Geometrical aspects of integrable systems

Emanuele Fiorani
Department of Mathematics and Informatics, University of Camerino
62032 Camerino (MC), Italy
emanuele.fiorani@unicam.it

Abstract

We review some basic theorems on integrability of Hamiltonian systems, namely the Liouville-Arnold theorem on complete integrability, the Nekhoroshev theorem on partial integrability and the Mishchenko-Fomenko theorem on noncommutative integrability, and for each of them we give a version suitable for the noncompact case. We give a possible global version of the previous local results, under certain topological hypotheses on the base space. It turns out that locally affine structures arise naturally in this setting.

Keywords: Integrable Hamiltonian systems; global action-angle coordinates; locally affine structures.

1 Introduction

There are some well known theorems treating integrable Hamiltonian systems on a $2n$-dimensional symplectic manifold $(Z, \Omega)$: the so called Liouville-Arnold theorem (or Liouville-Mineur-Arnold theorem) for completely integrable systems (see e.g. [18, 24]), the Nekhoroshev theorem (or Poincaré-Lyapounov-Nekhoroshev theorem) for partially integrable systems (see [16]), and the Mishchenko-Fomenko theorem for noncommutative (i.e. non-Abelian) systems (see [4, 14]). All of them state the existence of a neighborhood of a compact invariant submanifold (which is a torus $T^n$) which is topologically trivial (i.e. a product), and the existence of particular adapted coordinates on it, called action-angle coordinates. These theorems have been extended to the case of noncompact invariant submanifold (which is a cylinder $\mathbb{R}^{n-h} \times T^h$); see [7, 8, 9, 12]. Sections 2, 3 and 4 intend to compare the compact and noncompact situations.

If invariant submanifolds are compact, topological obstructions to the global triviality of the whole $Z$ and to the global existence of action-angle coordinates have been studied (see [2, 3, 15]). Here we aim to extend this study to the case of noncompact invariant submanifolds; a possible approach is shown in Section 5 and follows that of [10]. The main idea is to provide a sufficient condition in order that a principal bundle with Abelian structure group be trivial.

Section 6 is devoted to the interplay between integrability and locally affine structures. The motivation is to clarify the meaning of a certain connection naturally arising on the invariant submanifolds; see for instance [5, 24]. For this purpose it is needed to study in some details the relation between flat and
parallizable manifolds and, in this context, the role of the torsion of a flat connection is considered. Some peculiarity of spheres and Euclidean spaces are outlined.

Throughout this paper all manifolds are smooth and paracompact and all maps and structures are smooth. \((Z,\Omega)\) is a \(2n\)-dimensional connected symplectic manifold and \((C^\infty(Z),\{\})\) is the Poisson algebra of (real) functions on \(Z\). Given a function \(F : Z \to \mathbb{R}\), its Hamiltonian vector field \(\Omega^\sharp dF\) is indicated with \(X_F\).

\section{Complete integrability}

Perhaps Theorem 1 below is the prototype of integrability theorems; it treats Hamiltonian systems with the maximum possible number of (independent) functions in involution. This theorem, together with further discussions, can be found for instance in the books \[18, 24\] which are standard references.

**Definition 1.** A subset \((F_1,\ldots,F_n)\) of \((C^\infty(Z),\{\})\) is called a completely integrable system on \(Z\) if at every point of \(Z\) \(dF_1,\ldots,dF_n\) are linearly independent and \(\{F_i, F_j\} = 0 \forall i, j = 1,\ldots,n\).

**Theorem 1.** Let us assume the following:

(i) \(F = (F_1,\ldots,F_n) : Z \to \mathbb{R}^n\) is a completely integrable system;

(ii) There is a compact component \(M\) of a fiber of \(F\).

Then:

(I) There exists a (saturated) neighborhood \(U\) of \(M\) such that \(F|_U\) is a trivial principal bundle with structure group the torus \(T^n\), and \(M\) is one of its fibers;

(II) Given standard coordinates \((\varphi^\lambda)\) on \(T^n\), the neighborhood \(U\) is provided with Darboux coordinates \((I_\lambda,\varphi^\lambda)\) such that

\[\Omega = dI_\lambda \wedge d\varphi^\lambda\]

and \(F_\lambda = F_\lambda(I_\mu)\) on \(U\).

**Remark 1.**

(i) If \(F : A \to B\) is a submersion, the compact components of the fibers of \(F\) make up an open subset of \(A\); see the discussion on page 358 of \[22\].

(ii) A submersion \(F : A \to B\) (\(B\) connected) with compact fibers having constant (finite) number of components is a bundle; indeed it can be shown that, under these hypotheses, \(F\) is a proper map. Alternatively, if the fibers of \(F\) are diffeomorphic to a Euclidean space \(\mathbb{R}^n\) then \(F\) is a bundle. See \[13\].

Thus (i) and (ii) can be used to give a simple proof of Theorem 1.
(iii) In contrast with (ii), a submersion with fibers diffeomorphic to a product $\mathbb{R}^n \times K$, where $K$ is a compact manifold, need not be a bundle. See [19].

Example 1.2.2, in which a submersion onto $\mathbb{R}$ with fibers all diffeomorphic to $\mathbb{R} \times S^1$ is shown to have no trivial neighborhood of $0 \in \mathbb{R}$.

It is possible to remove the compactness hypothesis for $M$ in the previous theorem in the following way; see [7, 8] for a proof and further discussions.

**Theorem 2.** Let us assume the following:

(i) $F = (F_1, \ldots, F_n) : Z \to \mathbb{R}^n$ is a completely integrable system;

(ii) The fibers of $F$ are connected and mutually diffeomorphic;

(iii) Each Hamiltonian vector field $X_{F_\lambda}$ is complete.

Then:

(I) $F$ is a bundle with typical fiber the cylinder

$$\mathbb{R}^{n-h} \times T^h;$$

moreover there is a covering of $F(Z)$ with simply connected domains $\{V_\alpha\}$ such that over each $V_\alpha$ $F$ is a principal bundle with structure group $[7]$;

(II) Given standard coordinates $(y^\lambda)$ on $[4]$, each fiber of $F$ has a (saturated) neighborhood $U$ provided with Darboux coordinates $(I^\lambda, y^\lambda)$ such that

$$\Omega = dI^\lambda \wedge dy^\lambda$$

and $F_\lambda = F_{\lambda}(I^\mu)$ on $U$.

**Remark 2.** (i) If we remove the completeness hypothesis (iii) in Theorem 2 then the fibers of $F$ can be much more complicated than $\mathbb{R}^{n-h} \times T^h$; see [6] where it is shown that if we take for $F$ a polynomial with two complex variables, say $(z_1, z_2) \in \mathbb{C}^2 \simeq \mathbb{R}^4$, then the functions Re($F$) and Im($F$) are in involution with respect to a certain symplectic structure on $\mathbb{R}^4$. Thus we can have generic Riemann surfaces as fibers of such an $F$.

(ii) If one of the fibers of $F$ is compact, then it must be a torus and by (i) of Remark 3 there is a saturated neighborhood of it made up of such compact fibers; moreover each $X_{F_\lambda}$ is complete on them. Thus all the hypotheses of Theorem 2 are verified and we can see Theorem 1 as a special case of Theorem 2.

3 Partial integrability

Theorem 3 below treats the case of a Hamiltonian system with a number of (independent) functions in involution lesser than the maximum possible. It first appeared in the article of N. N. Nekhoroshev [10] but a more detailed proof of
it, together with discussions and some worked out examples, can be found in [11]. Moreover, Proposition 27.8 of [22] is a related result: it gives a sufficient condition for a foliation with a compact leaf to be a trivial bundle around that leaf.

**Definition 2.** Let \( k \) be an integer, \( 1 \leq k \leq n \). A subset \((F_1, \ldots, F_k)\) of \((C^\infty(Z), \{\})\) is called a partially integrable system on \( Z \) if at every point of \( Z \) \( dF_1, \ldots, dF_k \) are linearly independent and \( \{F_i, F_j\} = 0 \), \( \forall i, j = 1, \ldots, k \).

**Theorem 3.** Let us assume the following:

(i) \( F = (F_1, \ldots, F_k) : Z \rightarrow \mathbb{R}^k \) is a partially integrable system;

(ii) There is a compact leaf \( M \) of the foliation spanned by the Hamiltonian vector fields \( X_{F_1}, \ldots, X_{F_k} \);

(iii) There is a closed curve \( \gamma \) on \( M \) such that \( D\Psi_\gamma \) does not have 1 as an eigenvalue, where \( \Psi_\gamma \) is the Poincaré map corresponding to \( \gamma \).

Then:

(I) There exists a submanifold \( N \) of \( Z \), \( N \) of dimension \( 2k \) and \( N \supseteq M \), such that \( F|_N \) is a trivial principal bundle with structure group \( T^n \), and \( M \) is one of its fibers;

(II) Given standard coordinates \((\varphi^\lambda)\) on \( T^k \), \( N \) is provided with Darboux coordinates \((I^\lambda, \varphi^\lambda)\) such that

\[
\Omega = dI^\lambda \wedge d\varphi^\lambda
\]

and \( F_\lambda = F_\lambda(I_{\mu}) \) on \( N \).

It is possible to remove the compactness hypothesis for the leaf \( M \) in the previous theorem in the following way; see [8, 12] for a proof and further discussions.

**Theorem 4.** Let us assume the following:

(i) \( F = (F_1, \ldots, F_k) : Z \rightarrow \mathbb{R}^k \) is a partially integrable system;

(ii) The leaves of the foliation spanned by the Hamiltonian vector fields \( X_{F_1}, \ldots, X_{F_k} \)

are mutually diffeomorphic and are the fibers of a submersion \( F : Z \rightarrow B \);

(iii) Each Hamiltonian vector field \( X_{F_\lambda} \) is complete.

Then:

(I) \( F \) is a bundle with typical fiber the cylinder

\[
\mathbb{R}^{k-h} \times T^h;
\]

moreover there is a covering of \( F(Z) \) with simply connected domains \( \{V_\alpha\} \) such that over each \( V_\alpha \) \( F \) is a principal bundle with structure group \( \mathbb{Z} \);
Given standard coordinates \((y^\lambda)\) on \(\mathcal{F}\), each fiber of \(\mathcal{F}\) has a (saturated) neighborhood \(U\) provided with Darboux coordinates \((I^\lambda, p_A, q^A, y^\lambda)\) such that
\[
\Omega = dI^\lambda \wedge dy^\lambda + dp_A \wedge dq^A
\]
and \(F_\lambda = F_\lambda(I_{\mu})\) on \(U\).

Remark 3. (i) If \(k = n\) this theorem reduces to Theorem \(\mathcal{F}\) about complete integrability.

(ii) The Hamiltonian vector fields \(\{X^I_{\lambda} = \frac{\partial}{\partial y^\lambda}\}\) are obviously complete, but the Hamiltonian vector fields \(\{X_{p_A} = \frac{\partial}{\partial q^A}\}\) and \(\{X_{q^A} = -\frac{\partial}{\partial p_A}\}\) may fail to be complete.

If \(\{X_{p_A}\}\) are complete, then the functions \((I^\lambda, p_A)\) verifies the hypotheses of Theorem \(\mathcal{F}\) about complete integrability on \(U\). Moreover if also \(\{X_{q^A}\}\) are complete, then the functions \((I^\lambda, p_A, q^A)\) verifies the hypotheses of Theorem \(\mathcal{F}\) below about noncommutative integrability on \(U\).

This is indeed more than it is needed; see part (ii) of Remark \(\mathcal{F}\).

4 Noncommutative integrability

In the previous sections we considered subsets of \((C^\infty(Z), \{}\) of at most \(n\) (independent) functions which commute, i.e. generate a commutative Lie algebra.

Now we want to treat the case of a number of functions greater than \(n\); of course in this case they cannot be mutually commuting (if they are independent).

The following theorem goes in this direction; it first appeared in the the article of A. S. Mishchenko and A. T. Fomenko \(\mathcal{F}\) in the special case in which the functions \(F_1, \ldots, F_k\) generate a certain finite dimensional Lie algebra, in general not commutative (here \(n \leq k < 2n\)). Theorem \(\mathcal{F}\) in the present form (which does not need this restriction) can be found in \(\mathcal{F}\).

Theorem 5. Let us assume the following:

(i) \(F = (F_1, \ldots, F_k) : Z \to \mathbb{R}^k\) is a submersion with compact and connected fibers (here \(n \leq k < 2n\));

(ii) \(\{F_\lambda, F_\mu\} = s_{\lambda\mu}(F)\) with the \(k \times k\) matrix \((s_{\lambda\mu})\) of constant rank \(2(k - n)\).

Then:

(I) \(F\) is a bundle with typical fiber the torus \(\mathbb{T}^{2n-k}\); moreover there is a covering of \(F(Z)\) with simply connected domains \(\{V_\alpha\}\) such that over each \(V_\alpha\) \(F\) is a principal bundle with structure group \(\mathbb{T}^{2n-k}\);

(II) Given standard coordinates \((\varphi^i)\) on \(\mathbb{T}^{2n-k}\), each fiber of \(F\) has a (saturated) neighborhood \(U\) provided with Darboux coordinates \((I_i, p_A, q^A, \varphi^i)\) such that
\[
\Omega = dI_i \wedge d\varphi^i + dp_A \wedge dq^A
\]
and \( F_\lambda = F_\lambda(I_i, p_A, q^A) \) on \( U \).

Given a Hamiltonian function \( \mathcal{H} \) of the system, it depends only on coordinates \( (I_i) \) on \( U \).

It is possible to remove the compactness hypothesis for the fibers of \( F \) in the previous theorem in the following way; the proof given here is an improvement of the proof that can be found in [9] together with further discussions and a worked out example.

**Theorem 6.** Let us assume the following:

(i) \( F = (F_1, \ldots, F_k) : Z \to \mathbb{R}^k \) is a submersion with connected and mutually diffeomorphic fibers (here \( n \leq k < 2n \));

(ii) \( \{F_\lambda, F_\mu\} = s_{\lambda\mu}(F) \) with the \( k \times k \) matrix \( (s_{\lambda\mu}) \) of constant rank \( 2(k - n) \).

(iii) Each Hamiltonian vector field \( X_{F_\lambda} \) is complete.

Then:

(I) \( F \) is a bundle with typical fiber the cylinder

\[
\mathbb{R}^{2n-k-h} \times \mathbb{T}^h,
\]

moreover there is a covering of \( F(Z) \) with simply connected domains \( \{V_\alpha\} \) such that over each \( V_\alpha \) \( F \) is a principal bundle with structure group \( \mathbb{Z} \).

(II) Given standard coordinates \( (y^i) \) on \( Z \), each fiber of \( F \) has a (saturated) neighborhood \( U \) provided with Darboux coordinates \( (I_i, p_A, q^A, y^i) \) such that

\[
\Omega = dI_i \wedge dy^i + dp_A \wedge dq^A
\]

and \( F_\lambda = F_\lambda(I_i, p_A, q^A) \) on \( U \).

Given a Hamiltonian function \( \mathcal{H} \) of the system, it depends only on coordinates \( (I_i) \) on \( U \).

**Proof.** \( F \) is a submersion, so \( F(Z) \) is an open subset of \( \mathbb{R}^k \) and we can choose standard coordinates \( (x_\lambda) \) on it.

Since \( \forall \lambda, \mu = 1, \ldots k \) we have \( \{F_\lambda, F_\mu\} = s_{\lambda\mu}(F) \), it is possible to provide \( F(Z) \) with a Poisson structure \( \{\} \) whose Poisson tensor is just

\[
s_{\lambda\mu}(x) \frac{\partial}{\partial x_\lambda} \wedge \frac{\partial}{\partial x_\mu}
\]  

(4)

and \( F \) is a Poisson morphism.

Since \( \forall x \in F(Z) \) we have \( \text{rank}(s_{\lambda\mu}(x)) = 2(k - n) \), around each \( x \in F(Z) \) there are \( 2n - k \) independent Casimir functions \( \{C_i\} \) of \( \{\} \) such that the Hamiltonian vector fields \( \{X_{F\cdot C_i}\} \) span the fibers of \( F \). See [24] for these facts, where \( \{\} \) is called coinduced Poisson structure.
Now if we consider the Hamiltonian vector field \( \{ X_{F_\lambda} \} \) instead of the functions \( \{ F_\lambda \} \), from (ii) it follows that

\[
[X_{F_\lambda}, X_{F_\mu}] = -\frac{\partial s_{\lambda \mu}}{\partial x^\alpha} X_{F_\alpha}
\]

i.e. on each fiber of \( F \) they generate a *finite* dimensional Lie algebra; then each linear combination of \( \{ X_{F_\lambda} \} \) is complete; see [17]. In particular, each Hamiltonian vector field of the functions \( \{ F^*C_i \} \)

\[
X_{F^*C_i} = \frac{\partial C_i}{\partial F_\alpha} X_{F_\alpha}
\]

is complete.

Then it is possible to apply the previous Theorem 4 to the partial integrable system on \( Z \) given by \( \{ F^*C_i \} \).

**Remark 4.** (i) If \( k = n \) this theorem reduces to Theorem 2 about complete integrability.

(ii) The Hamiltonian vector fields \( \{ X_{I_i} = \frac{\partial}{\partial y^i} \} \) are obviously complete, thus the functions \( (I_i) \) verifies the hypotheses of Theorem 3 about partial integrability on \( U \) if we take as \( \mathcal{F} \) the projection \((I_i, p_A, q_A, y^i) \mapsto (I_i, p_A, q_A)\).

The Hamiltonian vector fields \( \{ X_{p_A} = \frac{\partial}{\partial q^A} \} \) and \( \{ X_{q^A} = -\frac{\partial}{\partial p_A} \} \) may fail to be complete. If \( \{ X_{p_A} \} \) are complete, then the functions \( (I_i, p_A) \) verifies the hypotheses of Theorem 2 about complete integrability on \( U \).

In any case, the completeness hypothesis (iii) here is used only to guarantee that each field \( X_{F^*C_i} \) in (5) is complete.

## 5 Integrability and global action-angle coordinates

So far all the theorems stated the existence of action-angle coordinates on a (saturated) neighborhood of any invariant manifold. It is natural now to investigate the global picture: for the compact case standard references are the articles of P. Dazord and T. Delzant [2], J. J. Duistermaat [3] and N. N. Nekhoroshev [15].

In what follows there is a possible way to treat the case of noncompact invariant manifolds; Theorem 8, 9 and 10 can be found in [10]; let us note that Lemma 1 and Theorem 11 here establish a basic sufficient condition for a principal bundle with Abelian structure group to be trivial, and permit to simplify the proofs of the theorems contained in [10].

**Lemma 1.** Let \( P \to B \) be a principal bundle and let us assume that its structure group is a product \( G_1 \times G_2 \).

Then it is canonically isomorphic to the product principal bundle \( P_1 \times P_2 \to B \) where \( P_1 \to B \) is the quotient principal bundle \( P/G_2 \to B \) and \( P_2 \to B \) is \( P/G_1 \to B \).
Proof. Using the fact that we have canonical projections of $G_1 \times G_2$ onto its factors $G_1$ and $G_2$, the proof is a straightforward check.

Theorem 7. Let $P \to B$ be a principal bundle and let us assume the following:

(i) The structure group $G$ is Abelian;
(ii) The base $B$ verifies $H^2(B, \mathbb{Z}) = 0$.

Then the principal bundle $P \to B$ is trivial.

Proof. Since the structure group $G$ is Abelian, it is isomorphic to $\mathbb{R}^n$ or to the cylinder $\mathbb{R}^{n-h} \times T^h$, where $n = \dim G$ and $h \geq 1$. In the first case, the result follows immediately from the existence of a global section; see [23].

In the second case, it is possible to reduce the group $\mathbb{R}^{n-h} \times T^h$ to its maximal compact subgroup $T^h$; then, by Lemma [11] it is isomorphic to the product of $h$ copies of principal $U(1)$-bundles over the same base $B$. Since $H^2(B, \mathbb{Z}) = 0$, each of these bundles has trivial first Chern class and consequently it is trivial; see [21]. Thus the original principal bundle is trivial.

The following theorems are global versions of Theorem 4 and 6 about partial and noncommutative integrability.

Theorem 8. Let us assume the following:

(i) $F = (F_1, \ldots, F_k) : Z \to \mathbb{R}^k$ is a partially integrable system;
(ii) The leaves of the foliation spanned by the Hamiltonian vector fields $X_{F_1}, \ldots, X_{F_k}$ are mutually diffeomorphic and are the fibers of a submersion $F : Z \to B$;
(iii) Each Hamiltonian vector field $X_{F_\lambda}$ is complete;
(iv) The base $B$ is simply connected and $H^2(B, \mathbb{Z}) = 0$.

Then:

(I) $F$ is a trivial principal bundle with structure group the cylinder $\mathbb{R}^{k-h} \times T^h$;
(II) Given standard coordinates $(y^\lambda)$ on $\mathbb{R}^{k-h} \times T^h$, $Z$ is provided with coordinates $(I_\lambda, x^A, y^\lambda)$ such that

$$
\Omega = dI_\lambda \wedge dy^\lambda + \Omega_A^\lambda dI_\lambda \wedge dx^A + \Omega_{AB} dx^A \wedge dx^B
$$

and $F_\lambda = F_\lambda(I_\mu)$ on $Z$.

Theorem 9. Let us assume the following:

(i) $F = (F_1, \ldots, F_k) : Z \to \mathbb{R}^k$ is a submersion with connected and mutually diffeomorphic fibers (here $n \leq k < 2n$);
(ii) $\{F_\lambda, F_\mu\} = s_{\lambda\mu}(F)$ with the $k \times k$ matrix $(s_{\lambda\mu})$ of constant rank $2(k - n)$.
(iii) Each Hamiltonian vector field $X_{F_{\lambda}}$ is complete;

(iv) There is an open subset $V$ of $F(Z)$ admitting $2n - k$ independent Casimir functions of the coinduced Poisson structure on $F(Z)$ as in the proof of Theorem 6 which is simply connected and such that $H^2(V, Z) = 0$.

Then:

(I) The restriction $F|V : F^{-1}(V) \to V$ is a trivial principal bundle with structure group the cylinder $\mathbb{R}^{2n-k-h} \times T^h$;

(II) Given standard coordinates $(y^i)$ on $\mathbb{R}^{2n-k-h} \times T^h$, $F^{-1}(V)$ is provided with coordinates $(I_{\lambda}, x^A, y^\lambda)$ such that

$$\Omega = dI_{\lambda} \wedge dy^\lambda + \Omega_{AB} dx^A \wedge dx^B$$

and $F_{\lambda} = F_{\lambda}(I_{\mu}, x^A)$ on $F^{-1}(V)$.

Given a Hamiltonian function $\mathcal{H}$ of the system, it depends only on coordinates $(I_{\lambda})$ on $F^{-1}(V)$.

Remark 5. The proof of Theorem 6 shows that each $x \in F(Z)$ has a neighborhood $V$ admitting $2n - k$ independent Casimir functions; thus the result holds over any such $V$ which is simply connected and such that $H^2(V, Z) = 0$.

The following theorem is a global version of Theorem 2; note that in this case the set of hypotheses is simpler than in the previous cases in that it requires only topological properties on the image $F(Z)$ of the original submersion $F$.

**Theorem 10.** Let us assume the following:

(i) $F = (F_1, \ldots, F_n) : Z \to \mathbb{R}^n$ is a completely integrable system;

(ii) The fibers of $F$ are connected and mutually diffeomorphic;

(iii) Each Hamiltonian vector field $X_{F_{\lambda}}$ is complete;

(iv) $F(Z)$ is simply connected and $H^2(F(Z), Z) = 0$.

Then:

(I) $F$ is a trivial principal bundle with structure group the cylinder $\mathbb{R}^{n-k-h} \times T^h$;

(II) Given standard coordinates $(y^i)$ on $\mathbb{R}^{n-k-h} \times T^h$, $Z$ is provided with Darboux coordinates $(I_{\lambda}, y^\lambda)$ such that

$$\Omega = dI_{\lambda} \wedge dy^\lambda$$

and $F_{\lambda} = F_{\lambda}(I_{\mu})$ on $Z$.

Remark 6. Since in hypothesis (iv) we assume $F(Z)$ to be simply connected, $H^2(F(Z), Z) = 0$ is equivalent to $\pi_2(F(Z)) = 0$, i.e. (iv) can be restated by requiring $F(Z)$ to be 2-connected. The same holds for the base manifold $B$ in Theorem 8 and for the neighborhood $V$ in Theorem 9.
6 Integrability and locally affine structures

In what follows $M$ is a connected and $m$-dimensional manifold; we recall the following two standard definitions.

**Definition 3.** Let $\nabla$ be a linear connection on $M$ and let $R_\nabla$, $T_\nabla$ be its curvature and torsion tensors respectively;

- $\nabla$ is flat when $R_\nabla = 0$;
- $\nabla$ is a locally affine structure on $M$ when $R_\nabla = 0$ and $T_\nabla = 0$.

**Definition 4.** Let $\{X_1, \ldots, X_m\}$ be a parallelization of $M$ and let $X_i, Y$ be vector fields on $M$, $Y = b^j X_j$ with $b^j$ global smooth functions on $M$. The expression

$$\nabla_X Y = X(b^j)X_j$$  \hspace{1cm} (6)

is easily seen to define a linear connection on $M$, called the connection of the parallelization.

The basic properties of the connection (6) are summarized in the following theorem.

**Theorem 11.** Let $\{X_1, \ldots, X_m\}$ be a parallelization of $M$ and let $\nabla$ be as in (6). Then:

(I) $\nabla$ is a flat linear connection;

(II) $T_\nabla(X_i, X_j) = -[X_i, X_j]$;

(III) Each $X_i$ is a geodesic vector fields for $\nabla$, i.e. each integral curve of $X_i$ is a geodesic of $\nabla$;

(IV) Given another parallelization $\{\bar{X}_1, \ldots, \bar{X}_m\}$ of $M$, it defines the same $\nabla$ if and only if $\bar{X}_j = G^i_j X_i \forall j = 1, \ldots, m$, with $(G^i_j)$ some constant invertible matrix.

**Proof.** Statement (I) can be found in [20], but they all follow from direct computations. In particular, to prove (IV) it is sufficient to consider two parallelizations $\{X_i\}$ and $\{\bar{X}_j\}$ of $M$ and observe that there must be a relation $\bar{X}_j = G^i_j(x)X_i$ between the two, with $(G^i_j(x))$ an invertible matrix of smooth functions on $M$. Then the condition that their associated connections $\nabla$ and $\overline{\nabla}$ must be equal leads to the equivalent condition $X_k(G^i_j(x)) = 0 \forall i, j, k = 1, \ldots, m$ and $\forall x \in M$; the facts that $\{X_i\}$ is a basis at every point and that $M$ is connected yields that the matrix $(G^i_j)$ must be constant. \[ \square \]

The following theorem states that when $M$ is simply connected also the converse of Theorem 11 holds.

**Theorem 12.** Let us assume the following:

(i) There is a flat linear connection $\nabla$ on $M$;

(ii) $\nabla$ is a locally affine structure on $M$ when $R_\nabla = 0$ and $T_\nabla = 0$.

Then $M$ is simply connected.

**Proof.** Statement (i) is a direct consequence of the definition.

(ii) $\nabla$ is flat when $R_\nabla = 0$; hence by (I) $\nabla$ is a parallel connection.

By (II) $\nabla$ is a locally affine structure on $M$ when $R_\nabla = 0$ and $T_\nabla = 0$.

By (III) each $X_i$ is a geodesic vector fields for $\nabla$, i.e. each integral curve of $X_i$ is a geodesic of $\nabla$.

Hence $\nabla$ is a flat linear connection.

By (IV) $\nabla$ defines a flat linear connection on $M$.

Therefore $M$ is simply connected.

\[ \square \]
(ii) $M$ is simply connected.

Then there is a parallelization $\{X_1, \ldots, X_m\}$ of $M$ such that its connection is $\nabla$.

Proof. The proof can be found in [20], but also a direct proof can be given. Let us fix a point $x_0 \in M$ and a basis $\{X_1, \ldots, X_m\}$ of $T_{x_0}M$. Since $\nabla$ is flat and $M$ simply connected, it is possible, through parallel transport, to move this basis to any point $x \in M$ getting a basis of $T_xM$ and the result does not depend on the path followed. Thus we get a parallelization of $M$ still indicated with $\{X_1, \ldots, X_m\}$; it can be seen that it is smooth and $\nabla X_i X_j = 0$ $\forall i, j = 1, \ldots, m$. It follows that, given any vector field $Y = b^j X_j$ on $M$ with $b^j$ global smooth functions on $M$, we get for the original connection $\nabla$ the formula $\nabla X_i Y = X_i (b^j X_j)$, which coincides with the formula (6) for the connection of the parallelization $\{X_1, \ldots, X_m\}$. 

Remark 7. Let us consider the following relation on the set of all the parallelizations of a manifold $M$: two parallelizations $\{X_1, \ldots, X_m\}$ and $\{\bar{X}_1, \ldots, \bar{X}_m\}$ are equivalent if and only if there is a constant invertible matrix $(G^j_i)$ such that

$$\bar{X}_j = G^j_i X_i \quad \forall j = 1, \ldots, m. \tag{7}$$

It is clear that (7) is an equivalence relation. From Theorem 11 and 12 it follows that on a simply connected manifold $M$, flat linear connections $\nabla$ are in 1-to-1 correspondence with equivalence classes of (7).

We can improve Theorem 11 and 12 by considering the torsion of $\nabla$.

Theorem 13. Let $\{X_1, \ldots, X_m\}$ be a parallelization of $M$ with commuting vector fields and let $\nabla$ be as in (6); then $\nabla$ is a locally affine structure on $M$.

Conversely, let $\nabla$ be a locally affine structure on $M$ simply connected; then there is a parallelization $\{X_1, \ldots, X_m\}$ of $M$ with commuting vector fields such that its connection is $\nabla$.

Proof. It all follows from Theorem 11 and 12 remembering the formula $T_\nabla(X_i, X_j) = -[X_i, X_j]$. 

Remark 8. (i) Let us consider $\{X_1, \ldots, X_m\}$ a parallelization of $M$ with commuting and complete vector fields: in this case, besides the locally affine structure (6), we can obtain information about $M$. Indeed, the composition $\Phi$ of the flows of the vector fields $\{X_1, \ldots, X_m\}$ define a transitive action of $\mathbb{R}^m$ on $M$. Thus $M$ must be diffeomorphic to $\mathbb{R}^m$ modulo the discrete isotropy group of one of its point, which yields $\mathbb{R}^{m-h} \times T^h$ for some $h \in \{0, \ldots, m\}$. See for instance [15] for this standard argument. This situation occurred in in particular in all the theorems of the preceding sections.

(ii) Of course, also nonorientable manifolds can admit a locally affine structure: examples are the Möbius strip and the Klein bottle; they obviously are not simply connected.
It is known that among all the spheres $S^m$, the parallelizable ones are $S^1$, $S^3$ and $S^7$. Putting together the facts of this section, we obtain the following additional properties; we consider only the case $S^m$ with $m \geq 2$, the case $S^1$ being trivial.

**Theorem 14.** Let us consider the spheres $S^m$, $m \geq 2$.

(I) The only spheres admitting a flat linear connection are $S^3$ and $S^7$;

(II) No sphere $S^m$ admits a locally affine structure.

**Proof.** $S^3$ and $S^7$ are parallelizable, thus they admit a flat linear connection. Moreover, each $S^m$, $m \geq 2$, is simply connected but for $m \neq 3, 7$ they are not parallelizable; thus for $m \neq 3, 7$ they cannot admit a flat linear connection. This proves (I).

According to (I), there remain only $S^3$ and $S^7$ to be checked. Since they are simply connected, if they admitted a locally affine structure they would be parallelizable with commuting vector fields. Since they are compact, these vector fields would be complete and by (i) of Remark 8 they would be diffeomorphic to $T^3$ and $T^7$ respectively, which is impossible because they are not simply connected. This proves (II).

We can also obtain the following version of the Cartan-Hadamard theorem:

**Theorem 15.** Let us assume the following:

(i) There is a locally affine structure $\nabla$ on $M$;

(ii) $\nabla$ is geodesically complete, i.e. all of its geodesics can be defined on $\mathbb{R}$;

(iii) $M$ is simply connected.

Then there is a diffeomorphism $\Psi : M \to \mathbb{R}^m$ such that $\nabla$ corresponds under $\Psi$ to the standard locally affine structure of $\mathbb{R}^m$.

**Proof.** By Theorem 13 $M$ is parallelizable with commuting vector fields, which are complete by (III) of Theorem 11 then, after we fix a starting point $\alpha \in M$, the map $\Phi : \mathbb{R}^m \to M$ of (i) of Remark 8 must define a diffeomorphism because $M$ is simply connected. Moreover a computation shows that if $\Phi : t \mapsto p$ then

$$T_t \Phi : e_i \mapsto X_i(p) \quad \forall i = 1, \ldots, m$$

where $(e_1, \ldots, e_m)$ is the standard basis of $\mathbb{R}^m$. Thus it is enough to take as $\Psi$ the function $\Phi^{-1}$.

**Remark 9.** (i) More generally, we can state the following: no compact and simply connected manifold admits a locally affine structure; the proof is exactly the same as that of part (II) of Theorem 14. In particular, there we implicitly considered the standard differentiable structure of $S^7$, but the result of part (II) holds for all its exotic differentiable structures.
(ii) From Theorem 15 it follows that on $\mathbb{R}^m$, after a diffeomorphism, all the geodesically complete locally affine structures can be brought to the standard one. In particular, since there we implicitly considered the standard differentiable structure for $\mathbb{R}^4$, no exotic differentiable structure on $\mathbb{R}^4$ admits a geodesically complete locally affine structure.

It is well known that locally affine structures arise naturally with certain foliations of a symplectic manifold. We recall only the basic definition and construction; for further discussions we refer to [2, 4, 5, 25] and references therein for the following terminology and construction.

**Definition 5.** Let $\mathcal{F}$ be a foliation of the symplectic manifold $(Z, \Omega)$.

The polar foliation of $\mathcal{F}$ is, if it exists, the foliation $\mathcal{F}^\perp$ of $Z$ with the property that the tangent spaces of its leaves are the symplectic complements of the tangent spaces of the leaves of $\mathcal{F}$.

A foliation $\mathcal{F}$ which admits a polar foliation $\mathcal{F}^\perp$ is called symplectically complete.

If we have a symplectically complete foliation $\mathcal{F}$ of $Z$ with isotropic leaves, then these leaves admit a locally affine structure, which is given by

$$\nabla^\Omega_X Y = \Omega^\sharp L_X \Omega^\flat (Y)$$

where $X$, $Y$ are vector fields tangent to the leaves of $\mathcal{F}$.

**Remark 10.** It is readily observed that the fibers of the following submersions

(i) $F$ of Theorem 2

(ii) $\mathcal{F}$ of Theorem 4

(iii) $F$ of Theorem 6

define foliations of $Z$ which are isotropic and symplectically complete, so they admit the locally affine structure $\nabla^\Omega$ defined in (8).

It is natural to investigate the nature of the connection $\nabla^\Omega$ for the cases listed in Remark 10; the following theorem states that it is nothing else than the connection of appropriate parallelizations.

**Theorem 16.** Let us consider the submersions (i), (ii) and (iii) of Remark 10 and the connection $\nabla^\Omega$ defined on their fibers by (8); $\nabla^\Omega$ coincides with the connection $\nabla$ of the following parallelizations:

- For (i) the parallelization given by the Hamiltonian vector fields $\{X_{F_1}, \ldots, X_{F_n}\}$;
- For (ii) the parallelization given by the Hamiltonian vector fields $\{X_{F_1}, \ldots, X_{F_k}\}$;
- For (iii) the parallelization given by the Hamiltonian vector fields $\{X_{F^\ast C_1}, \ldots, X_{F^\ast C_{2n-k}}\}$ found at the end of the proof of Theorem 4.
Proof. To simplify the notations, let us call generically $X_i$ the Hamiltonian vector field of the function $F_i$; then this proof will adapt to all the three previous statements. Given a vertical vector $Y = b^j X_j$, we have

$$\nabla^\Omega_{X_k} Y = \Omega^\sharp L_{X_k} \Omega^\flat (b^j X_j) = \Omega^\sharp L_{X_k} (b^j dF_j) = \Omega^\sharp [i_{X_k} d(b^j dF_j)] = \Omega^\sharp [i_{X_k} (db^j \wedge dF_j)] = \Omega^\sharp [X_k(b^j) dF_j] = X_k(b^j) X_j = \nabla_{X_i} Y$$

where we have used that in our case $\Omega(X_i, X_j) = \{F_i, F_j\} = 0$. \qed

References

[1] D. Bambusi and G. Gaeta, On persistence of invariant tori and a theorem by Nekhoroshev, *MPEJ* 8 (2002), 1.

[2] P. Dazord and T. Delzant, Le problème général des variables actions-angles, *J. Diff. Geom.* 26 (1987), 223–251.

[3] J. J. Duistermaat, On global action-angle coordinates, *Comm. Pure Appl. Math.* 33 (1980), 687–706.

[4] F. Fassò, Superintegrable Hamiltonian systems: geometry and perturbations, *Acta Appl. Math.* 87 (2005), 93–121.

[5] F. Fassò and T. Ratiu, Compatibility of symplectic structures adapted to noncommutatively integrable systems, *J. Geom. Phys.* 27 (1998), 220.

[6] H. Flaschka, A remark on integrable Hamiltonian systems, *Phys. Lett. A* 131 (1988), 505–508.

[7] E. Fiorani, G. Giachetta and G. Sardanashvily, Geometric quantization of time-dependent completely integrable Hamiltonian systems, *J. Math. Phys.* 43 (2002), 5013–5025.

[8] E. Fiorani, G. Giachetta and G. Sardanashvily, An extension of the Liouville-Arnold theorem for the noncompact case, *Il Nuovo Cimento* B118 (2003), 307–317.

[9] E. Fiorani and G. Sardanashvily, Noncommutative integrability on noncompact invariant manifolds, *J. Phys. A: Math. Gen.* 39 (2006), 14035–14042.

[10] E. Fiorani and G. Sardanashvily, Global action-angle coordinates for completely integrable systems with noncompact invariant submanifolds, *J. Math. Phys.* 48 (2007), 32901.

[11] G. Gaeta, The Poincaré-Lyapunov-Nekhoroshev theorem, *Ann. Phys.* 297 (2002), 157–173.
[12] G. Giachetta, L. Mangiarotti and G. Sardanashvily, Bi-Hamiltonian partially integrable systems, *J. Math. Phys.* **44** (2003), 1984–1997.

[13] G. Meigniez, Submersions, fibrations and bundles, *Trans. Am. Math. Soc.* **354** (2002), 3771–3787.

[14] A. S. Mishchenko and A. T. Fomenko, Generalized Liouville method of integration of Hamiltonian systems, *Funct. Anal. Appl.* **12** (1978), 113–121.

[15] N. N. Nekhoroshev, Action-angle variables and their generalization, *Trans. Moscow Math. Soc.* **26** (1972), 180–198.

[16] N. N. Nekhoroshev, The Poincaré-Lyapunov-Liouville-Arnold theorem, *Funct. Anal. Appl.* **28** (1994), 128–129.

[17] R. S. Palais, A global formulation of the Lie theory of transformation groups, *Mem. Am. Math. Soc.* **22** (1994).

[18] V. I. Arnold (ed.), *Dynamical Systems III* (Springer, Berlin, 1988).

[19] G. Giachetta, L. Mangiarotti and G. Sardanashvily, *New Lagrangian and Hamiltonian Methods in Field Theory* (World Scientific, Singapore, 1997).

[20] W. Greub, S. Halperin and R. Vanstone, *Connections, Curvature and Cohomology (Vol.2)* (Academic Press, New York, 1973).

[21] P. Griffiths and J. Harris, *Principles of Algebraic Geometry* (John Wiley & Sons, New York, 1978).

[22] V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics* (Cambridge University Press, Cambridge, 1984).

[23] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry (Vol. 1)* (Interscience Publishers, New York, 1969).

[24] P. Libermann and C. M. Marle, *Symplectic Geometry and Analytical Mechanics* (Reidel, Dordrecht, 1987).

[25] N.M.J. Woodhouse, *Geometric quantization (2nd ed.)* (Clarendon Press, Oxford, 1994).