Generalized Solutions of a Nonlinear Parabolic Equation with Generalized Functions as Initial Data

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In [8] Brézis and Friedman prove that certain nonlinear parabolic equations, with the δ-measure as initial data, have no solution. However in [9] Colombeau and Langlais prove that these equations have a unique solution even if the δ-measure is substituted by any Colombeau generalized function of compact support. Here we generalize Colombeau and Langlais their result proving that we may take any generalized function as the initial data. Our approach relies on resent algebraic and topological developments of the theory of Colombeau generalized functions and results from [1].

Introduction

The necessity to prove the existence of solutions of equations may lead to the discovery of new and interesting mathematical structures. Observe that, in general, it takes time for these structures to be fully understood and appreciated by the mathematical community. One might say that the algebras of generalized numbers and functions, introduced in the eighties by J. F. Colombeau, are among these structures. They are natural environments where the multiplication of distributions is something well defined and equals the classical definition for $C^\infty$ functions. It should be noticed that regularization is definitely not the same as working in a Colombeau algebra. A Colombeau algebra is an algebraic and analytic environment!

It is well known that rather simple and non-pathological linear equations have no distributional solution. However, the Colombeau algebras are environments where the concept of derivation and that of solution of a P.D.E. can be generalized in a natural way allowing, in many cases, to prove the existence of new and interesting solutions for these equations (see [6] and [9]).

This was proved by Colombeau and stimulated research in this new field as one can see from the results obtained by the Austrian, Brazilian, French, Serbian and South African research groups and their collaborators (see [2] and [12]).

In the beginning of the eighties Brézis and Friedman showed that certain nonlinear parabolic equations have no solution if one chooses the initial data to be the δ-measure. These non existence results, new in those days, were considered rather surprising because of several facts carefully explained by Brézis and Friedman. An explanation for these non existence results was given in [9] by Colombeau and Langlais. Even more, they proved that the Brézis-Friedman equations do have a unique solution in the Colombeau algebra, as long as the initial data, which could be a distribution or Colombeau generalized function, had compact support. One natural question can be formulated: Is there still a solution if the initial data has non-compact support? Is this solution unique too?

Since Colombeau defined his new algebras the Theory of Colombeau Generalized Functions has undergone rapid developments. The algebraic and topological aspects of the theory were developed which, in their turn allowed further development of other parts of the theory and led, to the development of a differential calculus

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which behaves like classical calculus (see [4] and its reference list). A result obtained in [5] states that the set of

generalized functions of compact support is a dense ideal in the simplified Colombeau algebra.

To give an answer to the questions raised above we first generalize the result mentioned in the last paragraph,
i.e., we prove that the set of generalized functions of compact support is a dense ideal in the full Colombeau
algebra. Then, using the results of [1] on quasi-regular sets, we push further the topological stepping stone of
the theory. All this is done in sections 1 and 2. In the last section (section 3) we settle, in the positive, the two
questions raised above. Notation is mostly standard unless explicitly stated.

1 The completeness of the full Colombeau algebras

As said in the introduction, in order to solve our problem we will need to establish some topological facts
about the Colombeau algebras on the closure of an open set.

We begin by introducing a natural topology \( T_{\omega,b} \) on the algebra \( G(\omega) \), where \( \omega \) is a bounded open subset of
\( \mathbb{R}^m \). The basic facts about the algebra \( G(\Omega) \) are presented in [1] (for an arbitrary non-void open subset \( \Omega \) of
\( \mathbb{R}^m \)). The main results in this section are the completeness of \( (G(\omega), T_{\omega,b}) \) and, as an easy consequence, the
completeness of \( (G(\Omega), T_{\Omega}) \), where \( G(\Omega) \) (resp. \( T_{\Omega} \)) is defined in [1], 2.1 (resp. [3], Definitions 3.3 and 3.7
and Theorem 3.6)). In what follows, we will denote by \( E \) a non void open (resp. bounded open) subset of
\( \mathbb{R}^m \). We also set \( I_0 := [0, 1], I_\eta := [0, \eta], \eta \in 1, \mathbb{K} \) denotes \( \mathbb{R} \) or \( \mathbb{C} \), \( ||f|| := ||f||_\infty = sup \{ |f(x)| : x \in \omega \} \). If \( K \subset X \subset \mathbb{R}^m \), the symbol \( K \subset \subset X \) means that \( K \) is a compact subset of \( X \).

In the sequel we will use freely the partial order relation \( \leq \) on \( \mathbb{R} \) introduced in [3], Lemma 2.1 and Definition
2.2]. Let us recall that for a given \( \varphi \in D(\mathbb{R}^m; \mathbb{K}) \), \( \varphi \neq 0 \), we define

\[ i(\varphi) := \text{diam supp}(\varphi) \]

and it is easily seen that \( i(\varphi_\varepsilon) = \varepsilon i(\varphi) \) for all \( \varepsilon > 0 \). For every \( r \in \mathbb{R} \) the function [3, Ex.2.3]

\[ \tilde{\alpha}_r^* : \varphi \in \Lambda_0 \mapsto i(\varphi)^r \in \mathbb{R}_+^* \]

is moderated and therefore \( \alpha_r^* := \text{cl}(\tilde{\alpha}_r^*) \in \mathbb{R}_+^* \).

Since \( \omega \subset \subset \Omega \) the definitions of \( E_M(\omega) \) and \( N(\omega) \) (see [1], definition of \( E_M[X; \mathbb{K}] \) and Definition 1.2(c)))
becomes simpler: \( u \in E_M(\omega) \) means

\[ (M) \quad \forall \sigma \in \mathbb{N}^m \exists N \in \mathbb{N} \text{ such that } \forall \varphi \in \Lambda_N \exists c = c(\varphi) > 0 \text{ and } \eta = \eta(\varphi) \in I_\eta \text{ verifying } ||\partial^\sigma u(\varphi_\varepsilon, \cdot)|| \leq c\varepsilon^{-N}, \forall \varepsilon \in I_\eta; \]

and \( u \in N(\omega) \) means

\[ (N) \quad \forall \sigma \in \mathbb{N}^m \exists N \in \mathbb{N} \text{ and } \gamma \in \Gamma \text{ such that, } \forall q \geq N \text{ and, } \forall \varphi \in \Lambda_q \exists c = c(\varphi) > 0 \text{ and } \eta = \eta(\varphi) \in I_\eta \text{ verifying } ||\partial^\sigma u(\varphi_\varepsilon, \cdot)|| \leq c\varepsilon^{\gamma(q) - N}, \forall \varepsilon \in I_\eta; \]

In the remainder of this paper we shall use a fixed exhaustive sequence \( (\Omega_i)_{i \in \mathbb{N}} \) of open subsets of \( \Omega \) (see [3],
begin of section 2).

A natural topology on \( G(\omega) \) For a given \( f \in G(\omega) \), if \( \widehat{f} \in E_M(\omega) \) is any representative of \( f \) and \( \sigma \in \mathbb{N}^m \)
is arbitrary, we have

\[ \partial^\sigma \widehat{f}(\varphi, \cdot) \in C(\omega; \mathbb{K}), \forall \varphi \in \Lambda_0, \]

hence

\[ ||\partial^\sigma \widehat{f}(\varphi, \cdot)|| < +\infty, \forall \varphi \in \Lambda_0. \]
Lemma 1.1 For every $u \in \mathcal{E}_M[\varpi]$ and $\sigma \in \mathbb{N}^m$,

(a) The function

$$u_\sigma : \varphi \in A_0 \mapsto \|\partial^\sigma u(\varphi, \cdot)\| \in \mathbb{R}_+$$

is moderate (i.e., $u_\sigma \in \mathcal{E}_M(\mathbb{R})$ and so $cl(u_\sigma) \in \mathbb{R}_+$);

(b) If $v \in \mathcal{E}_M[\varpi]$ and $(u - v) \in \mathcal{N}[\varpi]$ then $(u_\sigma - v_\sigma) \in \mathcal{N}(\mathbb{R})$ (and so $cl(u_\sigma) = cl(v_\sigma)$).

Proof. Follows immediately from definitions and [1], Lemma 3.1.

Definition 1.2 Fix any $\sigma \in \mathbb{N}^m$. For each $f \in \mathcal{G}(\varpi)$ we set

$$\|f\|_\sigma := cl\left(\left.\frac{\partial^\sigma f(\varphi, \cdot)}{\partial^\sigma f(\varphi, \cdot)}\right|_{\varphi \in A_0} \mapsto \left.\|\partial^\sigma f(\varphi, \cdot)\| \right|_{\varphi \in \mathbb{R}_+} \in \mathbb{R}_+,$$

where $\hat{f}$ is an arbitrary representative of $f$. For each $r \in \mathbb{R}$ we define

$$W_{\sigma,r} = W_{\sigma,r}[0] := \{f \in \mathcal{G}(\varpi) | \|f\|_\kappa \leq \alpha^*_r, \forall \kappa \leq \sigma\}.$$

Remark 1.3 (a) If $\sigma \geq \sigma'$ (i.e. $\sigma_i \geq \sigma'_i$, $\forall \sigma \in \mathbb{N}^m$) then $W_{\sigma,r} \subset W_{\sigma',r}$ for each $r \in \mathbb{R}$;

(b) If $r > r'$ then $W_{\sigma,r} \subset W_{\sigma',r}$. So, in the filter basis $(W_{\sigma,r})$ $(\sigma \in \mathbb{N}^m$ and $r \in \mathbb{R})$ we can replace the condition $r \in \mathbb{R}$ by $r \in \mathbb{N}^*$.

In the sequel we will use the following trivial results about inequalities:

(a) If $x, y \in \mathbb{R}$ then $x \leq y$ in $\mathbb{R}$ if and only if $x \leq y$ in $\overline{\mathbb{R}}$;

(b) If $x, x_1, y$ and $y_1$ belong to $\overline{\mathbb{R}}$ with $0 \leq x \leq x_1$ and $0 \leq y \leq y_1$ then $0 \leq xy \leq x_1y_1$.

Lemma 1.4 For every $f, g \in \mathcal{G}(\varpi)$ we have

(a) $\|f + g\|_\sigma \leq \|f\|_\sigma + \|g\|_\sigma$;

(b) $\|fg\|_\sigma \leq \sum_{\kappa \leq \sigma} \binom{\sigma}{\kappa} \|f\|_{\kappa} \|g\|_{\sigma - \kappa}$.

Proof. (a) From Definition 1.2 we have $\|f + g\|_\sigma = cl(v)$, where

$$v : \varphi \in A_0 \mapsto \left.\|\partial^\sigma (\hat{f} + \hat{g})(\varphi, \cdot)\| \right|_{\varphi \in \mathbb{R}_+}$$

and $\hat{f}$ and $\hat{g}$ are any representatives of $f$ and $g$ respectively. From Definition 1.2 we have also $\|f\|_\sigma + \|g\|_\sigma = cl(u)$, where

$$u : \varphi \in A_0 \mapsto \left.\|\partial^\sigma \hat{f}(\varphi, \cdot)\| + \|\partial^\kappa \hat{g}(\varphi, \cdot)\| \right|_{\varphi \in \mathbb{R}_+}.$$

Since $v(\varphi) \leq u(\varphi)$ for every $\varphi \in A_0$, from [1], Lemma 2.1 and Definition 2.2) statement (a) follows.

(b) From Definition 1.2 it follows that $\|fg\|_\sigma = cl(v)$, where

$$v : \varphi \in A_0 \mapsto \left.\|\partial^\sigma (\hat{f} + \hat{g})(\varphi, \cdot)\| \right|_{\varphi \in \mathbb{R}_+}$$

($\hat{f}$ and $\hat{g}$ are arbitrary representatives of $f$ and $g$ respectively) and that $\sum_{\kappa \leq \sigma} \binom{\sigma}{\kappa} \|f\|_{\kappa} \|g\|_{\sigma - \kappa} = cl(u)$, where

$$u : \varphi \in A_0 \mapsto \sum_{\kappa \leq \sigma} \binom{\sigma}{\kappa} \left.\|\partial^\kappa \hat{f}(\varphi, \cdot)\| \right|_{\varphi \in \mathbb{R}_+} \|\partial^{\sigma - \kappa} \hat{g}(\varphi, \cdot)\| \in \mathbb{R}_+.$$

Now, from Leibnitz formula (see [1], [1.2], p.373) we have $v(\varphi) \leq u(\varphi)$ for all $\varphi \in A_0$ and therefore $cl(v) \leq cl(u)$, which is statement (b).
The two following lemmas will be useful and the proofs, which are trivial (apply \([3\), Lemma 2.1 and Example 2.3]), are omitted.

**Lemma 1.5**

(a) For every \(k,r \in \mathbb{R}_+^\ast\) we have \(k (\alpha_s^\ast)^2 \leq \alpha_s^\ast\) (resp. \(k \alpha_s^\ast \leq \alpha_s^\ast\)) whenever \(s > \frac{\tau}{2}\) (resp. \(s > r\));

(b) For every \(k,r \in \mathbb{R}_+^\ast\) and \(N \in \mathbb{N}\) there is \(s \in \mathbb{R}_+^\ast\) such that \(k \alpha_s^\ast \leq \alpha_r^\ast\) (it is sufficient to choose \(s \geq N + r + 1\)).

**Lemma 1.6**

For every \(g \in \mathcal{G}(\mathcal{W})\) and \(\sigma \in \mathbb{N}^m\) there are \(c > 0\) and \(N \in \mathbb{N}\) such that \(\|g\|_\kappa \leq c \alpha_s^\ast\) for each \(\kappa \leq \sigma\).

With the notation introduced in Definition [1.2] we have the following result.

**Theorem 1.7**

Let \(\omega\) be an open bounded subset of \(\mathbb{R}^m\). Then the set

\[
\mathcal{B}_{\omega,b} := \{W_{\sigma,r}| \sigma \in \mathbb{N}^m \text{ and } r \in \mathbb{R}_+^\ast\}
\]

is a filter basis on \(\mathcal{G}(\mathcal{W})\) which satisfies the seven conditions of \([3\), Proposition 1.2 (2\(\alpha\))] and so determine a topology \(\mathcal{T}_{\omega,b}\) on \(\mathcal{G}(\mathcal{W})\) which is compatible with its \(\mathbb{K}\)-algebra structure (here we assume that \(\mathbb{K}\) is endowed with its topology (see \([3\), Definition 2.10]). Moreover the topology \(\mathcal{T}_{\omega,b}\) is metrizable.

In the proof below of Theorem 1.7 for the sake of simplicity, we write \(B\) instead of \(\mathcal{B}_{\omega,b}\). Also we will use freely the notation \((\text{GA}_1), \ldots, (\text{AV}_H)\) for the seven condition in \([3\), Proposition 1.2 (2\(\alpha\)]). Note also that the proof below works for \(\mathcal{B}_{\omega,b}\).

**Proof.** In view of \([3\), Corollary 1.3] it is enough to show that \(B\) is a filter basis which satisfies the four conditions (\(\text{GA}_1\), \(\text{GA}_2\), \(\text{AV}_1\) and \(\text{AV}_H\)) of \([3\), Proposition 1.2] and that the topology \(\mathcal{T}_{\omega,b}\) determine by \(B\) induces on \(\mathbb{K}\) its own topology \(T\) (see \([3\), Definition 2.10]). That \(B\) is a filter basis follows at once since clearly \(B \neq \emptyset\) and \(\emptyset \notin B\) and, if \(W_{\sigma,r}, W_{\tau,s}\) are any two elements of \(B\) then, by defining \(\kappa := \max (\sigma, \tau)\) (i.e., \(\kappa_i := \max (\sigma_i, \tau_i), \forall i = 1, 2, \ldots, m\)) and \(\ell := \max (r, s) + 1\), from Remark [1.3] we have \(W_{\kappa,\ell} \subset W_{\sigma,r} \cap W_{\tau,s}\).

**Verification of** (\(\text{GA}_1\)):

Given any \(W_{\sigma,r} \in B\) it is enough to show that for any \(s > \frac{\tau}{2}\) we have \(W_{\sigma,s} \subset W_{\sigma,r} \subset W_{\sigma,r}\), which follows immediately from Lemma [1.2] (a) and Lemma [1.5] (a).

**Verification of** (\(\text{GA}_2\)):

Obvious since \(U = -U\) for all \(U \in B\).

**Verification of** (\(\text{AV}_H\)):

It suffices to prove that \(W_{\sigma,r}^2 \subset W_{\sigma,r}\) for each \(W_{\sigma,r} \in B\). Fixed any \(W_{\sigma,r} \in B\), for \(\kappa \in \mathbb{N}^m\), let:

\[
M := \max_{r \leq \kappa} \binom{\kappa}{r}; \quad M := \max M_{\kappa}; \quad p_{\kappa} := \text{number of terms of } \sum_{r \leq \kappa} \binom{\kappa}{r}; \quad p := \max_{\kappa \leq \sigma} p_{\kappa} \text{ and } k := M \cdot p.
\]

We must show that

\[
f, g \in W_{\sigma,r} \implies f g \in W_{\sigma,r}.
\]

Indeed, the assumption in [1.1] means that \(\|f\|_\lambda \leq \alpha_s^\ast\) and \(\|g\|_\lambda \leq \alpha_s^\ast\) for all \(\lambda \leq \sigma\) hence, from Lemma [1.4] (b) and Lemma [1.5] (a) [and the obvious fact: \(a, b \in \mathbb{R}_+^\ast, a \leq b, \lambda, \mu \in \mathbb{R}_+^\ast\) and \(\lambda \leq \mu \implies a\lambda \leq b\mu\)] we can conclude that for every \(\kappa \leq \sigma\)

\[
\|f g\|_\kappa \leq \sum_{r \leq \kappa} \binom{\kappa}{r} \|f\|_r \|g\|_{\kappa - r} \leq p_{\kappa} M_{\kappa} (\alpha_r^\ast)^2 \leq p M (\alpha_r^\ast)^2 \leq k (\alpha_r^\ast)^2 \leq \alpha_r^\ast
\]

since \(r > \frac{\tau}{2}\). Therefore (1) holds.

\footnote{Note that, in view of Remark [1.2] (b) the set \(\mathcal{B}_{\omega,b} := \{W_{\sigma,r}| \sigma \in \mathbb{N}^m \text{ and } r \in \mathbb{R}_+^\ast\}\) is another fundamental system of \(\mathcal{T}_{\omega,b}\)-neighborhoods of 0 in \(\mathcal{G}(\mathcal{W})\).}
**Verification of (AV')**: For given \( g \in \mathcal{G}(\varpi) \) and \( W_{\sigma,r} \in \mathcal{B} \), we will show that there exists \( W_{\tau,s} \in \mathcal{B} \) such that \( gW_{\tau,s} \subset W_{\sigma,r} \). Indeed, from Lemma 1.6 it follows that we can find \( c > 0 \) and \( N \in \mathbb{N} \) such that

\[
\|g\|_{\tau} \leq c\alpha_{-N}, \quad \forall \, \tau \leq \sigma. \tag{2}
\]

Let \( M, p, M_{r}, \alpha_{\sigma} \) be as in the preceding proof of (AV') and define \( k := pM_{c} \) (from (2)). Now, associated to \( k, \, r \) (which appears in \( W_{\sigma,r} \)) and \( N \in \mathbb{N} \) (which appears in (2)), from Lemma 1.5 (b) we find \( s := N + r + 1 \), for instance verifying

\[
k\alpha_{-N}.\alpha_{N+r+1} \leq \alpha_{r}. \tag{3}
\]

Next, we define \( W_{\tau,s} \) by \( \tau := \sigma \) and \( s := N + r + 1 \), that is, \( W_{\tau,s} := W_{\sigma,N+r+1} \) and it remains to prove that \( gW_{\tau,s} \subset W_{\sigma,r} \). In fact, fix any \( f \in W_{\tau,s} = W_{\sigma,N+r+1} \):

\[
\|f\|_{\tau} \leq \alpha_{N+r+1}, \quad \forall \, \tau \leq \sigma
\]

then, from Lemma 1.4 (b), (2), (4), (6) and the definition of \( k \), we get for every \( \kappa \leq \sigma \):

\[
\|gf\|_{\kappa} \leq \sum_{\tau \leq \kappa} \left( \frac{\kappa}{\tau} \right) \|g\|_{\tau} \|f\|_{\kappa-\tau} \leq pM \left( \alpha_{-N} \right) (\alpha_{N+r+1}) = k\alpha_{-N}.\alpha_{N+r+1} \leq \alpha_{r},
\]

hence \( gf \in W_{\sigma,r} \) and therefore (AV') holds.

Since \( \mathcal{B} \) satisfies (GA'), (GA''), (AV') and (AV') from [3, Corollary 1.3], we know that \( \mathcal{B} \) determines a topology \( \mathcal{T}_{\mathcal{B},b} \) on \( \mathcal{G}(\varpi) \) which is compatible with the ring structure of \( \mathcal{G}(\varpi) \). Moreover, it is clear (see [3, Definition 2.7]) that \( W_{\sigma,r} \cap \mathbb{K} = V_{\tau}[0] \) for every \( \sigma \in \mathbb{N}^{m} \) and \( r \in \mathbb{R}_{+}^{*} \), which implies that the topology induced by \( \mathcal{T}_{\mathcal{B},b} \) on \( \mathbb{K} \) coincides with the topology \( T \) (see [3, Definition 2.10]). Therefore, once more from [3, Corollary 1.3], we can conclude that \( \mathcal{T}_{\mathcal{B},b} \) is compatible with the structure of \( \mathbb{K} \)-algebra of \( \mathcal{G}(\varpi) \), where \( \mathbb{K} \) is endowed with its own topology \( T \). The topology \( \mathcal{T}_{\mathcal{B},b} \) is Hausdorff since obviously

\[
\|f\|_{\sigma} = 0, \quad \forall \, \sigma \in \mathbb{N}^{m} \implies f \equiv 0
\]

and hence \( \bigcap_{\sigma} W_{\sigma,r} = \{0\} \). Indeed, if \( f \in \bigcap_{\sigma} W_{\sigma,r} \) we have

\[
\|f\|_{r} \leq \alpha_{r}, \quad \forall \, \sigma, r \iff \|f\|_{\sigma} \in V_{\tau}[0], \quad \forall \, \sigma, r \iff \forall \, \sigma, \text{ we have } \|f\|_{\sigma} \in \bigcap_{r>0} V[0] = \{0\},
\]

since \( T \) is Hausdorff. Finally, it is clear that \( \mathcal{B}_{\mathcal{B},b} \) and \( \mathcal{B}_{\mathcal{B},b}^{\prime} \) generate the same filter (of all the \( \mathcal{T}_{\mathcal{B},b} \)-neighborhoods of 0) and since \( \mathcal{B}_{\mathcal{B},b} \) is countable, it follows that (see [2, chapter 9; 2 ed., section 1, n\O 4, Proposition 2]) \( \mathcal{T}_{\mathcal{B},b} \) is metrizable.

**Lemma 1.8** Let \( \Omega \) and \( \omega \) be two open subsets of \( \mathbb{R}^{m} \) such that \( \varpi \subset \subset \Omega \). Then the restriction map

\[
r = r_{\varpi}^{\Omega} : f \in \mathcal{G}(\Omega) \mapsto f|_{\varpi} \in \mathcal{G}(\varpi)
\]

is continuous with respect to the topologies \( T_{\Omega} \) and \( T_{\mathcal{B},b} \) on \( \mathcal{G}(\Omega) \) and \( \mathcal{G}(\varpi) \) respectively.

**Proof.** Fix an arbitrary \( W_{\sigma,r} \in \mathcal{B}_{\mathcal{B},b} \). It suffices to show that for any \( l \in \mathbb{N} \) such that \( \varpi \subset \subset \Omega_{l} \) we have \( r \left( W_{\sigma,r}^{l} \right) \subset W_{\sigma,r} \), which follows at once from the definitions (see the definition of \( W_{\sigma,r}^{l} \) in [3, Definition 3.3]).

In the next results we denote by \( \mathbb{K}_{\text{Top Alg}} \) the category whose object are the \( \mathbb{K} \)-topological algebras with the natural morphisms.
Proposition 1.9 The topology $T_\Omega$ on $\mathcal{G}(\Omega)$ is the initial topology (in the category $\text{KTop Alg}$) for the family of homomorphisms $(r_\ell)_{\ell \in \mathbb{N}}$ where

$$r_\ell = r_\ell^\Omega : f \in \mathcal{G}(\Omega) \mapsto f|_{\Omega_\ell} \in \mathcal{G}(\Omega_\ell)$$

and $\mathcal{G}(\Omega_\ell)$ is endowed with the topology $T_{\Omega_\ell}$, where $\mathcal{G}(\Omega)$ is the coarsest topology on $\mathcal{G}(\Omega)$, compatible with the structure of $\text{K}$-algebra of $\mathcal{G}(\Omega)$, for which all the maps $r_\ell \ (\ell \in \mathbb{N})$ are continuous.

Proof. Let denote by $T^*_\Omega$ the initial topology on $\mathcal{G}(\Omega)$ for the family $(r_\ell)_{\ell \in \mathbb{N}}$ (in the category $\text{KTop Alg}$). Then it is well known that $T^*_\Omega$ has a fundamental system of $0$-neighborhoods $\mathcal{B}^*_\Omega$ consisting of sets of the following type:

$$V = \bigcap_{1 \leq i \leq p} r_i^{-1} \left( W_{\sigma_i, r_i}^{(l_i)} \right)$$

where $p \in \mathbb{N}^*$, $(l_i)_{1 \leq i \leq p}$, $(\sigma_i)_{1 \leq i \leq p}$ and $(r_i)_{1 \leq i \leq p}$ are finite sequences in $\mathbb{N}$, $\mathbb{N}^m$ and $\mathbb{N}$ respectively and $W_{\sigma,r}^{(l)} \in \mathcal{B}_{\Omega,b}^*$ (here we need the upper index $(l)$ since we are working with the subset $W_{\sigma,r}$ of $\mathcal{G}(\Omega)$ which is denoted in $W_{\sigma,r}^{(l)}$). From Lemma 1.8 it follows that all the maps $r_\ell \ (\ell \in \mathbb{N})$ are continuous when $\mathcal{G}(\Omega)$ is endowed with the topology $T_\Omega$ and therefore $T^*_\Omega \leq T_\Omega$. In order to prove that $T^*_\Omega \leq T_\Omega$ it suffices to show that

$$\forall W_{\sigma,r}^* \in \mathcal{B}_\Omega \exists V \in \mathcal{B}^*_\Omega \text{ such that } V \subset W_{\sigma,r}^*.$$  

In fact, we can prove the following more precise statement

$$r_i^{-1} \left( W_{\sigma,r}^{(l)} \right) = W_{\sigma,r}^* \ (\forall \sigma, l, r);$$

note that the first member of (4) is of the type (5). Now it is clear that (4) follows immediately from the definitions. \hfill \Box

Lemma 1.10 Let $\omega$ and $\omega_1$ be two bounded open subsets of $\mathbb{R}^m$ such that $\omega_1 \subset \omega$. Then the restriction map (see [11], Definition 2.6)

$$r_{\omega_1} : f \in \mathcal{G}(\omega) \mapsto f|_{\omega_1} \in \mathcal{G}(\omega_1)$$

is continuous when $\mathcal{G}(\omega)$ and $\mathcal{G}(\omega_1)$ are endowed with the topologies $T_{\omega,b}$ and $T_{\omega_1,b}$ respectively.

Proof. For given $W_{\sigma,r}^1 \in \mathcal{B}_{\omega_1,b}$ one has that $r_{\omega_1}^{-1} (W_{\sigma,r}^1) \subset W_{\sigma,r}^1$, where

$$W_{\sigma,r} := \{ f \in \mathcal{G}(\omega) \mid \| f \|_\tau \leq \alpha_{\tau}, \ \forall \tau \leq \sigma \}$$

belongs to $\mathcal{B}_{\omega,b}$. \hfill \Box

The following result is an adaptation to our case of the category of the topological metrizable $\text{K}$-algebras of [11], Chapter 2, section 11, Proposition 3.

Proposition 1.11 Let $A$ be a metrizable topological ring, $(F_p)_{p \in \mathbb{N}}$ a sequence of metrizable topological $A$-algebras and assume that the condition below holds:

(a) If $p, q \in \mathbb{N}$ and $p \leq q$ then there exists a continuous homomorphisms of $A$-algebras $f_{pq} : F_q \to F_p$.

Now, for a given $A$-algebra $E$ we assume that the following two conditions hold:

(b) For each $p \in \mathbb{N}$ there exists an homomorphism of $A$-algebras $f_p : E \to F_p$ such that $p, q \in \mathbb{N}$ and $p \leq q$ implies $f_{pq} = f_{pq} \circ f_q$.

(c) If $(x_p)_{p \in \mathbb{N}} \in \prod_{p \in \mathbb{N}} F_p$ and for each $p, q \in \mathbb{N}$ with $p \leq q$ we have $f_{pq}(x_q) = x_p$, then there is $x \in E$ such that $f_p(x) = x_p$ for every $p \in \mathbb{N}$. Moreover, assume that $E$ is endowed with the initial topology $T$ for the sequence $(f_p)_{p \in \mathbb{N}}$ [in the category of the metrizable topological $A$-algebras] and that $T$ is metrizable.
The if every $F_p$ is complete, $E$ is complete.

**Proof.** Let $(y_\nu)_{\nu \in \mathbb{N}}$ be a Cauchy sequence in $E$ then by defining

$$y^p_\nu := f_p(y_\nu) \in F_p$$

(7)

it is clear that $(y^p_\nu)_{\nu \in \mathbb{N}}$ is a Cauchy sequence in $F_p$ for each $p \in \mathbb{N}$ and hence

$$\exists x_p := \lim_{\nu \to \infty} y^p_\nu \in F_p.$$  

(8)

Now, we will prove that $x_p = f_{pq}(x_q)$ whenever $p, q \in \mathbb{N}$ and $p \leq q$. Fix $p, q \in \mathbb{N}$ with $p \leq q$ arbitrary. Since $y^q_\nu \to x_q$ in $F_q$ the continuity of $f_{pq}$ shows that

$$f_{pq}(y^q_\nu) \to f_{pq}(x_q).$$

(9)

On the other hand, from (7) and the condition (b) we get

$$f_{pq}(y^p_\nu) = f_{pq}(f_q(y_\nu)) = f_p(y_\nu) = y^p_\nu \to x_p$$

(10)

and then, $f_{pq}(x_q) = x_p$ follows from (9) and (10). Therefore, the hypothesis (c) implies that there exists $x \in E$ such that

$$f_p(x) = x_p, \ \forall p \in \mathbb{N}.$$  

(11)

Next, fix an arbitrary $T$-neighborhood $W$ of 0 in $E$ that we can choose of the form $W := \bigcap_{1 \leq k \leq n} f_{pk}^{-1}(U_k),$ where $U_k$ is a 0-neighborhood in $F_{pk}$ ($1 \leq k \leq n$). From (8) it follows that for each $k = 1, 2, \ldots, n$ there is $l_k \in \mathbb{N}$ so that

$$\nu \geq l_k \implies (y^p_\nu - x_{pk}) \in U_k$$

(12)

and therefore, by defining $\nu_0 := \max_{1 \leq k \leq n} l_k$, we can conclude from (7), (11) and (12) that

$$\nu \geq \nu_0 \implies (y_\nu - x) \in W,$$

hence $\lim_{\nu \to \infty} y_\nu = x$.  

**Theorem 1.12** If $\omega$ is a bounded open subset of $\mathbb{R}^m$ then $G(\omega)$ endowed with the topology $T_{\omega,b}$ is complete.
Since $G$ is a sheaf, there exists $f \in G(\Omega)$ such that $f|_{\Omega_l} = f_l, \ \forall l \in \mathbb{N}$ and therefore

$$f|_{\Omega_{l+1}} = f_{l+1}|_{\Omega_{l+1}},$$

hence

$$f|_{\Omega_l} = (g_{l+1}|_{\Omega_{l+1}})|_{\Omega_l} = g_{l+1}|_{\Omega_l} = g_l, \ \forall l \in \mathbb{N},$$

which shows that condition (c) of Proposition 1.11 holds in our case.

We can then conclude that we have the following consequence of Proposition 1.11 and Theorem 1.12.

**Corollary 1.13** If $\Omega$ is an open subset of $\mathbb{R}^m$ then $G(\Omega)$ endowed with the topology $T_\Omega$ is complete.

The remainder of this section is essentially devoted to the rather long proof of Theorem 1.12 and to some auxiliary results which will need in the last section.

In order to prove Theorem 1.12 we start from some easy remarks which will be useful later.

**Remark 1.14** (a) Let $J$ be an ideal of a commutative ring $A$ with identity and assume that $\mathbb{Q} \subset A$. Then for each $a, b \in A/J$ and for every representative $\hat{a}$ of $a - b$ there are representatives $\hat{a}$ and $\hat{b}$ of $a$ and $b$ respectively such that $\hat{a} = (\hat{a} - \hat{b})$.

(b) Let $(x_\kappa)_{\kappa \geq 1}$ be a Cauchy sequence in an abelian topological metrizable group $G$ and $(W_\kappa)_{\kappa \geq 2}$ a sequence of $0$-neighborhoods in $G$ such that $W_{\kappa+1} \subset W_\kappa$ for all $\kappa \geq 2$. Then there is a subsequence $(x_{\kappa_i})_{\kappa_i \geq 1}$ such that $(x_{\kappa_i} - x_\kappa) \in W_{\kappa_i+1}$ for $\kappa_i \geq 1$.

In the result below, for $\kappa \in \mathbb{N}^m$ we will use the following function $\psi_\kappa$ defined by $\psi_\kappa(x) := (\kappa!)^{-1} x^\kappa, \forall x \in \mathbb{R}^m$ and, of course, the restriction $\psi_\kappa|_\kappa$ will be also denoted by $\psi_\kappa$. Note that $\partial^\kappa \psi_\kappa(x) \equiv 1, \forall x \in \mathbb{R}^m$ and $\partial^\kappa \psi_\kappa \in N[\mathbb{R}]$ for every $\hat{\kappa} \in N[\mathbb{R}]$ where $\partial^\kappa \psi_\kappa(\varphi, x) := \hat{\kappa}(\varphi) \psi_\kappa(x), \forall (\varphi, x) \in A_0 \times \mathbb{R}$.

In what follows we will often use the notation introduced in Definition 1.12 and Lemma 1.11.

**Lemma 1.15** Let $\omega$ be an open bounded subset of $\mathbb{R}^m$, $g \in G(\mathbb{R})$ and fix an arbitrary representative $\hat{g}$ of $g$. Then the statements below hold:

**I** Assume that $\kappa \in \mathbb{N}^m$ and that $\hat{R}$ is any representative of $\|g\|^\kappa_\kappa$. Then there are a representative $\hat{g}^{(\kappa)}$ of $g$ and $\hat{\kappa} \in N(\mathbb{R})$ verifying the following conditions:

(a) $$(\hat{g}^{(\kappa)}\kappa_\kappa) = \hat{R} \ (i.e. \ |\partial^\kappa \hat{g}^{(\kappa)}(\varphi, \cdot)| = \hat{R}(\varphi), \forall \varphi \in A_0);$$

(b) $\hat{g}^{(\kappa)} = \hat{g} + \hat{\kappa} \psi_\kappa$;

(c) $$(\hat{g}^{(\kappa)}\kappa_\kappa) = \hat{\kappa} + \hat{\kappa};$$

(d) If $g^{(\kappa)} := \hat{g} + \left(\hat{\kappa} \psi_\kappa \right)$, then $\hat{R} = (g^{(\kappa)}) + |\hat{\kappa}|$.

**II** For a given $(\sigma, r) \in \mathbb{N}^m \times \mathbb{N}^*$ the conditions below are equivalent:

(i) $\|g\|^\kappa_\kappa \leq \sigma^*_r$;

(ii) There exist $\hat{\kappa} \in N(\mathbb{R})$ and a representative $g^{(\kappa)}$ of $g$ such that

$$\left|\partial^\kappa g^{(\kappa)}(\varphi, \cdot)\right| + |\hat{\kappa}(\varphi)| \leq \sigma^*_r(\varphi), \forall \varphi \in A_0$$

(13)

and, moreover, $g^{(\kappa)}$ is given [as function of $\hat{\kappa}$ and of the representative $\hat{g}$ fixed at the beginning] by (I) (d).

**III** For a given $(\sigma, r) \in \mathbb{N}^m \times \mathbb{N}^*$ the conditions below are equivalent:

(i) $g \in W_{\sigma,r}$.
(ii) There exist a finite sequence \((g^{*(k)})_{k \leq \sigma}\) of representatives of \(g\) and a finite sequence \((\tilde{\theta}_k)_{k \leq \sigma}\) in \(\mathcal{N}(\mathbb{R})\) such that
\[
\left\| \partial^{*} g^{*(k)} (\varphi, \cdot) \right\| + \left| \tilde{\theta}_k (\varphi) \right| \leq \tilde{a}_r^* (\varphi), \forall \varphi \in A_0, \forall k \leq \sigma. \tag{14}
\]

Note that \((13)\) means that \(g^{*(k)} + \tilde{\theta}_k \leq \tilde{a}_r^*\) hence, from \((I)\) \((a), (c)\) and \((d)\) it is clear that \((13)\) is equivalent to
\[
\left( g^{*(k)} \right)_{k} = \tilde{g}_k + \tilde{\theta}_k \leq \tilde{a}_r^*. \tag{15}
\]

The notation \(g^{*(k)}, \tilde{g}^{*(k)}\) and \(\tilde{\theta}_k\) emphasize that these functions depend on \(\tilde{g}\) and \(\varphi\). Clearly \((14)\) is equivalent to
\[
\left( g^{*(k)} \right)_{k} (\varphi) = \tilde{g}_k (\varphi) + \tilde{\theta}_k (\varphi), \forall \varphi \in A_0, \forall k \leq \sigma. \tag{16}
\]

**Proof of Lemma 1.15.** \((I)\): Since \(\tilde{g}\) is a representative of \(g\), from the definition of \(\|g\|_{\varphi}\) (see Definition \(1.2\)) it follows that there is a unique \(\tilde{\theta}_k \in \mathcal{N}(\mathbb{R})\) such that
\[
\tilde{R} = \tilde{g}_k + \tilde{\theta}_k. \tag{17}
\]

Next, define \(\tilde{g}^{*(k)}\) by the identity in \((b)\) then for each \((\varphi, x) \in A_0 \times \mathcal{W}\) we have
\[
\partial^{*} \tilde{g}^{*(k)} (\varphi, x) = \partial^{*} \tilde{g} (\varphi, x) + \tilde{\theta}_k (\varphi)
\]

hence
\[
\left( g^{*(k)} \right)_{k} (\varphi) = \tilde{g}_k (\varphi) + \tilde{\theta}_k (\varphi), \forall \varphi \in A_0
\]

which proves statement \((c)\) and, from \((17)\), also statement \((a)\). Statement \((d)\) is also trivial since the definition of \(g^{*(k)}\) shows that
\[
g^{*(k)} + \left| \tilde{\theta}_k \right| \psi_k = \tilde{g} + \tilde{\theta}_k \psi_k
\]

hence
\[
\partial^{*} g^{*(k)} (\varphi, x) + \left| \tilde{\theta}_k (\varphi) \right| = \partial^{*} \tilde{g} (\varphi, x) + \tilde{\theta}_k (\varphi), \forall (\varphi, x) \in A_0 \times \mathcal{W}
\]

and so
\[
\left( g^{*(k)} \right)_{k} (\varphi) = \tilde{g}_k (\varphi) + \tilde{\theta}_k
\]

and therefore \((d)\) follows from \((17)\).

\((II)\): \((i) \Rightarrow (ii)\): Condition \((i)\) means (see [3], Lemma 2.1 (iii))] that there exists a representative \(\tilde{u}\) of \(\|g\|_{\varphi} - \alpha_r^*\) such that
\[
\tilde{u}(\varphi) \leq 0 \forall \varphi \in A_0. \tag{18}
\]

From Remark \(1.14(a)\) there are representatives \(\tilde{R}_1\) and \(\tilde{R}_2\) of \(\|g\|_{\varphi}\) and \(\alpha_r^*\) respectively such that \(\tilde{u} = \tilde{R}_1 - \tilde{R}_2\). Since \(\tilde{g}\) is a representative of \(g\) it follows that \(\tilde{g}_k\) is a representative of \(\|g\|_{\varphi}\) hence
\[
\tilde{n}_k := \tilde{R}_1 - \tilde{g}_k \in \mathcal{N}(\mathbb{R})
\]

which shows that \(\tilde{u} = \tilde{g}_k + \tilde{n}_k\) and therefore from \((18)\) we get
\[
\tilde{u}(\varphi) = \tilde{g}_k (\varphi) + \tilde{n}_k (\varphi) - \tilde{R}_2 (\varphi) \leq 0, \forall \varphi \in A_0. \tag{19}
\]

Since \(\tilde{h} := \tilde{a}_r^* - \tilde{R}_2 \in \mathcal{N}(\mathbb{R})\) it follows that \(\tilde{\theta}_k := \tilde{h} + \tilde{n}_k \in \mathcal{N}(\mathbb{R})\) hence
\[
\tilde{R} := \tilde{g}_k + \tilde{\theta}_k \tag{20}
\]
is a representative of $\|g\|_{\infty}$ and therefore, the proof of (I) implies that $\hat{g}^{(\nu)} := \hat{g} + \hat{\theta}_{\infty} \psi_{\infty}$ (see (I) (b)) satisfies

$$
\hat{R} = \left( \hat{g}^{(\nu)} \right)_{\infty} = \hat{g} + \hat{\theta}_{\infty}.
$$

Consequently

$$
\left( \hat{g}^{(\nu)} \right)_{\infty} - \hat{a}^* = \left( \hat{g} + \hat{\theta}_{\infty} \right) - \left( \hat{R}_2 + \hat{h} \right)
= \left( \hat{g} + \hat{h} + \hat{n}_{\infty} \right) - \left( \hat{R}_2 + \hat{h} \right)
= \hat{g} + \hat{n}_{\infty} - \hat{R}_2
$$

and therefore, from (19), we get

$$
\left( \hat{g}^{(\nu)} \right)_{\infty} (\nu) \leq \hat{a}^* (\nu), \forall \nu \in A_0.
$$

Finally, from (I) (d) we have $\hat{R} = \left( g^{(\nu)} \right)_{\infty} + \left| \hat{\theta}_{\infty} \right|$ which together with the first identity of (21) implies that (22) become

$$
\left( g^{(\nu)} \right)_{\infty} (\nu) + \left| \hat{\theta}_{\infty} \right| \leq \hat{a}^* (\nu), \forall \nu \in A_0,
$$

which proves (ii).

(ii) $\Rightarrow$ (i): Since $\left( g^{(\nu)} \right)_{\infty} + \left| \hat{\theta}_{\infty} \right|$ is a representative of $\|g\|_{\infty}$, by [3, Lemma 2.1 (iii)] the statement implies (i).

(III) From the definition of $W_{\sigma, r}$ condition (i) is equivalent to

$$(i') \quad \|g\|_{\infty} \leq \alpha^*_r, \forall \infty \leq \sigma.
$$

From (II), for each fixed $\infty \leq \sigma$, the inequality

$$
\|g\|_{\infty} \leq \alpha^*_r
$$

is equivalent to the statement

$$
\exists \infty \in N (\mathbb{R}), \text{ and } \exists \text{ a representative } g^{(\nu)} \text{ of } g \text{ such that (13) holds},
$$

hence (i') is equivalent to (III) (ii).

In what follows we will use the following notation

$$
\sigma_i := (i, i, \ldots, i) \in \mathbb{N}^m, \forall i \in \mathbb{N}.
$$

**Lemma 1.16** Let $(f_{\nu})_{\nu \geq 1}$ be a Cauchy sequence in $(G (\mathbb{T}), T_{\infty})$. Then there exists a subsequence $(f_{\nu_i})_{i \geq 1}$ of $(f_{\nu})_{\nu \geq 1}$ such that if $\hat{f}_{\nu_i}$ is an arbitrary representative of $f_{\nu_i}$ for each $i \geq 1$, then for each $i \geq 1$ there are a finite sequence $\left( R^i_{\theta_{\infty}} \right)_{\infty \leq \sigma_{i+1}}$ of representatives of $f_{\nu_{i+1}} - f_{\nu_i}$ and finite sequence $\left( \hat{\theta}_{\infty} \right)_{\infty \leq \sigma_{i+1}}$ in $N (\mathbb{R})$ such that

$$
\hat{f}_{\nu_{i+1}} - \hat{f}_{\nu_i} = R^i_{\theta_{\infty}} + \left( \left| \hat{\theta}_{\infty} \right| - \hat{\theta}_{\infty} \right) \psi_{\infty}, \forall i \geq 1, \forall \infty \leq \sigma_{i+1};
$$

$$
\|g^\infty R^i_{\theta_{\infty}} (\nu, \cdot)\| + \left| \hat{\theta}_{\infty} (\nu) \right| \leq \hat{a}^*_{i+1} (\nu), \forall i \geq 1, \forall, \infty \leq \sigma_{i+1}, \forall \nu \in A_0;
$$

$$
\|g^\infty (\hat{f}_{\nu_{i+1}} - \hat{f}_{\nu_i}) (\nu, \cdot)\| \leq 2.\hat{a}^*_{i+1} (\nu), \forall i \geq 1, \forall \infty \leq \sigma_{i+1}, \forall \nu \in A_0.
$$
Proof. Since \((W_{\sigma,i})_{i \geq 1}\) is a decreasing sequence of 0-neighborhoods in \((\mathcal{G}(\mathcal{W}), T_{\mathcal{W},b})\), from Remark 1.14 (b) it follows that we can find a subsequence \((f_{\nu_i})_{i \geq 1}\) of \((f_{\nu})_{\nu \geq 1}\) such that

\[
R_i := (f_{\nu_{i+1}} - f_{\nu_i}) \in W_{\sigma_{i+1},\sigma_i}, \forall \ i \geq 1.
\] (26)

Fix an arbitrary representative \(\hat{f}_{\nu_i}\) of \(f_{\nu_i}\) for each \(i \geq 1\). We shall apply Lemma 1.15 (III) and (26) to the representative

\[
\hat{R}_i := \hat{f}_{\nu_{i+1}} - \hat{f}_{\nu_i}
\]

of \(R_i\). Then we can find a finite sequence \(\left(\hat{\theta}_{i\kappa}\right)_{\kappa \leq \sigma_{i+1}}\) in \(N(\mathbb{R})\) and a finite sequence \(\left(R_i^{(\kappa)}\right)_{\kappa \leq \sigma_{i+1}}\) of representatives of \(R_i\) such that

\[
\left\|\partial^{\kappa} R_i^{(\kappa)}(\varphi,\cdot)\right\| + \left|\hat{\theta}_{i\kappa}(\varphi)\right| \leq \hat{\alpha}_{i+1}^{\bullet}(\varphi), \forall \ i \geq 1, \ \forall \ \varphi \in A_0, \ \text{and} \ \forall \ \kappa \leq \sigma_{i+1},
\]

which proves (24). Moreover, by the proof of Lemma 1.15 (III) we know that \(R_i^{(\kappa)}\) is given by Lemma 1.15 (I) (d), that is

\[
R_i^{(\kappa)} = \hat{R}_i + \left(\hat{\theta}_{i\kappa} - \left|\hat{\theta}_{i\kappa}\right|\right) \psi_{\kappa} (\kappa \leq \sigma_{i+1}, \ i \geq 1)
\]

hence, from the definition of \(\hat{R}_i\) we get

\[
\hat{f}_{\nu_{i+1}} - \hat{f}_{\nu_i} = R_i^{(\kappa)} + \left(\left|\hat{\theta}_{i\kappa}\right| - \hat{\theta}_{i\kappa}\right) \psi_{\kappa}
\]

which proves (23). Clearly from (23) it follows that

\[
\hat{\alpha}_{i+1}^{\bullet}(\varphi) - \hat{\theta}_{i\kappa}(\varphi) \geq 0, \ \forall \ i \geq 1, \ \forall \ \varphi \leq \sigma_{i+1}, \ \forall \ \varphi \in A_0
\] (27)

which implies at once that

\[
\hat{\alpha}_{i+1}^{\bullet}(\varphi) - \hat{\theta}_{i\kappa}(\varphi) \leq 2.\hat{\alpha}_{i+1}^{\bullet}(\varphi), \ \forall \ i \geq 1, \ \forall \ \varphi \leq \sigma_{i+1}, \ \forall \ \varphi \in A_0
\] (28)

[In fact, otherwise (28) would be false implying at once a contradiction with (27).] Now, from (28), (23) and (24) we can conclude that for all \((\varphi, x) \in A_0 \times \mathcal{W}, \ i \geq 1 \text{ and } \kappa \leq \sigma_{i+1}\) we have

\[
\partial^{\kappa} \left(\hat{f}_{\nu_{i+1}} - \hat{f}_{\nu_i}\right) (\varphi, x) = \partial^{\kappa} R_i^{(\kappa)} (\varphi, x) + \left|\hat{\theta}_{i\kappa} - \hat{\theta}_{i\kappa}\right| (\varphi),
\]

hence

\[
\left\|\partial^{\kappa} \left(\hat{f}_{\nu_{i+1}} - \hat{f}_{\nu_i}\right) (\varphi, \cdot)\right\| + \hat{\theta}_{i\kappa}(\varphi) = \left\|\partial^{\kappa} R_i^{(\kappa)} (\varphi, \cdot)\right\| + \left|\hat{\theta}_{i\kappa}(\varphi)\right| \leq \hat{\alpha}_{i+1}^{\bullet}(\varphi)
\]

which from (28) implies

\[
\left\|\partial^{\kappa} \left(\hat{f}_{\nu_{i+1}} - \hat{f}_{\nu_i}\right) (\varphi, \cdot)\right\| \leq \hat{\alpha}_{i+1}^{\bullet}(\varphi) - \hat{\theta}_{i\kappa}(\varphi) \leq 2.\hat{\alpha}_{i+1}^{\bullet}(\varphi), \ \forall \ i \geq 1, \ \forall \ \varphi \leq \sigma_{i+1} \text{ and } \forall \ \varphi \in A_0
\]

\[\Box\]

The application of Lemma 1.16 in the proof of Theorem 1.12 shall show that in fact, the important statement of Lemma 1.16 is (25), and that the other statements (23) and (24) are only preparatory results.

Proof of Theorem 1.12. Let \((f_{\nu})_{\nu \geq 1}\) be a Cauchy sequence in \((\mathcal{G}(\mathcal{W}), T_{\mathcal{W},b})\). We must shows that there exists \(f \in \mathcal{G}(\mathcal{W})\) such that \(f_{\nu} \xrightarrow{\nu \to \infty} f\) in the topology \(T_{\mathcal{W},b}\). To this end, we shall consider the subsequence \((f_{\nu_i})_{i \geq 1}\) of \((f_{\nu})_{\nu \geq 1}\) defined by (26) (see the proof of Lemma 1.16) and fix a sequence \(\left(\hat{f}_{\nu_i}\right)_{i \geq 1}\) where \(\hat{f}_{\nu_i}\)

is an arbitrary representative of \(f_{\nu_i}\) for each \(i \geq 1\). We shall use the sequence \(\left(\hat{f}_{\nu_i}\right)_{i \geq 1}\) to define an element
\[ \hat{f} \in E_M[\mathcal{O}] \] such that \( f_{\nu_i} \xrightarrow{i \to \infty} f := \text{cl}(\hat{f}) \in G(\mathcal{O}) \), and hence \( f_{\nu} \xrightarrow{\nu \to \infty} f \). So, from now on, we assume that the following are fixed: the subsequence \((f_{\nu_i})_{i \geq 1}\), a representative \(f_{\nu_i}\) of \(f_{\nu}\) for each \(i \geq 1\), \(R_i := f_{\nu_{i+1}} - f_{\nu_i}\) and \( \hat{R}_i := f_{\nu_{i+1}} - f_{\nu_i} \). Since \((\nu_i)_{i \geq 1}\) is strictly increasing it is clear that \((A_{\nu_i})_{i \geq 1}\) and \((I_{\nu_i}^{-1})_{i \geq 1}\) are strictly decreasing. For every \(i \geq 1\) and \(x \in \mathcal{O}\) we define

\[
\hat{u}_i(\varphi, x) := \begin{cases} 
\left( f_{\nu_{i+1}} - f_{\nu_i} \right)(\varphi, x), & \text{if } (\varphi, i(\varphi)) \in A_{\nu_i} \times I_{\nu_i}^{-1} \\
0, & \text{if } (\varphi, i(\varphi)) \notin A_{\nu_i} \times I_{\nu_i}^{-1} 
\end{cases}
\]

hence, for all \(i \geq 1\), \(x \in \mathcal{O}\) and \(\beta \in \mathbb{N}^m\) we have

\[
\partial^\beta \hat{u}_i(\varphi, x) = \begin{cases} 
\partial^\beta \hat{R}_i(\varphi, x), & \text{if } (\varphi, i(\varphi)) \in A_{\nu_i} \times I_{\nu_i}^{-1} \\
0, & \text{if } (\varphi, i(\varphi)) \notin A_{\nu_i} \times I_{\nu_i}^{-1} 
\end{cases}
\]

(29)

Obviously, we have \(\hat{u}_i \in E_M[\mathcal{O}]\) for all \(i \geq 1\). Define

\[
\hat{f}(\varphi, x) := \hat{f}_{\nu_1}(\varphi, x) + \sum_{i \geq 1} \hat{u}_i(\varphi, x), \quad \forall \ (\varphi, x) \in A_0 \times \mathcal{O}.
\]

(30)

Clearly, for every \((\varphi, x) \in A_0 \times \mathcal{O}\), the series in the second member of (30) is finite [indeed, note that for every \(\varphi \in A_0\) we have "or \(\varphi \notin A_{\nu_i}\) or \(\exists! s \in \mathbb{N}\) such that \(\varphi \in A_{\nu_i} \cap (\mathbb{C} \backslash A_{\nu_i+1})\)"], hence \(\hat{f}\) is well defined and furthermore,

\[
\forall \beta \in \mathbb{N}^m \text{ and } (\varphi, x) \in A_0 \times \mathcal{O} \text{ we have that } \partial^\beta \left( \sum_{i \geq 1} \hat{u}_i(\varphi, x) \right) = \sum_{i \geq 1} \partial^\beta \hat{u}_i(\varphi, x).
\]

(31)

Next we shall prove that

\[
\hat{f} \in E_M[\mathcal{O}].
\]

(32)

Initially, note that from (29) we get

\[
\left\| \partial^\beta \hat{u}_i(\varphi, x) \right\| \leq \left\| \partial^\beta \hat{R}_i(\varphi, x) \right\|, \quad \forall \varphi \in A_0, \ \forall i \geq 1, \ \forall \beta \in \mathbb{N}^m.
\]

(33)

Clearly, to prove (32) it suffices to show the moderateness of the function

\[
\hat{U} = \hat{U}_1 : (\varphi, x) \in A_0 \times \mathcal{O} \mapsto \sum_{i \geq 1} \hat{u}_i(\varphi, x) \in \mathbb{K}.
\]

So, we must show the following statement

\[
\text{For a fixed } \beta \in \mathbb{N}^m, \ \exists \ N \in \mathbb{N} \text{ such that } \forall \varphi \in A_N, \ \exists \ \tilde{c} = \tilde{c}(\varphi) > 0 \text{ and } \exists \ \tilde{\eta} = \tilde{\eta}(\varphi) \in \mathbb{I} \text{ verifying } \left\| \partial^\beta \hat{U}(\varphi, \cdot) \right\| \leq \tilde{c} \varepsilon^{-N}, \ \forall \ v \in \mathbb{I}_{\tilde{\eta}}.
\]

(34)

Indeed, for a given \(\beta \in \mathbb{N}^m\), choose \(r \in \mathbb{N}^*\) such that

\[
\beta \leq \sigma_{r+1}
\]

(35)

and consider the function \(\hat{U}_r\) defined by \(\hat{U}_r(\varphi, x) := \sum_{i \geq r} \hat{u}_i(\varphi, x), \ \forall (\varphi, x) \in A_0 \times \mathcal{O}\). By the same finiteness argument used proving (31) we have

\[
\partial^\beta \hat{U}_r(\varphi, x) = \sum_{i \geq r} \partial^\beta \hat{u}_i(\varphi, x), \quad \forall (\varphi, x) \in A_0 \times \mathcal{O}
\]

(36)
In view of (25) (see Lemma 1.16) applied to \( \hat{R}_r \) with \( \kappa = \beta \leq \alpha_{i+1}, \forall i \geq r \), see (35) we can write
\[
\left\| \partial^\beta \hat{R}_r (\varphi, \cdot) \right\| \leq 2 \bar{\alpha}^{\bullet}_{i+1} (\varphi), \forall i \geq r, \forall \varphi \in A_0. \tag{37}
\]
From (36) we have
\[
\left\| \partial^\beta \hat{U}_r (\varphi, x) \right\| \leq \sum_{i \geq r} \left| \partial^\beta \hat{u}_i (\varphi, x) \right|, \forall (\varphi, x) \in A_0 \times \bar{\omega}
\]
which together with (33) and (37) implies
\[
\left\| \partial^\beta \hat{U}_r (\varphi, \cdot) \right\| \leq \sum_{i \geq r} \left\| \partial^\beta \hat{u}_i (\varphi, \cdot) \right\|
\leq \sum_{i \geq r} \left\| \partial^\beta \hat{R}_i (\varphi, \cdot) \right\|
\leq 2 \sum_{i \geq r} \bar{\alpha}^{\bullet}_{i+1} (\varphi), \forall \varphi \in A_0. \tag{38}
\]
Now, define \( \eta = \eta (\varphi) := [2i (\varphi)]^{-1}, \forall \varphi \in A_0 \) then, since \( \varepsilon < \eta (\varphi) \) if and only if \( 1 - i (\varphi) \varepsilon > \frac{1}{2} \), we get
\[
2 \sum_{i \geq r} \bar{\alpha}^{\bullet}_{i+1} (\varphi) = \frac{2[i (\varphi) \varepsilon]^{r+1}}{1 - i (\varphi) \varepsilon} < 4 (i (\varphi))^{r+1} \varepsilon^{r+1} \leq 4 (i (\varphi))^{r+1} \varepsilon^{r-1}. \]
Thus, from (38) it follows that
\[
\left\| \partial^\beta \hat{U}_r (\varphi, \cdot) \right\| \leq c (\varphi) \varepsilon^{-1}, \forall \varphi \in A_0, \forall \varepsilon \in \mathscr{I}_\eta (\varphi) \text{ where } c (\varphi) := 4 (i (\varphi))^{r+1}, \forall \varphi \in A_0
\]
which obviously implies (34) since the finite sum \( (\hat{f}_{v_i} + \sum_{i \geq r} \hat{u}_i) \) is moderated. Hence \( U = \hat{U}_1 \) is moderated and (32) is proved. Finally, we will show that \( f_{v_i} \rightarrow f \) in the topology \( \mathcal{T}_{\mathcal{W}, b} \). Fix an arbitrary \( 0 \)-neighborhood \( W_{\lambda, p} (\lambda \in \mathbb{N}^m, p \in \mathbb{N}^*) \) in \( \mathcal{G} (\bar{\omega}) \), we must show that there exists \( \theta \in \mathbb{N} \) verifying
\[
t \geq \theta \Rightarrow (f - f_{v_{i+1}}) \in W_{\lambda, p}. \tag{39}
\]
In view of (30) and (31) we have
\[
\partial^\kappa \hat{f} (\varphi, x) = \partial^\kappa \hat{f}_{v_1} (\varphi, x) + \sum_{i \geq 1} \partial^\kappa \hat{u}_i (\varphi, x) \forall (\varphi, x) \in A_0 \times \bar{\omega}, \forall \kappa \in \mathbb{N}^m. \tag{40}
\]
For an arbitrary \( t \in \mathbb{N}^* \), if \( \varphi \in A_{v_t} \) verifies \( i (\varphi) \in I_{v_{t-1}} \) [and therefore \( (\varphi, i (\varphi)) \in A_{v_t} \times I_{v_{t-1}}, \forall i = 1, 2, ..., t \)] it is clear that
\[
\partial^\kappa \hat{f}_{v_t} (\varphi, x) + \sum_{i = 1}^{t} \partial^\kappa \hat{u}_i (\varphi, x) = \partial^\kappa \hat{f}_{v_{t+1}} (\varphi, x), \forall (\varphi, x) \in \bar{\omega}, \kappa \in \mathbb{N}^m,
\]
which shows that, for \( (\varphi, i (\varphi)) \in A_{v_t} \times I_{v_{t-1}} \), we can write (40) as
\[
\partial^\kappa \hat{f} (\varphi, x) = \partial^\kappa \hat{f}_{v_{t+1}} (\varphi, x) + \sum_{i \geq t+1} \partial^\kappa \hat{u}_i (\varphi, x) \forall (\varphi, x) \in \bar{\omega}, \kappa \in \mathbb{N}^m
\]
hence
\[
\partial^\kappa \left( \hat{f} - \hat{f}_{v_{t+1}} \right) (\varphi, x) = \sum_{i \geq t+1} \partial^\kappa \hat{u}_i (\varphi, x), \forall (\varphi, i (\varphi)) \in A_{v_t} \times I_{v_{t-1}}, \forall x \in \bar{\omega}, \forall \kappa \in \mathbb{N}^m
\]
and consequently
\[
\left\| \partial^\kappa \left( \hat{f} - \hat{f}_{v_{t+1}} \right) (\varphi, \cdot) \right\| \leq \sum_{i \geq t+1} \left\| \partial^\kappa \hat{u}_i (\varphi, \cdot) \right\|, \forall (\varphi, i (\varphi)) \in A_{v_t} \times I_{v_{t-1}}, \forall t \geq 1, \forall \kappa \in \mathbb{N}^m,
\]
which by (33) implies
\[ \| \partial^{\lambda} \left( \hat{f} - \hat{f}_{\nu_t+1} \right) (\varphi, \cdot) \| \leq \sum_{i \geq t+1} \| \partial^{\lambda} \hat{R}_i (\varphi, \cdot) \|, \forall (\varphi, i (\varphi)) \in A_{\nu_t} \times I_{\nu_t+1}, t \geq 1, \lambda \in \mathbb{N}^m. \tag{41} \]

Now, we choose \( \theta \in \mathbb{N}^* \) (see (39)) such that
\[ \theta > p \quad \text{and} \quad \lambda \leq \sigma_{t+1} (\leq \sigma_t + 1, \forall t \geq \theta). \tag{42} \]

Clearly (40) and (41) hold for any \( \lambda \in \mathbb{N}^m \) but now, in order to prove (39), it suffices to consider \( \lambda \leq \lambda \). Next, we apply (25) (see Lemma 1.16) for an arbitrary fixed \( \lambda \leq \lambda \), hence (by (42)) we have \( \lambda \leq \lambda \leq \sigma_{t+1} \leq \sigma_t + 1, \forall t \geq \theta \), which shows that the term \( \sigma_t + 1 \) in (25) should be replaced by \( \sigma_{t+1} \) for all \( t \geq \theta \). Therefore we can write
\[ \| \partial^{\lambda} \hat{R}_t (\varphi, \cdot) \| \leq 2 \hat{\alpha}_{t+1}^* (\varphi), \forall t \geq \theta, \forall \varphi \in A_0, \forall \lambda \leq \lambda. \]

[Note that in (25) all the parameters vary freely: “\( i \geq 1, \lambda \leq \sigma_t + 1, \varphi \in A_0 \)” but in the above application of (25), \( \lambda \) is fixed by the condition \( \lambda \leq \lambda \leq \lambda \) (see (42)) which implies \( \lambda \leq \sigma_t + 1, \forall t \geq \theta \), hence the above inequality is true for all \( t \geq \theta \).] and therefore (writing \( \varphi_t \) instead of \( \varphi \)):
\[ \| \partial^{\lambda} \hat{R}_t (\varphi_t, \cdot) \| \leq 2 [i (\varphi) \varepsilon]^{t+1}, \forall t \geq \theta, \forall \varphi \in A_0, \forall \lambda \leq \lambda. \]

Hence, we can find an upper bound for the second member of (41) in the following way: for \( \varphi \in A_0, \varepsilon \in I_{\eta(\varphi)} \) (\( \eta (\varphi) \) was already defined by \( \eta (\varphi) := [2i (\varphi)]^{-1}, \forall \varphi \in A_0 \)) and \( t \geq \theta \), we get
\[ \sum_{i \geq t+1} \| \partial^{\lambda} \hat{R}_i (\varphi, \cdot) \| \leq \sum_{i \geq t+1} \| \partial^{\lambda} \hat{R}_i (\varphi, \cdot) \| \leq 2 \sum_{i \geq t+1} [i (\varphi) \varepsilon]^{t+1} = 2 [i (\varphi) \varepsilon]^{\theta+2} \frac{1}{1 - i (\varphi) \varepsilon} < 4 [i (\varphi) \varepsilon]^{\theta+2} \]

and therefore we can write
\[ \sum_{i \geq t+1} \| \partial^{\lambda} \hat{R}_i (\varphi, \cdot) \| \leq 4 [i (\varphi) \varepsilon]^{\theta+2}, \forall \varphi \in A_0, \forall \varepsilon \in I_{\eta(\varphi)}, \forall t \geq \theta, \forall \lambda \leq \lambda. \tag{43} \]

Now, define \( \eta_t (\varphi) := \min \left( \eta (\varphi), [i (\varphi) \nu_t]^{-1} \right), \forall \varphi \in A_0, \) and \( t \geq 1 \). Then, by (41) and (43) we obtain:
\[ \| \partial^{\lambda} \left( \hat{f} - \hat{f}_{\nu_t+1} \right) (\varphi_t, \cdot) \| \leq 4 [i (\varphi) \varepsilon]^{\theta+2}, \forall \varphi \in A_0, \forall \varepsilon \in I_{\eta(\varphi)} \] for every \( t \geq \theta \), \( \forall \varphi \in A_0, \forall \varepsilon \leq \lambda \). (44)

Next, note that since \( \theta > p \) (see (42)) and \( \varepsilon \in I_{\eta(\varphi)} \) (which implies \( i (\varphi) \varepsilon < \frac{\lambda}{2} \)) it follows at once that
\[ 4 [i (\varphi) \varepsilon]^{\theta+2} < [i (\varphi) \varepsilon]^p, \]
which by (43) implies
\[ \| \partial^{\lambda} \left( \hat{f} - \hat{f}_{\nu_t+1} \right) (\varphi_t, \cdot) \| \leq [i (\varphi) \varepsilon]^p, \forall \varphi \in A_0, \forall \varepsilon \in I_{\eta(\varphi)} \] and \( \lambda \leq \lambda \). (45)

Since the above inequality means that
\[ \alpha_{p*} \left( \hat{f} - \hat{f}_{\nu_t+1} \right) (\varphi) \geq 0, \]
the statement (45) shows that for every \( t \geq \theta \) we have proved that
\[ \exists \ N := \nu_t \in \mathbb{N} \ such \ that \ \forall b > 0 \ and \ \forall \varphi \in A_N = A_{\nu_t} \ \exists \ \eta_t := \eta_t (\varphi) \in I_{\eta(\varphi)}, \forall \varepsilon \in I_{\eta(\varphi)} \ and \ \lambda \leq \lambda \]

and therefore (see [3, Lemma 2.1 (ii)]), (39) is proved. \qed
Proposition 1.17  The set
\[ G_c(\Omega) := \{ f \in G(\Omega) \mid \text{supp}(f) \subseteq \Omega \} \]
is a dense ideal of \((G(\Omega), T_\Omega)\).

Proof. Let \((\chi_l)_{l \geq 1}\) be a regularizing family associated to the fixed exhaustive sequence \((\Omega_l)_{l \geq 1}\) for \(\Omega\), that is, \(\chi_l \in D(\Omega_{l+1})\) and \(\chi_l|_{\Omega_l} \equiv 1\) for each \(l \geq 1\). Then it will suffice to show that for an arbitrary \(f \in G(\Omega)\) we have
\[ \chi_l f \rightarrow f \quad \text{in the topology } T_\Omega. \]
To this end fix an arbitrary 0-neighborhood \(W^\beta_{\nu, r}\) in \(G(\Omega)\) \((\beta \in \mathbb{N}^m\) and \(\nu, r \in \mathbb{N}^*)\) and fix any representative \(\hat{f}\) of \(f\). Let \(l_0 \in \mathbb{N}^*\) be such that \(\Omega_{l_0} \subset \Omega_l\) for all \(l \geq l_0\), then \((\chi_l f)|_{\Omega_l} = f|_{\Omega_l}, \forall l \geq l_0\)
hence (by setting \(\chi_l(\varphi, \cdot) := \chi_l(\cdot), \forall l \geq l_0, \) and \(\varphi \in A_0)\) we have
\[ \partial^\sigma \left( \chi_l \hat{f} - \hat{f} \right)(\varphi, \cdot)|_{\Omega_l} \equiv 0, \forall l \geq l_0, \forall \sigma \in \mathbb{N}^m\) and \(\forall \varphi \in A_0\)
which implies
\[ \| \partial^\sigma \left( \chi_l \hat{f} - \hat{f} \right)(\varphi, \cdot) \|_{\nu} \equiv 0, \forall l \geq l_0, \forall \sigma \leq \beta \text{ and } \forall \varphi \in A_0 \]
and therefore \((\chi_l f - f) \in W^\beta_{\nu, r}, \forall l \geq l_0. \)

Lemma 1.18  (a) The set \(V_r[x_0]\) (see [3, Definition 2.7]) is bounded in \((\mathbb{R}, T)\) for each \(x_0 \in \mathbb{R}\) and each \(r \in \mathbb{R}\).
(b) For given \(\mu \in \mathbb{R}\) and \(V_s = V_s[0], (s > 0)\) there is \(N \in \mathbb{N}\) such that \(\mu V_s \subset V_{-N}\).
(c) The set \(\{ V_{-N} | N \in \mathbb{N} \}\) is a fundamental system of bounded sets in \((\mathbb{R}, T)\).
(d) If \(X \subset G(\mathbb{R})\) satisfies the condition:

\[ (B) \ \forall \beta \in \mathbb{N}^m, \exists N \in \mathbb{N}\) such that \(\| u \|_{\sigma} \leq \alpha^*_N, \forall u \in X, \forall \sigma \leq \beta. \)

Then \(X\) is a bounded subset of \((G(\mathbb{R}), T_{\mathbb{R}, \infty})\).

Proof. In the proof of this result we will need the following characterization of the relation \(x \geq 0\) where \(x \in \mathbb{R}\) and \(\hat{x}\) is any representative of \(x\) (which is obviously equivalent to the conditions in [3, Lemma 2.1])
\[ \exists N \in \mathbb{N}\) such that \(\forall b > 0, \forall b > b_0 \text{ and } \forall \varphi \in A_N \]
\[ \exists \eta = \eta(b, \varphi) \in I \text{ verifying } \hat{x}(\varphi) \geq -\varepsilon b \forall \varepsilon \in I_{\eta}. \]  \(\text{(46)}\)

(a) case 1: \(x_0 = 0\) and \(r > 0\). Fix an arbitrary \(V_s\) with \(s > 0\); it suffices to show that there exists \(t > 0\) such that
\[ V_t V_r \subset V_s. \]  \(\text{(47)}\)
Indeed, it suffices to take \(t > \max(s - r, 0)\) by applying [3, Lemma 2.1 (i)] and then the proof of (47) follows at once.

\text{case 2:} \(x_0 = 0\) and \(r \leq 0\). Fix an arbitrary \(V_s\) with \(s > 0\); choose \(t > \max(s - r, 0)\); then the proof of (47) follows from (46) with \(b_0 = -r > 0\) if \(r < 0\) (the case \(r = 0\) is trivial).

\text{case 3:} \(x_0 \in \mathbb{R}\) and \(r \in \mathbb{R}\). Fix an arbitrary \(V_s\) with \(s > 0\); then it suffices to show that there is a \(T\)-neighborhood \(W\) of \(0\) such that
\[ W \cdot V_r [x_0] \subset V_s \]  \(\text{(48)}\)
It is easy to see that (48) follows from the continuity of the addition and multiplication in \((\mathbb{R}, T)\) and from the fact that \(V_r\) is bounded for each \(r \in \mathbb{R}\) (cases 1 and 2).
(b) If \( \hat{\mu} \) is any representative of \( \mu \) then there is \( N \in \mathbb{N} \) such that \( \forall \varphi \in A_N \ \exists \ c > 0 \ \text{and} \ \eta \in I \ \text{verifying} \)
\[
|\hat{\mu}(\varphi_x)| \leq c \varepsilon^{-N}, \ \forall \varepsilon \in \eta.
\]

Now, if \( \hat{x} \) is any representative of an arbitrary \( x \in V_s \) it suffices to apply \([3], \text{Lemma} \ 2.1 \ (i)\) to \( \hat{x} \) and to \( \hat{\mu} \hat{x} \) for to get \( |\mu x| \leq a^\ast \cdot N \).

(c) Let \( B \) be a bounded set in \( (\mathcal{G}(\Omega), \mathcal{T}_{\mathcal{G}}) \) then, for a given \( V_s \) with \( s > 0 \), there is \( V_t \) with \( t > 0 \) such that
\[
V_t B \subset V_s.
\]

Since \( 0 \in \text{Inv}(\mathbb{R}) \) there is \( \lambda^{-1} \in V_t \cap \text{Inv}(\mathbb{R}) \) and then, from the above inclusion, it follows that \( \lambda^{-1} B \subset V_s \) and therefore \( B \subset \lambda V_s \) and the conclusion follows from \( (b) \).

(d) Fix an arbitrary \( W_{\beta,t} \ (\beta \in \mathbb{N}^m, \ t \in \mathbb{R}^+) \), then we must show that there exists \( r > 0 \) such that
\[
V_r X \subset W_{\beta,t} \tag{49}
\]

From the assumption \((B)\), for the above fixed \( \beta \in \mathbb{N}^m, \ N \in \mathbb{N} \) such that
\[
||u||_\sigma \leq a^\ast \cdot N, \ \forall \ u \in X, \ \forall \sigma \leq \beta.
\]

We set \( r := N + t \). Fix \( \lambda \in V_r \), \( u \in X \) and representatives \( \hat{\lambda} \) and \( \hat{u} \) of \( \lambda \) and \( u \) respectively. From \((46)\), \( \lambda \in V_r \) means that
\[
\exists \ N_1 \in \mathbb{N} \ \text{such that} \ \forall \ b'_0 > 0, \ \forall \ b' > b'_0 \ \text{and} \ \forall \ \varphi \in A_{N_1} \ \exists \ \eta_1(2b', \varphi) \in I \ \text{verifying} \ \hat{\lambda} (\varphi_x) \leq i(\varphi) \varepsilon^r + c^{2b'} \varepsilon \in \eta_1. \tag{50}
\]

On the other hand, from the condition \((B)\), the relation \( u \in X \) means that
\[
\exists \ N' \in \mathbb{N} \ \text{such that} \ \forall \ b'_0 > 0, \ \forall \ b'' > b'_0 \ \text{and} \ \forall \ \varphi \in A_{N'}, \ \exists \ \eta = \eta(2b'', \varphi) \in I \ \text{verifying} \ ||\partial^\sigma \hat{u} (\varphi_x, \cdot) || \leq i(\varphi)^{-N} \varepsilon^{N} + c^{2b''} \varepsilon \in \eta_1, \ \forall \sigma \leq \beta. \tag{51}
\]

Next we are going to prove \((49)\), that is, \( \lambda u \in W_{\beta,t} \). We define \( N_0 := \max \{N', N_1\} \). Fix \( b > b_0 := \max \{r, t\}, \ \varphi \in A_0 \) and set
\[
\eta_* = \eta_1(b, \varphi) := \min (\eta(2b, \varphi), \eta_1(2b, \varphi)) \in I.
\]

We apply \((50)\) with \( b'_0 := \frac{1}{2} (N + b) \) and \((51)\) with \( b''_0 := \frac{1}{2} (b - r) \). From the definition of \( b \) it is clear that \( b > b'_0 \) and \( b > b''_0 \) hence \( \eta_* \) is well defined. Now it is trivial to see that there exists \( \eta \in I, \ \eta \leq \eta_* \) verifying
\[
||\partial^\sigma (\hat{\lambda} \hat{u}) (\varphi_x, \cdot) || \leq i(\varphi)^{t} \varepsilon^t + c^{b} \varepsilon, \ \forall \varepsilon \in \eta_1, \ \forall \sigma \leq \beta,
\]

that is, \( \lambda u \in W_{\beta,t} \). \hfill \Box

2 Two auxiliary results

Fix a bounded open subset \( \Omega \) of \( \mathbb{R}^m \), \( T \in \mathbb{R}^+ \) and set \( Q := \Omega \times [0, T] \subset \mathbb{R}^{m+1} \). For given \( f \in \mathcal{G}(\mathbb{Q}) \) and \( t_0 \in [0, T] \) we must give a sense to the “restriction”
\[
R := f|_{\Omega \times \{t_0\}}
\]

showing that \( R \) can be identified naturally to an element of \( \mathcal{G}(\mathbb{Q}) \).

Note that \( R \) does not a priori make sense since \( \text{int} (\Omega \times \{t_0\}) = \emptyset \) and hence \( \Omega \times \{t_0\} \) is not a quasi regular set (see \([11], \text{Definition} \ 1.1\)).

Fix a representative \( \hat{f} \in \varepsilon_M (\mathbb{Q}) \) of \( f \); then the restriction
The relation \( f \in \mathcal{W}_{\sigma',r} \) means that
\[
\| f \|_{\kappa'} \leq \beta'_s, \forall \kappa' \leq \sigma'.
\]  

Note that
\[
\kappa' \leq \sigma' = (\sigma, 0) \Rightarrow \kappa' = (\kappa_1, \ldots, \kappa_m, 0) = (\kappa, 0)
\]  

where \( \kappa := (\kappa_1, \ldots, \kappa_m) \) hence \( \kappa' \leq \sigma' \iff \kappa \leq \sigma \), therefore we can write (54) in the following way
\[
\| f \|_{\kappa'} \leq \beta'_s, \forall \kappa \leq \sigma.
\]
From the definition of $f_{t_0}$ we get
\begin{equation}
\left\| \partial^\nu f_{t_0} (\psi, -) \right\|_{\mathcal{J}} \leq \left\| \partial^\nu \hat{f} (\varphi, -) \right\|_{\mathcal{J}}, \quad \forall \varphi \in A_0 (m + 1), \psi := I_m^n (\varphi), \nu' \leq \nu'.
\end{equation}
(56)

Now, it is easily seen that the function (see [2], Lemma 3.1.1):
\begin{equation}
\psi \in A_0 (m) \mapsto \left\| \partial^\nu f_{t_0} (\psi, -) \right\|_{\mathcal{J}} \in \mathbb{R}_+
\end{equation}
is a representative of $\| f_{t_0} \|_{\mathcal{J}}$, and since
\begin{equation}
\psi \in A_0 (m) \mapsto \left\| \partial^\nu f_{t_0} (\psi, -) \right\|_{\mathcal{J}} \in \mathbb{R}_+
\end{equation}
is a representative of $\| f_{t_0} \|_{\mathcal{J}}$, it is clear from (56) and ([3], Lemma 2.1(iii)) that
\begin{equation}
\| f_{t_0} \|_{\mathcal{J}} \leq \| f \|_{\mathcal{J}}, \quad \forall \nu' \leq \nu' \iff \nu \leq \mu.
\end{equation}
(57)

Next, note that $\hat{\beta}^* \in \mathcal{E}_M (K, m + 1)$ hence $\hat{\beta}^* := \hat{\beta}^* \circ I_m^n \in \mathcal{E}_M (K, m)$ is a representative of $\beta^*$ in $\mathcal{E}_M (K, m)$ [since $\hat{\beta}^*$ is a representative of $B_m^n (\beta^*) \in \mathbb{K} (m)$, where $B_m^n : \mathbb{K} (m + 1) \to \mathbb{K} (m)$ is the natural isomorphism induced by the map $\hat{E}_m^n : w \in \mathcal{E}_M (K, m + 1) \mapsto w \circ I_m^n \in \mathcal{E}_M (K, m)$, see ([2], Lemma 3.1.1)]. Hence, in order to prove that $\beta^* \leq \alpha^*$, we can work with the representatives $\hat{\alpha}^*$ and
\begin{equation}
\hat{\beta}^* : \psi \in A_0 (m) \mapsto \iota \left( I_m^n (\psi) \right)^* \in \mathbb{R}_+,
\end{equation}
by applying ([3], Lemma 2.1(ii)). Indeed, take $N := 0$, fix $b > 0$ and $\psi \in A_0 (m)$ arbitrary. Since our hypothesis on $s$ implies
\begin{equation}
\lambda := \frac{sm}{m + 1} - r > 0
\end{equation}
it is obvious that there exists $\eta = \eta (\psi) \in I$ such that
\begin{equation}
L (\psi, \varepsilon) := i (\psi)^* - i \left( I_m^n (\psi) \right)^* \varepsilon^\lambda \geq 0, \quad \forall \varepsilon \in I_\eta.
\end{equation}

Since $(\hat{\alpha}^* - \hat{\beta}^*) (\psi_\varepsilon) = \varepsilon^\lambda L (\psi, \varepsilon), \forall \varepsilon > 0$, it follows that
\begin{equation}
(\hat{\alpha}^* - \hat{\beta}^*) (\psi_\varepsilon) \geq 0 > -\varepsilon^b, \quad \forall \varepsilon \in I_\eta
\end{equation}
hence $\beta^* \leq \alpha^*$. This inequality together ([54]) and ([57]) shows that $\| f_{t_0} \|_{\mathcal{J}} \leq \alpha^*_*, \forall \kappa \leq \sigma$, that is, $f_{t_0} \in W_{\sigma, \kappa}$ and ([53]) is proved.

The remainder of this section is devoted to the definition of a suitable topology on the algebra $\mathcal{G} (\partial \Omega \times [0, T])$. First, we present a definition of $\mathcal{G} (\partial \Omega \times X)$, where $X$ is a quasi-regular set in $\mathbb{R}^n$ and $\Omega$ is an open subset of $\mathbb{R}^m$ (this is necessary since in [11], Definition 3.7], two conditions obviously needed in the definition were omitted). The set $\mathcal{E}_M [\partial \Omega \times X]$ is defined as the set of all functions (where $A_0 = A_0 (m + n))$
\begin{equation}
\hat{u} : A_0 \times \partial \Omega \times X \to \mathbb{K}
\end{equation}
verifying the conditions below:

\begin{itemize}
\item[(M_1)] \((\xi \in \partial \Omega \mapsto \hat{u} (\varphi, \xi, t) \in \mathbb{K}) \in \mathcal{C} (\partial \Omega, \mathbb{K}), \forall \varphi \in A_0, \forall t \in X\)
\item[(M_II)] \((t \in X \mapsto \hat{u} (\varphi, \xi, t) \in \mathbb{K}) \in \mathcal{C}^\infty (X; \mathbb{K}), \forall \varphi \in A_0, \forall \xi \in \partial \Omega\)
\item[(M_III)] \(\forall \kappa \in \mathbb{N}^n, \forall K \subset \subset X \text{ and } H \subset \subset \partial \Omega \exists N \in \mathbb{N} \text{ such that } \forall \varphi \in A_N \exists \epsilon = \epsilon (\varphi) > 0 \text{ and } \eta = \eta (\varphi) \in I \text{ verifying } |\partial^\kappa \hat{u} (\varphi, \xi, t)| \leq \epsilon \epsilon^{-N}, \forall t \in K, \xi \in H \text{ and } \varepsilon \in I_\eta,\)
\end{itemize}
Clearly, the set $E_M[\partial \Omega \times X]$ with the pointwise operations is a $\mathbb{K}$-algebra and the set $N[\partial \Omega \times X]$ of all $\tilde{u} \in E_M[\partial \Omega \times X]$ verifying the condition:

\[
(N) \quad \forall \varphi \in \mathbb{K}, \forall K \subset X \text{ and } \forall H \subset \partial \Omega \exists N \in \mathbb{N} \text{ and } \gamma \in \Gamma \text{ such that } \forall q \geq N \text{ and } \forall \varphi \in A_{\Omega} \exists c = c(\varphi) > 0 \text{ and } \eta = \eta(\varphi) \in I \text{ verifying } |\partial^\nu \tilde{u}(\varphi, \xi, t)| \leq c\varepsilon^{\gamma(\varphi) - N}, \forall t \in K, \xi \in H \text{ and } \varepsilon \in I_{\eta}
\]

is an ideal of $E_M[\partial \Omega \times X]$. This allows to set

\[
G(\partial \Omega \times X) := \frac{E_M[\partial \Omega \times X]}{N[\partial \Omega \times X]}
\]

In the sequel we restrict attention to the present case: $\Omega$ is a bounded open subset of $\mathbb{R}^m$ and $X := [0, T] \subset \mathbb{R}_+ \quad (T > 0)$. Therefore, since $\partial \Omega$ and $X$ are compact sets, the conditions (MIII) and (N) can be simplified for it is enough to work with $K = X$ and $H = \partial \Omega$. In what follows, for the sake of simplicity, we set

\[
Q := \Omega \times [0, T] \quad \text{and } Q^* := \partial \Omega \times [0, T] .
\]

and we will define a topology on $G(Q^*)$. For a given $u \in G(Q^*)$ let $\tilde{u}$ be any representative of $u$ then, for every $\nu \in \mathbb{N}$ the function

\[
\tilde{u}_\nu(\varphi) : \varphi \in A_0 \mapsto \|\partial^\nu \tilde{u}(\varphi, \xi, t)\|_{Q^*} := \sup_{(\xi, t) \in Q^*} |\partial^\nu \tilde{u}(\varphi, \xi, t)| \in \mathbb{R}_+
\]

is moderate. If $u^* \in E_M[Q^*]$ and $u^*_\nu$ is defined as $\tilde{u}_\nu$ with $\tilde{u}$ replaced by $u^*$, it follows at once (see [3, Lemma 3.1]) that

\[
(\tilde{u} - u^*) \in N[Q^*] \Rightarrow (\tilde{u}_\nu - u^*_\nu) \in N(Q)
\]

which shows that $cl(\tilde{u}_\nu) \in \mathbb{R}_+$ is independent of the chosen representative chosen of $u$. Therefore, for each $\nu \in \mathbb{N}$, we have a function

\[
p_{\nu} : u \in G(Q^*) \mapsto cl(\tilde{u}_\nu) \in \mathbb{R}_+
\]

where $\tilde{u}$ is any representative of $u$. Clearly $p_{\nu}$ is a $G$-seminorm (see [3, Definition A.1]), that is, for all $u, v \in G(Q^*)$ and $\lambda \in \mathbb{K}$ we have

\[
p_{\nu}(u + v) \leq p_{\nu}(u) + p_{\nu}(v) \quad \text{and} \quad p_{\nu}(\lambda u) = |\lambda| p_{\nu}(u).
\]

It follows that the set $B_{Q^*}$ of the sets ($\nu \in \mathbb{N}, s \in \mathbb{R}$):

\[
N_{\nu, s} := \{u \in G(Q^*) | p_{\nu}(u) \leq s, \forall \lambda \leq \nu\}
\]

is a fundamental system of 0-neighborhoods for a topology $T_{Q^*}$ on $G(Q^*)$ which is compatible with the $\mathbb{K}$-algebra structure of $G(Q^*)$ (the proof of this statement, which consists in showing that $B_{Q^*}$ satisfies the seven condition of [3, Proposition 1.2 (28)], is easy but rather tedious and we do not give it here).

Now, fix $w \in G(Q)$ and any representative $\tilde{w} \in E_M[Q]$ of $w$. Then the restriction

\[
\tilde{u} := \tilde{w}|_{A_0 \times Q^*}
\]

is well defined and $\tilde{u} \in E_M[Q^*]$. Moreover, if $w^* \in E_M[Q^*]$ and $u^* := w^*|_{A_0 \times Q^*}$, it is clear that

\[
(\tilde{w} - u^*) \in N[Q^*] \Rightarrow (\tilde{u} - u^*) \in N[Q^*]
\]

which shows that we get a natural homomorphism of $\mathbb{K}$-algebras

\[
\rho = \rho_Q : w \in G(Q) \mapsto cl(\tilde{w}|_{A_0 \times Q^*}) \in G(Q^*) ,
\]

where $\tilde{w}$ is any representative of $w$. In the result below we assume that $G(Q^*)$ (resp. $G(Q)$) is endowed with the topology $T_{Q^*}$ (resp. $T_{Q^*}$, see Theorem 1.7).
Lemma 2.2 The above map \( \rho = \rho_Q \) is continuous.

Proof. Since the topology \( T_{\mathcal{Q}^b} \) (resp. \( T_{\mathcal{Q}^s} \)) is defined by the set (see Theorem 2.2)
\[
B_{\mathcal{Q}^b} := \{ W_{\sigma,r} | \sigma = (\sigma', \nu) \in \mathbb{N}^m \times \mathbb{N} \text{ and } r \in \mathbb{N}^+ \}
\]
(resp \( B_{\mathcal{Q}^s} := \{ N_{\nu,s} | \nu \in \mathbb{N} \text{ and } s \in \mathbb{N}^+ \} \)) it suffices to show that for an arbitrary given \( N_{\nu,s} \in B_{\mathcal{Q}^s} \) there is \( W_{\sigma,r} \in B_{\mathcal{Q}^b} \) such that
\[
w \in W_{\sigma,r} \Rightarrow \rho (w) \in N_{\nu,s}
\]
Fix \( N_{\nu,s} \) arbitrary in \( B_{\mathcal{Q}^s} \) then by defining \( \sigma := (0, \nu) \in \mathbb{N}^m \times \mathbb{N} \) and \( r := s \) it is easy to check that
\[
w \in W_{(0,\nu),s} \Rightarrow \rho (w) \in N_{\nu,s}.
\]
\[\square\]

3 An initial boundary value problem

In this section we shall use the following notation:

1. \( \mathbb{R}[x] \) denotes the ring of the polynomials in one variable with real coefficients;
2. \( \Omega \) is a non-void bounded open subset of \( \mathbb{R}^m, T \in \mathbb{R}^+_0 \) and hence \( Q := \Omega \times [0, T[ \) is a bounded open subset of \( \mathbb{R}^{m+1} ;\)
3. For the definition of \( C^\infty (Q) \) see [11, section 1];
4. If \( u \in C^\infty (Q) \) (resp. \( u_0 \in D(\Omega) \)) we set \( \|u\|_{C^k(Q)} := \sum_{|\sigma| \leq k} \|\partial^\sigma u\|_{C^0} \) (resp. \( \|u_0\|_{C_{2k+1}(\Omega)} := \sum_{|\sigma| \leq 2k+1} \|\partial^\sigma u_0\|_{C^0} \));
5. IBVP is an abbreviation for “initial boundary value problem”.

Next, for the benefit of the reader, we shall begin by presenting four results of [9] which we will need for to solve our problem.

Lemma 3.1 ([9, Lemma 1]) For any \( k \in \mathbb{N} \) there exists \( P_k \in \mathbb{R}[x] \) such that, if \( u_0 \in D(\Omega) \) then the solution \( u \in C^\infty (Q) \) of
\[
\begin{align*}
\begin{cases}
u_t - \Delta u + u^3 &= 0 \text{ in } Q \\
u |_{\Omega \times \{0\}} &= u_0 \\
u |_{\partial \Omega \times \{0\}} &= 0
\end{cases}
\end{align*}
\]
(59)
satisfies \( \|u\|_{C^k(Q)} \leq P_k \left( \|u_0\|_{C_{2k+1}(\Omega)} \right) \).

Theorem 3.2 ([9, Theorem 1]) For any given \( u_0 \in G_c(\Omega, \mathbb{R}) \) there exists a solution \( u \in G(\Omega) \) of
\[
\begin{align*}
\begin{cases}
u_t - \Delta u + u^3 &= 0 \text{ in } G(Q) \\
u |_{\Omega \times \{0\}} &= u_0 \text{ in } G(\Omega) \\
u |_{\partial \Omega \times \{0\}} = 0 \text{ in } G(\partial \Omega \times [0, T]).
\end{cases}
\end{align*}
\]
(60)

Lemma 3.3 ([9, Lemma 2]) Let \( v \in C^\infty (Q) \) be a solution of
\[
\begin{align*}
\begin{cases}
u_t - \Delta v + a_0 v &= f \text{ in } Q \text{ (with } f \in C^\infty (Q)) \\
u(x, 0) &= g(x) \text{ on } \Omega \text{ (with } g \in D(\Omega)) \\
u(x, t) &= h(x, t) \text{ in } \partial \Omega \times [0, T] \text{ (with } h \in C^\infty (\partial \Omega \times [0, T]))
\end{cases}
\end{align*}
\]
(61)
where \( T \in \mathbb{R}^+_0 \) and \( a_0(x, t) \geq 0 \) in \( Q \). Then for every \( k \in \mathbb{N} \) there exists \( P_k \in \mathbb{R}[x] \) with coefficients independent of \( a_0, f, g \) and \( h \), such that
\[
\|v\|_{C^k(Q)} \leq P_k \left( \|f\|_{C_{2k+1}(Q)} + \|g\|_{C_{2k}(\Omega)} + \|h\|_{C_{2k+1}(\Omega \times [0, T])} \right) P_k \left( \|a_0\|_{C_{2k+1}(Q)} \right).
\]
Therefore, from Theorem 3.2, for each $\Omega$ such that $\Omega$ are going to prove that the sequence $\Omega$ is a solution of (60) (with initial data $u_0$).

**Proof.** Since $u_0 \in \mathcal{G} (\Omega)$, from Proposition 3.1 it follows that there exists a sequence $(u_{0p})_{p \geq 1}$ in $\mathcal{G} (\Omega)$ such that

$$u_{0p} \xrightarrow{p \to \infty} u_0, \text{ in } (\mathcal{G} (\Omega), T_\Omega).$$

Therefore, from Theorem 3.2 for each $p \geq 1$ there exists $u_p \in \mathcal{G} (\Omega)$ such that $u_p$ is a solution of (60) with the corresponding initial data $u_{0p}$. Moreover, from the proof of Theorem 3.2 it follows that $u_{0p}$ has a representative $\hat{u}_{0p}$ such that $\hat{u}_{0p} (\psi, -) \in \mathcal{D} (\Omega)$, $\forall \psi \in A_0 (m)$, $p \geq 1$ and $u_p$ has a representative $\hat{u}_p$ verifying $\hat{u}_p (\varphi, -) \in \mathcal{C}^\infty (\Omega)$, $\forall \varphi \in A_0 (m + 1)$, $p \geq 1$ such that $\hat{u}_p (\varphi, -)$ is a classical solution of (59) with initial data $\hat{u}_{0p} (m^m + 1 (\varphi), -)$ [indeed, from Lemma 2.1, the second equality of (59):]

$$u_p \mid_{\mathbb{T} \times \{0\}} = u_{0p}$$

is intended as follows: a representative of $u_p \mid_{\mathbb{T} \times \{0\}}$ (resp. $u_{0p}$) is

$$R_{1} : (\psi, x) \in A_0 (m) \times \mathbb{T} \mapsto \hat{u}_p (m^m + 1 (\psi), x, 0) \in \mathbb{R} \text{ (resp. } \hat{u}_{0p})$$

hence, at the level of representatives, (*) can be written as

$$\hat{u}_p (m^m + 1 (\psi), x, 0) = \hat{u}_{0p} (\psi, x), \forall \psi \in A_0 (m), x \in \mathbb{T}$$

or equivalently:

$$\hat{u}_p (\varphi, x, 0) = \hat{u}_{0p} (m^m + 1 (\varphi), x), \forall \varphi \in A_0 (m + 1), x \in \mathbb{T}.$$

Therefore, from Lemma 3.1 it follows that for each $k \in \mathbb{N}$ there is $P_k \in \mathbb{R}[x]$ such that the inequality of Lemma 3.1 holds for all $p \geq 1$:

$$\| \hat{u}_p (\varphi, -) \|_{\mathcal{C}^k (\Omega)} \leq \| P_k (\hat{u}_{0p} (m^m + 1 (\varphi), -) \|_{\mathcal{C}^{2k+1} (\mathbb{T})}, \forall \varphi \in A_0 (m + 1).$$

(62)

Now, we have a sequence $(u_{pq})_{p \geq 1}$ in $\mathcal{G} (\Omega)$ of solutions of (60) corresponding to the initial data $(u_{0p})_{p \geq 1}$. We are going to prove that the sequence $(u_{pq})_{p \geq 1}$ converges to a $u \in \mathcal{G} (\Omega)$ which is the solution of our problem. Since $\Omega$ is a bounded open subset of $\mathbb{R}^{m+1}$, from Theorem 1.12 it follows that $(\mathcal{G} (\Omega), T_{\Omega, b})$ is complete and hence it suffices to show that $(u_{pq})_{p \geq 1}$ is a Cauchy sequence. In the sequel we need the following notation:

$$u_{pq} := u_p - u_q \quad \text{and} \quad \hat{u}_{pq} := \hat{u}_p - \hat{u}_q, (p, q \in \mathbb{N}).$$

The definition of $u_{pq}$ implies that $u_{pq}$ is a solution of the following IBVP:

$$\begin{align*}
\begin{cases}
(u_{pq})_t - \Delta u_{pq} + a_{pq} u_{pq} = 0 \text{ in } \mathcal{G} (\Omega) \\
u_{pq} \mid_{\mathbb{T} \times \{0\}} = u_{0p} - u_{0q} \text{ in } \mathcal{G} (\Omega) \\
u_{pq} \mid_{\partial T \times [0, T]} = 0 \text{ in } \mathcal{G} (\partial T \times [0, T])
\end{cases}
\end{align*}$$

where $a_{pq} := u_{pq}^2 + u_{pq}^2 + u_{pq} u_{pq}$. The definitions of $\hat{u}_p$ and $\hat{u}_{0p}$ shows that $\hat{u}_p (\varphi, -)$ is a classical solution to the IBVP (for each $\varphi \in A_0 (m + 1)$):

$$\begin{align*}
\begin{cases}
(\hat{u}_p (\varphi, -))_t - \Delta \hat{u}_p (\varphi, -) + \hat{a}_p (\varphi, -) \hat{u}_p (\varphi, -) = 0, \text{ in } \mathcal{C}^\infty (\Omega) \\
\hat{u}_p (\varphi, -) \mid_{\mathbb{T} \times \{0\}} = (\hat{u}_{0p} - \hat{u}_{0q}) (m^m + 1 (\varphi), -), \text{ in } \mathcal{D} (\Omega) \\
\hat{u}_p (\varphi, -) \mid_{\partial \Omega \times [0, T]} = 0, \text{ in } \mathcal{C}^\infty (\partial \Omega \times [0, T])
\end{cases}
\end{align*}$$

(63)
where \( \hat{a}_{pq} := \hat{a}_p^2 + \hat{a}_q^2 + \hat{a}_p \hat{a}_q = (\hat{a}_p + \frac{2}{3} \hat{a}_q)^2 + \frac{4}{3} \hat{a}_q^2 \geq 0 \). Next, by applying Lemma \( 3.3 \) to the solution \( \hat{a}_{pq} (\varphi, -) \) of \( (63) \), with \( f = h = 0 \) and \( g (x) := (\hat{a}_p - \hat{a}_q) (1_m^{m+1} (\varphi), x) \), we can conclude that for each \( k \in \mathbb{N} \) there is \( P_k \in \mathbb{R} [x] \) (with coefficients independent of \( a_{pq} \) and \( \hat{a}_p \)), such that for each \( \varphi \in A_0 (m + 1) \):

\[
\| \hat{a}_{pq} (\varphi, -) \|_{C^k (\overline{Q})} \leq \| (\hat{a}_p - \hat{a}_q) (1_m^{m+1} (\varphi), -) \|_{C^{2k+1} (\overline{Q})} \cdot P_k \left( \| \hat{a}_{pq} (\varphi, -) \|_{C^{2k+1} (\overline{Q})} \right).
\]

(64)

Now, the next steps of this proof are as follows: first, we shall prove that

\[
(u_p)_{p \geq 1} \text{ is a bounded sequence in } \left( \mathcal{G} (\overline{Q}), \mathcal{T}_{\mathcal{G}, b} \right),
\]

(65)

and then, by applying (64) and (65), we shall prove that

\[
(u_p)_{p \geq 1} \text{ is a Cauchy sequence in } \left( \mathcal{G} (\overline{Q}) \mathcal{T}_{\mathcal{G}, b} \right),
\]

(66)

from which our existence result follows. Indeed, from (66) and Theorem \( 1.12 \) we get

\[
\exists u := \lim_{p \to \infty} u_p \in \mathcal{G} (\overline{Q}).
\]

Since from the definitions of \( u_p \) and \( u_{0p} \) we have

\[
\begin{align*}
\begin{cases}
(u_p)_t - \Delta u_p + u_p^3 &= 0 \text{ in } \mathcal{G} (Q) \\
\hat{u}_p|_{\mathcal{T} \times \{0\}} &= u_{0p} \text{ in } \mathcal{G} (\overline{Q}) \\
\hat{u}_p|_{\partial \mathcal{T} \times [0, T]} &= 0 \text{ in } \mathcal{G} (\partial \mathcal{T} \times [0, T])
\end{cases}
\end{align*}
\]

(67)

and since the differential operator

\[
P : u \in \mathcal{G} (Q) \mapsto (u_t - \Delta u + u^3) \in \mathcal{G} (Q)
\]

is continuous [this follows from \( [13] \), Corollary 4.2] and from the continuity of the multiplication in \( (\mathcal{G} (\Omega), \mathcal{T}_\Omega) \) by taking limits in the first equality of (67), we obtain

\[
0 = \lim_{p \to \infty} P (u_p) = P (u) \text{ in } \mathcal{G} (Q).
\]

Moreover, by taking limits for \( p \to \infty \) in the two others equalities of (67), from Lemma \( 2.1 \) we get

\[
u|_{\mathcal{T} \times \{0\}} = u_0 \text{ in } \mathcal{G} (\overline{Q})
\]

and, from Lemma \( 2.2 \) we have

\[
u|_{\partial \mathcal{T} \times [0, T]} = 0
\]

hence, \( u \) is a solution of (60) with initial condition \( u_0 \in \mathcal{G} (\overline{Q}) \). It remains to prove (65) and (66).

**Proof of (65):** In view of Lemma \( 1.18 (d) \) it suffices to prove the following statement:

\[
\forall \beta \in \mathbb{N}^{m+1}, \exists N \in \mathbb{N} \text{ such that } \| u_p \|_\sigma \leq a^*_{N, \beta} \forall \sigma \leq \beta, p \geq 1.
\]

(68)

Fix any \( \beta \in \mathbb{N}^{m+1} \) and choose \( k \in \mathbb{N} \) such that \( k \geq |\beta| \), then \( k \geq |\sigma| \) for each \( \sigma \leq \beta \) and therefore, from (62) we have

\[
(\hat{u}_p)_\sigma (\varphi) \leq P_k \left( \| \hat{u}_{0p} (1_m^{m+1} (\varphi), -) \|_{C^{2k+1} (\overline{Q})} \right), \forall \varphi \in A_0 (m + 1).
\]

(69)

Next, it is easy to check that the class in \( \mathcal{R} \) of the moderated function defined by the second member of (69) is

\[
P_k \left( \sum_{|\sigma| \leq 2k+1} \| u_{0p} \|_\sigma \right);
\]
in other words for each $p \geq 1$:

$$
\text{cl} \left( \varphi \mapsto P_k \left( \| \bar{a}_{0p} (1_m^{m+1} (\varphi), -) \|_{C^{2k+1} (\mathbb{T})} \right) \right) = P_k \left( \sum_{|r| \leq 2k+1} \| a_{0p} \|_r \right).
$$

Indeed, this follows from definitions and from the following trivial remark: "If $v \in \mathcal{E}_M (\mathbb{R})$ and $P \in \mathbb{R}[x]$ then $P \circ v \in \mathcal{E}_M (\mathbb{R})$ and $\text{cl} \left( P \circ v \right) = P \left( \text{cl} (v) \right)$". Hence, from (69) and [3, Lemma 2.1 (iii)] we get

$$
\| u_p \|_\sigma \leq P_k \left( \sum_{|r| \leq 2k+1} \| u_{0p} \|_r \right), \ \forall \ \sigma \leq \beta, \ \forall \ p \geq 1.
$$

(70)

Assume now that $P_k (x) = \sum_{i=0}^{l} c_i x^i$ and $P_k^* (x) := \sum_{i=0}^{l} |c_i| x^i$. Then clearly we have

$$
x, y \in \mathbb{R} \text{ and } |x| \leq |y| \Rightarrow |P_k (x)| \leq P_k^* (|x|) \leq P_k^* (|y|).
$$

(71)

Now, we set

$$
t_p := \sum_{|r| \leq 2k+1} \| u_{0p} \|_r, \ \forall, \ p \geq 1,
$$

since $u_{0p} \xrightarrow{p \to \infty} u_0$ it is clear that

$$
t_p \xrightarrow{p \to \infty} t := \sum_{|r| \leq 2k+1} \| u_0 \|_r
$$

hence the set $T := \{ t_p | p \geq 1 \}$ is bounded in $(\mathbb{R}, T)$ which implies (see Lemma 1.18 (c)) that there exists $L \in \mathbb{N}$ such that $T \subset V_L$, that is,

$$
t_p = |t_p| \leq \alpha^*_{-L}, \ \forall \ p \geq 1.
$$

Therefore from (71):

$$
|P_k (t_p)| \leq P_k^* (|t_p|) \leq P_k^* (\alpha^*_{-L}), \ \forall \ p \geq 1.
$$

From Lemma 1.18 (c) there is $N \in \mathbb{N}$ such that $P_k^* (\alpha^*_{-L}) \leq \alpha^*_{-N}$ hence we can conclude that

$$
|P_k (t_p)| \leq \alpha^*_{-N}, \ \forall \ p \geq 1
$$

therefore, from (70) we obtain

$$
\| u_p \|_\sigma \leq \alpha^*_{-N}, \ \forall \ \sigma \leq \beta, \ \forall \ p \geq 1
$$

which proves (68) and hence (65).

Proof of (66): Fix any $W_{\beta,s}$ with $\beta \in \mathbb{N}^{m+1}$ and $s \in \mathbb{R}$ (see Definition 1.2) and choose $k \in \mathbb{N}$ such that $k \geq |\beta|$. From the definition of $\| \cdot \|_{C^{2k+1} (\mathbb{T})}$ and the previous remark on composition of a moderate function with a polynomial, we get

$$
\text{cl} \left( \varphi \mapsto P_k \left( \| \bar{a}_{pq} (\varphi, -) \|_{C^{2k+1} (\mathbb{T})} \right) \right) = P_k \left( \sum_{|r| \leq 2k+1} \| a_{pq} \|_r \right), \ \forall \ (p, q) \in \mathbb{N}^2.
$$

(72)

By applying (68) in the case $\beta^* := (2k+1, \ldots, 2k+1) \in \mathbb{N}^{m+1}$, since $|\sigma| \leq 2k + 1 \Rightarrow \sigma_i \leq \beta^*_i = 2k + 1, \ \forall \ i = 1, 2, \ldots, m$ and hence $\sigma \leq \beta^*$, we can conclude that there exists $N \in \mathbb{N}$ verifying

$$
\| u_p \|_\sigma \leq \alpha^*_{-N}, \ \forall \ |\sigma| \leq 2k + 1, \ p \geq 1.
$$
Therefore, from Lemma\[1.4\] by setting \( C := c \cdot \max_{\alpha \leq \sigma} \), where \( c := \text{Card} \{ \alpha \mid \alpha \leq \sigma \} \), we have

\[
\|u_p u_q\| \leq C \cdot \alpha_{-2N}^\bullet, \quad \forall \ |\sigma| \leq 2k + 1, \ (p, q) \in \mathbb{N}^2
\]

hence

\[
\|a_{pq}\|_\sigma \leq 3C \cdot \alpha_{-2N}^\bullet, \quad \forall \ |\sigma| \leq 2k + 1, (p, q) \in \mathbb{N}^2.
\]

Then, if \( N' = \text{Card} \{ \sigma \mid |\sigma| \leq 2k + 1 \} \) we get

\[
\sum_{|\sigma| \leq 2k + 1} \|a_{pq}\|_\sigma \leq 3N'C\alpha_{-2N}^\bullet, \quad \forall \ (p, q) \in \mathbb{N}^2.
\]

Let \( M \in \mathbb{N} \) such that \( M > 2N \) then \( 3N'C\alpha_{-2N}^\bullet \leq \alpha_{-M}^\bullet \). Hence the above inequality implies

\[
\sum_{|\sigma| \leq 2k + 1} \|a_{pq}\|_\sigma \leq \alpha_{-M}^\bullet, \quad \forall \ (p, q) \in \mathbb{N}^2
\]

and hence, from \( \text{(71)} \) it follows that

\[
\left| P_k \left( \sum_{|\sigma| \leq 2k + 1} \|a_{pq}\|_\sigma \right) \right| \leq P_k^* \left( \alpha_{-M}^\bullet \right), \quad \forall \ (p, q) \in \mathbb{N}^2.
\]

On the other hand for each \((p, q) \in \mathbb{N}^2\) we have

\[
\sum_{|\tau| \leq 2k} \|u_{0p} - u_{0q}\|_\tau = \text{cl} \left( \varphi \in A_0 (m + 1) \mapsto \sum_{|\tau| \leq 2k} \|\partial^\tau (\hat{u}_{0p} - \hat{u}_{0q}) (I_m^{m+1} (\varphi), -)\|_{\mathcal{T}} \right)
\]

and from the definition of \( \|\cdot\|_{c^8(\mathcal{T})} \) we obtain

\[
\sum_{|\sigma| \leq k} \|u_p - u_q\|_\sigma = \text{cl} \left( \varphi \mapsto \|\hat{u}_p - \hat{u}_q\| (\varphi, -)\|_{c^8(\mathcal{T})} \right)
\]

Now, from \( \text{(74)}, \ \text{(75)}, \ \text{(72)} \) and \( \text{[3]}, \ \text{Lemma 2.1 (iii)} \) we can conclude that \( \text{(64)} \) holds for classes:

\[
\sum_{|\sigma| \leq k} \|u_p - u_q\|_\sigma \leq \left( \sum_{|\tau| \leq 2k} \|u_{0p} - u_{0q}\|_\tau \right) \cdot P_k \left( \sum_{|\sigma| \leq 2k + 1} \|a_{pq}\|_\sigma \right), \quad \forall \ (p, q) \in \mathbb{N}^2
\]

hence

\[
\|u_p - u_q\|_\sigma \leq \left( \sum_{|\tau| \leq 2k} \|u_{0p} - u_{0q}\|_\tau \right) \cdot P_k \left( \sum_{|\sigma| \leq 2k + 1} \|a_{pq}\|_\sigma, \quad \forall \ |\sigma| \leq k, \ (p, q) \in \mathbb{N}^2.
\]

Obviously there exists \( L \in \mathbb{N} \) such that

\[
P_k^* \left( \alpha_{-M}^\bullet \right) = \left| P_k^* \left( \alpha_{-M}^\bullet \right) \right| \leq \alpha_{-L}^\bullet
\]

and since \( \sigma \leq \beta \Rightarrow |\sigma| \leq |\beta| \leq k \), we have

\[
\|u_p - u_q\|_{\sigma} \leq \left( \sum_{|\tau| \leq 2k} \|u_{0p} - u_{0q}\|_\tau \right) \cdot \alpha_{-L}^\bullet, \quad \forall \ \sigma \leq \beta, \ (p, q) \in \mathbb{N}^2.
\]
Since \((u_{0p})_{p \geq 1}\) is a Cauchy sequence there is \(\nu \in \mathbb{N}\) such that
\[
p, q \geq \nu \Rightarrow \sum_{|\tau| \leq 2k} \|u_{0p} - u_{0q}\|_{\nu} \leq \beta_{k}^{*},
\]
where \(\beta_{k}^{*}\) is represented by \(\beta_{k}^{*} : \psi \in A_{0} (m) \mapsto i (\psi)^{b} \in \mathbb{R}\) and \(b\) to be chosen conveniently. Clearly \(\beta_{k}^{*}\) can be represented by
\[
\psi_{0} \in A_{0} (m + 1) \mapsto \left(1_{m+1} (\varphi)\right)^{b} \in \mathbb{R}_{+},
\]
we will use this representative for to prove that
\[
b > \frac{m}{m+1} (L + s) \Rightarrow \alpha_{s-L}^{*} \beta_{k}^{*} \leq \alpha_{s}^{*}.
\]
Indeed, this follows at once from [3], Lemma 2.1 (i), (*) since for all \(\varphi \in A_{0} (m + 1)\) we have
\[
\left(\alpha_{s}^{*} - \alpha_{s-L}^{*} \beta_{k}^{*}\right) (\varphi_{\varepsilon}) = \varepsilon^{s} \left(C_{1} - C_{2} \varepsilon^{-b} \right) \geq 0
\]
for \(\varepsilon\) small enough \((C_{1}, C_{2} \) are two positive constants). As a consequence, from (76), (77) and (78) we get
\[
p, q \geq \nu \Rightarrow \|u_{p} - u_{q}\|_{\sigma} \leq \alpha_{s-L}^{*} \beta_{k}^{*} \leq \alpha_{s}^{*}, \quad \forall \sigma \leq \beta
\]
or equivalently
\[
p, q \geq \nu \Rightarrow (u_{p} - u_{q}) \in W_{\beta,s}.
\]
Finally, the uniqueness of the solution is obvious since this is precisely [9], Theorem 2. Indeed, in the proof of this result, the initial data \(u_{0}\) disappears and so, the compactness or not of \(\text{supp} (u_{0})\) is irrelevant. Therefore, this result holds in our case.

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