AHARONOV-BOHM EFFECT VS. DIRAC MONOPOLE: A-B ⇔ D

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Abstract. In the context of fiber bundle theory, we show that the existence of the Aharonov-Bohm connection implies the existence and uniqueness of the Dirac connection.

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1. Introduction

As is well known [1], the Aharonov-Bohm effect (A – B) [2,3] can be described in the trivial $U(1)$-bundle

$$\xi_{A-B} : U(1) \hookrightarrow \mathbb{C}^* \times U(1) \xrightarrow{pr_1} \mathbb{C}^*, \quad pr_1(z, \mu) = z$$

(\(\mathbb{C}^* = \mathbb{C} \setminus \{0\}, \mu = e^{i\varphi}, \varphi \in [0, 2\pi]\)), while the hypothetical $g = \frac{1}{2}$ magnetic charge or Dirac monopole (D) [4,5] can be described in the non trivial $U(1)$-bundle [6]

$$\xi_D : U(1) \hookrightarrow S^3 \xrightarrow{\pi_H} S^2$$

(Hopf bundle, [7]), where $\pi_H$ is the Hopf map

$$\pi_H(z_1, z_2) = \{ z_1/z_2, \quad z_2 \neq 0 \}$$

with

$$S^3 = \{(z_1, z_2) | |z_1|^2 + |z_2|^2 = 1 \} \subset \mathbb{C}^2$$

and

$$\Phi : S^2 \subset \mathbb{R}^3 \xrightarrow{\cong} \mathbb{C} \cup \{\infty\}, \quad \Phi(x_1, x_2, x_3) = \{ \frac{x_1 + ix_2}{1+ix_3}, \quad (x_1, x_2, x_3) \neq (0, 0, -1) \}$$

In terms of the Euler angles in $\mathbb{R}^3, \chi, \varphi \in [0, 2\pi], \theta \in [0, \pi]$, the $u(1) = \text{Lie}(U(1))$-valued $D$ connection on $S^3$ is given by [8]

$$\omega_D = \frac{i}{2}(d\chi + \cos\theta \ d\varphi)$$

with $D$ potentials on $S^2$

$$A_{D\pm} = \mp \frac{i}{2}(1 - \cos\theta) \ d\varphi,$$

and curvature $F_D = d\omega_D = \frac{i}{2}\sin\theta \ d\theta \wedge d\varphi$: $(-i)\times$ the magnetic field of the monopole, while the $A – B$ potentials on $\mathbb{C}^*$ (and global connection $A_{A-B}$ on $\mathbb{C}^* \times U(1)$ since $\xi_{A-B}$ is trivial) are given by [1]

$$A_{A-B\pm} = \mp \frac{i}{2} \ d\varphi = \mp \frac{i}{2} \frac{X_1dX_2 - X_2dX_1}{X_1^2 + X_2^2}$$

with $z = X_1 + iX_2 \in \mathbb{C}^*$ and $X_1, X_2$ the Cartesian coordinates on $\mathbb{R}^2^* \cong \mathbb{C}^*$; clearly, $A_{A-B\pm}$ are closed ($A_{A-B}$ is flat in its domain of definition, $z \neq 0$) but not exact 1-forms.

From (7) and (8),

$$A_{D\pm}|_{\theta = \pi/2} = A_{A-B\pm}$$

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which, in the context of bundle theory, tells us that the existence of the Dirac monopole implies the existence of the $A - B$ effect ("$D \Rightarrow A - B$"). The same conclusion has been arrived at in ref. [9], where the close relation between both phenomena was exhibited by showing that the $A - B$ bundle is equivalent (isomorphic) to the pull-back of the $D$ bundle by the inclusion $\iota : \mathbb{C}^* \to \mathbb{C} \cup \{\infty\}$, $\iota(z) = z$, between the corresponding base spaces:

$$\xi_{A - B} \cong \iota^*(\xi_D).$$

(10)

This fact immediately raises the question for the inverse implication, namely, if the existence of the $A - B$ effect implies, at least in the present mathematical sense, the existence of the Dirac monopole [10]. These monopoles, though yet not found in Nature, are predicted by grand unified [11] and string [12] theories. The purpose of the present note is to answer affirmatively the above question.

2. Pull-back of the $A - B$ potentials

If $N = (0, 0, 1)$ and $S = (0, 0, -1)$ are the north and south poles of $S^2$, then

$$\pi_H^{-1}(\{N, S\}) = \{(z_1, 0), \ |z_1| = 1\} \cup \{(0, z_2), \ |z_2| = 1\} \cong S^1 \times S^1 = T^2,$$

(11)

the 2-torus. If we "truncate" the $\xi_D$ bundle by defining the $U(1)$-bundle

$$\hat{\xi}_D : U(1) \hookrightarrow S^3 \setminus T^2 \xrightarrow{\pi_H^{-1}} S^2 \setminus \{N, S\} \cong \mathbb{C}^*,$$

(12)

the inclusion $\iota : \mathbb{C}^* \to S^2$ becomes the identity $\text{Id}_{\mathbb{C}^*}$, and we have the bundle map given by Diagram 1:

$$\begin{align*}
(C^* \times U(1)) \times U(1) & \xrightarrow{\iota \times \text{Id}_{U(1)}} (S^3 \setminus T^2) \times U(1) \\
\psi_0 \downarrow & \quad \downarrow \psi_D \\
C^* \times U(1) & \xrightarrow{\iota} S^3 \setminus T^2 \\
pr_1 \downarrow & \downarrow \pi_H \\
\mathbb{C}^* & \xrightarrow{\text{Id}_{\mathbb{C}^*}} \mathbb{C}^* \\
\text{Diagram 1}
\end{align*}$$

where $\psi_0$ and $\psi_D$ are the right actions of $U(1)$ on the corresponding total spaces,

$$\iota(z, \mu) = \frac{(z, 1)\mu}{||(z, 1)||},$$

(13)

and $|$ denotes the corresponding restrictions. It is clear that the "transitions" from $\omega_D$ and $A_{D\pm}$ to the restrictions $\omega_D|$ and $A_{D\pm}|$ respectively on $S^3 \setminus T^2$ and $\mathbb{C}^*$ are continuous, since they amount to the restriction of the domain of $\theta$ from $[0, \pi]$ to $(0, \pi)$.

Defining Hopf coordinates [13] $\{\eta, \xi_1, \xi_2\}$ on $S^3$:

$$(z_1, z_2) = (e^{i\xi_1} \sin \eta, e^{i\xi_2} \cos \eta), \ \eta \in [0, \pi/2], \ \xi_1, \xi_2 \in [0, 2\pi]$$

(14)

we obtain

$$\pi_H|_{\{(\eta, \xi_1, \xi_2)\}} = e^{i(\xi_1 - \xi_2)}\vartheta \eta,$$

with $\eta \in (0, \pi/2)$, which allows us to construct the pull-back $\beta \in \Omega^1(S^3 \setminus T^2; u(1))$ of $A_{A - B\pm} \in \Omega^1(\mathbb{C}^*; u(1))$ by $\pi_H|^{-1}$:

$$\beta|_{\eta} \begin{pmatrix}
\beta_{\eta} \\
\beta_{\xi_1} \\
\beta_{\xi_2}
\end{pmatrix} = \pm \begin{pmatrix}
\frac{\partial}{\partial \eta} (X_1 \circ \pi_H) \\
\frac{\partial}{\partial \xi_1} (X_1 \circ \pi_H) \\
\frac{\partial}{\partial \xi_2} (X_1 \circ \pi_H)
\end{pmatrix} \begin{pmatrix}
\frac{\partial}{\partial \eta} (X_2 \circ \pi_H) \\
\frac{\partial}{\partial \xi_1} (X_2 \circ \pi_H) \\
\frac{\partial}{\partial \xi_2} (X_2 \circ \pi_H)
\end{pmatrix} \begin{pmatrix}
A_{A - B1\pm} \\
A_{A - B2\pm}
\end{pmatrix}$$

(15)
\[
\begin{pmatrix}
\frac{\cos(\xi_1-\xi_2)}{\cos^2\eta} & \frac{\sin(\xi_1-\xi_2)}{\cos^2\eta} \\
-\sin(\xi_1-\xi_2)\tan\eta & \cos(\xi_1-\xi_2)\tan\eta
\end{pmatrix}
\begin{pmatrix}
A_{A-B1} \\
A_{A-B2}
\end{pmatrix}
= \begin{pmatrix} 0 & i/2 \\ i/2 & 0 \end{pmatrix}
\] (16)
i.e.
\[
\beta = \frac{i}{2}(d\xi_1 - d\xi_2).
\] (17)

From the relation between Hopf coordinates and Euler angles,
\[
(e^{i\xi_1} \sin \eta, e^{i\xi_2} \cos \eta) = (e^{i(\varphi+\chi)} \cos(\theta/2), e^{i(\varphi-\chi)} \sin(\theta/2))
\] (18)
one obtains
\[
\beta = \frac{i}{2}d\chi
\] (19)
i.e.
\[
\pi_H^*(A_{A-B}) = \omega_D(\theta = \pi/2).
\] (20)

3. Push-forward of the $A-B$ connection

The same relation between $A_{A-B}$ and $\omega_D(\theta = \pi/2)$ can be arrived at through the more direct path of pushing forward horizontal spaces of $A_{A-B}$ in $\xi_{A-B}$ into horizontal spaces of $\omega_D(\theta = \pi/2)$ in $\xi_D$. Since $Id_{C^*}$ is a diffeomorphism and $Id_{U(1)}$ is a group homomorphism (isomorphism), we are in the conditions of Proposition 6.1. in ref. [14]: given $A_{A-B}$ in $\xi_{A-B}$ there exist and is unique a connection $\omega$ in $\xi_D$ such that the horizontal subspaces of $A_{A-B}$ in $\mathbb{C}^* \times U(1)$ are mapped into the horizontal subspaces of $\omega$ in $\xi_D$ by $d\bar{\eta} \equiv \tilde{\eta}$. Here, we shall explicitly prove this fact and find that $\omega = \omega_D(\theta = \pi/2)$.

At any point $(z, e^{i\varphi})$ of $\mathbb{C}^* \times U(1)$, the horizontal space of $A_{A-B}$ is the kernel of (8). So, $(X_1 dX_2 - X_2 dX_1)(V_1 \frac{\partial}{\partial X_1} + V_2 \frac{\partial}{\partial X_2}) = X_1 V_2 - X_2 V_1 = 0$ implies
\[
V_2 = \frac{X_2}{X_1} V_1 \text{ for } X_1 \neq 0, \text{ and } V_1 = 0 \text{ for } X_1 = 0.
\] (21)

Since
\[
T_{(z, e^{i\varphi})}(\mathbb{C}^* \times U(1)) = T_{z, e^{i\varphi}}U(1) = \mathbb{C} \oplus \{te^{i(\varphi + \pi/2)}\}_{t \in \mathbb{R}},
\] (22)
the horizontal vectors at $(z, e^{i\varphi})$ are given by
\[
V \equiv (V_1, V_2, V_\varphi) = \{(V_1, \frac{X_2}{X_1} V_1, i e^{i\varphi}), \;X_1 \neq 0 \}
\] (23)
\[
(0, V_2, i e^{i\varphi}), \;X_1 = 0
\]

On the other hand, from the definition of $\bar{\eta}$ in eq. (13) and the definition of the Hopf coordinates on $S^3$, eq. (14), one obtains
\[
\bar{\eta}(z, e^{i\varphi}) = \bar{\eta}(X_1 + iX_2, e^{i\varphi}) \cong \bar{\eta}(X_1, X_2, e^{i\varphi}) = (\eta(X_1, X_2, \varphi), \xi_1(X_1, X_2, \varphi), \xi_2(X_1, X_2, \varphi))
\] (24)
\[
= (tg^{-1}(\sqrt{X_1^2 + X_2^2}), tg^{-1}\frac{X_2 \cos \varphi + X_1 \sin \varphi}{X_1 \cos \varphi - X_2 \sin \varphi}, \varphi),
\]
leading to $W = \bar{\eta}(V)$ with components
\[
\begin{pmatrix}
W_\eta \\
W_\xi_1 \\
W_\xi_2
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial \eta}{\partial \xi_1} & \frac{\partial \eta}{\partial \xi_2} & \frac{\partial \eta}{\partial \varphi} \\
\frac{\partial \xi_1}{\partial \xi_1} & \frac{\partial \xi_1}{\partial \xi_2} & \frac{\partial \xi_1}{\partial \varphi} \\
\frac{\partial \xi_2}{\partial \xi_1} & \frac{\partial \xi_2}{\partial \xi_2} & \frac{\partial \xi_2}{\partial \varphi}
\end{pmatrix}
\begin{pmatrix}
V_1 \\
V_2 \\
V_\varphi
\end{pmatrix}
= \begin{pmatrix}
\frac{X_1}{tg^2 \eta (1 + tg^2 \eta)} & \frac{X_2}{tg^2 \eta (1 + tg^2 \eta)} & 0 \\
\frac{-X_1}{tg \eta (1 + tg^2 \eta)} & \frac{X_2}{tg \eta (1 + tg^2 \eta)} & 1 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
V_1 \\
V_2 \\
V_\varphi
\end{pmatrix}
\] (25)
The relation between Hopf and Euler coordinates:

\[ tg \eta = cotg(\frac{\theta}{2}), \xi_1 = \frac{\varphi + \chi}{2}, \xi_2 = \frac{\varphi - \chi}{2}, \]  

allows to write

\[ \omega_D(\eta, \xi_1, \xi_2) = \frac{i}{1 + tg^2 \eta} \left( tg^2 \eta d\xi_1 - d\xi_2 \right). \]  

In particular,

\[ \omega_D(\pi/4, \xi_1, \xi_2) = \omega_D(\theta = \pi/2). \]  

The horizontal space at any point \((\eta, \xi_1, \xi_2) \in S^3 \setminus T^2\), being the kernel of \(\omega_D\), turns out to be

\[ H(\eta, \xi_1, \xi_2) = \{ (v_\eta, v_1, (tg^2 \eta)v_1), v_\eta, v_1 \in \mathbb{R}, \eta \in (0, \pi/2) \}. \]  

In particular,

\[ H(\pi/4, \xi_1, \xi_2) = \{ (v_{\pi/4}, v_1, v_1), v_{\pi/4}, v_1 \in \mathbb{R} \}. \]  

For \(X_1 \neq 0\), (25) leads to

\[ \begin{pmatrix} W_\eta \\ W_{\xi_1} \\ W_{\xi_2} \end{pmatrix} = \begin{pmatrix} X_1(1+tg^2 \eta) \\ ite^{i\phi} \\ ite^{i\phi} \end{pmatrix} \]  

while for \(X_1 = 0\), (25) leads to

\[ \begin{pmatrix} W_\eta \\ W_{\xi_1} \\ W_{\xi_2} \end{pmatrix} = \begin{pmatrix} X_2v_1 \\ (tg \eta)(1+tg^2 \eta) \\ ite^{i\phi} \end{pmatrix}, \]  

which belong to \(H(\pi/4, \xi_1, \xi_2)\). So, horizontal spaces of \(A_{A-B}\) are mapped into horizontal spaces of \(\omega_D|_{(\theta = \pi/2)}\).

4. Unique determination of \(\omega_D\)

By symmetry reasons, the unique \(\theta\)-dependent extensions \(\hat{\omega}\) of \(\omega\) are of the form \(sin \theta \, d\theta\), \(sin \theta \, d\varphi\), \(cos \theta \, d\varphi\), and \(cos \theta \, d\theta\). The first two lead to \(\hat{\omega}(\theta = \pi/2) = \frac{i}{2} (d\chi + d\theta)\) or \(\frac{i}{2} (d\chi + d\varphi)\) which are different from \(\omega_D|_{(\theta = \pi/2)}\), while the fourth one leads to \(\hat{\omega} = \frac{i}{2} (d\chi + d\sin \theta) = \frac{i}{2} d\chi'\), with \(\chi' = \chi + \sin \theta\), which is the same as \(\omega\). So, the unique \(\theta\)-dependent extension of \(\omega\) is the restriction to \(S^3 \setminus T^2\) of the Dirac connection:

\[ \hat{\omega}(\theta) = \omega_D|_{(\theta)}. \]  

Since \(\omega\) in \(\xi_D\) is uniquely determined by \(A_{A-B}\) in \(\xi_{A-B}\), its \(\theta\)-dependent extension \(\hat{\omega}\) is unique, and, as previously mentioned, the transition from \(\theta \in (0, \pi)\) to \(\theta \in [0, \pi]\) is continuous, then \(A_{A-B}\) uniquely determines \(\omega_D\). This ends the proof of the existence and uniqueness of the \(D\) connection from the existence of the \(A_{A-B}\) connection.

5. Final comment

The present note does not claim to prove the physical existence of the Dirac monopole, but only to reinforce this idea by showing that, at the mathematical level, in particular in the context of fiber bundle theory, the Aharonov-Bohm connection, relevant to the physically observed \(A-B\) effect, implies the existence and uniqueness of the connection which represents the till now hypothetical Dirac charge.

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