A probabilistic approach to consecutive pattern avoiding in permutations

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Consecutive pattern-avoiding permutations – Definition

Let \( \pi = (\pi_1 \ldots \pi_n) \in S_n \) permutation
and \( \sigma = (\sigma_1 \ldots \sigma_m) \in S_m \) pattern.

We consider \( n \gg m \).

**Reduction:** \( st(\pi_{i_1} \ldots \pi_{i_k}) = \tau \) if \( \tau \in S_k \) and \( \pi_{i_j} < \pi_{i_\ell} \iff \tau_j < \tau_\ell \).

\( \pi \) contains the consecutive pattern \( \sigma \) if \( \exists \ i \) such that \( st(\pi_{i+1} \ldots \pi_{i+m}) = \sigma \), otherwise it avoids \( \sigma \).

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**Example**

\( n = 5, \ m = 3: \)

\[ \pi = (15423) \quad \text{reduces to} \quad st(542) = (321) \]

\[ \pi = (15423) \quad \text{contains} \quad \sigma = (321) \]

\[ \pi = (15423) \quad \text{avoids} \quad \sigma = (123) \]
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\( \pi \) contains the consecutive pattern \( \sigma \) if \( \exists i \) such that \( st(\pi_{i+1} \ldots \pi_{i+m}) = \sigma \), otherwise it avoids \( \sigma \).

For any \( \sigma \in S_m \), we are interested in

\[
\alpha_n(\sigma) = |\{ \pi \in S_n : \pi \text{ avoids } \sigma \}|
\]

(Elizalde and Noy, 2003)
Consecutive pattern-avoiding permutations – Results

**CMP Conjecture** (Elizalde and Noy, 2003)

For any $\sigma \in S_m$, 
\[ \alpha_n(\sigma) \leq \alpha_n(12\ldots m). \]

**Theorem** (Elizalde, 2006)

For any $\sigma \in S_m$ the limit 
\[ \rho_\sigma = \lim_{n \to \infty} \left( \frac{\alpha_n(\sigma)}{n!} \right)^{1/n} \quad (\alpha_n(\sigma) \sim c \rho_\sigma^n n!). \]
Consecutive pattern-avoiding permutations – Results

**CMP Conjecture** (Elizalde and Noy, 2003)

For any $\sigma \in S_m$,

$$\rho_\sigma \leq \rho(12\ldots m)$$

**Theorem** (Elizalde, 2006)

For any $\sigma \in S_m$ the limit

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$(\alpha_n(\sigma) \sim c\rho^n_\sigma n!)$.
Theorem (Elisalde, 2012+ / P, 2012+)

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Theorem (Elizalde, 2006)

For any $\sigma \in S_m$ the limit

$$\rho_\sigma = \lim_{n \to \infty} \left( \frac{\alpha_n(\sigma)}{n!} \right)^{1/n} \quad (\alpha_n(\sigma) \sim c \rho_\sigma^n n!).$$
Fix $\sigma \in S_m$ and choose $\pi \in S_n$ uniformly at random.

Define for any $0 \leq i \leq n - m$, the event $A_i = \{st(\pi_{i+1} \ldots \pi_{i+m}) = \sigma\}$.

Example

$\sigma = (123), \pi = (193482576) \Rightarrow A_3$ but $A_1$

Then, the probability of $\pi$ is $\sigma$-avoiding is, $\Pr(\cap_{i=0}^{n-m} A_i)$.

Thus,

$$\alpha_n(\sigma) = \Pr(\cap_{i=0}^{n-m} A_i) n!$$

$$\rho_\sigma = \lim_{n \to \infty} \Pr(\cap_{i=0}^{n-m} A_i)^{1/n}.$$
Fix $A_i$. For any pattern $\sigma \in S_m$,

$$\Pr(A_i) = \frac{1}{m!}.$$

If they were independent...

$$\rho_{\sigma} = \lim_{n \to \infty} \Pr(\cap_{i=0}^{n-m} \overline{A_i})^{1/n} = \lim_{n \to \infty} \left( \prod_{i=0}^{n-m} \Pr(\overline{A_i}) \right)^{1/n} \sim 1 - \frac{1}{m!}.$$
Upper Bound on $\rho_{\sigma}$

**Theorem (P, 2012+)**

Let $\sigma \in S_m \backslash \{(12 \ldots m), (m \ldots 21)\}$, then

$$\rho_{\sigma} \leq 1 - \frac{1}{m!} + O\left(\frac{1}{m^2 \cdot m!}\right).$$

**Suen’s inequality**, If

$$\mu = \sum \Pr(A_i), \quad \Delta = \frac{1}{2} \sum_i \sum_{i \sim j} \Pr(A_i \land A_j) \quad \text{and} \quad \delta = \max_i \sum_{i \sim j} \Pr(A_j)$$

then,

$$\Pr(\cap A_i) \leq \exp\left(-\mu + \Delta e^{2\delta}\right).$$

We need to take care of $\Pr(A_i \land A_j)$: DEPENDS on the pattern.
Upper Bound on $\rho_\sigma$

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Number of permutations with no runs of length $m$,

$$\rho_{(12\ldots m)} \geq 1 - \frac{1}{m!} + O\left(\frac{1}{m \cdot m!}\right).$$

**Theorem (Elisalde, 2012+ / P, 2012+)**

CMP conjecture is true.
Lower Bound on $\rho_\sigma$

**Theorem (P, 2012+)**

Let $\sigma \in S_m$, then

$$\rho_\sigma \geq 1 - \frac{1}{m!} - O\left(\frac{m-1}{(m!)^2}\right).$$

One-sided Lovász Local Lemma,

Let $H$ be the dependency graph, if there exists an $x$ such that

$$\Pr(A_i) \leq x(1 - x)^{\Delta(H)}$$

then

$$\Pr(\cap \overline{A_i}) \geq (1 - x)^n.$$

We just care of $\Delta(H)$: does NOT depend on the pattern.
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TIGHT! $\implies (1, 2 \ldots m - 2, m, m - 1)$
How do most of the patterns behave?

**Theorem (P., 2012+)**

Let \( \sigma \in S_m \) chosen uniformly at random. For any \( 1 \leq k < m/2 \), we have

\[
\rho_\sigma \leq 1 - \frac{1}{m!} + O \left( \frac{4^{m-k}}{(m-k)!m!} \right),
\]

with probability at least \( 1 - \frac{2}{(k+1)!} - m2^{-m/2} \).
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![Diagram showing lower and upper bounds for different patterns](image)
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\[ F(t) = \Pr(\rho_\sigma < t) \]

\[ f(t) = \frac{\partial F}{\partial t} = ? \]
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