KNOT COBDIRISMS, TORSION, AND FLOER HOMOLOGY

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Abstract. Given a cobordism between two knots in the 3-sphere, our main result is an inequality involving torsion orders of the knot Floer homology of the knots, and the number of critical points of index zero, one, and two of the cobordism. This has a number of topological applications. In particular, the torsion order gives lower bounds on the bridge index and the band-unlinking number of a knot, and on the fusion number of a ribbon knot. It also gives a lower bound on the number of bands appearing in a ribbon concordance between two knots. We show the torsion order gives a lower bound on the number of minima appearing in a slice disk of a knot. Our bounds on the bridge index and fusion number are sharp for \( T_{p,q} \) and \( T_{p,q} \# T_{p,q} \), respectively.

Furthermore, we show the torsion order bounds a refinement of the cobordism distance on knots, which is a metric. As a special case, we can bound the number of band moves required to get from one knot to the other. We show knot Floer homology gives a lower bound on Sarkar’s ribbon distance, and give examples of ribbon knots with arbitrarily large ribbon distance from the unknot.

1. Introduction

The slice-ribbon conjecture is one of the key open problems in knot theory. It states that every slice knot is ribbon; i.e., admits a slice disk on which the radial function of the 4-ball induces no local maxima. It is clear from this conjecture that being able to bound the possible number of critical points of various indices on surfaces bounding knots is a hard and important question. In this paper, we use the torsion order of knot Floer homology to give bounds on the number of critical points appearing in knot cobordisms connecting two knots. As applications, we consider knot invariants that can be interpreted in terms of knot cobordisms, such as the band unlinking number or knots, and the fusion number of ribbon knots. See Section 3 for background on knot Floer homology and the link Floer TQFT, which we use in the proofs of our main results.

If \( K \) is a knot in \( S^3 \), we write \( \text{HFK}^- (K) \) for the minus version of knot Floer homology, which is a finitely generated module over the polynomial ring \( \mathbb{F}_2[v] \). The module \( \text{HFK}^-(K) \) decomposes non-canonically as

\[
\text{HFK}^-(K) \cong \mathbb{F}_2[v] \oplus \text{HFK}^-_{\text{red}}(K),
\]

where \( \text{HFK}^-_{\text{red}}(K) \) denotes the \( \mathbb{F}_2[v] \)-torsion submodule of \( \text{HFK}^-(K) \).

More generally, if \( M \) is an \( \mathbb{F}_2[v] \)-module, we define

\[
\text{Ord}_v(M) := \min \left\{ k \in \mathbb{N} : v^k \cdot \text{Tor}(M) = 0 \right\} \in \mathbb{N} \cup \{ \infty \}.
\]

Definition 1.1. If \( K \) is a knot in \( S^3 \), we define its torsion order as

\[
\text{Ord}_v(K) := \text{Ord}_v(\text{HFK}^-(K)).
\]

The module \( \text{HFK}^-_{\text{red}}(K) \) is annihilated by the action of \( v^k \) for sufficiently large \( k \), so \( \text{Ord}_v(K) \) is always finite. Our main result is the following:

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Theorem 1.2. Let $K_0$ and $K_1$ be knots in $S^3$. Suppose there is a connected knot cobordism from $K_0$ to $K_1$ with $m$ local minima, $b$ saddles, and $M$ local maxima. Then

$$\text{Ord}_v(K_0) \leq b - m + \max\{0, \text{Ord}_v(K_1) - M\}.$$ 

We now describe various topological applications of Theorem 1.2.

1.1. Ribbon concordances. A knot concordance with no maxima is called a ribbon concordance. The notion of ribbon concordance was introduced by Gordon [Gor81]. Suppose there is a ribbon concordance from $K_0$ to $K_1$ with $m$ local minima and $b$ saddles. One implication of Theorem 1.2 is that $\text{Ord}_v(K_0) \leq \text{Ord}_v(K_1)$, though we note that this also follows from previous work of the third author [Zem19a, Theorem 1.7]. If we reverse the roles of $K_0$ and $K_1$ in Theorem 1.2, we get that

$$\text{Ord}_v(K_1) \leq b + \max\{0, \text{Ord}_v(K_0) - b\}.$$ 

Hence, we obtain the following:

Corollary 1.3. Suppose that there is a ribbon concordance from $K_0$ to $K_1$ with $b$ saddles. Then either $b \leq \text{Ord}_v(K_0) = \text{Ord}_v(K_1)$, or $\text{Ord}_v(K_0) \leq \text{Ord}_v(K_1) \leq b$.

Hence, given knots $K_0$ and $K_1$ such that $\text{Ord}_v(K_0) \neq \text{Ord}_v(K_1)$, any ribbon concordance from $K_0$ to $K_1$ must have at least $\text{Ord}(K_1)$ saddles.

We can also apply Theorem 1.2 in the case that there is a ribbon cobordism $C$ of arbitrary genus from $K_0$ to $K_1$ with $m$ local minima and $b$ saddles. In that case, we obtain

$$\text{Ord}_v(K_0) \leq b - m + \max\{0, \text{Ord}_v(K_1)\} = b - m + \text{Ord}(K_1).$$

Since $\chi(C) = m - b$, we have

$$\text{Ord}_v(K_0) - \text{Ord}_v(K_1) \leq -\chi(C) = 2g(C).$$

Thus, we obtain the following corollary:

Corollary 1.4. Suppose there is a ribbon cobordism from $K_0$ to $K_1$ of genus-$g$. Then $\text{Ord}_v(K_0) - \text{Ord}_v(K_1) \leq 2g$.

1.2. The complexity of slice disks. Suppose $K$ is a slice knot with slice disk $D$. Viewing $D$ as a cobordism from the empty knot to $K$, suppose that $D$ has $m$ minima, $b$ bands, and $M$ maxima. By removing one of the minima, we obtain a concordance $C$ from the unknot to $K$, with $m - 1$ minima, $b$ bands, and $M$ maxima. We turn $C$ around to obtain a concordance from $K$ to the unknot with $M$ minima, $b$ bands and $m - 1$ maxima. Applying Theorem 1.2, we obtain that

$$\text{Ord}(K) \leq b - M + \max\{0, - (m - 1)\} = b - M = (b - M - m) + m = \chi(D) + m = m - 1.$$ 

Consequently, we obtain the following corollary:

Corollary 1.5. Suppose that $D$ is a slice disk for $K$, viewed as a cobordism from the empty knot to $K$, and let $m$ denote the number of minima of the slice disk $D$. Then

$$\text{Ord}_v(K) \leq m - 1.$$ 

1.3. The refined cobordism distance. If $K_0$ and $K_1$ are knots in $S^3$, we define the refined cobordism distance $d(K_0, K_1)$ as the minimum of the quantity $\max\{b - m, b - M\}$ over all connected, oriented knot cobordisms from $K_0$ to $K_1$, where $m$ is the number of local minima, $b$ is the number of saddles, and $M$ is the number of local maxima of the height function on the cobordism. The function $d$ is a metric on the set of knots in $S^3$, modulo isotopy; see Proposition 2.2. Furthermore, $d$ is a refinement of the standard cobordism distance on knots (i.e., the slice genus of $K_0 \# \overline{K_1}$). See Section 2 for more details. As a corollary of Theorem 1.2, we obtain the following:

Corollary 1.6. If $K_0$ and $K_1$ are knots in $S^3$, then

$$|\text{Ord}_v(K_0) - \text{Ord}_v(K_1)| \leq d(K_0, K_1).$$
Proof. From Theorem 1.2 we obtain that
\[ \text{Ord}_v(K_0) \leq b - m + \text{Ord}_v(K_1). \]
Consequently,
\[ \text{Ord}_v(K_0) - \text{Ord}_v(K_1) \leq b - m \leq \max\{b - m, b - M\}. \]
Reversing the roles of $K_0$ and $K_1$ yields the statement. \[\square\]

As $\max\{b - m, b - M\} \leq b$, the distance $d(K_0, K_1)$ is at most the number of bands appearing in any connected, oriented cobordism from $K_0$ to $K_1$. In particular, Corollary 1.6 gives a lower bound on the number of oriented band moves required to get from $K_0$ to $K_1$.

1.4. The band unlinking number. If $K$ is a knot, the unlinking number $u(K)$ is the minimum number of crossing changes one must perform until one obtains the unknot. The band-unlinking number $u_b(K)$ is the minimum number of (oriented) bands one must attach until one obtains an unknot. Since any crossing change can be obtained by attaching two bands, $u_b(K)$ is bounded above by $2u(K)$. That is,
\[ u_b(K) \leq 2u(K). \]

The band unlinking number, as well as an infinite family of variations, was described by Hoste, Nakanishi, and Taniyama [HNT90] (though the concept is classical; see e.g. Lickorish [Lic86]). In their terminology, attaching an oriented band is an $\text{SH}(2)$-move. They also studied the unoriented band unlinking number, which is often called the $H(2)$-unlinking number.

In our present work, we are interested in a variation, which we call the band-unlinking number, $ul_b(K)$, which is the minimum number of oriented band moves necessary to reduce $K$ to an unlink. Note that
\[ ul_b(K) \leq u_b(K). \]

More generally, the band unlinking and unlinking numbers are related to other topological invariants as follows:

\begin{equation}
2g_4(K) \leq 2g_r(K) \leq u_b(K) \leq u_b(K) \leq 2g_3(K).
\end{equation}

In Equation (1), $g_4$ is the slice genus, $g_r$ is the ribbon slice genus (the minimal genus of a knot cobordism from $K$ to the unknot with only saddles and local maxima), and $g_3$ is the Seifert genus. The inequality involving the Seifert genus is obtained by attaching bands corresponding to a basis of arcs for a minimal genus Seifert surface.

As a corollary to Theorem 1.2, we have the following:

**Corollary 1.7.** If $K$ is a knot in $S^3$, then
\[ \text{Ord}_v(K) \leq ul_b(K). \]

**Proof.** Let where $b = ul_b(K)$, and suppose that, after attaching the oriented bands $B_1, \ldots, B_b$, we obtain an $M$-component unlink. By capping $M - 1$ components of the unlink, we obtain a cobordism from $K$ to the unknot with $0$ local minima, $b$ saddles, and $M - 1$ local maxima. Theorem 1.2 implies
\[ \text{Ord}_v(K) \leq b, \]
completing the proof. \[\square\]

1.5. Ribbon knots and the fusion number. A knot $K$ is $S^3$ is smoothly slice if it bounds a smoothly embedded disk in $B^4$. A knot $K$ is ribbon if it bounds a smooth disk which has only index 0 and 1 critical points with respect to a radial height function on $B^4$. Equivalently, a knot $K$ is ribbon if it can be formed by attaching $n - 1$ bands to an $n$-component unlink. The fusion number $\text{Fus}(K)$ of a ribbon knot $K$ is the minimal number of bands required in any ribbon disk for $K$; see e.g. Miyazaki [Miy86]. Concerning the fusion number, we have the following consequence of Corollary 1.7:

**Corollary 1.8.** If $K$ is a ribbon knot in $S^3$, then
\[ \text{Ord}_v(K) \leq \text{Fus}(K). \]
Proof. If \( B_1, \ldots, B_b \) are the bands of a ribbon disk, then \( B_1, \ldots, B_b \) split \( K \) into an unlink. Consequently, \( \mu_b(K) \leq b \), so the statement follows from Corollary 1.7. \qed

1.6. The bridge index. If \( K \) is a knot in \( S^3 \), the bridge index of \( K \), denoted \( \text{br}(K) \), is the minimum over all diagrams \( D \) of \( K \) of the number of local maxima of \( D \) with respect to a height function on the plane. It is well known that there is a ribbon disk for \( K \) which has \( \text{br}(K) - 1 \) bands; see Figure 1.1. Consequently

\[
(2) \quad \mathcal{F}_{\text{us}}(K \# K) \leq \text{br}(K) - 1.
\]

Ozsváth and Szabó’s connected sum formula \([OS04, \text{Theorem 7.1}]\) implies

\[
(3) \quad \text{Ord}_v(K_1 \# K_2) = \max\{\text{Ord}_v(K_1), \text{Ord}_v(K_2)\}.
\]

Consequently, we obtain the following additional consequence of Corollary 1.7:

**Corollary 1.9.** If \( K \) is a knot in \( S^3 \), then

\[
\text{Ord}_v(K) \leq \text{br}(K) - 1.
\]

![Figure 1.1. Left: the standard ribbon disk for \( K \# K \) (in this illustration, \( K \) is a trefoil knot), immersed in \( S^3 \). Right: The corresponding \( \text{br}(K) - 1 \) bands attached to \( K \# K \) to obtain a \( \text{br}(K) \) component unlink.](image)

1.7. Sharpness and torus knots. As examples, we consider the positive torus knots \( T_{p,q} \). It is a classical fact due to Schubert [Sch54] that

\[
(4) \quad \text{br}(T_{p,q}) = \min\{p, q\}.
\]

Combining Equations (2) and (4), we obtain

\[
(5) \quad \mathcal{F}_{\text{us}}(T_{p,q} \# T_{p,q}) \leq \min\{p, q\} - 1.
\]

In Corollary 5.3, we show

\[
(6) \quad \text{Ord}_v(T_{p,q}) = \min\{p, q\} - 1.
\]

Equations (4) and (6) imply Corollaries 1.8 and 1.9 are sharp:

\[
\text{Ord}_v(T_{p,q}) = \text{br}(T_{p,q}) - 1 \quad \text{and} \quad \text{Ord}_v(T_{p,q} \# T_{p,q}) = \mathcal{F}_{\text{us}}(T_{p,q} \# T_{p,q}).
\]

1.8. Sarkar’s ribbon distance. We first introduce the torsion distance of two knots.

**Definition 1.10.** Let \( K \) and \( K' \) be knots in \( S^3 \). Then we define their torsion distance \( d_t(K, K') \) as

\[
\min\{d \in \mathbb{N} : v^d \mathcal{HFK}^-(K) \cong v^d \mathcal{HFK}^-(K')\}.
\]

Sarkar [Sar19] introduced the ribbon distance \( d_r(K, K') \) between knots \( K \) and \( K' \); see Section 6 for a precise definition. This is finite if and only if \( K \) and \( K' \) are concordant. He proved that Lee’s perturbation of Khovanov homology \([Lee05]\) gives a lower bound on the ribbon distance. We prove the following knot Floer homology analogue of Sarkar’s result:

**Theorem 1.11.** Suppose \( K \) and \( K' \) are knots in \( S^3 \). Then

\[
d_t(K, K') \leq d_r(K, K').
\]
Note that \( d_r(K,U) = \text{Ord}_v(K) \), where \( U \) denotes the unknot. Hence \( \text{Ord}_v(K) \leq d_r(K,U) \), and equations (3) and (6) imply that
\[
\min\{p,q\} - 1 = \text{Ord}_v(T_{p,q} \# T_{p,q}) \leq d_r(T_{p,q} \# T_{p,q},U).
\]
On the other hand, when \( K \) is ribbon, \( d_r(K,U) \leq \mathcal{Fus}(K) \). By equation (5), we obtain that
\[
d_r(T_{p,q} \# T_{p,q},U) = \min\{p,q\} - 1.
\]
As a consequence, \( d_r(K,U) \) can be arbitrarily large for ribbon knots \( K \), a fact that Sarkar was unable to establish using Khovanov homology; see [Sar19, Example 3.1].

**Remark 1.12.** It is easy to extend this computation to show that there are prime slice knots with determinant 1 that have arbitrarily large ribbon distance from the unknot. Kim [Kim10] showed that every knot \( K \) admits an invertible concordance \( C \) to a prime knot \( K' \) with the same Alexander polynomial, obtained by taking a certain satellite of \( K \). Let \( P \) and \( P' \) be decorations on \( K \) and \( K' \), respectively, choose a decoration \( \sigma \) on \( C \) compatible with these, and let \( C = (C,\sigma) \). By [JM16], the concordance map \( F_C \) is injective, and hence \( \text{Ord}_v(K) \leq \text{Ord}_v(K') \). When \( K = T_{p,q} \# T_{p,q} \) with \( p \) and \( q \) odd, then \( \det(K) = 1 \), and hence \( \det(K') = 1 \) as well.

1.9. **Previous bounds.** Bounding the fusion number is challenging, though there are some bounds already in the literature. A classical lower bound is provided by \( \text{rk}(H_1(\Sigma(K)))/2 \), where \( \Sigma(K) \) is the branched double cover of \( S^3 \) along \( K \), and \( \text{rk} \) denotes the smallest cardinality of a generating set; see Nakanishi and Nakagawa [NN82, Proposition 2] and Sarkar [Sar19, Section 3]. Following [Sar19, Example 3.1], if \( K \) is a ribbon knot with \( \det(K) \neq 1 \) (e.g., \( K = T_{2,3} \# T_{2,3} \)), and \( K_n \) is the connected sum of \( n \) copies of \( K \), then \( \mathcal{Fus}(K_n) \geq n \). This classical method fails when \( \det(K) = 1 \); e.g., for \( K = T_{p,q} \# T_{p,q} \) with \( p \) and \( q \) odd. Our methods allow us to show that \( \mathcal{Fus}(K) \) can be arbitrarily large even when \( \det(K) = 1 \); e.g., for \( K = T_{p,q} \# T_{p,q} \) when \( p \) and \( q \) are odd.

Kanenobu [Kan10, Theorem 4.3] proved a bound which involves the dimensions of \( H_1(\Sigma(K);\mathbb{Z}) \) and \( H_1(\Sigma(K);\mathbb{Z}_2) \). Mizuma [Miz06, Theorem 1.5] showed that if \( K \) is a ribbon knot which has Alexander polynomial 1 and whose Jones polynomial has non-vanishing derivative at \( t = -1 \), then \( K \) has fusion number at least 3. More recently, Aceto, Golla, and Lecuona [AGL18, Corollary 2.3] have given obstructions using the Casson–Gordon signature invariants of \( \Sigma(K) \). Note that these bounds do not give useful information for the ribbon knots \( K = T_{p,q} \# T_{p,q} \) for odd \( p \) and \( q \) since they involve \( H_1(\Sigma(K)) \), and \( \Sigma(T_{p,q} \# T_{p,q}) \) is the connected sum of the Brieskorn spheres \( \Sigma(2,p,q) \) and \( -\Sigma(2,p,q) \).

Alishahi [Ali17] and Alishahi–Eftekhary [AE18] have obtained bounds for the unknotting number using the torsion order of knot Floer homology and Lee’s perturbation of Khovanov homology, which are similar in flavor to our present work.

The work of Sarkar [Sar19] is the most similar to ours. Sarkar used the torsion order of the action on Lee’s perturbation of Khovanov homology to give a lower bound on the fusion number and the ribbon distance. We note that the torsion order of Khovanov homology is usually very small. Khovanov thin knots have torsion order at most 1. Prior to the work of Manolescu and Marengon [MM18], the largest known torsion order was 2. Their work exhibits a knot with torsion order at least 3. In contrast, the \( (p,q) \)-torus knot has knot Floer homology with torsion order \( \min\{p,q\} - 1 \); see Section 5.

One advantage of using \( \text{Ord}_v(K) \) to bound \( u_b(K) \) is computability. In particular, a program of Ozsváth and Szabó [OS] can quickly compute \( \text{HF}^{-}(K) \) and \( \text{Ord}_v(K) \).

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2. **A Refinement of the Cobordism Distance**

Suppose that \( K_0 \) and \( K_1 \) are knots in \( S^3 \). The standard cobordism distance between \( K_0 \) and \( K_1 \) is defined as the minimal genus of an oriented knot cobordism connecting \( K_0 \) and \( K_1 \); see Baader [Baa12]. We will write \( d_{\text{cob}}(K_0,K_1) \) for the standard cobordism distance. Equivalently, \( d_{\text{cob}} \)
can be defined in terms of the slice genus of $K_0 \# \overline{K}_1$. The distance $d_{\text{cob}}(K_0, K_1) = 0$ if and only if $K_0$ and $K_1$ are concordant, and hence descends to a metric on the knot concordance group. In this section, we describe a refinement of the standard cobordism distance, which is an actual metric on the set of knots in $S^3$, modulo isotopy.

**Definition 2.1.** If $S$ is a connected, oriented knot cobordism in $[0, 1] \times S^3$ from $K_0$ to $K_1$ with $m$ local minima, $b$ saddles, and $M$ local maxima, then we define the quantity $|S| \in \mathbb{Z}^{\geq 0}$ by the formula

$$ |S| := \max\{b - m, b - M\}. $$

We define the refined cobordism distance from $K_0$ to $K_1$ as

$$ d(K_0, K_1) := \min \{|S| : S \subseteq [0, 1] \times S^3 \text{ is a connected, oriented cobordism from } K_0 \text{ to } K_1 \}. $$

Note that

$$ 2d_{\text{cob}}(K_0, K_1) \leq d(K_0, K_1). $$

Indeed, if $S$ is a connected, oriented cobordism from $K_0$ to $K_1$ with $m$ local minima, $b$ saddles, and $M$ local maxima, then

$$ 2g(S) = b - m - M \leq |S|. $$

We now show that our refined cobordism distance is indeed a metric:

**Proposition 2.2.** The refined cobordism distance $d(K_0, K_1)$ defines a metric on the set of knots in $S^3$, modulo isotopy.

**Proof.** The distance from a knot to itself is 0, and symmetry is clear. Positive-definiteness follows from the fact that $b \geq m + M$ for any connected, oriented knot cobordism. Finally, the triangle inequality follows from the arithmetic inequality

$$ \max\{A + A', B + B'\} \leq \max\{A, B\} + \max\{A', B'\}. $$

There is another metric which commonly appears in the literature, the Gordian metric $d_G$, introduced by Murakami [Mur85]. The quantity $d_G(K_0, K_1)$ is the minimal number of crossing changes required to change $K_0$ into $K_1$. Since a crossing change may be realized with two band surgeries, we have

$$ d(K_0, K_1) \leq 2d_G(K_0, K_1). $$

### 3. Background on Knot and Link Floer Homologies

#### 3.1. The link Floer homology groups

Knot Floer homology is an invariant of knots in 3-manifolds defined by Ozsváth and Szabó [OS04], and independently Rasmussen [Ras03]. The construction was extended to links by Ozsváth and Szabó [OS08].

A *multi-based link* $\mathbb{L} = (L, w, z)$ consists of an oriented link $L$, equipped with two disjoint collections of basepoints, $w$ and $z$, satisfying the following:

1. $w$ and $z$ alternate as one traverses $L$.
2. Each component of $L$ has at least 2 basepoints.

To a multi-based link $\mathbb{L}$ in $S^3$, Ozsváth and Szabó associate several versions of the link Floer homology groups. The hat version is a bigraded $\mathbb{F}_2$-vector space $\widehat{HFL}(\mathbb{L})$. We will be mostly focused on the minus version, denoted $HFL^-(\mathbb{L})$,

which is a module over the polynomial ring $\mathbb{F}_2[v]$.

The link Floer groups are constructed by picking a Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ for $\mathbb{L}$. Write $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ for the attaching curves, and consider the two half-dimensional tori

$$ T_\alpha := \alpha_1 \times \cdots \times \alpha_n \quad \text{and} \quad T_\beta := \beta_1 \times \cdots \times \beta_n $$

in $\text{Sym}^n(\Sigma)$. The module $\overline{CFL}(\mathbb{L})$ is defined to be the free $\mathbb{F}_2$-module generated by the intersection points of $T_\alpha \cap T_\beta$. The module $\text{CFL}^-(\mathbb{L})$ is the free $\mathbb{F}_2[v]$-module generated by $T_\alpha \cap T_\beta$. The
differential \( \partial \) on \( \overline{CFL}(L) \) counts rigid pseudo-holomorphic disks in \( Sym^n(\Sigma) \) with zero multiplicity on \( w \cup z \). The differential on \( CFL^-(L) \) is given by

\[
\partial^- x = \sum_{y \in T_0} \sum_{\phi \in T_2(\Sigma)} \sum_{\mu(\phi) = 1} \#(M(\phi)/R) u^{n_{\phi}(\phi)} \cdot y,
\]

extended equivariantly over \( \mathbb{F}_2[v] \). The modules \( \overline{HFL}(L) \) and \( HFL^-(L) \) are the homologies of \( \overline{CFL}(L) \) and \( CFL^-(L) \), respectively.

The module \( HFL^-(L) \) decomposes (non-canonically) as

\[
HFL^-(L) \cong \bigoplus_{i=1}^{k} \mathbb{F}_2[v] \oplus HFL^\text{red}_-(L),
\]

where \( k = |w| = |z| \) and \( HFL^\text{red}_-(L) \) denotes the \( \mathbb{F}_2[v] \)-torsion submodule of \( HFL^-(L) \). Since \( HFL^-(L) \) admits a relative \( \mathbb{Z} \)-grading where \( v \) has grading +1 (the Alexander grading), the module \( HFL^\text{red}_-(L) \) is always isomorphic to a direct sum of modules of the form \( \mathbb{F}_2[v]/(v^i) \) for \( i \geq 0 \). In particular, \( v^l \) annihilates \( HFL^\text{red}_-(L) \) for all sufficiently large \( l \in \mathbb{N} \), hence \( \text{Ord}_v(K) \) is always finite.

If \( K = (K, w, z) \) is a doubly-based knot, then, by definition, the link Floer homology groups coincide with the knot Floer homology groups; i.e., \( HF^-(K) \cong HFL^-(K) \). Following standard notation, we will usually write \( HF^-(K) \) instead of \( HFL^-(K) \).

Ozsváth and Szabó’s connected sum formula [OS04, Theorem 7.1] implies that

\[
CFK^-(K_1 \# K_2) \cong CFK^-(K_1) \otimes_{\mathbb{F}_2[v]} CFK^-(K_2).
\]

Consequently, by the Künneth theorem for chain complexes over \( \mathbb{F}_2[v] \), we have

\[
\text{Ord}_v(K_1 \# K_2) = \max \{ \text{Ord}_v(K_1), \text{Ord}_v(K_2) \}.
\]

Ozsváth and Szabó also proved that the mirror of a knot has dual knot Floer homology:

\[
CFK^-(\overline{K}) \cong \text{Hom}_{\mathbb{F}_2[v]}(CFK^-(K), \mathbb{F}_2[v]).
\]

(The proof is the same as for the closed 3-manifold invariants; see Ozsváth and Szabó [OS06, Section 5.1].) Consequently,

\[
\text{Ord}_v(\overline{K}) = \text{Ord}_v(K).
\]

Combining equations (7) and (8), we obtain that

\[
\text{Ord}_v(K \# \overline{K}) = \text{Ord}_v(K),
\]

a result that we will use repeatedly.

### 3.2. The link Floer TQFT

We will be interested in the functorial aspects of link Floer homology. A **decorated link cobordism** between two multi-based links \( L_0 = (L_0, w_0, z_0) \) and \( L_1 = (L_1, w_1, z_1) \) is a pair \( \mathcal{F} = (S, \mathcal{A}) \), as follows:

1. \( S \) is a smooth, properly embedded, oriented surface in \( [0, 1] \times S^3 \) such that
   \[
   \partial S = (-\{0\} \times L_0) \cup (\{1\} \times L_1).
   \]
2. \( \mathcal{A} \subseteq S \) is a finite collection of properly embedded arcs, such that \( S \setminus \mathcal{A} \) consists of two disjoint subsurfaces, \( S_w \) and \( S_z \). Further, \( w \subseteq S_w \) and \( z \subseteq S_z \).

Figure 3.1 shows some examples of decorated link cobordisms.

To a decorated link cobordism \( \mathcal{F} \) from \( L_0 \) to \( L_1 \), there are cobordism maps

\[
\tilde{F}_\mathcal{F}: \overline{HFL}(L_0) \to \overline{HFL}(L_1) \quad \text{and} \quad F_\mathcal{F}: HFL^-(L_0) \to HFL^-(L_1).
\]

The construction of the map \( \tilde{F}_\mathcal{F} \) is due to the first author [Juh16], using the contact gluing map of Honda, Kazez, and Matić [HKM08]. The third author [Zem19b] subsequently gave an alternate construction which also works on the minus version. Their equivalence on the hat version was proven by the first and third authors [JZ18a, Theorem 1.4].
8 ANDRÁS JUHÁSZ, MAGGIE MILLER, AND IAN ZEMKE

The link cobordism maps satisfy a simple relation with respect to adding tubes:

**Lemma 3.1.** Suppose that $\mathcal{F} = (S, A)$ is a decorated link cobordism from $L_0$ to $L_1$. Suppose that $\mathcal{F}'$ is a decorated link cobordism obtained by adding a tube to $\mathcal{F}$, with both feet in the $S_z$ subregion of $S$; see Figure 3.1. Then

$$F_{\mathcal{F}'} = v \cdot F_{\mathcal{F}}.$$ 

A proof of Lemma 3.1 can be found in [JZ18b, Lemma 5.3].

![Figure 3.1. Stabilizing a surface in the $z$-subregion.](image)

4. Knot Floer homology and the cobordism distance

We begin with the main technical content of our theorem:

**Proposition 4.1.** Suppose that $S$ is a connected, oriented knot cobordism from $K_0$ to $K_1$ with $m$ local minima, $b$ saddles, and $M$ local maxima, and suppose that $\mathcal{F} = (S, A)$ is a decoration of $S$ such that the type-$w$ region is a regular neighborhood of an arc running from $K_0$ to $K_1$. Let $\mathcal{F}$ denote the cobordism from $K_1$ to $K_0$ obtained by vertically mirroring $\mathcal{F}$. Then

$$v^M \cdot F_{\mathcal{F}} \circ F_{\mathcal{F}} = v^{b-m} \cdot \text{id}_{HFK^{-}}(K_0).$$

**Proof.** We can rearrange the critical points of $S$ so that $S$ has a movie of the following form:

1. **(M-1)** $m$ births, which add $m$ unknots $U_1, \ldots, U_m$.
2. **(M-2)** $m$ fusion saddles $B_1, \ldots, B_m$, which merge $U_1, \ldots, U_m$ with $K_0$.
3. **(M-3)** $b - m$ additional saddles, along bands $B_{m+1}, \ldots, B_b$.
4. **(M-4)** $M$ deaths, corresponding to deleting unknots $U'_1, \ldots, U'_M$.

We can give a movie with 8 steps for $\mathcal{F} \circ \mathcal{F}$ by first playing (M-1)–(M-4) forward, and then playing them backward, in reverse order. The fourth step of this 8-step movie is to delete the unknots $U'_1, \ldots, U'_M$ via $M$ deaths. The fifth step is to add them back with $M$ births. Consider the cobordism $\mathcal{G}$ obtained by deleting these two levels. The cobordism $\mathcal{G}$ is obtained by attaching $M$ tubes to $HFK^{-}(K_0)$. Since the decoration of $\mathcal{G}$ is such that the type-$w$ region is a neighborhood of an appropriate arc from the incoming $K_0$ to the outgoing $K_0$, the cobordism $\mathcal{G}$ is obtained by attaching $M$ tubes to the $z$-subregion of $\mathcal{F} \circ \mathcal{F}$. Consequently, Lemma 3.1 implies that

$$F_{\mathcal{G}} = v^M \cdot F_{\mathcal{F} \circ \mathcal{F}}.$$ 

The cobordism $\mathcal{G}$ has the movie obtained by playing (M-1), (M-2), and (M-3) forward, and then playing them backward, in reverse order. The third and fourth steps of this movie describe $b - m$ tubes, added to a cobordism $\mathcal{G}'$, which is obtained by first playing (M-1) and (M-2), and then playing them backwards, in reversed order. By Lemma 3.1, we obtain

$$F_{\mathcal{G}} = v^{b-m} \cdot F_{\mathcal{G}'}.$$
Finally, \( G' \) is obtained by playing \((M-1)\) and \((M-2)\), and then playing them backwards, in reverse order. The births and deaths determine 2-spheres \( S_1, \ldots, S_m \), and the bands and their reverses determine tubes. Hence \( G' \) is the cobordism obtained by tubing in the spheres \( S_1, \ldots, S_m \) to the identity concordance \([0,1] \times K_0\). The proof of [Zen19a, Theorem 1.7] implies immediately that tubing in spheres in this manner does not affect the cobordism map, so
\[
F_{G'} = \text{id}_{HFK^-(K_0)}.
\]
Combining Equations (10), (11), and (12) yields the statement.

Our main theorem is now an algebraic consequence of Proposition 4.1:

**Theorem 1.2.** Suppose there is an oriented knot cobordism from \( K_0 \) to \( K_1 \) with \( m \) local minima, \( b \) saddles, and \( M \) local maxima. Then
\[
\text{Ord}_v(K_0) \leq b - m + \max\{0, \text{Ord}_v(K_1) - M\}.
\]

**Proof.** Let \( F \) denote the cobordism obtained by decorating \( S \) such that the \( \mathbf{w} \)-subregion is a regular neighborhood of an arc running from \( K_0 \) to \( K_1 \). Let \( \mathcal{F} \) denote the cobordism from \( K_1 \) to \( K_0 \) obtained by vertically mirroring \( K_0 \). Proposition 4.1 implies that
\[
v^M \cdot \mathcal{F}_{\mathcal{F} \circ F} = v^{b-m} \cdot \text{id}_{HFK^-(K_0)}.
\]
Since
\[
\mathcal{F}_{\mathcal{F} \circ F} = \mathcal{F} \circ F
\]
by the composition law, it follows that, if \( x \in \text{HFK}_{\text{red}}^-(K_0) \), then \( \mathcal{F}_{\mathcal{F} \circ F}(v^j \cdot x) = 0 \) if \( j \geq \text{Ord}_v(K_1) \). On the other hand, equation (13) implies that
\[
\mathcal{F}_{\mathcal{F} \circ F}(v^{l+M} \cdot x) = v^{b-m+l} \cdot x
\]
for all \( l \geq 0 \). Consequently, if \( l \geq \max\{0, \text{Ord}_v(K_1) - M\} \), then \( v^{b-m+l} \cdot x = 0 \). It follows that
\[
\text{Ord}_v(K_0) \leq b - m + \max\{0, \text{Ord}(K_1) - M\},
\]
as claimed.

5. **Knot Floer homology and torus knots**

An \( L \)-space is a rational homology 3-sphere \( Y \) such that \( \tilde{H}F(Y, s) \cong \mathbb{F}_2 \) for each \( s \in \text{Spin}^c(Y) \) (this is the smallest possible rank for a rational homology sphere). Lens spaces are examples of \( L \)-spaces. An \( L \)-space knot is a knot \( K \) in \( S^3 \) such that \( S^3_p(K) \) is an \( L \)-space for some \( p \in \mathbb{Z} \). If \( p, q > 0 \) are coprime, the torus knot \( T_{p,q} \) is an \( L \)-space knot since \( p \mathbf{q} + 1 \) surgery on \( T_{p,q} \) is the lens space \( L(pq \pm 1, q^2) \).

Ozsváth and Szabó [OS05, Theorem 1.2] proved that the knot Floer homology of an \( L \)-space knot is completely determined by its Alexander polynomial. Furthermore, they showed [OS05, Corollary 1.3] that the Alexander polynomial of an \( L \)-space knot can be written as
\[
\Delta_K(t) = \sum_{k=0}^{2n} (-1)^k t^{\alpha_k}
\]
for a decreasing sequence of integers \( \alpha_0, \ldots, \alpha_{2n} \). Their computation implies the following:

**Lemma 5.1.** If \( K \) is an \( L \)-space knot, and \( \alpha_0, \ldots, \alpha_{2n} \) are the non-zero degrees appearing in the Alexander polynomial of \( K \), written in decreasing order, then
\[
\text{Ord}_v(K) = \max\{\alpha_{i-1} - \alpha_i : 1 \leq i \leq 2n\}.
\]

**Proof.** We first describe Ozsváth and Szabó’s computation of \( CFK^\infty(K) \). Note that Ozsváth and Szabó only stated their computation for \( \tilde{H}FK(K) \), though their proof works for \( CFK^\infty(K) \); see [OSS17, Theorem 2.10]. Let \( d_1, \ldots, d_{2n} \) denote the gaps between the integers \( \alpha_0, \ldots, \alpha_{2n} \); i.e.,
\[
d_i := \alpha_{i-1} - \alpha_i.
\]
Ozsváth and Szabó proved that $\text{CFK}^\infty(K)$ is chain homotopy equivalent to the staircase complex with generators $x_0, \ldots, x_{2n}$ over $\mathbb{F}_2[U, U^{-1}]$, with the following differential:

\[ \partial x_{2k} = 0 \quad \text{and} \quad \partial x_{2k+1} = x_{2k} + x_{2k+2}. \]

Up to an overall shift, the $\mathbb{Z} \oplus \mathbb{Z}$-filtration is determined by the following:

- The element $x_{2k}$ has the same $j$-filtration as $x_{2k+1}$, but the $i$-filtration differs by $d_{2k+1}$.
- The element $x_{2k+2}$ has the same $i$-filtration as $x_{2k+1}$, but the $j$-filtration differs by $d_{2k+2}$.

See Figure 5.1 for a schematic of the staircase complex, as well as an example.

![Figure 5.1](image.png)

**Figure 5.1.** The generators of $\text{CFK}^\infty(K)$ on the left, for an L-space knot $K$. On the right is $\text{CFK}^\infty(T_{5,6})$. The symmetrized Alexander polynomial of $T_{5,6}$ is

\[ \Delta_{T_{5,6}}(t) = t^{10} - t^9 + t^5 - t^3 + 1 - t^{-3} + t^{-5} - t^{-9} + t^{-10}. \]

The minus version $\text{CFK}^{-}(K)$ can be read off from the above description of $\text{CFK}^\infty(K)$, as follows:

- There is one generator $y_i$ over $\mathbb{F}_2[v]$ for each $x_i$. The differential satisfies\[ \partial^- y_{2k} = 0 \quad \text{and} \quad \partial^- y_{2k+1} = v^{d_{2k+2}} \cdot y_{2k+2}. \]

Consequently, when $K$ is an L-space knot, \[ \text{Ord}_v(K) = \max\{d_{2k+1} : 0 \leq k \leq n-1\}. \]

Since the Alexander polynomial is symmetric, we have\[ d_{2k+1} = d_{2n-2k}, \] so \[ \text{Ord}_v(K) = \max\{d_i : 1 \leq i \leq 2n\}, \]

as claimed. □

We now need an elementary result concerning the Alexander polynomial of torus knots:

**Lemma 5.2.** If $p$ and $q$ are coprime, positive integers, then the first three terms of the symmetrized Alexander polynomial of $T_{p,q}$ are

\[ \Delta_{T_{p,q}}(t) = t^d - t^{d-1} + t^{d-\min(p,q)} + \cdots, \]

where $d = \frac{(p-1)(q-1)}{2}$.

**Proof.** Write

\[ \Delta_{T_{p,q}}(t) = t^{-d} \frac{(tpq - 1)(t - 1)}{(t^p - 1)(t^q - 1)}. \]

Canceling factors of $t - 1$ in Equation (14) and rearranging, we obtain

\[ t^d (t^{p-1} + \cdots + 1)(t^{q-1} + \cdots + 1) \Delta_{T_{p,q}}(t) = t^{pq} - 1 + t^{pq-2} + \cdots + 1. \]

It is a straightforward algebraic exercise to see that Equation (15) implies that the first three terms of $\Delta_{T_{p,q}}(t)$ are as claimed. □
We are now ready to show that our bounds in Corollaries 1.8 and 1.9 on the fusion number and the bridge index in terms of the torsion order are sharp:

**Corollary 5.3.** Let $T_{p,q}$ be a positive torus knot. Then

$$\text{Ord}_v(T_{p,q}) = \text{Ord}_v(T_{p,q}\#T_{p,q}) = \mathcal{F}_\text{us}(T_{p,q}\#T_{p,q}) = \text{br}(T_{p,q}) - 1 = \min\{p, q\} - 1.$$  

**Proof.** Combining Lemmas 5.1 and 5.2, we obtain that

$$\text{Ord}_v(T_{p,q}) \geq \min\{p, q\} - 1.$$  

On the other hand, $T_{p,q}\#T_{p,q}$ is ribbon, and hence equation (9) and Corollary 1.8 imply that

$$\text{Ord}_v(T_{p,q}) = \text{Ord}_v(T_{p,q}\#T_{p,q}) \leq \mathcal{F}_\text{us}(T_{p,q}\#T_{p,q}).$$  

By equations (2) and (4), we have

$$\mathcal{F}_\text{us}(T_{p,q}\#T_{p,q}) \leq \text{br}(T_{p,q}) - 1 = \min\{p, q\} - 1,$$  

and the result follows.  

\[\square\]

### 6. Sarkar’s ribbon distance and knot Floer homology

Following Sarkar [Sar19], if $K$ and $K'$ are concordant knots, then the ribbon distance $d_r(K, K')$ is the minimal $k$ such that there is a sequence of knots $K = K_0, K_1, \ldots, K_n = K'$ such that there exists a ribbon concordance connecting $K_i$ and $K_{i+1}$ (in either direction) with at most $k$ saddles. If $K$ and $K'$ are not concordant, we set $d_r(K, K') = \infty$. The ribbon distance satisfies the following properties:

1. $d_r(K, K') < \infty$ if and only if $K$ and $K'$ are concordant.
2. $d_r(K, K') = 0$ if and only if $K$ and $K'$ are isotopic.
3. $d_r(K, K') = d_r(K', K)$.
4. $d_r(K, K'') \leq \max\{d_r(K, K'), d_r(K', K'')\}$.

Furthermore, if $K$ is ribbon, then $d_r(K, U) \leq \mathcal{F}_\text{us}(K)$. Inspired by [Sar19, Theorem 1.1], we prove the following, which is equivalent to the statement in Section 1.8:

**Theorem 1.11.** Suppose $K$ and $K'$ are concordant knots, and let $d = d(K, K')$ denote their ribbon distance. Then

$$v^d \mathcal{HFK}^-(K) \cong v^d \mathcal{HFK}^-(K').$$

**Proof.** Since ribbon distance is defined by taking a sequence of ribbon concordances, it is sufficient to show that if there is a single ribbon concordance $C$ from $K$ to $K'$ with $n$ saddles, then

$$v^n \mathcal{HFK}^-(K) \cong v^n \mathcal{HFK}^-(K').$$

To prove Equation (16), we exhibit maps

$$F : v^n \mathcal{HFK}^-(K) \to v^n \mathcal{HFK}^-(K') \quad \text{and} \quad G : v^n \mathcal{HFK}^-(K') \to v^n \mathcal{HFK}^-(K),$$

and show that

$$F \circ G = \text{id}_{v^n \mathcal{HFK}^-(K')} \quad \text{and} \quad G \circ F = \text{id}_{v^n \mathcal{HFK}^-(K)}.$$

Let $C$ be the concordance from $K'$ to $K$ obtained by vertically mirroring $C$. We write $\overline{C}$ for a decoration of $C$ with two parallel dividing arcs, and $\overline{\mathcal{C}}$ for the mirrored decoration on $\overline{C}$. Let

$$F_0 : \mathcal{HFK}^-(K) \to \mathcal{HFK}^-(K') \quad \text{and} \quad G_0 : \mathcal{HFK}^-(K') \to \mathcal{HFK}^-(K)$$

denote the maps induced by $C$ and $\overline{C}$, respectively. Since $F_0$ and $G_0$ are $F_2[v]$-equivariant, we define $F$ and $G$ to be the restrictions of $F_0$ and $G_0$ to the images of $v^n$. A first application of Proposition 4.1 implies that $G_0 \circ F_0 = \text{id}_{\mathcal{HFK}^-(K)}$, so we easily obtain $G \circ F = \text{id}_{v^n \mathcal{HFK}^-(K)}$.

Next, Proposition 4.1 implies that

$$v^n \cdot (F_0 \circ G_0) = v^n \cdot \text{id}_{\mathcal{HFK}^-(K')}. $$

However this says exactly that $(F_0 \circ G_0)(v^n \cdot x) = v^n \cdot x$. Hence $F \circ G = \text{id}_{v^n \mathcal{HFK}^-(K')}$, completing the proof.  

\[\square\]
References

[AE18] Akram Alishahi and Eaman Eftekhary, Knot Floer homology and the unknotting number, 2018. e-print, arxiv:1810.05125.

[AGL18] Paolo Aceto, Marco Golla, and Ana G Recuana, Handle decompositions of rational homology balls and Casson-Gordon invariants, Proc. Amer. Math. Soc. 146 (July 2018), no. 9, 4059–4072.

[Ali17] Akram Alishahi, The Bar-Natan homology and unknotting number, 2017. e-print, arxiv:1710.07874.

[Baa12] Sebastian Baader, Scissor equivalence for torus links, Bull. Lond. Math. Soc. 44 (2012), no. 5, 1068–1078. MR2975163

[Gor81] Cameron Gordon, Ribbon concordance of knots in the 3-sphere, Math. Ann. 257 (1981), no. 2, 157–170. MR634459

[HKM08] Ko Honda, William Kazez, and Gordana Matić, Contact structures, sutured Floer homology and TQFT, 2008. e-print, arXiv:0807.2431.

[HN90] Jim Hoste, Yasutaka Nakanishi, and Kouki Taniyama, Unknotting operations involving trivial tangles, Osaka J. Math. 27 (1990), no. 3, 555–566. MR1075165

[Juh16] András Juhász, Cobordisms of sutured manifolds and the functoriality of link Floer homology, Adv. Math. 299 (2016), 940–1038. MR3519484

[JZ18a] András Juhász and Ian Zemke, Contact handles, duality, and sutured Floer homology, 2018. e-print, arXiv:1803.04401.

[JZ18b] Andras Juhász and Ian Zemke, Knot Floer homology obstructs ribbon concordance, 2019. e-print, arxiv:1902.04050.

[Mur85] Hitoshi Murakami, Some metrics on classical knots., Mathematische Annalen 270 (1985), 35–46.

[Nak10] Se-Goo Kim, Invertible knot concordances and prime knots, Honam Math. J. 32 (2010), no. 1, 157–165.

[Lee05] Eun Soo Lee, An endomorphism of the Khovanov invariant, Adv. Math. 197 (2005), no. 2, 554–586. MR2173845

[Miz95] Yoko Mizuma, An estimate of the ribbon number by the Jones polynomial, Osaka J. Math. 43 (2006), no. 2, 365–369.

[OS04] Peter Ozsváth and Zoltán Szabó, Holomorphic disks and knot invariants, Adv. Math. 186 (2004), no. 1, 58–116. MR2065507

[OS05] Peter Ozsváth and Zoltán Szabó, On knot Floer homology and lens space surgeries, Topology 44 (2005), no. 6, 1281–1300. MR2168576

[OS06] Peter Ozsváth and Zoltán Szabó, Holomorphic triangles and invariants for smooth four-manifolds, Adv. Math. 202 (2006), no. 2, 326–400.

[OS08] Peter Ozsváth and Zoltán Szabó, Holomorphic disks, link invariants and the multi-variable Alexander polynomial, Algebr. Geom. Topol. 8 (2008), no. 2, 615–692.

[OS] Peter Ozsváth and Zoltán Szabó, Knot Floer homology calculator. https://web.math.princeton.edu/~szabo/HFKcalc.html. Accessed: 2019-03-30.

[OS17] Peter Ozsváth, András Stipsicz, and Zoltán Szabó, Concordance homomorphisms from knot Floer homology, Adv. Math. 315 (2017), 366–426.

[Ras03] Jacob Rasmussen, Floer homology and knot complements, Ph.D. Thesis, 2003. arxiv:math/0306378.

[Sar19] Sucharit Sarkar, Ribbon distance and Khovanov homology, 2019. e-print, arxiv:1903.11095.

[Sch54] Horst Schubert, Über eine numerische knoteninvariante., Mathematische Zeitschrift 61 (1954/55), 245–288 (ger).

[Zem19a] Ian Zemke, Knot Floer homology obstructs ribbon concordance, 2019. e-print, arxiv:1902.04050.
