PERIODIC GIBBS MEASURES FOR THE POTTS–SOS MODEL ON A CAYLEY TREE

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We describe periodic Gibbs measures for the Potts–SOS model on a Cayley tree of order $k \geq 1$, i.e. a characterization of such measures with respect to any normal subgroup of finite index of the group representation of the Cayley tree.

Keywords: Cayley tree, configuration, Potts–SOS model, periodic Gibbs measure

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1. Introduction

A central problem in the theory of Gibbs measures (GMs) is to describe infinite-volume (or limiting) GMs corresponding to a given Hamiltonian. The existence of such measures for a wide class of Hamiltonians was established in the groundbreaking work of Dobrushin (see, e.g., [1]), but a complete analysis of the set of limiting GMs for a specific Hamiltonian is often a difficult problem.

Here, we consider models with a nearest-neighbor interaction on a Cayley tree (CT). Models on a CT were discussed in [2]–[5]. A classical example of such a model is the Ising model with two spin values [1]. It was considered in [4]–[8] and became a focus of active research in the first half of the 1990s and afterwards (see [8]–[14]). In [15], all translation-invariant GMs for the Potts model on the CT were described. Periodic and weakly periodic GMs for the Potts model were respectively studied in [16], [17] and in [18]. Translation-invariant and periodic GMs for the SOS model on the CT were studied in [19], [20].

Here, we consider the Potts–SOS model, a generalization of the Potts and SOS (solid-on-solid) models. Translation-invariant GMs for the Potts–SOS model on the CT were studied in [21], but periodic GMs have not yet been studied. We therefore study periodic GMs for this model.

2. Preliminaries and main facts

The Cayley tree. The CT $\Gamma^k$ of order $k \geq 1$ (see [8]) is an infinite tree, i.e., a graph without cycles and with exactly $k+1$ edges issuing from each vertex. Let $\Gamma^k = (V, L, i)$, where $V$ is the set of vertices of the tree, $L$ is the set of edges of the tree, and $i$ is the incidence function associating each edge $l \in L$ with its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then $x$ and $y$ are called nearest-neighbor vertices, and we write $l = \langle x, y \rangle$.

The distance $d(x, y)$ for $x, y \in V$ on the CT is defined as the minimum value $d$ for which there exist vertices $x = x_0, x_1, \ldots, x_{d-1}, x_d = y$ such that $\langle x_0, x_1 \rangle, \ldots, \langle x_{d-1}, x_d \rangle$ (in other words, the vertices are consecutive nearest neighbors).

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For a fixed \( x^0 \in V \),

\[
W_n = \{ x \in V \mid d(x, x^0) = n \}, \quad V_n = \{ x \in V \mid d(x, x^0) \leq n \},
\]

\[
L_n = \{ l = \langle x, y \rangle \in L \mid x, y \in V_n \}. \tag{1}
\]

It is known [8] that there is a one-to-one correspondence between the set \( V \) of vertices of the CT of order \( k \geq 1 \) and the group \( G_k \) that is a free product of \( k+1 \) cyclic groups \( \{e, a_i \}, i = 1, \ldots, k+1 \), of order two \( (a_i^2 = e, a_i^{-1} = a_i) \) with the generators \( a_1, \ldots, a_{k+1} \).

Let \( S(x) \) denote the set of “direct successors” of \( x \in G_k \). Let \( S_1(x) \) denote the set of all nearest-neighbor vertices of \( x \in G_k \), i.e., \( S_1(x) = \{ y \in G_k : \text{exists } \langle x, y \rangle \} \).

The model and a system vector-valued functional equations. Here, we give the main definitions and facts about the model. We consider models where the spin takes values in the set \( \Phi = \{0,1,\ldots,m\} \), \( m \geq 1 \). For \( A \subseteq V \), a spin configuration \( \sigma_A \) on \( A \) is defined as a function \( A \ni x \rightarrow \sigma_A(x) \in \Phi \). The set of all configurations coincides with \( \Omega_A = \Phi^A \). Let \( \Omega_V = \Omega \) and \( \sigma_V = \sigma \).

We define a periodic configuration as a configuration \( \sigma \in \Omega \) that is invariant under a subgroup of shifts \( K \subseteq G_k \) of finite index. More precisely, a configuration \( \sigma \in \Omega \) is said to be \( K \)-periodic if \( \sigma(yx) = \sigma(x) \) for any \( x \in G_k \) and \( y \in K \). For a given periodic configuration, the index of the subgroup is called the period of the configuration. A configuration that is invariant under all shifts is said to be translation-invariant.

The Hamiltonian of the Potts–SOS model with nearest-neighbor interaction has the form

\[
H(\sigma) = -J \sum_{\langle x, y \rangle \in L} |\sigma(x) - \sigma(y)| - J_p \sum_{\langle x, y \rangle \in L} \delta_{\sigma(x)\sigma(y)}, \tag{2}
\]

where \( J, J_p \in \mathbb{R} \) are nonzero coupling constants.

Let \( h: x \mapsto h_x = (h_{0,x}, h_{1,x}, \ldots, h_{m,x}) \in \mathbb{R}^{m+1} \) be a real vector-valued function of \( x \in V \setminus \{ x^0 \} \). We introduce probability distributions \( \mu^{(n)} \) on \( \Phi^V_n \) for a given \( n = 1, 2, \ldots \) defined by (here and hereafter, \( \beta = 1/T \) is the inverse temperature)

\[
\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp \left( -\beta H_n(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x),x} \right), \quad \sigma_n \in \Phi^V_n, \tag{3}
\]

where the related partition function \( Z_n \) can be expressed as

\[
Z_n = \sum_{\sigma_n \in \Phi^V_n} \exp \left( -\beta H_n(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x),x} \right). \tag{4}
\]

We say that a sequence of probability distributions \( \mu^{(n)} \) is consistent if for all \( n \geq 1 \) and \( \sigma_{n-1} \in \Phi^{V_{n-1}} \), we have

\[
\sum_{\omega_n \in \Phi^{V_n}} \mu^{(n)}(\sigma_{n-1} \lor \omega_n) = \mu^{(n-1)}(\sigma_{n-1}), \tag{5}
\]

where \( (\sigma_{n-1} \lor \omega_n) \in \Phi^{V_n} \) is a union of configurations \( \sigma_{n-1} \) and \( \omega_n \). If a probability distribution \( \mu^{(n)} \) on \( \Phi^{V_n} \) satisfies equality (5), then there exists a unique measure \( \mu \) on \( \Phi^V \) such that \( \mu(\sigma|V_n = \sigma_n) = \mu^{(n)}(\sigma_n) \) for all \( n \) and \( \sigma_n \in \Phi^V_n \).

Definition 1. A measure \( \mu \) satisfying the condition formulated above is called a splitting GM (SGM) corresponding to the Hamiltonian \( H \) and function \( h: x \mapsto h_x, x \neq x^0. \)
The following theorem [21] describes the condition on $h$ ensuring that a measure $\mu^{(n)}(\sigma_n)$ is consistent.

**Theorem 1.** Probability distributions $\mu^{(n)}(\sigma_n)$, $n = 1, 2, \ldots$, given by (3) are consistent if and only if for any $x \in V \setminus \{x^0\}$, the equality

$$
\begin{align*}
\sum_{y \in S(x)} F(h^*_y, m, \theta, r)
\end{align*}
$$

holds, where $\theta = e^{J_\beta}$, $r = e^{J_\beta}$, $h^*_x = (h_{0,x} - h_{m,x}, h_{1,x} - h_{m,x}, \ldots, h_{m-1,x} - h_{m,x})$, and the function $F(\cdot, m, \theta, r): \mathbb{R}^m \to \mathbb{R}^m$ is given by

$$
\begin{align*}
F(h, m, \theta, r) &= (F_0(h, m, \theta, r), F_1(h, m, \theta, r), \ldots, F_{m-1}(h, m, \theta, r)),
\end{align*}
$$

where

$$
\begin{align*}
F_i &= \log \frac{\sum_{j=0}^{m-1} \theta^{i-j} \rho^{\delta_{ij}} e^{h_j} + \theta^{m-i} \rho^{\delta_{mi}}}{\sum_{j=0}^{m-1} \theta^{m-j} \rho^{\delta_{mj}} e^{h_j} + r}, \quad i = 0, 1, \ldots, m-1,
\end{align*}
$$

and $h = (h_0, h_1, \ldots, h_m)$.

Hence, for any collection of functions satisfying functional equation (6), there exists a unique SGM because the correspondence is one-to-one.

Let $K$ be a subgroup of $G_k$.

**Definition 2.** A collection of vectors $h = \{h_x \in \mathbb{R}^m : x \in G_k\}$ is said to be $K$-periodic if $h_{y,x} = h_x$ for all $x \in G_k$ and $y \in K$. A $G_k$-periodic collection is said to be translation-invariant.

**Definition 3.** A GM is said to be $K$-periodic (translation-invariant) if it corresponds to a $K$-periodic (translation-invariant) collection $h$.

**Proposition 1.** An SGM $\mu$ is translation-invariant if and only if $h_{j,x}$ is independent of $x$: $h_{j,x} \equiv h_j$, $x \in V$, $j \in \Phi$.

**Proof.** The proof is straightforward.

**Proposition 2.** Any extreme GM is an SGM.

**Proof.** See Theorem 12.6 in [2] for the proof and the definition of “extreme.”

Translation-invariant SGMs were investigated in [22].

**Periodic SGMs.** Here, we study periodic solutions of functional equation (6), i.e., periodic SGMs. We describe periodic SGMs, i.e., characterize such measures with respect to any finite-index normal subgroup of $G_k$.

For the reader’s convenience, we recall some necessary notation. Let $K$ be a subgroup of index $r'$ in $G_k$, and let $G_k/K = \{K_0, K_1, \ldots, K_{r'-1}\}$ be the quotient group with the coset $K_0 = K$. For $x \in G_k$, let

$$
q_i(x) = |S_1(x) \cap K_i|, \quad i = 0, 1, \ldots, r'-1, \quad N(x) = |\{j : q_j(x) \neq 0\}|,
$$

where $|\cdot|$ is cardinality of the set. We set

$$
Q(x) = (q_0(x), q_1(x), \ldots, q_{r'-1}(x)).
$$
We note (also see [22]) that for every \( x \in G_k \), there is a permutation \( \pi_x \) of the coordinates of the vector \( Q(e) \) (where \( e \) is the identity of \( G_k \)) such that

\[
\pi_x Q(e) = Q(x).
\]  

(9)

Each \( K \)-periodic collection is given by

\[
\{ h_x = h_i, \ x \in K_i, \ i = 0, 1, \ldots, r' - 1 \}.
\]

By Theorem 1 (for \( m = 2 \)) and (9), the vector \( h_n, n = 0, 1, \ldots, r' - 1 \), satisfies the system

\[
h_n = \sum_{j=1}^{n(e)} q_{i_j}(e) F(h_{\pi_n(i_j)}; \theta, r) - F(h_{\pi_n(i_j_0)}; \theta, r), \quad j_0 = 1, \ldots, N(e),
\]

where the function \( h \mapsto F(h, m, \theta, r) \) defined in Theorem 1 now becomes

\[
h \mapsto F(h) = (F_0(h, \theta, r), F_1(h, \theta, r)),
\]

where

\[
F_0(h, \theta, r) = \log \frac{re^{\theta h_0} + \theta e^{h_1} + \theta^2}{\theta^2 e^{\theta h_0} + \theta e^{h_1} + r}, \quad F_1(h, \theta, r) = \log \frac{\theta e^{\theta h_0} + re^{h_1} + \theta}{\theta^2 e^{\theta h_0} + \theta e^{h_1} + r}.
\]

(11)

Proposition 3. If \( \theta \neq 1 \), then \( F(h) = F(l) \) if and only if \( h = l \).

Proof. Necessity. From \( F(h) = F(l) \), we obtain the system of equations

\[
\frac{re^{\theta h_0} + \theta e^{h_1} + \theta^2}{\theta^2 e^{\theta h_0} + \theta e^{h_1} + r} = \frac{re^{\theta l_0} + \theta e^{l_1} + \theta^2}{\theta^2 e^{\theta l_0} + \theta e^{l_1} + r}, \quad \frac{\theta e^{\theta h_0} + re^{h_1} + \theta}{\theta^2 e^{\theta h_0} + \theta e^{h_1} + r} = \frac{\theta e^{\theta l_0} + re^{l_1} + \theta}{\theta^2 e^{\theta l_0} + \theta e^{l_1} + r},
\]

where \( h = (h_0, h_1) \) and \( l = (l_0, l_1) \). We obtain

\[
(r - \theta^2)(\theta(e^{h_0 + l_1} - e^{h_1 + l_0}) + \theta(e^{h_1} - e^{l_1} + r + \theta^2)(e^{h_0} - e^{l_0})) = 0,
\]

\[
\theta^2(r - 1)(e^{h_1 + l_0} - e^{h_0 + l_1}) + \theta(r - \theta^2)(e^{h_0} - e^{l_0}) + (r^2 - \theta^2)(e^{h_1} - e^{l_1}) = 0.
\]

(13)

Using the fact that \( e^{h_0 + l_1} - e^{h_1 + l_0} = e^{l_1}(e^{h_0} - e^{l_0}) - e^{l_0}(e^{h_1} - e^{l_1}) \), we obtain the system of equations

\[
(r - \theta^2)(\theta e^{l_1} + \theta^2 + r)(e^{h_0} - e^{l_0}) + \theta(1 - e^{l_0})(e^{h_1} - e^{l_1}) = 0,
\]

\[
\theta(r - \theta^2 - \theta e^{l_1} + \theta)(e^{h_0} - e^{l_0}) + (r^2 - \theta^2 + e^{l_0}e^{h_1} - e^{l_1} - e^{l_1})(e^{h_1} - e^{l_1}) = 0.
\]

(14)

It follows that \( h_0 = l_0 \) and \( h_1 = l_1 \) in the case \( \theta \neq 1 \).

Sufficiency. The proof of sufficiency is straightforward.

Let \( G_k^{(2)} \) be the subgroup in \( G_k \) consisting of all words of even length. Clearly, \( G_k^{(2)} \) is a subgroup of index 2.

Theorem 2. Let \( K \) be a finite-index normal subgroup in \( G_k \). Then each \( K \)-periodic GM for the Potts-SOS model is either translation-invariant or \( G_k^{(2)} \)-periodic.
Proof. From (10), we see that

\[ F(h_{\pi_n(i_1)}) = \cdots = F(h_{\pi_n(i_N)}) \]

By Proposition 1, we hence have \( h_{\pi_n(i_1)} = \cdots = h_{\pi_n(i_N)} \). Therefore,

\[ h_x = \begin{cases} \hbar, & \text{if } x, y \in S_1(z), \ z \in K, \\ \lambda, & \text{if } x, y \in S_1(z), \ z \in G_{k} \setminus K. \end{cases} \]

Therefore, the measures are translation-invariant (if \( h = l \)) or \( G_k^{(2)} \)-periodic (if \( h \neq l \)). The proof is complete.

Let \( K \) be a finite-index normal subgroup in \( G_k \). What condition on \( K \) guarantees that each \( K \)-periodic GM is translation-invariant? We set \( I(K) = K \cap \{ a_1, \ldots, a_{k+1} \} \), where \( a_i, \ i = 1, \ldots, k + 1 \), are generators of \( G_k \).

**Theorem 3.** If \( I(K) \neq \emptyset \), then each \( K \)-periodic GM for the Potts-SOS model is translation-invariant.

**Proof.** We take \( x \in K \). We note that the inclusion \( xa_i \in K \) holds if and only if \( a_i \in K \). Because \( I(K) \neq \emptyset \), there is an element \( a_i \in K \). Therefore, \( K \) contains the subset \( Ka_i = \{ xa_i : x \in K \} \). By Theorem 2, we have \( h_x = h \) and \( h_{xa_i} = l \). Because \( x \) and \( xa_i \) belong to \( K \), it follows that \( h_x = h_{xa_i} = h = l \). Therefore, each \( K \)-periodic GM is translation-invariant. The theorem is proved.

Theorems 2 and 3 reduce the problem of describing a \( K \)-periodic GM with \( I(K) \neq \emptyset \) to describing the fixed points of \( kF(h, \theta, r) \), which describes a translation-invariant GM. If \( I(K) = \emptyset \), then this problem reduces to describing the solutions of the system:

\[ h = kF(l, \theta, r), \quad l = kF(h, \theta, r). \] (15)

We introduce the notation \( z_i = e^{h_i} \) and \( t_i = e^{t_i}, \ i = 0, 1 \). From (15), we then obtain

\[ z_0 = \left( \frac{rt_0 + \theta t_1 + \theta^2}{\theta^2 t_0 + \theta t_1 + r} \right)^k, \quad z_1 = \left( \frac{\theta t_0 + rt_1 + \theta}{\theta^2 t_0 + \theta t_1 + r} \right)^k, \]

\[ t_0 = \left( \frac{rz_0 + \theta z_1 + \theta^2}{\theta^2 z_0 + \theta z_1 + r} \right)^k, \quad t_1 = \left( \frac{\theta z_0 + rz_1 + \theta}{\theta^2 z_0 + \theta z_1 + r} \right)^k. \] (16)

From the first and third equations in (16), we obtain

\[ z_0^{1/k} - 1 = \left( \frac{z_0 - 1}{\theta^2 t_0 + \theta t_1 + r} \right), \quad t_0^{1/k} - 1 = \left( \frac{t_0 - 1}{\theta^2 z_0 + \theta z_1 + r} \right). \] (17)

We hence see that \( (z_0; t_0) = (1; 1) \) is a solution of system of equations (17) for every \( \theta, r, z_1, \) and \( t_1 \). In this case, from the second and fourth equations in (16), we obtain the system of equations

\[ z_1 = \left( \frac{2\theta + rt_1}{\theta^2 + \theta t_1 + r} \right)^k, \quad t_1 = \left( \frac{2\theta + rz_1}{\theta^2 + \theta z_1 + r} \right)^k. \] (18)

Let

\[ f(z_1) = \left( \frac{2\theta + rz_1}{\theta^2 + \theta z_1 + r} \right)^k. \]
Then system of equations (18) becomes

\[ f(f(z_1)) - z_1 = 0. \]  

(19)

Obviously, values \( z_1 \) satisfying \( f(z_1) = z_1 \) satisfy (19), and we therefore do not consider solutions of (19) for which \( f(z_1) = z_1 \). The remaining solutions correspond to \( G^{(2)}_k \)-periodic measure Gibbs that are not translation-invariant. In the case \( k = 2 \), from (19) after simplification, we obtain the quadratic equation

\[
(\theta^6 + 20^4 r + \theta^2 r^2 + r^4 + 2\theta r^3 + 2\theta^3 r^2)z_1^2 + \\
+ (2\theta^7 + 6\theta^5 r + 6\theta^3 r^2 + 6\theta r^3 - 4\theta^4 r^2 + 8\theta^2 r^2 + 2\theta^2 r^3 + 8\theta r^2 + r^4)z_1 + \\
+ 4\theta^2 r^2 + 4\theta^5 r + r^4 + 6\theta^4 r^2 + 8\theta^3 r^3 + \theta^8 + 4\theta^5 r + 8\theta^3 r^2 + 4\theta r^3 = 0.
\]

For this equation to have two positive real roots, the conditions \( D > 0 \) and \( b < 0 \), where

\[
D = (2\theta^7 + 6\theta^5 r + 6\theta^3 r^2 + 6\theta r^3 - 4\theta^4 r^2 + 8\theta^2 r^2 + 2\theta^2 r^3 + 8\theta r^2 + r^4)^2 - \\
- (\theta^6 + 2\theta^4 r + \theta^2 r^2 + r^4 + 2\theta r^3 + 2\theta^3 r^2) \\
\times (4\theta^2 r^2 + 4\theta^5 r + r^4 + 6\theta^4 r^2 + 8\theta^3 r^3 + \theta^8 + 4\theta^5 r + 8\theta^3 r^2 + 4\theta r^3),
\]

\[
b = 2\theta^7 + 6\theta^5 r + 6\theta^3 r^2 + 6\theta r^3 - 4\theta^4 r^2 + \theta^8 + 4\theta^5 r + 8\theta^3 r^2 + 8\theta^2 r^2 + 2\theta^2 r^3 + 8\theta r^2 + r^4
\]

must be satisfied. As a result, we obtain the following theorem.

**Theorem 4.** Let \( k = 2 \). If \( D > 0 \) and \( b < 0 \), then there exist at least two \( G^{(2)}_k \)-periodic (not translation-invariant) GMs for the Potts–SOS model. If \( D = 0 \) and \( b < 0 \), then there exists at least one \( G^{(2)}_k \)-periodic (not translation-invariant) GM for Potts–SOS model.

We show that the set \( \{(r, \theta) \in \mathbb{R}^2 : D \geq 0, b < 0\} \) is not empty. Indeed, let \( r = \theta^2 \). Then we have

\[
D = -16\theta^8(\theta^2 - 1)^2(3\theta^4 + 10\theta^3 + 6\theta^2 - 1), \quad b = 4\theta^4(\theta^4 + 5\theta^3 + 4\theta^2 - 1).
\]

If \( \theta < \theta_D \) (where \( \theta_D \approx 0.32359 \)), then we have \( D > 0 \) and \( b < 0 \), i.e., there exist at least two \( G^{(2)}_k \)-periodic (not translation-invariant) GMs. In that case, if \( \theta = \theta_D \), then it is obvious that \( D = 0 \) and \( b < 0 \), which means that there exists at least one \( G^{(2)}_k \)-periodic (not translation-invariant) GM for Potts–SOS model.

**Remark.** If \( k = 2 \) for the Potts model, then there is no periodic GM (see [16]), but for the Potts–SOS model, as shown in Theorem 4, such measures exist under certain conditions.

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