Loewy structure of $G_1T$-Verma modules of singular highest weights

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Abstract

Let $G$ be a reductive algebraic group over an algebraically closed field of positive characteristic, $G_1$ the Frobenius kernel of $G$, and $T$ a maximal torus of $G$. We show that the parabolically induced $G_1T$-Verma modules of singular highest weights are all rigid, determine their Loewy length, and describe their Loewy structure using the periodic Kazhdan-Lusztig $Q$-polynomials. We assume that the characteristic of the field is large enough that, in particular, Lusztig’s conjecture for the irreducible $G_1T$-characters holds.

Let $G$ be a reductive algebraic group over an algebraically closed field $k$ of positive characteristic $p$. The Frobenius kernel $G_1$ of $G$ is an analogue of the Lie algebra of $G$ in characteristic 0. To keep track of weights, we consider representations of $G_1T$ with $T$ a maximal torus of $G$. In this paper we study $G_1T$-Verma modules, standard objects of the theory.

Many years ago Henning Andersen and the second author of the present paper showed that the $G_1T$-Verma modules of $p$-regular highest weights are all rigid of Loewy length 1 plus the dimension of the flag variety of $G$, and described their Loewy structure using the periodic Kazhdan-Lusztig $Q$-polynomials [AK]. For that we assumed the validity of Lusztig’s conjecture on the irreducible characters for $G_1T$-modules, or rather Vogan’s equivalent version on the semisimplicity of certain $G_1T$-modules, modeling after Irving’s method [I85], [I88]. Lusztig’s conjecture is now a theorem for large $p$ as established by Andersen, Jantzen and Soergel [AJS]. Pushing their graded representation theory, with a machinery of Beilinson, Ginzburg and Soergel [BGS], we showed in [AbK] that the parabolic induction is graded on $p$-regular blocks, and determined the Loewy structure of parabolically induced $G_1T$-Verma modules of $p$-regular highest weights. In this paper we use Riche’s Koszulity of the $G_1$-block algebras [Ri] to uncover the structure of the parabolically induced $G_1T$-Verma modules of $p$-singular highest weights, to complete the entire picture.

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To describe our results precisely, let us introduce some notations. For simplicity we will assume throughout the paper that $G$ is simply connected and simple. Fix a Borel subgroup $B$ of $G$ containing $T$, and choose a positive system $R^+$ of $R$ such that the roots of $B$ are $-R^+$. Let $R^\circ$ denote the set of simple roots of $R^+$. Let $\Lambda$ denote the weight lattice of $T$ equipped with a partial order such that $\lambda \geq \mu$ iff $\lambda - \mu \in \sum_{\alpha \in R^+} N\alpha$. Put $p = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$. Let $W$ denote the Weyl group of $G$ relative to $T$, and let $W_\alpha = W \ltimes \mathbb{Z}R$ be the affine Weyl group with elements of $\mathbb{Z}R$ in $W_\alpha$ acting on $\Lambda$ by translations. We let $W_\alpha$ act on $\Lambda$ also via $x \cdot \lambda = px_\lambda(x + \rho) - \rho$, $x \in W_\alpha$, $\lambda \in \Lambda$. Let $R^\vee = \{ \alpha^\vee \mid \alpha \in R \}$ denote the set of coroots of $R$, and put $H_{\alpha,n} = \{ v \in \Lambda \otimes \mathbb{Z} \mid \langle v + \rho, \alpha^\vee \rangle = pn \}$, $\alpha \in R$ and $n \in \mathbb{Z}$. We call a connected component of $(\Lambda \otimes \mathbb{Z}R) \setminus \cup_{\alpha \in R, n \in \mathbb{Z}} H_{\alpha,n}$ an alcove. We say $\lambda \in \Lambda$ is $p$-regular if it belongs to an alcove, otherwise $\lambda$ is $p$-singular. Let also $\Lambda^+ = \{ \lambda \in \Lambda \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \forall \alpha \in R^+ \}$ the set of dominant weights. We let $A^+ = \{ v \mid \langle v + \rho, \alpha^\vee \rangle \in [0, p[ \forall \alpha \in R^+ \}$ denote the bottom dominant alcove. For a closed subgroup $H$ of $G$ we let $H_1$ denote its Frobenius kernel. Let $V = \text{ind}_{B_1T}^{G_1T}$ denote the induction functor [I, I.3] from the category of $B_1T$-modules to the category of $G_1T$-modules. The $G_1T$-simple modules are parametrized by their highest weights in $\Lambda$. We let $\hat{L}(\nu)$ denote the simple $G_1T$-module of highest weight $\nu \in \Lambda$. If $M$ is a finite dimensional $G_1T$-module, we will write $[M : \hat{L}(\nu)]$ for the composition factor multiplicity of $\hat{L}(\nu)$ in $M$.

A Loewy filtration of a finite dimensional $G_1T$-module $M$ is a filtration of $M$ of minimal length such that each of its subquotients is semisimple. The length of a Loewy filtration is uniform, called the Loewy length of $M$, denoted $\ell(M)$. Among the Loewy filtrations, the socle series of $M$ is defined by $0 < \text{soc}^1M < \text{soc}^2M < \cdots < \text{soc}^{\ell(M)}M = M$ with $\text{soc}^iM = \text{soc}M$, called the socle of $M$ which is the sum of simple submodules of $M$, and $\text{soc}^iM / \text{soc}^{i-1}M = \text{soc}(M / \text{soc}^{i-1}M)$ for $i > 1$. Also the radical series of $M$ is defined by $0 = \text{rad}^{\ell(M)}M < \cdots < \text{rad}^2M < \text{rad}^1M < M$ with $\text{rad}^1M = \text{rad}M$, called the radical of $M$ which is the intersection of the maximal submodules of $M$, and $\text{rad}^iM = \text{rad}(\text{rad}^{i-1}M)$ for $i > 1$. If $0 < M^1 < \cdots < M^{\ell(M)} = M$ is any Loewy filtration of $M$, $\text{rad}^{\ell(M) - i}M \leq M^i \leq \text{soc}^iM$ for each $i$. We say $M$ is rigid iff the socle and the radical series of $M$ coincide. We put $\text{soc}M = \text{soc}^iM / \text{soc}^{i-1}M$ and $\text{rad}M = \text{rad}^iM / \text{rad}^{i+1}M$.

In this paper we show

**Theorem:** Assume $p \gg 0$. Let $\nu \in \Lambda$ and let $N(\nu)$ denote the number of hyperplanes $H_{\alpha,n}$ on which $\nu$ lies. The $G_1T$-Verma module $\hat{V}(\nu)$ of highest weight $\nu$ is rigid of Loewy length $1 + \dim G/B - N(\nu)$. If $x \in W_a$ is such that $\nu$ belongs to the upper closure of $x \cdot A^+$ and if $\nu_0 = x^{-1} \cdot \nu$, the Loewy structure of $\hat{V}(\nu)$ is given by

$$
\sum_{x \in \mathbb{N}} q^{\frac{d(x, 0)}{2}} [\text{soc}_{i+1} \hat{V}(\nu) : \hat{L}(y \cdot \nu_0)]
$$

$$
= \begin{cases} 
Q^y \cdot A^+ \cdot A^+ & \text{if } y \in W_a \text{ with } y \cdot \nu_0 \text{ belonging to the upper closure of } y \cdot A^+,
0 & \text{else},
\end{cases}
$$

where $d(y, x)$ is the distance from the alcove $y \cdot A^+$ to the alcove $x \cdot A^+ [L80, 1.8]$ and $Q^y \cdot A^+ \cdot A^+$ is a polynomial from $[L80, 1.8]$.

For this theorem to hold, we assume $p \gg 0$ so that Lusztig’s conjecture for the irre-
ducible characters of $G_1T$-modules and also the conditions $[\text{Ri}]$ (10.1.1) and (10.2.1)] from $[\text{BMR06}]$ hold. While Fiebig $[\text{F}]$ gives an explicit lower, as crude as it may be, bound of $p$ for Lusztig’s conjecture to hold, a recent work of Williamson $[\text{W}]$ reveals that $p$ has, in general, to be much bigger than $h$ the Coxeter number of $G$, which was the original bound for the conjecture to hold. Compared to the restriction required for Lusztig’s conjecture to hold, the other conditions in $[\text{Ri}]$ are innocent.

We actually obtain, more generally, analogous results for parabolically induced module $\hat{\nabla}_P(L^\nu(\nu)) = \text{ind}_{P_1T}^P(L^\nu(\nu))$ with $L^\nu(\nu)$ denoting a simple $P_1T$-module of highest weight $\nu$ for a parabolic subgroup $P$ of $G$.

For a category $\mathcal{C}$ we will denote the set of morphisms from object $X$ to $Y$ in $\mathcal{C}$ by $\mathcal{C}(X,Y)$.

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1° Koszulity of the $G_1$-block algebras

Throughout the paper we will assume $p > h$ the Coxeter number of $G$ unless otherwise specified. In particular, $p \lambda \cap \mathbb{Z}R = p\mathbb{Z}R$. All modules we consider are finite dimensional over $\mathbb{k}$. Our basic strategy is to transport the known structure of a $G_1T$-block $\mathcal{C}(\lambda)$ of $p$-regular $\lambda \in \Lambda$ to an arbitrary block $\mathcal{C}(\mu)$ by the translation functor. For $p \gg 0$, thanks to $[\text{Ri}]$, the corresponding translation functor for the $G_1$-blocks is graded and the $G_1$-block algebras are all Koszul.

(1.1) For $\nu \in \Lambda$ let $\hat{L}(\nu)$ denote the simple $G_1T$-module of highest weight $\nu$, and $\hat{P}(\nu)$ the $G_1T$-projective cover of $\hat{L}(\nu)$. Let $\Omega$ be a $p$-regular orbit of $W_\alpha$ in $\Lambda$ and let $\mathcal{C}(\Omega)$ denote the corresponding $G_1T$-block. Thus $\mathcal{C}(\Omega) = \mathcal{C}(\nu)$, $\nu \in \Omega$, consists of $G_1T$-modules whose composition factors all have highest weights in $\Omega$. Let $\Omega'$ be a system of representatives of $\Omega$ under the translations by $p\mathbb{Z}R$, and let $\hat{P}(\Omega) = \bigsqcup_{\nu \in \Omega'} \hat{P}(\nu)$. Then $\bigsqcup_{\gamma \in p\mathbb{Z}R} \mathcal{C}(\Omega)(\hat{P}(\Omega) \otimes \gamma, \hat{P}(\Omega))$ forms a $p\mathbb{Z}R$-graded $\mathbb{k}$-algebra under the composition using the auto-functor $\otimes \gamma$, $\gamma \in p\mathbb{Z}R$, on $\mathcal{C}(\Omega)$. If we let $\hat{\mathcal{E}}(\Omega)$ denote its opposite algebra, $\bigsqcup_{\gamma \in p\mathbb{Z}R} \mathcal{C}(\Omega)(\hat{P}(\Omega) \otimes \gamma, ?)$ gives an equivalence of categories from $\mathcal{C}(\Omega)$ to the category of finite dimensional $p\mathbb{Z}R$-graded $\hat{\mathcal{E}}(\Omega)$-modules. Moreover, $\hat{\mathcal{E}}(\Omega)$ admits a $\mathbb{Z}$-grading compatible with its $p\mathbb{Z}R$-gradation $[AJS, 18.17]$. For $p$ large enough that Lusztig’s conjecture holds, $[AJS, 18.17]$ has proved that $\hat{\mathcal{E}}(\Omega)$ is Koszul with respect to its $\mathbb{Z}$-gradation. Let us state Lusztig’s conjecture in an equivalent form as follows: $\forall x, y \in W_\alpha$,

(L) $[\hat{\nabla}(x \bullet 0) : \hat{L}(y \bullet 0)] = Q^{y \bullet A^+, x \bullet A^+}(1),$

where $Q^{y \bullet A^+, x \bullet A^+}$ is a polynomial from $[L80, 1.8]$.

Assuming (L), let $\tilde{\mathcal{C}}(\Omega)$ denote the category of finite dimensional $(p\mathbb{Z}R \times \mathbb{Z})$-graded $\hat{\mathcal{E}}(\Omega)$-modules. For each $\nu \in \Omega'$ let $\hat{L}(\nu)$ be the lift of $\hat{L}(\nu)$ in $\tilde{\mathcal{C}}(\Omega)$ as a direct summand of
the degree 0 part of $\hat{E}(\Omega)$. If we denote the degree shift of objects in $\hat{C}(\Omega)$ by $[\gamma], \gamma \in p\mathbb{Z}R$, and by the $i$, $i \in \mathbb{Z}$, any simple of $\hat{C}(\Omega)$ may be written $\hat{L}(\nu)[\gamma](i)$, $\nu \in \Omega'$, $\gamma \in p\mathbb{Z}R$, and $i \in \mathbb{Z}$, in a unique way up to isomorphism. As $\hat{L}(\nu)[\gamma]$ is a lift of $\hat{L}(\nu + \gamma) = \hat{L}(\nu) \otimes 1$, we will also write $\hat{L}(\nu + \gamma)$ for $\hat{L}(\nu)[\gamma]$. For each $\nu \in \Omega$ the $G_1$-Verma module $\nabla(\nu)$ of highest weight $\nu$ admits a lift $\nabla(\nu)$ in $\hat{C}(\Omega)$ such that its socle is $\hat{L}(\nu)$. Likewise each projective $\hat{P}(\nu)$ admits a lift $\hat{P}(\nu)$ which is the projective cover of $\hat{L}(\nu)$.

1.2 Let $\Lambda_0 = \{ \nu \in \Lambda^+ \mid \langle \nu, \alpha^* \rangle < p \ \forall \alpha \in R^s\}$. For $\nu \in \Lambda$ we write $\nu = \nu^0 + p\nu_1$ with $\nu^0 \in \Lambda_0$ and $\nu_1 \in \Lambda$. We let $L(\nu^0)$ denote the simple $G$-module of highest weight $\nu^0$, which remains simple regarded as a $G_1$-module. All the simple $G_1$-modules are obtained thus. One has $\hat{L}(\nu) = L(\nu^0) \otimes 1$ and $\hat{P}(\nu) = \hat{P}(\nu^0) \otimes 1$, and $\hat{P}(\nu^0)$ provides the $G_1$-projective cover of $L(\nu^0)$, which we will denote by $P(\nu^0)$.

Let now $g$ denote the Lie algebra of $G$, $Ug$ the universal enveloping algebra of $g$, and $(Ug)_0$ the central reduction of $Ug$ with respect to the Frobenius central character $0$. As $(Ug)_0$ coincides with the the algebra of distributions of $G_1$, the representation theory of $G_1$ is equivalent to that of $(Ug)_0$. For each $\nu \in \Lambda$ let $(Ug)_0^\nu$ be the central reduction of $(Ug)_0$ with respect to the Harish-Chandra generalized character $\hat{\nu}$. This is the $G_1$-block component of $(Ug)_0$ associated to $\nu$. Let $B(\nu)$ denote the category of finite dimensional $(Ug)_0^\nu$-modules. The algebra $(Ug)_0^\nu$ is equipped with a $Z$-grading [Ri 6.3 and 10.2 line 16, p. 126]. We let $B^{gr}(\nu)$ denote the category of finite dimensional graded $(Ug)_0^\nu$-modules. Let $\Lambda(\nu) = \{(w \cdot \nu) \mid w \in W\}$, where $\hat{P}(\eta)$, $\eta \in \Lambda(\nu)$, admits a lift $P^{gr}(\eta)$ in $B^{gr}(\nu)$. Let $P^\nu = \bigsqcup_{\eta \in \Lambda(\nu)} P^{gr}(\eta)$ and set $E(\nu) = B(\nu)(P^\nu, P^\nu)^{op}$. As $P^\nu$ is a projective generator of $B(\nu)$ and as $E(\nu) = \prod_{i \in \mathbb{Z}} B^{gr}(\nu)(P^\nu(i), P^\nu)$ is equipped with a $Z$-gradation, $\langle i \rangle$ denoting the degree shift, $B(\nu)(P^\nu, ?)$ induces an equivalence from $B^{gr}(\nu)$ to the category of finite dimensional $Z$-graded $E(\nu)$-modules, which we will denote by $\hat{B}(\nu)$. For $p \gg 0$, thanks to [Ri 10.3], all $E(\nu)$ are Koszul by a careful choice of graded lift $P^{gr}(\eta), \eta \in \Lambda(\nu)$.

To be precise, let $I \subseteq R^s$ and let $P$ denote the corresponding standard parabolic subgroup of $G$ with the Weyl group $W_I = \{ s_\alpha \mid \alpha \in I\}$, where $s_\alpha$ is the reflection associated to $\alpha$. Let $\mu \in \Lambda$ lying in the closure $\overline{A^+}$ of the alcove $A^+$ such that $C_{W_\alpha}(y \cdot \mu) := \{ x \in W_\alpha \mid x y \cdot \mu = y \cdot \mu \} = W_I$ for some $y \in W_\alpha$. Let also $\lambda \in A^+$ such that $\langle y \cdot \lambda, \alpha^* \rangle = 0$ $\forall \alpha \in I$. If $p \gg 0$ so that the condition (L) holds, one can take each $P^{gr}(\eta), \eta \in \Lambda(\mu)$, to satisfy a certain condition [Ri 8.1(3)]. With this choice [Ri Th. 9.5.1] shows that the graded algebra $E(\lambda)$ is Koszul. For $\mu$ assume in addition to (L) two more conditions, which go as follows: the first one [Ri 10.1.1] coming from [BMR06 Lem. 1.10.9(ii)] reads, with $D^\lambda_{G/P}$ denoting the sheaf of PD-differential operators on $G/P$ twisted by the invertible sheaf $\mathcal{L}_{G/P}(\lambda)$,

\[(R1) \quad R^i \Gamma(G/P, D^\lambda_{G/P}) = 0 \ \forall i > 0.\]

With $(Ug)^\lambda$ denoting the central reduction of $Ug$ by the Harish-Chandra central character $\lambda$, the second condition [Ri 10.2.1] coming also from [BMR06 Lem. 1.10.9] reads that

\[(R2) \quad \text{the natural morphism } (Ug)^\lambda \rightarrow \Gamma(G/P, D^\lambda_{G/P}) \text{ is surjective.}\]

If $p \gg 0$ so that (L), (R1) and (R2) all hold, one can take each $P^{gr}(\eta), \eta \in \Lambda(\mu)$, to satisfy [Ri Th. 10.2.4], which makes $E(\mu)$ also Koszul [Ri Th. 10.3.1]. For any $\nu \in \Lambda$ there is by [BMR06 Lem. 1.5.2] some $\xi \in \Lambda$ such that $\nu + p\xi \in \overline{W_\alpha \cdot \mu}$ with $\mu$ as above. Thus under
the conditions (L), (R1) and (R2) we may assume that all $G_1$-block algebras $E(\nu)$ are Koszul. For each $\eta \in \Lambda(\nu)$ we denote by $\tilde{L}(\eta)$ the graded lift in $\tilde{B}(\nu)$ of $G_1$-simple $L(\eta)$ as a direct summand of $E(\nu)_0$. Let also $\tilde{P}(\eta) = \prod_{i \in \mathbb{Z}} B^{gr}(\nu)(P^{\nu}(i), P^{gr}(\eta))$ be a graded lift in $\tilde{B}(\nu)$ of $P(\eta)$ to form the projective cover of $\tilde{L}(\eta)$.

(1.3) Assume from now on throughout the rest of the paper that $p \gg 0$ so that all the conditions (L), (R1) and (R2) from (1.1) and (1.2) hold, unless otherwise specified. Fix also $\lambda$ and $\mu$ as in (1.2).

For our purposes, as tensoring with $p\eta$, $\eta \in \Lambda$, is an equivalence from the $G_1T$-block $\tilde{C}(\Gamma)$ of a $W_\alpha$-orbit $\Gamma$ to the $G_1T$-block $\tilde{C}(\Gamma + p\eta)$, we have only to determine the structure of parabolically induced $G_1T$-Verma modules of highest weight $x \cdot \mu$ with $\mu$ as above and $x \in W_\alpha$ such that $\langle xp, \alpha^+ \rangle \in ]0,p[ \forall \alpha \in R^w$.

If $\Omega = W_\alpha \cdot \lambda$, as $p > h$ by the standing hypothesis, $E(\lambda)$ coincides by the linkage principle with $E(\Omega)$ from (1.1) as $k$-algebras. As two $\mathbb{Z}$-gradations on the algebra must agree by their Koszulity [AJS F.2], there is no ambiguity about the functor from $\tilde{C}(\Omega)$ to $\tilde{B}(\lambda)$ forgetting the $p\mathbb{Z}$-$R$-gradation, which is compatible with the forgetful functor from the category of $G_1T$-modules to that of $G_1$-modules. Thus one has a commutative diagram of forgetful functors

$$
\begin{array}{ccc}
\tilde{C}(\Omega) & \longrightarrow & \tilde{B}(\lambda) \\
\bigg\downarrow & & \bigg\downarrow \\
B(\lambda) & \longrightarrow & \tilde{B}(\lambda).
\end{array}
$$

(1.4) For each $\nu \in \Lambda$ let $\text{pr}_\nu$ denote the projection from the category of finite dimensional $G_1$-modules to its $\nu$-block $B(\nu)$. For $\nu, \eta \in \Lambda^+$ recall from [BM08] the translation functor $T^\nu_\eta = \text{pr}_\eta(L((\eta - \nu)^+) \otimes ?) : B(\nu) \to B(\eta)$ with $(\eta - \nu)^+ \in W(\eta - \nu) \cap \Lambda^+$.

By [Ri] Prop. 5.4.3 and Th. 6.3.4] the adjoint translation functors $T^\mu_\lambda$ and $T^\lambda_\mu$ are graded to form a pair of functors $B^{gr}(\lambda) \xrightarrow{\sim} B^{gr}(\mu)$ such that graded $T^\mu_\lambda$ is right adjoint to graded $T^\lambda_\mu$. In turn, they induce a pair of graded functors, which we will denote by $\tilde{T}^\mu_\lambda$ and $\tilde{T}^\lambda_\mu$:

$$
\tilde{T}^\mu_\lambda = \prod_{i \in \mathbb{N}} B^{gr}(\lambda)(P^\lambda(i), T^\mu_\lambda ?) \circ (P^{\mu \otimes E(\mu)} ?) : \tilde{B}(\mu) \to \tilde{B}(\lambda),
$$

$$
\tilde{T}^\lambda_\mu = \prod_{i \in \mathbb{N}} B^{gr}(\mu)(P^\mu(i), T^\lambda_\mu ?) \circ (P^{\lambda \otimes E(\lambda)} ?) : \tilde{B}(\lambda) \to \tilde{B}(\mu)^{gr}.
$$

Thus $\tilde{T}^\mu_\lambda$ is right adjoint to $\tilde{T}^\lambda_\mu$.

Let $N(\nu)$ denote the number of hyperplanes $H_{\alpha,n}$ on which $\nu \in \Lambda$ lies, and put, in particular, $N = N(\lambda) = \dim G/B$, $N_P = N(\mu) = \dim G/P$. A crucial fact to our results is Riche’s [Ri] 10.2.8] that asserts for each $w \in W$ with $(w \cdot \mu)^0 \in \Lambda(\mu)$, i.e., $(w \cdot \mu)^0$ belonging to the upper closure of an alcove containing some $(w' \cdot \lambda)^0$, $w' \in W$, $T^\mu_\lambda P^{gr}((w \cdot \mu)^0) = P^{gr}((w' \cdot \lambda)^0)(N - N_P)$, and hence

$$
\tilde{T}^\mu_\lambda \tilde{P}((w \cdot \mu)^0) = \tilde{P}((w' \cdot \lambda)^0)(N - N_P).
$$


(1.5) For each \( \nu \in \Lambda \) let \( \hat{\pi}_\nu \) denote the projection from the category of finite dimensional \( G_1T \)-modules to the block \( \mathcal{C}(\nu) \). For \( \nu, \eta \in \bar{\Lambda}^+ \) one has as in (1.4) the translation functor \( \hat{T}_\nu^\eta = \hat{\pi}_\eta(L((\eta - \nu)^+) \otimes ?) : \mathcal{C}(\nu) \to \mathcal{C}(\eta) \) [J II.9.22]. Under the assumption \( p > h \), the functors \( T_\lambda^\nu \) and \( T_\lambda^\mu \) commute with the forgetful functors as in [BGS Lem. 4.3.2]:

\[
\begin{array}{ccc}
\mathcal{C}(\lambda) & \xrightarrow{\hat{T}_\lambda^\nu} & \mathcal{C}(\mu) \\
\downarrow & \circ & \downarrow \\
\mathcal{B}(\lambda) & \xrightarrow{T_\lambda^\mu} & \mathcal{B}(\mu).
\end{array}
\]

(1.6) Under the forgetful functors, \( \hat{\nabla} = \text{ind}_{B_1T}^{G_1T} \) yields an induction functor \( \hat{\nabla} = \text{ind}_{B_1}^{G_1} \) from the category of \( B_1 \)-modules to the category of \( G_1 \)-modules. If \( M \) is a \( G_1T \)-module, it is semisimple iff it is semisimple as \( G \)-module [J I.6.15]. Thus, in order to show that \( \hat{\nabla}(x \bullet \mu) \), \( x \in W_a \), is rigid, we have only to show that \( \hat{\nabla}(x \bullet \mu) \) is rigid.

For a facet \( F \) in \( \Lambda \otimes \mathbb{Z} \mathbb{R} \) with respect to \( W_a \) let \( \hat{F} \) denote its upper closure. As \( \hat{\nabla}(x \bullet \mu) = \hat{T}_\lambda^\nu \hat{\nabla}(x \bullet \lambda) \), \( \hat{T}_\lambda^\nu \hat{\nabla}(x \bullet \lambda) \in \hat{\mathcal{B}}(\mu) \) is a graded lift of \( \hat{\nabla}(x \bullet \mu) \), which we will denote by \( \tilde{\nabla}(x \bullet \mu) \langle i + N_P - N \rangle \) if \( x \bullet \mu \in x' \circ A^+ \), \( x' \in W_a \), and if \([\text{soc}_{i+1} \tilde{\nabla}(x \bullet \lambda) : \hat{L}(x' \bullet \lambda) \langle i \rangle \] \( \neq 0 \). As \( \tilde{\nabla}(x \bullet \mu) \) has a simple socle and a simple head, so does its lift, and hence the lift is rigid by [BGS Prop. 2.4.1]. There now follows the rigidity of \( \tilde{\nabla}(x \bullet \mu) \).

**Proposition:** All \( G_1T \)-Verma modules \( \tilde{\nabla}(\nu) \), \( \nu \in \Lambda \), are rigid.

(1.7) Let \( w \in W \) and put \( wB = wBw^{-1} \), \( \hat{\nabla}_w = \text{ind}_{B_1T}^{G_1T} \). If \( M \) is a \( G_1T \)-module, let \( wM \) denote the \( G_1T \)-module \( M \) with the \( G_1T \)-action twisted by \( w \), i.e., we let \( g \in G_1T \) act on \( m \in M \) by \( w^{-1}gw \). For each \( \nu \in \Lambda \) one has an isomorphism \( w\tilde{\nabla}(\nu) \simeq \tilde{\nabla}_w(w\nu) \) [J II.9.3]. Thus

**Corollary:** All \( \tilde{\nabla}_w(\nu) \), \( w \in W \), \( \nu \in \Lambda \), are rigid.

(1.8) Let \( J \subseteq R^a \), \( Q \) the standard parabolic subgroup of \( G \) associated to \( J \) with the Weyl group denoted \( W_J \), and let \( \hat{\nabla}_J = \text{ind}_{Q_1T}^{G_1T} \) denote the induction functor from the category of \( Q_1T \)-modules to the category of \( G_1T \)-modules. Let \( \nu \in \Lambda \) and let \( \hat{L}^J(\nu) \) denote the simple \( Q_1T \)-module of highest weight \( \nu \). Choose a \( p \)-regular \( \eta \in \Lambda \) such that \( \nu \) belongs to the upper closure of the \( W_{J,a} \)-alcove containing \( \eta \). Under the Lusztig conjecture (L) we have shown in [AbK 3.9] that \( \tilde{\nabla}_J(\hat{L}^J(\eta)) \) is graded, and in [AbK 2.3] that \( \hat{T}_\eta^J(\tilde{\nabla}_J(\hat{L}^J(\eta))) \simeq \tilde{\nabla}_J(\hat{L}^J(\nu)) \). As \( \tilde{\nabla}_J(\hat{L}^J(\nu)) \) has a simple head and socle [AbK 1.4], it follows again from [BGS Prop. 2.4.1] that

**Proposition:** All parabolically induced \( G_1T \)-Verma modules \( \tilde{\nabla}_J(\hat{L}^J(\nu)) \), \( \nu \in \Lambda \), are rigid.
2° The Loewy structure

Keep the notations from §1.

(2.1) For each ν ∈ Λ we will denote \( \hat{L}(ν) \) by \( \tilde{L}(ν) \) when regarded as a \( G_1 \)-module. Thus \( \tilde{L}(ν) = L(ν^0) \).

Lemma: Let \( x \in W_a \).

(i) One has

\[
\hat{T}_\nu^i \tilde{L}((x \cdot \lambda)^0) = \begin{cases} 
\tilde{L}((x \cdot \mu)^0)(N_P - N) & \text{if } x \cdot \mu \in x \cdot A^+, \\
0 & \text{else.}
\end{cases}
\]

(ii) If \( x \cdot \mu \in x \cdot A^+ \), one has \( \forall i \in \mathbb{N} \),

\[
\hat{T}_\nu^i \text{soc}^i \hat{\nabla}(x \cdot \lambda) = \text{soc}^i \hat{\nabla}(x \cdot \mu).
\]

Proof: (i) We may by (1.5) assume that \( x \cdot \mu \in x \cdot A^+ \) [8 II.7.15, 9.22.4], which occurs iff \( (x \cdot \mu)^0 \) lies in the upper closure of the alcove \( (x \cdot \lambda)^0 \) belongs to. Thus we are to show in that case that \( \hat{T}_\nu^i \tilde{L}((x \cdot \lambda)^0) = \tilde{L}((x \cdot \mu)^0)(N_P - N) \).

As \( \tilde{P}((x \cdot \mu)^0) \) (resp. \( \hat{P}((x \cdot \lambda)^0) \)) is a projective cover of \( \tilde{L}((x \cdot \mu)^0) \) (resp. \( \tilde{L}((x \cdot \lambda)^0) \)), we have for each \( n \in \mathbb{Z} \)

\[
\bar{B}(\mu)(\tilde{P}((x \cdot \mu)^0)\langle n \rangle, \hat{T}_\nu^i \tilde{L}((x \cdot \lambda)^0)) \simeq \bar{B}(\lambda)(\hat{T}_\nu^i \tilde{P}((x \cdot \mu)^0)\langle n \rangle, \tilde{L}((x \cdot \lambda)^0)) \\
\simeq \bar{B}(\lambda)(\hat{P}((x \cdot \lambda)^0)\langle n + N - N_P \rangle, \hat{L}((x \cdot \lambda)^0)) \quad \text{by (1.4.1)},
\]

which is nonzero iff \( n + N - N_P = 0 \), and hence the assertion follows.

(ii) Let \( \text{soc}_{G_1}^i \hat{\nabla}((x \cdot \lambda)^0), x \in W_a \), denote the \( i \)-th term of the \( G_1 \)-socle series of \( \hat{\nabla}((x \cdot \lambda)^0) \), which is just \( \text{soc}^i \hat{\nabla}(x \cdot \lambda) \) regarded as \( G_1 \)-module. As the socle series and the gradation over \( E(\lambda) \) (resp. \( E(\mu) \)) coincide on \( \nabla((x \cdot \lambda)^0) \) (resp. \( \nabla((x \cdot \mu)^0) \)) by [BGS Prop. 2.4.1], we see from (i) that \( \hat{T}_\nu^i \text{soc}_{G_1}^i \hat{\nabla}((x \cdot \lambda)^0) = \text{soc}_{G_1}^i \hat{\nabla}((x \cdot \mu)^0) \), and hence the assertion.

(2.2) \( \forall x, y \in W_a \), let \( Q^{g \cdot A^+, x \cdot A^+}(q) = \sum_j Q_j^{y,x} q^j \in \mathbb{Z}[q] \) be the periodic Kazhdan-Lusztig \( Q \)-polynomial from [L80]. Put \( Q^{y,x} = Q^{g \cdot A^+, x \cdot A^+}(q) \) for simplicity. Recall from [AK, AJS, 18.19]/[AbK 5.1, 2]

\[
\sum_{i \in \mathbb{N}} q^{\frac{d(y,x)-i}{2}} [\text{soc}_{i+1} \hat{\nabla}(x \cdot \lambda) : \hat{L}(y \cdot \lambda)] = \sum_{i \in \mathbb{N}} q^{\frac{d(y,x)-i}{2}} [\hat{\nabla}(x \cdot \lambda) : \hat{L}(y \cdot \lambda)\langle -i \rangle] = Q^{y,x},
\]

where \( d(y,x) = d(y \cdot A^+, x \cdot A^+) \) is the distance from the alcove \( y \cdot A^+ \) to the alcove \( x \cdot A^+ \) [L80]. Let \( W_a(\mu) = \{ x \in W_a \mid x \cdot \mu \in x \cdot A^+ \} \). For each \( x \in W_a(\mu) \), (2.1.ii) shows that

\[
\sum_{i \in \mathbb{N}} q^{\frac{d(y,x)-i}{2}} [\text{soc}_{i+1} \hat{\nabla}(x \cdot \mu) : \hat{L}(y \cdot \mu)] = \begin{cases} 
Q^{y,x} & \text{if } y \in W_a(\mu), \\
0 & \text{else.}
\end{cases}
\]
One can do the same with parabolically induced \( G_1T \)-Verma modules \( \hat{\nabla}_J(L^J(\nu)) \), \( J \subseteq R^s \), \( \nu \in \Lambda \), from (1.8), via \cite[2.3]{AbK}. Let \( W_{Ja} = W_J \ltimes \mathbb{Z}R_J \) denote the affine Weyl group for \( P_J \).

**Theorem:** Let \( \nu \in \Lambda \), \( x \in W_a \) such that \( \nu \in x \cdot A^+ \), and put \( \nu_0 = x^{-1} \cdot \nu \). Then
\[
\sum_{i \in \mathbb{N}} q^{d(y, \nu_0) - 1} [\text{soc}_{i+1} \hat{\nabla}_J(L^J(\nu)) : L(y \cdot \nu_0)]
= \begin{cases} 
\sum_{z \in W_{Ja}} Q_y \cdot A^+ \cdot A^+(\nu_0) \hat{P}_{z \cdot A^+} & \text{if } y \in W_a(\mu), \\
0 & \text{else},
\end{cases}
\]
where \( \hat{P}_{z \cdot A^+} \) is a \( \hat{P} \)-polynomial from \cite{Kad} for \( W_{Ja} \) and \( d(y, \nu_0) \) is the distance from \( z \cdot A^+ \) to \( x \cdot A^+ \) with respect to \( W_{Ja} \).

(2.4) Finally we determine the Loewy length of all parabolically induced \( G_1T \)-Verma modules. We first need analogues of \cite[Props. 3.2 and 3.3]{RS}.

Let \( \hat{\Delta}(\nu) = \text{Dist}(G_1) \otimes_{\text{Dist}(B_1)} \nu = \hat{\nabla}(\nu)^T \) the \( k \)-linear dual of \( \hat{\nabla}(\nu) \) twisted by the Chevalley anti-involution \( \tau \) of \( G \). We say a \( G_1T \)-module \( M \) admits a \( \hat{\nabla} \)-filtration iff there is a filtration \( 0 = \hat{\nabla}_M(\nu) \) of \( \hat{\Delta}(\nu) \) for \( M_0 = M_1 < \cdots < M_r = M \) of \( G_1T \)-modules with each \( M_i/M_{i-1} \simeq \hat{\nabla}(\nu_i) \) for some \( \nu_i \in \Lambda \), in which case one can arrange the filtration such that \( \nu_i \neq \nu_j \) if \( i > j \). Whenever \( M \) admits a \( \hat{\nabla} \)-filtration, we will assume that such a rearrangement has been done.

Let \( W_\nu = C_{W_a} \cdot \nu \) and take an alcove \( A \) in the closure of which \( \nu \) lies. Choose \( \eta \in \Lambda \) in \( A \). Let \( \eta^+ \) (resp. \( \eta^- \)) denote the highest (resp. lowest) weight in \( W_\nu \cdot \eta \). Let us also denote by \( \hat{T}_\nu^\eta : C(W_a \cdot \nu) \to C(W_a \cdot \eta) \) and \( \hat{T}_\eta^\nu : C(W_a \cdot \eta) \to C(W_a \cdot \nu) \) the associated translation functors.

**Lemma:** Assume \( p \gg 0 \) so that (L) holds.

(i) \( \hat{\Delta}(\eta^+) \leq \text{rad} N(\nu) \hat{T}_\nu^\eta \hat{\Delta}(\nu) \).

(ii) \( \hat{\Delta}(\eta^-) \leq \text{soc} N(\nu) + 1 \hat{\nabla}(\eta^+) \).

(iii) \( \ell \ell(\hat{T}_\nu^\eta \hat{\Delta}(\nu)) \geq 2N(\nu) + 1 \).

(iv) \( \forall M \in C(\nu), \ell \ell(\hat{T}_\nu^\eta M) \geq 2N(\nu) + \ell \ell(M) \).

**Proof:** (i) Recall from \cite[18.13]{AJS} that the translation functors \( \hat{T}_\eta^\nu \) and \( \hat{T}_\nu^\eta \) admit graded versions, denoted \( T_1 \) and \( T^* \), resp. If we let \( \hat{\Delta}(\eta) \) denote the graded version of \( \hat{\Delta}(\eta) \), \( T^* T_1 \hat{\Delta}(\eta^-) \) admits by \cite[18.15]{AJS} a filtration with the subquotients \( \hat{\Delta}(w \cdot \eta^-) / \text{o}(w \cdot \eta^-) \), \( w \in W_\nu \), where \( \text{o}(w \cdot \eta^-) \) denotes the number of hyperplanes \( H_{\alpha, n}, \alpha \in R^s, n \in \mathbb{Z} \), on which \( \nu \) lies and such that \( w \cdot \eta^- \) belongs to their positive sides \cite[15.13]{AJS}. Thus the graded version of \( \hat{\Delta}(\eta^+) = \text{hd} \hat{\Delta}(\eta^+) \) appears in \( T^* T_1 \hat{\Delta}(\eta^-) \) as \( \hat{\Delta}(\eta^+) \langle N(\nu) \rangle \) while that of \( \hat{\Delta}(\eta^-) = \text{hd} \hat{\Delta}(\eta^-) = \text{hd} \hat{T}_\nu^\eta \hat{\Delta}(\nu) \) appears as \( \hat{\Delta}(\eta^-) \). Under the assumption (L), \( \hat{\Delta}(\eta^-) \) is
graded over the Koszul algebra $\mathbb{E}(W_a \bullet \eta)$ from (1.1), and so therefore is $T^*T\Delta(\eta^-)$. As $T^\theta\Delta(\nu)$ has a simple socle and a simple head, its Loewy series coincides with the grading filtration up to degree shift by $\text{[BCS]}$. It follows that $\hat{L}(\eta^-)$ appears in $\text{rad}_{N(\nu)}T^\theta\Delta(\nu)$, and hence $\Delta(\eta^-) \leq \text{rad}_{N(\nu)}T^\theta\Delta(\nu)$.

(ii) Note first that the number of times $\hat{\nabla}(\eta^+)$ appears in the $\hat{\nabla}$-filtration of $\hat{P}(\eta^-)$ is by the translation principle equal to

\[ (1) \quad \hat{\nabla}(\eta^+) : \hat{L}(\eta^-) = 1. \]

Thus, if $r = \max\{i \in \mathbb{N} \mid \eta_i = \eta^+\}$ in the $\hat{\nabla}$-filtration $M^\bullet$ of $\hat{P}(\eta^-)$ with the suquotients $M^i/M^{i-1} \simeq \hat{\nabla}(\eta_i)$, one has $T^\theta\nabla(\nu) \leq M^r$. By (1) and by $\text{[AK]} 3.5$ there is unique $j \in \mathbb{N}$ such that $[\text{soc}_{j+1}M^r : \hat{L}(\eta^+)] = [\text{soc}_{j+1}\hat{\nabla}(\eta^+) : \hat{L}(\eta^-)] = 1$. As $[T^\theta\nabla(\nu) : \hat{L}(\eta^+)] \neq 0$, we must have $[\text{soc}_{j+1}\hat{T}^\theta\nabla(\nu) : \hat{L}(\eta^+)] = 1$. Then taking the $\tau$-dual yields that $[\text{rad}_j\hat{T}^\theta\Delta(\nu) : \hat{L}(\eta^+)] = 1$, and hence $j = N(\nu)$ by (i).

(iii) Consider a filtration of $T^\theta\Delta(\nu)$ with the subquotients $\hat{\Delta}(w \bullet \eta)$, $w \in W_\nu$. By the weight consideration $\hat{T}^\theta\hat{L}(\nu)$ must contain all the composition factors of $T^\theta\hat{\Delta}(\nu)$ isomorphic to $\hat{\Delta}(\eta^-)$.

On the other hand, $[\hat{\Delta}(w \bullet \eta^+) : \hat{L}(\eta^-)] = 1 \forall w \in W_\nu$ as in (1). Thus $T^\theta\hat{L}(\nu)$ contains a composition factor $\hat{L}(\eta^-)$ corresponding to one in each of $\hat{\Delta}(w \bullet \eta^+)$, $w \in W_\nu$. Consider the factor corresponding to the one in $\hat{\Delta}(\eta^+)$. Let $\theta \in G_1T\text{Mod}(\hat{\Delta}(\eta^+), T^\theta\hat{L}(\nu))$ be the restriction to $\hat{\Delta}(\eta^+)$ of the quotient $T^\theta\Delta(\nu) \rightarrow T^\theta\hat{L}(\nu)$. Then $\text{im} \theta \leq \text{rad}_{N(\nu)}T^\theta\hat{L}(\nu)$ by (i). As the composition factor $\hat{L}(\eta^-)$ comes from the one in $\text{rad}_{N(\nu)}\hat{\Delta}(\eta^+)$ by (ii), it lies in $\text{rad}_{N(\nu)}(\text{im} \theta)$. It follows that $2N(\nu) + 1 \leq \ell \ell(\text{im} \theta) + N(\nu) \leq \ell \ell(T^\theta\hat{L}(\nu))$.

(iv) Consider a nonsplit exact sequence $0 \rightarrow \hat{L}(y \bullet \nu) \rightarrow K \rightarrow \hat{L}(x \bullet \nu) \rightarrow 0$, $x, y \in W_a$, with $x \bullet \nu > y \bullet \nu$. There is an epi $\hat{\Delta}(x \bullet \nu) \twoheadrightarrow K$. As $T^\theta\Delta(\nu)$ has a simple head, so does $T^\theta\hat{K}$. In particular, $T^\theta\hat{K}$ is indecomposable, and so therefore is $(T^\theta\hat{K})^\tau \simeq T^\theta\hat{K}^\tau$.

We now argue by induction on $\ell \ell(M)$. We may assume $M$ has a simple head. Let $\hat{L}(x \bullet \nu) = \text{hd}M$, $x \in W_a$. Take a quotient $M/M'$ with $\text{rad}M > M' > \text{rad}^2M$ which fits in a short exact sequence $0 \rightarrow \hat{L}(y \bullet \nu) \rightarrow M/M' \rightarrow \hat{L}(x \bullet \nu) \rightarrow 0$ for some $y \in W_a$. As $T^\theta\Delta(\nu)$ is indecomposable, the exact sequence $0 \rightarrow T^\theta\Delta(\nu) \rightarrow T^\theta\hat{M} \rightarrow T^\theta\hat{L}(x \bullet \nu) \rightarrow 0$ cannot split. Thus $\ell \ell(T^\theta\hat{M}) \geq \ell \ell(T^\theta\Delta(\nu)) + 1$, as desired.

(2.5) Keep the notation of (2.4). Let $w_0$ denote the longest element of $W$. $\forall x \in W_a$, recall from $\text{[AK]} 3.4.2$ that

\[ (1) \quad \ell \ell \hat{P}(\nu) \geq 2 \ell \ell(\hat{\nabla}(w_0 \bullet \nu)) - 1 \geq 2N - 2N(\nu) + 1, \]

and from $\text{[AK]} 2.3$ that $\ell \ell \hat{\nabla}(w_0 \bullet \nu) \geq N - N(\nu) + 1$. Thus

\[
2N + 1 = \ell \ell \hat{P}(\eta^-) \quad \text{by [AK] 5.4}
\]

\[
= \ell \ell(T^\theta\hat{P}(\nu)) \geq \ell \ell(\hat{P}(\nu)) + 2N(\nu) \quad \text{by (2.4)}
\]

\[
\geq 2N - 2N(\nu) + 1 + 2N(\nu) = 2N + 1.
\]
It follows that $\ell \ell \hat{P}(\nu) = 2N - 2N(\nu) + 1$, and then $\ell \ell \hat{\nabla}(w_0 \bullet \nu) = N - N(\nu) + 1$ by (1).

As $\hat{\nabla}_w((w \bullet \nu)\langle w \rangle) \simeq w\hat{\nabla}(\nu) \otimes p(w \bullet 0) \forall w \in W$ by (1.5), we have

$$\ell \ell \hat{\nabla}_w((w \bullet \nu)\langle w \rangle) = 1 + N - N(\nu) = 1 + \dim G/B - N(\nu).$$

Let us also record

Theorem: Assume $p \gg 0$ so that (L) holds. $\forall \nu \in \Lambda$, $\ell \ell (\hat{P}(\nu)) = 2N - 2N(\nu) + 1$.

(2.6) Recall the notation of (1.8). To find the Loewy length of $\hat{\nabla}_J(\hat{L}^J(\nu))$, we first recall some identities from $[\text{AbK}]$. These hold without restrictions on $p$. Let $w_J$ denote the longest element of $W_J$ and put $w^J = w_0 w_J$. Let $\nu \in \Lambda$. We will write $\nu\langle w \rangle$, $w \in W$, for $\nu + (p-1)(w \bullet 0)$.

One can reformulate $[\text{AbK}, 1.4]$ as an isomorphism $\text{hd}\hat{\nabla}_J(\hat{L}^J(\nu)) \simeq L((w^J \bullet \nu)^0) \otimes_k p\{(w^J)^{-1} \bullet (w^J \bullet \nu)^1\}$. Also, $[\text{AbK} 4.5]$ carries over to arbitrary $\nu \in \Lambda$: $\text{hd}w^J\hat{\nabla}_J(\hat{L}^J(\nu)) \simeq \hat{L}(w^J \bullet \nu) \otimes_k \{-p(w^J \bullet 0)\}$. We then have a commutative diagram from $[\text{AbK} 4.6.1]

\[
\begin{array}{c}
\hat{\nabla}_{w^J}((w^J \bullet \nu)\langle w^J \rangle) \otimes \{-p(w^J \bullet 0)\} \\
\downarrow \phi_{w^J} \otimes \{-p(w^J \bullet 0)\} \\
\hat{\nabla}(w^J \bullet \nu) \otimes \{-p(w^J \bullet 0)\}
\end{array}
\]

\[
\begin{array}{c}
\hat{L}(w^J \bullet \nu) \otimes \{-p(w^J \bullet 0)\} \quad \hat{\nabla}_{w^J}(\hat{L}^J(\nu))
\end{array}
\]

and another from $[\text{AbK} 4.6.3]

\[
\begin{array}{c}
\hat{\nabla}_{w_0}((w^J \bullet \nu)\langle w_0 \rangle) \otimes \{-p(w^J \bullet 0)\} \\
\phi'_{w^J} \otimes \{-p(w^J \bullet 0)\} \\
\hat{\nabla}_{w^J}((w^J \bullet \nu)\langle w^J \rangle) \otimes \{-p(w^J \bullet 0)\}
\end{array}
\]

If we write $w^J = s_{i_1} \ldots s_{i_n}$ in a reduced expression with $s_i$ denoting the reflection associated to the simple root $\alpha_i$, the homomorphism $\phi_{w^J} : \hat{\nabla}_{w^J}((w^J \bullet \nu)\langle w^J \rangle) \rightarrow \hat{\nabla}(w^J \bullet \nu)$ is the composite

$$\hat{\nabla}_{s_{i_1} \ldots s_{i_n}}((w^J \bullet \nu)\langle s_{i_1} \ldots s_{i_n} \rangle) \rightarrow \hat{\nabla}_{s_{i_1} \ldots s_{i_{n-1}}}((w^J \bullet \nu)\langle s_{i_1} \ldots s_{i_{n-1}} \rangle) \rightarrow \ldots \rightarrow \hat{\nabla}_{s_{i_1} s_{i_2}}((w^J \bullet \nu)\langle s_{i_1} s_{i_2} \rangle) \rightarrow \hat{\nabla}_{s_{i_1}}((w^J \bullet \nu)\langle s_{i_1} \rangle) \rightarrow \hat{\nabla}((w^J \bullet \nu))$$

with each $\hat{\nabla}_{s_{i_1} \ldots s_{i_r}}((w^J \bullet \nu)\langle s_{i_1} \ldots s_{i_r} \rangle)$ bijective iff $\langle w^J \bullet \nu + \rho, s_{i_1} \ldots s_{i_{r-1}} \alpha_j \rangle \equiv 0 \mod p$ $[\text{AK} 2.2]$. Thus, if we put $R^+(w) = \{\alpha \in R^+ \mid w\alpha < 0\}$, $w \in W$, and $R^+ = \{\alpha \in R^+ \mid \nu + \rho, \alpha \rangle \equiv 0 \mod p\}$, then $\phi_{w^J} \otimes \{-p(w^J \bullet 0)\}$ annihilates $\text{soc}((w^J)^{-1} \bullet R^+(w) \cap R^+_\nu)_{\hat{\nabla}_J((w^J \bullet \nu)\langle w^J \rangle) \otimes \{-p(w^J \bullet 0)\}}$, and hence

$$\ell \ell w^J \hat{\nabla}_J(\hat{L}^J(\nu)) \geq \ell (w^J) - |R^+(w^J) \cap R^+_\nu| + 1.$$
Likewise, $\phi'_{w^J} \otimes \{-p(w^J \bullet 0)\}$ annihilates $soc^{\ell(w_J) - |(R^+ \setminus R^+(w^J)) \cap R^+_\nu|} \hat{\nabla}_{w_0}((w^J \bullet \nu)\langle w_0 \rangle) \otimes \{-p(w^J \bullet 0)\}$.

We now assume (L) again. As $\ell \hat{\nabla}_{w_0}((w^J \bullet \nu)\langle w_0 \rangle) = N - N(\nu) + 1$ by (2.5),

$$\ell \hat{\nabla}_{w_0}((w^J \bullet \nu)\langle w_0 \rangle) \leq N - N(\nu) + 1 - \{\ell(w_J) - |(R^+ \setminus R^+(w^J)) \cap R^+_\nu|\}.$$  

As $N(\nu) = |R^+(w^J) \cap R^+_\nu| + |(R^+ \setminus R^+(w^J)) \cap R^+_\nu|$, it now follows from (1) and (2) that

$$\ell \hat{\nabla}_{w_0}((w^J \bullet \nu)\langle w_0 \rangle) = \ell(w_J) - |R^+(w^J) \cap R^+_\nu| + 1 = 1 + \dim G/Q - |R^+(w^J) \cap R^+_\nu|$$

$$= |R^+(w^J)| - |R^+(w^J) \cap R^+_\nu| + 1 = 1 + |R^+(w^J) \setminus R^+_\nu|.$$  

Thus

**Theorem:** Assume $p \gg 0$ so that (L) holds. All $\hat{\nabla}_J((\hat{L}^J(\nu)))$, $J \subseteq R^e$, $\nu \in \Lambda$, have Loewy length $1 + |R^+(w^J) \setminus R^+_\nu|$.

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