THE GEOMETRY OF GENERALIZED LAMÉ EQUATION, I

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Abstract. In this paper, we prove that the spectral curve $\Gamma_n$ of the
generalized Lamé equation with the Treibich-Verdier potential

$$y''(z) = \left[ \sum_{k=0}^{3} n_k(n_k + 1)\wp(z + \omega^k_2|\tau) + B \right] y(z), \quad n_k \in \mathbb{Z}_{\geq 0}$$

can be embedded into the symmetric space $\text{Sym}^N E_\tau$ of the $N$-th copy
of the torus $E_\tau$, where $N = \sum n_k$. This embedding induces an addition
map $\sigma_n(\cdot|\tau)$ from $\Gamma_n$ onto $E_\tau$. The main result is to prove that the
degree of $\sigma_n(\cdot|\tau)$ is equal to $\sum_{k=0}^{3} n_k(n_k + 1)/2$.

This is the first step toward constructing the premodular form associated
with this generalized Lamé equation.

1. Introduction

Throughout the paper, we let $\tau \in \mathbb{H} = \{ \tau \in \mathbb{C} | \text{Im } \tau > 0 \}$, $E_\tau = \mathbb{C}/\Lambda_\tau$
be a flat torus with the lattice $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$, and $\wp(z) = \wp(z|\tau)$ be the
Weierstrass elliptic function with periods $\omega_1 = 1$, $\omega_2 = \tau$ and $\omega_3 = 1 + \tau$.
Let $\zeta(z) = \zeta(z|\tau) := -\int^{z}\wp(\xi|\tau)d\xi$ be the Weierstrass zeta function with
two quasi-periods $\eta_j(\tau)$, $j = 1, 2$:

$$\eta_j(\tau) = \zeta(z + \omega_j|\tau) - \zeta(z|\tau), \quad j = 1, 2,$$

and $\sigma(z) = \sigma(z|\tau)$ be the Weierstrass sigma function defined by $\sigma(z) := \exp \int^{z}\zeta(\xi)d\xi$. Notice that $\zeta(z)$ is an odd meromorphic function with simple
poles at $\Lambda_\tau$ and $\sigma(z)$ is an odd entire function with simple zeros at $\Lambda_\tau$. For
$z \in \mathbb{C}$ we denote $[z] := z \pmod{\Lambda_\tau} \in E_\tau$. For a point $[z]$ in $E_\tau$ we often
write $z$ instead of $[z]$ to simply notations when no confusion arises.

In this paper, we consider the complex second order ODE:

$$y''(z) = (I_n(z; \tau) + B)y(z), \quad z \in \mathbb{C},$$

where $B \in \mathbb{C}$ and

$$I_n(z; \tau) := \sum_{k=0}^{3} n_k(n_k + 1)\wp(z + \omega^k_2|\tau)$$

with $\omega_0 = 0$ and $n_k \in \mathbb{Z}_{\geq 0}$ for all $k$. We also denote $I_n(z; B, \tau) := I_n(z; \tau) + B$. The $I_n(z; \tau)$ is called the Treibich-Verdier potential. In [24, 25] Treibich
and Verdier proved that $I_n(z; \tau)$ is an algebro-geometric solution of the KdV
hierarchy equations or equivalently a finite-gap potential. Later Gesztesy and Weikard [9] generalized their result to prove that any Picard potential is an algebro-geometric solution of the KdV hierarchy equations.

We briefly recall the notion of algebro-geometric solutions. Let \( y_1(z; B), \ y_2(z; B) \) be two solutions of (1.2) and \( \Phi(z; B) := y_1(z; B)y_2(z; B) \). Then a direct computation show that \( \Phi \) satisfies the following third order ODE (called the second symmetric product equation of (1.2)):

\[
(1.4) \quad \Phi'''(z; B) - 4(I_n(z; \tau) + B)\Phi'(z; B) - 2I_n'(z; \tau)\Phi(z; B) = 0.
\]

Multiplying \( \Phi \) and integrating (1.4), we obtain that

\[
(1.5) \quad \Phi'(z; B)^2 - 2\Phi(z; B)\Phi''(z; B) + 4(I_n(z; \tau) + B)\Phi(z; B)^2
\]

is independent of \( z \). Let \( Q_n(B) = Q_n(B; \tau) \) denote the above expression of (1.5). Then \( I_n(z; \tau) \) is an algebro-geometric solution of the KdV hierarchy equations if \( Q_n(B) \) is a polynomial of \( B \) for some solution \( \Phi(z; B); \) see [9]. In this case, \( Q_n(B) \) is known as the spectral polynomial and \( \Gamma_n = \Gamma_n(\tau) := \{(B, W)|W^2 = Q_n(B; \tau)\} \) is called the spectral curve of the potential \( I_n(z; \tau) \).

When \( n = (n, 0, 0, 0) \), the potential \( n(n + 1)\varphi(z|\tau) \) is called the Lamé potential and

\[
(1.6) \quad y''(z) = (n(n + 1)\varphi(z|\tau) + B)y(z), \quad z \in \mathbb{C}
\]

is called the Lamé equation. The fact that the Lamé potential is a finite-gap potential was first discovered by Ince [12]. We refer the readers to the classic texts [11, 17, 26] and recent works [1, 2, 8, 16] for the Lamé equation. Therefore, (1.2) is called a generalized Lamé equation (GLE) in this paper. See [6, 10, 19, 20, 21, 22, 23] and references therein for recent developments of GLE (1.2).

In this paper, we want to study (1.2) from the aspect of monodromy representation. Clearly, (1.2) can be described as a Fuchsian equation defined on the torus \( E_\tau \) with four regular singularities at \( \frac{\omega_k}{2} \)'s. The local exponents, i.e. the roots of its indicial equation at \( \frac{\omega_k}{2} \) are \( -n_k, n_k + 1 \). Since \( n_k \in \mathbb{Z} \), it is well-known (cf. [10]) that \( I_n(z) \) is a Picard potential, i.e. any solution of (1.2) is meromorphic in \( \mathbb{C} \). By applying \( x = \varphi(z) \), (1.2) can be transformed a second order ODE on \( \mathbb{C}P^1 \) with four regular singular points at \( e_k \)'s and \( \infty \), where \( e_k := \varphi(\frac{\omega_k}{2}) \), \( k = 1, 2, 3 \). This is the well-known Heun equation with four singular points, which has been extensively studied since its isomonodromic deformation give arise to the famous Painlevé VI equation; see e.g. [13].

For GLE (1.2), we proved in [4] that the corresponding isomonodromic deformation equation is the elliptic form of the Painlevé VI equation. In this paper, we also denote GLE (1.2) by \( H(n, B, \tau) \).

The monodromy representation \( \rho_\tau \) of (1.2) is a group homomorphism from \( \pi_1(E_\tau) \) to \( SL(2, \mathbb{C}) \) because \( I_n(z; \tau) \) is a Picard potential. Since \( \pi_1(E_\tau) \) is abelian, the monodromy group is always abelian. Thus the monodromy group is comparably easier to compute for (1.2) on \( E_\tau \) than the Heun equation.
on \(\mathbb{CP}^1\). In terms of any linearly independent solutions \(y_1(z)\) and \(y_2(z)\), the monodromy group is generated by two matrices \(M_1, M_2 \in SL(2, \mathbb{C})\) satisfying

\[(1.7) \quad (y_1, y_2)(z + \omega_i) = (y_1(z), y_2(z))M_i, \quad i = 1, 2, \quad \text{and} \quad M_1M_2 = M_2M_1.\]

By (1.7), \(M_1\) and \(M_2\) can be normalized to satisfy one of the followings.

a) If \(\rho_\tau\) is completely reducible, then

\[(1.8) \quad M_1 = \begin{pmatrix} e^{-2\pi ir} & 0 \\ 0 & e^{2\pi ir} \end{pmatrix}, M_2 = \begin{pmatrix} e^{2\pi ir} & 0 \\ 0 & e^{-2\pi ir} \end{pmatrix}, \quad (r, s) \in \mathbb{C}^2.\]

See Section 2, where we will see that \((r, s) \notin \frac{1}{2}\mathbb{Z}^2\).

b) If \(\rho_\tau\) is not completely reducible, then

\[(1.9) \quad M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}, \quad C \in \mathbb{C} \cup \{\infty\}.\]

When \(C = \infty\), the monodromy matrices are understood as

\[(1.10) \quad M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.\]

The aforementioned spectral polynomial \(Q_n(B; \tau)\) also plays an important role for the monodromy representation: \(\rho_\tau\) is completely reducible if and only if \(Q_n(B; \tau) \neq 0\). Obviously, not all \(2 \times 2\) matrices of the form (1.8)-(1.9) are monodromy matrices of (1.2). Thus the following questions naturally arise:

1. If \(Q_n(B; \tau) \neq 0\), how to determine the monodromy data \((r, s)\)?
2. If \(Q_n(B; \tau) = 0\), how to determine the monodromy data \(C\)?

For the Lamé equation (1.6), in [2, 14, 15] Chai, Wang and the third author have constructed a premodular form \(Z_{n,r,s}(\tau)\) such that the monodromy matrices \(M_1, M_2\) of (1.6) at \(\tau = \tau_0\) with some \(B\) are given by (1.8) if and only if \(Z_{n,r,s}(\tau_0) = 0\). Therefore, the image of \(M_1, M_2\) for \(\rho_{\tau_0}\) is \(\{(r, s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2 | Z_{n,r,s}(\tau_0) = 0\}\). We note that \(Z_{n,r,s}(\tau)\) is holomorphic in \(\tau\) if \((r, s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2\). Moreover, \(Z_{n,r,s}(\tau)\) is a modular form of weight \(\frac{2(n+1)}{2}\) w.r.t. the principal congruence subgroup \(\Gamma(N)\) if \((r, s)\) is a \(N\)-torsion point; see [15]. Thus \(Z_{n,r,s}(\tau)\) is called a premodular form.

In this paper and the subsequent one [3], we want to extend the result in [15] to include the Trebich-Verdier potential. Precisely, we will establish the following theorem in [3]:

**Theorem 1.1.** There exists a premodular form \(Z_{n,r,s}(\tau)\) defined in \(\tau \in \mathbb{H}\) for any pair of \((r, s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2\) such that the followings hold.

(a) If \((r, s) = (\frac{k_1}{N}, \frac{k_2}{N})\) with \(N \in 2\mathbb{N} \geq 2\), \(k_1, k_2 \in \mathbb{Z} \geq 0\), and \(\text{gcd}(k_1, k_2, N) = 1\), then \(Z_{n,r,s}(\tau)\) is a modular form of weight \(\sum_{k=0}^{3} n_k(n_k + 1)/2\) with respect to the principal congruence subgroup \(\Gamma(N)\).

(b) For \((r, s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2\), \(Z_{n,r,s}(\tau_0) = 0\) for some \(\tau_0 \in \mathbb{H}\) if and only if there is \(B \in \mathbb{C}\) such that \(\text{GLE (1.2)}\) with \(\tau = \tau_0\) has its monodromy matrices \(M_1\) and \(M_2\) given by (1.8).
Following the ideas in [2, 15], the spectral curve $\Gamma_n(\tau)$ can be embedded into $\text{Sym}^N E_\tau := E_\tau^N/S_N$, the symmetric space of $N$-th copy of $E_\tau$, where $N := \sum_{k=0}^3 n_k$.

Obviously, $\text{Sym}^N E_\tau$ has a natural addition map to $E_\tau$: \{a_1, \cdots, a_N\} $\mapsto \sum_{i=1}^N a_i$. Then the composition give to a finite morphism $\sigma_n(\cdot | \tau) : \Gamma_n(\tau) \to E_\tau$, still called the addition map. The degree of $\sigma_n$ is defined as $\deg \sigma_n(\cdot | \tau) = \#\sigma_n^{-1}(z)$, $z \in E_\tau$, counted with multiplicity. Our main theorem in this paper is

**Theorem 1.2** (Theorem 6.6). Let $\tau \in \mathbb{H}$. Then the addition map $\sigma_n(\cdot | \tau) : \Gamma_n(\tau) \to E_\tau$ has degree $\sum_{k=0}^3 n_k(n_k + 1)/2$.

A corollary of Theorem 1.2 is that $\deg \sigma_n(\cdot | \tau)$ (the same as the weight of the premodular form in Theorem 1.1) is independent of $\tau$, which is not very obvious at the moment. For the case of the Lamé equation, Theorem 1.2 was proved in [15] by applying Theorem of the Cube for morphisms between varieties in algebraic geometry. But this method seems not work in the general case. Our strategy is to study the general class of ODE:

$$y''(z) = I_n(z; p, A, \tau)y(z),$$

where the potential $I_n(z; p, A, \tau)$ is given by

$$I_n(z; p, A, \tau) = \left[ \sum_{k=0}^3 n_k(n_k + 1)\varphi(z + \omega_k/2|\tau) + \frac{3}{4}(\varphi(z + p|\tau) + \varphi(z - p|\tau) + A(\zeta(z + p|\tau) - \zeta(z - p|\tau)) + B \right],$$

with $A \in \mathbb{C}$, $p \in E_\tau \setminus E_\tau[2]$, $E_\tau[2] := \{\omega_k|k = 0, 1, 2, 3\} + \Lambda_\tau$ and

$$B = A^2 - \zeta(2p)A - \frac{3}{4}\varphi(2p) - \sum_{k=0}^3 n_k(n_k + 1)\varphi(p + \omega_k/2|\tau).$$

The identity (1.13) is to guarantee that all the singular points of (1.11) are apparent i.e. all solutions of (1.11) are free of logarithmic singularity at any singular point. See [4, 5, 7, 18] for recent developments of (1.11). Like (1.2), we could associate a hyperelliptic curve $\Gamma_{n,p}(\tau) := \{(A, W)|W^2 = Q_{n,p}(A; \tau)\}$ and an addition map $\sigma_{n,p}$ with (1.11). These will be established in Sections 2 and 3. The reason we introduce (1.11) is that as $p \to \omega_k/2$, $k = 0, 1, 2, 3$, the limiting equation of (1.11) would be (1.2) with $n = n^+_k$, where $n^+_k$ is defined by replacing $n_k$ in $n$ with $n_k \pm 1$. Due to this relation with (1.2), we expect to have the following connection.

**Theorem 1.3** (Theorem 6.6). For $k \in \{0, 1, 2, 3\}$, there holds

$$\deg \sigma_{n,p}(\cdot | \tau) = \deg \sigma_{n^+_k}(\cdot | \tau) + \deg \sigma_{n^-_k}(\cdot | \tau).$$

The paper is organized as follows. In Section 2, we will give a brief review of the monodromy representation of (1.11). The similar argument also holds for (1.2). We will prove the existence of the embedding of $\Gamma_{n,p}$ and the addition map from $\Gamma_{n,p}$ onto $E_\tau$ in Section 3. In Sections 4 and 5,
we will study the limiting problem of (1.11) under two case: (i) fix \( p \) and \( A \to \infty \); (ii) \( p \to \frac{\pi}{2} \) and \( A(p) \to \infty \). This limit problem plays a crucial role in our study of the degree in Section 6, where Theorem 1.3 is proved and then Theorem 1.2 will be obtained by applying Theorem 1.3.

2. Monodromy representation

Let \( \mathbf{n} = (n_0, n_1, n_2, n_3) \), \( n_k \in \mathbb{C} \). For any fixed \( \tau \in \mathbb{H} \) and \( \pm p \notin E_\tau[2] := \{ \frac{\omega}{2} | k = 0, 1, 2, 3 \} + \Lambda_\tau \), we consider the following generalized Lamé equation GLE(\( \mathbf{n}, p, A, \tau \)) on the torus \( E_\tau \):

\[
y''(z) = I_\mathbf{n}(z; p, A, \tau) y(z) \quad \text{on} \quad E_\tau,
\]

where the potential \( I_\mathbf{n}(z; p, A, \tau) \) is given by

\[
I_\mathbf{n}(z; p, A, \tau) = \left[ \sum_{k=0}^{3} n_k (n_k + 1) \wp(z + \frac{\omega_k}{2} \tau) + \frac{3}{4} \wp(z + p | \tau) + \wp(z - p | \tau) + A(\zeta(z + p | \tau) - \zeta(z - p | \tau)) + B \right],
\]

with \( A \in \mathbb{C} \) and

\[
B = A^2 - \zeta(2p)A - \frac{3}{4} \wp(2p) - \sum_{k=0}^{3} n_k (n_k + 1) \wp(p + \frac{\omega_k}{2} | \tau).
\]

GLE(\( \mathbf{n}, p, A, \tau \)) is of Fuchsian type with singularities at \( S := E_\tau[2] \cup \{ \pm p \} \).

Let us briefly recall the associated monodromy representation. Let \( Y(z; \tau) = (y_1(z; \tau), y_2(z; \tau)) \) be a fundamental system of solutions of GLE(\( \mathbf{n}, p, A, \tau \)) near a fixed base point \( q_0 \notin S \). In general, \( Y(z; \tau) \) is multi-valued with respect to \( z \) and might have branch points at \( S \). For any loop \( \ell \in \pi_1(E_\tau \setminus S, q_0) \), there exists a matrix \( \rho_\tau(\ell) \in SL(2, \mathbb{Z}) \) such that \( \ell^* Y(z; \tau) = Y(z; \tau) \rho_\tau(\ell) \). Here \( \ell^* Y(z; \tau) \) denotes the analytic continuation of \( Y(z; \tau) \) along the loop \( \ell \). This induces a group homomorphism

\[
\rho_\tau : \pi_1(E_\tau \setminus S, q_0) \to SL(2, \mathbb{C}).
\]

which is called the monodromy representation of the GLE(\( \mathbf{n}, p, A, \tau \)).

The local exponent of GLE(\( \mathbf{n}, p, A, \tau \)) at \( \frac{\omega}{2} \) are \( -n_k, n_k + 1 \). In this paper, we are interested in the case \( n_k \in \mathbb{Z}_{\geq 0} \) (hereafter we always assume \( n_k \in \mathbb{Z}_{\geq 0} \) for all \( k \)), because in this case the local monodromy matrix at each \( \frac{\omega_k}{2} \) is \( I_2 \) and then the monodromy representation can be reduced to a homomorphism \( \rho_\tau : \pi_1(E_\tau \setminus \{ \pm p \}, q_0) \to SL(2, \mathbb{Z}) \). Let \( \gamma_\pm \in \pi_1(E_\tau \setminus S, q_0) \) be a simple loop encircling \( \pm p \) counterclockwise respectively, and \( \ell_j, j = 1, 2 \), be two fundamental cycles of \( E_\tau \) connecting \( q_0 \) with \( q_0 + \omega_j \) such that \( \ell_j \) does not intersect with \( \ell_p + \Lambda_\tau \) (here \( \ell_p \) is the straight segment connecting \( \pm p \)) and satisfies

\[
\gamma_+ \gamma_- = \ell_1 \ell_2 \ell_1^{-1} \ell_2^{-1} \quad \text{in} \quad \pi_1(E_\tau \setminus \{ \pm p \}, q_0).
\]

On the other hand, the local exponents at \( \pm p \) are \( -\frac{1}{2}, \frac{3}{2} \). Since (2.3) implies that \( \pm p \) are apparent singularities (i.e. non-logarithmic, see [4]), we have

\[
\rho_\tau(\gamma_\pm) = -I_2.
\]
Denote $\rho_p(\ell_j)$ by $M_j$. Then the monodromy group of $\text{GLE}(n, p, A, \tau)$ is generated by $\{-I_2, M_1, M_2\}$. Together with (2.5)-(2.6), we immediately obtain $M_1M_2 = M_2M_1$, which implies that there is always a solution denoted by $y_1(z) = y_1(z; A)$ being a common eigenfunction, i.e. $\ell_j^*y_1(z; A) = \varepsilon_jy_1(z; A)$, $j = 1, 2$. Therefore the monodromy representation is always reducible.

From the local exponents at $\frac{2\pi}{\tau}$ and $\pm p$, it is easy to see that $\frac{2\pi}{\tau}$ is not a branch point of $y_1(z)$ but $\pm p$ is a branch point with ramification index 2, i.e. $y_1(\pm p + e^{2\pi i}z) = -y_1(\pm p + z)$ if $|z| > 0$ is small. Then $y_1(z)$ can be viewed as a single-valued meromorphic function in $\mathbb{C}\backslash(\ell_p + \Lambda_\tau)$. Therefore, the analytic continuation of $y_1(z)$ along the fundamental cycles $\ell_j$ is the translation of $y_1$ by $\omega_j, j = 1, 2$, namely

$$y_1(z + \omega_j; A) = \ell_j^*y_1(z; A) = \varepsilon_jy_1(z; A), \quad j = 1, 2.$$  

(2.7)  

Since $\mathbb{C}\backslash(\ell_p + \Lambda_\tau)$ is symmetric about 0, $y_1(-z; A)$ is well-defined in $\mathbb{C}\backslash(\ell_p + \Lambda_\tau)$ and also a solution of the same $\text{GLE}(n, p, A, \tau)$. Define

$$y_2(z) = y_2(z; A) := y_1(-z; A) \quad \text{in } \mathbb{C}\backslash(\ell_p + \Lambda_\tau).$$  

Then by (2.7) we see that $y_2(z; A)$ is also a common eigenfunction, i.e.

$$y_2(z + \omega_j; A) = \ell_j^*y_2(z; A) = \varepsilon_j^{-1}y_2(z; A), \quad j = 1, 2.$$  

(2.9)  

If $\varepsilon_j \neq \pm 1$ for some $j$, then $y_1(z)$ and $y_2(z)$ are linearly independent. In general, $y_1(z)$ and $y_2(z)$ might be linearly dependent.

**Definition 2.1.** $\text{GLE}(n, p, A, \tau)$ is called completely reducible if its monodromy group acting on the 2-dimensional solution space has two linearly independent common eigenfunctions. Otherwise, it is called not completely reducible.

We will see later that $\text{GLE}(n, p, A, \tau)$ is completely reducible if and only if the above $y_1(z)$ and $y_2(z) := y_1(-z)$ are linearly independent.

The branch point $\pm p$ of $y_1(z)$ might cause trouble in analysis. To avoid it, we introduce

$$\Psi_p(z) := \frac{\sigma(z)}{\sqrt{\sigma(z - p)\sigma(z + p)}}.$$  

(2.10)  

By using the transformation law of $\sigma(z)$,

$$\sigma(z + \omega_j) = -e^{\eta_j(z+\frac{1}{2}\omega_j)}\sigma(z), \quad j = 1, 2,$$  

(2.11)  

we see that $\Psi_p(z)^2$ is an elliptic function. We have the following lemma.

**Lemma 2.2.** Recall that $\ell_j, j = 1, 2$ are the two fixed fundamental cycles of $E_\tau$ which do not intersect with $\ell_p + \Lambda_\tau$. Then the analytic continuation of $\Psi_p(z)$ along $\ell_j$ satisfies

$$\ell_j^*\Psi_p(z) = \Psi_p(z), \quad j = 1, 2.$$  

(2.12)
Proof. We only need to prove (2.12) in a small neighborhood $U$ of the base point $q_0$. Since $\ell_j \in \pi_1(E_\tau \setminus \{\pm [p]\}, q_0)$ does not intersect with $\ell_p + \Lambda_\tau$, $\Psi_p(z)$ can be viewed as a single-valued meromorphic function in $\mathbb{C}\setminus(\ell_p + \Lambda_\tau)$, and in this region we have
\[
\ell_j^* \Psi_p(z) = \Psi_p(z + \omega_j) = \pm \Psi_p(z), \quad z \in U,
\]
because $\Psi_p(z)^2 = \frac{\sigma(z)^2}{\sigma(z+p)\sigma(z-p)}$ is an elliptic function. Suppose
\begin{equation}
(2.13)
\Psi_p(z + \omega_j) = -\Psi_p(z), \quad z \in U
\end{equation}
holds true. By fixing $q_0$ and $\ell_j$, (2.13) always holds true as $p \to 0$ along $\ell_p$.
Note from (2.10) that for any $z \in \ell_j$, $\lim_{p \to 0} \Psi_p(z)$ is identical to either 1 or $-1$, but (2.13) implies that $\lim_{p \to 0} \Psi_p(z)$ for $z$ along $\ell_j$ contains both 1 and $-1$, a contradiction. \hfill $\square$

Classically, it has been known (cf. [26]) that the second symmetric product equation for any second order ODE play an important role. Let $\tilde{y}_1(z)$, $\tilde{y}_2(z)$ are any two solutions of GLE$(n, p, A, \tau)$ and set $\Phi(z) = \tilde{y}_1(z)\tilde{y}_2(z)$. Then $\Phi(z)$ satisfies the following third order ODE:
\begin{equation}
(2.14)
\Psi''(z) - 4I_n(z; p, A, \tau)\Psi'(z) - 2I''_n(z; p, A, \tau)\Phi(z) = 0.
\end{equation}
Recall (2.7)-(2.9) that $\tilde{y}_1(z)$ is an common eigenfunction and $\tilde{y}_2(z) = y_1(-z)$. Then $\Phi(z) := \tilde{y}_1(z)\tilde{y}_2(z)$ is a solution of (2.14), and also an even elliptic function due to (2.7)-(2.9). The following result was proved by Takemura [18], but we give a proof here for the convenience of readers because it plays a fundamental role in our theory.

**Proposition 2.3.** [18] The dimension of the space of even elliptic solutions to (2.14) is 1.

Proof. It is easy to see that the dimension of the space of even solutions to (2.14) is 2. Suppose the proposition is not true. Since we already know there is at least one even elliptic solution, the dimension of even elliptic solutions to (2.14) is 2, which implies any even solution of (2.14) must be elliptic.

Since the local exponents of GLE$(n, p, A, \tau)$ at 0 are $-n_0, n_0 + 1$, and $I_n(\cdot; p, A, \tau)$ is even, there are local solutions of the following form at 0:

\[
\tilde{y}_1(z) = z^{-n_0} \left( 1 + \sum_{j=1}^{\infty} a_j z^{2j} \right), \quad \tilde{y}_2(z) = z^{n_0+1} \left( 1 + \sum_{j=1}^{\infty} b_j z^{2j} \right).
\]

Then $\tilde{y}_j(z)^2$, $j = 1, 2$, are even solutions of (2.14) and hence even elliptic functions by our assumption. Define
\begin{equation}
(2.15)
\tilde{y}_j(z) := \frac{\tilde{y}_j(z)}{\Psi_p(z)}, \quad j = 1, 2,
\end{equation}
where $\Psi_p(z)$ is given by (2.10). Then $\tilde{y}_j(z)$ is a meromorphic function with poles at most at $E_\tau[2]$. Since $\tilde{y}_j(z)^2$ and $\Psi_p(z)^2$ are even elliptic, so do $\tilde{y}_j(z)^2$
for $j = 1, 2$. Assume

\[
\begin{pmatrix}
    \tilde{y}_1(z + \omega_i) \\
    \tilde{y}_2(z + \omega_i)
\end{pmatrix}
= \begin{pmatrix}
    a_i & b_i \\
    c_i & d_i
\end{pmatrix}
\begin{pmatrix}
    \tilde{y}_1(z) \\
    \tilde{y}_2(z)
\end{pmatrix}, \quad i = 1, 2
\]

for some \( \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \text{SL}(2, \mathbb{C}) \). Then

\[
\tilde{y}_1(z)^2 = \tilde{y}_1(z + \omega_i)^2 = a_i^2 \tilde{y}_1(z)^2 + 2a_i b_i \tilde{y}_1(z) \tilde{y}_2(z) + b_i^2 \tilde{y}_2(z)^2.
\]

So we have \( a_i^2 = 1, b_i = 0 \). Together with \( a_id_i - b_ic_i = 1 \), we see that \( d_i = a_i =: \varepsilon_i \in \{\pm 1\} \). Similarly, we could use \( \tilde{y}_2 \) to obtain \( c_i = 0 \). Hence

\[
\tilde{y}_k(z + \omega_i) = \varepsilon_i \tilde{y}_k(z)
\]

for \( k = 1, 2 \) and \( i = 1, 2 \).

This implies that \( \tilde{y}_1(z) \tilde{y}_2(z) \) is odd and elliptic. By the local exponent of \( \tilde{y}_1(z) \tilde{y}_2(z) \) at \( \frac{\pi i}{2} \) being one of \( \{-2n_k, 1, 2n_k + 2\} \) for \( k = 1, 2, 3 \), we see that \( \frac{\pi i}{2} \) is a simple zero of \( \tilde{y}_1(z) \tilde{y}_2(z) \) for \( k = 1, 2, 3 \). Thus the elliptic function \( \tilde{y}_1(z) \tilde{y}_2(z) \) has only a simple pole at \( z = 0 \), a contradiction. This proves the proposition. \( \square \)

Now we can apply Proposition 2.3 to answer the question proposed earlier.

**Proposition 2.4.** Let \( y_1(z) \) be a common eigenfunction of the monodromy representation of \( GLE(n,p,A,\tau) \). Then \( GLE(n,p,A,\tau) \) is completely reducible if and only if \( y_1(z) \) and \( y_2(z) := y_1(-z) \) are linearly independent.

**Proof.** Clearly \( y_2(z) \) is also a common eigenfunction. So the sufficient part is trivial. For the necessary part, since the monodromy is completely reducible, there exists another common eigenfunction \( y_3(z) \) which is linearly independent with \( y_1(z) \). Then \( y_3(z)y_3(-z) \) is also an even elliptic solution of \( 2.14 \). So Proposition 2.3 implies \( y_1(z)y_1(-z) = y_3(z)y_3(-z) \) up to a constant, which implies \( y_3(z) = cy_1(-z) \) for some constant \( c \neq 0 \). This proves that \( y_1(z) \) and \( y_2(z) = y_1(-z) \) are linearly independent. \( \square \)

Recall 2.7 and 2.9 that \( \varepsilon_j, \varepsilon_j^{-1} \) are the eigenvalues of \( M_j = \rho_{\tau}(\ell_j) \). Then Propositions 2.3 and 2.4 have the following consequence.

**Corollary 2.5.** If \( GLE(n,p,A,\tau) \) is completely reducible then \( (\varepsilon_1, \varepsilon_2) \notin \{\pm (1,1), \pm (1,-1)\} \).

Suppose \( GLE(n,p,A,\tau) \) is completely reducible and \( y_j(z), j = 1, 2 \), are its linearly independent common eigenfunctions in Proposition 2.4 then the monodromy matrix \( M_j \) can be written as

\[
M_1 = \begin{pmatrix}
    e^{-2\pi is} & 0 \\
    0 & e^{2\pi is}
\end{pmatrix}
\quad \text{and} \quad
M_2 = \begin{pmatrix}
    e^{2\pi ir} & 0 \\
    0 & e^{-2\pi ir}
\end{pmatrix},
\]

where \( (r, s) \in \mathbb{C}^2 \) satisfies \( (\varepsilon_1, \varepsilon_2) = (e^{-2\pi is}, e^{2\pi ir}) \). We call that \( (r, s) \) are the monodromy data of \( GLE(n,p,A,\tau) \) if it is completely reducible, and Corollary 2.5 implies

\[
(r, s) \notin \frac{1}{2}\mathbb{Z}^2.
\]
Later we will show how to compute $(r, s)$ from the zero set of the common eigenfunction $y_1(z)$.

The above argument shows that (2.14) has a unique even elliptic solution $\Phi(z)$ up to a constant, which is given by $\Phi(z) = y_1(z)y_2(z)$, where $y_1(z)$ is a common eigenfunction and $y_2(z) = y_1(-z)$. Furthermore, GLE$(n, p, A, \tau)$ is completely reducible if and only if $y_1(z)$ and $y_2(z)$ are linearly independent. This is where the even elliptic solution $\Phi$ plays the role which is of fundamental importance for GLE$(n, p, A, \tau)$. To determine the linear independence, we should consider the Wronskian of $y_1(z)$ and $y_2(z)$:

$$W := y_1(z)y_2'(z) - y_1'(z)y_2(z),$$

which is completely reducible if and only if $y_1(z)$ and $y_2(z)$ are linearly independent.

**Lemma 2.6.** Let $\Phi(z) = y_1(z)y_2(z)$ be the even elliptic solution of (2.14) and $W$ be the Wronskian of $y_1$ and $y_2$. Then

$$W^2 = \Phi'(z)^2 - 2\Phi''(z)\Phi(z) + 4I_n(z; p, A, \tau)\Phi(z)^2,$$

(2.19)

$$y_1(z) = \sqrt{\Phi(z)}\exp\int^z W \frac{d\xi}{2\Phi(\xi)}$$

**Proof.** Since $\Phi(z) = y_1(z)y_2(z)$ and $W = y_1'y_2 - y_1y_2'$, we have

$$\frac{y_1'}{y_1} = \frac{\Phi' + W}{2\Phi}, \quad \frac{y_2'}{y_2} = \frac{\Phi' - W}{2\Phi}$$

which implies (2.19) and

$$\frac{\Phi''}{2\Phi} - \frac{\Phi'}{2\Phi} + \Phi' = \left(\frac{y_1'}{y_1}\right)' = \frac{y_1''}{y_1} - \left(\frac{y_1'}{y_1}\right)^2 = I_n - \left(\frac{\Phi' + W}{2\Phi}\right)^2,$$

$$\frac{\Phi''}{2\Phi} - \frac{\Phi'}{2\Phi} - \Phi' = I_n - \left(\frac{\Phi' - W}{2\Phi}\right)^2.$$

Adding these two formulas together, we easily obtain (2.18).

To normalize the even elliptic solution $\Phi(z)$, we apply the following result due to Takemura [18].

**Theorem 2.A.** [18] Fix $\tau \in \mathbb{H}$ and $p \notin E_\tau[2]$. Then equation (2.14) has a unique even elliptic solution $\Phi_e(z; A)$ of the form

$$\Phi_e(z; A) = C_0(A) + \sum_{k=0}^{3} \sum_{j=0}^{nk-1} b_j^{(k)}(A)\varphi(z + \frac{\omega_k}{2})^{nk-j} + \frac{d(A)}{\varphi(z) - \varphi(p)},$$

(2.20)

such that the coefficients $C_0(A)$, $b_j^{(k)}(A)$ and $d(A)$ are all polynomials in $A$, and they do not have common zeros, and the leading coefficient of $C_0(A)$ is $\frac{1}{2}$. Moreover,

$$\deg_A C_0(A) > \max \left(\deg_A b_j^{(k)}(A), \deg_A d(A)\right).$$

The proof of Theorem 2.A is not difficult but a little tedious in computation. The expression of (2.20) is due to the fact that $\frac{d}{\varphi}$ and $\pm p$ might be poles of $\Phi(z)$ with order $-2n_k$ and $-1$ respectively (depending on $A$).
Theorem 2.7. There is a monic polynomial \( Q_{n,p}(A) = Q_{n,p}(A; \tau) \) in \( A \) such that the Wronskian of \( y_1(z; A), y_2(z; A) = y_1(-z; A) \), where \( \Phi_e(z; A) = y_1(z; A)y_2(z; A) \), satisfies
\[
W^2 = Q_{n,p}(A).
\]
Moreover, \( Q_{n,p}(A) \neq 0 \) if and only if \( GLE(n, p, A, \tau) \) is completely reducible.

Proof. By (2.2) and (2.3), we could write
\[
I_n(z; p, A, \tau) = A^2 + I_1(z)A + I_2(z).
\]
By Theorem 2.A, we have
\[
\Phi_e(z) = \frac{1}{2} A^g + \sum_{j=0}^{g-1} \varphi_j(z)A^j, \quad \text{where} \quad g := \deg C_0(A),
\]
Inserting these into (2.18), we easily obtain
\[
W^2 = A^{2g+2} + \sum_{j=0}^{2g+1} q_j(z)A^j,
\]
where \( q_j(z) \equiv c_j \) are independent of \( z \) for all \( j \) because \( W \) is independent of \( z \). Thus, \( W^2 = Q_{n,p}(A) \) is a monic polynomial in \( A \) of degree \( 2g + 2 \).

In view of Theorem 2.7, we define the hyperelliptic curve \( \Gamma_{n,p} = \Gamma_{n,p}(\tau) \) by
\[
(2.23) \quad \Gamma_{n,p}(\tau) := \{(A, W) | W^2 = Q_{n,p}(A; \tau)\}.
\]
We remark that the polynomials \( Q_{n,p}(A; \tau) \) might have multiple zeros. But we could prove that for each \( \tau \), \( Q_{n,p}(A; \tau) \) has distinct roots except for finitely many \( p \in E_\tau \setminus E_\tau[2] \). See e.g. our subsequent work [3].

Since \( \deg_q Q_{n,p}(A; \tau) \) is even, the curve \( \Gamma_{n,p}(\tau) \) has two points at infinity which are denoted by \( \infty_{\pm} \). Indeed, for a curve in \( \mathbb{C}^2 \) defined by \( y^2 = \prod_{i=1}^{2g+2}(x-x_i) \), to study its points at infinity, we let \( x = 1/x' \) and \( y = y'/x'^{g+1} \). Then the equation becomes \( y'^2 = \prod_{i=1}^{2g+2}(1-x'i) \). Thus \( (x', y') = (0, \pm 1) \) represents the two points at infinity and they are unramified. Hence
\[
(2.24) \quad \overline{\Gamma_{n,p}(\tau)} = \Gamma_{n,p}(\tau) \cup \{\infty_{\pm}\} \quad \text{is smooth at} \quad \infty_{\pm}.
\]

Remark 2.8. Given \( (A, W) \in \Gamma_{n,p} \), the unique \( \Phi_e(z; A) \) is the product of \( y_1(z; A) \) and \( y_2(z; A) = y_1(-z; A) \). If \( W \neq 0 \), then \( y_1(z; A) \) and \( y_2(z; A) \) are linearly independent. We rename \( y_1(z; A), y_2(z; A) \) by requiring that the Wronskian of \( y_1 \) and \( y_2 \) is equal to \( W \) (i.e. the Wronskian of \( y_2 \) and \( y_1 \) is \(-W\)). Then \( y_1(z; A) \) is unique up to a sign, and will be denoted by \( y_1(z; A, W) \). In particular, the zero set of \( y_1(z; A, W) \) is unique.
3. The embedding of $\Gamma_{n,p}$ in $Sym^NE_\tau$

The main purpose of this section is to define the addition map from $\Gamma_{n,p}$ to $E_\tau$. First we discuss the embedding of $\Gamma_{n,p}$ into $Sym^NE_\tau$, the symmetric $N$-th copy product of $E_\tau$, where $N = \sum_{k=0}^{3} n_k + 1$ in the sequel.

For any $a = (a_1, \cdots, a_N) \in \mathbb{C}^N$, we define $y_{a,c}(z) = y_{a,c}(z;p)$ by

\begin{equation}
(3.1) \quad y_{a,c}(z;p) := \frac{e^{cz} \prod_{i=1}^{N} \sigma(z-a_i)}{\sqrt{\sigma(z-p)\sigma(z+p)} \prod_{k=0}^{N} \sigma(z - \frac{\omega_k}{2})^{n_k}} \text{ for } c \in \mathbb{C}.
\end{equation}

**Proposition 3.1.** Fix $p \in E_\tau \setminus E_\tau[2]$ and $A \in \mathbb{C}$. Let $y_1(z;A,W)$ be the common eigenfunction determined in Remark 2.8 Then there always exists $a \in \mathbb{C}^N$ and $c \in \mathbb{C}$ such that $y_1(z;A,W) = y_{a,c}(z;p)$ up to a constant.

**Proof.** Since $y_1(z;A,W)$ is a common eigenfunction, we have

\begin{equation}
(3.2) \quad \ell_i^*y_1(z;A,W) = \varepsilon_i y_1(z;A,W) \text{ for some } \varepsilon_i \neq 0, i = 1, 2.
\end{equation}

As we discussed in Section 2 $y_1(z;A,W)$ has branch points at $\pm p$. Set $\tilde{y}(z) := y_1(z;A,W)/\Psi_p(z)$, where $\Psi_p(z)$ is defined in (2.10). Then $\tilde{y}(z)$ is a meromorphic function, and it follows from Lemma 2.2 and (3.2) that

\begin{equation}
(3.3) \quad \tilde{y}(z + \omega_i) = \varepsilon_i \tilde{y}(z), \quad i = 1, 2.
\end{equation}

Conventionally, a meromorphic function satisfying (3.3) is called an elliptic function of second kind with periods 1 and $\tau$. Then a classic theorem says that up to a constant, $\tilde{y}(z)$ can be written as

\begin{equation}
(3.4) \quad \tilde{y}(z) = \frac{e^{cz} \prod_{i=1}^{N} \sigma(z-a_i)}{\sigma(z) \prod_{k=0}^{N} \sigma(z - \frac{\omega_k}{2})^{n_k}},
\end{equation}

because $\tilde{y}(z)$ have poles at most at 0 with order $-n_0 - 1$ and at $\omega_k/2$ with order $-n_k, k = 1, 2, 3$. The proof is complete. \hfill \Box

**Remark 3.2.** A consequence of Remark 2.8 and Proposition 3.1 is that, up to a constant, the unique even elliptic solution $\Phi_e(z;A)$ in Theorem 2.2A can be written as

\begin{equation}
(3.5) \quad \Phi_e(z;A) = \frac{\prod_{i=1}^{N} \sigma(z-a_i)\sigma(z+a_i)}{\sigma(z-p)\sigma(z+p) \prod_{k=0}^{N} \sigma(z - \frac{\omega_k}{2})^{n_k} \sigma(z + \frac{\omega_k}{2})^{n_k}}.
\end{equation}

On the other hand, if $y_{a,c}(z)$ is a solution of GLE($n,p,A,\tau$), it is easy to see that (i) $y_{-a,-c}(z) = d_1 y_{a,c}(-z)$ (for some constant $d_1 \neq 0$) is also a solution of GLE($n,p,A,\tau$), (ii) $y_{a,c}(z)$ is a common eigenfunction of $M_j$’s. Together with Proposition 3.1, we have that (iii) $y(z)$ is a common eigenfunction if and only if $y(z)$ is of the form $y_{a,c}(z)$ up to a constant, and (iv) $y_2(z) := y_1(-z;A,W) = y_{-a,-c}(z;p)$ up to a constant. In particular, Proposition 2.4 implies that GLE($n,p,A,\tau$) is completely reducible if and only if $y_{a,c}(z)$ and $y_{-a,-c}(z) = d_1 y_{a,c}(-z)$ are linearly independent.
By applying the transformation law (2.11) of (3.9) to \( y \), we obtain the monodromy data \((r, s)\) in (2.16) if GLE(\( n, p, A, \tau \)) is completely reducible. These are proved in the following result. Denote \( \eta_3 := \eta_1 + \eta_2 \).

**Theorem 3.3.** Let \( p \notin E_r[2] \) and \( A \in \mathbb{C} \). Suppose GLE(\( n, p, A, \tau \)) is completely reducible with solution \( y_{a,c}(z) \) given in Proposition 3.1. Then the monodromy data \((r, s)\) in (2.16) and \( c = c(a) \) can be determined by \( a \) as follows:

\[
\sum_{i=1}^{N} a_i = r + s \tau + \sum_{k=1}^{3} \frac{n_k \eta_k}{2},
\]

\[
c(a) = r\eta_1 + s\eta_2 - \frac{1}{2} \sum_{i=1}^{N} (\zeta(a_i + p) + \zeta(a_i - p)) - \sum_{k=1}^{3} \frac{n_k \eta_k}{2},
\]

\[
c(a) = \frac{1}{2} \sum_{i=1}^{N} (\zeta(a_i + \frac{\eta_k}{2}) + \zeta(a_i - \frac{\eta_k}{2})) - \sum_{i=1}^{3} \frac{n_i \eta_i}{2} \text{ if } n_k \neq 0.
\]

Furthermore, \( a_i \notin E_r[2] \) for all \( i \) and \( a = \{a_1, \ldots, a_N\} \) must satisfy one of the following three alternatives:

(a-i) \( a_i \neq \pm p \) and \( a_i \neq \pm a_j \) for any \( i \neq j \), i.e. \( \pm p \) are simple poles of \( \Phi_e(z; A) \).

(a-ii) \( a_{N-1} = a_N = p, a_i \neq \pm p \) and \( a_i \neq \pm a_j \) for any \( i \neq j \leq N - 2 \), i.e. \( \pm p \) are simple zeros of \( \Phi_e(z; A) \).

(a-iii) \( a_{N-1} = a_N = -p, a_i \neq \pm p \) and \( a_i \neq \pm a_j \) for any \( i \neq j \leq N - 2 \), i.e. \( \pm p \) are simple zeros of \( \Phi_e(z; A) \).

**Proof.** Since GLE(\( n, p, A, \tau \)) is completely reducible, Remark 3.2 says that \( y_{a,c}(z) \) and \( y_{a,c}(-z) \) are linearly independent, which implies

\( y_{a,c}(z) \) and \( y_{a,c}(-z) \) has a pole of order \( -n_k \) at \( z = \frac{\omega_k}{2} \) if \( n_k > 0 \) and non-zero at \( z = \frac{\omega_k}{2} \) if \( n_k = 0 \).

Thus \( a_i \notin E_r[2] \) for all \( i \).

Recalling \( \tilde{y}(z) \) in (3.4) and \( \psi_p(z) \) in (2.10), we have

\[
y_{a,c}(z) = \tilde{y}(z)\psi_p(z).
\]

By applying the transformation law (2.11) of \( \sigma \) to \( \tilde{y}(z) \), we can determine \( \varepsilon_i \) in \( (3.2), (3.3) \) by

\[
\varepsilon_i = \exp \left( c \omega_i - \eta_i \left( \sum_{i=1}^{N} a_i - \sum_{k=1}^{3} \frac{n_k \omega_k}{2} \right) \right), \quad i = 1, 2.
\]
Define \((r, s) \in \mathbb{C}^2\) by

\[
-2\pi is := c - \eta_1 \left( \sum_{i=1}^{N} a_i - \sum_{k=1}^{3} \frac{n_k \omega_k}{2} \right)
\]

\[
2\pi ir := cr - \eta_2 \left( \sum_{i=1}^{N} a_i - \sum_{k=1}^{3} \frac{n_k \omega_k}{2} \right).
\]

Then \((\varepsilon_1, \varepsilon_2) = (e^{-2\pi is}, e^{2\pi ir})\), and it follows from \((3.2)\) that this \((r, s)\) is precisely the monodromy data in \((2.16)\). By the Legendre relation \(\tau \eta_1 - \eta_2 = 2\pi i\), we see that \((3.10)-(3.11)\) are equivalent to \((3.6)\) and \(c = r\eta_1 + s\eta_2\).

On the other hand, the second equality of \((3.7)\) can be proved by inserting the expression \((3.11)\) of \(y_{a,c}(z)\) into \(\text{GLE}(n, p, A, \tau)\) and computing the leading terms at singularities \(\pm p\). If \(n_k > 0\), then \(\frac{\omega_k}{k}\) is also a singularity. Then \((3.8)\) follows by inserting \((3.11)\) into \(\text{GLE}(n, p, A, \tau)\) and computing the leading terms at singularities \(\pm p\) and \(\frac{\omega_k}{k}\). We omit the details here.

To prove (i)-(iii), we recall Remark \((2.8)\) that \(\Phi_c(z; A) = y_{a,c}(z)y_{a,c}(-z)\).

Since \(\Phi_c(z; A)\) is even, it has the same local exponent \(\alpha\) at \(p\) and \(-p\), and \(\alpha \in \{-1, 1, 3\}\).

If \(\alpha = 3\), then the leading term of both \(y_{a,c}(z)\) and \(y_{a,c}(-z)\) near \(p\) is \((z - p)^{3/2}\), which implies that \(y_{a,c}(z)\) and \(y_{a,c}(-z)\) are linearly dependent, a contradiction.

If \(\alpha = -1\), i.e. \(\pm p\) are both simple poles of \(\Phi_c(z; A)\), then by \((3.3)\) we have \(a_j \neq \pm p\) in \(\mathbb{E}_r\) for all \(j\). Consequently, \(a_1, \ldots, a_N\) are all zeros of \(y_{a,c}(z)\) and so are all simple, i.e. \(a_i \neq a_j\) for \(i \neq j\). Since \(y_{a,c}(z)\) and \(y_{a,c}(-z)\) cannot have common zeros, we also have \(a_i \neq -a_j\) for all \(i, j\). This proves (a-i).

If \(\alpha = 1\), i.e. \(\pm p\) are both simple zeros of \(\Phi_c(z; A)\), one possibility is that the local exponent of \(y_{a,c}(z)\) at \(p\) is \(\frac{3}{2}\) and at \(-p\) is \(-\frac{1}{2}\). Then \((3.11)\) implies that \(-p \notin \{a_1, \ldots, a_N\}\) and there are exactly two elements in \(\{a_1, \ldots, a_N\}\) equal to \(p\) in \(\mathbb{E}_r\). By reordering \(a_1, \ldots, a_N\), (a-ii) is proved. Similarly, for the other possibility that the local exponent of \(y_{a,c}(z)\) at \(p\) is \(-\frac{1}{2}\) and at \(-p\) is \(\frac{3}{2}\), (a-iii) holds.

The proof is complete. \(\square\)

By the Legendre relation, the matrix \(\begin{pmatrix} 1 & \tau \\ \eta_1(\tau) & \eta_2(\tau) \end{pmatrix}\) is always invertible. Hence \((3.6)-(3.7)\) imply the monodromy data \((r, s)\) can be uniquely determined by \(a_1, \ldots, a_N\), which contain all the zeros of \(y_1(z; A, W)\). The next result is to show that \(A\) of \(\text{GLE}(n, p, A, \tau)\) can be expressed in terms of \(\{a_1, \ldots, a_N\}\) if Case (a-i) happens, i.e. \(\pm p \notin \{a_1, \ldots, a_N\}\).

**Proposition 3.4.** Let \(p \notin \mathbb{E}_r[2]\) and \(A \in \mathbb{C}\). Suppose \(\text{GLE}(n, p, A, \tau)\) is completely reducible and Case (a-i) in Theorem \((3.3)\) occurs, then

\[
A = \frac{1}{2} \sum_{i=1}^{N} (\zeta(a_i + p) - \zeta(a_i - p)) - \frac{1}{2} \zeta(2p)
\]


\[-\frac{1}{2} \sum_{k=0}^{3} n_k \left( \zeta(p + \frac{\omega_k}{2}) + \zeta(p - \frac{\omega_k}{2}) \right).\]

Again, (3.12) can be obtained directly by inserting the expression (3.1) of \( y_{a,c}(z) \) into GLE\((n, p, A, \tau)\) and computing the leading terms at singularities \( \pm \). We omit the proof here.

**Remark 3.5.** Theorem 2.A implies that for fixed \( \tau \) and \( p \notin E_\tau[2] \), there are only finite many \( A \)'s (i.e., zeros of the polynomial \( d(A) \)) such that Case (a-ii) or (a-iii) occurs. Therefore, if \( a \) is in Case (a-ii) or (a-iii), we could use a sequence of \( a^k \) of Case (a-i) to approximate \( a \). In particular, \( a_{N-1}^k \) and \( a_N^k \) converge to \( p \) for Case (a-ii). The fact that \( c(a^k) \) converges and (3.7) implies the sum \( \zeta(a_{N-1}^k - p) + \zeta(a_N^k - p) \) also converges, namely \( (a_{N-1}^k - p)^{-1} + (a_N^k - p)^{-1} \) tends to a finite limit as \( a^k \to a \). Then \( c(a) \) and \( A \) can be also expressed in terms of \( \{a_1, \ldots, a_{N-2}\} \) in Case (a-ii) and (a-iii).

Now we consider that GLE\((n, p, A_0, \tau)\) is not completely reducible. Then \( y_{a,c}(z; p) \) in Proposition 3.1 is the only common eigenfunction up to a constant.

**Theorem 3.6.** Let \( p \notin E_\tau[2] \) and \( A \notin \mathbb{C} \). Suppose that GLE\((n, p, A, \tau)\) is not completely reducible with solution \( y_{a,c}(z) \) given in Proposition 3.1. Then there exists \( (m_1, m_2) \in \frac{1}{2} \mathbb{Z}^2 \) such that

\[
\{a_1, \ldots, a_N\} \equiv \{-a_1, \ldots, -a_N\} \mod \Lambda_\tau,
\]

\[
\sum_{j=1}^{N} a_j = m_1 + m_2 \tau + \sum_{k=1}^{3} \frac{n_k \omega_k}{2} \in E_\tau[2],
\]

\[
c = m_1 \eta_1 + m_2 \eta_2.
\]

**Proof.** Remark 3.2 shows that \( y_{a,c}(-z) \) and \( y_{a,c}(z) \) are linearly dependent, so (3.13) follows trivially from the expression (3.1). Besides, (3.2) gives

\[
\ell_i^* y_{a,c}(z) = \varepsilon_i y_{a,c}(z), \quad \varepsilon_i \in \{\pm 1\}, \quad i = 1, 2,
\]

because the linear dependence of \( y_{a,c}(-z) \) and \( y_{a,c}(z) \) imply \( \varepsilon_i = \varepsilon_i^{-1} \). Consequently, (3.14)-(3.15) can be proved by the same way as (3.3)-(3.7), where \( (m_1, m_2) \in \frac{1}{2} \mathbb{Z}^2 \) follows from \( \varepsilon_i \in \{\pm 1\} \) for \( i = 1, 2 \).

Now we want to define the embedding of \( \Gamma_{n,p} \) into \( \text{Sym}^N E_\tau \), where \( N = \sum_{i=0}^{3} n_i + 1 \) and \( \text{Sym}^N E_\tau \) is the \( N \)-th symmetric product of \( E_\tau \). For any \( (A, W) \in \Gamma_{n,p} \), it follows from Remark 2.B that the solution \( y_1(z; A, W) \) is unique up to a sign. By (3.1) and Proposition 3.1 there is \( a = \{a_1, \ldots, a_N\} \) (unique mod \( \Lambda_\tau \)) such that \( y_1(z; A, W) = y_{a,c}(z) \), i.e., \( a \) gives all the zeros of \( y_1(z; A, W) \). Then we define a map \( i_{n,p} : \Gamma_{n,p} \to \text{Sym}^N E_\tau \) by

\[
i_{n,p}(A, W) := \{[a_1], \ldots, [a_N]\} \in \text{Sym}^N E_\tau,
\]
where as introduced in Section 1, \([a_i] := a_i \pmod{\Lambda_\tau} \in E_\tau\). The above argument shows that \(i_{n,p}\) is well-defined. Furthermore, if \(W \neq 0\), then we see from Remark 2.8 that

\[
(3.17) \quad i_{n,p}(A, -W) = \{-[a_1], \cdots, -[a_N]\}.
\]

**Proposition 3.7.** \(i_{n,p}\) is an embedding from \(\Gamma_{n,p}\) into \(\text{Sym}^N E_\tau\).

**Proof.** Suppose \(i_{n,p}(A, W) = i_{n,p}(\tilde{A}, \tilde{W}) = \{[a_1], \cdots, [a_N]\}\). Then (3.3) implies \(\Phi_e(z; A) = \Phi_e(z; \tilde{A})\) up to a constant and so \(A = \tilde{A}\) by (2.14). Together with Theorem 2.7, we have \(W^2 = \tilde{W}^2\). If \(W = 0\) then \(\tilde{W} = 0\). If \(W \neq 0\) then by (3.16) and (3.17) we also have \(W = \tilde{W}\). This proves the embedding of \(i_{n,p}\) \(\square\).

By applying \(i_{n,p}\), we could define \(\sigma_{n,p} : \Gamma_{n,p} \to E_\tau\) by

\[
(3.18) \quad \sigma_{n,p}(A, W) := \sum_{i=1}^N [a_i] - \sum_{k=1}^3 \frac{[n_k \omega_k]}{2},
\]

which is the composition of \(i_{n,p}\) and the addition map \(\{[a_1], \cdots, [a_N]\} \mapsto \sum_{i=1}^N [a_i] - \sum_{k=1}^3 \frac{[n_k \omega_k]}{2}\). Thus \(i_{n,p}\) is also called the *addition map*. Clearly

\[
\sigma_{n,p}(A, -W) = -\sum_{i=1}^N [a_i] - \sum_{k=1}^3 \frac{[n_k \omega_k]}{2} = -\sigma_{n,p}(A, W).
\]

The map \(\sigma_{n,p}\) is a holomorphic map (or a finite morphism) from \(\Gamma_{n,p}\) to \(E_\tau\) if the irreducible curve \(\Gamma_{n,p}\) is smooth (or if \(\Gamma_{n,p}\) is singular). Hence in both cases, the degree \(\text{deg} \sigma_{n,p} = \# \sigma_{n,p}^{-1}(z), z \in E_\tau\), is well-defined. How to calculate it is the main purpose of the paper.

**Definition 3.8.** We let \(Y_{n,p}(\tau)\) be the image of \(\Gamma_{n,p}(\tau)\) in \(\text{Sym}^N E_\tau\) under \(i_{n,p}\), and \(X_{n,p}(\tau)\) be the image of \(\{(A, W) \in \Gamma_{n,p} | W \neq 0\}\) under \(i_{n,p}\).

The next two section will discuss of the limiting of \(Y_{n,p}(\tau)\) for two case (i) \(A \to \infty\) and \(p \not\in E_\tau[2]\) is fixed; (ii) \(p \to \omega_k/2\).

4. **THE CLOSURE OF \(Y_{n,p}(\tau)\)**

The purpose of this section is to prove the following result.

**Theorem 4.1.** Let \(n = (n_0, n_1, n_2, n_3)\) where \(n_k \in \mathbb{Z}_{\geq 0}\). Then the followings hold.

(i) \(\overline{X_{n,p}(\tau)} = \overline{Y_{n,p}(\tau)} = Y_{n,p}(\tau) \cup \{\infty_+(p), \infty_-(p)\}\),

where

\[
(4.1) \quad \infty_{\pm}(p) := \left(0, \cdots, 0, \frac{\omega_1}{2}, \cdots, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \cdots, \frac{\omega_2}{2}, \frac{\omega_3}{2}, \cdots, \frac{\omega_3}{2}, \pm p \right).
\]

Furthermore, \(\overline{X_{n,p}(\tau)}\) is smooth at \(\infty_{\pm}(p)\).
(ii) The map \( i_{n,p} : \Gamma_{n,p} \to Y_{n,p} \) has a natural extension to \( \bar{i}_{n,p} : \Gamma_{n,p} \to Y_{n,p} \) by \( \bar{i}_{n,p}(\infty_{\pm}) := \infty_{\pm}(p) \), where \( \{ \infty_{\pm} \} := \Gamma_{n,p} \setminus \Gamma_{n,p} \) is given in \( [2,24] \).

We will prove a slight generalization of Theorem 4.1. Indeed, we want to study the limiting of \( Y_{n,p}(\tau) \) as \( p \to p_0 \notin E_r[2] \). For each \( p \) near \( p_0 \), we associate a \( A(p) \in \mathbb{C} \) and consider \( \text{GLE}(n,p,A(p),\tau) \). Letting \( a(p) = \{ [a_1(p)], \cdots, [a_N(p)] \} = i_{n,p}(A(p),W(p)) \) and \( c(p) := c(a(p)) = c \) given by Proposition 3.1, we study the limits of \( \{a(p)\} \) and \( c(p) \) as \( p \to p_0 \). Since \( a(p) \in \text{Sym}^N E_r \), up to a subsequence, we can always assume that \( a(p) \to a^0 = \{ [a^0_1], \cdots, [a^0_N] \} \) and

\[
(4.2) \quad a_j(p) \to a^0_j \quad \text{as} \quad p \to p_0, \quad \forall \, j.
\]

**Proposition 4.2.** Let \( p \to p_0 \notin E_r[2] \) and \( S := E_r[2] \cup \{ \pm [p_0] \} \). Then \( I_n(z;p,A(p),\tau) \) converges in \( E_r \setminus S \) if and only if \( A(p) \) converges if and only if the corresponding \( B(p) \) converges if and only if \( c(p) \) converges.

**Proof.** From \([22]\), it is easy to see that \( I_n(z;p,A(p),\tau) \) converges for \( z \in E_r \setminus S \) if and only if \( A(p) \) converges if and only if \( B(p) \) converges. To study its relation with \( c(p) \), we recall the solution \( y_{a(p),c(p)}(z;p) \) of \( \text{GLE}(n,p,A(p),\tau) \) given in Proposition 3.1. Then

\[
\frac{y'_{a(p),c(p)}(z;p)}{y_{a(p),c(p)}(z;p)} = c(p) + \sum_{j=1}^{N} \zeta(z - a_j(p)) - \sum_{k=0}^{3} n_k \zeta(z - \frac{\omega_k}{2}) - \frac{1}{2}(\zeta(z + p) + \zeta(z - p))
\]

and so

\[
(4.3) \quad I_n(z;p,A(p),\tau) = \frac{y'_{a(p),c(p)}(z;p)}{y_{a(p),c(p)}(z;p)} = \left( \frac{y'_{a(p),c(p)}}{y_{a(p),c(p)}} \right)' + \left( \frac{y'_{a(p),c(p)}}{y_{a(p),c(p)}} \right)^2 = c(p)^2 + 2c(p)D(z;a(p),p) + D(z;a(p),p)^2 + E(z;a(p),p),
\]

where

\[
(4.4) \quad D(z;a(p),p) := \left[ \sum_{j=1}^{N} \zeta(z - a_j(p)) - \sum_{k=0}^{3} n_k \zeta(z - \frac{\omega_k}{2}) \right] - \frac{1}{2}(\zeta(z + p) + \zeta(z - p))
\]

and \( E(z;a(p),p) = D'(z;a(p),p) \) is given by

\[
(4.5) \quad E(z;a(p),p) := \left[ \sum_{k=0}^{3} n_k \varphi(z - \frac{\omega_k}{2}) - \sum_{j=1}^{N} \varphi(z - a_j(p)) + \frac{1}{2}(\varphi(z + p) + \varphi(z - p)) \right].
\]

By \([12]\), both \( D(z;a(p),p) \) and \( E(z;a(p),p) \) converge uniformly for \( z \in E_r \setminus (\{ [a^0_j] \}_{j=1}^{N} \cup \{ \pm [p_0] \} \cup E_r[2]) \). Thus, \( I_n(z;p,A(p),\tau) \) converges if and only if \( c(p) \) converges.

Now we discuss the limit \( a^0 \) of \( a(p) \) as \( p \to p_0 \notin E_r[2] \). There are two cases: (i) the corresponding \( B(p) \to B_0 \in \mathbb{C} \); (ii) \( B(p) \to \infty \). Case (i) is easy, because \( B(p) \to B_0 \in \mathbb{C} \) implies \( A(p) \) converges to some \( A_0 \in \mathbb{C} \),
i.e. $\text{GLE}(n, p, A(p), \tau)$ converges to $\text{GLE}(n, p_0, A_0, \tau)$, and so $\lim_{p \to p_0} a(p) = a^0 \in Y_{n, p_0}(\tau)$. We discuss case (ii) in the following proposition.

**Proposition 4.3.** Suppose $a(p) \to a^0$ and $B(p) \to \infty$ as $p \to p_0 \notin E_\tau[2]$. Then $a^0 \in \{\infty + (p_0)\}$, where $\infty + (p_0)$ is defined in (4.1).

**Proof.** Since $B(p) \to \infty$, by Proposition 4.2, we have both $A(p)$ and $c(p) \to \infty$ as $p \to p_0$. We claim that $c(p)/A(p) \to \beta$ for some $\beta \in \mathbb{C} \setminus \{0\}$ as $p \to p_0$. In fact, by differentiating both expressions (2.1) and (4.3) of $I_n(z; p, A(p), \tau)$, we obtain

\begin{equation}
2c(p)E(z; a(p), p) = A(p)(\varphi(z - p) - \varphi(z + p)) + \text{other terms},
\end{equation}

where other terms remain bounded outside the singularities $\{(a^j)\}_{j=1}^N \cup \{\pm p_0\} \cup E_\tau[2]$ as $p \to p_0$. Therefore, $c(p)/A(p) \to \beta \neq 0$. Since $a^0 = \lim_{p \to p_0} a(p)$, (4.6) yields $2\beta E(z; a^0, p_0) = \varphi(z - p_0) - \varphi(z + p_0)$, and it follows from (4.1) that

\begin{equation}
2\beta \left[ \sum_{k=0}^{3} \varphi(z - a^j_k) - \sum_{j=1}^{N} \varphi(z - a^j_j) \right]
\end{equation}

\begin{equation}
= \varphi(z - p_0) - \varphi(z + p_0).
\end{equation}

From (4.1), we have two possibilities: One is $a^0 = \infty + (p_0)$ and $\beta = -1$; the other one is $a^0 = \infty - (p_0)$ and $\beta = 1$. This completes the proof. \qed

**Proof of Theorem 4.1.** Obviously, Theorem 4.1-(i) follows from Propositions 4.2-4.3 easily.

(ii). The above argument shows that $\lim_{A \to \infty} i_{n, p}(A, \pm W) = \infty \pm (p)$ (by renaming $W$ to $-W$ if necessary). As pointed out in (2.24), $\Gamma_{n, p} \setminus \Gamma_{n, p}$ consists of two points $\infty \pm := \lim_{A \to \infty} (A, \pm W)$. Thus the embedding $i_{n, p}$ has a natural extension to $\tilde{i}_{n, p}: \Gamma_{n, p} \to \Gamma_{n, \pm}$ by defining $\tilde{i}_{n, p}(\infty \pm) := \infty \pm (p)$. \qed

We have the following corollary.

**Corollary 4.4.** The map $\sigma_{n, p}$ can be extended to be a finite morphism (still denoted by $\sigma_{n, p}$) from the irreducible curve $\tilde{\Gamma}_{n, p}$ to $E_\tau$ such that

\[ \lim_{(A, \pm W) \to \infty \pm} \sigma_{n, p}(A, \pm W) = \pm [p]. \]

5. **The limiting of $Y_{n, p}(\tau)$ as $p \to \frac{\omega}{2}$**

5.1. **The counterpart of the above theory for $H(n, B, \tau)$**. Before stating the main result of this section, we recall the following generalized Lamé equation (denoted by $H(n, B, \tau)$)

\begin{equation}
y''(z) = I_n(z; B, \tau)y(z), \quad z \in \mathbb{C}.
\end{equation}

where

\begin{equation}
I_n(z; B, \tau) := \sum_{k=0}^{3} n_k(n_k + 1)\varphi(z + \frac{\omega}{2}\tau) + B,
\end{equation}
and \( n_k \in \mathbb{Z}_{>0} \) for all \( k \).

There is a counterpart of the theory about GLE(\( n, p, A, \tau \)) established in previous sections for \( H(\mathbf{n}, B, \tau) \), and the proof is simpler due to the absence of singularities \( \pm p \notin E_r[2] \). Therefore, we only write down the conclusions without any details of the proofs. Remark that part of the statements listed below can be found in [10, 19, 21] and references therein. In this section, we denote \( \bar{N} := \sum_k n_k \).

(i) Any solution of \( H(\mathbf{n}, B, \tau) \) is meromorphic in \( \mathbb{C} \). The corresponding second symmetric product equation (1.4) has a unique even elliptic solution \( \Phi_e(z; B) \) expressed by

\[
\hat{\Phi}_e(z; B) = \hat{C}_0(B) + \sum_{k=0}^{3} \sum_{j=0}^{n_k-1} \hat{b}^{(k)}_j(B) \varphi(z + \frac{j}{2})^{n_k-j}
\]

where \( \hat{C}_0(B), \hat{b}^{(k)}_j(B) \) are all polynomials in \( B \) with \( \deg \hat{C}_0 > \max_{j,k} \deg \hat{b}^{(k)}_j \) and the leading coefficient of \( \hat{C}_0(B) \) being \( \frac{1}{2} \). Moreover, \( \hat{\Phi}_e(z; B) = \hat{y}_1(z; B) \) \( \hat{y}_1(-z; B) \) where \( \hat{y}_1(z; B) \) is a common eigenfunction of the monodromy matrices of \( H(\mathbf{n}, B, \tau) \) and up to a constant, can be written as

\[
\hat{y}_1(z; B) = \hat{y}_n(a) := \frac{e^{c(n\tau)} \prod_{i=1}^{\bar{N}} a_i}{\prod_{k=0}^{3} \sigma(z - \frac{j}{2})^{n_k}}.
\]

(ii) Let \( \hat{W} \) be the Wronskian of \( \hat{y}_1(z; B) \) and \( \hat{y}_1(-z; B) \) (and denote \( \hat{y}_1(z; B) \) by \( \hat{y}_1(z; B, \hat{W}) \)), then \( \hat{W}^2 = Q_n(B; \tau) \), where

\[
Q_n(B; \tau) := \hat{\Phi}_e'(z; B) 2\hat{\Phi}_e(z; B) \hat{\Phi}_e''(z; B) + 4I_n(z; B, \tau)\hat{\Phi}_e(z; B)^2
\]

is a polynomial in \( B \) with odd degree and independent of \( z \). Define the hyperelliptic curve \( \Gamma_n(\tau) \) by

\[
\Gamma_n(\tau) := \{(B, \hat{W}) | \hat{W}^2 = Q_n(B; \tau)\}.
\]

Then the map \( i_n : \Gamma_n(\tau) \rightarrow \text{Sym}^\bar{N}E_\tau \) defined by

\[
i_n(B, \hat{W}) := \{[a_1], \cdots, [a_\bar{N}]\}
\]

is an embedding, where \( \{[a_1], \cdots, [a_\bar{N}]\} \) is uniquely determined by \( \hat{y}_1(z; B, \hat{W}) \) via (5.1). Let \( Y_n(\tau) \) be the image of \( \Gamma_n(\tau) \) in \( \text{Sym}^\bar{N}E_\tau \) under \( i_n \), and \( X_n(\tau) \) be the image of \( \{(B, \hat{W}) \in \Gamma_n | \hat{W} \neq 0\} \) under \( i_n \).

(iii) \( X_n(\tau) = Y_n(\tau) \cup \{\infty_0\} \) where

\[
\infty_0 := \left(0, \cdots, 0, \frac{\omega_1}{2}, \cdots, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \cdots, \frac{\omega_2}{2}, \frac{\omega_3}{2}, \cdots, \frac{\omega_3}{2}\right).
\]

Furthermore, the map \( i_n : \Gamma_n \rightarrow Y_n(\tau) \) has a natural extension to \( \bar{i}_n : \Gamma_n \rightarrow \bar{Y}_n(\tau) \) by \( \bar{i}_n(\infty) := \infty_0 \) where \( \{\infty\} := \Gamma_{n,p} \setminus \Gamma_{n,p} \).
(iv) The embedding $i_n$ also induces the addition map $\sigma_n : \Gamma_n \to E_\tau$ by

$$
\sigma_n(B, \hat{W}) := \sum_{i=1}^{N_n} [a_i] \cdot \sum_{k=1}^{3} \left( \frac{n_k^+}{2} \right).
$$

Then $\sigma_n$ can be extended as a finite morphism from the irreducible curve $\Gamma_n(\tau)$ to $E_\tau$ such that

$$
\lim_{(B, \hat{W}) \to \infty} \sigma_n(B, \hat{W}) = 0.
$$

5.2. The limit of $Y_{n,p}$ as $p \to \frac{\phi}{2}$. For any $n = (n_0, n_1, n_2, n_3)$, we define

$$
n_k^\pm := (n_{k,0}^\pm, n_{k,1}^\pm, n_{k,2}^\pm, n_{k,3}^\pm),
$$

where $n_{k,i}^\pm = n_i$ for $i \neq k$ and $n_{k,k}^\pm = n_k \pm 1$. Recall $N = \sum_k n_k + 1$. In the following, we always identify a point $a = \{[a_1], \cdots, [a_{N-2}]\} \in Y_n$ by $a = \{[a_1], \cdots, [a_{N-2}], [\frac{n_{k}^+}{2}], [\frac{n_{k}^-}{2}]\}$.

The main result of this section is the following theorem:

**Theorem 5.1.** Let $n = (n_0, n_1, n_2, n_3)$ with $n_k \geq 0$ for all $k$. Then

$$
\lim_{p \to \frac{\phi}{2}} Y_{n,p}(\tau) = Y_{n_k^}\tau(\tau) \cup Y_{n_k^-}(\tau).
$$

In the following, we will give the complete proof for the case $p \to 0$. For other cases the proof is similar. Let $p$ be a sequence of points in $E_\tau$ and a sequence $(A(p), W(p)) \in \Gamma_{n,p}$ with the corresponding $\{[a_1(p)], \cdots, [a_N(p)]\} \in Y_{n,p}$ satisfying $([a_1(p)], \cdots, [a_N(p)]) \to ([a_1], \cdots, [a_N])$. We assume that up to a subsequence if necessary, $A(p) \to A_0 \in \mathbb{C} \cup \{\infty\}$ as $p \to 0$. Then the same proof as Proposition 1.2 implies

$$
I_{\infty}(z; A(p), \tau) \text{ converges if and only if } c(a(p)) \text{ converges if and only if } B(p) \text{ converges as } p \to 0.
$$

Indeed, we have a more precise statement. Recall $e_k := \varphi(\frac{\phi}{2})$ for $k = 1, 2, 3$.

**Proposition 5.2.** $B(p)$ is convergent as $p \to 0$ if and only if

$$
A(p) = A^\pm(p) := \alpha^\pm_0 p^{-1} + \alpha^\pm_1 p + o(p)
$$

with

$$
\alpha^+_0 := -\left(\frac{1}{4} + n_0\right), \quad \alpha^-_0 := \frac{3}{4} + n_0
$$

and some $\alpha^\pm_1 \in \mathbb{C}$. Furthermore, if (5.8) holds, then $I_{\infty}(z; A^\pm(p), \tau)$ converges to $I_{\infty}(z; B^\pm, \tau)$, where $B^\pm$ and $\alpha^\pm_1$ are related by

$$
B^\pm = \mp(2n_0 + 1)\alpha^\pm_1 - \sum_{k=1}^{3} n_k(n_k + 1)e_k.
$$

**Proof.** Recall (2.3) and it yields

$$
B(p) = A(p)^2 - \zeta(2p)A(p) - \frac{3}{4}\varphi(2p) - \sum_{k=0}^{3} n_k(n_k + 1)\varphi(p + \frac{\phi}{2}).
$$
\[
\begin{align*}
&= \frac{1}{p^2} \left( (A(p)p)^2 - \frac{A(p)p}{2} - \frac{3}{16} - n_0(n_0 + 1) \right) \\
&= \frac{(A(p)p - \alpha_0^+)(A(p)p - \alpha_0^-)}{p^2} - \sum_{k=1}^{3} n_k(n_k + 1)e_k + o(1),
\end{align*}
\]

where \( \alpha_0^\pm \) is defined in (5.9). From here, we easily see that \( B(p) \) converges as \( p \to 0 \) if and only if \( A(p) \) satisfies (5.8).

Now suppose (5.8) holds, then (5.11) implies \( B(p) \to B^\pm \) as \( p \to 0 \). Furthermore,

\[
\begin{align*}
n_0(n_0 + 1)\varphi(z) + \frac{3}{4}(\varphi(z + p) + \varphi(z - p)) + A(p)(\zeta(z + p) - \zeta(z - p)) \\
= n_0(n_0 + 1)\varphi(z) + \frac{3}{4}\varphi(z) - 2A(p)p\varphi(z) + o(1) \\
&\to [n_0(n_0 + 1) + \frac{3}{4} = 2\alpha_0^\pm] \varphi(z) = n_{0,0}^\pm = n_{0,0}^+ + 1)\varphi(z),
\end{align*}
\]

where \( n_{0,0}^\pm \) is \( n_0 \pm 1 \). Thus \( I_n(z;p,A^\pm(p),\tau) \) converges to \( I_{n_0}^\pm(z;B^\pm,\tau) \). \( \Box \)

Now, we discuss the limit \( a^0 \) of \( a(p) \) as \( p \to 0 \). We divide two cases to discuss: (i) \( B(p) \to B_0 \in \mathbb{C} \), and (ii) \( B(p) \to \infty \).

**Proposition 5.3.** Suppose \( B(p) \to B_0 \in \mathbb{C} \) as \( p \to 0 \). Then either

(i) \( a^0 \in Y_{n+} \tau \), where \( n^+ = n_0^+ = (n_0 + 1, n_1, n_2, n_3) \); or

(ii) \( a^0 = \{[a_0^0], \ldots, [a_0^{N-2}], 0, 0\} \) and \( \{[a_0^0], \ldots, [a_0^{N-2}]\} \in Y_{n-} \tau \), where \( n^- = n_0^- = (n_0 - 1, n_1, n_2, n_3) \).

**Proof.** By Proposition (5.2) we see that \( I_n(z;p,A(p),\tau) \) converges to either (i) \( I_{n+}(z;B^+,\tau) \) or (ii) \( I_{n-}(z;B^-,\tau) \).

For the case (i), because \( c(a(p)) \to c_0 \) for some \( c_0 \in \mathbb{C} \) by (5.4), we see that

\[
y_{a(p),c(a(p))}(z;p) \to y_{a^0}(z) := \frac{e^{c_0z} \prod_{j=1}^{N} \sigma(z - a_j^0)}{\prod_{k=1}^{3} \sigma(z - \frac{2a_k}{2})^{n_k}},
\]

and \( y_{a^0}(z) \) is a solution to \( H(n^+,B^+,\tau) \). This implies \( a^0 \in Y_{n+} \tau \).

For the case (ii), we also have (5.12) but \( y_{a^0}(z) \) is a solution to \( H(n^-,B^-,\tau) \). It follows from (5.1) that at least two of \( \{[a_0^0], \ldots, [a_0^{N-2}]\} \) must be 0. After a rearrangement of the index, we have might assume

\[
a^0 = \{[a_0^0], \ldots, [a_0^{N-2}], 0, 0\}
\]

Then we have \( \{[a_0^0], \ldots, [a_0^{N-2}]\} \in Y_{n-} \tau \). \( \Box \)

**Remark 5.4.** In the following, we always identify a point \( \{[a_0^0], \ldots, [a_0^{N-2}]\} \) in \( Y_{n-} \tau \) with the point \( \{[a_0^0], \ldots, [a_0^{N-2}], 0, 0\} \) in \( \text{Sym}^N \mathbb{R} \).

Next, we consider the case \( I_n(z;p,A(p),\tau) \) diverges, i.e. \( B(p) \to \infty \) as \( p \to 0 \).
Proposition 5.5. Suppose $B(p) \to \infty$ as $p \to 0$. Then

$$a(p) \to a^0 = \infty(0)$$

where

$$\infty(0) := \left(0, \ldots, 0, \frac{\omega_1}{2}, \ldots, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \ldots, \frac{\omega_2}{2}, \frac{\omega_3}{2}, \ldots, \frac{\omega_3}{2}, 0\right).$$

Proof. Since $B(p) \to \infty$, we see from (5.7) that $I_n(z; p, A(p), \tau)$ diverges and $c(a(p)) \to \infty$ as $p \to 0$. By differentiating both expressions (5.11) and (4.3) of $I_n(z; p, A(p), \tau)$ and considering $p \to 0$, we obtain

$$2c(a(p))E(z; a(p), p) + 2D(z; a(p), p)E(z; a(p), p) + E'(z; a(p), p)$$

$$= \sum_{k=0}^{3} n_k(n_k + 1)\varphi'(z + \frac{\omega_k}{2}) + \frac{3}{4}(\varphi'(z + p) + \varphi'(z - p)) + A(p)[-2\varphi'(z)p + O(p^3)].$$

Then we divide our discussion into two cases.

Case 1. Suppose $A(p)p \to \beta \in \mathbb{C}$ up to a subsequence. Then the RHS of (5.14) is uniformly convergent outside the singularities $E_r[2]$. Since $c(a(p)) \to \infty$, we have

$$E(z; a(p), p) \to 0$$

uniformly outside $E_r[2]$. So we see from (4.5) that

$$\sum_{k=0}^{3} n_k\varphi(z - \frac{\omega_k}{2}) - \sum_{j=1}^{N} \varphi(z - a_j^0) + \varphi(z) = 0.$$

This implies that there are $(n_0 + 1)$ of $a_j^0$ equal to 0 and $n_k$ of $a_j^0$ equal to $\frac{\omega_k}{2}$ for $k \in \{1, 2, 3\}$, namely $a^0 = \infty(0)$.

(ii) Suppose $A(p)p \to \infty$. By dividing $A(p)p$ on both sides of (5.14), we see that

$$\frac{2c(a(p))}{A(p)p} E(z; a(p), p) = \frac{-2D(z; a(p), p)E(z; a(p), p) - E'(z; a(p), p)}{A(p)p}$$

$$+ \frac{1}{A(p)p} \left[ \sum_{k=0}^{3} n_k(n_k + 1)\varphi'(z + \frac{\omega_k}{2}) \right] - 2\varphi'(z) + O(p^2).$$

Since the RHS of (5.16) converges uniformly to $-2\varphi'(z)$ outside $E_r[2]$, we have either $\frac{c(a(p))}{A(p)p} \to \beta \in \mathbb{C}$ or $\frac{c(a(p))}{A(p)p} \to \infty$. Suppose $\frac{c(a(p))}{A(p)p} \to \beta \in \mathbb{C}$. Letting $p \to 0$, it follows from (5.16) and (4.5) that

$$\beta \left[ \sum_{k=0}^{3} n_k\varphi(z - \frac{\omega_k}{2}) - \sum_{j=1}^{N} \varphi(z - a_j^0) + \varphi(z) \right] = -\varphi'(z).$$
But the RHS of (5.17) has a pole of order 3 at $z = 0$ while the LHS does not, a contradiction. Thus, $\frac{c(a(p))}{A(p)p} \to \infty$ and then we obtain (5.15) again, which gives $a^0 = \infty(0)$. This completes the proof.

**Corollary 5.6.** Let $(A(p), W(p)) \in \Gamma_{n,p}$. Assume that $\sigma_{n,p}(A(p), W(p)) \to [u_0] \neq 0$ as $p \to 0$. Then $I_n(z; p, A(p), \tau)$ converges as $p \to 0$.

**Proof.** Let $a(p) = i_{n,p}(A(p), W(p))$ be the corresponding point in $Y_{n,p}(\tau)$ and we may assume $a(p) \to a^0$. Suppose $I_n(z; p, A(p), \tau)$ diverges as $p \to 0$. Then Proposition 5.5 says that $a^0 = \infty(0)$, which implies $[u_0] = 0$, a contradiction.

Now we are ready to prove Theorem 5.1.

**Proof of Theorem 5.1.** We prove the case for $p \to 0$ and the other three cases are similar. Let $p \to 0$. By Propositions 5.3 and 5.5 for any $a(p) \in Y_{n,p}(\tau)$, $a(p) \to a^0 \in Y_{n^+}(\tau) \cup Y_{n^-}(\tau)$ as $p \to 0$. Conversely, we want to show that any point $a \in Y_{n^+}(\tau) \cup Y_{n^-}(\tau)$ could be a limit point of some $a(p) \in Y_{n,p}(\tau)$ as $p \to 0$.

First we note that $\infty(0)$ is the limit of $\infty_{\pm}(p)$ as $p \to 0$.

Given any $a = \{[a_1], \cdots, [a_N]\} \in Y_{n^+}(\tau)$ and let $i_{n^+}(B, W) = a$ for some $B$. For $p$ close to $0$, we set $A(p) := -(1/4 + n_0)p^{-1} + \alpha^+_1p$, where $\alpha^+_1$ is given by (5.10):

$$\alpha^+_1 := \frac{-B + \sum_{k=1}^{3} n_k(n_k + 1)e_k}{2n_0 + 1},$$

and $B(p)$ by (2.9). Let $(A(p), \pm W(p)) \in \Gamma_{n,p}$ and $\pm a(p) = i_{n,p}(A(p), \pm W(p))$. By Propositions 5.2, 5.3 $I_n(z; p, A(p), \tau)$ converges to $I_{n^+}(z; B, \tau)$ and so $\{\pm a(p)\}$ converges to $\{\pm a\}$. By a similar argument, we could prove that any point of $Y_{n^-}(\tau)$ can be approximated by $Y_{n,p}(\tau)$ as $p \to 0$. This completes the proof.

6. THE DEGREE OF THE ADDITION MAP

In previous sections, we have defined the addition map $\sigma_{n,p}(\cdot | \tau)$ and $\sigma_n(\cdot | \tau)$ from $\Gamma_{n,p}(\tau)$ and $\Gamma_n(\tau)$ onto $E_\tau$ respectively. Since $\Gamma_{n,p}(\tau)$ and $\Gamma_n(\tau)$ are irreducible algebraic curves, it is an elementary fact that the degrees of $\sigma_{n,p}(\cdot | \tau)$ and $\sigma_n(\cdot | \tau)$ are well-defined. The purpose of this section is to prove the following result.

**Theorem 6.1.** Let $n = (n_0, n_1, n_2, n_3)$ with $n_k \in \mathbb{Z}_{\geq 0}$ and $p \in E_\tau \setminus E_\tau[2]$. Then

\begin{equation}
\deg \sigma_n = \sum_{k=0}^{3} \frac{n_k(n_k + 1)}{2},
\end{equation}

and

\begin{equation}
\deg \sigma_{n,p} = \sum_{k=0}^{3} n_k(n_k + 1) + 1.
\end{equation}
To compute the degree of the map \( \varphi(\sigma_{n,p}(A, W)|\tau) \). Since \( \sigma_{n,p}(A, -W) = -\sigma_{n,p}(A, W) \), \( \varphi(\sigma_{n,p}(A, W)|\tau) \) depends on \( A \) only and is a meromorphic function of \( A \in \mathbb{C} \). Together with the facts that (i) \( \varphi(\sigma_{n,p}(-W)|\tau) \) has finitely many poles \( A \)'s and (ii) \( \sigma_{n,p}(A, W) \to \pm[p] \) by Corollary 4.3, and so

\[
\varphi(\sigma_{n,p}(A, W)|\tau) \to \varphi(p|\tau) \quad \text{as} \quad A \to \infty \quad \text{for any} \quad \tau \quad \text{and} \quad p \not\in E_\tau[2],
\]

we conclude that there are coprime polynomials \( P_j(A, p; \tau) \in \mathbb{C}[A], \quad j = 1, 2 \), such that

\[
\varphi(\sigma_{n,p}(A, W)|\tau) = \frac{P_1(A, p; \tau)}{P_2(A, p; \tau)}.
\]

Indeed, a more delicate argument shows \( P_j(A, p; \tau) \in \mathbb{Q}[e_k(\tau), \varphi(p|\tau), \varphi'(p|\tau)] \) \( [A] \); see e.g. [18, Proposition 3.1]. Here \( e_k(\tau) := \varphi(\frac{\tau^{k+1}}{2^{k+1}})|\tau \), \( k = 1, 2, 3 \). Write

\[
P_1(A, p; \tau) = \sum_{k=0}^{m_1} a_k(p; \tau) A^k \quad \text{and} \quad P_2(A, p; \tau) = \sum_{k=0}^{m_2} b_k(p; \tau) A^k,
\]

where \( a_k(p; \tau), b_k(p; \tau) \in \mathbb{Q}[e_k(\tau), \varphi(p|\tau), \varphi'(p|\tau)] \) with \( a_{m_1}(p; \tau) \neq 0 \) and \( b_{m_2}(p; \tau) \neq 0 \).

**Lemma 6.2.** Let \( \tau \in \mathbb{H} \) and \( p \not\in E_\tau[2] \). Then under the above notations, the followings hold.

1. \( m_1 = m_2 =: m \) and

\[
a_{m_1}(p; \tau) = \varphi(p|\tau).
\]

2. If \( b_m(p; \tau) \neq 0 \), then

\[
\deg_A P_1(A, p; \tau) \leq \deg_A P_2(A, p; \tau) = \deg \sigma_{n,p} = m.
\]

Furthermore,

\[
\deg_A P_1(A, p; \tau) = \deg_A P_2(A, p; \tau) \iff \varphi(p|\tau) \neq 0.
\]

**Proof.** (1) Take any \( \tau \) and \( p \not\in E_\tau[2] \) such that \( \varphi(p|\tau) \neq 0 \), \( a_{m_1}(p; \tau) \neq 0 \) and \( b_{m_2}(p; \tau) \neq 0 \). Then it follows from (6.3) and (6.4) that \( m_1 = m_2 \) (denote it by \( m \)) and (6.5) holds. Since both sides of (6.3) are meromorphic in \( (\tau, p) \), we conclude that (6.5) holds for all \( \tau \) and \( p \not\in E_\tau[2] \).

(2) Fix any \( \tau \) and \( p \not\in E_\tau[2] \) such that \( b_m(p; \tau) \neq 0 \). Then \( \deg_A P_2 = m \geq \deg_A P_1 \) and (6.7) follows from (6.5). It suffices to prove \( \deg \sigma_{n,p} = m \).

Denote \( \bar{m} := \deg \sigma_{n,p} \). Take any \( \sigma_0 \notin E_\tau[2] \cup \{\pm[p]\} \) such that

\[
\sigma_{n,p}^{-1}(\sigma_0) = \{(A_1, W_1), \cdots, (A_{\bar{m}}, W_{\bar{m}})\}
\]

consists of \( \bar{m} \) distinct elements in \( \Gamma_{n,p}(\tau) \) and the polynomial

\[
P_1(\cdot, p; \tau) - \varphi(\sigma_0|\tau)P_2(\cdot, p; \tau) \text{ has only simple zeros}.
\]
If \( W_i = 0 \) for some \( i \), then it follows from Theorem 3.6 that \( \sigma_0 = \sigma_{n,p}(A_i, W_i) \in E_\tau[2] \), a contradiction. Hence \( W_i \neq 0 \) for all \( i \). If \( A_i = A_j \) for some \( i \neq j \), then \( W_i = -W_j \). By using

\[
\sigma_0 = \sigma_{n,p}(A_j, W_j) = \sigma_{n,p}(A_i, -W_i) = -\sigma_{n,p}(A_i, W_i) = -\sigma_0 \text{ in } E_\tau,
\]
we have \( \sigma_0 \in E_\tau[2] \), a contradiction again. Thus these \( A_i \)'s are \( \tilde{m} \) distinct roots of \((6.9)\), namely

\[
\deg_A(P_1(A, p; \tau) - \varphi(\sigma_0|\tau)P_2(A, p; \tau)) \geq \tilde{m}.
\]

Assume by contradiction that \( \deg_A(P_1 - \varphi(\sigma_0|\tau)P_2) > \tilde{m} \). By \((6.9)\), the polynomial \( P_1 - \varphi(\sigma_0|\tau)P_2 \) has another zero \( A_{\tilde{m} + 1} \notin \{A_1, \ldots, A_{\tilde{m}}\} \). Then the points \( (A_{\tilde{m} + 1}, \pm W_{\tilde{m} + 1}) \in \Gamma_n \) satisfy

\[
\varphi(\sigma_{n,p}(A_{\tilde{m} + 1}, \pm W_{\tilde{m} + 1})|\tau) = \frac{P_1(A_{\tilde{m} + 1}, p; \tau)}{P_2(A_{\tilde{m} + 1}, p; \tau)} = \varphi(\sigma_0|\tau),
\]

i.e. either \( \sigma_{n,p}(A_{\tilde{m} + 1}, W_{\tilde{m} + 1}) = \sigma_0 \) or \( \sigma_{n,p}(A_{\tilde{m} + 1}, -W_{\tilde{m} + 1}) = \sigma_0 \), which is a contradiction with \((6.8)\) and \( A_{\tilde{m} + 1} \notin \{A_1, \ldots, A_{\tilde{m}}\} \). This proves

\[
\deg_A(P_1(A, p; \tau) - \varphi(\sigma_0|\tau)P_2(A, p; \tau)) = \tilde{m}
\]

hold for almost \( \sigma_0 \notin E_\tau[2] \cup \{\pm |p|\} \), which implies

\[
m = \max\{\deg_A P_1, \deg_A P_2\} = \tilde{m} = \deg \sigma_{n,p}.
\]

The proof is complete. \( \square \)

**Lemma 6.3.** Let \( \tau_0 \in \mathbb{H} \) and \( p_0 \notin E_\tau_0[2] \). If at least one of \( \{a_m(p_0; \tau_0), \ldots, a_0(p_0; \tau_0), b_m(p_0; \tau_0), \ldots, b_0(p_0; \tau_0)\} \) is not zero, then \( b_m(p_0; \tau_0) \neq 0 \).

**Proof.** Assume by contradiction that \( b_m(p_0; \tau_0) = 0 \). By our assumption and Lemma 6.2(1), we can take \( \sigma_0 \notin E_\tau_0[2] \cup \{\pm |p_0|\} \) such that

\[
P(A, p_0; \tau_0) := P_1(A, p_0; \tau_0) - \varphi(\sigma_0|\tau_0)P_2(A, p_0; \tau_0)
\]

is a nonzero polynomial with \( \deg P(A, p_0; \tau_0) \leq m - 1 \). Let \( \tau_\ell, p_\ell \notin E_\tau_\ell[2] \) such that \( (p_\ell, \tau_\ell) \to (p_0, \tau_0) \) as \( \ell \to \infty \), \( b_m(p_\ell; \tau_\ell) \neq 0 \) and \( \sigma_0 \notin E_\tau_\ell[2] \cup \{\pm |p_\ell|\} \) for all \( \ell \). Then Lemma 6.2(2) gives \( \deg \sigma_{n,p}(\cdot|\tau_\ell) = m \), so

\[
\sigma_{n,p}(\sigma_0|\tau_\ell) = \{(A_{\ell,1}, W_{\ell,1}), \ldots, (A_{\ell,m}, W_{\ell,m})\}
\]

and \( A_{\ell,i}, i = 1, \ldots, m, \) are all the zeros of

\[
P(A, p_\ell; \tau_\ell) := P_1(A, p_\ell; \tau_\ell) - \varphi(\sigma_0|\tau_\ell)P_2(A, p_\ell; \tau_\ell).
\]

Since \( \deg P(A, p_0; \tau_0) \leq m - 1 \), there is some \( i \) such that \( A_{\ell,i} \to \infty \) as \( \ell \to \infty \). Then Proposition 6.3 implies that the corresponding \( (a_1(A_{\ell,i}, p_\ell, \tau_\ell), \ldots, a_N(A_{\ell,i}, p_\ell, \tau_\ell)) \) of \( (A_{\ell,i}, W_{\ell,i}) \) converges to \( \infty(p_0) \), which yields

\[
\sigma_0 = \lim_{\ell \to \infty} \sum_{j=1}^{N} a_j(A_{\ell,i}, p_\ell, \tau_\ell) - \sum_{k=1}^{3} \frac{n_k \omega_k}{2} = \pm p_0,
\]

a contradiction to the choice of \( \sigma_0 \). This proves \( b_m(p_0; \tau_0) \neq 0 \). \( \square \)

**Lemma 6.4.** \( \deg \sigma_{n,p}(\cdot|\tau) = m \) is independent of \( \tau \in \mathbb{H} \) and \( p \notin E_\tau[2] \).
Proof. By Lemma 6.2 (2), it suffices to prove that \( \deg \sigma_{n,p_0}(\cdot | \tau_0) = m \) for any \( \tau_0, p_0 \notin E_\tau[2] \) satisfying \( b_m(p_0; \tau_0) = 0 \).

Case 1. At least one of the coefficients of \( P_i(A, p; \tau_0) \) is not identical 0 in \( p \).

Then we may divide \( P_i(A, p; \tau_0) \) by a common factor \( (p - p_0)^\ell \) such that all the coefficients of the new \( \tilde{P}_i (A, p; \tau_0) := P_i(A, p; \tau_0)/(p - p_0)^\ell \) are holomorphic at \( p_0 \) and at least one of them do not vanish at \( p_0 \). Then Lemma 6.3 implies \( b_m(p_0; \tau_0) \neq 0 \), where \( \tilde{b}_m(p; \tau) := b_m(p; \tau)/(p - p_0)^\ell \), and so the same proof as Lemma 6.2 (2) shows that \( \deg \sigma_{n,p_0}(\cdot | \tau_0) = m \).

Case 2. All the coefficients of \( P_i(A, p; \tau_0) \) are identical 0 in \( p \).

Then we may divide \( P_i(A, p; \tau) \) by a factor \( (\tau - \tau_0)^\ell \) such that Case 1 holds, which again implies that \( \deg \sigma_{n,p_0}(\cdot | \tau_0) = m \). The proof is complete.

For the addition map \( \sigma_n(\cdot | \tau) \) from \( H(n, B, \tau) \), we have the similar result: There are coprime polynomials \( \tilde{P}_i(B; \tau) \in \mathbb{Q}[e_k(\tau)][B] \) such that

\[
\varphi(\sigma_n(B, \tilde{W})|\tau) = \frac{\tilde{P}_1(B; \tau)}{\tilde{P}_2(B; \tau)}.
\]

Since (5.6) says \( \sigma_n(B, \tilde{W}) \to 0 \) as \( B \to \infty \), we have \( \deg_B \tilde{P}_1(B; \tau) > \deg_B \tilde{P}_2(B; \tau) \). Takemura [22, Proposition 3.2] proved that

\[
\deg_B \tilde{P}_1(B; \tau) = \deg_B \tilde{P}_2(B; \tau) + 1 \text{ for any } \tau \in \mathbb{H},
\]

\[
\varphi(\sigma_n(B, \tilde{W})|\tau) = \frac{4B}{[\sum_{k=0}^3 n_k(n_k + 1)]^2} + O(1) \text{ as } B \to \infty \text{ for fixed } \tau.
\]

Write

\[
\tilde{P}_1(B; \tau) = \sum_{k=0}^{\tilde{m}_1} \hat{a}_k(\tau)B^k \quad \text{and} \quad \tilde{P}_2(B; \tau) = \sum_{k=0}^{\tilde{m}_2} \hat{b}_k(\tau)B^k,
\]

where \( \hat{a}_k(\tau), \hat{b}_k(\tau) \in \mathbb{Q}[e_k(\tau)] \) with \( \hat{a}_{\tilde{m}_1}(\tau) \neq 0 \) and \( \hat{b}_{\tilde{m}_2}(\tau) \neq 0 \). Then we have

**Lemma 6.5.** Under the above notations, the followings hold.

1. \( \tilde{m}_2 + 1 = \tilde{m}_1 =: \tilde{m} \) and

\[
\frac{\hat{a}_{\tilde{m}}(\tau)}{\hat{b}_{\tilde{m}-1}(\tau)} = \frac{4}{[\sum_{k=0}^3 n_k(n_k + 1)]^2} =: C(n).
\]

2. \( \deg \sigma_n(\cdot | \tau) = \tilde{m} \) is independent of \( \tau \in \mathbb{H} \). Furthermore,

\[
\deg_B \tilde{P}_1(B; \tau) = \tilde{m} = \deg \sigma_n(\cdot | \tau) \text{ if } \hat{a}_{\tilde{m}}(\tau) \neq 0.
\]

**Proof.** The proof is similar to those of Lemmas 6.2 [6.4] with minor modifications; we omit the details here. \( \square \)

Recall Theorem 5.1 that

\[
\lim_{p \to \frac{\tau}{Y_{n,p}(\tau)}} Y_{n,p}(\tau) = Y_{n,k}(\tau) \cup Y_{n,k}(\tau).
\]
Thus it is reasonably expected that the degree of $\sigma_{n,p}$ should be the sum of the degree of $\sigma_{n,k}^+$ and $\sigma_{n,k}^-$. The following result confirms this conjecture, which plays a crucial role in the proof of Theorem 6.1.

**Theorem 6.6.** Let $n = (n_0, n_1, n_2, n_3)$ where $n_k \in \mathbb{Z}_{\geq 0}$ and $p \in E_{\tau} \setminus E_{\tau}[2]$. Then for each $k = 0, 1, 2, 3$, we have

$$\deg \sigma_{n,p} = \deg \sigma_{n,k}^+ + \deg \sigma_{n,k}^-.$$ 

Before giving the proof of Theorem 6.6, we would like to apply it to prove Theorem 6.1.

**Proof of Theorem 6.1.** It was proved by Wang and the third author [15] that

(6.15) $$\deg \sigma_{(n_0,0,0,0)} = \frac{n_0(n_0 + 1)}{2}, \quad n_0 \in \mathbb{Z}_{\geq 0}.$$ 

Note that $(0, 0, 0, 0)$ and $(-1, 0, 0, 0)$ gives the same GLE, so (6.15) also holds for $n_0 = -1$. Then by Theorem 6.6, we have

(6.16) $$\deg \sigma_{(n_0,0,0,0),p} = \deg \sigma_{(n_0+1,0,0,0)} + \deg \sigma_{(n_0-1,0,0,0)}$$

$$= \frac{(n_0 + 1)(n_0 + 2)}{2} + \frac{(n_0 - 1)n_0}{2}$$

$$= n_0(n_0 + 1) + 1.$$ 

Suppose that

$$\deg \sigma_{(n_0,k,0,0)} = \frac{n_0(n_0 + 1)}{2} + \frac{k(k + 1)}{2}$$

for all $0 \leq k \leq n_1 - 1$. We claim that

(6.17) $$\deg \sigma_{(n_0,n_1,0,0)} = \frac{n_0(n_0 + 1)}{2} + \frac{n_1(n_1 + 1)}{2}.$$ 

We note that the following identities hold

$$\deg \sigma_{(n_0,n_1,0,0)} + \deg \sigma_{(n_0,n_1-2,0,0)}$$

$$= \deg \sigma_{(n_0,n_1-1,0,0),p}$$

$$= \deg \sigma_{(n_0-1,n_1-1,0,0)} + \deg \sigma_{(n_0+1,n_1-1,0,0)},$$

where the first identity follows from Theorem 6.6 with $k = 1$ and the second one follows from the $k = 0$. Hence

(6.18) $$\deg \sigma_{(n_0,n_1,0,0)}$$

$$= \deg \sigma_{(n_0-1,n_1-1,0,0)} + \deg \sigma_{(n_0+1,n_1-1,0,0)} - \deg \sigma_{(n_0,n_1-2,0,0)}$$

$$= \frac{(n_0 - 1)n_0}{2} + \frac{(n_1 - 1)n_1}{2} + \frac{(n_0 + 1)(n_0 + 2)}{2} + \frac{(n_1 - 1)n_1}{2}$$

$$- \frac{n_0(n_0 + 1)}{2} - \frac{(n_1 - 2)(n_1 - 1)}{2}$$

$$= \frac{n_0(n_0 + 1)}{2} + \frac{n_1(n_1 + 1)}{2}.$$ 

This proves (6.17).
Now we claim that
\begin{equation}
\deg \sigma_{(n_0, n_1, 0, 0), p} = n_0(n_0 + 1) + n_1(n_1 + 1) + 1.
\end{equation}
Indeed, a direct consequence of (6.17) and Theorem 6.6 imply that
\begin{equation}
\deg \sigma_{(n_0, n_1, 0, 0), p} = (n_0 + 1)(n_0 + 2) + n_1(n_1 + 1) + (n_0 - 1)n_0 + n_1(n_1 + 1) + 2 = n_0(n_0 + 1) + n_1(n_1 + 1) + 1.
\end{equation}
This proves (6.19).

By the same argument as (6.18) and (6.20), we could derive the formulas (6.1)-(6.2) for any $n = (n_0, n_1, n_2, n_3)$ via induction.

The rest of this section is devoted to the proof of Theorem 6.6 for $k = 0$ (the other cases $k \in \{1, 2, 3\}$ can be proved in an analogous way and we omit the details). Denote $n_0^\pm = n^\pm$ and
\[
\deg \sigma_{n, p} = m, \quad \deg \sigma_{n^+} = m^+, \quad \deg \sigma_{n^-} = m^-.
\]
Our goal is to prove $m = m^+ + m^-$. Recalling Lemmas 6.4 and 6.5 in the sequel we fix $\tau$ (so we will omit the notation $\tau$ freely) such that
\[
\hat{a}_{m^+}(\tau) \neq 0 \quad \text{and} \quad b_m(\tau) \neq 0.
\]
Then $\varphi(\sigma_{n^\pm}(B, \hat{W}))$ (see (6.11) and (6.14)) can be rewritten as
\begin{equation}
\varphi(\sigma_{n^\pm}(B, \hat{W})) = C(n^\pm) \prod_{i=1}^{m^\pm}(B - \hat{B}_i^{0\pm}) \prod_{i=1}^{m^\pm-1}(B - \hat{B}_i^{\infty \pm}),
\end{equation}
where $\hat{B}_i^{0\pm}$'s and $\hat{B}_i^{\infty \pm}$'s are all the zeros of $\hat{P}_1(B)$ and $\hat{P}_2(B)$ (corresponding to $n^\pm$) respectively. On the other hand, since $b_m(\tau) \neq 0$, we take any sequence $p \to 0$ such that $b_m(p) \neq 0$ and $\varphi(p) \neq 0$, then $\varphi(\sigma_{n, p}(A, W))$ (see (6.4) and (6.5)) can be rewritten as
\begin{equation}
\varphi(\sigma_{n, p}(A, W)) = \varphi(p) \prod_{i=1}^{m}(A - A_i^0(p)) \prod_{j=1}^{m}(A - A_j^\infty(p)),
\end{equation}
where $A_i^0(p)$'s and $A_j^\infty(p)$'s are all the zeros of $P_1(A, p)$ and $P_2(A, p)$ respectively.

To prove $m = m^+ + m^-$, we need to study the asymptotics of $A_i^0(p)$'s and $A_j^\infty(p)$'s as $p \to 0$. By Corollary 5.6 and Proposition 5.2 $A_i^0(p)$ satisfies the asymptotic formula (5.8) as $p \to 0$. We denote $A_i^0(p)$ by $A_i^{0+}(p)$ if
\begin{equation}
A_i^{0+}(p) = -\left(\frac{1}{4} + n_0\right)p^{-1} + \alpha_i^{0+} p + o(p) \quad \text{for some } \alpha_i^{0+} \in \mathbb{C},
\end{equation}
and by $A_i^{0-}(p)$ if
\begin{equation}
A_i^{0-}(p) = \left(\frac{3}{4} + n_0\right)p^{-1} + \alpha_i^{0-} p + o(p) \quad \text{for some } \alpha_i^{0-} \in \mathbb{C}.
\end{equation}
We assume that there are $A_0^+(p), \ldots, A_{m_1}^+(p)$ and $A_1^-(p), \ldots, A_{m_2}^-(p)$ (counted with multiplicity), so $m = m_1 + m_2$. Let $B_i^{0+}(p)$ be the coefficient of $\text{GLE}(n,p, A_i^{0\pm}(p))$. Then by (6.10), we have

$$B_i^{0\pm}(p) \to B_i^{0\pm} \text{ as } p \to 0,$$

where

(6.25) $$B_i^{0+} := -(2n_0 + 1)\alpha_i^{0+} - \sum_{k=1}^{3} n_k(n_k + 1)e_k,$$

(6.26) $$B_i^{0-} := -(2n_0 + 1)\alpha_i^{0-} - \sum_{k=1}^{3} n_k(n_k + 1)e_k.$$

Also by Proposition 5.2, $\text{GLE}(n,p, A_i^{0\pm}(p)) \to H(n^\pm, B_i^{0\pm})$ as $p \to 0$. Since (6.22) says $\phi(\sigma_{n,p}(A_i^{0\pm}(p), W_i^{0\pm}(p))) = 0$ for each $p$, we have

$$\phi(\sigma_{n}(B_i^{0\pm}, W_i^{0\pm})) = \lim_{p \to 0} \phi \left(\sigma_{n,p}(A_i^{0\pm}(p), W_i^{0\pm}(p))\right) = 0.$$

Recalling $\{\hat{B}_1^{0+}, \ldots, \hat{B}_{m_1}^{0+}\}$ and $\{\hat{B}_1^{0-}, \ldots, \hat{B}_{m_2}^{0-}\}$ in (6.21), our above argument yields the following corollary.

**Corollary 6.7**. $B_i^{0+} \in \{\hat{B}_1^{0+}, \ldots, \hat{B}_{m_1}^{0+}\}$ for each $i = 1, 2, \ldots, m_1$ and $B_j^{0-} \in \{\hat{B}_1^{0-}, \ldots, \hat{B}_{m_2}^{0-}\}$ for each $j = 1, 2, \ldots, m_2$.

Later we will prove $m_1 = m^+$ and $m_2 = m^-$. Indeed, our proof will imply $\{B_1^{0+}, \ldots, B_{m_1}^{0+}\} = \{\hat{B}_1^{0+}, \ldots, \hat{B}_{m_1}^{0+}\}$ and $\{B_1^{0-}, \ldots, B_{m_2}^{0-}\} = \{\hat{B}_1^{0-}, \ldots, \hat{B}_{m_2}^{0-}\}$. See (6.48) below.

Next we the asymptotics of $A_i^{\infty}(p)$'s as $p \to 0$. Since (6.22) says

(6.27) $$\sigma_{n,p}(A_i^{\infty}(p), W_i^{\infty}(p)) = 0 \text{ for each } p,$$

it might happen that the corresponding $\text{GLE}(n,p, A_i^{\infty}(p))$ does not converge as $p \to 0$. If $\text{GLE}(n,p, A_i^{\infty}(p))$ converges, again $A_i^{\infty}(p)$ satisfies the asymptotic formula (5.8) as $p \to 0$, and we denote it by $A_i^{\infty+}(p)$ if

(6.28) $$A_i^{\infty+}(p) = -\left(\frac{1}{4} + n_0\right)p^{-1} + \alpha_i^{\infty+}p + o(p) \text{ for some } \alpha_i^{\infty+} \in \mathbb{C},$$

and by $A_i^{\infty-}(p)$ if

(6.29) $$A_i^{\infty-}(p) = \left(\frac{3}{4} + n_0\right)p^{-1} + \alpha_i^{\infty-}p + o(p) \text{ for some } \alpha_i^{\infty-} \in \mathbb{C}.$$

If $\text{GLE}(n,p, A_i^{\infty}(p))$ does not converge, then there are three alternatives of $A_i^{\infty}(p)$ up to a subsequence, according to Proposition 5.2

(D-i) $A_i^{\infty}(p) = A_i^{\infty+}(p)$ where

$$A_i^{\infty+}(p) = -\left(\frac{1}{4} + n_0\right)p^{-1} + \alpha_i^{\infty+}p + o(1), \alpha_i^{\infty+} \neq 0,$$

(D-ii) $A_i^{\infty}(p) = A_i^{\infty2}(p)$ where

$$A_i^{\infty2}(p) = \left(\frac{3}{4} + n_0\right)p^{-1} + \alpha_i^{\infty2}p + o(1), \alpha_i^{\infty2} \neq 0,$$
(D-iii) $A_i^\infty(p) = A^\infty_i(p)$ such that

$$A_i^\infty(p) p \to \beta \in \mathbb{C} \cup \{\infty\} \setminus \{(-\frac{1}{4} + n_0), (\frac{3}{4} + n_0)\}.$$ 

The key step is to prove the following result.

Lemma 6.8. Recall (6.27) that $\sigma_{n,p}^{-1}(0) = \{(A_i^\infty(p), W_i^\infty(p))\}_{i=1}^m$. Then there are $m_1 - 1$ of $\{\text{GLE}(n, p, A_i^\infty(p))\}_{i=1}^m$ converge to $H(n^+, B)$ for some $B$, and $m_2 - 1$ of them converge to $H(n^-, B)$ for some $B$, and the rest two (say $A_{m_1-1}^\infty(p)$ and $A_{m_2}^\infty(p)$) do not converge. Moreover, $A_{m_1-1}^\infty(p)$ and $A_{m_2}^\infty(p)$ belong to the case (D-iii) with $A_i^\infty(p) p \to \beta_i \in \mathbb{C} \setminus \{(-\frac{1}{4} + n_0), (\frac{3}{4} + n_0)\}$, $i = m - 1, m$ as $p \to 0$.

Proof. Step 1. We prove that there are some $i$'s such that $\text{GLE}(n, p, A_i^\infty(p))$ does not converge as $p \to 0$.

Suppose $\text{GLE}(n, p, A_i^\infty(p))$ are convergent for all $i$. Suppose $k_1$ of them converge to $H(n^+, B)$ and $k_2$ of them converge to $H(n^-, B)$. Then we rewrite (6.22) as

$$\varphi \left( \sigma_{n,p}(A, W) \right) = \varphi(p) \frac{\prod_{i=1}^{m_1} (A - A^{0+}_i(p)) \cdot \prod_{i=1}^{m_2} (A - A^{0-}_i(p))}{\prod_{i=1}^{k_1} (A - A^{\infty+}_i(p)) \cdot \prod_{i=1}^{k_2} (A - A^{\infty-}_i(p))}.$$ 

Fix $\sigma_0 \in E_{r\setminus E_{r}[2]}$ such that $\varphi(\sigma_0) \notin \{0, \infty\}$. The same proof of Theorem 2.1 implies that there exist $\sigma^p \in E_{r\setminus E_{r}[2]}$ and

$$\sigma_{n,p}(A^\infty(p), W^\infty(p)) \in \sigma_{n,p}^{-1}(\sigma^p)$$ 

with $\lim_{p \to 0} \sigma^p = \sigma_0$,

such that $\text{GLE}(n, p, A^\infty(p))$ converges to $H(n^+, B^\pm)$ for some $B^\pm$, namely

$$A^\infty(p) = -\left(\frac{1}{4} + n_0\right) p^{-1} + \alpha^+ p + o(p)$$ 

for some $\alpha^+ \in \mathbb{C}$, and

$$A^\infty(p) = \left(\frac{3}{4} + n_0\right) p^{-1} + \alpha^- p + o(p)$$ 

for some $\alpha^- \in \mathbb{C}$.

Recalling (6.23)-(6.24) and (6.28)-(6.29), we remark that

$$\alpha^+ \notin \{\alpha^0_i\}_i \cup \{\alpha^{\infty+}_j\}_j, \quad \alpha^- \notin \{\alpha^0_i\}_i \cup \{\alpha^{\infty-}_j\}_j,$$

namely $A^\pm(p) - A^{0\pm}_i(p) = (\alpha^\pm - \alpha_i^0) p + o(p)$ with $\alpha^\pm - \alpha_i^0 \neq 0$ for any $i$ and so do $A^\pm(p) - A^{\infty\pm}(p)$. For example, if $\alpha^+ = \alpha_i^0$ for some $i$, then $\text{GLE}(n, p, A^{0+}_i(p))$ also converges to $H(n^+, B^\pm)$, the same limit as that of $\text{GLE}(n, p, A^{\infty+}(p))$. This yields from (6.31) that

$$0 = \varphi \left( \sigma_{n,p} \left( A^{0+}_i(p), W^{0+}_i(p) \right) \right) = \varphi \left( \sigma_{n,p} \left( B^+, W^+ \right) \right)$$ 

$$= \lim_{p \to 0} \varphi \left( \sigma_{n,p} \left( A^+(p), W^+(p) \right) \right) = \lim_{p \to 0} \varphi(\sigma^+_p) = \varphi(\sigma_0) \notin \{0, \infty\},$$

a contradiction.

Now by inserting $A = A^+(p)$ into (6.30), we easily see from (6.23)-(6.24), (6.28)-(6.29) and (6.32) that

$$\varphi(\sigma_0) + o(1) = \varphi(\sigma^+_p) = \frac{p^{m_1-m_2-2} \cdot O(1)}{p^{k_1-k_2} \cdot O(1)}$$ 

as $p \to 0$. 

(6.35)
Here different from the notation $O(1)$, we use the notation $\mathcal{O}(1)$ to denote various quantities depending on $p$ which is uniformly bounded away from 0 and $\infty$ as $p \to 0$. However, inserting $A = A^-(p)$ into (6.30) leads to

\begin{equation}
\varphi(\sigma_0) + o(1) = \varphi(\sigma_p^-) = \frac{p^{m_2-m_1-2} \cdot \mathcal{O}(1)}{p^{k_2-k_1} \cdot \mathcal{O}(1)},
\end{equation}

which contradicts with (6.35). This proves Step 1, namely there must exist

$A_i^\infty(p) \in \mathbb{A}^\infty := \{A_1^\infty(p), \ldots, A_m^\infty(p)\}$

such that $\text{GLE}(n, p, A_i^\infty(p))$ does not converge. Suppose there are $\ell_1, \ell_2$ and $\ell_3, A_i^\infty(p)$'s satisfying (D-i), (D-ii) and (D-iii) respectively. Then we rewrite (6.32) as

\begin{equation}
\varphi(\sigma_{n,p}(A, W)) = \frac{\varphi(p) \prod_{i=1}^{m_1} (A - A_i^{0+}(p)) \cdot \prod_{i=1}^{m_2} (A - A_i^{0-}(p))}{\prod_{i=1}^{k_1} (A - A_i^{\infty+}(p)) \cdot \prod_{i=1}^{k_2} (A - A_i^{\infty-}(p)) \cdot \prod_{i=1}^{k_3} (A - A_i^{\infty i}(p))}.
\end{equation}

**Step 2.** We prove that $(\ell_1, \ell_2, \ell_3) \in \{(0, 0, 2), (0, 2, 1), (2, 0, 1)\}$. By the asymptotics (6.32)-(6.33) and (D-i)-(D-iii), we have

\begin{equation}
\begin{cases}
A^+(p) - A_i^{\infty_1}(p) = -\alpha_i^{\infty_1} + o(1), & \alpha_i^{\infty_1} \neq 0, \\
A^+(p) - A_i^{\infty_2}(p) = -(1 + 2n_0)p^{-1} + O(1), & \alpha_i^{\infty_2} \neq 0,
\end{cases}
\end{equation}

\begin{equation}
\begin{cases}
A^+(p) - A_i^{\infty_1}(p) = (1 + 2n_0)p^{-1} + O(1), & \alpha_i^{\infty_1} \neq 0, \\
A^+(p) - A_i^{\infty_2}(p) = -\alpha_i^{\infty_2} + o(1), & \alpha_i^{\infty_2} \neq 0,
\end{cases}
\end{equation}

and

\begin{equation}
\frac{\prod_{i=1}^{\ell_3} (A^+(p) - A_i^{\infty_3}(p))}{\prod_{i=1}^{\ell_3} (A^-(p) - A_i^{\infty_3}(p))} = \mathcal{O}(1).
\end{equation}

Again by inserting $A = A^\pm(p)$ into (6.37) and using (6.31), we have the following identity

\begin{equation}
\varphi(\sigma_0) = \frac{p^{m_2-m_1-2} \cdot \mathcal{O}(1)}{p^{k_2-k_1-\ell_2} \prod_{i=1}^{\ell_3} (A^+(p) - A_i^{\infty_3}(p))} + o(1)
\end{equation}

\begin{equation}
\quad = \frac{p^{m_2-m_1-2} \cdot \mathcal{O}(1)}{p^{k_2-k_1-\ell_1} \prod_{i=1}^{\ell_3} (A^-(p) - A_i^{\infty_3}(p))} + o(1).
\end{equation}

Together with (6.40), we obtain

\begin{equation}
2 (m_1 - m_2) = 2 (k_1 - k_2) + (\ell_1 - \ell_2),
\end{equation}

which implies $\ell_1 - \ell_2$ is even. Since $\varphi(\sigma_0) \neq 0$, (6.41) and $|A^+(p) - A_i^{\infty_3}(p)| \geq |p|^{-1} \cdot \mathcal{O}(1)$ also yield

\begin{equation}
|p|^{-\ell_3} \leq \mathcal{O}(1) \cdot \prod_{i=1}^{\ell_3} |A^+(p) - A_i^{\infty_3}(p)|
\end{equation}

\begin{equation}
\quad = \mathcal{O}(1) \cdot |p|^{m_1-m_2-2-(k_1-k_2-\ell_2)} = \mathcal{O}(1) \cdot |p|^\frac{\ell_1+\ell_2}{2} - 2,
\end{equation}
which implies $\frac{\ell_1 + \ell_2}{2} - 2 \leq -\ell_3$, namely

\[(6.44) \quad 0 \leq \ell_1 + \ell_2 + 2\ell_3 \leq 4.\]

From here and that $\ell_1 - \ell_2$ is even, we see that

\[(\ell_1, \ell_2, \ell_3) = (0, 0, 1), (0, 0, 2), (0, 2, 0), (0, 2, 1),
(2, 0, 0), (2, 0, 1), (1, 1, 0), (1, 1, 1), (1, 3, 0), (3, 1, 0).\]

On the other hand, we have

\[(6.45) \quad m_1 + m_2 = m = k_1 + k_2 + \ell_1 + \ell_2 + \ell_3.\]

From here and (6.42), we can only have

\[(\ell_1, \ell_2, \ell_3) = (0, 0, 2), (0, 2, 1), (2, 0, 1), (1, 1, 0).\]

The case $(\ell_1, \ell_2, \ell_3) = (1, 1, 0)$ is impossible by (6.41). This proves Step 2.

**Step 3.** We prove that $(\ell_1, \ell_2, \ell_3) \neq (0, 2, 1), (2, 0, 1)$ and so $(\ell_1, \ell_2, \ell_3) = (0, 0, 2)$.

Suppose $(\ell_1, \ell_2, \ell_3) = (0, 2, 1)$. Then it follows from (6.42) and (6.45) that $k_1 = m_1 - 1, k_2 = m_2 - 2$. Inserting these into (6.41) leads to $A^\pm(p) - A^\infty(p) = \mathcal{O}(1) \cdot p^{-1}$, so

\[(6.46) \quad A^\infty(p)p \rightarrow \beta_1 \in \mathbb{C}\setminus\{(-\frac{1}{4} + n_0), \frac{3}{4} + n_0\}.\]

To get a contradiction, we take any $B$ and consider $A(p)$ such that $\text{GLE}(n, p, A(p)) \rightarrow \text{H}(n^-, B)$ as $p \rightarrow 0$. The existence of such $A(p)$ is proved in Theorem 5.1 which has the following asymptotics

\[A(p) = (\frac{3}{4} + n_0) p^{-1} + \alpha^- p + o(p)\]

where

\[\alpha^- := \frac{B + \sum_{k=1}^{3} n_k(n_k + 1)e_k}{2n_0 + 1}.\]

Furthermore, the proof of Theorem 5.1 also implies $\varphi(n, p, A(p), W(p)) \rightarrow \varphi(n^-, B, \hat{W})$. By (6.37) and $(\ell_1, \ell_2, \ell_3) = (0, 2, 1)$ we have

\[
\varphi(n, p, A(p), W(p)) = \frac{\varphi(p) \prod_{i=1}^{m_1}(A(p) - A_0^+(p)) \prod_{i=1}^{m_2}(A(p) - A_0^-(p)) (A(p) - A_1^\infty(p))^{-1}}{\prod_{i=1}^{m_1+1}(A(p) - A_i^\infty(p)) \prod_{i=1}^{m_2+2}(A(p) - A_i^\infty(p)) \prod_{i=1}^{m_3}(A(p) - A_i^\infty(p))}
\]

Inserting the asymptotics of $A(p)$, (6.23)-(6.24), (6.28)-(6.29), (D-2) and (6.20) to the above formula and letting $p \rightarrow 0$, we easily obtain

\[(6.47) \quad \varphi(n^-, B, \hat{W}) = \frac{\prod_{i=1}^{m_2}(B - B_i^0)}{(1 + 2n_0)(\frac{3}{4} + n_0 - \beta_1)\alpha^2 \alpha_2^\infty \prod_{i=1}^{m_2+2}(B - B_i^\infty)},\]

where $B_i^0 = \lim_{p \rightarrow 0} B_i^0(p)$ is given in (6.20) and similarly for $B_i^\infty = \lim_{p \rightarrow 0} B_i^\infty(p)$. However, (6.47) contradicts with (6.11) and (6.12). This proves $(\ell_1, \ell_2, \ell_3) \neq (0, 2, 1)$. Similarly we can prove $(\ell_1, \ell_2, \ell_3) \neq (2, 0, 1)$ and so $(\ell_1, \ell_2, \ell_3) = (0, 0, 2)$. 

Step 4. We complete the proof.

Since \((\ell_1, \ell_2, \ell_3) = (0, 0, 2)\), then \((6.42)\) and \((6.45)\) imply \(k_1 = m_1 - 1\) and \(k_2 = m_2 - 1\). Inserting these into \((6.41)\) leads to \(\prod_{i=1}^{2}(A^\pm(p) - A_1^\infty(p)) = \mathcal{O}(1) \cdot p^{-2}\) and so \(A_1^\infty(p)p \rightarrow \beta_i \in \mathbb{C}\{-(\frac{1}{4} + n_0), (\frac{3}{4} + n_0)\}\). The proof is complete. \qed

We are in the position to prove Theorem 6.6.

Proof of Theorem 6.6. We prove the case \(k = 0\) (the other cases \(k \in \{1, 2, 3\}\) are similar). As pointed out before, we only need to prove \(m_1 = m_1\) and \(m_2 = m_2\).

Since by Lemma 6.8, we can rewrite \((6.22)\) as

\[
\varphi(\sigma_{n,p}(A, W)) = \frac{\varphi(p) \prod_{i=1}^{m_1}(A - A_1^{0+}(p)) \cdot \prod_{i=1}^{m_2}(A - A_1^{0-}(p))}{\prod_{i=1}^{m_1}(A - A_1^{\infty+}(p)) \cdot \prod_{i=1}^{m_2-1}(A - A_1^{\infty-}(p)) \cdot \prod_{i=m-1}^{m}(A - A_1^{\infty}(p)).}
\]

Then by repeating the argument of Step 3 in Lemma 6.8, we obtain

\[
(6.48)\quad \varphi(\sigma_{n^-}(B, \hat{W})) = \hat{C}(n^-) \frac{\prod_{i=1}^{m_2}(B - B_i^{0-})}{\prod_{i=1}^{m_2-1}(B - B_i^{\infty-})},
\]

where \(\hat{C}(n^-)\) is a nonzero constant. Comparing this with \((6.21)\), we conclude \(m_2 = m^- = \deg \sigma_{n^-}\). Similarly we can prove \(m_1 = m^+ = \deg \sigma_{n^+}\). In conclusion,

\[
\deg \sigma_{n^+} + \deg \sigma_{n^-} = m_1 + m_2 = \deg \sigma_{n,p}.
\]

This completes the proof. \qed

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