TORIC ASPECTS OF THE FIRST EIGENVALUE

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Abstract. In this paper we study the smallest non-zero eigenvalue \( \lambda_1 \) of the Laplacian on toric Kähler manifolds. We find an explicit upper bound for \( \lambda_1 \) in terms of moment polytope data. We show that this bound can only be attained for \( \mathbb{CP}^n \) endowed with the Fubini-Study metric and therefore \( \mathbb{CP}^n \) endowed with the Fubini-Study metric is spectrally determined among all toric Kähler metrics. We also study the equivariant counterpart of \( \lambda_1 \) which we denote by \( \lambda_T^1 \). It is the the smallest non-zero eigenvalue of the Laplacian restricted to torus-invariant functions. We prove that \( \lambda_T^1 \) is not bounded among toric Kähler metrics thus generalizing a result of Abreu-Freitas on \( S^2 \). In particular, \( \lambda_T^1 \) and \( \lambda_1 \) do not coincide in general.

1. Introduction

Toric Kähler manifolds are very symmetric Kähler manifolds for which there is a concrete parametrization of the space of Kähler metrics. More concretely, they are symplectic manifolds admitting a Hamiltonian action from a maximal torus and endowed with a compatible, torus invariant Riemannian metric giving rise to an integrable complex structure. The underlying symplectic manifold is completely characterized by a combinatorial object which is a convex polytope called moment polytope arising as the image of the moment map for the torus action. Toric Kähler metrics are parametrized by convex functions on that moment polytope satisfying certain properties as we will discuss in section 2. Toric Kähler manifolds have played a crucial role in studying important questions in geometry. In [10], Donaldson was able to fully characterize those toric Kähler surfaces which admit constant scalar curvature thus settling an important conjecture in Kähler geometry in the toric context for real dimension 4. There has been a lot of interest in studying toric spectral geometry and, in particular, inverse spectral questions in this toric context as well (see [3], [13]).

Given a Riemannian manifold \((M, g)\), the Riemannian metric determines a Beltrami-Laplace operator whose smallest non-zero eigenvalue, which we also refer to as the first eigenvalue, and is denoted by \( \lambda_1(g) \), carries a surprising amount of geometric information. There has been great deal of effort put into finding sharp bounds for \( \lambda_1 \) with geometric meaning (see [6]). Bourguignon–Li–Yau found an upper bound for \( \lambda_1 \) for Kähler manifolds endowed with a full holomorphic embedding into projective space. Polterovich (see [21]) looked at boundedness of \( \lambda_1 \) in the context of symplectic manifolds. He showed, in particular, that there

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are symplectic manifolds admitting compatible Riemannian metrics whose $\lambda_1$ is arbitrarily large. One of the questions we want to address here is: “are there geometric bounds on $\lambda_1(M, g)$ where $M$ is a toric manifold and $g$ is a toric Kähler metric on it?” As it turns out, one can always use Bourguignon–Li–Yau’s result in the toric context, and we use it to give an explicit bound for $\lambda_1$ in terms of moment polytope data. More precisely, we prove the following theorem.

**Theorem 1.1.** Let $(M^{2n}, \omega)$ be a toric symplectic manifold endowed with a toric Kähler structure whose Riemannian metric we denote by $g$. Let $P \subset \mathbb{R}^n$ be its moment polytope. There is an integer, $k_0(P) \geq 1$ such that for any $k \geq k_0(P)$

$$\lambda_1(g) \leq \frac{2nk(N_k + 1)}{N_k} \left(1 + \frac{Cn \int_M c_1 \wedge \omega^{n-1}}{k \int_M \omega^n}\right),$$

where $C$ can be taken to be the supremum of the norms of the primitive normal vectors to facets of $P$ and $N_k + 1 = \sharp(P \cap \mathbb{Z}^n/k)$. If $P$ is integral (i.e its vertices lies in $\mathbb{Z}^n$), then we have a finer bound given by

$$\lambda_1(g) \leq \frac{2n(N + 1)}{N},$$

where $N + 1 = \sharp(P \cap \mathbb{Z}^n)$ is the number of integer points in $P$.

We will make $k_0(P)$ explicit ahead. The Fubini-Study metric realizes the bound in the above theorem. In fact we show that this is the only toric Kähler metric that does saturate this bound in the integral case.

**Theorem 1.2.** Let $(M^{2n}, \omega)$ be an integral toric symplectic manifold endowed with a toric Kähler structure whose Riemannian metric we denote by $g$. Let $N + 1$ be the number of integer points in the moment polytope of $M$. If

$$\lambda_1(g) = \frac{2n(N + 1)}{N},$$

then $M$ is equivariantly symplectomorphic to $\mathbb{C}P^n$ and this symplectomorphism takes $g$ into the Fubini-Study metric on $\mathbb{C}P^n$.

It was previously known (see [6]) that the Fubini-Study metric on $\mathbb{C}P^n$ is determined by the spectrum among all Kähler metrics on $\mathbb{C}P^n$ compatible with the standard complex structure. It was also proved by Tanno (see [22]) that, if a Kähler manifold of real dimension less than 12 has the same spectrum as $\mathbb{C}P^n$ with the Fubini-Study metric, then it is holomorphically isometric to it. A simple consequence of the above theorem is that the spectrum of the Laplacian of a toric Kähler metric on an integral toric manifold determines if the manifold is $\mathbb{C}P^n$ endowed with the Fubini-Study metric.

**Corollary 1.3.** An integral toric Kähler manifolds which has the same spectrum as $(\mathbb{C}P^n, \omega_{FS}, J_0)$ is holomorphically isometric to it.

Another interesting question is that of spectrally characterizing either constant scalar-curvature, extremal or Kähler-Einstein toric Kähler metrics. In [13] the authors prove that the equivariant spectrum determines if a toric Kähler metric
has constant scalar curvature. A variation of the argument there would show that the equivariant spectrum also determines if a metric is extremal.

Going back to the first eigenvalue, there are various bounds that one can write down for toric \(\text{Kähler} \) manifolds using Bourguignon–Li–Yau’s bound, see \( \S 4.1 \). It would be interesting to see what the best bound is for a given toric manifold, once we fix the polytope. In particular, one could hope to improve the bound in Theorem 1.1 for special classes of manifolds (monotone, Fano..) or special classes of metrics say extremal toric \(\text{Kähler} \) metric, or \(\text{Kähler-Einstein} \) metrics. In [4] the authors prove that a toric \(\text{Kähler-Einstein} \) manifold whose connected component of automorphism group is a torus is never \(\lambda_1\)-\textit{extremal}, where \(\lambda_1\)-\textit{extremal} means extremal for the first eigenvalue with respect to local variations in the \(\text{Kähler} \) metrics space. Hence, in general, we cannot expect a toric \(\text{Kähler-Einstein} \) metric to saturate fine bounds. Another natural candidate to consider is a balanced metric when it exists, see discussion \( \S 4.1 \).

However, the \(\text{Kähler-Einstein} \) property is somewhat reflected in the first eigenvalue. In fact, one can prove an improvement and a converse of Matsushima Theorem 1.1 in \( \S 2.2 \). We show that a toric \(\text{Kähler} \) metric is \(\text{Kähler-Einstein} \) if and only if the coordinates of its moment map are eigenfunctions for \(\lambda_1\).

**Proposition 1.4.** Let \((M,\omega,T)\) be a compact symplectic toric orbifold with moment map \(x : M \to t^*\). Then \((M,g,J,\omega,T)\) is a \(\text{Kähler-Einstein} \) toric orbifold with Einstein constant \(\lambda\) if and only if, up to an additive constant, the moment map satisfies

\[
2\lambda\langle x, b \rangle = \Delta^g \langle x, b \rangle \quad \forall b \in t,
\]

and \(2\lambda\) is the smallest non-vanishing eigenvalue for the \(\text{Kähler-Einstein} \) orbifold toric metric.

Matsushima’s theorem implies that a necessary condition for a toric \(\text{Kähler} \) metric to be \(\text{Kähler-Einstein} \) is that its \(\lambda_1\) be a multiple eigenvalue with multiplicity at least equal to half the dimension of the manifold. What’s more, it follows from the above proposition that one can see if a metric is \(\text{Kähler-Einstein} \) by simply checking if its moment map coordinates are eigenfunctions for \(2\lambda\).

On a toric manifold endowed with a torus invariant metric one can consider a toric version of \(\lambda_1\) namely \(\lambda_1^T\) defined to be the smallest non-zero invariant eigenvalue of the Laplacian i.e. the smallest eigenvalue of the Laplacian restricted to torus invariant functions. We clearly have \(\lambda_1 \leq \lambda_1^T\). In [3], Abreu–Freitas studied \(\lambda_1^T\) for the simplest toric manifold, namely \(S^2\) with the usual \(S^1\) action by rotations around an axis. They proved it was unbounded (both above and below) among \(S^1\)-invariant metrics. In this paper we generalize their results, by using an original approach for the upper bound, on all toric manifolds. We are able to prove the following.

\[\text{It is possible that this result was previously known but the authors did not find a reference for it in the literature and thus state it and prove it.}\]
Theorem 1.5. Let \((M, \omega, T)\) be a compact symplectic toric orbifold, let \(\mathcal{K}^T_\omega\) be the set of all toric Kähler metrics on \((M, \omega, T)\).

\[
\inf\{\lambda_1(g) \mid g \in \mathcal{K}^T_\omega\} = 0.
\]

and

\[
\sup\{\lambda^T_1(g) \mid g \in \mathcal{K}^T_\omega\} = +\infty.
\]

Combining Theorem 1.1 and 1.5, we see that there are toric Kähler manifolds for which \(\lambda_1\) does not coincide with \(\lambda^T_1\). For toric Kähler–Einstein metrics, it follows from Matsushima Theorem \[19\] that \(\lambda_1 = \lambda^T_1\) as there are invariant eigenfunctions for \(\lambda_1\). It would be interesting to characterize those toric Kähler manifolds for which this occurs. Given a weight vector \(m \in \mathbb{Z}^n\), one could also define \(m\)-equivariant \(\lambda_1\) which we denote by \(\lambda^m_1\) as the lower non-vanishing eigenvalue of the Laplacian restricted to the set of \(m\)-equivariant functions

\[
\{f \in C^\infty(M, \mathbb{C}) : f(e^{i\theta} p) = e^{i\theta \cdot m} f(p), \quad \forall p \in M, \theta \in \mathbb{R}^n\}.
\]

One could prove a similar result in this setting and again it would be interesting to understand which metrics have \(\lambda_1 = \lambda^m_1\) and how this depends on \(w\). Note that \(\lambda^T_1 = \lambda^0_1\). Recently, in \[18\], Hall-Murphy proved that on any toric manifold \(\lambda^T_1\) restricted to the class of toric Kähler metrics whose scalar curvature is non-negative is bounded and this generalizes another result in \[3\].

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2. Background

2.1. Toric Kähler geometry. This section does not contain all the ingredients of symplectic toric geometry needed in subsequent sections, we only lay down the notation and refer to the classical references for this theory (in particular for proofs of what is claimed in this section) like \[1\] \[2\] \[8\] \[11\] \[16\] \[17\] \[20\].

Let \((M^{2n}, \omega, T^n)\) be a compact toric symplectic orbifold. It admits a moment map \(x : M \to P \subset t^*\) where \(t = \text{Lie } T\) is the Lie algebra of \(T\) and \(t^*\) is its dual such that for all \(a \in t\)

\[-d\langle x, a \rangle = \omega(X_a, \cdot)\]

where \(X_a\) is the vector field on \(M\) induced by the 1-parameter subgroup associated to \(a\). The image of \(x\), that we denote \(P\), is called the moment polytope. It is a convex simple (i.e. its vertex are the intersection of exactly \(n\)–facets) polytope in \(t^*\). Let \(\nu = \{\nu_1, \ldots, \nu_d\}\) be a set of vectors in \(t\) which are normal to the facets of \(P\) and inward pointing. Let \(\Lambda\) be a lattice in \(t\) such that \(T = t/\Lambda\). If
ν ⊂ Λ, the triple (P, ν, Λ) is rational and if, each subset of vectors in ν normal to facets meeting at any given vertex, forms a basis of Λ, then we say that (P, ν, Λ) is Delzant. The Delzant–Lerman–Tolman correspondence states that compact toric symplectic orbifolds are in one to one correspondence with rational labelled polytopes and are smooth if and only if the rational labelled polytopes is Delzant.

In this text, we often identify t with R^n and Λ with Z^n.

Definition 2.1. Let (P, ν) be a labelled polytope. The functions L_1, ..., L_d ∈ Aff(t^*, R) are said to be the defining functions of (P, ν) if P = {x ∈ t^* | L_k(x) ≥ 0} and dL_k = ν_k.

On the pre-image of the interior of the polytope ˚M = x−1(˚P), the action of T is free. The action–angle coordinates (x, θ) = (x_1, ..., x_n, θ_1, ..., θ_n) are local coordinates on ˚M used to (locally) identify ˚M with ˚P × T where the first projection coincides with the moment map and

\[ \omega = \sum_{i=1}^{n} dx_i \wedge d\theta_i. \]  

(2)

The space of compatible T–invariant Kähler metrics on (M, ω, T) is parametrized by the set of symplectic potentials which is denoted by \( \mathcal{S}(P, \nu) \) (up to the addition of an affine linear function). The set \( \mathcal{S}(P, \nu) \) is defined as the subset of functions \( u \in C^\infty(P, \mathbb{R}) \cap C^0(P, \mathbb{R}) \), where ˚P denotes the interior of P, such that

(i) \( u - \frac{1}{2} \sum_{k=1}^{d} L_k \log L_k \in C^\infty(P, \mathbb{R}); \)

(ii) the restriction of u to ˚P is strictly convex;

(iii) for each face F of P, the restriction of u to ˚F (the relative interior of F) is strictly convex.

Given \( u \in \mathcal{S}(P, \nu) \), the metric defined by

\[ g_u = \sum_{i,j=1}^{n} u_{ij} dx_i \otimes dx_j + u^{ij} d\theta_i \otimes d\theta_j \]  

where \( u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j} \) and \( (u^{ij}) = (u_{ij})^{-1} \), is a t–invariant Kähler metric on ˚P × T ≃ ˚M compatible with ω. Conditions (i), (ii), (iii) ensure that \( g_u \) is the restriction of a smooth metric on M. For convenience, we denote \( H^u_{ij} = u^{ij} \), \( G^u_{ij} = u_{ij} \), \( H^u = (H^u_{ij}) \) and \( G^u = (G^u_{ij}) \). One can prove that any toric Kähler structure on (M, ω) can be written using a symplectic potential in \( \mathcal{S}(P, \nu) \) as above.

Abreu [1] computed the curvature of a compatible Kähler toric metric, \( g_u \), in terms of its symplectic potential \( u \). The scalar curvature of \( g_u \) is the pull-back by \( x \) of the function

\[ \text{scal}_u = -\sum_{i,j=1}^{n} \frac{\partial^2 H^u_{ij}}{\partial x_i \partial x_j}. \]  

(4)

\(^2\)To recover the original convention introduced by Lerman and Tolman in the rational case, take \( m_k \in \mathbb{Z} \) such that \( \frac{1}{m_k} \nu_k \) is primitive in Λ so (P, m_1, ..., m_d, Λ) is a rational labelled polytope.
Moreover, the Ricci curvature is

\[ \rho^{g_u} = \frac{-1}{2} \sum_{i,l,k} H_{i,l,k}^u dx_k \wedge d\theta_l. \]

2.2. Kähler–Einstein metrics and moment map as eigenfunctions of the Laplacian. Let \((M, g_u, J, \omega, T)\) be a compact Kähler toric manifold with moment map \(x\) and denote by \(\Delta^u\), the Laplacian with respect to the Riemannian metric \(g_u\). Recall that \((M, g_u, J, \omega)\) is Kähler–Einstein if there exists \(\lambda\) such that \(\lambda \omega = \rho^{g_u}\), where \(\rho^{g_u}\) is the Ricci form of the Chern and Levi connection. We say that \(\lambda\) is the Einstein constant. In the compact toric setting, \(\lambda > 0\).

**Proposition 2.2.** Let \((M, \omega, T)\) be a compact symplectic toric orbifold with moment map \(x : M \to t^*\). Then \((M, g_u, J, \omega, T)\) is a Kähler–Einstein toric orbifold with Einstein constant \(\lambda\) if and only if, up to an additive constant, the moment map satisfies

\[ 2\lambda \langle x, b \rangle = \Delta^u \langle x, b \rangle \quad \forall b \in t. \]

Matsushima Theorem [19] already states that on a given a Kähler–Einstein manifolds, of Einstein constant \(\lambda\), the eigenvalues of the Laplacian are bounded below by \(2\lambda\). Proposition 2.2 is a finer converse of this: for a toric Kähler metric if the moment map coordinates are eigenfunctions for the same eigenvalue then the metric is Kähler–Einstein (and then, by Matsushima’s theorem, this common eigenvalue is the smallest one).

**Proof.** We have

\[ \Delta^u = -\sum_{i,j=1}^n G_{ij} \frac{\partial^2}{\partial \theta_i \partial \theta_j} + \frac{\partial}{\partial x_i} \left( H_{ij} \frac{\partial}{\partial x_j} \right), \]

so that

\[ d\Delta^u \langle x, b \rangle = -\sum_{i,j,k=1}^n H_{i,j,k}^u b_j dx_k = -2\rho^{g_u} \langle x_b, \cdot \rangle \quad \forall b \in t \]

using (5). From (8), we see that \(\Delta^{g_u} x\) is a moment map for \(2\rho^{g_u}\) and, in the Kähler–Einstein case \(\lambda \omega = \rho^{g_u}\), this implies that \(2\lambda x - \Delta^{g_u} x = \alpha \in t^*\) is constant. Thus \(x - \alpha \frac{2}{\lambda}\) satisfies (1).

The converse is also a simple computation. Indeed, assuming (1), we have

\[ \Delta^{g_u} x_i = -\sum_{j=1}^n \frac{\partial H_{ij}}{\partial x_j} = 2\lambda x_i \]
for \(i = 1, \ldots, n\). Inserting this in (5), we get

\[
\rho^{g_u}(\cdot, \cdot) = \frac{-1}{2} \sum_{i, l, k=1}^{n} H_{li, ik} dx_k \wedge d\theta_l
\]

\[
= \frac{1}{2} \sum_{l, k=1}^{n} \frac{\partial}{\partial x_k} (2\lambda x_l) dx_k \wedge d\theta_l
\]

\[
= \lambda \sum_{k=1}^{n} dx_k \wedge d\theta_k = \lambda \omega,
\]

as in (2).

3. THE FIRST INVARIANT EIGENVALUE \(\lambda_T^1\)

3.1. Minimizing \(\lambda_T^1\). The goal of this subsection is to show the first part of Theorem 1.5. With the notation introduced in Section 2, Theorem 1.5 would follow from

\[
\inf_{u \in S(P, \nu)} \{\lambda_1(g_u)\} = 0.
\]

An easy computation shows that for any \(T\)-invariant function

\[
\int_M g_u(\nabla^{g_u} f, \nabla^{g_u} f) dv_{g_u} = \int_{T^n} d\theta_1 \wedge \cdots \wedge d\theta_n \int_P H^u(df, df) dx_1 \wedge \cdots \wedge dx_n.
\]

Here \(df\) denotes the differential of \(f\) seen as a function on \(P\). We fix coordinates on \(T^*\) and, by translating if necessary, we assume that \(\int_P x_i d\varpi = 0\) where we have set \(d\varpi = dx_1 \wedge \cdots \wedge dx_n\). The Rayleigh characterization of the first eigenvalue tells us that for any \(i = 1, \ldots, n\)

\[
\lambda_1(g_u) \leq \frac{\int_P H^u(dx_i, dx_i) d\varpi}{\int_P x_i^2 d\varpi} = \frac{\int_P u_{ii} d\varpi}{\int_P x_i^2 d\varpi}
\]

with equality if and only if \(x_i\) is an eigenfunction of the Laplacian \(\Delta^{g_u}\). Since the denominator does not depend on \(u\), to show (10), it is sufficient to show that we can find \(u \in S(P, \nu)\) with arbitrarily small \(u_{ii}\), as Abreu and Freitas did for \(S^1\)-invariant metrics on \(S^2\) in [3].

Take any \(u_o \in S(P, \nu)\) and for any positive real number \(c > 0\) put \(u_c = u_o + cx_i^2\). First, we will show that \(u_{ii}^c\) decreases when \(c\) increases. We have \(\text{Hess } u_c = \text{Hess } u_o + cE_i\) where \(E_i = (\delta_{li}\delta_{ki})_{1 \leq l, k \leq n}\) and \(\delta_{li}\) being the Kronecker symbol. In particular,

\[
\det \text{Hess } u_c = \det \text{Hess } u_o + c \det M_{ii}
\]

where \(M_{lk}\) denotes the \((l, k)\)-minor matrix of \(\text{Hess } u_o\). Note that \(M_{ii}\) is positive definite on each point of the interior \(\bar{P}\) since it corresponds to the restriction of the metric \(g_{u_o}\) (as a metric on \(\bar{P}\)) to the orthogonal space to \(\frac{\partial}{\partial x_i}\). In particular for
any $c > 0$, formula (12) gives $\det \text{Hess} u_c > 0$. Now, since the $(i, i)$-minor matrices of $\text{Hess} u_o$ and $\text{Hess} u_c$ are the same we have

$$u_c^{ii} = \frac{\det M_{ii}}{\det \text{Hess} u_o + c \det M_{ii}}$$

Thus, $u_c^{ii} \to 0$ when $c \to +\infty$.

Now, we will show that $u_c \in \mathcal{S}(P, \nu)$ for all $c > 0$ by verifying each of the conditions $(i)$, $(ii)$ and $(iii)$ of the definition, see [2]:

(i) $u_c - \frac{1}{2} \sum_{k=1}^{d} L_k \log L_k = cx_i^2 + (u_o - \frac{1}{2} \sum_{k=1}^{d} L_k \log L_k)$ is smooth since $u_o \in \mathcal{S}(P, \nu)$;

(ii) let $x \in \hat{P}$, $(\text{Hess} u_c)_x$ is positive definite if all its eigenvalues are positive. The smallest eigenvalues of $(\text{Hess} u_c)_x$ is

$$\text{min} \left\{ \frac{(\text{Hess} u_c)_x v, v}{(v, v)} : (\nu, v) \neq (v, v) \right\} = \text{min} \left\{ \frac{((\text{Hess} u_o)_x v, v)}{(v, v)} + c \frac{v_i^2}{(v, v)} \right\}$$

$$\geq \text{min} \left\{ \frac{((\text{Hess} u_o)_x v, v)}{(v, v)} \right\} > 0;$$

(iii) let $F$ be a face of $P$ and $x \in \hat{F}$ (we may assume that $F$ is not a vertex since in this case there is nothing to check). Again, the smallest eigenvalues of $(\text{Hess} (u_c)_x)$ is

$$\text{min} \left\{ \frac{(\text{Hess} u_c)_x v, v}{(v, v)} : (\nu, v) \neq (v, v) \right\} = \text{min} \left\{ \frac{((\text{Hess} u_o)_x v, v)}{(v, v)} + c \frac{v_i^2}{(v, v)} \right\}$$

$$\geq \text{min} \left\{ \frac{((\text{Hess} u_o)_x v, v)}{(v, v)} \right\} > 0.$$

Hence $u_c \in \mathcal{S}(P, \nu)$ for all $c > 0$ and $\lambda_1(g_{u_c}) \to 0$ when $c \to +\infty$. This proves (LU).

3.2. Maximizing $\lambda_1^T$. The goal of this subsection is to show the second part of Theorem [13]. Let $(P, \nu)$ be the labelled moment polytope of a symplectic toric orbifold $(M, \omega, T)$. Without loss of generality, we assume, in this section, that $0 \in \hat{P}$. In particular, the defining functions $L_k(x) = \langle x, \nu_k \rangle + c_k$ satisfy $L_k(0) = c_k > 0$. Let $u_o \in \mathcal{S}(P, \nu)$ be the Guillemin potential, that is,

$$u_o = \frac{1}{2} \sum_{i=1}^{d} L_i \log L_i - L_i$$

and $G_o = \text{Hess} u_o$ and $H_o = (\text{Hess} u_o)^{-1}$. Choosing coordinates and an inner product $(\cdot, \cdot)$, we see $G_o$ and $H_o$ as matrices. For $s > 1$, we denote $u_s^o$, the Guillemin potential of $sP$ which is the dilation of $P$ by an $s$-factor. The defining affine-linear functions of $sP$ are $L_k^s = \langle x, \nu_k \rangle + sc_k$. Consider the following family of functions on $P$

$$u^s = u_o - \frac{u^o_s}{s}.$$
We will show that $u^s \in S(P, \nu)$. Since $u^s_o$ is smooth on $P$ when $s > 1$, to show that $u^s \in S(P, \nu)$ it is sufficient to show that $G^s = \text{Hess } u^s$ is positive definite on $\hat{P}$. This is clear since $L_k^s(x) > L_k(x)$ on $P$ and

$$G^s = \frac{1}{2} \sum_{k=1}^{d} \left( \frac{1}{L_k} - \frac{1}{sL_k^s} \right) \nu_k \otimes \nu_k.$$ 

In [3], the authors show that for the 2-sphere, $\lambda^T_1(g_{u^s}) \nearrow +\infty$ when $s$ goes to 1. We will use another approach to show that the same holds in higher dimension. The rough idea is that, since $u^s \to 0$ uniformly on $P$, the eigenvalues of the inverse of its Hessian should, in some way, tend to infinity and thus the Rayleigh quotient of any function should go to infinity. We write

- $G^s_o = \text{Hess } u^s_o$, and $H^s_o = (G^s_o)^{-1}$
- $G^s = \text{Hess } u^s$ and $H^s = (G^s)^{-1}$.

We start by proving the following simple lemma.

**Lemma 3.1.** For any $f \in C^1(P)$

$$\int_P H^s(df, df) d\bar{\omega} \geq \int_P H_o(df, df) d\bar{\omega}. \quad (16)$$

In particular, the variation $\lambda^T_1(s) := \lambda^T_1(g_{u^s})$ is bounded below by $\lambda_1(g_{u_o})$.

**Proof.** First note that the eigenvalues of $H_o G^s$ are, strictly smaller than 1 on $\hat{P}$. In fact, if $\lambda$ is an eigenvalue for $H_o G^s$ and $u$ is the corresponding eigenvector, $G^s u = \lambda G^s_o u$, so that

$$\sum_{k=1}^{d} \left( \frac{1 - \lambda}{L_k(x)} - \frac{1}{sL_k^s(x)} \right) (\nu_k, u)^2 = 0,$$

which is not possible if $\lambda \geq 1$. Since $G^s = G_o - \frac{1}{s} G^s_o$, we have

$$H^s = \left( Id_n + \sum_{k=1}^{\infty} \left( \frac{1}{s} H_o G^s_o \right)^k \right) H_o,$$

on the interior of $P$. From this expression, we get that for any $f \in C^1(P)$,

$$\int_P H^s(df, df) d\bar{\omega} \geq \int_P H_o(df, df) d\bar{\omega}. \quad (17)$$

We are now in a position to prove that for the family of metrics determined by $u^s$, $\lambda^T_1$ is unbounded.

**Proposition 3.2.** $\sup\{\lambda^T_1(s) \mid s > 1\} = +\infty$. 

Proof. Assume that $\lambda^T_1(s)$ is also bounded above by a constant, say $\kappa > 0$, then, we can find a sequence $s_k \to 1^+$ such that $\lambda^T_1(s_k)$ converges to some $\lambda > 0$.

Consider a sequence $f_{s_k} \in C^\infty(M)^T = C^\infty(P)$ of eigenfunctions

$$\Delta g_{s_k} f_{s_k} = \lambda^T_1(s_k) f_{s_k}$$

normalized such that $\|f_{s_k}\|_{L^2} = \int_P (f_{s_k})^2 d\omega = 1$. Note that the inequality (16) implies that the Sobolev norms of $\{f_{s_k}\}$ in $H^1(M, g_{uo})$ are bounded above by $\kappa + 1$.

Indeed, combining the hypothesis and (16), we have

$$\kappa > \lambda^T_1(s_k) = \int_P H^s(df_{s_k}, df_{s_k}) d\omega \geq \|\nabla_{g_{uo}} f_{s_k}\|_{g_{uo}}^2.$$  

Consequently, there exists a subsequence, that we still index by $s$ for simplicity, of eigenfunctions $f_s \in C^\infty(M)^T$ converging in the $L^2(M, g_{uo})$ topology to some function $f \in L^2(M)$. We have $\|f\|_{L^2} = 1$, $\int_P f(x) d\omega = 0$.

A straightforward calculation yields

$$G^s(x) = \frac{s - 1}{2s} \sum_{k=1}^d \left( \frac{L_k(x) + sc_k}{L_k(x)L^*_k(x)} \right) \nu_k \otimes \nu_k,$$

and thus, for any $x \in \mathring{P}$,

$$B_x := \lim_{s \to 1^+} \frac{G^s(x)}{s - 1} = \frac{1}{2} \sum_{k=1}^d \left( \frac{L_k(x) + c_k}{L_k(x)^2} \right) \nu_k \otimes \nu_k$$

is positive definite and depends smoothly on $x \in \mathring{P}$. For $x \in \mathring{P}$, let

$$A_x = \lim_{s \to 1^+} (s - 1) H^s(x)$$

be the inverse of $B_x$.

Let $K$ be a compact subset of $\mathring{P}$. The integral

$$\int_K H^s(df_s, df_s) d\omega$$

can be written as

(18)

$$\int_K A_x \left( \frac{df_s}{\sqrt{s - 1}}, \frac{df_s}{\sqrt{s - 1}} \right) d\omega + \int_K ((s - 1) H^s - A_x) \left( \frac{df_s}{\sqrt{s - 1}}, \frac{df_s}{\sqrt{s - 1}} \right) d\omega.$$  

Now for any $\epsilon > 0$

$$\left| \int_K ((s - 1) H^s - A_x) \left( \frac{df_s}{\sqrt{s - 1}}, \frac{df_s}{\sqrt{s - 1}} \right) d\omega \right|$$

$$\leq \sup_K ((s - 1) H^s - A_x) \int_K \left| \frac{df_s}{\sqrt{s - 1}} \right|^2 d\omega$$

$$\leq \epsilon \int_K \left| \frac{df_s}{\sqrt{s - 1}} \right|^2 d\omega$$
when $s$ is sufficiently close to 1. On $K$, the symmetric bilinear form $A$ is positive definite and its smallest eigenvalue $\mu$ is strictly positive. Hence, on $K$, the norm of $A_x$ is equivalent to the Euclidean norm i.e.

$$\int_K A \left( \frac{df_s}{\sqrt{s-1}}, \frac{df_s}{\sqrt{s-1}} \right) d\varpi \geq \Gamma_K \int_K \left| \frac{df_s}{\sqrt{s-1}} \right|^2 d\varpi$$

is bounded. The inequality

$$\int_K H^s(df_s, df_s) d\varpi \leq \kappa,$$

implies

$$\int_K A_x \left( \frac{df_s}{\sqrt{s-1}}, \frac{df_s}{\sqrt{s-1}} \right) d\varpi + \int_K ((s-1)H^s - A_x) \left( \frac{df_s}{\sqrt{s-1}}, \frac{df_s}{\sqrt{s-1}} \right) d\varpi \leq \kappa,$$

but

$$\int_K A_x \left( \frac{df_s}{\sqrt{s-1}}, \frac{df_s}{\sqrt{s-1}} \right) d\varpi + \int_K ((s-1)H^s - A_x) \left( \frac{df_s}{\sqrt{s-1}}, \frac{df_s}{\sqrt{s-1}} \right) d\varpi \geq (\Gamma_K - \epsilon) \int_K \left| \frac{df_s}{\sqrt{s-1}} \right|^2 d\varpi$$

and we conclude that

$$\frac{\Gamma_K}{2} \int_K \left| \frac{df_s}{\sqrt{s-1}} \right|^2 d\varpi \leq \kappa.$$

Using the Poincaré inequality, there exists $C_K$, a constant depending only on $K$, such that

$$C_K \int_K \left| \frac{df_s}{\sqrt{s-1}} \right|^2 d\varpi \geq \frac{1}{s-1} \int_K \left( f - \frac{\int_K f d\varpi}{\int_K d\varpi} \right)^2 d\varpi.$$

However, since $f_s \rightarrow f$ in the $L^2$-topology on $P$ and on $K$,

$$\int_K \left( f - \frac{\int_K f d\varpi}{\int_K d\varpi} \right)^2 d\varpi \rightarrow \int_K \left( f - \frac{\int_K f d\varpi}{\int_K d\varpi} \right)^2 d\varpi.$$

But

$$0 \leq \int_K \left( f - \frac{\int_K f d\varpi}{\int_K d\varpi} \right)^2 d\varpi \leq \frac{2(s-1)C_K \kappa}{\Gamma_K} \rightarrow 0,$$

and so it must tend to zero. This implies that

$$\int_K \left( f - \frac{\int_K f d\varpi}{\int_K d\varpi} \right)^2 d\varpi = 0,$$

and $f$ is a constant on $K$. Since $K$ is arbitrary, $f$ is constant on $P$. But $\int_P f = 0$ so that $f$ must be identically zero which contradicts $\int_P f^2 = 1$. \qed

This proposition proves the second part of Theorem [1.5]. We have thus proved Theorem [1.5].
4. Bounds on \( \lambda_1 \) for toric manifolds

4.1. The Bourguignon–Li–Yau bound of an integral polytope. Given a complex projective manifold \((M, J, L)\) where \((M, J)\) is complex manifold and \(L \to M\) is a complex line bundle giving a fixed embedding \(\Phi : M \hookrightarrow \mathbb{CP}^N \cong \mathbb{P}(H^0(M, L))^*\), in [7], Bourguignon–Li–Yau gave a bound on the first eigenvalue of any Kähler metric \(\omega\), compatible with \(J\). The bound depends only on the dimension of \(M\), the Kähler class \(\left[\omega\right] \in H^2(M, \mathbb{R})\) and the embedding class \([\Phi^*\omega_{FS}] = 2\pi c_1(L)\). The aim of this subsection is to discuss and review the result, as well as apply it to integral toric manifolds. We start by stating the main result of [7].

**Theorem 4.1** (Bourguignon–Li–Yau). Let \(M^n\) be a compact complex manifold and let \(\Phi : M \to \mathbb{CP}^N\) be a holomorphic immersion such that \(\Phi(M)\) is not contained in any hyperplane in \(\mathbb{CP}^N\). Then for any Kähler metric \(\omega\) on \(M\), compatible with the given complex structure

\[
\lambda_1(M, \omega) \leq \frac{2n(N+1) \int_M \Phi^*\omega_{FS} \wedge \omega^{n-1}}{N \int_M \omega^n},
\]

where \(\omega_{FS} = i\partial\bar{\partial} \log(|Z_0|^2 + \cdots + |Z_N|^2)\) is the Fubini-Study form on \(\mathbb{CP}^N\).

Given a complex projective manifold \((M, J, L)\), we say that an immersion \(\Phi : M \to \mathbb{CP}^N\) is full if its image is not contained in any hyperplane of \(\mathbb{CP}^N\). Given a full holomorphic immersion, we set

\[
B([\Phi], [\omega]) = \frac{2n(N+1) \int_M \Phi^*\omega_{FS} \wedge \omega^{n-1}}{N \int_M \omega^n}.
\]

It is clear that \(B([\Phi], [\omega])\) only depends on the \(H^{1,1}(M, \mathbb{R})\)-cohomology classes \([\omega]\) and \([\Phi^*\omega_{FS}]\).

Arezzo–Ghigi–Loi generalized Theorem 4.1 to provide a bound on the first eigenvalue of Kähler manifolds admitting a Gieseker stable bundle, see [3]. In [21], Polterovich used the Bourguignon–Li–Yau Theorem to give a bound on \(\lambda_1\) for all rational symplectic manifold whose symplectic class \(\frac{1}{2\pi}[\omega]\) is in \(H^2(M, \mathbb{Q})\). In the toric case, this theorem can be applied directly to provide (various) bounds on the first eigenvalue of compact toric Kähler manifolds. Indeed, given a toric compact Kähler manifold \((M, \omega, J, T)\) it is known, see for e.g. [16], that \(H^2_{ds}(M) = H^{1,1}_d(M)\). Hence, one can pick a symplectic form \(\tilde{\omega}\), compatible with \(J\) and lying in an integral and very ample class. Using Kodaira’s embedding Theorem, we know that there exists a full embedding \(\Phi : M \to \mathbb{CP}^N\) such that \([\Phi^*\omega_{FS}] = [\tilde{\omega}]\). Hence, by Theorem 4.1, \(\lambda_1(\omega, J) \leq B(\Phi, [\omega])\). The point of Theorem 4.1 is to give a bound that depends only (and explicitly) on the polytope.

**Remark 4.2.** Using Bourguignon–Li–Yau Theorem, we can get a finer upper bound for \(\lambda_1(M, \omega)\), namely

\[
\inf \{ B([\Phi], [\omega]) \mid \Phi : M \to \mathbb{CP}^N \text{ full holomorphic immersion} \}.
\]
Many natural questions arise: given $\Omega = [\omega]$, is this infimum reached for some immersion? If so, is this immersion minimal or balanced? Note that in the Riemannian case, there is an Embedding Theorem due to Colin-de-Verdière and El Soufi–Ilias (see [14]) concerning $\lambda_1$–extremal metrics. These Riemannian metrics are essentially defined as critical points of the map $g \mapsto \lambda_1(g)$ on the space of Riemannian metrics with fixed total volume. In that case, the aforementioned authors showed that an orthonormal basis of the first eigenspace provides a minimal embedding into a sphere $S^N$ such that the standard round metric on $S^N$ pulls-back to the extremal one.

**Definition 4.3.** Given a compact symplectic toric manifold $(M, \omega, T)$ with moment polytope $P$ we say that $P \subset \mathfrak{t}^*$ is integral if its vertices lie in the dual of the lattice $\Lambda \subset \mathfrak{t}$ of circle subgroups of the torus $T$.

It is well known that integral polytopes correspond to symplectic toric manifolds whose cohomology class is integral.

Any symplectic toric manifold admits compatible complex structures (see [8, 16]) and two such compatible complex structures are biholomorphic. A symplectic toric manifold whose cohomology class is integral carries a holomorphic line bundle $L$ whose first Chern class is the class of the symplectic form. It is known that this holomorphic line bundle defines a full holomorphic embedding $\Phi_u : M \hookrightarrow \mathbb{CP}^N$ where $N$ is the number of lattice points in $P$. The embedding is associated to a basis for $H^0(L)$ namely $\{e^{m_0}1, m \in \Lambda \cap P\}$ where $1$ is a reference holomorphic section of $L$. The embedding is defined by

$$\Phi(z) = [e^{m_0} : \cdots : e^{m_N}]$$

where $m_0, \ldots, m_N$ are the lattice points in $P$. One can express such an embedding in action-angle coordinates.

$$\Phi_u(x, \theta) = [\Phi_{u,m_0}(z) : \cdots : \Phi_{u,m_N}(z)],$$

where $\Phi_{m_0}(z) = e^{m_0 \cdot z}$ and $z = \frac{\partial u}{\partial x} + i\theta$ is the complex coordinate on $M$. If we take the Guillemin potential, $u_0 = \frac{1}{2} \sum_{k=1}^{d} L_k \log L_k$ then

$$\Phi_{u_0}(x, \theta) = [\Pi_{i=1}^{d} L_k(x)^{\omega^{m_0}_{\mathfrak{t}^*}} e^{i m_0 \cdot \theta} : \cdots : \Pi_{i=1}^{d} L_k(x)^{\omega^{m_N}_{\mathfrak{t}^*}} e^{i m_N \cdot \theta}].$$

It is known, see [16] that $[\Phi_{u_0, \omega_{FS}}] = [\omega]$.

**Remark 4.4.** It is known, see for example [9], that for two distinct symplectic potentials $u, u_0 \in \mathcal{S}(P, \nu)$ the map

$$\gamma_{u,u_0} : P \times T \longrightarrow P \times T$$

defined by $\gamma_{u,u_0}(x, e^{i\theta}) = ((du_0)^{-1}(d_x u), e^{i\theta})$ extends as an equivariant diffeomorphism on $M$ sending $J_u$ to $J_{u_0}$ and $\gamma_{u,u_0}^* \omega = \omega + dd^c h$ where $h \in C^\infty(M)^T$. We have $\gamma_{u,u_0}^* \Phi_{u_0} = \Phi_u$ and, since the space of symplectic potentials is convex the map $\gamma_{u,u_0}$ preserves cohomology classes and $[\Phi_{u_0, \omega_{FS}}] = [\omega]$.

**Remark 4.5.** Together with $\Phi_u$ comes an embedding $\phi : T \hookrightarrow \mathbb{T}^{N+1}$ induced by the map linear map $\phi_* : \mathfrak{t} \rightarrow \mathbb{R}^{N+1}$, taking $\theta \in \mathfrak{t}$ to

$$(\theta \cdot \alpha_0, \ldots, \theta \cdot \alpha_N) \in \mathbb{R}^{N+1}.$$
so that the maps $\Phi_u$ are $\phi$–equivariant embeddings.

Applying the Bourguignon–Li–Yau theorem to $(M^{2n}, \omega, g_u, J_T)$, we get that
\begin{equation}
\lambda_1(\omega, g_u) \leq \frac{4n(N + 1)}{N} \int_M \Phi_u^* H \wedge \omega^{n-1} = \frac{2n(N + 1)}{N},
\end{equation}
where $N + 1$ is the number of lattice points in $P$.

**Remark 4.6.** Observe that taking $kP$ for $k \in \mathbb{N}^*$ and $k \geq 2$ implies that the right hand side of (21) decreases to $2n$. However, the left hand side decreases quickly as well since $\lambda_1(k\omega, kg_u) = \frac{1}{k} \lambda_1(\omega, g_u)$. Hence, in each rays of Kähler cone in $H^{1,1}(M, \mathbb{Z})$ there is an optimal class, the primitive class, on which we may apply the bound $B(\Phi_u, [\omega])$.

**Remark 4.7.** The Bourguignon–Li–Yau bound is an integer if and only if $N = 2$, $N = n$ or $N = 2N$. The two first cases imply $M$ is a projective space and the last one gives $\lambda_1(\omega, g_u) \leq 2n + 1$. Note that, in this last case, the first eigenvalue of a Kähler–Einstein metric, which is $2\lambda$ by Proposition 1.4 where $\lambda = 2\pi c_1(M)/[\omega]$, cannot reach this bound whenever $[\omega]$ is integral.

### 4.2. A bound on $\lambda_1$ for toric manifolds.

Let $(M^{2n}, \omega, g, J)$ be a compact toric Kähler manifold. The cohomology class of $\frac{\omega}{2\pi}$ is integral if and only if $P$ is integral. In this subsection we will not assume that $P$ is integral. We start by defining an integer $k_0(P)$ associated to $P$. Let $k$ be a fixed integer. Consider the lattice $\mathbb{Z}^n/k \cap P$. Let $P_k$ be the convex hull of $\mathbb{Z}^n/k \cap P$. It is a convex polytope in $\mathbb{R}^n$ contained in $P$. As $k$ tends to infinity the lattice $\mathbb{Z}^n/k \cap P$ becomes finer and eventually $P_k$ will look combinatorially like $P$ i.e. it will have the same number of facets and of vertices as $P$.

**Definition 4.8.** Let $P$ be a Delzant polytope. Set
$$L_i^{\text{min}} = \min \{ L_i(m), m \in P \cap \mathbb{Z}^n/k \}.$$  
We define $k_0(P)$ to be smallest integer $k \geq 1$ such that $P_k$, the convex hull of $\mathbb{Z}^n/k \cap P$ can be written as
$$P_k = \{ x \in P : L_i \geq L_i^{\text{min}}, i = 1, \cdots d \}.$$  
For any integer $k$, we set $N_k = \sharp(\mathbb{Z}^n/k \cap P) - 1$.

This implies in particular that if $k \geq k_0(P)$, $P_k$ has the same number of facets as $P$. Note that if $P$ is integral, $k_0(P) = 1$.

**Theorem 4.9.** Let $(M^{2n}, \omega)$ be a toric symplectic manifold endowed with a toric Kähler structure whose Riemannian metric we denote by $g$. Let $P$ be its moment polytope. For any $k \geq k_0(P)$,
$$\lambda_1(g) \leq \frac{2nk(N_k + 1)}{N_k} \left( 1 + \frac{Cn \int_M c_1 \wedge \omega^{n-1}}{k \int_M \omega^n} \right),$$
where $C$ can be taken to be the supremum of the norms of the primitive vectors normals to facets of $P$. If $P$ is integral then we have a finer bound given by
$$\lambda_1(g) \leq \frac{2n(N + 1)}{N},$$
where $N + 1$ is the number of integer points in $P$.

The theorem is essentially a consequence of the Bourguignon–Li–Yau bound on the first eigenvalue of projective Kähler manifolds. By Kodaira’s embedding theorem, a multiple of a rational symplectic form on a Kähler manifold gives rise to a holomorphic embedding into projective space and Polterovich’s theorem from [21] follows from this. In fact, any Kähler toric manifold admits an embedding into projective space even in the case where its symplectic form is not rational. This can be seen as a consequence of the fact the complex structure of a toric symplectic manifold does not depend on the full moment polytope but only on the fan it determines. Two moment polytopes that are close determine the same fan and therefore an irrational moment polytope can be perturbed into a rational one without changing the complex structure. We will make this precise in the proof of the theorem so as to have effective bounds to use when applying Bourguignon–Li–Yau’s result.

**Proof.** Let $k \geq k_0(P)$. Consider the set $P \cap \mathbb{Z}^n / k$. We set

$$L_{i}^{\min} = \min\{L_i(m), m \in P \cap \mathbb{Z}^n / k\}.$$  

The polytope $P_k$ is given by

$$P_k = \{x \in P : L_i \geq L_{i}^{\min}, i = 1, \ldots, d\}$$

$$= \{x \in P : \langle x, \nu_i \rangle + c_i - L_{i}^{\min} \geq 0, i = 1, \ldots, d\}.$$  

$P_k$ is a rational polytope and $kP_k$ is an integral polytope. This uses the fact that $P$ is a Delzant polytope. A Delzant polytope is integral if and only if its vertices are in $\mathbb{Z}^n$ as this is equivalent to the $c_i$’s being integral.

We will build an explicit embedding of $M$ into $\mathbb{C}P^{N_k}$. On an open dense subset of $M$ we have symplectic coordinates which we denote by $(x, \theta) \in P \times \mathbb{T}^n$. In these coordinates our metric can be described by a symplectic potential $u \in S(P, \nu)$. The potential $u$ is given by $u = u_o + v$ where $v$ is smooth in a neighborhood of $P$ and $u_o = \frac{1}{2} \sum_{i=1}^{d} L_i \log L_i - L_i$ is the symplectic potential of the Guillemin metric on $M$. We also have complex coordinates in the same open dense subset of $M$ and the two are related by Legendre transform. More specifically the complex coordinate $z = y + i\theta$ is given by

$$y = \frac{\partial u}{\partial x} = \frac{1}{2} \sum_{i=1}^{d} \nu_i \log L_i + \frac{\partial v}{\partial x},$$

Set

$$\Phi_{u,k}(x, \theta) = \left[\left(\prod_{i=1}^{d} L_i^{\frac{c_i - L_{i}^{\min}}{L_i}}\right)^{e^{km \cdot z}}\right]_{m \in P \cap \mathbb{Z}^n / k},$$

where we think of $z = u_x + i\theta$ as in (20) and $\left(\prod_{i=1}^{d} L_i^{\frac{c_i - L_{i}^{\min}}{L_i}}\right)^{e^{km \cdot z}}$ are the homogeneous coordinates of $\Phi_{u,k}(x, \theta)$. The first thing to do is check that this embedding is well defined. We have

$$e^{km \cdot z} = e^{ikm \cdot \theta} e^{km \cdot u_x} = e^{ikm \cdot \theta} e^{km \cdot \nu_x} \prod_{i=1}^{d} L_i^{km \cdot \nu_i},$$
Therefore
\[ \Phi_{u,k}(x, \theta) = \left[ \prod_{i=1}^{d} L_{i}^{L_{i}} \left( \sum_{\nu} \alpha_{i} \nu_{i} \right) e^{km \cdot z} e^{i k m \cdot \theta} \right]_{m \in \mathbb{P} \cap \mathbb{Z}^n / k}, \]

It is clear that the homogeneous coordinates of \( \Phi_{u,k} \) do not vanish over the interior of \( P \). In fact, the function \( \Phi_{u,k} \) extends to \( \partial P \) even though \( \theta \) does not. To see why this is so, let \( x \in \partial P \). Without loss of generality assume that \( L_{1}(x) = \cdots = L_{r}(x) = 0 \) and suppose that \( x \) is in the interior of the face \( F \) defined by \( L_{1}(x) = \cdots = L_{r}(x) = 0 \). Then, we should check that
\[
\Phi_{u,k}(x, \theta) = \Phi_{u,k} \left( x, \theta + \sum_{l=1}^{r} \alpha_{l} \nu_{l} \right),
\]
because the Lie algebra of the subgroup of \( \mathbb{T}^{n} \) that fixes the points in \( x^{-1}(F) \) is spanned by \( \nu_{1}, \cdots, \nu_{r} \). The only homogeneous coordinates that do not vanish at all \( x \) are those corresponding to \( m \)'s such that \( L_{i}(m) = L_{i}^{\text{min}} \) for \( i = 1, \cdots, r \). Let \( m_{a} \) be such an \( m \). Then
\[
e^{i \theta + \sum_{l=1}^{r} \alpha_{l} \nu_{l} - c_{l} + L_{i}^{\text{min}}} = e^{i \theta - m_{a} \nu_{l} + \sum_{l=1}^{r} \alpha_{l} \nu_{l} - c_{l} + L_{i}^{\text{min}}}
\]

It is clear that \( \Phi_{u,k} \) is holomorphic because on the interior of \( P \) it coincides with
\[
\left[ e^{km \cdot z} \right]_{m \in \mathbb{P} \cap \mathbb{Z}^n / k},
\]
which is expressed in terms of complex coordinates as a holomorphic function.

It is also clear that \( \Phi_{u,k} \) is injective and that, because \( \{ e^{km \cdot z} \} \) is a linearly independent set of functions, the image of \( \Phi_{u,k} \) is not contained in any hyperplane.

At this point we can apply Bouguignon–Li–Yau’s theorem \[4.1\] to conclude that
\[
\lambda_{1}(M, \omega) \leq \frac{2n(N_k + 1) \int_{P} \Phi_{u,k}^{*} \omega_{FS} \wedge \omega^{n-1}}{N_{k} \int_{P} \omega^{n}}.
\]

We proceed to calculate the cohomology class \( [\Phi_{u,k}^{*} H] \). We have
\[
\Phi_{u,k}^{*} \omega_{FS} = \partial \bar{\partial} \log \left( \sum_{m \in \mathbb{P} \cap \mathbb{Z}^n / k} |e^{km \cdot z}|^{2} \right)
\]
As in the integral case, see Remark \[4.4\], \( [\Phi_{u,k}^{*} \omega_{FS}] = [\Phi_{u,k}^{*} \omega_{FS}] \) for any pair of symplectic potentials \( u \) and \( u_{o} \). So it is enough to consider the Guillemin potential
\[ u_o = \frac{1}{2} \sum_{i=1}^{d} (L_i \log L_i - L_i) \] and this gives \( y = \text{Re}(z) = \frac{1}{2} \sum_{i=1}^{d} \nu_i \log L_i \). Replacing in equation (24)

\[
\Phi_{u_o,k}^* \omega_{FS} = \partial \bar{\partial} \log \left( \sum_{m \in P \cap \mathbb{Z}^d / k} e^{2k \cdot y} \right)
\]

\[
= \partial \bar{\partial} \log \left( \sum_{m \in P \cap \mathbb{Z}^d / k} \prod_{i=1}^{d} L_i^{k \cdot \nu_i} \right)
\]

\[
= \partial \bar{\partial} \log \left( \sum_{m \in P \cap \mathbb{Z}^d / k} \prod_{i=1}^{d} L_i^{k(L_i(m) - L_i^{\text{min}})} \right) + k \sum_{i=1}^{d} (-c_i + L_i^{\text{min}}) \partial \bar{\partial} \log L_i
\]

As we have seen before

\[
\sum_{m \in P \cap \mathbb{Z}^d / k} \prod_{i=1}^{d} L_i^{k(L_i(m) - L_i^{\text{min}})}
\]

is nowhere vanishing in \( \bar{P} \) so that the cohomology class of \( \Phi_{u_o,k}^* \omega_{FS} \) is that of

\[
k \sum_{i=1}^{d} (-c_i + L_i^{\text{min}}) \partial \bar{\partial} \log L_i.
\]

Let \( E_i \) be the cohomology class that is Poincaré dual to the divisor in \( M \) whose image under the moment map is the facet \( i \). Then

\[
[\partial \bar{\partial} \log L_i] = E_i
\]

and we conclude that \([\Phi_{u_o,k}^* \omega_{FS}] = k \sum_{i=1}^{d} (-c_i + L_i^{\text{min}}) E_i \). It is well know (see [16]) that \([\omega] = - \sum_{i=1}^{d} c_i E_i \) therefore

\[
[\Phi_{u_o,k}^* \omega_{FS}] = k \left( [\omega] + \sum_{i=1}^{d} L_i^{\text{min}} E_i \right).
\]

Replacing in equation (23) we see that

\[
\lambda_1(M, \omega) \leq \frac{2n(N_k + 1)}{N_k} \left( k + \frac{k \int_P \sum_{i=1}^{d} L_i^{\text{min}} E_i \wedge \omega^{n-1}}{\int_P \omega^n} \right)
\]

Because \( \mathbb{Z}^d / k \) gives a grid in \( P \) of side length \( 1/k \) we have

\[
L_i^{\text{min}} = L_i(m_0) = L_i(m_0) - L_i(x), \quad x \in L_i^{-1}(0),
\]

hence \( L_i^{\text{min}} = |(m_0 - x) \cdot \nu_i| \leq |\nu_i| n/k \) so that

\[
\int_P \sum_{i=1}^{d} L_i^{\text{min}} E_i \wedge \omega^{n-1} \leq \frac{nC}{k} \int_P \sum_{i=1}^{d} E_i \wedge \omega^{n-1},
\]
where $C$ is a constant. Because $c_1(M) = \sum_{i=1}^{d} E_i$ it follows that

$$\lambda_1(M,\omega) \leq \frac{2nk(N_k + 1)}{N_k} \left( 1 + \frac{nC \int_P c_1(M) \wedge \omega^{n-1}}{k \int_P \omega^n} \right).$$

It is clear from the above argument that if $P$ is integral one can take $k = 1$ and the bound becomes

$$\lambda_1(M,\omega) \leq \frac{2n(N + 1)}{N}.$$

\[\square\]

4.3. The equality case in Bourguignon–Li–Yau’s bound. The goal of this subsection is to study Kähler toric metrics which saturate the bound in Theorem 4.9. We give a quick overview of Bourguignon, Li and Yau’s proof in [7] as this proof will be important to us.

**Proof of Theorem 4.1**, see [7]. Recall that the first eigenspace of $(\mathbb{C}P^N, \omega_{FS})$ has a basis given by the functions $[Z] \mapsto \Psi_{ij}(Z) - \delta_{ij} \frac{\omega_{n+1}}{N+1}$ where for $i,j \in \{0,1,\ldots,N\}$,

$$\Psi_{ij}(Z) = \sum_{k=0}^{N} \frac{Z_i Z_j}{|Z_k|^2}$$

is one component of the $SU(N+1)$ moment map $\Psi : \mathbb{C}P^N \hookrightarrow \mathfrak{su}_{N+1}^*$. The main step of the proof in [7], is to show that, given a full embedding $\Phi : M \hookrightarrow \mathbb{C}P^N$, there exists a unique $B \in SL(N+1, \mathbb{C})$ such that $B^* B > 0$ and

$$\int_M \omega^n \int_M (\Psi_{ij} \circ B \circ \Phi)(p) \omega^n = \frac{\delta_{ij}}{N+1}.$$

Said differently, $(B \circ \Phi)^* \omega_{FS}$ is $(\omega^n/n!)$–balanced, (see [12]). To simplify the notation we write $\omega$–balanced instead of $(\omega^n/n!)$–balanced.

Denote $f_{ij}^B = \Psi_{ij} \circ B \circ \Phi - \frac{\delta_{ij}}{N+1} \in C^\infty(M)$. The Rayleigh principle implies that

$$\lambda_1(M,\omega) \int_M (f_{ij}^B) \frac{\omega^n}{n!} \leq \int_M |\nabla_{\omega} f_{ij}^B| \frac{\omega^n}{n!}$$

with equality if and only if $f_{ij}^B$ is an eigenfunction of $\Delta$ for the eigenvalue $\lambda_1(M,\omega)$.

Taking the sum over $i,j = 0,\ldots,N$ the left hand side of (26) gives

$$\frac{N}{N+1} \int_M \frac{\omega^n}{n!}$$

thanks to (25). Noticing that $g(\nabla f, \nabla f) \omega^n = n Re(df \wedge d^c f \wedge \omega^{n-1})$, that the form $\sum_{i,j} df_{ij}^B \wedge d^c f_{ij}^B$ is real and that, on $\mathbb{C}P^N$,

$$\sum_{i,j} d\Psi_{ij} \wedge d^c \Psi_{ij} = 2\omega_{FS},$$

(27)
the right hand side of (26) gives
\[
\sum_{i,j} \int_M |\nabla \omega f_{ij}^B| \frac{\omega^n}{n!} = \sum_{i,j} \int_M df_{ij}^B \wedge d^c f_{ij}^B \wedge \frac{\omega^{n-1}}{(n-1)!} = \frac{2}{(n-1)!} \int_M (B \circ \Phi)^* \omega_{FS} \wedge \omega^{n-1}.
\]
Finally, \(B^* \omega_{FS}\) and \(\omega_{FS}\) are in the same cohomology class on \(\mathbb{C}P^N\), this concludes the proof. \(\square\)

Remark 4.10. The equality case in (19) implies the equality case in each inequality (26) and then that each function \(f_{ij}^B\) is an eigenfunction of \(\Delta\) for the first eigenvalue.

In the toric context and for the embedding \(\Phi_u\) we get the following refinement of Bourguignon–Li–Yau’s result on the existence of balanced metrics.

Lemma 4.11. Let \((M, \omega, g_u, J_u, T)\) be a toric Kähler manifold with integral polytope \(P \subset \mathfrak{t}^*\) and corresponding embedding \(\Phi_u : M \hookrightarrow \mathbb{C}P^N\). There exists a diagonal matrix \(B = diag(\alpha_0, \ldots, \alpha_N) \in \text{GL}(N+1, \mathbb{R})\) with \(\alpha_i > 0\) and \(\text{tr}B = 1\) satisfying the condition (25).

Proof. First, observe that when \(i \neq j\), the function \(\Psi_{ij} \circ \Phi_u\) integrates to 0 on \(M\) since it does on each orbit of \(T\). Hence to prove the lemma, we only have to prove that there exists \(\alpha = (\alpha_0, \ldots, \alpha_N) \in \mathbb{R}^{N+1}_{>0}\) such that, for \(i = 0, \ldots, N,\)
\[
\psi_{u,i}(\alpha) := \alpha_i^2 \int_M \frac{|Z_i|^2}{\sum_{j=0}^N \alpha_j^2 |Z_j|^2} \frac{\omega^n}{n!} = \frac{1}{N+1}
\]
where \(Z_i = \Phi_{u,m_i}(x, \theta)\) see (20). Let \(\Sigma\) be the simplex defined by
\[
\Sigma := \left\{ X \in \mathbb{R}^{N+1} \mid \sum_{i=0}^N X_i = 1, X_i > 0 \right\}.
\]
Since \(\sum_{i=1}^N \psi_{u,i}(\alpha) = \text{vol}(M) < +\infty\) and each component \(\psi_{u,i}(\alpha) \geq 0\) for each \(\alpha \in \mathbb{R}^{N+1}_{>0}\), by the dominated convergence lemma, the map \(\psi_u\) can be extended continuously to \(\overline{\Sigma}\). Hence, we see \(\psi_u = (\psi_{u,0}, \ldots, \psi_{u,N})\) as a continuous map
\[
\psi_u : \Sigma \longrightarrow \Sigma
\]
from the closed simplex \(\Sigma\) to itself. It is obvious that \(\psi_u\) maps \(\partial \Sigma\) to \(\partial \Sigma\).

To prove the lemma we need to prove that \(\psi_u\) is surjective which will follow if we prove that the restriction \(\psi_u : \partial \Sigma \to \partial \Sigma\) has non-trivial degree.

Like in [7], instead of integrating on \(M\), we integrate on \(\mathbb{C}P^N\) with the measure \(d\mu_u\) defined to be the pushforward of the measure on \(M\) defined by the metric. This is possible because \(\Phi_u\) is a full embedding. Hence
\[
\psi_{u,i}(\alpha) = \alpha_i^2 \int_{\mathbb{C}P^N} \frac{|Z_i|^2}{\sum_{j=0}^N \alpha_j^2 |Z_j|^2} d\mu_u.
\]
Now if we consider the volume form $d\mu_o$ induced by the Fubini-Study metric on $\mathbb{C}P^N$, the corresponding map $\psi_o : \partial \Sigma \to \partial \Sigma$ with components

$$
\psi_{o,i}(\alpha) = \alpha_i^2 \int_{\mathbb{C}P^N} \frac{|Z_i|^2}{\sum_{j=0}^N \alpha_j^2 |Z_j|^2} d\mu_o
$$

and its restriction $\psi_o : \partial \Sigma \to \partial \Sigma$ are bijections. Now $\psi_t = t\psi_u + (1-t)\psi_o$ is a family of continuous maps from $\Sigma$ to itself preserving the boundary. Then the degree does not depend on $t$ and is non trivial for $t = 0$. □

**Remark 4.12.** A straightforward corollary of this lemma is that the $\omega$–balanced metric is toric as soon as $\omega$ is toric.

Theorem 1.2 is a consequence of two propositions we state below and which we prove using the following observation. Assume that the first eigenvalue of $(M, \omega, g_u, J_u, T)$ reaches Bourguignon, Li and Yau’s bound, i.e $\lambda_1(g_u) = \frac{2n(N+1)}{N}$. By Remark 4.10 and Lemma 4.11, there exists a set of $N+1$ real positive numbers \{\alpha_k\}_{k \in P \cap \mathbb{Z}^n}$ such that for each $m,k \in P \cap \mathbb{Z}^n$, the function $\Psi_{mk} - \delta_{mk}/(n+1)$ is an eigenfunction of eigenvalue $\frac{2n(N+1)}{N}$ where

$$
\Psi_{mk} = \frac{\alpha_k \alpha_m Z_m Z_k}{\sum_j |\alpha_j Z_j|^2}
$$

is seen as a function of $(x, \theta)$. Here we write

$$
Z_m = e^{(u_x + i\theta) \cdot m},
$$

where $u_x = \frac{\partial u}{\partial x}$. We assume, without loss of generality, that $0 \in P \cap \mathbb{Z}^n$. We normalize the $\alpha$’s so as to have $\alpha_0 = 1$ instead of $\sum_{k \in P \cap \mathbb{Z}^n} \alpha_k = 1$ as in the previous lemma. We recall that

(28) \[ \int_M \Psi_{mk} \frac{\omega^n}{n!} = \delta_{mk} \int_M \frac{\omega^n}{n!}. \]

Observe that for each pair $m,k \in P \cap \mathbb{Z}^n$

(29) \[ \Delta \Psi_{mk} = \Delta(\alpha_m Z_m \Psi_{0k}) = \alpha_m Z_m \Delta \Psi_{0k} - 2\alpha_m \langle dZ_m, d\Psi_{0k} \rangle \]

since $\Delta(Z_m) = 0$ where $\langle \cdot, \cdot \rangle$ denotes the inner product induced by $g_u$ on the cotangent bundle of $M$.

When $k = 0 \neq m$, identity (29) becomes

$$
\frac{2n(N+1)}{N} \Psi_{m0} = \frac{2n(N+1)}{N} \alpha_m Z_m (\Psi_{00} - 1/n + 1) - 2\alpha_m \langle dZ_m, d\Psi_{00} \rangle
$$

$$
= \frac{2n(N+1)}{N} \Psi_{m0} - \frac{2n(N+1)\alpha_m Z_m}{N(n+1)} - 2\alpha_m \langle dZ_m, d\Psi_{00} \rangle,
$$

and we get

$$
\frac{2n(N+1)\alpha_m Z_m}{N(n+1)} = -2\alpha_m \langle dZ_m, d\Psi_{00} \rangle.
$$
Developing the right hand side in action angle coordinates, using (3), we have

\[
\frac{2n(N + 1)\alpha_m Z_m}{N(n + 1)} = -2\alpha_m \langle dZ_m, d\Psi_{00} \rangle
\]

\[
= -2\alpha_m \sum_{i,j=1}^n H_{ij} \partial_{x_i} Z_m \partial_{x_j} \Psi_{00}
\]

\[
= \frac{4\alpha_m Z_m \sum_{k \in W} \sum_{s,t=1}^n u_{ts} m_s k_t |\alpha_k Z_k|^2}{(\sum_{k \in W} |\alpha_k Z_k|^2)^2}
\]

where \(W = P \cap \mathbb{Z}^n\). Dividing both sides by \(2\alpha_m Z_m\), we end up with

\[
\frac{n(N + 1)}{N(n + 1)} = \frac{2 \sum_{k \in W} \sum_{s,t=1}^n u_{ts} m_s k_t |\alpha_k Z_k|^2}{(\sum_{k \in W} |\alpha_k Z_k|^2)^2} = -(d\Psi_{00})(m)
\]

**Proposition 4.13.** Let \((\omega, g_u, J_u, T_u)\) be a toric Kähler structure on \(\mathbb{CP}^n\). Assume that \(\lambda_1 = 2(n + 1)\) i.e. assume that the first eigenvalue of the Laplacian reaches the Bourguignon–Li–Yau bound. Then, the toric Kähler metric \(g_u\) is the Fubini-Study metric on \(\mathbb{CP}^n\).

**Proof.** The moment polytope \(P\) of \(\mathbb{CP}^n\) is a simplex and if \((\omega, g_u, J_u, T_u)\) saturates the bound then \(P\) is primitive as explained in Remark 4.6. So one can suitably normalize it so that it has integer vertices \(0, e_1, \ldots, e_n\) where \(e_i\) is the vector in \(\mathbb{R}^n\) whose \(i\)-th component is 1 and all others are zero. In the notation above, \(N = n\) and we identify \(W = P \cap \mathbb{Z}^n\) with \(\{0, 1, \ldots, n\}\). Equation (30) implies that, for all \(m = 1, \ldots, n\),

\[
\frac{n(N + 1)}{N(n + 1)} = \frac{2 \sum_{m,k=1}^n u_{mk} |\alpha_k Z_k|^2}{(1 + \sum_{k=1}^n |\alpha_k Z_k|^2)^2} = -\frac{\partial \Psi_{00}}{\partial x_m}
\]

where \(Z_k = e^{(u_k + i\theta_k)}\) for \(k = 1, \ldots, n\). Hence

\[
\Psi_{00} = K - \sum_{k=1}^n x_k
\]

for some constant \(K\). The additive constant is fixed \(K = 1\) by the integration constraint (28).

In the case \(m = k \neq 0\), identity (29) gives

\[
2(n + 1)(\Psi_{mm} - 1/n + 1) = \alpha_m Z_m 2(n + 1)\Psi_{0m} - 2\alpha_m \langle dZ_m, d\Psi_{0m} \rangle
\]

that is

\[-2 = -2\alpha_m \langle dZ_m, d\Psi_{0m} \rangle.
\]

Developing the right hand side using (31), we get

\[-2 = -2\alpha_m \langle dZ_m, d\Psi_{0m} \rangle = 2|\alpha_m Z_m|^2 \left( \frac{2u_{mm}}{\sum_i |\alpha_i Z_i|^2} - 1 \right) = -2\frac{\partial \Psi_{mm}}{\partial x_m}.
\]

Therefore, for each \(m > 0\)

\[
\frac{\partial \Psi_{mm}}{\partial x_m} = 1
\]
In the case \(0 \neq m \neq k \neq 0\), identity (29) gives
\[
2(n + 1)\Psi_{mk} = \alpha_m Z_m 2(n + 1)\Psi_{0k} - 2\alpha_m \langle dZ_m, d\Psi_{0k} \rangle,
\]
that is
\[
0 = -2\alpha_m \langle dZ_m, d\Psi_{0k} \rangle.
\]
Developing the right hand side using (31), we get
\[
0 = -2\alpha_m \langle dZ_m, d\Psi_{0k} \rangle = 2\alpha_m \alpha_k Z_m Z_k \left( \frac{2(u_{mk})}{\sum_i |\alpha_i Z_i|^2} - 1 \right)
\]
Therefore, for each \(m, k > 0\) and \(k \neq m\), we have
\[
\frac{\partial \Psi_{mm}}{\partial x_k} = 0.
\]
Together with (32) it gives
\[
(33) \quad \Psi_{mm} = x_m
\]
where again the additive constant is fixed by the integration constraint (28) on \(\Psi_{mm}\). We conclude from Remark (4.10) that the components of the moment map are eigenfunction for the same eigenvalue and thus, by Proposition 2.2 \((g_u, J_u)\) is Kähler-Einstein. By uniqueness of extremal toric metric [15], \((g_u, J_u)\) is the Fubini-Study metric.

We can also prove that if a toric manifold admits a toric Kähler metric for which the embedding given by the integral points of the moment polytope saturates the Bourguignon–Li–Yau bound, then that manifold must be \(\mathbb{CP}^n\) and it follows from the above proposition that the metric must be the Fubini-Study metric.

**Proposition 4.14.** Let \((M, \omega, g_u, J_u)\) be a toric Kähler manifold with integral polytope \(P\). Assume that \(\lambda_1(g_u) = \frac{2n(N+1)}{N}\) with \(N = \#(P \cap \mathbb{Z}^n) - 1\). Then \(P\) is the standard simplex and \(M\) is (equivariantly symplectomorphic to) \(\mathbb{CP}^n\).

**Proof.** Again we assume that the origin lies in \(P \cap \mathbb{Z}^n\), more precisely, up to an integral invertible affine transformation, we may assume that \(P\) is standard at the origin i.e. the facets that meet at 0 have normals \(e_1, \cdots, e_n\). In particular, the vertices of \(P\) adjacent to the origin, say \(m_1, \cdots, m_n\), are each an integral multiple of an element of a dual basis of \(e_1, \cdots, e_n\) respectively.

Under the hypothesis of the proposition and with respect to the notation above, Equations (29) and (30) hold for points in \(P \cap \mathbb{Z}^n\). Suppose there is \(m \in P \cap \mathbb{Z}^n\) a vertex distinct from the origin and from \(m_1, \cdots, m_n\). Then, there exist \(a_1, \cdots, a_n\) such that \(m = \sum_{l=1}^n a_l m_l\).

Equation (30) holds for \(m\) as well as for \(m_1, \cdots, m_n\). So we must have
\[
\frac{n(N + 1)}{N(n + 1)} = -(d\Psi_{00})(m) = -\sum_{l=1}^n a_l (d\Psi_{00})(m_l) = \left( \sum_{l=1}^n a_l \right) \frac{n(N + 1)}{N(n + 1)}
\]
which implies \(\sum_{l=1}^n a_l = 1\). So \(m\) lies on a facet of the simplex of vertices \(0, m_1, \cdots, m_n\) which contradicts convexity unless \(P\) is that simplex.

To sum up we proved Theorem 1.2.
TORIC ASPECTS OF THE FIRST EIGENVALUE

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TORIC ASPECTS OF THE FIRST EIGENVALUE

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ABSTRACT. In this paper we study the smallest non-zero eigenvalue $\lambda_1$ of the Laplacian on toric Kähler manifolds. We find an explicit upper bound for $\lambda_1$ in terms of moment polytope data. We show that this bound can only be attained for $\mathbb{CP}^n$ endowed with the Fubini-Study metric and therefore $\mathbb{CP}^n$ endowed with the Fubini-Study metric is spectrally determined among all toric Kähler metrics. We also study the equivariant counterpart of $\lambda_1$ which we denote by $\lambda_T^1$. It is the smallest non-zero eigenvalue of the Laplacian restricted to torus-invariant functions. We prove that $\lambda_T^1$ is not bounded among toric Kähler metrics thus generalizing a result of Abreu-Freitas on $S^2$. In particular, $\lambda_T^1$ and $\lambda_1$ do not coincide in general.

1. INTRODUCTION

Toric Kähler manifolds are very symmetric Kähler manifolds for which there is a concrete parametrization of the space of Kähler metrics. More concretely, they are symplectic manifolds admitting an effective Hamiltonian action from a maximal torus and endowed with a compatible, torus invariant Riemannian metric giving rise to an integrable complex structure. The underlying symplectic manifold is completely characterized by a combinatorial object which is a convex polytope called moment polytope arising as the image of the moment map for the torus action. Toric Kähler metrics are parametrized by convex functions on that moment polytope satisfying certain properties as we will discuss in section 2. Toric Kähler manifolds have played a crucial role in studying important questions in geometry. Mabuchi was one the first to study their Kähler Geometry in [24]. In [13], Donaldson was able to fully characterize those toric Kähler surfaces which admit constant scalar curvature thus settling an important conjecture in Kähler geometry in the toric context for real dimension 4. There has been a lot of interest in studying toric spectral geometry and, in particular, inverse spectral questions in this toric context as well (see [4], [16]).

Given a Riemannian manifold $(M, g)$, the Riemannian metric determines a Beltrami-Laplace operator whose smallest non-zero eigenvalue, which we also refer to as the first eigenvalue, and is denoted by $\lambda_1(g)$, carries a surprising amount of geometric information. There has been great deal of effort put into finding sharp bounds for $\lambda_1$ with geometric meaning (see [7]). In [23], Hersch discovered an upper bound for $\lambda_1$ for metrics on $S^2$. Bourguignon–Li–Yau found an upper

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bound for $\lambda_1$ for Kähler manifolds endowed with a full holomorphic embedding into projective space \[9\]. This result has been extended in some ways to Kähler manifolds carrying Gieseker stable bundle \[6\] by allowing maps to Hermitian symmetric spaces \[8\]. Polterovich (see \[28\]) looked at boundedness of $\lambda_1$ in the context of symplectic manifolds. He showed, in particular, that there are symplectic manifolds admitting compatible Riemannian metrics whose $\lambda_1$ is arbitrarily large. One of the questions we want to address here is: “are there geometric bounds on $\lambda_1(M, g)$ where $M$ is a toric manifold and $g$ is a toric Kähler metric on it?” As it turns out, one can always use Bourguignon–Li–Yau’s result in the toric context, and we use it to give an explicit bound for $\lambda_1$ in terms of moment polytope data. More precisely, we prove the following theorem.

**Theorem 1.1.** Let $(M^{2n}, \omega)$ be a toric symplectic manifold endowed with a toric Kähler structure whose Riemannian metric we denote by $g$. Let $P \subset \mathbb{R}^n$ be its moment polytope. There is an integer, $k_0(P) \geq 1$ such that for any $k \geq k_0(P)$

$$\lambda_1(g) \leq \frac{2nk(N_k + 1)}{N_k},$$

where $N_k + 1 = \sharp(P \cap \mathbb{Z}^n/k)$. If $P$ is integral (i.e its vertices lie in $\mathbb{Z}^n$), then we have a finer bound given by

$$\lambda_1(g) \leq \frac{2n(N + 1)}{N},$$

where $N + 1 = \sharp(P \cap \mathbb{Z}^n)$ is the number of integer points in $P$.

We will make $k_0(P)$ explicit ahead. The Fubini-Study metric realizes the bound in the above theorem. In fact we show that this is the only toric Kähler metric that does saturate this bound in the integral case.

**Theorem 1.2.** Let $(M^{2n}, \omega)$ be an integral toric symplectic manifold endowed with a toric Kähler structure whose Riemannian metric we denote by $g$. Let $N + 1$ be the number of integer points in the moment polytope of $M$. If

$$\lambda_1(g) = \frac{2n(N + 1)}{N},$$

then $M$ is equivariantly symplectomorphic to $\mathbb{CP}^n$ and this symplectomorphism takes $g$ into the Fubini-Study metric on $\mathbb{CP}^n$.

It was previously known (see \[7\]) that the Fubini-Study metric on $\mathbb{CP}^n$ is determined by the spectrum among all Kähler metrics on $\mathbb{CP}^n$ compatible with the standard complex structure. It was also proved by Tanno (see \[29\]) that, if a Kähler manifold of real dimension less than 12 has the same spectrum as $\mathbb{CP}^n$ with the Fubini-Study metric, then it is holomorphically isometric to it. A simple consequence of the above theorem is that the spectrum of the Laplacian of a toric Kähler metric on an integral toric manifold determines if the manifold is $\mathbb{CP}^n$ endowed with the Fubini-Study metric.

**Corollary 1.3.** An integral toric Kähler manifolds which has the same spectrum as $(\mathbb{CP}^n, \omega_{FS}, J_0)$ is holomorphically isometric to it.
Another interesting question is that of spectrally characterizing either constant scalar-curvature, extremal or Kähler-Einstein toric Kähler metrics. In [16] the authors prove that the equivariant spectrum determines if a toric Kähler metric has constant scalar curvature. A variation of the argument there would show that the equivariant spectrum also determines if a metric is extremal.

Going back to the first eigenvalue, there are various bounds that one can write down for toric Kähler manifolds using Bourguignon–Li–Yau’s bound, see §4.1. It would be interesting to see what the best bound is for a given toric manifold, once we fix the polytope. In particular, one could hope to improve the bound in Theorem 1.1 for special classes of manifolds (monotone, Fano..) or special classes of metrics say extremal toric Kähler metric, or Kähler-Einstein metrics. In [5] the authors prove that a toric Kähler–Einstein manifold whose connected component of automorphism group is a torus is never λ1–extremal, where λ1–extremal means extremal for the first eigenvalue with respect to local variations in the Kähler metrics space. Hence, in general, we cannot expect a toric Kähler–Einstein metric to saturate fine bounds. Another natural candidate to consider is a balanced metric when it exists, see discussion §4.1.

However, the Kähler–Einstein property is somewhat reflected in the first eigenvalue. In fact, one can prove an improvement and a converse of Matsushima Theorem [25] in §2.2. We show that a toric Kähler metric is Kähler-Einstein if and only if the coordinates of its moment map are eigenfunctions for \( \lambda_1 \).

Proposition 1.4. Let \((M,\omega,T)\) be a compact symplectic toric orbifold with moment map \(x : M \rightarrow t^*\). Then \((M,g,J,\omega,T)\) is a Kähler–Einstein toric orbifold with Einstein constant \(\lambda\) if and only if, up to an additive constant, the moment map satisfies
\[
2\lambda(x,b) = \Delta g(x,b) \quad \forall b \in t.
\]
In this case, \(2\lambda\) is the smallest non-vanishing eigenvalue for the Kähler–Einstein orbifold toric metric.

Matsushima’s theorem implies that a necessary condition for a toric Kähler metric to be Kähler-Einstein is that its \(\lambda_1\) be a multiple eigenvalue with multiplicity at least equal to half the dimension of the manifold. What’s more, it follows from the above proposition that one can see if a metric is Kähler–Einstein by simply checking if its moment map coordinates are eigenfunctions for \(2\lambda\).

On a toric manifold endowed with a torus invariant metric one can consider a toric version of \(\lambda_1\) namely \(\lambda^T_1\) defined to be the smallest non-zero invariant eigenvalue of the Laplacian i.e. the smallest eigenvalue of the Laplacian restricted to torus invariant functions. We clearly have \(\lambda_1 \leq \lambda^T_1\). In [4], Abreu–Freitas studied \(\lambda^T_1\) for the simplest toric manifold, namely \(S^2\) with the usual \(S^1\) action by rotations around an axis. They proved it was unbounded (both above and below) among \(S^1\)-invariant metrics. In this paper we generalize their results, by using an original approach for the upper bound, on all toric manifolds. We are able to prove the following.

\(^1\)It is possible that this result was previously known but the authors did not find a reference for it in the literature and thus state it and prove it.
**Theorem 1.5.** Let \((M, \omega, T)\) be a compact symplectic toric orbifold, let \(\mathcal{K}_\omega^T\) be the set of all toric Kähler metrics on \((M, \omega, T)\). Then,

\[\inf \{ \lambda_1^T(g) \mid g \in \mathcal{K}_\omega^T \} = 0.\]

and

\[\sup \{ \lambda_1^T(g) \mid g \in \mathcal{K}_\omega^T \} = +\infty.\]

Combining Theorem 1.1 and 1.5, we see that there are toric Kähler manifolds for which \(\lambda_1\) does not coincide with \(\lambda_1^T\). For toric Kähler–Einstein metrics, it follows from Matsushima Theorem [25] that \(\lambda_1 = \lambda_1^T\) as there are invariant eigenfunctions for \(\lambda_1\). It would be interesting to characterize those toric Kähler manifolds for which this occurs. Given a weight vector \(m \in \mathbb{Z}^n\), one could also define \(m\)-equivariant \(\lambda_1\) which we denote by \(\lambda_1^m\) as the lower non-vanishing eigenvalue of the Laplacian restricted to the set of \(m\)-equivariant functions

\[\{ f \in C^\infty(M, \mathbb{C}) : f(e^{i\theta}p) = e^{i\theta \cdot m}f(p), \quad \forall p \in M, \theta \in \mathbb{R}^n \}.\]

One could prove a similar result in this setting and again it would be interesting to understand which metrics have \(\lambda_1 = \lambda_1^m\) and how this depends on \(m\). Note that \(\lambda_1^T = \lambda_1^0\). Recently, in [22], Hall-Murphy proved that on any toric manifolds \(\lambda_1^T\) restricted to the class of toric Kähler metrics whose scalar curvature is non-negative is bounded and this generalizes another result in [4].

The paper is organized as follows. In section 2 we quickly review some basic facts about toric manifolds and their toric Kähler metrics. The reader is encouraged to consult the references for more details and proofs. We also give a proof of Proposition 1.4. In section 3 we study \(\lambda_1^T\) and generalize Abreu-Freitas’ result to prove Theorem 1.5. Section 4 deals with \(\lambda_1\) and there we prove Theorems 1.1 and 1.2.

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## 2. Background

### 2.1. Toric Kähler geometry.** This section does not contain all the ingredients of symplectic toric geometry needed in subsequent sections, we only lay down the notation and refer to the classical references for this theory (in particular for proofs of what is claimed in this section) like [1] [3] [11] [14] [20] [21] [27].

Let \((M^{2n}, \omega, T^n)\) be a compact toric symplectic orbifold. It admits a moment map \(x : M \to P \subset \mathfrak{t}^*\) where \(t = \text{Lie } T\) is the Lie algebra of \(T\) and \(\mathfrak{t}^*\) is its dual such that for all \(a \in \mathfrak{t}\)

\[-d(x, a) = \omega(X_a, \cdot)\]

where \(X_a\) is the vector field on \(M\) induced by the 1–parameter subgroup associated to \(a\). The image of \(x\), that we denote \(P\), is called the moment polytope. It is a convex simple (i.e. its vertex are the intersection of exactly \(n\)–facets) polytope in \(\mathfrak{t}^*\).
Definition 2.1. Consider $P \subset \mathfrak{t}^*$ a simple polytope, $\nu = \{\nu_1, \ldots, \nu_d\}$ a set of vectors in $\mathfrak{t}$ which are normal to the facets of $P$ and inward pointing. Let $\Lambda$ be the lattice in $\mathfrak{t}$ such that $T = \mathfrak{t}/\Lambda$. If $\nu \subset \Lambda$, the triple $(P, \nu, \Lambda)$ is called a labelled rational polytope. If each subset of vectors in $\nu$, normals to facets meeting at a vertex, forms a basis of $\Lambda$, then we say that $(P, \nu, \Lambda)$ is Delzant.

The Delzant–Lerman–Tolman correspondence states that compact toric symplectic orbifolds are in one to one correspondence with rational labelled polytopes and are smooth if and only if the rational labelled polytopes is Delzant.

In this text, we often identify $\mathfrak{t}$ with $\mathbb{R}^n$ and $\Lambda$ with $\mathbb{Z}^n$.

Definition 2.2. Let $(P, \nu)$ be a labelled polytope. The functions $L_1, \ldots, L_d \in \text{Aff}(\mathfrak{t}^*, \mathbb{R})$ are said to be the defining functions of $(P, \nu)$ if $P = \{x \in \mathfrak{t}^* | L_k(x) \geq 0\}$ and $\frac{dL_k}{dx} = \nu_k$.

Let $\hat{P}$ denote the interior of $P$. On the pre-image of the interior of the polytope $\hat{M} = x^{-1}(\hat{P})$, the action of $T$ is free. The action–angle coordinates $(x, \theta) = (x_1, \ldots, x_n, \theta_1, \ldots, \theta_n)$ are local coordinates on $\hat{M}$ used to (locally) identify $\hat{M}$ with $\hat{P} \times T$ where the first projection coincides with the moment map and

$$\omega = \sum_{i=1}^n dx_i \wedge d\theta_i. \tag{2}$$

As it is shown in [2], the space of compatible $T$–invariant Kähler metrics on $(M, \omega, T)$ is parametrized by the set of symplectic potentials which is denoted by $S(P, \nu)$ (up to the addition of an affine linear function). The set $S(P, \nu)$ is defined as the subset of functions $u \in C^\infty(\hat{P}, \mathbb{R}) \cap C^0(P, \mathbb{R})$, such that

(i) $u - \frac{1}{2} \sum_{k=1}^d L_k \log L_k \in C^\infty(P, \mathbb{R})$;

(ii) the restriction of $u$ to $\hat{P}$ is strictly convex;

(iii) for each face $F$ of $P$, the restriction of $u$ to $\hat{F}$ (the relative interior of $F$) is strictly convex.

Definition 2.3. The Guillemin potential $u_o \in S(P, \nu)$ is

$$u_o = \frac{1}{2} \sum_{i=1}^d (L_k \log L_k - L_k). \tag{3}$$

It corresponds to the Kähler toric metric on $(M, \omega)$ obtained via the Kähler reduction of $C^d$, see [20].

Given $u \in S(P, \nu)$, the metric defined by

$$g_u = \sum_{i,j=1}^n u_{ij} dx_i \otimes dx_j + u^{ij} d\theta_i \otimes d\theta_j. \tag{4}$$

\(^2\text{To recover the original convention introduced by Lerman and Tolman in the rational case, take } m_k \in \mathbb{Z} \text{ such that } \frac{1}{m_k} \nu_k \text{ is primitive in } \Lambda \text{ so } (P, m_1, \ldots, m_d, \Lambda) \text{ is a rational labelled polytope.}\)
where \( u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j} \) and \( (u^{ij}) = (u_{ij})^{-1} \), is a \( t \)-invariant Kähler metric on \( \hat{P} \times T \cong \hat{M} \) compatible with \( \omega \). Conditions (i), (ii), (iii) ensure that \( g_u \) is the restriction of a smooth metric on \( M \). For convenience, we denote \( H^u_{ij} = u^{ij} \), \( G^u_{ij} = u_{ij} \), \( H^u = (H^u_{ij}) \) and \( G^u = (G^u_{ij}) \). One can prove that any toric Kähler structure on \( (M, \omega) \) can be written using a symplectic potential in \( S(P, \nu) \) as above (see [2]).

Remark 2.4. Given \( u \in S(P, \nu) \) the map \( \frac{\partial u}{\partial x^i} : \hat{P} \to t \) is a diffeomorphism (because \( u \) is strictly convex) and the coordinates \( z = y + i\theta \), where \( y = \frac{\partial u}{\partial x^i} \), are local complex coordinates on \( \hat{M} \). See [19, §A1.3]) for instance.

Note that in [20], \( u_{o} \) is defined to be \( u_{o} = \frac{1}{2} \sum_{i=1}^{d} (L_k \log L_k) \). This will yield the same metric as the metric we define via formula [4] and complex coordinates which are related to the ones in [20] by an overall translation.

Abreu [1] computed the curvature of a compatible Kähler toric metric, \( g_u \), in terms of its symplectic potential \( u \). The scalar curvature of \( g_u \) is the pull-back by \( x \) of the function

\[
\text{scal}_u = - \sum_{i,j=1}^{n} \frac{\partial^2 H^u_{ij}}{\partial x_i \partial x_j}.
\]

Moreover, the Ricci curvature is

\[
\rho^{g_u} = - \frac{1}{2} \sum_{i,l,k} H^u_{il,ik} dx_k \wedge d\theta_l.
\]

(See for instance [20] where the above formula is proved in the more general context of almost Kähler metrics.)

2.2. Kähler–Einstein metrics and moment map coordinates as eigenfunctions of the Laplacian. Let \( (M, g_u, J, \omega, T) \) be a compact Kähler toric manifold with moment map \( x \) and denote by \( \Delta^u \), the Laplacian with respect to the Riemannian metric \( g_u \). Recall that \( (M, g_u, J, \omega) \) is Kähler–Einstein if there exists \( \lambda \) such that \( \lambda \omega = \rho^{g_u} \omega \) where \( \rho^{g_u} \omega \) is the Ricci form of the Chern and Levi connection. We say that \( \lambda \) is the Einstein constant. In the compact toric setting, \( \lambda > 0 \).

Next we proceed to prove Proposition (1.4)

Proof. Expressing the Laplacian (i.e \( \Delta^g = - \text{Div}^g \text{grad}^g \)) in the action angle coordinates (4), we get

\[
\Delta^u = - \sum_{i,j=1}^{n} \left[ G_{ij} \frac{\partial^2}{\partial \theta_i \partial \theta_j} + \frac{\partial}{\partial x_i} \left( H_{ij} \frac{\partial}{\partial x_j} \right) \right],
\]

so that

\[
d\Delta^u(x, b) = - \sum_{i,j,k=1}^{n} H_{ij,ik} b_j dx_k = - 2 \rho^{g_u}(X_b, \cdot) \quad \forall b \in t
\]

using (3). From (8), we see that \( \Delta^{g_u} x \) is a moment map for \( 2 \rho^{g_u} \) and, in the Kähler–Einstein case \( \lambda \omega = \rho^{g_u} \), this implies that \( 2 \lambda x - \Delta^{g_u} x = \alpha \in t^* \) is constant. Thus \( x - \frac{\lambda}{2} x \) satisfies (1).
The converse is also a simple computation. Indeed, assuming \( (1) \), we have
\[
\Delta^{g_u} x_i = -\sum_{j=1}^{n} \frac{\partial H_{ij}}{\partial x_j} = 2\lambda x_i
\]
for \( i = 1, \ldots, n \). Inserting this in \( (6) \), we get
\[
\rho^{g_u} (\cdot, \cdot) = -\frac{1}{2} \sum_{i,l,k=1}^{n} H_{li,ik} dx_k \wedge d\theta_l
\]
\[
= \frac{1}{2} \sum_{l,k=1}^{n} \left( \frac{\partial}{\partial x_k} (2\lambda x_l) \right) dx_k \wedge d\theta_l
\]
\[
= \lambda \sum_{k=1}^{n} dx_k \wedge d\theta_k = \lambda \omega,
\]
as in \( (2) \). \( \square \)

3. The first invariant eigenvalue \( \lambda_1^T \)

3.1. Minimizing \( \lambda_1^T \). The goal of this subsection is to show the first part of Theorem \( 1.5 \). With the notation introduced in Section 2, the first part of Theorem \( 1.5 \) would follow from
\[
(10) \quad \inf_{u \in S(P, \nu)} \{ \lambda_1(g_u) \} = 0.
\]
An easy computation shows that for any \( T \)-invariant function
\[
\int_M g_u (\nabla^{g_u} f, \nabla^{g_u} f) dv_{g_u} = \int_{T^n} d\theta_1 \wedge \cdots \wedge d\theta_n \int_P H^u(df, df) dx_1 \wedge \cdots \wedge dx_n.
\]
Here \( df \) denotes the differential of \( f \) seen as a function on \( P \). We fix coordinates on \( T^* \) and, by translating if necessary, we assume that \( \int_P x_i d\omega = 0 \) where we have set \( d\omega = dx_1 \wedge \cdots \wedge dx_n \). The Rayleigh characterization of the first eigenvalue tells us that for any \( i = 1, \ldots, n \)
\[
(11) \quad \lambda_1(g_u) \leq \frac{\int_P H^u(dx_i, dx_i) d\omega}{\int_P x_i^2 d\omega} = \frac{\int_P u^{ii} d\omega}{\int_P x_i^2 d\omega}
\]
with equality if and only if \( x_i \) is an eigenfunction of the Laplacian \( \Delta^{g_u} \). Since the denominator does not depend on \( u \), to show \( (10) \), it is sufficient to show that we can find \( u \in S(P, \nu) \) with arbitrarily small \( u^{ii} \), as Abreu and Freitas did for \( S^1 \)-invariant metrics on \( S^2 \) in \( [4] \).

Take any \( u_o \in S(P, \nu) \) and for any positive real number \( c > 0 \) put \( u_c = u_o + \frac{c}{2} x_i^2 \). First, we will show that \( u_c^{ii} \) decreases when \( c \) increases. We have \( \text{Hess } u_c = \text{Hess } u_o + cE_i \) where \( E_i = (\delta_{li}\delta_{ki})_{1 \leq l,k \leq n} \) and \( \delta_{li} \) being the Kronecker symbol. In particular,
\[
(12) \quad \det \text{Hess } u_c = \det \text{Hess } u_o + c \det M_{ii}
\]
where \( M_{lk} \) denotes the \((l,k)\)-minor matrix of \( \text{Hess } u_o \). Note that \( M_{ii} \) is positive definite at each point in \( \hat{P} \) since it corresponds to the restriction of the metric \( g_{u_o} \).
(as a metric on \( \tilde{P} \)) to the orthogonal space to \( \frac{\partial}{\partial x_i} \) with respect to the Euclidean metric. In particular for any \( c > 0 \), formula (12) gives \( \det \text{Hess} u_c > 0 \). Now, since the \((i,i)\)-minor matrices of \( \text{Hess} u_o \) and \( \text{Hess} u_c \) are the same we have

\[
u^i_c = \frac{\det M_{ii}}{\det \text{Hess} u_o + c \det M_{ii}}
\]

Thus, \( u^i_c \to 0 \) when \( c \to +\infty \).

Now, we will show that \( u_c \in \mathcal{S}(P,\nu) \) for all \( c > 0 \) by verifying each of the conditions \((i),(ii)\) and \((iii)\) of the definition, see [2]:

(i) \( u_c - \frac{1}{2} \sum_{k=1}^d L_k \log L_k = cx^2 + (u_o - \frac{1}{2} \sum_{k=1}^d L_k \log L_k) \) is smooth since \( u_o \in \mathcal{S}(P,\nu); \)

(ii) let \( x \in \tilde{P} \), (Hess \( u_c \)) is positive definite because it is the sum of a positive definite matrix namely Hess \( u_o \) with a semi-positive definite matrix.

(iii) let \( F \) be a face of \( P \) and \( x \in F \). The restriction of (Hess \( u_c \)) to the tangent space to \( F \) is again the sum of a positive definite form namely Hess \( u_o|_F \) with a semi-positive definite form.

Hence \( u_c \in \mathcal{S}(P,\nu) \) for all \( c > 0 \) and \( \lambda_1(g_u) \to 0 \) when \( c \to +\infty \). This proves (10).

3.2. Maximizing \( \lambda^T_1 \). The goal of this subsection is to show the second part of Theorem 1.5. Let \((P,\nu)\) be the labelled moment polytope of a symplectic toric orbifold \((M,\omega,T)\). Without loss of generality, we assume, in this section, that \( 0 \in \tilde{P} \). In particular, the defining functions \( L_k(x) = \langle x,\nu_k \rangle + c_k \) satisfy \( L_k(0) = c_k > 0 \). Let \( u_o \in \mathcal{S}(P,\nu) \) be the Guillemin potential, that is,

\[
u_o = \frac{1}{2} \sum_{i=1}^d L_i \log L_i - L_i
\]

and \( G_o = \text{Hess} u_o \) and \( H_o = (\text{Hess} u_o)^{-1} \). Choosing coordinates and an inner product \( \langle \cdot,\cdot \rangle \), we see \( G_o \) and \( H_o \) as matrices. For \( s > 1 \), we denote \( u^s_o \), the Guillemin potential of \( sP \) which is the dilation of \( P \) by an \( s \)-factor. The defining affine-linear functions of \( sP \) are \( L^s_k = \langle x,\nu_k \rangle + sc_k \). Consider the following family of functions on \( P \)

\[
u^s = \nu_o - \frac{u^s_o}{s}.
\]

We will show that for \( s > 1 \), \( u^s \in \mathcal{S}(P,\nu) \). Since \( u^s_o \) is smooth on \( P \) when \( s > 1 \), to show that \( u^s \in \mathcal{S}(P,\nu) \) we need to show that \( G^s = \text{Hess} u^s \) is positive definite on \( \tilde{P} \). This is clear since \( L^s_k(x) > L_k(x) \) on \( P \) and

\[
G^s = \frac{1}{2} \sum_{k=1}^d \left( \frac{1}{L_k} - \frac{1}{sL_k^s} \right) \nu_k \otimes \nu_k.
\]

Given a face of the polytope \( F \), a similar argument using only the \( L_i \)'s which do not vanish identically over \( F \), would show that the restriction of \( Hess(u^s) \) to \( F \) is positive definite on \( F \).
Indeed, if $\lambda$ is an eigenvalue for $H_o$ and $u$ is the corresponding eigenvector, then $G_o^s u = \lambda G_o u$, so that
\[
\sum_{k=1}^d \left( \frac{1}{L_k^s(x)} - \frac{\lambda}{L_k(x)} \right) \langle \nu_k, u \rangle^2 = 0,
\]
which is not possible if $\lambda \geq 1$ since $L_k^s(x) > L_k(x)$ on $P$. Because $H_o G_o^s$ is symmetric as each $H_o$ and $G^s$ are, this implies that $||H_o G_o^s|| < 1$. Since $G^s = G_o - \frac{1}{s} G_o^s$ and $s > 1$, we have
\[
H^s = \left( \text{Id}_n + \sum_{k=1}^\infty \left( \frac{1}{s} H_o G_o^s \right)^k \right) H_o,
\]
on the interior of $P$. From this expression, we get that the inequality (14) holds for any $f \in C^1(P)$ and $s > 1$.}

We are now in a position to prove that for the family of metrics determined by $u^s$, $\lambda_1^T$ is unbounded.

**Proposition 3.2.** $\sup \{ \lambda_1^T(s) \mid s > 1 \} = +\infty$.

**Proof.** Assume that $\lambda_1^T(s)$ is also bounded above by a constant, say $\kappa > 0$, then, we can find a sequence $s_k \to 1^+$ such that $\lambda_1^T(s_k)$ converges to some $\lambda > 0$.

Consider a sequence $f_{s_k} \in C^\infty(M)^T = C^\infty(P)$ of eigenfunctions
\[
\Delta_{g^s_{s_k}} f_{s_k} = \lambda_1^T(s_k) f_{s_k}
\]
normalized such that $\|f_{s_k}\|_{L^2} = \int_P (f_{s_k})^2 d\omega = 1$. Note that the inequality (14) implies that the Sobolev norms of $\{f_{s_k}\}$ in $H^1(M, g_{u_0})$ are bounded above by $\kappa + 1$. Indeed, combining the hypothesis and (14), we have
\[
\kappa > \lambda_1^T(s_k) = \int_P H^s(df_{s_k}, df_{s_k}) d\omega \geq \|\nabla_{g_{u_0}} f_{s_k}\|_{g_{u_0}}^2.
\]
Consequently, there exists a subsequence, that we still index by \( s \) for simplicity, of eigenfunctions \( f_s \in C^\infty(M)^T \) converging in the \( L^2(M, g_{u_0}) \) topology to some function \( f \in L^2(M) \). We have \( \|f\|_{L^2} = 1 \), \( \int_P f(x) d\omega = 0 \).

A straightforward calculation yields

\[
G^s(x) = \frac{s - 1}{2s} \sum_{k=1}^{d} \left( \frac{L_k(x) + sc_k}{L_k(x)L_k^*(x)} \right) \nu_k \otimes \nu_k,
\]

and thus, for any \( x \in \hat{P} \),

\[
B_x := \lim_{s \to 1^+} \frac{G^s(x)}{s - 1} = \frac{1}{2} \sum_{k=1}^{d} \left( \frac{L_k(x) + c_k}{L_k(x)^2} \right) \nu_k \otimes \nu_k
\]

is positive definite and depends smoothly on \( x \in \hat{P} \). For \( x \in \hat{P} \), let

\[
A_x = \lim_{s \to 1^+} (s - 1)H^s(x)
\]

be the inverse of \( B_x \). Notice that \( (s - 1)H^s(x) \) converges to its limit uniformly on compact subsets in \( P \).

Let \( K \) be a compact subset of \( \hat{P} \). The integral

\[
\int_K H^s(df_s, df_s) d\omega
\]

can be written as

\[
\int_K A_x \left( \frac{df_s}{\sqrt{s - 1}}, \frac{df_s}{\sqrt{s - 1}} \right) d\omega + \int_K ((s - 1)H^s - A_x) \left( \frac{df_s}{\sqrt{s - 1}}, \frac{df_s}{\sqrt{s - 1}} \right) d\omega.
\]

Now for any \( \epsilon > 0 \)

\[
\left| \int_K ((s - 1)H^s - A_x) \left( \frac{df_s}{\sqrt{s - 1}}, \frac{df_s}{\sqrt{s - 1}} \right) d\omega \right| \\
\leq \sup_K \|(s - 1)H^s - A_x\| \int_K \left| \frac{df_s}{\sqrt{s - 1}} \right|^2 d\omega \\
\leq \epsilon \int_K \left| \frac{df_s}{\sqrt{s - 1}} \right|^2 d\omega
\]

when \( s \) is sufficiently close to 1. On \( K \), the symmetric bilinear form \( A \) is positive definite and its smallest eigenvalue is strictly positive. Hence, on \( K \), the norm of \( A_x \) is equivalent to the Euclidean norm i.e.

\[
\int_K A \left( \frac{df_s}{\sqrt{s - 1}}, \frac{df_s}{\sqrt{s - 1}} \right) d\omega \geq \Gamma_K \int_K \left| \frac{df_s}{\sqrt{s - 1}} \right|^2 d\omega
\]

for some constant \( \Gamma_K \). The inequality

\[
\int_K H^s(df_s, df_s) d\omega \leq \kappa,
\]
implies
\[ \int_K A_x \left( \frac{df_s}{\sqrt{s-1}}, \frac{df_s}{\sqrt{s-1}} \right) d\varpi + \int_K ((s-1)H^s - A_x) \left( \frac{df_s}{\sqrt{s-1}}, \frac{df_s}{\sqrt{s-1}} \right) d\varpi \leq \kappa, \]
but
\[ \int_K A_x \left( \frac{df_s}{\sqrt{s-1}}, \frac{df_s}{\sqrt{s-1}} \right) d\varpi + \int_K ((s-1)H^s - A_x) \left( \frac{df_s}{\sqrt{s-1}}, \frac{df_s}{\sqrt{s-1}} \right) d\varpi \geq (\Gamma_K - \epsilon) \int_K \left| \frac{df_s}{\sqrt{s-1}} \right|^2 d\varpi \]
and we conclude that
\[ \frac{\Gamma_K}{2} \int_K \left| \frac{df_s}{\sqrt{s-1}} \right|^2 d\varpi \leq \kappa. \]

Using the Poincaré inequality, there exists $C_K$, a constant depending only on $K$, such that
\[ C_K \int_K \left| \frac{df_s}{\sqrt{s-1}} \right|^2 d\varpi \geq \frac{1}{s-1} \int_K \left( f_s - \frac{\int_K f_s d\varpi}{\int_K d\varpi} \right)^2 d\varpi. \]
However, since $f_s \to f$ in the $L^2$-topology on $P$ and on $K$,
\[ \int_K \left( f_s - \frac{\int_K f_s d\varpi}{\int_K d\varpi} \right)^2 d\varpi \longrightarrow \int_K \left( f - \frac{\int_K f d\varpi}{\int_K d\varpi} \right)^2 d\varpi. \]
But
\[ 0 \leq \int_K \left( f_s - \frac{\int_K f_s d\varpi}{\int_K d\varpi} \right)^2 d\varpi \leq \frac{2(s-1)C_K \kappa}{\Gamma_K} \longrightarrow 0, \]
when $s \to 1^+$. This implies that
\[ \int_K \left( f - \frac{\int_K f d\varpi}{\int_K d\varpi} \right)^2 d\varpi = 0, \]
and $f$ is a constant on $K$. Since $K$ is arbitrary, $f$ is constant on $P$. But $\int_P f = 0$ so that $f$ must be identically zero which contradicts $\int_P f^2 = 1$. \[ \square \]

This proposition proves the second part of Theorem 1.5. We have thus proved Theorem 1.5.

4. Bounds on $\lambda_1$ for toric manifolds

4.1. The Bourguignon–Li–Yau bound of an integral polytope. Consider a complex projective manifold $(M,J,L)$ where $(M,J)$ is complex manifold and $L \to M$ is a very ample line bundle giving a fixed embedding $\Phi : M \hookrightarrow \mathbb{C}P^N \simeq \mathbb{P}(H^0(M,L))^*$. In [9], Bourguignon–Li–Yau gave a bound on the first eigenvalue of any Kähler metric $\omega$, compatible with $J$. The bound depends only on the dimension of $M$, the Kähler class $[\omega] \in H^2(M,\mathbb{R})$ and the embedding class $[\Phi^*\omega_{FS}] = 2\pi c_1(L)$. The aim of this subsection is to discuss and review the result, as well as apply it to integral toric manifolds. We start by stating the main result of [9].
Theorem 4.1 (Bourguignon–Li–Yau). Let $M^n$ be a compact complex manifold and let $\Phi : M \to \mathbb{CP}^N$ be a holomorphic immersion such that $\Phi(M)$ is not contained in any hyperplane in $\mathbb{CP}^N$. Then for any Kähler metric $\omega$ on $M$, compatible with the given complex structure

\begin{equation}
\lambda_1(M, \omega) \leq \frac{2n(N + 1) \int_M \Phi^* \omega_{FS} \wedge \omega^{n-1}}{N \int_M \omega^n},
\end{equation}

where $\omega_{FS} = i \partial \bar{\partial} \log(|Z_0|^2 + \cdots + |Z_N|^2)$ is the Fubini-Study form on $\mathbb{CP}^N$.

We say that an immersion $\Phi : M \to \mathbb{CP}^N$ is full if its image is not contained in any hyperplane of $\mathbb{CP}^N$. Given a full holomorphic immersion, we set

\[ B(\Phi, [\omega]) = \frac{2n(N + 1) \int_M \Phi^* \omega_{FS} \wedge \omega^{n-1}}{N \int_M \omega^n}. \]

It is clear that $B(\Phi, [\omega])$ only depends on the $H^{1,1}(M, \mathbb{R})$–cohomology classes $[\omega]$ and $[\Phi^* \omega_{FS}]$.

Arezzo–Ghigi–Loi generalized Theorem 4.1 to provide a bound on the first eigenvalue of Kähler manifolds admitting a Gieseker stable bundle, see [6]. In [28], Polterovich used the Bourguignon–Li–Yau Theorem to give a bound on $\lambda_1$ for all symplectic manifolds whose symplectic class $\frac{1}{2\pi}[\omega]$ lies in $H^2(M, \mathbb{Q})$. In the toric case, this theorem can be applied directly to provide (various) bounds on the first eigenvalue of compact toric Kähler manifolds. Indeed, given a toric compact Kähler manifold $(M, \omega, J, T)$ it is known, see for e.g. [10, 20], that $H^2_{dR}(M) = H^1_{\partial}(M)$. Hence, one can pick a symplectic form $\tilde{\omega}$, compatible with $J$ and lying in an integral and very ample class. Using Kodaira’s embedding Theorem, we know that there exists a full embedding $\Phi : M \to \mathbb{CP}^N$ such that $[\Phi^* \omega_{FS}] = [\tilde{\omega}]$. Hence, by Theorem 4.1 $\lambda_1(\omega, J) \leq B(\Phi, [\omega])$. The input of Theorem 1.1 is to give a bound that depends only (and explicitly) on the polytope.

Remark 4.2. Using Bourguignon–Li–Yau Theorem, we can get a finer upper bound for $\lambda_1(M, \omega)$, namely

\[ \inf \left\{ B([\Phi], [\omega]) \mid \Phi : M \to \mathbb{CP}^N \text{ full holomorphic immersion} \right\}. \]

Many natural questions arise: given $\Omega = [\omega]$, is this infimum reached for some immersion ? If so, is this immersion minimal or balanced ? Note that in the Riemannian case, there is an Embedding Theorem due to Colin-de-Verdière and El Soufi–Ilias (see [17]) concerning $\lambda_1$–extremal metrics. These Riemannian metrics are essentially defined as critical points of the map $g \mapsto \lambda_1(g)$ on the space of Riemannian metrics with fixed total volume. In that case, the aforementioned authors showed that an orthonormal basis of the first eigenspace provides a minimal embedding into a sphere $S^N$ such that the standard round metric on $S^N$ pulls-back to the extremal one.

Definition 4.3. Given a compact symplectic toric manifold $(M, \omega, T)$ with moment polytope $P$ we say that $P \subset \mathfrak{t}^*$ is integral if its vertices lie in the dual of the lattice $\Lambda \subset \mathfrak{t}$ of circle subgroups of the torus $T$. 

It is well known that integral polytopes correspond to symplectic toric manifolds whose cohomology class is integral. Moreover, any such toric manifold admits a compatible toric Kähler structure and an explicit equivariant holomorphic full embedding into some projective space, see for e.g. Theorem 6.1.5. We will apply Bourguignon–Li–Yau Theorem using this embedding. For the sake of completeness we recall some facts about this embedding.

Any symplectic toric manifold admits compatible complex structures (see [11, 20]). An example of such a complex structure is given by the Guillemin metric \( g = g_{u_o} \) corresponding to the Guillemin potential \( u_o \) (see Definition 2.3). Moreover, any two such compatible complex structures are biholomorphic (see Remark 4.4 below).

We start with a smooth Kähler toric manifold \((M, g, \omega, J)\) whose cohomology class \([\omega]\) is integral and such that \( g = g_u \) for some symplectic potential \( u \in \mathcal{S}(P, \nu) \). On the underlying toric variety \((M, J)\) the class \([\omega]\) corresponds to an ample divisor which is then very ample \([10, \text{Theorem 6.1.5}]\). More precisely, \((M, J)\) carries a holomorphic line bundle \( L \) whose first Chern class is \([\omega]\) and which defines a full holomorphic embedding \( \Phi_u : M \hookrightarrow \mathbb{C}P^N \) where \( N + 1 \) is the number of lattice points in \( P \). The embedding is associated to a basis of \( H^0(L) \) namely \( \{ e^{m_1} \cdot 1, m \in \Lambda^* \cap P \} \) where \( 1 \) is a reference holomorphic section of \( L \) and may be defined by

\[
\Phi_u(z) = [e^{m_0 \cdot z} : \cdots : e^{m_N \cdot z}],
\]

where \( m_0, \ldots, m_N \) are the lattice points in \( P \) and \( z = y + i\theta \) are local holomorphic coordinates on \( M \), see Remark 2.4.

One can express such an embedding in action-angle coordinates via \( z = y + i\theta = \frac{\partial u}{\partial x} + i\theta \) and for convenience we denote each coordinate

\[
\Phi_u(x, \theta) = [e^{m_0 \cdot \frac{\partial u}{\partial x}} e^{m_0 \cdot i\theta} : \cdots : e^{m_N \cdot \frac{\partial u}{\partial x}} e^{m_N \cdot i\theta}],
\]

because \( e^{m_0 \cdot z} = e^{m_0 \cdot \frac{\partial u}{\partial x}} e^{m_0 \cdot i\theta} \).

Remark 4.4. It is known, see for example [12], that for two distinct symplectic potentials \( u, u_o \in \mathcal{S}(P, \nu) \) the map

\[
\gamma_{u, u_o} : P \times T \longrightarrow P \times T
\]
defined by \( \gamma_{u, u_o}(x, e^{i\theta}) = ((\frac{\partial u}{\partial x})^{-1} \frac{\partial u}{\partial x}, e^{i\theta}) \) extends as an equivariant diffeomorphism on \( M \) sending \( J_u \) to \( J_{u_o} \) and \( \gamma_{u, u_o}^* \omega = \omega + dd^c h \) where \( h \in C^\infty(M)^T \).

Remark 4.5. It has been proved by Guillemin in [20] that \( [\Phi_{u, o}^* \omega_{FS}] = [\omega] \). Moreover, considering the diffeomorphism \( \gamma_{u, u_o} \in \text{Diffeo}(M) \) of Remark 4.4, we have

\[
\gamma_{u, u_o}^* \Phi_{u_o} = \Phi_u
\]

Since the space of symplectic potentials is convex (i.e \( u_t = tu + (1 - t)u_o \in \mathcal{S}(P, \nu) \) for \( t \in [0, 1] \)) the map \( \gamma_{u, u_o} \) lies in the connected component of the identity in \( \text{Diffeo}(M) \). In particular, it preserves cohomology classes and \( [\Phi_{u, o}^* \omega_{FS}] = [\omega] \).

Remark 4.6. Together with \( \Phi_u \) comes an embedding \( \phi : T \hookrightarrow \mathbb{R}^{N+1} \) induced by the linear map \( \phi : t \rightarrow \mathbb{R}^{N+1} \), taking \( \theta \in t \) to

\[
(\theta \cdot m_0, \ldots, \theta \cdot m_N) \in \mathbb{R}^{N+1}
\]
which is clearly injective. The maps $\Phi_u$ are $\phi$–equivariant embeddings.

It follows from Remark 4.5 that the class of $[\Phi_0^*\omega_{FS}]$ does not depend on the chosen symplectic potential $u \in \mathcal{S}(P, \nu)$. It also follows that for any $u \in \mathcal{S}(P, \nu)$, $\Phi_u$ is a full holomorphic embedding iff $\Phi_{u_0}$ is.

Let us consider the Guillemin potential $u_0$.

**Proposition 4.7.** Given a symplectic toric manifold, the map $\Phi_{u_0}$ above is a well defined full holomorphic embedding.

This is a well known fact. We write the details down here for the reader’s convenience. See [20] and [10] for more on this.

**Proof.** Since $u_0 = \frac{1}{2} \sum_{k=1}^{d} (L_k \log L_k - L_k)$, then substituting

$$y = \frac{\partial u_0}{\partial x} = \frac{1}{2} \sum_{k=1}^{d} (\log L_k(x)) \nu_k,$$

we get that

$$\Phi_{u_0}(x, \theta) = [\Pi_{i=1}^{d} L_k(x) \frac{m \cdot \nu_k}{2} e^{i m \theta} : \cdots : \Pi_{i=1}^{d} L_k(x) \frac{m \cdot \nu_k}{2} e^{i m \theta}]_m \in \mathcal{P} \cap \mathbb{Z}^n,$$

where we identify $t \simeq \mathbb{R}^n$ and $\Lambda^* \simeq \mathbb{Z}^n$.

- The homogeneous coordinates of $\Phi_{u_0}$ do not vanish simultaneously. It is clear that the homogeneous coordinates of $\Phi_{u_0}$ do not vanish over the interior of $P$. Let $x \in \partial P$. Without loss of generality assume that $L_1(x) = \cdots = L_r(x) = 0$ and suppose that $x$ is in the interior of the face $F$ defined by $L_1(x) = \cdots = L_r(x) = 0$. Because $P$ is integral there is also a point $m \in F \cap \mathbb{Z}^n$ so that $L_1(m) = \cdots = L_r(m) = 0$ and the $m$-th coordinate does not vanish.

- $\Phi_{u_0}$ is globally defined on $M$. The application $\theta : \hat{M} \to \mathbb{R}^n/\mathbb{Z}^n$ is well defined on the interior of $P$ but not on the boundary. Let $x \in \partial P$ say $L_1(x) = \cdots = L_r(x) = 0$. Then, we should check that

$$\Phi_{u_0}(x, \theta) = \Phi_{u_0} \left( x, \theta + \sum_{i=1}^{r} \alpha_i \nu_i \right),$$

because the Lie algebra of the subgroup of $\mathbb{T}^n$ that fixes the points in $x^{-1}(F)$ is spanned by $\nu_1, \cdots \nu_r$. The only homogeneous coordinates that do not vanish at $x$ are those corresponding to $m$’s such that $L_i(m) = 0$ for $i = 1, \cdots, r$. Let $m_a$ be such a point. Then $\sum_{i=1}^{r} \alpha_i \nu_i = \sum_{i=1}^{r} \alpha_i (-c_i)$ because $m_a \cdot \nu_i = -c_i$ is independent
of \( m \), that is each component of \( \Phi_u(x, \theta + \sum_{l=1}^r \alpha_l \nu_l) \) is \( \left(e^{\frac{i}{2} \sum_{l=1}^r \alpha_l (-\nu_l)}\right) \) times the corresponding component of \( \Phi_u(x, \theta) \) and thus \( \Phi_u(x, \theta + \sum_{l=1}^r \alpha_l \nu_l) = \Phi_u(x, \theta) \).

- \( \Phi_u \) is holomorphic because on the interior of \( P \) it coincides with \([e^{m \cdot z}]_{m \in \mathbb{N} \cap \mathbb{Z}^n}\), which is expressed in terms of complex coordinates as a holomorphic function.

- \( \Phi_u \) is an injective immersion. It is clear over the interior of \( P \) thanks to Remark 4.6. Over the boundary, we may assume that \( P \) is standard around one of its vertices so that \( L_1 = x_1 - a_1, \ldots, L_n = x_n - a_n \). In this case by reordering if necessary we may assume that \( m_1 - m_0 = (1, 0, \ldots, 0), \ldots, m_n - m_0 = (0, \ldots, 0, 1) \). To prove injectivity of \( \Phi_u \) and its derivative it is enough to prove injectivity of \( z \rightarrow (e^{(m_1 - m_0)z}, \ldots, e^{(m_n - m_0)z}) \).

But the above is simply \( z \rightarrow (e^{z_1}, \ldots, e^{z_n}) \) which is injective modulo \( 2\pi i \mathbb{Z}^n \) as expected and has injective derivative. \( \square \)

Applying the Bourguignon–Li–Yau theorem to \((M^{2n}, \omega, g_u, J, T)\), we get that

\[
\lambda_1(\omega, g_u) \leq \frac{2n(N+1)}{N} \int_M \Phi_u^* \omega_{FS} \land \omega^{n-1} \frac{\omega^n}{\omega^n} = \frac{2n(N+1)}{N}
\]

where \( N+1 \) is the number of lattice points in \( P \).

Remark 4.8. Observe that taking \( kP \) for \( k \in \mathbb{N}^* \) and \( k \geq 2 \) implies that the right hand side of (20) decreases to \( 2n \). However, the left hand side decreases quickly as well since \( \lambda_1(k\omega, kg_u) = \frac{1}{k} \lambda_1(\omega, g_u) \). Hence, in each rays of Kähler cone in \( H^{1,1}(M, \mathbb{Z}) \) there is an optimal class, the primitive class, on which we may apply the bound \( \text{B}(\Phi_u, [\omega]) \).

Remark 4.9. The Bourguignon–Li–Yau bound is an integer if and only if \( N = 2, N = n \) or \( N = 2n \). The first two cases imply \( M \) is a projective space and the last one gives \( \lambda_1(\omega, g_u) \leq 2n + 1 \). Note that, in this last case, the first eigenvalue of a Kähler–Einstein metric, which is \( 2\lambda \) by Proposition 1.4 where \( \lambda = 2\pi c_1(M)/[\omega] \), cannot reach this bound whenever \( [\omega] \) is integral.

4.2. A bound on \( \lambda_1 \) for toric manifolds. Let \((M^{2n}, \omega, g, J)\) be a compact toric Kähler manifold. The cohomology class of \( \frac{\omega}{2\pi} \) is integral if and only if \( P \) is integral. In this subsection we will not assume that \( P \) is integral. We start by defining an integer \( k_0(P) \) associated to \( P \). Let \( k \) be a fixed integer. Consider the lattice \( \mathbb{Z}^n/k \cap P \).

Definition 4.10. Let \( P \) be a Delzant polytope. Set

\[
L_{i,k} = \min\{L_i(m), m \in P \cap \mathbb{Z}^n/k\}.
\]
Let \( P_k \) be the polytope defined by the inequalities \( L_i(x) \geq L_{i,k}^{\text{min}} \) i.e.
\[
P_k = \{ x \in P : L_i \geq L_{i,k}^{\text{min}}, \, i = 1, \ldots, d \}.
\]
Note that if \( P_k \) is a non-empty polytope with \( d \) facets then these facets are parallel to those of \( P \).

We want to show that as \( k \) tends to infinity the lattice \( \mathbb{Z}^n/k \cap P \) becomes finer and eventually \( P_k \) will look combinatorially like \( P \).

**Lemma 4.11.** Let \( P \) be a simple compact Delzant polytope
\[
P = \{ x \in \mathbb{R}^n : L_i \geq 0, \, i = 1, \ldots, d \}.
\]
Let \( L_{i,k}^{\text{min}} \) and \( P_k \) be defined as above then \( L_{i,k}^{\text{min}} \to 0 \) as \( k \) tends to infinity and \( P_k \) has the same combinatorial type as \( P \) for \( k \) large enough.

**Proof.** Assume without loss of generality that \( i = 1 \) i.e. we want to prove that \( L_{1,k}^{\text{min}} \to 0 \) as \( k \) tends to infinity. Choose a vertex of \( P \) say \( a = (a_1, \ldots, a_n) \) on the first facet of \( P \) that is \( L_1(a) = 0 \). Assume furthermore that the first \( n \) facets of \( P \) meet at \( a \). This is possible as we can simply relabel the facets. Since \( P \) is Delzant then there is \( A \in SL(n, \mathbb{Z}) \) such that \( AP \) is standard at \( a \). That is
\[
AP = \{ x \in \mathbb{R}^n : x_1 - \tilde{a}_1 \geq 0, \ldots, x_n - \tilde{a}_n \geq 0, \tilde{L}_{n+1}(x) \geq 0, \ldots, \tilde{L}_d(x) \geq 0 \},
\]
where \( \tilde{L}_{n+1}, \ldots, \tilde{L}_d \) are affine functions and \( \tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_n) = Aa \). Note that \( \tilde{L}_i(\tilde{a}) > 0 \) for all \( i = n + 1, \ldots, d \) because \( \{ \tilde{a} \} \) is the intersection of the first \( n \) facets of \( AP \). Pick \( a_k = (a_{1,k}, \ldots, a_{n,k}) \in \mathbb{Z}^n \) in the following way
\[
\frac{a_{i,k} - 1}{k} \leq \tilde{a}_i \leq \frac{a_{i,k}}{k}, \forall i = 1, \ldots, n.
\]
Now \( a_k/k \) satisfies the first \( n \) inequalities in the definition of \( AP \) by construction. On the other hand \( |a_k/k - \tilde{a}| \leq n/k \) so that when \( k \) is sufficiently large and \( i > n \), \( \tilde{L}_i(a_k) \) is close to \( \tilde{L}_i(\tilde{a}) > 0 \) hence \( \tilde{L}_i(a_k) > 0 \) for large enough \( k \). This implies \( a_k/k \in AP \) therefore \( a_k/k \in \mathbb{Z}^n/k \cap AP \). But \( A^{-1}a_k/k \in \mathbb{Z}^n/k \cap P \) where we have used the fact that \( A \in SL(n, \mathbb{Z}) \). Also \( |A^{-1}a_k/k - a| \leq ||A^{-1}||n/k \) and therefore tends to zero as \( k \) tends to infinity. It follows that \( L_1(A^{-1}a_k/k) \to L_1(a) = 0 \) because \( L_1 \) is a continuous function. Since \( L_{1,k}^{\text{min}} \leq L_1(A^{-1}a_k/k) \) and is positive, we prove our claim.

\[ \square \]

**Definition 4.12.** We define \( k_0(P) \) to be smallest integer \( k \geq 1 \) such that \( P_k \) has the same combinatorial type as \( P \).

For any integer \( k \), we set \( N_k = \#(\mathbb{Z}^n/k \cap P) - 1 \). Note that if \( P \) is integral, \( k_0(P) = 1 \). Recall that we write \( L_i(x) = x \cdot \nu_i + c_i \).

**Lemma 4.13.** Let \( P \) be a Delzant polytope and \( k \geq k_0(P) \) then \( kP_k \) is an integral Delzant polytope such that \( N_k + 1 = \#((kP_k) \cap \mathbb{Z}^n) \).

**Proof.** For \( i \in \{1, \ldots, d\} \), we denote \( \tilde{c}_{i,k} = c_i - L_{i,k}^{\text{min}} \) so that \( P_k = \{ x \in \mathbb{R}^n : x \cdot \nu_i + \tilde{c}_{i,k} \geq 0, \, \text{for} \, \ i = 1, \ldots, d \} \). Then
\[
kP_k = \{ x \in \mathbb{R}^n : (x/k) \cdot \nu_i + \tilde{c}_{i,k} \geq 0, \, \text{for} \, \ i = 1, \ldots, d \}
\]
\[
= \{ x \in \mathbb{R}^n : x \cdot \nu_i + k\tilde{c}_{i,k} \geq 0, \, \text{for} \, \ i = 1, \ldots, d \}.
\]
Therefore, \( kP_k \) has the same normal inward vectors as \( P \) and is Delzant iff \( P \) is. We are using the fact that if \( k \geq k_0(P) \) so that \( kP_k, P_k \) and \( P \) have the same combinatorial type. Moreover \( k\tilde{c}_{i,k} \in \mathbb{Z} \). Indeed, the compacity of \( P \) implies that there exists \( m/k \in P \) such that \( m \in \mathbb{Z}^n \) and \( L_{i,k}^{\text{min}} = L_i(m/k) \) and thus
\[
k\tilde{c}_{i,k} = k(c_i - L_{i,k}^{\text{min}}) = k(c_i - L_i(m/k)) = k(c_i - ((m/k) \cdot \nu_i + c_i)) = -m \cdot \nu_i \in \mathbb{Z}.
\]

Hence, \( kP_k \) is an integral Delzant polytope. Finally, \( m \in ((kP_k) \cap \mathbb{Z}^n) \) if and only if \( \frac{m}{k} \in ((P_k) \cap \mathbb{Z}^n/k) \subset (P \cap (\mathbb{Z}^n/k)) \), thus \( \sharp ((kP_k) \cap \mathbb{Z}^n) \leq \sharp (P \cap (\mathbb{Z}^n/k)) \). Now any point in \( (P \cap (\mathbb{Z}^n/k)) \backslash (P_k \cap (\mathbb{Z}^n/k)) \) would contradict the minimality of \( L_{i,k}^{\text{min}} \) for some \( i \in \{1, \ldots, d\} \). Hence \( \sharp ((kP_k) \cap \mathbb{Z}^n) = \sharp ((P_k) \cap (\mathbb{Z}^n/k)) = \sharp (P \cap (\mathbb{Z}^n/k)) \). \( \square \)

We are now in a position to prove Theorem 1.1

**Proof.** (of Theorem 1.1) At this point we can apply Bourguignon–Li–Yau’s theorem [17] to conclude that
\[
\lambda_1(M, \omega) \leq \frac{2n(N_k + 1) \int_P \Phi_{u,k}^* \omega_{FS} \wedge \omega^{n-1}}{N_k \int_P \omega^n},
\]
where \( \Phi_{u,k} \) is the embedding (18) associated to \( kP_k \). Now it is well known that the symplectic class associated to \( kP_k \) is \( k \) times the one associated to \( P_k \). Then, see Remark 4.5, we have \( \Phi_{u,k}^* \omega_{FS} = k[\omega_k] \) where \( \omega_k \) is the symplectic form determined by \( P_k \). Recall that \( P_k = \{ x \in P : L_i \geq L_{i,k}^{\text{min}}, i = 1, \ldots, d \} \) has the same combinatorial type as \( P \), by assumption on \( k \). In particular, \( P \) and \( P_k \) share the same normal inward vectors (hence the same fan) but the constant parts of the affine-linear functions defining \( P_k \) are \( c_i - L_{i,k}^{\text{min}} \) for \( i = 1, \ldots, d \).

Let \( E_i \) be the cohomology class that is Poincaré dual to the divisor in \( M \) whose image under the moment map is the facet \( F_i \). Then
\[
\frac{1}{2\pi i} [\partial \bar{\partial} \log L_i] = E_i
\]
(see [20], Theorem 6.2 for a statement of this elementary fact).

It is well known (see [20]) that \( \frac{[\omega]}{2\pi} = \sum_{i=1}^d c_i E_i \). Applying this to \( P_k \) we get
\[
\frac{[\Phi_{u,k}^* \omega_{FS}]}{2\pi} = k(c_i - L_{i,k}^{\text{min}})E_i = k \left( [\omega] - \sum_{i=1}^d L_{i,k}^{\text{min}} E_i \right).
\]
Replacing in equation (22) we see that
\[
\lambda_1(M, \omega) \leq \frac{2n(N_k + 1)}{N_k} \left( k - k \int_P \sum_{i=1}^d L_{i,k}^{\text{min}} E_i \wedge \omega^{n-1} \right). \int_P \omega^n
\]

Because \( E_i \) is Poincaré dual to the pre-image of the \( i \)-th facet, which we denote by \( D_i \), we have that
\[
\int_P E_i \wedge \omega^{n-1} = \int_{D_i} \omega^{n-1} = \text{vol}_{n-1}(D_i) > 0.
\]

It follows that
\[
\lambda_1(M, \omega) \leq \frac{2nk(N_k + 1)}{N_k}.
\]
It is clear from the above argument that if $P$ is integral one can take $k = 1$ and the bound becomes
\[ \lambda_1(M, \omega) \leq \frac{2n(N + 1)}{N}. \]
This proves Theorem 1.1. □

4.3. The equality case in Bourguignon–Li–Yau’s bound. The goal of this subsection is to study Kähler toric metrics which saturate the bound in Theorem 1.1. We give a quick overview of Bourguignon, Li and Yau’s proof in [9] as this proof will be important to us.

Sketch of proof of Theorem 4.1, see [9]. Recall that the first eigenspace of $(\mathbb{C}P^N, \omega_{FS})$ has a basis given by the functions $[Z] \mapsto \Psi_{ij}(Z) - \frac{\delta_{ij}}{N+1}$ where for $i, j \in \{0, 1, \ldots, N\}$,
\[ \Psi_{ij}(Z) = \frac{Z_i Z_j}{\sum_{k=0}^{N} |Z_k|^2} \]
is one component of the $SU(N+1)$ moment map $\Psi : \mathbb{C}P^N \hookrightarrow \mathfrak{su}^*_{N+1}$. The main step of the proof in [9], is to show that, given a full embedding $\Phi : M \hookrightarrow \mathbb{C}P^N$, there exists a unique $B \in SL(N+1, \mathbb{C})$ such that $B^* \omega > 0$ and
\[ \int_M \omega^n \int_M (\Psi_{ij} \circ B \circ \Phi)(p) \omega^n = \frac{\delta_{ij}}{N+1}. \]
Said differently, $(B \circ \Phi)^* \omega_{FS}$ is $(\omega^n/n!)$-balanced, (see [15]). To simplify the notation we write $\omega$–balanced instead of $(\omega^n/n!)$–balanced.

Denote $f^B_{ij} = \Psi_{ij} \circ B \circ \Phi - \frac{\delta_{ij}}{N+1} \in C^\infty(M)$. The Rayleigh principle implies that
\[ \lambda_1(M, \omega) \int_M |f^B_{ij}|^2 \frac{\omega^n}{n!} \leq \int_M |\nabla^\omega f^B_{ij}|^2 \omega^n \]
with equality if and only $f^B_{ij}$ is an eigenfunction of $\Delta$ for the eigenvalue $\lambda_1(M, \omega)$.

Taking the sum over $i, j = 0, \ldots, N$ the left hand side of (25) gives
\[ \lambda_1(M, \omega) \frac{N}{N+1} \int_M \omega^n \]
thanks to (24). Noticing that $g(\nabla f, \nabla \overline{f})\omega^n = n \text{Re}(df \wedge d\overline{f} \wedge \omega^{n-1})$, that the form $\sum_{i,j} df^B_{ij} \wedge d^c f^B_{ij}$ is real and that, on $\mathbb{C}P^N$,
\[ \sum_{i,j} d\Psi_{ij} \wedge d^c \overline{\Psi}_{ij} = 2 \omega_{FS}, \]
the right hand side of (25) gives
\[ \sum_{i,j} \int_M |\nabla^\omega f^B_{ij}|^2 \frac{\omega^n}{n!} = \sum_{i,j} \int_M df^B_{ij} \wedge d^c f^B_{ij} \wedge \omega^{n-1} \frac{1}{(n-1)!} = \frac{2}{(n-1)!} \int_M (B \circ \Phi)^* \omega_{FS} \wedge \omega^{n-1}. \]
Finally, $B^* \omega_{FS}$ and $\omega_{FS}$ are in the same cohomology class on $\mathbb{C}P^N$, this concludes the proof. □

Remark 4.14. The equality case in (16) implies the equality case in each inequality (25) and then that each function $f^B_{ij}$ is an eigenfunction of $\Delta$ for the first eigenvalue.
In the toric context and for the embedding $\Phi_u$ we get the following refinement of Bourguignon–Li–Yau’s result on the existence of balanced metrics.

**Lemma 4.15.** Let $(M, \omega, g_u, J_u, T)$ be a toric Kähler manifold with integral polytope $P \subset t^*$ and corresponding embedding $\Phi_u : M \hookrightarrow \mathbb{C}P^N$. There exists a diagonal matrix $B = \text{diag}(\alpha_0, \ldots, \alpha_N) \in GL(N + 1, \mathbb{R})$ with $\alpha_i > 0$ and $\text{tr}B = 1$ satisfying the condition (24).

**Proof.** First, observe that when $i \neq j$, the function $\Psi_{ij} \circ \Phi_u$ integrates to 0 on $M$ since it does on each orbit of $T$. Hence to prove the lemma, we only have to prove that there exists $\alpha = (\alpha_0, \ldots, \alpha_N) \in \mathbb{R}^{N+1}_{>0}$ such that, for $i = 0, \ldots, N$,

$$
\psi_{u,i}(\alpha) := \alpha_i^2 \int_M \frac{|Z_i|^2}{\sum_{j=0}^N \alpha_j^2 |Z_j|^2 n!} \omega^n = \frac{1}{N + 1}
$$

where $Z_i = \Phi_{u,m_i}(x, \theta)$ see (18). Let $\Sigma$ be the simplex defined by

$$
\Sigma := \left\{ X \in \mathbb{R}^{N+1} \mid \sum_{i=0}^N X_i = 1, \, X_i > 0 \right\}.
$$

Because $M$ is full in $\mathbb{C}P^N$, $\sum_{j=0}^N \alpha_j^2 |Z_j|^2$ does not vanish identically on $M$ (otherwise the $M$ would be contained in a positive codimension subvariety). Since $\sum_{i=1}^N \psi_{u,i}(\alpha) = \text{vol}(M) < +\infty$ and each component $\psi_{u,i}(\alpha) \geq 0$ for each $\alpha \in \mathbb{R}^{N+1}$, by the dominated convergence lemma, the maps $\psi_{u,i}$ can be extended continuously to $\overline{\Sigma}$. Hence, we see $\psi_u = (\psi_{u,0}, \ldots, \psi_{u,N})$ as a continuous map

$$
\psi_u : \overline{\Sigma} \longrightarrow \overline{\Sigma}
$$

from the closed simplex $\overline{\Sigma}$ to itself. It is obvious that $\psi_u$ maps $\partial \Sigma$ to $\partial \Sigma$.

To prove the lemma we need to prove that $\psi_u$ is surjective which will follow if we prove that the restriction $\psi_u : \partial \Sigma \rightarrow \partial \Sigma$ has non-trivial degree.

Like in [9], instead of integrating on $M$, we integrate on $\mathbb{C}P^N$ with the measure $d\mu_u$ defined to be the push forward of the measure on $M$ defined by the metric. This is possible because $\Phi_u$ is a full embedding. Hence

$$
\psi_{u,i}(\alpha) = \alpha_i^2 \int_{\mathbb{C}P^N} \frac{|Z_i|^2}{\sum_{j=0}^N \alpha_j^2 |Z_j|^2} d\mu_u.
$$

Now if we consider the volume form $d\mu_o$ induced by the Fubini-Study metric on $\mathbb{C}P^N$, the corresponding map $\psi_o : \partial \Sigma \rightarrow \partial \Sigma$ with components

$$
\psi_{o,i}(\alpha) = \alpha_i^2 \int_{\mathbb{C}P^N} \frac{|Z_i|^2}{\sum_{j=0}^N \alpha_j^2 |Z_j|^2} d\mu_o
$$

and its restriction $\psi_o : \partial \Sigma \rightarrow \partial \Sigma$ are bijections. Now $\psi_t = t\psi_u + (1 - t)\psi_o$ is a family of continuous maps from $\overline{\Sigma}$ to itself preserving the boundary. The degree of the $\psi_t$ does not depend on $t$ and is non trivial for $t = 0$. \(\square\)

**Remark 4.16.** A straightforward corollary of this lemma is that the $\omega$-balanced metric is toric as soon as $\omega$ is toric.
Theorem 1.2 is a consequence of two propositions we state below and which we prove using the following observation. Assume that the first eigenvalue of \((M, \omega, g_u, J_u, T)\) reaches the Bourguignon, Li and Yau’s bound, i.e \(\lambda_1(g_u) = \frac{2n(N+1)}{N}\). By Remark 4.14 and Lemma 4.15 there exists a set of \(N + 1\) real positive numbers \(\{\alpha_k\}_{k \in P \cap \mathbb{Z}^n}\) such that for each \(m, k \in P \cap \mathbb{Z}^n\), the function \(\Psi_{mk} = \delta_{mk}/(N + 1)\) is an eigenfunction of eigenvalue \(\frac{2n(N+1)}{N}\) where

\[
\Psi_{mk} = \frac{\alpha_k \alpha_m Z_m Z_k}{\sum_j |\alpha_j Z_j|^2}
\]

is seen as a function of \((x, \theta)\). Here we write

\[
Z_m = e^{(u_x + i\theta)m},
\]

where \(u_x = \frac{\partial}{\partial x}\). We assume, without loss of generality, that \(0 \in P \cap \mathbb{Z}^n\). Hence the image of \(\Phi_u\), see (18), lies in the set where \(Z_0 \neq 0\) and we work on this set.

We normalize the \(\alpha_k\)’s so as to have \(\sum_{k \in P \cap \mathbb{Z}^n} \alpha_k = 1\) as in the previous lemma. We recall that

\[
\int_M \Psi_{mk} \omega_n = \delta_{mk} \int_M \omega_n/n!.
\]

Observe that for each pair \(m, k \in P \cap \mathbb{Z}^n\)

\[
\Delta \Psi_{mk} = \Delta(\alpha_m Z_m \Psi_{0k}) = \alpha_m Z_m \Delta \Psi_{0k} - 2\alpha_m \langle dZ_m, d\Psi_{0k} \rangle
\]

since \(\Delta(Z_m) = 0\) where \(\langle \cdot, \cdot \rangle\) denotes the inner product induced by \(g_u\) on the cotangent bundle of \(M\).

When \(k = 0 \neq m\), identity (28) becomes

\[
\frac{2n(N+1)}{N} \Psi_{m0} = \frac{2n(N+1)}{N} \alpha_m Z_m (\Psi_{00} - \frac{1}{N + 1}) - 2\alpha_m \langle dZ_m, d\Psi_{00} \rangle
\]

\[
= \frac{2n(N+1)}{N} \Psi_{m0} - \frac{2n\alpha_m Z_m}{N} - 2\alpha_m \langle dZ_m, d\Psi_{00} \rangle,
\]

and we get

\[
\frac{2n\alpha_m Z_m}{N} = -2\alpha_m \langle dZ_m, d\Psi_{00} \rangle.
\]

Developing the right hand side in action angle coordinates, using (4), we have

\[
\frac{2n\alpha_m Z_m}{N} = -2\alpha_m \langle dZ_m, d\Psi_{00} \rangle
\]

\[
= -2\alpha_m \sum_{i,j=1}^n H_{ij} \partial_x_i Z_m \partial_x_j \Psi_{00}
\]

\[
= \frac{4\alpha_m Z_m \sum_{k \in W} \sum_{s,t=1}^n u_{ts} m_s k_t |\alpha_k Z_k|^2}{(\sum_{k \in W} |\alpha_k Z_k|^2)^2}
\]

where \(W = P \cap \mathbb{Z}^n\). Dividing both sides by \(2\alpha_m Z_m\), we end up with

\[
\frac{n}{N} = \frac{2 \sum_{k \in W} \sum_{s,t=1}^n u_{ts} m_s k_t |\alpha_k Z_k|^2}{(\sum_{k \in W} |\alpha_k Z_k|^2)^2} = -(d\Psi_{00})(m)
\]
**Proposition 4.17.** Let \((\omega, g_u, J_u, T)\) be a toric Kähler structure on \(\mathbb{CP}^n\). Assume that \(\lambda_1 = 2(n+1)\) i.e. assume that the first eigenvalue of the Laplacian reaches the Bourguignon–Li–Yau bound. Then, the toric Kähler metric \(g_u\) is the Fubini-Study metric on \(\mathbb{CP}^n\).

**Proof.** The moment polytope \(P\) of \(\mathbb{CP}^n\) is a simplex and if \((\omega, g_u, J_u, T)\) saturates the bound then \(P\) is primitive as explained in Remark 4.8. So one can suitably normalize it so that it has integer vertices 0, \(e_1, \ldots, e_n\) where \(e_i\) is the vector in \(\mathbb{R}^n\) whose \(i\)-th component is 1 and all others are zero. In the notation above, \(N = n\) and we identify \(W = P \cap \mathbb{Z}^n\) with \(\{0, 1, \ldots, n\}\). Equation (29) implies that, for all \(m = 1, \ldots, n\),

\[
1 = 2 \sum_{m,k=1}^{n} u_{mk} |\alpha_k Z_k|^2 \left(1 + \sum_{k=1}^{n} |\alpha_k Z_k|^2\right)^{-2} = \frac{\partial \Psi_{00}}{\partial x_m}
\]

where \(Z_k = e^{(u_k + i\theta_k)}\) for \(k = 1, \ldots, n\). Hence

\[
\Psi_{00} = K - \sum_{k=1}^{n} x_k
\]

for some constant \(K\). The additive constant is fixed to be \(K = 1\) by the integration constraint (27).

In the case \(m = k \neq 0\), identity (28) gives

\[
2(n + 1)(\Psi_{mm} - 1/n + 1) = \alpha_m Z_m 2(n + 1)\Psi_{0m} - 2\alpha_m \langle dZ_m, d\Psi_{0m} \rangle
\]

that is

\[
-2 = -2\alpha_m \langle dZ_m, d\Psi_{0m} \rangle.
\]

Developing the right hand side using (30), we get

\[
-2 = -2\alpha_m \langle dZ_m, d\Psi_{0m} \rangle = 2|\alpha_m Z_m|^2 \left(\sum_{i} |\alpha_i Z_i|^2 - 1\right) = -2 \frac{\partial \Psi_{mm}}{\partial x_m}.
\]

Therefore, for each \(m > 0\)

\[
\frac{\partial \Psi_{mm}}{\partial x_m} = 1
\]

In the case \(0 \neq m \neq k \neq 0\), identity (28) gives

\[
2(n + 1)\Psi_{mk} = \alpha_m Z_m 2(n + 1)\Psi_{0k} - 2\alpha_m \langle dZ_m, d\Psi_{0k} \rangle,
\]

that is

\[
0 = -2\alpha_m \langle dZ_m, d\Psi_{0k} \rangle.
\]

Developing the right hand side using (30), we get

\[
0 = -2\alpha_m \langle dZ_m, d\Psi_{0k} \rangle = 2\alpha_m \alpha_k Z_m Z_k \left(\sum_{i} |\alpha_i Z_i|^2 - 1\right) = -2 \frac{\alpha_m \alpha_k Z_m Z_k}{|\alpha_m Z_m|^2} \left(\frac{\partial \Psi_{mm}}{\partial x_k}\right).
\]

Therefore, for each \(m, k > 0\) and \(k \neq m\), we have

\[
\frac{\partial \Psi_{mm}}{\partial x_k} = 0.
\]
Together with (31) it gives
\begin{equation}
\Psi_{mm} = x_m
\end{equation}
where again the additive constant is fixed by the integration constraint (27) on \( \Psi_{mm} \). We conclude from Remark (4.14) that the components of the moment map are eigenfunction for the same eigenvalue and thus, by Proposition 1.4 \((g_u, J_u)\) is Kähler-Einstein. By uniqueness of extremal toric metric \[13\], \((g_u, J_u)\) is the Fubini-Study metric. \[\square\]

We can also prove that if a toric manifold admits a toric Kähler metric for which the embedding given by the integral points of the moment polytope saturates the Bourguignon–Li–Yau bound, then that manifold must be \( \mathbb{CP}^n \) and it follows from the above proposition that the metric must be the Fubini-Study metric.

**Proposition 4.18.** Let \((M, \omega, g_u, J_u)\) be a toric Kähler manifold with integral polytope \( P \). Assume that \( \lambda_1(g_u) = \frac{2n(N+1)}{N} \) with \( N = \#(P \cap \mathbb{Z}^n) - 1 \). Then \( P \) is the standard simplex and \( M \) is (equivariantly symplectomorphic to) \( \mathbb{CP}^n \).

**Proof.** Again we assume that the origin lies in \( P \cap \mathbb{Z}^n \), more precisely, up to an integral invertible affine transformation, we may assume that \( P \) is standard at the origin i.e. the facets that meet at 0 have normals \( e_1, \cdots, e_n \). In particular, the vertices of \( P \) adjacent to the origin, say \( m_1, \cdots m_n \), are each an integral multiple of an element of a dual basis of \( e_1, \cdots, e_n \) respectively.

Under the hypothesis of the proposition and with respect to the notation above, Equations (28) and (29) hold for points in \( P \cap \mathbb{Z}^n \). Suppose there is \( m \in P \cap \mathbb{Z}^n \) a vertex distinct from the origin and from \( m_1, \cdots m_n \). Then, there exist \( a_1, \cdots a_n \) such that \( m = \sum_{l=1}^n a_l m_l \).

Equation (29) holds for \( m \) as well as for \( m_1, \cdots m_n \). So we must have
\[
\frac{n}{N} = -(d \Psi_{00})(m) = -\sum_{l=1}^n a_l (d \Psi_{00})(m_l) = \left( \sum_{l=1}^n a_l \right) \frac{n}{N}
\]
which implies \( \sum_{l=1}^n a_l = 1 \). So \( m \) lies on a facet of the simplex of vertices 0, \( m_1, \cdots m_n \) which contradicts convexity unless \( P \) is that simplex. \[\square\]

To sum up we proved Theorem 1.2.

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