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SUBEXPONENTIALITY OF DENSITIES OF INFINITELY DIVISIBLE DISTRIBUTIONS

MUNEYA MATSUI

Abstract. We show the equivalence of three properties for an infinitely divisible distribution: the subexponentiality of the density, the subexponentiality of the density of its Lévy measure and the tail equivalence between the density and its Lévy measure density, under monotonic-type assumptions on the Lévy measure density. The key assumption is that tail of the Lévy measure density is asymptotic to a non-increasing function or is eventually non-increasing. Our conditions are novel and cover a rather wide class of infinitely divisible distributions. Several significant properties for analyzing the subexponentiality of densities have been derived such as closure properties of convolution, convolution roots and asymptotic equivalence and the factorization property. Moreover, we illustrate that the results are applicable for developing the statistical inference of subexponential infinitely divisible distributions which are absolutely continuous.

1. Introduction

Studies on the subexponentiality of infinitely divisible distributions have been initiated by Embrecht et al. [6], where the subexponentiality of one-sided distributions was completely characterized. Pakes [15] extended the result into distributions on the real line. More general $\gamma$-subexponentiality $\gamma \geq 0$ (see e.g. [19, p.369]) has intensively investigated by Embrechts and Goldie [5], Pakes [15] and Watanabe [19] (see a comprehensive literature in the introduction in [19]).

However, on the subexponentiality of densities of infinitely divisible distributions there are only a few works. Watanabe and Yamamuro [21] investigated the class of self-decomposable distributions, and Watanabe [20] studied the subexponential densities on the half-line. Shimura and Watanabe [17] treated the compound Poisson case on the positive half. As stated in Watanabe [20] “the subexponentiality of a density is a stronger and more difficult property than the subexponentiality of a distribution.” Besides, we treat two-sided distributions which are harder to handle than one-sided ones. The results have been applied to characterize the tail asymptotics for the density of the supremum of a random walk, which is closely related with classical ruin theory and queuing theory ([9, Section 5], [17, Section 4]).

One of our motivations here is an application in statistics. Infinitely divisible distributions provide finite dimensional distributions of Lévy processes, the processes which have found numerous applications in meteorology, seismology, telecommunications, finance and insurance, and still attract a lot of attention (see [1]). Thus, the related statistical methods have been intensively studied. In applications such as statistical modelings or statistical estimations of Lévy processes, densities are often more convenient than distributions to handle. Therefore, further studies on tail properties of densities such as subexponentiality are desirable. Indeed, the tail condition is crucial for asymptotics of various estimation methods (see the argument in Section 5).

In this paper we characterize the subexponentiality of densities of infinitely divisible distributions on the whole real line. We establish the tail equivalence between the absolutely continuous part of an infinitely divisible distribution and the density of the corresponding Lévy measure. Furthermore, we show that this equivalence implies that the equivalence in subexponentiality, and vice versa. Our

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key is to assume a kind of monotonic property on the Lévy measures, that is “asymptotic to a non-increasing function” (abbreviated by a.n.i.) property or “eventually non-increasing” (abbreviated by e.n.i.) property. The latter assumption is a bit stronger, but we could derive stronger results. Notice that the regularly varying functions with negative indices satisfy the a.n.i. property ([2, p.23]). Moreover, these two properties are covered by a rather wide class of Lévy measures, and indeed both are shared by all self-decomposable distributions.

Our strategy is to reduce the problem to the compound Poisson case. We add an extra measure to the original Lévy measure and make another infinitely divisible distribution. With this operation right-tail asymptotics of both the density and the Lévy density are unchanged. Then we decompose this new distribution by truncation and symmetrization of the updated Lévy measure into a negligible distribution in the tail and a symmetric unimodal compound Poisson, i.e. the new version is obtained by convolution of the two distributions. Both the tail of the original density and that of the original Lévy density are approximated with those of symmetric unimodal compound Poisson. In this compound Poisson both the absolutely continuous part and Lévy density satisfy the a.n.i. property, so that we make full use of the a.n.i. property to solve the problem. In the derivation process, we prove various properties for the tail of an infinitely divisible distribution with the a.n.i. assumption: the closure properties of [convolution, convolution roots and asymptotic equivalence] and the factorization property, which are crucial for analyzing tail asymptotics and which have their own interest.

We apply our results to the consistency proof of the maximum likelihood estimation (MLE for abbreviation) for an absolutely continuous infinitely divisible distribution. Usually an explicit expression for the density is unavailable for this class, while properties of the density such as boundedness and tail behavior (and sometimes continuity) are crucial in both the definition and asymptotics of MLE. Our proof depends only on the Lévy density and we do not touch the genuine density or distribution. Therefore, by our results we could extend the scope of MLE to a rather wide subclass of infinitely divisible distributions beyond particular ones with explicit densities. We believe that our results would be useful tools for other estimation methods than MLE.

In Section 2, notation and definitions are formulated. We state main results in Section 3 together with the closure/factorization properties of the subexponential density under assumption of a.n.i. The main results are based on the compound Poisson case treated in Section 4. In Section 5, a statistical application is provided. The proofs of the main results are given in Section 6, and technical auxiliary proofs are summarized in Appendices A and B.

2. Preliminaries

Let \( F, G, H \) be probability distribution functions on \( \mathbb{R} \) and denote by \( F \ast G \) the convolution of \( F \) and \( G \):

\[
F \ast G(x) = \int_{-\infty}^{\infty} F(x-y)G(dy)
\]

and denote by \( F^{*n} \) the \( n \)th convolutions with itself. The tail probability of \( F \) is denoted by \( \overline{F}(x) = 1 - F(x) \). Let \( f, g, h \) be the corresponding probability density functions on \( \mathbb{R} \) and we use the same notations for the convolution as those for distributions, e.g.

\[
f \ast g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy \quad \text{or} \quad f^{*n}(x) \quad \text{for the} \ n\text{th convolution.}
\]

We say that \( F \) on \( \mathbb{R} \) is long-tailed, denoted by \( F \in \mathcal{L} \), if \( \overline{F}(x) > 0 \) for all \( x \) and

\[
\lim_{x \to \infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1 \quad \text{for any fixed} \quad y > 0.
\]

In addition we call that \( F \) is subexponential on \( \mathbb{R} \), denoted by \( F \in \mathcal{S} \), if \( F \in \mathcal{L} \) and

\[
\lim_{x \to \infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} = 2.
\]
The class $S$ was introduced by [3] and it is known that $S$ includes regularly varying functions. It should be noted that if $F$ is a distribution on $\mathbb{R}_+ = [0, \infty)$, then the condition (2.1) solely implies $F \in S$ and we do not need the assumption $F \in \mathcal{L}$, since $F \in S$ automatically satisfies $F \in \mathcal{L}$ (cf. [9, Lemma 3.2]). Throughout the paper, for functions $\alpha, \beta : \mathbb{R} \to \mathbb{R}_+$, $\alpha(x) \sim \beta(x)$ means that $\lim_{x \to \infty} \alpha(x)/\beta(x) \to 1$.

In this paper we study the corresponding characteristics for densities.

**Definition 2.1.** (i) The density $f$ of $F$ is (right-side) long-tailed, denoted by $f \in \mathcal{L}$, if there exists $x_0 > 0$ such that $f(x) > 0$, $x \geq x_0$ and for any fixed $y > 0$ $f(x + y) \sim f(x)$.

(ii) The density $f$ of $F$ is (right-side) subexponential on $\mathbb{R}$, denoted by $S$, if $f \in \mathcal{L}$ and $f^{*2}(x) \sim 2f(x)$.

(iii) The density $f$ of $F$ is weakly (right-side) subexponential on $\mathbb{R}$, denoted by $S_{\alpha}$, if $f \in \mathcal{L}$ and the function $f_{\alpha}(x) = \mathbb{1}_{\mathbb{R}_+}(x)f(x)/F(0)$, $x \in \mathbb{R}$ is subexponential, i.e. $f_{\alpha} \in S$.

Here $f_{\alpha}$ is the density of the conditional distribution $F_{\alpha}$ of $F$ on $\mathbb{R}_+$. For a distribution $F$, it is known that

$$F \in S \iff F_{\alpha} \in S \iff F^{+} \in S,$$

where $F^{+}$ is the distribution given by $F^{+}(x) = F(x)$ for $x \geq 0$ and $F^{+}(x) = 0$ for $x < 0$ (see Corollary 2.1 of [15], Lemma 3.4 of [9]). However, for a probability density $f$ the situation is different, i.e. unless the support of $f$ is bounded below, we could not have $f \in S \iff f \in S_{\alpha}$ without additional conditions ([9, p.83]). Therefore, we assume one of the following two assumptions, under which $f \in S \iff f \in S_{\alpha}$ ([9, Lemma 4.13]), and which are key tools in this paper.

**Definition 2.2.** (i) We say that a density $f : \mathbb{R} \to \mathbb{R}_+$ is asymptotic to a non-increasing function (a.n.i. for short) if $f$ is locally bounded and positive on $[x_0, \infty)$ for some $x_0 > 0$, and

$$\sup_{t \geq x} f(t) \sim f(x) \quad \text{and} \quad \inf_{x_0 \leq t \leq x} f(t) \sim f(x).$$

(ii) We say that a density $g : \mathbb{R} \to \mathbb{R}_+$ is eventually non-increasing (e.n.i. for short) if there exists $x_0 > 0$ such that for $x \geq x_0$ $g(x) > 0$ and $g(x)$ is non-increasing.

Notice that the a.n.i. property includes the e.n.i. property, and the former is satisfied by the regularly varying functions with negative indices [2, p.23].

We will investigate properties of the above sort, particularly on infinitely divisible distributions $\mu$ on $\mathbb{R}$. The characteristic function (ch.f.) of $\mu$ is

$$\hat{\mu}(z) = \exp \left\{ \int_{-\infty}^{\infty} (e^{izy} - 1 - izy\mathbb{1}_{\{|y| \leq 1\}}) \nu(dy) + iaz - \frac{1}{2}b^2z^2 \right\},$$

where $a \in \mathbb{R}$, $b \geq 0$ and $\nu$ is the Lévy measure satisfying $\nu\{0\} = 0$ and $\int_{-\infty}^{\infty} (1 \wedge x^2)\nu(dx) < \infty$.

Throughout this paper, we always assume that the Lévy measure $\nu$ of $\mu$ has a density, and we denote by $ID(\mathbb{R})$ the class of all infinitely divisible distributions on $\mathbb{R}$.

3. **Main Results**

We separate the cases depending on weather $\nu(\mathbb{R}) < \infty$ or $\nu(\mathbb{R}) = \infty$. The former implies that $\mu$ is a compound Poisson plus Gaussian (e.g. [18, Ch.IV, Theorem 4.1.8], [13, Lemma 2.13]). Since we always assume a density for the Lévy measure, the latter implies that the purely non-Gaussian part of $\mu$ is absolutely continuous (e.g. [16, Theorem 27.7] with $l = 1$). Note that we use notation $g$ also for the (non-proper) density of a Lévy measure.

**Theorem 3.1.** Let $\mu \in ID(\mathbb{R})$ with $\nu(dx) = g(x)dx$ such that $\nu(\mathbb{R}) < \infty$. Denote the non-Gaussian part $\mu'$, which is a $\gamma$-shifted compound Poisson given by

$$\mu'(dx) = e^{-\gamma} \delta_{\gamma}(dx) + (1 - e^{-\gamma})f(x - \gamma)dx, \quad \gamma \in \mathbb{R},$$

where $\delta_{\gamma}$ is a point mass at $\gamma$.


where \( \delta_\gamma \) is Dirac measure at \( \gamma \), \( f \) is a proper density and \( \lambda > 0 \) is the Poisson parameter. Suppose that \( g \) is bounded and a.n.i. Then the following are equivalent.

(i) \( f \in \mathcal{S} \)

(ii) \( g \in \mathcal{S}_+ \)

(iii) \( g \in \mathcal{L} \) \& \( \lim_{x \to \infty} f(x)/g(x) = \lambda/(1 - e^{-\lambda}) \).

Theorem 3.1 directly follows from the compound Poisson case (Theorem 4.1).

**Theorem 3.2.** Let \( \mu \in \text{ID}(\mathbb{R}) \) with \( \nu(dx) = g(x)dx \) such that \( \nu(\mathbb{R}) = \infty \). Let \( f_0(x) \) be a density of \( \mu_0 \in \text{ID}(\mathbb{R}) \) with \( a = b = 0 \) and \( \nu(dx) = 1_{\{|x| \leq 1\}} g(x) dx \). Suppose that \( g_1(x) = 1_{\{x > 1\}} g(x)/\nu(1, \infty) \) is bounded, and there exists \( \gamma > 0 \) such that

\[
\lim_{x \to \infty} e^{\gamma x} f_0(x) = 0.
\]

For a density \( f \) of \( \mu \) we consider the following properties.

(i) \( f \) is a.n.i. \& \( f \in \mathcal{S}_+ \)

(ii) \( g_1 \in \mathcal{S}_+ \)

(iii) \( g_1 \in \mathcal{L} \) \& \( \lim_{x \to \infty} f(x)/g_1(x) = \nu((1, \infty)) \).

(a) If \( g \) is a.n.i., then we can choose \( f \) such that (ii) \( \Leftrightarrow \) (iii) implies (i).

(b) If \( g \) is e.n.i., then we can choose \( f \) such that (i), (ii) and (iii) are equivalent.

The proof is given in Section 6, which depends on the compound Poisson case (Section 4). Obviously \( \mathcal{S}_+ \) of the two theorems is replaced with \( \mathcal{S} \).

**Remark 3.3.** (i) Theorem 3.2 (and Theorem 3.4) holds regardless of values of \( b \), and thus Gaussian density is convoluted in \( f \) when \( b > 0 \). Generally it is difficult to separate non-Gaussian part from \( f \), since it also includes the small jump part density \( f_0 \) as a convoluted element, which is not heavy tail.

(ii) Theorem 3.2 covers many important subclasses of \( \text{ID}(\mathbb{R}) \), such as self-decomposable distributions whose Lévy densities \( g \) are non-increasing on \((0, \infty)\) and non-decreasing on \((-\infty, 0)\) (Jurek [11]). Indeed, it includes all self-decomposable cases of [21].

(iii) Assume that \( g(x) = \lvert x \rvert^{-1} k(x) \) where \( k(x) \geq 0 \) is non-decreasing on \((-\infty, 0)\) and non-increasing on \((0, \infty)\) such that \( \int_{-\infty}^{\infty} \lvert x \rvert^2 \wedge 1 k(x)/\lvert x \rvert dx < \infty \). Then the density \( f_0 \) of \( \mu_0 \in \text{ID}(\mathbb{R}) \) satisfies (3.4).

This is a self-decomposable distribution.

(iv) Notice that the a.n.i. or e.n.i. property of \( g_1 \in \mathcal{S} \) does not imply that \( g_1 \) is regularly varying. Recall from [9, p.86] (cf. [21, p.1042]) that the density of the standard log normal distribution or the Weibull distribution with parameter \( \alpha \in (0, 1) \) is subexponential and e.n.i., but it is not regularly varying. Both distributions are known to be self-decomposable (cf. [18, p.360, p.414]).

(v) We could easily find non self-decomposable examples which are covered by Theorem 3.2. Indeed, assume that the tail of Lévy density \( g \) is given by the density of the standard log normal distribution or the Weibull distribution with parameter \( \alpha \in (0, 1) \), and further assume that \( g \) is not monotone. Then, \( \mu \in \text{ID}(\mathbb{R}) \) with this Lévy density \( g \) is not self-decomposable, while \( g_1 \in \mathcal{S} \).

We could remove several conditions in Theorem 3.2 by assuming the absolute integrability of \( \hat{\mu} \) (cf. [20, Theorem 2]). In our case we require a stronger condition, since we treat the two sided case. Define a spectrally positive version \( \hat{\mu}_+ \) by

\[
\hat{\mu}_+(z) = \exp \left\{ \int_0^\infty (e^{izx} - 1 - izx 1_{\{0 < x \leq 1\}}) \nu(dx) \right\}.
\]

We assume the absolute integrability of \( \hat{\mu}_+ \). Although the proof is made by a minor change to that of Theorem 3.2, the result is convenient in applications and we state the result as a theorem.
**Theorem 3.4.** Let \( \mu \in \text{ID}(\mathbb{R}) \) with and \( \nu(dx) = g(x)dx \) such that \( g_1(x) \) is bounded. Suppose that \( f_\to \| \mu_\tau(z)dz < \infty \), which implies \( f_\to \| \mu(z)dz < \infty \), so that \( \mu \) has a bounded continuous density \( f \). Then the following relations hold between the properties (i), (ii) and (iii) of Theorem 3.2.

(a) If \( g \) is a.n.i., then (ii) \( \iff \) (iii) implies (i).

(b) If \( g \) is e.n.i., then (i), (ii) and (iii) are equivalent.

The proof is given in the end of Section 6.

**3.1. Known properties of \( f \) in \( \mathcal{L}, \mathcal{S}_+ \) or \( \mathcal{S} \).** We introduce known or easily-derived properties of \( f \) in \( \mathcal{L}, \mathcal{S}_+ \) or \( \mathcal{S} \). Since we handle a compound Poisson distribution, which is not absolutely continuous, we introduce a generalized density,

\[
\tilde{f}(x) = p\delta(x) + qf(x), \quad p + q = 1, \quad p, q \geq 0,
\]

where \( f \) is a proper density and \( \delta(x) \) is the Dirac delta function (see [12] for treatment of \( \delta(x) \)). Here \( f \) is an ordinary function and it does not include \( \delta(x) \). Notice that by direct calculations \( f \in \mathcal{L} \iff \tilde{f} \in \mathcal{L} \) and \( f \in \mathcal{S} \iff \tilde{f} \in \mathcal{S} \) clearly hold. Moreover \( f \in \mathcal{S}_+ \iff \tilde{f} \in \mathcal{S}_+ \) follows, which is equivalent to \( f \in \mathcal{S} \iff \tilde{f} \in \mathcal{S} \) if \( f \) is on \( \mathbb{R}_+ \). The properties of \( \tilde{f} \) rely heavily on the following crucial concept by [9, Definition 2.18], which the class \( \mathcal{L} \) enjoys and which we will use frequently.

**Definition 3.5.** Given a strictly positive non-decreasing function \( \alpha \), an ultimately positive function \( \beta \) is called \( \alpha \)-insensitive if

\[
\sup_{|y| \leq \alpha(x)} |\beta(x+y) - \beta(x)| = o(\beta(x)) \quad \text{as} \quad x \to \infty, \quad \text{uniformly in} \quad |y| \leq \alpha(x).
\]

The following is known for \( f \), but easily extended for \( \tilde{f} \) with slight modifications in the proof.

**Lemma 3.6.** (i) [9, Lemma 2.19]. Let \( \tilde{f} \in \mathcal{L} \) and then there exists a function \( \alpha \) such that \( \alpha(x) \to \infty \) as \( x \to \infty \) and \( \tilde{f} \) is \( \alpha \)-insensitive.

(ii) [9, Proposition 2.20]. Given a finite collection of \( \tilde{f}_1, \ldots, \tilde{f}_n \in \mathcal{L} \) we may choose a single function \( \alpha \) increasing to infinity w.r.t. which each of functions \( \tilde{f}_i \) is \( \alpha \)-insensitive.

(iii) [9, Theorem 4.2, Corollary 4.5]. Let \( \tilde{f} \in \mathcal{L} \), then

\[
\liminf_{x \to \infty} \tilde{f} * \tilde{g}(x)/\tilde{f}(x) \geq 1.
\]

Moreover, \( \tilde{f}^{*n} \in \mathcal{L} \) and

\[
\liminf_{x \to \infty} \tilde{f}^{*n}(x)/\tilde{f}(x) \geq n.
\]

(iv) [9, Theorem 4.3]. Let \( \tilde{f}, \tilde{g} \in \mathcal{L} \), then \( \tilde{f} * \tilde{g} \in \mathcal{L} \).

(v) cf. [9, Proofs of Lemmas 4.12 and 4.13]. Let \( \tilde{f} \in \mathcal{S} \) and is \( \alpha \)-insensitive such that \( \alpha(x) \to \infty \) as \( x \to \infty \). Then

\[
\int_{-\infty}^{-\alpha(x)} \tilde{f}(x-y)\tilde{f}(y)dy = o(\tilde{f}(x)) \quad \text{and} \quad \int_{\alpha(x)}^{x-\alpha(x)} \tilde{f}(x-y)\tilde{f}(y)dy = o(\tilde{f}(x)).
\]

(vi) [9, Theorem 4.8]. If \( \tilde{f} \in \mathcal{S}_+ \) and \( \tilde{g}(x) \sim c\tilde{f}(x) \) with \( c > 0 \), then \( \tilde{g} \in \mathcal{S}_+ \).

We only give the proof for (v).

**Proof of Lemma 3.6 (v).** We take an insensitive function \( \alpha \) for \( \tilde{f} \) and write

\[
\frac{\tilde{f}^{*2}(x)}{\tilde{f}(x)} = 2 \int_{-\infty}^{-\alpha(x)} \frac{\tilde{f}(x-y)\tilde{f}(y)}{\tilde{f}(x)}dy + 2 \int_{-\alpha(x)}^{0} \frac{\tilde{f}(x-y)\tilde{f}(y)}{\tilde{f}(x)}dy + \int_{0}^{x-\alpha(x)} \frac{\tilde{f}(x-y)\tilde{f}(y)}{\tilde{f}(x)}dy.
\]

\[
\quad \quad \quad \quad \quad = 2I_1(x) + 2I_2(x) + I_3(x).
\]
By the property of $\delta(x)$ and $\tilde{f} \in \mathcal{L} \iff f \in \mathcal{L}$, we have
\[
I_2(x) = p + q \int_{-\alpha(x)}^{\alpha(x)} f(x-y)f(y)/f(x)dy \to 1,
\]
as $x \to \infty$. Thus other integrals $I_1, I_3 \geq 0$ should converge to zero. \hfill \Box

The next result is Kesten’s type bound for densities.

**Lemma 3.7** ([9, Theorem 4.11] & [8, Theorem 2]). Let $f \in S_+$ be bounded. Suppose that $f$ is a density on $\mathbb{R}_+$ or that there exist $x_0 > 0$ and $K > 0$ such that
\[
f(x+y) \leq Kf(x), \quad x > x_0, y > 0.
\]
Then for any $\varepsilon \in (0,1)$, there exist $C_\varepsilon > 0$ and $x_\varepsilon > 0$ such that
\[
f^n(x) \leq C_\varepsilon(1 + \varepsilon)^n f(x), \quad x > x_\varepsilon, n \in \mathbb{N}.
\]

3.2. **New properties of $f$ in $\mathcal{L}$, $S_+$ or $S$ with a.n.i. condition.** We close this section with the following new results, which are crucial for proving main theorems and which are significant in themselves for analyzing tail behavior of densities under the a.n.i. assumption. Proofs are given in Appendix B.

**Lemma 3.8.** Let $f$ be a density with a.n.i. property. Then
\[
\lim_{x \to \infty} f(x)/\alpha(x) = 1.
\]
Conversely, if $f(x) \to 0$ as $x \to \infty$, then (3.8) implies the a.n.i. property of $f$. In particular, $f \in \mathcal{L}$ with the condition (3.8) implies that $f$ is a.n.i.

**Lemma 3.9** (Asymptotic equivalence). Suppose $\tilde{f} \in S$ and
\[
\lim_{x \to \infty} \tilde{f}(x)/g(x) = c \quad \text{for some} \quad c \in (0, \infty).
\]
If $\tilde{g}$ or $\tilde{f}$ is a.n.i., or $\tilde{g}(-x) = O(\tilde{f}(-x))$ then $\tilde{g} \in S$.

**Proposition 3.10** (Convolution and factorization). Let $h = f * \tilde{g}$ be the convolution of a density $f$ and a generalized density $\tilde{g}$.
(i) Let $f \in S_+$ and $\tilde{g}(x) = o(f(x))$. If $f$ is a.n.i. or $g(-x) = O(f(-x))$ as $x \to \infty$, then $h \in S_+$ and $h(x) \sim f(x)$.
(ii) Let $h \in S_+$ and $\tilde{g}(x) = o(h(x))$. If $f$ is a.n.i. and $[h$ is a.n.i. or $\tilde{g}(x) = o(f(x))]$, then $f \in S_+$ and moreover $h(x) \sim f(x)$.

**Corollary 3.11** (Factorization). Let $h = f * \tilde{g}$. If $h \in S_+$, $\tilde{g}(x) = o(e^{-\varepsilon x})$ for some $\varepsilon > 0$ and $f$ is a.n.i., then $f \in S_+$ and indeed, $h(x) \sim f(x)$.

**Theorem 3.12** (Convolution root). Assume that $\tilde{f}^n \in S_+$ for some $n \in \mathbb{N}$, and $\tilde{f}^n$ is a.n.i. If $\tilde{f}^{*k}, k = 1, \ldots, n-1$ are a.n.i. or if $f \in \mathcal{L}$, then $\tilde{f} \in S$.

**Remark 3.13.** (i) For positive-half densities, the long-tailed property $\mathcal{L}$ plays the most fundamental role for deriving various tail properties (see [20]). One could see in above that the a.n.i. property could play a role of $\mathcal{L}$ in several cases especially for densities on $\mathbb{R}$. However, it remains to be seen whether the a.n.i. characteristic has similar properties to that of $\mathcal{L}$ such as closedness under convolution. An open question is that under what conditions the a.n.i. property is retained.
(ii) Watanabe and Yamamuro [22] have proved that the class of subexponential densities is neither closed under asymptotic equivalence nor closed under convolution roots. Therefore in Lemma 3.9 and Theorem 3.12, we need additional conditions other than the subexponentiality.
4. The compound Poisson case

Recall that we always assume a density for the Lévy measure. Let $g$ be a density on $\mathbb{R}$. For $\lambda > 0$ define the compound Poisson probability measure by

$$
\mu(dx) = e^{-\lambda} \delta_0(dx) + (1 - e^{-\lambda}) f(x) dx,
$$

where

$$
f(x) = (e^\lambda - 1)^{-1} \sum_{n=1}^{\infty} (\lambda^n/n!) g^n(x).
$$

We call $f$ the proper absolutely continuous part of $\mu$.

**Theorem 4.1.** Suppose that $g$ or equivalently $f$ is bounded and $g$ is a.n.i. The following assertions are equivalent.

(i) $f \in \mathcal{S}$

(ii) $g \in \mathcal{S}_+$

(iii) $g \in \mathcal{L}$ & $\lim_{x \to \infty} f(x)/g(x) = \lambda/(1 - e^{-\lambda})$.

Obviously $\mathcal{S}_+$ of the theorem can be replaced with $\mathcal{S}$. For the proof, we consider the two-sided extension of [17], and borrow and extend several useful tools in [17]. The study on the equivalence of above type has been initiated in the distribution version [6, Theorem 3]. However, direct tracing of the idea in [6, Theorem 3] is quite difficult and we need to make new passes. Indeed, even in the positive-half density case of [17], several new tools have been invented to overcome the difficulty.

The following auxiliary results specific to a generalized density with a delta function part are useful, which are interesting and important in themselves. Proofs for these results are rather technical and we give them in Appendix B.

**Proposition 4.2** (Factization). Let $\tilde{h} = \tilde{g} * \tilde{f} \in \mathcal{S}$ such that $\tilde{g}(x) = p_g \delta(x) + (1 - p_g) g(x)$ with $p_g \in (2^{-1}, 1)$. Suppose that $\tilde{h}$ is a.n.i. or $\tilde{f}(x) = p_f \delta(x) + (1 - p_f) f(x)$ with $p_f \in (0, 1)$. Then, $\tilde{g}(x) = o(\tilde{h}(x))$ implies $\tilde{f} \in \mathcal{S}$, and indeed $\tilde{h}(x) \sim \tilde{f}(x)$.

**Proposition 4.3** (Convolution root). Let $\tilde{f}(x) = p \delta(x) + (1 - p) f(x)$. If $p \in (2^{-1/(n-1)}, 1)$ then $\tilde{f}^* \in \mathcal{S}$ implies $\tilde{f} \in \mathcal{S}$.

**Corollary 4.4** (Factization, compound Poisson). Let $\tilde{h} = \tilde{g} * \tilde{f} \in \mathcal{S}$ such that $\tilde{g}$ is a compound Poisson. Suppose that $\tilde{h}$ is a.n.i. or $\tilde{f}(x) = p \delta(x) + (1 - p) f(x)$ with $p \in (0, 1)$. Then $\tilde{g} = o(\tilde{h}(x))$ implies $\tilde{f} \in \mathcal{S}$ and indeed $\tilde{h}(x) \sim \tilde{f}(x)$.

We make a remark about “steutel conjecture” for a compound Poisson distribution [6, p.340]. For the generalized density $\tilde{f}$ of $\mu$ with ch.f. $\tilde{\mu}(z)$, we denote by $\tilde{f}^{*\alpha}$ the generalized density given by ch.f. $\tilde{\mu}(z)^{\alpha}$ for any positive real number $\alpha$. If $\tilde{f}$ is a compound Poisson with Lévy density $\lambda g(x)$, then so is $\tilde{f}^{*\alpha}$ with Lévy density $\lambda \alpha g(x)$. Due to Theorem 4.1 when $g$ is bounded and a.n.i., then $\tilde{f} \in \mathcal{S}$ implies $\tilde{f}^{*\alpha} \in \mathcal{S}$ for all $\alpha > 0$. By applying Proposition 4.3 without assuming such conditions for $g$, we have the following.

**Corollary 4.5** (Steutel conjecture, compound Poisson). If $\tilde{f} \in \mathcal{S}$ is the generalized density of a compound Poisson with parameter $\lambda < \log 2$. Then $\tilde{f}^{*\alpha} \in \mathcal{S}$ for every rational $\alpha > 0$ and $\tilde{f}^{*\alpha}(x)/\tilde{f}(x) \to \alpha$ as $x \to \infty$.

**Proof of Theorem 4.1.** Let $c$ be a generic positive constant whose value is not of interest, and denote the Laplace transform of $f$ by $L[f](z) = \int_0^\infty e^{zx} f(x) dx$. 

(i) implies (ii) and (iii)
Since $g$ is a proper density, we may choose $c_1 > 0$ such that $G(c_1) = \Lambda_1 < \log 2/\lambda$ and define another compound Poisson by

$$
\tilde{\mu}_1(z) = \exp \left( \lambda \Lambda_1 \int_{c_1}^{\infty} (e^{izy} - 1) g_1(y) dy \right), \quad g_1(x) = g(x)/\Lambda_1,
$$

so that

$$
\mu = \mu_1 * \mu_2, \quad i.e. \quad \hat{\mu}(z) = \tilde{\mu}_1(z)\tilde{\mu}_2(z),
$$

where

$$
\tilde{\mu}_2(z) = \exp \left( \lambda \Lambda_2 \int_{-\infty}^{c_1} (e^{izy} - 1) g(y)/\Lambda_2 dy \right), \quad \Lambda_2 = G(c_1).
$$

Let $\tilde{f}_1$ and $\tilde{f}_2$ be generalized densities of $\mu_1$ and $\mu_2$ respectively. By Lemma A.1 $\tilde{f}_2(x) = o(e^{-\gamma x})$ with some $\gamma > 0$, so that $\tilde{f}_2(x) = o(\tilde{f}(x))$. Now apply Corollary 4.4 with $(\tilde{h}, \tilde{g}, \tilde{f})$ there be $(\tilde{f}, \tilde{f}_2, \tilde{f}_1)$ here and obtain $\tilde{f}_1 \in S(= S_\pm)$ and $\tilde{f}_1(x) \sim \tilde{f}(x)$.

For the proper densities $f_1$ of $\tilde{f}_1$, since $\tilde{f}_1 \in S \Leftrightarrow f_1 \in S$, we will see that $f_1 \in S_+$ implies $g_1 \in S_+$. This part is totally due to [17, Theorem 1] and for consistency we give a proof. Write

$$
f_1(x) = (e^{\lambda \Lambda_1} - 1)^{-1} \sum_{n=1}^{\infty} ((\lambda \Lambda_1)^n/n!) g_1^{*n}(x)
$$

whose Laplace transform is

$$
L_{f_1}(z) = (e^{\lambda \Lambda_1}L_{g_1}(z) - 1)/(e^{\lambda \Lambda_1} - 1),
$$

so

$$
\lambda \Lambda_1 L_{g_1}(z) = \log \left( 1 - (1 - e^{\lambda \Lambda_1})L_{f_1}(z) \right).
$$

Since $e^{\lambda \Lambda_1} - 1 < 1$, we have

$$
\lambda \Lambda_1 L_{g_1}(z) = - \sum_{n=1}^{\infty} n^{-1}(1 - e^{\lambda \Lambda_1})^n L_{f_1}^n(z)
$$

and thus

$$
\lambda_1 \Lambda_1 g_1(x) \overset{a.e.}{=} - \sum_{n=1}^{\infty} n^{-1}(1 - e^{\lambda \Lambda_1})^n f_1^{*n}(x) =: \lambda_1 \Lambda_1 \tilde{g}_1(x),
$$

where $a.e.$ implies that the equality holds $a.e. x \in \mathbb{R}$.

We derive $\tilde{g}_1 \in S_+$ and the tail equivalence between $f_1$ and $\tilde{g}_1$. Then using (4.11) we prove $g_1 \in S_+$. Take a sufficiently small $\varepsilon > 0$ such that $(e^{\lambda - 1})(1 + \varepsilon) < 1$. By Lemma 3.7 there exists $C_\varepsilon$ such that $f_1^{*n}(x) \leq C_\varepsilon(1 + \varepsilon)^n f_1(x)$ for $x$ sufficiently large. Applying the dominated convergence in (4.12) we obtain

$$
\lim_{x \to \infty} \tilde{g}_1(x)/f_1(x) = (1 - e^{-\lambda \Lambda_1})/\lambda \Lambda_1),
$$

so that by Lemma 3.6 (vi) $\tilde{g}_1 \in S_+$. Now write (4.11) as

$$
(e^{\lambda \Lambda_1} - 1)f_1(x)/\tilde{g}_1(x) - \lambda \Lambda_1 g_1(x)/\tilde{g}_1(x) = \sum_{n=2}^{\infty} ((\lambda \Lambda_1)^n/n!) \tilde{g}_1^{*n}(x)/\tilde{g}_1(x)
$$

$$
= \sum_{n=2}^{\infty} ((\lambda \Lambda_1)^n/n!) \tilde{g}_1^{*n}(x)/\tilde{g}_1(x)
$$
and let $x \rightarrow \infty$. Again by the dominated convergence we obtain by (4.13) that $\tilde{g}_1(x) \sim g_1(x)$. Thus we prove $g_1 \in S_+$ and $g_1(x) \sim (1 - e^{-\lambda_1})/(\lambda_1!) f_1(x)$. Now we see

$$
\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{f(x)}{f_1(x)} \frac{g_1(x)}{g(x)} = \lambda/(1 - e^{-\lambda}),
$$

which is (iii).

Finally we prove $g \in S$. Since $g \in \mathcal{L}$ follows from $\Lambda_1 g_1(x) \sim g(x)$, we take an insensitive function $\alpha$ for both $g$ and $f$ (cf. Lemma 3.6 (ii)). Write

$$
\frac{g\ast^2(x)}{g(x)} = \left(2 \int_{-\infty}^{\alpha(x)} + \int_{\alpha(x)}^{x - \alpha(x)} + 2 \int_{-\infty}^{-\alpha(x)} \right) \frac{g(x - y)g(y)}{g(x)} dy
=: I_1(x) + I_2(x) + I_3(x).
$$

Obviously $\lim_{x \rightarrow \infty} I_1(x) = 2$, and due to $g_1 \in S_+$ and $\Lambda_1 g_1(x) \sim g(x)$ we have

$$
I_2(x) \leq c \int_{-\alpha(x)}^{x - \alpha(x)} \frac{g_1(x - y)g_1(y)}{g_1(x)} dy \rightarrow 0 \quad \text{as} \quad x \rightarrow \infty,
$$

where Lemma 3.6 (v) is used. For $I_3$ we use (4.14), and moreover, we notice from (4.10) that $f(x) \geq (e^\lambda - 1)^{-1} g(x)$ for all $x \in \mathbb{R}$. Due to Lemma 3.6 (v) together with $f \in S$,

$$
I_3(x) \leq c \int_{-\infty}^{-\alpha(x)} \frac{f(x - y)f(y)}{f(x)} dy \rightarrow 0 \quad \text{as} \quad x \rightarrow \infty.
$$

Thus (ii): $g \in S$ follows.

(iii) implies (ii)

We write

$$
g\ast^2(x) = (2/\lambda^2 e^\lambda) \{(1 - e^{-\lambda})f(x) - e^{-\lambda} \sum_{n \neq 2} (\lambda^n/n!) g^n(x)\}.
$$

Dividing this by $g$ and taking $\limsup$ on both sides, we have by Fatou’s lemma

$$
\limsup_{x \rightarrow \infty} \frac{g\ast^2(x)}{g(x)} \leq (2/\lambda^2) e^\lambda \left(\lambda - e^{-\lambda} \sum_{n \neq 2} \liminf_{x \rightarrow \infty} \frac{\lambda^n}{n!} \frac{g^n(x)}{g(x)}\right) = 2,
$$

where we use Lemma 3.6 (iii). Now Lemma 3.6 (iii) with $n = 2$ implies $g \in S$.

(ii) implies (iii) and (i)

Notice that the a.n.i. property of $g$ implies the condition (3.7) of Lemma 3.7. Thus applying $g^n(x)/g(x) \rightarrow n$ and the dominated convergence to (4.10), we show that (ii) implies (iii). Then (ii) and (iii) together with the a.n.i. property of $g$ yield $f \in S$ by Lemma 3.9.

5. Application in the asymptotic theory of statistics

We apply our results to the consistency proof of the maximum likelihood estimation (MLE for short) for $\mu \in \text{ID} (\mathbb{R})$ which is absolutely continuous. MLE is the most important estimation in statistics and stands as the benchmark for other estimation methods. For simplicity we put $a = b = 0$ in $\tilde{\mu}(z)$ of (2.3) and assume that $\tilde{\mu}(z)$ is absolutely integrable.

Let $f(x; \theta)$ be the density of $\mu$ with $\theta$ a parameter vector and $g(x; \theta)$ be a density of the corresponding Lévy measure $\nu$. Let $(X_1, \ldots, X_n)$ be a random sample from $f(x; \theta_0)$ with $\theta_0 \in \Theta$ where $\Theta$ is a compact parameter space. Define the likelihood function

$$
M_n(\theta) = n^{-1} \sum_{i=1}^{n} \log f(X_i; \theta).
$$
MLE $\hat{\theta}_n$ maximizes the function $\theta \mapsto M_n(\theta)$. We say that a function $\alpha(x; \theta)$ is identifiable if $\alpha(\cdot; \theta) \neq \alpha(\cdot; \theta')$ every $\theta \neq \theta' \in \Theta$, i.e. $\alpha(x; \theta) \overset{\text{a.e.}}{=} \alpha(x; \theta')$ does not hold. For convenience, we only consider the symmetric or positive-half case, but we can easily generalize the result in the non-symmetric two-sided case. We use the function $g_1$ defined in Theorems 3.4.

**Proposition 5.1.** Let $\mu \in \text{ID}(\mathbb{R})$ given by (2.3) with $a = b = 0$ such that $\hat{\mu}_+(z)$ is absolutely integrable. Let $g(x; \theta)$ be a symmetric or positive-half density of $\nu$. Suppose (i): $g(x; \theta)$ is identifiable, $\theta \mapsto g(x; \theta)$ is continuous in $\theta$ for every $x$, and $\int (\sup_{\theta \in \Theta} |\log g_1(x; \theta)|)|g_1(x; \theta_0)|dx < \infty$ with $\Theta$ a compact set such that $\theta_0 \in \Theta$. Suppose (ii): $g_1(x; \theta)$ is bounded and a.n.i., and $g_1 \in S$. Then MLE $\hat{\theta}_n$ satisfies $\hat{\theta}_n \xrightarrow{P} \theta_0$.

The condition (i) comes from those required for consistency of MLE, while the condition (ii) guarantees $g_1(x) \sim cf(x)$ through Theorem 3.4.

**Remark 5.2.** The proof is done only with the Lévy density $g$ and we do not touch the genuine density $f$. Generally the explicit expression for $f$ of $\mu \in \text{ID}(\mathbb{R})$ is unavailable. Nevertheless, the estimation methods by $f$ such as MLE are computationally feasible through its ch.f. Moreover, there exist many ch.f. based estimators which are comparable with those by $f$ (cf. [7, 25]). For the construction and the theoretical justification (asymptotics) of these estimators the properties of $f$, including the tail asymptotics, are inevitable. However, deriving the necessary properties of $f$ from ch.f. are often not so easy, and therefore such validity has been proved only for well studied $\mu \in \text{ID}(\mathbb{R})$ such as stable laws (e.g. [4, 14]). By our results we could extend the scope of these estimators to a rather wide subclass of $\text{ID}(\mathbb{R})$, which is crucial in applications.

As a next step we are trying to prove the asymptotic normality of MLE for $\mu \in \text{ID}(\mathbb{R})$ which is absolutely continuous, though it would be much harder than consistency because derivatives of $f$ and $g$ w.r.t. $\theta$ are involved.

**Proof.** We check the condition of [24, Theorem 5.7]. Let $M(\theta) = \mathbb{E} \log f(X; \theta)$. For consistency $\hat{\theta}_n \xrightarrow{P} \theta_0$, we need two conditions: the deterministic condition

$$
\sup_{\theta: |\theta - \theta_0| \geq \varepsilon} M(\theta) < M(\theta_0)
$$

and the stochastic condition

$$
\sup_{\theta \in \Theta} |M_n(\theta) - M(\theta)| \xrightarrow{P} 0.
$$

Since $\mu \in \text{ID}(\mathbb{R})$ is uniquely defined by the Lévy measure, the identifiability of $g$ implies that of $f$. Thus (5.15) follows from Lemma 5.35 of [24]. The condition (5.16) is implied by two conditions [24, p.46]: $\theta \mapsto \log f(x; \theta)$ are continuous for every $x$ and they are dominated by an integrable envelop function.

For the former condition we use the inversion formula and evaluate

$$
|f(x; \theta) - f(x; \theta')| = \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} e^{-izx} (\hat{\mu}(z; \theta) - \hat{\mu}(z; \theta'))dz \right|.
$$

Since $\hat{\mu}(z; \theta)$, $\theta \in \Theta$ is absolutely integrable, we have a dominant integrable function for the integrand. Moreover for each $z \in \mathbb{R}$, we have

$$
|\hat{\mu}(z; \theta) - \hat{\mu}(z; \theta')| = |\hat{\mu}(z; \theta)| \int_{-\infty}^{\infty} (e^{izy} - 1 - izy1_{|y| \leq 1})(g(y; \theta) - g(y; \theta'))dydz - 1 \to 0
$$
as $\theta' \to \theta$. Indeed, since for $z \in \mathbb{R}$

$$
\left| \int_{-\infty}^{\infty} (e^{izy} - 1 - izy1_{|y| \leq 1})(g(y; \theta) - g(y; \theta'))dy \right|
$$

$$
\leq \int_{|y| > 1} |e^{izy} - 1||g(y; \theta) - g(y; \theta')|dy + \int_{|y| \leq 1} |e^{izy} - 1 - izy||g(y; \theta) - g(y; \theta')|dy.
$$
\[ M + cz^2 \int_{|y| \leq 1} y^2|g(y; \theta) - g(y; \theta')|dy < \infty, \quad M > 0, \]

(cf. [16, Eq.(9.9)]) and the dominated convergence works. For the latter condition, since \( g(\pm x; \theta) \sim f(\pm x; \theta) \) by Theorem 3.4, for \( x \) large enough \( \sup_{\theta \in \Theta} |\log f(\pm x; \theta)| \leq c \sup_{\theta \in \Theta} |\log g(\pm x; \theta)| \) for some constant \( c > 0 \). Moreover, \( f(x; \theta) \) is continuous in \( x \). Thus \( E[\sup_{\theta \in \Theta} |\log f(X; \theta)|] < \infty \) is implied by the last condition of (i). Now (5.16) is proved. \( \square \)

6. Proofs of main theorems

Since the proof of Theorem 3.4 is given by a minor change to that of Theorem 3.2, we see the latter proof first and then state the difference.

6.1. Proof of Theorem 3.2. In the proof of part [(i) implies (ii) and (iii) when \( g \) is e.n.i.], we reduce the problem to the compound Poisson case (Theorem 4.1) by applying the factorization/convolution property (Proposition 3.10) to \( \mu \). For this we need the a.n.i. property for the targeting compound Poisson. Our strategy is to consider a symmetric unimodal version \( \mu_s \) of \( \mu \) without Gaussian part, which is the compound Poisson.

Recalling that \( g \) is e.n.i., so that there exists \( c_1 > 1 \) such that for \( x > c_1 \) \( g(x+y) \leq g(y) \) for any \( y > 0 \), we may define

\[
g_s(x) = \begin{cases} \frac{g(x)}{\lambda_s} & \text{if } x > c_1 \\ \frac{g(c_1)}{\lambda_s} & \text{if } -c_1 < x \leq c_1 \\ \frac{g(-x)}{\lambda_s} & \text{if } x < -c_1, \end{cases}
\]

where \( \lambda_s = 2\overline{\gamma}(c_1) + 2c_1 g(c_1) \), and denote the corresponding compound Poisson by \( \mu_s \), namely

\[
\hat{\mu}_s(z) = \exp \left( \lambda_s \int_{-\infty}^{\infty} (e^{izy} - 1)g_s(y)dy \right).
\]

The proper absolutely continuous part \( f_s \) of \( \mu_s \) is given by (4.10) with \( \lambda \) and \( g \) replaced with \( \lambda_s \) and \( g_s \). Our goal is to approximate the tail of \( f \) by that of \( f_s \), and apply Theorem 4.1 through \( f_s \).

**Proposition 6.1.** Suppose the conditions before the three assertions of Theorem 3.2 and that \( g \) is e.n.i. Then, \([f \text{ is a.n.i. & } f \in S_+] \leftrightarrow f_s \in S_+\) and moreover,

\[
\lim_{x \to \infty} f_s(x)/f(x) = (1 - e^{-\lambda_s})^{-1}.
\]

We need some preparation before the proof. We use the relation

\[
\mu \ast \mu_a = \mu_s \ast \mu_r,
\]

where \( \mu_a \) and \( \mu_r \) are a compound Poisson and an infinitely divisible distribution respectively defined by

\[
\hat{\mu}_a(z) = \exp \left( \lambda_s \Lambda_a \int_{-\infty}^{c_1} (e^{izy} - 1)g_s(y)/\Lambda_a dy \right), \quad \Lambda_a = (\overline{\gamma}(c_1) + 2c_1 g(c_1))/\lambda_s,
\]

\[
\hat{\mu}_r(z) = \exp \left\{ \int_{-\infty}^{c_1} (e^{izy} - 1 - izy1_{|y| \leq 1})g(y)dy + iaz - \frac{1}{2}b^2 y^2 \right\}.
\]

**Proof.** First we characterize the tails of the proper absolute continuous part \( f_a \) of \( \mu_a \) and the density \( f_r \) of \( \mu_r \), namely for some \( \gamma > 0 \) we show

\[
\lim_{x \to \infty} e^{\gamma x}(f_a(x) + f_r(x)) = 0.
\]
We study the case $a = b = 0$ in $\mu_r$ and then generalize the result. Since $\mu_a$ is a compound Poisson which satisfies the condition of Lemma A.1 $f_a(x) = o(e^{-\gamma x})$ is immediate. We decompose $\tilde{\mu}_r(z)$ into

$$\tilde{\mu}_r(z) = \exp \left\{ \left( \int_{-\infty}^{-1} + \int_{-1}^{1} \int_{1}^{c_1} \right) \left( e^{izy} - 1 - izy 1_{|y| \leq 1} \right) g(y) dy \right\} =: \tilde{\mu}_r(z) \tilde{\mu}_r(z) \tilde{\mu}_r(z).$$

Consider the proper absolutely continuous part $f_{r_3}$ of $\mu_{r_3}$. Since $g$ is bounded on $[1, c_1]$, by exactly the same logic as for $f_2$ in the proof of Lemma A.1, we have $f_{r_3}(x) = o(e^{-\gamma x})$. Since $\mu_{r_1}$ is a compound Poisson with a non-positive support, again by the same reasoning as in the proof of Lemma A.1 (cf. (A.25)), the absolutely continuous part $f_{r_13}$ of $\mu_{r_1} \ast \mu_{r_3}$ satisfies $f_{r_13}(x) = o(e^{-\gamma x})$.

Now consider the convolution of $\mu_{r_2}(dx) = f_0(x)dx$ and $\mu_{r_1} \ast \mu_{r_3}(dx) := e^{-c_1} \delta_0(dx) + (1 - e^{-c_1}) f_{13}(x) dx$, which yields

$$e^{\gamma x} f_r(x) = e^{\gamma x} e^{-c_13} f_0(x) + (1 - e^{-c_13}) \int_{-\infty}^{x/2} e^{\gamma (x-y)} f_0(x-y) e^{\gamma y} f_{13}(y) dy$$

$$+ (1 - e^{-c_13}) \int_{-\infty}^{x/2} e^{\gamma (x-y)} f_{13}(x-y) e^{\gamma y} f_0(y) dy.$$}

Recall that both $\mu_{r_2}$ and $\mu_{r_1} \ast \mu_{r_3}$ have $e^{\gamma x}$ moment [16, Theorem 25.3], and so do $f_{13}$ and $f_0$. Moreover, both $e^{\gamma x} f_0(x)$ and $e^{\gamma x} f_{13}(x)$ converge to $0$ as $x \to \infty$. Thus by the dominated convergence we have $\lim_{x \to \infty} e^{\gamma x} f_r(x) = 0$.

Finally let $f_G$ the density of Gaussian part plus shift, and then

$$e^{\gamma x} f_r \ast f_G(x) = \int_{-\infty}^{x/2} \left\{ e^{\gamma (x-y)} f_r(x-y) e^{\gamma y} f_G(y) + e^{\gamma (x-y)} f_G(x-y) e^{\gamma y} f_r(y) \right\} dy < \infty$$

follows by exactly the same way as before.

**Proof of sufficiency** ($\Rightarrow$).

We derive the tail behavior of the density $f_{+a}$ of $\mu \ast \mu_a$. Observe that

(6.21) 

$$f_{+a}(x) = f \ast f_a(x),$$

where $f_a(x) = e^{-\lambda_a} \delta(+) + (1 - e^{-\lambda_a}) f_0(x)$. Recall that $f \in \mathcal{L}(\in \mathcal{S}_+)$, $f$ is a.n.i. and $f_a(x) = o(e^{-\gamma x})$ for some $\gamma$, so that $f_a(x) = o(f(x))$. Thus Proposition 3.10 (i) yields $f_{+a} \in \mathcal{S}_+$ and $f_{+a}(x) \sim f(x)$. Moreover, $f_{+a} \in \mathcal{S}$ by Lemma 3.9.

Next we see the relation (6.18). Noticing that $\mu_r$ is absolutely continuous, we have

(6.22) 

$$f_{+a}(x) = e^{-\lambda_s} f_r(x) + (1 - e^{-\lambda_s}) f_r \ast f_s(x).$$

Rewrite this as

$$f_{ar}(x) = \frac{f_{+a}(x) - e^{-\lambda_s} f_r(x)}{1 - e^{-\lambda_s}} = f_r \ast f_s(x),$$

and then by a direct calculation $f_{ar} \in \mathcal{S}$. Since $f_r(x) = o(e^{-\gamma x})$ and $f_s$ is a.n.i., we see from Corollary 3.11 that $f_s \in \mathcal{S}_+$ and $f_{ar}(x) \sim f_s(x)$. Hence

$$\lim_{x \to \infty} \frac{f_s(x)}{f(x)} = \lim_{x \to \infty} \frac{f_{+a}(x)}{f(x)} \cdot \frac{f_{ar}(x)}{f_{+a}(x)} \cdot \frac{f_s(x)}{f_{ar}(x)} = (1 - e^{-\lambda_s})^{-1}.$$}

**Proof of the necessity** ($\Leftarrow$).

Since $f_r(x) = o(f_s(x))$, due to Proposition 3.10 (i) we have $f_r \ast f_s \in \mathcal{S}_+$ and $f_r \ast f_s(x) \sim f_s(x)$. Thus from (6.22)

$$\lim_{x \to \infty} \frac{f_{+a}(x)}{f_s(x)} = 1 - e^{-\lambda_s}.$$
Then Lemma 3.9 implies $f_{+a} \in \mathcal{S}$, and indeed $f_{+a}$ is a.n.i. by Lemma 3.8. Recall that $\tilde{f}_a$ is a compound Poisson and in view of (6.20) $\tilde{f}_a(x) = o(e^{-\gamma x})$ so that $\tilde{f}_a(x) = o(f_{+a}(x))$. Now due to Corollary 4.4 we have $f \in \mathcal{S}$ and moreover, $f(x) \sim f_{+a}(x) \sim f_{s}(x)(1-e^{-\lambda x})$. Again by Lemma 3.8 $f$ is a.n.i.

\textbf{Proof of Theorem 3.2.} (i) implies (ii) and (iii) under the condition that $g$ is e.n.i.

In this part we work on $f_{s}$ and $g_{s}$. Recall that $g_{s}$ is bounded and symmetric unimodal, and so is $f_{s}$ due to the relation (4.10) with $\lambda, g$ and $f$ respectively replaced with $\lambda_{s}, g_{s}$ and $f_{s}$. By Proposition 6.1 $f_{s} \in \mathcal{S}_{+}$ (so that $f_{s} \in \mathcal{S}$) and $f(x) \sim (1-e^{-\lambda x})f_{s}(x)$. We apply Theorem 4.1 with $\lambda = \lambda_{s}$ to $f_{s}$ and $g_{s}$, which yields $g_{s} \in \mathcal{S}_{+}$ and

$$
\lim_{x \to \infty} \frac{f(x)}{g_{s}(x)} = \lim_{x \to \infty} \frac{f(x)}{f_{s}(x)} \cdot \frac{f_{s}(x)}{g_{s}(x)} = \lambda_{s}.
$$

Since $g_{s}(x) \sim g(x)/\lambda_{s}$, we have $f(x) \sim g(x) \sim g_{1}(x)\nu((1, \infty))$ and $g \in \mathcal{L}$, which implies (iii). Now (ii) follows from Lemma 3.9.

(iii) implies (ii) under the condition that $g$ is a.n.i.

Define $g_{u}(x) = g(x)1_{\{x \geq c_{1}\}}/\Lambda_{u}$ where $\Lambda_{u} = \mathbb{C}(c_{1})$ and consider the compound Poisson $\mu_{u}$ with ch.f.

$$
\hat{\mu}_{u}(z) = \exp \left\{ \Lambda_{u} \int_{c_{1}}^{\infty} (e^{i z y} - 1)g_{u}(y)dy \right\},
$$

and the proper absolutely continuous part

$$
f_{u}(x) = (e^{\Lambda_{u}} - 1)^{-1} \sum_{n=1}^{\infty} (\Lambda_{u}^{n}/n!)g_{u}^{*n}(x).
$$

By Fubini the density $f$ of $\mu = \mu_{r} \ast \mu_{u}$ has an expression

$$
f(x) = e^{-\Lambda_{u}}f_{r}(x) + e^{-\Lambda_{u}} \sum_{n=1}^{\infty} (\Lambda_{u}^{n}/n!)g_{u}^{*n} \ast f_{r}(x),
$$

where $f_{r}$ is the density of $\mu_{r}$. We write

\begin{equation}
(6.23) \quad \frac{g_{u}^{*2} \ast f_{r}(x)}{g_{u}(x)} = e^{\Lambda_{u}} \frac{2}{\Lambda_{u}^{2}} \frac{f(x)}{g_{u}(x)} - \frac{2}{\Lambda_{u}^{2}} f_{r}(x) - \frac{2}{\Lambda_{u}^{2}} \sum_{n 

\text{and observe the limit behavior when } x \to \infty. \text{ Since } g_{u}(x) \sim g(x)/\Lambda_{u}, \text{ due to the second condition of (iii) together with } g(x) \sim g_{1}(x)\nu((1, \infty)),

$$
\lim_{x \to \infty} e^{\Lambda_{u}} \frac{2}{\Lambda_{u}^{2}} \cdot \frac{f(x)}{g_{u}(x)} = \lim_{x \to \infty} e^{\Lambda_{u}} \frac{2}{\Lambda_{u}^{2}} \cdot \frac{f_{r}(x)}{g_{u}(x)} \sim g(x) = \Lambda_{u} \frac{2}{\Lambda_{u}}
$$

Since $f_{r}(x) = o(e^{-\gamma x})$ for some $\gamma > 0$ and $g_{u} \in \mathcal{L}$, the second quantity vanishes. Moreover, by Lemma 3.6 (iii)

\begin{equation}
(6.24) \quad \lim_{x \to \infty} \frac{g_{u}^{*n} \ast f_{r}(x)}{g_{u}(x)} \geq \lim_{x \to \infty} \frac{g_{u}^{*n}(x)}{g_{u}(x)} \lim_{x \to \infty} \frac{g_{u}^{*n} \ast f_{r}(x)}{g_{u}(x)} \geq n.
\end{equation}

Thus, Fatou’s lemma yields

$$
\lim_{x \to \infty} \frac{\sum_{n \neq 2}^{\infty} \Lambda_{u}^{n} g_{u}^{*n} \ast f_{r}(x)}{g_{u}(x)} \geq \sum_{n \neq 2}^{\infty} \Lambda_{u}^{n} \lim_{x \to \infty} \frac{g_{u}^{*n} \ast f_{r}(x)}{g_{u}(x)} \geq \Lambda_{u}(e^{\Lambda_{u}} - \Lambda_{u}).
$$

Considering lim sup in (6.23) with above results including (6.24) with $n = 2$, we have

$$
\lim_{x \to \infty} \frac{g_{u}^{*2} \ast f_{r}(x)}{g_{u}(x)} = 2.
$$
Hence, now observe that $f_r$.

Finally, since $\mu_r$, the density satisfies $f_r(x) = \int x g(x) dx$.

We make use of $g_u$ and $f_u$ in the part (iii) of Lemma 3.6 (iii) implies (ii). By definition $g \in S_+$ implies $g_u \in S_+$ and $g(x) \sim g_u(x) \Lambda_u$. Moreover, $g_u$ is bounded and a.n.i. Hence, by Theorem 4.1 (ii) implies (i), $f_u \in S_+$ and $f_u$ is a.n.i. In view of the relation

$$f(x) = f_u * f_r(x) = e^{-\Lambda_u} f_r(x) + (1 - e^{-\Lambda_u}) f_r * f_u(x),$$

since $f_r(x) = o(e^{-\gamma x})$ implies $f_r(x) = o(f_u(x))$, Proposition 3.10 (i) yields $f(x) \sim (1 - e^{-\Lambda_u}) f_u(x)$. Thus (i) follows from Lemmas 3.8 and 3.9.

6.2. Proof of Theorem 3.4. We need the following lemma, which characterizes the tail of the density $f_r$ of $\mu_r$ in (6.19).

Lemma 6.2. Suppose that $\int_{-\infty}^{\infty} |\hat{\mu}_+(z)| dz < \infty$ and then for sufficiently large $c_1$ of (6.17) the density $f_r$ of $\mu_r$ satisfies $\lim_{x \to \infty} e^{\gamma x} f_r(x) = 0$ for any $\gamma > 0$.

Proof. We prepare the spectrally positive version $\mu_{r+}$ of $\mu_r$ by

$$\hat{\mu}_{r+}(z) = \exp \left\{ \int_0^{c_1} (e^{izy} - 1 - izy \mathbf{1}_{|y| \leq 1}) g(y) dy \right\}.$$ 

First we see $\int_{-\infty}^{\infty} |\hat{\mu}_+(z)| dz < \infty \iff \int_{-\infty}^{\infty} |\hat{\mu}_{r+}(z)| dz < \infty$, but the proof is only a reproduction of that for Lemma 10 (i) of [20] and we omit it. We show that $\int_{-\infty}^{\infty} |\hat{\mu}_{r+}(z)| dz < \infty$ implies that its density satisfies $f_{r+}(x) = o(e^{-\gamma x})$. Although this part is again quite similar to that of Lemma 10 (ii) of [20], since we treat a spectrally positive case, we briefly state the outline. Because $\int_{-c_1}^{c_1} e^{\gamma x} g(x) dx < \infty$, by [16, Theorem 25.3], we obtain $C_r := \int_{-\infty}^{\infty} e^{\gamma x} f_{r+}(x) dx < \infty$. Thus, we may define the exponential tilt $\hat{\mu}_{r+}$ on $\mathbb{R}$ as $\mu_{r+}(dx) = C_r^{-1} e^{\gamma x} f_{r+}(x) dx$. Then due to [13, Theorem 3.9] (cf. [16, Ex. 33.15] and [20, Lemma 7]), $\mu_{r+}^\gamma$ still belongs to $ID(\mathbb{R})$ given by (2.3) with Lévy-Khintchine triplet

$$a = \int_0^{1} (e^{\gamma x} - 1) x g(x) dx, \quad b = 0 \quad \text{and} \quad \nu(dx) = \mathbf{1}_{|x| \leq c_1} e^{\gamma x} g(x) dx.$$

Now observe that

$$|\hat{\mu}_{r+}^\gamma(z)| = |\hat{\mu}_{r+}(z)| \exp \left\{ \int_0^{c_1} (\cos(z x) - 1)(e^{\gamma x} - 1) \nu(dx) \right\} \leq |\hat{\mu}_{r+}(z)|.$$

Hence $\hat{\mu}_{r+}^\gamma$ is absolutely integrable, and the Riemann-Lebesgue lemma implies $f_{r+}(x) = o(e^{-\gamma x})$.

Finally, since $\mu_r = \mu_{r+} \ast \mu_-$ with $\hat{\mu}_{-}(z) := \hat{\mu}(z)/\hat{\mu}_{+}(z)$, and $\mu_{r+}$ has a bounded continuous density $e^{\gamma x} f_{r+}(x) = \int_{-\infty}^{\infty} e^{\gamma(x-y)} f_{r+}(x-y) e^{\gamma y} \mu_-(dy) \leq c \int_{-\infty}^{\infty} e^{\gamma y} \mu_-(dx) < \infty$.

Thus we have $f_r(x) = o(e^{-\gamma x})$. \hfill \Box
Proof of Theorem 3.4. Notice that the conditions of Theorem 3.4 includes the conditions other than (3.4) of Theorem 3.2. In view of the proof of Proposition 6.1, the condition (3.4) is used only for deriving \( f_r(x) = o(e^{-\gamma x}) \). Thus Proposition 6.1 holds under the conditions of Theorem 3.4 by Lemma 6.2. Moreover, in the proof of Theorem 3.2, (3.4) appears implicitly only through the fact: \( f_r(x) = o(e^{-\gamma x}) \). Thus, we could reuse the proof of Theorem 3.2 for that of Theorem 3.4. \( \square \)

Appendix A. Auxiliary results and proofs for Section 4

Throughout this section let \( c \) be a positive constant whose value may differ depending on context. Firstly we state an auxiliary result for the proof of Theorem 4.1 and then go to proofs of propositions and corollaries in Section 4.

Lemma A.1. Let \( \mu \) be a compound Poisson with ch.f.

\[
\hat{\mu}(z) = \exp \left( \lambda \int_{-\infty}^{c_1} (e^{izy} - 1)g(y)dy \right),
\]

where \( g \) is bounded. Then for any \( c_1 > 0 \) there exists \( \gamma > 0 \) such that the absolutely continuous part \( f \) of \( \mu \) satisfies \( f(x) = o(e^{-\gamma x}) \).

Proof. We decompose \( \hat{\mu}(z) \) into

\[
\hat{\mu}(z) = \exp \left\{ \lambda \left[ \int_{-\infty}^{0} + \int_{0}^{c_1} \right] (e^{izy} - 1)g(y)dy \right\} =: \hat{\mu}_1(z)\hat{\mu}_2(z).
\]

First we consider the compound Poisson \( \mu_2 \) and let \( \Lambda_2 = G(c_1) - G(0) < \infty \). We write the proper absolutely continuous part as

\[
f_2(x) = (e^{\Lambda_2} - 1)^{-1} \sum_{n=1}^{\infty} ((\lambda \Lambda_2)^n / n! \) \( g_2^n(x),
\]

where \( g_2(x) = \Lambda_2^{-1} g(x) \mathbf{1}_{\{0 \leq x \leq c_1\}} \). Since \( g \) is bounded, \( f_2(x) \) is bounded as well. Recall that for any \( \gamma > 0 \), \( \int_{0}^{\infty} e^{\gamma x} \mu_2(dx) < \infty \) ([16, Theorem 25.3]). So from e.g. [16, Ex. 33.15] (cf. [20, Lemma 7] and [13, Theorem 3.9]), we can define the exponential tilt \( (\mu_2)_\gamma \) of \( \mu_2 \) as

\[
(\mu_2)_\gamma(dx) = \frac{e^{\gamma x}}{\int_{0}^{\infty} e^{\gamma x} \mu_2(dx)} \mu_2(dx).
\]

Then \( (\mu_2)_\gamma \) is again the compound Poisson with the proper absolutely continuous part

\[
f_\gamma(x) = (e^{\Lambda_\gamma} - 1)^{-1} \sum_{n=1}^{\infty} ((\lambda \Lambda_\gamma)^n / n! \) \( g_\gamma^n(x),
\]

with \( \Lambda_\gamma = \int_{0}^{c_1} e^{\gamma x} g(x)dx \) and \( g_\gamma(x) = (\Lambda_2/\Lambda_\gamma)e^{\gamma x} g_2(x) \). For any \( x > 0 \), the right-hand side is well defined. Since the support of \( g_\gamma^n \) is included in the interval \([0, nc_1]\), we have

\[
\lim_{x \to \infty} e^{\gamma x} f_\gamma(x) = \lim_{x \to \infty} c f_\gamma(x) = c \lim_{x \to \infty} \sum_{nc_1 \geq x} ((\lambda \Lambda_\gamma)^n / n! \) \( g_\gamma^n(x) = 0.
\]

Now since \( \mu_1 \) is a compound Poisson with non-negative support, it suffices to check that for the absolutely continuous part \( f_1 \) of \( \mu_1 \),

\[
e^{\gamma x} f_1 * f_2(x) = \int_{-\infty}^{\infty} e^{\gamma(x-y)} f_2(x-y)e^{\gamma y} f_1(y)dy \leq o(1) \int_{-\infty}^{0} e^{\gamma y} f_1(y)dy.
\]

Thus we may take some \( \gamma > 0 \) such that \( f(x) = o(e^{-\gamma x}) \). \( \square \)
Proof of Proposition 4.2. With the form $\tilde{g}(x)$, we may write
\begin{equation} \tag{A.26} \label{A.26} 1 = p_g \tilde{f}(x)/\tilde{h}(x) + (1 - p_g) \tilde{f} \ast g(x)/\tilde{h}(x), \end{equation}
and put
\begin{equation*} \overline{C} := \limsup_{x \to \infty} \tilde{f}(x)/\tilde{h}(x) \quad \text{and} \quad \underline{C} := \liminf_{x \to \infty} \tilde{f}(x)/\tilde{h}(x), \end{equation*}
which are well-defined since $\tilde{h}(x) \geq p_g \tilde{f}(x)$ and the Dirac delta parts disappear for $x > 0$.

Assume the first condition. Take an insensitive function $\alpha$ for $\tilde{h}$ and consider
\begin{equation*} \frac{\tilde{f} \ast g(x)}{h(x)} = \left( \int_{-\alpha(x)}^{\alpha(x)} + \int_{-\alpha(x)}^{\infty} \int_{\alpha(x)}^{\infty} \right) \frac{\tilde{f}(x - y)g(y)}{h(x)} dy 
=: I_1(x) + I_2(x) + I_3(x). \end{equation*}
By Fatou’s lemma and $\tilde{h} \in \mathcal{L}$
\begin{equation} \tag{A.27} \liminf_{x \to \infty} I_2(x) = \liminf_{x \to \infty} \int_{-\alpha(x)}^{\alpha(x)} \frac{\tilde{f}(x - y)}{h(x)} \frac{\tilde{h}(x - y)}{h(x)} g(y) dy 
\geq \int_{-\infty}^{\infty} \liminf_{x \to \infty} \frac{\tilde{f}(x - y)}{h(x - y)} 1_{\{y \in [-\alpha(x), \alpha(x)]\}} g(y) dy \geq \overline{C}, \end{equation}
where we may take a continuous $\alpha(x)$ if needed. Again by Fatou’s lemma
\begin{equation} \tag{A.28} \limsup_{x \to \infty} I_2(x) \leq \int_{-\infty}^{\infty} \limsup_{x \to \infty} \frac{\tilde{f}(x - y)}{h(x)} 1_{\{y \in [-\alpha(x), \alpha(x)]\}} g(y) dy \leq \overline{C}. \end{equation}
Since $\tilde{h}(x) \geq p_g \tilde{f}(x)$ by (A.26) and $\tilde{g}(x) = o(\tilde{h}(x))$ we have
\begin{equation} \tag{A.29} \limsup_{x \to \infty} I_3(x) = \limsup_{x \to \infty} \int_{-\infty}^{\infty} \frac{\tilde{f}(x - y)}{h(x - y)} \frac{\tilde{h}(x - y)}{h(x)} p_g^{-1} \tilde{g}(y) dy 
\leq \limsup_{x \to \infty} \frac{g(x)}{h(x)} \limsup_{x \to \infty} \int_{-\infty}^{\infty} \frac{\tilde{h}(x - y) \tilde{h}(y)}{h(x)} dy = 0. \end{equation}
By the a.n.i. property of $\tilde{h}$, it follows that
\begin{equation} \tag{A.30} \limsup_{x \to \infty} I_1(x) = \limsup_{x \to \infty} \int_{-\infty}^{0} \frac{\tilde{f}(x - y)}{h(x)} \frac{\tilde{h}(x - y)}{h(x)} g(y) dy 
\leq c \limsup_{x \to \infty} \int_{-\infty}^{0} g(y) dy = 0. \end{equation}
Collecting (A.27)-(A.30), we obtain
\begin{equation} \tag{A.31} \liminf_{x \to \infty} \frac{\tilde{f} \ast g(x)}{h(x)} \geq \underline{C} \quad \text{and} \quad \limsup_{x \to \infty} \frac{\tilde{f} \ast g(x)}{h(x)} \leq \overline{C}. \end{equation}
Now taking $\liminf_{x \to \infty}$ and $\limsup_{x \to \infty}$ on both sides of (A.26), we have
\begin{equation} \tag{A.32} p_g \overline{C} + (1 - p_g) \underline{C} \leq 1 \leq p_g \overline{C} + (1 - p_g) \underline{C} \end{equation}
and thus
\begin{equation*} 0 \leq (1 - 2p_g)(\overline{C} - \underline{C}). \end{equation*}
The assumption $p_g \in (2^{-1}, 1)$ implies $\overline{C} = \underline{C}$. Moreover from (A.32), $\overline{C} = \underline{C} = 1$. Then noticing $p_g \tilde{f}(x) \leq \tilde{h}(x)$ for all $x \in \mathbb{R}$, we have $\tilde{f} \in \mathcal{S}$ from Lemma 3.9.

Next assume the second condition. In view of the expression
\begin{equation} \tag{A.33} \tilde{h}(x) = p_g p_f \delta(x) + (1 - p_f) p_g \tilde{f}(x) + (1 - p_g) p_f g(x) + (1 - p_g)(1 - p_f) \tilde{f} \ast g(x), \end{equation}
we notice that \( \tilde{h}(x) \geq cg(x) \) and \( \tilde{h}(x) \geq cf(x) \) hold for all \( x \in \mathbb{R} \). From the first part proof, it suffices to show that \( \limsup_{x \to \infty} I_1(x) = 0 \) where the a.n.i. property of \( \tilde{h} \) is used. However, by above inequalities

\[
\limsup_{x \to \infty} I_1(x) = \limsup_{x \to \infty} \int_{-\infty}^{-\alpha(x)} \frac{f(x - y) g(y) \tilde{h}(x - y) \tilde{h}(y)}{\tilde{h}(x)} dy \\
\leq c \limsup_{x \to \infty} \int_{-\infty}^{-\alpha(x)} \frac{\tilde{h}(x - y) \tilde{h}(y)}{\tilde{h}(x)} dy = 0.
\]

\[ \square \]

Proof of Proposition 4.3. Observe that

\[ (A.34) \quad \tilde{f}^{kn}(x) = \sum_{k=0}^{n} \binom{n}{k} (1 - p)^k p^{n-k} f^k(x), \]

where \( f^0(x) = \delta(x) \) and define

\[ \liminf_{x \to \infty} f(x)/\tilde{f}^{sn}(x) = C \quad \text{and} \quad \limsup_{x \to \infty} f(x)/\tilde{f}^{sn}(x) = \overline{C}, \]

which are well-defined since \( \tilde{f}^{sn}(x) \geq n(1-p)p^{n-1}f(x) \). We show by induction that

\[ (A.35) \quad \liminf_{x \to \infty} f^{*k}(x)/\tilde{f}^{sn}(x) \geq kC \quad \text{and} \quad \limsup_{x \to \infty} f^{*k}(x)/\tilde{f}^{sn}(x) \leq k\overline{C} \]

hold. Since the proof for the lim sup part is similar, we only consider the lim inf part. Suppose that (A.35) holds with \( k - 1 \), \( k \geq 2 \) and consider

\[
\frac{f^{*k}(x)}{\tilde{f}^{sn}(x)} = \left( \int_{-\alpha(x)}^{-\alpha(x) + \alpha(x)} + \int_{-\alpha(x)}^{\alpha(x)} + \int_{-\infty}^{-\alpha(x)} + \int_{\alpha(x)}^{+\infty} \right) \frac{f(x - y) f^{*(k-1)}(y)}{\tilde{f}^{sn}(x)} dy \\
=: I_1(x) + I_2(x) + I_3(x) + I_4(x) + I_5(x),
\]

where \( \alpha \) is an insensitive function for \( \tilde{f}^{sn} \). By Fatou’s lemma and \( \tilde{f}^{sn} \in \mathcal{L} \),

\[ (A.36) \quad \liminf_{x \to \infty} I_1(x) = \liminf_{x \to \infty} \int_{-\alpha(x)}^{\alpha(x)} \frac{f(x - y) \tilde{f}^{sn}(x - y) f^{*(k-1)}(y)}{\tilde{f}^{sn}(x)} dy \\
\geq \int_{-\alpha(x)}^{\alpha(x)} \liminf_{x \to \infty} \frac{f(x - y)}{\tilde{f}^{sn}(x - y)} \{ y \in [-\alpha(x), \alpha(x)] \} f^{*(k-1)}(y) dy \geq C
\]

and by the induction hypothesis

\[ (A.37) \quad \liminf_{x \to \infty} I_2(x) = \liminf_{x \to \infty} \int_{-\alpha(x)}^{\alpha(x)} \frac{f^{*(k-1)}(x - y) \tilde{f}^{sn}(x - y) f^{*(k-1)}(y)}{f^{sn}(x)} dy \\
\geq \int_{-\alpha(x)}^{\alpha(x)} \liminf_{x \to \infty} \frac{f^{*(k-1)}(x - y) f^{*(k-1)}(y)}{f^{sn}(x)} dy \geq (k - 1)C.
\]

Moreover, since \( \tilde{f}^{sn}(x) \geq n(1-p)p^{n-1}f(x) + \binom{n}{k-1} (1-p)^{k-1} p^{n-k+1} f^{*(k-1)}(x) \) for all \( x \in \mathbb{R} \),

\[ (A.38) \quad \limsup_{x \to \infty} I_3(x) \leq c \limsup_{x \to \infty} \int_{-\alpha(x)}^{\alpha(x)} \frac{\tilde{f}^{sn}(x - y) \tilde{f}^{sn}(y)}{\tilde{f}^{sn}(x)} dy = 0
\]

by Lemma 3.6 (v), while by exactly the same logic,

\[ (A.39) \quad \limsup_{x \to \infty} (I_4(x) + I_5(x)) \leq c \limsup_{x \to \infty} \int_{-\alpha(x)}^{\alpha(x)} \frac{\tilde{f}^{sn}(x - y) \tilde{f}^{sn}(y)}{\tilde{f}^{sn}(x)} dy = 0.
\]
Now collecting (A.36)-(A.39) we obtain (A.35). Then recalling (A.34) we observe that

\[
1 \geq \limsup_{x \to \infty} n(1-p)p^{n-1} f(x) / \bar{f}^n(x) \\
+ \liminf_{x \to \infty} \sum_{k=2}^{n} \binom{n}{k} (1-p)^k p^{n-k} f^k(x) / \bar{f}^n(x)
\]

\[
\geq n(1-p)p^{n-1} \bar{C} + \sum_{k=2}^{n} \binom{n}{k} (1-p)^k p^{n-k} k \bar{C}
\]

\[
= n(1-p) \{ p^{n-1} \bar{C} + (1-p^{-1}) \bar{C} \}
\]

and

\[
1 \leq \liminf_{x \to \infty} n(1-p)p^{n-1} f(x) / \bar{f}^n(x) \\
+ \limsup_{x \to \infty} \sum_{k=2}^{n} \binom{n}{k} (1-p)^k p^{n-k} f^k(x) / \bar{f}^n(x)
\]

\[
\leq n(1-p) \{ p^{n-1} \bar{C} + (1-p^{n-1}) \bar{C} \},
\]

which together yield

\[
0 \leq n(1-p)(1-2p^{n-1})(\bar{C} - C).
\]

From the condition \(2^{-1/(n-1)} < p, \bar{C} = C\) should hold. Then, since \(f(x) \leq c \bar{f}^n(x)\) for all \(x \in \mathbb{R}\), \(\bar{f} \in S\) follows from Lemma 3.9.

\[\Box\]

Proof of Corollary 4.4. We assume non-degeneracy for \(\bar{g}\), since otherwise the proof is obvious. Let \(\lambda\) be the Poisson parameter of \(\bar{g}\). If \(\lambda < \log 2\), then since \(e^{-\lambda} > 2^{-1}\) the result is immediate from Proposition 4.2. If \(\lambda \geq \log 2\), we take an integer \(n\) such that \(\lambda/n < \log 2\), and define a compound Poisson density

\[
\bar{g}_1/n(x) = e^{-\lambda/n} \delta(x) + (1 - e^{-\lambda/n}) \bar{g}_1/n(x)
\]

with \(\bar{g}_1/n\) be the properly absolutely continuous part such that \(\bar{g} = \bar{g}_1/n\). Notice that since

\[
\bar{g}(x) = (e^{-\lambda/n} \delta + (1 - e^{-\lambda/n}) \bar{g}_1/n)^n(x) \geq ne^{-\lambda(n-1)/n}(1 - e^{-\lambda/n}) \bar{g}_1/n(x)
\]

and thus \(\bar{g}(x) = o(\bar{h}(x))\) implies \(\bar{g}_1/n(x) = o(\bar{h}(x))\). Moreover, the coefficient \(e^{-\lambda k/n} p\) of the delta part \(\delta\) in \(\bar{g}_1/n * \bar{f}\) satisfies \(e^{-\lambda k/n} p \in (0,1)\), \(k = 1, \ldots, n\). Now we apply Proposition 4.2 to \(\bar{g}_1/n * (\bar{g}_1/n^{n-1} * \bar{f})\) with \(\bar{g}_1/n\) be the negligible part, and obtain that \(\bar{g}_1/n^{n-1} * \bar{f} \in S\) and \(\bar{h}(x) \sim \bar{g}_1/n^{n-1} * \bar{f}(x)\). Here, if \(\bar{h}\) is a.n.i., then by Lemma 3.8, \(\bar{g}_1/n^{n-1} * \bar{f}(x)\) is also a.n.i. We iterate this step until we reach \(\bar{f} \in S\) and \(\bar{h}(x) \sim \bar{f}(x)\).

\[\Box\]

Proof of Corollary 4.5. Take \(\alpha = n^{-1}, n \in \mathbb{N}\), so that \((\bar{f}^\alpha)^n = \bar{f}\) and the Poisson parameter of \(\bar{f}^\alpha\) is \(\lambda/n\). The coefficient of \(\delta\) of \(\bar{f}^\alpha\) satisfies \(e^{-\lambda/n} > 2^{-1/n} > 2^{-1/(n-1)}\). Thus by Proposition 4.3, \(\bar{f}^\alpha \in S\) and \(\lim_{x \to \infty} \bar{f}^\alpha(x)/f(x) = \alpha\). This implies that the result holds for any rational \(\alpha > 0\).

\[\Box\]

Appendix B. Proofs for results in Section 3

Throughout this section \(c\) denotes a positive constant whose value is not of interest.

Proof of Lemma 3.8. Obviously the a.n.i. property implies (3.8), and we show the converse.

\[
1 \leq \sup_{t \geq x} f(t) / f(x) = \sup_{t \geq x} f(t) \alpha(x) / f(x) \leq \sup_{t \geq x} f(t) \cdot \alpha(x) / f(x) \to 1
\]
as \( x \to \infty \). Moreover, since \( f \) is positive on \([x_0, \infty)\) for some \( x_0 > 0 \) and \( f(x) \to 0 \) as \( x \to \infty \), there exists \( y_x > x \) such that

\[
1 \geq \inf_{x_0 \leq t \leq x} \frac{f(t)}{f(x)} = \inf_{x \leq t \leq y_x} \frac{f(t)}{\alpha(t)f(x)} \geq \inf_{x \leq t \leq y_x} \frac{f(t)}{\alpha(t)} \cdot \frac{\alpha(x)}{f(x)}.
\]

Letting \( y_x \to \infty \) and then \( x \to \infty \), we obtain the result. Finally it suffices to notice that \( f \in \mathcal{L} \) implies \( f(x) \to 0 \) as \( x \to \infty \) (see [9, p.76]).

**Proof of Lemma 3.9.** Obviously \( \tilde{g} \in \mathcal{L} \). We take a non-decreasing function \( 0 < \alpha(x) < x/2 \) such that \( f \) and \( \tilde{g} \) are \( \alpha \)-insensitive (see Lemma 3.6). Write

\[
\frac{\tilde{g}^2(x)}{\tilde{g}(x)} = \left(2 \int_{-\infty}^{-\alpha(x)} + 2 \int_{-\alpha(x)}^{0} + \int_{0}^{\alpha(x)} \right) \frac{\tilde{g}(x-y)\tilde{g}(y)}{\tilde{g}(x)} dy.
\]

By the dominated convergence the second term converges to 2. If necessary, put \( \tilde{g}(x) = p\delta(x) + qg(x) \) and avoid the delta function when applying the dominated convergence. Moreover, due to (3.9), the third term is bounded by

\[
c \int_{\alpha(x)}^{x-\alpha(x)} \frac{\tilde{f}(x-y)\tilde{f}(y)}{f(x)} dy,
\]

which is \( o(1) \) according to Lemma 3.6 (v). We see that the first term is negligible. For the case \( \tilde{g}(-x) = O(\tilde{f}(-x)) \) as \( x \to \infty \), (3.9) together with Lemma 3.6 (v) implies

\[
\int_{-\infty}^{-\alpha(x)} \frac{\tilde{g}(x-y)\tilde{g}(y)}{\tilde{g}(x)} dy \leq c \int_{-\infty}^{-\alpha(x)} \frac{\tilde{f}(x-y)\tilde{f}(y)}{f(x)} dy \to 0 \quad \text{as} \quad x \to \infty.
\]

If \( g \) is a.n.i. (or \( \tilde{f} \) is a.n.i.), then the first term is bounded by

\[
\sup_{t \geq x+\alpha(x)} g(t) \int_{-\infty}^{-\alpha(x)} \tilde{g}(y) dy \quad \text{or} \quad c \sup_{t \geq x+\alpha(x)} f(t) \int_{-\infty}^{-\alpha(x)} \tilde{f}(y) dy \to 0,
\]

as \( x \to \infty \). Thus we prove the assertion. \( \square \)

**Proof of Proposition 3.10.** (i) Owning to Lemma 3.9, it suffices to see

\[
\lim_{x \to \infty} \frac{f * \tilde{g}(x)}{f(x)} = 1.
\]

Take an insensitive function \( \alpha(x) \) for \( f \) such that \( 0 < \alpha(x) < x/2 \), and write

\[
\frac{f * \tilde{g}(x)}{f(x)} = \left( \int_{-\infty}^{-\alpha(x)} + \int_{-\alpha(x)}^{0} + \int_{0}^{\alpha(x)} + \int_{\alpha(x)}^{\infty} \right) \frac{f(x-y)}{f(x)} \tilde{g}(y) dy
\]

\[
=: I_1(x) + I_2(x) + I_3(x) + I_4(x).
\]

Since \( f \in \mathcal{L} \), \( \lim_{x \to \infty} I_2(x) = 1 \) (If it is needed, write \( \tilde{g}(x) = p\delta(x) + qg(x) \) and apply the calculation rule for \( \delta \)). We consider the first condition. Since \( f \) is a.n.i. and \( f \in \mathcal{L} \),

\[
I_1(x) \leq \sup_{t \geq x+\alpha(x)} f(t) \int_{-\infty}^{-\alpha(x)} \tilde{g}(y) dy \to 0 \quad \text{as} \quad x \to \infty.
\]

Moreover, since \( \tilde{g}(x) = o(f(x)) \) and \( f \in \mathcal{L} \), in view of \( I_1 \) and \( I_2 \) with \( \tilde{g} \) replaced by \( f \), we have

\[
I_4(x) = \int_{-\infty}^{-\alpha(x)} \frac{\tilde{g}(x-y)}{f(x-y)} f(y) dy \leq o(1) \left( \int_{-\infty}^{\alpha(x)} + \int_{-\infty}^{\infty} \right) \frac{f(x-y)}{f(x)} \tilde{g}(y) dy \to 0
\]
as $x \to \infty$. Next since $\bar{g}(x) = o(f(x))$ and $f \in S_+$ (cf. Lemma 3.6 (v))

$$I_3(x) = \int_{-\alpha}^{x-\alpha} \frac{f(x-y)\bar{g}(y)}{f(x)} f(y)dy \leq c \int_{-\alpha}^{x-\alpha} \frac{f_+(x-y)}{f_+(x)} f_+(y)dy \to 0$$

as $x \to \infty$. Thus we obtain (B.40). Finally, for the second case, it suffices to show that $\lim_{x \to \infty} I_1(x) = 0$. Since $\alpha(x) \to \infty$ and $\bar{g}(-x) = O(f(-x))$,

$$I_1(x) \leq c \int_{-\alpha}^{x-\alpha} \frac{f(x-y)f(y)}{f(x)} dy \to 0$$

follows from Lemma 3.6 (v).

(ii) Let $z \in \mathbb{R}$. For sufficiently large $c > 0$,

$$h(x+z) \geq \int_c^x \bar{g}(x+z-y)f(y)dy \geq \inf_{y \in [c,x]} f(y) \int_z^{x+z-c} \bar{g}(y)dy.$$

Then, since $f$ is a.n.i.

$$\liminf_{x \to \infty} \frac{h(x)}{f(x)} = \lim_{z \to -\infty} \liminf_{x \to \infty} \frac{h(x)}{h(x+z)} \inf_{y \in [c,x]} \frac{f(y)}{f(x)} \geq \lim_{z \to -\infty} \int_z^{\infty} \bar{g}(y)dy = 1,$$

which implies

(B.41) \quad $\limsup_{x \to \infty} f(x)/h(x) \leq 1$.

We will prove

(B.42) \quad $\liminf_{x \to \infty} f(x)/h(x) \geq 1$.

Take an insensitive function $0 < \alpha(x) < x/2$ for $h$ and write

$$1 = \left( \int_{-\infty}^{x-\alpha} + \int_{x-\alpha}^{x-\alpha(x)} + \int_{x-\alpha(x)}^{\infty} \right) \frac{f(x-y)\bar{g}(y)}{h(x)} dy$$

$$= I_1(x) + I_2(x) + I_3(x) + I_4(x).$$

Here we may take $\alpha(x)$ to be continuous (see [9, p.20]). Since $f$ is a.n.i. and $h \in L$

$$I_2(x) \leq \frac{h(x-\alpha(x))}{h(x)} \frac{f(x-\alpha(x))}{h(x-\alpha(x))} \frac{\sup_{y \geq x-\alpha(x)} f(y)}{f(x-\alpha(x))} \int_{-\alpha(x)}^{\alpha(x)} \bar{g}(y)dy.$$

The terms other than $f(x-\alpha(x))/h(x-\alpha(x))$ converge to 1, and we have

$$\liminf_{x \to \infty} I_2(x) \leq \liminf_{x \to \infty} f(x)/h(x).$$

For $I_1$, we use the a.n.i. property of $f$ and (B.41), i.e. for sufficiently large $x > 0$

$$I_1(x) = \frac{f(x+\alpha(x))}{h(x)} \int_{-\infty}^{x-\alpha(x)} \frac{f(x-y)}{f(x+\alpha(x))}\bar{g}(y)dy \leq (1+\epsilon) \int_{-\alpha(x)}^{x-\alpha(x)} \bar{g}(y)dy \to 0 \quad \text{as } x \to \infty.$$

Next, we consider

$$I_3(x) = \int_{\alpha(x)}^{x-\alpha(x)} \frac{f(x-y)\bar{g}(y)}{h(x-y)h(y)} h(x-y)h(y) dy.$$

Since we have $\bar{g}(x) = o(h(x))$, (B.41) and $h \in S_+$,

$$I_3(x) \leq c \int_{\alpha(x)}^{x-\alpha(x)} \frac{h_+(x-y)h_+(y)}{h_+(x)} dy \to 0 \quad \text{as } x \to \infty.$$
holds by Lemma 3.6 (v). We study $I_4$. If $h \in \mathcal{L}$ is a.a.i. and then since $\bar{g}(x) = o(h(x))$,

$$I_4(x) = \int_{-\infty}^{\alpha(x)} \frac{\bar{g}(x-y)}{h(x-y)} \frac{h(x) - h(x - \alpha(x))}{h(x)} f(y)dy$$

$$\leq o(1) \sup_{y \geq x - \alpha(x)} \frac{h(y)}{h(x - \alpha(x))} \int_{-\infty}^{\alpha(x)} f(y)dy \to 0 \quad \text{as} \quad x \to \infty.$$ 

When $\bar{g}(x) = o(f(x))$, by (B.41) and the a.a.i. property of $f$, we have

$$I_4(x) = \int_{-\infty}^{-\alpha(x)} \frac{\bar{g}(x-y)}{f(x-y)} \cdot f(x + \alpha(x)) \frac{h(x + \alpha(x))}{h(x)} f(y)dy$$

$$+ \int_{-\alpha(x)}^{\alpha(x)} \frac{\bar{g}(x-y)}{h(x-y)} \frac{h(y)}{h(x)} f(y)dy$$

$$\leq o(1) \int_{-\infty}^{\alpha(x)} f(y)dy \to 0 \quad \text{as} \quad x \to \infty.$$ 

Thus in view of above results, (B.42) follows. Now by Lemma 3.9 together with the a.a.i. property of $f$, we obtain $f \in \mathcal{S}_+$. 

Proof of Corollary 3.11. Since $h \in \mathcal{L}$ implies that $\lim_{x \to \infty} e^{\varepsilon x} h(x) = \infty$ for any $\varepsilon > 0$, $\bar{g}(x) = o(h(x))$ holds. Thus the condition of Proposition 3.10 (ii) is satisfied. 

Proof of Theorem 3.12. The case $\tilde{f}^s k$ are a.a.i. We start by deriving

$$(B.43) \quad \liminf_{x \to \infty} \frac{\tilde{f}^{s n}(x)}{\tilde{f}(x)} \geq n.$$ 

Let $z < 0$. We take a uniform constant $x_0 > 0$ such that for $x \geq x_0$, $\tilde{f}^s k(x), 1 \leq k \leq n - 1$ satisfy the condition (2.2). Then for $x > x_0$ sufficiently large, we have

$$\tilde{f}^{s n}(x + (n-1)z) = \int_{(n-1)z/2}^{x} \tilde{f}^{s(n-1)}(x/2 + (n-1)z - y) \tilde{f}(y + x/2)dy$$

$$+ \int_{(n-1)z/2}^{x} \tilde{f}(x/2 + (n-1)z - y) \tilde{f}^{s(n-1)}(y + x/2)dy$$

$$\geq \int_{x_0}^{x/2} \tilde{f}^{s(n-1)}(x/2 + (n-1)z - y) \tilde{f}(y + x/2)dy$$

$$+ \int_{x_0}^{x/2+(n-2)z} \tilde{f}(x/2 + (n-1)z - y) \tilde{f}^{s(n-1)}(y + x/2)dy$$

$$\geq \tilde{f}(x)(1 - \varepsilon^x_1) \int_{(n-1)z}^{x/2+(n-1)z-x_0} \tilde{f}^{s(n-1)}(y)dy$$

$$+ \tilde{f}^{s(n-1)}(x) \int_{(n-1)z}^{x/2+(n-1)z-x_0} \tilde{f}(y)dy$$

$$=: I^{x}_{n-1}(z) \tilde{f}(x) + J^{x}_{n-1}(z) \tilde{f}^{s(n-1)}(x + (n-2)z),$$

where $\varepsilon^x_1, \delta^x_{n-1} \in (0, 1)$ are small constants such that $\varepsilon^x_1, \delta^x_{n-1} \to 0$ as $x \to \infty$. We further take small constants $\varepsilon^x_k, \delta^x_{k} \in (0, 1), k = 2, \ldots, n-2$ such that $\varepsilon^x_k, \delta^x_{k} \to 0$ as $x \to \infty$, and we successively apply the inequality and reach

$$(B.44) \quad \tilde{f}^{s n}(x + (n-1)z) \geq \tilde{f}(x) \sum_{k=1}^{n} I^{x}_{n-k}(z) \prod_{\ell=1}^{k-1} J^{x}_{n-\ell}(z),$$
where \( I^x_0(z) := 1 \) and \( \prod_{k=1}^{n-1} J^x_{n-k}(z) := 1 \). Since all \( I^x_k(z) \), \( J^x_k(z) \), \( 1 \leq k \leq n-1 \) satisfy
\[
\lim_{z \to -\infty} \lim_{x \to \infty} I^x_k(z) = \lim_{z \to -\infty} \lim_{x \to \infty} J^x_k(z) = 1,
\]
by \( \tilde{f}^{sn} \in \mathcal{L} \) we have
\[
\liminf_{x \to \infty} \frac{\tilde{f}^{sn}(x)}{\tilde{f}(x)} = \liminf_{z \to -\infty} \frac{\tilde{f}^{sn}(x + (n-1)z)}{\tilde{f}(x)} \geq \liminf_{z \to -\infty} \liminf_{x \to \infty} \sum_{k=1}^{n} I^x_{n-k}(z) \prod_{\ell=1}^{k-1} J^x_{n-\ell}(z) = n.
\]

Next we prove the other direction,
\[
\limsup_{x \to \infty} \frac{\tilde{f}^{sn}(x)}{\tilde{f}(x)} \leq n.
\]

This time we take \( z > 0 \) and observe that
\[
\tilde{f}^{sn}(x + (n-1)z) = \int_{-\infty}^{z} \tilde{f}^{s(n-1)}(x + (n-1)z - y) \tilde{f}(y) dy
\]
\[
\quad + \left( \int_{-\infty}^{(n-1)z} + \int_{(n-1)z}^{x+(n-2)z} \right) \tilde{f}(x + (n-1)z - y) \tilde{f}^{s(n-1)}(y) dy
\]
\[
\quad \leq \tilde{f}^{s(n-1)}(x + (n-2)z)(1 + \tilde{\varepsilon}^x_{n-1}) \int_{-\infty}^{z} \tilde{f}(y) dy
\]
\[
\quad + \tilde{f}(x)(1 + \tilde{\varepsilon}^x_1) \int_{-\infty}^{(n-1)z} \tilde{f}^{s(n-1)}(y) dy + \tilde{J}^x_{n-1}(z)
\]
\[
\quad =: \tilde{f}^{s(n-1)}(x + (n-2)z) \tilde{f}^{x}_{1}(z) + \tilde{f}(x) \tilde{J}^x_{n-1}(z) + \tilde{J}^x_{n-1}(z),
\]
where \( \tilde{\varepsilon}^x_k \in (0, 1) \), \( k = 1, \ldots, n-1 \) are small constants such that \( \tilde{\varepsilon}^x_k \to 0 \) as \( x \to \infty \). We successively apply the inequality above and obtain
\[
\tilde{f}^{sn}(x + (n-1)z) \leq \sum_{k=1}^{n-2} I^x_{n-k}(z) \left( \tilde{f}^{x}_{1}(z) \right)^{k-1} + 2 \left( \tilde{f}^{x}_{1}(z) \right)^{n-1} \tilde{f}(x)
\]
\[
\quad + \sum_{k=1}^{n-1} J^x_{n-k}(z) \left( \tilde{f}^{x}_{1}(z) \right)^{k-1},
\]
where \( \left( \tilde{f}^{x}_{1}(z) \right)^{0} := 1 \). We introduce a non-decreasing function \( 0 < \alpha(x) < x/2 \) such that \( \tilde{f}^{sn} \) is \( \alpha \)-insensitive (Lemma 3.6 (i)) and put \( (n-1)z = \alpha(x) \), so that \( z = \alpha'(x) := \alpha(x)/(n-1) \). Notice that \( \alpha' \) is again an insensitive function for \( \tilde{f}^{sn} \) (cf. [9, p.20]). Obviously
\[
\lim_{x \to \infty} \tilde{J}^{x}_{k}(\alpha'(x)) = \lim_{x \to \infty} \left( 1 + \tilde{\varepsilon}^x_{n-k} \right) \int_{-\infty}^{\alpha'(x)k} \tilde{f}^{sk}(y) dy = 1.
\]
We show that
\[
\limsup_{x \to \infty} \tilde{J}^{x}_{k}(\alpha'(x)) = o(\tilde{f}^{sn}(x)) \quad \text{as} \quad x \to \infty.
\]

First we see that
\[
\limsup_{x \to \infty} \tilde{f}^{sk}(x) / \tilde{f}^{sn}(x) \leq 1 \quad \text{for} \quad 1 \leq k \leq n-1.
\]

For \( v > 0 \), we write
\[
\tilde{f}^{sn}(x - v) \geq \int_{-v}^{v} \tilde{f}^{sk}(x - v - y) \tilde{f}^{s(n-k)}(y) dy \geq \inf_{y \in [x-2v, x]} \tilde{f}^{sk}(z) \int_{-v}^{v} \tilde{f}^{s(n-k)}(y) dy.
\]
Since \( \tilde{f}^*k \) is a.n.i., it follows that
\[
\liminf_{x \to \infty} \frac{\tilde{f}^{*n}(x)}{\tilde{f}^{*k}(x)} = \lim_{v \to \infty} \liminf_{x \to \infty} \frac{\tilde{f}^{*n}(x)}{\tilde{f}^{*n}(x-v)} \geq \lim_{v \to \infty} \int_{-v}^{v} \tilde{f}^{*(n-k)}(y)dy = 1.
\]

We return to \( \tilde{J}_k^z \) and observe that for sufficiently large \( z > 0 \)
\[
\tilde{J}_k^z(z) = \int_z^x \tilde{f}(x+z-y)\tilde{f}^{*k}(y+(k-1)z)dy \leq c \int_{z/2}^{x-z/2} \tilde{f}^{*n}(x-y)\tilde{f}^{*n}(y)dy,
\]
where we use (B.48) and the a.n.i. property of \( \tilde{f}^{*n} \). Then recalling that \( \alpha' \) is an insensitive function of \( f^{*n} \) and so is \( \alpha'/2 \), we have by Lemma 3.6 (v),(3.6), that
\[
\tilde{J}_k^z(\alpha'(x)) \leq c \int_{\alpha'(x)/2}^{x-\alpha'(x)/2} \tilde{f}^{*n}(x-y)\tilde{f}^{*n}(y)dy = o(\tilde{f}^{*n}(x)).
\]

Now in view of (B.46) putting \( z = \alpha'(x) \), we have
\[
1 \leq \lim_{x \to \infty} \inf_{f^{*n}(x)} \frac{\tilde{f}(x)}{f^{*n}(x+\alpha(x))} \left[ \sum_{k=1}^{n-2} \bar{J}_{n-k}(\alpha'(x)) \{ \bar{f}_1'(\alpha'(x)) \}^{k-1} + 2 \{ \bar{f}_1'(\alpha'(x)) \}^{n-1} \right] + \lim_{x \to \infty} \sup_{f^{*n}(x)} \frac{\sum_{k=1}^{n-1} \bar{J}_{n-k}(\alpha'(x)) \{ \bar{f}_1'(\alpha'(x)) \}^{k-1}}{f^{*n}(x+\alpha(x))}
\leq \lim_{x \to \infty} \inf_{f^{*n}(x)} \frac{\tilde{f}(x)}{f^{*n}(x)} \lim_{x \to \infty} \left[ \sum_{k=1}^{n-2} \bar{J}_{n-k}(\alpha'(x)) \{ \bar{f}_1'(\alpha'(x)) \}^{k-1} + 2 \{ \bar{f}_1'(\alpha'(x)) \}^{n-1} \right] + o(1)
= n \cdot \lim_{x \to \infty} \inf_{f^{*n}(x)} \frac{\tilde{f}(x)}{f^{*n}(x)},
\]
where we use (B.47) in the second step. Thus we obtain (B.45). Finally we apply Lemma 3.9 to the fact that \( \lim_{x \to \infty} \tilde{f}^{*n}(x)/\tilde{f}(x) = n \) and obtain the result.

**The case** \( f \in \mathcal{L} \). Since \( \tilde{f}^{*k} \in \mathcal{L} \) for \( k = 1, \ldots, n \) (cf. Lemma 3.6 (iv)), we may take a single non-decreasing function \( 0 < \alpha(x) < x/2 \) such that \( \tilde{f}^{*k} \) is \( \alpha \)-insensitive (Lemma 3.6 (ii)). We decompose the integral form for \( \tilde{f}^{*n} = \tilde{f}^{*(n-1)} \ast \tilde{f} \) and write for \( x > 0 \)
\[
1 = \left( \int_{-\infty}^{-\alpha(x)} + \int_{-\alpha(x)}^{0} + \int_{0}^{x-\alpha(x)} + \int_{x-\alpha(x)}^{x+\alpha(x)} + \int_{x+\alpha(x)}^{\infty} \right) \frac{\tilde{f}(y)}{\tilde{f}^{*n}(x)} \tilde{f}^{*(n-1)}(x-y)dy
=: I_1(x) + \cdots + I_5(x).
\]

We start with \( I_2(x) \). Since \( \tilde{f}^{*(n-1)} \in \mathcal{L} \) and \( \tilde{f}^{*n} \in \mathcal{S} \), we have for \( |y| \leq \alpha(x) \)
\[
(B.49) \lim_{x \to \infty} \sup_{f^{*n}(x)} \frac{\tilde{f}^{*(n-1)}(x-y)}{f^{*n}(x)} = \lim_{x \to \infty} \sup_{f^{*n}(x)} \frac{\tilde{f}^{*(n-1)}(x)}{f^{*n}(x)} = \lim_{x \to \infty} \sup_{f^{*n}(x)} \frac{\tilde{f}^{*(n-1)}(x)}{f^{*n}(x)} \tilde{f}^{*(n-1)}(x) \leq 1 - n^{-1}.
\]

Recalling the form \( \tilde{f}(x) = p\delta(x) + qf(x) \), we have by the property of \( \delta \) that
\[
\limsup_{x \to \infty} I_2(x) \leq p \limsup_{x \to \infty} \frac{\tilde{f}^{*(n-1)}(x)}{f^{*n}(x)} + q \limsup_{x \to \infty} \frac{\tilde{f}^{*(n-1)}(x)}{f^{*n}(x)} \tilde{f}(y)dy
\leq p(1 - n^{-1}) + q(1 - n^{-1}) \int_{-\infty}^{\infty} f(y)dy = 1 - n^{-1},
\]
where in the second term, we use Fatou’s lemma, which is possible since the delta function is not involved. It follows from (B.49) and a.n.i. property of $\tilde{f}^{sn}$ that

$$I_1(x) = \int_{-\infty}^{-\alpha(x)} \frac{\tilde{f}^{s(n-1)}(x-y)}{\tilde{f}^{sn}(x-y)} \tilde{f}^{sn}(x-y) \tilde{f}(y) dy \leq c \int_{-\infty}^{-\alpha(x)} \tilde{f}(y) dy \to 0$$

as $x \to \infty$. For $I_3(x)$, noticing an expression

$$I_3(x) = \int_{-\infty}^{x-\alpha(x)} \frac{\tilde{f}^{s(n-1)}(x-y)}{\tilde{f}^{sn}(x-y)} \tilde{f}^{sn}(x-y) \tilde{f}(y) dy,$$

we apply (B.49) and Lemma 3.6 (iii) respectively to the first and the second terms of the integrand. Then Lemma 3.6 (v) yields $\lim_{x \to \infty} I_3(x) = 0$. By the dominated convergence

$$\lim_{x \to \infty} \frac{f(x)}{\tilde{f}^{sn}(x)} \int_{-\alpha(x)}^{\infty} \frac{\tilde{f}(x-y)}{f(x)} \tilde{f}^{s(n-1)}(y) dy = \lim_{x \to \infty} \frac{\tilde{f}(x)}{\tilde{f}^{sn}(x)}.$$

Here to apply the dominated convergence avoiding $\delta$, if necessary, write

$$\tilde{f}^{s(n-1)}(y) = (p\delta + qf)^{s(n-1)}(y) = \sum_{k=1}^{n-1} n_{-1} C_k f^{*k}(y) q^{n-1-k} + p^{n-1} \delta(y)$$

and take a similar approach as for $I_2(x)$.

Finally we apply Lemma 3.6 (iii) and the a.n.i. property of $\tilde{f}^{sn}$ to $I_5$, and obtain

$$I_5(x) = \int_{-\infty}^{-\alpha(x)} \frac{\tilde{f}(x-y)}{\tilde{f}^{sn}(x-y)} \tilde{f}^{sn}(x-y) \tilde{f}^{s(n-1)}(y) dy,$$

$$\leq n^{-1} \int_{-\infty}^{x-\alpha(x)} \tilde{f}^{s(n-1)}(y) dy \to 0 \text{ as } x \to \infty.$$

Now correcting above bounds, we reach

$$1 = \lim_{x \to \infty} \left( \sum_{i=1}^{5} I_i(x) \right) \leq \lim_{x \to \infty} I_4(x) + \lim_{x \to \infty} \sup_{i \neq 4} I_i(x) = \lim_{x \to \infty} \tilde{f}(x) / \tilde{f}^{sn}(x) + 1 - n^{-1},$$

which is equal to (B.45). By Lemma 3.6 (iii), we obtain

$$\lim_{x \to \infty} \tilde{f}^{sn}(x) / \tilde{f}(x) = n.$$

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Department of Business Administration, Nanzan University, 18 Yamazato-cho, Showa-ku, Nagoya 466-8673, Japan.

*Email address: mmuneya@gmail.com*