ON A CONJECTURE OF KELLY ON (1,3)-REPRESENTATION OF SYLVESTER-GALLAI DESIGNS

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Abstract. We give an exact criterion of a conjecture of L. M. Kelly to hold true which is stated as follows. If there is a finite family \( \Sigma \) of mutually skew lines in \( \mathbb{R}^d \), \( d \geq 4 \) such that the 3-flat spanned by every two lines in \( \Sigma \), contains at least one more line of \( \Sigma \), then we have that all of the lines of \( \Sigma \) are contained in a single 3-flat if and only if the arrangement of 3-flats is central. Finally, this article leads to an analogous question for higher dimensional skew affine spaces, where we prove that, for \( (2,5) \)-representations of Sylvester-Gallai designs in \( \mathbb{R}^6 \), the analogous statement does not hold.

1. Historical Introduction and the Main Theorem

In 1893, J. J. Sylvester [17] posed the following problem that has attracted considerable attention.

Problem 1.1. Let \( n \) given points have the property that the line joining any two of them passes through a third point of the set. Must the \( n \) points lie on a line?

For a few decades this problem was forgotten and forty years later, in 1933, Erdős asked this question [8], which was solved by a fellow countryman T. Gallai and others [9], (also see [3] for a survey).

Let \( K \) be a field and \( \mathbb{P}^n(K) \) be the \( n \)-dimensional projective space. We say a finite set \( A \subseteq \mathbb{P}^n(K) \) of points is a Sylvester-Gallai configuration (SGC), if for any distinct points \( x, y \in A \) there exists \( z \in A \) such that \( x, y, z \) are distinct collinear points. An easy generalization of the Sylvester’s problem gives the following theorem.

Theorem 1.2 (Sylvester-Gallai Theorem). If \( A \) is a finite noncollinear set of points in the \( n \)-dimensional projective space \( \mathbb{P}^n(\mathbb{R}) \) then there exists a line through exactly two points of \( A \), that is, every SGC in \( \mathbb{P}^n(\mathbb{R}) \) is collinear.

It is a classical fact that the nine inflection points of a non-degenerate cubic curve in \( \mathbb{P}^2(\mathbb{C}) \) constitute an SGC. Serre [15] asked whether an SGC in \( \mathbb{P}^n(\mathbb{C}) \) must be coplanar, that is, it must lie in a two dimensional complex plane. This has been solved by L. M. Kelly [11] using an inequality of F. Hirzebruch [10] involving the number of incidences of points and lines in \( \mathbb{P}^2(\mathbb{C}) \). Y. Miyaoka [13] and S. T. Yau [18] have

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proved that, for complete non-singular algebraic surfaces $S$ of general type which are minimal, the inequality
\[ c_1^2(S) \leq 3c_2(S), \]
where $c_1^2(S), c_2(S)$ are the Chern numbers of $S$. Using the Miyaoka-Yau inequality, F. Hirzebruch \[10\] has deduced that, for an arrangement of $k$ lines in the complex projective plane with
\[ t_k = t_{k-1} = 0, \]
the inequality
\[ t_2 + t_3 \geq k + t_5 + 2t_6 + 3t_7 + \cdots, \]
where $t_r$ is the number of points where exactly $r$ lines meet. Using Hirzebruch inequality and the fact that, for a field $K$ of characteristic zero, any non-linear SGC in $\mathbb{P}^2(K)$ cannot be contained in three concurrent lines intersecting at a point of SGC, the following theorem is deduced.

**Theorem 1.3 (Kelly).** Every SGC in $\mathbb{P}^n(\mathbb{C})$ is coplanar.

Also generalizing an SGC, more liberal representations of them have been studied where points are represented by affine or projective subspaces of dimension $m$ and lines are represented by affine or projective subspaces of dimension $n$ in a higher dimensional affine or projective space respectively. Such a representation is called an $(m,n)$-representation and a Sylvester-Gallai design.

**Definition 1.4.** Let $F$ be a field and $0 \leq m \leq n, 1 \leq r$ be three integers. Let
\[ C = \{A_1, A_2, \ldots, A_r\} \]
be a finite set of $m$-dimensional affine subspaces in $\mathbb{P}^{n+1}$. Suppose it satisfies the SGC property, that is, given any two distinct affine subspaces $A_i, A_j$ there exists another distinct affine subspace $A_k$ such that $A_k$ is contained in the flat spanned by $A_i, A_j$, that is, in the smallest affine space containing both $A_i, A_j$. Then we say such a configuration $C$ of subspaces as a Sylvester-Gallai design.

In the hope to obtain an elementary solution to Question \[15\] of Serre, L. M. Kelly in 1986 has mentioned the following conjecture in \[11\] regarding a $(1,3)$-representation in four dimensional euclidean space.

**Conjecture 1.5.** If there is a finite family $\Sigma$ of mutually skew lines in $\mathbb{R}^4$ such that the $3$-flat spanned by every two lines in $\Sigma$, contains at least one more line of $\Sigma$, then the lines of $\Sigma$ are contained in a single $3$-flat.

Three years later in 1989, the authors E. Boros, Z. Füredi and L. M. Kelly \[2\] have disproved Conjecture 1.5 by actually giving a $(1,3)$-representation of the seven point projective plane $\mathbb{P}^2(\mathbb{F}_2)$ in $\mathbb{R}^4$ where points of $\mathbb{P}^2(\mathbb{F}_2)$ are represented by mutually skew lines in $\mathbb{R}^4$ and lines of $\mathbb{P}^2(\mathbb{F}_2)$ are represented by $3$-flats in $\mathbb{R}^4$. We will mention in Example 2.2 of this article, plenty of such $(1,3)$-representations.

In a further effort to obtain an elementary proof of Serre’s Question \[15\], the authors in \[2\] have mentioned the following conjecture.
Conjecture 1.6. If a finite family of pairwise skew lines in \( \mathbb{R}^4 \) is such that the 3-flat spanned by any two of the lines contains at least two more, then the family is contained in a single 3-flat.

However again in 1992, J. Bokowski and J. Richter-Gebert \[1\] have constructed a (1,3)-representation of the thirteen point projective plane \( \mathbb{P}^2(\mathbb{F}_3) \) in \( \mathbb{R}^4 \) negating Conjecture 1.6. Then it is believed for a while that the answer to Serre’s Question \[15\] perhaps does require deeper algebraic properties of complex numbers. More than a decade later in 2006, N. Elkies, L. M. Pretorius, K. J. Swanepoel \[7\] have given an elementary proof of Theorem 1.3. They in fact have proved much more and thereby have concluded the quest for an elementary proof. In 2014 another elementary proof of Theorem 1.3 is obtained by Z. Dvir, S. Saraf, A. Wigderson \[6\] which is completely different from \[7\].

1.1. The statement of the Main Theorem

Conjecture 1.5 in \[11\] has been forgotten after there had been an attempt to correct by L. M. Kelly in \[2\]. This article now settles in a better manner enhancing the importance of this conjecture which has been made by L. M. Kelly in \[11\]. The main theorem concerning Conjecture 1.5 is stated as follows.

**Theorem 1.7 (Main Theorem).** Let \( d \geq 4 \) be an integer. Suppose there is a finite family \( \Sigma \) of mutually skew lines in \( \mathbb{R}^d \) such that the 3-flat spanned by every two lines in \( \Sigma \), contains at least one more line of \( \Sigma \). Then we have that the lines of \( \Sigma \) are contained in a single 3-flat if and only if the arrangement of the 3-flats is central, that is, the collection of all the 3-flats in \( \mathbb{R}^d \) spanned by every two lines in \( \Sigma \) contains a common point.

More about recent incidence and arrangement problems in particular of Sylvester-Gallai-Type can be found in P. Brass, W. O. J. Moser and J. Pach \[4\] along with a huge list of references at the end section 7.2 on pages 307-310.

2. On small (1,3)-representations in \( \mathbb{R}^4 \)

We start this section with the following lemma which is an easy observation.

**Lemma 2.1.** For \( 1 \leq n \leq 6 \), if any \( n \) mutually skew lines in \( \mathbb{R}^4 \) satisfies that the 3-flat spanned by every two lines contains a third, then all the \( n \) lines are contained in a single 3-flat in \( \mathbb{R}^4 \).

**Proof.** The proof is immediate. Also, see \[12\] Theorem 3.1].

For \( n = 7 \), the only (1,3)-representation scenario in \( \mathbb{R}^4 \) which requires further examination is the (1,3)-representation of the seven-point projective plane \( \mathbb{P}^2(\mathbb{F}_2) \). Also refer to Theorem 3.2 in L. M. Kelly, S. Nwanpka \[12\]. We discuss this example below.

**Example 2.2.** Let \( L_i, 1 \leq i \leq 7 \) be mutually skew lines in \( \mathbb{R}^4 \) which is a (1,3)-representation of the Fano plane. Let

\[
H_1 \supset L_1 \cup L_2 \cup L_3, \quad H_2 \supset L_1 \cup L_4 \cup L_5, \quad H_3 \supset L_1 \cup L_6 \cup L_7, \quad H_4 \supset L_2 \cup L_4 \cup L_6,
\]
where \( H_i, 1 \leq i \leq 7 \) are hyperplanes in \( \mathbb{R}^4 \). Suppose the hyperplanes \( H_i, 1 \leq i \leq 7 \) are all distinct and form a generic hyperplane arrangement in \( \mathbb{R}^4 \). Then we can recover the lines from the hyperplanes as follows. We have

\[
L_1 = H_1 \cap H_2 \cap H_3, L_2 = H_1 \cap H_4 \cap H_5, L_3 = H_1 \cap H_6 \cap H_7, L_4 = H_2 \cap H_4 \cap H_6, L_5 = H_2 \cap H_5 \cap H_7, L_6 = H_3 \cap H_4 \cap H_7, L_7 = H_3 \cap H_5 \cap H_6.
\]

We begin with a generic hyperplane arrangement \( H_i, 1 \leq i \leq 7 \) and define \( L_i \) using the hyperplanes. It is clear that the lines \( L_i, 1 \leq i \leq 7 \) do not intersect since the intersection of any five distinct hyperplanes is empty. The coplanarity conditions on any pair of lines \( L_i, L_j, i \neq j \) is a polynomial condition on the coefficients defining the hyperplanes. So, if we assume the coefficients defining the hyperplanes are more generic enough we obtain plenty of \((1,3)\)-representations of the Fano plane. This proves that main Theorem L.7 does not hold true, if the hyperplane arrangement is not central.

Now we prove the following proposition which is the first non-trivial case of main Theorem L.7.

**Proposition 2.3.** A \((1,3)\)-representation of the Fano plane where the 3-flats spanned by any two mutually skew lines are all linear subspaces containing origin in \( \mathbb{R}^4 \) does not exist.

**Proof.** Let \( L_i, 1 \leq i \leq 7 \) be mutually skew lines in \( \mathbb{R}^4 \) which is a \((1,3)\)-representation of the Fano plane. Let

\[
H_1 \supseteq L_1 \cup L_2 \cup L_3, H_2 \supseteq L_1 \cup L_4 \cup L_5, H_3 \supseteq L_1 \cup L_6 \cup L_7, H_4 \supseteq L_2 \cup L_4 \cup L_6, H_5 \supseteq L_2 \cup L_5 \cup L_7, H_6 \supseteq L_3 \cup L_4 \cup L_7, H_7 \supseteq L_3 \cup L_5 \cup L_6
\]

where \( H_i, 1 \leq i \leq 7 \) are hyperplanes in \( \mathbb{R}^4 \). These are the 3-flats spanned by the skew lines. If \( H_i = H_j \) for some \( 1 \leq i \neq j \leq 7 \) then we have

\[
H_1 = H_2 = H_3 = H_4 = H_5 = H_6 = H_7.
\]

So assume that all the \( H_i \) are distinct and hence for \( 1 \leq i \neq j \leq 7 \),

\[
\dim_{\mathbb{R}} (H_i \cap H_j) = 2.
\]

We also have that

\[
0 \notin \bigcap_{i=1}^{7} H_i.
\]

Assume, without loss of generality, that \( L_1, L_2, L_3, L_4, L_6, L_7 \) do not pass through origin and \( L_5 \) may or may not pass through origin. Then we have

\[
2 = \dim_{\mathbb{R}} (H_1 \cap H_2 \cap H_3) = \dim_{\mathbb{R}} (H_1 \cap H_4 \cap H_5) = \dim_{\mathbb{R}} (H_1 \cap H_6 \cap H_7) = \dim_{\mathbb{R}} (H_2 \cap H_4 \cap H_6) = \dim_{\mathbb{R}} (H_3 \cap H_5 \cap H_6) = \dim_{\mathbb{R}} (H_3 \cap H_4 \cap H_7) = 2
\]

and

\[
\dim_{\mathbb{R}} (H_2 \cap H_5 \cap H_7) \leq 2.
\]
Let \( v_i \in \mathbb{R}^4 \) be a non-zero normal vector of \( H_i, 1 \leq i \leq 7 \). Then each of the following six sets \( \{v_1, v_2, v_3\}, \{v_1, v_4, v_5\}, \{v_1, v_6, v_7\}, \{v_2, v_4, v_6\}, \{v_3, v_5, v_6\}, \{v_3, v_4, v_7\} \) span a two dimensional space in \( \mathbb{R}^4 \). If \( v_1, v_3, v_6 \) are linearly dependent then

\[
H_1 \cap H_3 \cap H_6 = H_1 \cap H_3 = H_1 \cap H_6 = H_3 \cap H_6,
\]

which implies that \( L_1, L_3, L_7 \) are coplanar which is a contradiction. So \( v_1, v_3, v_6 \) are linearly independent. If \( v_1, v_3, v_6, v_4 \) are linearly independent then by applying a suitable linear transformation in \( GL_4(\mathbb{R}) \) we can assume that \( v_1 = (1, 0, 0, 0), v_3 = (0, 1, 0, 0), v_6 = (0, 0, 1, 0), v_4 = (0, 0, 0, 1) \). Then we have \( v_2 = (a, b, 0, 0) \) with \( ab \neq 0 \) and similarly \( v_2 = (0, 0, x, y) \) with \( xy \neq 0 \) which is contradiction. So \( v_4 \) is linearly dependent on the linear independent set \( \{v_1, v_3, v_6\} \). Let us assume again that \( v_1 = (1, 0, 0, 0), v_3 = (0, 1, 0, 0), v_6 = (0, 0, 1, 0), v_4 = (a, b, 0, 0) \). We observe that the product \( \alpha \beta \gamma \neq 0 \). For example, if \( \gamma = 0 \) then

\[
H_1 \cap H_3 \cap H_4 = H_1 \cap H_3 = H_1 \cap H_4 = H_3 \cap H_4.
\]

This implies that \( L_1, L_2, L_6 \) are coplanar which is a contradiction. Similarly \( \alpha \neq 0 \neq \beta \). So we get now that

\[
v_2 = (c, d, 0, 0) = cv_1 + dv_3 = pv_6 + qv_4 = (q \alpha, q \beta, q \gamma + p, 0)
\]

for some \( c, d, p, q \in \mathbb{R}^* = \mathbb{R}\setminus\{0\} \). So \( p = -q \gamma \) and \( v_2 = (q \alpha, q \beta, 0, 0) \). Similarly we have \( v_7 = (r \alpha, 0, r \gamma, 0), v_5 = (0, s \beta, s \gamma, 0) \) for some \( r, s \in \mathbb{R}^* \). Now the following matrix

\[
\begin{pmatrix}
q \alpha & q \beta & 0 & 0 \\
r \alpha & 0 & r \gamma & 0 \\
0 & s \beta & s \gamma & 0
\end{pmatrix}
\]

has rank three because \( qrs \alpha \beta \gamma \neq 0 \). This implies that

\[
H_2 \cap H_5 \cap H_7 = L_5
\]

is one dimensional and \( L_5 \) must pass through origin. Also note that the line \( L = \{t(0, 0, 0, 1) \mid t \in \mathbb{R}\} \) is perpendicular in \( \mathbb{R}^4 \) to all \( v_i, 1 \leq i \leq 7 \). Hence we have

\[
L \subset \bigcap_{i=1}^{7} H_i.
\]

This implies that \( L = L_5 \). Now it follows that \( L_5 \) is coplanar with all \( L_i, 1 \leq i \leq 7 \) which is a contradiction. Hence the proposition follows. \( \square \)

Now as a consequence, we have the following corollary when \( |\Sigma| = 7 \) in Theorem [1,7] for \( d = 4 \).

**Corollary 2.4.** Suppose there is a finite family \( \Sigma \) of seven mutually skew lines in \( \mathbb{R}^4 \) such that the 3-flat spanned by every two lines in \( \Sigma \), contains at least one more line of \( \Sigma \). Then we have that the lines of \( \Sigma \) are contained in a single 3-flat if and only if the hyperplane arrangement of 3-flats is central.
3. Proof of the Main Theorem

We prove main Theorem 1.7 in this section. Here we give an elegant and brief proof of the main theorem and give another longer proof in the appendix section.

Proof. So let us assume that we have a finite set of pairwise skew lines in \( d \)-dimensional space, \( d \geq 4 \), such that the 3-flat spanned by any two lines contains the origin. If we replace each line by the span of itself together with the origin, we obtain a finite collection of 2-dimensional subspaces (or 1-dimensional if one of the original lines already passed through the origin) with the property that any two of the 2-dimensional subspaces span a 3-dimensional subspace, or equivalently, any two of them intersect in a line. If we interpret this in terms of the \((d-1)\)-dimensional projective space obtained by projection from the origin, we obtain a finite collection of lines (including perhaps a point) such that any two of the lines have a common point. It is then easy to see that either all the lines lie in the same 2-flat of the projective space, or all the lines intersect in a common point \( p \). Interpreting this in the original \( d \)-space, we obtain the following: Given a finite family of pairwise skew lines in real \( d \)-space \((d \geq 4)\) such that the 3-flat spanned by any two lines contains the origin, then either all the lines lie in the same 3-flat, or there is a line \( L \) through the origin such that \( L \) and any of the given lines lie in a 2-flat.

Going back to the \((d-1)\)-dimensional situation, we see that in the second case, where all the lines intersect in the same point \( p \), we can form the \((d-2)\)-dimensional projective space by projecting from \( p \), and now the original hypothesis that the 3-flat through any two of the original lines contains another line, becomes that the line through any two points contains a third point. Then the original Sylvester-Gallai theorem in \((d-2)\)-dimensional space immediately gives that all these points lie on the same line, and in the original \( d \)-dimensional space we obtain the all the lines lie in the same 3-space. This completes the proof of the main theorem in the paper. □

4. On (2,5)-Representations

We mention a definition first.

**Definition 4.1.** Let \( V \) be a finite dimensional vector space over a field \( \mathbb{K} \). We say two affine spaces \( W_1, W_2 \subset V \) are skew if

\[
\dim_{\mathbb{K}}(W_1) + \dim_{\mathbb{K}}(W_2) + 1 = \dim_{\mathbb{K}}(F(W_1, W_2))
\]

where \( F(W_1, W_2) = \{tw_1 + (1-t)w_2 \mid w_i \in W_i, i = 1, 2, t \in \mathbb{K}\} \) is the flat spanned by \( W_1 \) and \( W_2 \), that is, the smallest affine space containing both \( W_1 \) and \( W_2 \).

The analogous result of main Theorem 1.7 for (2,5)-representations is not true as we mention in the following theorem. The proof of this theorem is given later at the end of this section.

**Theorem 4.2.** Let \( \Sigma \) be a finite family of mutually skew affine spaces in \( \mathbb{R}^6 \) of dimension 2. Suppose the 5-dimensional flat of every two affine spaces in \( \Sigma \) is a linear subspace containing origin and contains at least one more affine space of \( \Sigma \). Then all the affine spaces in \( \Sigma \) need not be entirely contained in a single 5-dimensional flat.
There have been some results in the direction of this theorem. Instead of \( \Sigma \subset R^6 \), we suppose \( \Sigma \subset R^d \) with \( d \geq 11 \). Using the bound given in Theorem 1.6 in [5], it can be shown that the set \( \Sigma \) is contained in a 10-dimensional space. This can be obtained as follows. Assume that \( \Sigma \subset \subset \) it can be shown that the set \( \Sigma \) is contained in a 10-dimensional space. This can be supposed \( \Sigma \subset \subset \subset \) we suppose \( \Sigma \subset \subset \subset \), which intersect pairwise at the origin. Now, we complexify these linear subspaces to get 3-dimensional linear subspaces in \( C^{d+1} \) intersecting pairwise at origin. Here we apply Theorem 1.6 in [5], to obtain the dimension bound of the span of all 3-dimensional spaces obtained from \( \Sigma \) to be 11. This bound will reduce to 10 after intersecting with the hyperplane \( z_{d+1} = 1 \). This bound holds even if we assume that the 5-dimensional flats of any two skew affine spaces in \( \Sigma \) need not contain origin in \( R^d \). Also there is some work done on Sylvester-Gallai configurations in computer science related fields by N. Saxena, C. Seshadri [14] and most recently in 2020 by A. Shpilka [16].

We mention another related theorem.

**Theorem 4.3.** Suppose we have a finite family \( \Sigma \) of linear 2-dimensional subspaces of \( R^5 \) such that any two linear subspaces in \( \Sigma \) intersect only at origin. Also suppose that, in the 4-dimensional subspaces spanned by two linear subspaces of \( \Sigma \) there exists at least one more linear subspace from \( \Sigma \). Then \( \Sigma \) need not be contained in a single 4-dimensional subspace.

**Proof.** Consider the more generic enough hyperplane arrangement of Example 2.2 in \( R^4 \) where the seven mutually skew lines corresponding to the fano plane do not lie in a single three dimensional subspace. Place this arrangement in the subset \( R^4 = \{ z_5 = 1 \} \subset R^5 \). Now consider the cones of these seven lines in \( \{ z_5 = 1 \} \) over origin in \( R^5 \). Then we get a family \( \widetilde{\Sigma} = \{ P_1, P_2, \ldots, P_7 \} \) of seven planes in \( R^5 \) which pairwise intersect at the origin. So we obtain that all the seven planes cannot lie in a single four dimensional spaces. Otherwise upon intersecting with \( \{ z_5 = 1 \} \), this implies that all the seven lines in a single three dimensional space which is a contradiction. \( \square \)

We prove Theorem 4.2

**Proof.** Let \( \widetilde{\Sigma} = \{ P_1, P_2, \ldots, P_7 \} \) be the example obtained in the support of Theorem 4.3. Now let \( L = \{ t(0,0,0,0,0,1) | t \in R \} \). Lift the arrangement under the inverse image of the projection \( \pi : R^6 \rightarrow R^5, \pi(x,y,z,t,u,v) = (x,y,z,t,u) \). So we have \( C_i = \pi^{-1}(P_i), 1 \leq i \leq 7 \) the three dimensional spaces such that \( C_i \cap C_j = L, 1 \leq i \neq j \leq 7 \) which are obtained from the fano plane \( \mathbb{P}^2(F_2) \) whose \((1,3)\)-representation comes from a more generic enough hyperplane arrangement in \( R^4 \). We can find two dimensional affine subspaces \( A_i \subset C_i \subset R^6, 1 \leq i \leq 7 \) such that \( A_i \cap L = \{ (0,0,0,0,0,t_i) \}, t_i \neq 0, t_i \neq t_j, 1 \leq i \neq j \leq 7 \) which are mutually skew and \( C_i \) is the cone of \( A_i \) in \( R^6 \). Then the flat spanned by \( A_i \) and \( A_j \) for \( 1 \leq i \neq j \leq 7 \) is exactly \( C_i + C_j \) which passes through origin and is 5-dimensional. The flats \( C_i + C_j, 1 \leq i \neq j \leq 7 \) do not lie in a single five dimensional space in \( R^6 \). \( \square \)
In the appendix section below, we give another slightly longer proof of main Theorem 1.7, illustrating in detail, how the methods which are used in proving Lemma 2.1 for small values $1 \leq n \leq 6$, and how the methods which are used in proving Corollary 2.4 can be directly and naturally extended, and argued upon, especially in Theorems 5.2, 5.4 to prove later, main Theorem 1.7 in general.

5. Appendix: Another Proof of Main Theorem 1.7

In this section, we first give a proof of the main theorem when $d = 4$ and later prove it in general. Now we prove a lemma.

Lemma 5.1. Let $\Sigma$ be a finite family of mutually skew lines in $\mathbb{R}^d$ such that the 3-flat spanned by every two lines in $\Sigma$, contains at least one more line of $\Sigma$. Then either all the lines are contained in a single 3-flat or every line is contained in at least three different 3-flats.

Proof. Let $L$ be a line which is contained in exactly two 3-flats $H_1$ and $H_2$. Then we have a partition of the set

$$\Sigma \{L\} = \Sigma_1 \cup \Sigma_2$$

where the line $L$ and all the lines in $\Sigma_1$ are contained in the 3-flat $H_1$ and the same line $L$ and all the lines in $\Sigma_2$ are contained in another 3-flat $H_2$. Now we have $|\Sigma_i| \geq 2$, $i = 1, 2$. Let $L_i \in \Sigma_i$, $i = 1, 2$. Then the 3-flat $H \notin \{H_1, H_2\}$ which is spanned by $L_1$ and $L_2$ must contain a third line from $\Sigma$. So either $H$ contains two lines from the set $\Sigma_1 \cup \{L\}$ or two lines from the set $\Sigma_2 \cup \{L\}$. This implies either $H = H_1$ or $H = H_2$ which is a contradiction. Hence the lemma follows. $\square$

We state a very important theorem which is required to prove main Theorem 1.7 for $d = 4$.

Theorem 5.2. Suppose there is a finite family $\Sigma = \{L_1, L_2, \ldots, L_n\}$ of mutually skew lines in $\mathbb{R}^4$ such that the 3-flat spanned by every two lines in $\Sigma$, contains at least one more line of $\Sigma$. Let $\{H_1, H_2, \ldots, H_m\}$ be the distinct 3-flats and $m \geq 2$. If the hyperplane arrangement of the 3-flats is central, that is, $O \in \bigcap_{i=1}^{m} H_i$, then there exists a line $L \subset \mathbb{R}^4$ different from $L_i$ and passing through $O$ such that the line $L_i$ and $L$ are coplanar for each $1 \leq i \leq n$.

We first prove the main theorem for $d = 4$ using the Theorem 5.2 and then prove Theorem 5.2.

Proof of the main theorem for $d = 4$. Suppose there are at least two distinct 3-flats. Using Theorem 5.2 let $O$ be the origin and

$$L = \{t(0, 0, 0, 1) \mid t \in \mathbb{R}\}.$$ 

Let $P_i$ be the plane containing origin spanned by the lines $L$ and $L_i$ for $1 \leq i \leq n$. We have by $P_i \neq P_j$, $1 \leq i \neq j \leq n$ because the lines $L_i, 1 \leq i \leq n$ are mutually skew. Now we project $\mathbb{R}^4$ perpendicular to the line $L$ onto $\mathbb{R}^3$. Let

$$\pi : \mathbb{R}^4 \longrightarrow \mathbb{R}^3,$$ 


\[ \pi(x, y, z, t) = (x, y, z) \] be the projection. Then we get a finite configuration of \( n \) points
\[
\{ \pi(P_i) \mid 1 \leq i \leq n \} \subset \mathbb{P}^2(\mathbb{R}).
\]
The points \( \{ \pi(P_i) \mid L_i \in \Lambda \} \) are collinear in \( \mathbb{P}^2(\mathbb{R}) \) where \( \Lambda \) is the set of lines in a 3-flat spanned by a pair of mutually skew lines. Hence the configuration forms an SGC in \( \mathbb{P}^2(\mathbb{R}) \). Now by using the basic Sylvester-Gallai theorem we conclude that all points \( \pi(P_i), 1 \leq i \leq n \) are collinear in \( \mathbb{P}^2(\mathbb{R}) \). Hence all the lines \( L_1, L_2, \ldots, L_n \) lie in one three dimensional space containing origin which is a contradiction to the existence of at least two distinct 3-flats. This completes the proof of the main theorem. \[ \square \]

We prove Theorem 5.2.

Proof. Let \( L_1, L_2, \ldots, L_n \) be \( n \) mutually skew lines in \( \mathbb{R}^4 \) for some \( n \geq 7 \). In fact we can even assume that \( n \geq 8 \) using Corollary [2.4] because there is at least two distinct 3-flats, all of them containing origin. So using Lemma 5.1 we have that, each line \( L_i, 1 \leq i \leq n \) is in at least three different 3-flats. Let us assume that the 3-flats of all these lines are \( H_1, H_2, \ldots, H_m \) which are given as follows, without loss of generality, by gathering as much definite set theoretic knowledge as possible about the configuration.

\[
H_1 \leadsto \{ L_1, L_2, L_3, \ldots, L_{a_1} \} = \Sigma_1
\]
\[
H_2 \leadsto \{ L_1, L_{a_1+1}, L_{a_1+2}, \ldots, L_{a_2} \} = \Sigma_2
\]
\[
H_3 \leadsto \{ L_1, L_{a_2+1}, L_{a_2+2}, \ldots, L_{a_3} \} = \Sigma_3
\]
\[ \vdots \]
\[
H_{b_1} \leadsto \{ L_1, L_{a_{b_1-1}+1}, L_{a_{b_1-1}+2}, \ldots, L_{a_{b_1}} = \Sigma_{b_1}
\]
\[
H_{b_1+1} \leadsto \{ L_2, L_{a_{b_1}+1}, \ldots \} = \Sigma_{b_1+1}
\]
\[
H_{b_1+2} \leadsto \{ L_2, L_{a_{b_1}+2}, \ldots \} = \Sigma_{b_1+2}
\]
\[ \vdots \]
\[
H_{b_2} \leadsto \{ L_2, \ldots \} = \Sigma_{b_2}
\]
\[
H_{b_2+1} \leadsto \{ L_3, L_{a_{b_2}+1}, \ldots \} = \Sigma_{b_2+1}
\]
\[
H_{b_2+2} \leadsto \{ L_3, L_{a_{b_2}+2}, \ldots \} = \Sigma_{b_2+2}
\]
\[ \vdots \]
\[
H_{b_3} \leadsto \{ L_3, \ldots \} = \Sigma_{b_3}
\]
\[ \vdots \]
\[
H_{b_{a_1-1}} \leadsto \{ L_{a_1-1}, \ldots \} = \Sigma_{b_{a_1-1}}
\]
\[
H_{b_{a_1-1}+1} \leadsto \{ L_{a_1}, L_{a_1+1}, \ldots \} = \Sigma_{b_{a_1-1}+1}
\]
\[
H_{b_{a_1-1}+2} \leadsto \{ L_{a_1}, L_{a_1+2}, \ldots \} = \Sigma_{b_{a_1-1}+2}
\]
The notation for example $H_1 \leftrightarrow \{L_1, L_2, L_3, \ldots, L_a\} = \Sigma_1$ means that $H_1$ is the 3-flat spanned by any two lines in the set $\Sigma_1$ and the set $\Sigma_1$ gives rise to all pairs of lines, each of which, span the 3-flat $H_1$. So in particular $H_1$ contains the lines $L_i, 1 \leq i \leq a_1$. Here we observe the following.

$L_1 \subset \{H_1, H_2, \ldots, H_{b_1}\} = \Delta_1$
$L_2 \subset \{H_1, H_{b_1+1}, \ldots, H_{b_2}\} = \Delta_2$
$L_3 \subset \{H_1, H_{b_2+1}, \ldots, H_{b_1}\} = \Delta_3$

$L_{a_1} \subset \{H_1, H_{b_{a_1-1}+1}, \ldots, H_{b_{a_1}}\} = \Delta_{a_1}$
$L_{a_1+1} \subset \{H_2, H_{b_1+1}, H_{b_2+1}, \ldots, H_{b_{a_1-1}+1}, \ldots\} = \Delta_{a_1+1}$
$L_{a_1+2} \subset \{H_2, H_{b_1+2}, H_{b_2+2}, \ldots, H_{b_{a_1-1}+2}, \ldots\} = \Delta_{a_1+2}$

$L_n \subset \{H_{b_1}, \ldots\} = \Delta_n$.

The notation $L_1 \subset \{H_1, H_2, \ldots, H_{b_1}\} = \Delta_1$ means that $L_1$ is contained in any 3-flat $H \in \Delta_1$ and the set $\Delta_1$ has all the 3-flats which contain $L_1$. Here we have $b_1 = k, a_k = n$ to avoid notation of repeated subscripts. We have

$|\Sigma_i \cap \Sigma_j| \leq 1, 1 \leq i \neq j \leq m$

and each $H_i$ is the 3-flat spanned by any two lines in $\Sigma_i, 1 \leq i \leq m$. We have that all the $H_i, 1 \leq i \leq m$ are distinct and hence

$\dim_{\mathbb{R}}(H_i \cap H_j) = 2, 1 \leq i \neq j \leq m$.

We also have that

$0 \in \bigcap_{i=1}^{m} H_i$.

We also assume that $L_1, L_2, \ldots, L_{n-1}$ does not pass through origin and $L_n$ may or may not pass through origin. Hence we have $\dim_{\mathbb{R}}(\bigcap_{j \in \Delta_i} H_j) = 2, 1 \leq i \leq n - 1$ and $\dim_{\mathbb{R}}(\bigcap_{j \in \Delta_n} H_j) \leq 2$.

**Claim 5.3.** Let $v_j$ be a normal vector of the hyperplane $H_j, 1 \leq j \leq m$. The set $\{v_1, v_{b_1}, v_{b_2}\}$ is linearly independent and $v_j, 1 \leq j \leq m$ is linearly dependent on the set $\{v_1, v_{b_1}, v_{b_2}\}$. 
Proof of Claim. For $1 \leq i \leq n - 1$, the set $\{v_j \mid H_j \in \Delta_i\} \subset \mathbb{R}^4$ spans a two dimensional space. Now the set $\{v_1, v_{b_1}, v_{b_2}\}$ is linearly independent. Suppose not, then $H_1 \cap H_{b_1} \cap H_{b_2} = H_1 \cap H_{b_1} = H_1 \cap H_{b_2}$. This implies the lines $L_1, L_2$ are coplanar which is a contradiction. Now the vector $v_{b_3}$ is linearly dependent on the set $\{v_1, v_{b_1}, v_{b_2}\}$. Suppose not then by applying a suitable linear transformation in $GL_4(\mathbb{R})$ we can assume that $v_1 = (1, 0, 0, 0), v_{b_1} = (0, 1, 0, 0), v_{b_2} = (0, 0, 1, 0), v_{b_3} = (0, 0, 0, 1)$. Since $n \geq 4$ we have $\dim_\mathbb{R}(\cap_{j \in \Delta_i} H_j) = 2, i = 1, 2, 3$ and hence $v_2 = (a, b, 0, 0), v_{b_3} = (c, 0, d, 0), v_{b_2+1} = (e, 0, 0, f)$ with $a, b, c, d, e, f \in \mathbb{R}\{0\}$. Now the rank of the matrix

$$
\begin{pmatrix}
  a & b & 0 & 0 \\
  c & 0 & d & 0 \\
  e & 0 & 0 & f
\end{pmatrix}
$$

is three. Hence the space $H_2 \cap H_{b_1+1} \cap H_{b_2+1}$ is exactly one dimensional and contains the line $L_{a_1+1}$. But $n > a_1 + 1$ and the line $L_{a_1+1}$ does not pass through origin which is a contradiction. So $v_{b_3}$ is linearly dependent on the set $\{v_1, v_{b_1}, v_{b_2}\}$. Similarly by applying the same argument, we conclude that $v_{b_j}$ is linearly dependent on the set $\{v_1, v_{b_1}, v_{b_2}\}$ for $3 \leq j \leq a_1$ since $H_2 \cap H_{b_1+1} \cap H_{b_{j-1}+1}$ contains the line $L_{a_{j-1}}$. Now $v_i$ is linearly dependent on $\{v_1, v_{b_1}, v_{b_2}\}$ for $b_{i-1} + 1 \leq i \leq b_j - 1, 3 \leq j \leq a_1$ and hence we obtain that $v_i$ is linearly dependent on the set $\{v_1, v_{b_1}, v_{b_2}\}$ for $1 \leq i \leq b_{a_1}$. Now we observe the following. For any $2 \leq i \leq n$ there exists a unique $j \in \{1, 2, \cdots, b_i\}$ such that $L_i \subset H_j$. Similarly for $1 \leq i \leq n, i \neq 2$ there exists a unique $j \in \{1, b_1 + 1, b_1 + 2, \cdots, b_2\}$ such that $L_i \subset H_j$. Now we observe that the set $\Delta_n \subset \bigcup_{i=1}^{n-1} \Delta_i$. So by applying the dual argument we get that, for $a_1 + 1 \leq i \leq n - 1$, the set $\Delta_i$ contains two hyperplanes one from the set $\Delta_1$ and one from the set $\Delta_2$ different from $H_1$. The normal vector of any hyperplane in $\Delta_i$ is linearly dependent on the normal vectors of those two hyperplanes in $\Delta_i$ coming from $\Delta_1$ and $\Delta_2$. So for $b_{a_1} + 1 \leq i \leq m, v_i$ is linearly dependent on the set $\{v_1, v_{b_1}, v_{b_2}\}$. This can be concluded irrespective of the apriori fact that $\dim_\mathbb{R}(\cap_{j \in \Delta_n} H_j) = 0$ or $1$ or $2$. This completes the proof of the claim. \hfill $\square$

Without loss of generality by applying a linear transformation in $GL_4(\mathbb{R})$, let $v_1 = (1, 0, 0, 0), v_{b_1} = (0, 1, 0, 0), v_{b_2} = (0, 0, 1, 0)$. Then we get using the claim that, the line

$$
\{t(0, 0, 0, 1) \mid t \in \mathbb{R}\} = L \subset \bigcap_{i=1}^{m} H_i.
$$

Now for $1 \leq i \leq n$, the line $L_i$ and the line $L$ are contained in a two dimensional plane $H_r \cap H_s$ for any $r \neq s$ such that $H_r, H_s \in \Delta_i$. So in that plane the line $L_i$ and $L$ are coplanar. Now $L \neq L_i, 1 \leq i \leq n - 1$ because $L$ passes through origin and $L_i$ does not. We also have that $L \neq L_n$. This completes the proof of Theorem 5.2. \hfill $\square$

Now we state the analogous statement of Theorem 5.2 in higher dimensions.

**Theorem 5.4.** Let $d \geq 4$ be an integer. Suppose there is a finite family $\Sigma = \{L_1, L_2, \cdots, L_n\}$ of mutually skew lines in $\mathbb{R}^d$ such that the 3-flat spanned by every two lines in $\Sigma$, contains at least one more line of $\Sigma$. Let $\{H_1, H_2, \cdots, H_m\}$ be the
distinct 3-flats and \( m \geq 2 \). If the arrangement of 3-flats is central, that is, \( O \in \bigcap_{i=1}^{m} H_i \), then there exists a line \( L \subset \mathbb{R}^d \) different from \( L_i \) and passing through \( O \) such that the line \( L_i \) and \( L \) are coplanar for each \( 1 \leq i \leq n \).

We first prove the main theorem using this Theorem 5.4.

**Proof of the main theorem.** This proof is similar to the proof of the main theorem for the value \( d = 4 \) given in the previous section. \( \square \)

We prove Theorem 5.4.

**Proof.** Just similar to the proof of Theorem 5.2, here also we have a similar set theoretic knowledge of the lines and their 3-flats. We assume as usual that \( O \) is the origin and \( L_i, 1 \leq i \leq n - 1 \) do not pass through origin and \( L_n \) may or may not pass through origin. Here we have that all the 3-flats \( H_i, 1 \leq i \leq m \) are distinct and \( \dim(\mathbb{R}(H_i \cap H_j)) = 2 \) for all \( i \neq j \) such that \( H_i, H_j \in \Delta_p \) for any \( 1 \leq p \leq n - 1 \) and in general we have \( \dim(\mathbb{R}(H_i \cap H_j)) \leq 2 \) unlike the case when \( d = 4 \). It follows that \( \dim(\bigcap_{j \in \Delta_i} H_j) = 2, 1 \leq i \leq n - 1 \) and \( \dim(\bigcap_{j \in \Delta_n} H_j) \leq 1 \). Now we prove the following claim.

**Claim 5.5.** Let \( V_i = H_i^\perp \subset \mathbb{R}^d, 1 \leq i \leq m \). Then

1. \( \dim(\mathbb{R}(V_1 + V_{b_1} + V_{b_2})) = l - 1. \)
2. \( V_i \not\subset V_1 + V_{b_1} + V_{b_2} \) for \( 1 \leq i \leq m \).

**Proof of Claim.** We prove (1) first. The space \( (V_1 + V_{b_1} + V_{b_2}) = (H_1 \cap H_{b_1} \cap H_{b_2})^\perp \). We have

\[
\dim(\mathbb{R}(H_1 \cap H_{b_1} \cap H_{b_2})) = \dim(\mathbb{R}(H_1 \cap H_{b_1} \cap H_{b_2})) - \dim(\mathbb{R}(H_2 \cap H_{b_1} \cap H_{b_2})).
\]

Now for \( i = 1, 2, H_1 \cap H_{b_i} \) is the plane containing \( L_i \) and the origin \( O \). Hence the space \( (H_1 \cap H_{b_1}) + (H_1 \cap H_{b_2}) \) is exactly the 3-flat \( H_1 \) spanned by the lines \( L_1 \) and \( L_2 \) containing \( O \) which is therefore three dimensional. So we have \( \dim(\mathbb{R}(H_1 \cap H_{b_1} \cap H_{b_2})) = 1 \) and hence \( \dim(\mathbb{R}(V_1 + V_{b_1} + V_{b_2})) = l - 1. \)

Now we prove (2). First we observe that for \( 1 \leq p \leq n - 1, i \neq j, r \neq s, i, j, r, s \in \Delta_p \) the \( V_i + V_j = V_r + V_s, \dim(\mathbb{R}(V_i + V_j)) = l - 2. \) This is because \( H_i \cap H_j = H_r \cap H_s = \bigcap_{t \in \Delta_p} H_t \) and its dimension is 2. Now we have the following sequence of subspace inclusions

\[
V_{b_3} \subset V_1 + V_{b_1} + V_{b_2}
\]

Similarly we get \( V_{b_j} \subset V_1 + V_{b_1} + V_{b_2} \) for \( 3 \leq j \leq a_1 \). This implies that \( V_i \not\subset V_1 + V_{b_1} + V_{b_2} \) for \( 1 \leq i \leq b_1 \). We have \( \Delta_n \subset \bigcup_{i=1}^{n-1} \Delta_i \) and for \( a_1 + 1 \leq i \leq n - 1, \) the set \( \Delta_i \) contains two 3-flats one from the set \( \Delta_1 \) and one from the set \( \Delta_2 \) different from \( H_1. \) Hence \( V_i \not\subset V_1 + V_{b_1} + V_{b_2}, 1 \leq i \leq m. \) This proves the claim. \( \square \)
Coming to the proof of Theorem 5.4, let \( L = (V_1 + V_b + V_{b_2})^\perp \subseteq \bigcap_{i=1}^m H_i \). For \( 1 \leq i \leq n \), the line \( L \) and \( L_i \) are coplanar in the plane \( H_r \cap H_s \) for \( r \neq s, r, s \in \Delta_i \). Clearly \( L \neq L_i, 1 \leq i \leq n - 1 \) and by coplanarity of \( L \) with say \( L_1 \), we must have \( L \neq L_n \). This proves Theorem 5.4.

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