Killing-Poisson tensors on Riemannian manifolds

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Abstract. We introduce a new class of Poisson structures on a Riemannian manifold. A Poisson structure in this class will be called a Killing-Poisson structure. The class of Killing-Poisson structures contains the class of symplectic structures, the class of Poisson structures studied in (Differential Geometry and its Applications, Vol. 20, Issue 3 (2004), 279–291) and the class of Poisson structures induced by some infinitesimal Lie algebra actions on Riemannian manifolds. We show that some classical results on symplectic manifolds (the integrability of the Lie algebroid structure associated to a symplectic structure, the non exactness of a symplectic structure on a compact manifold) remain valid for regular Killing-Poisson structures.

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1 Introduction and main results

In the present paper, we pursue our investigations on the interactions between Riemannian geometry and Poisson geometry initialized in [1], [2] and

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[3], by introducing a new class of Poisson structures. A Poisson structure in this new class will be called a **Killing-Poisson structure**. We show that this terminology is appropriate by pointing out that there is a notion of Killing multi-vector fields as a generalization of Killing vector fields (cf. Proposition 1.1) and, by definition, a Killing-Poisson structure is a Poisson structure whose associated bivector field is a Killing bivector field. We show that the class of Killing-Poisson structures contains the class of symplectic structures, the class of Poisson structures studied in [2] and the class of Poisson structures induced by some infinitesimal Lie algebras actions on Riemannian manifolds. We show that some classical results on symplectic manifolds (the integrability of the Lie algebroid structure associated to a symplectic structure, the non exactness of a symplectic structure on a compact manifold) remain valid for regular Killing-Poisson structures.

To state our results, we first explain some fundamental notions which we use in this paper.

Recall that a Poisson structure on a manifold $M$ is an $\mathbb{R}$-bilinear Lie bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$ satisfying the Leibniz rule

$$\{f, gh\} = \{f, g\}h + g\{f, h\}, \quad \text{for all } f, g, h \in C^\infty(M).$$

For a function $f \in C^\infty(M)$, the derivation $H_f = \{f,\cdot\}$ is called the **Hamiltonian vector field** of $f$. If $H_f = 0$, we call $f$ a **Casimir function**. It follows from the Leibniz rule that there exists a bivector field $\pi \in \Gamma(\wedge^2 TM)$ such that

$$\{f, g\} = \pi(df, dg);$$

the Jacobi identity for $\{\cdot,\cdot\}$ is equivalent to the condition $[\pi, \pi] = 0$, where $[\cdot,\cdot]$ is the Schouten-Nijenhuis bracket, see e.g. [13].

In local coordinates $(x_1, \ldots, x_n)$ the tensor $\pi$ is determined by the matrix

$$\pi_{ij}(x) = \{x_i, x_j\}.$$

The rank of this matrix is called the rank of $\pi$ at $x$. A Poisson structure is called **regular** if the rank of $\pi$ is constant on $M$. If this matrix is invertible a each $x$, then $\pi$ is called nondegenerate or **symplectic**. In this case, the local matrices $(\omega_{ij}) = (-\pi_{ij})^{-1}$ define a global 2-form $\omega \in \Omega^2(M)$, and the condition $[\pi, \pi] = 0$ is equivalent to $d\omega = 0.$
We denote by $\pi_# : T^*M \to TM$ the anchor map given by $\beta(\pi_#(\alpha)) = \pi(\alpha, \beta)$, and by $[, ]_\pi$ the Koszul bracket given by

$$[\alpha, \beta]_\pi = L_{\pi_#(\alpha)}\beta - L_{\pi_#(\beta)}\alpha - d(\pi(\alpha, \beta)), \quad \alpha, \beta \in \Omega^1(M).$$

The distribution $\text{Im}\pi_#$ is integrable and defines a singular foliation $\mathcal{F}$. The leaves of $\mathcal{F}$ are symplectic immersed submanifolds of $M$. The foliation $\mathcal{F}$ is called the symplectic foliation associated to the Poisson structure. The anchor map and the Koszul bracket define the Lie algebroid structure associated to $\pi$. The Poisson structure is called integrable if this Lie algebroid structure integrates to a Lie groupoid structure. (For a detailed explanation of the integrability of Lie algebroids, see [5], [6] etc.)

The main tool to be used extensively in this paper is the metric contravariant connection associated canonically to a couple of a Riemannian metric and a Poisson tensor. General contravariant connections associated to a Poisson structure have recently turned out to be useful in several areas of Poisson geometry. Contravariant connections were defined by Vaismann [13] and were analyzed in detail by Fernandes [7]. This notion appears extensively in the context of noncommutative deformations (see [12], [8] and [9]). One can consult [7] for the general properties of contravariant connections.

Let $(M, \pi)$ be a Poisson manifold and $V \to M$ a vector bundle over $M$. A contravariant connection on $V$ with respect to $\pi$ is a map $\mathcal{D} : \Omega^1(M) \times \Gamma(M, V) \to \Gamma(M, V)$, $(\alpha, s) \mapsto \mathcal{D}_\alpha s$ satisfying the following properties:

1. $\mathcal{D}_\alpha s$ is linear over $C^\infty(M)$ in $\alpha$:

$$\mathcal{D}_{f\alpha_1 + h\alpha_2} s = f\mathcal{D}_{\alpha_1} s + h\mathcal{D}_{\alpha_2} s, \quad f, h \in C^\infty(M);$$

2. $\mathcal{D}_\alpha s$ is linear over $\mathbb{R}$ in $s$:

$$\mathcal{D}_\alpha(as_1 + bs_2) = a\mathcal{D}_\alpha s_1 + b\mathcal{D}_\alpha s_2, \quad a, b \in \mathbb{R};$$

3. $\mathcal{D}$ satisfies the following product rule:

$$\mathcal{D}_\alpha(fs) = f\mathcal{D}_\alpha s + \pi_#(\alpha)(f)s, \quad f \in C^\infty(M).$$
The curvature of a contravariant connection $\mathcal{D}$ is formally identical to the usual definition

$$K(\alpha, \beta) = \mathcal{D}_\alpha \mathcal{D}_\beta - \mathcal{D}_\beta \mathcal{D}_\alpha - \mathcal{D}_{[\alpha, \beta]}\pi.$$ 

We call $\mathcal{D}$ flat if $K$ vanishes identically.

If $V = T^* M$, one can define the torsion $T$ of $\mathcal{D}$ by

$$T(\alpha, \beta) = \mathcal{D}_\alpha \beta - \mathcal{D}_\beta \alpha - [\alpha, \beta] \pi.$$ 

A contravariant connection $\mathcal{D}$ is called a $\mathcal{F}$-connection if it satisfies the following property

$$\pi_\#(\alpha) = 0 \quad \Rightarrow \quad \mathcal{D}_\alpha = 0;$$ 

$\mathcal{D}$ will be called a $\mathcal{F}_{\text{reg}}$-connection if $\mathcal{D}$ is a $\mathcal{F}$-connection on the regular open set where the rank of $\pi$ is locally constant.

$\mathcal{F}$-connections were introduced by Fernandes in [7]; they will appear extensively in this paper.

**Remark.**

1. The definition of a contravariant connection is similar to the definition of an ordinary (covariant) connection, except that cotangent vectors have taken the place of tangent vectors. One can translate many definitions, identities and proof for covariant connections to contravariant connections simply by exchanging the roles of tangent and cotangent vectors and replacing Lie Bracket with Koszul bracket. Nevertheless, Fernandes pointed out in [7] that some classical results on covariant connections are not true any more for general contravariant connections and, however, those results remain valid for $\mathcal{F}$-connections.

For a couple of a Poisson tensor and a Riemannian metric there exists an unique torsion-free contravariant connection which preserves the metric. It appeared first in [1] and, recently, has turned out to be useful in the context of noncommutative deformations (see [8] and [9]).

Let $M$ be a pseudo-Riemannian manifold and $\pi$ a Poisson tensor on $M$. We denote by $<, >$ the metric when it measures the length of 1-forms.

The **metric contravariant connection** associated to $(\pi, <, >)$ is the unique contravariant connection $\mathcal{D}$ with respect to $\pi$ such that:
1. The metric $<\cdot, \cdot>$ is parallel with respect to $\mathcal{D}$ i.e.,

$$\pi_\#(\alpha). < \beta, \gamma > = < \mathcal{D}_\alpha \beta, \gamma > + < \beta, \mathcal{D}_\alpha \gamma >;$$

2. $\mathcal{D}$ is torsion-free.

One can define $\mathcal{D}$ by the Koszul formula

$$2 < \mathcal{D}_\alpha \beta, \gamma > = \pi_\#(\alpha). < \beta, \gamma > + \pi_\#(\beta). < \alpha, \gamma > - \pi_\#(\gamma). < \alpha, \beta > + < [\gamma, \alpha]_\pi, \beta > + < [\gamma, \beta]_\pi, \alpha > + < [\alpha, \beta]_\pi, \gamma > .$$

In [2] and [3] we sited and studied the following notion of compatibility between a Riemannian metric and a Poisson tensor. A Poisson tensor $\pi$ on a Riemannian manifold $(M, <\cdot,\cdot>)$ is called compatible with the metric if:

(P1) The Poisson tensor $\pi$ is parallel with respect to the metric contravariant connection $\mathcal{D}$ associated to $(\pi, <\cdot,\cdot>)$ i.e., $\mathcal{D}\pi = 0$.

We pointed out in [2] and [3] that a Poisson tensor $\pi$ for which (P1) holds satisfies the following properties:

(P2) The divergence of $\pi$ with respect to the Levi-Civita connection vanishes.

(P3) For any open subset $U$ in $M$ and for any function $f \in C^\infty(U)$ such that $i_{df}\pi = 0$, the gradient vector field $\nabla f$ preserves $\pi$ i.e., $L_{\nabla f} \pi = 0$.

In Section 3, we will recall the definition of the divergence of a multi-vector field with respect to a covariant connection.

Note that (P2) is equivalent to the fact that the Riemannian density is invariant by any hamiltonian vector field and then $\pi$ is an unimodular Poisson structure (see [15]).

One can remark that the properties (P2) and (P3) are not specific to bivector fields and make sense for any multi-vector field on a Riemannian manifold. In particular, what can one tell about a vector field on a Riemannian manifold which satisfies (P2) and (P3)? The following proposition gives an answer to this question.

**Proposition 1.1** A vector field $X$ on a Riemannian manifold $(M, g)$ is a Killing vector field if and only if the following assertions hold:
1. The divergence of $X$ with respect to the Levi-Civita connection vanishes.

2. For any open set $U \subset M$ and any function $f \in C^\infty(U)$ such that $X(f) = 0$, $[X, \nabla f] = 0$ (the gradient field of $f$ given by $g(\nabla f, Y) = df(Y)$).

**Proof.** Suppose that $X$ is a Killing vector field. It is a classical result on Killing vector fields that the divergence of $X$ vanishes. On other hand, for any function $f$ such that $X(f) = 0$ and for any vector field $Y$, we have

$$0 = L_Xg(Y, \nabla f) = X.(Y(f) - [X,Y](f) - g(Y, [X, \nabla f]) = -g(Y, [X, \nabla f]),$$

and then $[X, \nabla f] = 0$.

Conversely, suppose that $X$ is a vector field which satisfies the two assertions above. We will show that $L_Xg$ vanishes on the dense open set $U_1 \cup U_2$ where $U_1 = \{m \in M| X(m) \neq 0\}$ and $U_2$ is the interior of $\{m \in M| X(m) = 0\}$. The vector field $X$ vanishes on $U_2$ and hence $L_Xg(m) = 0$ for any $m \in U_2$. Let $m$ be a point in $U_1$. Choose a local coordinates $(x_1, \ldots, x_n)$ such that $X = \frac{\partial}{\partial x_1}$. The functions $x_2, \ldots, x_n$ satisfy $X(x_i) = 0$ and then $[X, \nabla x_i] = 0$ for $i = 2, \ldots, n$.

Let us compute $L_Xg$ on the local frame $(X, \nabla x_2, \ldots, \nabla x_n)$. We have, for $i = 2, \ldots, n$,

$$L_Xg(X, \nabla x_i) = X.X(x_i) = 0.$$

$$L_Xg(\nabla x_i, \nabla x_j) = X.\nabla x_i(x_j) = \nabla x_i.X(x_j) = 0.$$

To conclude, it remains to show that $L_Xg(X, X) = X.g(X, X) = 0$. Indeed, we put $E_1 = \frac{X}{|X|}$ and we orthonormalize $(\nabla x_2, \ldots, \nabla x_n)$ to get an orthonormal frame $(E_1, \ldots, E_n)$. Note that, from $X.g(\nabla x_i, \nabla x_j) = 0$, we get $[E_i, X] = 0$ for $i = 2, \ldots, n$.

The vanishing of the divergence of $X$ gives

$$0 = \sum_{i=1}^n g(\nabla_{E_i}X, E_i),$$

where $\nabla$ is the Levi-Civita connection associated to the metric. Since, for $i = 2, \ldots, n$,

$$g(\nabla_{E_i}X, E_i) = g(\nabla_X E_i, E_i) = \frac{1}{2}X.g(E_i, E_i) = 0,$$
we get
\[ 0 = -\frac{1}{2|X|^2}X.g(X, X), \]
which completes the proof. q.e.d.

According to this proposition, it is natural to put the following definition.

**Definition 1.1** A Poisson structure on a Riemannian manifold \((M, g)\) will be called a **Killing-Poisson structure** if the associated bivector field \(\pi\) is a Killing-Poisson tensor i.e.,

1. the divergence of \(\pi\) with respect to the Levi-Civita connection vanishes,
2. for any open set \(U \subset M\) and any function \(f \in C^\infty(U)\) such that \(f\) is a Casimir function,
\[ L_{\nabla}f\pi = 0.\]

**Remark.**

1. Let \(\pi\) be an invertible Poisson tensor on a Riemannian 2n-manifold \((M, g)\). The Riemannian volume \(\mu_g\) satisfies \(\mu_g = f \wedge^n \omega\) where \(\omega\) is the symplectic form associated to \(\pi\) and \(f\) a function. It is easy to check that \(\pi\) is a Killing-Poisson tensor if and only if \(df = 0\). Hence, for any invertible Poisson tensor \(\pi\) on a manifold \(M\) and for any Riemannian metric \(g\) on \(M\), there exists a function \(\phi\) such that \(\pi\) is a Killing-Poisson tensor with respect to \(e^\phi g\).

2. One can check easily that a bivector field on a Riemannian manifold which is parallel with respect to the Levi-Civita connection is a Killing-Poisson tensor.

This paper is organized as follows. In Section 2, we show that \((\mathcal{P}3)\) is equivalent to:

\((\mathcal{P}3')\) the metric contravariant connection associated to \((\pi, <, >)\) is a \(\mathcal{F}^{reg}\)-connection.

We show that a Poisson structure for which there exists a Riemannian metric such that \((\mathcal{P}3')\) holds possesses the following properties.
Theorem 1.1 Let \((M, g)\) be a Riemannian manifold endowed with a regular Poisson tensor \(\pi\) such that the metric contravariant connection associated to \((\pi, g)\) is a \(\mathcal{F}\)-connection. Then:

1. the metric \(g\) is a bundle-like metric for the symplectic foliation,
2. for any open subset \(U\) in \(M\) and for any vector field \(X\) on \(U\) which is perpendicular to the symplectic foliation, \(X\) is a foliated vector field if and only if \(X\) is a Poisson vector field.

Theorem 1.2 Let \((M, \pi)\) be a regular Poisson manifold. If there exists a Riemannian metric such that the corresponding metric contravariant connection is a \(\mathcal{F}\)-connection, then the Lie algebroid structure associated to \(\pi\) is integrable.

Note that the conclusions of Theorem 1.1 and Theorem 1.2 hold, in particular, for any regular Killing-Poisson structure.

In Section 3, we give some general properties of Killing-Poisson structures and show the following results.

Theorem 1.3 Let \(M\) be a Riemannian manifold endowed with a Poisson tensor such that \(D\pi = 0\). Then \(\pi\) is a Killing-Poisson tensor.

This theorem shows that the class of Killing-Poisson structures is large. For instance in [4], the author showed that the dual \(G^*\) of a Lie algebra \(G\) carries a Riemannian metric compatible with the canonical linear Poisson structure if and only if the Lie algebra is a semi-direct product of an abelian Lie algebra and an abelian ideal.

Theorem 1.4 Let \(G\) be a Lie algebra and \(r \in \wedge^2 G\) an unimodular solution of the classical Yang-Baxter equation. Let \(\Gamma : G \rightarrow \mathcal{X}(M)\) be a locally free action of \(G\) on a Riemannian manifold \((M, g)\) such that, for any \(u \in G\), \(\Gamma(u)\) is a Killing vector field. Then \(\pi := \Gamma(r)\) is a Killing-Poisson tensor.

Corollary 1.1 Any unimodular left-invariant Poisson tensor on a Lie group \(G\) is a Killing-Poisson tensor with respect to any right-invariant Riemannian metric on \(G\).
A left-invariant Poisson tensor on a Lie group is unimodular if its value at the unity is an unimodular solution of the classical Yang-Baxter equation. Unimodular solutions of the classical Yang-Baxter equation will be defined in Section 3.

In Section 4, we study the exactness of Killing-Poisson tensors and show the following result.

**Theorem 1.5** Let \( \pi \) be a non trivial regular Killing-Poisson tensor on a compact Riemannian manifold. Then \( \pi \) cannot be exact.

Section 5 is devoted to the characterization of Killing-Poisson tensors on Riemannian 3-manifolds. In particular, we will characterize all Killing-Poisson tensors on \( \mathbb{R}^3 \) endowed with the Euclidean metric.

## 2 On metric contravariant \( F \)-connections

In this section, we give some properties of Poisson structures for which there exists a Riemannian metric such that the corresponding metric contravariant connection is a \( F \)-connection, we prove Theorem 1.2 and Theorem 1.3 and we show the equivalence between \( (P3) \) and \( (P3') \).

Let \( M \) be a Riemannian manifold and \( \pi \) a Poisson tensor on \( M \). We denote by \( g \) (resp. by \( < , > \)) the Riemannian metric when it measures the length of vectors (resp. the length of covectors). \( \mathcal{D} \) denotes the metric contravariant connection associated to \( (\pi, g) \), and \( \# : T^*M \to TM \) the isomorphism associated to the Riemannian metric. We denote by \( \mathcal{F} \) the symplectic foliation and, for any open set \( U \subset M \), \( \mathcal{X}(\mathcal{F}, U)^\perp \) denotes the space of orthogonal foliated vector fields. A vector field \( X \in \mathcal{X}(U) \) belongs to \( \mathcal{X}(\mathcal{F}, U)^\perp \) if, for any vector field \( Y \in \mathcal{X}(U) \) tangent to the foliation \( \mathcal{F} \), \( [X, Y] \) is tangent to \( \mathcal{F} \) and \( g(X, Y) = 0 \).

For any open subset \( U \subset M \), we consider \( Z^0(U) = \{ f \in C^\infty(U); i_d f \pi = 0 \} \) the space of Casimir functions on \( U \) and \( Z^1(U) = \{ \alpha \in \Omega^1(U); [\alpha, \beta]_\pi = 0 \ \forall \beta \in \Omega^1(U) \} \) the center of the Lie algebra \( \Omega^1(U) \) endowed with the Koszul bracket. One can see easily that \( Z^1(U) \) is the space of basic 1-forms relative to the restriction of \( \mathcal{F} \) to \( U \) i.e.,

\[
Z^1(U) = \{ \alpha \in \Omega^1(U); \pi_\#(\alpha) = 0 \ \text{and} \ i_{\pi_\#(\beta)}d\alpha = 0 \ \forall \beta \in \Omega^1(U) \}.
\]
For $\alpha \in \Omega^1(U)$ and \( f, h \in C^\infty(U) \), we have
\[
L_{\#(\alpha)}(df, dh) = \#(\alpha).\pi(df, dh) - \pi(L_{\#(\alpha)}df, dh) - \pi(df, L_{\#(\alpha)}dh) \\
= \#(\alpha).\pi(df, dh) + \#(df). < \alpha, df > - \pi_{\#}(df). < \alpha, dh > \\
= \#(\alpha).\pi(df, dh) + < D_{dh}\alpha, df > - < D_{df}\alpha, dh > + < \alpha, D_{dh}df - D_{df}dh > \\
= < D_{dh}\alpha, df > - < D_{df}\alpha, dh > .
\]

So we get the formula
\[
L_{\#(\alpha)}(\beta, \gamma) = < D_{\gamma}\alpha, \beta > - < D_{\beta}\alpha, \gamma >, \quad \alpha, \beta, \gamma \in \Omega^1(U) .
\]

**Proposition 2.1** Let \( M \) be a Riemannian manifold, \( \pi \) a Poisson tensor on \( M \) and \( U \) an open set such that \( D \) is a \( \mathcal{F} \)-connection on \( U \). Then, we have:

1. for any \( \alpha, \beta \in \Omega^1(U) \),
   \[
   \alpha \in \text{Ker}\pi_\# \quad \Rightarrow \quad D_\beta \alpha \in \text{Ker}\pi_\# ;
   \]
2. for any \( \alpha, \beta \in \Omega^1(U) \)
   \[
   \alpha \in \text{Ker}\pi_\#^\perp \quad \Rightarrow \quad D_\beta \alpha \in \text{Ker}\pi_\#^\perp ,
   \]
   where \( \text{Ker}\pi_\#^\perp \) is the orthogonal of \( \text{Ker}\pi_\# \).

**Proof.**
1. If \( \pi_\#(\alpha) = 0 \), \( D_\alpha \beta = 0 \) and then \( \pi_\#(D_\beta \alpha) = \pi_\#([\beta, \alpha]_\pi) = [\pi_\#(\alpha), \pi_\#(\beta)] = 0 \).
2. Suppose that \( \alpha \in \text{Ker}\pi_\#^\perp \) and \( \pi_\#(\gamma) = 0 \). We have
   \[
   < D_\beta \alpha, \gamma > = \pi_\#(\beta). < \alpha, \gamma > - < D_\beta \gamma, \alpha > = 0
   \]
   since \( \pi_\#(D_\beta \gamma) = 0 \) according to 1. \quad q.e.d.

**Proposition 2.2** Let \( M \) be a Riemannian manifold, \( \pi \) a Poisson tensor on \( M \) and \( U \) an open set such that \( D \) is a \( \mathcal{F} \)-connection on \( U \). Then
\[
Z^1(U) = \{ \alpha \in \Omega^1(U); \pi_\#(\alpha) = 0 \text{ and } D\alpha = 0 \},
\]
\[
= \{ \alpha \in \Omega^1(U); \pi_\#(\alpha) = 0 \text{ and } L_{\#(\alpha)}\pi = 0 \} .
\]
**Proof.** The first equality comes from the fact that, if $\mathcal{D}$ is a $\mathcal{F}$-connection and $\alpha \in \Omega^1(M)$ such that $\pi\#(\alpha) = 0$, we have $[\alpha, \beta]_{\pi} = -D_{\beta}\alpha$ for any $\beta \in \Omega^1(U)$.

From (1), we have $D\alpha = 0$ implies $L_{\#(\alpha)}\pi = 0$ and then $Z^1(U) \subset \{\alpha \in \Omega^1(U); \pi\#(\alpha) = 0 \}$.

Suppose now that $\pi\#(\alpha) = 0$ and $L_{\#(\alpha)}\pi = 0$. From Proposition 2.1, we get $\pi\#(D_{\beta}\alpha) = 0$ for any $\beta \in \Omega^1(U)$. On other hand, let $p \in U$ and $\gamma \in T_p^*M$ such that $\pi\#(\gamma) = 0$, by using (1), we get for any $\beta \in \Omega^1(U)$

$$<D_{\beta}\alpha, \gamma >= <D\gamma, \beta > = 0$$

and hence $D_{\beta}\alpha \in Ker\pi\# \cap Ker\pi\#$ which implies $D\alpha = 0$. q.e.d.

**Proposition 2.3** Let $M$ be a Riemannian manifold, $\pi$ a Poisson tensor on $M$ and $U$ an open set such that $\mathcal{D}$ is a $\mathcal{F}$-connection on $U$ and the rank of $\pi$ is constant on $U$. Then

$$\mathcal{X}(\mathcal{F}, U)^\perp = \#(Z^1(U)).$$

**Proof.** If $\alpha \in Z^1(U)$, $\#(\alpha)$ is perpendicular to the symplectic foliation and, according to Proposition 2.2, $\#(\alpha)$ is a Poisson vector field and then a foliated vector field.

Let $X \in \mathcal{X}(\mathcal{F}, U)^\perp$. Since the Poisson tensor is regular in $U$, there exists an unique $\alpha \in \Omega^1(U)$ such that $\pi\#(\alpha) = 0$ and $X = \#(\alpha)$. The vector field $\#(\alpha)$ is foliated if and only if, for any $f \in C^\infty(U)$ and for any $h \in Z^0(U)$, $[\#(\alpha), H_f](h) = 0$. This is equivalent to $H_f. <\alpha, dh > = 0$ which is equivalent to $<D_d\alpha, dh > = 0$ since $Ddh = 0$. From Proposition 2.1, we deduce that $D\alpha = 0$ and hence $\alpha \in Z^1(U)$, by Proposition 2.2, which completes the proof. q.e.d.

Let us prove now Theorem 1.1 and Theorem 1.2.

**Proof of Theorem 1.1.** 1. Recall that the metric $g$ is bundle-like for the foliation $\mathcal{F}$ if it has the following property: for any open set $U$ of $M$ and for all vector fields $X, Y \in \mathcal{X}(\mathcal{F}, U)^\perp$, the function $g(X, Y)$ is a Casimir function. Let $X, Y \in \mathcal{X}(\mathcal{F}, U)^\perp$. According to Proposition 2.3, $X = \#(\alpha)$ and $Y = \#(\beta)$ where $\alpha, \beta \in Z^1(U)$. According to Proposition 2.2, $D\alpha = D\beta = 0$ and then, for any $f \in C^\infty(U)$,

$$H_f. <\alpha, \beta > = <D_d\alpha, \beta > + <\alpha, D_d\beta > = 0$$

11
and the assertion follows.

2. The result is a consequence of Proposition 2.2 and Proposition 2.3. q.e.d.

**Proof of Theorem 1.2.** Let $M$ be a Riemannian manifold and $\pi$ a regular Poisson tensor on $M$ such that the metric contravariant connection $D$ is a $\mathcal{F}$-connection. For any symplectic leaf $S$, we have

$$T^*M|_S = \text{Ker}\pi|_S \oplus \text{Ker}\pi^\perp|_S,$$

where $\text{Ker}\pi$ denotes the kernel of $\pi_#$ and $\text{Ker}\pi^\perp$ its orthogonal. For any local sections $\alpha$ and $\beta$ of $T^*M|_S$, we define $[\alpha, \beta]\pi|_S$ by

$$[\alpha, \beta]\pi|_S = [\tilde{\alpha}, \tilde{\beta}]\pi|_S,$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are two extensions of $\alpha$ and $\beta$. In a similar way, we define $D_{\alpha\beta}$.

If $\alpha, \beta$ are two local sections of $\text{Ker}\pi^\perp|_S$, from Proposition 2.1, $D_{\alpha\beta}$ and $D_{\beta\alpha}$ are sections of $\text{Ker}\pi^\perp|_S$ and hence $[\alpha, \beta]\pi|_S$ is also a section of $\text{Ker}\pi^\perp|_S$. We deduce that the inverse $\pi_{\#}^{-1} : TS \to \text{Ker}\pi^\perp|_S$ of the isomorphism $\pi_{\#} : \text{Ker}\pi^\perp|_S \to TS$ satisfies

$$[\pi_{\#}^{-1}(X), \pi_{\#}^{-1}(Y)]\pi|_S = \pi_{\#}^{-1}([X, Y]) \quad \forall X, Y \in \Gamma(TS).$$

Hence $\pi_{\#}^{-1} : TS \to \text{Ker}\pi^\perp|_S$ is a splitting of the anchor map which is compatible with the Lie bracket. According to [5] Corollary 5.2 (iii), we get the result. q.e.d.

Let us give now some characterizations of metric contravariant $\mathcal{F}^{reg}$-connections.

**Proposition 2.4** Let $M$ be a Riemannian manifold and $\pi$ a Poisson tensor on $M$. Then the following assertions are equivalent:

1. For any open set $U \subset M$ and for any function $f \in Z^0(U)$, $Ddf = 0$.
2. For any open set $U \subset M$ and for any $\alpha \in Z^1(U)$, $D\alpha = 0$.
3. For any open set $U \subset M$ and for any function $f \in Z^0(U)$, the gradient vector field $\nabla f$ is a Poisson vector field.
4. For any open set $U \subset M$ and for any $\alpha \in Z^1(U)$, $\#(\alpha)$ is a Poisson vector field.

5. The metric contravariant connection $\mathcal{D}$ associated to the metric and $\pi$ is a $\mathcal{F}^{\text{reg}}$-connection.

**Proof.** We denote by $O^{\text{reg}}$ the dense open set where the rank of $\pi$ is locally constant. Suppose that $\mathcal{D}$ is a $\mathcal{F}^{\text{reg}}$-connection. Let $U \subset M$ be an open set and $f \in Z^0(U)$. We have $df \in Z^1(U)$ and then, according to Proposition 2.2, $\mathcal{D}df$ vanishes in $U \cap O^{\text{reg}}$ and hence in $U$. Moreover, from (1), $L_{\nabla_f} \pi = 0$. We have shown that $5 \Rightarrow 1 \Rightarrow 3$. In a similar way, we have $5 \Rightarrow 2 \Rightarrow 4$. On the other hand, we have obviously $4 \Rightarrow 3$. We will establish that $3 \Rightarrow 5$ and the proposition will follow. Suppose that 3 holds. Let $p$ be a regular point of $\pi$ and $\alpha \in T^*_p M$ such that $\pi_{\#}(\alpha) = 0$. There exists an open set $U$ and $f \in Z^0(U)$ such that $df(p) = \alpha$. For $\beta, \gamma \in \Omega^1(U)$, we have

$$< \mathcal{D}_\alpha \beta, \gamma > = < \mathcal{D}_f \beta, \gamma > (p) = < \mathcal{D}_\beta df, \gamma > (p) = < \mathcal{D}_\gamma df, \beta > (p)$$

$$= < \mathcal{D}_d \gamma, \beta > (p) = - < \mathcal{D}_d \beta, \gamma > (p) = - < \mathcal{D}_\alpha \beta, \gamma >$$

and hence $\mathcal{D}_\alpha \beta = 0$ and the implication follows. \(\text{q.e.d.}\)

### 3 Killing-Poisson structures

In this section, we recall the definition of the divergence of a multi-vector field with respect to a covariant connection and we prove Theorem 1.3 and Theorem 1.4.

Let $M$ be a smooth manifold and $Q$ a $q$-vector field, that is, $Q \in \Gamma(\wedge^q TM)$ ($q \geq 0$). Let

$$c : \Omega^1(M) \times \Gamma(\wedge^q TM) \to \Gamma(\wedge^{q-1} TM)$$

denote the contraction. Let $\nabla$ be a covariant connection on $M$. Then the $(q - 1)$-vector field $\text{div}_\nabla Q$ given by

$$\text{div}_\nabla(Q) = c(\nabla P)$$

is called the divergence of $Q$ associated with $\nabla$. 

13
It is shown that if $\nabla$ is the Levi-Civita connection on an orientable Riemannian manifold and $Q = X$ is a vector field, $\text{div}_\nabla X$ is the usual divergence of $X$ with respect to the Riemannian volume $\mu_g$, i.e., $L_X \mu_g = (\text{div}_\nabla X) \mu_g$.

Although the divergence of a $q$-vector field depends on the choice of the connection $\nabla$, we often omit $\nabla$ and write $\text{div} Q$ for $\text{div}_\nabla Q$.

if $\nabla$ preserves a volume form $\mu$, we have

$$d(i_Q \mu) = -(-1)^q i_{\text{div} Q} \mu.$$  \hfill (2)

On other hand, if $\pi$ is a Poisson tensor on $M$, we have for any $f \in C^\infty(M)$,

$$\text{div}\pi(f) = \text{div} H_f$$  \hfill (3)

and then

$$\text{div} \pi = 0 \Leftrightarrow d(i_\pi \mu) = 0 \Leftrightarrow L_{H_f} \mu = 0 \forall f \in C^\infty(M).$$  \hfill (4)

So we get the following characterization of Killing-Poisson structures.

**Proposition 3.1** Let $M$ be a Riemannian manifold and $\pi$ a Poisson tensor on $M$. Then $\pi$ is a Killing-Poisson tensor if and only if the following assertions hold:

1. for any $f \in C^\infty(M)$, $L_{H_f} \mu_g = 0$ ($\mu_g$ stands of the Riemannian density),

2. the metric contravariant connection associated to $\pi$ and the Riemannian metric is a $F_{reg}$-connection.

Let us prove now Theorem 1.3.

**Proof of Theorem 1.3.** In [2] Section 4, we have shown that $D\pi = 0$ implies that $\text{div} \pi = 0$.

Let $U$ be an open subset of $M$, $f \in Z^0(U)$ and $\alpha \in \Omega^1(U)$. The condition $D\pi = 0$ implies that the conclusions of Proposition 2.1 hold and then $\pi_\#(D_\alpha df) = 0$.

Let $p$ be regular point in $U$, $V$ a neighborhood of $p$ where the rank of $\pi$ is constant and $\gamma \in \Omega^1(V)$ such that $\pi_\#(\gamma) = 0$. There exists $h \in Z^0(V)$ such that $\alpha(p) = dh(p) + \alpha_0$ with $\alpha_0 \in \text{Ker}\pi_\#$. Hence

$$<D_\alpha df, \gamma>(p) = <D_\gamma \alpha, \gamma>(p) = -<\alpha, D_\gamma \gamma>(p)$$

$$= -<dh, D_\gamma \gamma>(p) + <\alpha_0, D_\gamma \gamma>(p) = 0,$$
where \( < dh, D \gamma > = 0 \) from the definition of \( D \) and \( < \alpha_0, D \gamma > = 0 \) from Proposition 2.1. Hence \( D df = 0 \) and then \( D \) is an \( F^{reg} \)-connection according to Proposition 2.4. q.e.d.

To prove Theorem 1.4, we recall some facts about solutions of the classical Yang-Baxter equation.

Let \((G, [\ , \ ])\) be a Lie algebra. Recall that a solution of the classical Yang-Baxter equation is a bivector \( r \in \wedge^2 G \) such that
\[
[r, r](\alpha, \beta, \gamma) = \alpha([r(\beta), r(\gamma)]) + \beta([r(\gamma), r(\alpha)]) + \gamma([r(\alpha), r(\beta)]),
\]
where \( r : G^* \rightarrow G \) denotes also the linear map given by \( \alpha(r(\beta)) = r(\alpha, \beta) \).

A solution \( r \) of the classical Yang-Baxter equation is called unimodular if the subalgebra \( im r \) is unimodular i.e., for any \( u \in im r \), the trace of the endomorphism \( ad_u : im r \rightarrow im r, v \mapsto [u, v] \), vanishes.

Let \( G \rightarrow \mathcal{X}(M) \) be an action of a Lie algebra \( G \) on a manifold \( M \) i.e., a morphism of Lie algebras from \( G \) to the Lie algebra of vector fields on \( M \). Any solution \( r \) of the classical Yang-Baxter equation defines a Poisson tensor \( \Gamma(r) \) on \( M \).

Let us prove now Theorem 1.4.

**Proof of Theorem 1.4.**

1. **The divergence of \( \Gamma(r) \) vanishes**

The solution \( r \) defines on \( im r \) a 2-form \( \omega_r \) by
\[
\omega_r(u, v) = r(r^{-1}(u), r^{-1}(v)),
\]
where \( r^{-1}(u) \) denotes any antecedent of \( u \). The 2-form \( \omega_r \) is nondegenerate and symplectic i.e.,
\[
\omega_r(u, [v, w]) + \omega_r(v, [w, u]) + \omega_r(w, [u, v]) = 0, \quad u, v, w \in im r.
\]

There exists a basis \((e_1, \ldots, e_n, f_1, \ldots, f_n)\) of \( im r \) such that the symplectic form \( \omega_r \) is given by
\[
\omega_r = \sum_{i=1}^n e_i^* \wedge f_i^*.
\]

Since \( I_m r \) is unimodular, then for any \( z \in I_m r \), the trace of \( ad_z \) is zero which is equivalent to
\[
\sum_{i=1}^{n} (\omega_r([z, e_i], f_i) + \omega_r(e_i, [z, f_i])) = 0.
\]
This relation is equivalent to
\[
\sum_{i=1}^{n} \omega_r(z, [e_i, f_i]) = 0
\]
and hence to
\[
\sum_{i=1}^{n} [e_i, f_i] = 0.
\]
Denotes by \( \mu_g \) the Riemannian volume. Since \( L_{\Gamma(e_i)}\mu_g = L_{\Gamma(f_i)}\mu_g = 0 \) for \( i = 1, \ldots, n \), we have
\[
d(i_r, \mu_g) = d \left( \sum_{i=1}^{n} i_{\Gamma(e_i)} \wedge i_{\Gamma(f_i)} \mu_g \right)
= \sum_{i=1}^{n} \left( i_{\Gamma(e_i)} \mu_g - i_{\Gamma(f_i)} L_{\Gamma(e_i)} \mu_g - i_{\Gamma(f_i)} L_{\Gamma(f_i)} \mu_g \right)
= i_{\Gamma(\sum_{i=1}^{n} [e_i, f_i])} \mu_g = 0
\]
and hence \( \text{div} \Gamma(r) = 0 \).

2. The metric contravariant connection is a \( \mathcal{F}^{reg} \)-connection

If \( r = \sum_{i,j} a_{ij} u_i \wedge u_j \), we put, for \( \alpha, \beta \in \Omega^1(P) \),
\[
\mathcal{D}^r_{\alpha \beta} := \sum_{i,j} a_{ij} \alpha(U_i) L_{U_j} \beta,
\]
where \( U_i = \Gamma(u_i) \). We get a map \( D^r : \Omega^1(P) \times \Omega^1(P) \to \Omega^1(P) \) which is the metric contravariant connection associated to the Poisson tensor \( \Gamma(r) \) and the Riemannian metric. On can check easily that \( \mathcal{D}^r \) is a \( \mathcal{F}^{reg} \)-connection since the action is locally free. \( \text{q.e.d.} \)
4 Killing-Poisson tensors on a compact Riemannian manifold cannot be exact

A Poisson manifold \((M, \pi)\) is called exact if there exists a vector field \(X\) such that \([X, \pi] = \pi\). The vector field \(X\) is called a Liouville vector field. If \(\pi\) comes from a symplectic form \(\omega\), the condition \([X, \pi] = \pi\) is equivalent to the exactness of the form \(\omega\). Although a compact symplectic manifold cannot be exact, there do exist compact regular Poisson manifolds admitting Liouville vector fields (see [10], [11] and [14]). In this section, we show that, like symplectic structures on compact manifolds, a non trivial regular Killing-Poisson tensor on a compact Riemannian manifold cannot be exact.

We begin by giving some general results on unimodular exact Poisson tensors. Recall that a Poisson manifold \((M, \pi)\) is called unimodular if there exists a volume form \(\mu\) on \(M\) such that any hamiltonian vector field \(Hf\) preserves the volume form i.e., \(L_{Hf}\mu = 0\).

Let \((M, \pi)\) be a Poisson manifold and \(X\) a Liouville vector field on \(M\). We have obviously, for any hamiltonian vector field \(Hf\),

\[
[X, Hf] = Hf + HX(f), \quad f \in C^\infty(P). 
\]

(5)

**Proposition 4.1** Let \((M, \pi)\) be an unimodular exact Poisson manifold. Then, for any invariant volume form \(\mu\), any Liouville vector field \(X\) and for each \(n \in \mathbb{N}\), we have

\[
L_X(i_\wedge^n \pi \mu) = di_X i_\wedge^n \pi \mu = (n + \text{div}_\mu X)i_\wedge^n \pi \mu. 
\]

(6)

**Proof.** We have obviously \([X, \wedge^n \pi] = n \wedge^n \pi\) and then

\[
ni_\wedge^n \pi \mu = i_{[X, \wedge^n \pi]} \mu = L_X \circ i_\wedge^n \pi \mu - i_\wedge^n \pi \circ L_X \mu = L_X \circ i_\wedge^n \pi \mu - \text{div}_\mu X(i_\wedge^n \pi \mu).
\]

This gives the relation. q.e.d.

**Proposition 4.2** Let \((M, \pi)\) be a compact unimodular regular non trivial Poisson manifold. Then, for any Liouville vector field \(X\), there exists a point in \(M\) where \(X\) is transverse to the symplectic foliation.

**Proof.** Let \((M, \pi)\) be a compact unimodular regular Poisson manifold of rank \(2q\) (\(q > 0\)). Suppose that there exists a Liouville vector field \(X\) which is
everywhere tangent to the symplectic foliation. This implies that the multi-vector field \( X \wedge (\wedge^9 \pi) \) vanishes identically. From (6), we get that, for any invariant volume form \( \mu \), the form \((q + \text{div}_\mu X)i_{\wedge^9 \pi \mu} \) vanishes also. The form \( i_{\wedge^9 \pi \mu} \) being a transverse volume form to the symplectic foliation, we get \( \text{div}_\mu X = -q \) which is a contradiction with \( \int_M \text{div}_\mu X \mu = 0 \). q.e.d.

We are able now to give a proof of Theorem 1.5.

**Proof of Theorem 1.5.** Let \( X \) be a Liouville vector field. The vector field \( X \) splits \( X = X^t + X^\perp \) where \( X^t \) is tangent to the symplectic foliation and \( X^\perp \) is perpendicular to the symplectic foliation. From (5), \( X \) is a foliated vector field which implies that \( X^\perp \) is a foliated vector field and hence a Poisson vector field according to Theorem 2.1. This implies that \( X^t \) is a Liouville vector field which it contradicts Proposition 4.2. q.e.d.

5 Killing-Poisson structures on 3-dimensional Riemannian manifolds

We begin by giving a description of Killing-Poisson tensors on a Riemannian surface. Let \( S \) be a connected orientable Riemannian surface and \( \pi \) a Poisson tensor on \( S \). If \( \mu \) is the Riemannian volume, we have that \( \text{div} \pi = 0 \) if and only if the function \( i_{\pi} \mu \) is constant. This mean that \( \pi \) is either trivial or symplectic. Hence a non trivial Killing-Poisson tensor on \( S \) is symplectic.

**Theorem 5.1** Let \( M \) be a 3-dimensional oriented Riemannian manifold and \( \pi \) a bivector field on \( M \). We denote by \( \mu \) the Riemannian volume. Then the following assertions are equivalent:

1. \( \pi \) is a Killing-Poisson tensor.
2. \( \mathcal{D} \pi = 0 \).
3. The 1-form \( \alpha = i_{\pi} \mu \) satisfies:

\[
\begin{align*}
  d\alpha &= 0 \quad \text{and} \quad d <\alpha, \alpha> + \delta(\alpha)\alpha = 0, \\
  \text{where} \, \delta(\alpha) &= -\text{div}(\#(\alpha)).
\end{align*}
\]
Proof. Recall (see [13]) that for any multi-vector fields $Q$ and $R$ on $M$, we have

$$i_{[Q,R]}\mu = (-1)^{(|Q|-1)(|R|-1)} \left( i_Qdi_R\mu - (-1)^{|Q|}di_Qi_R\mu + (-1)^{(|Q|-1)|R|+|Q|i_Rdi_Q\mu} \right)$$

where $|Q|$ is the degree of the multi-vector field $Q$.

From this relation, we get for $X$ a vector field and $\pi$ is a bivector field:

$$i_{[\pi,\pi]}\mu = di_\pi\wedge\pi\mu - 2i_\pi di_\pi\mu,$$  \hspace{1cm} (8)

$$i_{[X,\pi]}\mu = i_Xdi_\pi\mu + di_Xi_\pi\mu - (\text{div}_\mu X)i_\pi\mu.$$ \hspace{1cm} (9)

We will prove now the equivalence $1 \iff 2$.

We have seen, in Theorem 1.3, that $2 \implies 1$.

Conversely, suppose that $\pi$ is a Killing-Poisson tensor. The 1-form $\alpha = i_\pi\mu$ satisfies $\pi\#(\alpha) = 0$ and $d\alpha = 0$ and then $\alpha \in Z^1(U)$. From Proposition 2.2, we get $\mathcal{D}\alpha = 0$. Since $\mathcal{D}\mu = 0$, we get $\mathcal{D}\pi = 0$ which completes the proof of the equivalence.

We will prove now the equivalence $1 \iff 3$.

Suppose that $\pi$ is a Killing-Poisson tensor. We have $d\alpha = 0$ and $\alpha \in Z^1(U)$ and, from Proposition 2.2, $[\#(\alpha), \pi] = 0$. Then, from (9), we get the relation

$$d < \alpha, \alpha > + \delta(\alpha)\alpha = 0.$$

Suppose now that 3 holds. From (8), we get $[\pi, \pi] = 0$ and then $\pi$ is a Poisson tensor, $\text{div}\pi = 0$ and also $[\#(\alpha), \pi] = 0$. Let $U$ be an open set and $f \in Z^0(U)$. We will show that $\nabla f$ is a Poisson vector field and the result follows.

We denote by $U^{reg}$ the open set intersection of $U$ with the dense open set of regular point of $\pi$. Let $p \in U^{reg}$. We have two case:

First case: $\alpha(p) = 0$ and then $\pi$ vanishes on a neighborhood of $p$ and hence $[\nabla f, \pi](p) = 0$.

Second case: $\alpha(p) \neq 0$ and then $\alpha$ does not vanish on a neighborhood of $p$ and hence $df = h\alpha$ where $h$ is a local function. Since $\alpha \in Z^1(M)$, $h$ is a Casimir function and then $[\nabla f, \pi] = h[\#(\alpha), \pi] = 0$. This completes the proof. \hspace{1cm} q.e.d.

**Corollary 5.1** Let $M$ be an oriented Riemannian 3-manifold such that $H^1_{dR}(M) = 0$ and let $\pi$ be a bivector field on $M$. The bivector field $\pi$ is a Killing-Poisson tensor if and only if there exists $f \in C^\infty(M)$ such that $i_\pi\mu = df$ and

$$d < df, df > + \Delta(f)df = 0$$

19
where $\Delta$ is the Beltrami-Hodge Laplacian acting on functions.

**Examples.** According to Corollary 5.1, a bivector field $\pi$ on $\mathbb{R}^3$ is a Killing-Poisson tensor with respect to the Euclidian metric if and only if

$$\pi = \frac{\partial f}{\partial z} \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + \frac{\partial f}{\partial x} \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$$

where $f \in C^\infty(\mathbb{R}^3)$ verifies

$$d < df, df > + \Delta(f) df = 0. \quad (E)$$

The polynomial functions of degree 2 solutions of $(E)$ are

$$f(x, y, z) = (a + c)x^2 + (a + b)y^2 + (b + c)z^2 - 2\sqrt{bc}xy + 2\sqrt{ab}xz + 2\sqrt{ac}yz$$

where $a, b, c$ are real constants with the same sign. This gives all linear Killing-Poisson structures on $\mathbb{R}^3$ endowed with the Euclidian metric.

One can check easily that a function $f(x, y, z) = g(r)$ where $r = x^2 + y^2 + z^2$ is a solution of $(E)$ if and only if the function $g$ satisfies the differential equation

$$2ry'' - y' = 0.$$

Then $g(r) = ar^{\frac{3}{2}}$ where $a$ is a constant. Hence, the Poisson tensor

$$\pi = \sqrt{x^2 + y^2 + z^2} \left( z \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + x \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} \right)$$

is a Killing-Poisson tensor on $\mathbb{R}^3$. Remark that $\pi = \sqrt{x^2 + y^2 + z^2} \pi_{so(3)}$ where $\pi_{so(3)}$ is the Lie-Poisson structure on the dual of the Lie algebra $so(3)$.

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