Quantum Scattering From Classical Field Theory

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Abstract

We show that scattering amplitudes between initial wave packet states and certain coherent final states can be computed in a systematic weak coupling expansion about classical solutions satisfying initial value conditions. The initial value conditions are such as to make the solution of the classical field equations amenable to numerical methods. We propose a practical procedure for computing classical solutions which contribute to high energy two particle scattering amplitudes. We consider in this regard the implications of a recent numerical simulation in classical $SU(2)$ Yang-Mills theory for multiparticle scattering in quantum gauge theories and speculate on its generalization to electroweak theory. The generalization of our results to complex trajectories allows its application to any wave packet to coherent state transition. Finally, we discuss the relevance of these results to the issues of baryon number violation and multiparticle scattering at high energies.

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1 Introduction

The behavior of quantum scattering amplitudes at high center of mass energies, \( E \sim E_* \equiv m/g^2 \), is an open question currently under investigation. Here, \( m \) is a mass scale characterizing the particle excitations of the theory, and \( g \) is a small coupling constant controlling the nonlinear interactions. It has been speculated that multiparticle scattering from initial states with only a small number of particles could be unsuppressed at these energies. A second related issue is whether anomalous processes, such as baryon number violation in electroweak theory where \( E_* \approx \mathcal{O}(M_w/\alpha_w) \sim 10 \text{ TeV} \), are unsuppressed at energies of the same order \([1]\).

Standard perturbation theory is clearly inadequate to address either of these issues. Recent investigations have focused on expansions about nonperturbative solutions of classical field equations, such as instantons \([1, 2]\). However, none of the previously proposed solutions provides a stationary point for a controlled semiclassical expansion at the energies of interest, \( E \sim E_* \). In general, the semiclassical expansion about the proposed instanton-like saddlepoints is controlled only at low energies and provides little or no information at \( E \sim E_* \).

In this paper, we propose a method for finding classical solutions which will prove suitable saddlepoints of scattering amplitudes even at \( E \sim E_* \). The equations and boundary conditions which must be satisfied are dramatically simplified compared to a previous approach to the same problem by Mattis, McLerran and Yaffe \([3]\). In particular, the resulting equations and boundary conditions for the classical saddlepoint are no longer of the integro-differential type. Rather, the equations of motion are source-free,

\[
\frac{\delta S}{\delta \phi(x)} = 0
\]

while the boundary conditions are of the initial value type, which can be integrated forward in time using standard numerical algorithms. This is also a departure from the approach of Khlebnikov, Rubakov and Tityakov \([2, 4]\), who derive similar source-free equations, but for an initial coherent state, rather than a two particle state.

We discuss the implications of some recent numerical simulations in classical field theory for high energy two-particle scattering. In particular, in 4-dimensional pure SU(2) Yang-Mills theory, Gong, Matinyan, Müller and Trayanov \([5]\) have found an instability of high frequency standing waves of pure glue to decay to low frequency modes. When interpreted within our formalism, this result leads to the speculation that multiparticle scattering in SU(2) gauge theories may be unsuppressed (at least exponentially) at high energies. We give a generalization of our method to complex trajectories which allows their application to essentially any wave packet - coherent state scattering amplitude. We finally discuss how our formulation relates to some previous work by other authors on high energy multiparticle and anomalous processes in electroweak theory \([3, 4]\).
2 Formalism

We wish to calculate, in the stationary phase approximation, the on-shell \( S \)-matrix element between an initial two particle state and an arbitrary final state \( \{ \rangle \). The result will be a boundary value problem which determines the classical field which is the stationary point of the \( S \)-matrix element.

To this end, we express the kernel of the \( S \)-matrix in a basis of coherent states, first used in this context in \cite{4}. The initial and final states are defined by sets of complex variables \( a \equiv \{ a_\ell \}, \ b^* \equiv \{ b^*_\ell \} \), respectively. A coherent state \( | a \rangle \) is an eigenstate of the annihilation operator \( \hat{a}_\ell \): \( \hat{a}_\ell | a \rangle = a_\ell | a \rangle \). The relevance of coherent states for the scattering problem stems from the fact that upon differentiation they generate all momentum eigenstates (see eq.2.10 below).

For simplicity, we illustrate the case of a single real massive scalar field with self-coupling \( g^2 \) in 3+1 dimensions. The results can be generalized to the case of gauge fields. First, we express the transition amplitude from an initial coherent state \( | a \rangle \) at time \( T_i \) to a final coherent state \( | b^* \rangle \) at time \( T_f \), in terms of the path integral by inserting a complete set of eigenstates of the field operator on the initial and final time slices (see figure 1)

\[ \langle b^* | U | a \rangle = \int d\phi_i d\phi_f \langle b^* | \phi_f \rangle \langle \phi_f | U | \phi_i \rangle \langle \phi_i | a \rangle . \]  

\( U \) is the evolution operator between time \( T_i \) and \( T_f \). A “position” eigenstate of the field operator \( \phi \) is denoted \( | \phi \rangle \) and \( \phi_{i,f} = \phi(T_{i,f}) \). Then, from (2.1), we obtain the \( S \)-matrix kernel in a compact form in terms of path integrals

\[ \langle b^* | S | a \rangle \equiv S[b^*, a] = \lim_{T_i,T_f \to \pm \infty} \int d\phi_f d\phi_i e^{B_f} e^{B_i} \int_{\phi_i}^{\phi_f} D\phi \ e^{iS[\phi]} , \]

where \( S[\phi] \) is the action functional. The path integral appearing here is over fields obeying the boundary conditions \( \phi(T_{i,f}) = \phi_{i,f} \). The functional \( B_f \) is

\[ B_f [b^*, \phi_f] = -\frac{1}{2} \int d^3k \ b_\ell b^*_\ell e^{2i\omega_k T_f} - \frac{1}{2} \int d^3k \ \omega_k \phi_f(k) \phi_f(-k) \]

\[ + \int d^3k \sqrt{2\omega_k} e^{i\omega_k T_f} b^*_k \phi_f(-k) , \]

in terms of which the wave functional of the final coherent state is

\[ \langle b^* | \phi_f \rangle \equiv \exp( B_f [b^*, \phi_f] ) . \]

Similarly, the functional \( B_i \) is

\[ B_i [a, \phi_i] = -\frac{1}{2} \int d^3k \ a_\ell a_{-\ell} e^{-2i\omega_k T_i} - \frac{1}{2} \int d^3k \ \omega_k \phi_i(k) \phi_i(-k) \]

\[ + \int d^3k \sqrt{2\omega_k} e^{-i\omega_k T_i} a_k \phi_i(k) . \]

The authors of \cite{3} extremize momentum-space Greens functions. See Section 6.
is related to the wave functional of the initial coherent state

\[ \langle \phi_i | a \rangle \equiv \exp \left( B_i [a, \phi_i] \right) . \] (2.6)

Here the 3-dimensional Fourier transform is defined

\[ \phi_{i,f}(\vec{k}) = \int \frac{d^3x}{(2\pi)^{3/2}} e^{i\vec{k} \cdot \vec{x}} \phi(T_{i,f}, \vec{x}) , \] (2.7)

and is related to the residue of the 4-dimensional Fourier transform when the field reduces to a plane wave superposition as \(|T_{i,f}| \to \infty\). [4]

\[ \begin{aligned}
\text{Figure 1: Spacetime, with two slices.}
\end{aligned} \]

The kernel (2.2) is a generating functional for S-matrix elements between any initial and final \(N\) particle states, by functional differentiation with respect to arbitrary \(a_k\) and \(b^*_k\). We now use this fact to construct a kernel for scattering from initial two particle states. We define an initial two particle (wave packet) state at \(t = T_i\)

\[ |\vec{p},-\vec{p}\rangle \equiv \int d^3k \alpha_R(\vec{k}) \hat{a}_k^\dagger \int d^3k' \alpha_L(\vec{k}') \hat{a}_{k'}^\dagger |0\rangle , \] (2.8)

where \(\hat{a}_k^\dagger\) is a creation operator, and \(\alpha_{R,L}(\vec{k})\) are arbitrary smearing functions of \(\vec{k}\), localized around some reference momenta \(\vec{p}\) and \(-\vec{p}\) respectively. The wave packets are normalized so that \[2\int d^3k |\alpha_{R,L}(\vec{k})|^2 = 1 . \] (2.9)

The regime relevent to the high energy multiparticle scattering problem is where, in the limit \(g^2 \to 0\), the wave packet is localized around momenta \(|p| \sim m/g^2\) with a characteristic width, \(\Delta p \sim m\).

\[ \text{We use the non-relativistically normalized commutator } \left[ \hat{a}_{k'}, \hat{a}_k^\dagger \right] = \delta^3(\vec{k} - \vec{k}') . \] (2.8)
This state can be generated by functional differentiation of the coherent state \(|a\rangle\) with respect to \(a_k\)

\[
|\vec{p}, -\vec{p}\rangle = \int d^3\vec{k} \ d^3\vec{k}' \ \alpha_R(\vec{k}) \ \alpha_L(\vec{k}') \ \frac{\delta}{\delta a_k} \ \frac{\delta}{\delta a_{k'}} \ |a\rangle \bigg|_{a=0} . \tag{2.10}
\]

So, differentiating under the functional integral, the S-matrix element between the two particle state (2.10) and any final state \(|\{b_k^*\}\rangle\) involves the following functional at \(t = T_i\)

\[
\frac{\delta}{\delta a_k} \frac{\delta}{\delta a_{k'}} \exp \left( B_i [a, \phi_i] \right) \bigg|_{a=0} = \left. \exp \left( -\frac{i}{2} \int d^3 k \ \omega_k \ \phi_i(\vec{k}) \ \phi_i(-\vec{k}) \right) \right). \tag{2.11}
\]

after dropping a term which vanishes in the limit \(T_i \to -\infty\). The last factor here is simply the normalization of the initial position eigenstate

\[
\exp \left( -\frac{1}{2} \int d^3 k \ \omega_k \ \phi_i(\vec{k}) \ \phi_i(-\vec{k}) \right) . \tag{2.12}
\]

We combine this with the smearing functions and finally obtain an S-matrix kernel for the scattering of two wave packets into arbitrary final states,

\[
S[b^*, 2] = \lim_{T_i, T_f \to \mp \infty} \int d\phi_f \ d\phi_i \ \alpha_R \cdot \phi_i \ \alpha_L \cdot \phi_i \ e^{B_f [b, \phi_f]} + B_i[0, \phi_i] \ \int_{\phi_i}^{\phi_f} D\phi \ e^{iS[\phi]} , \tag{2.13}
\]

where we have denoted the initial state (2.10) by \(\text{“2”}\). Here we have used the following compact notation for the initial state factors

\[
\alpha \cdot \phi_i \equiv \int d^3 k \ \sqrt{2\omega_k} \ \alpha(k) \ \phi_i(k) \ e^{-i\omega_k T_i} . \tag{2.14}
\]

We now derive the boundary value problem obeyed by the classical field in the stationary phase approximation of (2.13). The stationary phase approximation of ordinary integrals suggests that the initial state factors be included in the extremization of the path integral \(\Gamma\). These factors will certainly impact the stationary phase in the kinematic regime relevant to the high energy scattering problem, where the initial wave packets are peaked around \(|p| \sim m/g^2\), as expansions around the instanton indicate \([4]\). More generally though, we will show in Section 4 that this procedure provides a consistent weak coupling expansion at any fixed center of mass energy.

We include initial state factors in a straightforward manner by first exponentiating the initial state factors into an “effective action”, so that

\[
S[b^*, 2] = \lim_{T_i, T_f \to \mp \infty} \int d\phi_f \ d\phi_i \ D\phi \ e^{\Gamma} , \tag{2.15}
\]

\[\text{3The Stirling approximation to the Gamma function is a classic example.}\]
where the effective action $\Gamma$ is

$$\Gamma [\phi] = \ln \alpha_R \cdot \phi_i \alpha_L \cdot \phi_i + B_i [0, \phi_i] + iS [\phi] + B_f [b^*, \phi_f],$$  \hspace{1cm} (2.16)$$
after dropping a term which vanishes as $T_i \to -\infty$.

We can now derive the boundary value problem by varying the effective action. Varying the entire exponent $\Gamma$ with respect to $\phi(x)$ for $T_i < t < T_f$ gives the source-free equations of motion

$$\frac{\delta S}{\delta \phi(x)} = 0.$$  \hspace{1cm} (2.17)$$
Varying the entire exponent with respect to $\phi_i(k)$, gives

$$i \dot{\phi}_i(k) + \omega_k \phi_i(k) = \sqrt{2} \omega_k \left( \frac{\alpha_R(k)}{\alpha_R \cdot \phi_i} + \frac{\alpha_L(k)}{\alpha_L \cdot \phi_i} \right) e^{-i\omega_k T_i}.$$  \hspace{1cm} (2.18)$$
The first term on the left hand side comes from a surface term in the action $S$. The other terms come from variation of the wave functional at $t = T_i$. This boundary condition involves both the positive and negative frequency parts of the field, unlike the boundary condition which arises for an initial coherent state $|a\rangle$, which depends on the negative frequency component of $\phi$ only [2].

The boundary condition (2.18) at the initial time slice is rather complicated. However, it can be simplified since a real field $\phi$ may be written in the asymptotic region $t = T_i \to -\infty$ as a plane wave superposition

$$\phi_i(k) = \frac{1}{\sqrt{2} \omega_k} \left( u_k e^{-i\omega_k T_i} + u^*_k e^{i\omega_k T_i} \right).$$  \hspace{1cm} (2.19)$$
Equation (2.18) then reduces to the requirement

$$u_k = \frac{\alpha_R(k) + \alpha_L(k)}{\left( 1 + \int d^3 k' \alpha_R(k') \alpha_L(k') \right)^{1/2}},$$  \hspace{1cm} (2.20)$$
using the normalization (2.9). This solution is consistent with physical intuition, the classical field reducing to the initial particles at early times. The overlap of the left- and right- moving wave packets in the denominator is very small for narrow high energy wave packets.

If the field is real, then the negative frequency part equals the complex conjugate of the positive frequency part, and the field $\phi_i$ is determined as in (2.19) and (2.20). The real initial condition can be integrated forward, and uniquely determines the final field $\phi_f$. We say more about the case of complex stationary points of real fields in Section 5.

It remains to be seen in what sense it is a good approximation to include the initial state prefactors in the stationary phase calculation. We will demonstrate in Section 4 that
the stationary phase solution which results from this approach provides a controlled weak coupling expansion of the scattering amplitude.

Finally, varying with respect to $\phi_f(k)$ gives

$$ -i \dot{\phi}_f(k) + \omega_k \phi_f(k) = \sqrt{2 \omega_k} b^*_k e^{i\omega_k T_f} . $$

(2.21)

Again with a free field form as in (2.19), this boundary condition may be re-expressed as

$$ \phi_f(\vec{k}) = \frac{1}{\sqrt{2 \omega_k}} \left( b_k e^{-i\omega_k T_f} + b^*_k e^{i\omega_k T_f} \right) . $$

(2.22)

We see that the precise scattering amplitude which is extremized is only determined at the end of the calculation, by the asymptotic form of the classical solution in the far future, $T_f \to \infty$. For the initial conditions given by (2.18), the scattering amplitude corresponds to a transition from wave packet states to a final coherent state described by $|b^*\rangle$ from (2.21). Note that neither a wave packet state\footnote{Except in the limit of $\alpha_R(\vec{k}) \sim \delta^3(\vec{k} - \vec{p})$, where it reduces to a plane wave.} nor a coherent state are eigenstates of the Hamiltonian, so that neither have a definite energy.

When the classical field satisfies the equation of motion (2.17) and the boundary conditions (2.18) and (2.21), there are no initial and final state corrections to leading order in the semiclassical expansion. This is not the case for the instanton, which satisfies the equation of motion but not the correct boundary conditions. Instead, there are linear terms in the fluctuation about the instanton and these generate initial and final state corrections. We will discuss this further in Sections 4.

3 The Program

The results of the previous section establish an important fact:

The classical field obtained by starting with two incident wave packets and evolving them forward using (2.17) is the dominant contribution to some two particle scattering amplitude, in the semiclassical approximation.

In the case of a purely Minkowskian classical solution, the result for the corresponding scattering amplitude will not contain an exponential suppression. Thus, if a real time classical trajectory can be found which connects wave packet initial conditions to an “interesting” final state, for example a multiparticle or fermion number violating state, our results imply that the corresponding scattering amplitude is unsuppressed. Classical solutions which either exist in complex time or are themselves complex are relevant to classically forbidden transitions and will be discussed in Section 5.
The boundary conditions given by (2.19) and (2.20) are amenable to straightforward numerical integration. The boundary conditions specify \( \phi(x, T_i) \) and \( \dot{\phi}(x, T_i) \), which are sufficient to construct \( \phi(x, t) \) for all subsequent \( t > T_i \), given a discretization of the equations of motion (2.17). In principle, it is possible to generate an infinite number of classical trajectories, each relevant to a particular initial wave packet scattering amplitude.

Of course, it is not clear \textit{a priori} that the final state which results from the classical evolution will be one which is interesting to the problem of multiparticle production or anomalous baryon number violation. However, the above observation summarizes the relevance for scattering amplitudes of computations by Rajagopal and Turok \cite{6}, and also Goldberg, Nash and Vaughn \cite{7}. Rajagopal and Turok studied the classical scattering of wave packets in the Abelian Higgs model, while Goldberg et al. studied the classical dynamics of \( \phi^4 \) theory. Neither group found that energy was readily transferred from high frequency to low frequency modes, a signal which would indicate the production of a final state with many (soft) particles. Therefore, the scattering amplitudes which are dominated by their classical trajectories are probably not relevant to multiparticle production or fermion number violation.

On the other hand, the initial indications on the behavior of non-Abelian classical trajectories seem more promising. Some recent results on the classical behavior of pure \( SU(2) \) Yang-Mills theory by Gong, Matinyan, Müller and Trayanov \cite{5} may provide insight into solutions of the boundary value problem presented above. These authors have considered the classical stability of a stationary mono-color wave in Yang-Mills theory (in \( A_0^c = 0 \) gauge):

\[
A_i^c(x, t) = \delta_{i3} \delta_{c3} A \cos k_0 x \cos \omega_0 t,
\]

where \( c \) is a color and \( i \) is a spatial index. Small amplitude variations of the field in directions of different color are found to lead to an instability with long wavelength.

The Hamiltonian evolution of a perturbed standing wave is equivalent to the evolution of the initial conditions relevant to the scattering of two perturbed plane waves in \( SU(2) \) gauge theory. The discovery of an instability of the initial configuration to decay into long wavelength modes implies the existence of a classical trajectory connecting initial high energy plane waves to long wavelength modes in the final state. It therefore suggests that the corresponding \( 2 \rightarrow \text{many} \) gluon scattering amplitude may be unsuppressed! Unfortunately, the relevance of weak coupling calculations to pure gauge theory is unclear, due to the asymptotic freedom of the theory. It is possible that arguments similar to those from deep inelastic scattering may be used to justify the semiclassical result. In the limit that all of the energy scales (including the energies of the many “soft” outgoing gluons) are large compared to the intrinsic mass scale of the theory \( \Lambda_{SU(2)} \), it seems plausible that the relevant running coupling constant is small. It is also important to determine whether the instability persists for wave packet initial conditions, as the plane wave limit is never achieved at an actual accelerator.

\[\text{NB - A travelling wave is stable in Yang-Mills theory.}\]
Another class of trajectories in pure $SU(2)$ Yang-Mills theory are those found by Farhi et al. \[9\]. These solutions in the spherical ansatz correspond to spherical shells of glue that originate at infinity, collapse inward, and bounce back to infinity, leaving behind some fractional topological charge. There are some open questions concerning such solutions - in particular, their stability to perturbations which are outside the spherical ansatz, and their relevance to fermion number violation. However, in the absence of fermions, they can be interpreted within our formalism as extremizing scattering amplitudes between spherically symmetric initial and final quantum states.

Of course, our eventual goal is to address the electroweak theory, where the Higgs mechanism cuts off the infrared growth of the coupling constant. It would be extremely interesting if simulations of wave packet scattering could be performed in a spontaneously broken $SU(2)$ gauge theory.

Clearly, there is much numerical work to be done to address these issues.

4 Small Fluctuations

In this section, we construct the perturbative expansion around the classical solution of the BVP. We will demonstrate that corrections to the leading contribution are systematically suppressed in powers of the small coupling constant for any fixed energy. The resulting small coupling expansion differs from the usual one (e.g.– instanton) and the subtleties involved will be discussed at the end of this section.

As usual, we proceed by expanding the quantum field $\phi$ around a classical background field $\phi^c$, which is a solution to the classical boundary value problem derived in Section 2.

$$\phi(x) = \phi^c(x; \mu) + \nu(x) \ . \quad (4.1)$$

The classical solution depends in general on a collection of collective coordinates denoted here by $\mu$, corresponding to the invariances of the scattering amplitude $[3, 10]$. We expect invariances of translation and scale size to be broken by the choice of $\alpha_{R,L}(\vec{k})$; however, $\mu$ may include gauge coordinates in a gauge theory. Equation (4.1) involves an expansion on the initial and final time slices, which we can express in momentum space as

$$\phi_{i,f}(\vec{k}) = \phi^c_{i,f}(\vec{k}; \mu) + \nu_{i,f}(\vec{k}) \quad \text{for} \quad t = T_{i,f} \ , \quad (4.2)$$

where $\phi^c_{i,f}(\vec{k})$ is known explicitly for any given initial and final wave packets, from (2.19) and (2.22).

Note that the classical solution is of order $O(g^0)$, since it is non-trivial in the limit as $g \to 0$. In fact, when $g$ vanishes, the background field is simply the initial condition propagated forward in time with the free Hamiltonian,

$$\phi^c(t, \vec{k}) = \frac{1}{\sqrt{2\omega_k}} \left( u_k e^{-i\omega_k t} + u^*_k e^{i\omega_k t} \right) \ , \quad (4.3)$$
where \( u_k \) is as in (2.20). In this limit, the two wave packets simply pass through each other. For non-vanishing \( g \), the classical field \( \phi^c \) will have some complicated dependence on \( g \), reflecting the non-linearity of the theory and the external boundary conditions.

The kernel of the S-matrix (2.13) is composed of three functional integrals which we write as

\[
S[b^*, 2] = \int d\phi f \, d\phi_i \, \alpha_R \cdot \phi_i \, \alpha_L \cdot \phi_i \, e^{B_f[b^*, \phi_f]} + B_i[0, \phi_i] \int^{\phi_f}_{\phi_i} D\phi \, e^{iS[\phi]} \tag{4.4}
\]

where the limit \( |T_{i,f}| \to \infty \) is understood. Now we formally expand each of the terms in (4.4) using (4.1). The action becomes

\[
S[\phi^c + \nu] = S[\phi^c] + \int d^3x \, \nu(x) \, \dot{\phi}^c(x) \bigg|_{T_i}^{T_f} + S_2[\nu] + S_{int}[\nu] , \tag{4.5}
\]

after integrating by parts and retaining the surface terms in the time direction, and using the equations of motion (2.17). All of the non-quadratic dependence is contained in the interaction terms \( S_{int} \simeq O(\nu^3, \nu^4) \) which are suppressed by powers of \( g^2 \). Here \( S_2 \) is the part of the action quadratic in the fluctuation field, which can be expressed as

\[
S_2[\nu] = -\frac{1}{2} \int d^3x \, \nu(x) \, \dot{\nu}(x) \bigg|_{T_i}^{T_f} - \frac{1}{2} \int d^4x \, \nu(x) \, \Delta[x] \, \nu(x) , \tag{4.6}
\]

in terms of the operator \( \Delta \) of quadratic fluctuations in the \( \phi^c \) background,

\[
\delta^4(x - y) \Delta[x] \equiv \frac{\delta S}{\delta \phi(x) \, \delta \phi(y)} \bigg|_{\phi^c} = \delta^4(x - y) \left\{ \partial^2 \mu + V''(\phi^c) \right\} . \tag{4.7}
\]

Expanding the boundary functionals gives

\[
B_f\left[b^*, \phi^c_f + \nu_f\right] = -i \int d^3k \, \nu_f(k) \, \dot{\phi}^c_f(-k) + B_f\left[b^*, \phi^c_f\right] + B_f[0, \nu_f] , \tag{4.8}
\]

after using the boundary condition (2.21), and

\[
B_i[0, \phi^c_i + \nu_i] = -i \int d^3k \, \nu_i(k) \, \dot{\phi}^c_i(-k) + B_i[0, \phi^c_i] + B_i[0, \nu_i] - \frac{\alpha_R \cdot \nu_i}{\alpha_R \cdot \phi^c_i} - \frac{\alpha_L \cdot \nu_i}{\alpha_L \cdot \phi^c_i} , \tag{4.9}
\]

after using the boundary condition (2.18). The \( \dot{\phi}_{i,f} \) terms in (4.8) and (4.9) cancel with the boundary terms in the action (4.3). Note the appearance of additional terms linear in \( \nu_i \) in (4.9), which arise because we are expanding around a stationary point of the effective action (2.16), instead of the action. Note also that the terms independent of \( \nu_{i,f} \) are the overlaps of the classical field with the initial and final states, respectively.
Now, substituting (4.5), (4.8) and (4.9) in (4.4), we obtain
\[ S[b^a, 2] = \int d\mu \ e^{B_f[b^a, \phi_i]} + B_i[0, \phi_i] + iS[\phi^e] \ Z[\phi^c] \left[ 1 + \mathcal{O}(g^2) \right] \] (4.10)
where the quadratic integral over fluctuations is contained in
\[ Z[\phi^c] \equiv \int d\nu D\nu \ \{ 1 + \alpha_R \cdot \nu_i \} \ (1 + \alpha_L \cdot \nu_i) \ e^{B_f[0, \nu_f] + B_i[0, \nu_i] - \alpha_R \cdot \nu_i - \alpha_L \cdot \nu_i + iS_2[\nu]} , \]
(4.11)
after using \( \alpha_{R,L} \cdot \phi_i^c = 1 \) in the initial state factors and in the exponent. An integration over collective coordinates has been factored out of (4.10) and the remaining functional integral involves only fluctuations orthogonal to any zero modes of the classical background.

The \( \mathcal{O}(g^2) \) corrections result from the interactions in \( S_{\text{int}} \). Expanding \( \exp(iS_{\text{int}}[\nu]) \) inside (4.4) yields the leading term plus corrections which have the form of vacuum to vacuum loops. The vertices used to construct these loops each carry a suppression of \( g^2 \). The propagators inside these loops are the usual Feynman propagators evaluated in the classical background \( \phi^c \). In the instanton case \([2, 11]\), dangerous initial state corrections arose from the residues of propagators in the instanton background, which displayed an energy dependence \( \sim g^2 s \), where \( s \) is the center of mass energy of the collision and could be parametrically of order \( 1/g^4 \).

Note that it is only the residue of the instanton propagator that is ill-behaved and not the propagator itself. Since the corrections in our case are proportional to loop integrals rather than the residues of individual propagators \([2, 11]\), we expect them to remain subleading even at high energies.

The Gaussian path integral appearing in (4.11) can now be done exactly. We first factor out the quadratic prefactor by defining a more general functional integral,
\[ Z[j] \equiv \int d\nu_f D\nu_f \ \{ 1 + \alpha_R \cdot \nu_i \} \ \{ 1 + \alpha_L \cdot \nu_i \} \ e^{B_f[0, \nu_f] + B_i[0, \nu_i] - \alpha_R \cdot \nu_i - \alpha_L \cdot \nu_i + iS_2[\nu]} , \] (4.12)
in terms of which (4.11) may be written
\[ Z[\phi^c] = \left\{ 1 + \alpha_R \cdot \frac{\delta}{\delta j} \right\} \left\{ 1 + \alpha_L \cdot \frac{\delta}{\delta j} \right\} Z[j] \bigg|_{j=j^*} , \] (4.13)
where we will set the arbitrary current \( j(k) \) equal to
\[ j^*(k) = \sqrt{2 \omega_k} \ (\alpha_R(k) + \alpha_L(k)) \ e^{-i\omega_k T_i} , \] (4.14)
\(^6\)It is worth remembering that since we are working directly in Minkowski space (instead of beginning in Euclidean space and analytically continuing back), the Feynman boundary conditions on Greens functions are only obtained through some form of regularization, such as an \( i\epsilon \) prescription.
after the functional differentiation. The stationary point of (4.12) is a “classical” fluctuation field $\nu^c$ satisfying
\[ \Delta [x] \nu^c(x) = 0 , \tag{4.15} \]
with boundary conditions
\[ i \dot{\nu}^c_i(k) + \omega_k \nu^c_i(k) = j(k) \quad \text{and} \quad -i \dot{\nu}^c_f(k) + \omega_k \nu^c_f(k) = 0 . \tag{4.16} \]
The solution of this linear homogeneous boundary value problem can be expressed in terms of a Greens function
\[ \nu(x) = \int d^3q \, G(\vec{x}, \vec{q}; t) \, j(\vec{q}) , \tag{4.17} \]
where $G$ obeys
\[ \Delta [x] \, G(\vec{x}, \vec{x}; t) = 0 , \tag{4.18} \]
with boundary conditions in terms of its Fourier transform
\[ i \dot{G}(\vec{k}, \vec{q}; T_i) + \omega_k \, G(\vec{k}, \vec{q}; T_i) = \delta^3(\vec{k} - \vec{q}) \tag{4.19} \]
and
\[ -i \dot{G}(\vec{k}, \vec{q}; T_f) + \omega_k \, G(\vec{k}, \vec{q}; T_f) = 0 . \tag{4.20} \]
Using the above expressions, we obtain a compact expression for (4.12) entirely in terms of this Greens function, evaluated at $t = T_i$, and the known initial state source.
\[ Z[j] = \text{det}^{-1/2}[\Delta] \exp\left( \frac{1}{2} \int d^3k \, d^3q \, j(\vec{k}) \, G(\vec{k}, \vec{q}; T_i) \, j(\vec{q}) \right) . \tag{4.21} \]
The exponent appearing here is due entirely to the boundary functional $B_i$ in (4.12). All other terms vanish on the stationary phase solution (4.17). With this result, we obtain the Gaussian integral (4.13) by functional differentiation and setting $j = j^*$.
\[ Z[\phi^c] = \text{det}^{-1/2}[\Delta] . \tag{4.22} \]
Several terms in (4.13) have vanished in the limit of $T_i \to \infty$, most notably initial state factors. This is because both $j^*(k)$ and our inner product (2.14) contain rapidly oscillating factors $e^{-i\omega_k T_i}$. This rapid oscillation allows the application of the Riemann-Lebesgue lemma in the limit $T_i \to -\infty$, which guarantees that for any function $f(k)$ whose Fourier transform exists,
\[ \lim_{T_i \to -\infty} \int dk \, f(k) \, e^{-i\omega_k T_i} = 0 . \tag{4.23} \]
Here the functional determinant $\text{det}^{-1/2}[\Delta]$ is actually the product of three determinants: the standard 4-dimensional determinant of the operator $\Delta$ satisfying Feynman boundary conditions, and two 3-dimensional determinants representing the edge fluctuations $\nu(\vec{x}, T_{i,f})$. 

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The latter determinants can be expressed in terms of a homogeneous Greens function similar to $G$ described above.

We have demonstrated that the kernel of the S-matrix \((1.4)\) is obtained to leading order in $g^2$ from the action of the classical solution and the determinant of quadratic fluctuations around it.

$$S[b^*, 2] = \int \mu e^{B_i[\Phi^*, \phi^*_f]} + B_i[0, \phi^*_f] + iS[\phi^*_f] \det^{-1/2}[\Delta] \left[ 1 + \mathcal{O}(g^2) \right].$$ \hspace{1cm} (4.24)

The classical factors here are known functions, depending only on the initial and final boundary values of the classical field, \((2.19)\) and \((2.22)\). Thus, we may further reduce \((4.24)\) to

$$S[b^*, 2] = e^{-1} \int \mu e^{\tilde{N}/2} + iS[\phi^*_f] \det^{-1/2}[\Delta] \left[ 1 + \mathcal{O}(g^2) \right].$$ \hspace{1cm} (4.25)

where $\tilde{N} \equiv \int d^3k \; b^*_k b_k$ is the average number of particles in the final coherent state, which depends implicitly on the classical field. The origin of this factor is the use of coherent states which are not normalized. The standard normalization for coherent states which we use here is

$$\langle b^* | b^* \rangle = e^{\tilde{N}}.$$ \hspace{1cm} (4.26)

The correctly normalized S-matrix element would not include this factor, leaving the classical action and the determinant.

There are no additional tree-level radiative corrections to \((4.25)\), as there are in an expansion around a stationary point which does not obey the correct boundary conditions (c.f.– the instanton \([2, 4]\)). This is a direct result of the fact that our classical solution obeys both the correct equation of motion and the correct boundary conditions.

It is worth noting that our semiclassical expansion differs from the usual one in a crucial way. Our classical background depends on the coupling constant itself. In the usual instanton calculation, the field equations and boundary conditions can be made independent of $g$ by rescaling $\phi \rightarrow \tilde{\phi} = g\phi$. In our case, since the boundary conditions are fixed by the wave packet shapes $\alpha_{R,L}(k)$, the equations of motion for $\phi$ contain a dependence on $g$ which cannot be scaled away. It is clear that a change in $g$ will lead to a different nonlinear evolution of $\phi$ and hence a different classical trajectory. In the limit of strictly zero coupling constant, the evolution of the initial wave packet is clearly trivial, and we get only a contribution to the “diagonal” part of the S-matrix (i.e.– no scattering). Therefore, in order to obtain a nontrivial result, we have to take $g$ large enough to find an interesting classical trajectory, and yet small enough to justify the semiclassical $g^2$ expansion derived above. We have not demonstrated that the two requirements can be satisfied simultaneously in any theory. This can only be checked \textit{a posteriori} by computing the size of the subleading $\mathcal{O}(g^2)$ corrections in the background of a proposed trajectory. Of course, our experience with perturbation theory leads us to believe that for $g^2 \ll 1$ the corrections will be small. Thus, a classical trajectory found using weakly coupled equations of motion is likely to provide a controlled approximation.
5 Complex Trajectories

In the preceding discussion, we have focused on classical trajectories dependent on a real time variable. These are relevant to classically allowed amplitudes, such as those for multiparticle scattering or transitions at energies well above an energy barrier in configuration space. As we noted previously, the existence of a real time trajectory implies an unsuppressed scattering amplitude for the corresponding transition. However, there are many instances in which classically disallowed transitions are of interest. In particular, baryon number violation in the electroweak theory at energies less than the sphaleron energy \( E_s \sim M_w/\alpha_w \sim 10 \text{ TeV} \) is classically forbidden and therefore cannot be described by the same type of real time trajectories. The application of our formalism to such cases therefore requires some generalization, which we will describe in this section.

Consider the initial value problem given by (2.17) and (2.18). Since both the initial field values \( \phi_i(x) \) and their time derivatives \( \dot{\phi}_i(x) \) are given, the classical field has a well defined energy. Since real time trajectories conserve energy, it is clear that a suitable trajectory will not exist for some choices of initial and final conditions. In particular, in the electroweak theory for \( E_{cl} < E_s \), there are no real time trajectories which connect initial and final conditions with different topological charge.

\[ \text{Im } t \]

\[ iT \]

\[ \text{Re } t \]

Figure 2: Complex time contour

In order to find such trajectories, previous authors \[12, 13, 14\] have proposed complex time paths, which involve alternately Minkowskian, Euclidean and Minkowskian paths joined together at two times, say 0 and \( iT \) as in figure 2. In this approach, \( t = 0 \) and \( t = iT \) must be turning points (\( \dot{\phi}(\vec{x}, T_i) = 0 \) for all \( \vec{x} \)) of the field equations with given initial conditions. As is familiar from one dimensional quantum mechanics \[13\], complex time contours describe an incident configuration which reaches a turning point, and then tunnels under the barrier in Euclidean time, returning to real time at the second turning point upon reaching the other side of the barrier. The field \( \phi(x) \) can be taken to be real on the entire contour because
of the existence of the two turning points. The action is then purely imaginary on the Euclidean part of the contour and yields the familiar exponential tunneling suppression.

\[ |\langle f | i \rangle|^2 \sim e^{-2 \text{Im} S}. \quad (5.1) \]

As a matter of practice, one might try to find a suitable complex time trajectory by integrating our initial conditions (2.20) forward in real time in hopes that a turning point is encountered. Then, the time variable should be Wick rotated \( t \rightarrow ix_4 \) and the turning point configuration integrated forward in Euclidean time until another turning point is encountered. Finally, the contour should be rotated back to real time and the second turning point integrated forward until it achieves its asymptotic plane wave state. The final state \( |b^*\rangle \) can be read off from (2.22).

However, in the most general case, there is no guarantee of encountering turning points. Here we propose a more general method for extremizing S-matrix elements for which there is not a corresponding real, Minkowskian trajectory. Rather than complexifying time, we instead search for a general complex saddlepoint configuration satisfying (2.17) and (2.18).

Our motivation stems from the usual method of steepest descents applied to a real integral of the form

\[ I(h) = \int_{x_a}^{x_b} dx \ e^{\frac{1}{h} f(x)}. \quad (5.2) \]

In general, the asymptotic series in \( h \) for this integral can be obtained by expanding about a saddlepoint of the function \( f(x) : f'(x)|_{x^*} = 0 \). However, the saddlepoint \( x^* \) is often complex, and to apply the method, one must first deform the original real integral into the complex plane (see figure 3). In our case, we first think of the Feynman Path Integral as the

\footnote{A fact guaranteed in one dimensional QM, but not in higher dimensions or in field theory \[14\].}
product of a large number of regular integrals by discretizing spacetime:
\[
\int D\phi \ e^{iS[\phi]} \rightarrow \prod_x d\Phi_x \ e^{iS(\Phi_x, \Phi_{x\pm 1})}. \tag{5.3}
\]
For a real field \(\phi(x)\) each integral in the product is a real integral like (5.2). We generalize
the method of steepest descents by allowing the variable of each integration, \(\Phi_x\), to become
complex. There should be no obstruction to this generalization as the function \(S(\Phi_x, \Phi_{x\pm 1})\)
is analytic in the variables \(\Phi_x\). This procedure leads us to search for a complex trajectory \(\{\Phi_x\}\) (or in
the continuum, \(\phi^c(x)\)) satisfying (2.17), (2.18) and (2.21). The integral over small fluctuations about \(\phi^c(x)\) must be performed along the path of steepest descent in
configuration space. In other words, along small \(\nu(x)\) fluctuations which keep the imaginary
part of \(iS[\phi^c(x) + \nu(x)]\) constant.

For a complex field, there is no longer a relation between the positive and negative energy
Fourier components. Without loss of generality, we can write
\[
\phi_i(\vec{p}) = \frac{1}{\sqrt{2\omega_p}} \left( u_p \ e^{-i\omega_p T_i} + v_p \ e^{i\omega_p T_i} \right), \tag{5.4}
\]
where \(u_p, v_p\) are independent complex functions. The initial boundary condition (2.18) is satisfied by
\[
u_p \ = \ \frac{\alpha_R(\vec{p})}{\int d^3k \, \alpha_R(\vec{k}) \, v_k} + \frac{\alpha_L(\vec{p})}{\int d^3k \, \alpha_L(\vec{k}) \, v_k}, \tag{5.5}
\]
where \(v_k\) is arbitrary. The initial configurations which satisfy (5.4), (5.3) have negative
frequency modes which “resemble” the wave packet \(\alpha_L + \alpha_R\) (up to complex multiplicative
factors) and positive frequency modes which are arbitrary.

It is then clear that there are an infinite number of complex trajectories which can result
from evolving (5.4) using (2.17). The final quantum state \(|b^*\rangle\) which results from a given
trajectory depends only on the positive frequency component of the trajectory in the far
future (see Eq. (2.21)). The multiplicity of trajectories allows a very large set of scattering
amplitudes to be addressed within our formalism. In fact, it seems in general there always
exists a complex trajectory which satisfies (2.17), (2.18) and (2.21) simultaneously for any
\(\alpha(k)\) and \(b^*\). This is because the initial and final conditions (2.18) and (2.21) each correspond
to conditions on a combination of \(\phi(x)|_{T_i}\) and \(\dot{\phi}(x)|_{T_i}\) and \(\phi(x)|_{T_f}\) and \(\dot{\phi}(x)|_{T_f}\), respectively
(mixed Dirichlet-Neumann boundary conditions). Since the equations of motion (2.17) are
second order in time, this mixed set of boundary conditions should be enough to specify a
unique solution.

Since the trajectories found here are complex, the action \(iS[\phi^c]\) may contain a real sup-
pression factor. This is to be expected, as some of the trajectories will correspond to clas-
sically disallowed tunnelling transitions. Thus, although potentially any S-matrix element
can be approximated in this way, the result for particular trajectories may be exponentially
small.
Nostalgia

Here we comment on the relationship between our approach and some previous work by other authors on high energy multi-particle and anomalous processes in electroweak theory. First, we will recall the expansion around the constrained instanton in electroweak theory \[2, 4\]. This approach neglects effects of initial and final states on the saddlepoint and results in a low energy \(E \ll E_* \approx M_w/\alpha_w\) approximation to the total two-particle cross-section. Then, we remark on a previous approach to account for the impact of initial and final states on the saddlepoint \[3\] and comment on the formal resemblance which it bears to our approach.

A few of the properties of the electroweak instanton will make clear the relationship to our work: (i) The electroweak instanton is a solution of the Euclidean equations of motion. So, it will be necessary to rotate our real time formalism to Euclidean space in the analysis above, \(t \to ix_4\). We would then be considering a saddlepoint approximation to an on-shell truncated Euclidean Greens function. (ii) The electroweak instanton has finite Euclidean action, \(S = 8\pi^2/g^2\), and therefore satisfies vacuum boundary conditions. So, it is a suitable saddlepoint point for a Greens function with external fields only insofar as the effect of initial and final states on the saddlepoint are neglected (dropping the right hand side of (2.18) and (2.21)).

Since the instanton does not obey the correct boundary conditions to be a saddlepoint of a scattering amplitude \[3\], linear terms in the fluctuation expansion do not cancel, and the expansion of the scattering amplitude entails corrections which are formally large \(O(1/g^2)\) in the exponent. Fortunately, these corrections are under control and calculable for sufficiently small energies, \(E \ll E_* \) \[3\]. The effect of the final states can be taken into account, in a perturbative expansion in powers of \(x \equiv E/E_* \ll 1 \) \[4\]. For instance, the total inclusive cross section in the one-instanton sector is given by the well-known result

\[
\sigma_{tot}(E) \sim \exp \left\{ \frac{16\pi^2}{g^2} \left( -1 + c_1 x^{4/3} + c_2 x^2 + O\left(x^{8/3}\right) \right) \right\},
\]

in the limit \(E/E_* = fixed\) and \(g \to 0\). Here \(~\) implies that only the exponential behavior of the cross-section is shown. The first term here is just twice the instanton action, the ’tHooft suppression for vacuum tunnelling, while the next term indicates the exponential growth of the cross-section at low energies \[1\]. The higher order terms are determined from all tree-level corrections to the many soft \((E \simeq M_w)\) final state particles, and the first few coefficients \(c_i\) are known. Thus, provided the energy is sufficiently low \(E \ll E_*\), tree-level corrections to the final state can be described semiclassically.

It has not been similarly demonstrated that the effect of initial state corrections, involving many loops, can be described semiclassically. However, some results indicate that these corrections may run counter to naive intuition and the initial two particle state may be described semiclassically as well \[1\] \[3\] \[2\]. As we noted in Section 4, initial state corrections in the instanton expansion appear as factors of the residue of the propagator in the instanton.
background. The residue of the propagator has hard high energy behavior \((\sim g^2 s)\) where \(s\) is the center of mass energy \([11]\), so that initial state corrections can contribute at the \textit{semiclassical} level when \(s\) is of order \(1/g^4\). Mueller \([11]\) has shown that they contribute first at order \((E/E_*)^{10/3}\) to \((6.1)\).

By contrast, in our formalism the classical background field is constructed to solve the correct multi-particle boundary value problem derived in Section 2. So, there are no additional tree-level corrections in the limit \(E = \text{fixed}\) while \(g \rightarrow 0\). All corrections are formally suppressed by powers of \(g\), as shown in Section 4. The size of subleading corrections to our result depend on the properties of the Feynman propagator in the background of our proposed classical trajectory, \(\phi^c\). It would therefore be very interesting to understand the high energy behavior of the propagator in our background field. Also, for the purposes of direct comparison to \((6.1)\), it would be interesting to obtain its behavior in the same limit, where \(E/E_* = \text{fixed}\) while \(g \rightarrow 0\). We have reason to expect that the propagator behaves less severely than the instanton propagator in this limit, due to the lack of translational zero modes of the background field \([3]\).

A previous approach to account for the impact of initial and final states on the saddle-point \([3]\) bears a resemblance to ours but results in a different classical equation. We discuss it here for the purposes of comparison. These authors extremized Minkowskian \(n\)-point Greens functions of electroweak gauge fields

\[
G^{(n)}(p_1, p_2, \ldots) = \int D A A^{\mu_1 a_1}(p_1) \cdots A^{\mu_n a_n}(p_n) e^{iS[A]/g^2}.
\]

(6.2)

It is not clear to us whether the Greens function is the proper quantity to extremize to yield information about the corresponding scattering amplitude (and ultimately the cross-section). It may be that extremizing the entire Greens function, rather than its LSZ projection, is too strict a requirement.

An equation of motion for the saddlepoint of \((6.2)\) can be derived in a manner with formal similarity to our own. To do so, first express the \(n\) fields as an exponential, using \(A = \exp \ln A\), and then extremize the exponent. This yields \([3]\)

\[
\frac{\delta S}{\delta A^{\mu b}(x)} = i \sum_i \delta^{\mu \nu} \delta^{a_i b} \frac{e^{ip_i \cdot x}}{A^{\mu a_i}(p_i)}.
\]

(6.3)

This is a non-linear integro-differential field equation, depending on both the field \(A(x)\) and its Fourier transform \(A(p)\). Not much is known about equations of this type, and despite much effort \((6.3)\) defies solution \([15]\). It appears more difficult to solve in practice than the boundary value problem we have derived \((2.17), (2.19)\) and \((2.21)\). Of course, we do not yet have an explicit solution to our boundary value problem, but we hope to have established a practical procedure for computing one.
7 Conclusions

In this paper, we have established a connection between certain classical trajectories and corresponding quantum S-matrix elements in a weakly coupled field theory. The connection is made via a semiclassical approximation which appears to be controlled at small coupling, regardless of the scattering energy. This is in contrast to methods which involve expansions about instanton-like trajectories (i.e. satisfying vacuum boundary conditions), which have been shown to break down at nonperturbative energy scales $E \sim m/g^2$. Real Minkowskian trajectories can be found by the straightforward time integration of a well-defined initial value problem which is determined by the initial quantum state. Complex Minkowskian trajectories require the implementation of mixed boundary conditions at both initial and final times. This procedure, while more difficult in practice, can be guaranteed in principle to describe any wave packet to coherent state transition specified by the complex functions $\alpha(k), b^*_k$. The formalism described here can thus be used to compute intrinsically nonperturbative scattering amplitudes in a variety of field theories.

Some examples discussed here are the problem of baryon number violation in the electroweak theory and multiparticle scattering in gauge and scalar theories. The discovery of nontrivial or “interesting” real classical trajectories in these theories can now be related to unsuppressed scattering amplitudes which are potentially observable experimentally. Alternatively, nontrivial complex trajectories, of which there are an infinite number, allow the calculation of a much larger class of scattering amplitudes, including some that are classically forbidden and therefore involve tunneling. We hope that our results will stimulate future numerical work in this area, particularly the search for nontrivial trajectories.

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