The rainbow connection number of the power graph of a finite group

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Abstract

This paper studies the rainbow connection number of the power graph \( \Gamma_G \) of a finite group \( G \). We determine the rainbow connection number of \( \Gamma_G \) if \( G \) has maximal involutions or is nilpotent, and show that the rainbow connection number of \( \Gamma_G \) is at most three if \( G \) has no maximal involutions. The rainbow connection numbers of power graphs of some nonnilpotent groups are also given.

Key words: rainbow path; rainbow connection number; finite group; power graph.

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1 Introduction

Given a connected graph \( \Gamma \), denote by \( V(\Gamma) \) and \( E(\Gamma) \) the vertex set and edge set, respectively. Define a coloring \( \zeta : E(\Gamma) \to \{1, 2, \ldots, k\}, k \in \mathbb{N} \), where adjacent edges may be colored the same. A path \( P \) is rainbow if any two edges in \( P \) are colored distinct. If \( \Gamma \) has a rainbow path from \( u \) to \( v \) for each pair of vertices \( u \) and \( v \), then \( \Gamma \) is rainbow-connected under the coloring \( \zeta \), and \( \zeta \) is called a rainbow \( k \)-coloring of \( \Gamma \). The rainbow connection number of \( \Gamma \), denoted by \( rc(\Gamma) \), is the minimum \( k \) for which there exists a rainbow \( k \)-coloring of \( \Gamma \).

The rainbow connection number of a graph \( \Gamma \) was introduced by Chartrand et al. [6]. It was showed in [7, 23] that computing \( rc(\Gamma) \) is NP-hard. Actually, it has been proved in [23], that for any fixed \( t \geq 2 \), deciding if \( rc(\Gamma) = t \) is NP-complete. Some topics on restrict graphs are as follows: oriented graphs [8], graph products [15], hypergraphs [4], corona graphs [9], line graphs [21], Cayley graphs [22], dense graphs [20] and sparse random graphs [12]. Most of the results and papers that dealt with it can be found in [19].

In this paper we study the rainbow connection number of the power graph of a finite group. We always use \( G \) to denote a finite group with the identity \( e \). The power graph \( \Gamma_G \) has the vertex set \( G \) and two distinct elements are adjacent if one is a power of the other. Chakrabarty et al. [5] introduced the power graph of a
semigroup. Recently, many interesting results on power graphs have been obtained, see [2, 3, 10, 11, 16–18]. A detailed list of results and open questions on power graphs can be found in [1].

In $G$, an involution $x$ is maximal if any cyclic subgroup does not contain $x$ except $\langle x \rangle$. Denote by $M_G$ the set of all maximal involutions of $G$. We shall use $M_G$ to discuss the rainbow connection number of $\Gamma_G$.

This paper is organized as follows. In Section 2 we express $rc(\Gamma_G)$ in terms of $|M_G|$ if $M_G \neq \emptyset$. In Section 3 we show that $rc(\Gamma_G) \leq 3$ if $M_G = \emptyset$. In particular, we determine $rc(\Gamma_G)$ if $G$ is nilpotent. The rainbow connection numbers of power graphs of some non-nilpotent groups are also given.

2 $M_G \neq \emptyset$

In this section we shall prove the following theorem.

**Theorem 2.1** Let $G$ be a finite group of order at least 3. Then

$$rc(\Gamma_G) = \begin{cases} 3, & \text{if } 1 \leq |M_G| \leq 2; \\ |M_G|, & \text{if } |M_G| \geq 3. \end{cases}$$

We begin with the following lemma.

**Lemma 2.2** $rc(\Gamma_G) \geq |M_G|$.

**Proof.** Let $M_G = \{z_1, \ldots, z_m\}$. Observe that $e$ is the unique vertex adjacent to $z_i$ in $\Gamma_G$, where $i = 1, \ldots, m$. Hence, for each pair of maximal involutions $z_i$ and $z_j$, the path from $z_i$ to $z_j$ is unique, which is $\langle z_i, e, z_j \rangle$. Suppose $\zeta$ is a rainbow $k$-coloring of $\Gamma_G$. Then $|\langle \{z_i, e\} : i = 1, \ldots, m \rangle| = m$, and so $k \geq m$, as desired. \hfill $\square$

For $x \in G$, let $[x] = \{y \in G : \langle y \rangle = \langle x \rangle\}$. Then $\{[x] : x \in G\}$ is a partition of $G$.

**Lemma 2.3** $rc(\Gamma_G) \leq \max\{|M_G|, 3\}$.

**Proof.** Suppose that $\{[x_1], \ldots, [x_s]\}$ and $\{[x_{s+1}], \ldots, [x_{s+t}]\}$ are partitions of $\{x \in G : |x| \text{ is even at least } 4\}$ and $\{x \in G : |x| \text{ is odd at least } 3\}$, respectively. For $1 \leq i \leq s$, let $u_i$ be the involution in $\langle x_i \rangle$. Write $M_G = \{z_1, \ldots, z_m\}$ and

$$E_1 = \{[x, x_i] : x \in \bigcup_{i=1}^{s+t} ([x_i] \setminus \{x_i\}) \cup \{u_i, x_i\} : i = 1, \ldots, s\},$$

$$E_2 = \{[e, x_i] : i = 1, \ldots, s + t\} \cup (\bigcup_{i=1}^s \{u_i, x\} : x \in [x_i] \setminus \{x_i\}) \cup (\bigcup_{j=1}^t \{e, z_j\} : j = 1, \ldots, m\}.$$

The sets of edges $E_1, E_2$ and $\{[e, z_j] : j = 1, \ldots, m\}$ are showed in Figure 1.

Let $k = \max\{|M_G|, 3\}$. Define a coloring

$$\zeta : E(\Gamma_G) \to \{1, \ldots, k\}, \quad f \mapsto \begin{cases} i, & \text{if } f \in E_i, \quad \text{where } i = 1, 2, 3; \\ j, & \text{if } f = \{e, z_j\}, \quad \text{where } j = 1, \ldots, m. \end{cases}$$

In order to get the desired inequality, we only need to show that $\zeta$ is a rainbow $k$-coloring of $\Gamma_G$. Pick a pair of non-adjacent vertices $v$ and $w$ of $\Gamma_G$. It suffices
to find a rainbow path from \( v \) to \( w \) under the coloring \( \zeta \). If \( \zeta(\{e, v\}) \neq \zeta(\{e, w\}) \), then \((v, e, w)\) is a desired rainbow path. Now suppose \( \zeta(\{e, v\}) = \zeta(\{e, w\}) \). Then \( \{v, w\} \not\subseteq (M_G \cup \{e\}) \). Without loss of generality, assume \( v \in V(\Gamma_G) \setminus (M_G \cup \{e\}) \). As shown in Figure 1, there exists a vertex \( v' \in V(\Gamma_G) \setminus (M_G \cup \{e\}) \) such that 

\[
\{\zeta(\{e, v\}), \zeta(\{e, v'\}), \zeta(\{v, v'\})\} = \{1, 2, 3\},
\]

which implies that \((v, v', e, w)\) is a rainbow coloring, as desired.

Combining Lemmas 2.2 and 2.3, we get the following.

**Proposition 2.4** If \(|M_G| \geq 3\), then \(rc(\Gamma_G) = |M_G|\).

For a prime \( p \), let \( s_p(G) \) denote the number of subgroups of order \( p \) in \( G \).

**Lemma 2.5** ([13, Section 4, I]) Let \( p \) be a prime dividing the order of \( G \). Then

\[ s_p(G) \equiv 1 \pmod{p}. \]

**Lemma 2.6** Let \( p \) be a prime dividing \( |G| \). If \( rc(\Gamma_G) = 2 \), then \( s_p(G) = 1 \).

**Proof.** Suppose for the contrary that \( s_p(G) \neq 1 \). It follows from Lemma 2.5 that \( s_p(G) \geq 3 \). Let \( \langle y_1 \rangle, \langle y_2 \rangle \) and \( \langle y_3 \rangle \) be pairwise distinct subgroups of order \( p \) in \( G \). Note that, for \( i \neq j \), there is no cyclic subgroup containing \( \langle y_i \rangle \) and \( \langle y_j \rangle \). Hence, the path from \( y_i \) to \( y_j \) with length 2 is unique, which is \((y_i, e, y_j)\). For any rainbow \( k \)-coloring \( \zeta \) of \( \Gamma_G \), we deduce that \( \zeta(\{e, y_1\}), \zeta(\{e, y_2\}) \) and \( \zeta(\{e, y_3\}) \) are pairwise distinct, which implies that \( k \geq 3 \), contrary to \( rc(\Gamma_G) = 2 \). \( \square \)

By Lemmas 2.2, 2.3 and 2.6, we get the following result.

**Proposition 2.7** If \(|M_G| = 2\), then \(rc(\Gamma_G) = 3\).

**Proposition 2.8** If \(|G| \geq 3\) and \(|M_G| = 1\), then \(rc(\Gamma_G) = 3\).

**Proof.** It follows from Lemma 2.3 that \( rc(\Gamma_G) \leq 3 \). Suppose for the contrary that \( rc(\Gamma_G) \leq 2 \). If \( rc(\Gamma_G) = 1 \), then \( \Gamma_G \) is a complete graph, and so \( G \) is a cyclic group.
of prime power order by [5, Theorem 2.12], contrary to $|G| \geq 3$ and $|M_G| = 1$. In the following assume that $rc(\Gamma_G) = 2$.

Suppose that $G$ is a 2-group. By Lemma 2.6, the involution is unique, which implies that $G$ is cyclic or generalised quaternion by [14, Theorem 5.4.10 (ii)], a contradiction.

Suppose that $|G|$ has a prime divisor $p$ at least 3. Let $x$ be an element of $G$ with $|x| = p$. Write $M_G = \{z\}$. It follows from Lemma 2.6 that $\langle x \rangle$ and $\langle z \rangle$ are normal subgroups in $G$. Note that $\langle x \rangle \cap \langle z \rangle = \langle e \rangle$. So $\langle x \rangle \langle z \rangle$ is a cyclic group, contrary to the fact that $z$ is maximal. \hfill \Box

**Proof of Theorem 2.1:** It follows from Propositions 2.4, 2.7 and 2.8. \hfill \Box

For $n \geq 3$, let $D_{2n}$ be the dihedral group of order $2n$, and let $\mathbb{Z}_2^n$ be the elementary abelian 2-group. Note that $M_{D_{2n}}$ consists of $n$ reflections and $M_{\mathbb{Z}_2^n}$ consists of nonidentity elements. By Theorem 2.1, we get the followings.

**Example 1** For $n \geq 3$, we have $rc(\Gamma_{D_{2n}}) = n$ and $rc(\Gamma_{\mathbb{Z}_2^n}) = 2^n - 1$.

### 3 $M_G = \emptyset$

In this section we study the rainbow connection number of $\Gamma_G$ when $G$ has no maximal involutions.

For a positive integer $n$, let $D(n)$ be the set of all divisors of $n$. Denote by $\phi$ the Euler’s totient function. In view of [24, Part VIII, Problem 45], one has $\phi(n) \geq |D(n)| - 2$. For $x \in G$, recall that $[x] = \{y \in G : \langle y \rangle = \langle x \rangle\}$. Write

$$E_1(\langle x \rangle) = \bigcup_{i=1}^{[D(n)]-2} \{x_i, y : y \in \langle x \rangle, |y| = d_i\},$$

where $[x] = \{x_1, \ldots, x_{\phi(|x|)}\}$ and $D(n) = \{1, d_1, \ldots, d_{[D(n)]-2}, |x|\}$. See Figure 2.

![Diagram](image)

Figure 2: The partition of $V(\Gamma_{\langle x \rangle})$ and the set of edges $E_1(\langle x \rangle)$

**Theorem 3.1** Let $G$ be a finite group with no maximal involutions.

(i) If $G$ is cyclic, then

$$rc(\Gamma_G) = \begin{cases} 
1, & \text{if } |G| \text{ is a prime power;} \\
2, & \text{otherwise.}
\end{cases}$$

(ii) If $G$ is noncyclic, then $rc(\Gamma_G) = 2$ or 3.
Proof. (i) Write $G = \langle x \rangle$. If $|x|$ is a prime power, then $\Gamma_G$ is a complete graph by [5, Theorem 2.12], and so $rc(\Gamma_G) = 1$. Now suppose that $|x|$ is not a power of any prime. Then $rc(\Gamma_G) \geq 2$. With reference to (1), write $E_1 = E_1(\langle x \rangle)$. It is clear that $E_1 \subseteq E(\Gamma_G)$. Let $E_2 = E(\Gamma_G) \setminus E_1$. Define a coloring

$$
\zeta : E(\Gamma_G) \to \{1, 2\}, \quad f \mapsto i \text{ if } f \in E_i.
$$

In order to get the desired result, we only need to show that $\zeta$ is a rainbow 2-coloring. For any pair of nonadjacent vertices $v$ and $w$, there exist distinct indices $i$ and $j$ in $\{1, \ldots, |D(\langle x \rangle)| - 2\}$ such that $|v| = d_i$ and $|w| = d_j$. It follows from Figure 2 that $(v, x, w)$ is a rainbow path under the coloring $\zeta$, as desired.

(ii) It is immediate from Lemma 2.3. \hfill $\square$

We first give two examples for computing $rc(\Gamma_G)$ when $G$ is noncyclic with no maximal involutions. The generalized quaternion group which is given by

$$Q_{4n} = \langle x, y : x^n = y^2, x^{2n} = 1, y^{-1}xy = x^{-1}, \quad n \geq 2. \quad (2)$$

Example 2 If $n$ is odd, then $rc(\Gamma_{Q_8 \times \mathbb{Z}_n}) = 2$.

Proof. There are exactly three maximal cyclic subgroup in $Q_8 \times \mathbb{Z}_n$, which we denote by $\langle x_1 \rangle$, $\langle x_2 \rangle$ and $\langle x_3 \rangle$. It is easy to see that $|x_1| = |x_2| = |x_3| = 4n$. Let $C$ be a subgroup of order $2n$ in $\langle x_1 \rangle$. Then $C = \langle x_i \rangle \cap \langle x_j \rangle$ for $1 \leq i < j \leq 3$. Write $D(n) = \{d_1, \ldots, d_t\}$. Let $B_i$, $C_i$ and $D_i$ be the set of generators of the subgroup of order $4d_i$ in $\langle x_1 \rangle$, $\langle x_2 \rangle$ and $\langle x_3 \rangle$, respectively. Consequently, we have

$$V(\Gamma_{Q_8 \times \mathbb{Z}_n}) = C \cup \bigcup_{i=1}^{t} (B_i \cup C_i \cup D_i),$$

$$E(\Gamma_{Q_8 \times \mathbb{Z}_n}) = E(\Gamma_{\langle x_1 \rangle}) \cup E(\Gamma_{\langle x_2 \rangle}) \cup E(\Gamma_{\langle x_3 \rangle}).$$

The partition of $V(\Gamma_{Q_8 \times \mathbb{Z}_n})$ is showed in Figure 3, where $u$ is the unique involution.

![Figure 3: The partition of $V(\Gamma_{Q_8 \times \mathbb{Z}_n})$ and the set of edges $E_1'$](image)

With reference to (1), there exists a unique vertex $x_3' \in [x_3]$ such that $\{u, x_3'\} \in E_1(\langle x_3 \rangle)$. Write

$$E_1' = \bigcup_{i=1}^{t} \{\{e, x\} : x \in B_i\} \cup \{\{u, x\} : x \in C_i\},$$

$$E_1 = E_1' \cup E_1(\langle x_1 \rangle) \cup E_1(\langle x_2 \rangle) \cup (E_1(\langle x_3 \rangle) \setminus \{\{u, x_3'\}\}).$$

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It is clear that \( E_1 \subseteq E(\Gamma_{Q_8 \times Z_n}) \). Write \( E_2 = E(\Gamma_{Q_8 \times Z_n}) \setminus E_1 \). Define a coloring

\[
\zeta : E(\Gamma_{Q_8 \times Z_n}) \rightarrow \{1, 2\}, \quad f \mapsto k \text{ if } f \in E_k.
\]

For \( i = 1, 2, 3 \), let \( \Delta_i \) be the subgraph of \( \Gamma_{(x_i)} \) induced on \( V(\Gamma_{(x_i)}) \setminus \{e, u\} \). Similar to the proof of Theorem 3.1 (i), we deduce that \( \zeta|_{E(\Delta_i)} \) is a rainbow 2-coloring of \( \Delta_i \). If vertices \( v \) and \( w \) satisfy \( u \notin \{v, w\} \) and \( \{v, w\} \not\subseteq V(\Delta_i) \) for any \( i \in \{1, 2, 3\} \), then \( (v, e, u, w) \) or \( (v, u, w, e) \) is a rainbow path under \( \zeta \) from Figure 3. If \( v \) is a vertex that is not adjacent to \( u \), there exists a vertex \( x''_3 \in [x_3] \setminus \{x'_3\} \) such that \( \{x''_3, v\} \in E_1(\langle x_3 \rangle) \), and so \( (u, x''_3, v) \) is a rainbow path under \( \zeta \). It follows that \( \zeta \) is a rainbow 2-coloring of \( \Gamma_{Q_8 \times Z_n} \). We accomplish the proof. \qed

**Example 3** If \( n \geq 3 \), then \( rc(\Gamma_{Q_{4n}}) = 3 \).

**Proof.** With reference to (2), we have \( y^{-1} = x^n y \) and \( (x^i y)^{-1} = x^{2n-i} y \) for \( i \in \{1, \ldots, n-1\} \), which implies that

\[
V(\Gamma_{Q_{4n}}) = \{e, x, \ldots, x^{2n-1}\} \cup \left( \bigcup_{i=0}^{n-1} \{x^i y, (x^i y)^{-1}\} \right),
\]

\[
E(\Gamma_{Q_{4n}}) = E(\Gamma_{(x)}) \cup \bigcup_{i=0}^{n-1} E(\Gamma_{(x^i y)}),
\]

as shown in Figure 4. It follows from Theorem 3.1 that \( rc(\Gamma_{Q_{4n}}) = 2 \) or 3. Suppose

![Figure 4: \( \Gamma_{Q_{4n}} \)](image)

for the contrary that there exists a rainbow 2-coloring \( \zeta \) of \( \Gamma_{Q_{4n}} \).

Assume that \( n = 3 \). Without loss of generality, let \( \zeta(\{e, x^2\}) = 1 \). Then \( \zeta(\{e, x^i y\}) = 2 \) for \( i \in \{0, 1, 2\} \). Hence, for \( 0 \leq i < j \leq 2 \), the rainbow path from \( x^i y \) to \( x^j y \) is \( (x^i y, x^3, x^j y) \), which implies that \( \zeta(\{y, x^3\}) \), \( \zeta(\{x y, x^3\}) \) and \( \zeta(\{x^2 y, x^3\}) \) are pairwise distinct, a contradiction. Therefore \( rc(\Gamma_{Q_{12}}) = 3 \).

In the following, assume that \( n \geq 4 \). Let \( \Delta \) be the induced subgraph of \( \Gamma_{Q_{4n}} \) on the vertices \( \{e, x, y, xy, x^2 y, x^3 y, x^n\} \). Then \( \zeta|_{E(\Delta)} \) is a rainbow 2-coloring of \( \Delta \).

**Claim.** There exists a rainbow path from \( e \) to \( x^n \) with length 2 under \( \zeta|_{E(\Delta)} \) in \( \Delta \). In fact, if \( \zeta|_{E(\Delta)}(\{e, x^i y\}) = \zeta|_{E(\Delta)}(\{x^i y, x^n\}) \) for each \( i \in \{0, 1, 2, 3\} \), then there exist two distinct indices \( j \) and \( k \) in \( \{0, 1, 2, 3\} \) such that

\[
\zeta|_{E(\Delta)}(\{e, x^i y\}) = \zeta|_{E(\Delta)}(\{x^i y, x^n\}) = \zeta|_{E(\Delta)}(\{e, x^k y\}) = \zeta|_{E(\Delta)}(\{x^k y, x^n\}),
\]
which implies that there is no rainbow path from \( x^j y \) to \( x^k y \) under \( \zeta|_{E(\Delta)} \) in \( \Delta \), a contradiction. Hence, the claim is valid.

Let \( \Delta_0 \) be the graph obtained from \( \Delta \) by deleting the edge \( \{e, x_n\} \). Then \( \Delta_0 \) is isomorphic to the complete bipartite graph \( K_{2,5} \). By Claim, we have \( rc(K_{2,5}) = 2 \), contrary to [6, Theorem 2.6].

For a noncyclic group \( G \) with no maximal involutions, it is difficult for us to determine which groups \( G \) satisfy \( rc(\Gamma_G) = 2 \). However, we give a sufficient condition.

**Proposition 3.2** If \( G \) is a group of order \( p^n q \) for positive integer \( n \), where \( p, q \) are distinct primes and \( p < q \), such that the following conditions hold, then \( rc(\Gamma_G) = 2 \).

(i) Each Sylow \( p \)-subgroup is cyclic and the Sylow \( q \)-subgroup is unique.

(ii) The intersection of all Sylow \( p \)-subgroups is of order \( p^{n-1} \).

(iii) \( p^{n-1} \geq q \).

**Proof.** Note that the number of Sylow \( p \)-subgroups is \( q \). Suppose that \( \{P_1, \ldots, P_q\} \) is the set of all Sylow \( p \)-subgroups, and \( Q \) is the unique Sylow \( q \)-subgroup. Then \( \bigcap_{i=1}^q P_i \) and \( Q \) are cyclic and normal in \( G \). Hence, there exists an element \( x \) of order \( p^{n-1}q \) such that \( (\bigcap_{i=1}^q P_i)Q = \langle x \rangle \), and so the set of all cyclic subgroups of \( G \) is

\[
\{P_1, \ldots, P_q\} \cup \{\langle y \rangle : y \in \langle x \rangle\}.
\]

For \( 1 \leq i \leq q \), let \( A_i \) be the set of all generators of \( P_i \). By (iii) we choose pairwise distinct elements \( u_1, \ldots, u_{q-1} \) in \( (\bigcap_{i=1}^q P_i) \setminus \{e\} \). With reference to (1), write

\[
E'_1 = \{e, y\} : y \in \bigcup_{i=1}^q A_i \cup \bigcup_{i=1}^{q-1} \{\{u_i, y\} : y \in A_i\},
\]

\[
E_1 = E'_1 \cup E_1(\langle x \rangle).
\]

The set \( E'_1 \) is showed in Figure 5. It is clear that \( E_1 \subseteq E(\Gamma_G) \). Let \( E_2 = E(\Gamma_G) \setminus E_1 \).

![Figure 5: \( V(\Gamma_G) \) and the set of edges \( E'_1 \)](image)

Define a coloring

\[
\zeta : E(\Gamma_G) \longrightarrow \{1, 2\}, \quad f \mapsto k \text{ if } f \in E_k.
\]

In order to get the desired result, we only need to show that \( \zeta \) is a rainbow 2-coloring of \( \Gamma_G \). It follows from Theorem 3.1 that \( \zeta|_{E(\Gamma_{\langle x \rangle})} \) is a rainbow 2-coloring
of $\Gamma_{\langle x \rangle}$. Pick any pair of nonadjacent vertices $z$ and $w$ such that $\{z, w\} \not\subseteq V(\Gamma_{\langle x \rangle})$. It suffices to find a rainbow path from $z$ to $w$ under $\zeta$. Without loss of generality, assume that $z \in \bigcup_{i=1}^{q} A_i$. If $w \in \bigcup_{i=1}^{q} A_i$, then there exist indices $i$ and $j$ in $\{1, \ldots, q\}$ with $i < j$ such that $z \in A_i$ and $w \in A_j$, and so $(z, u_i, w)$ is a desired rainbow path. If $w \in V(\Gamma_{\langle x \rangle})$, then $(z, e, w)$ is a desired rainbow path.

By Proposition 3.2, we have the following example.

**Example 4** Let $G = \langle a, b : a^{27} = b^7 = e, a^{-1}ba = b^2 \rangle \cong \mathbb{Z}_{27} \rtimes \mathbb{Z}_7$. Then $rc(\Gamma_G) = 2$.

The following sufficient condition for $rc(\Gamma_G) = 3$ is immediate from Theorem 3.1 and Lemma 2.6.

**Proposition 3.3** Suppose that $G$ is a noncyclic group with no maximal involutions. If there exists a prime $p$ dividing $|G|$ such that the subgroup of order $p$ in $G$ is not unique, then $rc(\Gamma_G) = 3$.

Finally, we determine the rainbow connection number of the power graph of a nilpotent group.

**Corollary 3.4** Let $G$ be a noncyclic nilpotent group with no maximal involutions. Then

$$rc(\Gamma_G) = \begin{cases} 2, & \text{if } G \text{ is isomorphic to } Q_8 \times \mathbb{Z}_n \text{ for some odd number } n; \\ 3, & \text{otherwise.} \end{cases}$$

**Proof.** It follows from Theorem 3.1 that $rc(\Gamma_G) = 2$ or 3. Suppose $rc(\Gamma_G) = 2$. Then for any prime $p$ dividing $|G|$, the subgroup of order $p$ in $G$ is unique by Proposition 3.3. By [14, Theorem 5.4.10 (ii)], the Sylow $p$-subgroups are cyclic for any odd prime $p$, which implies that 2 is a divisor of $|G|$ and the Sylow 2-subgroup is isomorphic to $Q_{2^m}$ for $m \geq 3$. Hence we get $G \cong Q_{2^m} \times \mathbb{Z}_n$ for some odd number $n$. Let $H$ be a subgroup of $G$ that is isomorphic to $Q_{2^m}$.

**Claim.** For any pair of nonadjacent vertices $x$ and $y$ of $\Gamma_H$, there does not exist a vertex in $G \setminus H$ adjacent to both $x$ and $y$ in $\Gamma_G$. Suppose for the contrary that $\{\{x, z\}, \{y, z\}\} \subseteq E(\Gamma_G)$ for some $z \in G \setminus H$. Then $x = z^s$ and $y = z^t$ for some integers $s$ and $t$, which implies that $x, y \in \langle z \rangle$. Note that $|x|$ and $|y|$ are powers of 2. Therefore $x$ and $y$ are adjacent, a contradiction. Hence, the claim is valid.

By Claim, one gets $rc(\Gamma_H) = 2$. It follows from Example 3 that $m = 3$, and so $G \cong Q_8 \times \mathbb{Z}_n$. By Example 2, we get the desired result.

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