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Colombo, Leonardo J.; de Marina, Hector Garcia

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A Variational Integrator for the distance-based formation control of multi-agent systems

Leonardo J. Colombo * Hector García de Marina **

* Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM), Calle Nicolas Cabrera 13-15, 28049, Madrid, Spain. (e-mail: leo.colombo@icmat.es)
** unmanned Aerial Systems Center, The Maersk McKinney Møller Institute, University of Southern Denmark, Odense, Denmark. (e-mail: hgm@mmmi.sdu.dk)

Abstract: Distance-based formation control of second order agents can be seen as a physical system of particles linked by springs, whose evolution can be described by a Lagrangian function. An interesting family of geometric integrators, called variational integrators, is defined by using discretizations of the Hamilton’s principle of critical action. The variational integrators preserve some geometric features such as the symplectic structure, they preserve the momentum map, and the evolution of the system’s energy presents a good (bounded) behavior. We derive variational integrators that can be employed in the context of distance-based formation control algorithms. In particular, we provide an accurate numerical integrator with a lower computational cost than traditional solutions. Consequently, we can provide a faster identification of regions of attraction for desired distance-based shapes, and more computationally efficient estimation algorithms like Kalman filters that employ distance-based controllers as prediction models. We use a formation consisting of four autonomous planar agents as an example and benchmark to test and compare the performances of the proposed variational integrator.

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1. INTRODUCTION

Formation control algorithms have emerged as powerful tools for the usage of multi-agent systems as surveyed by Oh et al. (2015). Recently, researchers have shown an increased interest in employing distributed controllers for multi-agent systems, where the agents only rely on local information for executing their designated tasks as reported by Cao et al. (2013). In particular, distributed multi-agent systems present higher robustness and need of fewer resources per agent than centralized systems. One branch of formation control is devoted to the control of inter-agent distances based on the rigidity theory, as shown by Anderson et al. (2008), which allows stabilizing specific geometrical shapes described by the group of agents. Sun et al. (2018) study conservation and associated decay laws in distance-based formation control of second order agents seen as a classical physical system. We consider a more general class of systems by describing the dynamics of agents in the formation through a Lagrangian function together with non-conservative (dissipative) forces, in a similar mathematical description as it was recently proposed for the optimal control of multiple agents avoiding collision by means of artificial (conservative) potential functions by Colombo and Dimarogonas (2018). In such a work the authors also construct variational integrators but for a different problem, in this work we study a formation problem where the formation is achieved by consider external (non-conservative) external forces.

Variational integrators are derived from a discrete variational principle as it has been surveyed by Marsden and West (2001); Hairer et al. (2006). These integrators retain some of the main geometric properties of the continuous systems, such as symplecticity, momentum conservation (as long as the symmetry survives the discretization procedure), and good (bounded) behavior of the energy associated to the system. These class of numerical methods have been applied to a wide range of problems in optimal control, constrained systems, power systems, nonholonomic systems, and systems on Lie groups. For more details we refer to Campos et al. (2015); Ober-Blöbaum et al. (2011); Jiménez et al. (2013); Leyendecker et al. (2010); Cortés (2002); Kobilarov and Marsden (2011).

In order to help actual implementations of distributed multi-agent systems in formation control, we need to aim at relaxing the requirements in sensors and computational cost (energy efficiency) per agent as much as possible. For example, it is common to count on distance and directional sensors for the formation control of aerial robots, but agents can estimate distances by only having information from directional sensors as Schiano and Tron (2018) have shown with a Kalman filter, where predictions of future states are required based on a known model. As we have discussed, this model in the context of distance-based formation control can be a Lagrangian system. Therefore, we can exploit the properties of such a system via the proposed variational integrators. In particular, agents can employ them for their estimation algorithms to save...
energy consumption since they have a lower computational cost than traditional numerical solutions like Runge-Kutta, and without compromising accuracy (Euler integrator). In fact, the accuracy in a simulation is also crucial when a multi-agent system can consists of a significant number of agents and links, i.e., the bigger the number of initial conditions, the bigger the sensitivity for the agents’ trajectories. For example, desired shapes in distance-based control are locally stable and their analytic region of attraction is rather conservative. Therefore, it is important for the identification of bigger regions of attraction to have an accurate simulation of the trajectories, but without dramatically increasing the computational cost with the number of agents.

In this paper, we introduce a framework to study the formation control of multi-agent systems. We start by constructing a geometric integrator based on the discretization of an extension of the Lagrange-d’Alembert principle for a single agent, and we continue to a multi-agent system based on distance-based formation control algorithms. We may understand the problem as a Lagrangian system subject to conservative forces coming from the potential whose minimum corresponds to the desired distance-based shape, and external dissipative forces.

The paper is organized as follows. We first in Section 2 introduce discrete mechanics and variational integrators. In Section 3 we derive, from Lagrange-d’Alembert principle for mechanical systems subject to external forces, the dynamics for formation control of multiple agents. Section 4 is devoted to the construction of distributed formation control algorithms obtained by employing a variational discretization of the variational principle. We illustrate and compare the effectiveness of the proposed discrete integration method with numerical experiments together with some conclusions in Section 5.

2. DISCRETE MECHANICS AND VARIATIONAL INTEGRATORS

Let $Q$ be a $n$-dimensional differentiable manifold with local coordinates $(q^A)$, with $1 \leq A \leq n$, the configuration space of a mechanical system. Denote by $TQ$ its tangent bundle with induced local coordinates $(q^A, \dot{q}^A)$. Given a Lagrangian function $L : TQ \rightarrow \mathbb{R}$, its Euler-Lagrange equations are

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^A} \right) - \frac{\partial L}{\partial q^A} = 0, \quad 1 \leq A \leq n. \tag{1}$$

These equations determine a system of implicit second-order differential equations in general. If we assume that the Lagrangian is regular, that is, the $n \times n$ matrix $\left( \frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^A} \right)$ is non-singular, the local existence and uniqueness of solutions is guaranteed for any given initial condition.

A discrete Lagrangian is a differentiable function $L_d : Q \times Q \rightarrow \mathbb{R}$, which may be considered as an approximation of the action integral defined by a continuous regular Lagrangian $L : TQ \rightarrow \mathbb{R}$. That is, given a time step $h > 0$ small enough,

$$L_d(q_0, q_1) \approx \int_0^h L(q(t), \dot{q}(t)) \, dt,$$

where $q(t)$ is the unique solution of the Euler-Lagrange equations for $L$ with boundary conditions $q(0) = q_0$ and $q(h) = q_1$.

We construct the grid $\mathcal{F} = \{ t_k = kh \mid k = 0, \ldots, N \}$, with $Nh = T$ and define the discrete path space $\mathcal{P}_d(Q) := \{ q_d : \{ t_k \}_{k=0}^N \rightarrow Q \}$. We identify a discrete trajectory $q_d \in \mathcal{P}_d(Q)$ with its image $q_d = \{ q_k \}_{k=0}^N$, where $q_k := q_k(t_k)$. The discrete action $\mathcal{A}_d : \mathcal{P}_d(Q) \rightarrow \mathbb{R}$ for this sequence of discrete paths is calculated by summing the discrete Lagrangian on each adjacent pair, and it is defined by

$$\mathcal{A}_d(q_d) = \mathcal{A}_d(q_0, \ldots, q_N) := \sum_{k=0}^{N-1} L_d(q_k, q_{k+1}). \tag{2}$$

We would like to point out that the discrete path space is isomorphic to the smooth product manifold which consists of $(N + 1)$ copies of $Q$. The discrete action inherits the smoothness of the discrete Lagrangian and the tangent space $\mathcal{T}_d \mathcal{P}_d(Q)$ at $q_d$ is the set of maps $v_{q_d} : \{ t_k \}_{k=0}^N \rightarrow TQ$ such that $TQ \circ v_{q_d} = q_d$ which will be denoted by $v_{q_d} = (\{ q_k, \dot{q}_k \})_{k=0}^N$, where $TQ : TQ \rightarrow Q$ is the canonical projection.

The discrete variational principle, or Cadzow’s principle (Cadzow (1970)), states that the solutions of the discrete system determined by $L_d$ must extremize the action sum given fixed points $q_0$ and $q_N$. Extremizing $\mathcal{A}_d$ over $q_k$ with $1 \leq k \leq N - 1$, we obtain the following system of difference equations

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = 0. \tag{3}$$

These equations are usually called the discrete Euler-Lagrange equations. Given a solution $\{ q_k^* \}_{k=0}^N$ of eq.(3) and assuming the regularity hypothesis, i.e., the matrix $(D_1 L_d(q_k, q_{k+1}))$ is regular, it is possible to define implicitly a (local) discrete flow $\mathcal{T}_d : \mathcal{F}_d \subseteq Q \times Q \rightarrow Q$ by $\mathcal{T}_d(q_{k-1}, q_k) = (q_k^*, q_{k+1})$ from (3), where $\mathcal{F}_d$ is a neighborhood of the point $(q_{k-1}, q_k)$.

3. LAGRANGE-D’ALEMBERT PRINCIPLE FOR MULTI-AGENT DISTANCE BASED FORMATION CONTROL

Consider a set $\mathcal{N}$ consisting of $s$ free agents evolving on a configuration manifold $Q$ with dimension $n$. We denote by $q_i \in Q$ the configurations (positions) of agent $i \in \mathcal{N}$, with local coordinates $q_i^A, A = 1, \ldots, n$, and by $q \in Q^s$ the stacked vector of positions where $Q^s$ represents the cartesian product of $s$ copies of $Q$.

The neighbor relationships are described by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ denotes the set of nodes and the set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ denotes the set of ordered edges of the graph. The set of neighbors for agent $i$ is defined by $\mathcal{N}_i = \{ j \in \mathcal{N} : (i, j) \in \mathcal{E} \}$. We define the elements of the incidence matrix $B \in Q^{s \times |\mathcal{E}|}$ for $\mathcal{G}$ by

$$b_{jk} = \begin{cases} +1 & \text{if } i = e_{jk}^{\text{tail}} \\ -1 & \text{if } i = e_{jk}^{\text{head}} \\ 0 & \text{otherwise} \end{cases}$$

where $e_{jk}^{\text{tail}}$ and $e_{jk}^{\text{head}}$ denote the tail and head nodes, respectively, of the edge $e_{jk}$, i.e., $e_{jk} = (e_{jk}^{\text{tail}}, e_{jk}^{\text{head}})$.

The dynamics of each agent is determined by a Lagrangian system on $TQ$, that is, the motion of the agent $i \in \mathcal{N}$ is described by $L_i : TQ \rightarrow \mathbb{R}$ and its dynamics is given by the Euler-Lagrange equations for $L_i$, i.e.,

$$\frac{d}{dt} \left( \frac{\partial L_i}{\partial \dot{q}_i^A} \right) - \frac{\partial L_i}{\partial q_i^A} = 0.$$

Usually, the Lagrangian takes the form of kinetic minus potential energy, $L_i(q_i, \dot{q}_i) = K(q_i, \dot{q}_i) - V(q_i) = \frac{1}{2} \dot{q}_i^T M(q_i) \dot{q}_i - V(q_i)$ for some (positive definite) mass matrix $M(q_i)$.
In addition, the agent \( i \in \mathcal{N} \) may be influenced by a non-conservative force (conservative forces maybe included into the potential energy \( V_i \)), which is a fibered map \( F_i : TQ \to T^*Q \). For instance, \( F_i \) can describe a virtual linear damping between two agents. At a given position and velocity, the force will act against variations of the position (virtual displacements). A consequence of the Lagrange-d’Alembert principle (or principle of virtual work), an extension to external non conservative forces of Hamilton’s principle (or principle of least action), establishes that the natural motions of the system are those paths \( q : [0, T] \to T^*Q \) satisfying
\[
\delta \int_0^T L_i(q_i, \dot{q}_i) \, dt + \int_0^T F_i(q_i, \dot{q}_i) \delta q_i \, dt = 0
\]
for null boundary variations \( \delta q_i(0) = \delta q_i(T) = 0 \). The first term is the action variation, while the second is known as virtual work since \( F_i(q_i, \dot{q}_i) \delta q_i \) is the virtual work done by the force field \( F_i \) with a virtual displacement \( \delta q_i \). Lagrange-d’Alembert principle leads to the forced Euler-Lagrange equations
\[
\frac{d}{dt} \left( \frac{\partial L_i}{\partial \dot{q}_i^k} \right) - \frac{\partial L_i}{\partial q_i^k} = F_i(q_i, \dot{q}_i).
\]
If the Lagrangian is regular, it induces a well defined map, the Lagrangian flow, \( F_i : TQ \to TQ \) by \( F_i(q_0, q_0) : = (q_i(t_i), \dot{q}_i(t_i)) \) where \( q_i \in \mathcal{C}^2([0, T], Q) \) is the unique solution of the Euler-Lagrange equation with initial condition \((q_0, \dot{q}_0) \in TQ \).

Next, consider the Lagrangian function for the formation problem \( \mathcal{L} : (TQ)^s \to \mathbb{R} \) given by
\[
\mathcal{L}(q, \dot{q}) = \sum_{i=1}^{s} L_i(\pi_i(q_i), \tau_i(q_i)) \quad (4)
\]
where \((TQ)^s = \prod_{i=1}^{s} TQ \), \( \pi_i : Q^s \to Q \) is the canonical projection from \( Q^s \) over its \( i \)th-factor and \( \tau_i : (TQ)^s \to TQ \) is the canonical projection from \((TQ)^s \) over its \( i \)th-factor.

For the motivation commented in the introduction, we particularize our analysis to purely kinematic agents and when the configuration manifold \( Q \) is a Riemannian manifold. To control the shape of the formation we introduce the potential functions \( V_{ij} : Q \times Q \to \mathbb{R} \)
\[
V_{ij}(q_i, q_j) = \frac{1}{4} \left( |q_i - q_j|^2 - d_{ij}^2 \right)^2, \quad (5)
\]
where \(|\cdot|\) is a norm on \( Q \) induced by the Riemannian metric on \( Q \) (and therefore inducing a distance on \( Q \)), \( d_{ij} \) denotes the relative position between agents \( i \) and \( j \), and \( d_{ij} \) is the desired distance between agents \( i \) and \( j \) for the edge \( \delta E_k = (i, j) \). If we are interested in stabilizing a specific geometrical shape, then the incidence matrix (3) and the set of desired distances can be determined by the rigidity theory as reviewed by Anderson et al. (2008). Note that the potential (5) acts as an elastic potential, where the resting length of the spring is \( d_{ij} \). Indeed, the form of (5) is not unique, and it can be given by other similar expressions as it was discussed by Sun et al. (2016).

Therefore, the Lagrangian \( \mathcal{L} : (TQ)^s \to \mathbb{R} \) takes the form
\[
\mathcal{L}(q, \dot{q}) = \sum_{i=1}^{s} \left( K_i(\pi_i(q_i), \tau_i(q_i)) - \frac{1}{2} \sum_{j \in \mathcal{N}_i} V_{ij}(\pi_i(q_i), \pi_j(q_i)) \right). \quad (6)
\]
Note that the factor \( \frac{1}{2} \) in the potential function in (6) comes from the fact that \( V_{ij} \equiv V_{ji}, \) i.e., for each virtual spring with elastic potential (5) we have an agent at each of the tips of the spring.

If each agent \( i \in \mathcal{N} \) is subject to external non-conservative forces, the dynamics for the formation problem is determined by an extension of Lagrange-d’Alembert principle for a single agent to multiple agents by considering the Lagrangian function \( \mathcal{L} \). The following result describes the (continuous-time) dynamics for the formation problem:

**Proposition 3.1.** Let \( \mathcal{L} : (TQ)^s \to \mathbb{R} \) be the Lagrangian function defined in (6) and \( F : (TQ)^s \to (T^*Q)^s \) be external forces. The curve \( q \in \mathcal{C}^2(Q^s) \) satisfies \( \delta \mathcal{A}(q) = 0 \) for the action functional defined by
\[
\delta \mathcal{A}(q) = \int_0^T \sum_{i=1}^{s} K_i(\pi_i(q_i), \tau_i(q_i)) - \frac{1}{2} \sum_{j \in \mathcal{N}_i} V_{ij}(\pi_i(q_i), \pi_j(q_i)) \, dt + \int_0^T \sum_{i=1}^{s} F(\pi_i(q_i), \tau_i(q_i)) \, dt
\]
if and only if, for variations of \( q \in Q^s \) with fixed endpoints and the virtual work done by the forces when the path \( q(t) \) is only varied by \( \delta q(t) \), \( q \) is a solution of the forced Euler-Lagrange equations for \( \mathcal{L} \):
\[
\frac{d}{dt} \left( \frac{\partial K_i}{\partial q_i^k} \right) - \frac{\partial K_i}{\partial q_i^k} - \frac{1}{2} \sum_{j \in \mathcal{N}_i} \frac{\partial V_{ij}}{\partial q_i^k} \delta q_i^k \, dt
\]
\[
+ \int_0^T \sum_{i=1}^{s} F(\pi_i(q_i), \tau_i(q_i)) \delta q_i \, dt.
\]
Since this must holds for all variations \( \delta q_i \), with \( i = 1, \ldots, s \) we get the desired result.

**Example 3.1.** We consider four agents evolving on \( Q = \mathbb{R}^2 \) endowed with the Euclidean Riemannian metric, with local coordinates \( q_i = (x_i, y_i) \) and each one with unit mass. The graph defining the neighbor’s relations is depicted in the following diagram:

![Diagram](2)

The set of neighbors for each agent are \( \mathcal{N}_1 = \{2, 3, 4\} \), \( \mathcal{N}_2 = \{1, 3\} \), \( \mathcal{N}_3 = \{1, 2, 4\} \), and \( \mathcal{N}_4 = \{1, 3\} \).

We choose (5) as potential functions, and the external forces are given by a linear damping, i.e., \( F_i(q_i, \dot{q}_i) = -\kappa q_i \) with \( \kappa \in \mathbb{R}^+ \).

Denoting \( \Gamma_{ij} = ((x_i - x_j)^2 + (y_i - y_j)^2 - d_{ij}^2) \) and using Proposition 3.1, the dynamics for the formation of the four planar agents, is given by the following set of second-order nonlinear equations
\[
\begin{aligned}
\dot{x}_i &= -\kappa x_i - \sum_{j \in \mathcal{N}_i} (x_i - x_j) F_{ij}, \\
\dot{y}_i &= -\kappa y_i - \sum_{j \in \mathcal{N}_i} (y_i - y_j) F_{ij}, \quad i \in \mathcal{N}
\end{aligned}
\] 

(7)

4. A VARIATIONAL INTEGRATOR FOR MULTI-AGENT FORMATION CONTROL

The key idea of variational integrators is that the variational principle is discretized rather than the resulting equations of motion. As we did in Section 2, we discretize the state space \( TQ \) as \( Q \times Q \) and consider a discrete Lagrangian \( L^d : Q \times Q \rightarrow \mathbb{R} \) and discrete "external forces" \( F^\pm_i : Q \times Q \rightarrow T^*Q \) approximating the continuous time Lagrangian action and non-conservative external forces for the agent \( i \in \mathcal{N} \), respectively, given by

\[
\begin{aligned}
\int_{t_k}^{t_{k+1}} L(q(t), \dot{q}(t)) dt &\simeq L^d_i(q_k, q_{k+1}) \\
\int_{t_k}^{t_{k+1}} F_i(q(t), \dot{q}(t)) \delta q dt &\simeq F^+_i(q_k, q_{k+1}) \delta q_k \\
&\quad + F^-_i(q_k, q_{k+1}) \delta q_{k+1}.
\end{aligned}
\]

(8)

(9)

Note that \( F_i^\pm \) are not external forces, physically speaking. They are in fact momentum, since \( F_i^\pm \) are defined by a discretization of the work done by the force \( F_i \). The idea behind the \( \pm \) is that for a fixed \( i \in \mathcal{N} \), one needs to combine the two discrete forces to give a single one-form \( F_i : Q \times Q \rightarrow T^*Q \) defined by

\[
F_{i,t}(q_0, q_1)(\delta q_0, \delta q_1) = F^+_i(q_0, q_1) \delta q_1 + F^-_i(q_0, q_1) \delta q_1.
\]

It is well known that, for a single agent (see Marsden and West (2001) Section 4.2.1, by deriving the discrete variational principle using (8) and (9), one can derive the forced discrete Euler-Lagrange equations

\[
\begin{aligned}
0 = &D_1 L^d_i(q_k, q_{k+1}) + D_2 L^d_i(q_{k-1}, q_k) \\
&+ F^+_i(q_k, q_{k+1}) + F^-_i(q_{k-1}, q_k)
\end{aligned}
\]

(10)

for \( k = 1, \ldots, N-1 \), and where \( D_j \) stands for the partial derivative with respect to the \( j \)-th component of \( L^d_i \). The previous equations define the integration scheme \( (q'_{k-1}, q'_{k+1}) \rightarrow (q_k', q_{k+1}') \).

The situation for multi-agent systems can be described in a similar fashion as for a single agent. Corresponding to the configuration manifold \( Q \), the discrete state space is defined by \((Q \times Q)^N \) which is locally isomorphic to \((TQ)^N \). For a constant time-step \( h \in \mathbb{R}^+ \), a path \( q : [0, t_N] \rightarrow Q \) is replaced by a discrete path \( q_d = \{q_k\}_{k=0}^N \) where \( q_k = [q_k^1, \ldots, q_k^N] = q_d(tk) = q_d(t_0 + kh) \). Let \( C_d(Q') = \{q_d : \{t_k\}_{k=0}^N \rightarrow Q' \} \) be the space of discrete paths on \( Q' \). Given that \( C_d(Q') \) is isomorphic to \( s(N+1) \) copies of \( Q \) it can be endowed with a product manifold structure. Define the discrete action sum \( \mathcal{A}_d : C_d(Q') \rightarrow \mathbb{R} \) by

\[
\mathcal{A}_d(q_d) = \sum_{i=1}^s \sum_{k=0}^{N-1} L^d_i(q_k, q_{k+1}) \\
+ \sum_{i=1}^s \sum_{k=0}^{N-1} F^+_i(q_k, q_{k+1}) \delta q_k + F^-_i(q_k, q_{k+1}) \delta q_{k+1}
\]

(11)

where, as for a single agent, we are using that

\[
\begin{aligned}
\int_{t_k}^{t_{k+1}} L(q(t), \dot{q}(t)) dt &= \int_{t_k}^{t_{k+1}} \sum_{i=1}^s L_i(q_i(t), \tau_i(q(t))) dt \\
&\simeq \sum_{i=1}^s \int_{t_k}^{t_{k+1}} F_i(q_i(t), \tau_i(q(t))) \delta q_i dt \\
&\simeq \sum_{i=1}^s \left( F^+_i(q_k, q_{k+1}) \delta q_k \\
&\quad + F^-_i(q_k, q_{k+1}) \delta q_{k+1} \right).
\end{aligned}
\]

where \( L^d : (Q \times Q)^s \rightarrow \mathbb{R} \) be the discrete Lagrangian defined in (12). A discrete path \( q_d^{k} \) at any \( k \in \mathbb{N} \) extremizes the discrete actions \( \mathcal{A}_d \) if and only if it is a solution for the forced discrete Euler LaGrange equations

\[
\begin{aligned}
D_2 L^d_i(q_{k-1}, q_k) + F^+_i(q_k, q_{k+1}) &= -D_1 L^d_i(q_k, q_{k+1}) \\
- F^-_i(q_k, q_{k+1})
\end{aligned}
\]

for \( k = 1, \ldots, N-1 \), \( i = 1, \ldots, s \) where \( F^+_i \) and \( F^-_i \) are given by

\[
\begin{aligned}
F^+_i(q_k, q_{k+1}) &= \sum_{i=1}^s F^+_i(q_k, q_{k+1}), \\
F^-_i(q_k, q_{k+1}) &= \sum_{i=1}^s F^-_i(q_k, q_{k+1}).
\end{aligned}
\]

Proof: Variations of the action sum (11), after a shift in the index for the discrete external force \( F^+_i \), reads

\[
\mathcal{A}_d = \delta \sum_{k=0}^{N-1} L^d(q_k, q_{k+1})
\]

\[
= D_1 L^d(q_0, q_1) \delta q_0 + D_2 L^d(q_{N-1}, q_N) \delta q_N
\]

\[
+ \sum_{k=1}^{N-1} (D_1 L^d(q_k, q_{k+1}) + D_2 L^d(q_{k-1}, q_k)) \delta q_k
\]

\[
+ \sum_{k=1}^{N-1} (F^-_i(q_k, q_{k+1}) + F^+_i(q_{k-1}, q_k)) \delta q_{k+1}
\]

Requiring its stationarity for all \( \delta q_k \) \( k = 0, \ldots, N \) and \( \delta q_0 = \delta q_N = 0 \) yields the forced discrete Euler LaGrange equations

\[
\begin{aligned}
D_2 L^d_i(q_{k-1}, q_k) + F^+_i(q_k, q_{k+1}) &= -D_1 L^d_i(q_k, q_{k+1}) \\
- F^-_i(q_k, q_{k+1})
\end{aligned}
\]

for \( k = 1, \ldots, N-1 \) and \( i = 1, \ldots, s \).

Remark 4.1. When we consider the Lagrangian functions

\[
L_i(q_i, \dot{q}_i) = K_i(q_i, \dot{q}_i) - \frac{1}{2} \sum_{j \in \mathcal{N}_i} V_{ij}(q_i, q_j)
\]

the discrete forcer Euler-LaGrange equations are given by

\[
\begin{aligned}
0 = &D_2 K^d_i(q_{k-1}, q_k) - \frac{1}{2} \sum_{j \in \mathcal{N}_i} D^2 V_{ij}^d(q_{k-1}, q_k, q_{k+1}, q'_j) \\
&+ D_1 K^d_i(q_k, q_{k+1}) + F^+_i(q_k, q_{k+1}) + F^-_i(q_k, q_{k+1})
\end{aligned}
\]

for \( k = 1, \ldots, N-1 \) and \( i = 1, \ldots, s \), where \( D^1_i \) and \( D^2_i \) denote the derivative to the virtual springs \( V_{ij}^d \) with respect to the first and second arguments of the agent \( i \in \mathcal{N} \), respectively. Note
that the unique partial derivatives that appear in the potential functions are with respect to the agent \( i \in \mathcal{N} \) since in the variational principle, variations are only taken with respect to \( q_i \) and \( i \neq j \).

**Example 4.2.** Consider the situation of Example 3.1 with four kinematic agents on the plane.

The velocities are discretised by simple finite-difference expressions, i.e., \( \dot{q}_i = \frac{q_{i,k+1} - q_i}{h} \) for \( t \in [t_k, t_{k+1}] \). The discrete Lagrangian \( L^d : (\mathbb{R}^3)^4 \to \mathbb{R} \) is given by setting the trapezoidal discretization for the Lagrangian \( L = \sum_{i=1}^{N} L_i(q_i, \dot{q}_i) \),

\[
L^d(q_k^d, q_{k+1}^d) = \frac{h}{2} L_i \left( \frac{q_{k+1}^d - q_k^d}{h} + \frac{q_k^d - q_{k-1}^d}{h} \right)
\]

that is,

\[
L^d(q_k^d, q_{k+1}^d) = \frac{1}{2h} (q_{k+1}^d - q_k^d)^2 - \frac{h}{4} \sum_{j \neq k} (V_{ij}^d(q_k^d, q_j^d) + V_{ji}^d(q_{k+1}^d, q_{j+1}^d)),
\]

where, \( h \) is the fixed time step and \( q_k = (q_{x_k}^1, q_{y_k}^2, q_{k}^3) \), with \( q_k = (x_k^i, y_k^i) \). The discrete potential functions \( V_{ij}^d \) are given by

\[
V_{ij}^d(q_k^d, q_j^d) = \frac{1}{4} ((x_{k}^i - x_{j}^i)^2 + (y_{k}^i - y_{j}^i)^2 - d_{ij}^2)^2.
\]

Note that in the first term of the trapezoidal rule, the discretization chosen corresponds to a forward finite-difference and in the second term to a backward finite-difference.

The external forces \( F_i(q_i, \dot{q}_i) = -\kappa \dot{q}_i \) are discretized also by using the trapezoidal discretization,

\[
F_{i,d}^d(q_k^d, q_{k+1}^d) = \frac{h}{2} F_i \left( q_k^d, \frac{q_{k+1}^d - q_k^d}{h} + \frac{q_k^d - q_{k-1}^d}{h} \right)
\]

that is \( F_{i,d}^d(q_k^d, q_{k+1}^d) = -\kappa (q_{k+1}^d - q_k^d) - F_{i,d}^d(q_k^d, q_{k-1}^d) = -\kappa (q_k^d - q_{k-1}^d) \).

Using that

\[
D_1L^d(q_k^d, q_{k+1}^d) = -\frac{1}{h} (q_{k+1}^d - q_k^d) - \frac{h}{4} \sum_{j \neq k} \frac{\partial V_{ij}^d(q_k^d, q_j^d)}{\partial q_j^d},
\]

\[
D_2L^d(q_k^d, q_{k+1}^d) = \frac{1}{h} (q_{k+1}^d - q_k^d) - \frac{h}{4} \sum_{j \neq k} \frac{\partial V_{ij}^d(q_k^d, q_j^d)}{\partial q_k^d},
\]

\[
\frac{\partial V_{ij}^d}{\partial x_k^i}(x_k^i, x_{j}^i, y_k^i, y_{j}^i) = \Gamma_{ij}^x(x_k^i - x_{j}^i),
\]

\[
\frac{\partial V_{ij}^d}{\partial y_k^i}(x_k^i, x_{j}^i, y_k^i, y_{j}^i) = \Gamma_{ij}^y(y_k^i - y_{j}^i),
\]

where \( \Gamma_{ij}^x = \langle (x_k^i - x_{j}^i)^2 + (y_k^i - y_{j}^i)^2 - d_{ij}^2 \rangle \), the forced discrete Euler Lagrange equations are given by

\[
q_{k+1}^d = \kappa q_k^d + \frac{2}{1 + kh} q_k^d + \kappa \sum_{j \neq k} \Gamma_{ij}^d (q_k^d - q_j^d)
\]

with \( \kappa = \frac{k h - 1}{1 + k h} \), \( \kappa \), \( \Gamma_{ij}^d(q_k^d - q_j^d) \), for \( k = 1, \ldots, N - 1 \) and \( i = 1, 2, 3, 4 \), that is.

\[
\begin{align*}
q_{k+1}^1 &= G_{k-1}^1 - \kappa h \left( \Gamma_{12}^d(q_k^1 - q_2^1) + \Gamma_{13}^d(q_k^1 - q_3^1) + \Gamma_{14}^d(q_k^1 - q_4^1) \right), \\
q_{k+1}^2 &= G_{k-1}^2 - \kappa h \left( \Gamma_{21}^d(q_k^2 - q_1^2) + \Gamma_{23}^d(q_k^2 - q_3^2) \right), \\
q_{k+1}^3 &= G_{k-1}^3 - \kappa h \left( \Gamma_{31}^d(q_k^3 - q_1^3) + \Gamma_{32}^d(q_k^3 - q_2^3) \right), \\
q_{k+1}^4 &= G_{k-1}^4 - \kappa h \left( \Gamma_{41}^d(q_k^4 - q_1^4) + \Gamma_{43}^d(q_k^4 - q_3^4) \right).
\end{align*}
\]

(14)

where \( G_{k-1} = G(q_{k-1}^d, q_k^d) = \kappa_0 q_k^d + \frac{2}{1 + k h} q_k^d \).

Note that the previous equations are a set of \( 8(N - 1) \) for the \( 8(N + 1) \) unknowns \( \{x_k^i\}_{k=0}^N \) and \( \{y_k^i\}_{k=0}^N \) with \( i = 1, 2, 3, 4 \). Nevertheless the boundary conditions on initial positions and velocities of the agents \( (x_0^i, y_0^i) = (x_i(0), y_i(0)) \), \( (v_{0i}^x, v_{0i}^y) = (x_i(0), y_i(0)) \) contribute sixteen extra equations that convert eq. (13) in a nonlinear root finding problem of \( 8(N - 1) \) and the same amount of unknowns.

Equations (13) define the integration scheme by means of discrete flow \( Y_j : (\mathbb{R}^2 \times \mathbb{R}^2)^4 \to (\mathbb{R}^2 \times \mathbb{R}^2)^4 \) by

\[
Y_{j,d}^d(q_k^d, q_{k+1}^d) = (q_k^d, q_{k+1}^d),
\]

with \( q_k = (q_k^1, q_k^2, q_k^3, q_k^4) \), with \( q_k = (x_k^i, y_k^i) \).

To start the algorithm we use the boundary conditions for the first two steps, that is,

\[
x_0^i = x_i(0), \quad y_0^i = y_i(0), \quad x_1^i = h v_{x0}^i + x_0^i = h x_i(0) + x_i(0), \quad y_1^i = h v_{y0}^i + y_0^i = h y_i(0) + y_i(0),
\]

where we used that \( v_{0i}^x = \frac{q_{1i}^d - q_{0i}^d}{h} \), \( v_{0i}^y = (v_{0i}^x, v_{0i}^y) \). Note that the Lagrangians \( L_i \) are regular because the Hessian matrix with respect to the velocities is the identity. The total energy of each agent \( E_i : \mathcal{T} \mathcal{Q} \to \mathbb{R} \) is given by

\[
E_i(q_i, \dot{q}_i) = \frac{1}{2} ||\dot{q}_i||^2 + \frac{1}{2} \sum_{j \neq i} V_{ij}(q_i, q_j).
\]

Using the trapezoidal rule and \( E_i \), the discrete energy function \( E_i^d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \) is given by

\[
E_i^d(q_k^d, q_{k+1}^d) = \frac{1}{2h} ((x_{k+1}^i - x_k^i)^2 + (y_{k+1}^i - y_k^i)^2) + \frac{h}{4} \sum_{j \neq k} ((x_k^i - x_j^i)^2 + (y_k^i - y_j^i)^2 - d_{ij}^2)^2.
\]

**5. NUMERICAL EXPERIMENTS AND CONCLUSIONS**

In this section we compare the performance of the discretization in (14) and the Euler discretization of (7) since both methods are similar in terms of computational cost per time step. Indeed, other methods like Runge-Kutta can give excellent results in terms of accuracy. However, one needs to evaluate the differential equation (7) several times per discrete step depending on the desired accuracy, hence increasing the computational cost. We continue the Examples 3.1 and 4.3 by choosing a square as desired shape that is defined by \( d_{12} = d_{23} = d_{34} = 10 \) and \( d_{13} = 10 \sqrt{2} \). We set \( \kappa = 5 \) for the dissipating forces. We arbitrarily choose the following initial positions \( q_0 = [5.03 -6.56 2.02 2.22 10.91 -1.53 -2.33 12.28] \), and we set the initial velocities to zero.

While the Euler method starts to be stable, i.e., the solution does not diverge to infinity, at \( h = 0.005 \) seconds, it presents a
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