \rightarrow 0,
\]

where \( d : \mathcal{O}(\ast p_\infty) \rightarrow \mathcal{K}_X(\ast p_\infty) \) is the exterior differentiation. Taking cohomologies we have

\[
H^1(X, \mathbb{C}) \simeq H^0(X, \text{d}\mathcal{O}(\ast p_\infty))/dH^0(X, \mathcal{O}(\ast p_\infty)).
\]

(57)

We remark that \( H^0(X, \text{d}\mathcal{O}(\ast p_\infty)) \) is the space of locally exact meromorphic one forms with a pole only at \( p_\infty \). Thus this space is generated, as a vector space, by holomorphic one forms and the differentials of the second kind with a pole only at \( p_\infty \).

We realize \( H^1(X, \mathbb{C}) \) as in the right hand side of (57).

Let \( \{du_i^{(1)}, du_i^{(2)}\} \) be the basis of \( H^1(X, \mathbb{C}) \) which is dual to \( \{\alpha_i, \beta_i\} \). Namely they satisfy

\[
\int_{\alpha_j} du_i^{(1)} = \delta_{ij}, \quad \int_{\alpha_j} du_i^{(2)} = 0, \\
\int_{\beta_j} du_i^{(1)} = 0, \quad \int_{\beta_j} du_i^{(2)} = \delta_{ij}.
\]

(58)
We set
\[
\hat{\Omega}(p_1, p_2) = d_{p_1} \log \frac{E(p_1, p_2)}{E(p_1, p_\infty)} - 2\pi i \sum_{k=1}^{g} dv_k(p_1) \int_{p_1}^{p_2} du_k^{(2)}. \tag{59}
\]

We show that this \(\hat{\Omega}(p_1, p_2)\) satisfies properties (i) - (iv).

Since the first term in the right hand side is invariant when \(p_1\) goes round \(\alpha_i\) or \(\beta_i\) cycles due to the transformation rule of the theta function, (i) is obvious. Since \(E(p_1, p_2)\) has a unique simple zero at \(p_1 = p_2\), (iii) and (iv) are also obvious. Let us prove (ii). We have to show that \(\hat{\Omega}(p_1, p_2)\) is invariant when \(p_2\) goes round \(\alpha_i\) or \(\beta_i\).

If \(p_2\) goes round \(\alpha_j\) then \(\hat{\Omega}(p_1, p_2)\) is invariant due to the transformation rule of the theta function and (58). Next consider the case where \(p_2\) goes round \(\beta_j\). Then the integral of the normalized holomorphic differential changes as
\[
\int_{p_1}^{p_2} dv \mapsto \int_{p_1}^{p_2} dv + \Omega e_j,
\]
where \(e_j = ^t(0, ..., 1, ..., 0)\) is the \(j\)-th unit vector and \(\Omega\) is the normalized period matrix. Since
\[
\theta[\xi_0](\int_{p_1}^{p_2} dv + \Omega e_j) \quad = \quad e^{-2\pi i (e''_j) - \pi i \Omega jj - 2\pi i \int_{p_1}^{p_2} dv_j \theta[\xi_0](\int_{p_1}^{p_2} dv)},
\]
we have
\[
d_{p_1} \log \frac{E(p_1, p_2)}{E(p_1, p_\infty)} \mapsto d_{p_1} \log \frac{E(p_1, p_2)}{E(p_1, p_\infty)} + 2\pi i dv_j(p_1). \tag{60}
\]

On the other hand we have
\[
\int_{p_1}^{p_2} du_k^{(2)} \mapsto \int_{p_1}^{p_2} du_k^{(2)} + \delta_{jk}.
\]

Thus
\[
-2\pi i \sum_{k=1}^{g} dv_k(p_1) \int_{p_1}^{p_2} du_k^{(2)} \mapsto -2\pi i \sum_{k=1}^{g} dv_k(p_1) \int_{p_1}^{p_2} du_k^{(2)} - 2\pi i dv_j(p_1). \tag{61}
\]

By (60) and (61) \(\hat{\Omega}(p_1, p_2)\) is invariant when \(p_2\) goes round \(\beta_j\). Thus the property (ii) is proved.

Next let us prove (56) by defining \(d\tilde{r}_i\) explicitly. Taking \(d_{p_2}\) of (59) we get
\[
d_{p_2} \hat{\Omega}(p_1, p_2) = \omega(p_1, p_2) - 2\pi i \sum_{j=1}^{g} dv_j(p_1) du_j^{(2)}(p_2). \tag{62}
\]
Using \( dv_i = \sum_{j=1}^{g} c_{ij} d w_j \), \( c_{ij} = ((2\omega_1)^{-1})_{ij} \), we have

\[
2\pi i \sum_{j=1}^{g} dv_j(p_1) du_j^{(2)}(p_2) = \sum_{j=1}^{g} d w_j(p_1) d \tilde{r}_j(p_2),
\]

\[
d \tilde{r}_j := 2\pi i \sum_{k=1}^{g} c_{kj} d u_k^{(2)}.
\]

Since \( d \tilde{r}_j \) is a locally exact meromorphic one form with a pole only at \( p_\infty \) by construction, the relation (56) is proved. □

**Corollary 5** \( \{d w_i, d \tilde{r}_i\} \) is a symplectic basis of \( H^1(X, \mathbb{C}) \) with respect to the intersection form \( \cdot \) :

\[
d w_i \cdot d w_j = d \tilde{r}_i \cdot d \tilde{r}_j = 0, \quad d w_i \cdot d \tilde{r}_j = \delta_{ij}.
\] (63)

**Proof.** The computation of the intersection is the same as that in Proposition 3 in [24] using Proposition 6. □

Let us define the period matrices \( \eta_i, i = 1, 2 \) by

\[
-2\eta_1 = \left( \int_{\alpha_j} d \tilde{r}_i \right), \quad -2\eta_2 = \left( \int_{\beta_j} d \tilde{r}_i \right),
\]

and set

\[
M = \left( \begin{array}{cc} \omega_1 & \omega_2 \\ \eta_1 & \eta_2 \end{array} \right).
\]

As a consequence of (63) \( M \) satisfies the Riemann bilinear relation of the following form [24]

\[
^t M \left( \begin{array}{cc} 0 & I_g \\ -I_g & 0 \end{array} \right) M = -\frac{\pi i}{2} \left( \begin{array}{cc} 0 & I_g \\ -I_g & 0 \end{array} \right).
\]

It guarantees the consistency of the transformation rule of sigma functions [4].

The following corollary shows that we can determine \( \eta_1, \eta_2 \) directly from \( \tilde{\omega}(p_1, p_2) \).

**Corollary 6** The following relations are valid:

\[
\int_{\alpha_j} \tilde{\omega}(p_1, p_2) = \sum_{i=1}^{g} d w_i(p_1)(-2\eta_{1,ij}),
\] (64)

\[
\int_{\beta_j} \tilde{\omega}(p_1, p_2) = \sum_{i=1}^{g} d w_i(p_1)(-2\eta_{2,ij}),
\] (65)

where the integrals are taken with respect to the variable \( p_2 \).
8.2 Klein form

In defining sigma functions we need a meromorphic bilinear differential \( \tilde{\omega}(p_1, p_2) \) which is independent of the choice of canonical homology basis. The Klein differential \( [14, 12] \) is one of such differentials. Let us recall its definition.

Let \( S \) be the set of half characteristics \( \varepsilon = \{\varepsilon', \varepsilon''\} \), \( \varepsilon', \varepsilon'' \in (\frac{1}{2} \mathbb{Z}^g) / \mathbb{Z}^g \) such that \( \theta[\varepsilon](0|\Omega) \) does not vanish and \( N(S) \) denote the number of elements of \( S \). Then the Klein form \( \omega_K(p_1, p_2) \) is defined by

\[
\omega_K(p_1, p_2) = \omega(p_1, p_2) + \frac{1}{N(S)} \sum_{i,j=1}^g \ dv_i(p_1) dv_j(p_2) \frac{\partial^2}{\partial z_i \partial z_j} \log F(z)|_{z=0},
\]

\[
F(z) = \prod_{\varepsilon \in S} \theta[\varepsilon](z|\Omega).
\]

Using the transformation formula of the theta function under the action of the symplectic group, it can be proved that \( \omega_K \) is independent of the choice of canonical homology basis \([12, 18, 14]\).

We can easily compute periods of Klein form \( \omega_K(p_1, p_2) \) using

\[
\int_{\alpha_j} \omega(p_1, p_2) = 0, \quad \int_{\beta_j} \omega(p_1, p_2) = 2\pi i dv_j(p_1),
\]

where integrals are with respect to the variable \( p_2 \). The result is given by \([64] \) and \([65] \) with

\[
\eta_1 = t_1 \omega_1^{-1} \Lambda, \quad \eta_2 = -\frac{\pi}{2} t_1 \omega_1^{-1} + t_1 \omega_1^{-1} \Lambda \Omega,
\]

where \( \Lambda = (\Lambda_{ij}) \) is given by

\[
\Lambda_{ij} = -\frac{1}{4N(S)} \frac{\partial^2}{\partial z_i \partial z_j} \log F(z)|_{z=0}.
\]

These \( \eta_i \)'s are nothing but the ones given in \([18] \).

8.3 Definition of sigma function

Take any \( \tilde{\omega}(p_1, p_2) \) of the form

\[
\tilde{\omega}(p_1, p_2) = \omega(p_1, p_2) + \sum_{i,j=1}^g c_{ij} dv_{w_i}(p_1) dv_{w_j}(p_2),
\]

where \( c_{ij} = c_{ji} \) is a constant independent of the choice of canonical homology basis. Define the period matrices \( \eta_i \) by \([64] \) and \([65] \). Let \( \varepsilon = \{\varepsilon', \varepsilon''\}, \varepsilon', \varepsilon'' \in \mathbb{R}^g \) be the characteristics of \( e \):

\[
e = q_1 + \cdots + q_{g-1} - \Delta = \Omega \varepsilon' + \varepsilon''.
\]
and \( u = t(u_{w_1}, \ldots, u_{w_g}) \). Notice that if we change the choice of canonical homology basis \( \{\alpha_i, \beta_i\} \), then \( \Delta, \Omega \) and the Abel-Jacobi map are changed. Consequently \( \varepsilon \) also depends on the choice of canonical homology basis.

By Corollary 3 we know
\[
\partial_{A_0} \theta[\varepsilon](0 | \Omega) \neq 0.
\]
 Taking this quantity as a normalization constant we define the sigma function as follows.

**Definition 9** The sigma function associated with \( (X, \{\alpha_i, \beta_i\}, p_\infty, z, e, \{du_w\}, \tilde{\omega}) \) is defined by
\[
\sigma[\varepsilon](u) = \exp\left( \frac{1}{2} t u \tilde{\omega}_1^{-1} u \right) \frac{\theta[\varepsilon][(2\omega_1)^{-1} u | \Omega]}{c_0 \partial_{A_0} \theta[\varepsilon](0 | \Omega)},
\]
where \( c_0 = \pm 1 \) is given by \([10]\).

### 8.4 Modular invariance

**Theorem 13** The function \( \sigma[\varepsilon](u) \) is independent of the choice of \( \{\alpha_i, \beta_i\} \) and has the series expansion of the form
\[
\sigma[\varepsilon](u) = s_\lambda(t)|_{t_{w_i} = u_{w_i}} + \text{higher weight terms},
\]
where \( \lambda \) is given by \([7]\).

**Proof.** The series expansion \([68]\) follows from Theorem \([10]\). The invariance under the change of the canonical homology basis is proved based on the following transformation formula for the theta function in \([13]\).

Let
\[
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]
be an element of the symplectic group \( Sp(2g, \mathbb{Z}) \), where \( A, B, C, D \) are integral matrices of degree \( g \), and \( z = t(z_1, \ldots, z_g) \). We set
\[
\tilde{\Omega} = (A\Omega + B)(C\Omega + D)^{-1}, \quad \tilde{\varepsilon} = t(C\Omega + D)^{-1} z, \quad \tilde{e} = \frac{1}{2} \text{diag}(C' D).
\]

where \( \text{diag}(\cdot) \) denotes the column vector whose components are the diagonal entries of the matrices in \( \cdot \). Then
\[
\theta[\varepsilon](\tilde{\varepsilon} | \tilde{\Omega}) = \gamma(\det(C\Omega + D)) e^{\pi \varepsilon^t (C\Omega + D)^{-1} C \varepsilon} \theta[\varepsilon](z | \Omega),
\]
where \( \gamma \) is some 8-th root of unity.
Let \( \alpha = t(\alpha_1, ..., \alpha_g) \), \( \beta = t(\beta_1, ..., \beta_g) \). Let
\[
T = \begin{pmatrix} D & C \\ B & A \end{pmatrix}
\]
be an element of \( Sp(2g, \mathbb{Z}) \). We change the canonical homology basis by \( T \):
\[
\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} = T \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.
\]
(72)

Notice that \( T \) is an element of \( Sp(2g, \mathbb{Z}) \) if and only if \( M \) is an element of \( Sp(2g, \mathbb{Z}) \).

Let \( \tilde{\omega}_i, \tilde{\eta}_i \) be the period matrices corresponding to \( \{\tilde{\alpha}_i, \tilde{\beta}_i\} \). Since the bilinear differential \( \tilde{\omega}(p_1, p_2) \) does not depend on the choice of a canonical homology basis we have, by computation,
\[
\begin{align*}
\tilde{\omega}_1 &= \omega_1 t D + \omega_2 t C, \\
\tilde{\omega}_2 &= \omega_1 t B + \omega_2 t A, \\
\tilde{\eta}_1 &= \eta_1 t D + \eta_2 t C, \\
\tilde{\eta}_2 &= \eta_1 t B + \eta_2 t A.
\end{align*}
\]

Then the normalized period matrix \( \tilde{\Omega} \) corresponding to \( \{\tilde{\alpha}_i, \tilde{\beta}_i\} \) becomes
\[
\tilde{\Omega} = t\Omega = t(\tilde{\omega}_1^{-1}\tilde{\omega}_2) = (A\Omega + B)(C\Omega + D)^{-1}.
\]

Using Riemann’s vanishing theorem we see that Riemann’s constant \( \delta \) changes to (see also [12])
\[
\tilde{\delta} = t(C\Omega + D)^{-1}\delta - \tilde{\Omega}\zeta' - \zeta'',
\]
where \( \zeta', \zeta'' \in (1/2\mathbb{Z}^g)/\mathbb{Z}^g \) are given by
\[
\begin{pmatrix} \zeta' \\ \zeta'' \end{pmatrix} = \frac{1}{2} \text{diag} \left( \begin{pmatrix} C^t D \\ A^t B \end{pmatrix} \right).
\]

Then \( e \) transforms to \( \tilde{e} = \Omega \tilde{e}' + \tilde{e}'' \) with \( \tilde{e} \) being given by (70).

Notice that
\[
(2\tilde{\omega}_1)^{-1} = t(C\Omega + D)^{-1}(2\omega_1)^{-1}.
\]

Therefore if we change the canonical homology basis to \( \{\tilde{\alpha}_i, \tilde{\beta}_i\} \) by (72), the theta function \( \theta[\tilde{e}]((2\omega_1)^{-1}u|\Omega) \) changes to
\[
\theta[\tilde{e}](t(C\Omega + D)^{-1}(2\omega_1)^{-1}u|\tilde{\Omega}),
\]
where \( \tilde{\Omega}, \tilde{e} \) are given by (69), (70). Thus the formula (71) can be applied. Then we have
\[
\theta[\tilde{e}]((2\tilde{\omega}_1)^{-1}u|\tilde{\Omega}) = \gamma (\det(C\Omega + D))^{1/2} e^{\pi i u t(2\omega_1)^{-1}(C\Omega + D)^{-1}C(2\omega_1)^{-1}u}
\times \theta[\tilde{e}]((2\omega_1)^{-1}u|\Omega).
\]
(73)
Applying $\partial_{A_0}$ to (73) and set $u = 0$. Then, using (1) or (2) of Corollary 3 we have

$$\partial_{A_0} \theta[\tilde{\varepsilon}](0 | \tilde{\Omega}) = \gamma (\det(C\Omega + D))^{1/2} \partial_{A_0} \theta[\varepsilon](0 | \Omega).$$

(74)

On the other hand, as shown in [4], we have

$$\frac{1}{2} t u \tilde{\eta}_1 \tilde{\omega}_1^{-1} u = \frac{1}{2} t u \eta_1 \omega_1^{-1} u - \pi^t u (2\omega_1)^{-1} (C\Omega + D)^{-1} C(2\omega_1)^{-1} u. \tag{75}$$

By (73), (74), (75) we have

$$\exp(\frac{1}{2} t u \tilde{\eta}_1 \tilde{\omega}_1^{-1} u) \frac{\theta[\tilde{\varepsilon}](2\tilde{\omega}_1^{-1} u | \tilde{\Omega})}{\partial_{A_0} \theta[\varepsilon](0 | \Omega)}$$

$$= \exp(\frac{1}{2} t u \eta_1 \omega_1^{-1} u) \frac{\theta[\varepsilon](2\omega_1^{-1} u | \Omega)}{\partial_{A_0} \theta[\varepsilon](0 | \Omega)},$$

which shows that $\sigma[\varepsilon](u)$ does not depend on the choice of canonical homology basis. □

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