On the number of $q$-ary quasi-perfect codes with covering radius 2

Alexander M. Romanov

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Abstract
In this paper we present a family of $q$-ary nonlinear quasi-perfect codes with covering radius 2. The codes have length $n = q^m$ and size $M = q^n - m - 1$ where $q$ is a prime power, $q \geq 3$, $m$ is an integer, $m \geq 2$. We prove that there are more than $qq^c$ nonequivalent such codes of length $n$, for all sufficiently large $n$ and a constant $c = \frac{1}{q} - \varepsilon$.

Keywords Quasi-perfect codes · Generalized Reed–Muller codes · Galois geometry · Switching construction

Mathematics Subject Classification 94B05 · 94B25 · 05B25

1 Introduction

Let $F_q^n$ be a vector space of dimension $n$ over the finite field $F_q$, where $q$ is a prime power. An arbitrary subset $\mathcal{C}$ of $F_q^n$ is called a $q$-ary error correcting code (briefly a $q$-ary code). The length of a code $\mathcal{C} \subseteq F_q^n$ is the dimension of the vector space $F_q^n$. We assume that the all-zero vector $\mathbf{0}$ always belongs to the code, unless otherwise specified. A code is called linear if it is a subspace of $F_q^n$. Otherwise, the code is called nonlinear. The dimension of a linear code is the dimension of the subspace, denoted by $\dim(\mathcal{C})$. The vectors belonging to a code $\mathcal{C}$, we will call codewords. The Hamming weight of a vector $x \in F_q^n$ is the number of nonzero coordinate positions of $x$, denoted by $wt(x)$. The Hamming distance between two vectors $x, y \in F_q^n$ is the number of coordinate positions in which they differ, denoted by $d(x, y)$. The minimum distance of a code $\mathcal{C}$ is the smallest Hamming distance between two different codewords of $\mathcal{C}$, denoted by $d(\mathcal{C})$.

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Alexander M. Romanov
rom@math.nsc.ru

1 Sobolev Institute of Mathematics, 4 Acad. Koptyug Avenue, Novosibirsk, Russia 630090
Define the packing radius $e(\mathcal{C})$ of a code $\mathcal{C} \subseteq \mathbb{F}_q^n$. Let $e(\mathcal{C}) = \lfloor (d - 1)/2 \rfloor$, where $d$ is the minimum distance of $\mathcal{C}$. We also define the covering radius $\rho(\mathcal{C})$ of a code $\mathcal{C}$. Let

$$\rho(\mathcal{C}) = \max_{x \in \mathbb{F}_q^n} \min_{c \in \mathcal{C}} d(x, c).$$

A code $\mathcal{C} \subseteq \mathbb{F}_q^n$ is called perfect if $\rho(\mathcal{C}) = e(\mathcal{C})$. If $\rho(\mathcal{C}) = e(\mathcal{C}) + 1$, then $\mathcal{C}$ is called quasi-perfect. See [9,p. 19].

Two codes $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathbb{F}_q^n$ are said to be equivalent if there exists a vector $v \in \mathbb{F}_q^n$ and a $n \times n$ monomial matrix (or generalized permutation matrix) $P$ over $\mathbb{F}_q$ such that

$$\mathcal{C}_2 = \{(v + cP) \mid c \in \mathcal{C}_1\}.$$

A codeword $c = (c_1, c_2, \ldots, c_n)$ is called even if $\sum_{i=1}^n c_i = 0$. A code is called even if it contains only even codewords.

We will use the standard notation $(n, M, d; \rho)_q$ to denote a $q$-ary code of length $n$, size $M$, minimum distance $d$, and covering radius $\rho$.

In [14] Vasil’ev used the switching construction to construct a family of binary nonlinear perfect codes with packing radius $e = 1$ and length $n = 2^m - 1$, $m \geq 4$. In [14] Vasil’ev proved that there are more than $q^{q^{cn}}$ nonequivalent such codes of length $n$, for all sufficiently large $n$ and a constant $c = \frac{1}{2} - \varepsilon$. In [5] Etzion and Vardy used the switching construction to construct binary nonlinear perfect codes of full-rank. In [11] Phelps and Villanueva used the switching construction to construct $q$-ary nonlinear perfect codes with ranks of different sizes.

In this paper we use the switching construction to construct a family of $q$-ary nonlinear quasi-perfect codes with covering radius $\rho = 2$. The codes we offer are of length $n = q^m$ and size $M = q^{n-m-1}$ where $q$ is a prime power, $q \geq 3$, $m$ is an integer, $m \geq 2$. In this paper we prove that there are more than $q^{q^{cn}}$ nonequivalent such codes of length $n$, for all sufficiently large $n$ and a constant $c = \frac{1}{q} - \varepsilon$.

In [12], using the concatenation construction, a family of $q$-ary even quasi-perfect codes with covering radius $\rho = 2$ was constructed. The codes proposed in [12] have length $n = q^m$ and size $M = q^{n-m-1}$ where $q$ is a prime power, $q \geq 3$, $m$ is an integer, $m \geq 4$. It was shown [12] that there are more than $q^{q^{mn}}$ nonequivalent such codes of length $n$, for all sufficiently large $n$ and a constant $c$.

For $q = 2$, the construction from [12] is the concatenation construction of binary extended perfect codes proposed by Phelps in [10].

### 2 Generalized Reed–Muller codes

Let $\mathbb{F}_q[X_1, X_2, \ldots, X_m]$ be the algebra of polynomials in $m$ variables over the field $\mathbb{F}_q$. Let a polynomial $f$ be in $\mathbb{F}_q[X_1, X_2, \ldots, X_m]$, then by $\deg(f)$ we denote its total degree. Let $AG(m, q)$ be the $m$-dimensional affine space over the field $\mathbb{F}_q$. Let $n = q^m$ and the points $P_1, P_2, \ldots, P_n$ of $AG(m, q)$ be arranged in some fixed order. Let $r$ be an integer such that $0 \leq r \leq (q - 1)m$. Then the generalized Reed–Muller code [7] of order $r$ over the field $\mathbb{F}_q$ is the following subspace:

$$RM_q(r, m) = \{ (f(P_1), f(P_2), \ldots, f(P_n)) \mid f \in \mathbb{F}_q[X_1, X_2, \ldots, X_m], \deg(f) \leq r \}.$$

The code $RM_q(r, m)$ has the following parameters ([7], [1,Theorem 5.5]):

1. the length is $q^m$;
2. the dimension is
\[
\sum_{i=0}^{r} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \binom{i - kq + m - 1}{i - kq}
\]  
(1)

3. the minimum distance is
\[
(q - b)q^{m - a - 1},
\]  
(2)

where \( r = (q - 1)a + b \) and \( 0 \leq b < q - 1 \).

In what follows, we are only interested in the generalized Reed–Muller codes \( RM_q(r, m) \) of order \( r = (q - 1)m - 2 \). In the binary case, the code \( RM_q(r, m) \) of order \( r = (q - 1)m - 2 \) is the extended Hamming code of length \( n = 2^m \). The dimension of the extended Hamming code of length \( n = 2^m \) is \( n - m - 1 \) and the minimum distance is 4.

**Proposition 1** Let \( q \geq 3 \), \( m \geq 1 \), and \( n = q^m \). Then the dimension of the generalized Reed–Muller code \( RM_q((q - 1)m - 2, m) \) is \( n - m - 1 \) and the minimum distance is 3.

**Proof** If a code \( \mathcal{C} \) belongs to the class of generalized Reed–Muller codes, then the dual code \( \mathcal{C}^\perp \) also belongs to this class [3]. Assume that the length of a code \( \mathcal{C} \) is \( n \). Then
\[
\dim(\mathcal{C}) + \dim(\mathcal{C}^\perp) = n.
\]  
(3)

From [1, Theorem 5.8], for \( r < (q - 1)m \), we have
\[
RM_q(r, m)^\perp = RM_q((q - 1)m - 1 - r, m).
\]
Therefore, the order of the code dual to the code \( RM_q((q - 1)m - 2, m) \) is 1. By formulas (1) and (3), the dimension of the code \( RM_q((q - 1)m - 2, m) \) is \( n - m - 1 \).

Now we show that for all \( m \geq 1 \) the minimum distance of the generalized Reed–Muller code \( RM_q((q - 1)m - 2, m) \) is 3. Since
\[
r = (q - 1)m - 2 = (q - 1)a + b \quad \text{and} \quad 0 \leq b < q - 1,
\]
for \( q \geq 3 \), we have \( a = m - 1 \) and \( b = q - 3 \). Therefore, by the formula (2), the minimum distance of the generalized Reed–Muller code \( RM_q((q - 1)m - 2, m) \) is 3. \( \square \)

**Proposition 2** The generalized Reed–Muller code \( RM_q((q - 1)m - 2, m) \) is a \( q \)-ary quasi-perfect code with covering radius \( \rho = 2 \).

**Proof** The first order generalized Reed–Muller code is a linear two-weight code [3]. According to [2, Theorem 5.10], the code dual to a linear two-weight code has covering radius \( \rho = 2 \). Therefore, the generalized Reed–Muller codes of order \( (q - 1)m - 2 \) are quasi-perfect codes with covering radius \( \rho = 2 \). \( \square \)

From [13, Theorem 3.11] it follows that linear quasi-perfect codes are uniformly packed codes. Uniformly packed codes attain the Johnson bound and are optimal codes [13, Theorem 1.3].

### 3 Switching construction

The parity-check matrix \( H \) of \( RM_q((q - 1)m - 2, m) \) is \((m + 1) \times q^m \) matrix that contains the all-unity vector of length \( n = q^m \) and all transposed vectors of \( \mathbb{F}_q^m \), see [3]. Let \( H = [h_1, h_2, \ldots, h_n] \), where \( h_1, h_2, \ldots, h_n \) are columns of the matrix \( H \). The affine space
AG(m, q) will be regarded as the incidence geometry. Points $P_1, P_2, \ldots, P_n$ of the affine space $AG(m, q)$ correspond to the coordinates of the vector space $\mathbb{F}_q^n$. The vectors of $\mathbb{F}_q^n$ are points of $AG(m, q)$ and cosets of k-dimensional linear subspaces of $\mathbb{F}_q^n$ are k-dimensional affine subspaces $AG(m, q)$. The lines are 1-dimensional affine subspaces.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_q^n$, then the support of the vector $\mathbf{x}$ is the set

$$\text{supp}(\mathbf{x}) = \{i \mid x_i \neq 0\}.$$ 

By Proposition 1, the minimum distance of the code $RM_q((q - 1)m - 2, m)$ is 3 if $q \geq 3$. A codeword of weight 3 of the code $RM_q((q - 1)m - 2, m)$ we will call a triple. Let $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ be a triple and $\text{supp}(\mathbf{c}) = \{i, j, k\}$. Then

1. the triple $\mathbf{c}$ lies on a line $l$ if $\{P_i, P_j, P_k\} \subseteq l$;
2. the corresponding columns $\mathbf{h}_i, \mathbf{h}_j, \mathbf{h}_k$ are linearly dependent, i.e.,

$$c_i \mathbf{h}_i + c_j \mathbf{h}_j + c_k \mathbf{h}_k = 0.$$ 

Let $i \in \{1, \ldots, n\}$, then by $\mathcal{A}_i$ we denote a subspace spanned by the set of all triples of $RM_q((q - 1)m - 2, m)$ having 1 in the $i$th coordinate. By definition, the minimum distance of $\mathcal{A}_i$ is 3.

Define the $q$-analogue of the natural number $m$ by $[m]_q = 1 + q + \cdots + q^{m-1}$.

**Proposition 3** Let $q \geq 3$, $m \geq 1$, and $n = q^m$. Let $\mathcal{A}_i \subseteq RM_q((q - 1)m - 2, m)$. Then for any $i \in \{1, 2, \ldots, n\}$ the dimension of $\mathcal{A}_i$ is $n - [m]_q - 1$, where $[m]_q$ is the $q$-analogue of the natural number $m$.

**Proof** For $q \geq 3$, every line of the affine space $AG(m, q)$ is in $RM_q((q - 1)m - 2, m)$, see [3, 8]. The generalized Reed–Muller code $RM_q((q - 1)m - 2, m)$ is spanned by its minimum-weight vectors [3, 4]. In an affine space $AG(m, q)$, each point has $\frac{q^n}{q-1}$ lines passing through it, and each line contains $q$ points. For every two distinct points, there is exactly one line that contains both points. Thus, for each line, there are $q - 2$ linear independent triples that lie on this line. Therefore, the number of linear independent triples that generate $\mathcal{A}_i$ is $(q - 2)\frac{q^n}{q-1} = n - [m]_q - 1$. \hfill $\square$

For $m = 1$, generalized Reed–Muller codes are extended Reed-Solomon codes [9, p. 296]. For $m = 1$, the affine space $AG(1, q)$ is a line, and for $\mathcal{A}_i \subseteq RM_q(q - 3, 1)$,

$$\dim(RM_q(q - 3, 1)) = \dim(\mathcal{A}_i) = q - 2.$$ 

**Proposition 4** Let $q \geq 3$, $m \geq 1$. Let $\mathbf{c}$ be a triple and $\mathbf{c} \in \mathcal{A}_i \subseteq RM_q((q - 1)m - 2, m)$. Then $\mathbf{c}$ lies on a line passing through the point $P_i$.

**Proof** If the triples lie on same line, then their linear combination lies on the same line. In an affine space $AG(m, q)$, the intersection of any two distinct lines contains exactly one point. Therefore, triples that lie on distinct lines passing through the point $P_i$ have only one common point $P_i$ and their linear combination contains more than three points. \hfill $\square$

By Proposition 2, the code $RM_q((q - 1)m - 2, m)$ is a $q$-ary quasi-perfect code with covering radius $\rho = 2$. For a given code $RM_q((q - 1)m - 2, m)$ and $t = 0, 1, 2$, define

$$RM_q^{(t)}((q - 1)m - 2, m) = \{\mathbf{x} \in \mathbb{F}_q^n \mid d(\mathbf{x}, RM_q((q - 1)m - 2, m)) = t\}.$$ 

The sets $RM_q^{(t)}((q - 1)m - 2, m)$ are called the subconstituents of $RM_q((q - 1)m - 2, m)$.

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Proposition 5 Let $q \geq 3$, $m \geq 2$. Let $x \in RM_q^2((q - 1)m - 2, m)$, $wt(x) = 2$, and $\text{supp}(x) = \{j, k\}$. Let $i \neq j$, $i \neq k$, and the points $P_i$, $P_j$, $P_k$ are collinear. Then there is a triple $c \in \mathcal{R}_i \subset RM_q((q - 1)m - 2, m)$ such that $\text{supp}(c) = \{i, j, k\}$ and $d(c, x) = 2$.

Proof Consider $x = (x_1, x_2, \ldots, x_n) \in RM_q^2((q - 1)m - 2, m)$, $n = q^m$. Let $wt(x) = 2$ and $\text{supp}(x) = \{j, k\}$. Then there are $q - 2$ linear independent triples that lie on a line $l$ passing through the points $P_j$ and $P_k$. Let $i \neq j, i \neq k$, and the points $P_i$, $P_j$, $P_k$ are collinear. Then there is a triple $c \in (\mathcal{R}_i \subset RM_q((q - 1)m - 2, m))$ such that $\text{supp}(c) = \{i, j, k\}$. For some scalar $\lambda \in \mathbb{F}_q \setminus \{0\}$, either $xy = \lambda x_j$ or $x_k = \lambda x_k$, and $d(x, \lambda e) = 2$. □

Let $e_i$ denote the vector of length $n$ having all components equal to zero, except the $i$th component which contains a one. Let $x \in RM_q((q - 1)m - 2, m)$, $\lambda \in \mathbb{F}_q \setminus \{0\}$. Let $\mathcal{R}_i + x + \lambda e_i$ be a translate of $\mathcal{R}_i + x$. We shall define a switching to be the process of the replacing the coset $\mathcal{R}_i + x$ with the translate $\mathcal{R}_i + x + \lambda e_i$.

Theorem 1 Let $q \geq 3$, $m \geq 2$, and $n = q^m$. Let $\mathcal{R}_i + x \subset RM_q((q - 1)m - 2, m)$. Then

$$\mathcal{C}' = \left(RM_q((q - 1)m - 2, m) \setminus (\mathcal{R}_i + x) \right) \cup (\mathcal{R}_i + x + \lambda e_i)$$

is a $q$-ary nonlinear quasi-perfect code with parameters $(n, q^{m-1}, 3; 2)_q$, for any $\lambda \in \mathbb{F}_q \setminus \{0\}$ and for any $i \in \{1, 2, \ldots, n\}$.

Proof It is easy to show that $\mathcal{C}'$ is nonlinear. Let us show that $d(\mathcal{C}') = 3$. Let $y \in (\mathcal{R}_i + x)$ and $z \in RM_q((q - 1)m - 2, m) \setminus (\mathcal{R}_i + x)$. Assume that $d(y + \lambda e_i, z) = 2$. Then $d(\lambda e_i, z - y) = 2$. Thus, $z - y$ is a triple that lies on a line passing through the point $P_i$. So $z - y \in \mathcal{R}_i$ and $z \in \mathcal{R}_i + x$. We obtained a contradiction. Therefore, the minimum distance of the code $\mathcal{C}'$ is 3.

Now let us show that $\rho(\mathcal{C}') = 2$. Let $y \in RM_q^1((q - 1)m - 2, m)$. In this case, it is obvious that there exists $c \in \mathcal{C}'$ such that $d(c, y) \leq 2$.

Let $y \in RM_q^2((q - 1)m - 2, m)$. Then there exists $c \in RM_q((q - 1)m - 2, m)$ such that $d(c, y) = 2$. If $c \in RM_q((q - 1)m - 2, m) \setminus (\mathcal{R}_i + x)$, then $c \in \mathcal{C}'$. Let $c \in (\mathcal{R}_i + x)$. Since $wt(y - c) = 2$, let $ supp(y - c) = \{j, k\}$.

1. If the points $P_i, P_j, P_k$ are not collinear, then, by Proposition 4, there exists a triple $c' \notin \mathcal{R}_i$ such that $d(c, y, c') = 2$.

2. If the points $P_i, P_j, P_k$ are collinear. If $i = j$ or $i = k$, then it is obvious that there is a triple $c' \in \mathcal{R}_i$ such that $d(y, c' + c + \lambda e_i) = 2$. If $i \neq j$, $i \neq k$, then, by Proposition 5, there exists a triple $c' \in \mathcal{R}_i$ such that $d(y, c' + c + \lambda e_i) = 2$. □

Theorem 2 Let $q \geq 3$, $m \geq 2$, and $n = q^m$. Let $\mathcal{R}_i + x_t \subset RM_q((q - 1)m - 2, m)$, $1 \leq t \leq q^{m_1} - m$, $(\mathcal{R}_i + x_t) \cap (\mathcal{R}_i + x_{t'}) = \emptyset$ for all $1 \leq t_1 \leq q^{m_1} - m, 1 \leq t_2 \leq q^{m_1} - m, t_1 \neq t_2$. Then

$$\mathcal{C}' = \bigcup_{t=1}^{q^{m_1} - m} (\mathcal{R}_i + x_t + \lambda_te_i)$$

is a $q$-ary quasi-perfect code with parameters $(n, q^{m-1}, 3; 2)_q, \lambda_t \in \mathbb{F}_q, i \in \{1, 2, \ldots, n\}$. Springer
The number of cosets $\mathcal{R}_t + x_t$ is $q^{[m]q^m-m}$. The cosets $\mathcal{R}_t + x_t$ form a partition of $RM_q((q-1)m-2, m)$. Let us show that $d(\mathcal{C}) = 3$. Let $y \in (\mathcal{R}_t + x_{t_1} + \lambda_{t_1} e_t)$, $z \in (\mathcal{R}_t + x_{t_2} + \lambda_{t_2} e_t)$, and $(\mathcal{R}_t + x_{t_1}) \cap (\mathcal{R}_t + x_{t_2}) = \emptyset$. Since $(y - \lambda_{t_1} e_t) \in (\mathcal{R}_t + x_{t_1} + (\lambda_{t_1} - \lambda_{t_2}) e_t)$, 

$(z - \lambda_{t_2} e_t) \in (\mathcal{R}_t + x_{t_2}) \subset RM_q((q-1)m-2, m) \setminus (\mathcal{R}_t + x_{t_1})$,

by Theorem 1 we have $d((y - \lambda_{t_1} e_t), (z - \lambda_{t_2} e_t)) \geq 3$.

Now let us show that $\rho(\mathcal{C}) = 2$. Let $y \in RM_q((q-1)m-2, m)$. Then there exists $c \in RM_q((q-1)m-2, m)$ such that $d(c, y) = 2$. Let $c \in R_i + x_t$. Since $wt(y - c) = 2$, let $supp(y - c) = \{j, k\}$. Assume that the points $P_i, P_j, P_k$ are collinear. If $i = j$ or $i = k$, then it is obvious that there is a triple $c' \in R_i$ such that $d(y, c' + c + \lambda_i e_t) = 2$ and $c' + c \in R_i + x_t$. If $i \neq j, i \neq k$, then, by Proposition 5, there exists a triple $c' \in R_i$ such that $d(y, c' + c + \lambda_i e_t) = 2$ and $c' + c \in \mathcal{R}_t + x_t$.

If points $P_i, P_j, P_k$ are not collinear, then, by Proposition 4, there exists a triple $c' \notin \mathcal{R}_i$ such that $d(y, c' + c) = 2$ and $c' + c \notin \mathcal{R}_i + x_t$. 

\[\square\]

4 Lower bound

Theorem 3 Let $q \geq 3$, $m \geq 2$, and $n = q^m$. Then there are more than $q^{[m]q^m-m-n(m+2)}$ nonequivalent $q$-ary quasi-perfect codes with parameters $(n, q^{n-m-1}, 3; 2)_q$.

Proof The number of words of length $q^{[m]q^m-m}$ over $\mathbb{F}_q$ is $q^{[m]q^m-m}$. Hence, by Theorem 2, the number of different quasi-perfect codes with parameters $(n, q^{n-m-1}, 3; 2)_q$ is $q^{[m]q^m-m}$. The number of monomial matrices of size $n \times n$ over $\mathbb{F}_q$ is $(q-1)^n n!$. Any equivalence class contains not more than $(q-1)^n n! \cdot q^n$ different quasi-perfect codes with parameters $(n, q^{n-m-1}, 3; 2)_q$. Therefore, there are more than $q^{[m]q^m-m-n(m+2)}$ nonequivalent $q$-ary quasi-perfect codes with parameters $(n, q^{n-m-1}, 3; 2)_q$.

Corollary 1 Let $q \geq 3$, $m \geq 2$, and $n = q^m$. Then there are more than $q^{cn}$ nonequivalent $q$-ary nonlinear quasi-perfect codes with parameters $(n, q^{n-m-1}, 3; 2)_q$ for all sufficiently large $n$ and a constant $c = \frac{1}{q} - \varepsilon$.

References

1. Assmus E.F., Key J.D.: Polynomial codes and finite geometries. In: Pless V.S., Huffman W.C., Brualdi R.A. (eds.) Handbook of Coding Theory, vol. II, pp. 1269–1344. Elsevier, Amsterdam (1998).
2. Delsarte P.: An algebraic approach to the association schemes of coding theory. Philips Res. Rep. Suppl. 10 (1973).
3. Delsarte P., Goethals J.M., MacWilliams F.J.: On generalized Reed-Muller codes and their relatives. Inf. Control 16, 403–442 (1970).
4. Ding P., Key J.D.: Minimum-weight codewords as generators of generalized Reed-Muller codes. IEEE Trans. Inf. Theory 46, 2152–2157 (2000).
5. Etzion T., Vardy A.: Perfect binary codes: constructions, properties, and enumeration. IEEE Trans. Inf. Theory 40, 754–763 (1994).
6. Goethals J.M., van Tilborg H.C.A.: Uniformly packed codes. Philips Res. Rep. 30, 9–36 (1975).
7. Kasami T., Lin S., Peterson W.W.: New generalizations of the Reed-Muller codes. Part I: primitive codes. IEEE Trans. Inf. Theory 14, 189–199 (1968).
8. Kasami T., Lin S., Peterson W.W.: Polynomial codes. IEEE Trans. Inf. Theory 14, 807–814 (1968).
9. MacWilliams F.J., Sloane N.J.A.: The Theory of Error-Correcting Codes. North-Holland Publishing Co., Amsterdam (1977).
10. Phelps K.T.: A general product construction for error correcting codes. SIAM J. Algebraic Discret. Methods 5, 224–228 (1984).
11. Phelps K.T., Villanueva M.: Ranks of $q$-ary 1-perfect codes. Des. Codes Cryptogr. 27, 139–144 (2002).
12. Romanov A.M.: On perfect and Reed-Muller codes over finite fields. Probl. Inf. Transm. 57, 199–211 (2021).
13. van Tilborg H.C.A.: Uniformly packed codes. PhD Thesis, University of Tech., Eindhoven (1976).
14. Vasil’ev Yu.L.: On nongroup close-packed codes. Probl. Kybern. 8, 337–339 (1962).

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