Modular reduction of the Steinberg lattice of the general linear group

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1 Introduction

We are concerned with a conjecture formulated by Gow [3] regarding the reduction modulo $\ell$ of the Steinberg lattice of the general linear group $G = \text{GL}_n(q)$. Here $n \geq 2$, the underlying finite field $F_q$ has characteristic $p$, and $\ell$ is a prime different from $p$.

Let $U$ be subgroup of $G$ consisting of all upper triangular matrices having 1’s along the main diagonal. Write $H$ for the diagonal subgroup of $G$, and set $B = UH$. Let $\mathcal{P}$ stand for the set of all subgroups of $G$ that contain $B$, i.e. the standard parabolic subgroups.

The Steinberg character, say $\chi$, of $G$ is a complex irreducible character that may be characterized as the only constituent of the permutation character $1^G_B$ that is not a constituent of $1^G_P$ for any $P$ in $\mathcal{P}$ different from $B$. This follows from Steinberg’s [4] own determination of $\chi$, and later work by Curtis [1] expressing $\chi$ as an alternating sum of permutation characters $1^G_P$.

An explicit realization of $\chi$ was obtained by Steinberg in [5]. He considers the element $e$ in the integral group algebra $\mathbb{Z}G$, defined by

$$e = \left( \sum_{\sigma \in S_n} \text{sign}(\sigma)\sigma \right) \sum_{b \in B} b,$$

where the symmetric group $S_n$ is viewed as a subgroup of $G$. Then the left ideal $I = \mathbb{Z}Ge$ is a $\mathbb{Z}G$-lattice of rank $|U|$ and $\mathbb{Z}$-basis $\{ue \mid u \in U\}$, affording $\chi$. Furthermore, Steinberg shows that $I/\ell I$ is an irreducible $F_\ell G$-module if and only if $\ell \not| [G : B]$. One then is faced with the interesting problem of trying to find a composition series for the $G$-module $I/\ell I$ over $F_\ell$, or, rather, a suitably large extension thereof, when $\ell \mid [G : B]$.

Given that $\chi$ may be realized over $\mathbb{Z}$, restriction to $I$ of the canonical bilinear form $\mathbb{Z}G \times \mathbb{Z}G \to \mathbb{Z}$, defined by $(g, h) \mapsto \delta_{g,h}$, yields a $G$-invariant symmetric bilinear form.
\[ f : I \times I \rightarrow \mathbb{Z} \] with zero radical. Such a form is unique up to scaling, and Gow normalizes \( f \) so that \( f(e, e) = |S_n| \).

We replace the rational integers \( \mathbb{Z} \) in the above construction by a local principal ideal domain \( R \) of characteristic 0 and maximal ideal \( \ell R \), containing a primitive \( p \)-th root of unity. If \( \zeta_p \) is a complex primitive \( p \)-th root of unity, we may take \( R \) to be the localization of \( \mathbb{Z}[\zeta_p] \) at a prime ideal lying above the unramified prime \( \ell \). The residue field \( K = R/\ell R \) has characteristic \( \ell \) and a primitive \( p \)-th root of unity.

The left ideal \( I = RG e \) of \( RG \) is an \( RG \)-lattice of rank \( |U| \) and \( R \)-basis \( \{ue \mid u \in U\} \), affording \( \chi \). Note that \( U \) acts on \( I \) via the regular representation.

Gow uses the form \( f : I \times I \rightarrow R \) to produce \( RG \)-submodules \( I(k) \) of \( I \), defined by

\[ I(k) = \{ x \in I \mid f(x, I) \subseteq \ell^k R \}, \quad k \geq 0. \]

Consider the \( KG \)-module \( \overline{T} = I/\ell I \) and its \( KG \)-submodules \( \overline{I(k)} = (I(k) + \ell I)/\ell I \), \( k \geq 0 \). This produces the filtration for \( \overline{T} \):

\[ \overline{T} = \overline{I(0)} \supseteq \overline{I(1)} \supseteq \overline{I(2)} \supseteq \ldots \quad (1) \]

As \( \overline{T} \) has finite dimension \( |U| \) over \( K \), only finitely many of the factors \( M(k) = \overline{I(k)}/\overline{I(k+1)} \) appearing in (1) are non-zero. By carefully examining the form \( f \), Gow is able to determine the exact non-negative integers \( k \) such that \( M(k) \neq (0) \). We will discuss this matter in great detail later.

Based on information from tables for \( n \leq 10 \), Gow conjectures that all non-zero \( M(k) \) are irreducible \( KG \)-modules. This would effectively produce a composition series for \( \overline{T} \).

Our contribution to this problem is the following. We start by showing that the \( M(k) \) are completely reducible \( KG \)-modules. Next, we prove that the irreducible constituents of a non-zero \( M(k) \) must be self-dual. This is in complete agreement with Gow’s conjecture. It follows that each irreducible constituent of \( \overline{T} \) is self-dual, and we prove that these are pairwise non-isomorphic. We also show that \( \overline{T} \) itself is self-dual if only if \( \overline{T} \) is irreducible.

Our main tool in obtaining these results is the following: if \( M \) is a self-dual \( KG \)-module with no repeated linear characters of \( U \), then \( M \) is completely reducible and all its irreducible constituents are self-dual.
Finally we produce an irreducibility criterion for $M(k)$, and illustrate its use with a particular case of Gow’s conjecture, as described below.

Let $\kappa_1 = \nu_\ell([G : B])$, the $\ell$-valuation of $[G : B]$, and set $S_1 = \overline{I(\kappa_1)}$. It is known that $S_1$ is irreducible and equal to the socle of $\overline{I}$. Moreover, all terms $\overline{I(s)}$, $s > \kappa_1$, are known to be $(0)$. Let $\kappa_2 = \nu_\ell([G : P])$, where $P$ is a minimal standard parabolic subgroup of $G$, and set $S_2 = \overline{I(\kappa_2)}$. Any term of (1) lying strictly between $S_1$ and $S_2$ is known to be equal to $S_1$. Moreover, one has $S_2/S_1 \neq (0)$ if and only if $\ell \mid q + 1$.

What we show is that if $\ell$ divides $q + 1$ then $S_2/S_1$ is indeed an irreducible $KG$-module.

Our calculations seem to indicate that $S_2/S_1$ equals the socle $I/S_1$. More generally, for an arbitrary prime $\ell$ dividing $[G : B]$, we wonder if the distinct terms of (1) are in fact $I = S_\omega \supset \cdots \supset S_2 \supset S_1 \supset 0$, where $S_{i+1}/S_i$ is the socle of $\overline{I}/S_i$.

2 The underlying root system

We digress here to develop some notation. Fix a real Euclidean space with orthonormal basis $e_1, \ldots, e_n$. Then

$$\Phi = \{e_i - e_j \mid 1 \leq i \neq j \leq n\}$$

is a root system in the hyperplane orthogonal to $e_1 + \cdots + e_n$. We use the abbreviated notation

$$[i, j] = e_i - e_j, \quad 1 \leq i \neq j \leq n.$$

The set

$$\Pi = \{[i, i + 1] \mid 1 \leq i < n\}$$

is a fundamental system for $\Phi$, and the associated system of positive roots is

$$\Phi^+ = \{[i, j] \mid 1 \leq i < j \leq n\}.$$

Let $W$ stand for the Weyl group of $\Phi$. We identify $W$ with the symmetric group $S_n$ via the action of $W$ on $\{e_1, \ldots, e_n\}$. If $r = [i, j] \in \Phi^+$ then the reflection $w_r \in W$ is identified
with the transposition \((i, j) \in S_n\). As already mentioned, we view \(S_n\), and hence \(W\), as a subgroup of \(G\).

If \(r = [i, j] \in \Phi\) then \(a \mapsto t_r(a) = I + aE^{ij}\) is a group isomorphism from \(F_q^+\) into

\[X_r = \{I + aE^{ij} \mid t \in F_q\}\]

Here \(I\) stands for the \(n \times n\) identity matrix and \(E^{ij}\) is the \(n \times n\) matrix having a single non-zero entry, namely a 1, in the \(ij\)-th position. We observe that

\[wt_r(a)w^{-1} = t_{w(r)}(a), \quad w \in W, r \in \Phi, a \in F_q. \quad (2)\]

Suppose now \(n \geq 3\) and let \(r, s \in \Pi\) be distinct non-orthogonal roots. Then \(r + s \in \Phi^+\) and both \(X_r\) and \(X_s\) commute elementwise with \(X_{r+s}\). For group elements \(x, y\) we denote the commutator \([x, y] = xyx^{-1}y^{-1}\). For \(a, b \in F_q\) we have

\[[t_r(a), t_s(b)] = t_{r+s}((-1)^{\nu(r,s)}ab), \quad (3)\]

where \(\nu(r, s) = 0\) if \(r = [i, i + 1]\) and \(s = [i + 1, i + 2]\) for some \(i\), while \(\nu(r, s) = 1\) if \(s = [i, i + 1]\) and \(r = [i + 1, i + 2]\) for some \(i\).

To any subset \(J\) of \(\Pi\) we associate the standard parabolic subgroup \(P_J\) of \(G\), i.e. the subgroup of \(G\) generated by \(B\) and all \(w_r, r \in J\).

3 A result of Gelfand and Graev

We will require a modified version of a result due to Gelfand and Graev, originally proven in the context of complex representations [2].

This and the following section are the only ones in which the notation already introduced will be modified. This will allow for more generality. Here \(K\) will stand for an arbitrary field, subject solely to the condition of having a primitive \(p\)-th root of unity if the characteristic of \(K\) is different from \(p\). We will also allow here the case \(n = 1\).

To stress their dependence on \(n\), we let \(G(n) = \text{GL}_n(q)\) and \(U(n) = \text{U}_n(q)\), the upper unitriangular subgroup of \(G(n)\).
We consider the subgroups \( H(n) \), \( L(n) \) and \( A(n) \) of \( G(n) \), defined as follows. For \( n \geq 2 \) they respectively consists of all matrices of the form

\[
\begin{pmatrix}
1 & u \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
X & 0 \\
0 & 1
\end{pmatrix}
\text{ and }
\begin{pmatrix}
X & u \\
0 & 1
\end{pmatrix},
\]

where \( X \in G(n-1) \) and \( u \) is a vector in the column space \( F_q^{n-1} \). Note that \( A(n) = H(n) \rtimes L(n) \), where \( H(n) \) is canonically isomorphic to \( F_q^{n-1} \), and \( L(n) \) to \( G(n-1) \). Moreover, we have

\[
\begin{pmatrix}
X & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & u \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
X & 0 \\
0 & 1
\end{pmatrix}^{-1} =
\begin{pmatrix}
1 & Xu \\
0 & 1
\end{pmatrix}.
\]

We observe that \( U(n) \) is a subgroup of \( A(n) \). We further define \( H(1) \), \( L(1) \) and \( A(1) \) to be the trivial subgroups of \( G(1) \). For \( n > 1 \) we will view \( A(n-1) \) as canonically embedded in \( L(n) \). Under this embedding \( U(n-1) \) becomes a subgroup of \( U(n) \), and we have the decomposition \( U(n) = H(n) \rtimes U(n-1) \).

**3.1 Theorem** Any non-zero module \( M \) for \( G(n) \) or \( A(n) \) over \( K \), whether finite or infinite dimensional, has a one dimensional subspace that is \( U(n) \)-invariant.

**Proof.** The result is clear if \( K \) has characteristic \( p \), for in this case the only irreducible \( U(n) \)-module is the trivial one. Suppose henceforth that \( K \) has characteristic different from \( p \) and that \( K \) possesses a \( p \)-root of unity different from 1.

Since any \( G(n) \)-module is automatically an \( A(n) \)-module, it suffices to prove the theorem when \( M \) is an \( A(n) \)-module. We show this by induction on \( n \).

The group \( U(1) \) being trivial, it acts trivially on \( M \), so any one dimensional subspace of \( M \) will do. Suppose next that \( n > 1 \) and the result is true for all \( 1 \leq m < n \). Note that \( H(n) \) is a finite elementary abelian \( p \)-group. Our assumption on \( K \) implies that

\[
M = \bigoplus_{\lambda} M_{\lambda},
\]

where

\[
M_{\lambda} = \{ y \in M \mid hy = \lambda(h)y \text{ for all } h \in H(n) \},
\]
and $\lambda$ runs through all group homomorphisms $H(n) \rightarrow K^*$.

Suppose first that $H(n)$ acts trivially on $M$. We consider $M$ as a module for $L(n)$ and, as mentioned above, we view $A(n-1)$ embedded as a subgroup of $L(n)$. By inductive hypothesis there is a one dimensional subspace of $M$ that is invariant under $U(n-1)$, and therefore under $U(n) = H(n)U(n-1)$.

Suppose next that $M_\lambda \neq (0)$ for one or more non-trivial group homomorphisms $\lambda : H(n) \rightarrow K^*$. There is a right action of $L(n)$ on the set of all non-trivial group homomorphisms $\mu : H(n) \rightarrow K^*$ given by $\mu^x(h) = \mu(xhx^{-1})$ for all $h \in H(n)$ and $x \in L(n)$. We claim that this action is transitive. Indeed, our assumption on $K$ yields a non-trivial linear character $\nu : F_q^+ \rightarrow K^*$. This gives a non-trivial linear character $\epsilon : H(n) \rightarrow K^*$ defined by

$$\epsilon \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \nu(u_{n-1}), \text{ for } u = \begin{pmatrix} u_1 \\ \vdots \\ u_{n-1} \end{pmatrix} \in F_q^{n-1}.$$

As $\nu$ is non-trivial, formula (4) ensures that the stabilizer of $\epsilon$ in $L(n)$ is exactly $A(n-1)$, again viewed as subgroup of $L(n)$. Now the index $[L(n) : A(n-1)] = q^n - 1$, which is equal to the number of non-trivial group homomorphisms $H(n) \rightarrow K^*$. This proves the claim.

Now $M_\lambda \neq (0)$, so $xM_\lambda \neq (0)$ for all $x \in L(n)$. But clearly $x^{-1}M_\lambda = M_{\lambda x}$. By transitivity $M_\epsilon \neq (0)$. Moreover, $M_\epsilon$ is invariant under $A(n-1)$, the stabilizer of $\epsilon$. By inductive hypothesis there is a one dimensional subspace of $M_\epsilon$ that is invariant under $U(n-1)$, and therefore under $U(n) = H(n)U(n-1)$.

$\Box$

3.2 Corollary  Let $M$ be a non-zero $KG$-module. Suppose $M$ has a one dimensional $U$-invariant subspace $L$ that is contained in the $KG$-module generated by any other one dimensional $U$-invariant subspace of $M$. Then the socle of $M$ is irreducible and is equal to the $KG$-submodule generated by $L$. Thus $M$ irreducible if and only if it is completely reducible.

Proof. Let $N$ be an irreducible $KG$-submodule of $M$. From theorem 3.1 we know that $N$
has a one dimensional $U$-invariant subspace of $M$. By hypothesis $N$ contains $L$. Hence $N$ equals the $KG$-module generated by $L$, say $S$, which is then irreducible. Thus $M$ has only one irreducible submodule, namely $S$, so the socle of $M$ is irreducible and equals $S$. □

4 Complete reducibility of self-dual modules

As previously mentioned, here we also make modifications to our general conventions.

4.1 Theorem Let $K$ be a field, $G$ a group, and $U$ a subgroup of $G$ satisfying: (a) any non-zero $KG$-module has a one dimensional $U$-invariant subspace; (b) if a $KG$-module admits $\lambda : U \to K^*$ as a linear character then it also admits $\lambda^{-1}$ (for instance, a field $K$ containing a primitive $p$-th root of unity if the characteristic of $K$ is different from $p$, $G = GL_n(q)$ and $U = U_n(q)$, where now $n \geq 1$).

Let $M$ be a self-dual (and hence finite dimensional) $KG$-module. Suppose that $M$, when viewed as a $KU$-module, has no repeated irreducible constituents of dimension one. Then $M$ is a completely reducible (and multiplicity free) $KG$-module. Moreover, all submodules of $M$ are self-dual as well.

Proof. By assumption there is an isomorphism of $KG$-modules $\phi : M \to M^*$. To a $KG$-submodule $N$ of $M$ we associate the $KG$-submodule $N^\perp$ of $M$ defined as follows:

$$N^\perp = \{x \in M \mid \phi(x)(N) = 0\}.$$

Let $P = N \cap N^\perp$ and note that $P \subseteq P^\perp$. We claim that $P = (0)$. Suppose not. Then by (a) there is a one dimensional $U$-invariant subspace $L$ of $P$. Then $U$ acts upon $L$ via a group homomorphism, say $\lambda : U \to K^*$. By (b) there is a one dimensional subspace of $P$ upon which $U$ acts via $\lambda^{-1}$. This is the same way in which $U$ acts upon $L^*$. Thus $L^*$ is an irreducible constituent of the $KU$-module $P$, and hence of $P^\perp$.

Now from $L \subseteq P$ we get the $KU$-epimorphism $P^* \to L^*$. Likewise, the inclusion $P \subseteq M$ yields the $KG$-epimorphism $M^* \to P^*$, which can be combined with the $KG$-isomorphism $M \to M^*$ to produce the $KG$-epimorphism $M \to M^* \to P^*$ with kernel $P^\perp$.  

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All in all, this yields a $KU$-epimorphism $M/P^\perp \to L^*$. Hence the multiplicity of $L^*$ as an irreducible constituent of the $KU$-module $M$ is at least two. This contradiction proves that $P = (0)$.

Since $N \cap N^\perp = (0)$ and, as noted above, $M/N^\perp \cong N^*$, we deduce $M = N \oplus N^\perp$. This shows that $M$ is completely reducible.

The fact that $M$ is multiplicity free follows at once from (a) and the fact that $M$ has no repeated linear characters of $U$.

To see that $N$ is also self-dual, we consider the linear map $\phi_N : N \to N^*$ defined by $\phi_N(x)(y) = \phi(x)(y)$ for all $x, y \in N$. It is a $KG$-homomorphism with kernel $N \cap N^\perp = (0)$. Since $N$ and $N^*$ have the same dimension, $\phi_N$ is an isomorphism.

It remains to verify that the given example works. Clearly (a) is just theorem 3.1, while (b) can be confirmed as follows. Let $N$ be $KG$-module with a one dimensional $U$-invariant subspace $L$ upon which $U$ acts via the group homomorphism $\lambda : U \to K^*$. Recall that $H$ stands for the diagonal subgroup of $G$. Given $h \in H$, the subspace $hL$ of $N$ is also one dimensional. Furthermore, for $u \in U$ we have $uhL = hh^{-1}uhL = \lambda(h^{-1}uh)hL$, so $hL$ is $U$-invariant and is acted upon by $U$ via a group homomorphism, which we denote by $h\lambda$. Consider the special element $h = \text{diag}(-1,1,-1,1,...)$ of $H$. For any $r \in \Pi$ and $a \in F_q$ we have

$$h^{-1}t_r(a)h = t_r(-a) = t_r(a)^{-1}.$$  

Since $\lambda$ is determined by its effect on the fundamental root subgroups $X_r$, $r \in \Pi$, it follows that $h\lambda = \lambda^{-1}$, the inverse character of $\lambda$. Thus $U$ acts upon $hL$ via $\lambda^{-1}$, as required.

\[\square\]

5 The Steinberg lattice I

We return to the $RG$-lattice $I$ and its $\ell$-modular reduction $\overline{T}$. Given $x \in I$, we let $\overline{x} = x + \ell I \in \overline{T}$. The set $\{u\overline{x} | u \in U\}$ is a $K$-basis of $\overline{T}$, so $U$ acts upon $\overline{T}$ via the regular representation.

To a group homomorphism $\lambda : U \to R^*$ we associate the set $J(\lambda)$ of all $r \in \Pi$ such
that \( \lambda \) is non-trivial on \( X_r \). Let \( P_{J(\lambda)} \) be the corresponding standard parabolic subgroup of \( G \). We also associate to \( \lambda \) the element \( E_\lambda \) of \( RG \) defined by

\[
E_\lambda = \sum_{u \in U} \lambda(u)ue.
\]

Clearly \( E_\lambda \neq (0) \) and \( xE_\lambda = \lambda(-x)E_\lambda \) for all \( x \in U \). Thus \( E_\lambda \) spans the rank one submodule of \( I \) upon which \( U \) acts via \( \lambda^{-1} \).

Let \( c_\lambda = \nu([G : P_{J(\lambda)}]) \). Through skillful calculations, Gow shows that

\[
E_\lambda \in I(c_\lambda), \quad E_\lambda \notin I(c_\lambda + 1).
\] (5)

At this point Gow asserts that

\[
\overline{E_\lambda} \in \overline{I(c_\lambda)}, \quad \overline{E_\lambda} \notin \overline{I(c_\lambda + 1)}.
\] (6)

This crucial fact is true, but it does not follow automatically from (5). Indeed, we know from [3] that \( f(x, y) = 1 \) for some \( x, y \in I \), so \( x \) belongs to \( I(0) \) but not to \( I(1) \). Then \( z = \ell x \) belongs to \( I(1) \) and not to \( I(2) \), but \( \overline{z} = 0 \), so \( \overline{z} \) does belong to \( \overline{I(2)} \).

A special argument is required to justify (6). We digress here to supply the missing details. These are intimately related to the fact that each non-zero \( M(k) \) is a self-dual \( KG \)-module.

Let \( m = |U| \). As \( f \) has zero radical, we see that 0 is not an elementary divisor of \( f \). Thus, these are of the form \( \ell^{a_1}, ..., \ell^{a_m} \), where the \( a_1 \leq \cdots \leq a_m \) are non-negative integers and, by above, \( a_1 = 0 \). Let \( \{x_1, ..., x_m\} \) and \( \{y_1, ..., y_m\} \) be \( R \)-bases of \( I \) chosen so that \( f(x_i, y_j) = \ell^{a_i} \delta_{ij} \).

For ease of notation set \( c = c_\lambda \). We identify \( \ell^c R/\ell^{c+1} R \) with \( K = R/\ell R \) via the map

\[
r + \ell R \mapsto \ell^c r + \ell^{c+1} R, \quad r \in R.
\] (7)

The \( R \)-bilinear form \( f : I \times I \to R \) gives rise to a well-defined \( K \)-bilinear form, say \( f_c : \overline{I(c)} \times \overline{I(c)} \to K \), as follows

\[
f_c(\overline{x}, \overline{y}) = f(x, y) + \ell^{c+1} R, \quad x, y \in I(c).
\] (8)
Clearly \( \overline{I(c+1)} \) is contained in the radical of \( f_c \). This naturally produces a \( K \)-bilinear form \( f_c : M(c) \times M(c) \to K \). A basis \( B_1 \) for \( M(c) \) is formed by all \( \overline{x_i + I(c+1)} \), if any, such that \( a_i = c \). A basis \( B_2 \) is obtained by taking the corresponding \( \overline{y_i + I(c+1)} \). Taking into account the identification (7) and the definition (8), we see that the matrix of \( f_c \) relative the pair of bases \((B_1, B_2)\) is simply the identity matrix of size \( \dim_K M(c) \) (at this point this is possibly zero). Thus \( f_c \) is non-degenerate, so the radical of \( f_c \) is precisely \( I(c+1) \).

In particular, \( M(c) \) is self-dual.

Why is \( M(c) \neq (0) \)? Of course, this would follow from Gow’s statement (6). Why is this statement true? Well, by above this is equivalent to \( \overline{E_\lambda} \) not being in radical of \( f_c \). Since the group homomorphism \( \lambda^{-1} \), inverse to \( \lambda \), is associated to the same standard parabolic subgroup as \( \lambda \), it follows that \( E_{\lambda^{-1}} \) also belongs to \( I(c) \). We claim that \( f_c(\overline{E_\lambda}, \overline{E_{\lambda^{-1}}}) \neq 0 \), thereby justifying (6). Indeed, by virtue of lemma 3.1 and theorem 3.6 of [3] we have

\[
f_c(\overline{E_\lambda}, \overline{E_{\lambda^{-1}}}) = |U||[G : P_{J(\lambda)}]| + \ell^{c+1}R.
\]

Since \( c = \nu_\ell([G : P_{J(\lambda)}]) \) and \( \ell \nmid |U| \), our claim is established.

Taking into account the above discussion and theorem 3.1, we obtain the following result due to Gow.

**5.1 Theorem**  Let \( k \geq 0 \). Then a factor \( M(k) \) of (1) is not zero if and only if \( k = \nu_\ell([G : P]) \) for some standard parabolic subgroup \( P \) of \( G \). Moreover, in this case:

(a) A linear character \( \lambda : U \to K^* \) enters \( M(k) \) if and only if \( \nu_\ell([G : P_{J(\lambda)}]) = k \), in which case it enters only once.

(b) The \( KG \)-module \( M(k) \) is self-dual.

**6 The Steinberg lattice II**

Here we discuss some of the consequences of the results obtained in the previous sections.

We let \( \mathcal{A} \) stand for the set of all \( \nu_\ell([G : P]) \) as \( P \) ranges through \( \mathcal{P} \).

**6.1 Theorem**  The \( KG \)-module \( \overline{I} \) is multiplicity free.
Proof. This follows from theorem 3.1 and the fact that $U$ acts on $\mathbf{T}$ via the regular representation, where $\ell \nmid |U|$.

\[\square\]

6.2 Note The above result does not hold, in general, for the $\ell$-modular reduction of the Steinberg lattice of other classical groups, as no linear character a Sylow $p$-subgroup may be present in a given composition factor for such a group (cf. Example 5.4 of [3]). However, we do want to point out that the multiplicity of the two non-equivalent Weil modules found in [6] to be constituents of $\mathbf{T}$ when $\ell = 2$ for the symplectic group $\text{Sp}_{2n}(q)$, $q$ odd, is indeed one. This is so because a linear character of the type described above is present in each Weil module (cf. section 4 of [6]).

6.3 Theorem The $K^G$-module $I$ is self-dual if and only if it is irreducible.

Proof. If $I$ is irreducible then $I(1) = (0)$, so $f_0$ is a non-degenerate $G$-invariant bilinear form on $I$, whence $I$ is self-dual. Conversely if $I$ is self-dual, then theorem 4.1 implies that $I$ is completely reducible. But the socle of $I$ is known to be irreducible, so $I$ itself is irreducible.

\[\square\]

6.4 Theorem Let $k \in A$. Then $M(k)$ is a self-dual, completely reducible, non-zero $K^G$-module, each of whose irreducible constituents is also self-dual.

Proof. We know from theorem 5.1 that $M(k)$ is non-zero and self-dual. The remaining assertions follow from theorem 4.1.

\[\square\]

6.5 Theorem The irreducible constituents of the $K^G$-module $I$ are self-dual.

Proof. Taking into account (1), theorem 5.1 implies that each irreducible constituent of $I$ must be a constituent of one of the $K^G$-modules $M(k)$, $k \in A$. By theorem 6.4 each constituent of such $M(k)$ is self-dual, as required.

\[\square\]
We next produce an irreducibility criterion for $M(k)$. Given $P \in \mathcal{P}$, we see that $H$ acts transitively on the set of group homomorphisms $\lambda : U \to K^*$ associated to $P$. We denote by $E_P$ a fixed representative from the $H$-orbit of all $E_\lambda$, with $\lambda$ associated to $P$. Note that $E_P$ generates the same $RG$-submodule of $I$ as any other representative, so whether $E_P$ belongs to a given $I(k)$ or not depends only on $P$ and not on the chosen representative.

6.6 Theorem (Irreducibility Criterion) Let $k \in A$. Then $M(k)$ is an irreducible $KG$-module if and only if there exists $P \in \mathcal{P}$ such that $\nu_\ell([G : P]) = k$ and such that for any other $Q \in \mathcal{P}$ satisfying $\nu_\ell([G : Q]) = k$, the image of $E_P$ in $M(k)$ belongs to the $KG$-submodule of $M(k)$ generated by the image of $E_Q$.

Proof. Necessity is clear. We know from theorem 6.4 that $M(k)$ is completely reducible. Thus sufficiency follows from the above discussion and corollary 3.2. □

We wish to apply this criterion to confirm a particular case of Gow’s conjecture. First we need to make sure that the hypotheses of our criterion are met. This requires three subsidiary results: one involving the index in $G$ of certain parabolic subgroups, and two more describing some identities in the group algebra $RG$.

6.7 Lemma Suppose that $\ell$ divides $q + 1$. Let $k = \nu_\ell([G : B]) - \nu_\ell(q + 1)$. Then $k \in A$ and the only standard parabolic subgroups $P$ satisfying $\nu_\ell([G : P]) = k$ are those with associated fundamental subset equal to either $J = \{r\}$, with $r \in \Pi$, or $J = \{r, s\}$, with $r, s \in \Pi$ distinct and non-orthogonal to each other. The last type exists only if $n \geq 3$.

Proof. Let $J \subseteq \Pi$ and set $Q = P_J$, $s = \nu_\ell([G : Q])$. If $J = \emptyset$ then $s = k + \nu_\ell(q + 1) > k$. If $J = \{r\}$ for some $r \in \Pi$ then $s = k$. If $J = \{r, s\}$ for distinct $r, s \in \Pi$ then either $r \not\perp s$, in which case $[Q : B] = (q + 1)(q^2 + q + 1)$ and $s = k - \nu_\ell(q^2 + q + 1) = k - 0 = k$, or else $r \perp s$, in which case $[Q : B] = (q + 1)^2$ and $s = k - \nu_\ell(q + 1) < k$.

If $J \supseteq \{r, s, t\}$ for distinct $r, s, t \in \Pi$ then either $r, s, t$ are all orthogonal to each other, in which case $(q + 1)^3 = [P_{\{r,s,t\}} : B]$ divides $[Q : B]$, or two of them are non-orthogonal to each other but orthogonal to the third, in which case $(q + 1)^2(q^2 + q + 1) = [P_{\{r,s,t\}} : B]$.
divides \([Q : B]\), or else one of them is non-orthogonal to the other two, in which case 
\((q+1)^2(q^2+q+1)(q^2+1) = [P_{r,s,t} : B]\) divides \([Q : B]\). In all cases \(s < k\).

\[\square\]

7 Three identities in \(RG\)

Given a group homomorphism \(\lambda : F_q^+ \to R^*\) and \(r \in \Pi\) we associate to them the group homomorphism \(\lambda[r] : U \to R^*\) defined to be trivial on \(X_s\) for \(r \neq s \in \Pi\) and satisfying 
\(t_r(a) \mapsto \lambda(a)\) for all \(a \in F_q\).

For \(r \in \Pi\), we clearly have 
\[w_r e = -e.\] (9)

Moreover, formula (17) of [5] gives 
\[w_r t_r(a)e = t_r(-a^{-1})e - e, \quad a \neq 0.\] (10)

Given a subset \(Y\) of \(G\) we let \(\hat{Y}\) stand for the element \(\sum y\) of \(RG\). If \(r \in \Phi\) we write \(\hat{X}_r\) rather than \(\hat{X}_r\). From (9) and (10) we obtain 
\[w_r \hat{X}_r e = \hat{X}_r e - (q+1)e, \quad r \in \Pi.\] (11)

Finally, if \(\lambda : F_q^+ \to R^*\) is a group homomorphism we set \(\hat{X}_{\lambda(r)} = \sum_{a \in F_q} \lambda(a)t_r(a).\)

7.1 Theorem Assume \(n \geq 3\). Let \(\lambda : F_q^+ \to R^*\) be a non-trivial group homomorphism. Suppose that \(r, s \in \Pi\) are distinct and non-orthogonal. Then 
\[\hat{X}_r w_r \hat{X}_s w_s E_{\lambda[r]} = E_{\lambda[s]} = \hat{X}_r \hat{X}_{r+s} w_r w_s E_{\lambda[r]}\] (12)

Moreover, if \(\mu : F_q^+ \to R^*\) is a non-trivial group homomorphism then 
\[\hat{X}_{\mu(r)} w_r \hat{X}_s w_s E_{\mu[r]} = (q^2 + q + 1)E_{\mu[r] \lambda[s]} = \hat{X}_{\mu(r)} \hat{X}_{r+s} w_r w_s E_{\lambda[r]}.\] (13)

7.2 Theorem Let \(n \geq 3\) and let \(\lambda, \mu\) be non-trivial group homomorphisms \(F_q^+ \to R^*\). Suppose that \(r, s \in \Pi\) are distinct and non-orthogonal. Then 
\[\hat{X}_r w_r \hat{X}_s w_s E_{\lambda[r] \mu[s]} = E_{\lambda[s]} = \hat{X}_r \hat{X}_{r+s} w_r w_s E_{\lambda[r] \mu[s]}\] (14)

It will be better to postpone these two technical proofs until the end of the paper.
8 Irreducibility of $S_2/S_1$

8.1 Theorem Suppose $\ell$ divides $q + 1$. Let $S_1$ stand for the socle of $T$. Starting at $S_1$ and going up in the filtration (1), let $S_2$ be the first term strictly containing $S_1$. Let $M = S_2/S_1$. Then $M$ is an irreducible $KG$-module.

Proof. By (12) all $E_{P_J}$, where $J = \{r\}$ and $r \in \Pi$, generate the same $RG$-submodule of $I$. By (14) this submodule is contained in the one generated by any $E_{P_J}$, where $J = \{r, s\}$ and $r, s \in \Pi$ are distinct and non-orthogonal. By lemma 6.7 the index in $G$ of these standard parabolic subgroups has the same $\ell$-valuation, which is shared by no other type of standard parabolic subgroup. Now apply theorem 6.6 with $P = P_J$, $J = \{r\}$.

□

9 Proof of the identities in $RG$

Proof of theorem 7.1. We will make repeated and implicit use of (2), (9), (10) and (11) throughout.

The first and third terms (12) are clearly equal, a comment which also applies to (13). We are thus reduced to proving the second equality in both (12) and (13).

Let $R = U_{w_{r+s}}$, i.e. the group of all $u \in U$ such that $w_{r+s}uw_{r+s}^{-1} \in U$. It equals the product -taken in any order- of all root subgroups $X_t$, where $t \in \Phi^+$ is different from $r$, $s$ and $r+s$. We easily see that $R$ is a normal subgroup of $U$. Thus every element of $X_rX_{r+s}$ commutes with $\hat{R}$, so $\hat{X}_r$ and $\hat{X}_{r+s}$ also commute with $\hat{R}$. Using (2) we verify that $w_r$ and $w_s$ conjugate $\hat{R}$ back into itself. We will implicitly use all these facts below.

It is easy to see that $U$ acts on the right hand side of (12) via $\lambda[s]^{-1}$. Hence this right hand side must be a scalar multiple of $E_{\lambda[s]}$. This would be enough for our purposes, provided the scalar is not 0 modulo $\ell R$. It is not clear why that should be the case. In fact, a similar remark applies to (13), and in this case the right hand side does become 0 modulo $\ell R$ when $\ell$ divides $q^2 + q + 1$. This is to be expected, since, modulo $\ell I$, $E_{\mu[r]\lambda[s]}$ lies strictly above $E_{\lambda[r]}$ in the filtration (1) when $\ell | q^2 + q + 1$. 

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Returning to the proof, note that, by definition

\[ E_{\lambda[r]} = \hat{R}\hat{X}_{r+s}\lambda(r)s. \]

Therefore

\[ w_s E_{\lambda[r]} = \hat{R}\hat{X}_r\lambda(r+s)s - (q + 1)\hat{R}\hat{X}_r\lambda(r+s)s. \] (15)

We first work on the second of the summands appearing on the right hand side of (15).

As \( X_r \) and \( X_{r+s} \) commute elementwise, we have

\[ w_r\hat{X}_r\lambda(r+s)s = w_r\hat{X}_r\lambda(r+s)s = \hat{X}_r\lambda(s)r - (q + 1)\hat{X}_r\lambda(s)s. \] (16)

Multiplying (16) on the left by \(- (q + 1)\hat{X}_{r+s}R\) we get

\[ \hat{X}_{r+s}w_r\left( -(q + 1)\hat{R}\hat{X}_r\lambda(r+s)s \right) = (q + 1)^2\hat{R}\hat{X}_{r+s}\lambda(s)r - (q + 1)E_{\lambda[s]}. \] (17)

We now turn our attention to the first summand appearing in (15). We have

\[ \hat{X}_r\hat{X}_r\lambda(r+s)s = \sum_{\alpha,\beta,\gamma} t_r(\alpha)t_{r+s}(\beta)t_s(\gamma)\lambda(\beta)e = \sum_{\alpha,\beta,\gamma} t_{r+s}(\beta)t_r(\alpha)t_s(\gamma)\lambda(\beta)e, \] (18)

since \( X_{r+s} \) and \( X_r \) commute elementwise. By the commutator formula (3)

\[ \sum_{\alpha,\beta,\gamma} t_{r+s}(\beta)t_r(\alpha)t_s(\gamma)\lambda(\beta)e = \sum_{\alpha,\beta,\gamma} t_r(\beta)t_{r+s}(\gamma)t_s(\alpha)\lambda(\beta + (-1)^{\zeta(r,s)}\alpha)e, \] (19)

where \( \zeta(r,s) \) is 0 or 1 and depends only on the pair \((r,s)\). Thus (18) and (19) give

\[ w_r\hat{X}_r\lambda(r+s)s = \hat{X}_r\lambda(s)s + \sum_{\alpha \neq 0,\beta,\gamma} t_s(\beta)t_{r+s}(\gamma)\lambda(\beta + (-1)^{\zeta(r,s)}\alpha)e \left( t_r(-\alpha^{-1})e - e \right). \] (20)

The second summand in (20) equals

\[ \sum_{\alpha \neq 0,\beta,\gamma} t_s(\beta)t_{r+s}(\gamma)t_r(-\alpha^{-1})\lambda(\beta + (-1)^{\zeta(r,s)}\alpha)e - \sum_{\alpha \neq 0,\beta,\gamma} t_s(\beta)t_{r+s}(\gamma)\lambda(\beta + (-1)^{\zeta(r,s)}\alpha)e. \]

Using \( \sum_{\delta} \lambda(\delta) = 0 \) we easily verify that left multiplication of \( \hat{X}_{r+s} \) by each of these two summands is 0. Therefore (20) gives

\[ \hat{X}_{r+s}w_r\hat{R}\hat{X}_r\lambda(r+s)s = -q\hat{R}\hat{X}_{r+s}\lambda(s)e. \] (21)
Going back to (15) and taking into account (17) and (21) we obtain

\[ \hat{X}_{r+s} w_r w_s E_{\lambda[r]} = (q^2 + q + 1) \hat{R} \hat{X}_{r+s} \hat{X}_{\lambda(s)} e - (q + 1) E_{\lambda[s]}, \]  

(22)

Respectively multiplying (21) on the left by \( \hat{X}_r \) and \( \hat{X}_{\mu(r)} \) yields (12) and (13).

\[ \square \]

**Proof of theorem 7.2.** As in the previous result, we only need to prove the second equality. All remarks made earlier about \( R = U_{w_{r+s}}^+ \) are still valid. The implicit use of (2), (9), (10) and (11) remains in effect.

By definition

\[ E_{\lambda[r]\mu[s]} = \hat{R} \hat{X}_{r+s} \hat{X}_{\lambda(r)} \hat{X}_{\mu(s)} e. \]

We let

\[ A = \hat{X}_{r+s} \hat{X}_{\lambda(r)} \hat{X}_{\mu(s)} e = \sum_{\alpha, \beta, \gamma} t_{r+s}(\alpha) t_r(\beta) t_s(\gamma) \lambda(\beta) \mu(\gamma) e. \]

Then

\[ w_s A = -\hat{X}_r \hat{X}_{\lambda(r+s)} e + \sum_{\gamma \neq 0, \alpha, \beta} t_r(\alpha) t_{r+s}(\beta) t_s(\gamma) \lambda(\beta) \mu(-\gamma^{-1}) e - \hat{X}_r \hat{X}_{\lambda(r+s)} e \sum_{\gamma \neq 0} \mu(\gamma). \]

But

\[ -\sum_{\gamma \neq 0} \mu(\gamma) = \mu(0) = 1, \]

so

\[ w_s A = \sum_{\gamma \neq 0, \alpha, \beta} t_r(\alpha) t_{r+s}(\beta) t_s(\gamma) \lambda(\beta) \mu(-\gamma^{-1}) e \]

\[ = \sum_{\gamma \neq 0, \alpha, \beta} t_{r+s}(\beta) t_s(\gamma) t_r(\alpha) \lambda(\beta + (-1)^{\zeta(r,s)} \alpha) \mu(-\gamma^{-1}) e, \]

where again \( \zeta(r, s) \) is 0 or 1 and depends only on the pair \((r, s)\). Now

\[ w_r w_s A = A_1 + A_2 + A_3, \]

where

\[ A_1 = -\sum_{\gamma \neq 0, \beta} t_s(\beta) t_{r+s}(\gamma) \lambda(\beta) \mu(-\gamma^{-1}) e; \]

\[ A_2 = \sum_{\gamma \neq 0, \alpha \neq 0, \beta} t_s(\beta) t_{r+s}(\gamma) t_r(\alpha) \lambda(\beta - (-1)^{\zeta(r,s)} \alpha^{-1}) \mu(-\gamma^{-1}) e, \]

\[ 16 \]
A_3 = - \sum_{\gamma \neq 0, \beta} t_s(\beta) t_{r+s}(\gamma) \mu(-\gamma^{-1}) a_{\beta, \gamma} e

and

a_{\beta, \gamma} = \sum_{\alpha \neq 0} \lambda(\beta + (-1)\zeta(r,s)\alpha) \alpha \gamma.

For \gamma \neq 0 and any \beta \in F_q we have

-a_{\beta, \gamma} = \lambda(\beta),

so A_1 + A_3 = 0. Hence

w_r w_s A = A_2.

Thus

\hat{X}_{r+s} w_r w_s A = \sum_{\delta, \gamma \neq 0, \alpha \neq 0, \beta} t_s(\beta) t_{r+s}(\gamma + \delta) t_r(\alpha) \lambda(\beta - (-1)\zeta(r,s)\alpha^{-1}\gamma) \mu(-\gamma^{-1}) e.

Making the change of variable \epsilon = \delta + \gamma we obtain

\hat{X}_{r+s} w_r w_s A = \sum_{\epsilon, \alpha \neq 0, \beta} t_s(\beta) t_{r+s}(\epsilon) t_r(\alpha) b_{\alpha, \beta} e,

where

b_{\alpha, \beta} = \sum_{\gamma \neq 0} \lambda(\beta - (-1)\zeta(r,s)\alpha^{-1}\gamma) \mu(-\gamma^{-1}).

We may write this in the form

\hat{X}_{r+s} w_r w_s A = \hat{X}_{r+s} \sum_{\alpha \neq 0, \beta} t_s(\beta) t_r(\alpha) b_{\alpha, \beta} e

= \hat{X}_{r+s} \sum_{\alpha \neq 0, \beta} t_r(\alpha) t_s(\beta) b_{\alpha, \beta} e.

The last equality holds because \hat{X}_{r+s} absorbs all commutators arising from elements of X_r and X_s. Multiplying by \hat{X}_r and making a suitable change of variable we get

\hat{X}_r \hat{X}_{r+s} w_r w_s A = \hat{X}_{r+s} \sum_{\nu, \beta} t_r(\nu) t_s(\beta) c_{\beta} e,

where

c_{\beta} = \sum_{\gamma \neq 0} \mu(-\gamma^{-1}) d_{\beta, \gamma}

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and
\[ d_{\beta,\gamma} = \sum_{\alpha \neq 0} \lambda(\beta - (-1)^{\zeta(r,s)} \alpha^{-1} \gamma). \]

For \( \gamma \neq 0 \) and any \( \beta \) we have
\[ d_{\beta,\gamma} = -\lambda(\beta). \]

Therefore
\[ c_{\beta} = -\lambda(\beta) \sum_{\gamma \neq 0} \mu(-\gamma^{-1}) = \lambda(\beta). \]

Hence,
\[ \hat{X}_r \hat{X}_{r+s} w_r w_s A = \hat{X}_{r+s} \sum_{\nu,\beta} t_r(\nu) t_s(\beta) \lambda(\beta)e = \hat{X}_{r+s} \hat{X}_r \hat{X}_\lambda e. \]

Recalling the meaning of \( A \) and multiplying through by \( \hat{R} \) we obtain (14).

\[ \square \]

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