The Number of Perfect Matchings in Möbius Ladders and Prisms

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Abstract

The 1970s conjecture of Lovász and Plummer that the number of perfect matchings in any 3-regular graph is exponential in the number of vertices was proved in 2011 by Esperet, Kardoš, King, Král’, and Norine. We give the exact formula for the number of perfect matchings in two families of 3-regular graphs. In the graph consisting of a $2n$-cycle with diametric chords (also known as the Möbius ladder $M_n$ and a Harary graph) and in the cartesian product of the cycle $C_n$ with an edge (called the cycle prism), the number of matchings is the sum of the Fibonacci numbers $F_{n-1}$ and $F_{n+1}$, plus two more for the Möbius ladder when $n$ is odd and for the cycle prism when $n$ is even.

1 Introduction

A matching in a graph is a set of edges such that every vertex belongs to at most one edge in the set. A matching is perfect if every vertex belongs to exactly one of its edges. The counting of perfect matchings in a graph is a fundamental and challenging problem in graph theory and applications. For bipartite graphs, it is a basic #P-complete problem (Valiant [15]). For algorithmic and applied aspects, see [7] and [11].

Petersen [12] showed that every 3-regular graph without cut-edges (also called “bridges”) has at least one perfect matching (a cut-edge is an edge whose deletion increases the number of components of the graph). Lovász and Plummer conjectured in the 1970s that the minimum number of perfect matchings in 3-regular bridgeless $n$-vertex graphs grows exponentially with $n$; this later appeared as Conjecture 8.1.8 in their extensive book [10] on matchings. After various partial results on the conjecture, including [2, 4, 6, 9, 14, 16], it was finally proved in full by Esperet, Kardoš, King, Král’, and Norine [3], published in 2011.

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Theorem 1 ([3]). Every 3-regular bridgless loopless multigraph with \( n \) vertices has at least \( 2^{n/3656} \) perfect matchings.

Plesnık [13] proved that if deleting any \( k-2 \) edges of a \( k \)-regular \( n \)-vertex loopless multigraph with even order leaves a connected subgraph, then every subgraph obtained by deleting at most \( k-1 \) edges has a perfect matching. In particular, every edge belongs to a perfect matching. The result in [3] builds on this by showing that either every edge lies in at least \( 2^{n/3656} \) perfect matchings, or there are two matchings whose symmetric difference has at least \( n/3656 \) components (thus yielding \( 2^{n/3656} \) perfect matchings in the graph).

Since \( 2^{1/3656} \) is a rather small base for the exponential, there remains interest in obtaining stronger guarantees or exact formulas for the number of perfect matchings in special families of 3-regular graphs. Our purpose in this note is to determine the exact number of perfect matchings in two families of graphs that have been of interest in other contexts. To our knowledge, these are the first infinite families of 3-regular bridgeless graphs whose perfect matching have been counted.

We consider first the Möbius ladder \( M_n \), consisting of a \( 2n \)-cycle plus chords joining opposite vertices. Very similar arguments handle the prism \( C'_n \), consisting of two \( n \)-cycles with edges added joining corresponding vertices. Let \( F_n \) denote the \( n \)th Fibonacci number, defined by \( F_1 = F_2 = 1 \) and \( F_n = F_{n-1} + F_{n-2} \) for \( n \geq 2 \). For \( M_n \) and \( C'_n \), the number of perfect matchings is \( F_{n-1} + F_{n+1} \), except that two additional special matchings occur in \( M_n \) when \( n \) is odd and in \( C'_n \) when \( n \) is even.

It is well-known that \( F_n/F_{n-1} \to (1 + \sqrt{5})/2 \). Thus our formula implies that the number of perfect matchings in \( M_n \) and in \( C'_n \) grows exponentially with base about 1.618. The base \( 2^{1/3656} \) for the general lower bound from [3] is about 1.0001896. Our results suggest that there may be a lower bound for the number of perfect matchings in 3-regular bridgeless graphs that is exponential with a larger base than that known. They also show that the guaranteed base cannot be larger than 1.618.

It would be interesting to extend this study to the generalized Petersen graph \( G_{k,n} \) with \( n > 2k \). The vertices consist of the outer vertices \( u_0, \ldots, u_{n-1} \) and the inner vertices \( v_0, \ldots, v_{n-1} \), viewing the index set as \( \mathbb{Z}_n \). The edges consist of the cycle \( u_0, \ldots, u_{n-1} \) on the outer vertices, the edges \( u_iv_i \) for \( i \in \mathbb{Z}_n \), and all edges of the form \( v_iv_{i+k} \) on the inner vertices. The graph is 3-regular when \( n > 2k \). The prism \( C'_n \) is the special case \( G_{1,n} \), and the famous Petersen graph is \( G_{2,5} \). Conceivably the techniques here can be used to attack \( G_{k,n} \), but already when \( k = 2 \) there seem to be many types of perfect matchings to count.

2 The Results

The Möbius ladder \( M_n \) has that name because it is obtained from the “ladder graph” \( P_n \square K_2 \) (the cartesian product of the \( n \)-vertex path \( P_n \) and the complete graph \( K_2 \)) by adding two
edges making the end of each copy of \( P_n \) adjacent to the beginning of the other copy. This has the effect of “twisting” the strip formed by the ladder into a Möbius band. Note that since \( M_n \) is 2n-cycle with \( n \) pairwise crossing chords, it is nonplanar when \( n \geq 3 \). The special case \( M_3 \) is known as the Wagner graph.

One reason \( M_n \) is of interest is that it is the case \( k = 3 \) of the Harary graph \( H_{k,2n} \). A graph is \( k \)-edge-connected if deletion of any \( k-1 \) edges leaves a spanning connected subgraph. Since minimum degree at least \( k \) is needed, such a graph with \( p \) vertices must have at least \([kp/2]\) edges. Harary [5] introduced \( H_{k,p} \) to achieve equality. When \( kp \) is even, \( H_{k,p} \) consists of \( p \) vertices on a circle, with each vertex adjacent to the \( k \) closest vertices in each direction, plus the edges joining diametrically opposite vertices when \( k \) is odd (when \( kp \) is odd the construction is somewhat different).

As stated in the introduction, the solution to our problem involves the Fibonacci numbers. A thorough discussion of this sequence appears in [8]. It has long been known (see [1], for example), that \( F_{n+1} \) is the number of lists from the alphabet \( \{1,2\} \) that sum to \( n \). We use this model to count the perfect matchings in \( M_n \).

**Theorem 2.** The 2n-vertex Möbius ladder \( M_n \) has exactly \( F_{n-1} + F_{n+1} + \epsilon \) perfect matchings, where \( \epsilon \) is 0 for even \( n \) and 2 for odd \( n \).

**Proof.** View the vertex set as \( \{u_i: i \in \mathbb{Z}_{2n}\} \), with the neighbors of \( u_i \) being \( \{u_{i-1}, u_{i+1}, u_{i+n}\} \). Call \( u_{i+n} \) the antipode of \( i \). In a perfect matching, \( u_i \) is matched to its antipode or to a neighbor along the cycle.

First suppose that for some \( i \), exactly one of \( \{u_i u_{i+1}, u_{i+n} u_{i+n+1}\} \) is in the matching. We claim that this holds for all \( i \). If not, then for some \( i \) the matching contains exactly one of \( \{u_i u_{i+1}, u_{i+n} u_{i+n+1}\} \) but both or neither of \( \{u_{i-1} u_i, u_{i+n-1} u_{i+n}\} \). If both, then \( u_i \) or \( u_{i+n} \) is covered twice by the matching. If neither, then \( u_i \) or \( u_{i+n} \) is not covered by the matching.

When \( n \) is odd, there are two matchings where all pairs of opposite cycle edges disagree. When \( n \) is even there are none. This case accounts for the \( \epsilon \) in the theorem statement.

We may henceforth assume that an edge \( u_i u_{i+1} \) is in the matching if and only if \( u_{i+n} u_{i+n+1} \) is in the matching. As we move along one half of the cycle increasing vertices by 1, a vertex is matched to its antipode (covering one vertex on the cycle) or to a neighbor on the path (covering two vertices on the cycle), and the same pattern occurs for the antipodal vertices.

We apply this observation after grouping the remaining matchings by which edge contains \( u_0 \). Those using \( u_0 u_n \) correspond to 1,2-lists with sum \( n-1 \), covering \( u_1 \) through \( u_{n-1} \). Thus the number of perfect matchings using \( u_0 u_n \) is \( F_n \).

If \( u_0 u_1 \) is used, then also \( u_n u_{n+1} \) is used, and the completions of the matching correspond to the 1,2-lists with sum \( n-2 \). The same number use \( u_0 u_{2n-1} \), by reflection. Thus the number of perfect matchings is \( F_n + 2F_{n-1} + \epsilon \), which equals \( F_{n+1} + F_{n-1} + \epsilon \). \( \square \)
Theorem 3. The $2n$-vertex prism $C'_n$ has exactly $F_{n-1} + F_{n+1} + \epsilon$ perfect matchings, where $\epsilon$ is 0 for odd $n$ and 2 for even $n$.

Proof. The vertex set consists of $u_i$ and $v_i$ for $i \in \mathbb{Z}_n$. The edges are $u_iu_{i+1}$, $u_iv_i$, and $v_iv_{i+1}$ for all $i$. As in Theorem 2 if for any $i$ a matching contains exactly one of $u_iu_{i+1}$ and $v_iv_{i+1}$, then this holds for all $i$, since it cannot hold for $i$ and fail for $i-1$.

If a perfect matching uses exactly one of $\{u_iu_{i+1}, v_iv_{i+1}\}$ for all $i$, then no edge $u_jv_j$ is used, and the two $n$-cycles are covered separately. There are two such perfect matchings when $n$ is even, none when $n$ is odd. This accounts for the $\epsilon$ in the theorem statement.

We may henceforth assume that $u_iu_{i+1}$ is used if and only if $v_iv_{i+1}$ is used, for all $i$. Along increasing indices, an edge $u_iv_i$ covers one index, and a pair $\{u_iu_{i+1}, v_iv_{i+1}\}$ covers two. Grouping the remaining matchings by which edge contains vertex $u_0$, those containing the edge $u_0v_0$ correspond to 1,2-lists with sum $n-1$, those containing $u_0u_1$ or $u_0u_{n-1}$ both correspond to 1,2-lists with sum $n-2$. Hence the number of perfect matchings is again $F_n + 2F_{n-1} + \epsilon$, which equals $F_{n+1} + F_{n-1} + \epsilon$. □

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