NON-DEGENERATE MAPS AND SETS

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Abstract. We construct certain non-degenerate maps and sets, mainly in the complex-analytic category.

1. Introduction

Let $X$ be a variety or an analytic space. A subset of resp. a map to $X$ is degenerate if it resp. its image is contained in a proper sub-variety/analytic subspace of $X$. Clearly it is easy to find degenerate maps and subsets, but sometimes it is maybe not so easy to find non-degenerate objects. In this paper our goal is to construct maps and infinite subsets which are as non-degenerate as possible.

2. Summary

The main results of this article are the following:

Concerning the existence of non-degenerate maps and sets in the complex-analytic category:

• Let $X$ be an irreducible complex space and $S \subset X$ a countable subset. Then there exists a holomorphic map $f$ from the unit disk $\Delta$ to $X$ with $S \subset f(\Delta)$.

• For every irreducible complex space $X$ there exists a holomorphic map from the unit disk $\Delta$ to $X$ with dense image. More general: If $X$ and $Y$ are irreducible complex spaces and $X$ admits a non-constant bounded holomorphic function, then there exists a holomorphic map from $X$ to $Y$ with dense image.

• Let $X$ be an irreducible complex space, $\dim(X) > 0$. Then there exists an infinite subset $\Gamma \subset X$ such that $\Gamma \cap Z$ is finite for every closed analytic subset $Z \subset X$.

• Every holomorphic map $f : \mathbb{C} \to \mathbb{C}^n$ can be approximated uniformly on compact sets by a sequence $f_n$ of holomorphic maps from $\mathbb{C}$ to $\mathbb{C}^n$ with dense image.

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The corresponding statements in the real-analytic category are true as well.

Furthermore, in the real-analytic category we have the following statement (for which there is no complex-analytic analogue):

- Let $X$ be a real analytic manifold, $\dim(X) > 1$, and $D, D'$ discrete subsets. Then every bijection $f_0 : D \to D'$ extends to a real-analytic diffeomorphism of $X$.

Finally we have an algebraic analogue of one of the complex-analytic statements:

- Let $X$ be a variety defined over a field $k$ such that the set of $k$-rational points $X(k)$ is dense in the Zariski topology. Assume $\dim(X) > 0$.

  Then there exists an infinite subset $\Gamma \subset X(k)$ such that $\Gamma \cap Z$ is finite for every subvariety $Z \subset X$.

3. PREPARATIONS

All manifolds are assumed to be paracompact, Hausdorff, finite-dimensional and connected.

We start by recalling some standard facts on real analytic maps.

A crucial instrument for our purposes is Grauert’s embedding theorem (see [8]):

**Theorem 1.** Every real-analytic manifold can be embedded into some $\mathbb{R}^n$.

Here an embedding is a proper injective immersion.

Using the standard embedding of $\mathbb{R}^n$ into the Stein complex manifold $\mathbb{C}^n$, one obtains the following immediate consequence:

**Corollary 1.** Every real analytic manifold admits a real-analytic embedding (as a closed real-analytic submanifold) into a Stein complex manifold, namely into $\mathbb{C}^n$.

Grauert’s embedding theorem is related to the following approximation theorem ([8], see also [10]):

**Theorem 2.** Let $X, Y$ be real-analytic manifolds. Then $C^\omega_S(X, Y)$ is dense in $C^\infty_S(X, Y)$.

Here $C^\omega_S(X, Y)$ and $C^\infty_S(X, Y)$ are endowed with the Whitney topology (sometimes called strong topology).

Let us assume that $X$ and $Y$ are equipped with some Riemannian metrics.
Now a subbasis for this topology is given by the collection of all sets
\[ V(k, f, \epsilon) = \{ F : X \to Y \mid ||D^k(F - f)(x)|| < \epsilon(x) \ \forall x \in X \} \]
where \( k \in \mathbb{N}, f \in C^\infty(X, Y) \) and \( \epsilon \in C^0(X, \mathbb{R}^+) \). Here \( D^k \) denotes the \( k \)-th derivative and the norm on \( D^k \) is induced by the Riemannian metric. See [10] for more information on this topology.

In this article we will always use this topology if we deal with smooth or real-analytic functions or maps. In contrast, we use the topological of locally uniform convergence if we deal with holomorphic maps.

The above approximation result admits a variant with interpolation ([16], Thm 3.3, p.128):

**Theorem 3.** Let \( X, Y \) be real-analytic manifolds and \( Z \) a closed analytic subset of \( X \). Let \( g : Z \to Y \) be a real-analytic map and \( F \) denote
the set of all maps \( f : X \to Y \) whose restriction to \( Z \) coincides with \( g \).

Then \( C^\omega_S(X, Y) \cap F \) is dense in \( C^\infty_S(X, Y) \cap F \).

We will use this result in the case where \( Z \) is a discrete subset.

**Corollary 2.** Let \( M \) be a real analytic manifold.

Then there exists a real-analytic map \( f : \mathbb{R} \to M \) with dense image.

**Proof.** Let \( S \subset M \) be a countable dense subset of \( M \) and \( \zeta : \mathbb{N} \to S \) a bijection. Now \( \mathbb{N} \) is a discrete (hence real-analytic) subset of \( \mathbb{R} \). Thus \( \zeta : \mathbb{N} \to S \subset M \) extends to a real-analytic map \( f : \mathbb{R} \to M \). Finally \( f(\mathbb{R}) \) is dense in \( M \) because it contains \( S \).

**Lemma 1.** Let \( X \) be a complete Riemannian manifold. Then the set of
diffeomorphisms is open in the set of self-maps \( C^\infty(X, X) \) with respect
to the Whitney topology.

**Proof.** See [10], Ch.2, Thm.1.7. \( \square \)

### 4. Automorphisms and discrete subsets

**Theorem 4.** Let \( X \) be a real analytic manifold of dimension at least two, \( D \) and \( D' \) discrete subsets.

Then every bijection \( f_0 : D \to D' \) extends to a real analytic diffeomorphism \( f \) of \( X \).

**Proof.** Since \( \dim(X) \geq 2 \), the complement of a real curve is connected. Therefore it is possible to find a family of disjoint smooth curves \( \rho_\gamma : [0, 1] \to X \ (\gamma \in D) \) with \( \rho_\gamma(0) = \gamma \) and \( \rho_\gamma(1) = f_0(\gamma) \). We choose disjoint open and relatively compact neighbourhoods \( U_\gamma \) of the curves \( \rho_\gamma([0, 1]) \). Next on each \( U_\gamma \) we choose a smooth vector field \( v_\gamma \) with support contained inside \( U_\gamma \) such that \((v_\gamma)_{\rho_\gamma(t)} = \gamma'_\gamma(t)\) for all \( \gamma \in D \)
and \( t \in [0, 1] \). Then we define a global vector field \( v \) by stipulating that \( v_x = (v_x)_\gamma \) if \( x \in U_\gamma \) and \( v_x = 0 \) if \( x \not\in \bigcup_\gamma U_\gamma \). Note that the support \( \text{supp}(v) \) is contained in a disjoint union of relatively compact open subsets. This implies that \( v \) is globally integrable. Now \( \exp_v(1) \) is a smooth diffeomorphism of \( X \) which extends \( f_0 \). By theorem 3 there is a real-analytic map \( f \) arbitrarily close to \( \exp_v(1) \) such that \( f|_D = f_0 \).

Finally, since \( \exp_v(1) \) is a diffeomorphism and diffeomorphism are open in the Whitney topology, we may require that \( f \) is a diffeomorphism.

\[ \square \]

**Remark.** If \( \dim(X) = 1 \) (i.e. \( X \simeq \mathbb{R} \) or \( X \simeq S^1 \)), the statement of the theorem still holds, provided we assume that \( f_0 \) extends to a homeomorphism of \( X \).

**Remark.** The corresponding statement for complex manifolds is wrong, even for \( X \simeq \mathbb{C}^n \). Rosay and Rudin proved that there are infinite discrete subsets \( D, D' \subset \mathbb{C}^n \) for all \( n \in \mathbb{N} \) such that there exists no holomorphic automorphism of \( \mathbb{C}^n \) mapping \( D \) to \( D' \) (\cite{15}). In \cite{17} we generalized this result to the case where \( \mathbb{C}^n \) is replaced by an arbitrary Stein manifold.

**Question 1.** Given a complex manifold \( X \) (\( \dim(X) > 0 \)), is it always possible to find discrete subsets \( D, D' \subset X \) of the same cardinality such that there exists no holomorphic automorphism \( \phi \) of \( X \) with \( \phi(D) = D' \)?

We believe that the answer is positive. See \cite{17} for a more thorough discussion of this question.

5. **Maps: Disks with dense image**

5.1. **First Approach.** We show that every complex space admits a dense disc.

**Theorem 5.** Let \( Z \) be an irreducible complex space and let \( S \subset Z \) be a countable subset (not necessarily discrete).

Then there exists a holomorphic map \( F \) from the unit disk \( \Delta = \{ z \in \mathbb{C} : |z| < 1 \} \) to \( Z \) such that \( S \subset F(\Delta) \).

**Proof.** Let \( \sigma : \tilde{Z} \to Z \) be a desingularization (\cite{10,3}) and \( \tilde{S} \subset \tilde{Z} \) a countable subset with \( \sigma(\tilde{S}) = S \). Without loss of generality we may assume that \( S \) (hence also \( \tilde{S} \)) is an infinite subset. Let \( \zeta : \mathbb{N} \to \tilde{S} \) be a bijection. Because \( \tilde{Z} \) is connected, this map \( \zeta \) extends to a \( C^\infty \)-map \( \zeta_1 : \mathbb{R} \to \tilde{Z} \). Using thm. 3 it follows that there is a real-analytic map \( f_0 : \mathbb{R} \to \tilde{Z} \) with \( f_0|_\mathbb{N} = \zeta \). Since analytic maps are locally given by
convergent power series, this map $f_0$ extends to a holomorphic map $f_1$ defined an some open neighborhood $W$ of $\mathbb{R}$ in $\mathbb{C}$. Denoting the natural projection from $X = \mathbb{C} \times \tilde{Z}$ onto the second factor $\tilde{Z}$ by $pr_2$ we define $F : W \to Z$ as $F = \sigma \circ pr_2 \circ f_1$. Note that $S \subset F(W)$. By shrinking $W$, if necessary, we may assume that $W \neq \mathbb{C}$ and that $W$ is simply-connected. Then Riemann’s mapping theorem (see e.g. [14]) implies that there is a biholomorphic map $\Delta \simeq W$.

**Corollary 3.** Let $X$ be an irreducible complex space. Then there exists a holomorphic map $F$ from the unit disk $\Delta = \{ z \in \mathbb{C} : |z| < 1 \}$ to $X$ such that $F(\Delta) = X$.

**Proof.** Choose a dense countable subset $S \subset X$ and invoke the preceding theorem. □

**Remark.** In particular, it is always possible to connect two points in a complex manifold by one disk.

Recall that for a complex manifold $X$ its Kobayashi pseudodistance $d_X$ is defined via chains of disks (see [11], [12]). Since any two points can be connected by one disk, one may also define a map $d' : X \times X \to \mathbb{R}^+$ using just single disks instead of disk chains. However, in this case it is not clear whether $d'$ fulfills the triangle inequality. Thus it is preferable to use disk chains.

**Question 2.** Given a complex manifold $X$, is the set of all holomorphic maps from the unit disk $\Delta$ to $X$ with dense image dense in the space of all holomorphic maps from $\Delta$ to $X$?

**Question 3.** Is there an analog of theorem 4 for spaces over non-archimedean complete fields like e.g. the $p$-adic numbers?

### 5.2. An alternative approach to dense disks.

By results of Forinaess and Stout ([6], [7]) it is known that for every complex manifold $X$ of dimension $n$ there exists a surjective holomorphic map from the polydisk $\Delta^n$ to $X$. Thus in order to prove that there exists a holomorphic map from the disk $\Delta$ to $X$ with dense image, it suffices to construct a holomorphic map from $\Delta$ to $\Delta^n$ with dense image. This can be done quite explicitly: First we recall that $z \mapsto -i^{w+1} \frac{w-1}{w-1}$ maps $\Delta$ biholomorphically to the upper half plane $H^+ = \{ z : \Im(z) > 0 \}$.

Now let $\lambda_1, \ldots, \lambda_{2n}$ be positive real numbers which are linearly independent over $\mathbb{Q}$. We consider the map $\phi : H^+ \to \Delta^{2n}$ given by

$$\phi : z \mapsto (e^{i\lambda_1 z}, \ldots, e^{i\lambda_{2n} z})$$

Fix a positive real number $\tau > 0$. Now for any $z \in H^+$ with $\Im(z) = \tau$ we have $|e^{i\lambda_j z}| = e^{\Re(i\lambda_j z)} = e^{-\lambda_j \tau}$. Thus the real line $i\tau + \mathbb{R}$ maps into
the totally real \( n \)-dimensional torus given as
\[
\{(v_1, \ldots, v_n) : |v_k| = e^{-\lambda_k \tau} \forall k\}.
\]

The linear independance of the \( \lambda_j \) over \( \mathbb{Q} \) ensures that the image is dense.

Therefore the closure of \( \phi(H^+) \) in \( \Delta^{2n} \) consists of all \( v = (v_1, \ldots, v_j) \) such that either all \( v_j \) are zero or \( \lambda_j \log |v_k| = \lambda_k \log |v_j| \) for all \( j, k \).

As a next step we consider
\[
\eta : v \mapsto \frac{1}{2}(v_1 + v_2, v_3 + v_4, v_5 + v_6, \ldots).
\]

First, let us remark that given \( r \geq s > 0 \) a complex number \( z \) can be written as sum \( z = z_1 + z_2 \) with \( |z_1| = r \) and \( |z_2| = s \) if and only if \( r + s \geq |z| \geq r - s \). Now observe that \( \lim_{\tau \to \infty} e^{-\lambda \tau} = 1 \) for \( \lambda \in \mathbb{R}^+ \). As a consequence, for every \( (w_1, \ldots, w_n) \in (\Delta \setminus \{0\})^n \) there is a number \( \tau >> 0 \) such that
\[
\frac{1}{2} (e^{-\lambda_{2k} \tau} + e^{-\lambda_{2k-1} \tau}) > |z_k| > \frac{1}{2} (|e^{-\lambda_{2k} \tau} - e^{-\lambda_{2k-1} \tau}|)
\]
for all \( k \in \{1, \ldots, n\} \) and consequently \( (\Delta \setminus \{0\})^n \) is contained in the image \( \eta(\Phi(H^+)) \). Thus \( \eta \circ \Phi : H^+ \to \Delta^n \) has dense image.

In order to give a completely explicit example:
\[
z \mapsto \left( \frac{1}{2} (e^{\frac{z}{\sqrt{2}}} + e^{\sqrt{2} \frac{z}{\sqrt{2}}}), \frac{1}{2} (e^{\sqrt{3} \frac{z}{\sqrt{3}}} + e^{\sqrt{3} \frac{z}{\sqrt{3}}}) \right)
\]
defines holomorphic map from \( \Delta \) to \( \Delta^2 \) with dense image.

5.3. Other source manifolds. Instead of considering maps from the unit disk we may investigate the same question for other manifolds.

**Definition 1.** A connected complex manifold \( X \) is called **universally dominating** if for every irreducible complex space \( Y \) there exists a holomorphic map from \( X \) to \( Y \) with dense image.

Taking \( Y = \Delta \), it is clear that a universally dominating complex manifold admits a non-constant bounded holomorphic function. In fact this necessary condition is also sufficient.

**Theorem 6.** A connected complex manifold \( X \) is universally dominating (in the sense of def. 1) if and only if there exists a non-constant bounded holomorphic function on \( X \).

The crucial part of the proof is the lemma below.

**Lemma 2.** Let \( U \) be a bounded connected open subset of \( \mathbb{C} \).

Then there exists a holomorphic map \( \rho : U \to \Delta \) with dense image.
Proof. First we recall the theory of “Ahlfors maps”. (see [2], p. 49.) If \( \Omega \) is a connected bounded domain in \( \mathbb{C} \) with smooth boundary (often called “of finite type”) and \( p \in \Omega \), then there is a unique holomorphic map \( \rho : \Omega \to \Delta \), called “Ahlfors map” with the following properties:

1. \( \rho(p) = 0 \),
2. \( \rho'(p) \in \mathbb{R}^+ \) and 
3. \( |\rho'(p)| \geq |f'(p)| \) for all holomorphic maps \( f : \Omega \to \Delta \) with \( f(p) = 0 \).

For a bounded domain \( \Omega \subset \mathbb{C} \) with smooth boundary this map \( \rho \) is surjective and in fact even proper.

This theory implies in particular:

Let \( q \in \Delta \) with \( \min\{1 - |q|, |q|\} > \epsilon > 0 \) and let \( W = \{z \in \Delta : |z - q| > \epsilon\} \).

Then there exists a holomorphic map \( \rho : W \to \Delta \) with \( |\rho'(0)| > 1 \).

Indeed, if we take \( p = 0 \) and compare \( \rho \) with \( f = \text{id}_\Delta|_W \), then the unicity of the Ahlfors map \( \rho \) implies \( |\rho'(p)| > |f'(0)| = 1 \).

Now let us consider an arbitrary bounded domain \( U \), making no assumptions about its boundary. Choose a basepoint \( p \in U \).

Let \( \mathcal{F} \) denote the family of all holomorphic maps \( f \) from \( U \) to \( \Delta \) with \( f(p) = 0 \) and let \( \alpha = \sup_{f \in \mathcal{F}} |f'(p)| \in \mathbb{R} \cup \{+\infty\} \). Let \( f_n \in \mathcal{F} \) be a sequence with \( \lim |f_n(p)| = \alpha \). Using the theorem of Montel we may assume that the sequence of holomorphic maps \( f_n \) converges to a holomorphic map \( \rho \). By these arguments we see:

For every bounded domain \( U \subset \mathbb{C} \) and every point \( p \in U \) there exists a holomorphic map \( \rho : U \to \Delta \) with \( \rho(p) = 0 \) such that \( |\rho'(p)| \geq |f'(p)| \) for every holomorphic map \( f : U \to \Delta \) with \( f(p) = 0 \).

Unlike an Ahlfors map for a bounded domain with smooth boundary, in general this map \( \rho \) is not necessarily surjective. For instance, consider \( U = \Delta \setminus \{1/2\} \) and \( p = 0 \). In this case we have \( \rho = \lambda \text{id}_\Delta|_U \) for some \( \lambda \in \mathbb{C} \) with \( |\lambda| = 1 \).

However, we claim that \( \rho \) has always dense image. Indeed, let us assume that the image is not dense. Then we can find a relatively compact ball \( B_\epsilon(q) \subset \Delta \) such that \( B_\epsilon(q) \cap \rho(U) = \emptyset \). But then we obtain a contradiction to the maximality property of \( \rho \) by considering the composition of \( \rho \) with the Ahlfors map of \( \Delta \setminus B_\epsilon(q) \).

□

Proof of the theorem. Let \( f \) be a non-constant bounded holomorphic function on \( X \). The image \( f(X) \) is a bounded open subset of \( \mathbb{C} \). By the preceding lemma there is a holomorphic map \( \rho \) from \( f(X) \) to \( \Delta \) with dense image. Thus there is a holomorphic map from \( X \) to \( \Delta \) with dense image, namely \( \rho \circ f \).
Let $Y$ be an irreducible complex space. Thanks to cor. 3 there is a holomorphic map $F : \Delta \to Y$ with dense image. Then $F \circ \rho \circ f : X \to Y$ is a holomorphic map with dense image. \qed

6. Entire Curves

6.1. Entire Curves in the Affine Space. The purpose of this section is to prove the following statement.

Proposition 1. For a holomorphic map $f : \mathbb{C} \to \mathbb{C}^n$ let $J_d(f) : \mathbb{C} \to \mathbb{C}^{nk}$ be defined as $J_d(f) = (f, f', f'', \ldots)$.

Let $d \in \mathbb{N} \cup \{0\}$ and $\Phi_d \subset Hol(\mathbb{C}, \mathbb{C}^n)$ the set of all holomorphic maps from $\mathbb{C}$ to $\mathbb{C}^n$ for which $(J_d f)(\mathbb{C}) = \mathbb{C}^{nk}$.

Then $\Phi_d$ is dense in $Hol(\mathbb{C}, \mathbb{C}^n)$ with respect to the topology of locally uniform convergence.

To prepare the proof of the proposition, we develop some lemmata.

Lemma 3. Let $p \in \mathbb{C}$ and $|p| > r > 0$ and $\epsilon > 0$.

Then there exists a polynomial $P \in \mathbb{C}[X]$ with $P(p) = 1$ and $|P(z)| \leq \epsilon$ for all $z \in B_r = \{z \in \mathbb{C} : |z| \leq r\}$.

Proof. Choose $N \in \mathbb{N}$ such that $(\frac{r}{|p|})^N < \epsilon$ and take $P(z) = \left(\frac{z}{p}\right)^N$. \qed

Lemma 4. Let $p \in \mathbb{C}$, $|p| > r > 0$, $\epsilon > 0$, $d \in \mathbb{N}$ and $(a_0, \ldots, a_n) \in \mathbb{C}^{n+1}$.

Then there exists a polynomial $P \in \mathbb{C}[X]$ such that

1. $|P(z)| < \epsilon$ whenever $|z| \leq r$ and
2. $P^{(k)}(p) = a_k$ for $k = 0, \ldots, d$.

(Here $P^{(k)}$ denotes the $k$-th derivative of $P$.)

Proof. We define recursively polynomials $P_m$ ($m = 0, \ldots, d$) such that

1. $|P_m(z)| < \frac{m+1}{d+1} \epsilon$ whenever $|z| \leq r$ and
2. $P^{(k)}(p) = a_k$ for $k = 0, \ldots, m$.

The existence of $P_0$ follows from the preceding lemma. Now let us assume that $P_{m-1}$ has been constructed. Define

$$Q_m(z) = \left(a_m - P_{m-1}^{(m)}(p)\right) \frac{1}{m!} (z - p)^m$$

and choose $R_m$ such that $R_m(p) = 1$ and $|R_m(z)| < \frac{1}{d+1} \inf_{|w| < r} |Q_m(w)|^{-1}$ for all $z \in \Delta_r$.

Then $P_m = P_{m-1} + Q_m R_m$ has the desired properties. Finally set $P = P_d$. \qed
Lemma 5. Let $D \subset \mathbb{C}$ be a discrete subset with $\min\{|z| : z \in D\} > r > 0$, $d \in \mathbb{N} \cup \{0\}$, $f : D \to \mathbb{C}^{d+1}$ a map and $\epsilon > 0$.

Then there exists a holomorphic function $F$ on $\mathbb{C}$ with $J_d F(\gamma) = f(\gamma)$ for all $\gamma \in D$ and

$$\max_{|z| \leq r} |F(z)| \leq \epsilon$$

Proof. Fix an enumeration $n \mapsto \gamma_n$ of $D$ such that $|\gamma_{n+1}| \geq |\gamma_n|$ for all $n \in \mathbb{N}$. Choose a strictly increasing sequence $r_n$ with $r < r_n < |\gamma_n|$ and $\lim r_n = +\infty$. (E.g. choose $r_1 = (r + |\gamma_1|)/2$ and $r_{n+1} = (r_n + |\gamma_n|)/2$ for $n \geq 1$.) Now we define recursively a sequence of functions $F_n$ as follows: $F_0 = 0$. Assume $F_1, \ldots, F_{n-1}$ are already defined. Let $Q_n$ be a polynomial with $(J_d Q_n)(\gamma_j) = 0$ for $j < n$ and $(J_d Q_n)(\gamma_n) = f(\gamma_n) - (J_d F_{n-1})(\gamma_n)$. Choose $\delta$ such that $\delta ||Q_n||_{B_n} < 2^{-n+1} \epsilon$. Due to lemma 4 there is a polynomial $P_n$ with $P_n(\gamma_n) = 1$, $P_n(k)(\gamma_n) = 0$ for $1 \leq k \leq d$ and $||P_n||_{B_n} < \delta$. Now define $F_n(z) = F_{n-1}(z) + P_n(z) Q_n(z)$.

Observe that

$$(J_d F_n)(\gamma_n) = (J_d F_{n-1})(\gamma_n) + (J_d (P_n Q_n))(\gamma_n) = f(\gamma_n)$$

and

$$(J_d F_n)(\gamma_j) = (J_d F_{n-1})(\gamma_j) + (J_d (P_n Q_n))(\gamma_j) = (J_d F_{n-1})(\gamma_j)$$

for $j < n$. Hence $(J_d F_n)(\gamma_j) = f(\gamma_j)$ for all $j \leq n$ by induction.

Since $||P_n Q_n||_{B_n} < 2^{-n+1} \epsilon$, the sequence of functions $F_n$ converges to a global holomorphic function $F$. By construction $F$ has the desired properties. \qed

Now we can prove the proposition.

Proof of the proposition. We have to show that for every $R, \epsilon > 0$, $f \in Hol(\mathbb{C}, \mathbb{C}^n)$ there exists a map $\phi \in \Phi$ with $||\phi - f||_{B_R} < \epsilon$.

Let $D = \{n \in \mathbb{N} : n > R\}$ and fix a bijection $\xi : D \to (\mathbb{Q}[i])^n$.

By lemma 5 there is a holomorphic map $g : \mathbb{C} \to \mathbb{C}^n$ with $g(n) = \xi(n) - f(n)$ for all $n \in D$ and $||g||_{B_R} < \epsilon$. Now $\phi = g + f$ is a map from $\mathbb{C}$ to $\mathbb{C}^n$ with dense image such that $||\phi - f||_{B_R} < \epsilon$. \qed

6.2. Entire Curves in Projective Varieties. Next we discuss dense entire curves in certain projective varieties. As a preparation we prove the lemma below.

Lemma 6. Let $Z \subset \mathbb{C}^n$ be an algebraic subvariety of codimension at least 2. Then there exists a dominant morphism $F : \mathbb{C}^n \to \mathbb{C}^n \setminus Z$.

Proof. Let $e_1, \ldots, e_n$ denote the standard basis of $\mathbb{C}^n$, $H_i$ the quotient vector space $\mathbb{C}^n / \mathbb{C} e_i$ and $\pi : \mathbb{C}^n \to H_i$ the natural linear projection.
Let \( Z_i \) denote the closure of \( \pi_i(Z) \). Then \( Z_i \) is an algebraic subvariety of codimension at least one.

Without loss of generality we may assume that none of the \( Z_i \) contains the origin \( 0 \in H_i \). Now we choose polynomial functions \( P_i \) on \( H_i \) such that \( P_i|_{Z_i} \equiv 0 \) and \( P_i(0) = 1 \). For each \( i \in \{1, \ldots, n\} \) and \( \zeta \in \mathbb{C} \) we obtain an automorphism \( \phi_{i,\zeta} : v \mapsto v + \zeta P_i(\pi_i(v))e_i \). Now for \( t = (t_1, \ldots, t_n) \in \mathbb{C}^n \) we define an automorphism \( \Phi_t \) of \( \mathbb{C}^n \) by

\[
\Phi_t : v \mapsto \phi_{n,t_n} \circ \cdots \circ \phi_{1,t_1}(v)
\]

By the construction each of the \( \phi_{i,\zeta} \) stabilizes \( Z \) pointwise. Combined with \( 0 \notin Z \) and the invertibility of \( \Phi_t \) we obtain that \( \Phi_t(0) \) is never contained in \( Z \). Therefore we obtain an algebraic morphism \( F : \mathbb{C}^n \to \mathbb{C}^n \setminus Z \) by

\[
F(t) = \Phi_t(0).
\]

The definition of \( \Phi \) also implies that \( F(\zeta e_i) = \zeta e_i \) for every \( i \in \{1, \ldots, n\} \), \( \zeta \in \mathbb{C} \). It follows that \( DF \) has maximal rank at 0. Thus \( F \) must be dominant. \( \square \)

**Proposition 2.** Let \( X \) be a unirational complex projective variety. Then there exists a holomorphic map \( f : \mathbb{C} \to X \) with dense image.

**Proof.** By the definition of unirationality we obtain a dominant rational map \( f_0 : \mathbb{C}^n \to X \). The indeterminacy locus \( E(f_0) \) is an algebraic subvariety of codimension at least two and \( f_0 \) is regular outside \( E(f_0) \). By lemma \( \square \) there is a dominant morphism \( g : \mathbb{C}^n \to \mathbb{C}^n \setminus E(f_0) \). Thus we obtain a dominant morphism \( h : \mathbb{C}^n \to X \) via \( h = f_0 \circ g \). By prop. \( \square \) there is a holomorphic map \( \phi : \mathbb{C} \to \mathbb{C}^n \) with dense image. Therefore \( f = h \circ \phi : \mathbb{C} \to X \) is a holomorphic map with dense image. \( \square \)

**Question 4.** Let \( X \) be a complex projective variety. Is \( X \) special in the sense of Campana \( [5] \) iff there exists a holomorphic map \( f : \mathbb{C} \to X \) with dense image?

A positive answer would in particular require that there is such a map \( f : \mathbb{C} \to X \) for every \( K3 \)-surface \( X \). By the results of Buzzard and Lu \( [4] \) there is a holomorphic map \( F : \mathbb{C}^2 \to X \) with dense image if \( X \) is an elliptic \( K3 \)-surface or a Kummer surface. Combined with prop. \( \square \) this implies that there is a holomorphic map \( f : \mathbb{C} \to X \) with dense image for these two special kinds of \( K3 \)-surfaces. However, for arbitrary \( K3 \)-surfaces this is still an open question.

6.3. Other ground fields. In this section we have rarely used special properties of the field of complex numbers. Thus the results mostly are valid for arbitrary fields with an absolute value (like e.g. the field \( \mathbb{Q}_p \).
of $p$-adic numbers, or $\Omega_p$ or the field $\mathbb{F}_q((t))$ of formal Laurent series over some finite field $\mathbb{F}_q$.

However, the following restrictions are necessary:

- If the ground field is not locally compact, “uniform convergence on compact sets” must be replaced by “uniform convergence on bounded sets” and the condition “$D$ is discrete in $\mathbb{C}$” has to be replaced by “$\{z \in D : |z| < R\}$ is finite for all $R > 0$”.

- If the characteristic $p$ of the field is positive, one cannot divide by $p$ and the $p$-th derivative is always zero. Hence prop. 4 and the lemmata 4 and 5 remain valid only in the case where $d$ is smaller than the characteristic.

- A dominant morphism between algebraic varieties has necessarily dense image only if the ground field is algebraically closed. Thus one needs to assume that the ground field is algebraically closed for prop. 2 and lemma 6.

7. Non degenerate Sets

7.1. Analytic case.

Proposition 3. Let $X$ be an irreducible complex (resp. real) analytic space. Then there exists an infinite subset $\Gamma \subset X$ such that $Z \cap \Gamma$ is finite for every closed analytic subset $Z$ of $X$.

Proof. It suffices to consider the real case.

We need some preparations. First we recall that

$$\arctan : \mathbb{R} \to [-1, +1[$$

is a proper real analytic map.

Next we choose a point $x \in X \setminus \text{Sing}(X)$. Define $D_r = \{x \in \mathbb{R}^d : ||x|| < r\}$. Fix a real-analytic coordinate chart on an open neighbourhood $U$ of $x$ in $X \setminus \text{Sing}(X)$ as

$$\zeta : U(x) \sim D_2$$

Let $d = \dim(X)$. Due to cor. 2 there exists a real-analytic map

$$\phi : \mathbb{R} \to \partial D_1 = \{x \in \mathbb{R}^d : ||x|| = 1\}$$

with dense image. Let $F : \mathbb{R} \to U \subset X$ be the real-analytic map given by

$$F : t \mapsto \zeta^{-1}(\arctan(t)\phi(t))$$

We claim that $F(\mathbb{R}) \supset \zeta^{-1}(\partial D_1)$. Indeed, for every $x \in \partial D_1$ there is a sequence $t_n \in \mathbb{R}$ with $\lim t_n = +\infty$ and $\lim \phi(t_n) = x$, since $\phi(\mathbb{R})$ is dense in $\partial D_1$. This implies $\lim F(t_n) = x$, because $\lim \arctan(t_n) = 1$. On the other hand $F(\mathbb{R}) \cap \zeta^{-1}(\partial D_1) = \{\}$. 

Let \( \Gamma_0 \) be a bounded infinite subset of \( \mathbb{R} \) and define our infinite subset \( \Gamma \) as
\[
\Gamma = F(\Gamma_0).
\]

In order to prove the proposition, we have to show that every infinite subset of \( \Gamma \) is Zariski dense in \( X \).

Let \( \Gamma' \) be an infinite subset of \( \Gamma \) and \( A \) the smallest closed real analytic subset of \( X \) containing \( \Gamma' \). We want to show that \( A = X \).

Because \( \Gamma \) is contained in the compact set \( \zeta^{-1}(D_1) \) the analytic set \( A \) has only finitely many irreducible components. Hence there is no loss in generality in assuming that \( A \) is irreducible. Since \( \Gamma_0 \) is bounded, there is an accumulation point \( q \) of \( \Gamma' \) with \( ||\zeta(q)|| < 1 \). Thus \( F^{-1}(\Gamma') \subset F^{-1}(A) \) contains an accumulation point \( p = F^{-1}(q) \). By the identity principle this implies that \( F(\mathbb{R}) \subset A \). Therefore \( F(\mathbb{R}) \subset A \). However, \( F(\mathbb{R}) \) contains \( C = \zeta^{-1}(\partial D_1) \) and \( C \) is a real-analytic hypersurface.

Because of the irreducibility of \( A \) we thus obtain that either \( A = C \) or \( A = X \). But \( A = C \) is impossible, since \( C \) does not contain any element of \( \Gamma \). Hence \( A = X \). Thus every infinite subset \( \Gamma' \) of \( \Gamma \) is dense in \( X \) with respect to the analytic Zariski topology. It follows that for every proper closed analytic subset \( Z \subset X \) the intersection \( Z \cap \Gamma \) must be finite. \( \square \)

7.2. Algebraic case.

**Proposition 4.** Let \( V \) be a variety defined over a field \( k \). Assume that \( V(k) \) is Zariski-dense.

Then there exists an infinite subset \( \Gamma \subset V(k) \) such that for every proper subvariety \( Z \) of \( V \) the intersection \( \Gamma \cap Z \) is finite.

**Proof.** Let \( \text{Div}^+(V) \) the set of all effective reduced \( k \)-Weil divisors on \( V \). Using Chow schemes, \( \text{Div}^+(V) \) can be exhausted by an increasing sequence of \( k \)-varieties \( C_n \). We define the universal space \( U_{n,m} \subset C_n \times V^m \) by the condition that \( (\theta; x_1, \ldots, x_m) \in U_{n,m} \) iff all the points \( x_i \) are contained in the support of the Weil divisor \( D_\theta \) indexed by \( \theta \). Note that \( U_{n,m} \) is of codimension at least \( m \). Let \( \rho_{n,m} : U_{n,m} \to V^m \) be the natural projection. Its generic fiber dimension does not exceed \( \dim(C_n) - m \).

(Here the dimension of the empty space is defined as \(-\infty\).) We define \( \Omega_{n,m} \subset V^m(k) \) as the set of \( k \)-rational points where the fiber dimension of \( \rho_{n,m} \) is less or equal \( \dim(C_n) - m \). Thus for \( m > \dim(C_n) \) the fiber \( \rho_{n,m}^{-1}(S) \) is empty for every \( S \in \Omega_{n,m} \). Observe that \( \Omega_{n,m} \) contains a Zariski open subset of \( V^m(k) \). Let \( S = (s_1, \ldots, s_m) \in \Omega_{n,m} \). We may choose a point in each irreducible component of the fiber \( \rho_{n,m}^{-1}(S) \). Let \( \Theta_1, \ldots, \Theta_l \) denote the corresponding Weil divisors. Then for every
$x \in V \setminus \bigcup_{i=1}^{l} \Theta_i$ we have
\[(S, x) = (s_1, \ldots, s_m, x) \in \Omega_{n,m+1},\]
because the fiber $\rho_{n,m+1}^{-1}(S, x)$ is a subvariety of $\rho_{n,m}^{-1}(S)$ which by construction does not contain any of the $(\Theta_i, S, x)$. In particular, if $S \in \Omega_{n,m}$, then there exists a Zariski open subset $W$ of $V$ such that $(S, x) \in \Omega_{n,m+1}$ for every $x \in W(k)$. Moreover $W(k) \neq \emptyset$, because $V(k)$ is assumed to be Zariski dense in $V$.

Therefore it is possible to choose recursively a sequence $\gamma_j \in V(k)$ with the following property:
If $n \leq r \leq j$ and if $S$ is a finite subset of $\{\gamma_r, \ldots, \gamma_j\}$, then $S \in \Omega_{n,\#S}$.
(If $\gamma_1, \ldots, \gamma_{j-1}$ are already chosen, this property defines an open subset of $V$ from which we have to choose $\gamma_j$.)

Now this property is equivalent to the following statement:
Let $S$ be a finite subset of $\Gamma = \{\gamma_j : j \in \mathbb{N}\}$ with $r = \min\{j : \gamma_j \in S\}$.
Then $S \in \Omega_{n,\#S}$ for all $n \leq r$.

Recall that for $m > \dim(C_n)$ the fiber $\rho_{n,m}^{-1}(S)$ is empty for all $S \in \Omega_{n,m}$. It follows that if $S$ is a finite subset of $\Gamma$ and $r = \min\{j : \gamma_j \in S\}$, then $S$ is not contained in the support for any Weil divisor parametrized by $\theta \in C_n$ for any $n$ with $n \leq r$ and $\#(S) > \dim(C_n)$.

Let $Z$ be a proper subvariety of $V$. Then $Z$ is contained in a Weil divisor parametrized by an element $\theta \in C_n$ for some $n$. Now the above arguments imply that the cardinality of a finite subset $S$ of $(\Gamma \cap Z) \setminus \{\gamma_1, \ldots, \gamma_{n-1}\}$ is bounded by $\dim(C_n)$. Hence $\#(\Gamma \cap Z) \leq \dim(C_n) + n - 1$. □

**Remark.** For $k = \mathbb{C}$ this statement can also be deduced from the analytic analogue stated above. However, already for proper subfields $k \subsetneq \mathbb{C}$ this is not just a corollary, because in this case the analytic statement only yields the existence of such a subset $\Gamma$ in $V(\mathbb{C})$, but not that $\Gamma$ can be chosen inside $V(k)$.

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