Connecting Weighted Automata and Recurrent Neural Networks through Spectral Learning

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Abstract

In this paper, we unravel a fundamental connection between weighted finite automata (WFAs) and second-order recurrent neural networks (2-RNNs): in the case of sequences of discrete symbols, WFAs and 2-RNNs with linear activation functions are expressively equivalent. Motivated by this result, we build upon a recent extension of the spectral learning algorithm to vector-valued WFAs and propose the first provable learning algorithm for linear 2-RNNs defined over sequences of continuous input vectors. This algorithm relies on estimating low rank sub-blocks of the so-called Hankel tensor, from which the parameters of a linear 2-RNN can be provably recovered. The performances of the proposed method are assessed in a simulation study.

1 Introduction

Many tasks in natural language processing, computational biology, reinforcement learning, and time series analysis rely on learning with sequential data, i.e. estimating functions defined over sequences of observations from training data. Weighted finite automata (WFAs) and recurrent neural networks (RNNs) are two powerful and flexible classes of models which can efficiently represent such functions. On the one hand, WFAs are tractable, they encompass a wide range of machine learning models (they can for example compute any probability distribution defined by a hidden Markov model (HMM) [10] and can model the transition and observation behavior of partially observable Markov decision processes [33]) and they offer appealing theoretical guarantees. In particular, the so-called spectral methods for learning HMMs [19], WFAs [3, 4] and related models [16, 6], provide an alternative to EM based algorithms that is both computationally efficient and consistent. On the other hand, RNNs are remarkably expressive models — they can represent any computable function [32] — and they have successfully tackled many practical problems in speech and audio recognition [17, 24, 13], but their theoretical analysis is difficult. Even though recent work provides interesting results on their expressive power [21, 37] as well as alternative training algorithms coming with learning guarantees [31], the theoretical understanding of RNNs is still limited.

In this work, we bridge a gap between these two classes of models by unraveling a fundamental connection between WFAs and second-order RNNs (2-RNNs): when considering input sequences of discrete symbols, 2-RNNs with linear activation functions and WFAs are one and the same, i.e. they are expressively equivalent and there exists a one-to-one mapping between the two classes (moreover, this mapping conserves model sizes). While connections between finite state machines (e.g. deterministic
finite automata) and recurrent neural networks have been noticed and investigated in the past (see e.g., [14, 25]), to the best of our knowledge this is the first time that such a clear equivalence between linear 2-RNNs and weighted automata is explicitly formalized. This result naturally leads to the observation that linear 2-RNNs are a natural generalization of WFAs (which take sequences of discrete observations as inputs) to sequences of continuous vectors, and raises the question of whether the spectral learning algorithm for WFAs can be extended to linear 2-RNNs. The second contribution of this paper is to show that the answer is in the positive: building upon the spectral learning algorithm for vector-valued WFAs introduced recently in [29], we propose the first provable learning algorithm for second-order RNNs with linear activation functions. Our learning algorithm relies on estimating sub-blocks of the so-called Hankel tensor, from which the parameters of a 2-linear RNN can be recovered using basic linear algebra operations. One of the key technical difficulties in designing this algorithm resides in estimating these sub-blocks from training data where the inputs are sequences of continuous vectors. We leverage multilinear properties of linear 2-RNNs and the fact that the Hankel sub-blocks can be reshaped into higher-order tensors of low tensor train rank (a result we believe is of independent interest) to perform this estimation efficiently using matrix sensing and tensor recovery techniques. As a proof of concept, we validate our theoretical findings in a simulation study on toy examples where we experimentally compare different recovery methods and investigate the robustness of our algorithm to noise and rank mis-specification.

Related work. Combining the spectral learning algorithm for WFAs with matrix completion techniques (a problem which is closely related to matrix sensing) has been theoretically investigated in [5]. The connections between tensors and RNNs have been previously leveraged to study the expressive power of RNNs in [21] and to achieve model compression in [37, 36, 34]. Exploring the relationship between RNNs and automata for interpretability purposes has been explored in [35] and the ability of RNNs to learn classes of formal languages has been investigated in [2]. The predictive state RNN model introduced in [11] is closely related to 2-RNNs and the authors propose to use the spectral learning algorithm for predictive state representations to initialize a gradient based algorithm; their approach however comes without theoretical guarantees. Lastly, a provable algorithm for RNNs relying on the tensor method of moments has been proposed in [31] but it is limited to first-order RNNs with quadratic activation functions (which do not encompass linear 2-RNNs).

The proofs of the results given in the paper can be found in the supplementary material.

2 Preliminaries

In this section, we first present basic notions of tensor algebra before introducing second-order recurrent neural network, weighted finite automata and the spectral learning algorithm. We start by introducing some notation. For any integer \( k \) we use \([k]\) to denote the set of integers from 1 to \( k \). We use \([l]\) to denote the smallest integer greater or equal to \( l \). For any set \( S \), we denote by \( S^* = \bigcup_{k \in \mathbb{N}} S^k \) the set of all finite-length sequences of elements of \( S \) (in particular, \( \Sigma^* \) will denote the set of strings on a finite alphabet \( \Sigma \)). We use lower case bold letters for vectors (e.g. \( v \in \mathbb{R}^{d_1} \)), upper case bold letters for matrices (e.g. \( M \in \mathbb{R}^{d_1 \times d_2} \)) and bold calligraphic letters for higher order tensors (e.g. \( T \in \mathbb{R}^{d_1 \times d_2 \times d_3} \)). We use \( e_i \) to denote the \( i \)th canonical basis vector of \( \mathbb{R}^{d} \) (where the dimension \( d \) will always appear clearly from context). The \( d \times d \) identity matrix will be written as \( I_d \). The \( i \)th row (resp. column) of a matrix \( M \) will be denoted by \( M_{i,:} \) (resp. \( M_{:,i} \)). This notation is extended to slices of a tensor in the straightforward way. If \( v \in \mathbb{R}^{d_1} \) and \( v' \in \mathbb{R}^{d_2} \), we use \( v \otimes v' \in \mathbb{R}^{d_1 \times d_2} \) to denote the Kronecker product between vectors, and its straightforward extension to matrices and tensors. Given a matrix \( M \in \mathbb{R}^{d_1 \times d_2} \), we use \( \text{vec}(M) \in \mathbb{R}^{d_1 \times d_2} \) to denote the column vector obtained by concatenating the columns of \( M \). The inverse of \( M \) is denoted by \( M^{-1} \), its Moore-Penrose pseudo-inverse by \( M^\dagger \), and the transpose of its inverse by \( M^{-\top} \); the Frobenius norm is denoted by \( \| M \|_F \) and the nuclear norm by \( \| M \|_* \).

Tensors. We first recall basic definitions of tensor algebra; more details can be found in [22]. A tensor \( T \in \mathbb{R}^{d_1 \times \cdots \times d_p} \) can simply be seen as a multidimensional array \( \{ T_{i_1, \ldots, i_p} : i_n \in [d_n], n \in [p] \} \). The mode-\( n \) fibers of \( T \) are the vectors obtained by fixing all indices except the \( n \)th one, e.g. \( T_{i_1, \ldots, i_n, \cdot, \ldots, \cdot} \in \mathbb{R}^{d_1 \times \cdots \times d_{n-1} \times d_{n+1} \times \cdots \times d_p} \). The mode-\( n \) matricization of \( T \) is the matrix having the mode-\( n \) fibers of \( T \) as columns and is denoted by \( T^{(n)} \in \mathbb{R}^{d_n \times \cdots \times d_{n-1} \times d_{n+1} \times \cdots \times d_p} \). The vectorization of a tensor is defined by \( \text{vec}(T) = \text{vec}(T_{(1)}) \). In the following \( T \) always denotes a tensor of size \( d_1 \times \cdots \times d_p \).
The mode-$n$ matrix product of the tensor $\mathcal{T}$ and a matrix $X \in \mathbb{R}^{m \times d_n}$ is a tensor denoted by $\mathcal{T} \times_n X$. It is of size $d_1 \times \cdots \times d_{n-1} \times m \times d_{n+1} \times \cdots \times d_p$ and is defined by the relation $Y = \mathcal{T} \times_n X \iff Y_{(n)} = X \mathcal{T}_{(n)}$. The mode-$n$ vector product of the tensor $\mathcal{T}$ and a vector $v \in \mathbb{R}^{d_n}$ is a tensor defined by $\mathcal{T} \cdot_n v = \mathcal{T} \times_n v^T \in \mathbb{R}^{d_1 \times \cdots \times d_{n-1} \times d_{n+1} \times \cdots \times d_p}$. It is easy to check that the $n$-mode product satisfies $(\mathcal{T} \times_n A) \times_n B = \mathcal{T} \times_n BA$ where we assume compatible dimensions of the tensor $\mathcal{T}$ and the matrices $A$ and $B$.

Given strictly positive integers $n_1, \cdots, n_k$ satisfying $\sum_i n_i = p$, we use the notation $(\mathcal{T})_{\langle n_1, n_2, \ldots, n_k \rangle}$ to denote the $k$th order tensor obtained by reshaping $\mathcal{T}$ into a tensor1 of size $(\prod_{i=1}^{n_1} d_i) \times (\prod_{i=2}^{n_2} d_{n_1+i}) \times \cdots \times (\prod_{i=k}^{n_k} d_{n_{k-1}+i})$. In particular we have $(\mathcal{T})_{\langle p \rangle} = \text{vec}(\mathcal{T})$ and $(\mathcal{T})_{\langle 1, p-1 \rangle} = \mathcal{T}(1)$.

A rank $R$ tensor train (TT) decomposition [26] of a tensor $\mathcal{T} \in \mathbb{R}^{d_1 \times \cdots \times d_p}$ consists in factorizing $\mathcal{T}$ into the product of $p$ core tensors $\mathcal{G}_1 \in \mathbb{R}^{d_1 \times R}$, $\mathcal{G}_2 \in \mathbb{R}^{R \times d_2 \times R}$, $\cdots$, $\mathcal{G}_{p-1} \in \mathbb{R}^{R \times d_{p-1} \times R}$, $\mathcal{G}_p \in \mathbb{R}^{R \times d_p}$, and is defined2 by

$$\mathcal{T}_{i_1, \ldots, i_p} = \sum_{r_1, \ldots, r_{p-1}=1}^{R} (\mathcal{G}_1)_{i_1,r_1} (\mathcal{G}_2)_{r_1,i_2,r_2} (\mathcal{G}_3)_{r_2,i_3,r_3} \cdots (\mathcal{G}_{p-1})_{r_{p-2},i_{p-1},r_{p-1}} (\mathcal{G}_p)_{r_{p-1},i_p}$$

for all indices $i_1 \in [d_1], \ldots, i_p \in [d_p]$; we will use the notation $\mathcal{T} = [\mathcal{G}_1, \ldots, \mathcal{G}_p]$ to denote such a decomposition. A tensor network representation of this decomposition is shown in Figure 1. While the problem of finding the best approximation of TT-rank $R$ of a given tensor is NP-hard [18], a quasi-optimal SVD based compression algorithm (TT-SVD) has been proposed in [26]. It is worth mentioning that the TT decomposition is invariant under change of basis in the sense that, for any invertible matrix $M$ and any core tensors $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_p$, we have $[\mathcal{G}_1, \ldots, \mathcal{G}_p] = [\mathcal{G}_1 \times_2 M^{-T}, \mathcal{G}_2 \times_1 M \times_3 M^{-T}, \cdots, \mathcal{G}_{p-1} \times_1 M \times_3 M^{-T}, \mathcal{G}_p \times_1 M]$.

**Second-order RNNs.** A second-order recurrent neural network (2-RNN) [15, 27, 23] with $n$ hidden units can be defined as a tuple $M = (h_0, A, \Omega)$ where $h_0 \in \mathbb{R}^n$ is the initial state, $A \in \mathbb{R}^{n \times n}$ is the transition tensor, and $\Omega \in \mathbb{R}^{p \times n}$ is the output matrix, with $d$ and $p$ being the input and output dimensions respectively. A 2-RNN maps any sequence of inputs $x_1, \ldots, x_k \in \mathbb{R}^d$ to a sequence of outputs $y_1, \ldots, y_k \in \mathbb{R}^p$ defined by

$$y_t = z_2(\Omega h_t) \text{ with } h_t = z_1(A \bullet_1 x_t \bullet_2 h_{t-1}) \quad \text{for } t = 1, \ldots, k$$

where $z_1 : \mathbb{R}^n \to \mathbb{R}^n$ and $z_2 : \mathbb{R}^p \to \mathbb{R}^p$ are activation functions. Alternatively, one can think of a 2-RNN as computing a function $f_M : (\mathbb{R}^d)^* \to \mathbb{R}^p$ mapping each input sequence $x_1, \ldots, x_k$ to the corresponding final output $y_k$. While $z_1$ and $z_2$ are usually non-linear component-wise functions, we consider in this paper the case where both $z_1$ and $z_2$ are the identity, and we refer to the resulting model as a linear 2-RNN. For a linear 2-RNN $M$, the function $f_M$ is multilinear in the sense that, for any integer $l$, its restriction to the domain $(\mathbb{R}^d)^l$ is multilinear. Another useful observation is that linear 2-RNNs are invariant under change of basis: for any invertible matrix $P$, the linear 2-RNN $M = (P^{-T} h_0, A \times_1 P \times_3 P^{-T}, \Omega)$ is such that $f_M^{P} = f_M$. We will say that a linear 2-RNN $M$ with $n$ states is minimal if its number of hidden units is minimal (i.e. any linear 2-RNN computing $f_M$ has at least $n$ hidden units).

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1Note that the specific ordering used to perform matricization, vectorization and such a reshaping is not relevant as long as it is consistent across all operations.

2The classical definition of the TT-decomposition allows the rank $R$ to be different for each mode, but this definition is sufficient for the purpose of this paper.

3
We start by unraveling a fundamental connection between vv-WFAs and linear 2-RNNs: vv-WFAs more precisely, the WFA $f$ with $\alpha \in \mathbb{R}^n$ is the initial weights vector, $\Omega \in \mathbb{R}^{P \times n}$ is the matrix of final weights, and $A_\sigma \in \mathbb{R}^{n \times n}$ is the transition matrix for each symbol $\sigma$ in a finite alphabet $\Sigma$. A vv-WFA $A$ computes a function $f_A : \Sigma^* \rightarrow \mathbb{R}^d$ defined by

$$f_A(x) = \Omega(A^{x_1}A^{x_2} \cdots A^{x_k})^\top \alpha$$

for each word $x = x_1x_2 \cdots x_k \in \Sigma^*$. We call a vv-WFA minimal if its number of states is minimal. Given a function $f : \Sigma^* \rightarrow \mathbb{R}^p$ we denote by $\text{rank}(f)$ the number of states of a minimal vv-WFA computing $f$ (which is set to $\infty$ if $f$ cannot be computed by a vv-WFA).

The spectral learning algorithm for vv-WFAs relies on the following fundamental theorem relating the rank of a function $f : \Sigma^* \rightarrow \mathbb{R}^d$ to its Hankel tensor $H \in \mathbb{R}^{\Sigma^* \times \Sigma^* \times P}$, which is defined by $H_{u,v,;} = f(uv)$ for all $u, v \in \Sigma^*$.

**Theorem 1** ([29]). Let $f : \Sigma^* \rightarrow \mathbb{R}^d$ and let $H$ be its Hankel tensor. Then $\text{rank}(f) = \text{rank}(H_{1})$.

The vv-WFA learning algorithm leverages the fact that the proof of this theorem is constructive: one can recover a vv-WFA computing $f$ from any low rank factorization of $H_{1}$. In practice, a finite sub-block $H \in \mathbb{R}^{P \times S \times P}$ of the Hankel tensor is used to recover the vv-WFA, where $P, S \subset \Sigma^*$ are finite sets of prefixes and suffixes forming a complete basis for $f$, i.e. such that $\text{rank}(H_{1}) = \text{rank}(H_{1})$. More details can be found in [29].

3 A Fundamental Relation between WFAs and Linear 2-RNNs

We start by unraveling a fundamental connection between vv-WFAs and linear 2-RNNs: vv-WFAs and linear 2-RNNs are expressively equivalent for representing functions defined over sequences of discrete symbols. More formally, we have the following theorem.

**Theorem 2.** Any function that can be computed by a vv-WFA with $n$ states can be computed by a linear 2-RNN with $n$ hidden units on sequences of one-hot vectors (i.e. canonical basis vectors) can be computed by a WFA with $n$ states.

More precisely, the WFA $A = (\alpha, \{A_\sigma\}_{\sigma \in \Sigma}, \Omega)$ with $n$ states and the linear 2-RNN $M = (\alpha, A, \Omega)$ with $n$ hidden units, where $A \in \mathbb{R}^{n \times n}$ is defined by $A_\sigma := A_\sigma^\sigma$ for all $\sigma \in \Sigma$, are such that $f_A(\sigma_1, \cdots, \sigma_k) = f_M(x_1, x_2, \cdots, x_k)$ for all sequences of input symbols $\sigma_1, \cdots, \sigma_k \in \Sigma$, where for each $i \in [k]$ the input vector $x_i \in \mathbb{R}^d$ is the one-hot encoding of the symbol $\sigma_i$.

This result first implies that linear 2-RNNs defined over sequence of discrete symbols (using one-hot encoding) can be provably learned using the spectral learning algorithm for WFAs/vv-WFAs; indeed, these algorithms have been proved to return consistent estimators. Furthermore, Theorem 2 reveals that linear 2-RNNs are a natural generalization of classical weighted automata to functions defined over sequences of continuous vectors instead of discrete symbols. This spontaneously raises the question of whether the spectral learning algorithms for WFAs and vv-WFAs can be extended to the general setting of linear 2-RNNs; we show that the answer is in the positive in the next section.

4 Spectral Learning of Linear 2-RNNs

In this section, we extend the learning algorithm for vv-WFAs to linear 2-RNNs, thus providing the first provable learning algorithm for linear second-order RNNs.

4.1 Recovering a 2-RNN from Hankel Tensors

We first present an identifiability result showing how one can recover a linear 2-RNN computing a function $f : (\mathbb{R}^d)^* \rightarrow \mathbb{R}^p$ from observable tensors extracted from some Hankel tensor associated with $f$. Intuitively, we obtain this result by reducing the problem to the one of learning a vv-WFA. This is done by considering the restriction of $f$ to canonical basis vectors; loosely speaking, since the domain of this restricted function is isomorphic to $[d]^*$, this allows us to fall back onto the setting of sequences of discrete symbols.
Given a function $f : (\mathbb{R}^d)^* \to \mathbb{R}^p$, we define its Hankel tensor $\mathcal{H}_f \in \mathbb{R}^{[d]^* \times [d]^* \times p}$ by

$$(\mathcal{H}_f)_{i_1 \cdots i_s \cdots j_1 \cdots j_r} = f(e_{i_1}, \cdots, e_{i_s}, e_{j_1}, \cdots, e_{j_r}) \quad \text{for all } i_1, \cdots, i_s, j_1, \cdots, j_r \in [d],$$

which is infinite in two of its modes. It is easy to see that $\mathcal{H}_f$ is also the Hankel tensor associated with the function $\tilde{f} : [d]^* \to \mathbb{R}^p$ mapping any sequence $i_1 i_2 \cdots i_k \in [d]^*$ to $f(e_{i_1}, \cdots, e_{i_k})$. Moreover, in the special case where $f$ can be computed by a linear 2-RNN, one can use the multilinearity of $f$ to show that $f(x_1, \cdots, x_k) = \sum_{i_1, \cdots, i_k=1}^d (x_1)_{i_1} \cdots (x_k)_{i_k} \tilde{f}(i_1 \cdots i_k)$, making it intuitively clear that one can learn $f$ by learning a v-v-WFA computing $\tilde{f}$ using the spectral learning algorithm. That is, given a large enough sub-block $\mathcal{H}_f \in \mathbb{R}^{p \times S \times p}$ of $\mathcal{H}_f$ for some prefix and suffix sets $\mathcal{P}, S \subseteq [d]^*$, one should be able to recover a v-v-WFA computing $\tilde{f}$ and consequently a linear 2-RNN computing $f$ using Theorem 2. For the sake of clarity, we present the learning algorithm for the particular case where there exists an $L$ such that the prefix and suffix sets consisting of all sequences of length $L$, i.e. $\mathcal{P} = S = [d]^L$, forms a complete basis for $\tilde{f}$. This assumption allows us to present in a simpler way all the key elements of the algorithm, the technical details needed to lift this assumption are given in the supplementary material.

For any integer $l$, we define the finite tensor $\mathcal{H}_f^{(l)} \in \mathbb{R}^{d \times \cdots \times d \times p}$ of order $l + 1$ by

$$(\mathcal{H}_f^{(l)})_{i_1 \cdots i_s : j_1 \cdots j_r} = f(e_{i_1}, \cdots, e_{i_s}) \quad \text{for all } i_1, \cdots, i_s \in [d].$$

Observe that for any integer $l$, the tensor $\mathcal{H}_f^{(l)}$ can be obtained by reshaping a finite sub-block of the Hankel tensor $\mathcal{H}_f$. When $f$ is computed by a linear 2-RNN, we have the useful property that, for any integer $l$,

$$f(x_1, \cdots, x_l) = \mathcal{H}_f^{(l)} \bullet_1 x_1 \bullet_2 \cdots \bullet_l x_l \quad (2)$$

for any sequence of inputs $x_1, \cdots, x_l \in \mathbb{R}^d$ (which can easily be shown using the multilinearity of $f$). Before formalizing the identifiability result for linear 2-RNNs in Theorem 3 below, we remark that the tensors $\mathcal{H}_f^{(l)}$ are of low tensor train rank. Indeed, for any $l$, it is easy to check that $\mathcal{H}_f^{(l)} = \left[ A \bullet_1 \alpha, A \bullet_2 \cdots, A \bullet_l \Omega \right]$ (the tensor network representation of this decomposition is shown in Figure 2). This property will be particularly relevant to the learning algorithm we design in the following section, but it is also a fundamental relation that deserves some attention on its own: it implies in particular that, beyond the classical relation between the rank of the Hankel matrix $H_f$ and the number states of a minimal WFA computing $f$, the Hankel matrix possesses a deeper structure intrinsically connecting weighted automata to the tensor train decomposition. While this relation have been noticed previously (see e.g. [8, 9, 28]), we believe that it could be successfully leveraged to design more efficient learning algorithms (both in terms of sample and computational complexity) for WFA and related models.

**Theorem 3.** Let $f : (\mathbb{R}^d)^* \to \mathbb{R}^p$ be a function computed by a minimal linear 2-RNN with $n$ hidden units and let $L$ be an integer such that $\text{rank}(\mathcal{H}_f^{(2L)})_{\langle L, L+1 \rangle} = n$.

Then, for any $P \in \mathbb{R}^{d^L \times n}$ and $S \in \mathbb{R}^{n \times d^L \times p}$ such that $\mathcal{H}_f^{(2L)}_{\langle L, L+1 \rangle} = PS$, the linear 2-RNN $M = (\alpha, A, \Omega)$ defined by

$$\alpha = (S^\top \mathcal{H}_f^{(L)})_{\langle L+1 \rangle}, \quad A = (\mathcal{H}_f^{(2L+1)})_{\langle L+1, L+1 \rangle} \times_1 P^\top \times_3 (S^\top)^\top, \quad \Omega^\top = P^\top (\mathcal{H}_f^{(L)})_{\langle L, 1 \rangle}$$

is a minimal linear 2-RNN computing $f$. 

Figure 2: Tensor network representation of the TT decomposition of the Hankel tensor $\mathcal{H}_f^{(4)}$ induced by a linear 2-RNN $(\alpha, A, \Omega)$. 

$$\mathcal{H}_f^{(4)} = \begin{array}{cccc}
  \alpha & A & A & A \\
  1 & 2 & 3 & 4 \\
  \end{array}$$
First observe that such an integer $L$ exists under the assumption that $\mathcal{P} = \mathcal{S} = [d]^L$ forms a complete basis for $\hat{f}$. It is also worth mentioning that a necessary condition for $\text{rank}(\mathcal{H}_f^{(2L)})_{\ell_2^L} = n$ is that $d^L \ge n$, i.e. $L$ must be of the order $\log_d(n)$.

### 4.2 Hankel Tensors Recovery from Linear Measurements

We showed in the previous section that, given the Hankel tensors $\mathcal{H}_f^{(L)}$, $\mathcal{H}_f^{(2L)}$ and $\mathcal{H}_f^{(2L+1)}$, one can recover a linear 2-RNN computing $f$ if it exists. This readily implies that the class of functions that can be computed by linear 2-RNNs is learnable in Angluin’s exact learning model [1] where one has access to an oracle that can answer membership queries (e.g. what is the value computed by the target $f$ on $(x_1, \ldots, x_k)$?) and equivalence queries (e.g. is my current hypothesis $h$ equal to the target $f$?). While this fundamental result is of significant theoretical interest, assuming access to such an oracle is unrealistic. In this section, we show that a stronger learnability result can be obtained in a more realistic setting, where we only assume access to randomly generated input/output examples $((x_1^{(n)}, x_2^{(n)}, \ldots, x_t^{(n)}), y_n) \in (\mathbb{R}^d)^* \times \mathbb{R}^p$ where $y_n = f(x_1^{(n)}, x_2^{(n)}, \ldots, x_t^{(n)})$.

The key observation is that such an input/output example $((x_1^{(n)}, x_2^{(n)}, \ldots, x_t^{(n)}), y_n)$ can be seen as a linear measurement of the Hankel tensor $\mathcal{H}_f^{(l)}$. Indeed, we have

$$y_n = f(x_1^{(n)}, x_2^{(n)}, \ldots, x_t^{(n)}) = \mathcal{H}_f^{(l)} \bullet_1 x_1 \bullet_2 \cdots \bullet_t x_t = (\mathcal{H}_f^{(l)})_{\ell_2^l}^\top x^{(n)}$$

where $x^{(n)} = x_1^{(n)} \otimes \cdots \otimes x_t^{(n)} \in \mathbb{R}^{d^l}$. Hence, by regrouping $N$ output examples $y_n$ into the matrix $Y \in \mathbb{R}^{N \times p}$ and the corresponding input vectors $x^{(n)}$ into the matrix $X \in \mathbb{R}^{N \times d^l}$, one can recover $\mathcal{H}_f^{(l)}$ by solving the linear system $Y = X(\mathcal{H}_f^{(l)})_{\ell_2^l}$, which has a unique solution whenever $X$ is of full column rank. This naturally leads to the following theorem, whose proof relies on the fact that the matrix $X$ will be of full column rank whenever $N \ge d^l$ and the components of each $x_i^{(n)}$ for $i \in [l], n \in [N]$ are drawn independently from a continuous distribution over $\mathbb{R}^d$ (w.r.t. the Lebesgue measure).

**Theorem 4.** Let $(\ell_0, \mathcal{A}, \Omega)$ be a minimal linear 2-RNN with $n$ hidden units computing a function $f : (\mathbb{R}^d)^* \rightarrow \mathbb{R}^p$, and let $L$ be an integer\(^3\) such that $\text{rank}(\mathcal{H}_f^{(2L)})_{\ell_2^{L+1}} = n$.

Suppose we have access to 3 datasets $D_l = \{(x_1^{(n)}, x_2^{(n)}, \ldots, x_t^{(n)}, y_n)\}_{n=1}^{N_l} \subset (\mathbb{R}^d)^l \times \mathbb{R}^p$ for $l \in \{L, 2L, 2L+1\}$ where the entries of each $x_i^{(n)}$ are drawn independently from the standard normal distribution and where each $y_n = f(x_1^{(n)}, x_2^{(n)}, \ldots, x_t^{(n)})$.

Then, whenever $N_l \ge d^l$ for each $l \in \{L, 2L, 2L+1\}$, the linear 2-RNN $M$ returned by Algorithm 1 with the least-squares method satisfies $f_M = f$ with probability one.

A few remarks on this theorem are in order. The first observation is that the 3 datasets $D_L$, $D_{2L}$ and $D_{2L+1}$ can either be drawn independently or not (e.g. the sequences in $D_L$ can be prefixes of the sequences in $D_{2L}$ but it is not necessary). In particular, the result still holds when the datasets $D_l$ are constructed from a unique dataset $S = \{(x_1^{(n)}, x_2^{(n)}, \ldots, x_T^{(n)}), (y_1^{(n)}, y_2^{(n)}, \ldots, y_T^{(n)})\}_{n=1}^{N}$ of input/output sequences with $T \ge 2L + 1$, where $y_t^{(n)} = f(x_1^{(n)}, x_2^{(n)}, \ldots, x_T^{(n)})$ for any $t \in [T]$.

Observe that having access to such input/output training sequences is not an unrealistic assumption: for example when training RNNs for language modeling the output $y_t$ is the next symbol conditional probability vector, and for classification tasks the output is the one-hot encoded label for all time steps. Lastly, when the outputs $y_n$ are noisy, one can solve the least-squares problem $\|Y - X(\mathcal{H}_f^{(l)})_{\ell_2^l}\|_{F}^{2}$ to approximate the Hankel tensors; we will empirically evaluate this approach in Section 5 and we defer its theoretical analysis in the noisy setting to future work.

While the least-squares method is sufficient to obtain the theoretical guarantees of Theorem 3, it does not leverage the low rank structure of the Hankel tensors $\mathcal{H}_f^{(l)}$. We now propose three alternative recovery methods to leverage this structure, whose efficiency will be assessed in a simulation study in Section 5 (deriving improved sample complexity guarantees using these methods is left

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\(^3\)Note that the theorem can be adapted if such an integer $L$ does not exists (see supplementary material).
We perform two experiments. In the first one, we randomly generated a linear 2-RNN with 5 dimensions. These two methods are implemented in lines 8-12 of Algorithm 1. There, the resulting function can be computed by a 2-RNN with a simple arithmetic function that computes the sum of the running differences between the two inputs)

\[ f(x_1, x_2) = x_1 - x_2 \]

...recovery setting in practice). The two methods are used in the noiseless setting (though it could be extended to the noisy recovery setting in practice).

Algorithm 1 2RNN-SL: Spectral Learning of linear 2-RNNs

Input: Three training datasets \( D_L, D_{2L}, D_{2L+1} \) with input sequences of length \( L, 2L \) and \( 2L + 1 \) respectively, a recovery_method, rank \( R \) and learning rate \( \gamma \) (for IHT/TIHT).

1. \( \text{for } l \in \{ L, 2L, 2L + 1 \} \) do
2. \( \) From the dataset \( D_l = \{(x^{(1)}_i, x^{(2)}_i, \cdots, x^{(n)}_i, y_n)\}_{n=1}^{N_l} \subset (\mathbb{R}^d)^l \times \mathbb{R}^p, \) build \( X \in \mathbb{R}^{N_l \times d^l} \) with rows \( x^{(a)}_i \otimes x^{(b)}_i \otimes \cdots \otimes x^{(n)}_i \) for \( n \in [N_l] \) and \( Y \in \mathbb{R}^{N_l \times d^p} \) with rows \( y_n \) for \( n \in [N_l] \).
3. \( \) if recovery_method = "Least-Squares" then
4. \( \) \( H_l = \arg \min_{T \in \mathbb{R}^{d \times \cdots \times d \times p}} \|X(T)\|_{(l,1)} \) subject to \( X(T)_{(l,1)} = Y. \)
5. \( \) else if recovery_method = "Nuclear Norm Minimization" then
6. \( \) \( H_l = \arg \min_{T \in \mathbb{R}^{d \times \cdots \times d \times p}} \|X(T)\|_{(l,1)} \) subject to \( X(T)_{(l,1)} = Y. \)
7. \( \) else if recovery_method = "IHT" or "TIHT" then
8. \( \) Initialize \( H_l(0) \in \mathbb{R}^{d \times \cdots \times d \times p} \) to 0.
9. \( \) repeat
10. \( \) \( H_l(l+1) = H_l(l+1) + \gamma X^T (Y - X(H_l(l+1))) \)
11. \( \) \( H_l(l+1) = \text{project}(H_l(l+1), R) \) (using either SVD for IHT or TT-SVD for TIHT).
12. \( \) until convergence.
13. \( \) end if.
14. \( \) end for.
15. \( \) Let \( (H_l(2l))_{(l,1+1)} \) be a rank \( R \) factorization.
16. \( \) Return the linear 2-RNN \( (h_0, A, \Omega) \) where
\[ \alpha = (S^\dagger)^T (H_l(2l))_{(l,1+1)}^T, \quad A = ((H_f(2l+1))_{(l,1,1+1)})^T 	imes_1 P^\dagger \times_3 (S^\dagger)^T, \quad \Omega = P^\dagger (H_f(2l))_{(l,1)}^T \]

for future work). In the noiseless setting, we first propose to replace solving the linear system \( Y = X(H_l(1)) \) with a nuclear norm minimization problem (see line 6 of Algorithm 1), thus leveraging the fact that \( (H_l(l+1))_{(l,1)} \) is potentially of low matrix rank. We also propose to use iterative hard thresholding (IHT) [20] and its tensor counterpart TIHT [30], which are based on the classical projected gradient descent algorithm and have shown to be robust to noise in practice. These two methods are implemented in lines 8-12 of Algorithm 1. There, the \( \text{project} \) method either projects \( (H_l(l+1))_{(l,1)} \) onto the manifold of low rank matrices using SVD (IHT) or projects \( H_l(l+1) \) onto the manifold of tensors with TT-rank \( R \) (TIHT).

5 Experiments

In this section, we perform experiments on synthetic data to compare how the choice of the recovery method (LeastSquares, NuclearNorm, IHT and TIHT) affects the sample efficiency of the learning algorithm\(^4\), and to assess the robustness of 2RNN–SL to noise and the robustness of the IHT and TIHT recovery methods to rank mis-specification.

We perform two experiments. In the first one, we randomly generated a linear 2-RNN with 5 units computing a function \( f : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) by drawing the entries of all parameters \( (h_0, A, \Omega) \) independently from the normal distribution \( \mathcal{N}(0, 0.2) \). The training data consists of 3 independently drawn sets \( D_l = \{(x^{(1)}_i, x^{(2)}_i, \cdots, x^{(n)}_i, y_n)\}_{n=1}^{N_l} \subset (\mathbb{R}^d)^l \times \mathbb{R}^p \) for \( l \in \{ L, 2L, 2L + 1 \} \) with \( L = 2 \), where each \( x^{(a)}_i \sim \mathcal{N}(0, 1) \) and where the outputs can be noisy, i.e. \( y_n = f(x^{(1)}_i, x^{(2)}_i, \cdots, x^{(n)}_i) + \xi_n \) where \( \xi_n \sim \mathcal{N}(0, \sigma^2) \) for some noise variance \( \sigma^2 \). In the second experiment, the goal is to learn a simple arithmetic function that computes the sum of the running differences between the two components of a sequence of 2-dimensional vectors, i.e. \( f(x_1, \cdots, x_k) = \sum_{i=1}^k v^T x_i \) where \( v^T = (-1, 1) \). The 3 training datasets are generated using the same process as above and a constant entry equal to one is added to all the input vectors to encode a bias term (one can check that the resulting function can be computed by a 2-RNN with 2 hidden units).

\(^4\)The NuclearNorm method was only used in the noiseless setting (though it could be extended to the noisy recovery setting in practice).
We run the experiments for different sizes of datasets $N$ ranging from 20 to 5,000 (we set $N_1 = N_2 = N_{2L+1} = N$) and we compare the different methods in terms of mean squared error\(^3\) (MSE) on a test set of size 5,000 containing sequences of length 6 generated in the same way as the training data (note that the training data only contains sequences of lengths up to 5).

The results are reported in Figure 3 and 4 where we see that all recovery methods lead to consistent estimates of the target function given enough training data. This is the case even in the presence of noise (in which case more samples are needed to achieve the same accuracy, as expected). We can also see that IHT and TIHT are overall more sample efficient than the other methods (especially with noisy data), showing that taking the low rank structure of the Hankel tensors into account is profitable. Moreover, TIHT tends to perform better than its matrix counterpart, confirming our intuition that leveraging the tensor train structure is beneficial. Lastly, as one can expect, when the rank parameter $R$ is over-estimated IHT and TIHT still converge to the target but they need more samples (when the rank parameter was underestimated both algorithms did not learn at all). The running time of all methods were overall of the same order.

6 Conclusion and Future Directions

In this paper, we proposed the first provable learning algorithm for second-order RNNs with linear activation functions. The elaboration of this meaningful learnability result relied on showing that linear 2-RNNs are a natural extension of vv-WFAs, and on extending the vv-WFA spectral learning algorithm to linear 2-RNNs. We believe that the results presented in this paper open a number of exciting and promising research directions on both the theoretical and practical perspectives. We first plan to use the spectral learning estimate as a starting point for gradient based methods to train 2-RNNs. More precisely, since linear 2-RNNs can be thought of as 2-RNN using LeakyRelu activation functions with negative slope 1, one can use a linear 2-RNN as initialization before gradually reducing the negative slope parameter during training. The extension of spectral learning to linear 2-RNNs also opens the door to scaling up the classical spectral method to problem with large size discrete alphabets (which is a known caveat of the spectral algorithm for WFAs) since it allows one to use low dimensional embeddings of large vocabularies (using e.g. word2vec or latent semantic analysis). From the theoretical perspective, we plan on deriving learning guarantees for linear 2-RNNs in the noisy setting (e.g. using the PAC learnability framework). Even though it is intuitive that such guarantees should hold (given the continuity of all operations used in our algorithm), we believe

\(^3\)Using the IHT/TIHT methods, it seldom happens on small datasets that 2RNN-SL returns aberrant models (due to numerical instabilities). We used the following scheme to circumvent this issue: when the training MSE of the hypothesis was greater than the one of the zero function, the zero function was returned instead.
that such an analysis may entail results of independent interest. In particular, analogously to the matrix case studied in [7], obtaining rate optimal convergence rates for the recovery of the low TT-rank Hankel tensors from rank one measurements is an interesting direction; such a result could for example allow us to improve the generalization bounds provided in [5] for spectral learning of general WFAs.
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A Proofs

A.1 Proof of Theorem 2

**Theorem.** Any function that can be computed by a vv-WFA with $n$ states can be computed by a linear 2-RNN with $n$ hidden units. Conversely, any function that can be computed by a linear 2-RNN with $n$ hidden units on sequences of one-hot vectors (i.e. canonical basis vectors) can be computed by a WFA with $n$ states.

More precisely, the WFA $A = (\alpha, \{A^\sigma\}_{\sigma \in \Sigma})$ with $n$ hidden units, and the linear 2-RNN $M = (\alpha, A, \Omega)$ with $n$ hidden units, where $A \in \mathbb{R}^{n \times \Sigma \times n}$ is defined by $A_{\sigma_i, \sigma_{i+1}} = A^\sigma$ for all $\sigma \in \Sigma$, are such that

$$f_A(\sigma_1 \cdots \sigma_k) = f_M(x_1, x_2, \cdots, x_k) \text{ for all sequences of input symbols } \sigma_1, \cdots, \sigma_k \in \Sigma,$$

where for each $i \in [k]$ the input vector $x_i \in \mathbb{R}^d$ is the one-hot encoding of the symbol $\sigma_i$.

**Proof.** We first show by induction on $k$ that, for any sequence $\sigma_1 \cdots \sigma_k \in \Sigma^*$, the hidden state $h_k$ computed by $M$ (see Eq. (1)) on the corresponding one-hot encoded sequence $x_1, \cdots, x_k \in \mathbb{R}^d$ satisfies $h_k = (A_{\alpha_1 \cdots A_{\alpha_k}})^\top \alpha$. The case $k = 0$ is immediate. Suppose the result true for sequences of length up to $k$. One can check easily check that $A \bullet_2 x_i = A^\sigma_i$ for any index $i$. Using the induction hypothesis it then follows that

$$h_{k+1} = A \bullet_1 h_k \bullet_2 x_{k+1} = A^\sigma_{k+1} \bullet_1 h_k = (A^\sigma_{k+1})^\top h_k$$

$$= (A^\sigma_k)^\top (A^\sigma_1 \cdots A^\sigma_k)^\top \alpha = (A^\sigma_1 \cdots A^\sigma_k)^\top \alpha.$$

To conclude, we thus have

$$f_M(x_1, x_2, \cdots, x_k) = \Omega h_k = \Omega (A^\sigma_1 \cdots A^\sigma_k)^\top \alpha = f_A(\sigma_1 \sigma_2 \cdots \sigma_k).$$

\hfill \Box

A.2 Proof of Theorem 3

**Theorem.** Let $f : (\mathbb{R}^d)^* \to \mathbb{R}^p$ be a function computed by a minimal linear 2-RNN with $n$ hidden units and let $L$ be an integer such that $\text{rank}(\left(\mathcal{H}_{\mathcal{f}}^{(2L)}\right)_{\{L+1\}}) = n$.

Then, for any $P \in \mathbb{R}^{d \times n}$ and $S \in \mathbb{R}^{n \times d^2}$ such that $(\mathcal{H}_{\mathcal{f}}^{(2L)})_{\{L+1\}} = PS$, the linear 2-RNN $M = (\alpha, A, \Omega)$ defined by

$$\alpha = (S^\top \left(\mathcal{H}_{\mathcal{f}}^{(L)}\right)_{\{L+1\}})^\top, \quad A = \left(\left(\mathcal{H}_{\mathcal{f}}^{(2L+1)}\right)_{\{L+1\}}\right)_{\{L+1\}} P \times_3 (S^\top)^\top, \quad \Omega^T = P^\top \left(\mathcal{H}_{\mathcal{f}}^{(L)}\right)_{\{L+1\}}$$

is a minimal linear 2-RNN computing $f$.

**Proof.** Let $P \in \mathbb{R}^{d \times n}$ and $S \in \mathbb{R}^{n \times d^2}$ be such that $(\mathcal{H}_{\mathcal{f}}^{(2L)})_{\{L+1\}} = PS$. Define the tensors

$$P^* = [A \bullet_1 \alpha^*, A^*, \cdots, A^*, I_n] \in \mathbb{R}^{d \times \cdots \times d \times n} \quad \text{and} \quad S^* = [I_n, A^*, \cdots, A^*, \Omega^*] \in \mathbb{R}^{n \times d \cdots \times d \times p}$$

of order $L + 1$ and $L + 2$, respectively, and let $P^* = (P^*)_{\{L+1\}} \in \mathbb{R}^{d \times n}$ and $S = (S^*)_{\{L+1\}} \in \mathbb{R}^{n \times d^2}$. Using the identity $\mathcal{H}_{\mathcal{f}}^{(j)} = \left[ A \bullet_1 \alpha, A^*, \cdots, A^*, \Omega^* \right]_{\{j\}}$ for any $j$, one can easily check the following identities:

$$(\mathcal{H}_{\mathcal{f}}^{(2L)})_{\{L+1\}} = P S^*, \quad (\mathcal{H}_{\mathcal{f}}^{(2L+1)})_{\{L+1\}} = A^* \times_1 P^* \times_3 (S^*)^\top,$$

$$(\mathcal{H}_{\mathcal{f}}^{(L)})_{\{L+1\}} = P^* (\Omega^*)^\top, \quad (\mathcal{H}_{\mathcal{f}}^{(L)})_{\{L+1\}} = (S^*)^\top \alpha.$$

Let $M = P^* P^*$. We will show that $\alpha = M^{-\top} \alpha^* = A^* \times_1 M \times_3 M^{-\top}$ and $\Omega = M \Omega^*$, which will entail the results since linear 2-RNN are invariant under change of basis (see Section 2). First observe that $M^{-1} = S^* S^\top$. Indeed, we have $P^* P^* S^* S^\top = P^* (\mathcal{H}_{\mathcal{f}}^{(2L)})_{\{L+1\}} S^\top = P^* P S S^\top = I$ where we used the fact that $P$ (resp. $S$) is of full column rank (resp. row rank) for the last equality.
Then, whenever \( \Omega \) is an algebraic subvariety of \( N \) is a proper algebraic subvariety of \( \mathbb{R}^{d_l} \), we denote by \( H \) that \( \Omega \) is a proper algebraic subvariety of \( \mathbb{R}^{d_l} \), and hence has Lebesgue measure zero [12, Section 2.6.5].

Suppose we have access to datasets \( D_l = \{(x_1^{(n)}, x_2^{(n)}, \ldots, x_{n_l}^{(n)}) : y_n \} \), where each \( x_i^{(n)} \) are drawn independently from the standard normal distribution and where each \( y_n = f(x_1^{(n)}, x_2^{(n)}, \ldots, x_{n_l}^{(n)}) \).

Then, whenever \( N_l \geq d_l \) for each \( l \in \{L, 2L, 2L + 1\} \), the linear 2-RNN \( M \) returned by Algorithm 1 with the least-squares method satisfies \( f_M = f \) with probability one.

Proof. We just need to show for each \( l \in \{L, 2L, 2L + 1\} \) that, under the hypothesis of the Theorem, the Hankel tensors \( \mathcal{H}^{(l)} \) computed in line 4 of Algorithm 1 are equal to the true Hankel tensors \( \mathcal{H}^{(l)} \) with probability one. Recall that these tensors are computed by solving the least-squares problem

\[
\mathcal{H}^{(l)} = \arg\min\limits_{T \in \mathbb{R}^{d \times \cdots \times d \times p}} \|X(T) - Y\|_F^2,
\]

where \( X \in \mathbb{R}^{N_l \times d_l} \) is the matrix with rows \( x_1^{(n)} \otimes x_2^{(n)} \otimes \cdots \otimes x_{n_l}^{(n)} \) for each \( n \in [N_l] \). Since \( X(\mathcal{H}^{(l)}) = Y \) and since the solution of the least-squares problem is unique as soon as \( X \) is of full column rank, we just need to show that this is the case with probability one when the entries of the vectors \( x_i^{(n)} \) are drawn at random from a standard normal distribution. The result will then directly follow by applying Theorem 3.

We will show that the set

\[
S = \{(x_1^{(n)}, \ldots, x_{n_l}^{(n)}) \mid n \in [N_l], \dim(\text{span}(\{x_1^{(n)} \otimes x_2^{(n)} \otimes \cdots \otimes x_{n_l}^{(n)}\})) < d_l^4\}
\]

has Lebesgue measure 0 in \((\mathbb{R}^{d_l})^{N_l} \simeq \mathbb{R}^{d_lN_l}\) as soon as \( N_l \geq d_l \), which will imply that it has probability 0 under any continuous probability, hence the result. For any \( S = \{(x_1^{(n)}, \ldots, x_{n_l}^{(n)})\} \), we denote by \( X_S \in \mathbb{R}^{N_l \times d_l} \) the matrix with rows \( x_1^{(n)} \otimes x_2^{(n)} \otimes \cdots \otimes x_{n_l}^{(n)} \). One can easily check that \( S \in S \) if and only if \( X_S \) is of rank strictly less than \( d_l^4 \), which is equivalent to the determinant of \( X_S^\top X_S \) being equal to 0. Since this determinant is a polynomial in the entries of the vectors \( x_i^{(n)} \), \( S \) is an algebraic subvariety of \( \mathbb{R}^{d_lN_l} \). It is then easy to check that the polynomial \( \det(X_S^\top X_S) \) is not uniformly 0 when \( N_l \geq d_l \). Indeed, it suffices to choose the vectors \( x_i^{(n)} \) such that the family \( \{x_1^{(n)} \otimes x_2^{(n)} \otimes \cdots \otimes x_{n_l}^{(n)}\}_{n=1}^{N_l} \) spans the whole space \( \mathbb{R}^{d_l^4} \) (which is possible since we can choose arbitrarily any of the \( N_l \geq d_l \) elements of this family), hence the result. In conclusion, \( S \) is a proper algebraic subvariety of \( \mathbb{R}^{d_lN_l} \) and hence has Lebesgue measure zero [12, Section 2.6.5].

\[\Box\]

\[\Box\]

Note that the theorem can be adapted if such an integer \( L \) does not exists (see supplementary material).
B  Lifting the simplifying assumption

We now show how all our results still hold when there does not exist an $L$ such that \( \text{rank}((\mathcal{H}_{11}^{(2L)})_{\langle L, L+1 \rangle}) = n \). Recall that this simplifying assumption followed from assuming that the sets $\mathcal{P} = \mathcal{S} = [d]^L$ form a complete basis for the function $\tilde{f} : [d]^* \to \mathbb{R}^p$ defined by $\tilde{f}(i_1 i_2 \cdots i_k) = f(e_{i_1}, e_{i_2}, \cdots, e_{i_k})$. We first show that there always exists an integer $L$ such that $\mathcal{P} = \mathcal{S} = \cup_{l \leq l L}[d]^l$ forms a complete basis for $\tilde{f}$. Let $M = (\alpha^*, \mathcal{A}^*, \Omega^*)$ be a linear 2-RNN with $n$ hidden units computing $f$ (i.e. such that $f_M = f$). It follows from Theorem 2 and from the discussion at the beginning of Section 4.1 that there exists a vv-WFA computing $\tilde{f}$ and it is easy to check that $\text{rank}(\tilde{f}) = n$. This implies $\text{rank}((\mathcal{H}_{11}(f))_{\langle 1 \rangle}) = n$ by Theorem 1. Since $\mathcal{P} = \mathcal{S} = \cup_{l \leq l L}[d]^l$ converges to $[d]^*$ as $l$ grows to infinity, there exists an $L$ such that the finite sub-block $\mathcal{H}_f \in \mathbb{R}^{P \times S \times P}$ of $\mathcal{H}_f \in \mathbb{R}^{[d]^* \times [d]^* \times P}$ satisfies $\text{rank}((\mathcal{H}_{11}(f))_{\langle 1 \rangle}) = n$, i.e. such that $\mathcal{P} = \mathcal{S} = \cup_{l \leq l L}[d]^l$ forms a complete basis for $\tilde{f}$.

Now consider the finite sub-blocks $\tilde{\mathcal{H}}_f^+ \in \mathbb{R}^{P \times [d]^* \times P}$ and $\tilde{\mathcal{H}}_f^- \in \mathbb{R}^{P \times P}$ of $\mathcal{H}_f$ defined by

$$
(\tilde{\mathcal{H}}_f^+)^{u,i,v} = \tilde{f}(uiv), \quad \text{and } (\tilde{\mathcal{H}}_f^-)^{u,i,v} = f(u)
$$

for any $u \in \mathcal{P} = \mathcal{S}$ and any $i \in [d]$. One can check that Theorem 3 holds by replacing mutatis mutandi $\mathcal{H}_f^{(2L)}_{\langle L, L+1 \rangle}$ by $\mathcal{H}_f^{(1)}$, $\mathcal{H}_f^{(2L+1)}_{\langle l, l+1 \rangle}$ by $\tilde{\mathcal{H}}_f^+$, $\mathcal{H}_f^{(L)}_{\langle l, l \rangle}$ by $\tilde{\mathcal{H}}_f^-$ and $\mathcal{H}_f^{(L)}_{\langle L+1 \rangle}$ by vec($\tilde{\mathcal{H}}_f^-)$. 

To conclude, it suffices to observe that both $\tilde{\mathcal{H}}_f^+$ and $\tilde{\mathcal{H}}_f^-$ can be constructed from the entries for the tensors $\mathcal{H}_f^{(l)}$ for $1 \leq l \leq 2L + 1$, which can be recovered (or estimated in the noisy setting) using the techniques described in Section 4.2 of the main paper.

We thus showed that linear 2-RNNs can be provably learned even when there does not exist an $L$ such that $\text{rank}((\mathcal{H}_{11}^{(2L)})_{\langle L, L+1 \rangle}) = n$. In this setting, one needs to estimate enough of the tensors $\mathcal{H}_f^{(l)}$ to reconstruct a complete sub-block $\tilde{\mathcal{H}}_f$ of the Hankel tensor $\mathcal{H}$ (along with the corresponding tensor $\tilde{\mathcal{H}}_f^+$ and matrix $\tilde{\mathcal{H}}_f^-$) and recover the linear 2-RNN by applying Theorem 3. In addition, one needs to have access to sufficiently large datasets $D_l$ for each $l \in [2L + 1]$ rather than only the three datasets mentioned in Theorem 4. However the data requirement remains the same in the case where we assume that each of the datasets $D_l$ is constructed from a unique training dataset $S = \{((x_1^{(n)}, x_2^{(n)}, \cdots, x_T^{(n)}), (y_1^{(n)}, y_2^{(n)}, \cdots, y_T^{(n)}))\}_{n=1}^N$ of input/output sequences.