UNIQUE PRODUCT GROUPS AND CONGRUENCE SUBGROUPS

WILLIAM CRAIG AND PETER A. LINNELL

Abstract. We prove that a uniform pro-$p$ group with no nonabelian free subgroups has a normal series with torsion-free abelian factors. We discuss this in relation to unique product groups. We also consider generalizations of Hantzsche-Wendt groups.

1. Introduction

The group $G$ is called a unique product group if given two nonempty finite subsets $X, Y$ of $G$, then there exists at least one element $g \in G$ which has a unique representation $g = xy$ with $x \in X$ and $y \in Y$. It is easy to see that a unique product group is torsion free, though the converse is not true in general [13, 11].

There has been much interest recently in determining which torsion-free groups are unique product groups [1, 7, 6, 14]. Original motivation for studying unique product groups was the Kaplansky zero divisor conjecture, namely that if $k$ is a field and $G$ is a torsion-free group, then $kG$ is a domain. This and other early results are described in [10, §13.1], which we summarize here. The group $G$ is called a two unique product group if given two nonempty finite subsets $X, Y$ of $G$ with $|X| + |Y| \geq 3$, there exist at least two elements in $G$ which have a unique representation of the form $xy$ with $x \in X$ and $y \in Y$. There is also the trivial units conjecture, namely that if $k$ is a field and $G$ is a torsion-free group, then $kG$ has only trivial units, i.e. the only units of $kG$ are of the form $ag$ with $0 \neq a \in k$ and $g \in G$. It is shown in [10, Lemma 13.1.9] that if $G$ is a two unique product group, then $kG$ has only trivial units. It is clear that if $G$ is a two unique product group, then $G$ is a unique product group. However Strojnowski [15, Theorem 1] showed that all unique product groups are two unique product groups. He also proved that if $G$ fails to be a unique product group, then there exists a nonempty finite subset $X$ of $G$ such that no element of $G$ has a unique representation $xy$ with $x, y \in X$, and it follows easily from Strojnowski’s proof that the finite subset $X$ can be chosen to be arbitrarily large.

If $H \lhd G$ are groups and $H$ and $G/H$ are unique product groups, then $G$ is a unique product group [10, 13.1.8]. Also it is not difficult to see that direct limits of unique product groups are unique product groups. Since $Z$ is a unique product group, it follows that if there is a normal series $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ such that $G_{i+1}/G_i$ is torsion-free abelian for all $i$, then $G$ is a unique product group.

Let $p$ be a prime. We shall use the notation $Z_p$ for the $p$-adic integers, $Q_p$ for the $p$-adic numbers, and $GL_n(Z_p)$ for the ring of invertible $n$ by $n$ matrices over the $p$-adic integers.

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Suppose \( G \) is a uniform pro-\( p \) group [4, Definition 4.1]. Good examples of pro-\( p \) groups are congruence subgroups. Thus \( \{ A \in \text{GL}_n(\mathbb{Z}_p) \mid A \equiv 1 \mod p \} \) for \( p \) odd, and \( \{ A \in \text{GL}_n(\mathbb{Z}_2) \mid A \equiv 1 \mod 4 \} \) are uniform pro-\( p \) groups [4, Theorem 5.2]. Then \( kG \) is a domain for all fields \( k \) of characteristic 0 or \( p \) [5, Theorem 1.3, remark to Proposition 6.4], [17, Theorem 8.7.8] and [4, Corollary 7.26] (though it is unknown whether this remains true for fields of nonzero characteristic not equal to \( p \)). We make the following conjecture:

**Conjecture.** If \( G \) is a uniform pro-\( p \) group, then \( G \) is a unique product group.

We cannot answer this conjecture, however we verify that if \( H \) is a virtually solvable subgroup of \( G \), then \( H \) is a unique product group. In fact we show that \( H \) has an invariant series \( 1 = H_0 \lhd H_1 \lhd \cdots \lhd H_n = H \) where \( H_i \lhd H \) and \( H_{i+1}/H_i \) is torsion free abelian for all \( i \). We will also show that if \( H \) is virtually nilpotent, then \( H \) is nilpotent, and also if \( H \) is virtually abelian, then \( H \) is abelian. More precise statements are given in Theorems 2.4, 2.5 and Corollary 2.7. The group \( \langle x, y \mid x^{-1}y^2xy^2 \rangle \), often called the Hantzsche-Wendt group, was shown by Promislow [11] to be a nonunique product group. This group is virtually abelian but not abelian, so cannot be a subgroup of a uniform pro-\( p \) group. In the final section we consider some generalizations of the Hantzsche-Wendt group.

**Motivation for this paper** is a result concerning amenable groups with a locally invariant order [8, Theorem 1.1]. A locally invariant order \( < \) on the group \( G \) is a strict partial order \( < \) on \( G \) with the property that for all \( x, y \in G \) with \( y \neq 1 \), either \( xy > x \) or \( xy^{-1} > x \). Now a group with a locally invariant order is a unique product group. However if the group happens also to be amenable, then \( G \) has the much stronger property of being locally indicable.

**Notation.** We shall use the notation \( \mathbb{Z}(G) \) for the center of the group \( G \), \( \text{Fitt}(G) \) for the Fitting subgroup of \( G \) [16, p. 46], and \( \mathbb{N} \) will denote the positive integers \( \{1, 2, \ldots \} \).

## 2. Uniform pro-\( p \) groups

**Proposition 2.1.** Let \( G \) be a uniform pro-\( p \) group, let \( 0 \neq n \in \mathbb{Z} \), and let \( x, y \in G \). If \( x^n = y^n \), then \( x = y \).

**Proof.** This follows from [4, Theorem 4.17]. \( \square \)

**Lemma 2.2.** Let \( G \) be a virtually solvable residually finite \( p \)-group. Then \( G \) is solvable.

**Proof.** Let \( H \) be a normal solvable subgroup of finite index in \( G \), let \( \hat{G} \) denote the pro-\( p \) completion of \( G \), and let \( K \) denote the closure of \( H \) in \( \hat{G} \). Then \( K \) is a closed solvable subgroup of finite index \( \hat{G} \), so \( \hat{G}/K \) is a finite pro-\( p \) group and therefore \( \hat{G}/K \) is a finite \( p \)-group. It follows that \( \hat{G} \) is solvable, which completes the proof because \( G \) is a subgroup of \( \hat{G} \). \( \square \)

Let \( \mathcal{X} \) denote one of the following classes of groups:

- All nilpotent groups of class at most \( c \) for a fixed nonnegative integer \( c \).
- All nilpotent groups.
- All solvable groups.

Of course nilpotent groups of class at most 1 (case \( c = 1 \) above) is the class of abelian groups.
Lemma 2.3. Let $G$ be a virtually $X$-subgroup of a uniform group $U$. Then there is a closed uniform virtually $X$-subgroup $T$ of $U$ containing $G$.

Proof. Since the closure of $G$ is a virtually $X$-subgroup of $U$, we may assume that $G$ is closed. Then $G$ is a finitely generated pro-$p$ group by [4, Theorem 3.8] and therefore it has a characteristic uniform subgroup $K$ of finite index by [4, Corollary 4.3]. We now consider the Lie algebra of $U$, as described in [4, Chapter 4]. Then [4, Proposition 4.31(i)] shows that $K$ is a $\mathbb{Z}_p$-Lie subalgebra of $U$. Let $T/K$ denote the torsion subgroup of $U/K$; note that $T$ is a $\mathbb{Z}_p$-Lie subalgebra of $U$. Since $U$ is a finitely generated free $\mathbb{Z}_p$-module by [4, Theorem 4.17], we see that $T/K$ is finite and in particular, that $T$ is a virtually $X$-subgroup of $U$ because $K$ is virtually $X$. Also [4, Lemma 4.14(ii)] shows that $G \subseteq T$. We now see from [4, Proposition 7.15(i)] that $T$ is a closed uniform subgroup of $U$, and the result follows. □

Theorem 2.4. Let $G$ be a virtually abelian subgroup of a uniform pro-$p$ group. Then $G$ is abelian.

Proof. Suppose that $G$ is a subgroup of a uniform pro-$p$ group, and that $G$ is virtually abelian but not abelian. By Lemma 2.3 we may assume that $G$ is uniform. Let $B$ be a normal abelian subgroup of finite index in $G$. Replacing $B$ with its closure in $G$, which is still a normal abelian subgroup, we may assume that $B$ is closed, and then $G/B$ is a finite pro-$p$ group, that is a finite $p$-group. Now let $g \in G$ and $b \in B$. Then $g^r \in B$ where $q$ is a power of $p$ and we see that $g^rb = bg^q$. Therefore $g^q = (bgb^{-1})^q$ and we deduce from Proposition 2.1 that $g = bgb^{-1}$. This shows that $B \subseteq Z(G)$. Now if $g, h \in G$, then $g^r h = hg^r$ where $r$ is a power of $p$, hence $(hgh^{-1})^r = h^r$ and consequently $hgh^{-1} = g$ by Proposition 2.1. Thus $G$ is abelian as required. □

Theorem 2.5. Let $c$ be a nonnegative integer and let $G$ be a virtually nilpotent subgroup of class at most $c$ of a uniform pro-$p$ group. Then $G$ is nilpotent of class at most $c$.

Proof. By Lemma 2.3 we may assume that $G$ is uniform. By hypothesis, there is a normal subgroup $N$ of $G$ such that $N$ is nilpotent of class at most $c$ and of finite index in $G$. Replacing $N$ with its closure in $G$, we may assume that $N$ is closed in $G$. Since $Z(N)$ is closed in $G$ and since $G$ is powerful, we see that $G/Z(N)$ is a powerful pro-$p$ group [4, Chapter 3]. By [4, Theorem 4.20], the elements of finite order in $G/Z(N)$ form a finite subgroup $T/Z(N)$ which is normal in $G/Z(N)$ such that $G/T$ uniform. Then $NT/T$ is a normal subgroup of finite index in $G/T$ and has nilpotency class at most $c−1$. Now since $T/Z(N)$ is finite, $T$ is virtually abelian and we see that $T$ is abelian by Theorem 2.4. Now let $t \in T$. Then $t^q \in Z(N)$ for some power $q$ of $p$. Therefore if $h \in NT$, then $(hth^{-1})^q = h^q t h^{-1} = t^q$ and since $T$ is a torsion-free abelian group, we deduce that $hth^{-1} = t$. This shows $T \subseteq Z(NT)$. Now let $t \in T$ and $g \in G$. Then $g^r \in N$ where $r$ is a power of $p$ and we see that $g^r t = tg^r$, consequently $(tg^r)^q = g^r$. Therefore $tg^r t^{-1} = g$ by Proposition 2.1 and we conclude that $t \in Z(G)$. By induction $G/T$ is nilpotent of class at most $c−1$, and the result follows. □

Theorem 2.6. Let $G$ be a virtually solvable subgroup of a uniform pro-$p$ group. Then Fitt($G$) is nilpotent and then $G/$Fitt($G$) is torsion-free abelian.
Proof. By Lemma 2.5 we may assume that $G$ is uniform. Let $F = \text{Fitt}(G)$. Using [10] Theorem 7.19, we see that $G$ is a linear group over $\mathbb{Q}_p$ and it now follows from [16] Theorem 8.2(ii) that $F$ is nilpotent. Since the closure of a nilpotent subgroup is nilpotent, we see that $F$ is closed in $G$. By [10] Theorem 3.6, $G/F$ is virtually abelian. Let $T/F$ denote the elements of finite order in $G/F$. Since $G/F$ is a powerful pro-$p$ group [4] Chapter 3, $T/F$ is a finite normal subgroup of $G/F$ and $G/T$ is uniform [4] Theorem 4.20. By Theorem 2.5, $G/T$ is abelian and by Theorem 2.6, $T$ is nilpotent. The result follows.

Applying the Tit’s alternative, we obtain

Corollary 2.7. Let $G$ be a subgroup of a uniform pro-$p$ group and suppose $G$ contains no nonabelian free subgroups. Then $\text{Fitt}(G)$ is nilpotent and $G/\text{Fitt}(G)$ is torsion-free abelian.

Proof. By [4] Theorem 7.10, $G$ is a linear group over $\mathbb{Q}_p$. We now apply the Tit’s alternative [10] Corollary 10.17 to deduce that $G$ is virtually solvable. The result now follows from Theorem 2.6

3. Hantzsche-Wendt groups

Definition. Define the combinatorial generalized Hantzsche-Wendt group $G_n$ by

$$G_n = \langle x_1, \ldots, x_n \mid x_i^{-1}x_j^2x_ix_j^{-1} \forall i \neq j \rangle$$

Call any group isomorphic to some $G_n$ a CHW group for shorthand. Note that the Hantzsche-Wendt group is isomorphic to $G_2$. Of course $G_0 = 1$ and $G_1 \cong \mathbb{Z}$.

Lemma 3.1. Let $A_n = \langle x_1^2, x_2^2, \ldots, x_n^2 \rangle$, a subgroup of $G_n$. Then $A_n \triangleleft G_n$ and $A_n$ is free abelian on $\{x_1^2, \ldots, x_n^2\}$.

Proof. The relations $x_i^{-1}x_j^2x_ix_j^{-1} = 1$ imply that $x_i^{-1}x_j^2=x_j^{-2}$, hence $A_n$ is closed under conjugation and $A_n \triangleleft G_n$. The identity $x_i^{-1}x_j^2x_ix_j^{-1} = 1$ is equivalent to $x_i^2x_j^2 = x_j^{-2}x_i$. Furthermore, inverting both sides of this equation yields $x_j^{-2}x_i^{-1} = x_i^{-1}x_j^{-2}$, and multiplying this on the right and left by $x_i$ yields $x_ix_j^{-2} = x_j^{-1}x_i$. Thus we may write

$$x_i^2x_j^2 = x_i(x_ix_j^2) = (x_ix_j^{-2})x_i = x_j^{-1}x_i^2.$$

Therefore, all of the generators of $A_n$ commute, and it remains only to show that the generators of $A_n$ are linearly independent. Suppose by way of contradiction that the generators satisfy the relation $(x_i^2)^{m_1}\cdots(x_n^2)^{m_n}$, where the $m_i$ are integers, not all zero. Without loss of generality, we may assume $m_1 \neq 0$. Let $F_n$ denote the free group on $\{x_1, \ldots, x_n\}$, let $D = \langle a, b \mid a^2, b^2 \rangle$ denote the infinite dihedral group, and consider the epimorphism $\theta: F_n \to D$ defined by $\theta x_1 = ba$ and $\theta x_i = b$ for all $i \geq 2$. Since $\theta(x_i^{-1}x_j^2x_ix_j^{-1}) = 1$ for all $i \neq j$, we see that $\theta$ induces a homomorphism $\phi: G_n \to D$. Then $\phi(x_i^2) = (ba)^2$ and $\phi(x_i^2) = b^2 = 1$ for $i \geq 2$. Since $ba$ has infinite order, we have a contradiction and the result follows.

We need the following well known result.

Lemma 3.2. Let $n \in \mathbb{N}$, let $G = \langle x_1, \ldots, x_n \mid x_1^2 = \cdots = x_n^2 \rangle$, and let $N = \langle x_1x_2, \ldots, x_{n-1}x_n \rangle$. Then $N \triangleleft G$, $|G/N| = 2$, and $N$ is freely generated by $\{x_1x_2, \ldots, x_{n-1}x_n\}$. Furthermore any nontrivial element of finite order in $G$ is conjugate to one of the $x_i$.  


Proof. We have a well-defined surjective homomorphism \( \theta : G \to (\mathbb{Z}/2\mathbb{Z})^n \) defined by \( x_i \mapsto 1 \). Clearly \( N \subseteq \ker \theta \). It is easily checked that \( N < G \) and that \( |G/N| \leq 2 \). Therefore \( N = \ker \theta \) and we deduce that \( |G/N| = 2 \). We now apply the Kurosh subgroup theorem \[3\] Theorem I.7.8 and \[2\] Exercise 3, p. 212. We find that \( N \) is torsion free and any nontrivial finite subgroup of \( G \) is conjugate to one of the \( x_i \). Furthermore \( N \) is free of rank \( n - 1 \), and the result follows. □

**Theorem 3.3.** \( G_n \) is torsion free for all nonnegative integers \( n \).

**Proof.** Let \( A_n \) be defined as in Lemma 3.1. Then \( G_n/A_n \cong \langle x_1, \ldots, x_n \mid x_1^2, \ldots, x_n^2 \rangle \).

By Lemma 3.2, this group has a free subgroup \( N/A_n \) of index 2 freely generated by the images of \( x_1x_2, \ldots, x_{n-1}x_n \) in \( G_n/A_n \), and any nontrivial finite subgroup of \( G_n/A_n \) is conjugate to \( \langle A_nx_i \rangle/A_n \) for some \( i \). Therefore, to prove that \( G_n \) is torsion free, it will suffice to show that \( \langle A_n, x_i \rangle \) is torsion free for each \( i \). By symmetry, we need only show this for some particular \( i \), so we set \( i = 1 \) and prove that \( B := \langle A_n, x_1 \rangle = \langle x_1, x_2^2, \ldots, x_n^2 \rangle \) is torsion free.

Now set \( D = \langle x_2^2, \ldots, x_n^2 \rangle \). Since \( A_n \) is torsion free and \( D \leq A_n \), we have that \( D \) is torsion free. It is also an immediate corollary of the relations on \( G_n \) that \( D < B \), and then we see that \( B/D = \langle Dx_1 \rangle \). Since \( A_n \) is free abelian on \( \{ x_1^2, \ldots, x_n^2 \} \), we find that \( Dx_1^2 \) has infinite order in \( A_n/D \) and we deduce that \( B/D \cong \mathbb{Z} \), which is torsion free. Since both \( D \) and \( B/D \) are torsion free, it follows that \( B \) is torsion free, and from previous arguments we conclude that \( G_n \) is torsion free. □

**Proposition 3.4.** If \( 1 \leq m \leq n \) are integers, then \( G_m \) embeds in \( G_n \).

**Proof.** Let \( A_m = \langle x_1^2, \ldots, x_m^2 \rangle \), a free abelian normal subgroup of \( G_m \) and \( A_n = \langle x_1^2, \ldots, x_n^2 \rangle \), a free abelian normal subgroup of \( G_n \), by Lemma 3.1. Let \( f : G_m \to G_n \) be the natural homomorphism given by \( f(x_i) = x_i \), and consider the homomorphism \( f' : G_m/A_m \to G_n/A_n \) induced by \( f \), so \( f'(A_mx_i) = A_nx_i \). Since \( G_m/A_m \cong \langle x_1, \ldots, x_m \mid x_1^2, \ldots, x_m^2 \rangle \) and \( G_n/A_n \cong \langle x_1, \ldots, x_n \mid x_1^2, \ldots, x_n^2 \rangle \), it is clear that \( f' \) is one-to-one and we deduce that \( \ker f \subseteq A_m \), so it remains to prove that \( f \) is injective on \( A_m \). But \( A_m \) is free abelian on \( \{ x_1^2, \ldots, x_m^2 \} \) and \( A_n \) is free abelian on \( \{ x_1^2, \ldots, x_n^2 \} \) by Lemma 3.1 and the result follows. □

**Corollary 3.5.** For all integers \( n \geq 2 \), \( G_n \) is a nonunique product group.

**Proof.** It is well known that \( G_2 \) is a nonunique product group \[11\]. Since \( G_2 \) is isomorphic to a subgroup of \( G_n \) by Proposition 3.4, the result follows. □
\( \beta_i^2 = e_i \), where \( e_i \) is the \( i \)th standard basis element of \( \mathbb{R}^n \). Then \( \beta_i^2 = e_i \), and we have for all \( i \neq j \) that
\[
\beta_i^{-1} \beta_j^2 \beta_i = (B_i^{-1}, -B_i^{-1}b_i)(1, e_j)(B_i, b_i) = (1, B_i^{-1}b_i + B_i^{-1}e_j - B_i^{-1}b_i) = (1, B_i^{-1}e_j) = (1, -e_j) = \beta_j^{-2}
\]
and therefore \( \beta_i^{-1} \beta_j^2 \beta_i \beta_j^2 = 1 \), so the \( \beta_i \) satisfy the relations on \( G_n \). A second consequence of this representation of \( \Gamma \) is that \( \Gamma \cap \mathbb{R}^n = \Lambda \).

**Theorem 3.6.** Let \( n \) be odd. Then there is a surjective homomorphism \( \Phi : G_{n-1} \to \Gamma \) for all generalized HW groups \( \Gamma \) of dimension \( n \), and \( \ker \Phi \) is a free group.

**Proof.** Let \( n = 2k + 1 \) for some nonnegative integer \( k \) and that the group \( \Gamma \) is a generalized HW group of dimension \( n \). Define the map \( \Phi : G_{n-1} \to \Gamma \) by \( \Phi(x_1) = \beta_1 \), which is a homomorphism since the generators of \( \Gamma \) satisfy \( \beta_i^{-1} \beta_j^2 \beta_i \beta_j^2 = 1 \) whenever \( i \neq j \). It follows from the definition of \( \Phi \) that \( \{\beta_1, \ldots, \beta_{n-1}\} \subseteq \Phi(G_{n-1}) \). To show that \( \Phi \) is onto \( \Gamma \), it suffices to show that \( \beta_n \in \{\beta_1, \ldots, \beta_{n-1}\} \).

Set \( I = \{1, 2, \ldots, 2k\} \). Then by Proposition 2.1, we have that there exists \( j \not\in I \) such that \( |\{i \in I \mid [b_i]_j = \frac{1}{2}\}| \) is odd. Since \( I \) is defined as a subset of \( \{1, 2, \ldots, n\} \), it follows that \( j = 2k + 1 = n \). Now let \( \pi : I \to I \) be a permutation, and define \( P = \prod_{i=1}^{2k} \beta_{\pi(i)} = \prod_{i=1}^{2k} (B_{\pi(i)}, b_{\pi(i)}) \). From repeated application of the multiplication of these pairs, we obtain
\[
P = (B_{\pi(1)}B_{\pi(2)} \cdots B_{\pi(2k)}, b_{\pi(1)} + B_{\pi(1)}b_{\pi(2)} + B_{\pi(2)}b_{\pi(3)} + \cdots + B_{\pi(2(n-1))}b_{\pi(2k)}).
\]
Define \( v_x = \left( \sum_{i=1}^{x-1} B_{\pi(i)} \right) b_{\pi(x)} \). Now the \( B_i \) commute (since they are diagonal matrices) so we have \( \prod_{i=1}^{2k} B_{\pi(i)} = \prod_{i=1}^{2k} B_i = B_n \), and therefore \( P = \left( B_n, \sum_{i=1}^{2k} v_i \right) \). We now wish to show that for some permutation \( \pi \), the \( j \)th entry of \( \sum_{i=1}^{2k} v_i \) is equal to \( \frac{1}{2} \). Then for each \( v_i \) we have
\[
[v_i]_n = [B_{\pi(1)} \cdots B_{\pi(i-1)} b_{\pi(i)}]_n = (-1)^{i-1} [b_{\pi(i)}]_n
\]
since the \( (n, n) \)th entry of \( B_i \) is equal to \(-1\) for all \( i \in I \). Therefore we have
\[
\left[ \sum_{i=1}^{2k} v_i \right]_n = \sum_{i=1}^{2k} (-1)^{i-1} [b_{\pi(i)}]_n = [b_{\pi(1)} - b_{\pi(2)} - \cdots - b_{\pi(2k)}]_n.
\]
Earlier it was shown that if \( J = \{i \in I \mid [b_i]_j = \frac{1}{2}\} \), then \( |J| \) is odd. If \( \pi \) satisfies \( \pi(\{1, 2, \ldots, |J|\}) = J \), which we may assume without loss of generality, then this implies \( [b_{\pi(1)} - b_{\pi(2)} - \cdots - b_{\pi(2k)}]_n = \frac{1}{2} \). Therefore we may write
\[
P = (B_n, a_1e_1 + a_2e_2 + \cdots + a_ne_n)
\]
where \( 2a_i \in \mathbb{Z} \) for each \( i \) and \( a_\in \pm \frac{1}{2} \). Therefore,
\[
P^2 = (1, B_n(a_1e_1 + \cdots + a_ne_n) + a_1e_1 + \cdots + a_ne_n) = \pm e_n.
\]
Thus \( e_n \in \langle \beta_1, \ldots, \beta_{n-1} \rangle \), and it follows that \( \Lambda \subseteq \langle \beta_1, \ldots, \beta_{n-1} \rangle \). Now, \( \beta_1 \beta_2 \cdots \beta_n \in \mathbb{R}^n \) because \( B_1B_2 \cdots B_n = I \), and since \( \Gamma \cap \mathbb{R}^n = \Lambda \), we have \( \beta_1 \beta_2 \cdots \beta_n \in \Lambda \subseteq \langle \beta_1, \ldots, \beta_{n-1} \rangle \). Thus, \( \beta_n \in \langle \beta_1, \ldots, \beta_{n-1} \rangle \), and therefore \( \Phi \) must be onto \( \Gamma \).
Write $A_{n-1} = \langle x_2^1, \ldots, x_{n-1}^2 \rangle$ and $K = \ker \Phi$. Since $A_{n-1}$ is abelian by Lemma 3.1, $A_{n-1} \leq G_{n-1}$ and $\beta^2_i = e_i$, we see that $K \cap A_{n-1} = 1$ and therefore $K$ is isomorphic to a subgroup of $G_{n-1}/A_{n-1} \cong \langle x_1^1, \ldots, x_{n-1}^2 \rangle$. Furthermore $K$ is torsion free by Theorem 3.3 and we conclude that $K$ is free by the Kurosh subgroup theorem [3, Theorem I.7.8]. □

Remark. We note that the groups $G_n$ satisfy the Kaplansky zero divisor conjecture. In fact if $k$ is any field, then $kG_n$ can be embedded in a division ring. One way to see this is as follows. Since $G_m$ embeds in $G_n$ for $m \leq n$ by Lemma 3.4, we may assume that $n$ is odd. Also $G_n$ is torsion free by Theorem 3.3. Then by Theorem 3.6 there is a normal free subgroup $K$ of $G_n$ such that $G_n/K$ is isomorphic to an HW group of dimension $n$. Since an HW group is virtually abelian, we may apply [9, Theorem 1.5], at least in the case $k = \mathbb{C}$. However the arguments of [9, §4] apply for any field $k$.

References

[1] William Carter. New examples of torsion-free non-unique product groups. J. Group Theory, 17(3):445–464, 2014.
[2] Daniel E. Cohen. Combinatorial group theory: a topological approach, volume 14 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1989.
[3] Warren Dicks and M. J. Dunwoody. Groups acting on graphs, volume 17 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1989.
[4] J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal. Analytic pro-p groups, volume 61 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, second edition, 1999.
[5] Daniel R. Farkas and Peter A. Linnell. Congruence subgroups and the Atiyah conjecture. In Groups, rings and algebras, volume 420 of Contemp. Math., pages 89–102. Amer. Math. Soc., Providence, RI, 2006.
[6] D. Gruber, A. Martin, and M. Steenbock. Finite index subgroups without unique product in graphical small cancellation groups. Bull. Lond. Math. Soc., 47(4):631–638, 2015.
[7] Steffen Kionke and Jean Raimbault. On geometric aspects of diffuse groups. Doc. Math., 21:873–915, 2016. With an appendix by Nathan Dunfield.
[8] Peter Linnell and Dave Witte Morris. Amenable groups with a locally invariant order are locally indicable. Groups Geom. Dyn., 8(2):467–478, 2014.
[9] Peter A. Linnell. Division rings and group von Neumann algebras. Forum Math., 5(6):561–576, 1993.
[10] Donald S. Passman. The algebraic structure of group rings. Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1977.
[11] S. David Promislow. A simple example of a torsion-free, nonunique product group. Bull. London Math. Soc., 20(4):392–394, 1988.
[12] Bartosz Putrycz. Commutator subgroups of Hantzsche-Wendt groups. J. Group Theory, 10(3):401–409, 2007.
[13] Eliyahu Rips and Yoav Segev. Torsion-free group without unique product property. J. Algebra, 108(1):116–126, 1987.
[14] Markus Steenbock. Rips-Segev torsion-free groups without the unique product property. J. Algebra, 438:337–376, 2015.
[15] Andrzej Strojnowski. A note on u.p. groups. Comm. Algebra, 8(3):231–234, 1980.
[16] B. A. F. Wehrfritz. Infinite linear groups. An account of the group-theoretic properties of infinite groups of matrices. Springer-Verlag, New York-Heidelberg, 1973. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 76.
[17] John S. Wilson. Profinite groups, volume 19 of London Mathematical Society Monographs. New Series. The Clarendon Press, Oxford University Press, New York, 1998.
