On Renormalization Group Flows in Four Dimensions

Zohar Komargodski ♠♥ and Adam Schwimmer ♠

♠ Weizmann Institute of Science, Rehovot 76100, Israel
♥ Institute for Advanced Study, Princeton, NJ 08540, USA

We discuss some general aspects of renormalization group flows in four dimensions. Every such flow can be reinterpreted in terms of a spontaneously broken conformal symmetry. We analyze in detail the consequences of trace anomalies for the effective action of the Nambu-Goldstone boson of broken conformal symmetry. While the $c$-anomaly is algebraically trivial, the $a$-anomaly is “non-Abelian,” and leads to a positive-definite universal contribution to the $S$-matrix element of $2 \to 2$ dilaton scattering. Unitarity of the $S$-matrix results in a monotonically decreasing function that interpolates between the Euler anomalies in the ultraviolet and the infrared, thereby establishing the $a$-theorem.
1. Introduction

One of the fundamental questions about quantum field theory is whether the Renormalization Group (RG) flux is reversible. Namely, if there exist two conformal field theories $A, B$ such that one can flow from $A$ to $B$ but also from $B$ to $A$. The answer in two-dimensional field theories has been long known to be negative \[1\], but a corresponding result for four-dimensional field theories has so far eluded us.

In the case of two space-time dimensions, Zamolodchikov has established the existence of a monotonically decreasing function, $C$, interpolating between the central charges of the UV and IR CFTs. This not only proves that the RG flux is irreversible, but also provides an effective measure for the number of degrees of freedom, such that as we integrate out high momentum modes this number decreases.

There are several conceivable ways to generalize to 4d. One such proposal by Cardy \[2\] (see also \[3,4\]) states that one should consider the integral of the energy-momentum tensor over the four-sphere

$$a \sim \int_{S^4} \langle T_\mu^\mu \rangle .$$ \tag{1.1}

Cardy’s conjecture is that the quantity (1.1) decreases as we flow. \[1\]

The trace of the stress tensor in four-dimensional conformal field theories is nonzero in curved spaces (see \[5\] and references therein) due to the trace anomalies $a$ and $c$

$$T_\mu^\mu = aE_4 - cW^2_{\mu\nu\rho\sigma} ,$$ \tag{1.2}

where $E_4$ is the Euler density and $W^2_{\mu\nu\rho\sigma}$ is the Weyl tensor squared. \[2\] Hence, in the UV and IR CFTs the integral over the four-sphere (1.1) isolates the $a$-anomalies $a_{UV}$ and $a_{IR}$, respectively. Cardy’s conjecture thus implies that, in particular,

$$a_{IR} < a_{UV} .$$ \tag{1.3}

Generally speaking, one can study the quantities $a_{UV}, a_{IR}$ only in a limited set of examples. Perturbative fixed points serve as one realm of theories where (1.3) can be put to test, and it has always been found to hold. With supersymmetry the situation

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1. Another proposal for a measure of degrees of freedom in four dimensions is due to Appelquist-Cohen-Schmaltz \[3\]. We will have nothing to add to this conjecture here.

2. Interesting bounds on the values $a$ and $c$ may assume at fixed points have been devised recently \[4,5\].
becomes much better and many more examples can be examined. This is due to duality \cite{3}, the important relation between $a$ and the superconformal $R$-symmetry \cite{10,11}, and the breakthrough of $a$-maximization \cite{12} (see also \cite{13,14}). All the known examples have yielded results consistent with (1.3). The $a$-theorem has been also widely discussed in the context of holography, starting from \cite{15,16}. See also the recent illuminating study in \cite{17}.

Needless to say, it is important to know if (1.3) is indeed a true property of quantum field theory in four dimensions. Very much like ’t Hooft’s anomaly matching, this can lead to strong constraints on the dynamics of gauge theories, the patterns of symmetry breaking, and other questions that can be relevant to particle physics in the near future.

For chiral symmetries that are conserved along the flow, ’t Hooft argued that all the anomalies should match. Recall that one cancels the anomalies by introducing very weakly interacting (spectator) fields into the system and then argues that consistency of the IR theory implies that the anomalies in the IR should reproduce those in the UV.

One cannot quite repeat this argument for the conformal (alternatively, Weyl) symmetry, since every RG flow violates it explicitly. In the deep UV and the deep IR the symmetries of the problem are enhanced to the conformal group, but the conformal symmetry is broken throughout the bulk of the flow. Therefore, the anomalies at high energies generally differ from those at low energies. In this sense, that there is any possible regularity in these anomalies (1.3) is surprising.

In this paper we develop a formalism which allows to follow this violation of conformal symmetry along the flow. As a result, we prove the inequality (1.3) for all unitary RG flows. We also establish a monotonically decreasing interpolating function between $a_{UV}$ and $a_{IR}$.

The main idea is to use a dilaton spectator field in order to reinterpret every massive RG flow as if it results from spontaneously broken conformal symmetry. This can be done while keeping the dilaton fluctuations arbitrarily weakly coupled to the matter theory. We then study carefully the effective action of the dilaton field and show that one particular four-derivative term in this theory is related to the $a$-anomaly. This special term in the dilaton effective action is reminiscent of the topological term in pion physics \cite{18,19}. Finally, we show that this term can be isolated by computing a $2 \rightarrow 2$ $S$-matrix element and it satisfies a positive-definite dispersion relation, establishing our main claim. This also gives rise to a monotonically decreasing function interpolating between $a_{UV}$ and $a_{IR}$. Morally speaking, our dilaton field can be thought of as a cousin of ’t Hooft’s spectators,
however, many other aspects of the analysis have no analogs in the argument for anomaly
matching of global symmetries.

The plan of this paper is as follows. In section 2 we discuss the rules of writing
diff×Weyl invariant actions for the dilaton field in general curved backgrounds. We also
discuss the anomaly functional and show that there is a particular four-derivative term
which is uniquely determined by the $a$-anomaly. (More precisely, $a$ fixes the result of a
particular low energy $S$-matrix calculation.) As a warm-up exercise that highlights some
of the important ingredients in our construction, we prove a special case of the $a$-theorem
in section 3. In section 4 we explain how every massive RG flow can be reinterpreted as
coming from spontaneously broken conformal symmetry and prove the strongest version of
the $a$-theorem. Various open questions and further research directions are briefly discussed
in section 5. Some of the conventions used throughout this paper are summarized in
appendix A. In appendix B we illustrate the methods of section 4 for the flow of a free
massive field.

2. The Theory of the Dilaton: Invariant Terms and Anomalous Functionals

2.1. Invariant Terms

Consider a spontaneously broken CFT. Then, by the Nambu-Goldstone theorem, there
is a massless particle, the dilaton $\tau$. Effective actions for it follow the same rules as in any
theory of spontaneously broken symmetry. (One can also use current algebra techniques
to derive the same results.) It is easy to organize such actions by introducing a space-time
metric $g_{\mu\nu}$ and demanding that the theory is invariant under diff×Weyl transformations,
where Weyl transformations act as

$$g_{\mu\nu} \rightarrow e^{2\sigma} g_{\mu\nu} , \quad \tau \rightarrow \tau + \sigma . \quad (2.1)$$

Diffeomorphisms act as usual, with the dilaton being a space-time scalar. We will often
denote $\hat{g} = e^{-2\tau} g_{\mu\nu}$. The combination $\hat{g}$ transforms as a metric under diffeomorphisms
and is Weyl invariant.

The most general theory up to two derivatives is:

$$f^2 \int d^4 x \sqrt{-\det \hat{g}} \left( \Lambda + \frac{1}{6} \hat{R} \right) , \quad (2.2)$$
where we have defined $\hat{R} = \hat{g}^{\mu\nu} R_{\mu\nu}[\hat{g}]$. $f$ is the “decay constant” of the spontaneously broken conformal theory. The cosmological constant term $\Lambda$ leads to a scale-invariant potential for the dilaton. There is nothing wrong with it by itself, except that if the dilaton really comes from spontaneously broken conformal symmetry the vacuum degeneracy cannot be lifted and hence $\Lambda = 0$. (This is the well-known statement that the cosmological constant is zero in vacua that break the conformal symmetry spontaneously.)

Since we are ultimately interested in the Minkowskian theory, let us evaluate the kinetic term with $g_{\mu\nu} = \eta_{\mu\nu}$. Using integration by parts we get

$$S = f^2 \int d^4 x e^{-2\tau} (\partial \tau)^2 \, .$$

This describes a free massless particle, albeit in a somewhat strange choice of variables. One can use the field redefinition

$$\varphi = 1 - e^{-\tau}$$

(2.4)

to bring the kinetic term into canonical form. The utility of the variable $\tau$ will be seen later. The equation of motion is

$$\square \tau = (\partial \tau)^2 \, .$$

(2.5)

One can also study terms in the effective action with more derivatives. With four derivatives, one has three independent (dimensionless) coefficients

$$\int d^4 x \sqrt{-\hat{g}} \left( \kappa_1 \hat{R}^2 + \kappa_2 \hat{R}_{\mu\nu}^2 + \kappa_3 \hat{R}_{\mu\nu\rho\sigma}^2 \right) \, .$$

(2.6)

It is implicit that indices are raised and lowered with $\hat{g}$. (We have not included the Pontryagin term as well as $\square \hat{R}$ in (2.6) since they both integrate to zero.) This basis of interactions is somewhat inconvenient for our purposes. Recall the expressions for the Euler density $\sqrt{-g} E_4$ and the Weyl tensor squared

$$E_4 = R_{\mu\nu\rho\sigma}^2 - 4 R_{\mu\nu}^2 + R^2 \, , \quad W_{\mu\nu\rho\sigma}^2 = R_{\mu\nu\rho\sigma}^2 - 2 R_{\mu\nu}^2 + \frac{1}{3} R^2 \, .$$

(2.7)

We can thus choose instead of (2.6) a different parameterization

$$\int d^4 x \sqrt{-\hat{g}} \left( \kappa'_1 \hat{R}^2 + \kappa'_2 \hat{E}_4 + \kappa'_3 \hat{W}_{\mu\nu\rho\sigma}^2 \right) \, .$$

(2.8)

We immediately see that the $\kappa'_2$ term is a total derivative. If we set $g_{\mu\nu} = \eta_{\mu\nu}$, then $\hat{g}_{\mu\nu} = e^{-2\tau} \eta_{\mu\nu}$ is conformal to the flat metric and hence also the $\kappa'_3$ term does not play...
any role as far as the dilaton interactions in flat space are concerned. Consequently, terms in the flat space limit arise solely from $\tilde{R}^2$. A straightforward calculation yields

$$\int d^4x \sqrt{-\tilde{g}}\tilde{R}^2 \bigg|_{g_{\mu\nu}=\eta_{\mu\nu}} = 36 \int d^4x (\Box \tau - (\partial \tau)^2)^2 .$$ (2.9)

The combination on the right hand side of (2.9) vanishes by the equation of motion of the two-derivative theory (2.5). (Equivalently, using the variable $\varphi$ defined in (2.4) the left hand side of (2.9) is seen to be proportional to $(\Box \varphi)^2$, which vanishes on-shell.) The mistake in using the zeroth order equation of motion is higher order in the derivative expansion. Thus, the diff×Weyl invariant terms in the Lagrangian do not yield a genuine tree-level four-derivative interaction. Or, said more invariantly, there is no $s^2 + t^2 + u^2$ term in the low momentum expansion of the scattering amplitude of four dilatons.

2.2. Anomalous Functionals

We should also contemplate functionals of $g_{\mu\nu}, \tau$ which are not diff×Weyl invariant. This is important because the physical theories we will discuss in the next sections have trace anomalies and we will need to match these anomalies with functionals of $g_{\mu\nu}, \tau$.

The most general anomalous variation one needs to consider takes the form

$$\delta_\sigma S_{\text{anomaly}} = \int d^4x \sqrt{-g}\sigma \left( cW_{\mu\nu\rho\sigma}^2 - aE_4 + b'R \right) .$$ (2.10)

The question is then how to write a functional $S_{\text{anomaly}}$ that reproduces this anomaly. (Note that $S_{\text{anomaly}}$ is only defined modulo diff×Weyl invariant terms.) Without the field $\tau$ one must resort to non-local expressions, but in the presence of the dilaton one has a local action.

The term $b'$ is uninteresting to us because it can be accounted for by a local functional that only depends on $g$ and not on $\tau$ (hence, $b'$ is not associated to an anomaly; it is simply a contact term in the two-point function of stress tensors that we may or may not want to include). To verify this one can easily check that

$$\delta_\sigma \int R^2 \sim \int \sigma \Box R .$$ (2.11)

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3 A more conceptual way of seeing that (2.9) had to vanish on-shell comes by noting that, in the absence of the cosmological constant term in (2.2), the equation of motion for the dilaton is just the trace of the Einstein equation, hence, the Ricci scalar vanishes on-shell.
Hence, the coefficient \( b' \) does not affect the matrix elements of \( \tau \) in flat space. We will henceforth set \( b' = 0 \) in this paper. To reintroduce \( b' \) one simply adds \( \sim \int d^4x \sqrt{-g} R^2 \) to the anomaly functional.

It is a little tedious to solve (2.10), but the procedure is straightforward in principle. (See also [20–23] and references therein for other approaches to the problem.) We first replace \( \sigma \) on the right-hand side of (2.10) with \( \tau \)

\[
S_{\text{anomaly}} = \int d^4x \sqrt{-g} \left( cW^2_{\mu\nu\rho\sigma} - aE_4 \right) + \cdots .
\]  

(2.12)

While the variation of this includes the sought-after terms (2.10), as the \( \cdots \) suggest, this cannot be the whole answer because the object in parenthesis is not Weyl invariant. Hence, we need to keep fixing this expression with more factors of \( \tau \) until the procedure terminates. Note that \( \sqrt{-g}W^2_{\mu\nu\rho\sigma} \), being the square of the Weyl tensor, is Weyl invariant, and hence we do not need to add any fixes proportional to the \( c \)-anomaly in (2.12). This makes the \( c \)-anomaly “Abelian” in some sense. The “non-Abelian” structure coming from the Weyl variation of \( E_4 \) is the key to our construction. The \( a \)-anomaly is therefore quite distinct algebraically from the \( c \)-anomaly.

The final expression for \( S_{\text{anomaly}} \) is

\[
S_{\text{anomaly}} = -a \int d^4x \sqrt{-g} \left( \tau E_4 + 4 \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \partial_\mu \tau \partial_\nu \tau - 4(\partial \tau)^2 \Box \tau + 2(\partial \tau)^4 \right)
+ c \int d^4x \sqrt{-g} W^2_{\mu\nu\rho\sigma}.
\]  

(2.13)

Note that even when the metric is flat, self-interactions of the dilaton survive. This is analogous to what happens in pion physics when the background gauge fields are set to zero [18,19]. These interaction terms in flat space-time are forced on us by the “non-Abelian” structure of the \( a \)-anomaly.

Since the self interactions which survive \( g_{\mu\nu} = \eta_{\mu\nu} \) are four-derivative interactions, we can use the equation of motion (2.5) to simplify things. Hence, we get that the anomaly contribution is

\[
S_{\text{anomaly}} \bigg|_{g_{\mu\nu} = \eta_{\mu\nu}} \approx 2a \int d^4x (\partial \tau)^4 .
\]  

(2.14)

Above, the \( \approx \) symbol means that we have used the leading order equation of motion. Physical observables such as the \( S \)-matrix are invariant under using the equations of motion, so we loose nothing by utilizing (2.5) at the order of four derivatives.
Let us conclude what we have found in this and the previous subsection. No diff×Weyl invariant terms give rise to the scattering of four dilatons at the order of four derivatives. Such a contribution, however, arises from the anomaly functional, and its coefficient is fixed by the $a$-anomaly.

This important conclusion is independent of whether or not there is a $b'$ term in the anomaly (2.10), because $b'$ can always be canceled by a local counterterm (2.11) that does not affect dilaton scattering in flat space. It is also independent of our choice of the anomaly functional (2.13) because all such choices differ by diff×Weyl invariant terms, and the latter have no effect on flat-space dilaton scattering at the level of four derivatives. We therefore arrived at a low energy theorem for dilaton scattering. (Analogous to some soft pion theorems.)

3. The $a$-Theorem for Motion on the Moduli Space

Consider a conformal field theory, CFT$_{UV}$ (with anomalies $a_{UV}, c_{UV}$), which has a moduli space of vacua $\mathcal{M}$. In all of them but the one at the “origin” conformal symmetry is spontaneously broken. Let us study the physics in one of these conformal symmetry-breaking vacua. Generally, at the scale of breaking $f$ there are massive particles, but there can be some massless particles too. In fact, the dilaton, which is the Nambu-Goldstone boson of conformal symmetry breaking, must be massless. In addition to it, there may be a nontrivial IR CFT, denoted CFT$_{IR}$ (with anomalies $a_{IR}, c_{IR}$). The situation is summarized in Fig.1.
Fig. 1: The shaded region represents the moduli space of the UV CFT. An operator $\mathcal{O}$ obtains a VEV of order $f$ and breaks the conformal symmetry spontaneously. This results in a flow to a low energy theory containing the massless dilaton and possibly a non-trivial conformal field theory, CFT$_{IR}$.

Moduli spaces of vacua might not be easy to find in non-supersymmetric theories, but such moduli spaces are ubiquitous in supersymmetric theories and the RG flows one can trigger by turning on VEVs for moduli may lead to intricate IR CFTs. It is therefore not at all obvious a priori that $a_{IR} < a_{UV}$ is satisfied.

In conformal field theory, whether it is spontaneously broken or not, the total energy-momentum tensor is traceless, in other words, it satisfies the operator equation $T^\mu_\mu = 0$. Anomalies show up in curved space (alternatively, as contact terms in special correlation functions). In this situation, there is no operatorial violation of the Ward identity $T^\mu_\mu = 0$, only a $c$-number violation due to anomalies. Therefore, the total anomalies in the UV and IR must agree because of the usual arguments for anomaly matching [23]. This does not mean that the $a$- and $c$-anomalies of CFT$_{UV,IR}$ match, rather, that the difference must be compensated for by the dilaton\footnote{4 This situation is somewhat reminiscent of the role of the Liouville field in non-critical string theory.}.

\footnote{4 This situation is somewhat reminiscent of the role of the Liouville field in non-critical string theory.}
The matching of the total anomaly forces the effective action in the IR to take the form

\[ S_{IR}[g_{\mu\nu}] = \text{CFT}_{IR}[g_{\mu\nu}] + \frac{1}{6} f^2 \int d^4x \sqrt{-g} \hat{R} + \frac{\kappa}{36} \int d^4x \sqrt{-g} \hat{R}^2 + \kappa' \int d^4x \sqrt{-g} \hat{W}_{\mu\nu\rho\sigma}^2 \\
- (a_{UV} - a'_{IR}) \int d^4x \sqrt{-g} \left( \tau E_4 + 4(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) \partial_\mu \tau \partial_\nu \tau - 4(\partial \tau)^2 \Box \tau + 2(\partial \tau)^4 \right) \\
+ (c_{UV} - c'_{IR}) \int d^4x \sqrt{-g} \tau W_{\mu\nu\rho\sigma}^2 + \cdots. \]

(3.1)

The cosmological constant term from (2.2) is zero (\( \Lambda = 0 \)) since we are working on the moduli space and hence there cannot be a potential for the dilaton. The constant \( f \) is the physical decay constant of the dilaton. In addition, we have defined \( a'_{IR} = a_{IR} + a_{scalar} \), \( c'_{IR} = c_{IR} + c_{scalar} \). The reason is that in addition to the explicit tree-level contribution contained in the effective action, the dilaton contributes to the total anomalies like any massless scalar via loop diagrams.\(^5\)

The different terms in the effective action (3.1) may be generated by integrating out the massive modes. The most important feature in (3.1) is, of course, that the anomaly functional comes with prescribed coefficients to match the total anomalies.\(^6\)

We would like to examine the effective action (3.1) for a flat space-time metric \( g_{\mu\nu} = \eta_{\mu\nu} \). The result is

\[ S_{IR} = \text{CFT}_{IR} \\
+ \int d^4x \left( f^2 e^{-2\tau} (\partial \tau)^2 + \kappa \left( \Box \tau - (\partial \tau)^2 \right)^2 + (a_{UV} - a'_{IR}) \left( 4(\partial \tau)^2 \Box \tau - 2(\partial \tau)^4 \right) \right) + \cdots. \]

(3.2)

We see that the difference between the \( a \)-anomalies \( a_{UV} - a'_{IR} \) appears in front of some specific four-derivative terms, and other terms with four derivatives appear to be multiplied by an unknown coefficient \( \kappa \). As we have explained in section 2, these two types of contributions are neatly disentangled when one considers the \( S \)-matrix for \( \tau \tau \)

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\(^5\) We thank R. Myers for a discussion.

\(^6\) In general we need to cancel the \( b' \) anomaly (2.10) too. Even if we choose it to vanish in \( \text{CFT}_{UV} \), it would generally be nonzero in \( \text{CFT}_{IR} \). This must be taken into account by adding a term proportional to \( \sqrt{g} R^2 \) in (3.1). This contribution does not affect our discussion so we omit it.
scattering. The leading contribution to the scattering amplitude is fixed by the difference of the \( a \)-anomalies (recall the usual relation \( s + t + u = 0 \))

\[
A(s, t) = \frac{a_{UV} - a'_{IR}}{f^4} (s^2 + t^2 + u^2) + \cdots . \tag{3.3}
\]

Higher order contributions, encompassed by the \( \cdots \), may be model-dependent.

It has been known for a while that low energy contributions to scattering elements can be sometimes evaluated by dispersion relations. This follows from analyticity. For instance, certain subleading operators in the chiral Lagrangian are known to have positive coefficients \[24\]. (See also \[25\] for a discussion in the context of \( WW \) scattering.) This positivity constraint has been also shown to be closely related to the absence of low energy superluminal modes \[26\]. Many other constraints of this kind exist and have applications for a wide range of problems, see for instance \[27\] for one such example.

After using the equation of motion in (3.2), the quartic term one remains with is proportional to \((a_{UV} - a'_{IR})(\partial \tau)^4\). This operator is the simplest example for the superlumina behavior discussed in \[26\]. The absence of superluminal modes immediately implies that \(a'_{IR} \leq a_{UV}\) (and thus \(a_{IR} < a_{UV}\)). We will now study more closely the analytic structure and establish the stronger inequality \(a'_{IR} < a_{UV}\), along with constructing a sum rule for the difference \(a_{UV} - a'_{IR}\). This also leads to a monotonically decreasing function that interpolates between \(a_{UV}\) and \(a'_{IR}\).

Consider the scattering of four dilatons, with momenta \(p_1, p_2, p_3, p_4\) such that they are all on-shell \(p_i^2 = 0\) and of course \(\sum_{i=1}^{4} p_i = 0\). We assume that we are in the forward limit \(t = 0\) where

\[
p_1 = -p_3 \text{ , } \quad p_2 = -p_4 . \tag{3.4}
\]

The amplitude for this scattering process thus becomes (3.3)

\[
A(s) = \frac{2(a_{UV} - a'_{IR})}{f^4} s^2 + \mathcal{O}(s^4) . \tag{3.5}
\]

The last step is to consider the amplitude \(A/s^3\) and write a dispersion relation for it. There are branch cuts both at positive and negative \(s\). Negative \(s\) cuts correspond to physical states in the \(u\) channel, and the symmetry \(s \leftrightarrow u\) renders these contributions identical to the ones for positive \(s\). In addition, \(A/s^3\) has a pole at the origin which selects the coefficient \(a_{UV} - a'_{IR}\). Hence, by closing the contour we can write a dispersion relation:

\[
a_{UV} - a'_{IR} = \frac{f^4}{\pi} \int_{s' > 0} ds' \frac{Im A(s')}{s'^3} . \tag{3.6}
\]
Here $ImA(s')$ denotes the imaginary part of the amplitude. This discontinuity is positive definite because it satisfies $ImA(s) = s\sigma(s)$, where $\sigma(s)$ is the total cross section for the scattering of $\tau\tau$. We conclude that $a'_I < a_U$. This also provides a natural function that decreases along the flow. Define a scale-dependent “$a$-anomaly”

$$a(\mu) \equiv a_U - \frac{f^4}{\pi} \int_{s' > \mu} ds' \sigma(s') \frac{s'}{s'^2}.$$  

(3.7)

This decreases monotonically as a function of $\mu$ and interpolates between $a_U$ at $\mu \to \infty$ and $a'_I$ at $\mu \to 0$. This proves the $a$-theorem for this class of theories.

4. Renormalization Group Flows as Spontaneously Broken Conformal Symmetry and the $a$-Theorem

In the previous section we have discussed renormalization group flows which are triggered by turning on VEVs for operators that parameterize the moduli space of some conformal field theory. However, the most interesting case to consider is a CFT$_{UV}$ which is deformed by some relevant operator(s) $M^{4-\Delta}O_\Delta$. (The operator can also be marginally relevant, like the gauge coupling in QCD.) This sets off a flow to some IR physics, which in the deep low energy limit is described by a possibly nontrivial CFT$_{IR}$. We summarize this in Fig.2.

![Fig.2: Starting from some CFT at high energies, we add a relevant operator $O_\Delta$ and flow to a new CFT in the deep infrared.](image)

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7 We have been a little cavalier in manipulating the dispersion relation above, not explaining why it converges. If the dispersion relation had diverged this would have meant that the difference between the anomalies needs a subtraction (i.e. a counterterm). This is clearly not the case since $a_U - a_I$ is a physical quantity.
Consider the flow of the matter theory described in Fig.2. The matter theory is described by an action functional for some fields \( \Phi_i \) and some mass parameters \( M_i \)

\[
S_{\text{matter}} = S_{\text{matter}}[\Phi_i, M_i] . \tag{4.1}
\]

Upon coupling this theory to a background metric \( g_{\mu\nu} \) the partition function is guaranteed to be diffeomorphism invariant by virtue of the conservation of the stress tensor. However, Weyl invariance is violated by the \( a \)- and \( c \)-anomalies \((1.2)\), as well as the explicit mass parameters which induce a nonzero \( (T^{\text{matter}})^{\mu}_{\mu} \) in flat space. The nonzero \( (T^{\text{matter}})^{\mu}_{\mu} \) due to the mass parameters \( M_i \) is referred to as the \emph{operatorial anomaly}, to be distinguished from the \( c \)-number anomalies which do not enter the operator equation for \( (T^{\text{matter}})^{\mu}_{\mu} \) in flat space (the \( c \)-number anomalies only manifest themselves via contact terms in special correlation functions).

What precludes one from matching straightforwardly the anomalies of CFT\(_{UV} \) \((a_{UV}, c_{UV})\) and CFT\(_{IR} \) \((a_{IR}, c_{IR})\) is the operatorial anomaly in \( (T^{\text{matter}})^{\mu}_{\mu} \). However, the operatorial anomaly can be very easily removed with the aid of a dilaton (alternatively, a conformal compensator). Denoting \( \Omega \equiv e^{-\tau} \), we replace every mass scale according to \( M_i \rightarrow M_i \Omega \). We also add a kinetic term for this dilaton (replacing the metric in \((2.2)\) by the flat metric) such that the theory becomes

\[
S = S_{\text{matter}}[\Phi_i, M_i \Omega] + f^2 \int d^4 x (\partial \Omega)^2 . \tag{4.2}
\]

This theory is now void of operatorial Weyl anomalies, in other words, this theory satisfies the operator equation

\[
T^{\mu}_{\mu} = 0 . \tag{4.3}
\]

Operator equations, by definition, hold at separated points, and this is how \((4.3)\) is to be interpreted.

So far we have not said much about the dimensionful scale \( f \). Since it appears as the coefficient of the kinetic term of the dilaton, we see that the physical dilaton fluctuations couple to matter fields by inverse powers of \( f \) and thus if we take

\[
M_i \ll f , \tag{4.4}
\]

the coupling between the dilaton and the matter sector is arbitrarily weak.

To recapitulate, we have seen that the operator anomaly can be canceled with a dilaton whose fluctuations couple weakly to the matter fields. Thus, setting the dilaton
to its VEV ($\Omega = 1$) the original matter theory is recovered, and it flows as depicted in Fig.2, perturbed only by the infinitesimal coupling to the dilaton field. Hence, the deep IR theory consists of CFT$_{IR}$ supplemented by the decoupled dilaton field.

Our theory (4.2) also needs to be defined properly at high energies. We can introduce a cutoff $\Lambda_{UV}$ which satisfies $\Lambda_{UV} \gg M_i$. All momenta are restricted to satisfy $p^2 \ll \Lambda_{UV}^2$. We will now see that at high energies (namely $M_i^2 \ll p^2 \ll \Lambda_{UV}^2$) our system (4.2) consists of CFT$_{UV}$ plus the additional dilaton, very weakly coupled to CFT$_{UV}$. Indeed, this is true since in the limit (4.4) the physical dilaton interactions with operators in the conformal theory are various marginal operators suppressed by $M_i/f$. Consequently, the fact that these operators are not necessarily exactly marginal plays no role to leading order. In other words, there are logarithms which are higher order in $M_i/f$, and since we work to leading order in this expansion, it is consistent to treat the ultraviolet theory as consisting of CFT$_{UV}$ plus a decoupled dilaton.

To summarize, in the limit (4.4) the introduction of the dilaton modifies the flow in Fig.2 in a trivial way. The UV now effectively consists of CFT$_{UV}$ together with a dilaton, and analogously, the IR consists of CFT$_{IR}$ plus the dilaton. In between the UV ($M_i^2 \ll p^2 \ll \Lambda_{UV}^2$) and the IR ($p^2 \ll M_i^2$) the flow proceeds as in Fig.2, essentially unperturbed by the dilaton.

However, the presence of this innocuous dilaton gives us an important handle on anomalies. Since our theory now satisfies the Ward identity (4.3), the total $a$- and $c$-trace anomalies of (4.2) must match between the UV and IR. As in section 3, this does not mean that the anomalies of CFT$_{UV}$ and CFT$_{IR}$ match, rather, that together with the dilaton the total anomaly agrees. Indeed, since along the flow the dilaton is weakly coupled to the matter theory, various effective operators involving the dilaton field are generated upon integrating out the matter fields.

Since we assume (4.4), we are only interested in the leading terms in $1/f$. To leading order in this expansion it is sufficient to integrate out the matter fields while the dilaton sits on external lines. (Internal lines of the dilaton unavoidably suppress the diagrams by further powers of $1/f$.) Then, the diagrams one needs to compute in order to find the dilaton couplings at low energies are depicted schematically in Fig.3.

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8 This procedure we have invoked for canceling the operatorial anomaly has analogs in many other contexts. In general, every explicit symmetry breaking can be reinterpreted as spontaneous breaking by adding a massless field with an arbitrarily large decay constant (and hence weakly coupled to the matter fields, such that the essential dynamics is intact).
The matching of the total anomaly constrains the effective action in the IR. To make it a little clearer where the different terms come from we introduce some background metric $g_{\mu\nu}$. Up to terms with four derivatives acting on the dilaton the most general allowed effective theory is

$$S_{IR}[g_{\mu\nu}] = \text{CFT}_{IR}[g_{\mu\nu}] + \frac{1}{6} f^2 \int d^4x \sqrt{-g} \hat{R} + \frac{\kappa}{36} \int d^4x \sqrt{-g} \hat{R}^2 + \kappa' \int d^4x \sqrt{-\hat{g}} W_{\mu\nu\rho\sigma}^2$$

$$- (a_{UV} - a_{IR}) \int d^4x \sqrt{-g} \left( \tau E_4 + 4 (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) \partial_\mu \tau \partial_\nu \tau - 4 (\partial \tau)^2 \Box \tau + 2 (\partial \tau)^4 \right)$$

$$+ (c_{UV} - c_{IR}) \int d^4x \sqrt{-g} \tau W_{\mu\nu\rho\sigma}^2 .$$

(4.5)

The coefficients of the anomalous functional are fixed so that the total anomaly of (4.5) matches the UV anomalies $a_{UV}, c_{UV}$. In contrast to section 3, the dilaton is present both in the UV and in the IR, hence, its contribution to the anomalies via loop diagrams cancels from the matching. Note that we have not included a cosmological constant term, although this is generally generated by the flow. Indeed, the physical cosmological constant can always be tuned to zero by including an appropriate bare vacuum energy term in (4.2). (In addition, the coupling of the dilaton to matter may affect the normalization of the kinetic term of the dilaton, but to not clutter the notation we have denoted by $f$ the physical decay constant already in (4.2)).

Since at this stage the physics is identical to what we have already discussed in great length in section 3, the consequences are similar too. Namely, the difference between the
\( \alpha \)-anomalies is isolated by the leading contribution to the \( 2 \to 2 \) \( S \)-matrix element at low energies. In the forward kinematics, the scattering amplitude at low energies is given by

\[
A(s) = \frac{2(a_{UV} - a_{IR})}{f^4} s^2 + O(s^4). \tag{4.6}
\]

This leads to the sum rule

\[
a_{UV} - a_{IR} = \frac{f^4}{\pi} \int_{s' > 0} ds' \frac{\sigma(s')}{s'^2}, \tag{4.7}
\]

with \( \sigma(s') \) the (manifestly positive-definite) cross section for the scattering of two dilatons. Note that the cross section goes as \( 1/f^4 \) in the limit (4.4), but the prefactor \( f^4 \) in (4.7) ensures the result stays finite (as it should) no matter how large \( f \) is. We can also construct an obvious interpolating function which is monotonically decreasing, as in (3.7)

\[
a(\mu) \equiv a_{UV} - \frac{f^4}{\pi} \int_{s' > \mu} ds' \frac{\sigma(s')}{s'^2}. \tag{4.8}
\]

This completes the proof of the \( \alpha \)-theorem. An explicit example that demonstrates the ideas in this section is worked out in appendix B.

5. Discussion and Open Questions

Recall that in two dimensions there is a “dispersive” proof of the \( C \)-theorem due to [28]. Some aspects of our result look analogous to the corresponding discussion in two dimensions, especially, the role played by the sum rule. There have been many previous attempts to utilize various sum rules and dispersion relations to shed light on Cardy’s conjecture, but one crucial difference between our approach and previous attempts is that four-point functions play a pivotal role in our discussion. The scattering of \( 2 \to 2 \) dilatons, contains, after all, data from the correlator of four traces of the stress tensor \( \langle T_\mu T_\nu T_\mu T_\nu \rangle \). Such objects have nice positivity properties, unlike three-point functions (which are commonly used to define the \( \alpha \)- and \( c \)-anomalies). It would be instructive to reformulate our proof in the language of correlation functions, never referring to the auxiliary dilaton.\[9\]

Let us now allude briefly to some additional questions our analysis raises:

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\[9\] In the case of the chiral anomaly, one can indeed avoid the spectator fields altogether and analyze the correlation functions and contact terms in great detail [29,30].
1) One central ingredient is the “non-Abelian” structure of the Euler anomaly. This leads to the universal $2 \rightarrow 2$ scattering in flat space. It would be interesting to understand better the algebraic (cohomological) structure of this phenomenon.

2) We have constructed a monotonic decreasing function that interpolates between $a_{UV}$ and $a_{IR}$. However, we have not addressed the question of whether the RG flow is a gradient flow or not. The four-dilatons coupling (2.14), which is the hero of our story, is obviously related to the four-point function of $T_{\mu}$. Hence, one may speculate that the evolution from the UV to the IR is associated to a positive-definite quartic differential, rather than the gradient flow in two dimensions. Being a fundamental property of four-dimensional RG flows, this is clearly worth addressing. The study of simple examples could be instrumental in attacking this question.

3) We have not tried to make contact with Cardy’s original proposal for $\int_{S^4} \langle T_{\mu} \rangle$ as the quantity that monotonically decreases along the flow. Of course, in the UV and IR this object coincides with $a_{UV}, a_{IR}$, respectively. But can it be related to our construction at some intermediate scale $\mu$ as well? Note that our discussion of the $a$-anomaly uses local methods (i.e. an effective action approach), while the integral over the sphere is a global quantity. A better understanding of the relationship between these two approaches is sorely needed.

4) It would be interesting to understand the effective action of the dilaton on the moduli space of $\mathcal{N} = 4$ Yang-Mills theory (see [31] for a discussion of some aspects of this problem) and to compare with expectations from strong coupling [32].

5) The flat space-time dilaton self-interaction coming from (2.13) may have an interesting manifestation in holography. One could consider conformal symmetry breaking in AdS spaces and try to identify the (perhaps geometric) reason for the universality of the coefficient of this interaction.

6) While the generalization of our results to trace anomalies in 6d [33,34] seems feasible, the question of what happens when the number of dimensions is odd remains open. A proposal for a measure of degrees of freedom in 3d has been given by [17], and some further evidence for it in supersymmetric theories is discussed in [35]. Such a theorem in three dimensions could have applications for condensed matter systems.

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10 One can verify that in the case of conformal symmetry breaking by a $D3$ brane localized in the radial direction of $AdS_5$, the correct four-derivative dilaton interaction is captured by the DBI action. We thank O. Aharony, J. Maldacena, and S. Theisen for pointing this out to us.
The conjecture in three dimensions again concerns itself with the partition function over the round three-sphere. We certainly cannot prove this theorem yet, but we would like to offer some intuition for why such a result is not inconceivable. If a quantity is to decrease in every RG flow, it must be constant on conformal manifolds. Changing the location on the conformal manifold corresponds to taking a derivative with respect to a coordinate on the conformal manifold, thus one has to calculate a one-point function on $S^3$. But since $S^3$ can be stereographically projected onto $R^3$, one-point functions on $S^3$ of all the primary operators (besides the unit) are zero. Hence, the partition function on the sphere is constant on conformal manifolds.

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Appendix A. Conventions

The signature we use is $(+,-,-,-)$. The energy momentum tensor can be extracted from the action by using

$$T^{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}, \quad T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}.$$  \hspace{1cm} (A.1)

This implies that under Weyl variations, $\delta g_{\mu\nu} = 2\sigma g_{\mu\nu}$,

$$\sqrt{-g}T^\mu_\mu = -\frac{\delta S}{\delta \sigma}.$$  \hspace{1cm} (A.2)

\footnote{11} We thank D. Jafferis for a crucial discussion of these matters.

\footnote{12} This is satisfied by $C$ in two dimensions and also by $a$ in four dimensions. The latter follows trivially from our general results. Another proof of this is given by repeating verbatim the argument given in the text, replacing $R^3, S^3$ by $R^4, S^4$ (and taking a derivative with respect to log $R$).
We also define

\[ R^\rho_{\lambda\mu\nu} = \partial_\mu \Gamma^\rho_{\lambda\nu} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\mu\kappa} \Gamma^\kappa_{\nu\lambda} - \Gamma^\rho_{\nu\kappa} \Gamma^\kappa_{\mu\lambda} \], \tag{A.3} \]

which can be contracted to give \( R_{\lambda\nu} = R^\mu_{\lambda\mu\nu}, R = g^{\lambda\nu} R_{\lambda\nu} \). The Euler tensor is defined as

\[ E_4 = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2, \tag{A.4} \]

and the Weyl tensor squared satisfies

\[ W^2_{\mu\nu\rho\sigma} = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3} R^2. \tag{A.5} \]

The trace anomaly is then given by

\[ T_\mu^\mu = aE_4 - cW^2. \tag{A.6} \]

Real scalars contribute to the anomalies \((a, c) = \frac{1}{90(8\pi)^2}(1, 3)\), Weyl fermion: \((a, c) = \frac{1}{90(8\pi)^2}(11/2, 9)\), gauge field: \((a, c) = \frac{1}{90(8\pi)^2}(62, 36)\). (The calculation leading to these values is reviewed in [36], where additional references can be found too.) The variation of the action is then given by

\[ \delta_\sigma S = \int d^4x \sqrt{-g} \sigma \left( cW^2 - aE_4 \right), \tag{A.7} \]

with the positive values of \(a, c\) quoted above.

Appendix B. A Free Massive Field

As an illustration of the procedure outlined in section 4, we discuss the flow of a free massive field, coupled to the “compensator” (dilaton).

The action is simply

\[ S = \int d^4 x \left( \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} M^2 \Phi^2 \right). \tag{B.1} \]

We introduce a dimensionless compensator field \( \Omega \), with expectation value \( \Omega = 1 \) such that the action becomes

\[ S = \int d^4 x \left( f^2 \partial_\mu \Omega \partial^\mu \Omega + \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} M^2 \Omega^2 \Phi^2 \right). \tag{B.2} \]
The action (B.2), which is now interactive, should be understood in the presence of an ultraviolet cutoff \( \Lambda \). We will restrict always all the momenta \( p \) to the range \( p^2 \ll \Lambda^2 \). We would not like the compensator to modify the flow of the free massive field, so we take

\[
 f \gg M .
\]  

(B.3)

In this limit the physical coupling between \( \Omega \) and \( \Phi \) is arbitrarily weak.

With these assumptions, at large momenta the theory behaves as a conformal theory since the ultraviolet logarithms (due to the quartic interaction) are suppressed by positive powers of \( M/f \). We will always compute quantities only to leading order in \( M/f \) and so the theory can be treated as if it is exactly conformally invariant at energies \( M \ll E \ll (\Lambda, f) \). (We remark that such an embedding of the free massive field in a conformal theory is explicitly realized on the Coulomb branch of the \( \mathcal{N} = 4 \) supersymmetric gauge theory at weak coupling where the range \( p^2 \ll \Lambda^2 \) is extended to infinity since the ultraviolet logarithms cancel due to supersymmetry.)

We proceed now to study the broken phase of the theory, i.e. when \( \Omega \) gets a vacuum expectation value

\[
 < \Omega > = 1 .
\]  

(B.4)

Expanding the field \( \Omega \) around the VEV, we define the fluctuation \( \varphi \) through

\[
 \Omega = e^{-\tau} = 1 - \varphi .
\]  

(B.5)

The action becomes now:

\[
 S = \int d^4x \left( f^2 \partial_\mu \varphi \partial^\mu \varphi + \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} M^2 \Phi^2 - M^2 J \Phi^2 \right) ,
\]  

(B.6)

where

\[
 J = \left( \frac{1}{2} \varphi^2 - \varphi \right) .
\]  

(B.7)

We are interested in the effective action of the \( \varphi \) field after integrating out the massive \( \Phi \). Since the action is quadratic in \( \Phi \) this is exhausted by the one-loop diagrams of Fig.4. Our limit (B.3) guarantees that diagrams where the \( \varphi \) circulates in the loops are suppressed and can thus be neglected.
Fig. 4: We integrate out the massive scalar $\Phi$ and obtain an effective action for $J$, which can then be translated to $\varphi$ via (B.7).

The infrared action of the $\varphi$ field is given by an expansion in powers of the momenta $p^2$. In particular, we are interested in the $p^4$ scale independent terms which should be related to the anomaly.

We illustrate the method of our calculation for the simplest $J^2$ term. The prototypical Feynman integral with two external momenta $p_1, p_2$ is given by:

$$M^4 \delta^{(4)} (p_1 + p_2) \int \frac{d^4 q}{(q^2 - M^2) \left[ (q - p_1)^2 - M^2 \right]}.$$  \hspace{1cm} (B.8)

One combines the propagators using the Feynman trick and expands to order $p^4$. The remaining ultraviolet convergent integral is

$$3M^4 \left( p_1^2 \right)^2 \delta (p_1 + p_2) \int_0^1 d\alpha \alpha^2 (1 - \alpha)^2 \int \frac{d^4 q}{(q^2 - M^2)^4}.$$ \hspace{1cm} (B.9)

This evaluates to

$$\frac{i \pi^2}{60} \left( p_1^2 \right)^2.$$ \hspace{1cm} (B.10)
Taking into account all the symmetry factors, one finds that in order to reproduce this four-derivative \( J^2 \) term via a contribution to the effective action we must include
\[
\frac{\pi^2}{60 (2\pi)^3} J \Box^2 J. \quad (B.11)
\]

Similarly, we calculate the Feynman diagrams associated to three and four external \( J \) fields. We expand these diagrams to four derivatives. Substituting (B.7) we get the four-derivative effective action for \( \varphi \) (neglecting terms with more than four \( \varphi \)s)
\[
\frac{1}{2880\pi^2} \left[ \left( (\nabla \varphi)^2 \right)^2 + 9\varphi^2 (\Box \varphi)^2 + 6\varphi (\nabla \varphi)^2 \Box \varphi + 2 (\nabla \varphi)^2 \Box \varphi + 6\varphi (\Box \varphi)^2 + 3 (\Box \varphi)^2 \right]. \quad (B.12)
\]

The expression (B.12) fits the terms we expect from (4.5) (recall (2.4)). There is a contribution from the invariant piece proportional to \( \kappa \) in (4.5) and also the terms associated to the anomaly functional are present. Indeed, the theory (B.1) evolves from a CFT\(_{UV}\) containing a free massless field to an empty theory in the IR. So we expect to obtain (4.5) with
\[
a_{UV} - a_{IR} = \frac{1}{5760\pi^2}, \quad (B.13)
\]
which is exactly the right value contained in (B.12).

As in the general construction of section 4, the change in the \( a \)-anomaly of a free massive field (B.13) can be represented by the dispersion relation (4.7). The integral over the branch cut runs from the threshold \( 4m^2 \) to infinity and one can construct the interpolating function (4.8) explicitly.
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