(p, q)-Beta Functions and Applications in Approximation

Gradimir V. Milovanović, Vijay Gupta and Neha Malik

Abstract. In the present paper, we consider (p, q)-analogue of the Beta operators and using it, we propose the integral modification of the generalized Bernstein polynomials. We estimate some direct results on local and global approximation. Also, we illustrate some graphs for the convergence of (p, q)-Bernstein-Durrmeyer operators for different values of the parameters p and q using MATHEMATICA package.

Mathematics Subject Classification (2010). Primary 33B15; Secondary 41A25.

Keywords. (p, q)-Beta function, (p, q)-Gamma function, Bernstein polynomial, Durrmeyer variant, direct results.

1. Introduction

In the last decade, the generalizations of several operators to quantum variant have been introduced and their approximation behavior have been discussed [see for instance [1], [5], [8], [12], [10] etc]. The further generalization of quantum calculus is the post-quantum calculus, denoted by (p, q)-calculus. Recently, some researchers started working in this direction (cf. [2], [6], [13], [16]). Some basic definitions and theorems, which are mentioned below may be found in these papers and references therein.

\[ [n]_{p,q} := p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + pq^{n-2} + q^{n-1} = \frac{p^n - q^n}{p - q}. \]

The (p, q)-factorial is given by \([n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}, \quad n \geq 1, \quad [0]_{p,q}! = 1.\) The (p, q)-binomial coefficient satisfies

\[ \left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!}, \quad 0 \leq k \leq n. \]

This paper was supported by the Serbian Ministry of Education, Science and Technological Development (No. #O1174015).
The \((p,q)\)-power basis is defined as
\[
(x \ominus a)_p^n = (x - a)(px - qa)(p^2x - q^2a) \cdots (p^{n-1}x - q^{n-1}a).
\]
The \((p,q)\)-derivative of the function \(f\) is defined as
\[
D_{p,q} f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \quad x \neq 0.
\]
As a special case when \(p = 1\), the \((p,q)\)-derivative reduces to the \(q\)-derivative.

The \((p,q)\)-derivative fulfills the following product rules
\[
D_{p,q}(f(x)g(x)) = f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x),
\]
\[
D_{p,q}(f(x)g(x)) = g(qx)D_{p,q}f(x) + f(px)D_{p,q}g(x).
\]

Let \(f\) be an arbitrary function and \(a \in \mathbb{R}\). The \((p,q)\)-integral of \(f(x)\) on \([0,a]\) (see [15]) is defined as
\[
\int_0^a f(x) \, dp,qx = (q-p) a \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f \left( \frac{p^k}{q^{k+1}} a \right) \quad \text{if} \quad \left| \frac{p}{q} \right| < 1
\]
and
\[
\int_0^a f(x) \, dp,qx = (p-q) a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f \left( \frac{q^k}{p^{k+1}} a \right) \quad \text{if} \quad \left| \frac{q}{p} \right| < 1.
\]
The formula of \((p,q)\)-integration by part is given by
\[
\int_a^b f(px) \, D_{p,q}g(x) \, dp,qx = f(b) g(b) - f(a) g(a)
\]
\[
- \int_a^b g(qx) \, D_{p,q}f(x) \, dp,qx \quad (1.1)
\]
Very recently Gupta and Aral [7] proposed the \((p,q)\) analogue of usual Durrmeyer operators by considering some other form of \((p,q)\) Beta functions, which is not commutative. In the present article, we define different \((p,q)\)-variant of Beta function of first kind and find an identity relation with \((p,q)\)-Gamma functions. It is observed that \((p,q)\)-Beta functions may satisfy the commutative property, by multiplying the appropriate factor while choosing \((p,q)\) Beta function. As far as the approximation is concerned, order is important in post-quantum calculus. We propose a generalization of Durrmeyer type operators and establish some direct results.
2. \((p, q)\)-Gamma and \((p, q)\)-Beta Functions

**Definition 2.1** ([14]). Let \(n\) is a nonnegative integer, we define the \((p, q)\)-Gamma function as

\[
\Gamma_{p,q}(n+1) = \frac{(p \odot q)^n_{p,q}}{(p - q)^n} = [n]_{p,q}!, \quad 0 < q < p.
\]

**Definition 2.2.** Let \(m, n \in \mathbb{N}\), we define \((p, q)\)-Beta integral as

\[
B_{p,q}(m, n) = \int_0^1 x^{m-1} (1 \otimes qx)_{p,q}^{n-1} \, dp,qx. \quad (2.1)
\]

**Theorem 2.3.** The \((p, q)\)-Gamma and \((p, q)\)-Beta functions fulfil the following fundamental relation

\[
B_{p,q}(m, n) = p^{(n-1)(2m+n-2)/2} \frac{\Gamma_{p,q}(m) \Gamma_{p,q}(n)}{\Gamma_{p,q}(m+n)}, \quad (2.2)
\]

where \(m, n \in \mathbb{N}\).

**Proof.** For any \(m, n \in \mathbb{N}\) since

\[
B_{p,q}(m, n) = \int_0^1 x^{m-1} (1 \otimes qx)_{p,q}^{n-1} \, dp,qx,
\]

using [14] for \(f(x) = (x/p)^{m-1}\) and \(g(x) = -(1 \otimes x)_{p,q}^n/[n]_{p,q}\) with the equality \(D_{p,q}(1 \otimes x)^n = -[n]_{p,q} (1 \otimes qx)_{p,q}^{n-1}\) we have

\[
B_{p,q}(m, n) = \frac{[m-1]_{p,q}}{p^{m-1}[n]_{p,q}} \int_0^1 x^{m-2} (1 \otimes qx)_{p,q}^n \, dp,qx = \frac{[m-1]_{p,q}}{p^{m-1}[n]_{p,q}} B_{p,q}(m-1, n+1). \quad (2.3)
\]

Also we can write for positive integer \(n\)

\[
B_{p,q}(m, n+1) = \int_0^1 x^{m-1} (1 \otimes qx)_{p,q}^{n} \, dp,qx
\]

\[= \int_0^1 x^{m-1} (1 \otimes qx)_{p,q}^{n-1} (p^{n-1} - q^n x) \, dp,qx
\]

\[= p^{n-1} \int_0^1 x^{m-1} (1 \otimes qx)_{p,q}^{n-1} \, dp,qx - q^n \int_0^1 x^{m} (1 \otimes qx)_{p,q}^{n-1} \, dp,qx
\]

\[= p^{n-1} B_{p,q}(m, n) - q^n B_{p,q}(m+1, n).
\]
Using \(2.3\), we have
\[
B_{p,q}(m, n + 1) = p^{n-1} B_{p,q}(m, n) - q^n \frac{|m|_{p,q}}{p^m |n|_{p,q}} B_{p,q}(m, n + 1),
\]
which implies that
\[
B_{p,q}(m, n + 1) = p^{n-m-1} \frac{p^n - q^n}{p^{n+m} - q^{n+m}} B_{p,q}(m, n).
\]
Further, by definition of \((p, q)\) integration
\[
B_{p,q}(m, 1) = \int_0^1 x^{m-1} d_{p,q} x = \frac{1}{|m|_{p,q}}.
\]
We immediately have
\[
B_{p,q}(m, n) = p^{n+m-2} \frac{p^n - q^n}{p^{n+m-1} - q^{n+m-1}} B_{p,q}(m, n - 1)
= p^{n+m-2} \frac{p^n - q^n}{p^{n+m-1} - q^{n+m-1}} p^{n+m-3} \frac{p^{n-2} - q^{n-2}}{p^{n+m-2} - q^{n+m-2}} B_{p,q}(m, n - 2)
= p^{n+m-2} \frac{p^n - q^n}{p^{n+m-1} - q^{n+m-1}} p^{n+m-3} \frac{p^{n-2} - q^{n-2}}{p^{n+m-2} - q^{n+m-2}} \cdots
\times p^m \frac{p - q}{p^{m+1} - q^{m+1}} B_{p,q}(m, 1)
= p^{m+(m+1)+\cdots+(m+n-2)} \frac{(p \ominus q)_{p,q}^{n-1}}{(p^m \ominus q^m)_{p,q}^n} (p - q)
= p^{(n-1)(2m+n-2)/2} \frac{(p \ominus q)_{p,q}^{n-1}}{(p^m \ominus q^m)_{p,q}^n} (p - q).
\]
Following [14], we have \((a \ominus b)_{p,q}^{n+m} = (a \ominus b)_{p,q}^n (ap^n \ominus bq^n)_{p,q}^m\) thus \(2.4\) leads to
\[
B_{p,q}(m, n) = p^{\frac{(n-1)(2m+n-2)}{2}} \frac{(p \ominus q)_{p,q}^{n-1}}{(p^m \ominus q^m)_{p,q}^n} (p - q)
= p^{\frac{(n-1)(2m+n-2)}{2}} \frac{(p \ominus q)_{p,q}^{n-1}}{(p - q)_{p,q}^{n-1}} \frac{(p \ominus q)_{p,q}^{m-1}}{(p - q)_{p,q}^{m-1}} (p - q)
= p^{\frac{(n-1)(2m+n-2)}{2}} \frac{(p \ominus q)_{p,q}^{n-1}}{(p - q)_{p,q}^{n-1}} \frac{(p \ominus q)_{p,q}^{m-1}}{(p - q)_{p,q}^{m-1}} \frac{(p \ominus q)_{p,q}^{m+n-1}}{(p - q)_{p,q}^{m+n-1}}
= p^{\frac{(n-1)(2m+n-2)}{2}} \frac{\Gamma_{p,q}(m) \Gamma_{p,q}(n)}{\Gamma_{p,q}(m+n)}.
\]
This completes the proof of the theorem.
Remark 2.4. The following observations have been made for $(p,q)$ Beta functions:

- For $m,n \in \mathbb{N}$, we have
  \[ B_{p,q}(m,n+1) = p^{n-1}B_{p,q}(m,n) - q^n B_{p,q}(m+1,n). \]

- The $(p,q)$-Beta integral defined by (2.1) is not commutative. In order to make commutative, we may consider the following form
  \[ \tilde{B}_{p,q}(m,n) = \int_0^1 p^{m(m-1)/2}x^{m-1}(1 \ominus qx)^{n-1}_{p,q}d_{p,q}x. \]

For this form, $(p,q)$-Gamma and $(p,q)$-Beta functions fulfill the following fundamental relation
\[ \tilde{B}_{p,q}(m,n) = p^{(2mn+m^2+n^2-3m-3n+2)/2} \frac{\Gamma_{p,q}(m)\Gamma_{p,q}(n)}{\Gamma_{p,q}(m+n)}, \tag{2.5} \]
where $m,n \in \mathbb{N}$. Obviously for form (2.5), we get $\tilde{B}_{p,q}(m,n) = \tilde{B}_{p,q}(n,m)$.

3. $(p,q)$ Bernstein-Type Operators and Moments

For $n \in \mathbb{N}$ and $k \geq 0$, we have the following identity, which can be easily verified using the principle of mathematical induction:
\[ \sum_{k=0}^{n} \binom{n}{k}_{p,q} p^{k(k-1)/2}x^k(1 \ominus x)^{n-k}_{p,q} = p^{n(n-1)/2}. \tag{3.1} \]

Using the above identity, we consider the $(p,q)$-analogue of Bernstein operators for $x \in [0,1]$ and $0 < q < p \leq 1$ as
\[ B_{n,p,q}(f,x) = \sum_{k=0}^{n} b_{n,k}^{p,q}(1,x)f \left( \frac{p^{n-k}\binom{k}{p,q}}{\binom{n}{p,q}} \right), \tag{3.2} \]
where the $(p,q)$-Bernstein basis is defined as
\[ b_{n,k}^{p,q}(1,x) = \binom{n}{k}_{p,q} p^{k(k-1)-n(n-1)/2}x^k(1 \ominus x)^{n-k}_{p,q}. \]

Remark 3.1. Other form of the $(p,q)$-analogue of Bernstein polynomials has been recently considered by Mursaleen et al. [13].

Remark 3.2. Using the identity (3.1) and the following recurrence relation (for $m \geq 1$, $U_{n,m}^{p,q}(x) = B_{n,p,q}(e_m,x) = B_{n,p,q}(t^m,x)$):
\[ \binom{n}{p,q} U_{n,m+1}^{p,q}(px) = p^n x(1 - px)D_{p,q}[U_{n,m}^{p,q}(x)] + \binom{n}{p,q}px U_{n,m}^{p,q}(px), \]
the $(p,q)$-Bernstein polynomial satisfy
\[ B_{n,p,q}(e_0,x) = 1, \quad B_{n,p,q}(e_1,x) = x, \quad B_{n,p,q}(e_2,x) = x^2 + \frac{p^{n-1}x(1-x)}{\binom{n}{p,q}}, \]
where $e_i = t^i$, $i = 0,1,2$. 
Recently, Gupta and Wang (see [9]) discussed the $q$-variant of certain Bernstein-Durrmeyer type operators. We now extend these studies and propose the following $(p, q)$-Bernstein-Durrmeyer operators based on $(p, q)$-Beta function.

For $x \in [0, 1]$ and $0 < q < p \leq 1$, the $(p, q)$-analogue of Bernstein-Durrmeyer operators is defined as

$$D_{n}^{p,q}(f, x) = [n + 1]_{p,q} \sum_{k=1}^{n} p^{-(n-k+1)(n+k)/2} b_{n,k}^{p,q}(1,x)$$

$$\times \int_{0}^{1} b_{n,k-1}^{p,q}(t) f(t) d_{p,q} t + b_{n,0}^{p,q}(1,x) f(0), \quad (3.3)$$

where $b_{n,k}^{p,q}(1,x)$ is defined by (3.2) and

$$b_{n,k}^{p,q}(t) = \left[ \frac{n}{k} \right]_{p,q} t^{k}(1 \ominus qt)^{n-k}_{p,q}.$$ 

It may be remarked here that for $p = 1$, these operators will not reduce to the $q$-Durrmeyer operators; but for $p = q = 1$, these will reduce to the Durrmeyer operators.

**Lemma 3.3.** Let $e_m = t^m$, $m \in \mathbb{N} \cup \{0\}$, then for $x \in [0, 1]$ and $0 < q < p \leq 1$, we have

$$D_{n}^{p,q}(e_0, x) = 1, \quad D_{n}^{p,q}(e_1, x) = \frac{p[n]_{p,q} x}{[n + 2]_{p,q}},$$

$$D_{n}^{p,q}(e_2, x) = \frac{(p + q)p^{n+1}[n]_{p,q} x}{[n + 2]_{p,q}[n + 3]_{p,q}} + \frac{([n]_{p,q} - p^{n-1}) p^2 q[n]_{p,q} x^2}{[n + 2]_{p,q}[n + 3]_{p,q}}.$$ 

**Proof.** Using (2.2) and (2.1) and Remark 3.2, we have

$$D_{n}^{p,q}(e_0, x) = [n + 1]_{p,q} \sum_{k=1}^{n} p^{-(n+1-k)(n+k)/2} b_{n,k}^{p,q}(1,x)$$

$$\times \int_{0}^{1} \left[ \frac{n}{k - 1} \right]_{p,q} t^{k-1}(1 \ominus qt)^{n+1-k}_{p,q} d_{p,q} t + b_{n,0}^{p,q}(1,x)$$

$$= [n + 1]_{p,q} \sum_{k=1}^{n} p^{-(n+1-k)(n+k)/2} b_{n,k}^{p,q}(1,x) \left[ \frac{n}{k - 1} \right]_{p,q}$$

$$\times B_{p,q}(k, n - k + 2) + b_{n,0}^{p,q}(1,x)$$

$$= [n + 1]_{p,q} \sum_{k=1}^{n} p^{-(n+1-k)(n+k)/2} b_{n,k}^{p,q}(1,x) \frac{[n]_{p,q}[k - 1]_{p,q}!}{[n + 1 - k]_{p,q}!}$$

$$\times p^{(n+1-k)(n+k)/2} \frac{[k - 1]_{p,q}[n - k + 1]_{p,q}!}{[n + 1]_{p,q}!} + b_{n,0}^{p,q}(1,x)$$

$$= B_{n,p,q}(1,x) = 1.$$
Next, applying Remark 3.2, we have

\[
D_{p,q}^n(e_1, x) = [n + 1]_{p,q} \sum_{k=1}^{n} p^{-(n+1-k)(n+k)/2} b_{n,k}^{p,q}(1, x) \\
\times \int_0^1 \left[ \begin{array}{c} n \\ k-1 \end{array} \right]_{p,q} \ t^k \ (1 \otimes qt)^{n+1-k} d_{p,q} \ t \\
= [n + 1]_{p,q} \sum_{k=1}^{n} p^{-(n+1-k)(n+k)/2} b_{n,k}^{p,q}(1, x) \\
\times \left[ \begin{array}{c} n \\ k-1 \end{array} \right]_{p,q} B_{p,q}(k + 1, n - k + 2) \\
= [n + 1]_{p,q} \sum_{k=1}^{n} p^{-(n+1-k)(n+k)/2} b_{n,k}^{p,q}(1, x) \left[ \begin{array}{c} n \\ k-1 \end{array} \right]_{p,q} \\
\times p^{(n+1-k)(n+k+2)/2} \frac{k!_{p,q} [n-k+1]_{p,q}!}{[n+2]_{p,q}!} \\
= \sum_{k=1}^{n} p^{n-k+1} b_{n,k}^{p,q}(1, x) \frac{k!_{p,q}}{[n+2]_{p,q}} \\
= \frac{p[n]_{p,q}}{[n+2]_{p,q}} \sum_{k=1}^{n} b_{n,k}^{p,q}(1, x) \frac{p^{n-k} [k]_{p,q}!}{[n]_{p,q}} = \frac{p[n]_{p,q} x}{[n+2]_{p,q}}.
\]

Further, using the identity \([k+1]_{p,q} = p^k + q[k]_{p,q}\) and by Remark 3.2, we get

\[
D_{p,q}^n(e_2, x) = [n + 1]_{p,q} \sum_{k=1}^{n} p^{-(n+1-k)(n+k)/2} b_{n,k}^{p,q}(1, x) \\
\times \int_0^1 \left[ \begin{array}{c} n \\ k-1 \end{array} \right]_{p,q} \ t^{k+1} \ (1 \otimes qt)^{n+1-k} d_{p,q} \ t \\
= [n + 1]_{p,q} \sum_{k=1}^{n} p^{-(n+1-k)(n+k)/2} b_{n,k}^{p,q}(1, x) \\
\times \left[ \begin{array}{c} n \\ k-1 \end{array} \right]_{p,q} B_{p,q}(k + 2, n - k + 2) \\
= [n + 1]_{p,q} \sum_{k=1}^{n} p^{-(n+1-k)(n+k)/2} b_{n,k}^{p,q}(1, x) \left[ \begin{array}{c} n \\ k-1 \end{array} \right]_{p,q} \\
\times p^{n-1-k+4} \frac{k!_{p,q} [n-k+1]_{p,q}!}{[n+3]_{p,q}!} \\
= \sum_{k=1}^{n} p^{2(n-k+1)} b_{n,k}^{p,q}(1, x) \frac{k!_{p,q} [k+1]_{p,q}}{[n+2]_{p,q} [n+3]_{p,q}}.
\]

Lemma 3.5. Let
\[ D^{n,q}_p(e_2, x) = \sum_{k=1}^{n} p^{2(n-k+1)} t^{p,q}_{n,k}(1, x) \frac{[k]_{p,q}([k]_{p,q})}{[n+2]_{p,q}[n+3]_{p,q}} \]
\[ = \frac{p^{n+2}[n]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} \sum_{k=1}^{n} t^{p,q}_{n,k}(1, x) \frac{p^{-k}[k]_{p,q}}{[n]_{p,q}} \]
\[ + \frac{p^{2}q(n)^2}{[n+2]_{p,q}[n+3]_{p,q}} \sum_{k=1}^{n} b^{p,q}_{n,k}(1, x) \left( \frac{p^{-k}[k]_{p,q}}{[n]_{p,q}} \right)^2 \]
\[ = \frac{p^{n+2}[n]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} + \frac{p^{2}q[n]^2}{[n+2]_{p,q}[n+3]_{p,q}} \left( x^2 + \frac{p^{-1}x(1-x)}{[n]_{p,q}} \right) \]
\[ = \frac{p^{n+2}[n]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} + \frac{p^{2}q[n]^2}{[n+2]_{p,q}[n+3]_{p,q}} \left[ x^2 + \frac{p^{-1}q[n]p,q x(1-x)}{[n+2]_{p,q}[n+3]_{p,q}} \right]. \]

Remark 3.4. Using above lemma, we can obtain the following central moments:

1° \[ D^{p,q}_n((t-x), x) = \frac{(p[n]_{p,q} - [n+2]_{p,q})x}{[n+2]_{p,q}} \]

2° \[ D^{p,q}_n((t-x)^2, x) = \frac{(p+q)p^{n+1}[n]_{p,q}x}{[n+2]_{p,q}[n+3]_{p,q}} \]
\[ + \frac{[([n]_{p,q} - p^{-1})p^2[n]_{p,q} - 2p[n]_{p,q}[n+3]_{p,q} + [n+2]_{p,q}[n+3]_{p,q}]x^2}{[n+2]_{p,q}[n+3]_{p,q}}. \]

Lemma 3.5. Let \( n \) be a given natural number, then
\[ D^{p,q}_n((t-x)^2, x) \leq \frac{6}{[n+2]_{p,q}} \left( \varphi^2(x) + \frac{1}{[n+2]_{p,q}} \right), \]
where \( \varphi^2(x) = x(1-x), \ x \in [0, 1]. \)

Proof. In view of Lemma 3.3 we obtain
\[ D^{p,q}_n((t-x)^2, x) = \frac{(p+q)p^{n+1}[n]_{p,q}x}{[n+2]_{p,q}[n+3]_{p,q}} \]
\[ + \frac{[([n]_{p,q} - p^{-1})p^2[n]_{p,q} - 2p[n]_{p,q}[n+3]_{p,q} + [n+2]_{p,q}[n+3]_{p,q}]x^2}{[n+2]_{p,q}[n+3]_{p,q}}. \]

By direct computations, using the definition of the \((p,q)\)-numbers, we get
\((p+q)p^{n+1}[n]_{p,q} = p^{n+1}(p+q)(p^{-1} + p^{-2} + \cdots + pq^{n-2} + q^{n-1}) \geq 0\)
for every \( q \in (q_0, 1). \)

Furthermore, the expression
\((p+q)p^{n+1}[n]_{p,q} + ([n]_{p,q} - p^{-1})p^2[n]_{p,q} - 2p[n]_{p,q}[n+3]_{p,q} + [n+2]_{p,q}[n+3]_{p,q}\)
is equal to
\[
(p + q)p^{n+1}[n]_{p,q} + p^2q^n[n]_{p,q}^2 - p^{n+1}q[n]_{p,q} \\
-2p[n]_{p,q}(p^{n+2} + qp^{n+1} + q^2p^n + q^3[n]_{p,q}) \\
+(p^{n+1} + qp^n + q^2[n]_{p,q})(p^{n+2} + qp^{n+1} + q^2p^n + q^3[n]_{p,q}) \leq 6.
\]
In conclusion, for \( x \in [0, 1] \), we have
\[
D_{n,p,q}((t - x)^2, x) \leq \frac{6}{[n + 2]_{p,q}} \delta_n^2(x), \tag{3.4}
\]
which was to be proved. \( \square \)

4. Local and Global Estimates

In this section, we estimate some direct results, viz., local and global approximation in terms of modulus of continuity.

Our first main result is a local theorem. For this, we denote
\[
W^2 = \{ g \in C[0, 1] : g'' \in C[0, 1] \},
\]
for \( \delta > 0 \), a functional is defined as
\[
K_2(f, \delta) = \inf \{ \| f - g \| + \eta \| g'' \| : g \in W^2 \},
\]
where norm-\( \| . \| \) denotes the uniform norm on \( C[0, 1] \). Following the well known inequality due to DeVore and Lorentz \[3\], there exists an absolute constant \( C > 0 \) such that
\[
K_2(f, \delta) \leq C \omega_2(f, \sqrt{\delta}), \tag{4.1}
\]
where the second order modulus of smoothness for \( f \in C[0, 1] \) is defined as
\[
\omega_2(f, \sqrt{\delta}) = \sup_{0 < h \leq \sqrt{\delta}} \sup_{x + h \in [0, 1]} |f(x + h) - f(x)|.
\]
The usual modulus of continuity for \( f \in C[0, 1] \) is defined as
\[
\omega(f, \delta) = \sup_{0 < h \leq \delta} \sup_{x + h \in [0, 1]} |f(x + h) - f(x)|.
\]

**Theorem 4.1.** Let \( n > 3 \) be a natural number and let \( 0 < q < p \leq 1 \), \( q_0 = q_0(n) \in (0, p) \) be defined as in Lemma 3.5. Then, there exists an absolute constant \( C > 0 \) such that
\[
|D_{n,p,q}^p(f, x) - f(x)| \leq C \omega_2 \left( f, \left[ n + 2 \right]_{p,q}^{-1/2} \delta_n(x) \right) + \omega \left( f, \frac{2x}{[n + 2]_{p,q}} \right),
\]
where \( f \in C[0, 1] \), \( \delta_n^2(x) = \varphi^2(x) + \frac{1}{[n + 2]_{p,q}} \), \( \varphi^2(x) = x(1 - x) \), \( x \in [0, 1] \) and \( q \in (q_0, 1) \).

**Proof.** For \( f \in C[0, 1] \) we define
\[
\tilde{D}_{n,p,q}(f, x) = D_{n,p,q}(f, x) + f(x) - f(\frac{p[n]_{p,q}x}{[n + 2]_{p,q}}).
\]
Then, by Lemma 3.3 we immediately observe that
\[ \tilde{D}_n^{p,q}(1, x) = D_n^{p,q}(1, x) = 1 \]
and
\[ \tilde{D}_n^{p,q}(t, x) = D_n^{p,q}(t, x) + x - \frac{p[n]_{p,q}x}{n + 2]_{p,q}} = x. \]

By applying Taylor’s formula
\[ g(t) = g(x) + (t - x)g'(x) + \int_x^t (t - u)g''(u) du, \]
we get
\[
\begin{align*}
\tilde{D}_n^{p,q}(g, x) &= g(x) + \tilde{D}_n^{p,q}\left(\int_x^t (t - u)g''(u) du, x\right) \\
&= g(x) + D_n^{p,q}\left(\int_x^t (t - u)g''(u) du, x\right) \\
&\quad - \int_x^t \left(\frac{p[n]_{p,q}x}{n + 2]_{p,q}} - u\right) g''(u) du.
\end{align*}
\]

Thus, \(|\tilde{D}_n^{p,q}(g, x) - g(x)|\)
\[
\leq D_n^{p,q}\left(\left|\int_x^t |t - u| \cdot |g''(u)| du\right|, x\right) + \int_x^t \left|\frac{p[n]_{p,q}x}{n + 2]_{p,q}} - u\right| |g''(u)| du
\]
\[
\leq D_n^{p,q}\left((t - x)^2, x\right)\|g''\| + \left(\frac{p[n]_{p,q}x}{n + 2]_{p,q}} - x\right)^2 \|g''\|. \tag{4.2}
\]

Also, we have
\[
D_n^{p,q}\left((t - x)^2, x\right) + \left(\frac{p[n]_{p,q}x}{n + 2]_{p,q}} - x\right)^2
\]
\[
\leq \frac{6}{n + 2]_{p,q}} \left(\varphi^2(x) + \frac{1}{n + 2]_{p,q}}\right) + \left(\frac{p[n]_{p,q} - [n + 2]_{p,q}x}{n + 2]_{p,q}}\right)^2
\]
\[
\leq \frac{10}{n + 2]_{p,q}} \left(\varphi^2(x) + \frac{1}{n + 2]_{p,q}}\right). \tag{4.3}
\]

Hence, by (4.2) and with the condition \(n > 3\) and \(x \in [0, 1]\), we have
\[
|\tilde{D}_n^{p,q}(g, x) - g(x)| \leq \frac{10}{n + 2]_{p,q}} \delta_n^2(x) \|g''\|. \tag{4.4}
\]
Furthermore, for \( f \in C[0, 1] \) we have \( ||D^{p,q}_n(f, x)|| \leq ||f|| \), thus

\[ |\tilde{D}^{p,q}_n(f, x)| \leq |D^{p,q}_n(f, x)| + |f(x)| + \left| f \left( \frac{p[n\cdot x]}{n + 2} \right) \right| \leq 3||f||. \tag{4.5} \]

for all \( f \in C[0, 1] \).

Now, for \( f \in C[0, 1] \) and \( g \in W^2 \), we obtain

\[ |D^{p,q}_n(f, x) - f(x)| \]

\[ = \left| \tilde{D}^{p,q}_n(f, x) - f(x) + f \left( \frac{p[n\cdot x]}{n + 2} \right) - f(x) \right| \]

\[ \leq |\tilde{D}^{p,q}_n(f - g, x)| + |\tilde{D}^{p,q}_n(g, x) - g(x)| + |g(x) - f(x)| \tag{4.6} \]

\[ \leq 4 ||f - g|| + \frac{10}{n + 2} \delta_n^2(x)||g''|| + \omega \left( f, \frac{p[n\cdot x] - [n + 2] \cdot x}{n + 2} \right), \]

where we have used (4.4) and (4.5). Taking the infimum on the right hand side over all \( g \in W^2 \), we obtain at once

\[ |D^{p,q}_n(f, x) - f(x)| \leq 10K_2 \left( f, \frac{1}{n + 2} \delta_n^2(x) \right) + \omega \left( f, \frac{2x}{n + 2} \right). \]

Finally, in view of (4.1), we find

\[ |D^{p,q}_n(f, x) - f(x)| \leq C \omega_2 \left( f, [n + 2]^{-1/2} \delta_n(x) \right) + \omega \left( f, \frac{2x}{n + 2} \right). \]

This completes the proof of the theorem. \( \square \)

The weighted modulus of continuity of second order is defined as:

\[ \omega^2_\varphi(f, \sqrt{\delta}) = \sup_{0 < h \leq \sqrt{\delta}} \sup_{x \in [0, 1]} |f(x + h\varphi(x)) - 2f(x) + f(x - h\varphi(x))| \]

where \( \varphi(x) = \sqrt{x(1 - x)}. \) The corresponding \( K \)-functional is defined by

\[ K_{2,\varphi}(f, \delta) = \inf \{ \|f - g\| + \delta \|\varphi^2 g''\| + \delta^2 \|g''\| : g \in W^2(\varphi) \} \]

where

\[ W^2(\varphi) = \{ g \in C[0, 1] : g' \in AC_{loc}[0, 1], \varphi^2 g'' \in C[0, 1] \} \]

and \( g' \in AC_{loc}[0, 1] \) means that \( g \) is differentiable and \( g' \) is absolutely continuous on every closed interval \( [a, b] \subset [0, 1] \). It is well-known due to Ditzian-Totik (see [4] p. 24, Theorem 1.3.1) that

\[ K_{2,\varphi}(f, \delta) \leq C \omega^2_\varphi(f, \sqrt{\delta}) \tag{4.7} \]

for some absolute constant \( C > 0 \). Moreover, the Ditzian-Totik moduli of first order is given by

\[ \omega_\psi(f, \delta) = \sup_{0 < h \leq \delta} \sup_{x \in [0, 1]} |f(x + h\psi(x)) - f(x)|, \]

where
where $\psi$ is an admissible step-weight function on $[0,1]$.

Now, we state our next main result, i.e., the global estimate.

**Theorem 4.2.** Let $n > 3$ be a natural number and let $0 < q < p \leq 1$, $q_0 = q_0(n) \in (0,p)$ be defined as in Lemma 3.5. Then, there exists an absolute constant $C > 0$ such that

$$
\|D_{n}^{p,q}f - f\| \leq C \omega_2^{q_0}(f, [n+2]^{1/2}) + \tilde{\omega}_{\psi}(f, [n+2]^{-1}),
$$

where $f \in C[0,1]$, $q \in (q_0, 1)$, and $\psi(x) = x$, $x \in [0,1]$.

**Proof.** We again consider

$$
\tilde{D}_{n}^{p,q}(f, x) = D_{n}^{p,q}(f, x) + f(x) - f\left(\frac{p[n]_{p,q}x}{[n+2]_{p,q}}\right),
$$

where $f \in C[0,1]$. Now, using Taylor’s formula, we have

$$
g(t) = g(x) + (t - x) g'(x) + \int_{x}^{t} (t - u) g''(u) \, du.
$$

Using (3.3), we obtain

$$
\tilde{D}_{n}^{p,q}(g, x) = g(x) + D_{n}^{p,q}\left(\int_{x}^{t} (t - u) g''(u) \, du, x\right)
$$

$$
- \int_{x}^{t} \left(\frac{p[n]_{p,q}x}{[n+2]_{p,q}} - u\right) g''(u) \, du.
$$

Thus,

$$
|\tilde{D}_{n}^{p,q}(g, x) - g(x)| \leq D_{n}^{p,q}\left(\int_{x}^{t} |t - u||g''(u)| \, du \bigg|, x\right)
$$

$$
+ \int_{x}^{t} \left|\frac{p[n]_{p,q}x}{[n+2]_{p,q}} - u\right| |g''(u)| \, du. \tag{4.8}
$$

Since $\delta_n^2$ is concave on $[0,1]$, therefore, for $u = t + \tau(x - t)$, $\tau \in [0,1]$, the following estimate holds:

$$
\frac{|t - u|}{\delta_n^2(u)} = \frac{\tau|x - t|}{\delta_n^2(t + \tau(x - t))} \leq \frac{\tau|x - t|}{\delta_n^2(t) + \tau(\delta_n^2(x) - \delta_n^2(t))} \leq \frac{|t - x|}{\delta_n^2(x)}.
$$

Thus, using (4.8), we obtain
\[ |\tilde{D}_{n}^{p,q}(g, x) - g(x)| \]
\[ \leq D_{n}^{p,q} \left( \left( \int_{x}^{t} \frac{t-u}{\delta_{n}^{2}(u)} \, du \right) \delta_{n}^{2} g'' \right) + \left| p[n]_{p,q} x \right|_{n+2}^{n+1} p[n]_{p,q} x - u \right| \delta_{n}^{2} g'' \]
\[ \leq \frac{1}{\delta_{n}^{2}(x)} D_{n}^{p,q}((t-x)^{2}, x) \delta_{n}^{2} g'' + \frac{1}{\delta_{n}^{2}(x)} \left( \frac{p[n]_{p,q} x}{[n+2]_{p,q}} - x \right)^{2} \delta_{n}^{2} g''. \]

For \( x \in [0, 1] \), in view of (4.3) and
\[ \delta_{n}^{2}(x) |g''(x)| = |\varphi^{2}(x) g''(x)| + \frac{1}{[n+2]_{p,q}} |g''(x)| \leq \|\varphi^{2} g''\| + \frac{1}{[n+2]_{p,q}} \|g''\|, \]
we have
\[ |\tilde{D}_{n}^{p,q}(g, x) - g(x)| \leq \frac{5}{[n+2]_{p,q}} \left( \|\varphi^{2} g''\| + \frac{1}{[n+2]_{p,q}} \|g''\| \right). \quad (4.9) \]

Using the fact that \([n]_{p,q} \leq [n+2]_{p,q}, (4.5)\) and (4.9), for \( f \in C[0, 1] \), we get
\[ |D_{n}^{p,q}(f, x) - f(x)| \leq |\tilde{D}_{n}^{p,q}(f - g, x)| + |\tilde{D}_{n}^{p,q}(g, x) - g(x)| + |g(x) - f(x)| \]
\[ + \left| f \left( \frac{p[n]_{p,q} x}{[n+2]_{p,q}} \right) - f(x) \right| \]
\[ \leq 4 \|f - g\| + \frac{10}{[n+2]_{p,q}} \|\varphi^{2} g''\| + \frac{10}{[n+2]_{p,q}} \|g''\| \]
\[ + \left| f \left( \frac{p[n]_{p,q} x}{[n+2]_{p,q}} \right) - f(x) \right| . \]

On taking the infimum on the right hand side over all \( g \in W_{2}^{2}(\varphi) \), we obtain
\[ |D_{n}^{p,q}(f, x) - f(x)| \leq 10K_{2,\varphi} \left( f, \frac{1}{[n+2]_{p,q}} \right) + \left| f \left( \frac{p[n]_{p,q} x}{[n+2]_{p,q}} \right) - f(x) \right| . \quad (4.10) \]

Moreover,
\[ \left| f \left( \frac{p[n]_{p,q} x}{[n+2]_{p,q}} \right) - f(x) \right| = \left| f \left( x + \psi(x) \cdot \frac{x (p[n]_{p,q} - [n+2]_{p,q})}{[n+2]_{p,q} \psi(x)} \right) - f(x) \right| \]
\[ \leq \sup_{t, t+\psi(t)} \left| f \left( t + \psi(t) \cdot \frac{x (p[n]_{p,q} - [n+2]_{p,q})}{[n+2]_{p,q} \psi(x)} \right) - f(t) \right| \]
\[ \leq \tilde{\omega}_{\varphi} \left( f, \frac{x (p[n]_{p,q} - [n+2]_{p,q})}{[n+2]_{p,q} \psi(x)} \right) \]
\[ \leq \tilde{\omega}_{\varphi} \left( f, \frac{1}{[n+2]_{p,q}} \right) = \tilde{\omega}_{\varphi} \left( f, \frac{1}{[n+2]_{p,q}} \right). \]
Hence, by (4.10) and (4.7), we finally get

$$\| D_{n}^{p,q} f - f \| \leq C \omega_{2}^{\phi}(f, [n + 2]_{p,q}^{-1/2}) + \omega_{\psi}(f, [n + 2]_{p,q}^{-1}).$$

This completes the proof of the theorem. □

**Remark 4.3.** For $q \in (0, 1)$ and $p \in (q, 1]$ it is obvious that $\lim_{n \to \infty} [n]_{p,q} = \frac{1}{p-q}$. An example of such choice for $p, q$ depending on $n$ is recently given in [7].

**Example.** Now, we show comparisons and some illustrative graphs for the convergence of $(p, q)$-analogue of Bernstein-Durrmeyer operators $D_{n}^{p,q}(f, x)$ for different values of the parameters $p$ and $q$, such that $0 < q < p \leq 1$.

For $x \in [0, 1]$, $p = 0.5$ and $q = 0.4$, the convergence of the difference of the operators $D_{n}^{p,q}(f, x)$ to the function $f$, where $f(x) = 9x^{2} - 4x + 5$, for different values of $n$ is illustrated in Fig. 1.

![Figure 1](image1.png)

**Figure 1.** Graphics of the difference $x \mapsto D_{n}^{0.5,0.4}(f, x) - f(x)$ for $x \in [0, 1]$, when $f(x) = 9x^{2} - 4x + 5$ and $n = 5, 10, 15$ and $n = 100$

For $x \in [0, 1]$, $p = 0.5$ and $q = 0.4$, the convergence of the difference of the operators $D_{n}^{p,q}(f, x)$ to the function $f$, where $f(x) = (x + 1)^{2}\sin(10\pi x/3)$ and $n = 5, 10, 15$ and $n = 100$ is illustrated in Fig. 2.

![Figure 2](image2.png)

**Figure 2.** Graphics of the difference $x \mapsto D_{n}^{0.5,0.4}(f, x) - f(x)$ for $x \in [0, 1]$, when $f(x) = (x + 1)^{2}\sin(10\pi x/3)$ and $n = 5, 10, 15$ and $n = 100
The convergence of the difference of the operators $D_{n}^{p,q}(f, x)$ to the function $f$, where $f(x) = (x + 1)^{2} \sin \left(\frac{10}{3} \pi x\right)$ for different values of $n$ and $x \in [0, 1]$ is illustrated in Fig. 2.

**Example.** For the function $f(x) = 9x^2 - 4x + 5$ (and $p = 0.5$, $q = 0.4$), the limit of $D_{n}^{0.5,0.4}(f, x)$, when $n \to +\infty$, is $f^*(x) = 5 + (124/25)x + (576/25)x^2$. Graphics of $D_{n}^{0.5,0.4}(f, x) - f^*(x)$, for $n = 10, 15, 20, 50$, are presented in Fig. 3.

![Figure 3. Graphics of $D_{n}^{0.5,0.4}(f, x) - f^*(x)$, when $f(x) = 9x^2 - 4x + 5$ and $f^*(x) = 5 + (124/25)x + (576/25)x^2$, for $n = 10, 15, 20, 50$.](image)

5. Better Approximation

About a decade ago, King [11] proposed a technique to obtain better approximation for the well known Bernstein polynomials. In this technique, these operators approximate each continuous function $f \in [0, 1]$, while preserving the function $e_2(x) = x^2$. These were basically compared with estimates of approximation by Bernstein polynomials. Various standard linear positive operators preserve $e_0$ and $e_1$, i.e., preserve constant and linear functions, but this approach helps in reproducing the quadratic functions as well.

So, using King’s technique, we modify the operators (3.3) as follows:

$$D_{n,p,q}^*(f, x) = [n + 1]_{p,q} \sum_{k=1}^{n} p^{-(n-k+1)(n+k)/2} b_{n,k}^{p,q}(1, r_n(x))$$

$$\times \int_{0}^{1} b_{n,k-1}^{p,q}(t)f(t)dt + b_{n,0}^{p,q}(1, r_n(x))f(0),$$

where $r_n(x) = \frac{[n+2]_{p,q,x}}{p[n]_{p,q}}$ and $x \in I_{n,p,q} = \left[0, \frac{[n+2]_{p,q}}{p[n]_{p,q}}\right]$. Then, we have

$$D_{n,p,q}^*(e_0, x) = 1, \quad D_{n,p,q}^*(e_1, x) = x,$$
\[ D_{n,p,q}^*(e_2, x) = \frac{(p+q)p^n x}{[n+3]_{p,q}} + \frac{([n]_{p,q} - p^{n-1})q [n+2]_{p,q} x^2}{[n]_{p,q} [n+3]_{p,q}}. \]

Now, Theorem 4.1 can be modified as:

**Theorem 5.1.** Let \( n > 3 \) be a natural number and let \( 0 < q < p \leq 1, q_0 = q_0(n) \in (0,p) \) be defined as in Lemma 3.5. Then, there exists an absolute constant \( C > 0 \) such that

\[ |D_{n,p,q}^*(f, x) - f(x)| \leq C \omega_2 \left(f, \sqrt{\delta_{n,p,q}^*(x)}\right), \]

where \( x \in I_{n,p,q} = \left[0, \frac{[n+2]_{p,q}}{p[n]_{p,q}}\right] \), \( q \in (q_0, 1) \), and

\[
\delta_{n,p,q}^*(x) = \frac{(p+q)p^n x}{[n+3]_{p,q}} + \frac{([n]_{p,q} - p^{n-1})q [n+2]_{p,q} - [n]_{p,q} [n+3]_{p,q} x^2}{[n]_{p,q} [n+3]_{p,q}}.
\]

The proof is on similar lines, so we omit the details.

**Example.** We compare the convergence of \((p,q)\)-analogue of Bernstein-Durrmeyer operators \( D_{n,p,q}^n(f, x) \) with the operators \( D_{n,p,q}^*(f, x) \). We have considered the same function as in the previous example.

For \( x \in \left[0, \frac{[n+2]_{p,q}}{p[n]_{p,q}}\right], p = 0.5 \) and \( q = 0.4 \), the convergence of the difference of the operators \( D_{n,p,q}^*(f, x) \) to the function \( f \), where \( f(x) = 9x^2 - 4x + 5 \), for different values of \( n \) is illustrated in Fig. 4.

![Figure 4](image-url)  
**Figure 4.** Graphics of the difference \( x \mapsto D_{n,0.5,0.4}^*(f, x) - f(x) \) for \( x \in [0, [n+2]_{p,q}/p[n]_{p,q}] \), when \( f(x) = 9x^2 - 4x + 5 \) and \( n = 5, 10, 15 \) and \( n = 100 \).
References

[1] A. Aral, V. Gupta, R. P. Agarwal, Applications of q Calculus in Operator Theory, Springer 2013.

[2] T. Acar, (p,q)-Generalization of Szász-Mirakyan Operators, arXiv preprint arXiv:1505.06839.

[3] R. A. DeVore, G. G. Lorentz, Constructive Approximation, Springer, Berlin 1993.

[4] Z. Ditzian, V. Totik, Moduli of Smoothness, Springer, New York 1987.

[5] V. Gupta, Some approximation properties on q-Durrmeyer operators, Appl. Math. Comput. 197 (1) (2008), 172–178.

[6] V. Gupta, (p,q)-genuine Bernstein Durrmeyer operators, Bollettino dell’Unione Matematica Italiana, in press, doi: 10.1007/s40574-016-0054-4.

[7] V. Gupta, A. Aral, Bernstein Durrmeyer operators based on two parameters, Facta Univ. Ser. Math. Inform. 31 (1) (2016), 79–95.

[8] V. Gupta, Z. Finta, On certain q Durrmeyer operators, Appl. Math. Comput. 209 (2009), 415–420.

[9] V. Gupta, H. Wang, The rate of convergence of q-Durrmeyer operators for 0 < q < 1, Math. Methods Appl. Sci. 31 (16) (2008), 1946–1955.

[10] V. Gupta, R. P. Agarwal, Convergence Estimates in Approximation Theory, Springer, 2014.

[11] J. P. King, Positive linear operators which preserves x^2, Acta Math. Hung. 99 (2003), 203–208.

[12] G. V. Milovanović, A. S. Cvetković, An application of little 1/q-Jacobi polynomials to summation of certain series, Facta Univ. Ser. Math. Inform. 18 (2003), 31–46.

[13] M. Mursaleen, K. J. Ansari, A. Khan, On (p,q)-analogue of Bernstein operators, Appl. Math. Comput. 266 (2015) 874–882.

[14] P. N. Sadjang, On the (p,q)-Gamma and the (p,q)-Beta functions, arXiv 1506.07394v1. 22 Jun 2015.

[15] P. N. Sadjang, On the fundamental theorem of (p,q)-calculus and some (p,q)-Taylor formulas, arXiv:1309.3934 [math.QA].

[16] V. Sahai, S. Yadav, Representations of two parameter quantum algebras and p,q-special functions, J. Math. Anal. Appl. 335 (2007), 268–279.

Gradimir V. Milovanović
Serbian Academy of Sciences and Arts
Kneza Mihaila 35, 11000 Beograd, Serbia
& State University of Novi Pazar
Novi Pazar, Serbia
e-mail: gvm@mi.sanu.ac.rs

Vijay Gupta
Department of Mathematics
Netaji Subhas Institute of Technology
Sector 3 Dwarka, New Delhi-110078, India
e-mail: vijaygupta2001@hotmail.com
Neha Malik
Department of Mathematics
Netaji Subhas Institute of Technology
Sector 3 Dwarka, New Delhi-110078, India
e-mail: neha.malik_nm@yahoo.com