Reverse reconciliation protocols
for quantum cryptography with continuous variables

Frédéric Grosshans and Philippe Grangier
Laboratoire Charles Fabry de l’Institut d’Optique (CNRS UMR 8501) F-91403 Orsay, France

We introduce new quantum key distribution protocols using quantum continuous variables, that are secure against individual attacks for any transmission of the optical line between Alice and Bob. In particular, it is not required that this transmission is larger than 50%. Though squeezing or entanglement may be helpful, they are not required, and there is no need for quantum memories or entanglement purification. These protocols can thus be implemented using coherent states and homodyne detection, and they may be more efficient than usual protocols using quantum discrete variables.

I. INTRODUCTION

A. Coherent QKD protocols

In the presently very active field of continuous variable quantum information processing, a stimulating question is whether quantum continuous variables (QCV) may provide a valid alternative to the usual “single photon” quantum key distribution (QKD) schemes. Many recent proposals to use QCV for QKD (for a short review see [1]) are based upon the use of “non-classical” light beams, such as squeezed light, or entangled pairs of light beams. We have shown recently [1] that there is actually no need for squeezed light: an equivalent level of security may be obtained by simply generating and transmitting random distributions of coherent states. More precisely, we have shown in [1] that a whole family of secure protocols can be obtained by using either coherent states, squeezed states, or entangled EPR beams, provided that the transmission of the line is larger than 50% (i.e. the losses are less than 3 dB). The security of these protocols is related to the no-cloning theorem [2,3], and non-classical features like squeezing or EPR correlations have no influence on the achievable secret key rate. The 3dB loss limit of these cryptography protocols makes their security demonstration quite intuitive, but there exist in principle multiples ways for Alice and Bob to go beyond this limit, using e.g. entanglement purification [4].

In this paper we present new protocols that are secure for any value of the line transmission. The basic idea is to use reverse reconciliation protocols, that is, Alice will try to guess what was received by Bob, rather than Bob trying to guess what was sent by Alice. In that case, Alice can always guess better than the eavesdropper Eve: this is the basic reason for the security of these new protocols.

B. Direct and reverse reconciliation protocols

In the first step of a generic QKD protocol, Alice prepares a quantum state and sends it to Bob, who makes a measurement on this state. Alternatively, Alice and Bob may share two EPR-correlated systems, and they both make measurement on their parts. In order to warrant security, Alice and Bob must randomly choose to use different measurement basis, and the transmitted data will be significant only when their basis are compatible. After the quantum exchange, they have thus to agree on a common measurement basis, and discard the wrong measurements. At the end of this step, Alice and Bob (and the potential eavesdropper Eve) know a set of correlated measurements, that we will call the “key elements”.

As a second step, Alice reveals some randomly chosen samples of the data that she sent, and Bob reveals his corresponding measurements. These samples allow them to measure some relevant parameters of the quantum channel, e.g. the error rate and the transmission, that is called “channel gain” for QCV protocols. Knowing the correlations between their key elements, Alice and Bob can evaluate the amount of information they share ($I_{AB}$), and the information the eavesdropper Eve can have about their values ($I_{AE}$ and $I_{BE}$). Therefore they can evaluate the size of the secret key they will generate at the end of the protocol. If Eve knows too much, the size of this secret key will be 0, and Alice and Bob will abort the protocol at this step.

Now comes the crucial step of reconciliation, where Alice and Bob will use classical communications to extract a common key from their correlated key elements, revealing as little information as possible to a third party ignoring these key elements. This step usually uses parity-based algorithms like Cascade. There are actually two main options for doing the reconciliation [5]:

Direct Reconciliation (DR). Alice sends correction information and Bob corrects his key elements to have the same values as Alice. Alice knows from the previous step the minimum amount of information she’s got to reveal at this step. If the reconciliation protocol is perfect, it keeps $I_{AB} - I_{AE}$ constant. At the end of this step, Alice and Bob know a common bit string of length $I_{AB}$, and Eve knows $I_{AE}$ bits of this string (slightly more if the reconciliation protocol is not perfect). It will provide a usable secret key if $I_{AB} - I_{AE} > 0$ at the beginning. We call this “direct reconciliation” (DR) because Bob is reconstructing what was sent by Alice, and the classical information flow at this step has the same direction as the initial quantum information flow.
Direct reconciliation is quite intuitive, and it was used in the coherent state QCV protocol that we proposed recently. However, it is not secure as soon as the quantum channel efficiency falls below 50%. Intuitively, Eve could simulate the losses by a beam splitter and look one output port of this beamsplitter. It seems obvious that, if she keeps the biggest part of the beam sent by Alice (i.e. if she simulate losses higher than 50%), she can extract more information from this beam than Bob ($I_{AE} > I_{AB}$), thus forbidding any secret key generation. This limitation is actually not specific to QCV: a "direct" version of BB84 would be a protocol where Bob would try to fill the "empty slots" where he did not get any photon. It’s straightforward to show that this protocol only works when the losses are smaller than 3 dB.

Reverse Reconciliation (RR). We will thus consider "reverse" reconciliation (RR) protocols, where Bob sends correction information and Alice corrects her key elements to have the same values as Bob. Since Bob gives the correction information (also to Eve), this type of reconciliation keeps $I_{AB} - I_{BE}$ constant, and will provide a usable key if $I_{AB} - I_{BE} > 0$. We call it "reverse reconciliation" (RR) because Alice adapts herself to what was received by Bob.

In a noiseless BB84 with finite line transmission, this step corresponds to Bob giving to Alice his "empty slots" where he did not get any photon, and Alice removing the bits she sent at these slots to have the same key. Obviously it is also possible to make a reconciliation protocol using two way communications, but it can be shown that reverse reconciliation is optimum for a coherent state protocol when there is no excess noise in the transmission line (see below). Therefore two-ways protocols will not be considered further in the present paper.

Finally, the last step of a practical QKD protocol is that Alice and Bob perform privacy amplification to filter out Eve’s information. Since this step is based on an evaluation of the amount of information collected by Eve on the reconciled key, a crucial requirement is to get a bound on $I_{AE}$ for DR, and on $I_{BE}$ for RR. For a coherent state protocol, the DR bound was given in ref. [1], and leads to a security limit for a 50% line transmission as said above. We will now establish the RR bound, and we will show that it is not associated with a minimum value of the line transmission.

II. ENTANGLING CLONER

A. Definition

To eavesdrop a reverse reconciliation scheme, as described above, Eve needs to guess the results of Bob’s measurement. We will call entangling cloner a system allowing her to do so, because this kind of system can be described a cloner creating two quantum-correlated output: Eve keeps one of them and sends the other to Bob. Let $(x_n, p_n)$ be the input quadratures of the entangling cloner and $(x_E, p_E)$ the quadratures of its two outputs. A good entangling cloner should minimize the conditional variances $V_{x\mid x_E}$ and $V_{p\mid p_E}$.

Alice and Bob should assume Eve uses the best possible entangling cloner, knowing the Alice-Bob channel quality. This channel can be described by

\begin{align}
  x_B &= g_x(x_n + B_x) \\
  p_B &= g_p(p_n + B_p),
\end{align}

with ($N_0$ is the shot noise variance)

\begin{align}
  \langle x_n^2 \rangle &= \langle p_n^2 \rangle \equiv VN_0 \geq N_0 \\
  \langle B^2 \rangle &= \chi_x N_0 \\
  \langle x_n B_x \rangle &= \langle p_n B_p \rangle = 0
\end{align}

B. Heisenberg inequalities on Alice and Eve’s conditional variances

For reverse reconciliation protocols, Alice needs to evaluate $x_B$. Her estimator can be noted $\alpha x_n$, with $\alpha \in \mathbb{R}$; Eve’s estimator for $p_B$ will be $\varepsilon p_E$. Their error will be

\begin{align}
  x_B|A,\alpha &= x_B - \alpha x_n \\
  p_B|E,\varepsilon &= p_B - \varepsilon p_E.
\end{align}

The commutator $[x_B|A,\alpha, p_B|E,\varepsilon]$ of these two quantities is then equal to

\begin{equation}
  [x_B, p_B] - \alpha [x_n, p_B] - \varepsilon [x_B, p_E] + \alpha \varepsilon [x_n, p_E].
\end{equation}

We have therefore

\begin{equation}
  [x_B|A,\alpha, p_B|E,\varepsilon] = [x_B, p_E]
\end{equation}

\begin{equation}
  \langle x_B^2|A,\alpha \rangle \langle p_E^2|E,\varepsilon \rangle \geq N_0^2
\end{equation}

The conditional variances obey by definition the following relations :

\begin{align}
  V_{x_B|x_A} &= \min_{\alpha} \left\{ \langle x_B^2|A,\alpha \rangle \right\} \\
  V_{p_B|p_E} &= \min_{\varepsilon} \left\{ \langle p_B^2|E,\varepsilon \rangle \right\}
\end{align}

and equation [10] leads to

\begin{equation}
  V_{x_B|x_A} V_{p_B|p_E} \geq N_0^2, \text{ i.e. } V_{p_B|p_E} \geq \frac{N_0^2}{V_{x_B|x_A}}.
\end{equation}

By exchanging the roles of $x$ and $p$ one obtains similarly

\begin{equation}
  V_{p_B|p_A} V_{x_B|x_E} \geq N_0^2, \text{ i.e. } V_{x_B|x_E} \geq \frac{N_0^2}{V_{p_B|p_A}}.
\end{equation}

These inequalities mean that Alice and Eve cannot jointly know more about Bob’s field than allowed by the Heisenberg principle.
C. Alice’s conditional variance

If Alice creates the field \((x_{\text{in}}, p_{\text{in}})\), we can write
\[
x_{\text{in}} = x_A + A_x
\]
\[
p_{\text{in}} = p_A + A_p
\]  
(15)
(16)
where \(x_A\) \((p_A)\) is Alice’s best estimation of \(x_{\text{in}}\) \((p_{\text{in}})\) and
\[
\langle A_x^2 \rangle = \langle A_p^2 \rangle = sN_0
\]  
(17)
where \(s\) is the amount of squeezing used by Alice to generate this field. We have then
\[
s \geq \frac{1}{V}. \tag{18}
\]
The correlation coefficients are equal to
\[
\langle p_A^2 \rangle = (V - s)N_0 \tag{19}
\]
\[
\langle p_B^2 \rangle = G_p(V + \chi_p)N_0 \tag{20}
\]
\[
\langle p_{AB} \rangle = g_p \langle p_A^2 \rangle, \tag{21}
\]
and allow us to calculate Alice’s conditional variance on Bob’s measurement :
\[
V_{p_B|p_A} = \langle p_B^2 \rangle - \frac{\langle p_{AB} \rangle^2}{\langle p_A^2 \rangle} = G_pV N_0 + G_p N_0 - G_p V N_0 + G_p sN_0 = G_p(\chi_p + s)N_0 \tag{22}
\]
A similar calculation leads to the symmetric relation
\[
V_{x_B|x_A} = G_x(\chi_x + s)N_0 \tag{23}
\]
These equations and the constraint \((18)\) on the squeezing give finally
\[
V_{p_B|p_A} \geq V_{p_B|p_A,\text{min}} = G_p(\chi_p + \frac{1}{V})N_0 \tag{24}
\]
\[
V_{x_B|x_A} \geq V_{x_B|x_A,\text{min}} = G_x(\chi_x + \frac{1}{V})N_0 \tag{25}
\]

D. Eve’s conditional variance

The output–output correlations of an entangling cloner, described e.g. by \(V_{p_B|p_E}\), should only depend on the density matrix of the field \((x_{\text{in}}, p_{\text{in}})\) at its input, and not on the way this field was built. The inequality \((18)\) has thus to be fulfilled for every physically allowed value of \(V_{x_B|x_A}\), given the density matrix of the field \((x_{\text{in}}, p_{\text{in}})\). If we look for a boundary to Eve’s knowledge by using eq. \((18)\), we have thus to use the tightest limit on \(V_{x_B|x_A}\), that is given by \(V_{x_B|x_A,\text{min}}\) according to \((25)\). Obviously the same reasoning holds for \(V_{p_B|p_A}\), with the corresponding tightest limit \(V_{p_B|p_A,\text{min}}\).

We have then
\[
V_{x_B|x_E} \geq V_{x_B|x_E,\text{min}} = \frac{N_0}{G_p(\chi_p + 1/V)} \tag{26}
\]
and, similarly
\[
V_{p_B|p_E} \geq V_{p_B|p_E,\text{min}} = \frac{N_0}{G_x(\chi_x + 1/V)} \tag{27}
\]

E. Implementation

In a practical QKD scheme Alice and Bob will give the same roles to \(x\) and \(p\). Assuming therefore that \(G_x = G_p = G\) and \(\chi_x = \chi_p = \chi\), the two bounds \((26, 27)\) reduce to a single one, and it is possible to explicitly describe an entangling cloner achieving this limit. We will consider here only the case where \(G < 1\), but the limit is tight for any \(G\). The entangling cloner can then be sketched as follows: Eve uses a beamsplitter with a transmission \(G\) to split up part of the Alice-Bob transmitted signal, and she injects into the other input port a field \(E_1\), with the right variance to induce a noise of variance \(G\chi N_0\) at Bob’s end. One has therefore:
\[
\langle x_{E_1}^2 \rangle = \frac{G\chi N_0}{1-G}, \quad \langle p_{E_1}^2 \rangle = \frac{G\chi N_0}{1-G} \tag{28}
\]
Eve should know the maximum about this injected field \(E_1\), and will therefore use an half-pair of EPR-correlated beams, so that she does perform an “entangling” attack. We can then write
\[
x_{E_1} = x_{\text{known}} + x_{\text{unknown}} \tag{29}
\]
where \(x_{\text{known}}\) stand for Eve’s best estimation of \(x_{E_1}\), given by the measure of its brother-beam, and \(x_{\text{unknown}}\) stand for the noise she cannot know. We have
\[
\langle x_{\text{unknown}}^2 \rangle = \frac{N_0^2}{\langle x_{E_1}^2 \rangle} = \frac{(1-G)N_0}{G\chi} \tag{30}
\]
\[
\langle x_{\text{known}}^2 \rangle = \langle x_{E_1}^2 \rangle - \langle x_{\text{unknown}}^2 \rangle \tag{31}
\]
Eve also use an output port of the beamsplitter to measure the field \(E_2\), which gives her information about the input field :
\[
x_{E_2} = gx_{E_1} - \sqrt{1-G}x_{\text{in}}. \tag{32}
\]
She can cancel a part of the noise induced by \(E_1\) by substracting the part proportional to \(x_{\text{known}}\). Thus she knows
\[
x_{E_2} = gx_{\text{unknown}} - \sqrt{1-G}x_{\text{in}}. \tag{33}
\]
We also have
\[
x_B = gx_{\text{in}} + \sqrt{1-G}x_{E_1}. \tag{34}
\]
where Eve already knows the part proportional to $x_{\text{known}}$, injected with $x_{\text{E}1}$ and she only needs to guess
\[ x_B' = g x_{\text{in}} + \sqrt{1 - G} x_{\text{unknown}} \] (35)
from $x_{E2}'$. We have therefore
\[ V_{x_B|x_{E1},x_{E2}} = V_{x_B'|x_{E2}'} . \] (36)
The calculation of the quantities $\langle x_{B}'^2 \rangle$, $\langle x_{E2}'^2 \rangle$, $\langle x_{E2}' x_B' \rangle$ lead straightforwardly to the conditional variance:
\[ V_{x_B'|x_{E2}'} = \frac{N_0}{G x + G/V} = V_{x_B|x_{E,\text{min}}} \] (37)
showing that the entangling cloner does reach the lower limit of $\boxed{23, 27}$.

III. REVERSE CRYPTOGRAPHY

A. Tolerable noise

In a reverse quantum cryptography protocol, Eve’s power is limited by the values of $V_{x_B|x_{E,\text{min}}}$ and $V_{p_B|p_{E,\text{min}}}$ given by $\boxed{23, 27}$. In the following, we will assume that a “perfect” Eve is able to reach that limit:
\[ V_{x_B|x_{E}} = V_{x_B|x_{E,\text{min}}} = \frac{N_0}{G p (\chi p + 1/V)} \] (38)
\[ V_{p_B|p_{E}} = V_{p_B|p_{E,\text{min}}} = \frac{N_0}{G s (\chi s + 1/V)} \] (39)
Reverse cryptography is possible when
\[ V_{x_B|x_{E}} > V_{x_B|x_{A}} \] (40)
or $V_{p_B|p_{E}} > V_{p_B|p_{A}}$ (41)
Combining equations $\boxed{22, 23}$ and $\boxed{8}$, the above conditions become
\[ (G x_{\chi x} + G_x s) (G_p \chi p + G_s p) < 1 \] (42)
\[ (G_p \chi p + G_p s) (G_x \chi x + G_s e) < 1 . \] (43)
These inequalities give the general conditions for the security of a reverse reconciliation protocol. For simplicity reasons, we will assume in the following everything is symmetric in $(x, p)$, i.e. $G_x = G_p = G$ and $\chi_x = \chi_p = \chi$, so that these conditions simplify into:
\[ (G \chi + G s)(G \chi + G/V) < 1 . \] (44)
Any experimental implementation of this protocol should however estimate these parameters from statistical tests, which are likely not to be exactly symmetric.

B. Secret information rates

The conditions $\boxed{14}$ can directly be translated into an information rate by using Shannon’s formula:
\[ I_{BA} = \frac{1}{2} \log_2 \frac{\langle x_B'^2 \rangle}{V_{B|A}} \] (45)
\[ I_{BE} = \frac{1}{2} \log_2 \frac{\langle x_B'^2 \rangle}{V_{B|E}} \] (46)
The secret information rate, for a reverse reconciliation protocol is therefore
\[ \Delta I = I_{BA} - I_{BE} = \frac{1}{2} \log_2 \frac{V_{B|E}}{V_{B|A}} \] (47)
\[ \Delta I = \frac{1}{2} \log_2 \frac{1}{(G \chi + G s)(G \chi + G s)} \] (48)

C. High-modulation limit

At the high modulation limit, $V \to \infty$, and for any squeezing $s$ the condition $\boxed{14}$ becomes
\[ G \chi (G \chi + G s) < 1 \] (49)
\[ (G \chi)^2 + G s (G \chi) - 1 < 0 \] (50)
\[ \Delta = G^2 s^2 + 4 \] (51)
We have therefore
\[ G \chi < \frac{1}{2} \left( \sqrt{G^2 s^2 + 4} - G s \right) \] (52)
The inequality $\boxed{52}$ gives the maximum tolerable added noise for a reverse protocol to be secure. The corresponding limit is less stringent for low $s$ values, i.e. for strong squeezing. The more squeezing Alice uses, the more noise reverse cryptography can tolerate.

Let us consider the case of a lossy transmission line with $G \leq 1$. The added noise is always bigger than a minimal value, $G \chi \geq (1 - G)$, this inequality being saturated if there is no excess noise (in that case the only added noise is vacuum noise). In that case where the noise is only due to losses, $\boxed{52}$ becomes
\[ 1 - G < \frac{1}{2} \left( \sqrt{G^2 s^2 + 4} - G s \right) \] (53)
It is straightforward to check that this inequality holds for arbitrary high losses ($G \to 0$), even for coherent states ($s = 1$). Therefore reverse reconciliation provides a simple way to extend the coherent state protocols of ref. $\boxed{1}$ into the high-loss regime.
IV. IMPLEMENTATIONS

In this section we consider protocols with the same quantum communication part as in ref. \[1\], but we assume that reverse reconciliation is used. The main advantage is that the 3dB loss limit for security goes away as explained above. Here we give more quantitative estimates of the security thresholds.

A. EPR vs coherent beams

If Alice uses EPR beams (or modulates a maximally squeezed beam), \(s = 1 \sqrt{V}\), but Alice and Bob have information only every second transmission, since they don’t always choose the same measurement basis.

\[
\Delta I_{\text{EPR}} = \frac{1}{4} \log_2 \frac{1}{(G\chi + \frac{1}{V})^2} = \frac{1}{2} \log_2 \frac{1}{G\chi + \frac{1}{V}} \geq 0 \quad (54)
\]

If the added noise only comes from losses, \(G \leq 1\) and \(G\chi = 1 - G\). In that case equation (54) becomes

\[
\Delta I_{\text{EPR,losses}} = \frac{1}{2} \log_2 \frac{1}{1 - G(1 - \frac{1}{V})} \geq 0 \quad (55)
\]

For coherent beams, no squeezing is used, therefore \(s = 1\) and the mutual informations are not dependent of the basis choice. We have thus

\[
\Delta I_{\text{coh}} = \frac{1}{2} \log_2 \frac{1}{(G\chi + \frac{1}{V})} (G\chi + G) \quad (56)
\]

\[
\Delta I_{\text{coh}} = \Delta I_{\text{EPR}} - \frac{1}{2} \log_2 G(1 + \chi) \quad (57)
\]

Since \(G\chi \geq (1 - G)\), with the equality iff the noise is minimal and \(G \leq 1\), we obtain

\[
\Delta I_{\text{coh}} \leq \Delta I_{\text{EPR}}, \quad (58)
\]

both secret rates being equal if and only if the noise comes only from losses. As in \[6\], squeezing does not improve the secret rate for losses only, but this is no more true in presence of excess noise.

B. Strong losses

Assuming strong losses, \(G \ll 1\), no excess noise, and a large initial modulation, eqs. (55) and (57) become

\[
\Delta I_{\text{EPR,losses}} = \Delta I_{\text{coh,losses}} \approx \frac{G}{2 \ln 2} \quad (59)
\]

This secret rate can be compared with BB84’s rate, which is \(\frac{1}{4} G\bar{n}\), with \(\bar{n} = 1\) for single photons and \(\bar{n} \ll 1\) for weak coherent pulses. Even if BB84 uses two modes of the electromagnetic field, it is slightly less efficient than our reversed continuous variable protocols, but the order of magnitude is the same (for strong losses).

Taking for instance a 100 km line with 20 dB loss \((G = 0.01)\) and a reasonable modulation \((V \approx 10)\), the secret key rate is \(6.5 \cdot 10^{-3}\) bit/symbol. For the same parameters, the secret key rate for QDV QKD with an ideal single photon source would be at best \(5 \cdot 10^{-3}\) bit/time slot, and would be one order of magnitude smaller using attenuated light pulses with \(\bar{n} = 0.1\), even with perfect detectors. Actually it is zero with state-of-the-art QDV systems at 1550 nm. It is also noticeable that with a “symbol rate” of a few MHz that should be easy to achieve, the QCV secret key rate after 100 km is more than 10 kbits/sec, while it is simply zero for QDV.

More realistically, one should take into account possible excess noise in the line. Defining the excess noise as \(\epsilon = \chi - (1 - G)/G\), it is simple to show that the reverse protocols are secure as long as \(\epsilon < (V - 1)/(2V) \sim 1/2\) for coherent states, and \(\epsilon < (V - 1)/V \sim 1\) for EPR beams. This shows again that it is possible to use coherent states, though EPR beams are more robust indeed.

V. CONCLUSION

In this paper we have shown that reverse reconciliation protocols can be used to extract a secret key from the exchange of coherent, squeezed or EPR beams between Alice and Bob. The key is secure against individual attacks for any transmission of the optical line between Alice and Bob, provided that the excess noise (noise beyond the loss-induced vacuum noise) is not too large.

Squeezing makes these protocols more robust against the excess noise, but it is not absolutely required. It can be shown \[8\] that reverse reconciliation is optimum for a coherent states protocol with no excess noise in the transmission line, but this is not always the case: for instance, direct reconciliation may be better for high line transmission and large excess noise. It may therefore be possible to optimize further the secret bit rate by using two-ways reconciliation protocols.

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