Scaling with a modified Wilson action which suppresses $Z_2$ artifacts in SU(2) lattice gauge theories

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Abstract

A modified Wilson action which suppresses plaquettes which take negative values is used to study the scaling behavior of the string tension. The use of the $\beta_E$ scheme gives good agreement with asymptotic two loop results.
1 Introduction

The simplest and most popular choice of a gauge lattice action is the one proposed by Wilson, which for the group $SU(2)$ reads:

$$S_W(U_l) = \beta \sum_{\Box} (1 - \frac{1}{2} \text{Tr} \ U_{\Box}),$$  \hspace{1cm} (1)$$

where $\beta = 4/g^2$, $U_l \in SU(2)$ is the link variable defined on the link $l \equiv (x, \mu)$, while $\Box \equiv (x, \mu\nu)$ refers to the location and orientation of the corresponding plaquette. $U_{\Box}$ is the standard plaquette variable:

$$U_{\Box} \equiv U_{x,\mu\nu} = U_{x,\mu} U_{x+\mu,\nu} U_{x+\nu,\mu} U_{x,\nu}.$$  \hspace{1cm} (2)$$

In any Monte Carlo simulation the crucial question is whether the field configurations generated provide an adequate representation of the continuum physics. The naive continuum limit can only be obtained for $\frac{1}{2} \text{Tr} \ U_{\Box} \approx 1$. Especially any lattice configurations which locally have some $U_{\Box}$’s where $\frac{1}{2} \text{Tr} \ U_{\Box} \approx -1$ are lattice artifacts which a priori have nothing to do with continuum configurations. A complete suppression of such configurations should not change any continuum physics. For the values of $\beta$ where one performs the Monte Carlo simulations such local $Z_2$ fluctuations are not at all rare if $S_W$ is used. It is thus very important to make sure that these fluctuations have no impact on continuum observables. The easiest way to do so is to modify the lattice action in such a way that these small-scale fluctuations are suppressed, without influencing plaquettes where $\text{Tr} \ U_{\Box} > 0$. This can be done by modifying the Wilson action as follows [1]:

$$S = S_W + S_\lambda,$$  \hspace{1cm} (3)$$

where $S_W$ is the standard Wilson action (1), while $S_\lambda$ is

$$S_\lambda = \lambda \sum_{\Box} [1 - \text{sgn}(\text{Tr} \ U_{\Box})].$$  \hspace{1cm} (4)$$

The action (3) was studied for $\lambda = \infty$ in [2] while the phase structure in the $(\beta, \lambda)$ plane was studied in [1]. For a fixed $\lambda$ one would expect the same continuum limit for $\beta \to \infty$ and also the same identification $\beta = 4/g^2$. However, as shown in [1] there is a marked difference between, say, Creutz ratios, measured for $\lambda = 0$ (the Wilson action) and for $\lambda = 0.5$ in the case of $\beta = 2.5$. One would expect the difference to be even more pronounced with increasing $\lambda$. This illustrates that plaquettes with negative values may play an important role for the range of $\beta$’s where the scaling of pure $SU(2)$ lattice gauge theories is usually studied, and one could be worried about
the relation to continuum physics. The fact that one gets acceptable agreement with continuum scaling relations could be fortuitous since one would expect the action $S$ to reproduce continuum physics better for large $\lambda$. One purpose of this article is to show that we indeed get the correct scaling relations for large $\lambda$ and that there is no reason to expect the plaquettes with negative values which appear in the Wilson action to play any important role for $\beta \geq 2.2$. Another purpose is to test whether the $\lambda$ modification of the action may improve the approach to scaling for the reasons mentioned above.

2 Numerical method

The numerical simulations were performed using a standard Metropolis algorithm to update the gauge fields, combined with an overrelaxation algorithm to decrease the autocorrelations. For every Metropolis sweep we performed 4 overrelaxation steps. Lattice size $12^d$ with periodic boundary conditions was used and we measured all Wilson loops up to the size $6 \times 6$. In order to improve the statistics we modified the configurations using the Parisi trick \[3\] before measuring. Unlike in the case of $\lambda = 0$ the integration involved in the Parisi trick had to be performed numerically and was thus costly in computer time. But the gain in the statistics did more than compensate for that.

The choice of parameters was $\lambda = 10$, which was sufficient to suppress all negative plaquettes, and $\beta \in [1.0, 2.5]$. We found a scaling window for $\beta \in [1.5, 1.9]$ and concentrated the simulations in that interval. Usually 1000 sweeps were used for thermalization and the number of sweeps used for measuring, for various $\beta$, is shown in table 1. Measurements were performed every fifth sweep and the errors were estimated by using the jackknife method.
3 Scaling of the string tension

The \( \beta \)-values in the scaling window are even smaller than the ones used for the standard Wilson action. If one wants to test scaling by comparing with the perturbative two-loop result using the identification \( \beta = 4/g^2 \) it is doomed to fail. We clearly need a suitable effective coupling which we can use instead of \( g^2 \). Here we will use the so-called \( \beta_E \) scheme \[3, 4, 5\]. This scheme is simple and well suited to deal with “perturbations” of the Wilson action, like the ones given by (3)-(4). Let us define

\[
\beta_E = \frac{3}{4} \frac{1}{1 - \langle \frac{1}{2} \text{Tr} U \rangle}.
\]

From weak coupling expansions (\( \beta \to \infty, \ \lambda \) fixed), it is seen that

\[
\beta_E \to \beta \quad \text{for} \quad \beta \to \infty.
\]

It is consistent to use the \( g_E = 4/\beta_E \) in the two loop \( \beta \)-function since \( g_E \) is a function of \( g \) and the two first coefficients of the \( \beta \)-function are invariant under a non-singular coupling constant redefinition. The use of \( \beta_E \) in the two loop \( \beta \)-function is known to diminish scaling violations in a variety of situations: The string tension in \( SU(2) \) and \( SU(3) \) gauge theories, the mass gap in the \( SU(2) \) and \( SU(3) \) chiral models and the deconfining transition temperature \( T_c \), again for \( SU(2) \) and \( SU(3) \) gauge theories, just to mention some. We will now show that the method works well for the modified Wilson action with \( \lambda = 10 \).

Let us first note that for \( \lambda = 10 \) the \( \beta \)-range \( 1.4 – 1.9 \) is mapped into the \( \beta_E \)-range \( 1.837 – 2.104 \), which should compared to the corresponding map for the ordinary Wilson action where the \( \beta \)-range \( 2.2 – 2.5 \) is mapped into the \( \beta_E \)-range \( 1.741 – 2.154 \). From the measurements of Wilson loops mentioned in the last section we extract the string tension using the simplest method: If \( W(R, T) \) denotes a \( R \times T \) Wilson loop we first form

\[
V_{\text{eff}}(R, T) = \log[\frac{W(R, T)}{W(R, T - 1)}].
\]

For \( T \to \infty \) \( V_{\text{eff}}(R, T) \) should go to the ground state static quark potential \( V_{qq}(R) \). We have available \( T \leq 6 \) and the results agree within error bars for \( T = 5 – 6 \). We have used these values as \( V_{qq}(R) \) for \( R \leq 6 \). Next \( V_{qq}(R) \) is fitted to the form:

\[
V_{qq}(R) = C - \frac{E}{R} + \sigma R
\]

and the error bars for \( \sigma \) are mainly coming from systematic errors arising from fits with various lower cuts in \( R \). It is clear that with the limited range of \( R \) and \( T \)'s
available to us nothing will be gained by applying some of the many more elaborate schemes of fitting which are available.

\[ \log \sigma(\beta_E) \] is finally plotted in fig. 1 against \( \beta_E \). The curve shown is the one obtained from the two-loop \( \beta \)-function, according to which the scaling of the lattice string tension \( \sigma(\beta_E) = \sigma_{\text{cont}} a^2(\beta_E) \) is governed by the two-loop scaling of the lattice spacing \( a(\beta_E)^2 \):

\[
a^2(\beta_E) = \Lambda^{-2} \left( \frac{6\pi^2}{11} \beta_E \right)^{102/121} \exp \left[ -\frac{6\pi^2}{11} \beta_E \right]. \tag{9}
\]

It is clear that scaling is reasonable well satisfied in the given range of \( \beta_E \).

It is interesting to compare these results with the corresponding ones obtained by using the ordinary Wilson action. The range of \( \beta \) which gives the same range of \( \beta_E \) as were used for the modified action will be \( 2.3 \leq \beta \leq 2.5 \). In this range it is well known that naive application of scaling does not work particularly well, and to get a fair comparison with the results for the modified action we use again the \( \beta_E \) coupling constant transformation. We now follow the same method as in the case of the modified action and extract the string tension\[1\]. The result is shown as a function of \( \beta_E \) in fig. 1. The use of \( \beta_E \) has improved the scaling considerably compared to the use of \( \beta \) as a coupling constant, as already mentioned, and it is apparent from fig.1 that agreement with the two-loop scaling is as good for the Wilson action as for the modified action.

4 Conclusions

The suppression of negative valued plaquettes should not change the continuum limit of pure \( SU(2) \) lattice gauge theories since these configurations are pure lattice artifacts. Naively one might even expect a smoother approach to the continuum limit if such a suppression is implemented. The modified Wilson action \([3]\) indeed suppresses negative valued plaquettes for large values of \( \lambda \) and it is possible to approach continuum physics for much smaller values of \( \beta \). However, for these small values of \( \beta \) the relation between the continuum coupling constant \( g^2 \) and \( \beta \) is not a simple one and in order to extract the scaling behavior one has to use a modified coupling constant closer reflecting the physics of the system. The \( \beta_E \)-scheme is such a prescription and using it we have found good agreement with continuum physics. From these results it seems clear that the theory with modified Wilson action belongs to the same universality class as the theory defined by the ordinary Wilson action when \( \beta \) is sufficiently large. The worry mentioned in \([1]\) is thus ruled

\[1\]The data used for \( \beta = 2.4 \) and 2.5 are taken from \([3]\).
out. In addition it seems that the approach to scaling is not dramatically improved compared to the situation for ordinary Wilson action. Indirectly this indicates that negative plaquette excitations play no important role in the scaling region for the ordinary Wilson action, at least when we discuss observables like the string tension. However, one would expect that the modified Wilson action is considerable better when it comes to the measurements of topological objects like instantons.

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**Figure 1:** The measured string tension using the modified action (upper curve) and the ordinary Wilson action (lower curve) as functions of $\beta_E$. 