Fixed points, bounded orbits and attractors of planar flows

Héctor BARGE and José M.R. SANJURJO*

Departamento de Geometría y Topología
Universidad Complutense de Madrid
28040 Madrid, Spain
hbarge@ucm.es

Departmento de Geometría y Topología
Universidad Complutense of Madrid
28040 Madrid, Spain
jose.sanjurjo@mat.ucm.es

Affectionately dedicated to José María Montesinos-Amilibia.

ABSTRACT

In this paper we provide a dynamical characterization of isolated invariant continua which are global attractors for planar dissipative flows. As a consequence, a sufficient condition for an isolated invariant continuum to be either an attractor or a repeller is derived for general planar flows.

2010 Mathematics Subject Classification: 34C35, 34D23, 37C10, 37C25, 37C70.
Key words: Attractor, Fixed point, Bounded orbit, Dissipative flow, Isolated invariant set.

1. Introduction

In this paper we are concerned with the study of planar flows \( \varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \). In particular, we provide a dynamical characterization of isolated invariant continua which are global attractors for planar dissipative flows. This characterization is inspired by a result of Alarcón, Guíñez and Gutiérrez about dissipative planar embeddings with only one fixed point (see [1]). Moreover we will derive a sufficient condition for a

*The authors are partially supported by project MTM2012-30719.
planar continuum to be an attractor or a repeller provided that it contains all the
fixed points of $\varphi$.

We shall use through the paper the standard notation and terminology in
the theory of dynamical systems. In particular we shall use the notation
$\gamma(x)$ for the trajectory of the point $x$, i.e. $\gamma(x) = \{xt | t \in \mathbb{R}\}$. By the omega-limit of a point $x$
we understand the set $\omega(x) = \bigcap_{t>0} x[0,t)$ while the negative omega-limit is the set $\omega^*(x) = \bigcap_{t<0} x(-\infty,t]$. An invariant compactum $K$ is stable if every neighborhood
$U$ of $K$ contains a neighborhood $V$ of $K$ such that $V[0,\infty) \subset U$. Similarly, $K$ is negatively stable if every neighborhood $U$ of $K$ contains a neighborhood $V$ of $K$ such that $V(-\infty,0] \subset U$. An invariant compactum $K$ is said to be attracting provided that there exists a neighborhood $U$ of $K$ such that $\omega(x) \subset K$ for every $x \in U$. In an analogous way, $K$ is said to be repelling provided that there exists a neighborhood $U$ of $K$ such that $\omega^*(x) \subset K$ for every $x \in U$. An attractor (or asymptotically stable compactum) is an attracting stable set and a repeller is a repelling negatively stable set.

If $K$ is an attracting set, its region (or basin) of attraction $A$ is the set of all points
$x \in M$ such that $\omega(x) \subset K$. An attracting set $K$ is globally attracting provided that $A$ is the whole phase space. If $K$ is an attractor and $A$ is the whole phase space, then $K$ is said to be a global attractor (or globally asymptotically stable compactum). For
the reader interested in a detailed treatment of attracting sets we recommend [14] and [18].

Through this paper we shall deal with a special kind of invariant compacta, the so-called
isolated invariant sets (see [5, 6, 7, 17] for details). These are compact invariant sets $K$ which possess an isolating neighborhood, i.e. a compact neighborhood $N$ such that $K$ is the maximal invariant set in $N$. For instance, attractors and repellers are isolated invariant sets. We shall make use of the next result which states that isolated globally attracting continua for planar flows are stable.

**Theorem 1.1 (Morón, Sánchez-Gabites and Sanjurjo [14])** Every connected
isolated globally attracting set $K$ in $\mathbb{R}^2$ is a global attractor.

A special kind of isolating neighborhoods shall be useful in the sequel, the so-called
isolating blocks, which have good topological properties. More precisely, an isolating block $N$ is an isolating neighborhood such that there are compact sets $N^i, N^o \subset \partial N$, called the entrance and the exit sets, satisfying

1. $\partial N = N^i \cup N^o$;

2. for each $x \in N^i$ there exists $\varepsilon > 0$ such that $x[-\varepsilon,0) \subset M - N$ and for each $x \in N^o$ there exists $\delta > 0$ such that $x(0,\delta] \subset M - N$;

3. for each $x \in \partial N - N^i$ there exists $\varepsilon > 0$ such that $x[-\varepsilon,0) \subset \mathring{N}$ and for every $x \in \partial N - N^o$ there exists $\delta > 0$ such that $x(0,\delta] \subset \mathring{N}$. 

2
These blocks form a neighborhood basis of $K$ in $M$. If the flow is differentiable, the isolating blocks can be chosen to be differentiable manifolds which contain $N^i$ and $N^o$ as submanifolds of their boundaries and such that $\partial N^i = \partial N^o = N^i \cap N^o$. In particular, for flows defined on $\mathbb{R}^2$, the exit set $N^o$ is a disjoint union of a finite number of intervals $J_1, \ldots, J_m$ and circumpherences $C_1, \ldots, C_n$ and the same is true for the entrance set $N^i$.

The dynamical structure near isolating invariant sets shall play an important role in this paper and it is described by the

**Theorem 1.2 (Ura-Kimura-Egawa [20, 8])** Let $M$ be a locally compact separable metric space and $\varphi$ a flow on $M$. Suppose $K \neq M$ is a non-empty isolated invariant compactum. Then, one and only one of the following alternatives holds:

1. $K$ is an attractor;
2. $K$ is a repeller;
3. There exist points $x \in M - K$ and $y \in M - K$ such that $\emptyset \neq \omega(x) \subset K$ and $\emptyset \neq \omega^r(y) \subset K$.

We shall also make use of a classical result of C. Gutiérrez about smoothing of 2-dimensional flows.

**Theorem 1.3 (Gutiérrez [10])** Let $\varphi : M \times \mathbb{R} \to M$ be a continuous flow on a compact $C^\infty$ two-manifold $M$. Then there exists a $C^1$ flow $\psi$ on $M$ which is topologically equivalent to $\varphi$. Furthermore, the following conditions are equivalent:

1. any minimal set of $\varphi$ is trivial;
2. $\varphi$ is topologically equivalent to a $C^2$ flow;
3. $\varphi$ is topologically equivalent to a $C^\infty$ flow.

By a trivial minimal set we understand a fixed point, a closed trajectory or the whole manifold if $M$ is the 2-dimensional torus and $\varphi$ is topologically equivalent to an irrational flow. We readily deduce from Gutiérrez’ Theorem applied to the Alexandrov compactification of the plane that continuous flows $\varphi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ are topologically equivalent to $C^\infty$ flows.

Some basic results about planar vector fields such as, the Poincaré-Bendixson Theorem, the Tubular Flow Theorem and the elementary properties of transversal sections shall be used through the paper. Two good references covering this material are the book of Hirsch, Smale and Devaney [12] and the monograph of Palis and de Melo [15]. In addition, a form of homotopy theory, namely shape theory, which is the most suitable for the study of global topological properties in dynamics, will be occasionally used. Although shape theory is not necessary to understand the paper, we recommend to the reader the references [4], which contains an exhaustive treatment of the subject and [19], which covers some dynamical applications of this theory.
2. Planar dissipative systems and isolated invariant continua

We start this section by recalling the definition of dissipative flow. Let $M$ be a locally compact metric space and $\varphi : M \times \mathbb{R} \to M$ a flow on $M$. The flow $\varphi$ is said to be dissipative if $\omega(x) \neq \emptyset$ for every $x \in M$ and $\bigcup_{x \in M} \omega(x)$ has compact closure. If the phase space $M$ is not compact, dissipativeness is equivalent to $\{\infty\}$ being a repeller of the extended flow $\hat{\varphi} : (M \cup \{\infty\}) \times \mathbb{R} \to M \cup \{\infty\}$ to the Alexandrov compactification of $M$ leaving $\infty$ fixed (See [9, 11, 19]), and therefore to the existence of a global attractor for $\varphi$.

The following result gives a relation between global asymptotic stability of a fixed point and the non-existence of additional fixed points in the case of discrete dynamical systems.

**Theorem 2.1 (Alarcón-Guíñez-Gutiérrez [1], Ortega-Ruiz del Portal [16])**

Assume that $h \in H^+ (\text{orientation preserving homeomorphisms of } \mathbb{R}^2)$ is dissipative and $p$ is an asymptotically stable fixed point of $h$. The following conditions are equivalent:

1. $p$ is globally asymptotically stable;
2. $\text{Fix}(h) = p$ and there exists an arc $\gamma \subset S^2$ with end points at $p$ and $\infty$ such that $h(\gamma) = \gamma$.

The proof in [1] is based on Brouwer’s theory of fixed point free homeomorphisms of the plane. Ortega and Ruiz del Portal give in [16] an alternative proof based on the theory of prime ends.

Inspired by Theorem 2.1, the authors prove in [2] that for continuous and dissipative dynamical systems the result is satisfied even if the fixed point $p$ is substituted by a connected attractor $K$ which contains every fixed point of the flow. We prove in our next result that the asymptotical stability condition can be dropped from the hypothesis, obtaining in this way a simple characterization of global attractors of dissipative planar flows.

**Theorem 2.2** Let $K$ be an isolated invariant continuum of a dissipative flow $\varphi$ in $\mathbb{R}^2$. The following conditions are equivalent:

1. $K$ is a global attractor;
2. There are no fixed points in $\mathbb{R}^2 - K$ and there exists an orbit $\gamma$ connecting $\infty$ and $K$ (i.e. such that $||\gamma(t)|| \to \infty$ when $t \to -\infty$ and $\omega(\gamma) \subset K$).

**Proof.** By the Gutiérrez Theorem [10] we can assume that the flow $\varphi$ is differentiable. Since $\varphi$ is dissipative, given $x \in \mathbb{R}^2$ its $\omega$-limit is non-empty and compact. Moreover, by the Poincaré-Bendixson Theorem either $\omega(x)$ contains fixed points and, hence, $\omega(x) \cap K \neq \emptyset$ or $\omega(x)$ is a periodic orbit. If $\omega(x)$ is a periodic orbit then $K$ is not
contained in its interior since, in that case, \( \gamma \) would meet \( \omega(x) \), which is impossible. Therefore, if \( \omega(x) \) is not contained in \( K \), then \( K \) is in the exterior of \( \omega(x) \) and, moreover, \( \omega(x) \) being a periodic orbit, there must exist a fixed point in its interior. Hence this point belongs to \( K \), which is a contradiction. We conclude that if \( \omega(x) \) is a periodic orbit then \( \omega(x) \subseteq K \).

If \( \omega(x) \) is not a periodic orbit then \( \omega(x) \cap K \neq \emptyset \) and we shall prove that, in fact, \( \omega(x) \subseteq K \). We suppose, to get a contradiction, that there exists \( y \in \omega(x) \setminus K \). By hypothesis \( y \) is not a fixed point and, thus, we can take a local section \( I \) containing \( y \) and meeting transversally the trajectory of \( y \). Since \( y \notin K \) we can assume that \( I \cap K = \emptyset \). It is a well-known fact that the trajectory of \( x \) meets \( I \) infinitely many times. We consider two consecutive points of intersection \( x_1 = xt_1 \) and \( x_2 = xt_2 \) with \( x_1, x_2 \in I \), \( 0 < t_1 < t_2 \) and \( x[t_1, t_2] \cap I = \{ x_1, x_2 \} \). Then the set \( C = x[t_1, t_2] \cup J \), where \( J \) is the subinterval of \( I \) bounded by \( x_1 \) and \( x_2 \), is a simple closed curve which, by the Jordan Theorem, decomposes \( \mathbb{R}^2 \) into two connected components \( U \) and \( V \). If \( U \) is the bounded component then \( U \) is either positively or negatively invariant by \([12]\). Then, a simple argument involving again the Poincaré-Bendixson Theorem, leads to the existence of a fixed point in \( U \) which, by hypothesis, belongs to \( K \). Now, the intersection of \( K \) with \( C \) is empty, which implies that \( K \subseteq U \cup V \) and, \( K \) being connected, that \( K \subseteq U \). If \( U \) is negatively invariant, the trajectory \( \gamma \) linking \( \infty \) with \( K \) cannot enter in \( U \) since the only possibility would be through \( J \), which is an exit set. This makes it impossible that \( \omega(\gamma) \subseteq K \) and we get a contradiction with the hypothesis. If \( U \) is positively invariant then an easy argument shows that \( y \in \omega(\gamma) \) in contradiction with the assumption. This proves that \( \omega(x) \subseteq K \) for every \( x \in \mathbb{R}^2 \) and, as a consequence, \( K \) is a globally attracting set. Since \( K \) is isolated, by Theorem 1.1 \( K \) must be stable, i.e. a global attractor. This establishes the implication 2. \( \Rightarrow \) 1. The converse implication is straightforward.

3. Attractors, repellers and bounded orbits

We present in this section a result which gives a readily testable sufficient condition for a planar compactum to be either an attractor or a repeller.

**Theorem 3.1** Let \( K \) be an isolated invariant continuum of a flow \( \varphi \) in \( \mathbb{R}^2 \). Suppose that there is a closed disk \( D \) containing \( K \) in its interior such that there are no fixed points in \( D \setminus K \) and that there is an orbit \( \gamma \) completely contained in \( D \setminus K \). Then \( K \) is either an attractor or a repeller. Moreover, \( K \) has trivial shape.

**Proof.** We can assume again that \( \varphi \) is differentiable. Since \( \Upsilon \subseteq D \) we have that \( \omega(\gamma) \subseteq D \) and \( \omega^*(\gamma) \subseteq D \). We start by proving that there exists an orbit \( \Gamma \) in \( D \setminus K \) satisfying the additional condition that either \( \omega(\Gamma) \subseteq K \) or \( \omega^*(\Gamma) \subseteq K \). As a consequence of the Poincaré-Bendixson Theorem and the hypothesis of the present theorem we have that either \( \omega(\gamma) \cap K \neq \emptyset \) or \( \omega(\gamma) \) is a periodic orbit not meeting \( K \), and the same can be said for \( \omega^*(\gamma) \). If \( \omega(\gamma) \) is a periodic orbit not meeting \( K \)
then $K$ is in its interior and, by the Ura-Kimura Theorem, there exists a point $x$, also in the interior of $\omega(\gamma)$, with $\omega(x) \subset K$ or $\omega^*(x) \subset K$, and the same happens if $\omega^*(\gamma)$ is a periodic orbit not meeting $K$. Hence, in both cases $\Gamma$ can be taken as the trajectory of $x$. On the other hand, we will prove that the possibility that both intersections, $\omega(\gamma) \cap K$ and $\omega^*(\gamma) \cap K$, are non-empty can never happen. Suppose, to get a contradiction, that $\omega(\gamma) \cap K \neq \emptyset$ and $\omega^*(\gamma) \cap K \neq \emptyset$. Take an isolating block $N$ of $K$. By [6] $N$ can be chosen to be a topological closed disk with $i$ holes, one for every bounded component of $\mathbb{R}^2 - K$. We suppose that $\gamma$ is in the unbounded component $U$ (the argument being only slightly different in the other case) and consider the only circle $C \subset \partial N$ which is contained in $U$. Then, there exists a point $x \in C \cap \gamma$ leaving $N$ and returning to $N$ after a time $t \neq 0$, i.e. such that $xt \in C$ and $x(0, t) \cap N = \emptyset$. The possibility that the time $t$ be positive or negative is irrelevant in this construction. Consider the arc $A$ in $C$ with extremes $x$ and $xt$ such that the topological circle $x[0, t] \cup A$ does not contain $K$ in its interior. This arc can be mapped to the unit interval $I = [0, 1]$ of the real line by a homeomorphism $h : A \rightarrow I$. If we take the point $x_1 \in A$ corresponding to the center of $I$ then $x_1$ must leave $N$ (in the past or in the future) and return again since, otherwise, the Theorem of Poincaré-Bendixson would imply the existence of a fixed point in the disk limited by $x[0, t] \cup A$. Hence, we can repeat the operation with $x_1[0, t_1] \cup A_1$, where $A_1$ is an arc in $A$ with extremes $x_1$ and $x_1 t_1$ and the topological circle $x_1[0, t_1] \cup A_1$ does not contain $K$ in its interior. Now take $x_2 \in A_1$ corresponding to the middle point of $h(A_1)$ and repeat the construction. In this way we obtain a sequence $A \supset A_1 \supset A_2 \supset \ldots$ of arcs whose intersection $\bigcap_{i=1}^{\infty} A_i$ consists of one point $p \in \partial N$. The orbit of $p$ defines an internal tangency to $\partial N$, which is in contradiction with the properties of isolating blocks. We get from this contradiction that either $\omega(\gamma) \cap K = \emptyset$ or $\omega^*(\gamma) \cap K = \emptyset$ and, as a consequence, one of the two limits is a periodic orbit. Therefore, it follows from the remarks at the beginning of the proof that there exists an orbit $\Gamma$ in $D - K$ satisfying the additional condition that either $\omega(\Gamma) \subset K$ or $\omega^*(\Gamma) \subset K$.

Suppose that $\omega(\Gamma) \subset K$. Then, $\omega^*(\Gamma)$ is a periodic orbit containing $K$ in its interior. Let $V$ be the interior of $\omega^*(\Gamma)$ and consider the flow restricted to $\overline{V}$. An elementary argument involving local sections again shows that $\omega^*(\Gamma)$ is a repellor for $\varphi|\overline{V}$ and, as a consequence, the restriction of $\varphi$ to $V$ is a dissipative flow. Then, using an arbitrary homeomorphism between $V$ and $\mathbb{R}^2$ we can define a dissipative flow in $\mathbb{R}^2$ conjugated to $\varphi|V$ and satisfying the conditions of Theorem 22. We deduce from that theorem that $K$ is an attractor of $\varphi$ whose basin of attraction, $V$, is an open topological disk. Hence, $K$ has trivial shape by [13]. In the dual situation (when $\omega^*(\Gamma) \subset K$ and $\omega(\Gamma)$ is a periodic orbit containing $K$ in its interior), which could be discussed analogously using the reverse flow, it follows that $K$ is a repellor with trivial shape.

From Theorem 3.1 it follows:

**Corollary 3.1** Let $K$ be an isolated invariant continuum of a flow $\varphi$ in $\mathbb{R}^2$. Suppose
that $K$ contains all the fixed points of $\varphi$ and that there exists a bounded orbit $\gamma$ in $\mathbb{R}^2 - K$. Then $K$ is either an attractor or a repeller. Moreover, $K$ has trivial shape.

**Proof.** The set $K \cup \gamma$ is compact and as a consequence there exists a closed disk $D$ such that $K \cup \gamma \subset D$. Then, Theorem 3.1 applies since the bounded orbit $\gamma \subset D - K$, and $D - K$ does not contain fixed points by assumption.

**Remark** The assumptions about the existence of a disk $D$ such that there is an entire orbit contained in $D - K$ in Theorem 3.1 and the existence of a bounded orbit in $\mathbb{R}^2 - K$ in Corollary 3.1 are unavoidable. For instance, consider the flow $\varphi$ induced by the linear system

$$\begin{align*}
\dot{x} &= x \\
\dot{y} &= -y
\end{align*}$$

The origin $(0,0)$ is a fixed point which is isolated as an invariant set and there are neither fixed points nor other bounded orbits in $\mathbb{R}^2 - \{(0,0)\}$. In this case, $\{(0,0)\}$ is a saddle and hence, it is neither an attractor nor a repeller.

As a consequence of Corollary 3.1 and [2, Theorem 12] we obtain the following dichotomy for dissipative flows:

**Corollary 3.2** Let $K$ be an isolated invariant continuum of a dissipative flow $\varphi$ in $\mathbb{R}^2$. Suppose that $K$ contains all the fixed points of $\varphi$, then $K$ has trivial shape and is either an attractor or a repeller. Moreover, if $K$ is a repeller then there exists an attractor $K^* \subset \mathbb{R}^2 - K$ which is either a limit cycle or homeomorphic to a closed annulus bounded by two limit cycles.

**Proof.** The dissipativeness of $\varphi$ guarantees the existence of a global attractor $K'$ and as a consequence $K \subset K'$. Suppose $K' \neq K$, since otherwise we have nothing to prove. Let $x \in K' - K$, the orbit $\gamma(x)$ is a bounded orbit being contained in the invariant compactum $K'$. Then, Corollary 3.1 ensures that $K$ is either an attractor or a repeller. This proves the first part of the statement.

Suppose that $K$ is a repeller and consider the flow $\varphi|K'$, i.e. the restriction of $\varphi$ to the global attractor. The continuum $K$ is also a repeller for $\varphi|K'$ and then there exists an invariant compactum $K^* \subset K'$ such that the pair $(K^*, K')$ is an attractor-repeller decomposition of $\varphi|K'$. Besides, the invariant compactum $K^*$ is an attractor for $\varphi$ since $K^*$ is an attractor for $\varphi|K'$ and $K'$ is an attractor. The region of attraction of $K^*$ agrees with $\mathbb{R}^2 - K$ since $K$ is a repeller and $(K^*, K')$ is an attractor-repeller decomposition of the restriction of $\varphi$ to the global attractor $K'$. Moreover, $\mathbb{R}^2 - K$ is connected $K$ being of trivial shape [14] and hence so is $K^*$ by [13] and [4]. We have proved that $K^*$ is a connected attractor which does not contain fixed points, thus by [2, Theorem 12] it must be either a limit cycle or homeomorphic to a closed annulus bounded by two limit cycles.
References

[1] B. Alarcón, V. Guíñez and C. Gutiérrez: Planar embeddings with a globally attracting fixed point. Nonlinear Anal. 69 (2008), 140–150.

[2] H. Barge and J. M. R. Sanjurjo: Unstable manifold, Conley index and fixed points of flows. J. Math. Anal. Appl. 420 (2014), no. 1, 835–851.

[3] N. P. Bhatia and G. P. Szego: Stability Theory of Dynamical Systems. Grundlehren der Mat. Wiss. 16, Springer, Berlin, 1970.

[4] K. Borsuk: Theory of Shape. Monografie Matematyczne 59, Polish Scientific Publishers, Warsaw, 1975.

[5] C. Conley: Isolated invariant sets and the Morse index. CBMS Regional Conference Series in Mathematics 38 (Providence, RI: American Mathematical Society) 1978.

[6] C. Conley and R. W. Easton: Isolated invariant sets and isolating blocks. Trans. Amer. Math. Soc. 158 (1971) 35–61.

[7] R. W. Easton: Isolating blocks and symbolic dynamics. J. Diff. Equations 17 (1975) 96–118.

[8] I. Egawa: A remark on the flow near a compact invariant set. Proc. Japan Acad. 49 (1973) 247–251.

[9] B. M. Garay: Uniform persistence and chain recurrence. J. Math. Anal. Appl. 139 (1989) 372–381.

[10] C. Gutiérrez: Smoothing continuous flows on two-manifolds and recurrences. Ergod. Th. & Dynam. Sys. 6 (1986) 17–44.

[11] J. K. Hale: Stability and gradient dynamical systems. Rev. Mat. Complut. 17 (2004) 7–57.

[12] M. W. Hirsch, S. Smale and R. L. Devaney: Differential Equations, Dynamical Systems, and an Introduction to Chaos. Third Edition. Elsevier/Academic Press, Amsterdam, 2013.

[13] L. Kapitanski and I. Rodnianski: Shape and Morse theory of attractors. Comm. Pure Appl. Math. 53 (2000) 218–242.

[14] M. A. Morón, J. J. Sánchez Gabites and J. M. R. Sanjurjo: Topology and dynamics of unstable attractors. Fund. Math. 197 (2007) 239–252.

[15] J. Palis, W. de Melo: Geometric Theory of Dynamical Systems. An introduction. Springer-Verlag, New York-Berlin 1982.

[16] R. Ortega and F. R. Ruiz del Portal: Attractors with vanishing rotation number. J. Eur. Math. Soc. 13 (2011) 1569–1590.

[17] D. Salamon: Connected simple systems and the Conley index of isolated invariant sets. Trans. Amer. Math. Soc. 291 (1985) 1–41.

[18] J. J. Sánchez-Gabites: Unstable attractors in manifolds. Trans. Amer. Math. Soc. 362 (2010) 3563–3589.

[19] J. M. R. Sanjurjo: On the fine structure of the global attractor of an uniformly persistent flow. J. Diff. Equations 252 (2012) 4886–4897.

[20] T. Ura and I. Kimura: Sur le courant exterieur a une region invariante. Theoreme de Bendixson. Comm. Mat. Univ. Sancii Pauli 8 (1960) 23–39.