Quantum advantage by weak measurements

Lin-Ping Huai,1,2 Bo Li,2 Shi-Xian Qu,1,* and Heng Fan2†

1Theoretical Computational Institute, School of Physics and Information Technology, Shaanxi Normal University, Xi’an 710062, China
2Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China

Weak measurements may result in extra quantity of quantumness of correlations compared with standard projective measurement on a bipartite quantum state. We show that the quantumness of correlations by weak measurements can be consumed for information encoding which is only accessible by coherent quantum interactions. Then it can be considered as a resource for quantum information processing and can quantify this quantum advantage. We conclude that weak measurements can create more valuable quantum correlation.

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INTRODUCTION

Quantum information processing offers protocols and algorithms which may surpass their classical counterparts [1]. We generally consider that those quantum advantages are due to unique features of quantum world such as the superposition and the quantumness of correlations like entanglement and quantum discord. Indeed, entanglement and quantum discord are shown to be resources for some quantum information tasks. However, different criteria are used to distinguish quantum from classical worlds and various measures are applied in quantifying the quantum correlations. Entanglement is a well studied quantum correlation and it plays a clear role in Bell inequalities, teleportation and superdense coding etc. The history of entanglement study can trace back to Einstein, Podolsky, Rosen who considered this spooky action at a distance in quantum mechanics [2].

However, entanglement seems not account for all quantum correlation in bipartite system since the absence of entanglement does not eliminate all signatures of quantum behaviour. Henderson and Vedral [3], Ollivier and Zurek [4] proposed a quantity of nonclassical correlation, which is named as quantum discord, to measure the discrepancy of total and classical correlation. It is thus a measure of quantumness of correlation. Quantum discord is shown to be present in a mixed state quantum computation algorithm, the deterministic quantum computation with one qubit (DQC1), which offers the quantum advantage but with negligible entanglement. It is then considered that quantum discord plays a key role in this algorithm. Alternatively, some other proposals are also discussed DQC1 [5]. In the past years, great progress have been made in understanding entanglement and various measures of quantum correlations [3, 4, 6–14].

Recently, it is found that discord consumption can be related with the quantum advantage for encoding information [15]. This is a clear evidence that quantum discord can be considered as a resource for quantum information processing like the role of entanglement in teleportation. In this protocol, quantum advantage is upper bounded by the consumption of discord during the encoding information and is lower bounded by the difference of discord consumption and classical correlation. Thus, quantum discord consumed during encoding can quantify the advantage about coherent interactions. This operational significance of discord has enlightened our knowledge of quantum correlation other than entanglement.

On the other hand, quantum discord is quantum measurement dependent. Its quantity depends on the positive operator valued measurement (POVM) or projection measurement [16]. We know that quantum measurement is one of the cornerstones of quantum mechanics. Instead of the strong measurement such as the projection measurement which will cause completely collapsing of the wavefunction of a quantum state, a measurement scheme called “weak measurement”, which only induces partial collapsing of the quantum state, is indispensable [14–18]. As we known, weak measurement plays a key role in the fundamentals of quantum mechanics and has practical applications [19, 20]. Now, it seems natural to consider a weak measurement scheme for quantum discord [21]. It is understandable that the accessible information by weak measurement will be smaller than that of a projection measurement which is strong, so the residue correlation will become larger. The result is that we may obtain extra quantumness of correlation by weak measurement. While the standard quantum discord is shown to be a valuable resource, then the question is whether this new defined quantumness of correlation which is larger is also useful. In this paper, we will provide a positive answer to this question! We can prove that for two-qubit pure state and Bell-diagonal state, this super discord induced by weak measurement can indeed be consumed during the encoding information and therefore can be regarded also as a precious resource in this quantum protocol. Our finding reveals that more resources of quantum correlation might be explored by weak measurement. This highlights the fundamental role of weak measurements in studying quantum correlations.

This paper is organized as follows. In Section II, we first give a brief review of the definition of quantum advantage, the formulism of the projective measurement and weak measurement. In Section III, we prove that super discord by weak measurement can achieve the best quantum advantage, for which the quantum correlation is completely consumed, in the case of general pure entangled states and the Bell-diagonal states. Section IV is the summary and discussion.
QUANTUM CORRELATION WITH DIFFERENT MEASUREMENTS

For a bipartite quantum state \( \rho_{ab} \) shared by Alice and Bob, in general, the total correlation between the two subsystems is measured by mutual information,

\[
I(\rho_{ab}) = S(\rho_a) + S(\rho_b) - S(\rho_{ab}),
\]

where \( S(\rho_x) = -\text{Tr}(\rho_x \log \rho_x) \) is the von Neumann entropy, \( \rho_x \) is the density matrix of system \( x \). Here, \( \rho_a \) and \( \rho_b \) are the reduced matrices of the state \( \rho_{ab} \). \( \rho_{ab} = \text{Tr}_b(\rho_{ab}) = \text{Tr}_a(\rho_{ab}) \). Classical correlation accessible by measurement is defined as \( J(\rho_{a,b}) = S(\rho_a) - \min_{\{\Pi_i\}} \sum_i p_i S(\rho_{a|\Pi_i}) \), where \( \{\Pi_i\} \) is the set of all possible measurements that performed locally only on subsystem \( b \) on Bob’s side. So the quantum discord \( \delta(a|b) \) is defined as the difference between those two correlations

\[
\delta(a|b) = I(\rho_{ab}) - J(\rho_{a,b}).
\]

For normal quantum discord, the measurement acting on one side refers to the POVM or projective measurement. A POVM with \( d \) (\( d \) is the dimension of system) outcomes is a \( d \)-tuple of operators \( (\Pi_i^0, \Pi_i^1, ..., \Pi_i^{d-1}) \), and \( \Pi_i^0 \geq 0, \sum_i \Pi_i = I \). The conditional state \( \rho_{a|\Pi_i} = \text{Tr}_b(\Pi_i \rho_{ab})/p_i \), \( p_i = \text{Tr}_b(\Pi_i \rho_{ab}) \) is the probability of the outcome \( i \). For two-qubit system, we know that the best measurement that Bob can perform to get information about Alice’s system \( a \) is a projective measurement onto the state of system \( b \). Suppose \( \{\Pi_i\} \) are the elements of projective measurement, we can assume \( \Pi_i = |i\rangle \langle i|, i = 0, 1 \), and \( \sum_i \Pi_i = \sum_i \Pi_i^0 \Pi_i = I \).

Very recently, Gu et al. \[13\] consider the operational significance of discord consumption during encoding information within state \( \rho_{ab} \), where the discord is obtained by the POVM. In this paper, we mainly follow this scheme but apply it to the more general case. Next, we present the details of the scheme. Alice encodes an arbitrary feasible \( K \) with probability \( p_k \) onto her subsystem \( a \) by application of unitary operator \( U_k \). After encoding, the state becomes \( \tilde{\rho}_{ab} = \sum_k p_k U_k \rho_{ab} U_k^\dagger \) and is given to Bob. Bob uses some decoding protocols to get the best possible estimate of the encoded data \( K \) from \( \tilde{\rho}_{ab} \). The Holevo accessible information can be used for this scheme.

Before proceed, we consider that Bob has no access to \( b \) but can access \( a \), the amount of information accessible to Bob is given by, \( I_0 = S(\tilde{\rho}_a) - S(\rho_a) \), here \( \tilde{\rho}_a = \text{Tr}_b(\tilde{\rho}_{ab}) = \sum_k p_k U_k \rho_{ab} U_k^\dagger \). This case can be considered as that Bob cannot access any of the correlations between the two subsystems. Next, the scheme is divided into two different cases. First, Bob can perform any measurements on the whole system \( \rho_{ab} \) to retrieve the encoded random variable \( K \). For this case, \( I_q - I_0 \) stands for the information that Bob can obtain, where \( I_q = S(\tilde{\rho}_{ab}) - S(\rho_{ab}) \). Second, Bob is allowed to implement only the local measurement on \( a \) or \( b \). If Bob firstly measures on system \( b \) then \( a \), he can obtain the maximal information defined as, \( \tilde{T}_c = \sup_{\Pi_i}(\sum_i p_i S(\tilde{\rho}_{a|\Pi_i}) - \sum_i p_i S(\rho_{a|\Pi_i})) \), here \( S(\tilde{\rho}_{a|\Pi_i}) \) is the quantum conditional entropy. Alternatively, Bob first measures on \( a \) then \( b \), \( \tilde{T}_c \) is the information that Bob can obtain and \( \tilde{T}_c \leq I_0 + S(\rho_a) - \min_{\{\Pi_i\}} \sum_i S(\rho_{a|\Pi_i}) \), \( \{\Pi_i\} \) means measurement on system \( a \). We take the maximum value between \( \tilde{T}_c \) and \( \tilde{T}_c \) as the optimal information \( I_c \).

We can find that \( \Delta I = I_q - I_c \) is the difference of obtained information between coherent measurements and local measurements. It can be considered as the extra quantum advantage which is induced by only coherent interactions in the whole process. It is proven that the following inequality must hold for any \( \rho_{ab} \) under any feasible measurements,

\[
\Delta \delta(a|b) - \tilde{J}(\rho_{ab}) \leq \Delta I \leq \Delta \delta(a|b).
\]

We know that \( \Delta \delta(a|b) \) and \( \tilde{J}(\rho_{ab}) \) represent the amount of discord consumed during encoding and classical correlation after encoding respectively. If there is no discord between Alice and Bob, then the quantum advantage cannot exist. If the encoding is the maximal encoding, we have \( I(\sum_k p_k U_k \rho_{ab} U_k^\dagger) = 0 \), \( \rho_{ab} = \tilde{\rho}_a \otimes \tilde{\rho}_b \), \( \tilde{\rho}_a \) is a maximally mixed state regardless of \( \rho_b \). So the classical correlation of the state after encoding \( \tilde{J} \) is zero. For the maximal encoding, the quantum advantage is exactly equal to the discord consumed during encoding. For this case, we consider the quantum advantage as the best quantum advantage. When Alice and Bob have no discord at the beginning, the quantum advantage is meaningless. In that way, coherent interactions will be no use for Bob to retrieve the encoded information.

Aharonov-Albert-Vaidman \[22\] came up with the weak measurement formalism, where the system interacts weakly with the measured tool. The system is disturbed weakly and the coherence may not be destroyed completely. This is in sharp contrast with the projective measurement which is strong and results in complete decoherence of the system. However, any projective measurement can be decomposed into a sequence of weak measurements, which change the quantum state gradually and in the end result in the same outcomes. Further more, any measurement can be generated by weak measurements, so weak measurements are universal \[23\]. Weak measurement plays both a fundamental role in quantum theory and can have physical applications \[24–28\]. Quantum discord can also be defined for weak measurement and it is shown to be larger than the standard quantum discord based on projective measurements \[21\].

For weak measurement, a measurement with \( n \)-tuples of outcomes can be simplified to having two outcomes, and the weak measurement operators are given as \[23\]

\[
\hat{P}(x) = \sqrt{X_-} \hat{P}_1 + \sqrt{X_+} \hat{P}_2,
\]

\[
\hat{P}(-x) = \sqrt{X_-} \hat{P}_1 + \sqrt{X_+} \hat{P}_2,
\]

where, \( x \in R \), representing the strength of the measurement process, and \( X_- = \frac{1 - \tan x}{2}, \quad X_+ = \frac{1 + \tan x}{2} \), if \( x = \varepsilon \), when
where $|\varepsilon| \ll 1$, $\hat{P}(\pm x)$ is weak measurement, $\hat{P}_1$ and $\hat{P}_2$ are orthogonal projectors with $\hat{P}_1 + \hat{P}_2 = \hat{I}$, $\hat{P}_2^2(x) + \hat{P}_2^2(-x) = \hat{I}$, $\hat{I}$ is the identity. For the superposition state $|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\varphi} \sin \frac{\theta}{2} |1\rangle$, we can let $\hat{P}_1 = |\psi\rangle \langle \psi|$, $\hat{P}_2 = |\psi^\perp\rangle \langle \psi^\perp|$, where $|\psi^\perp\rangle$ are orthogonal to $|\psi\rangle$. Explicitly, the operators $\hat{P}_1$ and $\hat{P}_2$ are

$$\hat{P}_1 = \cos^2 \frac{\theta}{2} |0\rangle\langle 0| + e^{i\varphi} \sin \frac{\theta}{2} \cos \frac{\theta}{2} |0\rangle\langle 1|$$

$$+ e^{i\varphi} \sin \frac{\theta}{2} \cos \frac{\theta}{2} |1\rangle\langle 0| + \sin^2 \frac{\theta}{2} |1\rangle\langle 1|,$$

$$\hat{P}_2 = \cos^2 \frac{\theta}{2} |1\rangle\langle 1| + e^{i\varphi} \sin \frac{\theta}{2} \cos \frac{\theta}{2} |0\rangle\langle 1|$$

$$- e^{i\varphi} \sin \frac{\theta}{2} \cos \frac{\theta}{2} |0\rangle\langle 0| + \sin^2 \frac{\theta}{2} |0\rangle\langle 0|.$$

Under the condition of the maximal encoding, we will show that for different measurements, the best quantum advantages are different.

**THE BEST QUANTUM ADVANTAGE IN SPECIFIC FAMILIES OF STATES**

We introduce the generalized Pauli matrices $U_{m,n} = X^m Z^n$, $m,n = 0,\ldots, d-1$ in $d$ dimension, where $X|j\rangle = |j+1\text{mod}d\rangle$, $Z|j\rangle = e^{2\pi i/4} |j\rangle$, $w = e^{2\pi i/4}$, here $\{|j\rangle\}_{j=0}^{d-1}$ is an orthonormal basis. These operators constitute a basis of unitary operators, see for example [29]. When $d = 2$, we recover the standard Pauli matrices,

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = iXZ = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For a general pure entangled state of bipartite system

$$|\phi\rangle_{ab} = \sum_i \sqrt{\lambda_i} |i_a\rangle |i_b\rangle,$$

(5)

where $\lambda_i$ are Schmidt coefficients and satisfy $\sum_i \lambda_i = 1$.

*The two-qubit pure state.*—In the following, we will restrict our attention on two-dimensional case. The two-qubit pure state studied is $|\phi\rangle = \sqrt{\lambda_0} |00\rangle + \sqrt{\lambda_1} |11\rangle$. We first consider the projective measurement acting on the system, and follow the scheme presented in Ref[[15]]. The quantum discord of $|\phi\rangle$ is given as,

$$\delta(a|b) = S(\bar{q}_a) = -\lambda_0 \log \lambda_0 - \lambda_1 \log \lambda_1.$$  

(6)

We consider the encoding as applying Pauli matrices and identity with equal probability, so the state after encoding becomes as,

$$\bar{q}_{ab} = \frac{1}{2} \sum_{m,n=0}^1 (X^m Z^n \otimes I) \bar{q}_{ab} (X^m Z^n \otimes I)^\dagger$$

$$\overset{\text{I}_q}{\Rightarrow} \frac{1}{2} \otimes (\lambda_0 |0\rangle_b \langle 0| + \lambda_1 |1\rangle_b \langle 1|).$$

(7)

From Eq.(7), we can find, $\tilde{J} = \tilde{\delta} = 0$, where $\tilde{J}$ and $\tilde{\delta}$ are the classical correlation and the discord of the state after encoding, respectively. The amount of discord consumed during encoding is $\Delta \delta = \delta - \tilde{\delta} = -\lambda_0 \log \lambda_0 - \lambda_1 \log \lambda_1$. Next we calculate quantum advantage, here the measurement acts on $b$ of the state $\bar{q}_{ab}$, the minimum of condition entropy is given as $\min_{\{n_i\}} \sum_i p_i S(\bar{q}_{ab}(n_i)) = 1$. From the encoding scheme, we obtain

$$I_q = 1 - \lambda_0 \log \lambda_0 - \lambda_1 \log \lambda_1,$$

$$I_e = 1.$$

Thus the quantum advantage is obtained as,

$$\Delta I_p = -\lambda_0 \log \lambda_0 - \lambda_1 \log \lambda_1,$$

(8)

We can find that the consumption of quantum discord $\Delta \delta$ is equal to quantum advantage. Thus it is the best quantum advantage which corresponds to the maximum encoding.

As an example, for maximally entangled pure state $|\phi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, we have $\bar{q}_{ab} = \frac{I}{2} \otimes \frac{I}{2}$, $I_p = 2$, $I_e = 1$, the best quantum advantage and the consumption of quantum discord is 1.

Now let us turn to the weak measurement. The measurement operator acting on the system is given as

$$I_a \otimes \hat{P}_b(x) = A(x,\theta)|00\rangle\langle 00| + |10\rangle\langle 10| + B(x,\theta,\varphi)$$

$$|00\rangle\langle 01| + |10\rangle\langle 11| + C(x,\theta,\varphi)|01\rangle\langle 01| + |11\rangle\langle 11|,$$

(9)

where $A(x,\theta) = \sqrt{X_-} \cos^2 \frac{\theta}{2} + \sqrt{X_+} \sin^2 \frac{\theta}{2}, B(x,\theta) = (\sqrt{X_-} - \sqrt{X_+}) e^{-i\varphi} \sin \frac{\theta}{2} \cos \frac{\theta}{2}, C(x,\theta) = B^*(x,\theta), D(x,\theta) = \sqrt{X_-} \sin^2 \frac{\theta}{2} + \sqrt{X_+} \cos^2 \frac{\theta}{2}$. Similarly we can obtain the other coefficients when $x$ is negative, the probability is, $p(\pm x) = \text{Tr}(\hat{P}_b(\pm x)^\dagger \hat{P}_b(\pm x) \rho) = \frac{1}{2}(1 - (\lambda_0 - \lambda_1) \tanh (\pm x) \cos \theta)$. After measurement the eigenvalues are

$$k_1(\pm x) = \frac{1}{2}(1 + \sqrt{1 - \frac{\lambda_0 \lambda_1}{p(\pm x)^2 \cosh^2 x}}),$$

$$k_2(\pm x) = \frac{1}{2}(1 - \sqrt{1 - \frac{\lambda_0 \lambda_1}{p(\pm x)^2 \cosh^2 x}}).$$

Thus the quantum conditional entropy for weak measurement is given as

$$S_w(\bar{q}_{ab}(M_x(\pm x))) = p(x) S(\bar{q}_{ab}(M_x(\pm x))) + p(-x) S(\bar{q}_{ab}(M_x(-\pm x))) = 1.$$  

According to the encoding scheme presented in Eq.(7), Bob implements now weak measurement, the maximal information obtained by Bob by different coherent or local measurements, $I_q$ and $I_e$, are given as

$$I_q = -2\lambda_0 \log \lambda_0 - 2\lambda_1 \log \lambda_1,$$

$$I_e = -\lambda_0 \log \lambda_0 - \lambda_1 \log \lambda_1 + p(x) |k_1(\pm x) \log k_1(\pm x|$$

$$+ k_2(\pm x) \log k_2(\pm x) + p(-x) |k_1(-\pm x) \log k_1(-\pm x)$$

$$+ k_2(-\pm x) \log k_2(-\pm x).$$
The best quantum advantage which is the difference of the above two quantities can be found as,

\[ \Delta I_w = -\lambda_0 \log \lambda_0 - \lambda_1 \log \lambda_1 - p(x) \left[ k_1(x) \log k_1(x) + k_2(x) \log k_2(x) \right] - p(-x) \left[ k_1(-x) \log k_1(-x) + k_2(-x) \log k_2(-x) \right]. \tag{10} \]

This quantity depends on the initial pure entangled state and the strength of the weak measurement. When the state is a maximally entangled state, \( I_q = 2, I_c^\pm = 1 + X_- \log X_- + X_+ \log X_+ \). The best quantum advantage is

\[ \Delta I_w = 2 - X_- \log (2X_-) - X_+ \log (2X_+). \tag{11} \]

In Fig[1] we plot different cases of quantum advantage in terms of the measurement strength \( x \). One can see that quantum advantage revealed by weak measurement is always larger than that when only projective measurement is performed. Also, quantum advantage is gradually diminishing with the increasing of measurement strength \( x \). When \( x \) is larger than approximately 2.7, the difference between \( \Delta I_w \) and \( \Delta I_p \) is negligible, which means the coherent interaction does not provide much help to Bob. One may notice that the maximally entangled state has much more best quantum advantage than the other pure states for both weak measurement and normal projective measurement. So we may conclude that weak measurement can reveal much more quantum advantage than normal projective measurement for general bipartite pure entangled state.

The two-qubit Bell diagonal state.— For the two-qubit Bell diagonal states \[ (10) \]

\[ \theta_{ab} = \frac{1}{4} \left( I + \sum_{j=1}^3 c_j \sigma_j \otimes \sigma_j \right), \tag{12} \]

where \( c_j \in \mathbb{R} \), when \( |c_1| = |c_2| = |c_3| = c \), it is a Werner state, in case \( |c_1| = |c_2| = |c_3| = 1 \), it is one Bell state. Consider the optimumal projective measurement, the conditional entropy is \[ S(\rho_a | (\Pi^a_i)) = \frac{1 - c}{2} \log (1 - c) + \frac{1 + c}{2} \log (1 + c), \]

where \( c = \max \{|c_1|, |c_2|, |c_3|\} \). By maximal encoding, the state becomes, \( \rho_{ab} = \frac{I}{2} \otimes \frac{I}{2} \). By methods similar to the previous, we have \( I_q \) and \( I_c \) as follows,

\[ I_q = \frac{1}{4} \left[ (1 - c_1 - c_2 - c_3) \log (1 - c_1 - c_2 - c_3) + (1 + c_1 + c_2 - c_3) \log (1 + c_1 + c_2 - c_3) + (1 + c_1 - c_2 + c_3) \log (1 + c_1 + c_2 + c_3) + (1 - c_1 + c_2 + c_3) \log (1 - c_1 + c_2 + c_3) \right], \]

\[ I_c^\pm = 1 + \frac{1 - c}{2} \log \frac{1 - c}{2} + \frac{1 + c}{2} \log \frac{1 + c}{2}. \]

The difference is the best quantum advantage, and it is written as,

\[ \Delta I_p = \frac{1}{4} \left[ (1 - c_1 - c_2 - c_3) \log (1 - c_1 - c_2 - c_3) + (1 + c_1 + c_2 - c_3) \log (1 + c_1 + c_2 - c_3) + (1 + c_1 - c_2 + c_3) \log (1 + c_1 + c_2 + c_3) + (1 - c_1 + c_2 + c_3) \log (1 - c_1 + c_2 + c_3) \right] - \frac{1 - c}{2} \log (1 - c) - \frac{1 + c}{2} \log (1 + c). \tag{13} \]

For weak measurements, the probability is \( p(x) = \frac{1}{2} \),

\[ I_q = \frac{1}{4} \left[ (1 - c_1 - c_2 - c_3) \log (1 - c_1 - c_2 - c_3) + (1 + c_1 + c_2 - c_3) \log (1 + c_1 + c_2 - c_3) + (1 + c_1 - c_2 + c_3) \log (1 + c_1 - c_2 + c_3) + (1 - c_1 + c_2 + c_3) \log (1 - c_1 + c_2 + c_3) \right], \]

\[ I_c^\pm = 1 + \frac{1}{2} \left[ \lambda_1(x) \log \lambda_1(x) + \lambda_2(x) \log \lambda_2(x) + \lambda_1(-x) \log \lambda_1(-x) + \lambda_2(-x) \log \lambda_2(-x) \right], \]

where the eigenvalues are

\[ \lambda_1(\pm x) = \frac{1 + \sum c_j \tanh(\pm x)}{2}, \quad n_1 = \sin \theta \cos \psi, \quad n_2 = \sin \theta \sin \psi, \quad n_3 = \cos \theta. \]

The best quantum advantage is

\[ \Delta I_w = \frac{1}{4} \left[ (1 - c_1 - c_2 - c_3) \log (1 - c_1 - c_2 - c_3) + (1 + c_1 + c_2 - c_3) \log (1 + c_1 + c_2 - c_3) + (1 + c_1 - c_2 + c_3) \log (1 + c_1 - c_2 + c_3) + (1 - c_1 + c_2 + c_3) \log (1 - c_1 + c_2 + c_3) \right] - \frac{1}{2} \left[ \lambda_1(x) \log \lambda_1(x) + \lambda_2(x) \log \lambda_2(x) + \lambda_1(-x) \log \lambda_1(-x) + \lambda_2(-x) \log \lambda_2(-x) \right]. \tag{14} \]
The best quantum advantage is a function of $x$ and $\theta$ for Bell-diagonal state with $c_1 = 0.15$, $c_2 = 0.03$, $c_3 = 0.7$, $\varphi = 0$. The purple surface represent only using weak measurement, the green plane at $0.045$ represent quantum advantage using projective measurement.

In Fig. 2, the quantum advantage induced by weak measurement is greater than that normal projective measurement. When $\theta = n\pi (n = 0, \pm 1, ...)$, $x$ is larger than approximately 3, the quantum advantage with weak measurement is close to that with normal projective measurement. Thus we summarize that for Bell-diagonal states, weak measurement reveal much more quantum advantage than the normal projective measurement. It is worth mentioning that for weak measurement, $\theta$ possesses a periodic behavior (such as $\theta = 0, \pm \pi, ..., n\pi$).

For Werner state, i.e., $|c_1| = |c_2| = |c_3| = c$. It is a mixture of the fully mixed state with probability $1 - p$ and a singlet state $|\psi\rangle$ with probability $p$:

$$\varrho_{ab} = c|\psi\rangle\langle\psi| + \frac{I}{4}(1 - c),$$

(15)

here $|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$. Depending on the encoding protocol mentioned above, we obtain the best quantum advantage with the normal projective measurement and weak measurement respectively,

$$\Delta I_p = \frac{1 + 3c}{4} \log (1 + 3c) + \frac{1 - c}{4} \log (1 - c) - \frac{1 + c}{2} \log (1 + c),$$

(16)

$$\Delta I_w = \frac{3(1 - c)}{4} \log (1 - c) + \frac{1 + 3c}{4} \log (1 + 3c) - \frac{1 - c \tanh x}{2} \log \left(\frac{1 - c \tanh x}{2}\right) - \frac{1 + c \tanh x}{2} \log \left(\frac{1 + c \tanh x}{2}\right) - 1.$$

(17)

In Fig. 3, we plot quantum advantage for Werner state, the best quantum advantage by using weak measurements with fixed strength $x = 0.7$ is greater than the one with projective measurements for the whole class of states, since the entanglement and quantum discord is monotonously increasing along with the increase of parameter $c$. One can also say that quantum advantage is a monotone function verses entanglement and quantum discord for Werner state. Meanwhile, in Fig. 4, the state parameter $c = 0.4$, when with weak measurement, quantum advantage is gradually diminishing along with the measurement strength $x$, when $x$ is smaller, quantum advantage is always larger than that when only projective measurement is performed. But if $x$ is larger than around 2.5, $\Delta I_w$ approach to $\Delta I_p$, which means the coherent interactions almost have no help to Bob. These further illustrate that weak measurement can reveal much more quantum advantage than normal projective measurement for Werner state.
SUMMARY AND DISCUSSIONS

In summary, weak measurement is of fundamental interest in quantum science. We know that weak measurement may result in additional quantumness of correlations. We show that this is not only a quantity which has the meaning of quantification of quantum correlation, but also it may be the valuable resource in quantum information processing. One can see that all of the quantum correlation based on weak measurement can be consumed for encoding information, it may demonstrate the quantum advantage in comparing coherent measurements with local measurements. Our result provides an additional application of weak measurement in quantum information processing.

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* Electronic address: sxqu@snnu.edu.cn
† Electronic address: hfan@iphy.ac.cn

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