On the decay problem for the Zakharov and Klein–Gordon–Zakharov systems in one dimension

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Abstract. We are interested in the long time asymptotic behaviour of solutions to the scalar Zakharov system

\[ \begin{align*}
  iu_t + \Delta u &= nu, \\
  n_{tt} - \Delta n &= \Delta|u|^2
  \end{align*} \]

and the Klein–Gordon–Zakharov system

\[ \begin{align*}
  u_{tt} - \Delta u + u &= -nu, \\
  n_{tt} - \Delta n &= \Delta|u|^2
  \end{align*} \]

in one dimension of space. For these two systems, we give two results proving decay of solutions for initial data in the energy space. The first result deals with decay over compact intervals assuming smallness and parity conditions \((u \text{ odd})\). The second result proves decay in far field regions along curves for solutions whose growth can be dominated by an increasing \(C^1\) function. No smallness condition is needed to prove this last result for the Zakharov system. We argue relying on the use of suitable virial identities appropriate for the equations and follow the technics of [22,24] and [33].

1. Introduction

In this work, we are concerned with the one-dimensional Zakharov system

\[ \begin{align*}
  iu_t + \Delta u &= nu, \\
  \alpha^{-2}n_{tt} - \Delta n &= \Delta|u|^2
  \end{align*} \]  \tag{1.1}

with initial data

\[ u(t = 0, x) = u_0(x), \quad n(t = 0, x) = n_0(x), \quad n_t(t = 0, x) = n_1(x). \]

where \(u(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}, n(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) and \(\alpha > 0\).

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We are also interested in the Klein–Gordon–Zakharov system in one dimension

\[
\begin{align*}
c^{-2}u_{tt} - \Delta u + c^2 u &= -nu, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \\
\alpha^{-2}n_{tt} - \Delta n &= \Delta|u|^2, \quad (t, x) \in \mathbb{R} \times \mathbb{R},
\end{align*}
\]

(1.2)

with initial data

\[
\begin{align*}
u(t = 0, x) &= u_0(x), \quad u_t(t = 0, x) = u_1(x) \\
n(t = 0, x) &= n_0(x), \quad n_t(t = 0, x) = n_1(x).
\end{align*}
\]

where \(u(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, n(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \alpha > 0, \ c > 0.\)

The Zakharov systems are simplified models for the description of long-wavelength small-amplitude Langmuir oscillations in a ionized plasma [50]. Langmuir waves are rapid oscillations of the electron density; electrons and ions oscillate out of phase. Zakharov equations model the nonlinear interactions between the mean mode of the ionic fluctuations of density in the plasma \(n\) and the changing amplitude of electric field \(u\), which varies slowly compared to the unperturbed plasma frequency. The constant \(\alpha\) is the ion sound speed and \(c\) is the plasma frequency.

In the subsonic limit \((\alpha \to \infty)\), in which density perturbations are changing slowly, the term \(n_{tt}\) of the wave equation in (1.1) is negligible. This would imply that the Langmuir waves follow the cubic NLS equation

\[
iu_t + \Delta u + |u|^2 u = 0.
\]

If one considers \(\tilde{u} = e^{ic^2 t}\) in (1.2), then it follows that

\[
c^{-2}\tilde{u}_{tt} - 2i\tilde{u}_t - \Delta \tilde{u} = -n\tilde{u}, \\
\alpha^{-2}n_{tt} - \Delta n = \Delta|\tilde{u}|^2.
\]

Thus, formally, in the high-frequency limit (that is, taking \(c \to \infty\)) of the Klein–Gordon–Zakharov (1.2), the Zakharov system is recovered.

These high-frequency and subsonic limits were extensively studied in [1, 39, 47] and [27–30]. See, also, [4, 43, 46] for more details on the physical derivation.

The Zakharov system (1.1) preserves the mass \(\|u(t)\|_{L^2(\mathbb{R})} = \|u(0)\|_{L^2(\mathbb{R})}\) and the energy

\[
H_S(t) := \int_\mathbb{R} |\nabla u(t, x)|^2 + \frac{1}{2}\left(\|n(t, x)|^2 + \frac{1}{\alpha^2}\|D^{-1}n_t(t, x)|^2\right) + n(t, x)|u(t, x)|^2 dx,
\]

where \(D = \sqrt{-\Delta}\). The Klein–Gordon–Zakharov system (1.2) preserves the following energy, as well:

\[
H_{KG}(t) = \int_\mathbb{R} c^2|u(t, x)|^2 + |\nabla u(t, x)|^2 + \frac{1}{c^2}|u_t(t, x)|^2 + \frac{1}{2}|n(t, x)|^2 \\
+ \frac{1}{2}\|\alpha D|^{-1}n_t(t, x)|^2 + n(t, x)|u(t, x)|^2 dx = H_{KG}(0).
\]
The system (1.1) in one dimension is globally well-posed for initial data in $H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times \hat{H}^{-1}(\mathbb{R})$, where

$$w \in \hat{H}^s$$

if there exists $v : \mathbb{R}^d \to \mathbb{R}^d$ such that $w = \nabla \cdot v$ and $\|w\|_{\hat{H}^s} = \|v\|_{H^{s+1}}$.

The first approach in this regard was presented by Sulem and Sulem in [45], where they stated local well-posedness of (1.1) for dimensions $d = 1, 2, 3$ and initial data $(u_0, n_0, n_1) \in H^m(\mathbb{R}^d) \times H^{m-1}(\mathbb{R}^d) \times \left(H^{m-2} \cap \hat{H}^{-1}\right)(\mathbb{R}^d)$, $m \geq 3$.

Using the Brezis-Gallouët inequality

$$\|u\|_{L^\infty} \lesssim 1 + \|u\|_{H^1}(\ln(1 + \|\Delta u\|_{L^2})),$$

valid if $u \in H^2(\mathbb{R}^2)$, Added and Added in [1] improved [45] to global well-posedness for small initial data in the 2-dimensional case. The local well posedness result ($d = 1, 2, 3$) was refined by Ozawa and Tsutsumi [40], for data $(u_0, n_0, n_1) \in H^2 \times H^1 \times L^2$, and by Colliander [7] for $(u_0, n_0, n_1) \in H^1 \times L^2 \times \hat{H}^{-1}$. In fact, in [7], using a priori estimates on the $H^1$-norm of $u$, Colliander shows global well-posedness for small data in the one-dimensional case. Finally, on [41], Pecher proves that the Zakharov system is globally well posed for rough data $(u_0, n_0, n_1) \in H^s \times L^2 \times \hat{H}^{-1}$, $1 > s > 9/10$ for dimension $d = 1$ without any smallness condition. More results on local and global well-posedness for other dimensions and more general nonlinearities are stated in [5,8,10,32], and on the torus in [3].

Regarding well-posedness for system (1.2), the first result was presented in [37], where the authors followed a method based on the theory of normal forms to prove that (1.2) in dimensions $d = 1, 2, 3$ with $c = \alpha = 1$, admits a unique global solutions for small initial data with rather restrictive regularity conditions. Also, they show the existence of global solutions that tend to behave asymptotically (when $t \to \infty$) as the free solutions for the 3-dimensional case.

Using Sobolev invariant spaces, Tsutaya [48], improved the regularity conditions of the global existence result in [37] for dimension $d = 3$. The low-frequency case ($0 < \alpha < 1$) in three dimensions was addressed in [38], where the authors rely on the different propagation speed (normalized $c = 1$ in the Klein–Gordon equation, while assumed $0 < \alpha < 1$ in the wave equation) to prove local and then global well-posedness for small initial data in the energy space:

$$(u_0, u_1, n_0, n_1) \in H^1 \times L^2 \times L^2 \times \hat{H}^{-1}.$$

Following the idea stated in [38], Ohta and Todorova [36] extended the well-posedness result for all $\alpha > 0$ and $d < 3$. The high-frequency, subsonic case in dimension $d = 3$ was later treated by Masmoudi and Nakanishi in [31,32], where they prove well-posedness in the energy space under the assumption $\alpha < c$.

In this paper, we are interested in the decay of solutions to systems (1.1) and (1.2).
It is known that for dimension $d = 2, 3$, there exist solutions to (1.1) that decay to zero in the energy space $H^1 \times L^2 \times \dot{H}^{-1}$. Indeed, Ozawa and Tsutsumi [40], Ginibre and Velo [11], and Shimomura [44], proved existence and uniqueness of asymptotically free solutions of (1.1) by solving the system with final data given at $t \to \infty$, instead of the initial value problem; that is, for $u_+$ and $n_+$ free solutions of the Schrödinger and wave equations, respectively,

$$\|u(t) - u_+(t)\|_{H^1} + \|\nabla n(t) - \nabla n_+\|_{L^2} + \|\partial_t n(t) - \partial_t n_+(t)\|_{L^2} \to 0, \ as \ t \to \infty.$$ 

Similar results were also obtained for the 3-dimensional Klein–Gordon–Zakharov (1.2). In fact, Ozawa, Tsutaya and Tsutsumi [37] proved the existence of global solutions that behave asymptotically as free solutions in space

$$\sum_{j=0,1} \|\partial_j^j (u(t) - u_+(t))\|_{H^{52-j}} + \sum_{j=0,1} \|\partial_j^j (n(t) - n_+(t))\|_{H^{51-j}} \to 0 \ as \ t \to \infty.$$ 

In [14], Guo and Nakanishi prove that radially symmetric solutions for the Zakharov system in $d = 3$ with small energy do scatter. By using generalized Strichartz estimates for the Schrödinger equation, Guo, Lee, Nakanishi and Wang in [13] were able to improve [14] by showing scattering of small solutions without the radial assumption.

Following the idea in [14], Guo, Nakanishi and Wang [15] proved scattering in the energy space for radially symmetric solutions with small energy for the system (1.2) in three dimensions, as well. In [16], they continue the study of global dynamics of radial solutions in three dimensions and find a dichotomy between scattering and blow-up. More specifically, relying on virial identities, they show that if the initial data is radially symmetric and its energy is below the energy of the ground state then the solution to (1.2) can either (for both $i = 1, 2$):

- scatter when $J_i(u_0) \geq 0$, or
- blow up in finite time when $J_i(u_0) < 0$,

where $J_i$ are scaling derivative of the static Klein–Gordon energy:

$$J_1(v) = \int |u|^2 + |\nabla u|^2 - |u|^4 \, dx \quad \text{and} \quad J_2(v) = \int |\nabla u|^2 - \frac{3}{4} |u|^4 \, dx.$$ 

Blow-up solutions to (1.1) in two dimensions were addressed by Glangetas and Merle in [12] as well, proving the existence of solutions that blow-up. The behaviour of radially symmetric solutions of (1.1) was also studied in [25]. Using virial identities, Merle in [25] showed blow up at either finite or infinity time for radially symmetric solutions to (1.1) that satisfy $E_s < 0$ for $d = 2, 3$. In [26], Merle improved the results in [25] by presenting lower estimates on the blow-up of the Zakharov system in the 2-dimensional case.

Notice that all positive decay/scattering results above mentioned do not deal with the case $d = 1$.

From now on, we consider the one-dimensional case. In the following subsections, we introduced a reduction of order for the Zakharov and Klein–Gordon–Zakharov systems and present the main results of this work.
1.1. Main results for Zakharov system

In order to simplify the computations, from now on we consider system (1.1) with \( \alpha = 1 \), although the analysis still works for \( \alpha \neq 1 \). With the purpose of reducing (1.1) into a first-order system, we introduce the real function \( v \) such that:

\[
\begin{align*}
    iu_t + u_{xx} &= nu, \\
    n_t + v_x &= 0, \\
    v_t + \left( n + |u|^2 \right)_x &= 0,
\end{align*}
\]

and

\[
u(t = 0, x) = u_0(x), \quad n(t = 0, x) = n_0(x), \quad v(t = 0, x) = v_0.
\]

Such supposition is possible because we study the Zakharov system (1.1) in the Hamiltonian case, meaning that we assume that there exists \( v_0 \in L^2(\mathbb{R}) \) such that \(- \nabla \cdot v_0 = n_t(0)\); property that is preserved by the flow. This way, to consider \((u, n, v) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times H^{-1}(\mathbb{R})\) solution of (1.1) is equivalent to study \((u, n, v) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})\) solution to (1.4).

The system (1.4) preserves:

- **Mass:**
  \[
  M_s(t) := \int_\mathbb{R} |u(t, x)|^2 \, dx = M_s(0),
  \]

- **Energy:**
  \[
  E_s(t) := \int_\mathbb{R} |u_s(t, x)|^2 + \frac{1}{2} \left( |n(t, x)|^2 + |v(t, x)|^2 \right) + n(t, x)|u(t, x)|^2 \, dx = E_s(0),
  \]

- **Momentum:**
  \[
  P_s(t) := \text{Im} \int_\mathbb{R} u(t, x) \overline{u_s(t, x)} \, dx - \int_\mathbb{R} v(t, x) n(t, x) \, dx = P_s(0).
  \]

In the present work, we show decay for solutions of (1.4) in one dimension in two different ways. On one hand, we prove decay on any compact interval for solutions to (1.4) under parity assumptions \( u \) odd. On the other hand, we are able to show decay, without any oddness condition but with sufficient regularity (\( \| u(t) \|_{H^2} \in L^\infty \)), in regions along curves outside the “light cone”.

**Theorem 1.1.** Let \((u, n, v) \in C \left( \mathbb{R}^+, H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}) \right)\) be a solution of (1.4) such that \( u \) is odd and satisfies, for some \( \varepsilon > 0 \) small,

\[
\sup_{t \geq 0} \| u(t) \|_{H^1(\mathbb{R})} < \varepsilon.
\]

Then, for every compact interval \( I \subset \mathbb{R}, \)

\[
\lim_{t \to \infty} \| u(t) \|_{L^\infty(I)} + \| u(t) \|_{L^2(I)} + \| n(t) \|_{L^2(I)} + \| v(t) \|_{L^2(I)} = 0.
\]
Remark 1.1. Asking for $u$ to be odd implies necessarily for $n$ to be even. This property is preserved by the flow.

Remark 1.2. The fact that $u$ is odd allows to rule out solitary waves. The first result regarding solitary waves was stated by Wu in [49], where he proves existence and orbital stability of solutions

$$u(t, x) = e^{-i\omega t} e^{i q(x - ct)} u_{\omega, c}(x - ct),$$  \hspace{1cm} (1.10)$$
and

$$n(t, x) = n_{\omega, c}(x - ct),$$  \hspace{1cm} (1.11)$$
for

$$u_{\omega, c}(x) = \sqrt{\frac{(4\omega + c^2)(1 - c^2)}{2}} \text{sech} \left( \frac{\sqrt{4\omega + c^2} x}{2} \right),$$

$$n_{\omega, c}(x) = \left( 2\omega + \frac{c^2}{2} \right) \text{sech}^2 \left( \frac{\sqrt{4\omega + c^2} x}{2} \right), \quad q = \frac{c}{2},$$
satisfying

$$4\omega + c^2 \geq 0 \quad \text{and} \quad 1 - c^2 > 0.$$  

Angulo and Banquet [2] studied existence of periodic travelling wave forms such as (1.10)-(1.11), in this case for $u_{\omega, c}$ and $n_{\omega, c}$ being periodic functions, and prove their orbital stability as well. See also [9,18,19,35,51] for other results on solitary waves for generalized Zakharov systems.

Remark 1.3. We do not prove decay in the energy space $H^1 \times L^2 \times \dot{H}^{-1}(\mathbb{R})$. This is because uncontrolled $H^2$-terms emerge when considering semi-norm $\dot{H}^1$ for the solution $u$ of the Schrödinger equation. We show $L^\infty$ decay instead.

Remark 1.4. The result also holds for a generalized Zakharov system when adding a potential term $|u|^p u$ in the Schrödinger equation. See [24] for the details on how to treat the new nonlinear term.

Remark 1.5. In [23,42], the authors study the asymptotic behaviour of the Zakharov-Rubenchik system. They prove that solutions blow up if the energy is negative and give instability results for the solitary wave in the case $d = 3$. Such system is of special interest since in the supersonic limit it has been proven that it converges to (1.1).

The proof of Theorem 1.1 is based on the use of suitable virial identities. The argument follows from [21,22], where the authors deal with the Klein Gordon case. The idea is to argue as in [24], where a functional adapted to the momentum for the nonlinear Schödinger equation was considered. Unfortunately, the identity used for the
NLS equation, which allows us to conclude, it is not appropriate in this case. Instead, as in [25,26], we need to work with a virial identity that comes from the quantity

\[ P(t) := \text{Im} \int_{\mathbb{R}} u(t,x) \overline{u}_x(t,x) \, dx - \int_{\mathbb{R}} v(t,x) n(t,x) \, dx. \]

Such virial has an uncontrolled term that we manage by adding the condition (1.8).

Our second result deals with decay in far field regions along curves.

**Theorem 1.2.** Let \((u, n, v)\) satisfy (1.4).

\(a\) If \((u, n, v) \in C(\mathbb{R}^+, H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}))\), then, for any \(\mu \in C^1(\mathbb{R})\) satisfying \(\mu(t) \gtrsim t \log(t)^{1+\delta}, \delta > 0\),

\[ \lim_{t \to \infty} \|u(t)\|_{L^2(|x| \sim \mu(t))} = 0. \] (1.12)

\(b\) If \((u, n, v) \in C(\mathbb{R}^+, H^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}))\) and there exists \(f(t) \in C^1(\mathbb{R})\) a non-decreasing function such that

\[ \|u(t)\|_{H^2(\mathbb{R})} \lesssim f(t), \] (1.13)

then, for any \(\mu \in C^1(\mathbb{R})\) satisfying \(\mu(t) \gtrsim t \log(t)^{1+\delta} f(t), \delta > 0\),

\[ \lim_{t \to \infty} \|u(t)\|_{H^1(|x| \sim \mu(t))} + \|n(t)\|_{L^2(|x| \sim \mu(t))} + \|v(t)\|_{L^2(|x| \sim \mu(t))} = 0. \] (1.14)

A direct consequence of the proof of Theorem 1.2 is the following result for the NLS equation:

**Corollary 1.3.** Let \(u(t) \in H^1(\mathbb{R})\) be a solution of the nonlinear Schrödinger equation

\[ iu_t + u_{xx} \pm |u|^{p-1}u = 0, \]

where \(1 < p < 5\), with initial data \(u(t = 0, x) = u_0\) satisfying \(\|u(t = 0)\|_{H^1(\mathbb{R})} < \infty\). Then,

\[ \lim_{t \to \infty} \|u(t)\|_{L^2(|x| \sim \mu(t))} = 0. \]

The proof of Theorem 1.2 follows an argument recently introduced by Muñoz, Ponce and Saut in [33], where they deal with the long time behaviour of intermediate long wave equation. This method proves to be independent of the integrability of the equation and does not need size restriction. However, when dealing with the Zakharov system, because of the presence of uncontrolled \(H^2\)-terms in the dynamics of the \(H^1\)-norm of \(u\), we need the additional condition (1.13). Note that such condition allows as to obtain decay of the \(\| \cdot \|_{H^1}\)-norm, which was not present in results established in [33].
1.2. Main results for Klein–Gordon–Zakharov system

We will consider system (1.2) with \( \alpha = c = 1 \), although the computations still hold for different values of \( \alpha, c \in \mathbb{R} \). As we did for the Zakharov system (1.1), we reduce (1.2) by introducing a real function \( v \) satisfying \( -\nabla \cdot v = n_t \) for all \( t \geq 0 \). That is, we get a new first order system,

\[
\begin{align*}
    u_{tt} - u_{xx} + u &= -nu, \\
    n_t + v_x &= 0, \\
    v_t + \left( n + |u|^2 \right)_x &= 0,
\end{align*}
\]

and

\[
\begin{align*}
    u(t = 0, x) &= u_0(x), & u_t(t = 0, x) &= u_1(x) \\
    n(t = 0, x) &= n_0(x), & v(t = 0, x) &= v_0(x).
\end{align*}
\]

The system (1.15) preserves:

- Energy:
  
  \[
  E_{KG}(t) := \int_{\mathbb{R}} \left( |u(t, x)|^2 + |u_x(t, x)|^2 + |u_t(t, x)|^2 + \frac{1}{2} \left( |n(t, x)|^2 + |v(t, x)|^2 \right) + n(t, x)|u(t, x)|^2 \right) \, dx = E_{KG}(0),
  \]

- Momentum:
  
  \[
  P_{KG}(t) := \int_{\mathbb{R}} u_t(t, x)u_x(t, x) \, dx - \frac{1}{2} \int_{\mathbb{R}} v(t, x)n(t, x) \, dx = P_{KG}(0).
  \]

As we did for (1.4), we prove decay of solutions to (1.15) in two different ways: over compact intervals of time and over far field regions along curves. Our result for compact intervals is the following:

**Theorem 1.4.** Let \((u, u_t, n, v) \in C\left( \mathbb{R}^+, H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}) \right)\) be a solution of (1.4) such that \( u \) is odd and satisfies

\[
\sup_{t \geq 0} \|u(t)\|_{H^1(\mathbb{R})} \leq \varepsilon \quad \text{and} \quad \sup_{t \geq 0} \|u_t(t)\|_{L^2(\mathbb{R})} \leq C
\]

for some \( C > 0 \) and \( \varepsilon > 0 \) small. Then, for every compact interval \( I \subset \mathbb{R} \),

\[
\lim_{t \to \infty} \|u(t)\|_{H^1(I)} + \|u_t(t)\|_{L^2(I)} + \|n(t)\|_{L^2(I)} + \|v(t)\|_{L^2(I)} = 0.
\]

**Remark 1.6.** The oddness condition rules out solitary waves. Indeed, solitary waves of (1.15) exist and they are orbitally stable. They were first introduced by Chen in [6], where he stated that solitons of the form

\[
    u(t, x) = e^{-i\omega t}e^{iq(x-ct)}u_{\omega, c}(x - ct),
\]
\( n(t, x) = n_{\omega,c}(x - ct), \)

with

\[
\begin{align*}
  u_{\omega,c}(x) &= \sqrt{2(1 - c^2 - \omega^2)} \sech \left( \frac{\sqrt{1 - c^2 - \omega^2}}{1 - c^2} x \right), \\
  n_{\omega,c}(x) &= -2 \frac{(1 - c^2 - \omega^2)}{1 - c^2} \sech^2 \left( \frac{\sqrt{1 - c^2 - \omega^2}}{1 - c^2} x \right), \\
  q &= \omega c \frac{1 - c^2}{1 - c^2}.
\end{align*}
\]

exists when the real constants \( \omega \) and \( c \) satisfy \( 1 - c^2 - \omega^2 > 0 \). There also exist solitary waves \((u_{\omega,c}, (u_t)_{\omega,c}, n_{\omega,c}, v_{\omega,c})\) of (1.15) of the form

\[
\begin{align*}
  u_{\omega,c}(x) &= \sqrt{2(1 - c^2 - \omega^2)} \sech \left( \frac{\sqrt{1 - c^2 - \omega^2}}{1 - c^2} x \right) e^{i \frac{\omega c}{1 - c^2} x}, \\
  (u_t)_{\omega,c} &= \left( i \omega + c \frac{\partial}{\partial x} \right) u_{\omega,c}(x), \\
  n_{\omega,c}(x) &= -2 \frac{(1 - c^2 - \omega^2)}{1 - c^2} \sech^2 \left( \frac{\sqrt{1 - c^2 - \omega^2}}{1 - c^2} x \right), \\
  v_{\omega,c}(x) &= 2c \frac{(1 - c^2 - \omega^2)}{1 - c^2} \sech^2 \left( \frac{\sqrt{1 - c^2 - \omega^2}}{1 - c^2} x \right),
\end{align*}
\]

with \( 1 - 2c^2 - 2\omega^2 < 0 \) and are orbitally stable [6].

**Remark 1.7.** The result holds when considering cubic nonlinear KGZ system, that is, when adding an additional term \(|u|^2 u\) in the Klein–Gordon equation. We do not address this case in the proof, but it follows naturally from the analysis of the nonlinear term in [21].

The proof of this results follows more closely the idea in [21]. Indeed, we construct a virial identity that comes from the momentum:

\[
P_{KG} = \int_{\mathbb{R}} u_t(t, x) u_x(t, x) dx - \frac{1}{2} \int_{\mathbb{R}} v(t, x) n(t, x) dx.
\]

But, since there are uncontrolled terms involving \( u \) and \( u_t \) in the identity from the potential, we need to consider \( u_t \) uniformly bounded. Notice that we obtain now decay in the whole energy norm (that is, even for the \( H^1 \)-norm).

The last theorem is devoted to the decay of the solutions to (1.15) in regions along curves outside the light cone:

**Theorem 1.5.** If \((u, u_t, n, v) \in C(\mathbb{R}^+, H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}))\) is a solution to (1.15) such that

\[
\sup_{t \geq 0} \|u(t)\|_{H^1(\mathbb{R})} \leq \varepsilon \quad (1.20)
\]
for some $0 < \varepsilon \leq 1$, then, for any $\mu \in C^1(\mathbb{R})$ satisfying $\mu(t) \gtrless t \log(t)^{1+\delta}$, $\delta > 0$,

$$
\lim_{t \to \infty} \|u(t)\|_{H^1(|x| \sim \mu(t))} + \|u_t(t)\|_{L^2(|x| \sim \mu(t))} + \|n(t)\|_{L^2(|x| \sim \mu(t))} + \|v(t)\|_{L^2(|x| \sim \mu(t))} = 0.
$$

(1.21)

**Notation**

We introduce

$$
\|u(t)\|_{L^2(\mathbb{R})}^2 := \int_{\mathbb{R}} \text{sech}(x) |u(t, x)|^2 dx,
$$

$$
\|u(t)\|_{H^1(\mathbb{R})}^2 := \int_{\mathbb{R}} \text{sech}(x) \left( |u_x(t, x)|^2 + |u(t, x)|^2 \right) dx,
$$

(1.22)

as the weighted $L^2$-norm and $H^1$-norm.

This paper is organized as follows. In Sect. 2 we prove Theorem 1.1; the virial argument is given in Subsect. 2.1. Section 3 is devoted to the proof of Theorem 1.2. Sections 4 and 5 contain the KGZ system results, Theorems 1.4 and 1.5, respectively.

2. Decay on compact intervals for Zakharov

This section is devoted to the proof of Theorem 1.1. Before we begin with the virial analysis, we give the following result, which states boundness of the energy norm for every solution to (1.4) with finite energy, It will be useful also in Sect. 3.

**Lemma 2.1.** Let $(u, n, v) \in C\left(\mathbb{R}^+, H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})\right)$ be a solution of (1.4) such that $E_s < \infty$ and $M_s < \infty$. Then, there exists $K_s > 0$ ($K_s$ depending only on the initial data) such that

$$
\int_{\mathbb{R}} \left( |u_x(t, x)|^2 + |u(t, x)|^2 + |v(t, x)|^2 + |n(t, x)|^2 \right) dx \leq K_s.
$$

(2.1)

**Proof.** We have that

$$
\int_{\mathbb{R}} |u_x|^2 + \frac{1}{2} \left( |v|^2 + |n|^2 \right) dx
$$

$$
= \int_{\mathbb{R}} |u_x|^2 + \frac{1}{2} \left( |v|^2 + |n|^2 \right) + 2n|u|^2 dx - 2 \int_{\mathbb{R}} n|u|^2 dx
$$

$$
\leq \int_{\mathbb{R}} |u_x|^2 + \frac{1}{2} \left( |v|^2 + |n|^2 \right) + 2n|u|^2 dx + 2 \int_{\mathbb{R}} |n||u|^2 dx.
$$

Using Young inequality for products, for $\epsilon > 0$ we get

$$
\int_{\mathbb{R}} |u_x|^2 + \frac{1}{2} \left( |v|^2 + |n|^2 \right) dx
$$

$$
\leq \int_{\mathbb{R}} |u_x|^2 + \frac{1}{2} \left( |v|^2 + |n|^2 \right) + 2n|u|^2 dx + \frac{1}{\epsilon} \int_{\mathbb{R}} |n|^2 dx + \epsilon \int_{\mathbb{R}} |u|^4 dx.
$$

(2.2)
Now, Gagliardo-Nirenberg inequality [20,34], implies that
\[
\int_{\mathbb{R}} |u|^4 dx \leq C_{GN} \|u_x\|_{L^2(\mathbb{R})} \|u\|_{L^2(\mathbb{R})}^3 \leq \frac{C_{GN}}{2} \|u_x\|^2_{L^2(\mathbb{R})} + \frac{C_{GN}}{2} \|u\|_{L^2(\mathbb{R})}^6,
\]
where
\[
C_{GN} = \frac{\sqrt{3}}{3}.
\]
Then, going back to (2.2) and taking, for instance, \(\epsilon = 2\), we obtain
\[
\int_{\mathbb{R}} |\nabla u|^2 + \frac{1}{2} \left(|v|^2 + |n|^2\right) dx \leq 2E_s(0) + \frac{\sqrt{3}}{6}M_s(0)^3
\]
Which means that there exists a constant \(K_s\) depending on \(M_s(0)\) and \(E_s(0)\) such that (2.1) holds. \(\square\)

2.1. Virial argument

Step 1: Virial identity.
We now introduce suitable virial identities that allow us to work out our argument. Let \(\varphi \in C^\infty(\mathbb{R})\) be a bounded real function. Define
\[
I(t) = \text{Im} \int_{\mathbb{R}} \varphi(x)u(t,x)\overline{u}_x(t,x) dx - \int_{\mathbb{R}} \varphi(x)v(t,x)n(t,x) dx.
\]
Then, we have the following:

**Lemma 2.2.** (Virial identity) Let \((u, n, v)\) be a solution to (1.4). Then,
\[
-\frac{d}{dt} I(u(t)) = 2 \int_{\mathbb{R}} \varphi' (x) |u_x(t,x)|^2 dx - \frac{1}{2} \int_{\mathbb{R}} \varphi''(x)|u(t,x)|^2 dx
+ \int_{\mathbb{R}} \varphi' (x)n(t,x)|u(t,x)|^2
d+
\frac{1}{2} \int_{\mathbb{R}} \varphi' (x)|n(t,x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}} \varphi' |v(t,x)|^2 dx.
\]

**Proof.** We compute
\[
\frac{d}{dt} I(u(t)) = \text{Im} \int_{\mathbb{R}} \varphi(x)u_t(t,x)\overline{u}_x(t,x) dx + \text{Im} \int_{\mathbb{R}} \varphi(x)u(t,x)\overline{u}_{tx}(t,x) dx
- \int_{\mathbb{R}} \varphi(x)v_t(t,x)n(t,x) dx - \int_{\mathbb{R}} \varphi(x)v(t,x)n_t(t,x) dx.
\]
Integrating by parts,
\[
\frac{d}{dt} I(t) = 2\text{Im} \int_{\mathbb{R}} \varphi(x)u_t(t,x)\overline{u}_x(t,x) dx - \text{Im} \int_{\mathbb{R}} \varphi'(x)u(t,x)\overline{u}_t(t,x) dx
- \int_{\mathbb{R}} \varphi(x)v_t(t,x)n(t,x) dx - \int_{\mathbb{R}} \varphi(x)v(t,x)n_t(t,x) dx
= -2\text{Re} \int_{\mathbb{R}} \varphi(x)i u_t(t,x)\overline{u}_x(t,x) dx - \text{Re} \int_{\mathbb{R}} \varphi'(x)u(t,x)\overline{u}_t(t,x) dx
- \int_{\mathbb{R}} \varphi(x)v_t(t,x)n(t,x) dx - \int_{\mathbb{R}} \varphi(v(t,x)n_t(t,x) dx.
\]
Now, since \((u, n, v)\) is a solution of the system \((1.4)\),
\[
\frac{d}{dt} I(t) = \int_{\mathbb{R}} \phi(x) \left( |u_x(t, x)|^2 \right)_x \, dx - \int_{\mathbb{R}} \phi(x) n(t, x) \left( |u(t, x)|^2 \right)_x \, dx \\
+ \Re \int_{\mathbb{R}} \phi'(x) u(t, x) \overline{u}_x x(t, x) \, dx \\
- \int_{\mathbb{R}} \phi'(x) |u(t, x)|^2 n(t, x) \, dx + \int_{\mathbb{R}} \phi(x) \left( |u(t, x)|^2 + n(t, x) \right)_x \, n(t, x) \, dx \\
+ \frac{1}{2} \int_{\mathbb{R}} \phi(x) \left( |v(t, x)|^2 \right)_x \, dx.
\]

We integrate by parts once more and get
\[
\frac{d}{dt} I(t) = -2 \int_{\mathbb{R}} \phi'(x) |u_x(t, x)|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}} \phi''(x) \left( |u(t, x)|^2 \right)_x \, dx \\
- \int_{\mathbb{R}} \phi'(x) n(t, x) |u(t, x)|^2 \, dx \\
- \int_{\mathbb{R}} \phi'(x) |n(t, x)|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}} \phi(x) \left( |n(t, x)|^2 \right)_x \, dx \\
- \frac{1}{2} \int_{\mathbb{R}} \phi'(x) |v(t, x)|^2 \, dx
\]
\[
= -2 \int_{\mathbb{R}} \phi'(x) |u_x(t, x)|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}} \phi''(x) |u(t, x)|^2 \, dx \\
- \int_{\mathbb{R}} \phi'(x) n(t, x) |u(t, x)|^2 \, dx \\
- \frac{1}{2} \int_{\mathbb{R}} \phi'(x) |n(t, x)|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}} \phi'(x) |v(t, x)|^2 \, dx.
\]

Then, the identity follows. \(\square\)

**Step 2: Estimations of the terms on the virial.**

In order to find a more compact expression of \((2.3)\), we define the bilinear form
\[
B(u) = 2 \int_{\mathbb{R}} \phi' |u_x|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}} \phi'' |u|^2 \, dx.
\]

Then identity \((2.3)\) turns into
\[
- \frac{d}{dt} I(u(t)) = B(u) + \int_{\mathbb{R}} \phi' n |u|^2 + \frac{1}{2} \int_{\mathbb{R}} \phi' |n|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}} \phi' |v|^2 \, dx.
\]

Following the argument in \([24]\), for \(\lambda > 0\), let us take \(\phi(x) = \lambda \tanh \left( \frac{x}{\lambda} \right)\) and \(\omega(x) = \sqrt{\phi'(x)}\). Notice that if \(u = u_1 + i u_2\), where \(u_1, u_2\) are real functions, then \(B(u) = B(u_1) + B(u_2)\). We take \(\eta = u_i, i = 1, 2\), and find estimations for \(B(\eta)\). From now on, we are going to assume \(u\) odd, which implies that \(\eta\) is also odd. Note that, by integration by parts,
\[
\int_{\mathbb{R}} (\omega \eta_x)^2 \, dx = \int_{\mathbb{R}} \phi' (\eta_x)^2 \, dx + \int_{\mathbb{R}} \omega \omega' (\eta_x)^2 \, dx + \int_{\mathbb{R}} (\omega')^2 \eta^2 \, dx
\]
\[
\int_{\mathbb{R}} \varphi'(\eta x)^2 \, dx - \int_{\mathbb{R}} \omega \omega'' \eta^2 \, dx,
\]

It follows that
\[
\int_{\mathbb{R}} \varphi'(\eta x)^2 \, dx = \int_{\mathbb{R}} (\omega \eta)^2 \, dx + \int_{\mathbb{R}} \frac{\omega''}{\omega} (\omega \eta)^2 \, dx. \tag{2.6}
\]

On the other hand, we can re-write \( \varphi''' = (\omega^2)'' = 2 \left( \omega \omega'' + (\omega')^2 \right) \), and obtain
\[
\int_{\mathbb{R}} \varphi''' \eta^2 \, dx = 2 \int_{\mathbb{R}} \left( \frac{\omega''}{\omega} + \frac{(\omega')^2}{\omega^2} \right) (\omega \eta)^2 \, dx. \tag{2.7}
\]

Thus, from (2.6) and (2.7),
\[
B(\eta) = 2 \int_{\mathbb{R}} (\omega \eta)^2 \, dx - \int_{\mathbb{R}} \left( \frac{(\omega')^2}{\omega^2} - \frac{\omega''}{\omega} \right) (\omega \eta)^2 \, dx.
\]

Since \( \omega(x) = \text{sech} \left( \frac{x}{\lambda} \right) \), then
\[
B(\eta) = 2 \int_{\mathbb{R}} (\omega \eta)^2 \, dx - \frac{1}{\lambda^2} \int_{\mathbb{R}} \text{sech}^2 \left( \frac{x}{\lambda} \right) (\omega \eta)^2 \, dx.
\]

Now, introducing a new variable \( \zeta = \omega \eta \), we set
\[
B(\zeta) = 2 \int_{\mathbb{R}} \zeta^2 \, dx - \frac{1}{\lambda^2} \int_{\mathbb{R}} \text{sech}^2 \left( \frac{x}{\lambda} \right) \zeta^2 \, dx
\]
so that
\[
B(\zeta) = B(\eta).
\]

At this point, we would like to prove that the bilinear \( B \) is coercive, which would imply
\[
B(\eta) = B(\zeta) \gtrsim \| \xi_x \|^2_{L^2(\mathbb{R})} = \| (\omega \eta)_x \|^2_{L^2(\mathbb{R})}
\]
That way, we could have an estimation for the bilinear part on (2.5), \( B(u) \), using the weighted norm \( \| \cdot \|_{H^1_\omega} \).

**Lemma 2.3.** (See [24]) Let \( \zeta \in H^1(\mathbb{R}) \) be odd. Then,
\[
B(\zeta) \geq \frac{3}{2} \int_{\mathbb{R}} \zeta_x^2 \, dx.
\]

We refer to [24, Proposition 2.2] for the details of the proof.

Finally, to conclude the analysis of the linear term \( B(u) \), we need to bound this term by \( \| u \|_{H^1_\omega(\mathbb{R})} \).
Lemma 2.4. (See [24]) Let \( u \in H^1(\mathbb{R}) \) be odd, \( u = u_1 + iu_2 \). Then there exists a positive constant \( c_0 < 1 \) such that

\[
B(u_1) + B(u_2) \geq c_0 \|u\|^2_{H^1_0(\mathbb{R})} \tag{2.8}
\]

We omit the proof, see [24, Lemma 2.3].

**Step 3: Conclusion of the argument.**

The key ingredient of the virial argument is the subsequent proposition:

**Proposition 2.5.** Let \((u, n, v)\) be a solution of (1.4) such that \( u \) is odd and satisfies (1.8), for \( \epsilon > 0 \) sufficiently small. Then, there exists \( C > 0 \) such that

\[
\int_0^\infty \|u(t)\|^2_{H^1_0(\mathbb{R})} + \frac{1}{2} \|v(t)\|^2_{L^2_0(\mathbb{R})} + \frac{1}{2} \|n(t)\|^2_{L^2_0(\mathbb{R})} dt \leq C. \tag{2.9}
\]

In particular,

\[
\int_0^\infty \|u_x(t)\|^2_{L^2_0(\mathbb{R})} + \frac{1}{2} \|v(t)\|^2_{L^2_0(\mathbb{R})} + \frac{1}{2} \|n(t)\|^2_{L^2_0(\mathbb{R})} dt \leq C. \tag{2.10}
\]

**Proof.** From (2.5) and (2.8), we have that

\[
-\frac{d}{dt} I(t) \geq c_0 \|u(t)\|^2_{H^1_0(\mathbb{R})} + \frac{1}{2} \int_\mathbb{R} \phi'(v^2 + n^2) dx + \int_\mathbb{R} \phi'n|u|^2 dx. \tag{2.11}
\]

Now, following the idea of the proof of Lemma 2.1, by Young inequality, for some \( \epsilon > 0 \) we get

\[
\int_\mathbb{R} \phi'|n||u|^2 dx \leq \frac{1}{2\epsilon} \int_\mathbb{R} \phi'n^2 dx + \frac{\epsilon}{2} \int_\mathbb{R} \phi'|u|^4 dx. \tag{2.12}
\]

At this point, we need to absorb the negative terms using the weighted-norm (1.22). Since \( u \) is odd,

\[
\int_\mathbb{R} \operatorname{sech}^2 \left( \frac{x}{\lambda} \right) |u|^4 dx = 2 \int_0^\infty \operatorname{sech}^2 \left( \frac{x}{\lambda} \right) |u|^4 dx
\]

\[
= 2 \int_0^\infty \operatorname{sech}^{-2} \left( \frac{x}{\lambda} \right) \operatorname{sech}^4 \left( \frac{x}{\lambda} \right) |u|^4
\]

\[
\simeq \int_0^\infty e^{2x/\lambda} \operatorname{sech}^4 \left( \frac{x}{\lambda} \right) |u|^4 dx.
\]

With a slight abuse of notation, set \( \zeta(t, x) := \operatorname{sech} \left( \frac{x}{\lambda} \right) u(t, x) \). Note that \( \zeta(t, 0) = 0 \) and vanishes at infinity \( \forall t \in \mathbb{R} \). Then, integrating by parts,

\[
\int_0^\infty e^{2x/\lambda} |\zeta|^4 dx = -\frac{\lambda}{2} \int_0^\infty e^{2x/\lambda} \left( |\zeta|^4 \right)_x dx
\]

\[
= -2\lambda \operatorname{Re} \int_0^\infty e^{2x/\lambda} |\zeta|^2 \bar{\zeta} \zeta_x dx.
\]
Hence,
\[ \int_{0}^{\infty} e^{2x/\lambda} |\xi|^4 \, dx = -2\lambda \operatorname{Re} \int_{0}^{\infty} e^{x/\lambda} |\xi| \xi_x \left( e^{x/\lambda} |\xi| \right) \, dx \]
\[ \lesssim \|u\|_{L^\infty(\mathbb{R})} \operatorname{Re} \int_{0}^{\infty} e^{x/\lambda} |\xi| \xi_x \, dx \]
\[ \lesssim \|u\|_{L^\infty(\mathbb{R})} \int_{0}^{\infty} e^{x/\lambda} |\xi|^2 \xi_x \, dx. \]

By Young’s inequality,
\[ \int_{0}^{\infty} e^{2x/\lambda} |\xi|^4 \, dx \]
\[ \lesssim \|u\|_{L^\infty(\mathbb{R})} \int_{0}^{\infty} |\xi_x|^2 \, dx + \|u\|_{L^\infty(\mathbb{R})} \int_{0}^{\infty} e^{2x/\lambda} |\xi|^4 \, dx \]
\[ \simeq \|u\|_{L^\infty(\mathbb{R})} \int_{0}^{\infty} |\xi_x|^2 \, dx + \|u\|_{L^\infty(\mathbb{R})} \int_{0}^{\infty} \operatorname{sech}^{-2} \left( \frac{x}{\lambda} \right) \operatorname{sech}^4 \left( \frac{x}{\lambda} \right) |u|^4 \, dx \]
\[ = \|u\|_{L^\infty(\mathbb{R})} \int_{0}^{\infty} |\xi_x|^2 \, dx + \|u\|_{L^\infty(\mathbb{R})} \int_{0}^{\infty} \operatorname{sech}^2 \left( \frac{x}{\lambda} \right) |u|^4 \, dx. \]

By Sobolev’s embedding and (1.8) with $0 < \varepsilon < 1$, this actually means that
\[ \int_{\mathbb{R}} \operatorname{sech}^2 \left( \frac{x}{\lambda} \right) |u|^4 \, dx \lesssim \varepsilon \int_{0}^{\infty} |(\omega u)_x|^2 \, dx. \]

From Lemma 2.3 and Lemma 2.4, we obtain
\[ \int_{\mathbb{R}} \operatorname{sech}^2 \left( \frac{x}{\lambda} \right) |u|^4 \, dx \lesssim c_0 \varepsilon \|u\|_{H^1_0(\mathbb{R})}^2. \]

Then, going back to (2.11), taking $\varepsilon = 2$ and $\varepsilon$ sufficiently small, one gets
\[ -\frac{d}{dt} I \gtrsim \|u(t)\|_{H^1_0(\mathbb{R})}^2 + \|n(t)\|_{L^2_0(\mathbb{R})}^2 + \|v(t)\|_{L^2_0(\mathbb{R})}^2. \]

Now, we integrate in time over $[0, \tau]$ for $\tau > 0$,
\[ \int_{0}^{\tau} \|u(t)\|_{H^1_0(\mathbb{R})}^2 + \|n(t)\|_{L^2_0(\mathbb{R})}^2 + \|v(t)\|_{L^2_0(\mathbb{R})}^2 \, dt \lesssim |I(\tau)| + |I(0)|. \]

Thanks to Lemma 2.1, one obtains that
\[ |I(t)| \leq \|u(t)\|_{H^1(\mathbb{R})}^2 + \|n(t)\|_{L^2(\mathbb{R})}^2 + \|v(t)\|_{L^2(\mathbb{R})}^2 \leq K_s, \quad \forall t \geq 0. \]

Finally, taking $\tau \to \infty$, we conclude. \hfill \Box

2.2. Proof of Theorem 1.1:

Now, we proceed to conclude the proof of Theorem 1.1.
Let \( \phi \in C^\infty(\mathbb{R}) \). Using equation (1.4), we can compute
\[
\frac{d}{dt} \int_\mathbb{R} \phi \left( |u|^2 + |v|^2 + |n|^2 \right) dx = -2\text{Im} \int_\mathbb{R} \phi' u \overline{u}_x dt
\]
\[
- 2 \int_\mathbb{R} \phi v \left( |u|^2 + n \right)_x dx - 2 \int_\mathbb{R} \phi n v_x dx.
\]
By integration by parts, one obtains
\[
\frac{d}{dt} \int_\mathbb{R} \phi \left( |u|^2 + |v|^2 + |n|^2 \right) dx
\]
\[
= -2\text{Im} \int_\mathbb{R} \phi' u \overline{u}_x dt + 2 \int_\mathbb{R} \phi' v n dx - 2 \int_\mathbb{R} \phi v \left( |u|^2 \right)_x dx.
\]
This implies that
\[
\frac{d}{dt} \int_\mathbb{R} \phi \left( |u|^2 + |v|^2 + |n|^2 \right) dx
\]
\[
= -2\text{Im} \int_\mathbb{R} \phi' u \overline{u}_x dx + 2 \int_\mathbb{R} \phi' v n dx - 4\text{Re} \int_\mathbb{R} \phi v u \overline{u}_x dx.
\]
(2.13)

From Hölder inequality and (1.8), taking \( \phi(x) = \text{sech}(x) \), we can conclude that
\[
\frac{d}{dt} \left( \|u(t)\|^2_{L^2_w(\mathbb{R})} + \frac{1}{2} \|v(t)\|^2_{L^2_w(\mathbb{R})} + \frac{1}{2} \|n(t)\|^2_{L^2_w(\mathbb{R})} \right)
\]
\[
\leq 2 \int_\mathbb{R} \text{sech}(x) \left( |u|^2 + |u_x|^2 + |v|^2 + |n|^2 \right) dx
\]
\[
+ 2\varepsilon \int_\mathbb{R} \text{sech}(x) \left( |v|^2 + |u_x|^2 \right) dx
\]
\[
\lesssim \|u(t)\|^2_{H^1_w(\mathbb{R})} + \frac{1}{2} \|n(t)\|^2_{L^2_w(\mathbb{R})} + \frac{1}{2} \|v(t)\|^2_{L^2_w(\mathbb{R})}.
\]
(2.14)

By (2.10) we have that there exists a sequence \( \{t_n\} \subset \mathbb{R}, t_n \to \infty \) such that
\[
\|u(t_n)\|^2_{L^2_w(\mathbb{R})} + \frac{1}{2} \|v(t_n)\|^2_{L^2_w(\mathbb{R})} + \frac{1}{2} \|n(t_n)\|^2_{L^2_w(\mathbb{R})} \to 0.
\]

We integrate (2.14) over \([t, t_n]\), for some \( t \in \mathbb{R} \) and take \( t_n \to \infty \).
\[
\|u(t)\|^2_{L^2_w(\mathbb{R})} + \frac{1}{2} \|v(t)\|^2_{L^2_w(\mathbb{R})} + \frac{1}{2} \|n(t)\|^2_{L^2_w(\mathbb{R})}
\]
\[
\lesssim \int_t^{t_n} \|u(s)\|^2_{H^1_w(\mathbb{R})} + \frac{1}{2} \|n(s)\|^2_{L^2_w(\mathbb{R})} + \frac{1}{2} \|v(s)\|^2_{L^2_w(\mathbb{R})} ds.
\]

Finally, taking \( t \to \infty \), in view of (2.10), we obtain
\[
\lim_{t \to \infty} \|u(t)\|^2_{L^2_w(\mathbb{R})} + \|v(t)\|^2_{L^2_w(\mathbb{R})} + \|n(t)\|^2_{L^2_w(\mathbb{R})} = 0.
\]
(2.15)

To show decay of the \( L^\infty \)-norm, we use the following claim, proven in [24]:
Claim 1. For every interval $I$ there exists $\tilde{x}(t) \in I$ such that, as $t$ tends to infinity,

$$|u(t, \tilde{x}(t))|^2 \to 0.$$ 

Now, if $x \in I$, by Fundamental Theorem of calculus and Hölder’s inequality

$$|u(t, x)|^2 - |u(t, \tilde{x}(t))|^2 = \int_{\tilde{x}(t)}^{x} (|u|^2)_x \, dx \leq 2 \int_{\tilde{x}(t)}^{x} |u||u_x| \, dx$$

$$\leq 2 \|u(t)\|_{L^2(I)} \|u_x(t)\|_{L^2(I)}.$$ 

Then,

$$|u(t, x)|^2 \lesssim |u(t, \tilde{x}(t))|^2 + 2 \|u(t)\|_{L^2(I)} \|u_x(t)\|_{L^2(I)}, \quad \forall x \in I. \quad (2.16)$$

Using Lemma 2.1, we get

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^1(\mathbb{R})} < \infty.$$ 

Hence, taking $t \to \infty$ in (2.16), from Claim 1 and (2.15), we get that

$$|u(t, x)|^2 \to 0, \quad \forall x \in I.$$ 

3. Decay in regions along curves for Zakharov

From now on, let us assume $\lambda, \mu \in C^1(\mathbb{R})$ are functions depending on time. For $\varphi \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, define

$$K(t) = \frac{1}{2} \int_{\mathbb{R}} \varphi \left( \frac{x + \mu(t)}{\lambda(t)} \right) |u(t, x)|^2 \, dx,$$

and

$$J(t) = \int_{\mathbb{R}} \varphi \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left( |u_x(t, x)|^2 + \frac{1}{2} |v(t, x)|^2 \right.$$ 

$$\left. + \frac{1}{2} |n(t, x)|^2 + n(t, x)|u(t, x)|^2 + |u(t, x)|^2 \right) \, dx.$$ 

As we did in the previous section, we obtain a virial identity from which we are going to construct the argument.

Lemma 3.1. Let $(u, n, v) \in H^1 \times L^2 \times L^2$ a solution to (1.4). Then,

a) 

$$\frac{d}{dt} K(t) = \frac{1}{\lambda(t)} \text{Im} \int_{\mathbb{R}} \varphi' \left( \frac{x + \mu(t)}{\lambda(t)} \right) \bar{u}(t, x) u_x(t, x) \, dx$$

$$+ \frac{\mu'(t)}{2\lambda(t)} \int_{\mathbb{R}} \varphi' \left( \frac{x + \mu(t)}{\lambda(t)} \right) |u(t, x)|^2 \, dx$$

$$- \frac{\lambda'(t)}{2\lambda(t)} \int_{\mathbb{R}} \varphi' \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left( \frac{x + \mu(t)}{\lambda(t)} \right) |u(t, x)|^2 \, dx. \quad (3.1)$$
The proof follows from (2.13).

Proof: The proof follows from (2.13).

Let us consider \( \varphi \in C^2(\mathbb{R}) \) a decreasing bounded function such that \( \varphi(s) = 1 \) for \( s \leq -1 \) and \( \varphi(s) = 0 \) for \( s \geq 0 \). Thus, we get that \( \text{supp}(\varphi') \subset [-1, 0] \) and

\[
\varphi'(s) \leq 0, \quad \varphi'(s)s \geq 0 \quad \forall s \in \mathbb{R}.
\]

Now, we want to take \( \lambda \) such that \( \lambda^{-1} \) integrates finite in time over an interval such as \( [T, \infty), T > 0 \). With this idea in mind, define, for \( \delta > 0 \) and \( t \geq 2, \lambda(s) = t \log^{1+\delta}(t) \) and \( \mu(t) = \lambda(t) \). In fact, we are considering \( \mu = \lambda \) but in the following we will see that is possible to take \( \mu(t) \gtrsim \lambda(t) \) and the computations would still work. Then, for \( t \geq 2 \) we have

\[
\frac{\lambda'(t)}{\lambda(t)} = \frac{\mu'(t)}{\lambda(t)} \geq \frac{1}{t} + \frac{1+\delta}{t \log(t)}.
\]

So now, whereas \( \lambda^{-1}(t) \) is integrable in \([2, \infty], \frac{\lambda'}{\lambda} \) is not.

3.1. First part of the proof of Theorem 1.2

In this subsection, we prove (1.12). The idea is to take advantage of the non-integrability of \( \frac{\lambda'}{\lambda} \) (and of \( \frac{\mu'}{\mu} \), as well) by using the virial identity from \( K(t) \). Let us rearrange the terms in (3.1):

\[
\frac{\lambda'(t)}{2 \lambda(t)} \int_{\mathbb{R}} \varphi' \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left( \frac{x + \mu(t)}{\lambda(t)} \right) |u(t, x)|^2 dx
- \frac{\mu'(t)}{2 \lambda(t)} \int_{\mathbb{R}} \varphi' \left( \frac{x + \mu(t)}{\lambda(t)} \right) |u(t, x)|^2 dx
= -\frac{d}{dt} K(t) + \frac{1}{\lambda(t)} \text{Im} \int_{\mathbb{R}} \varphi' \left( \frac{x + \mu(t)}{\lambda(t)} \right) \bar{u}(t, x) u_x(t, x) dx
\]

(3.3)
Notice that, because of our election of $\varphi$, each term on the LHS is positive. Now, our aim is to control the RHS so that we get integrability on that part of the equation. Indeed, computing the integral of the RHS over $[2, \infty]$, from Lemma 2.1 we have that

$$
\int_2^\infty \frac{1}{\lambda(\tau)} \Im \int_\mathbb{R} \varphi' \left( \frac{x + \mu(t)}{\lambda(\tau)} \right) \bar{u}(\tau, x) u_x(\tau, x) dx d\tau
$$

$$
\leq \int_2^\infty \frac{1}{\lambda(\tau)} \|u(\tau)\|_{L^2(\mathbb{R})} \|u_x(\tau)\|_{L^2(\mathbb{R})} d\tau
$$

$$
\lesssim \int_2^\infty \frac{1}{\lambda(\tau)} d\tau < \infty.
$$

Thus, we integrate equation (3.3) in time and obtain:

$$
\int_2^\infty \frac{\lambda'(\tau)}{2\lambda(\tau)} \int_\mathbb{R} \left[ \varphi' \left( \frac{x + \mu(t)}{\lambda(\tau)} \right) \left( \frac{x + \mu(t)}{\lambda(\tau)} \right) - \varphi' \left( \frac{x + \mu(t)}{\lambda(t)} \right) \right] |u(t, x)|^2 dx d\tau < \infty.
$$

(3.4)

This implies the existence of a sequence $\{t_n\} \subset \mathbb{R}$, $t_n \to \infty$, such that

$$
\int_\mathbb{R} \left[ \varphi' \left( \frac{x + \mu(t_n)}{\lambda(t_n)} \right) \left( \frac{x + \mu(t_n)}{\lambda(t_n)} \right) - \varphi' \left( \frac{x + \mu(t_n)}{\lambda(t)} \right) \right] |u(t_n, x)|^2 dx \to 0.
$$

(3.5)

Now, take $\phi \in C_0^1(\mathbb{R})$ such that $\phi(s) \in [0, 1]$ for all $s \in \mathbb{R}$, $\text{supp}(\phi) = [-3/4, -1, 4]$, satisfying

$$
\phi(s) \lesssim |\varphi'(s)| \quad \text{and} \quad |\phi'(s)| \lesssim |\varphi'(s)| \quad \text{for all} \quad s \in \mathbb{R}.
$$

Then, if we consider $\phi$ instead $\varphi$ in (3.1), we obtain

$$
\left| \frac{d}{dt} \int_\mathbb{R} \phi \left( \frac{x + \mu(t)}{\lambda(t)} \right) |u(t, x)|^2 dx \right|
$$

$$
\lesssim \frac{1}{\lambda(t)} + \frac{\mu'(t)}{2\lambda(t)} \int_\mathbb{R} \phi' \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left( x + \mu(t) \right) \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left( \frac{x + \mu(t)}{\lambda(t)} \right) |u(t, x)|^2 dx
$$

$$
\lesssim \frac{1}{\lambda(t)} + \frac{\mu'(t)}{2\lambda(t)} \int_\mathbb{R} \phi' \left( \frac{x + \mu(t)}{\lambda(t)} \right) |u(t, x)|^2 dx.
$$

Integrating over $[t, t_n]$ and taking $t_n \to \infty$, one gets from (3.5) that

$$
\left| \int_\mathbb{R} \phi \left( \frac{x + \mu(t)}{\lambda(t)} \right) |u(t, x)|^2 dx \right| \lesssim \int_t^\infty \frac{1}{\lambda(\tau)} d\tau
$$

$$
+ \int_t^\infty \frac{\mu'(\tau)}{2\lambda(\tau)} \int_\mathbb{R} \phi' \left( \frac{x + \mu(\tau)}{\lambda(\tau)} \right) |u(\tau, x)|^2 dx d\tau.
$$

Thus, because (3.4) holds, we can take $t \to \infty$ and conclude that

$$
\lim_{t \to \infty} \left| \int_\mathbb{R} \phi \left( \frac{x + \mu(t)}{\lambda(t)} \right) |u(t, x)|^2 dx \right| \lesssim 0
$$

Finally, the region of the convergence comes from the fact that $-\frac{3}{4} \leq \frac{x + \mu(t)}{\lambda(t)} - \frac{1}{4}$ is equivalent to $x \sim \mu(t)$. 
3.2. Second part of the proof of Theorem 1.2

This section deals with the proof of (1.14). The idea of the proof is the same as before, but since our aim is to show decay of the solution \((u, n, v)\), we need to consider the virial identity (3.2). As before, we re-write the terms in (3.2) and get

\[
\frac{\lambda'(t)}{2 \lambda(t)} \int_{\mathbb{R}} \varphi' \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left( 2|x|^2 + 2n|u|^2 + |u|^2 + |v|^2 + |n|^2 \right)(t, x) dx \\
- \frac{\mu'(t)}{2 \lambda(t)} \int_{\mathbb{R}} \varphi' \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left( 2|x|^2 + 2n|u|^2 + |u|^2 + |v|^2 + |n|^2 \right)(t, x) dx \\
= -\frac{d}{dt} J(t) + \frac{1}{\lambda(t)} \text{Im} \int_{\mathbb{R}} \varphi' \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left( 2\text{Im} u_{xx} + \overline{u}u_x \right)(t, x) dx \\
+ \frac{1}{\lambda(t)} \int_{\mathbb{R}} \varphi' \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left( 2\text{Im} n\overline{u}u_x + (n + |u|^2)v \right)(t, x) dx. \tag{3.6}
\]

Just as before, we need to take \(\lambda\) such that \(\lambda^{-1}\) integrates finite in time over an interval \([T, \infty), T > 0\). Then, for \(\delta > 0\) and \(t \geq 2\), we define \(\lambda(s) = t \log^{1+\delta}(t) f(t)\) and \(\mu(t) = \lambda(t)\) (although the computations will still work for \(\mu(t) \gtrsim \lambda(t)\)). Then, for \(t \geq 2\) we have

\[
\frac{\lambda'(t)}{\lambda(t)} = \frac{\mu'(t)}{\lambda(t)} \geq \frac{1 + \delta}{t} + \frac{1}{t \log(t)}.
\]

Notice that, if \(\|u(t)\|_{H^2(\mathbb{R})} \leq C\) for all \(t \geq 0\) for some \(C > 0\), then we take \(f(t) = C\) and get that \(\lambda(t)^{-1} = \frac{1}{t \log^{1+\delta}(t) f(t)} \approx \frac{1}{t \log^{1+\delta}(t)}\). Furthermore, in the case \(\|u(t)\|_{H^2(\mathbb{R})}\) increasing infinitely, there would exist \(T > 0\) and \(C > 0\) such that \(\frac{1}{f(t)} \leq C\) for \(t \geq T\). Then, either way, we get that there exists \(T \geq 2\)

\[
\lambda(t)^{-1} \lesssim \frac{1}{t \log^{1+\delta}(t)}, \quad \text{for } t > T.
\]

Thus, we get that \(\lambda^{-1}\) integrates finite over \([T, \infty)\), while \(\frac{\lambda'}{\lambda}\) does not.

We estimate each term on the RHS of (3.2) after integrating in time. From Lemma 2.1 we have that

\[
\int_{T}^{\infty} \frac{2}{\lambda(\tau)} \int_{\mathbb{R}} \varphi' \left( \frac{x + \mu(\tau)}{\lambda(\tau)} \right) \overline{u}_x(\tau, x)u_{xx}(\tau, x) dx d\tau \lesssim \int_{T}^{\infty} \frac{1}{\lambda(\tau)} \|u_x(\tau)\|_{L^2(\mathbb{R})} \|u_{xx}(\tau)\|_{L^2(\mathbb{R})} d\tau \\
\lesssim \int_{T}^{\infty} \frac{f(\tau)}{\lambda(\tau)} d\tau < \infty.
\]

The same way, we get

\[
\int_{T}^{\infty} \frac{2}{\lambda(\tau)} \int_{\mathbb{R}} \varphi' \left( \frac{x + \mu(\tau)}{\lambda(\tau)} \right) \overline{u}(\tau, x)u_x(\tau, x) dx d\tau < \infty.
\]
Also, from Lemma 2.1 and Sobolev embeddings,
\[
\int_T^\infty \frac{2}{\lambda(\tau)} \text{Im} \int_\mathbb{R} \frac{\phi'}{\lambda(\tau)} \left( \frac{x + \mu(\tau)}{\lambda(\tau)} \right) n(\tau, x) \hat{u}(\tau, x) u_x(\tau, x) dx dt \\
\lesssim \| u(t) \|_{L^\infty(\mathbb{R})} \int_T^\infty \frac{2}{\lambda(\tau)} \int_\mathbb{R} |n(\tau, x)|^2 + |u_x(\tau, x)|^2 dx dt \\
\lesssim \int_T^\infty \frac{2}{\lambda(\tau)} dt \leq \infty,
\]
\[
\int_T^\infty \frac{1}{\lambda(\tau)} \int_\mathbb{R} \phi' \left( \frac{x + \mu(\tau)}{\lambda(\tau)} \right) n(\tau, x) v(\tau, x) dx dt \\
\lesssim \int_T^\infty \frac{1}{\lambda(\tau)} \int_\mathbb{R} |n(\tau, x)|^2 + |v(\tau, x)|^2 dx dt < \infty
\]
and
\[
\int_T^\infty \frac{1}{\lambda(\tau)} \int_\mathbb{R} \phi' \left( \frac{x + \mu(\tau)}{\lambda(\tau)} \right) |u(\tau, x)|^2 v(\tau, x) dx dt \\
\lesssim \| u(t) \|_{L^\infty(\mathbb{R})} \int_T^\infty \frac{1}{\lambda(\tau)} \int_\mathbb{R} |u(\tau, x)|^2 + |v(\tau, x)|^2 dx dt < \infty.
\]
Consequently, integrating (3.6) over \([T, \infty]\), one obtains
\[
\int_T^\infty \frac{\lambda'(\tau)}{2\lambda(\tau)} \int_\mathbb{R} \phi' \left( \frac{x + \mu(\tau)}{\lambda(\tau)} \right) \left( \frac{x + \mu(\tau)}{\lambda(\tau)} \right) \\
\left( 2|u_x|^2 + 2n|u|^2 + |u|^2 + |v|^2 + |n|^2 \right) (\tau, x) dx \\
- \frac{\mu'(\tau)}{2\lambda(\tau)} \int_\mathbb{R} \phi' \left( \frac{x + \mu(\tau)}{\lambda(\tau)} \right) \\
\left( 2|u_x|^2 + 2n|u|^2 + |u|^2 + |v|^2 + |n|^2 \right) (\tau, x) dx d\tau < \infty.
\] (3.7)

Furthermore,
\[
\int_T^\infty \frac{\mu'(\tau)}{2\lambda(\tau)} \int_\mathbb{R} \phi' \left( \frac{x + \mu(\tau)}{\lambda(\tau)} \right) \\
\left( 2|u_x|^2 + |u|^2 + |v|^2 + |n|^2 \right) (\tau, x) dx dt < \infty.
\] (3.8)

Indeed, from (3.7) and Young inequality for products,
\[
\infty > \int_T^\infty \frac{\lambda'(\tau)}{2\lambda(\tau)} \int_\mathbb{R} \phi' \left( \frac{x + \mu(\tau)}{\lambda(\tau)} \right) \left( \frac{x + \mu(\tau)}{\lambda(\tau)} \right) \\
\left( 2|u_x|^2 + 2n|u|^2 + |u|^2 + |v|^2 + |n|^2 \right) (\tau, x) dx
\]
\[
- \frac{\mu'(\tau)}{2\lambda(\tau)} \int_{\mathbb{R}} \varphi' \left( \frac{x + \mu(\tau)}{\lambda(\tau)} \right) \left( 2|u_x|^2 + 2n|u|^2 + |u|^2 + |v|^2 + |n|^2 \right)(\tau, x) dx \, dt
\]
\[
\geq \int_{T}^{\infty} \frac{\lambda'(\tau)}{2\lambda(\tau)} \int_{\mathbb{R}} \varphi' \left( \frac{x + \mu(\tau)}{\lambda(\tau)} \right) \left( 2|u_x|^2 + |u|^2 + |v|^2 + \frac{1}{2}|n|^2 - 2|u|^4 \right)(\tau, x) dx \]
\[
- \frac{\mu'(\tau)}{2\lambda(\tau)} \int_{\mathbb{R}} \varphi' \left( \frac{x + \mu(\tau)}{\lambda(\tau)} \right) \left( 2|u_x|^2 + |u|^2 + |v|^2 + \frac{1}{2}|n|^2 - 2|u|^4 \right)(\tau, x) dx \, dt.
\]

By Sobolev embedding and (3.4), we have that
\[
\int_{T}^{\infty} \frac{\mu'(\tau)}{\lambda(\tau)} \int_{\mathbb{R}} \left( \varphi' \left( \frac{x + \mu(\tau)}{\lambda(\tau)} \right) \left( \frac{x + \mu(\tau)}{\lambda(\tau)} \right) - \varphi' \left( \frac{x + \mu(\tau)}{\lambda(\tau)} \right) \right) |u(\tau, x)|^4 dx \, dt
\]
\[
\lesssim \int_{T}^{\infty} \frac{\mu'(\tau)}{\lambda(\tau)} \int_{\mathbb{R}} \left( \varphi' \left( \frac{x + \mu(\tau)}{\lambda(\tau)} \right) \left( \frac{x + \mu(\tau)}{\lambda(\tau)} \right) \right) |u(\tau, x)|^2 dx \, dt < \infty.
\]

Then, (3.8) holds. Thus, there exists a sequence \( \{t_n\} \), \( t_n \to \infty \) satisfying
\[
\int_{\mathbb{R}} \left| \varphi' \left( \frac{x + \mu(t_n)}{\lambda(t_n)} \right) \right| \left( |u_x|^2 + |u|^2 + |v|^2 + |n|^2 \right)(t_n, x) dx \to 0. \tag{3.9}
\]
Furthermore, using Sobolev embedding and Lemma 2.1, we have from (3.9) that
\[
\left| \int_{\mathbb{R}} \varphi' \left( \frac{x + \mu(t_n)}{\lambda(t_n)} \right) n(t_n, x)|u(t_n, x)|^2 dx \right|
\]
\[
\lesssim \int_{\mathbb{R}} \left| \varphi' \left( \frac{x + \mu(t_n)}{\lambda(t_n)} \right) \right| |n(t_n, x)|^2 dx + \int_{\mathbb{R}} \left| \varphi' \left( \frac{x + \mu(t_n)}{\lambda(t_n)} \right) \right| |u(t_n, x)|^2 dx \to 0.
\]

Then,
\[
\left| \int_{\mathbb{R}} \varphi' \left( \frac{x + \mu(t_n)}{\lambda(t_n)} \right) \left( |u_x|^2 + |u|^2 + |v|^2 + |n|^2 + n|u|^2 \right)(t_n, x) dx \right| \to 0. \tag{3.10}
\]

We argue as before and consider \( \phi \in C_0^1(\mathbb{R}) \) such that \( \phi(s) \in [0, 1] \) for all \( s \in \mathbb{R} \),
\[
\text{supp}(\phi) = [-3/4, -1, 4],
\]
\[
\phi(s) \lesssim |\phi'(s)| \text{ and } |\phi'(s)| \lesssim |\phi'(s)| \text{ for all } s \in \mathbb{R}.
\]
One gets,
\[
\left| \frac{d}{dt} \int_{\mathbb{R}} \phi \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left( |ux|^2 + \frac{1}{2} |u|^2 + \frac{1}{2} |n|^2 + \frac{1}{2} |v|^2 + n |u|^2 \right) (t,x) \, dx \right| \\
\lesssim \frac{2}{\lambda(t)} \int_{\mathbb{R}} \phi' \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left| \left( \bar{u}_x u_{xx} + n \bar{u}_x \right) \right| (t,x) \, dx \\
+ \frac{1}{\lambda(t)} \int_{\mathbb{R}} \phi' \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left| \left( \bar{u}_x + (n + |u|^2) v \right) \right| (t,x) \, dx \\
+ \frac{\mu'(t)}{2 \lambda(t)} \int_{\mathbb{R}} \phi' \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left| \left( 2 |u_x|^2 + 2n |u|^2 + |u|^2 + |v|^2 + |n|^2 \right) \right| (t,x) \, dx \\
\lesssim \frac{1}{\lambda(t)} + \frac{\mu'(t)}{2 \lambda(t)} \int_{\mathbb{R}} \phi' \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left| \left( 2 |u_x|^2 + 2n |u|^2 + |u|^4 + |u|^2 + |v|^2 \right) \right| (t,x) \, dx 
\]
Integrate over \([t, t_n]\), \(t \geq T\) and take \(t_n \to \infty\). Thanks to (3.10), we obtain
\[
\left| \int_{\mathbb{R}} \phi \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left( |ux|^2 + \frac{1}{2} |u|^2 + \frac{1}{2} |n|^2 + \frac{1}{2} |v|^2 + n |u|^2 \right) (t,x) \, dx \right| \\
\lesssim \int_t^\infty \frac{1}{\lambda(\tau)} \, d\tau + \int_t^\infty \frac{\mu'(\tau)}{2 \lambda(\tau)} \int_{\mathbb{R}} \phi' \left( \frac{x + \mu(\tau)}{\lambda(\tau)} \right) \left| \left( 2 |u_x|^2 + n |u|^2 + |u|^4 + |u|^2 + |v|^2 \right) \right| (\tau, x) \, dx \, d\tau 
\]
Now, as in subsect. 3.1, taking \(t \to \infty\), we obtain
\[
\lim_{t \to \infty} \left| \int_{\mathbb{R}} \phi \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left( |ux|^2 + \frac{1}{2} |u|^2 + \frac{1}{2} |n|^2 + \frac{1}{2} |v|^2 + n |u|^2 \right) (t,x) \, dx \right| \leq 0 
\]
Note that by Hölder inequality and Lemma 2.1,
\[
\int_{\mathbb{R}} \phi \left( \frac{x + \mu(t)}{\lambda(t)} \right) (n |u|^2) (t,x) \, dx \lesssim \|n(t)\|_{L^2(\mathbb{R})} \left( \int_{\mathbb{R}} \phi \left( \frac{x + \mu(t)}{\lambda(t)} \right) |u(t,x)|^2 \, dx \right)^{1/2} \\
\lesssim \left( \int_{\mathbb{R}} \phi \left( \frac{x + \mu(t)}{\lambda(t)} \right) |u(t,x)|^2 \, dx \right)^{1/2} 
\]
Thanks to (1.12), we conclude the proof.

4. Decay on compact intervals for Klein–Gordon–Zakharov

Before we present the proof of Theorems 1.4 and 1.5, we give an estimation of the energy norm of a solution for (1.15), that will be useful in the following.

**Lemma 4.1.** Let \((u, u_t, n, v) \in C\left( \mathbb{R}^+; H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}) \right)\) be a solution of (1.4) such that \(E_{KG} < \infty\) and it satisfies (1.18) for some \(C > 0\) and \(\varepsilon > 0\) not necessarily small. Then, there exists \(K_{KG} > 0\) such that
\[
\int_{\mathbb{R}} |u_t(t,x)|^2 + |u_x(t,x)|^2 + |u(t,x)|^2 + |n(t,x)|^2 + |v(t,x)|^2 \, dx \leq K_{KG} 
\]
Proof. We write
\[
\frac{1}{2} \int_{\mathbb{R}} |u_t|^2 + |u_x|^2 + |u|^2 + |n|^2 + |v|^2 \, dx \\
\leq \int_{\mathbb{R}} |u_t|^2 + \frac{1}{2} |u_x|^2 + |u|^2 + \frac{1}{2} |n|^2 + |v|^2 + 2n|u|^2 \, dx - 2 \int_{\mathbb{R}} n|u|^2 \, dx.
\]
The first integral in the RHS can be bounded by the energy. Thus, we need to control the remaining term. By Young inequality and Gagliardo-Nirenberg inequality \[20, 34\], for \(\epsilon > 0\) we have that
\[
2 \int_{\mathbb{R}} nu^2 \, dx \leq \frac{1}{\epsilon} \int_{\mathbb{R}} n^2 \, dx + \frac{1}{2} \int_{\mathbb{R}} u^4 \, dx \leq \frac{1}{\epsilon} \int_{\mathbb{R}} n^2 \, dx + \epsilon C_{GN} \|u_x\|_{L^2(\mathbb{R})} \|u\|_{L^2(\mathbb{R})}^3,
\]
where
\[
C_{GN} = \frac{\sqrt{3}}{3}.
\]
Then, taking \(\epsilon = 2\), we get
\[
2 \int_{\mathbb{R}} nu^2 \, dx \leq \frac{1}{2} \int_{\mathbb{R}} n^2 \, dx + 2 \frac{\sqrt{3}}{3} \|u_x\|_{L^2(\mathbb{R})} \|u\|_{L^2(\mathbb{R})}^3
\]
\[
\leq \frac{1}{2} \int_{\mathbb{R}} n^2 \, dx + \frac{\sqrt{3}}{3} \int_{\mathbb{R}} u_x^2 \, dx + \frac{\sqrt{3}}{3} \|u\|_{L^2(\mathbb{R})}^6.
\]
Finally, this means that
\[
\frac{1}{2} \int_{\mathbb{R}} |u_t|^2 + |u_x|^2 + |u|^2 + |n|^2 + |v|^2 \, dx \leq 2E_0 + \frac{\sqrt{3}}{3} \|u\|_{L^2(\mathbb{R})}^6.
\]
Thanks to (1.18), we conclude. \(\square\)

4.1. Virial argument

During this section, we are going to consider \((u, n, v)\) a solution such that \(u\) is odd and satisfies (1.18), for some \(C > 0\) and \(\epsilon\) small. As in Sect. 2, let \(\varphi \in C^\infty(\mathbb{R})\) a bounded real function and define
\[
I(t) = 2 \int_{\mathbb{R}} \varphi(x)u_x(t, x)u_t(t, x) \, dx - \int_{\mathbb{R}} \varphi(x)v(t, x)n(t, x) \, dx
\]
\[
+ \int_{\mathbb{R}} \varphi'(x)u(t, x)u_t(t, x) \, dx.
\]
We get the following virial identity:

Lemma 4.2. (Virial Identity) Let \((u, u_t, n, v)\) be a solution to (1.15). Then,
\[
- \frac{d}{dt} I(t) = 2 \int_{\mathbb{R}} \varphi' u_x^2 \, dx - \frac{1}{2} \int_{\mathbb{R}} \varphi'' u^2 \, dx + \frac{1}{2} \int_{\mathbb{R}} \varphi' n^2 \, dx
\]
\[
+ \frac{1}{2} \int_{\mathbb{R}} \varphi' v^2 \, dx + \int_{\mathbb{R}} \varphi' u^2 n \, dx.
\] (4.1)
Proof. Using equation (1.15), we compute
\[
\frac{d}{dt} I(t) = 2 \int_\mathbb{R} \varphi u_x u_t dx + 2 \int_\mathbb{R} \varphi u_x u_{xx} dx - 2 \int_\mathbb{R} \varphi u_x u dx \\
- \int_\mathbb{R} \varphi n u_x u dx + \int_\mathbb{R} \varphi \left(n + u^2\right) u dx \\
+ \int_\mathbb{R} \varphi u v_x dx + 2 \int_\mathbb{R} \varphi u_t u_t dx + \int_\mathbb{R} \varphi u u_x u_{xx} dx \\
- \int_\mathbb{R} \varphi' u u dx - \int_\mathbb{R} \varphi' u u dx \\
= \int_\mathbb{R} \varphi(u^2)_x dx + \int_\mathbb{R} \varphi(u^2)_x dx \\
- \int_\mathbb{R} \varphi(u^2)_x dx - \int_\mathbb{R} \varphi n(u^2)_x dx + \frac{1}{2} \int_\mathbb{R} \varphi(n^2)_x dx \\
+ \int_\mathbb{R} \varphi(u^2)_x dx + \frac{1}{2} \int_\mathbb{R} \varphi((v^2)_x dx + \int_\mathbb{R} \varphi' u^2 dx \\
+ \int_\mathbb{R} \varphi' u u x dx - \int_\mathbb{R} \varphi' u^2 dx - \int_\mathbb{R} \varphi' u^2 dx.
\]

We integrate by parts
\[
\frac{d}{dt} I(t) = -2 \int_\mathbb{R} \varphi' u^2 dx - \frac{1}{2} \int_\mathbb{R} \varphi' n^2 dx - \frac{1}{2} \int_\mathbb{R} \varphi' v^2 dx - \int_\mathbb{R} \varphi'' u_x u dx - \int_\mathbb{R} \varphi' u^2 dx \\
= -2 \int_\mathbb{R} \varphi' u^2 dx + \frac{1}{2} \int_\mathbb{R} \varphi''' u^2 dx - \frac{1}{2} \int_\mathbb{R} \varphi' n^2 dx - \frac{1}{2} \int_\mathbb{R} \varphi' v^2 dx - \int_\mathbb{R} \varphi' u^2 dx.
\]

Notice that now we have a very similar virial identity to the one obtained in Subsect. 2.1. In fact, the RHS is the same. Then, we are entitled to use the estimations for the bilinear part of (2.3). Indeed, we can write
\[
- \frac{d}{dt} I(t) = B(u) + \frac{1}{2} \int_\mathbb{R} \varphi' n^2 dx + \frac{1}{2} \int_\mathbb{R} \varphi' v^2 dx + \int_\mathbb{R} \varphi' u^2 ndx,
\]

where $B$ is defined in (2.4). Just as before, for $\lambda > 0$, consider $\varphi(x) = \lambda \tanh(x/\lambda)$ and $\omega(x) = \sqrt{\varphi'(x)}$. Finally, using the arguments in Subsect. 2.1 and by estimation (2.8), we have that
\[
- \frac{d}{dt} I(t) \gtrsim \|u(t)\|_{H^1_0(\mathbb{R})}^2 + \frac{1}{2} \int_\mathbb{R} \varphi' n^2 dx + \frac{1}{2} \int_\mathbb{R} \varphi' v^2 dx + \int_\mathbb{R} \varphi' u^2 ndx, \quad (4.2)
\]

where $\|\cdot\|_{H^1_0(\mathbb{R})}$ is the weighted-norm introduced in (1.22).

In order to conclude the argument, we present the following proposition:

**Proposition 4.3.** Let $(u, n, v)$ be a solution of (1.15). Then, there exists $C > 0$ such that
\[
\int_0^\infty \|u(t)\|_{L^2_0(\mathbb{R})}^2 + \|u(t)\|_{H^1_0(\mathbb{R})} + \|n(t)\|_{L^2_0(\mathbb{R})} + \|v(t)\|_{L^2_0(\mathbb{R})} dt \leq C.
\]

**Proof.** Thanks to (4.2) and (1.18), the proof follows as in Proposition 2.5. \qed
4.2. Conclusion of the proof

Let $\phi$ be a $C^\infty(\mathbb{R})$ function to be defined later. Then, since $(u, n, v)$ is a solution to equation (1.15),

$$
\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}} \phi(x) \left( |u|^2 + |u_t|^2 + |u_x|^2 + |n|^2 + |v|^2 \right) (t, x) dx
$$

$$
= \int_{\mathbb{R}} \phi(x) u_t(t, x) u_x(t, x) dx - \int_{\mathbb{R}} \phi(x) n(t, x) u_t(t, x) u(t, x) dx + \int_{\mathbb{R}} \phi(x) n(t, x) u_x(t, x) dx - \int_{\mathbb{R}} \phi(x) v(t, x) (n + |u|^2) (t, x) dx.
$$

After integration by parts, one gets

$$
\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}} \phi(x) \left( |u|^2 + |u_t|^2 + |u_x|^2 + |n|^2 + |v|^2 \right) (t, x) dx
$$

$$
= - \int_{\mathbb{R}} \phi'(x) u_t(t, x) u_x(t, x) dx - \int_{\mathbb{R}} \phi(x) n(t, x) u_t(t, x) u(t, x) dx + \int_{\mathbb{R}} \phi'(x) v(t, x) n(t, x) dx + 2 \int_{\mathbb{R}} \phi(x) v(t, x) u(t, x) u_x(t, x) dx. \tag{4.3}
$$

Thus, if we take $\phi(x) = \text{sech}(x)$, we have that

$$
\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}} \text{sech}(x) \left( |u|^2 + |u_t|^2 + |u_x|^2 + |n|^2 + |v|^2 \right) (t, x) dx
$$

$$
\lesssim \|u(t)\|^2_{L^2_0(\mathbb{R})} + \|u_t(t)\|^2_{L^2_0(\mathbb{R})} + \|v(t)\|^2_{L^2_0(\mathbb{R})} + \|n(t)\|^2_{L^2_0(\mathbb{R})}.
$$

Proposition 4.3 implies the existence of a sequence $\{t_n\} \subset \mathbb{R}, t_n \to \infty$, such that

$$
\|u(t_n)\|^2_{H^1_0(\mathbb{R})} + \|u_t(t_n)\|^2_{L^2_0(\mathbb{R})} + \|v(t_n)\|^2_{L^2_0(\mathbb{R})} + \|n(t_n)\|^2_{L^2_0(\mathbb{R})} \to 0.
$$

Then, we integrate over $[t, t_n]$, take $t_n \to \infty$ and obtain

$$
\|u(t)\|^2_{H^1_0(\mathbb{R})} + \|u_t(t)\|^2_{L^2_0(\mathbb{R})} + \|v(t)\|^2_{L^2_0(\mathbb{R})} + \|n(t)\|^2_{L^2_0(\mathbb{R})}
$$

$$
\lesssim \int_{t}^{\infty} \|u(\tau)\|^2_{H^1_0(\mathbb{R})} + \|u_t(\tau)\|^2_{L^2_0(\mathbb{R})} + \|v(\tau)\|^2_{L^2_0(\mathbb{R})} + \|n(\tau)\|^2_{L^2_0(\mathbb{R})} d\tau.
$$

Thanks to Proposition 4.3, the RHS of the last equation is finite. Consequently, we can take $t \to \infty$ and conclude the proof.
5. Decay in regions along curves for Klein–Gordon–Zakharov

To construct the virial identity, consider \( \varphi \in C^2(\mathbb{R}) \) a bounded real function and \( \lambda, \mu \in C^1(\mathbb{R}) \) functions depending on time. We define

\[
J(t) = \frac{1}{2} \int_{\mathbb{R}} \varphi\left(\frac{x + \mu(t)}{\lambda(t)}\right) \left(|u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2}|v|^2 + \frac{1}{2}|n|^2 + n|u|^2\right)(t, x) dx.
\]

Lemma 5.1. Let \((u, n, v)\) be a solution to (1.15). Then,

\[
\frac{d}{dt} J(t) = \frac{1}{\lambda(t)} \int_{\mathbb{R}} \varphi'\left(\frac{x + \mu(t)}{\lambda(t)}\right) \left(\frac{1}{2} vn + \frac{1}{2} v|u|^2 - u_t u_x\right)(t, x) dx
\]

\[
+ \frac{\mu'(t)}{2\lambda(t)} \int_{\mathbb{R}} \varphi'\left(\frac{x + \mu(t)}{\lambda(t)}\right) \left(|u_x| + |u|^2 + |u_t|^2 + \frac{1}{2}|v|^2 + \frac{1}{2}|n|^2 + n|u|^2\right)(t, x) dx
\]

\[
- \lambda'(t) \frac{1}{2\lambda(t)} \int_{\mathbb{R}} \varphi'\left(\frac{x + \mu(t)}{\lambda(t)}\right) \left(\lambda(t) + x + \mu(t)\right) \left(|u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2}|v|^2 + \frac{1}{2}|n|^2 + n|u|^2\right)(t, x) dx.
\]

We skip the proof of Lemma 5.1, since it follows from (4.3).

5.1. Proof Theorem 1.5

As we did in Sect. 3, we consider \( \varphi \in C^2(\mathbb{R}) \) a decreasing function satisfying \( \varphi(s) = 1 \) for \( s \leq -1 \) and \( \varphi(s) = 0 \) for \( s \geq 0 \). It follows that \( \text{supp}(\varphi') \subset [-1, 0) \) and

\[
\varphi'(s) \leq 0, \quad \varphi'(s)s \geq 0 \quad \forall s \in \mathbb{R}.
\]

Also, for \( \delta > 0 \), we take \( \lambda(t) = t \log^{1+\delta}(t) \) and \( \mu(t) = \lambda(t) \). Just as before, we are going to take advantage of the fact that \( \lambda^{-1} \) is integrable in time over the time interval \([T, \infty)\), for some \( T \geq 2 \), while \( \frac{\lambda'}{\lambda} \) is not.

We re-arrange the virial identity (5.1) as:

\[
\frac{\lambda'(t)}{2\lambda(t)} \int_{\mathbb{R}} \varphi'\left(\frac{x + \mu(t)}{\lambda(t)}\right) \left(\lambda(t) + x + \mu(t)\right) \left(|u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2}|v|^2 + \frac{1}{2}|n|^2 + n|u|^2\right)(t, x) dx
\]

\[
- \frac{\mu'(t)}{2\lambda(t)} \int_{\mathbb{R}} \varphi'\left(\frac{x + \mu(t)}{\lambda(t)}\right) \left(|u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2}|v|^2 + \frac{1}{2}|n|^2 + n|u|^2\right)(t, x) dx
\]
\begin{equation}
\frac{d}{dt} J(t) + \frac{1}{\lambda(t)} \int_{\mathbb{R}} \phi' \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left( \frac{1}{2} v n + \frac{1}{2} v |u|^2 - u_t u_x \right) (t, x) dx.
\end{equation} (5.1)

We have that, by Gagliardo-Nirenberg inequality and Lemma 4.1

\begin{align*}
\int_{\mathbb{R}} \phi' \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left( \frac{1}{2} v n + \frac{1}{2} v |u|^2 - u_t u_x \right) (t, x) dx \\
\lesssim \|v(t)\|_{L^2(\mathbb{R})}^2 + \|n(t)\|_{L^2(\mathbb{R})}^2 + \|u_x(t)\|_{H^1(\mathbb{R})}^2 + \|u_t(t)\|_{L^2(\mathbb{R})}^2 + \|u(t)\|_{L^2(\mathbb{R})}^6 \leq K,
\end{align*}

where $K > 0$, is a constant depending on the energy and the $L^2$-norm of $u$. Thus, we integrate equation (5.1) in time over $[2, \infty)$ and get that

\begin{align*}
\int_2^{\infty} \frac{\lambda'(t)}{2\lambda(t)} \int_{\mathbb{R}} \phi' \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left( \frac{x + \mu(t)}{\lambda(t)} \right) \\
\left( |u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2} |v|^2 + \frac{1}{2} |n|^2 + n |u|^2 \right) (t, x) dx \\
- \frac{\mu'(t)}{2\lambda(t)} \int_{\mathbb{R}} \phi' \left( \frac{x + \mu(t)}{\lambda(t)} \right) \\
\left( |u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2} |v|^2 + \frac{1}{2} |n|^2 + n |u|^2 \right) (t, x) dx dt < \infty.
\end{align*}

Note that by Sobolev embeddings and (1.20),

\begin{align*}
- \int_{\mathbb{R}} \phi \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left( |n| |u|^2 \right) (t, x) dx \\
\geq - \int_{\mathbb{R}} \phi \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left( \frac{1}{3} |n(t, x)|^2 + \frac{3}{4} |u(t, x)|^4 \right) dx \\
\geq - \int_{\mathbb{R}} \phi \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left( \frac{1}{3} |n(t, x)|^2 + \frac{3}{4} |u(t, x)|^2 \right) dx.
\end{align*} (5.2)

Then, we argue as in Subsect. 3.2 and obtain that there exists a sequence of time $\{t_n\}$, $t_n \to \infty$, such that,

\begin{equation}
\int_{\mathbb{R}} \phi' \left( \frac{x + \mu(t_n)}{\lambda(t_n)} \right) \left( |u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2} |v|^2 + \frac{1}{2} |n|^2 + n |u|^2 \right) (t_n, x) dx \to 0.
\end{equation} (5.3)

As in Sect. 3, we consider $\phi \in C^1_0(\mathbb{R})$ such that $\phi(s) \in [0, 1]$ for all $s \in \mathbb{R}$, supp$(\phi) = [-3/4, -1, 4]$, $\phi(s) \lesssim |\phi'(s)|$ and $|\phi'(s)| \lesssim |\phi'(s)|$ for all $s \in \mathbb{R}$. 

Following the computations for $\varphi$, one gets,
\[
\left| \frac{d}{dt} \int_{\mathbb{R}} \phi \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left( |u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2} |n|^2 + \frac{1}{2} |v|^2 + n|u|^2 \right)(t, x) dx \right| 
\leq \frac{1}{\lambda(t)} \int_{\mathbb{R}} \left| \phi' \left( \frac{x + \mu(t)}{\lambda(t)} \right) \right| \left( \frac{1}{2} vn + \frac{1}{2} v|u|^2 - u_t u_x \right)(t, x) dx
\]
\[+ \frac{\mu'(t)}{2\lambda(t)} \int_{\mathbb{R}} \left| \phi' \left( \frac{x + \mu(t)}{\lambda(t)} \right) \right| \left( |u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2} |n|^2 + \frac{1}{2} |v|^2 + n|u|^2 \right)(t, x) dx.
\]
Integrate over $[t, t_n]$, $t \geq T$. Thanks to (5.3), we obtain
\[
\left| \int_{\mathbb{R}} \phi \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left( |u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2} |n|^2 + \frac{1}{2} |v|^2 + n|u|^2 \right)(t, x) dx \right|
\leq \int_{t}^{\infty} \frac{1}{\lambda(\tau)} d\tau + \int_{t}^{\infty} \frac{\mu'(\tau)}{2\lambda(\tau)} \int_{\mathbb{R}} \left| \phi' \left( \frac{x + \mu(\tau)}{\lambda(\tau)} \right) \right| \left( |u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2} |n|^2 + \frac{1}{2} |v|^2 + n|u|^2 \right)(\tau, x) dxd\tau
\]
Now, taking $t \to \infty$, we obtain
\[
\lim_{t \to \infty} \left| \int_{\mathbb{R}} \phi \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left( |u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2} |n|^2 + \frac{1}{2} |v|^2 + n|u|^2 \right)(t, x) dx \right| \leq 0
\]
Consequently, taking into account (5.2), one gets
\[
\lim_{t \to \infty} \int_{\mathbb{R}} \phi \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left( |u_x|^2 + \frac{1}{4} |u|^2 + |u_t|^2 + \frac{1}{6} |n|^2 + \frac{1}{2} |v|^2 \right)(t, x) dx
\leq \lim_{t \to \infty} \int_{\mathbb{R}} \phi \left( \frac{x + \mu(t)}{\lambda(t)} \right) \left( |u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2} |n|^2 + \frac{1}{2} |v|^2 + n|u|^2 \right)(t, x) dx = 0.
\]
Then, (1.21) follows.

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