Incidence bicomodules, Möbius inversion, and a Rota formula for infinity adjunctions

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Abstract

In the same way decomposition spaces, also known as unital 2-Segal spaces, have incidence (co)algebras, and certain relative decomposition spaces have incidence (co)modules, we identify the structures that have incidence bi(co)-modules: they are certain augmented double Segal spaces subject to some exactness conditions. We establish a Möbius inversion principle for (co)modules, and a Rota formula for certain more involved structures called Möbius bicomodule configurations. The most important instance of the latter notion arises as mapping cylinders of infinity adjunctions, or more generally of adjunctions between Möbius decomposition spaces, in the spirit of Rota’s original formula.

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Introduction

The theory of Möbius categories, developed by Leroux ([16], 1976) generalises the theory for locally finite posets ([20] Rota, 1966), and Cartier–Foata finite-decomposition monoids ([5], 1969) which admit incidence (co)algebras and a Möbius inversion principle. It has recently been generalised to infinity categories and decomposition spaces by Gálvez-Carrillo, Kock, and Tonks ([8] and [9]).

A classical formula of Rota [20, Theorem 1] compares the Möbius functions across a Galois connection. The original goal of the present work, thought to be a routine exercise, was to generalise this formula to infinity adjunctions. It turned out a lot of machinery was required to do this in a satisfactory way, and developing this machinery ended up as a substantial contribution, warranting the present paper: they are general constructions in the theory of decomposition space/2-Segal spaces, concerning bicomodules which are of interest not just in combinatorics but also in representation theory, in connection with Hall algebras.
Before stating and proving this formula for categories, let us recall some definitions.

**Incidence coalgebras**

Given a small category $X$, write $X_0$ for its set of objects and $X_1$ for its set of arrows. Let $Q_{X_1}$ be the free vector space on $X_1$. We say a category $X$ is *locally finite* if each morphism $f : x \rightarrow z$ in $X$ admits only finitely many two-step factorisations $x \xrightarrow{g} y \xrightarrow{h} z$. This condition guarantees that the comultiplication on $Q_{X_1}$, given by

$$
\Delta : Q_{X_1} \rightarrow Q_{X_1} \otimes Q_{X_1}
$$

$$
f \mapsto \sum_{hg = f} g \otimes h
$$

is well defined. The counit $\delta : Q_{X_1} \rightarrow \mathbb{Q}$ is given by $\delta(\text{id}_x) = 1$, and $\delta(f) = 0$ else.

The *incidence algebra* $I_X$ is the linear dual, $(\text{Lin}(Q_{X_1}, \mathbb{Q}), * , \delta)$ with the convolution product:

$$(\alpha * \beta)(f) = \sum_{hg = f} \alpha(g) \beta(h),$$

where $\alpha, \beta \in I_X$ and $f \in Q_{X_1}$.

The *zeta function* $\zeta_X : Q_{X_1} \rightarrow \mathbb{Q}$ is defined by $\zeta_X(f) = 1$ for all $f \in Q_{X_1}$.

Define $\Phi_{\text{even}} : Q_{X_1} \rightarrow \mathbb{Q}$ to be the number of even-length factorisations of a morphism, without identities, and $\Phi_{\text{odd}} : Q_{X_1} \rightarrow \mathbb{Q}$ to be the number of odd-length factorisations, without identities. A category is *Möbius* if it is locally finite and $\Phi_{\text{even}}$ and $\Phi_{\text{odd}}$ are finite.

**Theorem** (Content, Lemay, Leroux [6]). If $X$ is a Möbius category then the zeta function is invertible, and the inverse, called the Möbius function, is given by $\mu = \Phi_{\text{even}} - \Phi_{\text{odd}}$.

**Rota formula for categories**

A classical formula due to Rota ([20], 1964) compares the Möbius functions of two posets related by a *Galois connection*; it generalises to adjunctions between Möbius categories as follows.

**Theorem** (Rota formula). Let $X$ and $Y$ be Möbius categories, and let $F : X \rightleftarrows Y : G$ be an adjunction, $F \dashv G$. Then for all $x \in X, y \in Y$,

$$
\sum_{\substack{x' \in X \\ f : x \rightarrow x' \\ Fx' = y}} \mu_X(f) = \sum_{\substack{y' \in Y \\ g : y' \rightarrow y \\ Gy = x}} \mu_Y(g).
$$

The reader is not expected to read the following elementary proof, but only notice that it looks like an associativity formula for a convolution product, except that the arrows live in different categories.
Proof of the Rota formula.

\[
\sum_{x' \in X} \mu_X(x \xrightarrow{f} x') = \sum_{x' \in X} \mu_X(x \xrightarrow{f} x') \delta_Y(Fx' \xrightarrow{h} y)
\]

\[
= \sum_{x' \in X} \mu_X(x \xrightarrow{f} x') \left( \sum_{y' \in Y} \zeta_Y(Fx' \xrightarrow{h'} y') \mu_Y(y' \xrightarrow{g} y) \right)
\]

\[
= \sum_{x' \in X} \mu_X(x \xrightarrow{f} x') \zeta_X(x' \xrightarrow{k'} Gy') \mu_Y(y' \xrightarrow{g} y) \quad \text{by adjunction}
\]

\[
= \sum_{y' \in Y} \delta_X(x \xrightarrow{k} Gy') \mu_Y(y' \xrightarrow{g} y) = \sum_{y' \in Y} \mu_Y(y' \xrightarrow{g} y).
\]

\[\square\]

In the main result of the present paper, theorem 4.5.2, we write this formula as

\[\mu_X \star_l \delta_Y = \delta_X \star_r \mu_Y,\]

with the following more conceptual proof, valid also for adjunctions between Möbius decomposition spaces:

\[\mu_X \star_l \delta_Y = \mu_X \star_l (\zeta \star_r \mu_Y) = (\mu_X \star_l \zeta) \star_r \mu_Y = \delta_X \star_r \mu_Y,\]

referring to certain left and right convolution actions. Two ingredients are necessary to make sense of this pleasing proof: one is to exhibit the data necessary to induce bicomodules and establish that adjunctions constitute an example. The other is
to establish a Möbius inversion principle for (co)modules, a notion which has not previously been considered in the literature, to the knowledge of the author.

Although the motivation and the statement of the theorem belongs to combinatorics, the setting for this work and the tools employed are from simplicial homotopy theory, in the style of [8], [9], working with ∞-groupoids, homotopy pullbacks, mapping spaces, fibrations and fibre sequences. One technical novelty compared to [8] and [9] is that we exploit general simplicial maps between decomposition spaces, not just CULF ones, and introduce the notion of adjunction between decomposition spaces. Another is that the notion of mapping cylinder is exploited systematically: on one hand locally to model the shapes needed to index the various configurations, and on the other hand globally, as infinity mapping cylinders.

Outline of the paper

We begin in section 1 with a brief review of needed notions from the theory of ∞-categories, with an emphasis on decomposition spaces. In section 2, following Walde [21] and Young [22], we first explain how to obtain a comodule in the context of decomposition spaces.

**Proposition 2.1.1.** If \( f : C \to X \) is a CULF map between two simplicial ∞-groupoids such that \( C \) is Segal and \( X \) is a decomposition space, then the span

\[
C_0 \xleftarrow{d_1} C_1 \xrightarrow{(f_1,d_0)} X_1 \times C_0
\]

induces on the slice ∞-category \( S/C_0 \) the structure of a left \( S/X_1 \)-comodule, and the span

\[
C_0 \xleftarrow{d_0} C_1 \xrightarrow{(d_1,f_1)} C_0 \times X_1.
\]

induces on \( S/C_0 \) the structure of a right \( S/X_1 \)-comodule.

The data needed to obtain a comodule is called a **comodule configuration**. In order to obtain a bicomodule structure, an augmented bisimplicial ∞-groupoid Segal in each direction is required, which is furthermore required to be stable, see subsection 2.3. This stability condition is a pullback condition on certain squares, and is a ∞-categorical reformulation of the notion of [3], suitable for ∞-groupoids.

**Theorem 2.4.1.** Let \( B \) be an augmented stable double Segal space, and such that the augmentation maps are CULF. Suppose moreover \( X := B_{-1} \) and \( Y := B_{-1} \) are decomposition spaces. Then the spans

\[
B_{0,0} \xleftarrow{e_1} B_{1,0} \xrightarrow{(u,e_0)} X_1 \times B_{0,0}
\]

and

\[
B_{0,0} \xleftarrow{d_0} B_{0,1} \xrightarrow{(d_1,v)} B_{0,0} \times Y_1
\]

induce on \( S/B_{0,0} \) the structure of a bicomodule over \( S/X_1 \) and \( S/Y_1 \).

An augmented bisimplicial ∞-groupoid satisfying the conditions of the theorem is called a **bicomodule configuration**.

In section 3, we first show that any correspondence of decomposition spaces gives rise to a bicomodule configuration. We then introduce the notion of **cartesian**
and **cocartesian fibration** of decomposition spaces, adapting a homotopy-invariant definition for \(\infty\)-categories which can be found in [1]. We construct a category of shape like \(\Delta/\Delta^1\), but with certain extra diagonal maps, and show how to obtain from any (co)cartesian fibration a comodule configuration with such extra diagonal maps and sections. We define an **adjunction** between decomposition spaces \(X\) and \(Y\) to be a simplicial map between decomposition spaces \(p: M \to \Delta^1\) which is both a cartesian and a cocartesian fibration, equipped with equivalences \(X \simeq M\{0\}\) and \(Y \simeq M\{1\}\).

In section 4, we define left and right convolution actions \(\star_l\) and \(\star_r\) dual to the comodule structures. The following is a consequence of theorem 2.4.1.

**Proposition 4.3.1.** Given a bicomodule configuration, the left and right convolutions satisfy the associative law (in a strong homotopy sense)

\[
\alpha \star_l (\theta \star_r \beta) \simeq (\alpha \star_l \theta) \star_r \beta.
\]

We then establish in subsection 4.4 a Möbius inversion principle for complete comodules. Let \(C \to Y\) be a right comodule configuration such that the simplicial \(\infty\)-groupoid \(C\) is augmented and with new bottom degeneracies \(s_{-1} : C_{n-1} \to C_n\) which are sections to \(d_0\). We say it is complete (subsection 4.2) if the sections \(s_{-1}\) are monomorphisms.

For a complete decomposition space \(Y\), the spans \(Y_1 \leftarrow \overrightarrow{Y}_n \to 1\), where \(\overrightarrow{Y}_n\) is the full subgroupoid of simplices with all principal edges nondegenerate, define linear functors, the **Phi functors** \(\Phi_n : S/Y_1 \to S\). We also put \(\Phi_{\text{even}} := \sum_{n \text{ even}} \Phi_n\), and \(\Phi_{\text{odd}} := \sum_{n \text{ odd}} \Phi_n\).

The zeta functor \(\zeta : S/C_0 \to S\) is the linear functor defined by the span \(C_0 \leftarrow C_0 \to 1\), and \(\delta^R : S/C_0 \to S\) is the linear functor given by the span \(C_0 \leftarrow C_{-1} \to 1\). We define \(\delta^L\) similarly for left comodule configurations.

**Theorem 4.4.3 and 4.4.4.** Given \(C \to Y\) a complete right comodule configuration and \(D \to X\) a complete left comodule configuration, then

\[
\zeta^C \star_r \Phi^Y_{\text{even}} = \delta^R + \zeta^C \star_r \Phi^Y_{\text{odd}},
\]

\[
\Phi^X_{\text{even}} \star_l \zeta^D = \delta^L + \Phi^X_{\text{odd}} \star_l \zeta^D.
\]

In subsection 4.5, we establish a Möbius inversion principle at the algebraic level. To this end, we need to impose some finiteness conditions in order to take homotopy cardinality. Define the **Möbius functions** as the homotopy cardinalities \(|\mu^Y| := |\Phi^Y_{\text{even}}| - |\Phi^Y_{\text{odd}}|\) and \(|\mu^X| := |\Phi^X_{\text{even}}| - |\Phi^X_{\text{odd}}|\).

**Theorem 4.5.1.** Given \(C \to Y\) a right Möbius comodule configuration and \(D \to X\) a left Möbius comodule configuration,

\[
|\zeta^C| \star_r |\mu^Y| = |\delta^R|, \quad |\mu^X| \star_l |\zeta^D| = |\delta^L|.
\]
Finally we can extend the Rota formula to bicomodules with Möbius inversion for both comodules, called Möbius bicomodule configurations. Combining Proposition 4.3.1 and Theorem 4.5.1, we obtain the main theorem of the present paper:

**Theorem 4.5.2.** Given a Möbius bicomodule configuration $B$ with $X := B_{\ast,-1}$ and $Y := B_{-1,\ast}$, we have

$$|\mu^X| \ast_l |\delta^R| = |\delta^L| \ast_r |\mu^Y|,$$

where $\delta^R$ is the linear functor given by the span

$$B_{0,0} \leftarrow Y_0 \rightarrow 1,$$

and $\delta^L$ is the linear functor given by the span

$$B_{0,0} \leftarrow X_0 \rightarrow 1.$$

The motivating example, treated in subsection 4.6 shows that any (co)cartesian fibration $p : M \rightarrow \Delta^1$ such that $M$ is a complete decomposition space gives rise to a complete left (or right) comodule configuration:

**Lemma 4.6.3.** Given an adjunction of decomposition spaces in the form of a bicartesian fibration $p : M \rightarrow \Delta^1$, suppose moreover that $M$ is a Möbius decomposition space. Then the bicomodule configuration extracted from this data is Möbius. In particular, we have the Rota formula for the adjunction $p$:

$$|\mu^X| \ast_l |\delta^R| = |\delta^L| \ast_r |\mu^Y|.$$ 

When specialised to the case of a classical adjunction between 1-categories, this is the classical Rota formula from page 2.

**Acknowledgements**

The author would like to thank Joachim Kock not only for suggesting to investigate this Rota formula but also for help and support all along the project, and also Christian Sattler for useful remarks. The author was supported by PhD grant attached to MTM2013-42293-P, and grant number MTM2016-80439-P of Spain.

1 Preliminaries

We work in the $\infty$-category of $\infty$-groupoids, denoted $S$, following the notation of [8]. Our $\infty$-categories are quasi-categories; the theory of quasi-category has been substantially developed by Joyal [13, 14] and Lurie [17]. An $\infty$-groupoid is an $\infty$-category in which all morphisms are invertible. They are precisely Kan complexes: simplicial sets in which every horn admits a filler (and not only the inner ones). However, since the infinity category theory needed here is elementary, it is possible to work with infinity categories model-independently.
1.1 Pullbacks and fibres

The main tool used throughout this paper are pullbacks. We use the following standard lemma many times.

**Lemma 1.1.1.** Given a prism diagram of ∞-groupoids

\[
\begin{array}{ccc}
X & \longrightarrow & X' \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Y'
\end{array}
\quad \longrightarrow
\quad
\begin{array}{ccc}
X'' & \longrightarrow & X' \\
\downarrow & & \downarrow \\
Y'' & \longrightarrow & Y''
\end{array}
\]

in which the right-hand square is a pullback. Then the outer rectangle is a pullback if and only if the left-hand square is.

**Remark 1.1.2.** We talk about a prism, it is a $\Delta^1 \times \Delta^2$-diagram, so consisting of three squares and two triangles. We have not drawn the square whose horizontal sides are composites of the horizontal arrows. The triangles are not drawn either, they are the fillers that exist by the axioms of ∞-categories. For a proof, see Lurie [17, Lemma 4.4.2.1] for the dual case of pushouts.

Given a map of ∞-groupoid $p : X \to S$, an object $s \in S$, the fibre $X_s$ of $p$ over $s$ is the pullback

\[
\begin{array}{ccc}
X_s & \longrightarrow & X \\
\downarrow & & \downarrow \\
1 & \longrightarrow & S
\end{array}
\]

A map of ∞-groupoids is a **monomorphism** when its fibres are $(-1)$-groupoids, that is, are either empty or contractible. If $f : X \to Y$ is a monomorphism, then there is a complement $Z := Y \setminus X$ such that $Y \simeq X + Z$; a monomorphism is essentially an equivalence from $X$ onto some connected components of $Y$.

1.2 Slices and linear functors

Recall that the objects of the slice ∞-category $S/I$ are maps of ∞-groupoids with codomain $I$. Pullback along a morphism $f : J \to I$, defines an functor $f^* : S/J \to S/I$. This functor is right adjoint to the functor $f_! : S/I \to S/J$ given by post-composing with $f$. A span $I \leftarrow M \rightarrow J$ induces a functor between the slices by pullback and postcomposition

\[
S/I \xrightarrow{f^*} S/M \xrightarrow{q_!} S/J.
\]

A functor is **linear** if it is homotopy equivalent to a functor induced by a span. The following Beck-Chevalley rule holds for ∞-groupoids: for any pullback square

\[
\begin{array}{ccc}
J & \longrightarrow & I \\
\downarrow & & \downarrow \\
V & \longrightarrow & U
\end{array}
\]

\[
\begin{array}{ccc}
J & \longrightarrow & I \\
\downarrow & & \downarrow \\
V & \longrightarrow & U
\end{array}
\]
the functors $p_! f^*, g_! q^*: S/I \to S/V$ are naturally homotopy equivalent (see [12] for the technical details regarding coherence of these equivalences). By the Beck-Chevalley rule, the composition of two linear functor is linear. For an extended treatment of linear functors and homotopy linear algebra, we refer to [11].

1.3 Simplicial infinity-groupoids, Segal condition

We consider the functor $\infty$-category

$$\text{Fun}(\Delta^{op}, S)$$

whose objects are functors from the $\infty$-category $\Delta^{op}$ to the $\infty$-category $S$. The simplicial identities are not strictly commutative squares, but equipped with homotopies, those homotopies are to then be coherently linked by higher homotopies and so on.

A simplicial $\infty$-groupoid $X$ is called a Segal space if the following squares are pullbacks, for all $n > 0$:

$$
\begin{array}{ccc}
X_{n+1} & \xrightarrow{d_0} & X_n \\
\downarrow^{d_{n+1}} & & \downarrow^{d_n} \\
X_n & \xrightarrow{d_0} & X_{n-1}.
\end{array}
$$

1.4 Decomposition spaces

The simplex category $\Delta$ has an active-inert factorisation system. A morphism $[m] \to [n]$ is active (also called generic) if it preserves endpoints: $g(0) = 0, g(m) = n$. A morphism is inert (also called free) if it is distance preserving: $f(i+1) = f(i) + 1$, for $0 \leq i \leq m - 1$. The active maps are generated by the codegeneracy maps and the inner coface maps, and the inert maps are generated by the outer coface maps $d^\perp := d_0$ and $d^\top := d^n$.

A decomposition space $X : \Delta^{op} \to S$ is a simplicial $\infty$-groupoid such that the image of any pushout diagram in $\Delta$ of an active map $g$ along an inert map $f$ is a pullback of $\infty$-groupoids. It is enough to check that the following squares are pullbacks:

$$
\begin{array}{ccc}
X_{n+1} & \xrightarrow{s_{k+1}} & X_{n+2} \\
\downarrow^{d_\perp} & & \downarrow^{d_\perp} \\
X_n & \xrightarrow{s_k} & X_{n+1}.
\end{array}
\quad
\begin{array}{ccc}
X_{n+1} & \xrightarrow{s_k} & X_{n+2} \\
\downarrow^{d_\top} & & \downarrow^{d_\top} \\
X_n & \xrightarrow{s_k} & X_{n+1}.
\end{array}
$$

The notion of decomposition space was introduced by Gálvez-Carrillo, Kock, and Tonks [8], and independently by Dyckerhoff and Kapranov [7] under the name unital 2-Segal space. It can be seen as an abstraction of posets. It is precisely the condition required to obtain a counital coassociative comultiplication on $S_{/X_1}$, see also [19] for the exact role played by the decomposition space condition. Since the motivation in the present paper comes from combinatorics, we follow the terminology of [8]; for a survey motivated by combinatorics, see [10].
Proposition 1.4.1 ([7, Proposition 2.3.3], [8, Proposition 3.5]). Every Segal space is a decomposition space.

There are plenty of examples of decomposition spaces which are not Segal, e.g. Schmitt’s Hopf algebra of graphs, which is a running example in [9].

1.5 Incidence (co)algebras

For any decomposition space $X$, we get an incidence algebra $S_{/X_1}$. The span $X_1 \leftarrow X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1$ defines a linear functor, the comultiplication:

$$\Delta : S_{/X_1} \to S_{/X_1 \times X_1}$$

$$(T \xrightarrow{t} X_1) \mapsto (d_2,d_0)_*(t).$$

The span $X_1 \leftarrow X_0 \xrightarrow{z} 1$ defines a linear functor, the counit:

$$\delta : S_{/X_1} \to S$$

$$(T \xrightarrow{t} X_1) \mapsto z \circ s^*_0(t).$$

The up-to-coherent-homotopy coassociativity follows from the decomposition space axioms, see [8, sections 5 and 7] or [19, subsection 4.3] for a proof. We obtain a coalgebra $(S_{/X_1}, \Delta, \delta)$ called the incidence coalgebra.

The category $S_{/I}$ plays the role of the vector space with basis $I$. The presheaf category $S^I$ can be considered the linear dual of the slice category $S_{/I}$ (see [11] for the precise statements and proofs). If $X$ is a decomposition space, the coalgebra structure on $S_{/X_1}$ therefore induces an algebra structure on $S^{X_1}$. In details, the convolution product of two linear functors $F,G : S_{/X_1} \to S$, given by the spans $X_1 \leftarrow M \to 1$ and $X_1 \leftarrow N \to 1$, is the composite of their tensor product $F \otimes G$ with the comultiplication:

$$F \ast G : S_{/X_1} \xrightarrow{\Delta} S_{/X_1} \otimes S_{/X_1} \xrightarrow{F \otimes G} S \otimes S \simeq S,$$

where the tensor product $F \otimes G$ is given by the span $X_1 \times X_1 \leftarrow M \times N \to 1$. The neutral element for convolution is

$$\delta : S_{/X_1} \to S$$

defined by the span $X_1 \leftarrow X_0 \to 1$.

1.6 Conservative ULF functors

A map $f : X \to Y$ of simplicial spaces is cartesian on an arrow $[n] \to [k]$ in $\Delta$ if the naturality square for $F$ with respect to this arrow is a pullback. It is called a right fibration if it is cartesian on $d_\perp$ and on all active maps, and is called a left fibration if it is cartesian on $d_\top$ and on all active maps.

A simplicial map $f : X \to Y$ is conservative if it is cartesian with respect to codegeneracy maps.
It is ULF (unique lifting of factorisations) if it is cartesian with respect to inner coface maps

\[
\begin{array}{ccc}
X_n & \xrightarrow{s_i} & X_{n+1} \\
\downarrow f_n & & \downarrow f_{n+1} \downarrow \,
\end{array}
\]

\[0 \leq i \leq n.\]

We write $\text{CULF}$ for conservative and ULF, that is cartesian on all active maps. The CULF functors induce coalgebra homomorphisms between the incidence algebras. They play an essential role in [8] and [9] as a natural notion of morphism between decomposition spaces, but the present paper deals with general simplicial maps.

## 2 Bicomodules

### 2.1 Comodules

The theory of modules in the context of decomposition spaces has been developed by Walde [21] in 2016, and independently by Young [22] in 2016, both in the context of Hall algebras. They call them relative 2-Segal spaces. Here we give a conceptual way to reformulate their definitions using linear functors.

Given a map between two simplicial $\infty$-groupoids $f : C \to X$, the span $C_0 \leftarrow C_1 \xrightarrow{(f_1,d_0)} X_1 \times C_0$ defines a linear functor $\gamma_l : S_{/C_0} \to S_{/X_1} \otimes S_{/C_0}$, and the span $C_0 \leftarrow C_1 \xrightarrow{(d_1,f_1)} C_0 \times X_1$ defines a linear functor $\gamma_r : S_{/C_0} \to S_{/C_0} \otimes S_{/X_1}$.

**Proposition 2.1.1.** Let $f : C \to X$ be a map between two simplicial $\infty$-groupoids. Suppose moreover that $C$ is Segal, $X$ is a decomposition space and the map $f : C \to X$ is CULF, then the span

\[
C_0 \xleftarrow{d_1} C_1 \xrightarrow{(f_1,d_0)} X_1 \times C_0
\]

induces on the slice $\infty$-category $S_{/C_0}$ the structure of a left $S_{/X_1}$-comodule, and the span

\[
C_0 \xleftarrow{d_0} C_1 \xrightarrow{(d_1,f_1)} C_0 \times X_1
\]

induces on $S_{/C_0}$ the structure of a right $S_{/X_1}$-comodule.

The data needed to obtain a comodule is called a *comodule configuration*, that is a CULF map from a Segal space to a decomposition space.
Remark 2.1.2. The relevance of the Segal condition on $C$ and the CULF condition on $f$ can be explained individually as follows. It is standard that for a category $C$, the coalgebra of arrows $C_1$ coacts on $C_0$: the coaction (from the right) is given by $b \mapsto \sum_{f:a \to b} a \otimes f$. Coassociativity of this coaction is equivalent to the Segal condition. Now a CULF map $C \to X$ defines a coalgebra homomorphism, and in this way, also $X_1$ coacts on $C_0$, by “corestriction of coscalars”.

Proof. We want to prove that the map $\gamma_l$ is a left $S_{/X_1}$-coaction. The desired homotopy coherent diagram

$$
\begin{array}{cccc}
S_{/C_0} & \xrightarrow{\gamma_l} & S_{/X_1 \times C_0} \\
\downarrow & & \downarrow \\
S_{/X_1 \times C_0} & \xrightarrow{\Delta \otimes \mathrm{Id}} & S_{/X_1 \times X_1 \times C_0}
\end{array}
$$

is induced by the solid spans in the diagram

$$
\begin{array}{cccc}
C_0 & \xleftarrow{d_1} & C_1 & \xrightarrow{(f_1,d_0)} & X_1 \times C_0 \\
\downarrow & & \downarrow & & \downarrow \\
C_1 & \xleftarrow{d_1} & C_2 & \xrightarrow{(d_2 f_1, d_0)} & X_1 \times C_1 \\
\downarrow & & \downarrow & & \downarrow \\
X_1 \times C_0 & \xleftarrow{d_1 \otimes \mathrm{Id}} & X_2 \times C_0 & \xrightarrow{(d_2, d_0) \otimes \mathrm{Id}} & X_1 \times X_1 \times C_0.
\end{array}
$$

The coassociativity will follow from Beck-Chevalley equivalences if we have the two pullbacks indicated in the diagram. The upper right-hand square is a pullback if and only if its composite with the second projection is a pullback. This composite outer square is a pullback because $C$ satisfies the Segal condition. Similarly, the lower left-hand square is a pullback if its composite with the first projection is a pullback. This composite outer square is a pullback because $f : C \to X$ is CULF.

Finding an equivalence is not enough, but the higher coherence can be established using similar techniques used in [8] or [19] to prove the coassociativity of the comultiplication defined on any decomposition space.

For categories, given a functor $f : C \to D$, define the mapping cylinder (or collage in [15]) $M_f$ to be the category where objects are either objects of $C$ or objects of $D$ and

$$
\Hom_{M_f}(x, y) = \begin{cases} 
\Hom_C(x, y) & \text{if } x, y \in C, \\
\Hom_D(x, y) & \text{if } x, y \in D, \\
\Hom_D(f(x), y) & \text{if } x \in C, y \in D, \\
\emptyset & \text{else}.
\end{cases}
$$

There exists a unique $p : M_f \to \Delta^1$ such that $p^{-1}(0) = C$ and $p^{-1}(1) = D$. This is moreover a cocartesian fibration, the cocartesian lift for $x \in C$ being given by
Id_{f(x)} \in \text{Map}_{M}(x, f(x)). The shape of a comodule configuration is that of $\text{(Id)}^{\text{op}}$, where $M_{\text{id}}$ is the mapping cylinder of the identity of $\Delta$.

Let $\Delta_{\text{bot}}$ be the simplex category of finite linear orders with a specified bottom element, and bottom-preserving monotone maps. The shape of the mapping cylinder of the functor $j : \Delta \to \Delta_{\text{bot}}$ freely adding a bottom element is the following. The shape of $\text{(M}_j\text{)}^{\text{op}}$ is:

\[
\begin{array}{cccccc}
X_0 & \overset{d_1}{\underset{d_0}{\rightarrow}} & X_1 & \overset{d_2}{\underset{d_0}{\rightarrow}} & X_2 & \cdots \\
C_{-1} & \overset{u}{\longleftarrow} & C_0 & \overset{d_1}{\underset{d_0}{\rightarrow}} & C_1 & \overset{d_2}{\underset{d_0}{\rightarrow}} & C_2 & \cdots \\
\end{array}
\]

This is the shape of what we call a right pointed comodule configuration: it is a comodule configuration $C \to X$ such that the Segal space $C$ is augmented, and with new bottom sections $s_{-1} : C_{n-1} \to C_n$. The importance of the pointing (the extra bottom degeneracy maps) is that it makes possible to formulate the notion of completeness and the condition locally finite length, see section 4.5 below; it guarantees the existence of a filtration on the associated comodule (see [9, section 6] for a similar argument), which is of independent interest.

### 2.2 Augmented bisimplicial infinity-groupoids

We shall establish conditions under which left and right comodule structures define a bicomodule. The main objects of interest are augmented bisimplicial $\infty$-groupoids subject to conditions, which are formulated in terms of pullbacks. We consider the functor $\infty$-category

\[\text{Fun}(\Delta^{\text{op}} \times \Delta^{\text{op}}, S)\]

whose objects are functors from the $\infty$-category $\Delta^{\text{op}} \times \Delta^{\text{op}}$ to the $\infty$-category $S$. The simplicial identities for a bisimplicial $\infty$-groupoid

\[B : \Delta^{\text{op}} \times \Delta^{\text{op}} \to S\]

are not strictly commutative squares, but homotopy coherent diagrams; a square comes equipped with an homotopy satisfying higher coherences.

A double Segal space is a bisimplicial $\infty$-groupoid satisfying the Segal condition for each restriction $\Delta^{\text{op}} \times \{[n]\} \to S$ (the columns) and $\{[n]\} \times \Delta^{\text{op}} \to S$ (the rows).

An augmented bisimplicial $\infty$-groupoid has in addition $\infty$-groupoids $B_{i,-1}$ and $B_{-1,i}$ of $(-1)$-simplices. We consider the functor $\infty$-category

\[\text{Fun}(\Delta_i^{\text{op}} \times \Delta_i^{\text{op}} \setminus \{-1, -1\}, S)\]

whose objects are augmented bisimplicial $\infty$-groupoids.
Remark 2.2.1. The shape of an augmented bisimplicial ∞-groupoid is also $(\Delta/\Delta^1)^{op}$. We denote $[i, j]$ the object given by the map $\Delta^{i+1+j} \to \Delta^1$ sending the $i+1$ first vertices to 0 and the others to 1. We allow $i$ or $j$ to be equal to $-1$ but not both. Maps $[i, j] \to [k, l]$ are given by the inclusions respecting the horizontal map. For example, the object $[2, 1]$ can be drawn as follows

\[
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array}
\]

where the horizontal maps lie over the map in $\Delta^1$.

We can draw $[i, j]$ as a column of $i+1$ black dots followed by $j+1$ white dots. Maps send black dots to black dots and white dots to white dots, without crossing.

We use the following notation for an augmented bisimplicial ∞-groupoid. We denote $d_k : B_{i,j} \to B_{i,j-1}$ and $e_l : B_{i,j} \to B_{i-1,j}$ the face maps, and $s_k : B_{i,j-1} \to B_{i,j}$ and $t_l : B_{i-1,j} \to B_{i,j}$ the degeneracy maps; $u$ and $v$ are the augmentation maps.

An augmented double Segal space satisfies that rows and columns are Segal. If we suppose that augmentations are CULF and $B_{•,-1}$ and $B_{-1,•}$ are decomposition spaces, we can use proposition 2.1.1 to obtain comodules:

**Proposition 2.2.2.** Let $B$ be an augmented double Segal space, and such that the augmentation maps are CULF. Suppose moreover $B_{•,-1}$ and $B_{-1,•}$ are decomposition spaces. Then the span

$B_{0,0} \xleftarrow{e_0} B_{1,0} \xrightarrow{(u,e_0)} B_{1,-1} \times B_{0,0}$

induces on $S/B_{0,0}$ the structure of a left comodule over $S/B_{1,-1}$, and the span

$B_{0,0} \xleftarrow{d_0} B_{0,1} \xrightarrow{(d_1,v)} B_{0,0} \times B_{-1,1}$

induces on $S/B_{0,0}$ the structure of a right comodule over $S/B_{-1,1}$.

**Proof.** This follows from proposition 2.1.1. □
2.3 Stability

We say a bisimplicial $\infty$-groupoid is stable if the following squares are pullbacks:

\[
\begin{array}{ccc}
B_{i-1,j-1} & \xleftarrow{d_k} & B_{i-1,j} \\
\downarrow e_l & & \downarrow e_l \\
B_{i,j-1} & \xleftarrow{d_k} & B_{i,j}
\end{array} \quad \begin{array}{ccc}
B_{i-1,j-1} & \xrightarrow{s_k} & B_{i-1,j} \\
\downarrow t_l & & \downarrow t_l \\
B_{i,j-1} & \xrightarrow{s_k} & B_{i,j}
\end{array}
\]

for all values of the indices except for $d_\perp$ along $e^\top$ and $d^\top$ along $e_\perp$.

**Remark 2.3.1.** A bisimplicial $\infty$-groupoid is stable if it satisfies all the following properties:

- $s_k : B_{i,j-1} \to B_{i,j}$ is a cartesian natural transformation, for all $0 \leq k \leq j - 1$;
- $d_k, k \neq \top, \perp$, is a cartesian natural transformation;
- $d^\top$ is a left fibration;
- $d_\perp$ is a right fibration.

**Remark 2.3.2.** Bergner, Osorno, Ozornova, Rovelli, and Scheimbauer introduced the notion of stable double category (bisimplicial set) in [3]: they define a double category to be stable if every square is uniquely determined by its span of source morphisms and, independently by its cospan of target morphisms. The present definition is a categorical reformulation of their notion suitable for $\infty$-groupoids. The motivation for the terminology is the following example. Let $C$ be a stable $\infty$-category (see [18, Chapter 1]). Define a double Segal space $B$ where $B_{0,0}$ is the $\infty$-groupoid of objects of $C$, where $B_{0,1}$ is the $\infty$-groupoid of arrows of $C$ (as in the fat nerve), and $B_{1,1}$ is the $\infty$-groupoid of pullback squares (equivalently, pushout squares). More generally, $B_{m,n}$ is the $\infty$-groupoid of $(\Delta^m \times \Delta^n)$-diagrams in $C$ for which all the rectangles are pullbacks (and hence pushouts). This is a stable bisimplicial $\infty$-groupoid (which of course is a double Segal space). This is almost by definition: since we only took pullback and pushout squares, they are determined by their sources by pushout or their targets by pullback, in the sense of our definition.

**Lemma 2.3.3.** Let $B$ be a double Segal space. Suppose we have the two following pullbacks:

\[
\begin{array}{ccc}
B_{0,0} & \xleftarrow{d_0} & B_{0,1} \\
\uparrow e_0 & & \uparrow e_0 \\
B_{1,0} & \xleftarrow{d_0} & B_{1,1}
\end{array} \quad \begin{array}{ccc}
B_{0,0} & \xrightarrow{d_1} & B_{0,1} \\
\uparrow e_1 & & \uparrow e_1 \\
B_{1,0} & \xrightarrow{d_1} & B_{1,1}
\end{array}
\]
then the double Segal space is stable.

Proof. First, the second pullback implies that every square with top maps is a pullback. Indeed, in the cube

```
B_{0,0} \leftarrow \downarrow d_{\top} \uparrow e_{\top} \downarrow \downarrow \downarrow \downarrow d_{\bot} \downarrow \downarrow \downarrow \downarrow B_{0,1}
```

```
B_{0,1} \leftarrow \downarrow d_{\top} \uparrow e_{\top} \downarrow \downarrow \downarrow \downarrow d_{\bot} \downarrow \downarrow \downarrow \downarrow B_{0,2}
```

```
B_{1,0} \leftarrow \downarrow d_{\top} \uparrow e_{\top} \downarrow \downarrow \downarrow \downarrow d_{\bot} \downarrow \downarrow \downarrow \downarrow B_{1,1}
```

```
B_{1,1} \leftarrow \downarrow d_{\top} \uparrow e_{\top} \downarrow \downarrow \downarrow \downarrow d_{\bot} \downarrow \downarrow \downarrow \downarrow B_{1,2}
```

the top and bottom squares are pullbacks because every row is Segal, and the back square is a pullback by hypothesis. Thus the rectangle consisting of bottom and back is a pullback because bottom and back squares are; front is a pullback because top and rectangle are. By induction, suppose the squares

```
B_{i-1,j-1} \leftarrow \downarrow d_{\top} \uparrow e_{\top} \downarrow \downarrow \downarrow \downarrow d_{\bot} \downarrow \downarrow \downarrow \downarrow B_{i,j}
```

are pullback, we can form cubes with the top and bottom faces pullbacks thanks to the Segal condition, and the back square is a pullback by hypothesis. This proves that every square involving top maps are pullbacks. Starting with the first pullback, we prove in the same way that every square involving bottom maps are pullbacks.

Now we want to prove that the following square is a pullback, for $0 < i < j$,

```
B_{k-1,j-1} \leftarrow \downarrow d_{\top} \uparrow e_{\top} \downarrow \downarrow \downarrow \downarrow d_{\bot} \downarrow \downarrow \downarrow \downarrow B_{k,j}
```

We compose with a $e_{\top}d_{\top}$ square

```
B_{k-1,j-2} \leftarrow \downarrow d_{\top} \uparrow e_{\top} \downarrow \downarrow \downarrow \downarrow d_{\bot} \downarrow \downarrow \downarrow \downarrow B_{k,j-2}
```

```
B_{k-1,j-1} \leftarrow \downarrow d_{\top} \uparrow e_{\top} \downarrow \downarrow \downarrow \downarrow d_{\bot} \downarrow \downarrow \downarrow \downarrow B_{k,j-1}
```

```
B_{k-1,j} \leftarrow \downarrow d_{\top} \uparrow e_{\top} \downarrow \downarrow \downarrow \downarrow d_{\bot} \downarrow \downarrow \downarrow \downarrow B_{k,j}
```

and use simplicial identities ($e_{\top}e_{i} \simeq e_{i}e_{\top}$) to replace $e_{i}$ by $e_{\top}$.
If \( i = j - 1 \), then the top square is a pullback, thus the rectangle is a pullback. Coming back to the previous diagram, the rectangle and top squares are pullback, thus the bottom square is a pullback. If \( i < j - 1 \), we now replace the new \( e_i d_{\top} \) square, and so on, until \( e_i \) become a \( e_{\top} \) (note that \( e_0 \) is never a \( e_{\top} \)):

\[
\begin{array}{c}
B_{k-1,j-2} \xleftarrow{d_{\top}} B_{k,j-2} \\
\uparrow e_i & \uparrow e_i \\
B_{k-1,j-1} \xleftarrow{d_{\top}} B_{k,j-1} \\
\uparrow e_{\top} & \uparrow e_{\top} \\
\vdots & \vdots \\
\uparrow e_{\top} & \uparrow e_{\top} \\
B_{k-1,j-2} \xleftarrow{d_{\top}} B_{k,j-2} \\
\uparrow e_{\top} & \uparrow e_{\top} \\
B_{k-1,j-1} \xleftarrow{d_{\top}} B_{k,j-1} \\
\uparrow e_{\top} & \uparrow e_{\top} \\
\vdots & \vdots \\
B_{k-1,j} \xleftarrow{d_{\top}} B_{k,j}.
\end{array}
\]

The whole rectangle is a pullback because all the squares are, and thus the rectangle consisting of all squares but the top one, and with \( e_i \) maps in the top square, is a pullback. Continuing these manipulations, we come back to the original square and prove it is a pullback.

We can do the same proof with bottom maps. We can also replace \( d_{\top} \) in the new previous pullback squares and obtain the remaining pullbacks involving the face maps.

For squares with face and degeneracy maps, we use the following strategy: in the diagram

\[
\begin{array}{c}
B_{00} \xrightarrow{s_0} B_{01} \xrightarrow{d_1} B_{00} \\
\uparrow e_{\perp} & \uparrow e_{\perp} & \uparrow e_{\perp} \\
B_{10} \xrightarrow{s_0} B_{11} \xrightarrow{d_1} B_{10}.
\end{array}
\]

the map \( s_0 \) is a section of \( d_1 \), then the long edge is an identity. The right-hand square is a pullback (it is one of the two pullback in the hypothesis). Thus the left-hand
square is a pullback. We can proceed in the same way for the other degeneracy maps.

There remains the case of squares involving only degeneracy:

\[
\begin{array}{ccc}
B_{ij} & \xrightarrow{s_0} & B_{i,j+1} \\
\downarrow{s_1} & & \downarrow{s_k} \\
B_{i+1,j} & \xrightarrow{s_0} & B_{i+1,j+1}.
\end{array}
\]

We again glue on the right a square with face map such that the long edge is an identity and use once again the lemma 1.1.1.

\[\Box\]

2.4 Bicomodules

Theorem 2.4.1. Let \(B\) be an augmented stable double Segal space, and such that the augmentation maps are CULF. Suppose moreover \(X := B_{\bullet,-1}\) and \(Y := B_{-1,\bullet}\) are decomposition spaces. Then the spans

\[
B_{0,0} \xleftarrow{e_1} B_{1,0} \xrightarrow{(u,e_0)} X_1 \times B_{0,0}
\]

and

\[
B_{0,0} \xleftarrow{d_0} B_{0,1} \xrightarrow{(d_1,v)} B_{0,0} \times Y_1
\]

induce on \(S_{/B_0,0}\) the structure of a bicomodule over \(S_{/X_1}\) and \(S_{/Y_1}\).

A bisimplicial \(\infty\)-groupoid satisfying the conditions of the theorem is called a bicomodule configuration.

Proof. The left and right comodule structures were established in proposition 2.2.2. The desired homotopy coherent diagram

\[
\begin{array}{ccc}
S_{/B_0,0} & \xrightarrow{\gamma} & S_{/B_{1,-1} \times B_0,0} \\
\downarrow{\gamma r} & & \downarrow{\text{Id} \oplus \gamma r} \\
S_{/B_{0,0} \times B_{-1,1}} & \xrightarrow{\gamma \circ \text{Id}} & S_{/B_{1,-1} \times B_{0,0} \times B_{-1,1}}
\end{array}
\]

is induced by the solid spans in the diagram

\[
\begin{array}{ccc}
B_{0,0} & \xleftarrow{e_1} & B_{1,0} \xrightarrow{(u,e_0)} B_{1,-1} \times B_{0,0} \\
\downarrow{d_0} & & \downarrow{\text{Id} \oplus d_0} \\
B_{0,1} & \xleftarrow{d_0} & B_{1,1} \xrightarrow{(u,d_0,e_0)} B_{1,-1} \times B_{0,1}
\end{array}
\]

\[
\begin{array}{ccc}
B_{0,0} \times B_{-1,1} & \xleftarrow{e_1 \circ \text{Id}} & B_{1,0} \times B_{-1,1} \xrightarrow{(u,e_0) \circ \text{Id}} B_{1,-1} \times B_{0,0} \times B_{-1,1}.
\end{array}
\]
The homotopy commutativity of the squares follows one again from the new augmentation simplicial identities. The upper-right hand square is a pullback if and only if its composite with the second projection is a pullback and, similarly, the lower-left hand square is a pullback if and only if its composite with the first projection is a pullback. These composite outer squares are pullbacks due to the stability condition.

The higher coherence can also be established using the same techniques as in [8] or [19]; this requires all the stability pullbacks.

3 Correspondences, fibrations, and adjunctions

3.1 Decomposition space correspondences

A correspondence is by definition a decomposition space $M$ with a map to the simplex category $\Delta$ considered as a simplicial $\infty$-groupoid.

We consider the slice $\infty$-category $\text{Cat}_{\Delta^1}$. It contains in particular $\Delta/\Delta^1$, whose objects are $[i,j]$, see remark 2.2.1. There is now a natural notion of nerve in this context. Given a correspondence $p : M \to \Delta$, the relative nerve $N_{\Delta^1} : \text{Cat}_{\Delta^1} \to \text{Fun}((\Delta/\Delta^1)^{op}, S)$ of $p$ is the augmented bisimplicial $\infty$-groupoid given by $B_{i,j} := N_{\Delta^1}(p)_{i,j} = \text{Map}_{\Delta^1}([i,j], p)$, where $[i,j]$ is given by the map $\Delta^1 \to \Delta$ sending the $i+1$ first vertices to 0 and the others to 1. It is allowed for $i$ or $j$ to be equal to $-1$ but not both.

From the nerve definition, the following square is a standard mapping-space fibre sequence for slices:

$$
\begin{array}{ccc}
B_{i,j} & \longrightarrow & \text{Map}(\Delta^{i+1+j}, M) \\
\downarrow & & \downarrow_{\text{post } p} \\
1 & \longrightarrow & \text{Map}(\Delta^{i+1+j}, \Delta^1).
\end{array}
$$

Proposition 3.1.1. Given a decomposition space correspondence $p : M \to \Delta^1$, the bisimplicial $\infty$-groupoid $B$ described above enjoys the following properties:

1. it is Segal in both directions;
2. it is stable;
3. it is augmented;
4. these augmentations are CULF.

To prove these properties, we will use the following lemmas. Proofs can be found in [4] in the setting of 1-groupoids, but work also for $\infty$-groupoids.

Lemma 3.1.2. Given a diagram such that top and bottom are two fibre sequences

$$
\begin{array}{ccc}
F & \longrightarrow & E & \longrightarrow & B \\
\downarrow & & \downarrow & & \downarrow_q \\
F' & \longrightarrow & E' & \longrightarrow & B'.
\end{array}
$$

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If \( q \) is an equivalence, then the left-hand square is a pullback.

**Lemma 3.1.3.** Given a diagram such that horizontal maps form fibre sequences

\[
\begin{array}{ccc}
F_4 & \\ & \searrow \swarrow & \\
E_4 & \\ & \searrow \swarrow & \\
B_4 & \\
\end{array}
\begin{array}{ccc}
F_3 & \\ & \searrow \swarrow & \\
E_3 & \\ & \searrow \swarrow & \\
B_3 & \\
\end{array}
\begin{array}{ccc}
F_2 & \\ & \searrow \swarrow & \\
E_2 & \\ & \searrow \swarrow & \\
B_2 & \\
\end{array}
\begin{array}{ccc}
F_1 & \\ & \searrow \swarrow & \\
E_1 & \\ & \searrow \swarrow & \\
B_1. & \\
\end{array}
\]

Suppose the vertical middle square (involving \( E_i, 1 \leq i \leq 4 \)) is a pullback, and suppose \( u \) and \( v \) are equivalences, then the left vertical square is a pullback.

**Proof of Proposition 3.1.1.** (1) Segal in both directions means: for any \( i \), the squares

\[
\begin{array}{ccc}
B_{n+1,i} & \xrightarrow{\epsilon_0} & B_{n,i} \\
\downarrow e_{n+1} & & \downarrow e_n \\
B_{n,i} & \xrightarrow{\epsilon_0} & B_{n-1,i} \\
\end{array}
\begin{array}{ccc}
B_{i,n+1} & \xrightarrow{d_0} & B_{i,n} \\
\downarrow d_{n+1} & & \downarrow d_n \\
B_{i,n} & \xrightarrow{d_0} & B_{i,n-1} \\
\end{array}
\]

are pullbacks.

The \( \infty \)-groupoids \( B_{n,i} \), for \( n \geq 0 \) are also given by the following fibre sequences:

\[
\begin{array}{ccc}
B_{n,i} & \xrightarrow{j} & \text{Map}(\Delta^{n+1+i}, M) \\
\downarrow & & \downarrow R_{n+1+i} \\
1 & \xrightarrow{id} & \text{Map}(\Delta^1, \Delta^1),
\end{array}
\]

the right-hand map \( R_{n+1+j} \) sends \( \sigma \in \text{Map}(\Delta^{n+1+j}, M) \) to \( p \circ \sigma \circ \rho_{n+1} \) where \( \rho_{n+1} : \Delta^1 \to \Delta^{n+1+j} \) maps the arrow in \( \Delta^1 \) to the \( (n + 1) \)st edge of \( \Delta^{n+1+j} \), that is \( \rho_{n+1} = (d^1)^n (d^\top)^j \). Indeed, in the diagram

\[
\begin{array}{ccc}
B_{i,j} & \xrightarrow{j} & \text{Map}(\Delta^{i+1+j}, M) \\
\downarrow & & \downarrow \text{post } p \\
1 & \xrightarrow{\beta_{i,j}} & \text{Map}(\Delta^{i+1+j}, \Delta^1) \\
\downarrow & & \downarrow \text{pre } \rho_{i+1} \\
1 & \xrightarrow{id} & \text{Map}(\Delta^1, \Delta^1),
\end{array}
\]

the bottom square is a pullback, because the fibre of the right bottom map is contractible, thus the whole rectangle is a pullback by lemma 1.1.1.

Using the lemma 3.1.2, we only have to check that the front square in the cube
is a pullback, and apply lemma 1.1.1. But the squares

\[ \mathcal{M}_{n+1+i} \xrightarrow{d_{n+1}} \mathcal{M}_{n+1+i} \]
\[ \mathcal{M}_{n+i} \xrightarrow{d_n} \mathcal{M}_{n+i} \]

are pullbacks because \( \mathcal{M} \) is a decomposition space, \( d_\perp \) is an inert map and \( d_{n+1} \) and \( d_n \) are always inner coface maps thus active maps. For the remaining squares, we use that the squares

\[ \mathcal{M}_{i+1+n+1} \xrightarrow{d_{i+1}} \mathcal{M}_{i+1+n} \]
\[ \mathcal{M}_{i+1+n} \xrightarrow{d_\tau} \mathcal{M}_{i+1+n-1} \]

are also pullbacks because \( \mathcal{M} \) is a decomposition space.

(2) To establish the stability condition, by lemma 2.3.3 it is enough to prove that the two following squares are pullbacks:

\[ B_{0,0} \xleftarrow{d_0} B_{0,1} \]
\[ B_{0,0} \xleftarrow{d_1} B_{0,1} \]

\[ B_{1,0} \xleftarrow{d_0} B_{1,1} \]
\[ B_{1,0} \xleftarrow{d_1} B_{1,1} \]

We can prove this with the same strategy used above. The decomposition space axioms used here are that the followings squares are pullbacks

\[ \mathcal{M}_0 \xrightarrow{s_0} \mathcal{M}_1 \]
\[ \mathcal{M}_0 \xrightarrow{s_0} \mathcal{M}_1 \]

\[ \mathcal{M}_1 \xrightarrow{s_0} \mathcal{M}_2 \]
\[ \mathcal{M}_1 \xrightarrow{s_0} \mathcal{M}_2 \]

(3) and (4) The augmentations \( Y_j := B_{-1,j} \) and \( X_i := B_{i,-1} \) are also given by the following fibre sequences
where the map \( S \) sends \( \sigma \) to \( p \circ \sigma \circ (d^\top)^j \), and the map \( T \) sends \( \sigma \) to \( p \circ \sigma \circ (d^\bot)^i \).

Since the following squares commute

\[
\begin{array}{ccc}
\text{Map}(\Delta^i, M) & \xrightarrow{(d^\top)^j} & \text{Map}(\Delta^j, M) \\
\downarrow S & & \downarrow T \\
\text{Map}(\Delta^0, \Delta^1), & & \text{Map}(\Delta^0, \Delta^1),
\end{array}
\]

it is enough to define maps \( B_{i,j} \to Y_j \) and \( B_{i,j} \to X_i \).

The augmentation maps are CULF: we need to prove that the back squares of the following cubes are pullbacks:

We can apply the lemma 3.1.3 since the front square is a pullback because \( M \) is a decomposition space.

To summarise, given a decomposition space correspondence \( p : M \to \Delta^1 \), we get a bicomodule configuration and then \( S_{f_{B_{i,0}}} \) is a bicomodule by theorem 2.4.1.
3.2 Cocartesian and cartesian fibrations of decomposition spaces

Ayala and Francis [1] formulate a homotopy-invariant definition of cartesian and cocartesian fibrations so it can be equally well formulated in any model for ∞-categories. We adapt here those definitions to decomposition spaces.

Let $p: X \rightarrow Y$ be a simplicial map between decomposition spaces. A morphism $\Delta^1 \xrightarrow{s,t} X$ is $p$-cocartesian if the diagram of coslices of decomposition spaces

$$
\begin{array}{ccc}
\Delta^1 & \rightarrow & X \\
\downarrow & & \downarrow \\
p \Delta^1 & \rightarrow & Y
\end{array}
$$

is a pullback, where the coslice $s/X$ is given by pullback of lower decalage $\text{Dec}_\perp(X)$:

$$
\begin{array}{ccc}
(s/X)_n & \rightarrow & \text{Dec}_\perp(X)_n \\
\downarrow & & \downarrow_{(d_\perp)^n+1} \\
1 & \rightarrow & X_0,
\end{array}
$$

similarly the coslice $a/X$ is given by pullback of $\text{Dec}_\perp(\text{Dec}_\perp(X))$

$$
\begin{array}{ccc}
(a/X)_n & \rightarrow & \text{Dec}_\perp(\text{Dec}_\perp(X))_n \\
\downarrow & & \downarrow_{(d_\perp)^n+1} \\
1 & \rightarrow & X_1,
\end{array}
$$

and the functor $a/X \rightarrow s/X$ is given by $\text{Dec}_\perp(d_\perp)$, where the simplicial map $d_\perp: \text{Dec}(X) \rightarrow X$ is given by the original $d_0$.

The functor $p: X \rightarrow Y$ is a cocartesian fibration if any diagram of solid arrows

$$
\begin{array}{ccc}
\Delta^0 & \rightarrow & X \\
d_1 & \rightarrow & Y \\
\Delta^1 & \rightarrow & Y
\end{array}
$$

admits a $p$-cocartesian diagonal filler.

Similarly, a morphism $\Delta^1 \xleftarrow{s,t} X$ is $p$-cartesian if the diagram of slice decomposition spaces

$$
\begin{array}{ccc}
X_{/t} & \rightarrow & X_{/a} \\
\downarrow & & \downarrow \\
Y_{/pt} & \rightarrow & Y_{/pa}
\end{array}
$$

is a pullback, where the slice $X_{/t}$ is given by pullback of the upper decalage $\text{Dec}_\top(X)$, the slice $X_{/a}$ is given by pullback of $\text{Dec}_\top(\text{Dec}_\top(X))$, and the functor $X_{/a} \rightarrow X_{/t}$ is given by $\text{Dec}_\top(d_\top) = d_{\top-1}$, where $d_\top: \text{Dec}_\top(X) \rightarrow X$ is given by the original $d_\top$.

The functor $p: X \rightarrow Y$ is a cartesian fibration if any diagram of solid arrows
admits a $p$-cartesian diagonal filler.

**Bisimplex category with diagonal maps**

We define $[i, j]: M_{i,j} \to \Delta^1$ to be the canonical projection from the mapping cylinder of $\phi_{i,j} := (d^T)^j \Delta^i \to \Delta^{i+j}$; it is a cocartesian fibration. They assemble into a category, denoted $\overline{\Delta}/\Delta^1$, of shape like $\Delta/\Delta^1$, but with extra diagonal maps $d: [i-1, j] \to [i, j-1]$ given by inclusion. These satisfy new simplicial identities: $\sigma_k d = d\sigma_{k+1}$, $0 < k \leq j$, where $\sigma_k$ are degeneracy maps “on $j$” (horizontal) or “on $i$” (vertical) and $d$ are diagonal maps, and similarly with face maps: $d\delta_k = \delta_k d$, $0 \leq k \leq j$, where $\delta_k$ are horizontal face maps or vertical face maps. For example, we can draw $[2, 1]$ as follows

\[ \begin{array}{cccc}
& & \uparrow & \\
& \uparrow & & \uparrow \\
\downarrow & & \downarrow & \\
& & \downarrow & \\
& & \uparrow & \\
\end{array} \]

where the horizontal maps lie over the map in $\Delta^1$. It is like a cocartesian version of the earlier drawing of remark 2.2.1.

**Remark 3.2.1.** We can draw $[i, j]$ as a column of $i+1$ black dots followed by $j+1$ white dots. Where arrows in $\overline{\Delta}/\Delta^1$ send black dots to black dots and white dots to white dots (without crossing), in $\overline{\Delta}/\Delta^1$ we allow moreover to map white dots to black dots.

There is a natural notion of nerve in the context of cocartesian fibrations over $\Delta^1$: given a cocartesian fibration $p: M \to \Delta^1$ between decomposition spaces, define the cocartesian nerve $N_{\text{cocart}}: \text{Cat}_{\text{cocart}}^{\overline{\Delta}/\Delta^1} \to \text{Fun}(\overline{\Delta}/\Delta^1, \text{S})$ by $N_{\text{cocart}}(p)_{i,j} := \text{Map}_{\text{cocart}}^{\overline{\Delta}}([i, j], M)$, the mapping space preserving cocartesian arrows.

Similarly to the previous subsection 3.1, we get a bicomodule configuration and $S_{/\text{B}0,0}$ is a bicomodule over $S_{/X_1}$ and $S_{/Y_1}$. We have here moreover diagonal maps $B_{i,j-1} \to B_{i-1,j}$ and new sections $s_{-1}: B_{i,j-1} \to B_{i,j}$, for $i \geq 0$ given by the composition with a diagonal map. That is $S_{/\text{B}0,0}$ is pointed as a right comodule over $S_{/Y_1}$.
3.3 Adjunctions of decomposition spaces

An adjunction between decomposition spaces $X$ and $Y$ is a simplicial map between decomposition spaces $p : M \to \Delta^1$ which is both a cartesian and a cocartesian fibration together with equivalences $X \simeq M_{[0]}$ and $Y \simeq M_{[1]}$.

4 “Möbius inversion” for comodules and a Rota formula

4.1 Finiteness and cardinality

An $\infty$-groupoid $X$ is locally finite if at each base point $x$ the homotopy groups $\pi_i(X, x)$ are finite for $i \geq 1$ and are trivial for $i$ sufficiently large. It is called finite if furthermore it has only finitely many components. We denote $\mathcal{F}$ (following the notation of [9]) the $\infty$-category of finite $\infty$-groupoids. A map is finite if each fibre is finite. A pullback of any homotopy finite map is again finite. A span $I \overset{p}{\leftarrow} M \overset{q}{\rightarrow} J$ and the corresponding linear functor $S_{/I} \to S_{/J}$ are finite if the map $p$ is finite.

A decomposition space $X$ is locally finite if $X_1$ is locally finite and both $s_0$ and $d_1$ are finite maps [9, subsection 7.4].

Proposition 4.1.1 ([11, proposition 4.3]). Let $I, J, M$ be locally finite $\infty$-groupoids and $I \overset{p}{\leftarrow} M \overset{q}{\rightarrow} J$ a finite span. Then the induced finite linear functor $S_{/I} \to S_{/J}$ restricts to $\mathcal{F}_{/I} \to \mathcal{F}_{/J}$.

The cardinality [2] of a finite $\infty$-groupoid $X$ is the alternating product of cardinalities of the homotopy groups

$$|X| = \sum_{x \in \pi_0(X)} \prod_{k=1}^{\infty} |\pi_k(X, x)|^{(-1)^k}.$$ 

For a locally finite $\infty$-groupoid $S$, there is a notion of cardinality $|\cdot| : \mathcal{F}/S \to \mathbb{Q}_{\pi_0 S}$ sending a basis element $\sigma s \gamma$ to the basis element $\delta_s = |\sigma s \gamma|$.

For any locally finite decomposition space $X$, we can take the cardinality of the linear functors $\delta : \mathcal{F}_{/X_1} \to \mathcal{F}$ and $\Delta : \mathcal{F}_{/X_1} \to \mathcal{F}_{/X_1 \times X_1}$ to obtain a coalgebra structure

$$\mathbb{Q}_{\pi_0 X_1} \xrightarrow{|\delta|} \mathbb{Q}$$
\[ \mathbb{Q}_{\pi_0 X_1} \xrightarrow{|\Delta|} \mathbb{Q}_{\pi_0 X_1} \otimes \mathbb{Q}_{\pi_0 X_1} \]
called the *numerical incidence coalgebra* of \( X \), see [9, subsection 7.7].

### 4.2 Completeness and Möbius condition

A decomposition space is called *complete* if \( s_0 : X_0 \to X_1 \) is a monomorphism [9, section 2]. Since \( s_0 \) is a monomorphism, we can identify \( X_0 \) with a \( \infty \)-subgroupoid of \( X_1 \). We denote \( X_n \) its complement: \( X_1 = X_0 + X_a \). More generally, recall the word notation from [9]: consider the alphabet with three letters \( \{0, 1, a\} \); 0 indicates degenerate edges \( s_0(x) \in X_1 \), \( a \) denotes edges specified to be nondegenerate, and 1 denotes unspecified edges. For \( w \) a word of length \( n \) in this alphabet, define

\[
X^w = \prod_{i \in w} X_i \subset (X_1)^n.
\]

This inclusion is full since \( X_a \subset X_1 \) is full by completeness.

Denote by \( X_w \) the \( \infty \)-groupoid of \( n \)-simplices whose principal edges have the types indicated in the word \( w \), that is the full subgroupoid of \( X_n \) given by the following pullback

\[
\begin{array}{ccc}
X_w & \longrightarrow & X_n \\
\downarrow & & \downarrow \\
X^w & \longrightarrow & (X_1)^n.
\end{array}
\]

We define \( \overset{\longrightarrow}{X}_n = X_{a \cdots a} \subset X_n \) to be the full subgroupoid of simplices with all principal edges nondegenerate.

For a complete decomposition space, the spans \( X_1 \overset{d_{1}}{\longrightarrow} \overset{\longrightarrow}{X}_n \to 1 \) define linear functors, the *Phi functors* \( \Phi_n : \mathcal{S}/X_1 \to \mathcal{S} \).

We also put \( \Phi_{\text{even}} := \sum_{n \text{ even}} \Phi_n \), and \( \Phi_{\text{odd}} := \sum_{n \text{ odd}} \Phi_n \).

The incidence algebra of a decomposition space contains the *zeta functor*

\[
\zeta : \mathcal{S}/X_1 \to \mathcal{S}
\]
given by the span \( X_1 \leftarrow X_1 \to 1 \).

**Theorem 4.2.1 ([9, Theorem 3.8]).** For a complete decomposition space, the following Möbius inversion holds:

\[
\zeta \ast \Phi_{\text{even}} \simeq \delta + \zeta \ast \Phi_{\text{odd}} \\
\simeq \Phi_{\text{even}} \ast \zeta \simeq \delta + \Phi_{\text{odd}} \ast \zeta.
\]
This is however not enough to allow the Möbius inversion formula to descend to the vector space level. A complete decomposition space $X$ is of locally finite length \[9\] if every edge $f \in X_1$ has finite length, that is, the fibres $(\overrightarrow{X}_n)_f$ of $d_1^{(n)} : \overrightarrow{X}_n \to X_1$ over $f$ are empty for $n$ sufficiently large.

A Möbius decomposition space \[9\] is a decomposition space which is locally finite and of locally finite length; the fibre $(\overrightarrow{X}_n)_f$ is finite (eventually empty). It follows that the map

$$\sum_n d_1^{n-1} : \sum_n \overrightarrow{X}_n \to X_1$$

is finite; by proposition 4.1.1, the Phi functors descend to

$$\Phi_n : F_{/X_1} \to F$$

and we can take cardinality to obtain functions $|\Phi_n| : \pi_0(X_1) \to \mathbb{Q}$.

Finally, we can take cardinality of the abstract Möbius inversion formula of 4.2.1, see \[9\] for a complete exposition.

**Theorem 4.2.2** ([9, Theorem 8.9]). If $X$ is a Möbius decomposition space, then the cardinality of the zeta functor, $|\zeta| : \mathbb{Q}_{\pi_0 X_1} \to \mathbb{Q}$, is convolution invertible with inverse $|\mu| := |\Phi_{\text{even}}| - |\Phi_{\text{odd}}|:

$$|\zeta| * |\mu| = |\delta| = |\mu| * |\zeta|.$$

### 4.3 Right and left convolutions

We introduce left and right convolution actions as dual to the comodule structure. Explicitly, given a right comodule configuration, we get a right comodule over $S/Y_1$.

The right convolution $\theta *_r \beta$ of the two functors $\theta : S_{/C_0} \to S$ and $\beta : S_{/Y_1} \to S$, given by the spans $C_0 \leftarrow M \to 1$ and $Y_1 \leftarrow N \to 1$, is the composite of $\theta \otimes \beta$ with the right coaction $\gamma_r$:

$$\theta *_r \beta : S_{/C_0} \xrightarrow{\gamma_r} S_{/C_0} \otimes S_{/Y_1} \xrightarrow{\theta \otimes \beta} S,$$

where the tensor product $\theta \otimes \beta$ is given by the span $C_0 \times Y_1 \leftarrow M \times N \to 1$.

Similarly, given a left comodule configuration, we can define the left convolution $\alpha *_l \theta$ of $\alpha : S_{/X_1} \to S$ and $\theta : S_{/C_0} \to S$:

$$\alpha *_l \theta : S_{/C_0} \xrightarrow{\gamma_l} S_{/X_1} \otimes S_{/C_0} \xrightarrow{\alpha \otimes \theta} S.$$

If we have a bicomodule configuration, then the following associativity formula is a consequence of the compatibility of coactions (theorem 2.4.1).

**Proposition 4.3.1.** Given a bicomodule configuration, the convolutions defined above satisfy

$$\alpha *_l (\theta *_r \beta) \simeq (\alpha *_l \theta) *_r \beta.$$
4.4 “Möbius inversion” for (co)modules

Let $C \to Y$ be a comodule configuration. The zeta functor

$$\zeta^C : S/C_0 \to S$$

is the linear functor defined by the span

$$C_0 \leftarrow C_0 \to 1.$$  

We are interested in the “invertibility” of this functor under the convolution actions.

Let $C \to Y$ be a right pointed comodule configuration. The augmented simplicial $\infty$-groupoid $C$ is an object of the functor $\infty$-category

$$\text{Fun}(\Delta^\text{op}_\text{bot}, S)$$

where $\Delta^\text{bot}$ is the simplex category of finite linear orders with a specified bottom element, and with monotone maps preserving the bottom element. The forgetful functor $\Delta^\text{bot} \to \Delta$ is right adjoint to the functor $j : \Delta \to \Delta^\text{bot}$ adding a bottom element.

A right pointed comodule configuration $f : C \to Y$ is complete if the new degeneracies $s_{-1} : C_{n-1} \to C_n$ are monomorphisms.

In this situation, we define $\overrightarrow{C}_n = C_{a...a} \subset C_n$ to be the full subgroupoid of simplices with all principal edges nondegenerate. It is given by the pullback diagram

$$\begin{array}{ccc}
C_{a...a} & \rightarrow & C_n \\
\downarrow & & \downarrow \\
C_0^a \times Y^{a...a} & \rightarrow & C_0 \times Y_1^n.
\end{array}$$

The principal edges of the $\infty$-groupoid $C_n$ consist of an element in $C_0$ given by $(d_\top)^n$, and $n$ edges in $Y_1$, the principal edges of the image of $C_n$ by $f$.

Define

$$\delta^R : S/C_0 \to S$$

to be the linear functor given by the span

$$C_0 \leftarrow_{s_{-1}} C_{-1} \to 1$$

and define the right Phi functors

$$\Phi^R_n : S/C_0 \to S$$

to be the linear functors given by the spans

$$C_0 \leftarrow \overrightarrow{C}_n \to 1.$$  

If $n = -1$, $\overrightarrow{C}_{-1} = C_{-1}$ (by convention) and $\Phi^R_{-1}$ is the linear functor $\delta^R$.

**Lemma 4.4.1.** The following square is a pullback:
Proof. Let $n = |v|$. The square is the top rectangle of the diagram

$$
\begin{array}{c}
C_{1v} \rightarrowtail C_1 \\
\downarrow \quad \downarrow \\
C_0 \times Y_v \rightarrowtail C_0 \times Y_1.
\end{array}
$$

The bottom square and left-hand rectangle are pullbacks by definition of $Y_v$ and $C_{1v}$, hence the top left-hand square is a pullback. The right-hand square is a pullback because the augmentation map $C \rightarrow Y$ is CULF. Hence the top rectangle, which is the desired square, is a pullback.

Given a complete decomposition space $Y$, we denote $\Phi_Y^n : S/Y_\ast \rightarrow S$ the usual Phi functors, see 4.1 above.

**Proposition 4.4.2.** The right Phi functors satisfy

$$
\zeta^C \ast_r \Phi_Y^n \simeq \Phi_{n-1}^R + \Phi_n^R.
$$

**Proof.** Compute the convolution action $\zeta^C \ast_r \Phi_Y^n$ by lemma 4.4.1 as:

$$
\begin{array}{c}
C_0 \\
\downarrow \quad \downarrow \\
C_1 \leftarrowtail C_1a\ldots a \\
\downarrow \quad \downarrow \\
C_0 \times Y_1 \leftarrowtail C_0 \times \overrightarrow{Y}_n \rightarrowtail 1.
\end{array}
$$

But $C_{1a\ldots a} \simeq C_{0a\ldots a} + C_{aa\ldots a} \simeq \overrightarrow{C}_{n-1} + \overrightarrow{C}_n$. This is an equivalence of $\infty$-groupoids over $C_0$ and the resulting span is $\Phi_{n-1}^R + \Phi_n^R$. 

Denote

$$
\Phi_Y^{\text{even}} := \sum_{n \text{ even}} \Phi_Y^n, \quad \Phi_Y^{\text{odd}} := \sum_{n \text{ odd}} \Phi_Y^n.
$$

The previous proposition implies the following “Möbius inversion” formula.

**Theorem 4.4.3.** Given $C \rightarrow Y$ a complete right pointed comodule configuration,

$$
\zeta^C \ast_r \Phi_Y^{\text{even}} \simeq \delta^R + \zeta^C \ast_r \Phi_Y^{\text{odd}}.
$$
Proof. The two linear functors are equivalent to the sum of the right Phi functors:
\[ \zeta^C \ast_r \Phi^Y_{\text{even}} \simeq \Phi^R_{-1} + \Phi^R_0 + \Phi^R_1 + \cdots \simeq \delta^R + \zeta^C \ast_r \Phi^Y_{\text{odd}}. \]

We can also define a left pointed comodule configuration \( D \to X \), with new top sections instead of bottom: we consider instead the mapping cylinder of \( \Delta \to \Delta_{\text{top}} \), where \( \Delta_{\text{top}} \) is the simplex category of finite linear orders with a specified top element, and with monotone maps preserving the top element. A left pointed comodule configuration is complete if the new degeneracies \( t_{\top+1} : D_{n-1} \to D_n \) are monomorphisms. Similarly, we define the left Phi functors and \( \delta^L \) using \( t_{\top+1} \) and \( d_{\top} \) and we obtain the following formula.

**Theorem 4.4.4.** Given \( D \to X \) a complete left pointed comodule configuration,
\[ \Phi^X_{\text{even}} \ast_l \zeta^D \simeq \delta^L + \Phi^X_{\text{odd}} \ast_l \zeta^D. \]

### 4.5 Möbius bicomodule configurations and the Rota formula

In order to take homotopy cardinality to recover the usual Möbius inversions, we need to impose some finiteness conditions. We adapt the approach of [9] summarised in 4.1 and 4.2 above.

A right Möbius comodule configuration is a complete right pointed comodule configuration \( C \to Y \) such that the decomposition space \( Y \) is Möbius and the augmented comodule is Möbius, that is

- \( C \) is locally finite: the groupoid \( C_0 \) is finite and both \( s_{-1} \) and \( d_0 \) are finite maps;
- \( C \) is of locally finite length: every edge has a finite length, that is for all \( a \in C_0 \), the fibres of \( d^{(n)}_0 : \overrightarrow{C}_n \to C_0 \) over \( a \) are empty for \( n \) sufficiently large.

Similarly we define a left Möbius comodule configuration to be a complete left pointed comodule configuration \( D \to X \) such that the decomposition space \( X \) is Möbius and the augmented comodule is Möbius, using \( t_{\top+1} \) and \( d_{\top} \).

Under these conditions, the Phi functors descend to
\[ \Phi^R_n : \mathcal{F}/C_0 \to \mathcal{F} \]
and we can now take the cardinality of the "Möbius formulas" (Theorems 4.4.3 and 4.4.4).

**Theorem 4.5.1.** Given \( C \to Y \) a right Möbius comodule configuration and \( D \to X \) a left Möbius comodule configuration,
\[ |\zeta^C \ast_r |\mu^Y| = |\delta^R|, \quad |\mu^X| \ast_l |\zeta^D| = |\delta^L|, \]
where \( |\mu^Y| := |\Phi^Y_{\text{even}}| - |\Phi^Y_{\text{odd}}| \) and \( |\mu^X| := |\Phi^X_{\text{even}}| - |\Phi^X_{\text{odd}}| \).
A Möbius bicomodule configuration is a bicomodule configuration with two pointings such that both left and right comodule configurations are Möbius. It hence has extra degeneracy maps in both directions, extra bottom degeneracy maps in horizontal direction and extra top degeneracy maps in vertical direction.

Note that given a Möbius bicomodule configuration \( B \), the zeta functors are defined only on the \( \infty \)-groupoid \( B_{0,0} \) and then are the same for the two comodules. In both cases it is given by the span

\[
B_{0,0} \leftarrow Y_0 \rightarrow 1.
\]

**Theorem 4.5.2.** Given a Möbius bicomodule configuration \( B \) with \( X := B_{\cdot,-1} \) and \( Y := B_{-1,\cdot} \), we have

\[
|\mu^X| \star_l |\delta^R| = |\delta^L| \star_r |\mu^Y|,
\]

where \( \delta^R \) is the linear functor given by the span

\[
B_{0,0} \leftarrow Y_0 \rightarrow 1,
\]

and \( \delta^L \) is the linear functor given by the span

\[
B_{0,0} \leftarrow X_0 \rightarrow 1.
\]

**Proof.** Using the Möbius formulas at the algebraic level from Theorem 4.5.1, and the associativity of the convolution actions from Proposition 4.3.1, we compute

\[
|\mu^X| \star_l |\delta^R| = |\mu^X| \star_l (|\zeta| \star_r |\mu^Y|) \\
= (|\mu^X| \star_l |\zeta|) \star_r |\mu^Y| \\
= |\delta^L| \star_r |\mu^Y|.
\]

\[\square\]

### 4.6 Main motivating example

We saw in section 3.2 that given a cocartesian fibration \( p : \mathcal{M} \rightarrow \Delta^1 \) between decomposition spaces, we obtain that \( B_{0,0} \) is a right comodule configuration, with diagonal maps \( B_{i,j} \rightarrow B_{i} \rightarrow B_{j} \) and new sections \( s_{-1} : B_{i,j} \rightarrow B_{i,j-1} \), for \( i \geq 0 \) given by the composition with a diagonal map.

**Lemma 4.6.1.** Given a cocartesian fibration \( p : \mathcal{M} \rightarrow \Delta^1 \) between decomposition spaces, suppose moreover that \( \mathcal{M} \) is complete. Then the associated right pointed comodule configuration is complete.

**Proof.** The new sections will be monomorphisms if the following square is a pullback:

\[
\begin{array}{ccc}
B_{i,j-1} & \xrightarrow{id} & B_{i,j-1} \\
\downarrow{id} & & \downarrow{s_{-1}} \\
B_{i,j-1} & \xrightarrow{s_{-1}} & B_{i,j}.
\end{array}
\]

by assumption, \( \mathcal{M} \) is a complete decomposition space, hence all degeneracy maps are monomorphisms, and we can apply lemma 3.1.3, to obtain the desired pullbacks. \[\square\]
Instantiating the general definitions from 4.4, the zeta functor
\[ \zeta : S_{/B_{0,0}} \to S \]
is given by the span
\[ B_{0,0} \leftarrow B_{0,0} \to 1, \]
and the functor
\[ \delta^R : S_{/B_{0,0}} \to S \]
is defined by the span
\[ B_{0,0} \leftarrow B_{-1,0} \to 1. \]

The right comodule configuration being complete, we get a “Möbius inversion” formula (theorem 4.4.3):
\[ \zeta \star_r \Phi^Y_{\text{even}} = \delta^R + \zeta \star_r \Phi^Y_{\text{odd}}, \]
where \( Y := B_{-1,\bullet} \).

Similarly, given a cartesian fibration \( p : \mathcal{M} \to \Delta^1 \) between decomposition spaces, we obtain a left pointed comodule configuration.

**Lemma 4.6.2.** Given a cartesian fibration \( p : \mathcal{M} \to \Delta^1 \) between decomposition spaces, suppose moreover that \( \mathcal{M} \) is complete. Then the left pointed comodule configuration is complete.

The functor
\[ \delta^L : S_{/B_{0,0}} \to S \]
is given by the span
\[ B_{0,0} \leftarrow B_{-1,0} \to 1. \]
This leads to the “Möbius inversion” formula
\[ \Phi^X_{\text{even}} \star_l \zeta = \delta^L + \Phi^X_{\text{odd}} \star_l \zeta. \]

Given an adjunction between decomposition spaces, that is a simplicial map \( \mathcal{M} \to \Delta^1 \) which is both cartesian and cocartesian, and suppose that \( \mathcal{M} \) is complete, then we just obtained two “Möbius inversion” formulas.

**Lemma 4.6.3.** Given an adjunction of decomposition spaces in the form of a bicartesian fibration \( p : \mathcal{M} \to \Delta^1 \), suppose moreover that \( \mathcal{M} \) is a Möbius decomposition space. Then the bicomodule configuration extracted from this data is Möbius. In particular, we have the Rota formula for the adjunction \( p \):
\[ |\mu^X| \star_l |\delta^R| = |\delta^L| \star_r |\mu^Y|. \]

**Proof.** The needed finiteness conditions of \( B_{0,\bullet} \) and \( B_{\bullet,0} \) follow from the fact that the \( \infty \)-groupoids and maps are obtained by pullbacks and that a pullback of a finite map is finite. \( \Box \)
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