Weighted Power Lomax Distribution and its Length Biased Version: Properties and Estimation Based on Censored Samples

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Abstract

In this paper, a weighted version of the power Lomax distribution referred to the weighted power Lomax distribution is introduced. The new distribution comprises the length biased and the area biased of the power Lomax distribution as new models as well as contains an existing model as the length biased Lomax distribution as special model. Essential distributional properties of the weighted power Lomax distribution are studied. Maximum likelihood and maximum product spacing methods are proposed for estimating the population parameters in cases of complete and type-II censored samples. Asymptotic confidence intervals of the model parameters are obtained. A sample generation algorithm along with a Monte Carlo simulation study is provided to demonstrate the pattern of the estimates for different sample sizes. Finally, a real-life data set is analyzed as an illustration and its length biased distribution is compared with some other lifetime distributions.

Key Words: Power Lomax distribution; Maximum product spacing; Approximate confidence intervals; Coverage probability; Monte Carlo simulation

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1. Introduction

Weighted distributions (WDs) are handled in studies associated with reliability, biomedicine, meta-analysis, econometrics, survival analysis, renewal processes, physics, ecology and branching processes which are found in Zelen and Feinleib (1969), Patil and Ord (1976), Patil and Rao (1978), Gupta and Keating (1986), Oluyede (1999) and references therein. In fact, these distributions arise in practice when observations from a sample are recorded with unequal probabilities.

Suppose that $T$ is a nonnegative random variable with the probability density function (pdf) $f_w(t)$. The pdf of the weighted random variable $T$ is defined by:

$$f_w(t) = \frac{\omega(t) g(t)}{E(\omega(T))}, \quad t > 0,$$

(1)

where $\omega(t)$ is a nonnegative weight function and $E(\omega(T)) > 0$. Different choices of $\omega(t)$ give different WDs, that is, for $\omega(t) = t^s, s > 0$, the pdf in (1) is called as WD of order $s$. Also, for $s=1$ or $s=2$, the pdf (1) is called as the length-biased (size-biased) and the area-biased distributions, respectively.

The Lomax (Lo) distribution is an important model in lifetime analysis. It has been widely applied in a variety of contexts; analysis of income and wealth data (Harris (1968)), modelling business failure data (Atkinson and Harrison (1978)) and, biological sciences (Holland et al. (2006)), model firm size and queuing
problems (Corbellini et al. (2010)) and reliability modelling and life testing (Hassan and Al-Ghamdi (2009) and Hassan et al. (2016)).

In recent times, many generalizations and extensions of the Lo distribution have been provided by many authors, including exponentiated Lo distribution (Abdul-Moniem and Abdel-Hameed (2012)), beta Lo distribution, Kumaraswamy Lo distribution and McDonald Lo distribution (Lemonte and Cordeiro (2013)), gamma-Lo distribution (Cordeiro et al. (2013)), Weibull Lo (WLo) distribution (Tahir et al. (2015)), Gumbel-Lo distribution (Tahir et al. (2016)), power Lo distribution (Rady et al. (2016)), exponentiated Lomax geometric distribution (Hassan and Abdelghafar (2017)), power Lo Poisson distribution (Hassan and Nassr (2018)), exponentiated Weibull-Lo distribution (Hassan and Abd-Allah (2018)), inverse power Lo distribution (Hassan and Abd-Allah (2019)), inverse exponentiated Lo distribution (Hassan and Mohamed (2019a)), Weibull inverse Lo distribution (Hassan and Mohamed (2019b)), type II half logistic Lo distribution (Hassan et al. (2020a), Zubair Lomax (Bantan et al. (2020))) and truncated power Lomax (Hassan et al. (2020b)) among others.

The power Lo (PLO) distribution has been proposed by Rady et al. (2016) as a new extension of the Lo distribution with an extra shape parameter. The pdf of the PLO distribution with shape parameters $a$, $b$ and scale parameter $c$ is defined by

$$g(t) = abc^s t^{b-1} (c + t^b)^{-a-1}, \quad t, a, b, c > 0. \tag{2}$$

The cumulative distribution function (cdf) of the PLO distribution is as follows:

$$G(t) = 1 - c^s (c + y^b)^{-a}. \tag{3}$$

The $s^{th}$ moment corresponding to (2) is given by

$$\mu'_s = a c^b B\left(1, (s/b), 1+(s/b)\right) = a c^b D_s, \tag{4}$$

where $D_s = B\left(1- (s/b), 1+ (s/b)\right)$, $s = 1, 2, 3, ...$ and $B(., .)$ is the beta function.

In view of the importance of the PLO distribution as well as the idea of the WD, we introduce a weighted version of the PLO distribution called the weighted PLO (WPLO) distribution. The WPLO distribution can (i) be viewed as an alternate model to some new extended forms of the Lo and generalized exponential distributions. (ii) hold both inverted bathtub and decreasing hazard rate and (iii) have wider applications in some areas. We discuss the estimation of the population parameter via maximum likelihood (ML) and maximum product spacing (MPS) methods in the case of complete and type II censoring (TIIC) samples. Application to real data for its length biased version is considered.

The rest of the article is organized as follows. The weighted version for PLO distribution is described in Section 2. Section 3 gives moments and related measures, entropy measure and stochastic ordering for WPLO distribution. Section 4 deals with the point and approximate confidence interval (CI) of the model parameters based on the ML and MPS procedures. A simulation study is presented in Section 5. Real data illustration is described in Section 6 for studying the application of the length biased PLO (LBPLO) distribution. In the end, concluding remarks are implemented.

2. Weighted Power Lomax Distribution

Here, we obtain the WPLO distribution by considering the weight function $o(t) = t^s$, and using (2) by substituting them in (1), as appear in the following definition

**Definition 1**: A nonnegative continuous random variable, $T$, is said to follow the WPLO distribution with parameters $a$, $b$, and $s$ if its pdf is of the form:

$$f_w(t) = abc^s b D_s^{-1} t^{s+b-1} (c + t^b)^{-a-1}, \quad t, a, b, c > 0, \quad s = 1, 2, ..., \tag{5}$$

where $a$ and $b$ are shape parameters and $D_s = B\left(1- (s/b), 1+ (s/b)\right)$, $s = 1, 2, 3, ...$ A random variable with pdf (5) will be denoted by $T \sim (a, b, c, s)$. The associated cdf of the WPLO distribution is given by:

$$F_w(t) = D_s^{-1} \gamma \left(1- (s/b), 1+ (s/b), b t^s / (c + t^b)\right), \quad t, a, b, c > 0, \quad s = 1, 2, ..., \tag{6}$$

where $\gamma \left(1- (s/b), 1+ (s/b), b t^s / (c + t^b)\right) = \int_0^{t^s / (c + t^b)} (1-y)^{(s/b)-1} y^1 dy$, is the incomplete beta function.

Some special sub-models can be obtained from (5) as follows

i. For, $s = 1$ in (5), we get LBPLo distribution as a new model.

ii. For, $s = 2$ in (5), we get area biased PLO (ABPLO) distribution as a new model.

iii. For, $s = 1, b = 1$ in (5), we get length biased Lo (LBLO) distribution (Ahmad et al. (2016)).

iv. For, $s = 2, b = 1$ in (5), we get area biased Lo (ABLO) distribution as a new model.
The survival function \( sf \) of the WPLo distribution is then,
\[
F_w(t) = 1 - D^{-1}_s(a - (s/b), 1 + (s/b), t^b/(c + t^b)).
\]

The hazard rate function \( hrf \) is
\[
h_w(t) = \frac{bc^{-s}D^{-1}_s(s+s+b-1(c+t^b)^{-a-1}}{1 - D^{-1}_s(a - (s/b), 1 + (s/b), t^b/(c + t^b))}.
\]

Fig. 1, Fig. 2 and Fig. 3 show the pdfs and hrfs for WPLo distribution for some choices values of \( a, b, c \) and \( s \).
From the above figures, we conclude that the pdf of the WPLo distribution and their particular cases (LBPLo and ABPLo) can take different shapes according to some values of $a$, $b$, $c$ and $s$. Also, their hrf can be increasing, decreasing, reversed J-shaped, and up-side down shapes.

3. Main Properties

In this section, some structural properties of the WPLo distribution are discussed.

3.1 Moments and Associated Measures

The moments of a random variable are necessary in statistical analysis, especially in applied works. The $r^{th}$ moment, $\mu'_r$, for $r=1,2,\ldots$ of the WPLo distribution is obtained directly by using the pdf (5), hence, we have

$$\mu'_r = c^r \beta_{r+1} \left\{ \frac{r+s}{b} + 1, a - \left( \frac{r+s}{b} \right) \right\}, \quad ab-s > r, \quad r=1,2,\ldots$$

Similarly, the $r^{th}$ central moment of a given random variable $T$, can be defined as

$$\mu_r = E(T - \mu_1)^r = \sum_{i=0}^{r} (-1)^i \left( \begin{array}{c} r \\vdash \end{array} \right) (\mu'_1)^i \mu''_{r-i}.$$  

The coefficient of skewness (SK) and coefficient of kurtosis (KU) are defined by

$$SK = \frac{\mu'_3}{\mu'_2^{3/2}}, \quad KU = \frac{\mu'_4}{\mu'_2^{2}}.$$  

Thus, numerical values of the $\mu'_1$, variance ($\sigma^2$), coefficient of variation (CV), SK and KU of the WPLo distribution for some certain values of parameters are listed in Table 1.

| Parameters | $\mu'_1$ | $\sigma^2$ | CV   | SK    | KU   |
|------------|--------|-----------|------|-------|------|
| $a=2$, $b=3$, $c=2$, $s=1$ | 1.260  | 0.382     | 0.490 | 2.362 | 24.949 |
| $a=3$, $b=3$, $c=2$, $s=2$ | 1.172  | 0.213     | 0.394 | 1.411 | 8.562  |
| $a=4$, $b=3$, $c=0.5$, $s=1$ | 0.556  | 0.043     | 0.373 | 0.771 | 4.484  |
| $a=4$, $b=3$, $c=0.5$, $s=2$ | 0.633  | 0.049     | 0.351 | 0.899 | 5.084  |
| $a=3$, $b=4$, $c=0.5$, $s=1$ | 0.68   | 0.043     | 0.306 | 0.645 | 4.351  |
| $a=5$, $b=6$, $c=3$, $s=3$ | 0.974  | 0.029     | 0.174 | 0.051 | 3.242  |
| $a=6$, $b=5$, $c=3$, $s=3$ | 0.949  | 0.035     | 0.197 | 0.110 | 3.197  |
| $a=7$, $b=3$, $c=2$, $s=3$ | 0.858  | 0.062     | 0.290 | 0.510 | 3.633  |

Table 1 shows that the LBPLo and ABPLo distributions are positively skewed and leptokurtic for selected values of parameters.

Further, a simple formula for the $p^{th}$ incomplete moment of $T$, say $\mu'_p(y) = E(T^p \mid T < y)$, is obtained as follows:

$$\mu'_p(y) = c^p \beta_{p+1} \left\{ \frac{p+s}{b} + 1, a - \left( \frac{p+s}{b} \right) b^{y/c+b} \right\}, \quad p=1,2,\ldots,$$

where $\beta(\ldots, y/(c+y))$ is the incomplete beta function. For $p=1$, we get the first incomplete moment. The essential applications of the first incomplete moment are the Lorenz and Bonferroni curves.

3.2 Residual and Reversed Life Functions

The residual life plays vital role in many situations like life testing and reliability theory. The $m^{th}$ moment of the residual life (RL) is defined by:

$$\zeta_m(x) = E[(T-x)^m \mid T > x] = \frac{1}{F(x)} \int_x^\infty (t-x)^m f(t) dt.$$  

Employing the binomial expansion and pdf, $f_a(t)$, then $\zeta_m(x)$ can be written as follows:

$$\zeta_m(x) = \frac{1}{F_a(x)} \sum_{j=0}^{m} \frac{m}{j} (-x)^{m-j} \int_x^\infty \left( \frac{b}{b} \right)^{j+1} (c+t)^{-(a+j)} dt.$$  

So, after simplification the $m^{th}$ moment of the RL of the WPLo distribution is obtained as follows:

$$
\xi_n(x) = \frac{1}{F_n(x)} \sum_{j=0}^{m} \left( \begin{array}{c} m \\ j \end{array} \right) (-x)^{m-j} \cdot \frac{j}{b} \cdot D_n^{-1}(\rho) \left( \frac{j+s}{b} + 1, a - \left( \frac{c+\lambda^b}{c+x^b} \right) \right)
$$

where $\rho = \left( \frac{1}{c+x^b} \right)$ is the incomplete beta function and $F_n(x)$ is the sf of the WPLo distribution. For $m = 1$, we get the mean RL of the WPLo distribution which represents the expected additional life length for a unit which is alive at age $x$.

The $n^{th}$ moment of the reversed RL, say $\xi_n(x) = E[(T-x)^n | T \leq x]$, for $x > 0$ and $n = 1, 2, ...$ uniquely determines $F(t)$. Therefore, the $n^{th}$ moment of the reversed RL of $T$ is defined by

$$
\xi_n(x) = \frac{1}{F(x)} \int_0^x (t-x)^n f(t) dt.
$$

Using pdf (5), the $n^{th}$ moment of the reversed RL of the WPLo distribution is as follows

$$
\xi_n(x) = \frac{1}{F_n(x)} \sum_{j=0}^{n} \left( \begin{array}{c} n \\ j \end{array} \right) (-x)^{n-j} \cdot \frac{j}{b} \cdot D_n^{-1}(\rho) \left( \frac{j+s}{b} + 1, a - \left( \frac{c+\lambda^b}{c+x^b} \right) \right).
$$

For $n = 1$, we get the mean waiting time also called the mean reversed RL which represent the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0,x)$.

### 3.3 Quantile function

The $q^{th}$ quantile, $t_q$ (also called the percentile of order $q$) of the WPLo distribution can be obtained from (6) as follows

$$
F(t_q) = D_n^{-1}(\rho) \left( a - (s/b), 1 + (s/b), \lambda^b / (c + t^b) \right) - q = 0.
$$

It is a complex equation so by using iteration technique as a Newton-Raphson we obtain the quartiles. Further, from (7), the values of $t_q$ for $q \sim$ uniform $(0,1)$ provides the random values generated from the WPLo distribution.

### 3.4 Rényi Entropy

For a certain random phenomenon under study, it is important to quantify the uncertainty associated with the random variable of interest. One of the most famous measures used to quantify the variability of random variable is the Rényi entropy. It is defined for $\lambda > 0$ and $\lambda \neq 1$, as follows

$$
E_\lambda(T) = \frac{1}{1-\lambda} \log \int_0^\infty f(t)^\lambda dt.
$$

Using pdf (5), the Rényi entropy of the WPLo distribution can be written as follows

$$
E_\lambda(T) = \frac{1}{1-\lambda} \log \int_0^\infty b^\lambda c^{\lambda(a/b)} D_n^{-1}(\lambda^b a/b) \cdot D_n^{-1}(\lambda^b b/c) \cdot D_n^{-1}(\lambda^b) \cdot D_n^{-1}(\lambda^b c) dt.
$$

After simplification, the Rényi entropy of the WPLo distribution is given by

$$
E_\lambda(T) = \frac{1}{1-\lambda} \log \left[ b^{\lambda-1} c^{\lambda-1} D_n^{-1}(\lambda^b a/b) \cdot D_n^{-1}(\lambda^b b/c) \cdot D_n^{-1}(\lambda^b) \cdot D_n^{-1}(\lambda^b c) \right].
$$

### 3.5 Stochastic ordering

Shaked and Shanthikumar (2007) mentioned that, for independent random variables $T$ and $W$ with cdfs $F_T$ and $F_W$ respectively, $T$ is said to be smaller than $W$ if the following ordering holds:

- **Stochastic order** ($T \leq_s W$) if $F_T(t) \geq F_W(t)$ for all $t$.
- **Likelihood ratio order** ($T \leq_l W$) if $f_T(t)/f_W(t)$ is decreasing in $t$.
- **Hazard rate order** ($T \leq h W$) if $h_T(t) \geq h_W(t)$ for all $t$.
- **Mean residual life order** ($X \leq_{mrl} Y$) if $m_T(t) \geq m_W(t)$ for all $t$.

We have the following chain of implications among the various partial orderings discussed above:
Let \[ T \leq_{fr} W \Rightarrow T \]
\[ \downarrow \]
\[ T \leq_{sr} W \]

**Theorem 1:** Let \( T \sim \text{WPLo distribution} (a_1, b_1, c_1, s_1) \) and \( W \sim \text{WPLo distribution} (a_2, b_2, c_2, s_2) \) If, \( a_1 \geq a_2, b_1 \geq b_2, c_1 \geq c_2, \) and \( s_1 \geq s_2, \) then \( T \leq_{fr} W, T \leq_{bs} W, T \leq_{ml} W, \) and \( T \leq_{sr} W. \)

**Proof**
It is sufficient to show \( f_T(t)/f_W(t) \) is a decreasing function of \( t; \) therefore,
\[
\frac{d}{dt} \log \frac{f_T(t)}{f_W(t)} = \frac{(a_1 + 1)b_1 t^{b_1 - 1} - (a_2 + 1)b_2 t^{b_2 - 1}}{c_1 + t^{b_1}} \leq \frac{(s_1 + b_1 - 1)}{t} - \frac{(s_2 + b_2 - 1)}{t} + \frac{(a_2 + 1)b_2 t^{b_2 - 1}}{c_2 + t^{b_2}}.
\]

Now if \( a_1 \geq a_2, b_1 \geq b_2, c_1 \geq c_2, \) and \( s_1 \geq s_2, \) then \( \frac{d}{dt} \log \frac{f_T(t)}{f_W(t)} \leq 0, \) which implies that \( W \) is stochastically greater than \( T \) with respect to likelihood ratio order i.e., \( T \leq_{fr} W. \) Similarly, we can conclude for \( T \leq_{bs} W, T \leq_{ml} W, \) and \( T \leq_{sr} W. \)

### 4. Parameter Estimation

This section provides the ML and MPS estimators of the population parameters for WPLo distribution via complete and TIIC. Further, the asymptotic CI for model parameters is given.

#### 4.1. ML Estimators

The ML population parameter estimators for the WPLo distribution are derived in case of TIIC and complete samples. Let \( T_1, T_2, \ldots, T_n \) be independent and identically WPLo distribution random variables representing the lifetimes of \( n \) independent units. In TIIC case, only the first prefixed \( k \) \((k \leq n)\) failures, say \( T_{i(1)}, T_{i(2)}, \ldots, T_{i(k)} \) are observed. These failures correspond to the first \( k \) order statistics of the random sample \( T_1, T_2, \ldots, T_n. \) The log-likelihood function, denoted by \( \ln \ell, \) for the WPLo distribution based on TIIC, is obtained as follows:

\[
\ln \ell = C_1 + k \ln \left( \Gamma(a + 1) \right) - k \ln \left( \Gamma \left( a - \frac{s}{b} \right) \right) - k \ln \left( \Gamma \left( 1 + \frac{s}{b} \right) \right) + k(a - \frac{s}{b}) \ln c + k \ln b + (s + b - 1) \sum_{i=1}^{n-k} \ln(t_i) - (a + 1) \sum_{i=1}^{k} \ln(c + t_i^b) + (n-k) \left( \Xi(a,b,c,s) \right),
\]

where \( C_1 = \frac{n!}{n-k}! \), \( \Xi(a,b,c,s) = \ln \left( 1 - D_{x}^{-1} \gamma \left( a - \frac{s}{b}, 1 + \frac{s}{b}, t^b \right) \right), \) and for simplicity we write \( t_i \) instead of \( t_{i(i)}. \) The partial derivatives with respect to \( a, b, c \) and \( s \) are obtained as follows:

\[
\frac{\partial \ln \ell}{\partial a} = k \Psi(a) - k \Psi(a - \frac{s}{b}) + k \ln c - \sum_{i=1}^{k} \ln(c + t_i^b) + \frac{(n-k) \Xi(a,b,c,s)}{\partial a},
\]

\[
\frac{\partial \ln \ell}{\partial b} = \frac{ks}{b^2} \Psi(1 - \frac{s}{b}) - \frac{ks}{b^2} \Psi(a - \frac{s}{b}) + \frac{k}{b} \ln c + \frac{k}{b} \ln(t_i) - \sum_{i=1}^{k} \frac{(a + 1)t_i^b \ln t_i}{(c + t_i^b)} + \frac{(n-k) \Xi(a,b,c,s)}{\partial b},
\]

\[
\frac{\partial \ln \ell}{\partial c} = \frac{k}{c} \left( a - \frac{s}{b} \right) - (a + 1) \sum_{i=1}^{k} \frac{1}{c + t_i^b} + \frac{(n-k) \Xi(a,b,c,s)}{\partial c},
\]

and,

\[
\frac{\partial \ln \ell}{\partial s} = \frac{k}{b} \Psi(a - \frac{s}{b}) - \frac{k}{b} \Psi(1 - \frac{s}{b}) - \frac{k}{b} \ln c + \sum_{i=1}^{k} \ln(t_i) + \frac{(n-k) \Xi(a,b,c,s)}{\partial s},
\]

where \( \Psi(e) = \Gamma'(e)/\Gamma(e) \) is the digamma function. The ML estimators of the unknown parameters of the WPLo distribution can be obtained by solving the following non-linear equations: \( \partial \ln \ell/\partial a = 0, \partial \ln \ell/\partial b = 0, \partial \ln \ell/\partial c = 0, \) and \( \partial \ln \ell/\partial s = 0. \) Unfortunately these equations cannot be solved analytically, so numerical technique is employed. Additionally, the ML estimators of parameters are obtained by solving the non-linear Equations (8)-(11) for \( k = n \) in case of complete sample.
4.2 Maximum Product Spacing Estimator

The MPS technique was presented by Cheng and Amin (1979) and independently notable with Ranneby (1984) as an alternative method to ML estimation technique for continuous distributions. The product spacing’s under the considered TIIC scheme can be written as

\[ D(a,b,c,s) = \sum_{i=1}^{k-1} \left( F(x_{(i)}) - F(x_{(i+1)}) \right)^{a-1}, \]

where \( F(x_{(0)}) = 0 \) and \( F(x_{(k+1)}) = 1 \). The MPS estimators can be obtained from maximizing \( D(a,b,c,s) \) with respect to \( a, b, c \) and \( s \) subject to the constraint \( ab > s \). Further, under complete sample, the MPS estimates are obtained by maximizing (12) for \( k = n \).

4.3 Approximate Confidence Interval

It is known that under regularity condition that the asymptotic distribution of ML estimators of elements of unknown parameters \( a, b, c \) and \( s \) is given by

\[ (\hat{a} - a), (\hat{b} - b), (\hat{c} - c), (\hat{s} - s) \rightarrow N(0, I^{-1}(a,b,c,s)), \]

where \( I^{-1}(a,b,c,s) \) is the variance covariance matrix of population parameters; \( a, b, c \) and \( s \). The elements of Fisher information matrix are obtained for complete and TIIC scheme. Therefore, the two-sided approximate \( 100 \) percent limits for the ML estimators of population parameters for \( a, b, c \) and \( s \) are obtained, respectively, as follows:

\[ L_a = \hat{a} - Z_{a/2} \sqrt{\text{var}(\hat{a})}, \quad U_a = \hat{a} + Z_{a/2} \sqrt{\text{var}(\hat{a})}, \]

\[ L_b = \hat{b} - Z_{b/2} \sqrt{\text{var}(\hat{b})}, \quad U_b = \hat{b} + Z_{b/2} \sqrt{\text{var}(\hat{b})}, \]

\[ L_c = \hat{c} - Z_{c/2} \sqrt{\text{var}(\hat{c})}, \quad U_c = \hat{c} + Z_{c/2} \sqrt{\text{var}(\hat{c})}, \]

\[ L_s = \hat{s} - Z_{s/2} \sqrt{\text{var}(\hat{s})}, \quad U_s = \hat{s} + Z_{s/2} \sqrt{\text{var}(\hat{s})}, \]

where \( Z \) is the \( 100(1-p/2)^{th} \) standard normal percentile and \( \text{var(.)}'s \) denote the diagonal elements of variance covariance matrix corresponding to the model parameters.

5. Simulation Study

Here, we give up with a numerical study to assess the attitude of the ML and MPS estimates of the WPLo distribution and their length biased version based on complete sample and TIIC scheme. The algorithm used here is done via R package and the steps are summarized as follows:

- 1000 random sample of sizes \( n = 50, 100 \) and 200 are generated from the WPLo distribution by solving numerically Equation (7) under complete and TIIC.
- The number of failure items; \( k \), based on TIIC are selected as 60\%, 80\% and 100\% (complete sample).
- Exact values of parameters are chosen as; Case 1 = \((a = 2.5, b = 1.5, c = 0.75, s = 1)\), Case 2 = \((a = 2.5, b = 1.5, c = 0.75, s = 2)\), Case 3 = \((a = 3, b = 0.9, c = 3, s = 1)\) and Case 4 = \((a = 3, b = 2, c = 0.5, s = 3)\).
- The ML estimates of the model parameters are obtained by solving the non-linear Equations (8)-(11) based on complete \((k=n)\) and TIIC scheme. Also, the MPS estimates of the population parameters are obtained by maximizing Equation (12) with respect to \( a, b, c \) and \( s \).
- The average length (AL) of CIs with confidence level 0.95 for all samples sizes and the corresponding coverage probability (CP) are computed.
- The absolute bias (AB), mean square errors (MSE), AL and CP at \( \alpha = 0.05 \) of all estimates are calculated.
- The numerical outcomes of the simulated data are listed in Tables 2, 3, 4 and 5.

| Table 2: AB, MSE, AL and CP of ML and MPS estimates under complete and TIIC in Case 1 |
|----------------------------------|------------------|----------------|----------------|----------------|------------------|------------------|------------------|
| \( n \) | \( k \) | ML | | | | MPS | | |
| | | AB | MSE | AL | CP | AB | MSE | AL | CP |
| 1 | 50 | | | | | | | |
| a | 0.0722 | 0.3158 | 0.5546 | 95.4 | 0.0952 | 0.2829 | 0.5509 | 97.5 |
| b | 0.2156 | 0.2459 | 1.1565 | 93.7 | 0.0226 | 0.1254 | 1.3857 | 96.1 |
| c | 0.2115 | 0.1042 | 0.4331 | 95.1 | 0.2268 | 0.1401 | 0.4320 | 94.7 |
| 0.8 | a | 0.6764 | 0.4902 | 0.7083 | 94.9 | 0.1045 | 0.4836 | 0.75619 | 96.5 |
| b | 0.9111 | 1.0028 | 1.6301 | 94.9 | 0.1473 | 0.2662 | 1.9392 | 95.6 |
| c | 0.3260 | 0.1258 | 0.5476 | 95.4 | 0.3141 | 0.4834 | 0.5433 | 94.7 |
| 0.6 | a | 0.8293 | 1.1887 | 0.7954 | 95.0 | 0.1075 | 0.7800 | 0.7438 | 95.9 |
\[ a = 2.5, \ b = 1.5, \ c = 0.75, \ s = 1 \]

| \( n \) | \( k \) | \( a \) | \( b \) | \( c \) | \( s \) | \( AB \) | \( MSE \) | \( AL \) | \( CP \) | \( AB \) | \( MSE \) | \( AL \) | \( CP \) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 100 | 0.8 | b  | 0.9309 | 3.7500 | 95.3 | 0.1005 | 0.341 | 0.249 | 0.599 | 0.798 | 0.341 | 0.249 |
| 200 | 0.8 | c  | 0.8901 | 3.7500 | 95.3 | 0.1005 | 0.341 | 0.249 | 0.599 | 0.798 | 0.341 | 0.249 |

**Table 3:** AB, MSE, AL and CP of ML and MPS estimates under complete and TIIIC in Case 2

\[ a = 2.5, \ b = 1.5, \ c = 0.75, \ s = 2 \]

| \( n \) | \( k \) | Bias | MSE | AL | CP | Bias | MSE | AL | CP |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1   | a   | 0.4396 | 0.3901 | 3.7000 | 95.9 | 0.0555 | 0.3403 | 2.4814 | 95.9 |
| 50  | b   | 0.1226 | 0.0818 | 2.7563 | 95.0 | 0.0862 | 0.0718 | 1.6102 | 93.9 |
| 1   | c   | 0.4860 | 0.2578 | 1.5760 | 95.6 | 0.1791 | 0.1515 | 1.3723 | 95.6 |
| 100 | a   | 0.5763 | 0.4107 | 2.9457 | 94.2 | 0.2311 | 0.4092 | 2.5978 | 98.2 |
| 200 | b   | 0.1371 | 0.0850 | 2.2317 | 94.3 | 0.2376 | 0.1353 | 2.1552 | 93.8 |
| 1   | c   | 0.6107 | 0.3751 | 1.3952 | 94.7 | 0.1688 | 0.1532 | 1.3851 | 95.5 |
| 1   | a   | 0.2875 | 0.5018 | 2.1985 | 94.2 | 0.2151 | 0.5043 | 2.4171 | 96.6 |
| 50  | b   | 0.2943 | 0.1635 | 2.3300 | 95.7 | 0.2242 | 0.1359 | 2.1792 | 94.7 |
| 1   | c   | 0.6832 | 0.4671 | 1.9395 | 94.4 | 0.1479 | 0.1628 | 1.4732 | 95.6 |
| 100 | a   | 0.8246 | 0.3648 | 3.8582 | 97.0 | 0.0866 | 0.3453 | 2.2794 | 95.5 |
| 200 | b   | 0.0986 | 0.0439 | 1.6195 | 94.7 | 0.0058 | 0.0561 | 0.9694 | 96.0 |
| 1   | c   | 0.5263 | 0.2971 | 1.5556 | 96.5 | 0.0679 | 0.0548 | 0.8787 | 96.3 |

Weighted Power Lomax Distribution and its Length Biased Version: Properties and Estimation Based on Censored Samples
### Table 4: AB, MSE, AL and CP of ML and MPS estimates under complete and TIIC in Case 3

| n   | k   | AB     | MSE | AL  | CP | AB     | MSE | AL  | CP |
|-----|-----|--------|-----|-----|----|--------|-----|-----|----|
| 50  | 0.8 | 1      | a   | 0.2628 | 0.4976 | 2.5676 | 93  | 0.1678 | 0.1472 | 1.3533 | 95.1 |
|     |     | b     | 0.0240 | 0.0507 | 0.8777 | 94.4 | 0.0457 | 0.0190 | 0.5097 | 95.7 |
|     |     | c     | 0.3197 | 0.6879 | 3.0014 | 97.3 | 0.3038 | 0.3948 | 2.1569 | 93.1 |
|     | 0.6 | 1      | a   | 0.5731 | 0.5417 | 2.6675 | 95.2 | 0.2382 | 0.3485 | 2.3105 | 94.2 |
|     |     | b     | 0.4636 | 0.2928 | 2.0944 | 95.6 | 0.0790 | 0.0227 | 0.5894 | 96   |
|     |     | c     | 0.5052 | 1.3781 | 4.1559 | 97.8 | 0.3122 | 1.2196 | 4.1545 | 95   |
| 100 | 0.8 | 1      | a   | 1.2169 | 1.5360 | 2.9412 | 95.8 | 0.0185 | 0.3486 | 2.9344 | 94.2 |
|     |     | b     | 0.9362 | 1.0561 | 2.6623 | 95.7 | 0.0062 | 0.0313 | 0.6931 | 95.3 |
|     |     | c     | 1.5916 | 3.1306 | 4.4317 | 95.9 | 0.2740 | 1.8039 | 4.3483 | 94.5 |

### Table 5: AB, MSE, AL and CP of ML and MPS estimates under complete and TIIC in Case 4

| n   | k   | AB     | MSE | AL  | CP | AB     | MSE | AL  | CP |
|-----|-----|--------|-----|-----|----|--------|-----|-----|----|
| 50  | 0.8 | 1      | a   | 0.2068 | 0.0679 | 2.7121 | 95.5 | 0.1499 | 0.0503 | 1.7183 | 97.5 |
|     |     | b     | 0.0527 | 0.2058 | 1.7673 | 95.4 | 0.1599 | 0.2037 | 1.3003 | 94.0 |
|     |     | c     | 0.2001 | 0.1157 | 2.8500 | 94.3 | 0.0635 | 0.1086 | 1.2681 | 95.4 |
|     |     | s     | 0.1412 | 0.1069 | 5.1009 | 96.0 | 0.2146 | 0.0975 | 3.2931 | 95.6 |
|     | 0.6 | 1      | a   | 0.2503 | 0.0816 | 3.5403 | 96.9 | 0.2126 | 0.0760 | 2.9112 | 97.3 |
|     |     | b     | 0.4026 | 0.2854 | 1.3773 | 94.9 | 0.3161 | 0.2720 | 1.0887 | 95.3 |
|     |     | c     | 0.3788 | 0.1446 | 1.1298 | 93.7 | 0.0792 | 0.0876 | 1.1186 | 95.4 |
|     |     | s     | 0.1522 | 0.1112 | 3.1635 | 94.6 | 0.0419 | 0.1539 | 2.8750 | 95.8 |
|     | 1   | 1      | a   | 0.7795 | 0.0697 | 2.3446 | 96.5 | 0.1595 | 0.0493 | 2.0764 | 95.7 |
|     |     | b     | 0.0661 | 0.0718 | 1.0182 | 96.7 | 0.1138 | 0.0685 | 1.5467 | 95.9 |
|     |     | c     | 0.1138 | 0.1239 | 1.8636 | 95.8 | 0.0285 | 0.0399 | 0.7750 | 95.9 |
|     | 100 | 0.8   | a   | 0.2601 | 0.0795 | 0.5201 | 97.1 | 0.2096 | 0.0638 | 1.2710 | 96.0 |
|     |     | b     | 0.3846 | 0.2149 | 1.0154 | 95.1 | 0.2325 | 0.2034 | 1.0921 | 94.7 |
|     |     | c     | 0.3777 | 0.1433 | 0.2975 | 95.3 | 0.0562 | 0.0410 | 0.1763 | 95.5 |
|     |     | s     | 0.1862 | 0.0974 | 0.9004 | 95.1 | 0.0871 | 0.3363 | 2.2487 | 95.6 |
|     | 0.6 | 0.8   | a   | 0.0710 | 0.0920 | 0.3337 | 94.1 | 0.2313 | 0.0836 | 0.9165 | 94.3 |

**Weighted Power Lomax Distribution and its Length Biased Version: Properties and Estimation Based on Censored Samples**
\[ a = 3, \ b = 2, \ c = 0.5, \ s = 3 \]

From the numerical outcomes listed in Tables 2–5 we conclude the following:

- The MSE of the MPS estimates of \( a \) is less than the corresponding of the ML estimates based on complete and TIIC in all cases (see for example Fig. 4 and Fig. 5).

- The MSE for MPS and ML estimates of \( c \) decreases as the value of \( k \) increases in Case 2 (ABPLo distribution) at \( n = 200 \). The MSE of the ML estimate of \( c \) is greater than the corresponding MSE of the MPS estimate of \( c \) (see Fig. 6).

- Fig. 7 demonstrates that the CP for the MPS and ML estimates for \( a \) increases as the value of \( k \) increases. Also, the CP of the MPS estimates is greater than the corresponding CP of the ML estimate in Case 1 (LBPLo distribution) at \( n = 50 \).

- Fig. 8 shows that the CP of the ML estimate of \( c \) increases as the value of \( k \) increases in Case 1 (LBPLo distribution) at \( n = 100 \).
The MSE for the ML estimate of \( b \) in Case 1 (LBPLo distribution) decreases as \( n \) increases. Also it shows that as the value of \( k \) increases, the MSE for the ML estimate of \( b \) decreases for all sample sizes (see Fig. 9).

![Fig. 8: The CP of \( c \) when \( n = 100 \) in Case 1 (LBPLo distribution)](image)

The MSE of the MPS estimate of \( a \) in Case 1 (LBPLo distribution) decreases as \( n \) increases. Also as the value of \( k \) increases, the MSE for the MPS estimate of \( a \) decreases for all \( n \) as shown in Fig. 10.

![Fig. 10: MSE for the MPS estimate of parameter \( a \) in Case 1 (LBPLo distribution)](image)

Fig. 11 illustrates that the CP for ML estimate of parameter \( c \) in Case 2 (ABPLo distribution) increases as \( n \) increases. Also it shows that as the value of \( k \) increases, the CP for the ML estimate of \( a \) increases for all \( n \).

Generally, the MPS of all parameters are preferable than the corresponding ML estimates in almost most of the situations. As the censoring level; \( k \) and sample size increase the MSE and AL of all estimates decrease.

The CP is very close to the considered significance level for all sample size.

6. Application to Real Data

To demonstrate the adequacy of one special model of the WPLo distribution, namely; LBPLo distribution, is done using the beta Lomax (BLo) distribution, Weibull Lomax (WLo) distribution Kumaraswamy Lomax (KLo) distribution, exponentiated generalized Lomax (EGLo), gamma Lomax (GLo), log gamma Lomax (LGaLo), exponentiated Lomax (ELo) and generalized exponential (GE). The following criteria are utilized to detect the distribution with the best fit: negative log-likelihood (−LL) value, Akaike information criteria (AIC), Bayesian information criteria (BIC), consistent AIC (AICC), and Hannan and Quinn information criteria (HQIC).

The first data relating to the strengths of 1.5 cm glass fibres which was obtained by workers at the UK National Physical Laboratory are used. The data have previously been used by Smith and Naylor (1987), Merovci et al. (2016), Oguntunde et al. (2017), Khaleel et al. (2018) and Oguntunde et al. (2018a); (2018b). The first observations are as follows:

- 0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25
- 1.27, 1.28, 1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50
- 1.50, 1.55, 1.52, 1.53, 1.54, 1.55, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63
- 1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.70, 1.68, 1.68
- 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82, 1.84, 1.84
- 2.00, 2.01, 2.24.

The second civil engineering data consisting of 85 hailing times observations Kotz and van Dorp (2004). The second observations are as follows...
The performances of the LBPLo distribution with the other competing distributions are shown in Tables 6 and 7.

**Table 6: The performances of the LBPLo model with some competing distributions**

| Model | -LL     | AIC     | AICC    | BIC     | HQIC    |
|-------|---------|---------|---------|---------|---------|
| LBPLo | 14.979  | 37.95   | 38.648  | 46.531  | 41.33   |
| Blo   | 24.017  | 56.0355 | 56.726  | 64.608  | 59.411  |
| KuLo  | 16.338  | 40.676  | 41.365  | 49.248  | 44.042  |
| EGLo  | 31.502  | 71.005  | 71.698  | 79.578  | 74.377  |
| GLo   | 26.99   | 59.98   | 60.387  | 66.409  | 62.509  |
| LgaLo | 30.289  | 68.574  | 69.265  | 77.152  | 71.95   |
| Elo   | 31.456  | 69.912  | 70.316  | 75.345  | 71.441  |
| GE    | 31.384  | 66.766  | 66.967  | 71.054  | 68.458  |

**Table 7: The performances of the LBPLo model with some competing distributions**

| Model | -LL     | AIC     | AICC    | BIC     | HQIC    |
|-------|---------|---------|---------|---------|---------|
| LBPLo | 130.5025| 269.005 | 269.505 | 278.776 | 272.935 |
| Blo   | 133.3107| 274.621 | 275.121 | 284.392 | 278.554 |
| KuLo  | 132.7561| 273.512 | 274.012 | 283.232 | 277.442 |
| EGLo  | 146.6422| 301.284 | 301.784 | 311.055 | 305.214 |
| GLo   | 138.3625| 282.724 | 283.021 | 290.053 | 285.673 |
| LgaLo | 159.2354| 326.47  | 326.979 | 336.241 | 330.4   |
| Elo   | 137.4985| 280.997 | 281.293 | 288.325 | 283.944 |
| GE    | 137.3389| 278.677 | 278.824 | 283.563 | 280.642 |

Finally, in order to assess whether the proposed model is appropriate for the above mentioned data, we display the visualization of the estimated pdfs in Figures 12–13. As seen we suggest that the fit of the LBPLo model performs better than the other competitive distributions.

![Fig. 12: Estimated pdfs for the first data](image1)

![Fig. 13: Estimated pdfs for the second data](image2)

7. **Concluding Remarks**

In this paper, we propose a new weighted distribution related to power Lomax model, named as weighted power Lomax distribution. The new model contains some new distributions besides it contains existing distribution. Main properties of the weighted power Lomax distribution are discussed. Based on complete and Type II censoring samples, the point and approximate confidence intervals of parameters are derived depend on maximum likelihood and maximum product spacing procedures. Due to the complicated forms of the non-linear equations, the Monte Carlo simulation study is done to assess the behaviour of the estimates for different sample sizes. Based on simulation study, we conclude that the maximum product spacing estimates are preferable than the maximum likelihood estimates in approximately most of the situations. Finally, for illustrative purpose, a real life data set is analyzed and compared with other lifetime distributions.
Conflict of Interest
The authors declare that they have no conflict of interest.

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