Black holes and Bhargava’s invariant theory

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Abstract
Attractor black holes in type II string compactifications on $K3 \times T^2$ are in correspondence with equivalence classes of binary quadratic forms. The discriminant of the quadratic form governs the black hole entropy, and the count of attractor black holes at a given entropy is given by a class number. Here, we show this tantalizing relationship between attractors and arithmetic can be generalized to a rich family, connecting black holes in supergravity and string models with analogous equivalence classes of more general forms under the action of arithmetic groups. Many of the physical theories involved have played an earlier role in the study of ‘magical’ supergravities, while their mathematical counterparts are directly related to geometry-of-numbers examples in the work of Bhargava et al. This paper is dedicated to the memory of Peter Freund. The last section is devoted to some of MG’s personal reminiscences of Peter Freund.

Keywords: attractor black holes, magical supergravities, geometry of numbers

1. Introduction

Studies of BPS states provide one of the few windows we have into the structure of non-perturbative field theory and string theory. It is therefore interesting to ask if underlying (perhaps non-manifest) mathematical structures can be uncovered from studies of BPS spectra. One set of hints in this direction appears in the papers [1, 2], where connections between arithmetic and the study of attractor black holes [3] were discussed. A particularly striking observation in these papers relates the numbers of attractor black holes at a fixed entropy in $K3 \times T^2$ compactification of type II strings to class numbers of binary quadratic forms with negative discriminant.

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These numbers are defined as follows. Consider quadratic forms
\[ ax^2 + bxy + cy^2, \]
which have integral coefficients and are positive definite, so that in particular the discriminant
\[ D = b^2 - 4ac \]
is negative. The arithmetic group \( SL(2, \mathbb{Z}) \) acts on such quadratic forms by simply acting on \((x, y)\) by the \( SL(2, \mathbb{Z}) \)-transformation. The number \( H(D) \) of such \( SL(2, \mathbb{Z}) \)-equivalence classes at a fixed discriminant \( D \) defines the (Hurwitz) class number, which in turn may immediately be related to class numbers of imaginary quadratic fields. For example, if \( D \) is squarefree, \( H(D) \) is simply the class number\(^4\) of \( \mathbb{Q}[\sqrt{D}] \).

The low energy effective supergravity describing the \( K^3 \times T^2 \) compactification of type II superstring belongs to the family of \( N = 4 \) Maxwell–Einstein supergravity theories that describes the coupling of \( N = 4 \) supergravity to \( N = 4 \) vector multiplets with the global symmetry group \( SO(n, 6) \times SU(1, 1) \)---realizing the specific case \( n = 22 \). Moore argues in [1, 2] that given a BPS black hole with quantized electric and magnetic charges \((q, p)\) transforming in the \((n + 6, 2)\) representation of \( SO(n, 6) \times SU(1, 1) \), one may associate the \( SL(2, \mathbb{Z}) \)-equivalence class of binary quadratic forms with
\[ a = \frac{1}{2}p^2, \quad b = p \cdot q, \quad c = \frac{1}{2}q^2. \]
This association arises naturally from the geometry of the attractor \( K^3 \times T^2 \) associated to the black hole with those charges. Then \( H(D) \) gives the number of distinct attractor black holes at a fixed value of the discriminant \( D \), which coincides with the quartic invariant of \( SO(n, 6) \times SU(1, 1) \) and also governs the supergravity black hole entropy:
\[ S = \sqrt{p^2q^2 - (p \cdot q)^2}. \]
This relationship, and its extension to the class groups which endow the equivalence classes at fixed \( D \) with a group multiplication law [1, 2], was reviewed recently in [4].

It is natural to wonder whether this connection is an accident of special coincidences about the geometry of \( K^3 \) or if it might presage a more general set of relations between BPS black holes and structures in number theory. Here, we give evidence for the latter viewpoint by generalizing this observation to a wider class of supergravity theories, many of which arose originally in the study of ‘magical’ supergravities in the early 1980s [5, 6]. Many of these theories, in turn, admit embeddings into string theory\(^5\). In particular, we highlight the striking observation that the special points of arithmetic interest in many of Bhargava’s geometry-of-numbers examples are in exact correspondence with special points from the point of view of supergravity via the attractor flow.

The organization of this note is as follows. In the next section, we describe examples of the physical theories of interest. In each, the classification of attractor points is equivalent to the problem of counting equivalence classes of certain forms modulo the action of an arithmetic

\[^4\] Strictly speaking, the Hurwitz class number also has some fractional contributions to correctly account for automorphisms.

\[^5\] More precisely, we are only concerned with the geometry of the vector multiplet moduli space here. So realizations of these theories with extra purely neutral hypermultiplets will constitute UV completions of our picture. In what follows, whether we know an explicit string embedding or not, we will assume that the supergravity duality group is broken to a corresponding arithmetic group in the full theory. This is certainly what happens when a known string embedding exists.
group; the black hole entropy is then further controlled by a suitable arithmetic invariant. In several cases, this corresponding algebraic structure also appears in the study of Bhargava’s geometry of numbers as originally studied in the context of higher composition laws [7], and the class numbers governing attractor degeneracies have been of independent interest in the study of arithmetic statistics\(^6\). We conclude by discussing some natural directions for future exploration. In an appendix we describe in more detail the connection of the structures we study to Jordan algebras.

2. Physical theories and mathematical structures

In this section, we describe in telegraphic terms some of the supergravity theories (often with string embedding) which figure in the sequel. They give rise to mathematical structures which can be related to the work of Bhargava, as discussed in a different language in e.g. the work of Krutelevich [9]; see particularly table 1 there. We should stress that Krutelevich uses the language of Jordan algebras and related Freudenthal triple systems which underlie the symmetry groups and geometries of all extended Maxwell–Einstein supergravity theories with symmetric target spaces in \(d = 5, 4, 3\), in particular the magical supergravity theories [5], as well as \(N > 4\) simple supergravity theories. For a review and references on the subject we refer to [10].

2.1. Example one

Consider pure \(5d\) \(N = 2\) Poincare supergravity, or more generally a \(5d\) \(N = 2\) theory with no vector multiplets (but perhaps coupled to hypermultiplets), a theory with eight supercharges. Upon compactification on a circle, one obtains \(4d\) \(N = 2\) supergravity with a single vector multiplet (coming from Kaluza–Klein reduction of the graviton on the circle, plus its superpartners) plus, perhaps, neutral hypers\(^7\). The cubic form that defines this theory is simply \(N = X^3\) which leads to the prepotential

\[ F = X^3 / X_0 \]

in four dimensions.

The black holes in this theory are characterized by two electric charges (under the matter vector multiplet and the graviphoton), and the corresponding magnetic charges. They transform in the spin \(s = 3/2\) representation of an \(SL(2, \mathbb{R})\) duality group of the \(4d\) supergravity, as discussed in [11]. We should note that the attractor flows for BPS black holes in general homogeneous supergravity theories, including this theory, were studied in [13, 14]. The attractor flows for non-supersymmetric black holes in this theory were studied in detail in [12].

In a non-perturbative completion of this theory, one expects \(SL(2, \mathbb{R})\) to be demoted to \(SL(2, \mathbb{Z})\). There is a U-duality invariant formula for the black hole entropy in such a theory, given in e.g. [15]. We find it to be in beautiful correspondence with the following construction.

Consider a binary cubic form

\[ F(x, y) = ax^3 + bx^2y + cxy^2 + dy^3. \] (1)

It has a discriminant given by

\[ D = 18abcd + b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2. \] (2)

\(^6\)For an accessible introduction to this subject, see e.g. [8].

\(^7\)We should note that there is another \(d = 4\) \(N = 2\) Maxwell–Einstein supergravity with one vector multiplet which does not have an uplift to five dimensions. Its quartic invariant is given by the square of a quadratic form.
By a simple change of variables, 
\[ a = -\xi_0/3, \quad b = \xi_1/3, \quad c = -\eta_1/3, \quad d = \eta_0/3 \]
it takes the form
\[ \tilde{I}_4 = -\tilde{I}_4/3 \]
where
\[ \tilde{I}_4 = (\xi_0^2)(\eta_0^2) + 4(\xi_1)^3(\eta_0) + 2\xi_0\eta_0\xi_1\eta_1 - 1/3(\xi_1)^2(\eta_1)^2 - 4/27(\xi_0)(\eta_1)^3 \]
is the $SL(2, \mathbb{R})$ invariant quartic form corresponding to equation (3.40) of [15]. This in turn governs the entropy of 4d charged black holes in our supergravity theory.

There is a natural action of $SL(2, \mathbb{Z})$ on $(x, y)$ which induces an action of $SL(2, \mathbb{Z})$ on the set of binary cubic forms. Defining equivalence classes as we did in the case of binary quadratic forms, we obtain class numbers for cubic forms $h_3(D)$. These were studied by Davenport in 1951 [16]. These count the inequivalent attractor black holes (at fixed entropy) in the physical theory we are describing. The promotion of these class numbers to orders of class groups is described in [7].

The geometry of numbers of this example proceeds by an intriguing quadratic map from binary cubic forms to binary quadratic forms originally investigated by Eisenstein [17]. We refer to [18] for a review of Eisenstein’s work and further references on the subject, which we explain here to indicate the special point structure as well as the fundamental importance of the quadratic transformation of the charge lattice.

Indeed, given a binary cubic form $F(x, y)$ as in equation (1) with discriminant $D$ one can associate with it a closely-related binary quadratic form $Q_F(x, y)$ essentially given by the Hessian of $F$, so that the association is naturally $SL(2, \mathbb{Z})$-equivariant. To be more explicit, it is convenient to restrict our cubic forms slightly and instead take
\[ F(x, y) = ax^3 + 3bx^2y + 3cxy^2 + dy^3, \]
with $a, b, c, d$ still all integers. Then the associated quadratic
\[ Q_F(x, y) = Ax^2 + Bxy + Cy^2 \]
is given by
\[ A = b^2 - ac, \quad B = bc - ad, \quad C = c^2 - bd \]
and the discriminant $D_Q$ of the quadratic form is related to the discriminant $D$ of the cubic form as
\[ D_Q = B^2 - 4AC = -D/27. \]

As mentioned, the mapping from $F(x, y)$ into $Q_F(x, y)$ commutes with the $SL(2, \mathbb{Z})$ action, so we have a map from equivalence classes of integral binary cubic forms at a given discriminant to those of binary quadratic forms at (roughly) the same discriminant. It is obvious interest to understand if this map is one-to-one and to characterize its image. We mention here only the simplest case, of $D_Q \equiv 0 \pmod{4}$ and $D_Q/4$ squarefree, in which case the map is indeed injective but the image is striking: one obtains exactly those points of order 3 in the class group $\text{Cl}(\mathbb{Q}(\sqrt{D_Q}))$. 

4
Of particular interest is the formula for the growth of the class numbers \( h_3(D) \) with \( D \). Davenport proves that

\[
\sum_{N=1}^{D} h_3(N) = \frac{\pi^2}{108} D + O(D^{15}).
\]

It would be interesting to reproduce this formula directly from the perspective of supergravity counts of black hole attractors.

Regarding a string construction of the pure supergravity theory with 8 supercharges in five dimensions, its possible existence is discussed (as a ‘fantasy island’) in [19] but more completely analyzed in [20, 21] as a consistent truncation of type II supergravity (while not directly arising as a massless sector of a string compactification).

2.2. Example two

In a similar spirit, one can consider a theory of 4d \( N = 2 \) supergravity coupled to two vector multiplets that descends from \( N = 2 \) supergravity coupled to one vector multiplet in \( d = 5 \). The cubic norm that defines this theory is \( \mathcal{N} = X^2 Y \) which leads to the prepotential \( X^2 Y / X_0 \) in four dimensions. The scalar manifold of the 4d supergravity is the symmetric space

\[
\mathcal{M}_4 = \frac{SO(2,2)}{U(1) \times U(1)}
\]

The discrete U-duality group is \( SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z}) \) under which the six charges (including electric and magnetic graviphoton charge) transform in the (3, 2) representation.

The mathematical structure here is that of \textbf{pairs} of binary quadratic forms, which are exchanged by one of the \( SL(2, \mathbb{Z}) \) symmetries. The other acts on both (simultaneously) as described in the natural action on binary quadratic forms above. A higher composition on pairs of binary quadratic forms is described in [7].

Given such a pair

\[
ax^2 + bxy + cy^2 \\
Ax^2 + Bxy + Cy^2
\]

there are three natural discriminants one can define; those of each of the quadratic forms \( \Delta_{1,2} \), and the codiscriminant

\[
\Delta_c = bB - 2aC - 2cA.
\]

The problem of counting equivalence classes of such forms (up to action of the arithmetic group) for triples of these quantities has been discussed by Morales [22]. This is a more refined count than is natural in supergravity, where the entropy would be determined by the single duality invariant

\[
\Delta_c^2 - 4\Delta_1 \Delta_2.
\]

Hence, (sums of) the class numbers of Morales govern the attractor counts of black holes in this supergravity theory.

2.3. Example three

We can consider the famous STU model with three vector multiplets coupled to 4d \( N = 2 \) supergravity. This theory descends from the 5d Maxwell–Einstein supergravity with two vector
multiplets defined by the cubic norm \(XYZ\) that was first studied in [23]. A nice discussion of BPS black holes in this model can be found in [24]. A promotion of this model to string theory is described in [25] (where it is basically \(N = 2\) example D) and in [26].

The prepotential in \(d = 4\) is \(XYZ/X_0\) and is generally denoted as

\[
F = \text{STU}
\]

in the gauge \(X_0 = 1\). The model has an \(SL(2,\mathbb{Z})^3\) duality symmetry. The 8 electric and magnetic charges transform in the \((2,2,2)\) representation of the duality group.

The algebraic structure involved here is

\[
V_2 \otimes V_2 \otimes V_2
\]

with \(V_2\) the two dimensional representation of \(SL(2,\mathbb{Z})\). Explicit actions of the duality group on electric and magnetic charges can be found in [24]. In a suitable basis of the electric and magnetic charges, the duality invariant entropy is again given by

\[
S = \sqrt{p^2 q^2 - (p \cdot q)^2}.
\]

Again, class numbers are determined by the numbers of elements in \(V_2^3\) modulo the action of \(SL(2,\mathbb{Z})^3\), and give the numbers of distinct attractor points at different values of the entropy.

### 2.4. Example four

There is a similar story relating counts of BPS black holes to binary quartic forms. Consider the family of forms

\[
f(x, y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4.
\]

Again, there is a natural \(SL(2,\mathbb{Z})\) symmetry that acts on \((x, y)\) and induces an action on the space of forms. There are now two invariants:

\[
I(f) = 12ae - 3bd + c^2
\]

\[
J(f) = 72ace + 9bcd - 27ad^2 - 27eb^2 - 2c^3.
\]

The discriminant of \(f\) is

\[
\Delta(f) = \frac{1}{27} \left(4I(f)^3 - J(f)^2\right).
\]

The discriminant \(\Delta(f)\) has the form of the entropy of the 5d uplift of an extremal black hole of \(N = 2\) Maxwell–Einstein supergravity with one vector multiplet in \(d = 4\), except for the fact that it involves an extra parameter in addition to four charges (two electric and two magnetic) with respect to the 4d vector fields. This is easily established by setting the integer \(e\) in the binary cubic form equal to zero. We then find that the discriminant takes the form

\[
\Delta(f)|_{e=0} = d^2 D
\]

It is also easy to construct Calabi–Yau threefolds with \(h^{1,1} = 3\) and a complex structure moduli space governed by a prepotential of this form; we thank J Bryan for discussions of this point.

The \(SL(2,\mathbb{Z})\) factors are broken slightly to congruence subgroups in the known string embeddings, which would cause minor modification to the discussion below.
where $D$ is the discriminant of the binary cubic form given above. The quartic invariant $I_4$ that defines the entropy of charged black holes given in equation (3.40) of [5] can be written in a more symmetrical way between the electric and magnetic charges, via the identification

$$
\xi_0 = -q_0 = -9a, \quad \eta_0 = p_0 = d/3, \quad \xi_1 = q_1/3 = b/3, \quad \eta_1 = 3p_1 = c.
$$

Then the quartic invariant takes the form

$$
I_4 = \frac{1}{9} \left( \frac{-D}{3} \right)^2 - 4(27p_0 q_1^2 + 3p_1 q_1^2 (p_1)^2).
$$

The quadratic and cubic invariants $I(f)$ and $J(f)$ associated with the binary quartic form then take the following form in terms of these variables and the extra parameter $e$

$$
I(f) = -\frac{4}{3}eq_0 - 9p_0 q_1 + 9(p_1)^2
$$

$$
J(f) = 3 \left( -9e(q_1)^2 - 9q_0(p_0)^2 + 8eq_0 p_1 + 27p_1 q_0 q_1 - 18(p_1)^3 \right).
$$

Now for $e = 0$ the discriminant $\Delta$ describes the duality invariant form governing the entropy of a 5d uplift of an extremal black hole of a 4d, $N = 2$ Maxwell–Einstein supergravity with one vector multiplet, since

$$
\Delta|_{e=0} = -27(p_0)^2 I_4. \quad (6)
$$

This follows from the general result that the entropy of a spinning charged black hole (or ring) that is an uplift of 4d extremal black hole with charges $p_i, q_i$ has the form [27–29]

$$
S_{5d} = \sqrt{\left( N_3(Q_i) - J^2 \right)}. \quad (7)
$$

where

$$
Q_i = p^0 q_i + \frac{1}{2} C_{ijk} p^j p^k
$$

$$
N_3(Q_i) = \frac{1}{6} C_{ijk} Q_i Q_j Q_k
$$

$$
J = \frac{1}{2} \left( p^0(p^0 q_0 + p^j q_j) + \frac{1}{3} C_{ijk} p^j p^k \right). \quad (8)
$$

Then the entropy of the corresponding 4d BPS black hole is given by

$$
S_{4d} = \frac{2\pi}{|p_0|} \sqrt{N_3(Q_i) - J^2},
$$

where $J$ is the 5d angular momentum.

Therefore the invariant $J(f)|_{e=0}$ describes the 5d angular momentum of the spinning black hole in our example. We should note that the general black hole solution of pure $N = 2$ supergravity in five dimensions involving six parameters, namely the four electromagnetic charges, mass and angular momentum was given in [30] to which we refer for references on earlier work on the subject. However the general solution has not been written in terms of an invariant corresponding to the discriminant $\Delta$ with non-vanishing parameter $e$. On the basis of the above analysis we predict that there exists a five parameter family of extremal black ring solutions of pure $N = 2$ 5d supergravity whose entropy is described by the invariants $J(f), I(f)$ and $\Delta(f)$.
As was shown in [31] for fixed invariants $I(f)$ and $J(f)$ there exists a single orbit of $SL(2, \mathbb{R})$ in the space of quartic forms $F(x, y)$ if $\Delta(f) < 0$ and the orbit lies in the subspace of quartic forms with one pair of complex roots. For fixed invariants $I(f)$ and $J(f)$ with positive discriminant $\Delta(f) > 0$ one finds three different orbits of $SL(2, \mathbb{R})$. Two of these orbits lie in the subspace where the quartic form admits two pairs of complex roots while the third orbit lies in the subspace with real roots only.

Aspects of the asymptotics of class numbers $h(I, J)$ of binary quartic forms with given values of $I, J$ are determined by Bhargava and Shankar in [31]. Theorem 1.6 of that paper tells us the following. If we define

$$H(I, J) = \max |I^3|, |J^2/4|,$$

then

$$\sum_{H(I, J) < X} h(I, J) \sim \zeta(2)X^{5/6} + O(X^{3/4 + \epsilon}).$$

Again, it would be nice to present a physics proof.

2.5. Magical supergravities

In this section we will give an example related to one of the magical supergravity theories [5, 6] and indicate how this example could be extended to the other magical supergravity theories, that are briefly reviewed in the appendix.

Consider the extension of example 1 to the case of ternary quadratic forms [39]

$$Q(x, y, z) = ax^2 + by^2 + cz^2 + uyz + vzx + wxy. \tag{7}$$

The (half)-discriminant of $Q(x, y, z)$ is given by

$$\Delta_Q = \frac{1}{2}Z = 4abc + uvw - au^2 - bv^2 - cw^2 \tag{8}$$

where

$$Z = \begin{pmatrix} 2a & w & u \\ w & 2b & v \\ u & v & 2c \end{pmatrix}. \tag{9}$$

Therefore the matrix $Z$ can be identified with the charges of extremal black hole solutions in real magical supergravity defined by the Jordan algebra $J^3_R$ of $3 \times 3$ symmetric matrices and the discriminant $\Delta_Q$ determines their entropy. Under the action of 5d U-duality group $SL(3, \mathbb{R})$ of real magical supergravity on $(x, y, z)$ the ternary quadratic form transforms as follows:

$$g : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = g^{-1} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow Q(x', y', z') = g \cdot Q(x, y, z) \tag{10}$$

where $g \in SL(3, \mathbb{R})$. One finds that the discriminant, that determines the entropy, is left invariant under this action of $SL(3, \mathbb{R})$:

$$\Delta[Q(x', y', z')] = \text{Det}(g) \Delta[Q(x, y, z)] = \Delta[Q(x, y, z)]. \tag{11}$$
The prepotential of the real magical supergravity in four dimensions is given by

\[ F \sim - i \frac{\det(X)}{X^0}, \]

where \( X \in J^R_3 \). The U-duality group of the four dimensional supergravity is \( Sp(6, \mathbb{R}) \) with the maximal compact subgroup \( U(3) \). The entropy of 4d BPS black hole solutions is given by the quartic invariant of the Freudenthal triple system associated with \( J^R_3 \) [10, 36, 37]. This vector multiplet moduli space is \( Sp(6, \mathbb{R})/U(3) \) and corresponds to the item 9 in table 1 of [9]. The charges transform in the 14 dimensional irreducible representation of the arithmetic group, and the relevant highest weight module \( V(\omega_3) \) is identified in [9] and serves as the analogue of the binary forms in our previous examples. The Fourier coefficients of modular forms on \( PGSp(6) \) and their connection to quaternion rings were studied in [40].

We should note that both Krutelevich and Bhargava in the cited work [9] restrict themselves to split real forms of the corresponding U-duality groups. For the real magical supergravity the U-duality groups in five, four and three dimensions are all of the split form. However the case of real magical supergravity does not appear in Bhargava’s work and Krutelevich suggests that it could produce a new example of a space with higher composition law. Here we identify the corresponding supergravity theory as the real magical supergravity theory which can be obtained by orbifolding the quaternionic magical supergravity [5] constructed by Sen and Vafa from superstring theory by orbifolding using the dual pair method [25].

In Krutelevich’s table the rows 5,6 and 7 correspond to Jordan algebras of \( 3 \times 3 \) Hermitian matrices over the split composition algebras with associated symmetry groups \( SL(6, \mathbb{R}) \), \( SO(6, 6) \) and \( E_{7(7)} \). Of these only the one corresponding to split complex numbers appears in Bhargava’s work. Its symmetry group is \( SL(6, \mathbb{R}) \). The split exceptional Jordan algebra is associated with the maximal \( N = 8 \) supergravity with its 4d U-duality group \( E_{7(7)} \). Maximal supergravity can be truncated to a purely bosonic theory with \( SO(6, 6) \) symmetry or \( SL(6, R) \) symmetry. Since they are not supersymmetric one can not have BPS black holes in these truncated theories.

On the other hand maximal supergravity can be truncated to quaternionic magical supergravity with \( SO'(12) \) symmetry and 15 vector multiplets with target space \( SO'(12)/U(6) \) in four dimensions. This is the theory obtained by Sen and Vafa from superstring theory by orbifolding. Quaternionic theory can further be truncated to the complex magical supergravity with \( SU(3, 3) \) symmetry and 9 vector multiplets [5, 6].

\[ M_{\text{vector}} = SU(3, 3; \mathbb{Z})\backslash SU(3, 3)/\left( SU(3) \times SU(3) \times U(1) \right) \]

The moduli space is 9 (complex) dimensional, and the prepotential is given by

\[ F \sim - i \frac{\det(X)}{X^0}, \]

where \( X \) is a three-by-three Hermitian matrix. It descends from the complex magical supergravity in 5d with the scalar manifold

\[ M_5 = SL(3, \mathbb{C})/SU(3). \]

This model was known to have a string theory embedding, as a \( \mathbb{Z}_3 \) orbifold of \( T^6 \) compactification before the work of Sen and Vafa and was discussed in detail in [32].

Underlying Jordan algebras of magical supergravity theories are all Euclidean defined by \( 3 \times 3 \) Hermitian matrices over the four division algebras. This and the above mentioned facts suggest that Krutelevich’s work and related Bhargava’s higher composition laws may be
extended along magical lines by replacing the split Jordan algebras with the Euclidean Jordan algebras. Their symmetry groups will be different real forms of the split real forms appearing in Krutelevich’s table including the exceptional Jordan algebra which underlies the octonionic magical supergravity with target space $E_{7(-25)}/E_6 \times U(1)$ in four dimensions.

We should also stress the important fact for all the rows in Krutelevich’s table that have a supergravity realization the last column lists symmetry groups that are isomorphic to the three dimensional supergravity theories obtained by dimensional reduction. These groups have been proposed as spectrum generating symmetry groups of $4d$ black holes\(^{10}\). Furthermore the Fourier coefficients of modular forms associated with the minimal unitary representations of the three dimensional U-duality groups are expected to describe the degeneracies of $4d$ black holes\([13, 29]\). For the examples discussed above these groups are $G_2(2), SO(4, 3), SO(4, 4)$ and $F_4(4)$ corresponding to $d = 4N = 2$ supergravity coupled to one, two, three and six vector multiplets, respectively.

3. Discussion

In this note, we outlined a connection between special points in geometry-of-numbers examples and special points in supergravity. If one wishes to count these points, one matches class numbers characterizing forms with fixed values of various arithmetic invariants and BPS black holes in supergravity and string theory. While we provided a few examples in section 2, it should be clear that similar considerations extend to many further constructions. Extension of our results to other supergravity theories, in particular those that have stringy extensions, and possible extensions along the lines of example four above will be left to future investigations.

In the prototype case involving binary quadratic forms and BPS black holes on $K3 \times T^2$, it has been observed that the generating function

$$
\sum_D H(D) q^D
$$

(with suitable constant term) provides the holomorphic piece of a weight 3/2 mock modular form\([33]\). It would be very interesting to relate the black hole counting problems above to modular objects in a similar manner. In fact, the general statements of Kudla–Millson theory, as described in\([35]\), encapsulate the count of attractor black holes in our example 2 in a degree 2 weight 2 Siegel form studied by Kohnen\([34]\). Kohnen’s $Sp(4, \mathbb{R}) = \text{Spin}(3, 2)$ symmetry is a subgroup of the U-duality group $SO(4, 3)$ of the corresponding three dimensional supergravity theory that was proposed as spectrum generating symmetry group of the $4d$ supergravity as mentioned above.

The results here suggest many other natural questions:

- Can we find proofs of the asymptotic results governing class numbers (some of which were presented above) using physical arguments?
- The theories most closely tied to these arithmetic variants seem to be $N \geq 2$ supersymmetric theories which enjoy a certain finiteness property (no instanton corrections to the prepotential). Can we demonstrate a connection of all such $N = 2$ theories to the theory of arithmetic invariants? Can we turn this around and propose a classification of such theories in terms of invariant theory?

\(^{10}\) See the lectures\([10]\) and the references therein.
Most ambitiously, one would like to find a version of this story which holds in quantum geometry, e.g. for counts of BPS black holes in the mirror quintic. This should involve a suitable generalization of the various classical number theoretic notions that played a role here. Progress on this front would be most interesting.

4. Remembering Peter Freund (M G)

Scientifically I got to be introduced to Peter Freund for the first time when I received a preprint of his paper on ‘quark parastatistics and color gauging’ (mid 1970s) in which he coined the term GG statistics referring to my work with my advisor Feza Gürsey on the connection between color quarks and octonions. Personally I got to meet him for the first time in a workshop at the University of Washington organized by Tony Zee at the beginning of 1980s. The participants of that workshop were all housed in a sorority building on the campus of the university. One of the most memorable events of the workshop was the five course dinner that John Iliopoulos cooked, in the kitchen of the sorority, for all the participants which was certainly worthy of some Michelin stars. The evening was capped by Peter singing the Kindertotenlieder, the song cycle of Mahler, in the accompaniment of another physicist at the piano. It was an emotional performance. Afterwards Peter told me that singing the Kindertotenlieder moves him always very deeply.

My next encounter with Peter was at the University of Chicago where I gave a seminar on my work on magical supergravity theories which found a very receptive audience in him. I appreciated greatly the stimulating discussions I had with him after my talk. Peter’s work on Kaluza–Klein supergravity, in particular his paper with Rubin on $S^7$ compactification of 11 dimensional supergravity, had a major impact in the field including some of my own work. My work on the construction of the spectra IIB supergravity on $S^5$ (with Marcus), of 11d supergravity on $S^7$ (with Warner) and on $S^5$ (with van Nieuwenhuizen and Warner), by a simple tensoring procedure from some fundamental supermultiplets, represent some of the earliest work on AdS/CFT dualities within the framework of Kaluza–Klein supergravity in a true Wignerian sense. All this work was done during the summer of 1984 in Aspen. Next door to us Mike Green and John Schwarz were busy working on their famous anomaly cancellation mechanism in superstring theory. I do not recall seeing Peter in Aspen that summer. Maybe we simply did not overlap.

Peter attended a conference in memory of my advisor Feza Gürsey, who was his friend, that we organized on the campus of my alma mater, Bogazici University (formerly Robert College) during the summer of 1994. He clearly enjoyed his visit to Istanbul, the conference and the historical sites. It was a pleasure to be in his company. He always emanated a sense of enthusiasm and excitement whatever he talked about with his baritone voice.

Last time I saw him was at Gunnar Nordström Symposium on theoretical physics at the University of Helsinki, Helsinki, Finland on August 28, 2003. Peter had very broad interests in physics as well as outside physics and has made many important original contributions to theoretical physics. Outside of his work on Kaluza–Klein supergravity he is probably best known for his work on p-adic strings.

Peter Freund belonged to that endangered species of scholarly gentleman physicists of the Old World. I will always remember him fondly.

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Appendix

The examples we considered in this paper correspond to the \( N = 2 \) Maxwell–Einstein supergravity theories with zero, one, two and five vector multiplets in \( d = 5 \) and their \( 4d \) counterparts. Five dimensional \( N = 2 \) Maxwell–Einstein supergravities with symmetric target spaces \( G/H \), such that \( G \) is a symmetry of the Lagrangian, are in one-to-one correspondence with the Euclidean Jordan algebras of degree three \([5,6]\). The \( C \)-tensor \( C_{IJK} \) appearing in the coupling

\[
C_{IJK} F^I \wedge F^J \wedge A^K
\]

in these theories determines their Lagrangian uniquely and is given by the cubic norm of the underlying Jordan algebra \( J \). The Jordan algebras of degree three admit 3 idempotents \( e_1, e_2 \) and \( e_3 \) with the identity element given by

\[
I = e_1 + e_2 + e_3
\]

which corresponds to the bare graviphoton in the corresponding supergravity theory. Hence the pure \( N = 2 \) supergravity is described by the norm

\[
\mathcal{N}_3(XI) = X^3.
\]

The example 2 we considered above corresponds to the \( N = 2 \) Maxwell–Einstein supergravity described by the cubic norm

\[
\mathcal{N}_3(Xe_1 + Ye_2 + Ye_3) = XY^2
\]

and the STU model in five dimensions is defined by

\[
\mathcal{N}_3(Xe_1 + Ye_2 + Ze_3) = XYZ
\]

as found in [23]. The real magical supergravity in 5\( d \) is defined by the Jordan algebra \( J_R^3 \) of three-by-three real symmetric matrices and the norm of an element \( J \) is given by its determinant

\[
\mathcal{N}_3(J) = \det(J)
\]

By the action of the automorphism group \( H \) of the underlying Jordan algebra every element of \( J \) can be brought to the form \( (Xe_1 + Ye_2 + Ze_3) \) of the STU model. For the real magical supergravity the automorphism group of \( J_R^3 \) is \( SO(3) \). This shows clearly that there exist natural extensions of the invariants we studied to more general classes of invariants related to the arithmetic subgroups of the global symmetry groups of the \( N = 2 \) supergravity theories. For \( N = 4 \) Maxwell–Einstein supergravities in 5\( d \), the underlying Jordan algebras of degree three are non-Euclidean. Similarly the symmetries of maximal \( N = 8 \) supergravity are given by the symmetries of the non-Euclidean exceptional Jordan algebra of Hermitian 3 \( \times \) 3 matrices over
the split octonions [36]. Under dimensional reduction to four dimensions the correspondence between Jordan algebras of degree three and 5d supergravities goes over to a correspondence between Freudenthal triple systems $F(J)$ associated with the Jordan algebras $J$ of degree three and 4d supergravities [10, 36, 37].

Orbits of extremal black hole solutions of supergravity theories under the action of continuous U-duality groups $G$ have been studied extensively beginning with the work of [36]. For those theories that have stringy extensions the relevant orbits are with respect to the discrete U-duality groups which are typically the maximal arithmetic subgroups $G(\mathbb{Z})$ of $G$ [38, 41]. Motivated by the works of [36, 38] Krutelevich considered the integral version of Freudenthal’s construction of exceptional groups and studied their connection to higher composition laws of Bhargava [9]. Of all the examples discussed in this paper only the example 4 does not appear in Krutelevich’s table.

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References

[1] Moore G W 1998 Attractors and arithmetic (arXiv:hep-th/9807056)
[2] Moore G W 1998 Arithmetic and attractors (arXiv:hep-th/9807087)
[3] Ferrara S, Kallosh R and Strominger A 1995 $N = 2$ extremal black holes Phys. Rev. D 52 R5412
[4] Ferrara S and Kallosh R 1996 Universality of supersymmetric attractors Phys. Rev. D 54 1525–34
[5] Ferrara S and Kallosh R 1996 Supersymmetry and attractors Phys. Rev. D 54 1514
[6] Strominger A 1996 Microscopic entropy of $N = 2$ extremal black holes Phys. Lett. B 383 39
[7] Bhargava M 2004 Higher composition laws I: a new view on Gauss composition, and quadratic generalizations Ann. Math. 159 217
[8] Yao Xiao S 2015 Lecture notes for Bhargavaology learning seminar https://theconsciousmathematician.files.wordpress.com/2015/02/bhargavaology-learning-seminar.pdf
[9] Krutelevich S 2007 Jordan algebras, exceptional groups, and Bhargava composition J. Algebra 314 924
[10] Günaydin M 2010 Lectures on spectrum generating symmetries and U-duality in supergravity, extremal black holes, quantum attractors and harmonic superspace Springer Proc. Phys. 134 31
[11] Mizoguchi S and Ohta N 1998 More on the similarity between $D = 5$ simple supergravity and M theory Phys. Lett. B 441 123–32
[12] Gaiotto D, Li W and Padi M 2007 Non-supersymmetric attractor flow in symmetric spaces J. High Energy Phys. JHEP12(2007)093
[13] Günaydin M, Neitzke A, Pionne B and Waldron A 2006 BPS black holes, quantum attractor flows and automorphic forms Phys. Rev. D 73 084019
[14] Günaydin M, Neitzke A, Pionne B and Waldron A 2007 Quantum attractor flows J. High Energy Phys. JHEP09(2007)056
[15] Günaydin M, Neitzke A, Pavlyk O and Pionne B 2008 Quasi-conformal actions, quaternionic discrete series and twistors: $SU(2, 1)$ and $G_2(2)$ Commun. Math. Phys. 283 169
[16] Davenport H 1951 On the class-number of binary cubic forms (I) J. Lond. Math. Soc. 26 183–92
[17] Eisenstein G 1844 Untersuchungen über die cubischen Formen mit zwei Variabeln (German) J. Reine Angew. Math. 27 89–104
Eisenstein G 1844 Théorèmes sur les formes cubiques et solution d’une équation du quatrième degré à quatre indéterminées (French) *J. Reine Angew. Math.* **27** 75–9

[18] Hoffman W J and Morales J 2000 Arithmetic of binary cubic forms *Enseign. Math.* **46** 61–94

[19] Dabholkar A and Harvey J String islands 1998 (arXiv:hep-th/9809122)

[20] Mizoguchi S and Schröder G 2000 On discrete U-duality in M-theory *Class. Quantum Grav.* **17** 835–70

[21] Mizoguchi S 2001 On asymmetric orbifolds and the $D = 5$ no-modulus supergravity *Phys. Lett.* B 523 351–6

[22] Morales J 1991 The classification of pairs of binary quadratic forms *Acta Arith.* **26** 183–92

[23] Günaydin M, Sierra G and Townsend P K 1985 Gauging the $d = 5$ Maxwell/Einstein supergravity theories: more on Jordan algebras *Nucl. Phys.* B 253 573

[24] Behrndt K, Kallosh R, Rahmfeld J, Shmakova M and Wong W 1996 STU black holes and string triality (arXiv:hep-th/9608059)

[25] Sen A and Vafa C 1995 Dual pairs of type II string compactification *Nucl. Phys.* B **455** 165

[26] Gregori A, Kounnas C and Marios Petropoulos P 1999 Non-perturbative triality in heterotic and type II $N = 2$ strings *Nucl. Phys.* B 553 108

[27] Gaiotto D, Strominger A and Yin X 2006 5D black rings and 4D black holes *J. High Energy Phys.* JHEP02(2006)023

[28] Gaiotto D, Strominger A and Yin X 2006 New connections between 4D and 5D black holes *J. High Energy Phys.* JHEP02(2006)024

[29] Pioline B 2005 BPS black hole degeneracies and minimal automorphic representations *J. High Energy Phys.* JHEP08(2005)071

[30] Tomizawa S and Mizoguchi S 2013 General Kaluza–Klein black holes with all six independent charges in five-dimensional minimal supergravity *Phys. Rev.* D **87** 024027

[31] Bhargava M and Shankar A 2015 Binary quartic forms having bounded invariants, and the boundedness of the average rank of elliptic curves *Ann. Math.* **181** 191

[32] Ferrara S, Fre P and Soriani P 1992 On the moduli space of the $T 6/Z 3$ orbifold and its modular group $T 6/Z 3$ orbifold and its modular group *Class. Quantum Grav.* **9** 1649

[33] Kachru S and Tripathy A 2017 Black holes and Hurwitz class numbers *Int. J. Mod. Phys.* D **26** 1742003

[34] Kohnen W 1993 Class numbers, Jacobi forms and Siegel–Eisenstein series of weight 2 on Sp2 ($\mathbb{Z}$) *Math. Z.* **213** 75

[35] Kachru S and Tripathy A 2018 BPS jumping loci are automorphic *Commun. Math. Phys.* **360** 919

[36] Ferrara S and Günaydin M 1998 Orbits of exceptional groups, duality and BPS states in string theory *Int. J. Mod. Phys.* A **13** 2075

[37] Günaydin M, Koepsell K and Nicolai H 2001 Conformal and quasiconformal realizations of exceptional lie groups *Commun. Math. Phys.* **221** 57

[38] Maldacena J M, Moore G W and Strominger A 1999 Counting BPS black holes in toroidal type II string theory (arXiv:hep-th/9903163)

[39] Gross B H and Lucianovic M W 2009 On cubic rings and quaternion rings *J. Number Theor.* **129** 1468–78

[40] Lucianovic M W Quaternion rings, ternary quadratic forms, and Fourier coefficients of modular forms on PGSp(6) 2003 PhD Thesis Harvard University, ProQuest LLC, Ann Arbor, MI

[41] Benjamin N, Kachru S and Tripathy A 2017 Counting spinning dyons in maximal supergravity: the Hodge-elliptic genus for tori *Lett. Math. Phys.* **107** 2081