Mathematical modeling of rotating disk states

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Abstract. We consider the problem of a rapidly rotating disk in the elastic-plastic state. The piecewise linear plasticity condition in general form is chosen. It is believed that all plastic curves have the common point of intersection which corresponds to uniaxial tension. For external parameters, we obtain the conditions that determine the probability of inception of plastic zones. It is shown that plastic zones could incept in the center of the disk and/or on the boundary of it. The problem in the plastic zone is statically determinate. The case when the plastic zone occupies some central part (core) of the disk, where one regime of plastic condition is fulfilled, is considered. In order to estimate the stress state inside the elastic zone of the disk, equivalent stress which is equal to the chosen plasticity function is defined. In order to define the relationship between plastic deformations and stresses, the piecewise linear plastic potential being equal to the plasticity function is chosen. The plastic incompressible body is considered. The associated flow rule can be integrated so that the problem of getting displacements turns into quasistatic one. The problem of determining displacements in the plastic region leads to a first-order differential equation with respect to the radial component of the displacement vector. Therefore the continuity condition for displacements at the elastic-plastic boundary and the assumption that the displacements in the center of the disk are equal to zero leads to an overdetermined problem. So, only the continuity condition for displacements at the elastoplastic boundary is accepted. It is assumed that plastic deformations at the elastic-plastic boundary are equal to zero. It is shown that displacements at the center of the disk are equal to zero automatically for all piecewise linear conditions of plasticity apart from the Tresca yield criterion. For the Schmidt-Ishlinskii yield criterion, all deformations at the center of the disk attain finite values. Meanwhile, for other piecewise linear conditions, plastic deformations at the center of the disk attain infinitely large values. This explains the discontinuity of displacements at the center of the disk for the Tresca yield criterion. The calculation results are presented as graphs of stresses, displacements, and deformations.

1. Introduction
Rotating discs are essential elements of many machines. The main loads acting on the discs are centrifugal forces that occur during rotation. Mathematical modeling, which allows determining stresses and strains from centrifugal forces, is an important stage of many engineering works. Analytical and numerical solutions to the problems of a rotating disk are considered in many books on the theory of elasticity [1] and plasticity [2–4] as well as in numerous engineering papers.

If we talk about the mode of a homogeneous isotropic elasto-plastic body, then we can highlight the results of researches obtained within the theory of plastic flow with the associated law of plastic flow and in the framework of the deformation theory. In most papers, the Tresca yield criterion and strain hardening of a material using the theory of plastic flow [5, 6] or the
deformation theory of plasticity [3] are considered. Another popular approach is to employ the Mises yield criterion and strain hardening of a material within the framework of the deformation theory of plasticity [7] and plastic flow [8]. Within the model of elastic/perfectly plastic material, several important papers have recently been published: Toussi and Farimani [9] (for the Mises yield criterion and the theory of plastic deformation) and Nejad et al. [10] (for the Mises yield criterion and the flow law associated with it). The most recent paper by Lomakin et al. [11] introduces also in analytical treatment the Mises yield criterion combined with its associated flow rule. By Aleksandrova [12] the complete solution of stress-strain state of a rotating disc has been then developed for practical engineering applications.

The plasticity regime of the Tresca yield criterion was considered in [5], which provides the statement that the radial component of the stress tensor inside the side surface of the disk is equal to zero. It was shown that, in the frame of the theory of plastic flow, the choice of this yield criterion leads to a discontinuity of displacements or stresses at the elasto-plastic boundary. This property is determined by the choice of one of two considered alternative boundary conditions for displacements. These conditions are equality of displacements in the center of the disk to zero and continuity of displacements at the elasto-plastic boundary. Since in [5] the reason for the overdetermination of the problem of determining displacements is not analyzed, then a linear differential equation of the first order has to be solved in order to determine the displacements in the plastic region. In order to provide continuity of the displacement field Gamer in later papers [6] applies the Tresca yield condition, which takes into account isotropic hardening. It provides continuity of displacements in the center of the disk. The problem of determining stresses in the plastic region is statically indeterminate. Following Gamer’s approach, a lot of papers have been published. The modern treatment of a rotating disk problem by this approach is reflected in [13].

So, the main objective of the present paper is to analyze the use of various piecewise linear and smooth plasticity functions and flow theory to solve the rotating disk problem, as well as to compare the determined displacement and deformation fields for the selected plasticity functions. The novelty of the current research is that it is proved that the Tresca yield condition can be employed to solve problems similar to the rotating disk problem.

For piecewise-smooth plastic potentials for the ideal elasto-plastic body model, when more than one plasticity regime is fulfilled in the plastic region, singular regimes lead to a rupture of plastic strains at the boundary of the regime change [14]. The corresponding jump in the work of stresses in plastic strains, when passing through singular points of the plasticity curve, can be represented as a sharp change in the “structure” of the plastic body prescribed by the model with piecewise smooth plastic potentials. In the framework of the Hencky deformation theory, the fields of displacements and stresses are continuous when we solve the problem of a rotating disk [2].

2. Statement of the problem

The problem of a rotating thin disk of constant thickness is considered in the framework of plane-stress state. Cylindrical coordinate system $\rho \theta z$ (where the axis $z$ passes through the center of the disk $\rho = 0$ and the plane $z = 0$ represents the mid-plane of the disk) is assumed to be suitable for the geometric representation of such a disk. The outer contour of the disk $\rho = b$ is subjected to compressive external pressure $p_b$. The physical model of the disk given as isotropic elastic/perfectly-plastic.

3. Non-dimensional variables

The paper is written in non-dimensional form. All variables of length measure are scaled to the outer radius of the disk $b$. The values of stress measure are scaled to the yield stress in tension $k$. 

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4. The choice of yield criterion
We will consider the case when the plasticity function is piecewise linear with respect to the main stresses. Then, for an axially symmetric plane-stress state, the yield condition is written as
\[
F = \max_{i=1, \ldots, n} \left\{ \alpha_i \sigma_\theta + \beta_i \sigma_\rho + \gamma_i \sigma_z \right\} = k,
\]
\[\sigma_z = 0,
\]
where \( F \) is the function of plasticity, \( \sigma_\rho, \sigma_\theta, \sigma_z \) is the main stresses, \( k \) is the yield stress, \( n \) is the number of non-singular plastic regimes. Without loss of the generality, let us assume \( \alpha_i + \beta_i + \gamma_i = 0 \), that is, the plasticity function, does not depend on the first invariant of the stress tensor.

The coordinates \( \sigma_\rho^{(i)}, \sigma_\theta^{(i)} \) of the vertices of the plastic polygon (1) are determined by the formulas
\[
\sigma_\rho^{(i)} = \frac{(\alpha_i - \alpha_{i+1})k}{\alpha_i \beta_{i+1} - \alpha_{i+1} \beta_i}, \quad \sigma_\theta^{(i)} = \frac{(\beta_{i+1} - \beta_i)k}{\alpha_i \beta_{i+1} - \alpha_{i+1} \beta_i}.
\]

While the yield stress under uniaxial tension is chosen as the yield stress, then on the plane with coordinates \((\sigma_\rho, \sigma_\theta)\) all plastic curves of normally isotropic bodies are located between the Tresca plastic hexagon (the yield criterion of maximum shear stress) [15] and the Schmidt–Ishlinskii hexagon (the yield criterion of the maximum reduced stress) [16–18].

Figure 1. shows the Treska and Schmidt–Ishlinskii hexagons, as well as a dodecagon defined by the equation
\[
\frac{\eta}{2} (|\sigma_\theta - \sigma_\rho| + |\sigma_\theta| + |\sigma_\rho|) + \frac{1-\eta}{4} (|2\sigma_\theta - \sigma_\rho| + |2\sigma_\rho - \sigma_\theta| + |\sigma_\theta + \sigma_\rho|) = k,
\]
when parameter \( \eta = 0.5 \ (\eta \in [0, 1]) \).

Equation (3) defines one of the two possible types of twelve plasticity squares; when a singular or nonsingular regime of yield conditions is activated at a point \( A \) (Figure 1). The choice of condition (3) is due to the fact that in the future we consider the case when the point \( A \) (Figure 1) has to be singular.

For certain values of the parameters \( \alpha_i, \beta_i, \gamma_i \) yield conditions (1) and (3) are alternative: the same plastic curve is determined by them.

The choice of the piecewise linear yield condition (1) allows performing integration of the ratios of so called associated law of plastic flow.
5. Determination of stresses and deformations in plastic zone

For the present case, the equilibrium equation is written in the form [1]

\[ \rho \frac{d\sigma_\rho}{d\rho} + \sigma_\rho - \sigma_\theta + m\rho^2 = 0, \]  

where the parameter \( m \) characterizes the action of inertia. The choice of piecewise linear plasticity functions (1) leads to a linear system of equations (1), (4) for determining stresses. The solution to this system is

\[ \sigma_\rho = \frac{k}{\alpha_i + \beta_i} - \frac{\alpha_i m \rho^2}{3\alpha_i + \beta_i} + \frac{C}{\rho(\alpha_i + \beta_i)/\alpha_i}, \quad \sigma_\theta = \frac{k}{\alpha_i + \beta_i} + \frac{\beta_i m \rho^2}{3\alpha_i + \beta_i} - \frac{\beta_i C}{\alpha_i \rho(\alpha_i + \beta_i)/\alpha_i}. \]  

Elastic deformations in the plastic region are determined through stresses according to the linear Hooke law [1]:

\[ E \varepsilon^e_\rho = \sigma_\rho - \nu \sigma_\theta, \quad E \varepsilon^e_\theta = \sigma_\theta - \nu \sigma_\rho, \quad E \varepsilon^e_z = -\nu (\sigma_\rho + \sigma_\theta), \]  

where \( E \) is Young’s modulus and \( \nu \) is the Poisson coefficient.

According to the associated law of plastic flow [19] for each nonsingular regime for condition (1), we have

\[ \frac{d\varepsilon^p_\rho}{\partial f/\partial \sigma_\rho} = \frac{d\varepsilon^p_\theta}{\partial f/\partial \sigma_\theta} = \frac{d\varepsilon^p_z}{(\partial F/\partial \sigma_z)|_{\sigma_z=0}}, \quad f = F|_{\sigma_z=0}. \]  

In the case of small strains, an additive representation of the total strains through elastic and plastic strains takes place, therefore

\[ \varepsilon^p_\rho = \varepsilon_\rho - \varepsilon^e_\rho, \quad \varepsilon^p_\theta = \varepsilon_\theta - \varepsilon^e_\theta, \quad \varepsilon^p_z = -\varepsilon^e_z \quad (\varepsilon_z = 0). \]  

Total deformations are determined by radial displacement according to the Cauchy relations [1]:

\[ \varepsilon_\rho = \frac{du}{d\rho}, \quad \varepsilon_\theta = \frac{u}{\rho}. \]  

From the formulas (6), (8)–(10) follows the equation for radial displacement

\[ \frac{dE u}{d\rho} = \frac{\partial f/\partial \sigma_\rho}{\partial f/\partial \sigma_\theta} \left( \frac{E u}{\rho} - E \varepsilon^e_\theta \right) + E \varepsilon^e_\rho. \]  

For the yield condition (1)

\[ \frac{\partial f/\partial \sigma_\rho}{\partial f/\partial \sigma_\theta} = \frac{\beta_i}{\alpha_i}. \]
Using formulas (5), (6), (12), we solve equation (11) and get
\[ Eu = \frac{(1 - \nu)k}{\alpha_i + \beta_i} \rho - (\beta_i^2 + 2\nu\alpha_i\beta_i + \alpha_i^2) \left( \frac{m\rho^2}{9\alpha_i^2 - \beta_i^2} + \frac{C}{\alpha_i\beta_i} \rho^{-\frac{\beta_i}{\alpha_i}} \right) + C_1\rho^{-\frac{\beta_i}{\alpha_i}}. \] (13)

According to (6), (9), (10), we take into account (5), (13) and obtain formulas for elastic deformations
\[ E\varepsilon^p = \frac{1 - \nu}{\alpha_i + \beta_i} k - (\alpha_i + \nu\beta_i) \left( \frac{m\rho^2}{3\alpha_i + \beta_i} - \frac{C}{\alpha_i} \rho^{-\frac{\alpha_i + \beta_i}{\alpha_i}} \right), \] (14)
\[ E\varepsilon^\theta = \frac{1 - \nu}{\alpha_i + \beta_i} k + (\beta_i + \nu\alpha_i) \left( \frac{m\rho^2}{3\alpha_i + \beta_i} - \frac{C}{\alpha_i} \rho^{-\frac{\alpha_i + \beta_i}{\alpha_i}} \right), \] for total deformations
\[ E\varepsilon^p = \frac{1 - \nu}{\alpha_i + \beta_i} k - (\beta_i^2 + 2\nu\alpha_i\beta_i + \alpha_i^2) \left( \frac{3m\rho^2}{9\alpha_i^2 - \beta_i^2} + \frac{C}{2\alpha_i} \rho^{-\frac{\alpha_i + \beta_i}{\alpha_i}} \right) + \frac{\beta_i}{\alpha_i} C_1\rho^{-\frac{\beta_i - \alpha_i}{\alpha_i}}, \] (15)
\[ E\varepsilon^\theta = \frac{1 - \nu}{\alpha_i + \beta_i} k - (\beta_i^2 + 2\nu\alpha_i\beta_i + \alpha_i^2) \left( \frac{m\rho^2}{9\alpha_i^2 - \beta_i^2} + \frac{C}{\alpha_i\beta_i} \rho^{-\frac{\alpha_i + \beta_i}{\alpha_i}} \right) + C_1\rho^{-\frac{\beta_i - \alpha_i}{\alpha_i}}, \]
for plastic deformations
\[ E\varepsilon^p = \frac{\nu}{9\alpha_i + \beta_i} + \frac{\alpha_i + 3\beta_i}{9\alpha_i^2 - \beta_i^2} \beta_i m\rho^2 - \frac{\alpha_i^2 - \beta_i^2}{2\alpha_i} C \rho^{-\frac{\alpha_i + \beta_i}{\alpha_i}} + \frac{\beta_i}{\alpha_i} C_1\rho^{-\frac{\beta_i - \alpha_i}{\alpha_i}}, \] (16)
\[ E\varepsilon^\theta = \frac{3 + \nu}{9\alpha_i + \beta_i} + \frac{(1 + 3\nu)\alpha_i}{9\alpha_i^2 - \beta_i^2} \beta_i m\rho^2 - \frac{\alpha_i^2 - \beta_i^2}{2\alpha_i} C \rho^{-\frac{\alpha_i + \beta_i}{\alpha_i}} + \frac{\beta_i}{\alpha_i} C_1\rho^{-\frac{\beta_i - \alpha_i}{\alpha_i}}, \]
\[ E\varepsilon^p = \frac{3 + \nu}{9\alpha_i + \beta_i} + \frac{(1 + 3\nu)\alpha_i}{9\alpha_i^2 - \beta_i^2} \beta_i m\rho^2 - \frac{\alpha_i^2 - \beta_i^2}{2\alpha_i} C \rho^{-\frac{\alpha_i + \beta_i}{\alpha_i}} + C_1\rho^{-\frac{\beta_i - \alpha_i}{\alpha_i}}. \]

For the Tresca yield criterion regimes (when either \( \alpha_i = 0 \), or \( \beta_i = 0 \), or \( \alpha_i + \beta_i = 0 \), and also when \( 9\alpha_i^2 - \beta_i^2 = 0 \)), the desired formulas for stresses, strains and displacements can be obtained from (5), (13)–(16). In this case, it is necessary to perform the passage to the limit when either \( \alpha_i \to 0 \), or \( \beta_i \to 0 \), or \( \alpha_i + \beta_i \to 0 \), or \( 9\alpha_i^2 - \beta_i^2 \to 0 \), on the condition that the values \( C \) and \( C_1 \) (determined by the boundary conditions) depend on the parameters \( \alpha_i, \beta_i \).

A simpler option for obtaining the desired formulas follows directly from the solution of system (1), (4), where in (1) the indicated cases are taken into account; \( \alpha_i = 0, \beta_i = 0, \alpha_i + \beta_i = 0, 9\alpha_i^2 - \beta_i^2 = 0 \). So, when \( \alpha_i + \beta_i = 0 \), we have
\[ \sigma = \frac{k \ln(\rho)}{\alpha_i} - \frac{m\rho^2}{2} + C, \quad \sigma = \frac{k (1 + \ln(\rho))}{\alpha_i} - \frac{m\rho^2}{2} + C, \]
\[ Eu = (1 - \nu) \left( \frac{k \ln(\rho)}{\alpha_i} - \frac{m\rho^2}{4} + C \right) \rho - C_1 \rho, \]
\[ E\varepsilon^p = (1 - \nu) \left( \frac{k \ln(\rho)}{\alpha_i} - \frac{m\rho^2}{4} + C \right) - \frac{k \nu}{\alpha_i}, \quad E\varepsilon^\theta = (1 - \nu) \left( \frac{k \ln(\rho)}{\alpha_i} - \frac{m\rho^2}{4} + C \right) + \frac{k}{\alpha_i}, \]
\[ E\varepsilon^p = (1 - \nu) \left( \frac{k \ln(\rho)}{\alpha_i} - \frac{3m\rho^2}{4} + C \right) \rho - C_1 \rho^2, \quad E\varepsilon^\theta = (1 - \nu) \left( \frac{k \ln(\rho)}{\alpha_i} - \frac{m\rho^2}{4} + C \right) \rho + C_1 \rho^2. \]

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\[ E\varepsilon^\rho = \frac{k}{\alpha_i} \frac{1 - \nu}{4} m\rho^2 - \frac{C_1}{\rho^2}, \quad E\varepsilon^\rho_\theta = -\frac{k}{\alpha_i} \frac{1 - \nu}{4} m\rho^2 + \frac{C_1}{\rho^2}. \]

If \(3\alpha_i - \beta_i = 0\), then

\[ \sigma_\rho = \frac{k}{4\alpha_i} - \frac{m\rho^2}{6} + C, \quad \sigma_\theta = \frac{k}{4\alpha_i} + \frac{m\rho^2}{2} + 3C, \quad \rho_\theta = \rho_\rho \]

\[ E\varepsilon^\rho = \frac{1 - \nu}{4\alpha_i} k - (1 + 3\nu) \left( \frac{m\rho^2}{6} - \frac{C}{\rho^4} \right), \quad E\varepsilon^\rho_\theta = \frac{1 - \nu}{4\alpha_i} k - (3 + \nu) \left( \frac{m\rho^2}{6} - \frac{C}{\rho^4} \right), \]

\[ E\rho = \frac{1 - \nu}{4\alpha_i} k - 5 + 3\nu \left( \frac{m\rho^2}{3} \ln(\rho) + \frac{C}{\rho^4} \right) + \rho^3 C_1, \quad E\varepsilon^\rho = -\left( \frac{3 + \nu}{2} + (5 + 3\nu) \ln(\rho) \right) m\rho^2 + \frac{4C}{\rho^4} + 3\rho^2 C_1, \]

\[ E\varepsilon^\rho_\theta = \left( \frac{3 + \nu}{2} + (5 + 3\nu) \ln(\rho) \right) \frac{m\rho^2}{3} + \frac{4C}{\rho^4} + \rho^2 C_1. \]

If \(3\alpha_i + \beta_i = 0\), then we have

\[ \sigma_\rho = (C - m\ln(\rho))\rho^2 - \frac{k}{2\alpha_i}, \quad \sigma_\theta = 3(C - m\ln(\rho))\rho^2, \quad \rho_\theta = \rho_\rho \]

\[ E\varepsilon^\rho = (1 - 3\nu)(C - m\ln(\rho))\rho^2 - \left( \frac{1 - \nu}{2\alpha_i} \right) k, \quad E\varepsilon^\rho_\theta = (3 - \nu)(C - m\ln(\rho))\rho^2 - \left( \frac{1 - \nu}{2\alpha_i} \right) k, \]

\[ E\rho = \left( \frac{5 - 3\nu}{3} \right) \left( C - m\ln(\rho) + \frac{m}{6} \right) \rho^2 - \left( \frac{1 - \nu}{2\alpha_i} \right) k + \frac{C_1}{\rho^4}, \quad E\varepsilon^\rho = (5 - 3\nu) \left( C - m\ln(\rho) - \frac{m}{6} \right) \rho^2 - \left( \frac{1 - \nu}{2\alpha_i} \right) k + 3C_1, \]

\[ E\varepsilon^\rho_\theta = \left( \frac{5 - 3\nu}{3} \right) \left( C - m\ln(\rho) + \frac{m}{6} \right) \rho^2 - \left( \frac{1 - \nu}{2\alpha_i} \right) k + \frac{C_1}{\rho^4}, \quad E\varepsilon^\rho_\theta = \left( \frac{4C}{3} + \frac{4}{3} m\ln(\rho) + \frac{(5 - 3\nu)m}{18} \right) \rho^2 + \frac{C_1}{\rho^4}. \]

If \(\beta_i = 0\), then we have [4]:

\[ \sigma_\rho = \frac{k}{\alpha_i} - \frac{m\rho^2}{3} + \frac{C}{\rho}, \quad \sigma_\theta = \frac{k}{\alpha_i}, \quad \rho_\theta = \rho_\rho \]

\[ E\varepsilon^\rho = \frac{1 - \nu}{\alpha_i} k - \frac{m\rho^2}{3} + \frac{C}{\rho}, \quad E\varepsilon^\rho_\theta = \frac{1 - \nu}{\alpha_i} k + \frac{m\rho^2}{3} - \nu C, \quad \rho_\theta = \rho_\rho \]

\[ E\rho = \frac{1 - \nu}{\alpha_i} k - \frac{m\rho^2}{3} + \frac{C}{\rho}, \quad E\varepsilon^\rho_\theta = \frac{1 - \nu}{\alpha_i} k - \frac{m\rho^2}{3} + \frac{C\ln(\rho) + C_2}{\rho}, \quad \rho_\theta = \rho_\rho. \]
\[ E_{\rho}^p = 0, \quad E_{\theta}^p = -\frac{1 + 3\nu}{9}m\rho^2 + \frac{C(\nu + \ln(\rho)) + C_2}{\rho}. \]  

Moreover, if \( \alpha_i = 0 \), then we have

\[
\begin{align*}
\sigma_\rho &= \frac{k}{\beta_i}, \quad \sigma_\theta = \frac{k}{\beta_i} + m\rho^2, \\
E_{\rho}^e &= \frac{1 - \nu}{\beta_i}k - \nu m\rho^2, \quad E_{\theta}^e = \frac{1 - \nu}{\beta_i}k + m\rho^2, \\
E_u &= \frac{1 - \nu}{\beta_i}k\rho + \frac{m\rho^3}{3} + C, \\
E_{\rho}^p &= \frac{1 - \nu}{\beta_i}k + m\rho^2, \quad E_{\theta}^p = \frac{1 - \nu}{\beta_i}k + \frac{m\rho^2}{3} + \frac{C}{\rho}, \\
E_{\rho}^p &= (3 + \nu)m\rho^2, \quad E_{\theta}^p = 0.
\end{align*}
\]

Depending on the value of the external parameters, the plastic region may occupy the central part of the disk \( 0 \leq \rho \leq c_1 \), and/or the external part of the disk \( c_2 \leq \rho \leq b \) [20]. The boundaries between elastic and plastic regions \( \rho = c_1, \rho = c_2 \) are unknown in advance and can be determined during the process of solving the problem.

6. Regime AB

We will consider the case when the plastic region occupies some internal part of the disk. There one regime of plasticity is activated

\[ f = \alpha \sigma_\theta + \beta \sigma_\rho = k, \quad \alpha \in [0.5, 1], \quad \beta = 1 - \alpha. \]  

Since we are considering one regime, hereinafter, the parameters are not indexed. The boundaries of the allowable values of the parameters \( \alpha, \beta \) in (22) are determined from the condition that at the plane of stresses \( \sigma_\rho, \sigma_\theta \) the straight line \( \alpha \sigma_\theta + \beta \sigma_\rho = k \) must pass through the point \( \sigma_\rho = \sigma_\theta = k \) (the condition of symmetry of the stress field at the center of the disk) and lie inside the angle formed by the straight lines; \( \sigma_\rho = \sigma_\theta = k \) (the Tresca yield condition regime) and \( \sigma_\rho + \sigma_\theta = 2k \) (the Schmidt–Ishlinskii yield condition regime).

If the parameter \( \alpha \in (0.5, 1] \), then the singular regime \( A \) will always be activated (Figure 1) at the point \( \rho = 0 \) which is in the plastic state. In the future, we will be interested in this particular case.

For definiteness, we assume that in the deviatorial plane the plastic curve forms a dodecagon. Then, for the singular regime \( B \) (Figure 1), formulas (2) are written as

\[
\sigma^{(1)}_\rho = \frac{k}{1 + \alpha}, \quad \sigma^{(1)}_\theta = \frac{2k}{1 + \alpha}.
\]  

The parameter \( \eta \) in (3) is determined by the parameter \( \alpha \) by the formula \( \eta = 2\alpha - 1 \).

In the plastic region \( 0 \leq \rho \leq c_1 \), one regime \( AB \) will be activated on the condition that at the elasto-plastic boundary \( \sigma^{(1)}_\rho |_{\rho = c_1} \leq \sigma_\rho |_{\rho = c_1} \). Checking this condition, as well as finding the radius of the elasto-plastic boundary \( \rho = c_1 \), is performed during the implementation of the algorithm of the problem solution.

The boundaries of the values of the parameter \( \alpha \), which are indicated in (22), allow us to establish that the exponent \( (\alpha + \beta)/\alpha = 1/\alpha > 0 \) in formulas (5). Therefore, considering that the stresses at a point \( \rho = 0 \) cannot take on an infinitely large value, it should be assumed in
formulas (5) that \( C = 0 \). The same result \((C = 0)\) follows from the symmetry of the stress field in the center of the disk

\[ \sigma_{\rho=0} = \sigma_{\theta=0}. \]  

(24)

Hence, in the region \( 0 \leq \rho \leq c_1 \)

\[ \sigma_{\rho} = k - \frac{\alpha m \rho^2}{2\alpha + 1}, \quad \sigma_{\theta} = k + \frac{(1-\alpha)m \rho^2}{2\alpha + 1}. \]  

(25)

Hereinafter, \( \beta = 1 - \alpha \) is taken into account. Then formula (13) is written as

\[ Eu = (1-\nu)k \rho + \frac{(2\alpha(1-\alpha)(1-\nu)-1)m \rho^3}{(1+2\alpha)(4\alpha-1)} + C_1 \rho^{1/\alpha - 1}. \]  

(26)

Using (8)–(10), (25), (26) we obtain formulas for the components of the total strain tensor

\[ E\varepsilon_{\rho} = (1-\nu)k + 3 \frac{2\alpha(1-\alpha)(1-\nu)-1}{(1+2\alpha)(4\alpha-1)} m \rho^3 + \frac{1-\alpha}{\rho^{1/\alpha - 2}} C_1, \]  

(27)

\[ E\varepsilon_{\theta} = (1-\nu)k + 2 \frac{2\alpha(1-\alpha)(1-\nu)-1}{(1+2\alpha)(4\alpha-1)} m \rho^3 + \rho^{1/\alpha - 2} C_1, \]  

and plastic deformations

\[ E\varepsilon_{\rho}^p = \frac{1-\alpha}{1-4\alpha} \left( \frac{3-2\alpha}{1+2\alpha} + \nu \right) m \rho^3 + \frac{1-\alpha}{\rho^{1/\alpha - 2}} C_1, \]  

(28)

\[ E\varepsilon_{\theta}^p = \frac{\alpha}{1-4\alpha} \left( \frac{3-2\alpha}{1+2\alpha} + \nu \right) m \rho^3 \rho^{1/\alpha - 2} C_1. \]  

For the regime (22), if \( \alpha \in [0.5, 1) \), then \( \frac{1}{\alpha} - 1 > 0 \), therefore, according to formula (26), the continuity condition for displacements in the center of the disk \( u|_{\rho=0} = 0 \) is fulfilled for any final value \( C_1 \). In this case, the value \( C_1 \) in (26) is determined from the continuity condition of displacements at the elasto-plastic boundary or from the equality of plastic deformations at this boundary to zero (if the loading process is active). So, if we choose the conditions \( \varepsilon_{\rho}^p|_{\rho=c_1} = \varepsilon_{\theta}^p|_{\rho=c_1} = 0 \), then from (28) we obtain

\[ C_1 = \frac{\alpha}{4\alpha-1} \left( \nu + \frac{3-2\alpha}{1+2\alpha} \right) mc_1^{4-1/\alpha}. \]  

(29)

If \( \alpha = 1 \), then

\[ C_1 = \frac{(1+3\nu)mc_1^3}{9}. \]

7. Displacement determination at the elasto-plastic boundary

If we denote displacement at the boundary \( \rho = c_1 \) by \( u_1 \), then we obtain form the continuity condition of displacements at the elasto-plastic boundary \( [u]|_{\rho=c_1} = 0 \) and formula (26) that

\[ C_1 = \left( Eu_1 - (1-\nu)c_1 k + \frac{2\alpha(\alpha-1)(1-\nu)+1}{(2\alpha+1)(4\alpha-1)}mc_1^3 \right) c_1^{(\alpha-1)/\alpha}. \]  

(30)

Combining (29) and (30), we obtain the formula of radial displacements at the elastoplastic boundary

\[ Eu_1 = (1-\nu)kc_1^2 + \frac{1-(1-\nu)\alpha}{(1+2\alpha)} mc_1^3. \]
8. The limit state of the disk
Since only the regime (22) of condition (3) is considered, the condition \( \sigma^B_\rho \leq \sigma^B_{\rho=c_1} \) must be satisfied at the elasto-plastic boundary.

In the case when \( \sigma^B_{\rho=c_1} = \sigma^B_\rho \), taking into account formulas (23) and (25), we find that the largest value of the radius of the elasto-plastic boundary is

\[
C_1 = \sqrt{\frac{(1 + 2\alpha)k}{(1 + \alpha)m}}.
\]

If the disk is in the limiting state \( (c_1 = b) \), then it follows from (27) that

\[
m = \frac{(1 + 2\alpha)k}{(1 + \alpha)b^2}.
\]

In this case, according to (25), we have the formula for the pressure at the outer boundary \( \rho = b \)

\[
p_b = \frac{\alpha mb^2}{1 + 2\alpha} - k.
\]

To determine the value of \( C_1 \) in formulas (26)–(28) at the boundary \( \rho = b \), the value of displacements or components of plastic deformation should be set as \( \varepsilon^p_\rho = \varepsilon^p_\theta = 0 \). For example, if the components \( \varepsilon^p_\rho = \varepsilon^p_\theta = 0 \) are at the boundary \( \rho = b \) (on plastic strains at the boundary is zero), then

\[
C_1 = \frac{1 + 3\nu}{9} mb^3, \quad E\varepsilon^p_\rho|_{\rho=b} = (1 - \nu)k\rho - \frac{1}{9}m\rho^3 - \frac{1 + 3\nu}{9} mb^3.
\]

If we assume that \( u = u_b \) on the boundary \( \rho = b \), then

\[
C_1 = \frac{1}{9} mb^3 - (1 - \nu)kb + E u_b, \quad E\varepsilon^p_\theta|_{\rho=b} = -\frac{1 + 3\nu}{9} m\rho^2 + \frac{mb^3/9 - (1 - \nu)kb + E u_b}{\rho}.
\]

9. The Tresca yield criterion
For the Tresca yield criterion regime \( \sigma_\theta = k, 0 \leq \sigma_\rho \leq k \) formulas for stresses, displacements, and strains follow from (17)–(21), when the parameter \( \alpha_i = 1 \).

From (19), taking into account the limited displacements in the center of the disk, the radial displacements in the plastic region will be determined by the formula

\[
E u = (1 - \nu)k\rho - \frac{1}{9}m\rho^3 + C_1
\]

and plastic deformations according to formulas (21)

\[
E\varepsilon^p_\rho = 0, \quad E\varepsilon^p_\theta = -\frac{1 + 3\nu}{9} m\rho^2 + \frac{C_1}{\rho}.
\]

From formulas (32) and (33) it follows that \( E u|_{\rho=0} = C_1 \). If \( C_1 \neq 0 \), then \( u|_{\rho=0} \neq 0 \) and

\[
\lim_{\rho \to 0} E\varepsilon^p_\theta = \infty.\]

From the condition of incompressibility of plastic deformations it follows that \( \varepsilon^p_z = -\varepsilon^p_\theta \); this equality shows that in the center of the discs there is a strong thinning and the point \( \rho = 0 \) in the disc disappears. These conclusions do not contradict any physical principles.

In [5] it was assumed that \( u|_{\rho=0} = 0 \). In this case, an overdetermined problem with respect to the boundary conditions is obtained. Therefore, in [5], two options of choosing the boundary conditions were considered:
1) The displacements in the center of the disk are equal to zero \( u|_{\rho=0} = 0 \), the continuity conditions of stresses at the elasto-plastic boundary \([\sigma_\rho]|_{\rho=c_1} = |\sigma_\theta||_{\rho=c_1} = 0\) and the boundary condition on the side surface of the disk \( \sigma_\rho|_{\rho=b} = 0 \). But the continuity condition of displacements at the elasto-plastic boundary \([u]|_{\rho=c_1} = 0\) is not used (square brackets indicate the jump of the considered value when it crosses the elasto-plastic boundary \( \rho = c_1 \)).

2) The fulfillment of two conditions for displacements \( u|_{\rho=0} = 0 \), \( [u]|_{\rho=c_1} = 0 \) and for radial stresses \([\sigma_\rho]|_{\rho=c_1} = 0\), \( \sigma_\rho|_{\rho=b} = 0\). The condition \([\sigma_\theta]|_{\rho=c_1} = 0\) is not used. It will be shown below that the condition \( u|_{\rho=0} = 0 \) for regime (22) (when \( \alpha = 1 \)) is not reasonable. Therefore, the value \( C_1 \) is determined from the condition. \([u]|_{\rho=c_1} = 0\) or \( \varepsilon_{\theta}^0|_{\rho=c_1} = 0\).

10. **Elastic region**

In the region of the elastic state of the disk, stresses are determined by the formulas [1]:

\[
\sigma_\rho = -\frac{3 + \nu}{8} m \rho^2 + A - \frac{B}{\rho^2}, \quad \sigma_\theta = -\frac{1 + 3\nu}{8} m \rho^2 + A + \frac{B}{\rho^2}. \tag{34}
\]

The values \( A \) and \( B \) are determined by the continuity condition of stresses at the elasto-plastic boundary \( \rho = c_1 \):

\[
A = k + \left( \nu - \frac{4\alpha - 1}{2\alpha + 1} \right) \frac{mc^2}{4}, \quad B = \left( \nu + \frac{4\alpha - 1}{2\alpha + 1} \right) \frac{mc^4}{8}. \tag{35}
\]

Let us denote \( \rho_c = -\sigma_\rho|_{\rho=c_1} \). If and are determined by the continuity condition of the radial stress at the elasto-plastic boundary and the boundary condition \( \sigma_\rho|_{\rho=b} = -\rho_b \), then the values \( A \) and \( B \) will be calculated by the formulas:

\[
A = \frac{1}{8} (c_1^2 + b^2)(\nu + 3)m + \frac{p_c c_1^2 - \rho_b b^2}{b^2 - c_1^2}, \quad B = \left( \frac{\nu + 3}{8} m + \frac{p_c - \rho_b}{b^2 - c_1^2} \right) b^2 c_1^2.
\]

According to Hooke’s law, taking into account formulas (34), (35), we obtain deformations

\[
E \varepsilon_\rho = \frac{3 + \nu}{8} \left( \nu \mu - 1 \right) m \rho^2 + (1 - \nu) A - \frac{(1 + \nu)B}{\rho^2}, \quad E \varepsilon_\theta = \frac{3 + \nu}{8} \left( \mu - \nu \right) m \rho^2 + (1 - \nu) A + \frac{(1 + \nu)B}{\rho^2}
\]

and displacements

\[
E u = \frac{3 + \nu}{8} \left( \mu - \nu \right) m \rho^3 + (1 - \nu) A \rho - \frac{(1 + \nu)B}{\rho}
\]

in the region of elastic state.

11. **Equivalent stress**

To estimate the magnitude of the stress state in the elastic region, we determine the equivalent stress by the formula \( \sigma_{eq} = f \).

If the disk is in an elastic state, then the condition (24) must be satisfied at the point \( \rho = 0 \).

At the boundary \( \rho = b \), the radial stress is \( \sigma_\rho|_{\rho=b} = -\rho_b \). For these boundary conditions, the stresses will be determined by the formulas:

\[
\sigma_\rho = \frac{3 + \nu}{8} m (b^2 - \rho^2) - \rho_b, \quad \sigma_\theta = \frac{3 + \nu}{8} m (b^2 - \mu \rho^2) - \rho_b, \quad \mu = \frac{1 + 3\nu}{3 + \nu}. \tag{36}
\]

From (36) \( \sigma_\theta \) can be expressed as \( \sigma_\rho \) and other parameters:

\[
\sigma_\theta = \mu \sigma_\rho + (1 - \nu) \left( \frac{mb^2}{4} - \frac{2\rho_b}{3 + \nu} \right). \tag{37}
\]
Since the condition $\sigma_{eq} \leq k$ must be fulfilled in this case, then on the plane $(\sigma_{\theta}, \sigma_{\rho})$ the hodograph of the stress vector, defined by (37), is a straight line segment located inside the curve $\sigma_{eq} = k$. Since the Poisson coefficient $\nu$ belongs $[0, 0.5]$, therefore, the parameter $\mu$ belongs $[1/3, 5/7]$, the function $\sigma_{\theta} = \sigma_{\theta}(\sigma_{\rho})$ is monotonically increasing. From (37) it also follows that an increase or decrease in the parameters $m$ and $p_b$ leads to opposite effects, and the slope of the straight line (37) to the abscissa axis depends on the parameter $\nu$: it increases with increasing parameter $\nu$.

12. Boundaries of derivation of the plastic region

In the plane with coordinates $(\sigma_{\theta}, \sigma_{\rho})$, line (37) intersects the line $\sigma_{eq} = k$ at two points; one of them has coordinates $\sigma_{\theta} = k, \sigma_{\rho} = k$ which are possible stresses in the center of the disk.

The condition for the plastic region beginning has the form $\max_{\rho \in [0, b]} \sigma_{eq} = k$.

From formulas (22) and (36) we get

$$\sigma_{eq} = \frac{3 + \nu}{8} m(b^2 - (\alpha(\mu - 1) + 1)\rho^2) - p_b. \quad (38)$$

For allowable values of the Poisson’s coefficient and parameter $\alpha$ values, specified in condition (22), we have $\alpha(\mu - 1) + 1 > 0$. Therefore, the function $\sigma_{eq}(\rho)$ is monotonically decreasing and has the greatest value at the point $\rho = 0$. From (38) and the condition $\sigma_{eq}|_{\rho=0} = k$ follows the formula for the minimum value of the parameter $m$ when the point $\rho = 0$ of the disk goes into the plastic state

$$\begin{cases} m = m_0 = \frac{8(k + p_b)}{(3 + \nu)b^2}, \\ p_b \in (-k, p_{max}). \end{cases} \quad (39)$$

For a disk being in an elastic state, the condition $\sigma_{eq}(\rho) < k$ must be satisfied at any point. Therefore, the value $p_{max}$ is determined by the condition $\sigma_{eq}|_{\rho=b} = k$. The maximum allowable pressure $p_b = p_{max}$ value depends on the choice of a particular yield condition.

If $p_b = -k$, then $m = 0$. In this case, the stress field determined by formulas (36) will be uniform $\sigma_{\rho} = \sigma_{\theta} = k$; the disk is in the limiting state and the regime of full plasticity is activated.

13. Elasto-plastic boundary

In the case where the derivation of the formulas for the components of the stress tensor in the elastic region $c_1 \leq \rho \leq b$ takes into account the continuity conditions of stresses at the elasto-plastic boundary, we can obtain the equation to find the radius of the elasto-plastic boundary from the boundary condition at the boundary $\rho = b$. Taking into account formulas (34) and (35), from the condition $\sigma_{\rho}|_{\rho=b} = -p_b$ we get

$$c_1 = \sqrt{b - 2b\sqrt{\frac{2(2\alpha + 1)(k + p_b) - \alpha mb^2}{(4\alpha + \nu(2\alpha + 1) - 1)m}}} \quad (40)$$

In the formula (40), the parameters $m$ and $p_b$ cannot take on any value.

If the radial pressure at the elasto-plastic boundary assumes the greatest permissible value $\sigma_{\rho}|_{\rho=c_1} = \sigma_{\rho}^{(B)}$ (if we consider only the AB regime Figure 1), then it follows from (39) and (40) that the radius of the elasto-plastic boundary is determined by the formula

$$c_1 = \sqrt{\frac{k(1 + 2\alpha)}{m(1+\alpha)}}.$$


14. The Hershey–Hosford yield criterion

In the papers of Hershey [21] and Hosford [22], it was proposed that yield criterion is written as

\[(\sigma_1 - \sigma_2)^{2n} + (\sigma_2 - \sigma_3)^{2n} + (\sigma_3 - \sigma_1)^{2n} = 2k^{2n}, \tag{41}\]

where \(\sigma_i\) is the eigenvalue of the stress tensor. When \(n = 1\) then the condition (41) is the same as the Mises yield criterion. When \(n \to \infty\) then the condition (29) is the same as the Tresca yield criterion. Therefore, the case is of interest to compare the solution of the problem of a rotating disk when the Tresca yield criterion and the Hershey–Hosford yield criterion are considered. For a plane stress state, the condition (41) can be written as

\[f(\sigma_{\rho}, \sigma_{\theta}) = \left(\frac{(\sigma_{\rho} - \sigma_{\theta})^{2n} + \sigma_{\theta}^{2n} + \sigma_{\rho}^{2n}}{2}\right)^{1/2n} = k. \tag{42}\]

For condition (29), taking into account the boundary condition (24), the allowed values of parameter \(p_b \in [-k, -\sigma_{\rho}^{(\text{min})}]\), where \(\sigma_{\rho}^{(\text{min})}\) it is determined by the system

\[
\begin{align*}
\begin{cases}
 f(\sigma_{\rho}, \sigma_{\theta}) = k, \\
 \frac{\partial f}{\partial \sigma_{\theta}} = 0.
\end{cases}
\end{align*}
\]

It is obvious that if \(n \to \infty\) in (42), then \(\sigma_{\rho}^{(\text{min})} \to k.\)

The plasticity function in (41) has a number of advantages in comparison with piecewise linear plasticity functions. Firstly, it does not contain singular points, due to which plastic deformations undergo a discontinuity at the boundary of the change of nonsingular plasticity regimes. Secondly, the Hershey–Hosford function can be expressed in terms of the basic invariants of the stress deviator [23].

Since, for piecewise linear yield conditions, plastic strains are determined by relations (8), we assume that plastic strains are also associated with stresses by the plastic strain law

\[\varepsilon^p = \psi \frac{\partial F(\sigma)}{\partial \sigma}. \tag{43}\]

where \(\sigma\) is the tensor of stresses, \(\varepsilon^p\) is the tensor of plastic strains, and \(\psi\) is an indefinite coefficient.

Stresses in the plastic region are determined by the solution of Cauchy problem

\[
\begin{align*}
\begin{cases}
 \rho \frac{d\sigma_{\rho}}{d\rho} + \sigma_{\rho} - \sigma_{\theta} + m\rho^2 = 0, \\
 \frac{d\sigma_{\theta}}{d\rho} + \frac{\partial f/\partial \sigma_{\rho}}{\partial f/\partial \sigma_{\theta}} \frac{d\sigma_{\rho}}{d\rho} = 0, \\
 \sigma_{\rho}|_{\rho=0} = \sigma_{\theta}|_{\rho=0} = k.
\end{cases}
\end{align*}
\]

Taking into account (6) and (8), from (42) we obtain the equation for displacement in the plastic region

\[
\frac{dE_u}{d\rho} = \frac{\partial f/\partial \sigma_{\rho}}{\partial f/\partial \sigma_{\theta}} \left( \frac{E_u}{\rho} - \sigma_{\theta} + \nu\sigma_{\rho} + \sigma_{\rho} - \nu\sigma_{\theta} \right).
\]

From (24) and (41) it follows that

\[
\lim_{\rho \to 0} \frac{\partial f/\partial \sigma_{\rho}}{\partial f/\partial \sigma_{\theta}} = 1. \tag{46}
\]
Assuming that deformations are limited in the center of the disk, from (45) and (46) we obtain

$$\lim_{\rho \to 0} \frac{u}{\rho} = \lim_{\rho \to 0} \frac{du}{d\rho} = \alpha.$$ 

Thus, in a small neighborhood $\delta$ of the point $\rho = 0$, the solution of the equation (45) has the form $u = k\rho$, and in the region $\delta \leq \rho \leq c_1$ we solve the problem

$$\begin{cases}
\frac{dE_u}{d\rho} = \frac{\partial f}{\partial \sigma\rho} \left( \frac{E_u}{\rho} - \sigma_\theta + \nu \sigma_\rho \right) + \sigma_\rho - \nu \sigma_\theta, \\
u|_{\rho=\delta} = k\delta.
\end{cases}$$

An indefinite quantity is determined by (for example) the condition that the plastic deformations at the elasto-plastic boundary $\rho = c_1$ are equal to zero.

Problem (44), (47) is solved numerically.

15. Numerical results

Figures 2 and 3 show plots of the distribution of stresses, strains, and displacements in a disk (when one plastic regime is activated in the plastic zone $0 \leq \rho \leq c_1$).

**Figure 2.** $\alpha = 0.5128$, $m = 2.8$, $p_0 = 0$, $c_1 = 0.3981$, a) complete deformations, displacements, b) plastic deformations, c) circumferential and equivalent stress, d) stress vector hodograph.

**Figure 3.** $\alpha = 0.99$, $m = 2.8$, $p_0 = 0$, $c = 0.6274$, a) complete deformations, displacements, b) plastic deformations, c) circumferential and equivalent stress, d) stress vector hodograph.
Figure 4. $m = 3.3$, $c_1 = 0.64$, $\nu = 0.2$ plastic, complete deformations and displacements.

Figure 5. $\delta = 10^{-20}$, $k = 5.9727$, $m = 2.8$, $p_b = 0$, $c = 0.6274$, $a)$ complete deformations and displacements, $b)$ plastic strains, $c)$ circumferential and equivalent stress, $d)$ stress vector hodograph.

The numerical results show that for the values of the parameter $\alpha = 0.5 - \delta$, where $\delta$ is an arbitrarily small positive value, a sharp change in plastic deformations takes place in a small neighborhood of the point $\rho = 0$, while the displacements in the center of the disk remain equal to zero. If the parameter $\alpha = 1 - \delta$, then in some small neighborhood of the point $\rho = 0$ there is a sharp increase in the value of the radial displacement, while $\lim_{\delta \to 0} u(\delta)|_{\rho=0} \neq 0$. It corresponds to the solution of the problem choosing the Tresca yield criterion.

Figure 4 shows plots of plastic deformations and displacements for the regime ($\alpha = \beta = 0.5$) of the Schmidt–Ishlinskii yield criterion.

Figure 5 shows plots of stresses, displacements, and deformations when the parameter $n = 400$, which almost provides the coincidence of the plastic curve (42) with the Tresca hexagon. However, curve (42) does not have singular points.

The experiments performed show that for all regimes of piecewise-linear yield conditions different from the regime $\sigma_\theta + \sigma_\rho = 2k$, in some neighborhood of the center of the disk, the plastic strains will not be small, which contradicts the assumption that the strains are small for the chosen mathematical model.

16. Conclusion

The choice of the associated law of plastic flow in solving the problem of a rotating elasto-plastic disk leads to an infinitely large increase in plastic deformations in the center of the disk for all regimes of piecewise linear plasticity functions except Schmidt–Ishlinskii. Colladin
highlights this property of the Tresca yield condition \cite{24}: “For a solid disc the mechanism \(\dot{\varepsilon}_\theta = \dot{u}/\rho, \dot{\varepsilon}_\rho = 0, \dot{\varepsilon}_z = -\dot{u}/\rho\), in \(\rho = 0\), gives singularities in the strain increments at the centre, which can be interpreted as a tendency for the disc to “thin” so much as to produce a small hole very quickly. These singularities are in fact a consequence of the precise angularity of the Tresca yield condition. If a small “rounding” of the edge could be allowed, the singularity would disappear, because in the immediate vicinity of the centre the stress components \(\sigma_\rho\) and \(\sigma_\theta\) are very close. This somewhat curious state of affairs should not be regarded as reflecting discredit on the Tresca condition; it must always be remembered that our main aim here is to predict bursting speeds”.

Not only is the statement made in \cite{24} valid for the Tresca yield condition, but also for all piecewise linear yield conditions different from the Schmidt–Ishlinskii yield condition.

For all piecewise linear yield conditions except for the Tresca yield criterion, the displacements in the center of the disk are equal to zero. In the framework of the theory of plastic flow, when we consider the Tresca yield criterion, the problem of determining displacements in the disk is overdetermined. Since we assume that the continuity condition of displacements is fulfilled at the elasto-plastic boundary, then for the Tresca yield criterion we have a jump (gap) of displacements in the center of the disk. The infinitely large increase in plastic deformations in the center of a rotating disk within the framework of the plastic flow theory for all piecewise linear potentials, except for Schmidt–Ishlinskii, contradicts the assumption regarding small deformations.

The choice of smooth exponential plasticity functions \cite{21, 22} makes it possible to approximate piecewise-linear plasticity functions with a sufficient degree of accuracy. Thereby it leads to the “smoothing” of singular points, and plastic deformations in the center of the disk will have finite values.

It is worth noting that when several regimes are activated in the plastic region of the disk, singular regimes can take place only during the transition from one smooth regime to another, and not in some extended region, which is suggested by the generalized associated law of plastic flow. Therefore, in the framework of the plastic flow theory we have discontinuities of plastic deformations at these boundaries. This property also manifests itself in solving other problems.

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