The Fujita phenomenon in exterior domains under dynamical boundary conditions

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Abstract. The Fujita phenomenon for nonlinear parabolic problems
\[ \partial_t u = \Delta u + u^p \] in an exterior domain of \( \mathbb{R}^N \) under
dissipative dynamical boundary conditions \( \sigma \partial_t u + \partial_\nu u = 0 \) is investigated in the superlinear case. As in the case of Dirichlet
boundary conditions (see [2] and [9]), it turns out
that there exists a critical exponent \( p = 1 + \frac{2}{N} \) such that blow-up of positive solutions always occurs for subcritical exponents,
whereas in the supercritical case global existence can occur for small non-negative initial data.

Keywords: nonlinear parabolic problems, dynamical boundary conditions, global solutions

1. Introduction

Let \( \Omega \) be an exterior domain of \( \mathbb{R}^N \), that is to say a connected open set \( \Omega \) such that \( \overline{\Omega}^c \) is a bounded
domain when \( N \geq 2 \), and in dimension one, \( \Omega \) is the complement of a real closed interval. We always
suppose that the boundary \( \partial \Omega \) is of class \( C^2 \). The outer normal unit vector field is denoted by \( \nu : \partial \Omega \to \mathbb{R}^N \)
and the outer normal derivative by \( \partial_\nu \). Let \( p \) be a real number with \( p > 1 \) and \( \varphi \) be a continuous
function in \( \overline{\Omega} \).

Consider the following nonlinear parabolic problem

\[
\begin{ Alignat* } { 2 }
\partial_t u &= \Delta u + u^p & \text{in } & \overline{\Omega} \times (0, \infty), \\
B_\sigma(u) := \sigma \partial_t u + \partial_\nu u &= 0 & \text{on } & \partial \Omega \times (0, \infty), \\
u(\cdot, 0) &= \varphi & \text{in } & \overline{\Omega}.
\end{ Alignat* }
\] (1)

The aim of this paper is to show that the well-known Fujita phenomenon in the case of \( \Omega = \mathbb{R}^N \) (see [6])
and in the case of Dirichlet boundary conditions (see [2] and [9]) still holds for the dynamical boundary
conditions. One can notice that dynamical boundary conditions \( B_\sigma(u) = 0 \) with \( \sigma \equiv 0 \) correspond to the
Neumann boundary conditions, which case has been discussed by Levine and Zhang [8]. It is already
known, by Bandle, vonBelow and Reichel in [1], that for \( p \in (1, 1 + \frac{2}{N}) \), also for \( p = 1 + \frac{2}{N} \) if \( N \geq 3 \),
and for constant coefficient \( \sigma \in [0, \infty) \), all positive solutions of (1) blow up in finite time. In addition,
if the complement is star-shaped there exist global positive solutions of class \( C^1 \) for \( p > 1 + \frac{2}{N} \) by [1].

Our purpose is to show the existence of global positive solutions of Problem (1) for sufficiently small
initial data in the supercritical case \( p > 1 + \frac{2}{N} \) for any exterior domain. Moreover, our condition on \( \sigma \)
is more general. Throughout, we shall assume the dissipativity condition

\[ \sigma \geq 0 \quad \text{on } \partial \Omega \times (0, \infty) \] (2)
and dealing with classical solutions
\[ \sigma \in C^1(\partial \Omega \times (0, \infty)). \] (3)

The initial data is always supposed to be continuous, non-trivial, bounded, non-negative in \( \Omega \), and vanishing at infinity
\[ \varphi \in C(\overline{\Omega}), \quad 0 < \| \varphi \|_{\infty} < \infty, \quad \varphi \geq 0, \quad \lim_{\|x\|_2 \to \infty} \varphi(x) = 0. \] (4)

In the case \( \Omega = \mathbb{R}^N \), the boundary condition is dropped and the result is well known by the classical paper of Fujita [6]. Thus, we will suppose \( \Omega \neq \mathbb{R}^N \).

2. Preliminaries

First, we give the definition of positive solution which is understood along this paper.

**Definition 2.1.** A positive solution of Problem (1) is a positive function \( u : (x, t) \mapsto u(x, t) \) of class \( C(\overline{\Omega} \times [0, T)) \cap C^{2,1}(\overline{\Omega} \times (0, T)) \), satisfying
\[
\begin{cases}
\partial_t u = \Delta u + u^p & \text{in } \overline{\Omega} \times (0, T),
\theta \sigma(u) := \sigma \partial_t u + \partial_u u = 0 & \text{on } \partial \Omega \times (0, T),
\theta(u, 0) = \varphi & \text{in } \overline{\Omega},
\end{cases}
\]
where \( \varphi \) is a function, given in \( C(\overline{\Omega}) \). The time \( T \in [0, \infty) \) is the maximal existence time of the solution \( u \). If \( T = \infty \), the solution \( u \) is called global.

From [2], if \( T < \infty \), \( u \) blows up in finite time, that is to say
\[ \lim_{t \nearrow T} \sup_{x \in \overline{\Omega}} u(x, t) = \infty. \]

Note that for initial data \( \varphi \) of class \( C^2(\overline{\Omega}) \), the solution \( u \) is \( C^{2,1}(\overline{\Omega} \times [0, T)) \), whereas \( u \in C(\overline{\Omega} \times [0, T)) \cap C^{2,1}(\overline{\Omega} \times (0, T)) \) if \( \varphi \) is only continuous in \( \overline{\Omega} \). Then, let us recall a standard procedure to construct solutions of Problem (1) in outer domains for uniformly bounded and continuous initial data \( \varphi \). Let \( B(0, R) \) be the ball centered at the origin of radius \( R > 0 \) such that \( \overline{\Omega}^c \subset B(0, R) \). For any \( n \in \mathbb{N} \), we set \( B_n := B(0, R + n) \) and \( \Omega_n := \Omega \cap B_n \). The boundary of \( \Omega_n \) is decomposed into two disjoint open sets
\[ \partial \Omega_n = \partial \Omega \cup \partial B_n. \]

Define also an increasing sequence of initial data \( (\varphi_n)_{n \in \mathbb{N}^*} \) such that
\[
\begin{align*}
0 \leq \varphi_n & \leq \varphi & \text{in } \overline{\Omega}_n,
\varphi_n & = 0 & \text{on } \partial B_n,
\varphi_n & = \varphi & \text{in } \overline{\Omega}_{n-1},
\end{align*}
\] (5)
and consider the following problem with mixed boundary conditions

\[
\begin{align*}
\partial_t u &= \Delta u + u^p & \text{in } \overline{\Omega}_n \times (0, \infty), \\
B_\sigma(u) := \sigma \partial_t u + \partial_\nu u &= 0 & \text{on } \partial \Omega \times (0, \infty), \\
u(\cdot, 0) &= \varphi & \text{in } \overline{\Omega}_n, \\
\end{align*}
\]

\[(P(n))\]

Let \( z \) be the maximal solution of

\[
\begin{align*}
\dot{z} &= z^p, \\
z(0) &= \|\varphi\|_\infty,
\end{align*}
\]

with maximal existence time \( t_0 = \frac{1}{(p-1)\|\varphi\|_\infty^p} \). It is known from [4] that, for each \( n \in \mathbb{N}^* \), Problem \((P(n))\) has a solution \( u_n \in C(\overline{\Omega}_n \times [0, T_n]) \cap C^{2,1}(\Omega_n \times (0, T_n)) \), where \( T_n \) is the maximal existence time of \( u_n \). Moreover, by comparison principle from [3], we have, for any \( n \in \mathbb{N}^* \), \( 0 \leq u_n \leq u_{n+1} \) and \( u_n(\cdot, t) \leq z(t) \) in \( \overline{\Omega}_n \), so we have also \( t_0 \leq T_n \). Hence we obtain a sequence \( (u_n)_{n \in \mathbb{N}^*} \) of functions in \( C(\overline{\Omega}_n \times [0, t_0]) \cap C^{2,1}(\Omega_n \times (0, t_0)) \). Then, standard arguments based on a priori estimates for the heat equation imply \( u_n \to u \) in the sense of \( C^{2,1}(\Omega \times (0, t_0)) \) as \( n \to \infty \), where \( u \) is a positive solution of Problem (1), see [1] and [7]. Moreover, since \( u_n \) vanishes on \( \partial B_n \) for each \( n \in \mathbb{N}^* \), the solution \( u \) vanishes at infinity

\[
\lim_{\|x\|_2 \to \infty} u(x, t) = 0 \quad \forall t \in (0, T).
\]

Note that \( t_0 \) is only a lower bound for the maximal existence time of solutions \( u_n \) and \( u \), and it is possible that the times \( T_n \) and \( T \) are infinite. Indeed, results on blow-up for problems under dynamical boundary conditions from [5] cannot be applied to the problems \((P(n))\) with mixed boundary conditions because their solutions \( u_n \) vanish on a part of the boundary.

3. Global existence in dimension \( \mathcal{N} \geq 3 \)

Throughout this section, we consider supercritical exponent \( p \),

\[ p > 1 + \frac{2}{\mathcal{N}}. \]

Our technique will be to construct a function that bounds from above each solution \( u_n \) of Problem \((P(n))\). This will give us a sequence \( (u_n)_{n \in \mathbb{N}^*} \) of global solutions in \( C(\overline{\Omega}_n \times [0, \infty)) \cap C^{2,1}(\overline{\Omega}_n \times (0, \infty)) \), thus the solution \( u \) of (1) must be global too. We will proceed by using the solution of the Neumann problem

\[
\begin{align*}
\partial_t v &= \Delta v + v^p & \text{in } \overline{\Omega} \times (0, \infty), \\
\partial_\nu v &= 0 & \text{on } \partial \Omega \times (0, \infty), \\
v(\cdot, 0) &= \psi & \text{in } \overline{\Omega},
\end{align*}
\]

\[(6)\]

with \( \psi \) verifying (4). In [8], Levine and Zhang proved that Problem \((6)\) admits global positive solutions for sufficiently small initial data. We show that the solution \( v \) of Problem \((6)\) bounds from above the
solution $u$ of Problem (1) if the initial data are well ordered ($\varphi \leq \psi$) and if $\psi$ satisfies the following hypotheses

$$\psi \in C^2(\Omega),$$

(7)

and for every $n \in \mathbb{N}^*$

$$\Delta \psi_n + \psi_n^p \geq 0 \quad \text{in } \Omega_n,$$

(8)

where $(\psi_n)_{n \in \mathbb{N}^*}$ is the sequence of truncated initial data introduced in (5). We need a technical lemma, similar to Lemma 2.1 of [4].

**Lemma 3.1.** Let $\psi$ be a function satisfying (4), (7) and (8). For every $n \in \mathbb{N}^*$, the solution $v_n$ of Problem $(P(n))$ under the Neumann boundary conditions and with the truncated initial data $\psi_n$ verifies:

$$\partial_t v_n \geq 0 \quad \text{in } \Omega_n \times (0, T_{v_n}),$$

where $T_{v_n}$ is the maximal existence time of $v_n$.

**Proof.** The function $v_n$ is solution of the following problem

$$\begin{cases}
\partial_t v_n = \Delta v_n + v_n^p & \text{in } \Omega_n \times (0, T_{v_n}), \\
\partial \nu v_n = 0 & \text{on } \partial \Omega \times (0, T_{v_n}), \\
v_n = 0 & \text{on } \partial B_n \times (0, T_{v_n}), \\
v_n(\cdot, 0) = \psi_n & \text{in } \Omega_n,
\end{cases}$$

in the bounded domain $\Omega_n$. From (4) and from the strong maximum principle in [3], we claim

$$v_n > 0 \quad \text{on } (\Omega_n \cup \partial \Omega) \times (0, T_{v_n}).$$

Then, by regularity results from [7], we obtain $v_n \in C^{2,2}(\Omega_n \times (0, T_{v_n}))$, and for $y = \partial_t v_n \in C^{2,1}(\Omega_n \times (0, T_{v_n}))$ we have

$$\begin{cases}
\partial_t y = \Delta y + pv_n^{p-1}y & \text{in } \Omega_n \times (0, T_{v_n}), \\
\partial \nu y = 0 & \text{on } \partial \Omega \times (0, T_{v_n}), \\
y = 0 & \text{on } \partial B_n \times (0, T_{v_n}),
\end{cases}$$

and $y(\cdot, 0) \geq 0$ in $\Omega_n$ thanks to (8). By the comparison principle in [3], we conclude: $y \geq 0$ in $\Omega_n \times (0, T_{v_n})$. □

**Lemma 3.2.** Let a coefficient $\sigma$ verifying (2) and (3), two functions $\varphi$ and $\psi$ satisfying (4) and $\psi$ with (7) and (8). If

$$\varphi \leq \psi \quad \text{in } \Omega,$$

(9)
then Problem (1) with initial data \( \varphi \) admits a solution \( u \) verifying
\[
\begin{align*}
u &\leq v \quad \text{in } \overline{\Omega} \times (0, T_v),
\end{align*}
\]
and
\[
0 < T_v \leq T \leq \infty,
\]
where \( v \) is solution of Problem (6) with initial data \( \psi \), of maximal existence time \( T_v \).

**Proof.** We consider the sequences of truncated solutions \((u_n)_{n \in \mathbb{N}^*}\) and \((v_n)_{n \in \mathbb{N}^*}\) respectively associated to the solutions \( u \) and \( v \). Let \( n \in \mathbb{N}^* \). First, we show that \( u_n \leq v \) in \( \overline{\Omega}_n \times [0, T_v) \). By construction (5), we have \( \psi_n \leq \psi \) in \( \overline{\Omega}_n \). Since \( v \) is a positive solution of Problem (6), it satisfies
\[
\begin{align*}
\partial_t v &\geq \Delta v + v^p \quad \text{in } \overline{\Omega}_n \times (0, T_v), \\
\partial_\nu v &\geq 0 \quad \text{on } \partial \Omega \times (0, T_v), \\
v &\geq 0 \quad \text{on } \partial B_n \times (0, T_v), \\
v(\cdot, 0) &\geq \psi_n \quad \text{in } \overline{\Omega}_n.
\end{align*}
\]
As \( v_n \) is a positive solution of \((P(n))\) under Neumann boundary conditions, we obtain from the comparison principle in [3]
\[
v_n \leq v \quad \text{in } \overline{\Omega}_n \times [0, \tau) \quad (10)
\]
for all \( 0 < \tau < \min\{T_{v_n}, T_v\} \). We deduce \( T_v \leq T_{v_n} \). Then, we show that \( u_n \leq v_n \) in \( \overline{\Omega}_n \times [0, T_{v_n}) \). The previous lemma ensures that \( \partial_t v_n \geq 0 \). From (2), we obtain
\[
\sigma \partial_t v_n + \partial_\nu v_n \geq 0 \quad \text{on } \partial \Omega \times (0, T_{v_n}).
\]
Next, \( v_n \) is a positive solution of \((P(n))\), and thanks to (9), \( v_n \) verifies
\[
\begin{align*}
\partial_t v_n &\geq \Delta v_n + v_n^p \quad \text{in } \overline{\Omega}_n \times (0, T_{v_n}), \\
\sigma \partial_t v_n + \partial_\nu v_n &\geq 0 \quad \text{on } \partial \Omega \times (0, T_{v_n}), \\
v_n &\geq 0 \quad \text{on } \partial B_n \times (0, T_{v_n}), \\
v_n(\cdot, 0) &\geq \varphi_n \quad \text{in } \overline{\Omega}_n.
\end{align*}
\]
Again, by the comparison principle in [3] and by definition of \( u_n \), we obtain
\[
\begin{align*}
u_n &\leq v_n \quad \text{in } \overline{\Omega}_n \times [0, \tau) \quad (11)
\end{align*}
\]
for all \( 0 < \tau < \min\{T_n, T_{v_n}\} \), and hence \( T_{v_n} \leq T_n \). From Eqs (10) and (11), we have \( T_v \leq T_n \) and \( u_n \leq v \) in \( \overline{\Omega}_n \times [0, T_v) \). Thus the solution \( u \) of Problem (1), obtained as the limit of the sequence \((u_n)_{n \in \mathbb{N}^*}\) with the procedure described in section 2, verifies \( u \leq v \) in \( \overline{\Omega} \times [0, T_v) \) and \( T_v \leq T \). \( \Box \)

Now, we just have to choose an initial data \( \psi_n \) sufficiently small such that Problem (6) admits a global positive solution (see [8]), and satisfying (7) and (8).
**Theorem 3.3.** Under conditions (2)–(4), Problem (1) admits global positive solutions for sufficiently small initial data. Moreover, some of these solutions vanish at infinity.

**Proof.** An initial data \( \varphi \) verifying (4) and with \( \varphi \leq \psi_\ast \) in \( \overline{\Omega} \) allows us to conclude thanks to Lemma 3.2. □

**Remark 3.4.** One can notice that only the dissipativity and the regularity of the coefficient \( \sigma \) are needed. We are not obliged to impose any restriction like \( \sigma \) bounded or \( \partial_t \sigma \equiv 0 \). Moreover, the hypotheses (7) and (8) on the initial data \( \psi \) of Problem (6) are strictly technical and do not concern the initial data \( \varphi \) of Problem (1).

### 4. Global existence in lower dimension

In this case, we can not use Levine and Zhang’s result because it is proved only for dimension \( N \geq 3 \): they used some estimates for Green’s functions, specific to dimension \( N \geq 3 \). We need an additional hypothesis on the coefficient \( \sigma \). There exists a constant \( \varsigma \in [0, \infty) \) such that

\[
\forall (x, t) \in \partial \Omega \times [0, \infty): \sigma(x, t) \leq \varsigma. \tag{12}
\]

We begin with the case of dimension 2. Until now, Bandle–von Below–Reichel’s lemma, concerning star-shaped domains, is the best result.

**Lemma 4.1 ([1], Lemma 28).** Suppose \( \sigma \) is a positive constant. If \( \Omega^C \) is star-shaped with respect to the origin and if \( \min_{\partial \Omega} |x \cdot \nu(x)| \geq \sigma N \), then there exist positive global solutions of Problem (1), which vanish at infinity, for sufficiently small initial data.

This allows us to deduce the following result for problems with mixed boundary conditions.

**Corollary 4.2.** Suppose conditions (2), (3) and (12). Let \( y \in \partial \Omega \). There exists a neighborhood \( N_y \) of \( y \) relatively open in \( \partial \Omega \) such that the following parabolic problem with mixed boundary conditions

\[
\begin{aligned}
\partial_t u = \Delta u + u^p & \quad \text{in } \overline{\Omega} \times (0, \infty), \\
B_\sigma(u) = 0 & \quad \text{on } N_y \times (0, \infty), \\
u = 0 & \quad \text{on } \partial \Omega \setminus N_y \times (0, \infty), \\
u(\cdot, 0) = \varphi & \quad \text{in } \overline{\Omega}
\end{aligned}
\]

admits global positive solutions which vanish at infinity, for sufficiently small initial data \( \varphi \) satisfying (4).

**Proof.** Let \( \mu \) be a vector in \( \mathbb{R}^N \) such that the scalar product between the vector \((y + \mu)\) and the outer normal unit vector at \( y \) satisfies

\[
(y + \mu) \cdot \nu(y) < -\varsigma N. \tag{13}
\]
Then, as the mapping $(\partial \Omega \ni z \mapsto (z + \mu) \cdot \nu(z) \in \mathbb{R})$ is continuous, the above inequality remains true on an open neighborhood $N_y \subseteq \partial \Omega$ of $y$. We obtain the statement of the corollary by using the comparison principle from [3] and the function $U$ defined on $\overline{\Omega} \times [0, \infty)$ by

$$U(x, t) = A(t + 1)^{-\gamma} \exp \frac{-\|x + \mu\|^2}{4(t + 1)},$$

with $A = \frac{1}{2}(\frac{N}{2} - \frac{1}{p-1})^{1/(p-1)}$ and $\gamma = \frac{1}{p-1}$. It is clear that $U \geq 0$, belongs to $C^{2,1}(\overline{\Omega} \times [0, \infty))$ and satisfies:

$$\partial_t U = \left(-\gamma \frac{N}{2(t + 1)} + \frac{\|x + \mu\|^2}{4(t + 1)^2}\right)U,$$

$$\Delta U = \left(-\frac{N}{2(t + 1)} + \frac{\|x + \mu\|^2}{4(t + 1)^2}\right)U,$$

$$\partial_\nu U = \left(-\frac{x \cdot \nu(x)}{2(t + 1)}\right)U.$$

We have

$$\partial_t U - \Delta U = \left(-\gamma \frac{N}{t + 1} + \frac{N}{2(t + 1)}\right)U = \left(-\frac{2\gamma + N}{2(t + 1)}\right)U,$$

and by definition of the constants $A$ and $\gamma$, we obtain

$$\partial_t U - \Delta U - U^p \geq 0 \quad \text{in} \quad \overline{\Omega} \times [0, \infty).$$

Then, on $\partial \Omega$, we have

$$\sigma \partial_t U + \partial_\nu U = \left(-\frac{2\sigma\gamma - (x + \mu) \cdot \nu(x)}{2(t + 1)} + \frac{\sigma\|x + \mu\|^2}{4(t + 1)^2}\right)U.$$

Since $p > 1 + \frac{2}{N}$, using (12) and ignoring the non-negative term $\frac{\sigma\|x + \mu\|^2}{4(t + 1)^2}$, we can reduce the last equation to

$$\sigma \partial_t U + \partial_\nu U \geq \left(-\frac{\sigma N - (x + \mu) \cdot \nu(x)}{2(t + 1)}\right)U.$$

Thanks to (13) we obtain $B_{N}(U) \geq 0$ in $N_y \times [0, \infty)$. And we have $U \geq 0$ on $\partial \Omega \setminus N_y \times (0, \infty)$. An initial data $\varphi$ with $\varphi \leq U(\cdot, 0)$ in $\overline{\Omega}$ permits to conclude. \hfill \Box

In the case of dimension one, we use the fact that $\Omega$ is not connected. Let us write $\Omega = \mathbb{R} \setminus [a, b]$ with $a < b$ in $\mathbb{R}$, and let $V$ be the function defined in $\overline{\Omega} \times [0, \infty)$ by

$$V(x, t) = \begin{cases} 
A(t + 1)^{-\gamma} \exp \frac{-\|x + \mu_1\|^2}{4(t + 1)} & \text{if } x \leq a, \\
A(t + 1)^{-\gamma} \exp \frac{-\|x + \mu_2\|^2}{4(t + 1)} & \text{if } x \geq b,
\end{cases}$$
with $A$ and $\gamma$ like in Corollary 4.2, $\mu_1$ and $\mu_2$ in $\mathbb{R}$ such that

$$-(a + \mu_1) - \varsigma \geq 0$$

and

$$(b + \mu_2) - \varsigma \geq 0.$$ 

As $\nu(a) = 1$ and $\nu(b) = -1$, we obtain with (12)

$$\sigma \partial_t V + \partial_{\nu} V \geq 0 \text{ on } \{a\} \cup \{b\} \times [0, \infty).$$

Following the proof of Corollary 4.2, we obtain this result.

**Theorem 4.3.** Under conditions (2)–(12), $N = 1$ and $p > 3$, Problem (1) admits global positive solutions vanishing at infinity, for sufficiently small initial data $\varphi$.

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