SKELETONS OF STABLE MAPS II: SUPERABUNDANT GEOMETRIES

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ABSTRACT. We implement new techniques involving Artin fans to study the realizability of tropical stable maps in superabundant combinatorial types. Our approach is to understand the skeleton of a fundamental object in logarithmic Gromov–Witten theory – the stack of prestable maps to the Artin fan. This is used to examine the structure of the locus of realizable tropical curves and derive 3 principal consequences. First, we prove a realizability theorem for limits of families of tropical stable maps. Second, we extend the sufficiency of Speyer’s well-spacedness condition to the case of curves with good reduction. Finally, we demonstrate the existence of liftable genus 1 superabundant tropical curves that violate the well-spacedness condition.

1. INTRODUCTION

Central to the application of tropical techniques to questions in algebraic geometry are so-called lifting theorems. That is, given a “synthetic” tropical object, such as a weighted balanced polyhedral complex, one must understand whether this object is the tropicalization of an algebraic variety. We deal in this paper with the case of curves. The tropical lifting question in this setting asks, when does an embedded tropical curve in $\mathbb{R}^n$ arise as the tropicalization of an algebraic curve in a torus over a non-archimedean field. This question becomes highly nontrivial in the so-called superabundant case and has been the primary obstacle to the application of tropical curve counting in high genus settings. Superabundance is a combinatorial phenomenon exhibited by tropical curves that implies nontrivial obstructions to deforming certain logarithmic maps from curves. The earliest results in the area of superabundant realization theorems is due to Speyer, who observed a subtle combinatorial condition guaranteeing the realizability of superabundant genus 1 tropical curves [30]. While there has since been substantial additional work in the area, the general question remains mysterious [7, 15, 18, 22, 24, 23, 28, 32].

In this note, we use recent technical breakthroughs in non-archimedean geometry and the theory of logarithmic maps to provide a conceptual framework in which to understand superabundance and use this to give simple proofs of three new results for lifting tropical curves. The same framework allows us to provide some insight into the structure of realizability conditions more globally – the locus of realizable tropical curves is given by a union of bend loci of collections of tropical polynomials in the edge lengths of the tropical curve.

1.1. Statement of results. All valued fields appearing in this paper will have equicharacteristic zero. Throughout, $\mathcal{C}$ will be used to denote an abstract tropical curve\(^1\). We say that a parametrized tropical curve $[\mathcal{C} \to \mathbb{R}^n]$ is realizable if there exists a smooth curve $C$ over a non-archimedean field and a map $[C \to \mathbb{G}_m^n]$ whose tropicalization is $[\mathcal{C} \to \mathbb{R}^n]$.

\(^1\)We use $\mathcal{C}$, $C$ and $\mathcal{C}$ to denote families of curves, single curves, and tropical curves respectively, choosing notation that best approximates the shape of these objects as found in the wild. We thank Dan Abramovich for this most creative of suggestions.
Theorem A. Let \([\Gamma_t \rightarrow \mathbb{R}^n]\) for \(t \in [0, 1)\) be a continuously varying family of parametrized tropical curves. Assume that \([\Gamma_t \rightarrow \mathbb{R}^n]\) is realizable for all \(t\). If \([\Gamma_1 \rightarrow \mathbb{R}^n]\) is the limit of this family in the moduli space of parametrized tropical curves, then \([\Gamma_1 \rightarrow \mathbb{R}^n]\) is realizable.

Our next result extends the reach of Speyer’s well-spacedness condition to the case of elliptic curves with good stable reduction, see Definition 3.3.1.

Theorem B. Let \([\Gamma \rightarrow \mathbb{R}^n]\) be a parametrized tropical curve of genus 1 with a unique genus 1 vertex \(v\). Assume that the star of \(v\) is realizable. If \([\Gamma_1 \rightarrow \mathbb{R}^n]\) is well-spaced, then \([\Gamma \rightarrow \mathbb{R}^n]\) is realizable.

The work of Speyer shows that the well-spacedness condition is sufficient for realizability of tropical genus 1 curves. He also proves that this condition is also necessary, with the restriction that the curve is trivalent. The following result shows that outside the trivalent case, well-spacedness can be violated.

Theorem C. Let \(n \geq 3\). There exist superabundant parametrized genus 1 tropical curves \([\Gamma \rightarrow \mathbb{R}^n]\) that lift to algebraic curves but violate the well-spacedness condition.

The new point of view taken in this paper is to attempt to understand the realizability locus inside the moduli space of all parametrized tropical curves as a global tropical geometric object. We do so by studying a fundamental object in logarithmic Gromov–Witten theory – the space of logarithmic prestable maps to the Artin fan. This is inspired by the insights of Abramovich–Wise, Gross–Siebert, and Ulirsch. We are led to the following result.

Let \(X\) be a toric variety with fan \(\Delta\) and let \(\mathcal{L}_g^\circ(X)\) denote the moduli space of maps from smooth pointed genus \(g\) curves into \(X\) with fixed contact orders with the toric boundary along smooth marked points. In Section 2.2 a generalized extended cone complex \(T_\Gamma(\Delta)\) is constructed which parametrizes tropical stable maps with the analogous discrete data.

Theorem D. There is a continuous tropicalization map
\[ \text{trop} : \mathcal{L}_g^\circ(X)^{an} \rightarrow T_\Gamma(\Delta), \]
compatible with evaluation maps and forgetful maps to the moduli space of curves. The locus in \(T_\Gamma(X)\) parametrizing the set of realizable tropical curves is a closed polyhedral set. Given a cell \(\sigma \in T_\Gamma(X)\), after passing to a finite cover of \(\sigma\) by a cone \(\tilde{\sigma}\), the locus of realizable curves in \(\tilde{\sigma}\) is the union of bend loci of collections of tropical polynomials.

1.2. Further discussion. A number of experts have made the informal observation that the condition appearing in Speyer’s realizability theorem – that the minimum of a collection of numbers occurs at least twice – resembles the tropical variety of a tropical ideal. We view Theorem D as giving a simple and rigorous explanation for this phenomenon.

While the proof of our lifting theorems only rely on compactness of the realizable locus, the tropical structure is useful for applications to enumerative geometry. This is illustrated, for instance, by the results of Len and the author [19], in which the polyhedral structure of the realizability locus is used to derive multiplicities for tropical curve counting.

The above Theorem D also contributes to the study of tropical moduli spaces, which have received considerable interest in recent years. These results have aimed at an improved conceptual understanding of information contained in tropical moduli spaces, including applications to enumerative geometry [10, 11, 27] and to the geometry and topology of moduli spaces [9, 12, 13]. The novelty of the present paper is that the tropicalization of \(\mathcal{L}_g^\circ(X)\) is studied as the tropicalization of
a map to a certain toroidal stack—the stack of prestable maps to the Artin fan. This is in sharp contrast to recent results on tropicalizations for moduli spaces, which have used toroidal structures on the spaces themselves.

The well-spacedness condition has inspired a great deal of research. However, to our knowledge, the results of Baker–Payne–Rabinoff [7, Theorem 6.9] are the only known nontrivial necessary conditions for realizability for non-maximally degenerate tropical curves. By studying the limits of non-superabundant families of curves and applying Theorem A, one can obtain sufficient conditions for lifting tropical curves in non-maximally degenerate situations, i.e. when vertices carry nonzero genus. Furthermore, Theorem B exhibits the first instance of a sufficient condition for the realizability for non-maximally degenerate superabundant curves.

In addition to the work of Baker–Payne–Rabinoff mentioned above, Katz extracts a number of necessary conditions for realizability stemming directly from the logarithmic deformation theory of maps to toric varieties [18]. A similar approach is used by Cheung, Fantini, Park, and Ulirsch to prove that in a large range of cases, non-superabundance is a sufficient condition for realizability [15]. Note however, that limits of non-superabundant tropical curves can often become superabundant. As a result, Theorem A extends the reach of these theorems as well.

The first part of this project [27], which was a chapter in author’s doctoral dissertation, studies the tropicalization of the moduli space of logarithmic maps to toric varieties in genus 0, and it is in this sense that the present paper is a sequel. Superabundance never appears in the genus 0 setting, and the analogue of Theorem D can be used to derive a number of consequences concerning the geometry of the space of maps. We refer to loc. cit. for details. We also note that a similar polyhedrality result as above has been proved by Tony Yu in the context of non-archimedean analytic Gromov–Witten theory using methods that are quite different from ours [38, 39].

An implicit goal of this paper is to demonstrate the usefulness of the perspective on tropicalization arising from logarithmic prestable maps to Artin fans arising from the work of Abramovich, Ulirsch, and Wise [5, 35, 36] and the insights of Gross and Siebert [17]. Indeed, accepting this technical input, the reader will note that the proofs of our realizability theorems follow from reductions to existing theorems in tropical geometry.

Prerequisites. We assume familiarity with the fundamental concepts of Berkovich geometry and logarithmic structures. A rapid overview of the relevant concepts may be found in the preceding article [27, Section 2]. We refer the reader to two excellent recent surveys in this area by Abramovich, Chen, and their collaborators [3, 4].

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2. Tropicalization

2.1. Logarithmic stable maps. Our central object of study is the moduli space of genus $g$ curves in a projective toric variety $X$ in a fixed curve class, meeting each torus invariant divisor $D_\rho$ at marked points with prescribed contact orders. We work with the compactification of this space of
ramified curves, provided by the Abramovich–Chen–Gross–Siebert theory of logarithmic stable maps \([2, 14, 17]\).

More formally, let \(X\) be a projective toric variety with dense torus \(T\), corresponding to a fan \(\Delta\) in the vector space \(N_\mathbb{R}\) spanned by the cocharacters of \(T\). Fix integers \(g\) and \(n\). We study the moduli space \(\mathcal{L}(X)\) of families of minimal stable logarithmic morphisms 
\[
f : (C, p_1, \ldots, p_n) \to X,
\]
from nodal pointed genus \(g\) curves \((C, p_1, \ldots, p_n)\). Minimality is a condition on the characteristic monoids of the log structure on the base of the family. We refer the reader to \([17, \text{Section 1}]\) for an explicit discussion, to \([2, \text{Section 2}]\) for a conceptual discussion, and to \([28, \text{Section 3.1}]\) for a quick working definition of the concept. Stability amounts to stability of the underlying map – all contracted rational components must have at least 3 special points and all contracted elliptic components must have at least 1 special point. We will also fix the contact orders on \(C\). The contact order will be recorded by a function 
\[
c : \{p_1, \ldots, p_n\} \to \Delta(\mathbb{N}),
\]
where \(\Delta(\mathbb{N})\) denotes the integral points of the fan \(\Delta\). If \(c(p_i) = e \cdot v_\rho\) where \(v_\rho\) is a primitive generator of the ray corresponding to \(D_\rho\), then the curve has contact order \(e\) along \(D_\rho\) at \(p_i\). Note that by Fulton–Sturmfels’ description of the Chow cohomology of a complete toric variety by Minkowski weights \([16]\), the data of \(c\) determines an operational curve class \(f_\star[C] \in A_1(X; \mathbb{Z})\).

We package the discrete data by the symbol \(\Gamma = (g, n, c)\). Let \(\mathcal{L}_\Gamma(X)\) denote the locus of the space of maps where the logarithmic structure is trivial. The following result is established by Abramovich–Chen–Gross–Siebert.

**Theorem 2.1.1.** The moduli space \(\mathcal{L}_\Gamma(X)\) of minimal logarithmic stable maps with discrete data \(\Gamma\) is a proper logarithmic algebraic stack with projective coarse moduli space.

### 2.2. Tropical stable maps

The purpose of this section is to construct the relevant parameter space for tropical stable map, which will serve as the target of our tropicalization map.

**Definition 2.2.1.** A pre-stable \(n\)-pointed tropical curve of genus \(g\) is a metric graph \(\Gamma\) with a chosen underlying finite graph model \(G(\Gamma)\) and two additional data:

1. A genus function 
   \[
g(-) : \Gamma \to \mathbb{N},
\]
   with finite support.
2. A labeling of a subset of the 1-valent genus 0 vertices of \(\Gamma\) by the set \(\{p_1, \ldots, p_n\}\).

The genus of \(\Gamma\) is defined to be 
\[
g(\Gamma) = h^1(\Gamma) + \sum_{v \in \Gamma} g(v).
\]

Marked leaf-edges are metrized as \([0, \infty]\), infinitely long marked internal edges are metrized as two glued intervals \([0, \infty] \cup [0, \infty]\). We will refer to a curve \(\Gamma\) whose metric is non-singular away from the leaves as smooth.

Recall that a morphism \(\phi : \Sigma_1 \to \Sigma_2\) between polyhedral complexes is a map on the underlying point sets such that each polyhedron in \(\Sigma_1\) is mapped to a unique polyhedron in \(\Sigma_2\).
Definition 2.2.2. A tropical stable map from a smooth tropical curve is a continuous and proper morphism of polyhedral complexes

\[ f : (\mathcal{C}, \{p_1, \ldots, p_n\}) \to (\Delta, \partial \Delta) \]

where \((\mathcal{C}, \{p_1, \ldots, p_n\})\) is a smooth \(n\)-marked abstract tropical curve, such that the following conditions are satisfied.

(TSM1) Each edge \(e \in \mathcal{C}\) is mapped to \(\Delta\) with integer slope, denoted \(w_e\). This is referred to as the expansion factor of \(f\) along \(e\).

(TSM2) The map \(f\) is balanced in the usual sense, i.e. at all points of \(\mathcal{C}\) the sum of the derivatives of \(f\) in each tangent direction is zero.

(TSM3) The map \(f\) is stable. That is, if \(p \in \mathcal{C}\) has valence 2, then the image of \(\text{Star}(v)\) is not contained in the relative interior of a single cone of \(\Delta\).

We will often suppress the markings from the notation, and simply write the map as \([\mathcal{C} \to \Delta]\). The curve class \(\beta\) of \([C \to \Delta]\) is defined to be the recession fan of its image.

The following definition indexes the deformation class of a tropical stable map and is obtained by dropping the data of the lengths of the edges of \(\mathcal{C}\).

Definition 2.2.3. The combinatorial type of a tropical stable map \([\mathcal{C} \to \Delta]\) is the data

(CT1) The finite graph model \(G(\mathcal{C})\) underlying \(\mathcal{C}\).

(CT2) For each vertex \(v \in G(\mathcal{C})\), a cone \(\sigma_v \in \Delta\) containing the image of \(v\).

(CT3) For each edge \(e\), the slope \(w_e\) of \(f\) restricted to \(e\) and the primitive vector \(u_e\) in the direction of \(f(e)\).

We recall the following well known proposition, see [24, Proposition 2.1] and [27, Proposition 3.2.1].

Proposition 2.2.4. Let \(\Theta\) be the combinatorial type of a tropical stable map \([\mathcal{C} \to \Delta]\). The set of all tropical curves \([\mathcal{C} \to \Delta]\) together with a chosen finite graph model \(G(\mathcal{C})\), is parametrized by a cone \(\sigma_\Theta\). Fixing the curve class associated to \(\Theta\), there are only finitely many combinatorial types.

Following [21, Proposition 2.14], the overvalence of a type \(\Theta\) is defined as

\[ \text{ov}(\Theta) = \sum_{p : \text{val}(p) \geq 4} \text{val}(p) - 3. \]

Well-known calculations show that the expected dimension of this deformation space of tropical maps is

\[ \dim \sigma_\Theta = (\dim(\Delta) - 3)(1 - g) + n - \text{ov}(\Theta), \]

If cycles of \(\mathcal{C}\) are mapped into proper affine subspaces of \(|\Delta|\) of high codimension, the dimension of the actual deformation space will be larger than the above expected dimension.

Definition 2.2.5. A combinatorial type \(\Theta\) is said to be superabundant if the dimension of \(\sigma_\Theta\) is strictly larger than the expected dimension.

Following [1, 31] there is a natural compactification of any cone \(\sigma\) with integral structure. If \(S_\sigma\) denotes the dual monoid, the cone \(\sigma\) can be recovered as the space of monoid homomorphisms \(\text{Hom}(S_\sigma, \mathbb{R}_{\geq 0})\), while the compactification is defined to be

\[ \overline{\sigma} := \text{Hom}(S_\sigma, \mathbb{R}_{\geq 0} \cup \{\infty\}). \]
Fixing a combinatorial type $\Theta$, the extended cone $\sigma_{\Theta}$ naturally parametrizes tropical stable maps to $\Delta$ from curves with infinitely long internal edges. We will have no need to work with such maps directly.

**Definition 2.2.6.** An isomorphism between two tropical stable maps maps $[\square_1 \to \Delta]$ and $[\square_2 \to \Delta]$ is an isometry of graphs commuting with the map:

$$
\begin{array}{ccc}
\square_1, p_1, \ldots, p_n & \rightarrow & \square_2, q_1, \ldots, q_n \\
\sigma & \rightarrow & \Delta.
\end{array}
$$

An isomorphism of a stable map with itself is said to be an automorphism. Similarly, an automorphism of the combinatorial type $\Theta$ is an automorphism of the underlying finite graph $G(\square)$ preserving the expansion factors and the cones associated to each vertex.

The deformation spaces for fixed combinatorial types described above form the local models for the space of tropical stable maps. The globalization is achieved by what are now standard techniques, introduced by Abramovich, Caporaso, and Payne [1].

Given any combinatorial type $\Theta$, the faces of $\sigma_{\Theta}$ are naturally identified with maps from tropical curves where some edges of $G(\square)$ have been contracted. Sequences of such edge contraction give natural morphisms between the faces. The moduli space of tropical stable maps is defined to be

$$
T_\Gamma(\Delta) := \lim_{\Theta: \text{stable}} (\sigma_{\Theta}, j_{\Theta}).
$$

In the colimit above, $j_{\Theta}$ ranges over all identifications of faces induced by edge contractions or automorphisms of combinatorial types.

The topological space constructed above has the structure of a generalized extended cone complex in the sense of [1, Section 2]. By forming the union of the images of the non-compact cones $\sigma_{\Theta}$ in $T_\Gamma(\Delta)$, we obtain the moduli space $T_{\text{pre}}(\Delta)$, parametrizing those maps with smooth source graph.

### 2.3. Prestable tropical maps.

For our later study, it will be convenient to relax the stability condition above. A prestable tropical map to $\Delta$ is a map $[\square \to \Delta]$ as in Definition 2.2.2, but possibly violating the stability condition (TSM3). In other words, $[\square \to \Delta]$ becomes a stable map after dropping finitely many 2-valent vertices from $\square$. There are infinitely many such destabilizations of any combinatorial type. However, there are no substantive changes to the construction above – prestable tropical curve of a fixed combinatorial type are still parametrized by a cone, though of arbitrarily high dimension. Gluing these cones together exactly as above, we obtain a generalized extended cone complex, locally of finite type,

$$
T_{\text{pre}}(\Delta) := \lim_{\Theta: \text{prestable}} (\sigma_{\Theta}, j_{\Theta}).
$$

The crucial technical connection between the logarithmic maps theory and tropical geometry comes from an elegant result of Gross and Siebert which we now recall. Let $[C \to X]$ be a logarithmic stable map over a logarithmic point $\text{Spec}(P \to \mathbb{C})$. Such a map is said to be minimal if the
logarithmic structure given by \( P \) on \( \text{Spec}(\mathbb{C}) \) coincides with the logarithmic structure obtained by pulling back the structure on the moduli space \( \mathcal{L}_\Gamma(X) \) via the underlying map
\[
\text{Spec}(\mathbb{C}) \rightarrow \mathcal{L}_\Gamma(X).
\]

The logarithmic stable map \([C \rightarrow X]\) has a well-defined \textit{combinatorial type}. The source graph \( \Gamma \) is taken to be the dual graph of \( C \), the vertices map to the cone \( \sigma \) dual to the stratum containing the generic point of the corresponding component, and the expansion factors along edges are uniquely determined by the contact order, see [17, Section 1.4] Theorem 2.3.1 ( [17, Remark 1.21]).

\textbf{Theorem 2.3.1} \( (\text{[17, Remark 1.21]}) \). A logarithmic stable map \([C \rightarrow X]\) over \( \text{Spec}(P \rightarrow \mathbb{C}) \) as above is minimal if and only if \( P \) is isomorphic to the dual monoid \( Q \) of the integral monoid in the cone of tropical curves whose combinatorial type is that of \([C \rightarrow X]\).

The same result holds for \( X \) replaced with its Artin fan \( \mathcal{A}_X = [X/T] \).

2.4. \textbf{Pointwise tropicalization for logarithmic stable maps}. Our next goal is to construct a map \( \text{trop} : \mathcal{L}_\Gamma^\circ(X)^\text{an} \rightarrow T_\Gamma^\circ(\Delta) \) from the analytification of the space of maps from smooth curves with prescribed contact orders, to the space of tropical maps.

Any point \( x \in \mathcal{L}_\Gamma^\circ(X)^\text{an} \) may be represented by a map 
\[
\text{Spec}(K) \rightarrow \mathcal{L}_\Gamma^\circ(X)^\text{an},
\]
where \( K \) is a valued field extension of \( \mathbb{C} \). After a ramified base change, the existence of the compactification \( \mathcal{L}_\Gamma(X) \) of \( \mathcal{L}_\Gamma^\circ(X) \) guarantees an extension to 
\[
\text{Spec}(R) \rightarrow \mathcal{L}_\Gamma(X)^\text{an}.
\]

By pulling back the universal curve and universal map we obtain a diagram 
\[
\begin{array}{ccc}
(C, s_1, \ldots, s_n) & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
\text{Spec}(R),
\end{array}
\]

where the \( s_i \) are horizontal sections determining marked points on the geometric fibers. By [6], this choice of model determines a retraction of the analytic generic fiber onto a tropical curve 
\[
\mathcal{C}_\eta^\text{an} \rightarrow \overline{\mathcal{C}},
\]
with a canonical continuous section 
\[
\overline{\mathcal{C}} \rightarrow \mathcal{C}_\eta^\text{an}.
\]
Similarly, there is a deformation retraction 
\[
X^\text{an} \rightarrow \overline{X},
\]
of the analytic toric variety onto its skeleton by [25, 31]. Composing the section map with the natural map 
\[
\mathcal{C}_\eta^\text{an} \rightarrow X^\text{an} \rightarrow \overline{X},
\]
we obtain a map \([f_{\text{trop}} : \overline{\mathcal{C}} \rightarrow \overline{X}]\). By [27, Theorem 3.3.2], this map is seen to be a tropical stable map from a smooth tropical curve.
2.5. **Tropicalization via the Artin fan.** Let $X$ continue to denote a projective toric variety. The *Artin fan* of $X$ is defined to be the stack quotient $\mathcal{A}_X = X/T$. Work of Ulirsch [35, 36] asserts that, at the level of underlying topological spaces, the generic fiber of the map $[X \to \mathcal{A}_X]$ is canonically identified with the extended tropicalization map

$$X^\text{an} \to \Delta,$$

constructed in [25, 31]. This is made precise by the following theorem.

Given a scheme or stack $Y$ defined over a valuation ring $R$, we will use $Y^\text{an}_\circ$ to denote Raynaud’s generic fiber functor applied to the formal completion of $Y$ along the maximal ideal of $R$. See [36, 35, 38] for background on generic fibers of algebraic stacks. Note that if $Y$ is proper, the generic fiber coincides with the Berkovich analytification, and will drop the $\circ$ in the subscript.

**Theorem 2.5.1** ([36, Theorem 1.1]). *There is a canonical identification of extended cone complexes given by $\mu_\Delta : |\mathcal{A}(\Delta)_{\circ}^\text{an}| \to N(\Delta)$, making the diagram*

\[ \begin{array}{ccc}
X^\text{an} & \xrightarrow{\mu_\Delta} & |\mathcal{A}_X^\text{an}_\circ| \\
\downarrow \text{trop} & & \downarrow \text{Stack Quotient} \\
\Delta & & \Delta
\end{array} \]

*commute.*

We obtain the following as an immediate corollary.

**Corollary 2.5.2.** *Let $f : C \to X$ be a logarithmic stable map with smooth generic fiber defined over a valuation ring $\text{Spec}(R)$ containing $C$. Let $\subset$ be the skeleton of $C$. Then, the tropicalization of $f$ coincides with the composite map*

$$\subset \to C^\text{an}_\eta \to X^\text{an} \to |\mathcal{A}_X^\text{an}_\circ|$$



2.6. **Maps to $\mathcal{A}_X$.** The purpose of this subsection is to establish a “global” version of Corollary 2.5.2 above. In [5], Abramovich and Wise introduce a stack of prestable logarithmic morphisms to the Artin fan $\mathcal{A}_X$ itself. Fixing the discrete data as before, we will use the following result of theirs.

**Theorem 2.6.1.** *The stack $\mathcal{L}_{\Gamma}^\text{pre}(\mathcal{A}_X)$ is a logarithmically smooth algebraic stack of locally finite type and dimension $3g - 3 + n$.***

The reader will note that the logarithmic smoothness of this stack is in sharp contrast to the geometry of $\mathcal{L}_\Gamma(X)$, which satisfies Murphy’s law\(^2\) in the sense of Vakil [37]. Indeed, this remarkable smoothness property was applied in [28] to show that every tropical stable map arose as the tropicalization of a family of stable maps to the Artin fan.

\(^2\)Taking $X = \mathbb{P}^n$ with its toric boundary and $\Gamma$ to have transverse contact orders, $\mathcal{L}_\Gamma(X)$ contains the moduli space of degree $d$ curves in projective space as a subscheme, and the fact follows immediately from [37, Theorem M1a].
In [31], Thuillier defines a generic fiber functor \((\cdot)^2\) from the category of locally finite type schemes over trivially valued fields to analytic spaces over trivially valued fields. This notion is closely related to Raynaud’s generic fiber in the nontrivially valued case. Given a locally finite type scheme \(X\), points of \(X^2\) are, by definition, equivalence classes of maps
\[
\text{Spec}(R) \to X,
\]
where \(R\) is the valuation ring of a valued field extension of the trivially valued base field \(k\). As a consequence of the valuative criteria for separatedness and properness, \(X^2\) is a compact Hausdorff analytic space. In [35, Section 5.2], this association \(X \mapsto X^2\) is extended to locally finite type algebraic stacks. Our application of this analytification is to the space \(\mathcal{L}_G^{\text{pre}}(X)\).

Any point of \(\mathcal{L}_G^{\text{pre}}(X)^2\) gives rise to a family of logarithmic prestable maps
\[
(\mathcal{C}, s_1, \ldots, s_n) \longrightarrow \mathcal{A}_X
\]
over a valuation ring extending \(\mathbb{C}\). Following the same procedure as in the previous section, passing to skeletons yields a map
\[
\tau : \mathcal{L}_G^{\text{pre}}(\mathcal{A}_X)^2 \to T^{\text{pre}}(\Delta).
\]
On the other hand, the stack \(\mathcal{L}_G^{\text{pre}}(X)\) is a toroidal (i.e. logarithmically smooth) stack in the lisse-étale topology. It now follows from [35, Theorem 1.2] that there exists a continuous deformation retraction of the analytic space onto an extended cone complex:
\[
P_{\mathcal{L}} : \mathcal{L}_G^{\text{pre}}(\mathcal{A}_X) \to \Sigma(\mathcal{L}_G^{\text{pre}}(\mathcal{A}_X)).
\]
Note that in loc. cit., the deformation retraction above is constructed for logarithmically smooth schemes with logarithmic structure defined on the étale site. There is no difficulty in extending this result to the logarithmically smooth Artin stacks appearing in our context. It can also be constructed purely by means of toroidal geometry in analogy with the Deligne–Mumford case, worked out in detail in [1, Section 6]. For the toroidal Artin stacks we use here, one needs to simply replace the étale open covers occurring in the former theory with smooth covers. The necessary developments in the non-archimedean stack theory were accomplished in [35].

The main result of this section is the relationship between the generalized extended cone complexes \(T^{\text{pre}}(\Delta)\) and \(\Sigma(\mathcal{L}_G^{\text{pre}}(\mathcal{A}_X))\).

**Theorem 2.6.2.** There is a commutative diagram of continuous morphisms
\[
\begin{array}{ccc}
\mathcal{L}_G^{\text{pre}}(\mathcal{A}_X)^2 & \xrightarrow{\text{trop}} & T^{\text{pre}}(\Delta) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\Sigma(\mathcal{L}_G^{\text{pre}}(\mathcal{A}_X)) & \xrightarrow{\text{trop}_{\Sigma}} & \Sigma(\mathcal{L}_G^{\text{pre}}(\mathcal{A}_X))
\end{array}
\]
where
\[
\text{trop}_{\Sigma} : \Sigma(\mathcal{L}_G^{\text{pre}}(X)) \to T^{\text{pre}}(X),
\]
is a finite morphism of generalized extended cone complexes and becomes an isomorphism upon restriction to any cell.
Proof. Consider a point \( p \in \mathcal{L}_\Gamma(\mathcal{A}_X) \) corresponding to a minimal logarithmic map \( [C \to \mathcal{A}_X] \). Suppose \( p \) is contained in \( W \), the locally closed stratum of the logarithmically smooth stack \( \mathcal{L}_\Gamma(\mathcal{A}_X) \) parametrizing maps of combinatorial type \( \Theta \). There is a smooth neighborhood \( U_\sigma \to \mathcal{L}_\Gamma(\mathcal{A}_X) \) containing \( p \) where \( U_\sigma = \text{Spec}(\mathbb{C}[S_\sigma]) \) is an affine toric variety. It follows from logarithmic smoothness of the moduli stack and the definition of the minimal log structure in [17, Section 1.4] that the monoid \( S_\sigma \) coincides with stalk of the minimal characteristic of \( [C \to \mathcal{A}_X] \). Moreover, by Theorem 2.3.1 there is a natural identification of the cones \( \sigma \Theta \) and \( \sigma \). Thus, the skeleton of \( U^{\Delta} \) is naturally identified with the extended cone \( \sigma \). The points of \( U^{\Delta} \) parametrize degenerations of logarithmic stable maps whose special fiber is identified with \([C \to \mathcal{A}_X]\). The local coordinate description of the retraction map \([33, Section 6]\) shows that the retraction
\[
U^{\Delta} \to \sigma \Theta
\]
coincides with the pointwise tropicalization map construction in Section 2.4. By construction, the skeleton \( \Sigma(\mathcal{L}(\mathcal{A}_X)) \) decomposes as
\[
\Sigma(\mathcal{L}_{\Gamma}^{pre}(\mathcal{A}_X)) = \bigsqcup_{W} \sigma_{\Theta}/\text{Aut}(\Theta).
\]
Similarly, the moduli space of prestable tropical maps decomposes as
\[
T_{\Gamma}^{pre}(\Delta) = \bigsqcup_{\Theta} \sigma_{\Theta}/\text{Aut}(\Theta).
\]
The skeleton \( \Sigma(\mathcal{L}(\mathcal{A}_X)) \) includes naturally into \( \mathcal{L}_{\Gamma}^{pre}(\mathcal{A}_X)^{\Delta} \), so by composing with the pointwise tropicalization map \( \text{trop} \), we obtain
\[
\text{trop}_\Sigma : \Sigma(\mathcal{L}_{\Gamma}^{pre}(\mathcal{A}_X)) \to T_{\Gamma}^{pre}(\Delta).
\]
By the above discussion, \( \text{trop}_\Sigma \) is an isomorphism upon restriction to any fixed cell of \( \Sigma(\mathcal{L}_{\Gamma}(\mathcal{A}_X)) \). It remains to analyze the strata of these two extended cone complexes. For this, we recall that given any logarithmic stable map \([f : C \to \mathcal{A}_X]\), there is a map in the category of fine but not necessarily saturated logarithmic stacks \( f^{\text{ras}} : C \to \mathcal{A}_X \) by the constructions of [14, Section 3.7] and [27, Section 3.6]. In particular, fix an the underlying map \([C \to \mathcal{A}_X]\). Given two logarithmic enhancements \( f_1 : C \to \mathcal{A}_X \) and \( f_2 : C \to \mathcal{A}_X \), the maps \( f^{\text{ras}}_1 \) and \( f^{\text{ras}}_2 \) coincide up to saturation. Since the saturation index is finite, it follows that for each combinatorial type \( \Theta \), there are finitely many strata \( W \) having type \( \Theta \). This ensures that the map \( \text{trop}_\Sigma \) is finite of degree equal to this index. The result follows. \( \square \)

We derive the first part of Theorem D as a corollary.

**Corollary 2.6.3.** There is a continuous tropicalization map
\[
\text{trop} : \mathcal{L}_{\Gamma}^{\Delta}(X)^{an} \to T_{\Gamma}(\Delta),
\]
compatible with evaluation maps and forgetful maps to the moduli space of curves.

**Proof.** Consider the sequence of maps
\[
\text{trop} : \mathcal{L}_{\Gamma}^{\Delta}(X)^{an} \to \mathcal{L}_{\Gamma}(X)^{\Delta} \to \mathcal{L}_{\Gamma}^{pre}(\mathcal{A}_X)^{\Delta} \to T_{\Gamma}^{pre}(\Delta).
\]
Since \( \mathcal{L}_{\Gamma}(X) \) is proper, its formal fiber coincides with its analytification, and first map is simply the analytification of the inclusion of an open subscheme. The second arrow is constructed by applying the functor \((\cdot)^{\Delta}\) for stacks to the natural map from \( \mathcal{L}_{\Gamma}(X) \) to \( \mathcal{L}_{\Gamma}(\mathcal{A}_X) \), see [35, Section 5]. The final map is the pointwise tropicalization constructed in the preceding theorem. The
composition is clearly continuous, since each arrow is continuous. Forgetful morphisms to \( \overline{\mathcal{M}}_{g,n} \) and evaluation maps to \( X \) are all logarithmic, so functoriality follows from Theorem 2.6.2 above and general results on functoriality for tropicalization [33, Theorem 1.1].

As a consequence, we rephrase the tropical lifting question as when does a tropical stable map \( [\mathcal{C} \to \Delta] \) lie in the image of the continuous map \( \text{trop} \) above?

3. PROOFS OF LIFTING THEOREMS

3.1. Polyhedrality. Let \( p = [C \to X] \) be a minimal logarithmic stable map and consider the associated map \( [C \to \mathscr{A}_X] \). Let \( Z \) be the local model in the smooth topology of the moduli space \( \mathcal{L}_T(X) \) near \( p \). After possibly shrinking \( Z \), the points of the compact analytic space \( Z^\Delta \) correspond to families over valuation rings of logarithmic stable maps whose special fiber is a pull back of \( [C \to X] \) along a logarithmic morphism

\[
\text{Spec}(\mathbb{R}_{\geq 0} \to \mathbb{C}) \to \text{Spec}(Q \to \mathbb{C}),
\]

where \( Q \) is the stalk of the minimal characteristic at \( p \). Let \( U = \text{Spec}(\mathbb{C}[Q]) \) be the local toric model for the moduli space \( \mathcal{L}_T^{\text{pre}}(\mathscr{A}_X) \) near \( [C \to \mathscr{A}_X] \), i.e. near the image of \( p \) under the natural map

\[
\mathcal{L}_T(X) \to \mathcal{L}_T^{\text{pre}}(\mathscr{A}_X).
\]

By the results of the previous section, the locus of realizable tropical curves having the combinatorial type dual to \( Q \) is the image of \( Z^\Delta \) under the natural morphism

\[
Z^\Delta \to U^\Delta \to \text{Hom}(Q, \mathbb{R}_{\geq 0}).
\]

The latter map is the retraction map of \( U^\Delta \) onto its skeleton. Polyhedrality now follows from a local application of the fundamental theorem of tropical geometry, see [8, 20, 26, 34]. This concludes the proof of Theorem D.

Remark 3.1.1. The essential content of the above theorem is that the realizability conditions are given by the lend loci of the equations that describe the moduli space of maps locally. By Vakil’s Murphy’s law, one should expect these equations to become arbitrarily complicated, and thus one should expect the same phenomenon on the piecewise linear side. A precise version of this statement will appear in future work.

3.2. Realizability of limits: Theorem A. Let \( [\mathcal{C}_t \to \Delta] \) for \( t \in [0, 1) \) be a continuously varying family of tropical stable maps. We have an induced moduli map

\[
[0, 1) \to T_1(\Delta),
\]

whose image lies in the locus of realizable tropical curves. Since the realizable locus is a closed set, it follows that the limiting map \( [\mathcal{C}_1 \to \Delta] \) also lies in this set, and the result follows immediately.

3.3. Well-spacedness for good reduction. We first recall the definition of Speyer’s well-spacedness condition. Note that any genus 1 abstract tropical curve \( \mathcal{C} \) has a unique cycle, possibly a single vertex of genus 1 or a self-loop at a vertex. A a genus 1 tropical stable map \( [\mathcal{C} \to \Delta] \) is said to be superabundant if the image of the unique loop \( L \) in \( |\Delta| \) is contained in a proper linear subspace.
**Definition 3.3.1.** Let \(|f : \mathbb{C} \to \Delta|\) be a superabundant genus 1 tropical stable map. Let \(H\) be a hyperplane containing the loop \(L\) and consider the subgraph \(\mathbb{C}_H\), the connected component of \(f^{-1}(H \cap \mathbb{C})\) containing \(L\). Denote the 1-valent vertices of \(\mathbb{C}_H\) by \(v_1, \ldots, v_k\) and by \(\ell_i\) the distance from \(v_i\) to \(L\). The map \(f\) is well-spaced with respect to \(H\) if the the minimum of the multiset of distances \(\{\ell_1, \ldots, \ell_k\}\) occurs at least twice.

The map \(f\) is said to be well-spaced if it is well-spaced with respect to every hyperplane containing \(L\).

3.4. **Proof of Theorem B.** Let \([f : \mathbb{C} \to \Delta]\) be a tropical stable map of genus 1 such that there is a point \(p \in \mathbb{C}\) with \(g(p) = 1\). Let \(\hat{\mathbb{C}}_t \to \Delta\) be the tropical stable map obtained by replacing \(p\) with a genus 0 vertex with a self-loop of length \(t \in \mathbb{R}_+\). By assumption \(\text{Star}(\mathbb{C}, p) \to \Delta\) is realizable, so by degenerating the genus 1 curve to a nodal curve, the map \(\text{Star}(\hat{\mathbb{C}}, p) \to \Delta\) is also realizable. A direct application of [30, Theorem 3.4] now ensures that \([\hat{\mathbb{C}}_t \to \Delta]\) is realizable. Letting \(t \to 0\) we obtain a continuous family of realizable tropical curves whose limit is \(f\). The result follows immediately from Theorem A.

\(\square\)

**Remark 3.4.1.** David Speyer has communicated to us a different approach to the proof of the above result using techniques akin to those in [30]. We briefly outline the argument here. Let \([\mathbb{C} \to \Delta]\) be a tropical stable map and \(v \in \mathbb{C}\) the genus 1 vertex. Let \(a_1, a_2, \ldots, a_m\) be the directions of the edges pointing outward from \(v\) and write

\[ a_i = (a_{i1}, a_{i2}, \ldots, a_{in}) \in \mathbb{Z}^n. \]

Let \(K\) be a non-archimedean field such that the vertices of \(\mathbb{C}\) map to points of \(\Delta\) that are rational over the value group. Let \(R\) be the valuation ring and and \(k\) the residue field. Since the link is realizable, we may build an elliptic curve \(E\) over \(k\) with points \(z_1, z_2, \ldots, z_m \in E(k)\) such that

\[ \sum a_i z_i = 0 \in E^n. \]

Choose a lifting of \(E\) to an elliptic curve \(\mathcal{E}\) over \(\text{Spec}(R)\) with reduction \(E\) over \(k\). By the arguments of [30, Section 7,8], to lift the map \([\mathbb{C} \to \Delta]\) we must lift the points \(\{z_i\}\) to the generic fiber of \(\mathcal{E}\) with prescribed relations, preserving \(\sum a_i z_i = 0\). This can be done by mimicking the calculations in Speyer’s original argument, where the lifts of the points \(z_i\) lie in the fibers of the reduction map \(\mathcal{E}(R) \to E(k)\). By [29, Theorem 6.4], since we work in residue characteristic 0, the fibers of this map are isomorphic to the additive group \(R^x\). The calculations are now exactly as in [30, Lemma 8.3 and Theorem 3.4] and the result follows.

3.5. **Proof of Theorem C.** Let \([\mathbb{C} \to \Delta]\) be a genus 1 tropical curve whose loop is contained in \(H\) and is well-spaced. Furthermore, assume that the minimum of the distances in Definition 3.3.1 is 0 and occurs exactly twice. That is, there are two vertices \(v_1\) and \(v_2\) of the loop \(L\) at which \(\mathbb{C}\) leaves \(H\). Choose a path in \(L\) between \(v_1\) and \(v_2\) and a family of tropical curves \([\mathbb{C}_t \to \Delta]\) for \(t \in [0, 1]\) such that the distance between \(v_1\) and \(v_2\) is \((1 - t)\). The limiting map \(\mathbb{C}_t \to \Delta\) cannot be well-spaced, but is realizable by Theorem A.

\(\square\)

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