Variational characterizations of weighted Hardy spaces and weighted \textit{BMO} spaces

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This paper obtains new characterizations of weighted Hardy spaces and certain weighted \textit{BMO} type spaces via the boundedness of variation operators associated with approximate identities and their commutators, respectively.

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1. Introduction and main results

Variational inequalities originated from the intention of improving the well-known Doob maximal inequality. Relied upon the work of Lépingle [26], Bourgain [5] obtained the corresponding variational estimates for the Birkhoff ergodic averages and pointwise convergence results. This work has set up a new research subject in harmonic analysis and ergodic theory. Afterwards, the study of variational inequalities has been spilled over into harmonic analysis, probability and ergodic theory. Particularly, the classical work of \(\rho\)-variation operators for singular integrals was given in [7], in which the authors obtained the \(L^p\)-bounds and weak type (1,1) bounds for \(\rho\)-variation operators of truncated Hilbert transform if \(\rho > 2\), and then extended to higher dimensional cases in [8]. For further studies, we refer readers to [9, 14, 17, 33, 34, 42], etc., for variation operators of singular integrals with rough kernels and weighted cases, [3, 10, 22, 31, 41, 42], etc., for variation operators of commutators.

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Here, we will focus on the variation operators associated with approximate identities. For the special case, the variation operators associated with heat and Poisson semigroups, Jones and Reinhold [25] and Crescimbeni et al. [12] independently established the $L^p$-bounds and weak type $(1,1)$ bounds with different approaches. Recently, Liu [32] generalized the results in [12, 25] to the variation operators associated to approximate identities and obtained a variational characterization of Hardy spaces. In this paper, one of our main purposes is to extend the results in [32] to the weighted cases, and give a new characterization of weighted Hardy spaces via variation inequalities associated with approximate identities. Meanwhile, we will also consider the weighted variation inequalities associated with commutators of approximate identities and aim to provide new characterizations of certain weighted $BMO$ spaces. Before stating our results, we first recall some relevant notation and definitions.

Given a family of complex numbers $a := \{a_t\}_{t \in I}$ with $I \subset (0, +\infty)$. For $\rho > 1$, the $\rho$-variation of $a$ is given by

$$\|a\|_{V_\rho} := \sup \left( \sum_{k \geq 1} |a_{t_k} - a_{t_{k+1}}|^\rho \right)^{1/\rho},$$

where the supremum is taken over all finite decreasing sequences $\{t_k\}$ in $I$. From the definition of $\rho$-variation, it is easy to check that

$$\sup_{t \in I} |a_t| \leq |a_{t_0}| + \|a\|_{V_\rho} \quad (1.1)$$

holds for arbitrary $t_0 \in I$.

Let $F := \{F_t\}_{t > 0}$ be a family of operators. Then the $\rho$-variation of the family $F$ is defined as

$$V_\rho(Ff)(x) := \|\{F_t f(x)\}_{t > 0}\|_{V_\rho}.$$ 

In particular, let $\phi \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \phi(x) \, dx = 1$, where $\mathcal{S}(\mathbb{R}^n)$ is the space of Schwartz functions. We consider the following family of operators

$$\Phi \ast f(x) := \{\phi_t \ast f(x)\}_{t > 0}, \quad (1.2)$$

where $\phi_t(x) := t^{-n} \phi(x/t)$. Then the $\rho$-variation of families $\Phi \ast f$ is defined by

$$V_\rho(\Phi \ast f)(x) = \sup_{\{t_k\} \downarrow 0} \left( \sum_{k \geq 1} |\phi_{t_k} \ast f(x) - \phi_{t_{k+1}} \ast f(x)|^\rho \right)^{1/\rho}. \quad (1.3)$$

It is well-known that the $L^p$-boundedness of $V_\rho(\Phi \ast f)$ implies the almost everywhere convergence of $\{f \ast \phi_t\}_{t > 0}$ as $t \to 0^+$ for every $f \in L^p(\mathbb{R}^n)$, without knowing the almost everywhere convergence property for $f$ in some dense subset in $L^p(\mathbb{R}^n)$, see [24, p. 90]. For $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\phi$ being as above, we define $(\Phi \ast f)_b$, the family
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1.1. Let \( \phi \in S(\mathbb{R}^n) \) with \( \int_{\mathbb{R}^n} \phi(x) \, dx = 1 \). For \( f \in S'(\mathbb{R}^n) \), define the maximal function \( M_\phi \) by

\[
M_\phi f(x) := \sup_{t > 0} |f \ast \phi_t(x)|.
\]

Then for \( \omega \in A_\infty \), the Muckenhoupt class, and \( 0 < p < \infty \), define the weighted Hardy spaces \( H^p(\omega) \) by

\[
H^p(\omega) := \{ f \in S'(\mathbb{R}^n) : M_\phi f \in L^p(\omega) \}
\]

with the quasi-norm \( \|f\|_{H^p(\omega)} := \|M_\phi f\|_{L^p(\omega)} \). When \( \omega \equiv 1 \), we denote \( H^p(\omega) \) by \( H^p(\mathbb{R}^n) \). It is well-known that the space \( H^p(\omega) \) is independent of the choice of \( \phi \) and \( H^p(\omega) = L^p(\omega) \) for \( p > 1 \) and \( \omega \in A_p \).

In [32], Liu showed that \( \mathcal{V}_\rho(\Phi \ast f) \) is bounded from the Hardy space \( H^p(\mathbb{R}^n) \) to \( L^p(\mathbb{R}^n) \), and obtained a characterization of \( H^p(\mathbb{R}^n) \) via \( \mathcal{V}_\rho(\Phi \ast f) \). Now we formulate our first main result, which is the weighted version of the corresponding result in [32], as follows.

**Theorem 1.1.** Let \( \phi \in S(\mathbb{R}^n) \) with \( \int_{\mathbb{R}^n} \phi(x) \, dx = 1 \), \( \rho > 2 \). Then,

(i) when \( n/(n+1) < p \leq 1 \) and \( \omega \in A_p(n+1)/n \), for \( f \in S'(\mathbb{R}^n) \) and any \( t > 0 \), \( f \in H^p(\omega) \) if and only if \( \phi_t \ast f \in L^p(\omega) \) and \( \mathcal{V}_\rho(\Phi \ast f) \in L^p(\omega) \), with

\[
\|\phi_t \ast f\|_{L^p(\omega)} + \|\mathcal{V}_\rho(\Phi \ast f)\|_{L^p(\omega)} \\
\lesssim \|f\|_{H^p(\omega)} \lesssim \|\phi_t \ast f\|_{L^p(\omega)} + \|\mathcal{V}_\rho(\Phi \ast f)\|_{L^p(\omega)};
\]
(ii) when $1 < p < \infty$ and $\omega \in A_p$,
\[
\|\mathcal{V}_{p}(\Phi \ast f)\|_{L^p(\omega)} \leq \left[ \omega \right]_{A_p}^{\max\{1, \frac{1}{p-1}\}} \|f\|_{L^p(\omega)}, \quad \forall f \in L^p(\omega); \tag{1.6}
\]
and for any $t > 0$, a function $f \in H^p(\omega)$ if and only if $\phi_t \ast f \in L^p(\omega)$ and $\mathcal{V}_p(\Phi \ast f) \in L^p(\omega)$, with
\[
\|\phi_t \ast f\|_{L^p(\omega)} + \|\mathcal{V}_p(\Phi \ast f)\|_{L^p(\omega)} 
\leq \|f\|_{H^p(\omega)} 
\leq \|\phi_t \ast f\|_{L^p(\omega)} + \|\mathcal{V}_p(\Phi \ast f)\|_{L^p(\omega)}.
\]
Here the implicit constants are independent of $f$ and $t$.

On the other hand, recalling the commutators
\[
[b, T]f(x) := b(x)Tf(x) - T(bf)(x),
\]
where $b$ is a given locally integrable function and $T$ is a linear operator, Coifman, Rochberg and Weiss [11] showed that $b \in BMO(\mathbb{R}^n)$ if and only if $[b, R_j]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, where $R_j$ is the $j$-th Riesz transform on $\mathbb{R}^n$, $j = 1, \ldots, n$. For $n = 1$, Bloom [4] extended the above result to the following weighted case:
\[
\|[b, H]\|_{L^p(\mu) - L^p(\lambda)} \simeq \|b\|_{BMO_{\nu}(\mathbb{R})} \quad \text{for} \quad 1 < p < \infty, \mu, \lambda \in A_p,
\]
where $H$ is the Hilbert transform, $\nu = (\mu/\lambda)^{1/p}$, and $BMO_{\nu}(\mathbb{R})$ is the following weighted $BMO$ space defined by
\[
BMO_{\nu}(\mathbb{R}) := \{b \in L^1_{\text{loc}}(\mathbb{R}) : \|b\|_{BMO_{\nu}(\mathbb{R})}
:= \sup_{B \subset \mathbb{R}^n} \frac{1}{\nu(B)} \int_B |b(y) - \langle b \rangle_B| \, dy < \infty\}
\]
Here the supremum is taken over all balls, $\langle b \rangle_B := |B|^{-1} \int_B b(y) \, dy$. Subsequently, a lot of attention has been paid to this topic. We refer to [1, 2, 18–21, 29] and therein references for recent works.

In addition, it is well known that when $b \in BMO(\mathbb{R}^n)$, and $T$ is a singular integral, $[b, T]$ may not map $H^1(\mathbb{R}^n)$ boundedly into $L^1(\mathbb{R}^n)$ (see [38, remark]). To investigate the $H^1 - L^1$ bound of $[b, T]$, Liang, Ky and Yang [30] introduced the following weighted $BMO$ type space
\[
BMO_\omega(\mathbb{R}^n) := \{b \in L^1_{\text{loc}}(\mathbb{R}) : \|b\|_{BMO_\omega(\mathbb{R}^n)} < \infty\}
\]
where
\[
\|b\|_{BMO_\omega(\mathbb{R}^n)} := \sup_{B \subset \mathbb{R}^n} \frac{1}{\omega(B)} \int_{B^c} \frac{\omega(x)}{|x - \langle b \rangle_B|^n} \, dx \int_{B} |b(y) - \langle b \rangle_B| \, dy
\]
for $\omega \in A_\infty$ and $\int_{\mathbb{R}^n} \frac{\omega(x)}{1 + |x|^n} \, dx < \infty$, the supremum is taken over all balls $B := B(x_B, r)$. It is clear that $BMO_\omega(\mathbb{R}^n) \subseteq BMO(\mathbb{R}^n)$ (see [30]). And in [30], the authors showed that for $\delta$-Calderón-Zygmund operator $T$ and $\omega \in A_{(n+\delta)/n}$, $b \in BMO_\omega(\mathbb{R}^n)$ if and only if $[b, T]$ is bounded from $H^1(\omega)$ to $L^1(\omega)$. 
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Inspired by the results above, it is natural to ask whether the corresponding characterizations can be given via variation operators of commutators associated with approximate identities. Our next theorems will give a positive answer to this question.

**Theorem 1.2.** Let $φ ∈ \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} φ(x) \, dx = 1$, $1 < p < ∞$, $μ, λ ∈ A_p$ and $ν = (μλ^{-1})^{1/p}$. If $φ$ or $b$ is a real-valued function, then for $ρ > 2$, the following statements are equivalent:

1. $f \mapsto V_ρ((Φ ⋆ f)_b)$ is bounded from $L^p(μ)$ to $L^p(λ)$;
2. $b ∈ BMO_ν(\mathbb{R}^n)$.

Moreover, we have the estimate

$$\|V_ρ((Φ ⋆ f)_b)\|_{L^p(λ)} \lesssim \|b\|_{BMO_ν(\mathbb{R}^n)} (|μ|_A_p |λ|_A_p)^{\max\{1, \frac{1}{p-1}\}} \|f\|_{L^p(μ)}.$$

**Theorem 1.3.** Let $φ ∈ \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} φ(x) \, dx = 1$, $ρ > 2$. Assume that $b ∈ BMO(\mathbb{R}^n)$, $ω ∈ A_1$ with $\int_{\mathbb{R}^n} ω(x) \frac{1+|x|^n}{dx} \, dx < ∞$. Then the following statements are equivalent:

1. $V_ρ((Φ ⋆ f)_b)$ is bounded from $H^1(ω)$ to $L^1(ω)$.
2. $b ∈ BMO_ω(\mathbb{R}^n)$.

**Remark 1.4.** Consider the heat semigroup $W := \{e^{tΔ}\}_{t>0}$ and the Poisson semigroup $P := \{e^{-t\sqrt{-Δ}}\}_{t>0}$ associated to $Δ = \sum_{i=1}^n \frac{∂^2}{∂x^2_i}$. Since the heat kernels $W_t(x) := (πt)^{-n/2} e^{-|x|^2/t}$ belongs to $\mathcal{S}(\mathbb{R}^n)$ and satisfies $\int_{\mathbb{R}^n} W_t(x) \, dx = 1$, so theorems 1.1–1.3 hold for the variation operators associated with $W$ and their commutators. Similarly, the same conclusions are true for the the variation operators associated with $P$ and their commutators.

The rest of the paper is organized as follows. After providing the weighted estimates of variation operators in §2, we will prove theorem 1.1 in §3. The proofs of theorems 1.2 and 1.3 will be given in §4 and 5, respectively.

We end this section by making some conventions. Denote $f \lesssim g$, $f \sim g$ if $f ≤ CG$ and $f ≤ g ≤ f$, respectively. For any ball $B := B(x_0, r) \subset \mathbb{R}^n$, $(f)_B$ means the mean value of $f$ over $B$, $χ_B$ represents the characteristic function of $B$, $\int_B ω(y)dy$ is denoted by $ω(B)$. For $a ∈ \mathbb{R}$, $[a]$ is the largest integer no more than $a$.

2. The weighted estimate of variation operators

In this section, we establish the weighted estimate of variation operators associated with approximations to the identity, which is useful in the proof of theorem 1.1. We begin with the following definition of $A_p$ weights.
A weight $\omega$ is a non-negative locally integrable function on $\mathbb{R}^n$. Let $1 < p < \infty$. We say that $\omega \in A_p$ if there exists a positive constant $C$ such that

$$[\omega]_{A_p} := \sup_Q \left( \frac{1}{|Q|} \int_Q \omega(y) \, dy \right) \left( \frac{1}{|Q|} \int_Q \omega(y)^{1-p'} \, dy \right)^{p-1} \leq C,$$

where $1/p + 1/p' = 1$ and the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. We recall that if $\omega \in A_p$, then for any $\lambda > 1$ and all balls $B$, we have $\omega(\lambda B) \leq c_n \lambda^{np} [\omega]_{A_p} \omega(B)$. When $p = 1$, we say that $\omega \in A_1$ if

$$[\omega]_{A_1} := \left\| \frac{M \omega}{\omega} \right\|_{L^{\infty}(\mathbb{R}^n)} < \infty,$$

where $M$ is the Hardy-Littlewood maximal operator. For $\omega \in A_\infty := \bigcup_{1 \leq p < \infty} A_p$, the $A_\infty$ constant is given by

$$[\omega]_{A_\infty} := \sup_{Q \subset \mathbb{R}^n} \frac{1}{\omega(Q)} \int_Q M(\chi_Q \omega)(x) \, dx.$$

Now we recall the definitions of dyadic lattice, sparse family and sparse operator; see, for example, [27, 28, 39]. Given a cube $Q \subset \mathbb{R}^n$, let $\mathcal{D}(Q)$ be the set of cubes obtained by repeatedly subdividing $Q$ and its descendants into $2^n$ congruent subcubes.

**Definition 2.1.** A collection of cubes $\mathcal{D}$ is called a dyadic lattice if it satisfies the following properties:

1. If $Q \in \mathcal{D}$, then every child of $Q$ is also in $\mathcal{D}$;
2. For every two cubes $Q_1, Q_2 \in \mathcal{D}$, there is a common ancestor $Q \in \mathcal{D}$ such that $Q_1, Q_2 \in \mathcal{D}(Q)$;
3. For any compact set $K \subset \mathbb{R}^n$, there is a cube $Q \in \mathcal{D}$ such that $K \subset Q$.

**Definition 2.2.** A subset $\mathcal{S} \subset \mathcal{D}$ is called an $\eta$-sparse family with $\eta \in (0, 1)$ if for every cube $Q \in \mathcal{S}$, there is a measurable subset $E_Q \subset Q$ such that $\eta |Q| \leq |E_Q|$, and the sets $\{E_Q\}_{Q \in \mathcal{S}}$ are mutually disjoint.

Let $\mathcal{S}$ be a sparse family. Define the sparse operator $T_\mathcal{S}$ by

$$T_\mathcal{S}f(x) := \sum_{Q \in \mathcal{S}} \langle |f| \rangle_Q \chi_Q(x). \quad (2.1)$$

Then for $\omega \in A_p$ with $1 < p < \infty$

$$\|T_\mathcal{S}f\|_{L^p(\omega)} \lesssim [\omega]_{A_p}^{\max\{1, \frac{1}{p'}\}} \|f\|_{L^p(\omega)}; \quad (2.2)$$

see [13] for $p = 2$ and [27, 35] for $p > 1$. Moreover, for $\omega \in A_1$ and $\alpha > 0$,

$$\alpha \omega(\{x \in \mathbb{R}^n : T_\mathcal{S}f(x) > \alpha\}) \lesssim [\omega]_{A_1} \log(e + [\omega]_{A_\infty}) \|f\|_{L^1(\omega)}; \quad (2.3)$$

see [15, 23].
We now give the following pointwise estimate of variation operators in terms of sparse operators.

**Lemma 2.3.** Let \( \rho > 2 \). Then for each \( f \in L^\infty_c(\mathbb{R}^n) \), there exist \( 3^n \) dyadic lattices \( D_j \) and sparse families \( S_j \subset D_j \) such that for a.e. \( x \in \mathbb{R}^n \),

\[
V_\rho(\Phi \ast f)(x) \lesssim 3^n \sum_{j=1}^{3^n} T_{S_j}(f)(x).
\]

To prove lemma 2.3, we introduce the maximal function \( M_{V_\rho(\Phi)} \) by

\[
M_{V_\rho(\Phi)}f(x) := \sup_{Q \ni x} \text{ess sup}_{\xi \in Q} V_\rho(\Phi \ast (f \chi_{\mathbb{R}^n \setminus 3Q}))(\xi).
\]

Since \( V_\rho(\Phi \ast f) \) is of weak type \((1,1)\) (see [32, theorem 2.6]), then arguing as in [27, theorem 3.1] (see also [27, remark 4.3]), lemma 2.3 readily follows if \( M_{V_\rho(\Phi)} \) is of weak type \((1,1)\). Therefore, we need only to settle the following result.

**Lemma 2.4.** For \( \rho > 2 \), \( M_{V_\rho(\Phi)} \) is bounded from \( L^1(\mathbb{R}^n) \) to \( L^{1,\infty}(\mathbb{R}^n) \).

**Proof.** For any \( f \in L^1(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \), we first show that for any cube \( Q \ni x \),

\[
V_\rho(\Phi \ast (f \chi_{\mathbb{R}^n \setminus 3Q}))(\xi) \lesssim Mf(x) + M_{1/2}(V_\rho(\Phi \ast f))(x) \quad \text{for a.e. } \xi \in Q, \tag{2.4}
\]

where \( M_{1/2}(f) := (M(|f|^{1/2}))^2 \), and the implicit constant is independent of \( f, x \) and \( \xi \).

Indeed, for any cube \( Q \ni x \), a.e. \( \xi \in Q \) and \( z \in Q \), we write

\[
V_\rho(\Phi \ast (f \chi_{\mathbb{R}^n \setminus 3Q}))(\xi) \leq \sup_{\{t_k\} \ni 0} \left( \sum_k \int_{\mathbb{R}^n \setminus 3Q} [(\phi_{t_k}(\xi - y) - \phi_{t_{k+1}}(\xi - y))
\]

\[
- (\phi_{t_k}(z - y) - \phi_{t_{k+1}}(z - y))f(y) \ dy \right)^{1/\rho}
\]

\[
+ V_\rho(\Phi \ast (f \chi_{3Q}))(z) + V_\rho(\Phi \ast f)(z)
\]

\[
=: J(\xi,z) + V_\rho(\Phi \ast (f \chi_{3Q}))(z) + V_\rho(\Phi \ast f)(z).
\]

By the Minkowski inequality and mean value theorem, we see that for given \( \xi, z \in Q \), \( y \in \mathbb{R}^n \setminus 3Q \) and some \( \theta \in (0,1) \),

\[
\|\{\phi_t(\xi - y) - \phi_t(z - y)\}_{t > 0}\|_{V_\rho}
\]

\[
\leq \sup_{\{t_k\} \ni 0} \left( \sum_k \int_{t_{k+1}}^{t_k} \frac{\partial}{\partial t}(\phi_t(\xi - y) - \phi_t(z - y)) \ dt \right)
\]

\[
\leq \int_0^\infty \left| \frac{\partial}{\partial t}(\phi_t(\xi - y) - \phi_t(z - y)) \right| \ dt.
\]
\[
\mathcal{V}_\rho(\Phi \ast f)(\xi) \lesssim M_f(x) + \inf_{z \in Q} |\mathcal{V}_\rho(\Phi \ast (f \chi_{3Q}))(z)| + \mathcal{V}_\rho(\Phi \ast f)(z)
\]

\[
\lesssim M_f(x) + \frac{1}{|Q|} \int_{Q} \mathcal{V}_\rho(\Phi \ast (f \chi_{3Q}))(z)^{1/2} \, dz
\]

\[
+ \left[ \frac{1}{|Q|} \int_{Q} \mathcal{V}_\rho(\Phi \ast f)(z)^{1/2} \, dz \right]^2
\]

\[
\lesssim M_f(x) + \frac{1}{|Q|} \int_{3Q} |f(y)| \, dy + M_{1/2}(\mathcal{V}_\rho(\Phi \ast f))(x)
\]

\[
\lesssim M_f(x) + M_{1/2}(\mathcal{V}_\rho(\Phi \ast f))(x),
\]

where in the last-to-second inequality, we used the Kolmogorov inequality and the weak type \((1, 1)\) of \(\mathcal{V}_\rho(\Phi \ast f)\) (see [32]). Therefore, (2.4) holds.

Recall that \(M_{1/2}(f)\) is bounded on \(L^{1, \infty}(\mathbb{R}^n)\) (see [37]). Then for \(f \in L^1(\mathbb{R}^n)\), we have \(\mathcal{V}_\rho(\Phi \ast f) \in L^{1, \infty}(\mathbb{R}^n)\) and

\[
\|M_{1/2}(\mathcal{V}_\rho(\Phi \ast f))\|_{L^{1, \infty}(\mathbb{R}^n)} \lesssim \|\mathcal{V}_\rho(\Phi \ast f)\|_{L^{1, \infty}(\mathbb{R}^n)}
\]

\[
\lesssim \|\mathcal{V}_\rho(\Phi)\|_{L^1(\mathbb{R}^n) \rightarrow L^{1, \infty}(\mathbb{R}^n)} \|f\|_{L^1(\mathbb{R}^n)}.
\]

This, together with (2.4) and the weak type \((1, 1)\) of \(M\), implies lemma 2.4. □

By lemmas 2.3, (2.2) and (2.3), we now obtain the following weighted estimate of variation operators associated with approximations to the identity.
3.1. Let

Definition

and a lemma that will be useful for us.

In this section, we give the proof of Theorem 1.1. Let us begin recalling a definition

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3.2. (cf. [Lemma Zhou 2.6.

Remark Theorem 2.5. If ω ∈ A_1, then for any f ∈ L^1(ω),

$$\|V_p(\Phi \ast f)\|_{L^p(\omega)} \lesssim [\omega]_{A_p}^{\max\{1, \frac{1}{p-1}\}} \|f\|_{L^p(\omega)}.$$ 

If ω ∈ A_1, then for any f ∈ L^1(ω),

$$\|V_p(\Phi \ast f)\|_{L^1, \infty(\omega)} \lesssim [\omega]_{A_1} \log(e + [\omega]_{A_{\infty}}) \|f\|_{L^1(\omega)}.$$ 

Remark 2.6.

(i) Theorem 2.5 is the quantitative weighted version of theorem 2.6 in [32];

(ii) Consider the heat semigroup $W := \{e^{t\Delta}\}_{t \geq 0}$ and the Poisson semigroup $P := \{e^{-t\sqrt{-\Delta}}\}_{t \geq 0}$. We remark that theorem 2.5 holds for the variation operators associated to $W$ and $P$, which can be regarded as the quantitative weighted version of the corresponding results in [12, 25].

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In this section, we give the proof of Theorem 1.1. Let us begin recalling a definition and a lemma that will be useful for us.

Definition 3.1. Let ω ∈ A_∞, $q_\omega = \inf\{q \in [1, \infty) : \omega \in A_q\}$, and $0 < p \leq 1$. Then for $q \in (q_\omega, \infty]$ and $s \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ with $s \geq [(q_\omega/p - 1)n]$, a function $a$ on $\mathbb{R}^n$ is called a $(p, q, s)_\omega$-atom if the following conditions hold:

1. $\text{supp } a \subset B(x_0, r)$;
2. $\|a\|_{L^q(\omega)} \leq \omega(B(x_0, r))^{1/q - 1/p}$;
3. $\int_{B(x_0, r)} a(x)x^\alpha \, dx = 0$ for every multi-index $\alpha$ with $|\alpha| \leq s$.

We now recall the following useful lemma on the boundedness criterion of an operator from $H^p(\omega)$ to $L^p(\omega)$ with $p \in (0, 1]$, established by Bownik, Li, Yang and Zhou [6].

Lemma 3.2. (cf. [6, theorem 7.2]) Let $\omega \in A_\infty$, $q_\omega = \inf\{q \in [1, \infty) : \omega \in A_q\}$, $0 < p \leq 1$, and $s \in \mathbb{Z}_+$ with $s \geq [(q_\omega/p - 1)n]$. Then there exists a unique bounded sublinear operator $T$ from $H^p(\omega)$ to $L^p(\omega)$, which extends $T$, if one of the following holds:

1. $q_\omega < q < \infty$ and $T : H^{p,q,s}_{f_{in}}(\omega) \to L^p(\omega)$ is a sublinear operator such that

$$\sup\{\|Ta\|_{L^p(\omega)} : a \text{ is any } (p, q, s)_\omega - \text{atom}\} < \infty,$$

where $H^{p,q,s}_{f_{in}}(\omega)$ is the space of all finite linear combinations of $(p, q, s)_\omega$-atoms.

2. $T$ is a sublinear operator defined on continuous $(p, \infty, s)_\omega$-atoms with the property that

$$\sup\{\|Ta\|_{L^p(\omega)} : a \text{ is any continuous } (p, \infty, s)_\omega - \text{atom}\} < \infty.$$
Now, we are in the position to prove theorem 1.1.

Proof of theorem 1.1. (i). When \( n/(n+1) < p \leq 1 \) and \( \omega \in A_{p(n+1)/n} \), we first assume that \( f \in H^p(\omega) \). Then by the definition of \( H^p(\omega) \), we see that \( \phi_t * f \in L^p(\omega) \) and \( \|\phi_t * f\|_{L^p(\omega)} \leq \|f\|_{H^p(\omega)} \). We now show that

\[
\|\mathcal{V}_\rho(\Phi * f)\|_{L^p(\omega)} \lesssim \|f\|_{H^p(\omega)}. \tag{3.1}
\]

Invoking lemma 3.2, it suffices to verify that for some \( \rho \), the Hölder inequality, theorem 2.5 and definition 3.1, we have

\[
\omega \in A_{p(n+1)/n},
\]

\[
\|\mathcal{V}_\rho(\Phi * f)\|_{L^p(\omega)} < \infty,
\tag{3.2}
\]

and there is a positive constant \( C \) such that for any \( (p,q,s)_\omega \)-atom \( a \),

\[
\|\mathcal{V}_\rho(\Phi * a)\|_{L^p(\omega)} \leq C. \tag{3.3}
\]

Moreover, assume that (3.3) holds first. By [6, theorem 6.2], for given \( f \in H^p_{fin}(\omega) \), there exist numbers \( \{\lambda_j\}_{j=1}^l \) and \( (p,q,s)_\omega \)-atoms \( \{a_j\}_{j=1}^l \) such that \( f = \sum_{j=1}^l \lambda_j a_j \) pointwise and \( \sum_{j=1}^l |\lambda_j|^p \lesssim \|f\|_{H^p(\omega)}^p \). Then

\[
\|\mathcal{V}_\rho(\Phi * f)\|_{L^p(\omega)} \leq \left( \sum_{j=1}^l |\lambda_j|^p \right)^{1/p} \lesssim \|f\|_{H^p(\omega)}.
\]

Therefore, it remains to show (3.3).

We assume that \( \text{supp} \ a \subset B := B(x_0,r) \) and denote \( \tilde{B} := 4B \). Then applying the Hölder inequality, theorem 2.5 and definition 3.1, we have

\[
\int_{\tilde{B}} \mathcal{V}_\rho(\Phi * a)(x)^p \omega(x) \, dx \leq [\omega(\tilde{B})]^{1-p/q_0} \|\mathcal{V}_\rho(\Phi * a)\|_{L^{q_0}(\omega)}^p

\lesssim [\omega(B)]^{1-p/q_0} \|a\|_{L^{q_0}(\omega)}^p \lesssim 1. \tag{3.4}
\]

On the other hand, using (2.5) and the vanishing condition of \( a \), we have

\[
\mathcal{V}_\rho(\Phi * a)(x) = \sup_{\{t_k\} \neq 0} \left( \sum_k \int_{\mathbb{R}^n} \left| (\phi_{t_k}(x-y) - \phi_{t_{k+1}}(x-y)) \right| a(y) \, dy \right)^{1/p}

\leq \int_B |a(y)||\{\phi_t(x-y) - \phi_{t+1}(x-y)\}|_{t>0} \mathcal{V}_\rho \, dy

\lesssim \int_B |a(y)||y - x_0| |x - x_0|^{n+1} \, dy, \quad \forall \ x \notin \tilde{B}. \tag{3.5}
\]
By $\omega \in A_{q_0}$ and definition 3.1, it yields that
\[
\left( \int_B |a(y)|^p \, dy \right)^{1/p} \lesssim \left( \int_B |a(y)|^{q_0 \omega(y)} \, dy \right)^{1/p} \lesssim \omega(B)^{1/p - q_0 \omega^{-1}} \omega(B)^{-p/q_0} |B|^p = \omega(B)^{-1} |B|^p ,
\]
and
\[
\omega(2^{j+2} B) \lesssim 2^{jnq_0 \omega(B)} , \quad j \in \mathbb{Z}^+ .
\]

Therefore,
\[
\int_{B^c} \mathcal{V}_\rho (\Phi \ast a)(x)^p \omega(x) \, dx \lesssim \sum_{j=1}^{\infty} \int_{2^{j+2} B \setminus 2^{j+1} B} \frac{r^p}{|x - x_0|^{(n+1)p}} \omega(x) \, dx \left( \int_B |a(y)|^p \, dy \right)^{1/p} \lesssim \omega(B)^{-1} |B|^p \sum_{j=1}^{\infty} 2^{-j(pn+p)} |B|^{-p} \omega(2^{j+2} B) \lesssim \sum_{j=1}^{\infty} 2^{-j(pn+p-nq_0)} \sim 1 .
\]

This, together with the estimate (3.4), implies (3.3) and completes the proof of (3.1).

Conversely, if $\phi_t \ast f \in L^p(\omega)$ and $\mathcal{V}_\rho (\Phi \ast f) \in L^p(\omega)$, then by (1.1), $f \in H^p(\omega)$ and
\[
\|f\|_{H^p(\omega)}^p = \|M_\phi f\|_{L^p(\omega)}^p \leq \|\phi_t \ast f\|_{L^p(\omega)}^p + \|\mathcal{V}_\rho (\Phi \ast f)\|_{L^p(\omega)}^p .
\]
This finishes the proof of (i).

(ii). When $1 < p < \infty$ and $\omega \in A_p$, it follows from theorem 2.5 that
\[
\|\mathcal{V}_\rho (\Phi \ast f)\|_{L^p(\omega)} \lesssim [\omega]_{A_p}^{\max\{1, \frac{1}{p-1}\}} \|f\|_{L^p(\omega)} , \quad \forall \, f \in L^p(\omega).
\]
That is, (1.6) holds. Moreover, if $\phi_t \ast f \in L^p(\omega)$ and $\mathcal{V}_\rho (\Phi \ast f) \in L^p(\omega)$, then by (1.1), $f \in H^p(\omega)$ and
\[
\|f\|_{H^p(\omega)} = \|M_\phi f\|_{L^p(\omega)} \leq \|\phi_t \ast f\|_{L^p(\omega)} + \|\mathcal{V}_\rho (\Phi \ast f)\|_{L^p(\omega)} .
\]
In converse, if $f \in H^p(\omega) = L^p(\omega)$, then by (1.6) and the $L^p(\omega)$-boundedness of $M_\phi$,
\[
\|\phi_t \ast f\|_{L^p(\omega)} + \|\mathcal{V}_\rho (\Phi \ast f)\|_{L^p(\omega)} \lesssim \|M_\phi f\|_{L^p(\omega)} + \|f\|_{L^p(\omega)} \lesssim \|f\|_{L^p(\omega)} \sim \|f\|_{H^p(\omega)} .
\]
This completes the proof of (ii). Theorem 1.1 is proved. \qed
4. The characterization of $BMO_{\nu}(\mathbb{R}^n)$

This section is concerned with the proof of theorem 1.2. We first recall the following relevant notation and the equivalent definition of $BMO_{\nu}(\mathbb{R}^n)$.

**Definition 4.1.** (cf. [29]) By a median value of a real-valued measurable function $f$ over a measure set $E$ of positive finite measure, we mean a possibly non-unique, real number $m_f(E)$ such that

$$\max(|\{x \in E : f(x) > m_f(E)\}|, |\{x \in E : f(x) < m_f(E)\}|) \leq |E|/2.$$ 

In order to introduce the equivalent definition of $BMO_{\nu}(\mathbb{R}^n)$, we recall the definition of local mean oscillation.

**Definition 4.2.** (cf. [29]) For a complex-valued measurable function $f$, we define the local mean oscillation of $f$ over a cube $Q$ by

$$a_{\tau}(f;Q) := \inf_{c \in \mathbb{C}} (|f - c\chi_Q|^{\ast}(\tau|Q|)) (0 < \tau < 1),$$

where $f^{\ast}$ denotes the non-increasing rearrangement of $f$.

For $\tau \in (0, \frac{1}{2n+2}]$, the following equivalent relation is valid:

$$\|f\|_{BMO_{\nu}(\mathbb{R}^n)} \sim \sup_Q \frac{|Q|}{\nu(Q)} a_{\tau}(f;Q). \tag{4.1}$$

We refer readers to [29, lemma 2.1] for more details.

Now we prove theorem 1.2.

**Proof of theorem 1.2.** We first show that (1) $\Rightarrow$ (2). Without loss of generality, we assume that $b$ and $\phi$ are real-valued, and $\phi(z) \geq 1$ for $z \in B(z_0, \delta)$, where $|z_0| = 1$ and $\delta > 0$ is a small constant. For any cube $Q$, denote by

$$P := Q - 10\sqrt{n}\delta^{-1}l_Q z_0$$

the cube associated with $Q$. For $0 < \tau < 1$, by the definition of $a_{\tau}(f;Q)$, there exists a subset $\tilde{Q}$ of $Q$, such that $|\tilde{Q}| = \tau|Q|$ and for any $x \in \tilde{Q}$,

$$a_{\tau}(b;Q) \leq |b(x) - m_b(P)|.$$ 

Then, by the definition of $m_b(P)$, there exist subsets $E \subset \tilde{Q}$ and $F \subset P$ such that

$$|E| = |\tilde{Q}|/2 = \tau|Q|/2, \ |F| = |P|/2 = |Q|/2,$$

and

$$a_{\tau}(b;Q) \leq |b(x) - b(y)|, \ \forall \ x \in E, \ y \in F,$$

and $b(x) - b(y)$ does not change sign in $E \times F$. Let

$$f(x) := \left(\int_F \mu(x) \, dx\right)^{-1/p} \chi_F(x).$$
Then,
\[
V_{\rho}((\Phi \ast f)_b)(x) = \sup_{t_k \downarrow 0} \left( \sum_{k=1}^{\infty} \left| \int_F (b(x) - b(y))(\phi_{t_k} (x - y) - \phi_{t_{k+1}} (x - y)) \, dy \right|^{\rho} \right)^{1/\rho} \\
\times \left( \int_F \mu(x) \, dx \right)^{-1/p} \\
\geq \lim_{t \to 0} \left| \int_F (b(x) - b(y))(\phi_{10\sqrt{n}\delta^{-1}l_Q}(x - y) - \phi_t (x - y)) \, dy \right| \\
\times \left( \int_F \mu(x) \, dx \right)^{-1/p}.
\]

For \( x \in E \subset Q, y \in F \subset P \), we have
\[
x - y \in 2l_QQ_0 + 10\sqrt{n}\delta^{-1}l_Qz_0 \subset (l_QQ_0)^c,
\]
\[
\frac{x - y}{10\sqrt{n}\delta^{-1}l_Q} \in \frac{\delta}{5\sqrt{n}}Q_0 + z_0 \subset B(z_0, \delta),
\]
where \( Q_0 \) is the cube centred at origin with side length 1. From this, for \( x \in E, y \in F \), we have the following estimates
\[
\phi_{10\sqrt{n}\delta^{-1}l_Q}(x - y) \gtrsim \frac{1}{|Q|} \phi \left( \frac{x - y}{10\sqrt{n}\delta^{-1}l_Q} \right) \geq \frac{1}{|Q|},
\]
and
\[
\lim_{t \to 0} |\phi_t(x - y)| = \lim_{t \to 0} \frac{1}{t^n} \left| \phi \left( \frac{x - y}{t} \right) \right| \lesssim \lim_{t \to 0} \frac{1}{t^n} \left( \frac{|x - y|}{t} \right)^{-n-1}
\]
\[
\lesssim \lim_{t \to 0} t(|x - y|)^{-n-1} = 0.
\]

Hence, for \( x \in E \),
\[
V_{\rho}((\Phi \ast f)_b)(x) \geq \lim_{t \to 0} \int_F |b(x) - b(y)||\phi_{10\sqrt{n}\delta^{-1}l_Q}(x - y) - \phi_t (x - y)| \, dy \\
\times \left( \int_F \mu(x) \, dx \right)^{-1/p} \\
\geq \int_F |b(x) - b(y)||\phi_{10\sqrt{n}\delta^{-1}l_Q}(x - y) - \phi_t (x - y)| \, dy \\
\times \left( \int_F \mu(x) \, dx \right)^{-1/p} \\
= \int_F |b(x) - b(y)||\phi_{10\sqrt{n}\delta^{-1}l_Q}(x - y)| \, dy \left( \int_F \mu(x) \, dx \right)^{-1/p} \\
\gtrsim a_{r}(b; Q) \left( \int_F \mu(x) \, dx \right)^{-1/p},
\]
which yields that
\[
\int_E V_\rho((\Phi \ast f)_b)(x) \, dx \gtrsim \tau |Q| a_\tau(b; Q) \left( \int_P \mu(x) \, dx \right)^{-1/p}.
\] (4.2)

On the other hand, by the Hölder inequality and (1) of theorem 1.2, we have
\[
\int_E V_\rho((\Phi \ast f)_b)(x) \, dx \leq \left( \int_E V_\rho((\Phi \ast f)_b)(x)^p \lambda(x) \, dx \right)^{1/p'} \left( \int_Q \lambda(x)^{-p'/p} \, dx \right)^{1/p'}
\]
\[
\lesssim \left( \int_Q \lambda(x)^{-p'/p} \, dx \right)^{1/p'}.
\]
This, together with (4.2) and $P \subset KQ$ for some $K > 0$, gives that
\[
a_\tau(b; Q) \lesssim \left( \frac{1}{|Q|} \int_Q \mu(x) \, dx \right)^{1/p} \left( \frac{1}{|Q|} \int_Q \lambda(x)^{-p'/p} \, dx \right)^{1/p'}.
\] (4.3)

Noting that
\[
\frac{1}{|Q|} \int_Q \mu(x) \, dx \lesssim \left( \frac{1}{|Q|} \int_Q \mu(x)^{1/(p+1)} \, dx \right)^{p+1}
\]
(see [29]), using Hölder’s inequality and $\mu = \nu^p \lambda$, we obtain
\[
\left( \frac{1}{|Q|} \int_Q \mu(x)^{1/(p+1)} \, dx \right)^{p+1} \leq \left( \frac{1}{|Q|} \int_Q \nu(x) \, dx \right)^p \left( \frac{1}{|Q|} \int_Q \lambda(x) \, dx \right).
\]
Thus, by (4.3) and $\lambda \in A_p$, we conclude that
\[
a_\tau(b; Q) \lesssim \left( \frac{1}{|Q|} \int_Q \nu(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q \lambda(x) \, dx \right)^{1/p} \left( \frac{1}{|Q|} \int_Q \lambda(x)^{-p'/p} \, dx \right)^{1/p'}
\]
\[
\lesssim \frac{1}{|Q|} \int_Q \nu(x) \, dx.
\]
This implies that $b \in BMO_\nu(\mathbb{R}^n)$ by choosing $\tau = 1/2^{n+2}$ and invoking (4.1).

Next, we show that (2) $\Rightarrow$ (1). Indeed, using lemma 2.4, following the standard steps of [28], there exist $3^n$ sparse families $\mathcal{S}_j$ such that
\[
V_\rho((\Phi \ast f)_b)(x) \lesssim \sum_{j=1}^{3^n} (T_{\mathcal{S}_j,b}(f)(x) + T_{\mathcal{S}_j,b}^*(f)(x)),
\] (4.4)
where
\[
T_{\mathcal{S},b}f(x) := \sum_{Q \in \mathcal{S}} |b(x) - \langle b \rangle_Q| \langle f \rangle_Q \chi_Q(x),
\]
\[
T_{\mathcal{S}_j,b}^*f(x) := \sum_{Q \in \mathcal{S}_j} \langle |b - \langle b \rangle_Q|f \rangle_Q \chi_Q(x).
\]
In [28], the authors proved that
\[
\|T_S b f + T^*_S b f\|_{L^p(\lambda)} \lesssim (|\mu| A_p \lambda A_p)^{\max\{1, \frac{1}{p-1}\}} \|b\|_{BMO_\nu(\mathbb{R}^n)} \|f\|_{L^p(\mu)},
\]
where $\mu, \lambda \in A_p$ ($1 < p < \infty$), $\nu = (\mu \lambda^{-1})^{1/p}$ and $b \in BMO_\nu(\mathbb{R}^n)$. This, together with (4.4), shows that (2) implies (1). Theorem 1.2 is proved. \hfill \Box

5. The characterization of $BMO_\omega(\mathbb{R}^n)$ spaces

This section is devoted to the proof of theorem 1.3. We first recall and establish two lemmas.

**Lemma 5.1.** (cf. [30]) Let $\hat{\phi} \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat{\phi}(x) = 1$ for all $x \in B(0,1)$, and $M_{\hat{\phi}}$ be defined as in (1.5). Suppose that $f$ is a measurable function such that $\text{supp} \, f \subset B := B(x_B, r)$ with some $x_B \in \mathbb{R}^n$ and $r \in (0, \infty)$. Then for all $x \notin B$,
\[
\frac{1}{|x - x_B|^n} \left| \int_B f(y) \, dy \right| \lesssim M_{\hat{\phi}}(f)(x).
\]

**Lemma 5.2.** Let $\omega \in A_q$ with $q \in (1, 1 + 1/n)$. Then for any $b \in BMO(\mathbb{R}^n)$ and $(1, q, s)_\omega$-atom $a$ with $s > 0$ and $\text{supp} \, a \subset B := B(x_B, r)$, there holds that
\[
\| (b - \langle b \rangle_B) \mathcal{V}_\rho(\Phi \ast a) \|_{L^1(\omega)} \lesssim \| b \|_{BMO(\mathbb{R}^n)}.
\]

**Proof.** We prove this lemma by considering the following two terms:
\[
I_1 := \int_{4B} |b(x) - \langle b \rangle_B| \mathcal{V}_\rho(\Phi \ast a) \omega(x) \, dx,
\]
and
\[
I_2 := \int_{(4B)^c} |b(x) - \langle b \rangle_B| \mathcal{V}_\rho(\Phi \ast a) \omega(x) \, dx.
\]

Note that for any $\omega \in A_\infty$, $q \in [1, \infty)$ and $B \subset \mathbb{R}^n$,
\[
\left[ \frac{1}{\omega(B)} \int_B |b(x) - \langle b \rangle_B|^q \omega(x) \, dx \right]^{1/q} \lesssim \| b \|_{BMO(\mathbb{R}^n)}. \tag{5.1}
\]

By Hölder’s inequality, theorem 2.5 and definition 3.1, we have
\[
I_1 \lesssim \left( \int_{4B} |b(x) - \langle b \rangle_B|^q \omega(x) \, dx \right)^{1/q'} \left( \int_{\mathbb{R}^n} \mathcal{V}_\rho(\Phi \ast a)(x)^q \omega(x) \, dx \right)^{1/q} \lesssim \omega(4B)^{1/q'} \| b \|_{BMO(\mathbb{R}^n)} \| a \|_{L^q(\omega)} \lesssim \omega(4B)^{1/q'} \| b \|_{BMO(\mathbb{R}^n)} \omega(B)^{-1/q'} \sim \| b \|_{BMO(\mathbb{R}^n)}.
\]
For $I_2$, noting that $\omega \in A_q$, $\omega(2^{j+1}B) \lesssim 2^{(j+1)n_q}\omega(B)$ and invoking the vanishing property of $a$, it follows from (3.5) and (5.1) that

$$I_2 \lesssim \int_{(4B)^c} |b(x) - \langle b \rangle_B| \int_B |a(y)| \frac{|y - x_B|}{|x - x_B|^{n+1}} dy \omega(x) dx$$

$$\lesssim \int_B |a(y)| \sum_{j=2}^\infty \int_{2^{j+1}B \setminus 2^j B} |b(x) - \langle b \rangle_B| \frac{|y - x_B|}{|x - x_B|^{n+1}} \omega(x) dx dy$$

$$\lesssim \left( \int_B |a(y)|^q \omega(y) dy \right)^{1/q} \left( \int_B \omega(y)(-q^{'} / q) dy \right)^{1/q^{'}}$$

$$\times \sum_{j=2}^\infty \int_{2^{j+1}B} \frac{r}{(2^j r)^{n+1}} \omega(y)(2^{j+1})dy \|b\|_{BMO(\mathbb{R}^n)}$$

$$\lesssim \frac{|B|}{\omega(B)} \sum_{j=2}^\infty 2^{-j(n+1)} \frac{\omega(2^{j+1}B)}{|B|} \|b\|_{BMO(\mathbb{R}^n)}$$

$$\lesssim \|b\|_{BMO(\mathbb{R}^n)} \sum_{j=2}^\infty 2^{-j(n+1-nq)} j \lesssim \|b\|_{BMO(\mathbb{R}^n)}.$$

Combining the estimates of $I_1$ and $I_2$, we finish the proof of lemma 5.2.

Now, we are in the position to prove theorem 1.3.

Proof of theorem 1.3. First, we show that (2) implies (1). In view of lemma 3.2, we only need to prove that for any $(1, \infty, s)$-atom $a$ with $s \geq 0$ and supp $a \subset B := B(x_B, r)$, there holds that

$$\|V_p((\Phi \ast a)_b)\|_{L^1(\omega)} \lesssim \|b\|_{BMO_{\omega}(\mathbb{R}^n)}.$$

Write

$$\|V_p((\Phi \ast a)_b)\|_{L^1(\omega)} \leq \|V_p((\Phi \ast ((b - \langle b \rangle_B)a))\|_{L^1(\omega)} + \|(b - \langle b \rangle_B)V_p(\Phi \ast a)\|_{L^1(\omega)}.$$

Since $(1, \infty, s)$-atom is $(1, q, s)$-atom and $\|b\|_{BMO(\mathbb{R}^n)} \lesssim \|b\|_{BMO_{\omega}(\mathbb{R}^n)}$, by lemma 5.2 and theorem 1.1, it suffices to show that $(b - \langle b \rangle_B)a \in H^1(\omega)$ with

$$\|(b - \langle b \rangle_B)a\|_{H^1(\omega)} \lesssim \|b\|_{BMO_{\omega}(\mathbb{R}^n)}.$$ (5.2)

We now show (5.2). For $x \not\in 2B$, note that

$$M_\phi((b - \langle b \rangle_B)a)(x) \leq \sup_{t > 0} t^{-n} \int_B |b(y) - \langle b \rangle_B||a(y)||\phi\left(\frac{x - y}{t}\right)| dy$$

$$\lesssim \sup_{t > 0} t^{-n} \int_B |b(y) - \langle b \rangle_B||a(y)|(1 + |x - y|/t)^{-n} dy$$

$$\lesssim \frac{1}{|x - x_B|^n} \int_B |b(y) - \langle b \rangle_B||a(y)| dy.$$
Hence, by the definition of $\mathcal{BMO}_\omega(\mathbb{R}^n)$ and $\|a\|_{L^\infty(\mathbb{R}^n)} \leq \omega(B)^{-1}$, we have

$$\int_{(2B)^c} M_\phi((b - \langle b \rangle_B)a)(x)\omega(x)\,dx$$

$$\lesssim \frac{1}{\omega(B)} \left( \int_{(2B)^c} \frac{\omega(x)}{|x - x_B|^n} \,dx \right) \left( \int_B |b(y) - \langle b \rangle_B|\,dy \right) \leq \|b\|_{\mathcal{BMO}_\omega(\mathbb{R}^n)}. \quad (5.3)$$

Meanwhile, by the $L^q(\omega)$-boundedness of $M_\phi$, $\|a\|_{L^\infty(\mathbb{R}^n)} \leq \omega(B)^{-1}$ and (5.1), we obtain

$$\int_{2B} M_\phi((b - \langle b \rangle_B)a)(x)\omega(x)\,dx \leq \left[ \int_{\mathbb{R}^n} M_\phi((b - \langle b \rangle_B)a)(x)^q\omega(x)\,dx \right]^{1/q}$$

$$\times \left( \int_{2B} \omega(x)\,dx \right)^{1/q'}$$

$$\lesssim \|(b - \langle b \rangle_B)a\|_{L^q(\omega)}^q \omega(2B)^{-1/q}$$

$$\lesssim \omega(B)^{-1/q} \left( \int_B |b(x) - \langle b \rangle_B|^q\omega(x)\,dx \right)^{1/q}$$

$$\lesssim \|b\|_{\mathcal{BMO}(\mathbb{R}^n)} \lesssim \|b\|_{\mathcal{BMO}_\omega(\mathbb{R}^n)}.$$

This, together with (5.3), implies (5.2), and proves that (2) \Rightarrow (1).

Next, we show that (1) \Rightarrow (2). For any ball $B := B(x_B, r)$, take $h := \text{sgn}(b - \langle b \rangle_B)$ and

$$a := \frac{1}{2\omega(B)}(h - \langle h \rangle_B)\chi_B.$$

Then supp $a \subset B$, $\|a\|_{L^\infty(\mathbb{R}^n)} \leq \omega(B)^{-1}$ and $\int_B a(y)\,dy = 0$. By lemma 5.2 and assumption (1) in theorem 1.3, we obtain

$$\|\mathcal{V}_\rho(\Phi \ast ((b - \langle b \rangle_B)a))\|_{L^1(\omega)} \leq \|\mathcal{V}_\rho(\Phi \ast a)_b\|_{L^1(\omega)} + \|b - \langle b \rangle_B\|_{\mathcal{BMO}(\mathbb{R}^n)}$$

$$\lesssim \|a\|_{H^1(\omega)} + \|b\|_{\mathcal{BMO}(\mathbb{R}^n)}.$$

Hence, by (i) of theorem 1.1 with $\lim_{t \to 0} \phi_t \ast f = f$ on $L^1(\omega)$ for $\omega \in A_1$ (see [36]) and (5.1),

$$\|(b - \langle b \rangle_B)a\|_{H^1(\omega)} \leq \|(b - \langle b \rangle_B)a\|_{L^1(\omega)} + \|\mathcal{V}_\rho(\Phi \ast ((b - \langle b \rangle_B)a))\|_{L^1(\omega)}$$

$$\lesssim \frac{1}{\omega(B)} \int_B |b(x) - \langle b \rangle_B|\omega(x)\,dx + \|a\|_{H^1(\omega)} + \|b\|_{\mathcal{BMO}(\mathbb{R}^n)}$$

$$\lesssim \|b\|_{\mathcal{BMO}(\mathbb{R}^n)} + \|a\|_{H^1(\omega)} + \|b\|_{\mathcal{BMO}(\mathbb{R}^n)} \lesssim 1.$$

Also, invoking lemma 5.1, for any $x \notin B$, we have

$$\frac{1}{2\omega(B)|x - x_B|^n} \int_B |b(y) - \langle b \rangle_B|\,dy$$

$$= \frac{1}{|x - x_B|^n} \int_B (b(y) - \langle b \rangle_B)a(y)\,dy \lesssim M_\phi((b - \langle b \rangle_B)a)(x).$$
Consequently,
\[
\left(\frac{1}{\omega(B)} \int_{B'} \frac{\omega(x)}{|x-x_B|^n} \, dx\right) \left(\int_B |b(y) - \langle b \rangle_B| \, dy\right) \lesssim \|M_\tilde{\varphi}((b - \langle b \rangle_B)a)\|_{L^1(\omega)} \\
\lesssim \|(b - \langle b \rangle_B)a\|_{H^1(\omega)},
\]
which implies that
\[
\|b\|_{BMO(\mathbb{R}^n)} \lesssim \sup_B \|(b - \langle b \rangle_B)a\|_{H^1(\omega)} \lesssim 1.
\]
This finishes the proof of the implication (1) ⇒ (2). Theorem 1.3 is proved. \(\square\)

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