Thermal fluctuations in a charged AdS black hole

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Abstract – In this paper, we will analyze the effects of thermal fluctuations on a charged anti-de Sitter (AdS) black hole. This will be done by analyzing the corrections to black-hole thermodynamics due to these thermal fluctuations. We will demonstrate that the entropy of this black hole gets corrected by a logarithmic term. We will also calculate other corrections to other important thermodynamic quantities for this black hole. Finally, we will use the corrected value of the specific heat to analyze the phase transition in this system.

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Introduction. – It is known that if entropy is not associated with a black hole, then the second law of thermodynamics will get violated [1,2]. This is because the absence of entropy for a black hole can cause a spontaneous reduction in the entropy of the universe. This will happen whenever an object with a finite entropy crosses the horizon. Thus, the black holes are taken to be maximum-entropy objects, i.e., black holes have more entropy than any other object of the same volume [3,4]. This maximum entropy is proportional to the area of the black hole [5], and this observation in turn has led to the development of the holographic principle [6,7]. In fact, almost all the approaches to quantum gravity predict the same relation between the area and the entropy of a black hole. This relation is given by \( S = \frac{A}{4} \), where \( S \) is the entropy of the black hole and \( A \) is the area. However, this maximum entropy of the black holes is expected to get corrected due to quantum fluctuations, and this is in turn expected to modify the holographic principle [8,9]. These quantum fluctuations become important as the black hole reduces its size by giving off Hawking radiation. So, as the black holes get smaller in size due to the Hawking radiation, quantum fluctuations are expected to correct the standard relation between the area and the entropy of a black hole.

Such corrections have been evaluated using several different approaches. The density of microstates for asymptotically flat black holes have been calculated using the non-perturbative quantum general relativity [10]. In this approach, the density of states has been obtained by counting the conformal blocks of a well-defined conformal field theory. It has thus been demonstrated that the non-perturbative quantum general relativity does reproduce the Bekenstein entropy for large black holes. However, it is also possible to calculate the leading-order corrections to this entropy using the non-perturbative quantum general relativity, and there results a logarithmic correction. The Cardy formula has been used to argue that such logarithmic corrections should occur in all the black holes whose microscopic degrees of freedom are described by a conformal field theory [11,12]. Such logarithmic corrections have also been obtained for three-dimensional BTZ black holes by computing the exact partition function [13]. Matter fields in the backgrounds of a black hole have been used to demonstrate that the leading-order corrections to the black-hole entropy are logarithmic corrections [14–16]. It has also been demonstrated that the string-theoretical effects can generate logarithmic corrections to the entropy of a black hole [17–20]. In fact, the corrections to the entropy of dilatonic black holes have been explicitly obtained [21], and they again turn out to be logarithmic corrections. Rademacher expansion of the partition function also has been used to obtain such corrections to the entropy of a black hole [22].

It may be known that recently it has been argued that the corrections to the black-hole entropy can come from the Planckian deformation of the spacetime, due to the generalized uncertainty [23]. It has been demonstrated that such corrections can lead to the existence of black-hole remnants, which can have important phenomenological consequences [24]. It is expected that
quantum fluctuations in the geometry will lead to thermal fluctuations in the thermodynamics of the black holes. It is possible to calculate the corrections to the entropy of a black hole by considering the thermal fluctuations around equilibrium [25, 26]. The corresponding canonical ensemble is stable, as the stability condition is equivalent to having a positive specific heat.

It is possible to construct various asymptotically AdS charged black-hole solutions [27–34]. In fact, it is possible to define a thermodynamic equilibrium, and thus obtain many related results for such black holes [35–38]. So, it is interesting to analyze the effect of thermal fluctuations for various thermodynamic quantities for such black holes. In fact, this can be further used for analysing phase transition in such systems. The phase transitions have also been studied in large AdS black holes [39]. In fact, there is a close analogy between the charged AdS black holes and van der Waals fluids [40, 41]. This analogy becomes even more clear in the extended phase space [42, 43]. This is because in this case the cosmological constant can be treated as a thermodynamical pressure. This can be used to write the first law of thermodynamics in terms of the cosmological constant. Finally, it is also possible to define a thermodynamic volume using this first law of thermodynamics. It will be interesting to see how this analysis changes when thermal fluctuations are taken into account. In this paper, we will use the leading-order form for the corrections to the entropy of a charged AdS black hole to analyze the phase transition of this system. This will be done by first calculating the corrections to the entropy of a charged black hole explicitly, and then using this to calculate other thermodynamical quantities in the system. Finally, that will be used for analyzing the phase transition of this system.

** Charged AdS black hole. ** In this section, we will consider a charged AdS black hole. The action for a four-dimensional asymptotically AdS spacetime coupled to Maxwell’s equations can be written as

\[ I = \int d^4 x \sqrt{-g} \left( R + \frac{6}{l^2} - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} \right), \]

where \( l \) is the AdS radius, and \( F_{\mu \nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \). The field equations for this system can now be written as \( G_{\mu \nu} = 3g_{\mu \nu}/l + F^{\mu \tau} F_\tau^\nu/2 - g_{\mu \nu} F^{\tau \rho} F_\tau_\rho/8 \), and \( \sqrt{-g} \nabla_\nu F^{\mu \nu} = 0 \). The metric for this charged AdS black hole can now be written as

\[ ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_k^2, \]

where

\[ d\Omega_k^2 = d\theta^2 + \frac{1}{k} \sin^2 \left( \sqrt{k} \theta \right) d\varphi^2, \]

and

\[ f = k - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{l^2}. \]

The black-hole horizon radius can be obtained by taking the real positive root of the following equation:

\[ r_+^4 + \beta^2 r_+^2 - 2M l^2 r_+ + l^2 Q^2 = 0. \]

In the path integral formalism, it is possible to calculate the amplitude for a field configuration to propagate to another field configuration. This can be done using a formalism called the Euclidean quantum gravity, where the temporal coordinate is rotated on the complex plane. Thus, the partition function for the charged AdS spacetime can be written as [39, 44–47]

\[ Z = \int Dg \; DA \exp(-I), \]

where \( I \rightarrow -iI \) is the Euclidean action for this system. It may be noted that this can be related to the statistical mechanical partition function [48, 49]

\[ Z = \int_0^{\infty} dE \; \rho(E) e^{-\beta E}, \]

where \( \beta \) is the inverse of the temperature. Now, the density of states can be written as

\[ \rho(E) = \frac{1}{2\pi i} \int_{\beta_0-i\infty}^{\beta_0+i\infty} d\beta \; e^{S(\beta)}, \]

where

\[ S = \beta E + \ln Z. \]

Usually this entropy is measured around the equilibrium temperature \( \beta_0 \), and all thermal fluctuations are neglected. The expression for the entropy, when all the thermal fluctuations are neglected is given by

\[ S_0 = \pi r_+^2. \]

Similarly, the temperature of the charged AdS black hole can be written as

\[ T = \frac{l^2 (r_+^2 - Q^2)}{4\pi^2 r_+^2}. \]

It is also possible to define a thermodynamic volume, in the absence of thermal fluctuations as [5]

\[ V = \frac{4}{3} \pi r_+^3. \]

However, it is possible to take these thermal fluctuations into account, and expand \( S(\beta) \) around the equilibrium temperature \( \beta_0 \) [25, 26],

\[ S = S_0 + \frac{1}{2} (\beta - \beta_0)^2 \left( \frac{\partial^2 S(\beta)}{\partial \beta^2} \right)_{\beta = \beta_0}, \]

where we have neglected higher-order corrections to the entropy. Now we can write the density of states as

\[ \rho(E) = \frac{e^{S_0}}{2\pi i} \int_{\beta_0-i\infty}^{\beta_0+i\infty} d\beta \; \exp \left( \frac{1}{2} (\beta - \beta_0)^2 \left( \frac{\partial^2 S(\beta)}{\partial \beta^2} \right)_{\beta = \beta_0} \right). \]
After changing the variables, this expression can be written as
\[ \rho(E) = \frac{e^{S_0}}{\sqrt{2\pi}} \left( \frac{\partial^2 S(\beta)}{\partial \beta^2} \right)_{\beta = \beta_0}^{-1/2}. \] (15)

Thus, we can write
\[ S = S_0 - \frac{1}{2} \ln \left( \frac{\partial^2 S(\beta)}{\partial \beta^2} \right)_{\beta = \beta_0}. \] (16)

It may be noted that this second derivative of the entropy is actually a fluctuation squared of the energy. It is possible to simplify this expression by using the fact that the microscopic degrees of freedom of a black hole can be calculated from a conformal field theory [11]. Thus, the entropy can be assumed to have the form \( S = a\beta^m + b\beta^n \), where \( a, b, m, n > 0 \) [12]. This has an extremum at the value \( \beta_0 = (n b / m a)^{1/(m+n)} \), and so expanding \( S \) around this extremum, we can demonstrate that [25,26]
\[ \left( \frac{\partial^2 S(\beta)}{\partial \beta^2} \right)_{\beta = \beta_0} = S_0 \beta_0^{-2}. \] (17)

Thus, we will use the corrected form for the entropy, which, neglecting higher-order corrections, can be written as
\[ S = S_0 - \frac{1}{2} \ln S_0 T^2, \] (18)
where we have used \( \beta_0 = T^{-1} \). It may be noted that the quantum fluctuations in the geometry of a black hole give rise to thermal fluctuations in the thermodynamics of a black hole. So, this correction term will only contribute sufficiently when the black hole is small and its temperature is large. This is because we can neglect the quantum fluctuations to the geometry of a large black hole. In fact, as the thermal fluctuations only become significant for objects with large temperature, and the temperature of a black hole increases as its size reduces, we expect this correction term to only contribute when the size of the black hole is sufficiently small. In this section, we analyzed some basic aspects of thermodynamics of the charged AdS black hole. In the next section, we will explicitly calculate the corrections to these thermodynamic quantities due to thermal fluctuations.

**Phase transition.** – In this section, we will use the form for the corrections to the entropy of a charged AdS black hole to obtain explicit expression for various thermodynamic quantities. Then, we will use these explicit values to study the phase transition in this system. Thus, we will use the corrected form for the entropy, neglecting higher-order corrections:
\[ S = S_0 - \frac{\alpha}{2} \ln S_0 T^2, \] (19)
where we have added the parameter \( \alpha \), to help track corrections coming from the thermal fluctuations. So, by setting \( \alpha = 0 \), we will recover the expression for the entropy without any corrections. Furthermore, if we can set \( \alpha = 1 \), we obtain the corrections due to thermal fluctuations. Thus, for large black holes whose temperature is very small, we can take the limit \( \alpha \to 0 \), and for small black holes whose temperature is sufficiently large, we can take the limit \( \alpha \to 1 \). We will now analyze the effect of this parameter \( \alpha \) on the thermodynamical stability of a black hole. Now using temperature and entropy given by eqs. (11) and (10), we can obtain the following expression:
\[ S = \pi r_+^2 - \alpha \ln[4(\pi l^2 r_+^2)] + \alpha \ln[4(\pi l^2 r_+^2)]. \] (20)

It is clear that the effect of the logarithmic correction is the reduction of the entropy. Specific heat at constant pressure and constant volume can be obtained as follows:
\[ C_p = \frac{4}{3} \frac{9n r_+^6 + (2\pi l^2 - 6\alpha)r_+^4 - \pi l^2 Q^2 r_+^2}{3Q^2 l^2 - l^2 r_+^4 + 3r_+^4}, \] (21)
and
\[ C_v = \frac{6\pi r_+^6 + (2\pi l^2 - 6\alpha)r_+^4 - 2\pi l^2 Q^2 r_+^2}{3Q^2 l^2 - l^2 r_+^4 + 3r_+^4}. \] (22)

We will use \( C_v \geq 0 \) to investigate the phase transition. The ratio of the above quantities is denoted by
\[ \gamma = \frac{2}{3} \frac{9n r_+^6 + (2\pi l^2 - 6\alpha)r_+^4 - \pi l^2 Q^2 r_+^2}{3\pi r_+^6 + (\pi l^2 - 3\alpha)r_+^4 - \pi l^2 Q^2 r_+^2 + \alpha Q^2 l^2} = \frac{2}{3} \left[ 1 + \frac{6\pi r_+^6 + (\pi l^2 - 3\alpha)r_+^4 + \alpha l^2 Q^2}{3\pi r_+^6 + (\pi l^2 - 3\alpha)r_+^4 - \pi l^2 Q^2 r_+^2 - \alpha Q^2 l^2} \right]. \] (23)

It is clear that if we have
\[ 6\pi r_+^6 + (\pi l^2 - 3\alpha)r_+^4 + \alpha l^2 Q^2 = 0, \] (24)
then \( \gamma = \frac{2}{3} \), which is smaller than the cases of the ideal classic gas \( \gamma = \frac{2}{3} \) and the extreme relativistic gas \( \gamma = \frac{4}{3} \). In the case of \( \alpha \to 0 \), the above condition reduces to \( r_+^2 = -\frac{l^2}{5\alpha} \), which seems illegible. Therefore, the existence of a logarithmic correction is necessary to have \( \gamma = \frac{2}{3} \). We can obtain the following expression for the total pressure:
\[ P = \frac{3\pi r_+^6 + (\pi l^2 - 3\alpha)r_+^4 - \pi l^2 Q^2 r_+^2 - \alpha Q^2 l^2}{8\pi l^2 r_+^4}. \] (25)

It may be noted that this is a decreasing function of \( \alpha \). Now using eqs. (11), (12) and (25), we obtain \( PV \propto T \), and it is satisfied at \( \alpha \to 0 \) or \( Q \to 0 \). So, for infinitesimal \( \alpha Q \), we can write
\[ PV = \frac{2}{3} \pi T. \] (26)

Then, we can find enthalpy as follows:
\[ H = \frac{3\pi r_+^4 + (2\pi l^2 - 6\alpha)r_+^2 + Q^2 l^2}{3\pi l^2 r_+^4}. \] (27)
\[ \Phi = \frac{3\alpha l^2 Q^2 - 2\alpha M^2 r_+ + l^2 (3\pi Q^2 + 2\alpha) r_+^4 - (3\pi l^2 - 13\alpha) r_+^4 - 9\pi r_+^6 Q}{6\pi r_+^4 (2r_+^4 + l^4 r_+^2 - Ml^2)} . \] 

(29)

which can be considered as the black-hole mass \[ M \] [50,51]. Hence we can use the first law of thermodynamics [52],

\[ dM = TdS + VdP + \Phi dQ, \]

to extract electric potential \[ \Phi = dM/dQ = dH/dQ. \] It is easy to find

\[ see \ eq. \ (29) \ above \]

which is an increasing function of \[ \alpha \] outside the inner horizon, while it is a decreasing function of \[ \alpha \] in the center of the black hole. This means that the effect of the corrected terms is increasing the electric potential at the horizon. On the other hand, in the center of the black hole, logarithmic corrections decrease the value of the electric potential.

Finally, we can find Helmholtz and Gibbs energies as follows:

\[ F = -\frac{3\pi r_+^6 + (3\pi l^2 - 18\alpha) r_+^4 - 9\pi l^2 Q^2 r_+^2 - 2\alpha Q^2 l^2}{12\pi l^2 r_+^4} \]
\[ - \frac{\alpha (l^2 Q^2 - l^2 r_+^2 - 3r_+^4)}{4\pi l^2 r_+^4} \ln(3r_+^4 + l^2 r_+^2 - l^2 Q^2) \]
\[ + \frac{\alpha (l^2 Q^2 - l^2 r_+^2 - 3r_+^4)}{4\pi l^2 r_+^4} \ln(4\sqrt{\pi} l^2 r_+^2) , \]

(30)

and

\[ G = +\frac{3\pi r_+^6 + (5\pi l^2 - 24\alpha) r_+^4 + 7\pi l^2 Q^2 r_+^2}{12\pi l^2 r_+^4} \]
\[ - \frac{\alpha (l^2 Q^2 - l^2 r_+^2 - 3r_+^4)}{4\pi l^2 r_+^4} \ln(3r_+^4 + l^2 r_+^2 - l^2 Q^2) \]
\[ + \frac{\alpha (l^2 Q^2 - l^2 r_+^2 - 3r_+^4)}{4\pi l^2 r_+^4} \ln(4\sqrt{\pi} l^2 r_+^2) . \]

(31)

All of these quantities are decreasing functions of \[ \alpha \], which means the logarithmic correction term decreases the energy:

\[ E = \frac{3\pi r_+^6 + (3\pi l^2 - 9\alpha) r_+^4 + 3\pi l^2 Q^2 r_+^2 + \alpha Q^2 l^2}{6\pi l^2 r_+^4} \]

(32)

Thus, the energy of the system is decreased by thermal fluctuations.

Now using the specific heat given by eq. (22), we can investigate the phase transition of the black hole. The specific heat can be written in terms of the horizon radius as a parameter, by using eqs. (5), (27) and (22):

\[ C_v = \frac{6\pi (2\alpha - 1) r_+^2 - \pi + 12\alpha}{6\pi r_+^4 + \pi - 18\alpha} . \]

(33)

where \( l = 1 \) assumed. An important consequence of the logarithmic correction is that the black hole is unstable for \( \alpha = 0 \) and stable for \( \alpha \geq \frac{1}{2} \). In fig. 1, we show behavior of the specific heat given by eq. (33), for several values of \( \alpha \). We can see that there are some stable regions \( r_+ \geq r_m \) corresponding to \( \alpha \geq \frac{1}{2} \). Furthermore, \( r_+ = r_m \) is the minimum value of the black-hole horizon radius where phase transition occurs. For example, at \( \alpha = 1.6 \) phase transition occurs at \( r_+ = 1 \). So, the black hole is completely stable for \( r_+ \geq 1 \), for instance \( Q = 1 \) and \( M = 2 \) gives stable black hole with \( r_+ \approx 1.25 \).

**Conclusions.** In this paper, we analyzed the effects of thermal fluctuations on a charged AdS black hole. This was done by analyzing the corrections to the thermodynamics of this charged AdS black hole due to thermal fluctuations. It was also demonstrated that the logarithmic term is the leading-order correction term to the entropy of this charged AdS black hole. We also calculated other corrections to other important thermodynamic quantities for this black hole. Finally, we used the corrected value of the specific heat to analyze the phase transition in this black hole. The corrections terms to the entropy may be related to the conformal field theory description of all black solutions [53]. It may be noted that asymptotically AdS spacetime has been studied because of its importance in the AdS/CFT correspondence [54]. According to the AdS/CFT correspondence, the string theory on AdS space is dual to a conformal field theory (CFT) on the boundary of that AdS space. It has also been demonstrated using the AdS/CFT correspondence that the asymptotically AdS black hole is dual to a strongly coupled gauge theory.
at finite temperature [55–58]. It is possible to analyze the strongly correlated condensed-matter physics using the AdS/CFT correspondence. In fact, a holographic model of superconductors has also been constructed from black-hole solutions using the AdS/CFT correspondence [59]. It will be interesting to analyze the CFT dual to thermal fluctuations on the AdS space, and discuss its possible applications for the strongly correlated condensed-matters systems.

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