ALGEBRAIC MONTGOMERY-YANG PROBLEM: THE NONCYCLIC CASE

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Abstract. Montgomery-Yang problem predicts that every pseudofree circle action on the 5-dimensional sphere has at most 3 non-free orbits. Using a certain one-to-one correspondence, Kollár formulated the algebraic version of the Montgomery-Yang problem: every projective surface $S$ with quotient singularities such that $b_2(S) = 1$ has at most 3 singular points if its smooth locus $S^0$ is simply-connected.

In this paper, we prove the conjecture under the assumption that $S$ has at least one noncyclic singularity. In the course of the proof, we classify projective surfaces $S$ with quotient singularities such that (i) $b_2(S) = 1$, (ii) $H_1(S^0, \mathbb{Z}) = 0$, and (iii) $S$ has 4 or more singular points, not all cyclic, and prove that all such surfaces have $\pi_1(S^0) \cong \mathbb{A}_5$, the icosahedral group.

1. Introduction

A pseudofree $S^1$-action on a sphere $S^{2k-1}$ is a smooth $S^1$-action which is free except for finitely many non-free orbits (whose isotropy types $\mathbb{Z}_{m_1}, \ldots, \mathbb{Z}_{m_n}$ have pairwise relatively prime orders).

For $k = 2$ Seifert [Se] showed that such an action must be linear and hence has at most two non-free orbits. In the contrast to this, for $k = 4$ Montgomery and Yang [MY] showed that given any pairwise relatively prime collection of positive integers $m_1, \ldots, m_n$, there is a pseudofree $S^1$-action on homotopy 7-sphere whose non-free orbits have exactly those orders. Petrie [P] proved similar results in all higher odd dimensions. This led Fintushel and Stern to formulate the following problem:

Conjecture 1.1 (FS87). (Montgomery-Yang Problem)

Let

$$S^1 \times S^5 \to S^5$$

be a pseudofree $S^1$-action. Then it has at most 3 non-free orbits.

The problem has remained unsolved since its formulation.

Pseudofree $S^1$-actions on 5-manifolds $L$ have been studied in terms of the 4-dimensional quotient orbifold $L/S^1$ (see e.g., [FS85], [FS87]). The following one-to-one correspondence was known to Montgomery, Yang, Fintushel and Stern, and recently observed by Kollár ([Kol05], [Kol08]):

Theorem 1.2. There is a one-to-one correspondence between:
(1) Pseudofree $S^1$-actions on 5 dimensional rational homology spheres $L$ with $H_1(L,\mathbb{Z}) = 0$.
(2) Smooth, compact 4-manifolds $M$ with boundary such that
   (a) $\partial M = \cup_i L_i$ is a disjoint union of lens spaces $L_i = S^3/\mathbb{Z}_{m_i}$,
   (b) the $m_i$ are relatively prime to each other,
   (c) $H_1(M,\mathbb{Z}) = 0$ and $H_2(M,\mathbb{Z}) \cong \mathbb{Z}$.

Furthermore, $L$ is diffeomorphic to $S^5$ iff $\pi_1(M) = 1$.

Using the one-to-one correspondence, Kollár formulated the algebraic version of the Montgomery-Yang problem as follows:

**Conjecture 1.3.** [Kol08] (Algebraic Montgomery-Yang Problem)
Let $S$ be a rational homology projective plane with quotient singularities, i.e., a normal projective surface with quotient singularities such that $b_2(S) = 1$. Assume that $S^0 := S \setminus \text{Sing}(S)$ is simply-connected. Then $S$ has at most 3 singular points.

In this paper, we verify the conjecture when $S$ has at least one noncyclic singularity. More precisely, we prove the following:

**Theorem 1.4.** Let $S$ be a rational homology projective plane with quotient singularities such that $\pi_1(S^0) = \{1\}$. Assume that $S$ has at least one noncyclic singularity. Then $|\text{Sing}(S)| \leq 3$.

We note that the condition $\pi_1(S^0) = \{1\}$ cannot be replaced by the weaker condition $H_1(S^0,\mathbb{Z}) = 0$. There are infinitely many examples of rational homology projective planes with exactly 4 quotient singularities, 3 cyclic, 1 noncyclic, such that $H_1(S^0,\mathbb{Z}) = 0$ but $\pi(S^0) \neq \{1\}$ ([Br] or [Kol08], Example 31). These examples are the global quotients

$$S_{I_m} := \mathbb{CP}^2/I_m = (\mathbb{CP}^2/\mathbb{Z})/\mathfrak{A}_5,$$

where $I_m \subset GL(2,\mathbb{C})$ is the group of order 120m in Brieskorn’s list (see Table 1), an extension of the icosahedral group $\mathfrak{A}_5 \subset PGL(2,\mathbb{C})$ by the cyclic group $Z \cong \mathbb{Z}_{2m}$, and the action of $I_m$ on $\mathbb{CP}^2$ is induced from the natural action on $\mathbb{C}^2$. We call $S_{I_m}$ a Brieskorn quotient.

On the other hand, it follows from the orbifold Bogomolov-Miyaoka-Yau inequality that every rational homology projective plane $S$ with quotient singularities such that $H_1(S^0,\mathbb{Z}) = 0$ has at most 4 singular points([Kol08], [HK], [Keu10]). Therefore, to prove Theorem 1.4 it is enough to classify rational homology projective planes $S$ with 4 quotient singularities, not all cyclic, such that $H_1(S^0,\mathbb{Z}) = 0$. It turns out that such a surface is deformation equivalent to a Brieskorn quotient.

**Theorem 1.5.** Let $S$ be a rational homology projective plane with 4 quotient singularities, not all cyclic, such that $H_1(S^0,\mathbb{Z}) = 0$. Then the following hold true.

(1) $S$ has 3 cyclic singularities of type $\mathbb{C}^2/\mathbb{Z}_2$, $\mathbb{C}^2/\mathbb{Z}_3$, $\mathbb{C}^2/\mathbb{Z}_5$, and one noncyclic singularity of type $\mathbb{C}^2/I_m$, where $I_m \subset GL(2,\mathbb{C})$ is the 2m-ary icosahedral group of order 120m (in Brieskorn’s notation). Furthermore, the 3 cyclic singularities are of type $\frac{1}{2}(1,1)$, $\frac{1}{3}(1,\alpha)$, $\frac{1}{5}(1,\beta)$, if the 3 branches of the dual graph of the noncyclic singularity are of type $\frac{1}{2}(1,1)$, $\frac{1}{3}(1,3-\alpha)$, $\frac{1}{5}(1,5-\beta)$ (see Table 4).
(2) $-K_S$ is ample.
The minimal resolution of $S$ can be obtained by starting with a minimal rational ruled surface and blowing up inside 3 of the fibres, i.e. the blowing up starts at three centers, one on each of the 3 fibres.

$S^0$ is deformation equivalent to $(\mathbb{CP}^2/I_m)^0$, where $I_m$ is determined by the noncyclic singularity of $S$ and its action on $\mathbb{CP}^2$ is induced by the natural action on $\mathbb{C}^2$. The deformation space has dimension 2.

$\pi_1(S^0) \cong A_5$, the alternating group of order 60.

We will prove Theorem 1.5. In the proof, we use the orbifold Bogomolov-Miyaoka-Yau inequality (Theorem 2.2 and 2.3) and a detailed computation for $(-1)$-curves on the minimal resolution $S'$ of $S$. The latter idea was used in [Keu08].

Remark 1.6. Consider a Brieskorn quotient $S_{I_m} := \mathbb{CP}^2/I_m = (\mathbb{CP}^2/\mathbb{Z})/\mathbb{A}_5$. The cone $\mathbb{CP}^2/\mathbb{Z}$ is the closure of the $\mathbb{A}_5$-universal cover of $S^0_{I_m}$. Note that the cone has no deformation. Thus the deformation of $S^0_{I_m}$ must correspond to a deformation of the $I_m$-action on $\mathbb{CP}^2$. This was pointed out to us by János Kollár. It is an interesting problem to describe explicitly such a deformation.

Throughout this paper, we work over the field $\mathbb{C}$ of complex numbers.

2. Algebraic Surfaces with Quotient Singularities

2.1. A singularity $p$ of a normal surface $S$ is called a quotient singularity if locally the germ is analytically isomorphic to $(\mathbb{C}^2/G, O)$ for some nontrivial finite subgroup $G$ of $GL_2(\mathbb{C})$ without quasi-reflections. Brieskorn classified such finite subgroups of $GL(2, \mathbb{C})$ [Bri]. Table 1 summarizes the result. Here we only explain the notation for dual graph.

$$< q, q_1 > := \text{the dual graph of the singularity of type } \frac{1}{q}(1, q_1)$$

$$< b; s_1, t_1; s_2, t_2; s_3, t_3 > := \text{the tree of the form}$$

$$< s_2, t_2 > \quad < s_1, t_1 > - b - < s_3, t_3 >$$

For more information about the table, we refer to the original paper of Brieskorn [Bri].

2.2. Let $S$ be a normal projective surface with quotient singularities and $f : S' \rightarrow S$ be a minimal resolution of $S$. It is well-known that quotient singularities are log-terminal singularities. Thus one can write

$$K_{S'} \equiv \sum_{p \in \text{Sing}(S)} f^* K_S - \sum_{p \in \text{Sing}(S)} D_p$$

where, for each singular point $p$, $D_p = \sum (a_j E_j)$ is an effective $\mathbb{Q}$-divisor supported on $f^{-1}(p) = \bigcup E_j$ with $0 \leq a_j < 1$. It implies that

$$K_S^2 = K_{S'}^2 - \sum_p D_p^2 = K_{S'}^2 + \sum_p D_pK_{S'}.$$
Lemma 2.1. If $-K_S$ is ample, then $C^2 \geq -1$ for any irreducible curve $C \subset S'$ not contracted by $f : S' \to S$.

Proof. Note that $C(f^*K_S) < 0$ and $C(\sum D_p) \geq 0$. Thus $CK_{S'} < 0$, and hence $C^2 \geq -1$. □

Also we recall the orbifold Euler characteristic

$$e_{orb}(S) := e(S) - \sum_{p \in Sing(S)} \left(1 - \frac{1}{|G_p|}\right)$$

where $G_p$ is the local fundamental group of $p$.

The following theorem, called the orbifold Bogomolov-Miyaoka-Yau inequality, is one of the main ingredients in the proof of our main theorem.

Theorem 2.2 ([S], [MI], [KNS], [Me]). Let $S$ be a normal projective surface with quotient singularities such that $K_S$ is nef. Then

$$K_S^2 \leq 3e_{orb}(S).$$

We also need the following weaker inequality, which also holds when $K_S$ is nef.

| Type | $G$ | $|G|$ | $G/[G,G]$ | Dual Graph $\Gamma_G$ |
|------|-----|------|------------|-------------------------|
| $A_{q,q_1}$ | $C_{q,q_1}$ | $q$ | $\mathbb{Z}_q$ | $< q,q_1 >$ |
| | | | | $0 < q_1 < q, (q,q_1) = 1$ |
| $D_{q,q_1}$ | $(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; D_q, D_q)$ | $4mq$ | $\mathbb{Z}_{2m} \times \mathbb{Z}_2$ | $< b; 2, 1; 2, 1; q, q_1 >$ |
| | | | | $m = (b - 1)q - q_1$ odd |
| | | | | $m = (b - 1)q - q_1$ even |
| $T_m$ | $(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; T, T)$ | $24m$ | $\mathbb{Z}_{3m}$ | $< b; 2, 1; 3, 2, 3, 2 >, m = 6(b - 2) + 1$ |
| | | | | $< b; 2, 1; 3, 1, 3, 1 >, m = 6(b - 2) + 2$ |
| $O_m$ | $(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; O, O)$ | $48m$ | $\mathbb{Z}_{2m}$ | $< b; 2, 1; 3, 2, 4, 3 >, m = 12(b - 2) + 1$ |
| | | | | $< b; 2, 1; 3, 1, 4, 3 >, m = 12(b - 2) + 2$ |
| | | | | $< b; 2, 1; 3, 2, 4, 1 >, m = 12(b - 2) + 4$ |
| | | | | $< b; 2, 1; 3, 1, 4, 1 >, m = 12(b - 2) + 6$ |
| | | | | $< b; 2, 1; 3, 2, 5, 3 >, m = 30(b - 2) + 1$ |
| | | | | $< b; 2, 1; 3, 2, 5, 3 >, m = 30(b - 2) + 3$ |
| | | | | $< b; 2, 1; 3, 2, 5, 3 >, m = 30(b - 2) + 5$ |
| | | | | $< b; 2, 1; 3, 2, 5, 3 >, m = 30(b - 2) + 7$ |
| | | | | $< b; 2, 1; 3, 2, 5, 3 >, m = 30(b - 2) + 9$ |
| | | | | $< b; 2, 1; 3, 2, 5, 3 >, m = 30(b - 2) + 11$ |
| | | | | $< b; 2, 1; 3, 2, 5, 3 >, m = 30(b - 2) + 13$ |
| | | | | $< b; 2, 1; 3, 2, 5, 3 >, m = 30(b - 2) + 15$ |
| | | | | $< b; 2, 1; 3, 2, 5, 3 >, m = 30(b - 2) + 17$ |
| | | | | $< b; 2, 1; 3, 2, 5, 3 >, m = 30(b - 2) + 19$ |
| | | | | $< b; 2, 1; 3, 2, 5, 3 >, m = 30(b - 2) + 21$ |
| | | | | $< b; 2, 1; 3, 2, 5, 3 >, m = 30(b - 2) + 23$ |
| | | | | $< b; 2, 1; 3, 2, 5, 3 >, m = 30(b - 2) + 25$ |
| | | | | $< b; 2, 1; 3, 2, 5, 3 >, m = 30(b - 2) + 27$ |
| | | | | $< b; 2, 1; 3, 2, 5, 3 >, m = 30(b - 2) + 29$ |
Theorem 2.3 ([KM]). Let $S$ be a normal projective surface with quotient singularities such that $-K_S$ is nef. Then
\[ 0 \leq e_{arb}(S). \]

2.3. Let $S$ be a normal projective surface with quotient singularities and $f : S' \rightarrow S$ be a minimal resolution of $S$. It is well-known that the torsion-free part of the second cohomology group,
\[ H^2(S', \mathbb{Z})_{\text{free}} := H^2(S', \mathbb{Z})/\text{torsion} \]
has a lattice structure which is unimodular. For a quotient singular point $p \in S$, let
\[ R_p \subset H^2(S', \mathbb{Z})_{\text{free}} \]
be the sublattice of $H^2(S', \mathbb{Z})_{\text{free}}$ spanned by the numerical classes of the components of $f^{-1}(p)$. It is a negative definite lattice, and its discriminant group
\[ \text{disc}(R_p) := \text{Hom}(R_p, \mathbb{Z})/R_p \]
is isomorphic to the abelianization $G_p/[G_p, G_p]$ of the local fundamental group $G_p$. In particular, the absolute value $|\det(R_p)|$ of the determinant of the intersection matrix of $R_p$ is equal to the order $|G_p/[G_p, G_p]|$. Let
\[ R = \bigoplus_{p \in \text{Sing}(S)} R_p \subset H^2(S', \mathbb{Z})_{\text{free}} \]
be the sublattice of $H^2(S', \mathbb{Z})_{\text{free}}$ spanned by the numerical classes of the exceptional curves of $f : S' \rightarrow S$. We also consider the sublattice
\[ R + \langle K_{S'} \rangle \subset H^2(S', \mathbb{Z})_{\text{free}} \]
spanned by $R$ and the canonical class $K_{S'}$. Note that
\[ \text{rank}(R) \leq \text{rank}(R + \langle K_{S'} \rangle) \leq \text{rank}(R) + 1. \]

Lemma 2.4 ([HK], Lemma 3.3). Let $S$ be a normal projective surface with quotient singularities and $f : S' \rightarrow S$ be a minimal resolution of $S$. Then the following hold true.

1. $\text{rank}(R + \langle K_{S'} \rangle) = \text{rank}(R)$ if and only if $K_S$ is numerically trivial.
2. $\det(R + \langle K_{S'} \rangle) = \det(R) \cdot K_S^2$ if $K_S$ is not numerically trivial.
3. If in addition $b_2(S') = 1$ and $K_S$ is not numerically trivial, then $R + \langle K_{S'} \rangle$ is a sublattice of finite index in the unimodular lattice $H^2(S', \mathbb{Z})_{\text{free}}$, in particular $|\det(R + \langle K_{S'} \rangle)|$ is a nonzero square number.

We denote the number $|\det(R + \langle K_{S'} \rangle)|$ by $D$, i.e., we define
\[ D := |\det(R + \langle K_{S'} \rangle)|. \]

The following will be also used in our proof.

Lemma 2.5. Let $S$ be a rational homology projective plane with quotient singularities such that $H_1(S^0, \mathbb{Z}) = 0$. Let $f : S' \rightarrow S$ be a minimal resolution. Then

1. $H^2(S', \mathbb{Z})$ is torsion free, i.e. $H^2(S', \mathbb{Z}) = H^2(S', \mathbb{Z})_{\text{free}},$
2. $R$ is a primitive sublattice of the unimodular lattice $H^2(S', \mathbb{Z}),$
3. $\text{disc}(R)$ is a cyclic group, in particular, the orders $|G_p/[G_p, G_p]| = |\det(R_p)|$ are pairwise relatively prime,
4. $K_S$ is not numerically trivial, i.e. $K_S$ is either ample or anti-ample,
5. $D = |\det(R)|K_S^2$ and $D$ is a nonzero square number.
(6) the Picard group $\text{Pic}(S')$ is generated over $\mathbb{Z}$ by the exceptional curves and a $\mathbb{Q}$-divisor $M$ of the form

$$M = \frac{1}{\sqrt{D}} f^* K_S + z,$$

where $z$ is a generator of $\text{disc}(R)$, hence of the form $z = \sum_{p \in \text{Sing}(S)} b_p e_p$ for some integers $b_p$, where $e_p$ is a generator of $\text{disc}(R_p)$.

Proof. (1), (2) and (3) are easy to see. cf. [Keu07], Proposition 2.3 and Lemma 3.4.

(4) Assume that $K_S$ is numerically trivial. Then $S'$ is an Enriques surface if all singularities are rational double points, and is a rational surface otherwise. If $S'$ is an Enriques surface, then $H_1(S^0, \mathbb{Z}) \neq 0$ since $H_1(S', \mathbb{Z}) = \mathbb{Z}/2$ (cf. Proposition 2.3 in [Keu07]). Thus $S$ is a rational surface, and

$$K_{S'} \equiv \frac{\text{num}}{\text{num}} - \sum_{p \in \text{Sing}(S)} D_p$$

with $D_p \not\equiv 0$ for some $p$. Note that $D_p$ defines an element of $R^+_p := \text{Hom}(R_p, \mathbb{Z})$ and the discriminant group $\text{disc}(R_p) := R^+_p / R_p$ has order $|\text{det}(R_p)|$. Thus $|\text{det}(R_p)|D_p \in R_p$ but $D_p \not\equiv 0$ if $D_p \not\equiv 0$. Now we see that

$$\left( \prod_p |\text{det}(R_p)| \right) K_{S'} \in R \subset H^2(S', \mathbb{Z})$$

but $K_{S'} \not\equiv R$. Hence the primitive closure $\bar{R}$ of $R$ in $H^2(S', \mathbb{Z})$ is not equal to $R$. Now by Lemma 2.5 in [Keu07], $H_1(S^0, \mathbb{Z}) \neq 0$.

(5) follows from (4) and Lemma 2.4.

(6) Note first that $\text{Pic}(S') = H^2(S', \mathbb{Z})$ and the sublattice $R \subset H^2(S', \mathbb{Z})$ generated by the exceptional curves is a primitive sublattice of corank 1. Let $R^\perp \subset H^2(S', \mathbb{Z})$ be the orthogonal complement of $R$. Note that $R^\perp$ is positive definite and of rank 1. Since $H^2(S', \mathbb{Z})$ is unimodular,

$$\text{det}(R^\perp) = |\text{det}(R)| = \prod_{p \in \text{Sing}(S)} |\text{det}(R_p)|.$$ 

Note that $f^* K_S \in R^\perp$. Thus $R^\perp$ is generated by

$$v = \frac{|\text{det}(R)|}{\sqrt{D}} f^* K_S,$$

and $\text{disc}(R^\perp)$ is generated by $\frac{1}{\sqrt{D}} f^* K_S$. Also note that

$$\text{disc}(R^\perp \oplus R) \cong (\mathbb{Z}/|\text{det}(R)|) \oplus (\mathbb{Z}/|\text{det}(R)|).$$

Thus $\text{Pic}(S')/(R^\perp \oplus R)$ is an isotropic subgroup of $\text{disc}(R^\perp \oplus R)$ of order $|\text{det}(R)|$, hence is generated by an element $M \in \text{disc}(R^\perp \oplus R)$ of order $|\text{det}(R)|$. Moreover $M$ is the sum of a generator of $\text{disc}(R^\perp)$ and a generator of $\text{disc}(R)$, since $\text{Pic}(S')$ is unimodular. By replacing $M$ by $kM$ for a suitable choice of an integer $k$, we get $M$ of the desired form

$$M = \frac{1}{\sqrt{D}} f^* K_S + \sum_{p \in \text{Sing}(S)} a_p e_p.$$
for some integers \( a_p \), \( 0 \leq a_p < |\det(R_p)| \). This implies that \( \text{Pic}(S') \) is generated over \( \mathbb{Z} \) by \( R, v \) and \( M \). Finally, note

\[ |\det(R)|M = v \pmod{R}, \]

a generator of \( R^\perp \). Thus \( \text{Pic}(S') \) is generated over \( \mathbb{Z} \) by \( R \) and \( M \). \qed

3. Proof of Theorem 1.5

Let \( S \) be a rational homology projective plane with 4 or more quotient singularities with \( H_1(S^0, \mathbb{Z}) = 0 \). By Lemma 2.3, the orders of the abelianized local fundamental groups are pairwise relatively prime. Thus by Theorem 2.3, one can see that \( S \) has 4 singular points and the 4-tuple of orders of the local fundamental groups must be one of the following:

1. \( (2,3,5,q) \), \( q \geq 7 \),
2. \( (2,3,7,q) \), \( 11 \leq q \leq 41 \),
3. \( (2,11,13) \).

Table 1 shows that all noncyclic singularities of type different from \( I_m \) have abelianized local fundamental groups of order divisible by 2 or 3.

Assume that one of the singularities is noncyclic. By Lemma 2.5(3), it must be of type \( I_m \) and the other 3 singularities are cyclic of order 2, 3 and 5, respectively. Here we recall that \( I_m \subset \text{GL}(2, \mathbb{C}) \) is the \( 2m \)-ary icosahedral group of order 120m.

Table 1 shows that there are 8 infinite cases of type \( I_m \).

There are two types of order 3, \( < 3, 2 > \) and \( < 3, 1 > \); three types of order 5, \( < 5, 4 > \), \( < 5, 3 > \), \( < 5, 2 > \) and \( < 5, 1 > \). Thus there are exactly 48 infinite cases for possible combinations of types of singularities. That is, there are exactly 48 infinite cases for \( R \), the sublattice of \( \text{Pic}(S') = H^2(S', \mathbb{Z}) \) generated by all exceptional curves, where \( f : S' \to S \) is a minimal resolution. In each of the 48 cases we compute \( D = |\det(R)|K_S^2 \) and check if \( D \) is a square number (see Lemma 2.5(5)), using elementary number theoretic arguments. There remain 8 infinite cases and 2 sporadic cases, as given in Table 2 and Table 3. In both tables, the entries of the column \( b \) are the possible values of \( b \) that make \( D \) a square number.

We will explain how to compute \( D \). First note that

\[ |\det(R)| = 2 \cdot 3 \cdot 5 \cdot m = 30m. \]

To compute \( K_S^2 \), we use the equality from (2.2)

\[ K_S^2 = K_S^2 + \sum_p D_p K_{S'}^1. \]

Note that \( S' \) has \( H^1(S', \mathcal{O}_{S'}) = H^2(S', \mathcal{O}_{S'}) = 0 \). Thus by Noether formula,

\[ K_{S'}^2 = 12 - \varepsilon(S') = 10 - b_2(S') = 9 - \mu \]

where \( \mu \) is the number of the exceptional curves of \( f \).

For each singular point \( p \), the coefficients of the \( \mathbb{Q} \)-divisor \( D_p \) can be obtained by solving the equations given by the adjunction formula

\[ D_p E = -K_{S'} + 2 + E^2 \]

for each exceptional curve \( E \subset f^{-1}(p) \). Once we know the coefficients, we can easily compute the intersection number \( D_p K_{S'} \).

We first rule out the two sporadic cases.
Lemma 3.1. The case $< 2, 1 > + < 3, 2 > + < 5, 4 > + < b; 2, 1; 3, 2; 5, 4 >$ does not occur.

Proof. In this case, $m = 30(b - 2) + 7 = 187$, so

$$|\det(R)| = 30 \cdot 187.$$ 

The number of exceptional curves $\mu = 13$, so $K_{S'}^2 = -4$, where $f : S' \to S$ is a minimal resolution. Let $p_1, p_2, p_3, p_4$ be the four singular points. Let $E_1, \ldots, E_6$ be
### Table 3.

| Type of $R$ | $D = |\text{det}(R)|K_S^2$ | $b$ |
|-------------|-----------------|-----|
| $< 2, 1 > + < 3, 1 > + < 5, 4 > + < b; 2, 1; 3, 2; 5, 4 >$ | $20(45b^2 - 390b + 593)$ | none |
| $< 2, 1 > + < 3, 1 > + < 5, 4 > + < b; 2, 1; 3, 2; 5, 3 >$ | $20(45b^2 - 264b + 326)$ | none |
| $< 2, 1 > + < 3, 1 > + < 5, 4 > + < b; 2, 1; 3, 1; 5, 4 >$ | $100(9b^2 - 606 + 74)$ | none |
| $< 2, 1 > + < 3, 1 > + < 5, 4 > + < b; 2, 1; 3, 2; 5, 2 >$ | $20(45b^2 - 246b + 275)$ | none |
| $< 2, 1 > + < 3, 1 > + < 5, 4 > + < b; 2, 1; 3, 1; 5, 3 >$ | $20(45b^2 - 174b + 157)$ | none |
| $< 2, 1 > + < 3, 1 > + < 5, 4 > + < b; 2, 1; 3, 2; 5, 1 >$ | $100(3b - 4)^2$ | $b \geq 2$ |
| $< 2, 1 > + < 3, 1 > + < 5, 4 > + < b; 2, 1; 3, 1; 5, 2 >$ | $20(45b^2 - 156b + 124)$ | none |
| $< 2, 1 > + < 3, 1 > + < 5, 4 > + < b; 2, 1; 3, 1; 5, 1 >$ | $20(45b^2 - 306 - 17)$ | none |
| $< 2, 1 > + < 3, 1 > + < 5, 3 > + < b; 2, 1; 3, 2; 5, 4 >$ | $4(225b^2 - 1410b + 1903)$ | none |
| $< 2, 1 > + < 3, 1 > + < 5, 3 > + < b; 2, 1; 3, 2; 5, 3 >$ | $4(15b^2 - 26)^2$ | $b \geq 2$ |
| $< 2, 1 > + < 3, 1 > + < 5, 3 > + < b; 2, 1; 3, 1; 5, 4 >$ | $4(225b^2 - 960b + 968)$ | none |
| $< 2, 1 > + < 3, 1 > + < 5, 3 > + < b; 2, 1; 3, 2; 5, 2 >$ | $4(15b^2 - 23)^2$ | $b \geq 2$ |
| $< 2, 1 > + < 3, 1 > + < 5, 3 > + < b; 2, 1; 3, 1; 5, 3 >$ | $4(225b^2 - 330b + 11)$ | none |
| $< 2, 1 > + < 3, 1 > + < 5, 3 > + < b; 2, 1; 3, 2; 5, 1 >$ | $4(225b^2 - 606 - 338)$ | none |
| $< 2, 1 > + < 3, 1 > + < 5, 3 > + < b; 2, 1; 3, 1; 5, 2 >$ | $4(225b^2 - 240b - 46)$ | none |
| $< 2, 1 > + < 3, 1 > + < 5, 3 > + < b; 2, 1; 3, 1; 5, 1 >$ | $4(225b^2 + 390b - 643)$ | none |
| $< 2, 1 > + < 3, 1 > + < 5, 1 > + < b; 2, 1; 3, 2; 5, 4 >$ | $4(15b - 29)^2$ | $b \geq 2$ |
| $< 2, 1 > + < 3, 1 > + < 5, 1 > + < b; 2, 1; 3, 2; 5, 3 >$ | $4(225b^2 - 240b - 278)$ | none |
| $< 2, 1 > + < 3, 1 > + < 5, 1 > + < b; 2, 1; 3, 1; 5, 4 >$ | $4(225b^2 - 420b + 86)$ | none |
| $< 2, 1 > + < 3, 1 > + < 5, 1 > + < b; 2, 1; 3, 2; 5, 2 >$ | $4(225b^2 - 150b - 317)$ | none |
| $< 2, 1 > + < 3, 1 > + < 5, 1 > + < b; 2, 1; 3, 1; 5, 3 >$ | $4(225b^2 + 210b - 763)$ | none |
| $< 2, 1 > + < 3, 1 > + < 5, 1 > + < b; 2, 1; 3, 2; 5, 1 >$ | $4(225b^2 + 480b - 1076)$ | $b = 2$ |
| $< 2, 1 > + < 3, 1 > + < 5, 1 > + < b; 2, 1; 3, 1; 5, 2 >$ | $4(225b^2 + 300b - 712)$ | none |
| $< 2, 1 > + < 3, 1 > + < 5, 1 > + < b; 2, 1; 3, 1; 5, 1 >$ | $4(225b^2 + 930b - 1201)$ | none |

The components of $f^{-1}(p_4)$ such that

\[
\begin{align*}
E_2 & - E_3 - E_6 - E_5 - E_4 \\
\frac{1}{E_1} & - E_2
\end{align*}
\]

Solving the equations given by the adjunction formula, we get

\[
K_{S'} = f^*K_S - \frac{93E_1 + 186E_6 + 62E_2 + 124E_3 + 112E_4 + 149E_5}{187}.
\]
It is easy to compute that
\[ K_S^2 = K_{S'}^2 + \frac{186 E_6 K_{S'} + 112 E_4 K_{S'}}{187} = -4 + \frac{186 \cdot 6 + 112}{187} = \frac{480}{187}. \]
Thus
\[ D = |\text{det}(R)|K_S^2 = 120^2. \]
Note that \( K_S > 3e_{\text{orb}}(S) \), so \(-K_S\) is ample by the orbifold Bogomolov-Miyaoka-Yau inequality. Thus \( S' \) is a rational surface, not minimal. Also note that the divisor \( M \) from Lemma 2.5(6) takes the form
\[ M = \frac{1}{120} f^* K_S + \sum_{p \in \text{Sing}(S)} a_p e_p. \]
Let \( C \) be a \((-1)\)-curve on \( S' \). By Lemma 2.5(6), \( C \) can be written as
\[ C = kM + r \]
for some integer \( k \) and some \( r \in R \), hence as
\[ C = \frac{k}{120} f^* K_S + C(1) + C(2) + C(3) + C(4) \]
where \( C(i) \) is a \( \mathbb{Q} \)-divisor supported on \( f^{-1}(p_i) \). Note that
\[ C^2 = (\frac{k}{120} f^* K_S)^2 + C(1)^2 + C(2)^2 + C(3)^2 + C(4)^2. \]
Since \((f^* K_S)C(i) = 0\) for all \( i \), we have
\[ (f^* K_S)C = (f^* K_S)(\frac{k}{120} f^* K_S) = \frac{k}{120} K_S^2 = \frac{4k}{187}. \]
Since \(-K_S\) is ample and \( C \notin R \), we see that \((f^* K_S)C < 0\), hence \( k < 0 \).
Note that \( K_{S'}C = -1 \). From the equality
\[ K_{S'}C = (f^* K_S)C - \frac{(93 E_1 + 186 E_6 + 62 E_2 + 124 E_3 + 112 E_4 + 149 E_5)C}{187}, \]
we get
\[ (93 E_1 + 186 E_6 + 62 E_2 + 124 E_3 + 112 E_4 + 149 E_5)C = 187 + 4k. \]
This is possible only if
\[ E_6 C = E_5 C = E_4 C = E_3 C = 0, \quad E_2 C = E_1 C = 1, \quad k = -8. \]
Since \( E_j C(4) = E_j C \) for \( j = 1, \ldots, 6 \), we obtain the coefficients of \( C(4) \) by solving the equations given by the above intersection numbers.
\[ C(4) = -\frac{106 E_1 + 133 E_2 + 79 E_3 + 5 E_4 + 15 E_5 + 25 E_6}{187} = E_1^* + E_2^*, \]
where \( E_j^* \in \text{Hom}(R_{p_i}, \mathbb{Z}) \) is the dual vector of \( E_j \). Thus
\[ C(4)^2 = (E_1^* + E_2^*)C(4) = -\frac{106 + 133}{187}. \]
Now we have
\[ \sum_{j \leq 3} C(j)^2 = C^2 - C(4)^2 - (-8f^* K_S)^2 = -1 + \frac{239}{187} - \frac{32}{15 \cdot 187} \geq 0 \]
which contradicts the negative definiteness of exceptional curves. \( \square \)
Lemma 3.2. The case \( < 2, 1 > + < 3, 1 > + < 5, 1 > + < 2, 2; 1, 3, 2; 5, 1 > \) does not occur.

Proof. The proof is similar to the previous case. In this case, \( m = 19 \) and \( \mu = 8 \), so \(|\det(R)| = 30 \cdot 19 \) and \( K_S^2 = 1 \). Let \( B_2, B_3 \) be the components of \( f^{-1}(p_2) \), \( f^{-1}(p_3) \), \( E_1, \ldots, E_5 \) be the components of \( f^{-1}(p_4) \) such that

\[
\begin{align*}
-\frac{2}{E_2} & - \frac{2}{E_3} - \frac{2}{E_5} - \frac{5}{E_4} \\
\frac{1}{E_1} & \quad \text{for some integer } K
\end{align*}
\]

Then

\[
K_{S'} = f^*K_S - \frac{B_2}{3} - \frac{3B_3}{5} - \frac{9E_1 + 6E_2 + 12E_3 + 15E_4 + 18E_5}{19}.
\]

\[
K_S^2 = \frac{28 \cdot 56}{15 \cdot 19}, \quad D = |\det(R)|K_S^2 = 56^2.
\]

Here again by the orbifold Bogomolov-Miyaoka-Yau inequality, \( -K_S \) is ample and \( S' \) is a rational surface, not minimal. Let \( C \) be a \((-1)\)-curve on \( S' \). Then

\[
C = \frac{k}{56}f^*K_S + C(1) + C(2) + C(3) + C(4)
\]

for some integer \( k \) and some \( \mathbb{Q} \)-divisor \( C(i) \) supported on \( f^{-1}(p_i) \).

Since \((f^*K_S)C = \frac{28k}{285} < 0\), we see that \( k < 0 \) and we get

\[
95B_2C + 171B_3C + 15(9E_1 + 6E_2 + 12E_3 + 15E_4 + 18E_5)C = 285 + 28k.
\]

This is impossible because \( k < 0 \) and \( E_jC \geq 0, B_iC \geq 0 \) for every \( i, j \).

\[
\square
\]

Lemma 3.3. For any of the 8 infinite cases, \(-K_S\) is ample.

Proof. For the 8 infinite cases, we compute \( K_S^2 \) as follows.

| Type of \( R \) | \( K_S^2 \) |
|---|---|
| \(< 2, 1 > + < 3, 2 > + < 5, 4 > + < b; 2, 1; 3, 1; 5, 1 >\) | \( \frac{30(b-1)^2}{309-31} \geq \frac{30}{29} \) |
| \(< 2, 1 > + < 3, 2 > + < 5, 2 > + < b; 2, 1; 3, 1; 5, 3 >\) | \( \frac{6(5b-7)^2}{5(309-43)} \geq \frac{16}{5} \) |
| \(< 2, 1 > + < 3, 2 > + < 5, 3 > + < b; 2, 1; 3, 1; 5, 2 >\) | \( \frac{6(5b-8)^2}{5(309-49)} \geq \frac{57}{55} \) |
| \(< 2, 1 > + < 3, 2 > + < 5, 1 > + < b; 2, 1; 3, 1; 5, 4 >\) | \( \frac{10(3b-4)^2}{5(309-41)} \geq \frac{57}{57} \) |
| \(< 2, 1 > + < 3, 1 > + < 5, 4 > + < b; 2, 1; 3, 2, 5, 1 >\) | \( \frac{2(15b-29)^2}{15(309-53)} \geq \frac{132}{106} \) |
| \(< 2, 1 > + < 3, 1 > + < 5, 2 > + < b; 2, 1; 3, 2, 5, 3 >\) | \( \frac{2(15b-23)^2}{15(309-47)} \geq \frac{98}{195} \) |
| \(< 2, 1 > + < 3, 1 > + < 5, 3 > + < b; 2, 1; 3, 2, 5, 2 >\) | \( \frac{2(15b-29)^2}{15(309-59)} \geq \frac{2}{15} \) |

In each case, \( e_{orb}(S) = -1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{120m} \leq \frac{5}{120} \). From the table we see that \( K_S^2 > 3e_{orb}(S) \), so \(-K_S\) is ample by the orbifold Bogomolov-Miyaoka-Yau inequality.

\[
\square
\]
This completes the proof of (1) and (2) of Theorem \ref{thm:main}. To prove the remaining part, we need to analyze \((-1)\)-curves on the minimal resolution \(S'\). Note that by Lemma\ref{lem:1.1} \(S'\) contains no \((-n)\)-curve with \(n \geq 2\) other than the exceptional curves of \(f : S' \to S\).

The following proposition will be proved case by case in the next section.

**Proposition 3.4.** If \(S\) has 4 singularities \(p_1, p_2, p_3, p_4\) of type \(< 2, 1 >, < 3, \alpha >, < 5, \beta >, < b; 2, 1; 3, 3 - \alpha; 5, 5 - \beta >, b \geq 2\), respectively, as in Table 4, then there are three mutually disjoint \((-1)\)-curves \(C_1, C_2, C_3\) on \(S'\) such that

1. each \(C_i\) intersects exactly 2 components of \(f^{-1}(p_1) \cup f^{-1}(p_2) \cup f^{-1}(p_3) \cup f^{-1}(p_4)\) with multiplicity 1 each,
2. \(C_1\) intersects the component of the branch \(< 2, 1 >\) of \(f^{-1}(p_4)\) and the component of \(f^{-1}(p_1)\), \(C_2\) intersects the terminal component of the branch \(< 3, 3 - \alpha >\) of \(f^{-1}(p_4)\) and one end component of \(f^{-1}(p_2)\), and \(C_3\) intersects the terminal component of the branch \(< 5, 5 - \beta >\) of \(f^{-1}(p_4)\) and one end component of \(f^{-1}(p_3)\) which is a \((-1)\)-curve if \(\beta = 2\) or 4, a \((-3)\)-curve if \(\beta = 3\), and a \((-5)\)-curve if \(\beta = 1\).

**Proposition 3.5.**

1. The surface \(S'\) can be blown down to the Hirzebruch surface \(F_b\). Conversely, \(S'\) can be obtained by starting with \(F_b\) and blowing up inside 3 of the fibres, i.e. the blowing up starts at three centers, one on each of the 3 fibres.

2. If two rational homology projective planes \(S_1\) and \(S_2\) have the same type of singularities \(< 2, 1 > + < 3, \alpha > + < 5, \beta > + < b; 2, 1; 3, 3 - \alpha; 5, 5 - \beta >, b \geq 2\), then \(S_1^0\) and \(S_2^0\) are deformation equivalent.

**Proof.** (1) By Proposition \ref{prop:3.4} there are three mutually disjoint \((-1)\)-curves \(C_1, C_2, C_3\) on \(S'\) satisfying (1) and (2) of Proposition \ref{prop:3.3}. By starting with them, we can blow down \(S'\) to \(F_b\). Furthermore, the blow up process from \(F_b\) to \(S'\) is carried out inside 3 of the fibres of \(F_b\).

(2) The blow up process from \(F_b\) to \(S'\) depends on the choice of three fibres, each with a point marked. The three marked points are the centers of the blowing up. The choice of three fibres is unique up to automorphisms of \(F_b\), while the choice of three points, one on each of the fixed three fibres, is not unique up to automorphisms of \(F_b\), but depends on a 2-dimensional moduli. \(\square\)

This completes the proof of (3) of Theorem \ref{thm:main}.

The following examples mentioned in Introduction were discussed in \cite{Kol08}, Example 31.

**Example 3.6.** Consider the \(2m\)-ary icosahedral group

\[ I_m \subset GL(2, \mathbb{C}) \]

of order 120m in Brieskorn’s list (Table 1). Let \(Z \subset I_m\) be its center, then \(Z \cong \mathbb{Z}_{2m}\) and \(I_m/Z \cong \mathfrak{A}_5 \subset PGL(2, \mathbb{C})\), the icosahedral group. Extend the natural \(I_m\)-action on \(\mathbb{C}^2\) to \(\mathbb{C}P^2\). The center acts trivially on the line at infinity and \(\mathbb{C}P^2/Z\) is a cone over the rational normal curve of degree 2m = |\(Z\)|. Then

\[ S_{I_m} := \mathbb{C}P^2/I_m = (\mathbb{C}P^2/Z)/\mathfrak{A}_5 \]

has 4 quotient singularities, one of type \(\mathbb{C}^2/I_m\) at the origin, three of order 2, 3, 5 at infinity. The fundamental group of \(S_{I_m}^0\) is \(\mathfrak{A}_5\). By Theorem \ref{thm:main} (1), the types
of the 3 cyclic singularities are determined by the types of the 3 branches of the non-cyclic singularity. By Proposition 3.4 and 3.5, its minimal resolution $S'_{I_m}$ can be blown down to the Hirzebruch surface $F_b$. Conversely, $S'_{I_m}$ can be obtained by starting with $F_b$ and blowing up inside 3 of the fibres. Here the 3 centers of the blowing up lie on a section of $F_b$.

In Proposition 3.5, the 3 centers of the blowing up lie on a section of $F_b$ if and only if the surface $S'$ is isomorphic to $S'_{I_m}$ for some $I_m$. This completes the proof of (4) and (5) of Theorem 1.5.

4. Proof of Proposition 8.21

As before, let $p_1, p_2, p_3, p_4$ be the singular points of $S$ of order 2, 3, 5, 120, respectively, and let $f: S' \to S$ be a minimal resolution. Let $R_{p_i}$ be the sublattice of $H^2(S', \mathbb{Z})$ generated by all exceptional curves contained in $f^{-1}(p_i)$.

Let $C$ be an irreducible curve on $S'$. By Lemma 2.5(6), $C$ can be written as $C = kM + r$ for some integer $k$ and some $r \in R$, hence as

\[(4.1) \quad C = \frac{k}{\sqrt{D}}f^*K_S + C(1) + C(2) + C(3) + C(4)\]

where $C(i)$ is a $\mathbb{Q}$-divisor supported on $f^{-1}(p_i)$ that is of the form

\[C(i) = a_ie_i + r_i\]

for some integer $a_i$ and some $r_i \in R_{p_i}$, where $e_i$ is a generator of the discriminant group $\text{disc}(R_{p_i})$.

Lemma 4.1. Let $C$ be an irreducible curve on $S'$ of the form (4.1).

1. $C(i)^2 = 0$ if and only if $C(i) = 0$ if and only if $C$ does not meet $f^{-1}(p_i)$.
2. $C(1)^2 = -\frac{1}{2}x$ for some integer $x \geq 0$.
   - $C(2)^2 = -\frac{1}{2}y$ if and only if $C$ meets with multiplicity 1 the component of $f^{-1}(p_2)$.
3. Assume that $p_2$ is of type $< 3, 2 >$. Then
   - $C(2)^2 = -\frac{4}{3}y$ if and only if $C$ meets with multiplicity 1 exactly one of the two components of $f^{-1}(p_2)$.
4. Assume that $p_3$ is of type $< 5, 4 >$. Then
   - $C(3)^2 \leq -\frac{4}{3}$ if $C(3) \neq 0$.
   - $C(3)^2 = -\frac{4}{9}$ if and only if $C$ meets with multiplicity 1 exactly one of the two end components of $f^{-1}(p_3)$.

Proof. (1) The first equivalence follows from the negative definiteness of exceptional curves. Note that $EC = EC(i)$ for any curve $E \subset f^{-1}(p_i)$.

The curve $C$ does not meet $f^{-1}(p_i) \iff EC = 0$ for any curve $E \subset f^{-1}(p_i) \iff EC(i) = 0$ for any curve $E \subset f^{-1}(p_i) \iff C(i) = 0$.

(2) is trivial.

(3) Let $E_1, E_2$ be the exceptional curves generating $R_{p_2}$. Take

\[e := \frac{E_1 + 2E_2}{3} = E_2^*\]

as a generator of $\text{disc}(R_{p_2})$. Then $C(2)$ is of the form $C(2) = ae + b_1E_1 + b_2E_2$ for some integers $a, b_1, b_2$, hence of the form $C(2) = se + tE_2$ for some integers $s, t$. We
have
\[ C(2)^2 = -\frac{2}{3}(s^2 - 3st + 3t^2). \]
It is easy to see that \( y := s^2 - 3st + 3t^2 = (s - 3t/2)^2 + 3t^2/4 \geq 0 \) for all \( s, t \).

\( C \) meets exactly one of the two components of \( f^{-1}(p_2) \) with multiplicity 1 \( \iff \) 
\((E_1C(2), E_2C(2)) = (1, 0) \) or \((0, 1) \iff C(2) = E_1^* = 2e + E_2 \) or \( C(2) = E_2^* = e \iff \( s, t \) = (2, 1) \) or \((1, 0) \Rightarrow C(2)^2 = -2/3. \) Conversely, if \( C(2)^2 = -2/3, \) then there are six solutions \( (s, t) = (\pm 1, 0), (\pm 2, 1), (\pm 1, 1) \) for \( y = (s - 3t/2)^2 + 3t^2/4 = 1. \) Since \( E_iC(2) = E_iC \geq 0 \) for \( i = 1, 2, \) there remain only two solutions \( (s, t) = (1, 0), (2, 1). \)

(4) Let \( E_1, E_2, E_3, E_4 \) be the exceptional curves generating \( R_{p_3}. \) Take
\[ e := -\frac{1}{5}E_1 + 2E_2 + 3E_3 + 4E_4 = E_4^* \]
as a generator of \( \text{disc}(R_{p_3}). \) Then \( C(3) \) is of the form \( C(3) = ae + b_1E_1 + b_2E_2 + b_3E_3 + b_4E_4 \) for some integers \( a, b_1, b_2, b_3, b_4, \) hence of the form \( C(3) = se + uE_2 + vE_3 + wE_4 \) for some integers \( s, u, v, w. \) We have
\[
C(3)^2 = -\frac{1}{4} s^2 - 2u^2 - 2v^2 - 2w^2 + 2sw + 2uv + 2vw = -\frac{1}{4} (s - \frac{u}{2})^2 + \frac{5}{2} (u - \frac{v}{2})^2 + \frac{15}{8} (v - \frac{w}{3})^2 + \frac{5}{48} w^2.
\]

To prove the first assertion, assume that
\[ (s - \frac{5w}{4})^2 + \frac{5}{2} (u - \frac{v}{2})^2 + \frac{15}{8} (v - \frac{w}{3})^2 + \frac{5}{48} w^2 < 1. \]
We need to show that \( (s, u, v, w) = (0, 0, 0, 0) \) to the inequality. If \( w = 0, \) then there is only one solution \( (s, u, v, w) = (0, 0, 0, 0) \) to the inequality. If \( w = \pm 1, \pm 2, \pm 3 \), no solution to the inequality. This proves the first assertion.

\( C \) meets exactly one of the two end components of \( f^{-1}(p_3) \) with multiplicity 1 \( \iff \) 
\((E_1C, E_2C, E_3C, E_4C) = (1, 0, 0, 0) \) or \((0, 0, 0, 1) \iff C(3) = E_1^* = 4e + E_2 + 2E_3 + 3E_4 \) or \( C(3) = E_4^* = e \iff (s, u, v, w) = (4, 1, 2, 3) \) or \((1, 0, 0, 0) \Rightarrow C(3)^2 = -4/5. \) Conversely, if \( C(3)^2 = -\frac{4}{5}, \) then
\[ (s - \frac{5w}{4})^2 + \frac{5}{2} (u - \frac{v}{2})^2 + \frac{15}{8} (v - \frac{w}{3})^2 + \frac{5}{48} w^2 = 1. \]

There are ten solutions \( (s, u, v, w) = (\pm 1, 0, 0, 0), (\pm 4, 1, 2, 3), (\pm 1, 1, 1, 1), (\pm 1, 0, 1, 1), (\pm 1, 0, 0, 0). \)
Since \( E_iC(3) = E_iC \geq 0 \) for \( i = 1, 2, 3, 4, \) there remain only two solutions \( (s, u, v, w) = (4, 1, 2, 3), (1, 0, 0, 0). \)

**Case 1:** \(< 2, 1 > + < 3, 2 > + < 5, 4 > + < b; 2, 1; 3, 1; 5, 1 >, b \geq 2. \) In this case, the number of exceptional curves \( \mu = 11, \) so \( K_{S'}^2 = -2. \) Let \( E_1, \ldots, E_4 \) be the components of \( f^{-1}(p_4) \) such that
\[ -3 E_2 - E_4 - E_3 \]
\[ \begin{vmatrix}
\frac{1}{2} \\
E_1
\end{vmatrix}
\]

We compute
\[
(4.2) \quad K_{S'} = f^*K_S - \frac{(15b - 16)E_1 + (20b - 21)E_2 + (24b - 25)E_3 + (30b - 32)E_4}{30b - 31}.
\]
Claim 4.1.1. Let $C$ be a $(-1)$-curve of the form (4.1). Suppose that $C$ meets $f^{-1}(p_4)$. Then it satisfies one of the following three cases:

| Case | $CE_4$ | $CE_3$ | $CE_2$ | $CE_1$ | $k$ |
|------|--------|--------|--------|--------|-----|
| (a)  | 0      | 0      | 0      | 1      | $-15$ |
| (b)  | 0      | 0      | 1      | 0      | $-10$ |
| (c)  | 0      | 1      | 0      | 0      | $-6$  |

Proof. We use the same argument as in the proof of Lemma 3.1. First note that

\[
(f^* K_S) C = \frac{1}{f^*} (f^* K_S)^2 = \frac{1}{30(b-1)}.
\]

Since $-K_S$ is ample and $C \not\subset R$, $(f^* K_S) C < 0$, so $k < 0$. Intersecting $C$ with $(4.2)$ we get

\[
C \{ (15b-16) E_1 + (20b-21) E_2 + (24b-25) E_3 + (30b-32) E_4 \} = (b-1) k + 30b - 31.
\]

This is possible only if $C$ satisfies one of the three cases (a), (b), (c), or the case (d) $CE_4 = 1, CE_3 = 0, CE_2 = 0, CE_1 = 0, b = 2, k = -1$.

In the last case, we compute $C(4) = E_4^* = -\frac{1}{29} (15E_1 + 10E_2 + 6E_3 + 30E_4)$, so $C(4)^2 = C_1 C(4) = -\frac{30}{29}$ and hence we get

\[
\sum_{j \leq 3} C(j)^2 = C^2 - C(4)^2 - (\frac{-1}{30} f^* K_S)^2 = -1 + \frac{30}{29} - \frac{1}{30 \cdot 29} > 0,
\]

contradicts the negative definiteness of exceptional curves. □

Claim 4.1.2. Let $C$ be a $(-1)$-curve of the form (4.1). Suppose that $C$ meets $f^{-1}(p_4)$. Then $C$ meets only one component of $f^{-1}(p_1) \cup f^{-1}(p_2) \cup f^{-1}(p_3)$, the intersection multiplicity is 1, and the component is

1. the component of $f^{-1}(p_1)$, if $C$ satisfies (a),
2. one of the two components of $f^{-1}(p_2)$, if $C$ satisfies (b),
3. one of the two end components of $f^{-1}(p_3)$, if $C$ satisfies (c).

Proof. Assume that $C$ satisfies (a). Then, $C(4) = E_4^* = E_1 C(4) = -\frac{15b-8}{30b-31}$, $C(1)^2 + C(2)^2 + C(3)^2 = C^2 - C(4)^2 - (\frac{15}{30(b-1)} f^* K_S)^2 = -\frac{1}{2}$.

By Lemma 4.1, $C(2) = C(3) = 0, C(4)^2 = -\frac{1}{2}$, and $C$ does not meet $f^{-1}(p_2) \cup f^{-1}(p_3)$, but meets the component of $f^{-1}(p_1)$ with multiplicity 1.

Assume that $C$ satisfies (b). Then, $C(4) = E_3^*, C(4)^2 = E_2 C(4) = -\frac{15b - 7}{30b - 31}$, $C(1)^2 + C(2)^2 + C(3)^2 = C^2 - C(4)^2 - (\frac{15}{30(b-1)} f^* K_S)^2 = -\frac{2}{5}$.

By Lemma 4.1, $C(1) = C(3) = 0, C(2)^2 = -\frac{2}{5}$, and $C$ does not meet $f^{-1}(p_1) \cup f^{-1}(p_3)$, but meets one of the two components of $f^{-1}(p_2)$ with multiplicity 1.

Assume that $C$ satisfies (c). Then, $C(4) = E_3^*, C(4)^2 = E_2 C(4) = -\frac{6b - 5}{30b - 31}$, $C(1)^2 + C(2)^2 + C(3)^2 = C^2 - C(4)^2 - (\frac{6b - 5}{30(b-1)} f^* K_S)^2 = -\frac{2}{5}$.

By Lemma 4.1, $C(1) = C(2) = 0, C(3)^2 = -\frac{2}{5}$, and $C$ does not meet $f^{-1}(p_1) \cup f^{-1}(p_2)$, but meets one of the end components of $f^{-1}(p_3)$ with multiplicity 1. □

\[
K_S^2 = \frac{30(b-1)^2}{30b-31}, \quad |\text{det}(R)| = 30 \cdot (30b-31), \quad D = |\text{det}(R)| K_S^2 = 30^2(b-1)^2.
\]
Claim 4.1.3. There are three, mutually disjoint, \((-1)\)-curves \(C_1, C_2, C_3\) satisfying (a), (b), (c) from Claim 4.1.1, respectively.

Proof. By Lemma 3.3 \(S'\) is a rational surface. Since \(K_{S'}^2 < 8\), \(S'\) contains a \((-1)\)-curve and can be blown down to a minimal rational surface \(F_n\) or \(\mathbb{CP}^2\).

Assume that there is no \((-1)\)-curve \(C \subset S'\) meeting \(f^{-1}(p_4)\). Then, since \(S'\) cannot contain a \((-l)\)-curve with \(l > 2\) other than the exceptional curves of \(f\) (Lemma 2.1), the configuration of \(f^{-1}(p_4)\) remains the same under the blow down process to \(F_n\) or \(\mathbb{CP}^2\). This is impossible, as the configuration would define a negative definite sublattice of rank 4 inside the Picard lattice of \(F_n\) or \(\mathbb{CP}^2\).

Assume that there is only one \((-1)\)-curve meeting \(f^{-1}(p_4)\). Then, the 3 components of \(f^{-1}(p_4)\) untouched by the \((-1)\)-curve remain the same under the blow down process and define a negative definite sublattice of rank 3 inside the Picard lattice of \(F_n\) or \(\mathbb{CP}^2\). This is impossible.

If there are only two \((-1)\)-curve meeting \(f^{-1}(p_4)\). Then the 2 components of \(f^{-1}(p_4)\) untouched by the two \((-1)\)-curves would remain the same under the blow down process and define a negative definite sublattice of rank 2 inside the Picard lattice of \(F_n\) or \(\mathbb{CP}^2\). Again, this is impossible.

For the mutual disjointness, we note that
\[
\begin{align*}
C_1 &= \frac{1}{30(b-1)} f^*K_S + C_1(1) + E_1, \\
C_2 &= \frac{10}{30(b-1)} f^*K_S + C_2(2) + E_2, \\
C_3 &= \frac{6}{30(b-1)} f^*K_S + C_3(3) + E_3.
\end{align*}
\]
A direct calculation shows that \(C_iC_j = 0\) for \(i \neq j\). \(\square\)

4.2. Case 2: \(< 2, 1 > + < 3, 2 > + < 5, 2 > + < b; 2, 1; 3, 1; 5, 3 >, b \geq 2\). In this case, \(\mu = 10\), so \(K_{S'}^2 = -1\). Let \(B_1, B_2\) be the components of \(f^{-1}(p_3)\), and \(E_1, \ldots, E_5\) be the components of \(f^{-1}(p_4)\) such that
\[
\begin{pmatrix}
-2 & -3 \\
B_1 - B_2 & E_2 - E_5 - E_4 - E_3 \\
1 & E_1 \end{pmatrix}
\]
Then
\[
K_{S'} = f^*K_S - \frac{1}{5}(B_1 + 2B_2) - \frac{1}{30(b-43)} \{(15b - 22)E_1 + (20b - 29)E_2 + (18b - 26)E_3 + (24b - 35)E_4 + (30b - 44)E_5\},
\]

\[
K_S^2 = \frac{6(5b - 7)^2}{5(30b - 43)}, \quad |\det(R)| = 30 \cdot (30b - 43), \quad D = 6^2(5b - 7)^2.
\]

We also compute the dual vectors,
\[
\begin{align*}
B_1^* &= -\frac{3B_1 + B_2}{5}, & B_2^* &= -\frac{B_1 + 2B_2}{5}, \\
E_1^* &= -\frac{1}{30b - 43} \{(15b - 14)E_1 + 5E_2 + 3E_3 + 9E_4 + 15E_5\}, \\
E_2^* &= -\frac{1}{30b - 43} \{5E_1 + (10b - 11)E_2 + 2E_3 + 6E_4 + 10E_5\}, \\
E_3^* &= -\frac{3E_1 + 2E_2 + (12b - 16)E_3 + (6b - 5)E_4 + 6E_5}{30b - 43}.
\end{align*}
\]
Claim 4.2.1. Let $C$ be a $(-1)$-curve of the form (4.1). Suppose that $C$ meets $f^{-1}(p_4)$. Then it satisfies one of the following three cases:

| Case | $CE_5$ | $CE_4$ | $CE_3$ | $CE_2$ | $CE_1$ | $CB_2$ | $CB_1$ | $k$ |
|------|--------|--------|--------|--------|--------|--------|--------|-----|
| (a)  | 0      | 0      | 0      | 0      | 1      | 0      | 0      | 15 |
| (b)  | 0      | 0      | 0      | 1      | 0      | 0      | 0      | 10 |
| (c)  | 0      | 0      | 1      | 0      | 0      | 0      | 1      | 6  |

Proof. First note that $(f^* K_S)C = \frac{k}{\sqrt{B}}(f^* K_S)^2 = (\frac{5b-7}{3(30b-43)})$. Since $-K_S$ is ample and $C \notin R$, we see that $k < 0$. Intersecting $C$ with $C_3$ we get 

\[(30b - 43)C(B_1 + 2B_2) + 5C((15b - 22)E_1 + (20b - 29)E_2 + (18b - 26)E_3 + (24b - 35)E_4 + (30b - 44)E_5) = (5b - 7)K + 5(30b - 43) < 5(30b - 43).

This is possible only if $C$ satisfies one of the three cases or the following case 

(d) $CE_5 = 0, CE_4 = 1, CE_3 = CE_2 = CE_1 = 0, CB_1 = 1, CB_2 = 0, b = 2, k = -1$.

In case (d), $C(3) = B_1$ and $C(4) = E_2 = \frac{1}{17}(9E_1 + 6E_2 + 7E_3 + 21E_4 + 18E_5)$, thus $C(1)^2 + C(2)^2 = C^2 - C(3)^2 - C(4)^2 - (\frac{15}{6(30b-43)})f^* K_S)^2 = -1 + \frac{3}{3} + \frac{2}{3} - \frac{1}{30b-43} > 0$, contradicts the negative definiteness of exceptional curves.

Claim 4.2.2. Let $C$ be a $(-1)$-curve of the form (4.1). Suppose that $C$ meets $f^{-1}(p_4)$. Then $C$ meets only one component of $f^{-1}(p_1) \cup f^{-1}(p_2) \cup f^{-1}(p_3)$, the intersection multiplicity is 1, and the component is 

(1) the component of $f^{-1}(p_1)$, if $C$ satisfies (a),

(2) one of the two components of $f^{-1}(p_2)$, if $C$ satisfies (b),

(3) the component $B_1$ of $f^{-1}(p_3)$, if $C$ satisfies (c).

Proof. Assume that $C$ satisfies (a). Then, $C(3) = 0$ and $C(4) = E_1^*$, so 

$C(1)^2 + C(2)^2 = C^2 - C(3)^2 - C(4)^2 - (\frac{15}{6(30b-43)})f^* K_S)^2 = -1$.

By Lemma 4.1 $C(2) = 0$ and $C(1)^2 = -\frac{1}{2}$.

Assume that $C$ satisfies (b). Then, $C(3) = 0$ and $C(4) = E_2^*$, so 

$C(1)^2 + C(2)^2 = C^2 - C(3)^2 - C(4)^2 - (\frac{15}{6(30b-43)})f^* K_S)^2 = -\frac{2}{3}$.

By Lemma 4.1 $C(1) = 0$ and $C(2)^2 = -\frac{2}{3}$.

Assume that $C$ satisfies (c). Then, $C(3) = B_1^* = \frac{3B_1}{30b-43}$ and $C(4) = E_3^*$, so 

$C(1)^2 + C(2)^2 = C^2 - C(3)^2 - C(4)^2 - (\frac{15}{6(30b-43)})f^* K_S)^2 = 0$.

By the negative definiteness, $C(1) = C(2) = 0$.

By the same proof as in the previous case, we see that there are three, mutually disjoint, $(-1)$-curves $C_1, C_2, C_3$ satisfying (a), (b), (c) from Claim 4.2.1, respectively.

4.3. Case: $< 2, 1 > + < 3, 2 > + < 5, 3 > + < b; 2, 1; 3, 1; 5, 2 >, b \geq 2$. In this case, $\mu = 10$, so $K_5^2 = -1$. Let $B_1, B_2$ be the components of $f^{-1}(p_3)$, and $E_1, \ldots, E_5$ be the components of $f^{-1}(p_4)$ such that

\[
\begin{array}{ccccccc}
 \frac{1}{3} & -2 & -3 & -\frac{b}{3} & -\frac{3}{3} & -\frac{2}{3} \\
 B_1 & B_2 & E_2 & -E_5 & -E_4 & -E_3 \\
\end{array}
\]

Then

\[
K_5' = f^* K_2 - \frac{1}{5}(B_1 + 2B_2) - \frac{1}{30b-43}\{(15b-19)E_1 + (20b-25)E_2 + (12b-15)E_3 + (24b-30)E_4 + (30b-38)E_5\},
\]
The same proof as in the previous cases shows that there are three, mutually dis-

tinct curves $C_1, C_2, C_3$ satisfying (a), (b), (c) from Claim 4.3.1, respectively.

The same proof as in the previous cases shows that there are three, mutually dis-

tinct curves $C_1, C_2, C_3$ satisfying (a), (b), (c) from Claim 4.3.1, respectively.
4.4. **Case 4:** $< 2, 1 > + < 3, 2 > + < 5, 1 > + < b; 2, 1; 3, 1; 5, 4 >, b \geq 2$. In this case, $\mu = 11$, so $K_S^* = -2$. Let $B$ be the component of $f^{-1}(p_3)$, and $E_1, \ldots, E_7$ be the components of $f^{-1}(p_4)$ such that

$$\begin{align*}
E_2 - E_7 - E_6 - E_5 - E_4 - E_3
\end{align*}$$

Then

$$\begin{align*}
E_2 - E_7 - E_6 - E_5 - E_4 - E_3
\end{align*}$$

$$(4.5)$$

$K_S^* = \frac{6(5b - 8)^2}{5(30b - 49)}.$$

We also compute the dual vectors,

$$\begin{align*}
E_1^* &= -\frac{1}{30b - 49}\{15b - 17\}E_1 + 5E_2 + 3E_3 + 6E_4 + 9E_5 + 12E_6 + 15E_7, \\
E_2^* &= -\frac{1}{30b - 49}\{15b - 17\}E_1 + 5E_2 + 3E_3 + 6E_4 + 9E_5 + 12E_6 + 15E_7, \\
E_3^* &= -\frac{1}{30b - 49}\{15b - 17\}E_1 + 5E_2 + 3E_3 + 6E_4 + 9E_5 + 12E_6 + 15E_7.
\end{align*}$$

Claim 4.4.1. Let $C$ be a $(-1)$-curve of the form $(4.1)$. Suppose that $C$ meets $f^{-1}(p_4)$. Then it satisfies one of the following three cases:

| Case | CE7 | CE6 | CE5 | CE4 | CE3 | CE2 | CE1 | CB | k |
|------|-----|-----|-----|-----|-----|-----|-----|----|---|
| (a)  | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0  | -15 |
| (b)  | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0  | -10 |
| (c)  | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 1  | -6  |

Proof. Since $(f^*K_S)C = \frac{(5b - 8)^k}{5(30b - 49)} < 0$, $k < 0$. Intersecting $C$ with $(4.5)$ we get

$$\begin{align*}
3(30b - 49)CB + 5C\{15b - 25\}E_1 + (20b - 33)E_2 + (24b - 40)E_3 + (18b - 30)E_5 + (24b - 40)E_6 + (30b - 50)E_7.
\end{align*}$$

This is possible only if $C$ satisfies one of the three cases, or one of the two cases:

| Case | CE7 | CE6 | CE5 | CE4 | CE3 | CE2 | CE1 | CB | k | b |
|------|-----|-----|-----|-----|-----|-----|-----|----|----|---|
| (d)  | 0   | 0   | 0   | 0   | 2   | 0   | 0   | 1  | -1 | 2 |
| (e)  | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 1  | -1 | 2 |

In Case (d), $C(3) = B^* = -\frac{1}{2}B$ and $C(4) = 2E_3^*$, thus

$$\begin{align*}
C(1)^2 + C(2)^2 &= C^2 - C(3)^2 - C(4)^2 = -\frac{1}{12}(f^*K_S)^2 = -1 + \frac{1}{2} + \frac{40}{11} - \frac{1}{30b - 49} > 0.
\end{align*}$$

In Case (e), $C(3) = -\frac{1}{2}B$ and $C(4) = E_4^* = -\frac{1}{12}(6E_1 + 4E_2 + 9E_3 + 18E_4 + 16E_5 + 14E_6 + 12E_7)$, thus

$$\begin{align*}
C(1)^2 + C(2)^2 &= C^2 - C(3)^2 - C(4)^2 = -\frac{1}{12}(f^*K_S)^2 = -1 + \frac{1}{2} + \frac{18}{11} - \frac{1}{30b - 49} > 0.
\end{align*}$$

Both contradict the negative definiteness of exceptional curves.

Claim 4.4.2. Let $C$ be a $(-1)$-curve of the form $(4.1)$. Suppose that $C$ meets $f^{-1}(p_4)$. Then $C$ meets only one component of $f^{-1}(p_1) \cup f^{-1}(p_2) \cup f^{-1}(p_3)$, the intersection multiplicity with the component is 1, and the component is

1. the component of $f^{-1}(p_1)$, if $C$ satisfies (a),
2. one of the two components of $f^{-1}(p_2)$, if $C$ satisfies (b),
3. the component $B$ of $f^{-1}(p_3)$, if $C$ satisfies (c).
Proof. Assume that $C$ satisfies (a). Then, $C(3) = 0$ and $C(4) = E_1^*$, so $C(4)^2 = -\frac{15b-17}{30b-49}$ and $C(1)^2 + C(2)^2 = C^2 - C(4)^2 - (\frac{15}{6(35-8)} f^* K_S)^2 = -\frac{1}{2}$. By Lemma 4.1, $C(2) = 0$ and $C(1)^2 = -\frac{1}{2}$.

Assume that $C$ satisfies (b). Then, $C(3) = 0$ and $C(4) = E_2^*$, so $C(1)^2 + C(2)^2 = -1 + \frac{10b-13}{30b-49} - (\frac{-10}{6(35-8)} f^* K_S)^2 = -\frac{2}{3}$. By Lemma 11, $C(1) = 0$ and $C(2)^2 = -\frac{2}{3}$.

Assume that $C$ satisfies (c). Then, $C(3) = -\frac{1}{5} B$ and $C(4) = E_3^*$, so $C(1)^2 + C(2)^2 = -1 + \frac{1}{5} + \frac{24b-38}{30b-49} - (\frac{-6}{6(35-8)} f^* K_S)^2 = 0$. By the negative definiteness, $C(1) = C(2) = 0$. □

The same proof as in the previous cases shows that there are three, mutually disjoint, $(-1)$-curves $C_1, C_2, C_3$ satisfying (a), (b), (c) from Claim 4.4.1, respectively.

4.5. Case 5: $< 2, 1 > + < 3, 1 > + < 5, 4 > + < b, 2, 1 ; 3, 2, 5, 1 >$, $b \geq 2$. In this case, $\mu = 11$, so $K^2_{S'} = -2$. Let $B$ be the component of $f^-(p_2)$, and $E_1, \ldots, E_5$ be the components of $f^-(p_4)$ such that

\[
\begin{array}{c}
-\frac{2}{E_2 - E_3 - E_5 - E_4} \\
\frac{1}{E_1}
\end{array}
\]

Then

\[
K_{S'} = f^* K_S - \frac{1}{5} B - \frac{1}{30b-41} \{(15b-21) E_1 + (10b-14) E_2 + (20b-28) E_3 + (24b-33) E_4 + (30b-42) E_5\},
\]

\[
K^2_S = \frac{10(3b-4)^2}{3(30b-41)}, \quad |\det(R)| = 30 \cdot (30b-41), \quad D = 10^2(3b-4)^2.
\]

We also compute the dual vectors,

\[
\begin{aligned}
E_1^* &= -\frac{1}{30b-41} \{(15b - 13) E_1 + 5 E_2 + 10 E_3 + 3 E_4 + 15 E_5\}, \\
E_2^* &= \frac{1}{30b-41} \{5 E_1 + (20b - 24) E_2 + (10b - 7) E_3 + 2 E_4 + 10 E_5\}, \\
E_3^* &= -\frac{1}{30b-41} \{5 E_1 + 2 E_2 + 4 E_3 + (6b - 7) E_4 + 6 E_5\}.
\end{aligned}
\]

Claim 4.5.1. Let $C$ be a $(-1)$-curve of the form (4.1). Suppose that $C$ meets $f^-(p_4)$. Then it satisfies one of the following three cases:

| Case | $CE_5$ | $CE_4$ | $CE_3$ | $CE_2$ | $CE_1$ | $CB$ | $k$ |
|------|--------|--------|--------|--------|--------|------|-----|
| (a)  | 0      | 0      | 0      | 0      | 1      | 0    | -15 |
| (b)  | 0      | 0      | 0      | 1      | 0      | 1    | -10 |
| (c)  | 0      | 1      | 0      | 0      | 0      | 0    | -6  |

Proof. Since $(f^* K_S) C = \frac{(3b-4) k}{3(30b-41)} < 0$, $k < 0$. Intersecting $C$ with (4.6), we get

\[
(30b - 41) CB + 3 C \{ (15b - 21) E_1 + (10b - 14) E_2 + (20b - 28) E_3 + (24b - 33) E_4 + (30b - 42) E_5 \} = (3b - 4) k + 3(30b - 41) < 3(30b - 41).
\]

This is possible only if $C$ satisfies one of the three cases, or one of the following three cases:

| Case | $CE_6$ | $CE_5$ | $CE_4$ | $CE_3$ | $CE_2$ | $CE_1$ | $CB$ | $k$ | $b$ |
|------|--------|--------|--------|--------|--------|--------|------|-----|-----|
| (d)  | 0      | 0      | 0      | 1      | 0      | 0      | 1    | -1  | 2   |
| (e)  | 0      | 0      | 0      | 0      | 2      | 0      | 1    | -1  | 2   |
| (f)  | 0      | 0      | 0      | 0      | 1      | 1      | 0    | -6  | 2   |
In Case (d), $C(2) = -\frac{1}{17}B$ and $C(4) = E_3^* = -\frac{1}{17}((10E_3 + 13E_2 + 26E_3 + 4E_4 + 20E_5)$, thus $C(1)^2 + C(3)^2 = C^2 - C(2)^2 - C(4)^2 - \left(\frac{21}{10}f^* K_S\right)^2 = -1 + \frac{1}{3} + \frac{17}{5} = \frac{1}{30} > 0$.

In Case (e), $C(2) = -\frac{4}{3}B$ and $C(4) = 2E_2^*$, thus $C(1)^2 + C(3)^2 = -1 + \frac{1}{3} + \frac{44}{19} = \frac{1}{30} > 0$.

In Case (f), $C(2) = 0$ and $C(4) = E_1^* + E_2^*$, thus $C(1)^2 + C(3)^2 = -1 + \frac{36}{30} = \frac{2}{3} > 0$. All contradict the negative definiteness of exceptional curves.

**Claim 4.5.2.** Let $C$ be a $(-1)$-curve of the form $\text{(4.1)}$. Suppose that $C$ meets $f^{-1}(p_4)$. Then $C$ meets only one component of $f^{-1}(p_1) \cup f^{-1}(p_2) \cup f^{-1}(p_3)$, the intersection multiplicity with the component is 1, and the component is

1. the component of $f^{-1}(p_1)$, if $C$ satisfies (a),
2. the component of $B$ of $f^{-1}(p_2)$, if $C$ satisfies (b),
3. one of the two end components of $f^{-1}(p_3)$, if $C$ satisfies (c).

**Proof.** Assume that $C$ satisfies (a). Then, $C(2) = 0$ and $C(4) = E_1^*$, so $C(4)^2 = -\frac{150}{30b-41}$, hence $C(1)^2 + C(3)^2 = C^2 - C(2)^2 - \left(\frac{-15}{10(3b-4)}f^* K_S\right)^2 = -\frac{1}{2}$. By Lemma 4.1, $C(3) = 0$ and $C(1)^2 = -\frac{1}{2}$.

Assume that $C$ satisfies (b). Then, $C(2) = -\frac{1}{17}B$ and $C(4) = E_2^*$, so $C(1)^2 + C(3)^2 = -1 + \frac{1}{3} + \frac{200}{30b-41} - \left(\frac{-10}{10(3b-4)}f^* K_S\right)^2 = 0$. By the negative definiteness, $C(1) = C(3) = 0$.

Assume that $C$ satisfies (c). Then, $C(2) = 0$ and $C(4) = E_3^*$, so $C(1)^2 + C(3)^2 = -1 + \frac{36}{30b-53} - \left(\frac{-6}{10(3b-4)}f^* K_S\right)^2 = -\frac{1}{2}$. By Lemma 4.1, $C(1) = 0$ and $C(3)^2 = -\frac{4}{17}$.

The same proof as in the previous cases shows that there are three, mutually disjoint, $(-1)$-curves $C_1, C_2, C_3$ satisfying (a), (b), (c) from Claim 4.5.1, respectively. In this case, $C_1 = \frac{-15}{10(3b-4)}f^* K_S + C_1(1) + E_1^*$, $C_2 = \frac{-10}{10(3b-4)}f^* K_S + C_2(2) + E_2^*$, $C_3 = \frac{-6}{10(3b-4)}f^* K_S + C_3(3) + E_3^*$.

**4.6. Case 6:** $< 2, 1 > + < 3, 1 > + < 5, 2 > + < b, 2; 1, 3; 2, 5, 3 >, b \geq 2$. In this case, $\mu = 10$, so $K_S^2 = -1$. Let $B$ be the component of $f^{-1}(p_2)$, $B_2, B_3$ be the components of $f^{-1}(p_3)$, and $E_1, \ldots, E_6$ be the components of $f^{-1}(p_4)$ such that

$$\begin{align*}
B_2 - B_3 & \quad -2 \\
E_2 - E_3 & \quad -2 \\
E_2 - E_3 & \quad -2 \\
E_2 - E_3 & \quad -2 \\
E_5 - E_6 & \quad -2 \\
\text{I} & \quad -2 \\
E_1 & \quad -2
\end{align*}$$

Then

$$K_{S'} = f^* K_S - \frac{1}{17}B - \frac{1}{30b-41}(B_2 + 2B_3) - \frac{1}{30b-53}\left\{((15b - 27)E_1 + (10b - 18)E_2 + (20b - 36)E_3 + (18b - 32)E_4 + (24b - 43)E_5 + (30b - 54)E_6\right\},$$

$$K_S^2 = \frac{2(15b - 26)^2}{15(30b - 53)} \quad |\det(R)| = 30 \cdot (30b - 53), \quad D = 2^2(15b - 26)^2.$$

We also compute the dual vectors,

$$\begin{align*}
B_2^* & = -\frac{3B_2 + B_3}{5} \\
B_3^* & = \frac{B_2 + 2B_3}{5}, \\
E_1^* & = -55(15b - 19)E_1 + 5E_2 + 10E_3 + 3E_4 + 9E_5 + 15E_6, \\
E_2^* & = -\frac{10}{30b-53}\left\{(15E_1 + 20b - 32)E_2 + (10b - 11)E_3 + 2E_4 + 6E_5 + 10E_6\right\}, \\
E_4^* & = -\frac{30b-53}{30b-53}\{3E_1 + 2E_2 + 4E_3 + (12b - 20)E_4 + (6b - 7)E_5 + 6E_6\}.
\end{align*}$$
Claim 4.6.1. Let $C$ be a $(-1)$-curve of the form (4.1). Suppose that $C$ meets $f^{-1}(p_1)$. Then it satisfies one of the following three cases:

| Case | $CE_6$ | $CE_5$ | $CE_4$ | $CE_3$ | $CE_2$ | $CE_1$ | $CB_3$ | $CB_2$ | $CB$ | $k$ |
|------|--------|--------|--------|--------|--------|--------|--------|--------|------|-----|
| (a)  | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0    | −15 |
| (b)  | 0      | 0      | 0      | 1      | 0      | 0      | 0      | 0      | 1    | −10 |
| (c)  | 0      | 0      | 1      | 0      | 0      | 0      | 0      | 1      | 0    | −6  |

Proof. Since $(f^*K_S)C = \frac{(15b−26)k}{15(30b−53)} < 0$, $k < 0$. Intersecting $C$ with (4.7) we get $(30b−53)C(5B + 3B_2 + 6B_3) + 15C\{(15b−27)E_1 + (10b−18)E_2 + (20b−36)E_3 + (18b−32)E_4 + (24b−43)E_5 + (30b−54)E_6\} = (15b−26)k + 15(30b−53).

This is possible only if $C$ satisfies one of the three cases, or one of the following five cases:

| Case | $CE_6$ | $CE_5$ | $CE_4$ | $CE_3$ | $CE_2$ | $CE_1$ | $CB_3$ | $CB_2$ | $CB$ | $k$ |
|------|--------|--------|--------|--------|--------|--------|--------|--------|------|-----|
| (d)  | 0      | 0      | 0      | 0      | 0      | 1      | 0      | 0      | 3    | −3  |
| (e)  | 0      | 0      | 0      | 0      | 1      | 0      | 1      | 0      | 1    | −3  |
| (f)  | 0      | 0      | 0      | 0      | 0      | 1      | 0      | 1      | 1    | −1  |
| (g)  | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 1      | 0    | −6  |
| (h)  | 0      | 0      | 0      | 1      | 0      | 0      | 0      | 1      | 0    | −6  |

In Case (d), $C(2) = 0, C(3) = 3B_5^*$ and $C(4) = E_2^*$, thus $C(1)^2 = C^2 - C(3)^2 - C(4)^2 - \left(\frac{-3}{5}f^*K_S\right)^2 = -1 + \frac{27}{5} + \frac{8}{7} - \frac{9}{30} > 0$.

In Case (e), $C(2) = 0, C(3) = B_3^* + B_4^* = -\frac{4B_2 + 3B_3}{5}$ and $C(4) = E_2^*$, thus $C(1)^2 = C^2 - C(3)^2 - C(4)^2 - \left(\frac{-6}{5}f^*K_S\right)^2 = -1 + \frac{5}{3} + \frac{8}{7} - \frac{9}{30} > 0$.

In Case (f), $C(2) = -\frac{1}{5}B, C(3) = B_3^*$ and $C(4) = E_1^*$, thus $C(1)^2 = C^2 - C(2)^2 - C(3)^2 - C(4)^2 - \left(\frac{-1}{8}f^*K_S\right)^2 = -1 + \frac{1}{3} + \frac{3}{5} + \frac{11}{7} - \frac{1}{30} > 0$.

In Case (g), $C(2) = 0, C(3) = B_2^*$ and $C(4) = E_3^* = -\frac{1}{5}(10E_1 + 9E_2 + 18E_3 + 4E_4 + 12E_5 + 20E_6)$, thus $C(1)^2 = C^2 - C(3)^2 - C(4)^2 - \left(\frac{-10}{5}f^*K_S\right)^2 = -1 + \frac{3}{5} + \frac{11}{7} - \frac{36}{30} > 0$.

All contradict the negative definiteness of exceptional curves. □

Claim 4.6.2. Let $C$ be a $(-1)$-curve of the form (4.1). Suppose that $C$ meets $f^{-1}(p_1)$. Then $C$ meets only one component of $f^{-1}(p_1) \cup f^{-1}(p_2) \cup f^{-1}(p_3)$, the intersection multiplicity with the component is 1, and the component is

1. the component of $f^{-1}(p_1)$, if $C$ satisfies (a),
2. the component $B$ of $f^{-1}(p_2)$, if $C$ satisfies (b),
3. the component $B_2$ of $f^{-1}(p_3)$, if $C$ satisfies (c).

Proof. Assume that $C$ satisfies (a). Then, $C(2) = C(3) = 0$ and $C(4) = E_1^*$, so $C(1)^2 = -\frac{15b−10}{500−63}$, hence $C(1)^2 = C^2 - C(4)^2 - \left(\frac{-15}{500−63}f^*K_S\right)^2 = -\frac{1}{2}$.

Assume that $C$ satisfies (b). Then, $C(2) = -\frac{1}{5}B, C(3) = 0$ and $C(4) = E_2^*$, so $C(1)^2 = -1 + \frac{1}{5} + \frac{30b−32}{500−63} - \left(\frac{-10}{500−63}f^*K_S\right)^2 = 0$. Hence $C(1) = 0$.

Assume that $C$ satisfies (c). In this case, $C(2) = 0, C(3) = B_2^*$ and $C(4) = E_3^*$, so $C(1)^2 = -1 + \frac{1}{5} + \frac{12b−20}{300−63} - \left(\frac{-6}{2(15b−26)}f^*K_S\right)^2 = 0$. Hence $C(1) = 0$. □

The same proof as in the previous cases shows that there are three, mutually disjoint, $(-1)$-curves $C_1, C_2, C_3$ satisfying (a), (b), (c) from Claim 4.6.1, respectively.
4.7. Case 7: $< 2, 1 > + < 3, 1 > + < 5, 3 > + < b; 2, 1; 3, 2; 5, 2 >$, $b \geq 2$. In this case, $\mu = 10$, so $K_S^2 = -1$. Let $B$ be the component of $f^{-1}(p_2)$, $B_2, B_3$ be the components of $f^{-1}(p_3)$, and $E_1, \ldots, E_6$ be the components of $f^{-1}(p_4)$ such that

$$B_2 - B_3 \quad E_2 - E_3 - E_6 - E_5 - E_4$$

Then

$$K_{S'} = f^*K_S - \frac{1}{3}B - \frac{1}{3}(B_2 + 2B_3) - \frac{1}{306 - 47}\{(15b - 24)E_1 + (10b - 16)E_2 + (20b - 32)E_3 + (12b - 19)E_4 + (24b - 38)E_5 + (30b - 48)E_6\},$$

$$K_S^2 = \frac{2(15b - 23)^2}{15(30b - 47)}, \quad |\det(R)| = 30 \cdot (30b - 47), \quad D = 2^2(15b - 23)^2.$$ We also compute the dual vectors,

$$B_3^* = \frac{-3B_2 + B_3}{3}, \quad B_3^* = \frac{-B_2 + 2B_3}{3},$$

$$E_1^* = -\frac{1}{306 - 47}\{(15b - 16)E_1 + 5E_2 + 10E_3 + 3E_4 + 6E_5 + 15E_6\},$$

$$E_2^* = -\frac{1}{306 - 47}\{5E_1 + (20b - 28)E_2 + (10b - 9)E_3 + 2E_4 + 4E_5 + 10E_6\},$$

$$E_3^* = -\frac{1}{306 - 47}\{3E_1 + 2E_2 + 4E_3 + (18b - 27)E_4 + (6b - 7)E_5 + 6E_6\}.$$ 

Claim 4.7.1. Let $C$ be a $(-1)$-curve of the form (4.1). Suppose that $C$ meets $f^{-1}(p_4)$. Then it satisfies one of the following three cases:

| Case | $CE_0$ | $CE_5$ | $CE_4$ | $CE_3$ | $CE_2$ | $CE_1$ | $CB_3$ | $CB_2$ | $CB$ | $k$ |
|------|--------|--------|--------|--------|--------|--------|--------|--------|------|-----|
| (a)  | 0      | 0      | 0      | 0      | 0      | 1      | 0      | 0      | 0    | -15 |
| (b)  | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 1    | -10 |
| (c)  | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0    | -6  |

Proof. Since $(f^*K_S)C = \frac{(15b - 23)^k}{15(30b - 47)} < 0$, $k < 0$. Intersecting $C$ with (4.8), we get $(30b - 47)C(5B + 3B_2 + 6B_3) + 15C((15b - 24)E_1 + (10b - 16)E_2 + (20b - 32)E_3 + (12b - 19)E_4 + (24b - 38)E_5 + (30b - 48)E_6) = (15b - 23)k + 15(30b - 47)$.

This is possible only if $C$ satisfies one of the three cases, or the case (d) $CE_0 = CE_5 = 0, CE_4 = 1, CE_3 = 0, CE_2 = 1, CE_1 = 0, CB_3 = 0, CB_2 = 1, CB = 0$, $b = 2$, $k = -3$.

In the last case, $C(2) = 0, C(3) = B_3^*$ and $C(4) = E_2^* + E_4^*$, thus

$$C(1)^2 = C^2 = C(3)^2 = C(4)^2 = (\frac{2}{15}f^*K_S)^2 = -1 + \frac{3}{9} + \frac{25}{2} - \frac{9}{15} > 0,$$

which contradicts the negative definiteness of exceptional curves. \qed

Claim 4.7.2. Let $C$ be a $(-1)$-curve of the form (4.1). Suppose that $C$ meets $f^{-1}(p_4)$. Then $C$ meets only one component of $f^{-1}(p_1) \cup f^{-1}(p_2) \cup f^{-1}(p_3)$, the intersection multiplicity with the component is 1, and the component is

(1) the component of $f^{-1}(p_1)$, if $C$ satisfies (a),
(2) the component $B$ of $f^{-1}(p_2)$, if $C$ satisfies (b),
(3) the component $B_3$ of $f^{-1}(p_3)$, if $C$ satisfies (c).

Proof. Assume that $C$ satisfies (a). Then, $C(2) = C(3) = 0$ and $C(4) = E_1^*$, so

$$C(4)^2 = -\frac{15b - 23}{306 - 47},$$

hence $C(1)^2 = C^2 - C(4)^2 = -\frac{15}{2(15b - 23)}f^*K_S)^2 = -\frac{1}{2}$.

Assume that $C$ satisfies (b). Then, $C(2) = -\frac{1}{2}B, C(3) = 0$ and $C(4) = E_2^*$, so

$$C(1)^2 = -1 + \frac{1}{9} + \frac{20b - 28}{306 - 47} - \frac{10}{2(15b - 23)}f^*K_S)^2 = 0.$$ Hence $C(1) = 0$. \qed
Assume that $C$ satisfies (c). In this case, $C(2) = 0$, $C(3) = B_1^+$ and $C(4) = E_1^+$, so $C(1)^2 = -1 + \frac{2}{3} + \frac{18b-27}{30b-59} - (\frac{-6}{2(15b-25)} f^*K_S)^2 = 0$. Hence $C(1) = 0$. □

The same proof as in the previous cases shows that there are three, mutually disjoint, $(-1)$-curves $C_1, C_2, C_3$ satisfying (a), (b), (c) from Claim 4.7.1, respectively.

**4.8. Case 8:** $< 2, 1 > > + < 3, 1 > + < 5, 1 > + < b, 2 >; 1; 3; 2; 5, 4 >, b > 2$. In this case, $\mu = 11$, so $K_S^2 = -2$. Let $B, B_2$ be the components of $f^{-1}(p_2), f^{-1}(p_3)$, and $E_1, \ldots, E_8$ be the components of $f^{-1}(p_4)$ such that

$$\frac{-2}{E_2} - \frac{-2}{E_3} - \frac{-b}{E_4} - \frac{-2}{E_5} - \frac{-2}{E_6} - \frac{-2}{E_7} - \frac{-2}{E_8} = 1$$

Then

$$K_{S'} = f^*K_S - \frac{1}{3}B - \frac{3}{5}B_2 - \frac{b-2}{30b-59} (15E_1 + 10E_2 + 20E_3 + 6E_4 + 12E_5 + 18E_6 + 24E_7 + 30E_8),$$

$$K_S^2 = \frac{2(15b-29)^2}{15(30b-59)} \cdot |\text{det}(R)| = 30 \cdot (30b-59), \quad D = 2^2(15b-29)^2.$$

We also compute the dual vectors,

$$E_1^* = -\frac{1}{30b-59} \{ (15b-22)E_1 + 5E_2 + 10E_3 + 3E_4 + 6E_5 + 9E_6 + 12E_7 + 15E_8 \},$$

$$E_2^* = -\frac{1}{30b-59} \{ (5E_1 + (20b-36)E_2 + (10b-13)E_3 + 2E_4 + 4E_5 + 6E_6 + 8E_7 + 10E_8) \},$$

$$E_4^* = -\frac{1}{30b-59} \{ (3E_1 + 2E_2 + 4E_3 + (24b-46)E_4 + (18b-33)E_5 + (12b-20)E_6 + (6b-7)E_7 + 6E_8) \}.$$

**Claim 4.8.1.** Let $C$ be a $(-1)$-curve of the form (1.1). Suppose that $C$ meets $f^{-1}(p_4)$. Then it satisfies one of the following three cases:

| Case | $CE_8$ | $CE_7$ | $CE_6$ | $CE_5$ | $CE_4$ | $CE_3$ | $CE_2$ | $CE_1$ | $CB_2$ | $CB$ | $k$ |
|------|--------|--------|--------|--------|--------|--------|--------|--------|--------|------|-----|
| (a)  | 0      | 0      | 0      | 0      | 0      | 0      | 1      | 0      | 0      | -15  |
| (b)  | 0      | 0      | 0      | 0      | 0      | 0      | 1      | 0      | 0      | -10  |
| (c)  | 0      | 0      | 0      | 0      | 1      | 0      | 0      | 0      | 1      | -6   |

**Proof.** Since $(f^*K_S)C = \frac{(15b-29)k}{15(30b-59)} < 0, k < 0$. Intersecting $C$ with (4.9) we get

$$(30b-59)(C(B + 9B_2) + 15b - 2)C(15E_1 + 10E_2 + 20E_3 + 6E_4 + 12E_5 + 18E_6 + 24E_7 + 30E_8) = (15b-29)k + 15(30b-59) < 15(30b-59).$$

This is possible only if $C$ satisfies one of the three cases, or the case

(d) $CB_2 = CB = 1, b = 2, k = -1, \ (CE_i \text{ are not determined}).$

In case (d), $C(2) = -\frac{1}{4}B$ and $C(3) = -\frac{1}{3}B_2$, thus

$$C(1)^2 + C(4)^2 = C^2 - C(2)^2 - C(3)^2 - \left(\frac{-6}{2(15b-25)} f^*K_S\right)^2 = -\frac{1}{9}.$$ 

Also note that in this case the sublattice $R_{p_4} \subset H^2(S', \mathbb{Z})$ generated by the components of $f^{-1}(p_4)$ is a negative definite unimodular lattice of rank 8. In particular, $R_{p_4} = R_{p_4}$, so $C(4) \in R_{p_4}$ and $C(4)^2$ is a non-positive even integer. By Lemma 4.4, $C(4)^2 = 0$. Thus $C$ does not meet $f^{-1}(p_4)$, contradicts the assumption. □

**Claim 4.8.2.** Let $C$ be a $(-1)$-curve of the form (1.1). Suppose that $C$ meets $f^{-1}(p_4)$. Then $C$ meets only one component of $f^{-1}(p_1) \cup f^{-1}(p_2) \cup f^{-1}(p_4)$, the intersection multiplicity with the component is 1, and the component is

(1) the component of $f^{-1}(p_1)$, if $C$ satisfies (a),
(2) the component $B$ of $f^{-1}(p_2)$, if $C$ satisfies (b),
(3) the component $B_2$ of $f^{-1}(p_3)$, if $C$ satisfies (c).

**Proof.** Assume that $C$ satisfies (a). Then, $C(2) = C(3) = 0$ and $C(4) = E_1^*$, so
$C(4)^2 = -\frac{15}{36} - \frac{22}{59}$, hence $C(1)^2 = C^2 - C(4)^2 - \left(\frac{-15}{2(15b-29)}f^*K_S\right)^2 = -\frac{1}{2}$.

Assume that $C$ satisfies (b). Then, $C(2) = -\frac{1}{2}B$, $C(3) = 0$ and $C(4) = E_2^*$, so
$C(1)^2 = -1 + \frac{1}{2} + \frac{36}{36} - \left(\frac{-15}{2(15b-29)}f^*K_S\right)^2 = 0$. Hence $C(1) = 0$.

Assume that $C$ satisfies (c). Then, $C(2) = 0$, $C(3) = -\frac{1}{2}B_2$ and $C(4) = E_4^*$, so
$C(1)^2 = -1 + \frac{1}{2} + \frac{36}{36} - \left(\frac{-15}{2(15b-29)}f^*K_S\right)^2 = 0$. Hence $C(1) = 0$. \qed

The same proof as in the previous cases shows that there are three, mutually dis-
joint, $(-1)$-curves $C_1, C_2, C_3$ satisfying (a), (b), (c) from Claim 4.8.1, respectively.

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