ON VOLUMES OF HYPERBOLIC ORBIFOLDS

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Abstract. In this paper we give a lower bound for the volume of a hyperbolic $n$-orbifold. Our main tool is H. C. Wang’s bound on the radius of a ball embedded in the fundamental domain of a lattice of a semisimple Lie group.

0. Introduction

Let $\mathbb{H}^n$ denote hyperbolic $n$-space. A hyperbolic $n$-orbifold $Q$ is a quotient $\mathbb{H}^n/\Gamma$, where $\Gamma$ represents a discrete group of orientation-preserving isometries. The purpose of this paper is to give a lower bound for the volume of a hyperbolic $n$-orbifold dependent on dimension only. That such a lower bound exists was proved for dimension two by Siegel [23] in a theorem closely related to a result on birational transformations of an algebraic curve due to Hurwitz [15]. The analogous result for dimension three was proved by Meyerhoff [20], motivated by work of Jørgensen [16] and Thurston [24]. The hyperbolic 2-orbifold of minimum volume was given in Siegel’s paper. The smallest volume hyperbolic 3-orbifold was identified by Gehring and Martin in [11].

A hyperbolic $n$-orbifold is: a manifold when $\Gamma$ does not contain elliptic elements; cusped when $\Gamma$ does contain parabolic elements; arithmetic when $\Gamma$ can be derived by a specific number-theoretic construction (see e.g. [2]). The existence of a lower bound for dimensions four and above was proved by Wang [26]. Much of what is known explicitly for these dimensions relate to the categories mentioned above and their various intersections. A lower bound for the volume of hyperbolic $n$-manifolds was constructed in [18]. The smallest cusped hyperbolic $n$-orbifolds of dimension less than ten can be found in [13]. The minimal volume arithmetic hyperbolic $n$-orbifolds are described in [2], [4] and [6].

In this article, hyperbolic $n$-space is viewed as a symmetric space, $\mathbb{H}^n = SO_o(n,1)/SO(n).$ We then define a Riemannian submersion $\pi : SO_o(n,1) \to \mathbb{H}^n$, where the fiber over any point in $\mathbb{H}^n$ is an embedded copy of $SO(n).$ The volume of a hyperbolic $n$-orbifold $Q = \mathbb{H}^n/\Gamma$ is defined to be the volume of the fundamental domain of $\Gamma$ in $\mathbb{H}^n.$ Hence, $\text{Vol}[Q] = \text{Vol}[SO_o(n,1)/\Gamma]/\text{Vol}[SO(n)].$ The study of the volume of a hyperbolic orbifold is therefore reduced to the study of the volume of the quotient of a lattice in a Lie group.

Let $G$ be a semisimple Lie group without compact factor. In [26], Wang showed that the volume of the fundamental domain of any discrete subgroup of $G$ can be bounded below by the volume of ball with a radius that depends only on the Lie group itself. An estimate on the curvature of $G$, together with a comparison theorem due to Gunther (see e.g. [10]),

The second author is partially supported by NSF grant DMS-0806016.
produces a lower bound for the volume of the embedded ball. The following theorem gives
our volume bound for hyperbolic orbifolds.

**Theorem 0.1.** Let $Q$ be a hyperbolic $n$-orbifold. Denote by $V(d, k, r)$ the volume of a ball
of radius $r$ in the complete simply connected Riemannian manifold of dimension $d$ with
constant curvature $k$. Then

$$\text{Vol}[Q] \geq \frac{V(d_0, k_0, r_0)}{\text{Vol}[SO(n)]},$$

where $d_0 = \frac{n^2 + n}{2}$, $k_0 = \frac{n^2 + 43n}{8}$ and $r_0 = .114 \sqrt{2(n-1)}$.

In the first and second sections of this paper, we revisit fundamental facts of semisimple
Lie groups leading to an upper bound on the sectional curvature of $SO_o(n, 1)$. The third
section of this paper outlines Wang’s crucial result. The fifth section of the article lists
several results on hyperbolic volume.

In Section 4, the bound of Theorem 0.1 is given as an explicit function of $n$. From this
formula, we get a lower bound of $2.804 \times 10^{-6}$ for hyperbolic 3-orbifolds, $2.568 \times 10^{-10}$ for
4-orbifolds and $3.144 \times 10^{-16}$ for 5-orbifolds. The bounds we achieve are not sharp. They
are smaller than the bounds for arithmetic $n$-orbifold ([2], [4], [6]), which are known to be
extremal in dimensions two and three and conjectured to be so for dimensions four and
above. However, our work improves upon the results of [1] and [18], which are the only
general orbifold and manifold bounds known to the authors.

**Remark 0.2.** The constant $k_0$ in Theorem 0.1 represents an upper bound on the sectional
curvatures of the Lie group $O(n, 1)$ proved in Proposition 2.7. From Proposition 2.6 and
[19], we expect that this value can be lowered significantly.

Intimately linked with hyperbolic volume is the size of symmetry groups of hyperbolic
manifolds. Specifically, any bound in one category immediate produces a bound in the
other. The quotient of a hyperbolic manifold $M$ by its group of orientation-preserving
isometries is an orientable hyperbolic orbifold (as long as $\pi_1(M)$ is not virtually abelian,
in which case Vol[$M$] is infinite). The following corollary is a direct analogue of Hurwitz’s
formula for groups acting on surfaces.

**Corollary 0.3.** Let $M$ be an orientable hyperbolic $n$-manifold. Let $H$ be a group of
orientation-preserving isometries of $M$. Then

$$|H| \leq \frac{\text{Vol}[M] \text{Vol}[SO(n)]}{V(d_0, k_0, r_0)}.$$

The Mostow-Prasad rigidity theorem [21], [22] implies that the group of isometries of a
finite volume hyperbolic $n$-manifold can be identified with Out($\pi_1(M)$). Hence, we have
the following ‘topological’ version of Corollary 0.3.

**Corollary 0.4.** Let $M$ be a finite volume orientable hyperbolic $n$-manifold. Let $H$ be a
subgroup of Out($\pi_1(M)$). Then

$$|H| \leq \frac{2 \text{Vol}[M] \text{Vol}[SO(n)]}{V(d_0, k_0, r_0)}.$$
1. The Curvature of Semisimple Lie Groups

In this section we review some general properties of semisimple Lie groups and derive the curvature formulas for the canonical metric. These formulas are of independent interest as we could not find them in the literature.

Let $G$ be a Lie group and denote by $\mathfrak{g}$ the Lie algebra of $G$. For $X \in \mathfrak{g}$, the adjoint action of $X$ is the endomorphism $\text{ad} X : \mathfrak{g} \to \mathfrak{g}$ defined by the Lie bracket $\text{ad} X(Y) = [X,Y]$. The Killing form on $\mathfrak{g}$ is a symmetric bilinear form given by $B(X,Y) = \text{trace}(\text{ad} X \circ \text{ad} Y)$. For all $X \in \mathfrak{g}$, $\text{ad} X$ is skew symmetric with respect to $B$; i.e.

$$B([X,Y], Z) = -B(Y, [X,Z]).$$

(1.1)

The Lie algebra $\mathfrak{g}$ is called semisimple if $B$ is non-degenerate. In this case (see e.g. [5, Page 75]) $\mathfrak{g}$ may be decomposed as $\mathfrak{p} \oplus \mathfrak{k}$ such that $B|\mathfrak{k}$ is negative definite and $B|\mathfrak{p}$ is positive definite, with bracket laws $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$. (1.2)

If $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}$, by the bracket laws $(\text{ad} X \circ \text{ad} Y)(\mathfrak{k}) \subseteq \mathfrak{p}$ and $(\text{ad} X \circ \text{ad} Y)(\mathfrak{p}) \subseteq \mathfrak{k}$. Hence $\text{trace}(\text{ad} X \circ \text{ad} Y) \equiv 0$, and therefore $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal under $B$. A Lie group $G$ is semisimple if its Lie algebra $\mathfrak{g}$ is semisimple.

A Cartan involution $\theta$ of a semisimple Lie algebra $\mathfrak{g}$ is a function defined on $\mathfrak{g}$ given by

$$\theta = \begin{cases} +1 & \text{on } \mathfrak{k} \\ -1 & \text{on } \mathfrak{p} \end{cases}$$

The positive definite bilinear form defined by

$$B_\theta(X,Y) = -B(X, \theta Y) \text{ for } X, Y \in \mathfrak{g}$$

induces a left invariant Riemannian metric on $G$. This metric will be referred to as the canonical metric for $G$ and denoted by $g_0$.

Given a connection $\nabla$ on the tangent bundle of a manifold, the curvature tensor is defined by

$$R(U,V)X = \nabla_U \nabla_V X - \nabla_V \nabla_U X - \nabla_{\nabla_U V} X.$$  

When $G$ is compact semisimple, $B$ is negative definite (no $\mathfrak{p}$ component) and $B_\theta = -B$ induces a biinvariant metric on $G$. With this canonical metric, the connection and curvature are simply given by (see e.g. [5 Cor. 3.19])

$$\nabla_X Y = \frac{1}{2}[X,Y],$$

(1.3)

$$\langle R(X,Y)Y, X \rangle = \frac{1}{4}\|X, Y\|^2.$$  

(1.4)
When $G$ is noncompact semisimple, $B_\theta$ is biinvariant only when restricted to $K$, the maximal compact subgroup of $G$. Note that the Lie algebra of $K$ is $\mathfrak{k}$. We now derive simple connection and curvature formulas with respect to the canonical metric for a noncompact semisimple Lie group.

First for any left invariant metric and left invariant vector fields $X, Y, Z, W$, the Koszul formula gives
\[
\langle \nabla_X Y, Z \rangle = \frac{1}{2} \left\{ \langle [X, Y], Z \rangle - \langle Y, [X, Z] \rangle - \langle X, [Y, Z] \rangle \right\}.
\] (1.5)

By left invariance
\[ X \langle \nabla_Y Z, W \rangle = 0. \]

Therefore
\[ \langle \nabla_X \nabla_Y Z, W \rangle = -\langle \nabla_Y Z, \nabla_Z W \rangle, \]
and
\[ \langle R(X, Y)Z, W \rangle = \langle \nabla_X \nabla_Y Z, W \rangle - \langle \nabla_Y Z, \nabla_X W \rangle - \langle \nabla_X [Y, Z], W \rangle. \] (1.6)

We will treat vector fields from $\mathfrak{k}$ and $\mathfrak{p}$ separately. From here on, $U, V, W, W' \in \mathfrak{k}$ and $X, Y, Z, Z' \in \mathfrak{p}$ denote left invariant vector fields. We have

**Lemma 1.1.** With respect to the canonical metric the subgroup $K$ is totally geodesic in $G$.

**Proof.** Since $B_\theta$ restricted to $K$ is biinvariant
\[ \langle \nabla_U U, V \rangle = 0. \]

By (1.5) and (1.2)
\[ \langle \nabla_U U, X \rangle = -\langle U, [U, X] \rangle = 0. \]

□

Now we compute the connections.

**Lemma 1.2.**
\[
\begin{align*}
\nabla_U V &= \frac{1}{2} [U, V], \quad (1.7) \\
\nabla_X Y &= \frac{1}{2} [X, Y], \quad (1.8) \\
\n\nabla_U X &= \frac{3}{2} [U, X], \quad \nabla_X U = -\frac{1}{2} [X, U]. \quad (1.9)
\end{align*}
\]

**Proof.** The first equation follows from Lemma 1.1. We will derive the last two equations, the proof of the second equation is similar.

By (1.5) and (1.2)
\[ \langle \nabla_U X, V \rangle = \frac{1}{2} \langle [U, X], V \rangle - \frac{1}{2} \langle X, [U, V] \rangle - \frac{1}{2} \langle U, [X, V] \rangle = 0 \]
and
\[ \langle \frac{3}{2} [U, X], V \rangle = 0. \]
Thus
\[ \langle \nabla_U X, V \rangle = \langle \frac{3}{2} [U, X], V \rangle \quad \forall \ V \in \mathfrak{k}. \quad (1.10) \]

Similarly, by (1.5), (1.1) and the relationship between \( B \) and \( B_\theta \),
\[ \langle \nabla_U X, Y \rangle = \frac{1}{2} \langle [U, X], Y \rangle - \frac{1}{2} \langle X, [U, Y] \rangle - \frac{1}{2} \langle U, [X, Y] \rangle \]
and
\[ \langle X, [U, Y] \rangle = -\langle [U, X], Y \rangle \]
and
\[ \langle U, [X, Y] \rangle = -B(U, [X, Y]) = B([X, U], Y) = \langle [X, U], Y \rangle. \]

Hence
\[ \langle \nabla_U X, Y \rangle = \frac{1}{2} \langle [U, X], Y \rangle + \frac{1}{2} \langle [U, X], Y \rangle + \frac{1}{2} \langle [U, X], Y \rangle = \langle \frac{3}{2} [U, X], Y \rangle \quad \forall \ Y \in \mathfrak{p}. \quad (1.11) \]

From (1.10) and (1.11) we have
\[ \nabla_U X = \frac{3}{2} [U, X]. \]

Finally,
\[ \nabla_X U = \nabla_U X + [X, U] = -\frac{1}{2} [X, U]. \]

For the curvatures we have

**Proposition 1.3.**
\[ \langle R(U, V)W, W' \rangle = \frac{1}{4} \left( \langle [U, W'], [V, W] \rangle - \langle [U, W], [V, W'] \rangle \right), \quad (1.12) \]
\[ \langle R(X, Y)Z, Z' \rangle = \frac{7}{4} \left( \langle [X, Z], [Y, Z'] \rangle - \langle [Y, Z], [X, Z'] \rangle \right), \quad (1.13) \]
\[ \langle R(U, X)V, Y \rangle = \frac{3}{4} \left( 3 \langle [U, V], [X, Y] \rangle + \langle [X, V], [U, Y] \rangle \right), \quad (1.14) \]
\[ \langle R(U, V)X, Y \rangle = \frac{3}{4} \left( \langle [U, X], [V, Y] \rangle - \langle [V, X], [U, Y] \rangle \right), \quad (1.15) \]
\[ \langle R(U, V)W, X \rangle = 0. \quad (1.16) \]
\[ \langle R(X, Y)Z, U \rangle = 0. \quad (1.17) \]

In particular,
\[ \langle R(U, V)V, U \rangle = \frac{1}{4} \| [U, V] \|^2, \quad (1.18) \]
\[ \langle R(X, Y)Y, X \rangle = -\frac{7}{4} \| [X, Y] \|^2, \quad (1.19) \]
\[ \langle R(U, X)X, U \rangle = \frac{1}{4} \| [U, X] \|^2. \quad (1.20) \]
Proof. We prove (1.13). The proofs of the remaining equations are similar.

By (1.6) and Lemma 1.2

\[
\langle R(X, Y)Z, Z' \rangle = \langle \nabla_X Z, \nabla_Y Z' \rangle - \langle \nabla_Y Z, \nabla_X Z' \rangle - \frac{1}{4}\langle [X, Y], [Z, Z'] \rangle - \frac{3}{2}\langle [X, Y], [Z, Z'] \rangle.
\]

Using the Jacobi identity

\[
[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0
\]

and (1.1) we have

\[
\langle [[X, Y], Z], Z' \rangle = \langle [X, [Y, Z]], Z' \rangle + \langle [Y, [Z, X]], Z' \rangle - B([[Y, Z], [X, Z']]) - B([[Z, X], [Y, Z']])
\]

\[
\langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle.
\]

And (1.13) follows. \(\square\)

2. The Semisimple Lie Group \(O(n,1)\)

Denote by \(GL(n, \mathbb{R})\) the real general linear group consisting of all nonsingular \(n\)-by-\(n\) real matrices with matrix multiplication as group operation. Let \(J \in GL(n+1, \mathbb{R})\) be given by

\[
J = \begin{pmatrix}
1 & & \\
& \ddots & \\
& & 1
\end{pmatrix}
\]

The Lorentz group of \(n+1\)-by-\(n+1\) matrices \(O(n,1)\) is defined by

\[
O(n,1) = \{ A \in GL(n+1, \mathbb{R}) : JA^TJ = A^{-1} \}.
\]

The Lorentz group is a matrix Lie group, it is a differentiable manifold where matrix multiplication is compatible with the smooth structure. In this section we study its Lie algebra structure and then we compute its canonical metric and curvature. We note here that the results of this section apply immediately to \(SO_o(n,1)\), the connected component of the identity of \(O(n,1)\).

For a matrix Lie group \(G\), its Lie algebra \(\mathfrak{g}\) is the set of all matrices \(X\) such that \(e^{tX} \in G\) for all real numbers \(t\). Let \(\mathfrak{o}(n,1)\) denote the Lie algebra of \(O(n,1)\). Then

\[
X \in \mathfrak{o}(n,1) \Rightarrow e^{tX} \in O(n,1) \Rightarrow J(e^{tX})^TJ = (e^{tX})^{-1}
\]

\[
J e^{tX} J = e^{-tX}
\]

\[
e^{tJX^TJ} = e^{-tX}
\]

\[
J X^TJ = -X
\]
Let $X = (a_{ij})$ be an $n+1$-by-$n+1$ matrix. The equation $JX^TJ = -X$ implies that $X$ has the form

$$
\begin{pmatrix}
0 & a_{12} & a_{13} & \ldots & a_{1,n+1} \\
-a_{12} & 0 & a_{23} & \ldots & a_{2,n+1} \\
-a_{13} & -a_{23} & 0 & \ddots & \\
\vdots & \ddots & \ddots & \ddots & \\
a_{1,n+1} & a_{2,n+1} & \ldots & 0 & 0
\end{pmatrix}.
$$

Let $e_{ij}$ denote a square matrix (of appropriate size) with 1 in the $ij$-position and 0 everywhere else.

**Definition 2.1.** For each $n$, let $\alpha_{ij} = e_{ij} - e_{ji}$, represent the $n+1$-by-$n+1$ matrix with 1 in the $ij$-position, $-1$ in the $ji$-position and 0 elsewhere. Similarly, let $\sigma_{ij} = e_{ij} + e_{ji}$ denote the $n+1$-by-$n+1$ matrix with 1 in the $ij$-position, 1 in the $ji$-position and 0 elsewhere.

The **standard basis** for $\mathfrak{o}(n,1)$, denoted by $\mathcal{B}$, consists of the following set of $n(n+1)/2$ matrices:

$$
\begin{array}{cccccc}
\alpha_{12} & \alpha_{13} & \alpha_{14} & \ldots & \alpha_{1n} & \sigma_{1,n+1} \\
\alpha_{23} & \alpha_{24} & \ldots & \alpha_{2n} & \sigma_{2,n+1} \\
\alpha_{34} & \ldots & \alpha_{3n} & \sigma_{3,n+1} \\
\vdots & & \ddots & \ddots & \ddots & \ddots \\
\alpha_{n-1,n} & \sigma_{n-1,n+1} \\
\alpha_{n,n} & \sigma_{n,n+1}
\end{array}
$$

The **Lie bracket** of a matrix Lie algebra is a binary operation

$$[\cdot, \cdot] : g \times g \to g$$

given by matrix multiplication

$$[X,Y] \mapsto XY - YX.$$ 

The Lie brackets of $\mathfrak{o}(n,1)$ are given by

**Proposition 2.2.** For $1 \leq i < j \leq n, 1 \leq k < l \leq n$, 

$$
[\alpha_{ij}, \alpha_{kl}] = \delta_{jk} \alpha_{il} + \delta_{jl} \alpha_{ki} + \delta_{ij} \alpha_{jk} + \delta_{ki} \alpha_{lj}
$$

$$
= \begin{cases}
\alpha_{il} & \text{if } j = k \\
\alpha_{ki} & \text{if } j = l \\
\alpha_{jk} & \text{if } i = l \\
\alpha_{ij} & \text{if } i = k \\
0 & \text{otherwise}
\end{cases}
$$

(2.21)
\[ [\alpha_{ij}, \sigma_{k,n+1}] = \delta_{kj}\sigma_{i,n+1} - \delta_{ik}\sigma_{j,n+1}, \quad (2.23) \]

\[
[\begin{array}{ll}
\alpha_{ij} & \sigma_{k,n+1} \\
\alpha_{kl} & \sigma_{j,n+1}
\end{array}] = \left[ \begin{array}{ll}
e_{ij} - e_{ji} & e_{kl} - e_{lk} \\
e_{kl} - e_{lk} & e_{ij} - e_{ji}
\end{array} \right] = \delta_{jk}\alpha_{il} + \delta_{jl}\alpha_{ki} + \delta_{ij}\alpha_{lk} + \delta_{ki}\alpha_{lj}.
\]

\[
[\sigma_{i,n+1}, \sigma_{j,n+1}] = \alpha_{ij} \quad (2.25)
\]

\textbf{Proof.} The proof of the first equation is given here. The proofs of the remaining identities are similar.

By the definition of \( \alpha_{ij} \) and the fact that \( e_{ij}e_{kl} = \delta_{jk}e_{il} \),

\[
[\alpha_{ij}, \alpha_{kl}] = [e_{ij} - e_{ji}, e_{kl} - e_{lk} - (e_{kl} - e_{lk})(e_{ij} - e_{ji})
\]

\[
= e_{ij}e_{kl} - e_{ij}e_{lk} - e_{ji}e_{kl} - e_{kl}e_{ij} + e_{lk}e_{ij} - e_{ik}e_{jl} + e_{kj}e_{il} - e_{jl}e_{kj} - \delta_{ij}\alpha_{kl} + \delta_{kl}\alpha_{ij}.
\]

\[
\square
\]

\textbf{Remark 2.3.} When deriving general conclusions from Proposition 2.2, it is useful to keep in mind the anti-commutativity of the Lie bracket and the fact that \( \alpha_{ji} = -\alpha_{ij} \).

Proposition 2.2 illustrates a Cartan decomposition \( \mathfrak{o}(n,1) = \mathfrak{k} \oplus \mathfrak{p} \), where

\( \mathfrak{k} = \text{span}\{\alpha_{ij}, 1 \leq i < j \leq n\}, \mathfrak{p} = \text{span}\{\sigma_{i,n+1}, 1 \leq i \leq n\} \)

satisfying the brackets law \( \mathfrak{l}(2.2) \). We note here that \( \mathfrak{k} = \mathfrak{so}(n) \), the Lie algebra of the Lie group \( SO(n) \). In turn, \( SO(n) \) is the maximal compact subgroup of \( O(n,1) \).

An explicit Cartan involution on \( \mathfrak{o}(n,1) \) can be given by

\[ \theta(X) = -X^T. \]

The following lemma describes the canonical metric for \( O(n,1) \)

\textbf{Lemma 2.4.} Let \( X, Y \in \mathfrak{B} \). Then

\[ \langle X, Y \rangle = \left\{ \begin{array}{ll}
2n - 2 & \text{if } X = Y \\
0 & \text{otherwise}
\end{array} \right. \]

\textbf{Proof.} The proof follows from a close study of Proposition 2.2 and Remark 2.3.

The set \( \mathfrak{B} \) is closed under the Lie bracket (modulo sign). Therefore, for any \( X \in \mathfrak{B} \) the entries of \( \text{ad} \ X \) are all \( 0, 1 \) or \( -1 \) and each column has at most one non-zero entry. Since bracket multiplication is determined by index, each row also has at most one non-zero entry. Furthermore, two standard basis elements have a non-zero Lie bracket if and only if they share exactly one index number. So if \( X \) has index \( ij \), \( \text{ad} \ X \) has exactly

\[ (n + 1 - i) + (j - 1) + (n + 1 - j) + (i - 1) - 1 - 1 = 2n - 2 \]
non zero entries.

Now assume $X = \alpha_{ij}$. For all $Y \in \mathfrak{B}$, $[X, Y] = Z \Rightarrow [X, Z] = -Y$. This implies that the $hg$ entry of $ad X$ is the negative of the $gh$ entry.

By definition
\[
\langle \alpha_{ij}, \alpha_{ij} \rangle = -B(\alpha_{ij}, \theta(\alpha_{ij})) = -B(\alpha_{ij}, \alpha_{ij}) = -\text{trace}(ad \alpha_{ij} \circ ad \alpha_{ij}).
\]

The $h$th diagonal entry of $ad \alpha_{ij} \circ ad \alpha_{ij}$ is the dot product of the $h$th row of $ad \alpha_{ij}$ with the $h$th column of $ad \alpha_{ij}$. If the only non-zero entry in the $h$th row of $ad \alpha_{ij}$ is a $1$ (resp. $-1$) in the $hg$-position then the only non-zero entry in the $h$th column of $ad \alpha_{ij}$ is a $-1$ (resp. $1$) in the $gh$-position. Hence, the $h$th diagonal entry of $ad \alpha_{ij} \circ ad \alpha_{ij}$ is $-1$. Thus,
\[
\langle \alpha_{ij}, \alpha_{ij} \rangle = -\left(\frac{-1 + -1 + \cdots + -1}{2n-2 \text{ times}}\right) = 2n - 2
\]

Similarly, $\langle \sigma_{ij}, \sigma_{ij} \rangle = 2n - 2$.

Let $X, Y \in \mathfrak{B}$ with $X \neq \pm Y$. If $ad X$ has a nonzero entry in the $hg$-position then the bracket of $X$ with the $h$th basis element is sent to the $g$th basis element. That is, there is some $V, W \in \mathfrak{B}$ such that
\[
[X, V] = \pm W.
\]
If, in addition, $ad Y$ has a nonzero entry in the $gh$-position, we may write
\[
[Y, W] = \pm V.
\]
Again, note that the Lie bracket of basis elements is determined by index. This forces
\[
X = \pm Y
\]
and we have a contradiction. Thus, all the diagonal entries of $ad X \circ ad Y$ are equal to zero. Therefore $\langle X, Y \rangle = 0$. \hfill \Box

**Corollary 2.5.** The matrix representation for the canonical metric of $O(n, 1)$ with respect to the standard basis of $\mathfrak{o}(n, 1)$ is the square $n(n+1)/2$ diagonal matrix
\[
\begin{pmatrix}
n & 2n-2 \\
2n-2 & 2n-2 \\
& & \ddots \\
& & & 2n-2
\end{pmatrix}.
\]

For any $X, Y \in \mathfrak{g}$, the sectional curvature of the planes spanned by $X$ and $Y$ is denoted and defined by
\[
K(X, Y) = \frac{\langle R(X, Y)Y, X \rangle}{\|X\|^2 \|Y\|^2 - \langle X, Y \rangle^2}.
\]
With respect to the canonical metric $g_0$, the quotient $\mathbb{H}^n = SO_o(n, 1)/SO(n)$ has constant sectional curvature $\frac{1}{2(n-1)}$. In order to have $\mathbb{H}^n$ with constant sectional curvature $-1$, we consider the metric $\tilde{g}_0 = \frac{1}{2(n-1)}g_0$.

**Proposition 2.6.** The sectional curvature of $O(n, 1)$ with respect to $\tilde{g}_0$ at the planes spanned by the standard basis is bounded above by $\frac{1}{4}$.

**Proof.** Since $\alpha_{ij}, \alpha_{kl}$ are orthogonal,
\[
K(\alpha_{ij}, \alpha_{kl}) = \frac{\langle R(\alpha_{ij}, \alpha_{kl})\alpha_{kl}, \alpha_{ij}\rangle}{\|\alpha_{ij}\|^2\|\alpha_{kl}\|^2}.
\]

By (1.18), Proposition 2.2 and Corollary 2.5
\[
K(\alpha_{ij}, \alpha_{kl}) = \frac{\|[\alpha_{ij}, \alpha_{kl}]\|^2}{4\|\alpha_{ij}\|^2\|\alpha_{kl}\|^2} \leq \frac{1}{4}.
\] (2.26)

Similarly
\[
K(\alpha_{ij}, \sigma_{k,n+1}) = \frac{\|[\alpha_{ij}, \sigma_{k,n+1}]\|^2}{4\|\alpha_{ij}\|^2\|\sigma_{k,n+1}\|^2} \leq \frac{1}{4}
\] (2.27)
and
\[
K(\sigma_{i,n+1}, \sigma_{j,n+1}) = -\frac{7\|[\sigma_{i,n+1}, \sigma_{j,n+1}]\|^2}{4\|\sigma_{i,n+1}\|^2\|\sigma_{j,n+1}\|^2} \leq 0.
\] (2.28)

\[\square\]

**Proposition 2.7.** The sectional curvature of $O(n, 1)$ with respect to $\tilde{g}_0$ is bounded above by
\[\frac{n(n-1)}{8} + 2\frac{n}{4} + 2\frac{6n}{4} + 2\frac{4n}{4} = \frac{n^2 + 43n}{8}.
\]

**Proof.** Again with $U, V \in \mathfrak{k}$ and $X, Y \in \mathfrak{p}$, we have by (1.16) and (1.17)
\[
\langle R(X + U, Y + V)Y + V, X + U \rangle = \langle R(X, Y)Y, X \rangle + \langle R(U, V)V, U \rangle + \langle R(U, Y)V, U \rangle + \langle R(X, V)V, X \rangle + 2\langle R(X, Y)V, U \rangle + 2\langle R(X, V)Y, U \rangle.
\]
Assume that $\|U + X\| = 1$, $\|V + Y\| = 1$ and $\langle U + X, V + Y \rangle = 0$. Write
\[
U = \sum_{i<j} a_{ij} \alpha_{ij}, \ V = \sum_{i<j} a'_{ij} \alpha_{ij}, \ X = \sum_{i=1}^{n} b_i \sigma_{i,n+1}, \ Y = \sum_{i=1}^{n} b'_i \sigma_{i,n+1}.
\]

Note that
\[
\sum_{i<j} |a_{ij}|^2, \ \sum_{i<j} |a'_{ij}|^2, \ \sum_{i=1}^{n} |b_i|^2, \ \sum_{i=1}^{n} |b'_i|^2 \leq 1.
\] (2.29)
By (1.18) and (2.22)

\[ \langle R(U, V) V, U \rangle = \frac{1}{4} \| \sum_{i<j} a_{ij} \alpha_{ij} + \sum_{k<l} a'_{kl} \alpha_{kl} \|^2 \]

\[ = \frac{1}{4} \sum_{i<j} \left( \sum_{k \neq i,j} a_{ik} a'_{kj} - a'_{ik} a_{kj} \right) \alpha_{ij} \|^2 \]

\[ = \frac{1}{4} \sum_{i<j} \left( \sum_{k \neq i,j} a_{ik} a'_{kj} - a'_{ik} a_{kj} \right)^2 \]

\[ \leq \frac{1}{4} \frac{n(n-1)}{2} = \frac{n(n-1)}{8}. \]

Similarly, by (1.20), (1.15), (1.14) and (2.23)

\[ \langle R(U, Y) Y, U \rangle = \frac{1}{4} \| \sum_{i<j} a_{ij} \alpha_{ij} + \sum_{k} b_i' \sigma_{k,n+1} \|^2 \]

\[ = \frac{1}{4} \sum_{k} \left( \sum_i a_{ki} b_i' \right) \sigma_{k,n+1} \|^2 \]

\[ = \frac{1}{4} \sum_{k} \left( \sum_i a_{ki} b_i' \right)^2 \]

\[ \leq \frac{1}{4} C_1, \]

\[ \langle R(X, Y) V, U \rangle \leq 6 \cdot \frac{n}{4}, \]

and

\[ \langle R(X, V) Y, U \rangle \leq 4 \cdot \frac{n}{4}. \]

\[ \square \]

3. H. C. Wang’s Result

A classical theorem due to Kazhdan-Margulis [17] states that every semisimple Lie group without compact factor has a neighborhood $U$ of the identity $e$ such that, given any discrete subgroup $\Gamma$ of $G$, there exists $g \in G$ with the property that $g \Gamma g^{-1} \cap U = \{ e \}$. H. C. Wang [26] undertook a quantitative study of the neighborhood $U$.

Let $G$ be a semisimple Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a Cartan decomposition. For each endomorphism $f : \mathfrak{g} \to \mathfrak{g}$, let $N(f) = \sup \{ \| f(X) \| : X \in \mathfrak{g}, \| X \| = 1 \}$. Define $C_1 = \sup \{ N(\mathrm{ad} X) : X \in \mathfrak{p}, \| X \| = 1 \}$, $C_2 = \sup \{ N(\mathrm{ad} X) : X \in \mathfrak{k}, \| X \| = 1 \}$. The number $R_G$ is the least positive zero of the real-valued function

\[ F(t) = \exp C_1 t - 1 + 2 \sin C_2 t - C_1 t/(\exp C_1 t - 1). \]
Let $\rho$ be the distance function on $G$ induced by the canonical metric. For any Lie group, a theorem of Zassenhaus [27] guarantees the existence of a neighborhood $U$ of the identity such that the subgroup generated by any subset of $U$ is either non-discrete or nilpotent. The following theorem (Theorem 3.2 in [26]) demonstrates the role of $R_G$ in the construction of an explicit Zassenhaus neighborhood for a semisimple Lie group.

**Theorem 3.1 (Wang).** Let $G$ be a semisimple Lie group. Let $e \in G$ denote the identity. Then for any discrete subgroup $\Gamma$ of $G$, the set

$\Theta = \{ g \in \Gamma : \rho(e, g) \leq R_G \}$

generates a nilpotent group.

Now, let $g_\pi$ be the totality of elements $X$ in $g$ such that the imaginary parts of all the eigenvalues of $\text{ad} X$ lie in the open interval $(-\pi, \pi)$ and let $G_\pi = \{ \exp X : X \in g_\pi \}$. In an earlier paper [25], Wang had proved that the restriction of the exponential map to $g_\pi$ is injective. Therefore the following proposition (Proposition 5.1 in [26]), derived from Theorem 3.1, establishes the fact that $R_G$ is less than the injectivity radius of $G$.

**Proposition 3.2 (Wang).** Let $G$ be a semisimple Lie group. Then the closed ball

$B = \{ x \in G : \rho(e, x) \leq R_G \}$

is contained in $G_\pi$.

We now give Wang’s quantitative version of the theorem of Kazhdan-Margulis (Theorem 5.2 in [26]). It shows that the volume of the fundamental domain of $\Gamma$ is larger than the volume of a $\rho$-ball with radius $R_G/2$.

**Theorem 3.3 (Wang).** Let $G$ be a semisimple Lie group without compact factor and $B = \{ x \in G : \rho(e, x) \leq R_G \}$. Then for any discrete subgroup $\Gamma$ of $G$, there exists $g \in G$ such that $B \cap g\Gamma g^{-1} = \{ e \}$.

The values of $R_G$ important for this paper are

$R_G = 277\sqrt{2}/1000$ when $G = SO_\sigma(3, 1)$

$R_G = 228\sqrt{2(n - 1)}/1000$ when $G = SO_\sigma(n, 1)$ for $n \geq 4$. \hspace{1cm} (3.30)

4. The Volume of Hyperbolic $n$-Orbifolds

A hyperbolic $n$-orbifold $Q = \mathbb{H}^n/\Gamma$ is the quotient of hyperbolic $n$-space by a discrete group of orientation-preserving isometries. The volume of $Q$ is defined to be the volume of the fundamental domain of $\Gamma$ in $\mathbb{H}^n$. In this section, we establish key facts about volume relating to Riemannian submersions and sectional curvature. We then produce a lower bound for the volume a hyperbolic orbifold of dimension greater than or equal to four.

Let $(M, g)$ and $(N, h)$ be Riemannian manifolds and $q : M \to N$ a surjective submersion. For each point $x \in M$ the tangent space $T_xM$ decomposes into the orthogonal direct sum

$T_xM = (\text{Ker} dq)^\perp_x \oplus (\text{Ker} dq)_x$.  

The map $q$ is said to be a Riemannian submersion if
$$g(X, Y) = h(dqX, dqY)$$ whenever $X, Y \in (\text{Ker } dq)^\perp_x$ for some $x \in M$.

**Lemma 4.1.** Let $K \to M \xrightarrow{q} N$ denote a fiber bundle where $q$ is a Riemannian submersion and $K$ is a compact and totally geodesic submanifold of $M$. Then for any subset $Z \subset N$,
$$\text{Vol}[q^{-1}(Z)] = \text{Vol}[Z] \cdot \text{Vol}[K]$$

**Proof.** Since $K$ is totally geodesic, the fibers of $q$ are isometric to each other. We have
$$d\text{Vol}_M = d\text{Vol}_N \cdot d\text{Vol}_K.$$ Thus $$\text{Vol}[q^{-1}(Z)] = \int_{q^{-1}(Z)} d\text{Vol}_M = \int_{q^{-1}(Z)} d\text{Vol}_K \cdot d\text{Vol}_N = \text{Vol}(K) \cdot \text{Vol}(Z).$$ □

Now fix a point $p \in \mathbb{H}^n$ and let $\pi: SO_o(n, 1) \to \mathbb{H}^n$ be defined by
$$\pi(s) = s(p) \quad \forall \ s \in SO_o(n, 1).$$

The map $\pi$ is a Riemannian submersion. The fiber over $p$, the set of orientation-preserving isometries of $\mathbb{H}^n$ that fix $p$, is an embedded copy of $SO(n)$. This also gives a Riemannian submersion
$$SO(n) \to SO_o(n, 1)/\Gamma \to \mathbb{H}^n/\Gamma = Q$$
with totally geodesic fibers on the smooth points of $Q$. Hence,
$$\text{Vol}[Q] = \frac{\text{Vol}[\pi^{-1}(Q)]}{\text{Vol}[SO(n)]} \cdot \frac{\text{Vol}[SO_o(n, 1)/\Gamma]}{\text{Vol}[SO(n)]}. \tag{4.31}$$

In [12, Page 399], the volumes of the classical compact groups are given explicitly. For the special orthogonal group, the volume with respect to the metric $\tilde{g}_0$ is given by
$$\text{Vol}[SO(n)] = \frac{2^{\frac{n^2+2n-2}{2}}\pi^{\frac{n^2}{4}}}{(n-2)!(n-4)! \cdots 1}. \tag{4.32}$$
Therefore we are left to consider $\text{Vol}[SO_o(n, 1)/\Gamma]$.

Denote by $V(d, k, r)$ the volume of a ball of radius $r$ in the complete simply connected Riemannian manifold of dimension $d$ with constant curvature $k$. A proof of the following comparison theorem can be found in [10, Theorem 3.101].

**Theorem 4.2** (Gunther). Let $M$ be a complete Riemannian manifold of dimension $d$. For $m \in M$, let $B_m(r)$ be a ball which does not meet the cut-locus of $m$.

If the sectional curvatures of $M$ are bounded above by a constant $b$, then
$$\text{Vol}[B_m(r)] \geq V(d, b, r).$$

**Proposition 4.3.** Let $\Gamma$ be a discrete subgroup of $SO_o(n, 1)$. Then
$$\text{Vol}[SO_o(n, 1)/\Gamma] \geq V(d_0, k_0, r_0)$$
where $d_0 = \frac{n^2 + n}{2}$, $k_0 = \frac{n^2 + 43n}{8}$ and $r_0 = 0.114 \sqrt{2(n-1)}$. 

Proof. From Definition 2.1 the dimension of $SO_0(n,1)$ is $(n^2 + n)/2$. By Proposition 2.7 the sectional curvatures of $SO_0(n,1)$ are bounded above by $(n^2/2 + 43n)/8$.

The Proposition then follows from Proposition 3.2, Theorems 3.3, 4.2 and (3.30). □

We conclude with an explicit version of our main theorem. Combining Proposition 4.3 with (4.32) gives a formula for the lower bound of a hyperbolic orbifold dependent only on dimension.

**Theorem 4.4.** Let $Q$ be a hyperbolic $n$-orbifold. Then the volume of $Q$ is bounded below by

$$2^{2n^2+n+6} |π|^4 \left( \frac{1}{n^2 + 43n} \right)^{[n^2+n]/4} \frac{(n-2)!(n-4)! \ldots 1}{\Gamma(n^2+n)} \int_0^{\min[0.057\sqrt{n^3+42n^2-43n},\pi]} \sin^{1+n^2-4/4} \rho \ d\rho.$$  

Proof. As an immediate consequence of Proposition 4.3 and (4.31), we have

$$\text{Vol}[Q] \geq \frac{V(d_0,k_0,r_0)}{\text{Vol}[SO(n)]}$$

For a given $k > 0$, the complete simply connected Riemannian manifold with constant curvature $k$ is the sphere of radius $k^{-1/2}$. By explicit computation we have

$$V(d,k,r) = 2(\pi/k)^{d/2} \int_0^{\min[\pi^{1/2},\pi]} \sin^{d-1} \rho \ d\rho.$$  

Direct substitution and (4.32) completes the proof. □

5. Volume Bounds

In this section, we list several known results on the volume of hyperbolic $n$-orbifolds. For ease of comparison, we approximate to two significant digits.

5.1. Cusped Hyperbolic Orbifolds. The smallest cusped hyperbolic 2-orbifold has volume $5.23 \times 10^{-1}$ [23]. The corresponding bound is $7.22 \times 10^{-2}$ in dimension three [20] and $6.85 \times 10^{-3}$ in dimension four [14]. Analogous results for all dimensions less than ten can be found in [13].

5.2. Hyperbolic Manifolds. That the smallest hyperbolic 2-manifold has area $4\pi$ is a classical result. More recently, it was shown in [9] that the Weeks manifold, with volume .94, is the hyperbolic 3-manifold of minimum volume. Both are arithmetically defined [7].

An explicit lower bound for the radius of a ball that can be embedded in every hyperbolic $n$-manifold was given in [13]. An error in that paper was later corrected in [8]. Using the corrected radius

$$\frac{0.0025}{17^{n/2}},$$  

one can obtain a lower bound for the volume of a hyperbolic $n$-manifold. In dimension three the bound is $1.33 \times 10^{-11}$. 
5.3. Arithmetic Orbifolds. As mentioned in the introduction, arithmetic hyperbolic $n$-orbifolds are conjectured to be extremal in terms of volume. This has been shown to be the case in dimensions two and three. The minimal volume arithmetic $n$-orbifolds were identified for all dimensions greater than or equal to four in [2], [3], [4]. The value for $n = 2r, r$ even is given by

$$
\omega_c(n) = \frac{4 \cdot 5^r r^2 + r/2}{(2r - 1)!!} \cdot (2\pi)^{r} \prod_{i=1}^{r} \frac{(2i - 1)!^2}{(2\pi)^{2i}} \zeta_{k_0}(2i).
$$

where $\zeta_{k_0}$ represents the Dedekind zeta function of the number field $k_0 = \mathbb{Q}[\sqrt{5}]$.

Hence, for dimension four ($r = 2$) we have $\omega_c(4) = 1.8 \times 10^{-3}$. The cited papers contain similar formulas for $n = 2r, r$ odd and $n = 2r - 1$.

ACKNOWLEDGMENTS

The first author is grateful to Francis Bonahon, Dick Canary and Daryl Cooper for their support and useful conversations. This project was initiated during the first author’s fellowship at the Mathematical Sciences Research Institute in the Fall of 2007.

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