HOOKS AND POWERS OF PARTS IN_partitions

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Abstract: This paper shows that the number of hooks of length $k$ contained in all partitions of $n$ equals $k$ times the number of parts of length $k$ in partitions of $n$. It contains also formulas for the moments (under uniform distribution) of $k$-th parts in partitions of $n$.

1. Introduction and Main Results

Many textbooks contain material on partitions. Two standard references are [A] and [S].

A partition of a natural integer $n$ with parts $\lambda_1, \ldots, \lambda_k$ is a finite decreasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ of natural integers $\lambda_1, \ldots, \lambda_k > 0$ such that $n = \sum_{i=1}^{k} \lambda_i$. We denote by $|\lambda|$ the content $n$ of $\lambda$. Partitions are also written as sums: $n = \lambda_1 + \cdots + \lambda_k$ and one uses also the (abusive) multiplicative notation

$$\lambda = 1^{\nu_1} \cdot 2^{\nu_2} \cdots n^{\nu_n}$$

where $\nu_i$ denotes the number of parts equal to $i$ in the partition $\lambda$.

A partition is graphically represented by its Young diagram obtained by drawing $\lambda_1$ adjacent boxes of identical size on a first row, followed by $\lambda_2$ adjacent boxes of identical size on a second row and so on with all first boxes (of different rows) aligned along a common first column. In the sequel we identify a partition with its Young diagram. A hook in a partition is a choice of a box $H$ in the corresponding Young diagram together with all boxes at the right of the same row and all boxes below of the same column. The total number of boxes in a hook is its hooklength, the number of boxes in a hook to the right of $H$ is its armlength and the number of boxes of a hook below $H$ is the leglength. The Figure below displays the Young diagram of the partition $(5, 4, 3, 1)$ of 13 together with a hook of length 4 having armlength 2 and leglength 1. We call the couple (armlength,leglength) of a hook its hooktype and denote it by $\tau = \tau(\alpha, k - 1 - \alpha)$ if its armlength is $\alpha$ and its leglength $k - 1 - \alpha$. Such a hook has hence total length $k$ and there are exactly $k$ different hooktypes for hooks of length $k$.

The partition $(5, 4, 3, 1)$ of 13 together with a hook of type $\tau(2, 1)$ and length 4.

Date: last modified Aug 29, 2001.

Math. Class.: 05A17 Keywords: Partition.
Let \(k\) be a natural integer and let \(\tau = \tau(\alpha, k - 1 - \alpha)\) be the hooktype of a hook of length \(k\) with armlength \(\alpha\) and leglength \(k - 1 - \alpha\). Given a partition \(\lambda\) of \(n\), set

\[
\tau(\lambda) = \#\{\text{hooks of type } \tau \text{ in (the Young diagram of) } \lambda\}
\]

and

\[
\tau(n) = \sum_{\lambda, |\lambda|=n} \tau(\lambda)
\]

where the sum is over all partitions of \(n\).

**Theorem 1.1.** One has

\[
\sum_{n=1}^{\infty} \tau(n) z^n = \frac{z^k}{1 - z^k} \prod_{i=1}^{\infty} \frac{1}{1 - z^i} = \sum_{\lambda=1,2,2...} \nu_k \, z^{|\lambda|}
\]

where the last sum is over all partitions of integers.

In other terms, the number of hooks of given type and length \(k\) appearing in all partitions of \(n\) equals the number of parts of length \(k\) in all partitions of \(n\).

This result implies in particular that the total number of hooks of given type \(\tau = \tau(\alpha, k - 1 - \alpha)\) occurring in all partitions of \(n\) depends only on the length \(k\) and not on the particular hooktype \(\tau(\alpha, k - 1 - \alpha)\) itself. Since there are exactly \(k\) distinct hooktypes for hooks of length \(k\), the total number of hooks of length \(k\) in partitions of \(n\) is given by the coefficient of \(z^n\) of the series

\[
\frac{z^k}{1 - z^k} \prod_{i=1}^{\infty} \frac{1}{1 - z^i}.
\]

For \(\alpha, \beta \geq 0\) two natural integers, define the \(q\)-binomial \((\alpha+\beta)_{\alpha, q}\) by

\[
\binom{\alpha+\beta}{\alpha}_q = \frac{\prod_{j=1}^{\alpha+\beta} (q^j - 1)}{\prod_{j=1}^{\alpha} (q^j - 1) \prod_{j=1}^{\beta} (q^j - 1)}.
\]

The \(q\)-binomial coefficient \((\alpha+\beta)_{\alpha, q}\) is a polynomial of degree \(\alpha \beta\) in \(q\) with the coefficient of \(q^n\) enumerating all partitions of \(n\) having at most \(\beta\) non-zero parts which are all of length at most \(\alpha\).

The main ingredient of the proof of Theorem 1.1 is the following result which is perhaps also of independent interest.

**Proposition 1.2.** One has for \(\alpha, \beta \in \mathbb{N}\)

\[
\binom{\alpha+\beta}{\alpha}_q = \frac{1}{1 - q^{\alpha+\beta+1}} \left( \sum_{i=0}^{\infty} q^i (\beta+1) \prod_{j=i+1}^{\beta+1} (1 - q^j) \right)^{-1}.
\]

As remarked previously, partitions of a natural integer \(n\) can be written in (at least) two different ways: either by considering the finite decreasing sequence

\[
\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)
\]

of its parts or by considering the vector

\[
\nu = (\nu_1, \nu_2, \ldots, \nu_n)
\]
proof of this.

It would be interesting to have a direct combinatorial

development. This last equality states for instance that the sum over all partitions of $n$ of the number of distinct parts arising with multiplicity at least $k$ equals the number of parts equal to $k$ in all partitions of $n$. Our proof of this fact uses generating series. It would be interesting to have a direct combinatorial proof of this.
The generating series and define \((\nu)\) using Theorem 1.3. More precisely, one has the following result:

\[
\frac{z^k}{1 - z^k} \prod_{i=1}^{\infty} \frac{1}{1 - z^i}.
\]

The coordinates of the vectors \(\lambda(n)\) and \(\nu(n)\) are then easily computed using Theorem 1.3. More precisely, one has the following result:

**Corollary 1.4.** (i) The \(k\)-th coordinate of the vector \(\lambda(n)\) is the coefficient of \(z^n\) in the generating series

\[
\prod_{i=1}^{\infty} \frac{1}{1 - z^i} \sum_{j=k}^{\infty} \frac{z^j}{1 - z^j}.
\]

(ii) The \(k\)-th coordinate of the vector \(\gamma(n)\) is the coefficient of \(z^n\) in the generating series

\[
\frac{(1 - z) z^k}{(1 - z^k)(1 - z^{k+1})} \prod_{i=1}^{\infty} \frac{1}{1 - z^i}.
\]

Given a partition

\[
\lambda = (\lambda_1, \ldots, \lambda_n) = (1^{\nu_1} \cdots n^{\nu_n})
\]

and an integer \(d \geq 0\) we introduce the vectors \((\lambda)\) and \((\nu)\) \(\in \mathbb{Z}^n\) by setting

\[
(\lambda) = \left(\frac{\lambda_1}{d}, \ldots, \frac{\lambda_n}{d}\right) \quad \text{and} \quad (\nu) = \left(\frac{\nu_1}{d}, \ldots, \frac{\nu_n}{d}\right)
\]

and define

\[
(\lambda(n)) = \sum_{|\lambda|=n} (\lambda) \quad \text{and} \quad (\nu(n)) = \sum_{|\nu|=n} (\nu)
\]

with coordinates

\[
(\lambda_k(n)) = \sum_{|\lambda|=n} (\lambda_k) \quad \text{and} \quad (\nu_k(n)) = \sum_{|\nu|=n} (\nu_k)
\]

The following example shows the vectors \((\lambda)\), \((\nu)\) and \((\gamma)\) associated to all five partitions of 4.

**Example:**

| Partition | \((1,1,1,1)\) | \((2,1,1)\) | \((2,2)\) | \((3,1)\) | \((4)\) |
|-----------|---------------|-------------|---------|---------|-------|
| \((\lambda)\) | (1, 1, 1, 1) | (2, 1, 1, 0) | (2, 2, 0, 0) | (3, 1, 0, 0) | (4, 0, 0, 0) |
| \((\nu)\) | (0, 0, 0, 0) | (1, 0, 0, 0) | (1, 1, 0, 0) | (3, 0, 0, 0) | (6, 0, 0, 0) |
| \((\gamma)\) | (0, 0, 0, 0) | (0, 0, 0, 0) | (0, 0, 0, 0) | (1, 0, 0, 0) | (4, 0, 0, 0) |

| Partition | \((1,1,1,1)\) | \((2,1,1)\) | \((2,2)\) | \((3,1)\) | \((4)\) |
|-----------|---------------|-------------|---------|---------|-------|
| \((\lambda)\) | (1, 1, 1, 1) | (2, 1, 1, 0) | (2, 2, 0, 0) | (3, 1, 0, 0) | (4, 0, 0, 0) |
| \((\nu)\) | (0, 0, 0, 0) | (1, 0, 0, 0) | (1, 1, 0, 0) | (3, 0, 0, 0) | (6, 0, 0, 0) |
| \((\gamma)\) | (0, 0, 0, 0) | (0, 0, 0, 0) | (0, 0, 0, 0) | (1, 0, 0, 0) | (4, 0, 0, 0) |
We have thus
\[
\binom{\lambda(4)}{d} = (12, 5, 2, 1), \quad \binom{\nu(4)}{2} = (11, 1, 0, 0), \quad \binom{\lambda(4)}{3} = (5, 0, 0, 0)
\]
\[
\binom{\nu(4)}{1} = (7, 3, 1, 1), \quad \binom{\nu(4)}{2} = (7, 1, 0, 0), \quad \binom{\nu(4)}{3} = (4, 0, 0, 0)
\]

The following probably well-known result allows easy computations of the vectors \(\binom{\lambda(n)}{d}\) and \(\binom{\nu(n)}{d}\).

**Proposition 1.5.** For any natural integer \(d \geq 0\), the \(k\)-th coefficients \(\binom{\lambda_k(n)}{d}\), respectively \(\binom{\nu_k(n)}{d}\) (extended by \(\binom{0}{d}\) for \(k > n\)) have generating series

\[
\sum_{n} \binom{\lambda_k(n)}{d} z^n = \left( \prod_{j=1}^{k-1} \frac{1}{1-z^j} \right) \left( \sum_{i=0}^{\infty} \binom{i}{d} z^i \right) \sum_{i=0}^{\infty} \frac{i}{d} z^i \prod_{j=1}^{\infty} \frac{1}{1-z^j}
\]

and

\[
\sum_{n} \binom{\nu_k(n)}{d} z^n = \left( \frac{z}{1-z^d} \right)^d \prod_{j=1}^{\infty} \frac{1}{1-z^j}
\]

**Remark 1.6.** One has

\[
\sum_{|\lambda|=n} \lambda^d = \sum_i i! \text{Stirling}_2(d,i) \binom{\lambda_k(n)}{i}
\]

and

\[
\sum_{|\nu|=n} \nu^d = \sum_i i! \text{Stirling}_2(d,i) \binom{\nu_k(n)}{i}
\]

where \(\text{Stirling}_2(d,i)\) denote Stirling numbers of the second kind, defined by \(x^d = \sum_i \text{Stirling}_2(d,i) x(x-1) \cdots (x-i+1)\).

Asymptotics are not so easy to work out from the formula for \(\binom{\lambda(n)}{d}\). Our last result is an equivalent expression for the above series on which asymptotics are easier to see.

We introduce the generating series \(\sigma_r(k)\) defined as

\[
\sigma_r(k) = \sum_{i=k}^{\infty} \left( \frac{z^i}{1-z^i} \right)^r
\]

for \(r \geq 1\) and \(k \geq 1\) natural integers. We consider the series \(\sigma_r(k)\) as being graded of degree \(r\) and define the homogeneous series \(S_d(k)\) of degree \(d\) by

\[
S_d(k) = \sum_{|1^{\nu_1} 2^{\nu_2} \cdots| = d} \frac{d!}{(\sum_i \nu_i)!} \left( \sum_i \binom{\nu_i}{i} \prod_{i=1}^{d} \frac{\binom{\sigma_i(k)}{i}^{\nu_i}}{i!} \prod_{j=1}^{d} \frac{(\sigma_j(k))^{\nu_j}}{j!}\right)
\]

(i.e. the coefficient of the homogeneous “monomial” series \(\sigma_\lambda(k) = \sigma_{\lambda_1}(k) \cdots \sigma_{\lambda_s}(k)\) equals the number of elements in the symmetric group on \(|\lambda|\) elements of the conjugacy class with cycle structure \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)\)).

We have then the following result.
Theorem 1.7. For any natural integers \(d \geq 1\) and \(k \geq 1\), we have
\[
\sum_{n}^{\infty} \left( \frac{\lambda_k(n)}{d} \right) z^n = \frac{S_d(k)}{d!} \left( \prod_{j=1}^{\infty} \frac{1}{1-z^j} \right).
\]

The first series \(S_i = S_i(k)\) are given in terms of \(\sigma_j = \sigma_j(k)\) as follows
\[
S_0 = 1, \\
S_1 = \sigma_1, \\
S_2 = \sigma_1^2 + \sigma_2, \\
S_3 = \sigma_1^3 + 3\sigma_1 \sigma_2 + 2\sigma_3 \\
S_4 = \sigma_1^4 + 6\sigma_1^2 \sigma_2 + 3\sigma_1 \sigma_3 + 8\sigma_1 \sigma_2 + 6\sigma_4 \\
S_5 = \sigma_1^5 + 10\sigma_1^3 \sigma_2 + 20\sigma_1^2 \sigma_3 + 15\sigma_1 \sigma_4 + 30\sigma_1 \sigma_2 \sigma_3 + 24\sigma_5
\]

Let us remark that the analogous statement of the Theorem 1.9 for the generating series \(\sum_{n}^{\infty} \binom{n}{d} z^n\) boils down to a trivial identity.

The formulas of Theorem 1.7 ease the computations of asymptotics (in \(n\)) for \(\lambda_k(n)\) and its moments and allow probably a rederivation of the results contained in [EL] and [VK].

2. Proofs

Proof of Proposition 1.2. Since \(\binom{\beta}{0}_q = 1\), for \(\alpha = 0\) the proposition boils down to the well-known formula for the geometric series
\[
\sum_{i=0}^{\infty} q^{i(\beta+1)} = \frac{1}{1-q^{\beta+1}}.
\]

The proposition is equivalent to the identity
\[
\left( \prod_{j=\alpha+1}^{\alpha+\beta+1} (1-q^j) \right) \left( \sum_{i=0}^{\infty} q^{i(\beta+1)} \prod_{j=i+1}^{\alpha+\beta+1} (1-q^j) \right) = \prod_{j=1}^{\beta} (1-q^j).
\]

Since the right-hand side of this expression depends only on \(\beta\), it is enough to show that the expression
\[
\left( \prod_{j=\alpha+1}^{\alpha+\beta+1} (1-q^j) \right) \left( \sum_{i=0}^{\infty} q^{i(\beta+1)} \prod_{j=i+1}^{\alpha+\beta} (1-q^j) \right) - \left( \prod_{j=\alpha}^{\alpha+\beta} (1-q^j) \right) \left( \sum_{i=0}^{\infty} q^{i(\beta+1)} \prod_{j=i+1}^{\alpha+\beta-1} (1-q^j) \right)
\]
equals zero for all \(\alpha \geq 1\).

Dividing by \(\prod_{j=\alpha+1}^{\alpha+\beta} (1-q^j)\) we get
\[
(1-q^{\alpha+\beta+1}) \left( \sum_{i=0}^{\infty} q^{i(\beta+1)} \prod_{j=i+1}^{\alpha+\beta} (1-q^j) \right) - (1-q^\alpha) \left( \sum_{i=0}^{\infty} q^{i(\beta+1)} \prod_{j=i+1}^{\alpha+\beta-1} (1-q^j) \right)
\]
\[
= \sum_{i=0}^{\infty} q^{i(\beta+1)} \prod_{j=i+1}^{\alpha+\beta-1} (1-q^j) - \sum_{i=0}^{\infty} q^{i\alpha+i(\beta+1)} \prod_{j=i+1}^{\alpha+\beta-1} (1-q^j)
\]
Proof of Theorem 1.1. Let us consider a hooktype $\tau = (\alpha, k - 1 - \alpha)$ of length $k$ with armlength $\alpha$ and leglength $k - 1 - \alpha$. The number

$$\tau(n) = \sum_{|\lambda| = n} \tau(\lambda)$$

equals the coefficient of $z^n$ in the generating series

$$\sum_{i=0}^{\infty} \left( \prod_{j=1}^{i} \frac{1}{1 - z^j} \right) z^{(i+1)(k-\alpha)+\alpha} P_{k-1-\alpha,\alpha}(z) \prod_{j=i+1+\alpha}^{\infty} \frac{1}{1 - z^j}$$

$$= \left( \prod_{j=1}^{\infty} \frac{1}{1 - z^j} \right) P_{k-1-\alpha,\alpha}(z) z^k \left( \sum_{i=0}^{\infty} z^{i(k-\alpha)} \prod_{j=i+1}^{\infty} (1 - z^j) \right)$$

(the factor $\left( \prod_{j=1}^{\infty} \frac{1}{1 - z^j} \right)$ accounts for all what happens below a hook $H$ of type $\tau$, the factor $z^{(i+1)(k-\alpha)+\alpha} P_{k-1-\alpha,\alpha}(z)$ accounts for rows involved in $H$ and the last factor $\prod_{j=i+1+\alpha}^{\infty} \frac{1}{1 - z^j}$ depends on what happens on rows above $H$) where $P_{\alpha,\beta}(z)$ denotes the generating series of partitions having at most $\alpha$ parts and all parts are of length at most $\beta$. The generating series $P_{\alpha,\beta}(q)$ is by definition the $q$-binomial coefficient

$$P_{\alpha,\beta}(q) = \binom{\alpha + \beta}{\alpha}_q = \frac{\prod_{j=1}^{\alpha+\beta} (q^j - 1)}{\prod_{j=1}^{\alpha} (q^j - 1) \prod_{j=1}^{\beta} (q^j - 1)}.$$

Applying Proposition 1.2 with $\beta = k - 1 - \alpha$ we have

$$\sum_n \tau(n) z^n = \left( \prod_{j=1}^{\infty} \frac{1}{1 - z^j} \right).$$

$$= \frac{1}{1 - z^k} \left( \sum_{i=0}^{\infty} z^{i(k-\alpha)} \prod_{j=i+1}^{\infty} (1 - z^j) \right) \left( \sum_{i=0}^{\infty} z^{i(k-\alpha)} \prod_{j=i+1}^{\infty} (1 - z^j) \right)$$

$$= \frac{z^k}{1 - z^k} \prod_{j=1}^{\infty} \frac{1}{1 - z^j}$$

which proves the first equality of the Theorem.
The last equality follows from the easy identities
\[
\sum_{\lambda=(1^t \ 2^t \ldots)} \nu_k \, z^{\lambda[j]} = \sum_{i \geq 1} i z^{ik} \prod_{1 \leq j \neq i} \frac{1}{1 - z^j} = \frac{z^k}{1 - z^k} \prod_{j=1}^{\infty} \frac{1}{1 - z^j}
\]
thus finishing the proof.

**Proof of Theorem 1.3.** The partition 1\( ^n \) yields the unique non-zero contribution to \( \lambda_n(n) \) and \( \gamma_n(n) \) and this contribution equals one in both cases. The partition \( n \) consisting of a unique part of length \( n \) yields the unique non-zero contribution to \( \nu_n(n) \) and this contribution equals again 1.

Given a partition \( \lambda = (\lambda_1, \ldots, \lambda_k) \) of \( n \), the *conjugate partition* \( \lambda^t = (\lambda_1^t, \ldots, \lambda_k^t) \) of \( \lambda \) is defined by
\[
\lambda_j^t = \# \{ i \mid \lambda_i \geq j \}
\]
(this corresponds to a reflection of the Young diagram of \( \lambda \) through the main diagonal \( y = -x \)). The difference \( \lambda_k - \lambda_{k+1} \) (where non-existing parts are considered as parts of length 0) equals hence the number \( \nu_k \) of parts having length \( k \) in the transposed partition \( \lambda^t = (1^{\nu_1^t} \ 2^{\nu^t_2} \ldots) \) of \( \lambda \). Summing over all partitions of \( n \) yields then the recursion relation \( \lambda_k(n) = \nu_k(n) + \lambda_{k+1}(n) \).

The proof of the equality \( \nu_k(n) = \gamma_k(n) + \nu_{k+1}(n) \) uses generating series. Introducing the numbers
\[
m_k(\lambda) = \# \{ i \mid \nu_i \geq k \}, \quad m_k(n) = \sum_{|\lambda|=n} m_k(\lambda)
\]
one has obviously \( \gamma_k(n) = m_k(n) - m_{k+1}(n) \). We have hence to show the equality \( m_k(n) = \nu_k(n) \) for \( 1 \leq k < n \) (the equalities \( m_n(n) = \nu_n(n) = 1 \) are easy).

We introduce the generating function
\[
\psi_k(y, z) = \sum_{\lambda} y^{m_k(\lambda)} z^{|\lambda|}.
\]
One has
\[
\psi_k(y, z) = \sum_{I \subset \{1,2,3,\ldots\}, |I|<\infty} \left( \prod_{i \in I} \frac{1}{1 - z^i} \right) \left( \prod_{1 \leq j \notin I} \frac{1 - z^{kj}}{1 - z^j} \right) = \prod_{j=1}^{\infty} \left( \frac{1 - z^j}{1 - z^j} \right) \left( \frac{1}{1 - z^j} - (1 - y) \frac{z^{kj}}{1 - z^j} \right).
\]
A small computation yields
\[
\frac{\partial \psi_k(y, z)}{\partial y} = \psi_k(y, z) \sum_{j=1}^{\infty} \frac{z^{kj}}{1 - (1 - y) z^{kj}}
\]
and we have hence
\[
\sum_{n=0}^{\infty} m_k(n) \, z^n = \frac{\partial \psi_k(1, z)}{\partial y} = \left( \prod_{j=1}^{\infty} \frac{1}{1 - z^j} \right) \frac{z^k}{1 - z^k} = \sum_{n=0}^{\infty} \nu_k(n) \, z^n
\]
(cf. Theorem 1.1 for the last equality) which finishes the proof.

QED
Corollary 1.4 results immediately from Theorem 1.3 and from the last equality in Theorem 1.1.

**Proof of Proposition 1.5.** A partition

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k = i \geq \lambda_{k+1} \geq \cdots)$$

with $$\lambda_k = i$$ of $$n = \sum_{j=1}^{\infty} \lambda_j$$ can be written as

$$\lambda = (i + (\lambda_1 - i) \geq i + (\lambda_2 - i) \geq \cdots \geq i + (\lambda_{k-1} - i) \geq i \geq \lambda_{k+1} \geq \cdots).$$

Such partitions are hence in bijection with pairs of partitions

$$\alpha = (\alpha_1 = (\lambda_1 - i) \geq \alpha_2 \geq \cdots \geq \alpha_{k-1} = (\lambda_{k-1} - i) \geq 0),$$

$$\omega = (\omega_1 = \lambda_{k+1} \geq \omega_2 = \lambda_{k+2} \geq \cdots)$$

with $$\alpha$$ having at most $$k - 1$$ non-zero parts and $$\omega$$ having all parts $$\leq i$$. The conjugate partition $$\alpha^t$$ of $$\alpha$$ has hence only parts $$\leq k - 1$$. Such a pair $$\alpha^t, \omega$$ of partitions yields hence a unique partition with $$\lambda_k = i$$ of the integer $$n = ki + \sum_{j=1}^{k-1} \alpha_j + \sum_{j=k+1}^{\infty} \omega_j$$ and contributes hence with \( (\binom{\lambda_i}{d}) \) to the $$k$$-th coordinate \( (\lambda_i) \) of \( (\lambda_i) \). Summing up over $$i \in \mathbb{N}$$ yields easily the generating series for \( (\lambda_i) \).

Considering the generating series for \( (\nu_i) \) one has

$$\sum_{n} (\nu_i) z^n = \left( \sum_{j} \binom{j}{d} z^j \right) \prod_{i \neq k} \frac{1}{1 - z^i}$$

and the (easy) equality

$$\sum_{j} \binom{j}{d} Z^j = \frac{1}{Z} \left( \frac{Z}{1 - Z} \right)^{d+1}$$

implies the result. QED

**Proof of Theorem 1.7.** Given a partition $$\lambda = (\lambda_1, \lambda_2, \ldots)$$ the definition

$$\lambda_i = \sharp \{ i \mid \lambda_i \geq k \}$$

for the $$k$$-th part of its transposed partition $$\lambda^t = (\lambda_1^t, \lambda_2^t, \ldots)$$ shows the equalities

$$\sum_{\lambda} y^\lambda z^{\lambda} = \sum_{\lambda} y^{(\lambda^t)_k} z^{\lambda} = \left( \prod_{i=1}^{k-1} \frac{1}{1 - z^i} \right) \left( \prod_{j=k}^{\infty} \frac{1}{1 - y z^j} \right).$$

Denote this series by $$\varphi_k(y, z)$$. An easy computation yields

$$\varphi_k(y, z) = \left( \prod_{i=1}^{\infty} \frac{1}{1 - z^i} \right) \prod_{j \geq k} \left( 1 - (y - 1) \frac{z^j}{1 - z^j} \right)^{-1}.$$
of formal power series to the last factor we get
\[ \prod_{j \geq k} \left( 1 - (y - 1) \frac{z^j}{1 - z^j} \right)^{-1} = \exp \left( \sum_{l=1}^{\infty} \frac{(y - 1)^l}{l} \sigma_l(k) \right) \]
\[ = \sum_{n=0}^{\infty} \frac{(y - 1)^n}{n!} \left( \sum_{(\nu_1, \nu_2, \ldots)} n! \frac{(\nu_1 \nu_2 \ldots)}{(\nu_1 \nu_2 \ldots)} \prod_{i} \left( \frac{\sigma_i(k)}{i} \right)^{\nu_i} \right) \]
\[ = \sum_{n=0}^{\infty} \frac{(y - 1)^n}{n!} S_n(k) . \]
We have thus
\[ d! \sum_{\lambda} \left( \frac{\lambda_k}{d} \right) z^{\lambda} = \frac{\partial^d \varphi_k}{\partial y^d}(1, z) = \left( \prod_{i=1}^{\infty} \frac{1}{1 - z^i} \right) S_d(k) \]
which finishes the proof by comparison with Proposition 1.5. QED

We thank S. Attal and M-L. Chabanol for comments and for their interest in this work.

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