Understanding Quantum Theory in Terms of Geometry

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Abstract

Understanding quantum theory in terms of a geometric picture sounds great. There are different approaches to this idea. Here we shall
present a geometric picture of quantum theory using the de-Broglie–Bohm causal interpretation of quantum mechanics. We shall show that it is possible to understand the key character of de-Broglie–Bohm theory, the quantum potential, as the conformal degree of freedom of the space–time metric. In this way, gravity should give the causal structure of the space–time, while quantum phenomena determines the scale. Some toy models in terms of tensor and scalar–tensor theories will be presented. Then a few essential physical aspects of the idea including the effect on the black holes, the initial Big–Bang singularity and nonlocality are investigated. We shall formulate a quantum equivalence principle according to which gravitational effects can be removed by going to a freely falling frame while quantum effects can be eliminated by choosing an appropriate scale. And we shall see that the best framework for both quantum and gravity is Weyl geometry. Then we shall show how one can get the de-Broglie–Bohm quantum theory out of a Weyl covariant theory. Extension to the case of many particle systems and spinning particles is discussed at the end.
1 Introduction and survey

In this century, physicists have been departed from 19th century physics, in two ways. The first was the generalization and bringing the old idea of frame independence or general covariance, in a manifest form. The result of this effort was the pioneer general relativity theory, in which the gravitational effects of matter are identified with the geometry of the space–time. The enigmatic character of this theory is just the above-mentioned property, i.e. the interconnection of gravity and general covariance. When one tries to make a general covariant theory, one is forced to include gravity.

The main root of this interconnection is the equivalence principle. According to the equivalence principle, it is possible to go to a frame in which gravity is locally absent, and thus the special theory of relativity is applicable locally. Now using the general covariance and writing down anything in a general frame, we will get the general relativity theory.\[1]\]

The second was the investigation of the quantal behavior of matter, that leads to the quantum theory, according to which a great revolution appeared in physics. In order to explain the atomic world, the quantum theory threw out two essential classical concepts, the principle of causality and the dogma of formulation of physics in terms of motion in space–time (motion dogma).
The first one is violated during a measurement process, while the second does not exist at any time.

After the appearance of quantum mechanics, it was proven that not only do the ordinary particles show quantal behavior but mediators of the fundamental forces also do so. In this way quantum electrodynamics, quantum chromodynamics and quantum flavor dynamics were born. But the construction of quantum gravitodynamics or quantum gravity, and its application to cosmology, is considerably very problematic\textsuperscript{[2]}. These difficulties may be mainly divided into two categories. Some of them are related to the conceptual problems of the standard quantum mechanics, while others are specific to gravity, and are in fact related to the classical features of gravity theory. The first category includes the measurement problem and the meaning of the wave function of the universe, while the vanishing of the hamiltonian which leads to the time independence of the wave function and nonrenormalizability, belong to the second category.

From a fundamental physical viewpoint, in contrast to the general theory of relativity which is the best theory for gravity, standard quantum mechanics is not the only satisfactory way of understanding the quantal behavior of matter. One of the best theories explaining the quantal behavior of matter but
remaining faithful to the principle of causality and the motion dogma, is the de 
Broglie–Bohm quantum theory. According to this theory, all the enigmatic 
quantal behavior of the matter results from a self-interaction of the particle. 
In fact, any particle which exerts a quantum force on itself can be expressed 
in terms of a quantum potential and which is derived from the particle wave 
function.

The celebrated property of the de Broglie–Bohm quantum theory is the fol-
lowing property. At anytime, even when a measurement is done, the particle is 
on the trajectory given by Newton’s law of motion, including the quantum force. 
During a measurement, the system is in fact a many-body system (including 
the particle itself, the probe particle, and the registering system particles). 
When one writes down the appropriate equation of motion of all the particles, and when one considers the very fact that we know nothing about the initial 
conditions of the registering system particles, one sees how the projection 
postulate of quantum mechanics came about. Accordingly the result of any 
measurement is one of the eigenvalues of the operator related to the measured 
quantity with some calculable probability distribution. The de-Broglie–Bohm 
quantum theory of motion, is a causal theory which although behaves as the

\[1\] There is a consistent de Broglie–Bohm quantum theory for a many particle system
Copenhagen quantum mechanics at the statistical level, it has non of the con-
ceptual difficulties of the standard quantum mechanics. It is well proved that
the causal theory reproduces all the results of the orthodox quantum theory[3],
as well as predicting some new results (such as time of tunneling through a
barrier[4]) which in principle lets the experiment to choose between the or-
thodox and the causal quantum theories. Perhaps the most important point
about the causal theory is that it presents a causal deterministic description
of the reality. So it looks very natural to make a quantum theory of gravity in
the spirit of the de-Broglie–Bohm quantum theory of motion. In the standard
form of this theory, the classical gravity should be viewed as a field. Then it is
possible to construct Bohmian metric trajectories. This is what is essentially
done in [5].

A way of struggling with quantum gravity is to use the minisuperspace of
the conformal degree of freedom of the space-time metric[6]. This approach
has several fruitfull results. It admits non-perturbative calculations, and it is
very useful for studying quantum cosmology. Because the isotropic and ho-
mogeneous space-time used in cosmology is conformally flat. In addition, by
including the effects of the back–reactions of the quantum variable (i.e. the
conformal factor) on the background metric, one arrives at some extended form
of Einstein’s equations. These semi-classical equations lead to non–singular cosmological solutions and they have the correct classical limit. In this approach, by merely quantizing one degree of freedom of the space-time metric, and by considering the back–reaction effects, the time independence problem is solved. This is achieved because of the extension of Einstein’s equations. But it must be noted that the physical meaning of the quantum variable, i.e. the conformal factor, is not clear in this approach. The non-singularity of the results of the above approach rests on the theorem[7] which states that for any singular metric, there is some appropriate conformal factor, in such a way that conformal metric is non-singular.

The present work tries to combine the de-Broglie–Bohm quantum theory of motion and gravity in a very different way. The foundation of this approach is the de-Broglie remark[8] that the quantum theory of motion for relativistic spinless particles is very similar to the classical theory of motion in a conformally flat space-time. The conformal factor is related to the Bohm’s quantum potential. We shall present a generalization and an appropriate formulation of this remark. That is to say, we geometrize Bohmian mechanics according to the de-Broglie remark. Then, it can be seen that the effects of gravity on geometry and the quantum effects on the geometry of the space-time are highly
coupled. In fact there are two contributions to the background metric the gravitational quantal effects of matter which constitute the energy–momentum tensor. Since in the evaluation of the quantal part the background metric is used, the gravitational and quantal contributions to the background metric are so highly coupled that no one without the other has any physical significance.

It must be pointed out here that as a result the conformal factor is meaningless as the ensemble density goes to zero and the geometry loses its meaning at this limit. This is a desired property, because it is in accord with Mach’s principle, which states that for an empty universe the space–time should be meaningless. In subsection 3.1 the authors, as a first step towards the formulation of the above conclusion, introduced the quantum conformal degree of freedom via the method of Lagrange multipliers. In this way there are a set of equations of motion describing the background metric, the conformal degree of freedom and the particle trajectory. A corollary of this theory is that one can always work in a gauge (classic gauge) in which no quantum effect be present or in a gauge (quantum gauge) in which the conformal degree of freedom of the space–time metric is identified with the quantum effect. This, in its turn, leads to dramatic departures from the classical prediction, when both the effects of gravity and quantum on geometry are considerable, i.e. around those areas of
the space-time which are singular according to the classical theory.

As a different approach in ref [10] the authors symmetrized the Brans–Dicke theory by a conformal transformation. And arrive to a particle interpretation suggesting that the quantum aspect of matter can be geometrized.

In [11], the conformal transformation was applied only to the space–time metric. Other quantities like mass, density and so on were assumed to possess no transformation. This is because the above conformal transformation which incorporates the quantum effects of matter into a specific conformal factor, is in fact a scale transformation. As the conformal transformation is more general than scale transformation which is used in [9], it seems preferable to make a conformal transformation, in which all physical quantities are transformed, instead of making only a scale transformation. In reference [11], it is shown that by the conformal transformation the equation of motion would be transformed to an equation in which there is no quantum effects. As a result, the geodesic equation would be changed to the one without the quantum force. This means that it is possible to have two identical pictures for investigating the quantal effects of matter in the curved space–time background. According to the first picture, the space–time metric contains only the gravitational effects of matter. The quantum effects affect the path of the particles via
the quantum force. In the second picture, the space–time metric is related to
the previous by a conformal factor and contains the gravitational and quantal
effects of matter.

This shows that the quantum as well as the gravitational effects of matter
have geometrical nature. The second picture mentioned above provides a uni-
fied geometrical framework for understanding the gravitational and quantum
forces. Accordingly, we call the conformal metric as the physical metric (con-
taining both gravity and quantum) and the other metric is the background
metric (including only gravity).

The above-mentioned theory,[9] has a problem. In this theory, it is assumed
that one deals with an ensemble of similar particles with density. In Bohm’s
theory, the quantum potential exists for a single particle as well as for an
ensemble. In the case of a single particle, the interpretation of the quantum
potential is in terms of an hypothetical ensemble. Note that in the above
theories, the ensemble is a real one, not an hypothetical one, because, the
energy–momentum tensor of the ensemble is appeared and has physical effects.
As we shall show in subsection (3.2)[12], we have solved this problem and the
theory would work both for a single particle and for an ensemble.

In subsection (3.2) we shall show that it is possible to make a pure tensor
theory for quantum gravity. As a result we shall show that the correct quantum conformal degree of freedom would be achieved, and that the theory works for a particle as well as for a real ensemble of the particle under consideration and that it includes the pure quantum gravity effects. We shall do all of these by trying to write the quantum potential terms in terms of geometrical parameters, not in terms of ensemble properties.

The important point about both references[9] and [11] is that in order to fix the relation of the conformal degree of freedom of the space–time metric and the quantum potential, the method of lagrange multiplier is used and in this way they are a little artificial. In subsection (4.1) we shall show[13] that in the framework of the scalar–tensor theories, it is possible to write an action principle, in which both gravitational and quantum contributions to the geometry are included and that the conformal degree of freedom of the space–time metric is fixed at the level of the equations of motion not needing the method of lagrange multiplier.

Next in subsection (4.2)[14] we attend to the double scalar case because in some theories such as superstring and Kaluza–Klein, it is more useful[15]. In both of these theories the gravitational interaction includes two other fields in addition to the metric field. In string frame(or Jordan frame)one of them is
coupled nonminimally to gravity as in the Brans–Dicke theory and the other is coupled minimally to gravity, but has a nontrivial coupling with the first scalar field. Note that, in these theories one can couple both the scalar fields minimally to gravity by a conformal transformation (Einstein frame). As a result, the question that the physical interpretation must be presented in which frame, is an open problem\cite{16}. On the other hand we shall show that using two scalar fields, one can relax this preassumption and on the equations of motion, the correct form of quantum potential will be achieved.

In subsections (5.1), (5.2) and (5.3) some general solutions are obtained. using these, the important question that if this quantum gravity theory leads to some new results, is investigated. That solutions are used for black holes and bigbang in subsections (5.4) and (5.5)\cite{12}. In subsection (5.6)\cite{17} we are interested in investigating whether this theory has anything to do with the cluster formation or clustering of the initial uniform distribution of matter in the universe. The problem of cluster formation is an important problem of cosmology and there are several ways to tackle with it\cite{18}. Here we don’t want to discuss those theories, and our claim is not that the present work is a good one. Here we only state that the cluster formation can also be understood in this way. It is a further task to decide if this work is in complete agreement
with experiment or not.

A special aspect of the quantum force is that it is highly nonlocal. This property, is an experimental matter of fact [19]. Since the mass field represents the conformal degree of freedom of the physical metric, quantum gravity is expected to be highly nonlocal. In the subsection (5.7) [20] this is shown explicitly for a specific problem.

From a different point of view it has been believed for a long time that the long range forces (i.e. electromagnetism and gravity) are different aspects of a unique phenomena. So they must be unified. Usually it is proposed that one must generalize Einstein’s general relativity theory to have a geometrical description of electromagnetic fields. This means to change the properties of the manifold of general relativity. Using higher dimensional manifolds[21], changing the compatibility relation between the metric and the affine connection[22] and using a non-symmetric metric[23] are some examples of the attempts towards this idea. In all the above approaches, the additional degrees of freedom correspond to the components of the electromagnetic potential. The second idea leads to the Weyl’s gauge invariant geometry. Apart from the electromagnetic aspects of Weyl geometry, it has some other applications. Some authors believe that Weyl geometry is a suitable framework for quantum gravity. E.g.
in a series of papers [24] a successful approach to Weyl quantum gravity and conformal sector in quantum gravity is presented. The authors have used an effective theory based on integrated conformal anomaly dynamics, in the infrared region. They also have considered a sigma model action which is the most general version of a renormalizable theory in four dimensions. They have investigated the phase structure and the infrared properties of conformal quantum gravity and then extend its results to higher derivative quantum gravity.

Also in ref[25] a new quantum theory is proposed on the basis of Weyl picture which is purely geometric. The observables are introduced as zero Weyl weight quantities. Moreover any weightful field has a Weyl conjugate such as complex conjugate of the state vector in quantum mechanics. By these dual fields, the probability can be defined. These are the elements of a consistent quantum theory which is equivalent to the standard quantum mechanics. Moreover it is shown that the quantum measurement and the related uncertainty would emerged from Weyl geometry naturally. In this theory when the curl of Weyl vector is zero, we arrive at the classical limit. By noting the transformation relation of Weyl vector, it is concluded that the change of length scale is only a quantum effect.
One more approach to geometrize quantum mechanics can be found in [26]. Here a modified Weyl–Dirac theory is used to join the particle aspects of matter and Weyl symmetry breaking. This is also a geometrization of quantum mechanics. Also one can find the relation of quantum potential, the basic character of Bohm’s theory, to the fundamental geometric properties, especially to the curvature of the space-time using Weyl geometry in [27]. Furthermore in [28] Sidharth considers the geometrical interpretation of quantum mechanics from the point of view of non-commutative non-integrable geometry.

In the present work we shall look at the conformal invariance at the quantum level. Does the quantum theory lead us to any characteristic length scale and thus break the conformal symmetry? Or conversely the quantum effects lead us to a conformal invariant geometry? In section (6) we shall discuss these questions in the context of the causal quantum theory proposed by Bohm [3] and use our new way of geometrization of quantum mechanics introduced in here. We emphasize that what we shall show that our specific geometrization of quantum mechanics procedure (based on Bohmian quantum mechanics) can be better understood in the Weyl framework. This is different from Weyl quantum gravity approaches like those of [24].

We shall show that the Weyl vector and the quantum effects of matter are
connected. We shall see how the conformal symmetry emerges naturally by considering quantum effects of matter. Finally in section (7) we show that the Weyl–Dirac theory is a suitable framework for identification of the conformal degree of freedom of the space–time with the Bohm’s quantum mass.

From a similar perspective, Quiros and et all [29] discuss the space–time singularity by the geometrical dual representation in general relativity. On this basis they emphasize on the Weyl integrable geometry as a consistent framework to describe the gravitational field.

Finally in section (8) we shall investigate possible extension of our results in two ways. First analyzing the case of many–particle systems and second, inclusion of spin.

2 The geometric nature of quantum potential

2.1 Non–relativistic de-Broglie–Bohm theory

The de-Broglie–Bohm quantum theory of motion[3] is a causal theory which although agrees with the Copenhagen quantum mechanics at the statistical level, it is able to determine the exact path of a particle. In this way it predicts all the physical quantities of a particle, deterministically. This theory
does not contain such difficulties as the reduction of the wave function, and so on [3]. Therefore, it seems more appropriate that in building a quantum theory in the presence of gravity, to use the de-Broglie–Bohm theory rather than the Copenhagen quantum mechanics. Because in this case some of the conceptual problems of the standard quantum mechanics appear more clearly [30].

Now, we make a glance at the de-Broglie–Bohm quantum theory of motion. It contains three postulates [3]. The first one states that for any particle, there is an objectively real field \( \psi(\vec{x}, t) \) which in the non-relativistic domain satisfies the Schrödinger equation. The second postulate presents the effect of the field on the particle. According to this postulate the linear momentum of the particle is given by the so-called guidance formula:

\[
\vec{p} = \vec{\nabla}(\hbar \times \text{phase of } \psi) \tag{1}
\]

One can show that the particle experiences the force

\[
\vec{F} = -\vec{\nabla}Q \tag{2}
\]

from the field \( \psi \), where \( Q \) is the quantum potential given by:

\[
Q = -\frac{\hbar^2}{2m} \frac{\nabla^2 |\psi|}{|\psi|} \tag{3}
\]

Finally the third postulate states that at the statistical level we have:

\[
\rho = \psi^* \psi = |\psi|^2 \tag{4}
\]
where $\rho$ is the ensemble density of particles.

A simple way to prove (2), is to make the canonical transformation $\psi = \sqrt{\rho} \exp(iS/\hbar)$ in the action for the Schrödinger equation. The equations of motion of $\rho$ and $S$ would be:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left( \rho \frac{\nabla S}{m} \right) = 0$$ (5)

$$\frac{\partial S}{\partial t} + \frac{|\nabla S|^2}{2m} + V + Q = 0$$ (6)

in which $V$ is the classical potential and $S$ is the Hamilton–Jacobi function. Equation (5) is the continuity equation provided (1) is satisfied. Under this condition the quantum Hamilton-Jacobi equation (6) is identical to (2).

### 2.2 Relativistic de-Broglie–Bohm theory

Extension of de-Broglie–Bohm causal theory of quantum phenomena to the relativistic case is a problematic matter. Essentially all the problems of the Copenhagen relativistic quantum mechanics can in principle be present in the de-Broglie–Bohm theory. There are at least three problems with Copenhagen relativistic quantum mechanics. They are:

- The problem of negative energy and space–like current densities for integer spins.
• The problem of defining probability distribution for many–particle systems.

• The conflict between measurement principle (which states that measurement is instantaneous) and the Poincarè transformations.

We shall discuss about the second problem at the end of this paper. The third one is essentially not a problem in de-Broglie–Bohm theory, provided the second problem is solved. This is because of the fact that measurement is not an instantaneous phenomena in this theory, it is a many–particle situation.

In a recent paper[31] we have shown that the first problem is not present in de-Broglie–Bohm theory, provided one sets the natural constraint that the theory should lead the correct non–relativistic limit. Let us make this point clear. Usually one gets a de-Broglie–Bohm version of a Copenhagen theory by writing the wave function in its polar form $\psi = |\psi| \exp(iS/\hbar)$ and decomposing the real and imaginary parts of the wave equation. Doing this with the Klein–Gordon equation leads to a quantum Hamilton–Jacobi equation:

$$\partial_{\mu}S \partial^{\mu}S = m^2 c^2 (1 + Q)$$

with the quantum potential defined as:

$$Q = \frac{\hbar^2}{m^2 c^2} \Box |\psi|$$
and the continuity equation:

$$\partial_\mu (\rho \partial^\mu S) = 0$$  \hspace{1cm} (9)

The above Hamilton–Jacobi equation (7) shows that in the relativistic case
the quantum potential is essentially the mass square. So one can define the
quantum mass of a particle as:

$$M^2 = m^2 (1 + Q)$$  \hspace{1cm} (10)

Since the quantum potential can be a negative number, in general the tachyonic
solutions would emerge. This is essentially related to the first problem noted
above. Although it can be shown that a non–tachyonic initial condition leads
to a global (in time) non–tachyonic solution\cite{S}, but the existence of tachyonic
solutions is a fatal problem.

It can be shown that the problem is that equation (7) is not the correct
relativistic equation of motion\cite{B1}. A correct relativistic quantum equation
of motion should not only be poincarè invariant but also has the correct non–
relativistic limit. In \cite{B1} we have shown that using these requirements one gets
the correct equation of motion as:

$$\partial_\mu S \partial^\mu S = M^2 c^2$$  \hspace{1cm} (11)
with

\[ \mathcal{M}^2 = m^2 \exp(Q) \]  

(12)

does this clearly is free from the above mentioned problem.

### 2.3 de-Broglie–Bohm theory in curved space–time

The extension to the case of a particle moving in a curved background is not very difficult. This can be done through the same way as writing any special relativistic relation in a general relativistic form. One should only change the ordinary differentiating \( \partial_\mu \) with the covariant derivative \( \nabla_\mu \) and change the Lorentz metric \( \eta_{\mu\nu} \) to the curved metric \( g_{\mu\nu} \).

Therefore the equations of motion for a particle (of spin zero) in a curved background are:

\[ \nabla_\mu (\rho \nabla^\mu S) = 0 \]  

(13)

\[ g^{\mu\nu} \nabla_\mu S \nabla_\nu S = \mathcal{M}^2 c^2 \]  

(14)

where

\[ \mathcal{M}^2 = m^2 \exp(Q) \]  

(15)

\[ Q = \frac{\hbar^2}{m^2 c^2} \frac{\Box_g |\psi|}{|\psi|} \]  

(16)
de-Broglie made the following interesting and fruitfull observation\cite{8}: The quantum Hamilton-Jacobi equation (14) can be written as:

$$m^2 \mathcal{M}^2 g_{\mu\nu} \nabla_\mu S \nabla_\nu S = m^2 c^2$$

(17)

From this relation it can be concluded that the quantum effects are identical with the change of the space-time metric from $g_{\mu\nu}$ to:

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \mathcal{M}^2 m^2 \Big( g_{\mu\nu} \Big)$$

(18)

which is a conformal transformation.

Therefore equation (17) can be written as:

$$\tilde{g}^{\mu\nu} \tilde{\nabla}_\mu S \tilde{\nabla}_\nu S = m^2 c^2$$

(19)

where $\tilde{\nabla}_\mu$ represents the covariant differentiation with respect to the metric $\tilde{g}_{\mu\nu}$. In this new curved space-time the other equation of motion, i.e. the continuity relation should be written as:

$$\tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \left( \rho \tilde{\nabla}_\nu S \right) = 0$$

(20)

The important conclusion we draw from this argumentation is that the presence of the quantum potential is equivalent to a curved space-time with its metric being given by (18). So in fact, we have the geometrization of the
quantum aspects of matter. In this way, it seems that there is a dual aspect to the role of geometry in physics. The space-time geometry sometimes looks like what we call gravity and sometimes looks like what we understand as quantal behaviours. Since the equations governing the space-time geometry are highly non-linear, the curvature due to the quantum potential may have a large influence on the classical contribution to the curvature of the space-time. This would be investigated in the following sections.

The particle trajectory can be derived from the guidance relation and by differentiating \((14)\) leading to Newton’s equation of motion:

\[
\mathcal{M} \frac{d^2 x^\mu}{d\tau^2} + \mathcal{M} \Gamma^\mu_{\nu\kappa} u^\nu u^\kappa = (c^2 g^{\mu\nu} - u^\mu u^\nu) \nabla_\nu \mathcal{M}.
\]  

(21)

using the above conformal transformation, Eq.\((21)\) reduces to the standard geodesic equation via the above conformal transformation.
3 A tensor model of the idea

3.1 The case of an ensemble of particles

A general relativistic system consisting of gravity and classical matter (relativistic particles without quantum effects) is determined by the action:

\[ A_{\text{no-quantum}} = \frac{1}{2\kappa} \int d^4 x \sqrt{-g} R + \int d^4 x \sqrt{-g} \frac{\hbar^2}{m} \left( \frac{\rho}{\hbar^2} \nabla_{\mu} S \nabla^{\mu} S - \frac{m^2}{\hbar^2 \rho} \right) \]  (22)

where \( \kappa = 8\pi G \) and hereafter we chose the units in which \( c = 1 \).

On the other hand as it was seen in the previous section, the de-Broglie remark leads to the conclusion that introducing the quantum potential, is equivalent to the introduction of a conformal factor \( \Omega^2 = \mathcal{M}^2/m^2 \) in the metric. So in order to introduce the rather than quantum effects of matter into the action (22), we make the aforementioned conformal transformation, instead of adding the quantum potential term.

Accordingly, we write our action with quantum effects as:

\[ A[\bar{g}_{\mu\nu}, \Omega, S, \rho, \lambda] = \frac{1}{2\kappa} \int d^4 x \sqrt{-\bar{g}} \left( \mathcal{R} \Omega^2 - 6 \nabla_{\mu} \Omega \nabla^{\mu} \Omega \right) \]

\[ + \int d^4 x \sqrt{-\bar{g}} \left( \frac{\rho}{m} \Omega^2 \nabla_{\mu} S \nabla^{\mu} S - m \rho \Omega^4 \right) + \int d^4 x \sqrt{-\bar{g}} \lambda \left[ \Omega^2 - \left( 1 + \frac{\hbar^2 \Box \sqrt{\rho}}{m^2 \sqrt{\rho}} \right) \right] \]  (23)

26
where a bar over any quantity means that it corresponds to no–quantum regime. Here we used only the first two terms of expansion of equation (15) to kip things simple. No physical change emerges considering all terms. In the above action, $\lambda$ is a Lagrange multiplier which is introduced to identify the conformal factor with its Bohmian value.

Here two problems must be noted. First, in the above action, we use $\tilde{g}_{\mu\nu}$ to raise or lower indices and to evaluate the covariant derivatives. Second, the physical metric (i.e. the metric containing the quantum effects of matter) is $g_{\mu\nu}$ given by $\Omega^2 \tilde{g}_{\mu\nu}$.

By the variation of the above action with respect to $\tilde{g}_{\mu\nu}$, $\Omega$, $\rho$, $S$ and $\lambda$ we arrive at the following relations as our quantum equations of motion:

1. The equation of motion for $\Omega$:

$$\bar{R}\Omega + 6\Box \Omega + \frac{2\kappa}{m} \rho \Omega (\nabla_\mu S \nabla^\mu S - 2m^2 \Omega^2) + 2\kappa \lambda \Omega = 0$$  \hspace{1cm} (24)

2. The continuity equation for particles:

$$\nabla_\mu (\rho \Omega^2 \nabla^\mu S) = 0$$  \hspace{1cm} (25)

3. The equation of motion for particles:

$$\left(\nabla_\mu S \nabla^\mu S - m^2 \Omega^2\right) \Omega^2 \sqrt{\rho} + \frac{\hbar^2}{2m} \left[\Box \left(\frac{\lambda}{\sqrt{\rho}}\right) - \lambda \frac{\Box \sqrt{\rho}}{\rho}\right] = 0$$  \hspace{1cm} (26)
4. The modified Einstein equations for $\tilde{g}_{\mu\nu}$:

$$
\Omega^2 \left[ \mathcal{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \mathcal{R} \right] - \left[ \tilde{g}_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right] \Omega^2 - 6 \nabla_\mu \Omega \nabla_\nu \Omega + 3 \tilde{g}_{\mu\nu} \nabla_\alpha \Omega \nabla^\alpha \Omega \\
+ \frac{2\kappa}{m} \rho \Omega^2 \nabla_\mu S \nabla_\nu S - \frac{\kappa}{m} \rho \Omega^2 \tilde{g}_{\mu\nu} \nabla_\alpha S \nabla^\alpha S + \kappa m \rho \Omega^4 \tilde{g}_{\mu\nu}
$$

$$
+ \frac{\kappa \hbar^2}{m^2} \left[ \nabla_\mu \sqrt{\rho} \nabla_\nu \left( \frac{\lambda}{\sqrt{\rho}} \right) + \nabla_\nu \sqrt{\rho} \nabla_\mu \left( \frac{\lambda}{\sqrt{\rho}} \right) \right] - \frac{\kappa \hbar^2}{m^2} \tilde{g}_{\mu\nu} \nabla_\alpha \left[ \frac{\nabla^\alpha \sqrt{\rho}}{\sqrt{\rho}} \right] = 0
$$

(27)

5. The constraint equation:

$$
\Omega^2 = 1 + \frac{\hbar^2}{m^2} \frac{\Box \sqrt{\rho}}{\sqrt{\rho}}
$$

(28)

As it is seen, the back–reaction effects of the quantum factor on the background metric are contained in those highly coupled equations. It may be noted that by combining (24) and (27) it is possible to arrive at a more simple relation instead of (24). If we take the trace of (27) and use (24), we have after some mathematical manipulations:

$$
\lambda = \frac{\hbar^2}{m^2} \nabla_\mu \left[ \frac{\nabla^\mu \sqrt{\rho}}{\sqrt{\rho}} \right]
$$

(29)

Before proceeding, some important point about this relation must be noted. If one tries to solve it via perturbation method, in terms of the small parameter:

$$
\alpha = \frac{\hbar^2}{m^2}
$$

(30)
by writing:

$$\lambda = \lambda^{(0)} + \alpha \lambda^{(1)} + \alpha^2 \lambda^{(2)} + \cdots$$ \hspace{1cm} (31)

and

$$\sqrt{\rho} = \sqrt{\rho^{(0)}} + \alpha \sqrt{\rho^{(1)}} + \alpha^2 \sqrt{\rho^{(2)}} + \cdots$$ \hspace{1cm} (32)

one gets:

$$\lambda^{(0)} = \lambda^{(1)} = \lambda^{(2)} = \cdots = 0$$ \hspace{1cm} (33)

So the perturbative solution of (29) is $\lambda = 0$ which is its trivial solution.

Therefore, our equations are:

$$\nabla_\mu \left( \rho \Omega^2 \nabla^{\mu} S \right) = 0$$ \hspace{1cm} (34)

$$\nabla_\mu S \nabla^{\mu} S = m^2 \Omega^2$$ \hspace{1cm} (35)

$$G_{\mu\nu} = -\kappa T^{(m)}_{\mu\nu} - \kappa T^{(\Omega)}_{\mu\nu}$$ \hspace{1cm} (36)

where $T^{(m)}_{\mu\nu}$ is the matter energy–momentum tensor and

$$\kappa T^{(\Omega)}_{\mu\nu} = \left[ g_{\mu\nu} \square - \nabla_\mu \nabla_\nu \right] \frac{\Omega^2}{\Omega^2} + 6 \frac{\nabla_\mu \Omega \nabla_\nu \Omega}{\omega^2} - 3 g_{\mu\nu} \frac{\nabla_\alpha \Omega \nabla^\alpha \Omega}{\Omega^2}$$ \hspace{1cm} (37)

and

$$\Omega^2 = 1 + \alpha \frac{\square \sqrt{\rho}}{\sqrt{\rho}}$$ \hspace{1cm} (38)

Note that relation (35) is, in fact, the Bohmian equation of motion, and if we write it in terms of the physical metric $g_{\mu\nu}$, it reads as $\nabla_\mu S \nabla^{\mu} S = m^2 c^2$. This is what we expect from de-Broglie’s conjecture.
3.2 The case of a single particle

In the previous subsection we have assumed that there is a real ensemble of the quantum particle. Now the question is what happens for the case of a single particle? To investigate this, we first examine how we can translate the quantum potential in a complete geometrical manner, i.e. we write it in a form that there is no explicit reference to matter parameters. Only after using the field equations can one deduce the original form of the quantum potential. This has the advantage of allowing our theory to work both for a single particle and an ensemble. Next, we write a special field equation as a toy theory and extract some of its consequences.

3.2.1 Geometry of the quantum conformal factor

Let us first ignore gravity and examine the geometrical properties of the conformal factor given by

\[ g_{\mu\nu} = e^{4\Sigma} \eta_{\mu\nu}, \quad e^{4\Sigma} = \frac{M^2}{m^2} = \exp \left( \alpha \frac{\Box \eta \sqrt{\rho}}{\sqrt{\rho}} \right) = \exp \left( \alpha \frac{\Box \eta \sqrt{|T|}}{\sqrt{|T|}} \right), \quad (39) \]

where \( T \) is the trace of the energy–momentum tensor\(^2\) and is substituted for \( \rho \) (as it is true for dust).

\(^2\)The absolute value sign is introduced to make the square root always meaningful.
Evaluating the Einstein’s tensor for the above metric, we have:

\[ G_{\mu\nu} = 4g_{\mu\nu}e^{\Sigma\Box}e^{-\Sigma} + 2e^{-2\Sigma}\partial_\mu\partial_\nu e^{2\Sigma} \]  

(40)

So as an ansatz, we suppose that in the presence of gravitational effects, the field equation have some form like:

\[ \mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = \kappa T_{\mu\nu} + 4g_{\mu\nu}e^{\Sigma\Box}e^{-\Sigma} + 2e^{-2\Sigma}\nabla_\mu\nabla_\nu e^{2\Sigma}. \]  

(41)

This equation is written in such a way that in the limit \( T_{\mu\nu} \rightarrow 0 \) the solution achieved.

Making the trace of the above equation one gets

\[ -\mathcal{R} = \kappa T - 12\Box\Sigma + 24(\nabla\Sigma)^2 \]  

(42)

which has the iterative solution:

\[ \kappa T = -\mathcal{R} + 12\alpha\Box\left(\frac{\Box\sqrt{\mathcal{R}}}{\sqrt{|\mathcal{T}|}}\right) + \cdots \]  

(43)

leading to

\[ \Sigma = \alpha\Box\sqrt{|T|} \simeq \alpha\frac{\Box\sqrt{|\mathcal{R}|}}{\sqrt{|\mathcal{R}|}} \]  

(44)

up to first order in \( \alpha \). Now we are ready to make a toy model.

### 3.2.2 Field equations of a toy quantum gravity

From the above equation we learn that \( T \) can be replaced with \( \mathcal{R} \) in the expression for the quantum potential or for the conformal factor of the space–
time metric. This replacement is in fact an important improvement, because the explicit reference to ensemble density is removed. This allows the theory to work for both a single particle and an ensemble.

So with a glance at Eq. (41) for our toy quantum–gravity theory, we assume the following field equations:

\[ G_{\mu\nu} - \kappa T_{\mu\nu} - Z_{\mu\nu\alpha\beta} \exp \left[ \frac{\alpha}{2} \Phi \right] \nabla^{\alpha} \nabla^{\beta} \exp \left[ - \frac{\alpha}{2} \Phi \right] = 0, \]  

(45)

where

\[ Z_{\mu\nu\alpha\beta} = 2 \left[ g_{\mu\nu} g_{\alpha\beta} - g_{\mu\alpha} g_{\nu\beta} \right] \]  

(46)

\[ \Phi = \frac{\Box \sqrt{|R|}}{\sqrt{|R|}}. \]  

(47)

Note that the number 2 and the minus sign of the second term of the last equation are chosen so that the energy equation derived later be correct. It would be very useful to take the trace of Eq. (45):

\[ \mathcal{R} + \kappa \mathcal{T} + 6 \exp \left[ \frac{\alpha \Phi}{2} \right] \Box \exp \left[ - \frac{\alpha \Phi}{2} \right] = 0. \]

(48)

In fact this equation represents the connection of Ricci scalar curvature of space–time and the trace of matter energy–momentum tensor. In the cases when a perturbative solution is admitted, i.e. when we can expand anything in terms of powers of \( \alpha \), one can find the relation between \( \mathcal{R} \) and \( \mathcal{T} \) perturbatively.
In the zeroth approximation one has the classical relation:

\[ R^{(0)} = -\kappa T. \]  

(49)

As a better approximation up to first order in \( \alpha \), one gets

\[ R^{(1)} = -\kappa T - 6 \exp \left[ \frac{\alpha \Phi^{(0)}}{2} \right] \Box \exp \left[ -\frac{\alpha \Phi^{(0)}}{2} \right], \]

(50)

where

\[ \Phi^{(0)} = \frac{\Box \sqrt{|\mathcal{T}|}}{\sqrt{|T|}}. \]

(51)

A better result can be obtained in the second order as

\[ R^{(2)} = -\kappa T - 6 \exp \left[ \frac{\alpha \Phi^{(0)}}{2} \right] \Box \exp \left[ -\frac{\alpha \Phi^{(0)}}{2} \right] - 6 \exp \left[ \frac{\alpha \Phi^{(1)}}{2} \right] \Box \exp \left[ -\frac{\alpha \Phi^{(1)}}{2} \right], \]

(52)

with

\[ \Phi^{(1)} = \frac{\Box \sqrt{|-\kappa T - 6 \exp[\alpha \Phi^{(0)}/2] \Box \exp[-\alpha \Phi^{(0)}/2]|}}{\sqrt{|-\kappa T \exp[\alpha \Phi^{(0)}/2] \Box \exp[-\alpha \Phi^{(0)}/2]|}}. \]

(53)

The energy relation can be obtained via taking the four divergence of the field equations. Using the fact that the divergence of Einstein’s tensor is zero, one gets

\[ \kappa \nabla^\nu \mathcal{T}_{\mu \nu} = \alpha R_{\mu \nu} \nabla^\nu \Phi - \frac{\alpha^2}{4} \nabla_\mu (\nabla \Phi)^2 + \frac{\alpha^2}{2} \nabla_\mu \Phi \Box \Phi. \]

(54)

For a dust with

\[ \mathcal{T}_{\mu \nu} = \rho u_\mu u_\nu \]

(55)
where $u_\mu$ is the velocity field. Assuming the conservation law for mass

$$\nabla^\nu (\rho M u_\nu) = 0$$  \hspace{1cm} (56)

up to first order in $\alpha$ one arrives at:

$$\frac{\nabla_\mu M}{M} = -\frac{\alpha}{2} \nabla_\mu \Phi$$  \hspace{1cm} (57)

or

$$M^2 = m^2 \exp(-\alpha \Phi),$$  \hspace{1cm} (58)

where $m$ is some integration constant. This is the correct relation of mass field and the quantum potential.

## 4 A scalar–tensor model of the idea

In the last section, the form of the quantum potential and its relation to the conformal degree of freedom of the space–time metric are assumed. The next step is to remove these assumptions and derive them using the equations of motion. In doing this we first include the conformal factor as a scalar field and then introduce another scalar field (quantum potential in fact). On the equations of motion the correct relation between quantum potential and conformal factor and also the form of the quantum potential would emerge.
4.1 Making the conformal factor dynamical[13]

We start from the most general scalar–tensor action:

$$\mathcal{A} = \int d^4x \left\{ \phi R - \frac{\omega}{\phi} \nabla^\mu \phi \nabla_\mu \phi + 2\Lambda \phi + \mathcal{L}_m \right\}$$  \hspace{1cm} (59)$$

in which $\omega$ is a constant independent of the scalar field, and $\Lambda$ is the cosmological constant. Also, it is assumed that the matter lagrangian is coupled to the scalar field. The equations of motion are:

$$\mathcal{R} + \frac{2\omega}{\phi} \Box \phi - \frac{\omega}{\phi^2} \nabla^\mu \phi \nabla_\mu \phi + 2\Lambda + \frac{\partial \mathcal{L}_m}{\partial \phi} = 0$$  \hspace{1cm} (60)$$

$$G^{\mu\nu} - \Lambda g^{\mu\nu} = -\frac{1}{\phi} T^{\mu\nu} - \frac{1}{\phi} [\nabla^\mu \nabla^\nu - g^{\mu\nu} \Box] \phi + \frac{\omega}{\phi^2} \nabla^\mu \phi \nabla^\nu \phi - \frac{1}{2} \frac{\omega}{\phi^2} g^{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi$$  \hspace{1cm} (61)$$

The scalar curvature can be evaluated from the contracted form of the latter equation, and it can be substituted in the relation (60). Then we have:

$$\frac{2\omega - 3}{\phi} \Box \phi = -\frac{T}{\phi} + 2\Lambda - \frac{\partial \mathcal{L}_m}{\partial \phi}$$  \hspace{1cm} (62)$$

The matter lagrangian for an ensemble of relativistic particles of mass $m$ is (without any quantum contribution):

$$\mathcal{L}_{m\text{(no-quantum)}} = \frac{\rho}{m} \nabla_\mu S \nabla^\mu S - \rho m$$  \hspace{1cm} (63)$$

This lagrangian can be generalized if one assumes that there is some interaction between the scalar field and the matter field. Here, for simplicity, it is
assumed that this interaction is in the form of powers of $\phi$. In order to bring in the quantum effects, one needs to add terms containing the quantum potential. Physical intuition leads us to the fact that it is necessary to assume some interaction between cosmological constant and matter quantum potential. This suggestion will be confirmed after obtaining all of the equations of motion. These arguments lead us to consider the matter lagrangian as:

$$L_m = \frac{\rho}{m} \phi^a \nabla^\mu S \nabla_\mu S - m \rho \phi^b - \Lambda (1 + Q)^c$$ (64)

in which the $a$, $b$, and $c$ constants have to be fixed later. Again we used the first two terms of equation (15) for simplicity. Therefore the energy–momentum tensor is:

$$\mathcal{T}^{\mu\nu} = -\frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \int d^4x \sqrt{-g} L_m = -\frac{1}{2} g^{\mu\nu} L_m + \frac{\rho}{m} \phi^a \nabla^\mu S \nabla_\nu S - \frac{1}{2} \Lambda c Q (1 + Q)^{c-1} g^{\mu\nu}$$

$$- \frac{1}{2} \alpha \Lambda c \nabla_a \sqrt{\rho} \nabla_\beta \left( \frac{(1 + Q)^{c-1}}{\sqrt{\rho}} \right) \left[ g^{\mu\nu} \alpha^\beta - g^{\alpha\mu} g^{\beta\nu} - g^{\beta\mu} g^{\alpha\nu} \right]$$ (65)

Using the matter lagrangian and contracting the above tensor, one can calculate the first and third terms in the relation (62). The other equations, the continuity equation and the quantum Hamilton–Jacobi equation, are expressed respectively as:

$$\nabla^\mu (\rho \phi^a \nabla_\mu S) = 0$$ (66)
\( \nabla^\mu S \nabla_\mu S = m^2 \phi^{b-a} - \frac{1}{2} \Lambda mc \frac{Q}{\rho \phi^a} (1 + Q)^{c-1} + \frac{1}{2} \Lambda mc \alpha \left( \frac{1 + Q}{\sqrt{\rho \phi}} \right)^{c-1} \)

(67)

To simplify the calculations, with due attention to the equation (62), one can choose \( \omega \) to be \( \frac{3}{2} \). Then a perturbative expansion for the scalar field and matter distribution density can be used as:

\[
\phi = \phi_0 + \alpha \phi_1 + \cdots
\]

(68)

\[
\sqrt{\rho} = \sqrt{\rho_0} + \alpha \sqrt{\rho_1} + \cdots
\]

(69)

In the zeroth order approximation, the scalar field equation gives:

\[
b = a + 1; \quad \phi_0 = 1
\]

(70)

In the first order approximation one gets:

\[
\alpha \phi_1 = \frac{c}{2} (1 - a) Q + \frac{a}{2} c \tilde{Q}
\]

(71)

in which:

\[
\tilde{Q} = \alpha \frac{\nabla_\mu \sqrt{\rho} \nabla^\mu \sqrt{\rho}}{\rho}
\]

(72)

Since the scalar field is the conformal factor of the space–time metric, and because of some arguments[9, 11] show that this field is a function of matter quantum potential, one might choose the constant \( a \) equal to zero. Then, the
scalar field is independent of $\tilde{Q}$ and we have:

$$\alpha \phi_1 = \frac{c}{2} Q$$  \hspace{1cm} (73)

Also the Bohmian equations of motion give:

$$\nabla_\mu S \nabla^\mu S = m^2 (1 + cQ/2) - \Lambda mc \frac{Q - \tilde{Q}}{\rho_0}$$  \hspace{1cm} (74)

It is necessary to choose $c = 2$ in order that the first term on the right hand side be the same as the quantum mass $M$. These choices for parameters $a$, $b$ and $c$ lead to the non–perturbative quantum gravity equations as follows:

$$\phi = 1 + Q - \frac{\alpha}{2} \Box Q$$  \hspace{1cm} (75)

$$\nabla^\mu S \nabla_\mu S = m^2 \phi - \frac{2\Lambda m}{\rho} (1 + Q)(Q - \tilde{Q}) + \frac{\alpha \Lambda m}{\rho} \left( \Box Q - 2\nabla_\mu Q \frac{\nabla^\mu \sqrt{\rho}}{\sqrt{\rho}} \right)$$  \hspace{1cm} (76)

$$\nabla^\mu (\rho \nabla_\mu S) = 0$$  \hspace{1cm} (77)

$$G^{\mu\nu} - \Lambda g^{\mu\nu} = -\frac{1}{\phi} T^{\mu\nu} - \frac{1}{\phi} \left[ \nabla^\mu \nabla^\nu - g^{\mu\nu} \Box \right] \phi + \frac{\omega}{\phi^2} \nabla^\mu \phi \nabla_\nu \phi - \frac{1}{2 \phi^2} g^{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi$$  \hspace{1cm} (78)

We conclude this section by pointing out some important hints:

- It is very interesting that in the framework of the scalar–tensor theories, one is able to derive all of the quantum gravity equations of motion without using the method of lagrange.
• In the suggested quantum gravity theory, the causal structure of the space–time \( g_{\mu\nu} \) is determined via equation (78). This shows that except for back–reaction terms of the quantum effects on \( g_{\mu\nu} \), the causal structure of the space–time is determined by the gravitational effects of matter. Quantum effects, determine directly the scale factor of the space–time, from the relation (75).

• It must be noted that the mass field given by the right hand side of the relation (76), consists of two parts. The first part which is proportional to \( \alpha \), is a purely quantum effect, and the second part which is proportional to \( \alpha \Lambda \), is a mixture of the quantum effects and the large scale structure introduced via the cosmological constant.

• In the present theory, the scalar field produces quantum force that appears on right hand side and violates the equivalence principle. Similarly, in Kalutza–Klein theory, the scalar field (dilaton) produces some fifth force leading to the violation of the equivalence principle [32].
4.2 Making the quantum potential dynamical

Using the findings of the previous subsection, one can write an appropriate action such that the conformal factor and quantum potential are both dynamical fields. In this way, the relation between the conformal factor and quantum potential, and also the dependence of quantum potential to the ensemble density are resulted at the first order of approximation. In this way one deals with a scalar–tensor theory with two scalar fields. Thus we start from the most general action:

$$\mathcal{A} = \int d^4x \sqrt{-g} \left[ \phi R - \omega \frac{\nabla_{\mu} \phi \nabla^{\mu} \phi}{\phi} - \frac{\nabla_{\mu} Q \nabla^{\mu} Q}{\phi} + 2\Lambda \phi + \mathcal{L}_m \right],$$

(79)

The cosmological constant generally has an interaction term with the scalar field. We prefer to use the matter Lagrangian:

$$\mathcal{L}_m = \frac{\rho}{m} \phi^a \nabla_{\mu} S \nabla^{\mu} S - m \rho \phi^b - \Lambda (1 + Q)^c + \alpha \rho (e^{\beta Q} - 1).$$

(80)

The first three terms of this Lagrangian are the same as those of the previous subsection. The last term is chosen in such a way, that satisfies two facts. It is necessary to have an interaction between the quantum potential field and the ensemble density, to have a relation between them via the equations of motion. Furthermore, this interaction is written such that in the classical limit, it vanishes.
Variation of the above action functional leads to the following equations of motion:

• the scalar field’s equation of motion

\[ R + \frac{2\omega}{\phi} \Box \phi - \frac{\omega}{\phi^2} \nabla^\mu \phi \nabla_\mu \phi + 2\Lambda \]
\[ + \frac{1}{\phi^2} \nabla^\mu Q \nabla_\mu Q + \frac{a}{m} \rho \phi^{a-1} \nabla^\mu S \nabla_\mu S - mb \phi^{b-1} = 0 \quad (81) \]

• the quantum potential’s equation of motion

\[ \frac{\Box Q}{\phi} - \frac{\nabla_\mu Q \nabla^\mu \phi}{\phi^2} - \Lambda c (1 + Q)^{c-1} + \alpha \beta e^{\beta Q} = 0 \quad (82) \]

• the generalized Einstein’s equation

\[ G^{\mu \nu} - \Lambda g^{\mu \nu} = -\frac{1}{\phi} T^{\mu \nu} - \frac{1}{\phi} [\nabla^\mu \nabla^\nu - g^{\mu \nu} \Box] \phi + \frac{\omega}{\phi^2} \nabla^\mu \phi \nabla^\nu \phi \]
\[ - \frac{\omega}{2\phi^2} g^{\mu \nu} \nabla^\alpha \phi \nabla_\alpha \phi + \frac{1}{\phi^2} \nabla^\mu Q \nabla^\mu Q - \frac{1}{2\phi^2} g^{\mu \nu} \nabla^\alpha Q \nabla_\alpha Q \quad (83) \]

• the continuity equation

\[ \nabla_\mu (\rho \phi^a \nabla^\mu S) = 0 \quad (84) \]

• the quantum Hamilton–Jacobi equation

\[ \nabla^\mu S \nabla_\mu S = m^2 \phi^{b-a} - \alpha m \phi^{b-a} (e^{\beta Q} - 1). \quad (85) \]
In Eq. (81), the scalar curvature and the term $\nabla^\mu S\nabla_\mu S$ can be eliminated using Eqs. (83) and (85). In addition, on using the matter Lagrangian and the definition of the energy–momentum tensor, one has

$$(2\omega - 3)\Box \phi = (a + 1)\rho \alpha (e^{\beta Q} - 1) - 2\Lambda (1 + Q)^c + 2\Lambda \phi - \frac{2}{\phi} \nabla_\mu Q \nabla^\mu Q,$$  

where the constant $b$ is chosen as $a + 1$ as in the previous subsection. We solve the Eqs. (82) and (86), using perturbative expansion with $\alpha$ as the expansion parameter:

$$Q = Q_0 + \alpha Q_1 + \cdots,$$  

$$\phi = 1 + \alpha Q_1 + \cdots,$$  

$$\sqrt{\rho} = \sqrt{\rho}_0 + \alpha \sqrt{\rho}_1 + \cdots,$$

where the conformal factor is chosen to be unity at the zeroth order of perturbation, because in the limit $\alpha \to 0$ Eq. (85) would lead to the classical Hamilton–Jacobi equation. Since by Eq. (85), the quantum mass is given by $m^2 \phi + \text{other terms}$, the first order term of $\phi$ is chosen to be $Q_1$ as it must be so according to the relation of quantum mass (15). Also we shall show that $Q_1$ would be equal to $\Box \sqrt{\rho}/\sqrt{\rho}$ plus some corrections, which is desired as we called $Q$ the quantum potential field.
At the zeroth order one gets

\[ \Box Q_0 - \Lambda c - \Lambda c(c - 1)Q_0 = 0, \]  
\[ \nabla_\mu Q_0 \nabla^\mu Q_0 = -\Lambda c Q_0 \]  

and at the first order:

\[ \nabla_\mu Q_0 \nabla^\mu Q_1 = \Box Q_1 - Q_1 \Box Q_0 - \Lambda c(c - 1)Q_1 + \beta \rho_0 e^{\beta Q_0} \]  
\[ (2\omega - 3)\Box Q_1 = (a + 1)\rho_0 e^{\beta Q_0} - 1 - 2\Lambda(c - 1)Q_1 \]

\[ - 4\nabla_\mu Q_0 \nabla^\mu Q_1 + 2Q_1 \nabla_\mu Q_0 \nabla^\mu Q_0. \]

On using Eqs. (90), (91) and (92), in Eq. (93), one get:

\[ \Box Q_1 + A(\rho_0)Q_1 + B(\rho_0) = 0, \]

where

\[ A(\rho_0) = \frac{-1}{1 + 2\omega}2\Lambda \left( 1 - c + 2e^2 \right) + 2c \left( c - \frac{3}{2} \right) Q_0 \]  
\[ B(\rho_0) = \frac{-1}{1 + 2\omega}[(a + 1)(e^{\beta Q_0} - 1) - 4\beta e^{\beta Q_0}]\rho_0. \]

Equation (94) can be solved iteratively. At the first iteration:

\[ Q_1^{(1)} = -\frac{B}{A} \]  
\[ 43 \]
and at the second and third iteration:

\[ Q^{(2)}_1 = \frac{1}{A} \Box \left( \frac{B}{A} \right) - \frac{B}{A}, \]  
\[ Q^{(3)}_1 = -\frac{1}{A} \Box \left( \frac{\Box B/A}{A} \right) + \frac{1}{A} \Box \left( \frac{B}{A} \right) - \frac{B}{A}. \]  

(98)

(99)

In order to have the correct dependence of the quantum potential on the ensemble density, it is sufficient to set:

\[ A = k_1 \sqrt{\rho_0}; \quad B = k_2 \rho_0 \]  

(100)

where \( k_1 \) and \( k_2 \) are two constants. This leads to the following expressions for the quantum potential up to the third order of iteration:

\[ Q^{(1)}_1 = -\frac{k_2}{k_1} \sqrt{\rho_0}, \]  
\[ Q^{(2)}_1 = \frac{k_2}{k_1^2} \sqrt{\rho_0} - \frac{k_2}{k_1} \sqrt{\rho_0}, \]  
\[ Q^{(3)}_1 = -\frac{k_2}{k_1^2} \frac{1}{\sqrt{\rho_0}} \frac{\Box \sqrt{\rho_0}}{\sqrt{\rho_0}} + \frac{k_2}{k_1^2} \frac{\Box \sqrt{\rho_0}}{\sqrt{\rho_0}} - \frac{k_2}{k_1} \sqrt{\rho_0}. \]

(101)

(102)

(103)

If the ensemble density be not much great, and it be so smooth that its higher derivatives be small, the result would be in agreement with the desired relation \( Q = \Box \sqrt{\rho_0}/\sqrt{\rho_0} \) provided we choose \( k_2 = k_1^2 = k \). Comparison of relations (95), (96) and (100) leads to

\[ a = 2\omega k, \]  

44
\[ \beta = \frac{2\omega k + 1}{4}, \quad (104) \]

\[ Q_0 = \frac{1}{c(2c - 3)} \left[ -\frac{2\omega k + 1}{2\Lambda} k\sqrt{\rho_0} - (2c^2 - c + 1) \right]. \]

The space–time dependence of \( \rho_0 \) can be derived from the relation (91).

We see that the except \( c \) and \( \omega \), all other constants are fixed. The other equations of motion which are not used in the perturbation procedure can be used to determine the space–time metric and the Hamilton–Jacobi function.

We conclude this section by emphasizing on the fact that in our present work, the quantum potential is a dynamical field. And, that solving perturbatively the equations of motion, one gets the correct dependence of quantum potential upon density plus some corrective terms.

We stop here and shall investigate some physical results in the next section and after that continue our way to construct a theory for geometrization of quantum effects in terms of Weyl geometry.

5 Some results of the idea

In this section we shall first study some general solutions of field equations and then look for some specific cases like black holes and big bang.
5.1 Conformally flat solution

Suppose we search for a solution which is conformally flat, and that the conformal factor is near unity. Such a solution is of the form:

\[ g_{\mu\nu} = e^{2\Sigma} \eta_{\mu\nu}; \quad \Sigma \ll 1. \]  

(105)

As a result one can derive the following relations by equation (41):

\[ R_{\mu\nu} = \eta_{\mu\nu} \Box \Sigma + 2 \partial_{\mu} \partial_{\nu} \Sigma \implies \mathcal{G}_{\mu\nu} = 2 \partial_{\mu} \partial_{\nu} \Sigma - 2 \eta_{\mu\nu} \Box \Sigma \]  

(106)

In order to solve for \( \Sigma \) one can use the relation (48), and solve it iteratively as it is discussed in the subsubsection (3.2.2). The result is

\[ R^{(0)} = -\kappa T \implies \Sigma^{(0)} = -\frac{\kappa}{6} \Box^{-1} T, \]  

(107)

\[ R^{(1)} = -\kappa T + 3\alpha \Box \frac{\sqrt{|T|}}{\sqrt{|T|}} \implies \Sigma^{(1)} = -\frac{\kappa}{6} \Box^{-1} T + \frac{\alpha}{2} \Box \frac{\sqrt{|T|}}{\sqrt{|T|}} \]  

(108)

and so on. Thus:

\[ \Sigma = \frac{\kappa}{6} \Box^{-1} T + \frac{\alpha}{2} \Box \frac{\sqrt{|T|}}{\sqrt{|T|}} \]

pure gravity pure quantum

+ higher terms including gravity – quantum interactions, (109)

where \( \Box^{-1} \) represents the inverse of the dalambertian operator. Note that the solution is in complete agreement with de Broglie–Bohm theory.
5.2 Conformally quantic solution

As a generalization of the solution found in the previous subsection, suppose we set
\[ g_{\mu\nu} = e^{2\Sigma} \bar{g}_{\mu\nu} = (1 + 2\Sigma) \bar{g}_{\mu\nu}; \quad \Sigma \ll 1. \quad (110) \]

One can evaluate the following relations:
\[ R_{\nu\rho} = \bar{R}_{\nu\rho} + \bar{g}_{\mu\nu} \Box \Sigma + 2(\bar{\nabla}_\nu \bar{\nabla}_\rho \Sigma + \bar{g}_{\nu\rho} \bar{\nabla}_\alpha \Sigma \bar{\nabla}^\alpha \Sigma - \bar{\nabla}_\nu \Sigma \bar{\nabla}_\rho \Sigma), \quad (111) \]
\[ G_{\nu\rho} = \tilde{G}_{\nu\rho} - 2\bar{g}_{\nu\rho} \Box \Sigma + 2\bar{\nabla}_\nu \bar{\nabla}_\rho \Sigma. \quad (112) \]

On using these relations in field equations (45) one gets the following solution:
\[ \tilde{G}_{\mu\nu} = \kappa \bar{T}_{\mu\nu}; \quad \Sigma = \frac{\alpha}{2} \Phi \quad (113) \]

provided the energy–momentum tensor be conformally invariant, so under this condition we have
\[ g_{\mu\nu}^{\text{quantum+gravity}} = (1 + \alpha \Phi) g_{\mu\nu}^{\text{gravity}} \quad (114) \]

5.3 Conformally highly quantic solution

Now we can generalize the result of the previous subsection. Suppose in the overlined metric there is no quantum effect, so that \( \bar{G}_{\mu\nu} = \kappa \bar{T}_{\mu\nu} \) and we know that the quantum effects could bring in via a conformal transformation like
\(g_{\mu\nu} = e^{2\Sigma} \bar{g}_{\mu\nu}\). Using field equations (45) and the transformation properties of the Einstein’s equation one gets

\[
2 \big( \Box + 2 \nabla_\alpha \Sigma \nabla^\alpha \Sigma \big) = \alpha \big( \Box + 2 \nabla_\alpha \Phi \nabla^\alpha \Sigma \big) - \frac{\alpha^2}{2} \nabla_\alpha \Phi \nabla^\alpha \Phi
\]

which has the solution

\[
\Sigma = \frac{\alpha}{2} \Phi.
\]

In the above solution it is assumed that the energy–momentum tensor is either zero or conformally invariant. So, under this condition, no matter how large the quantum effects, are the general solution is

\[
g_{\mu\nu}^{\text{quantum+gravity}} = e^{\alpha \Phi} g_{\mu\nu}^{\text{gravity}}.
\]

5.4 Black holes

Let us now use the above solutions to examine the quantum effects near the regions of space–time where the gravitational effects of matter are large. Black hole is the first examples we consider.

For a spherically symmetric black hole we have

\[
g_{\mu\nu}^{\text{gravity}} = \text{diag} \left( 1 - \frac{r_s}{r}, -\frac{1}{1 - \frac{r_s}{r}}, -r^2, -r^2 \sin^2 \theta \right)
\]

where \(r_s\) is the Schwartzchid radius. Using the fact that the Ricci scalar is zero for the above metric and the transformation properties of the Ricci scalar
under conformal transformations, we have

$$\mathcal{R} = 3\alpha e^{-\alpha \Phi} \left[ \Phi + \frac{\alpha}{2} \nabla_{\mu} \Phi \nabla^{\mu} \Phi \right]$$  \hfill (119)

The above equation is in fact a differential equation for the conformal factor and can be solved for different regimes, giving the following solution:

$$g_{\mu\nu}^{\text{quantum+gravity}} = g_{\mu\nu}^{\text{gravity}} \times \begin{cases} \exp \left( \frac{-\alpha r}{3} \right) & r \to 0, \\ \text{Constant} & r \to r_s, \\ \exp \left( \frac{r^2}{3\alpha} \right) & r \to \infty. \end{cases}$$  \hfill (120)

The conformal factor is plotted in Fig. (1-a). It can be seen that the above conformal factor does not remove the metric singularity at \( r = 0 \).

### 5.5 Initial singularity

Big bang singularity is another place where the gravitational effects are large.

For an isotropic and homogeneous (FRW) universe we have

$$g_{\mu\nu}^{\text{gravity}} = \text{diag} \left( 1, -\frac{a^2}{1-kr^2}, -a^2r^2, -a^2r^2 \sin^2 \theta \right)$$  \hfill (121)

As in the previous subsection we have

$$\mathcal{R} = e^{-\alpha \Phi} \left[ \tilde{\mathcal{R}} + 3\alpha \left( \Phi + \frac{\alpha}{2} \nabla_{\mu} \Phi \nabla^{\mu} \Phi \right) \right]$$  \hfill (122)
As $t \to 0$ one can solve the above equations approximately

$$g_{\mu\nu}^{\text{quantum+gravity}} = g_{\mu\nu}^{\text{gravity}} \exp \left( \frac{\alpha}{2t^2} \right)$$

(123)

so the universe scale is given by

$$a(t) = a^{\text{classic}}(t) \exp \left( \frac{\alpha}{4t^2} \right) = \sqrt{t} \exp \left( \frac{\alpha}{4t^2} \right).$$

(124)

As can be seen easily, the curvature singularity at $t = 0$ is removed because as time goes to zero, the universe scale goes to infinity. The behavior of the universe scale is plotted in Fig. (1-b).
5.6 Production of inhomogeneity

At this end we shall examine the quantum effects on the motion of a fluid of matter. As a more practical example we take this fluid to be the cosmological matter fluid in an FRW model. We shall see that the quantum effects can produce some inhomogeneities and thus produce cluster formation. The hydrodynamics equation is given by:

\[
\frac{\partial p}{\partial x^\nu} g^{\mu\nu} + \frac{1}{\sqrt{-g}} \left( \sqrt{-g} (p + \rho) U^\mu U^\nu \right) + \Gamma^\mu_{\nu\lambda} (p + \rho) U^\mu U^\nu = \rho (g^{\mu\nu} - U^\mu U^\nu) \frac{\partial \ln \mathcal{M}/m}{\partial x^\nu}
\]

where we have introduced the quantum force in the right hand side just as it is introduced in the equation of motion of a single particle (see equation (21)). It must be noted that the metric itself must be calculated from the corrected Einstein’s equations including the back–reaction terms. In fact one must solve the above equation and the metric equation simultaneously to obtain the metric and the density. We shall not do in this way because solving those equations is difficult. We shall do in a similar way. It is an iterative way and is based on the fundamental assumption of this theory, that is equation (18). As the first order of iteration, we consider the space–time metric as given by the classical Einstein’s equations (Robertson–Walker metric) and solve the above equation for the density, then using the result obtained, calculate the quantum metric.
using equation (18). Then the new metric can be used to obtain the density at the second order and so on.

In the comoving frame and with the assumption that the universe is in the dust mode \((p = 0)\) with the flat Robertson–Walker metric, we have from the equation (125):

\[
\frac{d\rho^{(1)}}{dt} + 3H\rho^{(1)} = 0 \quad (126)
\]

\[
\frac{\partial Q^{(1)}}{\partial x^i} = 0 \quad (127)
\]

where \(^{(1)}\) denotes the first order of iteration and \(H = \dot{a}/a\) is the Hubble’s parameter. The solution of the above two equations is:

\[
\rho^{(1)} = \frac{\mathcal{X}^{(1)}t^2}{2}; \quad Q^{(1)} = \text{constant} \quad (128)
\]

where \(\mathcal{X}^{(1)}\) should yet be determined. The constancy of the quantum potential leads to:

\[
Q^{(1)} = -\frac{\alpha}{2a^2} \frac{\nabla^2 \mathcal{X}^{(1)}}{\mathcal{X}^{(1)}} \quad (129)
\]

so that:

\[
\nabla^2 \mathcal{X}^{(1)} + \beta \mathcal{X}^{(1)} = 0 \quad (130)
\]

where:

\[
\beta = \frac{2Q^{(1)}a^2}{\alpha} \quad (131)
\]
This equation for $X^{(1)}$ can simply be solved either in the Cartesian coordinates or in the spherical ones. The solution is:

$$X^{(1)} = \sin \left( \sqrt{\frac{\beta}{3}} x \right) \sin \left( \sqrt{\frac{\beta}{3}} y \right) \sin \left( \sqrt{\frac{\beta}{3}} z \right)$$  \hspace{1cm} (132)

or:

$$X^{(1)} = \sum_{l,m} \left( a_{lm} j_l(\sqrt{\beta} r) + b_{lm} n_l(\sqrt{\beta} r) \right) Y_{lm}(\theta, \phi)$$ \hspace{1cm} (133)

This is the first order approximation. At the second order, one must use the equation (18) to change the scale factor $a^2$ to $a^2(1 + Q)$, and then from the relation (126) we have:

$$a^2(1 + Q) = \frac{t^2}{3} X^{(1)} - \frac{4}{3}$$  \hspace{1cm} (134)

So that:

$$Q^{(2)} = -1 + X^{(1)} - \frac{4}{3}$$  \hspace{1cm} (135)

and then using this form of the quantum potential in the relation (130) or (132) leads to the following approximation for the density:

$$\rho^{(2)} = \frac{1}{t^2} \sin^2 \left( \sqrt{\frac{\gamma}{3}} x \right) \sin^2 \left( \sqrt{\frac{\gamma}{3}} y \right) \sin^2 \left( \sqrt{\frac{\gamma}{3}} z \right)$$ \hspace{1cm} (136)

where:

$$\gamma = \frac{2a^2}{\alpha} \left( -1 + X^{(1)} - \frac{4}{3} \right)$$ \hspace{1cm} (137)
and $X^{(1)}$ is given by the relation (1.62). This procedure can be done order by order.

In the figures (2) the density at three times are shown and the clustering can be seen easily. These figures are plotted using the solution in the Cartesian coordinates.

In figures (3) the $(l,m) = (00)$ mode of equation (133) is shown at three time steps.

In figures (4) the $(11) \oplus (1 - 1)$ mode is shown at three time steps.

The second order solution (136) is shown in figure 5, where both large scale and small scale structures are shown.

It is important to note that the clustering can be seen in any of these figures. In the last figure, however, one observes that at the large scale the universe is homogeneous and isotropic, while at the small scale these symmetries are broken.

At the end, in order to see whether our results are in agreement with the observed clustering, the correlation function ($\xi(r)$) is obtained from the third order of iteration and is compared with the cases $\xi = (r/r_0)^{-\gamma}$ with $\gamma = 1.8$ and $\gamma = 3$ and with the standard result of a typical $P^3M$ code [18]. As it can be seen in figure 6 our results are in good agreement with the $P^3M$ code and
Our claim here is not that this theory is a good one for the cluster formation problem. But it is only claimed that in the framework of causal quantum theory, the quantum force may be a cause for the cluster formation.
Figure 4: Density at three stages of expansion for $(11) \oplus (1-1)$ mode

Figure 5: The second order solution. Note the fine structure at the magnified portion.
Figure 6: Comparison with observation and $P^3M$ code.
5.7 Non–locality [20]

Since the quantum potential in de-Broglie–Bohm theory has a highly non–local character [3], it is expected to see non–local behaviour of metric in this theory. In order to illustrate how nonlocal effects can appear in quantum gravity through quantum potential, suppose that matter distribution is localized and has spherical symmetry. Then, one has:

\[ \rho = \rho(t; r) \]  
\[ \Omega = \Omega(t; r) \]

(138)  
(139)

Suppose, furthermore, that matter is at rest:

\[ -\nabla_0 S = E(t; r) \quad \text{as} \quad r \rightarrow \infty \]  
\[ \nabla_i S = 0; \quad i = 1, 2, 3 \]

(140)  
(141)

One expects that at large \( r \), where there is no matter, the background metric would be of the Schwartzschild form. The validity of this approximation will be examined at the end. The equation of motion (35) relates \( E \) and \( \Omega \):

\[ E = \frac{m\Omega}{\sqrt{1 - r_s/r}} \]

(142)
In order to calculate the conformal factor $\Omega$, one needs the specific form of $\rho$. It must be a localized function at $r = 0$. So we choose it as:

$$\rho(t; r) = A^2 \exp[-2\beta(t)r^2] \quad (143)$$

Using the relation (28), the conformal factor can be simply calculated. This leads to:

$$\Omega^2 = 1 + \alpha[\beta'^2r^4 - \beta''r^2 + 4\beta^2r]$$

from which we get:

$$\Omega^2 \approx \alpha\beta'^2r^4 \quad as \quad r \to \infty \quad (144)$$

Now it is a simple task to examine that the continuity equation (34) is satisfied automatically as $r \to \infty$. This solution is an acceptable one, only if the generalized Einstein’s equations (36) are satisfied. This is so if $\mathcal{T}^{(\Omega)}_{\mu\nu} \to 0$ as $r \to \infty$. It can be shown that in the limit $r \to \infty$ we have:

$$\frac{\Box \Omega^2}{\Omega^2} = 2(\beta'/\beta')^2 + 2\beta'/\beta - 20/r \quad (145)$$

$$\frac{\nabla_0 \nabla_0 \Omega^2}{\Omega^2} = (\beta'/\beta)^2 + \beta'/\beta \quad (146)$$

$$\frac{\nabla_1 \nabla_1 \Omega^2}{\Omega^2} = 12/r^2 \quad (147)$$

$$\frac{\nabla_1 \nabla_0 \Omega^2}{\Omega^2} = (8\beta'/r\beta) \quad (148)$$

$$\left(\frac{\nabla_0 \Omega}{\Omega}\right)^2 = (\beta'/\beta)^2 \quad (149)$$
So provided that higher time derivatives of the scale factor of matter density \( \beta \) are small with respect to its first time derivative, that is:

\[
\frac{\dddot{\beta}}{\beta} \simeq 0; \quad \frac{\ddot{\beta}}{\beta} \simeq 0 \quad \text{and so on}
\] (151)

one has:

\[
\lim_{r \to \infty} T^{(\Omega)\nu}_{\mu} = 0
\] (152)

Also we have from (37):

\[
\lim_{r \to \infty} T^{(m)\nu}_{\mu} = 0
\] (153)

So at large distances \( g_{\mu\nu} \) satisfies Einstein’s equations in vaccum, \( G_{\mu\nu} = 0 \). Therefore, the solution (118) is acceptable. In this way we find a solution to the quantum gravity equations at large distances.

Consequently, if the time variation of \( \beta \) is small, the physical metric \( \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \) is given by:

\[
\lim_{r \to \infty} \tilde{g}_{\mu\nu} = \alpha \dot{\beta}^2 r^4 g^{(Shwarzschild)}_{\mu\nu}
\] (154)

An important points must be noted here. As it was shown, a change in matter distribution (due to \( \dot{\beta} \)) instantaneosely alters the physical metric. This is because of the appearance of \( \dot{\beta}(t) \) in equation (154) and it comes from the quantum potential term.
We conclude that the specific form of the quantum potential leads to the appearance of nonlocal effects in quantum gravity.

6 Generalized equivalence principle[33]

After presenting some results of the idea of coding quantum effects in the conformal factor of the space–time metric, let us come back to our way of constructing a theory of it. One of the new points of the present approach for geometrization of quantum effects is the dual role of geometry in physics. The gravitational effects determine the causal structure of space–time as long as quantum effects give its conformal structure. This does not mean that quantum effects have nothing to do with the causal structure, they can act on the causal structure through back–reaction terms appearing in the metric field equations[14, 13, 11]. We only mean that a dominant role in the causal structure belongs to the gravitational effects. The same is true for the conformal factor. The conformal factor of the metric is a function of the quantum potential and the mass of a relativistic particle is a field produced by quantum corrections to the classical mass. We have shown that the presence of the quantum potential is equivalent to a conformal mapping of the metric. Thus
in conformally related frames we feel different quantum masses and different curvatures. It is possible to consider two specific frames. One of them contains the quantum mass field (appeared in quantum Hamilton-Jacobi equation) and the classical metric as while in the other the classical mass (appeared in classical Hamilton-Jacobi equation) and the quantum metric are appeared. In other frames both the space–time metric and mass field have quantum properties. This argument motivates us to state that different conformal frames are identical pictures of the gravitational and quantum phenomena. Considering the quantum force, the conformally related frames aren’t distinguishable. This is just what happens when we consider gravity, different coordinate systems are equivalent. Since the conformal transformation change the length scale locally, we feel different quantum forces in different conformal frames. This is similar to general relativity in which general coordinate transformation changes the gravitational force at any arbitrary point. Here it may be appropriate to state a basic question. Does applying the above correspondence, between quantum and gravitational forces, and between the conformal and general coordinate transformations, means that the geometrization of quantum effects implies conformal invariance just as gravitational effects imply the general coordinate invariance?
To discuss this question, we recall what has been considered earlier in the development of general relativity. The general covariance principle leads to the identification of gravitational effects of matter with the geometry of the space–time. In general relativity the important fact which supports this identification is the equivalence principle. According to it, one can always remove the gravitational field at some point by a suitable coordinate transformation. Similarly, as we pointed out previously, according to our new approach to Bohmian quantum gravity, at any point (or even globally) the quantum effects of matter can be removed by a suitable conformal transformation. Thus in that point(s) matter behaves classically. In this way we can introduce a new equivalence principle calling it the conformal equivalence principle, similar to the standard equivalence principle. The latter interconnects gravity and general covariance while the former has the same role about quantum and conformal covariance. Both these principles state that there is no preferred frame, either coordinate or conformal. Since Weyl geometry welcomes conformal invariance and since it has additional degrees of freedom which can be identified with quantum effects, it provides a unified geometric framework for understanding the gravitational and quantum forces. In this way a pure geometric interpretation of quantum behavior can be built.
Due to these results, we believe that the de-Broglie–Bohm theory must receive increasing attention in quantum gravity. This theory has some important features. One of them is that the quantum effects appear independent of any preferred length scale (opposed to the standard quantum mechanics in which the Plank length is a characteristic length). This is one of the intrinsic properties of this theory which results from a special definition of the classical limit. Another important aspect is that the quantum mass of the particle is a field. This is needed for having conformal invariance, since mass has a non–zero Weyl weight. Also, according to the geodesic equation, the appearance of quantum mass justifies Mach’s principle which leads to the existence of interrelation between the global properties of the universe (space–time structure, the large scale structure of the universe, · · ·) and its local properties (local curvature, motion in a local frame, etc.). In the present theory, it can be easily seen that the space–time geometry is determined by the distribution of matter. A local variation of matter field distribution changes the quantum potential acting on the geometry. Thus the geometry is altered globally (in conformity with Mach’s principle). In this sense our approach to the quantum gravity is highly non–local as it is forced by the nature of the quantum potential. What we call geometry is only the gravitational and
quantum effects of matter. Without matter the geometry would be meaningless. Moreover in [13, 14] we have shown that it is necessary to assume an interaction term between the cosmological constant (large scale structure) and the quantum potential (local phenomena). These properties all justify Mach’s principle. It is shown in [13, 14] that the gravitational constant is in fact a field depending on the matter distribution through quantum potential.

All these arguments based on Bohmian quantum mechanics convince us that Weyl geometry is a suitable framework for geometrization of quantum mechanics.

7 Formulation of the idea in terms of Weyl Geometry[33]

7.1 Weyl–Dirac theory

A straightforward generalization of the Einstein–Hilbert action to Weyl geometry leads to a higher order theory[22]. Dirac[35] introduced a new action, called Weyl–Dirac action, by including a new field which is in fact a gauge function. It helped him to avoid higher order actions since fixing the gauge
function led to Einstein–Maxwell equations.

The Weyl–Dirac action is given by \[35\]:

\[
A = \int d^4x \sqrt{-g} \left( F_{\mu\nu} F^{\mu\nu} - \beta^2 W\mathcal{R} + (\sigma + 6)\beta_\mu\beta^\mu + \mathcal{L}_{\text{matter}} \right) \tag{155}
\]

Here \( F_{\mu\nu} \) is the curl of the Weyl four–vector \( \phi_\mu \), \( \sigma \) is an arbitrary constant, and \( \beta \) is a scalar field of weight \(-1\). The “;” represents covariant derivative under general coordinate and conformal transformations (Weyl covariant derivative) defined as:

\[
X_{\mu} = W\nabla_\mu X - \mathcal{N} \phi_\mu X \tag{156}
\]

where \( \mathcal{N} \) is the Weyl weight of \( X \). The equations of motion will then be:

\[
G^{\mu\nu} = -\frac{8\pi}{\beta^2} (T^{\mu\nu} + M^{\mu\nu}) + \frac{2}{\beta} (g^{\mu\nu} W\nabla_\alpha W\nabla_\beta - W\nabla_\mu W\nabla_\nu \beta) + \frac{1}{\beta^2} (4 \nabla^\mu \beta \nabla^\nu \beta - g^{\mu\nu} \nabla^\alpha \beta \nabla_\alpha \beta) + \frac{\sigma}{\beta^2} (\beta^{\mu} \beta^{\nu} - \frac{1}{2} g^{\mu\nu} \beta^{\alpha} \beta^{\alpha}) \tag{157}
\]

\[
W\nabla_\nu F^{\mu\nu} = \frac{1}{2} \sigma (\beta^2 \phi^\mu + \beta \nabla^\mu \beta) + 4\pi J^\mu \tag{158}
\]

\[
\mathcal{R} = -(\sigma + 6) \frac{W\Box \beta}{\beta} + \sigma \phi_\alpha \phi^\alpha - \sigma W\nabla^\alpha \phi_\alpha + \frac{\psi}{2\beta} \tag{159}
\]

where:

\[
M^{\mu\nu} = \frac{1}{4\pi} \left( \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} - F_{\alpha}^{\mu} F^{\nu\alpha} \right) \tag{160}
\]
and the energy–momentum tensor $T^{\mu\nu}$, the current density vector $J^\mu$ and the scalar $\psi$ are defined as:

$$8\pi T^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta \sqrt{-g} \mathcal{L}_{\text{matter}}}{\delta g_{\mu\nu}}$$  \hspace{1cm} (161)

$$16\pi J^\mu = \frac{\delta \mathcal{L}_{\text{matter}}}{\delta \phi^\mu}$$  \hspace{1cm} (162)

$$\psi = \frac{\delta \mathcal{L}_{\text{matter}}}{\delta \beta}$$  \hspace{1cm} (163)

On the other hand the equations of motion of matter and the trace of energy-momentum tensor can be obtained from the invariance of the action under the coordinate and gauge transformations. One can write them as:

$$W \nabla_\nu T^{\mu\nu} - T \nabla_\beta \frac{\nabla^\mu \beta}{\beta} = J_\alpha \phi^{\alpha\mu} - (\phi^\mu + \nabla^\mu \beta) W \nabla_\alpha J^\alpha$$ \hspace{1cm} (164)

$$16\pi T - 16\pi W \nabla_\mu J^\mu - \beta \psi = 0$$  \hspace{1cm} (165)

The first relation is only a geometrical identity (the Bianchi identity), and the second shows the mutual dependence of the field equations.

It must be noted that in the Weyl–Dirac theory, the Weyl vector does not couple to spinors, so $\phi^\mu$ cannot be interpreted as the electromagnetic potential\[36\]. Here we use the Weyl vector not as the electromagnetic field but only as part of the space–time geometry. The Weyl–Dirac formalism is adopted, and we shall see that the auxiliary field (gauge function) in Dirac’s
action represents the quantum mass field. In addition both the gravitation fields \((g_{\mu\nu} \text{ and } \phi_\mu)\) and the quantum mass field determine the space–time geometry.

### 7.2 Weyl–invariant quantum gravity

In this section we shall construct a theory for Bohmian quantum gravity which is conformally invariant in the framework of Weyl geometry. To begin with, note that if our model should consider massive particles, the mass must be a field. This is because mass has non–zero Weyl weight. This is in agreement with Bohm’s theory. As we argued previously a general Weyl invariant action is the Weyl–Dirac action, whose equations of motion are derived in the previous subsection. To simplify our model, we assume that the matter lagrangian does not depend on the Weyl vector, so that \(J_\mu = 0\). The equations of motion are now:

\[
G^{\mu\nu} = -\frac{8\pi}{\beta^2}(T^{\mu\nu} + M^{\mu\nu}) + \frac{2}{\beta}(g^{\mu\nu} W_{\alpha\beta} W_{\alpha\beta} - W^{\mu} W^{\nu} W_{\beta})
\]

\[
+ \frac{1}{\beta^2}(4\nabla^{\mu} \nabla^{\nu} - g^{\mu\nu} \nabla^{\alpha} \nabla_{\alpha}) + \frac{\sigma}{\beta^2}(\beta^{\mu} \beta^{\nu} - \frac{1}{2}g^{\mu\nu} \beta^{\alpha} \beta_{\alpha})
\]

(166)

\[
W^\nu F^{\mu\nu} = \frac{1}{2}\sigma(\beta^2 \phi^{\mu} + \beta \nabla^{\mu} \beta)
\]

(167)

\[
\mathcal{R} = -(\sigma + 6) \frac{W^{\mu}}{\beta^3} + \sigma \phi_\alpha \phi^\alpha - \sigma W^{\alpha} \phi_\alpha + \frac{\psi}{2\beta}
\]

(168)
and the symmetry conditions are:

\[ W \nabla_\nu T^{\mu\nu} - T \frac{\nabla^\mu \beta}{\beta} = 0 \]  
(169)

\[ 16\pi T - \beta \psi = 0 \]  
(170)

It must be noted that from equation (167) we have:

\[ W \beta \nabla_\mu \left( \beta^2 \phi^\mu + \beta \nabla^\mu \beta \right) = 0 \]  
(171)

so \( \phi_\mu \) is not independent of \( \beta \).

It is worthwhile to see whether or not this model has anything to do with the Bohmian quantum theory. We want to introduce the quantum mass field. Now we shall show that this field is proportional to the Dirac field. In order to see this, two conditions should necessary meet. Firstly, the correct dependence of the Dirac field on the trace of energy–momentum tensor and, secondly the correct appearance of the quantum force in the geodesic equation. Now note that using equations (167), (168), and (170) we have:

\[ \Box \beta + \frac{1}{6} \beta R = \frac{4\pi T}{3} \beta + \sigma \beta \phi^\alpha \phi_\alpha + 2(\sigma - 6) \phi^\gamma \nabla_\gamma \beta + \frac{\sigma}{\beta} \nabla^\mu \beta \nabla_\mu \beta \]  
(172)

This equation can be solved iteratively. Let we rewrite it as:

\[ \beta^2 = \frac{8\pi T}{R} - \frac{1}{R/6 - \sigma \phi^\alpha \phi_\alpha} \beta \Box \beta + \ldots \]  
(173)
The first and the second order solutions of this equation are:

\[
\beta^{(1)} = \frac{8\pi T}{R}
\]

\[
\beta^{(2)} = \frac{8\pi T}{R} \left( 1 - \frac{1}{R/6 - \sigma \phi \phi^\alpha} \right)
\]

To derive the geodesic equation we use the relation (169). Assuming that matter consist of dust with the energy–momentum tensor (59) and multiplying equation (169) by \( u_\mu \), we have:

\[
u^\mu W_\nu u^\mu = \frac{1}{\beta} (g^{\mu\nu} - u^\mu u^\nu) \nabla_\nu \beta
\]

Comparison of equations (175) and (176) with equations (7) and (21) shows that we have the correct equations of motions of Bohmian quantum theory, provided we identify:

\[
\beta \longrightarrow M
\]

\[
\frac{8\pi T}{R} \longrightarrow m^2
\]

\[
\frac{1}{\sigma \phi \phi^\alpha - R/6} \longrightarrow \alpha
\]

This shows that we have succesfully geometrized Bohmian quantum mechanics. The \( \beta \) field is in fact the Bohmian quantum mass field, and the quantum coupling constant \( \alpha \) which depends on \( \hbar \) is also a field. In fact it is highly related to geometrical properties of the space–time through the above relation.
Since a gauge transformation can transform a general space–time dependent Dirac field to a constant one, and vice-versa, it can be shown that quantum effects and the length scale of the space–time are closely related. To see this suppose we are in a gauge in which Dirac field is a constant. By applying a gauge transformation one can change it to a general space–time dependent function.

\[ \beta = \beta_0 \longrightarrow \beta = \beta(x) = \beta_0 \exp(-\Xi(x)) \quad (180) \]

This gauge transformation is defined as:

\[ \phi_\mu \longrightarrow \phi_\mu + \partial_\mu \Xi \quad (181) \]

So, the gauge in which the quantum mass is constant (and thus the quantum force is zero) and the gauge in which the quantum mass is space–time dependent are related to each other via a scale change. In other words, \( \phi_\mu \) in the two gauges differ by \(-\nabla_\mu(\beta/\beta_0)\). Since \( \phi_\mu \) is a part of Weyl geometry, and Dirac field represents the quantum mass, one concludes that the quantum effects are geometrized. One can see this fact also by referring to the equation 167 which shows that \( \phi_\mu \) is not independent of \( \beta \), so the Weyl vector is determined by quantum mass, and thus this geometrical aspect of the manifold is related to the quantum effects. In this way, the physical meaning of auxiliary Dirac
field is clarified, as while as a suitable model for geometrization of quantum mechanics is introduced.

7.3 Application to cosmology

Most of physicists believe in a non–zero cosmological constant because of two important reasons. It helps us to make the theoretical results to agree with observations. Moreover some topics, like large scale structure of the universe, dark matter, inflation, can be explored using it. On the other hand from astronomical observations, especially gravitational lensing, cosmological constant should be very small. \( |\Lambda| < 10^{-54}/cm^2 \) The fact that the cosmological constant is small produces some difficulties. How explain theoretically this value of the cosmological constant? (This question also applies to the gravitation coupling constant.) Moreover the cosmological constant is a measure of vacuum energy density. This includes some contribution from scalar fields, bare cosmological constant, quantum effect, and so on. But observed cosmological constant is more smaller than (120 order of magnitude less than) each one of the above contributions. This is the so–called cosmological constant puzzle (see \cite{37} and its references). Till now many mechanisms are presented to solve the problem.

One way to solve the problem is to give dynamical characters to gravi-
tational and cosmological constants in such a way that they decrease as the universe expands. Some works are done in [38] and [39]. In the former, a mechanism is presented using the WDW equation, while the latter, focuses on the breaking the conformal invariance. Two scales, cosmological and particle physics are introduced. And a dynamical conformal factor which relates them produces an effective time dependent cosmological constant.

We also use the conformal invariance, but in the conformal invariant framework of the present paper. Let’s choose a spatially flat Robertson–Walker metric:

$$ds^2 = a^2(\eta) \left[ d\eta^2 - dr^2 - r^2 d\Omega^2 \right]$$

(182)

where $a(\eta)$ is the scale factor, and assuming the universe is filled of a dust, the equations of motion of theory presented in the previous section now simplifies to:

$$3 \frac{\dot{a}^2}{a^4} - \frac{8\pi\rho}{\beta^2} + \frac{6}{\beta} \left( \frac{\dot{a}}{a} - \phi \right) \frac{\dot{\beta}}{a^2} + 3 \frac{\dot{\beta}^2}{\beta^2 a^2}$$

$$+ \frac{\sigma}{2\beta^2} \frac{(\dot{\beta} + \phi \beta)^2}{a^2} = 0$$

(183)

$$\dot{\beta} + \beta \phi = 0$$

(184)

$$-6 \frac{\ddot{a}}{a^3} - (\sigma + 6) \left( \frac{1}{\beta} \frac{d}{d\eta} \left( \frac{\dot{\beta}}{a^2} \right) + \frac{\dot{\beta}}{\beta a^2} \left( 4 \frac{\dot{a}}{a} - 10 \phi \right) \right)$$

$$+ \sigma \frac{\dot{\phi}^2}{a^2} - \sigma \frac{d}{d\eta} \left( \frac{\phi}{a^2} \right) - \sigma \frac{\dot{\phi}}{a^2} \left( 4 \frac{\dot{a}}{a} - 10 \phi \right) + \frac{\psi}{2\beta} = 0$$

(185)
where a dot over any quantity represents derivation with respect to time and we have chosen the gauge

$$\phi_\mu = (\phi, 0, 0, 0)$$  \hspace{1cm} (186)

And the symmetry conditions are:

$$\dot{\rho} + 3\rho \left( \frac{\dot{a}}{a} - \phi \right) - \rho \frac{\dot{\beta}}{\beta} = 0$$  \hspace{1cm} (187)

$$16\phi\rho - \beta\psi = 0$$  \hspace{1cm} (188)

Introducing the cosmological time as $dt = a d\eta$ and simplifying the relations, we finally have:

$$\rho a^3 \beta^2 = \text{constant}$$  \hspace{1cm} (189)

$$3\frac{a'^2}{a^2} - \Lambda_{eff} - 8\pi G_{eff}\rho = 0$$  \hspace{1cm} (190)

$$3\frac{a''}{a} + 3\frac{a'^2}{a^2} + 30\frac{\beta'^2}{\beta^2} + 9\frac{a'}{a} \frac{\beta'}{\beta} + 3\frac{\beta''}{\beta} - 4\pi G_{eff} \rho = 0$$  \hspace{1cm} (191)

where a ‘ over any quantity represents derivation with respect to the cosmological time and we have defined:

$$\Lambda_{eff} = -9\frac{\beta'^2}{\beta^2} - \frac{6\frac{a'}{a} \frac{\beta'}{\beta}}{}$$  \hspace{1cm} (192)

$$G_{eff} = \frac{1}{\beta^2} = \frac{1}{M^2}$$  \hspace{1cm} (193)

The above equations can simply solved resulting in:

$$H \sim t^{-1}$$  \hspace{1cm} (194)
\[ \Lambda_{\text{eff}} \sim t^{-2} \quad (195) \]

\[ G_{\text{eff}} \sim t^{-4/19} \quad (196) \]

where \( H \) is the Hubble constant. As the universe expands these quantities decrease in agreement with the above discussion. These constants have a small value at the current epoch as the observation suggests. It can be noticed that these time dependences are through the quantum mass field (\( \beta \) or \( \mathcal{M} \)). The quantum mass field changes with time as \( t^{2/19} \).

8 Extension of the results

In this section we shall look for the possibility of extending the theory in two lines. Manipulating many-particle systems and inclusion of spin.

8.1 Many-particle systems

Till now our discussion was about geometrization of quantum effects of single particle systems. What happens when one deals with a system consisting of more than one particle? First the quantum potential for many-particle systems in the non-relativistic case is given by \[3\]:

\[ Q = -\frac{\hbar^2}{2m} \frac{1}{|\psi|} \sum_{i=1}^{N} \nabla_i^2 |\psi| \quad (197) \]
Its generalization to the case of particles moving in an arbitrary space–time is clearly:

\[ Q = \frac{\hbar^2}{m^2} \frac{1}{|\psi|} \sum_{i=1}^{N} \Box_i^2 |\psi| \]  

(198)

A difficulty arises here. This quantum potential is defined in the configuration space, i.e. it is a function of \( x_1, x_2, \ldots x_N \). Simply putting the exponential of the quantum potential as the conformal factor of the space–time metric should not work because this makes the metric to live in the configuration space, a completely meaningless thing. To solve the problem we can generalize the idea in this way. Different particles of the many–particle system does not experience the same geometry. The \( i \)th particle sees the metric:

\[ g_{\mu\nu}(x_i) = \exp(Q) \bar{g}_{\mu\nu}|_{x_j=r_j \text{ for } j \neq i} \]  

(199)

where \( r_j \) denotes the Bohmian trajectories. In this way one can see that there is no problem in geometrization of quantum effects for many–particle systems.

### 8.2 Spin

There are at least two approaches to include spin in de-Broglie–Bohm theory.

First one can see the spin as the rotation of the particle around itself\(^3\). This

\(^{3}\text{This can lead to superluminal angular velocities.}\)
approach can easily be used here. Only one needs to add the rotational degrees
degrees of freedom for the motion of the particle. This is surely straightforward.

The other approach looks at the spin (just like the Copenhagen quantum
mechanics) as an internal degree of freedom. So the spin is coded in the wave
function. There is also no difficulty in adopting this approach. To explain this,
first note that using the Wigner formulation of any spin wave equation[40, 36],
one has a Dirac–like wave equation for any spin:

\[
(\gamma^\mu \partial_\mu)_{\alpha_1} \psi_{\alpha_2 \ldots \alpha_{2s}} = \frac{im}{\hbar} \psi_{\alpha_1 \ldots \alpha_{2s}}
\]  

(200)

where the Dirac spinor of rank 2s is completely symmetric in its spinor indices.
For zero mass we also have the condition \(\gamma^5\psi_{\alpha_2 \ldots \alpha_{2s}} = \psi_{\alpha_1 \ldots \alpha_{2s}}\). The velocity
field compatible with this wave equation is the Dirac current field \(j^\mu\). Now
one must add the continuity equation which can be seen as an equation for
defining the ensemble density \(\rho\), and define the quantum potential in terms of
it as usual. So the space–time metric is \(g_{\mu\nu} = \exp(Q)\bar{g}_{\mu\nu}\).

9 Conclusion

We have used an approach which is different from other existing ones, which
try to combine the gravitational and quantal effects. By the investigation of
the quantum effects of matter in the framework of Bohmian mechanics, we have shown that the motion of a particle with quantum effects is equivalent to its motion in a curved space-time. We have investigated the coupling of purely gravitational effects and purely quantal effects of the particle, by considering a general background space-time metric. The use of the de-Broglie–Bohm quantum theory of motion, instead of the standard Copenhagen quantum mechanics, has at least three advantages. First, that the inherent problems of the standard quantum mechanics are not present. Second, that the conceptual problems of the standard quantum gravity like the meaning of the wave-function, are circled. Finally, the equivalence of quantum effects of matter and a curved space-time, which is our most important result, is achieved through this point of view. This leads directly to the minisuperspace of conformal degree of freedom, in which, the conformal factor has now a clear physical meaning. Other problems like, time- dependence which is necessary to understand the evolution of the universe, consideration of the back–reaction effects, and so on, are also handled in the present theory. An important property of this theory is that the conservation laws are the same as those of the classical theory. As it is shown, if one applies this theory to the case of quantum cosmology, it is possible to solve the equations of motion nonperturbatively (i.e.
exactly). One sees that there is no singularity at small times (provided one assumes that the quantum coupling constant of radiation is negative). It is smeared out by the quantum effects. Two remarks are in order here. First, in principle, the above model can be applied to the non flat Friedmann universes and similar results would emerge. Second, in the present theory, the quantum effects are only those of matter. The effect of these quantal behaviours on the background space-time is achieved via the modified Einstein equations, i.e. via the back-reaction effects. This means that if one removes the matter, the quantum effects of the background geometry would disappear. In order to generalize the present theory to a fully theory of quantum gravity, the quantum effects of gravity must be included.

As explained before the keystone of Bohm’s theory is the quantum potential. Any particle is acted upon by a quantum force derived from the quantum potential. The quantum potential is itself resulted from some self-field of the particle, the wave function. Since the quantum potential is related only to the norm of the wave function and because of Born’s postulate asserting that the ensemble density of the particle under consideration is given by the square of the norm of the wave function, the quantum potential is obtained resulted from the ensemble density. The non understandable point of Bohm’s theory is
just this. How does a particle know about its hypothetical ensemble? When the hypothetical ensemble is a real one, i.e. when there are is large number of similar particles just like the particle under consideration, quantum potential can be understood. It is a kind of interaction between the particles in the real ensemble. But when one deals with only one particle the quantum potential is interaction with the other hypothetical particles!!!

On the other hand, quantum potential is highly related to the conformal degree of freedom of the space–time metric. In fact, the presence of the quantum force is just like having a curved space–time which is conformally flat and the conformal factor is expressed in terms of the quantum potential. In this way one sees that quantum effects are in fact geometric effects. Geometrization of the quantum theory can be done successfully, but still, there is the problem of the ensemble noted above.

Here we have shown that if one tries to geometrize the quantum effects in a purely metric way, the ensemble problem would be overcome. In addition it provides the framework for bringing in the purely quantum gravity effects.

A point about the geodesic equation must be noted here. In the background metric, this equation resembles the geodesic equation in Brans–Dicke theory. Consideration of the matter quantum effects, leads to the physical metric
in which a particle moves on the geodesic of Branse–Dicke theory written in Einstein gauge. This point supports the suggestion that the discussion of quantum gravity requires a scalar–tensor theory. Previously this was suggested when discussing Bohmian quantum gravity [41].

Next it is shown that it is possible to write a scalar–tensor theory which automatically leads to the correct equations of motion. This has the advantage that the conformal factor would be fixed by the equations of motion, and not by introducing a lagrangian multiplier by hand. We see that the matter distribution determines the local curvature of the space–time (in conformity with Mach’s principle). Furthermore, from the matter equation of motion one can see that the cosmological constant (a large scale structure constant) and the quantum potential\(^4\) are coupled together. This is another manifestation of the Mach’s principle.

Also we have constructed a scalar–tensor theory with two scalar fields, for which the equations of motion lead to the correct form of quantum potential. We have not only shown that quantum effects are geometrical in nature, but also derive the form of quantum potential. This specific form for quantum potential\[^4^4^4\] in Bohmian quantum mechanics, observable effects of quantum potential may appear at both large and small scales, depending on the shape of the ensemble density\[^3^3^3\].
potential is a result of the equations of motion.

It must be noted that in this theory both the scalar fields interact with the cosmological constant. Hence, the presence of the cosmological constant (even very small) is essential in order the theory works. Note that the interaction between the cosmological constant and quantum potential represents a connection between the large and small scale structures.

We presented a toy model, and investigated its solutions. Also it is shown that the initial singularity is removed by quantum effects.

Finally we saw that one can formulate a generalized equivalence principle which states that gravitation can be removed locally via an appropriate coordinate transformation, while quantum force can be removed either locally or globally via an appropriate scale transformation. So the natural framework of quantum and gravity is Weyl geometry. The most simplest Weyl invariant action functional is written out. It surprisingly leads to the correct Bohm’s equations of motion. When it applied to cosmology it leads to time decreasing cosmological and gravitational constants. A phenomena which is good for describing their small values.
References

[1] S. Weinberg, *Gravitation and Cosmology*, John Wiley and sons, New York, 1972.

[2] A. Ashtekar, and J. Stachel, Eds. *Conceptual problems of quantum gravity*, The center for Einstein’s studies, Boston University, 1991.

[3] D. Bohm, *Phys. Rev.* 85, 166, 1952; D. Bohm, *Phys. Rev.* 85, 180, 1952; D. Bohm, and B. J. Hiley, *The Undivided Universe*, Routledge, 1993; P. R. Holland, *The Quantum Theory of Motion*, Cambridge University Press, 1993.

[4] *Bohmian Mechanics and Quantum Theory: An Appraisal* Boston Studies in the philosophy of science, Vol. 184, Ed. J.T. Cushing, A. Fine, and S. Goldstein, Kluwer Academic Publishers, 1996.

[5] T. Horiguchi, *Mod. Phys. Lett. A.*, 9, 16, 1429, 1994; J.K. Glikman and J.C. Vink, *Class. Quant. Grav.*, 7, 901, 1990; A. Blaut and J.K. Gilkman, *Class. Quant. Grav.*, 13, 39, 1996; A. Valentini, PHD dissertation, 1992; J. Marto, P.V. Moniz, *Phys. Rev. D.*, 65, 023516, 2001; N. Pinto–Neto, A.
F. Velasco, arXiv:gr-qc/0001074; N. Pinto–Neto, and E. Sergio Santini, arXive:gr-qc/0302112.

[6] J.V. Narlikar and T. Padmanabhan, Phys. Rep., 100, 3, 151-200, 1983; T. Padmanabhan, Gravitation, Guage theories and the Early universe, eds., B.R. Iyer et. al., 373-404, Kluwer Academic Publishers, 1989.

[7] A.K. Kembhavi, Mon. Not. Roy. Soc. A185, 807, 1978.

[8] L. de-Broglie, Non-linear Wave Mechanics, translated by A.J. Knodel, Elsevier Publishing Company, 1960.

[9] F. Shojai, and M. Golshani, Int. J. Mod. Phys. A., 13, 4, 677, 1998.

[10] H. Motavali, H. Salehi, and M. Golshani, arXiv:hep-th/0011062; H. Motavali, and M. Golshani, arXiv:hep-th/0011064; H. Motavali, and M. Golshani, Int. J. Mod. Phys. A., 17, 3, 375, 2002.

[11] F. Shojai, A. Shojai, and M. Golshani, Mod. Phys. Lett. A., 13, 34, 2725, 1998.

[12] A. Shojai, Int. J. Mod. Phys. A., 15, 12, 1757, 2000.

[13] F. Shojai, A. Shojai, and M. Golshani, Mod. Phys. Lett. A., 13, 36, 2915, 1998.
[14] F. Shojai, and A. Shojai, *Int. J. Mod. Phys. A.*, **15**, 13, 1859, 2000.

[15] J.D. Barrow, and K.E. Kunze, arXive:gr-qc/9807040.

[16] V. Faraoni, E. Gunzig, and P. Nardone, arXive:gr-qc/9811047; J.J. Levin, *Phys. Rev. D.*, **51**, 1536, 1995.

[17] A. Shojai, and F. Shojai, *Space–time and Substance*, **2**, 3, 134, 2001.

[18] G. Börner, *The early Universe*, 3rd ed. Springer Verlag, Berlin Heidelberg, 1993.

[19] J.S. Bell, *Physics*, **1**, 195, 1965; J.F. Clauser, and A. Shimony, *Rep. Prog. Phys.*, **41**, 1881, 1978.

[20] A. Shojai, F. Shojai, and M. Golshani, *Mod. Phys. Lett. A.*, **13**, 37, 2965, 1998.

[21] P.S. Wesson, *Space, Time, Matter: Modern Kaluza–Klein Theory*, World Scientific, Singapore, 1990.; J.M. Overduin, P.S. Wesson, *Phys. Rep.*, **283**, 303, 1997.

[22] H. Weyl, *Sitzungsber. Peruss. Akad. Wiss*, 465, 1918; R. Adler, M. Bazin, and M. Schiffer, *Introduction to general relativity*, McGraw Hill Book Company, 1975.
[23] A. Einstein, *Rev. Mod. Phys.*, **20**, 35, 1948; E. Schrödinger, *Proc. Roy. Irish Akad.*, **56**, 13, 1954.

[24] S.D. Odintsov, *Zeit. Physik C*, **54**, 531, 1992; E. Elizalde, and S. D. Odintsov, *Int. J. Mod. Phys. D.*, **2**, 51, 1993; I. Anthoniadis and S. D. Odintsov, *Mod. Phys. Lett. A*, **8**, 979, 1993; E. Elizalde, A. Jacksenaev, S. D. Odintsov, and I. Shapiro, *Phys. Lett. B*, **328**, 297, 1994; E. Elizalde, A. Jacksenaev, S. D. Odintsov, and I. Shapiro, *Class. Quant. Grav.*, **12**, 1385, 1995; S. D. Odintsov, and R. Percacci, *Mod. Phys. Lett. A.*, **9**, 2041, 1994; E. Elizalde, and S. D. Odintsov, *Phys. Lett. B*, **334**, 33, 1994; E. Elizalde, S. D. Odintsov, and A. Romeo, *Int. J. Mod. Phys. A.*, **11**, 4435, 1996; E. Elizalde, S. D. Odintsov, and A. Romeo, *Nucl. Phys. B*, **462**, 315, 1996; L.N. Granda, and S. D. Odintsov, *Grav. Cos.*, **4**, 85, 1998.

[25] J.T. Wheeler, *Phys. Rev. D.*, **41**, 2, 431, 1990.

[26] W.R. Wood, and G. Papini, *A geometric approach to the quantum mechanics of de-Broglie–Bohm and Vigier*, in "The present status of quantum theory of light", Proc. of Symposium in honor of J.P. Vigier, York University, Aug. 27–30, 1995.

[27] E. Santamato, *Phys. Rev. D.*, **32**, 10, 2615, 1985.
[28] B.G. Sidhart, arXive:physics/0211012.

[29] I. Quiros, arXive:gr-qc/9905071; I. Quiros, R. Bonal, R. Cardenas, arXive:gr-qc/9908075; R. Bonal, I. Quiros, R. Cardena, arXive:gr-qc/0010010; I. Quiros, arXive:gr-qc/0004014.

[30] S.W. Hawking, *The Quantum Theory of the Universe*, in Intersection between elementary particle physics and cosmology, Vol. 1, eds., T. Piran and S. Weinberg, Jerusalem, 28 Dec. 1983 - 6 Jan. 1984, World Scientific, 1984.

[31] A. Shojai, and F. Shojai, *Physica Scripta*, 64, 413, 2001.

[32] Y.M. Cho, and D.H. Park, *Nuvou Cimento B*, 105, 817, 1990.

[33] F. Shojai, A. Shojai, *Gravitation and Cosmology*, 9, 3(35), 163, 2003.

[34] See for example: *Mach’s Principle: From Newton’s Bucket to Quantum Gravity*, eds. J.B. Barbour and H. Pfister, Birkhäuser, 1995.

[35] P.A.M. Dirac, *Proc. Roy. Soc. Lond. A.*, 333, 403, 1973; N. Rosen, *Fond. Phys.*, 12, 3, 213, 1982.

[36] E.A. Lord, *Tensors, Relativity and Cosmology*, Tata McGraw Hill publishing, co. Ltd., NewDelhi, 1976.
[37] S.M. Carroll, *Living Rev. Relativity*, 4, 2001, (online at: http://www.livingreviews.org/Articles/Volume4/2001-1Carroll/).

[38] T. Horiguchi, *Il Nuovo Cimento B*, 112, 9, 1227, 1997.

[39] Y. Bisabr, and H. Salehi, *Class. Quant. Grav.*, 19, 2369, 2002.

[40] H. Bargmann, and E.P. Wigner, *Proc. Nat. Acad. Sci.*, 34 (5), 211, 1946.

[41] F. Shojai, and M. Golshani, *Int. J. Mod. Phys. A.*, 13, 13, 2135, 1998.