\( U_n(q) \) acting on flags and supercharacters

Richard Dipper*, Qiong Guo**

*Institut für Algebra und Zahlentheorie
Universität Stuttgart, 70569 Stuttgart, Germany
E-mail: richard.dipper@mathematik.uni-stuttgart.de

** Institut für Algebra, Zahlentheorie und Diskrete Mathematik
Leibniz Universität Hannover, 30167 Hannover, Germany
E-mail: guo@math.uni-hannover.de

Abstract

Let \( U = U_n(q) \) be the group of lower unitriangular \( n \times n \)-matrices with entries in the field \( \mathbb{F}_q \) with \( q \) elements for some prime power \( q \) and \( n \in \mathbb{N} \). We investigate the restriction to \( U \) of the permutation action of \( GL_n(q) \) on flags in the natural \( GL_n(q) \)-module \( \mathbb{F}_n^q \). Applying our results to the special case of flags of length two we obtain a complete decomposition of the permutation representation of \( GL_n(q) \) on the cosets of maximal parabolic subgroups into irreducible \( \mathbb{C}U \)-modules.

1 Introduction

One way to define unipotent Specht modules for \( GL_n(q) \) in a characteristic free way is provided by James’ kernel intersection theorem [10, Theorem 15.19]: For a partition \( \lambda \) of \( n \), the unipotent Specht module \( S(\lambda) \) for \( GL_n(q) \) over any field \( K \) of characteristic not dividing \( q \) is defined as intersection of the kernels of all homomorphisms \( \phi : M(\lambda) \to M(\mu) \) for all partition \( \mu \) of \( n \) dominating \( \lambda \). Here \( M(\lambda) \) denotes the permutation representation of \( GL_n(q) \) acting by right translation on the set of all \( \lambda \)-flags in the natural \( GL_n(q) \)-module \( V = \mathbb{F}_n^q \), where \( \mathbb{F}_q \) is the field with \( q \) elements. If \( K = \mathbb{C} \) then \( S(\lambda) \) are irreducible unipotent \( GL_n(q) \)-modules appearing as irreducible constituents of the permutation module of \( GL_n(q) \) acting on flags in \( V \).

Many properties of \( S(\lambda) \) turn into properties of the Specht module \( S^\lambda \) for the symmetric group \( \mathfrak{S}_n \) by taking \( q \) to 1. For instance, the dimension of \( S^\lambda \) is given by a hook formula [9]. Replacing the occurring numbers in it by the corresponding \( q \)-numbers produces a polynomial with integral coefficients (the generic degree) whose evaluation at \( q \) is the dimension of the \( GL_n(q) \)-module \( S(\lambda) \), (see [10]). So it appears to be reasonable to suspect, that there is also a \( q \)-analogue in \( S(\lambda) \) for the standard basis theorem [8] for the \( \mathfrak{S}_n \)-module \( S^\lambda \). This states that to each standard \( \lambda \)-tableau \( t \) there is an element \( e_t \in S^\lambda \) such that \( \{ e_t \mid t \ \text{standard} \ \lambda \text{-tableau} \} \) is an integral basis of the Specht module \( S^\lambda \) of \( \mathfrak{S}_n \). In [7] it was conjectured, that the unipotent Specht modules

\begin{flushleft}
\textbf{Date:} December 11, 2014
\textbf{2010 Mathematics Subject Classification.} Primary 20C15, 20D15. Secondary 20C33, 20D20
\end{flushleft}

\begin{flushleft}
This work was partially supported by the DFG priority programme SPP 1388 in representation theory, no. 99028426
\end{flushleft}
S(\lambda) for GL_n(q) have standard basis defined over any field K of characteristic coprime to q as follows: For each standard \lambda-tableau \sigma there is attached a polynomial g_\sigma(x) \in \mathbb{Z}[x] with g_\sigma(1) = 1 and there are g_\sigma(q) many elements of S(\lambda) such that all these elements form a basis of S(\lambda) where \sigma runs through all the standard \lambda-tableaux. In \[2,14,7\] such a standard basis for S(\lambda) was constructed in the special case of 2-part partition \lambda.

The order of the unitriangular group \(U = U_n(q) \leq GL_n(q)\) of (lower) unitriangular matrices in \(GL_n(q)\) is a power of \(q\). Thus the group algebra \(KU\) is semisimple for all fields \(K\) of characteristic not dividing \(q\). In \[8\] decomposing the restriction of \(M(\lambda)\) to \(U\) completely into irreducible \(U\)-modules for 2-part partitions \(\lambda\), and then applying James’ kernel intersection theorem, the main results of \[4\] were reproved, giving the standard basis of \(S(\lambda)\) a representation theoretic interpretation. This new approach in \[8\] bears resemblance to Kirillov’s orbit method \[13\] and the supercharacter theory of \(U\) introduced by André \[1\] and Yan \[14\]. As a first step towards constructing standard bases of unipotent Specht modules, it is the main goal of this paper to make this remarkable connection precise and in particular generalize it to arbitrary compositions \(\lambda\) of \(n\).

We briefly outline the main results of the paper below:

The starting point in section 2 is based on a general construction, due to Jedlitschky \[11\]: If \(G\) is a finite group acting on an abelian group by automorphisms, and if \(f : G \to V\) is a 1-cocycle (see definition 2.3), then, using \(f\), one can define an action of \(G\) on the group algebra \(CV\) making \(CV\) into a monomial \(CG\)-module. It has been shown in \[11\] that \(CV\) is isomorphic to the regular representation of \(CG\) under this action, provided \(f\) is a bijective map. For example, we may choose \(G = U_n(q) = U, V = \text{Lie}(U) = \{u - 1 | u \in U\}\) under addition, and \(f : U \to V : u \mapsto u - 1\). Then the characters of \(G\) on the orbit modules arising from an orbit decomposition of the monomial \(C\)-basis of \(CV\) are precisely the André-Yan supercharacters in \([1,14]\).

Our first main result 2.9 deals with the more general case that \(f\) is surjective but not necessarily injective. Then the kernel of \(f\) is a subgroup \(H\) of \(G\) and the monomial \(CG\)-module \(CV\) is isomorphic to the permutation module \(\text{Ind}_H^G \mathbb{C}_H\), where \(\mathbb{C}_H\) denotes the trivial \(H\)-module. The advantage of this monomial linearisation of \(\text{Ind}_H^G \mathbb{C}_H\) lies in the fact that the monomial basis of \(CV\) decomposes in general in many orbits yielding a direct sum decomposition of \(\text{Ind}_H^G \mathbb{C}_H\) into smaller \(CG\)-modules.

In section 3 this is applied to semidirect products \(U_K = U_J \ltimes U_L\) of pattern subgroups \(U_K, U_J, U_L\) of \(U_n(q)\). Here a suitable surjective 1-cocycle \(f : U_K \to V_J = \text{Lie}(U_J) = \{x - 1 | x \in U_J\}\) always exists and 2.9 can be applied. Thus the monomial \(\mathbb{C}U_K\)-module \(\mathbb{C}V_J\) is isomorphic to \(\text{Ind}_{U_L}^{U_K} \mathbb{C}_{U_L}\), (Theorem 3.10). In Theorem 3.18 we give an explicit description of the monomial basis of \(\mathbb{C}V_J\) and the action of \(U_K\) on it.

In section 4 we exhibit a special basis (Theorem 4.10) of the permutation module \(M(\lambda)\), \(\lambda\) a composition of \(n\), by normal forms of \(\lambda\)-adapted bases of \(V\). Applying Mackey decomposition to the restriction to \(U\) of \(M(\lambda)\), we obtain a decomposition of \(\text{Res}_U^G M(\lambda)\) into \(\sigma\)-components \(M_\sigma\), where \(\sigma\) runs through the set \(\mathbb{R}Std(\lambda)\) of row standard \(\lambda\)-tableaux (Proposition 5.1). If \(d = d(\sigma) \in \mathbb{S}_n\) is the unique permutation taking the initial \(\lambda\)-tableau to \(\sigma \in \mathbb{R}Std(\lambda)\), the subgroup \(U_\sigma^d \cap U\) of \(U\) is a pattern subgroup \(U_K\) which in turn is a semidirect product \(U_J \ltimes U_L\) of the normal pattern subgroup \(U_J\) by a pattern subgroup \(U_L\) (see 5.6). Moreover the restriction of \(M(\lambda)\) to \(U_K\) is isomorphic to the trivial \(U_L\)-module induced to \(U_K\), hence the results of section 3 apply. Thus \(\text{Ind}_{U_L}^{U_K} \mathbb{C}_{U_J}\) is isomorphic to \(\mathbb{C}V_J, V_J = \{x - 1 | x \in U_J\} = \text{Lie}(U_J)\). The monomial basis in \(\mathbb{C}V_J\) and the action of \(U_K\) on that is explicitly exhibited in Theorem 5.8.
In section 6 the supercharacter theory of André and Yan [1,4] is reinspected as a special case of our approach.

In the remaining two sections we return to the special case of 2-part compositions $\lambda$. It was shown in [8] that in this case the orbit modules of $U_K$ acting on $\mathbb{C}V_J$ are irreducible and invariant under the extended action of $U$. Thus they are irreducible $\mathbb{C}U$-modules. Moreover in [8] the orbits were classified by verges, as defined in 7.3. In Theorem 8.17 we apply the results of the previous sections to identify the resulting irreducible $\mathbb{C}U$-orbit modules as certain distinguished irreducible constituents of monomial $\mathbb{C}U$-orbit modules affording supercharacters of $U$. As a consequence we obtain, that for 2-part partitions $\lambda$ the $s$-component $M_s$, $s$ a row standard $\lambda$-tableau, is multiplicity free. Moreover the multiplicities of irreducible constituents of $\text{Res}_{U_n}^{GL_n(q)} M(\lambda)$ are combinatorially determined and in particular independent of $q$.

It should be mentioned here, that there is yet another reason of inspecting the monomial representations of pattern subgroups of the form $U_d \cap U$ concerning supercharacters of $U_n(q)$: In [12] Le reduced the question of decomposing supercharacters of $U_n(q)$ into irreducibles to the decomposition of certain monomial representations of pattern subgroups $U^d \cap U$, $U = U_k(q)$, where $d$ is an element of the symmetric group $S_k$ and $k < n$. The hope is, that the monomial linearisation of permutation modules for pattern subgroups of the form $U^d \cap U$ of $U$ introduced here may provide a new tool for the investigation of supercharacters of the unitriangular group $U$.

2 Monomial linearisation

In this section we shall present a procedure which allows to transform under certain circumstances a transitive $G$-set, $G$ a finite group, into a monomial $G$-set which may decompose into many orbits. Thus this yields a decomposition of the corresponding permutation module into a direct sum of monomial $\mathbb{C}G$-modules. This procedure is based on work of Jedlitschky [11] and Kirillov’s orbit method [13]. For the moment, let $G$ be an arbitrary finite group.

For any field $K$ and any finite group $G$ we may identify the group algebra $K G$ with $K$-algebra $K G$ of functions from $G$ to $K$ where the product in $K G$ is given by

$$\tau \sigma (g) = \sum_{h \in G} \tau (h) \sigma (h^{-1} g)$$

for $\tau, \sigma \in K G$, $g \in G$. The identification of $K G$ and $K G$ is given by

$$\tau \mapsto \sum_{g \in G} \tau (g) g$$

for $\tau \in K G$. (2.1)

We next describe the permutation representation $K G$-module acting on the cosets of a subgroup $H$ of $G$ in terms of functions on $G$ as follows:

2.2 Lemma. Let $G$ be a finite group, $H \leq G$ and let $e = \sum_{h \in H} h$ and $M = e K G$. Then $M$ consists of functions from $G$ to $K$, which are constant on the right cosets of $H$ in $G$.

Proof. In $K G$ we have $eg = \sum_{h \in H} hg$ for any $g \in G$, and we obtain a $K$-basis of $e K G$ by \{eq | g \in G/H\}. By (2.1) $eg$ corresponds to the map from $G$ to $K$ which takes value 1 on the right coset $H g$ of $H$ in $G$ and zero on all the other right cosets. From this the Lemma follows immediately.

Now let $G$ be a finite group acting on a finite abelian group $(V, +)$ by automorphisms, the action of $g \in G$ denoted by $v \mapsto vg (v \in V)$.
2.3 Definition. A map \( f : G \to V \) is called a (right) 1-cocycle if \( f(xg) = f(x)g + f(g) \) for all \( x, g \in G \) holds. Similarly we call \( f \) a left 1-cocycle if \( f \) satisfies \( f(xg) = xf(g) + f(x) \) for all \( x, g \in G \).

We observe the group algebra \( \mathbb{C}V \cong \mathbb{C}^V \) of \( V \) becomes a right \( \mathbb{C}G \)-module setting
\[
(\tau, g)(v) = \tau(vg^{-1}) \quad \text{for } \tau \in \mathbb{C}^V, g \in G, v \in V
\]
where the action of \( G \) on \( \mathbb{C}^V \) is denoted by \((\tau, g) \mapsto \tau g\).

We denote \( \hat{V} \) to be the set of linear characters of \( V \), i.e. that is of group homomorphisms of \((V, +)\) into the multiplicative group \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \) of \( \mathbb{C} \). Note that \( \hat{V} \) is contained in \( \mathbb{C}^V \). Indeed under the identification of \( \mathbb{C}^V \) and \( \mathbb{C}V \), \( \chi \in \hat{V} \) is mapped to \(|V|e_\bar{\chi} \) where \( \bar{\chi} \) is the complex conjugate character of \( V \) and \( e_\bar{\chi} \) is the primitive idempotent of \( \mathbb{C}V \) affording \( \chi \). Since \( \{e_\chi \mid \chi \in \hat{V}\} \) is a basis of \( \mathbb{C}V \), we conclude that \( \hat{V} \) is a \( \mathbb{C} \)-basis of \( \mathbb{C}V \). Moreover one checks immediately:

2.4 Lemma. The action \((\tau, g) \mapsto \tau g\) of \( G \) on \( \mathbb{C}^V \) permutes \( \hat{V} \). So, for \( \chi \in \hat{V}, g \in G \) we have \( \chi g \in \hat{V} \).

Now let \( f : G \to V \) be a 1-cocycle. Then we have:

2.5 Theorem. \[11\] The map \( \alpha : \hat{V} \times G \to \mathbb{C}^* : (\chi, g) \mapsto \chi(f(g^{-1})) \) for \( \chi \in \hat{V}, g \in G \), satisfies \( \alpha(\chi, gh) = \alpha(\chi, g) \alpha(\chi, h) \) for \( \chi, g, h \in V \).

Proof. For convenience we provide the proof. Let \( \chi \in \hat{V}, g, h \in G \). Then
\[
\alpha(\chi, gh) = \chi(f((gh)^{-1})) = \chi(f(h^{-1}g^{-1})) = \chi(f(h^{-1})g^{-1}) + f(g^{-1}) = \chi(f(h^{-1})g^{-1})\chi(f(g^{-1})) = (\chi(g)f(h^{-1})\alpha(\chi, g) = \alpha(\chi, g) = \alpha(\chi, g, h) \alpha(\chi, g).
\]

This property of \( \alpha \) provides a coefficient system in general sense for constructing monomial action, hence we have proved:

2.6 Corollary. \[11\] The group algebra \( \mathbb{C}^V \) becomes a monomial \( \mathbb{C}G \)-module with monomial basis \( \hat{V} \) by setting
\[
\chi g = \alpha(\chi, g) \chi g = \chi(f(g^{-1})) \chi g
\]
for \( \chi \in \hat{V}, g \in G \).

Note that for 1-cocycle \( f : G \to V \) (left or right) we always have \( f(1_G) = 0 \in V \).

2.7 Lemma. Let \( f : G \to V \) be a 1-cocycle (or left 1-cocycle). Then \( \ker f = \{ g \in G \mid f(g) = 0 \in V \} \) is a subgroup of \( G \).

Proof. We check this for a right 1-cocycle \( f \), since the case of a left 1-cocycle is shown similarly. Let \( a, b \in \ker f \), then \( f(ab) = f(a)b + f(b) = 0 \cdot b + 0 = 0 \) proving \( ab \in \ker f \). Moreover
\[
0 = f(1) = f(aa^{-1}) = f(a)a^{-1} + f(a^{-1}) = f(a^{-1})
\]
showing \( a^{-1} \in \ker f \).

Let \( f : G \to V \) be a 1-cocycle. We define
\[
f^* : \mathbb{C}^V \to \mathbb{C}G : \tau \mapsto \tau \circ f \quad \text{for } \tau \in \mathbb{C}^V.
\]

2.8 Lemma. \[11\] Consider \( \mathbb{C}^V \) as a monomial \( \mathbb{C}G \)-module as defined in 2.6. Then \( f^* : \mathbb{C}^V \to \mathbb{C}G \cong \mathbb{C}G \) is a \( \mathbb{C}G \)-homomorphism. In particular the image of \( f^* \) is isomorphic to a right ideal \( \mathbb{C}G \).
Proof. Let \( \tau \in C^V, g \in G \) we have to show that \( f^*(\tau g) = f^*(\tau)g \). We may assume \( \tau = \chi \in \hat{V} \) since \( \hat{V} \) is a basis of \( C^V \). Now
\[
\begin{align*}
f^*(\chi g) = (\chi g) \circ f = \alpha(\chi, g)\chi \circ f.
\end{align*}
\]
Let \( h \in G \). Then
\[
\begin{align*}
(\alpha(\chi, g)\chi \circ f)(h) &= \alpha(\chi, g)\chi(g(f(h))) \\
&= \chi(f(g^{-1}))\chi(f(h)g^{-1}) \\
&= \chi(f(g^{-1}) + f(h)g^{-1}) = \chi(f(hg^{-1})) \\
&= f^*(\chi)(hg^{-1}) = (f^*(\chi)g)(h).
\end{align*}
\]
Hence \( f^*(\chi g) = f^*(\chi)g \).
\( \square \)

Note that \( f^* : C^V \to C^G \) is injective (surjective) if and only if \( f : G \to V \) is surjective (injective). In particular, if \( f : G \to V \) is surjective, the image \( f^* \) provides an isomorphism from the \( CG \)-module \( C^V \) to an right ideal of \( CG \). Indeed we have:

**2.9 Theorem.** Let \( G \) and \( V \) be finite groups, \( V \) abelian, and let \( f : G \to V \) be a surjective 1-cocycle. Let \( H \leq G \) be the kernel of \( f \). Then \( C^V \cong C^H \cong \text{Ind}_H^G C_H \) as \( CG \)-module, where \( C_H \) is the trivial \( H \)-module.

**Proof.** By \([22]\) it suffices to show that \( f^*(C^V) \subseteq C^G \) consists precisely of all functions from \( G \) to \( C \) which are constant on the cosets of \( H \) in \( G \). Let \( \tau : V \to C \) be a function, \( h \in H \) and \( g \in G \). Then \( f(h) = 0 \) and
\[
\begin{align*}
f^*(\tau)(hg) &= \tau(f(hg)) = \tau(f(h)g + f(g)) = \tau(f(g)) = f^*(\tau)g.
\end{align*}
\]
This shows that \( f^*(\tau) \) is constant on the cosets of \( H \) in \( G \) and hence \( f^* = f^*(C^V) \subseteq \text{Ind}_H^G C_H \).

For \( v \in V \) define \( \tau_v : V \to C : u \mapsto \delta_{vu} \), (so \( \{\tau_v \mid v \in V \} \) is a group basis of \( C^V \)). Note that for \( g \in G, h \in H, v \in V \) we have
\[
(f^*(\tau_v))(hg) = f^*(\tau_v)(g) = \tau_v(f(g)) = \begin{cases} 1 & \text{for } f(g) = v \\ 0 & \text{for } f(g) \neq v. \end{cases}
\]
So \( f^*(\tau_v) \) takes the coset \( Hg \) to 1 for \( f(g) = v \). Let \( a, b \in G \) and suppose that \( f(a) = f(b) \). Then \( f(ab^{-1}) = f(a)b^{-1} + f(b^{-1}) = f(b)b^{-1} + f(b^{-1}) = f(bb^{-1}) = f(1) = 0 \) showing that \( a \) and \( b \) are contained in the same coset of \( H \) in \( G \). We conclude that \( f^*(\tau_v) = 0 \) on all cosets of \( H \) in \( G \) different from \( Hg \) with \( f(g) = v \). This shows that \( f^* : C^V \to \text{Ind}_H^G C_H \) is surjective, by choosing for each \( v \in V \) an element \( g \in G \) such that \( f(g) = v \) using surjectivity of \( f \).
\( \square \)

### 3 Pattern subgroups

Throughout \( q \) is a power of some prime \( p \) and \( \mathbb{F}_q \) is the field with \( q \) elements. The \( n \)-dimensional \( \mathbb{F}_q \)-space \( \mathbb{F}_q^n \) is the natural module for the general linear group \( GL_n(q) \) (acting from right). The **root system** \( \Phi \) of \( GL_n(q) \) consists of all positions \( \{(i, j) \mid 1 \leq i, j \leq n, i \neq j\} \) of \( n \times n \)-matrices, and \( \Phi = \Phi^+ \cup \Phi^- \) with \( \Phi^+ = \{(i, j) \mid 1 \leq i < j \leq n\}, \Phi^- = \{(i, j) \mid 1 \leq j < i \leq n\} \) (positive and negative roots). The \( \mathbb{F}_q \)-algebra of \( n \times n \)-matrices over \( \mathbb{F}_q \) is denoted by \( M_n(q) \). For \( A \in M_n(q) \), \( 1 \leq i, j \leq n \), let \( A_{ij} \in \mathbb{F}_q \) be the entry at position \( (i, j) \) in \( A \). Thus \( A = \sum_{1 \leq i, j \leq n} A_{ij}e_{ij} \), where \( e_{ij} \) is the \( (i, j) \)-th matrix unit. In particular \( E = \sum_{1 \leq i \leq n} e_{ii} \) is the identity of \( GL_n(q) \).

For \( (i, j) \in \Phi, \alpha \in \mathbb{F}_q \), let \( x_{ij}(\alpha) = E + \alpha e_{ij} \). Then the **root subgroup** \( X_{ij} = \{x_{ij}(\alpha) \mid \alpha \in \mathbb{F}_q \} \) is isomorphic to the additive group \( (\mathbb{F}_q, +) \). Moreover, for \( (i, j), (k, l) \in \Phi \), the special type \( A \) case of Chevalley’s commutator formula (see e.g. \([3] \)) can easily be checked by direct calculation:

```
| i j | k l |
```

---

\[ \tag{3} \]

5
\[ [x_{ij}(\alpha), x_{kl}(\beta)] = x_{ij}(\alpha)^{-1} x_{kl}(\beta)^{-1} x_{ij}(\alpha) x_{kl}(\beta) = \begin{cases} 1 & \text{for } i \neq l, j \neq k \\ x_i(\alpha \beta) & \text{for } i \neq l, j = k \\ x_i(-\alpha \beta) & \text{for } i = l, j \neq k \end{cases} \]  

(The case \([x_{ij}(\alpha), x_{ji}(\beta)]\) is more complicated and is not needed in this paper). Recall that for \(A \in M_n(\mathbb{F}_q)\), \((i,j) \in \Phi, \alpha \in \mathbb{F}_q, x_{ij}(\alpha)A\) (respectively \(Ax_{ij}(\alpha)\)) is obtained from \(A\) by multiplying row \(j\) (column \(i\)) of \(A\) by \(\alpha\) and adding it to row \(i\) (column \(j\)).

A subset \(J \subseteq \Phi\) is \textbf{closed}, if \((i,j), (j,k) \in J\) and \((i,k) \in \Phi\) implies \((i,k) \in J\). Note that if \(J \subseteq \Phi^\pm(\Phi^-)\), the condition \((i,k) \in \Phi\) is automatically satisfied and can be omitted.

For \(A \in M_n(\mathbb{F}_q)\), the \textbf{support} \(\text{supp}(A)\) of \(A\) is the set of positions \((i,j), 1 \leq i, j \leq n\) such that \(A_{ij} \neq 0\). Thus \(A = \sum_{(i,j) \in \text{supp}(A)} A_{ij} e_{ij}\). If \(J \subseteq \Phi^-(\Phi^+)^-\) is closed, then \(U_J = \{ E + A | A \in M_n(\mathbb{F}_q), \text{supp}(A) \subseteq J \}\) is a subgroup, called \textbf{pattern subgroup}, of the \(p\)-Sylow subgroup \(U^- = U_n(q)(U^+)^-\) of \(GL_n(q)\) of lower (upper) unitriangular matrices. This follows easily from \((3.1)\). Moreover \(U_J\) is generated by the root subgroups \(X_{ij}\), \((i,j) \in J\). Indeed it is well known and follows easily from \((3.1)\) that fixing an arbitrary linear ordering on \(J\) and taking every product in this ordering, every element of \(U_J\) can be uniquely written as product:

\[
\prod_{(i,j) \in J} x_{ij}(\alpha_{ij}) \quad \text{with } \alpha_{ij} \in \mathbb{F}_q.
\]  

Obviously \(U = U^- = U_{\Phi^-}\) and \(U^+ = U_{\Phi^+}\). Inspecting \((3.1)\) one obtains immediately:

\[3.3 \text{ Lemma.} \quad J \subseteq I \subseteq \Phi^- (\Phi^+^-) \text{ closed. Then } U_J \text{ is a normal subgroup of } U_I \text{ if and only if the following hold: If } (i,j), (j,k) \in I \text{ and } (i,j) \text{ or } (j,k) \text{ is in } J, \text{ then } (i,k) \in J. \]

\[3.4 \text{ Lemma.} \quad \text{Let } L, J \subseteq \Phi^- \text{ be closed, } L \cap J = \emptyset. \text{ Suppose that for } (i,j), (j,k) \in \Phi^- \text{ such that one of these roots is contained in } L \text{ the other in } J \text{ we have always } (i,k) \in J. \text{ Then } K = L \cup J \subseteq \Phi^- \text{ is closed, } U_L, U_J, \subseteq U_K \text{ and } U_K \text{ is the semi-direct product of } U_J \text{ by } U_L. \text{ Let } M = \text{Ind}_{U_J}^{U_K} \mathbb{C} \text{ and } e \text{ be a } \mathbb{C}\text{-vector space generator of the trivial } U_L\text{-module } \mathbb{C}. \text{ Then } \{e u | u \in U_J\} \text{ is a } \mathbb{C}\text{-basis of } M \text{ on which } U_J \text{ acts by multiplication and } U_L \text{ by conjugation.}

\[\text{Proof.} \quad \text{One checks immediately that } K \text{ is closed in } \Phi^- \text{ and that } U_K = U_J \rtimes U_L. \text{ Obviously } (e u_1) u_2 = e u_1 u_2 \text{ for } u_1, u_2 \in U_J \text{ and } e u_1 = e u_1 l^{-1} = e u_1 l \text{ for } l \in U_L \text{ and } u \in U_J. \]

For any subset \(J \subseteq \Phi^-\) we set

\[V_J = \{ A \in M_n(q) | \text{supp} A \subseteq J \}. \]  

(3.5)

Note that if \(J\) is closed then \(V_J = \{ u - E | u \in U_J \}\) is the Lie algebra \(\text{Lie}(U_J)\) associated with the pattern subgroup \(U_J\) and is a (nilpotent) \(\mathbb{F}_q\)-subalgebra of \(M_n(q)\). Moreover, the multiplication rule \(e_{ij} e_{kl} = \delta_{jk} e_{il}\) for matrix units implies with \((3.3)\) immediately:

\[3.6 \text{ Lemma.} \quad \text{Let } J \subseteq K \subseteq \Phi^- \text{ be closed. Then } U_J \subseteq U_K \text{ if and only if } V_J \text{ is an ideal of } V_K. \]

Note that in the situation of \((3.6)\) with \(U_J \subseteq U_K\), the natural projection map \(\tilde{\rho} : V_K \to V_{K \setminus J} = \{ A \in M_n(q) | \text{supp} A \subseteq K \setminus J \}\) is an \(\mathbb{F}_q\)-algebra homomorphism with kernel \(V_J\).

We now apply the results of section \((2)\) to the following situation:
3.7 Hypothesis. Let \( J, L \subseteq \Phi^- \) be closed, \( J \cap L = \emptyset, K = J \cup L \) and \( U_J \subseteq U_K \). Thus \( K \subseteq \Phi^- \) is closed and \( U_K = U_J \times U_L, V_J \subseteq V_K \) and the natural projection 
\[
\tilde{\rho} : V_K \to V_L
\]
is an \( \mathbb{F}_q \)-algebra homomorphism with kernel \( V_J \) and \( \tilde{\rho}|_{V_L} = id_{V_L} \).

We define \( \rho : U_K \to U_L : u \mapsto \tilde{\rho}(u - E) + E \) for \( u \in U_K \). Obviously \( \rho \) is a group epimorphism with kernel \( U_J \). Note that \( \rho|_{U_L} = id_{U_L} \), so \( \rho \) is the natural projection of \( U_K = U_J \times U_L \) onto \( U_L \).

3.8 Lemma. Suppose 3.7 Then \( U_K \) acts on \( V_J \) as group of \( \mathbb{F}_q \)-vector space automorphisms, where the action \( \circ : V_J \times U_K \to V_J \) of \( U_K \) on \( V_J \) being defined by \( A \circ g = \rho(g^{-1})Ag \) for \( A \in V_J, g \in U_K \).

Proof. Let \( g \in U_K, A \in V_J \). Then 
\[
\rho(g^{-1})Ag = (\tilde{\rho}(g^{-1} - E) + E)A((g - E) + E)
\]
\[
= A + A(g - E) + \tilde{\rho}(g^{-1} - E)A + \tilde{\rho}(g^{-1} - E)A(g - E)
\]
is contained in \( V_J \), since \( g - E, \tilde{\rho}(g^{-1} - E) \in V_K \) and \( V_J \) is an ideal of \( V_K \). Since \( \rho \) is a group homomorphism the claim follows at once. \( \square \)

3.9 Lemma. Suppose 3.7 Then the map 
\[
f : U_K \to V_J : g \mapsto \rho(g^{-1})g - E
\]
is an 1-cocycle with kernel \( U_J \). Moreover \( f|_{U_J} \) is bijective and hence \( f \) in particular surjective.

Proof. Let \( g \in U_K \). Since \( \rho(g^{-1}) \) and \( g \) are contained in \( U_K \), we have \( f(g) = \rho(g^{-1})g - E \in V_K \) and hence 
\[
\tilde{\rho}(f(g)) + E = \rho(f(g) + E) = \rho(\rho(g^{-1})g) = \rho(\rho(g^{-1}))\rho(g)
\]
\[
= \rho(g^{-1})\rho(g) = E,
\]
since \( \rho(g^{-1}) \in U_L \) and \( \rho|_{U_K} = id_{U_K} \). Thus \( \tilde{\rho}(f(g)) = 0 \) and hence \( f(g) \in \ker \rho = V_J \).

Let \( x, g \in U_K \). Then 
\[
f(x \circ g + f(g) = \rho(g^{-1})(\rho(x^{-1})x - E)g + \rho(g^{-1})g - E
\]
\[
= \rho(g^{-1}x^{-1})xg - \rho(g^{-1})g + \rho(g^{-1})g - E
\]
\[
= \rho(g^{-1}x^{-1})xg - E = f(xg),
\]
thus \( f \) is a 1-cocycle.

Suppose \( 0 = f(g) = \rho(g^{-1})g - E \), then \( \rho(g^{-1})g = E \) and hence \( \rho(g) = g \), since \( \rho : U_K \to U_L \) is a group homomorphism. But \( \rho(g) = \tilde{\rho}(g - E) + E \) and hence \( 0 = f(g) \) is equivalent to \( \tilde{\rho}(g - E) = g - E \), and hence \( g - E \in V_L \) that is \( g \in U_L \). So \( \ker f = U_L \).

Finally for \( g \in U_J, \rho(g) = E \) and hence \( f(g) = g - E \). In particular \( f|_{U_J} : U_J \to V_J : g \mapsto g - E \) is a bijection and thus \( f \) is in particular surjective. \( \square \)

Using 2.6 and 2.9 we have shown:

3.10 Theorem. Assume 3.7 and let \( V = V_J \). Let \( f : U_K \to V_J \) be the 1-cocycle defined in 3.9 Then the group algebra \( \mathbb{C}V \cong \mathbb{C}^{\hat{V}} \) of the finite abelian group \( (V, +) \) is a right \( \mathbb{C}U_K \)-module satisfying the following:

1) \( U_K \) acts on the \( \mathbb{C} \)-basis \( \hat{V} = \text{Hom}((V, +), \mathbb{C}^*) \) monomially, the action of \( g \in U_K \) on \( \chi \in \hat{V} \) given by 
\[
\chi g = \chi(f(g^{-1}))\chi.g,
\]
where \( \chi.g \in \hat{V} \) is defined by \( \chi.g(A) = \chi(A \circ g^{-1}) \) for \( A \in V \).
2) The $\mathbb{C}U_K$-module $\mathbb{C}V \cong \mathbb{C}V$ is isomorphic to $\text{Ind}_{U_L}^{U_K} \mathbb{C}$, the trivial $\mathbb{C}U_L$-module $\mathbb{C} = \mathbb{C}U_L$ induced to $U_K$.
3) The restriction $\text{Res}_{U_J}^{U_K}(\mathbb{C}V)$ is isomorphic to the right regular $\mathbb{C}U_J$-module.
4) For $g \in U_L$ we have $f(g^{-1}) = 0$, hence $\chi(f(g^{-1})) = 1$. So $U_L$ acts on $\hat{V}$ by permutations. 

3.11 Remark. For any closed subset $J \subseteq \Phi^-$, the supercharacter theory ([1, 5, 14]) for the pattern subgroup $U_J$ of $U = U_n(q)$ is a special case of 3.10. To see this we choose in 3.10 $J = K$ and $L = \emptyset$. Thus, for $g \in U_J$, we have $\rho(g) = E$ and hence $f(g) = \rho(g^{-1})g - E = g - E$. By symmetry this is a left 1-cocycle too and by 3.10 $\mathbb{C}V = \mathbb{C}V$ (for the additive group $V$) becomes a right- and analogously a left $\mathbb{C}U_J$-module, in fact a $\mathbb{C}U_J$-bimodule, which is isomorphic to $\mathbb{C}U_J$, the regular $\mathbb{C}U_J$-bimodule. The decomposition of the $\mathbb{C}$-basis $V$ of $\mathbb{C}V$ under the right $U_J$-action yields the decomposition of the regular representation into transitive monomial subrepresentations affording precisely the supercharacters of $U_J$.

Recall from basic linear algebra, that multiplying $A \in M_n(q)$ from the left (right) by $x_{ij}(\alpha) = E + \alpha x_{ij}$ ($\alpha \in \mathbb{F}_q, 1 \leq i, j \leq n, i \neq j$) produces a matrix obtained from $A$ by adding $\alpha$ times row $j$ (column $i$) of $A$ to row $i$ (column $j$). For any set $J \subseteq \{(i, j) | 1 \leq i, j \leq n\}$ of positions and $V = V_J = \{A \in M_n(q) | \supp(A) \subseteq J\}$ we denote the projection which takes $A \in M_n(q)$ to $\sum_{(i,j) \in J} A_{ij} e_{ij} \in V_J$ by $\pi_J$. Thus $\pi_J$ is the natural $\mathbb{F}_q$-vector space projection of $M_n(q)$ onto $V_J$. Adding $\alpha$ times ($\alpha \in \mathbb{F}_q$) row (column) $i$ to row (column) $j$ in a matrix $A \in M_n(q)$ ($1 \leq i, j \leq n, i \neq j$) and following up with $\pi_J$, (so obtaining a matrix in $V_J$) is called truncated row (column) operation, from row (column) $i$ to row (column) $j$ along $V_J$.

We return to the setting of 3.7. We want to exhibit explicit formulas for the action of $U_K$ on $\hat{V}$, where again we set $V = V_J = \{A \in M_n(q) | \supp(A) \subseteq J\}$.

First we describe $\hat{V}$. Recall that $V$ is an $\mathbb{F}_q$-vector space with $\mathbb{F}_q$-basis $\{e_{ij} | (i, j) \in J\}$. Let $\{e_{ij} | (i, j) \in J\}$ be the dual $\mathbb{F}_q$-basis of the dual space $V^* = \text{Hom}_{\mathbb{F}_q}(V, \mathbb{F}_q)$. Thus $e_{ij}$ maps $A \in V$ to its $(i, j)$-th coordinate $A_{ij} \in \mathbb{F}_q$ for $(i, j) \in J$, and every linear function $\chi^* \in V^*$ on $V$ is a unique linear combination $\sum_{(i,j) \in J} B_{ij} e_{ij} = \chi^*$ with $B_{ij} \in \mathbb{F}_q$. Let $B = \sum_{(i,j) \in J} B_{ij} e_{ij} \in M_n(q)$ and denote $\chi^*_B | B \in V = V^*$.

We choose once for all a nontrivial linear character $\theta : (\mathbb{F}_q, +) \rightarrow \mathbb{C}^*$ of the additive group of the field $\mathbb{F}_q$ into the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. The following result is easily checked by direct calculation:

3.13 Lemma. For $\chi^* = \chi^*_B \in V^*, B \in V$, let $\chi_B : V \rightarrow \mathbb{C}^*$ be the composite map $\theta \circ \chi^*_B$. Then $\chi_B \in \hat{V}$ and $\chi_B \neq \chi_C$ for $B, C \in V, B \neq C$. In particular, $\hat{V} = \{\chi_B \mid B \in V\}$. 

For $B \in V, g \in U_K$, 2.24 implies that $\chi^*_B \cdot g = \chi_C$ for $C \in V$ satisfying $(\chi^*_B \cdot g)(e_{st}) = C_{st}$, where $(s, t) \in J$. So in order to determine $C$, we have to evaluate $\chi^*_B \cdot g$ at matrix unites $e_{st}$ with $(s, t) \in J$. For convenience we denote $C$ by $B \cdot g$. Then $\chi^*_B \cdot g = \chi^*_{B \cdot g}$.

3.14 Lemma. Suppose 3.7 and let $B \in V = V_J, g \in U_K$. Then $B \cdot g = \pi_J((\rho(g))^l B g^{-t})$, where on the right hand side we have ordinary matrix multiplication and $A^t$ denotes the transpose of $A \in M_n(q)$ and $g^{-t} = (g^{-1})^l = (g^l)^{-1}$.

Proof. Since $U_K$ is generated by root subgroups $X_{kl}$ with $(k, l) \in K = J \cup L$, we may assume $g = x_{kl}(\alpha)$ with $\alpha \in \mathbb{F}_q$. Let $(i, j), (s, t) \in J$, then

$$\beta_{ij} := (\epsilon_{ij} x_{kl}(\alpha))(e_{st}) = \epsilon_{ij} (e_{st} \circ x_{kl}(-\alpha)) = \epsilon_{ij} (\rho(x_{kl}(\alpha)) e_{st} x_{kl}(-\alpha)) = \epsilon_{ij} (\rho(\theta(\alpha)) e_{st} x_{kl}(-\alpha))$$

(3.15)
1. Case: Suppose \((k,l) \in L\). Then by \ref{3.7} definition of \(\rho\),
\[
\beta_{ij} = \epsilon_{ij}(x_{kl}(\alpha)e_{st}x_{kl}(-\alpha)) = \epsilon_{ij}((E + \alpha e_{kl})e_{st}(E - \alpha e_{kl})) \\
= \epsilon_{ij}(e_{st} - \alpha e_{st}e_{kl} + \alpha e_{kl}e_{st} - \alpha^2 e_{kl}e_{st}e_{kl})
\]
Since \((k,l),(s,t) \in \Phi^-\), we have \(l < k, t < s\) and hence \(t = k\) implies \(l < s\). Thus \(e_{kl}e_{st}e_{kl} = 0\) and
\[
\beta_{ij} = \epsilon_{ij}(e_{st} - \alpha e_{st}e_{kl} + \alpha e_{kl}e_{st}) \quad \text{if } t = k \\
= \left\{ \begin{array}{ll}
\epsilon_{ij}(e_{st} - \alpha e_{st}) = \delta_{is}\delta_{jt} - \alpha \delta_{is}\delta_{jt} & \text{if } t = k \\
\epsilon_{ij}(e_{st} + \alpha e_{kl}) = \delta_{is}\delta_{jt} + \alpha \delta_{is}\delta_{jt} & \text{if } s = l \\
\epsilon_{ij}(e_{st}) = \delta_{is}\delta_{jt} & \text{otherwise,}
\end{array} \right. \tag{3.16}
\]
Now \(\chi^*_B = \sum_{(i,j) \in J} B_{ij}\epsilon_{ij}\) and hence by \ref{3.15}:
\[
C_{st} = \chi_C^*(e_{st}) = (\chi^*_B \cdot x_{kl}(\alpha))(e_{st}) = \sum_{(i,j) \in J} B_{ij}(\epsilon_{ij}(e_{st} \circ x_{kl}(-\alpha))) \\
= \sum_{(i,j) \in J} B_{ij}\beta_{ij} \tag{3.16}
\]
Thus we obtain matrix \(C = B \cdot x_{kl}(\alpha)\) from \(B\) by adding \(-\alpha\) times column \(l\) to column \(k\), adding \(\alpha\) times row \(k\) to row \(l\) in \(B\) and projecting the resulting matrix into \(V = V_J\). By basic linear algebra this is indeed \(\pi_j(x_{ik}(\alpha)Bx_{ik}(-\alpha))\) as desired.

2. Case: Now suppose \((k,l) \in J\). Then
\[
\beta_{ij} = (\epsilon_{ij}(x_{kl}(\alpha))(e_{st}) = \epsilon_{ij}(e_{st}(E - \alpha e_{kl})) \\
= \left\{ \begin{array}{ll}
\epsilon_{ij}(e_{st} - \alpha e_{st}) = \delta_{is}\delta_{jt} - \alpha \delta_{is}\delta_{jt} & \text{if } t = k \\
\epsilon_{ij}(e_{st}) = \delta_{is}\delta_{jt} & \text{otherwise,}
\end{array} \right. \quad \text{and we conclude that}
\]
\(C = B \cdot x_{kl}(\alpha)\) is obtained from \(B\) by adding \(-\alpha\) times column \(l\) to column \(k\) and projecting the resulting matrix into \(V\) by \(\pi_j\). But this is again \(\pi_j(Bx_{ik}(-\alpha)) = \pi_j((\rho(x_{ik}(\alpha)))^tBx_{ik}(-\alpha)^t)\), since \(\rho(x_{ik}(\alpha)) = E\) for \((k,l) \in J\).

\begin{proof} \[ \square \]
\end{proof}

3.17 Definition. Let \(1 \leq j < i \leq n\), then \(-\epsilon_{ij}(A) = -A_{ij}\) for \(A \in V_J\), and hence \(\theta \circ (-\epsilon_{ij})(A) = \bar{\theta}(A_{ij})\), where \(\bar{z}\) denotes the complex conjugate of \(z \in \mathbb{C}\). From this it follows immediately, that \(\chi_{-\Lambda} = \overline{\chi_{\Lambda}}\), the complex conjugate character to \(\chi_{\Lambda}\) for \(A \in V_J\). So for \(A \in V_J\), we define \(e_A = q^{-|J|/2} \sum_{B \in V_J} \chi_B(A)B\). Thus \(q^{-|J|/2} \chi_{-\Lambda} \in \mathbb{C}^V\) is identified under \(\ref{2.1}\) with the primitive idempotent \(e_A \in \mathbb{C}(V_J^+, +)\) affording the linear character \(\overline{\chi_{\Lambda}} = \chi_{-\Lambda}\). In order to distinguish the idempotents \(e_A \in \mathbb{C}(V_J^+, +)\) from idempotents in the group algebra \(\mathbb{C}U_J\), we call \(e_A, A \in V_J\) “idempotent” indicating that it belongs to the Lie algebra of \(U_J\) rather than \(U_J\). Thus \(E_J = \{e_A \mid A \in V_J\}\) is called \textbf{idempotent basis} of \(\mathbb{C}(V_J^+, +)\).

Combining \ref{3.10} and \ref{3.14} we have shown:

3.18 Theorem. Suppose \ref{3.7} and let \(f : U_K \to V = V_J\) be the 1-cocycle defined in \ref{3.9}. Let \(A \in V = V_J\) and \(g = x_{ij}(\alpha) \in U_K, (k,l) \in K = J \cup L, \alpha \in \mathbb{F}_q\). Then \(\chi_A(x_{kl}(\alpha)) = \chi_B\), where \(B \in V\) is obtained from \(A\) by the truncated column operation along \(V\) from column \(l\) to column \(k\) using the factor \(-\alpha\), if \((k,l) \in J\). If \((k,l) \in \bar{J}\), then \(B\) is obtained from \(A\) by combined truncated row and column operation along \(V\) from column \(l\) to column \(k\), row \(k\) to row \(l\) with
factors $-\alpha$ and $\alpha$ respectively. Moreover

$$\chi_A g = \begin{cases} \theta(-\alpha A_{ij}) \chi_B & \text{for } (i, j) \in J \\ \chi_B & \text{for } (i, j) \in L. \end{cases}$$ (3.19)

As a consequence, we obtain a monomial action of $U_K$ on the idempotent basis $\mathcal{E}_J$ of $\mathbb{C}(V_J, +)$ (defined in 3.17) as follows:

$$e_A g = \begin{cases} \theta(\alpha A_{ij}) e_B & \text{for } (i, j) \in J \\ e_B & \text{for } (i, j) \in L. \end{cases}$$ (3.20)

**Proof.** Everything is already shown except (3.19) and (3.20). By 3.10

$$\chi_A g = \chi_A(f(g^{-1})) \chi_B = \theta(\chi^*_A(f(g^{-1}))) \chi_B.$$

For $(i, j) \in L$, $f(g^{-1}) = 0$ and the claim follows (comp. 3.10). So let $(i, j) \in J$. Then using 3.9 we obtain $\chi^*_A(f(g^{-1})) = \chi^*_A(g^{-1} - E) = \chi^*_A(-\alpha e_{ij}) = -\alpha A_{ij}$ and (3.19) follows. Then by the identification given in 3.17 (3.20) follows. \qed

**3.21 Remark.** As mentioned already in 3.11 the special case $L = \emptyset, J = K$ in 3.7 yields a supercharacter theory of the pattern subgroup $U_J$ of $U$. In this case the action of $U_J$ on $\hat{V}_J$ can entirely be described by (ordered) sequences of truncated column operations.

**3.22 Example.** We illustrate the idempotent $e_A \in \mathbb{C}V_J$ by a triangle, omitting superfluous zeros in the upper half of matrix $A \in V$. Let $n = 6$ and $V = V_J$ where $J$ is obtained by taking out column 2 and column 4 from $\Phi^-$. Let

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ A_{21} & 0 & 0 & 0 & 0 & 0 \\ A_{31} & 0 & 0 & 0 & 0 & 0 \\ A_{41} & A_{43} & 0 & 0 & 0 & 0 \\ A_{51} & A_{53} & 0 & 0 & 0 & 0 \\ A_{61} & A_{63} & 0 & 0 & 0 & 0 \end{pmatrix} \in V \quad \text{with } A_{ij} \in \mathbb{F}_q, \quad \text{for } (i, j) \in J.$$

Then we denote

$$e_A = \begin{array}{c} 0 \\ A_{21} \frac{\check{} \check{}}{\check{}} \check{0} \\ A_{31} \frac{\check{} \check{} \check{}}{\check{} \check{} \check{}} \check{0} \\ A_{41} \frac{\check{\check{\check{}}}}{\check{\check{\check{}}}} A_{43} \check{0} \\ A_{51} \frac{\check{\check{\check{}}}}{\check{\check{\check{}}}} A_{53} \frac{\check{\check{\check{}}}}{\check{\check{\check{}}}} \check{0} \\ A_{61} \frac{\check{\check{\check{}}}}{\check{\check{\check{}}}} A_{63} \frac{\check{\check{\check{}}}}{\check{\check{\check{}}}} A_{65} \check{0} \end{array}$$

and by (3.20) we have

$$e_A x_{53}(\alpha) = \theta(\alpha A_{53})$$

where $\beta = A_{65} - \alpha A_{63}$. 

10
3.23 Notation. In the situation of \([4,7]\) we denote the \(U_K\)-orbit of \(E_J\) containing \(e_A \in E_J, A \in V_J\) by \(O^J_A\). In the special case \(L = \emptyset, J = K = \Phi^+\), we drop the sub- and super index \(J\), that is \(V = V_{\Phi^-}, \mathcal{E} = E_{\Phi^-}\) and \(O^\Phi_A = O^\Phi_{\lambda^\Phi}\) for \(A \in V\).

4 \(\lambda\)-flags

For \(n \in \mathbb{N}\) we denote the symmetric group on \(n\) letters by \(\mathfrak{S}_n\). A composition \(\lambda\) of \(n\), denoted by \(\lambda \vdash n\), is a finite sequence \(\lambda = (\lambda_1, \ldots, \lambda_k)\) of non-negative integers \(\lambda_i \in \mathbb{N}\), \(1 \leq i \leq k\), whose sum is \(n\). For the definition of the Young diagram \([\lambda]\) associated with \(\lambda\), \(\lambda\)-tableaux and related combinatorial objects in the representation theory of \(\mathfrak{S}_n\) we refer to the standard literature, e.g. \([3]\). The set of \(\lambda\)-tableaux (resp. row standard \(\lambda\)-tableaux) is denoted by \(\text{Tabl}(\lambda)\), (resp. \(\text{RStd}(\lambda)\)). The initial \(\lambda\)-tableau \(t^\lambda\) is the \(\lambda\)-tableau, where the numbers \(1, \ldots, n\) are inserted in order along the rows down row by row. For \(s \in \text{Tabl}(\lambda)\), we denote the permutation \(w \in \mathfrak{S}_n\) with \(t^\lambda w = s\) by \(d(s)\). For \(1 \leq i \leq n\), \(s \in \text{Tabl}(\lambda)\), \(row_s(i) = m\) if the \(m\)-th row counted from the top in \(s\) contains \(i\).

For \(\lambda \vdash n\), the row stabilizer of \(t^\lambda\) in \(\mathfrak{S}_n\) is the standard Young subgroup \(\mathfrak{S}_\lambda = \mathfrak{S}_{\{1, \ldots, \lambda_1\}} \times \mathfrak{S}_{\{\lambda_1+1, \ldots, \lambda_1+\lambda_2\}} \times \cdots \cong \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_k} \subseteq \mathfrak{S}_n\), \(\lambda = (\lambda_1, \ldots, \lambda_k)\). For any \(s \in \text{Tabl}(\lambda)\), the row stabilizer of \(s\) is then \(d(s)^{-1}\mathfrak{S}_\lambda d(s)\). The set

\[ D_\lambda = \{d(s) \mid s \in \text{RStd}(\lambda)\} \quad (4.1) \]

is a set of right coset representatives of \(\mathfrak{S}_\lambda\) in \(\mathfrak{S}_n\), called distinguished coset representatives, (see e.g. \([2]\)).

We identify \(\mathfrak{S}_n\) with the Weyl group of \(GL_n(q)\), i.e. with the set of permutation matrices in \(GL_n(q)\). Thus \(w \in \mathfrak{S}_n\) acts in the natural basis \(e_1, \ldots, e_n\) of \(\mathbb{F}_q^n\) by \(e_iw = e_{iw}\). Moreover, for \(1 \leq i, j \leq n, \alpha \in \mathbb{F}_q\), one checks directly that: \(we_{ij} = e_{iw-i,j}\) and \(e_{ij}w = e_{ijw}\). Hence

\[ x_{ij}(\alpha)w = w^{-1}x_{ij}(\alpha)w = w^{-1}(E + \alpha e_{ij})w = E + \alpha w^{-1}e_{ij}w = x_{iw,jw}(\alpha). \quad (4.2) \]

The subgroup \(T = \{(\sum_{i=1}^n \alpha_i e_{ii} \mid \alpha_1, \ldots, \alpha_n \in \mathbb{F}_q \setminus \{0\}\} \) of diagonal matrices in \(GL_n(q)\) is called torus and denoted by \(T = T_n(q)\).

For \(\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n\) we define a \(\lambda\)-flag in \(V = \mathbb{F}_q^n\) to be a sequence

\[ V = V_0 > V_1 > \cdots > V_{k-1} > V_k = (0) \]

of subspaces \(V_i\) of \(V\) \((i = 0, \ldots, k)\) such that \(\lambda_i = \dim V_{i-1} - \dim V_i\) holds. We denote the set of \(\lambda\)-flags in \(V\) by \(F(\lambda)\). Obviously \(GL_n(q)\) acts on \(F(\lambda)\) by left- and right multiplication transitively on both sides.

We set \(\Lambda_i = \lambda_i + 1 + 2 + \cdots + \lambda_i\) for \(i = 0, 1, \ldots, k\). For the natural basis \(e_1, \ldots, e_n\) of \(V = \mathbb{F}_q^n\) set \(V_i = \langle e_{\Lambda_{i-1}}, e_{\Lambda_i+1}, \cdots, e_n \rangle\). Then the \(\lambda\)-flag \(V = V_0 > V_1 > \cdots > V_{k-1} > V_k = (0)\) is called standard \(\lambda\)-flag and its stabilizer in \(GL_n(q)\) acting from the right on \(F(\lambda)\) is the standard parabolic subgroup \(P_\lambda\). By general theory, the linear permutation module \(\mathbb{C}F(\lambda)\) spanned by \(F(\lambda)\) under the right action of \(GL_n(q)\) is isomorphic to the trivial \(\mathbb{C}P_\lambda\)-module induced to \(GL_n(q)\), denoted by \(\text{Ind}_{P_\lambda}^{GL_n(q)}\mathbb{C}\).

4.3 Definition. Let \(\lambda \vdash n\). Then the subsets

\[ J_\lambda = \{(i, j) \in \Phi \mid \text{row}_{\lambda^\Phi}(i) \leq \text{row}_{\lambda^\Phi}(j)\} \]

\[ J_\lambda^c = \{(i, j) \in \Phi \mid \text{row}_{\lambda^\Phi}(i) = \text{row}_{\lambda^\Phi}(j)\} \]

\[ J_\lambda^c = \{(i, j) \in \Phi \mid \text{row}_{\lambda^\Phi}(i) < \text{row}_{\lambda^\Phi}(j)\} \]

are closed subsets of \(\Phi\) and \(P_\lambda = \langle T, X_{ij} \mid (i, j) \in J_\lambda \rangle\) is the standard parabolic subgroup. It contains the upper Borel subgroup of invertible upper triangular matrices. Moreover \(L_\lambda = \)
\((T, X_{ij} \mid (i, j) ∈ J_λ^\infty)\) is a Levi subgroup of \(P_λ\) (standard Levi subgroup) isomorphic to \(GL_{λ_1}(q) \times \cdots GL_{λ_κ}(q)\), \(λ = (λ_1, \ldots , λ_κ) \vdash n\), and \(U_λ = U_{J_λ^\infty} \subseteq U^+\) is the unipotent radical of \(P_λ\). In particular, \(U_λ \subseteq P_λ\), \(U_λ \cap L_λ = (1)\) and \(P_λ = U_λL_λ = U_λ ∋ L_λ\).

For later use we set \(U_λ^− = U_{J_λ^\infty}\), where \(J_λ^\infty = \{(i,j) ∈ Φ \mid row_{λ}(i) > row_{λ}(j)\}\). Then \(P_λ^− = L_λU_λ^−\) is the standard parabolic subgroup containing \(B^−\), the Borel subgroup of all lower triangular invertible matrices.

Let \(F : V = V_0 > V_1 > \cdots > V_{k−1} > V_k = (0)\) be a \(λ\)-flag. A basis \(\{v_1, \ldots , v_n\}\) of \(V\) is \(F\)-adapted, if \(v_{k+1}, \ldots , v_n\) spans \(V_i\) \((i = 0, \ldots , k−1)\). Writing each \(v_i\) (uniquely) as linear combination \(v_i = ∑_{j=1}^n β_{ij} e_j\), we obtain an invertible matrix \(B ∈ M_n(q)\) with \(B_{ij} = β_{ij}\), whose row vectors are \(v_i\). We illustrate this as follows:

\[
B = \begin{cases} 
\text{compartment 1} & \text{basis of } V = V_0 \mod V_1 (∼ F_q^{λ_1}) \\
\vdots \\
\text{compartment } k−1 & \text{basis of } V_{k−2} \mod V_{k−1} (∼ F_q^{λ_{k−1}}) \\
\text{compartment } k & \text{basis of } V_{k−1} (∼ F_q^{λ_k}) 
\end{cases}
\]  

\((4.4)\)

Let \(1 ≤ i ≤ n\). If row \(i\) lies in the \(r\)-th compartment of \(B\) for some \(r \in \{1, 2, \ldots , k\}\), then we denote: \(\text{comp}_B(i) = r\).

Using basic linear algebra one sees easily that multiplying \(B\) from the left by an element \(g\) of \(P_λ\) means carrying out a sequence of the following types of elementary row operations on \(B\):

1. Multiplication of row by a nonzero scalar.
2. Adding a multiple of row \(j\) to row \(i\) for \(1 ≤ i < j ≤ n\).
3. Adding a multiple of row \(j\) to row \(i\) for \(1 ≤ j < i ≤ n\), provided \(i\) and \(j\) are rows of the same compartment of \(B\).
4. Interchanging row \(i\) and row \(j\) provided they are in the same compartment of \(B\).

As a consequence, the row vectors \(x_1, \ldots , x_n\) of \(Bg\) are again an \(F\)-adapted basis of \(V\), that is left multiplication by elements of \(P_λ\) can be interperated as base change of \(F\)-adapted basis of \(F_q^n\). Note that the row vectors of any invertible matrix \(B ∈ M_n(q)\) can be considered as \(F\)-adapted bases of some \(λ\)-flag \(F ∈ F(λ)\). Moreover base change of two \(F\)-adapted bases of \(F_q^n\) are always obtained by left multiplication with some \(g ∈ P_λ\). Thus we have

4.6 Lemma. \(P_λ\) acts by left multiplication on the set of \(F\)-adapted bases \(B ∈ GL_n(q)\) of \(V\) transitively inducing thus a bijection:

\[
\{\text{right } P_λ\text{-cosets in } GL_n(q)\} \cong \{\text{left } P_λ\text{-orbits on } GL_n(q)\} \leftrightarrow \mathcal{F}_λ = \{\text{λ-flags in } V = F_q^n\}.
\]

Obviously the sets of the left \(P_λ\)-orbits on \(GL_n(q)\) and the right cosets \(P_λg \ (g ∈ GL_n(q))\) are the same. The next result gives a detailed description of this. Recall that \(U = U^−\) is the lower untriangular group.

4.7 Proposition. Let \(λ \vdash n\). Then \(D_λ\) is a set of \(P_λ\)-coset representatives in \(GL_n(q)\).

Moreover for \(d ∈ D_λ\), \(P_λ d U = \bigcup_{u ∈ (U_λ^−)^d \cap U} P_λ d u\) is a decomposition of \(P_λ d U\) into right \(P_λ\)-cosets in \(GL_n(q)\). Thus \(\{d u \mid d ∈ D_λ, u ∈ (U_λ^−)^d \cap U\}\) is a set of right coset representatives of \(P_λ\) in \(GL_n(q)\).

\(Proof.\) Cf. [10, Theorem 7.5].
Since \( U^{-}_\lambda = (X_{ij} \mid i,j) \in \Phi, \text{row}_{\lambda}(i) > \text{row}_{\lambda}(j) \) and for \( 1 \leq i \leq n, i \) occupies the position in \( s = t^4d \) which is occupied by \( id^{-1} \) in \( t^4 \), we have:

4.8 Lemma. Let \( \lambda \vdash n, d \in D_\lambda, s = t^4d. \) Then \( (U^{-}_\lambda)^d \cap U = U_J, \) where \( J = \{(i,j) \in \Phi^{-} \mid \text{row}_{\lambda}(i) > \text{row}_{\lambda}(j)\}. \) □

Let \( \lambda, d \) and \( s \) as above. Next we shall describe the matrices in the subset \( du J \) of \( GL_n(q) \). Let \( u \in U_J \). We consider the row vectors of \( du \in GL_n(q) \) as an \( F \)-adapted basis of some \( \lambda \)-flag \( F \).

By 4.8, \( u_{ij} \neq 0 \) implies

\[
(i = j \text{ and } u_{ii} = 1) \quad \text{or} \quad (i > j \text{ and } \text{row}_{\lambda}(i) > \text{row}_{\lambda}(j)).
\] (4.9)

Note that the left action of \( u \) on \( s \) is just permuting the rows of \( u \). Thus \( (du)_{kr} = u_{kd,r} \) for all \( (k,r) \in \Phi \). So \( (du)_{kl} \neq 0 \) implies by (4.9)

\[
(k = r \text{ and } (du)_{k, kd} = 1) \quad \text{or} \quad (kd > r \text{ and } \text{row}_{\lambda}(kd) > \text{row}_{\lambda}(r)).
\]

Therefore one can see immediately that the values on the positions \( (k, kd) \) in \( du \) are 1’s and these are the most right hand side nonzero entries in each row \( k \in \{1, \ldots, n\} \). Now assume \( (du)_{kr} \neq 0 \) and \( kd > r \). Recall that for \( 1 \leq i \leq n, i \) occupies the position in \( s \) which is occupied by \( id^{-1} \) in \( t^4 \). Thus \( \text{row}_{\lambda}(kd) > \text{row}_{\lambda}(r) \) if and only if \( \text{row}_{\lambda}(k) > \text{row}_{\lambda}(kd) \) and hence the position \( (k, r) \) lies in a lower compartment than the position \( (rd^{-1}, r) \), on which sits a one. So we have shown:

4.10 Theorem. For \( \lambda \vdash n \) we denote

\[ X_\lambda = \{ du \mid d \in D_\lambda, u \in (U^{-}_\lambda)^d \cap U \}, \]
called \( \lambda \)-normal matrices. Then any \( A \in X_\lambda \) satisfies:

1) There exists \( d \in D_\lambda \) such that \( A_{i,id} = 1 \) is the last nonzero entry in row \( i \) of \( A \). We call \( A_{i,id} = 1 \) \textit{the last one}, for \( 1 \leq i \leq n \). Define tab \( A = t^4d \in \text{RStd}(\lambda) \).
2) \( A_{r,id} \neq 0 \) for \( 1 \leq i \leq n \) implies \( r = i \) (and then \( A_{r,id} = 1 \)) or \( \text{comp}_A(r) > \text{comp}_A(i) \), that is, the compartment containing row \( r \) is lower than the compartment containing row \( i \).

4.11 Remark. By 4.7, \( X_\lambda \) is a set of right coset representatives of \( P_\lambda \) in \( GL_n(q) \). By 4.8 (more precisely, using the row operations in 4.5), each left \( P_\lambda \)-orbit \( P_\lambda A \subseteq GL_n(q) \) contains precisely one row reduced form (normal form) \( A^\lambda = P_\lambda A \cap X_\lambda \). So from now on, each \( \lambda \)-flag can be denoted by an unique element in the set \( X_\lambda \).

4.12 Definition. Let \( \lambda \vdash n \). Define a right action of \( GL_n(q) \) on \( X_\lambda \) by

\[
\bullet : (X_\lambda, GL_n(q)) \to X_\lambda : (A, g) \mapsto A \bullet g = (Ag)^\lambda \quad \text{for } A \in X_\lambda, g \in GL_n(q).
\]

4.13 Lemma. Let \( \langle X_\lambda \rangle \) be the vector space over \( \mathbb{C} \) with basis \( X_\lambda \). Then under the “\( \bullet \)”-action, \( \langle X_\lambda \rangle \) becomes a permutation module isomorphic to \( M(\lambda) := \text{Ind}_{P_\lambda}^{GL_n(q)} \mathbb{C} \), where the isomorphism is given by \( A \in X_\lambda \mapsto \overline{\text{tab}}_A, \) with \( \overline{\text{tab}}_A = \sum_{h \in P_\lambda} h \).

5 The \( s \)-components of \( \text{Res}_U^{GL_n(q)} M(\lambda) \)

Let \( \lambda \vdash n \) and \( M(\lambda) = \text{Ind}_{P_\lambda}^{GL_n(q)} \mathbb{C} \cong \mathbb{C}\langle \overline{\text{tab}}_\lambda \rangle_\lambda \) be as defined in 4.13. By 4.7, \( D_\lambda \) is a set of \( P_\lambda-U \) double coset representatives. We apply Mackey decomposition to obtain a first decomposition:

5.1 Proposition. Let \( \lambda \vdash n \). Then

1) \( \text{Res}_{U}^{GL_n(q)} M(\lambda) = \bigoplus_{d \in D_\lambda} \text{Ind}_{P_\lambda \cap U}^{P_\lambda} \mathbb{C}\overline{\text{tab}}d. \)
2) Let \( d \in D_\lambda \) and \( s = t^id \in RStd(\lambda) \). Then the \( U \)-permutation module \( M_s = \text{Ind}_{P_\lambda}^{U} \cap U \mathcal{C} T_d \) has a \( \mathcal{C} \)-basis \( \{ T_d u \mid u \in (U_\lambda^d)^d \cap U \} \). □

With \( s \in RStd(\lambda) \), we call \( M_s \) introduced in [5.1] s-component of \( M(\lambda) \) (in [5] called s-batch). Under the identification given in [4.13] its basis is given by \( \mathcal{X}_s = \{ du \mid u \in (U_\lambda^d)^d \cap U \} \), where \( d = d(s) \in D_\lambda \). For such an \( u \), the matrix \( du \in M_\lambda(q) \) is obtained from \( u \) by reordering the rows \( (r_1, \ldots, r_n) \) of the matrix \( u \) to \( (r_{1d}, \ldots, r_{nd}) \). Note that \( 1d, \ldots, nd \) are precisely the entries of \( s \in RStd(\lambda) \) from left to right in the rows of \( s \) and going the rows from top down.

In the following we encode \( u \) and \( du \) in one matrix as follows: We relabel in matrix \( du \) the rows top down by \( (1d, \ldots, nd) \). With these new row labels the entry at position \((i, j)\) of \( du \) coincides with \( u_{ij} \in F_q \). Moreover note that the rows of \( du \) then are automatically divided into \( s \)-compartments. So \( u \) may be recovered immediately from \( du \) by reordering rows of \( du \) in its natural order.

5.2 Example. Let \( \lambda = (2, 2, 2) \vdash 6 \) and \( s = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{pmatrix} \). Then under the new labeling we have

\[
\mathcal{X}_s = \begin{bmatrix}
1 & 1 \\
3 & 0 & 0 & 1 \\
2 & * & 1 \\
4 & * & 0 & * & 1 \\
5 & * & * & * & 1 \\
6 & * & * & * & 0 & 1 \\
\end{bmatrix}, \quad \lambda \in F_q
\]

Recall that \( U = U^- = \{ (i, j) \mid (i, j) \in \Phi \} \). Next we want to describe some pattern subgroups in \( U^- \). Let \( s \in RStd(\lambda) \), \( d = d(s) \in D_\lambda \). Recall that \( P_\lambda \) is generated by the torus \( T \) and root subgroups \( X_{ij} \) with \((i, j) \in J_\lambda \) defined in [4.3] Thus \( X_{ij} \subseteq P_\lambda^d \) if and only if \((id^{-1}, jd^{-1}) \in J_\lambda \) that is \( \text{row}_\lambda(id^{-1}) \leq \text{row}_\lambda(jd^{-1}) \) or equivalently, if and only if \( \text{row}_\lambda(i) \leq \text{row}_\lambda(j) \). This proves the following result:

5.3 Lemma. Let \( \lambda \vdash n \) and let \( s \in RStd(\lambda), d = d(s) \in D_\lambda \). Then:

1) \( P_\lambda^d \cap U = \{ X_{ij} \mid 1 \leq j < i \leq n, \text{row}_\lambda(i) \leq \text{row}_\lambda(j) \} \)
2) \( L_\lambda^d \cap U = \{ X_{ij} \mid 1 \leq j < i \leq n, \text{row}_\lambda(i) = \text{row}_\lambda(j) \} = (L_\lambda \cap U)^d \)
3) \( U_\lambda^d \cap U = \{ X_{ij} \mid 1 \leq j < i \leq n, \text{row}_\lambda(i) < \text{row}_\lambda(j) \} \)
4) \( U_\lambda^d \cap U = \{ X_{ij} \mid 1 \leq j < i \leq n, \text{row}_\lambda(i) \geq \text{row}_\lambda(j) \} \)

Proof. 1) through 3) are immediate from [5.3]

4) Let \( X_{ij} \subseteq U^d \cap (U, (i, j) \in \Phi \). Since \( X_{ij} \subseteq U \), we have \( i > j \). Now \( X_{ij} \subseteq U^d \) if and only if \( X_{id^{-1}jd^{-1}} = X_{id^{-1}jd^{-1}} \subseteq U \), hence \( id^{-1} > jd^{-1} \). So \( \text{row}_\lambda(i) \leq \text{row}_\lambda(jd^{-1}) = \text{row}_\lambda(j) \). Now suppose \( 1 \leq j < i \leq n \) and \( \text{row}_\lambda(i) \geq \text{row}_\lambda(j) \). So \( X_{ij} \subseteq U \). If \( \text{row}_\lambda(id^{-1}) = \text{row}_\lambda(i) > \text{row}_\lambda(j) = \text{row}_\lambda(jd^{-1}) \), we have \( id^{-1} > jd^{-1} \) yielding \( X_{id^{-1}jd^{-1}} = X_{id^{-1}jd^{-1}} \subseteq U \) and hence \( X_{ij} \subseteq U \). Conversely, suppose \( \text{row}_\lambda(id^{-1}) = \text{row}_\lambda(i) = \text{row}_\lambda(jd^{-1}) = \text{row}_\lambda(j) \). Since \( j < i \) by assumption and \( s \in RStd(\lambda) \), \( j \) is the left of \( i \) in \( s \) and hence \( jd^{-1} \) is to the left of \( id^{-1} \) in \( t^i \) implying \( jd^{-1} < id^{-1} \). Thus \( X_{id^{-1}jd^{-1}} = X_{id^{-1}jd^{-1}} \subseteq U \) and hence \( X_{ij} \subseteq U^d \) in this case as well.
All subsets of $\Phi^-$ below are obviously closed.

5.5 Definition. Let $\lambda \vdash n, s \in \text{RStd}_\lambda, d = d(s)$. Define
1. $P = P(s) = \{(i, j) \in \Phi^- | \text{row}_s(i) \subseteq \text{row}_s(j)\}$
2. $L = L(s) = \{(i, j) \in \Phi^- | \text{row}_s(i) = \text{row}_s(j)\} \subseteq P$
3. $I = I(s) = \{(i, j) \in \Phi^- | \text{row}_s(i) < \text{row}_s(j)\} \subseteq P$
4. $K = K(s) = \{(i, j) \in \Phi^- | \text{row}_s(i) > \text{row}_s(j)\} \supseteq L$
5. $J = J(s) = \{(i, j) \in \Phi^- | \text{row}_s(i) > \text{row}_s(j)\} \subseteq K$

5.6 Proposition. Keeping the notation introduced in [5.5] and [5.6] we have:
1. $I \cap U = U_P, L \cap U = U_L, U^d \cap U = U_J, (U^-)^d \cap U = U_J$
2. $U_J, U_L \leq U_P, \text{U}_I \leq U_P, \text{U}_I \cap U_L = 1$ and $U_P = U_I U_L$ (semidirect product).
3. $U_J, U_L \leq U_K, U_J \leq U_K, U_J \cap U_L = 1$ and $U_K = U_J U_L$ (semidirect product).
4. $U_P \cap U_K = U_L$.
5. $U = U_P U_J, U_P \cap U_J = 1$.
6. $U_L$ is conjugate by $d^{-1}$ to $L_P \cap U$. In particular, $U_L$ is isomorphic to the direct product of the full unitriangular groups $\text{U}_\lambda(q)$, $1 \leq i \leq k, \lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$.

Proof. 1) This follows immediately from [4.8] and [5.4].
2) $I, L \subseteq P$, hence $U_I, U_L \leq U_P$. Moreover $I \cap L = 0$, hence $U_I \cap U_L = 1$ and $I \cup L = P$, hence $U_P = U_I U_L$ by [3.2]. Finally $I \subseteq P$ satisfies the assumption of [3.3] hence $U_I \leq U_P$ and $U_P$ is the semidirect product of $U_I$ by $U_L$.
3) This is shown similarly.
4) $P \cap K = L$ implies $U_P \cap U_K = U_L$ by [3.2].
5) $P \cap J = \emptyset, P \cup J = \Phi^-$ implies the claim with [3.2].
6) This follows immediately from [5.4] part 2).

Next we investigate the action of $U^d \cap U = U_K \leq U$ on the $s$-component $M_s$ of $\text{Res}_{U}^{GL_n(q)} M(\lambda)$, $\lambda \vdash n, s \in \text{RStd}(\lambda), d = d(s)$.

5.7 Proposition. Let $\lambda \vdash n, s \in \text{RStd}(\lambda), d = d(s)$. With the notation of [5.6] the following holds:

$$\text{Res}_{U_K}^{U} M_s \cong \text{Ind}_{U_L}^{U_K} C.$$ 

Proof. This is Mackey decomposition again: First note that $U = U_P U_J \leq U_P U_K \leq U$ implies $U_P U_K = U$, i.e. there is only one $U_P U_K$ double coset in $U$ (with representative 1), hence

$$\text{Res}_{U_K}^{U} M_s = \text{Res}_{U_K}^{U} \text{Ind}_{U_P}^{U} C \cong \text{Ind}_{U_K \cap U_P}^{U_K} \text{Res}_{U_K \cap U_P}^{U_K} C = \text{Ind}_{U_L}^{U_K} C,$$

by Mackey decomposition and [5.6] part 4).

Observe that [5.6]part 3) implies that $J, L \subseteq K$ satisfy the hypothesis [3.7]. Thus we may apply the construction of section [3] to $U_K = U_J \rtimes U_L$. In particular, $\text{Ind}_{U_K}^{U} C$ is isomorphic to $\mathbb{C} \mathcal{E}_J$, where $\mathcal{E}_J$ is the idempotent basis of the group algebra $\mathbb{C} \mathcal{V}_J$ of the additive group $\mathcal{V}_J = \{u - E | u \in U_J\}$. The monomial action of $U_K$ on $\mathcal{E}_J$ is described in [3.18].

The set $\mathcal{X}_s := \{du - d | u \in U_J\} \cong V_J$ as $\mathbb{F}_q$-vector space, since $d(u - E) = du - d$. For $A \in V_J$, $dA$ is again obtained from $A$ by reordering the rows. Since $M_s$ has basis $\mathcal{X}_s$, the $\mathbb{C}$-basis $\mathcal{E}_J$ of $\text{Res}_{U_K}^{U} M_s \cong \text{Ind}_{U_L}^{U_K} C$ may be expanded by linear combination of $\mathcal{X}_s$. 

15
Applying left multiplication by \( d \) to matrices in \( V_J \), where rows of \( dA, A \in V_J \) are relabeled as in 5.2 and extending this action by linearity to \( C(V_J, +) \), we turn the idempotents \( e_A \in \mathcal{E}_J, A \in V_J \) into idempotents \( e_{dA} \in \mathbb{C}(X_\mathbb{F}_q^0, +) \), such that the \( \mathbb{C} \)-span of \( \{e_{dA} \mid A \in V_J\} \) is an \( U_K \)-module isomorphic to \( \mathbb{C}X_\mathbb{F}_q \).

5.8 Theorem. Let \((i, j) \in K, \alpha \in \mathbb{F}_q, A \in V_J \). Then

\[
e_{dA}x_{ij}(\alpha) = \begin{cases} \theta(A_{ij}\alpha)e_{dB}, & \text{for } (i, j) \in J \\ e_{db}, & \text{for } (i, j) \in L \end{cases}
\]

where \( dB \) is obtained from \( dA \) by adding \( -\alpha \) times column \( j \) to column \( i \) of \( dA \) and setting all entries of the resulting matrix to zero, which do not belong to \( J \), if \((i, j) \in J \). If \((i, j) \in L \), we obtain \( dB \) similarly by the combined truncated row and column operation. Moreover if \((i, j) \in L \) and belongs to highest (lowest) compartment in matrices in \( X_\mathbb{F}_q \), then \( x_{ij}(\alpha) \) acts by truncated column (row) operation alone.

Proof. Everything follows directly from 3.18 besides the last claim. Let \((i, j) \in L \) such that \((i, j) \) is a position in the highest \( s \)-compartment of \( dA \), that is, \( \text{rows}_s(i) = \text{rows}_s(j) = 1 \). By construction for any position \((k, l) \in \Phi^- \) satisfying \( \text{rows}_s(k) = \text{rows}_s(l) = 1 \) we have \((k, l) \notin J \) by Definition 5.26. In particular all entries in \( dA \) in the highest compartment are zero. Thus only the truncated column operation adding \( -\alpha \) times column \( j \) to column \( i \) in \( dA \) and truncating can change \( dA \) for the calculation of \( e_{dA}x_{ij}(\alpha) \). Now assume \( \lambda = (\lambda_1, \ldots, \lambda_k) \vdash n \) and \((i, j) \in L \) with \( \text{rows}_s(i) = \text{rows}_s(j) = k \). Then \((i, j) \) is a position in the lowest \( s \)-compartment of \( dA \). By 4.10 all entries in columns \( i \) and \( j \) in \( dA \) are zero and hence truncated column operation adding \( -\alpha \) times column \( j \) to column \( i \) will not change \( dA \). Thus the claim follows.

5.9 Proposition. Let \( \lambda = (\lambda_1, \ldots, \lambda_k) \vdash n \) with \( \lambda_k \neq 0 \). Let \( s \in RStd(\lambda) \) and set \( J_k = \{ (i, j) \in J \mid \text{rows}_s(i) = k \} \). Then \( J_k \) is closed in \( \Phi^- \), and \( U_{J_k} \) is an abelian normal subgroup of \( U_J \) which acts on \( e_{dA} \in \mathbb{C}(X_\mathbb{F}_q^0, +) \) by a linear character, \( d = d(s) \in D_\Lambda \).

Proof. Since \( J = \{ (i, j) \in \Phi^- \mid \text{rows}_s(i) > \text{rows}_s(j) \} \) by definition, we obtain by direct calculation that \( J_k \) is closed in \( \Phi^- \) and \( U_{J_k} \) is an abelian normal subgroup of \( U_J \). Let \((i, j) \in J_k, \alpha \in \mathbb{F}_q \) and \( A \in V_J \). Since row \( i \) lies in the \( k \)-th compartment of \( dA \), which is a lowest compartment, by 4.10 the \( i \)-th column of \( dA \) is a zero column. Hence by 5.8 \( e_{dA}x_{ij}(\alpha) = \theta(A_{ij}\alpha)e_{dA} \), which is a scalar action as desired.

5.10 Remark. Henceforth we identify for fixed \( d \in D_\Lambda \) the spaces \( V_J \) and \( X_\mathbb{F}_q^0 \) and think of matrices \( dA \in X_\mathbb{F}_q^0 \) as elements of \( (U^-_\lambda)^d \cap U \) with reordered rows keeping the original labeling of those as in 5.2. Thus for idempotents \( e_A \in \mathcal{E}_J \) we think as well \( A \) to be a matrix in \( V_J \) with reordered rows, sorting the rows of \( A \) into consecutive compartments, each of those corresponding to a row in the tableau \( s = t^\lambda d \).

5.11 Example. Let \( \lambda, s \) as defined in 5.2 and \( d = d(s) \). Then \( J(s) = \{ 1 \} \). Let \( J = J_l = L = L(s) = \{ (1, 3), (2, 4), (5, 6) \} \). Let

\[
A + E = \begin{pmatrix}
1 & 1 & \phantom{0} & \phantom{0} \\
0 & 1 & \phantom{0} & \phantom{0} \\
0 & 0 & 1 & \phantom{0} \\
A_{41} & 0 & A_{43} & 1 \\
0 & A_{52} & 0 & 0 \\
0 & A_{62} & A_{63} & 0 \\
\end{pmatrix} \in U_J = (U^-_\lambda)^d \cap U, \quad \text{then we denote}
\]

16
We remark that the “last ones” of \( d(A + E) = dA + d \) are located in the first empty positions of each row. By 5.8 for \( \alpha \in \mathbb{F}_q \) we have:

\[
\begin{align*}
\left[ e_{dA}.x_{43}^{ij}(\alpha) \right] = & & \theta(\alpha A_{43}) \\
1 & & 0 & & 0 \\
2 & & 0 & & A_{41} & & 0 & & A_{43} \\
3 & & 0 & & A_{52} & & 0 & & 0 \\
4 & & 0 & & A_{62} & & A_{63} & & -\alpha A_{63} & & 0 \\
5 & & & & & & & & & & & & & & \\
6 & & & & & & & & & & & & & &
\end{align*}
\]

6 Supercharacters of \( U \)

The supercharacter theory of André \cite{Andre} and Yan \cite{Yan} is the special case of the construction in the previous section taking \( \lambda = (1^n) \vdash n \) and \( w = 1 \in D_\lambda \). Then \( K = J = \Phi^- \), \( L = \emptyset \) and the 1-cocycle \( f : U \rightarrow V = V_{\Phi^-} = \text{Lie}(U) \) is given by \( f(u) = u - E \in V \). Note that this is a left 1-cocycle as well yielding a left action of \( U \) on \( C(V, +) \). Indeed \( C(V, +) \) is then a \( CU \)-bimodule isomorphic to the regular \( CU \)-bimodule \( CU \times CU \).

We write \( E = \mathcal{E}_{\Phi^-} \), the set of lidemptons arising from \( C(V, +) \). To distinguish this special case notationally from other cases we denote now the lidempotent affording \( \chi_{-A} \), \( A \in V \), by \( [A] \) instead of \( e_A \). For the convenience of the reader, we collect some well-known facts on the monomial \( C(U) \)-bimodule \( C(V, +) \) consisting of lidemptons. For details and proof we refer to \cite{Dong, Wan}.

6.1 Lemma. Let \( A \in V \), \( 1 \leq j < i \leq n \) and \( \alpha \in \mathbb{F}_q \), then

\[
[A]x_{ij}(\alpha) = \theta(\alpha A_{ij})[A. x_{ij}(\alpha)],
\]

where \( A. x_{ij}(\alpha) \) is obtained from \( A \) by adding \(-\alpha \) times column \( j \) to column \( i \) (from left to right) and setting nonzero entries in the resulting matrix at position on or to the right of the diagonal to zero. This is called truncated column operation, (comp. \cite{Dong}). Similarly the left operation of \( x_{ij}(\alpha) \) on the lidempton basis \( \{ [A] | A \in V \} \) of \( CV \) can be described by a truncated row operation from down up, the coefficient in \( C \) being again \( \theta(\alpha A_{ij}) \).

6.2 Definition. A subset \( p = \{(i_1, j_1), \ldots, (i_k, j_k)\} \subseteq \Phi^- \) is called a main condition set if \( p_x = \{i_1, \ldots, i_k\} \) and \( p_y = \{j_1, \ldots, j_k\} \) are sets of \( k \) many pairwise different indices in \( \{1, \ldots, n\} \). So \( p \) picks from each row and each column of \( n \times n \)-matrices at most one position. We call
Thus \( [A] \in \mathcal{E} \) is a verge if \( \text{supp}(A) = \mathfrak{p} \subseteq \Phi^- \) is a main condition set and call then the elements of \( \mathfrak{p} \) main conditions.

It is easy to see that verge idempotents correspond to the "basic characters" defined by André. Indeed it can be shown that each \( U-U \)-biorbit of \( \mathcal{E} \) contains exactly one verge \( [A] \in \mathcal{E} \) and all right orbits contained in the biorbit generated by the verge \( [A] \in \mathcal{E} \) afford identical characters of \( U \). Those are precisely the supercharacters. Distinct biorbits afford orthogonal characters and hence each irreducible character of \( U \) is irreducible constituent of precisely one supercharacter, (comp. [6, 2.13]).

We denote for \( [A] \in \mathcal{E}, A \in V \), the right orbit containing \( [A] \) by \( \mathcal{O}_A \).

6.3 Definition. Let \( 1 \leq j < i \leq n \). The **hook arm** \( h_{ij}^a \) centred at \( (i, j) \) consists of all positions \( (i, k) \in \Phi^- \) strictly to the right of \( (i, j) \), thus \( h_{ij}^a = \{(i, k) \mid j < k < i\} \), and the **hook leg** \( h_{ij}^l \) is the set of positions \( (l, j) \in \Phi^- \) strictly above \( (i, j) \), thus \( h_{ij}^l = \{(l, j) \mid j < l < i\} \). Finally the **hook** \( h_{ij} \) centred at \( (i, j) \) is defined to be \( h_{ij} = h_{ij}^a \cup h_{ij}^l \cup \{(i, j)\} \). Let \( \mathfrak{p} \subseteq \Phi^- \) be a main condition set. The hooks centred at positions in \( \mathfrak{p} \) are called **main hooks**.

6.4 Theorem. [6] Let \( [A] \in \mathcal{E} \) be a verge idempotent with \( \text{supp}(A) = \mathfrak{p}_A = \{(i_1, j_1), \ldots, (i_k, j_k)\} \subseteq \Phi^- \). Then the right projective stabilizer of \( [A] \) in \( U \), that is the set \( \{u \in U \mid [A]u = \lambda[A], \text{ for some } \lambda \in \mathbb{C}^*\} \), denoted by \( P_{\text{stab}}[A] \), is a pattern subgroup \( U_R \) with

\[
R = \{(r, s) \in \Phi^- \mid s \notin \{j_1, \ldots, j_k\}\} \cup \{(r, j_\nu) \mid 1, \ldots, k, i_\nu \leq r \leq n\}.
\]

Thus \( \mathfrak{p}_A \subseteq R \) and \( R^\circ = R \setminus \mathfrak{p}_A \) is closed. Moreover \( U_{R^\circ} \) acts trivially on \( [A] \), \( U_{R^\circ} \leq U_R \) and \( U_R / U_{R^\circ} \cong X_{i_1,j_1} \times \cdots \times X_{i_k,j_k} \) acting on \( [A] \) by the linear character \( \theta_A = \theta_1 \times \theta_2 \times \cdots \theta_k \), where \( \theta_\nu : X_{i_\nu, j_\nu} \to \mathbb{C}^* \) sends \( x_{i_\nu,j_\nu}(\alpha) \) to \( \theta(\alpha A_{i_\nu,j_\nu}) \in \mathbb{C}^* \) for \( \alpha \in \mathbb{F}_q, \nu = 1, \ldots, k \). \( \square \)

Thus \( R \) consists of all positions in \( \Phi^- \) in zero columns of \( A \) together with all positions on and below the positions in \( \mathfrak{p}_A \).

6.5 Definition. Let \( [A] \in \mathcal{E} \) be a verge with \( \text{supp}(A) = \mathfrak{p}_A \subseteq \Phi^- \). \( U_R = P_{\text{stab}}[A] \) is defined as above. We define \( \hat{R} \) to be \( R \) combined with all positions on hook legs, such that the corresponding subgroups change in \( [A] \) only the the values at a hook intersection acting from the right. We illustrate as follows:

\[
\text{R = positions in zero columns plus positions not above z marked by } \times
\]
\[
z = \text{main conditions}
\]
\[
\circ = \text{main hook intersections}
\]
\[
\square = \text{positions in } \hat{R} \setminus R
\]

The figure illustrates a verge \([A] = \ldots\) with its right orbit \( R \) containing the elements \((i, j), (b, j), (i, b)\) and \( \hat{R} \) containing the additional elements \((i, z), (z, z), (z, b)\).
For example $X_{ij}$ acting on $[A]$ will only change the entry at the main hook intersection $(i, b)$.

It was shown in \[6, 4.5\], that $\widehat{\mathcal{R}}$ is a closed subset of $\Phi^-$ with $U_{\mathcal{R}_-}, U_{\mathcal{R}} \subseteq U_{\mathcal{R}}$ and that $U_{\mathcal{R}}/U_{\mathcal{R}^o} \cong X_{i_1j_1} \times \cdots X_{i_kj_k}$ is a central subgroup of $U_{\mathcal{R}}/U_{\mathcal{R}^o}$.

6.7 Definition. Let

$$\mathfrak{p} = \{(i_1, j_1), \ldots, (i_k, j_k)\} \subseteq \Phi^-$$

being a main condition set. We call $\mathfrak{p}$ completely hook disconnected if $\mathfrak{p}_x = \{i_1, \ldots, i_k\}$ and $\mathfrak{p}_y = \{j_1, \ldots, j_k\}$ are disjoint. Thus, in this case, $(i_1, \ldots, i_k, j_1, \ldots, j_k)$ is a subset of $\{1, \ldots, n\}$ of $2k$ many pairwise different indices.

Thus if $[A] \in \mathcal{E}$ is a verge with completely disconnected condition set $\mathfrak{p} = \text{supp}(A)$, no main hooks of $\mathfrak{p}$ meet at the diagonal, that is the following does not occur:

$$(a, i), (i, j) \in \mathfrak{p}$$

For our main application for section\[6\] this condition is automatically satisfied and is the special case of hook disconnected main condition sets defined in \[6, 5.1\]. We state here the main result of \[6\] for those, as far as they are needed here.

6.8 Results. \[6\] Let $\mathfrak{p} \subseteq \Phi^-$ be a (completely) hook disconnected condition set and $[A] \in \mathcal{E}$ be a verge with $\text{supp}(A) = \mathfrak{p}$. Then the following holds:

1) $\widehat{\mathcal{R}}^- = \widehat{\mathcal{R}} \setminus \mathfrak{p}$ is closed in $\Phi^-$ with $U_{\mathcal{R}^-} \subseteq U_{\mathcal{R}}$.
2) $\mathrm{End}_{\mathcal{C}}(\mathcal{O}_A) \cong \mathcal{C}(U_{\mathcal{R}^-}/U_{\mathcal{R}}) \cong \mathcal{C}(U_{\mathcal{R}_-}/U_{\mathcal{R}^o}) = \mathcal{C}H$.
3) $U_{\mathcal{R}}/U_{\mathcal{R}^o} \cong H \times X_{i_1j_1} \times \cdots X_{i_kj_k}$. If $S$ is an irreducible $\mathcal{C}H$-module, extending the action of $H$ on $S$ by the linear character $\theta_A$ defined in \[6, 4\] and letting $U_{\mathcal{R}^o}$ act trivially yields an irreducible $\mathcal{C}U_{\mathcal{R}^-}$-module $\hat{S}$ such that $[A]\hat{S} \cong \hat{S}$ and $[A]\hat{S} CU \cong \text{Ind}_{U_{\mathcal{R}^-}}^{U_{\mathcal{R}}} \hat{S}$ is an irreducible constituent of $\mathcal{C}O_A$.
4) $S \mapsto S \mapsto [A]\hat{S} CU$ is a multiplicity preserving bijection between the irreducible constituents of the group algebra $\mathcal{C}H$ and those of the $\mathcal{C}U$-module $\mathcal{C}O_A$.

6.9 Remark. So, in particular, choosing $S$ to be the trivial $\mathcal{C}H$-module, $[A]\hat{S} CU \subseteq \mathcal{C}O_A$ is a unique irreducible constituent (of multiplicity one) of $\mathcal{C}O_A$, isomorphic to $\text{Ind}_{U_{\mathcal{R}}}^{U_{\mathcal{R}}}(\mathcal{C}\epsilon_\lambda)$, where $\epsilon_\lambda$ is the primitive central idempotent of $\mathcal{C}U_{\mathcal{R}^-}$ affording the linear character $\theta_A$ on $U_{\mathcal{R}^-}/U_{\mathcal{R}^o}$ and the trivial character on $H$. In particular, $\text{Stab}_H(\epsilon_\lambda) = U_{\mathcal{R}^-} \subseteq U_{\mathcal{R}} = \text{P}\text{Stab}_H(\epsilon_\lambda)$.

7 Two part compositions

In this section we apply the general method of section\[5\] to the special case that $\lambda = (n - m, 1) \leq m \leq n - 1$ is a composition of $n$ into two parts. Thus $P_\lambda \leq GL_n(q)$ is a maximal parabolic subgroup and all maximal parabolic subgroups of $GL_n(q)$ are conjugate to some $P_\lambda, \lambda = (n - m, m)$ $\vdash n$. Moreover, if $\lambda = (n - m, m) \vdash n$ then $\mu = (m, n - m) \vdash n$ too and it is a well known fact, that $\text{Ind}_{P_\lambda}^{GL_n(q)}(\mathcal{C}) \cong \text{Ind}_{P_\mu}^{GL_n(q)}(\mathcal{C})$. Thus, in the following, we may always assume that $\lambda = (n - m, m)$ is indeed a partition of $n$, that is $m \leq n - m$. Note that the set $\mathcal{F}(\lambda)$ of $\lambda$-flags is the set of $m$-dimensional $\mathbb{F}_q$-subspaces of $\mathbb{F}_q^n$. By \[4, 11\] we may identify $\mathcal{F}(\lambda)$
by matrices in $\mathfrak{X}_\lambda = \{du \mid d \in \mathcal{D}_\lambda, u \in (U^-)^d \cap U\}$ where each matrix in $\mathfrak{X}_\lambda$ is divided into two compartments, the upper compartment of $A \in \mathfrak{X}_\lambda$ consisting of the first $(n - m)$ and the lower one of the last $m$ rows of $A$. For $s \in RStd(\lambda)$, we denote the second row of $s$ by $\underline{s}$ thus $s = (i_1, \ldots, i_m)$ with $1 \leq i_1 < i_2 < \cdots < i_m \leq n$. Obviously $\underline{s}$ determines $s$ uniquely the first row of $s$ consisting of all numbers $i \in \{1, \ldots, n\}$ with $i \notin \underline{s}$ ordered from left to right by the natural ordering of $N$. Note that by our convention introduced in section 5 the rows in the lower compartment of $A \in \mathfrak{X}_\lambda$ are labelled by $i_1, \ldots, i_m$, (comp. 5.10). Fix $s \in RStd(\lambda), d = d(s)$ and recall Definition 5.5. In particular:

$$\begin{align*}
L &= L(s) = \{(i, j) \in \Phi^- \mid i, j \in \underline{s} \text{ or } i, j \notin \underline{s}\} \\
J &= J(s) = \{(i, j) \in \Phi^- \mid i \in \underline{s}, j \notin \underline{s}\} \\
K &= K(s) = L \cup J.
\end{align*}$$

Let $e_A \in \mathcal{E}_J$, then $A + E \in U_J$. Then all entries in the first compartment of $A$ are zero, and hence $e_A$ is entirely determined by the lower compartment. Thus, in illustrations, we may omit the first compartment.

7.2 Remark. By 5.9 $U_J$ is abelian and acts on the idempotents in $\mathcal{E}_J$ by linear characters. Moreover $L$ splits into $L_1 = \{(i, j) \in \Phi^- \mid i, j \notin \underline{s}\}$ and $L_2 = \{(i, j) \in \Phi^- \mid i, j \in \underline{s}\}$, the positions in $L_1$ belonging to the upper and in $L_2$ to the lower compartment in matrices in $\mathfrak{X}_s$. By 5.8 $U_L = U_{L_1} \times U_{L_2}$, $U_{L_1} \cong U_{n - m}(q), U_{L_2} \cong U_m(q)$, where $U_{L_1}$ acts by truncated column and $U_{L_2}$ by truncated row operations on $\mathcal{E}_J$ permuting $\mathcal{E}_J$. For $e_A \in \mathcal{E}_J$ we denote the $U_K$-orbit in $\mathcal{E}_J$ containing $e_A$ by $\mathcal{O}_A^J$ as in 6.2.3.

Throughout this section we shall use the setting and notation introduced above without further notice. The permutation module $Res_{U}^{GL_n(q)} M(\lambda)$ has been investigated in 3.2, 7 and 8 using the basis of $M(\lambda)$ consisting of $\lambda$-flags. The basis $\mathfrak{X}_\lambda$ was constructed there. Comparing Proposition 5.8 with section 2 of 8 shows that indeed our construction here contains the exposition in 8 as a special case.

For the convenience of the reader we summarize the relevant results on $M_\lambda$ shown in 8.

7.3 Results. (1) 8, 2.5.6 Each $U_K$-orbit $\mathcal{O}_J^J$ of $\mathcal{E}_J$ contains a unique idempotent $e_A, A \in V_J$, such that in each row and in each column of $A$ there is at most one non-zero entry. Similarly as in 6.2 we call such idempotent **verge (idempotent)** of $\mathcal{E}_J$ and define $\text{main}(\mathcal{O}_J^J) = \text{main}(e_A) = \text{supp}(A)$. Note, that then $p = p_A = \text{supp}(A)$ is a main condition set in $\Phi^-$ as defined in 6.2. In particular, (putting the rows of $A$ again in the natural order) $[A] \in \mathcal{E}$ is a verge in the idempotent basis of $\mathbb{C}(V_{q^k}^-, +)$.

(2) Let $e_A \in \mathcal{E}_J$ be a verge and $p = p_A = \text{main}(e_A)$ its main condition set. Recall from 6.2 the definition of $p_x = \{i \in \{1, \ldots, n\} \mid \exists 1 \leq j < i : (i, j) \in p\}$ and $p_y = \{j \in \{1, \ldots, n\} \mid \exists j < i \leq n : (i, j) \in p\}$. Let $\mu = (n - k, k) \vdash n, t \in RStd(\mu)$. We say $p$ fits the $t$-component $M_t$ of $Res_{U}^{GL_n(q)} M(\mu)$, if $p_x \subseteq \underline{1}$ and $p_y \cap \underline{1} = \emptyset$.

(3) 8, 2.5.10] Let $e_A \in \mathcal{E}_J$ be a verge with main condition set $p = p_A \subseteq \Phi^-$. i) Define $L_1^0 = \{(i, j) \in L_1 \mid j \notin p_x \text{ or } \exists (b, j) \in p \text{ with } b < i\}$. Thus $L_1^0$ consists of all positions $(i, j)$ in $L_1$ in the upper compartment of $dA, d = d(s) \in \mathcal{D}_\lambda$, where either column $j$ is a zero column (if and only if $j \notin p_x$) or there is a main condition $(b, j)$ in column $j \in p_y$, above position $(i, j)$ in $A$ (in the natural order of rows of $A$).

Note that in both cases $(i, j) \in R^n$ and hence $X_{ij} \in Stab_U[A]$ by 6.4. We illustrate the case $j \in p_y$. Let $\alpha \in \mathbb{F}_q$. 

20
Acting by $x_{ij}(\alpha)$:

\[
\begin{array}{ccc}
  j & b & i \\
  i & \circ & (b, i) \\
  (b, j) \in p & \odot & (b, i) \\
\end{array}
\]

only non-zero entry in column $j$ of $A$

upper compartment of $dA$

lower compartment of $dA$

If $(b, j) \in p$ with $b < i$, then $(b, i) \notin \Phi^-$ and hence $(b, i) \notin J$. Now $e_A x_{ij}(\alpha) = e_B$, where $B$ is obtained from $A$ by adding $-\alpha$ times column $j$ to column $i$ and projecting the resulting matrix into $V_j$. Since $(b, i) \notin J$ we conclude that $e_A x_{ij}(\alpha) = e_A$. Indeed $L_0^1$ is closed in $\Phi^-$ and $U_{L_0^1} = \text{Stab}_{U_{L_1}}(e_A)$.

ii) Define $L_2^0 = \{(i, j) \in L_2 | i \notin p_z \text{ or } \exists (i, v) \in p \text{ with } v > j,\}$. Thus $L_2^0$ consists of positions $(i, j)$ in $L_2$ in the lower compartment of $dA$, where either contained in a zero row of $A$ (if and only if $i \notin p_z$) or the main condition in row $i$ is to the right of $(i, j)$. Again we illustrate the situation in the second case $i \in p_z$ (omitting from $dA$ the upper compartment): Let $\alpha \in F_q$:

Acting by $x_{ij}(\alpha)$:

\[
\begin{array}{ccc}
  j & v & i \\
  i & \circ (j, v) & (i, v) \in p \\
\end{array}
\]

lower compartment of $dA$

Again, since $j < v$ and $(j, v)$ is the only position being changed in $dA$, when adding $\alpha$ times row $i$ to row $j$. But this entry at position $(j, v)$ is set back to zero by truncation. Thus $e_A x_{ij}(\alpha) = e_A$, indeed $L_2^0 \subseteq \Phi^-$ is closed and $U_{L_2^0} = \text{Stab}_{U_{L_2}}(e_A)$.

iii) The stabilizer of $e_A$ in $U_K$ is not a pattern subgroup in general. To see this consider the following situation: Let $(s, i), (t, j) \in p$ with $1 \leq j < i < t < s \leq n$ and $\alpha \in F_q$:

Acting by $x_{ij}(\alpha)$:

\[
\begin{array}{ccc}
  j & i & t \\
  t & (t, i) & (s, i) \\
  s & (t, j) & \odot (t, i) \\
\end{array}
\]

Acting by $x_{st}(\alpha)$:

\[
\begin{array}{ccc}
  j & i & t \\
  s & (t, i) & (s, i) \\
\end{array}
\]

lower compartment of $dA$
Then \((t, i) \in J\) is a main hook intersection. Note that \(i, j \notin \bullet\), \(s, t \in \bullet\) hence \((i, j) \in L_1, (s, t) \in L_2\) are in the upper respectively in the lower compartment. Acting by \(x_{ij}(\alpha) (\alpha \in \mathbb{F}_q)\) on \(e_{ij}\) adds \(-\alpha\) times column \(j\) to column \(i\) and hence inserts \(-\alpha A_{ij}\) into position \((t, i)\). Acting by \(x_{st}(\beta) (\beta \in \mathbb{F}_q)\) adds \(\alpha\) times row \(s\) to row \(t\) and hence inserts \(\beta A_{si}\) into position \((t, i)\). In both cases, the entry at position \((t, i)\) is the only one which is changed. Note that by assumption \(A_{ij} \neq 0 \neq A_{si}\) and hence choosing \(\beta = \frac{A_{ji}}{A_{ij}} \in \mathbb{F}_q\) we have \(e_{ij}(\alpha)x_{st}(\beta) = e_{ij}\), that is \(x_{ij}(\alpha)x_{st}(\beta) \in \text{Stab}_{U_K}(e_A)\).

Note that inspecting 6.4 we see that \((i, j) \in \mathcal{R} \setminus \mathcal{R}\) and hence \(X_{ij} \leq U_K\).

(4) \(\textbf{8}, 2.5.14\) Let \(O^J \subseteq E_J\) be an \(U_K\)-orbit. Then \(C O^J\) is an irreducible \(\mathbb{C} U_K\)-module.

(5) Recall from \(5.7\) and \(3.10\) that \(\text{Res}^U_K M_s \cong \mathbb{C} E_J\), where \(M_s = \text{Ind}^U_{U_P} C \overline{\tau} d\) by \(5.1\) \(d = d(s) \in D_J, U_P = P^J \cap U\) (by \(5.9\)) and \(\overline{\tau} = \sum_{h \in P^J} h\). An isomorphism from \(\mathbb{C} E_J\) to \(\text{Res}^U_K M_s\) can be described explicitly by \(f^*: \mathbb{C} (V_J, \tau) \to \mathbb{C} U_K: \tau \mapsto \tau \circ f\) for \(\tau \in \mathbb{C} V_J \cong \mathbb{C} (V_J, \tau)\), see \(2.8\). Here \(f : U_K \to V_J\) is the \(1\)-cocycle of \(3.9\) Recall that \(f|_{U_J} : U_J \to V_J\) is given as \(f(u) = u - E\) by the proof of \(3.9\). Setting \(U_L = q^{-|J|}\sum_{x \in U_L} x\), we have now \(f^*(A) = \hat{U}_L(A + E) \in \mathbb{C} U_K\), since \(U_L = \ker(f)\) by \(2.9\) Hence

\[
\begin{align*}
\hat{f}(e_A) &= f^*(q^{-|J|}\sum_{B \in V_J} \chi_A(B) B) = q^{-|J|}\hat{U}_L \sum_{u \in U_J} \chi_A(u - E)u
\end{align*}
\]

Observe that \(U_L \subseteq U_P \subseteq P^J\), \(M_s = \text{Ind}^U_{U_P} C \overline{\tau} d = P^J \mathbb{C} U\), we see that \(\overline{\tau} d = \overline{\tau} d \hat{U}_P = \overline{\tau} d \hat{U}_L\), where \(\hat{U}_P = q^{-|P|}\sum_{x \in U_P} x\). Therefore we may identify \(f^*(e_A)\) with \(\sum_{x \in U_J} \overline{\tau} d \chi_A(u - E)u \in P^J \mathbb{C} U \cong M_s\). Let \(O^J \subseteq E_J\) be an \(U_K\)-orbit, \(\hat{O}^J \subseteq P^J \mathbb{C} U\) its image under the identifications, then it was shown in \(\text{8}, 2.6.2\), that \(C \hat{O}^J \subseteq P^J \mathbb{C} U\) is \(U\)-invariant. Since \(C \hat{O}^J \cong C O^J\) is an irreducible \(\mathbb{C} U_K\)-module, we conclude that the \(U_K\)-action on \(C O^J\) can be extended to \(U\) yielding the irreducible \(\mathbb{C} U\)-module \(C O^J \cong C \hat{O}^J\).

\(\square\)

### 7.4 Remark.

For a given main condition \(p \subseteq \Phi^*\), it might fit many irreducible orbit modules for different components and even for different 2-part partitions. For example: Let \(0 \neq \alpha \in \mathbb{F}_q\).

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
3 & \ast & \square & 1 & & 6 \\
5 & \ast & 0 & \ast & 1 & \\
6 & & 0 & 0 & 1 & \\
\end{array}
\]

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
2 & \ast & 1 & & & 6 \\
5 & 0 & \ast & \ast & 1 & \\
6 & 0 & 0 & 1 & \\
\end{array}
\]

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
2 & \ast & 1 & & & 6 \\
5 & 0 & \ast & \ast & 1 & \\
6 & 0 & 0 & 1 & \\
\end{array}
\]

\[
\lambda = (3, 3), \quad t = 1 \quad 2 \quad 4 \quad 3 \quad 5 \quad 6 \\
\lambda = (3, 3), \quad t = 1 \quad 3 \quad 4 \quad 2 \quad 5 \quad 6 \\
\lambda = (4, 2), \quad t = 1 \quad 3 \quad 4 \quad 6 \quad 2 \quad 5 \\
\]

The three orbits above have the same main condition set \(p = (5, 1)\) with the filling \(\alpha\) and the same dimension \(q^3\).

In \(2.3\) part (4) and (5) we have seen that the \(\mathbb{C}\)-spaces spanned by orbits \(O^J \subseteq E_J\) are irreducible \(\mathbb{C} U\)-modules. The next theorem states, that these \(\mathbb{C} U\)-modules depend not really on \(O^J \subseteq E_J\) or even on the 2-part partition \(\lambda = (n - m, m)\), but only on the main condition set \(p = \text{main}(O^J)\) and the non-zero values of \(A\) at positions in \(p\) for the unique verge idempotent \(e_A \in O^J\).
7.5 Theorem. \[8, 3.1.30\] Let \(\lambda, \mu\) be 2-part partitions of \(n\) and let \(s \in \text{RStd}(\lambda), t \in \text{RStd}(\mu)\).

Let \(e_A \in \mathcal{E}_{J(s)}\) and \(e_B \in \mathcal{E}_{J(t)}\) be vertices with \(A = B\) in \(M_n(q)\). Thus \(\text{main}(O_A(\lambda)) = \text{supp}(A) = \text{main}(O_B(\mu)) = \text{supp}(B) = \text{main}(O_B(\mu))\) and \(A_{ij} = B_{ij}\) for all \((i, j) \in \text{supp}(A)\). Then \(\text{CO}_A(J(s)) \cong \text{CO}_B(J(t))\) as \(CU\)-modules.

As a consequence of the main result \[8, 7.5\] of this paper we obtain that the converse of Theorem \[8, 7.3\] holds as well, that is \(\text{CO}_A(J(s)) \cong \text{CO}_B(J(t))\) if and only if \(A = B\) in \(M_n(q)\).

7.6 Remark. Let \(\lambda = (n - m, m)\) be a partition of \(n\), \(s \in \text{RStd}(\lambda)\) and \(\mathcal{O}^J \subseteq \mathcal{E}_J\) an \(U_K\)-orbit, \(J = J(s)\) as above. Let \(p = \{(i_1, j_1), \ldots, (i_k, j_k)\}\) be \(\text{main}(\mathcal{O}^J)\) the set of main conditions for the unique vertex \(e_A\) \((A \in V_J)\) in \(\mathcal{O}^J = O_A^J\). Since \(p \subseteq J = \{(i, j) \in \Phi^- | i \in \mathcal{O}, j \notin \mathcal{O}\}\) we conclude that \(p_x \cap p_y = \{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_k\} = \emptyset\). Thus \(p \subseteq \Phi^-\) is a completely hook disconnected main condition set. Moreover \(k \leq m\). Let \(\mathcal{O} = \{i_{k+1}, \ldots, i_m\}\), then \(\mu = (n - k, k)\) is as well a partition of \(n\) and \(t \in \text{RStd}(\lambda)\) with \(t = (i_1, \ldots, i_k)\) (assuming as we always do, that \(1 \leq i_1 < i_2 < \cdots < i_k \leq n\)). It is obvious that \(p\) fits \(t\) as well and \(A \in V_{J(t)}\) as well. By \[8, 7.3\] \(CO_A(J(s)) \cong CO_B(J(t))\). This indeed works for arbitrary completely hook disconnected main condition sets \(p = \{(i_1, j_1), \ldots, (i_k, j_k)\} \subseteq \Phi^-\). Since then \(p_x \cap p_y = \emptyset\), we always have \(2k \leq n\) and hence \((n - k, k)\) is a partition of \(n\), and there is a unique row standard \(\lambda\)-tableau \(t\) with \(t = p_x = (i_1, \ldots, i_k)\). We remark that \(A\) is in fact standard, that is increasing in the columns as well. This is not hard to see, but is needed in this paper.

If \(|p| = k\) and \(\lambda = (n - k, k)\), each row of the lower compartment of \(A\) \((e_A \in \mathcal{E}_J\) a vertex, \(p = \text{supp}(A)\)) carries a condition. We say in this case that the corresponding orbit module \(\text{CO}_A^J\) has full condition set.

8 \(\text{CO}_A^J\) as constituent of \([A]\wedge U\)

Let \(p = \{(i_1, j_1), (i_2, j_2), \ldots, (i_m, j_m)\} \subseteq \Phi^-\), \(1 \leq i_1 < i_2 < \cdots < i_m \leq n\) be a completely hook disconnected main condition set in \(\Phi^-\), and let \(A \in M_n(q)\) be such that \(\text{supp}(A) = p\). Let \(\lambda = (n - m, m)\) and let \(s\) be the unique row standard \(\lambda\)-tableau with \(\mathcal{O} = \{i_1, \ldots, i_m\}\). Thus, in view of \[8, 7.6\] \(e_A \in \mathcal{E}_J, J = J(s)\) as in \[8, 7.4\] is a vertex, \(\text{main}(e_A) = p\) and \(\text{CO}_A^J\) has full condition set.

Being an irreducible \(CU\)-module by \[8, 7.3\] part (5), \(\text{CO}_A^J\) must occur as irreducible constituent of \([B]\wedge U\) for precisely one vertex \([B] \in \mathcal{E} = \mathcal{E}_{\Phi^-}, B \in \text{Lie}(U) = \{u - E | u \in U\}\). We shall show that \(B = A \in V_J \subseteq \text{Lie}(U)\). However, as we shall see, a \(CU\)-homomorphism from \(\text{CO}_A^J\) into \(\text{CO}_A, (O_A = O_A^J)\) will not take \(e_A \in O_A^J\) to \([A] \in O_A\).

Throughout this section \(A \in V_J, J = J(s), \text{supp}(A) = p\). We use the notation of the previous sections freely. In particular \(J = J(s), L = L(s)\) and \(K = K(s)\) are as in \[8, 7.1\] Recall from \[8, 7.2\] that \(L\) splits into closed subsets \(L_1 = \{(i, j) \in \Phi^- | i, j \notin \mathcal{O}\}\) and \(L_2 = \{(i, j) \in \Phi^- | i, j \in \mathcal{O}\}\) of \(\Phi^-\).

8.1 Definition. Throughout let \(f = \hat{U}_{L_2}\) be the trivial idempotent of \(U_{L_2}\), that is \(f = q^{-|L_2|} \sum_{x \in U_{L_2}} x\), and set \(\hat{e}_A = e_A f \in \text{CO}_A^J\).

8.2 Lemma. \(\hat{e}_A \neq 0\). Thus \(\hat{e}_A \wedge U = \text{CO}_A^J\).

Proof. Recall from \[8, 7.3\] part (3) ii), that \(\text{Stab}_{U_{L_2}}(e_A) = U_{L_2}\). Choose the following linear ordering of \(L_2\) and take all products of elements \(x_{ij}(\alpha), (i, j) \in L_2\) in this fixed order: First we take the roots \((i, j) \in L_2^0\) in an arbitrary ordering, then the remaining positions \((i, j)\) in \(L_2\) along columns top down and rows from left right. Thus \((i, j) < (a, b)\) in \(L_2 \setminus L_2^0\) implies \(i, j, a, b \in \mathcal{O}\), the main conditions in row \(i\) and \(a\) are to the left of \((i, j)\) and \((a, b)\) respectively, and \(j < b\) or \(j = b\) and \(i < a\).
$x_{ij}(\alpha), (i, j) \in L_2 \setminus L_0^0$, acts on $e_A$ by adding $\alpha$ times row $i$ to row $j$ and hence changing only position $(j, k) \in J$ ($k \notin \mathfrak{g}$). In the order above, the root subgroups fill the positions in $J$ on the top of main conditions from top down. As a consequence, $e_A f = q^{-|L_2| + |L_0^0|}$ times the sum of all idempotents $e_{i,j}$, where $B$ runs through the set of all matrices in $V_d$, coinciding with $A$ except the positions in $J$ in columns of and above the main conditions, which are filled by arbitrary entries from $F_q$. This proves $e_A f = q^{-|L_2| + |L_0^0|} \sum e_{i,j} \neq 0$ proving the claim. □

Now let $H \leq \text{Pst}_{U} (\hat{e}_A) = \{u \in U | \hat{e}_A u = \lambda u \hat{e}_A, \exists \lambda u \in \mathbb{C}^*\}$. Then $\lambda : H \to \mathbb{C}^*: u \mapsto \lambda u$ is a linear character of $H$ and we have a natural epimorphism of $CU$-modules

$$\mu : \text{Ind}_{U}^{U \cap \mathbb{C}^*} \hat{e}_A \to \mathbb{C}^n_{A}: \hat{e}_A u \mapsto \hat{e}_A u$$

(8.3) for any $u \in U$. We shall show that $U_{\hat{R}} \leq \text{Pst}_{U} (\hat{e}_A)$, where $\hat{R} \subseteq \Phi^{-}$ is defined in (6.4) for orbit $O_A = \mathcal{O}_A^{\Phi} \subseteq \mathcal{E}_{\Phi}^{-}$ containing the verge idempotent $[A]$. Then we prove that $U_{\hat{R}}$ acts on $\hat{e}_A$ by the linear character afforded by $\hat{e}_A$ defined in (6.9). Then (6.9) says in particular that $\text{Ind}_{U_{\hat{R}}}^{U, \mathbb{C}^*} \hat{e}_A$ is irreducible, proving that $\mu$ in (8.3) with $H = U_{\hat{R}}$ must be an isomorphism. This implies $\text{Pst} \hat{e}_A = U_{\hat{R}}$ and identifies the irreducible $\mathbb{C}U$-module $\mathbb{C}^n_{A}$ as the unique irreducible constituent in $\mathbb{C}^n_{A} = [A] \cap U_{\hat{R}}$ corresponding to the trivial module of $\mathbb{C}(U_{\hat{R}})_{n}$ extended to $\text{End}_{\mathbb{C}U}(\mathbb{C}^n_{A})$ by $\theta_A$ as in (6.8) part 3).

8.4 Lemma. $U_{L_2} \leq \text{Stab}_{U} (\hat{e}_A)$ and $L_2 \in R$. Moreover $U_{L_2} \leq \text{Stab}_{U}[A]$ as well.

Proof. By construction $\hat{e}_A u e_A = e_A u = \hat{e}_A f = \hat{e}_A$ for all $u \in U_{L_2}$. So $U_{L_2} \leq \text{Stab}_{U} (\hat{e}_A)$. For $(i, j) \in L_2$, we have $i, j \not\in \mathfrak{g}$ and hence $j \not\in \mathfrak{p}_f \subseteq J$. Thus column $j$ is a zero column in $A$ and hence $(i, j) \in R$ by (6.4).

Let $(a, b) \in L_2$, $(i, j) \in L_1$. Then $a, b \not\in \mathfrak{g}, i, j \not\in \mathfrak{g}$ and hence $a \neq j$ and $b \neq i$. by (3.1) $X_{ab}$ and $X_{ij}$ commute and hence $U_L = U_{L_1} \times U_{L_2}$. In particular this implies

8.5 Lemma. $U_{L_1} = \text{Stab}_{U_{L_1}} (\hat{e}_A)$, Moreover $L_1 \subseteq R$ and hence $U_{L_1} \leq \text{Stab}_{U}[A]$.

Proof. This follows from (7.3) part (3) i).

8.6 Definition. Let $L_1$ be the set of all positions $(i, j) \in L_1$ such that $e_A x_{ij}(\alpha) (\alpha \in \mathbb{F}_q)$ change entries at main hook intersections. Thus $(i, j) \in L_1$ if and only if $i, j \in \mathfrak{p}_f$ and there exist $s, t \in \mathfrak{p}_f$ such that $n > s > t > i > j \geq 1$ and $(s, i), (t, j) \in \mathfrak{p}$.

8.7 Lemma. Let $(i, j) \in L_1$. Then $X_{ij} \leq \text{Stab}_{U} (\hat{e}_A)$. Moreover $(i, j) \in \hat{R} \setminus R = \hat{R}^{-} \setminus R$.

Proof. By definition of $L_1$ and (7.3) part (3) iii) we find $(s, t) \in L_2$ and $\beta \in \mathbb{F}_q^*$ such that $e_A x_{st}(\beta) = e_A x_{ij}(\alpha)$ for $\alpha \in \mathbb{F}_q^*$. Thus

$$e_A x_{ij}(\alpha) = e_A f x_{ij}(\alpha) = e_A x_{ij}(\alpha) f = e_A x_{st}(\beta) f = e_A f = e_A,$$

since $x_{st}(\beta) f = x_{st}(\beta) = f$. Thus $X_{ij} \leq \text{Stab}_{U} (\hat{e}_A)$.

Now $[A] x_{ij}(\alpha) = \theta_A x_{ij}(\alpha) [B] = [B] \in \mathcal{O}_A$, since $A_{ij} = 0$, where $B$ is obtained from $A$ by adding $-\alpha$ times column $j$ to column $i$ in $A$ and
projecting the resulting matrix into \( \text{Lie}(U) \). Obviously this is the same matrix \( B \) occurring in \( e_A x_{ij}(\alpha) = e_b \in O^J_A \), differing from \( A \) only on the main hook intersection \((t, i) \in J \). By 5.4 \((i, j) \in \tilde{R} \setminus R = \tilde{R}^1 \setminus R^0 \), as desired.

Now let \( J^0 \subseteq J \) be the set of all positions \((a, j) \) in the lower compartment of \( dA \) in column \( j \) which are in zero columns of \( A \) if \( j \not\in \mathfrak{g} \) or, there is a main condition \((b, j) \) above or on it. Thus \( J^0 = \{(a, j) \in J \mid j \not\in \mathfrak{p} \} \) or \( \exists (b, j) \in \mathfrak{p} \) with \( b \leq a \).

8.8 Lemma. Let \((a, j) \in J^0 \). Then \( X_{a,j} \subseteq \text{Pstab}_U(\hat{e}_A) \) and \( X_{a,j} \subseteq \text{Stab}_U(\hat{e}_A) \) if and only if \((a, j) \not\in \mathfrak{p} \). Moreover \((a, j) \in \mathfrak{R} \) and \((a, j) \in \mathfrak{R}^0 \) if and only if \((a, j) \not\in \mathfrak{p} \). Finally \( X_{a,j} \) acts on \( \hat{e}_A \) by the same linear character as on \([A] \in \mathcal{E} \).

**Proof.** Recall from the proof of 8.2 that \( \hat{e}_A = e_A f = q^{-|La|+|L^0|} \sum B e_B \), where \( B \) runs through the set of all matrices in \( V_J \) coinciding with \( A \) at positions in \( \mathfrak{p} \), having zero entries at all positions in zero columns of \( A \) and in columns of \( A \) with main conditions below those. Let \((a, j) \in J, \alpha \in \mathbb{F}_q \), then by 7.2 and 5.18

\[
e_A x_{aj}(\alpha) = \theta(\alpha B_{aj}) e_B = \begin{cases} e_B & \text{for } (a, j) \not\in \mathfrak{p} \\ \theta(\alpha A_{aj}) e_B & \text{for } (a, j) \in \mathfrak{p} \end{cases}
\]

for such matrices \( B \), since then \( B_{aj} \neq 0 \) only if \((a, j) \in \mathfrak{p} \) and then \( B_{aj} = A_{aj} \). Thus

\[
\hat{e}_A x_{aj}(\alpha) = \begin{cases} e_A & \text{for } (a, j) \not\in \mathfrak{p} \\ \theta(\alpha A_{aj}) e_A & \text{for } (a, j) \in \mathfrak{p} \end{cases}
\]

(8.9)

By 5.4 again \( X_{a,j} \in \mathfrak{R}^0 \), if \((a, j) \not\in \mathfrak{p} \) and if \((a, j) \in \mathfrak{p} \) then \([A] x_{aj}(\alpha) = \theta(\alpha A_{aj}) [A] \).

Let \( I = I(\mathfrak{s}) = \{(i, j) \in \Phi^- \mid i \not\in \mathfrak{g}, j \in \mathfrak{g} \} \) (compare 5.3 part 3)). Then \( I \) is closed in \( \Phi^- \) and \( \Phi^- \) is the disjoint union of \( K \) and \( I \). So \( U = U_K U_I = U_I U_K \) (but in general neither \( U_K \) nor \( U_I \) is normal in \( U \)). In general \( U_I \) does not act monomially on the idempotent basis \( O^J_A \) of \( \text{CO}^J_A \), but, as we shall show, \( U_I \) is contained in \( \text{Stab}_U(\hat{e}_A) \). To prove this recall \( M_\mathfrak{s} \cong \mathbb{C}E_J \) has \( \mathbb{C} \)-basis \( \{P_\lambda d| u \in U_J \}, d = d(\mathfrak{s}) \in D_\lambda, U_J = (U^-_\lambda)^d \cap U \), by 6.1 part 2) and 5.6 part 1). Clearly the image of \( \hat{e}_A = e_A f \) in \( M_\mathfrak{s} \) under the isomorphism \( f^*: \mathbb{C}E_J \rightarrow M_\mathfrak{s} \) in \( \mathfrak{s} \) part 5) is contained in \( M_\mathfrak{g} \) which is generated as \( \mathbb{C} \)-vector space by \( \{P_\lambda d u| u \in U_J \} \), where \( P_\lambda = \sum_{x \in P_\lambda} x \). We show for \( g \in U_I \), that \( P_\lambda d u g = P_\lambda d u f \), proving \( m \mid g = m \mid f, \forall m \in M_\mathfrak{s} \). From this follows in particular \( \hat{e}_A g = \hat{e}_A \), that is \( U_I \subseteq \text{Stab}_U(\hat{e}_A) \).

We first inspect matrices of the form \( du \) for \( u \in U_J, v \in U_{L_2} \). Recall that we may think of \( du \) as \( u \) with reordered rows, dividing \( u \) into two compartments, the rows of the lower one labelled by \( i_1 < \cdots < i_m \), those of the upper one by the numbers \( 1 \leq j \leq n \) not contained in \( \mathfrak{p}_x = \{i_1, \ldots, i_m\} \) in their natural order. Note that with these convention, the “last ones” of 4.10 coming from the diagonal ones in \( u \) are at position \((i, i) \in du \) for \( 1 \leq i \leq n \).

Let \( u \in U_{J_a}(a, b) \in L_2 \) and \( \beta \in \mathbb{F}_q \). Then \( dux_{ab}(\beta) \) is obtained from \( du \) by adding \( \beta \) times column \( a \) to column \( b \). Note that in \( du \) the only non-zero entries in columns \( b \) and \( a \) are the “last ones” at positions \((b, b) \) and \((a, a) \), since \( u \in U_J, J = \{(r, s) \in \Phi^- \mid r \not\in \mathfrak{g}, s \not\in \mathfrak{g} \} \). So the matrix \( dux_{ab}(\beta) \) coincides with \( du \) in all positions but positions \((a, b) \), on which the entry of \( dux_{ab}(\beta) \) is \( \beta \).
\[ \text{Let } \mathfrak{D}_{L_2} \text{ be the set of all matrices in } M_n(q) \text{ by placing arbitrary values from } \mathbb{F}_q \text{ in } du \text{ at positions } (a, b) \in L_2. \] We have shown:

8.10 Lemma. Keeping the notation introduced above we have

\[ du_j = q^{-|L_2|} \sum_{v \in U_{L_2}} dw = q^{-|L_2|} \sum_{y \in \mathfrak{D}_{L_2}} y. \]

□

Fix \( u \in U_J \) and define \( \mathfrak{D}_{L_2} \subseteq M_n(q) \) as in 8.10. Let \( y \in \mathfrak{D}_{L_2} \), \((i, j) \in I \) and \( \alpha \in \mathbb{F}_q \). Then we have:

8.11 Lemma. There exists \( z \in P_\lambda \) (depending on \( x_{ij}(\alpha) \)) such that \( zy_{ij}(\alpha) = \tilde{y} \in \mathfrak{D}_{L_2} \).

Proof. Since \((i, j) \in I \) we have \( i \notin s, j \in s \). Note that \( yx_{ij}(\alpha) \) is obtained from \( y \) by adding \( \alpha \) times column \( i \) to column \( j \) in \( y \). Note further, that the only non-zero entries of \( y \) besides the last ones are all in the lower compartment and on positions \((s, t)\) with \( s > t \) and \( s \in p_x \), i.e. the “last one” at position \((s, s)\) belongs to the lower compartment. So:

Thus \( yx_{ij}(\alpha) \) has \( \alpha \) at position \((i, j)\). Now since \( j < i \), \( X_{ij} \subseteq U^+ \subseteq P_\lambda \), hence \( x_{ji}(\alpha)y_{ij}(\alpha) \) is obtained from \( yx_{ij}(\alpha) \) by adding \(-\alpha\) times row \( j \) to row \( i \), removing entry \( \alpha \) at position \((i, j)\) again. This might introduce non-zero entries in row \( i \) to the left of position \((i, j)\) But those can be removed by row operations coming from multiplication from the left by element \( x_{si}(\gamma), \gamma \in \mathbb{F}_q \), where either \( s \notin s \) (so \( X_{si} \subseteq L_\lambda \)) or \( X_{si}(\sigma) \subseteq U'_x \), the unipotent radical of \( P_\lambda \), if \( s \in s, s < j \). Note that besides the last one at position \((i, i)\), column \( i \) of \( y \) has non-zero
entries only at positions \((b, i)\) with \(i < b \in p\), that is to the left of the last one at position \((b, b)\), therefore the resulting matrix \(\hat{y} = zyx_{ij}(\alpha), z \in \mathbb{P}_A\), differs from \(y\) only at positions in column \(j\) below position \((j, j)\), which all belong to \(L_2\). Then \(\hat{y} \in D_{L_2}\) again, as desired. □

8.13 Corollary. Let \(u \in U_j\) and let \(D_{L_2}\) be defined for \(u\) as above. Then \(P_\chi dufg = P_\chi du\) for all \(g \in U_j\).

Proof. Since \(U_j\) is generated by \(X_{ij}, (i, j) \in I\), we may assume \(g = x_{ij}(\alpha)\) for some \((i, j) \in I\) and \(\alpha \in \mathbb{F}_q\). By 8.11 we have for each \(y \in D_{L_2}\), \(P_\chi yx_{ij}(\alpha) = P_\chi z^{-1} \hat{y} = P_\chi \hat{y}\). Clearly \(y \to \hat{y}\) is a permutation of \(D_{L_2}\) and hence by 8.10

\[
P_\chi dufgx_{ij}(\alpha) = q^{-|L_2|} \sum_{y \in D_{L_2}} P_\chi yx_{ij}(\alpha) = q^{-|L_2|} \sum_{y \in D_{L_2}} P_\chi \hat{y} = P_\chi du.\]

Now our desired result follows immediately, observing that \(f^*(\hat{e}_A) \in M_S\):

8.14 Corollary. Let \(g \in U_I\). Then \(\hat{e}_A g = \hat{e}_A\) and hence \(g \in Stab_{\nu}(\hat{e}_A)\). □

Let \((i, j) \in I\), then \(i \notin \mathfrak{g}\) and \(j \in \mathfrak{g}\). In particular, since \(p \subseteq J = \{(a, b) \in \Phi^- | a \in \mathfrak{g}, b \notin \mathfrak{g}\}\), \((r, j) \notin p\) for all \(j < r \leq n\) and hence column \(j\) is a zero column in \(A\). By 6.3 \(I \subseteq R^0 \subseteq \mathcal{R}\). Indeed we have the following:

8.15 Lemma. \(\tilde{R} = L_2 \cup L_0^1 \cup L_1^1 \cup J^0 \cup I\) with \(\tilde{R} \setminus R = L_1^1, R = L_2 \cup L_0^1 \cup J^0 \cup I\) and \(R^0 = L_2 \cup L_1^0 \cup I \cup (J^0 \setminus p)\).

Proof. We have already seen in the previous results that the right hand sides are contained in the left hand sides of all equalities in the lemma. For \((s, i), (t, j) \in p\) with \(s > t > i > j\) there is a main hook intersection \((t, i) \in J\), since \(t \in \mathfrak{g}\) and \(i \notin \mathfrak{g}\). From this follows immediately that \(R \setminus \mathcal{R} = L_1^1\). Moreover since \(p \subseteq J^0\), it suffices to check \(R^0 \subseteq L_2 \cup L_1^0 \cup I \cup (J^0 \setminus p)\).

Let \((i, j) \in R^0\). If \(j \in \mathfrak{g}\), then \((i, j) \in I\) if \(i \notin \mathfrak{g}\). So let \(i \in \mathfrak{g}\). Then \((i, j) \in L_2\). Now suppose \(j \notin \mathfrak{g}\). If \(i \in \mathfrak{g}\), then \((i, j) \in J\). If \(j \notin \mathfrak{g}\), then \(j\) is a zero column in \(A\) and \((i, j) \in J^0\). If \(j \in \mathfrak{g}\) then there exists a main condition \((b, j) \in p\) with \(b < a\) by the definition of \(R^0\) and hence \((i, j) \in J^0\) by the definition of \(J^0\). Finally let \(i, j \notin \mathfrak{g}\). Then \((i, j) \in L_1\). If \(j \notin \mathfrak{g}\), then column \(j\) is a zero column in \(A\) and \((i, j) \in L_0^1\) by 7.3 part (3) i). If \(j \in \mathfrak{g}\), there is a main condition \((a, j) \in p\) in column \(j\) above \((i, j)\), that is \(a < i\) by the definition of \(R^0\) and hence \((i, j) \in L_0^1\) again by 7.3 part (3) i). □

Recall that \(p = \{(i_1, j_1), \ldots, (i_m, j_m)\}\) is completely hook disconnected. Thus we may apply the results in 6.3 to the \(U\)-orbit module \(\mathcal{O}_A = [A]CU\). Recall from 6.9 that there exists a linear character \(\psi_A\) whose restriction to \(U_{R_-} \subseteq U_{\mathfrak{g}}\) is trivial and which is \(\theta_A\) defined in 8.4 on \(U_{\mathfrak{g}}/U_{R_-} \cong X_{i_1j_1} \times \cdots \times X_{i_mj_m}\). Note that by 8.8 and 8.15 this is precisely the linear character of \(U_{\mathfrak{g}}\) afforded by \(C\hat{e}_A\). Let \(e_A \in \text{CU}_{\mathfrak{g}}\) the primitive idempotent such that \(C\hat{e}_A\) affords \(\psi_A\) and set \([A] = [A]e_A \in \mathcal{O}_A\). Then \(S = [A]CU \leq \mathcal{O}_A\) is the induced \(CU\)-module \(\text{Ind}_{U_{\mathfrak{g}}}^{U_{\mathfrak{g}}} [A]\) and is irreducible. Since \(C\hat{e}_A \cong [A]C\hat{e}_A\) as \(CU\)-modules, we conclude that \(\text{Ind}_{U_{\mathfrak{g}}}^{U_{\mathfrak{g}}} [A]C\hat{e}_A\) is irreducible too and hence the map \(\mu\) in 8.8

\[
\mu : \text{Ind}_{U_{\mathfrak{g}}}^{U_{\mathfrak{g}}} C\hat{e}_A \to \hat{e}_A C\hat{e}_A = \mathcal{O}_A^I
\]

is an \(CU\)-module isomorphism. Thus we have shown:
8.17 Theorem. Let $\lambda = (n-m,m)$ be a composition of $n$, $s \in RStd(\lambda)$, $J = J(s)$ and let $e_s \in \mathcal{E}_J$ be a verge with $p = \text{supp}(A)$. Then $\mathcal{C}O_A^{J} = e_A \mathcal{C}U$ is an irreducible $\mathcal{C}U$-module isomorphic to the unique irreducible constituent of $[A] \mathcal{C}U = \mathcal{C}O_A$ corresponding to the trivial representation of $U_{\Phi^-}$ extended to $U_{\Phi}$ by the linear character $\theta_A$ of $X_{(i,j) \in p} X_{ij}$. Conversely, if $p \subseteq \Phi^-$ is a set of completely hook disconnected main conditions, $A \in M_n(q)$ with $\text{supp}(A) = p$, then the unique irreducible constituent of $\mathcal{C}O_A$ described above is isomorphic to $\mathcal{C}O_A^{J}$, where $\lambda = (n-|p|,|p|)$. □

There are several sequences of 8.17:

8.18 Consequences. 1) Let $\lambda, \mu$ be 2-part partitions of $n$ and let $s \in RStd(\lambda), t \in RStd(\mu)$. Let $e_s \in \mathcal{E}_{J(s)}$ and $e_t \in \mathcal{E}_{J(t)}$ be verge idempotents. Then $\mathcal{C}O_A^{J(s)} \cong \mathcal{C}O_B^{J(t)}$ if and only if $A = B$.

2) Let $\lambda = (n-m,m) \models n$ and let $s \in RStd(\lambda)$. Then $M_s$ is multiplicity free, its irreducible constituents being of the form $\mathcal{C}O_A^{J}$, $J = J(s)$, where $A \in V_J$ satisfies: $p = \text{supp}(A)$ is a main condition set with $p_r \subseteq \bar{\lambda}, p_r \cap \bar{\lambda} = \emptyset$. By [8, 2.5.10] $\text{dim}_C(\mathcal{C}O_A^{J}) = |\mathcal{O}_A^{J}|$ depends only on $p$, not on $A \in V_J$ with $\text{supp}(A) = p$, and hence there are $(q-1)^{|p|}$ many orbits $\mathcal{O}_A^{J} \subseteq \mathcal{E}_J$ with $\text{supp}(A) = p$ for a given completely hook disconnected main condition set fitting $s \in RStd(\lambda)$. As consequence, the number of irreducible constituents of $M_s$ of fixed dimension $q^s$ is a polynomial in $(q-1)$ with integral, non-negative coefficients independent of $q$.

3) Each permutation representation of $GL_n(q)$ on the cosets of a maximal parabolic subgroup is isomorphic to some $M(\lambda)$ of some partition $\lambda = (n-m,m)$ of $n$. Let $\mathfrak{M}_\lambda$ be the set of completely hook disconnected main condition sets $p$, which fit at least one row standard $\lambda$-tableaux and for $p \in \mathfrak{M}_\lambda$, let $k_{p,\lambda}$ be the number of distinct $s \in RStd(\lambda)$ such that $p$ fits $s$. Then $k_{p,\lambda}$ is independent of $q$ and

$$\text{Res}^{GL_n(q)}_U M(\lambda) = \bigoplus_{p \in \mathfrak{M}_\lambda} \bigoplus_{A \in M_n(q)} ([A] e_A \mathcal{C}U)^{k_{p,\lambda}}.$$

References

[1] C.A.M. ANDRÉ, “Basic characters of the unitriangular group” , J. Algebra, 175, 287-319, (1995).

[2] R. W. CARTER, “Simple groups of Lie type”, John Wiley & Sons, London, New York, Sydney, Toronto, (1972).

[3] M. BRANDT, “On unipotent Specht modules of general linear groups”, PhD Thesis, Universität Stuttgart, (2004).

[4] M. BRANDT, R. DIPPER, G. JAMES AND S. LYLE, “Rank polynomials”, Proc. London Math. Soc., (3). 98, 1-18, (2009).

[5] P. DIACONIS AND I. M. ISSAC, “Supercharacters and superclasses for algebra groups”, Trans. Amer. Math. Soc., 360(5), 2359–2392, (2008).

[6] R. DIPPER AND Q. GUO, “Irreducible constituents of minimal degree in supercharacters of the finite unitriangular groups”, J. Pure Appl. Algebra (2014), http://dx.doi.org/10.1016/j.jpaa.2014.09.016
[7] R. Dipper and G. James, “On Specht modules for general linear groups”, J. Algebra, 275, 106–142, (2004).

[8] Q. Guo, “On the $U$-module structure of the unipotent Specht modules of finite general linear groups, preprint, [arXiv:1304.4370v2].

[9] G. James, “The representation theory of the symmetric groups”, Lecture Notes in Math., 308, Springer-Verlag, Berlin and New York, (1973).

[10] G. James, “Representations of general linear groups”, LMS Lecture Notes, 94, (1984).

[11] M. Jedlitschky, “Decomposing André-Neto supercharacters for Sylow $p$-subgroups of type $D$”, PhD. Thesis, Universität Stuttgart, (2013).

[12] Tung Le, “Supercharacters and pattern subgroups in the upper triangular groups”, Proceedings of the Edinburgh Mathematical Society, 56, 177-186, (2013).

[13] A. A. Kirillov, “Lectures on the orbit method”, AMS, 64, (2004).

[14] N. Yan, “Representations of finite unipotent linear groups by the method of Clusters”, [arXiv:1004.2674v1], (2010).