Dimensionality reduction of SDPs through sketching

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August 1, 2017

We show how to sketch semidefinite programs (SDPs) using positive maps in order to reduce their dimension. More precisely, we use Johnson-Lindenstrauss transforms to produce a smaller SDP whose solution preserves feasibility or approximates the value of the original problem with high probability. These techniques allow to improve both complexity and storage space requirements. They apply to problems in which the Schatten 1-norm of the matrices specifying the SDP and of a solution to the problem is constant in the problem size. Furthermore, we provide some no-go results which clarify the limitations of positive, linear sketches in this setting. Finally, we discuss numerical examples to benchmark our methods.

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1. Introduction

Semidefinite programs (SDPs) are a prominent class of optimization problems \cite{LA16}. They have applications across different areas of science and mathematics, such as discrete optimization \cite{WA02} or control theory \cite{BEFB94}.

However, although there are many different algorithms that solve an SDP up to an error $\epsilon$ in a time that scales polynomially with the dimension and logarithmically with $\epsilon^{-1} \ [\text{Bub15}]$, solving large instances of SDPs still remains a challenge. This is not only due to the fact that the number and cost of the iterations scale superquadratically with the dimension for most algorithms to solve SDPs, but also due to the fact that the memory required to solve large instances is beyond current capabilities. This has therefore motivated research on algorithms that can solve SDPs, or at least obtain an approximate solution, with less memory requirements. One such example is the recent \cite{YUAC17}, where ideas similar to ours were applied to achieve optimal storage requirements necessary to solve a certain class of SDPs. While their work proposes a new way to solve an SDP using linear sketches, our approach relies on standard convex optimization methods.

In this work, we develop algorithms to estimate the value of an SDP with linear inequality constraints and to determine if a given linear matrix inequality (LMI) is feasible or not. These algorithms convert the original problem to one of the same type, but of smaller dimension, which we call the sketched problem. Subsequently, this new problem can be solved with the same techniques as the original one, but potentially using less memory and achieving a smaller runtime. Therefore, we call this a black box algorithm. With high probability an optimal solution to the sketched problem allows us to infer something about the original problem.

In the case of LMIs, if the sketched problem is infeasible, we obtain a certificate that the original problem is also infeasible. If the sketched problem is feasible, we are able to infer that the original problem is either “close to feasible” or feasible with high probability, under some technical assumptions.

In the case of estimating the value of SDPs, we are able to give an upper bound that holds with high probability and a lower bound on the value of the SDP from the value
of the sketched problem, again under some technical assumptions. For a certain class of SDPs, which includes the so-called semidefinite packing problems [IPS05], we are able to find a feasible point of the original problem which is close to the optimal point and most technical aspects simplify significantly. For this class it can be checked whether this feasible point is indeed optimal.

Our algorithms work by conjugating the matrices that define the constraints of the SDP with Johnson-Lindenstrauss transforms [Woo14], thereby preserving the structure of the problem. Similar ideas have been proposed to reduce the memory usage and complexity of solving linear programs [VPL15]. While those techniques aim to reduce the number of constraints, our goal is to reduce the dimension of the matrices involved.

Unfortunately, the dimension of the sketch needed to have a fixed error with high probability scales with the Schatten 1-norm of the constraints and that of an optimal solution to the SDP, which significantly restricts the class of problems to which these methods can be applied. We are able to show that one cannot significantly improve this scaling and that one cannot sketch general SDPs using linear maps.

This paper is organized as follows: in Section 2, we fix our notation and recall some basic notions from matrix analysis, Johnson-Lindenstrauss transforms, semidefinite programs and convex analysis which we will need throughout the paper. We then proceed to show how to sketch the Hilbert-Schmidt scalar product with positive maps in Section 3. We apply these techniques in Section 4 to show how to certify that certain LMIs are infeasible by showing the infeasibility of an LMI of smaller dimension. In Section 5, we apply similar ideas to estimate the value of an SDP with linear inequality constraints by solving an SDP of lower dimension. This is followed by a discussion of the possible gains in the complexity of solving these problems and for the memory requirements in Section 6. Furthermore, we make some numerical simulations in Section 7 to benchmark our findings by applying our techniques to a problem from the field of optimal designs of experiments and to a random LMI with matrices sampled from the Gaussian unitary ensemble.

2. Preliminaries

We begin by fixing our notation. For brevity, we will write the set \( \{1, \ldots, d\} \) as \([d]\). The set of \( d \times D \) matrices over some field \( \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\} \) will be written as \( \mathcal{M}_{d,D}(\mathbb{K}) \) and just \( \mathcal{M}_d(\mathbb{K}) \) if \( d = D \). We will often omit the underlying field if it is not relevant for the statement. We will denote by \( \mathcal{M}_d^{sym} \) the set of symmetric \( d \times d \) matrices in the real case and the set of Hermitian matrices in the complex case. For \( A \in \mathcal{M}_d \), \( A^T \) will denote the transpose of \( A \) in the real case and the Hermitian conjugate in the complex case. To avoid cumbersome notation and redundant theorems, we will prove most of the statements only for real matrices. However, note that all statements translate to the complex case in a straightforward fashion. We will state most of the definitions just for real matrices, but it should be clear how to generalize them to the complex case. For \( A \in \mathcal{M}_d^{sym} \) we will write \( A \geq 0 \) if \( A \) is positive semidefinite. We will denote the cone of \( d \times d \) positive semidefinite matrices by \( \mathcal{S}_d^+ \) and its interior, the positive definite matrices,
by $S_d^{++}$.

**Definition 2.1** (Schatten $p$-norm and Hilbert-Schmidt scalar product). Let $A \in M_d$ and $s_1, \ldots, s_d$ be its singular values. We define the Schatten $p$-norm of $A$ for $p \geq 1$, denoted by $\|A\|_p$, to be given by

$$\|A\|_p^p = \sum_{i=1}^d s_i^p$$

and for $p = \infty$ by $\|A\|_{\infty} = \max_{1 \leq i \leq d} s_i$. The Schatten 2–norm is induced by the Hilbert-Schmidt scalar product, which is given by

$$\langle A, B \rangle_{HS} = \text{Tr}(A^T B)$$

for $A, B \in M_d$.

We will sometimes refer to the case $p = 2$ as the Hilbert-Schmidt (HS) norm and $p = \infty$ as the operator norm.

**Definition 2.2** (Positive Map). A linear map $\Phi : M_D \to M_d$ is called positive if $\Phi(S_D^+) \subseteq S_d^+$. The structure of the set of positive maps is still not very well understood [Sto13]. For our purposes, however, we will only need maps of the form $\Phi(X) = XSS^T$ with $S \in M_{d,D}$, which are easily seen to be positive.

We will adopt the standard Big $O$ notation for the asymptotic behavior of functions. That is, for two functions $f, g : \mathbb{R} \to \mathbb{R}$, we will write $g = O(f)$ if there exists a constant $M > 0$ such that for all $x > x_0$ we have $|g(x)| \leq M|f(x)|$. Analogously, we write $g = \Omega(f)$ if there exists a constant $M > 0$ such that for all $x > x_0$ we have $|g(x)| \geq M|f(x)|$.

The following families of matrices will play a crucial role for our purposes:

**Definition 2.3** (Johnson-Lindenstrauss transform). A random matrix $S \in M_d(D)$ is a Johnson-Lindenstrauss transform (JLT) with parameters $(\epsilon, \delta, k)$ if with probability at least $1 - \delta$, for any $k$-element subset $V \subseteq K^D$, for all $v, w \in V$ it holds that

$$|\langle Sv, Sw \rangle - \langle v, w \rangle| \leq \epsilon \|v\|_2 \|w\|_2.$$

Note that one usually only demands that the norm of the vectors involved is distorted by at most $\epsilon$ in the definition of JLTs, but this is equivalent to the definition we chose by the polarization identity. There are many different examples of JLTs in the literature and we refer to [Woo14] and references therein for more details. Most of the constructions of JLTs focus on real matrices, but in Section 4 we show how to lift some of these results to cover complex matrices. The most prominent JLT is probably the following:

**Theorem 2.4.** Let $0 < \epsilon, \delta < 1$ and $S = \frac{1}{\sqrt{d}} R \in M_{d,D}(\mathbb{R})$, where the entries of $R$ are i.i.d. standard Gaussian random variables. If $d = \Omega(\epsilon^{-2}\log(k\delta^{-1}))$, then $S$ is an $(\epsilon, \delta, k)$-JLT.
Proof. We refer to [Woo14, Lemma 2.12] for a proof. □

The main drawback of using Gaussian JLTs is that these are dense matrices. We will denote by $\text{nnz}(X)$ the number of nonzero elements of a matrix $X \in \mathcal{M}_D$. As we later want to compute products of the form $SXST^T$, it will be advantageous to have a sparse $S$ to speed up the computation of this product. The computational cost of forming this product will most often be the bottleneck of our algorithms. Fortunately, there has been a lot of recent work on sparse JLTs. In particular, we have the following almost optimal result.

**Theorem 2.5** (Sparse JLT [KN14, Section 1.1]). There is an $(\epsilon, \delta, k)$-JLT $S \in \mathcal{M}_d, D$ with $d = \mathcal{O}(\epsilon^{-2} \log(k\delta^{-1}))$ and $s = \mathcal{O}(\epsilon^{-1} \log(k\delta^{-1}))$ nonzero entries per column.

*Proof.* We refer to [KN14, Section 1.1] for a proof and remark that the proof is constructive. □

Given some JLT $S \in \mathcal{M}_d, D$, the positive map $\Phi : \mathcal{M}_D \rightarrow \mathcal{M}_d$, $X \mapsto SXST^T$ will be called the sketching map and $d$ the sketching dimension.

We will now fix our notation for semidefinite programs. Semidefinite programs are a class of optimization problems in which a linear functional is optimized under linear constraints over the set of positive semidefinite matrices. We refer to [LA16] for an introduction to the topic. There are many equivalent ways of formulating SDPs. In this work, we will assume w.l.o.g. that the SDPs are given in the following form:

**Definition 2.6** (Sketchable SDP). Let $A, B_1, \ldots, B_m \in \mathcal{M}^{\text{sym}}_D$ and $\gamma_1, \ldots, \gamma_m \in \mathbb{R}$. We will call the constrained optimization problem

\[
\begin{align*}
\text{maximize} & \quad \text{Tr}(AX) \\
\text{subject to} & \quad \text{Tr}(B_iX) \leq \gamma_i, \quad i \in [m] \\
& \quad X \geq 0,
\end{align*}
\]

(1)

a sketchable SDP.

Sometimes we will also refer to a sketchable SDP as the original problem. We will see later how to approximate the value of these SDPs. SDPs have a rich duality theory, in which, instead of optimizing over positive semidefinite matrices that satisfy certain constraints, one optimizes over the points that satisfy a linear matrix inequality (LMI). The dual problem of a sketchable SDP is given by the following:

\[
\begin{align*}
\text{minimize} & \quad \langle c, \gamma \rangle \\
\text{subject to} & \quad \sum_{i=1}^m c_i B_i - A \geq 0 \\
& \quad c \in \mathbb{R}_+^m,
\end{align*}
\]

(2)

where $\gamma \in \mathbb{R}^m$ is the vector with coefficients $\gamma_i$. Here, $\mathbb{R}_+^m = \{ x \in \mathbb{R}^m : x_i \geq 0 \}$. SDPs and LMIs will be called feasible if there is at least one point satisfying all the constraints,
otherwise we will call it infeasible. A sketchable SDP will be called strictly feasible if there is a point $X \geq 0$ such that all the constraints in (1) are satisfied with strict inequality.

Under some conditions the primal problem (1) and the dual problem (2) have the same value. One widely used sufficient condition is Slater’s condition [LA16], which asserts that if we have a strictly feasible point of full rank for the primal problem and if the dual problem is feasible, then the primal and the dual have the same value and there is an optimal solution to the dual problem.

We will need some standard concepts from convex analysis. Given $a_1, \ldots, a_n \in V$ for a vector space $V$, we denote by conv$\{a_1, \ldots, a_n\}$ the convex hull of the points. By cone$\{a_1, \ldots, a_n\}$ we will denote the cone generated by these elements and a convex cone $C$ will be called pointed if $C \cap -C = \{0\}$.

3. Sketching the Hilbert-Schmidt product with positive maps

One of our main ingredients to sketch an SDP or LMI will be a random positive map $\Phi : \mathcal{M}_D \to \mathcal{M}_d$ that preserves the Hilbert-Schmidt scalar product with high probability. We demand positivity to assure that the structure of the SDP or LMI is preserved.

Below, we first consider the example $\Phi(X) = XSS^T$ with $S$ a JLT.

Lemma 3.1. Let $B_1, \ldots, B_m \in \mathcal{M}_D^{\text{sym}}$ and $S \in \mathcal{M}_{d,D}$ be an $(\epsilon, \delta, k)$-JLT with $\epsilon \leq 1$ and $k$ such that

$$k \geq \sum_{i=1}^m \text{rank} B_i.$$  

Then

$$\mathbb{P} \left[ \forall i, j \in [m] : |\text{Tr} \left( SB_i S^T SB_j S^T \right) - \text{Tr} \left( B_i B_j \right) | \leq 3 \epsilon \|B_i\|_1 \|B_j\|_1 \right] \geq 1 - \delta.$$  

(3)

Proof. Observe that the eigenvectors of the $B_i$ corresponding to nonzero eigenvalues of the $B_i$ form a subset of cardinality at most $k$ of $\text{K}^D$. Let $A, B \in \{B_1, \ldots, B_m\}$. As $S$ is an $(\epsilon, \delta, k)$-JLT, with probability at least $1 - \delta$ we have for all normalized eigenvectors $a_i$ of $A$ and $b_j$ of $B$ that

$$\|\langle a_i, Sb_j \rangle - |\langle a_i, b_j \rangle| \leq \epsilon$$

by the reverse triangle inequality. We also have that for any $a_i, b_j$

$$\|Sa_i\|_2 \cdot \|Sb_j\|_2 \leq \sqrt{1 + \epsilon},$$

again by the fact that $S$ is a JLT. As $\epsilon \leq 1$ and by the Cauchy-Schwarz inequality, it follows that

$$|\langle a_i, Sb_j \rangle| + |\langle a_i, b_j \rangle| \leq 3$$

and hence

$$|\langle a_i, Sb_j \rangle|^2 - |\langle a_i, b_j \rangle|^2 | \leq 3 \epsilon.$$  

(4)
Now let $\lambda_i$ and $\mu_j$ be the eigenvalues of $A$ and $B$, respectively. We have:

$$\left| \text{Tr} \left( SAS^T SBS^T \right) - \text{Tr} (AB) \right| = \sum_{i,j=1}^{D} \lambda_i \mu_j(|\langle Sa_i, Sb_j \rangle|^2 - |\langle a_i, b_j \rangle|^2) \leq 3\epsilon \sum_{i,j=1}^{D} |\lambda_i| |\mu_j| = 3\epsilon \|A\|_1 \|B\|_1$$

with probability at least $1 - \delta$. As $A, B$ were arbitrary, the claim follows.

The scaling of the error with the Schatten 1-norm of the matrices involved in Lemma 3.1 is highly undesirable and the estimates used to prove it are admittedly crude. We note that a similar estimate was proved in [SH15]. Moreover, just using the fact that $M_{sym}^D(R) \simeq R^{D(D+1)/2}$ as a Hilbert space, we could use an $(\epsilon, \delta, k)$-JLT $L$ for $R^{D(D+1)/2}$ and isometrically embed the resulting vector into a symmetric matrix. Denoting this transformation by $\tilde{L}$, we obtain

$$\mathbb{P} \left[ \forall i, j \in [n] : \left| \text{Tr} \left( \tilde{L}(B_i)\tilde{L}(B_j) \right) - \text{Tr} (B_iB_j) \right| \leq 3\epsilon \|B_i\|_2 \|B_j\|_2 \right] \geq 1 - \delta.$$

That is, if only demand the sketching map to map symmetric matrices to symmetric matrices, we clearly obtain a better scaling of the error with this procedure. Note, however, that the map $L$ may not be positive, one of the requirements to later sketch SDPs. The next theorem shows that a scaling of the error with the Schatten 2-norm of the matrices involved is not possible with positive maps if we want to achieve a non-trivial compression.

**Theorem 3.2.** Let $\Phi : M_D \rightarrow M_d$ be a random positive map such that with strictly positive probability for any $Y_1, \ldots Y_{D+1} \in M_D$ and $0 < \epsilon < \frac{1}{4}$ we have

$$\left| \text{Tr} \left( \Phi (Y_i)^T \Phi (Y_j) \right) - \text{Tr} (Y_i^T Y_j) \right| \leq \epsilon \|Y_i\|_2 \|Y_j\|_2. \quad (5)$$

Then $d = \Omega(D)$.

**Proof.** We refer to Appendix B for a proof.

One could hope to achieve a better bound for low rank matrices, but we note that this does not significantly improve our bound, as for $A \in M_D$ of rank $r$ we have $\|A\|_1 \leq \sqrt{r} \|A\|_2$. That is, by choosing an $(\epsilon, \delta, k)$-JLT, we may ensure that inequality (5) holds with the HS norm if the rank of the matrices involved is bounded by $r \ll d$. This just increases the dimension of the involved JLT matrices by a factor of $r^2$ if we have the usual $\epsilon^{-2}$ dependence on the dimension for the JLTs. It remains open if one could achieve a better compression for a sublinear number of matrices.
4. Sketching linear matrix inequality feasibility problems

In this section we will show how to use JLTs to certify that certain linear matrix inequalities (LMI) are infeasible by showing that an LMI of smaller dimension is infeasible. We will consider inequalities like the ones in the following lemma:

**Lemma 4.1.** Let \(A, B_1, \ldots, B_m \in \mathcal{M}_D^{\text{sym}} \setminus \{0\}\) such that
\[
\sum_{i=1}^{m} c_i B_i - A \not\geq 0
\]
for all \(c \in \mathbb{R}_+^m\). Suppose further that
\[
\Lambda = \text{cone}\{B_1, \ldots, B_m\}
\]
is pointed and \(\Lambda \cap S_D^+ = \{0\}\). Then there exists a \(\rho \in S_D^+\) such that for all \(i \in [m]\)
\[
\text{Tr}(\rho B_i) < 0, \quad \text{Tr}(-A\rho) < 0 \quad \text{and} \quad \text{Tr}(\rho) = 1.
\]

**Proof.** Let \(E = \text{conv}\{-A, B_1, \ldots, B_m\}\). We will show that \(S_D^+ \cap E = \emptyset\). Suppose there exists an \(X = -p_0 A + \sum_{i=1}^{m} p_i B_i \in S_D^+ \cap E\) with \(p \in [0,1]^{m+1}\). If \(p_0 > 0\), we could rescale \(X\) by \(p_0^{-1}\) and obtain a feasible point for (6), a contradiction. If \(p_0 = 0\) and \(X \neq 0\), this would in turn contradict \(\Lambda \cap S_D^+ = \{0\}\). And if \(X = 0\), the cone \(\Lambda\) would not be pointed. From these arguments it follows that \(0 \not\in E\). The set \(E\) is therefore closed, convex, compact and disjoint from the convex and closed set \(S_D^+\). We may thus find a hyperplane that strictly separates \(S_D^+\) from \(E\). That is, \(\rho \in \mathcal{M}_D^{\text{sym}}\) such that w.l.o.g. \(\text{Tr}(\rho X) \geq 0\) for all \(X \in S_D^+,\) as \(0 \in S_D^+,\) and \(\text{Tr}(Y\rho) < 0\) for all \(Y \in E\). As \(\text{Tr}(\rho X) \geq 0\) for all \(X \geq 0\), it follows that \(\rho\) is positive semidefinite and it is clear that by normalizing \(\rho\) we may choose \(\rho\) with \(\text{Tr}(\rho) = 1\). \(\square\)

The main idea is now to show that under these conditions we may sketch the hyperplane in a way that it still separates the set of positive semidefinite matrices and the sketched version of the set \(\{\sum_{i=1}^{m} \gamma_i B_i - A | \gamma_i \geq 0\}\).

**Theorem 4.2.** Let \(A, B_1, \ldots, B_m \in \mathcal{M}_D^{\text{sym}} \setminus \{0\}\) such that
\[
\sum_{i=1}^{m} c_i B_i - A \not\geq 0
\]
for all \(c \in \mathbb{R}_+^m\). Suppose further that
\[
\Lambda = \text{cone}\{B_1, \ldots, B_m\}
\]
is pointed and \(\Lambda \cap S_D^+ = \{0\}\). Moreover, let \(\rho \in S_D^+\) be such that for all \(i \in [m]\)
\[
\text{Tr}(\rho B_i) < 0, \quad \text{Tr}(-A\rho) < 0 \quad \text{and} \quad \text{Tr}(\rho) = 1.
\]
Set
\[ \epsilon = \frac{1}{2} \min \left\{ \frac{|\text{Tr}(\rho B_1)|}{\|B_1\|_1}, \ldots, \frac{|\text{Tr}(\rho B_m)|}{\|B_m\|_1}, \frac{|\text{Tr}(\rho A)|}{\|A\|_1} \right\} \]
and take \( S \in \mathcal{M}_{d,D} \) to be an \((\epsilon, \delta, k)\)-JLT. Here,

\[ k \geq \text{rank } A + \text{rank } \rho + \sum_{i=1}^{m} \text{rank } B_i. \]

Then
\[ \sum_{i=1}^{m} c_i S B_i S^T - SAS^T \not\geq 0 \quad (7) \]
for all \( c \in \mathbb{R}_+^m \), with probability at least \( 1 - \delta \).

**Proof.** It should first be noted that \( \rho \) exists and \( \epsilon > 0 \) by Lemma 4.1. The matrix \( \rho \) defines a hyperplane that strictly separates the set
\[ E = \left\{ \sum_{i=1}^{m} c_i B_i - A \mid c \in \mathbb{R}_+^m \right\} \]
and \( S^+_D \). We will now show that \( S \rho S^T \) strictly separates the sets
\[ E_S = \left\{ \sum_{i=1}^{m} c_i S B_i S^T - SAS^T \mid c \in \mathbb{R}_+^m \right\} \]
and \( S^+_d \) with probability at least \( 1 - \delta \), from which the claim follows. Note that by our choice of \( \rho \) and \( \epsilon \), it follows from Lemma 3.1 that we have
\[ \text{Tr} \left( S \rho S^T B_i S^T \right) \leq \text{Tr} (\rho B_i) + \epsilon \|B_i\|_1 < 0 \]
with probability at least \( 1 - \delta \) and similarly for \(-A\) instead of \( B_i \). Therefore, it follows that \( \text{Tr}(Z S \rho S^T) < 0 \) for all \( Z \in E_S \). As \( S \rho S^T \) is a positive semidefinite matrix, it follows that \( \text{Tr} \left( Y S \rho S^T \right) \geq 0 \) for all \( Y \in S^+_d \). We have therefore found a strictly separating hyperplane for \( E_S \) and \( S^+_D \) and the LMI (7) is infeasible. \( \square \)

Theorem 4.2 suggests a way of sketching feasibility problems of the form
\[ \sum_{i=1}^{m} c_i B_i - A \geq 0, \quad c \in \mathbb{R}_+^m. \quad (8) \]
If we are interested in the case in which it is infeasible, we investigate if the LMI
\[ \sum_{i=1}^{m} c_i S B_i S^T - S A S^T \geq 0, \quad c \in \mathbb{R}^m_+ \] (9)
is feasible, for \( S \) a suitably chosen JLT as in Theorem 4.2. If Equation (9) is infeasible, we know that Equation (8) is infeasible, as any choice of a feasible \( c \) leads to a feasible \( c \) for Equation (9). Moreover, it follows from Theorem 4.2 that if the cone spanned by the \( B_i \) is well-behaved enough and the JLT is suitably chosen, it only happens with very low probability that Equation (9) is feasible and Equation (8) is not. To obtain more concrete bounds on the probability that the original problem is feasible, one would need to know the parameter \( \epsilon \), which is not possible in most applications. We emphasize that this is a black box algorithm. That is, we can decide whether a large instance of an LMI is infeasible by showing that a smaller instance of an LMI is infeasible. In Section 6 we will discuss the implications for the complexity and memory usage of the last theorems.

5. Approximating the value of semidefinite programs through sketching

We will now show how to approximate with high probability the value of a sketchable SDP by conjugating the target matrix and the matrices that describe the constraints with JLTs and subsequently solving a smaller instance of an SDP. The next theorem shows that in general it is not possible to approximate the value of a sketchable SDP using linear sketches with high probability and that we need to make further assumptions on the problem to achieve a non-trivial compression using linear maps.

**Theorem 5.1.** Let \( \Phi : \mathcal{M}_{2D} \to \mathbb{R}^d \) be a random linear map such that for all sketchable SDPs there exists an algorithm which allows us to estimate the value of an SDP up to a constant factor \( 1 \leq \tau < \frac{2}{\sqrt{3}} \) given the sketch \( \{ \Phi(A), \Phi(B_1), \ldots, \Phi(B_m) \} \) with probability at least \( 9/10 \). Then \( d = \Omega(D^2) \).

**Proof.** By the duality relations for Schatten \( p \)-norms, it is easy to see that the value of the SDP
\[
\begin{align*}
\text{maximize} & \quad \text{Tr} (A X) \\
\text{subject to} & \quad \text{Tr} (X) \leq 1 \\
& \quad X \geq 0
\end{align*}
\] (10)

with
\[
A = \begin{pmatrix}
0 & G \\
G^T & 0
\end{pmatrix}
\]
is twice the operator norm of the matrix \( G \in \mathcal{M}_D \). In [Woo14, Theorem 6.5] it was shown that any algorithm that estimates the operator norm of a matrix from a linear sketch
with probability larger than $9/10$ must have sketch dimension $\Omega(D^2)$. As the sketch \( \{ \Phi(A), \Phi(1) \} \) would thus allow to sketch the operator norm, the assertion follows.

The above result remains true even if we restrict to SDPs that have optimal points with small Schatten 1-norm and low rank. This follows from the fact that the SDP given in Equation (10) has an optimal solution with rank 1 and trace 1.

We may even restrict to SDPs whose value scales sublinearly. To see that, notice that to show that the operator norm cannot be sketched, \cite{Woo14} constructs two families of random matrices whose operator norm is of order $\sqrt{D}$ (cf. \cite[Lemma 6.3]{Woo14}). As the class of SDPs considered here covers this problem, we obtain the claim.

As we will see soon, the main hurdle to sketch SDPs and thus overcome the last no-go theorem is that we also need to suppose that the matrices that define the constraints and the target function have a small Schatten 1-norm, not only one optimal solution. To this end, we define:

**Definition 5.2 (Sketched SDP).** Let $A, B_1, \ldots, B_m \in \mathcal{M}_D^{\text{sym}}$, $\eta, \gamma_1, \ldots, \gamma_m \in \mathbb{R}$ and $\epsilon > 0$. Given that an optimal point $X^* \in S_D^+$ of the sketchable SDP defined through these matrices satisfies $\text{Tr}(X^*) \leq \eta$ and given a random matrix $S \in \mathcal{M}_{d,D}$, we call the optimization problem

\[
\begin{align*}
\text{maximize} & \quad \text{Tr}(SAS^T Y) \\
\text{subject to} & \quad \text{Tr}(SB_i S^T Y) \leq \gamma_i + \mu \|B_i\|_1, \quad i \in [m] \\
Y & \geq 0
\end{align*}
\]

(11)

with $\mu = 3\epsilon \eta$ the sketched SDP.

The motivation for defining the sketched SDP is given by the following theorem.

**Theorem 5.3.** Let $A, B_1, \ldots, B_m \in \mathcal{M}_D^{\text{sym}}$, $\eta, \gamma_1, \ldots, \gamma_m \in \mathbb{R}$ and $\epsilon > 0$. Denote by $\alpha$ the value of the sketchable SDP and assume it is attained at an optimal point $X^*$ which satisfies $\text{Tr}(X^*) \leq \eta$. Moreover, let $S \in \mathcal{M}_{d,D}$ be an $\epsilon, \delta, k$-JLT, with

\[
k \geq \text{rank } X^* + \text{rank } A + \sum_{i=1}^{m} \text{rank } B_i.
\]

Let $\alpha_S$ be the value of the sketched SDP defined by $A, B_i$ and $S$. Then

\[
\alpha_S + 3\epsilon \|A\|_1 \geq \alpha
\]

with probability at least $1 - \delta$.

**Proof.** It follows from Lemma 3.1 that $SX^*S^T$ is a feasible point of the sketched SDP with probability at least $1 - \delta$. Again by Lemma 3.1, we have that

\[
\text{Tr}(SAS^T SX^*S^T) \geq \text{Tr}(AX^*) - 3\epsilon \|A\|_1.
\]
It then follows that $\alpha_S + 3\epsilon \eta \|A\|_1 \geq \alpha$. \hfill $\Box$

In Section 6, we will discuss the implications for memory usage and complexity of approximating the value of an SDP.

Note that the optimal value of an SDP is not necessarily attained. We could also demand $X^*$ to be only close to optimality which would slightly increase the error made by the sketch. Since this makes notation more cumbersome, we assume the existence of such an optimal point.

Although it is not customary to assume a bound on the Schatten 1-norm of an optimal solution to an SDP, it is common to assume that for example the solution lies in a given ellipsoid when using the ellipsoid method to solve SDPs [LA16]. From such assumptions it is straightforward to derive bounds on the HS norm of the solution. If we are also given that an optimal solution to the SDP is low rank, the HS norm gives a good upper bound on the Schatten 1-norm, as remarked after the proof of Lemma 3.1. Solutions of low rank of SDPs have been extensively studied over the past years and there are many results available in the literature which guarantee that the optimal solution to an SDP has low rank. In general, it has been shown [Bar95, Pat98] that if we have $m$ constraints and the SDP is feasible, there is an optimal solution with rank at most $r = \lfloor \sqrt{8m + 1} - 1 \rfloor / 2$.

Notice that Theorem 5.3 does not rule out the possibility that the value of the sketched problem is much larger than that of the sketchable SDP. To investigate this issue, we introduce the following:

**Definition 5.4 (Relaxed SDP).** Let $A, B_1, \ldots, B_m \in M_{D,D}^{\text{sym}}$, $\eta, \gamma_1, \ldots, \gamma_m \in \mathbb{R}$ and $\epsilon > 0$. Given that an optimal point $X^*$ of the sketchable SDP defined through these matrices satisfies $\text{Tr} (X^*) \leq \eta$, we call the optimization problem

$$\begin{align*}
\text{maximize} & \quad \text{Tr} (AX) \\
\text{subject to} & \quad \text{Tr} (B_i X) \leq \gamma_i + \mu \|B_i\|_1, \quad i \in [m] \\
& \quad X \geq 0
\end{align*}$$

(12)

with $\mu = 3\epsilon \eta$ the relaxed SDP.

Notice that, given a feasible point $Y$ to the sketched SDP, $S^T Y S$ is a feasible point for the relaxed problem by the cyclicity of the trace. It follows that if the values of the original and the relaxed are close, the values of the original and the sketched problem are close as well. We formalize this intuition and prove the following bound in Appendix C.

**Theorem 5.5.** We are in the setting of Definition 2.6. Assume that there exists an $X_0 > 0$ such that all the constraints of the sketchable SDP are strictly satisfied and that the dual problem is feasible. Then the value of the sketched SDP $\alpha_S$ is bounded by

$$\alpha_S \leq \alpha(0) + \epsilon C_1 \left( \alpha(0) - \text{Tr} (AX_0) \right) / C_2.$$

12
Here

\[ C_1 = \max \left\{ 3\eta \|B_i\|_1 \mid i \in [m] \right\}, \]
\[ C_2 = \min \left\{ (\gamma_i - \text{Tr} (B_iX_0)) \mid i \in [m] \right\}, \]

where \( \eta \geq \text{Tr} (X^*) \) for an optimal point \( X^* \) of the sketchable SDP.

**Proof.** We refer to Appendix C for the proof. \( \square \)

Note that this statement is not probabilistic and holds regardless of the choice of the sketching map. It could also be the case that \( \text{Tr} (AX_0) \) itself gives a better lower bound on the value than the one in Theorem 5.5. One can therefore say that Theorem 5.5 guarantees that in general the value of the sketched SDP cannot differ significantly from the value of the original one if we have feasible points which are not too close to the boundary. Combining Theorem 5.3 and Theorem 5.5 it is possible to pick \( \epsilon \) small enough to have an arbitrarily small additive error under some structural assumptions on the SDP. That is, we need bounds on the Schatten 1-norms of \( A \) and \( B_i \), be given a strictly feasible point for the relaxed SDP and a bound on the Schatten 1-norm of an optimal solution to the sketchable SDP.

In the case that all the \( \gamma_i > 0 \) for a sketchable SDP we may obtain a bound on the value and an approximate solution to it in a much simpler way. This class includes the so-called semidefinite packing problems [IPS05], where we have in addition that \( B_i \geq 0 \).

Note that we may set all \( \gamma_i = 1 \) w.l.o.g. by dividing \( B_i \) by \( \gamma_i \). We then obtain:

**Theorem 5.6.** For a sketchable SDP with \( \gamma_i = 1 \) and \( \kappa = \max_{i \in [m]} \|B_i\|_1 \), we have that

\[ \frac{\alpha_S}{1 + \nu} \leq \alpha, \]  

where \( \nu = 3\epsilon \eta \kappa \). Moreover, denoting by \( X^*_S \) an optimal point of the sketched SDP, we have that \( \frac{1}{1+\nu} S^T X^*_S S \) is a feasible point of the sketchable SDP that attains this lower bound. Furthermore, if \( \|B_i\|_1 = \kappa \) for all \( i \in [m] \) it can be checked if this lower bound is the optimal value.

**Proof.** The lower bound in Equation (13) follows immediately from the cyclicity of the trace, as \( \frac{1}{1+\nu} S^T X^*_S S \) is a feasible point of the sketchable SDP. Given an optimal solution \( \{c_i\}_{i \in [m]} \) to the dual of the sketched SDP and that \( \|B_i\|_1 = \kappa \) for all \( i \in [m] \), it is possible to check if the lower bound given by the sketched SDP is indeed optimal as follows. Slater’s condition holds for the sketched and sketchable SDP, as \( \vartheta I \) is a strictly feasible point for \( \vartheta > 0 \) small enough. If we have

\[ \sum_{i=1}^m c_i B_i - A \geq 0, \]

then the obtained feasible point is indeed optimal for the sketchable SDP by strong duality. \( \square \)
It is possible to relax the condition that all the Schatten 1-norms of the matrices that define the constraints are the same and still obtain a lower bound for which it can be checked whether it is indeed optimal. To achieve this, it is necessary to modify all the constraints of the sketched SDP, as in Equation (11), to \(1 + \mu \kappa\) instead of \(1 + \mu \|B_i\|_1\).

In the primal picture, we will find an optimal point of the sketched SDP through the sketched SDP whenever there is a \(Y \in S_D^+\) such that \(S^TYS = X^*\), for \(X^*\) an optimal point of the sketched SDP. Note that for semidefinite packing problems it is possible to derive a bound on the Schatten 1-norm of an optimal solution in a straightforward way.

**Lemma 5.7.** Let \(B_1, \ldots, B_m \in \mathcal{M}_D\) be positive semidefinite matrices such that the smallest strictly positive eigenvalue of \(\sum_{i=1}^m B_i\) is given by \(\lambda\). Then for a sketched SDP with constraints \(B_i, \gamma_1, \ldots, \gamma_m \in \mathbb{R}_+, A \succeq 0\) and finite value there exists an optimal point \(X^*\) such that

\[
\Tr(X^*) \leq \frac{1}{\lambda} \sum_{i=1}^m \gamma_i. \tag{14}
\]

**Proof.** As we assume that the SDP has a finite value, we may restrict to solutions whose support is contained in the support of \(\sum_{i=1}^m B_i\). Denote by \(P\) the projection onto the support of \(\sum_{i=1}^m B_i\). We then have

\[
\lambda P \leq \sum_{i=1}^m B_i. \tag{15}
\]

Conjugating both sides with \((X^*)^{\frac{1}{2}}\) and taking the trace we obtain

\[
\lambda \Tr(X^*) \leq \sum_{i=1}^m \Tr(X^* B_i),
\]

as we supposed w.l.o.g. that the support of \(X^*\) is contained in the support of \(\sum_{i=1}^m B_i\). As \(X^*\) is a feasible point, we have \(\Tr(X^* B_i) \leq \gamma_i\) and we obtain the claim. \(\square\)

Of all the assumptions we needed for the results of Theorem 5.3, the bound on the Schatten 1-norm of an optimal solution to the SDP is arguably the most difficult to show, as bounds of this form are not readily available in the literature. Moreover, some SDPs have \(\Tr(X) \leq \eta\) as constraint and would be natural candidates to apply these methods to. As \(\Tr(\mathbb{I}) = D\) we will not be able to obtain any non-trivial compression with the scheme discussed so far. However, if one is only interested in obtaining an upper bound on the value of the SDP, it is still possible to have \(\mathbb{I}\) as a constraint or as the target matrix and achieve a non-trivial compression.

**Theorem 5.8.** Let \(A, B_1, \ldots, B_{m-1} \in \mathcal{M}_D^{\text{sym}}, B_m = \mathbb{I}\). Further, let \(\gamma_1, \ldots, \gamma_{m-1} \in \mathbb{R}, \gamma_m = \eta\) and \(\epsilon > 0\). Denote the value of the sketchable SDP by \(\alpha\) and assume it is attained
at an optimal point $X^\ast$. Moreover, let $S \in \mathcal{M}_{d,D}$ be an $(\epsilon, \delta, k)$-JLT, with

$$k \geq \text{rank } X^\ast + \text{rank } A + \sum_{i=1}^{m} \text{rank } B_i.$$ 

Let $\alpha'_S$ be the value of the modified sketched SDP defined by $A$, $B_i$ and $S$, given by

$$\begin{align*}
\text{maximize} & \quad \text{Tr} (SAS^T X) \\
\text{subject to} & \quad \text{Tr} (SB_iS^T X) \leq \gamma_i + \mu \|B_i\|_1, \quad i \in [m - 1] \\
& \quad \text{Tr} (X) \leq (1 + \epsilon) \eta, \\
& \quad X \geq 0
\end{align*}$$

with $\mu = 3\epsilon\eta$. Then

$$\alpha'_S + 3\epsilon\eta \|A\|_1 \geq \alpha$$

with probability at least $1 - \delta$.

**Proof.** The proof is essentially the same as the one of Theorem 5.3 as we have that $|\text{Tr} (SX^\ast S^T) - \text{Tr} (X^\ast)| \leq 3\epsilon \text{Tr} (X^\ast)$ and so $SX^\ast S^T$ is a feasible point with probability at least $1 - \delta$. \(\square\)

The main difference between the modified sketched SDP and the sketched SDP is that here we do not conjugate the identity with $S$, only the other constraints. With this, we do not have that $S^T X^\ast S$ is a feasible point of the relaxed SDP, but we do not need the assumption $\text{Tr} (X^\ast) \leq \eta$ to obtain an upper bound. It should be clear from Theorem 5.8 that we may also optimize over the trace, i.e. $A = I$, without conjugating with $S$ and still have an upper bound and non-trivial compression.

We may therefore summarize the results of this section as follows. If we want to obtain an upper bound on the value of the sketchable SDP with our techniques, it is necessary to have upper bounds on the Schatten 1-norm of all the matrices that define the constraints, the target matrix and of an optimal solution. We may then choose a JLT of suitable dimension to solve the sketched SDP, whose value will allow us to infer an upper bound to the original problem with high probability. If we are additionally given a strictly feasible point of the sketchable SDP or if we are solving a semidefinite packing problem, we also obtain a lower bound on the value of the sketchable SDP in terms of the sketched one. In the case of semidefinite packing problems, we even obtain a feasible point of the sketchable SDP whose value is close to the sketched value. If we are not given a bound on the Schatten 1-norm of an optimal solution, we may impose it as a constraint as in Theorem 5.8 and obtain an upper bound on the value of the sketchable SDP constrained to points which have their Schatten 1-norm bounded by $\eta$. Although we are able to drop the assumption on the Schatten 1-norm of an optimal solution, we are not able to prove that this upper bound cannot differ significantly from the true value in this case.
6. Complexity and memory gains

In this section, we will discuss how much we gain by considering the sketched SDP instead of the sketchable SDP. We focus on the results of Section 5, but the discussion carries over to the results of Section 4. Throughout this section we will assume that we are guaranteed that the Schatten 1−norm of an optimal solution to our SDP and of the matrices that define the constraints is $O(1)$. It is therefore Theorem 5.3 for which we need a sketch of appropriate size. The theorem states that we need an $(\epsilon, \delta, k)$-JLT for the upper bound on $\alpha$ to hold with probability at least $1 - \delta$. As stated in Theorem 5.3, we can choose a sketching matrix $S \in \mathcal{M}_{d,D}$ with $d = O(\epsilon^{-2} \log(k\delta^{-1}))$ and $s = O(\epsilon^{-1} \log(k\delta^{-1}))$ nonzero entries per column. Here $k$ is as in Theorem 5.3. The cost of generating $S$ is at most $O(dD)$, which will be of smaller order than the necessary matrix multiplications. We will therefore not take this cost into account for the rest of the analysis.

One could argue that one needs to know the Schatten 1-norm of the different matrices that define our constraints for estimating the value or obtaining more concrete bounds for the feasibility problems. We will, however, suppose that an upper bound on the Schatten 1-norm of an optimal solution and the constraints is given or that this can be computed in a time which is $O(D^2)$. This is the case if for example we have a semidefinite packing problem, where the matrices are positive semidefinite and we can compute their Schatten 1-norm in $O(D)$ time.

To generate the sketched SDP, we need to compute $m + 1$ matrices of the form $SB_iS^T$, where $B \in \mathcal{M}_D$. Each of this computations needs $O(\max \{ \text{nnz}(A), \text{nnz}(B_1), \ldots, \text{nnz}(B_m) \})$ operations. In the worst case, when all matrices $\{ A, B_1, \ldots, B_m \}$ are dense and have full rank, this becomes $O(mD^2 \log(mD))$ operations to generate the sketched SDP for fixed $\epsilon$ and $\delta$.

Let us collect these considerations in a proposition:

**Proposition 6.1.** Let $A, B_1, \ldots, B_m \in \mathcal{M}^{\text{sym}}_D, \gamma_1, \ldots, \gamma_m \in \mathbb{R}$ of a sketchable SDP be given. Furthermore, let $z := \max \{ \text{nnz}(A), \text{nnz}(B_1), \ldots, \text{nnz}(B_m) \}$ and $\text{SDP}(m, d, \zeta)$ be the complexity of solving a sketchable SDP (up to accuracy $\zeta$) of dimension $d$. Then a number of

\[ O(\max \{ z, D\epsilon^{-2} \log(k\delta^{-1}) \} \epsilon^{-1} m \log(k\delta^{-1}) + \text{SDP}(m, \epsilon^{-2} \log(k\delta^{-1}), \zeta)) \]

operations is needed to generate and solve the sketched SDP, where $k$ is defined as in Theorem 5.3.

It is easy to see that we can parallelize computing the matrices $SB_iS^T$. Typically, the costs of forming the sketched matrices $SB_iS^T$ dominates the overall complexity. For example, using the ellipsoid method [GS88, Chapter 3], the complexity of solving an SDP becomes $\text{SDP}(m, D, \zeta) = O(\max \{ m, D^2 \} D^6 \log(1/\zeta))$ (cf. [Bub15, p.250]). Assuming that $\epsilon, \delta$ and $\zeta$ are fixed, we need $O(\max \{ m, D^2 \} D^6)$ operations to solve the sketchable SDP, compared to $O(mD^2 \log(mD))$ operations to obtain an approximate solution via first forming the sketched problem and then solving it. Admittedly, the
ellipsoid method is not used in practice, but using interior point methods, we still need
\[ \text{SDP}(m, D, \zeta) = \mathcal{O}(\max \{ m^3, D^2m^2, mD^{\omega} \} D^{0.5} \log(D/\zeta)) \] operations \cite[Chapter 5]{dik2002}, where \( \omega \) is the exponent of matrix multiplication. The best known algorithms achieve \( \omega \approx 2.37 \) \cite{LG14}. If the SDP can be sketched, doing so gives a speedup as long as the complexity of solving the SDP directly is \( \Omega(mD^{2+\mu}) \), where \( \mu > 0 \).

A great advantage is that for the sketched problem, we only need to store \( m+1 \) matrices of size \( d \times d \) instead of \( D \times D \). We collect this in a proposition.

**Proposition 6.2.** Let \( A, B_1, \ldots, B_m \in M_{D}^{\text{sym}}, \gamma_1, \ldots, \gamma_m \in \mathbb{R} \) be a sketchable SDP. Then we need only store \( \mathcal{O}(me^{-4}\log(mk/\delta)^2) \) entries for the sketched problem, where \( k \) is defined as in Theorem 5.3.

### 7. Applications and numerical examples

#### 7.1. Estimating the value of a semidefinite packing problem

Inspired by \cite{Sag11} we will test our techniques on an SDP stemming from the field of optimal design of experiments. The problem is the following: an experimenter wishes to estimate the quantity \( \langle c, \theta \rangle \), where \( \theta \in \mathbb{R}^D \) is an unknown \( D \)-dimensional parameter and \( c \in \mathbb{R}^D \) is given. To this end, one is given linear measurements of the parameter \( y_i = A_i^T \theta \), up to a (centered) measurement noise for \( A_i \in M_D \). We refer to \cite{Puk06} for more details on the topic. To find the amount of effort to spend on the \( i \)-th experiment to minimize the variance is given by the SDP

\[
\begin{align*}
\text{maximize} & \quad \text{Tr} (cc^T X) \\
\text{subject to} & \quad \text{Tr} (M_i X) \leq 1, \quad i \in [m] \\
& \quad X \succeq 0
\end{align*}
\]

with \( M_i = A_i^T A_i \). This problem always admits optimal solutions of rank 1 \cite{Sag11}. We generated random instances of this SDP in the following way:

1. We sampled four matrices \( A_i \) distributed as follows: the first three rows of \( A_i \) are sampled independently from the uniform distribution on the unit sphere in \( \mathbb{R}^D \). The other \( D-3 \) rows of \( A_i \) are set to 0.

2. Given the \( A_i \), we generate \( c \) by getting four samples \( k_1, k_2, k_3, k_4 \) from the uniform distribution on \( \{1, 2, 3\} \) and four samples \( x_1, x_2, x_3, x_4 \) from the standard normal distribution. \( c \) is then given by \( \sum_{i=1}^{4} x_i(A_i)_{k_i} \), where \( (A_i)_{k_i} \) is the \( k_i \)-th row of \( A_i \).

This gives matrices \( M_i \) of rank 3 almost surely and Schatten 1-norm equal to 3. The fact that \( c \) is a linear combination of the rows of \( A_i \) ensures that the problem is bounded, as can be easily seen by looking at the dual problem \cite{Sag11}. Note that this is a semidefinite packing problem, so we are able to use the results of Theorem 5.6 to obtain a lower bound. There exists an optimal solution whose Schatten 1-norm is bounded by 8 with very high
| $D$  | $d$ | Value | Error L.B. | Error U.B. | M.R.T. Sketchable [s] | M.R.T Sketch [s] |
|------|-----|-------|-----------|-----------|------------------------|-----------------|
| 100  | 10  | 2.52  | 0.0880    | 0.156     | 1.27                   | 0.324           |
| 100  | 20  | 2.50  | 0.00      | 0.250     | 1.17                   | 0.305           |
| 200  | 20  | 2.69  | 0.00      | 0.269     | 6.50                   | 0.299           |
| 200  | 40  | 2.53  | 0.00      | 0.00102   | 6.82                   | 0.375           |
| 500  | 50  | 2.55  | 0.00      | 0.255     | 98.0                   | 0.453           |
| 500  | 100 | 2.66  | 0.00      | 0.266     | 97.6                   | 1.23            |
| 700  | 70  | 2.57  | 0.00      | 0.257     | 557                    | 1.38            |
| 700  | 140 | 2.49  | 0.00      | 0.249     | 548                    | 3.53            |

Table 1: For each combination of the sketchable dimension ($D$) and dimension of the sketch ($d$) we have generated 40 instances of the SDP in Equation (17). Here “M.R.T.” stands for mean running time, “L.B.” stands for lower bound and “U.B.” for upper bound and each column shows the mean of the sample. The column “Value” stands for the optimal value of the sketchable SDP.

We can see that, excluding the case where the sketching dimension was 10, we were able to find feasible points which were numerically indistinguishable from being optimal by using our sketching methods. Moreover, the time needed to find an optimal solution was smaller by 1 or 2 orders of magnitude.

### 7.2. Linear matrix inequality feasibility problems

We will now apply our techniques to an LMI feasibility problem. Let $G_i \in \mathcal{M}_d$, $i \in [m]$, be random matrices sampled independently from the Gaussian unitary ensemble (GUE). We used sparse JLTs with sparsity parameter $s = 1$ to obtain faster matrix multiplications to form the sketches. We define the error of the lower bound to be given by $\alpha - \frac{1}{1+\eta} \alpha_S$ and of the upper to be $\alpha_S - \alpha$. To solve the SDP given in Equation (17) we used cvx, a package for specifying and solving convex programs.

As we can see, the results of Table 1 show that, excluding the case where the sketching dimension was 10, we were able to find feasible points which were numerically indistinguishable from being optimal by using our sketching methods. Moreover, the time needed to find an optimal solution was smaller by 1 or 2 orders of magnitude.
is feasible. Using standard techniques from random matrix theory, we show that Equation (19), and so Equation (18), is feasible with high probability for $\alpha > \frac{2}{\sqrt{m}}$ and infeasible with high probability for $\alpha < \frac{2}{\sqrt{m}}$ in case $m \ll d'$.

We refer to Appendix Section D for a proof of this claim. This therefore allows us to quantify “how close to feasible” the LMI inequality is in terms of how close $\alpha$ is to $\frac{2}{\sqrt{m}}$ and to know whether the LMI was feasible or not. We will choose $d' \ll D$, as this way we avoid having a Schatten 1-norm of the matrices that define the LMI which is of the same order as the dimension. The technique we used to solve Equation (18) is the same as discussed in Section 4. That is, we will check for the feasibility of

$$\sum_{i=1}^{m} t_i SVG_i(\alpha)V^*S^T - SVV^*S^T \geq 0, \quad t \in \mathbb{R}^m$$

for a complex Gaussian JLT $S$. That is, $S = \frac{1}{\sqrt{2}}(S_1 + iS_2)$ with $S_1$ and $S_2$ independent Gaussian JLTs. We refer to Section A for a proof that this choice of random matrices indeed gives a JLT with the same scaling of the parameters as the real one. We refer to Theorem [D.4] for a proof that the LMI defined in Equation (20) satisfies the assumptions of Theorem [4.2] with high probability. To solve the SDP given in Equation (20) we used cvx, a package for specifying and solving convex programs [GB14, GB08]. The results are summarized in Table 2. We can observe that by using our methods we were able to show that the LMI is infeasible in a much smaller running time or even show that certain LMI were infeasible when a direct computation was not possible due to memory constraints in most choices of the parameters. In some cases it was, however, necessary to increase the sketch dimension to show that the inequality was infeasible.

8. Conclusion

We have shown how to obtain sketches of the HS product using positive maps obtained from JLTs, how to apply these to show that certain LMI are infeasible and to obtain approximations of the value of certain SDPs. In some cases, these techniques can lead to significant improvements in the runtime necessary to solve the instances of the SDPs and significant gains in the memory needed to solve them. However, the class of problems to which these techniques can be applied is significantly restricted by the fact that the matrices that define the constraints of the problems and a solution must have Schatten 1-norms which do not scale with the dimension for them to be advantageous. Moreover, the no-go theorems proved here show that one cannot significantly improve our results using positive linear maps to sketch the HS norm or to approximate the value of SDPs.

Acknowledgements

We would like to thank Ion Nechita for helpful discussions. A.B. acknowledges support from the ISAM Graduate Center at Technical University of Munich. D.S.F. acknowled-
| $D$ | $d$ | $d'$ | $\alpha$ | M.R.T. Original [s] | M.R.T. Sketch [s] | Error Rate |
|-----|-----|-----|--------|------------------|------------------|------------|
| 200 | 50  | 100 | 0.3    | 452              | 4.65             | 0          |
| 200 | 50  | 100 | 0.4    | 407              | 4.31             | 0          |
| 200 | 50  | 100 | 0.5    | 383              | 5.71             | 0          |
| 200 | 50  | 100 | 0.6    | 407              | 6.46             | 0          |
| 200 | 100 | 100 | 0.3    | 449              | 66.3             | 0          |
| 200 | 100 | 100 | 0.4    | 344              | 86.6             | 0          |
| 200 | 100 | 100 | 0.5    | 393              | 102              | 0          |
| 200 | 100 | 100 | 0.6    | 457              | 91.4             | 0          |
| 400 | 50  | 200 | 0.3    | -                | 4.02             | 0          |
| 400 | 50  | 200 | 0.4    | -                | 7.04             | 0          |
| 400 | 50  | 200 | 0.5    | -                | 3.39             | 0.975      |
| 400 | 50  | 200 | 0.6    | -                | 2.35             | 1.0        |
| 400 | 100 | 200 | 0.3    | -                | 114              | 0          |
| 400 | 100 | 200 | 0.4    | -                | 122              | 0          |
| 400 | 100 | 200 | 0.5    | -                | 118              | 0          |
| 400 | 100 | 200 | 0.6    | -                | 115              | 0          |

Table 2: For each combination of the dimension of the image of the random isometry ($D$), dimension of the domain of the random isometry ($d'$), dimension of the sketch ($d$) and $\alpha$ we have generated 40 instances of the random LMI in Equation (18) with $m = 9$. Here “M.R.T.” stands for mean running time. The error rate gives the ratio of infeasible problems that were not detected to be infeasible by sketching. A dash in the running time means that we were not able to solve the LMI because we ran out of memory.
edges support from the graduate program TopMath of the Elite Network of Bavaria, the TopMath Graduate Center of TUM Graduate School at Technical University of Munich. D.S.F. is supported by the Technical University of Munich – Institute for Advanced Study, funded by the German Excellence Initiative and the European Union Seventh Framework Programme under grant agreement no. 291763.

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A. Complexifying Johnson-Lindenstrauss transforms

In this Appendix we will generalize some of the results we need concerning JLTs to complex vector spaces. We will see that, up to a constant, most of the statements that hold in the real case also hold in the complex case. We will consider matrices of the form \( \frac{1}{\sqrt{2d}} (S + iT) \), with \( S, T \) independent and with i.i.d. entries that are sub-gaussian and show that they give us complex JLTs. Note that these constructions clearly give sparse JLTs if the real JLTs used are sparse.

Definition A.1 (Sub-Gaussian Distribution). The probability distribution of a random variable \( X \) is called sub-gaussian if there exist \( C, v > 0 \) such that for all \( t > 0 \)

\[
P(\|X\| > t) \leq C e^{-vt^2}.
\]

A random variable is sub-gaussian if and only if for \( p \geq 1 \),

\[
E[|X|^p] = O(p^{\frac{p}{2}})
\]

and the sub-gaussian norm of \( X \) is defined as:

\[
\|X\|_{\psi_2} = \sup_{p \geq 1} p^{-\frac{1}{2}} E(|X|^p)^{\frac{1}{p}}.
\] (21)

See for example [Ver12, Section 5.2.3] for more details on this. The main ingredient to show how to generalize JLTs to the complex case will be the following theorem.

Theorem A.2 ([RV13, Theorem 2.1]). Let \( A \in \mathcal{M}_{d,D}(\mathbb{R}) \) be fixed. Consider a random vector \( X = (X_1, \ldots, X_D) \), where \( X_i \) are independent random variables satisfying \( E[X_i] = 0 \), \( E[X_i^2] = 1 \) and \( \|X_i\|_{\psi_2} \leq K \). Then for any \( t \geq 0 \), we have

\[
P(\|AX\|_2 - \|A\|_{HS} > t) \leq 2 \exp\left[-\frac{ct^2}{K^4 \|A\|_\infty^2}\right].
\]

Using this, we have a different way of proving the Johnson-Lindenstrauss lemma which generalizes to the complex case. We follow the proof of [RV13, Theorem 3.1].

Lemma A.3. Let \( S, T \in \mathcal{M}_{d,D}(\mathbb{R}) \) have independent sub-gaussian entries with \( E[X_{ij}] = 0 \), \( E[X_{ij}^2] = 1 \) and \( \|X_{ij}\|_{\psi_2} \leq K \), for all \( X \in \{S, T\} \). Then for \( d = \mathcal{O}(e^{-2 \ln(2/\delta)}) \), we have

\[
P\left(\left\|\frac{S + iT}{\sqrt{2d}} x\right\|_2 \in (1 \pm \epsilon)\|x\|_2\right) \geq 1 - \delta
\]

for any fixed \( x \in \mathbb{C}^D \).
**Proof.** Define the linear operator \( \Phi : \mathcal{M}_{d,D}(\mathbb{C}) \to \mathbb{C}^d, \ G \mapsto Gx \), where \( x \in \mathbb{C}^d \) is a fixed vector. We use the standard isomorphisms \( \mathcal{M}_{d,D}(\mathbb{C}) \cong \mathbb{R}^{2dD} \) and \( \mathbb{C}^d \cong \mathbb{R}^{2d} \) and denote by \( \tilde{\Phi} \) the map \( \Phi \) composed with these isomorphisms, which is now a linear map from a real vector space to another real vector space. Moreover, observe that a matrix of the form \( X + iY \in \mathcal{M}_{d,D}(\mathbb{C}) \) with \( X, Y \in \mathcal{M}_{d,D}(\mathbb{R}) \) is mapped to \( (X,Y) \) under the isomorphism. The map \( \tilde{\Phi} \) will play the role of \( A \) in the statement of Theorem A.2. It is straightforward to compute the norms involved in the statement, as explained in [RV13, Section 3.1]. We have

\[
\|\tilde{\Phi}\|_2^2 = 2d\|x\|_2^2, \quad \|\tilde{\Phi}\|_\infty^2 = \|x\|_2^2, \quad \|\tilde{\Phi}((S,T)^T)\|_2^2 = 2\|(S+iT)x\|_2^2.
\]

As \( S, T \) have sub-gaussian entries, the vector \( (S,T) \) satisfies the assumptions of Theorem A.2 and the statement follows.

Unfortunately, the sparse JLTs discussed in Theorem 2.5 are not of this form. The entries of these JLTs are not independent from each other, one of the assumptions of Lemma A.3.

**B. Proof of Theorem 3.2**

**Theorem 3.2.** Let \( \Phi : \mathcal{M}_D \to \mathcal{M}_d \) be a random positive map such that with positive probability for any \( Y_1, \ldots, Y_{D+1} \in \mathcal{M}_D \) and \( 0 < \epsilon < \frac{1}{4} \) we have

\[
|\text{Tr} (\Phi(Y_i)^T \Phi(Y_j)) - \text{Tr} (Y_i^T Y_j)| \leq \epsilon \|Y_i\|_2 \|Y_j\|_2. \tag{22}
\]

Then \( d = \Omega(D) \).

**Proof.** Let \( \{e_i\}_{1 \leq i \leq D} \) be an orthonormal basis of \( \mathbb{C}^D \) and define \( X_i = e_i e_i^T \). As Equation (22) is satisfied with positive probability, there must exist a positive map \( \Phi : \mathcal{M}_D \to \mathcal{M}_d \) such that Equation (22) is satisfied for \( Y_i = X_i, i \in [D] \), and \( Y_{D+1} = 1 \). As the \( X_i \) are orthonormal w.r.t. the Hilbert-Schmidt scalar product and by the positivity of \( \Phi \) we have for \( i, j \in [D] \)

\[
\text{Tr} (\Phi(X_i)\Phi(X_j)) \in \begin{cases} [0, \epsilon], & \text{for } i \neq j \\ [1-\epsilon, 1+\epsilon], & \text{for } i = j. \end{cases} \tag{23}
\]

Define the matrix \( A \in \mathcal{M}_D \) with \( (A)_{ij} = \text{Tr} (\Phi(X_i)\Phi(X_j)) \) for \( i, j \in [D] \). It is clear that \( A \) is Hermitian and that its entries are positive. We have

\[
\sum_{i,j \in [D]} A_{ij} = \text{Tr} (\Phi(1)\Phi(1)) \in \left([1-\epsilon)D, (1+\epsilon)D\right].
\]

As \( A_{ii} \geq (1-\epsilon) \), it follows that

\[
\sum_{i \neq j} A_{ij} \leq 2\epsilon D. \tag{24}
\]
Let

\[ J = \{ (i, j) \in [D] \times [D] \mid i \neq j, A_{ij} \leq \frac{1}{D} \}. \]

It follows from Equation (24) that \(| \{ (i, j) \in [D] \times [D] \mid i \neq j, (i, j) \notin J \} | \leq 2D^2 \epsilon\) and so

\[ |J| \geq (1 - 2\epsilon)D^2 - D. \]

Since for \((i, j) \in J\) also \((j, i) \in J\), we can write \(J = (I \times I) \setminus \{ (i, i) \mid i \in I \}\) for \(I \subseteq [D]\). Thus, we infer for \(D \geq 2\)

\[ |J| = |I|(|I| - 1) \geq (1 - 2\epsilon)D^2 - D \geq \left( \frac{1}{2} - 2\epsilon \right)D^2. \]

From this it follows that

\[ |I|^2 \geq |I|(|I| - 1) \geq \left( \frac{1}{2} - 2\epsilon \right)D^2, \]

and we finally obtain

\[ |I| \geq \sqrt{1/2 - 2\epsilon D}. \]

Notice that it follows from Equation (23) that we may rescale all the \(X_i\) to \(X'_i\) such that \(\text{Tr} (\Phi(X'_i)^2) = 1\) and the pairwise scalar product still satisfies \(\text{Tr} (\Phi(X'_i)\Phi(X'_j)) \leq \frac{1}{1 - \epsilon}\) for \((i, j) \in J\). If there is an \(N \in \mathbb{N}\) such that \(d > \sqrt{1/2 - 2\epsilon D}\) for all \(D \geq N\), the claim follows. We therefore now suppose that \(d \leq \sqrt{1/2 - 2\epsilon D}\). Hence, \(d \leq |I|\) by Equation (25). By the positivity of \(\Phi\) and the fact that the \(X'_i\) are positive semidefinite, we have that \(\Phi(X'_i)\) is positive semidefinite. In [Wol12, Proposition 2.7] it is shown that for any set \(\{P_i\}_{i \in I}\) of \(|I| \geq d\) positive semidefinite matrices in \(\mathcal{M}_d\) such that \(\text{Tr} (P_i^2) = 1\) we have that

\[ \sum_{i \neq j} \text{Tr} (P_i P_j)^2 \geq \frac{|I| - d}{|I| - 1} d^2. \]

By the definition of the set \(J\), we have that

\[ \sum_{(i, j) \in J} \text{Tr} (X'_i X'_j)^2 \leq \frac{|J|}{(1 - \epsilon)^2 D^2} \leq \frac{1}{|1 - \epsilon|^2}, \]

as \(|J| \leq D^2\). From Equation (25) it follows that

\[ \frac{1}{(1 - \epsilon)^2} \geq \left( \frac{\sqrt{1/2 - 2\epsilon D}}{d} - 1 \right)^2 \]

and after some elementary computations we finally obtain

\[ d \geq \frac{(1 - \epsilon) \sqrt{1/2 - 2\epsilon}}{2 - \epsilon} D. \]
C. Lower bound on the value of SDPs through sketching

We will obtain lower bounds on the value of the sketchable SDP in terms of the value of the sketched SDP through continuity bounds on the relaxed SDP. As the continuity bound we use is for SDPs given in equality form, we begin by giving an equivalent formulation of a sketchable SDP with equality constraints. The method of using duality to derive perturbation bounds on a convex optimization problem used here is standard and we refer to [BV04, Section 5.6] for a similar derivation. Given a sketchable SDP, define the maps $\Phi : \mathcal{M}_D \to \mathcal{M}_m$

$$\Phi(X) = \sum_{j=1}^{m} \text{Tr} (B_j X) e_j e_j^T$$

for $\{e_j\}_{j=1}^{m}$ an orthonormal basis of $\mathbb{R}^m$ and $\Psi : \mathcal{M}_{D+m} \to \mathcal{M}_m$

$$\Psi \left( \begin{bmatrix} X & * \\ * & Z \end{bmatrix} \right) = \Phi(X) + [Z_{jj}]_j.$$  

With the help of the matrix

$$G = \sum_{j=1}^{m} \gamma_j e_j e_j^T,$$  \hspace{1cm} (26)

the sketchable SDP can be written in equality form as

$$\begin{align*}
\text{maximize} & \quad \text{Tr} \left( \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X & * \\ * & Z \end{bmatrix} \right) \\
\text{subject to} & \quad \Psi \left( \begin{bmatrix} X & * \\ * & Z \end{bmatrix} \right) = G \\
& \quad \begin{bmatrix} X & * \\ * & Z \end{bmatrix} \succeq 0,
\end{align*}$$  \hspace{1cm} (27)

where * are submatrices which are not relevant for our discussion. The dual problem of Equation (27) may be written as

$$\begin{align*}
\text{minimize} & \quad \sum_{j=1}^{m} \gamma_j Y_{jj} \\
\text{subject to} & \quad \sum_{j=1}^{m} B_j Y_{jj} \succeq A \\
& \quad Y_{jj} \succeq 0, \quad \forall j \in [m].
\end{align*}$$  \hspace{1cm} (28)
Lemma C.1. If there is an $X > 0$ such that the sketchable SDP is satisfied with strict inequality and the dual problem is feasible, then both the primal problem given in Equation (27) and dual in Equation (28) are feasible, there is no duality gap and there is a dual solution which attains the optimal value. Furthermore, this condition is equivalent to Slater’s condition.

Proof. If we can show equivalence with Slater’s condition, the first statement follows automatically. Slater’s condition for the primal problem is the following. If there is a $\begin{bmatrix} X & * \\ * & Z \end{bmatrix} > 0$ which satisfies the constraints, then the duality gap is zero and there is a $Y$ which attains the optimal value. For such a block matrix, we also need $X > 0$, $Z > 0$. Hence, we can formulate Slater’s condition with off-diagonal entries $* = 0$. We observe

$$
\Psi \left( \begin{bmatrix} X & * \\ * & Z \end{bmatrix} \right) = G \iff \begin{bmatrix} \gamma_1 - \text{Tr} (B_1 X) \\ \vdots \\ \gamma_m - \text{Tr} (B_m X) \end{bmatrix} = Z > 0.
$$

Hence $X > 0$ satisfies the constraints with strict inequality. The converse is clear.

Now we can bound the optimal solution to Equation (11). We denote by $A(\epsilon)$ the feasible set of the relaxed SDP as in Definition 5.4 for some $\epsilon > 0$. With this notation, $A(0)$ is the feasible set for the primal problem. Analogously, we denote by $\alpha(\epsilon)$ the optimal value of the relaxed problem, by $\alpha(0)$ the optimal value of the sketchable SDP.

Lemma C.2. Assume that there exists $X_0 \in A(0)$ such that $X_0 > 0$ and the constraints are strictly satisfied. Then

$$
\alpha(0) \leq \alpha(\epsilon) \leq \alpha(0) + \langle \bar{\epsilon}, y^* \rangle,
$$

where $y^*$ is an optimal solution to the dual problem to the sketchable SDP and $\bar{\epsilon} \in \mathbb{R}^m$ with $\bar{\epsilon}_i = 3\epsilon \eta \|B_i\|$.

Proof. The first inequality is obvious, since any $X \in A(0)$ is also in $A(\epsilon)$. By Lemma C.1 strong duality holds and there is a $y^* \succeq 0$ which achieves the optimal value. Hence

$$
\alpha(0) = \sum_{j=1}^{m} y_j^* \gamma_j
$$

$$
\geq \sum_{j=1}^{m} y_j^* \gamma_j - \text{Tr} \left( \sum_{i=1}^{m} y_i^* B_i - A \right) X, \quad X \succeq 0
$$

$$
= \text{Tr} (AX) - \sum_{i=1}^{m} y_i^* \left[ \text{Tr} (B_i X) - \gamma_i \right]. \quad (29)
$$
The first line holds by duality. If we take the supremum over \( X \in \mathcal{A}(\epsilon) \), we infer
\[
\alpha(0) \geq \alpha(\epsilon) - \sum_{i=1}^{m} y_i^* \tilde{\epsilon}_i,
\]

since \( y_i^* \geq 0 \).

**Corollary C.3.** Assume that there exists \( X_0 \in \mathcal{A}(0) \) such that \( X_0 > 0 \) and the constraints are strictly satisfied. Then
\[
\alpha(\epsilon) \leq \alpha(0) + \left[ \max_{i \in [m]} \tilde{\epsilon}_i \right] \left( \alpha(0) - \text{Tr}(AX_0) \right) / \left( \min_{k} (\gamma_k - \text{Tr}(B_k X_0)) \right),
\]

where \( \tilde{\epsilon} \in \mathbb{R}^m \) is defined as in Lemma C.2.

**Proof.** By Equation (29), we have that
\[
\alpha(0) \geq \text{Tr}(AX_0) - \sum_{i=1}^{m} y_i^* [\text{Tr}(B_i X_0) - \gamma_i].
\]

Since \( [\text{Tr}(B_i X_0) - \gamma_i] < 0 \), it follows that
\[
\sum_{i=1}^{m} y_i^* \leq (\alpha(0) - \text{Tr}(AX_0)) / \min_{i \in [m]} [\gamma_i - \text{Tr}(B_i X_0)].
\]

With \( \langle \tilde{\epsilon}, y^* \rangle \leq \left[ \max_{i \in [m]} \epsilon_i \right] \sum_{i=1}^{m} y_i^* \) for \( y^* \geq 0 \), the Corollary follows. \( \square \)

**Theorem C.4.** Assume that there exists \( X_0 \in \mathcal{A}(0) \) such that \( X_0 > 0 \) and the constraints are strictly satisfied. Then the value of the sketched SDP \( \alpha_S \) is bounded by
\[
\alpha_S \leq \alpha(0) + \epsilon C_1 (\alpha(0) - \text{Tr}(AX_0)) / C_2.
\]

Here
\[
C_1 = \max \left\{ 3\eta \|B_i\|_1 \mid i \in [m] \right\},
C_2 = \min \left\{ (\gamma_i - \text{Tr}(B_i X_0)) \mid i \in [m] \right\},
\]

where \( \eta = \text{Tr}(X^*) \) for an optimal point \( X^* \) of the sketchable SDP.

**Proof.** The key step is to recognize that \( \text{Tr}(SCS^T X) \) is equal to \( \text{Tr}(CS^T XS) \) by the cyclicity of the trace. Thus, the relaxed SDP gives an upper bound for the sketched SDP. The theorem then follows by Corollary C.3. \( \square \)

Note that this result is not probabilistic and holds regardless of the sketching matrix \( S \) used.
D. Random feasibility problems

In this section, we investigate under which conditions the convex hull of $m$ random GUE [AGZ10, Section 2.2] matrices shifted by a multiple of the identity both contains a positive semidefinite matrix and the cone they define is pointed. This is used in Section 7.2.

Let $G_i \in \mathcal{M}_{d'}$, $i \in [m]$, be random matrices sampled independently from the GUE. This means that $G_i$ is Hermitian and that

$$(G_i)_{kl} = \begin{cases} (G_i)_{kl} = Y_k & \text{for } k = l, \\ (G_i)_{kl} = Z_{kl} & \text{for } k > l, \end{cases}$$

where $Y_k$ is a real normal random variable with mean 0 and variance 1 and $Z_{kl}$ is a complex normal random variable with mean 0 and variance 1. Since we will need similar matrices with different variance, we will call this distribution GUE($0, 1$). Consider the rescaled and shifted matrices $\tilde{G}_i(\alpha) = \frac{1}{\sqrt{d'}} G_i + \alpha \mathbb{I}$ for some $\alpha \in \mathbb{R}$. We call the convex hull of these matrices $X_\alpha := \text{conv} \left( \left\{ \tilde{G}_i(\alpha) \mid i \in [m] \right\} \right)$.

Lemma D.1. Let $\alpha \geq \frac{2}{\sqrt{m}}(1 + \epsilon)$. Then

$$\mathbb{P} \left[ X_\alpha \cap S_{d'}^{++} \neq \emptyset \right] \geq 1 - C e^{-2d'^{3/2}/C},$$

where $C$ is a numerical constant independent of $m$, $d'$, $\epsilon$.

Proof. Let $t \in \mathbb{R}^m$ such that $\sum_{i=1}^m t_i = 1$. By the definition of the GUE,

$$G(t) := \sum_{i=1}^m t_i G_i$$

is again in GUE($0, \sum_{i=1}^m t_i^2$). By [LR10, Theorem 1] and the fact that the GUE is invariant under unitary transformations, it holds that

$$\mathbb{P} \left[ \lambda_{\min}(G(t)/\sqrt{d'}) \leq -2\|t\|_2(1 + \epsilon) \right] \leq C e^{-2d'^{3/2}/C}.$$ 

The expression $-2\|t\|_2(1 + \epsilon)$ is maximized by $t_{\min} = (1/m, \ldots, 1/m)$, for which we obtain the value $-2/\sqrt{m}$ (see [Bha97, Remark II.3.7]). Hence, for $\alpha \geq 2(1 + \epsilon)/\sqrt{m}$, we infer that

$$\mathbb{P} \left[ \lambda_{\min}(G(t_{\min})/\sqrt{d'} + \alpha \mathbb{I}) \leq 0 \right] = \mathbb{P} \left[ \lambda_{\min}(G(t_{\min})/\sqrt{d'}) \leq -\alpha \right] \leq \mathbb{P} \left[ \lambda_{\min}(G(t_{\min})/\sqrt{d'}) \leq -\frac{2}{\sqrt{m}}(1 + \epsilon) \right] \leq C e^{-2d'^{3/2}/C}.$$
As $\lambda_{\min}(G(t_{\min})/\sqrt{d'} + \alpha \mathbb{1}) > 0$ implies that $G(t_{\min})/\sqrt{d'} + \alpha \mathbb{1}$ is positive definite, the assertions follows.

**Lemma D.2.** Let $\alpha \leq \frac{2}{\sqrt{m}}(1 - \epsilon)$ with $\epsilon \in (0, \frac{1}{2})$. Then

$$P \left[ X_{\alpha} \cap S_{d'}^+ = \emptyset \right] \geq 1 - m \left(1 + \frac{8\sqrt{m}}{\epsilon}\right)^m C^4 e^{-\frac{1}{4}(8 + d' \epsilon^{3/2})\epsilon^{3/2}d'},$$

where $C$ is a numerical constant independent of $m$, $d'$, $\epsilon$.

**Proof.** Take an $\epsilon/(4\sqrt{m})$-net $N$ for the $\ell_1$-sphere $S_{1}^{m-1}$ in $\mathbb{R}^m$. This means that for all $t \in S_{1}^{m-1}$, there is an $s \in N$ such that $\|t - s\|_1 \leq \epsilon/(4\sqrt{m})$. It can be shown as in [Ver12, Section 5.2.2] that we can choose $N$ such that $|N| \leq (1 + 8\sqrt{m}/\epsilon)^m$. Now assume that

$$\lambda_{\max}(G_i/\sqrt{d'}) \leq 2(1 + \epsilon) \land \lambda_{\min}(G_i/\sqrt{d'}) \geq -2(1 + \epsilon) \quad \forall i \in [m].$$

This implies that $\|G_i/\sqrt{d'}\|_\infty \leq 2(1 + \epsilon)$. By Weyl’s perturbation theorem [Bha97, Theorem II.2.6], it follows that for $t \in S_{1}^{m-1} \cap \mathbb{R}^m^+$:

$$\|\lambda_{\min}(G(t)/\sqrt{d'}) - \lambda_{\min}(G(s)/\sqrt{d'})\|_\infty \leq \sum_{i=1}^m |t_i - s_i| \|G_i/\sqrt{d'}\|_\infty \leq 2(1 + \epsilon)\|t - s\|_1 \leq \frac{\epsilon}{\sqrt{m}}.$$

Now assume further that $\lambda_{\min}(G(s)/\sqrt{d'} + \alpha \mathbb{1}) \leq -\epsilon/\sqrt{m}$ for all $s \in N$. Then clearly $X_{\alpha} \cap S_{d'}^+ = \emptyset$ by the above. We thus have to estimate the probability with which our assumptions are met. Using a union bound, we obtain

$$P \left[ X_{\alpha} \cap S_{d'}^+ = \emptyset \right] \geq P \left\{ \lambda_{\min}(\alpha \mathbb{1} + G_i/\sqrt{d'}) \leq -\frac{\epsilon}{\sqrt{m}} \right\} \forall s \in N \land \left\{ \|G_i/\sqrt{d'}\|_\infty \leq 2(1 + \epsilon) \right\} \forall i \in [m]$$

$$\geq 1 - \prod_{s \in N} P \left[ \lambda_{\min}(G(s)/\sqrt{d'}) \geq -\frac{2}{\sqrt{m}} (1 - \frac{\epsilon}{2}) \right] \prod_{i \in [m]} P \left[ \lambda_{\min}(G_i/\sqrt{d'}) \leq -2(1 + \epsilon) \right] \prod_{i \in [m]} P \left[ \lambda_{\max}(G_i/\sqrt{d'}) \geq 2(1 + \epsilon) \right].$$
Using again [LR10, Theorem 1] we infer that for all \( i \in [m] \)
\[
\begin{align*}
\mathbb{P} \left[ \lambda_{\min} \left( \frac{G_i}{\sqrt{d'}} \right) \leq -2(1 + \epsilon) \right] &\leq Ce^{-2d'\epsilon^{3/2}/C}, \\
\mathbb{P} \left[ \lambda_{\max} \left( \frac{G_i}{\sqrt{d'}} \right) \geq 2(1 + \epsilon) \right] &\leq Ce^{-2d'\epsilon^{3/2}/C}, \\
\mathbb{P} \left[ \lambda_{\min} \left( \frac{G(s)}{\sqrt{d'}} \right) \geq -\frac{2}{\sqrt{m}} \left( 1 - \frac{\epsilon}{2} \right) \right] &\leq \mathbb{P} \left[ \lambda_{\min} \left( \frac{G(s)}{\sqrt{d'}} \right) \geq -2\|s\|_2 \left( 1 - \frac{\epsilon}{2} \right) \right] \\
&\leq Ce^{-2d'(\epsilon/2)^{3/2}/C}.
\end{align*}
\]

For the last estimate, we have used that \( G(s) \) is again a GUE element with different variance (cf. proof of Lemma D.1). Combining this with the estimates concerning \(|N|\), the assertion follows. \( \square \)

We obtain as a corollary that the cone generated by the \( \tilde{G}(\alpha) \) is pointed with high probability if \( m \ll d' \).

**Corollary D.3.** Let \( \alpha \leq \frac{2}{\sqrt{m}}(1-\epsilon) \) for \( \epsilon \in \left( 0, \frac{1}{2} \right) \) and denote by \( C_\alpha = \text{cone} \{ \tilde{G}_1(\alpha), \ldots, \tilde{G}_m(\alpha) \} \). Then
\[
\mathbb{P} [C_\alpha \cap -C_\alpha = \{ 0 \}] \geq 1 - m \left( 1 + \frac{8\sqrt{m}}{\epsilon} \right) C_4 e^{-\frac{1}{2}\epsilon(8 + d'\epsilon^{3/2})/C^{3/2}d'},
\]
where \( C \) is a numerical constant independent of \( m, d', \epsilon \).

**Proof.** We know from Lemma D.2 that we have the same lower bound for the probability of the event \( \mathcal{A} = \{ X_\alpha \cap S_{d'}^+ = \emptyset \} \). But note that the event \( \mathcal{A} \) implies that the cone is pointed. That is because if the cone was not pointed, there would exist \( \gamma, \mu \in \mathbb{R}_+^{m} \setminus \{0\} \) such that
\[
\sum_{i=1}^{m} \gamma_i \tilde{G}_i(\alpha) = -\sum_{i=1}^{m} \mu_i \tilde{G}_i(\alpha)
\]
and so
\[
\frac{1}{m} \sum_{i=1}^{m} (\gamma_i + \mu_i) \sum_{i=1}^{m} (\mu_i + \gamma_i) G_i(\alpha) = 0,
\]
which implies that \( X_\alpha \cap S_{d'}^+ \neq \emptyset \). \( \square \)

**Theorem D.4.** Consider the LMI
\[
\sum_{i=1}^{m} t_i \tilde{G}_i(\alpha) - I.
\]

Let \( \epsilon \in \left( 0, \frac{1}{7} \right) \). Then for \( \alpha \geq \frac{2}{\sqrt{m}}(1 + \epsilon) \), this LMI is feasible with probability at least
\[
1 - Ce^{-2d'\epsilon^{3/2}/C}.
\]
Moreover, for $\alpha \leq \frac{2}{\sqrt{m}}(1 + \epsilon)$, this LMI is infeasible, the cone generated by the $\tilde{G}(\alpha)$ is pointed and we have $\mathcal{X}_\alpha \cap \mathcal{S}_d^+ = \emptyset$ with probability at least

$$1 - m \left(1 + \frac{8\sqrt{m \epsilon}}{\epsilon}\right)^m C^4 e^{-\frac{1}{32}(8+d'\epsilon^{3/2})\epsilon^{3/2}d'}.$$  \hfill \text{(31)}$$

Proof. For the first assertion, by Lemma D.1 $\mathcal{X}_\alpha$ contains a positive definite element $\sum_{i=1}^m r_i \tilde{G}_i(\alpha)$ with probability lower bounded by the expression in Equation (30). Then $G(\mu r)$ is feasible for $\mu \in \mathbb{R}_+$ large enough. For the second assertion, we note that the LMI being infeasible is equivalent to $\mathcal{X}_\alpha$ not containing a positive definite element. From Lemma D.2 the lower bound in (31) for the probability that the LMI is infeasible. Moreover, the fact that the same lower bound holds for the probability that the cone is pointed follows from Corollary D.3. \hfill \square

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