Quantum particles trapped in a position-dependent mass barrier; a d-dimensional recipe

Omar Mustafa\(^1\) and S. Habib Mazharimousavi\(^2\)
Department of Physics, Eastern Mediterranean University, G Magusa, North Cyprus, Mersin 10, Turkey
\(^1\)e-mail: omar.mustafa@emu.edu.tr
\(^2\)e-mail: habib.mazhari@emu.edu.tr

July 18, 2018

Abstract

We consider a free particle, \(V(r) = 0\), with a position-dependent mass \(m(r) = 1/(1 + \varsigma^2 r^2)^2\) in the d-dimensional Schrödinger equation. The effective potential turns out to be a generalized Pöschl-Teller potential that admits exact solution.

PACS numbers: 03.65.Ge, 03.65.Fd, 03.65.Ca

1 Introduction

Quantum mechanical particles endowed with position-dependent mass (PDM), \(M(r)\), have attracted attention and inspired intense research activities over the years [1-16]. They form interesting and useful models for many physical problems. They are used, for example, in the energy density many-body problem [2], in the determination of the electronic properties of the semiconductors [3] and quantum dots [4], in quantum liquids [5], in the \(^3\)He clusters [6] and metal clusters [7], in the Bohmian approach to quantum theory [8], full/partial Gaussian wave-packet revival inside an infinite potential [9], etc.

In addition to its practical applicability side, conceptual problems of delicate nature erupt in the study of quantum mechanical systems with position-dependent mass (e.g., momentum operator does not commute with \(m(r)\), uniqueness of kinetic energy operator, etc.). Comprehensive discussion on such issues could be found in, e.g., [8,10, and related references therein].

On the other hand, the exact solvability of the d-dimensional Schrödinger equation may very well form the major ingredient for methods that are based on a Liouvillean-type change of variables (cf., e.g., [17-19]) such as the point canonical transformation (PCT) method (cf., e.g., [13,14]). Where, in such
methodologies, the exact solution of the so-called reference Schrödinger equation (eigenvalues and eigenfunctions) is mapped into an exact solution (eigenvalues and eigenfunctions) of the so-called target Schrödinger equation (cf, e.g. [1,14]).

In a recent study, we have introduced a \( d \)-dimensional regularization of the PCT-method for some PDM-quantum mechanical particles [1]. Therein, inter-dimensional degeneracies associated with the isomorphism between the angular momentum quantum number \( \ell \) and dimensionality \( d \) are incorporated through the central repulsive/attractive core \( \ell (\ell + 1)/r^2 \rightarrow \ell_d (\ell_d + 1)/r^2 \) (where \( \ell_d = \ell + (d-3)/2 \) for \( d \geq 2 \)) of the spherically symmetric Schrödinger equation (cf, e.g., [20], Gang in [10] and Quesne in [10] for more details).

In this work, we study "free" particles, \( V(r) = 0 \), trapped in their own position-dependent mass barriers (hence, labeled as quasi-free particles), where inter-dimensional degeneracies remain intact in the \( d \)-dimensional radial Schrödinger equation (in atomic units \( h = m_o = 1 \), the position-dependent mass \( M(r) = m_o, m(r) \), and with \( \alpha = \gamma = 0 \) and \( \beta = -1 \) in Eq.(1.1) of Tanaka in [10]).

\[
\left\{ \frac{d^2}{dr^2} - \frac{\ell_d (\ell_d + 1)}{r^2} + \frac{m'(r)}{m(r)} \left( \frac{d-1}{2r} - \frac{d}{dr} \right) + 2m(r) E \right\} R_{n_r, \ell}(r) = 0. \tag{1}
\]

Where \( n_r = 0, 1, 2, \cdots \) is the radial quantum number, and \( m'(r) = dm(r)/dr \). Moreover, the \( d = 1 \) can be obtained through \( \ell_d = -1 \) and \( \ell_d = 0 \) for even and odd parity, \( P = (-1)^{\ell_d+1} \), respectively (cf, e.g., Mustafa and Znojil in [20]). Nevertheless, the inter-dimensional degeneracies associated with the isomorphism between angular momentum \( \ell \) and dimensionality \( d \) builds up the ladder of excited states for any given \( n_r \) and nonzero \( \ell \) from the \( \ell = 0 \) result, with that \( n_r \), by the transcription \( d \rightarrow d + 2\ell \). That is, if \( E_{n_r, \ell}(d) \) is the eigenvalue in \( d \)-dimensions then

\[
E_{n_r, \ell}(2) \equiv E_{n_r, \ell-1}(4) \equiv \cdots \equiv E_{n_r,1}(2\ell) \equiv E_{n_r,0}(2\ell + 2) \tag{2}
\]

for even \( d \), and

\[
E_{n_r, \ell}(3) \equiv E_{n_r, \ell-1}(5) \equiv \cdots \equiv E_{n_r,1}(2\ell + 1) \equiv E_{n_r,0}(2\ell + 3) \tag{3}
\]

for odd \( d \). For more details on inter-dimensional degeneracies the reader may refer to a sample of references in [20].

With the PCT-method in point (cf, e.g., [1]), a substitution of the form \( R(r) = g(r) \phi(q(r)) \) in (1) would result in \( g(r)^2 q'(r) = m(r) \), manifested by the requirement of a vanishing coefficient of the first-order derivative of \( \phi(q(r)) \) (hence a one-dimensional form of Schrödinger equation is achieved), and \( q'(r)^2 = m(r) \) to avoid position-dependent energies-multiplicity (i.e., \( 2E m(r) / q'(r)^2 \Rightarrow 2E \)). Hence,

\[
q(r) = \int_r^\infty \sqrt{m(t)}dt \implies g(r) = m(r)^{1/4}. \tag{4}
\]

This in effect implies

\[
\left\{ -\frac{1}{2} \frac{d^2}{dq^2} + V_{eff}(q(r)) \right\} \phi_{n_r, \ell_d}(q(r)) = E_d \phi_{n_r, \ell_d}(q(r)) , \tag{5}
\]
with an effective potentials

\[ V_{\text{eff}}(q(r)) = \frac{\ell_d(\ell_d + 1)}{2r^2m(r)} - U_d(r) \]  

(6)

where

\[ U_d(r) = \frac{m''(r)}{8m(r)} - \frac{7m'(r)^2}{32m(r)^3} + \frac{m'(r)(d-1)}{4r m(r)^2}. \]  

(7)

2 Consequences of an asymptotically vanishing mass settings as \( r \rightarrow \infty \)

A "free" particle with an asymptotically vanishing position-dependent mass \( m(r) = 1/(1 + \varsigma^2 r^2)^2 \) would experience an effective potential

\[ V_{\text{eff}}(q(r)) = \frac{\varsigma^2}{2} \left[ \frac{\varsigma(\varsigma - 1)}{\sin^2(\varsigma q)} + \frac{\lambda(\lambda - 1)}{\cos^2(\varsigma q)} \right] - \frac{\varsigma^2}{2}, \]  

(8)

where

\[ q(r) = \frac{1}{\varsigma} \arctan(\varsigma r) \Rightarrow \varsigma r = \tan(\varsigma q), \]  

(9)

and

\[ U_d(r) = - (\varsigma^2 d) \tan^2(\varsigma q) + \frac{\varsigma^2}{2}(1 - 2d) \]  

(10)

In such settings, Eq. (5) reads

\[ \left\{ -\frac{1}{2} \frac{d^2}{dq^2} + \frac{\varsigma^2}{2} \left[ \frac{\varsigma(\varsigma - 1)}{\sin^2(\varsigma q)} + \frac{\lambda(\lambda - 1)}{\cos^2(\varsigma q)} \right] \right\} \phi_{n_r,\ell_d}(q) = \varepsilon \phi_{n_r,\ell_d}(q), \]  

(11)

where

\[ \varsigma(\varsigma - 1) = l_d(l_d + 1), \ \lambda(\lambda - 1) = l_d(l_d + 1) + 2d \]  

and \( \varepsilon = E + \frac{1}{2} \varsigma^2. \)  

(12)

Equation (11) is obviously a standard one-dimensional form of Schrödinger equation with a generalized Pöschl-Teller effective potential which admits exact solution of the form

\[ \epsilon_{n_r} = \frac{\varsigma^2}{2} (\varsigma + \lambda + 2n_r)^2 \]  

(13)

\[ \phi_{n_r,\ell_d}(q) = C \sin^{\varsigma}(\varsigma q) \cos^{\lambda}(\varsigma q) \ {}_2F_1(-n_r, \varsigma + \lambda + n_r, \varsigma + \frac{1}{2}; \sin^2(\varsigma q)) \]  

(14)

with \( \varsigma, \lambda > 1, \phi_{n_r,\ell_d}(0) = 0 \) and \( \phi_{n_r,\ell_d}(\frac{\pi}{\varsigma}) = 0, \) as reported by Salem and Montemayor (see Eq.(4.7) in [21]). This in turn would lead to

\[ E_{n_r,\ell_d} = \frac{\varsigma^2}{2} [(c + \frac{1}{2} \Delta + 2n_r)^2 - 1]; \ \Delta = \sqrt{(2l_d + 1)^2 + 8d} \]  

(15)
\[ R_{\ell d}(r) = \tilde{C} \rho^{l+1} (1 + \rho^2)^{-\frac{1}{2}(2l_d+5+\Delta)} \ _2F_1(-n_r, c + \frac{\Delta}{2} + n_r, c; \frac{\rho^2}{1 + \rho^2}) \]  

where \( \rho = \varsigma r \), and \( c = l_d + \frac{3}{2} \).

However, for \( \kappa = 0, 1 \) (a requirement suggested by relation (12) when \( l_d = 0, -1 \)) the effective potential in (11) collapses into

\[ V_{\text{eff}}(q(r)) = \frac{\varsigma^2 \lambda(\lambda - 1)}{2 \cos^2(\varsigma q)}. \]  

Which admits an exact solution

\[ E_{n_r} = 2\varsigma^2(n_r + \frac{\lambda}{2})^2 - \frac{\varsigma^2}{2} \]  

and consequently

\[ E_{n_r,0} = 2\varsigma^2(n_r + \frac{\lambda}{2})^2 - \frac{\varsigma^2}{2} \]  

where \( \lambda = (1 + \Delta) / 2 \).

### 3 Concluding Remarks

In this letter, we considered a quasi-free particle with an asymptotically vanishing position-dependent mass \( m(r) = 1 / (1 + \varsigma^2 r^2)^2 \) and radial potential \( V(r) = 0 \) (i.e., "free" particle in this sense). We have shown that under these settings the particle experiences an effective potential of the form of a Pöschl-Teller, Eq.(8). The exact solution of which is mapped to match the attendant settings of our quasi-free particle with the above mentioned position-dependent mass.
References

[1] O Mustafa and S.H Mazharimousavi 2006 "Point canonical transformation d-dimensional regularization" (arXiv: math-ph/0602044)

[2] A Puente and M Casas Comput. Mater Sci. 2 (1994) 441

[3] Bastard G 1988 "Wave Mechanics Applied to Semiconductor Heterostructures", Les Editions de Physique, Les Ulis

[4] L I Serra and E Lipparini Europhys. Lett. 40 (1997) 667

[5] F Arias de Saaverda, J Boronat, A Polls, and A Fabrocini Phys. Rev. B 50 (1994) 4248

[6] M Barranco, M Pi, S.M Gatica, E.S Hernandez, and J. Navarro Phys. Rev. B 56 (1997) 8997

[7] A Puente, L I Serra, and M Casas Z. Phys. D 31 (1994) 283

[8] A R Plastino, M Casas and A. Plastino Phys. Lett. A281 (2001) 297 (and related references therein)

[9] A Schmidt Phys. Lett. A (2006) (in press)

[10] T Tanaka J. Phys. A 39 (2006) 219
    C Quesne, Ann. Phys. 321 (2006) 1221
    C Gang, Phys. Lett. A 329 (2004) 22
    A R Plastino, A Rigo, M Casas, F Garcia, and A. Plastino Phys. Rev. A 60 (1999) 4318

[11] S H Dong and M. Lozada-Cassou Phys. Lett. A 337 (2005) 313
    I O Vakarchuk J. Phys. A; Math and Gen 38 (2005) 4727
    C Y Cai, Z Z Ren and G X Ju Commun. Theor. Phys. 43 (2005) 1019

[12] B Bagchi, A Banerjee, C Quesne and V M Tkachuk J. Phys. A; Math and Gen 38 (2005) 2929
    J Yu and S H Dong Phys. Lett. A 325 (2004) 194
    L Dekar, L Chetouani and T F Hammann J. Math. Phys. 39 (1998) 2551

[13] C Quesne and V M Tkachuk J. Phys. A; Math and Gen 37 (2004) 4267
    L Jiang, L Z Yi, and C S Jia Phys. Lett. A 345 (2005) 279
    A D Alhaidari Int. J. Theor. Phys. 42 (2003) 2999

[14] A D Alhaidari Phys. Rev. A 66 (2002) 042116

[15] R De, R Dutt and U Sukhatme J. Phys. A; Math and Gen 25 (1992) L843
[16] G Junker J. Phys. A; Math and Gen 23 (1990) L881

[17] J Liouville J. Math. Pure Appl. 1 (1837) 16

[18] M Znojil and G Lévai J. Math. Phys. 42 (2001) 1996
M Znojil ”$\mathcal{PT}$-symmetric form of the Hulthén potential” (2000) (arXiv: math-ph/0002017)

[19] M Znojil and G Lévai Phys. Lett. A 271 (2000) 327

[20] D R Herschbach et al "Dimensional Scaling in Chemical Physics" (Kluwer Academic Publishers 1993, Dordrecht, Netherlands.)
D R Herschbach J. Chem. Phys. 84 (1986) 838
H Taseli J. Math. Chem. 20 (1996) 235
O Mustafa and M Znojil J. Phys. A; Math and Gen. 35 (2002) 8929
O Mustafa and M Odeh J. Phys. A; Math and Gen. 32 (1999) 6653
O Mustafa and M Odeh J. Phys. A; Math and Gen. 33 (2000) 5207
M M Nieto Am. J. Phys. 47 (1979) 1067

[21] L D Salem and R Montemayor Phys. Rev. A 47 (1993) 105
S Flügge 1974, Practical Quantum Mechanics, Springer, Berlin.