On diagonal actions of branch groups and the corresponding characters

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Abstract

We introduce a notion of absolute non-free action and show that every weakly branch group acts absolutely non-free on the boundary of the associated rooted tree. Using this result and the symmetrized diagonal actions we show that every branch group has at least countably many different ergodic totally non-free actions (in the terminology of Vershik) and therefore at least countably many II$_1$ factor representations.

1 Introduction.

Totally non-free actions were introduced recently by A. Vershik as an important tool to study factor representations and associated characters of groups (see [24] and [25]). Such actions were used around 1980 in two different contexts: to study characters of symmetric group $S(\infty)$ (see [16]) and for construction of groups of Burnside type (see [9]). The understanding that the action defined in [9] is totally non-free (and, moreover, completely non-free) came around 2000 (see [1]). The fact that the group $G = \langle a, b, c, d \rangle$ constructed in [9] and most of the groups from the family $G_\omega$, $\omega \in \{0, 1, 2\}^\mathbb{N}$, constructed in [10] and generalizing the group $G$, has branch structure was established in [10]. But the class of branch groups was formally defined later. One of the main significances of this class is the fact that (just-infinite) branch groups constitute one of the three subclasses on which the class of just-infinite

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groups splits (11). Also, branch groups may possess many remarkable properties, for instance be amenable but not elementary amenable groups, or have an intermediate growth (between polynomial and exponential).

Weakly branch groups is a more general class of groups acting on spherically homogeneous rooted trees. They were introduced around 2002 after the discovery that the so called Basilica group is a weakly branch group. This group is interesting due to the fact that this is the first example of amenable but not subexponentially amenable group and is an iterated monodromy group of the map \( f(z) = z^2 - 1 \) (15). The account of results and methods of theory of branch and weakly branch groups can be find in [11] and [2].

By definition, a factor representation is a unitary representation of a group that generates a von Neumann algebra that is a factor (i.e. its center consists of scalar operators). The theory of factor representations of groups is closely related to the theory of representations of factors. Both these areas have long history. An important notion associated with a factor representation is a notion of a character. Character of a factor representation \( \pi \) is defined by \( \chi(g) = \text{tr}(\pi(g)) \), where tr is the trace on the factor generated by representation \( \pi \). In fact, representation \( \pi \) can be reconstructed from \( \chi \).

A character is a complex valued function on \( G \) constant on conjugacy classes, positive definite and normalized: \( \chi(e) = 1 \) for the unit \( e \) of the group. Indecomposable characters are extreme points in the simplex of characters and determine factor representation up to quasi-equivalence. In some important cases there are uncountably many (16), or countably many of indecomposable characters (21), and they allow complete description.

On the other hand, there are interesting groups with poor set of indecomposable characters. For instance, Chevalier groups over an infinite discrete field (19) and simple Higman-Thompson groups (5) admit only two characters: the trivial and the regular one.

The old idea coming from studies of characters of \( S(\infty) \) and presented in [16] recently got a new life after A. Vershik had attracted the mathematics community to the notions of totally non-free and extremely non-free actions (two notions coincide for the class of countable groups). Such actions are characterized by the property that on the set of full measure of the phase space different points have different stabilizers, or by the fact that sigma-algebra generated by the sets Fix(\( g \)) of fixed points of elements \( g \in G \) is the whole sigma algebra \( \Sigma \) of measurable subsets of the phase space.

Observe that if a group \( G \) acts by measure-preserving transformations on
a probability space \((X, \Sigma, \mu)\) then the function \(\mu(\text{Fix}(g))\) is a character. The approach based on use of totally non-free actions to study characters and factor representation is quite promising and is based on the result of Vershik from \([24]\) (see Theorem \(\theta\)) which implies that characters corresponding to totally non-free actions are indecomposable. Our main results are stated in Section \(2.4\). A corollary of them is that every weakly branch group has at least countably many different ergodic totally non-free actions, and hence at least countably many indecomposable characters (i.e. at least countably many II\(_1\) factor representations).

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2 Preliminaries.
In this section we give necessary preliminaries on representation theory, groups acting on rooted trees and totally non-free actions, and formulate the main results.

2.1 Factor representations and characters.
We start by briefly recalling some important notions from Theory of Operator Algebras (see \([3]\), \([14]\) and \([15]\) for details).

Definition 1. An algebra \(\mathcal{M}\) of operators acting in a Hilbert space is called a von Neumann (or shortly \(W^*\)-) algebra if it is closed in the weak operator topology. A \(W^*\)-algebra \(\mathcal{M}\) is called a factor if its center is trivial.

Notice that by disintegration theorem (see \([22]\), Theorem 8.21), each von Neumann algebra can be written as a direct integral of factors. In this paper we will be concerned only with finite type \(W^*\)-algebras, that is those which admit a faithful finite trace (normal, normalized, positive-definite function on the algebra).

A unitary representation \(\pi\) of a group \(G\) is called factor representation if the \(W^*\)-algebra \(\mathcal{M}_\pi\) generated by operators \(\pi(g), g \in G\), is a factor.

Definition 2. A character of a group \(G\) is a function \(\chi : G \to \mathbb{C}\) satisfying the following properties:
(1) $\chi(g_1g_2) = \chi(g_2g_1)$ for any $g_1, g_2 \in G$;

(2) the matrix $\{\chi(g_ig_j^{-1})\}_{i,j=1}^{n}$ is nonnegative-definite for any integer $n \geq 1$ and any elements $g_1, \ldots, g_n \in G$;

(3) $\chi(e) = 1$, where $e$ is the identity of $G$.

A character $\chi$ is called **indecomposable** if it cannot be represented in the form $\chi = \alpha \chi_1 + (1 - \alpha) \chi_2$, where $0 < \alpha < 1$ and $\chi_1, \chi_2$ are distinct characters.

The character given by $\chi(g) = 1$ for all $g \in G$ is called the trivial character. The character $\chi(g) = \delta_{g,e}$, where delta stands for the Kronecker delta-symbol, is called the regular character. Notice that the trivial character is indecomposable for any group $G$. The regular character is indecomposable if and only if the group $G$ is ICC (all conjugacy classes except of the class of the unit element are infinite).

Two unitary representations $\pi_1, \pi_2$ of $G$ are called **quasi-equivalent** if there is a von Neumann algebra isomorphism $\omega : M_{\pi_1} \rightarrow M_{\pi_2}$ such that $\omega(\pi_1(g)) = \pi_2(g)$ for all $g \in G$.

There is a bijection between indecomposable characters on $G$ and classes of quasi-equivalence of **finite type** factor representations of a group $G$ via Gelfand-Naimark-Siegel (abbreviated “GNS”) construction. The GNS-construction associates to a character $\chi$ on $G$ a triple $(\pi, H, \xi)$, where $\pi$ is a unitary representation of $G$ acting in a Hilbert space $H$ and $\xi$ is unit vector in $H$ such that $(\pi(g)\xi, \xi) = \chi(g)$ for all $g \in G$.

Moreover, $\xi$ is cyclic (linear combinations of the vectors of the form $m\xi$, where $m \in M_\pi$, are dense in $H$) and separating ($A\xi = 0$ for $A \in M_\pi$ implies $A = 0$) for $M_\pi$. The function

$$\text{tr}(m) = (m\xi, \xi), \quad \text{where } m \in M_\pi,$$

is a trace on $M_\pi$.

One of important classes of examples of characters arises from group actions. Namely, let $(X, \mu)$ be a probability space with a measure preserving action of a group $G$ on it. Then the function

$$\chi(g) = \mu(\text{Fix}(g)), \quad \text{where } \text{Fix}(g) = \{x \in X : gx = x\},$$
is a character. The corresponding representation can be constructed in the following way. Denote by \( \mathcal{R} \) the orbit equivalence relation on \( X \). For \( A \subset X^2 \) and \( x \in X \) set \( A_x = A \cap (X \times \{x\}) \). Introduce a measure \( \nu \) on \( \mathcal{R} \subset X^2 \) by
\[
\nu(A) = \int_X |A_x| d\mu(x).
\]
Notice that also \( \nu(A) = \int_X |A^y| d\mu(y) \), where \( A^y = A \cap (\{y\} \times X) \).

**Definition 3.** The (left) groupoid representation of \( G \) is the representation \( \pi \) in \( L^2(\mathcal{R}, \nu) \) defined by
\[
(\pi(g)f)((x, y)) = f(g^{-1}x, y).
\]
Denote by \( \xi \) the unit vector \( \xi(x, y) = \delta_{x,y} \in L^2(\mathcal{R}, \nu) \), where \( \delta_{x,y} \) is the Kronecker delta-symbol. It is straightforward to verify that
\[
(\pi(g)\xi, \xi) = \mu(\text{Fix}(g)).
\]

Denote by \( \mathcal{M}_\pi \) the \( W^* \)-algebra generated by the operators of the representation \( \pi \). In this paper we are also interested in another algebra. Namely, for a function \( m \in L^\infty(X, \mu) \) introduce operators \( m_l : L^2(\mathcal{R}, \nu) \rightarrow L^2(\mathcal{R}, \nu) \) by
\[
(m_l f)(x, y) = m(x) f(x, y).
\]
Denote by \( \mathcal{M}_\mathcal{R} \) the \( W^* \)-algebra generated by \( \mathcal{M}_\pi \) and operators \( m_l, m \in L^2(X, \mu) \). This algebra is sometimes referred to as Murray-von Neumann or Krieger algebra. Observe that for an ergodic action of a group \( G \) the algebra \( \mathcal{M}_\mathcal{R} \) is a factor of finite type (see [23], Theorem 2.10, or [7], Proposition 2.9(2)) and vector \( \xi \) is cyclic and separating for \( \mathcal{M}_\mathcal{R} \) (see [7], Proposition 2.5).

### 2.2 Groups acting on rooted trees.

In this section we give all necessary preliminaries on groups acting on rooted trees. We refer the reader to [12], [13] and [18] for the details. In this paper we will focus on regular rooted trees. Some of the results of the present paper are true for more general classes of rooted trees (for instance, for spherically homogeneous rooted trees), however, the proofs become more technical.
Let $A_d$ be a finite alphabet and $d \geq 2$ be the number of elements of $A_d$. The vertex set of the regular rooted tree $T = T_d$ determined by $A_d$ is the set of finite words over $A_d$. Two vertices $v$ and $w$ are connected by an edge if one can be obtained from another by concatenation of a letter from $A_d$ at the end of the word. The root of the tree is the empty word $\emptyset$.

**Definition 4.** The automorphism group $\text{Aut}(T)$ is the group of all graph isomorphisms of the tree $T$ onto itself.

Notice that elements of $\text{Aut}(T)$ preserve the root and the levels of $T$. Denote by $V_n$ the set of vertices of the $n$-th level of $T$, that is the set of words of lengths $n$ over $A$. For a vertex $v$ of $T$ denote by $T_v$ the subtree of $T$ with the root vertex $v$. By definition, the boundary $X = \partial T$ of the rooted tree $T$ is the set of all infinite paths without backtracking in $T$ starting from the root. The set $X$ can be naturally identified with the set of all infinite sequences over $A_d$ supplied by the product topology. For $v \in V_n$ let $X_v$ be the set of all paths from $X$ passing through $v$. Equip $X$ with the uniform Bernoulli measure, so that $\mu(X_v) = \frac{1}{d^n}$ for each $v \in V_n$.

Let $G < \text{Aut}(T)$. For $n \in \mathbb{N}$ and a vertex $v \in V_n$ denote by $\text{St}_G(v)$ the subgroup of all elements from $g$ which fix $v$ and by $\text{St}_G(n)$ the subgroup of all elements of $G$ which fix each vertex of $V_n$:

$$\text{St}_G(v) = \{g \in G : gv = v\}, \quad \text{St}_G(n) = \bigcap_{v \in V_n} \text{St}_G(v).$$

Observe that $\text{St}_G(n)$ are normal subgroups of finite index in $G$. By definition, the rigid stabilizer $\text{rist}_G(v)$ of a vertex $v$ is the set of elements $g \in G$ which act trivially on the complement of the subtree $T_v$. The rigid stabilizer of $n$-th level is defined by

$$\text{rist}_G(n) = \langle \text{rist}_G(v) : v \in V_n \rangle.$$

**Definition 5.** A group $G < \text{Aut}(T)$ is called *branch* if it acts transitively on each level $V_n$ of $T$ and for each $n$ $\text{rist}_G(n)$ is a finite index subgroup in $G$. A group $G < \text{Aut}(T)$ is called *weakly branch* if it acts transitively on each level $V_n$ of $T$ and $\text{rist}_G(n)$ is infinite for each $n$ (equivalently, $\text{rist}_G(v)$ is nontrivial for each vertex $v$).

Observe that every branch group is weakly branch. From Proposition 6.5 in [13] it follows that $(G, \partial T, \mu)$ is ergodic iff the action is level transitive. The same holds for the properties of minimality and topological transitivity. In particular, weakly branch groups act on $(\partial T, \mu)$ ergodically.
2.3 Nonfree actions.

Let \((X, \Sigma, \mu)\) be a Lebesgue space (see [20] or [8] for definition and properties of a Lebesgue space). Neglecting a subset of measure 0 without loss of generality we will assume that \((X, \Sigma, \mu)\) is isomorphic to an interval with Lebesgue measure, a finite or countable set of atoms, or a disjoint union of both. A countable collection \(\mathcal{F}\) of measurable subsets of \(X\) separates points of \(X\) if for all \(x \neq y \in X\) there exists \(A \in \mathcal{F}\) such that \(x \in A, y \notin A\) or \(y \in A, x \notin A\). By definition, sigma-algebra generated by a collection of measurable sets \(\mathcal{F}\) and by measure \(\mu\) consists of all measurable sets \(B \subset X\) for which there exists a set \(A\) in the Borel sigma-algebra generated by \(\mathcal{F}\) such that \(\mu(B \Delta A) = 0\) (see [20]). There is a countable collection \(\mathcal{F}\) of measurable subsets of \(X\) separating points of \(X\). By Theorem on Bases (see [20], p. 22), \(\mathcal{F}\) is a basis, that is in addition to separability, it generates \(\Sigma\).

For a measurable automorphism \(g\) of \(X\) denote by Fix\((g)\) the set of fixed points of \(g\): Fix\((g) = \{x \in X : gx = x\}\). In [24] Vershik introduced two notions associated with non-free actions:

**Definition 6.** A measure preserving action of a countable group \(G\) on a Lebesgue space \((X, \Sigma, \mu)\) is called **totally nonfree** if the collection of sets Fix\((g)\), \(g \in G\) generates the sigma algebra \(\Sigma\).

**Definition 7.** A measure preserving action of a group \(G\) on a Lebesgue space \((X, \Sigma, \mu)\) is called **extremely nonfree** if there exists \(A \subset X, \mu(A) = 1\) such that for each \(x, y \in A, x \neq y\) one has St\(_G(x) \neq St_G(y)\).

Vershik in [24] showed that for a countable group \(G\) these two notions coincide:

**Theorem 8.** For a countable group \(G\) an action on a Lebesgue space is totally nonfree if and only if it is extremely nonfree.

Totally non-free actions are important due to the following important result of Vershik from [24]:

**Theorem 9.** Let \(G\) act on \((X, \Sigma, \mu)\) measure-preserving and ergodically. Let \(\pi\) be the corresponding groupoid representation. Then the groupoid algebra \(\mathcal{M}_\pi\) and the Krieger-Murray-von Neumann algebra \(\mathcal{M}_R\) coincide if and only if the action of \(G\) is totally non-free.

Since for an ergodic action of \(G\) the algebra \(\mathcal{M}_R\) is a factor, we obtain:
Corollary 10. Let $G$ act on $(X, \Sigma, \mu)$ measure-preserving, ergodically and totally non-freely. Then the function $\chi(g) = \mu(\text{Fix}(g))$ is an indecomposable character on $G$.

For the following fact see e.g. [12], Proposition 2.2.

**Proposition 11.** Let $G < \text{Aut}(T)$ be weakly branch and $\mu$ be the invariant Bernoulli measure on $X = \partial T$. Then the action of $G$ on $(X, \mu)$ is totally non-free.

### 2.4 Main results.

Let $G$ act measure preserving on a Lebesgue measure space $(X, \Sigma, \mu)$. Consider the space $(X^n, \Sigma^n, \mu^n)$ (the $n$-th Cartesian power of $(X, \Sigma, \mu)$). Here $\Sigma^n$ is the sigma-algebra on $X^n$ generated by the sets of the form $A_1 \times A_2 \times \cdots \times A_n$, where $A_i \in \Sigma$. Denote by $\text{diag}_n$ the diagonal action of $G$ on $(X^n, \mu^n)$:

$$\text{diag}_n(g)(x_1, \ldots, x_n) = (gx_1, \ldots, gx_n).$$

Notice that this action cannot be totally non-free, unless $\mu$ is concentrated in one point or $n = 1$, since

$$\text{Fix}(\text{diag}_n(g)) = \text{Fix}(g)^n = \text{Fix}(g) \times \text{Fix}(g) \times \cdots \times \text{Fix}(g) \subset X^n$$

for all $g \in G$ and sets of this form cannot generate the sigma-algebra $\Sigma^n$ if $n > 1$ and $\mu$ is not concentrated at one point.

Observe that $\text{diag}_n$ commutes with the action of symmetric group $\text{Sym}(n)$ permuting factors of $X^n = X \times \cdots \times X$ and thus factors through the quotient $X^n_{\text{Sym}} := X^n / \text{Sym}(n)$. The measure $\mu^n$ projects naturally to an invariant measure $\mu^n_{\text{Sym}}$ on $X^n_{\text{Sym}}$. Denote by $\gamma_n : X^n \to X^n_{\text{Sym}}$ the quotient map and by $\Sigma_n$ the sigma algebra of measurable sets on $X^n_{\text{Sym}}$. Denote by $S_n(A) := \gamma_n^{-1}(\gamma_n(A))$ the symmetrization of a subset $A \subset X^n$, that is the minimal subset of $X^n$ containing $A$ and invariant under the action of $\text{Sym}(n)$.

Now, let $G$ be a group acting on a rooted tree $T$, $X = \partial T$, $\mu$ be the uniform Bernoulli measure on $X$ and $\Sigma$ be the sigma-algebra of all measurable subsets of $X$. We are ready to formulate the main results of the paper:

**Theorem 12.** For any countable weakly branch group $G$ and any $n$ the action $\text{diag}_n$ of $G$ on $(X^n_{\text{Sym}}, \Sigma^n_{\text{Sym}}, \mu^n_{\text{Sym}})$ is totally non-free.
Recall that the action of a weakly branch group on the boundary of the corresponding rooted tree because of the level transitivity is ergodic with respect to the unique invariant measure. It is not hard to see that for $n \geq 2$ the action $\text{diag}_n$ of $G$ on $(X_{n\text{Sym}}, \Sigma_{n\text{Sym}}, \mu_{n\text{Sym}})$ is not ergodic. Nevertheless, the following holds:

**Theorem 13.** For any countable branch group $G$ the action $\text{diag}_n$ of $G$ on $(X_{n\text{Sym}}, \Sigma_{n\text{Sym}}, \mu_{n\text{Sym}})$ splits into countably many ergodic components.

Further, let $G$ be a branch group. Denote by $\mathcal{E}_n$ the set of ergodic components of the action $\text{diag}_n$ of $G$ on $X_{n\text{Sym}}$. For $\alpha \in \mathcal{E}_n$ denote by $X_\alpha$ the corresponding subset of $X_{n\text{Sym}}$, $\Sigma_\alpha$ the sigma algebra of measurable subsets of $X_\alpha$ and $\mu_\alpha$ the normalized restriction of $\mu_{n\text{Sym}}$ onto $X_\alpha$ (so that $\mu_\alpha(X_\alpha) = 1$). Since by Theorem 12 the sets $\text{Fix}(\text{diag}_n(g)), g \in G$ generate the sigma algebra $\Sigma_{n\text{Sym}}$, the sets $\text{Fix}(\text{diag}_n(g)) \cap X_\alpha, g \in G$ for each $\alpha$ generate the sigma algebra $\Sigma_\alpha$. Thus, the action of $G$ on $(X_\alpha, \Sigma_\alpha, \mu_\alpha)$ is totally non-free. Introduce a function $\chi_\alpha$ on $G$ by

$$\chi_\alpha(g) = \mu_\alpha(\text{Fix}(\text{diag}_n(g)) \cap X_\alpha).$$

From Corollary 10 we obtain

**Corollary 14.** The characters $\chi_\alpha$ are indecomposable.

Also, we prove the following:

**Theorem 15.** The characters $\chi_\alpha$, where $\alpha \in \mathcal{E}_n$ and $n \in \mathbb{N}$, are pairwise distinct.

**Corollary 16.** Every branch group has at least countably many indecomposable characters and countably many pairwise not isomorphic totally non-free actions.

Another result of this paper concerns quasi-regular representations and commensurator subgroups.

**Definition 17.** Let $G$ be a countable group, $H < G$ and $G/H$ be the set of right cosets in $G$ by $H$. A quasi-regular representation $\rho_{G/H}$ of $G$ is the unitary representation acting on $l^2(G/H)$ by:

$$(\rho_{G/H}(g)f)(x) = f(g^{-1}x), \quad x \in G/H.$$
Definition 18. Let $H \leq G$. The commensurator of $H$ is the subgroup of $G$ defined by

$$\text{comm}_G(H) = \{g \in G : H \cap gHg^{-1} \text{ has finite index in } H \text{ and } gHg^{-1}\}.$$ 

Mackey [17] showed the following

Theorem 19. Let $H$ be a subgroup of an infinite discrete countable group $G$. Then the quasi-regular representation $\rho_{G/H}$ is irreducible if and only if $\text{comm}_G(H) = H$.

In [1] Bartholdi and Grigorchuk showed that for every weakly branch group $G$ acting on a rooted tree $T$ the action of $G$ on $(\partial T, \Sigma, \mu)$ is completely non-free (i.e. the stabilizers $\text{St}_G(x)$ are pairwise distinct for all $x \in X$) and all representations $\rho_x = \rho_{G/\text{St}_G(x)}$, $x \in \partial T$, are irreducible. We prove

Theorem 20. Let $G$ be an arbitrary countable group acting ergodically and totally non-freely on a Lebesgue space $(X, \Sigma, \mu)$. Then the quasi-regular representations $\rho_x$ are irreducible for almost all $x \in X$, and therefore $\text{comm}_G(\text{St}_G(x)) = \text{St}_G(x)$ for almost all $x \in X$.

3 Non-free actions and the corresponding characters.

As some details of proofs of Theorems 8 and 9 are only sketched in [24] we give here proofs of these Theorems in full details.

Proof of Theorem 8 Assume that an action of $G$ on $(X, \Sigma, \mu)$ is extremely nonfree. Let $A$ be the set from the definition of the extremely non-free action. Let $x, y \in A, x \neq y$. Then there exists $g \in G$ for which $x \in \text{Fix}(g), y \notin \text{Fix}(g)$. Thus the collection of sets $\text{Fix}(g), g \in G$ separates points from $A$. By Theorem on bases from [20] (see page 22) the collection of sets $\text{Fix}(g), g \in G$ is a basis for the Lebesgue space $(A, \Sigma \cap A, \mu|_A)$, that is it generates the sigma algebra of all measurable sets on $A$, and so on $X$ as well. Here $\Sigma \cap A = \{B \in \Sigma : B \subset A\}$ and $\mu|_A$ is the restriction of the measure $\mu$ on the subset $A$.

Further, assume that the action of $G$ on $(X, \Sigma, \mu)$ is totally nonfree. Denote by $\Xi$ the partition on $X$ generated by the sets $\text{Fix}(g), g \in G$. The atoms
of this partition are of the form

\[ \bigcap_{g \in G} A_g, \text{ where } A_g = \text{Fix}(g) \text{ or } A_g = X \setminus \text{Fix}(g) \text{ for each } g \in G. \]

In other words, \( x, y \) belong to the same atom \( \xi \) of the partition \( \Xi \) if and only if \( \text{St}_G(x) = \text{St}_G(y) \). Since \( \text{Fix}(g), g \in G \), generate the sigma algebra of measurable sets on \((X, \Sigma, \mu)\), each measurable set \( B \) can be represented as \( B = \bigcup_i B_i \), where \( B_i \) is a union of some atoms of the partition \( \Xi \) and \( \mu(B) = 0 \).

Let \( B_i, i \in \mathbb{N} \), be a basis of measurable sets on \((X, \Sigma, \mu)\). Let \( B_i = B_i \setminus \text{Fix}(g) \) for each \( g \in G \). By definition of the basis, the sets \( B_i \) separate points of \( X \). In particular, for each \( x \neq y \in X \setminus K \) there exists \( B_j \) such that \( x \in B_j, y \notin B_j \). Then \( x \in B_j \Xi, y \notin B_j \Xi \). It follows that \( x, y \) are not in the same atom of \( \Xi \) and \( \text{St}_G(x) \neq \text{St}_G(y) \), which finishes the proof.

Before proving Theorem 9 we need to do some preliminary work. Similarly to the left groupoid representation of \( G \) introduce the right groupoid representation \( \tilde{\pi} \) of \( G \) in \( L^2(\mathcal{R}, \nu) \) by

\[ (\tilde{\pi}(g) f)((x, y)) = f(x, g^{-1}y). \]

For a function \( m \in L^\infty(X, \mu) \) introduce operator \( m_r : L^2(\mathcal{R}, \nu) \to L^2(\mathcal{R}, \nu) \) by

\[ (m_r f)(x, y) = m(y)f(x, y). \]

Denote by \( \mathcal{M}_\tilde{\pi} \) the \( W^* \)-algebra generated by \( \mathcal{M}_\tilde{\pi} \) and operators \( m_r, m \in L^2(\mathcal{R}, \nu) \). Notice that \( \mathcal{M}_\tilde{\pi} = \mathcal{M}_\tilde{\pi}^\prime \) and \( (\mathcal{M}_\tilde{\pi})^\prime = \mathcal{M}_\tilde{\pi} \) (5, Proposition 2.5). As before, let \( \xi = \delta_{x,y} \in L^2(\mathcal{R}, \nu) \).

**Remark 1.** For every \( g \in G \) one has: \( \pi(g)\xi = \tilde{\pi}(g^{-1})\xi \). In particular, vector \( \xi \) is cyclic for \( \mathcal{M}_\pi \) if and only if it is cyclic for \( \mathcal{M}_\tilde{\pi} \).

Introduce a unitary representation \( \rho \) of \( G \times G \) in \( L^2(\mathcal{R}, \nu) \) by

\[ \rho(g_1, g_2) = \pi(g_1)\tilde{\pi}(g_2). \]

**Proposition 21.** The following assertions are equivalent:
1) \( \mathcal{M}_x = \mathcal{M}_\mathcal{R} \);
2) \( \rho \) is irreducible;
3) the unit vector \( \xi = \delta_{x,y} \) is cyclic for \( \mathcal{M}_\pi \) (equivalently, for \( \mathcal{M}_\tilde{\pi} \)).
Proof. 1) ⇒ 2). Assume that $\mathcal{M}_\pi = \mathcal{M}_R$. Then by symmetry $\mathcal{M}_{\bar{\pi}} = \mathcal{M}_{\bar{R}}$. Notice that the commutant of $\rho$ is a subset of

$$\mathcal{M}'_\pi \cap \mathcal{M}'_{\bar{\pi}} = \mathcal{M}'_R \cap \mathcal{M}'_{\bar{R}} = \mathcal{M}'_R \cap \mathcal{M}_R = \mathbb{C}\text{Id},$$

where $\text{Id}$ is the identity operator and $\mathbb{C}\text{Id}$ is the set of scalar operators in $L^2(\mathcal{R}, \nu)$. Therefore, $\rho$ is irreducible.

2) ⇒ 3). Assume that $\rho$ is irreducible. Then $\xi$ is cyclic with respect to the algebra generated by $\rho$. Since $\rho((g, h))\xi = \pi(gh^{-1})\xi$ for all $g, h$ it follows that $\xi$ is cyclic with respect to $\mathcal{M}_\pi$.

3) ⇒ 1). Since $\mu$ is invariant with respect to $G$, the modular operator and the modular automorphism group corresponding to the trace $\text{tr}$ on $\mathcal{M}_R$ are trivial (see e.g. [7], Proposition 2.8). By Theorem 4.2 from [23] there exists a conditional expectation $E : \mathcal{M}_R \to \mathcal{M}_\pi$, that is a linear map such that

1) $\|E(x)\| \leq \|x\|$ and $E(x)^*E(x) \leq E(x^*x)$ for all $x \in \mathcal{M}_R$;

2) $E(x) = x$ for all $x \in \mathcal{M}_\pi$;

3) $\text{tr} \circ E = \text{tr}$;

4) $E(axb) = aE(x)b$ for all $a, b \in \mathcal{M}_\pi$ and $x \in \mathcal{M}_R$.

It follows that for all $x \in \mathcal{M}_R$ one has

$$\|E(x)\xi\|^2 = \text{tr}(E(x)^*E(x)) \leq \text{tr}(E(x^*x)) = \text{tr}(x^*x) = \|x\xi\|^2.$$

This implies that the map

$$\mathcal{M}_R\xi \to \mathcal{M}_\pi\xi, \quad x\xi \to E(x)\xi, \quad x \in \mathcal{M}_R$$

is well defined and extends to a bounded linear operator $E$ on $L^2(\mathcal{R}, \nu)$ of norm $\|E\| \leq 1$. Moreover, $E$ is identical on the cyclic hull of $\xi$ under $\mathcal{M}_\pi$. Therefore, if $\xi$ is cyclic with respect to $\mathcal{M}_\pi$, then $E = \text{Id}$, $E(x)\xi = x\xi$ and thus $E(x) = x$ for all $x \in \mathcal{M}_R$. This implies that $\mathcal{M}_\pi = \mathcal{M}_R$. □

Now we are ready to prove Theorem 9.

12
Proof of Theorem 9. Assume that the action of $G$ is not totally nonfree. Let $\mathcal{U}_G$ be the sigma-algebra generated by Borel sets $\text{Fix}(g), g \in G$ and by measure $\mu$. Then by Theorem 8 the sigma-algebra $\mathcal{U}_G$ is strictly smaller than the sigma-algebra $\Sigma$ of all measurable subsets of $X$. Let $G$ act on $\mathcal{R}$ by $(x, y) \rightarrow (gx, y)$. Consider the embedding $I : X \rightarrow \mathcal{R}$ given by $I(x) = (x, x)$. Denote by $\mathcal{U}^G_\mathcal{R}$ the minimal $G$-invariant sigma-algebra on $\mathcal{R}$ containing $I(\mathcal{U}_G)$ and all zero measure subsets of $(\mathcal{R}, \nu)$. Observe that for all $A \in \mathcal{U}_G$ and $g \in G$ one has

$$gI(A) \cap I(X) = I(\text{Fix}(g) \cap A) \in I(\mathcal{U}_G).$$

It follows that

$$\mathcal{U}^G_\mathcal{R} \cap I(X) = I(\mathcal{U}_G).$$

In particular, $\mathcal{U}^G_\mathcal{R}$ is strictly smaller than the sigma-algebra of all measurable subsets of $(\mathcal{R}, \nu)$. Since $\xi$ is $\mathcal{U}^G_\mathcal{R}$-measurable, we obtain that it is not cyclic in $L^2(\mathcal{R}, \nu)$, therefore $\mathcal{M}_\pi$ does not coincide with $\mathcal{M}_\mathcal{R}$. This proves the statement of Theorem 9 in one direction.

Assume that the action of $G$ on $(X, \mu)$ is totally nonfree. By Proposition 21 to prove the other direction of the statement of Theorem 9 it is sufficient to show that $\xi$ is cyclic under $\mathcal{M}_\pi$. Let $\eta \in L^2(\mathcal{R}, \nu)$ and $(\eta, \pi(g)\xi) = 0$ for all $g \in G$. We need to show that $\eta = 0$. Observe that the collection of subsets

$$C_g = \{(gx, x) : x \in X\} \subset \mathcal{R}$$

separates almost all points of $\mathcal{R}$. Indeed, assume that $(x_1, y_1), (x_2, y_2) \in C_g$ for some $g$. Almost surely $\text{St}_G(x_1) \neq \text{St}_G(x_2)$. Let $h \in \text{St}_G(x_1) \setminus \text{St}_G(x_2)$. Then $(x_1, y_1) \in C_{hg}$, but also $(hx_2, y_2) \in C_{hg}$, and thus $(x_2, y_2) \notin C_{hg}$.

Further, by Theorem on bases from [20] (see page 22) the collection of sets $\text{Fix}(g), g \in G$, is a basis (modulo zero measure) for the Lebesgue space $(\mathcal{R}, \nu)$. Thus

$$\int_{C_g} \eta(p) d\nu(p) = (\eta, \pi(g)\xi) = 0$$

for all $g \in G$ implies that $\eta(p) = 0$ for almost all $p \in \mathcal{R}$ which finishes the proof.
4 Absolutely non-free actions of weakly branch groups.

In this section we introduce and study a notion of an absolutely non-free action and, using this notion, prove Theorem 12. Let $A \Delta B$ stand for symmetric difference of the sets $A$ and $B$.

**Definition 22.** An action of a group $G$ on a measure space $(X, \Sigma, \mu)$ is called absolutely non-free if for every measurable set $A$ and every $\epsilon > 0$ there exists $g \in G$ such that $\mu(\text{Fix}(g) \Delta A) < \epsilon$.

Obviously, every absolutely non-free action is totally non-free. Indeed, totally non-freeness of an action means only that any measurable set can be approximated arbitrarily well by finite unions of finite intersections of the sets of the form $\text{Fix}(g)$, where $g \in G$. Thus, total non-freeness (and so extreme non-freeness as well) is a weaker condition, than absolute non-freeness. We will show that any weakly branch group acts on the boundary of the corresponding regular rooted tree absolutely non-free. In fact, we will prove even stronger statement. Denote by $\text{supp}(g)$ the set of points $x \in X = \partial T$ such that $gx \neq x$. Thus, $\text{supp}(g) = X \setminus \text{Fix}(g)$.

**Proposition 23.** Let $G$ be a weakly branch group, $T$ be the corresponding regular rooted tree, $X = \partial T$ and $\mu$ be the unique invariant measure on $X$. Then for any clopen subset $A \subset X$ and any $\epsilon > 0$ there exists $g \in G$ such that $\text{supp}(g) \subset A$ and $\mu(A \setminus \text{supp}(g)) < \epsilon$.

In particular, the action of $G$ on $(\partial T, \Sigma, \mu)$ is absolutely non-free.

First we prove a combinatorial lemma.

**Lemma 24.** Let $n \in \mathbb{N}$ and $H < \text{Sym}(n)$ be a subgroup acting transitively on $\{1, 2, \ldots, n\}$. Let $A$ be a subset of $\{1, \ldots, n\}$ such that for all $g \neq h \in H$ one has $|g(A) \Delta h(A)| \leq |A|$, where $|A|$ is the cardinality of $A$. Then $|A| > n/2$.

**Proof.** Set $k = |A|$. Then for all $h \neq g \in H$ one has

$$|h(A) \cap g(A)| = \frac{1}{2}(|h(A)| + |g(A)| - |h(A) \Delta g(A)|) \geq k/2.$$
For each \( h \in H \) introduce a vector \( \xi_h \in \mathbb{C}^n \) by the rule:
\[
\xi_h = (x_1, \ldots, x_n), \quad \text{where} \quad x_i = \begin{cases} 
0, & \text{if } i \notin h(A), \\
1, & \text{if } i \in h(A). 
\end{cases}
\]

Let \((\cdot, \cdot)\) be the standard scalar product in \( \mathbb{C}^n \) and \( \| \cdot \| \) be the corresponding norm. Then \( \| \xi_h \| = k \) and \( (\xi_h, \xi_g) = |h(A) \cap g(A)| \geq k/2 \) for all \( g \neq h \in H \).
It follows that
\[
\| \sum_{h \in H} \xi_h \| \geq \frac{k}{2}m(m + 1), \quad \text{where} \quad m = |H|.
\]

On the other hand, the group \( H \) acts on \( \mathbb{C}^n \) by permuting coordinates and \( g(\xi_h) = \xi_{gh} \) for all \( g, h \in H \). Since \( H \) acts transitively on \( \{1, \ldots, n\} \) and the vector \( \sum_{h \in H} \xi_h \) is fixed by \( H \), we obtain:
\[
\sum_{h \in H} \xi_h = \left( \frac{km}{n}, \frac{km}{n}, \ldots, \frac{km}{n} \right), \quad \| \sum_{h \in H} \xi_h \| = \frac{k^2m^2}{n}.
\]

It follows that \( km \geq n \frac{m+1}{2} \) and \( k > \frac{n}{2} \).

Let \( d \geq 2 \) be the valency of the regular rooted tree \( T \). The proof of Proposition 23 is based on the following:

**Lemma 25.** If \( G \) is weakly branch then for every vertex \( v \) there exists \( g \in G \) with \( \text{supp}(g) \subset X_v \) such that \( \mu(\text{supp}(g)) \geq \frac{1}{d} \mu(X_v) \).

**Proof.** For an element \( h \in G \) let \( l(h) = \max\{l : h \in \text{St}_G(l)\} \}. \) Fix a vertex \( v \) of the tree. Since \( G \) is weakly branch there exist \( g \neq \text{Id} \) with \( \text{supp}(g) \subset X_v \).
Set
\[
L = \min\{l(g) : g \in G, g \neq \text{Id}, \text{supp}(g) \subset X_v\}.
\]

For \( h \in \text{Aut}(T) \) denote by \( \sigma_h \) the permutation induced by \( h \) on the set \( V_L \) of vertices of level \( L \) in \( T \). For a vertex \( v \in V_n, n \in \mathbb{N} \) and \( l \geq n \) set \( V_l(v) = T_v \cap V_l \). For \( g \in G \) denote by \( W(g) \) the set of vertices \( w \) from \( V_L(v) \) such that \( g \) induces a nontrivial permutation on \( V_{L+1}(w) \). Set
\[
k(g) = |W(g)|, \quad K = \max\{k(g) : g \in G, \text{supp}(g) \subset X_v\}.
\]

By the choice of \( L, K > 0 \). Fix an element \( g \in G \) with \( \text{supp}(g) \subset X_v \) such that \( k(g) = K \).
Further, since $G$ acts transitively on $V_L$, we can find a number $m$ and a collection of elements $H = \{h_1, h_2, \ldots, h_m\} \subset G$ such that the family $S = \{\sigma h_0, \sigma h_1, \ldots, \sigma h_m\}$ of transformations of $V_L$ forms a group preserving $V_L(v)$ and transitive on $V_L(v)$. Denote $g_i = h_i g h_i^{-1}$. One has:

$$W(g) = \sigma_n(W(g)), \ W(g, g_j) \supset W(g) \Delta W(g_j)$$

for all $i, j$. It follows that the set $W(g)$ together with the group $S$ restricted to $V_L(v)$ satisfy the conditions of Lemma 24. Therefore, $K = |W(g)| > \frac{1}{2}|V_L(v)|$. For each $w \in W(g) \subset V_L(v)$ the element $g$ induces a non-trivial permutation of $V_L(w)$, and thus $\text{supp}(g) \supset X_w$. This implies that $\mu(\text{supp}(g)) > \frac{1}{d} \mu(X_w)$.

**Proof of Proposition 23** Let $A$ be any clopen set and $g_0 = \text{Id}$. Construct by induction elements $g_n \in G, n = 0, 1, 2 \ldots$ such that $\text{supp}(g_n) \subset A$ and $\mu(A \setminus \text{supp}(g_n)) \leq \left(\frac{d}{d+1}\right)^n$. If $g_n$ is constructed choose vertices $v_1, \ldots, v_k$ such that $X_{v_j}$ are disjoint subsets of $A \setminus \text{supp}(g_n)$ and $\sum \mu(X_{v_j}) \geq \frac{d}{d+1} \mu(A \setminus \text{supp}(g_n))$. Using the lemma construct elements $h_1, \ldots, h_k$ such that $\text{supp}(h_j) \subset X_{v_j}$ and $\mu(\text{supp}(h_j)) \geq \frac{1}{d} \mu(X_{v_j})$. Set $g_{n+1} = g_n h_1 h_2 \ldots h_k$. Then $\mu(A \setminus \text{supp}(g_{n+1})) \leq \frac{d}{d+1} \mu(A \setminus \text{supp}(g_n))$, which finishes the proof.

**Proposition 26.** Assume that the action of $G$ on $(X, \Sigma, \mu)$ is absolutely non-free. Then for any $n$ the action $\text{diag}_n$ of $G$ on $(X^n, \Sigma^n, \mu^n)$ is totally non-free.

**Proof.** The sigma-algebra $\gamma_n^{-1}(\Sigma^n_{\text{Sym}})$ is generated by the sets of the form

$$S_n(A_1 \times A_2 \times \cdots \times A_n) = \bigcup_{s \in \text{Sym}(n)} A_{s(1)} \times A_{s(2)} \times \cdots \times A_{s(n)}, \quad (1)$$

where $A_i$ are disjoint measurable subsets of $X$. Set

$$B = \bigcup_{i=1, \ldots, n} A_i, \ B_j = B \setminus A_j, \ j = 1, \ldots, n.$$  

Then we have

$$S_n(A_1 \times A_2 \times \cdots \times A_n) = B^n \setminus \bigcup_{j=1, \ldots, n} B^n_j.$$  

Thus, $\gamma_n^{-1}(\Sigma^n_{\text{Sym}})$ is generated by the sets of the form $A^n$, where $A \subset X$ is measurable. Since $\text{diag}_n(g)$ is absolutely non-free, any subset of $X^n$ of the
form $A^n$, where $A \in \Sigma$, can be approximated by the measure $\mu^n$ with any precision by the sets of the form $\text{Fix}(\text{diag}_n(g)) = \text{Fix}(g)^n$. This finishes the proof.

Now, Theorem 12 follows immediately from Propositions 23 and 26.

5 Ergodic components of the diagonal actions.

In this section we prove Theorem 13.

Let $G$ be a branch group acting on a regular rooted tree $T$, $X = \partial T$ and $\mu$ be the unique invariant ergodic measure on $X$. Fix $n \in \mathbb{N}$. First observe that since $(X^n_{\text{Sym}}, \Sigma^n_{\text{Sym}}, \mu^n_{\text{Sym}})$ is a finite quotient of $(X^n, \Sigma^n, \mu^n)$ it is sufficient to show that the action $\text{diag}_n$ of $G$ on $(X^n, \Sigma^n, \mu^n)$ admits at most countably many ergodic components. For a level $k \geq \log_2 n$ and an $n$-tuple $v = (v_1, \ldots, v_n)$ of distinct vertices from $V_k$ set

$$X_v = X_{v_1} \times X_{v_2} \times \cdots \times X_{v_n} \subset X^n.$$

Observe that union of the subsets of the form $S_n(X_v)$ (see (1)) of $X^n$ for all levels $k$ and all $n$-tuples $v$ of $n$ distinct vertices from $V_k$ is dense in $X^n$. Introduce stabilizer of $X_v$ in $G$ by:

$$G_v = \{g \in G : \text{diag}_n(g)X_v = X_v\}.$$

Clearly, to prove Theorem 13 it is sufficient to show for each $k$ and each $v$ that for the action of $G_v$ the space $X_v$ splits into finitely many ergodic components.

Lemma 27. For each vertex $u$ of $T$ the action of $\text{rist}_G(u)$ on $X_u$ has finitely many ergodic components.

Proof. Let $u \in V_k$. Since $G$ is branch, the subgroup

$$\text{rist}_G(k) = \prod_{w \in V_k} \text{rist}_G(w)$$

is of finite index in $G$. Let $m = |G/\text{rist}_G(k)|$, $g_1, \ldots, g_m$ be representatives of the right cosets $G/\text{rist}_G(k)$ and $A \subset X_u$ be a measurable subset of positive

17
measure invariant under the action of \( \text{rist}_G(u) \). Then \( A \) is invariant under the action of \( \text{rist}_G(k) \) and

\[
B = \bigcup_{i=1}^m g(A_i)
\]

is invariant under the action of \( G \). It follows that \( \mu(B) = 1 \) and \( \mu(A) \geq \frac{1}{m} \). This shows that the action of \( \text{rist}_G(u) \) on \( X_u \) has at most \( m \) ergodic components.

Let \( v = (v_1, \ldots, v_n) \) be an \( n \)-tuple with pairwise distinct \( v_i \in V_k \). From Lemma 27 we obtain that the restriction of the action \( \text{diag}_n \) onto \( \prod_{i=1}^n \text{rist}_G(v_i) \) admits finitely many ergodic components in \( X_v \). Since

\[
G_v \supset \text{rist}_G(k) \supset \prod_{i=1}^n \text{rist}_G(v_i)
\]

we obtain that \( X_v \) has finitely many ergodic components with respect to the action \( \text{diag}_n \) of \( G_v \), which finishes the proof of Theorem 13.

In the next statement we show that Theorem 13 fails when replacing “branch” by “weakly branch”.

**Proposition 28.** There exists a weakly branch group \( G \) acting on a regular rooted tree \( T \) and \( n \in \mathbb{N} \) such that the action \( \text{diag}_n \) of \( G \) on \( (\partial T^n, \Sigma^n, \mu^n) \) has no ergodic components of positive measure.

**Proof.** Consider the binary rooted tree \( T_2 \). For \( v \in V_n \) denote by \( \sigma_v \) the switch of the branches of \( T_2 \) emitting from \( v \). Consider the group \( G \) generated by all elements of the form:

\[
1) \ \sigma_v, \ v \in V_l, \ l \text{ is even}; \quad 2) \ h_l = \prod_{v \in V_l} \sigma_v, \ l \text{ is odd}.
\]

Notice that \( G \) is weakly branch. Indeed, any vertex \( u \in V_n, \ n \in \mathbb{N} \), can be encoded by a finite sequence \( u(1), u(2), \ldots, u(n) \), where \( u(j) \in \{0, 1\} \) for all \( j \). Using the generators of \( G \) we can change any element of the sequence \( u(j) \) not affecting the other elements of the sequence. Thus, we can obtain any other sequence of 0-s and 1-s, which means that \( G \) is spherically transitive. On the other hand, for any vertex \( u \) and any vertex \( v \in T_u \) from an even level one has \( \sigma_v \in G \). Therefore, \( \text{rist}_G(u) \) is nonempty.
Observe that $G$ is not branch. Indeed, for any element $g \in \text{rist}_G(1) < \text{St}_G(1)$, the shortest representation of $g$ in terms of the generators of $G$ do not contain generators of the form 2), and thus $G/\text{rist}_G(1)$ is infinite.

Fix an even $n$. Let us show that the action $\text{diag}_n$ of $G$ on $(X^n, \Sigma^n, \mu^n)$, where $X = \partial T$, does not have an ergodic component $A$ such that $\mu^n(A) > 0$.

Fix $j$. For $v = (v_1, \ldots, v_n) \in V^n_{2j}$ set

$$r_i(v) = \sum_{p=1}^n v_p(2i) \pmod{2}, \quad r(v) = (r_1(v), \ldots, r_j(v)).$$

It is not hard to show that $r(v)$ is invariant under the action $\text{diag}_n$ of $G$. Indeed, for any $l$ and any $u \in V_l$ the element $\sigma_u$ changes only the $l + 1$-st coordinate of any vertex. Thus,

$$(\sigma_u(v_p))(2i) = v_p(2i) \text{ for all } l \neq 2i - 1 \text{ and all } p,$$

which implies that

$$r_i(\text{diag}_n(\sigma_u)(v)) = r_i(\text{diag}_n(h_l(v))) = r_i(v)$$

whenever $l \neq 2i - 1$. If $l = 2i - 1$, then $(h_l(v_p))(2i) = 1 - v_p(2i)$ for each $p$. Therefore, since $n$ is even, $r_i(h_l(v)) = r_i(v)$.

Let $r = (r_1, \ldots, r_j) \in \{0, 1\}^j$. Denote

$$Y(r) = \bigcup_{v \in V^n_{2j}, r(v) = r} Y_v.$$

Then sets $Y(r)$ are invariant under the action of $\text{diag}_n(G)$, form a partition of $X^n$ and are of measure $2^{-j}$ each. Thus, $X^n$ can be split into invariant subsets of arbitrarily small measure and so cannot have ergodic components of positive measure.

\section{Characters}

Fix a countable branch group $G$ acting on a regular rooted tree $T$. As before, let $X = \partial T$, let $\Sigma$ be the sigma-algebra of all measurable subsets of $X$ and let $\mu$ be the unique invariant ergodic measure on $X$. Fix $n \in \mathbb{N}$ and consider the action $\text{diag}_n$ of $G$ on $X^n_{\text{sym}}$. Recall that $\mathcal{E}_n$ stands for the set
of ergodic components of this action. By Theorem 13, \( \mathcal{E}_n \) is countable. For \( \alpha \in \mathcal{E}_n \) denote by \( X_\alpha \) the corresponding ergodic component, by \( \Sigma_\alpha \) the sigma algebra of measurable subsets of \( X_\alpha \) and by \( \mu_\alpha \) the normalized restriction of the measure \( \mu^n_{\text{Sym}} \) onto \( X_\alpha \) (so that \( \mu_\alpha(X_\alpha) = 1 \)). Introduce a function \( \chi_\alpha \) on \( G \) by

\[
\chi_\alpha(g) = \mu_\alpha(\text{Fix}(\text{diag}_n(g)) \cap X_\alpha).
\]

From Theorems 9, 12 and 13 we obtain that the characters \( \chi_\alpha \) are indecomposable (Corollary 10). Now we are ready to prove that these characters are pairwise distinct.

**Proof of Theorem 15.** Let \( \alpha \in \mathcal{E}_n \) and \( \beta \in \mathcal{E}_m \) for some \( n, m \). Assume that \( \chi_\alpha = \chi_\beta \). Then

\[
\mu_\alpha(\text{Fix}(\text{diag}_n(g))) = \mu_\beta(\text{Fix}(\text{diag}_m(g)))
\]

for all \( g \in G \). Set

\[
Y_\alpha = \gamma_n^{-1}(X_\alpha), \quad Y_\beta = \gamma_m^{-1}(X_\beta), \quad c_\alpha = \mu^n(Y_\alpha), \quad c_\beta = \mu^m(Y_\beta).
\]

By absolute non-freeness of the action of \( G \) on \( (X, \Sigma, \mu) \) (see Proposition 23) and the definition of the measures \( \mu_\alpha, \mu_\beta \) we obtain that

\[
\frac{1}{c_\alpha} \mu^n(A \cap Y_\alpha) = \frac{1}{c_\beta} \mu^m(A \cap Y_\beta) \quad \text{for all} \quad A \in \Sigma.
\]  

(3)

**Case 1:** \( n = m \) and \( \alpha \neq \beta \). Introduce inductively collections \( B_k \) of measurable subsets of \( X^n \) as follows. Set \( B_0 = \{ A^n : A \subset X \text{ measurable} \} \). Denote by \( B_{k+1} \) the set of subsets of the form \( B_1 \sqcup B_2 \), where \( B_1, B_2 \) are disjoint subsets from \( B_k \), or \( B_1 \setminus B_2 \), where \( B_1 \supset B_2 \) are subsets from \( B_k \). Clearly, \( B_{k+1} \supset B_k \) for all \( k \). From (3) using induction it is easy to show that

\[
\frac{1}{c_\alpha} \mu^n(C \cap Y_\alpha) = \frac{1}{c_\beta} \mu^m(C \cap Y_\beta) \quad \text{for any} \quad k \quad \text{and all} \quad C \in B_k.
\]

Let us show by induction that for any \( k \) there exists \( N(k) \) such that for any \( B_1, B_2 \in B_k \) one has:

\[
B_1 \cup B_2, B_1 \cap B_2, B_1 \setminus B_2 \in B_{N(k)}.
\]

Since \( B_1 \setminus B_2 = B_1 \setminus (B_1 \cap B_2) \) and \( B_1 \cup B_2 = (B_1 \setminus B_2) \sqcup (B_1 \cap B_2) \sqcup (B_2 \setminus B_1) \) it is sufficient to consider the \( B_1 \cap B_2 \) case only.
Base of induction, when \( k = 0 \), is obvious, since \( A_1 \cap A_2 = (A_1 \cap A_2)^n \).

Step of induction. Assume that the statement is true for \( k \). Let \( B_1, B_2 \in B_{k+1} \). Then for \( i = 1, 2 \) there exists \( B_{i1}, B_{i2} \in B_k \) such that either \( B_i = B_{i1} \cup B_{i2} \) or \( B_i = B_{i1} \setminus B_{i2} \) and \( B_{i1} \supset B_{i2} \). Assume for instance that

\[
B_1 = B_{11} \cup B_{12} \quad \text{and} \quad B_2 = B_{21} \setminus B_{22}, \quad \text{where} \quad B_{21} \supset B_{22}.
\]

Then

\[
B_1 \cap B_2 = ((B_{11} \cap B_{21}) \cup (B_{12} \cap B_{21})) \setminus ((B_{11} \cap B_{22}) \cup (B_{12} \cap B_{22})) \in B_{N(k)+2},
\]

since \( B_{1i} \cap B_{2j} \in B_{N(k)} \) for \( i, j = 1, 2 \). The other three cases can be treated similarly.

Since the sets of the form \( A^n, A \in \Sigma \) generate \( \gamma_n^{-1}(\Sigma_{\text{Sym}}^n) \) (see the proof of Proposition 26) we obtain that \( \frac{1}{c_\alpha} \mu^n(C \cap Y_\alpha) = \frac{1}{c_\beta} \mu^n(C \cap Y_\beta) \) for all \( C \in \gamma_n^{-1}(\Sigma_{\text{Sym}}^n) \). In particular, for \( C = Y_\alpha \) we get: \( \mu^n(Y_\alpha) = 0 \). Obtained contradiction finishes the proof in the case \( n = m \). **Case 2:** \( n \neq m \). Without loss of generality let \( n > m \). From (3) and the arguments above it is not hard to show that for any \( k \), any \( C_1, C_2, \ldots, C_k \in \Sigma \) and any expression \( F(B_1, B_2, \ldots, B_k) \) involving only operations of union, intersection and difference of the sets one has

\[
\frac{1}{c_\alpha} \mu^n(F(C_1^n, \ldots, C_k^n) \cap Y_\alpha) = \frac{1}{c_\beta} \mu^m(F(C_1^m, \ldots, C_k^m) \cap Y_\beta).
\]

Find pairwise disjoint sets \( A_1, \ldots, A_n \in \Sigma \) such that

\[
\mu^n(\text{Sym}(A_1 \times A_2 \times \cdots \times A_n) \cap Y_\alpha) > 0.
\]

Set \( k = n + 1 \), \( C_{n+1} = \bigcup_{i=1}^n A_i \) and \( C_j = C_{n+1} \setminus A_j \) for \( j = 1, \ldots, n \). Let

\[
F(B_1, \ldots, B_{n+1}) = B_{n+1} \setminus \bigcup_{j=1}^n B_j.
\]

Then \( F(C_1^n, \ldots, C_k^n) = \text{Sym}(A_1 \times A_2 \times \cdots \times A_n) \) and \( F(C_1^m, \ldots, C_k^m) = \emptyset \). From (4) we obtain that \( \mu^n(\text{Sym}(A_1 \times A_2 \times \cdots \times A_n) \cap Y_\alpha) = 0 \). This contradiction finishes the proof.

---

7 Irreducibility of quasi-regular representation.

Below we prove several other useful facts related to the groupoid construction and totally nonfree actions. In particular, we prove Theorem 20.
Proposition 29. The action of $G$ on a Lebesgue space $(X, \Sigma, \mu)$ is totally non-free if and only if the restriction of the conditional expectation $\mathcal{E} : \mathcal{M}_R \rightarrow \mathcal{M}_\pi$ (described in the proof of Proposition 21) onto $L^\infty(X, \mu)_l = \{ m_l : m \in L^\infty(X, \mu) \}$ is injective.

Proof. Assume that the action of $G$ on $(X, \Sigma, \mu)$ is totally non-free. Let $m \in L^\infty(X, \mu)$ be such that $\mathcal{E}(m_l) = 0$. Then by properties of $\mathcal{E}$ for every $g \in G$ one has:

$$\int_{\text{Fix}(g)} m(x) d\mu(x) = (\pi(g)m_l, \xi) = \text{tr}(\pi(g)m_l) = \text{tr}(\pi(g)\mathcal{E}(m_l)) = 0.$$

Since the sets $\text{Fix}(g), g \in G$ generate the sigma algebra of all measurable sets on $X$ this implies that $m(x) = 0$ for almost all $x \in X$. Thus, $\mathcal{E}|_{L^\infty(X, \mu)_l}$ is injective.

Assume that the action of $G$ on $(X, \Sigma, \mu)$ is not totally non-free. Then there exists a function $m \in L^\infty(X, \mu)$ non-zero on a set of positive measure such that

$$\int_{\text{Fix}(g)} m(x) d\mu(x) = 0 \text{ for all } g \in G.$$

We obtain that $\text{tr}(\pi(g)\mathcal{E}(m_l)) = 0$. It follows that $\text{tr}(A\mathcal{E}(m)) = 0$ for all $A \in \mathcal{M}_\pi$. In particular, $\text{tr}((\mathcal{E}(m_l))^*\mathcal{E}(m_l)) = 0$, and thus $\mathcal{E}(m_l) = 0$. So, $\mathcal{E}|_{L^\infty(X, \mu)_l}$ is not injective.

Observe that from Proposition 29 one can obtain a different proof of the first part of Theorem 9 (i.e. that the equality $\mathcal{M}_\pi = \mathcal{M}_R$ implies that the action of $G$ is totally non-free). Indeed, if $\mathcal{M}_\pi = \mathcal{M}_R$ then the conditional expectation $\mathcal{E} : \mathcal{M}_R \rightarrow \mathcal{M}_\pi$ is the identity map. In particular, it is injective. By Proposition 29 the action of $G$ on $(X, \Sigma, \mu)$ is totally nonfree.

Proof of Theorem 20. We have: $\mathcal{M}_\pi = \mathcal{M}_R$. By Mackey’s result, it is sufficient to show that for almost all $x \in X$ the representation $\rho_x = \rho_{G/\text{St}_G(x)}$ is irreducible. Consider the Krieger’s presentation of $\pi$ as a direct integral of quasi-regular representations:

$$\pi = \int_X \rho_y d\mu(y) \text{ acting in } \int_X l^2(Gy)d\mu(y).$$

The elements of $\int_X l^2(Gy)d\mu(y)$ can be viewed as square integrable functions

$$f : X \rightarrow \bigcup_{y \in X} l^2(Gy),$$
such that \( f_y \in l^2(Gy) \) for all \( y \), where \( f_y \) is the value of \( f \) at \( y \in X \). We have

\[
(\pi(g)f)_y = \rho_y(g)f_y, \quad (\bar{\pi}(g)f)_y = f_{g^{-1}y}, \quad (m_rf)_y = m(y)f_y
\]

for all \( g \in G, m \in L^\infty(X, \mu) \). Assume that there exists a subset \( B \) of \( X \) such that \( \mu(B) > 0 \) and \( \rho_y \) is reducible for all \( y \in B \). Then we can find an integrable collection of operators \( A_y \in \rho_y(G)' \), \( y \in X \), such that \( A_y \) is non-scalar for all \( y \in B \). We have:

\[
A := \int_X A_y d\mu(y) \in \mathcal{M}_\pi' \subset \mathcal{M}_\mathcal{R}.
\]

Obviously, \( A \) commutes with \( L^2(X, \mu)_r \). Since \( L^2(X, \mu)_r \subset \mathcal{M}_\mathcal{R} \) is a maximal commutative subalgebra (see [7], Proposition 2.9), \( A \in L^2(X, \mu)_r \). Thus, \( A = m_r \) for some \( m \in L^2(X, \mu) \) and \( A_y \) is multiplication by \( m(y) \) for almost all \( y \in X \). This contradicts to the choice of \( A \) and finishes the proof.

An interesting question is whether the opposite direction of Theorem 20 is true. Namely:

**Question.** Let \( G \) be arbitrary countable group acting ergodically by measure preserving transformations on a Lebesgue space \((X, \Sigma, \mu)\). Assume that \( \text{comm}_G(\text{St}_G(x)) = \text{St}_G(x) \) for almost all \( x \in X \). Is it always true under these conditions that the action of \( G \) is totally non-free?

Recall that \( \rho \) is a representation of \( G \times G \) in \( L^2(\mathcal{R}, \nu) \) defined by

\[
\rho((g_1, g_2)) = \pi(g_1)\bar{\pi}(g_2) \quad \text{for all} \quad g_1, g_2 \in G
\]

and \( \xi \in L^2(\mathcal{R}, \nu) \) is the unit vector given by \( \xi(x, y) = \delta_{x,y} \). By Vershik’s Theorem 9 and Proposition 21 to prove that the answer on the above question is positive it is sufficient to show that \( \rho \) is irreducible (equivalently, that the algebra \( \mathcal{M}_\rho \) contains orthogonal projections on every one-dimensional subspace of \( L^2(\mathcal{R}, \nu) \)). We are not able to prove the latter. However, we can prove a weaker statement which can be considered as a natural step towards proving the positive answer on the above question. Denote by \( P_\xi \) the orthogonal projection from \( L^2(\mathcal{R}, \nu) \) onto \( \mathbb{C}\xi \).

**Proposition 30.** Let \( G \) be a countable group acting ergodically by measure preserving transformations on a Lebesgue space \((X, \Sigma, \mu)\). If for almost all \( x \in X \) \( \text{comm}_G(\text{St}_G(x)) = \text{St}_G(x) \) then \( \xi \) is a unique (up to scalar multiplication) vector in \( L^2(\mathcal{R}, \nu) \) fixed by \( \rho((g, g)) \) for all \( g \in G \), and therefore \( P_\xi \in \mathcal{M}_\rho \).
Proof. Assume that for almost all \( x \in X \) \( \text{comm}_G(\text{St}_G(x)) = \text{St}_G(x) \) and there is a nonzero \( \rho \)-invariant vector \( \eta \perp \xi \) in \( L^2(\mathcal{R}, \nu) \). Let \( \eta = \eta_1 + \eta_2 \), where \( \eta_1 \) is supported by the diagonal \( \text{diag}(X) \subseteq \mathcal{R} \) and \( \eta_2 \) is supported by the complement of \( \text{diag}(X) \). Since \( \eta_1(gx, gx) = \eta_1(x, x) \) for almost all \( x \) and \( (G, X, \mu) \) is ergodic, we obtain that \( \eta_1 \) is constant almost everywhere on \( \text{diag}(X) \). Since \((\eta_1, \xi) = (\eta, \xi) = 0\), we obtain that \( \eta_1 = 0 \).

Further, since \( \text{comm}_G(\text{St}_G(x)) = \text{St}_G(x) \), for almost all \( x \) the following is true. For all \( y \in Gx, y \neq x \) either \( \text{St}_G(x)y \) or \( \text{St}_G(y)x \) is infinite. It follows that for almost all \( (x, y) \in \mathcal{R} \setminus \text{diag}(X) \) there exist infinitely many \( z \) such that either \( \eta_2(x, z) = \eta_2(x, y) \) or \( \eta_2(z, y) = \eta_2(x, y) \). By definition of the measure \( \nu \) we obtain that \( \eta_2 \) cannot be square integrable unless \( \eta_2 = 0 \). This contradiction finishes the proof.

Thus, \( P_\xi \) is the orthogonal projection onto the subspace of vectors in \( L^2(\mathcal{R}, \nu) \) fixed by \( \rho((g, g)) \) for all \( g \in G \). This implies that \( P_\xi \in M_\rho \).

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