K-Witt bordism in characteristic 2

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Abstract

This note provides a computation of the bordism groups of $K$-Witt spaces for fields $K$ with characteristic 2. We provide a complete computation for the unoriented bordism groups. For the oriented bordism groups, a nearly complete computation is provided as well a discussion of the difficulty of resolving a remaining ambiguity in dimensions equivalent to 2 mod 4. This corrects an error in the $\text{char}(K) = 2$ case of the author’s prior computation of the bordism groups of $K$-Witt spaces for an arbitrary field $K$.

In [1], an $n$-dimensional $K$-Witt space, for a field $K$, is defined to be an oriented compact $n$-dimensional PL stratified pseudomanifold $X$ satisfying the $K$-Witt condition that the lower-middle perversity intersection homology group $I^{m}_{k}(L; K)$ is 0 for each link $L^{2k}$ of each stratum of $X$ of dimension $n - 2k - 1$, $k > 0$. Following the definition of stratified pseudomanifold in [2], $X$ does not possess codimension one strata. Orientability is determined by the orientability of the top (regular) strata. This definition generalizes Siegel’s definition in [11] of $Q$-Witt spaces (called there simply “Witt spaces”). The motivation for this definition is that such spaces possess intersection homology Poincaré duality $I^{m}_{k}(X; K) \cong \text{Hom}(I^{m}_{n-k}(X; K), K)$.

The author’s paper [1] concerns $K$-Witt spaces and, in particular, a computation of the bordism theory $\Omega^{K-\text{Witt}}$ of such spaces. However, there is an error in [1] in the computation of the coefficient groups $\Omega^{K-\text{Witt}}_{4k+2}$ when $\text{char}(K) = 2$.

It is claimed in [1] that $\Omega^{K-\text{Witt}}_{4k+2} = 0$. When $\text{char}(K) > 2$, the null-bordism of a $4k + 2$ dimensional $K$-Witt space $X$ is established in [1] by following Siegel’s computation [11] for $Q$-Witt spaces by first performing a surgery to make the space irreducible and then performing

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1There is a minor error in [1] in that Witt spaces are stated to be irreducible, meaning that there is only a single top dimensional stratum. In general, this should not be part of the definition of a $K$-Witt space; cf. [11]. However, as every $K$-Witt space of dimension $> 0$ is bordant to an irreducible $K$-Witt space (see [11] page 1099), this error does not affect the bordism group computations of [1]. It is not true that every 0-dimensional $K$-Witt space is bordant to an irreducible $K$-Witt space, but in this dimension the computations all reduce to the manifold theory and the computations given for this dimension in [1] are also correct if one removes irreducibility from the definition.
a sequence of singular surgeries to obtain a space \( X' \) such that \( I^m H_{2k+1}(X'; K) = 0 \). The \( K \)-Witt null-bordism of \( X \) is the union of the trace of the surgeries from \( X \) to \( X' \) with the closed cone \( cX' \). One performs the singular surgeries on elements \([z] \in I^m H_{2k+1}(X; K)\) such that \([z] \cdot [z] = 0\), where \( \cdot \) denotes the Goresky-MacPherson intersection product \([2]\). As the intersection product is skew symmetric on \( I^m H_{2k+1}(X; K)\), such a \([z]\) always exists. The error in \([1]\) stems from overlooking that this last fact is not necessarily true in characteristic 2, where skew symmetric forms and symmetric forms are the same thing and so skew-symmetry does not imply \([z] \cdot [z] = 0\).

**Corrected computations.** To begin to remedy the error of \([1]\), we first observe that it remains true in characteristic 2 that the map\([3]\) \( w : \Omega^Z_{2k+2} \rightarrow W(Z_2) \) is injective, where \( W(Z_2) \) is the Witt group of \( Z_2 \) and \( w \) takes the bordism class \([X] \) to the class of the intersection form on \( I^m H_{2k+1}(X; Z_2) \). For \( k > 0 \), this fact can be proven as it is proven for \( w : \Omega^K_{4j} \rightarrow W(K) \), \( j > 0 \), in \([1]\): if one assumes that the intersection form on \( X \) represents 0 in \( W(Z_2) \) then the intersection form is split, in the language of \([7]\); see \([7]\) Corollary III.1.6]. And so \( I^m H_{2k+1}(X; Z_2) \) will possess an isotropic (self-annihilating) element by \([7]\) Lemma I.6.3. The surgery argument can then proceed as \( W(Z_2) \cong Z_2 \) (see \([7]\) Lemma IV.1.5]), it follows that \( \Omega^Z_{2k+2} \) is either 0 or \( Z_2 \).

This argument does not hold for \( 4k+2 = 2 \) as in this case the dimensions are not sufficient to guarantee that every middle-dimensional intersection homology class is representable by an irreducible element, which is necessary for the surgery argument; see \([1]\) Lemma 2.2]. However, all 2-dimensional Witt spaces must have at worst isolated singularities, and so in particular such a space must have the form \( X \cong (\Pi S_i) / \sim \), where the \( S_i \) are closed oriented surfaces and the relation \( \sim \) glues them together along various isolated points. But then \( X \) is bordant to \( \Pi S_i \). This can be seen via a sequence of pinch bordisms as defined by Siegel \([1]\) Section II] that pinch together the regular neighborhoods of sets of points of \( \Pi S_i \). To see that the bordism is via a Witt space, it is only necessary to observe that the link of the interior cone point in each such pinch bordism will be a wedge of \( S^2 \)s, and it is easy to compute that \( I^m H_1(\vee_i S^2; K) = 0 \) for any \( K \). But now, since all closed oriented surfaces bound, \( \Omega^Z_{2k+2} \) = 0. This special case was also over-looked in \([1]\), though this argument holds for any field \( K \) and is consistent with the claim of \([1]\) that \( \Omega^K_{2} = 0 \) for all \( K \).

Thus we have shown that \( w : \Omega^Z_{2k+2} \rightarrow W(Z_2) \cong Z_2 \) is an injection for \( k \geq 0 \), trivially

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2Recall from \([1]\) Corollary 4.3] that the bordism groups depend only on the characteristic of the field, so for characteristic 2 it suffices to consider \( K = Z_2 \).

3There is one other possible complication due to characteristic 2 that must be checked but that does not provide difficulty in the end: For characteristic not equal to 2, every split form is isomorphic to an orthogonal sum of hyperbolic planes \([7]\) Lemma I.6.3], and this appears to be used in the proof of Theorem 4.4 of \([1]\], which is heavily referenced in \([1]\). For characteristic 2, one can only conclude that a split form is isomorphic to one with matrix \( \begin{pmatrix} 0 & I \\ I & A \end{pmatrix} \) for some matrix \( A \). However, a detailed reading of the proof of \([1]\) Theorem 4.4, particularly page 1097] reveals that it is sufficient to have a basis \( \{\alpha, \beta, \gamma_1, \ldots, \gamma_{2m}\} \) such that \( \alpha \cdot \alpha = \alpha \cdot \gamma_i = 0 \) for all \( i \) and \( \alpha \cdot \beta = 1 \), and this is certainly provided by a form with the given matrix.

4Recall that \( Z_2 \)-Witt spaces are assumed to be \( Z \)-oriented, though see below for more on orientation considerations.
so for $k = 0$. Unfortunately, the question of surjectivity of $w$ in dimensions $4k + 2$ is more complicated and not yet fully resolved. We can, however, make the following observation: if $X$ is a $\mathbb{Z}_2$-Witt space of dimension $4k - 2$, then $w([X \times \mathbb{C}P^2]) = w([X])$. So if there is a non-trivial element of $\Omega^\mathbb{Z}_2 - \text{Witt}_{4k-2}$, then there is a non-trivial element of $\Omega^\mathbb{Z}_2 - \text{Witt}_{4k+2}$.

Putting this together with the computations from [1] of $\Omega^K - \text{Witt}^*$ in dimension $\not\equiv 4k + 2 \mod 4$ (which remain correct), we have the following theorem:

**Theorem 1.** For a field $K$ with $\text{char}(K) = 2$, $\Omega^K - \text{Witt} = \Omega^\mathbb{Z}_2 - \text{Witt}$, and for $k \geq 0$,

1. $\Omega^K_0 - \text{Witt} \cong \mathbb{Z}$,

2. for $k > 0$, $\Omega^K_{4k} - \text{Witt} \cong \mathbb{Z}_2$, generated by $[\mathbb{C}P^{2k}]$,

3. $\Omega^K_{4k+3} = \Omega^K_{4k+1} = 0$,

4. Either

   (a) $\Omega^K_{4k+2} = 0$ for all $k$, or

   (b) there exists some $N > 0$ such that $\Omega^K_{4k+2} = 0$ for all $k < N$ and $\Omega^K_{4k+2} \cong \mathbb{Z}_2$ for all $k \geq N$.

We will provide below some further discussion of the difficulties of deciding which case of (4) holds after discussing unoriented bordism.

**Remark.** Independent of the existence or value of $N$ in condition (4) of the theorem, the computations from [1] of $\Omega^*_K - \text{Witt}$ in dimension $\not\equiv 4k + 2 \mod 4$ (which remain correct), we have the following theorem:

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**Unoriented bordism.** Given the motivation to recognize spaces that possess a form of Poincaré duality, it seems reasonable to consider $K$-Witt spaces that are $K$-oriented. This has no effect when $\text{char}(K) \neq 2$, in which case $K$-orientability is equivalent to $\mathbb{Z}_2$-orientability as considered in [1]. But when $\text{char}(K) = 2$, all pseudomanifolds are $\mathbb{Z}_2$-orientable, which is equivalent to being $K$ orientable, and the Poincaré duality isomorphism $I^m H_k(X; K) \cong \text{Hom}(I^m H_{n-k}(X; K), K)$ holds for all such compact pseudomanifolds satisfying the $K$-Witt condition.

If we allow $K$-Witt spaces and $K$-Witt bordism using $K$-orientations, then for $\text{char}(K) = 2$ we are essentially talking about unoriented bordism, so to clarify the notation, let us denote the resulting bordism groups by $N^*_K - \text{Witt}$. These groups can be computed as follows:

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5Recall that the Künneth theorem holds within a single perversity when one term is a manifold, so we can compute the intersection forms of such product spaces in the usual way; see e.g. [6].

6Since these are geometric bordism groups, they vanish in negative degree.

7One could also define unoriented bordism groups of unoriented compact PL pseudomanifolds satisfying the $K$-Witt condition with $\text{char}(K) \neq 2$, but it is not clear how to study such groups by the present techniques, as there is no reason to expect that $I^m H_*(X; K)$ would satisfy Poincaré duality for such a space $X$. 

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Theorem 2. For a field $K$ with char$(K) = 2$ and for $i \geq 0$,

$$
\mathcal{N}_i^{K-Witt} \cong \begin{cases} 
\mathbb{Z}_2, & i \equiv 0 \pmod{2}, \\
0, & i \equiv 1 \pmod{2}.
\end{cases}
$$

Since writing [1], the author has discovered that this theorem is also provided without detailed proof by Goresky in [4, page 498]. We provide here the details:

Proof. It continues to hold that the local Witt condition depends only on the characteristic of $K$ for the reasons provided in [1], so we may assume $K = \mathbb{Z}_2$. To see that $\mathcal{N}_Z^{2-Witt} = 0$ for $n$ odd, we simply note that $X$ bounds the closed cone $\tilde{c}X$, which is a $\mathbb{Z}_2$-Witt space. The map $w : \mathcal{N}_Z^{2^k-Witt} \to W(\mathbb{Z}_2) \cong \mathbb{Z}_2$ is onto for each $k > 0$, as the intersection pairing on the $\mathbb{Z}_2$-coefficient middle-dimensional homology of the real projective space $\mathbb{R}P^{2k}$ corresponds to the generator of $W(\mathbb{Z}_2)$ represented by the matrix (1). Furthermore, $w$ is injective for $k > 1$ as in the preceding surgery argument, which does not rely on whether or not $X$ is oriented, only on the existence of the intersection pairing over $\mathbb{Z}_2$. In dimension 0, we have unoriented manifold bordism of points, so $\mathcal{N}_Z^{0-Witt} \cong \mathbb{Z}_2$. Finally, as in the argument above for $\Omega_2^{Z_2-Witt}$, the group $\mathcal{N}_Z^{2-Witt}$ must be isomorphic to $\mathbb{Z}_2$ as $w$ maps $\mathbb{R}P^2$ onto the non-trivial element of $W(\mathbb{Z}_2) \cong \mathbb{Z}_2$.

Remark. An even simpler version of the argument of [1] implies that as a generalized homology theory

$$
\mathcal{N}_n^{K-Witt}(X) \cong \bigoplus_{r+s=n} H_r(X; \mathcal{N}_s^{K-Witt})
$$

for char$(K) = 2$, as in this case one no longer needs a separate argument to handle the odd torsion that can arises in $H_n(X; \Omega_0^{K-Witt})$ as a result of $\Omega_0^{K-Witt} \cong \mathbb{Z}$ not being 2-primary.

Further discussion of oriented bordism. We next provide some results that demonstrate the difficulty of determining which case of item (4) of Theorem 1 holds.

We will first see that $w([M]) = 0$ for any $\mathbb{Z}$-oriented manifold: Since dimension mod 2 is the only invariant of $W(\mathbb{Z}_2)$, this is a consequence of the following lemma, recalling that for a manifold, $I^nH_*(M) = H_*(M)$.

Lemma. Let $M$ be a closed connected $\mathbb{Z}$-oriented manifold of dimension $4k + 2$. Then $\dim(H_{2k+1}(M; \mathbb{Z}_2)) \equiv 0 \pmod{2}$.

Proof. By the universal coefficient theorem,

$$
H_{2k+1}(M; \mathbb{Z}_2) \cong (H_{2k+1}(M) \otimes \mathbb{Z}_2) \oplus (H_{2k}(M) * \mathbb{Z}_2),
$$

As observed in the proof of [7, Lemma III.3.3], rank mod 2 yields a homomorphism $W(F) \to \mathbb{Z}_2$ for any field $F$. Since we know that $W(\mathbb{Z}_2) \cong \mathbb{Z}_2$ and that (1), which has rank 1, is a generator of $W(F)$ (it is certainly non-zero, using [7, Lemma I.6.3 and Lemma III.1.6]), it follows that rank mod 2 determines the isomorphism.
where the asterisk denotes the torsion product. Let $T_*(M)$ denote the torsion subgroup of $H_*(M)$, and let $T^2_*(M)$ denote $T_*(M) \otimes \mathbb{Z}_2 \cong T_*(M) \ast \mathbb{Z}_2$; the isomorphism follows from basic homological algebra because $T_*(M)$ is a finite abelian group. $T^2_*(M)$ is a direct sum of $\mathbb{Z}_2$ terms. Then $H_{2k+1}(M) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2^B \oplus T^2_{2k+1}(M)$, where $B$ is the $2k + 1$ Betti number of $M$, and $H_{2k}(M) \ast \mathbb{Z}_2 \cong T^2_{2k}(M)$. Thus $H_{2k+1}(M; \mathbb{Z}_2) \cong \mathbb{Z}_2^B \oplus T^2_{2k+1}(M) \oplus T^2_{2k}(M)$. Since $M$ is a closed $\mathbb{Z}$-oriented manifold, there is a nondegenerate skew-symmetric intersection form on $H_{2k+1}(M; \mathbb{Q})$, and so $B$ is even. Since $M$ is a closed $\mathbb{Z}$-oriented manifold, the nonsingular linking pairing $T_{2k+1}(M) \otimes T_{2k}(M) \to \mathbb{Q}/\mathbb{Z}$ gives rise to an isomorphism $T_{2k+1}(M) \cong \text{Hom}(T_{2k}(M), \mathbb{Q}/\mathbb{Z})$, and since $\text{Hom}(\mathbb{Z}_n, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}_n$, it follows that $T_{2k+1}(M) \cong T_{2k}(M)$. Therefore $T^2_{2k+1}(M) \cong T^2_{2k}(M)$. Thus $H_{2k+1}(M; \mathbb{Z}/2)$ consists of an even number of $\mathbb{Z}_2$ terms. 

\textbf{Remark.} Since the lemma utilizes only integral Poincaré duality and the universal coefficient theorem, it follows that, in fact, $w([X]) = 0$ for any IP space\footnote{Also called “intersection homology Poincaré spaces,” though this is perhaps a misnomer as “Poincaré spaces” are generally not required to be manifolds while IP spaces are still expected to be pseudomanifolds.} these are spaces that satisfy local conditions guaranteeing that intersection homology Poincaré duality holds over the integers and that a universal coefficient theorem holds (see \cite{[3][10]}). 

A slightly more elaborate argument demonstrates that it is also not possible to have $w([X]) \neq 0$ if $X$ is a $\mathbb{Z}$-oriented $\mathbb{Z}_2$-Witt space with at worst isolated singularities:

\textbf{Proposition.} Let $X$ be a closed $\mathbb{Z}$-oriented $4k + 2$-dimensional $\mathbb{Z}_2$-Witt space with at worst isolated singularities. Then $w([X]) = 0$.

\textbf{Proof.} Since $X$ has at worst point singularities, it follows from basic intersection homology calculations (see \cite{[2]} Section 6.1) that $H^{2n}H_{2k+1}(X; \mathbb{Z}_2) \cong \text{im}(H_{2k+1}(M; \mathbb{Z}_2) \to H_{2k+1}(M, \partial M; \mathbb{Z}_2))$, where $M$ is the compact $\mathbb{Z}$-oriented PL $\partial$-manifold obtained by removing an open regular neighborhood of the singular set of $X$. We will show that if $[z] \in \text{im}(H_{2k+1}(M; \mathbb{Z}_2) \to H_{2k+1}(M, \partial M; \mathbb{Z}_2))$, then the intersection product $[z] \cdot [z] = 0$. It follows that the intersection pairing on $H^{2n}H_{2k+1}(X; \mathbb{Z}_2)$ is split by \cite{[7]} Lemma III.1.1, since then there can be no non-trivial anisotropic subspace. This implies that $w([X]) = 0$ by the definition of the Witt group.

The following argument that $[z] \cdot [z] = 0$ was suggested by “Martin O” on the web site MathOverflow \cite{[2]}. By Poincaré duality, it suffices to show that $\alpha \cup \alpha = 0$, where $\alpha$ is the Poincaré dual of $[z]$ in $H^{2k+1}(M, \partial M; \mathbb{Z}_2)$. But now $\alpha \cup \alpha = S_q^{2k+1} \alpha = S_q S_q^{2k} \alpha = \beta^* S_q^{2k} \alpha$, where $\beta^*$ is the Bockstein associated with the sequence $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$ (see \cite{[5]} Section 4.1]). In the case at hand, this is the Bockstein $\beta^* : H^{4k+1}(M, \partial M; \mathbb{Z}_2) \to H^{4k+2}(M, \partial M; \mathbb{Z}_2)$. But this map is trivial. To see this, observe that there is a commutative
diagram

\[
\begin{array}{ccc}
H^{4k+1}(M, \partial M; \mathbb{Z}_2) & \xrightarrow{\beta^*} & H^{4k+2}(M, \partial M; \mathbb{Z}_2) \\
\cong & & \cong \\
H_1(M; \mathbb{Z}_2) & \xrightarrow{\beta_*} & H_0(M; \mathbb{Z}_2),
\end{array}
\]

where \(\beta_*\) is the homology Bockstein and the vertical maps are Poincaré duality. The existence of this diagram follows as in [8, Lemma 69.2]. But now \(\beta_* : H_1(M; \mathbb{Z}_2) \to H_0(M; \mathbb{Z}_2)\) is trivial, as the standard map \(\times 2 : H_0(M; \mathbb{Z}_2) \to H_0(M; \mathbb{Z}_4)\) is injective. \(\square\)

Hence any candidate to have \(w([X]) = 1\) must have singular set of dimension \(> 0\) and must not be an IP space. Given that all \(K\)-Witt spaces for \(\text{char}(K) \neq 2\) are \(K\)-Witt bordant to spaces with at worst isolated singularities [11, 1], it is unclear how to proceed to determine whether \(\mathbb{Z}_2\)-Witt spaces with \(w([X]) = 1\) exist. One method to prove that they do not would be to try to show “by hand” that every \(\mathbb{Z}_2\)-Witt space is \(\mathbb{Z}_2\)-Witt bordant to a space with at most isolated singularities, but the only proof currently known to the author of this fact for fields of other characteristics utilizes the bordism computations of [11, 1].

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