Scattering Theory for a Class of Radial Focusing Inhomogeneous Hartree Equations

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Abstract
This paper studies the asymptotic behavior of global solutions to the generalized Hartree equation

\[ i\dot{u} + \Delta u + (I_\alpha \ast | \cdot |^b |u|^p) |x|^b |u|^{p-2} u = 0. \]

Indeed, using a new approach due to (Dodson et al. Proc. Amer. Math. Soc. 145(11), 4859–4867, 2017), one proves the scattering of the above inhomogeneous Choquard equation in the mass-super-critical and energy sub-critical regimes with radial setting.

Keywords Inhomogeneous Choquard equation · Scattering

Mathematics Subject Classification (2010) 35Q55

1 Introduction

In this note, one considers the scattering of energy global solutions to the following focusing inhomogeneous generalized Hartree problem

\[
\begin{cases}
  i\dot{u} + \Delta u + (I_\alpha \ast | \cdot |^b |u|^p) |x|^b |u|^{p-2} u = 0; \\
  u(0, \cdot) = u_0.
\end{cases}
\]

(1.1)

In this paper, \(u : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C}\), where the spatial dimension is \(N \geq 3\). The above problem is said inhomogeneous because of the singular quantity \( | \cdot |^b \), where \(b < 0\). The Riesz-potential is the radial function

\[ I_\alpha : x \mapsto \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right) \pi^{\frac{N}{2}}} \frac{1}{|x|^{N-\alpha}}, \quad 0 \neq x \in \mathbb{R}^N. \]
Here and hereafter, one assumes the following conditions done in [1],
\[ \min\{\alpha, -b, -\alpha + N, b + N, 2 + \alpha + 2b\} > 0. \] (2.1)

The Eq. 1.1 models many physical phenomena. For instance, it arises in the study of the mean-field limit of large systems of non-relativistic bosonic atoms and molecules [7]. Moreover, it describes the propagation of electromagnetic waves in plasmas [3]. The fourth space dimensional case gives the so-called Schrödinger-Poisson equation, which appears in semi-conductor and quantum mechanics theories [14].

If \( u \) resolves the inhomogeneous Hartree equation (1.1), so is the family of time-space scaled functions
\[ u_\lambda = \lambda \frac{2+b+\alpha}{2(p-1)} u(\lambda^2 \cdot, \lambda \cdot), \quad \lambda > 0. \]

This gives the critical Sobolev exponent
\[ s_c := \frac{N}{2} - \frac{2+2b+\alpha}{2(p-1)}, \]
which satisfies the identity
\[ \|u_\lambda(t)\|_{H^s} = \lambda^{s_c} \|u(\lambda^2 t)\|_{H^s}. \]

The Schrödinger-Hartree equation (1.1) is said to be \( L^2 \) or mass-critical if \( s_c = 0 \), which is equivalent to \( p = 1 + \frac{\alpha+2+2b}{N} \). It is called energy-critical or \( \dot{H}^1 \)-critical if \( s_c = 1 \), which is equivalent to \( p = 1 + \frac{2+2b+\alpha}{N-2} \). This work is concerned with the mass super-critical and energy sub-critical regimes \( 0 < s_c < 1 \).

To the author knowledge, the inhomogeneous Choquard problem, was only treated in [1], where the existence of a unique energy local solution was proved via an adapted Gagliardo-Nirenberg type identity. In the focusing regime, a threshold of global existence versus finite time blow-up of solutions was obtained.

On the other hand, the particular case \( b = 0 \) in Eq. 1.1, was investigated in many directions. In fact, the local well-posedness in the energy sub-critical and mass-super-critical regime were obtained in [6]. The energy scattering of the defocusing global solutions was established in [18]. In [12], the asymptotic properties of standing waves, were investigated.

It is the aim of this paper to extend the previous work [1], by the study of the asymptotic behavior of the energy global solutions to the focusing Choquard problem (1.1). Indeed, one proves that any global solution to the Hartree equation (1.1) is asymptotic, for large time to a free solution.

The method used in this note, due to [4], is based on a scattering criterion [17] and Morawetz estimates. In [16] the scattering for the homogeneous non-linear Choquard problem, which corresponds to \( b = 0 \) in Eq. 1.1, was proved using the concentration-compactness-rigidity approach introduced in [10]. This result was revisited recently in [2] by giving an alternative proof.

The rest of this paper is organized as follows. The next section contains the main results and some technical tools needed in the sequel. Some variational estimates are given in section three. In section four, one establishes a scattering criterion. In the fifth section, a Morawetz estimate is proved. The last section is concerned with the proof of scattering. Finally, a variance identity and a Strichartz type estimate of the source term are proved in the Appendix.

Here and hereafter, one denotes for simplicity, the Lebesgue and Sobolev spaces
\[ L^r := L^r(\mathbb{R}^N), \quad H^1 := H^1(\mathbb{R}^N), \quad H_{rad}^1 := \{ f \in H^1, \ f(\cdot) = f(|\cdot|) \} \]
and the usual norms
\[ \| \cdot \|_r := \| \cdot \|_{L^r}, \quad \| \cdot \| := \| \cdot \|_2, \quad \| \cdot \|_{H^1} := \left( \| \cdot \|^2 + \| \nabla \cdot \|_2^2 \right)^{1/2}. \]

\( T^* > 0 \) denotes the lifespan of an eventual solution to Eq. 1.1. Finally, if \( x \) is a real number, \( x^+ \) denotes a number larger and near to \( x \) and \( x^- \) is a number less and close to \( x \).

## 2 Background and Main Results

In this section, one gives the goal of this manuscript and some estimates to be used later.

### 2.1 Preliminary

Here and hereafter denote the following quantities. For \( a, b \in \mathbb{R} \) and \( u \in H^1 \), let the real numbers
\[ p_* := 1 + \frac{\alpha + 2 + 2b}{N}, \quad p^* := 1 + \frac{\alpha + 2 + 2b}{N - 2}; \]
\[ B := Np - N - \alpha - 2b, \quad A := 2p - B. \]

Denote the operator \( (\cdot) := (1 + \nabla) \cdot \) and the source term
\[ N \cdot := N \cdot (x, u) := |x|^b |u|^{p - 2}(I_\alpha \ast | \cdot |^b |u|^p)u. \]

Take also a radial smooth function
\[ \psi \in C^\infty_0(\mathbb{R}^N), \quad \text{supp}(\psi) \subset \{|x| < 1\}, \quad \psi = 1 \text{ on } \{|x| < \frac{1}{2}\}, \quad \psi_R := \psi(\frac{\cdot}{R}). \]

The existence of ground states was established in [1, Theorem 3.2].

**Definition 2.1** A ground state of Eq. 1.1 is a solution to
\[ \Delta \phi - \phi = |x|^b |\phi|^{p - 2}(I_\alpha \ast | \cdot |^b |\phi|^p)\phi, \quad \phi \in H^1 - \{0\}, \tag{2.3} \]
and a minimizer to the problem
\[ m := \inf_{0 \neq u \in H^1} \left\{ (E + M)(u) \right\} \quad \text{s.t} \quad E(u) = 0 \] \[ \mathcal{I}(u) := \frac{4}{N} \left( \| \nabla u \|^2 - \frac{B}{2p} \int_{\mathbb{R}^N} |x|^b |u|^p (I_\alpha \ast | \cdot |^b |u|^p) \, dx \right). \]

The next estimates related to the inhomogeneous Hartree problem (1.1) were proved in [1, Theorem 4.1].

**Proposition 2.2** Let \( N \geq 1, b, \alpha \) satisfying Eq. 1.2 and \( 1 + \frac{2b + \alpha}{N} < p < p^* \). Then,
1. there exists a positive constant \( C(N, p, b, \alpha) \), such that for any \( u \in H^1 \),
\[ \int_{\mathbb{R}^N} |x|^b |u|^p (I_\alpha \ast | \cdot |^b |u|^p) \, dx \leq C(N, p, b, \alpha) \| u \|^A \| \nabla u \|^B; \]
2. moreover, there is \( \phi \) a solution to Eq. 2.3 such that the best constant in the above inequality satisfies,

\[
C(N, p, b, \alpha) = \frac{2p}{A} \left( \frac{A}{B} \right)^{\frac{p}{2}} \frac{1}{\|\phi\|^{2(p-1)}}. \tag{2.4}
\]

Here and hereafter, \( \phi \) denotes a solution to Eq. 2.3, which satisfies Eq. 2.4. The next scale invariant quantities [5] called respectively, mass-energy and mass-gradient describe a dichotomy of global existence and scattering versus non-global existence of energy solutions to the above Choquard problem.

\[
\mathcal{M}[u] := \frac{E[u]M[u]^{\frac{1-\frac{p}{2}}{s}}}{E[\phi]M[\phi]^{\frac{1-\frac{p}{2}}{s}}}, \quad \mathcal{G}[u] := \frac{\|\nabla u\|^{\frac{1-\frac{p}{2}}{s}}}{\|\nabla \phi\|^{\frac{1-\frac{p}{2}}{s}}}.
\]

**Remark 2.3** Using Pohozaev identities, the terms \( E[\phi]^{\frac{p}{s}} M[\phi]^{\frac{1-\frac{p}{2}}{s}} \) and \( \|\nabla \phi\|^{\frac{p}{s}} \|\phi\|^{\frac{1-\frac{p}{2}}{s}} \) are invariant of the choice of \( \phi \).

The inhomogeneous Choquard problem (1.1) is locally well-posed in the energy space [1, Theorem 5.2].

**Proposition 2.4** Let \( N \geq 3 \), \( b, \alpha \) satisfying Eq. 1.2, \( u_0 \in H^1 \) and \( 2 \leq p < p^* \). Then, there exists \( T > 0 \) and a unique local solution to the inhomogeneous Choquard problem (1.1),

\[
u \in C([0, T], H^1).
\]

Moreover,

1. the solution satisfies the mass and energy conservation laws

\[
M(u_0) = M(u(t)) := \int_{\mathbb{R}^N} |u(t, x)|^2 dx;
\]

\[
E(u_0) = E(u(t)) := \frac{1}{p} \int_{\mathbb{R}^N} (J_\alpha * |\cdot|^p |u(t)|^p)|x|^p |u(t, x)|^p dx;
\]

2. \( u \in L^q_{loc}((0, T), W^{1,r}), \forall (q, r) \in \Gamma \) (see Definition 2.9).

Let us close this sub-section with some notations in the spirit of [4]. Take, for \( R >> 1 \), the radial function defined on \( \mathbb{R}^N \) by

\[
a : x \mapsto \begin{cases} \frac{x^2}{2}, & \text{if } |x| \leq \frac{R}{2}; \\ \frac{R|x|}{2}, & \text{if } |x| > \frac{R}. \end{cases}
\]

Moreover, one assumes that in the centered annulus \( C(0, \frac{R}{2}, R) \),

\[
\partial_r a > 0, \quad \partial_r^2 a \geq 0 \quad \text{and} \quad |\partial^\alpha a| \leq C_{\alpha R} |\cdot|^{1-\alpha}, \quad \forall |\alpha| \geq 1.
\]

Here, \( \partial_r a := \frac{1}{|r|} \nabla a \) denotes the radial derivative. Note that on the centered ball of radius \( \frac{R}{2} \), one has

\[
a_{jk} = \delta_{jk}, \quad \Delta a = N \quad \text{and} \quad \Delta^2 a = 0.
\]

Moreover, for \( |x| > R \),

\[
a_{jk} = \frac{R}{|x|} \left( \delta_{jk} - \frac{x_j x_k}{|x|^2} \right), \quad \Delta a = \frac{(N-1)R}{|x|} \quad \text{and} \quad \Delta^2 a = 0.
\]
Define the variance potential

\[ V_a := \int_{\mathbb{R}^N} a(x)|u(., x)|^2 \, dx, \]

and the Morawetz action

\[ M_a := 2\Re \int_{\mathbb{R}^N} \bar{a}(\nabla u) \, dx = 2\Re \int_{\mathbb{R}^N} \bar{a}(\nabla a \nabla u) \, dx. \]

With a standard way, repeated index are summed and subscripts denote the partial derivatives.

### 2.2 Main Result

In this manuscript, one proves mainly the following scattering result.

**Theorem 2.5** Let \( N \geq 3 \) and \( b, \alpha \) satisfying Eq. 1.2, such that \( 2b > -1 \) if \( N = 3 \). Take \( p \in (p_*, p^*) \) such that \( p \geq 2 \) and \( u_0 \in H^1_{r,d} \) satisfying

\[ \max \{ M_{E}(u_0), G_{M}(u_0) \} < 1. \]

Then, there exist a unique global solution \( u \in C(\mathbb{R}, H^1) \) to Eq. 1.1, which scatters in both directions. Precisely, there exists \( u_\pm \in H^1 \) such that

\[ \lim_{t \to \pm \infty} \| u(t) - e^{it\Delta} u_\pm \|_{H^1} = 0. \]

**Remarks 2.6**

1. The global existence of energy solutions to Eq. 1.1 was proved in [1, Theorem 6.1];
2. the radial assumption allows to use the following Strauss inequality, which is available for any \( u \in H^1(\mathbb{R}^N) \) such that \( N \geq 2 \),

\[ \| u \|^2 H^1 \leq C_N \| u \|_{H^1}, \quad \text{almost everywhere}; \]
3. the restriction \( N \geq 3 \) is needed in the proof of the scattering criterion, precisely when using the dispersion of the free Schrödinger kernel;
4. the assumption \( 2b > -1 \) if \( N = 3 \) is needed only in the next scattering criterion, precisely in the proof of Lemma 4.3.

In order to prove the scattering, one needs the following scattering criterion introduced by [17, Theorem 1.1].

**Proposition 2.7** Let \( N \geq 3 \) and \( b, \alpha \) satisfying Eq. 1.2, such that \( 2b > -1 \) if \( N = 3 \). Take \( p \in (p_*, p^*) \) such that \( p \geq 2 \) and \( u \in C(\mathbb{R}, H^1_{r,d}) \) be a global radial solution to Eq. 1.1.

Assume that

\[ 0 < \sup_{t \geq 0} \| u(t) \|_{H^1} := E < \infty. \]

There exist \( R, \epsilon > 0 \) depending on \( E, N, p, b, \alpha \) such that if

\[ \liminf_{t \to \pm \infty} \int_{|x| < R} |u(t, x)|^2 \, dx < \epsilon, \]

then, \( u \) scatters for positive time.

The following Morawetz estimate stands for a standard tool to prove scattering.
Proposition 2.8 Let $N \geq 3$ and $b$, $\alpha$ satisfying Eq. 1.2. Take $p \in (p_*, p^*)$ such that $p \geq 2$ and a global radial solution to Eq. 1.1 denoted by $u \in C(\mathbb{R}, H^1_{rd})$. Then, for $T > 0$, $\delta > 0$ (given by Lemma 3.1) and $R := R(\delta, M(u), \phi)$ large enough, holds

$$
\frac{1}{T} \int_0^T \left( \int_{|x|<R} |u(t, x)|^{2Np/(2N+2\delta)} \right)^{\frac{2N+2\delta}{Np}} dt \lesssim \frac{R}{T} + R^{-\frac{B(N-1)}{2N}} + R^{-2}.
$$

2.3 Useful Estimates

The Strichartz estimate [9, Proposition 3.9] is an essential tool to estimate an eventual solution to Eq. 1.1 in Sobolev spaces.

Definition 2.9 Take $N \geq 3$ and $s \in [0, 1)$. A pair of real numbers $(q, r)$ is $s$-admissible if

$$
2 \leq q, r \leq \infty;
$$

$$
\frac{2N}{N-2s} \leq r < \frac{2N}{N-2};
$$

$$
N \left( \frac{1}{q} - \frac{1}{r} \right) = \frac{2}{q} + s.
$$

Take the set $\Gamma_s := \{(q, r), s\text{-admissible}\}$ and $\Gamma := \Gamma_0$.

Proposition 2.10 Let $N \geq 3$, $s \in [0, 1)$, $(q, r) \in \Gamma_s$, $(\tilde{q}, \tilde{r}) \in \Gamma_{-s}$ and a time slab $I \subset \mathbb{R}$. Then,

1. $\sup_{(q, r) \in \Gamma_s} \| e^{\Delta t} f \|_{L^q(I, L^r)} \lesssim \| f \|_{L^q}$;
2. $\| \int_0^t e^{i(-\Delta) t} h(\cdot, s) \, ds \|_{L^q(I, L^r)} \lesssim \| h \|_{L^q(I, L^r)}$.

Remark 2.11 Let the Schrödinger equation $i \dot{u} + \Delta u = h \chi_{|x|<1} + h \chi_{|x|>1} := h_1 + h_2$. Taking $i \dot{u}_i + \Delta u_i = h_i$, where $u := u_1 + u_2$, one gets the Strichartz estimate

$$
\| u \|_{L^q(I, L^r)} \lesssim \| u_0 \|_{L^q} + \| h \|_{L^q(I, L^q(I, u_1(1-\chi_{|x|>1}))} + \| h \|_{L^q(I, L^q(I, u_1(1-\chi_{|x|>1}))},
$$

where $(q, r) \in \Gamma_s$, $(\tilde{q}_i, \tilde{r}_i) \in \Gamma_{-s}$, $i \in \{1, 2\}$.

The next variance identity is established in the Appendix.

Proposition 2.12 Let $N \geq 3$ and $b$, $\alpha$ satisfying Eq. 1.2. Take $p \in (p_*, p^*)$ such that $p \geq 2$ and a local solution to Eq. 1.1 denoted by $u \in C_T(H^1)$. Then, holds on $[0, T]$,

$$
V''_a = M_a'
$$

$$
= 4 \int_{\mathbb{R}^N} \partial_l \partial_k a \Re(\partial_k u \partial_l \bar{u}) \, dx - \int_{\mathbb{R}^N} \Delta^2 a |u|^2 \, dx
$$

$$
+ 2 \left( \frac{2}{p} - 1 \right) \int_{\mathbb{R}^N} \Delta |x|^b |u|^p (I_\alpha * |.|^b |u|^p) \, dx
$$

$$
+ \frac{4b}{p} \int_{\mathbb{R}^N} x \nabla a |x|^b |u|^p (I_\alpha * |.|^b |u|^p) \, dx
$$

$$
+ \frac{4}{p} (\alpha - N) \int_{\mathbb{R}^N} |x|^b |u|^p \nabla a \left| \frac{1}{|.|^2} I_\alpha |.|^b |u|^p \right| \, dx.
$$

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In the rest of this sub-section, some standard estimates independent of the Choquard equation (1.1), are listed. The next Hardy-Littlewood-Sobolev inequality is essential in the estimates to be done later [11] (see also [15, Corollary 2.14]).

**Lemma 2.13** Let $N \geq 1$ and $0 < \lambda < 1 < s, r, q < \infty$.

1. If $\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} = 2$, then,
   \[
   \int_{\mathbb{R}^N \times \mathbb{R}^N} |x - y|^{-\lambda} A(x) B(y) \, dx \, dy \leq C(N, \lambda, s)\|A\|_r \|B\|_s, \quad \forall A \in L^r, \forall B \in L^s;
   \]

2. if $\frac{1}{q} + \frac{1}{r} + \frac{1}{s} = 1 + \frac{\lambda}{N}$, then,
   \[
   \|(I_{\lambda} * A) B\|_{r^*} \leq C(N, s, \lambda)\|A\|_s \|B\|_q, \quad \forall A \in L^s, \forall B \in L^q.
   \]

The following interpolation estimate will be useful [13].

**Proposition 2.14** Let $1 \leq q, r \leq \infty$ and $0 \leq s < m$. Then,
\[
\|(\Delta)^{\frac{N}{2}} \cdot \|_{p'} \leq \|(\Delta)^{\frac{m}{2}} \cdot \|_p \|_{1-\theta},
\]
for any $\theta \in [\frac{m}{2}, 1]$ satisfying
\[
\frac{1}{p} = \frac{s}{N} + \theta \left(\frac{1}{2} - \frac{1}{N}\right) + \frac{1-\theta}{q}.
\]

The next abstract result ends this section [8, Lemma 2.15].

**Lemma 2.15** Let $T > 0$ and $X \in C([0, T], \mathbb{R})$ such that
\[
X(t) \leq a + b(X(t))^\theta \quad \text{for any} \quad t \in [0, T],
\]
where $a, b > 0, \theta > 1, a < (1 - \frac{1}{\theta})(\theta b)^{\frac{1}{\theta - 1}}$ and $X(0) \leq (\theta b)^{\frac{1}{\theta - 1}}$. Then
\[
X(t) \leq \frac{\theta a}{\theta - 1} \quad \text{for any} \quad t \in [0, T].
\]

### 3 Variational Analysis

In this section, one collects some estimates needed in the proof of the main result.

**Lemma 3.1** Let $N \geq 3$ and $b, \alpha$ satisfying Eq. 1.2. Take $p \in (p_*, p^*)$ such that $p \geq 2$ and $u_0 \in H^1$ satisfying
\[
\max\{\mathcal{M}(u_0), \mathcal{G}(u_0)\} < 1.
\]

Then, there exists $\delta > 0$ such that the solution $u \in C(\mathbb{R}, H^1)$ to Eq. 1.1 satisfies
\[
\max\left\{\sup_{t \in \mathbb{R}} \mathcal{M}(u(t)), \sup_{t \in \mathbb{R}} \mathcal{G}(u(t))\right\} < 1 - \delta.
\]
Proof Denote the constant $C_{N, p, b, \alpha} := C(N, p, b, \alpha)$ given by Proposition 2.2. The inequality $\mathcal{M}E(u_0) < 1$ gives the existence of $\delta > 0$ such that

$$1 - \delta > \frac{M(u_0)^{1-\frac{1}{Nc}} E(u_0)}{M(\phi)^{1-\frac{1}{Nc}} E(\phi)} \geq \frac{M(u_0)^{1-\frac{1}{Nc}}}{M(\phi)^{1-\frac{1}{Nc}}} \left( \|\nabla u(t)\|^2 - \frac{1}{p} \int_{\mathbb{R}^N} |x|^b |u|^p (I_\alpha * |\cdot|^b |u|^p) \, dx \right) \geq \frac{M(u_0)^{1-\frac{1}{Nc}}}{M(\phi)^{1-\frac{1}{Nc}}} \left( \|\nabla u(t)\|^2 - \frac{C_{N, p, b, \alpha}}{p} \|u\|^A \|\nabla u(t)\|^B \right).$$

Thanks to Pohozaev identities, one has

$$\frac{B - 2}{B} \|\nabla \phi\|^2 = \frac{B - 2}{A} \|\phi\|^2 = E(\phi).$$

Thus,

$$1 - \delta > \frac{B - 2}{B} \frac{M(u_0)^{1-\frac{1}{Nc}}}{M(\phi)^{1-\frac{1}{Nc}}} \|\nabla \phi\|^2 \left( \|\nabla u(t)\|^2 - \frac{C_{N, p, b, \alpha}}{p} \|u\|^A \|\nabla u(t)\|^B \right) = \frac{B - 2}{B} \frac{M(u_0)^{1-\frac{1}{Nc}}}{M(\phi)^{1-\frac{1}{Nc}}} \|\nabla u(t)\|^2 - \frac{B - 2}{B} \frac{M(u_0)^{1-\frac{1}{Nc}}}{M(\phi)^{1-\frac{1}{Nc}}} \|\nabla \phi\|^2 \frac{2}{A} \left( \frac{B}{A} \right)^{\frac{2}{p-1}} \|\phi\|^{-2(p-1)} \|u\|^A \|\nabla u(t)\|^B.$$ 

So,

$$1 - \delta > \frac{B - 2}{B} \frac{M(u_0)^{1-\frac{1}{Nc}}}{M(\phi)^{1-\frac{1}{Nc}}} \|\nabla u(t)\|^2 - \frac{B - 2}{B} \frac{2}{A} \frac{M(u_0)^{1-\frac{1}{Nc}}}{M(\phi)^{1-\frac{1}{Nc}}} \|\nabla \phi\|^2 \left( \|\phi\|^2 \|\nabla \phi\|^B \right) - \frac{B - 2}{B} \frac{2}{A} \|u_0\|^A \|\nabla u(t)\|^2 \left( \|\phi\|^B \|\nabla \phi\|^B \right) - \frac{2}{A} \|u_0\|^A \|\nabla u(t)\|^2 \left( \|\phi\|^B \|\nabla \phi\|^B \right).$$

Using the equalities $s_\phi = \frac{B-2}{2(p-1)}$ and $\frac{B}{A} = \left( \frac{\|\nabla \phi\|}{\|\phi\|} \right)^2$, one has

$$1 - \delta > \frac{B - 2}{B} \frac{M(u_0)^{1-\frac{1}{Nc}}}{M(\phi)^{1-\frac{1}{Nc}}} \|\nabla u(t)\|^2 - \frac{B - 2}{B} \frac{2}{A} \frac{M(u_0)^{1-\frac{1}{Nc}}}{M(\phi)^{1-\frac{1}{Nc}}} \|\nabla \phi\|^2 \left( \frac{\|\phi\|^2}{\|\nabla \phi\|^B} \right)^{2} \|\phi\|^{-2(p-1)} = \frac{2}{A} \frac{2}{B - 2} \left( \frac{B - 2}{A} \right)^{\frac{2}{2(p-1)}} \|u_0\|^A \|\nabla u(t)\|^2 \left( \frac{\|\phi\|^B}{\|\nabla \phi\|^B} \right) - \frac{2}{A} \|u_0\|^A \|\nabla u(t)\|^2 \left( \frac{\|\phi\|^B}{\|\nabla \phi\|^B} \right).$$

Let the function on $[0, 1]$ denoted by $f(x) := \frac{B - 2}{B - 2} x^2 - \frac{2}{B - 2} x^B$, with first derivative $f'(x) = \frac{2B}{B - 2} x (1 - x^{B-2})$. Thus, with the table change of $f$ via the continuity of $t \rightarrow X(t) := \frac{u_0}{\|\phi\|^B} \|\nabla u(t)\|^B$, it follows that $X(t) < 1$ for any $t < T^*$. Thus, $T^* = \infty$ and there exists $\epsilon > 0$ near to zero such that $X(t) \in f^{-1}([0, 1 - \delta]) = [0, 1 - \epsilon]$. This finishes the proof. \qed
Let us prove a coercivity result on balls with large radius.

**Lemma 3.2** There exists $R_0 := R_0(\delta, M(u), \phi) > 0$ such that for every $R > R_0$,

$$\sup_{r \in \mathbb{R}} \|\psi_R u(y)\|^{1-\varepsilon_c} \|\nabla(\psi_R u(x))\|^\varepsilon_c < (1 - \delta) \|\phi\|^{1-\varepsilon_c} \|\nabla\phi\|^\varepsilon_c.$$

Moreover, there exists $\delta' > 0$ satisfying

$$\|\nabla(\psi_R u)\|^2 - \frac{B}{2p} \int_{\mathbb{R}^N} (I_\alpha \ast |\cdot|^\beta |\psi_R u|^p) |x|^\beta |\psi_R u|^p \, dx \geq \delta' ||\psi_R u||^{2\frac{2Np}{N+\alpha+2\beta}}.$$

**Proof** Taking account of Proposition 2.2, one gets

$$E(u) = \|\nabla u\|^2 - \frac{1}{p} \int_{\mathbb{R}^N} |x|^\beta |u|^p (I_\alpha \ast |\cdot|^\beta |u|^p) \, dx \geq \|\nabla u\|^2 \left(1 - \frac{C_{N,p,b,a}}{p} ||u||^A \|\nabla u\|^{B-2}\right) \geq \|\nabla u\|^2 \left(1 - \frac{C_{N,p,b,a}}{p} ||u||^{1-\varepsilon_c} \|\nabla u\|^{\varepsilon_c}(p-1)\right).$$

So, with Lemma 3.1, it follows that

$$E(u) \geq \|\nabla u\|^2 \left(1 - (1 - \delta) \frac{2}{A} \left(\frac{A}{B}\right)^\frac{\beta}{2} \|\phi\|^{-2}(p-1) [\|\phi\|^{1-\varepsilon_c} \|\nabla\phi\|^{\varepsilon_c}]^{(p-1)}\right) \geq \|\nabla u\|^2 \left(1 - (1 - \delta) \frac{2}{B} \left(\frac{\|\phi\|}{\|\nabla\phi\|}\right)^{B-2} \left[\|\nabla\phi\|^{\frac{B-2}{B}}\right]\right) \geq \|\nabla u\|^2 \left(1 - (1 - \delta) \frac{2}{B}\right).$$

Thus, using Sobolev injections with the fact that $p < p^*$, one gets

$$\|\nabla u\|^2 - \frac{B}{2p} \int_{\mathbb{R}^N} |x|^\beta |u|^p (I_\alpha \ast |\cdot|^\beta |u|^p) \, dx \geq \delta' ||u||^{2\frac{2Np}{N+\alpha+2\beta}}.$$

This gives the second part of the claimed Lemma provided that the first point is proved. Compute

$$\|\nabla(\psi_R u)\|^2 = \int_{\mathbb{R}^N} \left(\psi_R |\nabla u|^2 + |\nabla\psi_R|^2 |u|^2 + 2R(\hat{u}\nabla u)\psi_R \nabla\psi_R\right) \, dx \leq \int_{\mathbb{R}^N} \left(\psi_R |\nabla u|^2 + |\nabla\psi_R|^2 |u|^2 + \frac{1}{2} \nabla(|u|^2) \nabla(|\psi_R|^2)\right) \, dx \leq \int_{\mathbb{R}^N} \left(\psi_R |\nabla u|^2 + |\nabla\psi_R|^2 |u|^2 - |u|^2 (\psi_R \Delta\psi_R + |\nabla\psi_R|^2)\right) \, dx \leq \|\nabla u\|^2 + C \frac{||u||^2}{R^2}.$$
Then, one gets the proof of the first point and so the Lemma.

\section{Scattering Criterion}

This section is devoted to prove Proposition 2.7. By Lemma 3.1, \( u \) is bounded in \( H^\frac{1}{2} \). Take \( \varepsilon > 0 \) near to zero and \( R(\varepsilon) >> 1 \) to be fixed later. Define the Strichartz norm

\[ \| \cdot \|_{S^\varepsilon(I)} := \sup_{(q,r) \in \Gamma} \| \cdot \|_{L^q(I,L^r)}. \]

Let us give a technical result.

\begin{lemma}
Let \( N \geq 3 \), \( I \) a time slab, \( b, \alpha \) satisfying Eq. 1.2 and \( p \in (p_*, p^*) \). Thus, there exists \( \theta \in (0, 2p - 1) \) satisfying

\[ \| u - e^{i \Delta} u_0 \|_{S^\varepsilon(I)} \lesssim \| u \|_{L^\infty(I,H^1)}^{\theta} \| u \|_{S^\varepsilon(I)}^{2p-1-\theta}. \]

\end{lemma}

\begin{proof}
Take the real numbers

\[ a := \frac{2p - \theta}{1 - s_c}, \quad d := \frac{2p - \theta}{1 + (2p - 1 - \theta)s_c}; \]

\[ r := \frac{2N(2p - \theta)}{(N - 2s_c)(2p - \theta) - 4(1 - s_c)}. \]

The condition \( \theta = 0^+ \) gives via some computations that the previous pairs satisfy the admissibility conditions. Indeed, \( a > \frac{2}{1 - s_c} \) implies that \( r < \frac{2N}{N - 2s_c} \). Moreover,

\[ r = \frac{2N(2p - \theta)}{(N - 2s_c)(2p - \theta) - 4(1 - s_c)} > \frac{2N}{N - 2s_c}. \]

Thus,

\[ (a, r) \in \Gamma_{s_c}, \quad (d, r) \in \Gamma_{-s_c} \quad \text{and} \quad (2p - 1 - \theta)d = a. \]

Take two positive real numbers \( r_1, \mu \) satisfying

\[ 1 + \frac{\alpha}{N} = \frac{2}{\mu} + \frac{\theta}{r_1} + \frac{2p - \theta}{r}. \]

This gives

\[ \frac{2N}{\mu} = \alpha + N - \frac{N\theta}{r_1} - \frac{N(2p - \theta)}{r} = \alpha + N - \frac{N\theta}{r_1} - \frac{(N - 2s_c)(2p - \theta) - 4(1 - s_c)}{2} = -2b - \frac{N\theta}{r_1} + \theta \frac{2 + 2b + \alpha}{2(p - 1)}. \]

Taking account of Lemma 2.13 via Hölder estimates, one has

\[ \| \nabla \|_{L^d(I,L^r([|x|<1]))} := \| |x|^b |u|^{p-2} (I_\theta * \chi_{[|x|<1]} \cdot |^b |u|^p) u \|_{L^d(I,L^r([|x|<1]))} \lesssim \| |x|^b \|_{L^\mu([|x|<1])} \| u \|_{L^\infty(I,H^1)}^{\theta} \| u \|_{L^r(I)}^{2p-1-\theta}. \]
Then, choosing \( r_1 := \frac{2N}{N-2} \), one gets because \( p < p^* \),
\[
\frac{2N}{\mu} + 2b = \frac{\theta}{2(p-1)} \left( 2 + 2b + \alpha - (p-1)(N-2) \right) > 0.
\]

So, \(|x|^b \in L^\mu(|x| < 1)\). Then, by Sobolev injections, one gets
\[
\|N_1\|_{L^{\theta'}(I,L^{\theta'}(|x|<1))} \lesssim \|x|^b\|_{L^\mu(|x|<1)} \|u\|^\theta_{L^\infty(I,L^\theta)} \|u\|_{r}^{2p-1-\theta} \|L^{\theta'}(I)\.
\]
\[
\lesssim \|u\|^\theta_{L^\infty(I,L^\theta)} \|u\|_{r}^{2p-1-\theta}.
\]

Moreover, on the complementary of the unit ball, one takes \( r_1 := 2 \). So, because \( p > p^* \), one has
\[
\frac{2N}{\mu} + 2b = \frac{\theta}{2(p-1)} \left( 2 + 2b + \alpha - (p-1)(N-2) \right) < 0.
\]

So, \(|x|^b \in L^\mu(|x| > 1)\). The proof is achieved via Strichartz estimates.

The following estimate will be proved in the Appendix.

**Lemma 4.2** Let \( N \geq 3 \) and \( b, \alpha \) satisfying Eq. 1.2. Take \( p \in (p^*, p^*) \) such that \( p \geq 2 \) and \( u \in C(\mathbb{R}, H^1) \) be a global solution to Eq. 1.1. Then, there exist \( 2 < q_1, q_2 < \frac{2N}{N-2} \) and \( 0 < \theta_1, \theta_2 < 2p \) such that
\[
\|u - e^{i\Delta}u_0\|_{S_0(T)} \lesssim \|u\|_{L^\infty_T(L^{q_1})} \|\langle u \rangle\|_{S_0(T)}^{2p-\theta_1} + \|u\|_{L^\infty_T(L^{q_2})} \|\langle u \rangle\|_{S_0(T)}^{2p-\theta_2}.
\]

Using Strichartz estimates, write
\[
\|e^{i\Delta}u_0\|_{S_0(R)} \lesssim 1.
\]

Taking account of the hypothesis, there exist \( T > 0 \) such that
\[
\max\{\|e^{i\Delta}u_0\|_{S_0(T)}, \|u(T)\|_{L^2(|x|<R)}\} < \epsilon.
\]

Thanks to the Eq. 1.1, if \( \frac{1}{R} \lesssim \epsilon^{1+\beta} \) for some \( 0 < \beta << 1 \), one has
\[
|\int_{\mathbb{R}^N} \psi_R(x)|u(t, x)|^2 \, dx| \lesssim \frac{1}{R};
\]
\[
\sup_{t \in [T-\epsilon^{1+\beta}, T]} \int_{\mathbb{R}^N} \psi_R(x)|u(t, x)|^2 \, dx \lesssim \epsilon.
\]

Denote \( J_2 := [0, T - \epsilon^{1+\beta}] \) and \( J_1 := [T - \epsilon^{1+\beta}, T] \). The integral formula gives
\[
u(T) = e^{iT\Delta}u_0 + i \int_0^T e^{i(T-s)\Delta}N \, ds;
\]
\[
e^{i(-T)u(T) = e^{i\Delta}u_0 + i \int_{J_1} e^{i(-s)\Delta}N \, ds + i \int_{J_2} e^{i(-s)\Delta}N \, ds.
\]

Strichartz estimates via Lemma 4.1 give
\[
\|\int_{J_j} e^{i(-s)\Delta}N \, ds\|_{S_0(T,T)} \lesssim \|u\|_{L^\infty(J_1, H^1)} \|u\|_{S_0(J_j)}^{2p-1-\theta} \lesssim \|u\|_{S_0(J_j)}^{2p-1-\theta}.
\]
On the other hand, if \((q, r) \in \Gamma_c\) and \(2^* := \frac{2N}{N - T}\), one writes
\[
\|u\|_{L^\infty(J_1, L^r)} \leq \|\psi R u\|_{L^\infty(J_1, L^r)} + \|(1 - \psi R) u\|_{L^\infty(J_1, L^r)} \\
\leq \|\psi R u\|_{L^\infty(J_1, L^r)}^{\frac{2(N-1)}{2N - 2T}} \|\psi R u\|_{L^\infty(J_1, L^r)}^{\frac{2N - 2T}{2N - 2T}} + \|(1 - \psi R) u\|_{L^\infty(J_1, L^r)}^{\frac{2N - 2T}{2N - 2T}} \|(1 - \psi R) u\|_{L^\infty(J_1, L^r)}^{\frac{2}{2N - 2T}} \\
\lesssim \epsilon^{\frac{2(N-1)}{2N - 2T}} + (R^{\frac{2N-1}{2N - 2T}} \|u\|_{L^\infty(J_1, H^1)}^{1 - \frac{2}{N}}) \lesssim \epsilon^{\frac{2(N-1)}{2N - 2T}} + \epsilon^{\frac{2(N-1)}{2N - 2T}}.
\]

Thus,
\[
\| \int_{J_1} e^{i(-s)\Delta} \mathcal{N} \, ds \|_{S^{0c}(J_1)} \lesssim \sup_{(q, r) \in \Gamma_c} \|u\|_{L^q(J_1, L^r)}^{2p - 1 - \theta} \\
\lesssim \sup_{(q, r) \in \Gamma_c} (\|u\|_{L^\infty(J_1, L^r)} \epsilon^{-\frac{\theta}{q}})^{2p - 1 - \theta} \\
\lesssim \left( \left( \epsilon^{\frac{2(N-1)}{2N - 2T}} + \epsilon^{\frac{2(N-1)(1-\theta)}{2N - 2T}} \right) \epsilon^{-\frac{\theta}{q}} \right)^{2p - 1 - \theta} \\
\lesssim \left( \epsilon^{\frac{2(N-1)}{2N - 2T}} + \epsilon^{\frac{2(N-1)(1-\theta)}{2N - 2T}} \right)^{2p - 1 - \theta}.
\]

Take \((q, r)\) be an \(s_c\)-admissible pair. Let a real number \(\eta \in (\max(0, 1 - 2s_c), 1 - \frac{2s_c}{N})\) and the admissible couple \((c, d := \eta r)\) \(\in \Gamma\). Thus,
\[
\frac{1}{r} = \frac{\eta}{d}; \\
\frac{1}{q} = \frac{\eta}{c} + \frac{1 - \eta}{e}; \\
e := \frac{4(1 - \eta)}{N(1 - \eta) - 2s_c},
\]
with Strichartz estimate and an interpolation, write
\[
\| \int_{J_2} e^{i(-s)\Delta} \mathcal{N} \, ds \|_{L^q((T, \infty), L^r)} \lesssim \| \int_{J_2} e^{i(-s)\Delta} \mathcal{N} \, ds \|_{L^q((T, \infty), L^d(\mathbb{R}^N))} \| \int_{J_2} e^{i(-s)\Delta} \mathcal{N} \, ds \|_{L^1((T, \infty), L^\infty(\mathbb{R}^N))}.
\]
Using the dispersion of the free NLS kernel via Hardy-Littlewood-Sobolev inequality, there exists \(\gamma_1 > 0\), such that
\[
(II) := \| \int_{J_2} e^{i(-s)\Delta} \mathcal{N} \, ds \|_{L^1((T, \infty), L^\infty(\mathbb{R}^N))} \\
\lesssim \| \int_{J_2} \frac{1}{(-s)^\frac{N}{2}} \|\mathcal{N}\|_{L^1} \, ds \|_{L^1((T, \infty))} \\
\lesssim \|u\|_{L^\infty(\mathbb{H}^1)}^{2p - 1} \| \int_{J_2} \frac{ds}{(-s)^\frac{N}{2}} \|_{L^1((T, \infty))}^{1 - \eta} \\
\lesssim \|u\|_{L^\infty(\mathbb{H}^1)}^{2p - 1} \|(-T + \epsilon^{-\beta})^{-\frac{N}{2} - \frac{1}{2}} \|_{L^1((T, \infty))}^{1 - \eta} \\
\lesssim \epsilon^{\beta \left(\frac{N}{2} + 1 - e^{-\beta}\right)} \lesssim \epsilon^{\gamma_1}.
\]
Moreover,
\[
(I) := \| \int_j e^{i(-s)\Delta} N \, ds \|_{L^\infty((t, \infty), L^4(\mathbb{R}^N))}^\eta 
= \| e^{i\Delta}((-T + i\beta)\Delta u(T - e^{-\beta}) - u_0) \|_{L^\infty((t, \infty), L^4(\mathbb{R}^N))}^\eta 
\lesssim \| u \|_{L^\infty((t, \infty), L^2(\mathbb{R}^N))}^\eta.
\]
Thus,
\[
\| \int_j e^{i(-s)\Delta} N \, ds \|_{L^4((t, \infty), L^2')} \lesssim e^{\gamma t}.
\]
Taking account of Duhamel formula
\[
e^{i(-T)\Delta} u(T) = e^{i\Delta} u_0 + i \int_j e^{i(-s)\Delta} N \, ds + i \int_j e^{i(-s)\Delta} N \, ds.
\]
Thus, there exists \( \gamma > 0 \) such that
\[
\| e^{i(-T)\Delta} u(T) \|_{S^c(0, \infty)} = \| e^{i(-T)\Delta} u(T) \|_{S^c(t, \infty)} \lesssim e^{\gamma t}.
\]
So, with Lemma 4.1 via the absorption result Lemma 2.15, one gets
\[
\| u \|_{S^c(t, \infty)} \lesssim e^{\gamma t}.
\]
The next result, which will be proved in the Appendix, via the above estimate and Lemma 2.15, yield
\[
u, \nabla u \in S(\mathbb{R}).
\]

**Lemma 4.3** One has two cases.

1. Assume that \( N = 3 \) and \(-1 < 2b < 0 \). Then, there exist \( 0 < \theta < 2(p - 1) \) and \(-2b < \gamma < 1 \) such that one has the estimates
\[
\| \nabla \|_{S^c(I)} \lesssim \| u \|_{L^\infty(I, H^1)}^{\theta} \| u \|_{S^c(I)}^{2(p-1) - \theta} \| u \|_{S^c(I)};
\]
\[
\| \nabla^\perp \|_{S^c(I)} \lesssim \| u \|_{L^\infty(I, H^1)}^{\theta} \| u \|_{S^c(I)}^{2(p-1) - \theta} \| \nabla u \|_{S^c(I)} + \| u \|_{L^\infty(I, H^1)}^{1 - \gamma} \| u \|_{S^c(I)} \| \nabla u \|_{S^c(I)}^{2(p-1) - \theta + \gamma}.
\]

2. Assume that \( N > 3 \). Then, there exist \( 0 < \theta < 2(p - 1) \) such that one has
\[
\| \nabla \|_{S^c(I)} \lesssim \| u \|_{L^\infty(I, H^1)}^{\theta} \| u \|_{S^c(I)}^{2(p-1) - \theta} \| u \|_{S^c(I)};
\]
\[
\| \nabla^\perp \|_{S^c(I)} \lesssim \| u \|_{L^\infty(I, H^1)}^{\theta} \| u \|_{S^c(I)}^{2(p-1) - \theta} \| \nabla u \|_{S^c(I)}.
\]

With Lemma 4.2, one gets for \( u_+ := e^{-iT\Delta} u(T) + i \int_T^\infty e^{-is\Delta} N \, ds, \)
\[
\| u(t) - e^{it\Delta} u_+ \|_{H^1} = \| \int_T^\infty e^{i(t-s)\Delta} N \, ds \|_{H^1}
\lesssim \| (1 + \nabla) N \|_{S^c(t, \infty)}
\lesssim \| u \|_{L^\infty((t, \infty), H^1)}^{\theta_1} \| u \|_{S^c(t, \infty)}^{2p - \theta_1} + \| u \|_{L^\infty((t, \infty), H^1)}^{\theta_2} \| u \|_{S^c(t, \infty)}^{2p - \theta_2}.
\]
Thus, as \( t \to \infty, \)
\[
\| u(t) - e^{it\Delta} u_+ \|_{H^1} \to 0.
\]
This finishes the proof.
5 Morawetz Estimate

In this section, one proves Proposition 2.8 about a classical Morawetz estimate.

Proof of Proposition 2.8 Using the properties of $a$ via Cauchy-Schwarz inequality, one has

$$\sup_{t \in \mathbb{R}} M_a(t) \lesssim R.$$ 

Taking account of Proposition 2.12, write

$$M'_a = 4 \int_{\mathbb{R}^N} \partial_1 \partial_2 a \partial(\partial_1 u \partial_2 u) \, dx - \int_{\mathbb{R}^N} \Delta^2 a |u|^2 \, dx$$

$$+ \left( \frac{4}{p} - 2 \right) \int_{\mathbb{R}^N} |x|^b |u|^p \Delta a(I_\alpha * | \cdot |^b |u|^p) \, dx$$

$$+ \frac{4b}{p} \int_{\mathbb{R}^N} \nabla a |x|^b \cdot |u|^p (I_\alpha * | \cdot |^b |u|^p) \, dx$$

$$+ \frac{4}{p} (\alpha - N) \int_{\mathbb{R}^N} |x|^b |u|^p \nabla a \left( \frac{1}{|x|^2} I_\alpha * | \cdot |^b |u|^p \right) \, dx.$$ 

In the centered ball of radius $\frac{R}{2}$,

$$(I) := 4 \int_{|x| < \frac{R}{2}} \partial_1 \partial_2 a \partial(\partial_1 u \partial_2 u) \, dx - \int_{|x| < \frac{R}{2}} \Delta^2 a |u|^2 \, dx$$

$$+ \left( \frac{4}{p} - 2 \right) \int_{|x| < \frac{R}{2}} \Delta a |x|^b |u|^p (I_\alpha * | \cdot |^b |u|^p) \, dx$$

$$+ \frac{4b}{p} \int_{|x| < \frac{R}{2}} \nabla a |x|^b \cdot |u|^p (I_\alpha * | \cdot |^b |u|^p) \, dx$$

$$= 4 \int_{|x| < \frac{R}{2}} |\nabla u|^2 \, dx + 2N \left( \frac{2}{p} - 1 \right) \int_{|x| < \frac{R}{2}} |x|^b |u|^p (I_\alpha * | \cdot |^b |u|^p) \, dx$$

$$+ \frac{4b}{p} \int_{|x| < \frac{R}{2}} |x|^b |u|^p (I_\alpha * | \cdot |^b |u|^p) \, dx.$$ 

In the centered annulus $C(0, \frac{R}{2}, R)$, one has

$$(II) := 4 \int_{\frac{R}{2} < |x| < R} \partial_1 \partial_2 a \partial(\partial_1 u \partial_2 u) \, dx - \int_{\frac{R}{2} < |x| < R} \Delta^2 a |u|^2 \, dx$$

$$+ \left( \frac{4}{p} - 2 \right) \int_{\frac{R}{2} < |x| < R} \Delta a |x|^b |u|^p (I_\alpha * | \cdot |^b |u|^p) \, dx$$

$$+ \frac{4b}{p} \int_{\frac{R}{2} < |x| < R} \nabla a |x|^b \cdot |u|^p (I_\alpha * | \cdot |^b |u|^p) \, dx$$

$$\geq 4 \int_{\frac{R}{2} < |x| < R} \partial_1 \partial_2 a \partial(\partial_1 u \partial_2 u) \, dx - O \left( R \int_{\frac{R}{2} < |x| < R} \frac{|u|^2}{|x|^3} \, dx \right)$$

$$- CR \int_{\frac{R}{2} < |x| < R} |x|^b - 1 |u|^p (I_\alpha * | \cdot |^b |u|^p) \, dx$$

$$- C \int_{\frac{R}{2} < |x| < R} |x|^b |u|^p (I_\alpha * | \cdot |^b |u|^p) \, dx.$$
For $|x| > R$, write using the radial assumption on $u$,

$$(III) := 4 \int_{|x| > R} \partial_i \partial_k a \Re(\partial_k u \partial_i \bar{u}) \, dx - \int_{|x| > R} \Delta^2 a |u|^2 \, dx$$

$$+ \left( \frac{4}{p} - 2 \right) \int_{|x| > R} \Delta a |x|^b |u|^p (I_\alpha \ast | \cdot \cdot | |u|^p) \, dx$$

$$+ \frac{4b}{p} \int_{|x| > R} x \nabla a |x|^{b-2} |u|^p (I_\alpha \ast | \cdot \cdot | |u|^p) \, dx.$$

Now, let us define the sets

$$\Omega := \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N, \ \text{s.t.} \ \frac{R}{2} < |x| < R \} \cup \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N, \ \text{s.t.} \ \frac{R}{2} < |y| < R \};$$

$$\Omega' := \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N, \ \text{s.t.} \ \frac{R}{2} < |x| < R \} \cup \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N, \ \text{s.t} \ |x| < \frac{R}{2}, |y| > R \}.$$

Consider the term

$$(IV) := \int_{\mathbb{R}^N} |x|^b |u|^p \nabla a \left( \frac{1}{|x|^2} I_\alpha \ast | \cdot \cdot | |u|^p \right) \, dx$$

$$= \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (|x||y|)^b (|u(x)||u(y)|)^p (\nabla a(x) - \nabla a(y))(x - y) \frac{I_\alpha(x-y)}{|x-y|^2} \, dx \, dy$$

$$= \left( \int_{\Omega'} + \int_{|x|, |y| < \frac{R}{2}} + \int_{|x|, |y| > \frac{R}{2}} \right) \left( (|x||y|)^b (|u(x)||u(y)|)^p \nabla a(x)(x-y) \frac{I_\alpha(x-y)}{|x-y|^2} \right) \, dx \, dy.$$

Compute

$$(a) := \int_{\Omega'} \left( (|x||y|)^b (|u(x)||u(y)|)^p \nabla a(x)(x-y) \frac{I_\alpha(x-y)}{|x-y|^2} \right) \, dx \, dy$$

$$= \int_{|x| > R, y < \frac{R}{2}} \left( (|x||y|)^b (|u(x)||u(y)|)^p \nabla a(x)(x-y) \frac{I_\alpha(x-y)}{|x-y|^2} \right) \, dx \, dy$$

$$+ \int_{|y| > R, x < \frac{R}{2}} \left( (|x||y|)^b (|u(x)||u(y)|)^p \nabla a(x)(x-y) \frac{I_\alpha(x-y)}{|x-y|^2} \right) \, dx \, dy$$

$$= \int_{|x| > R, y < \frac{R}{2}} \left( (|x||y|)^b (|u(x)||u(y)|)^p \nabla a(x)(x-y) \frac{I_\alpha(x-y)}{|x-y|^2} \right) \, dx \, dy$$

$$= \int_{|x| > R, y < \frac{R}{2}} \left( (|x||y|)^b (|u(x)||u(y)|)^p \frac{R}{|x|}(x-y)(x-y) \frac{I_\alpha(x-y)}{|x-y|^2} \right) \, dx \, dy.$$
Furthermore,

\[
(b) = \frac{1}{2} \int_{|x| < \frac{R}{2}, |y| < \frac{R}{2}} \left( |x| |y|^b (|u(x)| |u(y)|)^p (\nabla a(x) - \nabla a(y)) (x - y) \frac{I_a(x - y)}{|x - y|^2} \right) dx \, dy \\
= \frac{1}{2} \int_{|x| < \frac{R}{2}, |y| < \frac{R}{2}} \left( |x| |y|^b (|u(x)| |u(y)|)^p (x - y) (x - y) \frac{I_a(x - y)}{|x - y|^2} \right) dx \, dy \\
= \frac{1}{2} \int_{|x| < \frac{R}{2}, |y| < \frac{R}{2}} \left( I_a(x - y) |x| |y|^b (|u(x)| |u(y)|)^p \right) dx \, dy \\
= \frac{1}{2} \int_{\mathbb{R}^N} \left( I_a * | \cdot |^b |\psi_R u|^p \right) |x| |y|^b |\psi_R u|^p \, dx.
\]

Moreover,

\[
(c) = \int_{\left\{ \frac{R}{4} < |x| < R \right\}} \int_{\mathbb{R}^N} \left( |x| |y|^b (|u(x)| |u(y)|)^p \nabla a(x) (x - y) \frac{I_a(x - y)}{|x - y|^2} \right) dx \, dy \\
= \int_{\left\{ \frac{R}{4} < |x| < R, |y| - |x| > \frac{R}{4} \right\}} \left( |x| |y|^b (|u(x)| |u(y)|)^p \nabla a(x) (x - y) \frac{I_a(x - y)}{|x - y|^2} \right) dx \, dy \\
+ \int_{\left\{ \frac{R}{4} < |x| < R, |y| - |x| < - \frac{R}{4} \right\}} \left( |x| |y|^b (|u(x)| |u(y)|)^p \nabla a(x) (x - y) \frac{I_a(x - y)}{|x - y|^2} \right) dx \, dy \\
= \mathcal{O} \left( \int_{\left\{|x| > \frac{R}{2}\right\}} |x|^b |u|^p (I_a * | \cdot |^b |u|^p) \, dx \right).
\]

The last term is

\[
(d) = \int_{\left\{|x|, |y| > R\right\}} \left( |x| |y|^b (|u(x)| |u(y)|)^p \nabla a(x) (x - y) \frac{I_a(x - y)}{|x - y|^2} \right) dx \, dy \\
= \frac{R}{2} \int_{\left\{|x|, |y| > R\right\}} \left( |x| |y|^b (|u(x)| |u(y)|)^p \left( \frac{x}{|x|} - \frac{y}{|y|} \right) (x - y) \frac{I_a(x - y)}{|x - y|^2} \right) dx \, dy \\
= \frac{R}{2} \int_{\left\{|x|, |y| > R\right\}} \left( |x| |y|^b (|u(x)| |u(y)|)^p \left( \frac{x - y}{|x|} + y \frac{|y| - |x|}{|x||y|} \right) (x - y) \frac{I_a(x - y)}{|x - y|^2} \right) dx \, dy \\
\lesssim \int_{\left\{|x| > R\right\}} |x|^b |u|^p (I_a * | \cdot |^b |u|^p) \, dx.
\]

Using the identity

\[
\int_{\mathbb{R}^N} \psi_R^2 |\nabla u|^2 \, dx = \int_{\mathbb{R}^N} \left( |\nabla (\psi_R u)|^2 + \psi_R \Delta \psi_R |u|^2 \right) \, dx,
\]
and regrouping the previous estimates, yields

\[
M'_a \geq 4 \int_{|x| < \frac{R}{2}} |\nabla (\psi_R u)|^2 \, dx + 4 \int_{|x| < \frac{R}{2}} \psi_R \Delta \psi_R |u|^2 \, dx \\
- \frac{2N}{p} \left( p - 2 - \frac{2b}{N} \right) \int_{|x| < \frac{R}{2}} |x|^b |u|^p (I_\alpha \ast |\cdot|^b |u|^p) \, dx \\
+ 4 \int_{\frac{R}{2} < |x| < R} \partial_i \partial_k a \Re (\partial_k u \partial_i \bar{u}) \, dx - O\left( R \int_{\frac{R}{2} < |x| < R} \frac{|u|^2}{|x|^3} \, dx \right) \\
- CR \int_{\frac{R}{2} < |x| < R} |x|^{b-1} |u|^p (I_\alpha \ast |\cdot|^b |u|^p) \, dx \\
- C \int_{\frac{R}{2} < |x| < R} |x|^b |u|^p (I_\alpha \ast |\cdot|^b |u|^p) \, dx \\
- O\left( R \int_{|x| > R} \frac{|u|^2}{|x|^3} \, dx \right) \\
+(N-1) \left( \frac{2}{p} - 1 \right) R \int_{|x| > R} |x|^{b-1} |u|^p (I_\alpha \ast |\cdot|^b |u|^p) \, dx \\
+ \frac{4 R b}{p} \int_{|x| > R} |x|^{b-1} |u|^p (I_\alpha \ast |\cdot|^b |u|^p) \, dx \\
+ \frac{4}{p} (\alpha - N) (IV).
\]

So,

\[
M'_a \geq 4 \int_{|x| < \frac{R}{2}} |\nabla (\psi_R u)|^2 \, dx - \frac{2N}{p} \left( p - 2 - \frac{2b}{N} \right) \int_{|x| < \frac{R}{2}} |x|^b |u|^p (I_\alpha \ast |\cdot|^b |u|^p) \, dx \\
- c \int_{\frac{R}{2} < |x| < R} |x|^b |u|^p (I_\alpha \ast |\cdot|^b |u|^p) \, dx - c \int_{|x| > \frac{R}{2}} |x|^{b-1} |u|^p (I_\alpha \ast |\cdot|^b |u|^p) \, dx \\
+ \frac{4}{p} (\alpha - N) (IV) - \frac{c}{R^2} M(u)
\geq 4 \int_{|x| < \frac{R}{2}} |\nabla (\psi_R u)|^2 \, dx - \frac{2N}{p} \left( p - 2 - \frac{2b}{N} \right) \int_{|x| < \frac{R}{2}} |x|^b |u|^p (I_\alpha \ast |\cdot|^b |u|^p) \, dx \\
- c \int_{\frac{R}{2} < |x| < R} |x|^b |u|^p (I_\alpha \ast |\cdot|^b |u|^p) \, dx - c \int_{|x| > \frac{R}{2}} |x|^{b-1} |u|^p (I_\alpha \ast |\cdot|^b |u|^p) \, dx \\
- \frac{4}{p} (N - \alpha) \left( \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha \ast |\cdot|^b |\psi_R u|^p)|x|^b |\psi_R u|^p \, dx + (a) + (c) + (d) \right) - \frac{c}{R^2} M(u).
\]
Then,

\[ M'_a \geq 4 \int_{|x| < \frac{R}{2}} |\nabla (\psi_R u)^2| \, dx - \frac{2B}{p} \int_{|x| < \frac{R}{2}} (I_\alpha * | \cdot | \psi_R u^p) |x|^b |\psi_R u|^p \, dx \]

\[ - \frac{c}{\frac{R^2}{2}} \int_{\frac{R}{2} < |x| < R} |x|^b |u|^p (I_\alpha * | \cdot | \psi_R u^p) \, dx - c \int_{|x| > \frac{R}{2}} |x|^{b-1} |u|^p (I_\alpha * | \cdot | \psi_R u^p) \, dx \]

\[ - 4 \frac{N - \alpha}{p} \left( (a) + (c) \right) - \frac{c}{R^2} M(u) \]

\[ \geq 4 \int_{|x| < \frac{R}{2}} |\nabla (\psi_R u)^2| \, dx - \frac{2B}{p} \int_{|x| < \frac{R}{2}} (I_\alpha * | \cdot | \psi_R u^p) |x|^b |\psi_R u|^p \, dx \]

\[ - \frac{c}{\frac{R^2}{2}} \int_{\frac{R}{2} < |x| < R} (|x||y|^b |(u(x)||u(y))|^p \left( \frac{R \frac{x}{|x|} - y}{|x| - y} \right) (x - y) \frac{I_\alpha(x - y)}{|x - y|^2} \right) \, dx \, dy \]

\[ - \frac{c}{R^2} M(u) - c \int_{|x| > \frac{R}{2}} |x|^b |u|^p (I_\alpha * | \cdot | \psi_R u^p) \, dx. \]

By Lemma 3.2, one gets

\[ M'_a \geq 4 \delta \| \psi_R u \|_{2p}^{2p} \frac{2Np}{N+2p} - \frac{c}{R^2} M(u) - c \int_{|x| > \frac{R}{2}} |x|^b |u|^p (I_\alpha * | \cdot | \psi_R u^p) \, dx \]

\[ - \frac{c}{\frac{R^2}{2}} \int_{\frac{R}{2} < |x| < R} (|x||y|^b |(u(x)||u(y))|^p \left( \frac{R \frac{x}{|x|} - y}{|x| - y} \right) (x - y) \frac{I_\alpha(x - y)}{|x - y|^2} \right) \, dx \, dy. \]

Moreover, since for \( R >> 1, \) on \( \{|x| > R, |y| < \frac{R}{2} \}, |x - y| \approx |x| > R >> \frac{R^2}{2} > |y|, \) one has

\( d = \frac{1}{x \in R \cap |y| < \frac{R}{2}} \left( (|x||y|^b |(u(x)||u(y))|^p \left( \frac{R \frac{x}{|x|} - y}{|x| - y} \right) (x - y) \frac{I_\alpha(x - y)}{|x - y|^2} \right) \, dx \, dy \]

\[ \lesssim \int_{\{|x| > R, |y| < \frac{R}{2} \}} (|x||y|^b |(u(x)||u(y))|^p \left( |x| + |y| \right) |x - y| \frac{I_\alpha(x - y)}{|x - y|^2} \, dx \, dy \]

\[ \lesssim \int_{\{|x| > R, |y| < \frac{R}{2} \}} (|x||y|^b |(u(x)||u(y))|^p I_\alpha(x - y) \right) \, dx \, dy \]

\[ \lesssim \int_{R^N} \int_{R^N} (\chi_{|x| > R} (|x||y|^b |(u(x)||u(y))|^p I_\alpha(x - y) \right) \, dx \, dy \]

\[ \lesssim \int_{R^N} \int_{R^N} (\chi_{|x| > R} (|x||y|^b |(u(x)||u(y))|^p I_\alpha(x - y) \right) \, dx \, dy. \]

Thanks to Lemma 2.13 via Hölder and Strauss estimates and Sobolev injections and the assumption \( p \in (p_\alpha, p^*) \), write for \( \mu := \left( \frac{N}{|b|} \right)^+, \)

\( d \lesssim \int_{R^N} \int_{R^N} (I_\alpha(x - y) \chi_{|x| > R} (|x||y|^b |(u(x)||u(y))|^p) \, dx \, dy \]

\[ \lesssim \|u\|^p \frac{2Np}{L^N + 2N} \|u\|^p \frac{2Np}{L^N + 2N} \]

\[ \lesssim \left( \int_{|x| > R} |u|^\frac{2Np}{N + 2N} \, dx \right)^\frac{N + 2N}{2N}. \]

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So,
\[
(d) \lesssim \left( \int_{|x| > R} |u|^2 \left( |x|^{-\frac{N-1}{2}} \|u\| \|\nabla u\|^{\frac{1}{2}} \right)^{2+\frac{2Np}{N+b}} \, dx \right)^{\frac{N+b+2}{2N}} \\
\lesssim \|u\|^{\frac{N+b+2}{N}} \frac{1}{R^\frac{B(N-1)}{2N}} \left( \|u\| \|\nabla u\| \right)^{\frac{B}{2N}} \\
\lesssim R^{-\frac{B(N-1)}{2N}}.
\]

Thus,
\[
4\delta' \|\psi_R u\|_{\frac{2Np}{N+b+2b}}^2 \leq M_a + \frac{c}{R^2} M(u) + c \int |x|^\frac{b}{2} \|u\|^p (I_\alpha \ast | \cdot |^b |u|^p) \, dx + R^{-\frac{B(N-1)}{2N}}.
\]

By the fundamental theorem of calculus and arguing as previously, one gets
\[
\frac{1}{T} \int_0^T \|\psi_R u\|_{\frac{2Np}{N+b+2b}}^2 \, ds \lesssim \left( \frac{\|M_a\|_{L^\infty([0,T])}}{T} + \frac{c}{R^2} \right) \int_0^T \int_{|x| > R} |x|^\frac{b}{2} \|u\|^p (I_\alpha \ast | \cdot |^b |u|^p) \, dx \, ds \\
+ R^{-\frac{B(N-1)}{2N}} + \frac{c}{R^2} \\
\lesssim \frac{R}{T} + R^{-\frac{B(N-1)}{2N}} + R^{-2}.
\]

The proof is ended. \(\square\)

Let us prove an energy evacuation.

**Lemma 5.1** Let \( N \geq 3 \) and \( b, \alpha \) satisfying Eq. 1.2. Take \( p \in (p_*, p^*) \) such that \( p \geq 2 \) and a radial global solution to Eq. 1.1 denoted by \( u \in C(\mathbb{R}, H^1_{\tau, d}) \). Then, there exists two sequences of real numbers \( t_n \to \infty \), \( R_n \to \infty \) such that
\[
\|u(t_n)\|_{L^{\frac{2Np}{N+b+2b}}(|x|<R_n)} \to 0 \quad \text{as} \quad n \to \infty.
\]

**Proof** By Proposition 2.8, if \( \frac{B(N-1)}{2N} > 2 \), taking \( R := \sqrt{T} \gg 1 \), one gets
\[
\frac{1}{T} \int_0^T \|u(s)\|_{L^{\frac{2Np}{N+b+2b}}(|x|<\sqrt{T})}^2 \, ds \leq \frac{1}{T} \int_0^T \|u(s)\|_{L^{\frac{2Np}{N+b+2b}}(|x|<\sqrt{T})}^2 \, ds \\
\lesssim \frac{1}{\sqrt{T}} + \frac{1}{T} \\
\lesssim \frac{1}{\sqrt{T}}.
\]

The proof follows with the mean value Theorem because \( u \in L^\infty(\mathbb{R}, L^{\frac{2Np}{N+b+2b}}) \). The second case \( \frac{B(N-1)}{2N} < 2 \) is similar. \(\square\)
6 Proof of the Main Result

This section is devoted to prove Theorem 2.5. Taking account of Theorem 6.1 in [1], it follows that \( T^* = \infty \) and

\[
\mathcal{G}\mathcal{M}(u(t)) < 1, \quad \forall t \in \mathbb{R}.
\]

Take \( \epsilon > 0, R > 0 \) given by Proposition 2.7 and two sequences of real numbers \( t_n, R_n \to \infty \) given by Lemma 5.1. Applying Hölder inequality for large \( n \) so that \( R_n > R \), one gets as \( n \to \infty \),

\[
\|u(t_n)\|_{L^2(|x|<R)} \leq (|\{x|<R\}|)^{\frac{2p}{2Np}} \|u(t_n)\|_{L^{\frac{2Np}{N+2p}}(|x|<R)}^{\frac{2p}{2Np}} \leq R^{\frac{2p}{2Np}} \|u(t_n)\|_{L^{\frac{2Np}{N+2p}}(|x|<R_n)} \leq \epsilon.
\]

The scattering follows by Proposition 2.7.

Appendix

This section is devoted to prove a variance identity and a global Strichartz type estimate.

Proof of Proposition 2.12

Let \( u \in C([0, T], H^1) \) be a local solution to Eq. 1.1. Testing the Eq. 1.1 by \( 2\bar{u} \), one gets

\[-2\mathcal{I}(\bar{u} \Delta u) = \partial_t(|u|^2)\).

Thus,

\[
V_a' = -2 \int_{\mathbb{R}^N} a\mathcal{I}(\bar{u} \Delta u) \, dx = 2\mathcal{I} \int_{\mathbb{R}^N} (\partial_k a \partial_k u) \bar{u} \, dx = M_a.
\]

Compute,

\[
\partial_t \mathcal{I}(\partial_k u\bar{u}) = \mathcal{I}(\partial_k \bar{u} u) + \mathcal{I}(\partial_k u \bar{u})
\]

\[
= \Re(i \bar{u} \partial_k \bar{u}) - \Re(i \partial_k u \bar{u})
\]

\[
= \Re(\partial_k \bar{u} (-\Delta u - \mathcal{N})) - \Re(\bar{u} \partial_k (-\Delta u - \mathcal{N}))
\]

\[
= \Re(\bar{u} \partial_k \Delta u - \partial_k \bar{u} \Delta u) + \Re(\bar{u} \partial_k \mathcal{N} - \partial_k \bar{u} \mathcal{N}).
\]

Thanks to the equality,

\[
\frac{1}{2} \partial_k \Delta (|u|^2) = 2\Re(\partial_k u \partial_k \bar{u}) + \Re(\bar{u} \partial_k \Delta u - \partial_k \bar{u} \Delta u),
\]

one has

\[
\int_{\mathbb{R}^N} \partial_k a \Re(\bar{u} \partial_k \Delta u - \partial_k \bar{u} \Delta u) \, dx = \int_{\mathbb{R}^N} \partial_k a \left( \frac{1}{2} \partial_k \Delta (|u|^2) - 2\Re(\partial_k u \partial_k \bar{u}) \right) \, dx
\]

\[
= 2 \int_{\mathbb{R}^N} \partial_1 \partial_k a \Re(\partial_k u \partial_k \bar{u}) \, dx - \frac{1}{2} \int_{\mathbb{R}^N} \Delta^2 a |u|^2 \, dx.
\]
Moreover,

\[
\mathcal{R}(\tilde{\partial}_k \mathcal{N}) = R\left( \tilde{\partial}_k [x \partial_x |u|^p]^{p-2} (I_a * | \cdot |^b |u|^p) u \right)
\]
\[
= R\left( \tilde{u} [\alpha - N] [x \partial_x |u|^p]^{p-2} \left( \frac{x_k}{1 - r^2} I_a * | \cdot |^b |u|^p \right) u \right) + [x \partial_x |u|^p]^{p-2} (I_a * | \cdot |^b |u|^p) \tilde{\partial}_k u
\]
\[
+ R\left( \tilde{u} (I_a * | \cdot |^b |u|^p) x_k [x \partial_x |u|^p]^{p-2} u + (p - 2) (I_a * | \cdot |^b |u|^p) x_k [\partial_k \tilde{u}] |u|^p \right)
\]
\[
= (\alpha - N) [x \partial_x |u|^p]^{p-2} (I_a * | \cdot |^b |u|^p) u + [x \partial_x |u|^p]^{p-2} (I_a * | \cdot |^b |u|^p) \tilde{\partial}_k u
\]
\[
+ b x_k [x \partial_x |u|^p]^{p-2} (I_a * | \cdot |^b |u|^p) + (p - 2) [x \partial_x |u|^p]^{p-2} (I_a * | \cdot |^b |u|^p) \tilde{\partial}_k u
\).

Thus,

\[
(A) := \int_{\mathbb{R}^N} \tilde{\partial}_k a \mathcal{R}(\tilde{\partial}_k \mathcal{N} - \partial_k \tilde{u} \mathcal{N}) \, dx
\]
\[
= b \int_{\mathbb{R}^N} \partial_k a x_k [x \partial_x |u|^p]^{p-2} (I_a * | \cdot |^b |u|^p) \, dx + \frac{p - 2}{2} \int_{\mathbb{R}^N} \tilde{\partial}_k a [x \partial_x |u|^p]^{p-2} \partial_k (|u|^p) \, dx
\]
\[
+ (\alpha - N) \int_{\mathbb{R}^N} \tilde{\partial}_k a [x \partial_x |u|^p]^{p-2} \left( \frac{x_k}{1 - r^2} I_a * | \cdot |^b |u|^p \right) \, dx
\]
\[
= b \int_{\mathbb{R}^N} \partial_k a x_k [x \partial_x |u|^p]^{p-2} (I_a * | \cdot |^b |u|^p) \, dx + \frac{p - 2}{2} \int_{\mathbb{R}^N} \tilde{\partial}_k a [x \partial_x |u|^p]^{p-2} \partial_k (|u|^p) \, dx
\]
\[
+ (\alpha - N) \int_{\mathbb{R}^N} \tilde{\partial}_k a [x \partial_x |u|^p]^{p-2} \left( \frac{x_k}{1 - r^2} I_a * | \cdot |^b |u|^p \right) \, dx
\).

Write

\[
(B) := \int_{\mathbb{R}^N} \tilde{\partial}_k a [x \partial_x |u|^p]^{p-2} \partial_k (|u|^p) \, dx
\]
\[
= - \int_{\mathbb{R}^N} \Delta a [x \partial_x |u|^p]^{p-2} (I_a * | \cdot |^b |u|^p) \, dx - (\alpha - N) \int_{\mathbb{R}^N} \tilde{\partial}_k a \left( \frac{x_k}{1 - r^2} I_a * | \cdot |^b |u|^p \right) [x \partial_x |u|^p] \, dx
\]
\[
- b \int_{\mathbb{R}^N} \partial_k a (I_a * | \cdot |^b |u|^p) x_k [x \partial_x |u|^p]^{p-2} \, dx.
\]

So,

\[
(A) = (\alpha - N) \int_{\mathbb{R}^N} \tilde{\partial}_k a \left( \frac{x_k}{1 - r^2} I_a * | \cdot |^b |u|^p \right) [x \partial_x |u|^p] \, dx + b \int_{\mathbb{R}^N} \partial_k a x_k [x \partial_x |u|^p]^{p-2} (I_a * | \cdot |^b |u|^p) \, dx
\]
\[
+ \frac{p - 2}{p} (B)
\]
\[
= (\alpha - N) \int_{\mathbb{R}^N} \tilde{\partial}_k a [x \partial_x |u|^p]^{p-2} \left( \frac{x_k}{1 - r^2} I_a * | \cdot |^b |u|^p \right) \, dx + b \int_{\mathbb{R}^N} \partial_k a x_k [x \partial_x |u|^p]^{p-2} (I_a * | \cdot |^b |u|^p) \, dx
\]
\[
- \frac{p - 2}{p} \left( \int_{\mathbb{R}^N} \Delta a [x \partial_x |u|^p]^{p-2} (I_a * | \cdot |^b |u|^p) \, dx + (\alpha - N) \int_{\mathbb{R}^N} \tilde{\partial}_k a [x \partial_x |u|^p]^{p-2} \left( \frac{x_k}{1 - r^2} I_a * | \cdot |^b |u|^p \right) \, dx
\]
\[
+ b \int_{\mathbb{R}^N} \partial_k a x_k [x \partial_x |u|^p]^{p-2} (I_a * | \cdot |^b |u|^p) \, dx \right)
\]
\[
= \frac{2}{p} (\alpha - N) \int_{\mathbb{R}^N} \tilde{\partial}_k a [x \partial_x |u|^p]^{p-2} \left( \frac{x_k}{1 - r^2} I_a * | \cdot |^b |u|^p \right) \, dx + b \int_{\mathbb{R}^N} \partial_k a x_k [x \partial_x |u|^p]^{p-2} (I_a * | \cdot |^b |u|^p) \, dx
\]
\[
- \frac{p - 2}{p} \int_{\mathbb{R}^N} \Delta a [x \partial_x |u|^p]^{p-2} (I_a * | \cdot |^b |u|^p) \, dx.
\]
Finally,
\[
V''_a = 4 \int_{\mathbb{R}^N} \partial_t \Delta_a \mathfrak{R} (\partial_t u \partial_t \bar{u}) \, dx - \int_{\mathbb{R}^N} \Delta^2 a |u|^2 \, dx - \frac{2(p-2)}{p} \int_{\mathbb{R}^N} \Delta a |x|^p |u|^p (I_\alpha * |.|^p |u|^p) \, dx + \frac{4}{p} \left((\alpha-N) \int_{\mathbb{R}^N} \partial_t a \frac{x_k}{|x|^2} I_\alpha * |.|^p |u|^p |x|^p |u|^p \, dx + b \int_{\mathbb{R}^N} \partial_t ax_k (I_\alpha * |.|^p |u|^p) |x|^{b-2} |u|^p \, dx \right).
\]

This completes the proof.

**Proof of Lemma 4.2**

Denote, for \( D \subset \mathbb{R}^N \), \( S_T(D) := \cap_{(q,r) \in \Gamma} L_T^q \left( L^r(D) \right) \). With Duhamel formula and Strichartz estimates, one gets

\[
(E) := \left\| -e^{-t \Delta} u_0 + u \right\|_{S(0,T)} \lesssim |x|^p |u|^{p-1} \left[ I_\alpha * |.|^p |u|^p \right] \left. \right|_{S_T^t(\{x|<1\})} + \left\| \nabla \left[ |x|^p |u|^{p-2} \left[ I_\alpha * |.|^p |u|^p \right] u \right] \right\|_{S_T^t(\{|x|<1\})}
\]

\[
+ \left\| \left[ |x|^p |u|^{p-1} \left[ I_\alpha * |.|^p |u|^p \right] \right] \left. \right|_{S_T^t(\{|x|>1\})} + \left\| \nabla \left[ |x|^p |u|^{p-2} \left[ I_\alpha * |.|^p |u|^p \right] u \right] \right\|_{S_T^t(\{|x|>1\})}
\]

Take \( \mu_1 := \left( \frac{N}{b} \right)^{-1} \), \( r_1 := \frac{2Np}{\alpha+N-\frac{N}{p}} \) and \((q_1, r_1) \in \Gamma\). Then, 1 + \frac{\alpha}{N} = \frac{2}{\mu_1} + \frac{2p}{r_1} and using Hölder and Hardy-Littlewood-Sobolev inequalities, one gets for \( \theta_1 := q_1 - 2 \),

\[
(A_1) := \left\| \mathcal{N}_1 \right\|_{S_T^t(\{|x|<1\})} \lesssim \left\| \left( I_\alpha * |.|^p |u|^p \right) |x|^p |u|^{p-1} \right\|_{L_T^q \left( L^r(\{|x|<1\}) \right)} \lesssim \left\| |x|^p \right\|_{L^q(\{|x|<1\})} \left\| u \right\|_{L_T^r \left( L^q(\{|x|<1\}) \right)}^2 \lesssim \left\| u \right\|_{L_T^r \left( L^q(\{|x|>1\}) \right)}^2 \left\| u \right\|_{L_T^r \left( L^q(\{|x|>1\}) \right)}^{1+\theta_1}
\]

The condition \( p > p_* \) gives \( (2p-1) q'_1 > q_1 \) and

\[
0 < \theta_1 < 2(p-1).
\]

Moreover, \( p \in (p_*, p^*) \) implies that \( 2 < r_1 < \frac{2N}{N-2} \). Now, compute

\[
(F) := |\nabla \left[ |x|^p |u|^{p-2} \left( I_\alpha * |.|^p |u|^p \right) u \right] | \lesssim \left| |x|^p |u|^{p-2} \left( I_\alpha * |.|^p |u|^p \right) \nabla u \right| + \left| |x|^p |u|^{p-1} \left( I_\alpha * |.|^p |u|^p \right) \nabla u \right| + \left| |x|^p |u|^{p-1} \left( I_\alpha * |.|^p |u|^p \right) \right| + \left| |x|^p |u|^{p-2} \left( I_\alpha * |.|^p |u|^p \right) \right|.
\]

The first and second terms are controlled as previously.

\[
(B_1) := \left\| |x|^p |u|^{p-2} \left( I_\alpha * |.|^p |u|^p \right) \nabla u \right\|_{S_T^t(\{|x|<1\})} + \left\| |x|^p |u|^{p-1} \left( I_\alpha * |.|^p |u|^p \right) \nabla |x|^p |u|^{p-2} \right\|_{S_T^t(\{|x|<1\})} \lesssim \left\| u \right\|_{L_T^r \left( L^q(\{|x|<1\}) \right)}^{2(p-1)-\theta_1} \left\| u \right\|_{L_T^r \left( L^q(\{|x|>1\}) \right)}^{1+\theta_1}.
\]

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Choosing \( \rho := \left( \frac{N}{1 - 2\beta} \right)^{-} \) and \( r_2 := \frac{2Np}{1 + N + \alpha - \frac{2\beta}{\rho}} = \left( \frac{2Np}{N + \alpha + 2\beta} \right)^{+} \), one has

\[
1 + \frac{\alpha}{N} = \frac{1}{r_2} + \frac{1}{\rho} + \frac{2(p - 1)}{r_2} + \frac{1}{r_2} - \frac{1}{N}.
\]

Since \( \frac{1}{\rho} := (\frac{1 - \beta}{N})^{+} + (\frac{-\beta}{N})^{+} := \frac{1}{a_1} + \frac{1}{a_2} \), by Lemma 2.13 and Hölder inequality, one gets

\[
(B_2) := \|x|^{\mu - 1}|u|^{p - 1}(I_\alpha \ast \chi)|_{|x| < 1}|x|^{\mu - 1}|u|^p \|_{L_r^{q'_2}(L^{q'_1}(|x| < 1))}
+ \|x|^{\mu - 1}|u|^{p - 1}(I_\alpha \ast \chi)|_{|x| < 1}|x|^{\mu - 1}|u|^p \|_{L_r^{q'_2}(L^{q'_1}(|x| < 1))}
\leq \| \|x\|^{\mu} \|_{L^{q_1}(|x| < 1)} \| |x|^{\mu - 1}\|_{L^{q_2}(|x| < 1)} \| u \|_{r_2}^{2(p - 1)} \| u \|_{N_{r_2} - r_2}^{q'_2(0, T)}
\leq \| u \|_{2(p - 1) - \theta}^{2(p - 1) - \theta} \| u \|_{L_r^{q'_2}(W^{1, r_2})},
\]

where \((1 + \theta_2)q'_2 = q_2\) and \((q_2, r_2) \in \Gamma\). The condition \( p \in (p_*, p^*)\) gives \(2 < q_2 < 2p\) which is equivalent to \(0 < \theta_2 < 2(p - 1)\).

The estimation of the terms in the complementary of the unit ball follow similarly. This finishes the proof.

**Proof of Lemma 4.3**

Let us start with proving the estimate

\[
\| \mathcal{N} \|_{\mathcal{S}'(I)} \lesssim \| u \|^\theta_{L_r^{\infty}(I, H^1)} \| u \|^{2(p - 1) - \theta}_{S^{(1)}(I)} \| u \|_{S^{(1)}(I)}.
\]

Take \(0 < \theta << 1\) and the pairs

\[
(q, r) := \left( \frac{4(p - 1)(2p - \theta)}{(2p - 1)B - \theta(B - 2)}, \frac{2N(p - 1)(2p - \theta)}{(2p - 1)(Np - B) + \theta(B - 2 - N(p - 1))} \right) \in \Gamma;
\]

\[
(a, r) := \left( \frac{2(p - 1)(2p - \theta)}{2p - B}, r \right) \in \Gamma_s;
\]

\[
(d, r) := \left( \frac{2(p - 1)(2p - \theta)}{2p(B - 1) - B + \theta(B - 2)}, r \right) \in \Gamma_{-s}.
\]

Take two real numbers \(\mu > 0\) and \(r_1 := \frac{2N}{N - 2}\) such that

\[
1 + \frac{\alpha}{N} = \frac{2p - \theta}{\mu} + \frac{\theta}{r_1}.
\]

By Lemma 2.13 and Hölder estimate, one obtains

\[
\| \mathcal{N}_1 \|_{L_r^{s'(|x| < 1)}} \leq \| |x|^{\mu} \|^{2}_{L_r^{\infty}(|x| < 1)} \| u \|_{r_1}^{\theta} \| u \|_{r}^{2p - 1 - \theta}.
\]

Then, \( p < p^* \) gives

\[
\frac{N}{\mu} + b = \frac{\theta}{2} \left( \frac{2 + b}{2(p - 1)} - \frac{N}{r_1} \right) = \frac{\theta}{2} \left( \frac{2 + \alpha + 2b}{p - 1} - N + 2 \right) > 0.
\]

Thus, one gets

\[
\| \mathcal{N}_1 \|_{L_r^{s'(|x| < 1)}} \lesssim \| u \|^{\theta}_{r_1} \| u \|_{r}^{2p - 1 - \theta} \lesssim \| u \|^{\theta}_{H^1} \| u \|_{r}^{2p - 1 - \theta}.
\]
So, taking account of the equality $\frac{1}{q'} = \frac{2(p-1) - \theta}{a} + \frac{1}{q}$, one gets
\[
\| N \|_{L^p'(I, L^p'(|x| < 1))} \leq \| u \|_{L^\infty(I, H^1)}^{\theta} \| u \|_{L^p(I, L^p')}^{2p - 2 - \theta} \| u \|_{L^p(I, L')}^{\theta} \leq \| u \|_{L^\infty(I, H^1)}^{\theta} \| u \|_{S(I)}^{2p - 2 - \theta} \| u \|_{S(I)}.
\]
The estimate on the complementary of the ball follows similarly. This proves the first point. Now, one estimates
\[
\| \nabla N \|_{S'(I)} \lesssim \| x |b| |u|^{p-1} (I_\alpha \ast | \cdot | b| |u|^{p-1} \nabla u) \|_{S(I)} + \| x |b| |u|^{p-1} (I_\alpha \ast | \cdot | b| |u|^{p} \nabla u) \|_{S'(I)} + \| x |b| |u|^{p-2} (I_\alpha \ast | \cdot | b| |u|^{p}) \nabla u \|_{S(I)} + \| x |b-1| |u|^{p-1} (I_\alpha \ast | \cdot | b| |u|^{p}) \nabla u \|_{S'(I)}
\]
\[
:= (I) + (II) + (III) + (IV).
\]
Using the above calculus, one has
\[
(I) + (III) \lesssim \| u \|_{L^\infty(I, H^1)}^{\theta} \| u \|_{S'(I)}^{2p - 2 - \theta} \| \nabla u \|_{S(I)}.
\]
The rest of the proof discusses two cases. First, let us take $N = 3$ and estimate the term
\[
(II_1) \lesssim \| x |b| |u|^{p-1} (I_\alpha \ast | x|_{|x| < 1} | \cdot | b-1| |u|^{p}) \|_{L^2(I, L^5 (|x| < 1))}
\]
\[
\lesssim \| x |b| |u|^{p-1} \|_{L^1(|x| < 1)} \| x |b| |u|^{p} \|_{L^2(|x| < 1)} \| u \|_{r_1}^{\theta} \| u \|_{r_2}^{(2p - 2) - \theta} \| u \|_{r_3}^{1 - \gamma} \| L^2(I)
\]
\[
\lesssim \| x |b| |u|^{p-1} \|_{L^1(|x| < 1)} \| x |b| |u|^{p} \|_{L^2(|x| < 1)} \| u \|_{r_1}^{\theta} \| \nabla u \|_{n}^{2(p-1)-\theta} \| \nabla u \|_{H^1}^{1-\gamma} \| L^2(I).
\]
Here,
\[
\frac{5}{6} + \frac{\alpha}{3} = \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{\theta}{r_1} + \frac{2p - 2 - \theta}{r_2} + \frac{\gamma}{r_3} + \frac{1 - \gamma}{6}
\]
\[
:= \frac{1}{\mu} + \frac{\theta}{r_1} + (2p - 2 - \theta + \gamma)(\frac{1}{n} - \frac{\gamma}{3}) + \frac{1 - \gamma}{6}.
\]
Choose
\[
-2b < \gamma < 1, \quad 0 < \epsilon < 2b + \gamma;
\]
\[
\theta := \frac{(2 - \epsilon + \gamma + 4b)(p - 1)}{2 + 2b};
\]
\[
\frac{6\theta}{r_1} := \frac{(2 + 2b + \alpha)\theta}{p - 1} + 2p - 4 - \theta + \gamma - \epsilon;
\]
\[
\frac{1}{n} := \frac{1}{3} + \frac{6(2 - 2 - \theta + \gamma)}{2 + 2b}.
\]
This gives $0 < \theta < 2(p - 1)$. The conditions $|x|^{b-1} \in L^{\mu_1}$ and $|x|^{b} \in L^{\mu_2}$ are satisfied if $\frac{N}{\mu} + 2b - 1 > 0$. Compute
\[
\frac{N}{\mu} + 2b - 1 = \frac{3}{\mu} + 2b - 1
\]
\[
= 2b - 1 + \alpha + \frac{5}{2} - 3\left(\frac{\theta}{r_1} + (2p - 2 - \theta + \gamma)(\frac{1}{n} - \frac{\gamma}{3}) + \frac{1 - \gamma}{6}\right)
\]
\[
= 2b + 1 + \alpha - \frac{3\theta}{r_1} - (2p - 2 - \theta + \gamma)(\frac{3}{n} - 1) + \frac{\gamma}{2}.
\]
Thus,
\[
\frac{N}{\mu} + 2b - 1 = 2b + 1 + \alpha - \frac{1}{2} \left(\frac{(2 + 2b + \alpha)\theta}{p - 1} + 2p - 4 - \theta + \gamma - \epsilon\right) \\
- (2p - 2 - \theta + \gamma)\frac{\theta}{2(2p - 2 - \theta + \gamma)} + \frac{\gamma}{2} \\
= 2b + 1 + \alpha - \frac{1}{2} \left(\frac{(2 + 2b + \alpha)\theta}{p - 1} + 2p - 4 - \theta\right) \\
= 2b + 2 + \alpha - (p - 1) + \frac{\theta}{2} - \frac{(2 + 2b + \alpha)\theta}{2(p - 1)} \\
= 2b + 2 + \alpha + 2(p - 1) - \theta - 3(p - 1) + \theta \left(3 - \frac{2 + 2b + \alpha}{2(p - 1)}\right) \\
= 2(p - 1) - \theta - 2(p - 1) \left(\frac{3}{2} - \frac{2 + 2b + \alpha}{2(p - 1)}\right) + \theta \left(\frac{3}{2} - \frac{2 + 2b + \alpha}{2(p - 1)}\right) \\
= (1 - s_c) \left(2(p - 1) - \theta\right) > 0.
\]
This gives
\[
(II_I) \lesssim \|u\|_{r_1}^\theta \|\nabla u\|_n^{2(p - 1) - \theta + \gamma} \|u\|_{H^\gamma}^{1 - \gamma} \|u\|_{L^2(I)}.
\]
Now, let the real numbers
\[
m := \frac{4(2p - 2 - \theta + \gamma)}{2p - 2 - \theta + \gamma - \epsilon} \quad \text{and} \quad q_1 := \frac{4\theta}{2 + \epsilon - (2p - 2 - \theta + \gamma)}.
\]
Thus,
\[
\left(m, n\right) \in \Gamma, \quad (q_1, r_1) \in \Gamma_{s_c}; \\
\frac{1}{2} = \frac{\theta}{q_1} + \frac{2p - 2 - \theta + \gamma}{m}.
\]
Thus, by Hölder estimate, one gets
\[
(II_I) \lesssim \|u\|_{r_1}^\theta \|\nabla u\|_n^{2(p - 1) - \theta + \gamma} \|u\|_{H^\gamma}^{1 - \gamma} \|u\|_{L^2(I)} \\
\lesssim \|u\|_{L^2(I), L^{r_1}(I)}^\theta \|\nabla u\|_n^{2(p - 1) - \theta + \gamma} \|u\|_{L^\infty(I), H^{\gamma}}^{1 - \gamma} \\
\lesssim \|u\|_{L^\infty(I), H^{\gamma}}^{1 - \gamma} \|u\|_{S^{\gamma}(I)}^\theta \|\nabla u\|_S^{2(p - 1) - \theta + \gamma}.
\]
The estimate on the complementary of the unit ball and the term \(IV\) follow similarly.
Let us treat the case \(N > 3\) and estimate the term \((II_1)\). Take the admissible pairs
\[
(q, r) := \left(\frac{2(2p - \theta)}{s_c(2p - \theta) + 2(1 - s_c)}, \frac{2N(2p - \theta)}{(N - 2s_c)(2p - \theta) - 4(1 - s_c)}\right) \in \Gamma; \\
(a, r) := \left(\frac{2p - \theta}{1 - s_c}, r\right) \in \Gamma_{s_c}.
\]
By Lemma 2.13 and Hölder estimate, one obtains
\[
(II_1) \lesssim \|x|^b |u|^{p - 1} (I_{t + \phi} \ast \chi)_{|x| < 1} : \|u\|_{L^p(I_{t + \phi} \ast \chi)_{|x| < 1}} \\
\lesssim \||x|^b |u|^{p - 1} \|\chi\|_{L^p_{|x|<1}} \||x|^b \|_{L^{p, 2}_{|x|<1}} \||u\|_{L^2_{|x|<1}}^{2 - \theta} \|u\|_{\frac{p, \gamma}{p - \theta}} \|\nabla u\|_{L^{p'}(I)} \\
\lesssim \||x|^b |u|^{p - 1} \|\chi\|_{L^p_{|x|<1}} \||x|^b \|_{L^{p, 2}_{|x|<1}} \||u\|_{L^2_{|x|<1}}^{\theta} \|u\|_{L^r_{|x|<1}}^{2 - \theta} \|\nabla u\|_{r} \|\nabla u\|_{L^{p'}(I)}.
\]
Here,
\[
1 + \frac{\alpha}{N} = \frac{1}{r} + \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{\theta}{2^p} + \frac{2p - 2 - \theta}{r} + \frac{1}{r} - \frac{1}{N}
\]
\[
:= \frac{1}{\mu} + \frac{\theta}{2^p} + \frac{2p - \theta}{r} - \frac{1}{N}.
\]

Compute
\[
\frac{N}{\mu} + 2b - 1 = \alpha + N + 2b - \theta \left( \frac{N}{2} - 1 \right) - \frac{N(2p - \theta)}{r}
\]
\[
= \alpha + N + 2b - \theta \left( \frac{N}{2} - 1 \right) - \frac{(N - 2s_c)(2p - \theta) - 4(1 - s_c)}{2}
\]
\[
= \alpha + N + 2b - \theta (1 - s_c) - (p - 1)(N - 2s_c) - (N - 2s_c) + 2(1 - s_c)
\]
\[
= \alpha + N + 2b - \theta (1 - s_c) - (2 + \alpha + 2b) + 2 - N
\]
\[
= \theta (1 - s_c) > 0.
\]

This implies, via the equality \( \frac{1}{q'} = \frac{1}{q} + \frac{2p - 2 - \theta}{a} \), that
\[
(II_1) \lesssim \|u\|_{L^\infty(I, H^1)}^{q'} \|\nabla u\|_{L^q(I)}^{q'}
\]
\[
\lesssim \|u\|_{L^\infty(I, H^1)}^{q'} \|\nabla u\|_{L^q(I, L^q)}^{q'}
\]
\[
\lesssim \|u\|_{L^\infty(I, H^1)}^{q'} \|\nabla u\|_{L^q(I, L^q)}^{q'}.
\]

This ends the proof.

Data Availability  The data sets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

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