Bayesian Latent-Normal Inference for the Rank Sum Test, the Signed Rank Test, and Spearman’s ρ

Johnny van Doorn*
Department of Psychological Methods, University of Amsterdam

Alexander Ly
Department of Psychological Methods, University of Amsterdam

Maarten Marsman
Department of Psychological Methods, University of Amsterdam

Eric-Jan Wagenmakers
Department of Psychological Methods, University of Amsterdam

April 24, 2018

Abstract

Bayesian inference for rank-order problems is frustrated by the absence of an explicit likelihood function. This hurdle can be overcome by assuming a latent normal representation that is consistent with the ordinal information in the data: the observed ranks are conceptualized as an impoverished reflection of an underlying continuous scale, and inference concerns the parameters that govern the latent representation. We apply this generic data-augmentation method to obtain Bayesian counterparts of three popular rank-based tests: the rank sum test, the signed rank test, and Spearman’s ρ.

Keywords: Rank-based testing, Bayes factor, data augmentation.

*Correspondence may be addressed to Johnny van Doorn (E-mail: J.B.vanDoorn@uva.nl), Department of Psychological Methods, University of Amsterdam, Valckeniersstraat 59, 1018XE Amsterdam, the Netherlands. The authors acknowledge support from the European Science Foundation (ERC grant 283876) and the Netherlands Organisation for Scientific Research
1 Introduction

Rank-based statistical procedures offer a range of advantages. First, they are robust to outliers and to violations of normality. Second, they are invariant under monotonic transformations, which is desirable when interest concerns a hypothesized concept (e.g., rat intelligence) whose relation to the measurement scale is only weakly specified (e.g., brain volume or log brain volume could be used as a predictor; without a process model that specifies how brain physiology translates to rat intelligence, neither choice is privileged). Third, many data sets are inherently ordinal (e.g., Likert scales, where survey participants are asked to indicate their opinion on, say, a 7-point scale ranging from ‘disagree completely’ to ‘agree completely’). Finally, rank-based procedures can be relatively efficient; when all parametric assumptions are met, the rank-based procedures discussed in this manuscript are only slightly less efficient than their fully parametric counterparts; when parametric assumptions are violated, however, rank-based procedures can be substantially more efficient (Hollander and Wolfe 1973).

Prominent rank-based tests include the Mann-Whitney-Wilcoxon rank sum test (i.e., the rank-based equivalent of the two-sample $t$-test), the Wilcoxon signed rank test (i.e., the rank-based equivalent of the paired sample $t$-test), and Spearman’s $\rho$ (i.e., a rank-based equivalent of the Pearson correlation coefficient). These ordinal tests were developed within the frequentist statistical paradigm, and Bayesian analogues have, to the best of our knowledge, not yet been proposed. We speculate that one of the reasons for the dearth of Bayesian development is that ordinal data lack a likelihood function. As stated by Jeffreys (1939, pp. 178-179) for the case of Spearman’s $\rho$:

“The rank correlation, while certainly useful in practice, is difficult to interpret. It is an estimate, but what is it an estimate of? That is, it is calculated from the observations, but a function of the observations has no relevance beyond the observations unless it is an estimate of a parameter in some law. Now what can this law be? [...] the interpretation is not clear.”

This difficulty can be overcome by postulating a latent, normally distributed level for the observed data. In other words, the rank data are conceptualized to be an impoverished
reflection of richer latent data that are governed by a specific likelihood function. The latent
normal distribution was chosen for computational convenience and ease of interpretation.
This general procedure is widely known as data augmentation (Tanner and Wong 1987; Albert and Chib
1993), and Bayesian inference for the parameters of interest (e.g., a
location parameter $\delta$ or an association parameter $\rho$) can be achieved using Markov chain
Monte Carlo (MCMC).

Below we first outline the general framework and then develop Bayesian counterparts
for three popular frequentist rank-based procedures: the rank sum test, the signed rank
test, and Spearman’s rank correlation. Each test is accompanied by a data example that
highlights the desirable properties of rank-based inference.

2 General Methodology

2.1 Latent Normal Models

Latent normal models were first introduced by Pearson (1900) as a means of modeling data
from a $2 \times 2$ cross-classification table. The method was later extended by Pearson and
Pearson (1922) to accommodate $r \times s$ tables. Instead of modeling the count data directly,
Pearson assumed a latent bivariate normal level with certain governing parameters. In
the case of cross-classification tables, the governing parameter is the polychoric correlation
coefficient (PCC) and refers to Pearson’s correlation on the bivariate, latent normal level.

A maximum likelihood estimator for the PCC was developed by Olsson (1979); Olsson
et al. (1982), and a Bayesian framework for the PCC was later introduced by Albert
(1992b). This idea was extended by Pettitt (1982) to rank likelihood models, where the
latent boundaries are not estimated but determined directly by the latent scores (e.g., Hoff
2009). Recently, van Doorn et al. (2017) applied this framework to Bayesian inference for
Kendall’s $\tau$.

In general, the latent normal methodology allows one to transform ordinal problems to
parametric problems. The resulting models now have a data-generating process and are
governed by easily interpretable parameters. A detailed sampling algorithm of the general
methodology is presented in the next section.
2.2 Posterior Distribution and Bayes Factor

Using Bayes’ rule, the joint posterior of the model parameters $\theta$ and latent normal values (i.e., $(z^x, z^y)$), given the data (i.e., $(x, y)$), can be decomposed as follows:

$$ P(z^x, z^y, \theta \mid x, y) \propto P(x, y \mid z^x, z^y) \times P(z^x, z^y \mid \theta) \times P(\theta). $$  \hspace{1cm} (1)

In the rank-based context, the likelihood refers to the marginal of $P(x, y \mid z^x, z^y)P(z^x, z^y \mid \theta)$ with respect to the augmented variables $z^x$ and $z^y$. From a generative perspective, parameters $\theta$ produce latent normal data $z^x$ and $z^y$, and these in turn yield ordinal data $x$ and $y$.

The first factor in the right-hand side of Equation (1) $P(x, y \mid z^x, z^y)$, consists of a set of indicator functions, presented below, that map the observed ranks to latent scores, such that the ordinal information (i.e., the ranking function) is preserved. This is similar to the approach of Albert (1992a) and Albert and Chib (1993), who sampled latent scores to binary or polytomous response data from a normal distribution that was truncated with respect to the ordinal information of the data. Consequently, across the MCMC iterations the ordinal information in the latent values remains constant and identical to that in the original data. For the value $z^x_i$, this means that its range is truncated by the lower and upper thresholds that are respectively defined as:

$$ a^x_i = \max_{j: x_j < x_i} (z^x_j) \hspace{1cm} (2) $$

$$ b^x_i = \min_{j: x_j > x_i} (z^x_j) \hspace{1cm} (3) $$

For example, suppose that on a particular MCMC iteration we wish to augment the observed ordinal value $x_i$ to a latent $z^x_i$; on the latent scale, the lower threshold $a^x_i$ is given by the maximum latent value associated with all $x$ lower than $x_i$, whereas the upper threshold $b^x_i$ is determined by the minimum latent value associated with all $x$ higher than $x_i$.

The second factor in the right-hand side of Equation (1) $P(z^x, z^y \mid \theta)$, is the bivariate
normal distribution of the latent scores given the model parameters $\theta$:

$$
\begin{pmatrix}
Z^X \\
Z^Y
\end{pmatrix} \sim \mathcal{N}
\begin{bmatrix}
\left( \begin{array}{c}
\mu_{zx} \\
\mu_{zy}
\end{array} \right),

\begin{bmatrix}
\sigma_{zx} & \rho_{zxz_y} \\
\rho_{zxz_y} & \sigma_{zy}
\end{bmatrix}
\end{bmatrix}.
$$

Finally, the third factor in the right-hand side of Equation 1, $P(\theta)$, refers to the prior distributions for the model parameters.

After obtaining the joint posterior distribution for $\theta$ by MCMC sampling, we can either focus on estimation and present the marginal posterior distribution for the single parameter of interest $\delta$, or we can conduct a Bayes factor hypothesis test and compare the predictive performance of a point-null hypothesis $H_0$ (in which the parameter of interest is fixed at a predefined value $\delta_0$) against that of an alternative hypothesis $H_1$ (in which $\delta$ is free to vary; [Kass and Raftery, 1995] [Jeffreys, 1961] [Etz and Wagenmakers, 2017]). For such nested models the Bayes factor be be easily obtained using the Savage-Dickey density ratio ([Dickey and Lientz, 1970] [Wagenmakers et al., 2010]), that is, the ratio of the posterior and prior ordinate for the parameter of interest $\delta$, under $H_1$, evaluated at the point $\delta_0$ specified under $H_0$:

$$
BF_{10} = \frac{p(\delta_0 | H_1)}{p(\delta_0 | \text{data}, H_1)}.
$$

3 Case 1: Wilcoxon Rank Sum Test

3.1 Background

The ordinal counterpart to the two-sample $t$-test is known as the Wilcoxon rank sum test (or as the Mann-Whitney-Wilcoxon U test). It was introduced by Wilcoxon (1945) and further developed by Mann and Whitney (1947), who worked out the statistical properties of the test. Let $x = (x_1, ..., x_{n_1})$ and $y = (y_1, ..., y_{n_2})$ be two data vectors that contain measurements of $n_1$ and $n_2$ units, respectively. The aggregated ranks $r^x, r^y$ (i.e., the ranking of $x$ and $y$ together) are defined as:

$$
r^x_i = \text{rank of } x_i \text{ among } (x_1, ..., x_{n_1}, y_1, ..., y_{n_2}),
$$
\[ r^y_i = \text{rank of } y_i \text{ among } (x_1, \ldots, x_{n_1}, y_1 \ldots y_{n_2}). \]

The test statistic \( U \) is then given by summing over either \( r^x \) or \( r^y \), and subtracting \( \frac{n_x(n_x+1)}{2} \) or \( \frac{n_y(n_y+1)}{2} \), respectively. In order to test for a difference between the two groups, the observed value of \( U \) can be compared to the value of \( U \) that corresponds to no difference. This point of testing is defined as \( \frac{n_1n_2}{2} \).

To illustrate the procedure, consider the following hypothetical example. In the movie review section of a newspaper, three action movies and three comedy movies are each assigned a star rating between 0 and 5. Let \( X = (4, 3, 1) \) be the star ratings for the action movies, and let \( Y = (2, 3, 5) \) be the star ratings for the comedy movies. The corresponding aggregated ranks are \( R^x = (5, 3.5, 1) \) and \( R^y = (2, 3.5, 6) \). The test statistic \( U \) is then obtained by summing over either \( R^x \) or \( R^y \), and subtracting \( \frac{3(3+1)}{2} = 6 \), yielding 3.5 or 5.5, respectively. Either of these values can then be compared to the null point which is equal to \( \frac{n_1n_2}{2} = 4.5 \).

An often used standardized effect size for \( U \) is the rank-biserial correlation, denoted \( \rho_{rb} \), which is the correlation coefficient used as a measure of association between a nominal dichotomous variable and an ordinal variable. The transformation is as follows:

\[ \rho_{rb} = 1 - \frac{2U}{n_1n_2}. \]  

The rank-biserial correlation can also be expressed as the difference between the proportion of data pairs where \( x_i > y_j \) versus \( x_i < y_j \) (Cureton 1956; Kerby 2014):

\[ \rho_{rb} = \frac{\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} Q(x_i, y_j)}{n_1n_2}, \]  

where \( Q(x_i, y_j) \) is the sign indicator function defined as

\[ Q(x_i, y_i) = \begin{cases} 
-1 & \text{if } x_i - y_i < 0 \\
+1 & \text{if } x_i - y_i > 0 
\end{cases}. \]

This provides an intuitive interpretation of the test procedure: each data point in \( x \) is compared to each data point in \( y \) and scored -1 or 1 if it is lower or higher, respectively.
In the movie ratings data example, there are three pairs for which \( x_i > y_j \), five pairs for which \( x_i < y_j \), and one pair for which \( x_i = y_j \), yielding a rank-biserial correlation coefficient of \( \frac{3 - 5}{9} = -0.22 \), which is an indication that comedy movies receive slightly more positive reviews.

Compared to the parametric two-sample \( t \)-test, the rank sum test does not suffer much in asymptotic relative efficiency (ARE), that is, the ratio of the number of observations necessary to achieve the same level of power (Lehmann, 1999). Specifically, the rank sum test has an ARE of \( \frac{3}{\pi} \approx 0.955 \) when the data are normally distributed (Hodges and Lehmann, 1956; Lehmann, 1975). Thus, even when the distributional assumption of the \( t \)-test holds, the rank sum test is still relatively efficient. The ARE increases as the data distribution grows more heavy-tailed, with a maximum value of infinity. In addition, results for other distributions include the logistic distribution (ARE = \( \frac{\pi^2}{9} \approx 1.097 \)), the Laplace distribution (ARE = 1.5), and the exponential distribution (ARE = 3); these ARE values > 1 indicate that the rank test outperforms the \( t \)-test (van der Vaart, 2000).

### 3.2 Sampling Algorithm

The data augmentation algorithm for the rank sum test follows the graphical model outlined in Figure 1. The ordinal information contained in the aggregated ranking constrains the corresponding values for the latent normal parameters \( Z^x \) and \( Z^y \) to lie within certain intervals (i.e., the ordinal information imposes truncation). The parameter of interest here is \( \delta \), the difference in location of the distributions for \( Z^x \) and \( Z^y \). We follow Jeffreys (1961) and assign \( \delta \) a Cauchy prior with scale parameter \( \gamma \). For computational simplicity, this prior is implemented as a normal distribution with an inverse gamma prior on the variance, where the shape parameter is set to 0.5 and the scale parameter is set to \( \gamma^2/2 \) (Liang et al., 2008; Rouder et al., 2009). The difference with earlier work is that we set \( \sigma \) to 1, as the rank data contain no information about the variance and the inclusion of \( \sigma \) in the sampling algorithm becomes redundant.

In order to sample from the posterior distributions of \( \delta, Z^x \) and \( Z^y \), we used Gibbs sampling (Geman and Geman, 1984). Specifically, the sampling algorithm for the latent \( \delta \) is as follows, at sampling time point \( s \):
Figure 1: The graphical model underlying the Bayesian rank sum test. The latent, continuous scores are denoted by $Z^x_i$ and $Z^y_i$, and their manifest rank values are denoted by $r^x_i$ and $r^y_j$. The latent scores are assumed to follow a normal distribution governed by the parameter $\delta$. This parameter is assigned a Cauchy prior distribution, which for computational convenience is reparameterized to a normal distribution with variance $g$ (which is then assigned an inverse gamma distribution).

1. For each $i$ in $(1, \ldots, n_x)$, sample $Z^x_i$ from a truncated normal distribution, where the lower threshold is $a^x_i$ given in (2) and the upper threshold is $b^x_i$ given in (3):

$$
(Z^x_i \mid z^x_i, z^y_i, \delta) \sim \mathcal{N}(a^x_i, b^x_i)(-\frac{1}{2}\delta, 1),
$$

where the subscripts of $\mathcal{N}$ indicate the interval that is sampled from.

2. For each $i$ in $(1, \ldots, n_y)$, the sampling procedure for $Z^y_i$ is analogous to step 1, with

$$
(Z^y_i \mid z^y_i, z^x_i, \delta) \sim \mathcal{N}(a^y_i, b^y_i)(\frac{1}{2}\delta, 1).
$$

3. Sample $\delta$ from

$$(\delta \mid z^x, z^y, g) \sim \mathcal{N}(\mu_\delta, \sigma_\delta),$$

where

$$
\mu_\delta = \frac{2g(n_y \overline{z^y} - n_x \overline{z^x})}{g(n_x + n_y) + 4},
$$

$$
\sigma_\delta^2 = \frac{4g}{g(n_x + n_y) + 4}.
$$

4. Sample $g$ from

$$(G \mid \delta) \sim \text{Inverse Gamma}
\left(1, \frac{\delta^2 + \gamma^2}{2}\right).$$
where $\gamma$ determines the scale (i.e., width) of the Cauchy prior on $\delta$.

Repeating the algorithm a sufficient number of times yields samples from the posterior distributions of $Z^x, Z^y,$ and $\delta$.

### 3.3 Data Example

Cortez and Silva (2008) gathered data from 395 students concerning their math performance (scored between 1 and 20) and their level of alcohol intake (self-rated on a Likert scale between 1 and 5). Students passed the course if they scored $\geq 10$, and we will test whether students who failed the course ($n_1 = 130$) had a higher self-reported alcohol intake than their peers who passed ($n_2 = 265$).

As alcohol intake was measured on a Likert scale, the data contain many ties and show extreme non-normality. These properties make this data set particularly suitable for the latent-normal rank sum test. The hypotheses can be specified as follows:

$$H_0 : \delta = 0,$$

$$H_1 : \delta \sim \text{Cauchy} \left(0, \frac{1}{\sqrt{2}} \right).$$

The null hypothesis posits that alcohol intake does not differ between the students who passed the course and those who failed. The alternative hypothesis posits the presence of an effect and assigns effect size a Cauchy distribution with scale parameter set to $1/\sqrt{2}$, as advocated by Morey and Rouder (2015). Figure 2 shows the resulting posterior distribution for $\delta$ under $H_1$ and the associated Bayes factor. The posterior median for $\delta$ equals $-0.049$, with a 95% credible interval that ranges from $-0.273$ to $0.169$. The corresponding Bayes factor indicates that the data are about 7.5 times more likely under $H_0$ than under $H_1$, indicating moderate evidence against the hypothesis that self-reported alcohol intake differentiates between students who did and who did not pass the math exam.
Figure 2: Do students who flunk a math course report drinking more alcohol? Results for
the Bayesian rank sum test as applied to the data set from Cortez and Silva (2008). The
dashed line indicates the Cauchy prior with scale \( \frac{1}{\sqrt{2}} \). The solid line indicates
the posterior distribution. The two grey dots indicate the prior and posterior ordinate at
the point under test, in this case \( \delta = 0 \). The ratio of the ordinates gives the Bayes
factor.

4 Case 2: Wilcoxon Signed Rank Test

4.1 Background

The ordinal counterpart to the paired samples \( t \)-test was proposed by Wilcoxon (1945),
who termed it the signed rank test. The test procedure involves taking the difference
scores between the two samples under consideration and ranking the absolute values. The
procedure may also be applied to one-sample scenarios by ranking the differences between
the one observed sample and the point of testing. These ranks are then multiplied by the
sign of the respective difference scores and summed to produce the test statistic \( W \). For
the paired samples \( t \)-test, let \( x = (x_1, ..., x_n) \) and \( y = (y_1, ..., y_n) \) be two data vectors each
containing measurements of the same \( n \) units, and let \( d = (d_1, ..., d_n) \) denote the difference
scores. For the one-sample \( t \)-test, this process is analogous, except \( y \) is replaced by the test
value. The test statistic is then defined as:

\[ W = \sum_{i} \left[ \text{rank}(|d_i|) \times Q(d_i) \right], \]

where \( Q \) is the sign indicator function given in [8].

To illustrate the procedure, consider the following hypothetical data example. Three students take a math exam, graded between 0 and 10, before and after receiving a tutoring session. Let \( X = (5, 8, 4) \) be their scores on the exam before the session, and let \( Y = (6, 7, 7) \) be their scores on the exam after the session. The difference scores, the ranks of the absolute difference scores, and the sign indicator function are presented in Table 1. In order to have a positive test statistic indicate an increase in scores, the difference scores are defined here as \((y_i - x_i)\). The test statistic \( W \) is then calculated by summing over the product of the fourth and fifth column: \(1.5 - 1.5 + 3 = 3.0\). This value indicates a slight increase in math scores after the tutoring session.

| \( i \) | \((y_i - x_i)\) | \(d_i\) | rank(|\(d_i|)\) | \(Q(\d_i)\) |
|---|---|---|---|---|
| 1 | 6 - 5 | 1 | 1.5 | 1 |
| 2 | 7 - 8 | -1 | 1.5 | -1 |
| 3 | 7 - 4 | 3 | 3 | 1 |

Table 1: The scores, difference scores, ranks of the absolute difference scores, and the sign indicator function \( Q \) for the hypothetical scenario where \( X = (5, 8, 4) \) are the initial scores on a math exam and \( Y = (6, 7, 7) \) are the scores on the exam after a tutoring session.

The signed rank test is similar to the sign test, where the procedure is to sum over the sign indicator function. The difference here is that the output of the sign indicator function is weighted by the ranked magnitude of the absolute differences. The signed rank test has a higher ARE than the sign test: a relative efficiency of \(\frac{3}{2}\) for all distributions (Conover, 1999).

For the one-sample scenario, the ARE of the signed rank test (compared to the fully parametric \(t\)-test) is similar to the ARE of the rank sum test for the unpaired two-sample scenario; for example, when the data follow a normal distribution the ARE equals \(\frac{3}{\pi}\). For other distributions, especially when these are heavy-tailed, the signed rank test outperforms the \(t\)-test (Lehmann, 1999; van der Vaart, 2000).
4.2 Sampling Algorithm

Figure 3: The graphical model underlying the Bayesian signed rank test. The latent, continuous difference scores are denoted by $Z^d_i$, and their manifest signed rank values are denoted by $d_i$. The latent scores are assumed to follow a normal distribution governed by parameter $\delta$. This parameter is assigned a Cauchy prior distribution, which for computational convenience is reparameterized to a normal distribution with variance $g$ (which is then assigned an inverse gamma distribution).

The data augmentation algorithm is similar to that of the rank sum test and is outlined in Figure 3. Here $d$ denotes the difference scores as ordinal manifestations of latent, normally distributed values $Z^d$. The parameter of interest is again the standardized location parameter $\delta$, which is assigned a Cauchy prior distribution with scale parameter $\gamma$. Similar to the rank sum sampling procedure, the variance of $Z^d$ is set to 1, as the ranked data contain no information about the variance. The computational complexity of sampling from the posterior distribution of $\delta$ is again reduced by introducing the parameter $g$. The Gibbs algorithm for the data augmentation and sampling $\delta$ is as follows, at sampling time point $s$:

1. For each value of $i$ in $(1, \ldots, n)$, sample $Z^d_i$ from a truncated normal distribution, where the lower threshold is $a^d_i$ given in (2) and the upper threshold is $b^d_i$ given in (3):

$$
(Z^d_i \mid z^d_i, \delta) \sim N(a^d_i, b^d_i | \delta, 1)
$$
2. Sample $\delta$ from 
\[(\delta \mid z^d, g) \sim \mathcal{N}(\mu_\delta, \sigma^2_\delta),\]
where 
\[\mu_\delta = \frac{gnz^d_i}{gn + 1}\]
\[\sigma^2_\delta = \frac{1}{n + \frac{1}{g}}\]

3. Sample $g$ from 
\[(g \mid \delta) \sim \text{Inverse Gamma}\left(1, \frac{\delta^2 + \gamma^2}{2}\right),\]
where $\gamma$ determines the scale (i.e., width) of the Cauchy prior on $\delta$.

Repeating the algorithm a sufficient number of times yields samples from the posterior distributions of $Z^d$ and $\delta$.

### 4.3 Data Example

[Thall and Vail (1990)] investigated a data set obtained by D. S. Salsburg concerning the effects of the drug progabide on the occurrence of epileptic seizures. During an initial eight week baseline period, the number of epileptic seizures were recorded in a sample of 59 epileptics. Next, the patients were given either a placebo (28 participants) or progabide (31 participants), and the number of epileptic seizures was recorded for another eight weeks. In order to accommodate the discreteness and non-normality of the data, Thall and Vail (1990) applied a log-transformation on the counts.

This log-transformation has a clear impact on the outcome of a parametric Bayesian $t$-test ([Morey and Rouder 2015]: BF$_{10} \approx 0.2$ for the raw data, whereas BF$_{10} \approx 2.95$ for the log-transformed data. Here we analyze the data with the signed rank test; because this test is invariant under monotonic transformations, the same inference will result regardless of whether or not the data are log-transformed.

The hypothesis specification here is similar to that of the previous setup:

\[H_0 : \delta = 0,\]
\[ \mathcal{H}_1 : \delta \sim \text{Cauchy} \left( 0, \frac{1}{\sqrt{2}} \right), \]

where the null hypothesis postulates that the effect is absent whereas the alternative hypothesis assigns effect size a Cauchy prior distribution.

Figure 4 shows the resulting posterior distribution for \( \delta \) under \( \mathcal{H}_1 \) and the associated Bayes factor. The posterior median for \( \delta \) equals 0.276, with a 95\% credible interval that ranges from \(-0.079\) to \(0.638\). The corresponding Bayes factor indicates that the data are about 1.55 times more likely under \( \mathcal{H}_0 \) than under \( \mathcal{H}_1 \), indicating that, for the purpose of discriminating \( \mathcal{H}_0 \) from \( \mathcal{H}_1 \), the data are almost perfectly uninformative.

Figure 4: Does progabide reduce the frequency of epileptic seizures? Results for the Bayesian signed rank test as applied to the data set presented in Thall and Vail (1990). The dashed line indicates the Cauchy prior with scale \( \frac{1}{\sqrt{2}} \). The solid line indicates the posterior distribution. The two grey dots indicate the prior and posterior ordinate at the point under test, in this case \( \delta = 0 \). The ratio of the ordinates gives the Bayes factor.

5 Case 3: Spearman’s \( \rho \)

5.1 Background

Spearman (1904) introduced the rank correlation coefficient \( \rho \) in order to overcome the
main shortcoming of Pearson’s product moment correlation, namely its inability to capture
monotonic but non-linear associations between variables. Spearman’s method first applies
the rank transformation on the data and then computes the product-moment correlation
on the ranks. Let \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) be two data vectors each containing
measurements of the same \( n \) units, and let \( r^x = (r^x_1, \ldots, r^x_n) \) and \( r^y = (r^y_1, \ldots, r^y_n) \) denote
their rank-transformed values, where each value is assigned a ranking within its variable.
This then leads to the following formula for Spearman’s \( \rho \):
\[
\rho_s = \frac{\text{Cov}_{r^x r^y}}{\sigma_{r^x} \sigma_{r^y}}.
\]

5.2 Sampling Algorithm

The graphical model in Figure 5 illustrates the data augmentation setup for inference on the
latent correlation parameter \( \rho \). The sampling method is a Metropolis-within-Gibbs algo-
rithm, where data augmentation is conducted with a Gibbs sampling algorithm as before,
but combined with a random walk Metropolis-Hastings sampling algorithm (Metropolis
et al., 1953; Hastings, 1970) to sample from the posterior distribution of \( \rho \) (see also van
Doorn et al., 2017).

The sampling algorithm for the latent correlation is as follows, at sampling time point
\( s \):

1. For each \( i \) in \((1, \ldots, n_x)\), sample \( Z_i^x \) from a truncated normal distribution, where
the lower threshold is \( a_i^x \) given in (2) and the upper threshold is \( b_i^x \) given in (3):
\[
(Z_i^x \mid z_i^y, z_i^y, \rho_{z^x, z^y}) \sim \mathcal{N}(a_i^x, b_i^x, \sqrt{1 - \rho_{z^x, z^y}^2}) T(a_i^x, b_i^x).
\]

2. For each \( i \) in \((1, \ldots, n_y)\), the sampling procedure for \( Z_i^y \) is analogous to step 1.

3. Sample a new proposal for \( \rho_{z^x, z^y} \), denoted \( \rho^* \), from the asymptotic normal approxi-
mation to the sampling distribution of Fisher’s \( z \)-transform of \( \rho \) (Fisher, 1915):
\[
tanh^{-1}(\rho^*) \sim \mathcal{N}\left( \tanh^{-1}(\rho^{s-1}), \frac{1}{\sqrt{(n-3)}} \right).
\]
The acceptance rate \( \alpha \) is determined by the likelihood ratio of \( (z^x, z^y | \rho^s) \) and \( (z^x, z^y | \rho^{s-1}) \), where each likelihood is determined by the bivariate normal distribution in (4):

\[
\alpha = \min \left( 1, \frac{P(z^x, z^y | \rho^s)}{P(z^x, z^y | \rho^{s-1})} \right). 
\]

Repeating the algorithm a sufficient number of times yields samples from the posterior distributions of \( z^x, z^y, \) and \( \rho_{z^x, z^y} \).

\[ Z_i^x \sim \text{Normal}(0, 1) \]
\[ Z_i^y \sim \text{Normal}(0, 1) \]
\[ \rho_{Z_i^x Z_i^y} \sim \text{Uniform}(-1, 1) \]
\[ r_i^x \leftarrow \text{Rank}(Z_i^x) \]
\[ r_i^y \leftarrow \text{Rank}(Z_i^y) \]

Figure 5: The graphical model underlying the Bayesian test for Spearman’s \( \rho \). The latent, continuous scores are denoted by \( Z_i^x \) and \( Z_i^y \), and their manifest rank values are denoted by \( r_i^x \) and \( r_i^y \). The latent scores are assumed to follow a normal distribution governed by parameter \( \rho \) (which is assigned a uniform prior distribution).

### 5.3 Transforming Parameters

The transition from Pearson’s \( \rho \) to Spearman’s \( \rho \) can be made using a statistical relation described in [Kruskal (1958)](1958). This relation, defined as

\[
\rho_S = \frac{6}{\pi} \sin^{-1} \left( \frac{\rho}{2} \right). 
\]

enables the transformation of Pearson’s \( \rho \) to Spearman’s \( \rho \) when the data follow a bivariate normal distribution. Since the latent data are assumed to be normally distributed, this means that the posterior samples for Pearson’s \( \rho \) can be easily transformed to posterior samples for Spearman’s \( \rho \).
5.4 Data Example

We return to the data set from Cortez and Silva (2008) and examine the possibility that math grades (ranging from 0 to 20) are associated with the quality of family relations (self-reported on a Likert scale that ranges from 1−5). The hypotheses are specified as follows,

\[ \mathcal{H}_0 : \rho = 0, \]
\[ \mathcal{H}_1 : \rho \sim \text{Uniform}\([-1, 1]\), \]

where the null hypothesis specifies the lack of an association between the two variables and the alternative hypothesis assigns the degree of association a uniform prior distribution (e.g., Jeffreys, 1961; Ly et al., 2016).

Figure 6 shows the resulting posterior distribution for \( \rho \) under \( \mathcal{H}_1 \) and the associated Bayes factor. The posterior median for \( \rho \) equals −0.079, with a 95% credible interval that ranges from −0.186 to 0.032. The corresponding Bayes factor indicates that the data are about 5.1 times more likely under \( \mathcal{H}_0 \) than under \( \mathcal{H}_1 \), indicating moderate evidence against an association between math performance and the quality of family ties.

6 Concluding Comments

This article outlined a general methodology for applying conventional Bayesian inference procedures to ordinal data problems. Latent normal distributions are assumed to generate impoverished rank-based observations, and inference is done on the model parameters that govern the latent normal level. This idea, first proposed by Pearson (1900), yields all the advantages of ordinal inference including robustness to outliers and invariance to monotonic transformations. Moreover, the methodology also handles ties in a natural fashion, which is important for coarse data such as provided by popular Likert scales.

By postulating a latent normal level for the observed rank data, the advantages of ordinal inference can be combined with the advantages of Bayesian inference such as the ability to update uncertainty as the data accumulate, the ability to quantify evidence, and the ability to incorporate prior information. It should be stressed that, even though our
Figure 6: Is performance on a math exam associated with the quality of family relations? Results for the Bayesian version of Spearman’s $\rho$ as applied to the data set from Cortez and Silva (2008). The dashed line indicates the uniform prior distribution, and the solid line indicates the posterior distribution. The two grey dots indicate the prior and posterior ordinate at the point under test, in this case $\rho = 0$. The ratio of the ordinates gives the Bayes factor.

Examples used default prior distributions, the proposed methodology is entirely general in the sense that it also applies to informed or subjective prior distributions.

For computational convenience and easy of interpretation, our framework used latent normal distributions. This is not a principled limitation, however, and the methodology would work for other families of latent distributions as well (e.g., Albert 1992b).

In sum, we have presented a general methodology to conduct Bayesian inference for ordinal problems, and illustrated its potential by developing Bayesian counterparts to three popular ordinal tests: the rank sum test, the signed rank test, and Spearman’s $\rho$. Supplementary material, including R-code for each method and the example data used, is available at https://osf.io/gny35/. In the near future we intend to make these tests available in the open-source software package JASP (e.g., JASP Team 2017, jasp-stats.org), which we hope will further increase the possibility that the tests are used to analyze ordinal data sets for which the traditional parametric approach is questionable.
References

Albert, J. H. (1992a). Bayesian estimation of normal ogive item response curves using gibbs sampling. *Journal of educational statistics*, 17(3):251–269.

Albert, J. H. (1992b). Bayesian estimation of the polychoric correlation coefficient. *Journal of statistical computation and simulation*, 44:47–61.

Albert, J. H. and Chib, S. (1993). Bayesian analysis of binary and polychotomous response data. *Journal of the American statistical Association*, 88(422):669–679.

Conover, W. (1999). *Practical Nonparametric Statistics*. Wiley, 3rd edition.

Cortez, P. and Silva, A. M. G. (2008). Using data mining to predict secondary school student performance.

Cureton, E. (1956). Rank-biserial correlation. *Psychometrika*, 21:287–290.

Dickey, J. M. and Lientz, B. P. (1970). The weighted likelihood ratio, sharp hypotheses about chances, the order of a Markov chain. *The Annals of Mathematical Statistics*, 41:214–226.

Etz, A. and Wagenmakers, E.-J. (2017). J. B. S. Haldane’s contribution to the Bayes factor hypothesis test. *Statistical Science*, 32:313–329.

Fisher, R. A. (1915). Frequency distribution of the values of the correlation coefficient in samples from an indefinitely large population. *Biometrika*, pages 507–521.

Geman, S. and Geman, D. (1984). Stochastic relaxation, Gibbs distributions and the Bayesian restoration of images. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 6:721–741.

Hastings, W. (1970). Monte Carlo samplings methods using Markov chains and their applications. *Biometrika*, 57:97–109.

Hodges, J. and Lehmann, E. (1956). The efficiency of some nonparametric competitors of the t-test. *Annals of Mathematical Statistics*, 27:324–335.
Hoff, P. (2009). *A First Course in Bayesian Statistical Methods*. Springer-Verlag, New York.

Hollander, M. and Wolfe, D. (1973). *Nonparametric Statistical Methods*. Wiley, New York, 3rd edition.

JASP Team (2017). JASP (Version 0.8.3) [Computer software].

Jeffreys, H. (1939). *Theory of Probability*. Oxford University Press, Oxford, UK, 1 edition.

Jeffreys, H. (1961). *Theory of Probability*. Oxford University Press, Oxford, UK, 3rd edition.

Kass, R. E. and Raftery, A. E. (1995). Bayes factors. *Journal of the American Statistical Association*, 90:773–795.

Kerby, D. (2014). The simple difference formula: an approach to teaching nonparametric correlation. *Innovative teaching*, 3:1–9.

Kruskal, W. (1958). Ordinal measures of association. *Journal of the American Statistical Association*, 53:814–861.

Lehmann, E. (1975). *Nonparametrics: Statistical methods based on ranks*. Holden-Day series in probability and statistics. Holden-Day, Inc., London ; New York, 1st edition.

Lehmann, E. (1999). *Elements of Large Sample Theory*. Springer.

Liang, F., German, R. P., Clyde, A., and Berger, J. (2008). Mixtures of g priors for Bayesian variable selection. *Journal of the American Statistical Association*, 103:410–424.

Ly, A., Verhagen, A. J., and Wagenmakers, E.-J. (2016). Harold Jeffreys’s default Bayes factor hypothesis tests: Explanation, extension, and application in psychology. *Journal of Mathematical Psychology*, 72:19–32.

Mann, H. and Whitney, D. (1947). On a test of whether one of two random variables is stochastically larger than the other. *Annals of Mathematical Statistics*, 18:50–60.
Metropolis, N., Rosenbluth, A., Rosenbluth, M., Teller, A., and Teller, E. (1953). Equation of state calculations by fast computing machines. *Journal of Chemical Physics*, 21:1087–1092.

Morey, R. D. and Rouder, J. N. (2015). BayesFactor 0.9.11-1. Comprehensive R Archive Network.

Olssen, U. (1979). Maximum likelihood estimation of the polychoric correlation coefficient. *Psychometrika*, 44:443–460.

Olssen, U., Drasgow, F., and Dorans, N. (1982). The polyserial correlation coefficient. *Psychometrika*, 47:443–460.

Pearson, K. (1900). Mathematical contributions to the theory of evolution. vii. on the correlation of characters not quantitatively measurable. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 195(262-273):1–405.

Pearson, K. and Pearson, E. (1922). On polychoric coefficients of correlation. *Biometrika*, 14:127–156.

Pettitt, A. (1982). Inference for the linear model using a likelihood based on ranks. *Journal of the Royal Statistical Society. Series B*, 44:234–243.

Rouder, J., Speckman, P., Sun, D., Morey, R., and Iverson, G. (2009). Bayesian t tests for accepting and rejecting the null hypothesis. *Psychonomic Bulletin & Review*, 16:225–237.

Spearman, C. (1904). The proof and measurement of association between two things. *The American journal of psychology*, 15(1):72–101.

Tanner, M. A. and Wong, W. H. (1987). The calculation of posterior distributions by data augmentation. *Journal of the American statistical Association*, 82(398):528–540.

Thall, P. F. and Vail, S. C. (1990). Some covariance models for longitudinal count data with overdispersion. *Biometrics*, pages 657–671.
van der Vaart, A. (2000). *Asymptotic Statistics*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press.

van Doorn, J., Ly, A., Marsman, M., and Wagenmakers, E.-J. (2017). Bayesian estimation of kendall’s tau using a latent normal approach. *arXiv preprint arXiv:1703.01805*.

Wagenmakers, E.-J., Lodewyckx, T., Kuriyal, H., and Grasman, R. (2010). Bayesian hypothesis testing for psychologists: A tutorial on the Savage–Dickey method. *Cognitive Psychology*, 60:158–189.

Wilcoxon, F. (1945). Individual comparisons by ranking methods. *Biometrics Bulletin*, 1:80–83.