Radius of $\gamma$-Spirallikeness of order $\alpha$ for some Special functions

Sercan Kazımoğlu · Kamaljeet Gangania

Received: date / Accepted: date

Abstract In this paper, we establish the radius of $\gamma$-Spirallike of order $\alpha$ of certain well-known special functions. The main results of the paper are new and natural extensions of some known results.

Keywords $\gamma$-Spirallike functions · Radii of starlikeness and convexity · Wright and Mittag-Leffler functions · Legendre polynomials · Lommel and Struve functions · Ramanujan type entire functions

Mathematics Subject Classification (2010) 30C45 · 30C80 · 30C15

1 Introduction

Let $\mathcal{A}$ be the class of analytic functions normalized by the condition $f(0) = 0 = f'(0) - 1$ in the unit disk $\mathbb{D} := \mathbb{D}_1$, where $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$. We say that a function $f \in \mathcal{A}$ is $\gamma$-Spirallike of order $\alpha$ if and only if

$$\text{Re} \left( e^{-i\gamma} \frac{zf'(z)}{f(z)} \right) > \alpha \cos \gamma,$$

where $\gamma \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right)$ and $0 \leq \alpha < 1$. We denote the class of such functions by $\mathcal{S}_\gamma^\alpha$. We also denote its convex analog, that is the class $\mathcal{C}\mathcal{S}_\gamma^\alpha$ of convex $\gamma$-spirallike functions of order $\alpha$, which is defined below

$$\text{Re} \left( e^{-i\gamma} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right) > \alpha \cos \gamma.$$
The class $S_\gamma^p(0)$ was introduced by Spacek [24]. Each function in $CS_\gamma^p(\alpha)$ is univalent in $D$, but they do not necessarily be starlike. Further, it is worth to mention that for general values of $\gamma(|\gamma| < \pi/2)$, a function in $CS_\gamma^p(0)$ need not be univalent in $D$. For example: $f(z) = i(1 - z)i - i \in CS_{\pi/4}^p(0)$, but not univalent. Indeed, $f \in CS_\gamma^p(0)$ is univalent if $0 < \cos \gamma < 1/2$, see Robertson [23] and Pfaltzgraff [20]. Note that for $\gamma = 0$, the classes $S_\gamma^p(\alpha)$ and $CS_\gamma^p(\alpha)$ reduce to the classes of starlike and convex functions of order $\alpha$, given by

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad \text{and} \quad \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha,$$

which we denote by $S^*(\alpha)$ and $C(\alpha)$, respectively.

In the recent past, connections between the special functions and their geometrical properties have been established in terms of radius problems [1, 2, 3, 5, 6, 7, 8, 9, 10, 25]. In this direction, behavior of the positive roots of a special function and the Laguerre-Pólya class play an evident role. A real entire function $L$ maps real line into itself is said to be in the Laguerre-Pólya class $LP$, if it can be expressed as follows:

$$L(x) = cx^m e^{-ax^2 + \beta x} \prod_{k \geq 1} \left( 1 + \frac{x}{x_k} \right) e^{-\frac{\beta x}{x_k}},$$

where $c, \beta, x_k \in \mathbb{R}$, $a \geq 0$, $m \in \mathbb{N} \cup \{0\}$ and $\sum x_k^{-2} < \infty$, see [2], [12, p. 703], [18] and the references therein. The class $LP$ consists of entire functions which can be approximated by polynomials with only real zeros, uniformly on the compact sets of the complex plane and it is closed under differentiation.

The $S^*(\alpha)$-radius, which is given below

$$\sup \{ r \in \mathbb{R}^+ : \text{Re} \left( \frac{zg'(z)}{g(z)} \right) > \alpha, z \in D_r \}$$

and similarly, $C(\alpha)$-radius has recently been obtained for some normalized forms of Bessel functions [1, 3, 6] (see Watson’s treatise [26] for more on Bessel function), Struve functions [1, 2], Wright functions [7], Lommel functions [1, 2], Legendre polynomials of odd degree [9] and Ramanujan type entire functions [10]. For their generalization to Ma-Minda classes [19] of starlike and convex functions, we refer to see [13, 16].

With the best of our knowledge, $S_\gamma^p(\alpha)$-radius and $CS_\gamma^p(\alpha)$-radius for special functions are not handled till date. Therefore, in this paper, we now aim to derive the radius of $\gamma$-Spirallike of order $\alpha$, which is given below

$$R_{sp}(g) = \sup \left\{ r \in \mathbb{R}^+ : \text{Re} \left( e^{-i\gamma} \frac{zg'(z)}{g(z)} \right) > \alpha \cos \gamma, z \in D_r \right\}$$

and also the radius of convex $\gamma$-Spirallike of order $\alpha$, which is

$$R_{csp}(g) = \sup \left\{ r \in \mathbb{R}^+ : \text{Re} \left( e^{-i\gamma} \left( 1 + \frac{zg'(z)}{g(z)} \right) \right) > \alpha \cos \gamma, z \in D_r \right\}.$$
2 Wright functions

Let us consider the generalized Bessel function given by

\[ \Phi(\kappa, \delta, z) = \sum_{n \geq 0} \frac{z^n}{n! \Gamma(n\kappa + \delta)} \]

where \( \kappa > -1 \) and \( z, \delta \in \mathbb{C} \), named after E. M. Wright. The function \( \Phi \) is entire for \( \kappa > -1 \). From [7, Lemma 1, p. 100], we have the Hadamard factorization

\[ \Gamma(\delta)\Phi(\kappa, \delta, -z^2) = \prod_{n \geq 1} \left( 1 - \frac{z^2}{\zeta_{\kappa,\delta,n}^2} \right), \quad (2.1) \]

where \( \kappa, \delta > 0 \) and \( \zeta_{\kappa,\delta,n} \) is the \( n \)-th positive root of \( \Phi(\kappa, \delta, -z^2) \) and satisfies the interlacing property:

\[ \tilde{\zeta}_{\kappa,\delta,n} < \zeta_{\kappa,\delta,n} < \tilde{\zeta}_{\kappa,\delta,n+1} < \zeta_{\kappa,\delta,n+1}, \quad (n \geq 1) \quad (2.2) \]

where \( \tilde{\zeta}_{\kappa,\delta,n} \) is the \( n \)-th positive root of the derivative of the function

\[ \Psi_{\kappa,\delta}(z) = z^\delta \Phi(\kappa, \delta, -z^2). \]

Since \( \Phi(\kappa, \delta, -z^2) \not\in A \), therefore we choose the normalized Wright functions:

\[ \begin{cases} f_{\kappa,\delta}(z) = [z^\delta \Gamma(\delta)\Phi(\kappa, \delta, -z^2)]^{1/\delta} \\ g_{\kappa,\delta}(z) = z^\delta \Gamma(\delta)\Phi(\kappa, \delta, -z^2) \\ h_{\kappa,\delta}(z) = z^\delta \Gamma(\delta)\Phi(\kappa, \delta, -z). \end{cases} \quad (2.3) \]

For brevity, we write \( W_{\kappa,\delta}(z) := \Phi(\kappa, \delta, -z^2) \).

Theorem 1 Let \( \kappa, \delta > 0 \). The radius of \( \gamma \)-Spirallikeness for the functions \( f_{\kappa,\delta}, g_{\kappa,\delta} \) and \( h_{\kappa,\delta} \) are the smallest positive roots of the following equations:

(i) \( r W_{\kappa,\delta}'(r) + \delta (1 - \alpha) \cos \gamma W_{\kappa,\delta}(r) = 0 \)

(ii) \( r W_{\kappa,\delta}'(r) + (1 - \alpha) \cos \gamma W_{\kappa,\delta}(r) = 0 \)

(iii) \( \sqrt{\gamma} W_{\kappa,\delta}(\sqrt{r}) + 2 (1 - \alpha) \cos \gamma W_{\kappa,\delta}(\sqrt{r}) = 0 \)

in \( |z| < (0, \zeta_{\kappa,\delta,1}), (0, \zeta_{\kappa,\delta,1}) \) and \((0, \tilde{\zeta}_{\kappa,\delta,1}) \), respectively.

Proof Using (2.1), we obtain the following by the logarithmic differentiation of (2.3):

\[ \begin{aligned} \frac{zf_{\kappa,\delta}'(z)}{f_{\kappa,\delta}(z)} &= 1 + \frac{1}{\delta} \frac{z^{2\delta}}{W_{\kappa,\delta}(z)} = 1 - \frac{1}{\delta} \sum_{n \geq 1} \frac{2z^{2\delta}}{\zeta_{\kappa,\delta,n}^{2\delta}} \\ \frac{zg_{\kappa,\delta}'(z)}{g_{\kappa,\delta}(z)} &= 1 + \frac{2\delta}{W_{\kappa,\delta}(z)} = 1 - \sum_{n \geq 1} \frac{z^{2\delta}}{\zeta_{\kappa,\delta,n}^{2\delta}} \\ \frac{zh_{\kappa,\delta}'(z)}{h_{\kappa,\delta}(z)} &= 1 + \frac{z\gamma}{W_{\kappa,\delta}(\sqrt{z})} = 1 - \sum_{n \geq 1} \frac{z^{1 + \gamma}}{\zeta_{\kappa,\delta,n}^{1 + \gamma}} \end{aligned} \quad (2.4) \]

We need to show that the following inequalities for \( \alpha \in [0, 1) \) and \( \gamma \in (-\frac{1}{2}, \frac{1}{2}) \),

\[ \text{Re} \left( e^{-i\gamma} \frac{zf_{\kappa,\delta}'(z)}{f_{\kappa,\delta}(z)} \right) > \alpha \cos \gamma, \quad \text{Re} \left( e^{-i\gamma} \frac{zg_{\kappa,\delta}'(z)}{g_{\kappa,\delta}(z)} \right) > \alpha \cos \gamma \]

(2.5)
and
\[ \text{Re} \left( e^{-i\gamma} \frac{zh'_{\kappa,\delta}(z)}{h_{\kappa,\delta}(z)} \right) > \alpha \cos \gamma \]
are valid for \( z \in \mathbb{D}_{r_{\kappa,\delta}} \), \( z \in \mathbb{D}_{r_{\kappa,\delta}} \), and \( z \in \mathbb{D}_{r_{\kappa,\delta}} \) respectively, and each of the above inequalities does not hold in larger disks. It is known [11] that if \( z \in \mathbb{C} \) and \( \lambda \in \mathbb{R} \) are such that \( |z| \leq r<\lambda \), then
\[ \text{Re} \left( \frac{z}{\lambda - z} \right) \leq \frac{|z|}{|\lambda - |z||} \leq |z| \lambda. \]  
(2.6)

Then the inequality
\[ \text{Re} \left( \frac{z^2}{\zeta_{\kappa,\delta,n} - z^2} \right) \leq \frac{|z|^2}{\zeta_{\kappa,\delta,n} - |z|^2} \]
holds for every \( |z| < \zeta_{\kappa,\delta,1} \). Therefore, from (2.4) and (2.6), we have
\[ \text{Re} \left( e^{-i\gamma} \frac{zf'_{\kappa,\delta}(z)}{f_{\kappa,\delta}(z)} \right) = \text{Re} \left( e^{-i\gamma} \sum_{n \geq 1} \frac{2z^2}{\zeta_{\kappa,\delta,n} - z^2} \right) \]
\[ \geq \cos \gamma - \frac{1}{\delta} \sum_{n \geq 1} \frac{2z^2}{\zeta_{\kappa,\delta,n} - z^2} \geq \cos \gamma - \frac{1}{\delta} \sum_{n \geq 1} \frac{|z|^2}{\zeta_{\kappa,\delta,n} - |z|^2} \]
\[ = \frac{|z|^2 f_{\kappa,\delta}(|z|)}{f_{\kappa,\delta}(|z|)} + \cos \gamma - 1. \]  
(2.7)

Equality in each of the above inequalities (2.9) holds when \( z = r \). Thus, for \( r \in (0, \zeta_{\kappa,\delta,1}) \) it follows that
\[ \inf_{z \in \mathbb{B}_r} \left\{ \text{Re} \left( e^{-i\gamma} \frac{zf'_{\kappa,\delta}(z)}{f_{\kappa,\delta}(z)} - \alpha \cos \gamma \right) \right\} = \frac{|z|^2 f_{\kappa,\delta}(|z|)}{f_{\kappa,\delta}(|z|)} + (1 - \alpha) \cos \gamma - 1. \]

Now, the mapping \( \Theta : (0, \zeta_{\kappa,\delta,1}) \longrightarrow \mathbb{R} \) defined by
\[ \Theta(r) = \frac{rf'_{\kappa,\delta}(r)}{f_{\kappa,\delta}(r)} + (1 - \alpha) \cos \gamma - 1 = (1 - \alpha) \cos \gamma - 1 - \frac{1}{\delta} \sum_{n \geq 1} \frac{2r^2}{\zeta_{\kappa,\delta,n} - r^2} \]
is strictly decreasing since
\[ \Theta'(r) = -\frac{1}{\delta} \sum_{n \geq 1} \left( \frac{4r \zeta_{\kappa,\delta,n}}{\zeta_{\kappa,\delta,n}^2 - r^2} \right)^2 < 0 \]
for all \( \delta > 0 \). On the other hand, since
\[ \lim_{r \searrow 0} \Theta(r) = (1 - \alpha) \cos \gamma > 0 \quad \text{and} \quad \lim_{r \nearrow \zeta_{\kappa,\delta,1}} \Theta(r) = -\infty, \]
Radius of \( \gamma \)-Spirallikeness of order \( \alpha \) for some Special functions

in view of the minimum principle for harmonic functions imply that the corre-
spending inequality for \( f_{\kappa,\delta} \) in (2) for \( \delta > 0 \) holds if and only if \( z \in D_{rsp}(f_{\kappa,\delta}) \),
where \( r_{sp}(f_{\kappa,\delta}) \) is the smallest positive root of equation
\[
\frac{r f''_{\kappa,\delta}(r)}{f'_{\kappa,\delta}(r)} = 1 - (1 - \alpha) \cos \gamma
\]
which is equivalent to
\[
\frac{1}{\delta} \frac{W'_\kappa(z)}{W_\kappa(z)} = -(1 - \alpha) \cos \gamma,
\]
situated in \((0, \zeta_{\kappa,\delta,1})\). Reasoning along the same lines, proofs of the other parts
follows. \(\square\)

Remark 1 Taking \( \gamma = 0 \) in Theorem 1 yields \([7, \text{Theorem 1}]\).

In the following, we deal with convex analogue of the class of \( \gamma \)-spirallike
functions of order \( \alpha \).

Theorem 2 Let \( \kappa, \delta > 0 \) and the functions \( f_{\kappa,\delta}, g_{\kappa,\delta} \) and \( h_{\kappa,\delta} \) as given in
(2.3). Then
(i) the radius \( R_{sp}^c(f_{\kappa,\delta}) \) is the smallest positive root of the equation
\[
\frac{r \Psi''_{\kappa,\delta}(r)}{\Psi'_{\kappa,\delta}(r)} \left(\frac{1}{\delta} - 1\right) \frac{r \Psi'_{\kappa,\delta}(r)}{\Psi_{\kappa,\delta}(r)} + (1 - \alpha) \cos \gamma = 0.
\]
(ii) the radius \( R_{sp}^c(g_{\kappa,\delta}) \) is the smallest positive root of the equation
\[
r g''_{\kappa,\delta}(r) + (1 - \alpha) \cos \gamma g'_{\kappa,\delta}(r) = 0.
\]
(iii) the radius \( R_{sp}^c(h_{\kappa,\delta}) \) is the smallest positive root of the equation
\[
r h''_{\kappa,\delta}(r) + (1 - \alpha) \cos \gamma h'_{\kappa,\delta}(r) = 0.
\]

Proof We first prove the part (i). From (2.1), (2.3) and using the Hadamard
representation \( \Gamma(\delta)^{-1} \sum_{n \geq 1} \left(1 - \frac{z^2}{\zeta_{\kappa,\delta,n}^2}\right) \), (see \([7, \text{Eq. 7}]\)), we have
\[
1 + \frac{z f''_{\kappa,\delta}(z)}{f'_{\kappa,\delta}(z)} = 1 + \frac{z \Psi''_{\kappa,\delta}(z)}{\Psi'_{\kappa,\delta}(z)} + \left(\frac{1}{\delta} - 1\right) \frac{z \Psi'_{\kappa,\delta}(z)}{\Psi_{\kappa,\delta}(z)}
= 1 - \sum_{n \geq 1} \frac{2 z^2}{\zeta_{\kappa,\delta,n}^2} - \frac{1}{\delta} \sum_{n \geq 1} \frac{2 z^2}{\zeta_{\kappa,\delta,n}^2}
\]
and for \( \delta > 1 \), using the following inequality of \([11]\):
\[
\left|\frac{z}{y - z} - \lambda \frac{z}{x - z}\right| \leq \frac{|z|}{y - |z|} - \lambda \frac{|z|}{x - |z|}, \quad (x > y > r \geq |z|)
\]
(2.8)
with $\lambda = 1 - 1/\delta$, we get
\[
\left| \frac{zf_{k,\delta}'(z)}{f_{k,\delta}(z)} \right| \leq \frac{rf_{k,\delta}'(r)}{\Psi_{k,\delta}(r)} = -\frac{r\Psi_{k,\delta}'(r)}{\Psi_{k,\delta}(r)} - \left( 1 - \frac{1}{\delta} \right) \frac{r\Psi_{k,\delta}'(r)}{\Psi_{k,\delta}(r)}.
\]
Also, using the inequality $\|x - y\| \leq |x - y|$ and the relation in (2.2), we see that for $\delta > 0$
\[
\left| \frac{zf_{k,\delta}''(z)}{f_{k,\delta}'(z)} \right| \leq -\frac{rf_{k,\delta}''(r)}{f_{k,\delta}'(r)},
\]
holds in $|z| = r < \zeta_{\kappa,\delta,1}$. Therefore, we have
\[
\Re \left( e^{-i\gamma} \left( 1 + \frac{zf_{k,\delta}''(z)}{f_{k,\delta}'(z)} \right) \right)
= \Re(e^{-i\gamma}) - \Re \left( e^{-i\gamma} \left( \sum_{n \geq 1} \frac{2z^2}{\zeta_{k,\delta,n}^2 - z^2} + \left( 1 - \frac{1}{\delta} \right) \sum_{n \geq 1} \frac{2z^2}{\zeta_{k,\delta,n}^2 - z^2} \right) \right)
\geq \cos \gamma - \sum_{n \geq 1} \frac{2z^2}{\zeta_{k,\delta,n}^2 - z^2} + \left( 1 - \frac{1}{\delta} \right) \sum_{n \geq 1} \frac{2z^2}{\zeta_{k,\delta,n}^2 - z^2}
\geq \cos \gamma + \frac{rf_{k,\delta}''(r)}{f_{k,\delta}'(r)} \tag{2.9}
\]
hold for $\delta > 1$. Observe that these inequalities also hold for $\delta > 0$. Equality in the each of the above inequalities (2.9) holds when $z = r$. Thus, for $r \in (0, \zeta_{\kappa,\delta,1})$ it follows that
\[
\inf_{z \in D^r} \left\{ \Re \left( e^{-i\gamma} \left( 1 + \frac{zf_{k,\delta}''(z)}{f_{k,\delta}'(z)} \right) - \alpha \cos \gamma \right) \right\} = (1 - \alpha) \cos \gamma + \frac{|z|f_{k,\delta}''(|z|)}{f_{k,\delta}'(|z|)}.
\]
Now, the proof of part (i) follows on similar lines as of Theorem 1.

For the other parts, note that the functions $g_{\kappa,\delta}$ and $h_{\kappa,\delta}$ belong to the Laguerre-Pólya class $\mathcal{LP}$, which is closed under differentiation, their derivatives $g_{\kappa,\delta}'$ and $h_{\kappa,\delta}'$ also belong to $\mathcal{LP}$ and the zeros are real. Thus assuming $\tau_{\kappa,\delta,n}$ and $\eta_{\kappa,\delta,n}$ are the positive zeros of $g_{\kappa,\delta}'$ and $h_{\kappa,\delta}'$, respectively, we have the following representations:
\[
g_{\kappa,\delta}'(z) = \prod_{n \geq 1} \left( 1 - \frac{z^2}{\tau_{\kappa,\delta,n}^2} \right) \quad \text{and} \quad h_{\kappa,\delta}'(z) = \prod_{n \geq 1} \left( 1 - \frac{z}{\eta_{\kappa,\delta,n}} \right),
\]
which yield
\[
1 + \frac{zf_{k,\delta}''(z)}{g_{k,\delta}'(z)} = 1 - \sum_{n \geq 1} \frac{2z^2}{\tau_{k,\delta,n}^2 - z^2} \quad \text{and} \quad 1 + \frac{zf_{k,\delta}''(z)}{h_{k,\delta}'(z)} = 1 - \sum_{n \geq 1} \frac{z}{\eta_{k,\delta,n} - z}.
\]
Further, reasoning along the same lines as in Theorem 1, the result follows at once. \[ \square \]

**Remark 2** Taking $\gamma = 0$ in Theorem 2 yields [7, Theorem 5].
3 Mittag-Leffler functions

In 1971, Prabhakar [21] introduced the following function

\[ M(\mu, \nu, a, z) := \sum_{n \geq 0} \frac{(a)_n z^n}{n! \Gamma(\mu n + \nu)}, \]

where \((a)_n = \frac{\Gamma(a + n)}{\Gamma(a)}\) denotes the Pochhammer symbol and \(\mu, \nu, a > 0\).

The functions \(M(\mu, 1, 1, z)\) and \(M(1, 1, 1, z)\) were introduced and studied by Wiman and Mittag-Leffler, respectively. Now let us consider the set \(W_b = A(W_c) \cup B(W_c)\), where

\[ W_c := \left\{ \left( \frac{1}{\mu}, \nu \right) : 1 < \mu < 2, \nu \in [\mu - 1, 1] \cup [\mu, 2] \right\} \]

and denote by \(W_i\), the smallest set containing \(W_b\) and invariant under the transformations \(A, B\) and \(C\) mapping the set \(\left\{ \left( \frac{1}{\mu}, \nu \right) : \mu > 1, \nu > 0 \right\}\) into itself and are defined as:

\[ A : \left( \frac{1}{\mu}, \nu \right) \rightarrow \left( \frac{1}{2\mu}, \nu \right), \quad B : \left( \frac{1}{\mu}, \nu \right) \rightarrow \left( \frac{1}{2\mu}, \mu + \nu \right), \]

\[ C : \left( \frac{1}{\mu}, \nu \right) \rightarrow \begin{cases} \left( \frac{1}{\mu}, \nu - 1 \right), & \text{if } \nu > 1; \\ \left( \frac{1}{\mu}, \nu \right), & \text{if } 0 < \nu \leq 1. \end{cases} \]

Kumar and Pathan [17] proved that if \(\left( \frac{1}{\mu}, \nu \right) \in W_i\) and \(a > 0\), then all zeros of \(M(\mu, \nu, a, z)\) are real and negative. From [5, Lemma 1, p. 121], we see that if \(\left( \frac{1}{\mu}, \nu \right) \in W_i\) and \(a > 0\), then the function \(M(\mu, \nu, a, -z^2)\) has infinitely many zeros, which are all real and have the following representation:

\[ \Gamma(\nu) M(\mu, \nu, a, -z^2) = \prod_{n \geq 1} \left( 1 - \frac{z^2}{\lambda_{\mu, \nu, a, n}} \right), \]

where \(\lambda_{\mu, \nu, a, n}\) is the \(n\)-th positive zero of \(M(\mu, \nu, a, -z^2)\) and satisfy the interlacing relation

\[ \xi_{\mu, \nu, a, n} < \lambda_{\mu, \nu, a, n} < \xi_{\mu, \nu, a, n+1} < \lambda_{\mu, \nu, a, n+1} \quad (n \geq 1), \]

where \(\xi_{\mu, \nu, a, n}\) is the \(n\)-th positive zero of the derivative of \(z^\nu M(\mu, \nu, a, -z^2)\). Since \(M(\mu, \nu, a, -z^2) \notin \mathcal{A}\), therefore we consider the following normalized forms (belong to the Laguerre-Pólya class):

\[
\begin{align*}
L_{\mu, \nu, a}(z) &= \left[ z^\nu \Gamma(\nu) M(\mu, \nu, a, -z^2) \right]^{1/\nu}, \\
g_{\mu, \nu, a}(z) &= z \Gamma(\nu) M(\mu, \nu, a, -z^2), \\
h_{\mu, \nu, a}(z) &= z \Gamma(\nu) M(\mu, \nu, a, -z).
\end{align*}
\]

(3.1)

For brevity, write \(L_{\mu, \nu, a}(z) := M(\mu, \nu, a, -z^2)\). Now proceeding similarly as in Section 2, we obtain the following results:
**Theorem 3** Let \((\frac{1}{n}, \nu) \in W_i, a > 0.\) Then the radius of \(\gamma\)-Spiralliteness of order \(\alpha\) for the functions \(f_{\mu,\nu,a}, g_{\mu,\nu,a}\) and \(h_{\mu,\nu,a}\) given by (3.1) are the smallest positive roots of the following equations:

(i) \(rL_1'(r) + \delta (1 - \alpha) \cos \gamma L_1(a(r) = 0\)

(ii) \(rL_2'(r) + (1 - \alpha) \cos \gamma L_2(a(r) = 0\)

(iii) \(\sqrt{r}L_3'(\sqrt{r}) + 2 (1 - \alpha) \cos \gamma L_3(a(\sqrt{r}) = 0\)

in \(|z| < (0, \lambda_{\mu,\nu,a,1}), (0, \lambda_{\mu,\nu,a,1})\) and \((0, \lambda_{\mu,\nu,a,1})\), respectively.

**Proof** Using (3.1), we obtain after the logarithmic differentiation:

\[
\begin{align*}
\frac{zf'_{\mu,\nu,a}(z)}{f_{\mu,\nu,a}(z)} &= 1 - \frac{1}{\nu} \sum_{n \geq 1} \frac{2z^2}{\lambda_{\mu,\nu,a,n}^2 - z^2}, \\
\frac{g'_{\mu,\nu,a}(z)}{g_{\mu,\nu,a}(z)} &= 1 - \sum_{n \geq 1} \frac{2z^2}{\lambda_{\mu,\nu,a,n}^2 - z^2}, \\
\frac{h'_{\mu,\nu,a}(z)}{h_{\mu,\nu,a}(z)} &= 1 - \sum_{n \geq 1} \frac{2z}{\lambda_{\mu,\nu,a,n} - z}. \\
\end{align*}
\]

We need to show that the following inequalities for \(\alpha \in [0, 1)\) and \(\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})\),

\[
\text{Re}\left(e^{-i\gamma} \frac{zf'_{\mu,\nu,a}(z)}{f_{\mu,\nu,a}(z)}\right) > \alpha \cos \gamma, \quad \text{Re}\left(e^{-i\gamma} \frac{g'_{\mu,\nu,a}(z)}{g_{\mu,\nu,a}(z)}\right) > \alpha \cos \gamma
\]

and

\[
\text{Re}\left(e^{-i\gamma} \frac{h'_{\mu,\nu,a}(z)}{h_{\mu,\nu,a}(z)}\right) > \alpha \cos \gamma
\]

are valid for \(z \in D_{r,p}(f_{\mu,\nu,a}), z \in D_{r,p}(g_{\mu,\nu,a})\) and \(z \in D_{r,p}(h_{\mu,\nu,a})\) respectively, and each of the above inequalities does not hold in larger disks. Since using (2.6)

\[
\text{Re}\left(\frac{z^2}{\lambda_{\mu,\nu,a,n}^2 - z^2}\right) \leq \left|\frac{z^2}{\lambda_{\mu,\nu,a,n}^2 - z^2}\right| \leq \frac{|z|^2}{\lambda_{\mu,\nu,a,n}^2 - |z|^2}
\]

holds for every \(|z| < \lambda_{\mu,\nu,a,1}\). Therefore, from (3.2) and (3.4), we have

\[
\text{Re}\left(e^{-i\gamma} \frac{zf'_{\mu,\nu,a}(z)}{f_{\mu,\nu,a}(z)}\right) = \text{Re}\left(e^{-i\gamma}\right) - \frac{1}{\nu} \text{Re}\left(e^{-i\gamma} \sum_{n \geq 1} \frac{2z^2}{\lambda_{\mu,\nu,a,n}^2 - z^2}\right)
\]

\[
\geq \cos \gamma - \frac{1}{\nu} \left|e^{-i\gamma} \sum_{n \geq 1} \frac{2z^2}{\lambda_{\mu,\nu,a,n}^2 - z^2}\right|
\]

\[
\geq \cos \gamma - \frac{1}{\nu} \sum_{n \geq 1} \frac{2|z|^2}{\lambda_{\mu,\nu,a,n}^2 - |z|^2}
\]

\[
= \frac{|z|^2}{f_{\mu,\nu,a}(|z|)} + \cos \gamma - 1.
\]

(3.5)
Equality in the each of the above inequalities (3.5) holds when \( z = r \). Thus, for \( r \in (0, \lambda_{\mu,\nu,1}) \) it follows that

\[
\inf_{z \in \mathbb{D}_r} \left\{ \Re \left( e^{-i\gamma} \frac{zf_{\mu,\nu,a}(z)}{f_{\mu,\nu,a}(z)} - \alpha \cos \gamma \right) \right\} = \frac{|z| f'_{\mu,\nu,a}(|z|)}{f_{\mu,\nu,a}(|z|)} + (1 - \alpha) \cos \gamma - 1.
\]

Now, the mapping \( \Theta : (0, \lambda_{\mu,\nu,1}) \rightarrow \mathbb{R} \) defined by

\[
\Theta(r) = \frac{r f''_{\mu,\nu,a}(r)}{f_{\mu,\nu,a}(r)} + (1 - \alpha) \cos \gamma = (1 - \alpha) \cos \gamma - \frac{1}{\nu} \sum_{n \geq 1} \left( \frac{2\nu^2}{\lambda_{\mu,\nu,a,n}^2 - r^2} \right).
\]

is strictly decreasing since

\[
\Theta'(r) = -\frac{1}{\nu} \sum_{n \geq 1} \left( \frac{4r \lambda_{\mu,\nu,a,n}}{(\lambda_{\mu,\nu,a,n}^2 - r^2)^2} \right) < 0
\]

for all \( \nu > 0 \). On the other hand, since

\[
\lim_{r \downarrow 0} \Theta(r) = (1 - \alpha) \cos \gamma > 0 \quad \text{and} \quad \lim_{r \uparrow \lambda_{\mu,\nu,1}} \Theta(r) = -\infty,
\]

in view of the minimum principle for harmonic functions imply that the corresponding inequality for \( f_{\mu,\nu,a} \) in (3.3) for \( \nu > 0 \) holds if and only if

\[
z \in \mathbb{D}_{r_{sp}(f_{\mu,\nu,a})},
\]

where \( r_{sp}(f_{\mu,\nu,a}) \) is the smallest positive root of equation

\[
\frac{r f''_{\mu,\nu,a}(r)}{f_{\mu,\nu,a}(r)} = 1 - (1 - \alpha) \cos \gamma
\]

which is equivalent to

\[
\frac{1}{\nu} \frac{z L''_{\mu,\nu,a}(z)}{L_{\mu,\nu,a}(z)} = - (1 - \alpha) \cos \gamma,
\]

situated in \( (0, \lambda_{\mu,\nu,1}) \). Reasoning along the same lines, proofs of the other parts follows. \( \square \)

**Remark 3** Taking \( \gamma = 0 \) in Theorem 3 yields [5, Theorem 1].

In the following, we derive the result for the convex analog proceeds on similar lines as Theorem 2.

**Theorem 4** Let \( \left( \mu, \nu \right) \in W_\nu \), \( a > 0 \). Let the functions \( f_{\mu,\nu,a} \), \( g_{\mu,\nu,a} \) and \( h_{\mu,\nu,a} \) be given by (3.1). Then

(i) the radius \( R_{sp}^c(f_{\mu,\nu,a}) \) is the smallest positive root of the equation

\[
r f''_{\mu,\nu,a}(r) + (1 - \alpha) \cos \gamma f'_{\mu,\nu,a}(r) = 0.
\]

(ii) the radius \( R_{sp}^c(g_{\mu,\nu,a}) \) is the smallest positive root of the equation

\[
r g''_{\mu,\nu,a}(r) + (1 - \alpha) \cos \gamma g'_{\mu,\nu,a}(r) = 0.
\]

(iii) the radius \( R_{sp}^c(h_{\mu,\nu,a}) \) is the smallest positive root of the equation

\[
r h''_{\mu,\nu,a}(r) - (1 - \alpha) \cos \gamma h'_{\mu,\nu,a}(r) = 0.
\]

**Remark 4** Taking \( \gamma = 0 \) in Theorem 4 yields [5, Theorem 3].
4 Legendre polynomials

The Legendre polynomials $P_n$ are the solutions of the Legendre differential equation

$$((1 - z^2)P_n'(z))' + n(n + 1)P_n(z) = 0,$$

where $n \in \mathbb{Z}^+$ and using Rodrigues formula, $P_n$ can be represented in the form:

$$P_n(z) = \frac{1}{2^n n!} \frac{d^n (z^2 - 1)^n}{dz^n}$$

and it also satisfies the geometric condition $P_n(-z) = (-1)^n P_n(z)$. Moreover, the odd degree Legendre polynomials $P_{2n-1}(z)$ have only real roots which satisfy

$$0 = z_0 < z_1 < \cdots < z_{n-1} \quad \text{or} \quad -z_1 > \cdots > -z_{n-1}.$$  \hspace{1cm} (4.1)

Thus the normalized form is as follows:

$$P_{2n-1}(z) := \frac{P_{2n-1}(z)}{P'_{2n-1}(0)} = z + \sum_{k=2}^{2n-1} a_k z^k = a_{2n-1} z \prod_{k=1}^{n-1} (z^2 - z_k^2).$$  \hspace{1cm} (4.2)

**Theorem 5** Let $P_{2n-1}$ be given by (4.2). Then

(i) the radius $R_{cp}(P_{2n-1})$ is the smallest positive root of the equation

$$rP'_{2n-1}(r) + (1 - \alpha)P''_{2n-1}(r) = 0.$$

(ii) the radius of $\gamma$-Spirallikeness of order $\alpha$ for the normalized Legendre polynomial of odd degree is given by the smallest positive root of the equation

$$rP'_{2n-1}(r) + (1 - \alpha)P_{2n-1}(r) = 0.$$

**Proof** We prove first part and second part follows on same lines. From (4.2), upon the logarithmic differentiation, we have

$$1 + \frac{zP'_{2n-1}(z)}{P_{2n-1}(z)} = \frac{zP'_{2n-1}(z)}{P_{2n-1}(z)} - \frac{\sum_{k=1}^{n-1} 4z_k^2 z^2}{zP_{2n-1}(z)}.$$

where

$$\frac{zP'_{2n-1}(z)}{P_{2n-1}(z)} = 1 - \sum_{k=1}^{n-1} \frac{2z^2}{z_k^2 - z^2}.$$
Further, after using the inequality $||x|-|y|| \leq |x-y|$ and (4.1) for $|z| = r < z_1$, we see that
\[
\text{Re} \left( e^{-i\gamma} \left( 1 + \frac{zP''_{2n-1}(z)}{P'_{2n-1}(z)} \right) \right) = \text{Re} \left( e^{-i\gamma} - \text{Re} \left( e^{-i\gamma} \sum_{k=1}^{n-1} \frac{2iz^2}{z_k^2 - z^2} \right) - \text{Re} \left( e^{-i\gamma} \frac{\sum_{k=1}^{n-1} \frac{4iz^2}{(z_k - z)^2}}{1 - \sum_{k=1}^{n-1} \frac{2iz^2}{z_k^2 - z^2}} \right) \right) \geq \cos \gamma - \left| e^{-i\gamma} \sum_{k=1}^{n-1} \frac{2iz^2}{z_k^2 - z^2} \right| - \left| e^{-i\gamma} \frac{\sum_{k=1}^{n-1} \frac{4iz^2}{(z_k - z)^2}}{1 - \sum_{k=1}^{n-1} \frac{2iz^2}{z_k^2 - z^2}} \right| \geq \cos \gamma - \frac{n-1}{r^2} = \cos \gamma - \frac{\sum_{k=1}^{n-1} \frac{4iz^2}{(z_k - z)^2}}{1 - \sum_{k=1}^{n-1} \frac{2iz^2}{z_k^2 - z^2}} \equiv \cos \gamma + \frac{rP''_{2n-1}(r)}{P'_{2n-1}(r)}.
\]

Further, with similar reasoning as Theorem 3, result follows.

\( \square \)

\textbf{Remark 5} Taking $\gamma = 0$ in Theorem 5 yields [9, Theorem 2.2] and [9, Theorem 2.1].

\section{Lommel functions}

The Lommel function $L_{u,v}$ of first kind is a particular solution of the second-order inhomogeneous Bessel differential equation
\[
z^2w''(z) + zw'(z) + (z^2 - \nu^2)w(z) = z^{\nu+1},
\]
where $u \pm v \notin \mathbb{Z}^-$ and is given by
\[
L_{u,v} = \frac{z^{u+1}}{(u-v+1)(u+v+1)} \binom{u}{\frac{2}{2}}_2 F_2 \left( 1, \frac{u-v+3}{2}, \frac{u+v+3}{2}, \frac{-z^2}{4} \right),
\]
where $\binom{1}{-u \pm v - 3} \notin \mathbb{N}$ and $\binom{1}{2}$ is a hypergeometric function. Since it is not normalized, therefore we consider the following three normalized functions involving $L_{u,v}$:
\[
\begin{align*}
  f_{u,v}(z) &= ((u-v+1)(u+v+1)L_{u,v}(z))^{\frac{1}{u+1}}, \\
  g_{u,v}(z) &= (u-v+1)(u+v+1)z^{-u}L_{u,v}(z), \\
  h_{u,v}(z) &= (u-v+1)(u+v+1)z^{(1-u)/2}L_{u,v}(\sqrt{z}).
\end{align*}
\]

Authors in [1, 2] and [8] proved the radius of starlikeness and convexity for the following normalized functions expressed in terms of $L_{u-\frac{1}{2}, \frac{1}{2}}$:
\[
\begin{align*}
  f_{\frac{1}{2}, \frac{1}{2}}(z), \quad g_{\frac{1}{2}, \frac{1}{2}}(z) \quad \text{and} \quad h_{\frac{1}{2}, \frac{1}{2}}(z),
\end{align*}
\]
where $0 \neq u \in (-1, 1)$.

For brevity, we write these as $f_u, g_u$ and $h_u$, respectively and $L_{u-\frac{1}{2}, \frac{1}{2}} = L_u$. 

Theorem 6 Let \( u \in (-1, 1), u \neq 0 \). Let the functions \( f_u, g_u \) and \( h_u \) be given by (5.2). Then

(i) the radius \( R_u^{(i)}(f_u) \) is the smallest positive root of the equation
\[
rf_u''(r) + (1 - \alpha) \cos \gamma f_u'(r) = 0, \quad \text{if} \quad u \neq -1/2.
\]
(ii) the radius \( R_u^{(ii)}(g_u) \) is the smallest positive root of the equation
\[
r\gamma u''(r) + (1 - \alpha) \cos \gamma g_u'(r) = 0.
\]
(iii) the radius \( R_u^{(iii)}(h_u) \) is the smallest positive root of the equation
\[
rh_u''(r) + (1 - \alpha) \cos \gamma h_u'(r) = 0.
\]

Proof We begin with the first part. From (5.1), we have
\[
1 + \frac{zf_u''(z)}{f_u'(z)} = 1 + \frac{zL_u''(z)}{L_u'(z)} + \left( \frac{1}{u + \frac{1}{2}} - 1 \right) \frac{zL_u'(z)}{L_u(z)}. \tag{5.3}
\]
Also using the result \([8, \text{Lemma 1}],\) we have
\[
L_u(z) = \frac{z^{u+\frac{1}{2}}}{u(u+1)} \phi_0(z) = \frac{z^{u+\frac{1}{2}}}{u(u+1)} \prod_{n \geq 1} \left( 1 - \frac{z^2}{\tau_{u,n}^2} \right),
\]
where \( \phi_k(z) := _1F_2 \left( 1; \frac{u+k+2}{u+1}; \frac{-z^2}{\tau_{u,n}^2} \right) \) with conditions as mentioned in \([8, \text{Lemma 1}],\) and from the proof of \([8, \text{Theorem 3}],\) we see that the entire function \( \sum_{n \geq 1} \frac{z^{u+\frac{1}{2}}}{u+1} L_u'(z) \) is of order \( 1/2 \) and therefore, has the following Hadamard factorization:
\[
L_u'(z) = \frac{u+\frac{1}{2}}{u(u+1)} z^{u-\frac{1}{2}} \prod_{n \geq 1} \left( 1 - \frac{z^2}{\bar{x}_{u,n}^2} \right),
\]
where \( \tau_{u,n} \) and \( \bar{x}_{u,n} \) are the \( n \)-th positive zeros of \( L_u \) and \( L_u' \), respectively and interlace for \( 0 \neq u \in (-1, 1) \) (see \([8, \text{Theorem 1}],\) Now we can rewrite (5.3) as follows:
\[
1 + \frac{zf_u''(z)}{f_u'(z)} = 1 - \left( \frac{1}{u + \frac{1}{2}} - 1 \right) \sum_{n \geq 1} \frac{2z^2}{\tau_{u,n}^2 - z^2} \sum_{n \geq 1} \frac{2z^2}{\bar{x}_{u,n}^2 - z^2}.
\]
Let us now consider the case \( u \in (0, 1/2) \). Then using the inequality \(|x| - |y| \leq |x - y| \) for \( |z| = r < \tau_{u,1} < \tau_{u,1} \) we get
\[
\left| \frac{zf_u''(z)}{f_u'(z)} \right| \leq \left( \frac{1}{u + \frac{1}{2}} - 1 \right) \sum_{n \geq 1} \frac{2r^2}{\tau_{u,n}^2 - r^2} + \sum_{n \geq 1} \frac{2r^2}{\bar{x}_{u,n}^2 - r^2} = -rf_u''(r) \tag{5.4}
\]
and for the case \( u \in (1/2, 1) \), using the inequality (2.8) with \( \lambda = 1 - 1/(u+1/2) \), we also get
\[
\left| \frac{zf_u''(z)}{f_u'(z)} \right| \leq -rf_u''(r). \tag{5.5}
\]
which is same as (5.4). When \( u \in (-1, 0) \), then we proceed similarly substituting \( u \) by \( u - 1 \), \( \Phi_0 \) by \( \Phi_1 \), where \( \Phi_1 \) belongs to the Laguerre-Pólya class \( \mathcal{LP} \) and the \( n \)-th positive zeros \( \xi_{u,n} \) and \( \zeta_{u,n} \) of \( \Phi_1 \) and its derivative \( \Phi_1' \), respectively, interlace. Finally, replacing \( u \) by \( u + 1 \), we obtain the required inequality.

For \( 0 \neq u \in (-1, 1) \), the Hadamard factorization for the entire functions \( g_u' \) and \( h_u' \) of order \( 1/2 \) \cite[Theorem 3]{8} is given by

\[
\begin{align*}
g_u'(z) &= \prod_{n \geq 1} \left(1 - \frac{z^2}{\gamma_{u,n}^2}\right) \quad \text{and} \quad h_u'(z) = \prod_{n \geq 1} \left(1 - \frac{z}{\delta_{u,n}}\right), \tag{5.6}
\end{align*}
\]

where \( \gamma_{u,n} \) and \( \delta_{u,n} \) are \( n \)-th positive zeros of \( g_u' \) and \( h_u' \), respectively and \( \gamma_{u,1}, \delta_{u,1} < \tau_{u,1} \). Now from (5.1) and (5.6), we have

\[
\begin{align*}
1 + \frac{z g_u''(z)}{g_u'(z)} &= \frac{1}{2} - u + z \left(\frac{1}{2} - u\right) L_u'(z) + z L_u''(z) = 1 - \sum_{n \geq 1} \frac{2z^2}{\gamma_{u,n}^2 - z^2}, \\
1 + \frac{z h_u''(z)}{h_u'(z)} &= \frac{1}{2} \left(\frac{1}{2} - u + \sqrt{\frac{1}{2} - u}\right) L_u'(\sqrt{z}) + \sqrt{\frac{1}{2} - u} L_u'(\sqrt{z}) = 1 - \sum_{n \geq 1} \frac{z}{\delta_{u,n}^2 - z^2}. \tag{5.7}
\end{align*}
\]

Using the inequality \(||x| - |y|\| \leq |x - y|\) in (5.7) for \(|z| = r < \gamma_{u,1}\) and \(|z| = r = \delta_{u,1}\), we get

\[
\begin{align*}
\left|\frac{z g_u''(z)}{g_u'(z)}\right| &\leq \sum_{n \geq 1} \frac{2z^2}{\gamma_{u,n}^2 - r^2} = \frac{r g_u''(r)}{g_u'(r)}, \\
\left|\frac{z h_u''(z)}{h_u'(z)}\right| &\leq \sum_{n \geq 1} \frac{z}{\delta_{u,n}^2 - r^2} = \frac{r h_u''(r)}{h_u'(r)}. \tag{5.8}
\end{align*}
\]

Further, proceeding with the similar method as in Theorem 1, result follows.

With similar reasoning as Theorem 1, the proof of the following holds.

**Theorem 7** Let \( u \in (-1, 1) \), \( u \neq 0 \). Then the radius of \( \gamma \)-Spirallikeness of order \( \alpha \) for the functions \( f_u, g_u \) and \( h_u \) given by (5.2) are the smallest positive roots of the following equations:

(i) \( rf_u''(r) + ((1 - \alpha) \cos \gamma - 1)f_u(r) = 0 \)

(ii) \( rg_u''(r) + ((1 - \alpha) \cos \gamma - 1)g_u(r) = 0 \)

(iii) \( rh_u''(r) + ((1 - \alpha) \cos \gamma - 1)h_u(r) = 0 \)

in \((0, \tau_{u,1})\), \((0, \tau_{u,1})\) and \((0, \tau_{u,1}^2)\), respectively.

**Remark 6** Taking \( \gamma = 0 \), Theorem 7 reduces to \cite[Theorem 3]{8}.

### 6 Struve functions

The Struve function \( H_\beta \) of first kind is a particular solution of the second-order inhomogeneous Bessel differential equation

\[
z^2 w''(z) + zw'(z) + (z^2 - \beta^2)w(z) = \frac{4 \left(\frac{z}{2}\right)^{\beta+1}}{\sqrt{\pi z} \Gamma \left(\beta + \frac{1}{2}\right)}
\]
and have the following form:

\[ H_{\beta}(z) := \frac{(\frac{z}{2})^{\beta+1}}{\sqrt{\pi}^2 \Gamma(\beta + \frac{1}{2})} F_1 \left( \frac{3}{2}; \beta + \frac{3}{2}; -\frac{z^2}{4} \right), \]

where \( -\beta - \frac{3}{2} \notin \mathbb{N} \) and \( \Gamma \) is a hypergeometric function. Since it is not normalized, therefore we take the normalized functions:

\[
\begin{align*}
U_{\beta}(z) &= \sqrt{\pi} 2^{\beta} \beta^{\beta} + 1 \Gamma(\beta + \frac{3}{2}) H_{\beta}(z), \\
V_{\beta}(z) &= \sqrt{\pi} 2^{\beta} z^{-\beta} \Gamma(\beta + \frac{3}{2}) H_{\beta}(z), \\
W_{\beta}(z) &= \sqrt{\pi} 2^\beta z^{1-\beta} \Gamma(\beta + \frac{3}{2}) H_{\beta}(\sqrt{z}).
\end{align*}
\]

Moreover, for \( |\beta| \leq 1/2 \), \( H_{\beta} \) (see [4, Lemma 1]) and \( H'_{\beta} \) have the Hadamard factorizations [8, Theorem 4] given by

\[
H_{\beta}(z) = \frac{z^{\beta+1}}{\sqrt{\pi} 2^\beta \Gamma(\beta + \frac{3}{2})} \prod_{n \geq 1} \left( 1 - \frac{z^2}{z_{\beta,n}^2} \right)
\]

and

\[
H'_{\beta}(z) = \frac{(\beta + 1)z^\beta}{\sqrt{\pi} 2^\beta \Gamma(\beta + \frac{3}{2})} \prod_{n \geq 1} \left( 1 - \frac{z^2}{z_{\beta,n}^2} \right)
\]

where \( z_{\beta,n} \) and \( z_{\beta,n} \) are the \( n \)-th positive zeros of \( H_{\beta} \) and \( H'_{\beta} \), respectively and interlace [8, Theorem 2]. Thus from (6.2) with logarithmic differentiation, we obtain respectively

\[
\frac{z H'_{\beta}(z)}{H_{\beta}(z)} = (\beta + 1) - \sum_{n \geq 1} \frac{2z^2}{z_{\beta,n}^2} - z^2.
\]

and

\[
1 + \frac{z H''_{\beta}(z)}{H'_{\beta}(z)} = (\beta + 1) - \sum_{n \geq 1} \frac{2z^2}{z_{\beta,n}^2} - z^2.
\]

Also for \( |\beta| \leq 1/2 \), the Hadamard factorization for the entire functions \( V'_{\beta} \) and \( W'_{\beta} \) of order \( 1/2 \) [8, Theorem 4] is given by

\[
V'_{\beta}(z) = \prod_{n \geq 1} \left( 1 - \frac{z^2}{\eta_{\beta,n}^2} \right) \quad \text{and} \quad W'_{\beta}(z) = \prod_{n \geq 1} \left( 1 - \frac{z}{\sigma_{\beta,n}^2} \right),
\]

where \( \eta_{\beta,n} \) and \( \sigma_{\beta,n} \) are \( n \)-th positive zeros of \( V'_{\beta} \) and \( W'_{\beta} \), respectively. \( V'_{\beta} \) and \( W'_{\beta} \) belong to the Laguerre-Pólya class and zeros satisfy \( \eta_{\beta,1}, \sigma_{\beta,1} < z_{\beta,1} \). Now proceeding as in Theorem 6 using (6.1), (6.2), (6.3) and (6.4), we obtain the following results:

**Theorem 8** Let \( |\beta| \leq 1/2 \). Then the radii of \( \gamma \)-Spiral-like and \( \gamma \)-Spiral-like of order \( \alpha \) for the functions \( U_{\beta}, V_{\beta} \) and \( W_{\beta} \) given by (6.1) are the smallest positive roots of the following equations:
Radius of $\gamma$-Spirallikeness of order $\alpha$ for some Special functions

(i) $rU_\alpha'(r) + ((1 - \alpha) \cos \gamma - 1)U_\alpha(r) = 0$
(ii) $rV_\alpha'(r) + ((1 - \alpha) \cos \gamma - 1)V_\alpha(r) = 0$
(iii) $rW_\alpha'(r) + ((1 - \alpha) \cos \gamma - 1)W_\alpha(r) = 0$

in $(0, z_{\beta,1}), (0, z_{\beta,1})$ and $(0, z_{\beta,1}^2)$, respectively.

Remark 7 Taking $\gamma = 0$ in Theorem 8 gives [2, Theorem 2].

Theorem 9 Let $|\beta| \leq 1/2$. Let the functions $U_\beta, V_\beta$ and $W_\beta$ be given by (6.1). Then

(i) the radius $R_{sp}^c(U_\beta)$ is the smallest positive root of the equation

$$rU_\beta''(r) + (1 - \alpha) \cos \gamma U_\beta'(r) = 0.$$

(ii) the radius $R_{sp}^c(V_\beta)$ is the smallest positive root of the equation

$$rV_\beta''(r) + (1 - \alpha) \cos \gamma V_\beta'(r) = 0.$$

(iii) the radius $R_{sp}^c(W_\beta)$ is the smallest positive root of the equation

$$rW_\beta''(r) + (1 - \alpha) \cos \gamma W_\beta'(r) = 0.$$

Remark 8 Taking $\gamma = 0$ in Theorem 9 gives [8, Theorem 4].

7 On Ramanujan type entire functions

Ismail and Zhang [14] defined the following entire function of growth order zero for $\beta > 0$, called Ramanujan type entire function

$$A_p^{(\beta)}(c, z) = \sum_{n \geq 0} \frac{(c, p)_n p^{n^2/2} (p; p)_n}{(p; p)_n^2} z^n,$$

where $\beta > 0$, $0 < p < 1$, $c \in \mathbb{C}$, $(c; p)_0 = 1$ and $(c; p)_k = \prod_{j=0}^{k-1}(1 - cp^j)$ for $k \geq 1$, which is the generalization of both the Ramanujan entire function $A_p(z)$ and Stieltjes-Wigert polynomial $S_n(z; p)$ defined as (see [15, 22]):

$$A_p(-z) = A_p^{(1)}(0, z) = \sum_{n=0}^{\infty} \frac{p^{n^2/2}}{(p; p)_n} z^n$$

and

$$A_p^{(1/2)}(p^{-n}, z) = \sum_{m=0}^{\infty} \frac{(p^{-n}; p)_m p^{m^2/2}}{(p; p)_m} z^m = (p; p)_n S_n(zp^{(1/2)-n}; p).$$

Since $A_p^{(\beta)}(c, z) \not\in A$, therefore consider the following three normalized functions in $A$:

$$f_{\beta, p, c}(z) := \left(z^\beta A_p^{(\beta)}(-c, -z^2)\right)^{1/\beta}$$
$$g_{\beta, p, c}(z) := z A_p^{(\beta)}(-c, -z^2)$$
$$h_{\beta, p, c}(z) := z A_p^{(\beta)}(-c, -z),$$

(7.1)
where $\beta > 0$, $c \geq 0$ and $0 < p < 1$. From [10, Lemma 2.1, p. 4-5], we see that the function 
\[ z \to \Psi_{\beta,p,c}(z) := A_p^\beta(-c, -z^2) \]
has infinitely many zeros (all are positive) for $\beta > 0$, $c \geq 0$ and $0 < p < 1$. Let $\psi_{\beta,p,n}(c)$ be the $n$-th positive zero of $\Psi_{\beta,p,c}(z)$. Then it has the following Weiersstrass decomposition:
\[ \Psi_{\beta,p,c}(z) = \prod_{n \geq 1} \left( 1 - \frac{z^2}{\psi_{\beta,p,n}^2(c)} \right). \]  
(7.2)
Moreover, the $n$-th positive zero $\Xi_{\beta,p,n}(c)$ of the derivative of the following function
\[ \Phi_{\beta,p,c}(z) := z^\beta \Psi_{\beta,p,c}(z) \]
interlace with $\psi_{\beta,p,n}(c)$ and satisfy the relation
\[ \Xi_{\beta,p,n}(c) < \psi_{\beta,p,n}(c) < \Xi_{\beta,p,n+1}(c) < \psi_{\beta,p,n+1}(c) \]
for $n \geq 1$. Now using (7.1) and (7.2), we have
\[
\frac{zf_{\beta,p,n}(z)}{f_{\beta,p,c}(z)} = 1 + \frac{z^2}{\beta \Psi_{\beta,p,c}(z)} = 1 - \frac{1}{\beta} \sum_{n \geq 1} \frac{2z^2}{\psi_{\beta,p,n}^2(c) - z^2}; \quad (c > 0)
\]
\[
\frac{zg_{\beta,p,n}(z)}{g_{\beta,p,c}(z)} = 1 + \frac{z^2}{\Psi_{\beta,p,c}(z)} = 1 - \sum_{n \geq 1} \frac{2z^2}{\psi_{\beta,p,n}^2(c) - z^2};
\]
\[
\frac{zh_{\beta,p,n}(z)}{h_{\beta,p,c}(z)} = 1 + \frac{\sqrt{z}}{2 \Psi_{\beta,p,c}(\sqrt{z})} = 1 - \sum_{n \geq 1} \frac{z}{\psi_{\beta,p,n}^2(c) - z},
\]
where $\beta > 0$, $c \geq 0$ and $0 < p < 1$. Also, using (7.3) and the infinite product representation of $\Phi'$ [10, p. 14-15, Also see Eq. 4.6], we have
\[
1 + \frac{zf_{\beta,p,c}''(z)}{f_{\beta,p,c}'(z)} = 1 + \frac{z\Phi_{\beta,p,c}''(z)}{\Phi_{\beta,p,c}'(z)} \left( \frac{1}{\beta} - 1 \right) \frac{z\Phi_{\beta,p,c}'(z)}{\Phi_{\beta,p,c}(z)} = 1 - \sum_{n \geq 1} \frac{2z^2}{\Xi_{\beta,p,n}'(c) - z^2} - \left( \frac{1}{\beta} - 1 \right) \sum_{n \geq 1} \frac{2z^2}{\psi_{\beta,p,n}^2(c) - z^2}.
\]
As $(z\Psi_{\beta,p,c}(z))''$ and $h_{\beta,p,c}'(z)$ belong to $\mathcal{L}_P$. So suppose $\gamma_{\beta,p,n}(c)$ be the positive zeros of $g_{\beta,p,c}'(z)$ (growth order is same as $\Phi_{\beta,p,c}(z)$) and $\delta_{\beta,p,n}(c)$ be the positive zeros of $h_{\beta,p,c}'(z)$. Thus using their infinite product representations, we have
\[
1 + \frac{zg_{\beta,p,c}''(z)}{g_{\beta,p,c}'(z)} = 1 - \sum_{n \geq 1} \frac{2z^2}{\gamma_{\beta,p,n}^2(c) - z^2}
\]
\[
1 + \frac{zh_{\beta,p,c}''(z)}{h_{\beta,p,c}'(z)} = 1 - \sum_{n \geq 1} \frac{z}{\delta_{\beta,p,n}^2(c) - z}.
\]
Now proceeding similarly as done in the above sections, we obtain the following results:
Theorem 10 Let $\beta > 0$, $c \geq 0$ and $0 < p < 1$. Then the radii of $\gamma$-Spirallikeness of order $\alpha$ for the functions $f_{\beta,p,c}(z)$, $g_{\beta,p,c}(z)$ and $h_{\beta,p,c}(z)$ given by (7.1) are the smallest positive roots of the following equations:

(i) $r f'_{\beta,p,c}(r) + ((1 - \alpha) \cos \gamma - 1) f_{\beta,p,c}(r) = 0$

(ii) $r g'_{\beta,p,c}(r) + ((1 - \alpha) \cos \gamma - 1) g_{\beta,p,c}(r) = 0$

(iii) $r h'_{\beta,p,c}(r) + ((1 - \alpha) \cos \gamma - 1) h_{\beta,p,c}(r) = 0$

in $(0, \psi_{\beta,p,1}(c))$, $(0, \psi_{\beta,p,1}(c))$ and $(0, \psi_{\beta,p,1}^2(c))$, respectively.

We now conclude this section with the convex analog of Theorem 10.

Theorem 11 Let $\beta > 0$, $c \geq 0$ and $0 < p < 1$. Let the functions $f_{\beta,p,c}(z)$, $g_{\beta,p,c}(z)$ and $h_{\beta,p,c}(z)$ be given by (7.1). Then

(i) the radius $R_{sp}^c(f_{\beta,p,c}(z))$ is the smallest positive root of the equation

\[ r f''_{\beta,p,c}(r) + (1 - \alpha) \cos \gamma f'_{\beta,p,c}(r) = 0. \]

(ii) the radius $R_{sp}^c(g_{\beta,p,c}(z))$ is the smallest positive root of the equation

\[ r g''_{\beta,p,c}(r) + (1 - \alpha) \cos \gamma g'_{\beta,p,c}(r) = 0. \]

(iii) the radius $R_{sp}^c(h_{\beta,p,c}(z))$ is the smallest positive root of the equation

\[ r h''_{\beta,p,c}(r) + (1 - \alpha) \cos \gamma h'_{\beta,p,c}(r) = 0. \]

Statements and Declarations

- **Conflict of interest**: The authors declare that they have no conflict of interest
- **Availability of data and materials**: None
- **Authors’ contributions**: All authors contributed equally.

References

1. Aktaş, İ. Baricz, Á. and Orhan, H.: Bounds for radii of starlikeness and convexity of some special functions. Turkish J. Math. 42, 211–226 (2018) doi: 10.3906/mat-1610-41
2. Baricz, Á. Dimitrov, D.K., Orhan, H. and Yağmur, N.: Radii of starlikeness of some special functions. Proc. Amer. Math. Soc. 144, 3355–3367 (2016). doi: 10.1090/proc/13120
3. Baricz, Á. Kupán, P.A. and Szász, R.: The radius of starlikeness of normalized Bessel functions of the first kind. Proc. Amer. Math. Soc. 142, 2019–2025 (2014). doi: 10.1090/S0002-9939-2014-11902-2
4. Baricz, Á. Ponnusamy, S. and Singh, S.: Turán type inequalities for Struve functions. J. Math. Anal. Appl. 445, 971–984 (2017). doi: 10.1016/j.jmaa.2016.08.026
5. Baricz, Á. and Prajapati, A.: Radii of starlikeness and convexity of generalized Mittag-Leffler functions. Math. Commun. 25, 117–135 (2020).
6. Baricz, Á. and Szász, R.: The radius of convexity of normalized Bessel functions. Anal. Math. 41, 141–151 (2015). doi: 10.1007/s10476-015-0202-6
7. Baricz, Á. Toklu, E. and Kadioğlu, E.: Radii of starlikeness and convexity of Wright functions. Math. Commun. 23, 97–117 (2018).
8. Baricz, Á. and Yağmur, N.: Geometric properties of some Lommel and Struve functions. Ramanujan J. 42, 325–346 (2017). doi: 10.1007/s11139-015-9724-6
9. Bulut, S. and Engel, O.: The radius of starlikeness, convexity and uniform convexity of the Legendre polynomials of odd degree. Results Math. 74, Paper No. 48, 9 pp (2019). doi: 10.1007/s00025-019-0075-1
10. Deniz, E.: Geometric and monotonic properties of Ramanujan type entire functions. Ramanujan J. 55, 103–130 (2020). doi: 10.1007/s11139-020-00267-w
11. Deniz, E. and Szász, R.: The radius of uniform convexity of Bessel functions. J. Math. Anal. Appl. 453, 572–588 (2017). doi: 10.1016/j.jmaa.2017.03.079
12. Dimitrov, D.K. and Ben Cheikh, Y.: Laguerre polynomials as Jensen polynomials of Laguerre-Pólya entire functions. J. Comput. Appl. Math. 293, 703–707 (2017). doi: 10.1016/j.cam.2009.02.039
13. Gangania, K. and Kumar, S.S.: $S^*(\psi)$ and $C(\psi)$-radii for some special functions. Iran. J. Sci. Technol. Trans. A Sci. 46, 955–966 (2022).
14. Ismail, M.E.H. and Zhang, R.: $q$-Bessel functions and Rogers-Ramanujan type identities. Proc. Amer. Math. Soc. 146, 3633–3646 (2018). doi: 10.1090/proc/13978
15. Ismail, M.E.H.: Classical and quantum orthogonal polynomials in one variable. Encyclopedia of Mathematics and its Applications, 98, Cambridge University Press, Cambridge, 2005.
16. Kumar, S.S. and Gangania, K.: Subordination and radius problems for certain starlike functions. arXiv:2007.07816 (2020)
17. Kumar H and Pathan A.M. On the distribution of non-zero zeros of generalized Mittag-Leffler functions. International Journal of Engineering Research and Application, 6, 66–71 (2016).
18. Ya. Levin B. Lectures on Entire Functions. Translation of Mathematics Monographs, American Mathematical Society, Providence 150 (1996).
19. Ma, W.C. and Minda, D.: A unified treatment of some special classes of univalent functions. Proceedings of the Conference on Complex Analysis, Tianjin, Conf Proc Lecture Notes Anal, I Int Press. Cambridge, MA. 157–169 (1992).
20. Pfaltzgraff, J.A.: Univalence of the integral of $f(\lambda)$, Bull. London Math. Soc. 7, no. 3, 254–256 (1975).
21. Prabhakar, T.R.: A singular integral equation with a generalized Mittag-Leffler function in the kernel. Yokohama Math. J. 19, 7–15 (1971).
22. Ramanujan, S.: The lost notebook and other unpublished papers, Springer-Verlag, Berlin, 1988.
23. Robertson, M.S.: Univalent functions $f(z)$ for which $zf'(z)$ is spirallike. Michigan Math. J. 16, 97–101 (1969).
24. Spacek, L.: Contribution à la théorie des fonctions univalentes, Casop Pest. Mat.-Fys. 62, 12-19 (1933).
25. Kazımoğlu, S. and Deniz, E.: Radius Problems for Functions Containing Derivatives of Bessel Functions. Comput. Methods Funct. Theory (2022). https://doi.org/10.1007/s40315-022-00455-3
26. Watson, G.N.: A Treatise on the Theory of Bessel Functions, Cambridge University Press, 1944.