Uniform sets in a family with restricted intersections

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Abstract

Let $\mathcal{F}$ be a family of subsets of $[n] = \{1, \ldots, n\}$ and let $L$ be a set of nonnegative integers. The family $\mathcal{F}$ is $L$-intersecting if $|F \cap F'| \in L$ for every two distinct members $F, F' \in \mathcal{F}$; and $\mathcal{F}$ is $k$-uniform if all its members have the same size $k$. A large variety of problems and results in extremal set theory concern on $k$-uniform $L$-intersecting families. Many attentions are paid to finding the maximum size of a family among all $k$-uniform $L$-intersecting families with prescribed $n, k$ and $L$. In this paper, from another point of view, we propose and investigate the problem of estimating the maximum size of a member in a family among all uniform $L$-intersecting families with size $m$, here $n, m$ and $L$ are prescribed. Our results aim to find out more precise relations of $n, m, k$ and $L$.

Keywords: uniform intersecting family; Fisher’s inequality; Erdős-Ko-Rado theorem; extremal set theory

1 Introduction

For a positive integer $n$, we set $[n] = \{1, \ldots, n\}$. A family $\mathcal{F}$ of subsets of $[n]$ is an $m$-family if $\mathcal{F}$ has $m$ members; $\mathcal{F}$ is $k$-uniform if every member of $\mathcal{F}$ has size $k$; and for a set $L$ of nonnegative integers, $\mathcal{F}$ is $L$-intersecting if $|F \cap F'| \in L$ for every two distinct members $F$ and $F'$ in $\mathcal{F}$. Specially, if $L$ consists of all positive integers (i.e., $L = \mathbb{N}^*$), then an $L$-intersecting family is also called an intersecting family. We say $\mathcal{F}$ is uniform if $\mathcal{F}$ is $k$-uniform for some $k$.

Extremal set theory studies various types of intersecting families, see, e.g., [1 3 4 6 8 9 10 18 22 26 30 32 33 34 35 36]. Among them, a large variety of problems and results concern on $k$-uniform $L$-intersecting families. Many attentions are paid to finding the maximum size of a...
family among all \( k \)-uniform \( L \)-intersecting families with prescribed \( n, k \) and \( L \), see, e.g., \[5, 12, 13, 14, 30, 31, 33, 34\]. The first important result of this type is Fisher’s inequality.

**Theorem 1.1** (Fisher’s inequality, see \[5, 13\]). Let \( \mathcal{F} \) be a \( k \)-uniform \{l\}-intersecting \( m \)-family of distinct subsets of \([n] \), where \( l \geq 1 \) is an integer. Then \( m \leq n \).

The intersection set \( L \) in the above theorem consists of one positive integer. For \( L \) consists of more than one integer, in 1961, Erdős, Ko and Rado \[12\] proved the following classical result, which is now famous as EKR theorem and has a remarkable number of generalizations and analogues during the last half century, see, e.g., \[2, 10, 14, 15, 16, 17, 19, 21, 24, 25, 27, 28, 29, 35, 36\].

**Theorem 1.2** (Erdős, Ko and Rado \[12\]). Let \( \mathcal{F} \) be a \( k \)-uniform intersecting \( m \)-family of distinct subsets of \([n] \) with \( 1 \leq k \leq n/2 \). Then

\[
m \leq \left( \frac{n - 1}{k - 1} \right),
\]

and for \( 1 \leq k < n/2 \) the equality holds only if \( \mathcal{F} \) consists of all \( k \)-subsets with a common element.

For intersection sets consisting of \( s \) general nonnegative/positive integers, the following two results have been proved.

**Theorem 1.3** (Ray-Chaudhuri and Wilson \[30\]). Let \( \mathcal{F} \) be a \( k \)-uniform \( L \)-intersecting \( m \)-family of subsets of \([n] \), where \( L = \{l_1, \ldots, l_s\} \) is a set of \( s \) nonnegative integers. If \( \max \{l_1, \ldots, l_s\} \leq k - 1 \), then

\[
m \leq \left( \frac{n}{s} \right).
\]

**Theorem 1.4** (Hegedűs \[20\]). Let \( \mathcal{F} \) be a \( k \)-uniform \( L \)-intersecting \( m \)-family of subsets of \([n] \), where \( L = \{l_1, \ldots, l_s\} \) is a set of \( s \) positive integers with \( l_1 < l_2 < \cdots < l_s \). If \( n \geq \left( \frac{k^2}{l_{i+1}} \right) s + l_1 \), then

\[
m \leq \left( \frac{n - l_1}{s} \right).
\]

In the above results on \( k \)-uniform \( L \)-intersecting \( m \)-families of subsets of \([n] \), the authors fix \( n, k, L \) and then consider how large the size \( m \) of a family could be. In this paper we investigate this type problem in another direction. We make attempt to estimate what is the maximum size of a member in a family among all uniform \( L \)-intersecting \( m \)-families of subsets of \([n] \) with prescribed \( n, m \) and \( L \). For a better presentation, we assume in the following that the nonnegative integers in the considered intersection set \( L \) satisfy \( l_1 < l_2 < \cdots < l_s \). Since every two distinct members in \( \mathcal{F} \) has less than \( n \) common elements, we will also assume that \( l_s < n \). Define

\[
\kappa_L(n, m) = \max \{k: \text{there exits a } k\text{-uniform } L\text{-intersecting } m\text{-family of subsets of } [n] \},
\]

\[
\mu_L(n, k) = \max \{m: \text{there exits a } k\text{-uniform } L\text{-intersecting } m\text{-family of subsets of } [n] \}.
\]

We need to remark that in the definitions of \( \kappa_L(n, m) \) and \( \mu_L(n, k) \) the subsets in the family are required to be distinct. If there exists no \( k \)-uniform \( L \)-intersecting \( m \)-family of subsets of \([n] \) for any \( k \), then we define \( \kappa_L(n, m) = -\infty \). Note that if \( n \geq l_s + m \), then \( \kappa_L(n, m) \geq l_s + 1 \), as we can
construct an \(l_s\)-uniform \(L\)-intersecting \(m\)-family \(\{F_1, \ldots, F_m\}\) in which \(F_i = \{i\} \cup \{m+1, \ldots, m+l_s\}\) for each \(1 \leq i \leq m\). One can also see from the above definitions that

\[
\kappa_L(n, m) = \max\{k : \mu_L(n, k) \geq m\}.
\]

Alternatively, we can restate our problem as follows. Let \(\mathcal{H}\) be a uniform hypergraph with \(n\) vertices and \(m\) hyperedges such that the intersection of every two hyperedges has size in \(L\). For given \(n, m\) and \(L\), we want to know what is the maximum size of a hyperedge among all uniform hypergraphs satisfying the above conditions.

We now present an extension concept of the \(L\)-intersecting families. For an integer \(t \geq 2\), a family \(F\) is \(t\)-wise \(L\)-intersecting if the intersection of every \(t\) members in \(F\) has size in \(L\). So an \(L\)-intersecting family is 2-wise \(L\)-intersecting. We define

\[
\kappa^t_L(n, m) = \max\{k : \text{there exits a } k\text{-uniform } t\text{-wise } L\text{-intersecting } m\text{-family of subsets of } [n]\}. \quad (3)
\]

The rest of this paper is organized as follows. In the next section we study (2-wise) \(L\)-intersections for \(L\) consisting of one integer. We obtain exact values of \(\kappa^t_L(n, m)\) for \(1 \leq m \leq 4\), and afterwards, we present both a lower bound and an upper bound of \(\kappa^t_L(n, m)\) for general \(m\). In Section 3 we consider \(L\)-intersections for \(L = \{0, 1, \ldots, l_s\}\). In particular, we show that

\[
\lim_{n \to \infty} \frac{\kappa^t_{\{0,1\}}(n, n)}{\sqrt{n}} = 1.
\]

In Section 4 we consider \(t\)-wise \(L\)-intersections for general \(t \geq 2\) and \(L = \{l_1, \ldots, l_s\}\), we obtain an exact value of \(\kappa^t_L(n, m)\) for large \(n\). Section 5 is devoted to the proofs of the main results in Sections 2 and 3. In Section 6 we propose a problem for further research.

2 \(L\)-intersecting families with \(L = \{l\}\)

In this section we deal with the case that \(L\) is a singleton \(\{l\}\), where \(l \geq 0\) and \(n \geq l + m\). For convenience, we will write \(\kappa_l(n, m)\) for \(\kappa^1_L(n, m)\) in the following. It is easy to check that

\[
\kappa_l(n, m) = \left\lfloor \frac{n}{m} \right\rfloor \quad \text{for all } m \geq 1 \text{ and } n \geq m.
\]

So from now on we assume that \(l \geq 1\).

For the first case \(m = 1\), it is not difficult to see that the ground set \([n]\) forms a singleton family of maximum member size.

**Proposition 2.1.** \(\kappa_l(n, 1) = n\).

We will further obtain exact values of \(\kappa_l(n, m)\) for \(2 \leq m \leq 4\). Despite that the first two results \(\kappa_l(n, 1) = n\) and \(\kappa_l(n, 2) = \lfloor (n + l) / 2 \rfloor\) are not difficult to verify, the proofs for the cases \(m = 3\) and \(m = 4\) are somehow complicated.

**Theorem 2.1.** \(\kappa_l(n, 2) = \left\lfloor \frac{n + l}{2} \right\rfloor\) for all \(n \geq l + 2\).
Theorem 2.2. \( \kappa_i(n, 3) = \begin{cases} 
 \lfloor (n + l)/2 \rfloor, & l + 3 \leq n < 3l; \\
 l + \lfloor n/3 \rfloor, & n \geq \max\{3l, l + 3\}. 
\end{cases} \)

Theorem 2.3. \( \kappa_i(n, 4) = \begin{cases} 
 \lfloor (n + l)/2 \rfloor, & l + 4 \leq n < 2l; \\
 \lfloor (3n + 6l)/8 \rfloor, & \max\{2l, l + 4\} \leq n < 6l; \\
 \lfloor (n + 6l)/4 \rfloor, & n \geq 6l. 
\end{cases} \)

For general \( m \), we obtain a lower bound and an upper bound for \( \kappa_i(n, m) \). In the following inequality, we assume \( \kappa_i(n, m) = -\infty \) when \( l \) is negative.

Theorem 2.4. \( \kappa_i(n, m) \geq \max_{2 \leq l \leq m} \left\{ \kappa_{1-\left(\frac{n-2}{2}\right)} \left( n - \left( \frac{m-1}{i-j} \right) n \right) \right\} \) for all \( m \geq l \geq 1 \).

For the upper bound, we need some new necessary definitions and notations. Here we suppose that \( n, k, l \) are real numbers. Let \( X = [0, n] \) be the real interval and let \( \lambda \) be the Lebesgue measure on \( X \). If \( F = \{F_1, \ldots, F_m\} \) is a family of subsets of \( X \) such that

(i) \( \lambda(F_i) = k \) for all \( i \in [m] \), and

(ii) \( \lambda(F_i \cap F_j) = l \) for all distinct \( i, j \in [m] \),

then we call \( F \) a fractional \( \{l\} \)-intersecting \( k \)-uniform \( m \)-family of \( X \). For given real numbers \( n, l \) and integer \( m \), let \( \kappa_i^{\text{frac}}(n, m) \) be the largest real number \( k \) such that there exists a fractional \( \{l\} \)-intersecting \( k \)-uniform \( m \)-family of \( [0, n] \). We obtain the following result on \( \kappa_i^{\text{frac}}(n, m) \), which may be of independent interest.

Theorem 2.5. Let \( n > l \geq 0 \) be two real numbers and let \( m \geq 1 \) be an integer. Then

\[
\kappa_i^{\text{frac}}(n, m) = \left( \frac{m-1}{s-1} \right) \alpha + \left( \frac{m-1}{t-1} \right) \beta, 
\]

where

\[
s = \left\lfloor 1 + \sqrt{1 + 4m(m-1)l/n} \right\rfloor / 2, \\
t = \left\lfloor 1 + \sqrt{1 + 4m(m-1)l/n} \right\rfloor / 2, 
\]

and \((\alpha, \beta)\) is the solution of

\[
\begin{cases} 
\left( \frac{m}{s} \right) \alpha + \left( \frac{m}{t} \right) \beta = n; \\
\left( \frac{m-2}{s-2} \right) \alpha + \left( \frac{m-2}{t-2} \right) \beta = l. 
\end{cases} \quad (4)
\]

It is not difficult to verify that \( \kappa_i^{\text{frac}}(n, m) \) is an upper bound of \( \kappa_i(n, m) \).

Theorem 2.6. \( \kappa_i(n, m) \leq \kappa_i^{\text{frac}}(n, m) \) for all integers \( n > l \geq 0 \).

Here we remark that if the solution \((\alpha, \beta)\) of Equation \( 4 \) consists of two integers then the equality in Theorem 2.6 holds. It is not difficult to see that for any given integers \( l, m \), there are infinitely many integers \( n \) such that the solutions of Equation \( 4 \) are integers, it therefore follows that there are infinitely many examples showing the sharpness of the upper bound in Theorem 2.6.

3 \( L \)-intersecting families with \( L = \{0, 1, \ldots, l\} \)

In this section we deal with the case \( L = \{0, 1, \ldots, l\} \), where \( l < n \) is a positive integer. For convenience, we will write \( \kappa_{\leq l}(n, m) \) for \( \kappa_{(0, \ldots, l)}(n, m) \) in the following. We start with the following theorem by Deza, Erdős and Frankl.
Theorem 3.1 (Deza, Erdős and Frankl [9]). Let $s \leq k \leq n$ be positive integers, $L$ a set of $s$ nonnegative integers and $F$ an $L$-intersecting $k$-uniform family of subsets of $[n]$. Then there exists $n_0 = n_0(k, L)$ such that for $n > n_0$ we have

$$m = |F| \leq \prod_{i=1}^{s} \frac{n - l_i}{k - l_i}.$$ 

For the special case $L = \{0, \ldots, l\}$, we have the following result. We will give a simple proof for convenience.

Theorem 3.2. For fixed $n \geq k > l \geq 1$, we have $\mu_{\kappa_1}(n, k) \leq \frac{n(n-1) \cdots (n-l)}{k(k-1) \cdots (k-l)}$.

Proof. Suppose that $F = \{F_1, \ldots, F_m\}$ is a $\{0, \ldots, l\}$-intersecting $k$-uniform family of $[n]$. We will show that $m \leq \frac{n(n-1) \cdots (n-l)}{k(k-1) \cdots (k-l)}$. We use induction on $k$. If $k = l + 1$, then clearly $m = \binom{n}{k}$ and the assertion holds. So we assume that $k \geq l + 2$.

For $x \in [n]$, we set $F_x = \{F \in F : x \in F\}$. Clearly $\sum_{x \in [n]} |F_x| = \sum_{i=1}^{m} |F_i| = mk$. Note that $F'_x = \{F \setminus \{x\} : F \in F_x\}$ is a $\{0, \ldots, l-1\}$-intersecting $(k-1)$-uniform family of $[n] \setminus \{x\}$. By induction hypothesis,

$$|F_x| = |F'_x| \leq \frac{(n-1) \cdots (n-l)}{(k-1) \cdots (k-l)}.$$ 

It follows that $m = \sum_{x \in [n]} |F_x|/k \leq \frac{n(n-1) \cdots (n-l)}{k(k-1) \cdots (k-l)}$. \hfill $\square$

The above theorem in fact gives an upper bound for $\kappa_{\kappa_1}(n, m)$. We will make use of the following lower bound for special $n, m$.

Theorem 3.3. Let $l$ be a positive integer and let $p \geq l$ be a prime. Then $\mu_{\kappa_1}(p^{l+1}, p) \geq p^{l+1}$ and $\kappa_{\kappa_1}(p^{l+1}, p) \geq p$.

Proof. Let $X = \{(x, y) : x, y \text{ are integers with } 0 \leq x, y < p\}$.

We will find a $\{0, \ldots, l\}$-intersecting $p$-uniform $p^{l+1}$-family of $X$. Set $F = \{F_{a_0, \ldots, a_l} : 0 \leq a_i < p, 0 \leq i \leq l\}$, where

$$F_{a_0, \ldots, a_l} = \{(x, y) \in X : y \equiv a_0 + xa_1 + x^2a_2 + \cdots + x^la_l \pmod{p}\}.$$ 

We now show that any two members in $F$ have at most $l$ common elements. Suppose that

$$(x_0, y_0), \ldots, (x_l, y_l) \in F_{a_0, \ldots, a_l} \cap F_{b_0, \ldots, b_l}.$$ 

Let

$$v = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_l \end{pmatrix}, \quad X = \begin{pmatrix} 1 & x_0 & \cdots & x^l_0 \\ 1 & x_1 & \cdots & x^l_1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_l & \cdots & x^l_l \end{pmatrix}, \quad \alpha = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_l \end{pmatrix}, \quad \beta = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_l \end{pmatrix}.$$ 

Then $v \equiv X\alpha \pmod{p}$, $v \equiv X\beta \pmod{p}$, and $X(\alpha - \beta) \equiv 0 \pmod{p}$. 

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Since $\alpha \neq \beta$, we have that $X$ is irreversible (in the field $F_p$). That is

$$p \mid \det A = \prod_{0 \leq i < j \leq l} (x_j - x_i).$$

Since $p$ is a prime, there exist $i, j$ such that $p \mid (x_j - x_i)$. Since $0 \leq x_i, x_j \leq p - 1$, we have $x_i = x_j$ and $y_i = y_j$.

Now we deal with the case $L = \{0, 1\}$ and $m = n$. It is worth noting that $\kappa_{\leq 1}(n, n)$ is a non-decreasing function. Note also that any two distinct ($\lfloor n/2 \rfloor + 1$)-subsets of $[n]$ have at least two common elements. So $\kappa_{\leq 1}(n, n) \leq \lfloor n/2 \rfloor$ is a trivial upper bound. But this bound is far from being sharp. We shall show that $\kappa_{\leq 1}(n, n) = \Theta(\sqrt{n})$ in the following. For any real number $x$, let $p(x)$ be the smallest prime which is not less than $x$.

**Theorem 3.4.** \[
\left\lfloor \frac{n}{p(\sqrt{n})} \right\rfloor \leq \kappa_{\leq 1}(n, n) \leq \sqrt{n - \frac{3}{4}} + \frac{1}{2}, \quad \text{and thus} \quad \lim_{n \to \infty} \frac{\kappa_{\leq 1}(n, n)}{\sqrt{n}} = 1.
\]

The sharpness of the upper bound can be deduced from the result below, and the lower bound can be reached when, e.g., $n$ is a square of a prime.

**Theorem 3.5.** Let $q$ be a prime power. Then

$$\kappa_{\leq 1}(n, n) = \begin{cases} 
q, & \text{if } n \in [q^2, q^2 + q]; \\
q + 1, & \text{if } n = q^2 + q + 1.
\end{cases}$$

### 4 $t$-wise $L$-intersecting families

This section is devoted to $t$-wise $L$-intersecting families with general $t \geq 2$ and general intersection set $L = \{l_1, \ldots, l_s\}$. We first give a lower bound and an upper bound on $\kappa^t_L(n, m)$.

**Theorem 4.1.** Let $L = \{l_1, \ldots, l_s\}$ with $l_s > l_{s-1} > \cdots > l_1 \geq 0$, $n \geq l_s + \frac{m}{t-1}$ and $m \geq t$. Then

$$\left\lfloor \frac{(n - l_s)(t - 1)}{m} \right\rfloor + l_s \leq \kappa^t_L(n, m) \leq \left\lfloor \frac{n(t - 1)}{m} + \frac{l_s}{m} \binom{m}{t} \right\rfloor.$$

**Proof.** Let $A$ be a subset of $[n]$ of size $l_s$, say $A = \{n, n - 1, \ldots, n - l_s + 1\}$. Let $B = \{B_1, \ldots, B_m\}$ be a uniform family of subsets of $[n - l_s]$ such that every element of $[n - l_s]$ appears in at most $t - 1$ members of $B$. It is not difficult to see that the family $B$ exists with each $B_i$ has size

$$\left\lfloor \frac{(n - l_s)(t - 1)}{m} \right\rfloor \geq 1.$$

Now let $F = \{F_1, \ldots, F_m\}$ with

$$F_i = A \cup B_i, \quad \text{for each } 1 \leq i \leq m.$$

It is not difficult to see that $F$ is a $t$-wise $L$-intersecting family of $k$-subsets of $[n]$ with

$$k = \left\lfloor \frac{(n - l_s)(t - 1)}{m} \right\rfloor + l_s.$$
Theorem 4.2. Let \( F \) be a family of pairwise disjoint \( l_s \)-subsets of \([n]\). If \( m \geq t \) and \( n \geq \binom{m}{t} l_s \), then

\[
\kappa^t_L(n, m) = \left\lfloor \frac{n(t-1)}{m} \right\rfloor + l_s.
\]

Proof. By Theorem 4.1 it suffices to show that \( \kappa^t_L(n, m) \geq \left\lfloor \frac{n(t-1)}{m} \right\rfloor + l_s \). Let \( A = \{ A_T : T \subseteq [m], |T| = t \} \) be a family of pairwise disjoint \( l_s \)-subsets of \([n]\). The family \( A \) exists since \( n \geq \binom{m}{t} l_s \). Let \( B = \{ B_1, \ldots, B_m \} \) be a uniform family of subsets of \([n] \setminus (\bigcup A) \) such that every element of \([n] \setminus (\bigcup A) \) appears in at most \( t-1 \) members of \( B \). Then one can see that we can take \( B \) such that each \( B_i \) has size

\[
\left\lfloor \frac{n(t-1)}{m} \right\rfloor + \frac{l_s}{t} \binom{m}{t}.
\]

We construct a family \( F = \{ F_1, \ldots, F_m \} \) by letting

\[
F_i = B_i \cup \bigcup_{i \in T} A_T.
\]

It is not difficult to see that \( F \) is an \( L \)-intersecting family of \( k \)-subsets of \([n]\) with

\[
k = \left( \frac{m}{t} - 1 \right) l_s + \left\lfloor \frac{n(t-1)}{m} \right\rfloor + \left\lfloor \frac{l_s}{t} \binom{m}{t} \right\rfloor.
\]

Thus \( \kappa^t_L(n, m) \geq \left\lfloor \frac{n(t-1)}{m} \right\rfloor + l_s \binom{m}{t} \). \( \square \)

As a corollary of Theorem 4.2 taking \( t = 2 \), we have the following result.

Corollary 4.1. If \( n \geq \binom{m}{2} l \), then \( \kappa_L(n, m) = \kappa_{\overline{2}}(n, m) = \left\lfloor \frac{n}{m} + \frac{(m-1)!}{2} \right\rfloor. \)
5 Proofs of some main theorems

In this section we present the proofs of some theorems in Sections 2 and 3, namely, Theorems 2.1.2.5 in Section 2 and Theorems 3.1.3.5 in Section 3. In the following proof we do not require the members of the family $\mathcal{F}$ to be distinct. Note that under the above assumption the value of $\kappa_l(n, m)$ will not change when $n \geq l + m$.

Set $M = [m] = \{1, 2, \ldots, m\}$ and let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be an arbitrary $\{l\}$-intersecting $k$-uniform $m$-family of $[n]$. We define a function

$$\phi = \phi_{\mathcal{F}} : 2^M \to \mathbb{N}$$

such that $\phi(A)$ is the number of elements in $[n]$ that contained in each $F_i$ with $i \in A$ but not in any $F_j$ with $j \notin A$, i.e.,

$$\phi(A) = |\{a \in [n] : A = \{i : a \in F_i\}\}|, \text{ for all } A \subseteq M.$$  

By the definition of an $\{l\}$-intersecting uniform family of $[n]$, we have the following equations.

$$\begin{cases} 
\sum_{A \subseteq M} \phi(A) = n; \\
\sum_{x \in A} \phi(A) = \sum_{y \in B} \phi(B), \text{ for all } x, y \in M; \\
\sum_{x, y \in A} \phi(A) = l, \text{ for all } x, y \in M.
\end{cases}$$

We call a function $\phi : 2^M \to \mathbb{N}$ satisfying (5) an assignment (or exactly, an $(l, m, n)$-assignment), and the equivalent number $\sum_{x \in A} \phi(A)$ for each $x \in M$ is the value of $\phi$, denoted by $v(\phi)$. So every $\{l\}$-intersecting uniform family of $[n]$ corresponds to an assignment. On the other hand, for every $(l, m, n)$-assignment $\phi$, we can easily get an $\{l\}$-intersecting uniform family $\mathcal{F}$ of $[n]$ such that $\phi = \phi_{\mathcal{F}}$. So the problem to find largest size of the subsets in an $\{l\}$-intersecting uniform families of $[n]$, is transferred to maximize the value $v(\phi)$ among all $(l, m, n)$-assignments.

For two assignments $\phi_1$ and $\phi_2$, their difference $\tau = \phi_1 - \phi_2$ satisfies the following equations.

$$\begin{cases} 
\sum_{A \subseteq M} \tau(A) = 0; \\
\sum_{x \in A} \tau(A) = \sum_{y \in B} \tau(A), \text{ for all } x, y \in M; \\
\sum_{x, y \in A} \tau(A) = 0, \text{ for all } x, y \in M.
\end{cases}$$

We call a function $\tau : 2^M \to \mathbb{Z}$ satisfying (6) an extender. Note that the value of $\tau$ (the equivalent number $\sum_{x \in A} \tau(A)$) is $v(\tau) = v(\phi_1) - v(\phi_2)$. An extender with value $i$ is called an $i$-extender, and sometimes we call a 0-extender a regulator. Note that an extender image some subsets of $M$ to a negative number, whereas an assignment has only nonnegative objects.

Let $\phi$ be an assignment and let $\tau$ be an extender. If $\phi + \tau$ is also an assignment (i.e., $\tau(A) < 0$ implies $\phi(A) \geq -\tau(A)$ for all $A \subseteq M$), then we say that $\tau$ is compatible with $\phi$.

**Lemma 5.1.** An assignment $\phi$ has maximal value if and only if there exists no positive-extender $\tau$
that is compatible with $\phi$.

**Proof.** If there is another assignment $\phi'$ with $v(\phi') > v(\phi)$, then $\tau = \phi' - \phi$ is a positive extender compatible with $\phi$. If $\phi$ has a compatible positive extender $\tau$, then $\phi' = \phi + \tau$ is an assignment with $v(\phi') > v(\phi)$. \hfill $\Box$

We use $\text{rem}(n, m)$ to denote the remainder of $n$ divided by $m$.

**Proof of Theorem 2.1.** One can check by (6) that the extender $\tau_i$ in the following table is the only $i$-extender (for $m = 2$).

| A | 0 | 1 | 2 | M |
|---|---|---|---|---|
| $\tau_i$ | $-2i$ | $i$ | $i$ | 0 |

Let $\phi$ be an assignment such that $\phi(\emptyset) = \text{rem}(n + l, 2)$, $\phi(\{1\}) = \phi(\{2\}) = \lfloor (n - l)/2 \rfloor$ and $\phi(\{1, 2\}) = l$. It follows that there exists no positive-extender compatible with $\phi$. By Lemma 5.1 $\kappa_i(n, 2) = v(\phi) = \lfloor (n + l)/2 \rfloor$. \hfill $\Box$

Let $\tau$ and $\tau'$ be two positive-extenders. We write $\tau' \preccurlyeq \tau$ if $\tau'(A) < 0$ implies $\tau'(A) \geq \tau(A)$ for all $A \subseteq M$. If there are no other $\tau'$ with $\tau' \preccurlyeq \tau$, then $\tau$ is a critical extender.

**Lemma 5.2.** An assignment $\phi$ has maximal value if and only if there exists no critical positive-extender $\tau$ that is compatible with $\phi$.

**Proof.** Note that if $\tau' \preccurlyeq \tau$ and $\tau$ is compatible with $\phi$, then $\tau'$ is also compatible with $\phi$. Also note that if $\tau$ is not critical, then there is a critical extender $\tau'$ with $\tau' \preccurlyeq \tau$. The assertion now can be deduced by Lemma 6.1 immediately. \hfill $\Box$

**Proof of Theorems 2.2.** We first show that the positive extenders in the following table are the only critical extenders when $m = 3$.

| A | $\emptyset$ | 1 | 2 | 3 | 12 | 13 | 23 | M |
|---|---|---|---|---|----|----|----|---|
| $\tau_0$ | $-3$ | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| $\tau_1$ | $-2$ | 0 | 0 | 0 | 1 | 1 | 1 | $-1$ |
| $\tau_2$ | $-1$ | $-1$ | $-1$ | $-1$ | 2 | 2 | 2 | $-2$ |
| $\tau_3$ | 0 | $-2$ | $-2$ | $-2$ | 3 | 3 | 3 | $-3$ |

Let $\tau$ be an arbitrary positive extender. One can compute by (6) that

\[
\tau(\{1, 2\}) = \tau(\{1, 3\}) = \tau(\{2, 3\}) = -\tau(M),
\]

\[
\tau(\{1\}) = \tau(\{2\}) = \tau(\{3\}) = \tau(M) + v(\tau), \text{ and}
\]

\[
\tau(\emptyset) = -\tau(M) - 3v(\tau).
\]

Since $\tau$ is positive, we have $v(\tau) \geq 1$. If $\tau(M) \geq 1$, then $\tau(\emptyset) \leq -6$, implying that $\tau_0 \preccurlyeq \tau$. If $\tau(M) = -1$ and $v(\tau) \geq 2$, then $\tau(\emptyset) \leq -5$; if $\tau(M) = -2$ and $v(\tau) \geq 2$, then $\tau(\emptyset) \leq -4$; if $\tau(M) = -3$ and $v(\tau) \geq 2$, then $\tau(\emptyset) \leq -3$, implying that $\tau_0 \preccurlyeq \tau$. Suppose now $\tau(M) \leq -4$. If
\( v(\phi) \leq -\tau(M) - 2 \), then \( \tau(\{1\}) = \tau(\{2\}) = \tau(\{3\}) \leq -2 \), implying that \( \tau_3 \leq \tau \). If \( v(\phi) \geq -\tau(M) - 1 \), then \( \tau(\emptyset) \leq 2\tau(M) + 3 \leq -5 \), implying that \( \tau_0 \leq \tau \). It follows that \( \tau_0, \tau_1, \tau_2, \tau_3 \) are the only critical extenders.

Now we prove the assertion. If \( l \leq n < 3l \), then let
\[
\phi(A) = \begin{cases} 
\text{rem}(n - l, 2), & |A| = 0; \\
0, & |A| = 1; \\
\lfloor (n - l)/2 \rfloor, & |A| = 2; \\
\lceil (3l - n)/2 \rceil, & |A| = 3.
\end{cases}
\]
If \( n \geq 3l \), then let
\[
\phi(A) = \begin{cases} 
\text{rem}(n, 3), & |A| = 0; \\
\lfloor n/3 \rfloor - l, & |A| = 1; \\
l, & |A| = 2; \\
0, & |A| = 3.
\end{cases}
\]

One can check that all \( \tau_i, i = 0, \ldots, 3 \), are not compatible with \( \phi \). By Lemma 5.2, \( \phi \) has the maximum value, i.e., \( \kappa_i(n, 3) = v(\phi) \). We can therefore obtain the desired result.

An assignment (or extender) \( \phi \) is balanced if \( |A| = |B| \) implies \( \phi(A) = \phi(B) \). Clearly if \( m \leq 3 \), then every assignment is balanced. For \( m \geq 4 \), there will be unbalanced assignments.

Lemma 5.3. Suppose that there is a balanced assignment with maximum value among all assignments for a given \( m \). An assignment \( \phi \) has maximal value if and only if there exists no balanced critical positive-extender \( \tau \) that is compatible with \( \phi \).

Proof. Note that the difference of two balanced assignments is a balanced extender. The assertion can be obtained similarly as the analysis of Lemma 5.1.

Proof of Theorem 2.3. We first show that there is a balanced assignment for \( m = 4 \).

Claim 1. There is a balanced assignment \( \phi \) with maximum value among all assignments.

Proof. We will use the following regulators.

| \( A \) | \( \emptyset \) | 1 | 2 | 3 | 4 | 12 | 13 | 14 | 23 | 24 | 34 | 123 | 124 | 134 | 234 | \( M \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( \rho_0 \) | 1 | 0 | -1 | -1 | -1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | -1 | 0 |
| \( \rho_1 \) | 1 | 1 | 1 | -1 | -1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | -1 | -1 | 1 |
| \( \rho_2 \) | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | -1 | -1 | -1 | 1 |

Let \( \phi \) be a maximum-value assignment such that
\[
\Delta_\phi = \max_{|A|=3} \phi(A) - \min_{|A|=3} \phi(A)
\]
is as small as possible. It is sufficient to show that $\Delta_\phi = 0$. Let $x_i = \phi(M \setminus \{i\})$ for $i = 1, \ldots, 4$. One can compute by (6) that

$$\phi(\{i\}) = v(\phi) + \sum_{i=1}^{4} x_i - 3l - 2\phi(M) - x_1, \ i = 1, \ldots, 4.$$ 

If $x_1 > \max\{x_2, x_3, x_4\}$, then $\phi(\{1\}) < \phi(\{i\})$, $i = 2, 3, 4$, implying that $\phi(\{i\}) \geq 1$, $i = 2, 3, 4$. In this case $\rho_0$ is compatible with $\phi$. It follows that $\phi' = \phi + \rho_0$ has value $v(\phi') = v(\phi)$ and $\Delta_{\phi'} < \Delta_\phi$, a contradiction. If $x_1 = x_2 > \max\{x_3, x_4\}$, then $\phi(\{3\}) = \phi(\{4\}) \geq 1$ and $\phi\{1, 2\} \geq 1$. In this case $\rho_1$ is compatible with $\phi$. It follows that $\phi' = \phi + \rho_1$ has value $v(\phi') = v(\phi)$ and $\Delta_{\phi'} < \Delta_\phi$, a contradiction. If $x_1 = x_2 = x_3 > x_4$, then $\phi(\{4\}) \geq 1$. In this case $\rho_2$ is compatible with $\phi$. It follows that $\phi' = \phi + \rho_2$ has value $v(\phi') = v(\phi)$ and $\Delta_{\phi'} < \Delta_\phi$, a contradiction. The other cases are similarly. Thus we conclude that $\Delta_\phi = 0$. It follows from (6) that $\phi$ is balanced.

Now we list the following extenders. One can check that there exists no other balanced critical extender. We omit the details here.

| $A$ | $|A| = 0$ | $|A| = 1$ | $|A| = 2$ | $|A| = 3$ | $|A| = 4$ |
|-----|-----------|-----------|-----------|-----------|-----------|
| $\tau_0$ | $-2$ | $-1$ | $2$ | $-2$ | $2$ |
| $\tau_1$ | $0$ | $-2$ | $2$ | $-1$ | $0$ |
| $\tau_2$ | $-3$ | $0$ | $1$ | $-1$ | $1$ |
| $\tau_3$ | $2$ | $-3$ | $2$ | $0$ | $2$ |
| $\tau_4$ | $-1$ | $-1$ | $1$ | $0$ | $-1$ |
| $\tau_5$ | $-4$ | $1$ | $0$ | $0$ | $0$ |
| $\tau_6$ | $1$ | $-2$ | $1$ | $1$ | $-3$ |
| $\tau_7$ | $-2$ | $0$ | $0$ | $1$ | $-2$ |
| $\tau_8$ | $0$ | $-1$ | $0$ | $2$ | $-4$ |
| $\tau_9$ | $-1$ | $0$ | $-1$ | $3$ | $-5$ |
| $\tau_{10}$ | $0$ | $0$ | $-2$ | $5$ | $-8$ |

Now we construct an assignment $\phi$ as follows. If $l \leq n < 2l$, then let

$$\phi(A) = \begin{cases} \text{rem}(n - l, 2), & |A| = 0; \\ 0, & |A| = 1; \\ 0, & |A| = 2; \\ \lfloor (n - l)/2 \rfloor, & |A| = 3; \\ 2l - n + \text{rem}(n - l, 2), & |A| = 4. \end{cases}$$

If $2l \leq n < 6l - 5$, then let $n - 6l = -8q + r$, $0 \leq r < 8$, and

$$\phi(A) = \begin{cases} \text{rem}(r, 3), & |A| = 0; \\ 0, & |A| = 1; \\ l - 2q + \lfloor r/3 \rfloor, & |A| = 2; \\ q - \lfloor r/3 \rfloor, & |A| = 3; \\ r/3, & |A| = 4. \end{cases}$$
If \( n = 6l - 5 + r, \) \( 0 \leq r \leq 3, \) then let

\[
\phi(A) = \begin{cases} 
    r, & |A| = 0; \\
    0, & |A| = 1; \\
    l - 1, & |A| = 2; \\
    0, & |A| = 3; \\
    1, & |A| = 4.
\end{cases}
\]

If \( n = 6l - 1, \) then let

\[
\phi(A) = \begin{cases} 
    0, & |A| = 0; \\
    1, & |A| = 1; \\
    l - 1, & |A| = 2; \\
    0, & |A| = 3; \\
    1, & |A| = 4.
\end{cases}
\]

If \( n \geq 6l, \) then let

\[
\phi(A) = \begin{cases} 
    \text{rem}(n - 6l, 4), & |A| = 0; \\
    \lfloor (n - 6l)/4 \rfloor, & |A| = 1; \\
    l, & |A| = 2; \\
    0, & |A| = 3; \\
    0, & |A| = 4.
\end{cases}
\]

One can check that for each case, any \( \tau_i, i = 0, \ldots, 10, \) is not compatible with \( \phi. \) By Claim \( 1 \) and Lemma \( 5.3, \) \( \phi \) has maximum value, i.e., \( \kappa_l(n, 4) = v(\phi). \) One can compute the desired result. \( \square \)

**Proof of Theorem 2.4.** Let \( i \) be an arbitrary integer with \( 2 \leq i \leq m. \) Let \( \phi_i \) be an \((l - \binom{m - 2}{2}, m, n - \binom{m}{i})\)-assignment such that \( v(\phi_i) \) is maximum, and let \( \pi_i : 2^M \rightarrow \mathbb{N} \) be a function such that

\[
\pi_i(A) = \begin{cases} 
    1, & |A| = i; \\
    0, & \text{otherwise}.
\end{cases}
\]

Then \( \phi = \phi_i + \pi_i \) is an \((l, m, n)\)-assignment with \( v(\phi) = v(\phi_i) + \binom{m - 1}{i - 1}. \) Thus the assertion holds. \( \square \)

**Proof of Theorem 2.5.** Let \( \mathcal{F} = \{F_1, \ldots, F_m\} \) be a fractional \( \{l\}\)-intersecting uniform family of \( X. \) As in the previous case, we define a function \( \phi_\mathcal{F} : 2^M \rightarrow \mathbb{R}^+ \cup \{0\} \) such that

\[
\phi(A) = \lambda(\{x \in X : \{i \in M : x \in F_i\} = A\}), \text{ for all } A \subseteq M.
\]

Thus \( \phi = \phi_\mathcal{F} \) satisfies

\[
\begin{align*}
\sum_{A \subseteq M} \phi(A) &= n; \\
\sum_{x \in A} \phi(A) &= \sum_{y \in A} \phi(A), \text{ for all } x, y \in M; \\
\sum_{x, y \in A} \phi(A) &= l, \text{ for all } x, y \in M.
\end{align*}
\]
Now we will find the maximum value $v(\phi)$ among all assignments satisfying (7). Recall that $\phi$ is balanced if $|A| = |B|$ implies $\phi(A) = \phi(B)$ for all $A, B \subseteq M$.

Claim 2. There is a balanced assignment with maximum value.

Proof. Let $\phi$ be an assignment with maximum value, and let $\Omega$ be the symmetric group on $M$. For any $\sigma \in \Omega$, we define $\phi_{\sigma}$ as

$$\phi_{\sigma}(A) = \phi(\sigma(A)), \text{ for all } A \subseteq M.$$ 

Clearly $v(\phi_{\sigma}) = v(\phi)$, implying that $\phi_{\sigma}$ has maximum value for all $\sigma \in \Omega$. It follows that

$$\phi^* = \frac{1}{m!} \sum_{\sigma \in \Omega} \phi_{\sigma}$$

has value $v(\phi^*) = v(\phi)$, the maximum value as well. It is not difficult to see that $\phi^*$ is balanced. This proves the claim.

Now let $\phi$ be a balanced assignment with maximum value. For convenience, we define

$$\varphi : M^* = M \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$$

such that $\varphi(i) = \phi(A)$ for all $A$ of size $i$. Therefore

$$\begin{align*}
\sum_{i=0}^{m} \binom{m}{i} \varphi(i) &= n; \\
\sum_{i=2}^{m} \binom{m-2}{i-2} \varphi(i) &= l.
\end{align*} \tag{8}$$

Claim 3. There exists $s, 0 \leq s \leq m - 1$ such that $\varphi(i) = 0$ for all $i \in M^* \setminus \{s, s + 1\}$.

Proof. We need the following fact.

Lemma 5.4. Suppose that $0 \leq r < s < t \leq m$ are integers and $\alpha, \beta > 0$ are real numbers. If

$$\begin{align*}
\binom{m}{r}\alpha + \binom{m}{t}\beta &= \binom{m}{s}, \\
\binom{m-2}{r-2}\alpha + \binom{m-2}{t-2}\beta &= \binom{m-2}{s-2},
\end{align*} \tag{9}$$

then

$$\binom{m-1}{r-1}\alpha + \binom{m-1}{t-1}\beta < \binom{m-1}{s-1}.$$ 

Proof. If $r = 0$, then

$$\binom{m-1}{t-1}\beta = \binom{m-1}{t-1}\binom{m-2}{s-2}/\binom{m-2}{t-2} = \frac{s-1}{t-1}\binom{m-1}{s-1} < \binom{m-1}{s-1}.$$
Thus we assume that $r \geq 1$. By (9), we can solve that

$$
\alpha = \begin{vmatrix}
\frac{m}{s} & \frac{m}{t} \\
\frac{m-2}{s-2} & \frac{m-2}{t-2}
\end{vmatrix}, \quad \text{and} \quad \beta = \begin{vmatrix}
\frac{m}{r} & \frac{m}{t} \\
\frac{m-2}{r-2} & \frac{m-2}{t-2}
\end{vmatrix}.
$$

Thus the assertion is implied by

$$
\begin{vmatrix}
\frac{m-1}{r-1} \\
\frac{m}{s} & \frac{m}{t} \\
\frac{m-2}{s-2} & \frac{m-2}{t-2}
\end{vmatrix} + \begin{vmatrix}
\frac{m-1}{t-1} \\
\frac{m}{r} & \frac{m}{s} \\
\frac{m-2}{r-2} & \frac{m-2}{s-2}
\end{vmatrix} < \begin{vmatrix}
\frac{m}{r} \\
\frac{m}{s} \\
\frac{m-2}{r-2} & \frac{m-2}{t-2}
\end{vmatrix}.
$$

By taking a factor $\frac{m}{m-1} \frac{m-1}{r-1} \frac{m-1}{s-1} \frac{m-1}{t-1}$, we obtain

$$
\begin{vmatrix}
\frac{1}{s} & \frac{1}{t} \\
\frac{r}{s-1} & \frac{t}{t-1}
\end{vmatrix} + \begin{vmatrix}
\frac{1}{r} & \frac{1}{s} \\
\frac{r}{r-1} & \frac{s}{s-1}
\end{vmatrix} < \begin{vmatrix}
\frac{1}{r} & \frac{1}{t} \\
\frac{r}{r-1} & \frac{t}{t-1}
\end{vmatrix}.
$$

That is

$$
\frac{t-r}{s} < \frac{s-r}{t} + \frac{t-s}{r},
$$

which can be checked directly.

Now we prove the claim. Suppose that there are $0 \leq r, t \leq m$ with $t \geq r + 2$ such that $\varphi(r), \varphi(t) \neq 0$. Let $s$ be an integer such that $r < s < t$. Let $\alpha, \beta$ be the solution of (9). We take a coefficient $c$ such that $c\alpha \leq \varphi(r)$ and $c\beta \leq \varphi(t)$. Now we let $\varphi'$ be an assignment such that

$$
\varphi'(i) = \begin{cases}
\varphi(i) - c\alpha, & i = r; \\
\varphi(i) + c, & i = s; \\
\varphi(i) - c\beta, & i = t; \\
\varphi(i), & \text{otherwise}.
\end{cases}
$$

By Lemma 5.4 $\varphi'$ is an assignment with $v(\varphi') > v(\varphi)$, a contradiction.

**Claim 4.** There exist $0 \leq s, t \leq m$ with $\varphi(s), \varphi(t) > 0$ and satisfying that

$$
\left(\frac{m-2}{s-2}\right)\left(\frac{m}{s}\right) \leq \frac{l}{n} \quad \text{and} \quad \left(\frac{m-2}{t-2}\right)\left(\frac{m}{t}\right) \geq \frac{l}{n}.
$$
Moreover, the first inequality is strict if and only if the second inequality is strict.

Proof. If for all \( i \) with \( \varphi(i) > 0 \), either \( \frac{m - 2}{i} > \frac{1}{n} \); or \( \frac{m - 2}{t - 2} > \frac{1}{n} \) and there exists an integer \( t \) with \( \frac{m - 2}{t - 2} > \frac{1}{n} \), then by (8),

\[
\sum_{i=0}^{m} \frac{m - 2}{i} \varphi(i) > \sum_{i=1}^{m} \frac{m}{i} \frac{1}{n} \varphi(i) = 1,
\]
a contradiction. The second assertion can be proved similarly. \( \square \)

Now let \( r \) be the positive solution of \( \frac{m - 2}{r - 2} = \frac{1}{n} \). By Claims \( 3 \) and \( 4 \) if \( r \) is an integer, then \( s = t = r \); if \( r \) is not an integer, then \( s = \lfloor r \rfloor \) and \( t = \lceil r \rceil \). Now the theorem can be deduced by (8).

Proof of Theorem 3.4. We first show the limit part of the theorem. It is not difficult to check that the limit of the upper bound is one. For the limit of the lower bound, one can see that it is a consequence of the following lemma, which can be deduced from one result of Dusart in [11].

Lemma 5.5. \( \lim_{x \to \infty} \frac{p(x)}{x} = 1 \).

Now we show the upper bound and the lower bound of \( \kappa_{s_1}(n, n) \). For the upper bound, let \( F \) be a \( k \)-uniform \( \{0, 1\} \)-intersecting \( n \)-family of subsets of \( [n] \). By Theorem 3.2

\[
n \leq \frac{n(n-1)}{k(k-1)}.
\]

Thus we have \( k \leq \sqrt{n - 3/4} + 1/2 \).

For the lower bound, we first show the following claim.

Claim 5. Let \( p \) be a prime and let \( t < p \) be a positive integer. Then \( \kappa_{s_1}(p^2 - tp, p^2 - tp) \geq p - t \).

Proof. Set \( X = \{(x, y) : 0 \leq x, y < p\} \). From the proof of Theorem 5.3 we can see that the family \( F = \{F_{a,b} : 0 \leq a, b < p\} \) is a \( p \)-uniform \( \{0, 1\} \)-intersecting \( p^2 \)-family of subsets of \( X \), where

\[
F_{a,b} = \{(x, y) : y \equiv a + bx \pmod{p}\}.
\]

Let \( X' = \{(x, y) : 0 \leq x < p - t, 0 \leq y < p\} \). So \( X' \) is a subset of \( X \) of size \( p^2 - tp \). Let \( F' = \{F'_{a,b} : F_{a,b} \in F\} \). Clearly \( F' \) is a \( (p-t) \)-uniform \( \{0, 1\} \)-intersecting family with \( p^2 \geq p^2 - tp \) members. This implies that \( \kappa_{s_1}(p^2 - tp, p^2 - tp) \geq p - t \). \( \square \)

Now let \( p = p(\sqrt{n}) \) and \( t = p - \lfloor n/p \rfloor \). Thus \( n \geq p^2 - tp \). Recall that \( \kappa_{s_1}(n, n) \) is an increasing function for \( n \). By Claim 5

\[
\kappa_{s_1}(n, n) \geq \kappa_{s_1}(p^2 - tp, p^2 - tp) \geq p - t = \left\lfloor \frac{n}{p} \right\rfloor.
\]
**Definition 1.** A projective plane consists of a set of points, a set of lines, and a relation between points and lines called incidence, having the following properties:

1. Given any two distinct points, there is exactly one line incident with both of them;
2. Given any two distinct lines, there is exactly one point incident with both of them;
3. There are four points such that no line is incident with more than two of them.

It is not difficult to see that for every projective plane \( P \), there exists an integer \( q \) such that each point is incident with \( q + 1 \) lines and each line is incident with \( q + 1 \) points. Such an integer \( q \) is the order of \( P \). One can check that a projective plane of order \( q \) has \( q^2 + q + 1 \) points and \( q^2 + q + 1 \) lines. The following well-known result on the existence of finite projective planes will be used.

**Lemma 5.6** (see [7]). The projective plane of order \( q \) exists if \( q \) is prime power.

**Proof of Theorem 3.5.** Assume first that \( n = q^2 + q + 1 \). From Theorem 3.4 we have

\[
\kappa_{\leq 1}(q^2 + q + 1, q^2 + q + 1) \leq \sqrt{q^2 + q + \frac{1}{4} + \frac{1}{2}} = q + 1.
\]

Note that a projective plane of order \( q \) is a \((q+1)\)-uniform \(\{0,1\}\)-intersecting \((q^2 + q + 1)\)-family. Thus the equality holds in the above inequality.

Now assume that \( n \in [q^2, q^2 + q] \). From Theorem 3.4 we have \( \kappa_{\leq 1}(n, n) \leq q \). Let \( X = [n] = \{1, \ldots, n\} \) and let \( X' = \{1, \ldots, q^2 + q, q^2 + q + 1\} \). Since \( \kappa_{\leq 1}(q^2 + q + 1, q^2 + q + 1) = q + 1 \), there exists a \((q+1)\)-uniform family, say \( F' = \{F_1', \ldots, F_{q^2+q+1}'\} \) such that each two member of \( F \) intersects on at most one element. Assume without loss of generality that \( F_{q^2+q+1}' = \{q^2+1, \ldots, q^2+q+1\} \). For each \( 1 \leq i \leq n \), let \( F_i \) be a set obtained from \( F_i' \) by removing its largest number. Since \( |F_i' \cap F_{q^2+q+1}'| \leq 1 \), we have \( F_i \subseteq \{1, \ldots, q^2\} \subseteq X \). Clearly, \( F = \{F_1, \ldots, F_n\} \) is a \( q \)-uniform \(\{0,1\}\)-intersecting family. So \( \kappa_{\leq 1}(n, n) = q \) for \( n \in [q^2, q^2 + q] \).

\[\square\]

## 6 Concluding remarks

We conclude this paper by proposing a conjecture on estimating the maximum size of a member in a family among all uniform \( L \)-intersecting \( m \)-families of subsets of \([n] \) with \( m = n \) and \( L = \{0, 1, \ldots, l\} \).

**Conjecture 1.** Let \( l \geq 1 \) be an integer. Then \( \kappa_{\leq 1}(n, n) = (1 + o(1))n^{\frac{1}{l+1}} \), i.e.,

\[
\lim_{n \to \infty} \frac{\kappa_{\leq 1}(n, n)}{n^{1/(l+1)}} = 1.
\]

**Remark 1.** By Theorem 3.4 we have \( \lim_{n \to \infty} \frac{\kappa_{\leq 1}(n, n)}{\sqrt{n}} = 1 \), i.e., the conjecture holds for \( l = 1 \). Taking advantage of Theorem 3.1 we can obtain that

\[
\lim_{n \to \infty} \frac{\kappa_{\leq 1}(n, n)}{n^{1/(l+1)}} \leq 1.
\]

So it suffices to show that \( \lim_{n \to \infty} \frac{\kappa_{\leq 1}(n, n)}{n^{1/(l+1)}} \geq 1 \) for \( l \geq 2 \).
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