"SINGULARITIES" IN SPACETIMES WITH DIVERGING HIGHER-ORDER CURVATURE INVARIANTS

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After reviewing the definitions of classical and quantum singularities, it is shown by example that if zeroth-order curvature invariants are regular, a diverging higher-order curvature invariant does not necessarily imply the existence of a classical or a quantum singularity.

1. Introduction

A spacetime \((M, g)\) is a smooth, \(C^\infty\), paracompact, connected Hausdorff manifold \(M\) with a Lorentzian metric \(g\). Here we present three spacetimes \(^1\)–\(^3\) with regular zeroth-order curvature invariants but diverging higher-order invariants and illustrate with one spacetime \(^3\) (the Musgrave-Lake ST) that such a divergence does not necessarily foretell the existence of a "singularity" using the usual definitions.

2. Singularity Definitions

2.1. Classical Singularities

A classical singularity is indicated by incomplete geodesics or incomplete paths of bounded acceleration\(^4,5\) in a maximal spacetime. Since, by definition, a spacetime is smooth, all irregular points (singularities) have been excised; a singular point is a boundary point of the spacetime. There are 3 types of singularities\(^6\) quasi-regular (a mild, topological type), non-scalar curvature (diverging tidal forces on curves ending at the singularity; finite tidal forces on some nearby curves) and scalar curvature (diverging scalars – usually only consider \(C^0\) scalar polynomial (s.p.) invariants).

2.2. Quantum Singularities

A spacetime is QM (quantum-mechanically) nonsingular if the evolution of a test scalar wave packet, representing the quantum particle, is uniquely determined by the initial wave packet, manifold and metric, without having to put boundary conditions at the singularity\(^7\). Technically, a static ST is QM-singular if the spatial portion of the relevant wave operator, here the Klein-Gordon operator, is not essentially self-adjoint\(^8\) on \(C^\infty_0(\Sigma)\) in \(L^2(\Sigma)\) where \(\Sigma\) is a spatial slice.
3. Spacetimes with Diverging Higher-Order Curvature Invariants

As Musgrave and Lake say, “curvature invariants alone are not sufficient to probe the ‘physics’ of the solution.”

3.1. Kinnersley ‘photon rocket’

Bonnor\(^1\) analyzed the Kinnersley\(^9\) ‘photon rocket’ which has two-metric functions, the mass \(m = m(u)\) and the acceleration \(a = a(u)\), both functions of the radial null coordinate \(u\). He found that \(a(u)\) does not enter any zeroth-order s.p. curvature scalars, but it does enter into differential invariants. Thus, a singular acceleration ‘singularity’ would not show up on regular curvature invariants but are crucial for an adequate physical picture and predict a true physical singularity since they indicate incomplete, inextendible null geodesics.

3.2. Siklos Whimper Spacetimes

Siklos\(^2\) in 1976 considered the so-called “whimper” STs (“not with a bang, but with a whimper” as the poet T.S. Elliot wrote\(^10\)). These STs are geodesically incomplete and inextendible and thus classically singular. They possess \(C^0\) non-scalar curvature singularities; all zeroth-order s.p. invariants are regular. However, Siklos did find these STs to have diverging first-order curvature invariants.

3.3. Musgrave-Lake Spacetimes

Musgrave-Lake ST\(^3\) are static and spherically-symmetric with metric

\[
\text{d}s^2 = -(1 + r^{n+3/2})\text{d}t^2 + (1 + r^{n+3/2})\text{d}r^2 + r^2\text{d}\theta^2 + r^2\sin^2(\theta)\text{d}\phi^2
\]

(1)

where the coordinates have their usual ranges and the parameter \(n = 1, 2, 3, 4\). They have an anisotropic matter distribution and obey all energy conditions (weak, strong, dominate). Physically, they can be interpreted as a “thick shell” with a density and pressure that approach zero as \(r \to 0\) or as \(r \to \infty\). All \(C^0\) s.p. curvature invariants vanish at \(r = 0\). Differential invariants up to order \((n-1)\) are regular at \(r = 0\) while \(n^{th}\) order differential invariants diverge at \(r = 0\). However, there are no incomplete geodesics and hence no classical singularity in the usual sense. Observers following timelike and null geodesic paths feel nothing untoward at \(r = 0\). In particular, tidal forces do not diverge.

The quantum singularity structure of the Musgrave-Lake spacetime was also studied. The massive Klein-Gordon equation was solved, variables separated, and radial solutions approximated near \(r = 0\). Both radial solutions are square integrable, but

\(^*\) Actually, for the \(n = 1\) case, the tangential pressure is not \(C^1\) and this can be considered the physical reason why the first-order differential invariants diverge.\(^4\)
one does not form a proper solution of the three dimensional wave equation for the spatial wave function. This can be seen by combining each radial solution with the \( l = 0, m = 0 \) angular spherical harmonic, integrating over a spherical volume, and finding a nonzero value for one integral. Thus, the Musgrave-Lake spacetime is quantum mechanically non-singular.

4. Conclusions

Spacetimes with higher-order diverging invariants have interesting "singularity" structure. Further study of these and other cases is warrented.

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References

1. W.B. Bonnor, Class. Quantum Grav. 13, 277 (1996).
2. S.T.C. Siklos, "Singualrities, Invariants and Cosmology", PhD Thesis, unpublished (1976).
3. P. Musgrave and K. Lake Class. Quantum Grav. 12, L39 (1995).
4. S.W. Hawking and G.F.R. Ellis, The Large-Scale Structure of Spacetime (Cambridge University Press, 1973).
5. R. Geroch, Ann. Phys. 48, 526 (1968)
6. G.F.R. Ellis and B.G. Schmidt, Gen. Rel. Grav. 8, 915 (1977).
7. G.T. Horowitz and D. Marolf, Phys. Rev. D 52, 5670 (1995).
8. M. Reed and B. Simon, Functional Analysis (Academic Press, 1972); M. Reed and B. Simon, Fourier Analysis and Self-Adjointness (Academic Press, 1972).
9. W. Kinnersley, Phys. Rev. 186, 1335 (1968).
10. T.S. Eliot, "The Hollow Men," in e.g. The Wasteland and Other Poems (New York, Penguin, 1998).