E∞-STRUCTURES AND DIFFERENTIALS OF THE
ADAMS SPECTRAL SEQUENCE

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The Adams spectral sequence was invented by J.F. Adams [1] almost fifty years ago for calculations of stable homotopy groups of topological spaces and in particular of spheres. The calculation of differentials of this spectral sequence is one of the most difficult problem of Algebraic Topology. Here we consider an approach to solve this problem in the case of \( \mathbb{Z}/2 \) coefficients and find inductive formulas for the differentials. It is based on the \( A∞ \)-structures [2], \( E∞ \)-structures [3], [4], [5], [6] and functional homology operations [7], [8], [9]. This approach will be applied to the Kervaire invariant problem [10], [11].

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1. The Bousfield-Kan spectral sequences

Consider the Bousfield-Kan spectral sequence [12], which is one of the most general spectral sequence of the homotopy groups.

Let \( R \) be a field. Given a simplicial set \( Z \) denote by \( RZ \) the free simplicial \( R \)-module generated by \( Z \). There is the cosimplicial resolution

\[
R^* Z : \quad Z \xrightarrow{\delta^0} RZ \xrightarrow{\delta^0,\delta^1} R^2Z \rightarrow \cdots \rightarrow R^nZ \xrightarrow{\delta^0,\ldots,\delta^n} R^{n+1}Z \rightarrow \cdots
\]

This resolution was used by Bousfield and Kan [12] to construct the spectral sequence of the homotopy groups of \( Z \) with coefficients in \( R \).

The \( E^1 \)-term of this spectral sequence is expressed by the complex

\[
H_*(Z; R) \rightarrow H_*(RZ; R) \rightarrow \cdots \rightarrow H_*(R^{n-1}Z; R) \rightarrow H_*(R^nZ; R) \rightarrow \cdots
\]

Higher differentials are expressed by the homology operations

\[
d_m: H_*(R^{n-1}Z; R) \rightarrow H_*(R^{n+m-1}Z; R).
\]

In [7], [8] the homology operations were defined as partial and multi-valued mappings. However there is a general method to choose the homology operations to be usual homomorphisms. The corresponding theory was developed in [9]. Recall the main definitions.

For a chain complex \( X \) denote by \( X_* \) its homology, \( X_* = H_*(X) \). Fix chain mappings \( \xi: X_* \rightarrow X, \eta: X \rightarrow X_* \) and chain homotopy \( h: X \rightarrow X \) satisfying the relations

\[
x \circ h = Id, \quad d(h) = \xi \circ \eta - Id, \quad h \circ \xi = 0, \quad \eta \circ h = 0, \quad h \circ h = 0.
\]
Consider a sequence of mappings

\[ f^1: X^1 \to X^2, \ldots, f^n: X^n \to X^{n+1} \]

and define functional homology operations

\[ H_*(f^n, \ldots, f^1): X^1_* \to X^{n+1}_* \]

putting

\[ H_*(f^n, \ldots, f^1) = \eta \circ f^n \circ h \circ \cdots \circ f^1 \circ \xi. \]

Direct calculations show that the following relations are satisfied

\[
\sum_{i=1}^{n-1} (-1)^{n-i+1} H_*(f^n, \ldots, f^{i+1} \circ f^i, \ldots, f^1) = \\
\sum_{i=1}^{n-1} (-1)^{n-i} H_*(f^n, \ldots, f^{i+1}) \circ H_*(f^i, \ldots, f^1).
\]

Functional homology operations may be defined not only for the category of chain complexes but in some other situations, for example, for the category of simplicial modules.

Directly from the definition it follows that higher differentials of the Bousfield-Kan spectral sequence are expressed by the functional homology operations

\[ H_*(\delta, \ldots, \delta): H_*(R^{n-1} Z; R) \to H_*(R^{n+m-1} Z; R). \]

So we have

**Theorem 1.** The differentials of the Bousfield-Kan spectral sequence are expressed by the functional homology operations

\[ H_*(\delta, \ldots, \delta): H_*(R^{n-1} Z, R) \to H_*(R^{n+m-1} Z, R). \]

These operations determine on the \( E^1 \)-term new differential. The homology of the corresponding complex is isomorphic to the \( E^\infty \)-term of the spectral sequence.

As it was proved in [5] instead of the Bousfield-Kan cosimplicial object we may consider the following cosimplicial object

\[ F^*(C, RZ) : \ RZ \xrightarrow{\delta^0} CRZ \to \cdots \to C^{n-1} RZ \xrightarrow{\delta^0 \cdots \delta^n} C^n RZ \to \ldots, \]

where \( CRZ \) is the free commutative simplicial coalgebra generated by \( RZ \).

The \( E^1 \)-term of the corresponding spectral sequence is expressed by the complex

\[ H_*(Z; R) = \pi_*(RZ) \to \pi_*(CRZ) \to \cdots \to \pi_*(C^{n-1} RZ) \to \pi_*(C^n RZ) \to \ldots. \]

Directly from the definition it follows that higher differentials of this spectral sequence are expressed by the functional homology operations

\[ H_*[\delta, \ldots, \delta] : \pi_*(R^n RZ) \to \pi_*(C^n RZ) \to \ldots. \]
Moreover there is a cosimplicial mapping
\[
\begin{align*}
RZ & \longrightarrow R^2Z \longrightarrow \ldots \longrightarrow R^{n+1}Z \longrightarrow \ldots \\
\Downarrow & \quad \Downarrow \quad \Downarrow \\
RZ & \longrightarrow CRZ \longrightarrow \ldots \longrightarrow C^nRZ \longrightarrow \ldots,
\end{align*}
\]
inducing the isomorphism of the corresponding spectral sequences. So we have

**Theorem 2.** The differentials of the Bousfield-Kan spectral sequence of the cosimplicial object

\[
F^*(C, RZ) : \quad RZ \xrightarrow{\delta^0, \delta^1} CRZ \longrightarrow \cdots \longrightarrow C^{n-1}RZ \xrightarrow{\delta^0, \ldots, \delta^n} C^nRZ \longrightarrow \ldots
\]

are expressed by the functional homology operations

\[
H_*(\delta, \ldots, \delta) : \pi_*(C^nRZ) \rightarrow \pi_*(C^{n+m}RZ).
\]

These operations determine on the $E^1$-term a new differential. The homology of the corresponding chain complex is isomorphic to the $E^\infty$-term of this spectral sequence.

2. $E_\infty$-algebras and $E_\infty$-coalgebras

Recall that an operad in the category of chain complexes is a family $E = \{E(j)\}_{j \geq 1}$ of chain complexes $E(j)$ together with given actions of symmetric groups $\Sigma_j$ and operations

\[
\gamma : E(k) \otimes E(j_1) \otimes \cdots \otimes E(j_k) \rightarrow E(j_1 + \cdots + j_k)
\]

compatible with these actions and satisfying associativity relations [3], [4].

An operad $E = \{E(j)\}$ for which complexes $E(j)$ are acyclic and symmetric groups act on them freely is called an $E_\infty$-operad.

A chain complex $X$ is called an algebra (a coalgebra) over an operad $E$ or simply an $E$-algebra (an $E$-coalgebra) if there are given mappings

\[
\mu : E(k) \otimes \Sigma_k X^{\otimes k} \rightarrow X \quad (\tau : X \rightarrow Hom_{\Sigma_k}(E(k); X^{\otimes k}))
\]
satisfying some associativity relations.

Algebras (coalgebras) over an $E_\infty$-operad are called $E_\infty$-algebras ($E_\infty$-coalgebras).

Any operad in the category of chain complexes determines a monad $E$ and a comonad $\overline{E}$ by the formulas

\[
\begin{align*}
\overline{E}(X) &= \sum_j E(j, X), \quad E(j, X) = E(j) \otimes_{\Sigma_j} X^{\otimes j}; \\
\overline{E}(X) &= \prod_j E(j, X), \quad \overline{E}(j, X) = Hom_{\Sigma_j}(E(j), X^{\otimes j}).
\end{align*}
\]
An operad structure \( \gamma \) induces natural transformations
\[
\gamma: E \circ E \to E, \quad \tau: E \to E \circ E.
\]

An \( E \)-algebra (an \( E \)-coalgebra) structure on a chain complex \( X \) induces a mapping
\[
\mu: E(X) \to X \quad (\tau: X \to \overline{E}(X)).
\]
So to give on a chain complex \( X \) an \( E \)-algebra (\( E \)-coalgebra) structure is the same as to give on \( X \) an algebra (coalgebra) structure over the monad \( E \) (the comonad \( \overline{E} \)).

One of the most important example of an \( E_{\infty} \)-algebra is the singular cochain complex \( C^* (Y; R) \) of a topological space \( Y \).

Dually, the singular chain complex \( C_* (Y; R) \) of a topological space \( Y \) and the chain complex \( N(RZ) \) of a simplicial set \( Z \) are examples of \( E_{\infty} \)-coalgebras.

The homotopy theory of \( E_{\infty} \)-coalgebras was constructed in [5]. There were defined the homotopy groups of \( E_{\infty} \)-coalgebras. For the chain complex \( N(RZ) \) of a simplicial set \( Z \) these homotopy groups are isomorphic to the homotopy groups of \( Z \) with coefficients in \( R \).

For an \( E \)-coalgebra \( X \), using cosimplicial resolution \( F^*(E, E, X) : \tau \mapsto E(X) \to \cdots \to E^{n-1}(X) \to E^n(X) \to \cdots \),
there was constructed the spectral sequence of the homotopy groups of the \( E \)-coalgebra \( X \), [5].

Denote by \( X_* \) the homology of the complex \( X \) and by \( E_* \) the homology of a comonad \( E \). \( X_* \) will be \( E_* \)-coalgebra. There is the cosimplicial resolution
\[
F^*(E_*, E_*, X_*): E_*(X_*) \to E^2_*(X_*) \to \cdots \to E^n_*(X_*) \to E^{n+1}_*(X_*) \to \cdots.
\]

The \( E^1 \) term of the spectral sequence is expressed by the cobar construction
\[
F(E_*, X_*): X_* \to E_*(X_*) \to \cdots \to E^n_*(X_*) \to E^{n+1}_*(X_*) \to \cdots,
\]
obtained from the resolution by taking primitive elements. So there is the inclusion \( F(E_*, X_*) \to F(E_*, E_*, X_*) \).

The functional homology operations
\[
H_*(\delta, \ldots, \delta): E^n_*(X_*) \to E^{n+m}_*(X_*)
\]
determine new differential in the resolution and in the cobar construction. The corresponding complexes denote by \( \tilde{F}(E_*, E_*, X_*) \), \( \tilde{F}(E_*, X_*) \).

Note that the complex \( \tilde{F}(E_*, E_*, X_*) \) is a resolution of the complex \( X_* \) and there is the inclusion \( \tilde{F}(E_*, X_*) \to \tilde{F}(E_*, E_*, X_*) \).

**Theorem 3.** Differentials of the spectral sequence of the homotopy groups of an \( E \)-coalgebra \( X \) are determined by the functional homology operations
\[
H_*(\delta, \ldots, \delta): E^n_*(X_*) \to E^{n+m}_*(X_*)
\]
The homology of $\tilde{F}(E_\ast, X_\ast)$ is isomorphic to the $E^\infty$-term of the spectral sequence.

If $X$ is the normalized chain complex of a simplicial set $Z$, i.e. $X = N(RZ)$, then there is a mapping of cosimplicial objects

$$
\begin{array}{cccccccc}
N(RZ) & \longrightarrow & N(CRZ) & \longrightarrow & \ldots & \longrightarrow & N(C^nRZ) & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & \ldots & & \downarrow & & \ldots \\
X & \longrightarrow & E(X) & \longrightarrow & \ldots & \longrightarrow & E^n(X) & \longrightarrow & \ldots,
\end{array}
$$

inducing the isomorphism of the corresponding spectral sequences. So we have

**Theorem 4.** The differentials of the Bousfield-Kan spectral sequence of the homotopy groups of a simplicial set $Z$ are determined by the functional homology operations

$$
H_\ast(\delta, \ldots, \delta): E^m_\ast(Z_\ast) \to E^{n+m}_\ast(Z_\ast).
$$

The homology of $\tilde{F}(E_\ast, Z_\ast)$ is isomorphic to the $E^\infty$-term of the spectral sequence.

Note that the suspension $SX$ over an $E$-coalgebra $X$ is an $SE$-coalgebra and the above diagrams commute

$$
\begin{array}{cccccccc}
E & \longrightarrow & E \circ E & \longrightarrow & SX & \longrightarrow & SE(SX) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
SE & \longrightarrow & SE \circ SE & \longrightarrow & SX & \longrightarrow & S(E(X))
\end{array}
$$

Moreover from the expression of the homology $E_\ast$ of the comonad $E$ (see below) it follows that the mappings $\xi: E_\ast \to E$, $\eta: E \to E_\ast$, $h: E \to E$ may be chosen permutable with the suspension homomorphism $SE \to E$. Therefore constructed functional homology operations permute with the suspension homomorphism. Hence the following theorem is taken place.

**Theorem 5.** Functional homology operations giving higher differentials of the Bousfield-Kan spectral sequence permute with the suspension and hence are stable. They induce the differentials of the Adams spectral sequence of stable homotopy groups of a topological space.

3. The homology of an $E_\infty$-operad and the Milnor coalgebra

Let $E$ be an $E_\infty$-operad, $M$ be a graded module (over $\mathbb{Z}/2$). As it is known (see for example [6]), the homology $E_\ast(M)$ of the complex $E(M)$ is the polynomial algebra generated by the elements $e_{i_1} \cdots e_{i_k} x_m$, $1 \leq i_1 \leq \cdots \leq i_k$, $x_m \in M$ of dimensions $i_1 + 2i_2 + \cdots + 2^{k-1}i_k + 2^k m$.

The elements $e_{i_1} \cdots e_{i_k} x_m$ of $E_\ast(M)$ may be rewritten in the form

$$Q^{j_1} \cdots Q^{j_k} \otimes x_m; \quad j_1 \leq 2j_2, \ldots, j_{k-1} \leq 2j_k, m \leq j_k,$$

where

$$j_k = i_k + m,$$

$$j_{k-1} = i_{k-1} + i_k + 2m,$$

$$\cdots$$

$$i_k = i_{k-2} + 2i_{k-1} + 2^{k-2}i_{k-2} + 2^{k-1}m.$$
The sequences $Q^{j_1} \ldots Q^{j_k}$ give the elements of the Dyer-Lashof algebra $R$, [14], [15].

Given a graded module $M$ denote by $R \times M$ the quotient module of the tensor product $R \otimes M$ under the submodule generated by the elements $Q^{j_1} \ldots Q^{j_k} \otimes x_m$, $j_k < m$. The correspondence $M \mapsto R \times M$ determines the monad in the category of graded modules.

A graded module $M$ is called an unstable module over the Dyer-Lashof algebra if it is an algebra over the corresponding monad.

Dually, the homology $\overline{E}_*(M)$ of the complex $\overline{E}(M)$ is the free commutative coalgebra generated by the elements $e^{i_1} \ldots e^{i_k} x^m$, $1 \leq i_1 \leq \ldots \leq i_k$, $x^m \in M$ of dimensions $2^k m - (i_1 + 2i_2 + \ldots + 2^{k-1} i_k)$.

Regrading of the elements of $\overline{E}_*(M)$ leads to the Milnor coalgebra $K$. By definition $K$ is the polynomial algebra generated by the elements $\xi_i$, $i \geq 0$ of dimensions $2^i - 1$. A comultiplication

$$\nabla : K \to K \otimes K$$

on the generators $\xi_i$ is given by the formula

$$\nabla(\xi_i) = \sum_k \xi_{i-k}^{2^k} \otimes \xi_k.$$ 

On the other elements the comultiplication is determined by the Hopf relation.

Define the grading $deg(x)$ of elements $x \in K$ putting $deg(\xi_i) = 1$ and the grading of the product equal to the sum of the gradings of factors.

Given graded module $M$ denote by $K \times M$ the submodule of the tensor product $K \otimes M$ generated by the elements $x \otimes y$, $deg(x) = dim(y)$. The correspondence $M \mapsto K \times M$ determines the comonad $K$ in the category of graded modules.

A graded module $M$ is called an unstable comodule over the Milnor coalgebra if it is a coalgebra over the corresponding comonad.

Let $M$ be an unstable comodule over the Milnor coalgebra. There is a cosimplicial resolution

$$F^*(K, K, M) : M \to K \times M \to \cdots \to K^\times n \times M \to K^\times n+1 \times M \to \ldots$$

If $Y$ is a "nice" (in the sense of Massey-Peterson) space then the Bousfield-Kan spectral sequence turns to the Massey-Peterson spectral sequence. The $E^1$-term of this spectral sequence may be written in the form

$$F^*(K, Y_*) : Y_* \to K \times Y_* \to \cdots \to K^\times n \times Y_* \to \ldots,$$

where $Y_* = H_*(Y; \mathbb{Z}/2)$.

The functional homology operations determine on the Milnor coalgebra the structure of an $A_\infty$-coalgebra [9]. On the homology $Y_*$ of a topological space there is the structure of an $A_\infty$-comodule over the $A_\infty$-coalgebra $K$. The corresponding cobar construction denote $\tilde{F}(K, Y_*)$.

From the previous theorems it follows the next theorem.
**Theorem 6.** If $Y$ is a "nice" space then the homology of the cobar construction $\tilde{F}(\mathcal{K}, Y_\ast)$ is isomorphic to the $E^\infty$ term of the Massey-Peterson spectral sequence.

Besides the Milnor coalgebra $\mathcal{K}$ we will consider the stable Milnor coalgebra $\mathcal{K}_s$ for which $\xi_0 = 1$.

Given comodule $M$ over the stable Milnor coalgebra there is a cosimplicial resolution

$$F^\ast(K_s, K_s, M) : K_s \otimes M \to K_s^\otimes 2 \otimes M \to \cdots \to K_s^\otimes n \otimes M \to \cdots$$

Stabilization of the Bousfield-Kan spectral sequence leads to the Adams spectral sequence of stable homotopy groups of a topological space $Y$. $E^1$-term of this spectral sequence may be written in the form

$$F^\ast(K_s, Y_\ast) : Y_\ast \to K_s \otimes Y_\ast \to \cdots \to K_s^\otimes n \otimes Y_\ast \to \cdots$$

Since the functional homology operations are stable, the corresponding $A_\infty$-coalgebra structure on the Milnor coalgebra $\mathcal{K}$ also will be stable. Hence it induces an $A_\infty$-coalgebra structure on the stable Milnor coalgebra $\mathcal{K}_s$. Thus we have

**Theorem 7.** For a topological spaces $Y$ the homology of the cobar construction $\tilde{F}(\mathcal{K}_s, Y_\ast)$ is isomorphic to the $E^\infty$ term of the Adams spectral sequence of stable homotopy groups of the topological space $Y$.

4. $E_\infty$-structure on the Bousfield-Kan spectral sequence

Our aim here is to define $E_\infty$-structure on the Bousfield-Kan spectral sequence. To do it consider the following additional property of the operad $E$.

**Theorem 8.** Given $E_\infty$-operad $E$ there is a permutation mapping

$$T : E \circ E \to E \circ E,$$

commuting the following diagrams

$$
\begin{array}{ccc}
E^2 \circ E & \xrightarrow{TE \circ ET} & E \circ E^2 \\
2E & \downarrow & E \circ 2E \\
E \circ E & \xrightarrow{T} & E \circ E
\end{array}
$$

**Proof.** Given an $E_\infty$-operad $E$ we may construct an operad mapping $\nabla : E \to E \otimes E$ consisting of mappings $\nabla(j) : E(j) \to E(j) \otimes E(j)$ This mapping give on $E$ a Hopf operad structure. Denote $\nabla(j, i) : E(j) \to E(j)^{\otimes i}$ the iterations of these mappings $\nabla(j, 2) = \nabla(j)$. They are $\Sigma_j$-mappings, i.e.

$$\nabla(j)(x \sigma) = \nabla(j)(x) \sigma^{\otimes j}, \quad \sigma \in \Sigma_j$$

but are not commuting with permutations of factors of $E(j)^{\otimes i}$. However they may be extended till the mappings $\nabla(j, i) : E(i) \otimes E(j) \to E(j)^{\otimes i}$, compatible with the actions of symmetric groups $\Sigma_i$ and $\Sigma_j$.

Rewrite these mappings in the form

$$\nabla(j, i) : E(j) \to \overline{E}(j) \otimes E(j)^{\otimes i}$$
If the operad $E$ is chosen freely then they may be done compatible with an operad structure.

Passing to the dual mappings we obtain mappings

$$\nabla(j, i): E(i) \otimes E(j)^{\otimes i} \rightarrow \overline{E}(j).$$

Define mappings

$$T(j, i): E(j) \otimes E(i)^{\otimes j} \rightarrow E(i) \otimes E(j)^{\otimes i}$$

as the compositions

$$E(j) \otimes \overline{E}(i)^{\otimes j} \xrightarrow{\nabla(j) \otimes 1^{\otimes j}} E(j) \otimes E(j) \otimes \overline{E}(i)^{\otimes j} \rightarrow$$

$$\rightarrow E(j) \otimes \overline{E}(i)^{\otimes j} \otimes E(j) \xrightarrow{\nabla(j,i) \otimes 1} \overline{E}(i) \otimes E(j) \rightarrow$$

$$\xrightarrow{1 \otimes \nabla(j,i)} \overline{E}(i) \otimes \overline{E}(i) \otimes E(j)^{\otimes i} \xrightarrow{\nabla(i) \otimes 1^{\otimes i}} \overline{E}(i) \otimes E(j)^{\otimes i}.$$  

The family $T(j, i)$ determines the required permutation mapping

$$T: \overline{E} \circ E \rightarrow \overline{E} \circ E.$$ 

A chain complex $X$ will be called an $E_\infty$-Hopf algebra if there are given an $E_\infty$-algebra structure $\mu: \overline{E}(X) \rightarrow X$ and an $E_\infty$-coalgebra structure $\tau: X \rightarrow \overline{E}(X)$ such that the following diagram commutes

$$\begin{array}{ccc}
\overline{E}(X) & \xrightarrow{\mu} & X \\
E(\tau) \downarrow & & \downarrow = \\
\overline{E}\overline{E}(X) & \xrightarrow{T} & \overline{E}\overline{E}(X) \\
& \xrightarrow{\overline{E}(\mu)} & \overline{E}(X)
\end{array}$$

If a topological space $Y$ is an $E_\infty$-space then its singular chain complex $C_*(Y; R)$ will be an $E_\infty$-Hopf algebra. For example the singular chain complex of the infinite loop space is an $E_\infty$-Hopf algebra.

Let $X$ is an $E_\infty$-Hopf algebra. Then there is a mapping of augmented cosimplicial objects

$$\begin{array}{ccccccc}
\overline{E}(X) & \rightarrow & \overline{E}\overline{E}(X) & \rightarrow & \ldots & \rightarrow & \overline{E}\overline{E}^n(X) & \rightarrow & \ldots \\
\downarrow & & \downarrow & & \ldots & & \downarrow & & \\
X & \rightarrow & \overline{E}(X) & \rightarrow & \ldots & \rightarrow & \overline{E}^n(X) & \rightarrow & \ldots
\end{array}$$

Then the complex $F(\overline{E}, \overline{E}, X)$ will be $E_\infty$-algebra. Passing to the homology we obtain $E_\infty$-algebra structure on the complex $\tilde{F}(\overline{E}_*, \overline{E}_*, X_*)$. Thus we have

**Theorem 9.** If $X$ is an $E_\infty$-Hopf algebra, then the complex $\tilde{F}(\overline{E}_*, \overline{E}_*, X_*)$ possesses $E_\infty$-algebra structure.

**Corollary.** If $Y$ is a "nice" $E_\infty$-space then the complex $\tilde{F}(\overline{K}, \overline{K}, Y_*)$ possesses $E_\infty$-algebra structure.
This structure will be used in the further calculations of higher differentials of the Bousfield-Kan spectral sequence.

Note that on the cobar constructions $\tilde{F}(E_*, X_*)$, $\tilde{F}(\mathcal{K}, Y_*)$ there are no $E_\infty$-algebra structure.

Let us calculate the $E_\infty$-algebra structure on the Milnor coalgebra. As it was pointed out above for an $E_\infty$-operad $E$ there is the permuting mapping $T\colon E \circ \mathcal{E} \to E \circ \mathcal{E}$. It induces the permuting mapping $T_*\colon E_* \circ \mathcal{E} \to E_* \circ \mathcal{E}$, commuting diagrams

$$
\begin{array}{cccc}
E_*^2 \circ \mathcal{E} & \xrightarrow{T_* E_* \circ \mathcal{E} T_*} & E_*^2 \circ \mathcal{E} & \xrightarrow{T_*} & E_* \circ \mathcal{E} \\
\downarrow & & \downarrow & & \downarrow \\
E_* \circ \mathcal{E} & \xrightarrow{T_*} & E_* \circ \mathcal{E} & \xrightarrow{T_* \circ T_* \circ T_*} & E_*^2 \circ \mathcal{E}
\end{array}
$$

The permuting mapping $T_*$ induces the action $\mu_* \colon E_* \circ \mathcal{E} \to E_*$ and the dual coaction $\tau_* \colon E_* \to E_* \circ \mathcal{E}$.

Denote by $e_i \colon \mathcal{K} \to \mathcal{K}$ the operation on the Milnor coalgebra inducing by the restriction of $\mu_*$ on the elements $e_i$. From the commutative diagrams for the permuting mapping $T_*$ it follows

**Theorem 10.** The operations $e_i \colon \mathcal{K} \to \mathcal{K}$ satisfy the relations:

1. $e_0(x) = x^2$.
2. $e_i(xy) = \sum e_k(x)e_{i-k}(y)$.
3. $\nabla e_i(x) = \sum \xi_0^{-k}e_{i-k}(\xi_0^kx') \otimes e_k(x'')$, where $\sum x' \otimes x'' = \nabla(x)$.

Using these relations to calculate the operations $e_i$ it is sufficient to calculate only $e_1(\xi_0)$. Direct calculations show that $e_1(\xi_0) = \xi_1 \xi_0$. From the third relation it follows

**Theorem 11.** There are the following formulas

$$
e_i(\xi_k) = \begin{cases}
\xi_{m+k} \xi_k, & i = 2^m+k-2^k; \\
\xi_{m+k} \xi_{k-1}, & i = 2^m+k-2^k-2^{k-1}; \\
\vdots & \vdots \\
\xi_{m+k} \xi_0, & i = 2^m+k-2^k-\ldots-1;
\end{cases}
$$
in other cases.

Using the second relation we may obtain formulas for the operations $e_i$ on the products of the elements $\xi_k$.

Passing from the elements $e_i$ to the elements of the Dyer-Lashof algebra we obtain the action of the Dyer-Lashof algebra on the Milnor coalgebra. On the generators $\xi_i$ it is given by the formulas

$$Q^{i+2^k-1}(\xi_k) = \begin{cases}
\xi_{m+k} \xi_k, & i = 2^m+k-2^k; \\
\xi_{m+k} \xi_{k-1}, & i = 2^m+k-2^k-2^{k-1}; \\
\vdots & \vdots \\
\xi_{m+k} \xi_0, & i = 2^m+k-\ldots-1;
\end{cases}
in other cases.
On the other elements this action is determined by the Hopf relations

\[ Q^i(xy) = \sum Q^k(x)Q^{i-k}(y). \]

Besides the action of the Dyer-Lashof algebra on the Milnor coalgebra \( K \) there are \( \cup_i \)-products and an \( E_\infty \)-algebra structure. On the generators \( x \in K \) it is defined by the formulas

\[ x \cup_i x = e_i(x). \]

On the other elements \( \cup_i \)-products are defined by the relations

\[ x \cup_i y = y \cup_i x, \]

\[ (x_1x_2) \cup_i y = x_1(x_2 \cup_i y) + (x_1 \cup_i y)x_2. \]

Note that the stable Milnor coalgebra \( K_s \) has no action of the Dyer-Lashof algebra and has no \( E_\infty \)-algebra structure.

5. Homology operations for the operad \( E \).

Let \( \Delta^* = \{ \Delta^n \} \) denotes the cosimplicial object of the category of chain complexes, consisting of the chain complexes of the standard \( n \)-dimensional simplices. Let further \( F \) be a functor in the category of chain complexes for which there are given transformations

\[ \Delta^n \otimes F(X) \to F(\Delta^n \otimes X), \]

permuting with coface and codegeneracy operators. Such functor \( F \) will be called a chain functor.

A transformation \( \alpha: F' \to F'' \) of chain functor is a transformations of functors, commuting the diagrams

\[ \begin{array}{ccc}
\Delta^n \otimes F'(X) & \longrightarrow & F'(\Delta^n \otimes X) \\
1 \otimes \alpha & \downarrow & \downarrow \alpha \\
\Delta^n \otimes F''(X) & \longrightarrow & F''(\Delta^n \otimes X)
\end{array} \]

Given chain functor \( F \) we may consider mappings

\[ F(f): F(X) \to F(Y), \]

induced not only by chain mappings \( f: X \to Y \) of dimension zero but dimension \( n \) also. Namely given mapping \( f: X \to Y \) of dimension \( n \) we represent as the restriction of the mapping \( \tilde{f}: \Delta^n \otimes X \to Y \) on the \( n \)-dimensional generator \( u_n \in \Delta^n \). Then the required mapping

\[ F(f): F(X) \to F(Y) \]

of dimension \( n \) will be the restriction of the composition

\[ \Delta^n \otimes F(X) \to F(\Delta^n \otimes X) \xrightarrow{F(\tilde{f})} F(Y) \]
on the $n$-dimensional generator $u_n \in \Delta^n$.

Given chain functor $F$ denote by $F_*$ the functor, corresponding to a chain complex $X$ the graded module of its homology $F_*(X) = H_*(F(X))$. The functor $F_*$ is not only a functor. There are functional homology operations which assigns to sequences of chain mappings $f^1: X^1 \to X^2, \ldots, f^n: X^n \to X^{n+1}$ the mapping

$$F_*(f^n, \ldots, f^1) = H_*(F(f^n), \ldots, F(f^1)): F_*(X^1) \to F_*(X^{n+1}),$$

of dimension $n - 1$.

**Theorem 12.** Let $F$ be a chain functor. Then for any sequence of chain mappings $f^1: X^1 \to X^2, \ldots, f^n: X^n \to X^{n+1}$ the following formula is taken place

$$H_*(F(f^n), \ldots, F(f^1)) = \sum (-1)^\epsilon F_*(H_*(f^n, \ldots, f^{n+1}), \ldots, H_*(f^n, \ldots, f^1)),$$

where the sum is taken over $m$ and $n_1, \ldots, n_m$ such that $1 \leq n_1 < \cdots < n_m < n$.

**Proof.** Let $X$ be chain complex. We take the mapping $F_*(X_*) \to F(X)$ of choosing representatives as the composition

$$\xi(F): F_*(X_*) \to F(X_*), \quad F(*) : F(X_*) \to F(X).$$

Similary we take the projection $F(X) \to F_*(X_*)$ as the composition

$$F(\eta): F(X) \to F(X_*), \quad \eta(F): F(X_*) \to F_*(X_*).$$

We take the homotopy $H: F(X) \to F(X)$ as the sum

$$F(\xi) \circ h(F) \circ F(\eta) + F(h).$$

Substituting these mappings to the formula of functional homology operation we obtain the required formula.

Show that the functors $\underline{E}$, $\overline{E}$ corresponding to an $E_\infty$-operad $E$ are chain. To do it we define the family of mappings

$$\Delta^n \otimes \underline{E}(j; X) \to \underline{E}(j; \Delta^n \otimes X)$$

to be the compositions

$$\Delta^n \otimes \underline{E}(j; X) = \Delta^n \otimes E(j) \otimes \Sigma_j X^j \xrightarrow{1 \otimes \Delta^j \otimes 1} \Delta^n \otimes E(j) \otimes E(j) \otimes \Sigma_j X^j \xrightarrow{\tau \otimes 1 \otimes 1} \Delta^n \otimes E(j) \otimes \Sigma_j X^j \xrightarrow{\Delta^n \otimes 1 \otimes j} \Delta^n \otimes E(j) \otimes \Sigma_j (\Delta^n \otimes X)^j = \underline{E}(j; \Delta^n \otimes X),$$

where $\tau: \Delta^n \otimes E(j) \to \Delta^n \otimes j$ is an $E$-coalgebra structure on the complex $\Delta^n$.

Direct verification show that the required relations are satisfied.

Similary define mappings

$$\Delta^n \otimes \overline{E}(j; X) \to \overline{E}(j; \Delta^n \otimes X)$$

or, that is the same, mappings

$$E(j) \otimes \Delta^n \otimes \text{Hom}_{\mathbb{F} \text{-gr}}(F(X); X^\otimes 1) \to (\Delta^n \otimes X)^\otimes 1.$$
to be the compositions
\[
E(j) \otimes \Delta^n \otimes \text{Hom}_{\Sigma_j}(E(j); X^{\otimes j}) \xrightarrow{\nabla \otimes 1 \otimes 1} E(j) \otimes E(j) \otimes \Delta^n \otimes \text{Hom}_{\Sigma_j}(E(j); X^{\otimes j}) \\
\rightarrow E(j) \otimes \Delta^n \otimes \Delta^n \otimes \text{Hom}_{\Sigma_j}(E(j); X^{\otimes j}) \rightarrow (\Delta^n \otimes X)^{\otimes j}.
\]

Direct verification show that the required relations are satisfied.

Our aim is to calculate the functorial homology operations for the functors $E_*$, $\overline{E}_*$. It means that for any sequence
\[
f^1: X^1 \rightarrow X^2, \ldots, f^n: X^n \rightarrow X^{n+1}
\]
of chain mappings we need to calculate the mappings
\[
E_*(f^n, \ldots, f^1): E_*(X^1) \rightarrow E_*(X^{n+1}), \quad \overline{E}_*(f^n, \ldots, f^1): \overline{E}_*(X^1) \rightarrow \overline{E}_*(X^{n+1}).
\]

Consider firstly the functor $E(2; -)$ correspoding to a complex $X$ the complex
\[
E(2; X) = E(2) \otimes_{\Sigma_2} X \otimes X,
\]
where $E(2) - \Sigma_2$-free and acyclic complex with generators $e_i$ of dimensions $i$. A differential is defined by the formula
\[
d(e_i) = e_{i-1} + e_{i-1}T, \quad T \in \Sigma_2.
\]

The homology $E_*(2; -)$ of this functor, as it was pointed out above, is not only a functor but an $A_\infty$-functor. It means that for any sequence of chain mappings
\[
f^1: X^1 \rightarrow X^2, \ldots, f^n: X^n \rightarrow X^{n+1}
\]
there is the operation
\[
E_*(2; f^n, \ldots, f^1): E_*(2; X^1) \rightarrow E_*(2; X^{n+1}).
\]

Let us calculate these operations.

Note that for a chain complex $X$ there is an isomorphism
\[
E_*(2, X) \cong E_*(2, X_*).
\]

If $X_*$ is a graded module then $E_*(2; X_*)$ is the sum of two factors. The first factor is the quotient module $X_* \cdot X_*$ of the tensor product $X_* \otimes X_*$ up to permutation of factors. The second factor is the module generated by the elements of the form $e_i \times y_n, i \geq 1$ of dimensions $i + 2n$. The elements $y_n : y_n \in X_* \cdot X_*$ will be also denoted as $e_0 \times y_n$.

Let $\xi: X_* \rightarrow X$, $\eta: X \rightarrow X_*$, $h: X \rightarrow X$ are mappings giving a chain equivalence between $X$ and $X_*$. Denote by
\[
E(\xi): E(2, X_*) \rightarrow E(2, X), \quad E(\eta): E(2, X) \rightarrow E(2, X_*), \quad E(h): E(2, X) \rightarrow E(2, X).
\]
the mappings defined by the formulas

\[ E(\xi)(e_i \otimes y_1 \otimes y_2) = e_i \otimes \xi(y_1) \otimes \xi(y_2), \]
\[ E(\eta)(e_i \otimes x_1 \otimes x_2) = e_i \otimes \eta(x_1) \otimes \eta(x_2), \]
\[ E(h)(e_i \otimes x_1 \otimes x_2) = e_i \otimes (x_1 \otimes h(x_2) + h(x_1) \otimes \xi(y_2)) + e_{i-1} \otimes h(x_1) \otimes h(x_2). \]

It is clear they give a chain equivalence \( E(2, X) \approx E(2, X_*) . \)

Define mappings

\[ \xi(E) : E_*(2; X_*) \to E(2; X_*), \quad \eta(E) : E(2; X_*) \to E_*(2; X_*), \]
\[ h(E) : E(2; X_*) \to E(2; X_*). \]

To do it firstly we choose an ordering basis \( \{ y \} \) in \( X_* \). Then define the mapping \( \xi(E) \) putting

\[ \xi(E)(e_i \times y) = e_i \otimes y \otimes y; \quad \xi(E)(y_1 \cdot y_2) = e_0 \otimes (y_1 \otimes y_2), \quad y_1 \leq y_2. \]

Define the mapping \( \eta(E) \) putting

\[ \eta(E)(e_i \otimes y_1 \otimes y_2) = \begin{cases} 
  e_i \times y_1, & y_1 = y_2 \\
  y_1 \cdot y_2, & y_1 < y_2, i = 0 \\
  0, & \text{in other cases}
\end{cases} \]

Define the mapping \( h(E) \) putting

\[ h(E)(e_i \otimes y_1 \otimes y_2) = \begin{cases} 
  e_{i+1} \otimes y_2 \otimes y_1, & y_1 > y_2, \\
  0, & \text{in other cases}
\end{cases} \]

Direct calculations show that the required relations are satisfied.

The mappings

\[ E(\xi) \circ \xi(E) : E_*(2, X_*) \to E(2, X), \quad \eta(E) \circ E(\eta) : E(2, X) \to E_* (2, X_*), \]
\[ E(\xi) \circ h(E) \circ E(\eta) + E(h) : E(2, X) \to E(2, X) \]

give us a chain equivalence between \( E(2, X) \) and \( E_*(2, X_*) \).

From the general formula of functional homology operations for a chain functor it follows that for the functor \( E(2, -) \) the following formula is taken place

\[ E_*(2, f^n, \ldots, f^1) = \sum E_*(2, H_*(f^n, \ldots, f^{m+1}), \ldots, H_*(f^{n1}, \ldots, f^1)), \]

where the sum is taken over all \( m \) and \( n_1, \ldots, n_m \) such that \( 1 \leq n_1 \leq \cdots \leq n_m < n \).

For a graded module \( X \) with fixed ordering basis \( \{ x_1 \} \) define mappings \( p: X \otimes X \to X, q: X \to X \otimes X, r: X \otimes X \to X \otimes X \) putting \( q(x_i) = x_i \otimes x_i \) and

\[ p(x_i \otimes x_j) = \begin{cases} 
  x_i, & i = j; \\
  0, & i \neq j
\end{cases} \]
\[ r(x_i \otimes x_j) = \begin{cases} 
  x_j \otimes x_i, & i > j; \\
  0, & i \geq j
\end{cases} \]

For a sequence of mappings \( f^1: X^1 \to X^2, \ldots, f^n: X^n \to X^{n+1} \) of graded modules with ordering basis define the mapping \( (f^n, \ldots, f^1): X^1 \to X^{n+1} \) putting

\[ (f^n, \ldots, f^1) = p \circ (f^n) \otimes 2 \circ p \circ (f^{n-1}) \otimes 2 \circ \cdots \circ p \circ (f^1) \otimes 2 \circ q. \]
Directly from the definition of the homology operations it follows

**Theorem 13.** For a sequence of mappings \( f^1: X^1 \to X^2, \ldots, f^n: X^n \to X^{n+1} \) of graded modules the following formula is taken place

\[
E_*(2; f^n, \ldots, f^1)(e_i \times x) = e_{i+n-1} \times (f^n, \ldots, f^1)(x).
\]

To obtain the corresponding formula for the functor \( E_* \) it needs to use a monad structure \( \gamma_*: E_* \circ E_* \to E_* \) and the formula

\[
E_*(f^n, \ldots, f^1) \circ \gamma_* = \sum E_*(E_*(f^n, \ldots, f^{n+1}), \ldots, E_*(f^n, \ldots, f^1)),
\]

where the sum is taken over all \( m \) and \( n_1, \ldots, n_m \) such that \( 1 \leq n_1 < \cdots < n_m < n \).

Passing to the Dyer-Lashof algebra \( \mathcal{R} \) we obtain the operations

\[
\mathcal{R}(f^n, \ldots, f^1): \mathcal{R} \times X^1 \to \mathcal{R} \times X^{n+1},
\]

which on the generators \( Q^i \) are expressed by the formulas

\[
\mathcal{R}(f^n, \ldots, f^1)(Q^i \otimes x) = Q^{i+n-1} \otimes (f^n, \ldots, f^1)(x).
\]

Dually for the functor \( E_*(2; -) \) there is

**Theorem 14.** For a sequence of mappings \( f^1: X^1 \to X^2, \ldots, f^n: X^n \to X^{n+1} \) of graded modules the following formula is taken place

\[
E_*(2; f^n, \ldots, f^1)(e_i \times x) = \begin{cases} 
  e_{i+n-1} \times (f^n, \ldots, f^1)(x), & i \geq n-1 \\
  0, & \text{in other cases}
\end{cases}
\]

To obtain the corresponding formula for the functor \( E_* \) it needs to use a comonad structure \( \gamma_*: E_* \to E_* \circ E_* \) and the formula

\[
\gamma_* \circ E_*(f^n, \ldots, f^1) = \sum E_*(E_*(f^n, \ldots, f^{n+1}), \ldots, E_*(f^n, \ldots, f^1)) \circ \gamma_*,
\]

where the sum is taken over all \( m \) and \( n_1, \ldots, n_m \) such that \( 1 \leq n_1 < \cdots < n_m < n \).

Passing to the Milnor coalgebra \( \mathcal{K} \) we obtain the operations

\[
\mathcal{K}(f^n, \ldots, f^1): \mathcal{K} \times X^1 \to \mathcal{K} \times X^{n+1},
\]

which are expressed by the formulas

\[
\mathcal{K}(f^n, \ldots, f^1)(y \otimes x) = y \cdot \xi_1 \otimes (f^n, \ldots, f^1)(x).
\]

Consider operations associated with a comultiplication \( \nabla \) of the Milnor coalgebra \( \mathcal{K} \). Denote

\[
\nabla(n) = \nabla \otimes 1 \cdots \otimes 1 - \cdots + (-1)^{n-1} 1 \otimes \cdots \otimes 1 \otimes \nabla: \mathcal{K}^\times n \to \mathcal{K}^\times n+1.
\]

Direct calculations show that the operations

\[
\langle \nabla(n); \nabla \rangle: \mathcal{K} \to \mathcal{K}^\times n+1, \ n \geq 2,
\]


are trivial on the elements $\xi_i^{2k}$. However on the other elements these operations in general are not trivial. For example there is the formula

$$(\nabla(2), \nabla)(\xi_i \xi_j) = \xi_{j-i}^{2i} \xi_i^{2i} \otimes \xi_i \otimes \xi_i \otimes \xi_i, \ i < j.$$ 

Denote by $\tilde{\nabla}$ a comultiplication in the tensor product $\mathcal{K} \otimes \mathcal{K}$,

$$\tilde{\nabla} = (1 \otimes T \otimes 1)(\nabla \otimes \nabla).$$

Put

$$\tilde{\nabla}(n) = \tilde{\nabla} \otimes 1 \cdots \otimes 1 \cdots + (-1)^{n-1} 1 \otimes \cdots \otimes 1 \otimes \tilde{\nabla}: (\mathcal{K} \otimes \mathcal{K})^{\times n} \to (\mathcal{K} \otimes \mathcal{K})^{\times n+1}.$$ 

Consider the operations

$$(\pi^{\times n+1}, \tilde{\nabla}(n), \ldots, \tilde{\nabla)): \mathcal{K}^{\otimes 2} \to \mathcal{K}^{\times n+1}.$$ 

Its restriction on the elements $x \otimes x \in \mathcal{K} \otimes \mathcal{K}$ we denote by

$$\Psi^n: \mathcal{K} \to \mathcal{K}^{\times n+1}.$$ 

From the formula of a comultiplication in the Milnor coalgebra directly follows the formula

$$\Psi^1(\xi_n) = \sum_{i<j} \xi_{n-i}^{2i} \xi_{n-j}^{2j} \otimes \xi_i \xi_j,$$

or in more general case

$$\Psi^1(\xi_n^{2m}) = \sum_{i<j} \xi_{n-i}^{2i+m} \xi_{n-j}^{2j+m} \otimes \xi_i^{2m} \xi_j^{2m}.$$ 

In particular for the primitive elements $\xi_1^{2m} \in \mathcal{K}$ we have the formula

$$\Psi^1(\xi_1^{2m}) = \xi_1^{2m} \xi_0^{2m} \otimes \xi_1^{2m}.$$ 

Similary for the operation $\Psi^2$ we have the formula

$$\Psi^2(\xi_n^{2m}) = \sum_{i<j} \xi_{n-i}^{2i+m} \xi_{n-j}^{2j+m} \otimes \xi_i^{2k+m} \xi_j^{2k+m} \otimes \xi_k^{2m} \xi_l^{2m}.$$ 

In particular for the primitive elements $\xi_1^{2m} \in \mathcal{K}$ we have the formula

$$\Psi^2(\xi_1^{2m}) = 0.$$ 

And so on.

6. $\cup_\infty - A_\infty$-Hopf algebras

To calculate higher differentials of the Adams spectral sequence we need to use not only the action of the Dyer-Lashof algebra, but the $E_\infty$ structure. However
this structure is too complicate. Some of the calculations were made in [6]. Here we’ll use only a part of the $E_\infty$-structure consisting of $\cup_i$-products.

A chain complex $A$ will be called a $\cup_\infty$-algebra if there are given operations $\cup_i: A \otimes A \to A$, $i \geq 0$, called $\cup_i$-products, increasing dimensions by $i$ and satisfying the relation

$$d(x \cup_i y) = d(x) \cup_i y + x \cup_i d(y) + x \cup_{i-1} y + y \cup_{i-1} x.$$  

A differential coalgebra $K$ will be called a $\cup_\infty$-Hopf algebra if there are given $\cup_i$-products $\cup_i: K \otimes K \to K$ satisfying the distributivity relation

$$\nabla(x \cup_i y) = \sum_k (x' \cup_{i-k} T^k y') \otimes (x'' \cup_k y''),$$  

where $\nabla(x) = \sum x' \otimes x''$, $\nabla(y) = \sum y' \otimes y''$, $T: K \otimes K \to K \otimes K$ is the permutation mapping, $T^k$ it’s $k$-th iteration.

**Theorem 15.** The cobar construction $FK$ over a $\cup_\infty$-Hopf algebra $K$ is a $\cup_\infty$-algebra. Moreover $\cup_i$-products $\cup_i: FK \otimes FK \to FK$ uniquely determined by the formula

$$[x] \cup_i [y] = \left\{ \begin{array}{ll} [x \cup_{i-1} y], & i \geq 1, \\ [x, y], & i = 0. \end{array} \right.$$  

and the relations

$$(x_1 x_2) \cup_i [y] = (x_1 \cup_i [y]) x_2 + x_1 (x_2 \cup_i [y]),$$

$$(x_1 x_2) \cup_i (y_1 y_2) = \sum_k (x_1 \cup_{i-k} T^k y_1) (x_2 \cup_k y_2) +$$

$$+ (x_1 \cup_i (y_1 y_2)) x_2 + x_1 (x_2 \cup_i (y_1 y_2)) +$$

$$+ ((x_1 x_2) \cup_i y_1) y_2 + x_1 ((x_1 x_2) \cup_i y_2) +$$

$$+ x_1 (x_2 \cup_i y_1) y_2 + y_1 (x_1 \cup_i y_2) x_2,$$

where $x_1, x_2, y_1, y_2 \in FK$, $y \in K$, $i \geq 1$.

Indeed, the products $[x_1, \ldots, x_n] \cup_i [y]$ are determined by the first relation

$$[x_1, \ldots, x_n] \cup_i [y] = \sum_{k=1}^n [x_1, \ldots, x_k \cup_i y, \ldots, x_n]$$

From the second relation it follows that to define $\cup_i$-products in general case, i.e.

$$[x_1, \ldots, x_n] \cup_i [y_1, \ldots, y_m]$$

it is sufficient to define only $\cup_i$-products $[x] \cup_i [y_1, \ldots, y_m]$.

We have

$$d([y_1, \ldots, y_m] \cup_{i+1} [x]) = \sum_{k=1}^m [y_1, \ldots, d(y_k), \ldots, y_m] \cup_{i+1} [x] +$$

$$+ \sum_{k=1}^m [y_1, \ldots, y'_k, y''_k, \ldots, y_m] \cup_{i+1} [x] +$$

$$+ [y_1, \ldots, y_m] \cup_{i+1} ([d(x)] + [x', x'']) +$$

$$+ [x_1, \ldots, x_n] + [x] + [x] + [x_1, \ldots, x_n]$$
Thus the product \([x] \cup_i [y_1, \ldots, y_m]\) is expressed through already defined products and products of the elements lesser dimensions. Hence \(\cup_i\)-products are determined by induction.

So this theorem gives us the formulas for \(\cup_i\)-product in the cobar construction. However they are inductive and not so simple even in the case when higher \(\cup_i\)-products (\(i \geq 1\)) on \(K\) are trivial, i.e. when \(K\) is a commutative Hopf algebra.

An \(\cup_\infty\)-Hopf algebra \(K\) will be called commutative if the coproduct \(\nabla: K \rightarrow K \otimes K\) is commutative.

**Theorem 16.** The cobar construction \(\widetilde{F}K\) over a commutative \(\cup_\infty\)-Hopf algebra is a commutative \(\cup_\infty\)-Hopf algebra. So the cobar construction over a commutative \(\cup_\infty\)-Hopf algebra may be iterated.

**Proof.** Define the coproduct \(\nabla: \widetilde{F}K \rightarrow \widetilde{F}K \otimes \widetilde{F}K\) putting

\[
\nabla[x_1, \ldots, x_n] = \sum [x_{i_1}, \ldots, x_{i_p}] \otimes [x_{j_1}, \ldots, x_{j_q}],
\]

where the sum is taken over all \((p, q)\)-shuffles of \(1, 2, \ldots, n\). Direct calculations show that the required relations are satisfied.

Consider the question about the structure on the homology of a \(\cup_\infty\)-Hopf algebra.

Consider the question about the structure on the homology of a \(\cup_\infty\)-Hopf algebra.

It is clear that on the homology \(K_*\) of a \(\cup_\infty\)-Hopf algebra \(K\) there are \(\cup_\infty\)-algebra structure, consisting of the operations \(\cup_i: K_* \otimes K_* \rightarrow K_*\) and \(A_\infty\)-coalgebra structure, consisting of the operations \(\nabla_n: K_* \rightarrow K_*^{\otimes n+2}\).

But besides that there are another operations of the form \(\Psi_{i,n}: K_* \otimes K_* \rightarrow K_*^{\otimes n+2}\).

To describe these operations and relations between them we introduce the notion of a \(\cup_\infty - A_\infty\)-Hopf algebra.

An \(A_\infty\)-coalgebra \(K\) will be called an \(\cup_\infty - A_\infty\)-Hopf algebra if on the cobar construction \(\widetilde{F}K\) there is given \(\cup_\infty\)-algebra structure satisfying the relations

\[
(x_1 x_2) \cup_i [y] = (x_1 \cup_i [y]) x_2 + x_1 (x_2 \cup_i [y]),
\]

\[
(x_1 x_2) \cup_i (y_1 y_2) = \sum_k (x_1 \cup_{i-k} T^k y_1) (x_2 \cup_k y_2) +
+ (x_1 \cup_i (y_1 y_2)) x_2 + x_1 (x_2 \cup_i (y_1 y_2)) +
+ ((x_1 x_2) \cup_i y_1) y_2 + y_1 ((x_1 x_2) \cup_i y_2) +
+ x_2 (x_1 \cup_i y_2) y_1 + x_1 (x_1 \cup_i y_2) x_2,
\]

where \(x_1, x_2, y_1, y_2 \in \widetilde{F}K, y \in K, i \geq 1\).

**Theorem 17.** If \(K\) is a \(\cup_\infty\)-Hopf algebra then its homology \(K_* = H_*(K)\) is \(\cup_\infty - A_\infty\)-Hopf algebra and there is an equivalence of \(\cup_\infty\)-algebras \(\widetilde{F}K \sim \widetilde{F}K\).
Proof. It is known [16] that the homology $K_*$ of a differential coalgebra $K$ is $A_\infty$-coalgebra and there are algebra mappings $\xi: \tilde{F}K_* \to FK$, $\eta: FK \to \tilde{F}K_*$ and an algebra chain homotopy $h: FK \to FK$ such that $\eta \circ \xi = \text{Id}$, $d(h) = \xi \circ \eta - \text{Id}$. So we need to define on $\tilde{F}K_*$ the $\cup_i$-products. Put on generators $[x] \cup_i [y] = \eta(\xi([x] \cup_i \xi[y]))$.

On the other elements the $\cup_i$-products determines by the relations.

Applying this theorem to the Milnor coalgebra we obtain

**Theorem 18.** The Milnor coalgebra $\mathcal{K}$ possesses $\cup_\infty - A_\infty$-Hopf algebra structure. The homology of the corresponding cobar construction $\tilde{F}
\mathcal{K}$ is isomorphic to the $E^{\infty}$ term of the Adams spectral sequence.

7. Differentials of the Adams spectral sequence and the Kervaire invariant problem

Apply developed methods to calculate higher differentials of the Adams spectral sequence or, that is the same to calculate the differential in $\tilde{F}\mathcal{K}$.

Since any element of $\mathcal{K}$ may be obtained from $\xi_0$ by applying $\cup_i$-products, we have

**Theorem 19.** The formulas for $\cup_i$-products in the Milnor coalgebra and the relations for $\cup_\infty$-algebra structure in the cobar construction $\tilde{F}\mathcal{K}$ completely determine the differential in $\tilde{F}\mathcal{K}$.

However the formulas for the differential are inductive and very complicated. So the next step in the calculation of the differential is to replace the Milnor coalgebra $\mathcal{K}$ and the cobar construction $\tilde{F}\mathcal{K}$ by more simply objects.

To do it consider the filtration of $\tilde{F}\mathcal{K}$ putting the filtration of the elements $\xi_{i_1} \cdot \ldots \cdot \xi_{i_n} \in K$ to be equal $n$. Then the first term of the corresponding spectral sequence will be isomorphic to the polynomial algebra $PS^{-1}X$ over the module $X$ generated by the elements $\xi_{i_1}^{2^k}$. We’ll denote elements of $PS^{-1}X$ as elements of the cobar construction, i.e. $[x_1, \ldots, x_n], x_i \in X$.

Note that there is an algebra mapping $\eta: FK \to PS^{-1}X$, given by the formula

$$\eta[x] = \begin{cases} [\xi_{i_1}^{2^k}], & x = \xi_{i_1}^{2^k}, \\ 0, & \text{in other cases.} \end{cases}$$

The inverse mapping $\xi: PS^{-1}X \to FK$ may be given by the formula

$$\xi([x_1, \ldots, x_n]) = [x_1, \ldots, x_n], \ x_1 \leq \cdots \leq x_n.$$ 

It is not an algebra mapping, but if $x \leq y$ then $\xi(x \cdot y) = \xi(x) \cdot \xi(y)$.

From here and from the Perturbation Theory it follows

**Theorem 20.** The polynomial algebra $PS^{-1}X$ possesses a $\cup_\infty$-algebra structure. The homology of the corresponding complex $\tilde{PS}^{-1}X$ is isomorphic to the homology of $\tilde{F}K$ and hence to the $E^{\infty}$ term of the Adams spectral sequence.

This theorem gives us the inductive formulas for the differential and $\cup_i$-products in $\tilde{PS}^{-1}X$.

Since any element of $\tilde{PS}^{-1}X$ may be obtained from $[\xi_0]$ by applying $\cup_i$-products, we have
Theorem 21. The formulas for $\cup_i$-products in the module $X$ and the relations for $\cup_i$-products in $\tilde{P}S^{-1}X$ completely determine the differential in $\tilde{P}S^{-1}X$.

Following the Adams notations we put $h_n = [\xi^{2n}_1]$. Using the formula $h_n \cup_1 h_n = h_{n+1}$ we prove the following theorem.

Theorem 22. For the differential in $\tilde{P}S^{-1}X$ on the elements $h_n$ there is the next formula

$$d(h_n) = \sum_{i=0}^{n-1} \xi_i h_{n-1}^i.$$ 

Proof. It is clear that $d(h_1) = h_{-1}h_1$. For the element $h_2$ we have

$$d(h_2) = d(h_1 \cup_1 h_1) = (h_{-1}h_1 \cup_1 h_1 + h_1 \cup_1 (h_{-1}h_1) =$$

$$= (h_{-1}h_1) \cup_2 (h_{-1}h_1) = h_{-1}h_2 + h_0h_1^2.$$

Suppose the required formula is true for $n$ and prove it is true for $n + 1$. We have

$$d(h_{n+1}) = d(h_n \cup_1 h_n) = d(h_n) \cup_1 h_n + h_n \cup_1 d(h_n) =$$

$$= (\sum_{i=0}^{n-1} [\xi_i] h_{n-1}^i) \cup_2 \left(\sum_{i=0}^{n-1} [\xi_i] h_{n-i}^i\right) =$$

$$= \sum_{i=0}^{n} [\xi_i] h_{n+1-i}^i.$$ 

Apply this formula to solve the Kervaire invariant problem.

Theorem 23. For the elements $h_n^2$, $n \geq 4$, of the Adams spectral sequence there is the following formula

$$d_5(h_n^2) = h_1^2g_{n-4}h_{n-1}.$$ 

Proof. The differential on the elements $h_n^2$ is given by the formula

$$d(h_n^2) = d(h_n)h_n + h_n d(h_n) = d(h_n) \cup_1 d(h_n) =$$

$$= (h_{-1}h_n + h_0h_{n-1}^2 + \ldots) \cup_1 (h_{-1}h_n + h_0h_{n-1}^2 + \ldots) =$$

$$= h_{-1}h_n^2 + h_1h_{n-1}^4 + [\xi_2^2] h_{n-2}^8 + \ldots.$$ 

After the factorization over $h_{-1}$ we obtain the formula

$$d(h_n^2) = h_1h_{n-1}^4 + [\xi_2^2] h_{n-2}^8 + \ldots.$$ 

Write the differential $d$ as the sum $d = d_1 + d_2 + \ldots$, where $d_n$ increase the filtration by $n$. Note that the complex $\tilde{P}S^{-1}X$ with the differential $d_1$ is isomorphic to the $E^1$ term of the Adams spectral sequence. The corresponding homology is isomorphic to the $E^2$ term of this spectral sequence.

In the $E^1$ term there is the formula

$$d_1(h_n)\cup_1 [\xi^{2n-2}_1]h^2_2 + [\xi^{2n-1}_2]h^2_2 \cup_1 (\ldots) = h_1^4 + \ldots.$$
From this formula it follows that in the $E_2$ term the differential $d_3$ on the elements $h_n^2$ is equal to zero. Improve the elements $h_n^2$ and consider the elements

$$
\tilde{h}_n^2 = h_n^2 + h_1(h_{n-1}[\xi_2^{n-2}]^2 + h_{n-2}[\xi_2^{n-1}])
$$

The differential on the elements $[\xi_2^{2n-1}]$, $[\xi_2^{2n-2}]^2$ is given by the formula

\[
d[\xi_2^{2n-1}] = h_{n-1}h_n + h_0[\xi_2^{2n-2}]^2 + h_1h_{n-2}h_{n-1}[\xi_2^{n-2}] + \ldots,
\]
\[
d[\xi_2^{2n-2}]^2 = h_{n-2}h_n + h_3h_{n-1}[\xi_2^{n-3}]^4 + \ldots.
\]

Hence

\[
d(\tilde{h}_n^2) = h_1^2(h_{n-1}[\xi_2^{2n-3}]^4 + h_{n-3}[\xi_2^{2n-1}] + h_{n-2}h_{n-1}[\xi_2^{2n-2}]) + \ldots.
\]

Thus $d_i(\tilde{h}_n^2) = 0$ if $i \leq 4$ and

$$
d_5(\tilde{h}_n^2) = h_1^2(h_{n-1}[\xi_2^{2n-3}]^4 + h_{n-3}[\xi_2^{2n-1}] + h_{n-2}h_{n-1}[\xi_2^{2n-2}]).
$$

From the formula (*) it follows that in the $E_1$ term the element $h_{n-3}^4[\xi_2^{n-1}]$ is homological to the element $(h_{n-3}[\xi_2^{2n-4}]^2 + h_{n-4}[\xi_2^{2n-3}])h_{n-1}h_n$. Hence the element $d_5(\tilde{h}_n^2)$ is homological to the element

$$
h_1^2([\xi_2^{2n-3}]^4 + h_{n-3}[\xi_2^{2n-4}]^2h_n + h_{n-4}[\xi_2^{2n-3}]h_n + h_{n-2}h_{n-1}[\xi_2^{2n-2}])h_{n-1}.
$$

Direct calculations show that the expression in the parentheses is a cycle in $E_1$ of the filtration 4. The corresponding homology class in the $E^2$ term of the Adams spectral sequence is denoted by $g_{n-4}$.

So we have the formula

$$
d_5(h_n^2) = h_1^2g_{n-4}h_{n-1}.
$$

**Corollary.** The elements $h_n^2$ survive till the $E_6$ term of the Adams spectral sequence.

Indeed it is known that the element $g_0h_3$ is equal to zero in the $E_2$ term of the Adams spectral sequence. Hence the elements $g_{n-4}h_{n-1}$ also are equal to zero. Thus $d_5(h_n^2) = 0$ and the elements $h_n^2$ survive till the $E_6$ term.

**REFERENCES**

1. Adams J.F. On the structure and applications of the Steenrod algebra. Comm. Math. Helv. 1958, v.32, p.180–214.
2. Stasheff J.D. Homotopy associativity of $H$-spaces. Trans. Amer. Math. Soc. 1963, v. 108, N 2, p.275–312.
3. May J.P. The geometry of iterated loop spaces. Lect. Notes in Math. 1972, v.271.
4. Smirnov V.A. On the cochain complex of a topological space. Mat. Sbornik, 1981, v. 115, N 1, p.146–158.
5. Smirnov V.A. Homotopy theory of coalgebras. Izvestia AN SSSR, 1985, v.49, N.6, p.1302–1321.
6. Smirnov V.A. Secondary operations in the homology of the operad $E$. Izvestia RAN, 1992, v.56, N 2, p.449-468.
7. Steenrod N.E. Cohomology invariants of mappings. Ann. of Math. 1949, v.50, p.954–988.
8. Peterson F.P. Functional cohomology operations. Trans. Amer. Math. Soc. 1957, v.86, p.187–197.
9. Smirnov V.A. Functional homology operations and weak homotopy type. Mat. zametki, 1989, v.45, N.5, p.76–86.
10. Browder W. The Kervaire invariant of framed manifolds and its generalizations. Annals of Math. 1969, v. 90, p.157–186.
11. Barrat M.G., Jones J.D.S., Mahowald M.E. The Kervaire invariant and the Hopf invariant. Lecture Notes in Math. 1987, v. 1286, p. 135–173.
12. Bousfield A.K., Kan D.M. The homotopy spectral sequence of a spaces with coefficients in a ring. Topology, 1972, v.11, p.79–106.
13. Gugenheim V.K., Lambe L.A., Stasheff J.D. Perturbation theory in Differential Homological Algebra. Ill. J. of Math. 1991, v. 35, N 3, p.357–373.
14. Araki S., Kudo T. Topology of $H_n$-spaces and $H_n$-squaring operations. Mem. Fac. Sci. Kyusyu Univ., Ser. A, 1956, v.10, N 2, p.85–120.
15. Dyer E., Lashof R.K. Homology of iterated loop spaces. Amer. Jour. of Math. 1962, v.84, N 1, p.35–88.
16. Kadeishvili T.V. On the homology theory of fibre spaces. UMN. 1980. v. 35, N. 3. p. 183–188.

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