Magnetic resonance-based reconstruction method of conductivity and permittivity distributions at the Larmor frequency

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Abstract
Magnetic resonance electric properties tomography (MREPT) is a recent medical imaging modality for visualizing the electrical tissue properties of the human body using radio-frequency magnetic fields. It uses the fact that in magnetic resonance imaging (MRI) systems the eddy currents induced by the radio-frequency magnetic fields reflect the conductivity ($\sigma$) and permittivity ($\varepsilon$) distributions inside the tissues through Maxwell’s equations. The corresponding inverse problem consists of reconstructing the admittivity distribution ($\gamma = \sigma + i\omega\varepsilon$) at the Larmor frequency ($\omega/2\pi = 128$ MHz for a 3 Tesla MRI machine) from the positive circularly polarized component of the magnetic field $H(x, y, z)$. Previous methods are usually based on an assumption of local homogeneity ($\nabla\gamma \approx 0$) which simplifies the governing equation. However, previous methods that include the assumption of homogeneity are prone to artifacts in the region where $\gamma$ varies. Hence, recent work has sought a reconstruction method that does not assume local-homogeneity. This paper presents a new MREPT reconstruction method which does not require any local homogeneity assumption on $\gamma$. We find that $\gamma$ is a solution of a semi-elliptic partial differential equation with its coefficients depending only
on the measured data $H^+ := (H_2 + iH_1)/2$, which enable us to compute a blurred version of $\gamma$. To improve the resolution of the reconstructed image, we developed a new optimization algorithm that minimizes the mismatch between the data and the model data as a highly nonlinear function of $\gamma$. Numerical simulations are presented to illustrate the potential of the proposed reconstruction method.

Keywords: inverse problems, electrical property tomography, optimal control, MRI, Maxwell’s equations

(Some figures may appear in colour only in the online journal)

1. Introduction

Magnetic resonance imaging (MRI) system can be used to visualize both the conductivity, $\sigma$, and permittivity, $\epsilon$, distributions inside the human body, because the transmitted RF pulse is affected by the electrical properties of the body. Magnetic resonance electric property tomography (MREPT) uses a time-harmonic magnetic field inside an imaging object. The standard radio-frequency coil of the magnetic resonance scanner produces the field by feeding a sinusoidal current at the Larmor frequency. The time-harmonic magnetic field, denoted by $H = (H_1, H_2, H_3)$, reflects the admittivity $\gamma := \sigma + i\omega \epsilon$ of human tissues through the following arrangement of time-harmonic Maxwell’s equations [8] (see appendix C1):

$$-\Delta H = \nabla \ln \gamma \times [\nabla \times H] - i\omega \mu_0 \gamma H \quad \text{in } \Omega,$$

where $\mu_0 = 4\pi \times 10^{-7}$ H m$^{-1}$ is the magnetic permeability of free space, $\omega/2\pi$ is the Larmor frequency of the MRI scanner, and $\Omega$ denotes a three dimensional domain occupying an imaging object. Here, we use the fact that the magnetic permeability of the human body is approximately equal to $\mu_0$.

The positive rotating magnetic field, $H^+$, which is the component of the magnetic field $H$ in the direction $(1, i, 0)/2$ can be obtained by using B1 mapping technique from the data measured by clinical MRI scanners [11, 19, 20]. This is because the MR signal, denoted by $S$, contains partial information about the time-harmonic magnetic field $H$ in the following way

$$S_\gamma(r) \propto M(r) H^- (r) H^+(r) \frac{\sin(\alpha T |H^+(r)|)}{|H^+(r)|} \quad \text{for } r = (x, y, z) \in \Omega,$$

where $H^- = (H_2 - iH_1)/2$ is the negative rotating magnetic field, $M(r)$ is the standard MR magnitude image at position $r$, and $\alpha$ is a constant. Here, $\tau$ is the duration of the radio-frequency pulse that controls the intensity of the signal $S_\gamma$. Acquiring two MR signals $S_\gamma$ and $S_\alpha$ with suitably chosen $\gamma_1$ and $\gamma_2$, we can extract the $H^+$ data through (2) with the assumption that $H^+/|H^+| \approx H^-/|H^-|$ (see appendix C2). This data acquisition technique is called B1 mapping, and was first suggested by Haacke et al [2] in the early nineties. For details on the B1 mapping technique measuring $H^+$, we refer to numerous published works in the literature [1, 12, 15, 21, 22, 23].

The inverse problem of MREPT consists of reconstructing distributions of $\sigma$ and $\epsilon$ from $H^+$. To solve the inverse problem, we need to represent the distributions of $\sigma$ and $\epsilon$ with respect to the data $H^+$. By applying the inner-product with $a = (1, i, 0)/2$ to (1), we obtain
$$- \Delta H^+ = (\nabla \ln \gamma \times (\nabla \times \mathbf{H})) \cdot \mathbf{a} - i \omega \mu_0 \gamma H^+. \quad (3)$$

Under the assumption of the local homogeneity, $\nabla \gamma = 0$, the governing partial differential equation (3) directly gives the following simple relation between $\gamma$ and $H^+$;

$$- \Delta H^+ = - i \omega \mu_0 \gamma H^+ \quad \text{in } \Omega. \quad (4)$$

The most widely used MREPT reconstruction methods [4, 14, 16, 17, 18] are based on (4) as it gives the direct representation formula for $\gamma$ with respect to $H^+$.

However, when $\nabla \gamma$ is not small, the direct formula (5) produces serious reconstruction errors [9]. The local homogeneity assumption neglects the contribution of $\nabla \ln \gamma \times (\nabla \times \mathbf{H})$ in (3). Such reconstruction errors are rigorously analyzed in [9].

We need to remove the local homogeneity assumption to develop a reconstruction method. Recently, a reconstruction method [10] removing the assumption of $\gamma = 0$ has been developed, although it still requires the assumption of $\frac{\partial \gamma}{\partial n} = 0$. The method is based on finding that, under the assumption of longitudinal homogeneity, $\gamma$ is a solution of a semilinear elliptic partial differential equation (PDE) with coefficients that only depend on $H^+ [10].$

In this paper, with no assumption of local homogeneity for $\gamma$, we develop a new reconstruction method. We find that $\sigma$ and $\epsilon$ satisfy the elliptic partial differential equation,

$$\begin{cases} 
\nabla \cdot (G_2[H^+] \nabla \sigma) + G_1[\sigma, \epsilon, H^+] \cdot \nabla \sigma + G_0[\sigma, \epsilon, H^+] = 0 & \text{in } \Omega, \\
\nabla \cdot (G_2[H^+] \nabla \epsilon) + G_1[\sigma, \epsilon, H^+] \cdot \nabla \epsilon + G_0[\sigma, \epsilon, H^+] = 0 & 
\end{cases} \quad (6)$$

where $G_2[H^+]$ is a positive semi-definite matrix, and $G_1[\sigma, \epsilon, H^+]$ and $G_0[\sigma, \epsilon, H^+]$ are vector fields depending only on $\sigma, \epsilon$, and $H^+$. Hence, the distribution of $\sigma$ and $\epsilon$ can be obtained by solving equation (6). Unfortunately, $G_2[H^+]$ in (6) is degenerate, and thus requires the addition of the weighted diffusion, $\rho$, to the (3, 3) entry of $G_2[H^+]$ so that $G_2[H^+] + \rho \mathbf{e}_z \mathbf{e}_3$ is positive definite, where $\mathbf{e}_3 = (0, 0, 1)$. Thus, (6) yields blurred images of $\sigma$ and $\epsilon$ in the $z$-direction.

To improve the spatial resolution of the reconstructed image, we develop a technique of mathematical optimization for the parameters $\sigma$ and $\epsilon$. In the proposed adjoint-based optimization method, the conductivity and permittivity distributions are updated iteratively by a nonlinear optimization algorithm which minimizes the discrepancy function describing the $L^2$-mismatch between the forward model and the observed data. We compute the Fréchet derivatives of the discrepancy function with respect to $\sigma$ and $\epsilon$. This optimal control method requires a very good initial guess. Fortunately, we can obtain a good initial guess using (6). Several numerical simulations are carried out to show the validity of the proposed reconstruction method.

### 2. Governing equation for the admittivity reconstruction

We assume that an imaging object occupying a three-dimensional domain $\Omega$ with its boundary $\partial \Omega$ being of class $C^2$. The admittivity $\gamma$ is assumed to be a constant near the boundary; that is, $\gamma = \gamma_0$ in the region $\Omega_d = \{ x \in \Omega | \text{dist}(x, \partial \Omega) < d \}$ for some $d > 0$, where $\gamma_0 = \sigma_0 + i \omega \epsilon_0$ with $\sigma_0$ and $\epsilon_0$ being known reference quantities.
Let $H^s(\Omega)$ denote the standard Sobolev space of order $s$. We assume that the admittivity distribution $\gamma = \sigma + i \omega c$ belongs to the following admissible set $\mathcal{A}$:

$$
\mathcal{A} = \left\{ \gamma \in H^2(\Omega) \cap L^\infty_{\Delta, X}(\Omega) \left| \omega \mu_0 \| \gamma \|_{L^2} + 8 |\Omega|^{1/6} \left\| \frac{\nabla \gamma}{\gamma} \right\|_{L^2} < c_1, \gamma \right|_{\Omega^\prime} = \gamma_0 \right\},
$$
(7)

where $\Delta$, $X$ and $c_1$ are positive constants, $|\Omega|$ denotes the volume of $\Omega$, and

$$
L^\infty_{\Delta, X}(\Omega) = \left\{ \gamma \in L^\infty(\Omega) : \Delta < \Re\{\gamma\}, \Im\{\gamma\} < X \right\}.
$$

The admissible set $\mathcal{A}$ is designed to guarantee uniqueness of the equation (22) in section 3 and to prove theorem 3.1.

The inverse problem is to invert the map $H^+ \mapsto \gamma$ where $H^+$ represents the extracted data from measured MR signal in (2) and the relation between $H$ and $\gamma$ is given in (1). Noting that the component $H_z$ is known to be relatively small with a regular birdcage coil of MRI scanner [13], we assume $H_z = 0$.

To solve the inverse problem, we need to express $\gamma \mapsto H^+$ in terms of $H^+$ only using the governing equation (1). It follows from the result of [10] that the contribution of $H^-$ in (3) can be eliminated from the identity

$$
(\nabla \ln \gamma \times (\nabla \times H)) \cdot a = -\nabla \ln \gamma \cdot \left( \frac{\partial H^+}{\partial x} - i \frac{\partial H^+}{\partial y}, \frac{i \partial H^+}{\partial x} + \frac{\partial H^+}{\partial y}, \frac{\partial H^+}{\partial z} \right)
$$
(8)

for $H_z = 0$. Equation (3) with the above identity gives the following lemma [10].

**Lemma 2.1.** The $\gamma$ in (1) satisfies the following first-order partial differential equation

$$
\mathcal{L}H^+ \cdot \frac{\nabla \gamma}{\gamma} = i \omega \mu_0 \gamma H^+ = -\Delta H^+ \text{ in } \Omega,
$$
(9)

where $\mathcal{L}$ is the linear differential operator given by

$$
\mathcal{L} = \left( -\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}, -\frac{\partial}{\partial x} - \frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right).
$$
(10)

According to lemma 2.1, the inverse problem is reduced to solve the first-order partial differential equation (9) for $\gamma$. The paper [25] uses the convection equation (9) is proposed. Unfortunately, it will be very difficult to solve the first-order partial differential equation (9) because there is a fundamental drawback. As the direction vector field of $\mathcal{L}H^+$ is not a real-valued function, the method of characteristics can not be applied. Indeed, Hörmander [3] and Lewy [6] provided non-existence results for the first order partial differential equation with complex-valued coefficients. To be precise, the governing equation (9) for $H^+$ can be rewritten in the standard form $F(r) \cdot \nabla u(r) = f(r, u(r))$, where $F(r) = \mathcal{L}H^+(r)$, $u(r) = \ln \gamma(r)$, and $f(r, u(r)) = i \omega \mu_0 e^{i \omega \mu_0} H^+(r) - \Delta H^+(r)$ for $r$ in $\Omega$. According to the Cauchy–Kowalevski theorem [5], the equation $F(r) \cdot \nabla u(r) = f(r, u)$ with suitable initial data, can be locally solvable only when $f$ is analytic. On the other hand, this local solvability can not be guaranteed for general $f \in C^\infty$ from Lewy’s example [6]. This is why we do not use the model (9) to compute $\gamma$. 

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2.1. Elliptic equation for the admittivity

In this section, we prove that \( \sigma \) and \( \omega \) satisfy the elliptic partial differential equation (6) which is one of our main results in this paper. This key observation follows from long and careful computations. For convenience, we introduce the following functions

\[
\begin{align*}
P(H^+) &= \begin{pmatrix}
\frac{\partial H_x^+}{\partial x} & \frac{\partial H_y^+}{\partial x} & \frac{\partial H_z^+}{\partial x} \\
\frac{\partial H_x^+}{\partial y} & \frac{\partial H_y^+}{\partial y} & \frac{\partial H_z^+}{\partial y} \\
\frac{\partial H_x^+}{\partial z} & \frac{\partial H_y^+}{\partial z} & \frac{\partial H_z^+}{\partial z}
\end{pmatrix}, \\
Q(H^+) &= \begin{pmatrix}
\frac{\partial H_x^+}{\partial x} & -\frac{\partial H_y^+}{\partial y} & \frac{\partial H_z^+}{\partial y} \\
\frac{\partial H_x^+}{\partial y} & \frac{\partial H_y^+}{\partial x} & -\frac{\partial H_z^+}{\partial x} \\
\frac{\partial H_x^+}{\partial z} & \frac{\partial H_y^+}{\partial z} & \frac{\partial H_z^+}{\partial z}
\end{pmatrix}, \\
E[\eta, H^+] &= Q(H^+) \cdot \nabla(P[H^+] \cdot \nabla \eta) - P[H^+] \cdot \nabla(Q[H^+] \cdot \nabla \eta), \\
\xi[\sigma, \epsilon, R, I] &= \omega \mu_0 I \sigma^2 - \omega^3 \mu_0 I \epsilon^2 + 2 \omega^2 \mu_0 R \sigma \epsilon + \Delta R \sigma - \omega \Delta I \epsilon.
\end{align*}
\]

We are ready to derive the equations of \( \sigma, \epsilon \).

**Theorem 2.1.** Assume that \( H^+ \) is a solution of (9) corresponding to some \( \gamma \). Let \( P, Q, E, \xi \) be given in (11)–(14), respectively and \( A \) be given in (7). The 3 × 3 matrix \( A[H^+] \) is given by

\[
A[H^+] = \begin{pmatrix}
P_\sigma + P_\eta^2 & 0 & P_\sigma P_\eta + P_\eta Q_z \\
0 & P_\sigma + P_\eta^2 & P_\sigma P_\eta - P_\eta Q_z \\
P_\sigma P_\eta + P_\eta Q_z & P_\sigma P_\eta - P_\eta Q_z & P_\sigma^2 + Q_z^2
\end{pmatrix}
\]

where \( P[H^+] = (P_x, P_y, P_z) \), and \( Q[H^+] = (Q_x, Q_y, Q_z) \). Then, \( A[H^+] \) is positive semi-definite and moreover the solution \( (\sigma, \epsilon) \in A \times A \) of (9) solving the following semi-elliptic PDE:

\[
\begin{cases}
\nabla \cdot (A[H^+] \nabla \sigma) + F_0[H^+] \cdot \nabla \sigma = F_1[\sigma, \epsilon, H^+] \\
\nabla \cdot (A[H^+] \nabla (\omega \epsilon)) + F_0[H^+] \cdot \nabla (\omega \epsilon) = F_2[\sigma, \epsilon, H^+]
\end{cases}
\]

in \( \Omega \).

Here, \( F_0[H^+] \), \( F_1[\sigma, \epsilon, H^+] \), and \( F_2[\sigma, \epsilon, H^+] \) are given by

\[
F_0 = -\left( \nabla \cdot P[H^+] \right) P[H^+] - \left( \nabla \cdot Q[H^+] \right) Q[H^+],
\]

\[
F_1 = - P[H^+] \cdot \nabla \xi \bigg[ \sigma, \epsilon, H^+, -H^+ \bigg] + Q[H^+] \cdot \nabla \xi \bigg[ \sigma, \epsilon, H^+, -H^+ \bigg]
\]

\[
+ E[\omega \epsilon, H^+],
\]

\[
F_2 = - Q[H^+] \cdot \nabla \xi \bigg[ \sigma, \epsilon, H^+, -H^+ \bigg] - P[H^+] \cdot \nabla \xi \bigg[ \sigma, \epsilon, H^+, -H^+ \bigg]
\]

\[
- E[\sigma, H^+].
\]

The proof of the theorem is in appendix A.
2.2. Approximate solution

Using the elliptic partial differential equation (16) in theorem 2.1, we can compute a fairly good approximation of the true admittivity. Since the matrix $A$ in (16) is degenerate, we need a regularization strategy. By adding a regularization term $\rho e_3^T e_3$ to the matrix $A$, we can compute viscosity solution $\sigma^\rho$, $\epsilon^\rho$ of the elliptic partial differential equation (16):

\[
\begin{cases}
\nabla \cdot \left( (A[H^+] + \rho e_3^T e_3) \nabla \sigma^\rho \right) + F_0[H^+] \cdot \nabla \sigma^\rho &= F_1[\sigma^\rho, \epsilon^\rho, H^+] \\
\nabla \cdot \left( (A[H^+] + \rho e_3^T e_3) \nabla (\omega \epsilon^\rho) \right) + F_0[H^+] \cdot \nabla (\omega \epsilon^\rho) &= F_2[\sigma^\rho, \epsilon^\rho, H^+]
\end{cases}
\]

(20)

with the Dirichlet boundary condition $\sigma^\rho = \sigma_0$, $\epsilon^\rho = \epsilon_0$ on $\partial \Omega$, where $e_3 = (0, 0, 1)$, superposed $T$ denotes the transpose, and $\rho$ is a small positive constant. Note that the matrix $A + \rho e_3^T e_3$ is positive definite, since the eigenvalues of the matrix $A + \rho e_3^T e_3$ are

\[
\lambda_1 = P_s^2 + P_s^2
\]

\[
\lambda_2, \lambda_3 = \frac{(P_s^2 + P_s^2 + P_s^2 + Q_s^2 + \rho) \pm \sqrt{(P_s^2 + P_s^2 + P_s^2 + Q_s^2 + \rho)^2 - 4(P_s^2 + P_s^2)^2}}{2}
\]

(21)

Due to the use of $\rho > 0$, we can get the blurred admittivity image of the true distribution.

3. Adjoint-based optimization method

This section presents adjoint-based optimization method for finding admittivity distribution. We suggest an objective function of mathematical optimization and a steepest descent method is used to find optimal solution. To apply a steepest descent method, the derivative of the objective function is computed by solving adjoint differential equation from (9).

Let $H_m^+ \in H^d(\Omega)$ be the ideal data (measured data without noise and error) corresponding to the true admittivity $\gamma^* \in A$; hence $H_m^+$ satisfies

\[
LH_m^+ \cdot \frac{\nabla \gamma^*}{\gamma^*} - i\omega \mu_0 \gamma^* H_m^+ + \Delta H_m^+ = 0 \quad \text{in } \Omega.
\]

For $\gamma \in A$, let $H^+[\gamma]$ be a solution of the Dirichlet problem:

\[
\begin{cases}
LH^+[\gamma] \cdot \frac{\nabla \gamma}{\gamma} - i\omega \mu_0 \gamma H^+[\gamma] + \Delta H^+[\gamma] &= 0 \quad \text{in } \Omega, \\
H^+ &= H_m^+ \quad \text{on } \partial \Omega.
\end{cases}
\]

(22)

The equation (22) has a unique solution for properly chosen $c_1$ in the definition of $A$ in (7). Indeed, if $H_1^+$ and $H_2^+$ are solutions, than $\eta = H_1^+ - H_2^+$ satisfies

\[
0 = \int_\Omega |\nabla \eta|^2 dx - \left( \int_\Omega - \mathcal{L} \cdot \frac{\nabla \gamma}{\gamma} dx + \int_\Omega i\omega \mu_0 |\eta|^2 dx \right).
\]

If $c_1$ in (7) is sufficiently small, it follows from Sobolev inequality and Poincare inequality that

\[
0 \geq \int_\Omega |\nabla \eta|^2 dx - C_\epsilon \left( \int_\Omega |\nabla \eta|^2 dx + \int_\Omega |\nabla \eta|^2 dx \right).
\]

for $C_\epsilon < 1/2$. Hence, $0 = \int_\Omega |\nabla \eta|^2 dx$ proves uniqueness. From now on, we assume that $c_1$ is chosen so that (22) has a unique solution. Then, the map
\[ \gamma \in \mathcal{A} \mapsto H^s[\gamma] \] (23)

is well-defined.

We define the misfit function \( J[\gamma] \) of the variable \( \gamma = \sigma + i\omega\epsilon \) by the \( L^2 \)-norm of the difference between \( H^s[\gamma] \) in (22) and the measured data \( H_m^s \):

\[ J[\gamma] = \frac{1}{2} \int_\Omega \left| H^s[\gamma] - H_m^s \right|^2 \, d\mathbf{r}. \] (24)

Since \( J[\gamma] = \frac{1}{2} \left\| H^s[\gamma] - H_m^s \right\|_{L^2(\Omega)}^2 \), \( J[\gamma] \geq 0 \) and \( J[\gamma] \) has minimum 0 at \( H^s[\gamma] = H_m^s \).

In this minimization problem, we need to determine the Fréchet derivative of the misfit function \( J \) with respect to the variable \( \gamma \). Let \( \mathcal{A} \) be defined by

\[ \mathcal{A} = \left\{ \varphi \in H^2(\Omega) \cap L^2_{\text{div}}(\Omega) \mid \varphi|_{\partial\Omega} = 0 \right\}. \]

The following theorem proves the Fréchet differentiability of \( H^s[\gamma] \) under the assumption that \( c_1 < 1 \).

**Theorem 3.1.** Let \( c_1 < 1 \). For \( \gamma \in \mathcal{A} \), the map \( \gamma \mapsto H^s[\gamma] \) is Fréchet differentiable. Let \( \varphi \in \mathcal{A} \) be such that \( \gamma + \varphi \in \mathcal{A} \). The Fréchet derivative \( DH^s[\gamma](\varphi) \) at \( \varphi \) is given by the solution \( u \) of the following equation

\[
\begin{cases}
\mathcal{L}u \cdot \nabla - i\omega \mu_0 \gamma u + \Delta u &= - \left( \mathcal{L}H^s[\gamma] \cdot \nabla \left( \frac{\varphi}{\gamma} \right) - i\omega \mu_0 \varphi H^s[\gamma] \right) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega.
\end{cases}
\] (25)

The proof of the theorem is in appendix B.

The following theorem expresses the Fréchet derivative of \( J[\gamma] \).

**Theorem 3.2.** For \( \gamma = \sigma + i\omega\epsilon \in \mathcal{A} \), the Fréchet derivative of \( J[\gamma] \) at \( \varphi \in \mathcal{A} \) being such that \( \gamma + \varphi \in \mathcal{A} \) is given by

\[ DJ[\gamma](\varphi) = \Re \int_\Omega \varphi(\mathbf{r})g(\mathbf{r}) \, d\mathbf{r}, \] (26)

where

\[ g[\gamma] := \left( \frac{1}{\gamma} \nabla \cdot (\mu_0 \gamma pH^s[\gamma]) + i\omega \mu_0 \gamma H^s[\gamma] p \right) \] (27)

and \( p \) is the solution of the adjoint problem:

\[
\begin{cases}
\Delta p + \mathcal{L} \cdot \left( \frac{\nabla}{\gamma} \right) - i\omega \mu_0 \gamma p &= H^s[\gamma] - H_m^s \quad \text{in } \Omega, \\
p &= 0 \quad \text{on } \partial\Omega.
\end{cases}
\] (28)

**Proof.** To compute the Fréchet derivative of \( J[\gamma] \), we consider the perturbation \( J[\gamma + \varphi] - J[\gamma] \):
\[ J[\gamma + \varphi] - J[\gamma] = \frac{1}{2} \int_{\Omega} \left| H^+[\gamma + \varphi] - H_{w}^+ \right|^2 \, dx - \frac{1}{2} \int_{\Omega} \left| H^+[\gamma] - H_{w}^+ \right|^2 \, dx \]
\[ = \Re \int_{\Omega} w_\varphi \left( H^+[\gamma] - H_{w}^+ \right) \, dx + \frac{1}{2} \int_{\Omega} w_\varphi^2 \, dx, \tag{29} \]

where \( w_\varphi = H^+[\gamma + \varphi] - H^+[\gamma] \). So,
\[ \left| J[\gamma + \varphi] - J[\gamma] \right| - \Re \int_{\Omega} w_\varphi \left( H^+[\gamma] - H_{w}^+ \right) \, dx = \left| \frac{1}{2} \int_{\Omega} w_\varphi^2 \, dx \right|. \tag{30} \]

By (57),
\[ \left| \frac{1}{2} \int_{\Omega} w_\varphi^2 \, dx \right| = \frac{1}{2} \left\| w_\varphi \right\|_{L^1(\Omega)}^2 \leq C \left\| \varphi \right\|_{L^2(\Omega)} \left\| H^+[\gamma] \right\|_{L^2(\Omega)}. \]

Thus,
\[ \lim_{\varphi \to 0} \frac{\left| J[\gamma + \varphi] - J[\gamma] - \Re \int_{\Omega} w_\varphi \left( H^+[\gamma] - H_{w}^+ \right) \, dx \right|}{\left\| \varphi \right\|_{H^1(\Omega)}} = 0. \]

Therefore, the Fréchet derivative \( DJ[\gamma](\varphi) \) is \( \Re \int_{\Omega} w_\varphi \left( H^+[\gamma] - H_{w}^+ \right) \, dx \). Using the adjoint problem (28) with the homogeneous Dirichlet boundary condition, we get
\[ DJ[\gamma](\varphi) = \Re \int_{\Omega} \varphi \left( p \frac{\nabla \gamma}{\gamma} - i \omega \mu_0 \gamma \right) \, dx. \]

On integrating by parts, it follows that
\[ \int_{\Omega} \varphi \left( p \frac{\nabla \gamma}{\gamma} - i \omega \mu_0 \gamma \right) \, dx = \int_{\partial \Omega} \frac{\partial \varphi}{\partial n} \frac{\nabla \gamma}{\gamma} \cdot \nabla p \, ds - \int_{\partial \Omega} p \frac{\partial \varphi}{\partial n} \frac{\nabla \gamma}{\gamma} \, ds + \int_{\Omega} \Delta \varphi \, dx. \]

Moreover,\n\[ \int_{\Omega} \varphi \left( p \frac{\nabla \gamma}{\gamma} - i \omega \mu_0 \gamma \right) \, dx = \int_{\partial \Omega} \frac{\partial \varphi}{\partial n} \frac{\nabla \gamma}{\gamma} \cdot \nabla p \, ds - \int_{\partial \Omega} p \frac{\partial \varphi}{\partial n} \frac{\nabla \gamma}{\gamma} \, ds + \sum_{i=1}^{N} \int_{\Omega} \frac{\nabla \gamma}{\gamma} \cdot \nabla \varphi_{i} \, dx. \]

Hence,
\[ \frac{\partial \varphi}{\partial n} \frac{\nabla \gamma}{\gamma} = - \frac{\partial \varphi}{\partial n} \frac{\nabla \gamma}{\gamma} + \frac{\partial \varphi}{\partial n} \frac{\nabla \gamma}{\gamma} \cdot \nabla \varphi_{i} \, dx. \]

Note that \( w_\varphi \) satisfies the following identity:
\[ \mathcal{L} w_\varphi \cdot \frac{\nabla \gamma}{\gamma} - i \omega \mu_0 \gamma \varphi_\varphi + \Delta w_\varphi = - \mathcal{L} H^+[\gamma + \varphi] \cdot \nabla \left( \frac{\varphi}{\gamma} \right) + i \omega \mu_0 \varphi H^+[\gamma + \varphi]. \tag{31} \]

So,
\[ DJ[\gamma](\varphi) = \Re \int_{\Omega} p \left( - \mathcal{L} H^+[\gamma] \cdot \nabla \left( \frac{\varphi}{\gamma} \right) + i \omega \mu_0 \varphi H^+[\gamma] \right) \, dx. \]
Since \( \mathcal{L}H^{\gamma}[\gamma] \cdot \nabla \left( \frac{\varphi}{\gamma} \right) = \nabla \cdot \left( \frac{\varphi}{\gamma} \mathcal{L}H^{\gamma}[\gamma] \right) - \frac{\varphi}{\gamma} (\nabla \cdot \mathcal{L}H^{\gamma}[\gamma]), \)
\[
\int_{\Omega} p \left( - \mathcal{L}H^{\gamma}[\gamma] \cdot \nabla \left( \frac{\varphi}{\gamma} \right) \right) \, dr = - \int_{\Omega} p \nabla \cdot \left( \frac{\varphi}{\gamma} \mathcal{L}H^{\gamma}[\gamma] \right) \, dr + \int_{\Omega} \frac{\varphi}{\gamma} (\nabla \cdot \mathcal{L}H^{\gamma}[\gamma]) \, dr
\]
\[
= - \int_{\partial \Omega} p \left( \frac{\varphi}{\gamma} \mathcal{L}H^{\gamma}[\gamma] \right) \cdot n \, ds + \int_{\Omega} \nabla p \cdot \left( \frac{\varphi}{\gamma} \mathcal{L}H^{\gamma}[\gamma] \right) \, dr + \int_{\Omega} p \frac{\varphi}{\gamma} (\nabla \cdot \mathcal{L}H^{\gamma}[\gamma]) \, dr
\]
\[
= \int_{\Omega} \frac{\varphi}{\gamma} (\nabla p \cdot \mathcal{L}H^{\gamma}[\gamma] + p \nabla \cdot \mathcal{L}H^{\gamma}[\gamma]) \, dr = \int_{\Omega} \frac{\varphi}{\gamma} \nabla \cdot (p \mathcal{L}H^{\gamma}[\gamma]) \, dr.
\]

Therefore,
\[
DJ_{\gamma}[\varphi] = \Re \int_{\Omega} \varphi (r) g(r) \, dr,
\]
where
\[
g_{\gamma} := \left( \frac{1}{\gamma} \nabla \cdot (p \mathcal{L}H^{\gamma}[\gamma]) + i \omega \mu_0 H^{\gamma}[\gamma] p \right)
\]
and \( p[\gamma] \) is the solution of adjoint problem (28) which completes the proof. \( \square \)

It is worth mentioning that the smallness assumption on the bound \( c_1 \) defined in (7) ensures the well-posedness of (26) with homogeneous Dirichlet boundary condition.

In the next lemma, we rewrite the adjoint problem (28) as a second-order elliptic partial differential equation.

**Lemma 3.1.** For \( \gamma = \sigma + i \omega \varepsilon \), the adjoint problem (28) can be rewritten as
\[
\Delta p + G[\gamma] \cdot \nabla p - (i \omega \mu_0 \gamma + \Delta \ln \gamma) p = H^{\gamma}[\gamma] - H_m^m \text{ in } \Omega \quad (33)
\]
with the Dirichlet boundary condition \( p = 0 \) on \( \partial \Omega \), where
\[
G = \left( - \frac{\partial \ln \gamma}{\partial x} + i \frac{\partial \ln \gamma}{\partial y}, -i \frac{\partial \ln \gamma}{\partial x} + \frac{\partial \ln \gamma}{\partial y}, \frac{\partial \ln \gamma}{\partial z} \right).
\]

**Proof.** Denote by \( \nu := \ln \gamma \). Since the linear operator \( \mathcal{L} \) is given by
\[
\mathcal{L} = \left( - \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} - \frac{\partial}{\partial x} - \frac{\partial}{\partial y}, 0 \right) = -\nabla + i \left( \frac{\partial}{\partial y}, -\frac{\partial}{\partial x}, 0 \right).
\]
we obtain
\[ \mathcal{L} \cdot \left( \frac{\nabla \gamma}{\gamma} p \right) = \mathcal{L} \cdot (p \nabla v) = -\nabla \cdot (p \nabla v) + i \left( \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial x} \right) - \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial y} \right) \right) \cdot (p \nabla v) \]
\[ = -(p \Delta v + \nabla v \cdot \nabla p) + i \left( \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial x} \right) - \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial y} \right) \right) \]
\[ = -(\Delta v) p + G[\gamma] \cdot \nabla p. \]  
(34)

Hence, if we substitute (34) into the adjoint problem (28), we get the second-order elliptic partial differential equation (33) and the proof is complete.

4. Iterative reconstruction algorithm

In order to reconstruct \( \gamma \) which minimizes \( J[\gamma] \), we first use an iteration scheme to solve elliptic differential equation (20) in section 2. This method in section 2 aims to provide an approximate solution \( \gamma_0 \) which makes \( J[\gamma_0] \) small. Next, with this initial guess \( \gamma_0 \) we apply steepest descent method based on the adjoint approach in section 3. We should note a good initial guess is crucial for the adjoint based optimization problem.

For \( n = 0, 1, 2, 3, \cdots \), we compute the Fréchet derivative of the functional \( J, DJ[\gamma](\varphi) \) as in (26). Then, update \( \gamma_{n+1} \) by
\[ \gamma_{n+1} = \gamma_n - \beta_n \mathcal{F}_n, \]  
(35)
where the positive constant \( \beta_n \) can be chosen as \( \beta_n = \frac{J[\gamma_n]}{\| \mathcal{F}[\gamma_n] \|_2} \) for \( g \) in (32).

The proposed reconstruction procedure is as follows:

Step 1. For given data \( H^+ \), compute the matrix \( A[H^+] \) in (20).

Step 2. From the initial guess \( U_0^+ := (\sigma_0, \omega e_0)^T \), update the vector \( U_k^+ := (\sigma_k, \omega e_k)^T \) by solving the semi-elliptic PDE (20) with the Dirichlet boundary condition \( (\sigma_k, \omega e_k) = (\sigma^*, \omega^*) \) on \( \partial \Omega \), where \( \sigma^* \) and \( \omega^* \) are the true values.

Step 3. For a given tolerance \( \varepsilon_1 \), iterate step 2 until \( \| \gamma^k - \gamma^{k-1} \| \leq \varepsilon_1 \), where \( \gamma^k = \sigma_k + i \omega e_k \).

Step 4. Compute \( H^+[\gamma_n] \) from the given \( \gamma_n \), for \( n \geq 0 \). From (22), the following equation for \( H^+ \) can be obtained:
\[ \Delta H^+[\gamma_n] + G[\gamma_n] \cdot \nabla H^+[\gamma_n] - i \omega \mu_0 \gamma_n H^+[\gamma_n] = 0 \]  
(36)
with \( H^+ = H^+_n \) on \( \partial \Omega \) and \( G[\gamma_n] \) being the vector field in (33).

Step 5. Compute the functional \( J[\gamma_n] = \frac{1}{2} \int_\Omega |H^+[\gamma_n] - H_n^+|^2 d\Gamma \) and the function \( g[\gamma_n] \) given by (32).

Step 6. Update \( \gamma_n \):
\[ \gamma_{n+1} = \gamma_n - \frac{J[\gamma_n]}{\| \mathcal{F}[\gamma_n] \|_2} \mathcal{F}[\gamma_n] \]  
(37)
from \( J[\gamma_n] \) and \( g[\gamma_n] \) from step 5.

Step 7. For a given tolerance \( \varepsilon_2 \), repeat from step 4 to step 6 until \( \| \gamma_n - \gamma_{n-1} \| \leq \varepsilon_2 \).
5. Numerical simulations

In this section, we will present numerical simulation results from two models to validate the proposed algorithm. We followed steps in section 4 using Matlab R2010a with Intel(R) Core(TM) i7-2600 K CPU @ 3.40 GHz and 32 GB RAM. Roughly, it takes 15 hours for finding solution using our system; 5 hours for 10 iterations of step 2–3 and 10 hours for 10 iterations of step 4–6. In the first model, we set the domain Ω to be a cylindrical model where the admittivity distribution does not change along the z-direction. Figure 1 shows the simulation model, the conductivity values σ, and the relative permittivity values ε/ε0 in the domain, where ε = 8.85 × 10^{-12}[F m^{-1}] is the permittivity of free space. Figure 2 shows the real and imaginary parts of the given data, H^r[γ^*] in slice Ω0 = Ω ∩ {z = 0}, where γ^* is the true admittivity distribution.

In section 2.1, we proved that the solution of (16) is the blurred approximation of true admittivity. However, (16) is degenerate since the diffusion matrix A[H^r] is singular. So, we modified (16) to (20) by adding the regularization term ρε^T e_3. Figure 3 shows the determinant of A + ρε^T e_3, where ρ is 5% of the maximum of A_{33}, P^2 + Q^2 in (15) while the determinant of A is zero everywhere.

Figure 4 explains that the regularized semi-elliptic PDE is also degenerate near l = {(0, 0, z)|z ∈ R}. Assuming that z-axis is the direction of the main magnetic field of standard MR scanner with a birdcage coil, it is known that ∂H/∂z - i∂H/∂y ≈ 0 near the center axis of the coil. Due to this property, det(A[H^r] + ρε^T e_3) ≈ 0 near the centerer axis of the coil [28]. To avoid this, we segmented subdomain D near l, as shown in figure 4. In the subdomain Ω \ D, we applied the iteration method (20). Figure 4 illustrates solutions of (20), U_k = (σ_k, ε_k), in Ω \ D with various iteration numbers k. We set the initial values to be constant: σ_0 = 1 and ε_0 = 0. In order to check the convergence and the accuracy of the proposed algorithm (20), we plotted ∥γ_k - γ^*∥_L^2 and ∥γ_k - γ_k-1∥_L^2 with k = 1, 2, ⋯, 10 in figure 5. Figure 5 shows that the iteration method (20) converges as k increases and the error between true admittivity and reconstructed admittivity decreases. We choose U_3 to be the solution of the iterative algorithm. We used direct method (5) for the admittivity value γ in the segmented subdomain D. So, we let U_3 in Ω \ D with the direct method in D to be the initial guess of the iterative algorithm (37). Figure 6 illustrates the reconstructed conductivity and relative permittivity distribution by the steepest descent method (37). Figure 7 shows the functional J[γ_3] and the accuracy of the steepest descent method, 1/|S|∫_S |γ_3 - γ^*| dx, where S is the region of small anomalies 4,5,6,7,8 in figure 1. We defined the accuracy criterion to be ∫_S |γ_3 - γ^*| dx in order to see the performance of the steepest descent method only in the regions containing the small anomalies.

To illustrate the performance of the proposed method, we compared the reconstruction results with the true values in figure 8. Figure 9 shows the reconstructed images by the direct formula (5) and the proposed method. Figure 10 compares between the two methods, the direct formula (5) and the proposed method, for imaging small anomalies.

In the second numerical model, we simulate the model with admittivity changing along z-direction, i.e., ∂σ/∂z ≈ 0. The domain Ω is decomposed into two parts, Ω_− = Ω ∩ {z < 0} and Ω_+ = Ω ∩ {z ≥ 0}. In Ω_−, the admittivity distribution is the same as in model 1. However, the admittivity distribution in Ω_+ is different from model 1 such that ∂σ/∂z ≈ 0 in Ω_+. Figure 11 shows the second configuration model, the conductivity and the relative permittivity values in the domain. Figure 12 shows the reconstruction results using the direct method and the proposed method. Figure 13 shows the accuracy of the proposed method applied to the
second model. Figure 14 presents the reconstructed conductivity distribution of the second model in the slice of $\Omega_-$ using the proposed method. Figures 13 and 14 demonstrate that the proposed method works well in the case of $\frac{\partial}{\partial n} \neq 0$.

6. Concluding remarks

In this paper, we have developed an iterative novel scheme for reconstructing electrical tissue properties at the Larmor frequency from measurements of the positive rotating magnetic field. We first suggest the elliptic partial differential equation (20) which provides a blurred reconstructed image. By considering the blurred reconstructed image as an initial guess of the iteration for steepest descent method, the steepest descent method for finding the minimizer of the functional $J$ in (24) finds the final reconstruction admittivity. Note that our scheme does
not require a local homogeneity assumptions on $\gamma$ and allows to reconstruct inhomogeneous distributions accurately.

The direct reconstruction formula (4) works well within a locally homogeneous region [4, 14, 16, 17, 18], because the local homogeneity assumption of $\gamma$ allows to neglect the effect

Figure 3. Image of the determinant of the matrix $A[H^+] + \rho e_3^T e_3$ in the slice $\Omega_0$, where $\rho$ is 5% of the maximum of $A_{33}$, $P_3^T + Q_3^T$ in (15).

Figure 4. Reconstruction images obtained from the iterative scheme (20) with $k = 1, 2, 5, 10$ in the slice $\Omega_0$. (a)-(d): Images of the first row are reconstructed conductivity distribution. (e)-(h): Images of the second row are reconstructed relative permittivity distribution.
In theory, robustness of the direct reconstruction method at a given resolution is somehow related to the wavelength $\lambda \approx 2\pi \sqrt{|H'|/|\nabla^2 H'|}$, which is connected to the strength of the main magnetic field of MR scanner. But, experimental results show that there are serious artifacts where $|\nabla \ln \gamma|$ is large. With taking account of the wavelength, the direct method may not be reliable for evaluating $\gamma$ of small anomalies whose diameters are much less than the wavelength. The proposed method has better performance in these inhomogeneous regions where the effect of $\nabla \ln \gamma \times (\nabla \times H)$ is not ignored.

Most modern MREPT reconstructions are very sensitive to noise. In the proposed approach, the initial guess, which is obtained by solving the equation (20), is less sensitive to...
Figure 7. (a) Plot of the functional $J[\gamma_n]$. (b) Plot of $\frac{1}{|\mathcal{S}|} \int_{\mathcal{S}} \left| \frac{\gamma_n}{\gamma^b} - 1 \right| \, dx$ to show the accuracy of (20) with the iteration numbers $n = 0, 1, \ldots, 10$. (c) The small anomalies region $S$ in red.

Figure 8. (a) True conductivity distribution in the slice $\Omega_0$. (b) Reconstructed conductivity image by the semi-elliptic PDE (20) with $k = 3$ in the segmented slice $\Omega_0 \setminus D$. It is same in figure 4(b). (c) Reconstructed conductivity image by the steepest descent method (37) with $n = 10$ in the slice $\Omega_0$. It is same in figure 6(d). (d) The image of the error of (b) in $\Omega_0 \setminus D$. (e) The image of the error of (c) in $\Omega_0$. 

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Figure 9. (a) True conductivity in the slice $\Omega_0$. (b) Reconstructed conductivity obtained from the direct formula (5) in $\Omega_0$. (c) Reconstructed conductivity using the proposed method in $\Omega_0$.

Figure 10. (a) And (c), respectively, are images of the errors of conductivity and relative permittivity using the direct formula (5) in the slice $\Omega_0$. (b) and (d), respectively, are images of the errors of conductivity and relative permittivity using the proposed method in the slice $\Omega_0$.

Figure 11. Second model configuration (left) and table of the value of electrical property (right).
noise, whereas the adjoint method of finding the minimizer of (24) is sensitive to noise. Hence, effective denoising technique is needed for robust reconstruction, and this will be our future research topic.

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Appendix A. Proof of the theorem 2.1

First, compute the determinant of \( A - \lambda I \) to show that the matrix \( A \) is positive semi-definite:

\[
\text{det}(A - \lambda I) = -\lambda \left( \lambda - \left( p_x^2 + p_y^2 \right) \right) \left( \lambda - \left( p_x^2 + p_y^2 + p_z^2 + Q_z^2 \right) \right),
\]

where \( \text{det} \) denotes the determinant. Hence, all the eigenvalues of the matrix \( A \) are non-negative.

For simplicity, we denote \( H^+ \) and \( H^+ \) for the real and imaginary parts of \( H^+ \), i.e. \( H^+ = H^+ + iH^+ \) and let \( \gamma \) be the imaginary part of the admittance. We separate the equation (9) into two equations one of which is real part:
and the other is imaginary part of \((9)\):

\[
\Delta H^+_{t} \gamma + \Delta H^+_{t} \gamma = \left( \frac{\partial H^+_{t}}{\partial x} + \frac{\partial H^+_{t}}{\partial y} \right) \frac{\partial \gamma}{\partial x} + \left( \frac{\partial H^+_{t}}{\partial x} - \frac{\partial H^+_{t}}{\partial y} \right) \frac{\partial \gamma}{\partial y} \\
+ \frac{\partial H^+_{t}}{\partial z} \frac{\partial \gamma}{\partial z} + \frac{\partial H^+_{t}}{\partial z} \frac{\partial \gamma}{\partial z} \\
+ \omega \mu_0 \left( H^+_{t} (\sigma^2 - \gamma^2) - 2 H^+_{t} \sigma \gamma \right). \tag{40}
\]

Using the notations \(P\) in \((11)\) and \(Q\) in \((12)\), equation \((39)\) can be written as:

\[
P[H^+] \cdot \nabla \sigma + Q[H^+] \cdot \nabla \gamma + \kappa [\gamma, H^+] = 0 \text{ in } \Omega, \tag{41}
\]
where $\kappa := -\omega \mu_0 \left( 2 H^r \sigma^\gamma + H^r (\sigma^2 - \gamma^2) \right) - \Delta H^r_0 \sigma + \Delta H^r_0 \gamma$. Applying $P[H^r] \cdot \nabla$ and $Q[H^r] \cdot \nabla$ to equation (41), we obtain

$$P[H^r] \cdot \nabla \left( P[H^r] \cdot \nabla \sigma \right) + P[H^r] \cdot \nabla \left( Q[H^r] \cdot \nabla \gamma \right) + P[H^r] \cdot \nabla \kappa \left[ \gamma, H^r \right] = 0$$ \hspace{1cm} (42)

and similarly

$$Q[H^r] \cdot \nabla \left( P[H^r] \cdot \nabla \sigma \right) + Q[H^r] \cdot \nabla \left( Q[H^r] \cdot \nabla \gamma \right) + Q[H^r] \cdot \nabla \kappa \left[ \gamma, H^r \right] = 0.$$ \hspace{1cm} (43)

In a similar way, (40) can be written as

$$-Q[H^r] \cdot \nabla \sigma + P[H^r] \cdot \nabla \gamma + \zeta \left[ \gamma, H^r \right] = 0 \quad \text{in } \Omega$$ \hspace{1cm} (44)

where $\zeta := \omega \mu_0 \left( H^r (\sigma^2 - \gamma^2) - 2 H^r \sigma \gamma \right) - \Delta H^r_0 \gamma - \Delta H^r_0 \sigma$. Applying $P[H^r] \cdot \nabla$ and $Q[H^r] \cdot \nabla$ to (44), we have

$$-Q[H^r] \cdot \nabla \left( Q[H^r] \cdot \nabla \sigma \right) + Q[H^r] \cdot \nabla \left( P[H^r] \cdot \nabla \gamma \right) + Q[H^r] \cdot \nabla \zeta \left[ \gamma, H^r \right] = 0$$ \hspace{1cm} (45)
and
\[- P[H^+] \cdot \nabla \left( Q[H^+] \cdot \nabla \sigma \right) + P[H^+] \cdot \nabla \left( P[H^+] \cdot \nabla \gamma \right) \\
+ P[H^+] \cdot \nabla \zeta = 0. \quad (46)\]

Subtracting (42) from (45) yields
\[P[H^+] \cdot \nabla \left( P[H^+] \cdot \nabla \sigma \right) + Q[H^+] \cdot \nabla \left( Q[H^+] \cdot \nabla \sigma \right) = F_1[\sigma, \epsilon, H^+]. \quad (47)\]

where $F_1$ is given in (18). Similarly, (43) and (46) gives
\[P[H^+] \cdot \nabla \left( P[H^+] \cdot \nabla \gamma \right) + Q[H^+] \cdot \nabla \left( Q[H^+] \cdot \nabla \gamma \right) = F_2[\sigma, \epsilon, H^+] \quad (48)\]

with $F_2$ being given by (19). A direct computations show that equation (47) can be expressed as
\[
\nabla \cdot \left( \begin{bmatrix}
P_{\xi} + Q_{\xi}^2 & P_{\xi} P_{\eta} + Q_{\xi} & P_{\xi} P_{\zeta} + Q_{\xi} \\
P_{\eta} P_{\xi} + Q_{\eta} & P_{\eta}^2 + Q_{\eta}^2 & P_{\eta} P_{\zeta} + Q_{\eta} \\
P_{\zeta} P_{\xi} + Q_{\zeta} & P_{\zeta} P_{\eta} + Q_{\zeta} & P_{\zeta}^2 + Q_{\zeta}^2
\end{bmatrix} \nabla \sigma \\
+ F_0[H^+] \cdot \nabla \sigma = F_1[\sigma, \epsilon, H^+].
\]

(49)

Since $P_{\xi} = -Q_{\eta}$ and $P_{\eta} = Q_{\xi}$, $P_{\xi} P_{\eta} + Q_{\xi} Q_{\eta} = 0$, equation (49) can be rewritten as
\[
\nabla \cdot \left( A[H^+] \nabla \sigma \right) + F_0[H^+] \cdot \nabla \sigma = F_1[\sigma, \epsilon, H^+] \text{ in } \Omega. \quad (50)
\]

Similarly, (48) gives
\[
\nabla \cdot \left( A[H^+] \nabla (\omega \epsilon) \right) + F_0[H^+] \cdot \nabla (\omega \epsilon) = F_2[\sigma, \epsilon, H^+] \text{ in } \Omega. \quad (51)
\]

Hence, we complete deriving the equation (16).

\[\square\]

**Appendix B. Proof of the theorem 3.1**

First, remember Sobolev inequalities for $H^1(\Omega)$ [24] that
\[
\|w\|_{H^1(\Omega)} \leq 2 |\Omega|^{1/2} \|w\|_{H^1(\Omega)}. \quad (52)
\]

Then, defining
\[
w_{\varphi} := H^+[\gamma + \varphi] - H^+[\gamma] \in H^1(\Omega),
\]

it follows from (22) that
\[
\begin{aligned}
\mathcal{L}w_{\varphi} \cdot \nabla (\gamma + \varphi) - i \omega \mu_{0}(\gamma + \varphi) w_{\varphi} + \Delta w_{\varphi} = \\
- \left( \mathcal{L}H^+[\gamma] \cdot \nabla \left( \frac{\gamma^2}{\gamma + \varphi} \right) - i \omega \mu_{0}(\varphi H^+)[\gamma] \right) \text{ in } \Omega,
\end{aligned} \quad (53)
\]

\[\left. w_{\varphi} \right|_{\partial \Omega} = 0.
\]

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Therefore, we have
\[
\| w_\varphi \|_{H^1(\Omega)} \leq \left\| \mathcal{L} w_\varphi \cdot \frac{\nabla (\gamma + \varphi)}{\gamma + \varphi} - i \omega \mu_0 (\gamma + \varphi) w_\varphi \right\|_{L^2(\Omega)} + \left\| \mathcal{L} H^+ [\gamma] \cdot \nabla \left( \frac{\varphi}{\gamma} \right) - i \omega \mu_0 \varphi H^+ [\gamma] \right\|_{L^2(\Omega)}. \quad (54)
\]

By Hölder’s inequality and Sobolev embedding theorem (see (52)), the first term of the right-hand side of (54) can be estimated by
\[
\left\| \mathcal{L} w_\varphi \cdot \frac{\nabla (\gamma + \varphi)}{\gamma + \varphi} - i \omega \mu_0 (\gamma + \varphi) w_\varphi \right\|_{L^2(\Omega)} \leq c_1 \| w_\varphi \|_{H^2(\Omega)}. \quad (55)
\]

Combining (54) and (55), we have
\[
(1 - c_1) \| w_\varphi \|_{H^1(\Omega)} \leq \left\| \mathcal{L} H^+ [\gamma] \cdot \nabla \left( \frac{\varphi}{\gamma} \right) - i \omega \mu_0 \varphi H^+ [\gamma] \right\|_{L^2(\Omega)}. \quad (56)
\]

By Hölder’s inequality and Sobolev embedding theorem, there exists \( C' \) which is independent of \( \delta \) and \( H^+ [\gamma] \) such that
\[
\| w_\varphi \|_{H^2(\Omega)} \leq C' \| \varphi \|_{H^1(\Omega)} \| H^+ [\gamma] \|_{H^1(\Omega)} \quad (57)
\]
from (56) if \( c_1 < 1 \).

Since the data difference \( w_\varphi \) satisfies (53) and \( u \) is the solution of equation (25), the difference \( w_\varphi - u \in H^1(\Omega) \) satisfies
\[
\mathcal{L}(w_\varphi - u) \cdot \frac{\nabla \gamma}{\gamma} - i \omega \mu_0 \gamma (w_\varphi - u) + \Delta (w_\varphi - u) = - \left( \mathcal{L} w_\varphi \cdot \nabla \left( \frac{\varphi}{\gamma} \right) \frac{\gamma^2}{\gamma (\gamma + \varphi)} - i \omega \mu_0 \varphi w_\varphi + \mathcal{L} H^+ [\gamma] \cdot \nabla \left( \frac{\varphi}{\gamma} \right) \frac{\gamma}{\gamma + \varphi} \right). \quad (58)
\]

From the standard estimation of the Poisson equation, we have
\[
\| w_\varphi - u \|_{H^1(\Omega)} \leq \left\| \mathcal{L}(w_\varphi - u) \cdot \frac{\nabla \gamma}{\gamma} - i \omega \mu_0 \gamma (w_\varphi - u) \right\|_{L^2(\Omega)} + \left\| \mathcal{L} w_\varphi \cdot \nabla \left( \frac{\varphi}{\gamma} \right) \frac{\gamma^2}{\gamma (\gamma + \varphi)} - i \omega \mu_0 \varphi w_\varphi + \mathcal{L} H^+ [\gamma] \cdot \nabla \left( \frac{\varphi}{\gamma} \right) \frac{\gamma}{\gamma + \varphi} \right\|_{L^2(\Omega)}. \quad (59)
\]

Again, by Hölder’s inequality and Sobolev embedding theorem, the first term of the right-hand side of (59) can be estimated by
\[
\| \mathcal{L}(w_\varphi - u) \cdot \frac{\nabla \gamma}{\gamma} - i \omega \mu_0 \gamma (w_\varphi - u) \|_{L^2(\Omega)} \leq c_1 \| w_\varphi - u \|_{H^2(\Omega)}. \quad (60)
\]

Combining (59) and (60), we have
\[
(1 - c_1) \| w_\varphi - u \|_{H^1(\Omega)} \leq \left\| \mathcal{L} w_\varphi \cdot \nabla \left( \frac{\varphi}{\gamma} \right) \frac{\gamma^2}{\gamma (\gamma + \varphi)} - i \omega \mu_0 \varphi w_\varphi + \mathcal{L} H^+ [\gamma] \cdot \nabla \left( \frac{\varphi}{\gamma} \right) \frac{\gamma}{\gamma + \varphi} \right\|_{L^2(\Omega)}. \quad (61)
\]
By Hölder’s inequality and Sobolev embedding theorem, there exist $C_1$ and $C_2$ which are independent of $\delta$, $w_\gamma$ and $H^*[\gamma]$ such that
\[
\|w_\varphi - u\|_{H^2(\Omega)} \leq C_1 \|\varphi\|_{H^2(\Omega)} \|w_\varphi\|_{H^2(\Omega)} + C_2 \|\varphi\|_{H^2(\Omega)} \|H^*[\gamma]\|_{H^2(\Omega)}
\]  

(62)

from (61) if $c_1 < 1$.

By inequalities (57) and (62), it follows that
\[
\|w_\varphi - u\|_{H^2(\Omega)} \leq C'_1 \|\varphi\|_{H^2(\Omega)} \|H^*[\gamma]\|_{H^2(\Omega)} + C_2 \|\varphi\|_{H^2(\Omega)} \|H^*[\gamma]\|_{H^2(\Omega)},
\]

where $C'_1 = C'C$. Thus,
\[
\frac{\|H^*[\gamma + \varphi] - H^*[\gamma] - u\|_{H^2(\Omega)}}{\|\varphi\|_{H^2(\Omega)}} \to 0 \quad \text{as} \quad \|\varphi\|_{H^2(\Omega)} \to 0.
\]

Hence, $u$ is the Fréchet derivative of $H^*[\gamma]$ at $\varphi$, that is, $DH^*[\gamma](\varphi) = u$. \qed

Appendix C

C1. Derivation of the equation (1)

According to Maxwell’s equations,
\[
\begin{cases}
\nabla \times \mathbf{E} = -i\omega\mu_0 \mathbf{H} \\
\nabla \times \mathbf{H} = (\sigma + i\omega)\mathbf{E}
\end{cases}
\]

Applying the curl operation to $\nabla \times \mathbf{H} = (\sigma + i\omega)\mathbf{E}$, we get
\[
\nabla \times \nabla \times \mathbf{H} = \nabla \times ((\sigma + i\omega)\mathbf{E})
\]
\[
= \nabla(\sigma + i\omega) \times \mathbf{E} + (\sigma + i\omega)\nabla \times \mathbf{E}
\]
\[
= \nabla(\sigma + i\omega) \times (\nabla \times \mathbf{H}) - i\omega\mu_0(\sigma + i\omega)\mathbf{H}
\]

Since $\nabla \times \nabla \times \mathbf{H} = \nabla(\nabla \cdot \mathbf{H}) - \Delta \mathbf{H} = -\Delta \mathbf{H}$, finally we have the equation (1):
\[
-\Delta \mathbf{H} = \nabla \ln \gamma \times [\nabla \times \mathbf{H}] - i\omega\mu_0\gamma \mathbf{H}.
\]

More detailed explanation can be found in the book [11].

C2. A method to get $H^*$ from MR signal

For $\mathbf{r} \in \Omega$, the following MR signal $S_\gamma(\mathbf{r})$ is measurable quantity
\[
S_\gamma(\mathbf{r}) := \beta M_0(\mathbf{r}) H^-(\mathbf{r}) \left( \sin \left( \alpha \tau \left| H^+(\mathbf{r}) \right| \right) \frac{H^+(\mathbf{r})}{H^-(\mathbf{r})} \right),
\]

where $\alpha$ and $\beta$ are system-dependent constants [4, 27] and $\tau$ is the duration of the radio-frequency pulse. One simple method to extract $|H^+|$ is the double-angle method [12, 26], which uses
\[
\frac{S_{2\gamma}(\mathbf{r})}{S_\gamma(\mathbf{r})} = \frac{\sin \left( 2\alpha \tau \left| H^+(\mathbf{r}) \right| \right)}{\sin \left( \alpha \tau \left| H^+(\mathbf{r}) \right| \right)} = 2 \cos \left( \alpha \tau \left| H^+(\mathbf{r}) \right| \right).
\]
The phase of $\mathbf{H}^+$ can be measured by assuming $\frac{\mathbf{H}^+}{|\mathbf{H}^+|} \approx \frac{\mathbf{H}^+}{|\mathbf{H}^+|}$ and we can extract $\frac{\mathbf{H}^+}{|\mathbf{H}^+|}$ by

$$\frac{1}{2} \arg \left\{ S_\nu (\mathbf{r}) \right\} = \arg \left\{ \frac{H^+(\mathbf{r})}{|H^+(\mathbf{r})|} \right\}.$$

For details on the $B_1$ mapping techniques, we refer to the paper [8, 11, 19, 20].

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