Exploring the diffeomorphism-invariant Hilbert space of a scalar field

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Received 4 May 2007, in final form 31 July 2007
Published 30 August 2007
Online at stacks.iop.org/CQG/24/4601

Abstract
As a toy model for the implementation of the diffeomorphism constraint, the interpretation of the resulting states and the treatment of ordering ambiguities in loop quantum gravity, we consider the Hilbert space of spatially diffeomorphism-invariant states for a scalar field. We give a very explicit formula for the scalar product on this space and discuss its structure. Then we turn to the quantization of a certain class of diffeomorphism-invariant quantities on that space and discuss in detail the ordering issues involved. On a technical level these issues bear some similarity to those encountered in full loop quantum gravity.

PACS number: 04.60.Pp

1. Introduction

The space of spatially diffeomorphism-invariant states, $\mathcal{H}_{\text{diff}}$, is important in loop quantum gravity (LQG): it may be home to the physical states of the theory, and it is the space on which the Hamiltonian constraint, arguably the most important operator of the theory, is defined. We feel however that $\mathcal{H}_{\text{diff}}$ is not very well understood. For example, elements of $\mathcal{H}_{\text{diff}}$ are obtained by a group averaging procedure that is quite subtle [1, 2]. Very roughly speaking, they can be labeled by, among other things, diffeomorphism equivalence classes of graphs, i.e. objects that are hard to describe explicitly. Also the physical meaning of the states is rather unclear. One would need operators to 'probe' the states, operators that demonstrably correspond to classical quantities of interest. But besides the total spatial volume and the Hamilton constraint with constant lapse, there are no candidates for such operators.

Thiemann [3, 4] has given a prescription of how to quantize a large class of quantities on $\mathcal{H}_{\text{diff}}$, but there are issues (e.g. [5]) and ambiguities whose physical meaning and mathematical consequences are not clear. There are very interesting and encouraging results on some of them [6, 7]; see also [8] for a concise discussion of the ambiguities. But given the importance, specifically, of the correct implementation of the Hamiltonian constraint, we think it fair to say that one knows too little.
Here, we would like to start the consideration of a toy model in which the above-mentioned points can be studied with relative ease. More precisely, we will study a scalar field in the polymer representation [9–11]. We will not discuss the dynamics of the field. See [12] for a treatment of a physical scalar field coupled to gravity.

The basic field quantities derived from the canonical pair \((\phi, \pi)\) that are subject to quantization are

\[
T_{x,\alpha} = e^{i\alpha \phi(x)}, \quad \pi(f) = \int \pi(y) f(y), \quad \alpha \in \mathcal{I}.
\]

We will consider two cases, \(\mathcal{I} = \mathbb{Z}, \mathbb{R}\). In the former it may be more natural to consider \(T_{x,\alpha}\) as functions of the U(1) valued field \(T_{x,1}\), forgetting about \(\phi\) altogether.

In the first part of the paper we discuss the construction of the diffeomorphism-invariant Hilbert space for the field in analogy with that of LQG. Making a certain assumption on the class of allowed diffeomorphisms enables us to characterize states and a scalar product very explicitly. It turns out that the Hilbert space can be characterized as a Fock space in a natural way.

In the second part of the paper, we will be concerned with the quantization of certain diffeomorphism-invariant quantities,

\[
L_{\alpha} \equiv \int \pi(x) e^{i\alpha \phi(x)} \quad \alpha \in \mathcal{I}.
\]

One reason to consider these quantities is that they are easily expressed in terms of those in (1), and thus quantization may be expected to be relatively straightforward.

Another reason is that the ordering problems that can be expected for the quantization of (2) are analogous to (though much simpler than) those encountered for the ‘FEE’ term in the Hamiltonian constraint.

Finally, an important reason is that they form an algebra under Poisson brackets:

\[
\{L_{\alpha}, L_{\alpha'}\} = i(\alpha - \alpha')L_{\alpha + \alpha'}.
\]

For \(\mathcal{I} = \mathbb{Z}\) one recognizes the Witt algebra, the algebra of vector fields on the circle. Direct calculation confirms that in this case \(L_{\alpha}\) generate diffeomorphisms of the target U(1). For \(\mathcal{I} = \mathbb{R}\) one may therefore think of \(L_{\alpha}\) as generating ‘diffeomorphisms’ of \(\mathbb{R}\), but we will not bother to give this a technical meaning.

Thus for the quantities (2) there is a simple and thorough test of the quantization: is (3) reproduced (in appropriate commutation and adjointness relations)?

We will unfortunately not be able to find a complete solution to this quantization problem in the present paper. That certainly does not mean that there is none. The structures and obstruction we find will however show that this problem is subtle and merits further investigation.

The paper is organized as follows. Section 2 recalls the kinematical quantization of a scalar field in LQG. Section 3 gives an explicit description of the Hilbert space of diffeomorphism-invariant states \(\mathcal{H}_{\text{diff}}\) for the scalar field. In section 4, we will be concerned with the quantization of \(L_{\alpha}\). We end the paper with a discussion of results and possible future work in section 5. An appendix contains several longer computations.

2. Kinematical quantization

In this section, we recall the standard quantization of a scalar field used in LQG [9–11]. Note that there are other diffeomorphism-invariant representations [13, 14] that we will not make

1 As will become apparent when we introduce the representation for these quantities, it is mathematically more appropriate to describe \(\mathcal{I}\) as the Pontryagin dual of U(1) and of \(\mathbb{R}\) (the Bohr compactification of \(\mathbb{R}\)).

2 After all, in this case \(T_{x,\alpha}\) are not sufficient to reconstruct \(\phi\).
Exploring the diffeomorphism-invariant Hilbert space of a scalar field

use of. We will however use slightly non-standard notation: denote by \( \lambda, \lambda' \ldots \) functions from \( \Sigma \) to \( I \) that are non-zero at most at a finite number of points, and for a given point \( x \in \Sigma \), let \( \lambda(x), \lambda'(x) \ldots \) denote their values at \( x \). Let \( \text{Cyl} \) be the free linear space over such functions and write the generators as \( |\lambda\rangle, |\lambda'\rangle, \ldots \). An inner product on \( \text{Cyl} \) is given by

\[
\langle \lambda | \lambda' \rangle = \prod_{x \in \Sigma} \delta(\lambda_x, \lambda_x'),
\]

(4)

and linear extension, and \( \mathcal{H}_{kin} \) is the closure of \( \text{Cyl} \) under the associated norm. A representation of the basic quantities is given by

\[
\hat{T}_x, \lambda | \lambda \rangle = | \lambda + \delta_x \rangle, \quad \hat{\pi}(f) | \lambda \rangle = \sum_{x \in \Sigma} \lambda_x f(x) | \lambda \rangle,
\]

where we have denoted with \( \delta_x \) the Kronecker-delta \( \delta(\cdot, x) \).

Diffeomorphisms \( \varphi \) act as unitary operators \( U_\varphi \) on \( \mathcal{H}_{kin} \): denote with \( \varphi^* \lambda \) the pullback \((\lambda \circ \varphi)\) under diffeomorphisms, then \( U_\varphi | \lambda \rangle = | \varphi^{-1} \circ \lambda \rangle \).

3. The diffeomorphism-invariant Hilbert space \( \mathcal{H}_{diff} \)

In this section, we will define a Hilbert space of spatially diffeomorphism-invariant states for the scalar field. First, it might be surprising that there would be something non-trivial left after implementation of the diffeomorphism constraint\(^3\): after all, heuristic counting would suggest that since the phase space is coordinatized by two fields, the constraint hypersurface should be by one and the reduced phase space by none. In appendix A, we will however give a (heuristic) argument to the effect that the reduced phase space is in fact rather large.

We will define the Hilbert space \( \mathcal{H}_{diff} \) of diffeomorphism-invariant states in analogy with full LQG [1]. There it is part of the dual of \( \text{Cyl} \) equipped with a scalar product. The latter is defined using group averaging, which gives a map from \( \text{Cyl} \) to diffeomorphism-invariant elements of \( \text{Cyl}^* \). On a heuristic level that map is obtained by averaging over all diffeomorphisms,

\[
(\Gamma \Psi)(\Phi) = (\text{Vol}(\text{Diff}))^{-1} \int_{\text{Diff}} D\varphi \langle U_\varphi \Psi | \Phi \rangle.
\]

The actual formula is a bit more subtle. To spell it out let us introduce some concepts (we follow the exposition in [2]): let \( \text{Diff} \) be a group of diffeomorphisms (analytic, semi-analytic, etc), and given a graph \( \gamma \), denote by \( \text{Diff}_\gamma \) the subgroup of diffeomorphisms mapping \( \gamma \) onto itself. Narrowing it down even further, let \( \text{TDiff}_\gamma \) be the subgroup of \( \text{Diff} \) which is the identity on \( \gamma \). The quotient \( \text{GS}_\gamma \doteq \text{Diff}_\gamma / \text{TDiff}_\gamma \) is called the set of graph symmetries. With these definitions, the actual result for a function \( \Psi_\gamma \), cylindrical on \( \gamma \) is

\[
(\Gamma \Psi_\gamma)(\Phi) = \sum_{\varphi_1 \in \text{Diff}/\text{Diff}_\gamma} \frac{1}{|\text{GS}_\gamma|} \sum_{\varphi_2 \in \text{GS}_\gamma} \langle \varphi_1 \circ \varphi_2 | \Psi_\gamma \rangle | \Phi \rangle.
\]

(5)

The division by the order of \( \text{GS}_\gamma \) in (5) is necessary for consistency in the case of full LQG, but for the scalar field alone there is no clear justification. Thus, to be most general, we will consider all rigging maps

\[
(\Gamma_F \Psi_\gamma)(\Phi) \doteq \sum_{\varphi_1 \in \text{Diff}/\text{Diff}_\gamma} F(|\text{GS}_\gamma|) \sum_{\varphi_2 \in \text{GS}_\gamma} \langle \varphi_1 \circ \varphi_2 | \Psi_\gamma \rangle | \Phi \rangle
\]

(6)

\(^3\) We are grateful to T Thiemann for pointing this important issue out to us.
with $F(n)$ a strictly positive function on $\mathbb{N}$. Their images $\Gamma_F(Cyl)$ are all the same, but they induce different inner products

$$(\Gamma_F \Psi | \Gamma_F \Psi')_F = (\Gamma_F \Psi)(\Psi')$$

and hence different closures $\mathcal{H}_F^{\text{diff}}$. \footnote{To be precise and avoid confusion: $\mathcal{H}_F^{\text{diff}}$ are certainly all equivalent as Hilbert spaces, but they may differ as closures of $Cyl^*$.}

Our next task will be to describe the structure of these Hilbert spaces much more explicitly in the case of the scalar field. The results will be contained in proposition 3.2 and corollary 3.3. Before getting there, we need some more preparations: we have to talk about the class of diffeomorphisms that we admit. Several groups have been considered (piecewise linear, semi-analytic, analytic, smooth, etc.) and we will not describe them in any detail. Rather we will base the rest of the paper on the following assumption:

**Assumption 3.1.** For any two ordered sets $(p_1, \ldots, p_n), (p_1', \ldots, p_n')$ of $n$ points of $M$ there is $\varphi \in \text{Diff}$ such that $\varphi(p_i) = p_i', i = 1, \ldots, n$.

This assumption is certainly wrong for dim $M = 1$. For analytic diffeomorphisms, it seems hard to prove or disprove it, even in special cases. On the other hand it is probably true for the smooth category and dim $M > 1$. \footnote{Let us sketch a proof in the latter case: given ordered sets of $n$ points $(p_i), (p_i')$, we want to find $\varphi$ such that $\varphi(p_i) = p_i'$.}

The important consequence of the assumption is the following: the diffeomorphism-invariant information contained in a set of pairs of points of $\Sigma$ and elements of $\mathcal{I}$ is reduced to how many times each element of $\mathcal{I}$ showed up. This simplifies the description of the diffeomorphism-invariant Hilbert space drastically.

To make this precise let us introduce some notation: let $\mathcal{I}^*$ denote $\mathcal{I}\setminus 0$ and $\mathcal{N}$ the set of functions $\mathcal{N} : \mathcal{I}^* \rightarrow \mathbb{N}$, zero on all but finitely many elements of $\mathcal{I}^*$. Evaluation of such a function $\mathcal{N}$ on $\lambda \in \mathcal{I}^*$ will be denoted by $\mathcal{N}_\lambda$. Consider the free vector space $V_{\mathcal{N}}$ over the symbols $|\mathcal{N}\rangle, \mathcal{N} \in \mathcal{N}$ and equip it with a scalar product by stipulating

$$(\mathcal{N} | \mathcal{N}') = \prod_{\lambda \in \mathcal{I}^*} \mathcal{N}_\lambda \delta(\lambda, \lambda'),$$

and (sesqui-)linear extension. Completion gives a Hilbert space $V_{\mathcal{N}}$ with orthogonal basis $\{|\mathcal{N}\rangle\}_{\mathcal{N} \in \mathcal{N}}$.

We can identify elements of $V_{\mathcal{N}}$ with diffeomorphism-invariant elements of $Cyl^*$ as follows: for a basis state $|\lambda\rangle$, let

$$\mathcal{N}^{(\lambda)}_{\lambda'} = \sum_{\lambda \in \Sigma} \delta(\lambda, \lambda'),$$

which is nothing else than saying that $\mathcal{N}^{(\lambda)}_{\lambda'}$ is the number of points in the graph of $\lambda$ with label $\lambda'$. Then for $(\mathcal{N} | (\lambda))$ set

$$(\mathcal{N} | (\lambda)) = (\mathcal{N} | \mathcal{N}^{(\lambda)})$$

and extend linearly to all of $Cyl$ and anti-linear to all of $V_{\mathcal{N}}$.

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4 To be precise and avoid confusion: $\mathcal{H}_F^{\text{diff}}$ are certainly all equivalent as Hilbert spaces, but they may differ as closures of $Cyl^*$.

5 Let us sketch a proof in the latter case: given ordered sets of $n$ points $(p_i), (p_i')$, we want to find $\varphi$ such that $\varphi(p_i) = p_i'$. (1) We can find mutually non-intersecting continuous paths $e_i$ connecting $p_i$ with $p_i'$ (Proof by induction). (2) Blow up paths $e_i$ to mutually non-intersecting tubes $T_i$. (3) Construct to each $T_i$ a diffeomorphism $\varphi_i$ that is the identity outside $T_i$ and maps $p_i$ to $p_i'$. Then $\varphi = \varphi_1 \circ \varphi_2 \cdots$ does the job.
Proposition 3.2. Provided assumption 3.1 holds, the explicit form of the rigging map \( \Gamma_1 \) is
\[ \Gamma_1(\lambda) = (N^{(\lambda)})'. \]
The inner product (7) induced by this map coincides with the inner product (8).

The proof is contained in appendix B.

Let us define some operators on \( \mathcal{H}^{\text{diff}}_{\lambda} \) by giving their matrix elements in the basis (8):
\[
\hat{N}_a |N\rangle = N_a |N\rangle, \quad \hat{\tilde{N}} = \sum_{a \in I^*} \hat{N}_a, \quad \hat{\tilde{N}}! |N\rangle = \left( \sum_{a \in I^*} N_a \right) |N\rangle.
\]

One can easily check that \( \hat{N}_a \) (and hence \( \hat{\tilde{N}} \)) are symmetric on \( V_N \).

Corollary 3.3. Provided assumption 3.1 holds, the general rigging map is given by \( \Gamma_F(\lambda') = (N^{(\lambda')})' \). The resulting scalar product is
\[
(\cdot | \cdot)_F = (|1/\hat{\tilde{N}}\rangle | \cdot\rangle).
\]

Proof. It is clear from definition (6) of \( \Gamma_F \) that \( \Gamma_F = F(\hat{\tilde{N}}) \circ \Gamma_1 \). Then for the inner product (7) induced on \( V_N \) we have
\[
(\Gamma_F(\lambda)|\Gamma_F(\lambda')\rangle)_F = (\Gamma_F(\lambda)|\lambda')_F) \iff (F(\hat{\tilde{N}}! N^{(\lambda)}|F(\hat{\tilde{N}}! N^{(\lambda')})_F)
\]
\[
= (F(\hat{\tilde{N}}! N^{(\lambda)}|N^{(\lambda')})_F).
\]

But since the image of \( \Gamma_F \) is Cyl, and \( F(\hat{\tilde{N}}!) \) maps Cyl to itself and is invertible, we can conclude
\[
(\Psi |\Psi')_F = (\Psi |1/\hat{\tilde{N}}\Psi')
\]
for any \( \Psi, \Psi' \in \text{Cyl} \).

To finish our exposition, we define some additional natural operators on \( \mathcal{H}^{F_{\text{diff}}}_{\lambda} \): for \( \alpha \in I^* \),
\[
a_a |N\rangle = N_a |N - \delta_a\rangle, \quad a_a^\dagger |N\rangle = |N + \delta_a\rangle.
\]

\( a_a, a_a^\dagger \) are mutually adjoint. Furthermore one finds the familiar relations
\[
[a_a^\dagger, a_{a'}^\dagger] = [a_a, a_{a'}] = 0, \quad[a_a^\dagger, a_{a'}] = \delta(\alpha, \alpha') id
\]
(11)
\[
a_a^\dagger a_a = \hat{N}_a.
\]
(12)

for \( \alpha, \alpha' \in I^* \). Let us conclude with a few remarks.

1. The picture that we obtained is very simple and combinatorial. All that survives group averaging is multiplicity of labels. This means in particular that on the diffeomorphism-invariant level, the quantum theory for the scalar field has 'forgotten' about the dimension of the underlying spatial manifold \( \Sigma \) whenever assumption 3.1 holds.

2. There is a natural way to write \( \mathcal{H}^{\text{diff}}_\lambda \) as a Fock space: let \( h = L^2(I^*, d\mu_{\text{discr}}) \), where \( d\mu_{\text{discr}} \) is the discrete measure on \( I^* \). Then there is a natural isomorphism between the eigenspace to the eigenvalue \( n \) of \( \hat{\tilde{N}} \) and the symmetric tensor product \( \otimes_n^2 h \), and hence between \( \mathcal{H}^{\text{diff}}_\lambda \) and \( \mathcal{F}_S(h) \). The natural annihilation and creation operators on \( \mathcal{F}_S(h) \), evaluated on elements \( \delta(\cdot, \alpha) \) of \( h \) are mapped under this isomorphism to \( a_a, a_a^\dagger \) defined above.

However, we should point out that the interpretation of the \( \lambda \) labels is very different from that of the (Fourier-)\( h \) labels encountered in the standard Fock quantization of a free scalar field. \( \lambda \in I \) are representation labels for the group that plays the role of 'target space', whereas \( h \) of the standard quantization come from Fourier transform on the spatial manifold \( \Sigma \).
(3) We note that $F(\hat{N})^{1/2}$ is a unitary map from $\mathcal{H}_{\text{diff}}^1$ to $\mathcal{H}_{\text{diff}}^F$. Thus the annihilation and creation operators etc can be carried over from $\mathcal{H}_{\text{diff}}^1$ to any $\mathcal{H}_{\text{diff}}^F$. (However this is somewhat unnatural, as the unitary map does not come from a unitary map on the Hilbert space $\mathcal{H}$.)

(4) Finally equation (8) suggests that $\mathcal{H}_{\text{diff}}^1$ can be written as an infinite direct product of Hilbert spaces. We will not investigate this further.

4. Quantization on $\mathcal{H}_{\text{diff}}$

In this section, we will discuss the quantization of the quantities $L_\alpha$ in (2). That is we are looking for operators $\hat{L}_\alpha$ (on $\mathcal{H}_{\text{kin}}$ or $\mathcal{H}_{\text{diff}}^F$) such that

$$[\hat{L}_\alpha, \hat{L}_{\alpha'}] = (\alpha' - \alpha)\hat{L}_{\alpha+\alpha'}, \quad (\hat{L}_\alpha)^\dagger = \hat{L}_{-\alpha}. \quad (13)$$

The obvious difficulty of this endeavor is that $L_\alpha$ depend on both, configuration and momentum variables, and hence a choice of ordering will have to be made.

In the present paper, we will consider one particular ordering, the symmetric (or Weyl-) ordering. We will not define this order in any generality, but instead give an example from quantum mechanics that nevertheless contains all we need to know here: the symmetric ordering for a phase-space function $f(x)p$ is

$$\hat{f}(x)p = \{\hat{p}f(\hat{x}) + f(\hat{x})\hat{p}\} = \{f(\hat{x})\hat{p} + [\hat{f}(\hat{x})\hat{p}]\} = f(\hat{x})\hat{p} + \frac{1}{2}f'(\hat{x}). \quad (14)$$

It is instructive to start with trying to quantize $L_\alpha$ on $\mathcal{H}_{\text{kin}}$. Consideration of the actions of $\hat{\pi}(x)$ and $\hat{T}_{x,\lambda}$ on $\mathcal{H}_{\text{kin}}$, and ordering $\pi$ to the right gives an operator

$$\hat{S}_\alpha(\lambda) \doteq \sum_{x \in \Sigma} \lambda_x |\lambda + \alpha \delta^x\rangle, \quad (15)$$

where we have denoted with $\delta^x$ the Kronecker-delta $\delta(\cdot, x)$. A check on the commutation relation turns out positive: since

$$\hat{S}_\alpha \hat{S}_{\alpha'} |\lambda\rangle = \sum_{x \neq x'} \lambda_x\lambda_{x'} |\lambda + \alpha \delta^x + \alpha' \delta^{x'}\rangle + \sum_x \lambda_x (\lambda_{x' + \alpha'} |\lambda + (\alpha + \alpha') \delta^x\rangle,$$

we find that

$$[\hat{S}_\alpha, \hat{S}_{\alpha'}]|\lambda\rangle = \sum_x (\lambda_x (\lambda_{x' + \alpha'}) - \lambda_x (\lambda_{x' + \alpha})) |\lambda + (\alpha + \alpha') \delta^x\rangle = (\alpha' - \alpha) \hat{S}_{\alpha+\alpha'} |\lambda\rangle.$$  

So $\hat{S}_\alpha$ do fulfill the right commutation relations. What about the adjointness relation? We expect it not be fulfilled since we have ordered $\pi$ to the right. Symmetric order (14) would ask for the operator

$$\hat{L}_\alpha \doteq \frac{1}{2}(\hat{S}_\alpha + \hat{S}_{-\alpha}^\dagger).$$

($\hat{S}_\alpha$ corresponds to $f(\hat{x})\hat{p}$ in (14), so the above corresponds to the second equality in (14).) So what is $\hat{S}_\alpha$? Alas

**Proposition 4.1.** No element of Cyl is in the domain of definition of $\hat{S}_\alpha$ for $\alpha \neq 0$ (where we take $\text{dom}(\hat{S}_\alpha)$ to be Cyl).

**Proof.** Let $\alpha \neq 0$. Given any $|\lambda\rangle \in \text{Cyl}$, if we show that there are uncountably infinitely many $|\lambda'\rangle$ such that $\langle \lambda| \hat{S}_\alpha \lambda'\rangle \neq 0$ we are done, for then $\langle \lambda| \hat{S}_\alpha \rangle$ cannot be bounded.

Indeed, given $|\lambda\rangle$, choose $x$ such that $\lambda_x = 0$ and define $\lambda' = \lambda - \alpha \delta^x$. Then $\langle \lambda| \hat{S}_\alpha \lambda'\rangle = 1$. But there are uncountably many $x$ with $\lambda_x = 0$. \qed
Exploring the diffeomorphism-invariant Hilbert space of a scalar field

We can certainly guess where this failure comes from: one has to expect that $$\tilde{S}_{\alpha} = \tilde{S}_{\alpha} - \alpha \sum_x \tilde{T}_{x,-a}$$ if it existed, but it cannot because the sum over $$x \in \Sigma_1$$ does not converge on Cyl.

We will now consider representing $$L_\alpha$$ on $$H_{\text{diff}}$$. We start by noting that any operator $$a$$ on $$H_{\text{kin}}$$ that maps Cyl to Cyl automatically has a dual action on Cyl*: for $$l \in \text{Cyl}^*$$ set $$\tilde{a} l = l \circ a$$. This dual action preserves commutation relations in the sense that $$[\tilde{a}, \tilde{b}] = [\tilde{b}, a]$$. Furthermore, if the operator $$a$$ is diffeomorphism invariant then $$\tilde{a}$$ preserves $$V_N$$. Thus in particular for the operators $$\hat{S}_\alpha$$ (15) we get a dual action $$\tilde{S}_\alpha$$ on $$V_N$$. It is important to stress that at this point there is no need for any (choice of) rigging map. It will however become important once we consider adjoints.

What is the explicit form of $$\tilde{S}_\alpha$$?

Proposition 4.2. The action of $$\tilde{S}_\alpha$$ on $$V_N$$ can be expressed as

$$\tilde{S}_\alpha = -\alpha a^\dagger_{-\alpha} + \lambda \neq 0 (\lambda - \alpha) a^\dagger_{\lambda - \alpha} a_\lambda.$$ (16)

The proof of this proposition is contained in appendix C.

It will be convenient in the following to adopt the notation $$a_0 = a^\dagger_0 = \text{id}$$, the identity map. Then we can write more compactly

$$\tilde{S}_\alpha = \sum_\lambda (\lambda - \alpha) a^\dagger_{\lambda - \alpha} a_\lambda.$$ (16)

Now, in contrast to the situation on $$H_{\text{kin}}$$, $$\tilde{S}_\alpha$$ (their domain taken to be $$V_N$$) possess adjoints that are nontrivial on $$V_N$$, under all of the scalar products (indexed by $$F$$) considered in the preceding section. Let us start with $$F = 1$$: one finds immediately (we remind the reader that where we denote the adjoint for $$F = 1$$ with $$^\dagger$$)

$$\tilde{S}_\alpha = \tilde{S}_{\alpha} - \alpha \sum_\lambda a^\dagger_{\lambda} a_\lambda.$$ (16)

on $$V_N$$. Thus $$\tilde{S}_\alpha$$ do not yet fulfill the right adjointness relations. But that is no surprise as we have not yet ordered symmetrically. Let

$$\tilde{L}_\alpha = \frac{1}{2} (\tilde{S}_{-\alpha} + \tilde{S}_\alpha) = \tilde{S}_{\alpha} - \alpha \sum_\lambda a^\dagger_{\lambda} a_\lambda.$$ (16)

By definition we now have $$\tilde{L}_\alpha = \tilde{L}_{a - a}$$. However the commutation relations have to be checked. A short calculation shows that

$$[\tilde{L}_\alpha, \tilde{L}_{a}'] = (a' - \alpha) \tilde{L}_{a a'} + \frac{1}{2} \alpha a' (a^\dagger_{a - a} - a^\dagger_{a - a'}),$$

i.e. an anomalous term appears on the right-hand side.

We can follow the same steps, but for the more complicated situation $$F(n) \neq 1$$. In this case, the dual action (16) of $$\tilde{S}_\alpha$$ remains the same, however the scalar product, and hence the notion of adjoint, and of symmetric ordering, change. Let us denote the adjoint with respect to $$\langle \cdot | \cdot \rangle_F$$ for $$F \neq 1$$ with $$^\ast$$. Then symmetric ordering gives

$$\tilde{L}_\alpha = \frac{1}{2} (\tilde{S}_{-\alpha} + \tilde{S}_\alpha).$$

Again, adjointness relations are automatic. For the commutators we find

$$[\tilde{L}_\alpha, \tilde{L}_{a}'] = (a' - \alpha) \tilde{L}_{a a'} + \frac{1}{2} \alpha a' (a^\dagger_{a - a} - a^\dagger_{a - a'} - a_{a - a'}).$$

6 From here on, we will no longer explicitly mention that we consider all adjoint operators restricted to $$V_N$$. 

Proposition 4.3

\[ \left[ \hat{L}_{\alpha}, \hat{L}_{\alpha}^\prime \right] = (\alpha' - \alpha) \hat{L}_{\alpha+\alpha}^\prime + \frac{\alpha \alpha'}{4} (a_\alpha^\dagger (\Delta(\tilde{N}) + 1)a_{-\alpha} - a_{-\alpha}^\dagger (\Delta(\tilde{N}) + 1)a_\alpha), \]

where \( \Delta(n) = F((n+1)!)/F((n+2)! - F(n!)/F((n+1)!). \)

Let us finish with three remarks:

1. The standard choice \( F(n) = 1/n \) does not lead to a vanishing of the unwanted term, as in that case \( \Delta(n) + 1 = 1 + n + 2 - (n + 1) = 2 \neq 0. \)

2. One can ask the question if there is a valid (i.e. strictly positive) choice of \( F \) that leads to \( \Delta(n) + 1 = 0. \) It is not hard to solve the latter equation: let \( f(n) = F(n!) \). Then the equation immediately implies

\[ \frac{f(n)}{f(n+1)} = c_0 - n \]

for some \( c_0 \) whence

\[ f(n)^{-1} = f(0)^{-1} c_0 (c_0 - 1)(c_0 - 2) \cdots (c_0 - (n - 1)). \]

One sees immediately that there is no strictly positive solution. The best one can hope for is a non-negative solution, which is achieved by choosing for \( c_0 \) some integer \( N_0, \)

\[ \frac{1}{f(n)} = \begin{cases} f(0) N_0!/(N_0 - n)! & \text{for } n \leq N_0 \\ 0 & \text{else.} \end{cases} \]

Since \( 1/f(n) \) can actually become zero, the Hilbert space that one would obtain would be very different from \( \mathcal{H}_{\text{diff}} \) considered so far. (In particular, \( f(\tilde{N})^{-1/2} \) would not be a unitary map to \( \mathcal{H}_{\text{diff}}^{-1/2}. \) Also, it is in fact not immediately clear whether proposition 4.3 still holds in this case. We do however suspect that this choice would yield a representation of the relations (13). The reason is that there is a natural interpretation for this inner product: it seems to arise if, instead of \( \Sigma \) being a differentiable manifold, one allows \( \Sigma \) to be just a finite set, of \( N_0 \) points. Diffeomorphisms would be replaced by permutations, and group averaging can be carried out explicitly, leading to precisely the above inner product. Note however that \( (N_0 - n)! \) is not the number of graph symmetries (that would be \( n! \)) but the number of permutations that leave the graph invariant.

3. The unwanted term is really unwanted, because it is neither central nor can it be expressed in any simple way through \( \hat{L}_{\alpha}. \) It is thus not a simple ‘quantum correction’.

5. Closing remarks

In the preceding sections we have, under assumption 3.1, explicitly worked out the structure of \( \mathcal{H}_{\text{diff}}^{F} \) for a scalar field, both for the standard choice of the combinatorial factor \( F, \) as well as for a large class of other choices. Thereby we have seen that the role of the group averaging map is quite subtle: different maps give isomorphic Hilbert spaces, but they are not equal as spaces of linear functionals over Cyl. Thus if the latter is used for quantization, results in general depend on the map. We have also shown that there is a natural Fock structure on \( \mathcal{H}_{\text{diff}}^{1} \) (which can, by unitary equivalence, also be defined on any \( \mathcal{H}_{\text{diff}}^{F}. \))

Do these findings have any implications for the full theory? Certainly not immediately. For one thing, in the full theory the structure of \( \mathcal{H}_{\text{diff}} \) is much more complicated. Creating a new loop in a given state can happen in many ways, so that one does not expect to simply get away with standard annihilation and creation operators. There is knotting, the analog of which in the situation here would be to work with \( \Sigma \) one dimensional, i.e. without assumption 3.1.
Our results are not even directly applicable to a scalar field coupled to gravity, because $\mathcal{H}_{\text{diff}}$ in that case would not be the tensor product of the diff-invariant Hilbert space for gravity and that for the scalar field. But the results can give some inspiration for looking at the diff-invariant Hilbert space in LQG in new ways, e.g. building it up by performing simple operations on the vacuum, etc.

We have furthermore tried to represent the algebra of $L_{\alpha}$ on $\mathcal{H}_{\text{diff}}$ (and $\mathcal{H}_{\text{kin}}$), using symmetric ordering, without success. There are several attitudes that one can take towards this failure and its implications:

1) Symmetric ordering is not right—maybe a different ordering prescription can help? Indeed this is very possible. We have tried to start with an ordering like

$$L_{\alpha} = \int \text{sign}(\pi(x))\sqrt{\vert \pi \vert(x)}\exp(i\alpha\phi(x))\sqrt{\vert \pi \vert(x)},$$

but obtained more complicated correction terms. But maybe one can come up with a better prescription.

2) Maybe these quantities simply fail to exist at all as operators, on any of $\mathcal{H}_{\text{diff}}^F$? That seems to be plausible too. After all, at least on the kinematical Hilbert space one can understand the reason for the failure of the quantization very clearly: it is the failure of operators like

$$\int_S [\hat{\pi}(x), \exp i\alpha\hat{\phi}(x)]$$

to make sense. And there is no obvious reason why this problem should go away on the diffeomorphism-invariant level.

3) Maybe one should not care so much about the correction term or about the adjointness relations? It is certainly hard to argue about the importance of this term because the whole setup is very unphysical. We would like to repeat, however, that the correction term is not central, is not expressible in any straightforward way through $L_{\alpha}$’s, and therefore generates a whole bunch of new quantum objects that have no classical analog. Abandoning the adjointness relations does a very similar thing in one stroke, namely doubling the number of members of the algebra.

Do our results have implications for physically more interesting situations? We do not know. One thing that may be noted is that asking for diffeomorphism-invariant operators to be defined on the kinematical Hilbert space already (e.g. [1]) may be too much (see proposition 4.1). Furthermore it is certainly true that there are also somewhat similar ordering issues for physically very important operators such as the Hamiltonian constraint. The situation in that case is, however, also different in many respects (and it has been argued that violation of adjointness relations may be necessary in that case for other reasons), so direct conclusions cannot be drawn.

When comparing the present paper to the literature on the implementation of the Hamilton constraint, one should keep in mind that we asked for more than could be, and has been, asked in that case: since the algebra we considered here closed, we could ask for its implementation. Since the quantities were diffeomorphism invariant, we could seek the representation on $\mathcal{H}_{\text{diff}}$. Neither ‘luxuries’ are available for the Hamiltonian constraint. Asking for a lot makes it harder to succeed, and we saw that in the present situation it was hard to have both, anomaly freedom and correct adjointness relations. One can certainly ask for less, whence it would be easier to declare full success.

In any case we should stress again that the situation for the implementation of the Hamiltonian constraint is very different from that considered here, and moreover there are many suggestions (e.g. [15]) as to how to accomplish the former that differ on a technical and
conceptual level from the present approach. Therefore direct conclusions from our work here on the issue of the dynamics of LQG can unfortunately not be drawn.

We do however feel that seeing that things are not very straightforward even in a simple case such as considered here, should make one extremely cautious in more complicated situations.

The present work could be extended in many ways: maybe there is another quantization procedure that leads to more satisfying results? Can the analogy with quantization techniques used in full LQG be strengthened?

It would also be interesting to study the situation for \( \Sigma \) one dimensional. In this case, \( \mathcal{H}_{\text{diff}} \) is more complicated, and thus the previous analysis does not apply. In this case, there may even be some physical application, as one could use the likes of \( L_\alpha \) to build operators that resemble simple vertex operators for the bosonic string.

Acknowledgments

We would like to thank members of the Institute for Gravitational Physics at Penn State University for hospitality and discussions and also Thomas Thiemann for discussions and valuable input.

Appendix A. A look at the reduced phase space

The constraint is \( \pi(x) \partial_a \phi(x) = 0 \). So for \( (\pi, \phi) \) on the constraint surface \( \pi(x) = 0 \) or \( \phi(x) = \text{const.} \) at any given point \( x \). Identifying points on the constraint surface that can be mapped onto each other by the maps generated by the constraint gives the reduced phase space. How large is it?\(^7\) We will not answer this question directly, but at least for the smooth category we will exhibit enough points such that all the functions \( L_\alpha \) are necessary to tell them apart: given a sequence \( s = (v_i, \phi_i) \) of pairs of real numbers we construct a point on the constraint surface as follows: choose non-intersecting open subsets \( U_i \) of \( \Sigma \). Then choose a phase-space point \( (\phi_s, \pi_s) \) such that

\[
\pi_s|_{\Sigma - \bigcup_i U_i} = 0, \quad \int_{U_i} \pi = v_i, \quad \phi|_{U_i} = \phi_i.
\]

Such \( (\phi_s, \pi_s) \) obviously satisfies the constraint. Thus to each sequence \( s \) we find at least one point on the constraint surface and thus a point in the reduced phase space. Are these points distinct? We compute the quantities \( L_\alpha \) on \( (\phi_s, \pi_s) \) and find

\[
L_\alpha(\phi_s, \pi_s) = \sum_j v_j \exp(i c_j \alpha) =: F_s(\alpha).
\]

Let us assume \( s \) is chosen such that the above converges uniformly in \( \alpha \). Since \( L_\alpha \) are invariant under diffeomorphisms, there are at least as many points in the reduced phase as there are functions \( F_s \) of the above form, that is

- for \( \mathcal{I} = \mathbb{Z} \): at least as many points as functions on a circle with uniformly converging Fourier series.
- for \( \mathcal{I} = \mathbb{R} \): at least as many points as almost periodic functions.

It should also be clear that one indeed needs to consider all \( L_\alpha \) to separate the points thus obtained. (Certainly there are more points in the reduced phase space and \( L_\alpha \) are thus not separating all points.)

\(^7\) We are grateful to T Thiemann for posing this question.
Appendix B. Proof of proposition 3.2

Let us start by writing (6) in the case of \( F = 1 \) for two basis elements \(|\lambda\rangle, |\lambda'\rangle\):\(^8\)

\[
(\Gamma(\lambda))(|\lambda'\rangle) = \sum_{\{\phi_i\}\in\text{Diff}/\text{Diff}_y} \sum_{|\phi\rangle\in\text{GS}_y} \langle\phi_1 \ast \phi_2 \ast \lambda | \lambda'\rangle.
\] (B.1)

The kinematical scalar product in this formula will only be nonzero if \((\phi_1 \ast \phi_2 \ast \lambda) = \lambda'\) for all points \(x\). Thus it is important to understand when this can happen. Let us rewrite

\[
\lambda_x = \sum_i \lambda_i \delta(x_i, x),
\]

with \(\lambda_i \neq 0\) and \(x_i \neq x_j\) for \(i \neq j\). Let us also introduce the shorthand \(\gamma\) for the set of \(x_i\). We similarly decompose \(\lambda'\) and consider the second sum in (B.1): the action of elements of \(\text{Diff}_\gamma\), when restricted to \(\gamma\), can only consist of permutations of the points of \(\gamma\), and per our assumption 3.1 all the permutations do occur. Given two diffeomorphisms \(\phi_1, \phi_2 \in \text{Diff}_\gamma\) that reduce to the same permutation, then they are in the same equivalence class in \(\text{GS}_y\), as \(\phi_1 \circ \phi_2^{-1}\) is in \(\text{TDiff}, \) and \((\phi_1 \circ \phi_2^{-1}) \circ \phi_2 = \phi_1\). Hence \(\text{GS}_y\) is isomorphic to the permutation group \(\mathcal{P}_{|\gamma|}\) and in terms of the latter, the action on states is the obvious:

\[
|\lambda\rangle \mapsto |\pi \cdot \lambda\rangle, \quad \text{with} \quad \pi \cdot \lambda_x = \sum_i \lambda_i \delta(x_{\pi(i)}, x).
\]

So the second sum in (B.1) will essentially be over permutations in \(\mathcal{P}_{|\gamma|}\).

Now consider the sum over \(\text{Diff}/\text{Diff}_\gamma\) in (B.1): the kinematical scalar product will necessarily be zero if \(\phi_1\) does not map \(\gamma\) to \(\gamma'\). If \(|\gamma| \neq |\gamma'|\) then this can never happen, and (B.1) gives zero. On the other hand, if \(|\gamma| = |\gamma'|\), then per assumption 3.1 there exists at least one diffeomorphism mapping \(\gamma\) to \(\gamma'\). Let \(\phi_1\) be such a diffeomorphism. How big is \([\phi_1]\)\? Given \(\phi_1'\) that also maps \(\gamma\) to \(\gamma'\), then \((\phi_1')^{-1} \circ \phi_1\) is in \(\text{Diff}_\gamma\) and thus \(\phi_1'\) is already contained in \([\phi_1]\). Thus there is at most one element in the first sum of (B.1) that gives a nonzero contribution.

Altogether we find that (B.1) can be simplified to

\[
(\Gamma(\lambda))(|\lambda'\rangle) = \sum_{\pi \in \mathcal{P}_{|\gamma|}} \begin{cases} 
0 & \text{if } |\gamma| \neq |\gamma'| \\
\prod_i \delta(\lambda_{\pi(i)}, \lambda'_i) & \text{otherwise},
\end{cases}
\] (B.2)

(B.2) is nonzero only if the sequences \((\lambda_i), (\lambda'_i)\) are equal modulo permutations. Expressed more concisely, (B.2) is nonzero only if \(N^{(\lambda)}_{\mu}\) and \(N^{(\lambda')}_{\mu}\) are equal for all \(\mu \in \mathcal{I}\) (recall definition (9)). In that case, several terms in the sum may give nonzero contribution: the order of the stabilizer subgroup of permutations, for a set of objects with multiplicity \(n_1, n_2, n_3, \ldots\), is \(n_1!n_2!n_3!\ldots\). In our case this means that the sum actually gives \(\prod_\mu N^{(\lambda)}_{\mu}!\). Thus we can rewrite (B.2) as

\[
(\Gamma(\lambda))(|\lambda'\rangle) = \prod_{\mu \in \mathcal{I}} N^{(\lambda)}_{\mu}! \delta(N^{(\lambda)}_{\mu}, N^{(\lambda')}_{\mu}).
\]

But the right-hand side of the above equation is just \(N^{(\lambda')}|\lambda\rangle\). Since the equation holds for all basis elements, we have shown that indeed \(N^{(\lambda')} = (\Gamma(\lambda))\).

\(^8\) Since we will consider the case \(F = 1\) exclusively, we will here and in the following drop the corresponding subscript.
Appendix C. Proof of proposition 4.2

We want to show that the dual action of the operator $S_\alpha$ is given by (16). Since we possess a basis for $V_N$ and an orthonormal basis for Cyl, it is sufficient to verify on elements of these bases. More precisely, we would like to show that

$$\tilde{S}_\alpha N |(\lambda,)\rangle = (N) |\tilde{S}_\alpha |\lambda,)\rangle$$

for all $N, \lambda, \alpha$, and $\tilde{S}_\alpha$ defined by (16). We will prove in this the unimaginative way, i.e. by explicitly computing both sides and verifying that they are indeed equal in all cases.

Fix $\lambda, \alpha$, and $N$. From now on, let us also assume $\alpha \neq 0$, and come back to $\alpha = 0$ later.

We write $\lambda$ in the following way:

$$\lambda = \sum_i -\alpha \delta(x_i^{(0)}, x) + \sum_j \lambda_j \delta(x_j^{(i)}, x).$$

To make this decomposition unique, the data have to satisfy some obvious properties. To be clear, let us spell them out: we want $\lambda_i \neq \lambda_j$ for $i \neq j$, $\lambda_i \neq -\alpha$, $0$, all $n_j > 0$, and all $x_j^{(i)}$ different. In this notation

$$\hat{S}_\alpha |\lambda,) = \sum_i -\alpha |\lambda + \alpha \delta^{(m)}_i) + \sum_j \lambda_j |\lambda + \alpha \delta^{(i)}_j).$$

Also

$$|N^{(i)}_\alpha) = \left| n_0 \delta^{-\alpha} + \sum_j n_j \delta^\lambda_j \right).$$

Finally with expression (16) for $\tilde{S}_\alpha$,

$$\tilde{S}_\alpha |N) = -\alpha |N + \delta^{-\alpha}) + \sum_{\mu \neq 0} N_{\mu}(\mu - \alpha) |N + \delta^{\mu - \alpha} - \delta^\mu).$$

Thus we can write the right-hand side of (C.1)

$$(N |\Gamma(\tilde{S}_\alpha |\lambda,))) = -\alpha n_0 \left( N \left| n_0 - 1 \right) \delta^{-\alpha} + \sum_j n_j \delta^\lambda_j \right)$$

and for the left-hand side we find

$$(\tilde{S}_\alpha N |\Gamma(|\lambda,))) = -\alpha \left( N + \delta^{-\alpha} \left| n_0 \delta^{-\alpha} + \sum_j n_j \delta^\lambda_j \right)$$

We now have to show that these two expressions in fact evaluate to the same number. We do this term by term, and start with the first term in both expressions, i.e. the one with pre-factor $-\alpha$. The term in (C.2) can be written as

$$-\alpha n_0 N^{-\alpha} \delta(N^{-\alpha}, n_0 - 1) \prod_{\text{rest}}.$$
The term in (C.3) gives
\[-(\alpha(N_{\mu-\alpha} + 1)\delta(N_{\mu-\alpha} + 1, n_0) \cdot \prod \text{rest} = -(\alpha(N_{\mu-\alpha} + 1)N_{\mu-\alpha}\delta(N_{\mu-\alpha} \cdot n_0 - 1) \cdot \prod \text{rest},
\]
where we have used \(\delta(x, y + a) = \delta(x - a, y)\) and the fact that \(n_i\) can be replaced with \(N_{\mu} + 1\) thanks to the corresponding Kronecker delta. Thus it is equal to the term in (C.2), provided the ‘rest’ is the same. That however should be clear upon a quick inspection.

Now we proceed to comparing the rest of the terms in (C.2) and (C.3). What we have to do is: show that to each term in (C.2) there is an equal term in (C.3), and the rest of the terms in (C.3) vanish. To do this, let us first exclude certain ‘non-generic’ cases, and consider them separately, for reasons of notation: fix \(i\), and assume that \(\lambda_i + \alpha \neq \lambda_j\) for all \(j\), and also \(\lambda_i + \alpha \neq -\alpha\). Then we can partially expand the \(i\)th term in the sum in (C.2) as
\[
\lambda_i n_i \cdot N_{\mu-\alpha}! \delta(N_{\mu-\alpha}, n_0) \cdot N_{\lambda_i}! \delta(N_{\lambda_i}, n_i - 1) \cdot N_{\lambda_i+\alpha}! \delta(N_{\lambda_i+\alpha}, 1) \cdot \prod_{j \neq i} N_{\lambda_j}! \delta(N_{\lambda_j}, n_j) \cdot \prod \text{rest}.
\]

Compare this to the \(\mu = \lambda_i + \alpha\) term of (C.3), which expands as
\[
\lambda_i N_{\lambda_i+\alpha} \cdot N_{\mu-\alpha}! \delta(N_{\mu-\alpha}, n_0) \cdot (N_{\lambda_i} + 1) \delta(N_{\lambda_i} + 1, n_i)
\cdot \delta(N_{\lambda_i+\alpha} = 1) \delta(N_{\lambda_i+\alpha} - 1, 0) \cdot \prod_{j \neq i} N_{\lambda_j}! \delta(N_{\lambda_j}, n_j) \cdot \prod \text{rest}.
\]

That the latter expression is equal to the former can again be easily seen by the manipulations used before, and the fact that the ‘rest’ is in both cases just a product of all \(\delta(N_{\mu} \cdot 0)\) for \(\nu \neq -\alpha, \lambda_i, \lambda_i + \alpha, \lambda_j\).

Now we have to show that the rest of the terms in (C.3) vanish. Expand the \(\mu\)-term in (C.3) as
\[
N_{\mu} (\mu - \alpha) \cdot \delta(N_{\mu-\alpha} + 1, n_0\delta(-\alpha, \mu - \alpha) + \sum_n n_j \delta(\lambda_j, \mu - \alpha)) \cdot \prod \text{rest}.
\]

Now, \(n_0\delta(-\alpha, \mu - \alpha) + \sum_n n_j \delta(\lambda_j, \mu - \alpha)\) is zero as long as \(\mu \neq 0, \lambda_i + \alpha\) for any \(i\). There is actually no \(\mu = 0\) term in (C.3), and the \(\mu = \lambda_i + \alpha\) terms have already been dealt with. Thus for the remaining \(\mu\), the term contains the factor \(\delta(N_{\mu-\alpha} + 1, 0)\). Since \(N_{\mu-\alpha} \neq 0\), that factor is identically zero and the \(\mu\) term in the sum vanishes.

Now we turn to the non-generic cases:

1. \(\lambda_i + \alpha = -\alpha\). Let us examine the \(i\)-term in the sum (C.2). It can be expanded as
\[
n_i \lambda_i N_{\mu-\alpha}! \delta(N_{\mu-\alpha}, n_0 + 1) \cdot N_{\lambda_i}! \delta(N_{\lambda_i}, n_i - 1) \cdot \text{rest}.
\]

Similarly consider the \(\mu = \lambda_i + \alpha\) term in (C.3). It expands as
\[
N_{\mu-\alpha} \lambda_i (N_{\mu-\alpha} - 1)! \delta(N_{\mu-\alpha} - 1, n_0) \cdot (N_{\lambda_i} + 1) \delta(N_{\lambda_i} + 1, n_i) \cdot \text{rest},
\]

thus the \(i\)-term of (C.2) equals the \(\mu = \lambda_i + \alpha\) term in (C.3). Comparison of the other terms proceeds as in the generic case.

2. \(\lambda_i + \alpha = \lambda_j\). This case can be treated along similar lines as the case above: expand the product for the \(i\)-term in (C.2) and compare with the \(\mu = \lambda_i + \alpha\) term in (C.2). The rest of the terms coincide as shown in the non-generic case. We will refrain from giving the details.

Finally we have to treat the case \(\alpha = 0\): given \(\lambda\) we can write (using the same conventions as above)
\[
\lambda_i = \sum_j N_j \delta(\lambda_i, j, x).
\]
We also have \( \hat{\mathcal{S}}_0 |\lambda\rangle = \sum_i n_i \lambda_i |\lambda\rangle \) and hence
\[
(N | \hat{\mathcal{S}}_0 |\lambda\rangle) = \sum_i \lambda_i n_i (N |N^\lambda\rangle), \quad \text{and} \quad |N^\lambda\rangle = \left| \sum_i n_i \delta_i^\lambda \right|.
\]
We want to show that \( \hat{\mathcal{S}}_0 = \sum_{\mu \neq 0} |\mu\rangle a^\dagger_{\mu} a_{\mu} \), and the action of the latter is
\[
\sum_{\mu \neq 0} |\mu\rangle a^\dagger_{\mu} a_{\mu} |N\rangle = \sum_{\mu \neq 0} \mu N_{\mu} |N\rangle.
\]
Thus we have to show that
\[
\sum_{\mu \neq 0} \mu N_{\mu} \left( \sum_i n_i \delta_i^\lambda \right) = \sum_i \lambda_i n_i \left( \sum_i n_i \delta_i^\lambda \right).
\]
Both sides are zero if \( N \neq \sum_i n_i \delta_i^\lambda \). If instead equality holds in the last equation, then obviously \( \sum_{\mu \neq 0} \mu N_{\mu} = \sum_i \lambda_i n_i \), and hence we have proven the assertion.

Appendix D. Proof of proposition 4.3

Let \( f(n) = 1/F(n!) \). As in the main text, we will denote the adjoint with respect to \((\cdot|\cdot)_F\) for \( F \neq 1 \) with \( ^* \), the one for \( F = 1 \) with \( ^\dagger \). For any operator \( A \) that maps \( V_N \) to \( V_N \), they are related by
\[
A^* = f(\hat{N})^{-1} A^\dagger f(\hat{N}).
\]
In particular, they coincide on such operators that commute with \( \hat{N} \). For \( a^\dagger_\mu \) we find
\[
(a^\dagger_\mu)^* = \frac{f(\hat{N} + 1)}{f(\hat{N})} a_{\mu}.
\]
Therefore,
\[
\hat{S}_a^* = \hat{S}_a^\dagger - \alpha \left( \frac{f(\hat{N} + 1)}{f(\hat{N})} - 1 \right) a_{-a}
\]
and
\[
\hat{L}_a' = \hat{L}_a + \frac{\alpha}{2} \left( \frac{f(\hat{N} + 1)}{f(\hat{N})} - 1 \right) a_{-a} =: \hat{L}_a + \hat{\xi}_a.
\]
For the commutators containing \( \hat{\xi}_a \) we find
\[
[\hat{L}_a, \hat{\xi}_a'] = \frac{\alpha \alpha'}{4} a_{-a} \Delta(\hat{N}) a_{-a} + \frac{\alpha \alpha'}{2} \Delta(\hat{N}) a_{-a} a_{-a'} - \frac{\alpha}{2} (\alpha + \alpha') M(\hat{N}) a_{a+a'}
\]
with \( M(n) = f(n + 1)/f(n) - 1 \) and \( \Delta(n) = M(n) - M(0) \). Thus finally
\[
[\hat{L}_a', \hat{\xi}_a'] = [\hat{\xi}_a', \hat{\xi}_a] = [\hat{\xi}_a, \hat{\xi}_a'] + [\hat{L}_a, \hat{\xi}_a] + [\hat{\xi}_a, \hat{L}_a']
\]
\[
= (\alpha - \alpha') \hat{L}_{a+a'} + \frac{\alpha \alpha'}{4} (a_{-a} a_{-a'} - a_{-a'} a_{-a})
\]
\[
+ \frac{\alpha \alpha'}{4} (a_{-a} \Delta(\hat{N}) a_{-a'} - a_{-a'} \Delta(\hat{N}) a_{-a}) + (\alpha - \alpha') \hat{\xi}_{a+a'}
\]
\[
= (\alpha - \alpha') \hat{L}_{a+a'} + \frac{\alpha \alpha'}{4} (a_{-a} (\Delta(\hat{N}) + 1) a_{-a'} - a_{-a'} (\Delta(\hat{N}) + 1) a_{-a}).
\]
Exploring the diffeomorphism-invariant Hilbert space of a scalar field

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