QUADRATIC IRRATIONALS, GENERATING FUNCTIONS AND LÉVY CONSTANTS

ANNA BELOVA AND PETER HAZARD

Abstract. We show that the generating function corresponding to the sequence of denominators of the best rational approximants of a quadratic irrational is a rational function with integer coefficients. Consequently we can compute the Lévy constant of any quadratic irrational explicitly in terms of its partial quotients.

1. Introduction

1.1. Background. The aim of this article is to show the following theorem.

Theorem 1.1. Let \( \theta \in \mathbb{R} \setminus \mathbb{Q} \) be a quadratic irrational. For each \( n \), let \( p_n/q_n \) denote the \( n \)-th best rational approximant to \( \theta \). Then the generating functions

\[
F(z) = \sum_{n \geq 0} p_n z^n \quad G(z) = \sum_{n \geq 0} q_n z^n
\]  

are both rational with integer coefficients.

In fact, we will see that explicit expressions for both \( F \) and \( G \) may be given. If the continued fraction expansion of \( \theta \) is eventually periodic of period \( l \) and we let \( m = m_{\text{min}} + l \) where \( m_{\text{min}} \) denotes the minimal pre-period, then \( F \) and \( G \) are given respectively by the expression \((3.58)\) and \((3.59)\) below. The above theorem generalises the known case when \( \theta \) has pre-periodic continued fraction expansion of period one.

Using our result, we can easily derive the following results concerning the Lévy constant of a quadratic irrational. Recall that, given a real number \( \theta \) with \( n \)-th best rational approximant given by \( p_n/q_n \) for each \( n \), the Lévy constant of \( \theta \), when it exists, is given by the following expression

\[
\beta(\theta) = \lim_{n \to \infty} \frac{1}{n} \log q_n
\]  

Paul Lévy showed, following earlier work by A. Ya. Khintchine, that

\[
\beta(\theta) = \frac{\pi^2}{12 \log 2} \quad \text{Lebesgue-almost every } \theta
\]  

(See \[5] \[3] \[4] for more details.) It was shown by Jager and Liardet \[2\] that for every quadratic irrational, the Lévy constant exists. As an immediate corollary to
Theorem (1.1) above we get a new proof of the following result, which was implicitly contained in [2].

**Theorem 1.2.** Let \( \theta \in \mathbb{R} \setminus \mathbb{Q} \) be a quadratic irrational. Let \( l \) denote the (eventual) period of the simple continued fraction expansion of \( \theta \). Let \( M_\theta \) denote the element of \( \text{PSL}(2, \mathbb{Z}) \) corresponding to the simple continued fraction expansion of \( \theta \). Then

\[
\beta(\theta) = \frac{1}{l} \log \text{rad}(M_\theta) \tag{1.4}
\]

where, for every \( A \in \text{PSL}(2, \mathbb{C}) \), we denote by \( \text{rad}(A) \) the spectral radius of either of the linear transformations corresponding to \( A \).

(The description of \( M_\theta \) will be given in more detail below.)

### 1.2. Notation and Terminology

Let \( \mathbb{N} \) and \( \mathbb{N}_0 \) denote the set of positive and non-negative integers respectively. Denote the set of integers by \( \mathbb{Z} \) and, for each positive integer \( l \) let \( \mathbb{Z}_l \) denote the set of integers modulo \( l \), i.e. \( \mathbb{Z}_l = \mathbb{Z}/l\mathbb{Z} \). Let \( \mathbb{R} \) and \( \mathbb{C} \) denote, as usual, the real and complex number fields. Given an arbitrary polynomial \( \eta \), over either \( \mathbb{R} \) or \( \mathbb{C} \), we denote the discriminant by \( \text{discr}_\eta \).

Given \( \theta \in \mathbb{R} \), let \( \lfloor \theta \rfloor \) denote the integer part of \( \theta \), i.e., greatest integer less than or equal to \( \theta \), and let \( \{\theta\} = \theta - \lfloor \theta \rfloor \) denote the fractional part of \( \theta \).

Let \( S_l \) denote the symmetric group of size \( l \), i.e., the permutation group on a set of \( l \) elements. We will denote the signature of a permutation \( \upsilon \) in \( S_l \) by \( \epsilon(\upsilon) \).

(Recall, any permutation can be written, non-uniquely, as a product of adjacent transpositions \( (k \ k + 1) \), and the parity of the total number of such transpositions is the signature.) Such a permutation \( \upsilon \) can be expressed as a product of cycles. A single cycle will be expressed as \( (s_1, s_2, \ldots, s_k) \), for some \( s_1, s_2, \ldots, s_k \in \{1, 2, \ldots, l\} \) where \( \upsilon(s_j) = s_{j+1} \) for each \( j \), where addition is taken mod \( l \).

We denote the cardinality of a set \( S \) by \( \# S \).

### Acknowledgements

The authors would like to thank the Mathematics Department at Uppsala University and IME-USP for their continued hospitality and support. We thank Charles Tresser and Edson de Faria for reading an earlier draft of this article. We also thank Alby Fisher, Sinai Robins and Andreas Strömbergsson for their questions and comments.

### 2. Preliminaries

#### 2.1. Continued fractions

In this section we recall some basic properties of simple continued fraction expansions. The main aim is to set up notation for the rest of the paper. For more details we recommend that the reader consults [3][1].

**2.1.1. Best rational approximants**. Let \( \theta \in [0, 1] \setminus \mathbb{Q} \). We note that the discussion below can be carried out also for irrational points outside \([0, 1]\), with suitable modifications. However, for simplicity we restrict ourselves to the case \( \theta \in [0, 1]\setminus \mathbb{Q} \).

The simple continued fraction expansion of \( \theta \) is denoted by

\[
\theta = [a_1, a_2, \ldots] = \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_n + \cdots}}} \tag{2.1}
\]

where \( a_1, a_2, \ldots \) are positive integers called the partial quotients of the continued fraction. Define the \( n \)th convergent of \( \theta \) to be

\[
[a_1, a_2, \ldots, a_n] = \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_{n-1} + a_n}}} \tag{2.2}
\]
This is a rational number which we will express as \( p_n/q_n \), where \( p_n \) and \( q_n \) are positive integers having no common factors. The following property is satisfied for all \( n \in \mathbb{N} \):

\[
|\theta - \frac{p_n}{q_n}| \leq \inf_{\frac{a}{b} \in \mathbb{Q}} |\theta - \frac{a}{b}|
\]  

(2.3)

For this reason \( p_n/q_n \) is also called the \( nth \) best rational approximant of \( \theta \). Identifying \( \mathbb{C} \) with \( \mathbb{P}(\mathbb{C}^2) \setminus \{(1 : 0)\} \), equation (2.2) can be expressed in matrix form as

\[
\begin{bmatrix}
  p_n \\
  q_n
\end{bmatrix} = 
\begin{bmatrix}
  0 & 1 \\
  1 & a_1
\end{bmatrix} \begin{bmatrix}
  0 & 1 \\
  1 & a_2
\end{bmatrix} \cdots 
\begin{bmatrix}
  0 & 1 \\
  1 & a_{n-1}
\end{bmatrix} 
\begin{bmatrix}
  1 & 0
\end{bmatrix} 
\begin{bmatrix}
  1 & a_n
\end{bmatrix} 
\]  

(2.4)

Similarly

\[
\begin{bmatrix}
  p_{n-1} \\
  q_{n-1}
\end{bmatrix} = 
\begin{bmatrix}
  0 & 1 \\
  1 & a_1
\end{bmatrix} \begin{bmatrix}
  0 & 1 \\
  1 & a_2
\end{bmatrix} \cdots 
\begin{bmatrix}
  0 & 1 \\
  1 & a_{n-1}
\end{bmatrix} 
\begin{bmatrix}
  0 & 0
\end{bmatrix} 
\begin{bmatrix}
  1 & a_n
\end{bmatrix} 
\]  

(2.5)

Thus combining (2.4) and (2.5) we find

\[
\begin{bmatrix}
  p_{n-1} & p_n \\
  q_{n-1} & q_n
\end{bmatrix} = 
\begin{bmatrix}
  0 & 1 \\
  1 & a_1
\end{bmatrix} \begin{bmatrix}
  0 & 1 \\
  1 & a_2
\end{bmatrix} \cdots 
\begin{bmatrix}
  0 & 1 \\
  1 & a_{n-1}
\end{bmatrix} 
\begin{bmatrix}
  0 & 1 \\
  1 & a_n
\end{bmatrix} 
\]  

(2.6)

We therefore inductively get the following recurrence relation

\[
\begin{bmatrix}
  p_{n-1} & p_n \\
  q_{n-1} & q_n
\end{bmatrix} = 
\begin{bmatrix}
  p_{n-2} & p_{n-1} \\
  q_{n-2} & q_{n-1}
\end{bmatrix} \begin{bmatrix}
  0 & 1 \\
  1 & a_n
\end{bmatrix},
\begin{bmatrix}
  p_0 & p_1 \\
  q_0 & q_1
\end{bmatrix} = 
\begin{bmatrix}
  0 & 1 \\
  1 & a_1
\end{bmatrix} 
\]  

(2.7)

and, more generally, for any non-negative integer \( m \leq n-2 \),

\[
\begin{bmatrix}
  p_{n-1} & p_n \\
  q_{n-1} & q_n
\end{bmatrix} = 
\begin{bmatrix}
  p_{n-m-2} & p_{n-m-1} \\
  q_{n-m-2} & q_{n-m-1}
\end{bmatrix} \begin{bmatrix}
  0 & 1 \\
  1 & a_n
\end{bmatrix} \cdots 
\begin{bmatrix}
  0 & 1 \\
  1 & a_n
\end{bmatrix} 
\]  

(2.8)

For each suitable \( n \) and \( m \) define

\[
\begin{bmatrix}
  B_{n-m}^{(m)} & B_{n-m}^{(m+1)} \\
  A_{n-m}^{(m)} & A_{n-m}^{(m+1)}
\end{bmatrix} = 
\begin{bmatrix}
  0 & 1 \\
  1 & a_{n-m}
\end{bmatrix} \cdots 
\begin{bmatrix}
  0 & 1 \\
  1 & a_n
\end{bmatrix} 
\]  

(2.9)

Observe that the \( A_{n-m}^{(m)} \) and \( B_{n-m}^{(m)} \) are well-defined since we have the relation

\[
\begin{bmatrix}
  B_{n-m}^{(m)} & B_{n-m}^{(m+1)} \\
  A_{n-m}^{(m)} & A_{n-m}^{(m+1)}
\end{bmatrix} = 
\begin{bmatrix}
  0 & 1 \\
  1 & a_{n-m}
\end{bmatrix} \cdots 
\begin{bmatrix}
  0 & 1 \\
  1 & a_n
\end{bmatrix} 
\]  

(2.10)

This could equally be inferred from the corresponding dual relation

\[
\begin{bmatrix}
  B_{n-m}^{(m)} & B_{n-m}^{(m+1)} \\
  A_{n-m}^{(m)} & A_{n-m}^{(m+1)}
\end{bmatrix} = 
\begin{bmatrix}
  B_{n-m-1}^{(m)} & B_{n-m-1}^{(m+1)} \\
  A_{n-m-1}^{(m)} & A_{n-m-1}^{(m+1)}
\end{bmatrix} \begin{bmatrix}
  0 & 1 \\
  1 & a_n
\end{bmatrix} 
\]  

(2.11)

The following are well-known basic properties.

**Proposition 2.1.** Let \( \theta \in [0, 1) \setminus \mathbb{Q} \). Then the best rational approximants \( p_n/q_n \) satisfy the following

1. \( p_{n-1}q_n - p_nq_{n-1} = (-1)^n \) [Lagrange identity]
2. \( \frac{p_{n+1}}{q_{n+1}} = [a_n, a_{n-1}, \ldots, a_3, a_2] \)
3. \( \frac{p_{n-1}}{q_{n-1}} = [a_n, a_{n-1}, \ldots, a_2, a_1] \)
4. \( \frac{p_n}{q_{n+1}} = [a_1, a_2, \ldots, a_n, 1/\theta] \)
Proof. The first item follows by taking the determinant of (2.6). The second and third items follow by taking the transpose of (2.6). The last item follows by applying (2.6) to the vector \[
\begin{bmatrix} z \\
1 \end{bmatrix}.
\]

To summarize, the numerators and denominators of the convergents satisfy a recursion relation (2.6) which may be (re)stated as
\[
\begin{align*}
p_n &= a_n p_{n-1} + p_{n-2} & p_0 &= 0 & p_1 &= 1 \\
q_n &= a_n q_{n-1} + q_{n-2} & q_0 &= 1 & q_1 &= a_1
\end{align*}
\tag{2.12}
\]

and more generally
\[
\begin{align*}
p_n &= A_{n-1}^{(1)} p_{n-1} + B_{n-1}^{(1)} p_{n-2} & q_n &= A_{n-1}^{(1)} q_{n-1} + B_{n-1}^{(1)} q_{n-2} \\
&= A_{n-2}^{(2)} p_{n-2} + B_{n-2}^{(2)} p_{n-3} & &= A_{n-2}^{(2)} q_{n-2} + B_{n-2}^{(2)} q_{n-3} \\
& \vdots & & \vdots
\end{align*}
\tag{2.14}
\]

where \(A_{n-m}^{(m)}\) and \(B_{n-m}^{(m)}\) are positive integers satisfying the recurrence relations
\[
\begin{align*}
B_{n-m}^{(m)} &= A_{n-m+1}^{(m-1)} & A_{n-m}^{(m)} &= B_{n-m+1}^{(m-1)} + a_{n-m+1} A_{n-m+1}^{(m-1)} \\
B_n^{(0)} &= 0 & B_{n-1}^{(1)} &= 1 = A_n^{(0)}
\end{align*}
\tag{2.15}
\]

In fact, we have the following.

**Proposition 2.2.** Let \(\theta \in [0, 1) \setminus \mathbb{Q}\). Then the coefficients \(A_n^{(m)}\) satisfy the following
\[
\begin{align*}
A_{n-m}^{(m)} &= a_{n-m+1} A_{n-m+1}^{(m-1)} + A_{n-m+2}^{(m-2)} \\
A_{n-m}^{(m)} &= a_n A_{n-m}^{(m-1)} + A_{n-m}^{(m-2)}
\end{align*}
\tag{2.16, 2.17}
\]

Proof. The first equality is just an application of (2.15). For the second equality, multiply out the relation (2.11) and apply (2.15). \(\square\)

**Proposition 2.3.** Let \(\theta \in [0, 1) \setminus \mathbb{Q}\). Then the best rational approximants \(p_n/q_n\) satisfy the following
\[
\begin{align*}
p_{n+m+1} &= p_n A_{n}^{(m+1)} + p_{n-1} A_{n+1}^{(m)} \\
(-1)^{m+1} q_{n+1} &= q_n A_{n}^{(m+1)} + (-1)^m A_{n+1}^{(m)} \\
(-1)^{m+1} q_{n-m+1} &= q_n A_{n-m}^{(m+1)} + q_{n-m} A_{n}^{(m)} \\
(-1)^{m+1} q_{n-m} &= q_n A_{n-m}^{(m+1)} + q_{n-m} A_{n}^{(m)}
\end{align*}
\tag{2.18, 2.19, 2.20, 2.21}
\]

Proof. The first and third equality follows by applying the expression for \(B_{n-m}^{(m)}\) in (2.15) to (2.14). For the second and fourth equality, equations (2.8), (2.9), and (2.15), imply that
\[
\begin{bmatrix} p_{n-1} & p_n \\
q_{n-1} & q_n \end{bmatrix} = \begin{bmatrix} p_{n-2} & p_{n-1} \\
q_{n-2} & q_{n-1} \end{bmatrix} \begin{bmatrix} A_{n-m}^{(m-1)} & A_{n-m}^{(m)} \\
A_{n-m+1}^{(m-1)} & A_{n-m+1}^{(m)} \end{bmatrix}
\tag{2.22}
\]

The rightmost matrix factor on the right-hand side has determinant \((-1)^{m+1}\), as the matrix is the product of \(m+1\) matrices of determinant \(-1\). Thus, applying the inverse of the rightmost matrix to both sides and multiplying out gives the required equalities. \(\square\)
Proposition 2.4. Let \( \theta \in [0, 1) \setminus \mathbb{Q} \). Then
\[
A_{n-m}^{(m-1)} + A_{n-m}^{(m+1)} = (-1)^{n-m-1} (q_{n-m-1}p_{n-1} - p_{n-m-1}q_{n-1} - q_{n-m-2}p_n + p_{n-m-2}q_n)
\] (2.23)

Proof. Rearranging equation (2.22) from the preceding proposition by applying an appropriate inverse and applying Proposition 2.1(1) to compute the necessary determinant gives
\[
\begin{bmatrix}
A_{n-m}^{(m-1)} & A_{n-m}^{(m)} \\
A_{n-m-1}^{(m-1)} & A_{n-m-1}^{(m+1)}
\end{bmatrix} = (-1)^{n-m-1} \begin{bmatrix}
q_{n-m-1} & -p_{n-m-1} \\
-q_{n-m-2} & p_{n-m-2}
\end{bmatrix} \begin{bmatrix}
p_{n-1} & p_n \\
q_{n-1} & q_n
\end{bmatrix}
\] (2.24)

Taking the trace of both sides now gives the equation (2.23), as required. \(\square\)

2.1.2. Quadratic Irrationals. Let \( \theta \in [0, 1) \setminus \mathbb{Q} \) be a quadratic irrational. By this we will mean that \( \theta \) is an algebraic number with minimal polynomial \( \chi \) with (strict) degree two. A theorem of Lagrange [3, p. 56] implies that \( \theta \) has a pre-periodic simple continued fraction expansion. Hence, there exist positive integers \( a_1, a_2, \ldots, a_m, \ldots, a_{m+l} \) such that
\[
\theta = [a_1, a_2, \ldots, a_m, a_{m+1}, \ldots, a_{m+l}]
\] (2.25)

We call the minimal such \( l \) the period. We call any such \( m \) a preperiod and the least such preperiod the minimal preperiod of the simple continued fraction expansion.

Remark 2.1. Observe that, since \( a_{n+1} = a_n \) whenever \( n > m \), it follows from (2.9) that, for all non-negative integers \( m \) and \( n \) satisfying \( n - m > m \), we have
\[
A_{n-m+t-1}^{(m)} = A_{n-m-1}^{(m)}, \quad B_{n-m+t-1}^{(m)} = B_{n-m-1}^{(m)}
\] (2.26)

Therefore the following quantities are well-defined
\[
A_{m+k}^{(m)} = A_{m+k}^{(m)}, \quad B_{m+k}^{(m)} = B_{m+k}^{(m)} \quad \text{for any } k \geq 0, \ m + k = j \mod l
\] (2.27)

Let \( \theta^\circ = T^\circ (\theta) \), where \( T \) denotes the Gauss transformation. Then \( T^\circ (\theta^\circ) = \theta^\circ \). Thus \( \theta^\circ \) is a solution to the equation
\[
\theta^\circ \cdot m = \frac{1}{a_{m+1} + a_{m+2} + \ldots + a_{m+l} + \theta^\circ \cdot m}
\] (2.28)

and \( \theta \) can be expressed as
\[
\theta = \frac{1}{a_1 + a_2 + \ldots + a_m + \theta^\circ \cdot m}
\] (2.29)

These can be written in matrix form as
\[
\begin{bmatrix}
\theta^\circ \cdot m \\
1
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
1 & a_{m+1}
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
1 & a_{m+2}
\end{bmatrix} \ldots \begin{bmatrix}
0 & 1 \\
1 & a_{m+l}
\end{bmatrix} \begin{bmatrix}
\theta^\circ \cdot m \\
1
\end{bmatrix}
\] (2.30)

and
\[
\begin{bmatrix}
\theta \\
1
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
1 & a_1
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
1 & a_2
\end{bmatrix} \ldots \begin{bmatrix}
0 & 1 \\
1 & a_m
\end{bmatrix} \begin{bmatrix}
\theta^\circ \cdot m \\
1
\end{bmatrix}
\] (2.31)

where we identify matrices with their corresponding linear fractional transformations. Let
\[
M_l = \begin{bmatrix}
0 & 1 \\
1 & a_{m+1}
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
1 & a_{m+2}
\end{bmatrix} \ldots \begin{bmatrix}
0 & 1 \\
1 & a_{m+l}
\end{bmatrix}
\] (2.32)
and

\[
M_0 = \begin{bmatrix} 0 & 1 \\ 1 & a_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & a_2 \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ 1 & a_m \end{bmatrix}
\]

(2.33)

Let \( M_\theta = M_0 M_1 M_0^{-1} \). We call this the element of \( \text{PSL}(2, \mathbb{Z}) \) corresponding to continued fraction expansion of \( \theta \). We call \( M_1 \) the element of \( \text{PSL}(2, \mathbb{Z}) \) corresponding to the periodic part of the continued fraction expansion of \( \theta \). Then \( \theta \) is a solution of the fixed point equation \( M_\theta(z) = z \), \( \theta^{\wedge m} \) is a solution of the fixed point equation \( M_1(z) = z \), and \( \theta = M_0(\theta^{\wedge m}) \). From equation (2.7), equation (2.15), and applying definition (2.24) we have

\[
M_1 = \begin{bmatrix} B_{(m)}^{(l-1)} & B_{(m)}^{(l)} \\ A_{(m)}^{(l-1)} & A_{(m)}^{(l)} \end{bmatrix} = \begin{bmatrix} A_{(m+1)}^{(l-2)} & A_{(m+1)}^{(l-1)} \\ A_{(m)}^{(l-1)} & A_{(m)}^{(l)} \end{bmatrix} \quad (2.34)
\]

and

\[
M_0 = \begin{bmatrix} B_{(0)}^{(m-1)} & B_{(0)}^{(m)} \\ A_{(0)}^{(m-1)} & A_{(0)}^{(m)} \end{bmatrix} = \begin{bmatrix} A_{(0)}^{(m-2)} & A_{(0)}^{(m-1)} \\ A_{(0)}^{(m-1)} & A_{(0)}^{(m)} \end{bmatrix} \quad (2.35)
\]

To \( M_1 \), or equivalently to \( \theta^{\wedge m} \), there are two degree two polynomials which are naturally associated with it

\[
\chi_{\theta^{\wedge m}}(z) = z^2 - \left( A_{(m+1)}^{(l-2)} + A_{(m)}^{(l)} \right) z + (-1)^l \quad (2.36)
\]

\[
\omega_{\theta^{\wedge m}}(z) = A_{(m)}^{(l-1)} z^2 + \left( A_{(m)}^{(l)} - A_{(m+1)}^{(l-2)} \right) z - A_{(m+1)}^{(l-1)} \quad (2.37)
\]

The first is just the characteristic polynomial of \( M_1 \), and thus the roots are the eigenvalues of \( M_1 \). (Observe that det \( M_1 = (-1)^l \) as \( M_1 \) is a product of 1 matrices with determinant \(-1\).) The second is the minimal polynomial of \( \theta^{\wedge m} \), and thus the roots correspond to the eigenvectors of \( M_1 \). Notice that these two polynomials have the same discriminant, and thus the roots are rationally related.

In the same way we may associate a pair of degree two polynomials \( \chi_\theta \) and \( \omega_\theta \) to \( M_\theta \) (as the defining polynomials for eigenvalues and eigenvectors). Since the determinant and trace are conjugacy invariant we find that

\[
\chi_\theta(z) = \chi_{\theta^{\wedge m}}(z) \quad (2.38)
\]

Also observe that the roots of \( \omega_\theta \) are the image under \( M_0 \) of the roots of \( \omega_{\theta^{\wedge m}} \).

Thus it follows (noticing that \( A_{(1)}^{(m-2)} - A_{(0)}^{(m-1)} z \) is the denominator of \( M_0^{-1} \)) that

\[
\omega_\theta(z) = \left( A_{(1)}^{(m-2)} - A_{(0)}^{(m-1)} z \right)^2 \omega_{\theta^{\wedge m}}(M_0^{-1}(z)) \quad (2.39)
\]

3. Generating functions.

3.1. Generating functions associated to a simple continued fraction expansion. Given \( \theta \in [0, 1] \setminus \mathbb{Q} \), denote by \( p_n/q_n \) the \( n \)-th best rational approximant. Consider the generating functions

\[
F(z) = \sum_{n \geq 0} p_n z^n \quad G(z) = \sum_{n \geq 0} q_n z^n \quad (3.1)
\]
3.2. Generating functions associated to quadratic irrationals. We now focus on the case when $\theta \in [0, 1] \setminus \mathbb{Q}$ is a quadratic irrational. Thus, for some positive integers $a_1, a_2, \ldots, a_{m+1} \in \mathbb{N},$

$$\theta = [a_1, a_2, \ldots, a_m, a_{m+1}, \ldots, a_{m+l}]$$

(3.2)

As above, let $p_n/q_n$ denote the sequence of best rational approximants and denote the corresponding generating functions by

$$F(z) = \sum_{n \geq 0} p_n z^n \quad G(z) = \sum_{n \geq 0} q_n z^n$$

(3.3)

The aim of this section is to prove Theorem 1.1. Before doing so, let us set up the following notation. For $k \in \mathbb{N}_0$, define

$$I_k(z) = \sum_{0 \leq n \leq k} p_n z^n \quad J_k(z) = \sum_{0 \leq n \leq k} q_n z^n$$

(3.4)

For $m < j \leq m + l$, define

$$F_j(z) = \sum_{n = j \mod l}^{n > m} p_n z^n \quad G_j(z) = \sum_{n = j \mod l}^{n > m} q_n z^n$$

(3.5)

Given $k \in \mathbb{N}_0$, for $k = j \mod l$, where $m < j \leq m + l$, define

$$F_{(k)}(z) = F_j(z) \quad G_{(k)}(z) = G_j(z)$$

(3.6)

First consider the generating function $F(z)$. Observe that

$$F(z) = I_m(z) + \sum_{m < j \leq m + l} F_{(j)}(z)$$

(3.7)

More generally, for any $k \in \mathbb{N}_0$, we may split the generating function $F(z)$ as

$$F(z) = \sum_{0 \leq n \leq m+l} p_n z^n + \sum_{m < j \leq m + l} \left( \sum_{n = j \mod l}^{n > m+k} p_n z^n \right)$$

(3.8)

Fix $k \geq 1$. For $m < j \leq m + l$, using the expansion (2.14) and periodicity (2.27), we find that,

$$\sum_{n = j+k \mod l}^{n > m+k} p_n z^n = \sum_{n = j+k \mod l}^{n > m+k} \left( A_{n-k} P_{n-k} + B_{n-k} P_{n-k-1} \right) z^n$$

(3.9)

$$= A_{(j)} z^k \sum_{n = j \mod l}^{n > m+k} p_{n-k} z^{n-k} + B_{(j)} z^{k+1} \sum_{n = j-1 \mod l}^{n > m+k} p_{n-k-1} z^{n-k-1}$$

(3.10)

$$= A_{(j)} z^k \sum_{m = j \mod l}^{m \geq m+1} p_m z^m + B_{(j)} z^{k+1} \sum_{m = j-1 \mod l}^{m \geq m} p_m z^m$$

(3.11)

However, for $m < j \leq m + l$, we have the following equalities

$$\sum_{m = j \mod l}^{m \geq m+1} p_m z^m = F_{(j)}(z)$$

(3.12)
For each \( k \) gives \( F(z) \) and initial conditions. \( \) Let \( G(z) \) (In fact, the generating function of any sequence satisfying the recurrence relation for \( F(z) \) becomes

\[
F(z) = I_{m+k}(z) + B_{(m+1)}^{(k)} p_m z^{m+k+1} + \sum_{m<j\leq m+1} \left( A_{(j)}^{(k)} z^k F_{(j)}(z) + B_{(j)}^{(k)} z^{k+1} F_{(j-1)}(z) \right)
\]

This expression may be rewritten, again using (2.27), as

\[
F(z) = I_{m+k}(z) + B_{(m+1)}^{(k)} p_m z^{m+k+1} + \sum_{m<j\leq m+1} \left( A_{(j)}^{(k)} z^k + B_{(j)}^{(k)} z^{k+1} \right) F_{(j)}(z)
\]

Substituting the expression (3.16) in the left-hand side and rearranging therefore gives

\[
\sum_{m<j\leq m+1} \left( 1 - A_{(j)}^{(k)} z^k - B_{(j)}^{(k)} z^{k+1} \right) F_{(j)}(z) = I_{m+k}(z) - I_m(z) + B_{(m+1)}^{(k)} p_m z^{m+k+1}
\]

For each \( k \geq 1 \), exactly the same argument as for \( F(z) \) also yields the following relation for \( G(z) \),

\[
\sum_{m<j\leq m+1} \left( 1 - A_{(j)}^{(k)} z^k - B_{(j)}^{(k)} z^{k+1} \right) G_{(j)}(z) = J_{m+k}(z) - J_m(z) + B_{(m+1)}^{(k)} q_m z^{m+k+1}
\]

(3.17)

(In fact, the generating function of any sequence satisfying the recurrence relations (2.12) will satisfy an expression of the above type with appropriately chosen initial conditions.) Let

\[
F(z) = \begin{bmatrix} F_{(m+1)}(z) \\ F_{(m+2)}(z) \\ \vdots \\ F_{(m)}(z) \end{bmatrix}, \quad G(z) = \begin{bmatrix} G_{(m+1)}(z) \\ G_{(m+2)}(z) \\ \vdots \\ G_{(m)}(z) \end{bmatrix}
\]

and

\[
I(z) = \begin{bmatrix} I_{m+1}(z) - I_m(z) \\ I_{m+2}(z) - I_m(z) \\ \vdots \\ I_{m+1}(z) - I_m(z) \end{bmatrix}, \quad J(z) = \begin{bmatrix} J_{m+1}(z) - J_m(z) \\ J_{m+2}(z) - J_m(z) \\ \vdots \\ J_{m+1}(z) - J_m(z) \end{bmatrix}
\]

and, recalling the equations (2.15),

\[
K(z) = \begin{bmatrix} B_{(m+1)}^{(1)} z^{m+2} \\ B_{(m+1)}^{(2)} z^{m+3} \\ \vdots \\ B_{(m+1)}^{(1)} z^{m+1+1} \end{bmatrix} = \begin{bmatrix} A_{(m+2)}^{(0)} z^{m+2} \\ A_{(m+2)}^{(1)} z^{m+3} \\ \vdots \\ A_{(m+2)}^{(1)} z^{m+1+1} \end{bmatrix}
\]
Setting

\[
L(z) = \begin{bmatrix}
L_{11}(z) & L_{12}(z) & \cdots & L_{1l}(z) \\
L_{21}(z) & L_{22}(z) & \cdots &  \\
\vdots & \vdots & \ddots & \vdots \\
L_{l1}(z) & \cdots & \cdots & L_{ll}(z)
\end{bmatrix}
\]  

(3.21)

where

\[
L_{kj}(z) = 1 - A_{(m+j)k}^{(k)} z^k - B_{(m+j+1)k}^{(k)} z^{k+1}
\]  

(3.22)

the expressions [3.16] and [3.17] respectively become

\[
L(z)F(z) = I(z) + p_mK(z) \quad L(z)G(z) = J(z) + q_mK(z)
\]  

(3.23)

Let

\[
M(z) = \begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & 0 \\
-z^{-1} & z^{-1} & 0 & \cdots & \cdots & 0 \\
0 & -z^{-2} & z^{-2} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & -z^{-l+1} & z^{-l+1}
\end{bmatrix}
\]  

(3.24)

(This matrix encodes certain elementary row moves.) After recalling (cf. equations [2.15]) that we have the initial conditions \(B_{(k)}^{(1)} = 1, \ A_{(k)}^{(1)} = a_{k+1}\) and the relations

\[
B_{(m+j+1)}^{(k)} = A_{(m+j+2)}^{(k-1)} \\
A_{(m+j)}^{(k)} = B_{(m+j+1)}^{(k-1)} + A_{(m+j+1)}^{(1)} A_{(m+j+1)}^{(k)} = A_{(m+j+2)}^{(k-2)} + a_{m+j+1} A_{(m+j+1)}^{(k-1)}
\]  

(3.25) \hspace{1cm} (3.26)

we find that, since \(a_{m+1} = a_{m}\) and \(a_{m+l+1} = a_{m+1}\), the matrix \(N(z) = M(z)L(z)\) can be expressed in column form as

\[
\begin{bmatrix}
A_1 - a_{m+2} z A_2 - z^2 A_3 & A_2 - a_{m+3} z A_3 - z^2 A_4 & \cdots & A_l - a_{m+l+1} z A_1 - z^2 A_2
\end{bmatrix}
\]

\[
= \begin{bmatrix}
A_0^0 + A_1^2 + A_2^3 & A_0^1 + A_1^3 + A_2^4 & \cdots & A_0^l + A_1^{l+1} + A_2^{l+2}
\end{bmatrix}
\]  

(3.27)

where, throughout, addition in lower indices is taken modulo \(l\) and

\[
A_k^0 = A_k = \begin{bmatrix}
1 \\
A_{(m+k)}^{(1)} \\
\vdots \\
A_{(m+k)}^{(l-1)}
\end{bmatrix} \\
A_k^1 = -a_{m+k} z A_k \\
A_k^2 = -z^2 A_k
\]  

(3.28)

Define the \(l \times l\) matrix

\[
A = \begin{bmatrix}
A_1 & A_2 & \cdots & A_l
\end{bmatrix}
\]  

(3.29)

We will need the following preliminary result, which follows from a straightforward proof by induction.

**Proposition 3.1.** Let the matrix \(A\) be defined as above. Then \(\det A = 0\) if and only if the sequence \(a_{m+1}, a_{m+2}, \ldots, a_{m+l}\) has period strictly less than \(l\).
We are now in a position to give a proof of the main theorem. However, before proceeding we recommend that the reader consult Appendix A which contains the solution to linear equations with polynomial coefficients of a more general type.

**Remark 3.1.** The above analysis works for any pre-period \(m\), not just the minimal pre-period \(m_{\min}\). However, to achieve a rather succinct expression for the generating functions, it will become apparent that it is better to choose, for example, \(m = m_{\min} + 1\). (We will draw attention to the two places in the proof below where this is necessary.)

**Proof of Theorem 1.1.** By the preceding analysis above, the equations (3.23) hold. We will only consider the equation for \(F(z)\), as the case for \(G(z)\) is totally analogous.

Applying the matrix \(M(z)\) to both sides of the equation in (3.23), and recalling that \(N(z) = M(z)L(z)\), gives

\[
N(z)F(z) = M(z)(I(z) + p_mK(z))
\]

(3.30)

Recall that \(N(z)\) is expressible as (3.27). Thus, a straightforward application of the matrix \(N(z)\), together with (2.12) and (2.13) (recalling that \(A_{m+1}^k = A_m^{k+1}\) for each \(k\)) gives us the following expressions

\[
M(z)(I(z) + p_mK(z)) = p_{m+1}z^{m+1}A_1 + p_mz^{m+2}A_2
\]

(3.31)

where for these equalities we have used the initial conditions (2.12). Therefore the equation (3.30) which we must solve becomes

\[
N(z)F(z) = p_{m+1}z^{m+1}A_1 + p_mz^{m+2}A_2
\]

(3.32)

**Claim 1.** For each \(k \in \mathbb{Z}_q\) the equation

\[
N(z)E(z) = A_k
\]

(3.33)

has solution \(E_k(z)\) with entries \(E_{j,k}(z)\) given by

\[
E_{j,k}(z) = \frac{u_{j,k}(z)}{v(z)}
\]

(3.34)

where, if \(\kappa = \kappa(j,k)\) and \(\mu = \mu(j,k)\) are defined as in Theorem A.2, then

\[
u(j,k) = \begin{cases} 
  z^{(l-1)-\kappa}A_{m+k}^{(l-\kappa-1)} + z^{(l-1)+\mu}(-1)^{\mu}A_{m+k}^{(l-\mu-1)} & j \neq k - 1 \\
  z^{l-1}A_{m+k} & j = k - 1
\end{cases}
\]

(3.35)

and

\[
v(\xi) = 1 - (-1)^m(q_mp^l-1 - pmq_{m+1} - q_{m-1}p_{m+l} + p_{m-1}q_{m+1})\xi + (-1)^{l+1}\xi^2
\]

(3.36)

**Proof of Claim:** Since \(N(z)\) is expressible in the form (3.27) and \(\det A \neq 0\), we may apply Theorem A.2. More precisely, for \(j = 1, 2, \ldots, l\), if we take

\[
\gamma_j^0 = 1 \quad \gamma_j^1 = -a_{m+j+1} \quad \gamma_j^2 = -1
\]

(3.37)

we take \(C_j = A_j\), and we set \(t = k\), then Theorem A.2 implies that the solution, which we denote by \(E_k(z)\), exists and has entries \(E_{j,k}(z)\), \(j = 1, 2, \ldots, l\), given by

\[
E_{j,k}(z) = \frac{u_{j,k}(z)}{v(z)}
\]

(3.38)
where $v(z)$ and $u_{j,k}(z)$ are polynomials in the variable $z$. In fact, $u_{j,k}(z)$ is given by

$$u_{j,k}(z) = \begin{cases} 
  z^{(1-1)-\kappa} u_{j,k}^{\kappa,0} + z^{(1-1)+\mu} u_{j,k}^{0,\mu} & j \neq k - 1 \\
  z^{-1} u_{k-1,k}^{0,0} & j = k - 1
\end{cases} \quad (3.39)$$

where the coefficients $u_{j,k}^{\kappa,0}$ and $u_{j,k}^{0,\mu}$ satisfy the recurrence relations (A.37) and (A.38) respectively. Observe that the initial values for this recurrence satisfy

$$u_{k,k}^{1,0} = A_{(m+k)}^{(0)} \quad u_{k+1,k}^{1,0} = A_{(m+k)}^{(1)} \quad (3.40)$$

Observe that the recurrence relation (A.38), after making the substitutions (3.37) becomes (2.17). Moreover, the initial conditions (3.40) above agree with the initial conditions given by (2.17). Consequently, for all $j = 1, 2, \ldots, l$, we have

$$u_{j,k}^{\kappa,0} = A_{(m+k)}^{(l-\kappa-1)} \quad (3.41)$$

Next observe that

$$u_{k-2,k}^{0,1} = (-1)^{l-1} A_{(m+k-1)}^{(0)} \quad u_{k-3,k}^{0,2} = (-1)^{l-2} A_{(m+k-2)}^{(1)} \quad (3.42)$$

so the recurrence relation (A.38) together with these initial conditions (3.42) agree with the recurrence relation and initial conditions (2.16). Thus, for all $j = 1, 2, \ldots, l$, we have

$$u_{j,k}^{0,\mu} = (-1)^{\mu} A_{(m+k-l+\mu)}^{(l-\mu-1)} \quad (3.43)$$

(Note: here we have used the convention stated in Remark [3.1 above.] From this we get the expression for the numerator of $E_{j,k}$ in terms of $A_{(m+k)}^{(l-\kappa-1)}$ and $A_{(m+k-l+\mu)}^{(l-\mu-1)}$ given by (3.39).

Now consider the denominator of $E_{j,k}$. Theorem [A.1] tells us that $v$ is of the form

$$v(\xi) = v_0 + \xi v_1 + \xi^2 v_2 \quad (3.44)$$

where the coefficients are given by

$$v_0 = \prod_{1 \leq k \leq l} \gamma_0^k = 1 \quad v_2 = \prod_{1 \leq k \leq l} \gamma_2^k = (-1)^l \quad (3.45)$$

The equalities (3.41) and (3.43), and the hypothesis (3.37), together with Proposition [A.6] in the case $s = 1$, imply that

$$v_1 = u_{1,1}^{0,1} - a_{m+1}^{0,0} - u_{1,2}^{1,0} = -A_{(m+1)}^{(l-2)} - a_{m+1}^{1,1} A_{(m+1+1)}^{(l-1)} - A_{(m+1+2)}^{(l-2)} \quad (3.46)$$

(Recall that addition in lower indices is taken in $\mathbb{Z}_l$.) Applying the recurrence relation (2.16), after recalling that periodicity implies that $A_{(m)}^{(l)} = A_{(m)}^{(l)}$, we therefore find that

$$v_1 = -A_{(m+1)}^{(l-2)} - A_{(m)}^{(l)} \quad (3.47)$$

By applying Proposition [2.4] we get the equivalent expression

$$v_1 = -(-1)^m \left(q_m a_{m+1+1} - p_m a_{m+1} - q_{m-1} a_{m+1} + p_{m-1} q_{m+1} \right) \quad (3.48)$$

Thus equation (3.36) holds and the claim is shown. //
Linearity then gives us the solution in the particular case (3.32) above. Namely, the above claim in the case \( k = 1 \) and \( 2 \), followed by summing over all \( F_j \), we find that

\[
F(z) = z^{m+1} \frac{u(z)}{v(z)}
\]

(3.49)

where \( v \) is given by (3.30) and \( u \) is given by

\[
u(z) = p_{m+1} \sum_{1 \leq j \leq l} u_j,1(z) + p_m z \sum_{1 \leq j \leq l} u_j,2(z)
\]

(3.50)

Collecting terms of the same degree gives the expression, using the notation (3.39) used in Claim 1 above,

\[
u(z) = p_{m+1} u_{1,1}^{l-1,0} + \sum_{2 \leq j \leq l} z^{j-1} \left( p_{m+1} u_{j,1}^{l-j,0} + p_m u_{j,2}^{l-j+1,0} \right)
\]

(3.51)

\[
+ \sum_{i+1 \leq j \leq 2l-1} z^{j-1} \left( p_{m+1} u_{j,1}^{0,j-1} + p_m u_{j,2}^{0,j-1} \right) + z^{2l-1} p_m u_{2l,2}^{0,0,1-1}
\]

(Recall that the lower indices of \( u_{n,k,u} \) lie in \( \mathbb{Z}_4 \).) We investigate the four types of expressions in the above sum as follows. By the above (3.41)

\[
p_{m+1} u_{1,1}^{l-1,0} = p_{m+1} A_{m+k}^{(0)} = p_{m+1}
\]

(3.52)

For \( 2 \leq j \leq l \), applying (3.41) followed by equality (2.18) we find that

\[
p_{m+1} u_{j,1}^{l-j,0} + p_m u_{j,2}^{l-j+1,0} = p_{m+1} A_{m+1}^{(j-1)} + p_m A_{m+2}^{(j-2)}
\]

(3.53)

(3.54)

For \( l+1 \leq j \leq 2l-1 \), applying (3.43) followed by equality (2.19) gives us

\[
p_{m+1} u_{j,1}^{0,j-1} + p_m u_{j,2}^{0,j-1} = (-1)^{j-1} \left( p_{m+1} A_{m+1}^{(2l-j-1)} - p_m A_{m+1}^{(2l-j)} \right)
\]

(3.55)

\[
= (-1)^{l-1} p_{m+j-2l}
\]

(3.56)

(Note: here we have also used the convention stated in Remark 3.1 above.) Lastly, equality (3.43) gives

\[
p_{m} u_{l,2}^{0,l-1} = p_m (-1)^{l-1} A_{m+1}^{(0)} = (-1)^{l-1} p_m
\]

(3.57)

Together these imply that

\[
F(z) = \frac{\sum_{1 \leq j \leq l} p_{m+j} z^{j-1} + (-1)^{l-1} \sum_{l+1 \leq j \leq 2l} p_{m+j-2l} z^{j-1}}{1 - (-1)^{m} \left( q_m p_{m+l-1} - q_m q_{m+l-1} - q_m - p_{m+l-1} + p_m q_{m+l} \right) z^l + (-1)^{l-1} z^{2l}}
\]

(3.58)

The same argument also shows, mutatis mutandis, that

\[
G(z) = \frac{\sum_{1 \leq j \leq l} q_{m+j} z^{j-1} + (-1)^{l-1} \sum_{l+1 \leq j \leq 2l} q_{m+j-2l} z^{j-1}}{1 - (-1)^{m} \left( q_m p_{m+l-1} - q_m q_{m+l-1} - q_m - p_{m+l-1} + p_m q_{m+l} \right) z^l + (-1)^{l-1} z^{2l}}
\]

(3.59)
Since the matrix $A$ is nonsingular, it follows that a similar analysis can be made for an arbitrary right-hand side of equations (3.30). In particular, given any initial conditions to the recurrence relations (2.12) and (2.13).

**Corollary 3.1.** Take an arbitrary preperiodic sequence $a_j$ of positive integers of preperiod $l$. Let $r_n$ be an arbitrary sequence satisfying the recurrence relations given in (2.12) and (2.13). (For any choice of initial conditions.) Then the corresponding generating function is rational.

**Corollary 3.2.** Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ be a quadratic irrational. Let $M_1$ denote the element of $\text{PSL}(2, \mathbb{Z})$ corresponding to the periodic part of the continued fraction expansion of $\theta$. Denote the characteristic polynomial by $\chi$. The denominator of the corresponding generating functions $F$ and $G$ has the form $v(z^l)$, where

$$v(\xi) = \xi^2 \chi \left( \frac{1}{\xi} \right) \quad (3.60)$$

In particular, $\text{discr}_v = \text{discr}_\chi$.

**Proof.** Recall that $M_1$ is given by equation (2.51). Observe that $M_1$ is a product of $l$ matrices of determinant $-1$. Thus $\det M_1 = (-1)^l$. Next observe that by equations (3.47), (3.45) and (3.44)

$$v(\xi) = 1 - \left( A_{l+1}^{(l-2)} + A_{m}^{(l)} \right) \xi + (-1)^l \xi^2 \quad (3.61)$$

$$= 1 - \text{tr}(M_1) \xi + \det(M_1) \xi^2 \quad (3.62)$$

$$= \xi^2 \chi \left( \frac{1}{\xi} \right) \quad (3.63)$$

Since the discriminant of an arbitrary quadratic polynomial $\chi$ is invariant under the involution $\chi(\xi) \mapsto \xi^2 \chi(1/\xi)$, the result follows.

**Proof of Theorem 1.2.** By the Theorem of Jager and Liardet [2], the Lévy constant $\beta(\theta) = \lim_{n \to \infty} \frac{1}{n} \log q_n$ exists. By the Cauchy-Hadamard formula, and by continuity of the logarithm, $\frac{1}{\rho} = \lim_{n \to \infty} q_n^{1/n}$, where $\rho$ denotes the radius of convergence of the generating function $G$. Therefore $\beta(\theta) = -\log \rho$. Let $u$ and $v$ be the functions defined in the proof of Theorem 1.1 for the generating function $G$. Observe that the poles of $G$ must occur at solutions to $v(z^l) = 0$. Let $v_{\pm}$ denote the zeroes of $v$, where $\pm$ denotes the sign used in the quadratic formula. Thus

$$v_{\pm} = \frac{\tau \pm \sqrt{\tau^2 - 4(-1)^l}}{2(-1)^l} \quad (3.64)$$

where $\tau = \text{tr}M_1 = A_{l+1}^{(l-2)} + A_{m}^{(l)}$. Observe that both $v_-$ and $v_+$ are real. Also observe that the zero $v_{\min}$ with the smallest modulus ($v_-$ in the case when $l$ is even, and $v_+$ in the case when $l$ is odd) is always positive and, in fact, lies in the interval $(0, 1)$. Consequently $v_{\min}^{1/l}$ is a zero of $v(z^l)$, it is also positive real number, and moreover lies in the interval $(0, 1)$.

**Claim 2.** The polynomial $u(z)$ has no zeroes in $(0, 1)$. 
Proof of Claim: Rearranging the expression for numerator of $G(z)$ in (3.59) given above we have

$$u(z) = \sum_{1 \leq j \leq l} z^{j-1} (q_{m+j} + (-1)^{l}q_{m+j-l} z^{l})$$

(3.65)

Observe that $q_{k} > q_{j}$ for all $k > j > 0$. Thus, for all positive $j$ and all $z \in (0,1)$, we have the inequality $q_{m+j} > |q_{m+j-l} z^{l}|$ on the interval $(0,1)$ the polynomial $u$ can be expressed as a sum of positive terms, and the claim follows. / /

Consequently, $v_{\text{min}}^{1/l}$ is a pole of the generating function $G(z)$. Any other pole of $G(z)$ must lie on the union of circles

$$\left\{ |z| = |v_{-}|^{1/l} \right\} \cup \left\{ |z| = |v_{+}|^{1/l} \right\}$$

(3.66)

Thus $\rho = |v_{\text{min}}|^{1/l}$. However, by Corollary 3.2 above, it follows that $1/v_{-}$ and $1/v_{+}$ are zeroes of the characteristic polynomial $\chi$. Since $v_{\text{min}}$ is the zero of $v(\xi)$ of minimal modulus it follows that $1/v_{\text{min}}$ is the zero of $\chi(\xi)$ of maximal modulus, i.e., an eigenvalue of maximal modulus of the matrix $M$. The result follows. □

Appendix A. Matrix Computations.

A.1. The Setup. Let $C_{1}, C_{2}, \ldots, C_{l}$ be column vectors in $\mathbb{C}^{l}$ and define the $l \times l$ matrix

$$C = \begin{bmatrix} C_{1} & C_{2} & \cdots & C_{l} \end{bmatrix}$$

(A.1)

For $1 \leq r, s \leq l$, denote by

- $C'$ the matrix $C$ with the $r$th row removed.
- $C^{rs}$ the matrix $C$ with the $r$th row and $s$th column removed.
- $C_{s}$ the $s$th column of the matrix $C$.
- $C_{rs}$ the $r$th row of the $s$th column of the matrix $C$.
- $C_{s}'$ the column $C_{s}$ with the $r$th row removed.

For each $s = 1, 2, \ldots, l$ and $p = 0, 1, 2$, take a polynomial $c_{s}^{p}(z) \in \mathbb{C}[z]$ and define

$$C_{s,p}(z) = c_{s}^{p}(z)C_{s+p} \quad \forall s = 1, 2, \ldots, l, \quad \forall p = 0, 1, 2.$$  

(A.2)

(Thus, for instance, $C_{s,p}(z) = c_{s}^{p}(z)C_{s+p}$, where addition in the lower index is taken mod $l$.) Define

$$N_{s}(z) = C_{s,0}(z) + C_{s,1}(z) + C_{s,2}(z) \quad \forall s = 1, 2, \ldots, l$$

(A.3)

and

$$N(z) = \begin{bmatrix} N_{1}(z) & N_{2}(z) & \cdots & N_{l}(z) \end{bmatrix}$$

(A.4)

For $1 \leq r, s \leq l$, let

$$N^{rs}(z) = \begin{bmatrix} \hat{N}_{1}^{r}(z) & N_{2}^{r}(z) & \cdots & \hat{N}_{s}^{r}(z) & \cdots & N_{l}^{r}(z) \end{bmatrix}$$

(A.5)

where the hat denotes that the column is omitted. (Thus $N^{rs}(z)$ denotes the $(r, s)$th matrix minor of $N(z)$.)

---

1. i.e., the $(r, s)$th matrix minor of $C$.
2. i.e., the $(r, s)$th entry of $C$. 

---
A.2. Computation of a Determinant.

**Theorem A.1.** If
\[
c^p_s(z) = \gamma^p_s z^p \quad \text{some } \gamma^p_s \in \mathbb{C}, \quad \forall s = 1, 2, \ldots, l, \forall p = 0, 1, 2
\] (A.6)
then
\[
det N(z) = det C \times v(z^l)
\] (A.7)
where \(v\) is a quadratic polynomial
\[
v(\xi) = v_0 + v_1 \xi + v_2 \xi^2
\] (A.8)
with coefficients given by
- \(v_0 = \prod_{1 \leq k \leq l} \gamma^0_k\)
- \(v_1 = v_1(\gamma^0_1, \ldots)\) is a polynomial in \(\gamma^0_1, \ldots\), which is invariant under cyclic permutations in the lower index, but not a symmetric polynomial.
- \(v_2 = \prod_{1 \leq k \leq l} \gamma^2_k\)

To prove Theorem A.1, we need to set up the following notation and terminology. (This will be followed by some auxiliary propositions before we start the proof.) We will identify the index set \(\{1, 2, \ldots, l\}\) for the collection of columns \(C_1, C_2, \ldots, C_l\) with the cyclic group \(Z_l = \mathbb{Z}/l\mathbb{Z}\). Thus the index set \(\{1, 2, \ldots, l\}\) becomes endowed with addition modulo \(l\) and the cyclic order in a natural manner. We observe, importantly, that the notion of a closed (oriented) interval makes sense in this setting. Let
\[
\mathcal{T} = \{\tau: Z_l \rightarrow \{0, 1, 2\}\}
\] (A.9)
We will also represent \(\tau\) in \(\mathcal{T}\) by the corresponding string, over the alphabet \(\{0, 1, 2\}\), given by
\[
\tau(1)\tau(2)\cdots\tau(l)
\] (A.10)
As usual, we denote by \(0^r\) a (sub)string of \(00\cdots 0\) of length \(r\) and define \(1^r\) and \(2^r\) similarly.

**Remark A.1.** The reason for introducing this notation is the following. Since the determinant is multilinear, equation \[(A.3)\] implies that
\[
det N(z) = \sum_{\tau \in \mathcal{T}} \det [ C_{1;\tau(1)} \ C_{2;\tau(2)} \ \cdots \ C_{l;\tau(l)} ]
\] (A.11)
We need to compute each summand. For each \(\tau \in \mathcal{T}\), multilinearity of the determinant again, together with equation \[(A.2)\], implies that
\[
det [ C_{1;\tau(1)} \ C_{2;\tau(2)} \ \cdots \ C_{l;\tau(l)} ] = c(\tau) \det [ C_{1+\tau(1)} \ C_{2+\tau(2)} \ \cdots \ C_{l+\tau(l)} ]
\] (A.12)
where
\[
c(\tau) = \prod_{1 \leq k \leq l} c^{\tau(k)}_k
\] (A.13)
Thus we wish to determine (i) when the determinant on the right-hand side of \[(A.12)\] is zero, (ii) if the determinant on the right-hand side of \[(A.12)\] is non-zero, calculate \(c(\tau)\).
For \( \tau \in \mathcal{T} \), define \( v_\tau : \mathbb{Z}_4 \to \mathbb{Z}_4 \) by
\[
v_\tau(k) = k + \tau(k)
\] (A.14)
This gives a one-to-one correspondence between \( \mathcal{T} \) and the set
\[
\mathcal{U} = \{ v : \mathbb{Z}_4 \to \mathbb{Z}_4 \mid k \leq v(k) \leq k + 2 \mod 4, \forall k \in \mathbb{Z}_4 \}
\] (A.15)
Given \( v \in \mathcal{U} \), let \( \tau_v \) denote the corresponding element of \( \mathcal{T} \). We say that \( \tau \in \mathcal{T} \) is decreasing at \( k \in \mathbb{Z}_4 \) if
\[
\tau(k + 1) = \tau(k) - 1 \quad \text{or} \quad \tau(k + 2) = \tau(k) - 2
\] (A.16)
If \( \tau \) is not decreasing at any point we say it is non-decreasing. We say that \( v \in \mathcal{U} \) is non-decreasing if the corresponding \( \tau_v \) is non-decreasing. Let
- \( \mathcal{T}^+ \) denote the set of non-decreasing \( \tau \in \mathcal{T} \)
- \( \mathcal{U}^+ \) denote the set of non-decreasing \( v \in \mathcal{U} \).

**Proposition A.1.** For \( v \in \mathcal{U} \), \( v \in \mathcal{S}_4 \) only if \( v \in \mathcal{U}^+ \).

**Proof.** For \( k \in \mathbb{Z}_4 \), if \( \tau(k + 1) = \tau(k) - 1 \) then \( v(k + 1) = v(k) \). Similarly \( \tau(k + 2) = \tau(k) - 2 \) then \( v(k + 2) = v(k) \). In either case \( v \) is not injective. Thus decreasing \( v \) cannot be permutations.

Conversely, assume that \( v \in \mathcal{U} \) is not a permutation. Thus there exist distinct \( k_1, k_2 \in \mathbb{Z}_4 \) such that \( v(k_1) = v(k_2) \). Since \( k \leq v(k) \leq k + 2 \) for all \( k \in \mathbb{Z}_4 \), this implies that \( |k_1 - k_2| \leq 2 \). A case analysis now finishes the proof. \( \square \)

**Corollary A.1.** Let \( \tau \in \mathcal{T} \). If \( \tau \notin \mathcal{T}^+ \) then
\[
\det \begin{bmatrix} C_{1;\tau(1)} & C_{2;\tau(2)} & \cdots & C_{t;\tau(t)} \end{bmatrix} = 0
\] (A.17)
Otherwise \( \tau \in \mathcal{T}^+ \) and
\[
\det \begin{bmatrix} C_{1;\tau(1)} & C_{2;\tau(2)} & \cdots & C_{t;\tau(t)} \end{bmatrix} = (-1)^{\epsilon(v_\tau)} c(\tau) \det C
\] (A.18)
where \( \epsilon(v_\tau) \) denotes parity of the number of adjacent transpositions of the permutation \( v_\tau \).

**Lemma A.1.** Let \([s, t]\) be a closed (oriented) interval in \( \mathbb{Z}_4 \). If \( \tau \) is non-decreasing on \([s, t]\) then \( \tau \) has the form
\[
\tau(s) \tau(s + 1) \cdots \tau(t) = 0^\kappa 1^{\lambda_1} b 1^{\lambda_2} b \cdots 1^{\lambda_{r-1}} b 1^{\lambda_r} 2^\mu
\] (A.19)
where \( \kappa, \lambda_1, \lambda_2, \ldots, \lambda_r, \) and \( \mu \) are non-negative integers and \( b \) denotes the block \( b \) of length two given by
\[
b = 20
\] (A.20)

**Proof.** The key observation, together with its dual, is the following:
- if, in the string \( \tau(s) \tau(s + 1) \cdots \tau(t) \), there are two consecutive 2's, this cannot be preceded by either 0 or 1,
- if, in the string \( \tau(s) \tau(s + 1) \cdots \tau(t) \), there are two consecutive 0's, this cannot be preceded by either 2 or 1.

The reason for the first being that if \( \tau(r) = \tau(r + 1) = 2 \) then \( \tau(r + 2) = 1 \) implies \( \tau \) is decreasing at \( r + 1 \), while \( \tau(r + 2) = 0 \) implies \( \tau \) is decreasing at \( r \). The argument in the dual case is analogous.

As a consequence, by the non-decreasing property,
- any substring of 0's of length greater than one must be contained in a substring of 0's attached to the left boundary of \([s, t]\),
• any substring of 2’s of length greater than one must be contained in a substring of 2’s attached to the right boundary of \([s, t] \). Elsewhere in the string, 2 must be followed by 0 and 0 must be preceded by 2. (We now see the importance of the block \(b \).) □ 

**Proposition A.2.** For any \( \tau \in T^+ \), \( \sum_k \tau(k) = \eta \) for some \( \eta = \eta(\tau) \in \{0, 1, 2\} \).

**Proof.** First, observe that there is a unique \( \tau \) so that \( \sum_k \tau(k) = 0 \), namely \( \tau(k) = 0 \) for all \( k \). Likewise, there is a unique \( \tau \) so that \( \sum_k \tau(k) = 2l \), namely \( \tau(k) = 2 \) for all \( k \). Thus we just need to show that in all other cases \( \sum_k \tau(k) = l \). However, in all other cases, by Lemma A.1 above, as \( \tau \) is non-decreasing (and thus non-decreasing on all subintervals) we either have

\[
\tau(1)\tau(2) \cdots \tau(l) = 1^{\lambda_1}b_1\lambda_2b_2 \cdots 1^{\lambda_{r-1}}b_1\lambda_r \quad \text{(A.21)}
\]

or

\[
\tau(1)\tau(2) \cdots \tau(l) = 01^{\lambda_1}b_1\lambda_2b_2 \cdots 1^{\lambda_{r-1}}b_1\lambda_r \quad \text{(A.22)}
\]

for some non-negative integers \( \lambda_1, \lambda_2, \ldots, \lambda_r \). It follows trivially in either of these cases that \( \sum_k \tau(k) = l \). □

**Proof of Theorem A.1.** If hypothesis (A.6) is satisfied then

\[
c(\tau) = \gamma(\tau) \times z^{\sum_k \tau(k)} \quad \text{(A.23)}
\]

where

\[
\gamma(\tau) = \prod_{1 \leq k \leq l} \gamma_k^{\tau(k)} \quad \text{(A.24)}
\]

From equation (A.11), equation (A.12), the preceding Corollary A.1, we also get

\[
\det N(z) = \sum_{\tau \in T^+} \det \begin{bmatrix} C_{1: \tau(1)} & C_{2: \tau(2)} & \cdots & C_{l: \tau(l)} \end{bmatrix} \quad \text{(A.25)}
\]

\[
= \sum_{\tau \in T^+} \gamma(\tau) z^{\sum_k \tau(k)} \det \begin{bmatrix} C_{1+\tau(1)} & C_{2+\tau(2)} & \cdots & C_{l+\tau(l)} \end{bmatrix} \quad \text{(A.26)}
\]

\[
= \sum_{\tau \in T^+} (-1)^{e(\tau)} \eta(\tau) z^{\sum_k \tau(k)} \det \begin{bmatrix} C_1 & C_2 & \cdots & C_l \end{bmatrix} \quad \text{(A.27)}
\]

\[
= \det C \times v(z^l) \quad \text{(A.28)}
\]

Where, by Proposition A.2, \( v \) is the polynomial in \( z \) given by

\[
v(\xi) = \sum_{\tau \in T^+} (-1)^{e(\tau)} \eta(\tau) \xi^{\eta(\tau)} \quad \text{(A.29)}
\]

Moreover, Proposition A.2 shows that \( v \) is quadratic in \( \xi \). Observe that the only \( \tau \) where \( \eta \neq 1 \) are \( \tau = 0^l \) and \( \tau = 2^l \). Therefore \( v \) has the form

\[
v(\xi) = \xi^0 \gamma(0^l) + \xi^1 \sum_{\tau \in T^+: \tau \neq 0, 2} (-1)^{e(\tau)} \gamma(\tau) + \xi^2 \gamma(2^l) \quad \text{(A.30)}
\]

□
A.3. Solution to an inhomogeneous equation.

**Theorem A.2.** Fix \( t \in \mathbb{Z}_l \) and let \( C \) and \( N \) be as in the preceding section. Suppose that

\[
c_p^s(z) = \gamma_p^s z^p \quad \text{some } \gamma_p^s \in C, \quad \forall s = 1, 2, \ldots, l, \forall p = 0, 1, 2 \tag{A.31}
\]

and \( \det C \neq 0 \). Then the solution \( E(z) \) to the inhomogeneous equation

\[
N(z)E(z) = Ct \tag{A.32}
\]

exists and has entries \( E_s(z) \) given by

\[
E_s(z) = \frac{u_s(z)}{v(z)} \tag{A.33}
\]

where

1. \( v \) denotes the quadratic polynomial from Theorem A.1
2. if we define

\[
\kappa(s, t) = \begin{cases} 
1 - 1 & s = t \\
0 & s = t - 1 \\
1 & s = t - 2 \\
\#[s + 1, t - 1] & \text{otherwise}
\end{cases} \tag{A.34}
\]

and

\[
\mu(s, t) = \begin{cases} 
1 & s = t \\
0 & s = t - 1 \\
1 - 1 & s = t - 2 \\
\#[t - 1, s - 1] & \text{otherwise}
\end{cases} \tag{A.35}
\]

then \( u_s \) is the polynomial

\[
u_s(z) = \begin{cases} 
z^{(t-1)-\kappa(s, t)} + z^{(t-1)+\mu(s, t)} & s \neq t - 1 \\
z^{t-1}u_{t-1, t} & s = t - 1
\end{cases} \tag{A.36}
\]

where \( u_{s, t}^{\kappa, s} \) and \( u_{s, t}^{0, \mu} \) satisfy the recurrence relations

\[
u_{t-1, t}^{\kappa, s} = \frac{-1}{\gamma_s^2} \left( \gamma_s^{s-1}u_{s-1, t}^{\kappa+1, 0} + \gamma_s^2u_{s-2, t}^{\kappa+2, 0} \right) \\
u_{t-1, t}^{0, \mu} = \prod_{t+1 \leq k \leq t-1} \gamma_k^0 \\
u_{t, t}^{\kappa, s} = -\gamma_t^1 \prod_{t+2 \leq k \leq t-1} \gamma_k^0 \\
u_{t, t}^{0, \mu} = \frac{-1}{\gamma_t^2} \left( \gamma_t^{s+1}u_{s+1, t}^{0, \mu+1} + \gamma_t^0u_{s+2, t}^{0, \mu+2} \right) \\
u_{t-1, t}^{0, \mu} = \prod_{t-1 \leq k \leq t-3} \gamma_k^2 \\
u_{t, t}^{0, \mu} = -\gamma_t^{s-2} \prod_{t-1 \leq k \leq t-4} \gamma_k^2 \tag{A.37}
\]

and

This will basically follow from the computation of \( \det N(z) \) from Section A.2 together with Cramer’s rule. The hypothesis that \( \det C \neq 0 \) will also give us the following.

**Corollary A.2.** Let \( H \) be an arbitrary non-zero vector. Then the equation

\[
N(z)E(z) = H \tag{A.39}
\]
has a unique solution. If \( H = \sum_t h_tC_t \) then the solution is explicitly given by
\[
E_t(z) = \sum_t h_tE_t(z) \tag{A.40}
\]
where \( E_t(z) \) denote the solution from the preceding theorem.

In what follows, the notation and remarks initially mirror those of Section A.2. Throughout this section we fix \( t \in \mathbb{Z}_t \). For \( s \in \mathbb{Z}_t \), define
\[
T_s = \left\{ \tau : \mathbb{Z}_t \setminus \{s\} \to \{0, 1, 2\} \right\} \tag{A.41}
\]
We will also represent \( \tau \in T \) by the string over the alphabet \( \{0, 1, 2\} \) of length \( l - 1 \) given by
\[
\tau(s + 1)\tau(s + 2) \cdots \tau(s - 1) \tag{A.42}
\]
(Note that the initial and terminal index are different from Section A.2.)

**Remark A.2.** As in Section A.2, this notation is introduced for the following reason. From expression (A.3) together with multilinearity of the determinant, we find that
\[
\begin{align*}
\det\left[ \begin{array}{cccc}
N_1 & \cdots & N_{s-1} & C_t \\
\vdots & \ddots & \vdots & \vdots \\
N_s & \cdots & N_{s+t-1} & C_t \\
\end{array} \right] \\
= \sum_{\tau \in T_s} \det\left[ \begin{array}{cccc}
C_{1;\tau(1)} & \cdots & C_{s-1;\tau(s-1)} & C_t \\
\vdots & \ddots & \vdots & \vdots \\
C_{s+1;\tau(s+1)} & \cdots & C_{t;\tau(t)} \\
\end{array} \right] \tag{A.43}
\end{align*}
\]
For each \( \tau \in T_s \), multilinearity of the determinant and equality (A.2) give
\[
\begin{align*}
\det\left[ \begin{array}{cccc}
C_{1;\tau(1)} & \cdots & C_{s-1;\tau(s-1)} & C_t \\
\vdots & \ddots & \vdots & \vdots \\
C_{s+1;\tau(s+1)} & \cdots & C_{t;\tau(t)} \\
\end{array} \right] \\
= c(\tau) \det\left[ \begin{array}{cccc}
C_{1+\tau(1)} & \cdots & C_{s-1+\tau(s-1)} & C_t \\
\vdots & \ddots & \vdots & \vdots \\
C_{s+1+\tau(s+1)} & \cdots & C_{t+\tau(t)} \\
\end{array} \right] \tag{A.44}
\end{align*}
\]
where
\[
c(\tau) = \prod_{1 \leq k \leq t, k \neq s} c_k^{\tau(k)} \tag{A.45}
\]
Thus again, we will (i) ascertain for which \( \tau \) the determinant on the right-hand side of (A.44) is nonzero, and (ii) for these \( \tau \) we will compute \( c(\tau) \).

For \( \tau \in T_s \) define \( v_{\tau} : \mathbb{Z}_t \to \mathbb{Z}_t \) by
\[
v_{\tau}(k) = k + \tau(k) \tag{A.46}
\]
This gives a one-to-one correspondence between \( T_s \) and the set
\[
U_s = \left\{ v : \mathbb{Z}_t \setminus \{s\} \to \mathbb{Z}_t \mid k \leq v(k) \leq k + 2 \text{ mod } t, \forall k \in \mathbb{Z}_t \setminus \{s\} \right\} \tag{A.47}
\]
Given \( v \in U_s \), denote by \( \tau_v \) the element of \( T_s \) corresponding to \( v \).

As in Section A.2 we say that \( \tau \in T_s \) is \textit{decreasing} at \( k \in \mathbb{Z}_t \) if property (A.16) is satisfied. (Observe that, for certain \( k \), only one of the two statements in (A.16) may be defined.) Otherwise we say that \( \tau \) is \textit{non-decreasing}. Let
\[
T_{s,k}^+ = \left\{ \tau \in T_s : \tau \text{ non-decreasing and, whenever they are defined,} \right\} \tag{A.48}
\]
and denote by \( U_{s,k}^+ \) the corresponding subset of \( U_s \). Then it is straightforward to show that
\[
U_{s,k}^+ = \left\{ v \in U_s : v \text{ injective, and } v(k) \neq t, \forall k \in \mathbb{Z}_t \right\} \tag{A.49}
\]
Given \( v \in U_s \) we may extend it to \( Z_l \) via
\[
\bar{v}(k) = \begin{cases} 
  v(k) & k \in Z_l \setminus \{s\} \\
  t & k = s 
\end{cases} \tag{A.50}
\]
Thus, \( v \in U_t^{+} \) implies that \( v \) is a bijection between the sets \( Z_l \setminus \{s\} \) and \( Z_l \setminus \{t\} \), and we consequently get the following.

**Proposition A.3.** For \( v \in U_s \), \( \bar{v} \in S_l \) only if \( v \) is in \( U_t^{+} \).

Observe that the function \( \bar{v} \) denotes the lower index in the right-hand side of equation (A.44). The preceding proposition automatically implies the following.

**Corollary A.3.** Let \( \tau \in T_s \). If \( \tau \not\in T_t^{+} \) then
\[
\det \begin{bmatrix} 
  C_{1;\tau(1)} & \cdots & C_{s-1;\tau(s-1)} & C_{t} & C_{s+1;\tau(s+1)} & \cdots & C_{l;\tau(l)} 
\end{bmatrix} = 0 \tag{A.51}
\]
Otherwise \( \tau \in T_t^{+} \) and
\[
\det \begin{bmatrix} 
  C_{1;\tau(1)} & \cdots & C_{s-1;\tau(s-1)} & C_{t} & C_{s+1;\tau(s+1)} & \cdots & C_{l;\tau(l)} 
\end{bmatrix} 
= (-1)^{\epsilon(\bar{v})} c(\tau) \det C \tag{A.52}
\]
where \( \epsilon(\bar{v}) \) denotes parity of the number of adjacent transpositions of \( \bar{v} \).

This resolves the first problem (i). Now we wish to compute \( c(\tau) \) for \( \tau \in T_t^{+} \). The first step is to observe, by the conditions given in (A.44), that we have the following.

**Proposition A.4.** Let \( \tau \in T_t^{+} \). Then
\begin{enumerate}
  \item For \( s = t \), \( \tau(t-2)\tau(t-1) = 00 \) or \( 1^{12} \) \tag{A.53}
  \item For \( s = t - 1 \), \( \tau(t-2)\tau(t) = 1^{12} \) \tag{A.54}
  \item For \( s = t - 2 \), \( \tau(t-1)\tau(t) = 22 \) or \( 0^{12} \) \tag{A.55}
  \item For \( s \neq t, t - 1 \) or \( t - 2 \), either \( \tau(t-2)\tau(t-1)\tau(t) = 00^{12} \) \tag{A.56}
    or \( \tau(t-2)\tau(t-1)\tau(t) = 1^{12}22 \) \tag{A.57}
\end{enumerate}

\textbf{Proof.} \hfill \Box

For \( \tau \in T_t^{+} \) we require \( \tau \) to be non-decreasing on the interval \([s + 1, s - 1]\) in \( Z_l \). Therefore, applying Lemma [A.1] together with Proposition [A.4] we also get the following.

**Corollary A.4.** Let \( \tau \in T_t^{+} \).
\begin{enumerate}
  \item For \( s = t \), the string corresponding to \( \tau \) has the form
    \[ 0^{l-1} \text{ or } 1^{\lambda_1}b \cdots 1^{\lambda_{r-1}}b1^{\lambda_r}2 \] \tag{A.58}
  \item For \( s = t - 1 \), the string corresponding to \( \tau \) has the form
    \[ 1^{\lambda_1}b \cdots 1^{\lambda_{r-1}}b1^{\lambda_r} \] \tag{A.59}
\end{enumerate}
For \( s = t - 2 \), the string corresponding to \( \tau \) has the form
\[
2^{l-1} \text{ or } 0\lambda_1 b \ldots 1^{\lambda_{r-1}} b 1^{\lambda_r}
\]  
(A.60)

For \( s \neq t, t - 1, t - 2 \), the string corresponding to \( \tau \) either has the form
\[
0^\kappa 1^{\lambda_1} b \ldots 1^{\lambda_{r-1}} b 1^{\lambda_r} 2^\mu
\]  or
\[
1^{\lambda_1} b \ldots 1^{\lambda_{r-1}} b 1^{\lambda_r} 2^\mu
\]  
(A.61)

where
\[
\kappa = \# [s + 1, t - 1] \text{ and } \mu = \# [t - 1, s - 1]
\]  
(A.62)

Proof.  

We say that \( \tau \in T_{s,t}^+ \) is of type \((\kappa, \mu)\) if the string corresponding to \( \tau \) has the form
\[
0^\kappa 1^{\lambda_1} b \ldots 1^{\lambda_{r-1}} b 1^{\lambda_r} 2^\mu
\]  (A.64)

where \( \kappa, \lambda_1, \ldots, \lambda_r, \) and \( \mu, \) are non-negative integers. Observe that, since \( 0^{l-1} \) is the unique type \((l - 1, 0)\) and \( 2^{l-1} \) is the unique type \((0, l - 1)\), Corollary A.4 can be rephrased as follows: For \( \tau \in T_{s,t}^+ \),

1. if \( s = t \), then \( \tau \) is either of type \((l - 1, 0)\) or type \((0, 1)\)
2. if \( s = t - 1 \), then \( \tau \) is of type \((0, 0)\)
3. if \( s = t - 2 \), then \( \tau \) is either of type \((0, l - 1)\) and type \((1, 0)\)
4. if \( s \neq t, t - 1, t - 2 \), then \( \tau \) is either of type \((\kappa, 0)\) or type \((0, \mu)\), where
\[
\kappa = \# [s + 1, t - 1] \text{ and } \mu = \# [t - 1, s - 1].
\]

If we define \( \kappa(s, t) \) and \( \mu(s, t) \) as in (A.34) and (A.35) then this becomes the following.

**Corollary A.5.** \( T_{s,t}^+ \) consists exactly of types \((\kappa, 0)\) and types \((0, \mu)\).

**Proposition A.5.** Suppose that
\[
\gamma_s^p = \gamma_s^p z^p \text{ some } \gamma_s^p \in \mathbb{C} \quad \forall s = 1, 2, \ldots, l, \forall p = 0, 1, 2
\]  (A.65)

If we define
\[
b_{s,t}^{\kappa,0}(z) = \sum_{\tau \in T_{s,t}^+, \tau \text{ type } (\kappa,0)} (-1)^{e(\bar{\nu}_s)} c(\tau)(z)
\]  (A.66)
\[
b_{s,t}^{0,\mu}(z) = \sum_{\tau \in T_{s,t}^+, \tau \text{ type } (0,\mu)} (-1)^{e(\bar{\nu}_s)} c(\tau)(z)
\]  (A.67)

Then
\[
b_{s,t}^{\kappa,0}(z) = z^{(l-1)-\kappa} u_{s,t}^{\kappa,0} \quad b_{s,t}^{0,\mu}(z) = z^{(l-1)+\mu} u_{s,t}^{0,\mu}
\]  (A.68)

where \( u_{s,t}^{\kappa,0} \) and \( u_{s,t}^{0,\mu} \) satisfy respectively the recurrence relations (A.37) and (A.38).

**Remark A.3.** Observe that the function
\[
b_{s,t}^{0,0}(z) = \sum_{\tau \in T_{s,t}^+, \tau \text{ type } (0,0)} (-1)^{e(\bar{\nu}_s)} c(\tau)
\]  (A.69)
is used in both recurrence relations (A.37) and (A.38). Thus, it is implicitly taken in the statement of the above proposition that \( b_{t-1,t} \) has the form \( z^{t-1} u_{t-1,t}^0 \) and \( u_{t-1,t}^0 \) satisfies both recurrences.

**Proof.** First, let us show equation (A.68). Observe by equation (A.64), if \( \tau \) is of type \((\kappa, \mu)\) then
\[
\sum_{1 \leq k \leq t \land k \neq s} \tau(k) = 1 - 1 - \kappa + \mu \tag{A.71}
\]
Thus, \( \tau \) of type \((\kappa, 0)\) implies \( \deg_z c(\tau) = 1 - \kappa - 1 \); and \( \tau \) of type \((0, \mu)\) implies \( \deg_z c(\tau) = 1 - 1 + \mu \). Hence equation (A.68) is satisfied. We now prove the recurrence relations (A.37) and (A.38) by induction. First let us show (A.37). We start by considering the cases \( s = t, t + 1 \). For \( s = t \), there is a unique \( \tau \in T_{s,t}^{-} \). Namely
\[
\tau(t + 1) \tau(t + 2) \cdots \tau(t - 2) \tau(t - 1) = 0^{t-1} \tag{A.72}
\]
Therefore
\[
\gamma(\tau) = \gamma_{t+1} \gamma_{t+2} \cdots \gamma_{t-2} \gamma_{t-1} = \prod_{t+1 \leq k \leq t-1} \gamma_k^0 \tag{A.73}
\]
Also observe that \( \bar{\nu}_\tau = \text{id} \). Therefore \( \epsilon(\bar{\nu}_\tau) = 0 \). Hence, we find that
\[
\sum_{\tau \in T_{t,t}^{-}} (-1)^{\epsilon(\bar{\nu}_\tau)} \gamma(\tau) = (-1)^{\epsilon(\bar{\nu}_\tau)} \gamma(\tau) = \prod_{t+1 \leq k \leq t-1} \gamma_k^0 \tag{A.74}
\]
Next, for \( s = t + 1 \), there is a unique \( \tau \in T_{s,t}^{+} \) of type \((1 - 2, 0)\). Namely
\[
\tau(t + 2) \tau(t + 3) \cdots \tau(t - 1) \tau(t) = 0^{t-2}1 \tag{A.75}
\]
From which we see that
\[
\gamma(\tau) = \gamma_{t+2} \gamma_{t+3} \cdots \gamma_{t-1} \gamma_t = \left( \prod_{t+2 \leq k \leq t-1} \gamma_k^0 \right) \gamma_t^1 \tag{A.76}
\]
Also observe that \( \bar{\nu}_\tau \) is a single transposition given in cycle notation by \((t t + 1)\). Therefore \( \epsilon(\bar{\nu}_\tau) = 1 \). Thus, we find that
\[
\sum_{\tau \in T_{t+1,t}^{+}} (-1)^{\epsilon(\bar{\nu}_\tau)} \gamma(\tau) = (-1)^{\epsilon(\bar{\nu}_\tau)} \gamma(\tau) = - \left( \prod_{t+2 \leq k \leq t-1} \gamma_k^0 \right) \gamma_t^1 \tag{A.77}
\]
and the statement holds for \( s = t + 1 \). Now assume that the statement holds for \( t \leq r < s \). We wish to show the same holds for \( r = s \). Observe that we can decompose \( T_{s,t}^{+}(\kappa(s, t), 0) \) further into those \( \tau \) whose central string terminates with 1, and those whose central string terminates with \( b \), i.e., let
\[
T_{s,t}^{+}(\kappa(s, t), 0)^1 \triangleq \{ 0^s c_1 : \zeta(t) \cdots \zeta(s - 2) = 1^{\lambda_1 b_1 b_2 \cdots 1^{\lambda_r}} \text{ some } \lambda_1, \ldots \} \tag{A.78}
\]
\[
T_{s,t}^{+}(\kappa(s, t), 0)^0 \triangleq \{ 0^s c_b : \zeta(t) \cdots \zeta(s - 3) = 1^{\lambda_1 b_1 b_2 \cdots 1^{\lambda_r}} \text{ some } \lambda_1, \ldots \} \tag{A.79}
\]
Also observe that

- \( T_{s,t}^{+}(\kappa(s, t), 0)^1 \) is in one-to-one correspondence with \( T_{t-1,t}^{+}(\kappa(s, t) + 1, 0) \),
Moreover, if \( \tau \in T_{s,t}^+(\kappa(s,t),0)^0 \) corresponds to \( \tau' \in T_{s,1,t}^+(\kappa(s,t) + 1,0) \), then
\[
\gamma(\tau) = \tau_{s+1}^{0} \cdots \tau_{t-1}^{0} \tau_{t}^{0} \cdots \tau_{s-1}^{0} \\
= \left( \prod_{s+1 \leq k \leq t-1} \gamma_k^0 \right) \left( \prod_{t \leq k \leq s-2} \gamma_k^{(k)} \right) \gamma_{s-1}^1 \\
= \frac{\gamma_{s-1}^1}{\gamma_{s}^0} \left( \prod_{s \leq k \leq t-1} \gamma_k^0 \right) \left( \prod_{t \leq k \leq s-2} \gamma_k^{(k)} \right) \\
= \frac{\gamma_{s-1}^1}{\gamma_{s}^0} \gamma(\tau')
\]
Also, the corresponding permutations \( \bar{\nu}_\tau \) and \( \bar{\nu}_{\tau'} \) differ by a single transposition interchanging \( t \) and \( s \). (In cycle notation: \( \bar{\nu}_\tau = (t \, s) \bar{\nu}_{\tau'} \).) Since \( (t \, s) \) can be expressed as the product of \( 2 \# [t, s] - 3 \) adjacent transpositions \( (t \, t+1) (t+1 \, t+2) \cdots (s-2 \, s-1) (s-1 \, s) (s-2 \, s-1) \cdots (t \, t+1) \) it follows that \( \epsilon(\bar{\nu}_\tau) \) and \( \epsilon(\bar{\nu}_{\tau'}) \) have different parities.

Similarly, if \( \tau \in T_{s,t}^+(\kappa(s,t),0)^0 \) corresponds to \( \tau' \in T_{s-2,t}^+(\kappa(s,t) + 2,0) \), then
\[
\gamma(\tau) = \tau_{s+1}^{0} \cdots \tau_{t-1}^{0} \tau_{t}^{0} \cdots \tau_{s-1}^{0} \\
= \left( \prod_{s+1 \leq k \leq t-1} \gamma_k^0 \right) \left( \prod_{t \leq k \leq s-2} \gamma_k^{(k)} \right) \gamma_{s-2}^1 \cdot \gamma_{s-1}^0 \\
= \frac{\gamma_{s-2}^1 \gamma_{s-1}^0}{\gamma_{s}^0} \left( \prod_{s-1 \leq k \leq t-1} \gamma_k^0 \right) \left( \prod_{t \leq k \leq s-3} \gamma_k^{(k)} \right) \\
= \frac{\gamma_{s-2}^1 \gamma_{s-1}^0}{\gamma_{s}^0} \gamma(\tau')
\]
and the corresponding permutations \( \bar{\nu}_\tau \) and \( \bar{\nu}_{\tau'} \) differ by a single transposition interchanging \( t \) and \( s \). (In cycle notation: \( \bar{\nu}_\tau = (t \, s) \bar{\nu}_{\tau'} \).) Thus \( \epsilon(\bar{\nu}_\tau) \) and \( \epsilon(\bar{\nu}_{\tau'}) \) again have different parities.

Consequently, setting \( \kappa = \kappa(s,t) \) and observing that \( s = t + 1 - \kappa - 1 \),
\[
u_{s,t}^{(0)} = \sum_{\tau \in T_{s,t}^+(\kappa,0)} (-1)^{\epsilon(\bar{\nu}_\tau)} \gamma(\tau) \\
= \sum_{\tau \in T_{s,t}^+(\kappa,0)^1} (-1)^{\epsilon(\bar{\nu}_\tau)} \gamma(\tau) + \sum_{\tau \in T_{s,t}^+(\kappa,0)^0} (-1)^{\epsilon(\bar{\nu}_\tau)} \gamma(\tau) \\
= \sum_{\tau' \in T_{s-1,t}^+(\kappa+1,0)} (-1)^{\epsilon(\bar{\nu}_{\tau'})+1} \frac{\gamma_{s-1}^1}{\gamma_{s}^0} \gamma(\tau') \\
+ \sum_{\tau' \in T_{s-2,t}^+(\kappa+2,0)} (-1)^{\epsilon(\bar{\nu}_{\tau'})+1} \frac{\gamma_{s-2}^2}{\gamma_{s}^0} \gamma(\tau')
\]
\[
= \frac{1}{\gamma_{s}^{0}} \left( \gamma_{s-1}^1 u_{s-1,t}^{(\kappa+1,0)} + \gamma_{s-2}^2 u_{s-2,t}^{(\kappa+2,0)} \right)
\]
Now we consider the second recurrence relation (A.38). For $s = t - 2$ there exists a unique $\tau \in \mathcal{T}_{s,t}$. Namely
\[
\tau(t-1)\tau(t) \cdots \tau(t-4)\tau(t-3) = 2^{t-1}
\] (A.92)
Therefore
\[
\gamma(\tau) = \gamma^{\tau(t-1)}\gamma^{\tau(t)} \cdots \gamma^{\tau(t-4)}\gamma^{\tau(t-3)} = \prod_{t-1 \leq k \leq t-3} \gamma_k^2
\] (A.93)
Now observe that, since $\bar{\nu}$ can be written as $\nu^2$ where $\nu(k) = k+1 \mod t$, it follows that $\epsilon(\bar{\nu}) = 2\epsilon(\nu) = 2l - 2$. Hence
\[
\sum_{\tau \in \mathcal{T}_{s,t}^{1}, \tau \text{ type } (0,l-2)} (-1)^{\epsilon(\bar{\nu})} \gamma(\tau) = (-1)^{\epsilon(\bar{\nu})} \gamma(\tau) = \prod_{t-1 \leq k \leq t-3} \gamma_k^2
\] (A.94)
Now consider the case $s = t - 3$. Then there is a unique $\tau \in \mathcal{T}_{s,t}$ of type $(0,t-2)$. Namely
\[
\tau(t-2)\tau(t-1) \cdots \tau(t-5)\tau(t-4) = 12^{t-2}
\] (A.95)
From which, we find that
\[
\gamma(\tau) = \gamma^{\tau(t-2)}\gamma^{\tau(t-1)} \cdots \gamma^{\tau(t-5)}\gamma^{\tau(t-4)} = \gamma_{t-2}^1 \left( \prod_{t-1 \leq k \leq t-4} \gamma_k^2 \right)
\] (A.96)
Next, observe that $\bar{\nu}_\tau$ can be written, in cycle notation, as $(t \ t-1)\nu^2$ and thus $\epsilon(\bar{\nu}_\tau) = 2l - 1$. It follows that
\[
\sum_{\tau \in \mathcal{T}_{s,t}^{1}, \tau \text{ type } (0,t-2)} (-1)^{\epsilon(\bar{\nu})} \gamma(\tau) = (-1)^{\epsilon(\bar{\nu})} \gamma(\tau) = -\gamma_{t-2}^1 \left( \prod_{t-1 \leq k \leq t-4} \gamma_k^2 \right)
\] (A.97)
Now assume that the statement holds for $s + 1 \leq r < t$. We decompose $\mathcal{T}_{s,t}^+(0,\mu)$ into
\[
\mathcal{T}_{s,t}^+(0,\mu) = \{1 \cdot 2^\mu : \xi(s+2) \cdots \xi(t-2) = 1^{\lambda_1} b_1 \lambda_2 \cdots 1^{\lambda_r} \text{ some } \lambda_1, \ldots, \lambda_r\}
\] (A.98)
\[
\mathcal{T}_{s,t}^+(0,\mu) = \{b \cdot 2^\mu : \xi(s+3) \cdots \xi(t-2) = 1^{\lambda_1} b_1 \lambda_2 \cdots 1^{\lambda_r} \text{ some } \lambda_1, \ldots, \lambda_r\}
\] (A.99)
As before, we observe that
- $\mathcal{T}_{s,t}^+(0,\mu(s,t))^1$ is in one-to-one correspondence with $\mathcal{T}_{s+1,t}^+(0,\mu(s,t) + 1)$
- $\mathcal{T}_{s,t}^+(0,\mu(s,t))^0$ is in one-to-one correspondence with $\mathcal{T}_{s+2,t}^+(0,\mu(s,t) + 2)$
Moreover, if $\tau \in \mathcal{T}_{s,t}^+(0,\mu(s,t))^1$ corresponds to $\tau' \in \mathcal{T}_{s+1,t}^+(0,\mu(s,t) + 1)$ then
\[
\gamma(\tau) = \gamma^{\tau(s+1)} \cdots \gamma^{\tau(t-1)} \gamma^{\tau(t)} \cdots \gamma^{\tau(s-1)}
\] (A.100)
\[
= \gamma_{s+1}^{\tau(s+1)} \left( \prod_{s+2 \leq k \leq t-2} \gamma_k^{\xi(k)} \right) \left( \prod_{t-1 \leq k \leq s-1} \gamma_k^2 \right)
\] (A.101)
\[
= \frac{\gamma_{s+1}^{\tau(s+1)}}{\gamma_s^{\tau'}} \left( \prod_{s+2 \leq k \leq t-2} \gamma_k^{\xi(k)} \right) \left( \prod_{t-1 \leq k \leq s} \gamma_k^2 \right)
\] (A.102)
\[
= \frac{\gamma_{s+1}^{\tau(s+1)}}{\gamma_s^{\tau'}} \gamma(\tau')
\] (A.103)
and the corresponding permutations \( \bar{\upsilon}_\tau \) and \( \bar{\upsilon}_{\tau'} \) differ by a single transposition interchanging \( s \) and \( s + 1 \). (In cycle notation: \( \bar{\upsilon}_\tau = \bar{\upsilon}_{\tau'}( s, s + 1 ) \).) Thus \( \epsilon(\bar{\upsilon}_\tau) = \epsilon(\bar{\upsilon}_{\tau'}) = 1 + 1 = 2 \).

Similarly, if \( \tau \in \mathcal{T}^+_s(0, \mu(s, t))^0 \) corresponds to \( \tau' \in \mathcal{T}^+_s(0, \mu(s, t) + 2) \) then

\[
\gamma(\tau) = \gamma_{s+1}^{\tau(s+1)} \cdots \gamma_{t-1}^{\tau(t-1)} \gamma_t^{\tau(t)} \cdots \gamma_{s-1}^{\tau(s-1)}
\]

\[
= \gamma_{s+1}^2 \gamma_{s+2}^0 \left( \prod_{s+3 \leq k \leq t-2} \gamma_k^{\tau(k)} \right) \left( \prod_{t-1 \leq k \leq s-1} \gamma_k^{2} \right)
\]

\[
= \gamma_{s+1}^2 \gamma_{s+2}^0 \left( \prod_{s+3 \leq k \leq t-2} \gamma_k^{\tau(k)} \right) \left( \prod_{t-1 \leq k \leq s+1} \gamma_k^{2} \right)
\]

\[
= \gamma_{s+2}^0 \gamma_{s+2}^0 \gamma(\tau')
\]

and the corresponding permutations \( \bar{\upsilon}_\tau \) and \( \bar{\upsilon}_{\tau'} \) differ by a single transposition interchanging \( s \) and \( s + 2 \). (In cycle notation: \( \bar{\upsilon}_\tau = \bar{\upsilon}_{\tau'}( s, s + 2 ) \).) This transposition can be expressed as the product of three adjacent transpositions. Thus \( \epsilon(\bar{\upsilon}_\tau) = \epsilon(\bar{\upsilon}_{\tau'}) = 1 + 1 + 1 = 3 \). Consequently, setting \( \mu = \mu(s, t) \) and observing that \( s = t - 1 + \mu - 1 \),

\[
u_{s,t}^{0, \mu} = \sum_{\tau \in \mathcal{T}^+_s(0, \mu)} (-1)^{\epsilon(\bar{\upsilon}_\tau)} \gamma(\tau)
\]

\[
= \sum_{\tau \in \mathcal{T}^+_s(0, \mu)^0} (-1)^{\epsilon(\bar{\upsilon}_\tau)} \gamma(\tau) + \sum_{\tau \in \mathcal{T}^+_s(0, \mu)^0} (-1)^{\epsilon(\bar{\upsilon}_\tau)} \gamma(\tau)
\]

\[
= \sum_{\tau' \in \mathcal{T}^+_s(0, \mu+1)} (-1)^{\epsilon(\bar{\upsilon}_\tau') + 1} \gamma_{s+1}^1 \gamma_{s+2}^0 \gamma(\tau')
\]

\[
+ \sum_{\tau' \in \mathcal{T}^+_s(0, \mu+2)} (-1)^{\epsilon(\bar{\upsilon}_\tau') + 1} \gamma_{s+1}^0 \gamma_{s+2}^0 \gamma_{s+2}^0 \gamma(\tau')
\]

\[
= \frac{-1}{\gamma_{s+1}^1} \left( \gamma_{s+1}^0 \nu_{s+1, t}^{0, \mu+1} + \gamma_{s+2}^0 \nu_{s+2, t}^{0, \mu+2} \right)
\]

Using the preceding analysis we can relate the coefficients \( u_{s,t}^{0, \mu} \) and \( u_{s,t}^{0, \mu} \) to the coefficients of the quadratic polynomial \( v \) from Theorem A.7.

**Proposition A.6.** Let \( v \) denote the quadratic polynomial from Theorem A.7. Then, for any \( s \in \mathbb{Z}_4 \), the degree 1 coefficient \( v_1 \) of \( v \) satisfies the following relation

\[
v_1 = \gamma_{s+1}^0 u_{s,s+1}^{0, \mu} + \gamma_{s+1}^0 u_{s,s+1}^{0, \mu} + \gamma_{s+2}^0 u_{s,s+2}^{0, \mu+2}
\]

**Proof.** First let us comment on notation. Fix \( s \in \mathbb{Z}_4 \). Below we will denote elements of \( \mathcal{T}^+_s \) by \( \bar{\tau} \), for \( t = s, s+1, s+2 \), and elements of \( \mathcal{T}^+_s \) will be denoted by \( \tau \). Observe that,

\[
\{ \bar{\tau} \in \mathcal{T}^+_s: \tau \neq 0, 2 \} = \bigcup_{p=0,1,2} \{ \bar{\tau} \in \mathcal{T}^+_s: \tau \neq 0, 2, \tau(s) = p \}
\]

**Claim 3.** We have the following:
\{\bar{\tau} \in \mathcal{T}^+ : \bar{\tau} \neq 0, 2, \bar{\tau}(s) = 0\} \text{ is in one-to-one correspondence with } \mathcal{T}_{s,s}(0,1),
\{\bar{\tau} \in \mathcal{T}^+ : \tau \neq 0, 2, \bar{\tau}(s) = 1\} \text{ is in one-to-one correspondence with } \mathcal{T}_{s,s+1}^+(0,0),
\{\bar{\tau} \in \mathcal{T}^+ : \tau \neq 0, 2, \bar{\tau}(s) = 2\} \text{ is in one-to-one correspondence with } \mathcal{T}_{s,s+2}^+(1,0).

Proof of Claim: We prove the middle correspondence. The left-hand side can be identified with the set of strings satisfying the properties \(\tau(s - 1) \neq 2\) and \(\tau(s + 1) \neq 0\), i.e., those strings of the form
\[
\tau(s + 1) \cdots \tau(s - 1) = 1^{\lambda_1}b_1^{\lambda_2}b_1^{\cdots} \lambda_{r-1} b_1^{\lambda_r}
\]  
(A.114)
But strings of this form correspond to elements of \(\mathcal{T}_{s,s+1}^+(0,0)\). The other statements follow similarly. //

Under each of these correspondences we also trivially have that \(v_\bar{\tau} = \bar{v}_\tau\). In particular \(e(v_\bar{\tau}) = e(\bar{v}_\tau)\). We also have, for \(p = 0, 1, 2\), that if \(\bar{\tau} \in \mathcal{T}^+\) satisfies \(\bar{\tau}(s) = p\) then
\[
\gamma(\bar{\tau}) = \gamma_s^p \gamma(\tau)
\]  
(A.115)
Therefore, from the expression from Theorem [A.3]
\[
v_1 = \sum_{\bar{\tau} \in \mathcal{T}^+ : \bar{\tau} \neq 0, 2} (-1)^{\bar{v}(\bar{\tau})} \gamma(\bar{\tau})
\]  
(A.116)
\[
= \sum_{\bar{\tau} \in \mathcal{T}^+_{s,s+1}(0,1)} (-1)^{\bar{v}(\bar{\tau})} \gamma_s^0 \gamma(\tau) + \sum_{\bar{\tau} \in \mathcal{T}^+_{s,s+1}(0,0)} (-1)^{\bar{v}(\bar{\tau})} \gamma_s^1 \gamma(\tau)
\]  
(A.117)
\[
+ \sum_{\bar{\tau} \in \mathcal{T}^+_{s,s+2}(1,0)} (-1)^{\bar{v}(\bar{\tau})} \gamma_s^2 \gamma(\tau)
\]  
(A.118)
\[
= \gamma_s^0 u_{s,s} + \gamma_s^1 u_{s,s+1} + \gamma_s^2 u_{s,s+2}
\]  
\(
\)
We can also now prove the main result in this section.

Proof of Theorem [A.2]
Let
\[
u_s(z) = \sum_{\tau \in \mathcal{T}^+_s} (-1)^{\nu(\tau)} e(\tau)
\]  
(A.119)
Then by equations [A.43] and [A.44], we find that
\[
\det \begin{bmatrix} N_1 & \cdots & N_{s-1} & C_1 & N_{s+1} & \cdots & N_l \end{bmatrix}
\]  
(A.120)
\[
= \sum_{\tau \in \mathcal{T}_s} \det \begin{bmatrix} C_{1;\tau(1)} & \cdots & C_{s-1;\tau(s-1)} & C_2 & C_{s+1;\tau(s+1)} & \cdots & C_{l;\tau(l)} \end{bmatrix}
\]  
(A.121)
\[
= \sum_{\tau \in \mathcal{T}^+_s} (-1)^{\nu(\tau)} e(\tau) \det \begin{bmatrix} C_1 & C_2 & \cdots & C_l \end{bmatrix}
\]  
(A.122)
Theorem [A.1] implies that
\[
\det N(z) = \det C \times v(z^l)
\]  
(A.123)
Therefore, applying Cramer’s rule to equation [A.32], together with equations [A.122] and [A.123], a solution \(E(z)\) exists provided that \(\det C \neq 0\), and in this case the
sth entry is given by
\[ E_s(z) = \frac{u_s(z)}{v(z^l)} \tag{A.124} \]

By proposition \[A.3\]
\[ u_s(z) = \begin{cases} z^{(t-1)} - u_{s,t}^0,0 + z^{(t-1)} + \mu u_{s,t}^{0,\mu} & s \neq t - 1 \\ z^{t-1} u_{t-1,t}^0,0 & s = t - 1 \end{cases} \tag{A.125} \]

where \( u_{s,t}^{0,0} \) and \( u_{s,t}^{0,\mu} \) satisfy properties \[A.37\] and \[A.38\]. □

References

[1] G.H. Hardy and E.M. Wright. An introduction to the theory of number (6th ed.). Oxford University press, 2008.
[2] H. Jager and P. Liardet. Distributions arithmétiques des dénominateurs de convergents de fractions continues. Indag. Math., 50, (1988), 181–197.
[3] A. Ya. Khinchin. Continued Fractions, (transl. H. Eagle). Dover Publications, 1997.
[4] D. H. Lehmer. Note on an absolute constant of Khintchine. Amer. Math. Monthly, vol. 46, no. 3, (1939), 148–152.
[5] P. Lévy. Théorie de l’addition des variables aléatoire. Gauthier-Villars Paris, 1937.

Anna Belova, Mathematics Department, Uppsala University, Uppsala, Sweden
E-mail address: anna.belova@math.uu.se

Peter Hazard, Instituto de Matemática e Estatística, USP, São Paulo, SP, Brazil
E-mail address: pete@ime.usp.br