Anticipating stochastic equation of two-dimensional second grade fluids

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Abstract: In this paper, we consider a stochastic model of incompressible second grade fluids on a bounded domain of \( \mathbb{R}^2 \) driven by linear multiplicative Brownian noise with anticipating initial conditions. The existence and uniqueness of the solutions are established.

Key Words: Second grade fluids; Malliavin calculus; Anticipating Stratonovich integral; Skorohod integral

1 Introduction

In this article, we investigate the existence and uniqueness of solutions of the following anticipating stochastic equation of second grade fluids:

\[
\begin{aligned}
\frac{d(u - \alpha \Delta u)}{dt} + \left( -\nu \Delta u + \text{curl}(u - \alpha \Delta u) \times u + \nabla \mathfrak{P} \right) &= F(u, t) dt + (u - \alpha \Delta u) \circ \sigma dW, \quad \text{in } \mathcal{O} \times (0, T],
\text{div } u &= 0 \quad \text{in } \mathcal{O} \times (0, T],
\text{u} &= 0 \quad \text{in } \partial \mathcal{O} \times [0, T],
\text{u}(0) &= \xi \quad \text{in } \mathcal{O},
\end{aligned}
\]

(1.1)

where \( \mathcal{O} \) is a bounded domain of \( \mathbb{R}^2 \), simply-connected and open, with boundary \( \partial \mathcal{O} \) of class \( \mathcal{C}^{3,1} \). \( u = (u_1, u_2) \) and \( \mathfrak{P} \) represent the random velocity and modified pressure, respectively. \( \alpha, \sigma \) are positive constants and \( \nu \) is the kinematic viscosity. \( W \) is a one-dimensional standard Brownian motion defined on a complete filtered probability space \( (\Omega, \mathcal{F}, P) \) with the augmented Brownian filtration \( \{ \mathcal{F}_t \}_{t \geq 0} \). \( \xi \) is an \( \mathcal{F}_T \)-measurable random variable. The fluid is driven by external forces \( F(u, t) dt \) and the noise \( (u - \alpha \Delta u) \circ \sigma dW \), where the stochastic integral is understood in the sense of anticipating Stratonovich integrals.

We refer the reader to [7, 6, 8, 4, 5] for a comprehensive theory of the second grade fluids. These fluids are non-Newtonian fluids of differential type, they are admissible models of slow flow fluids such as industrial fluids, slurries, polymer melts, etc. They also have interesting connections with other fluid models, see [11, 2, 3]. For researchs on stochastic models of 2D second grade fluids, we refer to [12, 13, 15, 17, 16, 14].

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The consideration of the anticipating initial value is based on several aspects: random measurement errors, the stationary point of the stochastic dynamical system, substitution formulas of anticipating Stratonovich integrals. For more details, we refer to Mohammed and Zhang [9]. The difficulty in directly proving such a substitution theorem is that Kolmogorov continuity theorem fails within our infinite-dimensional setting. To solve this anticipating problem (1.1), we proceed with the following steps: firstly, we develop a simple chain rule of Malliavin derivative of Hilbert space-valued random variables and establish a product rule for the Skorohod integrals, see Lemma 4.1 and Proposition 4.1; secondly, we use Galerkin approximations to show that the solution of (1.1) with deterministic initial value is Malliavin differentiable, see Proposition 4.2; finally, combining the previous two steps, we easily obtain our main results. We believe that this method can also be used to solve the problem with anticipating initial value and linear multiplicative noise for more general framework of SPDE.

The organization of this paper is as follows. In Section 2, we introduce some preliminaries and notations. In Section 3, we formulate the hypotheses and state our main results. Section 4 is devoted to the proof of the main results.

Throughout this paper, $C, C(T), C(T, N), \ldots$ are positive constants depending on some parameters $T, N, \ldots$, whose value may be different from line to line.

2 Preliminaries

In this section, we will introduce some functional spaces, preliminaries and notations.

For $p \geq 1$ and $k \in \mathbb{N}$, we denote by $L^p(O)$ and $W^{k,p}(O)$ the usual $L^p$ and Sobolev spaces over $O$ respectively, and write $H^k(O) := W^{k,2}(O)$. We write $X = X \times X$ for any vector space $X$. The set of all divergence free and infinitely differentiable functions in $O$ is denoted by $C$. $V$ (resp. $H$) is the completion of $C$ in $H^1(O)$ (resp. $L^2(O)$), Let $((u, v)) := \int_O \nabla u \cdot \nabla v dx$, where $\nabla$ is the gradient operator. Denote $\|u\| := ((u, u))^{1/2}$. We endow the space $V$ with the norm $|\cdot|_V$ generated by the following inner product

$$(u, v)_V := (u, v) + \alpha((u, v)),$$

for any $u, v \in V$,

where $(\cdot, \cdot)$ is the inner product in $L^2(O)$ (in $H$). We also introduce the following space

$$\mathbb{W} := \{u \in V : \text{curl}(u - \alpha \Delta u) \in L^2(O)\},$$

and endow it with the semi-norm $|\cdot|_{\mathbb{W}}$ generated by the scalar product

$$(u, v)_{\mathbb{W}} := (\text{curl}(u - \alpha \Delta u), \text{curl}(v - \alpha \Delta v)).$$

In fact, $\mathbb{W} = H^3(O) \cap V$, and this semi-norm $|\cdot|_{\mathbb{W}}$ is equivalent to the usual norm in $H^3(O)$, the proof can be found in [5, 4].

Identifying the Hilbert space $V$ with its dual space $V^*$, via the Riesz representation, we consider the system (1.1) in the framework of Gelfand triple: $\mathbb{W} \subset V \subset \mathbb{W}^*$. We also denote by $\langle \cdot, \cdot \rangle$ the dual relation between $\mathbb{W}^*$ and $\mathbb{W}$ from now on.

Because the injection of $\mathbb{W}$ into $V$ is compact, there exists a sequence $\{e_i\}$ of elements of $\mathbb{W}$ which forms an orthonormal basis in $\mathbb{W}$, and an orthogonal system in $V$, such that

$$(u, e_i)_{\mathbb{W}} = \lambda_i(u, e_i)_V, \quad \text{for any } u \in \mathbb{W}, \quad (2.1)$$
where \(0 < \lambda_i \uparrow \infty\). Since \(\partial \mathcal{O}\) is of class \(C^{3,1}\), Lemma 4.1 in [3] implies that
\[
e_i \in \mathbb{H}^i(\mathcal{O}), \quad \forall i \in \mathbb{N}.
\] (2.2)

Define the Stokes operator by
\[
Au := -\mathbb{P} \Delta u, \quad \forall u \in D(A) = \mathbb{H}^2(\mathcal{O}) \cap \mathcal{V},
\]
where \(\mathbb{P} : L^2(\mathcal{O}) \to \mathbb{H}\) is the usual Helmholtz-Leray projection. Set \(\hat{A} := (I + \alpha A)^{-1}A\), it follows from [14] that \(\hat{A}\) is a continuous linear operator from \(\mathcal{W}\) onto itself, moreover,
\[
\langle \hat{A}u, v \rangle = \langle Au, v \rangle = ((u, v)), \quad \forall u \in \mathcal{W}, \ v \in \mathcal{V}.
\] (2.3)

Define the bilinear operator \(\hat{B}(\cdot, \cdot) : \mathcal{W} \times \mathcal{V} \to \mathcal{W}^*\) by
\[
\hat{B}(u, v) := (I + \alpha A)^{-1}\mathbb{P}(\text{curl}(u - \alpha \Delta u) \times v).
\] (2.4)

For simplicity, we write \(\hat{B}(u) := \hat{B}(u, u)\). We have the following estimates which can be found in [13]:
\[
\begin{align*}
&|\hat{B}(u, v)|_{\mathcal{W}^*} \leq C|u|_{\mathcal{W}}|v|_{\mathcal{V}}, \quad \forall u \in \mathcal{W}, \ v \in \mathcal{V} \\
&|\hat{B}(u, u)|_{\mathcal{W}^*} \leq C|u|_{\mathcal{W}}^2, \quad \forall u \in \mathcal{W}, \\
&\langle \hat{B}(u, v), v \rangle = 0, \quad \forall u, v \in \mathcal{W}, \\
&\langle \hat{B}(u, v), w \rangle = -\langle \hat{B}(u, w), v \rangle, \quad \forall u, v, w \in \mathcal{W}.
\end{align*}
\] (2.5)

Finally, we introduce some notations about Malliavin calculus (see e.g. [11]). Let \(V\) be a real separable Hilbert space, \(p \geq 1\), we denote by \(\mathcal{D}^{1,p}(V)\) the Malliavin Sobolev space of all \(\mathcal{F}_t\)-measurable and Malliavin differentiable \(V\)-valued random variables with Malliavin derivatives having \(p\)th-order moments. The Malliavin derivative of \(\xi\) will be a stochastic process denoted by \(\{\mathcal{D}_t \xi, 0 \leq t \leq T\}\). \(\mathcal{L}^{1,2}(V)\) is the class of \(V\)-valued processes \(u \in L^2([0, T] \times \Omega)\) such that \(u(t) \in \mathcal{D}^{1,2}(V)\) for almost all \(t\), and there exists a measurable version of the two-parameter process \(\mathcal{D}_t u(t)\) verifying \(E \int_0^T \int_0^T \|\mathcal{D}_s u(t)\|_V^2 \ ds dt < \infty\). Note that \(\mathcal{L}^{1,2}(V)\) is isomorphic to \(L^2([0, T]; \mathcal{D}^{1,2}(V))\). Let \(X \in \mathcal{L}^{1,2}(V)\), we denote by \(\mathcal{D}^+ X\) and \(\mathcal{D}^- X\) the element of \(L^1([0, T] \times \Omega; V)\) satisfying
\[
\lim_{m \to \infty} \int_0^T \sup_{0 \leq s \leq (s+1/n) \wedge T} E\|\mathcal{D}_s X_t - (\mathcal{D}^+ X)_s\|_V \ ds = 0,
\] (2.6)
\[
\lim_{m \to \infty} \int_0^T \sup_{0 \leq s \leq (s-1/n) \wedge T} E\|\mathcal{D}_s X_t - (\mathcal{D}^- X)_s\|_V \ ds = 0,
\] (2.7)
respectively. We denote by \(\mathcal{L}^{1,2}_1(V)\) the class of processes in \(\mathcal{L}^{1,2}(V)\) such that both (2.6) and (2.7) hold. From now on, for \(X \in \mathcal{L}^{1,2}_1(V)\) we write \((\nabla X)_t := (\mathcal{D}^+ X)_t + (\mathcal{D}^- X)_t\), and the Fréchet derivative is denoted by \(\mathcal{D}\). Let \(\mathcal{X}\) denote a class of random variables (or processes), we say that \(\xi \in \mathcal{X}_{\text{loc}}\) if there exists a sequence of \(\{\mathcal{O}_n, \xi^n\}, n \geq 1\) \(\subset \mathcal{F} \times \mathcal{X}\) such that \(\mathcal{O}_n \uparrow \mathcal{O}\) and \(\xi = \xi^n\) a.s. on \(\mathcal{O}_n\).
3 Hypotheses and results

Let $F : \mathbb{V} \times [0, T] \to \mathbb{V}$ be a given measurable map. We assume that:

(F1) For any $t \in [0, T]$,
\[
F(0, t) = 0,
\]
\[
|F(u_1, t) - F(u_2, t)|_{\mathbb{V}} \leq C_F |u_1 - u_2|_{\mathbb{V}}, \quad \forall u_1, u_2 \in \mathbb{V},
\]
where $C_F$ is a constant. In particular, we have
\[
|F(u, t)|_{\mathbb{V}} \leq C_F |u|_{\mathbb{V}}, \quad \forall u \in \mathbb{V}, t \in [0, T].
\]

(F2) $F$ is Fréchet differentiable with respect to the first variable, and the Fréchet derivative $D_F : \mathbb{V} \times [0, T] \to L(\mathbb{V})$ is continuous with respect to the first variable.

Set
\[
\hat{F}(u, t) := (I + \alpha A)^{-1} F(u, t).
\]

Applying $(I + \alpha A)^{-1}$ to the equation (1.1), we see that (1.1) is equivalent to the stochastic evolution equation:
\[
\begin{cases}
  du(t) + \nu \hat{A}u(t) dt + \hat{B}(u(t), u(t)) dt = \hat{F}(u(t), t) + u(t) \circ \sigma dW(t), \\
  u(0) = \xi \text{ in } \mathbb{W}.
\end{cases}
\]

where the stochastic integral is the anticipating Stratonovich integral.

Definition 3.1. A $\mathbb{V}$-valued continuous and $\mathbb{W}$-valued weakly continuous stochastic process $u$ is called a solution of the system (1.1), if the following two conditions hold:

(1) $u \in L^{1,2}_{1, loc}(\mathbb{V})$;

(2) for any $t \in [0, T]$, the following equation holds in $\mathbb{W}^*$ $P$-a.s.:
\[
\int_0^t u(s) \circ \sigma dW(s) = \int_0^t \sigma \nabla u(s) ds + \frac{\sigma}{2} \int_0^t (\nabla u)_s ds, \quad \forall t \in [0, T].
\]

Remark 3.1. To describe the class of anticipating Stratonovich integrable processes, the space $L^{1,2}_{1, loc}$ is often used. If $u = \{u(s), 0 \leq s \leq T\} \in L^{1,2}_{1, loc}$, then $u_{[0, t]}$ is also Stratonovich integrable for all $0 < t \leq T$. Moreover, this space has nice relationship between the Stratonovich and the Skorohod integrals (see Theorem 3.1.1 in [11]), in particular, we have

\[
\int_0^t u(s) \circ \sigma dW(s) = \int_0^t \sigma u(s) dW(s) + \frac{\sigma}{2} \int_0^t (\nabla u)_s ds, \quad \forall t \in [0, T].
\]

Now we can state the main result of this paper.

Theorem 3.1. Assume that (F1) and (F2) hold, $\xi$ is a $\mathbb{W} \cap H^4(O)$-valued $\mathcal{F}_T$-measurable random variable, and $\xi \in D^{1,2}_{loc}(\mathbb{W})$, then there exists a unique solution to the equation (3.1).

4 Proof of Theorem 3.1

We start with a lemma on a simple chain rule of Malliavin derivative of Hilbert-space valued random variables; next, we establish a product rule for the Skorohod integrals; then we use Galerkin approximations to show that the solutions of (1.1) with deterministic initial value are Mallivin differentiable; finally, we prove Theorem 3.1. For simplicity, we sometimes omit the parameter $\omega$ in the following when it is clear from the context.
Lemma 4.1. Let $G, K$ be real separable Hilbert spaces, $U$ is a subspace of $G$ and contains an orthonormal basis $\{e_i\}_{i=1}^\infty$ of $G$. Suppose that a random variable $\eta$ takes values in $U$ and $\eta \in D^{1,p}(G)$, $p > 1$, $||\eta||_G < \delta$. Consider a $K$-valued random field $u = \{u(f) : f \in G\}$ with continuously Fréchet differentiable paths on $U$ (i.e. the map $G \ni f \mapsto u(f, \omega) \in K$ is continuously Fréchet differentiable on $U \subset G$ for almost all $\omega \in \Omega$), such that $u(f) \in D^{1,r}(K)$, $r > 1$, for any $f \in U$, and the Malliavin derivative $Du(f)$ as a $L^2([0,T]) \otimes K$-valued random field has a continuous version on $U$. Suppose we have

$$
E \left[ \sup_{f \in \mathcal{U} \cap B^G_q} \left( \|u(f)\|_K + \|Du(f)\|_{L^2([0,T]) \otimes K} \right) \right] < \infty,
$$

$$
E \left[ \sup_{f \in \mathcal{U} \cap B^G_q} \|Du(f)\|_{L(G,K)}^q \right] < \infty,
$$

where $B^G_q := \{x \in G : \|x\|_G \leq \delta\}$, $1 \leq q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then $u(\eta) \in D^{1,r}(K)$, and

$$
D(u(\eta)) = \mathbb{D}(u(\eta))(D\eta) + (Du)(\eta). \tag{4.1}
$$

Proof. Let $\{\rho_i\}_{i=1}^\infty$ be an orthonormal basis in $K$. Set $\eta_m := \sum_{i=1}^m \langle \eta, e_i \rangle_G e_i$, and $u^n := \sum_{j=1}^n \langle u, \rho_j \rangle_K \rho_j$. Then by Lemma 3.2.3 in [11], we have

$$
D(u^n(\eta_m)) = \mathbb{D}(u^n)(\eta_m)(D\eta_m) + (Du^n)(\eta_m). \tag{4.2}
$$

Letting $n, m \to \infty$, we can show that the terms on the right of (4.2) converges to the corresponding terms in (4.1). Since the Malliavin derivative operator $D$ is closed, we conclude that $u(\eta) \in D^{1,r}(K)$ and (4.1) holds.

Next, we establish a precise product rule for the indefinite Skorohod integrals under very weak conditions, this formula is the main tool used in the proof of Theorem 3.1.

Proposition 4.1. Let $G$ be a real separable Hilbert space, Set $G^1 = G$, $G^2 = \mathbb{R}$. Consider processes of the form,

$$
X^i_t = X^i_0 + \int_0^t u^i_s dW_s + \int_0^t v^i_s ds, \quad i = 1, 2, \tag{4.3}
$$

where $u^i \in L^{1,2}_{1, loc}(G^i)$, $v^i$ is $G^i$-valued jointly measurable and $\int_0^T \|v^i_s\|_{G^i} ds < \infty$ a.s. $\omega \in \Omega$, $X^i \in L^{1,2}_{1, loc}(G^i)$ and $X^i_t \in D^{1,2}_{1, loc}(G^i)$ for all $t \in [0, T]$, $X^i$ and $u^i$ have versions which are $G^i$-valued continuous, then we have for any $t \in [0, T]$,

$$
X^1_t X^2_t = X^1_0 X^2_0 + \int_0^t X^1_s u^1_s dW_s + \int_0^t X^2_s v^1_s ds + \int_0^t X^1_s u^2_s dW_s + \int_0^t X^2_s v^2_s ds + \frac{1}{2} \int_0^t \langle \nabla X^1 \rangle_s u^2_s ds + \frac{1}{2} \int_0^t \langle \nabla X^2 \rangle_s u^1_s ds. \tag{4.4}
$$

Moreover, $X^1 X^2 \in L^{1,2}_{1, loc}(G)$ and

$$
(\nabla (X^1 X^2))_s = X^2_s (\nabla X^1)_s + X^1_s (\nabla X^2)_s. \tag{4.5}
$$

Remark 4.1. $X^i \in L^{1,2}_{1, loc}(G^i)$ implies that $X^i_t \in D^{1,2}_{1, loc}(G^i)$ for a.s. $t \in [0, T]$. Therefore, without the condition $X^i_t \in D^{1,2}_{1, loc}(G^i)$ for all $t \in [0, T]$, (4.4) holds only for a.s. $t \in [0, T]$. 

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Proof. We first use a localization argument to assume that \( u^i \in L^{1,2}(G^i) \), \( X^i \in L^{1,2}(G^i) \), \( \sup_{0 \leq t \leq T} \| X^i_t \|_{G^i} \leq k \), \( \sup_{0 \leq t \leq T} \| u^i_t \|_{G^i} \leq k \), \( \int_0^T \| u^i_s \|_{G^i} \, ds \leq k \), \( \text{for some fixed } k \in \mathbb{N} \). And also, for any fixed \( t > 0 \), let \( \{ 0 = t_0^i \leq t_1^i \leq \cdots \leq t_n^i = t \}_{n \geq 1} \) be a sequence of partitions of the interval \([0, t]\) such that \( r_n = \max_{0 \leq j \leq k_n} (t_{j+1}^n - t_j^n) \to 0 \) as \( n \to \infty \) and \( X^i(t^n_j) \in D^{1,2}(G^i) \) for each \( j = 1, \ldots, k_n \) and each \( n \in \mathbb{N} \). Then we note the identities:

\[
X^i_1 X^i_2 = X^i_0 X^i_0 + \sum_{j=0}^{k_n} X^i_2(t^n_j)(X^i_1(t^n_{j+1}) - X^i_1(t^n_j)) + \sum_{j=0}^{k_n} X^i_1(t^n_j)(X^i_2(t^n_{j+1}) - X^i_2(t^n_j))
\]

\[
+ \sum_{j=0}^{k_n} (X^i_1(t^n_{j+1}) - X^i_1(t^n_j))(X^i_2(t^n_{j+1}) - X^i_2(t^n_j)),
\] (4.6)

\[
X^i_1 X^i_2 = X^i_0 X^i_0 + \sum_{j=0}^{k_n} X^i_2(t^n_j+1)(X^i_1(t^n_{j+1}) - X^i_1(t^n_j)) + \sum_{j=0}^{k_n} X^i_1(t^n_j+1)(X^i_2(t^n_{j+1}) - X^i_2(t^n_j))
\]

\[
- \sum_{j=0}^{k_n} (X^i_1(t^n_{j+1}) - X^i_1(t^n_j))(X^i_2(t^n_{j+1}) - X^i_2(t^n_j)),
\] (4.7)

by the similar steps 1–5 as Theorem 3.2.2 in [11], we obtain the following formula from (4.6),

\[
X^i_1 X^i_2 = X^i_0 X^i_0 + \int_0^t X^i_1 u^i_1 \, dW_s + \int_0^t X^i_2 v^i_1 \, ds + \int_0^t X^i_1 u^i_2 \, dW_s + \int_0^t X^i_2 v^i_2 \, ds
\]

\[
+ \int_0^t u^i_1 u^i_2 \, ds + \int_0^t (D^- X^i_2) u^i_1 \, ds + \int_0^t (D^- X^i_1) u^i_2 \, ds.
\] (4.8)

Similarly, it follows from (4.7) that

\[
X^i_1 X^i_2 = X^i_0 X^i_0 + \int_0^t X^i_1 u^i_2 \, dW_s + \int_0^t X^i_2 v^i_1 \, ds + \int_0^t X^i_1 u^i_1 \, dW_s + \int_0^t X^i_2 v^i_2 \, ds
\]

\[
- \int_0^t u^i_1 u^i_2 \, ds + \int_0^t (D^+ X^i_2) u^i_1 \, ds + \int_0^t (D^+ X^i_1) u^i_2 \, ds.
\] (4.9)

Adding (4.8) and (4.9) and noticing that \( (\nabla X)_s := (D^+ X)_s + (D^- X)_s \), we obtain (4.4).

Obviously, \( X^i_1 X^i_2 \) is Malliavin differentiable and \( D_s(X^i_1 X^i_2) = X^i_1 D_s X^i_1 + X^i_2 D_s X^i_1 \), so it is easy to see that \( X^i_1 X^i_2 \in L^{1,2}(G) \).

\[
\int_0^T \sup_{s \leq t \leq (s+\frac{1}{n}) \wedge T} E \| X^i_1 D_s X^i_1 + X^i_1 D_s X^i_2 - X^i_2 (D^+ X^i)_s - X^i_1 (D^+ X^i)_s \|_{G} \, ds
\]

\[
= \int_0^T \sup_{s \leq t \leq (s+\frac{1}{n}) \wedge T} E \| X^i_1 (D_s X^i_1 - (D^+ X^i)_s) \|_{G} \, ds + \int_0^T \sup_{s \leq t \leq (s+\frac{1}{n}) \wedge T} E \| (X^i_2 - X^i_1) (D^+ X^i)_s \|_{G} \, ds
\]

\[
+ \int_0^T \sup_{s \leq t \leq (s+\frac{1}{n}) \wedge T} E \| X^i_1 (D_s X^i_2 - (D^+ X^i)_s) \|_{G} \, ds + \int_0^T \sup_{s \leq t \leq (s+\frac{1}{n}) \wedge T} E \| (X^i_1 - X^i_2) (D^+ X^i)_s \|_{G} \, ds
\]

\[
:= I_1 + I_2 + I_3 + I_4.
\]

Since \( \sup_{0 \leq t \leq T} \| X^i_t \|_{G^i} \leq k \) and \( X^i \in L^{1,2}(G^i) \), we have \( I_1 \to 0 \) as \( n \to \infty \).

\[
I_2 \leq E \int_0^T \sup_{s \leq t \leq (s+\frac{1}{n}) \wedge T} |X^i_t - X^i_s| \| (\nabla X^1)_s \|_{G} \, ds,
\]
by the continuity of $X^2$ and the dominated convergence theorem, it follows that $I_2 \to 0$ as $n \to \infty$. $I_3$ and $I_4$ also tend to zero by the same reason as $I_1$ and $I_2$. Therefore, we have

$$(D^+(X^1X^2))_s = X^2_s(D^+X^1)_s + X^1_s(D^+X^2)_s.$$ 

Similarly, we have

$$(D^-(X^1X^2))_s = X^2_s(D^-X^1)_s + X^1_s(D^-X^2)_s.$$ 

Hence, we obtain [4.5].

Let $Q(t) := \exp\{\sigma W(t)\}$. Consider the following system for each fixed $\omega \in \Omega$,

$$
\begin{cases}
    dv(t, f) = -v \, \hat{A}(v(t, f)) \, dt - Q(t) \, \hat{B}(v(t, f)) \, dt + \frac{1}{Q(t)} \, \hat{F}(Q(t)v(t, f), t) \, dt, & 0 < t \leq T,
    \\
v(0, f) = f & \text{in } \mathbb{W}.
\end{cases}
$$

(4.10)

The following lemma is taken from Proposition 4.1, 4.4 and 4.5 in [14].

**Lemma 4.2.** Assume that (F1) and (F2) are satisfied, then for any $f \in \mathbb{W}$, a.s. $\omega \in \Omega$, there exists a unique solution to (4.10). Furthermore, the solution map $(t, f, \omega) \mapsto v(t, f, \omega) \in \mathbb{W}$ is $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{W}) \otimes \mathcal{F}/\mathcal{B}(\mathbb{W})$-measurable and $\mathcal{F}_t$-adapted, and

$$
\sup_{t \in [0, T]} |v(t, f, \omega)|_{\mathbb{W}}^2 \leq C(T)|f|_{\mathbb{W}}^2.
$$

(4.11)

Moreover, for a.s. $\omega \in \Omega$, $\forall t \in [0, T]$, the map $v(t, \cdot, \omega) : \mathbb{W} \ni f \mapsto v(t, f, \omega) \in \mathbb{V}$ is continuously Fréchet differentiable on $\mathbb{W} \cap H^4(\Omega)$, and the following estimate holds

$$
\left\| \mathbb{D}v(t, f, \omega) (g) \right\|_{C([0, T]; \mathbb{V})} \leq C(T, \|Q\|_{\infty, T}, |f|_{\mathbb{W}}) |g|_{\mathbb{W}},
$$

(4.12)

where $\|Q\|_{\infty, T} := \sup_{0 \leq t \leq T} Q(t) < \infty$ for a.s. $\omega \in \Omega$.

By the classical Itô’s formula, we easily see that $Q(t)v(t, f)$ is a version of $u(t, f)$, where $u(t, f)$ is the solution of (3.1) with deterministic initial value $u(0) = f$. Therefore, it is natural to ask whether $Q(t)v(t, \xi)$ is a solution of (3.1) or not. In fact, the answer is affirmative. To illustrate this, by Lemma 1.1 and Proposition 4.1 it is necessary to show that $v(t, f) \in D^{1,2}_{loc}(\mathbb{V})$ and calculate $\mathbb{D}_t v(t, f)$ for $t \in [0, T]$. The uniqueness of solutions of (4.10) implies that

$$
v(t, f) = v^N(t, f) \quad \text{on } \Omega_N := \left\{ \omega : \sup_{0 \leq s \leq T} |W(s, \omega)| \leq N \right\},
$$

where $v^N(t, f)$ is the solution of an equation similar to (4.10) only replacing $Q(s)$ by

$$
Q^N(t) := \exp \left\{ \sigma \left[ (-N) \lor W(t) \land N \right] \right\}.
$$

Thus it suffice to prove that $v^N(t, f) \in D^{1,2}(\mathbb{V})$ for each fixed $N$. For this reason, we assume implicitly in the rest of this section that $Q = Q^N$. Noting that in this case

$$
\|Q\|_{\infty, T} := \sup_{0 \leq t \leq T} Q(t) < \exp(\sigma N),
$$

7
For any integer \( n \) the following finite dimensional equation

\[
D_r Q(s) = \begin{cases} 
\sigma \exp(\sigma W(s)) I_{[0,t]}(r) & \text{on } \Omega_N, \\
0 & \text{on } \Omega \setminus \Omega_N,
\end{cases}
\]

\[
\|DQ\|_{0,T} := \sup_{0 \leq r \leq T} \sup_{0 \leq s \leq T} |D_r Q(s)| \leq \sigma \exp(\sigma N).
\]

By Theorem 4.1.2 in [10], we see that the right of (4.16) is just the energy equation for Lemma 4.3.

We also need the following two lemmas.

**Lemma 4.3.** Assume that (F1) holds, \( v_n(t, f_n) \) is the solution of the equation (4.13), then

\[
\lim_{n \to \infty} E \int_0^T |v_n(t, f_n) - v(t, f)|^2_{\mathbb{W}} dt = 0,
\]

\[
\lim_{n \to \infty} E |v_n(t, f_n) - v(t, f)|^2_{\mathbb{W}} = 0, \quad \forall t \in [0, T].
\]

**Proof.** In fact, the proof of Proposition 4.1 in [14] implies that for a.s. \( \omega \in \Omega, \)

\[
v_n(t, f_n) \rightharpoonup v(t, f) \quad \text{weakly convergent in } \mathbb{W}, \quad \forall t \in [0, T].
\]

(4.8–4.9) in [14] imply that the following energy equation for \( v_n(t, f_n) \) holds:

\[
|v_n(t, f_n)|^2_{\mathbb{W}} = |f_n|^2_{\mathbb{W}} \left(e^{-\frac{2t}{\alpha}} + 2 \int_0^t K(v_n(s, f_n), s) e^{-\frac{2(s-t)}{\alpha}} ds\right),
\]

where

\[
K(v_n(s, f_n), s) := \left(\frac{\nu}{\alpha} \text{curl}(v_n(s, f_n)) + \text{curl}(F_Q(v_n(s, f_n), s)), \text{curl}(v_n(s, f_n) - \alpha \Delta v_n(s, f_n))\right).
\]

The convergence (4.13–4.15) in [14] also allow us to pass to the limit in (4.15) to obtain that

\[
\lim_{n \to \infty} |v_n(t, f_n)|^2_{\mathbb{W}} = |f|^2_{\mathbb{W}} e^{-\frac{2t}{\alpha}} + 2 \int_0^t K(v(s, f), s) e^{-\frac{2(s-t)}{\alpha}} ds.
\]

By Theorem 4.1.2 in [10], we see that the right of (4.16) is just the energy equation for \( v(t, f) \). Hence,

\[
\lim_{n \to \infty} |v_n(t, f_n)|^2_{\mathbb{W}} = |v(t, f)|^2_{\mathbb{W}},
\]

which together with (4.14) yield for a.s. \( \omega \in \Omega, \)

\[
v_n(t, f_n) \to v(t, f) \quad \text{strongly convergent in } \mathbb{W}, \forall t \in [0, T].
\]

Therefore, by (4.11) and the dominated convergence theorem, Lemma 4.3 follows immediately.

\[
\square
\]
Let \( f \in \mathbb{W} \), \( v(t, f) \) be the solution of (4.10). Consider the following random evolution equation:

\[
Y_r(t, f) = -\nu \int_0^t \tilde{A} Y_r(s, f) \, ds - \int_0^t D_r Q(s) \hat{B}(v(s, f), v(s, f)) \, ds \\
- \int_0^t Q(s) \hat{B}(Y_r(s, f), v(s, f)) \, ds - \int_0^t Q(s) \hat{B}(v(s, f), Y_r(s, f)) \, ds \\
+ \int_0^t D_r \left( \frac{1}{Q(s)} \right) \hat{F}(Q(s)v(s, f), s) \, ds \\
+ \int_0^t \frac{1}{Q(s)} \hat{D}(Q(s)v(s, f), s)v(s, f) D_r Q(s) \, ds \\
+ \int_0^t \hat{D}(Q(s)v(s, f), s) Y_r(s, f) \, ds. \tag{4.18}
\]

**Lemma 4.4.** Assume that (F1) and (F2) hold, then for each \( f \in \mathbb{W} \cap \mathbb{H}^4(\mathcal{O}), r \in [0, T] \), there exists a unique solution \( Y_r(\cdot, f) \in C([0, T]; \mathcal{V}) \cap L^\infty([0, T]; \mathcal{W}) \) to the equation (4.18). Moreover, the following estimates hold:

\[
\sup_{t \in [0, T]} \left| Y_r(t, f) \right|_V^2 \leq C(\|f\|_{\mathcal{W}}, T, N), \quad \forall r \in [0, T],
\]

\[
\sup_{t \in [0, T]} \left| Y_r(t, f) \right|_W^2 \leq C(\|f\|_{\mathcal{H}^4(\mathcal{O})}, \|f\|_{\mathcal{W}}, T, N), \quad \forall r \in [0, T].
\]

**Proof.** The proof of this lemma is similar to the proof of Proposition 4.1, Proposition 4.3 and Proposition 4.4 in [14], so we omit the details.

**Proposition 4.2.** Assume that (F1) and (F2) hold, then for each \( f \in \mathbb{W} \cap \mathbb{H}^4(\mathcal{O}), t \in [0, T] \), the solution \( v(t, f) \) of the equation (4.10) is Malliavin differentiable as a \( \mathcal{V} \)-valued random variable, and its Malliavin derivative \( D_r v(t, f) \) solves (4.18) for all \( t \in [0, T] \), a.s..

**Proof.** Let \( v_n(t, f_n) \) be the solution of the finite-dimensional random ordinary differential equation (4.11), it is known (see e.g. [11]) that \( v_n \) is Malliavin differentiable and the corresponding Malliavin derivative \( D_r v_n(t, f_n) \) satisfies the following random ODE:

\[
D_r v_n(t, f_n) = -\nu \int_0^t \tilde{A} D_r v_n(s, f_n) \, ds - \int_0^t D_r Q(s) \hat{B}(v_n(s, f_n), v_n(s, f_n)) \, ds \\
- \int_0^t Q(s) \hat{B}(D_r v_n(s, f_n), v_n(s, f_n)) \, ds - \int_0^t Q(s) \hat{B}(v_n(s, f_n), D_r v_n(s, f_n)) \, ds \\
+ \int_0^t D_r \left( \frac{1}{Q(s)} \right) \hat{F}(Q(s)v_n(s, f_n), s) \, ds \\
+ \int_0^t \frac{1}{Q(s)} \hat{D}(Q(s)v_n(s, f_n), s)v_n(s, f_n) D_r Q(s) \, ds \\
+ \int_0^t \hat{D}(Q(s)v_n(s, f_n), s) D_r v_n(s, f_n) \, ds, \tag{4.19}
\]

for all \( t \in [0, T] \). Since the Malliavin derivative operator \( D \) is closed, in view of Lemma 4.3 to prove the Proposition 4.2 it suffice to show that

\[
\lim_{n \to \infty} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \sup_{0 \leq r \leq T} \left| D_r v_n(t, f_n) - Y_r(t, f) \right|_V^2 \right] = 0.
\]
From (4.19) and (4.18), it follows that

\[
\frac{1}{2} |D_r v_n(t, f_n) - Y_r(t, f)|^2_Y \\
= -\nu \int_0^t \|D_r v_n(s, f_n) - Y_r(s, f)\|^2 ds \\
- \int_0^t D_r Q(s) \langle \hat{B}(v_n(s, f_n), v_n(s, f_n)) - \hat{B}(v(s, f), v(s, f)), D_r v_n(s, f_n) - Y_r(s, f) \rangle ds \\
- \int_0^t Q(s) \langle \hat{B}(D_r v_n(s, f_n), v_n(s, f_n)) - \hat{B}(Y_r(s, f), v(s, f)), D_r v_n(s, f_n) - Y_r(s, f) \rangle ds \\
- \int_0^t Q(s) \langle \hat{B}(v_n(s, f_n), D_r v_n(s, f_n)) - \hat{B}(v(s, f), Y_r(s, f)), D_r v_n(s, f_n) - Y_r(s, f) \rangle ds \\
+ \int_0^t D_r \left( \frac{1}{Q(s)} \right) \langle \hat{F}(Q(s)v_n(s, f_n), s) - \hat{F}(Q(s)v(s, f), s), D_r v_n(s, f_n) - Y_r(s, f) \rangle ds \\
+ \int_0^t \frac{D_r Q(s)}{Q(s)} \langle \nabla \hat{F}(Q(s)v_n(s, f_n), s)v_n(s, f_n) - \nabla \hat{F}(Q(s)v(s, f), s)v(s, f), \\
D_r v_n(s, f_n) - Y_r(s, f) \rangle ds \\
:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7. 
\]

Now we estimate these terms on the right of (4.20).

\[
I_2 = - \int_0^t D_r Q(s) \langle \hat{B}(v_n(s, f_n) - v(s, f), v_n(s, f_n)), D_r v_n(s, f_n) - Y_r(s, f) \rangle ds \\
- \int_0^t D_r Q(s) \langle \hat{B}(v(s, f), v_n(s, f_n) - v(s, f)), D_r v_n(s, f_n) - Y_r(s, f) \rangle ds \\
:= I_{2a} + I_{2b}. 
\]

By (2.13), we have

\[
|I_{2a}| \leq \|DQ\|_{\infty,T} \int_0^t \left| v_n(s, f_n) - v(s, f) \right|_W \left| v_n(s, f_n) \right|_W \left| D_r v_n(s, f_n) - Y_r(s, f) \right|_Y ds \\
\leq \frac{1}{2} \|DQ\|^2_{\infty,T} \int_0^t \left| v_n(s, f_n) - v(s, f) \right|_W^2 ds + \frac{1}{2} \int_0^t \left| D_r v_n(s, f_n) - Y_r(s, f) \right|_Y^2 \left| v_n(s, f_n) \right|_W^2 ds, \quad (4.22) \\
|I_{2b}| \leq \frac{1}{2} \|DQ\|^2_{\infty,T} \int_0^t \left| v_n(s, f_n) - v(s, f) \right|_W^2 ds + \frac{1}{2} \int_0^t \left| D_r v_n(s, f_n) - Y_r(s, f) \right|_Y^2 \left| v(s, f) \right|_W^2 ds. \quad (4.23) \\
\]

\[
I_3 = - \int_0^t Q(s) \langle \hat{B}(Y_r(s, f), v_n(s, f_n) - v(s, f)), D_r v_n(s, f_n) - Y_r(s, f) \rangle ds \\
- \int_0^t Q(s) \langle \hat{B}(D_r v_n(s, f_n) - Y_r(s, f)), v_n(s, f_n), D_r v_n(s, f_n) - Y_r(s, f) \rangle ds \\
:= I_{3a} + I_{3b}. 
\]
where

\[
|I_{3a}| \leq \|Q\|_{\infty,T} \int_0^t \left| Y_r(s, f) \right|_{\mathbb{W}} \left| D_r v_n(s, f) - Y_r(s, f) \right|_{\mathbb{V}} v_n(s, f) - v(s, f) \right|_{\mathbb{W}} ds
\]

\[
\leq \frac{1}{2} \|Q\|_{\infty,T}^2 \int_0^t \left| v_n(s, f) - v(s, f) \right|^2_{\mathbb{W}} ds + \frac{1}{2} \int_0^t \left| D_r v_n(s, f) - Y_r(s, f) \right|^2_{\mathbb{V}} Y_r(s, f) \right|_{\mathbb{W}}^2 ds,
\]

\[
|I_{3b}| \leq C \|Q\|_{\infty,T} \int_0^t \left| D_r v_n(s, f) - Y_r(s, f) \right|^2_{\mathbb{V}} v_n(s, f) \right|_{\mathbb{W}} ds.
\]

(4.25)

In the same way, we have

\[
I_4 = - \int_0^t Q(s) \left( \langle \hat{B}(v_n(s, f) - v(s, f), Y_r(s, f)), D_r v_n(s, f) - Y_r(s, f) \rangle \right) ds
\]

\[
- \int_0^t Q(s) \left( \langle \hat{B}(v_n(s, f), D_r v_n(s, f) - Y_r(s, f)), D_r v_n(s, f) - Y_r(s, f) \rangle \right) ds
\]

\[
I_{4a} + I_{4b},
\]

and

\[
|I_{4a}| \leq \|Q\|_{\infty,T} \int_0^t \left| v_n(s, f) - v(s, f) \right|_{\mathbb{W}} \left| D_r v_n(s, f) - Y_r(s, f) \right|_{\mathbb{V}} Y_r(s, f) \right|_{\mathbb{W}} ds,
\]

(4.28)

obviously, \(I_{4a}\) has the same estimate as \(I_{3a}\), and \(I_{4b} = 0\) due to (2.3). Note that

\[
\left| D_r \left( \frac{1}{Q(s)} \right) \right| |Q(s)| = \left| \frac{D_r Q(s)}{Q(s)} \right| \leq \sigma,
\]

thus, by (F1) we have

\[
|I_5| \leq C \int_0^t \left| D_r \left( \frac{1}{Q(t)} \right) \right| |Q(t)| \left| v_n(s, f) - v(s, f) \right|_{\mathbb{V}} \left| D_r v_n(s, f) - Y_r(s, f) \right|_{\mathbb{V}} ds
\]

\[
\leq C \int_0^t \left| v_n(s, f) - v(s, f) \right|^2_{\mathbb{V}} ds + C \int_0^t \left| D_r v_n(s, f) - Y_r(s, f) \right|^2_{\mathbb{V}} ds.
\]

(4.29)

The term \(I_6\) can be bounded as follows:

\[
I_6 = \int_0^t \frac{D_r Q(s)}{Q(s)} \left( \hat{D}(Q(s) v_n(s, f), s) (v_n(s, f) - v(s, f)), D_r v_n(s, f) - Y_r(s, f) \right) ds
\]

\[
+ \int_0^t \frac{D_r Q(s)}{Q(s)} \left( \left[ \hat{D}(Q(s) v_n(s, f), s) - \hat{D}(Q(s) v(s, f), s) \right] v(s, f),
\right.
\]

\[
\left. D_r v_n(s, f) - Y_r(s, f) \right)_{\mathbb{V}} ds
\]

\[
:= I_{6a} + I_{6b}.
\]

(4.30)

(F1) and (F2) imply that

\[
\left\| \hat{D}F(Q(s)v(s, f), s) \right\|_{L(V)} \leq C, \quad \forall s \in [0, T].
\]
Hence,

\[
|I_{6a}| \leq C \int_0^t \left| \frac{D_r Q(s)}{Q(s)} \right| |v_n(s, f_n) - v(s, f)|_V |D_r v_n(s, f_n) - Y_r(s, f)|_V ds,
\]

(4.31)

so \( I_{6a} \) has the same estimate as \( I_5 \).

\[
|I_{6b}| \leq \int_0^t \Psi(n, s) \times |v(s, f)|_V |D_r v_n(s, f_n) - Y_r(s, f)|_V ds
\leq \frac{1}{2} \int_0^t \Psi^2(n, s) ds + \frac{1}{2} \int_0^t |D_r v_n(s, f_n) - Y_r(s, f)|_V^2 |v(s, f)|_V^2 ds,
\]

(4.32)

where

\[
\Psi(n, s) = \left\| \mathbb{D} \hat{F}(Q(s) v_n(s, f_n), v) - \mathbb{D} \hat{F}(Q(s) v(s, f), v) \right\|_{L(V)}.
\]

Similarly,

\[
I_7 = \int_0^t \left( \mathbb{D} \hat{F}(Q(s) v_n(s, f_n), s) (D_r v_n(s, f_n) - Y_r(s, f)), D_r v_n(s, f_n) - Y_r(s, f) \right)_V ds
+ \int_0^t \left( \left[ \mathbb{D} \hat{F}(Q(s) v_n(s, f_n), s) - \mathbb{D} \hat{F}(Q(s) v(s, f), s) \right] Y_r(s, f),
D_r v_n(s, f_n) - Y_r(s, f) \right)_V ds
:= I_{7a} + I_{7b},
\]

where

\[
|I_{7a}| \leq C \int_0^t |D_r v_n(s, f_n) - Y_r(s, f)|_V^2 ds,
\]

(4.34)

\[
|I_{7b}| \leq \int_0^t \Psi(n, s) \times |Y_r(s, f)|_V |D_r v_n(s, f_n) - Y_r(s, f)|_V ds
\leq \frac{1}{2} \int_0^t \Psi^2(n, s) ds + \frac{1}{2} \int_0^t |D_r v_n(s, f_n) - Y_r(s, f)|_V^2 |Y_r(s, f)|_V^2 ds.
\]

(4.35)

Substituting the above estimates (4.21-4.35) into (4.20) gives

\[
\frac{1}{2} |D_r v_n(t, f_n) - Y_r(t, f)|_V^2 + \nu \int_0^t |D_r v_n(s, f_n) - Y_r(s, f)|_V^2 ds
\leq \int_0^t \Psi^2(n, s) ds + \left( C + \|DQ\|_{\infty, T}^2 \|Q\|_{\infty, T}^2 \right) \times \int_0^t |v_n(s, f_n) - v(s, f)|_W^2 ds
\]

\[
+ C \int_0^t |D_r v_n(s, f_n) - Y_r(s, f)|_V^2 \left( |v_n(t, f_n)|_W^2 + |v(t, f)|_W^2 + |Y_r(t, f)|_W^2 + \|Q\|_{\infty, T} |v_n(t, f_n)|_W + |v(t, f)|_V^2 + |Y_r(t, f)|_V^2 + 1 \right) ds.
\]

Applying Gronwall inequality, using Lemma 4.4 and Lemma 4.2 we obtain

\[
\sup_{0 \leq t \leq T} |D_r v_n(t, f_n) - Y_r(t, f)|_V^2 + 2\nu \int_0^T |D_r v_n(s, f_n) - Y_r(s, f)|_V^2 ds
\]
\[
\leq C(|f|_{H^4}, |f|_{\mathcal{W}}, T, N) \times \left( \int_0^T \Psi^2(n, s) \, ds + C(N) \int_0^T |v_n(s, f_n) - v(s, f)|^2_{\mathcal{W}} \, ds \right).
\]

Due to Lemma 4.3, 4.17 and the continuity of \(D F\) in (P2), applying the dominated convergence theorem, we conclude

\[
\lim_{n \to \infty} E \left[ \sup_{0 \leq r \leq T} \sup_{0 \leq t \leq T} |D_r v_n(t, f_n) - Y_r(t, f)|^2_{V} \right] = 0.
\]

\[\square\]

**Remark 4.2.** In the same way, we can obtain the fact: for any sequence \(h, h_n \in \mathcal{W} \cap H^4(O),\) and \(h_n \to h\) in \(\mathcal{W}\)-norm as \(n \to \infty,\) we have for a.s. \(\omega \in \Omega,\)

\[
\lim_{n \to \infty} \sup_{0 \leq r \leq T} \sup_{0 \leq t \leq T} |Y_r(t, h_n) - Y_r(t, h)|^2_{V} = 0.
\]

Owing to this, it follows that for each \(t \in [0, T],\)

\[
\lim_{n \to \infty} |D v(t, h_n) - D v(t, h)|^2_{L^2([0, T]) \otimes V} = \lim_{n \to \infty} \int_0^T |D_r v(t, h_n) - D_r v(t, h)|^2_{V} \, dr
\]
\[
\leq T \lim_{n \to \infty} \sup_{0 \leq r \leq T} \sup_{0 \leq t \leq T} |D_r v(t, h_n) - D_r v(t, h)|^2_{V}
\]
\[
= 0.
\]

Therefore, the \(L^2([0, T]) \otimes V\)-valued random field \(D v(t, h)\) has a continuous version.

**Proof of Theorem 3.1.** Existence. It follows from Lemma 4.1, Lemma 4.2, Proposition 4.2, Lemma 4.4 and Remark 4.2 that \(v(t, \xi) \in D_{v}^{1,2}(V)\) and

\[
D_s v(t, \xi) = D v(t, \xi)(D_s \xi) + (D_s v(t))(\xi),
\]

for every \(t \in [0, T].\) Since each term of (4.36) is continuous in \(t,\) and \((D_s v(t))(\xi) = 0\) for any \(t \leq s,\) by a localization argument and the dominated convergence theorem, we get \(v(\cdot, \xi) \in L_{1, loc}^{1,2}(V).\) On the other hand, obviously, \(Q \in L_1^{1,2}, (\nabla Q)_s = \sigma Q(s)\) and

\[
Q(t) = 1 + \int_0^t \sigma Q(s) \, dW(s) + \int_0^t \frac{1}{2} \sigma^2 Q\, ds.
\]

Note that

\[
v(t, \xi) = \xi + \int_0^t \left[ - \nu \widehat{A} v(s, \xi) - Q(t) \widehat{B}(v(s, \xi)) + Q^{-1}(s) \widehat{F}(Q(s)v(s, \xi), s) \right] dt, \quad t \in [0, T],
\]

therefore, we can apply Proposition 4.1 to obtain that \(v(t, \xi)Q(t)\) is a solution of (3.1).

Uniqueness. Let \(u\) be a solution of (3.1), define the process \(v(t) = u(t)Q^{-1}(t), t \in [0, T].\) By (3.2), Proposition 4.1 and the continuity of \(u,\) we immediately get that \(v\) satisfies the equation (4.37) for a.s. \(\omega \in \Omega.\) Now uniqueness of solutions for the equation (3.1) follows easily from the uniqueness of solutions for the equation (4.37).

\[\square\]
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