Entanglement–Coherence and Discord–Coherence analytical relations for X states

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Abstract
In this work, we derive analytical relations between Entanglement and Coherence as well as between Discord and Coherence, for Bell-diagonal states and for X states, evolving under the action of several noise channels: Bit Flip, Phase Damping and Depolarizing. We demonstrate that for these families, Coherence is the fundamental correlation, that is, Coherence is necessary for the presence of Entanglement and Discord.

Keywords Quantum correlations · Coherence · Entanglement · Discord

1 Introduction
Entanglement was the first quantum correlation known [9] and became a central subject of study, both theoretically and experimentally [5,12,14]. Later on, it was shown that there are other separable states that show non-classical behavior. Quantum Discord was presented in 2001 [22] and soon after other quantum correlations that relate to it [19]. Coherence, which is behind the interference phenomenon, has been a main topic in the framework of Quantum Optics. Recently, Baumgratz et al. [2] performed a quantitative characterization of Coherence, where quantum coherence is treated as a physical resource. Both Entanglement and Coherence are related to the phenomenon of quantum superposition; therefore, it was natural to try to understand qualitatively and quantitative the relationship between them [6,29,32] that after extended to the study of the relation between Coherence and Discord and other related quantum correlations [10,16,18].
In this work, we study analytical relations between entanglement and coherence and between Discord and Coherence for the set of Bell-diagonal states and for the more
general set of X states, both in the noiseless case and under local noisy channels. In Sect. 2, we do a geometric description of the Bell-diagonal states. In Sect. 3, we present the geometric measures of correlations. In Sect. 4, we find analytical dynamic relations between Entanglement and Coherence and between Discord and Coherence for the Bell-diagonal states evolving under the action of several noise channels: Bit Flip, Phase Damping and Depolarizing. In Sect. 5, we study the X states: we analyze their region of existence and its variations with the parameters that describes the X states and calculate the expression of the correlations for these states. In Sect. 6, we study the evolution of the correlations and their dynamical relations under the action of the same channels applied before. In Sect. 7, we discuss Coherence as the fundamental quantum correlation. In Sect. 8, we summarize and discuss the results.

2 Geometry of the Bell-diagonal states

The Bell-diagonal states are a paradigmatic class of states in which, due to their relative mathematical simplicity, it has been possible to study the behavior of various quantum correlations.

They can be written as:

$$\rho = \frac{1}{4} \left( I_2 \otimes I_2 + \sum_{i=1}^{3} r_i \sigma_i \otimes \sigma_i \right)$$

where $\sigma_i$ are the Pauli matrices and $r$ is the correlation vector:

$$r = r_1 \hat{i} + r_2 \hat{j} + r_3 \hat{k},$$

with $r_i = \text{Tr}(\rho \sigma_i \otimes \sigma_i)$, and the elements of the $\rho$ matrix can be directly related to the entries of the correlation vector as follows:

$$\rho_{11} = \rho_{44} = \frac{1}{4} (1 + r_3)$$
$$\rho_{22} = \rho_{33} = \frac{1}{4} (1 - r_3)$$
$$\rho_{14} = \frac{1}{4} (r_1 - r_2)$$
$$\rho_{23} = \frac{1}{4} (r_1 + r_2)$$

It is well-known the three-dimensional representation of the Bell-diagonal states as a function of the entries of the correlation vector [13], and there are several works studying the geometrical structure of several quantum correlation in its three-dimensional representation [15,17,25]. In Fig. 1, we can see the tetrahedron that delimits the zone of existence of the states, and the octahedron, which marks the region where the states
Fig. 1 Three-dimensional representation of the Bell-diagonal states. The tetrahedron limits the region of existence. Inside the tetrahedron, the octahedron confines the separables Bell-diagonal states are separable. Evolving under the action of different noise channels, the Bell-diagonal states describe a path within the tetrahedron [11].

3 Geometric measures of correlations

In order to quantify quantum correlations, we choose a geometric approach [1, 3, 4, 8, 26, 27, 31]. We determine how much Entanglement, Discord or Coherence a state $\rho$ posses, by its minimum distance to the set of states that does not posses that correlation, i.e.,:

$$
E(\rho) = \min_{\rho^{\text{sep}}} d(\rho, \rho^{\text{sep}}),
$$
$$
D(\rho) = \min_{\rho^{\text{cc}}} d(\rho, \rho^{\text{cc}}),
$$
$$
C(\rho) = \min_{\rho^{\text{inc}}} d(\rho, \rho^{\text{inc}}),
$$

where $\rho^{\text{sep}}$ belongs to the set of separables states (zero Entanglement), $\rho^{\text{cc}}$ to the set of classical states (zero Discord) and $\rho^{\text{inc}}$ to the set of incoherent states (zero Coherence). We choose to work with the Trace norm, which is a particular case of the $p$-norm:

$$
||A||_p^p := \text{Tr} \left( \sqrt[2p]{A^\dagger A} \right)^p,
$$

when $p = 1$. The Trace norm determines $d_1$ as a measure of distance:

$$
d_1(\rho, \sigma) = ||\rho - \sigma||
$$

With the previous considerations, in the next section, we will show the expressions of the three quantum correlations.
3.1 Entanglement

For the Entanglement, we have:

\[ E(\rho) = \min_{\rho_{\text{sep}} \in S} \frac{1}{2} ||\rho - \rho_{\text{sep}}||_1 = \min_{\rho_{\text{sep}} \in S} Tr|\rho - \rho_{\text{sep}}|, \]

where \( S \) is the set of separable states. For the particular case of X states, in a previous work [11], we found that:

\[ E(\rho) = 2 \max \left\{ 0, |\rho_{14}| - \sqrt{\rho_{22}\rho_{33}}, |\rho_{32}| - \sqrt{\rho_{11}\rho_{44}} \right\} \tag{4} \]

which is the Concurrence of the quantum state [33].

3.2 Discord

We will use the expression of the Discord for the Bell-diagonal states developed in [20,23]:

\[ D(\rho_{BD}) = \frac{r_{\text{int}}}{2}, \tag{5} \]

where \( r_{\text{int}} \) is the intermediate value of the \(|r_i|\).

3.3 Coherence

Applying the geometric definition for the Coherence, its expression is:

\[ C(\rho) = \min_{\rho_{\text{inc}} \in I} ||\rho - \rho_{\text{inc}}|| = \sum_{i,j} |\rho_{i,j}|. \]

where we used the fact that the nearest incoherent state is represented by the same \( \rho \) matrix, but with all the elements out of the diagonal nulls [2]. For the particular case of the X states, the Coherence is:

\[ C = |\rho_{14}| + |\rho_{32}| + |\rho_{41}| + |\rho_{23}|, \]

which under the assumption of all the entries of the matrix being reals, \( \rho_{14} = \rho_{41} \) and \( \rho_{32} = \rho_{23} \) it has the following expression:

\[ C = 2(|\rho_{14}| + |\rho_{32}|). \tag{6} \]
4 Dynamical relations for Bell-diagonal states

We will consider that our system evolves with each qubit in contact with its own environment. The evolution can be described by means of the Kraus operators [21,24] according to:

$$\rho_{AB}' = \sum_{i,j} (M_i^A \otimes M_j^B) \rho_{AB} (M_i^A \otimes M_j^B)^\dagger,$$

where $M_i^A, M_i^B$ are the Kraus operator acting on each qubit. By means of this tool, we will study the evolution of the two-qubit system under the action of several known quantum channels.

We start by writing the expression of the three correlations using the Trace Norm; the expression of the Entanglement can be obtained putting Eq. (3) in the expression of the Concurrence Eq. (4):

$$E(\rho) = \frac{1}{2} \max[0, |r_1 \pm r_2| - (1 \pm r_3)].$$

The expression of the Discord, Eq. (5):

$$D(\rho) = \text{int}[|r_1|, |r_2|, |r_3|]$$

And finally, for the Coherence we put Eq. (3) in the expression Eq. (6):

$$C(\rho) = \frac{1}{2} (|r_1 - r_2| + |r_1 + r_2|) = \max(|r_1|, |r_2|).$$

In the following subsections, we will study the dynamical relations between Entanglement, Discord and Coherence when the initial Bell-diagonal states evolve under the action of three known quantum channels: Bit Flip, Phase Damping and Depolarizing.

4.1 Bit flip

The Bit flip channel models the simplest type of error that suffers a qubit: it flips the state from $|0\rangle$ to $|1\rangle$ and reciprocally, with probability $(1 - p)$. The Kraus operators corresponding to this channel are:

$$M_1^{bf} = \sqrt{1 - p} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_2^{bf} = \sqrt{p} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Applying the Kraus operators to the initial Bell-diagonal state, it is easy to see that all the information of the evolution is contained in the vector of correlations.

$$\mathbf{r}' = r_1 \hat{i} + r_2 (p - 1)^2 \hat{j} + r_3 (p - 1)^2 \hat{k}$$
The Entanglement of the evolved state is:

\[ E(p) = \frac{1}{2} \max[0, |r_1 \pm r_2(p-1)^2| - |1 \pm r_3(p-1)^2|] \] (7)

The expression of the Discord:

\[ D(p) = \text{int}[|r_1|, |r_2|(p-1)^2, |r_3|(p-1)^2] \] (8)

And finally the Coherence:

\[ C(p) = \frac{1}{2}(|r_1 - r_2(p-1)^2| + |r_1 + r_2(p-1)^2|) = \max(|r_1|, |r_2|(p-1)^2). \] (9)

4.1.1 Entanglement and Coherence

In order to express the Entanglement as a function of Coherence, we have to discriminate two alternatives; in the region of the space where \( |r_1| > |r_2|(p-1)^2 \), Entanglement and Coherence are independent. On the other hand, where \( |r_1| < |r_2|(p-1)^2 \), we can express the Entanglement as a function of the Coherence:

\[ E = \max \left[ 0, \frac{1}{2} \left( |r_1 \pm \text{sign}(r_2)C| - |1 \pm \frac{r_3}{r_2} C| \right) \right]. \] (10)

4.1.2 Discord and Coherence

Looking for a functional relation between Discord and Coherence, we find that it is enough to analyze the initial values of the correlation vector. It is easy to see that this is only possible when \( |r_1| = \min(|r_1|, |r_2|, |r_3|) \). Starting with \( |r_1| < |r_2| < |r_3| \), as \( r \) evolves, we find two regions: one where \( D = C \) (when \( |r_1| < |r_2|(p-1)^2 \)), and another (when \( |r_2|(p-1)^2 < |r_1| \)) where Discord decays quadratically, \( D = |r_3|(p-1)^2 \) independent of Coherence, which remains constant, \( C = |r_1| \). When initially we have \( |r_1| < |r_3| < |r_2| \), we find three regions: one where \( D = \frac{|r_2|}{|r_2|} C \) (when \( |r_1| < |r_3|(p-1)^2 \)), a second region where Discord is constant, \( D = |r_1| \) independent of Coherence (when \( |r_3|(p-1)^2 < |r_1| \)), and a third region where the Discord decays quadratically, \( D = |r_2|(p-1)^2 \), while the Coherence remains constant, \( C = |r_1| \) (when \( |r_3|(p-1)^2 < |r_2|(p-1)^2 < |r_1| \)).

4.2 Phase damping

This channel describes a type of noise that is completely quantum: it is a process where quantum information is lost without loss of energy [21]. The Kraus operators
corresponding to this channel are:

\[ M_{0}^{pd} = \sqrt{1-p} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_{1}^{pd} = \sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_{2}^{pd} = \sqrt{p} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]

Applying the Kraus operators to the initial Bell-diagonal state, we obtain the expression of the evolved correlation vector:

\[ \mathbf{r}' = r_1(p - 1)^2 \hat{i} + r_2(p - 1)^2 \hat{j} + r_3 \hat{k}, \]

which allow us to calculate the expressions of the three correlations under this channel. For the Entanglement, we have:

\[ E(p) = \frac{1}{2} \max(0, |r_1 \pm r_2|(p - 1)^2 - |1 \pm r_3|) \quad (11) \]

The expression of the Discord is:

\[ D(p) = \text{int}([|r_1|(p - 1)^2, |r_2|(p - 1)^2, |r_3|]). \quad (12) \]

The Coherence has the following expression:

\[ C(p) = \frac{1}{2}(|r_1 - r_2| + |r_1 + r_2|)(p - 1)^2 = \max(|r_1|, |r_2|)(p - 1)^2. \quad (13) \]

### 4.2.1 Entanglement and Coherence

For this evolution, we can always find a relation between Entanglement and Coherence:

\[ E = \max \left[ 0, \frac{C|r_1 \pm r_2|}{2 \max(|r_1|, |r_2|)} - \frac{|1 \pm r_3|}{2} \right]. \quad (14) \]

### 4.2.2 Discord and Coherence

Taking into consideration the expression for the Discord:

\[ D = \text{int}([|r_1|(p - 1)^2, |r_2|(p - 1)^2, |r_3|]) \]

we see that it is possible to relate Discord with Coherence if $|r_1|$ or $|r_2|$ are the intermediate values. If the intermediate value is $|r_3|$, the Discord will be constant and independent of the Coherence.
When the intermediate value is \( r_1 \) or \( r_2 \), the expression of the Discord is:

\[
D = \frac{|r_i|C}{\max(|r_1|, |r_2|)}, \tag{15}
\]

where \( r_i = r_1, r_2 \), depending on which is the intermediate value.

### 4.3 Depolarizing

The depolarizing channel is a well-known channel; we can describe it by explaining its action on a qubit; with probability \((1 - p)\), the qubit remains unaffected, and with probability \(p\), it is depolarized [21].

\[
M_0^d = \sqrt{1 - p} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_1^d = \frac{p}{3} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
M_2^d = \frac{p}{3} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad M_3^d = \frac{p}{3} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

The expression of the evolved correlation vector under the depolarizing channel is:

\[
r' = r_1(p - 1)^2 + r_2(p - 1)^2 + r_3(p - 1)^2.
\]

The Entanglement, Discord and Coherence are expressed as:

\[
E(\rho) = \frac{1}{2} \max[0, |r_1 \pm r_2|(p - 1)^2 - |1 \pm r_3(p - 1)^2|], \tag{16}
\]
\[
D(\rho) = \text{int}[|r_1|, |r_2|, |r_3|](p - 1)^2, \tag{17}
\]
\[
C(\rho) = \max(|r_1|, |r_2|)(p - 1)^2. \tag{18}
\]

### 4.3.1 Entanglement and Coherence

Also in this case, we can easily write Entanglement as a function of Coherence.

\[
E = \frac{1}{2} \max \left[ 0, \frac{(|r_1 \pm r_2|C)}{\max(|r_1|, |r_2|)} - |1 \pm \frac{r_3C}{\max(|r_1|, |r_2|)}| \right],
\]

and again we find a linear relation between Entanglement and Coherence.

### 4.3.2 Discord and Coherence

As for the case of the previous channel, we can express Discord in function of Coherence:

\[
D = \frac{C|r_i|}{\max(|r_1|, |r_2|)},
\]
where \( r_i = r_1, r_2, r_3 \), depending which is the intermediate value. The difference with the previous channel is that for the Depolarizing channel, the intermediate value remains the same throughout the evolution because the three components of the correlations vector evolve the same way.

5 Geometry of X states: existence and correlations

In this section, we will work with the so-called X states, which their density matrix contains only nonzero elements along the main diagonal and anti-diagonal \([33]\). These states, which include the Bell-diagonal states we previously studied, are described by the following expression:

\[
\rho = \frac{1}{4} (I_4 + s \cdot \sigma_3 \otimes I + I \otimes c \cdot \sigma_3 + \sum_{j=1}^{3} r_j \sigma_j \otimes \sigma_j)
\]

Unlike Bell-diagonal states that only need three parameters to be described: \((r_1, r_2, r_3)\), to describe the X states, it is necessary to add two additional parameters: \(s\) and \(c\). In the following, it will be helpful to express the elements of the density matrix \(\rho_{i,j}\) as a function of the new parameters \(r_i, s\) and \(c\):

\[
\begin{align*}
\rho_{11} &= \frac{1}{4} (1 + r_3 + s + c) \\
\rho_{22} &= \frac{1}{4} (1 - r_3 + s - c) \\
\rho_{33} &= \frac{1}{4} (1 - r_3 - s + c) \\
\rho_{44} &= \frac{1}{4} (1 + r_3 - s - c) \\
\rho_{14} &= \rho_{41} = \frac{1}{4} (r_1 - r_2) \\
\rho_{23} &= \rho_{32} = \frac{1}{4} (r_1 + r_2)
\end{align*}
\]

To have a geometric 3D representation of the Bell-diagonal states is possible since they depend only on the three components of \(r\). The X states require five parameters to be described, so it is necessary to add two additional parameters: \(s\) and \(c\). In the following, it will be helpful to express the elements of the density matrix \(\rho_{i,j}\) as a function of the new parameters \(r_i, s\) and \(c\):

\[
\begin{align*}
|r_1 - r_2| &\leq \sqrt{(1 + r_3)^2 - (s + c)^2} \\
|r_1 + r_2| &\leq \sqrt{(1 - r_3)^2 - (s - c)^2}
\end{align*}
\]

These two inequalities determine the volume of the possible X states. By changing the values of \(s\) and \(c\), we can see how the region of existence is modified. In Fig. 2, we can see how the tetrahedron that determines the region of existence of the Bell-diagonal states is deformed as the values of \(s\) and \(c\) change. Just as the tetrahedron that
Fig. 2 In color red, the deformed tetrahedron for different values of s and c. The original tetrahedron was kept for comparison (Color figure online)

defines the allowed states is deformed, the octahedron defined by the states of zero entanglement also undergoes deformation. In Fig. 3, we show the transformation of the octahedron as s and c changes.

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5.2 Analytical expressions of correlations for X states

We start with the Entanglement, which as we saw in Eq. (4), for X states coincides with the expression of the Concurrence:

\[ E = \frac{1}{2} \max \left[ 0, |r_1 \pm r_2| - \sqrt{(1 \pm r_3)^2 - (s \pm c)^2} \right]. \]  \hspace{1cm} (19)
For the Discord, we will use the expression provided by [7]:

\[
D(\rho) = \begin{cases} 
\frac{|r_1|}{2} & \text{if } \Delta > 0, \\
\sqrt{\frac{r_2^2 + s^2}{\max(r_2^2, r_2^2 + s^2) - r_2^2 \min(r_2^2, r_1^2)}} & \text{if } \Delta \leq 0,
\end{cases}
\tag{20}
\]

where \(\Delta = r_3^2 - r_1^2 - s^2\).

Finally, the expression of Coherence is:

\[C = \max(|r_1|, |r_2|)\]

### 5.3 Regions of Discord

For X states, Discord a priori can have many different expressions in the different regions, depending on which are the maximums and minimums in Eq. (20) or which of the following relations, \(r_3^2 < r_1^2 + s^2\) or \(r_3^2 \geq r_1^2 + s^2\), is verified. Taking this into consideration, we found five regions:

1. \(r_3^2 > A\)
2. \((r_3^2 \leq A) \cap (\max(r_3^2, B) = r_3^2) \cap (\min(r_1^2, r_3^2) = r_3^2)\)
3. \((r_3^2 \leq A) \cap (\max(r_3^2, B) = r_3^2) \cap (\min(r_1^2, r_3^2) = r_1^2)\)
4. \((r_3^2 \leq A) \cap (\max(r_3^2, B) = B) \cap (\min(r_1^2, r_3^2) = r_3^2)\)
5. \((r_3^2 \leq A) \cap (\max(r_3^2, B) = B) \cap (\min(r_1^2, r_3^2) = r_1^2)\)

where \(A = r_1^2 + s^2\) and \(B = r_2^2 + s^2\). By substituting these results in Eq. (20), we find that in regions (1), (3) and (5), Discord has the same expression. Therefore, we have to consider three regions for Discord: **Region I**, defined by: \(|r_3| > |r_1|\) where the expression of the Discord is:

\[D = \frac{|r_1|}{2},\]

and we show it in Fig. 4.

**Region II**, defined by: \(|r_3| \leq |r_1|\) \(\cap (r_3^2 > r_2^2 + s^2)\) where the expression of Discord is,

\[D = \frac{|r_3|}{2},\]

and it is shown in Fig. 5.

**Region III** defined by: \(|r_3| \leq |r_1|\) \(\cap (r_3^2 \leq r_2^2 + s^2)\) where the expression of Discord is,

\[
D = \frac{1}{2} \sqrt{\frac{r_2^2 + s^2 + r_1^2 s^2}{r_1^2 - 3}}.
\tag{22}
\]

and in Fig. 6 we show it.
5.4 Analysis in the $s$–$c$ plane

The previous analysis was performed leaving the values of the parameters $s$ and $c$ fixed and varying $r$. In this section, we will analyze how the regions of existence change and how the correlations behave when we fix $r$ and vary the parameters $s$ and $c$.

In Figs. 7 and 8, we show the changes in the regions of existence and the behavior of Entanglement and Discord by varying $r_2$, on the $s$–$c$ plane. As can be seen, the region of existence contracts, until it forms a perfect square at $r_2 = 0$, and then stretches again until it becomes a line and then disappears. The images selected are with $r_1$ and $r_3$ fixed so that it is better understood how the region of existence is modified as $r_2$ is changed. Setting $r_2$ and $r_3$ fixed, and changing $r_1$, and fixing $r_1$ and $r_2$ and changing $r_3$ give similar results.
The Discord pattern is simple, because it does not depend on parameter $c$ and grows with $s$. On the other hand, Entanglement grows as $|s \pm c|$ increase, since:

$$|r_1 \pm r_2| - \sqrt{(1 \pm r_3)^2 - (s \pm c)^2}$$

for values of $r_i$ within the region of existence, showing growth in the longitudinal direction. Coherence was not included in this analysis since it is constant in this plane, because it is independent of $s$ and $c$.

### 6 Dynamical relations for X states

The description of the evolution of the X states is given by the evolution of the correlation vector, as in the case of the Bell-diagonal states, plus the evolution of the parameters $s$ and $c$. The only exception is made by the evolution under Phase Damping, in which the parameters $s$ and $c$ are not affected.

#### 6.1 Phase damping

The evolution of the correlation vector $\mathbf{r}$ under Phase Damping is given by:

$$\mathbf{r}' = r_1(p - 1)^2 \hat{i} + r_2(p - 1)^2 \hat{j} + r_3 \hat{k}.$$ 

#### 6.1.1 Entanglement

As we have already seen for the X states, each pair of values of $s$ and $c$ determines the zone of existence and the zero Entanglement octahedron. The expression of Entangle-
Fig. 7  Color maps of the Entanglement in the $s$–$c$ plane. The white region indicates the region of non-physical states (Color figure online)

(a) $r_1 = -0.9$, $r_2 = -0.08$, $r_3 = 0$

(b) $r_1 = -0.9$, $r_2 = -0.04$, $r_3 = 0$

(c) $r_1 = -0.9$ $r_2 = 0$ $r_3 = 0$

(d) $r_1 = -0.9$ $r_2 = 0.04$ $r_3 = 0$
**Fig. 8** Color maps of the Discord in the $s-c$ plane. The white region indicates the region of non-physical states (Color figure online)

(a) $r_1 = -0.9$, $r_2 = -0.08$, $r_3 = 0$

(b) $r_1 = -0.9$, $r_2 = -0.04$, $r_3 = 0$

(c) $r_1 = -0.9$, $r_2 = 0$, $r_3 = 0$

(d) $r_1 = -0.9$, $r_2 = 0.04$, $r_3 = 0$
ment is:

\[
E = \frac{1}{2} \max \left[ 0, |r_1 \pm r_2|(p - 1)^2 - \sqrt{(1 \pm r_3)^2 - (s \pm c)^2} \right].
\]

### 6.1.2 Coherence

The Coherence expression of a X state that evolves under phase damping is simple; it decreases monotonously and reaches the value 0 when \( p = 1 \):

\[
C = \max(|r_1|, |r_2|)(p - 1)^2.
\]

### 6.1.3 Discord

When a X state evolves under the Phase Damping channel, the expression of the Discord is expressed by Eq. (20):

\[
D(\rho) = \begin{cases} 
\frac{|r_1|}{2}(p - 1)^2 & \text{if } \Delta > 0, \\
\frac{1}{2} \sqrt{A_{\max}(r_2^2, B+s^2) - B \min(r_2^2, A)} & \text{if } \Delta \leq 0,
\end{cases}
\]

where \( \Delta = r_3^2 - s^2 - A \), \( A = r_1^2(p - 1)^4 \) and \( B = r_2^2(p - 1)^4 \). We showed from an analysis in Sect. 5.3 that there are three different regions for the Discord. Depending on the initial value of the vector \( \mathbf{r} \) and the parameters \( s \) and \( c \), the state in its evolution will cross some of those regions, as shown in the Fig. 9. Each color line represents a possible evolution of an X state, starting from different initial values of \( \mathbf{r} \) and the parameters \( s \) and \( c \).

The figure shows how the state evolves depending on the initial conditions of \( \mathbf{r} \). Evolution can take five possible paths: (1) to go through the three regions in descending order, Region 3, Region 2 and Region 1 (yellow path), (2) to go from Region 3 directly to Region 1 (red path), (3) always stay in Region 3 (blue path), (4) move from Region 2 to Region 1 (pink path) or (5) always stay in Region 1 (white path). Regions 1, 2 and 3 can be written as the following inequalities:

- **Region 1:** \( |r_3| > |r_1|(p - 1)^2 \)
- **Region 2:** \( \left( |r_3| \leq |r_1|(p - 1)^2 \right) \cap \left( r_3^2 > r_2^2(p - 1)^4 + s^2 \right) \)
- **Region 3:** \( \left( |r_3| \leq |r_1|(p - 1)^2 \right) \cap \left( r_3^2 \leq r_2^2(p - 1)^4 + s^2 \right) \)

Therefore, the quantum state will remain in Region 1 as long as it is verified that:

\[
p > 1 - \sqrt{\frac{|r_3|}{|r_1|}} = p_1,
\]
Fig. 9  Cut according to the plane $r_1 - r_2 = 0.2$. Region 1 is purple, Region 2 is green, and Region 3 is orange. The yellow line shows an evolution that cross through all three regions. In blue, the state always remains in Region 3. In red, it jumps from Region 3 to Region 1. In pink, from Region 2 to Region 1. In White, it always remains within Region 1 (Color figure online)

and the expression of the Discord in this region is:

$$D = \frac{|r_1|}{2}(p - 1)^2$$

(23)

Note that if $r_1 = 0$ the state always stay in Region 1.

The quantum state will remain in Region 2 as long as it is verified that:

$$\begin{cases} 
    p \leq 1 - \sqrt{\frac{|r_3|}{|r_1|}} = p_1 \\
    p > 1 - 4\sqrt{\frac{r_1^2 - s^2}{r_2}} = p_2,
\end{cases}$$

(24)

and the expression of the Discord in this region is:

$$D = \frac{|r_3|}{2}.$$

Note that if $|r_3| < |s|$, the state cannot be in Region 2.

Finally, it will remain in Region 3 as long as:

$$\begin{cases} 
    p \leq 1 - \sqrt{\frac{|r_3|}{|r_1|}} = p_1 \\
    p < 1 - 4\sqrt{\frac{r_1^2 - s^2}{r_2}} = p_2
\end{cases}$$

(25)

The expression of the Discord in this region is:

$$D = \frac{(p - 1)^2}{2} \sqrt{\frac{r_1^2 r_2^2 (p - 1)^4 - r_2^2 r_3^2 + r_1^2 s^2}{r_1^2 (p - 1)^4 - r_3^2 + s^2}}$$
If \( p < p_1 \) and \( r_2 = 0 \), the state will be in Region 2 or Region 3 depending on \(|r_3| > |s|\) or \(|r_3| \leq |s|\), respectively.

Looking at the figure, it is clear that the state in its evolution not necessarily crosses the three regions. We will clarify this point by giving some examples. Let us consider an initial state defined by the following values of its parameters: \( r_1 = -0.6, r_2 = 0.4, r_3 = 0.3, s = 0.2, c = 0.3 \). For these values, the state is in Region 3 and \( p_1 = 0.29 \), and \( p_2 = 0.25 \), so \( p_1 > p_2 \). This imply that the quantum state remains in Region 3 when \( p < p_2 \), it is in Region 2 when \( p_2 < p \leq p_1 \) and it is in the Region 1 for \( p > p_1 \). This behavior is showed in the yellow path in Fig. 9. Let us consider a second example with initial values \( r_1 = 0.5, r_2 = -0.2, r_3 = 0.3, s = 0.2, c = 0.3 \). In this case, \( p_1 = 0.33 \) and \( p_2 = -0.06 \). \( p_2 \) is excluded since \( p \) can never take negative values. This will imply that the state will never be in Region 3, since \( p > p_2 \forall p \). The state will be in Region 2 for \( 0 < p \leq p_1 \) and in Region 1 for \( p > p_1 \). This is illustrated in the red path. As a last example, consider the following initial conditions: \( r_1 = -0.6, r_2 = 0.4, r_3 = 0.7, s = 0.2, c = 0.3 \). In this situation, the state initially is in Region 1 and \( p_1 = -0.08 \), so \( p > p_1 \forall p \) and the state always remains in Region 1, as can be seen in the white path of the Fig. 9.

### 6.1.4 Entanglement and coherence

In the region outside the octahedron of the separable states, the Entanglement and the Coherence can be related by a linear function:

\[
E = \frac{1}{2} \max \left[ 0, \frac{|r_1 \pm r_2| C_{\max}(|r_1|, |r_2|) - \sqrt{(1 \pm r_3)^2 - (s \pm c)^2}}{\max(|r_1|, |r_2|)} \right].
\]

### 6.1.5 Discord and Coherence

In order to determine the relation between Discord and Coherence, the behavior of the Discord must be taken into account in each of the three regions defined above. In Region 1, the relation is linear:

\[
D = \frac{|r_1| C_{\max}}{2 \max(|r_1|, |r_2|)}
\]

In Region 2, there is no relation between Discord and Coherence because the Discord is constant and depends only of \( r_3 \): \( D = \frac{|r_3|}{2} \)

In Region 3, the relation is:

\[
D = \frac{1}{2} \sqrt{C^2 - \frac{r_1^2 r_2^2 C^2}{\max(|r_1|, |r_2|)^2} - \frac{r_2^2 r_3^2}{r_3^2 C^2 - r_3^2 + s^2}}.
\]
6.2 Bit flip

As the state evolves under this channel, we see that not only does $r$ change, but also parameters $s$ and $c$, as follows:

$$r' = r_1 \hat{i} + r_2(p - 1)^2 \hat{j} + r_3(p - 1)^2 \hat{k}$$

$$s' = s(p - 1) \quad c' = c(p - 1),$$

and this allows us to find the expression of the different correlations.

The Entanglement has the following expression:

$$E = \frac{1}{2} \max \left[ 0, |r_1 \pm r_2(p - 1)^2| \right] - \sqrt{(1 \pm r_3(p - 1)^2)^2 - (s \pm c)^2(p - 1)^2}$$

The expression of the Coherence is:

$$C = \max(|r_1|, |r_2|(p - 1)^2).$$

6.2.1 Discord

According to Eq. (20), the expression for the Discord for a X state evolving under the Bit Flip channel is:

$$D(\rho) = \begin{cases} 
\frac{|r_1|}{2} & \text{if } \Delta > 0, \\
\frac{1}{2} \sqrt{r_1^2 \max(A,C) - B \min(A,r_1^2)} & \text{if } \Delta \leq 0,
\end{cases}$$

where $A = r_3^2(p - 1)^4$, $B = r_2^2(p - 1)^4$, $C = r_2^2(p - 1)^4 + s^2(p - 1)^2$ and $\Delta = A - r_1^2 - s^2(p - 1)^2$.

Again, to study Discord, we have to analyze its behavior in three different regions, defined by the following relations:

Region 1: $|r_3|(p - 1)^2 > |r_1|$
Region 2: $\left(|r_3|(p - 1)^2 \leq |r_1|\right) \cap \left((r_3^2 - r_2^2)(p - 1)^2 > s^2\right)$
Region 3: $\left(|r_3|(p - 1)^2 \leq |r_1|\right) \cap \left((r_3^2 - r_2^2)(p - 1)^2 < s^2\right)$

Carrying out the same analysis as for the Phase Damping case, we arrived at the following results: the X state will be in Region 1 provided that:

$$\begin{cases} 
p < 1 - \sqrt{|r_1|/r_3^2}, \\
\text{if } r_3 \neq 0.
\end{cases}$$
In this region, the expression of Discord is:

\[ D = \frac{|r_1|}{2} \]

The state will remain in Region 2 as long as it is verified that

\[
\begin{cases}
  p \geq 1 - \sqrt{\frac{|r_1|}{|r_3|}} \\
  p < 1 - \sqrt{\frac{s^2}{r_3^2 - r_2^2}}
\end{cases}
\]

The expression of Discord in this region is:

\[ D = \frac{|r_3|}{2} (p - 1)^2 \]

Finally Region 3 is defined by:

\[
\begin{cases}
  p \geq 1 - \sqrt{\frac{|r_1|}{|r_3|}} \\
  p > 1 - \sqrt{\frac{s^2}{r_3^2 - r_2^2}}
\end{cases}
\]

Note that if \( r_3^2 \leq r_2^2 \), or \( r_3 = 0 \), the state is in Region 3. In this region, the Discord is expressed as:

\[
D = \frac{(p - 1)}{2} \sqrt{\frac{r_1^2 r_2^2 (p - 1)^2 - r_2^2 r_3^2 (p - 1)^6 + r_1^2 s^2}{r_1^2 - r_3^2 (p - 1)^4 + s^2 (p - 1)^2}}
\]

It should be noted that since the Bit Flip channel also changes the values of \( s \) and \( c \), it is not possible to show an evolution crossing the different regions, since for each value of \( s \) and \( c \) the entire volume that defines the space of X states as well as the interior regions varies.

### 6.2.2 Entanglement and Coherence

In order to relate the Entanglement and the Coherence, we must distinguish between two regions outside the region of separable states: Region A: \( |r_1| > |r_2|(p - 1)^2 \), in this region, the region the Coherence is constant: \( C = |r_1| \), and Region B: \( |r_1| < |r_2|(p - 1)^2 \), where we can relate Entanglement and Coherence as follows:

\[
E = \frac{1}{2} \max \left[ 0, |r_1 \pm C| - \sqrt{\left( 1 \pm \frac{r_3 C}{|r_2|} \right)^2 - (s \pm c)^2 \frac{C}{|r_2|}} \right].
\]
6.2.3 Discord and Coherence

By seeking to relate Discord to Coherence, we have to take into account the three regions of Discord. In Region 1, defined by \(|r_3|(p-1)^2 > |r_1|\), Discord has a constant value, \(D = \frac{|r_1|^2}{2}\) independent of the Coherence.

In Region 2, \(|r_2|(p-1)^2 < |r_1|\) is verified, so the Coherence takes a constant value \(C = |r_1|\) and Discord has the following expression: \(D = \frac{|r_3|(p-1)^2}{2}\); there is no relation between them in this region neither.

Within Region 3, in the area where \(|r_2|(p-1)^2 < |r_1|\), the Coherence is constant \(C = |r_1|\).

However, in the zone in which \(|r_2|(p-1)^2 > |r_1|\), Coherence takes the following expression: \(C = |r_2|(p-1)^2\) and therefore, we can relate it to Discord, as follows:

\[
D = \frac{1}{2} \sqrt{\frac{C (r_1^2 |r_2| s + r_2^2 |r_2|^2 C - r_3^2 C^3)}{r_1^2 |r_2|^2 + s^2 |r_3|^2 C - r_3^2 C^2}}.
\]

6.3 Depolarizing

By evolving under depolarizing channel, the correlation vector \(r, s\) and \(c\) change as:

\[
\begin{align*}
\mathbf{r}' &= r_1 (p-1)^2 \hat{i} + r_2 (p-1)^2 \hat{j} + r_3 (p-1)^2 \hat{k} \\
\mathbf{s}' &= s (p-1) \\
c' &= c (p-1) .
\end{align*}
\]

The expression of the Entanglement of the X state under this channel is:

\[
E = \frac{1}{2} \max \left[ 0, |r_1 \pm r_2| (p-1)^2 - \sqrt{\left(1 \pm r_3 (p-1)^2\right)^2 - (p-1)^2 (s \pm c)^2} \right].
\]

The Coherence is:

\[
C = \max(|r_1|, |r_2|) (p-1)^2 .
\]

6.3.1 Regions of Discord

The general expression for the Discord under this channel is:

\[
D(\rho) = \begin{cases} 
\frac{|r_1|^2 (p-1)^2}{2} \\
\frac{(p-1)^2}{2} \notag \\
\sqrt{\frac{r_3^2 A - r_2^2 (p-1)^2 \min(r_3^2, r_1^2)}{A - (p-1)^2 (r_1^2 - r_2^2 - \min(r_3^2, r_1^2))}}
\end{cases}
\]

where \(A = \max(r_3^2 (p-1)^2, r_2^2 (p-1)^2 + s^2)\). The upper expression corresponds to the condition \(r_1^2 (p-1)^2 - r_2^2 (p-1)^2 + s^2 < 0\) and the lower to \(r_1^2 (p-1)^2 - r_3^2 (p-1)^2 < 0\).
After a little mathematical manipulation, they are:

Region 1: \(|r_3| > |r_1|
Region 2: \(|r_3| \leq |r_1| \cap \left(r_3^2(p - 1)^2 > r_2^2(p - 1)^2 + s^2\right)
Region 3: \(|r_3| \leq |r_1| \cap \left(r_3^2(p - 1)^2 \leq r_2^2(p - 1)^2 + s^2\right)

In Region 1, the Discord is expressed as:

\[ D = \frac{|r_1|}{2}(p - 1)^2 \]

The state will remain in Region 2 as long as it is verified that:

\[
\begin{cases} 
|r_3| < |r_1| \\
|p| < 1 - \sqrt{\frac{s^2}{r_3^2 - r_2^2}} 
\end{cases}
\]

and the Discord in this region is expressed as:

\[ D = \frac{|r_3|}{2}(p - 1)^2. \]

If \(|r_3| \leq |r_2|\) the state will be in Region 3.

The quantum X state will remain in Region 3 as long as:

\[
\begin{cases} 
|r_3| < |r_1| \\
|p| > 1 - \sqrt{\frac{s^2}{r_3^2 - r_2^2}} 
\end{cases}
\]

and in this region Discord is expressed as:

\[ D = \frac{(p - 1)}{2} \sqrt{\frac{r_2^2(p - 1)^2(r_1^2 - r_3^2) + r_2^2s^2}{2r_2^2 - r_1^2 + r_3^2}}. \]

6.3.2 Entanglement and Coherence

It is also possible to relate in this case, Entanglement and Coherence:

\[
E = \frac{1}{2} \max \left[ 0, \frac{|r_1 \pm r_2|}{\max(|r_1|, |r_2|)} \right] - \sqrt{\left(1 + \frac{C}{\max(|r_1|, |r_2|)}\right)^2 - \frac{C(s \pm c)}{\max(|r_1|, |r_2|)}}.
\]
6.3.3 Discord and Coherence

To link Discord and Coherence we must analyze the behavior of both magnitudes in each of the regions that we already discussed. It is immediate to show that in regions 1 and 2 the relation is linear. In Region 1:

\[ D = \frac{|r_1|C}{2 \max(|r_1|, |r_2|)} \]

In Region 2:

\[ D = \frac{|r_3|C}{2 \max(|r_1|, |r_2|)} \]

In Region 3, the relation is a little bit complicated:

\[ D = \frac{1}{2} \sqrt{\frac{C(r_2^2(r_1^2 - r_3^2)C + \max(|r_1|, |r_2|)r_1^2s^2)}{(\max(|r_1|, |r_2|))^2(2r_2^2 - r_1^2 + r_3^2)}}. \]

7 Coherence as the fundamental correlation

As the theories of quantum correlations have been developed, attempts have been made to understand the possible relationships between them [10,18,28,32]. A reasonable question to ask is whether there is a correlation that is fundamental, that is, if there is a correlation whose presence is necessary for the others to exist. We begin by analyzing the expression of Coherence:

\[ C = \max(|r_1|, |r_2|), \]

We note that for Coherence to be canceled, it is necessary and sufficient that \( r_1 = r_2 = 0 \). If we impose that condition in the expression of the Entanglement:

\[ E = \frac{1}{2} \max \left[ 0, |r_1 \pm r_2| - \sqrt{(1 \pm r_3)^2 - (s \pm c)^2} \right], \]

we observe that necessarily the Entanglement is zero. Therefore we can affirm that:

\[ C = 0 \Rightarrow E = 0. \]

Making the same analysis with the expressions of Discord Eq. (20):

\[ D = \frac{|r_1|}{2} \quad \leftrightarrow \quad r_3^2 > r_1^2 + s^2 \]

(26)
using that \( r_1 = 0 \), we obtain,

\[
D = 0 \iff |r_3| > |s|
\]

For the complementary region, \( r_3^2 < r_1^2 + s^2 \):

\[
D = \frac{1}{2} \sqrt{\frac{r_1^2 \max(r_3^2, r_2^2 + s^2) - r_2^2 \min(r_3^2, r_1^2)}{\max(r_3^2, r_2^2 + s^2) - \min(r_3^2, r_1^2) + r_1^2 - r_2^2}}.
\]

imposing again \( r_1 = r_2 = 0 \), we obtain:

\[
D = 0 \iff |r_3| < |s|
\]

From these results, we can conclude that:

\[
C = 0 \Rightarrow E = 0 \quad C = 0 \Rightarrow D = 0
\]

What if the Entanglement is zero? What consequences does this have on Discord and Coherence? By canceling the Entanglement, in expression Eq. (19), we obtain the region of separable states, where we have already seen that there are states with Coherence and non-null Discord.

Now, we will analyze the consequences of nullifying Discord in the other correlations. We should consider the two regions of Discord. In the region \( r_3^2 > r_1^2 + s^2 \),

\[
D = \frac{|r_1|}{2}.
\]

When then have, that:

\[
D = 0 \iff r_1 = 0.
\]

Clearly for the states in which it is verified that \( |r_2| \neq 0 \), Discord being zero does not imply that Coherence is zero.

In the region \( r_3^2 > r_1^2 + s^2 \), the expression of the Discord is:

\[
D = \frac{1}{2} \sqrt{\frac{r_1^2 \max(r_3^2, r_2^2 + s^2) - r_2^2 \min(r_3^2, r_1^2)}{\max(r_3^2, r_2^2 + s^2) - \min(r_3^2, r_1^2) + r_1^2 - r_2^2}}.
\]

The condition for the Discord to be zero is:

\[
D = 0 \iff r_1^2 \max(r_3^2, r_2^2 + s^2) = r_2^2 \min(r_3^2, r_1^2)
\]

(27)

Here we must consider four cases:

1. \( \max(r_3^2, r_2^2 + s^2) = r_3^2 \) and \( \min(r_3^2, r_1^2) = r_3^2 \)
2. \( \max(r_3^2, r_2^2 + s^2) = r_2^2 + s^2 \) and \( \min(r_3^2, r_1^2) = r_3^2 \)
3. \( \max(r_3^2, r_2^2 + s^2) = r_3^2 \) and \( \min(r_3^2, r_1^2) = r_1^2 \)
4. \( \max(r_3^2, r_2^2 + s^2) = r_2^2 + s^2 \) and \( \min(r_3^2, r_1^2) = r_1^2 \)
By substituting each one in Eq. (27), we obtain:

1. \( r_1 = r_2 \)
2. \( r_3^2 = \frac{r_2^2}{r_2^2 + s^2} \)
3. \( r_1 = r_2 \)
4. \( s = 0 \)

Neither of these conditions implies zero Coherence or zero Entanglement. This result is very important and shows that for the X states, Coherence is the fundamental correlation.

All this results point in the same direction than Streltsov [29], who showed that from a coherent state it is possible to produce Entanglement, but that it is impossible to produce it from an incoherent state.

8 Conclusions

In this work, we derive analytical relations between Entanglement and Coherence as well as between Discord and Coherence, for Bell-diagonal states and for the X states, evolving under the action of several noise channels: Bit Flip, Phase Damping and Depolarizing. For the Bell-diagonal states, we found analytical relations between Entanglement and Coherence, and Discord and Coherence, for all cases. In particular, all relations between these correlations are linear. The study of the X states has the additional complexity that five parameters are required to describe the family, which modifies the region of existence: the tetrahedron of the Bell-diagonal states compresses and deforms as the two new parameters \( s \) and \( c \) change. For the X states we described the different regions that determine the expression of Discord, and in each of these regions, we discussed the existence of an analytical relation with Coherence. We found that Discord defines three different regions according to its analytical expression; there are regions where Discord remains constant regardless of Coherence, regions where it decays quadratically without being able to relate to it, and regions where it depends on Coherence in a more or less trivial way depending on the channel applied. Finally, we demonstrate that for the families of studied states, coherence is the fundamental correlation, that is, coherence is necessary for the presence of Entanglement and Discord. Our result points in the same line as previous results: Kok Chuan Tan et. al demonstrated that the Correlated Coherence in a bipartite system is necessary for the system to have Discord [30], and Streltsov et al. showed that Coherence is necessary to generate Entanglement [29].

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