LOCAL CONSTANT OF HEISENBERG REPRESENTATIONS

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Abstract. We can attach local constant for every representation of finite Galois group by Deligne and Langlands. Tate in [8] gives explicit formula for computation of local constant for abelian case but there is no such formula of local constant for any arbitrary representation of Galois group. In this paper we compute $\lambda$-function in odd case and give explicit formula of local constant for Heisenberg representations of Heisenberg group of order $p^3$, $p$ odd prime.

1. Introduction

Let $F$ be a non-Archimedean local field, $K$ a finite Galois extension of $F$ with Galois group $\text{Gal}(K/F)$, and $\rho: \text{Gal}(K/F) \to GL_C(V)$ be a complex representation of $\text{Gal}(K/F)$. Then for every $\rho$, we can associate a constant $W(\rho)$ with absolute value 1 by Langlands ([12]) and when the $\text{dim} \rho = 1$, it satisfies the following functional equation of $L$-function ([12], Theorem B):

$$L(s, \rho) = W(\rho)L(1 - s, \tilde{\rho})$$

where $\tilde{\rho}$ is the contragredient representation of $\rho$. This constant is called local constant of representation $\rho$.

The existence of this local constant is proved by Tate for one-dimensional representation in his thesis [9] and [8] and the general proof of the existence of the local constants is proved by Langlands (1970). He used local method which is very long and complicated (see [12]). In 1973, Deligne discovered a simpler proof using global methods in [10]. But in Deligne’s terminology this local constant $W(\rho)$ is $\epsilon_D(\rho, \psi_K, dx, 1/2)$, where $dx$ is the Haar measure on $K^+(\text{locally compact abelian group})$ which is self dual with respect to the additive character $\psi_K$ of $K$. Also Langlands denotes this local constant as $\epsilon_L(\rho, \psi)$. According to Tate ([9]), the Langlands factor $\epsilon_L(\rho, \psi)$ is $\epsilon_D(\rho\omega^{1/2}, \psi_K, dx_\psi)$ where $\omega$ denotes the normalized absolute value of $F$, i.e., $\omega(x) = |x|_F = q^{-v_F(x)}$, which we may consider as character of $F^\times$, and $dx_\psi$ is with

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respect to $\psi$ the self dual measure of $F^\times$. According to Tate ([8], pp.105) the relation among three variants of local constant or local epsilon factor is:

\begin{align}
W(\rho) = \epsilon_L(\rho, \psi) = \epsilon_D(\rho \omega_{1/2}, \psi_K, dx) \tag{1.2}
\end{align}

where $\psi$ is a canonical character of $F^+$. In $W(\rho)$ both $\psi$ and $dx$ are fixed.

Initially our motivation was to compute explicit formula of local constant for any arbitrary representation of finite Galois group. But it seems it is difficult. In this paper we compute local constant for Heisenberg representation as a special case. Langlands proves that the local constants are \textbf{extendible functions}. In this article we discuss the properties of extendible function which we use to compute the local constant of \textbf{Heisenberg representation} of finite Galois group.

\section{Notation and Preliminaries}

\subsection{Extendible function}

Let $G$ be any finite group. We denote $R(G)$ the set of all pairs $(H, \rho)$, where $H$ is a subgroup of $G$ and $\rho$ is a virtual representation of $H$. The group $G$ acts on $R(G)$ by means of

\begin{align*}
(H, \rho)^g &= (H^g, \rho^g), \; g \in G, \\
\rho^g(x) &= \rho(gxg^{-1}), \; x \in H^g := g^{-1}Hg
\end{align*}

Again denote by $\hat{H}$ the set of all one dimensional representations of $H$ and by $R_1(G)$ the subset of $R(G)$ of pairs $(H, \chi)$ with $\chi \in \hat{H}$. Here character $\chi$ of $H$ we mean always a \textbf{linear} character, i.e. $\chi : H \rightarrow \mathbb{C}^\times$.

Now define a function $\mathcal{F} : R_1(G) \rightarrow \mathcal{A}$, where $\mathcal{A}$ is a multiplicative group,

\begin{align}
\mathcal{F}(H, 1_H) = 1 \tag{2.1}
\end{align}

and

\begin{align}
\mathcal{F}(H^g, \chi^g) = \mathcal{F}(H, \chi) \tag{2.2}
\end{align}

for all $(H, \chi)$, where $1_H$ denotes the trivial representation of $H$.

Here a function $\mathcal{F}$ on $R_1(G)$ means a function which satisfies the equation (2.1) and (2.2).

A function $\mathcal{F}$ is said to be extendible if $\mathcal{F}$ can be extended to an $\mathcal{A}$-valued function on $R(G)$ satisfying:

\begin{align}
\mathcal{F}(H, \rho_1 + \rho_2) = \mathcal{F}(H, \rho_1)\mathcal{F}(H, \rho_2) \tag{2.3}
\end{align}
for all \((H, \rho_i) \in R(G), i = 1, 2\), and if \((H, \rho) \in R(G)\) with \(\dim \rho = 0\), and \(\Delta\) is a subgroup of \(G\) containing \(H\), then

\[
\mathcal{F}(\Delta, \text{Ind}_H^\Delta \rho) = \mathcal{F}(H, \rho),
\]

where \(\text{Ind}_H^\Delta \rho\) is the virtual representation of \(\Delta\) induced from \(\rho\). In general, let \(\rho\) be a representation of \(H\) with \(\dim \rho \neq 0\). Now we define a zero dimensional representation of \(H\) by \(\rho_0\) as:

\[
\rho_0 := \rho - \dim \rho.1_H.
\]

So \(\dim \rho_0\) is zero, then now we use the equation (2.4) for \(\rho_0\) and we have,

\[
\mathcal{F}(\Delta, \text{Ind}_H^\Delta \rho_0) = \mathcal{F}(H, \rho_0)
\]

now plug \(\rho_0 := \rho - \dim \rho.1_H\) in above equation and we have

\[
\mathcal{F}(\Delta, \text{Ind}_H^\Delta (\rho - \dim \rho.1_H)) = \mathcal{F}(H, \rho - \dim \rho.1_H) \quad \text{or,}
\]

\[
\frac{\mathcal{F}(\Delta, \text{Ind}_H^\Delta \rho)}{\mathcal{F}(\Delta, \text{Ind}_H^\Delta 1_H)^{\dim \rho}} = \frac{\mathcal{F}(H, \rho)}{\mathcal{F}(H, 1_H)^{\dim \rho}}
\]

Therefore,

\[
\mathcal{F}(\Delta, \text{Ind}_H^\Delta \rho) = \left\{ \frac{\mathcal{F}(\Delta, \text{Ind}_H^\Delta 1_H)}{\mathcal{F}(H, 1_H)} \right\}^{\dim \rho} \mathcal{F}(H, \rho)
\]

\[
= \lambda_H^\Delta (\mathcal{F})^{\dim \rho} \mathcal{F}(H, \rho).
\]

where

\[
\lambda_H^\Delta (\mathcal{F}) := \frac{\mathcal{F}(\Delta, \text{Ind}_H^\Delta 1_H)}{\mathcal{F}(H, 1_H)}
\]

but by the definition of \(\mathcal{F}\), we have \(\mathcal{F}(H, 1_H) = 1\), so we get the function

\[
\lambda_H^\Delta (\mathcal{F}) = \mathcal{F}(\Delta, \text{Ind}_H^\Delta 1_H)
\]

This \(\lambda_H^\Delta (\mathcal{F})\) is called Langlands \(\lambda\)-function which is independent of \(\rho\). A extendible function \(\mathcal{F}\) is called strongly extendible if it satisfies equation (2.3) and fulfills equation (2.4) for all \((H, \rho) \in R(G)\), and if the equation (2.4) is fulfilled only when \(\dim \rho = 0\), then \(\mathcal{F}\) is called weakly extendible function.

If \(\mathcal{F}\) is extendible, then there is a unique extension of \(\mathcal{F}\) to \(\mathcal{R}(F)\) which satisfies conditions (2.3) and (2.4) (cf. [8], pp.103 ). Again if \(H \leq G\) and \(\chi_H\) is a character of \(H\), then

\[
\mathcal{F}(H, \chi_H) = \mathcal{F}(H^g, \chi_H^g) \quad \text{where} \quad \chi_H^g(h) := \chi_H(ghg^{-1}) \quad \text{for all} \quad h \in H, \quad \text{conjugate of character} \quad \chi_H
\]

for some \(g \in G\) and \(H^g = g^{-1}Hg\).

**Example.** Langlands proves the local constants are weakly extendible function. The Artin
local constant is the strongly extendible function. (for more examples and details about extendible function, see [8] and [6]).

2.2. **Functoriality properties of** $\lambda$-**factor.** Let $H$ be a subgroup of a group $G$ and $\mathcal{F}$ an extendible function on $R_1(G)$, then we have the following properties of $\lambda$-factor.

1. If $g \in G$, then $\lambda_{g^{-1}Hg}^G(\mathcal{F}) = \lambda_H^G(\mathcal{F})$, where $H \subseteq G$.

2. If $H'$ is a subgroup of $H$ then $\lambda_{H'}^H(\mathcal{F}) = \lambda_{H'}/_{H'}^H(\mathcal{F})\lambda_{H'}^H(\mathcal{F})$, where $[H : H']$ is index of $H'$ in $H$.

3. If $H'$ is a normal subgroup of $G$ contained in $H$, then $\lambda_{H'}^H(\mathcal{F}) = \lambda_{H'/H'}^{G/H'}(\mathcal{F})$.

2.3. **Abelian Local Constants.** Let $F$ be a local field of characteristic 0, and $\chi$ a character of $F^\times$ of finite order, then the root number $W(\chi)$ is a complex number of absolute value 1, and its exact values is:

1. $F = \mathbb{C}$, $W(\chi) = 1$.
2. $F = \mathbb{R}$,

$W(\chi) = \begin{cases} 
1 & \text{if } \chi \text{ is trivial}, \\
-i & \text{otherwise, i.e., if } \chi(x) = \text{sgn}(x).
\end{cases}$

3. When $F$ non-Archimedean: Let $a(\chi)$ be the conductor of $\chi$ and $D_F$ the absolute different of $F$. Define $D(\chi) = a(\chi)D_F$ and choose $c \in F$ such that $c\mathcal{O}_F = D(\chi)$ (where $\mathcal{O}_F$ is ring of integers of $F$). Let $\psi_F$ be the additive character of $F^+$. Then:

$$W(\chi) = |a(\chi)|^{-1/2} \sum_{x \in \mathcal{O}_F^\times \mod \times a(\chi)} \bar{\chi}(c^{-1}x)\psi_F(c^{-1}x).$$

Here $N$ denotes the absolute norm of $a(\chi)$. We can also define our local constant by Gauss sum (see [7]):

$$W(\chi) = \begin{cases} 
N(a(\chi))^{-1/2}\tau(\bar{\chi}) & \text{when } \chi \text{ is ramified} \\
\chi(D_F) & \text{when } \chi \text{ is unramified}
\end{cases}$$

where $\tau(\chi)$ is the local Gauss sum of $\chi$.

**Remark 2.1.** If $F$ be non-Archimedean local field, and $\chi$ be a character of $\text{Gal}(\bar{\mathbb{Q}}_p/F)$. The local Galois Gauss sum $\tau(\chi)$ is defined by the formula:

$$\tau(\chi) = W(\chi)\sqrt{N(a(\chi))},$$
where $a(\chi)$ is the Artin conductor of $\chi$ and this square root is positive square root. We know from the properties of Artin conductors that $a(\bar{\chi}) = a(\chi)$. So we can define our local constant by Gauss sum and which is;

\[(2.11)\quad W(\chi) = \frac{\tau(\bar{\chi})}{\sqrt{N(a(\chi))}}.\]

### 3. Heisenberg representation

In this section we encounter the notion of Heisenberg representation and its properties. We follow the articles of E-W.Zink([2], [3]) on Heisenberg representations. Let $\rho$ be an irreducible representation of a (pro-)finite group $G$. Then $\rho$ is called a **Heisenberg representation** if it represents commutators by scalar operates, therefore higher commutators are represented by 1. Let $Z_{\rho}$ be a scalar group of $\rho$, i.e., $Z_{\rho} \subseteq G$ and $\rho(z) = \text{scalar matrix}$ for every $z \in Z_{\rho}$.

We can see the linear characters of $G$ are Heisenberg representations as degenerate special case.

If $C^3 G = G$, $C^{i+1}G = [C^i G, G]$ denotes the descending central series of $G$, the Heisenberg **property means** $C^3 G \subset \ker(\rho)$, and therefore $\rho$ determines a character $X$ on the alternating square of $A := G/C^2G$ such that:

\[(3.1)\quad \rho([\hat{a}_1, \hat{a}_2]) = X(a_1, a_2) \cdot E\]

for $a_1, a_2 \in A$ with lifts $\hat{a}_1, \hat{a}_2 \in G$. The equivalence class of $\rho$ is determined by the projective kernel $Z_{\rho}$ which has the property that $Z_{\rho}/C^2G$ is the radical of $X$ and by the character $\chi_{\rho}$ of $Z_{\rho}$ such that $\rho(g) = \chi_{\rho}(g) \cdot E$ for all $g \in Z_{\rho}$ and $E$ being the unit operator. Here $\chi_{\rho}$ is a $G$-invariant character of $Z_{\rho}$ which we call the central character of $\rho$.

We can prove that Heisenberg representations $\rho$ are fully characterized by the corresponding pair $(Z_{\rho}, \chi_{\rho})$. (cf. [3], Section 2)

**Proposition 3.1.** ([3], Proposition 4.2). The map $\rho \mapsto (Z_{\rho}, \chi_{\rho})$ is a bijection between equivalence classes of Heisenberg representations of $G$ and the pairs $(Z_{\rho}, \chi_{\rho})$ such that

(a) $Z_{\rho} \subseteq G$ is a coabelian normal subgroup,

(b) $\chi_{\rho}$ is a $G$-invariant character of $Z_{\rho}$,

(c) $X(\hat{g}_1, \hat{g}_2) := \chi_{\rho}(g_1 g_2 g_1^{-1} g_2^{-1})$ is a non-degenerate alternating character on $G/Z_{\rho}$

where $\hat{g}_1, \hat{g}_2 \in G/Z_{\rho}$ and their corresponding lifts $g_1, g_2 \in G$.

For pairs $(Z_{\rho}, \chi_{\rho})$ with the properties (a) – (c) the corresponding Heisenberg representation $\rho$ is determined by the identity:

\[(3.2)\quad \sqrt{|G : Z_{\rho}|} \cdot \rho = \text{Ind}_{Z_{\rho}}^G \chi_{\rho}.\]
Proof. (of equation (3.2))

Let $H$ be a maximal isotropic subgroup of $G$ and choose a character $\chi_H : H \to \mathbb{C}^\times$ such that $\chi_H|_{Z_{\rho}} = \chi_{\rho}$, then we have:

\[(3.3) \quad \rho = \text{Ind}_H^G \chi_H.\]

This induced representation from $\chi_H$ does not depend on the choice $H$ and the extension of $\chi_{\rho}$ to $H$. Again we know from transitivity of induction:

\[(3.4) \quad \text{Ind}_{Z_{\rho}}^G \chi_{\rho} = \text{Ind}_H^G \text{Ind}_{Z_{\rho}}^H \chi_{\rho}.\]

We also know that:

\[(3.5) \quad \text{Ind}_{Z_{\rho}}^H \chi_{\rho} = \chi_H \otimes \text{Ind}_{Z_{\rho}}^H 1_{Z_{\rho}} \quad \text{where} \quad 1_{Z_{\rho}} \text{ is trivial representation of } Z_{\rho}
= \sum \chi'_H \quad \text{all } \chi'_H \text{ which are extended from } \chi_{\rho}.\]

Here total number of $\chi'_H$ is exactly equal to $[H : Z_{\rho}]$. Putting this above result in equation (3.4) and we have:

\[(3.6) \quad \text{Ind}_{Z_{\rho}}^G \chi_{\rho} = \text{Ind}_H^G (\sum \chi'_H \quad \text{all } \chi'_H \text{ which are extension of } \chi_{\rho}),
= \sharp\{\text{no.of } \chi_H \text{ which are extended from } \chi_{\rho}\} \times \text{Ind}_H^G \chi_H
= [H : Z_{\rho}] \cdot \text{Ind}_H^G \chi_H
= [H : Z_{\rho}] \cdot \rho, \quad \text{since } \text{Ind}_H^G \chi_H = \rho.\]

We also know in our setting

\[(3.7) \quad [G : H] = [H : Z_{\rho}] = \sqrt{[G : Z_{\rho}]} = \text{dim } \rho.\]

From equation (3.6) and (3.7) we have our desire result, and which is:

\[(3.8) \quad \text{Ind}_{Z_{\rho}}^G \chi_{\rho} = \sqrt{[G : Z_{\rho}]} \cdot \rho
= \text{dim } \rho \cdot \rho.\]

Therefore, the equation (3.2) is proved. \qed
Remark 3.2. Let $\chi_\rho$ be the character of $Z_\rho$. All extensions $\chi_H \supset \chi_\rho$ are conjugate with respect to $G/H$. We can easily see this. Since we know $\chi_H \supset \chi_\rho$ and $\chi_H^g(h) = \chi_H(ghg^{-1})$, now we take $z \in Z_\rho$, such that:

$$\chi_H^g(z) = \chi_H(gzg^{-1}) = \chi_\rho(gzg^{-1}), \quad gzg^{-1} \in Z_\rho \text{ since } Z_\rho \text{ is normal subgroup of } G,$$

then

$$\chi_H^g(z') \quad \text{where } z' = gzg^{-1} \in Z_\rho.$$

Therefore $\chi_H^g$ are extensions of $\chi_\rho$ for all $g \in G/H$.

Remark 3.3. We can see similar concept of representation for Heisenberg group. If $\rho$ be a Heisenberg representation of a group $G$. Then the group

$$G/\ker \rho \cong \text{Im } \rho,$$

behaves like a Heisenberg group. Therefore the arguments regarding Heisenberg group are very similar as the arguments of Heisenberg representation. Here we see very briefly the Heisenberg group concept and its representations which have similar pattern with the quotient group $G/\ker \rho$. In the following paragraph we see the general notion of Heisenberg group and it’s irreducible representations.

Let $H$ be a separable locally compact group which is a central extension of an abelian group $P$ by the center $Z(G)$ (simply $Z$) of $H$. That is, we have an exact sequence

$$1 \to Z \to H \overset{\Pi}{\to} P \to 1$$

and the commutator subgroup $[H, H]$ of $H$ is a subgroup of $Z$. Let $\rho$ be an irreducible unitary representation of $H$ on a separable Hilbert space $V$, i.e., $\rho : H \to \text{Aut}(V)$.

Assume that $\Pi$ admits a continuous section map $a_H$ and $\Pi \circ a_H = \text{Id}_P$. We define $X : P \times P \to Z$ by

$$X(p_1, p_2) = a_H(p_1)a_H(p_2)a_H(p_1)^{-1}a_H(p_2)^{-1}.$$

This $X$ is well-defined and independent of choice of section map $a_H$. Moreover $X$ is an alternating bicharacter and $X(p, p) = 1$ for each $p \in P$. This $X$ determines $H$ up to unique isomorphism of central extensions. Let $\hat{P}$ denote the Pontryagin dual of $P$, then define a homomorphism

$$\varphi : P \to \hat{P}$$

by the setting $\varphi(x)(y) = X(x, y)$ for all $x, y \in P$. If $\varphi$ is an isomorphism, then $H$ is called a Heisenberg group.
Now let $M \leq P$ be a closed subgroup. If $X|_{M \times M} \equiv 1$, then $M$ is called isotropic subgroup and when it is maximal and isotropic, then it is called maximal isotropic subgroup.

**Theorem 3.4.** (Stone, Von Neumann, Mackey). Let $H$ be a Heisenberg group. Then, $H$ has a unique irreducible unitary representation \( \rho : H \to \text{Aut}(V) \)
such that \( \rho_z = C.id \) for all \( z \in Z \) and here \( C \in \mathbb{C}^\times \). This representation is called Heisenberg representation.

Let $M \leq P$ be a maximal isotropic subgroup and let $\chi \in \hat{\Pi}^{-1}(M)$ be a character such that $\chi|_Z = Id_Z$. Then we can get our Heisenberg representation from this character $\chi$ by induction and which is $\rho = \text{Ind}^H_{\hat{\Pi}^{-1}(M)} \chi$. Now if $\rho'$ is an irreducible representation with $\rho'|_Z = Id_Z$, then by Theorem 3.4. we have $\rho \cong \rho'$. For more informations about Heisenberg group and its representations, see [1].

**4. Local constant of Heisenberg representation**

This is the main section of this article. Our local constants are weakly extendible function, so to compute local constant for Heisenberg representation we have to compute $\lambda$-function and local constant for linear character. Let $\rho = \text{Ind}^G_H \chi$ be the Heisenberg representation of $G = \text{Gal}(L/F)$ where $L/F$ is a finite Galois extension of local field $F$. Here $H$ is maximal isotropic subgroup of $G$. Let $Z$ be the scalar group of the representation $\rho$.

**4.1. Computation of $\lambda$-function.** We know that local constant $W$ is a weak extension(by Langlands, cf. Theorem 1, [8]). Let $\theta$ be an irreducible representation of $H \leq G$ of dimensional $d$. Then we get by using equation (2.6)

\[
(4.1) \quad W(\text{Ind}^G_H \theta) = \frac{W(\text{Ind}^G_H 1_H)^d}{W(1_H)^d} \cdot W(\theta).
\]

But we know that $W(1_H) = 1$ and the Langlands $\lambda$-function( see equation (2.8)) $\lambda_H^G = W(\text{Ind}^G_H 1_H)$. Therefore we obtain:

\[
(4.2) \quad W(\text{Ind}^G_H \theta) = (\lambda_H^G)^d \cdot W(\theta).
\]

For orthogonal representations $\rho : G \to O(n)$ there is a well defined second Stiefel-Whitney class $s_2(\rho) \in H^2(G, \mathbb{Z}/2\mathbb{Z})$, and Deligne(cf. [11]) gives a procedure how to obtain out of $s_2(\rho)$ the invariant

\[
c(\rho) := \frac{W(\rho)}{W(\text{det}_\rho)}.
\]
We know $\rho = \text{Ind}_{H}^{G}1_{H}$ is orthogonal, then therefore we have:

$$\lambda_{H}^{G} := W(\text{Ind}_{H}^{G}1_{H}) = c(\text{Ind}_{H}^{G}1_{H}) \cdot W(\det \circ \text{Ind}_{H}^{G}1_{H}).$$

From example (III) on p.104 of [8], if $H = \text{Gal}(L/K) < G = \text{Gal}(L/F)$ corresponds to an extension $F \subset K \subset L$ of local fields then

$$\det \circ \text{Ind}_{H}^{G}1_{H} = \Delta_{K/F}$$

can be interpreted by class-field theory as a character of $F^{\times}$. It is then character of $F^{\times}$ corresponding to the quadratic extension $F(\sqrt{d_{K/F}})/F$, which is obtained by adjoining the square root of the discriminant $d_{K/F}$ of $K/F$.

Replacing Galois groups by the corresponding local fields we may write the local constant of finite extension $K/F$ as

$$\lambda_{K/F} = c(\text{Ind}_{K/F}1_{H}) \cdot W(\Delta_{K/F}),$$

where $c(\text{Ind}_{K/F}1_{H})$ is Deligne’s sign, and $\Delta_{K/F}$ is a quadratic character of $F^{\times}$ related to the discriminant.

**Lemma 4.1** If $H \leq G$ is normal subgroup and if $[G : H]$ is odd, then $\Delta_{K/F} \equiv 1$ and $\lambda_{K/F}^{2} = 1$.

**Proof.** If $H$ is a normal subgroup, then $\text{Ind}_{H}^{G}1_{H} = \text{Ind}_{H}^{G/H}1_{H}$ is the regular representation of $G/H$, hence $\det \circ \text{Ind}_{H}^{G}1_{H} = \Delta_{K/F}$ is the quadratic character of the group $G/H$. Therefore $\Delta_{K/F} \equiv 1$ if $G/H$ is of odd order and then $\lambda_{K/F}^{2} = 1$ since $\lambda_{K/F}^{2} = \Delta_{K/F}(-1)$.

**Lemma 4.2**

1. The Langlands’s $\lambda$ function is always a 4-th root of unity.
2. If $H \leq G$ is normal subgroup of odd index $[G : H]$, then $\lambda_{H}^{G} = 1$.
3. If there exists a normal subgroup $N$ of $G$ such that $N \leq H \leq G$ and $[G : N]$ odd, then $\lambda_{H}^{G} = 1$.

**Proof.** (1) We know that our local constant satisfies the following functional equation

$$W(\rho)W(\bar{\rho}) = \det_{\rho}(-1).$$

where $\rho$ is a representation of $G$. In this equation we plug the orthogonal representation $\rho = \text{Ind}_{H}^{G}1_{H}$ in the place of $\rho$ and we have

$$W(\text{Ind}_{H}^{G}1_{H})W(\overline{\text{Ind}_{H}^{G}1_{H}}) = \det_{\text{Ind}_{H}^{G}1_{H}}(-1)$$
Now
\[ \lambda^G_H(W)\lambda^G_H(W) = \det_{\text{Ind}^G_H 1_H}(-1) \quad \text{since } \text{Ind}^G_{H1} = \text{Ind}^G_H 1_H \]
\[ (\lambda^G_H(W))^2 = \det_{\text{Ind}^G_H 1_H}(-1) \]
\[ (\lambda^G_H(W))^4 = (\det_{\text{Ind}^G_H 1_H}(-1))^2 \]
\[ (\lambda^G_H(W))^4 = 1 \quad \text{since } \det_{\text{Ind}^G_H 1_H}(-1) \text{ is a sign} \]
so \( \lambda^G_H(W) \) is a 4-th root of unity.

(2) To prove (2) we use the expression (4.3)

(4.7)
\[ \lambda^G_H = W(\text{Ind}^G_{H1}H) = c(\text{Ind}^G_{H1}H) \cdot W(\det \circ \text{Ind}^G_{H1}H). \]

Since \( \rho = \text{Ind}^G_{H1}H \) is orthogonal we may compute \( c(\rho) \) by using \( s_2(\rho) \), and from Deligne (see Prop.2(v) on p.119, [8]) we know that \( c(\rho) = W(\rho)/W(\det \rho) \) is a sign. According to Deligne (see Theorem 3 on p.129, [8]) we have:
\[ c(\rho) = \text{cl}(s_2(\rho)) \]
if \( \rho \) is orthogonal. But the Stiefel-Whitney class \( s_2(\rho) \) is easy accessible only if \( \det \rho \equiv 1 \), and this is in general wrong for \( \rho = \text{Ind}^G_{H1}H \). But it is true for \( \rho = \text{Ind}^G_{H1}H \) if \( H \leq G \) is normal subgroup and \( [G:H] \) is odd (by using Lemma 4.1). Moreover we have then:
\[ s_2(\text{Ind}^G_{H1}H) \in H^2(G/H,\mathbb{Z}/2\mathbb{Z}) = \{1\}, \]
which implies that in (4.7) both factors are \( = 1 \), hence \( \lambda^G_H = 1 \).

(3) From \( N \leq H \leq G \) we obtain

(4.8)
\[ \lambda^G_N = \lambda^H_N \cdot (\lambda^G_H)^{[H:N]} \]

From (2) we obtain \( \lambda^G_N = \lambda^H_N = 1 \) because \( N \) is normal and the index \( [G:N] \) is odd, hence \( (\lambda^G_H)^{[H:N]} = 1 \). Finally this implies \( \lambda^G_H = 1 \) because \( \lambda^G_H \) is 4th root of unity and \( [H:N] \) is odd.

\[ \square \]

Lemma 4.3. Let \( F \) be a local field and \( \text{Gal}(E/F) \) be a local Galois group of odd order. If \( L \supset K \supset F \) be any finite extension inside \( E \), then \( \lambda_{L/K} = 1 \).

Proof. By the given condition \( |\text{Gal}(E/F)| \) is odd. Therefore the degree of extension \( [E:F] \) of \( E \) over \( F \) is odd. Let \( L \) be any arbitrary intermediate field of \( E/F \) which contains \( K/F \). Therefore here we have the tower of fields \( E \supset L \supset K \supset F \). Here the degree of extensions are
all odd since \([E : F]\) is odd. By assumption \(E/F\) is Galois, then also the extension \(E/L\) and \(E/K\) are Galois and \(H = \text{Gal}(E/L)\) is subgroup of \(G = \text{Gal}(E/K)\).

By the definition we have \(\lambda_{L/K} = \lambda_H^G\). If \(H\) is a normal subgroup of \(G\) then \(\lambda_H^G = 1\) because \(|G/H|\) is odd. But \(H\) need not be normal subgroup of \(G\) therefore \(L/K\) need not be Galois extension. By **Feit-Thompson Theorem** we know that every finite group of order odd is a solvable group. Therefore the Galois group \(G\) is a solvable group, so it has always a nontrivial normal subgroup. Let \(N\) be the largest normal subgroup of \(G\) contained in \(H\) and \(N\) can be written as:

\[
N = \cap_{g \in G} gHg^{-1}
\]

Therefore the fixed field \(E^N\) is the smallest normal extension of \(K\) containing \(L\). Now we have from properties of \(\lambda\)-function(cf. 2.2(2)),

\[
(4.9) \quad \lambda_H^G = \lambda_N^H \cdot (\lambda_H^G)^{[H:Z]}.
\]

This implies \((\lambda_H^G)^{[H:N]} = 1\) since \([H : N]\) and \([G : N]\) are odd and \(N\) is normal subgroup of \(G\) contained in \(H\). Therefore \(\lambda_H^G = 1\) because \(\lambda_H^G\) is 4th root of unity and \([H : N]\) is odd.

Then we may say \(\lambda_{L/K} = 1\) all possible cases if \([E^N : K]\) is odd. When the big extension \(E/F\) is odd then all intermediate extensions will be odd. Therefore the lemma is proved for all possible cases.

\(\square\)

**Remark 4.4.**

(i) If the Galois extension \(E/F\) is infinite then we say it is odd if \([K : F]\) is odd for all subextensions of finite degree. This means the profinite group \(\text{Gal}(E/F)\) can be realized as the projective limit of finite groups which are all odd order.

If \(E/F\) is Galois extension of odd order in this more general sense, then again we will have \(\lambda_{L/K} = 1\) in all cases where \(\lambda\)-function is defined.

(ii) In this above situation, all weak extensions are strong extensions, because

\[
W(\text{Ind}_H^G \rho_H) = (\lambda_H^G)^{\dim \rho_H}. W(\rho_H)
\]

\[
= W(\rho_H), \quad \text{since } \lambda_H^G = 1.
\]

**4.2. Local constant of Heisenberg representation.** Let \(\rho = \text{Ind}_H^G \chi_H\) be a Heisenberg representation of the Galois group \(G = \text{Gal}(L/F)\), where \(L/F\) be finite Galois extension of local field \(F\). Let \(Z\) be the scalar group of \(\rho\). We know that(see section 3),

\[
d := \dim \rho = [G : H] = [H : Z],
\]
and

\[(4.10) \quad d \cdot \rho = \text{Ind}_Z^G \chi_Z, \quad W(\rho)^d = \lambda_Z^G \cdot W(\chi_Z).
\]

We have a more direct expression for \(W(\rho)\) we obtained by fixing an extension \(\chi_H\) of \(\chi_Z\) onto the group \(H\). Then we have:

\[(4.11) \quad \rho = \text{Ind}_H^G \chi_H, \quad W(\rho) = \lambda_H^G \cdot W(\chi_H).
\]

But the general property (see 2.2(2)) of \(\lambda\)-function we have:

\[(4.12) \quad \lambda_Z^G = \lambda_H^G \cdot \lambda_{[H:Z]}^Z = \lambda_Z^H \cdot (\lambda_H^G)^d, \quad \lambda_H^G = \frac{(\lambda_Z^G)^{1/d}}{\lambda_{[H:Z]}^{1/d}}.
\]

Substituting this on the right side of equation (4.11) we find:

\[(4.13) \quad W(\rho) = (\lambda_Z^G)^{1/d} \cdot \frac{W(\chi_H)}{(\lambda_H^G)^{1/d}}.
\]

Now to check the equation (4.13) is compatible with equation (4.10), we need to prove the following Proposition:

**Proposition 4.5.** \(W(\chi_H)^d = W(\text{Ind}_Z^H \chi_Z)\).

**Proof.** We prove this above proposition by following few steps:

**Step 1.**

\[W(\text{Ind}_Z^H \chi_Z) = \prod_{\chi_H \supset \chi_Z} W(\chi_H)\]

**Proof.** We know that \(\chi_Z\) admits an extension to \(H\), then we will have:

\[(4.14) \quad \text{Ind}_Z^H \chi_Z = \chi_H \otimes \text{Ind}_Z^H 1_Z = \sum_{\chi_H \supset \chi_Z} \chi_H \implies W(\text{Ind}_Z^H \chi_Z) = \prod_{\chi_H \supset \chi_Z} W(\chi_H).\]

In the following two steps we will show that all factors on the right side of (4.14) are equal, i.e. \(W(\chi_H)\) does not depend on the choice of extension \(\chi_H \supset \chi_Z\).

**Step 2.**

\[W(\chi_H^g) = W(\chi_H) \quad \text{for all } g \in G/H.
\]

**Proof.** Fix one extension \(\chi_H \supset \chi_Z\). Then all other extensions of \(\chi_Z\) are given as \(\chi_H^g\) for \(g \in G/H\). We know that our local constant is an extendible function. So by the definition of extendible function (cf. equation (2.2)), we have

\[W(\chi_H^g) = W(\chi_H) \quad \text{for all } g \in G/H.
\]
Step 3.

\[ W_K(\chi_K^\sigma) = W_K(\chi_K) \quad \text{for all } \sigma \in \text{Gal}(K/F). \]

**Proof.** If \( G \supset H \supset Z \), then we get a field extension \( K \) of \( F \) corresponding to \( H \) such that \( F \subset K \subset L \) by Galois theory. Now by **class field theory**, we may consider \( \chi_H \) as a character of \( K^\times \), since \( H/[H,H] \cong K^\times \). Then we have the relation:

\[ \chi_H \longleftrightarrow \chi_K : K^\times \to \mathbb{C}^\times \quad \text{and} \quad \chi_H^g \longleftrightarrow \chi_K^g : K^\times \to \mathbb{C}^\times. \]

where \( g \in G/H \), \( \sigma \in \text{Gal}(K/F) \) and \( \chi_K^g(x) = \chi_K(\sigma(x)) \) and therefore we can consider

\[ \chi_H = \chi_K^g. \]

Then by Step 2 we have

\[ W(\chi_H^g) = W(\chi_H) = W_K(\chi_K) = W_K(\chi_K^\sigma), \quad \text{for all } \sigma \in \text{Gal}(K/F) \text{ and } g \in G/H. \]

We also can directly see \( W(\chi_K) = W(\chi_K^g) \) from equation (2.9) or from ([8], proposition 1, p.94/95).

Therefore we obtain,

\[
W(\text{Ind}_Z^H \chi_Z) = \prod_{\chi_{H} \supset \chi_Z} W(\chi_H) = \prod_{g \in G/H} W(\chi_H^g) = W(\chi_H)^d.
\]

Therefore the proposition is proved. \( \square \)

So the formula (4.13) is justified.

Therefore for computing local constant of Heisenberg representation we may use the formula (4.11) or (4.13) because both are compatible.

By the following theorem in general we can show directly that our local constant \( W(\chi_H) \) does not depend on \( H \).

**Theorem 4.6.** \( W(\chi_H) \) does not depend on \( H \).
Proof. In the Heisenberg setting we have $Z \triangleleft H \triangleleft G$, then the both quotient groups $G/H$ and $H/Z$ are abelian because $G$ is solvable group. By property of $\lambda$-function (cf. 2.2.(3)) we have here $\lambda^G_H = \lambda^{G/Z}_{H/Z}$. Let $\chi_{H/Z}$ be a linear character of $H/Z$, then we have:

\begin{align}
W(\text{Ind}^{G/Z}_{H/Z} \chi_{H/Z}) &= \lambda^{G/H}_{H/Z} \cdot W(\chi_{H/Z}) \\
&= \lambda^G_H \cdot W(\chi_{H/Z}) \quad \text{since } \lambda^{G/Z}_{H/Z} = \lambda^G_H \\
&= \lambda^G_H \cdot W(\chi_H) \quad \text{because } W(\chi_{H/Z}) = W(\chi_H) \text{ by inflativity of local constant.}
\end{align}

Moreover $G/Z$ is an abelian group because $Z$ includes $[G,G]$, therefore we can expression

\begin{align}
\text{Ind}^{G/Z}_{H/Z} \chi_{H/Z} = \sum_{\mu \in G/Z \atop \mu|_{H/Z} = \chi_{H/Z}} \mu.
\end{align}

Then

\begin{align}
W(\text{Ind}^{G/Z}_{H/Z} \chi_{H/Z}) = \prod_{\mu \in G/Z \atop \mu|_{H/Z} = \chi_{H/Z}} W(\mu).
\end{align}

Now take two distinct maximal isotropic subgroups $H_1, H_2$ of $G$, therefore $H_1/Z \cong H_2/Z$ and $\lambda^G_{H_1} = \lambda^G_{H_2} = \lambda^{G/Z}_{H_i/Z}(i = 1, 2)$. We have :

\begin{align}
W(\text{Ind}^{G/Z}_{H_i/Z} \chi_{H_i/Z}) = \prod_{\mu \in G/Z \atop \mu|_{H_i/Z} = \chi_{H_i/Z}} W(\mu) = W(\text{Ind}^{G/Z}_{H_2/Z} \chi_{H_2/Z}) \implies \\
\lambda^G_{H_i} \cdot W(\chi_{H_1}) = \lambda^G_{H_2} \cdot W(\chi_{H_2}) \\
W(\chi_{H_1}) = W(\chi_{H_2}).
\end{align}

This above equation shows that $W(\chi_H)$ does not depend on $H$, hence $\chi_H$. □

For Heisenberg group of order $p^3$, where $p$ is odd, we show directly that our local constant $W(\chi_H)$ does not depend on $H$ and hence on $\chi_H$ and it is a root of unity.

**Theorem 4.7.** Let $G$ be a non-abelian group of order $p^3$ where $p$ is an odd prime. Let $Z = Z(G)$ be the center of $G$ and $H$ be the maximal normal subgroup of $G$. The local constant $W(\chi_H)$ is a root of unity and local constant of Heisenberg representation $\rho$ is:

\begin{align}
W(\rho) = W(\chi_H), \quad \text{i.e., a root of unity.}
\end{align}
Proof. In this setting we have $Z \triangleleft H \triangleleft G$ and the order $|H| = p^2$ and $|Z| = p$. It can be proved that all irreducible representation of $G$ are Heisenberg representation. From Proposition 4.5, we have $W(\chi_H)^p = W(\text{Ind}_Z^H \chi_Z) = W(\chi_Z)$. By modified Modified Brauer induction due to Deligne ([10], Prop. 1.5) we know that:

For every virtual representation $\rho$ of a finite group $G$ there are subgroups $U_i$ of $G$, characters $\chi_i$ of $U_i$, and integers $n_i$ such that

\begin{equation}
(4.21) \quad \rho - \dim \rho \cdot 1_G \cong \sum_i n_i \text{Ind}_{U_i}^G (\chi_i - 1_{U_i}),
\end{equation}

where $1_G$ denotes the trivial representation of $G$. In this above equation we put $\rho = \text{Ind}_{U_i}^G 1_U$, and we have:

\begin{equation}
(4.22) \quad \text{Ind}_{U_i}^G 1_U - [G : U] \cdot 1_G \cong \sum_i n_i \text{Ind}_{U_i}^G (\chi_i - 1_{U_i})
\end{equation}

where certain $U_i \subset G$, $\chi_i \in \hat{U}_i$ and $n_i \in \mathbb{Z}$.

Hence

\begin{equation}
(4.23) \quad \lambda_U^G = \prod_i W(\chi_i)^{n_i}.
\end{equation}

Now we take $U = H$, then $U_i \subset G$ are $Z$, and $H$, then we have

\begin{equation}
(4.24) \quad \lambda_H^G = W(\chi_H)^{n_H} \cdot W(\chi_Z)^{n_Z}
\end{equation}

where $n_H, n_Z \in \mathbb{Z}$.

We know here $[G : H] = p$, a odd number and $H \triangleleft G$ a normal subgroup of $G$, therefore $\lambda_H^G = 1$ by Lemma 4.2(2) then therefore we have:

\begin{equation}
(4.25) \quad W(\chi_H)^{n_H + p \cdot n_Z} = \lambda_H^G = 1
\end{equation}

This implies $W(\chi_H)$ is a root of unity since $n_H + p \cdot n_Z \in \mathbb{Z}$. This shows that $W(\chi_H)$ does not depend on $H$.

Since we know our Heisenberg representation is: $\rho = \text{Ind}_H^G \chi_H$, then we have

\begin{align*}
W(\rho) &= W(\text{Ind}_H^G \chi_H) \\
&= \lambda_H^G \cdot W(\chi_H) \\
&= W(\chi_H) \quad \text{since } \lambda_H^G = 1.
\end{align*}

Therefore in this case our local constant $W(\rho)$ is a root of unity. □
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