Scattering and Blow up for the Two Dimensional Focusing Quintic Nonlinear Schrödinger Equation

Cristi Guevara and Fernando Carreon

Abstract. Using the concentration-compactness method and the localized virial type arguments, we study the behavior of $H^1$ solutions to the focusing quintic NLS in $\mathbb{R}^2$, namely,

$$i\partial_t u + \Delta u + |u|^4 u = 0, \quad (x, t) \in \mathbb{R}^2 \times \mathbb{R}.$$ 

Denoting by $M[u]$ and $E[u]$, the mass and energy of a solution $u$, respectively, and $Q$ the ground state solution to $-Q + \Delta Q + |Q|^4 Q = 0$, and assuming $M[u] E[u] < M[Q] E[Q]$, we characterize the threshold for global versus finite time existence. Moreover, we show scattering for global existing time solutions and finite or “weak” blow up for the complement region. This work is in the spirit of [KM06] and [DHR08, HR08a, HR10].

1. Introduction

Consider the focusing quintic nonlinear Schrödinger equation on $\mathbb{R}^2$

\begin{equation}
\begin{aligned}
&i\partial_t u + \Delta u + |u|^4 u = 0 \\
u(x, 0) = u_0(x) \in H^1(\mathbb{R}^2),
\end{aligned}
\end{equation}

where $u = u(x, t)$ is a complex-valued function in space-time $\mathbb{R}^2_t \times \mathbb{R}_t$.

The initial-value problem (1.1) is locally well-posed in $H^1_\ell$ (see Ginibre-Velo [GV79]). Let $I = (-T_*, T^*)$ be the maximal interval of existence in time of solutions to (1.1). Solutions to (1.1) on $(-T_*, T^*)$ satisfy mass conservation $M[u](t) = M[u_0]$, energy conservation $E[u](t) = E[u_0]$ and momentum conservation $P[u][t] = P[u_0]$, where

$$M[u](t) = \int_{\mathbb{R}^2} |u(x, t)|^2 dx,$$
\[ E[u](t) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u(x, t)|^2 dx - \frac{1}{6} \int_{\mathbb{R}^2} |u(x, t)|^6 dx, \]

\[ P[u](t) = Im \int_{\mathbb{R}^2} \bar{u}(x, t) \nabla u(x, t) dx. \]

The NLS equation has several symmetries and for the purpose of this paper we discuss two of them. If \( u(x, t) \) is a solution to (1.1), the Galilean invariant \( u_G \)
\[
(1.2) \quad u_G(x, t) = e^{ix_0 e^{-i\lambda t}} u(x - (x_0 + 2\xi_0 t), t)
\]
also is a solution.

Observe that for a fixed \( \lambda \in (0, \infty) \), if \( u(x, t) \) solves (1.1), then \( u_G(x, t) := \lambda^2 u(\lambda x, \lambda^2 t) \) solves (1.1). This scaling preserves the \( H^{1/2}(\mathbb{R}^2) \) norm, thus, the initial value problem (1.1) is known as an \( H^{1/2} \)-critical problem, hence, it is mass-supercritical and energy-subcritical. The purpose of this paper is to investigate global behavior of solutions (in time) for the Cauchy problem (1.1) with initial data \((u_0(x), u_1(x))\) applied the concentration-compactness and rigidity technique. The

Small data theory guarantees the global existence and scattering for solutions as opposed to scattering when these techniques are used to show some boundedness properties of solutions.

For studying long-term behavior of solutions in the energy-critical focusing case of NLS (1.3) (for \( p = \frac{4}{n-2} + 1 \), \( u_0 \in H^1(\mathbb{R}^n) \), and \( n = 3, 4, 5 \)), Kenig-Merle [KM06] applied the concentration-compactness and rigidity technique. The concentration-compactness method appears first in the context of wave equation in Gérard [Ger96] and NLS in Merle-Vega [MV98], which was later followed by Kerami [Ker01], and dates back to P.L. Lions [Lio84] and Brezis-Coron [BC85]. The localized variance estimates are due to F. Merle from mid 1980’s. In [KM06] the authors obtain a sharp threshold for scattering and finite time blow up for radial initial data for solutions with \( E[u] < E[W] \).

In the case of the 3d focusing cubic NLS (a mass-supercritical and energy-subcritical problem) equation with \( H^1 \) initial data this method was applied to obtain scattering for global existing solutions under the mass-energy threshold (i.e., \( M[u]E[u] < M[Q]E[Q] \) ) by Holmer-Roudenko for radial functions in [HR08a], Duyckaerts-Holmer-Roudenko for nonradial functions in [DHR08]. Duyckaerts-Roudenko in [DR10] obtain the characterization of all solutions at the threshold \( M[u]E[u] = M[Q]E[Q] \). Furthermore, for infinite variance nonradial solutions Holmer-Roudenko [HR10] established a version of the blow up result (in this paper refereed as “weak” blow up), meaning that either blow up occurs in finite time \( (T^* < +\infty) \), or \( T^* = +\infty \) and there exists a time sequence \( \{t_n\} \to +\infty \) such that \( \|\nabla u(t_n)\|_{L^2} \to +\infty \). This last result is the first application of the concentration compactness and rigidity arguments to establish the divergence property of solutions as opposed to scattering when these techniques are used to show some boundedness properties of solutions.
In the spirit of [DHR08, HR08a, HR10] we analyze the global behavior of solutions for the focusing quintic NLS in two dimensions \((1.1)\), denoted by \(\text{NLS}_5^+\(\mathbb{R}^2\)\). Note that \(u(x,t) = e^{itQ}(x)\) solves the equation \((1.1)\), provided \(Q\) solves
\[
(1.4) \quad -Q + \Delta Q + |Q|^4 Q = 0, \quad Q = Q(x), \quad x \in \mathbb{R}^2.
\]

From the theory of nonlinear elliptic equations denoted by Berestycki-Lions [BL83a, BL83b], it is known that the equation \((1.4)\) has infinite number of solutions in \(H^1(\mathbb{R}^2)\), but a unique solution of the minimal \(L^2\)-norm, which we denote again by \(Q(x)\). It is positive, radial, exponentially decaying (see [Tao06, Appendix B]) and is called the ground state solution.

Before stating our main result, we introduce the following notation:

- the renormalized gradient
  \[
  \mathcal{G}_u(t) := \frac{\|u\|_{L^2(\mathbb{R}^2)} \|\nabla u(t)\|_{L^2(\mathbb{R}^2)}}{\|Q\|_{L^2(\mathbb{R}^2)} \|\nabla Q\|_{L^2(\mathbb{R}^2)}}.
  \]
- the renormalized momentum
  \[
  \mathcal{P}[u] := \frac{P[u]\|u\|_{L^2(\mathbb{R}^2)}}{\|Q\|_{L^2(\mathbb{R}^2)} \|\nabla Q\|_{L^2(\mathbb{R}^2)}}.
  \]
- the renormalized Mass-Energy
  \[
  \mathcal{M}[u] := \frac{M[u]E[u]}{M[Q]E[Q]}.
  \]

**Remark 1.1 (Negative energy).** Note that it is possible to have initial data with \(E[u] < 0\) and the blowup from the dichotomy in Theorem A Part II (a) below applies. (It follows from the standard convexity blow up argument and the work of Glangetas-Merle [GM95].) Therefore, we only consider \(E[u] \geq 0\) in the rest of the paper.

The main result of this paper is the following

**Theorem A.** Let \(u_0 \in H^1(\mathbb{R}^2)\) and \(u(t)\) be the corresponding solution to \((1.1)\) in \(H^1(\mathbb{R}^2)\) with maximal time interval of existence \((-T_*, T^*)\). Assume
\[
(1.5) \quad \mathcal{M}[u] - 2\mathcal{P}[u] < 1.
\]

**I.** If
\[
(1.6) \quad \mathcal{G}_u^2(0) - \mathcal{P}[u] < 1,
\]

then
(a) \(\mathcal{G}_u(t) - \mathcal{P}[u] < 1\) for all \(t \in \mathbb{R}\), and hence, the solution is global in time (i.e., \(T_* = +\infty\)), moreover,
(b) \(u\) scatters in \(H^1(\mathbb{R}^2)\), this means, there exists \(\phi_\pm \in H^1(\mathbb{R}^2)\) such that
\[
\lim_{t \to \pm \infty} \|u(t) - e^{it\Delta} \phi_\pm\|_{H^1(\mathbb{R}^2)} = 0.
\]

**II.** If
\[
(1.7) \quad \mathcal{G}_u^2(0) - \mathcal{P}[u] > 1,
\]
then \(\mathcal{G}_u(t) - \mathcal{P}[u] > 1\) for all \(t \in (-T_*, T^*)\) and if
(a) \(u_0\) is radial or \(u_0\) is of finite variance, i.e., \(|x|u_0 \in L^2(\mathbb{R}^2)\), then the solution blows up in finite time in both time directions.
(b) If \(u_0\) non-radial and of infinite variance, then in the positive time direction either the solution blows up in finite time (i.e., \(T^* < +\infty\)) or there exists a sequence of times \(t_n \to +\infty\) such that \(\|\nabla u(t_n)\|_{L^2(\mathbb{R}^2)} \to \infty\). Similar statement holds for \(t < 0\).
To prove this theorem, we first reduce it to the solutions with zero momentum. This is possible by Galilean transformation (see Section 2.3), and thus, we only prove a reduced version of Theorem A, see the statement of Theorem A* in Section 2.3.

Our arguments follow [DHR08, HR07, HR08a, HR10] which considered the focusing NLS\(_3(\mathbb{R}^3)\), however, several non-trivial modifications had to be made. In particular,

- The range of the Strichartz exponents is adapted for the two dimensional case, as well as the range of admissible pairs for the Kato-type estimate \((2, \infty)\), see Section 2.1 and also Remarks 2.4 and 2.6.
- The pair \((2, \infty)\) is not \(\dot{H}^{1/2}(\mathbb{R}^2)\)-admissible (as oppose to \(\mathbb{R}^3\) as was used in [HR08a]), thus, when using Strichartz and Kato estimates, we have to avoid this end point pair. To do that we use various interpolation tricks on other admissible pairs \((p, r)\) with \(r < +\infty\), see Propositions 2.3 and 2.7.
- We also note that there is a minor error in [HR08a, Proposition 2.2] which we resolve in this paper, see also errata [HR08b]. Refer to Remarks 2.4, 2.6 and 2.8 discussing this matter.
- The ground state, its variational characterization and Pohozhaev identities are different for the NLS\(_3(\mathbb{R}^2)\) (see Subsections 2.2 and 7.1).
- A new argument to obtain blow up for the radial data when \(p = 5\) (Theorem A II part (a)) was obtained. The approach in [HR07] had a technical restriction, i.e., for \(n \geq 2\) the nonlinearity \(1 + \frac{4}{n} < p < \min\{5, 1 + \frac{4}{n-2}\}\), and thus, would not include the case \(p = 5\). Combining estimates on the \(L^6(\mathbb{R}^2)\) norm, the Gagliardo-Nierenberg estimate from [OT91] for radial functions and the conservation of the mass, we resolve this issue. (However, for \(n = 2\), showing blow up for \(p > 5\) for radial data is still open.)
- We explicitly state the linear and the nonlinear profile decompositions in Section 5 and “general” existence of wave operator (Proposition 3.5). General means in the sense that it can be applied later in both scattering and weak blow up parts of Theorem A. The nonlinear profile decomposition for the 3d cubic NLS is hidden in [DHR08, Propositions 2.1 and 6.1] as well in [KM06].

The structure of this paper is as follows: Section 2 reviews the local theory, the properties of the ground state and reduction of the problem with nonzero momentum to the case \(P[u] = 0\) via Galilean transformation for the equation (1.1). Section 3 states the blow up and scattering dichotomy results and existence of the wave operator for \(\text{NLS}_5^a(\mathbb{R}^2)\). In Section 4 we present the detailed proofs for the linear and nonlinear profile decompositions, these are the keys of the technique. And finally, in Sections 5 - 6, we prove Theorem A, both based on the concentration compactness machinery and localized virial identity, in particular, in Section 5 we prove scattering and in Section 6 we give the argument for the “weak” blow up (Theorem A II (b)).

The arguments, presented in this paper, can be extended to other mass- supercritical and energy-subcritical NLS cases and we will establish further generalizations elsewhere.
1.1. Notation. Throughout the paper, most of the $L^p$, $H^s$ and $\dot{H}^s$ norms are defined on $\mathbb{R}^2$, for example, $f \in L^p(\mathbb{R}^2)$ if $\|f\|_{L^p(\mathbb{R}^2)}^p = \int_{\mathbb{R}^2} |f(x)|^p dx < \infty$. In addition, we adopt the notation $X \lesssim Y$ whenever there exists some constant $c$, which does not depend on the parameters, so that $X \leq cY$. We denote $\text{NLS}(t)\psi(x)$ the solution to (1.1) with initial data $\psi(x)$.

2. Preliminaries

2.1. Local Theory. We first recall the Strichartz estimates (e.g., see Cazenave [Caz03], Keel-Tao [KT98], Foschi [Fos05]).

We say $(q,r)$ is $H^s - \text{Strichartz}$ admissible if

$$\frac{2}{q} + \frac{2}{r} = 1 - s \quad \text{with} \quad 2 \leq q,r \leq \infty \quad \text{and} \quad (q,r) \neq (2,\infty).$$

We will mainly consider $s = 0$ ($L^2$ admissible pairs) and $s = \frac{1}{2}$ ($\dot{H}^{1/2}$ admissible pairs). Let

$$\|u\|_{S(L^2)} = \sup_{(q,r)-L^2 \text{ admissible}} \|u\|_{L^q_t L^r_x},$$

where $\infty^{-}$ stands for any large real number. Define the Strichartz norm $S(\dot{H}^{1/2})$ as

$$\|u\|_{S(\dot{H}^{1/2})} = \sup_{(q,r)-\dot{H}^{1/2} \text{ admissible}} \|u\|_{L^q_t L^r_x},$$

Define the $S'(\dot{H}^{-1/2})$ norm

$$\|u\|_{S'(\dot{H}^{-1/2})} = \inf_{(q,r)-\dot{H}^{-1/2} \text{ admissible}} \|u\|_{L^{q'}_t L^{r'}_x},$$

where $q'$ and $r'$ are the conjugates of $q$ and $r$, respectively. In addition, the pair $(2^-,4^+)$ is $\dot{H}^{-\frac{1}{2}}$ admissible.

The standard Strichartz estimates [Caz03, KT98] are

$$\|e^{it\Delta} \phi\|_{S(L^2)} \leq c\|\phi\|_{L^2} \quad \text{and} \quad \left\| \int_0^t e^{i(t-\tau)\Delta} f(\tau) d\tau \right\|_{S(L^2)} \leq c\|f\|_{S'(L^2)}.$$  \hspace{1cm} (2.1)

By combining them with Sobolev embeddings yields

$$\|e^{it\Delta} \phi\|_{S(\dot{H}^{1/2})} \leq c\|\phi\|_{\dot{H}^{1/2}} \quad \text{and} \quad \left\| \int_0^t e^{i(t-\tau)\Delta} f(\tau) d\tau \right\|_{S(\dot{H}^{1/2})} \leq c\|D^{1/2} f\|_{S'(L^2)}.$$  \hspace{1cm} (2.2)
Also recall the Kato-Strichartz estimate \cite{Fos05}

\begin{equation}
\left\| \int_0^t e^{(t-s)\Delta} f(\tau) d\tau \right\|_{S(\tilde{H}^{1/2})} \leq c \|f\|_{S'(\tilde{H}^{-1/2})}.
\end{equation}

Note that the Kato-Strichartz estimate implies the second (inhomogeneous) estimate in \cite{2.2} by Sobolev embedding but not vice versa. The Kato estimate is essential in the long term perturbation argument.

**Lemma 2.1.** (Chain rule \cite{KPV93}) Suppose \( F \in C^1(\mathbb{C}) \) and \( 1 < p, q, p_1, p_2, q_2 < \infty, 1 < q_1 \leq \infty \) such that

\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.
\]

Then

\begin{equation}
\| D^{1/2} F(f) \|_{L^p_t L^q_x} \leq c \| F'(f) \|_{L^{p_1}_t L^{q_1}_x} \| D^{1/2} f \|_{L^{p_2}_t L^{q_2}_x}.
\end{equation}

**Lemma 2.2.** (Leibniz rule \cite{KPV93}) Let \( 1 < p, p_1, p_2, p_3, p_4 < \infty, \) such that

\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{p} = \frac{1}{p_3} + \frac{1}{p_4}.
\]

Then

\[
\| D^{1/2} (fg) \|_{L^p_t} \lesssim \| f \|_{L^{p_1}_t} \| D^{1/2} g \|_{L^{p_2}_t} + \| g \|_{L^{p_3}_t} \| D^{1/2} f \|_{L^{p_4}_t}.
\]

In what follows we will use the \( L^2 \)-admissible pairs \((6, 3)\) and \((3, 6)\); and the \( \tilde{H}^{1/2} \)-admissible pairs \((6, 12)\) and \((8, 8)\).

**Proposition 2.3.** (Small data). Suppose \( \|u_0\|_{\tilde{H}^{1/2}} \leq A \). There exists \( \delta_{sd} = \delta_{sd}(A) > 0 \) such that if \( \|e^{it\Delta} u_0\|_{S(\tilde{H}^{1/2})} \leq \delta_{sd} \), then \( u \) solving the NLS\(^+\) equation (1.1) is global in \( \tilde{H}^{1/2} \) and

\[
\| u \|_{S(\tilde{H}^{1/2})} \leq 2 \| e^{it\Delta} u_0 \|_{S(\tilde{H}^{1/2})}, \quad \| D^{1/2} u \|_{S(L^2)} \leq 2c \| u_0 \|_{\tilde{H}^{1/2}}.
\]

**Proof.** Define the map \( v \mapsto \Phi_{u_0}(u) \) via \( \Phi_{u_0}(u) = e^{it\Delta} u_0 + i \int_0^t e^{i(t-\tau)\Delta} |u|^4 u(\tau) d\tau \).

Let

\begin{equation}
B = \left\{ \| u \|_{S(\tilde{H}^{1/2})} \leq 2 \| e^{it\Delta} u_0 \|_{S(\tilde{H}^{1/2})}, \quad \| D^{1/2} u \|_{S(L^2)} \leq 2c \| u_0 \|_{\tilde{H}^{1/2}} \right\}.
\end{equation}

The argument is established by showing that \( \Phi_{u_0}(u) \) is a contraction in the ball \( B \). By triangle inequality and (2.2), we have

\[
\| \Phi_{u_0}(u) \|_{S(\tilde{H}^{1/2})} \leq \| e^{it\Delta} u_0 \|_{S(\tilde{H}^{1/2})} + \left\| \int_0^t e^{i(t-\tau)\Delta} |u|^4 u(\tau) d\tau \right\|_{S(\tilde{H}^{1/2})}
\]

\[
\leq \| e^{it\Delta} u_0 \|_{S(\tilde{H}^{1/2})} + c_1 \| D^{1/2} |u|^4 u \|_{S'(L^2)},
\]

where \( c_1 \) takes care of the constants from (2.2). Applying the triangle inequality followed by (2.2) and since \( \| D^{1/2} u_0 \|_{L^2} = \| u_0 \|_{\tilde{H}^{1/2}} \), we obtain

\[
\| D^{1/2} \Phi_{u_0}(u) \|_{S(L^2)} \leq \| e^{it\Delta} D^{1/2} u_0 \|_{S(L^2)} + \left\| \int_0^t e^{i(t-\tau)\Delta} D^{1/2} |u|^4 u(\tau) d\tau \right\|_{S(L^2)}
\]

\[
\leq c_1 \| D^{1/2} u_0 \|_{L^2} + c_1 \| D^{1/2} |u|^4 u \|_{S'(L^2)} \leq c_1 \| u_0 \|_{\tilde{H}^{1/2}} + c_1 \| D^{1/2} |u|^4 u \|_{S'(L^2)}.
\]

Then, we estimate the \( S'(L^2) \) norm by \( L^{\frac{5}{4}}_t L^{\frac{3}{2}}_x \) norm (the pair \((3, 6)\) is an \( L^2 \) admissible), apply Chain rule Lemma 2.1 followed by the Hölder’s inequality, and
finally, the \( L^5 \mathbb{L}^2 \) and \( L^6 \mathbb{L}^3 \) norms are estimated by the \( S(\dot{H}^{1/2}) \) norm and \( S(L^2) \) norm, respectively:

\[
\|D^{1/2}u\|^{4}_{S(L^2)} \leq \|D^{1/2}u\|^{2}_{L_x^5 L_t^\infty} \leq c_2 \|u\|_{L_x^6 L_t^3}^{4} \|D^{1/2}u\|_{L_x^5 L_t^\infty} \leq c_2 \|u\|_{S(\dot{H}^{1/2})} \|D^{1/2}u\|_{S(L^2)},
\]

where \( c_2 \) is the constant from (2.4). Thus, the conditions in (2.5) yield

\[
\|\Phi_{u_0}(\cdot)\|_{S(\dot{H}^{1/2})} \leq \|e^{it\Delta}u_0\|_{S(\dot{H}^{1/2})} + c_1 c_2 \|u\|_{S(\dot{H}^{1/2})} \|D^{1/2}u\|_{S(L^2)} \leq \left(1 + 32c_1 c_2 e\right) \|e^{it\Delta}u_0\|_{S(\dot{H}^{1/2})} \|e^{it\Delta}u_0\|_{S(\dot{H}^{1/2})},
\]

(2.6)

and

\[
\|D^{1/2}\Phi_{u_0}(\cdot)\|_{S(L^2)} \leq c_1 \|u_0\|_{\dot{H}^{1/2}} + c_1 c_2 \|u\|_{S(\dot{H}^{1/2})} \|D^{1/2}u\|_{S(L^2)} \leq c_1 \|u_0\|_{\dot{H}^{1/2}} \left(1 + 32c_2 e\right) \|e^{it\Delta}u_0\|_{S(\dot{H}^{1/2})}.
\]

(2.7)

Thus, (2.6) and (2.7) imply

\[
32C \|e^{it\Delta}u_0\|_{S(\dot{H}^{1/2})} \leq 1 \quad \text{and} \quad 32C \|e^{it\Delta}u_0\|_{S(\dot{H}^{1/2})} \leq 1
\]

and the contraction follows by letting \( C = \max\{c_1, c_1 c_2 e, c_2 e\} \) and choosing \( \delta_{sd} = \min\left\{ \frac{1}{32CA^2}, \frac{1}{\sqrt{32C}}, \frac{1}{\sqrt{32CA}} \right\} \).

\[ \square \]

**Remark 2.4.** (About the proof of Proposition 2.3) If we were to follow [HR08a] Proposition 2.1 directly, in the inhomogeneous Strichartz estimates, we would write \( \|D^{1/2}u\|_{L_x^\infty L_t^2} \), which would force us to estimate \( \|D^{1/2}u\|_{L_x^2 L_t^5} \). However, the pair (2.1) is not \( \dot{H}^{1/2} \)-admissible in \( \mathbb{R}^2 \). To avoid this problem, we choose the \( L^2 \)-admissible pair (3.6) with its conjugate pair \( \left( \frac{3}{2}, \frac{6}{n} \right) \), and estimate instead \( \|D^{1/2}u\|_{L_x^6 L_t^3} \).

**Proposition 2.5.** (\( H^1 \) scattering). Assume \( u_0 \in H^1 \), \( u(t) \) is a global solution to (1.1) with initial condition \( u_0 \), globally finite \( \dot{H}^{1/2} \) Strichartz norm \( \|u\|_{S(\dot{H}^{1/2})} < +\infty \) and uniformly bounded \( H^1 \) norm \( \sup_{t \in [0, +\infty)} \|u(t)\|_{H^1} \leq B \). Then there exists \( \phi_+ \in H^1 \) such that

\[
\lim_{t \to +\infty} \|u(t) - e^{it\Delta}\phi_+\|_{H^1} = 0,
\]

i.e., \( u(t) \) scatters in \( H^1 \) as \( t \to +\infty \). A similar statement holds for negative time.

**Proof.** Since \( u(t) \) solves (1.1) with initial datum \( u_0 \), we have the integral equation \( u(t) = e^{it\Delta}u_0 + i \int_0^t e^{i(t-\tau)\Delta}(|u|^4u)(\tau)d\tau \). Define

\[
\phi_+ = u_0 + i \int_0^{+\infty} e^{-i\tau\Delta}(|u|^4u)(\tau)d\tau.
\]

Then

\[
u(t) - e^{it\Delta}\phi_+ = -i \int_t^{+\infty} e^{i(t-\tau)\Delta}(|u|^4u)(\tau)d\tau.
\]

(2.9)
Estimating the $L^2$ norm of (2.9) by Strichartz estimates and Hölder’s inequality, we have
\[
\|u(t) - e^{it\Delta} \phi\|_{L^2} \lesssim \left\| \int_0^t e^{i(t-\tau)\Delta} \left(|u|^4 u\right)(\tau) d\tau \right\|_{S(\mathbb{L}^2)} \lesssim \|\|u\|_{L^\infty_t L^\infty_x} \|\|u\|_{L^2_t L^2_x}^4,
\]
and similarly, estimating the $H^1$ norm of (2.9), we obtain
\[
\|\nabla (u(t) - e^{it\Delta} \phi)\|_{L^2} \lesssim \left\| \int_0^t e^{i(t-\tau)\Delta} \left(\nabla(|u|^4 u)\right)(\tau) d\tau \right\|_{S(L^2)} \lesssim \|\|u\|_{L^\infty_t L^\infty_x} \|\|\nabla u\|_{L^2_t L^2_x}^4.
\]
The Leibnitz rule yields
\[
\|u(t) - e^{it\Delta} \phi\|_{H^1} \lesssim \|\|u\|^{4}(1+\nabla)u\|_{L^\infty_t L^2_x} \lesssim \|\|u\|_{L^6_t L^4_x} \|\|1+\nabla\|_{L^\infty_t L^2_x} \lesssim B \|\|u\|_{L^6_t L^4_x} \|_{L^2}.
\]
Note that the above estimate is obtained using the Hölder inequality with the split $\frac{6}{2} = \frac{3}{2} + \frac{3}{2}$ and $\frac{5}{2} = \frac{1}{2} + \frac{3}{2}$, and the hypothesis $\sup_{t \in [0, +\infty)}\|u(t)\|_{H^1} \leq B$. And as $t \to \infty$, $\|u\|_{L^6_t L^4_x} \to 0$, thus we obtain (2.8).

**Remark 2.6.** The above proof is a direct application of the strategy from [HR08a, Proposition 2.2], namely, we find that (2.9) is bounded in the $H^1$ norm by the Strichartz norm $S(L^2([t, \infty), \mathbb{R}^2))$, which diminishes to 0 as $t \to \infty$. However, this procedure fails in the case of NLS$_q(\mathbb{R}^2)$ as written in [HR08a, Proposition 2.2], since the pair considered there is $(\frac{2}{3}, 10)$ which is not an $L^2$–admissible Strichartz pair, since $q < \frac{5}{3} < 2$. In fact, the norm $\|\|u\|^{4}(1+\nabla)u\|_{L^\infty_t L^2_x}$ used in [HR08a, Proposition 2.2] will only allow pairs $(q, r)$ which are not $L^2$–admissible Strichartz pairs (the pair $(q', r')$ will not belong to the $S'(L^2)$ range). Thus, the original argument in [HR08a, Proposition 2.2] had an error. The issue is fixed in [HR08b] showing that for $\langle \nabla \rangle = (I - \Delta)^{1/2}$ the $\|\langle \nabla \rangle u\|_{S(L^2)}$ is bounded, and thus, $\|u(t) - e^{it\Delta} \phi\|_{H^1} \to 0$ as $t \to +\infty$.

**Proposition 2.7.** (Long time perturbation). For each $A > 0$, there exists $\epsilon_0 = \epsilon(A)$ and $c = c(A)$ such that the following holds. Let $u \in H^1_x$ for all $t$ and solve $i\partial_t u + \Delta u + |u|^4 u = 0$. Let $v \in H^1_x$ for all $t$ and define
\[
\tilde{v} = i\partial_t v + \Delta v + |v|^4 v.
\]
If $\|v\|_{S(H^{1/2})} \leq A$, $\|\tilde{v}\|_{S(H^{-1/2})} \leq \epsilon_0$ and $\|e^{i(t-t_0)\Delta}(u(t_0) - v(t_0))\|_{S(H^{1/2})} \leq \epsilon_0$, then $\|u\|_{S(H^{1/2})} < \infty$.

**Proof.** Define $w = u - v$, then $w$ solves
\[
w_t + \Delta w + F(v, w) - \tilde{v} = 0,
\]
where $F(v, w) = |w + v|^4(w + v) - |v|^4 v$. Since $\|v\|_{S(H^{1/2})} \leq A$, take a partition of $[t_0, \infty)$ with $N$ subintervals $I_j = [t_j, t_{j+1}]$ that satisfy $\|v\|_{S(H^{1/2}; I_j)} \leq \delta$ for a $\delta$ to be chosen later. Writing the integral equation for (2.10) in the interval $I_j$, we obtain
\[
w(t) = e^{i(t-t_j)\Delta} w(t_j) + i \int_{t_j}^t e^{i(t-\tau)\Delta} W(\tau) d\tau,
\]
where $W = F(v, w) - \tilde{v}$.
By applying Kato’s Strichartz estimate (2.3) on $I_j$, we obtain
\[ \|w\|_{S(\tilde{H}^{1/2};I_j)} \leq \|e^{i(t-t_j)\Delta}w(t_j)\|_{S(\tilde{H}^{1/2};I_j)} + c_1\|w\|_{S(\tilde{H}^{1/2};I_j)}, \]
where $c_1$ is the constant in (2.3) and
\[ \|w\|_{S(\tilde{H}^{1/2};I_j)} \leq \|F(v,w)\|_{S(\tilde{H}^{1/2};I_j)} + \|\tilde{c}\|_{S(\tilde{H}^{1/2};I_j)} \]
\[ \leq \|F(v,w)\|_{L_{t_j}^{q}L_{x}^{\frac{4}{q}}} + \|\tilde{c}\|_{S(\tilde{H}^{1/2};I_j)}, \]
here, the pair $(\frac{4}{q}, \frac{q}{r})$ is the conjugate to $(\frac{4}{5}, 6)$ which is $\tilde{H}^{-1/2}$ admissible. Using Hölder’s inequality and a simple fact that $(a+b)^4 \leq c(a^4 + b^4)$, we get
\[ \|F(v,w)\|_{L_{t_j}^{q}L_{x}^{\frac{4}{q}}} \leq \|w\|_{L_{x}^{4}L_{t_j}^{q}} \|v\|_{L_{x}^{4}L_{t_j}^{q}} \]
\[ \leq \|w\|_{L_{x}^{6}L_{t_j}^{4}} \left(\|w\|_{L_{x}^{6}L_{t_j}^{5}}^4 + \|v\|_{L_{x}^{6}L_{t_j}^{5}}^4\right) \]
\[ \approx \|w\|_{S(\tilde{H}^{1/2};I_j)} \left(\|w\|_{S(\tilde{H}^{1/2};I_j)}^4 + \|v\|_{S(\tilde{H}^{1/2};I_j)}^4\right). \]
Choosing $\delta < \min\left\{ 1, \frac{1}{2c_1}\right\}$ and $\|e^{i(t-t_j)\Delta}w(t_j)\|_{S(\tilde{H}^{1/2};I_j)} + c_1\epsilon_0 \leq \min\left\{ 1, \frac{1}{2\sqrt{c_1}}\right\}$, it follows that $\|w\|_{S(\tilde{H}^{1/2};I_j)} \leq 2\|e^{i(t-t_j)\Delta}w(t_j)\|_{S(\tilde{H}^{1/2};I_j)} + 2c_1\epsilon_0$.
Taking $t = t_{j+1}$, applying $e^{i(t-t_{j+1})\Delta}$ to both sides of (2.11) and repeating the Kato estimates, we obtain
\[ \|e^{i(t-t_{j+1})\Delta}w(t_{j+1})\|_{S(\tilde{H}^{1/2};I_j)} \leq 2\|e^{i(t-t_j)\Delta}w(t_j)\|_{S(\tilde{H}^{1/2};I_j)} + 2c_1\epsilon_0. \]
Iterating this process until $j = 0$, we obtain
\[ \|e^{i(t-t_{j+1})\Delta}w(t_{j+1})\|_{S(\tilde{H}^{1/2};I_j)} \leq 2^j\|e^{i(t-t_0)\Delta}w(t_0)\|_{S(\tilde{H}^{1/2})} + (2^j - 1)2c_1\epsilon_0 \leq 2^j + 2c_1\epsilon_0. \]
These estimates hold for all intervals $I_j$ for $0 \leq j \leq N - 1$, then $2^{N+2}c_1\epsilon_0 \leq \min\left\{ 1, \frac{1}{2\sqrt{c_1}}\right\}$, which determines how small $\epsilon_0$ has to be taken in terms of $N$ (as well as, in terms of $A$).

**Remark 2.8.** A direct application of [HR08a, Proposition 2.3] again is not possible, we would need to estimate $\|v\|_{L_t^4L_x^5}^5$, which is not an $L^2$-admissible norm in two dimensions. Therefore, we must use a pair $(q, r)$ with $r < +\infty$, which is possible, since it is not necessary to use a symmetric Strichartz norm $L_t^qL_x^r$ ($q \neq r$) as it was done in [HR08a, Proposition 2.3].

### 2.2. Properties of the Ground State.

Pohozaev identities imply:

(2.12) \[ \|Q\|_{L^6}^6 = 3\|Q\|_{L^2}^2 \]

(multiply (1.4) by $x \cdot \nabla Q$ and integrate over $x$) and

(2.13) \[ \|Q\|_{L^6}^6 = \|Q\|_{L^2}^2 + \|\nabla Q\|_{L^2}^2. \]

(multiply (1.4) by $Q$ and integrate over $x$). Substituting (2.13) and (2.12) into invariant quantities, we get

(2.14) \[ \|Q\|_{L^2}^2\|\nabla Q\|_{L^2} = \sqrt{2}\|Q\|_{L^2}^2 \quad \text{and} \quad M[Q]E[Q] = \frac{1}{2}\|Q\|_{L^2}^4. \]

The Gagliardo-Nirenberg estimate and the sharp constant $C_{GN} = \frac{3}{4\|Q\|_{L^2}^2}$

(2.15) \[ \|u\|_{L^6}^6 \leq C_{GN}\|u\|_{L^2}^2\|\nabla u\|_{L^2}^4, \]

where $C_{GN}$ is obtained from equality in (2.15) with $u$ replaced by $Q$, see [Wei82].
2.3. Properties of the Momentum. Let \( u \) be a solution of (1.1) with \( P[u] \neq 0 \). Take \( \zeta_0 \in \mathbb{R}^2 \) (chosen later) and let \( u_G \) be the Galilean transformation as in (1.2). Noting that \( \|\nabla w\|^2_{L^2} = |\zeta_0|^2 M[u] + 2 \zeta_0 \cdot P[u] + \|\nabla u\|^2_{L^2} \), and that \( M[w] = M[u] \), \( E[w] = \frac{1}{2} |\zeta_0|^2 M[u] + \zeta_0 \cdot P[u] + E[u] \), we minimize the above expressions to obtain the minimum at \( \zeta_0 = -\frac{P[u]}{M[u]} \), and hence, \( P[w] = \zeta_0 M[u] + P[u] = 0 \). We also have

\[
E[w] = E[u] - \frac{P^2[u]}{2M[u]} \quad \text{and} \quad \|\nabla w\|^2_{L^2} = \|\nabla u\|^2_{L^2} - \frac{P^2[u]}{M[u]}.
\]

Thus, \( \mathcal{M}E[w] = \mathcal{M}E[u] - 2P^2[u] < 1 \) and \( \|\nabla w\|^2_{L^2} \ll \|\nabla u\|^2_{L^2} - P^2[u] \). Therefore, if \( P[w] = 0 \), the conditions (1.5), (1.6) and (1.7) become

\[
\mathcal{M}E[w] < 1, \quad G_w(0) < 1, \quad \text{and} \quad G_w(0) > 1.
\]

The reduced version of Theorem A is the following

**Theorem A*. Let \( u_0 \in H^1(\mathbb{R}^2) \) and \( u(t) \) be the corresponding solution to (1.1) in \( H^1(\mathbb{R}^2) \) with maximal time interval of existence \((T_*, T^*)\). Assume \( P[u] = 0 \) and \( \mathcal{M}E[u] < 1 \).

I. If \( G_w(0) < 1 \), then

(a) \( G_u(t) < 1 \) for all \( t \in \mathbb{R} \), thus, the solution is global in time and

(b) \( u \) scatters in \( H^1(\mathbb{R}^2) \), this means, there is \( \phi_\pm \in H^1(\mathbb{R}^2) \) such that

\[
\lim_{t \to \pm \infty} \|u(t) - e^{it\Delta}\phi_\pm\|_{H^1(\mathbb{R}^2)} = 0.
\]

II. If \( G_w(0) > 1 \), then \( G_u(t) > 1 \) for all \( t \in (-T_*, T^*) \) and if

(a) \( u_0 \) is radial or \( u_0 \) is of finite variance, i.e., \( |x||u_0| \in L^2(\mathbb{R}^2) \), then the solution blows up in finite time in both time directions.

(b) If \( u_0 \) non-radial and of infinite variance, then the positive time direction either the solution blows up in finite time (i.e., \( T^* < +\infty \)) or there exists a sequence of times \( t_n \to +\infty \) such that \( \|\nabla u(t_n)\|_{L^2(\mathbb{R}^2)} \to \infty \). Similar statement holds for \( t < 0 \).

In the rest of the paper we shall assume that \( P[u] = 0 \) and prove Theorem A*.

Observe that bounding the energy \( E[u] \) above by the kinetic energy term, we obtain the upper bound in (2.16); using the definition of energy and the sharp Gagliardo-Nirenberg inequality (2.15) to bound the potential energy term, we obtain a bound from below in (2.16), combining, we have

\[
2G^2_w(t) - G^2_{\text{sc}}(t) \leq \mathcal{M}E[u] \leq 2G^2_w(t).
\]

We plot \( y = \mathcal{M}E[u] \) vs. \( G^2_w(t) \) using the restriction (2.16) in Figure 1. This plot contains the scenarios for global behavior of solutions given by Theorem A*.

3. Global versus Blow up Dichotomy

In this section we discuss the sharp threshold for the global existence and the finite time blow up of solutions for the \( \text{NLS}^{(0)}_3(\mathbb{R}^2) \). Theorem 2.1 and Corollary 2.5 of Holmer-Roudenko [HR07] proved the general case for the mass-supercritical and energy-subcritical \( \text{NLS} \) equations with \( H^1 \) initial data, thus, establishing Theorem A* I(a) and II(a) for finite variance data. Thus, we only discuss the case of radial initial data in part II(a). First we recall
Figure 1. Plot of $\mathcal{M}[u]$ against $G^2_t(t)$. The region above the line ABC and below the curve ADF are forbidden regions by (2.16). Global existence of solutions and scattering holds in the region ABD, which corresponds to Theorem A* part I. The region EDF explains Theorem A* part II (a) finite time blow up, and the “weak” blow up from Theorem A* part II (b). The characterization of solution on the line BDE and above is an open question.

Lemma 3.1 (Gagliardo-Nirenberg estimate for radial functions [OT91]). Let $u \in H^1(\mathbb{R}^2)$ be radially symmetric. Then for any $R > 0$, $u$ satisfies

$$
\|u(x)\|_{L^6(R<|x|)} \leq \frac{c}{R^2} \|u\|_{L^2(R<|x|)}^4 \|\nabla u\|_{L^2(R<|x|)}^2,
$$

and $c$ is an absolute constant.

Proof of Theorem A part II(a) (for radial functions.) Recall that the variance is given by $V(t) = \int |x|^2|u(x,t)|^2dx$. The standard argument for finite variance data is to examine the derivative of $V$ and show that

$$
\partial^2_t V(t) = 32E[u_0] - 8\|\nabla u(t)\|_{L^2}^2 < 0,
$$

which by convexity implies the finite time existence of solutions. To obtain a wider range of blow up solutions, there are more delicate arguments (see [Lus95], [HPR10]).

Here, for infinite variance radial data, the argument of localized variance is used following Ogawa-Tsutsumi technique in [OT91].

Let $\chi \in C^\infty(\mathbb{R}^2)$ be radial,

$$
\chi(r) = \begin{cases} 
  r^2 & 0 \leq r \leq 1 \\
  \text{smooth} & 1 < r < 4 \\
  c & 4 \leq r
\end{cases}
$$
such that $\partial^2_r \chi(r) \leq 2$ for all $r \geq 0$. Now, for $m > 0$ large, let $\chi_m(r) = m^2 \chi \left( \frac{r}{m} \right)$.

Define the localized variance $V(t) = \int \chi(x)|u(x,t)|^2 \, dx$ and consider

$$(3.2) \quad \partial^2_t V(t) = 4 \int \chi''(u)^2 - \int \Delta^2 \chi |u|^2 - \frac{4}{3} \int \Delta \chi |u|^{p+1}.$$ 

For $r \leq m$ it follows that $\Delta \chi_m(r) = 4$ and $\Delta^2 \chi_m(r) = 0$. Each of the three terms in the inequality (3.2) are bounded as follows:

- $4 \int \chi'' \nabla u^2 \leq 8 \int \nabla u^2,$
- $- \int \Delta^2 \chi_m |u|^2 \leq \frac{c_1}{m^2} \int_{m \leq |x| \leq 2m} |u|^2 \leq \frac{c_1}{m^2} \int_{m \leq |x|} |u|^2,$
- $- \int \Delta \chi_m |u|^{p+1} \leq -4 \int |u|^{p+1} + c_2 \int_{m \leq |x|} |u|^{p+1}.$

Thus, rewriting (3.2), we obtain

$$\partial^2_t V(t) \leq 32E[u] - 8\|\nabla u\|^2_{L^2} + \frac{c_1}{m^2}\|u\|^2_{L^2} + c_3\|u\|^6_{L^8(|x| \geq m)}$$

$$(3.3) \quad \leq 32E[u] - 8\|\nabla u\|^2_{L^2} + \frac{c_1}{m^2}\|u\|^2_{L^2} + \frac{c_4}{m^2}\|u\|^4_{L^4}\|\nabla u\|^2_{L^2},$$

where $\|u\|_{L^6(|x| \geq m)}$ was estimated using (3.1).

Let $\epsilon > 0$, to be chosen later, pick $m_1 > \left( \frac{c_1}{\epsilon E[Q]} \right)^{\frac{1}{2}} \|u\|_{L^2}$, $m_2 > \left( \frac{c_4}{2\epsilon} \right)^{\frac{1}{4}} \|u\|_{L^2}$ and $m = \max\{m_1, m_2\}$ to get

$$\partial^2_t V(t) < 32E[u] - (8 - \epsilon)\|\nabla u\|^2_{L^2} + \epsilon E[Q].$$

Furthermore, the assumptions $\mathcal{M}[u] < 1$ and $\mathcal{G}_u(0) > 1$ imply that there exists $\delta_1 > 0$ such that $\mathcal{M}[u] < 1 - \delta_1$ and there exists $\delta_2 = \delta_2(\delta_1)$ such that $\mathcal{G}_u(t) > (1 + \delta_2)$ for all $t \in I$. Multiplying both sides of (3.3) by $M[u_0]$, leads to

$$M[u_0]\partial^2_t V(t) < 32(1 - \delta_1)M[Q]E[Q] - (8 - \epsilon)(1 + \delta_2)\|Q\|^2_{L^2}\|\nabla Q\|^2_{L^2} + \epsilon M[Q]E[Q]$$

$$< 32(1 - \delta_1) - 4(8 - \epsilon)(1 + \delta_2) + \epsilon M[Q]E[Q],$$

the last inequality follows since $4E[Q] = \|\nabla Q\|^2_{L^2}$. Choosing $\epsilon < \frac{32(\delta_1 + \delta_2)}{5 + 4\delta_2}$ implies that the second derivative of the variance is bounded by a negative constant for all $t \in \mathbb{R}$, i.e., $\partial^2_t V(t) < -A$, and integrating twice over $t$, we have that $V(t) < -At^2 + Bt + C$. Thus, there exists $T$ such that $V(T) < 0$ which is a contradiction. Therefore, radially symmetric solutions of the type described in Theorem A* part II (a) must blow up in finite time.

In the rest of this section we establish useful estimates on solutions with initial gradient $\mathcal{G}_u(0) < 1$.

**Lemma 3.2.** (Lower bound on the convexity of the variance). Let $u_0 \in H^1$ satisfy (1.1) and $\mathcal{G}_u(0) < 1$. Assume $\delta > 0$ such that $\mathcal{M}[u_0] < (1 - \delta)$.

If $u$ is a solution to (1.1) with initial data $u_0$, then there exists $c_3 > 0$ such that for all $t \in \mathbb{R}$,

$$32E[u] - 8\|\nabla u(t)\|^2_{L^2} = 8\|\nabla u(t)\|^2_{L^2} - \frac{16}{3}\|u\|^6_{L^6} \geq c_3\|\nabla u(t)\|^2_{L^2},$$

in other words, for finite variance solutions, $\partial_t v(t) \geq c_3\|\nabla u(t)\|^2_{L^2}$. 

Similarly, there exists \( \|v\|_{L^2} \leq (1 - \delta_1)^2 \) for all \( t \in \mathbb{R} \). Let
\[
(3.5) \quad h(t) = \left(8\|\nabla u\|_{L^2}^2 - \frac{16}{3}\|u\|_{L^6}^6\right) - \frac{\|u\|_{L^2}^2}{\|Q\|_{L^2}^2}\|\nabla Q\|_{L^2}^2.
\]
By the Gagliardo-Nirenberg inequality (2.15) and the exact value of \( C_{GN} \), we get
\[
h(t) \geq 8G_n^2(t)(1 - G_n^2(t)).
\]
Setting \( g(y) = y^2(1 - y^2) \), it follows \( h(t) \geq 8g(G_n(t)) \).
We only consider \( g(y) \) in the range \([0, 1 - \delta_1]\). Thus, \( g(y) \geq c\delta^2 \), obtaining the result.

**Lemma 3.3.** (Equivalence of energy with the gradient). Let \( u_0 \in H^1 \) satisfy \( G_n(0) < 1 \) and \( \mathcal{M}[u_0] < 1 \). Then
\[
(3.6) \quad \frac{1}{4}\|\nabla u\|_{L^2}^2 \leq E[u] \leq \frac{1}{2}\|\nabla u\|_{L^2}^2.
\]

**Proof.** The first inequality is obtained by observing that the Gagliardo-Nirenberg inequality and the value of \( C_{GN} \), the Pohozhaev identity (2.14) and the hypothesis \( G_n(0) < 1 \) yield
\[
E[u] = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{6}\|u\|_{L^6}^6 \geq \frac{1}{2}\|\nabla u\|_{L^2}^2 \left(1 - \frac{C_{GN}}{6}\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2\right)
\]
\[
\geq \frac{1}{2}\|\nabla u\|_{L^2}^2 \left(1 - \frac{3}{8\|Q\|_{L^2}^2}\|\nabla Q\|_{L^2}^2\right) = \frac{1}{4}\|\nabla u\|_{L^2}^2,
\]
and the second inequality trivially follows from the definition of energy.

**Corollary 3.4.** Let \( u_0 \in H^1 \) satisfy \( G_n(0) < 1 \) and \( \mathcal{M}[u_0] < 1 \), then for all \( t, \omega = \sqrt{\mathcal{M}[u]} \), \( G_n(t) \leq \omega \), and
\[
16(1 - \omega^2)E[u] \leq 8(1 - \omega^2)\|\nabla u\|_{L^2}^2 \leq 8\|\nabla u\|_{L^2}^2 - \frac{16}{3}\|u\|_{L^6}^6.
\]

**Proof.** By the left inequality of (3.6), \( \|\nabla u\|_{L^2}^2 \leq 4E[u] \). Multiplying by mass of \( u_0 \) normalized by \( \|Q\|_{L^2}^2\|\nabla Q\|_{L^2}^2 \) and using that \( \|\nabla Q\|_{L^2}^2 = 4E[Q] \), we obtain \( G_n(t) \leq \omega \). Thus,
\[
V(t) = 8\|\nabla u\|_{L^2}^2 - \frac{16}{3}\|u\|_{L^6}^6 \leq 8\|\nabla u\|_{L^2}^2 \left(1 - \frac{2C_{GN}}{3}\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2\right)
\]
\[
\leq 8\|\nabla u\|_{L^2}^2 \left(1 - \omega^2\right) = 8(1 - \omega^2)\|\nabla u\|_{L^2}^2.
\]
The above estimate is obtained by combining the variance, Gagliardo-Nirenberg inequality, the exact value of \( C_{GN} \), the Pohozhaev identity and the estimate \( G_n(0) < \omega \), and applying Lemma 3.3 we obtain the left inequality, which completes the proof.

**Proposition 3.5.** (Existence of wave operator). Let \( \psi \in H^1(\mathbb{R}^2) \).
1. Then there exists \( v_+ \in H^1 \) such that for some \( -\infty < T^* < +\infty \) it produces a solution \( v(t) \) to NLS_{\psi}^+(\mathbb{R}^2) \) on time interval \( [T^*, \infty) \) such that
\[
(3.7) \quad \|v(t) - e^{it\Delta}\psi\|_{H^1} \to 0 \quad \text{as} \quad t \to +\infty.
\]
Similarly, there exists \( v_- \in H^1 \) such that for some \( -\infty < T_* < +\infty \) it produces a solution \( v(t) \) to NLS_{\psi}^-(\mathbb{R}^2) \) on time interval \( (-\infty, T_*) \) such that
\[
(3.8) \quad \|v(t) - e^{-it\Delta}\psi\|_{H^1} \to 0 \quad \text{as} \quad t \to +\infty.
\]
II. Suppose that for some $0 < \sigma < 1$

(3.9) \[ \frac{1}{2} \| \psi \|_{L^2}^2 \| \nabla \psi \|_{L^2}^2 < \sigma^2 M[Q]E[Q]. \]

Then there exists $v_0 \in H^1$ such that $v(t)$ solving NLS$_{+}^{4}(\mathbb{R}^{2})$ with initial data $v_0$ is global in $H^1$ with

(3.10) \[ M[v] = \| \psi \|_{L^2}^2, \quad E[v] = \frac{1}{2} \| \nabla \psi \|_{L^2}^2, \quad G_\sigma(t) \leq \sigma < 1 \]

Moreover, if $\| e^{it\Delta} \psi \|_{S(\dot{H}^{1/2})} \leq \delta_{sd},$ then $\| v(t) \|_{\dot{H}^{1/2}} \leq 2 \| \psi \|_{\dot{H}^{1/2}}$ and $\| v \|_{H^1} \leq 2 \| e^{it\Delta} \psi \|_{S(\dot{H}^{1/2})}.$

Proof. I. This is essentially Theorem 2 part (a) of Strauss [Str81a] adapted to the case $s = \frac{1}{d}$ $(d = 2$ and $p = 5)$ (see also Remark (36) and [Str81b] Theorem 17).

II. For this part, we want to find a solution to the integral equation

(3.12) \[ v(t) = e^{it\Delta} \psi_+ - i \int_{t}^{+\infty} e^{iT(t-\tau)} (|v|^4 v)(\tau) d\tau. \]

Note that for $T > 0$ from the small data theory (Proposition 2.3) there exists $\delta_{sd} > 0$ such that $\| e^{it\Delta} \psi_+ \|_{S(\dot{H}^{1/2};[T,\infty])} \leq \delta_{sd}.$ Thus, repeating the argument of Proposition 2.3 we first show that we can solve the equation (3.12) in $\dot{H}^{1/2}$ for $t \geq T$ with $T$ large. So this solution $v(t)$ is in $\dot{H}^{1/2}$, and hence, we have that $\| v \|_{S(\dot{H}^{1/2};[T,\infty])}$ is small for large $T >> 0.$ Now we will estimate $\| \nabla v \|_{S(\dot{L}^{2};[T,\infty])},$ which will also show that $v$ is in $H^1.$ Observe that for any $u \in H^1$

\[ \| \nabla (|u|^4 u) \|_{S(\dot{L}^2)} \leq \| \nabla (|u|^4 u) \|_{L^1_T L^2_x} \leq \| u \|_{L^4_T L^6_x}^4 \| \nabla u \|_{L^4_T L^6_x} \leq \| u \|_{S(\dot{H}^{1/2})} \| \nabla u \|_{S(\dot{L}^2)}. \]

Applying the Strichartz estimates (2.1) and Kato-Strichartz estimates (2.3) yields

\[ \| \nabla v \|_{S(\dot{L}^2;[T,\infty])} \leq c_1 \| \psi_+ \|_{L^2} + c_2 \| \nabla (|v|^4 v) \|_{S(\dot{L}^{2};[T,\infty])} \]

\[ \leq c_1 \| \psi_+ \|_{H^1} + c_3 \| \nabla v \|_{S(\dot{L}^{2};[T,\infty])} \| v \|_{S(\dot{H}^{1/2};[T,\infty])}^4 \]

Since $T$ can be chosen large, so that $c_3 \| v \|_{S(\dot{H}^{1/2};[T,\infty])} \leq \frac{1}{2},$ we get

\[ \| \nabla v \|_{S(\dot{L}^2;[T,\infty])} \leq 2c_1 \| \psi_+ \|_{H^1}. \]

Using this fact, we also get $\| \nabla (v(t) - e^{it\Delta} \psi_+) \|_{S(\dot{L}^{2};[T,\infty])} \leq \| \psi_+ \|_{H^1},$ since

\[ \| \nabla (v(t) - e^{it\Delta} \psi_+) \|_{S(\dot{L}^{2};[T,\infty])} \leq c_4 \| \nabla v \|_{S(\dot{L}^{2};[T,\infty])} \| v \|_{S(\dot{H}^{1/2};[T,\infty])}^4 + \| \psi_+ \|_{H^1}. \]

Thus,

\[ \lim_{T \to +\infty} \| \nabla (v(t) - e^{it\Delta} \psi_+) \|_{S(\dot{L}^{2};[T,\infty])} = 0. \]

So we showed that as $t \to +\infty,$ $v(t) \to e^{it\Delta} \psi_+$ in $H^1.$ In particular, this means that $v(t) \to e^{it\Delta} \psi_+$ in $L^2,$ hence

\[ M[v] \equiv \| v(t) \|_{L^2}^2 = \| e^{it\Delta} \psi_+ \|_{L^2}^2 = \| \psi_+ \|_{L^2}^2. \]

Moreover, Sobolev embedding implies $e^{it\Delta} \psi_+ \to 0$ in $L^6.$ Thus, $\| \nabla e^{it\Delta} \psi_+ \|_{L^2}$ is bounded, and

\[ E[v] = t \lim_{t \to +\infty} \left( \frac{1}{2} \| \nabla e^{it\Delta} \psi_+ \|_{L^2}^2 - \frac{1}{6} \| e^{i\Delta} \psi_+ \|_{L^6}^6 \right) = \frac{1}{2} \| \nabla \psi_+ \|_{L^2}^2. \]
From the hypothesis (3.9), we obtain
\[ M[u]E[u] = \frac{1}{2} \| \psi_+ \|_{L^2}^2 \| \nabla \psi_+ \|_{L^2}^2 < \sigma^2 M[Q]E[Q] \]
and so \( \mathcal{M}[u] < 1 \). Furthermore, \( \| \nabla v(t) \|_{L^2}^2 = \| \nabla e^{it\Delta} \psi_+ \|_{L^2}^2 = \| \nabla \psi_+ \|_{L^2}^2 \), and so,
\[
\lim_{t \to +\infty} \| \nabla v(t) \|_{L^2}^2 \| v \|_{L^2}^2 = \lim_{t \to +\infty} \| \nabla e^{it\Delta} \psi_+ \|_{L^2}^2 \| e^{it\Delta} \psi_+ \|_{L^2}^2 = \| \nabla \psi_+ \|_{L^2}^2 \| \psi_+ \|_{L^2}^2 < 2\sigma^2 M[Q]E[Q] = \sigma^2 \| \nabla Q \|_{L^2}^2 \| Q \|_{L^2}^2.
\]
Thus,
\[
\lim_{t \to +\infty} G_v(t) \leq \sigma < 1.
\]
For sufficiently large \( T > 0 \), we can get that \( G_v(T) < 1 \). Now we are in the assumption of Theorem A* part I (a), which shows that \( v(t) \) exists globally and evolving it from \( T \) back to 0, we will obtain the data \( v_0 \in H^1 \) as desired. □

4. Outline of Scattering via Concentration Compactness

The goal of this section is to outline the proof of scattering in \( H^1 \) for the global solution of (1.1), i.e., Theorem A (I. part b). The proof of the main steps will be given in Sections 5 and 6.

**Definition 4.1.** Suppose \( u_0 \in H^1 \) and let \( u \) be the corresponding \( H^1 \) solution to (1.1) and \([0, T^*) \) be the maximal (forward in time) interval of existence. We say that \( SC(u_0) \) holds if \( T^* = +\infty \) and \( \| u \|_{S(H^{1/2})} < \infty \). Note that if \( SC(u_0) \) holds, then together with Proposition 2.5 we obtain \( H^1 \) scattering of \( u(t) = NLS(t)u_0 \).

Our goal is to prove the following: if \( G_v(0) < 1 \) and \( \mathcal{M}[u] < 1 \), then \( SC(u_0) \) holds.

The hypotheses give an a priori bound for \( \| \nabla u(t) \|_{L^2} \) (by Theorem A part I), thus, the maximal forward time of existence is \( T = +\infty \). Therefore, it remains to show that the global-in-time \( H^{1/2} \) Strichartz norm is finite, i.e., \( \| u \|_{S(H^{1/2})} < \infty \). We prove this using the induction argument on the mass-energy threshold as in [KM06], [HR08a].

**Step 0: Small Data.** The equivalence of energy with the gradient from Lemma 2.3 yields
\[
\| u_0 \|_{H^{1/2}}^6 \leq (\| u_0 \|_{L^2} \| \nabla u_0 \|_{L^2})^3 \leq (4M[u]E[u])^{3/2}.
\]
If \( G_v(0) < 1 \) and \( M[u]E[u] < \frac{1}{4} \delta^4 \), then \( \| u_0 \|_{H^{1/2}} \leq \delta_{sd} \) and \( \| e^{it\Delta} u_0 \|_{S(H^{1/2})} \leq c\delta_{sd} \) by Strichartz estimates. Thus, the small data (Proposition 2.3) yields \( SC(u_0) \) condition.

This observation gives the basis for induction: we assume \( G_v(0) < 1 \). Then for small \( \delta > 0 \) such that \( M[u]E[u] < \delta \), \( SC(u_0) \) holds.

Define the supremum of all such \( \delta \) for which \( SC(u_0) \) holds, namely,
\[
(ME)_c = \sup \{ \delta \mid u_0 \in H^1 \text{ with the property:} \}
\]
\[
G_v(0) < 1 \quad \text{and} \quad M[u]E[u] < \delta \Rightarrow SC(u_0) \text{ holds} \}.
\]
We want to show that \( (ME)_c = M[Q]E[Q] \). Observe that \( u_0(x) = Q(x) \) does not scatter, and this is the solution such that \( G_Q(0) = 1 \) and \( M[u]E[u] = M[Q]E[Q] \). To be precise, one should consider \( G_u(0) \leq 1 \) in the definition of \( (ME)_c \), instead
of the strict inequality \( G_n(0) < 1 \). However, \( G_n(0)=1 \) only when \( ME[u] = 1 \) (see Figure 1 point D), thus, it suffices to consider the strict inequality \( G_n(0) < 1 \).

Assume that \( (ME)_c < M[Q]E[Q] \).

**Step 1**: Induction on the scattering threshold and construction of the “critical” solution. Since \( (ME)_c < M[Q]E[Q] \), we can find a sequence of initial data \( \{u_{n,0}\} \) in \( H^1 \) which will approach the threshold \( (ME)_c \) from above and produce solutions which do not scatter, namely, there exists a sequence \( \{u_{n,0}\} \in H^1 \) producing the NLS solution \( u_n(t) = NLS(t)u_{n,0} \) with

\[
G_{u_n}(0) < \sigma \text{ and } M[u_{n,0}]E[u_{n,0}] \searrow (ME)_c \text{ as } n \to \infty
\]

and \( \|u_n\|_{L(H^{1/2})} = +\infty \) (this is possible by definition of supremum of \( (ME)_c \)), i.e., \( SC(u_{n,0}) \) does not hold.

This sequence will allow us to construct (via profile decompositions) a “critical” solution of NLS\(^+_n(\mathbb{R}^2) \), denoted by \( u_c(t) \), that will lie exactly at the threshold \( (ME)_c \) and will not scatter, see Proposition 6.1.

**Step 2**: Localization properties of the critical solution. The critical solution \( u_c(t) \) will have the property that it is precompact in \( H^1 \), namely, \( K = \{u_c(t)\mid t \in [0, +\infty)\} \) is precompact in \( H^1 \) (Lemma 6.2), and its localization implies that for given \( \epsilon > 0 \), there exists an \( R > 0 \) and some path \( x(t) \) such that \( \|\nabla u(x,t)\|^2_{L^2(|x+x(t)| > R)} \leq \epsilon \) uniformly in \( t \). This combined with the zero momentum will give control on the growth of \( x(t) \) (Lemma 6.3). Note that in the radial case \( x(t) \equiv 0 \). On the other hand, such compact in \( H^1 \) solutions with the control on \( x(t) \), can only be zero solutions, by the rigidity theorem (Theorem 5.9), which contradicts the fact that \( u_c \) does not scatter. Therefore, such \( u_c \) does not exist and the assumption that \( (ME)_c < M[Q]E[Q] \) is not valid. This finishes the proof of scattering in Theorem A*.

In section 5 we proceed with the linear and nonlinear profile decompositions and in section 6 we give the proof of claims in Step 1 and Step 2.

**5. Profile decomposition**

This subsection contains the profile decomposition for linear and nonlinear flows for \( \text{NLS}^+_n(\mathbb{R}^2) \), analogous to the Keraani [Ker01], and a reordering of the decompositions that will be used in the proof of the “weak” blow up.

**Proposition 5.1.** (Linear Profile decomposition.) Let \( \phi_n(x) \) be a uniformly bounded sequence in \( H^1 \). Then for each \( M \in \mathbb{N} \) there exists a subsequence of \( \phi_n \) (also denoted \( \phi_{n_j} \)), such that, for each \( 1 \leq j \leq M \), there exist, fixed in \( n \), a profile \( \psi_j^\delta \) in \( H^1 \), a sequence \( \tau_{n_j}^j \) of time shifts, a sequence \( \tau_{n_j}^\delta \) of space shifts and a sequence \( \tilde{W}_{n,j}(x) \) of remainders \( \tilde{W}_{n,j}(x) \) in \( H^1 \), such that \( \phi_{n_j}(x) = \sum_{j=1}^M e^{-it_{n_j}^\delta} \psi_j^\delta (x - \tau_{n_j}^\delta) + \tilde{W}_{n,j}(x) \) with the properties:

- Pairwise divergence for the time and space sequences. For \( 1 \leq k \neq j \leq M \),

\[
\lim_{n \to \infty} |\tau_{n_j}^j - \tau_{n_k}^k| + |\tau_{n_j}^\delta - \tau_{n_k}^\delta| = +\infty.
\]

\footnote{Here, \( \tilde{W}_{n,j}(x) \) and \( \tilde{W}_{n,j}(x) \) represent the remainders for the linear and nonlinear profile decompositions, respectively.}
• **Asymptotic smallness for the remainder sequence**

\[
\lim_{M \to \infty} \left( \lim_{n \to \infty} \| e^{it\Delta} W^M_n \|_{S(H^{1/2})} \right) = 0.
\]

• **Asymptotic Pythagorean expansion.** For fixed \( M \in \mathbb{N} \) and any \( 0 \leq s \leq 1 \), we have

\[
\| \phi_n \|^2_{H^s} = \sum_{j=1}^{M} \| \psi^j \|^2_{H^s} + \| W^M_n \|^2_{H^s} + o_n(1).
\]

**Proof.** Let \( \phi_n \) be uniformly bounded in \( H^1 \), i.e., there exists \( 0 < c_1 \) such that \( \| \phi_n \|_{H^1} \leq c_1 \). Let \( (q, r) \) be \( H^{1/2} \) admissible pair. Interpolation and Strichartz estimates with \( \theta = \frac{2}{r-2} \), \( 0 < \theta < 1 \), \( r_1 = 2r \), and \( q_1 = \frac{4}{r-2} \) yield

\[
\| e^{it\Delta} W^M_n \|_{L^q_t L^r_x} \leq \| e^{it\Delta} W^M_n \|_{L^q_t L^r_x}^{1-\theta} \| e^{it\Delta} W^M_n \|_{L^\infty_t L^2_x}^\theta.
\]

The goal is to decompose a profile \( \phi_n \) as \( \sum_{j=1}^{M} e^{-it\Delta} \psi^j(x - x^j_n) + W^M_n(x) \) with \( \| W^M_n(x) \|_{H^{1/2}} \leq c_1 \). Since (5.4) holds, it suffices to show

\[
\lim_{M \to +\infty} \left[ \limsup_{n \to +\infty} \| e^{it\Delta} W^M_n \|_{L^\infty_t L^2_x} \right] = 0.
\]

**Construction of \( \psi^1 \):** Let \( A_1 = \limsup_{n \to +\infty} \| e^{it\Delta} \phi_n \|_{L^\infty_t L^2_x} \). If \( A_1 = 0 \), we are done by taking \( \psi^1 = 0 \) for all \( j \). Suppose that \( A_1 > 0 \) and \( c_1 = \limsup_{n \to +\infty} \| \phi_n \|_{H^1} \). Passing to a subsequence \( \phi_n \), it is shown that there exist sequences \( t^j_n \) and \( x^j_n \) and a function \( \psi^1 \in H^1 \), such that \( e^{it^j_n \Delta} \phi_n(\cdot + x^j_n) \to \psi^1 \) in \( H^1 \), such that

\[
8c_1^2 \| \psi^1 \|^2_{H^{1/2}} \geq A_1^2.
\]

Define \( W^M_n(x) = \phi_n(x) - e^{-it^j_n \Delta} \psi^1(x - x^j_n) \). Observe that \( e^{it^j_n \Delta} \phi_n(\cdot + x^j_n) \to \psi^1 \) in \( H^1 \), for any \( 0 \leq s \leq 1 \) it follows \( \langle \phi_n, e^{-it^j_n \Delta} \psi^1 \rangle_{H^s} = \langle e^{it^j_n \Delta} \phi_n, \psi^1 \rangle_{H^s} \to \| \psi^1 \|^2_{H^s} \).

Since \( \| W^M_n \|^2_{H^s} = \| \phi_n - e^{-it^j_n \Delta} \psi^1, \phi_n - e^{-it^j_n \Delta} \psi^1 \|^2_{H^s} \), we have

\[
\lim_{n \to +\infty} \| W^M_n \|^2_{H^s} = \lim_{n \to +\infty} \| e^{it\Delta} \phi_n \|^2_{H^s} - \| \psi^1 \|^2_{H^s}.
\]

Thus, taking \( s = 1 \) and \( s = 0 \) yields \( \| W^M_n \|_{H^1} \leq c_1 \).

**Construction of \( \psi^j \) for \( j \geq 2 \):** Inductively \( \psi^j \) are constructed from \( W^M_n \). Let \( M \geq 2 \). Suppose that \( \psi^j, x^j \), \( t^j_n \), and \( W^j_n \) are known for \( j \in \{1, \cdots, M-1\} \). Consider \( A_M = \limsup \| e^{it\Delta} \phi_n \|_{L^\infty_t L^2_x} \). If \( A_M = 0 \), then we are done (by taking \( \psi^j = 0 \) for \( j \geq M \)). Assume \( A_M > 0 \). Apply the previous step to \( W^{M-1}_n \), and let \( c_M = \limsup \| W^{M-1}_n \|_{H^1} \), thus, we obtain sequences (or subsequences) \( x^{M-1}_n, t^{M-1}_n \) and a function \( \psi^{M-1} \in H^1 \) such that

\[
e^{it^{M-1}_n \Delta} W^{M-1}_n(x + x^{M-1}_n) \to \psi^{M-1} \text{ in } H^1 \text{ and } 8c_M^2 \| \psi^{M-1} \|_{H^{1/2}} \geq A_M^2.
\]

Define \( W^M_n(x) = W^{M-1}_n(x) - e^{-it^{M-1}_n \Delta} \psi^{M-1}(x - x^{M-1}_n) \). Then (5.3) and (5.5) follow from induction, i.e., assume (5.3) holds at rank \( M-1 \). Expanding \( \| W^M_n \|^2_{H^s} =

---

2One could choose \( r_1 = kr \), for \( k > 1 \), thus, \( q_1 = \frac{2kr}{k-2} \) and \( \theta = \frac{4(k-1)}{kr-2} \) and \( 0 < \theta < 1 \), however, the choice of \( r_1 = 2r \) is analogous with [HR08a].

3Since the \( \psi^j \) are constructed inductively as in the proof of [HR08a Lemma 5.2] we omit the details.
By rearranging and reindexing, we can find (5.9) and applying the weak convergence, yields (5.3) at rank $M$.

To show condition (5.1), assume the statement is true for $j, k \in \{1, \ldots, M - 1\}$, that is $|t_n^j - t_n^k| + |x_n^j - x_n^k| \to +\infty$. Take $k \in \{1, \ldots, M - 1\}$, we want to show that $|t_n^j - t_n^k| + |x_n^j - x_n^k| \to +\infty$. Passing to a subsequence, assume $t_n^j - t_n^k \to t_M^j$ and $x_n^j - x_n^k \to x_M^j$ are finite. Then as $n \to \infty$

$$e^{it_n^j \Delta} W_n^{M-1}(x + x_n^j) = e^{it_n^j \Delta} (e^{it_n^j \Delta} W_n^{j-1}(x + x_n^j) - \psi^j(x + x_n^j))$$

$$- \sum_{k=j+1}^{M-1} e^{it_n^j \Delta} \psi^k(x + x_n^j - x_n^k).$$

The orthogonality condition (5.1) implies that the right hand side goes to 0 weakly in $H^1$, while the left side converges weakly to $\psi^M$, which is nonzero, contradiction. Then the orthogonality condition (5.1) holds for $k = M$. Since (5.3) holds for all $M$, we have $\|\phi_n\|^2_{H^1} \geq \sum_{j=1}^M \|\psi^j\|^2_{H^1} + \|W_n^M\|^2_{H^1}$. Thus, $c_M \leq c_1$. Taking $s = 1/2$, and the fact that for all $M, A_M > 0$, yields together with (5.6)

$$\sum_{M \geq 1} \left( \frac{A_M^3}{8\delta^2} \right)^2 \leq \sum_{n \geq 1} \|\psi^M\|^2_{H^{1/2}} \leq \limsup_n \|\phi_n\|^2_{H^{1/2}} \leq \infty.$$

Therefore, $A_M \to 0$ as $M \to \infty$, which implies (5.2). \(\square\)

**Proposition 5.2.** (Energy Pythagorean expansion). Under the hypothesis of Proposition 5.1, we have

(5.7) \quad \mathcal{E}[\psi_n] = \sum_{j=1}^M \|e^{-it_n^j \Delta} \psi^j\|^2_{H^1} + \mathcal{E}[W_n^M] + o_n(1).

**Proof.** By definition of $\mathcal{E}[u]$ and (5.3) with $s = 1$, it suffices to prove that for all $M \leq 1$, we have

(5.8) \quad \|\phi_n\|^6_{L^6} = \sum_{j=1}^M \|e^{-it_n^j \Delta} \psi^j\|^6_{L^6} + o_n(1).

**Step 1. Pythagorean expansion of a sum of orthogonal profiles.** Fix $M \geq 1$. We want to show that the condition (5.1) yields

(5.9) \quad \left\| \sum_{j=1}^M e^{-it_n^j \Delta} \psi^j(x - x_n^j) \right\|^6_{L^6} = \sum_{j=1}^M \|e^{-it_n^j \Delta} \psi^j\|^6_{L^6} + o_n(1).

By rearranging and reindexing, we can find $M_0 \leq M$ such that

(a) $t_n^j$ is bounded in $n$ whenever $1 \leq j \leq M_0$,

(b) $|t_n^j| \to \infty$ as $n \to \infty$ if $M_0 + 1 \leq j \leq M$.

For case (a) take a subsequence and assume that for each $1 \leq j \leq M_0$, $t_n^j$ converges (in $n$), then adjust the profiles $\psi^j$’s such that we can take $t_n^j = 0$. From (5.1) we have $|x_n^j - x_n^k| \to +\infty$ as $n \to \infty$, which implies

(5.10) \quad \left\| \sum_{j=1}^{M_0} \psi^j(x - x_n^j) \right\|^6_{L^6} = \sum_{j=1}^{M_0} \|\psi^j\|^6_{L^6} + o_n(1),
For case (b), i.e., for $M_0 \leq k \leq M$, $|t^n_k| \to \infty$ as $n \to \infty$, take $\tilde{\psi} \in \dot{H}^{5/6} \cap L^{6/5}$, thus, the Sobolev embedding and the $L^p$ space-time decay estimate yield
\[
\|e^{-it^k_n \Delta} \psi^k\|_{L^6} \leq c \|\psi^k - \tilde{\psi}\|_{\dot{H}^{5/6}} + \frac{c}{|t^k_n|^{2/3}} \|\tilde{\psi}\|_{L^6},
\]
and approximating $\psi^k$ by $\tilde{\psi} \in C^\infty_0$ in $\dot{H}^{5/6}$, we have
\[
(5.11) \quad \|e^{-it^k_n \Delta} \psi^k\|_{L^6} \to 0 \text{ as } n \to \infty.
\]
Thus, combining (5.10) and (5.11), we obtain (5.8).

Step 2. Finishing the proof. Note that
\[
\|W_n^{M^1}\|_{L^6} \leq \|W_n^{M^1}\|_{L^\infty L^6} \leq \|W_n^{M^1}\|_{L^{1/2} L^{6/1}} \|W_n^{M^1}\|_{L^{1/2} L^{6/1}}^2 \\
\leq \|W_n^{M^1}\|_{L^{1/2} L^{6/1}} \|W_n^{M^1}\|_{L^{1/2} H^1} \leq \|W_n^{M^1}\|_{L^{1/2} L^{6/1}} \sup_n \|\phi_n\|_{H^1},
\]
where in the last line we used the embeddings $H^1 \hookrightarrow \dot{H}^{5/6} \hookrightarrow L^{12}$ on $\mathbb{R}^2$. Thus, by (5.2) it follows that
\[
(5.12) \quad \lim_{M^1 \to +\infty} \left( \lim_{n \to +\infty} \|e^{it^1_n \Delta} W_n^{M^1}\|_{L^6} \right) = 0.
\]
Let $M \geq 1$ and $\epsilon > 0$. The sequence of profiles $\{\psi^n\}$ is uniformly bounded in $H^1$ and in $L^6$. Thus, (5.12) implies the sequence of remainders $\{W_n^M\}$ is also uniformly bounded in $L^6$. Thus, pick $M_1 \geq M$ and $N_1$ such that for $n \geq N_1$, we have
\[
(5.13) \quad \left| \|\phi_n - W_n^{M^1}\|_{L^6} - \|\psi_n\|_{L^6} \right| + \left| \|W_n^M - W_n^{M^1}\|_{L^6} - \|W_n^M\|_{L^6} \right| \\
\leq C \left( \sup_n \|\phi_n\|_{L^6} + \sup_n \|W_n^{M^1}\|_{L^6} \right) \|W_n^{M^1}\|_{L^6} \leq \frac{\epsilon}{3}.
\]
Choosing $N_2 \geq N_1$ such that $n \geq N_2$, then (5.9) yields
\[
(5.14) \quad \left| \|\phi_n - W_n^{M^1}\|_{L^6} - \sum_{j=1}^{M_1} \|e^{-it^1_n \Delta} \psi^j\|_{L^6} \right| \leq \frac{\epsilon}{3}.
\]
Since $W_n^M - W_n^{M^1} = \sum_{j=M+1}^{M_1} e^{-it^1_n \Delta} \psi^j(\cdot - x_n^j)$, by (5.9), there exist $N_3 \geq N_2$ such that $N_3 \leq n$,
\[
(5.15) \quad \left| \|W_n^M - W_n^{M^1}\|_{L^6} - \sum_{j=M+1}^{M_1} \|e^{-it^1_n \Delta} \psi^j\|_{L^6} \right| \leq \frac{\epsilon}{3}.
\]
Thus, for $N_3 \geq n$, (5.13), (5.14), and (5.15) yield
\[
(5.16) \quad \left| \|\phi_n\|_{L^6} - \sum_{j=1}^M \|e^{-it^1_n \Delta} \psi^j\|_{L^6} - \|W_n^M\|_{L^6} \right| \leq \epsilon,
\]
which concludes the proof.

**Proposition 5.3 (Nonlinear Profile decomposition).** Let $\phi_n(x)$ be a uniformly bounded sequence in $H^1(\mathbb{R}^2)$. Then for each $M \in \mathbb{N}$ there exists a subsequence of $\phi_n$, also denoted by $\phi_n$, for each $1 \leq j \leq M$, there exist a (same for all $n$) nonlinear
profile \( \tilde{\psi}^j \) in \( H^1(\mathbb{R}^2) \), a sequence of time shifts \( t_n^j \), and a sequence of space shifts \( x_n^j \) and in addition, a sequence \((n)\) of remainders \( \tilde{W}^M_n(x) \) in \( H^1(\mathbb{R}^2) \), such that

\[
\phi_n(x) = \sum_{j=1}^{M} \text{NLS}(-t_n^j) \tilde{\psi}^j(x - x_n^j) + \tilde{W}^M_n(x),
\]

where \((as n \to \infty)\)

(a) for each \( j \), either \( t_n^j = 0, t_n^j \to +\infty \) or \( t_n^j \to -\infty \),

(b) if \( t_n^j \to +\infty \), then \( \|\text{NLS}(-t)\tilde{\psi}^j\|_{S((0,\infty), \dot{H}^{1/2})} < +\infty \)

and if \( t_n^j \to -\infty \), then \( \|\text{NLS}(-t)\tilde{\psi}^j\|_{S((0,\infty), \dot{H}^{1/2})} < +\infty \),

(c) for \( k \neq j \), then \( |t_n^j - t_n^k| + |x_n^j - x_n^k| \to +\infty \).

The remainder sequence has the following asymptotic smallness property:

\[
\lim_{M \to \infty} \left( \lim_{n \to \infty} \|\text{NLS}(t)\tilde{W}^M_n\|_{S(\dot{H}^{1/2})} \right) = 0.
\]

For fixed \( M \in \mathbb{N} \) and any \( 0 \leq s \leq 1 \), we have the asymptotic Pythagorean expansion

\[
\|\phi_n\|^2_{H^s} = \sum_{j=1}^{M} \|\text{NLS}(-t_n^j)\tilde{\psi}^j\|^2_{H^s} + \|\tilde{W}^M_n\|^2_{H^s} + o_n(1)
\]

and the energy Pythagorean decomposition (note that \( E[\text{NLS}(-t_n^j)\tilde{\psi}^j] = E[\tilde{\psi}^j] \)):

\[
E[\phi_n] = \sum_{j=1}^{M} E[\tilde{\psi}^j] + E[\tilde{W}^M_n] + o_n(1).
\]

**Proof.** From Proposition 5.1 given that \( \phi_n(x) \) is a uniformly bounded sequence in \( H^1 \), we have

\[
\phi_n(x) = \sum_{j=1}^{M} e^{-it_n^j \Delta} \psi^j(x - x_n^j) + W^M_n(x)
\]
satisfying (5.1), (5.2), (5.3) and (5.7). We will choose \( M \in \mathbb{N} \) later. To prove this proposition, the idea is to replace a linear flow \( e^{it\Delta} \psi^j \) by some nonlinear flow.

Now for each \( \psi^j \) we can apply the wave operator (Proposition 3.3) to obtain a function \( \tilde{\psi}^j \in H^1 \), which we will refer to as the nonlinear profile (corresponding to the linear profile \( \psi^j \)) such that the following properties hold:

For a given \( j \), there are two cases to consider: either \( t_n^j \) is bounded, or \( |t_n^j| \to +\infty \).

**Case** \( |t_n^j| \to +\infty \): If \( t_n^j \to +\infty \), Proposition 3.3 Part I (3.7) implies that

\[
\|\text{NLS}(-t_n^j)\tilde{\psi}^j - e^{-it_n^j \Delta} \psi^j\|_{H^1} \to 0 \text{ as } t_n^j \to +\infty \text{ and so}
\]

\[
\|\text{NLS}(-t)\tilde{\psi}^j\|_{S((0,\infty), \dot{H}^{1/2})} < +\infty.
\]

Similarly, if \( t_n^j \to -\infty \), by (3.5) \( \psi^j \) we obtain \( \|\text{NLS}(-t_n^j)\tilde{\psi}^j - e^{-it_n^j \Delta} \psi^j\|_{H^1} \to 0 \) as \( t_n^j \to -\infty \), and hence,

\[
\|\text{NLS}(-t)\tilde{\psi}^j\|_{S((-\infty,0), \dot{H}^{1/2})} < +\infty.
\]

**Case** \( t_n^j \text{ is bounded} \) \((as n \to \infty)\): Adjusting the profiles \( \psi^j \) we reduce it to the case \( t_n^j = 0 \). Thus, (5.1) becomes \( |x_n^j - x_n^k| \to +\infty \) as \( n \to \infty \), and continuity of the linear flow in \( H^1 \), leads to \( e^{-it_n^j \Delta} \psi^j \to \psi^j \) strongly in \( H^1 \) as \( n \to \infty \). In this case, we simply let \( \tilde{\psi}^j = \text{NLS}(0) e^{-i(t_n^j \Delta) \psi^j} = e^{-it \Delta} \psi^j = \psi^j \)
Thus, in either case of sequence \( \{t_n^j\} \), we have a new nonlinear profile \( \tilde{\psi}^j \) associated to each original linear profile \( \psi^j \) such that

\[
\|\text{NLS}(-t_n^j)\tilde{\psi}^j - e^{-it_n^j\Delta}\psi^j\|_{H^1} \to 0 \quad \text{as} \quad n \to +\infty.
\]

Thus, we can substitute \( e^{-it_n^j\Delta}\psi^j \) by \( \text{NLS}(-t_n^j)\tilde{\psi}^j \) in (5.21) to obtain

\[
\phi_n(x) = \sum_{j=1}^{M} \text{NLS}(-t_n^j)\tilde{\psi}^j(x - x_n^j) + \tilde{W}_n^M(x),
\]

where

\[
\tilde{W}_n^M(x) = W_n^M(x) + \sum_{j=1}^{M} \left\{ e^{-it_n^j\Delta}\psi^j(x - x_n^j) - \text{NLS}(-t_n^j)\tilde{\psi}^j(x - x_n^j) \right\}
\]

The triangle inequality yields

\[
\|e^{it\Delta}\tilde{W}_n^M\|_{S(\dot{H}^{1/2})} \leq \|e^{it\Delta}W_n^M\|_{S(\dot{H}^{1/2})} + c \sum_{j=1}^{M} \|e^{-it_n^j\Delta}\psi^j - \text{NLS}(-t_n^j)\tilde{\psi}^j\|_{S(\dot{H}^{1/2})}.
\]

By (5.24) we have that \( \|e^{it\Delta}\tilde{W}_n^M\|_{S(\dot{H}^{1/2})} \leq \|e^{it\Delta}W_n^M\|_{S(\dot{H}^{1/2})} + c \sum_{j=1}^{M} a_n(1) \), and thus, \( \lim_{M \to \infty} \left( \lim_{n \to \infty} \|e^{it\Delta}\tilde{W}_n^M\|_{S(\dot{H}^{1/2})} \right) = 0 \). Now we are going to apply a nonlinear flow to \( \phi_n(x) \) and approximate it by a combination of “nonlinear bumps” \( \text{NLS}(t - t_n^j)\tilde{\psi}^j(x - x_n^j) \), i.e., \( \text{NLS}(t)\phi_n(x) \approx \sum_{j=1}^{M} \text{NLS}(t - t_n^j)\tilde{\psi}^j(x - x_n^j) \).

Obviously, this cannot hold for any bounded in \( H^1 \) sequence \( \{\phi_n\} \), since, for example, a nonlinear flow can introduce finite time blowup solutions. However, under the proper conditions we can use the long term perturbation theory (Proposition 2.7) to guarantee that a nonlinear flow behaves basically similar to the linear flow.

To simplify notation, introduce the nonlinear evolution of each separate initial condition \( u_{n,0} = \phi_n \): \( u_n(t,x) = \text{NLS}(t)\phi_n(x) \), the nonlinear evolution of each separate nonlinear profile (“bump”): \( v^j(t,x) = \text{NLS}(t)\tilde{\psi}^j(x) \), and a linear sum of nonlinear evolutions of “bumps”: \( \tilde{u}_n(t,x) = \sum_{j=1}^{M} v^j(t - t_n^j, x - x_n^j) \).

Intuitively, we think that \( \phi_n = u_{n,0} \) is a sum of bumps \( \tilde{\psi}^j \) (appropriately transformed) and \( u_n(t) \) is a nonlinear evolution of their entire sum. On the other hand, \( \tilde{u}_n(t) \) is a sum of nonlinear evolutions of each bump so we now want to compare \( u_n(t) \) with \( \tilde{u}_n(t) \).

Note that if we had just the linear evolutions, then both \( u_n(t) \) and \( \tilde{u}_n(t) \) would be the same.

Thus, \( u_n(t) \) satisfies \( i\partial_t u_n + \Delta u_n + |u_n|^4 u_n = 0 \), and \( \tilde{u}_n(t) \) satisfies \( i\partial_t \tilde{u}_n + \Delta \tilde{u}_n + |\tilde{u}_n|^4 \tilde{u}_n = \tilde{e}_n^M \), where \( \tilde{e}_n^M = |\tilde{u}_n|^4 \tilde{u}_n - \sum_{j=1}^{M} |v_n^j(t - t_n^j, \cdot - x_n^j)|^4 v_n^j(t - t_n^j, \cdot - x_n^j) \).

**Claim 5.4.** There exists a constant \( A \) independent of \( M \), and for every \( M \), there exists \( n_0 = n_0(M) \) such that if \( n > n_0 \), then \( \|\tilde{u}_n\|_{S(\dot{H}^{1/2})} \leq A \).

**Claim 5.5.** For each \( M \) and \( \epsilon > 0 \), there exists \( n_1 = n_1(M, \epsilon) \) such that if \( n > n_1 \), then \( \|\tilde{e}_n^M\|_{L^{12/5}_{t,x}} \leq \epsilon \).

We prove both claims at the end of this proof.
Note $\tilde{u}_n(0, x) - u_n(0, x) = \tilde{W}_n^M(x)$. Then for any $\tilde{\varepsilon} > 0$ there exists $M_1 = M_1(\tilde{\varepsilon})$ large enough such that for each $M > M_1$ there exists $n_2 = n_2(M)$ with $n > n_2$ implying

$$\|e^{it\Delta}(\tilde{u}_n(0) - u_n(0))\|_{S(\dot{H}^{1/2})} \leq \tilde{\varepsilon}. $$

Therefore, for $M$ large enough and $n = \max(n_0, n_1, n_2)$, since

$$e^{it\Delta}(\tilde{u}_n(0)) = e^{it\Delta}\left(\sum_{j=1}^{M} v_j(-t^j_n, x - x^j_n)\right),$$

which are scattering by (5.24), Proposition 2.7 implies $\|u_n\|_{S(\dot{H}^{1/2})} < +\infty$, a contradiction.

Coming back to the nonlinear remainder $\tilde{W}_n^M$, we estimate its nonlinear flow as follows (recall the notation of $\tilde{W}_n^M$, $W_n^M$ and $T^j$ in (5.26)):

By Strichartz estimates (2.1) and by the triangle inequality, we get

$$\|\text{NLS}(t)\tilde{W}_n^M\|_{S(\dot{H}^{1/2})} \leq \|e^{it\Delta}\tilde{W}_n^M\|_{S(\dot{H}^{1/2})} + \left\|\tilde{W}_n^M\right\|_{L^4_tL^8_x}^4 \left\|\tilde{W}_n^M\right\|_{S(\dot{H}^{1/2})}.$$

And

$$\left\|\tilde{W}_n^M\right\|_{L^4_tL^8_x}^4 \left\|\tilde{W}_n^M\right\|_{S(\dot{H}^{1/2})} \leq \left\|D^{1/2}\tilde{W}_n^M\right\|_{L^4_tL^8_x}^4 \left\|D^{1/2}T^j\right\|_{L^1_tL^6_x}^4 \leq c \sum_{j=1}^{M} \left\|T^j\right\|_{S(\dot{H}^{1/2})}^4 \left\|T^j\right\|_{S(L^2)}^4 \leq c \sum_{j=1}^{M} \left\|T^j\right\|_{S(\dot{H}^{1/2})} \left\|T^j\right\|_{S(\dot{H}^{1/2})}.$$

The $S'(\dot{H}^{-1/2})$ norm is bounded by $S'(L^2)$ norm which is estimated by $L^\frac{3}{2}_tL^6_x$ norm (the pair (3.6) is an $L^2$ admissible), apply Chain rule Lemma 2.1 followed by the Hölder’s inequality, and finally, the $L^5_tL^8_x$ and $L^6_tL^3_x$ norms are estimated by the $S'(H^{1/2})$ norm and $S(L^2)$ norm, respectively. And $\dot{H}^1 \hookrightarrow H^{1/2}$, yields (5.27). Hence,

$$\|\text{NLS}(t)\tilde{W}_n^M\|_{S(\dot{H}^{1/2})} \leq \|e^{it\Delta}\tilde{W}_n^M\|_{S(\dot{H}^{1/2})} + c \sum_{j=1}^{M} \left\|e^{-it\Delta}\psi^j - \text{NLS}(-it\tilde{n})\tilde{\psi}^j\right\|_{H^1}^5,$$

and by (5.24) the second term in (5.28) goes to zero as $n \to \infty$ and then applying (5.2) the first term in (5.28) goes to zero as $M \to \infty$, hence, we obtain

$$\lim_{n \to \infty} \|\text{NLS}(t)\tilde{W}_n^M\|_{S(\dot{H}^{1/2})} = 0 \quad \text{as} \quad M \to \infty.$$ 

Thus we proved (6.18) which completes the decomposition (5.17). This also gives (5.19).

Next, we obtain the Energy Pythagorean decomposition. We substitute the linear flow in Lemma 5.2 by the nonlinear and repeat the above long term perturbation argument to obtain

$$\|\phi_n\|_{L^6_t}^6 = \sum_{j=1}^{M} \|\text{NLS}(-t\tilde{n})\psi^j\|_{L^6}^6 + \|\tilde{W}_n^M\|_{L^6}^6 + o_n(1),$$

which yields the energy Pythagorean decomposition (5.20). The proof will be concluded after we prove the Claims (5.3) and (5.5).
Proof of Claim 5.4. We show that for a large constant $A$ independent of $M$ and if $n > n_0 = n_0(M)$, then $\|\tilde{u}_n\|_{S(H^{1/2})} \leq A$.

Let $M_0$ be a large enough such that $\|e^{it\Delta}(\overline{U}_n^M)\|_{S(H^{1/2})} \leq \delta_d$. Then, by (5.20), for each $j > M_0$, we have $\|e^{it\Delta}\phi^j\|_{S(H^{1/2})} \leq \delta_d$, thus, Proposition 3.5 yields $\|\psi\|_{S(H^{1/2})} \leq 2\|e^{it\Delta}\psi\|_{S(H^{1/2})}$ for $j > M_0$.

Recall the following inequality: for $a_j \geq 0$, $\left| \left( \sum_{j=1}^{M} a_j \right)^4 - \sum_{j=1}^{M} a_j^4 \right| \leq c_M \sum_{j \neq k} |a_j||a_k|^3$.

Then we have
\[ \|\tilde{u}_n\|_{L_t^8 L_x^8}^8 = \sum_{j=1}^{M_0} \|\psi\|_{L_t^8 L_x^8}^8 + \sum_{j=M_0+1}^{M} \|\psi\|_{L_t^8 L_x^8}^8 + \text{cross terms} \]
(5.30)
\[ \leq \sum_{j=1}^{M_0} \|\psi\|_{L_t^8 L_x^8}^8 + 2^8 \sum_{j=M_0+1}^{M} \|e^{it\Delta}\phi^j\|_{L_t^8 L_x^8}^8 + \text{cross terms}, \]
note that by (5.21) we have
\[ \|e^{it\Delta}\phi^j\|_{L_t^8 L_x^8}^8 = \sum_{j=1}^{M_0} \|e^{it\Delta}\phi^j\|_{L_t^8 L_x^8}^8 + 2^8 \sum_{j=M_0+1}^{M} \|e^{it\Delta}\phi^j\|_{L_t^8 L_x^8}^8 + \text{cross-terms}. \]

Observe that by (5.1) and taking $n_0 = n_0(M)$ large enough, we can consider $\{u_n\}_{n>n_0}$ and thus, make “the cross terms” $\leq 1$.

Then (5.31) and $\|e^{it\Delta}\phi_n\|_{L_t^8 L_x^8} \leq c \|\phi_n\|_{H^{1/2}} \leq c_1 \sum_{j=M_0+1}^{M} \|e^{it\Delta}\phi^j\|_{L_t^8 L_x^8}^8$ is bounded independent of $M$ provided $n > n_0$. Thus, if $n > n_0$, (5.30) yields $\|\tilde{u}_n\|_{L_t^8 L_x^8}$ is also bounded independent of $M$.

In a similar fashion, one can prove that $\|u_n\|_{L_t^8 L_x^8}$ is bounded independent of $M$ provided $n > n_0$. Interpolation between these exponents gives $\|\tilde{u}_n\|_{L_t^{12/5} L_x^{6/5}}$, which is as well bounded independent of $M$ for $n > n_0$. To close the argument, we apply Kato estimate (2.20) to the integral equation of $i\partial_t \tilde{u}_n + \Delta \tilde{u}_n + |\tilde{u}_n|^4 \tilde{u}_n = e^M \tilde{M}_n$. Using $\|e^M \tilde{M}_n\|_{S^{-1}(H^{-1/2})} \leq 1$ (Claim 5.5), as in Proposition 2.7, we obtain that $\|\tilde{u}_n\|_{S(H^{1/2})}$ is as well bounded independent of $M$ provided $n > n_0$. Thus, Claim 5.4 is proved.

Proof of Claim 5.5. The expansion of $e^M \tilde{M}_n$ consist of $\sim M^5$ cross terms of the form $\prod_{k=1}^{5} v^{j_k}(t - t_{j_k}^n, x - x_{j_k}^n)$, where not all five $j_k$’s are the same. Without lost of generalization, assume that a pair $j_1 \neq j_2$. We estimate simply by Hölder’s
\[ \| \prod_{k=1}^{5} v^{j_k}(t - t_{j_k}^n, \cdot - x_{j_k}^n) \|_{L_t^{12/5} L_x^{6/5}} \]
\[ \leq \| v^{j_1}(t - t_{j_1}^n, \cdot - x_{j_1}^n) v^{j_2}(t - t_{j_2}^n, \cdot - x_{j_2}^n) \|_{L_t^6 L_x^2} \prod_{m=3}^{5} \| v^{j_m}(t - t_{j_m}^n, \cdot - x_{j_m}^n) \|_{L_t^6 L_x^2}. \]

Note that either $\{t_{j_k}^n\} \to \pm \infty$ or $\{t_{j_k}^n\}$ is bounded.

If $\{t_{j_1}^n\} \to \pm \infty$, without loss of generalization assume $|t_{j_1}^n - t_{j_2}^n| \to \infty$ as $n \to \infty$ and by adjusting the profiles that $|x_{j_1}^n - x_{j_2}^n| \to 0$ as $n \to \infty$. Since
\[ v^{j_1}, v^{j_2} \in L^1_t L^6_x \hookrightarrow S(\dot{H}^{1/2}) \]

\[ \| v^{j_1} (t - (t^{j_2}_n - t^{j_2}_n), x) v^{j_2} (t, x) \|_{L^6_t L^3_x} \to 0. \]

If \( \{ t^{j_n}_n \} \) is bounded, without loss assume \( |t^{j_1}_n - t^{j_2}_n| \to 0 \) and \( |x^{j_1}_n - x^{j_2}_n| \to \infty \) as \( n \to \infty \), then \( \| v^{j_1} (t, x - (x^{j_1}_n - x^{j_2}_n)) v^{j_2} (t, x) \|_{L^6_t L^3_x} \to 0 \), since \( v^{j_1}, v^{j_2} \in L^1_t L^6_x \hookrightarrow S(\dot{H}^{1/2}) \). Thus, in either case we obtain Claim \( 5.3 \). This finishes the proof of Proposition \( 5.3 \). \[ \square \]

Observe that \( 5.19 \) gives \( \dot{H}^1 \) asymptotic orthogonality at \( t = 0 \) and the following lemma extends it to the bounded NLS flow for \( 0 \leq t \leq T \).

**Lemma 5.6.** (\( \dot{H}^1 \) Pythagorean decomposition along the bounded NLS flow.) Suppose \( \phi_n \) is a bounded sequence in \( \dot{H}^1 \). Let \( T \in (0, \infty) \) be a fixed time. Consider the nonlinear profile decomposition from Proposition \( 5.3 \). Denote \( \tilde{W}^M_n (t) \equiv \text{NLS}(t) \tilde{W}^M_n \). Then for all \( j \), the nonlinear profiles \( v^j (t) \equiv \text{NLS}(t) \tilde{v}^j \) exist up to time \( T \) and for all \( t \in [0, T] \).

\[ (5.32) \quad \| \nabla u_n (t) \|_{L^2_x}^2 = \sum_{j=1}^M \| \nabla v^j (t - t^{j_n}_n) \|_{L^2_x}^2 + \| \nabla \tilde{W}^M_n (t) \|_{L^2_x}^2 + o_n (1), \]

where \( o_n (1) \to 0 \) uniformly on \( 0 \leq t \leq T \).

**Proof.** Let \( M_0 \) be such that for \( M \geq M_0 \), we have \( \| \text{NLS}(t) \tilde{W}^M_n \|_{S(\dot{H}^{1/2})} \leq \delta_{sd} \) (as in Proposition \( 2.3 \)). Reorder the first \( M_0 \) profiles and denote by \( M_2 \), \( 0 \leq M_2 \leq M_0 \), such that

1. For each \( 1 \leq j \leq M_2 \), we have \( t^{j_n}_n = 0 \). Observe that if \( M_2 = 0 \), there are no \( j \) in this case.
2. For each \( M_2 + 1 \leq j \leq M_0 \), we have \( |t^{j_n}_n| \to \infty \). If \( M_2 = 0 \), then it means that there are no \( j \) in this case.

From Proposition \( 5.3 \) we have that \( v^j (t) \) for \( j > M_0 \) are scattering and for a fixed \( T \) and \( M_2 + 1 \leq j \leq M_0 \) we have \( \| v^j (t - t^{j_n}_n) \|_{S(\dot{H}^{1/2} [0, T])} \to 0 \) as \( n \to \infty \).

In fact, taking \( t^{j_n}_n \to +\infty \) and \( \| v^j (-t) \|_{S(\dot{H}^{1/2} [0, +\infty])} < \infty \), dominated convergence leads \( \| v^j (-t) \|_{L^6_{t, x} L^3_x} < \infty \), for \( q < \infty \), and consequently, \( \| v^j (t - t^{j_n}_n) \|_{L^6_{t, x} L^3_x} \to 0 \) as \( n \to \infty \). As \( v^j (t) \) has been constructed via the existence of wave operators to converge in \( \dot{H}^1 \) to a linear flow at \( \pm \infty \), the \( L^4_x \) decay of the linear flow together with the \( H^1 \) embedding yields \( \| v^j (t - t) \|_{L^6_{t, x} L^3_x} \to 0 \).

Let \( B = \max (1, \lim_n \| \nabla u_n (t) \|_{L^6_{t, x} L^3_x}) < \infty \). For each \( 1 \leq j \leq M_2 \), let \( T^j \leq T \) be the maximal forward time such that \( \| \nabla v^j \|_{L^6_{t, x} L^3_x} \leq 2B \). Denote by \( \tilde{T} = \min_{1 \leq j \leq M_2} T^j \), or \( \tilde{T} = T \) if \( M_2 = 0 \). It is sufficient to prove that \( 5.32 \) holds for \( \tilde{T} = T \), since then for each \( 1 \leq j \leq M_2 \) we will have \( T^j = T \), and therefore, \( \tilde{T} = T \).
Thus, let’s consider $[0,T]$. For each $1 \leq j \leq M_2$, we have

\begin{align*}
\|v^j(t)\|_{S(\dot{H}^{1/2};[0,T])} &\lesssim \|v^j\|_{L^\infty_{[0,T]}L^4_x} + \|v^j\|_{L^4_{[0,T]}L^{(4^*)'}_x} \\
&\lesssim \|v^j\|_{L^\infty_{[0,T]}L^2_x}^{1/2} \|v^j\|_{L^\infty_{[0,T]}L^\infty_x}^{1/2} + \|v^j\|_{L^4_{[0,T]}L^\infty_x} \|v^j\|_{L^4_{[0,T]}L^{(4^*)'}_x}^{1/2} \\
&\lesssim \|v^j\|_{L^\infty_{[0,T]}L^2_x}^{1/2} \|\nabla v^j\|_{L^\infty_{[0,T]}L^2_x}^{1/2} + \|v^j\|_{L^4_{[0,T]}L^\infty_x} \|\nabla v^j\|_{L^4_{[0,T]}L^2_x}^{1/2} \\
&\lesssim (\|v^j\|_{L^\infty_{[0,T]}L^2_x}^{1/2} + \|v^j\|_{L^4_{[0,T]}L^\infty_x}) \|\nabla v^j\|_{L^4_{[0,T]}L^2_x}^{1/2} \\
&\lesssim (\dot{T}^{1/2}) B,
\end{align*}

note that (5.33) comes from the “end point” admissible Strichartz norms ($L^1_tL^{(4^*)'}_x$ and $L^\infty_tL^4_x$) since all other $S(\dot{H}^{1/2})$ norms will be bounded by interpolation; (5.34) is obtained using Hölder’s inequality; the Sobolev’s embedding $\dot{H}^1 \hookrightarrow L^\infty$ and $\dot{H}^{1-(4^*)'/2} \hookrightarrow L^{(4^*)'}$ leads to (5.33); since $(4^*)'$ is large, we have the Sobolev’s embedding $\dot{H}^1 \hookrightarrow H^{1-(4^*)'/2}$ implying (5.36), and finally, since $\|v^j\|_{L^\infty_{[0,T]}L^2_x} = \|\psi\|_{L^2_x} \leq \lim_{n} \|\phi_n\|_{L^2_x}$ obtained from (5.19) with $s = 0$, we have (5.37).

As in proof of Proposition 5.3 set $\tilde{u}_n(t,x) = \sum_{j=1}^{M} v^j(t - t^n_j, x - x^n_j)$ and $\tilde{e}_n = i\partial_t \tilde{u}_n + \Delta \tilde{u}_n + |\tilde{u}_n|^4 \tilde{u}_n$. Thus, for $M > M_0$ we have

Claim 5.4. There exist a constant $A = A(\dot{T})$ independent of $M$, and for every $M$, there exists $n_0 = n_0(M)$ such that if $n > n_0$, then $\|\tilde{u}_n\|_{S(\dot{H}^{1/2};[0,T])} \leq A$.

Claim 5.5. For each $M$ and $\epsilon > 0$, there exists $n_1 = n_1(M, \epsilon)$ such that for $n > n_1$, then $\|\tilde{e}_n\|_{L^{12/5}_{t_x}L^{6/5}_{x}} \leq \epsilon$.

Remark 5.7. Note since $u(0) - \tilde{u}_n(0) = W_n^M$, there exists $M' = M'(\epsilon)$ large enough so that for each $M > M'$ there exists $n_2 = n_2(M)$ such that $n > n_2$ implies

$$\|e^{itA}(u(0) - \tilde{u}_n(0))\|_{S(\dot{H}^{1/2};[0,T])} \leq \epsilon.$$  

Thus, the long time perturbation argument\(^4\) (Proposition 2.7) gives us $\epsilon_0 = \epsilon_0(A)$. Selecting an arbitrary $\epsilon \leq \epsilon_0$, and from Remark 5.7 take $M' = M'(\epsilon)$. Now select an arbitrary $M > M'$ and take $n' = \max(n_0, n_1, n_2)$. Then combining Claims 5.4, 5.5, Remark 5.7 and Proposition 5.3 we obtain that for $n > n'(M, \epsilon)$ with $c = c(A) = c(\dot{T})$ we have

(5.38) \[ \|u_n - \tilde{u}_n\|_{S(\dot{H}^{1/2};[0,T])} \leq c(\dot{T}) \epsilon. \]

We will next prove (5.32) for $0 \leq t \leq \dot{T}$. Recall that $\|v^j(t - t^n_j)\|_{S(\dot{H}^{1/2};[0,T])} \rightarrow 0$ as $n \rightarrow \infty$ and for each $1 \leq j \leq M_2$, we have $\|\nabla v^j\|_{L^\infty_{[0,T]}L^2_x} \leq 2B$. By Strichartz

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\(^4\)Note that in Proposition 2.7 $T = +\infty$, while here, it is not necessary. However, $T$ does not form part of the parameter dependence since $\epsilon_0$ depends only on $A = A(T)$, not on $T$, that is, there will be dependence on $T$, but it is only through $A$.
estimates, $\|\nabla v^j(t-t_n^j)\|_{L^\infty_{[0,T]}L^2} \lesssim \|\nabla v^j(-t_n^j)\|_{L^\infty_{[0,T]}L^2}$, then

$$\|\nabla u_n(t)\|_{L^\infty_{[0,T]}L^2}^2 = \sum_{j=1}^{M_2} \|\nabla v^j(t)\|_{L^\infty_{[0,T]}L^2}^2 + \sum_{j=M_2+1}^{M} \|\nabla v^j(t-t_n^j)\|_{L^\infty_{[0,T]}L^2}^2 + o_n(1)$$

$$\lesssim M_2 B^2 + \sum_{j=M_2+1}^{M} \|\nabla NLS(-t_n^j)\|_{L^2}^2 + o(1)$$

$$\lesssim M_2 B^2 + \|\nabla \phi_n\|_{L^2}^2 + o_n(1) \lesssim M_2 B^2 + B^2 + o_n(1).$$

Using (5.38), we obtain

$$\|u_n - \bar{u}_n\|_{L^\infty_{[0,T]}L^6} \lesssim \|u_n - \bar{u}_n\|_{L^\infty_{[0,T]}L^2}^{2/3} \|u_n - \bar{u}_n\|_{L^\infty_{[0,T]}L^\infty}^{1/3}$$

$$\lesssim \|u_n - \bar{u}_n\|_{L^\infty_{[0,T]}L^2}^{1/6} \|\nabla(u_n - \bar{u}_n)\|_{L^\infty_{[0,T]}L^2}^{1/6} \|\nabla(u_n - \bar{u}_n)\|_{L^\infty_{[0,T]}L^2}^{1/3}$$

$$\lesssim \epsilon(T)^{1/6}(M_2 B^2 + B^2 + o(1))^{1/6} e^{1/3}.$$

Similar to the argument in the proof of (5.29), we establish that for $0 \leq t \leq \tilde{T}$

$$\|u_n(t)\|^6_{L^6} = \sum_{j=1}^{M} \|v^j(t-t_n^j)\|^6_{L^6} + \|\tilde{W}^M_n(t)\|^6_{L^6} + o_n(1).$$

Energy conservation and (5.20) give us

$$E[u_n(t)] = \sum_{j=1}^{M} E[v^j] + E[\tilde{W}^M_n] + o_n(1).$$

Combining (5.39) and (5.40) completes the proof. \qed

6. Proofs of claims in Step 1 and Step 2 for scattering

**Proposition 6.1.** (Existence of a critical solution.) There exists a global solution $u_c(t) \in H^1(\mathbb{R}^2)$ with initial datum $u_{c,0} \in H^1(\mathbb{R}^2)$ such that $\|u_{c,0}\|_{L^2} = 1$, $E[u_c] = (ME) c < M|Q|E[Q], \mathcal{G}_{u_c}(t) < 1$ for all $0 \leq t < +\infty$,

(6.1) \hspace{1cm} \text{and} \hspace{1cm} \|u_c\|_{S(H^{1/2})} = +\infty.

Note that the condition $E[u_c] = (ME) c < M|Q|E[Q]$ is equivalent to $\mathcal{M}[u_c] < 1$.

**Proof.** Consider a sequence of solutions $u_n(t)$ to NLS on $\mathbb{R}^2$ with corresponding initial data $u_{n,0}$ such that $\mathcal{G}_{u_n}(0) < 1$ and $M[u_n]E[u_n] \searrow (ME)_c$ as $n \to +\infty$, for which SC($u_{n,0}$) does not hold for any $n$.

Without loss of generality, rescale the solutions so that $\|u_{n,0}\|_{L^2} = 1$, thus

$$\|\nabla u_{n,0}\|_{L^2} < |Q| L^2 \|\nabla Q\|_{L^2} \text{ and } E[u_n] \searrow (ME) c.$$

By construction, $\|u_n\|_{S(H^{1/2})} = +\infty$. Note that the sequence $\{u_{n,0}\}$ is uniformly bounded on $H^1$. Thus, applying the nonlinear profile decomposition (Proposition 5.20), we have

(6.2) \hspace{1cm} u_{n,0}(x) = \sum_{j=1}^{M} \text{NLS}(-t_n^j)\psi^j(x - x_n^j) + \tilde{W}^M_n(x).
Now we will refine the profile decomposition property (b) in Proposition 5.3 by using part II of Proposition 3.5 (wave operator), since it is specific to our particular setting here.

Recall that in nonlinear profile decomposition we considered 2 cases when $|t_n| \to \infty$ and $|t_n|$ is bounded. In the first case, we can refine it to the following:

First note that we can obtain $\tilde{\psi}^j$ (from linear $\psi^j$) such that

$$\|\text{NLS}(-t_n^j)\tilde{\psi}^j - e^{-it_n^j\Delta} \psi^j\|_{H^1} \to 0 \quad \text{as} \quad n \to +\infty$$

with properties (3.10) and (5.11), since the linear profiles $\psi^j$'s satisfy

$$\sum_{j=1}^M \lim_{n \to +\infty} E[e^{-it_n^j\Delta}\psi^j] + \lim_{n \to +\infty} E[W_n^M] = \lim_{n \to +\infty} M[u_{n,0}] = (ME)_c,$$

thus, $M[\psi^j] \leq 1.$

Also,

$$\sum_{j=1}^M \lim_{n \to +\infty} E[e^{-it_n^j\Delta}\psi^j] + \lim_{n \to +\infty} E[W_n^M] = \lim_{n \to +\infty} E[u_{n,0}] = (ME)_c,$$

since each $E[e^{-it_n^j\Delta}\psi^j] \geq 0$ (Lemma 3.3), we have

$$\lim_{n \to +\infty} E[e^{-it_n^j\Delta}\psi^j] \leq (ME)_c$$

and thus,

$$\frac{1}{2} \|\psi^j\|_{L^2}^2 \|\nabla\psi^j\|_{L^2}^2 \leq M[\psi^j] \lim_{n \to +\infty} E[e^{-it_n^j\Delta}\psi^j] \leq (ME)_c.$$

The properties (3.10) for $\tilde{\psi}^j$ imply that $ME[\tilde{\psi}^j] < (ME)_c$, and thus we get that

$$\|\text{NLS}(t)\tilde{\psi}^j(\cdot - x_n^j)\|_{S(R^{1/2})} < +\infty.$$

This fact will be essential for the case 1 below. Otherwise, in the nonlinear decomposition (6.2) we also have the Pythagorean decomposition for mass and energy:

$$\sum_{j=1}^M \lim_{n \to +\infty} E[\tilde{\psi}^j] + \lim_{n \to +\infty} E[W_n^M] = \lim_{n \to +\infty} M[u_{n,0}] = (ME)_c,$$

so we have (6.1) with $\sigma = \frac{1}{\sqrt{2}}.$ Again, since each energy is greater than 0 (Lemma 3.9), for all $j$ we obtain

$$E[\tilde{\psi}^j] \leq (ME)_c.$$

Furthermore, $s = 0$ in (5.19) imply

$$\sum_{j=1}^M M[\tilde{\psi}^j] + \lim_{n \to +\infty} M[W_n^M] = \lim_{n \to +\infty} M[u_{n,0}] = 1.$$

We show that in the profile decomposition (6.2) either more than one profiles $\tilde{\psi}^j$ are non-zero, or only one profile $\tilde{\psi}^j$ is non-zero and the rest $(M-1)$ profiles are zero. The first case will give a contradiction to the fact that each $u_n(t)$ does not scatter, consequently, only the second possibility holds. That non-zero profile $\tilde{\psi}^j$
will be the initial data $u_{c,0}$ and will produce the critical solution $u_c(t) = \text{NLS}(t)u_{c,0}$, such that $\|u_c\|_{S(\dot{H}^{1/2})} = \infty$.

Case 1: More than one $\tilde{\psi}^j \neq 0$. For each $j$, (6.3) gives $M[\tilde{\psi}^j] < 1$ and for a large enough $n$, (6.4) and (6.5) yield
\[
M[\text{NLS}(t)\tilde{\psi}^j]E[\text{NLS}(t)\tilde{\psi}^j] = M[\tilde{\psi}^j]E[\tilde{\psi}^j] < (ME)_c.
\]
Recall (6.3), we have
\[
\|\text{NLS}(t - t^j)\tilde{\psi}^j (\cdot - x^j_n)\|_{S(\dot{H}^{1/2})} < +\infty, \quad \text{for large enough } n,
\]
and thus, the right hand side in (6.2) is finite in $S(\dot{H}^{1/2})$, since (5.18) holds for the remainder $\tilde{W}_n^M(x)$. This contradicts the fact that $\|\text{NLS}(t)u_{n,0}\|_{S(\dot{H}^{1/2})} = +\infty$.

Case 2: Thus, we have that only one profile $\tilde{\psi}^1$ is non-zero, renamed to be $\tilde{\psi}^1$,
\[
(6.6) \quad u_{c,0} = \text{NLS}(-t^1_n)\tilde{\psi}^1 (\cdot - x^1_n) + \tilde{W}^1_n,
\]
with $M[\tilde{\psi}^1] \leq 1$, $E[\tilde{\psi}^1] \leq (ME)_c$ and $\lim_{n \to +\infty} \|\text{NLS}(t)\tilde{W}^1_n\|_{S(\dot{H}^{1/2})} = 0$.

Let $u_c$ be the solution to $\text{NLS}^+\dot{u}(\mathbb{R}^2)$ with the initial condition $u_{c,0} = \tilde{\psi}^1$. Applying $\text{NLS}(t)$ to both sides of (6.6) and estimating it in $S(\dot{H}^{1/2})$, we obtain (by the nonlinear profile decomposition Proposition 5.3) that
\[
\|u_c\|_{S(\dot{H}^{1/2})} = \|\text{NLS}(t)\tilde{\psi}^1\|_{S(\dot{H}^{1/2})} = \lim_{n \to +\infty} \|\text{NLS}(t - t^1_n)\tilde{\psi}^1 (\cdot - x^1_n)\|_{S(\dot{H}^{1/2})} = +\infty,
\]
since by construction $\|u_n\|_{S(\dot{H}^{1/2})} = +\infty$, completing the proof. □

The proofs of the following Lemma 6.2, Lemma 6.3 and Proposition 6.4 are very close to the ones in [HR08a, DHR08b, HR10], and thus, we omit them.

**LEMA 6.2.** (Precompactness of the flow of the critical solution.) Assume $u_c$

as in Proposition 6.1 there is a continuous path $x(t)$ in $\mathbb{R}^2$ such that

\[
K = \{u_c(\cdot - x(t), t) | t \in [0, +\infty)\} \subset H^1.
\]

Then $K$ is precompact in $H^1$.

**LEMA 6.3.** Let $u(t)$ be a solution of (1.1) defined on $[0, +\infty)$ such that $P[u] = 0$ and either

a. $K = \{u(\cdot - x(t), t) | t \in [0, +\infty)\}$ precompact in $H^1$, or

b. for all $t$,
\[
(6.7) \quad \|u(t) - e^{i\theta(t)}Q(\cdot - x(t))\|_{H^1} \leq \epsilon_1
\]
for some continuous function $\theta(t)$ and $x(t)$. Then $\lim_{t \to +\infty} \frac{x(t)}{t} = 0$.

**COROLLARY 6.4.** (Precompactness of the flow implies uniform localization.) Assume $u$ is a solution to (1.1) such that

\[
K = \{u_c(\cdot - x(t), t) | t \in [0, +\infty)\}
\]
is precompact in $H^1$. Then for each $\epsilon > 0$, there exists $R > 0$, so that for all $0 \leq t < \infty$
\[
\int_{|x + x(t)| > R} |\nabla u(x, t)|^2 + |u(x, t)|^2 + |u(x, t)|^6 \, dx \leq \epsilon,
\]
and $\|u(t, \cdot - x(t))\|_{H^1(|x| \geq R)} \leq \epsilon$. Furthermore, $\|u(t, \cdot - x(t))\|_{H^1(|x| \geq R)} \leq \epsilon$. 

THEOREM 6.5. (Rigidity Theorem.) Let \( u_0 \in H^1 \) satisfy \( P[u_0] = 0 \), \( \mathcal{M} \mathcal{E}[u_0] < 1 \) and \( G_\mathcal{E}(0) < 1 \). Let \( u \) be the global \( H^1 \) solution of NLS \( u_0 \) with initial data \( u_0 \) and suppose that \( K = \{ u_0(\cdot - x(t), t) \mid t \in [0, +\infty) \} \) is precompact in \( H^1 \). Then \( u_0 \equiv 0 \).

PROOF. Let \( \phi \in C_0^\infty \) radial, such that \( \phi(x) = |x|^2 \) for \( |x| \leq 1 \) and vanishing for \( |x| \geq 2 \). For \( R > 0 \) define

\[
|z_R'(t)| \leq cR |\nabla u(t)||u(t)|dx 
\]

Then direct calculations yield \( z_R'(t) = 2 \Im \int R \nabla \phi(\frac{x}{R}) \cdot \nabla u(t) \bar{u}(t)dx \) and Hölder’s inequality leads to

\[
|z_R'(t)| \leq cR \int_{|x| \leq 2R} |\nabla u(t)||u(t)|dx \leq cR \|u(t)\|_{L^2} \|\nabla u(t)\|_{L^2}.
\]

Note that,

\[
z''(t) = 4 \int \phi''\left(\frac{|x|}{R}\right) |\nabla u|^2 - \frac{1}{R^2} \int \Delta^2 \phi\left(\frac{|x|}{R}\right) |u|^2 - \frac{4}{3} \int \Delta \phi\left(\frac{|x|}{R}\right) |u|^6.
\]

Since \( \phi \) is radial, we have

\[
z''(t) = 8 \int |\nabla u|^2 - \frac{16}{3} \int |u|^6 + A_R(u(t)),
\]

where

\[
A_R(u(t)) = 4 \int \left(\phi''\left(\frac{|x|}{R}\right) - 2\right) |\nabla u|^2 + 4 \int_{R \leq |x| \leq 2R} \phi''\left(\frac{|x|}{R}\right) |\nabla u|^2 - \frac{1}{R^2} \int \Delta^2 \phi\left(\frac{|x|}{R}\right) |u|^2 - \frac{4}{3} \int \left(\Delta \phi\left(\frac{|x|}{R}\right) - 4\right) |u|^6.
\]

Thus,

\[
A_R(u(t)) \leq c \int_{|x| \geq R} \left( |\nabla u(t)|^2 + \frac{1}{R^2} |u(t)|^2 + |u(t)|^6 \right) dx.
\]

Choosing \( R \) large enough, over a suitably chosen time interval \([t_0, t_1]\), with \( 0 \ll t_0 \ll t_1 < \infty \), it follows that

\[
|z''(t)| \geq 16(1 - \omega)E[u] - |A_R(u(t))|.
\]

In Corollary 6.4 take \( \epsilon = \frac{1 - \omega}{c} \), with \( c \) as in (6.12), we can take \( R_0 \geq 0 \) such that for all \( t \),

\[
\int_{|x+x(t)| > R_0} \left( |\nabla u(t)|^2 + |u(t)|^2 + |u(t)|^6 \right) \leq \frac{1 - \omega}{c} E[u].
\]

Thus combining (6.12), (6.13) and (6.14), taking \( R = R_0 + \sup_{t_0 \leq t \leq t_1} |x(t)| \) leads to the fact that for all \( t_0 \leq t \leq t_1 \),

\[
|z''(t)| \geq 8(1 - \omega)E[u].
\]

Choosing \( \gamma = (1 - \omega) \frac{E[u]}{\|u\|_{L^6}^2 \|\nabla u\|_{L^2}} \) and by Lemma 6.3 there exists \( t_0 \geq 0 \) such that for all \( t \geq t_0 \), we have \( |x(t)| \leq \gamma t \). Taking \( R = R_0 + \gamma t_1 \), we have that (6.15) holds for all \( t \in [t_0, t_1] \), then integrating it over this interval, we obtain

\[
|z_R'(t_1) - z_R'(t_0)| \leq 8(1 - \omega)E(t)(t_1 - t_0).
\]
Moreover, for all \( t \in [t_0, t_1] \)
\[
|z'_R(t)| \leq cR\|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2} \leq c\|Q\|_{L^2}\|\nabla Q\|_{L^2}(R_0 + \gamma t_1).
\]
Combining last two inequalities and letting \( t_1 \to +\infty \), yields \( E[u] = 0 \), which is a contradiction unless \( u(t) \equiv 0 \).

This finishes the first part of Theorem \( \Lambda^* \) (global existence and scattering).

7. Weak blowup via Concentration Compactness

In this section, we complete the proof of Theorem \( \Lambda^* \), i.e., we show the weak blow up part II (b). First, recall variational characterization of the ground state.

7.1. Variational Characterization of the Ground State. Proposition 7.1 is a restatement of Theorem 1.2 from \cite{Lio84}. It is adjusted for our case from Proposition 4.4 \cite{HR}. This Proposition shows that if a solution \( u(t, x) \) is close to \( Q \) in mass and energy, then it is close to \( Q \) in \( H^1 \), the phase and shift in space. The Proposition 7.2 is a variant of Proposition 4.1 \cite{HR}, rephrased for our case.

Proposition 7.1. There exists a function \( \epsilon(\rho) \) defined for small \( \rho > 0 \) with \( \lim_{\rho \to 0} \epsilon(\rho) = 0 \), such that for all \( u \in H^1(\mathbb{R}^2) \) with
\[
\|u\|_{L^6} - \|Q\|_{L^6} + \|u\|_{L^2} - \|Q\|_{L^2} + \|\nabla u\|_{L^2} - \|\nabla Q\|_{L^2} \leq \rho,
\]
there is \( \theta_0 \in \mathbb{R} \) and \( x_0 \in \mathbb{R}^2 \) such that
\[
\|u - e^{i\theta_0}Q(\cdot - x_0)\|_{H^1} \leq \epsilon(\rho). \tag{7.1}
\]

This Proposition shows that if a solution \( u(t, x) \) is close to \( Q \) in mass and energy, then it is close to \( Q \) in \( H^1 \), the phase and shift in space. The Proposition 7.2 is a variant of Proposition 4.1 \cite{HR}, rephrased for our case. Proposition 7.2. There exists a function \( \epsilon(\rho) \), such that \( \epsilon(\rho) \to 0 \) as \( \rho \to 0 \) satisfying the following: Suppose there is \( \lambda > 0 \) such that
\[
\left| \mathcal{M}[u] - (2\lambda^2 - \lambda^4) \right| \leq \rho \lambda^4 \tag{7.2}
\]
and
\[
\left| \mathcal{G}(u) - \lambda \right| \leq \rho \begin{cases} \lambda^3 & \text{if } \lambda \leq 1 \\ \lambda & \text{if } \lambda \geq 1 \end{cases}. \tag{7.3}
\]
Then there exist \( \theta_0 \in \mathbb{R} \) and \( x_0 \in \mathbb{R}^2 \) such that if \( \beta = \frac{M[u]}{M[Q]} \)
\[
\|u(x) - e^{i\theta_0}Q(\lambda(\beta^{-1/2}x - x_0))\|_{L^2} \leq \beta^{1/4} \epsilon(\rho), \quad \text{and}
\]
\[
\|\nabla \left[ u(x) - e^{i\theta_0}Q(\lambda(\beta^{-1/2}x - x_0)) \right]\|_{L^2} \leq \lambda \beta^{-1/4} \epsilon(\rho).
\]
The proof is similar to the one in \cite{HR} and we omit it.

7.2. Induction Step 0: Near Boundary Behavior. In order to prove the weak “blow up” we will employ the concentration compactness type argument. For establishing the divergence behavior and not scattering it was first developed in \cite{HR}.

Definition 7.3. Let \( \lambda > 0 \). The horizontal line for which \( M[u] = M[Q] \) and \( \frac{E[u]}{E[Q]} = 2\lambda^2 - \lambda^4 \) is called the “mass-energy” line for \( \lambda \) (See Figure 2).
Figure 2. For a given $\lambda > 0$ the horizontal line GH is referred as the “mass-energy” line for this $\lambda$. Observe that this horizontal line can intersect the parabola $y = 2G_u^2 - G_u^4$ twice, i.e., it can be a “mass energy” line for $0 < \lambda_1 < 1$ and $1 < \lambda_2 < \infty$, the first case produces solutions which are global and are scattering (by Theorem A* part I) and the second case produces solutions which either blow up in finite time or diverge in infinite time (“weak” blow up) as shown in Section 7.

Note that we either have $0 < \lambda < 1$ or $\lambda > 1$. Here we consider $\lambda > 1$.

We will begin showing that the renormalized gradient $G_u(t)$ cannot forever remain near the boundary if originally $G_u(0)$ is very close to it. Next we would like to show that $G_u(t)$ with initial condition $G_u(0) > 1$ close to the boundary on any “mass-energy” line with $\mathcal{M}[u] < 1$ will escape to infinity (along this line). To show this we assume to the contrary that for all solutions (starting from some mass-energy line corresponding to initial renormalized gradient $G_u(0) = \lambda_0 > 1$) are bounded in the renormalized gradient for all $t > 0$. And then conclude that this will lead to contradiction.

Theorem A* part II (a) yields $G_u(t) \geq 1$ for all $t \in \mathbb{R}$ whenever $G_u(0) \geq 1$ on the “mass-energy” line for some $\lambda > 1$. Thus, a natural question is whether $G_u(t)$ can be, with time, much larger than $\lambda$. We show (similar to [HR10] Proposition 5.1) that it can not.

Proposition 7.4. Fix $\lambda_0 > 1$. There exists $\rho_0 = \rho_0(\lambda_0) > 0$ (with the property that $\rho_0 \to 0$ as $\lambda_0 \searrow 1$), such that for any $\lambda \geq \lambda_0$, there is NO solution $u(t)$ of NLS (1.1) with $P[u] = 0$ satisfying $\|u\|_{L^2} = \|Q\|_{L^2}$, and $\frac{E[u]}{E[Q]} = 2\lambda^2 - \lambda^4$ (i.e., on any “mass-energy” line corresponding to $\lambda \geq \lambda_0$) with $\lambda \leq G_u(t) \leq \lambda(1 + \rho_0)$ for all $t \geq 0$. A similar statement holds for $t \leq 0$. 
Remark: Note that this statement claims uniform “non-closeness” to the boundary $DF$ (in Figure 2): if a solution lies on any “mass-energy” line $\lambda$ (with $\lambda \geq \lambda_0$) and $G_u(0)$ was close to the boundary $DF$, then eventually it will have to escape from this closeness, i.e., $G_u(t^*) > \lambda(1 + \rho_0)$ for some $t^* > 0$.

Proof. To the contrary, assume that there exists a solution $u(t)$ of (1.1) with $\|u\|_{L^2} = \|Q\|_{L^2}$, $\frac{E[u]}{E[Q]} = 2\lambda^2 - \lambda^4$ for some $\lambda > \lambda_0$ and $G_u(t) \in [\lambda, \lambda(1 + \rho_0)]$.

By continuity of the flow $u(t)$ and Proposition 7.2, there are continuous $x(t)$ and $\theta(t)$ such that

$$\|u(x) - e^{i\theta(t)}\lambda Q(\lambda(x - x(t)))\|_{L^2} \leq \epsilon(\rho),$$

(7.4) $$\|\nabla [u(x) - e^{i\theta(t)}\lambda Q(\lambda(x - x(t)))]\|_{L^2} \leq \lambda \epsilon(\rho).$$

(7.5) Define $R(T) = \max \{\max_{0 \leq t \leq T} |x(t)|, \log \epsilon(\rho)^{-1}\}$. Consider the localized variance (6.8). Note that $4\lambda^2 E[Q] = \lambda^2 \|\nabla Q\|_{L^2}^2 \leq \|\nabla u(t)\|_{L^2}^2$.

and since $\frac{E[u]}{E[Q]} = 2\lambda^2 - \lambda^4$, we have

$$z''_R = 32 E[u] - 8 \|\nabla u\|_{L^2}^2 + A_R(u(t)) \leq -32 E[Q] \lambda^2 (\lambda + 1)(\lambda - 1) + A_R(u(t)).$$

Let $T > 0$ and for the local virial identity (6.10) assume $R = 2R(T)$. Therefore, (6.3) and (7.5) assure that there exists $c_1 > 0$ such that

$$|A_R(u(t))| \leq c_1 \lambda^2 (\epsilon(\rho) + e^{-R(T)})^2 \leq c_1 \lambda^2 \epsilon(\rho)^2.$$ 

Taking a suitable $\rho_0$ small (i.e. $\lambda > 1$ is taken closer to 1), such that for $0 \leq t \leq T$, $\epsilon(\rho)$ is small enough, we get $z''_R(t) \leq -32 E[Q] \lambda^2 (\lambda + 1)(\lambda - 1)$. Integrating $z''_R(t)$ in time over $[0, T]$ twice, we obtain

$$\frac{z_R(T)}{T^2} \leq \frac{z_R(0)}{T^2} + \frac{z_R'(0)}{T} - 16 E[Q] \lambda^2 (\lambda + 1)(\lambda - 1).$$

Note sup$_x \phi(x)$ from (6.8), is bounded, say by $c_2 > 0$. Then from (6.8) we have $|z_R(0)| \leq c_2 R^2 \|u_0\|_{L^2}^2 = c_2 R^2 \|Q\|_{L^2}^2$, and by (5.9)

$$|z_R'(0)| \leq c_3 R \|u_0\|_{L^2} \|\nabla u_0\|_{L^2} \leq c_3 R \|Q\|_{L^2} \|\nabla Q\|_{L^2} \lambda(1 + \rho_0).$$

Taking $T$ large enough so that by Lemma (6.3) $\frac{R(T)}{T^2} < \epsilon(\rho)$, we estimate

$$\frac{z_{2R(T)}(T)}{T^2} \leq c_4 \left(\frac{R(T)^2}{T^2} + \frac{R(T)}{T} \right) - 16 E[Q] \lambda^2 (\lambda + 1)(\lambda - 1) \leq C(\epsilon(\rho)^2 + \epsilon(\rho)) - 16 E[Q] \lambda^2 (\lambda + 1)(\lambda - 1).$$

We can initially choose $\rho_0$ small enough (and thus, $\epsilon(\rho_0)$) such that $C(\epsilon(\rho)^2 + \epsilon(\rho)) < 8 E[Q] \lambda^2 (\lambda + 1)(\lambda - 1)$. We obtain $0 \leq z_{2R(T)}(T) < 0$, which is a contradiction, showing that our initial assumption about the existence of a solution to (1.1) with bounded $G_u(t)$ does not hold.

Fix $\lambda > \lambda_0 > 1$. Consider a solution $u(t)$ of (1.1) at the “mass-energy” line for this $\lambda$. We showed that any such solution cannot have a renormalized gradient $G_u(t)$ bounded near the boundary $DF$ for all time. We will show that $G_u(t)$, in fact, will tend to $+\infty$ (at least along an infinite time sequence). Again to the contrary assume that such solutions do have a uniform bound.
We say the property $\text{GBG}(\lambda, \sigma)$ holds\(^5\) if $\lambda \leq G_u(t) \leq \sigma$ for all $t \geq 0$, for some solutions on the “mass-energy” line for $\lambda$.

In other words, $\text{GBG}(\lambda, \sigma)$ is not true if for every solution $u(t)$ of (1.1) at the “mass-energy” line for $\lambda$ (for any $\lambda \leq \lambda_0 > 1$), such that $\lambda \leq \sigma < G_u(t)$ for some $t > 0$, then there exists $t^*$ such that $\sigma < G_u(t^*)$. Iterating, we conclude that, there exists a sequence $\{t_n\} \to \infty$ with $G_u(t_n) > \sigma_n$ for all $n$ (and $\sigma_n \to +\infty$).

Suppose $\text{GBG}(\lambda, \sigma)$ does not hold. Then for any $\sigma' < \sigma$ it does not hold either. This will allow us to induct on the $\text{GBG}$ notion.

**Definition 7.5.** Let $\lambda_0 > 1$. We define the critical threshold $\sigma_c$ by

$$\sigma_c = \sup\{\sigma | \sigma > \lambda_0 \text{ and } \text{GBG}(\lambda, \sigma) \text{ does NOT hold for all } \lambda \text{ with } \lambda_0 \leq \lambda \leq \sigma\}.$$  

Note that $\sigma_c = \sigma_c(\lambda_0)$ stands for “$\sigma$-critical”.

Notice that Proposition 7.4 implies that $\text{GBG}(\lambda, \lambda(1 + \rho_0(\lambda_0)))$ does not hold for all $\lambda \geq \lambda_0$.

**7.3. Induction argument.** Let $\lambda_0 > 1$, we would like to show that $\sigma_c(\lambda_0) = +\infty$. Let $u(t)$ be a solution to (1.1) with initial condition $u_{n,0}$ such that

$$M[u] = M[Q], \quad \frac{E[u]}{E[Q]} \leq 2\lambda_0^2 - \lambda_0^4 \quad \text{and} \quad G_u(t) > 1.$$

We want to show that there exists a sequence of times $\{t_n\} \to +\infty$ such that $\|\nabla u(t_n)\|_{L^2} \to \infty$. Assuming to the contrary, such sequence of times does not exist. Let $\lambda \geq \lambda_0$ be such that $\frac{E[\lambda]}{E[Q]} = 2\lambda^2 - \lambda^4$, and thus, there exists $\sigma < \infty$ such that $\lambda \leq G_u(t) \leq \sigma$ for all $t \geq 0$, i.e., $\text{GBG}(\lambda, \sigma)$ holds with $\sigma_c(\lambda_0) \leq \sigma < \infty$.

Now, we take $u(t) = u_c(t)$ to be the critical threshold solution given by Lemma 7.7 (see below). Then by Lemma 7.8 we have uniform concentration of $u_c(t)$ in time, which together with the localization property (Lemma 7.9) implies that $u_c(t)$ blows up in finite time, which contradicts the fact that $u_c(t)$ is bounded in $H^1$. As a result $u_c(t)$ cannot exist and this ends the proof of Theorem A*.

Before proceeding with the Existence Theorem we introduce the profile reordering (Lemma 7.6) which together with the nonlinear profile decomposition of the sequence $\{u_{n,0}\}$ will allow us to construct a “critical threshold solution” (see Existence of Threshold solution Lemma 7.7).

**Lemma 7.6.** (Profile reordering.) Suppose $\phi_n = \phi_n(x)$ is a bounded sequence in $H^1(\mathbb{R}^2)$. Assume that $M[\phi_n] = M[Q]$, $\frac{E[\phi_n]}{E[Q]} = 2\lambda_n^2 - \lambda_n^4$, such that $1 < \lambda_0 \leq \lambda_n$ and $\lambda_n \leq G_{\phi_n}(t)$ for each $n$. Apply Proposition 7.3 to sequence $\{\phi_n\}$ and obtain nonlinear profiles $\{\tilde{\psi}^j\}$ Then, these profiles $\tilde{\psi}^j$ can be reordered so that there exist $1 \leq M_1 \leq M_2 \leq M$ and

1. For each $1 \leq j \leq M_1$, we have $t^j_n = 0$ and $\psi^j(t) = \text{NLS}(t)\tilde{\psi}^j$ does not scatter as $t \to +\infty$. (In particular, there is at least one $j$ in this case.)
2. For each $M_1 + 1 \leq j \leq M_2$, we have $t^j_n = 0$ and $\psi^j(t)$ scatters as $t \to +\infty$.
   (If $M_1 = M_2$, there are no $j$ with this property.)
3. For each $M_2 + 1 \leq j \leq M$, we have $|t^j_n| \to \infty$ and $\psi^j(t)$ scatters as $t \to +\infty$. (If $M_2 = M$, there are no $j$ with this property.)

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\(^5\)GBG stands for globally bounded gradient.
PROOF. Pohozhaev identities and energy definition yield
\[
\frac{\|\phi_n\|^6_{L^6}}{\|Q\|^6_{L^6}} = 2G^2_{\phi_n}(t) - \frac{E[\phi_n]}{E[Q]} \geq \lambda^4_n \geq \lambda^4_0 > 1.
\]
Notice that if \( j \) is such that \( |t^j_n| \to \infty \), then \( L^6 \) scattering yields \( \|\text{NLS}(-t^j_n)\psi^j\|_{L^6} \to 0 \), and by (5.20) we have that \( \frac{\|\phi_n\|^6_{L^6}}{\|Q\|^6_{L^6}} \to 0 \). Therefore, there exist at least one \( j \) such that \( t^j_n \) converges as \( n \to \infty \). Without loss of generality, assume that \( t^j_n = 0 \), and reorder the profiles such that for \( 1 \leq j \leq M_2 \), we have \( t^j_n = 0 \) and for \( M_2 + 1 \leq j \leq M \), we have \( |t^j_n| \to \infty \).

It is left to prove that there is at least one \( j, 1 \leq j \leq M_2 \) such that \( v^j(t) \) is not scattering. Assume then for all \( 1 \leq j \leq M_2 \) we have that all \( v^j \) are scattering, and thus, \( \|v^j(t)\|_{L^6} \to 0 \) as \( t \to +\infty \). Let \( \epsilon > 0 \) and \( t_0 \) large enough such that for all \( 1 \leq j \leq M_2 \) we have \( \|v^j(t)\|^6_{L^6} \leq \epsilon/M^2 \). Using the \( L^6 \) orthogonality (5.39) along the NLS flow, and letting \( n \to +\infty \), we obtain
\[
\lambda^4_0\|Q\|^6_{L^6} \leq \|u_n(t)\|^6_{L^6} = \sum_{j=1}^{M_2} \|v^j(t_0)\|^6_{L^6} + \sum_{j=M_2+1}^{M} \|v^j(t_0 - t^j_n)\|^6_{L^6} + \|W^M_n(t)\|^6_{L^6} + o_n(1)
\]
\[
\leq \epsilon + \|W^M_n(t)\|^6_{L^6} + o_n(1).
\]
The last line is obtained, since \( \sum_{j=M_2+1}^{M} \|v^j(t_0 - t^j_n)\|^6_{L^6} \to 0 \) as \( n \to \infty \). This gives a contradiction. \( \square \)

**Lemma 7.7.** (Existence of the threshold solution.) There exists initial data \( u_{c,0} \) with \( M[u_{c}] = M[Q] \) and \( \lambda_0 \leq \lambda_c \leq \sigma_c(\lambda_0) \) such that \( u_{c}(t) \equiv \text{NLS}(t)u_{c,0} \) is global, \( E[u_{c}] = 2\lambda^2_c - \lambda^4_c \) and, moreover, \( \lambda_c \leq G_n(t) \leq \sigma_c \) for all \( t \geq 0 \).

**Proof.** Definition of \( \sigma_c \) implies the existence of sequences \( \{\lambda_n\} \) and \( \{\sigma_n\} \) with \( \lambda_0 \leq \lambda_n \leq \lambda_c \), \( \sigma_n \geq \sigma_c \) such that \( \text{BG}(\lambda_n, \sigma_n) \) is false. This means that there exists \( u_{n,0} \) with \( M[u] = M[Q], \frac{E[u_{n,0}]}{E[Q]} = 2\lambda^2_c - \lambda^4_c \) and \( \lambda_c \leq \frac{\|\nabla u\|^2}{\|u\|^2} = G_n(t) \leq \sigma_c \), such that \( u_n(t) = \text{NLS}(t)u_{n,0} \) is global.

Note that the sequence \( \{\lambda_n\} \) is bounded, thus we pass to a convergent subsequence \( \{\lambda_{n_k}\} \). Assume \( \lambda_{n_k} \to \lambda' \) as \( n_k \to \infty \), thus \( \lambda_0 \leq \lambda' \leq \sigma_c \).

We apply the nonlinear profile decomposition and reordering. In Lemma 7.6 let \( \phi_n = u_{n,0} \). Recall that \( v^j(t) \) scatters as \( t \to \infty \) for \( M_1 + 1 \leq j \leq M_2 \), and by Proposition 5.20 \( v^j(t) \) also scatter in one or the other time direction for \( M_2 + 1 \leq j \leq M \) and \( E[v^j] = E[\psi^j] \geq 0 \). Thus, by the Pythagorean decomposition for the nonlinear flow (5.20) we have \( \sum_{j=1}^{M_1} E[\psi^j] \leq E[\phi_n] + o_n(1) \). For at least one \( 1 \leq j \leq M_1 \), we have \( E[\psi^j] \leq \max\{\lim_n E[\phi_n], 0\} \). Without loss of generality, we may assume \( j = 1 \). Since \( 1 = M[\psi^1] \leq \lim_n M[\phi_n] = M[Q] = 1 \), it follows \( \mathcal{M}E[\psi^1] \leq \max\{\lim_n E[\phi_n], E[Q]\} \), thus, for some \( \lambda_1 \geq \lambda_0 \), we have \( \mathcal{M}E[\psi^1] = 2\lambda^2_c - \lambda^4_c \).

Recall \( \psi^1 \) is a nonscattering solution, thus \( G_{\psi^1}(t) > \lambda_c \), otherwise it will contradict Theorem A* Part 1 (b). We have two cases: either \( \lambda_1 \leq \sigma_c \) or \( \lambda_1 > \sigma_c \).

**Case 1.** \( \lambda_1 \leq \sigma_c \). Since the statement “\( \text{BG}(\lambda_1, \sigma_c - \delta) \) is false” implies for each \( \delta > 0 \), there is a nondecreasing sequence \( t_k \) of times such that \( \lim\{G_{\psi^1}(t_k)\} \geq \sigma_c \),
thus,
\[
\sigma_c^2 - o_k(1) \leq \lim [G_{v^1}(t_k)]^{\frac{2}{r}} \leq \frac{\|\nabla v^1(t_k)\|^2_{L^2}}{\|\nabla Q\|^2_{L^2}} \\
(7.6) \leq \frac{\sum_{j=1}^{M} \|\nabla v^1(t_k - t_n)\|^2_{L^2} + \|W_n^{M}(t_k)\|^2_{L^2}}{\|\nabla Q\|^2_{L^2}} \\
\leq \frac{\|\nabla u_n(t)\|^2_{L^2}}{\|\nabla Q\|^2_{L^2}} + o_n(1) \leq \sigma_c^2 + o_n(1).
\]

Taking \( k \to \infty \), we obtain \( \sigma_c^2 - o_n(1) = \sigma_c^2 + o_k(1) \). Thus, \( \|W_n^{M}(t_k)\|_{H^1} \to 0 \) and \( M[v^1] = M[Q] \). Then, Lemma 5.6 yields that for all \( t \),
\[
\frac{\|\nabla v^1(t)\|^2_{L^2}}{\|\nabla Q\|^2_{L^2}} \leq \lim_n \frac{\|u_n(t)\|^2_{L^2}}{\|\nabla Q\|^2_{L^2}} \leq \sigma_c.
\]

Take \( u_{c,0} = v^1(0) = \psi^1 \), and \( \lambda_c = \lambda_1 \).

Case 2. \( \lambda_1 \geq \sigma_c \). Note that \( \lambda_1^2 \leq \lim [G_{v^1}(t_k)]^{\frac{2}{r}} \). Thus, replacing (7.6) with this condition, taking \( t_k = 0 \) and sending \( n \to +\infty \), we obtain \( \lambda_1 \leq \sigma_c \), which is a contradiction. Thus, this case cannot happen. \( \square \)

Let’s assume \( u(t) = u_c(t) \) to be the critical solution provided by Lemma 7.7.

**Lemma 7.8.** There exists a path \( x(t) \) in \( \mathbb{R}^2 \) such that
\[
K = \{ u(\cdot - x(t), t) | t \geq 0 \} \subset H^1
\]
has a compact closure in \( H^1 \).

The proof of this Lemma follows closely to the proof of Lemma 9.1 in [HR10] and we omit it.

**Lemma 7.9** (Blow up for a priori localized solutions). Suppose \( u \) is a solution of the NLS on \( \mathbb{R}^2 \) at the mass-energy line \( \lambda > 1 \), with \( G_u(0) > 1 \). Select \( \kappa \) such that \( 0 < \kappa < \min(\lambda - 1, \kappa_0) \), where \( \kappa_0 \) is an absolute constant. Assume that there is a radius \( R \geq \kappa^{-1/2} \) such that for all \( t \), we have a localized gradient
\[
G_{u_R}(t) := \frac{\|u\|^2_{L^2(|x| \geq R)} \|\nabla u(t)\|^2_{L^2(|x| \geq R)}}{\|Q\|^2_{L^2(|x| \geq R)} \|\nabla Q\|^2_{L^2(|x| \geq R)}} \leq \kappa.
\]

Define \( \bar{r}(t) \) to be the scaled local variance: \( V_R(t) = \frac{32E[Q]|\lambda^2(\lambda^2 - 1 - \kappa)|}{32\bar{r}(t)} \), where \( r(t) \) is from (6.11). Then a blow up occurs in forward time before \( t_b \) (i.e., \( T^* < t_b \)), where \( t_b = V_R(0) + \sqrt{V_R(0)^2 + 2V_R(0)} \).

**Proof.** By the local virial identity (6.11),
\[
V_R(t) = \frac{32E[u] - 8\|\nabla u\|^2_{L^2} + A_R(u(t))}{32E[Q] (\lambda^2(\lambda^2 - 1 - \kappa))}
\]
where \( |A_R(u(t))| = \|\nabla u(t)\|^2_{L^2(|x| \geq R)} + \frac{1}{R^2}\|u(t)\|^2_{L^2(|x| \geq R)} + \|u(t)\|^6_{L^6(|x| \geq R)} \).

Note that, \( 4E[Q] = \|\nabla Q\|^2_{L^2} \) and definition of the mass-energy line yield
\[
\frac{32E[u] - 8\|\nabla u\|^2_{L^2}}{32E[Q]} = \frac{E[u]}{E[Q]} - \|\nabla u\|^2_{L^2} = 2\lambda^2 - \lambda^4 - |G_u(t)|^2
\]
In addition, we have the following estimates

\[ \| \nabla u(t) \|^2_{L^2(\{ |x| \geq R \})} \lesssim \kappa, \quad \frac{\| u(t) \|^2_{L^2(\{ |x| \geq R \})}}{R^2} = \frac{\| Q \|^2_{L^2}}{R^2} \lesssim \kappa, \]

(7.7)

\[ \| u(t) \|^2_{L^2(\{ |x| \geq R \})} \lesssim \| \nabla u(t) \|^2_{L^2(\{ |x| \geq R \})} \lesssim [G_{u_R}(t)]^2 (\| \nabla Q \|^2_{L^2} \| Q \|^2_{L^2}) \lesssim \kappa. \]

We used the Gagliardo-Nirenberg to obtain (7.7) and noticing that \( \| \nabla Q \|^2_{L^2} \) and \( \| Q \|^2_{L^2} \) we estimated by \( \kappa \) up to a constant. In addition, \( G_u(t) > 1 \), then \( \kappa \lesssim \kappa \| G_u(t) \|^2 \). Applying the above estimates, it follows

\[ V''_R(t) \lesssim \frac{2\lambda^2 - \lambda^4 - [G_u(t)]^2(1 - \kappa)}{\lambda^2(\lambda^2 - 1 - \kappa)}, \]

since \( G_u(t) \geq \lambda \), we obtain \( V''_R(t) \lesssim \frac{\lambda^2(1 + \kappa - \lambda^2)}{\lambda^2(\lambda^2 - 1 - \kappa)} \leq -1 \), which is a contradiction.

Now integrating in time twice gives \( V_R(t) \leq -t^2 + V'_R(0) + V_R(0) \). The positive root of the polynomial on the right-hand side is \( t_b = V'_R(0) + \sqrt{V''_R(0)^2 + 2V_R(0)} \). 

This finally finishes the proof of Theorem A*.

Note that Theorem A can be extended to other nonlinearities and dimensions except that one needs to deal carefully with fractional powers, Strichartz estimates and others implications from that. We address it elsewhere [Gue11].

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School of Mathematical and Statistical Sciences, Arizona State University, Tempe, Arizona, 85287

Current address: School of Mathematical and Statistical Sciences, Arizona State University, Tempe, Arizona, 85287
E-mail address: Cristi.guevara@asu.edu

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109
E-mail address: carreonf@umich.edu