ON ALTERNATIVE APPROXIMATING DISTRIBUTIONS IN THE MULTIVARIATE VERSION OF KOLMOGOROV’S SECOND UNIFORM LIMIT THEOREM

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Abstract. The aim of the present work is to show that recent results of the authors on the approximation of distributions of sums of independent summands by the infinitely divisible laws on convex polyhedra can be shown via an alternative class of approximating infinitely divisible distributions. We will also generalize the results to the infinite-dimensional case.

We would like to show that some of our recent results in [6] may be derived based on an alternative class of infinitely divisible distributions. We will also generalize the results to the infinite-dimensional case.

Let us first introduce some notation. Let \( \mathfrak{F}_d \) denote the set of probability distributions defined on the Borel \( \sigma \)-field of subsets of the Euclidean space \( \mathbb{R}^d \). Let \( \mathcal{D}_d \subset \mathfrak{F}_d \) be the set of infinitely divisible distributions. For \( F \in \mathfrak{F}_d \), we denote the corresponding distribution functions by \( F(b) \):

\[
F(b) = F\{(-\infty, b_1] \times \cdots \times (-\infty, b_d]\}, \quad b = (b_1, \ldots, b_d) \in \mathbb{R}^d.
\]

Let \( \mathcal{L}(\xi) \in \mathfrak{F}_d \) be the distribution of a \( d \)-dimensional random vector \( \xi \). Products and powers of measures are understood in the convolution sense:

\[
GH = G \ast H, \quad H^m = H^{m*}, \quad H^0 = E = E_0,
\]

where \( E_x \) is the distribution concentrated at a point \( x \in \mathbb{R}^d \). By \( c \) we denote absolute positive constants. Note that constants \( c \) can be different in different (or even in the same) formulas. If the corresponding constant depends on, say, \( s \), we write \( c(s) \).

Kolmogorov [3] posed the problem of estimating the accuracy of infinitely divisible approximation of distributions of sums of independent random variables, the distributions of which are concentrated on the short intervals of length \( \tau \leq 1/2 \) to within a small probability \( p \). The restriction on the distributions of summands is a non-asymptotic analogue of the classical infinitesimality (negligibility) condition for a triangular scheme of independent random variables. The bound for the rate of approximations may be considered as a quantitative improvement of the classical Khinchin theorem for the set of infinitely divisible distributions.

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being limit laws of the distributions of sums in a triangular scheme. Suppose that the distributions \( F_i \in \mathfrak{F}_d, i = 1, \ldots, n \), are represented as mixtures of probability distributions \( U_i, V_i \in \mathfrak{F}_d \):

\[
F_i = (1 - p_i) U_i + p_i V_i,
\]

where

\[
0 \leq p_i \leq 1, \quad \int x U_i \{dx\} = 0, \quad U_i \{\{x \in \mathbb{R}^d : \|x\| \leq \tau\}\} = 1, \quad \tau \geq 0,
\]

and \( V_i \) are arbitrary distributions. Denote

\[
p = \max_{1 \leq i \leq n} p_i, \quad F = \prod_{i=1}^{n} F_i.
\]

Kolmogorov \cite{8} proved that in the one-dimensional case, for \( d = 1 \), there exists an infinitely divisible distribution \( D \) such that

\[
L(F, D) \leq c \left( p^{1/5} + \tau^{1/2} \ln^{1/4} \frac{1}{\tau} \right),
\]

where

\[
L(F, D) = \inf \{ \varepsilon : F(b - \varepsilon) - \varepsilon \leq D(b) \leq F(b + \varepsilon) + \varepsilon, \quad \text{for all } x \in \mathbb{R} \},
\]

is the Lévy distance which metrizes the weak convergence of probability distributions.

This proves Khinchin’s theorem since weak convergence \( F \Rightarrow H \) implies weak convergence \( D \Rightarrow H \) as \( p \to 0 \) and \( \tau \to 0 \). The distribution \( H \) is infinitely divisible as a limit of infinitely divisible distributions \( D \). However, Kolmogorov’s inequality \cite{4} provides good infinitely divisible approximation for fixed small \( p \) and \( \tau \) even if the distributions of sums involved in the triangular scheme with \( p \to 0 \) and \( \tau \to 0 \) are not pre-compact.

Conditions (1)–(3) do not include any moment restrictions since \( V_i \) are arbitrary distributions. Note that the statement of Kolmogorov’s result \cite{8} is a little bit different, but it is not difficult to verify the equivalence of formulations. Later, Kolmogorov \cite{9} returned to this problem and proved the bound

\[
L(F, D) \leq c \left( p^{1/3} + \tau^{1/2} \ln^{1/4} \frac{1}{\tau} \right).
\]

Ibragimov and Presman \cite{7} have shown that it is possible to improve this inequality to

\[
L(F, D) \leq c \left( p^{1/3} + \tau^{2/3} \ln \frac{1}{\tau} \right).
\]

Finally, the optimal bound was derived in Zaitsev and Arak \cite{18}

\[
L(F, D) \leq c \left( p + \tau \ln \frac{1}{\tau} \right).
\]

The estimate \cite{8} was proved by Zaitsev. Moreover, as was shown by Arak, inequality \cite{8} is correct in order with respect to \( p \) and \( \tau \). As approximating laws, the so-called accompanying
infinitely divisible compound Poisson distributions were used. In 1986, a joint monograph by Arak and Zaitsev [1], containing a summary of these results, was published in Proceedings of the Steklov Institute of Mathematics.

Zaitsev [16] generalized inequality (8) to the multidimensional case. He has shown that, for $d \geq 1$,

$$L(F, D) \leq c(d) \left( p + \tau \ln \frac{1}{\tau} \right),$$

where

$$L(F, D) = \inf \{ \varepsilon : F(b - \varepsilon 1) - \varepsilon \leq D(b) \leq F(b + \varepsilon 1) + \varepsilon, \quad \text{for all } b \in \mathbb{R}^d \},$$

and $1 \in \mathbb{R}^d$ is the vector with all coordinates equal to one.

The multidimensional Lévy distance between distributions $G, H \in \mathcal{F}_d$ may be also defined as

$$L(G, H) = \inf \{ \lambda : L(G, H, \lambda) \leq \lambda \},$$

where

$$L(G, H, \lambda) = \sup \max_{b \in \mathbb{R}^d} \{ G(b) - H(b + \lambda 1), H(b) - G(b + \lambda 1) \}, \quad \lambda > 0. \quad (11)$$

The Prokhorov distance between distributions $G, H \in \mathcal{F}_d$ may be defined as

$$\pi(G, H) = \inf \{ \lambda : \pi(G, H, \lambda) \leq \lambda \},$$

where

$$\pi(G, H, \lambda) = \sup \max_{X} \{ G\{X\} - H\{X^\lambda\}, H\{X\} - G\{X^\lambda\} \}, \quad \lambda > 0,$$

and $X^\lambda = \{ y \in \mathbb{R}^d : \inf_{x \in X} \| x - y \| < \lambda \}$ is the $\lambda$-neighborhood of a Borel set $X$ (see, e.g., [19]).

Le Cam [10] proposed to use as a natural infinitely divisible approximation of $\prod_{i=1}^{n} F_i$ the accompanying compound Poisson distribution $\prod_{i=1}^{n} e(F_i)$, where

$$e(H) = e^{-1} \sum_{s=0}^{\infty} \frac{H^s}{s!}, \quad \text{for } H \in \mathcal{F}_d.$$

If $F = L(\xi) \in \mathcal{F}_d$ and $\mathbb{E} ||\xi||^2 < \infty$, then $\Phi(F) \in \mathcal{F}_d$ denotes below the Gaussian distribution with the same mean and covariance operator as $F$.

The following Theorem [11] is the main result of Zaitsev [16].

**Theorem 1.** Let conditions (1)–(3) be satisfied. Denote

$$D = \prod_{i=1}^{n} e(F_i),$$

(12)
Then, for any $\lambda > 0$,

$$L(F, D, \lambda) \leq c(d) \left( p + \exp \left( - \frac{\lambda}{c(d) \tau} \right) \right),$$

(13)

$$\pi(F, D, \lambda) \leq c(d) \left( p + \exp \left( - \frac{\lambda}{c(d) \tau} \right) \right) + \sum_{i=1}^{n} p_i^2.$$  

(14)

Hence,

$$L(F, D) \leq c(d) \left( p + \tau (|\ln \tau| + 1) \right),$$

(15)

$$\pi(F, D) \leq c(d) \left( p + \tau (|\ln \tau| + 1) \right) + \sum_{i=1}^{n} p_i^2.$$  

(16)

Inequalities (13)–(16) remain true after replacing $D$ by other approximating infinitely divisible distributions

$$D^* = \Phi \left( \prod_{i=1}^{n} ((1 - p_i) U_i + p_i E) \right) \prod_{i=1}^{n} e \left( (1 - p_i) E + p_i V_i \right)$$  

(17)

or

$$D^{**} = D_0 \prod_{i=1}^{n} e \left( (1 - p_i) E + p_i V_i \right),$$  

(18)

where $D_0$ is an arbitrary infinitely divisible distribution with spectral measure concentrated on the ball $\{ x \in \mathbb{R}^d : \|x\| \leq \tau \}$ and with the same mean and the same covariance operator as those of the distribution $\prod_{i=1}^{n} \left( (1 - p_i) U_i + p_i E \right)$.

**Remark 1.** Formally, similarly to the case $d = 1$, we consider Gaussian laws as infinitely divisible distributions with spectral measures concentrated at zero. Thus, the distribution $D_0$ may have a Gaussian component.

**Remark 2.** It is easy to see that the distributions $D$ and $D^*$ are particular cases of distribution $D^{**}$ with

$$D_0 = \prod_{i=1}^{n} e \left( (1 - p_i) U_i + p_i E \right) \quad \text{and} \quad D_0 = \Phi \left( \prod_{i=1}^{n} ((1 - p_i) U_i + p_i E) \right)$$

respectively.

**Remark 3.** The mean and the covariance operator of distribution $D_0$ may be not precisely equal to those of the distribution $\prod_{i=1}^{n} \left( (1 - p_i) U_i + p_i E \right)$ but may be just close to them. The additional remainder term will come from the estimation of the closeness of Gaussian laws $\Phi \left( D_0 \right)$ and $\Phi \left( \prod_{i=1}^{n} \left( (1 - p_i) U_i + p_i E \right) \right)$ (see, e.g., [2]).

Note that the estimation of $L(F, D, \lambda)$ and $\pi(F, D, \lambda)$ for all $\lambda > 0$ provides more information on the closeness of distributions $F$ and $D$ than the estimation of $L(F, D)$ and $\pi(F, D)$. For example, inequalities (15) and (16) are trivial for $\tau \geq 1$ while inequalities (13) and (14) are interesting for any $\tau > 0$. Moreover, the information containing in (13) and (14) remains invariant if we multiply the random vectors by a non-zero constant. However, inequalities
ESTIMATES IN KOLMOGOROV’S THEOREM

Equations (13) and (14) actually can be derived from inequalities (15) and (16) by varying normalizing factors (see [12] for details).

Kolmogorov [8, 9] has obtained actually the bounds for $L(F, D^*, \lambda), \lambda \geq 2\tau > 0$, in the case $d = 1$. Instead of (4) and (6), he has proved inequalities

\[
L(F, D^*, \lambda) \leq c\left(\frac{p^{1/5}}{\lambda} \ln^{1/2} \frac{\lambda}{\tau}\right),
\]

and

\[
L(F, D^*, \lambda) \leq c\left(\frac{p^{1/3}}{\lambda} \ln^{1/2} \frac{\lambda}{\tau}\right)
\]

respectively. The optimality of inequality (13) means that the case where $\lambda < 2\tau$ is trivial: if $\lambda < 2\tau$, then there exists $F$ from (3) such that $L(F, D, \lambda) \geq c$, for any $D \in \mathcal{D}_1$.

The proof of Theorem 1 is based on the following Lemmas 1–6.

**Lemma 1** (see [20]). Let $F, G, H \in \mathcal{F}_d$ be arbitrary distributions. Then $R(FH, GH) \leq R(F, G)$, where $R(\cdot, \cdot)$ is any of the distances $L(\cdot, \cdot)$, $\pi(\cdot, \cdot)$ or $\rho(\cdot, \cdot)$ ($\rho$ is the uniform distance between distribution functions). Moreover, $L(\cdot, \cdot) \leq \min\{\pi(\cdot, \cdot), \rho(\cdot, \cdot)\}$.

**Lemma 2** (Zaitsev [13]). Let the conditions of Theorem 1 be satisfied. Let

\[
G_i = (1 - p_i)E + p_iV_i, \quad H_i = (1 - p_i)U_i + p_iE, \quad i = 1, \ldots, n,
\]

and

\[
G = \prod_{i=1}^{n} G_i, \quad H = \prod_{i=1}^{n} H_i.
\]

Then

\[
\pi(F, GH) \leq c(d) \left(p + \tau(|\ln \tau| + 1)\right),
\]

and, for any $\lambda > 0$,

\[
\pi(F, GH, \lambda) \leq c(d) \left(p + \exp\left(-\frac{\lambda}{c(d)\tau}\right)\right).
\]

**Lemma 3** (Zaitsev [11, 16]). Assume that the distributions $G_i \in \mathcal{F}_d$ are represented as

\[
G_i = (1 - p_i)E + p_iV_i, \quad i = 1, \ldots, n,
\]

where $V_i \in \mathcal{F}_d$ are arbitrary distributions, $0 \leq p_i \leq p = \max_j p_j$,

\[
G = \prod_{i=1}^{n} G_i, \quad D = \prod_{i=1}^{n} c(G_i).
\]

Then

\[
\rho(G, D) \leq c(d)p,
\]
Lemma 4 (see [10, p. 186]). Let the conditions of Lemma 3 be satisfied. Then
\[ \pi(G, D) \leq \rho_{TV}(G, D) \leq \sum_{i=1}^{n} p_i^2, \] (25)
where
\[ \rho_{TV}(G, D) = \sup_X |G\{X\} - D\{X\}|, \]
is the distance in total variation and the supremum is taken over all Borel sets \( X \subset \mathbb{R}^d \).

Lemma 5 (see Zaitsev [14] or [15]). Let the conditions of Theorem 1 be satisfied with \( p = 0 \), that is
\[ \int x F_i\{dx\} = 0, \quad F_i\{\{x \in \mathbb{R}^d : \|x\| \leq \tau\}\} = 1, \quad \tau \geq 0. \] (26)
Then
\[ \pi(F, \Phi(F)) \leq c(d) \tau(|\ln \tau| + 1) \] (27)
and, for any \( \lambda > 0 \),
\[ \pi(F, \Phi(F), \lambda) \leq c(d) \exp \left( -\frac{\lambda}{c(d) \tau} \right). \] (28)

Lemma 6 (see Zaitsev [14] or [15]). Let \( D \) be an infinitely divisible distribution with spectral measure concentrated on the ball \( \{x \in \mathbb{R}^d : \|x\| \leq \tau\} \). Then
\[ \pi(D, \Phi(D)) \leq c(d) \tau(|\ln \tau| + 1) \] (29)
and, for any \( \lambda > 0 \),
\[ \pi(D, \Phi(D), \lambda) \leq c(d) \exp \left( -\frac{\lambda}{c(d) \tau} \right). \] (30)

Remark 4. The approximating distributions \( D^{**} \) were not included in the statement of [16, Theorem 1.1] but inequalities (13)–(16) are obviously extended to them in view of Lemmas 1 and 6.

Inequality (13) is equivalent to the validity of inequality
\[ \max \left\{ F\{P\} - D\{P_\lambda\}, \ D\{P\} - F\{P_\lambda\} \right\} \leq c(d) \left( p + \exp \left( -\frac{\lambda}{c(d) \tau} \right) \right) \] (31)
for any \( \lambda > 0 \) and for all sets \( P \) and \( P_\lambda \) of the form
\[ P = \{x \in \mathbb{R}^d : \langle x, e_j \rangle \leq b_j, \ j = 1, \ldots, d\}, \] (32)
and
\[ P_\lambda = \{x \in \mathbb{R}^d : \langle x, e_j \rangle \leq b_j + \lambda, \ j = 1, \ldots, d\}, \] (33)
where \( e_j \in \mathbb{R}^d \) are the vectors of the standard Euclidean basis, \(-\infty < b_j \leq \infty, \ j = 1, \ldots, d\).
It is easy to see that \( P^\lambda \subset P_\lambda \subset P_\lambda^{\sqrt{d+\varepsilon}} \) for \( \varepsilon > 0 \). Therefore, (31) is equivalent to the validity of inequality

\[
\max \left\{ F\{P\} - D\{P^\lambda\}, D\{P\} - F\{P^\lambda\}\right\} \leq c(d) \left( p + \exp \left( -\frac{\lambda}{c(d) \tau} \right) \right), \tag{34}
\]

for any \( \lambda > 0 \). In the paper of Götze, Zaitsev and Zaporozhets [6], it was shown that inequality (34) is valid for convex polyhedra \( P \in \mathcal{P}_m \) (see (35)) with \( c(d) \) replaced by \( c(m) \) depending only on \( m \), the number of half-spaces involved in the definition of a polyhedron \( P \).

In Theorem 3 of the present paper, we show that the same statement remains true after replacing \( D \) by approximating distributions \( D^* \) and \( D^{**} \) from (17) and (18). Thus, there is a freedom in the choice of \( D_0 \) in the definition of approximating distribution \( D^{**} \). The only restriction is that \( D_0 \) must be an infinitely divisible distribution with spectral measure concentrated on the ball \( \{ x \in \mathbb{R}^d : \| x \| \leq \tau \} \) and with the same mean and the same covariance operator as those of the distribution \( \prod_{i=1}^n \left( (1 - p_i)U_i + p_i E \right) \). The definition (10) of a multivariate version of the Lévy distance is actually not quite natural since the collection of sets \( P \) of the form (32) is not invariant with respect to rotation while the conditions of Theorem 1 are invariant. Therefore, inequality (31) remains true after replacing the sets \( P \) and \( P_\lambda \) by \( \mathbb{U}P \) and \( \mathbb{U}P_\lambda \), where \( \mathbb{U} \) is a unitary linear operator. A question is: how to define a multivariate version of the Lévy distance which can be used in more adequate bounds under the conditions of Theorem 1.

In the present paper, we give similar bounds for comparing quantities defined via multivariate polyhedra in Götze, Zaitsev and Zaporozhets [6].

For \( m \in \mathbb{N} \) we denote by \( \mathcal{P}_m \) the collection of sets \( P \subset \mathbb{R}^d \) representable in the form

\[
P = \{ x \in \mathbb{R}^d : \langle x, t_j \rangle \leq b_j, \ j = 1, \ldots, m \}, \tag{35}
\]

where \( t_j \in \mathbb{R}^d \) are the vectors satisfying \( \| t_j \| = 1, -\infty < b_j \leq \infty, \ j = 1, \ldots, m \). The elements of the set \( \mathcal{P}_m \) will be called convex polyhedra. They can be unbounded sets. For \( P \in \mathcal{P}_m \) defined in (35) and \( \lambda \geq 0 \), we denote

\[
P_\lambda = \{ x \in \mathbb{R}^d : \langle x, t_j \rangle \leq b_j + \lambda, \ j = 1, \ldots, m \}. \tag{36}
\]

By definition, \( P_\lambda \) is the intersection of closed \( \lambda \)-neighborhoods of half-spaces \( \{ x \in \mathbb{R}^d : \langle x, t_j \rangle \leq b_j \}, \ j = 1, \ldots, m \). Clearly, \( P^\lambda \subset P_\lambda \). However, \( P_\lambda \) may be essentially larger than \( P^\lambda \). For example, it is the case for \( m = 2 \), if \( 1 - \varepsilon < |\langle t_1, t_2 \rangle| < 1 \) with a small \( \varepsilon > 0 \). In this case the hyperplanes \( \{ x \in \mathbb{R}^d : \langle x, t_j \rangle b = b_j \}, j = 1, 2 \), are almost parallel and the point \( x_0 \) such that \( \langle x_0, t_2 \rangle b = b_j + \lambda, \ j = 1, 2 \), belongs to \( P_\lambda \) and is far from the set \( P \). In the proof of Theorem 3 below we will need, however, the inclusion \( P_\lambda \subset P^{\sqrt{\lambda}} \). For this purpose, we will modify the definition of \( P_\lambda \). It is evident that we can rewrite the definition of the polyhedron \( P \) adding in it extra restrictions

\[
P = \{ x \in \mathbb{R}^d : \langle x, t_j \rangle \leq b_j, \ j = 1, \ldots, m_0 \}, \tag{37}
\]

intersecting \( P \) with half-spaces \( H(t_j, b_j) = \{ x \in \mathbb{R}^d : \langle x, t_j \rangle \leq b_j \}, j = m + 1, \ldots, m_0 \). It will be the same polyhedron if \( P \subset H(t_j, b_j) \) for all \( j = m + 1, \ldots, m_0 \).
Similarly to (36), we denote
\[ P_\lambda = \{ x \in \mathbb{R}^d : \langle x, t_j \rangle \leq b_j + \lambda, \ j = 1, \ldots, m_0 \}. \]
This is the same notation, but here we considered \( P \) as an element of \( \mathcal{P}_{m_0} \). The polyhedron \( P_\lambda \) is again the intersection of closed \( \lambda \)-neighborhoods of half-spaces \( \{ x \in \mathbb{R}^d : \langle x, t_j \rangle \leq b_j \} \), \( j = 1, \ldots, m_0 \). The only difference is that in (38) we have more intersecting half-spaces. We choose these half-spaces with \( j = m + 1, \ldots, m_0 \) so that we “cut” points of \( P_\lambda \) which were far from the set \( P \) (see Lemma 7 below which is proved in Götte, Zaitsev and Zaporozhets [6]).

**Lemma 7.** Fix some \( m \in \mathbb{N} \) and \( \varepsilon > 0 \). Let the polyhedron \( P \in \mathcal{P}_m \) be defined in (35). Then there exist a \( c_{m, \varepsilon} \) depending on \( m \) and \( \varepsilon \) only, \( m_0 \in \mathbb{N} \), \( m_0 \leq c_{m, \varepsilon} \), \( t_j \in \mathbb{R}^d \) with \( \| t_j \| = 1 \), \( b_j \in \mathbb{R} \), \( j = m + 1, \ldots, m_0 \), such that, for any \( \lambda > 0 \),
\[ P_\lambda := \{ x \in \mathbb{R}^d : \langle x, t_j \rangle \leq b_j + \lambda, \ j = 1, \ldots, m_0 \} \subset P^{(1+\varepsilon)\lambda}. \]

The statement of Lemma 7 is almost evident for \( d = 2 \) and \( d = 3 \). Following [6], define, for \( m \in \mathbb{N} \), \( G, H \in \mathfrak{F}_d \),
\[ L_m(G, H) = \inf \{ \lambda : L_m(G, H, \lambda) \leq \lambda \}, \]
where
\[ L_m(G, H, \lambda) = \sup_{P \in \mathcal{P}_m} \max \{ G\{P\} - H\{P_\lambda\}, H\{P\} - G\{P_\lambda\} \}, \quad \lambda > 0. \]
Define also
\[ \pi_m(G, H) = \inf \{ \lambda : \pi_m(G, H, \lambda) \leq \lambda \}, \]
where
\[ \pi_m(G, H, \lambda) = \sup_{P \in \mathcal{P}_m} \max \{ G\{P\} - H\{P_\lambda\}, H\{P\} - G\{P_\lambda\} \}, \quad \lambda > 0. \]

**Remark 5.** With a fixed \( m \), it is easy to verify that \( L_m(\cdot, \cdot) \) is a distance in the space \( \mathfrak{F}_d \). An open question is to check that for \( \pi_m(\cdot, \cdot) \). For \( m > 1 \), it is problematic to prove or disprove the fulfillment of the triangle inequality. The difficulty is that the \( \lambda \)-neighborhood \( P_\lambda \) of a convex polyhedron \( P \) unlike \( P_\lambda \) generally speaking is not a convex polyhedron. It is also clear that the distance \( L_1(\cdot, \cdot) = \pi_1(\cdot, \cdot) \) metrizes weak convergence.

The following Theorems 2–4 are the main results of this paper.

**Theorem 2.** Let the conditions of Theorem 1 be satisfied. Then, for any \( m \in \mathbb{N} \),
\[ L_m(F, D) \leq c(m) \left( p + \tau(|\ln \tau| + 1) \right), \]
and
\[ L_m(F, D, \lambda) \leq c(m) \left( p + \exp \left( -\frac{\lambda}{c(m) \tau} \right) \right), \quad \lambda > 0. \]
Inequalities (11) and (12) remain true after replacing \( D \) by approximating distributions (17) and (18).
Theorem 3. Let the conditions of Theorem 1 be satisfied. Then, for any \( m \in \mathbb{N} \),
\[
\pi_m(F, D) \leq c(m) \left( p + \tau(\ln \tau + 1) \right),
\]
and
\[
\pi_m(F, D, \lambda) \leq c(m) \left( p + \exp \left( -\frac{\lambda}{c(m) \tau} \right) \right), \quad \lambda > 0.
\]
Inequalities (43) and (44) remain true after replacing \( D \) by approximating distributions \( D^* \) and \( D^{**} \) from (17) and (18).

Thus, the statement of Theorem 1 is generalized, since Theorems 2 and 3 deal with the values of distributions on convex polyhedra (35) whereas Theorem 1 corresponds to the sets (32). Note also that in Theorem 1 the constants depend on the dimension \( d \), while in Theorems 2 and 3 the constants depend only on \( m \) involved in the definition of polyhedra (35). Note that in Götze, Zaitsev and Zaporozhets [6] we have proved Theorems 2 and 3 for approximating distributions \( D \) only.

The proof of Theorem 2 is based on applying the \( m \)-variate version of Theorem 1. Indeed, the \( m \)-variate vectors with coordinates \( \langle \xi, t_j \rangle, \langle \eta, t_j \rangle, t_j \in \mathbb{R}^d, \|t_j\| = 1, j = 1, \ldots, m \), satisfy actually the same \( m \)-dimensional conditions as the random vectors \( \xi, \eta \in \mathbb{R}^d \) with compared \( d \)-dimensional distributions \( F \) and \( D \) from Theorem 1. Let \( A : \mathbb{R}^d \to \mathbb{R}^m \) be the linear operator mapping \( x \in \mathbb{R}^d \) to the vector with coordinates \( \langle x, t_j \rangle, j = 1, \ldots, m \). The vectors \( A\xi, A\eta \) satisfy the conditions of \( m \)-variate version of Theorem 1 with replacing \( \tau \) by \( \tau \sqrt{m} \).

This follows from inequality \( \|A\| \leq \sqrt{m} \). Thus, roughly speaking, from the known estimates of the distance \( L \) in space \( \mathbb{R}^m \) we derive estimates of the distance \( L_m \) in \( \mathbb{R}^d \). Theorem 3 will be derived from Theorem 2 with the help of Lemma 7.

It is not difficult to understand that the conditions of Theorems 2 and 3 are meaningful even for \( d = \infty \), that is, for distributions in the Hilbert space \( \mathbb{R}^\infty = H \). The definitions of \( L_m(\cdot, \cdot) \) and \( \pi_m(\cdot, \cdot) \) are applicable to such distributions without changes.

Theorem 4. The statements of Theorems 2 and 3 remain true for \( d = \infty \).

Theorem 4 can be considered as an adequate infinite-dimensional version of Kolmogorov’s second uniform limit theorem. Recall that inequality (8) (and hence inequalities (41) and (43)) are correct in order with respect to parameters \( p \) and \( \tau \).

It is possible, for example, to use Theorem 4 for comparing the distributions of random polygonal lines constructed via partial sums of independent random variables with distributions of accompanying processes with independent increments.

Remark 6. In the authors’ papers [4], some bounds for the distance \( \rho(\cdot, \cdot) \) were transferred to the distance \( \rho_m(\cdot, \cdot), m \in \mathbb{N} \), defined by equality
\[
\rho_m(F, G) = \sup_{P \in \mathcal{F}_m} |F\{P\} - G\{P\}|, \quad F, G \in \mathcal{F}_d.
\]
Proof of Theorem 2. Fix some polyhedron $P \in \mathcal{P}_m$:

$$P = \{x \in \mathbb{R}^d : \langle x, t_j \rangle \leq b_j, \; j = 1, \ldots, m \}.$$ 

where $t_j \in \mathbb{R}^d$, $\|t_j\| = 1$, $b_j \in \mathbb{R}$, $j = 1, \ldots, m$. Let $A : \mathbb{R}^d \to \mathbb{R}^m$ be a linear operator mapping as

$$x \mapsto y = (\langle x, t_1 \rangle, \ldots, \langle x, t_m \rangle).$$

Let $e_1, \ldots, e_m$ are the vectors of the standard Euclidean basis in $\mathbb{R}^m$. Consider the polyhedron $	ilde{P} \subset \mathbb{R}^m$ belonging to the class $\mathcal{P}_m^*$ of sets of the form (32) with $d = m$ and defined as

$$\tilde{P} = \{y \in \mathbb{R}^m : \langle y, e_j \rangle \leq b_j, \; j = 1, \ldots, m \}.$$

Since

$$\langle x, t_j \rangle = \langle x, A^* e_j \rangle = \langle Ax, e_j \rangle,$$

with adjoint operator $A^* : \mathbb{R}^m \to \mathbb{R}^d$, this implies that, for any random vector $\xi \in \mathbb{R}^d$, we have

$$\mathbb{P}[\xi \in P] = \mathbb{P}[A\xi \in \tilde{P}] \quad \text{and} \quad \mathbb{P}[\xi \in P_\lambda] = \mathbb{P}[A\xi \in \tilde{P}_\lambda].$$

where

$$P_\lambda = \{x \in \mathbb{R}^d : \langle x, t_j \rangle \leq b_j + \lambda, \; j = 1, \ldots, m \}.$$

Hence, for any random vectors $\xi, \xi' \in \mathbb{R}^d$ we have

$$\max\{\mathbb{P}[\xi \in P] - \mathbb{P}[\xi' \in P], \; \mathbb{P}[\xi' \in P] - \mathbb{P}[\xi \in P_\lambda]\}$$

$$= \max\{\mathbb{P}[A\xi \in \tilde{P}] - \mathbb{P}[A\xi' \in \tilde{P}_\lambda], \; \mathbb{P}[A\xi' \in \tilde{P}] - \mathbb{P}[A\xi \in \tilde{P}_\lambda]\}$$

$$\leq L(\mathcal{L}(A\xi), \mathcal{L}(A\xi'), \lambda), \quad (45)$$

where in the last step we used (11).

The distributions of $m$-variate vectors with coordinates $\langle \xi, t_j \rangle$, $\langle \eta, t_j \rangle$, $t_j \in \mathbb{R}^d$, $\|t_j\| = 1$, $j = 1, \ldots, m$, actually satisfy the same $m$-dimensional conditions as the distributions of random vectors $\xi, \eta \in \mathbb{R}^d$ with compared $d$-dimensional distributions $F$ and $D^{**}$, $D^*$ or $D$ from Theorem 11. Indeed, let $\alpha_i \in \mathbb{R}$, $X_i, Y_i \in \mathbb{R}^d$, $i = 1, \ldots, n$, be independent random variables and vectors such that

$$\mathbb{P}[\alpha_i = 1] = 1 - \mathbb{P}[\alpha_i = 0] = p_i, \quad \mathcal{L}(X_i) = U_i, \quad \mathcal{L}(Y_i) = V_i. \quad i = 1, \ldots, n. \quad (46)$$

Let

$$\xi_i = (1 - \alpha_i)X_i + \alpha_i Y_i, \quad i = 1, \ldots, n. \quad (47)$$
Then

$$L(ξ_i) = F_i = (1 - p_i)U_i + p_iV_i, \quad (48)$$

$$L(Åξ_i) = F_i^{(A)} = (1 - p_i)U_i^{(A)} + p_iV_i^{(A)}, \quad i = 1, \ldots, n. \quad (49)$$

Here and below for $W = L(ξ) ∈ ℱ_d$ we write $W(Aξ) = L(Aξ) ∈ ℳ_m$. If $W$ is an infinitely divisible distribution with spectral measure concentrated on the ball

$$\{ x ∈ R^d : ||x|| ≤ \tau \},$$

then $W^{(A)}$ is an infinitely divisible distribution with spectral measure concentrated on the ball $\{ x ∈ R^m : ||x|| ≤ \tau \sqrt{m} \}$. It suffices to verify that for $W = e(λEτe), \ λ ≥ 0, \ e ∈ R^d, ||e|| = 1$. Then $W^{(A)} = e(λE_{τke})$. It is easy to see that $(e(W))^{(A)} = e(W^{(A)})$. It remains to note that $||A|| ≤ \sqrt{m}$. Similarly, using (2), we see that

$$∫ x U_i^{(A)} \{ dx \} = 0, \quad U_i^{(A)} \{ x ∈ R^d : ||x|| ≤ \tau \sqrt{m} \} = 1, \quad i = 1, \ldots, n. \quad (50)$$

If the vectors $ξ, ξ'$ have the same covariance operators, then the covariance operators of the vectors $Åξ, Åξ'$ coinside too. Thus, the distributions $F^{(A)}, D^{(A)}, (D^{**})^{(A)}$ satisfy the conditions of $m$-variate version of Theorem 1 imposed on $F, D, D^{**}$ when replacing $τ$ by $τ \sqrt{m}$. Applying Theorem 1 we obtain, for any $λ > 0$,

$$L(F^{(A)}, (D^{**})^{(A)}, λ) ≤ c(m) \left( p + \exp \left( - \frac{λ}{c(m) \tau} \right) \right). \quad (51)$$

Using (45) and (51), we come to inequality

$$L_m(F, D^{**}, λ) ≤ \frac{c(m) \lambda}{c(m) \tau}, \quad λ > 0. \quad (52)$$

Recall that distributions $D$ and $D^*$ are particular cases of the distribution $D^{**}$. The second inequality of Theorem 2 follows now from (52). The first inequality follows from the second one by standard arguments. Theorem 2 is proved.

**Proof of Theorem 3** Fix some polyhedron $P ∈ ℋ_m$:

$$P = \{ x ∈ R^d : \langle x, t_j \rangle ≤ b_j, \ j = 1, \ldots, m \}. \quad (53)$$

It follows from Lemma 7 that it is possible to represent $P$ in the form

$$P = \{ x ∈ R^d : \langle x, t_j \rangle ≤ b_j, \ j = 1, \ldots, m_0 \}. \quad (54)$$

such that

$$P_{λ/2} ⊂ P^λ \quad \text{and} \quad m_0 ≤ N_m ∈ N,$$

where

$$P_{λ/2} = \left\{ x ∈ R^d : \langle x, t_j \rangle ≤ b_j + \frac{λ}{2}, \ j = 1, \ldots, m_0 \right\} \quad (55)$$
and the constant $N_m$ depends on $m$ only. Thus for any random vectors $\xi, \xi' \in \mathbb{R}^d$ we have

$$\max\{P[\xi \in P] - P[\xi' \in P^\lambda], P[\xi' \in P] - P[\xi \in P^\lambda]\}$$

$$\leq \max\{P[\xi \in P] - P[\xi' \in P_{\lambda/2}], P[\xi' \in P] - P[\xi \in P_{\lambda/2}]\}$$

$$\leq L_{N_m}\left(\mathcal{L}(\xi), \mathcal{L}(\xi'), \frac{\lambda}{2}\right).$$

Since this holds for any $P \in P_m$ we arrive at inequality

$$\pi_m(\cdot, \cdot, \lambda) \leq L_{N_m}\left(\cdot, \cdot, \frac{\lambda}{2}\right).$$

Thus, the second inequality of Theorem 3 follows from the second inequality of Theorem 2. The constants depending on $N_m$ may be treated as constants depending on $m$. The first inequality follows from the second inequality by standard reasoning. Theorem 3 is proved.

**Proof of Theorem 4.** Fix some polyhedron $P \in P_m$:

$$P = \left\{ x \in \mathbb{H} : \langle x, t_j \rangle \leq b_j, \; j = 1, \ldots, m \right\}.$$  

where $t_j \in \mathbb{H}, \|t_j\| = 1, b_j \in \mathbb{R}, j = 1, \ldots, m$. Let $L_t \subset \mathbb{H}$ be the linear span of vectors

$$\{t_j, j = 1, \ldots, m\}, \; k = \dim L_t \leq m,$$

and let $\mathbb{P}_t : \mathbb{H} \rightarrow L_t$ be the orthogonal projection operator on the subspace $L_t$. Consider the polyhedron $\bar{P} \subset L_t$ defined as

$$\bar{P} = \left\{ x \in L_t : \langle x, t_j \rangle \leq b_j, \; j = 1, \ldots, m \right\}.$$  

It is easy to see that, for any random vector $\zeta \in \mathbb{H}$, we have

$$\langle \mathbb{P}_t \zeta, t_j \rangle = \langle \zeta, t_j \rangle, \; j = 1, \ldots, m.$$  

Therefore,

$$P[\zeta \in P] = P[\mathbb{P}_t \zeta \in \bar{P}] \quad \text{and} \quad P[\zeta \in P^\lambda] = P[\mathbb{P}_t \zeta \in \bar{P}^\lambda],$$

where

$$\bar{P}^\lambda = \left\{ x \in L_t : \langle x, t_j \rangle \leq b_j + \lambda, \; j = 1, \ldots, m \right\}, \; \lambda > 0.$$  

Similarly, it is not difficult to show that $\{\zeta \in P^\lambda\}$ and $\{\mathbb{P}_t \zeta \in \bar{P}^\lambda\}$ are different descriptions of the same event. Therefore,

$$P[\zeta \in P^\lambda] = P[\mathbb{P}_t \zeta \in \bar{P}^\lambda].$$

For a better understanding of the situation, it is useful to mentally consider the case where $d = 3$ and $k = 2$.

The distributions of $k$-variate vectors $\mathbb{P}_t \xi, \mathbb{P}_t \eta \in L_t$ actually satisfy the same $k$-dimensional conditions as the distributions of random vectors $\xi, \eta \in \mathbb{H}$ with compared infinite-dimensional distributions $F$ and $D^{**}, D^*$ or $D$ from Theorem 4. In order to verify that, one should argue
like in the proof of Theorem 2 replacing operator \(A\) by operator \(P_t\) and using that \(\|P_t\| = 1\).

Applying Theorems 2 and 3, we obtain that, for any \(\lambda > 0\),

\[
\max\{P[P_t \xi \in \mathcal{P}] - P[P_t \eta \in \mathcal{P}], P[P_t \eta \in \mathcal{P}] - P[P_t \xi \in \mathcal{P}\} \leq c(m) \left(p + \exp \left( -\frac{\lambda}{c(m)\tau}\right) \right) \tag{56}
\]

and

\[
\max\{P[P_t \xi \in \mathcal{P}] - P[P_t \eta \in \mathcal{P}], P[P_t \eta \in \mathcal{P}] - P[P_t \xi \in \mathcal{P}\} \leq c(m) \left(p + \exp \left( -\frac{\lambda}{c(m)\tau}\right) \right) \tag{57}
\]

The statement of Theorem 4 follows now from (54)–(57). Theorem 4 is proved.

In our results, we assume, for simplicity, that

\[a_i = \int xU_i\{dx\} = 0, \quad i = 1, \ldots, n.\]

If we remove this assumption, then it will be valid again after replacing distributions \(F_i\) by distributions \(F_iE_{-a_i} = (1 - p_i)U_iE_{-a_i} + p_iV_iE_{-a_i}\). Of course, \(U_iE_{-a_i}\) is concentrated on the ball of larger radius \(2\tau\), but this does not imply any change of the rate of infinitely divisible approximation if we are not interested in numerical values of constants. In particular, applying inequalities (41)–(44), we get the bounds

\[L_m(F, \overline{D}) \leq c(m) \left(p + \tau(\ln \tau + 1)\right), \tag{58}\]

\[\pi_m(F, \overline{D}) \leq c(m) \left(p + \tau(\ln \tau + 1)\right), \tag{59}\]

and

\[L_m(F, \overline{D}, \lambda) \leq c(m) \left(p + \exp \left( -\frac{\lambda}{c(m)\tau}\right) \right), \quad \lambda > 0, \tag{60}\]

\[\pi_m(F, \overline{D}, \lambda) \leq c(m) \left(p + \exp \left( -\frac{\lambda}{c(m)\tau}\right) \right), \quad \lambda > 0, \tag{61}\]

where

\[\overline{D} = \prod_{i=1}^{n} E_{a_i}e(F_iE_{-a_i}).\]

Clearly, it is easy to write the corresponding analogues of approximating distributions (17) and (18) with the same rate of approximation as in (58)–(61).

The situation considered in Theorems 2 and 3 can be interpreted as a comparison of the sample containing independent observations of rare events with the Poisson point process which is obtained after a Poissonization of the initial sample (see [3, 17]).

Indeed, let \(Y_1, Y_2, \ldots, Y_n\) be independent not identically distributed elements of a measurable space \((Y, \mathcal{S})\). Assume that the set \(Y\) is represented as the union of two disjoint measurable sets: \(Y = Y_1 \cup Y_2\), with \(Y_1, Y_2 \in \mathcal{S}\), \(Y_1 \cap Y_2 = \emptyset\). We say that the \(i\)-th rare event occurs if \(Y_i \in Y_2\). Respectively, it does not occur if \(Y_i \in Y_1\).
Let $f : \mathcal{Y} \rightarrow \mathbb{R}^d$ be a Borel mapping and $F_i = \mathcal{L}(f(Y_i)), i = 1, 2, \ldots, n$. Then distributions $F_i \in \mathcal{F}_d$ can be represented as mixtures

$$F_i = (1 - p_i) U_i + p_i V_i,$$

where $U_i, V_i \in \mathcal{F}_d$ are conditional distributions of vectors $f(Y_i)$ given $Y_i \in \mathcal{Y}_1$ and $Y_i \in \mathcal{Y}_2$ respectively,

$$0 \leq p_i = P\{Y_i \in \mathcal{Y}_2\} = 1 - P\{Y_i \in \mathcal{Y}_1\} \leq 1.$$  

By definition, we deal with rare events whereas the quantity

$$p = \max_{1 \leq i \leq n} p_i$$

is small. In other words, this is the case if our rare events are sufficiently rare.

Denote

$$F = \prod_{i=1}^{n} F_i, \quad D = \prod_{i=1}^{n} e(F_i).$$

The sum

$$S = f(Y_1) + \cdots + f(Y_n)$$

has the distribution $F$. It is easy to see that $D$ is the distribution of

$$T = \sum_{i=1}^{n} \sum_{j=1}^{
u_i} f(Y_{i,j}),$$

where $Y_{i,j}$ and $\nu_i, i = 1, \ldots, n, j = 1, 2, \ldots$, are random elements in $\mathcal{Y}$ which are independent in aggregate such that $\mathcal{L}(Y_{i,j}) = \mathcal{L}(Y_i)$ and $\mathcal{L}(\nu_i) = e(E_1)$. Clearly, $e(E_1)$ is the Poisson distribution with mean 1.

Thus, the sum $T$ is defined similarly to $S$, but the initial sample $Y = (Y_1, Y_2, \ldots, Y_n)$ is replaced by its Poissonized version $\Pi = \{Y_{i,j} : i = 1, \ldots, n, j = 1, 2, \ldots, \nu_i\}$. Poissonization of the sample is known as one of the most powerful tools in studying empirical processes. The random set $\Pi$ may be considered as a realization of the Poisson point process on the space $\mathcal{Y}$ with intensity measure $\sum_{i=1}^{n} \mathcal{L}(Y_i)$. The important property of the Poisson point process is the space independence: for any pairwise disjoint sets $A_1, \ldots, A_m \in \mathcal{S}$, the random sets $\Pi \cap A_1, \ldots, \Pi \cap A_m \subset \mathcal{Y}$ are independent in aggregate. As a consequence, investigation of the Poisson point process $\Pi$ is much easier than studying the sample $Y$. One can use the independence property since the theory of independent objects is much more elaborated.

Let relations (62)–(67) be satisfied and let, for some $\tau \geq 0$,

$$U_i\{\{y \in \mathbb{R}^d : \|y\| \leq \tau\}\} = 1, \quad i = 1, 2, \ldots, n,$$

and the $V_i \in \mathcal{F}_d$ are arbitrary distributions. Define

$$a_i = \int_{\mathbb{R}^d} x U_i(dx), \quad i = 1, 2, \ldots, n.$$
Denote
\[ T^* = \sum_{i=1}^{n} \left( a_i + \sum_{j=1}^{\nu_i} (f(Y_{i,j}) - a_i) \right), \tag{69} \]
Then
\[ D = L(T^*) = \prod_{i=1}^{n} E_{a_i} e(F_{i - a_i}), \tag{70} \]
and
\[ T^* = T - \Delta, \quad \text{where} \quad \Delta = \sum_{i=1}^{n} (\nu_i - 1) a_i, \tag{71} \]
and \( \nu_i \) are i.i.d. Poisson with mean 1.

Theorem 2 implies the following assertions about the closeness of distributions \( F \) and \( D \), see (65).

**Theorem 5.** Let the above conditions be satisfied. Then, for any \( m \in \mathbb{N}, \lambda > 0 \) and \( P \in \mathcal{P}_m \) defined in (35), we have
\[ \max \{ F(P) - D(P_{2\lambda}), D(P) - F(P_{2\lambda}) \} \leq c(m) \left( p + \exp \left( -\frac{\lambda}{c(m)\tau} \right) \right) + \sum_{j=1}^{m} P\{|\langle \Delta, t_j \rangle| \geq \lambda\}, \tag{72} \]
where the polyhedron \( P_{2\lambda} \) is defined in (36).

**Proof.** Note that \( P_\lambda \in \mathcal{P}_m \) and \( (P_\lambda)_\lambda = P_{2\lambda} \). Using (60), we see that
\[ \max \{ F(P) - D(P_\lambda), D(P_\lambda) - F(P_{2\lambda}) \} \leq c(m) \left( p + \exp \left( -\frac{\lambda}{c(m)\tau} \right) \right). \tag{73} \]

By definition,
\[ D(P_\lambda) = P\{\langle T^*, t_j \rangle \leq b_j + \lambda, \ j = 1, \ldots, m\}, \tag{74} \]
\[ D(P) = P\{\langle T, t_j \rangle \leq b_j, \ j = 1, \ldots, m\}, \tag{75} \]
\[ D(P_{2\lambda}) = P\{\langle T, t_j \rangle \leq b_j + 2\lambda, \ j = 1, \ldots, m\}. \tag{76} \]

Using (71), (74), (75), we obtain inequality
\[ D(P) \leq D(P_\lambda) + \sum_{j=1}^{m} P\{|\langle \Delta, t_j \rangle| \geq \lambda\}. \tag{77} \]
Similarly, by (71), (74), (76), we have
\[ \overline{D}(P_\lambda) \leq D(P_{2\lambda}) + \sum_{j=1}^{m} P\{|\langle \Delta, t_j \rangle| \geq \lambda\}. \tag{78} \]

Inequality (72) follows now from (73), (77) and (78). Theorem 5 is proved.

Theorem 5 is a generalization of inequalities (15) and (16) of Theorem 9.
The probabilities $P\{|\langle \Delta, t_j \rangle| \geq \lambda\}$ may be estimated using Bernstein’s inequality, see inequality (17).

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