Generalized Mannheim Curves in Minkowski space-time $E^4_1$

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Abstract

In this paper, the definition of generalized spacelike Mannheim curve in Minkowski space-time $E^4_1$ is given. The necessary and sufficient conditions for the generalized spacelike Mannheim curve are obtained. Also, some characterizations of Mannheim curve are given.

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1 Introduction

The curves are a fundamental structure of differential geometry. An increasing interest of the theory of curves makes a development of special curves to be examined. A way to classification and characterization of curves is the relationship between the Frenet vectors of the curves. For example, Saint Venant proposed the question whether upon the surface generated by the principal normal of a curve, a second curve can exist which has for its principal normal of the given curve in 1845. This question was answered by Bertrand in 1850. He showed that a necessary and sufficient condition for the existence of such a second curve is that a linear relationship with constant coefficients exists between the first and second curvatures of the given original curve. The pairs of curves of this kind have been called Bertrand partner curves or more commonly Bertrand curves [10], [14], [2]. There are many works related with Bertrand curves in the Euclidean space and Minkowski space, [15]–[3]. Also, generalized Bertrand curves in Euclidean 4-space are defined and characterized in [6]. Another kind of associated curve have been called Mannheim curve and Mannheim partner curve. The notion of Mannheim curves was discovered by A. Mannheim in 1878. These curves in Euclidean 3-space are characterized in terms of the curvature and torsion as follows: A space curve is a Mannheim curve if and only if its
curvature $\kappa$ and torsion $\tau$ satisfy the relation

$$\kappa(s) = \alpha(\kappa^2(s) + \tau^2(s))$$

for some constant $\alpha$. The articles concerning Mannheim curves are rather few. In [13], a remarkable class of Mannheim curves is studied. General Mannheim curves in the Euclidean 3-space are obtained in [11]. Mannheim partner curves in Euclidean 3-space and Minkowski 3-space are studied and the necessary and sufficient conditions for the Mannheim partner curves are obtained in [5], [8]. Recently, Mannheim curves are generalized and some characterizations and examples of generalized Mannheim curves in Euclidean 4-space $E^4$ are given by [7].

In this paper, we study the generalized spacelike Mannheim partner curves in 4-dimensional Minkowski space-time. We will give the necessary and sufficient conditions for the generalized spacelike Mannheim partner curves.

2 Preliminaries

The basic concepts of the theory of curves in Minkowski space-time $E^4$ are briefly presented in this section. A more complete elementary treatment can be found in [1]. Minkowski space-time $E^4$ is an Euclidean space provided with the standard flat metric given by

$$\langle \, , \rangle = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

where $(x_1, x_2, x_3, x_4)$ is a rectangular coordinate system in $E^4$.

Since $\langle \, , \rangle$ is an indefinite metric, recall that a vector $v \in E^4_1$ can have one of the three causal characters: it can be spacelike if $\langle v, v \rangle > 0$ or $v = 0$, timelike if $\langle v, v \rangle < 0$ and null (lightlike) if $\langle v, v \rangle = 0$ and $v \neq 0$. Similarly, an arbitrary curve $c = c(s)$ in $E^4$ can locally be spacelike, timelike or null (lightlike) if all of its velocity vectors $c'(s)$ are, respectively, spacelike, timelike or null. The norm of $v \in E^4_1$ is given by $\|v\| = \sqrt{|\langle v, v \rangle|}$. If $\|c'(s)\| = \sqrt{|\langle c'(s), c'(s) \rangle|} \neq 0$ for all $s \in L$, then $C$ is a regular curve in $E^4_1$. A spacelike (timelike) regular curve $C$ is parameterized by arc-length parameter $s$ which is given by $c : L \to E^4_1$, then the tangent vector $c'(s)$ along $C$ has unit length, that is,

$$\langle c'(s), c'(s) \rangle = 1, \quad (\langle c(s), c(s) \rangle = -1)$$

for all $s \in L$.

Hereafter, curves are considered spacelike and regular $C^\infty$ curves in $E^4_1$. Let $e_1(s) = c'(s)$ for all $s \in L$, then the vector field $e_1(s)$ is spacelike and it is called spacelike unit tangent vector field on $C$.

The spacelike curve $C$ is called special spacelike Frenet curve if there exist three smooth functions $k_1, k_2, k_3$ on $C$ and smooth non-null frame field $\{e_1, e_2, e_3, e_4\}$ along the curve $C$. Also, the functions $k_1, k_2,$ and $k_3$ are called the first, the second, and the third curvature function on $C$, respectively. For the $C^\infty$ special spacelike Frenet curve $C$, the following Frenet formula is hold
Due to characters of Frenet vectors of the spacelike curve $C$, $\mu_i$ ($1 \leq i \leq 3$) are defined as in the following three subcases; **Case 1**: If $e_4$ is timelike, then

$$\mu_1 = \mu_2 = -1, \quad \mu_3 = 1$$

where $e_1, e_2, e_3$ and $e_4$ are mutually orthogonal vector fields satisfying equations

$$\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = \langle e_3, e_3 \rangle = 1, \quad \langle e_4, e_4 \rangle = -1.$$

**Case 2**: If $e_3$ is timelike, then $\mu_i$, $1 \leq i \leq 3$ are

$$\mu_1 = -1, \quad \mu_2 = \mu_3 = 1$$

where $e_1, e_2, e_3$ and $e_4$ are mutually orthogonal vector fields satisfying equations

$$\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = \langle e_4, e_4 \rangle = 1, \quad \langle e_3, e_3 \rangle = -1.$$

**Case 3**: If $e_2$ is timelike, then $\mu_i$, $1 \leq i \leq 3$ are

$$\mu_1 = \mu_2 = 1, \quad \mu_3 = -1$$

where $e_1, e_2, e_3$ and $e_4$ are mutually orthogonal vector fields satisfying equations

$$\langle e_1, e_1 \rangle = \langle e_4, e_4 \rangle = \langle e_3, e_3 \rangle = 1, \quad \langle e_2, e_2 \rangle = -1.$$

For $s \in L$, the non-null frame field $\{e_1, e_2, e_3, e_4\}$ and curvature functions $k_1$ and $k_2$ are determined as follows

1. **$1^{st}$ step** $e_1 (s) = e' (s)$
2. **$2^{nd}$ step** $k_1 (s) = ||e'_1 (s)|| > 0$
   $e_2 (s) = \frac{1}{k_1 (s)} e'_1 (s)$
3. **$3^{rd}$ step** $k_2 (s) = ||e'_2 (s) - \mu_1 k_1 (s) e_1 (s)|| > 0$
   $e_3 (s) = \frac{1}{k_2 (s)} (e'_2 (s) - \mu_1 k_1 (s) e_1 (s))$
4. **$4^{th}$ step** $e_4 (s) = \varepsilon \frac{1}{||e'_3 (s) - \mu_2 k_2 (s) e_2 (s)||} (e'_3 (s) - \mu_2 k_2 (s) e_2 (s))$

where $\varepsilon$ is taken $-1$ or $+1$ to make $+1$ the determinant of $\{e_1, e_2, e_3, e_4\}$, that is, the non-null orthonormal frame field is of positive orientation. The function $k_3$ is determined by

$$k_3 (s) = \langle e'_3 (s), e_4 (s) \rangle \neq 0.$$

So the function $k_3$ never vanishes.
In order to make sure that the spacelike curve $C$ is a special spacelike Frenet curve, above steps must be checked, from 1$^{st}$ step to 4$^{th}$ step, for $s \in L$.

At each point of spacelike curve $C$, a line $\ell_1$ in the direction of $e_2$ is called the first normal line, a line $\ell_2$ in the direction of $e_3$ is called the second normal line and a line $\ell_3$ in the direction of $e_4$ is called the third normal line.

Note that, according to three different case of spacelike curve $C$, $\ell_3$, $\ell_2$ and $\ell_1$ can be timelike, respectively, which are called second binormal, first binormal and principal normal line at each point of the spacelike curve $C$.

3 Generalized spacelike Mannheim curves in $E_4^1$

In $E_4^1$ the Bertrand curves and Mannheim curves are generalized by [6] and [7], respectively. In these regards, we have investigate generalization of spacelike Mannheim curves Minkowski space in $E_4^1$.

**Definition 3.1** A special spacelike curve $C$ in $E_4^1$ is a generalized spacelike Mannheim curve if there exists a special spacelike Frenet curve $C^*$ in $E_4^1$ such that the first normal line at each of $C$ is included in the plane generated by the second normal line and the third normal line of $C^*$ at the corresponding point under $\phi$. Here $\phi$ is a bijection from $C$ to $C^*$. The curve $C^*$ is called the generalized spacelike Mannheim mate curve of $C$.

By the definition, a generalized Mannheim mate curve $C^*$ is given by

$$c^*(s) = c(s) + \alpha(s)e_2(s), \ s \in L$$  \hspace{1cm} (3.1)

where $\alpha$ is a smooth function on $L$. Generally, the parameter $s$ isn’t an arc-length of $C^*$. Let $s^*$ be the arc-length of $C^*$ defined by

$$s^* = \int_0^s \left\| \frac{dc^*(s)}{ds} \right\| ds.$$

If a smooth function $f : L \rightarrow L$ is given by $f(s) = s^*$, then

$$\frac{dc^*(s)}{ds} = e_1(s) + \alpha'(s)e_2(s) + \alpha(s)\mu_1k_1(s)e_1(s) + \alpha(s)k_2(s)e_3(s)$$

$$= (1 + \mu_1\alpha(s)k_1(s))e_1(s) + \alpha'(s)e_2(s) + \alpha(s)k_2(s)e_3(s).$$

for $\forall s \in L$. Thus, we have

$$f'(s) = \frac{ds^*}{ds} = \left\| \frac{dc^*(s)}{ds} \right\| = \sqrt{(1 + \mu_1\alpha(s)k_1(s))^2 + \varepsilon_2(\alpha'(s))^2 + \varepsilon_3(\alpha(s)k_2(s))^2}$$

where $\varepsilon_i = \begin{cases} -1 , & e_i \text{ is timelike} \\ 1 , & e_i \text{ is spacelike} \end{cases}$, for $2 \leq i \leq 4$.

This means that, in the Case 1, $e_4$ is timelike and
\[ f' (s) = \sqrt{(1 - \alpha (s) k_1 (s))^2 + (\alpha' (s))^2 + (\alpha (s) k_2 (s))^2} \]

or in the Case 2, \( e_3 \) is timelike and

\[ f' (s) = \sqrt{(1 - \alpha (s) k_1 (s))^2 + (\alpha' (s))^2 - (\alpha (s) k_2 (s))^2} \]

or in the Case 3, \( e_2 \) is timelike and

\[ f' (s) = \sqrt{(1 + \alpha (s) k_1 (s))^2 - (\alpha' (s))^2 + (\alpha (s) k_2 (s))^2} \].

The spacelike curve \( C^* \) with arc-length parameter \( s^* \) is

\[ c^* : L^* \to E^4_1 \]
\[ s^* \to c^* (s^*) \).

For a bijection \( \phi : C \to C^* \) defined by \( \phi (c (s)) = c^* (f (s)) \), the reparametrization of \( C^* \) is

\[ c^* (f (s)) = c (s) + \alpha (s) e_2 (s) \]

where \( \alpha \) is a smooth function on \( L \).

**Theorem 3.1** If a special spacelike Frenet curve \( C \) in \( E^4_1 \) is a generalized spacelike Mannheim curve, then the first curvature function \( k_1 \) and the second curvature function \( k_2 \) of \( C \) satisfy the equality

\[ k_1 (s) = -\alpha (\mu_1 k_1^2 (s) + \mu_2 k_2^2 (s)) , \ s \in L \] (3.2)

where \( \alpha \) is a constant number and \( \mu_1 = \mu_2 = -1 \) when \( e_4 \) is timelike or \( \mu_1 = -1 , \mu_2 = 1 \) when \( e_3 \) is timelike or \( \mu_1 = \mu_2 = 1 \) when \( e_2 \) is timelike.

**Proof.** Let \( C \) be a generalized spacelike Mannheim curve and \( C^* \) be the generalized spacelike Mannheim mate curve of \( C \) with the diagram:

\[ f : L \to L^* \]
\[ \phi : E^4_1 \to E^4_1 . \]

A smooth function \( f \) is defined by \( f (s) = \int \| \frac{dc'}{ds} (s) \| ds = s^* \) and \( s^* \) is the arc-length parameter of \( C^* \). Also \( \phi \) is a bijection which is defined by \( \phi (c (s)) = c^* (f (s)) \). Thus, the spacelike curve \( C^* \) is reparametrized by

\[ c^* (f (s)) = c (s) + \alpha (s) e_2 (s) \] (3.3)
Thus, if we investigate the special cases separately, then we have

\[
\frac{d}{ds} \left( \langle \mathbf{e}_1 (f(s)), \mathbf{e}_3 (f(s)) \rangle \right) = 0
\]

is obtained.
On the other hand, since the first normal line at the each point of \( C \) is lying in the plane generated by the second normal line and the third normal line of \( C^* \) at the corresponding points under bijection \( \phi \), the vector field \( \mathbf{e}_2 (s) \) is given by

\[
\mathbf{e}_2 (s) = g (s) \mathbf{e}_1 (f(s)) + h (s) \mathbf{e}_3^* (f(s))
\]

where \( g \) and \( h \) are some smooth functions on \( L \). If we take into consideration

\[
\langle \mathbf{e}_1 (f(s)), g (s) \mathbf{e}_2^* (f(s)) + h (s) \mathbf{e}_3^* (f(s)) \rangle = 0
\]

and the equation (3.4), then we have \( \alpha' (s) = 0 \). So we rewrite the equation (3.4) as

\[
\frac{d}{ds} \left( \langle \mathbf{e}_1 (f(s)), \mathbf{e}_3 (f(s)) \rangle + \alpha \mathbf{k}_1 (s) \right) = 0
\]

that is,

\[
\mathbf{e}_1 (f(s)) = \frac{\alpha k_1 (s)}{f'(s)} \mathbf{e}_1 (s) + \frac{\alpha k_2 (s)}{f'(s)} \mathbf{e}_3 (s)
\]

where

\[
f'(s) = \sqrt{\left( 1 + \mu_1 \alpha k_1 (s) \right)^2 + \varepsilon_3 (\alpha k_2 (s))^2}, \quad \varepsilon_3 = \begin{cases} -1, & \text{e}_3 \text{ is timelike,} \\ 1, & \text{e}_3 \text{ is spacelike.} \end{cases}
\]

By taking differentiation both sides of the equations (3.5) with respect to \( s \),

\[
f'(s) k_1^* (f(s)) \mathbf{e}_2^* (f(s)) = \left( \frac{1+\mu_1 \alpha k_1 (s)}{f'(s)} \right) \mathbf{e}_1 (s) + \left( \frac{\alpha k_2 (s)}{f'(s)} \right) \mathbf{e}_3 (s) \]

is obtained for \( s \in L \). Since

\[
\langle \mathbf{e}_2^* (f(s)), g (s) \mathbf{e}_2^* (f(s)) + h (s) \mathbf{e}_3^* (f(s)) \rangle = 0,
\]

then in the equation (3.6) the coefficient of \( \mathbf{e}_2 (s) \) vanishes, that is,

\[
(1+\mu_1 \alpha k_1 (s)) k_1 (s) + \mu_2 \alpha k_2 (s) = 0.
\]

Thus, \( k_1 (s) = -\alpha (\mu_1 k_1^2 (s) + \mu_2 k_2^2 (s)) \) is satisfied. This completes the proof.

If we investigate the special cases separately, then we have

- in the Case 1:
  \[
k_1 (s) = \alpha (k_1^2 (s) + k_2^2 (s)),
\]

- in the Case 2:
  \[
k_1 (s) = \alpha (k_1^2 (s) - k_2^2 (s)),
\]

- in the Case 3:
  \[
k_1 (s) = -\alpha (k_1^2 (s) + k_2^2 (s)).
\]
Theorem 3.2 Let $C$ be a special spacelike Frenet curve in $E^4_1$ whose curvature functions $k_1$ and $k_2$ are non-constant functions and satisfy the equality $k_1 (s) = -\alpha (\mu_1 k_1^2 (s) + \mu_2 k_2^2 (s))$, where $\alpha$ is non-zero constant, for all $s \in L$. If the spacelike curve $C^*$ given by

$$c^* (s) = c (s) + \alpha e_2 (s)$$

is a special spacelike Frenet curve, then $C^*$ is a generalized spacelike Mannheim mate curve of $C$.

Proof. The arc-length parameter of $C^*$ is defined by

$$s^* = \int_0^s \left\| \frac{dc^* (t)}{ds} \right\| dt$$

for all $s \in L$. Under the assumption of

$$k_1 (s) = -\alpha (\mu_1 k_1^2 (s) + \mu_2 k_2^2 (s))$$

and after calculations for all cases, separately, we obtain

- in the Case 1: $f' (s) = \sqrt{1 - \alpha k_1 (s)}$,
- in the Case 2: $f' (s) = \sqrt{1 - \alpha k_1 (s)}$,
- in the Case 3: $f' (s) = \sqrt{1 + \alpha k_1 (s)}$.

Thus, we can generalize

$$f' (s) = \sqrt{1 + \mu_1 \alpha k_1 (s)}$$

for all $s \in L$.

By differentiating the equation $c^* (f (s)) = c (s) + \alpha e_2 (s)$ with respect to $s$, it is seen that

$$f' (s) e_1^* (f (s)) = (1 + \mu_1 \alpha k_1 (s)) e_1 (s) + \alpha k_2 (s) e_3 (s).$$

So, it is seen that

$$e_1^* (f (s)) = \left( \frac{1 + \mu_1 \alpha k_1 (s)}{\sqrt{1 + \mu_1 \alpha k_1 (s)}} e_1 (s) + \frac{\alpha k_2 (s)}{\sqrt{1 + \mu_1 \alpha k_1 (s)}} e_3 (s) \right)$$

(3.7)

for $s \in L$.

The differentiation of the last equation with respect to $s$ is

$$f' (s) k_1^* (f (s)) e_2^* (f (s)) = \left( \frac{1 + \mu_1 \alpha k_1 (s)}{\sqrt{1 + \mu_1 \alpha k_1 (s)}} \right)' e_1 (s)$$

$$+ \left( \frac{(1 + \mu_1 \alpha k_1 (s)) k_1 (s) + \mu_2 \alpha k_2^2 (s)}{\sqrt{1 + \mu_1 \alpha k_1 (s)}} \right)' e_2 (s)$$

$$+ \left( \frac{\alpha k_2 (s)}{\sqrt{1 + \mu_1 \alpha k_1 (s)}} \right)' e_3 (s) + \left( \frac{\alpha k_2 (s) k_3 (s)}{\sqrt{1 + \mu_1 \alpha k_1 (s)}} \right) e_4 (s).$$

(3.8)

According to our assumption,

$$\frac{(1 + \mu_1 \alpha k_1 (s)) k_1 (s) + \mu_2 \alpha k_2^2 (s)}{\sqrt{1 + \mu_1 \alpha k_1 (s)}} = 0$$

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is hold. Thus, the coefficient of $\mathbf{e}_2 (s)$ in the equation (3.8) is zero. It is seen from the equation (3.8), $\mathbf{e}_3^* (f (s))$ is given by linear combination of $\mathbf{e}_1 (s)$, $\mathbf{e}_3 (s)$ and $\mathbf{e}_4 (s)$. Also, from equation (3.7), $\mathbf{e}_4^* (f (s))$ is a linear combination of $\mathbf{e}_1 (s)$ and $\mathbf{e}_4 (s)$. Moreover, $C^*$ is a special spacelike Frenet curve that the vector $\mathbf{e}_2 (s)$ is given by linear combination of $\mathbf{e}_5^* (f (s))$ and $\mathbf{e}_4^* (f (s))$.

Therefore, the first normal line $C$ lies in the plane generated by the second normal line and third normal line of $C^*$ at the corresponding points under a bijection $\phi$ which is defined by $\phi (c (s)) = c^* (f (s))$. Thus, the proof of the theorem is completed.

**Remark 3.1** In 4-dimensional Minkowski space for a special spacelike Frenet curve $C$ with curvature functions $k_1$ and $k_2$ satisfying

$$k_1 (s) = -\alpha (\mu_1 k_2^2 (s) + \mu_2 \mu_3 k_2^2 (s)),$$

it is not clear that a smooth spacelike curve $C^*$ given by (3.1) is a special Frenet curve. So, it is unknown whether the reverse of Theorem 3.1 is true or false.

**Theorem 3.3** Let $C$ be a spacelike special curve in $E_4^1$ with non-zero third curvature function $k_3$. If there exists a spacelike special Frenet curve $C^*$ in $E_4^1$ such that the first normal line of $C$ is linearly dependent with the third normal line of $C^*$ at the corresponding points $c (s)$ and $c^* (s)$, respectively, under a bijection $\phi : C \to C^*$, then the curvatures $k_1$ and $k_2$ of $C$ are constant functions.

**Proof.** Let $C$ be a spacelike Frenet curve in $E_4^1$ with the Frenet frame field $\lbrace \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \rbrace$ and curvature functions $k_1$, $k_2$ and $k_3$. Also, we assume that $C^*$ be a spacelike special Frenet curve in $E_4^1$ with the Frenet frame field $\lbrace \mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*, \mathbf{e}_4^* \rbrace$ and curvature functions $k_1^*$, $k_2^*$ and $k_3^*$.

Let the first normal line of $C$ be linearly dependent with the third normal line of $C^*$ at the corresponding points $C$ and $C^*$, respectively. Then the parametrization of $C^*$ is

$$c^* (f (s)) = c (s) + \alpha (s) \mathbf{e}_2 (s) \quad (3.9)$$

for all $s \in L$. If $s^*$ is the arc-length parameter of $C^*$, then

$$s^* = \int_0^s \sqrt{(1 + \mu_1 \alpha k_1)^2 + \varepsilon_2 (\alpha' (s) + \varepsilon_3 (\alpha (s) k_2 (s))^2) \right) ds \quad (3.10)$$

where

$$\varepsilon_i = \begin{cases} -1 & \text{, } \mathbf{e}_i \text{ is timelike} \\ 1 & \text{, } \mathbf{e}_i \text{ is spacelike} \end{cases} \quad \text{for } 2 \leq i \leq 4$$

and

$$f : L \to L^*$$

$$s \mapsto f (s) = s^*.$$  

Moreover, $\phi : C \to C^*$ is a bijection given by $\phi (c (s)) = c^* (f (s))$.

By differentiating the equation (3.9) with respect to $s$ and using Frenet formulas, we have

$$f' (s) \mathbf{e}_1^* (f (s)) = (1 + \mu_1 \alpha (s) k_1 (s)) \mathbf{e}_1 (s) + \alpha' (s) \mathbf{e}_2 (s) + \alpha (s) k_2 (s) \mathbf{e}_3 (s). \quad (3.11)$$
Since $e_4^* (f (s)) = \mp e_2 (s)$, then
\[
(f' (s) e_4^* (f (s))) = \pm (1 + \mu_1 \alpha (s) k_1 (s)) e_1 (s) + \alpha' (s) e_2 (s)
\]
that is,
\[
0 = \mp \alpha' (s).
\]
It is easily seen that $\alpha$ is a constant number from the last equation. Thus, hereafter we can denote $\alpha (s) = \alpha$, for all $s \in L$.

From the equation (3.10), we get
\[
f' (s) = \sqrt{(1 + \mu_1 \alpha k_1 (s))^2 + \varepsilon_3 (\alpha k_2 (s))^2} > 0
\]
where
\[
\varepsilon_3 = \begin{cases} -1, & e_i \text{ is timelike} \\ 1, & e_i \text{ is spacelike} \end{cases}, \text{ for } 2 \leq i \leq 4.
\]

Then, we rewrite the equation (3.11) as follows;
\[
e_1^* (f (s)) = \left( \frac{1 + \mu_1 \alpha k_1 (s)}{f' (s)} \right) e_1 (s) + \left( \frac{\alpha k_2 (s)}{f' (s)} \right) e_3 (s).
\]

The differentiation of the last equation with respect to $s$ is
\[
f' (s) k_1^* (f (s)) e_2^* (f (s)) = \left( \frac{1 + \mu_1 \alpha k_1 (s)}{f' (s)} \right)' e_1 (s) + \left( \frac{k_1 (s) + \mu_1 \alpha k_1^2 (s) + \mu_2 \alpha k_2 (s)}{f' (s)} \right) e_2 (s)
\]
\[
+ \left( \frac{\alpha k_2 (s) k_1 (s)}{f' (s)} \right) e_3 (s) + \left( \frac{\alpha k_2 (s) k_2 (s)}{f' (s)} \right) e_4 (s).
\]

Since $\langle f' (s) k_1^* (f (s)) e_2^* (f (s)) \rangle$, $e_4^* (f (s))) = 0$ and $e_4^* (f (s)) = \mp e_2 (s)$ for all $s \in L$, we obtain
\[
k_1 (s) + \mu_1 \alpha k_1^2 (s) + \mu_2 \alpha k_2 (s) = 0
\]
is satisfied. Then,
\[
\alpha = -\frac{k_1 (s)}{\mu_1 k_1 (s) + \mu_2 k_2 (s)}
\]
is a non-zero constant number. Thus, from the equation (3.12), it is seen that
\[
e_2^* (f (s)) = \frac{1}{f' (s) K(s)} \left( \frac{1 + \mu_1 \alpha k_1 (s)}{f' (s)} \right)' e_1 (s) + \frac{1}{f' (s) K(s)} \left( \frac{\alpha k_2 (s)}{f' (s)} \right) e_3 (s)
\]
\[
+ \frac{1}{f' (s) K(s)} \left( \frac{\alpha k_2 (s) k_2 (s)}{f' (s)} \right) e_4 (s)
\]
where \( K(s) = k_1^2(f(s)) \) for all \( s \in L \). By differentiating the last equation with respect to \( s \), we obtain

\[
f'(s) \left[ \mu_1 k_1^2(f(s)) e_1^*(f(s)) + k_2^2(f(s)) e_3^*(f(s)) \right] = \left( \frac{1}{f'(s)K(s)} \left( \frac{1 + \mu_1 \alpha k_1(s)}{f'(s)} \right) \right) e_1(s)
+ \left( \frac{k_1(s)}{f'(s)K(s)} \left( \frac{1 + \mu_1 \alpha k_1(s)}{f'(s)} \right) \right)' e_2(s)
+ \left( \frac{1}{f'(s)K(s)} \left( \frac{\alpha k_2(s)}{f'(s)} \right) \right)' e_3(s)
+ \left( \frac{1}{f'(s)K(s)} \left( \frac{\alpha k_2(s)k_3(s)}{f'(s)} \right) \right)' e_4(s)
\]

for all \( s \in L \). If we take into consideration

\[
\langle f'(s) \left( \mu_1 k_1^2(f(s)) e_1^*(f(s)) + k_2^2(f(s)) e_3^*(f(s)) \right), e_4^*(f(s)) \rangle = 0
\]

and

\[
e_4^*(f(s)) = \mp e_2(s),
\]

then

\[
\mu_1 \alpha k_1(s) k_1'(s) f''(s) - k_1(s) (1 + \mu_1 \alpha k_1(s)) f''(s)
+ \mu_2 \alpha k_2(s) k_2'(s) f''(s) - \mu_2 \alpha k_2^2(s) f''(s) = 0.
\]

If we arrange the last equation, then we find

\[
\alpha (\mu_1 k_1(s) k_1'(s) + \mu_2 k_2(s) k_2'(s)) f''(s)
- (k_1 + \alpha (\mu_1 k_1^2(s) + \mu_2 k_2^2(s))) f''(s) = 0. \quad (3.14)
\]

Moreover, the differentiation of the equation (3.13) with respect to \( s \) is

\[
k_1'(s) + 2\alpha (\mu_1 k_1(s) k_1'(s) + \mu_2 k_2(s) k_2'(s)) = 0.
\]

From the above equation, we see

\[
- \frac{k_1'(s)}{2} = \alpha (\mu_1 k_1(s) k_1'(s) + \mu_2 k_2(s) k_2'(s)). \quad (3.15)
\]

If we substitute the equations (3.13) and (3.15) into the equation (3.14), we obtain

\[
- \frac{k_1'(s)}{2} = 0.
\]

Finally, we find that the first curvature function is constant (that is, positive constant).

Thus, from the equation (3.15) it is seen that the second curvature function \( k_2 \) is positive constant, too. This completes the proof.

In [9], a formula of parametric equation of Mannheim curve is given in \( E^3 \).

Moreover, the parametric equation of generalized Mannheim curve in \( E^4 \) is obtained in [7]. The following theorem gives a parametric representation of a generalized spacelike Mannheim curve with timelike second binormal vector in \( E^4_1 \).
Theorem 3.4 Let be a spacelike special curve defined by
\[
c(u) = \begin{bmatrix}
\alpha \int f(u) \sinh u \, du \\
\alpha \int f(u) \cosh u \, du \\
\alpha \int f(u) g(u) \, du \\
\alpha \int f(u) h(u) \, du
\end{bmatrix}
\]
for \( u \in I \subset \mathbb{R} \). Here \( \alpha \) is a non-zero constant number, \( g : I \to \mathbb{R} \) and \( h : I \to \mathbb{R} \) are any smooth functions and the positive valued smooth function \( f : I \to \mathbb{R} \) is given by
\[
f(u) = (1 + g^2(u) + h^2(u))^{-3/2}
\]
for \( u \in I \). Then the curvature functions \( k_1 \) and \( k_2 \) of \( C \) satisfy
\[
k_1(u) = \alpha \left( k_1^2(u) + k_2^2(u) \right)
\]
at the each point \( c(u) \) of \( C \).

Proof. Let \( C \) be a spacelike special curve defined by
\[
c(u) = \begin{bmatrix}
\alpha \int f(u) \sinh u \, du \\
\alpha \int f(u) \cosh u \, du \\
\alpha \int f(u) g(u) \, du \\
\alpha \int f(u) h(u) \, du
\end{bmatrix}, \quad u \in I \subset \mathbb{R}
\]
where \( \alpha \) is a non-zero constant number, \( g \) and \( h \) are any smooth functions. \( f \) is a positive valued smooth function. Thus, we obtain
\[
\dot{c}(u) = \begin{bmatrix}
\alpha f(u) \sinh u \\
\alpha f(u) \cosh u \\
\alpha f(u) g(u) \\
\alpha f(u) h(u)
\end{bmatrix}, \quad u \in I \subset \mathbb{R}
\]
(3.16)
where the subscript dot \( (.) \) denotes the differentiation with respect to \( u \).

The arc-length parameter \( s \) of \( C \) is given by
\[
s = \psi(u) = \int_{u_0}^{u} \| \dot{c}(u) \| \, du
\]
where \(\|\dot{c}(u)\| = \alpha f(u) \sqrt{1 + g^2(u) + h^2(u)}\).

If \(\varphi\) denotes the inverse function of \(\psi : I \rightarrow L \subset \mathbb{R}\), then \(u = \varphi(s)\) and

\[
\varphi'(s) = \left| \frac{d\varphi(s)}{ds} \right|^{-1} , \quad s \in I
\]

where the prime (') denotes the differentiation with respect to \(s\).

The unit tangent vector \(e_1(s)\) of the curve \(C\) at the each point \(c(\varphi(s))\) is given by

\[
e_1(s) = (1 + g^2(\varphi(s)) + h^2(\varphi(s)))^{-1/2} \begin{bmatrix} \sinh(\varphi(s)) \\ \cosh(\varphi(s)) \\ g(\varphi(s)) \\ h(\varphi(s)) \end{bmatrix}
\]

for all \(s \in L\). Some simplifying assumptions are made for the sake of brevity as follows:

- \(\sinh := \sinh(\varphi(s))\), \(\cosh := \cosh(\varphi(s))\)
- \(f := f(\varphi(s))\), \(g := g(\varphi(s))\), \(h := h(\varphi(s))\)
- \(\dot{g} := \dot{g}(\varphi(s)) = \left| \frac{dg(u)}{du} \right|_{u=\varphi(s)}\), \(\dot{h} := \dot{h}(\varphi(s)) = \left| \frac{dh(u)}{du} \right|_{u=\varphi(s)}\)
- \(\ddot{g} := \ddot{g}(\varphi(s)) = \left| \frac{d^2g(u)}{du^2} \right|_{u=\varphi(s)}\), \(\ddot{h} := \ddot{h}(\varphi(s)) = \left| \frac{d^2h(u)}{du^2} \right|_{u=\varphi(s)}\)
- \(\varphi' := \varphi'(s) = \left| \frac{d\varphi(s)}{ds} \right|_s\)

Then, we have

\[
\dot{A} = 2B , \quad \dot{B} = C + D , \quad \dot{C} = 2E , \quad \varphi' = \alpha^{-1} f^{-1} A^{-1/2}.
\]

So, we rewrite the equation (3.17) as

\[
e_1 := e_1(s) = A^{-1/2} \begin{bmatrix} \sinh \\ \cosh \\ g \\ h \end{bmatrix} . \quad (3.18)
\]

By differentiating the last equation with respect to \(s\), we find

\[
e'_1 = \varphi' = \begin{bmatrix} -\frac{1}{2} A^{-3/2} \dot{A} \sinh + A^{-1/2} \cosh \\ -\frac{1}{2} A^{-3/2} \dot{A} \cosh + A^{-1/2} \sinh \\ -\frac{1}{2} A^{-3/2} \dot{A} g + A^{-1/2} \dot{g} \\ -\frac{1}{2} A^{-3/2} \dot{A} h + A^{-1/2} \dot{h} \end{bmatrix} ,
\]

that is,
If we simplify $e_1'$ by the fact that $e_1' = -\varphi' A^{-1/2} \begin{bmatrix} A^{-1} B \sinh - \cosh \\ A^{-1} B \cosh - \sinh \\ A^{-1} Bg - q \\ A^{-1} Bh - h \end{bmatrix}$.

From the last equation, we obtain

$$k_1 := k_1 (s) = ||e_1' (s)|| = \varphi' A^{-1} | -A + AC - B^2 |^{1/2}.$$  

(3.20)

By the fact that $e_2 (s) = (k_1 (s))^{-1} e_1' (s)$, we have

$$e_2 := e_2 (s) = -A^{1/2} | -A + AC - B^2 |^{-1/2} \begin{bmatrix} A^{-1} B \sinh - \cosh \\ A^{-1} B \cosh - \sinh \\ A^{-1} Bg - q \\ A^{-1} Bh - h \end{bmatrix}.$$  

In order to get second curvature function $k_2$, we need to calculate $k_2 (s) = ||e_2' (s) - \mu_1 k_1 (s) e_1 (s)||$. It is seen from the above equation $\langle e_2 (s), e_2 (s) \rangle = 1$, that is, $e_2$ is spacelike. Thus, $\mu_1$ is equal to $-1$ and $k_2 (s) = ||e_2' (s) + k_1 (s) e_1 (s)||$.

After a long process of calculation, we have

$$e_2' + k_1 e_1 = \varphi' A^{-3/2} | -A + AC - B^2 |^{-3/2} \begin{bmatrix} (P + Q) \sinh - R \cosh \\ (P + Q) \cosh - R \sinh \\ Pg - Rq + Qg \\ Ph - Rh + Qh \end{bmatrix}.$$  

(3.21)

where

$$P = ( -A + AC - B^2)^2 + ( -A + AC - B^2 ) ( B^2 - AC - AD )$$
$$+ AB ( -B + AE - BD ),$$
$$Q = A^2 ( -A + AC - B^2 ),$$
$$R = A^2 ( -B + AE - BD ).$$  

(3.22)

If we simplify $P$ then we have

$$P = A^2 ( 1 - C + BE + D - CD ).$$

Thus, we rewrite the equations (3.22) and (3.23) as

$$e_2' + k_1 e_1 = \varphi' A^{1/2} | -A + AC - B^2 |^{1/2} \begin{bmatrix} (\tilde{P} + \tilde{Q}) \sinh - \tilde{R} \cosh \\ (\tilde{P} + \tilde{Q}) \cosh - \tilde{R} \sinh \\ \tilde{P}g - \tilde{R}q + \tilde{Q}g \\ \tilde{P}h - \tilde{R}h + \tilde{Q}h \end{bmatrix}.$$  

(3.23)
where
\[ \hat{P} = 1 - C + BE + D - CD, \]
\[ \hat{Q} = -A + AC - B^2, \]
\[ \hat{R} = -B + AE - BD. \] (3.24)

Consequently, from the equations (3.24) and (3.25), we find
\[ \|e'_2 + k_1e_1\|^2 = (\varphi')^2 A \left| -A + AC - B^2 \right|^{-3} \left( \hat{P} + \hat{Q} \right)^2 - \hat{R}^2 \]
\[ \quad + \hat{P}^2 (g^2 + h^2) + \hat{R}^2 \left( \tilde{g}^2 + \tilde{h}^2 \right) + \hat{Q}^2 \left( \tilde{g}^2 + \tilde{h}^2 \right) \]
\[ - 2\hat{P}\hat{R} \left( g\tilde{g} + h\tilde{h} \right) - 2\hat{R}\hat{Q} \left( \tilde{g}g + \tilde{h}h \right) + 2\hat{R}\hat{Q} \left( \tilde{g} + \tilde{h} \right). \]

If we substitute the abbreviations into the last equation, we get
\[ \|e'_2 + k_1e_1\|^2 = (\varphi')^2 A \left| -A + AC - B^2 \right|^{-3} \left[ \hat{P}^2 A + 2\hat{P}\hat{Q} + \hat{Q}^2 - \hat{R}^2 + \hat{R}^2 C \right. \]
\[ \left. + \hat{Q}^2 F - 2\hat{P}\hat{R}B - 2\hat{R}\hat{Q}E + 2\hat{R}\hat{Q}D \right]. \]

After substituting the equation (3.24) into the last equation and simplifying it, we have
\[ k_2^2 = \|e'_2 + k_1e_1\|^2 \]
\[ = (\varphi')^2 A \left| -A + AC - B^2 \right|^{-2} \left( -A + AC - B^2 \right) (1 + F) \]
\[ + (1 - C) (1 + D)^2 + 2BE (1 + D) - AE^2 \].

Moreover, from the equation (3.20) it is seen that
\[ k_1^2 = (\varphi')^2 A^{-2} \left| -A + AC - B^2 \right|. \]

The last two equation gives us
\[ k_1^2 + k_2^2 = (\varphi')^2 A^{-2} \left| -A + AC - B^2 \right|^{-2} \left( -A + AC - B^2 \right)^3 \]
\[ + A^3 \left( ( -A + AC - B^2 ) (1 + F) + (1 - C) (1 + D)^2 + 2BE (1 + D) - AE^2 \right). \]

By the fact \( \varphi' = \alpha^{-1} f^{-1} A^{-1/2} \), we obtain
\[ k_1^2 + k_2^2 = \alpha^{-2} f^{-2} A^{-3} \left| -A + AC - B^2 \right|^{-2} \left( -A + AC - B^2 \right)^3 \]
\[ + A^3 \left( ( -A + AC - B^2 ) (1 + F) + (1 - C) (1 + D)^2 + 2BE (1 + D) - AE^2 \right). \] (3.25)

and
\[ k_1 = \alpha^{-1} f^{-1} A^{-3/2} ( -A + AC - B^2 )^{1/2}. \] (3.26)
According to our assumption,
\[ f = (1 + g^2 + h^2)^{-3/2} \left| -1 - g^2 - h^2 + \dot{g}^2 + \dot{h}^2 + \left( \ddot{g}h - \ddot{g} \dot{h} \right)^2 \right|^{-5/2} \]
\[ \left| \left( -1 - g^2 - h^2 + \dot{g}^2 + \dot{h}^2 + \left( \ddot{g}h - \ddot{g} \dot{h} \right)^2 \right)^3 \right. \]
\[ - (1 + g^2 + h^2)^3 \left( g^2 + \dot{h}^2 \right)^2 \]
\[ \left. - \left( (gh - \dot{g} \dot{h}) + (\ddot{g}h - \ddot{g} \dot{h}) \right)^2 + (\ddot{g}h - \ddot{g} \dot{h})^2 \right|, \]
we obtain
\[ f = A^{-3/2} \left| -A + AC - B^2 \right|^{-5/2} \left( -A + AC - B^2 \right)^3 \]
\[ + A^3 \left( (1 + F) + (1 - C) (1 + D)^2 + 2BE (1 + D) - AE^2 \right) \].

If we substitute the above equations (3.25) and (3.26), we obtain
\[ k_1 = \alpha \left( k_1^2 + k_2^2 \right). \]

The proof is completed.
In the above equation \( \mu_1 = \mu_2 = -1 \) which is the special Case 1. This formula is the parametric equation of generalized spacelike Mannheim curve with timelike second binormal vector in the Minkowski space-time \( E_4 \).

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