1. Introduction

In a landmark 1982 paper [2], Atiyah and Bott studied the space of gauge equivalence classes of flat $SU(n)$ connections on an orientable surface with one fixed boundary component about which the holonomy was a generator of the center of the group. Using the Morse theory of the function which associated the norm square of the curvature to a connection, they obtained the group structure of the cohomology. Later Thaddeus [17] identified the ring structure of the cohomology when $n = 2$. Then Witten [19] identified the ring structure of the cohomology for general $n$ using methods from quantum field theory, which were subsequently made rigorous by Jeffrey and Kirwan [8].
Atiyah-Bott’s paper restricts to genus \( g \geq 2 \). In the case \( g = 2 \), in this paper we show that the cohomology ring can be identified with much more elementary methods, namely a particular Mayer-Vietoris sequence.

In [7], the first, second and fourth author studied Hamiltonian torus actions defined on the space of gauge equivalence classes of flat connections on a 2-manifold of genus 2. This space had been identified with \( \mathbb{C}P^3 \) by Narasimhan and Ramanan [14] using algebraic-geometric methods. The image of the moment map for this torus action is a tetrahedron, like the image of the moment map for the standard torus action on \( \mathbb{C}P^3 \). The authors of [7] attempted to obtain this identification using the methods of toric geometry, but were ultimately unsuccessful.

This paper should be accessible to a graduate student who has completed a first course in algebraic topology.

Let \( G = SU(2) \) and let \( T \) be its maximal torus of diagonal matrices. We will write elements of \( SU(2) \) as quaternions \( z + wj \) where \( z, w \in \mathbb{C} \) and \( |z|^2 + |w|^2 = 1 \).

Let \( \nu : SU(2) \times SU(2) \to SU(2) \) denote the commutator map. Note that its only singular value is the identity matrix \( I \).

Our results are as follows.

1. For regular values \( e^{i\theta} \neq I \) of \( \nu \), we describe a \( T \)-equivariant homeomorphism \( \nu^{-1}(e^{i\theta}) \to \mathbb{R}P^3 \), where \( T \) acts by pairwise conjugation on \( SU(2) \times SU(2) \) and left translation on \( \mathbb{R}P^3 = SO(3) \).
2. We find the cohomology ring and a cell decomposition of the space \( T := \nu^{-1}(I) \) of commuting pairs.
3. We compute the cohomology of \( M := \mu^{-1}(-I) \), where \( \mu : G^4 \to G \) is the product of commutators.
4. We give new calculations of the cohomology of \( A := M/G \), both as groups and as rings.

The group structure is due to Atiyah and Bott [2]. Atiyah and Bott’s research program was motivated by Morse theory, but completed using algebraic geometry. The ring structure is due to Thaddeus [17], who used methods from algebraic geometry and results from conformal field theory.

5. We calculate the cohomology of the total space of the prequantum line bundle over \( A \).
6. We also identify the transition functions of the induced \( SO(3) \) bundle \( M \to A \).

The layout of the paper is as follows.

In §2 we examine properties of the commutator map \( \nu \). For \( \theta \in [0, \pi] \) we introduce \( X_{\theta} := \nu^{-1}(e^{i\theta}) \), the space of pairs in \( SU(2) \) whose commutator equals \( e^{i\theta} \). For the regular values, corresponding to \( \theta > 0 \), in §3 we construct an explicit \( T \)-equivariant homeomorphism from \( X_{\theta} \) to \( \mathbb{R}P^3 \). Following this, in §4 we construct an explicit retraction from the complement of \( X_{\pi} \) to the space \( T := X_0 = \nu^{-1}(I) \) of commuting pairs. This allows us to use a Mayer-Vietoris sequence to identify the cohomology ring and a cell decomposition of \( T \). In §5 we study the space \( A = M/G \). We describe analogous retractions to the one in §4 which allow us to write \( A \) as the union of two open sets each of which is homotopy equivalent to \( T \). Using the resulting Mayer-Vietoris sequence we compute the cohomology groups \( H^*(A) \). In §6 we discuss the prequantum line bundle of \( A \). Using Mayer-Vietoris, we calculate the ring of the total space of a related line bundle which allows us
to obtain the ring structure of $H^*(A)$. In §7 we describe the 9-manifold $M$. We show that the bundle $M \to A$ has a local trivialization over two open sets, and find the transition function. Using Mayer-Vietoris, we calculate the cohomology ring of the restriction of the bundle to a certain submanifold of $A$ and later use this to compute $H^*(M)$. In §8 we recall results of Wall on 6-manifolds (see [18]) and show that the cohomology we computed for $A$ is consistent with the results in that paper. Finally, in §9 we conclude the calculation of $H^*(M)$ and the cohomology of the prequantum line bundle over $A$.

Notational note: To avoid conflict with $[\cdot,\cdot]$ which will denote “commutator” we use $[[X]]$ to denote “the equivalence class of $X$”.

2. The commutator map on $SU(2) \times SU(2)$

2.1. Preliminaries. Let $\mathbb{F}$ denote the finite field with 2 elements. Let $p : SU(2) \to SO(3) = \mathbb{R}P^3$ be the canonical projection. Let $T$ denote the maximal torus of diagonal matrices in $SU(2)$ and let $\bar{T} = p(T)$ denote the corresponding maximal torus of $\mathbb{R}P^3$. As topological groups, $T = \bar{T} = S^1$ and $p : T \to \bar{T}$ corresponds to the squaring map.

For $\theta \in [0, \pi]$, set:

$$W_\theta := \{(g, h) \in SU(2) \times SU(2) \mid [g, h] \sim e^{i\theta}\}$$

where “$\sim$” means “conjugate in $SU(2)$”;

$$X_\theta := \{(g, h) \in SU(2) \times SU(2) \mid [g, h] = e^{i\theta}\};$$

$$Y_\theta := \{g \in SU(2) \mid \exists h \in SU(2) \text{ with } [g, h] = e^{i\theta}\}.$$ 

For $S \subset [0, \pi]$ set $W_S := \cup_{\theta \in S} W_\theta$ and similarly define $X_S$ and $Y_S$. In particular, $W_{[0,\pi]} = SU(2) \times SU(2)$ and $X_{[0,\pi]} = \nu^{-1}(T)$ where $\nu : SU(2) \times SU(2) \to SU(2)$ is the commutator map.

In this section we consider $X_\theta$ for $\theta \in (0, \pi]$.

**Lemma 2.1.** (Meinrenken) $X_\theta \cong \mathbb{R}P^3$ for $\theta \in (0, \pi]$.

*Proof.* Since the only singular value of the commutator map $\nu : SU(2) \times SU(2) \to SU(2)$ is the identity matrix $I$, $\nu^{-1}(g)$ is homeomorphic to $\nu^{-1}(h)$ provided neither $g$ nor $h$ is $I$. In particular, $X_\theta = \nu^{-1}(e^{i\theta}) \cong \nu^{-1}(-1) = X_\pi$ if $\theta \neq 0$. Pick a basepoint $\ast \in X_\pi$. Define $\Phi : SU(2) \to X_\pi$ by $\Phi(g) := g \ast g^{-1}$. This gives a transitive action of $SU(2)$ on $X_\pi$ for which the stabilizer of each element is the center $\{I, -I\}$ of $SU(2)$. This $X_\pi \cong SO(3) = \mathbb{R}P^3$. \hfill $\square$

Since the above Lemma gives an explicit homeomorphism $X_\theta \to \mathbb{R}P^3$ only in the case $\theta = \pi$, in the next section we will proceed differently and give an explicit homeomorphism for arbitrary $\theta \in (0, \pi]$.

In preparation, in this section we give some properties of the commutator map.

**Lemma 2.2.** Let $\theta \in (0, \pi]$. Suppose $g = z + wj$ belongs to $Y_\theta$. Then $z = \pm |z| e^{i\theta/2}$.

*Proof.* Write $z = |z| e^{i\tau}$ in polar form. There exists $h$ such that $[g, h] = e^{i\theta}$. Then $gh^{-1} = e^{-i\theta} g$. Taking the trace of both sides gives $\cos(\tau) = \cos(\tau - \theta)$. Thus $\tau = \pm (\tau - \theta)$ (as elements of $S^1 = \mathbb{R}/(2\pi \mathbb{Z})$). Since $\theta \neq 0$, $\tau = -(\tau - \theta)$ so $2\tau = \theta$. That is, $\tau = \theta/2$ or $\theta/2 + \pi$. Therefore $z = |z| e^{i\tau} = \pm |z| e^{i\theta/2}$. \hfill $\square$
Similarly if \( h \in SU(2) \) such that \( \exists g \in SU(2) \) with \( [g,h] = e^{i\theta} \) then \( [h,g] = e^{-i\theta} \) so \( h \) has the form \( h = Qe^{-i\theta/2} + wj \) for some \( Q \in [-1,1] \), where \( w \in \mathbb{C} \) has norm \( \sqrt{1-|Q|^2} \).

**Corollary 2.3.** Suppose \((g,h) \in X_\theta \) where \( \theta \in (0,\pi] \). Then we have \((g,h) = (Pe^{i\theta/2} + Raj, Qe^{-i\theta/2} + Sbj) \) where \( P,Q \in [-1,1], R = \sqrt{1-|P|^2}, S = \sqrt{1-|Q|^2}, a,b \in S^1 \) . \( \square \)

**Corollary 2.4.** If \( \theta \neq 0 \) then \( Y_\theta \cong S^2 \).

**Proof.** Let \( g = Pe^{i\theta/2} + Raj \in Y_\theta \) where \( P \in [-1,1], R = \sqrt{1-|P|^2}, a = e^{i\alpha} \in S^1 \). A homeomorphism is given by \( g \mapsto (Ra,P) \) regarding the right hand side as an element of \( S^2 \) written in cylindrical coordinates. That is, \( g \mapsto (R \cos(\alpha), R \sin(\alpha), P) \in S^2 \subset \mathbb{R}^3 \). \( \square \)

Suppose \((Pe^{i\theta/2} + Raj, Qe^{-i\theta/2} + Sbj) \in X_\theta \) with \( \theta \neq 0 \). Since \( \dim X_\theta = \dim \mathbb{R}P^3 = 3 \) there must be some relation among \( P,Q,a,b \).

**Lemma 2.5 (Canonical Relation).** Let \((Pe^{i\theta/2} + Raj, Qe^{-i\theta/2} + Sbj) \in X_\theta \) with \( \theta \neq 0 \). Then
\[
(1) \quad PQ - vRS = e^{i\theta}(PQ - \bar{v}RS)
\]
where
\[
v := a\bar{b}.
\]

**Proof.** Expand \([Pe^{i\theta/2} + Raj, Qe^{-i\theta/2} + Sbj] = e^{i\theta} \) and equate either the coefficients of 1 or of \( j \). (Either results in the same equation.) \( \square \)

Write \( v = e^{i\phi} \). This defines \( \phi \) as an element of \( \mathbb{R}/(2\pi\mathbb{Z}) \). For \( P \neq 0 \), solving the canonical relation for \( Q \) (recalling that \( S = \sqrt{1 - Q^2} \)) gives

**Lemma 2.6.** Suppose \( \theta \neq 0 \). For \( P \neq 0 \),
\[
(2) \quad Q = \text{sgn}(P) \frac{KR}{\sqrt{P^2 + K^2 R^2}},
\]
where \( K(\phi) := \cos(\phi) - \cot(\theta/2) \sin(\phi) \).

**Proof.** If \( R = 0 \) then since \( \theta \neq 0 \), the canonical equation gives \( Q = 0 \) and the Lemma is satisfied. So assume \( R \neq 0 \).

Since we assumed \( P \neq 0 \), the canonical relation gives \( S \neq 0 \).

Set \( K := PQ/(RS) \). Recalling that \( S = \sqrt{1 - Q^2} \), we have
\[
Q = \text{sgn}(P) \frac{KR}{\sqrt{P^2 + K^2 R^2}}.
\]

The canonical relation gives
\[
K(1 - e^{i\theta}) = v - e^{i\theta}\bar{v}
\]
or equivalently
\[
K(1 - e^{i\theta})(e^{-i\theta/2}) = e^{-i\theta/2}(v - e^{i\theta}\bar{v}).
\]

Since
\[
(1 - e^{i\theta})e^{-i\theta/2} = e^{-i\theta/2} - e^{i\theta/2} = (-2i) \sin(\theta/2)
\]

and
\[
\cos(\phi) - \cot(\theta/2) \sin(\phi) = 0
\]

implies
\[
\phi = \frac{\theta}{2}
\]

Therefore
\[
Q = \text{sgn}(P) \frac{KR}{\sqrt{P^2 + K^2 R^2}}.
\]
we get
\[
\sin(\theta/2)K = e^{-i\theta/2}(v - e^{i\theta} \bar{v})/(-2i) = e^{-i\theta/2}(e^{i\theta}e^{-i\phi} - e^{i\phi})/(2i) \\
= (e^{i(\theta/2 - \phi)} - e^{-i(\theta/2 - \phi)})/(2i) = \sin(\theta/2 - \phi).
\]
Therefore
\[
K = \csc(\theta/2) \sin(\theta/2 - \phi) = \cos(\phi) - \cot(\theta/2) \sin \phi.
\]

\[\square\]

**Theorem 2.7.** For \(\theta \neq 0\) and \(0 < |P| < 1\), we have
\[
Q = \frac{\text{sgn}(P/K)}{\sqrt{1 + \frac{P^2 \sin^2(\theta/2)}{R^2 \sin^2(\theta/2 - \phi)}}}.
\]
When \(|P| = 1\), we have \(R = 0\) and the equation reduces to \(Q = 0\).

*Proof.* For \(\theta \neq 0\), \(|P| \in (0,1)\), equation (1) is equivalent to
\[
Q = \frac{\text{sgn}(P)}{\sqrt{P^2 + K^2 R^2}} = \frac{1}{\sqrt{1 + \frac{P^2}{R^2 R^2}}}
\]
where
\[
\sin(\theta/2)K = \sin(\theta/2 - \phi)
\]
Therefore
\[
Q = \text{sgn}(P/K)\frac{1}{\sqrt{1 + \frac{P^2}{R^2 R^2}}} = \text{sgn}(P/K)\frac{1}{\sqrt{1 + \frac{P^2 \sin^2(\theta/2)}{R^2 \sin^2(\theta/2 - \phi)}}}
\]
\[\square\]

If \(|P| = 1\), then \(R = 0\) and the equation implies \(Q = 0\). Similarly \(|Q| = 1\) implies \(P = 0\), however \(P = 0\) does not imply \(|Q| = 1\). Instead, for \(P = 0\) we have

**Lemma 2.8.** If \(\theta \neq 0\) then
\[
\{h \mid [a_j, h] = e^{i\theta}\} = \{Qe^{i\theta/2} + Sb_j \mid \text{either } S = 0 \text{ or } a^2 = b^2 e^{i\theta}\}.
\]

*Proof.* If \(P = 0\) then \(R = 1\). Substituting these into the canonical equation gives \(vS = e^{i\theta} \bar{v}S\).
Thus either \(S = 0\) or \(v = e^{i\theta} \bar{v}\). Since \(|v| = 1\), we have \(\bar{v} = v^{-1}\) so the latter condition says \(v^2 = e^{i\theta}\), or equivalently \(a^2 = b^2 e^{i\theta}\). \[\square\]

**2.2. Waves in the \((\phi, Q)\) cylinder.** Consider a fixed \(P\) with \(|P| \in (0,1]\). For each \(\theta > 0\) we get a periodic function
\[
Q_{\theta,P}(\phi) = \frac{\text{sgn}(P/K)}{\sqrt{1 + \frac{P^2 \sin^2(\theta/2)}{R^2 \sin^2(\theta/2 - \phi)}}},
\]
where by convention we set \(Q_{\theta,1}(\phi) \equiv 0\), which is the limiting function as \(|P| \to 1\).

**Definition 2.9.** We define a wave as a continuous map from \(S^1\) to \(S^1 \times [-1,1]\).
Let $\Gamma_{\theta,P}$ be one period of the graph of $Q_{\theta,P}$ (in the $(\phi,Q)$ cylinder) which we draw as a periodic function in the $(\phi,Q)$ cylinder. The curve $\Gamma_{\theta,P}$ (in the $(\phi,Q)$ cylinder) looks somewhat like a sine wave. As $\theta \to 0$ or $P \to 0$, the curves $\Gamma_{\theta,P}$ approach a square wave.

Extend the definition of $\Gamma_{\theta,P}$ to the case where $\theta = 0$ by letting it equal the limiting square wave.

As $P \to 0$ we also get a square wave but the resulting wave depends on whether $P$ approaches 0 from the left or the right. So we define $\Gamma_{\theta,0^+}$ as the limiting square wave as $P \to 0^+$ and similarly we have its reflection $\Gamma_{\theta,0^-}$.

The first plot shows three waves with $P = 1/\sqrt{2}$ illustrating $\theta = 1.2$, (the dashed graph), $\theta = 0.5$ (the dotted graph) and the limiting wave $\theta = 0$ (the solid line).

The second plot illustrates four waves with $\theta = \pi$. The dotted graph, nearest the axis, illustrates $P = 0.99$. At $P = 1$ it would become the axis. The dashed graph shows $P = 0.5$ and its reflection, the dash-dotted graph shows $P = -0.5$. The solid line is the square wave $\Gamma_{\pi,0^+}$. The wave $\Gamma_{\pi,0^-}$ (not shown) is its reflection about the axis.

We have oriented the curves counterclockwise if $P > 0$ and clockwise if $P < 0$. Taking the limit, we orient $\Gamma_{0^+,\phi}$ in the counterclockwise direction and $\Gamma_{0^-,\phi}$ in the clockwise direction.
3. **Explicit homeomorphism** $X_\theta \cong \mathbb{R}P^3$

Suppose $\theta \neq 0$. We give an explicit homeomorphism $\Phi_\theta : X_\theta \cong \mathbb{R}P^3$ generalizing the one in Lemma 2.1.

As per Corollary 2.3, the equation $[x, y] = e^{i\theta}$ requires that when written as quaternions, $x, y$ have the form $x = Pe^{i\theta/2} + R a$, $y = Q e^{-i\theta/2} + S b j$. Here $P, Q \in [-1, 1]$, $R = \sqrt{1 - P^2}$, $a \in S^1$, $S = \sqrt{1 - Q^2}$, $b \in S^1$. Furthermore, equation (1) of Lemma 2.5 must be satisfied, which determines $Q$ in terms of $P, a, b$. Thus generically the points of $X_\theta$ are parametrized by the values of $P, a, b$ or equivalently by $P, a,$ and $\phi$, where $\phi \in \mathbb{R}/(2\pi\mathbb{Z})$ is defined by $b = ae^{i\phi}$.

Generically a point in $\mathbb{R}P^3$ is also determined by three parameters. After picking a representative $z + wz$ with $|z|^2 + |w|^2 = 1$, the point $[[z + wz]] \in \mathbb{R}P^3$ is determined by the absolute value and argument of the ratio $z/w$ together with the argument of $z^2$, where we are using the notation $[[X]]$ for the equivalence class of $X$ introduced in the note at the end of §1. Replacing the representative by $-z - wz$ does not affect these parameters.

Recall that the space $X_\theta$ is generically parametrized by the parameters

$$P, R, a, Q, S, b$$

**Figure 2.** Four waves with $\theta = \pi$
with $P, Q, R, S \in \mathbb{R}$ satisfying $P^2 + R^2 = 1$ and $Q^2 + S^2 = 1$, and $a, b \in \mathbb{C}$ with $|a| = |b| = 1$, and satisfying the canonical relation (2.5). In order to define the inverse map
$$\Phi^{-1}_\theta : \mathbb{R}^3 \to X_\theta,$$
we need to specify $P, Ra, Q, Sb$ as functions of the parameters $z$ and $w$.

Let $[[z + w]]$ represent an element of $\mathbb{R}^3$ where $z, w \in \mathbb{C}$ with $|z|^2 + |w|^2 = 1$.

Write $z = Ze^{i\zeta}$, $w = We^{i\omega}$, and $\delta := \zeta - \omega$, where $\zeta$ is well defined only if $Z \neq 0$, $\omega$ when $W \neq 0$ and $\delta$ when $ZW \neq 0$.

**Step 1: Definition of $P, Ra$**

We define
$$P(z, w) := |z|^2 - |w|^2 = Z^2 - W^2 = 2Z^2 - 1$$
and
$$(Ra)(z, w) := -2e^{i\theta/2}zw.$$ where $P^2 + R^2 = 1$ and $|a| = 1$. The value of $a$ is uniquely determined only if $zw \neq 0$.

This concludes Step 1.

In Step 1 we showed how $P, Ra$ are specified as functions of $z$ and $w$. We now show how to determine $Q$ and $Sb$.

**Step 2: Definition of $Q, Sb$**

As in §2.2, the parameters $P$ and $\theta$ determine a wave $\Gamma_{\theta,P}$ in the $(\phi, Q)$ cylinder $C := S^1 \times [-1, 1]$. All the waves pass though the points $(\theta/2, 0)$ and $(\theta/2 + \pi, 0)$. Let $\pi : C \to S^2$ be the map which collapses $S^1 \times \{1\}$ to the north pole and $S^1 \times \{-1\}$ to the south pole. In detail,
$$\pi(\phi, Q) = (\sqrt{1 - |Q|^2} e^{i\phi}, Q)$$
where the right hand side describes a point in $S^2$ written in cylindrical coordinates. Notice that although the two curves $\Gamma_{0+,\theta}$ and $\Gamma_{0-,\theta}$ in $C$ differ (the latter is the reflection of the former), their images in $S^2$ under $\pi$ become the same. Indeed, each becomes the great circle passing through the poles and the antipodal points $\pi(\theta/2, 0)$ and $\pi(\theta/2 + \pi, 0)$.

**Step 2, Case I: $zw \neq 0$**

In case I, the quantity $a$ is well defined because $zw \neq 0$.

We define
$$s(z, w) := \text{Arg}(z/w).$$ The quantity $s$ is well defined because $zw \neq 0$.

Notice that $zw = 0$ if and only if $|P| = 1$. Therefore in case I we have $0 \leq |P| < 1$.

Recall that $b$ will be specified in terms of the parameter $\phi$ by $b = ae^{i\phi}$, so we need only to determine $Q$ and $\phi$.

Let
$$\gamma_{\theta,P} : [0, 2\pi] \to S^2$$
be the parametrization of the curve \( \pi(\Gamma_{\theta,P}) \) by normalized arc length, measured from \( \gamma_{\theta,P}(0) := \pi(\theta/2,0) \). To make this precise we need to decide whether increasing \( s \) moves clockwise or counterclockwise around the curve. Recall that in §2.2 we chose to orient the curve counterclockwise if \( P > 0 \) and clockwise if \( P < 0 \). Taking the limit, we oriented \( \pi(\Gamma_{P,\theta}) \) in the counterclockwise direction and \( \pi(\Gamma_{0,-\theta}) \) in the clockwise direction and obtained corresponding parametrizations \( \gamma_{0,\theta} \) and \( \gamma_{0,-\theta} \). (See §2.2 for the definition of \( \Gamma_{P,\theta} \).)

In the case \( 0 < |P| < 1 \), define \( \phi \) and \( Q \) by letting \((\sqrt{1 - Q^2} e^{i\phi}, Q)\) be the cylindrical coordinates of the point \( \gamma_{P,\theta}(s) \in \pi(\Gamma_{P,\theta}) \).

Extend the definition of \((\phi, Q)\) to the case \( P = 0 \) by using either \( \gamma_{0,\theta}(s) \) or equivalently \( \gamma_{0,-\theta}(s) \). These produce the same point since the curves are related by reflection about the equator and we have reversed orientation.

Set
\[
S(z, w) := \sqrt{1 - Q^2}.
\]
Set
\[
b(z, w) := ae^{-i\phi}.
\]

For \( zw \neq 0 \) define \( \Phi_{\theta}^{-1} : \mathbb{R} P^3 \rightarrow X_{\theta} \) by
\[
\Phi_{\theta}^{-1}([z + wj]) := (Pe^{i\theta/2} + Ra j, Q^{-i\theta/2} + Sb j)
\]
with the values of \( P, Ra, Q, Sb \) given by (3), (4), (8) and (9) above. By construction, the canonical relation will be satisfied, so the right hand side will lie in \( X_{\theta} \).

This concludes Step 2, Case I.

**Step 2, Case II: \( zw = 0 \).**

Suppose alternatively that \( zw = 0 \). Then either \( Z = 0 \) (which results in \( P = -1, R = 0 \)) or \( W = 0 \) (which results in \( P = 1, R = 0 \)). Since \(|P| = 1\), we will want to set \( Q = 0, S = 1 \) and then the canonical relation (Lemma 2.5) will be satisfied.

In Case II, we have neither a value for \( a \) nor one for \( s \) and thus cannot argue as in Case I. Instead we proceed as follows.

To complete the definition of \( \Phi_{\theta}^{-1} \) in case II, it remains to determine a value for \( b \) which makes the map \( \Phi_{\theta}^{-1} \) continuous.

Since \( zw = 0 \), either \( z = 0 \) or \( w = 0 \). Consider first the case \( z = 0 \). When \( z = 0 \) we have no value of \( \zeta \), but there is a well defined \( \omega \). Consider a point with \( z \) near zero but not equal to zero. This results in a point with \( R \) near 0 and \( P \) near \(-1\). Thus the graph \( \Gamma_{P,\theta} \) is close to the graph of the constant map \( Q = 0 \) (and since we are near the equator \( Q = 0 \) of the sphere \( S^2 \), the projection \( \pi \) has little effect on distances). In general, the relation between changes in \( s \) on the resulting value of \( \phi \) is complicated. However on the graph \( Q = 0 \), distances between points correspond exactly to changes in \( \phi \). When \( z \) is near zero, the wave is close to the straight line \( Q = 0 \), so a change in \( s \) produces the almost identical change in \( \phi \). Thus when \( z \) is near zero, we have
\[
\alpha - \beta \approx \phi \approx s = \text{Arg}(z/w) = \zeta - \omega,
\]
where \( a = e^{i\alpha} \) and \( b = e^{i\beta} \). We wish to determine the value of \( \beta \) which makes \( b := e^{i\beta} \) a continuous function of \( z \) and \( w \). The definition of \( Ra \) gives \( \alpha - \theta/2 = \zeta + \omega + \pi \). Therefore

\[
\beta \approx \alpha - \zeta + \omega = \zeta + \omega + \theta/2 - \zeta + \omega + \pi = 2\omega + \theta/2 + \pi.
\]

So when \( z = 0 \) we set

\[
(11) \quad \beta := 2\omega + \theta/2 + \pi
\]
to get a continuous map.

Similarly when \( w = 0 \) have \( P = 1 \) and for \( w \) near zero we have \( \phi \approx -s \). Therefore in this case we set

\[
(12) \quad \beta := 2\zeta + \theta/2 + \pi
\]
in this case to produce a continuous map.

In other words, in Case II we again define \( \Phi_{\theta}^{-1} \) as in (10) with the values of \( P, Ra \) given by (3) and (4) with \( b \) given by (11) or (12). This concludes Step 2, case II.

Notice that since \( |z|^2, |w|^2 \) and \( zw \) are quadratic, replacing the representative \( z + wj \) by \( -z - wj \) yields the same point. Therefore the map \( \Phi_{\theta}^{-1} \) is well defined. This concludes the definition of the map \( \Phi_{\theta}^{-1} \). It gives a homeomorphism from \( \mathbb{R}P^3 \) to \( X_\theta \).

If \( t \) lies in the maximal torus \( T \), then replacing \( [z + wj] \) by \( t[z + wj] \) produces no change in \( P \) and \( s \) and thus no change in \( Q \) or \( \phi \). However \( a \) gets replaced by \( at^2 \) and \( b \) gets replaced by \( bt^2 \). Thus \( \Phi_{\theta}^{-1} \) is \( T \)-equivariant with respect to translation action on \( \mathbb{R}P^3 \) and conjugation on \( X_\theta \).

Therefore we have shown

**Theorem 3.1.** There is a \( T \)-equivariant homeomorphism \( \Phi_{\theta} : X_\theta \cong \mathbb{R}P^3 \).

\[ \Box \]

### 4. Retraction

The only singular value of the commutator map \( \nu : SU(2) \times SU(2) \to SU(2) \) is the identity matrix \( I \). The space \( T := X_0 \) of commuting 2-tuples in \( SU(2) \) (which is not a manifold) has been studied in several places (see for example [1], [4], [6]).

Although we will provide explicit deformation retractions when needed, for motivation we note the following theorem which shows the existence of such retractions under some hypotheses.

**Theorem 4.1.** Let \( M \) be a smooth manifold and let \( f : M \to \mathbb{R} \) be smooth. Let \( c \) be an isolated critical value of \( f \). If \( f \) has no critical valuess in \( (c,d) \) then \( f^{-1}(c) \) is a deformation retract of \( f^{-1}([c,d]) \).

**Proof.** The result follows from the Remark 3.4 of [12] and the discussion in Chapter 3 of that book.

Applying Theorem 4.1 to the function \( \text{Trace} : X_{[0,\pi]} \to \mathbb{R} \) shows that there is a deformation retraction \( X_{[0,\pi]} \to T \). In this section we will describe two such deformation retractions \( X_{[0,\pi]} \to T \) with a view towards computing transition functions for a bundle \( M \to A \) to be defined later.
4.1. Gradient Flow. Let \( g = \text{su}(2) \), the Lie algebra of \( G \). Since \( G \) is compact, it has a \( G \)-invariant Riemannian metric defined by the trace form and the associated norm
\[
B(u, v) = -\text{tr}(uv), \quad ||u||^2 = B(u, u), \quad u, v \in g.
\]
where \( \text{tr} \) denotes the trace. This induces a \( G \)-invariant metric on \( G^2 \). Define the commutator map
\[
\nu : G^2 \rightarrow G, \quad \nu(g, h) = [g, h] = ghg^{-1}h^{-1}.
\]
Any vector in \( T_{(g, h)}G^2 \) can be identified with a vector in \( g^2 \) by left translation and we often perform our computation in \( g \). For \( u \in g \) and \( g \in G \), let
\[
u_g(u) = \text{Ad}(g)(u).
\]
A direct computation shows that
\[
d\nu_{(g, h)} : g^2 \rightarrow g, \quad d\nu_{(g, h)}(u, v) = (u^{h^{-1}} - u + v - v^{g^{-1}})hg.
\]
Notice that the map
\[
g \rightarrow g, \quad u \mapsto u^h - u
\]
has rank 2 if and only if \( u \neq \pm I \). Hence \( d\nu_{(g, h)} \) has full rank (rank 3) if and only if \( [g, h] \neq I \). Now we let
\[
f : G^2 \rightarrow \mathbb{R}, \quad f = \text{tr} \circ \nu.
\]
For \( (u, v) \in g^2 \), the gradient \( \nabla f \) at \( (g, h) \in G \), by definition, satisfies
\[
B(\nabla f, (u, v)) = \text{tr}(\nu(g, h)(d\nu_{(g, h)}(u, v))).
\]
Lemma 4.2. Suppose \( A \in G \). Then \( \text{tr}(Au) = 0 \) for all \( u \in g \) if and only if \( A = \pm I \).

Proof. This follows from direct computation. Let
\[
A = \begin{bmatrix} w_1 + iw_2 & -z_1 - iz_2 \\ z_1 - iz_2 & w_1 - iw_2 \end{bmatrix}, \quad u = \begin{bmatrix} i & b_1 - ib_2 \\ -b_1 - ib_2 & -i \end{bmatrix}.
\]
Then
\[
\text{tr}(Au) = -2rw_2 + 2b_1z_1 - 2b_2z_1.
\]
Hence if \( \text{tr}(Au) = 0 \) for all \( u \in g \), then
\[
z_1 = z_2 = w_2 = 0.
\]
Since \( A \in G \), \( \det(A) = 1 \) and this implies \( w_1 = \pm 1 \). Hence \( A = \pm I \). \( \square \)

It then follows from Equations (13, 14) that

Corollary 4.3. \( \nabla f = 0 \) if and only if \( \nu(g, h) = \pm I \). \( \square \)

For \( c \in [-2, 2] \), let \( M_{[a, b]} = f^{-1}([a, b]) \) and \( M_c = f^{-1}(c) \). Since \( G^2 \) is compact, \( M_c \) is compact since it is the inverse image of a closed set. Fix \( d \in (-2, 2) \) and let
\[
F : G^2 \times \mathbb{R} \rightarrow G^2
\]
be the gradient flow of \( f \) such that \( F(g, h, 0) \in M_d \). It then follows from Corollary 4.3 that
Corollary 4.4. Let \((g, h) \in M_c\) for \(c \in (-2, 2)\). Then there exists \((g', h') \in M_d\) and \(t\) such that \(F(g', h', t) = (g, h)\).

Proof. Suppose \(c < d\). Choose \(a\) such that \(d < a < 2\).

Since \(|\nabla f_{(g,h)}| > 0\) for all \((g, h) \in M_{[c,a]}\), the space \(M_{[c,a]}\) diffeomorphically retracts to \(M_{[d,a]}\) by the \((-\nabla f)\)-flow by Morse theory. Moreover \(M_c\) is diffeomorphic to \(M_d\) by this flow. The case of \(c < d\) is similar, but using the \((\nabla f)\)-flow instead. \(\Box\)

Starting at \((g, h) \in M_c\) with \(c \neq -2\), and again by Corollary 4.3, we have

\[
\lim_{t \to -\infty} F(g, h, t) \in M_{-2}, \quad \lim_{t \to \infty} F(g, h, t) \in M_2.
\]

Here the limit is pointwise with respect to the metric \(B\), but it is also uniform since \(G^2\) is compact. Let

\[
M_{-\infty} := \lim_{t \to -\infty} F(M_d, t), \quad M_{\infty} := \lim_{t \to \infty} F(M_d, t).
\]

It is immediate that both \(M_{-\infty} \subseteq M_{-2}\) and \(M_{\infty} \subseteq M_2\) are compact since both are images of the compact set \(M_d\) by the flow \(F\).

Theorem 4.5. \(M_{\infty} = M_2\).

Proof. \(M_2\) is a proper subvariety in \(G^2\) and \(G^2 \setminus M_2\) is open and dense in \(G^2\). Let \((a, b) \in M_2\), \(\epsilon > 0\) and \(\Gamma_{\epsilon}(a, b) \subset G^2\) the \(\epsilon\)-ball centered at \((a, b)\). Then \(\Gamma_{\epsilon}(a, b) \setminus M_2\) is open and dense in \(\Gamma_{\epsilon}(a, b)\). Since \(\lim_{r \to \infty} ||\nabla f_{(g,h,t)}|| = 0\) for \((g, t) \notin M_2\), there exists \(0 < \delta < \epsilon\) and \((g, h) \in \Gamma_{\epsilon}(a, b) \cap M_{2-d}\) such that \(\lim_{t \to \infty} F(g, h, t) \in \Gamma_{\epsilon}(a, b) \cap M_2\). By Corollary 4.4, there exists \((g', h') \in M_d\) and \(t\) such that

\[
F(g', h', t) = (g, h).
\]

Hence \(\lim_{t \to \infty} F(g', h', t) \in \Gamma_{\epsilon}(a, b) \cap M_2\). Hence \(\lim_{t \to \infty} F(M_d, t)\) is dense in \(M_2\). Since \(M_d\) is compact, \(\lim_{t \to \infty} F(M_d, t)\) is closed in \(M_2\). Hence \(\lim_{t \to \infty} F(M_d, t) = M_2\). \(\Box\)

4.2. Explicit Retraction. In this subsection we will consider another retraction \(r : X_{(0,\pi)} \to T\). Unlike the gradient flow retraction, the restriction of \(r\) to \(X_{\theta}\) is not onto for \(\theta > 0\). However the map \(r\) is explicit enough that we can use it to compute transition functions.

As before, define \(\phi\) by \(v = e^{i\phi}\) where \(v := ab\).

As in Subsection 2.2, we set \(C := S^1 \times [-1, 1]\). As before, for each pair \(P, \theta\) with \(|P| > 0\) we have a curve \(\Gamma_{\theta,P} \subset C\) passing through \((\theta/2, 0)\) while for \(P = 0\) we have curves \(\Gamma_{0+\theta}\) and \(\Gamma_{0-\theta}\) through \((\theta/2, 0)\).

Given \(\theta \in (0, \pi]\) and \(\alpha \in S^1\), each value of \(P \neq 0\) determines a unique wave \(\Gamma_{\theta,P}\) and each point \((\phi, Q)\) on that wave determines a point \((P e^{i\phi/2} + \sqrt{1 - P^2} a j, Q e^{-i\phi/2} + \sqrt{1 - Q^2} a e^{-i\phi})\) \(\in X_{\theta}\).

For \(|P| > 0\) we let \(\tilde{\gamma}_{\theta,P} : [0, 2\pi] \to C\) be the parametrization of the curve \(\Gamma_{\theta,P}\) by normalized arclength from \((\theta/2, 0)\) measured counterclockwise if \(P > 0\) and clockwise if \(P < 0\). Similarly in the case \(P = 0\) we have parametrizations of \(\Gamma_{\theta,0+}\) and \(\Gamma_{\theta,0-}\) oriented consistently with the preceding.

Define a homeomorphism \(\Psi_{\theta,\theta',P} : \Gamma_{\theta,P} \cong \Gamma_{\theta',P}\) by

\[
\Psi_{\theta,\theta',P}(\tilde{\gamma}_{\theta,P}(s)) := \tilde{\gamma}_{\theta',P}(s)
\]
That is, it takes a point on the curve $\Gamma_{\theta,P}$ to the corresponding point on the curve $\Gamma_{\theta',P}$ preserving the ratio
\[
\frac{\text{arc length of segment to point}}{\text{arc length of curve}}.
\]

For $(\phi, Q) \in \Gamma_{P,\theta}$ let $(\phi_{\theta,\theta'}, P, Q_{\theta,\theta'})$ be the coordinates (in the $(\phi, Q)$ cylinder) of $\Phi_{\theta,\theta',P}(\phi, Q) \in \Gamma_{\theta',P}$. For $t \in [0, 1]$, set $(\phi_{t\theta,P}, Q_{t\theta,P}) := \Psi_{t\theta,\theta',P}(\phi, Q)$ and set $S_{t\theta,P} := \sqrt{1-Q_{t\theta}^2}$.

We define a deformation $r : X_{[0,\pi]} \times I \to X_{[0,\pi]}$ with $\operatorname{Im} r_0 \subset X_0 = T$, where for a homotopy $H$ we use the convention $H_t(x) := H(x, t)$.

First consider $P \neq 0$. In this case we set
\[
r((Pe^{i\theta/2} + Raj, Qe^{-i\theta/2} + Sbj), t) := (Pe^{it\theta/2} + Raj, Qe^{-it\theta/2} + S_{t\theta,P}b_{t\theta,P}j).
\]
By construction $\operatorname{Im} r_0 \subset X_0 = T$.

**Lemma 4.6.** The map $r$ is continuous and extends (uniquely) to a continuous function $\overline{X_{[0,\pi]} \times I} \to \overline{X_{[0,\pi]}}$ with $r(\overline{X_{[0,\pi]}}, 0) \subset T$.

**Proof.** Set
\[
B := \{(Pe^{i\theta/2} + Raj, Qe^{-i\theta/2} + Sbj) \in X_{[0,\pi]} \mid P \neq 0\}.
\]
Note that the closure $\bar{B}$ equals the closure $\overline{X_{[0,\pi]}}$. By writing an explicit formula for $r$ in terms of integrals giving the arclengths, we see that the function $r$ is uniformly continuous on the domain $B \times I$. Therefore $r$ extends uniquely to $\overline{X_{[0,\pi]}}$. \qed

**Remark 4.7.** Observe that $\overline{X_{[0,\pi]}}$ is a proper subset of $X_{[0,\pi]}$. Its intersection with $X_0$ contains only points of the form $(P + Raj, Q + Sbj)$. Points $(g, h)$ where $g, h$ lie in the same maximal torus but do not have the above form are not in $\overline{X_{[0,\pi]} \cap X_0}$. Although $W_0 = X_0$, these points lie in $\overline{W_{[0,\pi]}}$ but not in $\overline{X_{[0,\pi]}}$.

We extend the domain of $r$ to $X_{[0,\pi]} \times I$ by setting $(r_t)|_{X_0} := \text{identity for all } t$ and thus obtain a deformation $X_{[0,\pi]} \times I \to X_{[0,\pi]}$ giving $X_{[0,\pi]} \simeq X_0$.

**Lemma 4.8.** For each $t$, the map $r_t : X_\theta \to X_{t\theta}$ is $T$-equivariant.

**Proof.** The $T$-action by $u \in T$ replaces $a$ and $b$ by $u^2a$, $u^2b$ respectively with no change in $\phi$ or the other parameters. Since the graphs $\Gamma_{t\theta,P}$ are unaffected by this action, the resulting point $\phi_{t\theta,P}$ is unaffected, so the resulting $(u^2a)e^{3i\phi_{t\theta,P}}$ has been multiplied by $u^2$. \qed

Summarizing, we have

**Theorem 4.9.** $r : X_{[0,\pi]} \times I \to X_{[0,\pi]}$ is $T$-equivariant, giving a $T$-equivariant strong deformation retraction $X_0 \to X_{[0,\pi]}$.

Similarly we have a $T$-deformation $r' : \overline{X_{[0,\pi]} \times I} \to \overline{X_{[0,\pi]}}$ with $\operatorname{Im}(r'_0) \subset X_\pi$ obtained by extending the map $r' : X_{[0,\pi]} \times I \to X_{[0,\pi]}$ given by
\[
r'(\overline{(Pe^{i\theta/2} + Raj, Qe^{-i\theta/2} + Sbj)}, t) :=
\]
\[
\overline{(Pe^{it\theta/2} + Raj, Qe^{-it\theta/2} + S_{t\theta,P}b_{t\theta,P}j)}.
\]

---

1By abuse of notation we omit the subscript $P$ from $Q_{\theta,P}$ and its analogues $S_{\theta,P}$ and $b_{\theta,P}$, writing $Y_{\theta}$ instead of $Y_{\theta,P}$. 
\[(Pe^{i(\theta+(1-t)\pi)})^{2} + Ra_j, Q_{\theta+(1-t)\pi} e^{-i(\theta+(1-t)\pi)/2} + S_{\theta+(1-t)\pi} b_{\theta+(1-t)\pi j}\]

We also define deformations \(r_{W} : W_{(0,\pi)} \times I \rightarrow W_{(0,\pi)}\) and \(r'_{W} : W_{(0,\pi)} \times I \rightarrow W_{(0,\pi)}\) by \(r_{W}(x', y', t) := gr(x, y, t)g^{-1}\) and \(r'_{W}(x', y', t) := gr'(x, y, t)g^{-1}\) respectively, where for \([x', y'] \in W_{\theta}\) we find \(g \in G/T\) and \((x, y) \in X_{\theta}\) such that \([x', y'] = g[x, y]g^{-1}\). Here it was necessary to restrict the definition of \(r_{W}\) to \(W_{(0,\pi)}\) since \(e^{i\pi}\) is central and so for \((x, y) \in W_{\pi}\) we do not get a well defined \(g \in G/T\). Similarly the domain of \(r'_{W}\) is restricted to \(W_{(0,\pi)}\).

The maps \(r_{W}\) and \(r'_{W}\) are well defined since \(r\) and \(r'\) are \(T\)-equivariant.

### 4.3. Cohomology of \(T\).

The homotopy type of the suspension \(\Sigma T\) has been given in several papers (see for example [1] and [4]), along with the group structure of \(H^{*}(T)\). Here we obtain the homotopy type of \(T\).

Recall \(W_{\theta} := \{(x, y) \in SU(2) \times SU(2) \mid [x, y] \sim e^{i\theta}\}\) (where “$$\sim$$” means “conjugate”). Then \(SU(2) \times SU(2) = \bigcup_{\theta} W_{\theta}\).

Notice that \(W_{0} = X_{0}\) and \(W_{\pi} = X_{\pi}\) since \(I\) and \(-I\) are in the center of \(SU(2)\).

**Lemma 4.10.** \(W_{\theta} \cong (X_{\theta} \times \mathbb{R}P^{3})/T\) where \(T\) acts diagonally by conjugation on the first factor and left translation by \(\bar{T} = p(T)\) on the second. After applying the the \(T\)-homeomorphism \(\Phi_{\theta}\), this can be stated equivalently as \(W_{\theta} \cong (\mathbb{R}P^{3} \times \mathbb{R}P^{3})/\bar{T}\) where \(\bar{T}\) acts diagonally.

**Proof.** Let \((x', y')\) belong to \(W_{\theta}\). Then \(\exists \ [g] \in \mathbb{R}P^{3}/\bar{T}\) such that \(g[x', y']g^{-1} = e^{i\theta}\). Set \(x := gx'g^{-1}, y := gy'g^{-1}\). Then \((x, y) \in X_{\theta}\). Define a homeomorphism

\[(15) \quad W_{\theta} \cong (X_{\theta} \times \mathbb{R}P^{3})/T\]

by \((x', y') \mapsto [[[\,(x, y), g]]]\. \square\]

**Corollary 4.11.** \(W_{\theta} \cong \mathbb{R}P^{3} \times S^{2}\)

**Proof.** Consider the fibration

\[\mathbb{R}P^{3} \rightarrow (\mathbb{R}P^{3} \times \mathbb{R}P^{3})/\bar{T} \xrightarrow{\pi_{2}} \mathbb{R}P^{3}/\bar{T}\]

This fibration has a retraction given by \((g, h) \mapsto g^{-1}h\) (which is well defined). Thus \((\mathbb{R}P^{3} \times \mathbb{R}P^{3})/\bar{T} \cong \mathbb{R}P^{3} \times S^{2} \cong \mathbb{R}P^{3} \times (\mathbb{R}P^{3}/\bar{T})\). \square

Note that the above homeomorphism is not canonical. For example, we could have projected onto the first factor instead of the second.

Write \(SU(2) \times SU(2) = U \cup V\) where \(U = W_{(0,\pi)}\), \(V = W_{(0,\pi)}\).

\(r_{W}\) and \(r'_{W}\) give strong deformation retractions \(T \cong X_{0} = W_{0} \rightarrow U\) and \(\mathbb{R}P^{3} \cong W_{\pi} = X_{\pi} \rightarrow V\).

Noting that \(\mathbb{R}P^{3}/\bar{T} \cong S^{2}\), we can now recover the cohomology of \(T = X_{0}\) from the Mayer-Vietoris sequence for \(SU(2) \times SU(2) = U \cup V\).
Lemma 4.12. The integral cohomology of $T$ is given by

$$H^q(T) = \begin{cases} 
\mathbb{Z} & q = 0, 2; \\
\mathbb{Z} \oplus \mathbb{Z} & q = 3; \\
\mathbb{Z}/2 & q = 4; \\
0 & \text{otherwise.}
\end{cases}$$

All cup products in $\tilde{H}^*(T)$ are zero.

The computation of the group structure of $H^*(T)$ has been done in several places in various ways. This group structure is given in [1] and [4], although the methods of those papers do not obtain the ring structure.

Proof. We calculate the groups $H^*(T)$ by means of the Mayer-Vietoris sequence for $SU(2) \times SU(2) = U \cup V$.

We have homotopy equivalences $U \simeq W_0 = X_0 = T$ and $V \simeq W_\pi = X_\pi \cong \mathbb{R}P^3$. Set $B := U \cap V = W_{(0,\pi)} \simeq \mathbb{R}P^3 \times S^2$.

First we calculate with $\mathbb{F}$ coefficients. With these coefficients $H^*(V) = \langle 1, v, v^2, v^3 \rangle$ where $|v| = 1$.

$H^*(SU(2) \times SU(2)) = \langle 1, w, w', ww' \rangle$ where $|w| = |w'| = 3$.

Also $H^*(B) = \langle 1, s, t, t^2, st, t^3, st^2, st^3 \rangle$ where $|s| = 2$, $t = |1|$. The Bockstein is given by $\beta(v) = v^2$, $\beta(t) = t^2$, $\beta(w) = \beta(w') = \beta(s) = 0$. See for example [13] (pp. 22 and 61) for further information about the Bockstein.

Let $j$ denote the inclusion $B \to U$, and let $j'$ denote the inclusion $B \to V$. Let $i$ denote the inclusion $U \to SU(2) \times SU(2)$.

\[
\begin{array}{ccccccccc}
0 & \rightarrow & H^1(U) \oplus H^1(V) & \rightarrow & H^1(B) & \rightarrow & \\
& & H^2(SU(2) \times SU(2)) \rightarrow & H^2(U) \oplus H^2(V) & \rightarrow & H^2(B) & \rightarrow & \\
& & & H^3(SU(2) \times SU(2)) \rightarrow & H^3(U) \oplus H^3(V) & \rightarrow & H^3(B) & \rightarrow & \\
& & & & H^4(SU(2) \times SU(2)) \rightarrow & H^4(U) \oplus H^4(V) & \rightarrow & H^4(B) & \rightarrow & \\
& & & & & H^5(SU(2) \times SU(2)) \rightarrow & H^5(U) \oplus H^5(V) & \rightarrow & H^5(B) & \rightarrow & \\
& & & & & & H^6(SU(2) \times SU(2)) \rightarrow & H^6(U) \oplus H^6(V) & \rightarrow & H^6(B) & \rightarrow & \\
& & & & & & & H^6(SU(2) \times SU(2)) \rightarrow & 0 & & & & & &
\end{array}
\]
Exactness gives us: $H^1(U) = 0$; $H^2(U) = \mathbb{F}$; $H^3(U) = \mathbb{F} + \mathbb{F} + \mathbb{F}$; $H^4(U) = \mathbb{F}$; $H^5(U) = 0$. (Here we use $+$ to distinguish from $\oplus$.) Let $a, b, s_1, s_2, c$ denote the generators in degrees 2, 3, 3, 3, 4 respectively.

Here $s_1 := i^*(w)$; $s_2 := i^*(w')$, so $\beta(s_1) = \beta(s_2) = 0$, where $\beta$ is the Bockstein.

In the above, the isomorphisms $H^2(B) \cong H^2(\mathbb{R}P^3 \times S^2) = \mathbb{F} + \mathbb{F}$ and $H^3(B) \cong H^3(\mathbb{R}P^3 \times S^2) = \mathbb{F} + \mathbb{F}$ depend upon a choice of homotopy equivalence $B \simeq \mathbb{R}P^3 \times S^2$, which, as noted earlier, involves a choice. With a suitable choice, we have $j^*(a) = s$ and $j^*(b) = st$ from which we conclude $\beta(a) = 0$ and $\beta(b) = c$. Therefore the integral cohomology groups are as stated.

For degree reasons, the only possible nonzero cup product in $\tilde{H}^* (\mathcal{T}; \mathbb{F})$ is $a^2$. Since $j^*(a^2) = (j^*(a))^2 = s^2 = 0$ and $j^*$ is an isomorphism on $H^1(\mathcal{T}; \mathbb{F})$, we see that $a^2 = 0$. \hfill $\Box$

Let $P^n(2)$ denote the Moore space $P^n(2) := S^{n-1} \cup_2 e^n = \Sigma^{n-2} \mathbb{R}P^2$.

**Corollary 4.13.** $\mathcal{T} \simeq S^2 \vee (\vee^2 S^3) \vee P^1(2)$

The fact that this holds after one suspension is given in several references, such as [4].

**Proof.** The only way to build a $CW$-complex with the cohomology groups above is the homotopy cofibre of some attaching map $f : P^3(2) \to S^2 \vee (\vee^2 S^3)$ which induces zero on $\tilde{H}^* (\mathcal{T}; \mathbb{F})$. The fact that $f^*$ is zero on $\tilde{H}^* (\mathcal{T}; \mathbb{F})$ requires that the composition $P^3(2) \xrightarrow{f} S^2 \vee (\vee^2 S^3) \to \vee^2 S^3$ be null, so $f$ is determined by the composition

$$P^3(2) \xrightarrow{f} S^2 \vee (\vee^2 S^3) \to S^2$$

which we denote as $\bar{f}$. Our calculation of the ring structure of $H^*(\mathcal{T}; \mathbb{F})$ in Lemma 4.12 gives that the cup products in the reduced cohomology $\tilde{H}^*(\mathcal{T}; \mathbb{F})$ of the homotopy cofibre of $\bar{f}$ are zero.

The group of homotopy classes of maps $[P^3(2), S^2]$ equals $\mathbb{Z}/2$ with the nonzero element given by $P^3(2) \xrightarrow{\text{pinch}} S^3 \xrightarrow{\eta} S^2$ where $\eta$ denotes the Hopf map. Since the homotopy fibre of $\eta$ is $\mathbb{C}P^2$ which has a nonzero cup square on $H^2(\mathcal{T}; \mathbb{F})$, naturality gives that the homotopy fibre of $P^3(2) \xrightarrow{\text{pinch}} S^3 \xrightarrow{\eta} S^2$ has a nonzero cup square on $H^2(\mathcal{T}; \mathbb{F})$, so $\bar{f}$ is not the nonzero element of $[P^3(2), S^2]$. Thus $\bar{f}$ is null and therefore $f$ is null. Hence $\mathcal{T}$ is the wedge, as claimed. \hfill $\Box$

### 4.4. Centralizers

If $g \in SU(2)$ is not $\pm 1$ the centralizer $Z_{SU(2)}(g)$ forms a maximal torus in $SU(2)$. It can be described explicitly as follows.

The Lie algebra $\mathfrak{g} := \mathfrak{su}(2)$ can be identified with the space of pure imaginary quaternions $\xi = \{xi + yj + zk\}$. The exponential map $\exp : \mathfrak{g} \to SU(2)$ is given by $\exp(\xi) = \cos(|\xi|) + \sin(|\xi|)\xi/|\xi|$. Let $\mathfrak{g} = \{\xi \in \mathfrak{su}(2) \mid 0 < \xi < |\pi|\}$. The exponential map is a homeomorphism $\exp : \mathfrak{g} \cong SU(2) \setminus \{I\}$.

For $g \in SU(2)$, $Z_{SU(2)}(g) = \{\exp(t\xi) \mid \exp(\xi) = g \text{ and } t \in \mathbb{R}\}$.

There are four obvious inclusions $SU(2) \to \mathcal{T}$ corresponding to the subsets of $\mathcal{T}$ where one of the entries is fixed at $\pm 1$. Since $H^3(\mathcal{T})$ has rank only 2, two pairs of these inclusions must be homotopic. An explicit homotopy is as follows.
Let $S^+_1 = \{(g, 1) \mid g \in SU(2)\}$ and let $S^-_1 = \{(g, -1) \mid g \in SU(2)\}$. Each $g \neq \pm 1$ in $\mathfrak{g}$ determines a one-parameter subgroup of $SU(2)$. We use this to define a homotopy. Define $J_1 : S^+_1 \times I \to \mathcal{T}$ by $J((g, 1), t) := (g, \exp(\pi t \xi / |\xi|))$ where $\xi = \exp^{-1}(g) \in \mathfrak{g}$. Then $J_1$ is a homotopy between the inclusions $SU(2) \to \mathcal{T}$ corresponding to $S^+_1$ and $S^-_1$. Similarly there is a homotopy $J_2$ between the inclusions corresponding to $S^+_2 = \{(1, g) \mid g \in SU(2)\}$ and $S^-_2 = \{(-1, g) \mid g \in SU(2)\}$.

Set $\mathfrak{S} = S^+_1 \cup S^+_2 = \{(g, h) \in \mathcal{T} \mid g = 1 \text{ or } h = 1\}$. Then $\mathfrak{S} \simeq S^3 \vee S^3$ and $\mathcal{T}/\mathfrak{S} \simeq S^2 \vee \Sigma^2 \mathbb{R}P^2$.

5. Atiyah space

Let $\mu : SU(2)^4 \to SU(2)$ denote the product of commutators map, $\mu((x, y, x', y')) := [x, y][x', y']$. Then $(-I)$ is a regular value of $\mu$ which is fixed by the conjugation action. Let $A$ denote the symplectic reduction $\mu/\!/SU(2)$. That is, $A$ is the 6-manifold given by $A := \mu^{-1}(-I)/SU(2)$, where $SU(2)$ acts by conjugation. We refer to $A$ as the “Atiyah space” since it was studied by Atiyah and Bott [2] who computed its cohomology groups using Morse theory. The Atiyah space corresponds to the moduli space of flat connections on a punctured 2-hole torus with holonomy $-1$ on the boundary circle of the puncture.

An algebraic variety corresponding to $A$ was studied by Newstead [16].

5.1. Retractions on $A$.

If $X$ represents an element of $A$, then $X = (x, y, x', y')$ for some $x, y, x', y'$ with $[x, y][x', y'] = -1$. There exists $\theta \in [0, \pi]$ such that $[x, y] \in W_\theta$, and the value of $\theta$ is independent of the choice of representative $X$. Moreover, one can choose the representative such that

$$[x, y] \in X_\theta.$$

With this choice of representative the condition $[x, y][x', y'] = -I$ implies that $[x', y'] \in X_{\theta'}$ where $\theta' := \pi - \theta$.

For $\theta \in [0, \pi]$ set

$$A_\theta := \{X \in A \mid [x, y] \in W_\theta \text{ for any representative } (x, y, x', y') \text{ of } X\}.$$

For $S \subset [0, \pi]$, set $A_S := \cup_{\theta \in S} A_\theta$. Notice that elements of $A_\theta$ have representatives in $X_\theta \times X_{\theta'}$.

**Lemma 5.1.**

a) For $\theta \in (0, \pi)$,

$$A_\theta \cong (X_\theta \times X_{\theta'})/T \cong \mathbb{R}P^3 \times (\mathbb{R}P^3/T) = \mathbb{R}P^3 \times S^2$$

b) $A_0 \cong (X_0 \times X_\pi)/SU(2) \cong X_0$

c) $A_\pi \cong (X_\pi \times X_0)/SU(2) \cong X_0$

**Proof.**

a) The first homeomorphism is clear, since for $X \in A_{(0, \pi)}$ any two representatives in $X_\theta \times X_{\theta'}$ differ by the action of $T$. For the second homeomorphism, after applying the $T$-homeomorphisms $X_\theta \cong \mathbb{R}P^3$, $X_{\theta'} \cong \mathbb{R}P^3$ we have $(\mathbb{R}P^3 \times \mathbb{R}P^3)/T$ where $T$ acts by left translation. Consider the fibration

$$\mathbb{R}P^3 \to (\mathbb{R}P^3 \times \mathbb{R}P^3)/T \to \mathbb{R}P^3/T$$
This fibration has a retraction given by \((g, h) \mapsto g^{-1}h\) (which is well defined).
Thus \(A_0 \cong \mathbb{R}P^3 \times \mathbb{R}P^3 / T \cong \mathbb{R}P^3 \times S^2\).
b) In this case, the stabilizer is all of \(SU(2)\) so
\[ A_0 \cong (X_0 \times X_\pi) / SU(2). \]
Since \(X_\pi / SU(2) = *\) the fibration
\[ X_0 \to (X_0 \times X_\pi) / SU(2) \to X_\pi / SU(2) \]
gives \(A_0 \cong X_0\).
Similarly we get part (c).

\[ \square \]

Notice that the second homeomorphism in (a) is non-canonical — we could equally well have reversed the roles of \(\theta\) and \(\theta'\). This is reflected in a lack of symmetry in the Mayer-Vietoris sequence below for \(H^*(A)\).

The existence of deformation retractions \(T \cong A_0 \cong A_{[0,\pi]}\) and \(T \cong A_\pi \cong A_{[0,\pi]}\) follows from Theorem 4.1. Furthermore, the methods of subsection 4.1 show that such retractions can be obtained via gradient flow.

In order to compute transition functions we will use another deformation retraction making use of the maps in subsection 4.2. Define a deformation as follows. For \(\theta \in (0, \pi)\) let \((x, y, x', y') \in X_\theta \times X_{\theta'}\) represent an element of \(A_\theta\). Define \(\rho: A_{[0,\pi]} \times I \to A_{[0,\pi]}\) with \(\text{Im} \rho_0 \subset A_0\) by
\[ \rho([[((x, y, x', y'), t]]) := \left[[\left(r((x, y), t), r'((x', y'), t)\right)]\right]. \]
Notice that \((x, y, x', y') \in X_\theta \times X_{\theta'}\) implies
\[ \mu\left(r((x, y), t), r'((x', y'), t)\right) = e^{i\theta}e^{i(t(\pi-\theta)+(1-t)\pi)} = e^{i\pi} = -I \]
That is, \(\rho([[[(x, y, x', y')]], t]) \in A_{\theta}.\)
If \(u(x, y, x', y')u^{-1}\) is another representative for \( [[([x, y, x', y'])]] \) in \(A_\theta = (X_\theta \times X_{\pi-\theta}) / T\), where \(u \in T\), then
\[ r(u(x, y)u^{-1}, t) = ur(x, y, t)u^{-1} \]
since \(r\) is \(T\)-equivariant and similarly
\[ r'\left(u(x', y')u^{-1}, t\right) = ur'(x', y', t)u^{-1}. \]
Thus \(\rho\) is well defined.

Extend \(\rho\) to \(A_0 \times I\) by projection onto the first factor to get a deformation \(\rho: A_{[0,\pi]} \times I \to A_{[0,\pi]}\).
Similarly we have a deformation \(\rho' : A_{[0,\pi]} \times I \to A_{[0,\pi]}\) with
\[ \text{Im} \rho'_0 \subset A_\pi. \]
5.2. Cohomology of $A$.

It turns out that $H^*(A)$ is torsion-free and its Betti numbers were computed by Atiyah and Bott [2] via Morse theory. In this section we will do the computation using Mayer-Vietoris.

As above, set $U := A_{(0, \pi)} \simeq A_0 \simeq \mathcal{T}$; $V := A_{(0, \pi]} \simeq A_\pi \simeq \mathcal{T}$. Set

$$\mathcal{S} = \{[(x, y, x', y')] \in A \mid \text{at least one of } x, y, \text{ is } I \} \cong S^3 \vee S^3$$

and

$$\mathcal{S}' = \{[(x, y, x', y')] \in A \mid \text{at least one of } x', y', \text{ is } I \} \cong S^3 \vee S^3.$$ Set $\bar{U} := U/\mathcal{S}$, $\bar{V} := V/\mathcal{S}'$. Set $\bar{A} := \bar{U} \cup \bar{V}$. Then $\bar{A} = A/\sim$ where $\{[(I, y_1, x', y')] \sim \{[(I, y_2, x', y')]\}$, and similarly with $I$ in the other positions the “partner” of the entry $I$ can be replaced by any other element of $SU(2)$. In view of Lemma 5.1, this can be expressed as $\bar{A} = A/\sim$ where $\{[(I, y, x', y')] \sim \{[(I, I, x', y')]\}$ and $\{[(x, y, x', I)] \sim \{[(x, y', x', I)]\}$ where $* = X_\pi/SU(2) = X_0/SU(2)$.

In fact, as we will see in section 8, $A$ is diffeomorphic to the connected sum

$$A \cong \bar{A} \# (S^3 \times S^3) \# (S^3 \times S^3).$$

Note that the composition $\mathcal{S} \to U \simeq \mathcal{T}$ induces an isomorphism on $H^3(\mathcal{S})$ and similarly $\mathcal{S}' \to V \simeq \mathcal{T}$ induces an isomorphism on $H^3(\mathcal{S}')$. Therefore $\bar{H}^*(\bar{U}) \cong \bar{H}^*(\bar{V}) \oplus \bar{H}^*(\mathcal{S})$ where $\bar{H}^q(\mathcal{S}) = \mathbb{Z} \oplus \mathbb{Z}$ if $q = 3$ and 0 otherwise. and similarly $\bar{H}^*(\bar{V}) \cong \bar{H}^*(\bar{U}) \oplus \bar{H}^*(\mathcal{S}')$.

We calculate the groups $H^*(\bar{A})$ by means of the Mayer-Vietoris sequence for $\bar{A} = \bar{U} \cup \bar{V}$ and then get the cohomology of $A$ from $H^*(A) = H^*(\bar{A}) \oplus H^*(\mathcal{S}) \oplus H^*(\mathcal{S}')$.

Set $B := \bar{U} \cap \bar{V} = U \cap V = A_{(0, \pi)} \cong \mathbb{R}P^3 \times S^2 \times (0, \pi)$.

First we calculate with $\mathbb{F}$ coefficients. With these coefficients

$$H^*(\bar{U}) = H^*(\bar{V}) = \langle 1, a, b, c \rangle$$

where $|a| = 2$, $|b| = 3$, $|c| = 4$, where

$$\bar{T}/\{(x, y) \in T \mid x = 1 \text{ or } y = 1\}.$$ Similarly $H^*(\bar{V}) = \langle 1, a', b', c' \rangle$ where $|a'| = 2$, $|b'| = 3$, $|c'| = 4$.

Also

$$H^*(B) = \langle 1, s, t, s^2, st, st^2, st^3 \rangle$$

where $|s| = 2$, $|t| = 1$.

The Bockstein is given by $\beta(b) = c$, $\beta(b') = c'$, $\beta(t) = t^2$, $\beta(a) = \beta(a') = \beta(s) = 0$.

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^1(B) & \longrightarrow & H^2(\bar{A}) & \longrightarrow & H^2(\bar{U}) \oplus H^2(\bar{V}) & \longrightarrow & H^2(B) \\
& \mathbb{F} & \mathbb{F} & \mathbb{F} & \mathbb{F} & \mathbb{F} & \mathbb{F} & \mathbb{F} & \mathbb{F} \\
& \mathbb{F} & \mathbb{F} & \mathbb{F} & \mathbb{F} & \mathbb{F} & \mathbb{F} & \mathbb{F} & \mathbb{F} \\
H^3(\bar{A}) & \longrightarrow & H^3(\bar{U}) \oplus H^3(\bar{V}) & \longrightarrow & H^3(B) \\
& \mathbb{F} & \mathbb{F} & \mathbb{F} & \mathbb{F} \\
& \mathbb{F} & \mathbb{F} & \mathbb{F} & \mathbb{F} \\
H^4(\bar{A}) & \longrightarrow & H^4(\bar{U}) \oplus H^4(\bar{V}) & \longrightarrow & H^4(B) \\
& \mathbb{F} & \mathbb{F} & \mathbb{F} & \mathbb{F} \\
\end{array}$$
Let \( j \) denote the inclusion \( B \to \tilde{U} \), and let \( j' \) denote the inclusion \( B \to \tilde{V} \). The group
\[
H^2(B) \cong H^2(S^2 \times \mathbb{R}P^3) = H^2(S^2) \oplus H^2(\mathbb{R}P^3),
\]
where the isomorphism involves a choice.

As in the computation of \( \mathcal{T} \), the map \( S^2 \to A_0 = \mathcal{T} \to \mathcal{T} = \tilde{U} \) has degree one on integral cohomology regardless of the choice. By symmetry, the map \( S^2 \to A_\pi \to \tilde{V} \) has degree one. However the maps \( \mathbb{R}P^3 \to A_0 \to \tilde{U} \) and \( \mathbb{R}P^3 \to A_\pi \to \tilde{V} \) depend on the choice. If we make the choice in which \( \mathbb{R}P^3 \to \tilde{U} \) is null homotopic, then \( \mathbb{R}P^3 \to \tilde{V} \) will induce an isomorphism on \( H^2(\ ; \mathbb{F}) \). With this choice, we have \( j^*(a) = s \) and \( (j')^*(a') = s + t^2 \) and we see that \( j^* \perp (j')^* \) is an isomorphism on \( H^2(\ ; \mathbb{F}) \).

With the above choice of \( B \cong S^2 \times \mathbb{R}P^3 \) we have \( j^*(b) = st; (j')^*(b') = t^3 + st \) and so \( j^* \perp (j')^* \) is an isomorphism on \( H^3(\ ; \mathbb{F}) \).

Applying the Bockstein gives \( j^*(c) = st^2; (j')^*(c') = st^2 \). In degree 4 the kernel is \( c + c' \). The cokernel is 0.

In degree 5, \( j^* \) and \( (j')^* \) are zero maps and the cokernel is \( st^3 \).

Thus we get

**Theorem 5.2.** The mod 2 cohomology of \( A \) is given by
\[
H^q(A; \mathbb{F}) = \begin{cases} 
\mathbb{F} & q = 0, 2, 4, 6; \\
\mathbb{F}^4 & q = 3; \\
0 & \text{otherwise}.
\end{cases}
\]

where
\[
H^2(A; \mathbb{F}) \text{ is generated by } \delta(t); \quad H^3(A; \mathbb{F}) \text{ is generated by } s_1, s_2, s'_1, s'_2; \\
\text{the generator of } H^4(A; \mathbb{F}) \text{ maps to } c + c'; \\
H^5(A; \mathbb{F}) \text{ is generated by } \delta(st^3).
\]

Here \( \delta \) is the connecting homomorphism. With integer coefficients this gives
\[
H^q(A) = \begin{cases} 
\mathbb{Z} & q = 0, 2, 4, 6; \\
\mathbb{Z}^4 & q = 3; \\
0 & \text{otherwise}.
\end{cases}
\]

This reproduces the result of Atiyah and Bott [2]. We will obtain the ring structure of \( H^*(A) \) in the next section.
6. Prequantum line bundle over \( A \)

The manifold \( A \) has a symplectic structure. It is an example of the reduction of a quasi-Hamiltonian \( G \)-space \[3\]. Letting \( \omega \) denote the symplectic form, to the symplectic manifold \((A, \omega)\) there is an associated prequantum line bundle. Line bundles are represented by elements of \( H^2(\cdot) \); the prequantum line bundle is represented by the cohomology class of \( H \omega \).

Let \( 1, x, s_1, s_2, s_3, s_4, y, z \) denote group generators of \( H^*(A) \) in degrees 0, 2, 3, 3, 3, 3, 4, 6, respectively. We may choose \( x \in H^2(A) \) to be the cohomology class represented by \( \omega \). Abusing notation, we will write \( H^*(A) = \langle 1, x, y, z \rangle \) using the same notation for elements of \( H^*(\tilde{A}) \) and their images under \( H^*(\tilde{A}) \rightarrow H^*(A) \). Let \( L \) denote the line bundle over \( A \) associated to \( x \). Thus the pullback of \( L \) to \( A \) is the prequantum line bundle of \( A \).

Let \( \tilde{A} \) denote the total space of \( L \). Letting \( \tilde{U}, \tilde{V}, B \) be as in \[5\], the restriction of \( L \) gives line bundles over those subspaces. Denote the total spaces by \( U, V, B \) respectively. The Mayer-Vietoris sequence for \( \tilde{A} \) tells us that \( H^2(\tilde{A}) \rightarrow H^2(\tilde{U}) \) and \( H^2(\tilde{A}) \rightarrow H^2(\tilde{V}) \) have degree 2. Thus the restrictions of \( L \) to \( U, V, B \) are classified by \( 2a, 2a', 2b, 2b', 2s, 2s' \), using the notation of \[5\].

The total space \( \tilde{B} \) of the bundle over \( B = S^2 \times \mathbb{R}P^3 \) is \( \mathbb{R}P^3 \times \mathbb{R}P^3 \). In fact we initially formed \( B \) in Lemma \[5.1\] as \((\mathbb{R}P^3 \times \mathbb{R}P^3)/T\). Therefore

\[
\begin{align*}
H^*(\tilde{B}; \mathbb{F}) &= \langle s, t, s^2, t^2, st, s^3, t^3, s^2t, st^2, s^3t, s^2t^2, s^3t^2, s^2t^3, s^3t^3 \rangle \\
&\text{with } s, t \in H^1(\mathbb{R}P^3; \mathbb{F}).
\end{align*}
\]

We compute first using \( \mathbb{F} \) coefficients. Letting \( v \) denote the preimage in \( H^*(\tilde{U}) \) of the generator of \( H^1(S^1) \), from the Serre spectral sequence we get \( \tilde{H}^*(\tilde{U}) = \langle v, a, va, b, vb, c, vc \rangle \) in degrees 1, 2, 3, 4, 4, 5 respectively, where the Bockstein is determined by \( \beta(v) = a, \beta(b) = c \) (which implies \( a = v^2 \) although we will not use this). Similarly we have \( \tilde{H}^*(\tilde{V}) = \langle v', a', va', b', vb', c', v'c' \rangle \). Taking inverse images of \( \tilde{A} = \tilde{U} \cup_B \tilde{V} \) under the bundle projection gives \( \tilde{A} = \tilde{U} \cup_B \tilde{V} \). The associated Mayer-Vietoris sequence with \( \mathbb{F} \) coefficients is

\[
\begin{align*}
H^1(\tilde{A}) &\longrightarrow H^1(\tilde{U}) \oplus H^1(\tilde{V}) \longrightarrow H^1(\tilde{B}) \\
&\begin{array}{|c|c|}
\hline
\text{F} & \text{F+F} \\
\hline
\end{array} \\
H^2(\tilde{A}) &\longrightarrow H^2(\tilde{U}) \oplus H^2(\tilde{V}) \longrightarrow H^2(\tilde{B}) \\
&\begin{array}{|c|c|}
\hline
\text{F} & \text{F+F} \\
\hline
\end{array} \\
H^3(\tilde{A}) &\longrightarrow H^3(\tilde{U}) \oplus H^3(\tilde{V}) \longrightarrow H^3(\tilde{B}) \\
&\begin{array}{|c|c|}
\hline
\text{F+P^3} & \text{F+P^3} \\
\hline
\end{array} \\
H^4(\tilde{A}) &\longrightarrow H^4(\tilde{U}) \oplus H^4(\tilde{V}) \longrightarrow H^4(\tilde{B}) \\
&\begin{array}{|c|c|}
\hline
\text{F+P^3} & \text{F+P^3} \\
\hline
\end{array} \\
H^5(\tilde{A}) &\longrightarrow H^5(\tilde{U}) \oplus H^5(\tilde{V}) \longrightarrow H^5(\tilde{B}) \\
&\begin{array}{|c|c|}
\hline
\text{F+P^3} & \text{F+P^3} \\
\hline
\end{array} \\
H^6(\tilde{A}) &\longrightarrow H^6(\tilde{U}) \oplus H^6(\tilde{V}) \longrightarrow H^6(\tilde{B}) \\
&\begin{array}{|c|c|}
\hline
\text{0} & \text{0} \\
\hline
\end{array}
\end{align*}
\]
The maps $H^*(\bar{U}) \to H^*(\bar{B})$ and $H^*(\bar{V}) \to H^*(\bar{B})$ are determined by the corresponding Mayer-Vietoris sequence for $\bar{A}$ and we get

Theorem 6.1.

$$H^q(\bar{A}; \mathbb{F}) = \begin{cases} \mathbb{F} & q = 0, 3, 4, 7; \\ 0 & \text{otherwise.} \end{cases}$$

Now consider the sequence with integer coefficients.

$$
\begin{array}{cccc}
H^1(\bar{A}) & H^1(\bar{U}) \oplus H^1(\bar{V}) & H^1(\bar{B}) \\
0 & 0 & 0 \\
H^2(\bar{A}) & H^2(\bar{U}) \oplus H^2(\bar{V}) & H^2(\bar{B}) \\
\mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/2 + \mathbb{Z}/2 \\
H^3(\bar{A}) & H^3(\bar{U}) \oplus H^3(\bar{V}) & H^3(\bar{B}) \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} + \mathbb{Z}/2 \\
H^4(\bar{A}) & H^4(\bar{U}) \oplus H^4(\bar{V}) & H^4(\bar{B}) \\
\mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/2 \\
H^5(\bar{A}) & H^5(\bar{U}) \oplus H^5(\bar{V}) & H^5(\bar{B}) \\
\mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/2 + \mathbb{Z}/2 \\
H^6(\bar{A}) & H^6(\bar{U}) \oplus H^6(\bar{V}) & H^6(\bar{B}) \\
0 & 0 & \mathbb{Z} \\
H^7(\bar{A}) & & 0 \\
\end{array}
$$

The segment $0 \to \text{coker} \delta \to H^4(\bar{A}) \to \ker \delta \to 0$ becomes

$$0 \to \mathbb{Z}/2 \to H^4(\bar{A}) \to \mathbb{Z}/2 \to 0$$

so we see that $H^4(\bar{A})$ has 4 elements. Our calculation with $\mathbb{F}$ coefficients shows that the mod 2 reduction $H^4(\bar{A}; \mathbb{F})$ of $H^4(\bar{A})$ has a single summand. Thus $H^4(\bar{A}) = \mathbb{Z}/4$.

Therefore the cohomology of $\bar{A}$ is given by

Theorem 6.2.

$$H^q(\bar{A}) = \begin{cases} \mathbb{Z} & q = 0, 7; \\ \mathbb{Z}/4 & q = 4; \\ 0 & \text{otherwise.} \end{cases}$$

We can now obtain the ring structure in $H^*(\bar{A})$. Let $1, x, s_1, s_2, s_3, s_4, y, z$ denote group generators of $H^*(\bar{A})$ in degrees 0, 2, 3, 3, 3, 4, 6, respectively. The elements $s_1, s_2, s_3, s_4$ are torsion-free elements of odd degree so they square to 0. By Poincaré duality, $s_1s_2 = s_3s_4 = xy = z$
with appropriate choices of signs of the generators. Write \( x^2 = \lambda y \), where we may choose the signs of the generators so that \( \lambda \geq 0 \).

Since we calculated above that \( H^1(\hat{A}) = 0 \), in the cohomology Serre spectral sequence for the principal fibration \( S^1 \to \hat{A} \to A \) we must have \( d(v) = x \), where \( v \) is a generator of \( H^1(S^1) \). Therefore \( d(vx) = x^2 = \lambda y \). Thus the spectral sequence gives \( H^4(\hat{A}) = \mathbb{Z}/\lambda \). Comparing this with our calculation above, we conclude that \( \lambda = 4 \). Thus we have

**Theorem 6.3.** As a ring \( H^*(A) = \langle x, s_1, s_2, s_3, s_4, y, z \rangle \) with the nontrivial products given by \( s_1 s_2 = s_3 s_4 = xy = z, x^2 = 4y \).

This agrees with the following result given by Thaddeus \[17\] using methods from algebraic geometry. More generally, Thaddeus shows

\[
x^m = (-1)^g2^{2g-2} \frac{m!}{(m-g+1)!} (2^{m-g+1} - 2) B_{m-g+1} z
\]

where \( B_k \) is the \( k \)th Bernoulli number and in our case, \( m = 3 \) and \( g = 2 \) and \( z \) is the volume form. This works out to \( x^3 = 4z \) as above.

7. The 9-manifold \( M := \mu^{-1}(-I) \)

As noted earlier, \((-I)\) is a regular value of the product of commutators map \( \mu \), so \( M := \mu^{-1}(-I) \) is a 9-manifold and \( A := M/SU(2) \) where \( SU(2) \) acts diagonally by conjugation. The stabilizer of the \( SU(2) \) action in \( A \) is the center of \( SU(2) \), so equivalently we have a free action of \( SO(3) = \mathbb{R}P^3 \) on \( M \), with \( A = M/\mathbb{R}P^3 \).

Let \( q : M \to A \) be the bundle projection. For \( \theta \in [0, \pi] \), set

\[ M_\theta := q^{-1}(A_\theta) = \{ (x, y, x', y') \in M \mid [x, y] \sim e^{i\theta} \text{ and } [x', y'] \sim -e^{-i\theta} \} \]

The deformation \( \rho : A_{[0, \pi]} \times I \to A_{[0, \pi]} \) is covered by a deformation \( \hat{\rho} : M_{[0, \pi]} \times I \to M_{[0, \pi]} \) with \( \text{Im} \hat{\rho}_0 \subset M_0 \) given by

\[ \hat{\rho}((x, y, x', y'), t) := (r_W(x, y, t), r'_W(x', y', t)) \]

Similarly \( \rho' : A_{[0, \pi]} \times I \to A_{[0, \pi]} \) is covered by a deformation \( \hat{\rho}' : M_{[0, \pi]} \times I \to M_{[0, \pi]} \) with \( \text{Im} \hat{\rho}'_0 \subset M_\pi \).

As in Section 5, set \( U := A_{[0, \pi]}, V := A_{[0, \pi]} \). Set \( \hat{U} := q^{-1}(U) = M_{[0, \pi]}, \hat{V} := q^{-1}(V) = M_{[0, \pi]} , \hat{S} := q^{-1}(S), \hat{S}' := q^{-1}(S'), \hat{U} := \hat{U}/\hat{S}, \hat{V} := \hat{V}/\hat{S}', \hat{B} := \hat{U} \cap \hat{V} = q^{-1}(U \cap V), \hat{M} := \hat{U} \cup \hat{V} = q^{-1}(\hat{A}) \).

7.1. Local Trivialization of \( M \to A \).

**Lemma 7.1.** The restrictions of the principal bundle \( \mathbb{R}P^3 \to M \to A \) to \( U \) and to \( V \) are trivial.

**Proof.** Since we have retractions \( U \to X_0 \) and \( V \to X_\pi \), it suffices to show that the restrictions of the bundle to \( X_0 \) and \( X_\pi \) are trivial.

\[ M_0 = \{ (x, y, x', y') \in SU(2)^4 \mid [x, y] = I \text{ and } [x', y'] = -I \} = X_0 \times X_\pi. \]

Under the conjugation action, \( X_\pi/SU(2) \) is isomorphic to \( \mathbb{R}P^3/SU(2) \) with translation action, which is a point. So the restriction of the bundle to \( X_0 \) is trivial. Similarly \( M_\pi = X_\pi \times X_0 \), and
under diagonal conjugation action the first factor becomes a point, so the restriction to $X_\pi$ is trivial. □

Let $\Delta : M_0 \to \mathbb{R}P^3$ be the composite

$$M_0 \cong X_0 \times X_\pi \xrightarrow{\pi_2} X_\pi \xrightarrow{\Phi_\pi} \cong \mathbb{R}P^3$$

where $\Phi_\theta$ is the homeomorphism discussed in Section 3. A trivialization of $q : \hat{U} \to U$ is given by

$$X \mapsto (\Delta \circ \hat{\rho}_0(X), q(X)) \in \mathbb{R}P^3 \times U.$$ 

Similarly a trivialization of $q : \hat{V} \to V$ is given by

$$X \mapsto (\Delta' \circ \hat{\rho}'_0(X), q(X)) \in \mathbb{R}P^3 \times V$$

where

$$\Delta : M_\pi \cong X_\pi \times X_0 \xrightarrow{\pi_1} X_\pi \xrightarrow{\Phi_\pi} \cong \mathbb{R}P^3.$$ 

Trivial principal bundles are in $1 - 1$ correspondence with maps from the base to the group. For $\theta \in (0, \pi)$ the transition function is given by

$$\tau([[((x, y, x', y'))]]) = (\Phi_\pi(r'(x', y', 0)))^{-1} \Phi_\pi(r'(x, y, 0)).$$

Note this is well defined since $r$, $r'$ and $\Phi_\pi$ are $T$-maps.

**Theorem 7.2.** The transition function $\tau : U \cap V \to \mathbb{R}P^3$ is homotopic to the function $[[((x, y, x', y'))]] \mapsto (\Phi_\theta(x', y'))^{-1}\Phi_\theta(x, y)$ for $[[((x, y, x', y'))]] \in U \cap V.$

**Proof.**

$$\tau([[((x, y, x', y'))]]) = (\Phi_\pi(r'(x', y', 0)))^{-1} \Phi_\pi(r'(x, y, 0)).$$

Define a homotopy $J : (U \cap V) \times I \to \mathbb{R}P^3$ by

$$J([[((x, y, x', y'))]], t) = (\Phi_{\theta+(1-t)\pi}(r'(x', y', t)))^{-1} \Phi_{\theta+(1-t)\pi}(r'(x, y, t)).$$

Then

$$J([[((x, y, x', y'))]], 1) = (\Phi_\theta(x', y'))^{-1} \Phi_\theta(x, y).$$

□

**Corollary 7.3.** On cohomology with $\mathbb{F}$ coefficients the transition function satisfies $\tau^*(v) = t$ where $v$ is the generator of $H^1(\mathbb{R}P^3; \mathbb{F})$ and $t$ is the generator of $H^1(B; \mathbb{F})$.

**Proof.** The preceding theorem shows that $\tau$ corresponds under the homeomorphism $B \cong (\mathbb{R}P^3 \times \mathbb{R}P^3)/T$ to the map $(g, h) \mapsto g^{-1}h$. Thus under the composition

$$\mathbb{R}P^3 \xrightarrow{i_2} S^2 \times \mathbb{R}P^3 \cong B \cong (\mathbb{R}P^3 \times \mathbb{R}P^3)/T \xrightarrow{\tau} \mathbb{R}P^3$$

$g$ drops out and we get the identity. Hence $\tau^*(v) = t.$ □
7.2. Cohomology of $\tilde{M}$.

Set $\tilde{B} := q^{-1}(B) \cong \mathbb{R}P^3 \times B$. Then $\tilde{M} = \tilde{U} \cup_B \tilde{V}$ where the bundle is trivial over $U$ and $V$. As before let

$$\tilde{T} = T/\{(x, y) \in T \mid x = 1 \text{ or } y = 1\}.$$ 

Then

$$\tilde{H}^*(\tilde{T}) = \langle a, b, c \rangle$$

with $|a| = 2$, $|b| = 3$, $|c| = 4$, $\beta(b) = c$.

First we consider $\tilde{H}^*(\tilde{U})$.

The transition function $\tau$ is given by $\tau : (b, g) \mapsto (\hat{\tau}_g(b))$ where the isomorphism is not canonical.

For $\tilde{H}^*(\tilde{V})$ we write the generators as $a'$, $b'$, $c'$, $w'$, $w'^2$, and $w'^3$.

For $\tilde{H}^*(\tilde{B})$, the ring generators are $s$ (in degree 2) and $t$ in degree 1 (from the base) and $v$ in degree 1 (from the fiber). As a ring

$$\tilde{H}^*(\tilde{B}) = \langle s, t, v \rangle$$

where $|s| = 2$, $|t| = 1$, $|v| = 1$, $s^2 = 0$, $t^4 = 0$, $v^4 = 0$.

Let $j : B \to U$ and $j' : B \to V$ denote the inclusions.

The transition function $\tau$ is the map $B \to \mathbb{R}P^3$ described in the previous subsection. Associated to a transition function is a self-homeomorphism $\tilde{\tau} : B \times \mathbb{R}P^3 \cong B \times \mathbb{R}P^3$ given by $(b, g) \mapsto (b, \tau(b)g)$.

**Lemma 7.4.** On mod 2 cohomology $\tilde{\tau}^*(v) = v + t$

**Proof.** $v \in H^1(\tilde{B}) \cong B \times \mathbb{R}P^3$ is defined as a preimage of the generator of $H^1(\mathbb{R}P^3)$ from the fibre. Equivalently, having chosen a trivialization $\tilde{B} \cong B \times \mathbb{R}P^3$, $v$ is the image of the generator of $H^1(\mathbb{R}P^3)$ under $\pi_2 : \tilde{B} \to \mathbb{R}P^3$. The composition $B \times \mathbb{R}P^3 = \tilde{B} \xrightarrow{\pi} \mathbb{R}P^3$ is given by $(b, g) \mapsto \tau(b)g$ so $\tilde{\tau}^*(v) = (1_{\mathbb{R}P^3})^*(v) + \tau^*(v) = v + t$. \qed

Recall from the Mayer-Vietoris sequence for $A$ that based on our choice of isomorphism: $j^*(a) = s$; $j^*(b) = st$; $j^*(c) = st^2$; $j^*(a') = s + t^2$; $j^*(b') = st + t^3$; $j^*(c') = st^2$.

Below, $f$ denotes $j \times 1_{\mathbb{R}P^3}$ while $f'$ denotes $(j' \times 1_{\mathbb{R}P^3}) \circ \tilde{\tau}$.

$$H^1(\tilde{M}) \xrightarrow{\text{products}} H^1(\tilde{U}) \overset{f}{\to} H^1(\tilde{V}) \overset{f}{\to} H^1(\tilde{B})$$
The map \(((j \times 1_{\mathbb{R}P^3}) \perp (j' \times 1_{\mathbb{R}P^3}) \circ \hat{\tau})^*\) is given by

1) \(w \mapsto v\)
\[w' \xrightarrow{j' \times 1_{\mathbb{R}P^3}} v \xrightarrow{\hat{\tau}} v + t\]

To compute \(((j' \times 1_{\mathbb{R}P^3}) \circ \hat{\tau})^* (w')\), we used that
\[H^1(\hat{B}) = H^1(S^2 \times \mathbb{R}P^3 \times \mathbb{R}P^3) = H^1(\mathbb{R}P^3 \times \mathbb{R}P^3),\]

while \(H^1(\hat{V}) = H^1(\hat{V} \times \mathbb{R}P^3) = H^1(\mathbb{R}P^3)\) since \(\hat{V}\) is simply connected. On \(H^1(\cdot)\), effectively the map is given by the multiplication map on \(\mathbb{R}P^3\), and so on cohomology \(w' \mapsto t + v\) using Corollary 6.1.

The above is an isomorphism.

2) \(f^*: a \mapsto s\)
\(f^*: w^2 \mapsto v^2\)
\((f')^*: a' \mapsto s + t^2\)
\((f')^*: (w')^2 \mapsto v^2 + t^2\)
\(\text{Ker} f = a + w^2 + a' + (w')^2\)
\(\text{Coker} f = vt\)

3) \(f^*: aw \mapsto sv\)
FLAT CONNECTIONS AND THE COMMUTATOR MAP FOR $SU(2)$

$f^* : b \mapsto st$
$f^* : w^3 \mapsto v^3$

$(f')^* : a'w' \mapsto (s + t^2)(v + t) = sv + st + t^2v + t^3$

$(f')^* : b' \mapsto st + t^3$

$(f')^* : (w')^3 \mapsto (v + t)^3 = v^3 + v^2t + vt^2 + t^3$

This is an isomorphism.

4) $f^* : aw^2 \mapsto sv^2$
$f^* : bw \mapsto stv$
$f^* : c \mapsto st^2$

$(f')^* : a'(w')^2 \mapsto (s + t^2)(v^2 + t^2) = sv^2 + st^2 + t^2v^2$

$(f')^* : b'(w')^2 \mapsto (st + t^3)(v + t) = stv + st^2 + vt^3$

$(f')^* : c' \mapsto st^2$

$\text{Ker} = < c + c' >$
$\text{Coker} = < v^3t >$

5) $f^* : aw^3 \mapsto sv^3$
$f^* : bw^2 \mapsto stv^2$
$f^* : cw \mapsto st^2v$

$(f')^* : a'(w')^3 \mapsto (s + t^2)(v + t)^3 = st^3 + st^2v + stv^2 + sv^3$

$+ t^2v^3 + t^3v^2$

$(f')^* : b'(w')^3 \mapsto (st + t^3)(v^2 + t^2) = stv^2 + st^3 + t^3v^2$

$(f')^* : c'(w')^3 \mapsto st^2(v + t)$

$= stv^2 + st^3$

This is an isomorphism.

6) $f^* : bw^3 \mapsto sv^3$
$f^* : cw^2 \mapsto st^2v^2$

$(f')^* : b'(w')^3 \mapsto (st + t^3)(v + t)^3 = stv^3 + st^2v^2 + st^3v + t^3v^3$

$(f')^* : c'(w')^3 \mapsto st^2(v + t)^2 = st^2v^2$

$\text{Ker} = < cw^2 + c'w^2 >$
$\text{Coker} = < t^3v^3 >$

7) $f^* : cw^3 \mapsto st^2v^3$

$(f')^* : c'(w')^3 \mapsto st^2(v + t)^3 = st^2v^3 + st^3v^2$

This is an isomorphism.

8) $\text{Coker} = < st^3v^3 >$
Therefore in $\tilde{H}^*(\tilde{M})$ we have generators

$$
\begin{align*}
& a_2 \rightarrow a + w^2 + a' + (w')^2 \\
& a_3 \leftarrow vt \\
& a_4 \rightarrow c + c' \\
& a_5 \leftarrow v^3t \\
& a_6 \rightarrow cw^2 + c'(w')^2 \\
& a_7 \leftarrow t^3v^3 \\
& a_9 \leftarrow st^3v^3
\end{align*}
$$

Thus

**Theorem 7.5.** We have

$$
H^q(\tilde{M}, \mathbb{F}) = \begin{cases} 
\mathbb{F} & q = 0, 2, 3, 4, 5, 6, 7, 9; \\
0 & \text{otherwise.}
\end{cases}
$$

Now consider the sequence with integer coefficients.

As earlier, $f$ denotes $j \times 1_{\mathbb{R}P^3}$ while $f'$ denotes $(j' \times 1_{\mathbb{R}P^3}) \circ \tau$.

$$
\begin{array}{llllllllll}
H^1(\tilde{M}) & \rightarrow & H^1(\hat{U}) \oplus H^1(\hat{V}) & \xrightarrow{(f \perp f')} & H^1(\hat{B}) & \rightarrow \\
H^2(\tilde{M}) & \rightarrow & H^2(\hat{U}) \oplus H^2(\hat{V}) & \xrightarrow{(f \perp f')} & H^2(\hat{B}) & \rightarrow \\
H^3(\tilde{M}) & \rightarrow & H^3(\hat{U}) \oplus H^3(\hat{V}) & \xrightarrow{(f \perp f')} & H^3(\hat{B}) & \rightarrow \\
H^4(\tilde{M}) & \rightarrow & H^4(\hat{U}) \oplus H^4(\hat{V}) & \xrightarrow{(f \perp f')} & H^4(\hat{B}) & \rightarrow \\
H^5(\tilde{M}) & \rightarrow & H^5(\hat{U}) \oplus H^5(\hat{V}) & \xrightarrow{(f \perp f')} & H^5(\hat{B}) & \rightarrow \\
H^6(\tilde{M}) & \rightarrow & H^6(\hat{U}) \oplus H^6(\hat{V}) & \xrightarrow{(f \perp f')} & H^6(\hat{B}) & \rightarrow \\
H^7(\tilde{M}) & \rightarrow & H^7(\hat{U}) \oplus H^7(\hat{V}) & \xrightarrow{(f \perp f')} & H^7(\hat{B}) & \rightarrow \\
\end{array}
$$
The segment $0 \to \text{coker } \delta \to H^4(\tilde{M}) \to \ker \delta \to 0$ becomes

$$0 \to \mathbb{Z}/2 \to H^4(\tilde{M}) \to \mathbb{Z}/2 \to 0$$

so we see that $H^4(\tilde{M})$ has 4 elements. Our calculation with $\mathbb{F}$ coefficients shows that the mod 2 reduction $H^4(\tilde{M}; \mathbb{F})$ of $H^4(\tilde{M})$ has a single summand. Thus $H^4(\tilde{M}) = \mathbb{Z}/4$. Similarly $H^6(\tilde{M}) = \mathbb{Z}/4$.

Therefore the cohomology of $\tilde{M}$ is given by

**Theorem 7.6.**

$$H^q(\tilde{M}) = \begin{cases} \mathbb{Z} & q = 0, 2, 7, 9; \\ \mathbb{Z}/4 & q = 4, 6; \\ 0 & \text{otherwise.} \end{cases}$$

We conclude this section by computing the ring structure of $H^*(\tilde{M}; \mathbb{F})$.

Using $\mathbb{F}$ coefficients, as a group, $H^*(\tilde{M}) = \langle 1, a_2, a_3, a_4, a_5, a_6, a_7, a_9 \rangle$.

Let $q : M \to A$ be the bundle projection.

Let $\iota : S^2 \to A$ denote the inclusion of the lowest degree cell of $A$. We may choose $\iota$ to be smooth.

Let $N \to S^2$ be the restriction of the bundle $q$ to $S^2$. Then $N$ is a 5-manifold and from the Serre spectral sequence we calculate $H^*(N) = \langle 1, a_2', a_3', a_5' \rangle$. By Poincaré duality $a_2'a_3' = a_5'$. Therefore naturality shows that $a_2a_3 = a_5$ in $H^*(\tilde{M})$.

Poincaré duality gives $a_4a_5 = a_9$. In other words, $a_2a_3a_4 = a_9$. In particular $a_2a_4 \neq 0$ and $a_3a_4 \neq 0$. Thus $a_2a_4 = a_6$ and $a_3a_4 = a_7$. By Poincaré duality we also have $a_2a_7 = a_9$ and $a_3a_6 = a_9$.

This describes all the cup products except for $a_2^2$ and $a_3^2$.

We next show $a_2^2 = 0$.

For a CW-complex $Y$, let $Y^{(k)}$ denote its $k$-skeleton. Let $\kappa_{\tilde{M}} : S^3 \to \tilde{M}^{(3)}$ denote the attaching map which produces the 4-skeleton of $\tilde{M}$ and let $\kappa_A : S^3 \to A^{(3)}$ denote the attaching map which produces that 4-skeleton of $A$. By naturality we have a diagram
The 3-skeleton of $\tilde{M}$ is $\tilde{M}^{(3)} = S^2 \vee S^3$. The group structure of $H^\ast(\tilde{M})$ tells us that the attaching map has the form $\kappa_{\tilde{M}} = \lambda \eta \oplus 4 \iota_3 \in \pi_3(S^2) \oplus \pi_3(S^3) \cong \mathbb{Z} \oplus \mathbb{Z}$ for some integer $\lambda$, where $\iota_3$ is the identity map on $S^3$.

The 3-skeleton of $\tilde{A}$ is $\tilde{A}^{(3)} = S^2$. The ring structure of $H^\ast(\tilde{A})$ tells us that the attaching map is $\kappa_{\tilde{A}} = 4 \eta \in \pi_3(S^2) \cong \mathbb{Z}$.

The long exact homotopy sequence of the bundle $\mathbb{R}P^3 \to \tilde{M} \to \tilde{A}$ tells us that $\pi_2(\tilde{M}) \to \pi_2(\tilde{A})$ is multiplication by 2, so $\tilde{M}^{(3)} \to \tilde{A}^{(3)}$ has the form $2 \iota_2 \perp \mu \eta$. Thus the diagram gives

$$4 \eta = \kappa_{\tilde{A}} = q \circ \kappa_{\tilde{M}} = (2\lambda + 4\mu)\eta.$$ 

It follows that $\lambda$ is even and hence $a^2 = 0$.

Finally we show that $a_3^2 = 0$.

A property of Steenrod operations is that $Sq^k(x) = x^2$ if $|x| = k$. Thus $a_3^2 = Sq^3(a_3) = Sq^1 Sq^2(a_3)$ from the Adem relations.

The Mayer-Vietoris sequence for $\tilde{M}$ gives $a_3 = \delta(vt)$, where $\delta$ is the connecting homomorphism. Therefore $Sq^2(a_3) = \delta(Sq^2(vt)) = \delta(Sq^2 vt + Sq^1 v Sq^1 t + v Sq^2 t) = \delta(0 + v^2 t^2 + 0) = 0$, since $v^2 t^2 = \text{Im}((j \times 1_{\mathbb{R}P^3}) \perp (j' \times 1_{\mathbb{R}P^3}) \circ \bar{\tau})^\ast$.

Hence $a_3^2 = 0$.

8. 6-MANIFOLDS

According to Wall, any simply connected 6-manifold $N$ is diffeomorphic to the connected sum $(S^3 \times S^3)^\# r \# \tilde{N}$. for some integer $r$ and some simply connected 6-manifold $\tilde{N}$ with $H^3(\tilde{N}) = 0$.

Further, Wall shows that simply connected 6-manifolds $\tilde{N}$ with $H^2(\tilde{N}) \cong \mathbb{Z}$ and $H^3(\tilde{N}) = 0$ are determined by:

(1) a positive integer $d$ given by $x^3 = dz$ where $x$ generates $H^2(\tilde{N})$ and $z$ generates $H^6(N)$ (which determines the cup products in $\tilde{N}$);

(2) an integer $p$ determined by $x p_1(\tilde{N}) = py$, where $p_1(\tilde{N}) \in H^4(Z)$ is the first Pontrjagin class of the tangent bundle of $\tilde{N}$ and $y$ is the generator of $H^4(N)$ whose sign is determined by $x^2 = dy$.

He shows that they must satisfy $p \equiv 4d$ modulo 24.

Note: Wall’s classification theorem is more general. The above is the restriction to our situation. In our case we have $A = (S^3 \times S^3)^\# (S^3 \times S^3)^\# \tilde{A}$ where $d(\tilde{A}) = 4$.

For a differentiable manifold $N$, let $\tau(N)$ denote the tangent bundle of $N$. According to Wall, $c_1(\tau(A)) = 2x$, $c_2(\tau(A)) = 12y$ and so $p_1(\tau(A)) = c_1^2 - 2c_2 = 4x^2 - 24y = 16y - 24y = -8y$.

Thus $d = 4$ and $p = -8$.

9. COHOMOLOGY OF $M$

In this section we use the results of [9] together with our earlier calculation of $H^\ast(\tilde{M})$ to obtain the cohomology of $M$. 

According to Wall, we can write $A = K \# \hat{A}$ where

$$K = (S^3 \times S^3) \# (S^3 \times S^3).$$

Consider the bundle $\mathbb{R}P^3 \to \hat{M} \to \hat{A}$. Set $K' := K \setminus \{\text{chart}\} \simeq \vee_4 S^3$. According to the Decomposition Theorem of [9], we have

$$H^q(M) \cong H^q(\hat{M}) \oplus H^q\left( (K' \times \mathbb{R}P^3) / (\ast \times \mathbb{R}P^3) \right)$$

for $0 < q < 9$. Since $(K' \times \mathbb{R}P^3) / (\ast \times \mathbb{R}P^3) \simeq \vee_4 (S^3 \vee \Sigma^3 \mathbb{R}P^3)$ we find that

$$H^q(M) = \begin{cases} 
\mathbb{Z} & q = 0, 2, 7, 9; \\
\mathbb{Z}^4 & q = 3; \\
\mathbb{Z}/4 & q = 4; \\
(\mathbb{Z}/2)^4 & q = 5; \\
\mathbb{Z}^4 \oplus \mathbb{Z}/4 & q = 6; \\
0 & \text{otherwise.} 
\end{cases}$$

We conclude by using the same method to calculate the cohomology of the total space $E(L_A)$ of the prequantum line bundle $L_A$ over $A$. Recall that $L_A$ is the pullback to $A$ of the line bundle $L_\text{discussed in [6].}$ Applying the theorem to $S^1 \to \hat{A} \to \hat{A}$ gives

$$H^q(E(L_A)) \cong H^q(\hat{A}) \oplus H^q\left( (K' \times S^1) / (\ast \times S^1) \right)$$

for $0 < q < 7$. Since $(K' \times S^1) / (\ast \times S^1) \simeq \vee_4 (S^3 \vee \Sigma^3 S^1)$ we find that

$$H^q(E(L_A)) = \begin{cases} 
\mathbb{Z} & q = 0, 7; \\
\mathbb{Z}^4 & q = 3; \\
\mathbb{Z}^4 \oplus \mathbb{Z}/4 & q = 4; \\
0 & \text{otherwise.} 
\end{cases}$$

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