Abstract. In this paper we investigate jump-diffusion processes in random environments which are given as the weak solutions to SDE’s. We formulate conditions ensuring existence and uniqueness in law of solutions. We investigate Markov property. To prove uniqueness we solve a general martingale problem for càdlàg processes. This result is of independent interest. Application of our results to a generalized exponential Levy model and a semi-Markovian regime switching model are presented in the last section.

Key words: Jump-diffusion, Stochastic differential equations, Markov switching.

AMS Subject Classification: 60H20, 60H30, 60J65, 60G55, 60G50

1. Introduction

In this paper we investigate properties of stochastic differential equation (SDE) which describes a behavior of some system in a random environment in the time interval $[0,T]$, $T < \infty$. We model this behavior by a process $Y$ being jump-diffusion which is a solution to some SDE. As an example of SDE considering in this paper, an SDE driven by a Lévy process $Z$ and some counting processes can be taken. This SDE has a nice interpretation so we describe it in detail. Let $Z$ be an $n$-dimensional Lévy process with the Lévy measure $\nu$ satisfying

$$\int_{\mathbb{R}^n} (\|x\|^2 \wedge 1) \nu(dx) < \infty.$$  

By $\mathcal{K} = \{1, \ldots, K\}$ we denote the set of indices of environments in which our system can stay. The set $\mathcal{K}$ can describe states of hybrid model, states of economy, rating classes in modeling credit risk etc. A stochastic process $C$ with values in $\mathcal{K}$ pointing out a type of environment in which our system lives. We assume that $C$ is a càdlàg process. Every change of $C$ results in change of drift and volatility of an SDE. Moreover, every jump of $C$ from $i$ to $j$ results in a jump of size $\rho_{i,j}$ of a system. Jumps of $C$ are described by SDE driven by counting processes. Therefore, the evolution of $(Y, C)$ can be described as a solution of the following SDE in $\mathbb{R}^d \times \mathcal{K}$:

$$
\begin{cases}
    dY_t = \mu(t, Y_{t-}, C_{t-})dt + \sigma(t, Y_{t-}, C_{t-})dZ_t + \sum_{i,j=1}^{K} \rho_{i,j}(t, Y_{t-}) \mathbb{1}_{(i \neq j)}(C_{t-})dN_t^{i,j}, \\
    dC_t = \sum_{i,j=1}^{K} (j - i) \mathbb{1}_{(i \neq j)}(C_{t-})dN_t^{i,j}, \\
    Y_0 = y, \quad C_0 = c \in \mathcal{K}.
\end{cases}
$$

For fixed $i, j \in \mathcal{K}$, the process $N^{i,j}$ is a counting point process with intensity $\lambda^{i,j}(t, Y_{t-})$, and

$$\Lambda(t, \cdot) : [0, T] \times \mathbb{R}^d \to \mathbb{R}_+$$

is a bounded continuous function in $(t, y)$. It means that, for the fixed $(i, j)$, the process

$$M^{i,j}_t := N^{i,j}_t - \int_{[0,t]} \lambda^{i,j}(u, Y_{u-})du$$

is a martingale. Moreover, we require that $N^{0,i}_0 = 0$ and processes $Z, N^{i,j}, i, j \in \mathcal{K}, i \neq j$, have no common jumps, i.e.,

$$\Delta Z_t \Delta N^{i,j}_t = 0 \quad \mathbb{P} \text{- a.s.},$$

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and for all \((i_1, j_1) \neq (i_2, j_2)\) we have

\[
\Delta N^{i_1,j_1}_t \Delta N^{i_2,j_2}_t = 0 \quad \text{P. a.s.}
\]

The coefficients in SDE (1.2) are measurable functions \(\mu(\cdot, \cdot, \cdot) : [0, T] \times \mathbb{R}^d \times \mathcal{K} \to \mathbb{R}^d\), \(\sigma(\cdot, \cdot, \cdot) : [0, T] \times \mathbb{R}^d \times \mathcal{K} \to \mathbb{R}^{d \times n}\) and \(\rho^{i,j}(\cdot, \cdot) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d\). This SDE is a non-standard one since driving noise depends on the solution itself, similarly as in Jacod and Protter [10] or in Becherer and Schweizer [2]. That is, the noise \((N^{i,j})_{i,j \in \mathcal{K}; j \neq i}\) is not given apriori and is also constructed, so the solution is not a pair \((Y, C)\) but a quadruple \((Y, C, (N^{i,j})_{i,j \in \mathcal{K}; j \neq i}, Z)\).

Therefore it is interesting to find a weak solution to (1.2) and to formulate conditions ensuring uniqueness in law of solutions. In our paper we investigate more general jump-diffusion processes than SDE (1.2). We construct a weak solution to this general SDE (2.1) (Thm. 2.2) and show Markov property of components \((Y, C)\) of this solution (Thm. 2.3). In Section 3, we show finiteness of 2m-moments of \(\sup_{t \in [0, T]} |Y_t|\) for \(Y\) being component of any weak solution to (2.1). Our goal is to prove uniqueness in law of solutions to SDE (2.1). To do this we use the notion of martingale problem. So, in Section 4, we consider and solve a general martingale problem for càdlàg processes. Results of this section are of independent interest. In Section 5, using the notion of martingale problem we solve the problem of uniqueness in law of solutions to (2.1) under some natural condition imposed on coefficients of SDE and intensity. Knowing that the components \((Y, C)\) of solution solve local martingale problem (Prop. 2.3) and that for any solution of local martingale problem holds \(\mathbb{E} \left( \sup_{t \leq T} |Y_t|^{2m} \right) < \infty\) (Thm. 2.2), we prove in Theorem 5.1 that a solution of local martingale problem is a solution of martingale problem. Next, we prove that a martingale problem is well-posed (Thm. 5.3) which implies uniqueness in law of solution to (2.1). Application of our results is presented in the last section on an example of semi-Markovian regime switching model and on an example of generalized exponential Levy model. This model is very useful in finance, see Cont and Voltchkova [1] or Jakubowski and Niewęgowski [11]. The model considered in our paper is related to the regime switching diffusion models with state-dependent switching which were considered in Becherer and Schweizer [2], Yin and Zhu [21, 22], Yin, Mao, Yuan and Cao [27] amongst others. Our model generalize jump-diffusions with state-dependent switching introduced by Xi and Yin [25, 26] and studied further by Yang and Yin [24]. Our main contribution is the presence of functions \(\rho^{i,j}\) in SDE (1.2), which allows to model the fact that the process \(Y\) jumps at the time of switching the regime \(C\). This is very important from the point of view of applications, because it adds extra flexibility to the model. For example, it gives us a possibility of introducing the dependence of intensity of jumps \(C\) at time \(t\) on the trajectory of process \(C\) up to time \(t-\). This is the case of semi-Markov processes, where \(\lambda^i_j\) at time \(t\) depends on time that process \(C\) spends in current state after the last jump. The process \((\text{say } Y^1)\) corresponding to this semi-Markovian dependence can be introduced in our framework by setting

\[
dY^1_t = dt - \sum_{i,j \in \mathcal{K}; j \neq j} Y^1_{t-} \mathbb{1}_i(C_{t-}) dN^{i,j}_t.
\]

Hence, allowing \(\lambda^i_j\) to be a (non-constant) function of \(Y^1\) we obtain a semi-Markov model.

2. Solutions to SDE’s defining jump-diffusions in random environments

2.1. Formulation of problem. Fix \(T < \infty\). We investigate a weak solution to a SDE on a time interval \([r, T]\), \(r < T\), given by

\[
\begin{align*}
\frac{dY_t}{dt} &= \mu(t, Y_t, C_t) dt + \sigma(t, Y_t, C_t) dW_t + \int_{|x| \leq a} F(t, Y_t, C_t, x) \Pi(dx, dt) \quad \text{a.s.}\ \\
&\quad + \int_{|x| > a} F(t, Y_t, C_t, x) \Pi(dx, dt) + \sum_{j=1}^{K} \rho^{j,j}(t, Y_t) \mathbb{1}_{(j \neq i)}(C_{t-}) dN^{i,j}_t, \\
\frac{dC_t}{dt} &= \sum_{j=1}^{K} \mathbb{1}_{j \neq i}(j - i) \mathbb{1}_i(C_{t-}) dN^{i,j}_t, \\
Y_r &= y, \quad C_r = c \in \mathcal{K},
\end{align*}
\]

where \(W\) is a standard \(p\)-dimensional Wiener process, \(a > 0\) is fixed, \(\Pi(dx, dt)\) is a Poisson random measure on \(\mathbb{R}^n \times [r, T]\) with the intensity measure \(\nu(dx) dt\), \(\nu\) is a Levy measure, and \(N^{i,j}\) are counting
point processes with intensities determined by $\lambda^{i,j}$, bounded nonnegative continuous functions in $(t, y)$, such that processes defined by (1.2) are martingales. By $\hat{\Pi}$ we denote the compensated measure of $\Pi$, i.e.,
$$
\hat{\Pi}(dx, dt) := \Pi(dx, dt) - \nu(dx)dt.
$$

The coefficients in SDE (2.1) are measurable, locally bounded, deterministic functions $\mu(\cdot, \cdot, \cdot): [0, T] \times \mathbb{R}^d \times \mathbb{K} \to \mathbb{R}^d$, $\sigma(\cdot, \cdot, \cdot): [0, T] \times \mathbb{R}^d \times \mathbb{K} \to \mathbb{R}^{d \times p}$, $\rho^{i,j}(\cdot, \cdot): [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, and $F(\cdot, \cdot, \cdot): [0, T] \times \mathbb{R}^d \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^d$ is such that the mapping

$$
(t, y, c) \mapsto \int_{\|x\| \leq a} |F(t, y, c, x)|^2 \nu(dx)
$$

is locally bounded. Moreover, we require that the Poisson random measure $\Pi$ and the processes $N^{i,j}$, $i, j \in \mathbb{K}$, $i \neq j$, have no common jumps, i.e., for every $t > r > 0$ and every $b > 0$,

$$
\int_r^t \int_{\|x\| > b} \Delta N^{i,j}_t \Pi(dx, du) = 0 \quad \mathcal{P} - a.s.
$$

and for all $(i_1, j_1) \neq (i_2, j_2)$,

$$
\Delta N^{i_1,j_1}_t \Delta N^{i_2,j_2}_t = 0 \quad \mathcal{P} - a.s.
$$

SDE (2.1) is a generalization of SDE (1.2), but has no such simple interpretation as (1.2). However (2.4) allows to describe the more complex systems. If

$$
F(t, y, c, x) = \sigma(t, y, c)x,
$$

then SDE (2.4) takes the form (1.2). The following processes play an important role in our considerations:

$$
H^i_t := \mathbb{1}_{\{C_t = i\}},
$$

$$
H^{i,j}_t := \sum_{r < u \leq t} \mathbb{1}_{\{C_u = i\}} \mathbb{1}_{\{C_u = j\}} = \sum_{r < u \leq t} H^i_u H^j_u
$$

for $i, j \in \mathbb{K}$, $i \neq j$. The random variable $H^i_t$ indicates a state in which $C$ is at the moment $t$, and $H^{i,j}_t$ counts a number of jumps of $C$ from $i$ to $j$ up to time $t$.

**Remark 2.1.** To distinct solutions started from different $y, c$ at different times $r \in [0, T]$ it is convenient to denote by $(Y^{r,y,c}_t, C^{r,y,c}_t)_{t \in [r, T]}$ a solution to SDE (2.1) started from $(y, c)$ at time $r$. Sometimes, for a notational convenience we drop $r, y, c$ in this notation if there is no confusion. By $\mathbb{P}_{r,y,c}$ we denote the law of $(Y^{r,y,c}_t, C^{r,y,c}_t)_{t \in [r, T]}$.

### 2.2. Existence of weak solutions to SDE’s in random environments

We prove the existence of a weak solution to SDE (2.1) using an argument of a suitable change of measure (cf. [2] or Kusuoka [14]). For a matrix $A \in \mathbb{R}^{d \times d}$, by $\|A\|$ we denote the matrix norm given by

$$
\|A\|^2 := \sum_{i=1}^d \sum_{j=1}^d |a_{i,j}|^2,
$$

and for simplicity of notation we use $N$ for $(N^{i,j})_{i,j \in \mathbb{K}, i \neq j}$.

**Theorem 2.2.** Assume that coefficients $\mu, \sigma, F$ satisfy conditions:

a) the linear growth condition: there exists a constant $K_1 > 0$ such that

$$
|\mu(t, y, c)|^2 + \|\sigma(t, y, c)\sigma(t, y, c)^\top\| + \int_{\|x\| \leq a} |F(t, y, c, x)|^2 \nu(dx) \leq K_1 (1 + |y|^2),
$$

b) the Lipschitz condition: there exists a constant $K_2 > 0$ such that

$$
|\mu(t, y_1, c) - \mu(t, y_2, c)|^2 + \|\sigma(t, y_1, c) - \sigma(t, y_2, c)\|^2 + \int_{\|x\| \leq a} |F(t, y_1, c, x) - F(t, y_2, c, x)|^2 \nu(dx) \leq K_2 |y_1 - y_2|^2,
$$

\text{JUMP-DIFFUSIONS 3}
c) for every \( c \in \mathcal{K} \) and every \( k \in \mathcal{K} \setminus \{ c \} \)

(Cont) \((t, y) \to F(t, y, c, x)\) for \( \|x\| > a \), and \((t, y) \to \rho^{c, k}(t, y)\) are continuous.

Then there exists a weak solution \((Y, C, W, \Pi, N)\) to SDE (2.1) on \([r, T]\), which is adapted, càdlàg, and moreover (2.3) and (2.4) are satisfied.

Proof. In the first step we consider a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) on which there exist independent processes: a standard Brownian motion \(W\), a Poisson random measure \(\Pi(dx, dt)\) with intensity measure \(\nu(dx)dt\) and the Poisson processes \((N^{i,j})_{i,j \in \mathcal{K}, \ i \neq j}\) with intensities equal to one. Let us consider, on this stochastic basis, a SDE for \([r, T]\):

\[
dY_t = \mu(t, Y_{t-}, C_{t-})dt + \sigma(t, Y_{t-}, C_{t-})dW_t + \int_{\|x\| \leq a} F(t, Y_{t-}, C_{t-}, x)\Pi(dx, dt)
\]

\[
+ \int_{\|x\| > a} F(t, Y_{t-}, C_{t-}, x)\Pi(dx, dt) + \sum_{i,j \in \mathcal{K}} \rho^{i,j}(t, Y_{t-})\mathbb{1}_{\{i\}}(C_{t-})dN^{i,j}_t,
\]

with \(Y_r = y, C_r = c \in \mathcal{K}\). For this SDE results on existence and uniqueness of solutions exist (see, e.g., Applebaum [1, Thm. 6.2.9]). So, under our assumptions, there exists the unique strong solution \((Y, C)\) to SDE (2.7), which is adapted and càdlàg. Let us define a new measure \(\mathbb{P}^\lambda\) by

\[
\frac{d\mathbb{P}^\lambda}{d\mathbb{P}} = Z_T := \mathcal{E}_T \left( \sum_{i,j \in \mathcal{K}} \int_{\|x\| \leq a} (\lambda^{i,j}(t, Y_{t-}) - 1)(dN^{i,j}_t - dt) \right),
\]

where \(\mathcal{E}\) denotes the Doléans-Dade exponential. The assumption that \(\lambda^{i,j}\) are non-negative bounded measurable functions for \(i, j \in \mathcal{K}\), \(i \neq j\) implies that \(\mathbb{P}^\lambda\) is a probability measure (see Brémaud [3, Thm. VI.2.T4]). Moreover, for each \(i, j\) the standard Poisson process \((N^{i,j})_{t \in [r, T]}\) under \(\mathbb{P}\) is a counting process under \(\mathbb{P}^\lambda\) with the \(\mathbb{P}^\lambda\)-compensator \((\int_{[r, t]} \lambda^{i,j}(u, Y_u^-)du)_{t \in [r, T]}\) (see [3, Thm. VI.2.T2]). Hence, we see that if \((Y, C)\) is a solution to (2.7), then the process \((Y, C, W, \Pi, N)\) solves (2.1) under \(\mathbb{P}^\lambda\). It remains to prove that processes have no common jumps, i.e., (2.3) and (2.4).

Let, for a fixed \(b > 0\),

\[
A^{i,j}_b := \left\{ \omega : \exists t \in [r, T] \text{ such that } \int_{r}^{t} \int_{\|x\| > b} \Delta N^{i,j}_u(\omega)\Pi(du, dx) \neq 0 \right\}.
\]

Independence of \(N^{i,j}\) and \(\Pi\) under \(\mathbb{P}\) yields that \(\mathbb{P}(A^{i,j}_b) = 0\) for each \(b > 0\). Absolute continuity of \(\mathbb{P}^\lambda\) with respect to \(\mathbb{P}\) implies that \(\mathbb{P}^\lambda(A^{i,j}_b) = 0\), so \(N^{i,j}\) and \(\Pi\) have no common jumps. In an analogous way we see that the processes \(N^{i,j}\) and \(N^{k,l}\) have no common jumps for \((i, j) \neq (k, l)\).

### 2.3. Markov property of solutions

At first, we are interested in Markov property of solutions to SDE (2.1). Note that in the formulation of SDE coefficients depend only on values of \((Y, C)\) at time \(t^-\), and since a driving noise can be considered as Poissonian type we can expect that a solution of this SDE possesses a Markov property. We start from the study of solution constructed in Theorem 2.2.

**Theorem 2.3.** The components \((Y, C)\) of the solution to the SDE (2.1) constructed in Theorem 2.2 have Markov property.

Proof. Let \((Y, C, W, (N^{i,j})_{i,j \in \mathcal{K}, \ i \neq j})\) be the solution to SDE (2.1) on \([r, T]\) constructed in Theorem 2.2. Fix \(t \geq r\), and let \(X\) be an arbitrary bounded random variable...
measurable with respect to $\sigma((Y_u, C_u) : T \geq u \geq t)$. To prove Markovianity it is sufficient to check that there exists a measurable function $f$ such that

$$(2.9) \quad E_{P^{\lambda,r}}(X | \mathcal{F}_t) = f(t, Y_t, C_t).$$

The density process defined by the Doleans-Dade exponential \cite{22} can be written explicitly (see Protter \cite{17} Thm. II.8.37) as

$$Z_t = \prod_{i,j \in K, i \neq j} \left(e^{\int_0^t (\lambda^{i,j}(u, Y_{u-}) - 1) du} \prod_{r < u \leq t} \left(1 + (\lambda^{i,j}(u, Y_{u-}) - 1) \Delta N^{i,j}_u \right) \right).$$

For convenience we introduce the following notation

$$L_{s,t} := \prod_{i,j \in K, i \neq j} \left(e^{\int_0^t (\lambda^{i,j}(u, Y_{u-}) - 1) du} \prod_{s < u \leq t} \left(1 + (\lambda^{i,j}(u, Y_{u-}) - 1) \Delta N^{i,j}_u \right) \right),$$

which allows to write the density in the shorter way

$$Z_T = L_{r,T} = L_{r,t} L_{t,T} = Z_t L_{t,T}.$$

Using similar arguments as in the proof of abstract Bayes formula we get

$$(2.10) \quad \mathbb{I}_{\{Z_t > 0\}} E_{P^{\lambda,r}}(X | \mathcal{F}_t) = \mathbb{I}_{\{Z_t > 0\}} E_{P}(XL_{t,T} | \mathcal{F}_t) = f(t, Y_t, C_t) \mathbb{I}_{\{Z_t > 0\}},$$

where the last equality follows from the fact that the random variable $XL_{t,T}$ is $P$ integrable and measurable with respect to $\sigma((W_u - W_t, \Pi(A, [t, u]), (N^{i,j}_u - N^{i,j}_t)_{i,j \in K, i \neq j}) : A \in \mathcal{B}(\mathbb{R}^n) : T \geq u \geq t)$ (see \cite{17} Thm. V.6.32). Equality (2.10) implies (2.9) since $\mathbb{P}^{\lambda,r}(Z_t = 0) = 0$. \hfill \square

Theorem 2.3 suggests that for any arbitrary solution to SDE (2.1) the process $(Y, C)$ is a time inhomogenous Markov process. We prove further, that this suggestion is true.

In the sequel we will use frequently the following technical result giving the canonical decomposition of a special semimartingale $(v(t, Y_t, C_t))_{t \in [r, T]}$, which plays a crucial role in what follows. This result is a consequence of Itô’s lemma for general semimartingales (see \cite{17} Thm. II.7.32). Let us denote by $C^{1,2} = C^{1,2}([0, T] \times \mathbb{R}^d \times K)$ - the space of all measurable functions $v : [0, T] \times \mathbb{R}^d \times K \to \mathbb{R}$ such that $v(\cdot, \cdot, k) \in C^{1,2}([0, T] \times \mathbb{R}^d)$ for every $k \in K$, and let $C^{1,2}_c$ be a set of functions $f \in C^{1,2}$ with compact support.

**Theorem 2.4.** Let $(Y, C, W, \Pi, N)$ be a solution to SDE (2.1) on $[r, T]$ and $v \in C^{1,2}$ be a function such that the mapping

$$(2.11) \quad (t, y, c) \mapsto \int_{\mathbb{R}^n} \left| v(t, y + F(t, y, x, c), c) - v(t, y, c) - \nabla v(t, y, c) F(t, y, c, x) \mathbb{I}_{\{\|x\| \leq a\}} \right| \nu(dx)$$

is locally bounded (i.e., bounded on compact sets), where $\nabla v$ denotes the vector of partial derivatives of $v$ with respect to components of $s$. Then the process $(v(t, Y_t, C_t))_{t \geq r}$ is a special semimartingale with the following (unique) canonical decomposition

$${dv(t, Y_t, C_t) = (\partial_t + A_t) v(t, Y_{t-}, C_{t-}) dt + \sum_i H^i_{t-} \nabla v(t, Y_{t-}, i) \sigma(t, Y_{t-}, i) dW_t}$$

$$+ \sum_i H^i_{t-} \int_{\mathbb{R}^n} (v(t, Y_{t-} + F(t, Y_{t-}, i, x), i) - v(t, Y_{t-}, i)) \Pi(dx, dt)$$

$$+ \sum_{i,j : i \neq j} (v(t, Y_{t-} + \rho^{i,j}(t, Y_{t-}), j) - v(t, Y_{t-}, i)) H^i_{t-} dM^{i,j}_{t-},$$

where $H^i_t = \mathbb{1}_{\{i\}} (C_t)$ and $A_t$ is defined by

$$(2.13) \quad \mathbb{A} v(y, c) := \nabla v(t, y, c) \mu(t, y, c) + \frac{1}{2} Tr \left( a(t, y, c) \nabla^2 v(t, y, c) \right)$$

$$+ \int_{\mathbb{R}^n} \left( v(t, y + F(t, y, c, x), c) - v(t, y, c) - \nabla v(t, y, c) F(t, y, c, x) \mathbb{I}_{\{\|x\| \leq a\}} \right) \nu(dx)$$
Here $a(t, y, c) := \sigma(t, y, c)(\sigma(t, y, c))^\top$, and by $\text{Tr}$ we denote the trace operator, and $\nabla^2 v$ is the matrix of second derivatives of $v$ with respect to the components of $s$.

The proof of this technical result is given in the appendix. From the above theorem we obtain a nice martingale property:

**Proposition 2.5.** Let $(Y, C)$ be components of a solution to SDE (2.11) on $[r, T]$. Then for any $f \in C_c^2$ the process

$$M^f_t := f(Y_t, C_t) - \int_r^t A_u f(Y_u-, C_u-) du$$

is an $\mathbb{F}$-local martingale on $[r, T]$.

**Proof.** Local boundedness of function (2.2) implies (2.11) for $f \in C_c^2$, so (2.12) holds. Therefore the process $M^f$ is a local martingale.

Since $M^f$ is $\mathbb{F}^{Y,C}$-adapted, we obtain immediately

**Corollary 2.6.** Under assumptions of Proposition 2.5 the process $M^f$ is an $\mathbb{F}^{Y,C}$-local martingale for any $f \in C_c^2$.

### 3. Moment estimates

In this section we prove finiteness of $2m$-moments of $\sup_{t \in [0,T]} |Y_t|$, which is the crucial fact in the proof of uniqueness in law of solutions to SDE (2.11). We stress that in this section we do not assume that $(Y, C)$ solves SDE but only that $(Y, C)$ solves a local martingale problem corresponding to the generator of (2.11). Throughout this section we make the following additional mild assumption:

**(LB)** The mapping

$$(t, y, c) \to \int_{|F(t, y, c, x)| > 1} |F(t, y, c, x)| \nu(dx)$$

is locally bounded.

In what follows we use the notation $z^\top = (z_1^\top, z_2)$, where $z_1 \in \mathbb{R}^d$ corresponds to coordinates of $Y$ and $z_2 \in \mathbb{R}$ corresponds to $C$. Let us introduce the following functions

$$(t, y, c) \to \tilde{b}(t, y, c) := \mu(t, y, c) + \int_{\mathbb{R}^d} F(t, y, c, x) \left( \mathbb{1}_{\{\|F(t, y, c, x)\| \leq 1\}} - \mathbb{1}_{\{\|x\| \leq a\}} \right) \nu(dx),$$

and measure $\mathcal{P}(t, y, c, \cdot)$ on $\mathbb{R}^d \times \mathbb{R}$ defined by

$$\mathcal{P}(t, y, c, dz_1, dz_2) := \nu_F(t, y, c, dz_1) \otimes \delta_{\{0\}}(dz_2) + \mathbb{1}_K(c) \sum_{k \in K \setminus \{c\}} \lambda_c^{s,k}(t, y) \delta_{\{\rho_{s,k}(t, y, c) > 0\}}(dz_1, dz_2),$$

where $\nu_F(t, y, c, \cdot)$ is a measure defined for $A \in \mathcal{B}(\mathbb{R}^d)$ by setting

$$\nu_F(t, y, c, A) := (\nu \circ F_{t,y,c}^{-1})(A) = \nu(\{x : F(t, y, c, x) \in A\}),$$

$\delta_a$ denotes Dirac measure at $a$.

**Theorem 3.1.** Assume that the law of a process $(Y, C)$ solves the local martingale problem for $(A_t)_{t \in [r, T]}$ given by (2.13). Then

i) The process $(Y, C)$ is a semimartingale with the characteristics $(\tilde{B}, \tilde{C}, \mathcal{P})$, with respect to the truncation function $h(z) := z \mathbb{1}_{\{\|z_1\| + |z_2| \leq 1\}}$, of the form

$$\tilde{B}_t = \int_0^t \left[ \tilde{b}(u, Y_u-, C_u-) \right] du, \quad \tilde{C}_t = \int_0^t \left[ a(u, Y_u-, C_u-) \right] du,$$
(3.4) \( \tilde{\nu}([0, t] \times A_1 \times A_2) := \int_{[0, t]} \int_{A_1 \times A_2} \tilde{\nu}(u, Y_u, C_u, dz_1, dz_2) du. \)

ii) The process \((Y, C)\) has the following decomposition
\[
\begin{bmatrix} Y_t \\ C_t \end{bmatrix} = \begin{bmatrix} Y_0 \\ C_0 \end{bmatrix} + \tilde{B}_t + \begin{bmatrix} Y_c \\ 0 \end{bmatrix} + \int_{[0, t]} \int_{\mathbb{R}^{d+1}} h(z) \tilde{\mu}(du, dz) + \int_{[0, t]} \int_{\mathbb{R}^{d+1}} (z - h(z)) \tilde{\nu}(du, dz),
\]
where \(Y^c\) is the martingale continuous part of \(Y\), \(\tilde{\nu}(dt, dz)\) is the measure associated with jumps of \((Y, C)\), and \(\tilde{\mu}(dt, dz) := \tilde{\nu}(dt, dz) - \tilde{\nu}(dz) dt\) is the compensated measure of jumps.

Proof. By our assumption, for \(v \in C^2_c\), the function
\[
I_v(t, y, c) := \int_{\mathbb{R}^n} \left( v(y + F(t, y, c, x), c) - v(y, c) - \nabla v(y, c) F(t, y, c, x) \mathbb{1}_{\{\|F(t, y, c, x)\| \leq 1\}} \right) \nu(dx)
\]
is bounded on compact sets. We can rewrite the generator \(A_t\) in the form
\[
A_t v(y, c) = \nabla v(y, c) \tilde{b}(t, y, c) + \frac{1}{2} \text{Tr} (a(t, y, c) \nabla^2 v(y, c))
\]
\[
+ \int_{\mathbb{R}^{d+1}} (v(y + z_1, c + z_2) - v(y, c) - \nabla_y v(y, c) h(z)) \tilde{\nu}(t, y, c, dz)
\]
where \(\tilde{b}\) is given by (3.2). Note that \(\tilde{b}\) is also bounded on compact sets. Note that any function \(v \in C^2_c(\mathbb{R}^d \times K)\) can be extended to a function \(v \in C^2(\mathbb{R}^{d+1})\). Therefore, we can extend \(A_t\) to an operator \(\tilde{A}_t\) acting on functions \(v \in C^2(\mathbb{R}^{d+1})\) by formula
\[
\tilde{A}_t v(y, c) = \nabla v(y, c) \tilde{b}(t, y, c) + \frac{1}{2} \text{Tr} (a(t, y, c) \nabla^2 v(y, c))
\]
\[
+ \int_{\mathbb{R}^{d+1}} (v(y + z_1, c + z_2) - v(y, c) - \nabla_y v(y, c) h(z)) \tilde{\nu}(t, y, c, dz)
\]
where \(\tilde{\nu}\) is given by (3.3) and for \(z^* = (z_1^*, z_2)\), \(z_1^* \in \mathbb{R}^d\), \(z_2 \in \mathbb{R}\),
\[
h(z) := z_1^1 \mathbb{1}_{\{|z_1| + |z_2| \leq 1\}}, \quad \text{and} \quad \nabla_y v(y, c) := \left[ \nabla v(y, c), \partial_c v(y, c) \right].
\]
Indeed, this formula gives the extension since by simple calculation we can see that for every \(c \in K\),
\[
\tilde{A}_t \tilde{v}(y, c) = A_t v(y, c),
\]
for any extension \(\tilde{v}\) of \(v\). Hence, by our assumption, we have that the process
\[
M^v_t := v(Y_t, C_t) - v(Y_0, C_0) - \int_0^t \tilde{A}_u v(Y_{u-}, C_{u-}) du
\]
is a local martingale for each \(v \in C^2(\mathbb{R}^{d+1})\). Applying Theorem 1 from Griengolichis and Mikulevicius [5] (cf. Jacod and Shirayev [12] Theorem II.2.42) we see that \((Y, C)\) is a semimartingale with the characteristics, calculated with respect to the truncation function \(h\), of the form
\[
\tilde{B}_t = \int_0^t \left[ \tilde{b}(u, Y_{u-}, C_{u-}) \right] du, \quad \tilde{C}_t = \int_0^t \left[ \begin{array}{c} a(u, Y_{u-}, C_{u-}) \\ 0 \\ 0 \end{array} \right] du,
\]
\[
\tilde{\nu}([0, t] \times A_1 \times A_2) := \int_{[0, t]} \int_{A_1 \times A_2} \tilde{\nu}(u, Y_u, C_u, dz_1, dz_2) du.
\]
Therefore, by Theorem II.2.34 [12], the process \((Y, C)\) has the canonical representation of the form
\[
\begin{bmatrix} Y_t \\ C_t \end{bmatrix} = \begin{bmatrix} Y_0 \\ C_0 \end{bmatrix} + \tilde{B}_t + \begin{bmatrix} Y_c \\ 0 \end{bmatrix} + \int_{[0, t]} \int_{\mathbb{R}^{d+1}} h(z) \tilde{\mu}(du, dz) + \int_{[0, t]} \int_{\mathbb{R}^{d+1}} (z - h(z)) \tilde{\nu}(du, dz)
\]
where \(Y^c\) is the martingale continuous part of \(Y\), \(\tilde{\nu}(dt, dz)\) is the measure of jumps of \((Y, C)\), and
\[
\tilde{\mu}(dt, dz) := \tilde{\nu}(dt, dz) - \tilde{\nu}(dz) dt
\]
Now we prove (3.10). Using (3.8) and applying the Itô lemma to the function

\[ b \]

Note that the function \( b \) satisfies (LB), (LG), and that there exists constants \( K_3 \) and \( K_4 \) such that for all \((t, y, c)\)

\[
\int_{||x|| > a} |F(t, y, c, x)|^2 \nu(dx) + |\rho \cdot k(t, y)|^2 \leq K_3(1 + |y|^2), \quad \text{for all } k \in \mathcal{K}, k \neq c,
\]

and

\[
\int_{\mathbb{R}^n} |F(t, y, c, x)|^2 \nu(dx) \leq K_4(1 + |y|^{2m}),
\]

then

\[
\mathbb{E} \left( \sup_{t \in [0, T]} |Y_t|^{2m} \right) < \infty.
\]

**Proof.** We present a proof for \( r = 0 \), the proof of general case is analogous. In the first step we recall that, by Theorem 3.1, any solution \((Y, C)\) to the martingale problem for \((\mathcal{A}_t)_{t \in [0, T]}\) is a semimartingale. Since \( h = (h_1, h_2) \), from (3.5) follows that

\[
\int_{[0, t]} \int_{\mathbb{R}^{d+1}} (z_1 - h_1(z)) d\mathcal{D}(u, y, c, dz) du
\]
is well defined. Therefore the semimartingale \( Y \) is special and has the following canonical decomposition

\[
Y_t = Y_0 + B_t + Y^c_t + \int_{[0, t]} \int_{\mathbb{R}^{d+1}} z_1 \mu(du, dz),
\]

where

\[
B_t := \int_0^t b(u, Y_{u-}, C_{u-}) du, \quad b(u, y, c) := \mu(\mu, y, c) + \int_{||x|| > a} F(u, y, c, x) \nu(dx) + \sum_{k \in \mathcal{K} \setminus \{c\}} \rho \cdot k(t, y) \lambda \cdot k(t, y).
\]

Note that the function \( b \) has also linear growth in \( s \), since \( \mu \) has linear growth in \( s \), by assumption. Thus, by the Cauchy-Schwarz inequality and (3.5) we have

\[
\int_{||x|| > a} F(u, y, c, x) \nu(dx) \leq \nu(||x|| \geq a) \int_{||x|| > a} |F(u, y, c, x)|^2 \nu(dx)
\]

\[
\leq \nu(||x|| \geq a) K(1 + |y|^2).
\]

Now, let

\[
\tau_n := \inf \{ t : |Y_t| > n \}, \quad Y^{*, 2m}_t := \sup_{0 \leq u \leq t} |Y_u|^{2m}.
\]

It is enough to show that there exists a constant \( K \) such that

\[
\mathbb{E} Y^{*, 2m}_{t \wedge \tau_n} \leq K \int_0^t \left( 1 + \mathbb{E} Y^{*, 2m}_{s \wedge \tau_n} \right) ds
\]

for each \( n \). Indeed, (3.10) and the Grownwall lemma imply

\[
\mathbb{E} Y^{*, 2m}_{t \wedge \tau_n} \leq K''(t),
\]

where \( K''(t) \) does not depend on \( n \), so using Fatou lemma we obtain (3.7), i.e.,

\[
\mathbb{E} Y^{*, 2m}_{T \wedge \tau_n} \leq K''(T) < \infty.
\]

Now we prove (3.10). Using (3.5) and applying the Itô lemma to the function \(|y|^{2m}\) and to the process \( Y \) we obtain

\[
|Y_t|^{2m} = |Y_0|^{2m} + A_t^1 + A_t^2 + M_t^c + M_t^d + D_t^1,
\]
where

\[ A^1_t := 2m \int_0^t |Y_u - |2m-2 Y_u^\top b(u, Y_u, C_u) du, \]

\[ A^2_t := \frac{1}{2} \int_0^t (2m|Y_u - |2m-2 Tr(a(u, Y_u, C_u))) + 2m(2m - 2)|Y_u - |2m-4 Tr(Y_u - Y_u^\top a(u, Y_u, C_u))) du, \]

\[ M^c_t := 2m \int_0^t |Y_u - |2m-2 Y_u^\top c du, \]

\[ M^d_t := 2m \int_0^t \int_{\mathbb{R}^d+1} |Y_u - |2m-2 Y_u^\top z_1 \hat{\mu}(du, dz), \]

\[ D^1_t := \int_0^t \int_{\mathbb{R}^n} (|Y_u - z_1|^2m - |Y_u - |2m-2m|Y_u - |2m-2 Y_u^\top z_1) \mu(du, dz), \]

and \( a(t, y, c) = \sigma(t, y, c)\sigma^\top(t, y, c). \) We treat each of the components of \(|Y_t \wedge \tau_n|^{2m}\) separately. We start with \( A^1. \) The following inequalities follow from the successive use of the Cauchy-Schwartz inequality, Young inequality, \([4,9]\), and (LG) condition

\[ |y|^{2m-2} |y^\top b(u, y, c)| = |y|^m |y|^{m-2} |y^\top b(u, y, c)| \leq \frac{1}{2} \left( |y|^{2m} + |y|^{2m-4} |y^\top b(u, y, c)|^2 \right) \]

\[ \leq \frac{1}{2} \left( |y|^{2m} + K_1 |y|^{2m-2} (1 + |y|^2) \right) \leq L_1 (1 + |y|^{2m}), \]

and yields that

\[ \mathbb{E} \sup_{s \leq t \wedge \tau_n} |A^1_s| \leq L_1 \mathbb{E} \int_0^{t \wedge \tau_n} (1 + |Y_u - |2m) du \leq L_1 \int_0^t \left( 1 + \mathbb{E} Y_u^{2m} \right) du. \]

To estimate \( A^2 \) we note that matrices \( yy^\top \) and \( a(u, y, c) \) are positive semi-definite, and hence the condition (LG) implies

\[ Tr \left( yy^\top a(u, y, c) \right) \leq Tr \left( yy^\top \right) Tr(a(u, y, c)) \leq K |y|^2 (1 + |y|^2). \]

Therefore

\[ 2m|y|^{2m-2} Tr(a(u, y, c)) + 2m(2m - 2)|y|^{2m-4} Tr(yy^\top a(u, y, c)) \leq 2m|y|^{2m-2} K(1 + |y|^2) + 2m(2m - 2)|y|^{2m-4} K|y|^2 (1 + |y|^2) \]

\[ \leq 2m(2m - 1)K |y|^{2m-2} (1 + |y|^2) \leq 2m(2m - 1)KL(1 + |y|^{2m}) \]

for some positive \( L. \) This gives

\[ \mathbb{E} \sup_{s \leq t \wedge \tau_n} |A^2_s| \leq L_2 \mathbb{E} \int_0^{t \wedge \tau_n} (1 + |Y_u - |2m) du \leq L_2 \int_0^t \left( 1 + \mathbb{E} Y_u^{2m} \right) du. \]

To estimate \( M^c \) we use the Burkholder-Davies-Gundy inequality (see, e.g., \([17, \text{Thm. IV.4.48}]\)), the Young inequality with \( \varepsilon (|ab| < \varepsilon a^2 + \frac{1}{4\varepsilon} b^2) \) and obtain

\[ \mathbb{E} \sup_{t \leq t \wedge \tau_n} \left| \int_0^v |Y_v - |2m-2 Y_v^\top dY_v^c \right| \leq C \mathbb{E} \left| \int_0^{t \wedge \tau_n} |Y_v - |4m-4 Y_v^\top a(v, Y_v, C_v) Y_v - dv \right| \]

\[ \leq C \mathbb{E} \left| \int_{t \wedge \tau_n} |Y_v^{2m}| \left| \int_0^v |Y_v - |2m-2 Tr(a(v, Y_v, C_v)) \right| dv \right| \]

\[ \leq C \mathbb{E} |Y_v^{2m}| + \frac{C}{4\varepsilon} \mathbb{E} \left| \int_0^{t \wedge \tau_n} |Y_v - |2m-2 Tr(a(v, Y_v, C_v)) \right| dv \]

\[ \leq C \mathbb{E} |Y_v^{2m}| + \frac{C}{4\varepsilon} \mathbb{E} \int_0^{t \wedge \tau_n} (1 + |Y_v - |2m) dv \leq C \mathbb{E} Y_v^{2m} + \frac{C}{4\varepsilon} \int_0^t \left( 1 + \mathbb{E} Y_v^{2m} \right) dv. \]
Hence, taking $\varepsilon = \frac{1}{8c_m}$, we have
\[ E \sup_{v \leq t \wedge \tau_n} |M^e| \leq \frac{1}{4} EY^{*,2m}_{t \wedge \tau_n} + L_3 \int_0^t (1 + EY^{*,2m}_{u \wedge \tau_n}) du. \]

To estimate $M^d$ we use again the Burkholder-Davies-Gundy and Young inequalities, which give
\[ E \sup_{v \leq t \wedge \tau_n} \left| \int_0^v \int_{R_{d+1}} |Y_u-|^{2m-2} Y_u^\top z_1 \mu(du,dz) \right| \leq CE \sqrt{\int_0^t \int_{R_{d+1}} |Y_u-|^{4m-2} |z_1|^2 \mu(du,dz)} \]
\[ \leq CE \sup_{v \leq t \wedge \tau_n} \int_0^v \int_{R_{d+1}} |Y_u-|^{2m-2} |z_1|^2 \mu(du,dz) \]
\[ \leq CE EY^{*,2m}_{t \wedge \tau_n} \sup_{v \leq t \wedge \tau_n} \int_0^v \int_{R_{d+1}} |Y_u-|^{2m-2} \left( |F(u,Y_u,-,C_u,-,x)|^2 \nu(dx) + \sum_{k \in \mathcal{K} \setminus C_u} |\rho_{C_u-k}(u,Y_u-)|^2 \chi_{C_u-k}(u,Y_u-) \right) du \]
\[ \leq CE EY^{*,2m}_{t \wedge \tau_n} + C \frac{K'}{4\varepsilon} \nu \int_0^v \left( 1 + |Y_u-|^{2m} \right) du \leq CE EY^{*,2m}_{t \wedge \tau_n} + C \frac{K'}{4\varepsilon} \nu \int_0^t \left( 1 + Y_u^{*,2m}_{u \wedge \tau_n} \right) du, \]
where the fourth inequality follows from (LG) and (3.3). Therefore, taking again $\varepsilon = \frac{1}{8c_m}$, we obtain
\[ E \sup_{v \leq t \wedge \tau_n} |M^d| \leq \frac{1}{4} EY^{*,2m}_{t \wedge \tau_n} + L_4 \int_0^t (1 + EY^{*,2m}_{u \wedge \tau_n}) du. \]

To estimate $D^1$ we use Taylor expansion for $f(y) = y^{2m}$ and we get
\[ |y + z_1|^{2m} - |y|^{2m} - 2m|y|^{2m-2}y^\top z_1 | \leq m(2m-1)|y + z_1|^{2m-2}|z_1|^2 \leq K^{iv} \left( |y|^{2m-2}|z_1|^2 + |z_1|^{2m} \right), \]
where $|\theta| \leq 1$. Let
\[ U(y,z_1) := |y|^{2m-2}|z_1|^2 + |z_1|^{2m}. \]

Then
\[ U(y,z_1) := |y|^{2m-2}|z_1|^2 + |z_1|^{2m}. \]

The function $U$ is nonnegative, $\nu(u,Y_u,-,C_u,-,d)du$ is the predictable projection of $\nu(du,dz)$ (see Thm. II.1.1.21)), so by Theorem II.1.1.8 [12],
\[ \left( \int_0^t \int_{R_{d+1}} U(Y_u,-,z_1) \nu(du,dz) \right) \leq \sup_{v \leq t \wedge \tau_n} \int_0^v \int_{R_{d+1}} U(Y_u,-,z_1) \nu(du,dz). \]

By (3.6) and (3.10),
\[ \int_0^t \int_{R_{d+1}} U(y,z_1) \nu(du,dz) = \int_0^t \int_{R_{d+1}} U(y,F(u,y,c,x)) \nu(dx) + \sum_{j \in \mathcal{K} \setminus j \neq c} \int_0^t U(y,\rho^{c-j}(u,y)) \chi^{c-j}(u,y) \leq K^{iv} (1 + |y|^{2m}). \]

Hence, and by (3.12) and (3.13), we have
\[ E \sup_{0 \leq u \leq t \wedge \tau_n} |D^1_u| \leq L_5 \int_0^t \left( 1 + EY^{*,2m}_{u \wedge \tau_n} \right) du. \]

Taking into account (3.11) and estimates of all summands of $|Y|^{2m}$ we obtain (3.10) with $K = 2(L_1 + \ldots + L_5)$. The proof is now complete.

**Corollary 3.3.** Condition (LG) and (3.3) imply
\[ E \left( \sup_{t \in [0,T]} |Y|^2_t \right) < \infty. \]

**Corollary 3.4.** Assume (3.5). If $\nu \equiv 0$, then (3.7) holds for every natural number $m$. 
Corollary 3.5. Assume
\begin{equation}
|\rho^{c,k}(t,y)|^2 \leq K_3(1 + |y|^2),
\end{equation}
and the following order of linear growth of $F$
\begin{equation}
|F(t,y,c,x)| \leq K_4(1 + |y|)
\end{equation}
for some function $K$. Then the conditions
\[
\int_{\|x\| \leq a} K^2(x) \nu(dx) < \infty,
\]
and
\begin{equation}
\int_{\mathbb{R}^n} K^{2m}(x) \nu(dx) < \infty
\end{equation}
for some natural $m$, imply (3.7) for that $m$.

Corollary 3.6. If the measure $\nu$ has a bounded support, (3.14) holds, and (3.15) is satisfied with $K$ being continuous, then (3.7) holds for every natural number $m$.

Corollary 3.7. If the measure $\nu$ has all moments, (3.14) holds, and (3.15) is satisfied with $K$ having polynomial growth, then (3.7) holds for every natural number $m$.

4. General martingale problem for càdlàg processes

In this section we consider a martingale problem connected with the law of components $(Y, C)$ of solution to (2.1). The law of $(Y, C)$ is a measure on an appropriate Skorohod space, since $(Y, C)$ is a càdlàg process as a solution to (2.1). So we consider the canonical space $\Omega = D_{[0,T]}(\mathbb{R}^d \times \mathcal{K})$, i.e. the space of càdlàg functions on $[0, T]$ with values in $\mathbb{R}^d \times \mathcal{K}$. On $\Omega$ we consider the coordinate process $\pi_t : (y, c) \mapsto (y_t, c_t)$, $t \in [0, T]$, and canonical filtration $\mathcal{F}$ generated by the coordinate process. Moreover, let $\mathbb{F}^r = (\mathcal{F}_s)_{s \geq r}$, $\mathcal{F}^r = \sigma(\mathcal{F}_s^r : s \geq r)$, where $\mathcal{F}_s^r = \sigma(\pi_t : t \in [r, s])$.

Let us denote by $C^2 = C^2(\mathbb{R}^d \times \mathcal{K})$ - the space of all measurable functions $v : \mathbb{R}^d \times \mathcal{K} \to \mathbb{R}$ such that $v(\cdot, k) \in C^2(\mathbb{R}^d)$ for every $k \in \mathcal{K}$, and let $C^2_c$ be a set of functions $f \in C^2$ with compact support.

Definition 4.1. We say that a law $\mathbb{P}$ on the space $(\Omega, \mathcal{F})$ solves a time-dependent (local) martingale problem started at time $r$ with initial distribution $\eta$ for a family of operators $A = (A_t)_{t \in [r,T]}$ if

1. the measure $\eta$ is a distribution of $(y_r, c_r)$,
2. for every function $v \in C^2_c$ the process
   \[
   M^v_t := v(y_t, c_t) - \int_r^t A_u v(y_u, c_u) \, du
   \]
is an $\mathbb{F}^r$ (local) martingale on $[r, T]$ with respect to $\mathbb{P}$.

We say that the (local) martingale problem for $A$ is well-posed if for any $r \in [0, T]$ and $(y, c) \in \mathbb{R}^d \times \mathcal{K}$ there exists exactly one solution to the martingale problem for $A$ started at time $r$ with initial distribution $\delta_{y, c}$ (cf. Stroock [20]).

Remark 4.2. In the case of diffusion processes, so in the case when the operators $(A_t)_{t \in [r,T]}$ are the second order differential operators, a solution to the local martingale problem for $(A_t)_{t \in [r,T]}$ is also a solution to the martingale problem for $(A_t)_{t \in [r,T]}$. This follows from the fact that for each $v \in C^2_c$ the function $A_t v$ is bounded and continuous, which implies that the local martingale $M^v$ is a true martingale. However, for processes with jumps this is not the case, unless you make quite restrictive assumption on coefficients that ensure boundedness of $A_t v$ for $v \in C^2_c$, see Kurtz [15] Thm. 3.1.

In the next proposition we formulate result which generalize the well known fact for diffusions (see, e.g. Kallenberg [13] Thm. 18.10) which states that the well-posedness of the martingale problem for $A$ implies the existence of solutions to the martingale problem with an arbitrary initial
distributions. Results of such type for Lévy type operators, under different assumptions, can be found e.g. in Stroock [20] or Ethier and Kurtz [7, Chapter 4].

**Proposition 4.3.** Assume that on the canonical space the martingale problem for \( A \) is well-posed. For an arbitrary probability measure \( \eta \) on \( \mathbb{R}^d \times \mathcal{K} \) the martingale problem for \( A \) started at \( r \in [0, T] \) with initial distribution \( \eta \) has the unique solution given by

\[
(4.1) \quad \mathbf{P}_r^\eta := \int_{\mathbb{R}^d \times \mathcal{K}} \mathbf{P}_{r,y,c} \eta(dy, dc),
\]

where \( \mathbf{P}_{r,y,c} \) is the unique solution to the martingale problem for \( A \) started at \( r \in [0, T] \) with initial distribution \( \delta_{(y,c)} \).

**Proof.** We start from the proof that the right hand side of (4.1) is well defined. Let

\[
\mathcal{P}_M := \{ \mathbf{P}_{r,y,c} : (y, c) \in \mathbb{R}^d \times \mathcal{K} \}.
\]

First, we prove that \( \mathcal{P}_M \) is a measurable subset of \( \mathcal{P} \) - the set of all probability measure on Skorochod space \( \Omega \). Let \( \mathcal{D} \) be a countable subset of \( C_c^\infty \) which is dense in \( C_c^\infty \), and

\[
\mathcal{Q} := \{ \mathbf{P} \in \mathcal{P} : M^v \text{ is a } \mathbf{P} \text{ martingale on } [r, T] \text{ for each } v \in \mathcal{D} \}.
\]

Note that for \( \mathbf{P} \in \mathcal{Q} \) the required martingale property is equivalent to the following countable many relations: for \( v \in \mathcal{D} \), \( s < t \) in \( \mathcal{Q} \cap [r, T] \cup \{ r, T \} \), \( A \in \mathcal{G}_s \)

\[
\mathbf{E}_v((M^r - M^t) \mathbb{1}_A) = 0,
\]

where \( \mathcal{G}_s \) is a countable set of generators of \( \mathcal{F}_s \). For fixed \( v \in \mathcal{D} \), \( s < t \) in \( \mathcal{Q} \cap [r, T] \cup \{ r, T \} \), and \( A \in \mathcal{G}_s \) consider the mapping

\[
Y_{s,t,v,A} : \mathcal{P} \mapsto \mathbb{R} \cup \{ \infty \}
\]

defined by

\[
Y_{s,t,v,A}(\mathbf{P}) = \left\{ \begin{array}{ll}
\mathbf{E}((M^r - M^t) \mathbb{1}_A), & \text{if } \mathbf{E}(|M^r - M^t| \mathbb{1}_A) < \infty, \\
\infty, & \text{if } \mathbf{E}(|M^r - M^t| \mathbb{1}_A) = \infty.
\end{array} \right.
\]

To show measurability of \( \mathcal{Q} \) it is sufficient to notice that \( Y_{s,t,v,A} \) is a measurable mapping. Indeed, measurability of \( Y_{s,t,v,A} \) yields that \( \mathcal{Q} \) is measurable since \( \mathcal{Q} \) can be represented as the following countable intersection

\[
\mathcal{Q} = \bigcap_{s,t \in \mathcal{Q} \cap [r, T] \cup \{ r, T \}} \bigcap_{A \in \mathcal{G}_s} Y_{s,t,v,A}^{-1}(0).
\]

Thus, it remains to show measurability of \( Y_{s,t,v,A} \). For \( N > 0 \) consider a mapping

\[
Z_N : \mathcal{P} \mapsto \mathbb{R}
\]

defined by

\[
Z_N(\mathbf{P}) := \mathbf{E}((|M^r - M^t| \mathbb{1}_A) \wedge N) \vee (-N) \mathbb{1}_A).
\]

We note that the mapping \( Z_N \) is measurable. Since on the set on which \( \lim N Z_N \) exists and is finite we have

\[
Y_{s,t,v,A}(\mathbf{P}) = \lim_N Z_N(\mathbf{P}),
\]

which implies measurability of \( Y_{s,t,v,A} \). Let

\[
\mathcal{R} := \{ \mathbf{P} \in \mathcal{P} : \exists (y, c) \in \mathbb{R}^d \times \mathcal{K} \text{ such that } \mathbf{P}(y_r = y, c_r = c) = 1 \},
\]

so \( \mathcal{R} \) is measurable (see Kallenberg [13, Lem. 1.36]). We note that

\[
\mathcal{P}_M = \mathcal{Q} \cap \mathcal{R},
\]

thus \( \mathcal{P}_M \) is measurable.

In order to prove that \( \mathbf{P}_r^\eta \) is well defined, we need to prove that the mapping

\[
f : \mathbb{R}^d \times \mathcal{K} \to \mathcal{P}_M, \quad \text{defined by } f(y, c) := \mathbf{P}_{r,y,c}
\]

is measurable. Let

\[
g : \mathcal{P}_M \to \mathbb{R}^d \times \mathcal{K}
\]
be defined by
\[ g(P_{r,y,c}) = (y, c). \]
The well-posedness of martingale problem, and measurability of \( P_M \) imply that \( g \) is a measurable bijection, and therefore by theorem of Kuratowski (see e.g. Kallenberg [13, Theorem A1.7]) \( g \) has measurable inverse which is equal to \( f \).

To end the proof of theorem we need to show that \( \mathbb{P}_\eta^y \) defined by (11), is the unique solution of the martingale problem for \( A \) for initial distribution \( \eta \) started at \( r \). This follows from the fact that the required martingale property is equivalent to the following countably many relations: for \( v \in \mathcal{D} \), \( s < t \in \mathbb{Q} \cap [r, T] \cup \{r, T\} \), \( A \in \mathcal{G}_s \)
\[ \mathbb{E}_{\mathbb{P}_\eta^y}( (M^v_t - M^v_r) \mathbb{1}_A) = 0, \]
which, by definition of \( \mathbb{P}_\eta^y \), can be written in the form
\[ \int_{\mathbb{R}^d \times \mathcal{K}} \mathbb{E}_{r,y,c}((M^v_t - M^v_r) \mathbb{1}_A) \eta(dy, dc) = 0. \]
This equality holds by the well-possedness of martingale problem. Therefore \( \mathbb{P}_\eta^y \) solves the required martingale problem. Now we consider the issue of uniqueness. Let \( Q_r^y \) be a solution to martingale problem started at \( r \) from distribution \( \eta \). For arbitrary \( v \in \mathcal{C}^\infty, s < t \in [r, T], A \in \mathcal{F}_s \) we have
\[ \mathbb{E}_{Q_r^y}( (M^v_t - M^v_r) \mathbb{1}_A | y_r, c_r) = 0. \]
Therefore \( Q_r^y(\cdot | y_r, c_r) \) is a solution to martingale problem started at \( r \) from distribution \( \delta_{(y_r, c_r)} \). This yields (by well posedness) that \( Q_r^y(\cdot | y_r, c_r) = P_{r,y,c} (\cdot) - Q_r^y \) a.s., and thus
\[ Q_r^y(\cdot) = \int_{\mathbb{R}^d \times \mathcal{K}} Q_r^y(\cdot | y_r = y, c_r = c) Q_r^y(y_r \in dy, c_r \in dc) \]
\[ = \int_{\mathbb{R}^d \times \mathcal{K}} P_{r,y,c}(\cdot) Q_r^y(y_r \in dy, c_r \in dc) \]
\[ = \int_{\mathbb{R}^d \times \mathcal{K}} P_{r,y,c}(\cdot) \eta(dy, dc) = P_r^y(\cdot), \]
where \( Q_r^y(\cdot | y_r = y, c_r = c) \) is a regular version of conditional probability \( Q_r^y(\cdot | y_r, c_r). \)
\[ \square \]

We can also prove that the family \( \{P_{r,y,c} \} \) is a Markov family.

**Proposition 4.4.** Assume that on the canonical space the martingale problem for \( A \) is well-posed. The family \( \{P_{r,y,c} : (r, y, c) \in [0, T] \times \mathbb{R}^d \times \mathcal{K} \} \) is a Markov family.

**Proof.** We have to prove that for arbitrary \( t \geq r \) and \( A \in \mathcal{F}_t = \sigma((y_u, c_u) : u \in [t, T]) \) we have
\[ P_{r,y,c}(A | \mathcal{F}_t) = P_{t,y,c,t}(A), \]
where by \( P_{r,y,c}(\cdot | \mathcal{F}_t) \) we denote the regular conditional probability of \( P_{r,y,c} \) with respect to \( \mathcal{F}_t := \sigma((y_u, c_u) : u \in [r, t]) \). For every \( \omega \in \mathcal{F}_t \) the probability measure \( P_{r,y,c}(\cdot | \mathcal{F}_t)(\omega) \) solves the martingale problem for \( A \) started at \( t \) from \( \delta_{(y_t(\omega), c_t(\omega))} \) (see e.g. Rogers and Williams [18, Thm. 21.1]). By uniqueness, we obtain
\[ P_{r,y,c}(A | \mathcal{F}_t) = P_{t,y,c,t}(A) \quad \text{for every } A \in \mathcal{F}_t, P_{r,y,c} \text{ a.s.} \]
which implies corresponding Markov family property. \[ \square \]

5. **Uniqueness in law of weak solutions**

In this section we prove uniqueness of finite-dimensional distributions of jump-diffusion under some assumptions on coefficients of SDE, and assumption on intensities. To prove this fact we use the martingale problem and prove that it is well-posed. As a consequence of Corollary 5.6 we see that the law of components \((Y, C)\) of a solution to SDE (2.1) solves a time dependent local martingale problem for the family of operators \((A_t)_{t \in [r, T]} \) (see e.g. [2, IV.7.A and IV.7.B]). If we prove that a solution to the local martingale problem is also a solution to the martingale problem and the martingale problem is well-posed, then these components constitutes a time inhomogeneous Markov
process and uniqueness in law of \((Y, C)\) holds (see e.g. [7, Section IV.4 Theorem 4.1 and Theorem 4.2] or Stroock [20, Theorem 4.3]). However, if the law of process \((Y, C)\) solves the local martingale problem for \(A\) given by (2.13), then the law of \((Y, C)\) does not necessarily solves the martingale problem for \(A\). Nevertheless,

**Theorem 5.1.** Assume that the law of \((Y, C)\) solves the local martingale problem for \((A_t)_{t \in [r, T]}\). If coefficients of \((A_t)_{t \in [r, T]}\) satisfies \((LG)\), and (3.5) or

\[
\int_{|y| \leq 1} \|F(t, y, c, x)\|^2 \nu(dx) \leq M,
\]

then the law of \((Y, C)\) solves the martingale problem for \((A_t)_{t \in [r, T]}\).

**Proof.** First assume that \((LG)\) and (5.1) holds and suppose that the law of \((Y, C)\) solves local martingale problem for \((A_t)_{t \in [r, T]}\). This implies that \(A_tf\) is bounded for \(f \in C^2\). Therefore, for any \(f \in C^2\), the local martingale \(M_f\) is bounded, so it is a martingale. This proves that if the law of \((Y, C)\) solves the local martingale problem for \((A_t)_{t \in [r, T]}\) then it is also a solution to the martingale problem for \((A_t)_{t \in [r, T]}\).

Now assume that \((LG)\) and (3.5) hold and suppose that the law of \((Y, C)\) solves the local martingale problem for \((A_t)_{t \in [r, T]}\). From Corollary 3.3 we infer that every solution to the local martingale problem for \((A_t)_{t \in [r, T]}\) satisfies

\[
\mathbb{E} \left( \sup_{t \in [r, T]} |Y_t|^2 \right) < K
\]

for some constant \(K > 0\). Moreover, for \(f \in C^2\) it follows from (3.5) and \((LG)\) that

\[
|A_t f(y, c)| \leq K_f (1 + |y|^2).
\]

This implies that the local martingale \(M_f\) has an integrable supremum and therefore is a martingale.

\(\square\)

**Remark 5.2.** The set of conditions \((LG)\) and (5.1) comparing to \((LG)\) and (3.5) is on the one hand more restrictive and on the other hand less restrictive because there is no assumptions on \(\rho\).

**Theorem 5.3.** Let continuous functions \(\lambda^{i,j}\), \(i \neq j\), \(i, j \in \mathcal{K}\), satisfy \((\Lambda)\) and

\[
(\Lambda_0) \quad \lambda^{i,j}(t, y) \equiv 0 \quad \text{or} \quad 0 < \lambda^{i,j}(t, y) \quad \forall t \in [0, T] \quad \forall s \in \mathbb{R}^d.
\]

Moreover assume \((\text{Lip})\), \((LG)\), \((\text{Cont})\), \((\text{LB})\) and (3.5). Then the martingale problem for \((A_t)_{t \in [0, T]}\), defined by (2.13), is well-posed.

**Proof.** Fix arbitrary \(r \in [0, T]\), \((y, c) \in \mathbb{R}^d \times \mathcal{K}\). Let \(\mathcal{A}_t^b\) be an operator defined by

\[
\mathcal{A}_t^b v(y, c) := \nabla v(t, y, c) \mu(t, y, c) + \frac{1}{2} \text{Tr} \left( a(t, y, c) \nabla^2 v(t, y, c) \right)
\]

\[
+ \int_{\mathbb{R}^d} \left( v(t, y + F(t, y, c, x), c) - v(t, y, c) - \nabla v(t, y, c) F(t, y, c, x) \mathbb{1}_{\{|x| \leq a\}} \right) \nu(dx)
\]

\[
+ \sum_{k \in \mathcal{K} \setminus \mathcal{c}} \left( v(t, y + \rho^{c,k}(t, y), k) - v(t, y, c) \right) b^{c,k},
\]

where we put \(b^{c,k} := \sup_{(t, y) \in [0, T] \times \mathbb{R}^d} \lambda^{c,k}(t, y)\). The operator \(\mathcal{A}_t^b\) is well defined on \(C^{1,2}_c\). Denote \(\mathcal{A}_t^b = (\mathcal{A}_t^b)_{t \in [r, T]}\). First, we consider the martingale problem for \(\mathcal{A}_t^b\). Take a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^b)\) on which there exist independent processes: a standard Brownian motion \(W\), a Poisson random measure \(\Pi(dx, dt)\) with intensity measure \(\nu(dx)dt\) and the Poisson processes \((N^{i,j})_{i,j \in \mathcal{K}, i \neq j}\) with intensities equal to \((b^{i,j})_{i,j \in \mathcal{K}, i \neq j}\) for \(b^{i,j} > 0\). If \(b^{i,j} = 0\), then we put \(N^{i,j} \equiv 0\).
On this stochastic basis we can construct for arbitrary \((y, c)\), under assumptions of theorem, a unique solution to the following SDE on \([r, T]\):

\[
dY_t = \mu(t, Y_{t-}, C_{t-})dt + \sigma(t, Y_{t-}, C_{t-})dW_t + \int_{\|x\| \leq a} F(t, Y_{t-}, C_{t-}, x)\Pi(dx, dt)
\]

\[
+ \int_{\|x\| \geq a} F(t, Y_{t-}, C_{t-}, x)\Pi(dx, dt) + \sum_{i, j, k \neq i} \rho^{i,j}(t, Y_{t-})\mathbb{1}_{\{i\}}(C_{t-})dN_t^{i,j},
\]

\[
dC_t = \sum_{i, j, k \neq i} (j - i)\mathbb{1}_{i}(C_{t-})dN_t^{i,j},
\]

\[
Y_r = y, \quad C_r = c.
\]

By Proposition 2.5 and Proposition 5.1 the law of \((Y, C)\) solves the martingale problem for \(A^b\), since SDE (5.3) is a special case of (2.1). The uniqueness of a strong solution to (5.3) implies pathwise uniqueness and therefore martingale problem for \(A^b\) is well-posed \([19, \text{Corollary 140}]\).

Let \(\overline{\mathbb{P}}_{r,y,c}^b\) denote the law of this unique solution on \(\overline{\Omega} = D_{[0,T]}(\mathbb{R}^d) \times D_{[0,T]}(\mathbb{K})\) endowed with \(\overline{\mathcal{F}}_{r,y,c}\) - completed canonical filtration \(\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{t \in [0,T]}\) generated by the coordinate process \(\pi_t : (y, c) \mapsto (y_t, c_t)\). Therefore, for every \(v \in C^2_c\), the process

\[
M_t^v := v(y_t, c_t) - \int_r^t A^b_v(y_{u-}, c_{u-})du
\]

is an \(\overline{\mathbb{F}}\)-martingale under \(\overline{\mathbb{P}}_{r,y,c}^b\). We now introduce on \(\overline{\Omega}\) the nonanticipating processes:

\[
H_t^{i,j}(y, c) := \sum_{r < s \leq t} \mathbb{1}_{\{c_{s-} = i\}}\mathbb{1}_{\{c_{s} = j\}}, \quad H_t^i(y, c) := \mathbb{1}_{\{c_t = i\}}.
\]

By definition, they depend only on the path of process \(c\). The process \(H_t^{i,j}\) is a counting process with the \(\overline{\mathbb{F}}\)-intensity given by \(\int_r^t H_{u-}^{i,b^{-j}}du\), i.e., the process

\[
M_t^{i,j} := H_t^{i,j} - \int_r^t H_{u-}^{i,b^{-j}}du
\]

is an \(\overline{\mathbb{F}}\)-martingale. Now we define a new probability measure \(\overline{\mathbb{P}}_{r,y,c}^\lambda\) on \((\overline{\Omega}, \overline{\mathcal{F}}_T)\) by the formula

\[
\frac{d\overline{\mathbb{P}}_{r,y,c}^\lambda}{d\overline{\mathbb{P}}_{r,y,c}^b} := Z_T := \mathbb{E}_T\left[\sum_{(i,j) \neq (b^{-j}, b^{i}) > 0} \int_{[r,T]} \left(\lambda^{i,j}(t, y_{r-}) \frac{b^{i,j}}{b^{-j}} - 1\right) (dH_t^{i,j} - H_{t}^{i,b^{-j}}dt)\right].
\]

By our assumptions, this is the probability measure equivalent to \(\overline{\mathbb{P}}_{r,y,c}^b\). We prove that \(\overline{\mathbb{P}}_{r,y,c}^\lambda\) solves our original martingale problem. To do this we check that for every \(v \in C^2_c\) the process

\[
\widehat{M}_t^v := v(y_t, c_t) - \int_r^t A^\lambda_v(y_{u-}, c_{u-})du
\]

is an \(\overline{\mathbb{F}}\)-martingale under \(\overline{\mathbb{P}}_{r,y,c}^\lambda\), or equivalently that \(\widehat{M}^\lambda Z\) is an \(\overline{\mathbb{F}}\)-martingale under \(\overline{\mathbb{P}}_{r,y,c}^b\), where \(Z_t := \mathbb{E}_{\overline{\mathbb{P}}_{r,y,c}^b} (Z_T|\overline{\mathcal{F}}_t)\). To prove this, we first decompose \(\mathbb{A}\) as

\[
\mathbb{A} = \mathbb{A}_b + \mathbb{B},
\]

where \(\mathbb{A}_b\) is defined by (5.2) and

\[
\mathbb{B} v(t, y, c) := \sum_{k \neq c} \left(v(t, y + \rho^{c,k}(t, y), k) - v(t, y, c)\right) (\lambda^{c,k}(t, y) - b^{c,k}).
\]

Obviously,

\[
\widehat{M}_t^v = M_t^v - \int_r^t B v(u, y_{u-}, c_{u-})du,
\]
where $M^v$ is given by (5.4), so $M^v$ is an $\mathbb{F}$-martingale under $\mathbb{F}_{r_2,y,c}$. Therefore, by integration by parts formula, we have

$$d(\hat{M}^v_t) = \hat{M}^v_t\,dZ_t + Z_{i-}d\hat{M}^v_{i-} - Z_{i-}Bv(t-, y_{i-}, c_{i-})\,dt + d[Z, \hat{M}^v]_t.$$  

Since $\left(\int Z_{i-}d\hat{M}^v_{i-} - \int Z_{i-}Bv(t-, y_{i-}, c_{i-})\,dt \right)_{t \in [0, T]}$ is a local martingale to obtain that $\hat{M}^v Z$ is a $\mathbb{F}_{r_2,y,c}$ local martingale it is enough to show that

$$d\left(\int Z_{i-}d\hat{M}^v_{i-} - \int Z_{i-}Bv(t-, y_{i-}, c_{i-})\,dt \right)_t = Z_{i-}Bv(t-, y_{i-}, c_{i-})\,dt.$$  

Let us denote by $\mathcal{H}$ the measure of jumps of $(y, c)$. By the Theorem 5.1 we find that the process $(y, c)$ is a semimartingale under $\mathbb{P}^\phi$, with the compensator of measure of jumps of $(y, c)$, denoted by $\mathcal{H}$, given by (5.3). Note that we have

$$d[Z, \hat{M}^v]_t = \Delta Z_t \Delta \hat{M}^v_t = \sum_{i,j \in K,} Z_{i-} \left( \frac{\lambda_{i,j}(t, y_{i-})}{b_{i,j}} - 1 \right) (v(t, y_{i-} + \Delta y_{i-}, i + \Delta c_{i-}) - v(t, y_{i-}, i)) \Delta^i_{i,j}.$$  

Since, for $j \neq i$,

$$\Delta^i_{i,j} = H^i_{i-} \mathbb{1}_{\{\Delta c_{i-} = j - i\}}$$

we have

$$H^i_{i-} = \int_{\mathbb{R}^{d+1}_t} (v(t, y_{i-} + \Delta y_{i-}, i + \Delta c_{i-}) - v(t, y_{i-}, i)) (v(t, y_{j-}, i)) \Delta^i_{i,j}.$$

Now we will show that

$$\int_{\mathbb{R}^{d+1}_t} (v(t, y_{i-} + \Delta y_{i-}, i + \Delta c_{i-}) - v(t, y_{i-}, i)) (v(t, y_{j-}, i)) \Delta^i_{i,j} \,dZ_{i-} = \int_{\mathbb{R}^{d+1}_t} (v(t, y_{i-} + \Delta y_{i-}, i + \Delta c_{i-}) - v(t, y_{i-}, i)) \Delta^i_{i,j} \,dt.$$  

Using formulae (5.1) and (5.3) defining the compensator $\mathcal{H}$ and its intensity $\mathcal{H}$ we see that we have to compute two integrals. The first integral is equal to

$$\int_{\mathbb{R}^{d+1}_t} (v(t, y_{i-} + \Delta y_{i-}, i + \Delta c_{i-}) - v(t, y_{i-}, i)) \Delta^i_{i,j} \,dZ_{i-} = \sum_{k \in K \setminus \{c_{i-}\}} b_{i,k} \delta_{\{\rho^{i,k}(t, y_{i-}, c_{-}), k - c_{i-}\}}(dz_{1}, dz_{2}).$$

and the second integral is equal to

$$\int_{\mathbb{R}^{d+1}_t} (v(t, y_{i-} + \Delta y_{i-}, i + \Delta c_{i-}) - v(t, y_{i-}, i)) \Delta^i_{i,j} \,dt = \sum_{k \in K \setminus \{i\}} (v(t, y_{i-} + \Delta y_{i-}, i + \Delta c_{i-}) - v(t, y_{i-}, i)) \Delta^i_{i,j} b_{i,k}.$$  

so (5.10) holds. Since

$$d[Z, \hat{M}^v]_t = \sum_{i,j \neq i} Z_{i-} \left( \frac{\lambda_{i,j}(t, y_{i-})}{b_{i,j}} - 1 \right) \left[ v(t, y_{i-} + \Delta y_{i-}, i + \Delta c_{i-}) - v(t, y_{i-}, i) \right] \,dt$$

and

$$\int_{\mathbb{R}^{d+1}_t} (v(t, y_{i-} + \Delta y_{i-}, i + \Delta c_{i-}) - v(t, y_{i-}, i)) \,dZ_{i-} = \sum_{k \in K \setminus \{c_{i-}\}} b_{i,k} \delta_{\{\rho^{i,k}(t, y_{i-}, c_{-}), k - c_{i-}\}}(dz_{1}, dz_{2}).$$

so (5.10) holds.
and, by (5.7),
\[
Z_t-Bv(t,y_t,c_t) = \sum_{i,j:j\neq i} Z_t \left( \frac{\lambda^{i,j}(t,y^{i,j})}{\rho^{i,j}} - 1 \right) \left( v(t,y^{i,j} + \rho^{i,j}(t,y^{i,j}), j) - v(t,y^{i,j}, i) \right) H_t^{i,j}(c) b^{i,j},
\]
using (5.11), (5.10) and (5.5) we get from (5.8) that
\[
d(M_t^v) = M_t^v dZ_t + Z_t dM_t^v
+ \sum_{i,j:j\neq i} Z_t \left( \frac{\lambda^{i,j}(t,y^{i,j})}{\rho^{i,j}} - 1 \right) \left( v(t,y^{i,j} + \rho^{i,j}(t,y^{i,j}), j) - v(t,y^{i,j}, i) \right) dM_t^{i,j}.
\]
Hence \( M^v \) is a \( \tilde{\mathcal{F}}^{r,y,c}_v \) local martingale. By assumption (LB) and Proposition 5.1, \( M^v \) is a true martingale under \( \mathcal{F}^{r,y,c} \). Similar argumentation shows that if \( Q^{r,y,c}_v \) solves the martingale problem for \( \mathcal{A} \), then \( dQ^{r,y,c}_v := Z_t^{-1} dQ^{r,y,c}_v \), where \( Z_T \) is given by (5.10), solves the martingale problem for \( \mathcal{A}^0 \).

This implies that the martingale problem for \( \mathcal{A} \) is well-posed. Contrary, let \( \tilde{\mathcal{F}}^{r,y,c}_1, \tilde{\mathcal{F}}^{r,y,c}_2 \) be two solutions of the martingale problem for \( \mathcal{A} \) and \( \tilde{\mathcal{F}}^{r,y,c}_1 \neq \tilde{\mathcal{F}}^{r,y,c}_2 \). Then \( \tilde{\mathcal{F}}^{r,y,c}_i \) given by \( d\tilde{\mathcal{F}}^{r,y,c}_i = Z_t^{-1} d\mathcal{F}^{r,y,c}_i \) for \( i = 1, 2 \) solve the martingale problem for \( \mathcal{A} \). This is a contradiction with well-posedness of the martingale problem for \( \mathcal{A}^0 \).

\[\square\]

6. Examples

Now, we present two examples illustrating how our results work: a generalized exponential Levy model and a semi-Markovian regime switching model.

**Example 1** (Generalized exponential Lévy models). This model generalize exponential Levy model described, e.g., in [5]. Consider the following SDE
\[
dY_t = Y_t \left( \sigma(C_t) dW_t + \int_\mathbb{R} e^{\sigma(C_t) x} (1 - 1 \Pi(dx, dt)) + \sum_{i,j,j \neq i} (e^{\rho^{i,j}} - 1) H_t^{i,j} dN_t^{i,j} \right)
\]
\[
dC_t = \sum_{i,j,j \neq i} (j - i) \mathbb{1}_{(i,j)}(C_t) dN_t^{i,j},
\]
where \( \sigma(i) \geq 0, \rho^{i,j} \in \mathbb{R}, N^{i,j} \) are independent Poisson processes with constant intensities \( \lambda^{i,j} > 0 \), \( \Pi(dx, dt) \) is a Poisson random measure with intensity measure \( \nu(dx)dt \) satisfying
\[
\int_{|x|>1} e^{2\sigma(i)x} \nu(dx) < \infty \quad \forall i \in K.
\]
Note that the coefficients of this SDE satisfy assumptions of Theorem 5.3 so there exists a solution unique in law. Moreover, by using the Itô lemma one can show that this unique solution is of the form:
\[
Y_t = Y_0 \exp \left( \int_0^t \sigma(C_{u-}) du + \int_0^t \sigma(C_{u-}) dZ_u + \sum_{i,j,j \neq i} \int_0^t \rho^{i,j} \mathbb{1}_{(i,j)}(C_{u-}) dN_u^{i,j} \right),
\]
where \( Z \) is a Lévy process with the Levy-Ito decomposition:
\[
Z_t = W_t + \int_0^t \int_{|x| \leq 1} x \Pi(dx, du) + \int_0^t \int_{|x| > 1} x \Pi(dx, du)
\]
and
\[
J(u) := -\frac{u^2}{2} + \int_\mathbb{R} e^{ux} - 1 + ux \mathbb{1}_{|x|<1} \nu(dx).
\]
Moreover, the coordinate \( C \) of the solution \((Y,C)\) is a Markov chain with the state space \( K \).
Example 2 (Semi-Markovian regime switching models). In this example we will illustrate how a feed-back mechanism in jumps of $Y$ and intensity of jumps of $C$ give in our framework extra flexibility in modelling. We present how semi-Markov switching processes can be embedded in our framework. Let us recall that semi-Markov nature of $C$ is reflected in the fact that the compensator of jumps from $i$ to $j$, $\lambda_{i,j}$ depends on time that process $C$ spends in current state after the last jump. We recall basic facts from theory of semi-Markov processes. The semi-Markov process $C$ is related with a pair $(X,T) = (X_n,T_n : n \geq 1)$ which is a homogenous Markov renewal process, i.e.

$$P(X_{n+1} = j, T_{n+1} - T_n \leq t|X_0,\ldots,X_n; T_0,\ldots,T_n) = P(X_{n+1}, T_{n+1} - T_n \leq t|X_n) = Q_{X_n,j}(t)$$

for every $t \geq 0$, $j \in K$. $Q$ is called a semi-Markov kernel. Let

$$P_{i,j} := \lim_{t \to \infty} Q_{i,j}(t).$$

It can be shown that $(X_n)$ is a homogenous Markov chain with one-step transition matrix $P = (P_{i,j})$. A semi-Markov process $C$ is defined by

$$C_t := X_{N_t},$$

where

$$N_t := \sup \{n : T_n \leq t\}.$$ 

In general, a semi-Markov process does not have Markov property, the Markov property holds only at times $(T_n)$. If we assume that the distribution of holding times, i.e.

$$F_{i,j}(t) := \frac{Q_{i,j}(t)}{P_{i,j}}$$

has a density $f_{i,j}$, then the related semi-Markov process considered as an MPP has intensity. It was shown in [9] that for semi-Markov processes the intensity of jumps from $i$ to $j$ depends on time that process $C$ spends after the last jump in a current state, and has the form

$$\lambda_{i,j}(\omega, t) = \lambda_{i,j}(R_t(\omega)) = \frac{P_{i,j}f_{i,j}(R_t(\omega))}{1 - \sum_m Q_{i,m}(R_t(\omega))},$$

where

$$R_t := t - T_{N_t}. \tag{6.1}$$

A semi-Markov regime switching process can be embedded in our framework by considering the following SDE

$$dS_t = S_t\left(rd_t + \sigma(C_t^-)dW_t + \int_{\mathbb{R}} (e^{\sigma(C_t^-)x} - 1)\Pi(dx,dt)\right)$$

$$dR_t = dt - \sum_{i,j \in K : j \neq i} R_t^i \mathbb{1}_{\{i\}}(C_t^-) dN^{i,j}_t$$

$$dC_t = \sum_{i,j \in K : j \neq i} (j - i) \mathbb{1}_{\{i\}}(C_t^-) dN^{i,j}_t,$$

where $W$ is a Wiener process, $N^{i,j}$ are the point processes with intensity functions given by

$$\lambda^{i,j}(z) := \frac{P_{i,j}f_{i,j}(z)}{1 - \sum_m Q_{i,m}(z)}.$$

$\Pi(dx,dt)$ is a Poisson random measure with an intensity measure $\nu(dx)dt$ satisfying

$$\int_{|x| > 1} e^{2\sigma(i)x^2} \nu(dx) < \infty \quad \forall i \in K.$$ \tag{6.2}

and $\sigma(i) \geq 0$. Note that the coefficients of this SDE satisfy assumptions of Theorem 5.3. Moreover, if functions $\lambda^{i,j}$ satisfy assumptions of this theorem, then there exists a solution unique in law. Since the process

$$\int_0^t \sum_{i,j \in K : j \neq i} \mathbb{1}_{\{i\}}(C_u^-) dN^{i,j}_u$$
counts the number of jumps of $C$, it is easy to see that the component $(R_t)$ represents process given by (6.1). Moreover, using the Itô lemma one can see that the process $S$ is of the form:

$$S_t = S_0 \exp \left( \int_0^t r + J(\sigma(C_u))du + \int_0^t \sigma(C_u) dZ_u \right),$$

where $Z$ is a Lévy process with the Lévy-Itô decomposition:

$$Z_t = W_t + \int_0^t \int_{|x| \leq 1} x\Pi(dx, du) + \int_0^t \int_{|x| > 1} x\Pi(dx, du)$$

and

$$J(u) := -\frac{u^2}{2} + \int_{\mathbb{R}} (e^{ux} - 1 + ux1_{\{|x|<1\}}) \nu(dx).$$

Moreover, the coordinate $C$ of the solution $(S, R, C)$ is a semi-Markov chain with the state space $\mathcal{K}$.

### 7. Appendix. Proof of Theorem 2.4

**Proof.** For the brevity, we use the following notation $\mu^i := \mu(t, Y_{t-}, i)$, $\sigma^i := \sigma(t, Y_{t-}, i)$, $F^i(x) := F(t, Y_{t-}, i, x)$, $\rho^{i,j} := \rho^{j,i}(t, Y_{t-})$, $\lambda^{i,j} := \lambda^{j,i}(t, Y_{t-})$, $v^i(t, x) := v(t, i, x)$. Let us recall that $H^i$, $H^{i,j}$ are given by (2.1), (2.9). By integration by parts formula

$$dv(t, Y_t, C_t) = \sum_i d(v(t, Y_t, i)H^i_t) = I_1 + I_2 + I_3,$$

where:

$$I_1 := \sum_i H^i_{t-}dv(t, Y_t, i), \quad I_2 := \sum_i v^i(t, Y_{t-})dH^i_t, \quad I_3 := \sum_i \Delta v^i(t, Y_t)\Delta H^i_t,$$

and $\Delta v^i(t, Y_t) := v^i(t, Y_t) - v^i(t, Y_{t-})$. We start from calculation of $I_2$.

$$I_2 = \sum_i v^i(t, Y_{t-})d \left( \sum_{j \neq i} (H^{j,i} - H^{i,j})_t \right) = \sum_{i, j \neq i} v^i(t, Y_{t-})dH^{j,i}_t - \sum_{i, j \neq i} v^i(t, Y_{t-})dH^{i,j}_t$$

$$= \sum_{i, j \neq i} v^j(t, Y_{t-})dH^{i,j}_t - \sum_{i, j \neq i} v^i(t, Y_{t-})dH^{i,j}_t = \sum_{i, j \neq i} (v^j(t, Y_{t-}) - v^i(t, Y_{t-}))dH^{i,j}_t.$$

Next, from the fact that $H^{i,j}$ and $\Pi$ have no common jumps and the form of $Y$ (see (2.1)) we note that

$$\Delta Y_t \Delta H^{i,j}_t = \rho^{i,j}_t \Delta H^{i,j}_t.$$

Therefore, by the same arguments as in $I_2$, we obtain

$$I_3 = \sum_{i, j \neq i} \left( (\Delta v^j(t, Y_t) - \Delta v^i(t, Y_t)) \Delta H^{i,j}_t \right)$$

$$= \sum_{i, j \neq i} \left( v^j(t, Y_{t-} + \rho^{i,j}_t) - v^j(t, Y_{t-}) - v^i(t, Y_{t-} + \rho^{i,j}_t) + v^i(t, Y_{t-}) \right) \Delta H^{i,j}_t.$$

Hence

$$I_2 + I_3 = \sum_{i, j \neq i} (v^j(t, Y_{t-} + \rho^{i,j}_t) - v^j(t, Y_{t-}) - v^i(t, Y_{t-} + \rho^{i,j}_t))dH^{i,j}_t.$$

Now we will deal with the first term $I_1$. By the Itô lemma we have

$$dv^i(t, Y_t) = \partial^2 v^i(t, Y_t)dt + \nabla v^i(t, Y_{t-})dY_t + \frac{1}{2} \text{Tr} \left( \sigma^i_t(\sigma^i_t)^\top \nabla^2 v^i(t, Y_{t-}) \right) dt$$

$$+ \Delta v^i(t, Y_t) - \nabla v^i(t, Y_{t-})\Delta Y_t,$$

where we use the fact that quadratic variation of continuous martingale part of $Y^j$ and $Y^k$ is equal to

$$d[Y^j, Y^k] = \sum_i H^i_{t-} \sum_{l=1}^n [\sigma^i_l(\sigma^i_l)^\top]_{j,k} dt = \sum_i H^i_{t-} [\sigma^i(\sigma^i)^\top]_{j,k} dt.$$
The jump $\Delta v^i(t, Y_i)$ is equal to

$$H^i_{t-} \Delta v^i(t, Y_t) = H^i_{t-} (v^i(t, Y_t) - v^i(t, Y_{t-}))$$

(7.2)

Moreover, we have

$$H^i_{t-} \sum_{j \neq i} (v^i(t, Y_{t-} + \rho^i_{t,j}) - v^i(t, Y_{t-})) \Delta H^{i,j}_t.$$ 

Using (7.2), (7.3) and (2.1) we get that single summand of $I_1$ is given by

$$H^i_{t-} dv^i(t, Y_{t-}) = H^i_{t-} \left( \partial_t v^i(t, Y_t)dt + \frac{1}{2} \text{Tr} \left( \sigma^i_t (\sigma^i_t)^\top \nabla^2 v^i(t, Y_{t-}) \right) \right) dt$$

$$+ H^i_{t-} \nabla v^i(t, Y_{t-}) \left( \mu^i_t dt + \sigma^i_t dW_t + \int_{\|x\| < a} F^i_t(x) \Pi(dx, dt) + \int_{\|x\| > a} F^i_t(x) \Pi(dx, dt) + \sum_{j \neq i} \rho^i_{t,j} dH^{i,j}_t \right)$$

$$+ H^i_{t-} \int_{\mathbb{R}^n} (v^i(t, Y_{t-} + F^i_t(x)) - v^i(t, Y_{t-}) - \nabla v^i(t, Y_{t-}) F^i_t(x)) \Pi(dx, dt)$$

$$+ H^i_{t-} \sum_{j \neq i} (v^i(t, Y_{t-} + \rho^i_{t,j}) - v^i(t, Y_{t-}) - \nabla v^i(t, Y_{t-}) \rho^i_{t,j}) dH^{i,j}_t.$$ 

In the above expression the jump part can be compensated, since $v$ satisfies (by assumption) integrability condition (2.11). Therefore the above expression can be rearranged in the following way.

$$H^i_{t-} dv^i(t, Y_t) = H^i_{t-} \left[ \left( \partial_t v^i(t, Y_t) + \nabla v^i(t, Y_{t-}) \mu^i_t + \frac{1}{2} \text{Tr} \left( \sigma^i_t (\sigma^i_t)^\top \nabla^2 v^i(t, Y_{t-}) \right) \right) dt 
+ \int_{\mathbb{R}^n} (v^i(t, Y_{t-} + F^i_t(x)) - v^i(t, Y_{t-}) - \nabla v^i(t, Y_{t-}) F^i_t(x) \Pi_{\{\|x\| < a\}}) v(dx) dt 
+ \sum_{j,j \neq i} (v^i(t, Y_{t-} + \rho^i_{t,j}) - v^i(t, Y_{t-}) \lambda^i_{t,j}) \lambda^i_{t,j} dt 
+ H^i_{t-} \nabla v^i(t, Y_{t-}) \sigma^i_t dW_t 
+ H^i_{t-} \int_{\mathbb{R}^n} (v^i(t, Y_{t-} + F^i_t(x)) - v^i(t, Y_{t-})) \Pi(dx, dt) 
+ H^i_{t-} \sum_{j \neq i} (v^i(t, Y_{t-} + \rho^i_{t,j}) - v^i(t, Y_{t-})) dM^{i,j}_t \right].$$

Hence and by (7.4)

$$dv(t, Y_t, C_t) = \sum_i H^i_{t-} \left( \partial_t v^i(t, Y_t) + \nabla v^i(t, Y_{t-}) \mu^i_t + \frac{1}{2} \text{Tr} \left( \sigma^i_t (\sigma^i_t)^\top \nabla^2 v^i(t, Y_{t-}) \right) 
+ \int_{\mathbb{R}^n} (v^i(t, Y_{t-} + F^i_t(x)) - v^i(t, Y_{t-}) - \nabla v^i(t, Y_{t-}) F^i_t(x) \Pi_{\{\|x\| < a\}}) v(dx) 
+ \sum_{j,j \neq i} (v^i(t, Y_{t-} + \rho^i_{t,j}) - v^i(t, Y_{t-}) \lambda^i_{t,j}) \lambda^i_{t,j} ) dt 
+ \sum_i H^i_{t-} (\nabla^j v^i(t, Y_{t-}) )^\top \sigma^i_t dW_t 
+ \sum_i H^i_{t-} \int_{\mathbb{R}^n} (v^i(t, Y_{t-} + F^i_t(x)) - v^i(t, Y_{t-})) \Pi(dx, dt) \right)$$

(7.4)
\[ \sum_{i,j \neq i} (v^j(t, Y_{t-} + \rho^{i,j} t) - v^i(t, Y_{t-})) H^i_{t-} dM^i_{t}. \]

which is precisely (2.12).

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