Stabilizability Theorems on Discrete-time Nonlinear Uncertain Systems *

Zhaobo Liu†, Chanying Li †

March 18, 2022

Abstract

This paper derives two stabilizability theorems for a basic class of discrete-time nonlinear systems with multiple unknown parameters. First, we claim that a discrete-time multi-parameter system is stabilizable if its nonlinear growth rate is dominated by a polynomial rule. Later, we find that a stabilizable multi-parameter system in discrete time is possible to grow exponentially fast. Meanwhile, optimality and closed-loop identification are also discussed in this paper.

1 Introduction.

Adaptive control of linear systems ([1], [2], [4], [9]) and nonlinear systems growing linearly ([21], [23]) has been a mature topic for decades, both in continuous time and discrete time. 

*This work was supported in part by the National Natural Science Foundation of China under Grants 11925109 and 11688101.
†Z. Liu and C. Li are with the Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, P. R. China. They are also with the School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, P. R. China. Corresponding author: Chanying Li (Email: cyli@amss.ac.cn).
But when it comes to systems whose output nonlinearities are faster than linearities, the similarities of adaptive control between continuous- and discrete-time systems disappear. Most continuous-time nonlinear systems can be globally stabilized by employing nonlinear damping or back-stepping techniques (10 and 11), however, its discrete-time counterpart is not that favored by fortune. It was found in 6 that even for the following basic discrete-time stochastic system

\[ y_{t+1} = \theta y_t^b + u_t + w_{t+1}, \quad \theta \text{ is an unknown scalar}, \tag{1} \]

the stabilizability is still possible to be failure. It showed that the system is stabilizable if and only if \( b < 4 \). This fundamental difficulty in discrete-time control was further confirmed by 22, where system (1) is extended to the multi-parameter case:

\[ y_{t+1} = \theta_1 y_t^{b_1} + \theta_2 y_t^{b_2} + \cdots + \theta_n y_t^{b_n} + u_t + w_{t+1}. \tag{2} \]

Work 22 established an “impossibility theorem” by providing a polynomial rule, which was proved, a decade later by 13, to be a necessary and sufficient condition of the stabilizability of system (2). This polynomial rule also serves as a critical stabilizability criterion for system (2) in the deterministic framework (see 14). Analogous phenomena arise in the adaptive control of discrete-time nonparametric nonlinear systems (18, 24, 28), semiparametric uncertain systems (8, 19), linear stochastic systems with unknown time-varying parameter processes (27), and continuous-time nonlinear systems with sampled observations (26). We refer the readers to 7, 17, 25 for other related works.

This paper is intended to extend the results of 13 and 22 to the following class of systems:

\[ y_{t+1} = \theta_1 f_1(y_t) + \theta_2 f_2(y_t) + \cdots + \theta_n f_n(y_t) + u_t + w_{t+1}. \tag{3} \]

We conjecture that system (3) is stabilizable if the nonlinear growths of \( f_1, \ldots, f_n \) are dominated by some power functions \( x_1^{b_1}, \ldots, x^{b_n} \) respectively, where \( b_1, \ldots, b_n \) satisfy the
polynomial rule referred. Comparing system (2) and system (3), a significant difference is that \( f_1, \ldots, f_n \) in (3) may be very close or intersect infinitely often, while functions \( x^{b_1}, \ldots, x^{b_n} \) with \( b_1 > \cdots > b_n > 0 \) in system (2) are away from each other when \( x \) is large. Intuitively, this will cause some obstacles in the closed-loop identification for system (3). And then, the stabilizability might be affected. By establishing an inequality on the minimal eigenvalue of the inverse information matrix in Proposition 3.1, we prove our conjecture in Theorem 2.1.

Theorem 2.1 requires that system (3) grows no faster than some power function. But this is not the growth rate limit for the stabilizability of system (3). For the scalar case \((n = 1)\), recall that [16] asserts \( f_1(x) = O(|x|^{b_1}) \) with \( b_1 < 4 \) is only required for a very tiny fraction of \( x \) in \( \mathbb{R} \), even if it grows exponentially fast for the other \( x \). Is it true for the multi-parameter case? We prove in this paper that a multi-parameter stabilizable system still has a chance to grow exponentially. With the help of the proposed inequality in Proposition 3.1, we again find that the stabilizability of system (3) can be achieved if \( f_1, \ldots, f_n \) are bounded on a tiny fraction of \( x \) in \( \mathbb{R} \), while these functions may grow exponentially fast for the other \( x \).

The paper is built up as follows. Section 2 presents two stabilizability theorems and Section 3 discusses the corresponding closed-loop identification. The proofs of the main results are included in Sections 4–5.

## 2 Global Stabilizability

We study the following discrete-time nonlinear system with multiple unknown parameters:

\[
y_{t+1} = \theta^\tau \phi(y_t) + u_t + w_{t+1}, \quad t \geq 0,
\]

where \( \theta = (\theta_1, \ldots, \theta_n)^\tau \in \mathbb{R}^n, n \geq 2 \) are unknown parameters, \( y_t, u_t, w_t \) are the output, input and noise signals, respectively. Assume that \( \phi = (f_1, \ldots, f_n)^\tau : \mathbb{R} \to \mathbb{R}^n \) is a known
measurable vector function, where $f_j \in C^n(E), 1 \leq j \leq n$, and $E$ is an open set in $\mathbb{R}$. We rewrite (4) as

$$y_{t+1} = \sum_{j=1}^{n} \theta_j f_j(y_t) + u_t + w_{t+1}, \quad t \geq 0,$$

and present the definition of stabilizability in the following sense.

**Definition 2.1.** System (5) is said to be almost surely globally stabilizable, if there exits a feedback control law

$$u_t \in F^y_t \triangleq \sigma\{y_i, 0 \leq i \leq t\}, \quad t = 0, 1, \ldots$$

such that for any initial conditions $y_0 \in \mathbb{R}$,

$$\sup_{t \geq 1} \frac{1}{t} \sum_{i=1}^{t} y_i^2 < +\infty, \quad a.s.$$  

We analyze our problem in some standard assumptions below.

**A1** The noise $\{w_t\}$ is an i.i.d random sequence with $w_1 \sim N(0, \sigma^2)$.

**A2** Parameter $\theta \sim N(\theta_0, I_n)$ is independent of $\{w_t\}$.

**A3** $f_1, \ldots, f_n$ are linearly independent on $E$.

**Remark 2.1.** We consider a typical case where $E = \mathbb{R}$. If $f_j \equiv 0$ for all $j \in [1, n]$, system (5) degenerates to $y_{t+1} = w_{t+1}$. Otherwise, with no loss of generality, let $f_1, \ldots, f_k$ be linearly independent on $\mathbb{R}, 1 \leq k \leq n$, such that every $f_l, l \in [k+1, n]$ is a linear combination of $f_1, \ldots, f_k$. Consequently, there are $n - k$ unit vectors $(x_{1,l}, \ldots, x_{k,l})^\tau$ satisfying $f_l(y) = \sum_{j=1}^{k} x_{j,l} f_j(y), l \in [k+1, n]$. Therefore, by letting

$$\theta'_j \triangleq \theta_j + \sum_{l=k}^{n} x_{j,l} \theta_l, \quad 1 \leq j \leq k,$$

system (5) becomes

$$y_{t+1} = \sum_{j=1}^{k} \theta'_j f_j(y_t) + u_t + w_{t+1}, \quad t \geq 0.$$  

This means it suffices to discuss system (6). So, Assumption A3 is a natural condition.
Our first theorem below extends the result of [13] to a more general situation.

**Theorem 2.1.** Under Assumptions A1–A3, system (5) is globally stabilizable if

\[ f_j(x) = O(|x|^{b_j}) + O(1), \quad 1 \leq j \leq n, \]

where \( b_1 > b_2 > \cdots > b_n > 0 \) are \( n \) numbers satisfying \( b_1 > 1 \) and

\[ P(x) = x^{n+1} - b_1 x^n + (b_1 - b_2) x^{n-1} + \cdots + b_n > 0, \quad x \in (1, b_1). \]  

**Example 2.1.** Under Assumptions A1–A2, consider system (5) with

\[ f_1(x) = x^2 \cos x \quad \text{and} \quad f_2(x) = x \sin x. \]

The images of \( f_1 \) and \( f_2 \) intersect each other infinitely many times. The stabilizability issue of such systems cannot be covered by the existing theory. Now, applying Theorem 2.1 with \( b_1 = 2 \) and \( b_2 = 1 \), we immediately conclude that the system is stabilizable.

For the sake of stabilizability, Theorem 2.1 requires that system (5) grows no faster than some power function. On the other hand, for the scalar-parameter case, [16] finds the corresponding system is possible to be stabilized when growing exponentially. But the number of the unknown parameters affects the allowed growth rate of a stabilizable system (see [13]). So, we wonder whether a multi-parameter stabilizable system still has a chance to grow exponentially? The following theorem gives an affirmative answer.

**Theorem 2.2.** Under Assumptions A1–A3, system (4) is globally stabilizable if

(i) for some \( k_1, k_2 > 0 \),

\[ \| \phi(x) \| \leq k_1 e^{k_2 |x|}, \quad \forall x \in \mathbb{R}; \]  

(ii) there exists a number \( L > 0 \) such that for \( S_L \triangleq \{ x : \| \phi(x) \| \leq L \} \),

\[ \lim_{l \to +\infty} \inf \frac{\ell(S_L \cap [-l, l])}{l} > 0, \]  

where \( \ell \) denotes the Lebesgue measure.
Clearly, $p_L \triangleq \liminf_{l \to +\infty} \frac{\ell(S_L \cap [-L, L])}{l}$ describes the proportion of $S_L$ in $\mathbb{R}$. Since $p_L > 0$ can be taken as small as one likes in Theorem 2.2, a stabilizable system may possess a very sparse $S_L$. We give an extreme example to illustrate it.

**Example 2.2.** Under Assumptions A1–A2, consider system (5) with

$$f_1(x) = 1 + e^x \cdot I_{\{\sin x > -0.999\}} \quad \text{and} \quad f_2(x) = e^{2x} \cdot I_{\{\sin x > -0.999\}}.$$  

Clearly, (11) holds for $L = 1$. The system is thus stabilizable by virtue of Theorem 2.2. We remark that this system grows exponentially fast on most part of the real line.

### 3 Closed-loop Identification

In order to achieve the stabilization of system (5), we employ the self-tuning regulator (STR) based on the least-squares (LS) algorithm. The standard LS estimate $\hat{\theta}_t$ for parameter $\theta$ can be recursively defined by

$$\begin{align*}
\hat{\theta}_{t+1} &= \hat{\theta}_t + \sigma^{-2} P_{t+1} \phi_t (y_{t+1} - u_t - \phi_t^T \hat{\theta}_t) \\
P_{t+1} &= P_t - (\sigma^2 + \phi_t^T P_t \phi_t)^{-1} P_t \phi_t \phi_t^T P_t, \quad P_0 = I_n, \\
\phi_t &\triangleq \phi(y_t), \quad t \geq 0
\end{align*}$$

(12)

where initial vectors $\hat{\theta}_0 = \theta_0$ and $\phi_0$ are taken random. In light of the “certainty equivalence principle”, the controller is designed as follows:

$$u_t = -\hat{\theta}_t^T \phi_t, \quad t \geq 0.$$  

(13)

We shall show in the next two sections that the LS-STR (12)–(13) is the desired stabilizing controller for both Theorems 2.1 and 2.2. Besides, during the control process, parameter $\theta$ can be identified simultaneously.
Theorem 3.1. Under the conditions of Theorem 2.1, the LS estimator is strong consistent in the closed-loop system (5), (12) and (13). More precisely, 

\[ \| \hat{\theta}_{t+1} - \theta \|^2 = O \left( \frac{\log t}{t} \right), \quad \text{a.s..} \]

Theorem 3.2. Under the conditions of Theorem 2.2, the LS estimator is strong consistent in the closed-loop system (4), (12) and (13). More precisely, 

\[ \| \hat{\theta}_{t} - \theta \|^2 = O \left( t^{-\frac{1}{2}} \right), \quad \text{a.s..} \]

It is worth mentioning that for our situation, the strong consistency of the LS estimates in the closed-loop system can be guaranteed by the stability of the system. We now discuss it in details.

Let \( \lambda_{\min}(t+1) \) be the minimal eigenvalue of \( P_{t+1}^{-1} \) defined in (12). Under Assumptions A1–A2, (3) and (20) imply

\[ \{ \lim_{t \to +\infty} \lambda_{\min}(t+1) = +\infty \} = \{ \lim_{t \to +\infty} \hat{\theta}_t = \theta \}. \] (14)

Furthermore, (12) and (5) point out that

\[ \| \hat{\theta}_{t+1} - \theta \|^2 = O \left( \frac{\log \left( \frac{1}{t} \sum_{i=0}^{t} \| \phi(y_i) \|^2 \right)}{\lambda_{\min}(t+1)} \right), \quad \text{a.s.,} \] (15)

which provides a powerful tool to estimate the convergence rate of \( \{ \hat{\theta}_t \}_{t \geq 0} \). By the help of (15), we can derive the following lemma with the proof stated in Appendix A.

Lemma 3.1. Under Assumptions A1–A2, let the closed-loop system (4), (12), (13) satisfy

\[ \| \phi(x) \| = O(|x|^b) + O(1), \quad b > 0. \] (16)

Then, there is a set \( D \) with \( P(D) = 0 \) such that

\[ \left\{ \sum_{i=1}^{t} y_i^2 = O(t) \right\} \backslash D \subseteq \left\{ \lim_{t \to +\infty} \frac{\lambda_{\min}(t+1)}{t} > 0 \right\} \cap \left\{ \lim_{t \to +\infty} \frac{1}{t} \sum_{i=1}^{t} y_i^2 = \sigma^2 \right\}, \] (17)

\[ \sum_{i=1}^{t} y_i^2 = O(t) \quad \text{implies} \quad \| \hat{\theta}_{t+1} - \theta \|^2 = O \left( \frac{\log t}{t} \right), \quad \text{a.s..} \] (18)
Remark 3.1. By virtue of (17) in Lemma 3.1,
\[ \sum_{i=1}^{t} y_i^2 = O(t) \] is equivalent to
\[ \lim_{t \to +\infty} \frac{1}{t} \sum_{i=1}^{t} y_i^2 = \sigma^2 \quad \text{a.s.,} \]
which is referred to as optimality (see [5]). Meanwhile, (18) suggests that Theorem 3.1 is a direct result of Theorem 2.1.

For systems growing exponentially, we remark that the proof of Theorem 2.2 indicates
\[ \lim_{t \to +\infty} \lambda_{\min}(t + 1) t > 0, \quad \text{a.s.} \quad (19) \]
Hence, (13) infers
\[ \| \hat{\theta}_{t+1} - \theta \|^2 = O \left( \log \left( 1 + \sum_{i=0}^{t} \| \phi(y_i) \|^2 \right) \right) = O \left( \frac{\sqrt{\sum_{i=1}^{t} y_i^2}}{t} \right), \quad \text{a.s.} \]
Consequently, it is straightforward that

Lemma 3.2. If the conditions of Theorem 2.2 hold, then in the closed-loop system (4), (12), (13),
\[ \sum_{i=1}^{t} y_i^2 = O(t) \] implies \[ \| \hat{\theta}_t - \theta \|^2 = O(t^{-\frac{1}{2}}), \quad \text{a.s.} \]

The remainder of the proof is thus focused on the stability of the closed-loop system (1), (12) and (13). We shall see that (19) plays a core role not only in the above closed-loop identification, but also in justifying Theorems 2.1 2.2. We close this section by presenting an important proposition, whose proof is included in Appendix A.

Proposition 3.1. Under Assumptions A1–A3, there is a constant \( M > 0 \) such that in the closed-loop system (4), (12), (13),
\[ \lim_{t \to +\infty} \frac{\lambda_{\min}(t + 1)}{t} \geq M \lim_{t \to +\infty} \frac{1}{t} \sum_{i=1}^{t} \frac{1}{\sigma_{i-1}} \quad \text{a.s.} \]
4 Proof of Theorem 2.1

For the closed-loop system (5), (12) and (13), one has

\[
\begin{align*}
P_{t+1}^{-1} &= I_n + \frac{1}{\sigma^2} \sum_{i=0}^{t} \phi_i \phi_i^T, \\
y_{t+1} &= \tilde{\theta}_t f(y_t) + w_{t+1},
\end{align*}
\]

(20)

where \(\tilde{\theta}_t \triangleq \theta - \hat{\theta}_t\), \(t \geq 0\). Since the LS algorithm (12) is exactly the standard Kalman filter in our case, it yields that \(\hat{\theta}_t = E[\theta | F_{y_t}]\) and \(P_t = E[\tilde{\theta}_t^T \tilde{\theta}_t | F_{y_t}]\). Hence, for each \(t \geq 0\), \(y_{t+1}\) possesses a conditional Gaussian distribution given \(F_{y_t}\). The conditional mean and variance are

\[
\begin{align*}
m_t &\triangleq E[y_{t+1} | F_{y_t}] = u_t + \tilde{\theta}_t \phi_t = 0 \\
\sigma_t^2 &\triangleq \text{Var}(y_{t+1} | F_{y_t}) = \sigma^2 + \phi_t^T P_t \phi_t = \sigma^2 \cdot \frac{|P_{t+1}^{-1}|}{|P_t^{-1}|}, \text{ a.s.}
\end{align*}
\]

(22)

So, we shall make efforts to prove \(\sup_t \sigma_t < +\infty\) almost surely. To this end, we provide several relevant lemmas.

**Lemma 4.1.** Under the conditions of Theorem 2.1, assume that events \(\{|P_t^{-1}| < (1 + \sigma^{-2})^t\}_{t \geq 1}\) occur finitely on some set \(D\) with probability \(P(D) > 0\), then

\[
\sup_t \sigma_t < +\infty, \text{ a.s. on } D.
\]

(23)

**Proof.** At first, we define some random matrices:

\[
\left\{
\begin{align*}
Q_0^{-1} &= I_n \\
Q_k^{-1} &= Q_{k-1}^{-1} + \frac{1}{\sigma^2} \cdot \phi_{t_{k-1}} \phi_{t_{k-1}}^T, \quad k \geq 1,
\end{align*}
\right.
\]

where the random subscript \(t_k\) with \(t_0 = 0\) satisfies

\[
\left\{
\begin{align*}
\phi_{t_k}^T Q_k \phi_{t_k} > \phi_{t_{k-1}}^T Q_{k-1} \phi_{t_{k-1}}, & \quad k \geq 1 \\
\phi_{t_k}^T Q_k \phi_{t_k} \leq \phi_{t_{k-1}}^T Q_{k-1} \phi_{t_{k-1}}, & \quad t_{k-1} < t < t_k
\end{align*}
\right.
\]

where the random subscript \(t_k\) with \(t_0 = 0\) satisfies
If \( \{t_k\} \) is a finite sequence, then there is a \( k \) such that
\[
\phi_t^* Q_{k+1} \phi_t \leq \phi_{t_k}^* Q_{k} \phi_{t_k}, \quad \forall \ t > t_k.
\]
Consequently,
\[
\sigma_t^2 = \sigma^2 \cdot \frac{|P_{t-1}|}{|P_t|} = \sigma^2 + \phi_t^* P_t \phi_t
\]
\[
\leq \sigma^2 + \phi_t^* Q_{k+1} \phi_t \leq \sigma^2 + \phi_{t_k}^* Q_{k} \phi_{t_k}, \quad \forall \ t > t_k,
\]
which leads to \( \sup_t \sigma_t < +\infty \).

Now, we assume that there exists a set \( D' \subset D \) with \( P(D') > 0 \) such that \( \{t_k\} \) is infinite on \( D' \). Clearly,
\[
\frac{|Q_{k-1}|}{|Q_{k-1}|} = \sigma^2 + \phi_{t_{k-1}}^* Q_{k-1} \phi_{t_{k-1}}
\]
\[
< \sigma^2 + \phi_{t_k}^* Q_k \phi_{t_k} = \frac{|Q_{k+1}|}{|Q_{k}|}, \quad k \geq 1.
\]
Similarly to [13, Lemma 3.1], we can prove that for any \( t \in (t_{k-1}, t_k] \),
\[
\frac{|P_{t+1}|}{|P_t|} \leq \frac{|Q_{k+1}|}{|Q_{k}|}.
\]
On the other hand, since
\[
\sum_{t=1}^{+\infty} P(|y_t| > \sigma_{t-1} \log t \mid \mathcal{F}_{t-1}^y) = \frac{1}{\sqrt{2\pi}} \sum_{t=1}^{+\infty} \int_{|x| \geq \log t} e^{-\frac{x^2}{2}} \, dx < +\infty,
\]
by Borel-Cantelli-Levy theorem, the events \( \{|y_t| > \sigma_{t-1} \log t\} \) occur only finite many times for \( t \geq 1 \). That is, for all sufficiently large \( t \),
\[
|y_t|^2 \leq \sigma_{t-1}^2 \log^2 t = \sigma^2 \cdot \frac{|P_{t-1}|}{|P_{t-1}|} \log^2 t, \quad \text{a.s.,}
\]
which infers that there exists a random number \( \gamma > 1 \) such that for all \( t \geq 0 \),
\[
|y_t|^2 \leq \gamma \cdot \sigma_{t-1}^2 \log^2 (t + 3), \quad \text{a.s.}
\]
Next, for $d \geq 1$, we define
\[
\alpha_d(j) \triangleq \frac{1}{\sigma^2} \cdot (f_1(y_{t_j})f_d(y_{t_j}), \ldots, f_n(y_{t_j})f_d(y_{t_j}))^\top, \quad j \geq 0,
\]
and $\alpha_d(-1) \triangleq e_d$, where $e_d$ is the $d$th column of the identity matrix $I_n$. Then,
\[
|Q_{k+1}^{-1}| = \det \left( \sum_{j=-1}^{k} \alpha_1(j), \ldots, \sum_{j=-1}^{k} \alpha_n(k) \right)
= \sum_{s_1, \ldots, s_n = -1}^{k} \det (\alpha_1(s_1), \ldots, \alpha_n(s_n)). \tag{25}
\]
If there exist two subscripts $d \neq d'$ such that $s_d = s_{d'} \neq -1$, we obtain
\[
\det (\alpha_1(s_1), \ldots, \alpha_n(s_n)) = 0,
\]
and hence
\[
|Q_{k+1}^{-1}| = \sum_{(s_1, \ldots, s_n) \in \mathcal{W}(k)} \det (\alpha_1(s_1), \ldots, \alpha_n(s_n)). \tag{26}
\]
Here, given integer $k \geq 1$ and positive integers $l_1 < \cdots < l_m$,
\[
\mathcal{W}(k) \triangleq \{(l_1, \ldots, l_n) : l_i \in \{-1, 0, \ldots, k\}, i \in [1, n];
\]
\[
l_i \neq l_i' \text{ if } i \neq i', \ l_i' \neq -1\},
\]
\[
\mathcal{H}_{k}^{(l_1, \ldots, l_m)} \triangleq \{(i_1, \ldots, i_k) : i_j \in \{l_1, \ldots, l_m\}, 1 \leq j \leq k; i_r \neq i_s \text{ if } r \neq s\}.
\]
Now, for any \((s_1, \ldots, s_n) \in \mathcal{W}(k)\),

\[
\det (\alpha_1(s_1), \ldots, \alpha_n(s_n)) \\
\leq \sum_{(l_1, \ldots, l_n) \in H_n^{(1, \ldots, n)}} \prod_{k \in [1,n], s_k \neq -1} \frac{1}{\sigma^2} \cdot |f_k(y_{sk}) f_k(y_{sk})| \\
\leq \sum_{(l_1, \ldots, l_n) \in H_n^{(1, \ldots, n)}} \prod_{k \in [1,n], s_k \neq -1} \frac{1}{\sigma^2} \cdot (L_1 + L_2 |y_k| = |b_k|) \cdot (L_1 + L_2 |y_k| = |b_k|) \\
\leq (L_1 + L_2)^{2n} \cdot \sum_{(l_1, \ldots, l_n) \in H_n^{(1, \ldots, n)}} \prod_{k \in [1,n], s_k \neq -1} \frac{1}{\sigma^2} \cdot \left(\gamma \cdot \log^2 (t_k + 3) \cdot \sigma^2 \cdot \frac{|P_{sk}^{-1}|}{|P_{sk}^{-1}|} \right)^{\frac{b_k}{2}} \\
\leq (L_1 + L_2)^{2n} \cdot \sum_{(l_1, \ldots, l_n) \in H_n^{(1, \ldots, n)}} \prod_{k \in [1,n], s_k \neq -1} \frac{1}{\sigma^2} \cdot \left(\gamma \cdot \log^2 (t_k + 3) \cdot \sigma^2 \cdot \frac{|Q_{sk}^{-1}|}{|Q_{sk}^{-1}|} \right)^{\frac{b_k}{2}} \\
\leq (L_1 + L_2)^{2n} \cdot (1 + \sigma^{2b_1 - 2} + \sigma^{2b_n - 2})^n \cdot n! \\
\cdot (\gamma \cdot \log^2 (t_k + 3))^{b_1 + \cdots + b_n} \cdot \prod_{i=1}^{n} \left(\frac{|Q_{k+1-i}^{-1}|}{|Q_{k-i}^{-1}|} \right)^{b_i},
\]

(27)

where \(Q_{-1} \triangleq I_n\), and \(L_1, L_2\) are two positive numbers satisfying

\[|f_j(x)| \leq L_1 + L_2 |x|^{b_j}, \quad \forall x \in \mathbb{R}, \ j \in [1,n].\]

By combining (26) and (27), we conclude

\[
|Q_{k+1}^{-1}| \leq (k + 2)^n \cdot (L_1 + L_2)^{2n} \cdot (1 + \sigma^{2b_1 - 2} + \sigma^{2b_n - 2})^n \\
\cdot n! \cdot (\gamma \cdot \log^2 (t_k + 3))^{b_1 + \cdots + b_n} \cdot \prod_{i=1}^{n} \left(\frac{|Q_{k+1-i}^{-1}|}{|Q_{k-i}^{-1}|} \right)^{b_i}.
\]

As a consequence, if \(|Q_{k+1}^{-1}| > t_k^{\log t_k}\) for all sufficiently large \(k\), then there must exist a
random number $t'_e$ for any given $\epsilon > 0$ such that

$$
(1 - \epsilon) \log |Q_{k+1}^{-1}|
\leq \sum_{i=1}^{n} b_i (\log |Q_{k+1-i}^{-1}| - \log |Q_{k-i}^{-1}|)
= b_1 \log |Q_k^{-1}| - \sum_{i=1}^{n-1} (b_i - b_{i+1}) \log |Q_{k-i}^{-1}| - b_n \log |Q_{k-n}^{-1}|, \quad k \geq t'_e.
$$

(28)

Define $z_k \triangleq \log |Q_{k+1}^{-1}|/\log |Q_k^{-1}|$, $k \geq 1$ and $z \triangleq \lim \inf_{k \to +\infty} z_k \geq 1$. Then inequality (28) is equivalent to

$$
1 - \epsilon + \sum_{i=1}^{n-1} (b_i - b_{i+1}) \frac{1}{\prod_{j=0}^{i} z_{k-j}} + b_n \frac{1}{\prod_{j=0}^{n} z_{k-j}} \leq b_1 \frac{1}{z_k}, \quad k \geq t'_e.
$$

Taking limit superior on both sides of the above inequality, we have

$$
1 - \epsilon + \sum_{i=1}^{n-1} (b_i - b_{i+1}) \frac{1}{z^{i+1}} + b_n \frac{1}{z^n} \leq b_1 \frac{1}{z}.
$$

Letting $\epsilon \to +\infty$ shows that $P(z) \leq 0$ and $z > 1$. This contradicts to the definition of $P(x)$. Hence, we immediately deduce that

$$
|Q_{k+1}^{-1}| \leq t_k^{\log t_k} \quad \text{i.o. a.s. on } D'.
$$

(29)

Similarly to (25)–(27), when $k$ is sufficiently large and satisfies $|Q_{k+1}^{-1}| \leq t_k^{\log t_k}$, for any
Given $t \in (t_k + 1, t_{k+1} + 1]$, we have

$$|P_t^{-1}| \leq (L_1 + L_2)^{2n} \sum_{(s_1, \ldots, s_n) \in W(t-1)} \sum_{(t_1, \ldots, t_n) \in H_n^{(1, \ldots, n)}} \prod_{k \in [1, n], s_k \neq 1} \frac{1}{ \sigma^2} \left( \frac{\gamma \cdot \log^2(s_k + 3) \cdot |P_{s_k}^{-1}|}{|P_{s_{k-1}}^{-1}|} \right)^{b_{s_{k-1}} - 1} \leq (L_1 + L_2)^{2n} \cdot (1 + \sigma^{2b_1 - 2} + \sigma^{2b_n - 2})^n \cdot (\gamma \cdot \log^2(t + 2)) \sum_{i=1}^n b_i (t + 1)^n \cdot n! \cdot \left( \frac{|Q_{k+1}^{-1}|}{|Q_k^{-1}|} \right) \sum_{i=1}^n b_i \leq (L_1 + L_2)^{2n} \cdot (1 + \sigma^{2b_1 - 2} + \sigma^{2b_n - 2})^n \cdot (\gamma \cdot \log^2(t + 2)) \sum_{i=1}^n b_i (t + 1)^n \cdot n! \cdot \sqrt[t]{k} \log t_k \sum_{i=1}^n b_i < (1 + \sigma^{-2})^t. $$

This together with (29) leads to

$$|P_t^{-1}| < (1 + \sigma^{-2})^t \quad \text{i.o. a.s. on } D'.$$

However, according to the assumption, events $\{|P_t^{-1}| < (1 + \sigma^{-2})^t\}_{t \geq 1}$ occur finitely on $D$, which arises a contradiction. Hence $\{t_k\}$ is finite on $D$ almost surely, and (23) follows. \qed

**Lemma 4.2.** Under Assumptions A1–A3, assume (16) holds and there is a set $D$ with $P(D) > 0$ such that

$$|P_t^{-1}| < (1 + \sigma^{-2})^t \quad \text{i.o. a.s. on } D. \quad (30)$$

Then,

$$\sup_{t} \sigma_t < +\infty, \quad \text{a.s. on } D. \quad (31)$$

**Proof.** Denote

$$F \triangleq \{t \geq 0 : |P_t^{-1}| \geq (1 + \sigma^{-2})^t, |P_{t+1}^{-1}| < (1 + \sigma^{-2})^{t+1}\}.$$

Firstly, by (30), for all sufficiently large $t$

$$|P_t^{-1}| < (1 + \sigma^{-2})^t, \quad \text{a.s. on } \{|F| < +\infty\} \cap D.$$
Let $\varepsilon = \frac{M}{3} \cdot \min\{(1 + \sigma)^{-1}, \sigma^3 : (1 + \sigma^2)^{-1}\}$. In view of Lemma 3.1, there is a random integer $t_1$ such that for all $t > t_1$,

$$
\lambda_{\min}(t + 1) \geq M \sum_{i=1}^{t} \frac{1}{\sigma_{i-1}} - \varepsilon t \geq M \cdot t \cdot \left( \frac{1}{\prod_{i=1}^{t} \sigma_{i-1}} \right)^{\frac{1}{t}} - \varepsilon t
$$

$$
= M \sigma \cdot t \cdot \left( \frac{1}{|P_t^{-1}|} \right)^{\frac{1}{t}} - \varepsilon t > M \sigma \cdot t \cdot \left( \frac{1}{(1 + \sigma^{-2})^t} \right)^{\frac{1}{t}} - \varepsilon t
$$

$$
> \varepsilon t, \quad \text{a.s. on } \{|F| < +\infty\} \cap D.
$$

In addition, for some integer $N > 0$,

$$
\varepsilon t > n(K_1 + K_2) + nK_2 \log^2 t \cdot (1 + \sigma)^{2b}, \quad \forall t \geq N, \tag{32}
$$

where constants $K_1, K_2$ satisfy $\|\phi(x)\| \leq K_1 + K_2|x|^b$ for $x \in \mathbb{R}$. Clearly, there exists a random integer $t_2 > t_1 + N$ such that $\sigma_{t_2} < 1 + \sigma$. Next, we show $\sigma_t < 1 + \sigma$ for all $t \geq t_2$ by induction on set $\{|F| < +\infty\} \cap D$.

Suppose that $\sigma_k < 1 + \sigma$ for some $k \geq t_2$, then (32) gives

$$
\sigma_{k+1}^2 = \sigma^2 \cdot \frac{|P_{k+1}^{-1}|}{|P_{k+1}^{-1}|} = \sigma^2 + \phi_{k+1}^T P_{k+1} \phi_{k+1} \leq \sigma^2 + \frac{\|\phi(y_{k+1})\|^2}{\lambda_{\min}(k + 1)}
$$

$$
\leq \sigma^2 + \frac{n(K_1 + K_2) + nK_2 |y_{k+1}|^{2b}}{\varepsilon k}
$$

$$
\leq \sigma^2 + \frac{n(K_1 + K_2) + nK_2 \log^2 k \cdot \sigma_k^{2b}}{\varepsilon k}
$$

$$
< (1 + \sigma)^2, \quad \text{a.s. on } \{|F| < +\infty\} \cap D.
$$

By induction, we know $\sigma_t < 1 + \sigma$ for all $t \geq t_2$. This means

$$
\sup_{t} \sigma_t < +\infty \quad \text{a.s. on } \{|F| < +\infty\} \cap D.
$$

So the remainder of the argument is focused on set $\{|F| = +\infty\} \cap D$. By Lemma 3.1 again, there exists a random integer $t_1'$ such that

$$
\lambda_{\min}(t + 1) \geq M \sum_{i=1}^{t} \frac{1}{\sigma_{i-1}} - \varepsilon t, \quad t > t_1', \quad \text{a.s.} \tag{33}
$$
On \( \{|F| = +\infty\} \cap D \), select an random integer \( k' \in F \) such that \( k' > t'_1 + N + 2 \). we now prove that for all \( t \geq k' \),

\[
\lambda_{\min}(t + 1) > \varepsilon t \quad \text{and} \quad \sigma_t < 1 + \sigma \quad \text{a.s. on} \quad \{|F| = +\infty\} \cap D.
\]  

(34)

As a matter of fact, for \( k' \in F \), we have

\[
\sigma_{k'}^2 = \sigma^2 \cdot \frac{|P_{k'-1}^{-1}|}{|P_{k'}^{-1}|} < \sigma^2 \cdot (1 + \sigma^{-2})^{k'-1} < (1 + \sigma)^2.
\]

Consequently, (33) yields

\[
\lambda_{\min}(k' + 1) \geq M \sum_{i=1}^{k'} \frac{1}{\sigma_{i-1}} - \varepsilon k' \geq M \cdot \sigma \cdot k' \cdot \left( \frac{1}{|P_{k'}^{-1}|} \right)^{\frac{1}{2k'}} - \varepsilon k'
\]

\[
\geq M \cdot \sigma \cdot k' \cdot \left( \frac{1}{P_{k'+1}^{-1}} \right)^{\frac{1}{2k'}} - \varepsilon k'
\]

\[
> M \cdot \sigma \cdot k' \cdot \left( \frac{1}{(1 + \sigma^{-2})^{k'+1}} \right)^{\frac{1}{2k'}} - \varepsilon k' \geq \varepsilon k'.
\]

Assume that for some \( j \geq k' \), (34) holds for all \( t \in [k', j] \). Then it follows that

\[
\sigma_{j+1}^2 = \sigma^2 \cdot \frac{|P_{j+2}^{-1}|}{|P_{j+1}^{-1}|} \leq \sigma^2 + \frac{\lambda_{\min}(j + 1)}{\lambda_{\min}(j + 1)} \leq \sigma^2 + \frac{m_1(K_1 + K_2) + m_1 K_2 |y_{j+1}|^{2b}}{\varepsilon_j}
\]

\[
\leq \sigma^2 + \frac{n(K_1 + K_2) + n K_2 \log^{2b} j \cdot \sigma_j^{2b}}{\varepsilon_j} < (1 + \sigma)^2 \quad \text{a.s. on} \quad \{|F| = +\infty\} \cap D.
\]

As a result,

\[
\lambda_{\min}(j + 2) \geq M \sum_{i=1}^{j+1} \frac{1}{\sigma_{i-1}} - \varepsilon(j + 1)
\]

\[
= M \sum_{i=1}^{k'} \frac{1}{\sigma_{i-1}} + M \sum_{k' < i \leq j+1} \frac{1}{\sigma_{i-1}} - \varepsilon(j + 1)
\]

\[
> M \cdot \sigma \cdot k' \cdot \left( \frac{1}{(1 + \sigma^{-2})^{k'+1}} \right)^{\frac{1}{2k'}} + \frac{M}{1 + \sigma} (j - k + 1) - \varepsilon(j + 1)
\]

\[
\geq \varepsilon(j + 1) \quad \text{a.s. on} \quad \{|F| = +\infty\} \cap D.
\]
Therefore, (34) is true for \( t = j + 1 \), and the induction is completed. So

\[
\sup_t \sigma_t < +\infty \quad \text{a.s. on } \{|F| = +\infty \} \cap D.
\]

To sum up, (31) holds as desired. \( \square \)

**Lemma 4.3.** Under Assumptions A1–A2, if (10) holds and \( \sup_t \sigma_t < +\infty \) a.s., then

\[
\sup_{t \geq 1} \frac{1}{t} \sum_{i=1}^t y_i^2 < +\infty, \quad \text{a.s.}
\]

*Proof.* Recall from [5, Lemma 3.1] that

\[
\sum_{i=0}^t \alpha_i = O \left( \log \left( 1 + \sum_{i=0}^t \phi_i^T \phi_i \right) \right), \quad \text{a.s.,}
\]

where \( \alpha_i \triangleq (1 + \phi_i^T P_i \phi_i)^{-1} (\bar{\theta} \phi_i)^2, \ i \geq 0 \). Therefore,

\[
\frac{1}{2} \sum_{i=1}^t y_i^2 - \sum_{i=0}^t w_{i+1}^2 \leq \sum_{i=0}^t (y_{i+1} - w_{i+1})^2 = \sum_{i=0}^t \alpha_i \frac{|P_{i+1}|}{|P_i|} = O \left( \sum_{i=0}^t \alpha_i \right)
\]

\[
= O \left( \log \left( 1 + \sum_{i=0}^t \phi_i^T \phi_i \right) \right).
\]

\[
\leq O \left( \log \left( 1 + k_i^2 \sum_{i=0}^t e^{2k_2 |y_i|} \right) \right)
\]

\[
\leq O \left( \log \left( 1 + k_i^2 e^{2k_2 |y_0|} + k_i^2 t \cdot e^{2k_2 \max_{1 \leq i \leq t} |y_i|} \right) \right)
\]

\[
= O(1) + O(\log t) + O \left( \left( \sum_{i=1}^t y_i^2 \right)^{\frac{1}{2}} \right), \quad \text{a.s.}
\]

Observe that \( \sum_{i=0}^t w_{i+1}^2 = O(t) \) as \( t \to +\infty \), then

\[
\frac{1}{2} \sum_{i=1}^t y_i^2 \leq O(t) + O \left( \left( \sum_{i=1}^t y_i^2 \right)^{\frac{1}{2}} \right),
\]

which implies \( \sum_{i=1}^t y_i^2 = O(t) \) almost surely. \( \square \)

With all the technique lemmas ready, Theorem 2.1 is straightforward.
Proof of Theorem 2.1. Taking account of Lemmas 4.1 and 4.2 we have

$$\sup_t \sigma_t < +\infty, \quad \text{a.s.,}$$

which leads to Theorem 2.1 directly by Lemma 4.3. □

5 Proof of Theorem 2.2

The proof is based on two lemmas below.

Lemma 5.1. If \( \lim_{t \to +\infty} \lambda_{\min}(t+1) > 0 \) and \( \sup_t \sigma_t = +\infty \) hold almost surely on a set \( D \) with \( P(D) > 0 \), then

$$\lim_{t \to +\infty} \sigma_t = +\infty, \quad \text{a.s. on } D.$$

Proof. Given a number \( z > \sigma^2 \), define

$$\Omega_{k+1} \triangleq \{ \sigma_k^2 \leq z, \sigma_{k+1}^2 \geq z \}, \quad k \geq 0.$$
Therefore,

\[
P(\Omega_{k+1} | \mathcal{F}_k^y) = P \left( \sigma^2 \cdot \frac{|P_{k+2}^{-1}|}{|P_{k+1}^{-1}|} \geq z, \sigma_k^2 \leq z \mid \mathcal{F}_k^y \right) \]

\[
\leq P \left( \sigma^2 + \frac{\|\phi(y_{k+1})\|^2}{\lambda_{\min}(k+1)} \geq z, \sigma_k^2 \leq z \mid \mathcal{F}_k^y \right) \]

\[
\leq P \left( |y_{k+1}| \geq \frac{1}{k^2} \log \left( \frac{(z - \sigma^2)\lambda_{\min}(k+1)}{k_1} \right), \sigma_k^2 \leq z \mid \mathcal{F}_k^y \right) \]

\[
= E \left\{ I \left\{ |y_{k+1}| \geq \frac{1}{k^2} \log \left( \frac{(z - \sigma^2)\lambda_{\min}(k+1)}{k_1} \right) \right\} \cdot I \{ \sigma_k^2 \leq z \mid \mathcal{F}_k^y \} \right\} \]

\[
= I \{ \sigma_k^2 \leq z \} \cdot E \left\{ I \left\{ |y_{k+1}| \geq \frac{1}{k^2} \log \left( \frac{(z - \sigma^2)\lambda_{\min}(k+1)}{k_1} \right) \right\} \mid \mathcal{F}_k^y \right\} \]

\[
= I \{ \sigma_k^2 \leq z \} \cdot \frac{1}{\sqrt{2\pi}} \int_{|x| \geq X_k} \frac{1}{k^2} \log \left( \frac{(z - \sigma^2)\lambda_{\min}(k+1)}{k_1} \right) e^{-\frac{x^2}{2}} \, dx \]

\[
\leq \frac{1}{\sqrt{2\pi}} \int_{|x| \geq X_k} e^{-\frac{x^2}{2}} \, dx = \frac{1}{\sqrt{2\pi}} \int_{|x| \geq X_k} e^{-\frac{x^2}{2}} \, dx, \quad (35) \]

where \( X_k \triangleq \frac{1}{\sqrt{2}} \frac{1}{k^2} \log \left( \frac{(z - \sigma^2)\lambda_{\min}(k+1)}{k_1} \right) \). Since \( \lim \inf_{k \to +\infty} \frac{\lambda_{\min}(k+1)}{k} > 0 \) implies \( \lim \inf_{k \to +\infty} \frac{X_k}{\log k} > 0 \), (35) yields

\[
\sum_{k=1}^{+\infty} P(\Omega_{k+1} | \mathcal{F}_k^y) \leq \sum_{k=1}^{+\infty} \frac{1}{\sqrt{2\pi}} \int_{|x| \geq X_k} e^{-\frac{x^2}{2}} \, dx < +\infty. \]

Taking account of Borel-Cantelli-Levy theorem, \( \{\Omega_k\} \) occur only finite times almost surely.

The rest of the proof is as the same of that for \([16, \text{Lemma 3.5}]\). \( \square \)

**Lemma 5.2.** Let \( \{A_k\}_{k \geq 1} \) be a sequence of events that \( A_k \triangleq \{y_k \in S_L\} \). Then, there exists a constant \( c > 0 \), which only depends on \( f_1, \ldots, f_n \), such that for all sufficiently large \( t \),

\[
\sum_{k=1}^{t} I_{A_k} \geq ct, \quad \text{a.s..} \quad (36) \]
Proof. Recall that (11) means there are two numbers \( q_1, q_2 > 0 \) such that

\[
\frac{\ell(S_L \cap [-l, l])}{l} > q_2 \quad \text{for} \quad l \geq q_1.
\]  

(37)

Since \( y_{i+1} \) is conditional Gaussian with the conditional mean \( m_i = 0 \) and variance \( \sigma_i^2 \) from (21) and (22), we compute

\[
P(A_{i+1}|F_y) = \frac{1}{\sqrt{2\pi}} \int_{|x\sigma_i| \in S_L} e^{-\frac{x^2}{2\sigma_i^2}} dx
\]

\[
\geq \frac{1}{\sqrt{2\pi}} \int_{|x\sigma_i| \in S_L, |x| \leq \frac{q_1}{\sigma}} e^{-\frac{x^2}{2\sigma_i^2}} dx
\]

\[
= \ell \left( x : |x\sigma_i| \in S_L, |x| \leq \frac{q_1}{\sigma} \right) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{q_1^2}{2\sigma_i^2}}
\]

\[
= \frac{\ell(S_L \cap [-q_1\sigma_i\sigma^{-1}, q_1\sigma_i\sigma^{-1}])}{\sigma_i} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{q_1^2}{2\sigma_i^2}}.
\]

Owing to (37) and \( \sigma_i \geq \sigma \), we immediately deduce \( P(A_{i+1}|F_y) > \frac{q_1 q_2}{\sqrt{2\pi}\sigma} e^{-\frac{q_1^2}{2\sigma^2}} \). Further, by applying the strong law of large numbers for the martingale differences, we have

\[
\sum_{k=1}^{t} (I_{A_k} - P(A_k|F_{y_{k-1}})) = o(1), \quad \text{a.s.}
\]

Then for all sufficiently large \( t \),

\[
\sum_{k=1}^{t} I_{A_k} \geq \sum_{k=1}^{t} P(A_k|F_{y_{k-1}}) - \frac{q_1 q_2}{2\sqrt{2\pi}\sigma} e^{-\frac{q_1^2}{2\sigma^2}} t \geq \frac{q_1 q_2}{2\sqrt{2\pi}\sigma} e^{-\frac{q_1^2}{2\sigma^2}} t.
\]

So Lemma 5.2 follows by letting \( c = \frac{q_1 q_2}{2\sqrt{2\pi}\sigma} e^{-\frac{q_1^2}{2\sigma^2}} \). \( \square \)

Proof of Theorem 2.2. Same as previous, we are going to show

\[
\sup_t \sigma_t < +\infty, \quad \text{a.s.}
\]  

(38)

Assume there is a set \( D \) with \( P(D) > 0 \) such that \( \sup_t \sigma_t = +\infty \) on \( D \). Note that \( y_k \in S_L \) infers

\[
\sigma_k^2 = \frac{|P_{k+1}^{-1}|}{|P_k^{-1}|} \leq \sigma^2 + \frac{\|\phi(y_k)\|^2}{\lambda_{\min}(k)} \leq \sigma^2 + L^2,
\]
then for any \( t \geq 1 \),
\[
\sum_{k=1}^{t} I_{\{\sigma_k \leq \sqrt{\sigma^2 + L^2}\}} \geq \sum_{k=1}^{t} I_{A_k},
\]
(39)
where \( I_{A_k}, k \in [1, t] \) are defined in Lemma 5.2. Take \( \varepsilon = \frac{1}{4}(\sigma^2 + L^2)^{-\frac{1}{2}}MC \). By Proposition 3.1 Lemma 5.2 and (39), for all sufficiently large \( t \), we have
\[
\lambda_{\min}(t + 2) \geq M \sum_{i=1}^{t+1} \frac{1}{\sigma_i - 1} - \varepsilon(t + 1)
\geq M \cdot \frac{ct}{\sqrt{\sigma^2 + L^2}} - \frac{1}{4}(\sigma^2 + L^2)^{-\frac{1}{2}}MC(t + 1)
\geq \frac{Mct}{2\sqrt{\sigma^2 + L^2}}, \quad \text{a.s.}
\]
This means \( \liminf_{t \to +\infty} \frac{\lambda_{\min}(t+1)}{t} > 0 \) almost surely. So, Lemma 5.1 gives
\[
\lim_{t \to +\infty} \sigma_t = +\infty, \quad \text{a.s. on } D.
\]
According to (39), it turns out that
\[
\limsup_{t \to +\infty} \frac{\sum_{k=1}^{t} I_{A_k}}{t} \leq \limsup_{t \to +\infty} \frac{\sum_{k=1}^{t} I_{\{\sigma_k \leq \sqrt{\sigma^2 + L^2}\}}}{t} = 0, \quad \text{a.s. on } D,
\]
which contradicts to (36). We thus conclude (38) and Lemma 4.3 applies. \( \square \)

A Proof of Lemma 3.1 and Proposition 3.1

To show Lemma 3.1, we need a simple fact.

Lemma A.1. Assume the conditions of Lemma 3.1 hold, then
\[
\sup_{t \geq 1} \frac{1}{t} \sum_{i=1}^{t} y_i^2 < +\infty \quad \text{is equivalent to} \quad \sup_{t} \sigma_t < +\infty \quad \text{a.s.}
\]

Proof. Denote
\[
W \triangleq \left\{ \sup_{t \geq 1} \frac{1}{t} \sum_{i=1}^{t} y_i^2 < +\infty \right\}.
\]

21
By Hadamard inequality, for all sufficiently large $t$,
\[
|P_t^{-1}| \leq \prod_{j=1}^{n} \left(1 + \frac{1}{\sigma^2} \sum_{i=0}^{t-1} f_{ij}^2(y_i)\right) = \prod_{j=1}^{n} \left(1 + O(t) + O\left(\sum_{i=0}^{t-1} y_i^2b\right)\right)
\leq \prod_{j=1}^{n} \left(1 + O(t) + O\left((\sum_{i=0}^{t-1} y_i^2)^b\right)\right) = O\left(t^{(b+1)n}\right), \quad \text{on } W. \tag{40}
\]

So, applying Lemma 4.2 infers
\[
\sup_t \sigma_t < +\infty \quad \text{a.s. on } W.
\]
Since the converse part is verified by Lemma 4.3, the lemma is proved.

**Proof of Lemma 3.1.** Let $W$ be defined in Lemma A.1 and $\varepsilon = \frac{M}{2\sigma}$. By Proposition 3.1,
\[
\lambda_{\min}(t + 1) \geq M \sum_{i=1}^{t} \frac{1}{\sigma_i} - M \frac{t}{2\sigma},
\]
\[
\geq M\sigma^{-1}t \left(\frac{1}{|P_t^{-1}|}\right)^{\frac{1}{2}} - M \frac{t}{2\sigma}, \quad \text{a.s. on } W, \tag{41}
\]
where $t$ is sufficiently large. Combining (40) and (41), we obtain
\[
\liminf_{t \to +\infty} \frac{\lambda_{\min}(t + 1)}{t} > 0, \quad \text{a.s. on } W. \tag{42}
\]

Define a martingale difference sequence
\[
Z_i = \frac{1}{i} (y_i^2 - E(y_i^2|F_{i-1}^y)) = \frac{1}{i} (y_i^2 - \sigma_i^2), \quad i \geq 1.
\]

In view of Lemma A.1,
\[
\sum_{i=1}^{+\infty} E(Z_i^2|F_{i-1}^y) = \sum_{i=1}^{+\infty} \frac{2\sigma_i^4}{i^2} < +\infty, \quad \text{a.s. on } W.
\]

By [2, Theorem 2.7], we deduce that $\sum_{i=1}^{+\infty} Z_i$ converges almost surely on $W$. This, together with Kronecker Lemma, leads to
\[
\sum_{i=1}^{t} (y_i^2 - \sigma_i^2) = o(t), \quad \text{a.s. on } W. \tag{43}
\]
By \((16), (24), (42)\) and Lemma A.1 as long as \(t\) is sufficiently large, \[
\sigma_{t+1}^2 \leq \sigma^2 + \frac{\|\phi(y_{t+1})\|^2}{\lambda_{\min}(t + 1)} = \sigma^2 + O\left(\frac{y_{t+1}^{2b} \log(t + 1)}{t}\right) = \sigma^2 + o(1), \quad \text{a.s. on } W.
\]
Recall that \(\sigma_t \geq \sigma\), therefore \[
\lim_{t \to \infty} \sigma_t = \sigma, \quad \text{a.s. on } W. \tag{44}
\]
According to (43) and (44), we derive \[
\lim_{t \to \infty} \frac{1}{t} \sum_{i=1}^{t} y_i^2 = \lim_{t \to \infty} \frac{1}{t} \sum_{i=1}^{t} \sigma_{i-1}^2 = \sigma^2, \quad \text{a.s. on } W,
\]
which together with (42) gives Lemma 3.1. \(\square\)

**Proof of Proposition 3.1.** The proof is a minor modification of that of [15]. So, we shall follow the notations of [15] and only write the differences. Let \(\delta, U_x\) and \(S_j(q)\) be defined in [15, Section 3.1]. Then, \[
\inf_{\|x\|=1} \ell \left( \{ y : |\phi^T(y)x| > \delta \} \cap \bigcup_{j=1}^{p} S_j(q) \right) > 0.
\]
For any \(x \in \mathbb{R}^\alpha\) with \(\|x\| = 1\), \[
P(y_i \in U_x | \mathcal{F}_{i-1}^y) = \frac{1}{\sqrt{2\pi}} \int_{s \sigma_{i-1} \in U_x} e^{-\frac{s^2}{2}} ds
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_{s \sigma_{i-1} \in U_x, |s| \leq \sigma^{-1} R} e^{-\frac{s^2}{2}} ds
\]
\[
\geq \frac{1}{\sqrt{2\pi}} \frac{\ell(U_x)}{\sigma_{i-1}} \cdot e^{-\frac{(\sigma^{-1} R)^2}{2}}
\]
\[
\geq \frac{1}{\sigma_{i-1}} \cdot \frac{e^{-\frac{(\sigma^{-1} R)^2}{2}}}{\sqrt{2\pi}} \inf_{\|x\|=1} \ell \left( \{ y : |\phi^T(y)x| > \delta \} \cap \bigcup_{j=1}^{p} S_j(q) \right)
\]
\[
= \frac{k_1}{\sigma_{i-1}}, \quad \text{(45)}
\]
where \( R = \text{dist}(\bigcup_{j=1}^{p} S_j(q)) \).

Now, we modify Section 3.3 of [15] to deduce out result. To this end, for any \( \epsilon > 0 \), recalling the definition of \( g_\epsilon \) in [15, Lemma 3.12], the strong large number laws for martingale differences shows that all \( g_\epsilon \in G_\epsilon \) fulfill

\[
\lim_{t \to \infty} \frac{1}{t} \sum_{i=1}^{t} g_\epsilon(i) + \epsilon = 0, \quad \text{a.s.}
\]

Since there are only finite \( U_\epsilon \) satisfying \( U_\epsilon \subset \bigcup_{j=1}^{p} S_j(q) \), we conclude

\[
\lim_{t \to \infty} \inf_{U_\epsilon \subset \bigcup_{j=1}^{p} S_j(q)} \frac{1}{t} \sum_{i=1}^{t} g_\epsilon(i) = -\epsilon, \quad \text{a.s.,}
\]

which, together with [15, Lemma 3.12(ii)], yields

\[
\liminf_{t \to +\infty} \inf_{\|x\|=1} \frac{1}{t} \sum_{i=1}^{t} g_x(i) \geq \liminf_{t \to +\infty} \inf_{\|x\|=1} \frac{1}{t} \sum_{i=1}^{t} g_\epsilon^x(i) \\
\geq \liminf_{t \to \infty} \inf_{U_x \subset \bigcup_{j=1}^{p} S_j(q)} \frac{1}{t} \sum_{i=1}^{t} g_\epsilon(i) \\
= -\epsilon, \quad \text{a.s.}
\]

Furthermore, by the arbitrariness of \( \epsilon \), we obtain

\[
\liminf_{t \to +\infty} \inf_{\|x\|=1} \frac{1}{t} \sum_{i=1}^{t} g_x(i) \geq 0 \quad \text{a.s..} \tag{46}
\]

Combining (46) and (15), for any given \( \epsilon > 0 \), there exists a random integer \( T > 0 \) such that for all \( t > T \),

\[
\frac{1}{t} \sum_{i=1}^{t} I_{\{y_i \in U_x\}} > \frac{1}{t} \sum_{i=1}^{t} P(y_i \in U_x|\mathcal{F}_{i-1}^y) - \frac{\sigma_i^2}{\delta^2} \cdot \epsilon \\
\geq \frac{1}{t} \sum_{i=1}^{t} \frac{k_i}{\sigma_{i-1}} - \frac{\sigma_i^2}{\delta^2} \cdot \epsilon.
\]

Then

\[
\lambda_{\min}(t+1) = \inf_{\|x\|=1} x^\top \left( I_n + \frac{1}{\sigma^2} \sum_{i=0}^{t} \phi_i \phi_i^\top \right) x \geq \frac{1}{\sigma^2} \sum_{i=1}^{t} (\phi_i^\top(y_i)x)^2 \\
\geq \frac{\delta^2}{\sigma^2} \cdot \left( \sum_{i=1}^{t} \frac{k_i}{\sigma_{i-1}} - \frac{\sigma_i^2}{\delta^2} \cdot \epsilon t \right) = \frac{\delta^2}{\sigma^2} \cdot k_1 \sum_{i=1}^{t} \frac{1}{\sigma_{i-1}} - \epsilon t.
\]

24
Let $M = \frac{\delta^2}{\sigma^2} \cdot k_1$ and Proposition 3.1 follows. □

References

[1] K. J. Åström and B. Wittemark, *Adaptive Control*, Addison-Wesley: Reading, MA, 1995.

[2] H. Chen and L. Guo, *Identification and Stochastic Adaptive Control*, Birkhauser: Boston, MA, 1991.

[3] F. Eicker, *Asymptotic normality and consistency of the least squares estimators for families of linear regressions*, Annals of Mathematical Statistics, 39 (1963), pp. 447–456.

[4] G. C. Goodwin, *Adaptive Filtering, Prediction and Control*, Prentice-Hall, Englewood Cliffs, NJ, 1984.

[5] L. Guo, *Convergence and logarithm laws of self-tuning regulators*, Automatica, 31 (1995), pp. 435–450.

[6] L. Guo, *On critical stability of discrete-time adaptive nonlinear control*, IEEE Transactions on Automatic Control, 42 (1997), pp. 1488–1499.

[7] L. Guo and C. Wei, *LS-based discrete-time adaptive nonlinear control Feasibility and limitations*, Science in China Series E-Technological Sciences, 39 (1996), pp. 255–269.

[8] C. Huang, and L. Guo, *On feedback capability for a class of semiparametric uncertain systems*, Automatica, 48 (2012), pp. 873–878.
[9] P. A. Ioannou and J. Sun, *Robust Adaptive Control*, Prentice-Hall: Englewood Cliffs, NJ, 1996.

[10] P. V. Kokotović and M. Arcak, *Constructive nonlinear control: Progress in the 90s*, in Proc. 14th IFAC World Congr., (1999), pp. 49–77.

[11] M. Krstić, I. Kanellakopoulos, and P. V. Kokotović, *Nonlinear and Adaptive Control Design*, Wiley, New York, 1995.

[12] T. L. Lai and C. Z. Wei, *Least squares estimates in stochastic regression models with applications to identification and control of dynamic systems*, Annals of Statistics, 10 (1982), pp. 154–166.

[13] C. Li and J. Lam, *Stabilization of discrete-time nonlinear uncertain systems by feedback based on LS algorithm*, SIAM Journal on Control and Optimization, 51 (2013), pp. 1128–1151.

[14] C. Li, L.-L. Xie, and L. Guo, *A polynomial criterion for adaptive stabilizability of discrete-time nonlinear systems*, Communications in Information and Systems, 6 (2006), pp. 273–298.

[15] Z. Liu and C. Li, *Asymptotic behavior of least squares estimator for nonlinear autoregressive models*, arXiv preprint arXiv:1909.06773, (2019).

[16] Z. Liu and C. Li, *Is it possible to stabilize discrete-Time parameterized uncertain systems growing exponentially fast?*, SIAM Journal on Control and Optimization, 57 (2019), pp. 1965–1984.

[17] Z. Liu and C. Li, *Stabilizability theorem of discrete-time nonlinear systems with scalar parameters*, Control Theory and Applications, 36 (2019), pp. 1929-1935.
[18] H. B. Ma, *Further results on limitations to the capability of feedback*, International Journal of Control, 81 (2008), pp. 21–42.

[19] V. Sokolov, *Adaptive stabilization of parameter-affine minimum-phase plants under lipschitz uncertainty*, Automatica, 73 (2016), pp. 64–70.

[20] J. Sternby, *On consistency for the method of least squares using martingale theory*, IEEE Transactions on Automatic Control, 22 (1997), pp. 346–352.

[21] G. Tao and P. V. Kokotovic, *Adaptive Control of Systems with Actuator and Sensor Nonlinearities*, John Wiley and Sons Inc., 1996.

[22] L.-L. Xie and L. Guo, *Fundamental limitations of discrete-time adaptive nonlinear control*, IEEE Transactions on Automatic Control, 44 (1999), pp. 1777–1782.

[23] L.-L. Xie and L. Guo, *Adaptive control of discrete-time nonlinear systems with structural uncertainties*, In: Lectures on Systems, Control, and Information, AMS/IP, (2000), pp. 49–90.

[24] L.-L. Xie and L. Guo, *How much uncertainty can be dealt with by feedback?*, IEEE Transactions on Automatic Control, 45 (2000), pp. 2203–2217.

[25] S. Xu and C. Li, *On instability of LS-based self-tuning systems with bounded disturbances*, Systems and Control Letters, 129 (2019), pp. 51–55.

[26] F. Xue and L. Guo, *On limitations of the sampled-data feedback for nonparametric dynamical systems*, Journal of Systems Science and Complexity, 15 (2002), pp. 225–249.

[27] F. Xue, L. Guo and M. Y. Huang, *Towards understanding the capability of adaptation for time-varying systems*, Automatica, 37 (2001), pp. 1551–1560.
[28] Y. X. ZHANG AND L. GUO, *A limit to the capability of feedback*, IEEE Transactions on Automatic Control, 47 (2002), pp. 687–692.