Analysis of additive generators of fuzzy operations represented by rational functions

T M Ledeneva
Department of Applied Mathematics, Informatics and Mechanics, Voronezh State University, 1, University Square, Voronezh, 394018, Russia
E-mail: ledeneva-tm@yandex.ru

Abstract. This article presents an approach for determining additive generators of commutative and associative operations. Its applicability for finding generators of triangular norms and conorms is shown. Conditions for the parameters of increasing generators that generate triangular conorms in the class of rational functions are determined.

1. Introduction
A purposeful approach to the formation of various fuzzy operations becomes possible by the introduction of triangular norms and conorms. Triangular norms appeared as a natural generalization of the classical ”triangle inequality” when considering the probability metric spaces and the corresponding axioms were proposed by Schweizer and Sclar in their studies [1]. The development of triangular norms and conorms, on the one hand, is based on the theory of functional equations. The foundations of this approach were laid in [2], and families of triangular norms and conorms that are obtained as solutions of functional equations (for example [3]) are widely known. The most important result is the characterization of certain classes of triangular norms by means of additive generators used to represent associative operations [2]. Another approach to the analysis of triangular norms and conorms is related to the theory of semigroups and allows them to be represented by means of ordinal sum. The main properties and representations of various families of triangular norms and conorms are presented in [4–6]. Among fuzzy operations, a large class consists of operations representable by the relation of two polynomials or, in a particular case, by a polynomial. The purpose of this article is to present a methodology for analyzing operations related to this class.

2. Materials and methods
2.1. Basic definitions [4, 7]
Definition 1. A triangular norm (t-norm) is a binary operation $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ such that for all $x, y, z \in [0, 1]$ we have the following properties:
   a) $T(x, y) = T(y, x)$ (commutativity),
   b) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity),
   c) $T(x, y) \leq T(x, z)$, if $y \leq z$ (monotonicity),
   d) $T(x, 1) = x$ (neutral element 1).
The system \( \langle [0,1], T, \leq \rangle \) is an Abelian semigroup with neutral element 1 and order relation \( \leq \). The triangular norm \( T(x,y) \) models the multiplying operators (intersection of fuzzy sets, conjunction).

**Definition 2.** A triangular conorm (t-conorm) is a binary operation \( S : [0,1] \times [0,1] \rightarrow [0,1] \) such that for all \( x, y, z \in [0,1] \) the commutativity, associativity, monotonicity properties and property \( S(x,0) = x \) are fulfilled.

The system \( \langle [0,1], S, \leq \rangle \) is an Abelian semigroup with neutral element 0 and order relation \( \leq \). The triangular conorm \( S(x,y) \) models the adding operators (union of fuzzy sets, disjunction).

Comparing Definitions 1 and 2 we can see that they differ only in the boundary conditions. The monotonicity with respect to the second argument together with the commutativity property causes monotonicity in both arguments. An important additional requirement is the continuity property.

**Definition 3.** A continuous, strictly monotonic in both arguments, triangular norm \( T \) that satisfies the condition \( T(x,x) < x \) (subidempotency) is called Archimedean.

**Definition 4.** A triangular conorm \( S(x,y) \) is called the Archimedean conorm if it is continuous, strictly monotonic in both arguments, and satisfies condition \( S(x,x) > x \) (superidempotency).

**Definition 5.** The pairs of triangular norms and conorms \( T(x,y) \) and \( S(x,y) \) are called dual ones if for them the equalities (de Morgan laws)

\[
S(x,y) = N(T(N(x),N(y))), \quad T(x,y) = N(S(N(x),N(y))),
\]

where \( N(x) \) is a strong negation function, are satisfied for all \( (x,y) \in [0,1]^2 \).

Examples of dual triangular norms and conorms \([4, 7]\) obtained by different authors are presented in Table 1. Most of them are rational functions. Classical operations \( T_M \) and \( S_M \) which have good characteristics from the mathematical point of view are most often used in applications. In particular, these are the only idempotent operations. Of the algebraic properties, only the laws of complementarity are not fulfilled for them. However, these operations are "strict", which affects the quality of fuzzy models, in particular, their sensitivity to the initial data. For the remaining pairs of fuzzy operations, the distributivity and idempotency laws are not fulfilled. Availability of parameters allows providing flexibility of fuzzy models and their adaptation to a specific task. However, as a rule, additional research is needed to determine the optimum values of the parameters.

### 2.2. Additive generators

We note that triangular norms and conorms are commutative and associative operations. Commutativity means that the result of the operation depends only on the values of the arguments and does not depend on their order. By associativity is understood the absence of hierarchy among the arguments. It is known that associative operations are represented by additive generators.

**Definition 6** ([7]). A strictly decreasing continuous function \( \varphi : [0,1] \rightarrow [0,\infty) \) such that \( \varphi(1) = 0 \) is called a decreasing generator.

A function \( \varphi^{-1} : [0,\infty) \rightarrow [0,1] \) such that

\[
\varphi^{-1}(x) = \begin{cases} 
\varphi^{-1}(x), & \text{if } x \in [0,\varphi(0)], \\
0, & \text{if } x \in (\varphi(0), \infty),
\end{cases}
\]

is called pseudoinverse function for \( \varphi(x) \).
Table 1. Some classes of fuzzy t-norms and t-conorms.

| N | $T(x, y)$ | $S(x, y)$ |
|---|---------|---------|
| 1 | $T_M(x, y) = \min(x, y)$ | $S_M(x, y) = \max(x, y)$ |
| 2 | $T_P(x, y) = xy$ | $S_P(x, y) = x + y - xy$ |
| 3 | $T_L(x, y) = \max(0, x + y - 1)$ | $S_L(x, y) = \min(1, x + y)$ |
| 4 | $T_D(x, y) = \begin{cases} 0, & \text{if } (x, y) \in [0, 1)^2 \\ \min(x, y), & \text{otherwise} \end{cases}$ | $S_D(x, y) = \begin{cases} 1, & \text{if } (x, y) \in [0, 1)^2 \\ \max(x, y), & \text{otherwise} \end{cases}$ |
| 5 | $T_0(x, y) = \frac{xy}{x + y - xy}$ | $S_0(x, y) = \frac{x + y - 2xy}{1 - xy}$ |
| 6 | $T_{\alpha}(x, y) = \frac{xy}{\alpha + (1 - \alpha)(x + y - xy)}$, $\alpha > 0$ | $S_{\beta}(x, y) = \frac{(\beta - 1)xy + x + y}{1 + \beta xy}$, $\beta > -1$ |
| 7 | $T_H(x, y) = \frac{xy}{h + (1 - h)(x + y - xy)}$, $h > 0$ | $S_H(x, y) = \frac{x + y + (h - 2)xy}{h + (h - 1)xy}$, $h > 0$ |
| 8 | $T_{\lambda}(x, y) = \max(0, x + y - 1 - \lambda(1 - x)(1 - y))$, $\lambda > -1$ | $S_{\lambda}(x, y) = \min(1, x + y + \lambda xy)$, $\lambda > -1$ |
| 9 | $T_w(x, y) = \max\left(0, \frac{x + y + wxy - 1}{1 + w}\right)$, $w > -1$ | $S_w(x, y) = \min\left(1, x + y - \frac{w}{1 - w}xy\right)$, $w > -1$ |

Definition 7 ([7]). A strictly increasing continuous function $\varphi : [0, 1] \to [0, \infty)$ such that $\varphi(0) = 0$ is called an increasing generator.

We note that in this case the pseudoinverse function looks like this:

$$\varphi^{-1}(x) = \begin{cases} \varphi^{-1}(x), & \text{if } x \in [0, \varphi(1)], \\ 1, & \text{if } x \in (\varphi(1), \infty), \end{cases}$$

In both cases for $[0, 1]$ we have $\varphi^{-1}(\varphi(x)) = x$.

It is known that a decreasing generator can be associated with an increasing generator and vice versa [4].

We have

Theorem 1 ([7], Characterization Theorem of t-norms). The binary operation $T : [0, 1]^2 \to [0, 1]$ is an Archimedean t-norm if and only if there exists a decreasing generator $t(x)$ such that for all $(x, y) \in [0, 1]^2$ we have

$$T(x, y) = t^{-1}(t(x) + t(y)).$$

(1)

There is an analogous representation for $(x, y) \in [0, 1]^2$ for Archimedean triangular conorms

$$S(x, y) = s^{-1}(s(x) + s(y)),$$

(2)

where $s(x)$ is an increasing generator.

Functions $t(x)$ and $s(x)$ are also called additive generators. They are determined up to a positive multiplicative constant.

Characterization of the main classes of additive generators is one of the most important problems of the functional representation of fuzzy operations [8]. It is closely related to the well-known Kolmogorov’s representation of associative operations [9].
2.3. The main classes of additive generators for commutative and associative operations representable by rational functions

We consider the class of binary fuzzy operations represented by rational functions — the relation of polynomials or, in a particular case, a polynomial where all variables are not to a higher power than the first, i.e.,

$$F(x, y) = \frac{a_0 + \alpha_1 x + \alpha_2 y + \alpha_3 xy}{b_0 + \beta_1 x + \beta_2 y + \beta_3 xy}, \quad (x, y \in [0, 1]).$$

Taking account of commutativity, from $\tilde{F}(x, y)$ we proceed to the following representation:

$$F(x, y) = \frac{a_0 + a_1 (x + y) + a_2 xy}{b_0 + b_1 (x + y) + b_2 xy} \quad (3)$$

This class of functions will be called an RF-class.

Within the framework of the study, the following statements are proved.

**Proposition 1** ([10]). The function $F(x, y)$ is associative under the following conditions:

\[
\begin{cases}
    b_1 a_0 + a_1^2 &= a_1 b_0 + a_2 b_0, \\
    b_2 a_1 + b_1^2 &= b_0 b_2 + a_2 b_1, \\
    a_1 b_1 &= a_0 b_2.
\end{cases}
\]

**Proposition 2** ([10]). Let $F(x, y) = \frac{a_0 + a_1 (x + y) + a_2 xy}{b_0 + b_1 (x + y) + b_2 xy}$ be an associative and commutative operation on $[0, 1]$, $b_0 + b_1 (x + y) + b_2 xy \neq 0$ and simultaneously the following values $\alpha = a_1 b_0 - a_0 b_1$, $\beta = a_2 b_0 - a_0 b_2$, $\gamma = a_2 b_1 - a_1 b_2$ are not equal to 0. Then there are the following classes $K_i$ of additive generators (determined up to a positive multiplicative constant):

1) $K_1$: if $\alpha = \beta = 0$ and $\gamma \neq 0$, then $\varphi_1(x) = \pm \frac{1}{\gamma x} + C$;
2) $K_2$: if $\alpha = \gamma = 0$, $\beta \neq 0$, then $\varphi_2(x) = \pm \frac{1}{\beta} \ln |x| + C$;
3) $K_3$: if $\alpha = 0$, $\beta \neq 0$, $\gamma \neq 0$, then $\varphi_3(x) = \pm \frac{1}{\beta} \ln \left| \frac{\gamma x}{\beta + \gamma x} \right| + C$;
4) $K_4$: if $\alpha \neq 0$, $\beta = \gamma = 0$, then $\varphi_4(x) = \pm \frac{1}{\alpha} x + C$;
5) $K_5$: if $\alpha \neq 0$, $\beta = 0$, $\gamma \neq 0$, then $\varphi_5(x) = \pm \frac{1}{\beta} \sqrt{\frac{\sqrt{\alpha}}{\gamma}} + C$; if $\frac{\alpha}{\gamma} > 0$,
6) $K_6$: if $\alpha \neq 0$, $\beta \neq 0$, $\gamma = 0$, then $\varphi_6(x) = \pm \frac{1}{\beta} \ln |\alpha + \beta x| + C$;
7) $K_7$: let $D = \beta^2 - 4\alpha\gamma$, then
   7.1) if $D > 0$, then $\varphi_7(x) = \pm \ln \left| 1 - \frac{2\sqrt{D}}{2\gamma x + \beta + \sqrt{D}} \right| + C$;
   7.2) if $D = 0$, then $\varphi_7(x) = \pm \frac{1}{\gamma x + \beta/2} + C$;
   7.3) if $D < 0$, then $\varphi_7(x) = \pm \arctg \frac{2\gamma x + \beta}{\sqrt{-D}} + C$.
When using the above formulas, the constant $C$ is found from the additional conditions: for a decreasing generator $\varphi(1) = 0$, for an increasing — $\varphi(0) = 0$.

**Example 1.** Consider a $t$-norm of the form $T_r(x, y) = \frac{xy}{r + (1 - r)(x + y - xy)}$ [7]. Here

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = 1, \quad b_0 = r, \quad b_1 = 1 - r, \quad b_2 = 1 - r.$$  

We compute $\alpha = 0$, $\beta = r$, $\gamma = 1 - r$. Thus, we have a decreasing generator $\varphi(x) = -\frac{1}{r} \ln \left| \frac{(1 - r)x}{r + (1 - r)x} \right| + C$ from the class $K_3$. Let us find the constant $C = \frac{1}{r} \ln |1 - r|$ from the condition $\varphi(1) = 0$. Substituting $C$, we get $\varphi(x) = -\frac{1}{r} \ln \left| \frac{x}{r + (1 - r)x} \right|$. We note that in [7] $r > 0$, then we finally obtain the generator $\varphi(x) = -\ln \left| \frac{x}{r + (1 - r)x} \right|$.

3. Results and discussion

3.1. Definition of commutative and associative functions from the RF-class based on known additive generators

Analyzing the obtained classes we can identify the following types of additive generators that generate commutative and associative operations in the class of rational functions of two variables:

$$\varphi_1(x) = k \cdot \frac{ax + b}{cx + d} + C, \quad \varphi_2(x) = k \cdot \ln \left| \frac{ax + b}{cx + d} \right| + C, \quad \varphi_3(x) = k \cdot \arctg \frac{ax + b}{cx + d} + C.$$  

where $k$ is a certain constant whose sign is essential.

In solving the inverse problem of finding the coefficients of the function (3) on the basis of additive generators, the following results were obtained.

**Proposition 3.** An additive generator $\varphi_1(x)$ generates a commutative and associative operation $F(x, y)$ of the form (3) with coefficients

$$a_0 = bd^2, \quad a_1 = ad^2, \quad a_2 = 2adc - bc^2, \quad b_0 = ad^2 - 2bcd, \quad b_1 = -bc^2, \quad b_2 = -ac^2.$$  

**Proposition 4.** When $ad - bc \neq 0$, the additive generator $\varphi_2(x)$ generates a commutative and associative operation $F(x, y)$ of the form (3) with coefficients

$$a_0 = \frac{db^2 - bd^2}{ad - bc}, \quad a_1 = \frac{bd(a - c)}{ad - bc}, \quad a_2 = \frac{da^2 - bc^2}{ad - bc}, \quad b_0 = \frac{ad^2 - cb^2}{ad - bc}, \quad b_1 = \frac{ac(d - b)}{ad - bc}, \quad b_2 = \frac{ac^2 - ca^2}{ad - bc}.$$  

**Proposition 5.** An additive generator $\varphi_3(x)$ generates a commutative and associative operation $F(x, y)$ of the form (3) with coefficients

$$a_0 = b(d^2 + b^2), \quad a_1 = a(d^2 + b^2), \quad a_2 = 2acd - b(c^2 - a^2), \quad b_0 = a(d^2 - b^2) - 2bcd, \quad b_1 = -b(a^2 + c^2), \quad b_2 = -a(a^2 + c^2).$$
Example 2. Consider a generator $\varphi(x) = -\ln \left| \frac{1 - \frac{1}{2}x}{1 + x} \right|$. In this case $a = -1$, $b = 2$, $c = 1$, $d = 1$. According to Proposition 4, we obtain the following coefficients:

$$a_0 = -2, \quad a_1 = 4, \quad a_2 = 1, \quad b_0 = 5, \quad b_1 = -1, \quad b_2 = 2.$$ 

The corresponding commutative and associative operation has the form

$$F(x, y) = \frac{-2 + 4(x + y) + xy}{5 - (x + y) + 2xy}.$$ 

We note that the generator $\varphi(x)$ is an increasing generator, $\varphi(0) = 0$, but the corresponding operation $F(x, y)$ is not a $t$-conorm. The reason for this fact is that the condition $F(0, x) = x$ is not considered.

Thus, there is the problem of characterizing such additive generators that generate precisely triangular norms and conorms.

3.2. Features of the representation of additive generators for triangular norms and conorms from the RF-class

When additional conditions provided by the definitions are fulfilled, the found functions $\varphi_1(x)$, $\varphi_2(x)$, $\varphi_3(x)$ become additive generators of triangular norms and conorms.

Let $\varphi(x)$ be an increasing generator. Then a condition $\varphi(0) = 0$ must be fulfilled and this condition allows us to find a constant $C$ and go to the following functions:

1) $\tilde{\varphi}_1(x) = \frac{(ad - bc)x}{d(cx + d)} = \frac{tx}{rx + 1}$, where $t = ad - bc$, $r = cd$, $l = d^2$;

2) $\tilde{\varphi}_2(x) = k \cdot \ln \left| \frac{(ax + b) \cdot d}{(cx + d) \cdot b} \right| = k \cdot \ln \left| \frac{1 + mx}{1 + nx} \right|$, where $m = a/b$, $n = c/d$;

3) $\tilde{\varphi}_3(x) = k \cdot \arctg \left( \frac{x(ad - bc)}{x(ab + dc) + (b^2 + d^2)} \right) = k \cdot \arctg \left( \frac{wx}{vx + w} \right)$,

where $u = ad - bc$, $v = ab + cd$, $w = b^2 + d^2$.

We note that since the generators are determined up to a positive multiplicative constant, actually, it is the sign of the constant that makes sense in formulas 2) and 3).

We have the following assertions.

Proposition 6. If $t > 0$ and $l \neq 0$, then $\tilde{\varphi}_1(x)$ is an increasing generator when the following conditions are fulfilled:

- $a) \ r > 0$ and $-1/r < 0$;  
- $b) \ r < 0$ and $-1/r > 1$.

There is no increasing generator for $k < 0$.

Proposition 7. The function $\tilde{\varphi}_2(x)$ is an increasing generator only if the following conditions are fulfilled:

- $a) \ k > 0$, $m > n$, $m > 0$, $n \in [-1, 0)$;  
- $b) \ k < 0$, $m < n$, $n > 0$, $m \in [-1, 0)$.

Proposition 8. The function $\tilde{\varphi}_3(x)$ where $w > 0$ is an increasing generator if $k \cdot u > 0$ and one of the conditions $a) \ -\frac{w}{v} < 0$ or $b) \ -\frac{w}{v} > 1$ is satisfied.

Example 3. Let us consider the function $\varphi_r(x) = -\ln \left( \frac{1 - x}{r + (1 - r)(1 - x)} \right)$ from [7] which is known as an increasing generator. Let us bring it to a form $\varphi_r(x) = \ln \left( \frac{1 + (r - 1)x}{1 - x} \right)$. It is
easy to verify that $\varphi_r(x)$ satisfies Proposition 7. Here $a = r - 1$, $b = 1$, $c = -1$, $d = 1$. On the basis of Proposition 4, we find the coefficients and obtain

$$S_r(x, y) = \frac{x + y + (r - 2)xy}{1 + (r - 1)xy}.$$  \hfill (4)

We note that the $t$-conorm corresponding to $\varphi_r(x)$ in [7] is represented in the form

$$S(x, y) = \frac{x + y + (r - 2)xy}{r + (r - 1)xy},$$

but this formula is wrong because it does not satisfy the definition ($S(x, 0) = \frac{x}{r} \neq x$). On the other hand, all the properties of the triangular conorm are satisfied for the function (4).

### 3.3. Additive generators for triangular conorms from the RF-class

Taking into account the property $S(x, 0) = x$, the $t$-conorms from the RF-class have the form

$$S(x, y) = \frac{x + y + a_2xy}{1 + b_2xy}.$$  

Taking into account the notations $p = a_2$, $q = b_2$ we consider a $t$-conorm of the form

$$S(x, y) = \frac{x + y + pxy}{1 + qxy}. \hfill (5)$$

We can see that the conditions for commutativity and associativity are satisfied.

An example of a $t$-conorm of the form (5) is, for example, the function (4) if we assume that $p = r - 2$, $q = r - 1$.

The study of the parameters included in (5) made it possible to obtain families of corresponding additive generators.

We have the following assertions.

**Proposition 9.** If $p = 0$ and $q \neq 0$, then the $t$-conorm $S_q(x, y) = \frac{x + y}{1 + qxy}$ is generated by increasing generators: a) $s_q(x) = \text{arcg}(-qx)$ when $q < 0$, b) $s_q(x) = \ln \left| \frac{1 + x + \sqrt{q}}{1 - x\sqrt{q}} \right|$ when $q \in (0, 1)$.

**Proposition 10.** If $p \neq 0$ and $q = 0$, then when $p > -1$ there is an increasing generator $s_p(x) = \frac{1}{p} \ln |1 + px|$ that generates a $t$-conorm $S_p(x, y) = x + y + pxy$.

**Proposition 11.** Let $p \neq 0$ and $q \neq 0$, $D = p^2 + 4q$.

- If $D > 0$, $p \in \left(-\sqrt{D}, \sqrt{D}\right)$, $q \in \left(0, \frac{p + \sqrt{D}}{2}\right)$ then there exists an increasing generator $s_{p,q}(x) = \ln \left| \frac{1 - \frac{2qx}{p - \sqrt{D}}}{1 - \frac{2qx}{p + \sqrt{D}}} \right|$ which generates a $t$-conorm (5).
- If $D = 0$, then there is no increasing generator.
- If $D < 0$, then when $q < 0$, then the increasing generator has the form

  $$s_{p,q}(x) = \text{arctg} \left( \frac{-Dx}{2 + px} \right),$$

  which generates $t$-conorm (5).
Example 4. Let us consider the function \( S_r(x, y) = \frac{x + y + (r - 2)xy}{1 + (r - 1)xy} \) again. In accordance with the representation (5), here we have \( p = r - 2, q = r - 1 \). We notice that \( q = p + 1 \).

Since, in general, \( p \neq 0, q \neq 0 \), in accordance with Proposition 11, we define \( D = p^2 + 4q = p^2 + 4(p + 1) = (p + 2)^2 > 0 \) and obtain

\[
s_{p,p+1}(x) = \ln \left| \frac{1 - \frac{2(p+1)x}{p - (p+2)}}{1 - \frac{2(p+1)x}{p + (p+2)}} \right| = \ln \left| \frac{(1 + (p+1)x)/(1-x)}{(1 + (p+1)x)/(1-x)} \right|.
\]

Substituting \( p = r - 2 \) we obtain an increasing generator

\[
s_{r-2,r-1}(x) = \ln \left| \frac{1 + (r-1)x}{1 - x} \right|
\]

which coincides with the generator \( \varphi_r(x) \) of Example 3 for \( x \in [0,1] \). From Proposition 11 it also follows that for the parameter \( r \) we must have the inequality \( r > 0 \).

Similar results were obtained for triangular norms.

3.4. Analysis of the main classes of additive generators for triangular conorms

There are no increasing generators in classes \( K_1, K_2 \) and \( K_3 \).

In the class \( K_4 \) there is an increasing generator \( s_L(x) = x \) which generates \( t \)-conorm \( S_L(x, y) \).

In the class \( K_5 \) for \( \alpha > 0 \) and \( \gamma > 0 \) we have increasing generator \( \varphi_1(x) = \arctg \left( x\sqrt{\gamma/\alpha} \right) \) that generates triangular conorm \( S_{\alpha,\gamma}(x, y) = \frac{x + y}{1 - \gamma/\alpha xy} \).

In the class \( K_6 \) we can identify the following situation \( \left\{ \begin{array}{ll}
\alpha > 0, & \beta > -\alpha \\
\alpha < 0, & \beta > |\alpha|/x
\end{array} \right. \), then the increasing generator has the form \( \varphi_1(x) = -\frac{1}{\beta} \ln \left( 1 + \frac{\beta}{\alpha} x \right) \) and the corresponding triangular conorm \( S_{\alpha,\beta}(x, y) = x + y + \frac{\beta}{\alpha} xy \), in this case, when \( \left\{ \begin{array}{l}
\beta = -\alpha \\
\beta > 0
\end{array} \right. \) we get \( S_P(x, y) \), \( s_P(x) = -\ln(1-x) \).

The class \( K_7 \) is represented by several types of additive generators. For their enumeration we denote \( D = \beta^2 - 4\alpha \gamma, v_1 = -\frac{\beta - \sqrt{D}}{2\gamma}, v_2 = -\frac{\beta + \sqrt{D}}{2\gamma} \). The following situations are possible:

a) if \( D > 0, \gamma > 0 \), \( \left\{ \begin{array}{ll}
v_1 < 0, & v_2 < 0 \\
v_1 > 1, & v_2 > 1
\end{array} \right. \) or \( \left\{ \begin{array}{l}
v_1 \in [0,1] \\
v_2 \in [0,1]
\end{array} \right. \); then the increasing generator has the form \( \varphi_1(x) = \ln \left( \frac{1 - x/v_2}{1 - x/v_1} \right) \), and the corresponding triangular conorm \( S_{v_1,v_2}(x, y) = \frac{v_1v_2(x + y) - (v_1 + v_2)xy}{v_1v_2 - xy} \);

b) if \( D > 0, \gamma < 0 \), \( \left\{ \begin{array}{l}
v_1 > 1, & v_2 > 1
\end{array} \right. \) or \( \left\{ \begin{array}{l}
v_1 \in [0,1] \\
v_2 \in [0,1]
\end{array} \right. \); then the increasing generator has the form \( \varphi_2(x) = -\varphi_1(x) \);

c) if \( D = 0 \) and \( \gamma > 0 \), then the increasing generator has the form \( \varphi_3(x) = 2x/\beta \left( x + \frac{\beta}{2\gamma} \right) \) and the corresponding conorm \( S_{\beta,\gamma}(x, y) = \frac{x + y + (2\beta/\gamma)xy}{1 - (4\gamma^2/\beta^2)xy} \).
d) if $D = 0$ and $\gamma < 0$, then the increasing generator has the form $\varphi_4(x) = -\varphi_3(x)$.

In the class $K_7$ with an additional constraint $\alpha + \beta + \gamma = 0$, where $\alpha \neq \gamma$ from $\varphi_1(x)$, taking into account the change of notation $\beta = -\gamma/\alpha$, we get a generator $s_\beta(x) = \ln \frac{1 + \beta x}{1 - x}$ that generates a $t$-conorm $S_\beta(x, y)$. If for $\varphi_5(x)$ we have an additional constraint $2\gamma/\beta = -1$, then we obtain a generator $s_{-1}(x)$ for a triangular conorm $S_{-1}(x)$.

We note that for triangular norms and conorms in general there are several additive generators that generate them.

4. Conclusion

In this paper, we consider a family of triangular norms and conorms that are representable in the class of rational functions. For such operations, additive generators are obtained based on the gradual implementation of axioms. Most of the presented results concern triangular conorms. However, similar results were also obtained for triangular norms. Additive generators for commutative and associative operations are of interest by themselves. These properties are defining ones in the axiomatics of aggregation functions.

References

[1] Schweizer B and Sklar A 1983 Probabilistic Metric Spaces (New York: North-Holland)
[2] Aczel J 1966 Lectures on Functional Equations and their Applications (New York: Academic Press)
[3] Frank M J 1979 Aequationes Math 19 194–226
[4] Klement E P, Mesiar R and Pap E 2004 Fuzzy Set and Systems 143 5–26
[5] Klement E P, Mesiar R and Pap E 2004 Fuzzy Set and Systems 145 439–454
[6] Klement E P, Mesiar R and Pap E 2004 Fuzzy Set and Systems 145 411–438
[7] Klir G J and Yuan B 1995 Fuzzy sets and fuzzy logic: theory and applications (New Jersey: Prentice Hall PTR)
[8] Klement E P, Mesiar R and Pap E 2004 Fuzzy Set and Systems 145 471–479
[9] Gini K 1970 Average values (Moscow: Statistics)
[10] Ledeneva T M 1997 News of Higher Educational Institutions. Mathematics 33–40