History-dependent Maximum Static Friction and Rectification of a System with a Few Particles in a Periodic Field

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Abstract

History-dependent maximum static friction is investigated using simple models with two and three particles in one-dimensional periodic potentials. In some situations, these systems possess two values of the maximum static friction, with that actually realized being determined by the direction of the slippage relative to the direction of the previous slippage. In particular, the maximum static friction for slippage in a given direction is smaller when the previous slippage was in the same direction than when it was in the opposite direction. Owing to this property, a particle can continue to move in a direction even in the case that it is subject to an external force whose direction change in time as in a ratchet system which is regarded as one of the simplest model of molecular motors. PACS number(s):

Static and dynamic frictions and the transition between them are universal phenomena commonly observed at the surfaces of macroscopic objects. There is a rich variety of such phenomena, and their characteristics depend on the physical properties of the surfaces as well as any lubricants that might exist between them. In some cases, frictions can depend on the system history. For example, the maximum static friction can depend on

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how long the two surfaces have been in contact. Also, in some case, dynamic friction with increasing slippage velocity is larger than that with decreasing velocity.

In this paper, the existence of history-dependent maximum static friction is investigated using simple systems. First, a system consisting of only two particles in a spatially periodic field is considered. In this system, the maximum static friction depends on the direction of the previous slippage. We show that this property can cause the system to exhibit rectification behavior. We also show that similar phenomena are observed in a system consisting of three particles in a spatially periodic field.

We study a one dimensional system containing two particles in which both particles are subject to a spatially periodic potential and one particle is subject to an external force $F_{ex}$ (Fig. 1). The motion of these two particles is described by the over-damped equations,

$$\dot{x}_1 = c_1 c_2 \sin(2\pi(x_1 - x_2)) + c_1 c_p \sin(2\pi x_1) + F_{ex}$$  \hspace{1cm} (1)$$

$$\dot{x}_2 = c_1 c_2 \sin(2\pi(x_2 - x_1)) + c_2 c_p \sin(2\pi x_2)$$ \hspace{1cm} (2)$$

where $x_i$ is the $i$th particle’s position. We set $c_1 = c_p = 1$ and $c_2 = c > 0$, and we consider the system to be defined for $0 \leq x_i < 1$, with periodic boundary conditions. This system belongs to a class of coupled phase oscillators systems that have been studied extensively.

The above system can be regarded as a simplified model of a physical system in which thin lubricants is spread uniformly between two objects with bumpy surfaces, and an external force acts on one of the two objects. In this situation, the motion of the contact points of the two objects and the motion of the lubricant particles are, approximately, modeled by the motion of the first and second particle, as described by Eqs. (1) and (2), respectively (Fig. 1). The condition $c_i > 0$ means that, as the interactions in the system, we consider only the effects of repulsive forces like the excluded volume effect which plays important roles in liquids, solids and gels.

In the following, we report the results of simulations employing above system whose purpose is to determine the relationships between the maximum static friction and observed
microscopic states. We use $F_{ex} = (1 - \cos(t/T))/2$ with large $T$, so that the external force varies smoothly and very slowly. The data points ($\times$) in Fig. 2 indicate values of the maximum static friction $R$, plotted as a function of $c$ ($0 \leq c \leq 1$) for two particle configurations, $x_1^b > x_2^b$ and $x_1^b < x_2^b$, before slippage. Here, $R$ is defined as the maximum value of $|F_{ex}|$ for which the first particle remains stuck as $|F_{ex}|$ increases, and $x_i^b$ is the $i$th particle’s position at the most recent time at which $F_{ex} = 0$. This $c$ - $R$ relation is divided into three regions: I) for $c < \hat{c}_{crit1} \approx 0.59$, $R$ is decreasing function of $c$ for all $x_1^b$, $x_2^b$, II) for $\hat{c}_{crit1} < c < \hat{c}_{crit2} \approx 0.66$, $R$ has two possible values, one realized for $x_1^b > x_2^b$ and one for $x_1^b < x_2^b$, and III) for $c > \hat{c}_{crit2}$, $R$ is an increasing function of $c$ for all $x_1^b$, $x_2^b$.

When $|F_{ex}(t)|$ decreases from some initial value greater than $R$, the first particle ceases slipping at a value of $|F_{ex}| = R_{stop}$. Through our simulations, we found relations between $R_{stop}$ and $R$ for each of the above described cases: $R_{stop} = R$ in case I), $R_{stop} < R$ in case III) as shown in Fig. 2, and $R_{stop} = R_{smaller}$ in case II), where $R_{smaller}$ is the smaller of the two values of $R$.

Figure 3 displays typical temporal evolutions of each particle’s velocity and position for cases I) and III), with $F_{ex}$ as given above. Here, in (a) $c = 0.2$ and $x_1^b < x_2^b$, in (b) $c = 0.2$ and $x_1^b > x_2^b$, in (c) $c = 0.8$ and $x_1^b < x_2^b$, and in (d) $c = 0.8$ and $x_1^b > x_2^b$. In this figure, the gray curves represent position and velocity of the first particle, and the black curves represent those of the second. In the situation depicted in Fig. 3 (a) and (b), $x_1 > x_2$ always holds just before the slippage of the first particle, independently of $x_1^b$ and $x_2^b$. This is due to the fact that if $F_{ex}$ is small, for sufficiently small $c_2$, it can be the case that the first particle is not able to cross the potential barrier, while it is able to cross the second particle. In fact this is the case in the situations described by Figs. 3(a) and (b). Contrastingly, in Figs. 3 (c) and (d), $x_1 < x_2$ always holds just before the slippage of the first particle, independently of $x_1^b$ and $x_2^b$. This follows from the fact that if $F_{ex}$ is small, for sufficiently large $c_2$, it can be the case that the first particle cannot cross the second particle, while it can cross the potential barrier. From these considerations, it is clear why $R$ has only a single value for each $c$ value in cases of I) and III).
By considering the balance equations obtained by setting \( \dot{x}_1 = 0 \) and \( \dot{x}_2 = 0 \) in Eqs. (1) and (2) and choosing the particle configuration before the slippage, we can obtain \( c-R \) curves plotted in Fig. 2. For example, the curve of \( R \) obtained as the maximum values of \( F_{ex} \) with \( \dot{x}_1 = 0 \) and \( \dot{x}_2 = 0 \) under the condition \( x_1 > x_2 \) is consistent with the numerical result for \( c < \dot{c}_{crit}^2 \), where \( R = -[\sin(2X) + c \sin(X)] \) with \( X = \arccos[(-c - (c^2 + 32)^{1/2})/8] \).

The line \( R = c \) is consistent with the numerical results for \( c > \dot{c}_{crit}^1 \), which is obtained from \( R = -[c \sin(2\pi(x_1 - x_2)) + \sin(2\pi x_1)]|_{x_1=0.5,x_2=0.75} \). Here \( x_1 = 0.5 \) and \( x_2 = 0.75 \) corresponds to the values for which the force of the second particle on the first under the condition \( x_1 < x_2 \) is maximal.

In case II), the maximum static friction is determined by the history, which direction did the first particle slip previously. This can be understood as follows. Figure 4 displays typical temporal evolutions of each particle’s velocity and position for case II). (Here, \( c = 0.63 \) and in (a) \( x_1^b < x_2^b \) and in (b) \( x_1^b > x_2^b \).) In contrast to cases I) and III), here, the particles do not cross as \( F_{ex} \) is increased until the first particle starts to slip. We also found that slippage begins at a later time for \( x_1^b < x_2^b \) than for \( x_1^b > x_2^b \). This means that \( R \) depends on the particle configuration: \( R \) for \( x_1^b < x_2^b \) is larger than that for \( x_1^b > x_2^b \). Moreover, the state with \( x_1 > x_2 \) is realized after slippage of the first particle whether \( x_1^b > x_2^b \) or \( x_1^b < x_2^b \). (Because of the symmetry of this system, \( x_1 < x_2 \) is necessary realized after slippage of the first particle if \( F_{ex} < 0 \).) This means that \( R \) for slippage in the same direction as the previous slippage is always smaller than that for the slippage in the opposite direction.

If \( F_{ex} > 0 \) and it varies slowly and smoothly in time, in case II), \( x_1 > x_2 \) is realized after slippage. We now explain the reason for this. Figure 4 (c) plots the temporal evolutions of each particle’s velocity and position for \( c = 0.63 \) with \( F_{ex} = 0.583 \), which is slightly larger than \( R_{smaller} \). If \( F_{ex} \) is constant and \( |F_{ex}| > R \), the first particle slips and the second particle oscillates periodically. As shown in Fig. 4 (c), the time required to switch from \( x_1 > x_2 \) to \( x_1 < x_2 \) is much longer than that to switch from \( x_1 < x_2 \) to \( x_1 > x_2 \) when \( F_{ex} \) is slightly larger than \( R_{smaller} \). If \( c \) is not large, the amplitude of the second particle’s oscillation is not large. In such a situation, \( x_2 \) cannot reach such a value \( (x_2 \sim 0.75) \) that the force of the
second particle on the first is large enough to balance $F_{ex}$ at $x_1 < x_2$. Hence, the system cannot remain in a state with $x_1 < x_2$ for an extended time if the first particle crosses over the potential barrier, in contrast to case III) depicted in Figs. 3 (c) and (d).

Also in case III), the second particle cannot reach at $x_2 \sim 0.75$ when $|F_{ex}|$ decreases from a value greater than $R$. However, in this case, the $x_2$ can reach such a value that the force of the second particle on the first at $x_1 < x_2$ balances $F_{ex}$ with $|F_{ex}| = R_{stop}$, in contrast to case II).

Since, in case II), the static maximum friction to the direction as the direction of previous slippage is smaller than that to the opposite direction, for this case, this system exhibits rectification behavior. Recently, rectification behavior has been observed, for example, in some ratchet systems that are simple models of molecular motors\textsuperscript{17–19}. In these models, with an external force that favors neither direction, particles can move in only one direction, which is completely determined by the shape of the potential.

Now, to demonstrate that our model can exhibit similar behavior, we consider the case $c = 0.63$ and $F_{ex} = 0.61 \sin(t/T)$, with sufficient large $T$. Thus, in this case, the external force acts symmetrically in both positive and negative directions. Here, we choose the value 0.61 because it is halfway between the two values of $R$ for $c = 0.63$. For this system, as shown in the left-hand side of Fig. 4 (d), the first particle can move only in one direction, even though $F_{ex}$ alternates between positive and negative values. However, if a sufficient strong force is applied instantaneously to the first particle in the direction opposite to its motion, its motion can reverse direction [see the right-hand side of Fig. 4 (d)]. This means we can control the slippage direction in the system. This is not possible in the ratchet systems proposed as models of molecular motors.

The phenomena discussed above can also be observed in systems with a larger number of degrees of freedom. As an example, we consider a system that consists of three particles in a spatially periodic field, with the motions of each particle obeying the equation,
\[ x_i = \sum_{i \neq j} c_i c_j \frac{1 + \cos(2\pi(x_i - x_j))}{2} \sin(2\pi(x_i - x_j)) + c_i c_p \frac{1 + \cos(2\pi x_i)}{2} \sin(2\pi x_i) + \delta_{i,1} F_{ex} \]  

(3)

The characteristic length of the interactions between particles in this system is shorter than that of the system described by (0.1) and (0.2). Again, we consider the system to be defined in the region \( 0 \leq x_i < 1 \) and use periodic boundary conditions. In the case that \( c_1 = c_2 = 1 \) and \( c_3 = c_p = c \), behavior similar to that exhibited in cases II) for the two-particle system is observed over a wide range of values of \( c \) with \( F_{ex} = F(1 - \cos(t/T))/2 \), for \( F > 0 \) and sufficiently large \( T \).

The solid curves in Fig. 5 represent \( R \) as a function of \( c \) \((0.1 \leq c \leq 1)\). As shown, \( R \) takes two values for each value of \( c \), depending on the relationships among the \( x_i \). Figures 6 (a), (b) and (c) displays typical temporal evolutions of each particle’s velocity and position with \( c = 0.5 \) for two particle configurations before slippage; the (1,2,3) configuration in which the particles are arranged in the order 1, 2, 3 with respect to the direction of \( F_{ex} \) (the direction of increasing \( x \) in this case), as in (a), and the (1,3,2) configuration, as in (b) and (c). Here, \( F = 0.5 \) in (a) and (b), and \( F = 0.36 \) in (c). By comparing (a) and (b), we find that \( R \) for the (1,3,2) configuration is larger than that for the (1,2,3) configuration. Also it is seen that the (1,2,3) configuration is always realized after slippage in both cases considered in (a) and (b). We are thus led to the conclusion that, as in the system with two particles, under a certain condition, \( R \) for the direction of the previous slippage is always smaller than \( R \) for the direction opposite to the previous slippage. For the three particle system, this condition is that \( F \) be larger than a particular value, which we discuss below.

As stated above, the (1,2,3) configuration is apparently always realized after slippage in the situations considered in Figs. 5 (a) and (b). If \( F \) is not too large, however, the (1,3,2) configuration can be preserved upon slippage. In fact, this is the case for the situation considered in Fig. 6 (c). The dotted curve in Fig. 5 represent critical values of \( F_{ex} \), below which the particle configuration is preserved upon slippage. Thus, in this system, the maximum strength of the external force at the slippage determines whether or not there is a
direction dependence of $R$. We found that with a properly chosen $F_{ex}$, this system too can exhibit rectification behavior.

In this paper, we have investigated the history dependence of the maximum static friction using simple systems consisting of two and three particles in a one-dimensional periodic potential. In these systems, we found that, in some cases, the maximum static friction can depend on the direction of the slippage, being smaller than this direction is the same as that of the previous slippage. By this property, a particle in this system can continue to slip along a single direction even in the case that the direction of the external force changes in time. This behavior is similar to that seen in ratchet systems regarded as one of simplest models of molecular motors. Analytical study of the systems considered here and further investigation of systems with three or more particles will be carried out in the future. The construction of a model that can realize more types of memory effects, for example aging of friction, is also an important future problem.

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FIGURES

Fig. 1. Schematic depiction of the system described by (0.1) and (0.2), consisting of two particles in a spatially periodic potential.

Fig. 2. (×) points and dotted lines represent maximum value of the static friction of the system $R$ as a function of $c$ for two conditions, $x_1^b > x_2^b$ and $x_1^b < x_2^b$, before slippage. The (×) points are the results of our simulation, and dotted lines were obtained analytically. (+) points represent $R_{\text{stop}}$ as a function of $c$ in case III).

Fig. 3. Typical temporal evolutions of the velocity and position of each particle for case with I) in (a) and (b) and case III) in (c) and (d) with slowly changing of $F_{ex}$. In (a) $c = 0.2$ and $x_1^b < x_2^b$, in (b) $c = 0.2$ and $x_1^b > x_2^b$, in (c) $c = 0.8$ and $x_1^b < x_2^b$, and in (d) $c = 0.8$ and $x_1^b > x_2^b$. The gray curves represent the first particle, and the black curves represent the second particle. The thickness of each curve is proportional to the value of $c_i$ for the particle to which it corresponds. The thin sinusoidal curve represents $F_{ex}$. The numbers at the right of the figures are the particle label.

Fig. 4. Typical temporal evolutions of the velocity and position of each particle for case II) with a slowly changing $F_{ex}$ in (a) and (b), a static $F_{ex}$ with the value 0.583 which is just larger than $R_{\text{smaller}}$ in (c), and a slowly changing $F_{ex} = 0.61 \sin(t/T)$ in (d). In (a) $c = 0.63$ and $x_1^b < x_2^b$, and in (b),(c) and (d) $c = 0.63$ and $x_1^b > x_2^b$. Shades and widths of the curves, and the numbers at the right of the figures have the same meanings as in Fig. 3.
Fig. 5. The solid curves represent typical maximum values of the static friction $R$ as the function of $c$ in the system consisting of three particles for two configurations, the (1,2,3) configuration (lower curve) and the (1,3,2) configuration (upper curve) before slippage. The dotted curve represents the critical values of $F_{ex}$, for which the (1,3,2) is preserved under slippage.

Fig. 6. Typical temporal evolutions of the velocity and position of each particle in the three particle system with $c = 0.5$. The shades and widths of the curves have the same meanings as in Fig. 3. The external forces $F_{ex}$ used here are $F_{ex} = (1 - \cos(t/T))/2$ in (a), (b), and (c). The particle configuration before slippage is (1,2,3) in (a) and (1,3,2) in (b) and (c).
