Kazhdan-Lusztig parameters and extended quotients

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Abstract

The Kazhdan-Lusztig parameters are important parameters in the representation theory of $p$-adic groups and affine Hecke algebras. We show that the Kazhdan-Lusztig parameters have a definite geometric structure, namely that of the extended quotient $T//W$ of a complex torus $T$ by a finite Weyl group $W$. More generally, we show that the corresponding parameters, in the principal series of a reductive $p$-adic group with connected centre, admit such a geometric structure. This confirms, in a special case, a recent geometric conjecture in [1].

In the course of this study, we provide a unified framework for Kazhdan-Lusztig parameters on the one hand, and Springer parameters on the other hand. Our framework contains a complex parameter $s$, and allows us to interpolate between $s = 1$ and $s = \sqrt{q}$. When $s = 1$, we recover the parameters which occur in the Springer correspondence; when $s = \sqrt{q}$, we recover the Kazhdan-Lusztig parameters.

1 Introduction

The Kazhdan-Lusztig parameters are important parameters in the representation theory of $p$-adic groups and affine Hecke algebras. We show that the Kazhdan-Lusztig parameters have a definite geometric structure, namely that of the extended quotient $T//W$ of a complex torus $T$ by a finite Weyl group $W$. More generally, we show that the corresponding parameters, in the principal series of a reductive $p$-adic group with connected centre, admit such a geometric structure. This confirms, in a special case, a recent geometric conjecture in [1].

In the course of this study, we provide a unified framework for Kazhdan-Lusztig parameters on the one hand, and Springer parameters on the other
hand. Our framework contains a complex parameter $s$, and allows us to interpolate between $s = 1$ and $s = \sqrt{q}$. When $s = 1$, we recover the parameters which occur in the Springer correspondence; when $s = \sqrt{q}$, we recover the Kazhdan-Lusztig parameters, see §5. Here, $q = q_F$ is the cardinality of the residue field of the underlying local field $F$.

Let $G$ denote a reductive split $p$-adic group with connected centre, maximal split torus $T$. Let $G, T$ denote the Langlands dual of $G, T$. Then the quotient variety $T/W$ plays a central role. For example, we have the Satake isomorphism

$$\mathcal{H}(G, \mathcal{K}) \cong \mathcal{O}(T/W)$$

where $\mathcal{O}(T/W)$ denotes the coordinate algebra of $T/W$, see [18, 2.2.1], and $\mathcal{H}(G, \mathcal{K})$ denotes the algebra (under convolution) of $\mathcal{K}$-bi-invariant functions of compact support on $G$, where $\mathcal{K} = G(0_F)$. In this article, we will show that the extended quotient plays a central role in the context of the Kazhdan-Lusztig parameters.

We will prove that the extended quotient $T//W$ is a model for the Kazhdan-Lusztig parameters, see §4. More generally, let

$$s = [T, \chi]_G$$

be a point in the Bernstein spectrum of $G$. We prove that the extended quotient $T//W^s$ attached to $s$ is a model of the corresponding parameters attached to $s$. This is our main result, Theorem 4.1. The principal series of a reductive $p$-adic group with connected centre has a definite geometric structure. The principal series is a disjoint union: each component is the extended quotient of the dual torus $T$ by the finite Weyl group $W^s$ attached to $s$. This confirms, in a special case, a recent geometric conjecture in [1].

We also show in §4 that our bijection is compatible with base change, in the special case of the irreducible smooth representations of $GL(n)$ which admit nonzero Iwahori fixed vectors.

The details of our interpolation between Springer parameters and Kazhdan-Lusztig parameters will be given in §5. Our formulation creates a projection

$$\pi_{\sqrt{q}} : T//W \to T/W$$

which provides a model of the infinitesimal character.

We conclude in §6 with some carefully chosen examples.

Since the crossed product algebra $\mathcal{O}(T) \rtimes W$ is isomorphic to

$$\mathbb{C}[X(T)] \rtimes W \cong \mathbb{C}[X(T) \rtimes W],$$

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we obtain a bijection

$$\text{Prim}\mathbb{C}[X(T) \rtimes W] \to T//W$$

where Prim denotes primitive ideals. By composing this bijection with the bijection $\mu$ in Theorem 4.1, we finally get a bijection

$$\text{Prim}\mathbb{C}[X(T) \rtimes W] \to \mathfrak{P}(G)$$

where $\mathfrak{P}(G)$ denotes the Kazhdan-Lusztig parameters. Let $I$ be a standard Iwahori subgroup in $G$ and let $\mathcal{H}(G, I)$ denote the corresponding Iwahori-Hecke algebra, i.e., the algebra (for the convolution product) of compactly supported $I$-biinvariant functions on $G$. The algebra is isomorphic to

$$\mathcal{H}(X(T) \rtimes W, q)$$

the Hecke algebra of the extended affine Weyl group $X(T) \rtimes W$, with parameter $q$. The simple modules of $\mathcal{H}(G, I)$ are parametrized by $\mathfrak{P}(G)$ [7].

Hence $\mathfrak{P}(G)$ provides a parametrization of the simple modules of both the Iwahori-Hecke algebra $\mathcal{H}(X(T) \rtimes W, q)$ and of the group algebra of $X(T) \rtimes W$ (that is, the algebra $\mathcal{H}(X(T) \rtimes W, 1)$).

Note that the existence of a bijection between these sets of simple modules was already proved by Lusztig (see for instance [9, p. 81, assertion (a)]). Lusztig’s construction needs to pass through the asymptotic Hecke algebra $J$, while we have replaced the use of $J$ by the use of the extended quotient $T//W$ (which is much simpler to construct).

2 Extended quotients

Let $\mathcal{O}(T)$ denote the coordinate algebra of the complex torus $T$. In noncommutative geometry, one of the elementary, yet fundamental, concepts is that of noncommutative quotient [8, Example 2.5.3]. The noncommutative quotient of $T$ by $W$ is the crossed product algebra

$$\mathcal{O}(T) \rtimes W.$$ 

This is a noncommutative unital $\mathbb{C}$-algebra. We need to filter this idea through periodic cyclic homology. We have an isomorphism

$$\text{HP}_*(\mathcal{O}(T) \rtimes W) \simeq H^*(T//W; \mathbb{C})$$

where $\text{HP}_*$ denotes periodic cyclic homology, $H^*$ denotes cohomology, and $T//W$ is the extended quotient of $T$ by $W$, see [3]. We recall the definition of the extended quotient $T//W$.

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Definition 2.1. Let
\[ \tilde{T} = \{(t, w) \in T \times W : w \cdot t = t\}. \]
The extended quotient is the quotient
\[ T//W := \tilde{T}/W \]
where \( W \) acts via \( \alpha(t, w) = (\alpha \cdot t, \alpha w \alpha^{-1}) \) with \( \alpha \in W \).

Let \( W(t) \) denote the isotropy subgroup of \( t \). Let \( \text{conj}(W(t)) \) denote the set of conjugacy classes in \( W(t) \), and let \([w]\) denote the conjugacy class of \( w \) in \( W(t) \). The map
\[
\{(t, w) : t \in T, w \in W(t)\} \to \{(t, c) : t \in T, c \in \text{conj}(W(t))\}
\]
\[(t, w) \mapsto (t, [w])\]
induces a canonical bijection
\[
\{(t, w) : t \in T, w \in W(t)\}/W \to \{(t, c) : t \in T, c \in \text{conj}(W(t))\}/W
\]
where \( W \) acts via \( \alpha(t, c) = (\alpha \cdot t, [\alpha x \alpha^{-1}]) \) with \( x \in c \).

Let \( \text{Irr}(W(t)) \) denote the set of equivalence classes of irreducible representations of \( W(t) \). A choice of bijection between \( \text{conj}(W(t)) \) and \( \text{Irr}(W(t)) \) then creates a bijection
\[ T//W \simeq \{(t, \tau) : t \in T, \tau \in \text{Irr}(W(t))\}/W \]
where \( W \) acts via \( \alpha(t, \tau) = (\alpha \cdot t, \alpha_*(\tau)) \). Here, \( \alpha_*(\tau) \) is the push-forward of \( \tau \) to an irreducible representation of \( W(\alpha \cdot t) \).

This leads us to

Definition 2.2. The extended quotient of the second kind is
\[ (T//W)_2 := \{(t, \tau) : t \in T, \tau \in \text{Irr}(W(t))\}/W \]

We then have a non-canonical bijection
\[ T//W \simeq (T//W)_2. \]

Let \( T^w \) denote the fixed set \( \{t \in T : w \cdot t = t\} \), and let \( Z(w) \) denote the centralizer of \( w \) in \( W \). We have
\[ T//W = \bigsqcup T^w/Z(w) \]
where one $w$ is chosen in each conjugacy class in $W$. Therefore $T//W$ is a complex affine algebraic variety. The number of irreducible components in $T//W$ is bounded below by $|\text{conj}(W)|$.

The Jacobson topology on the primitive ideal spectrum of $\mathcal{O}(T) \rtimes W$ induces a topology on $(T//W)_2$ such that the identity map

$$T//W \to (T//W)_2$$

is continuous. From the point of view of noncommutative geometry \cite{8}, the extended quotient of the second kind is a noncommutative complex affine algebraic variety.

The transformation groupoid $T \rtimes W$ is naturally an étale groupoid, see \cite{8} p. 45. Its groupoid algebra $\mathbb{C}[T \rtimes W]$ is the crossed product algebra

$$\mathcal{O}(T) \rtimes W.$$ 

In the groupoid $T \rtimes W$, we have

$$\text{source}(t, w) = t, \quad \text{target}(t, w) = w \cdot t$$

so that the set

$$\{(t, w) \in T \times W : w \cdot t = t\}$$

comprises all the arrows which are loops.

The decomposition of the groupoid $T \rtimes W$ into transitive groupoids leads naturally to Eqn. (1). The groupoid $T \rtimes W$ seems to be a bridge between $T//W$ and $(T//W)_2$.

In the context of algebraic geometry, the extended quotient is known as the inertia stack \cite{13}, in which case the notation is

$$I(T) := \tilde{T}, \quad [I(T)/W] := T//W.$$ 

3 The parameters for the principal series

Let $\mathcal{W}_F$ denote the Weil group of $F$, let $I_F$ be the inertia subgroup of $\mathcal{W}_F$. Let $\text{Frob} \subset \mathcal{W}_F$ denote a geometric Frobenius (a generator of $\mathcal{W}_F/I_F \simeq \mathbb{Z}$). We have $\mathcal{W}_F/I_F = \langle \text{Frob} \rangle$. We will think of this as a multiplicative group, with identity element 1.

Let $\mathfrak{P}(G)$ denote the set of conjugacy classes in $G$ of pairs $(\Phi, \rho)$ such that $\Phi$ is a morphism

$$\Phi: \mathcal{W}_F/I_F \rtimes \text{SL}(2, \mathbb{C}) \to G$$
which is admissible, i.e., $\Phi(1, -)$ is a morphism of complex algebraic groups, $\Phi(\text{Frob}, 1)$ is a semisimple element in $G$, and $\rho$ is defined in the following way.

We will adopt the formulation of Reeder [16]. Choose a Borel subgroup $B_2$ in $\text{SL}(2, \mathbb{C})$ and let $S_\Phi = \Phi(W_F \times B_2)$, a solvable subgroup of $G$. Let $B^\Phi$ denote the variety of Borel subgroups of $G$ containing $S_\Phi$. Let $G_\Phi$ be the centralizer in $G$ of the image of $\Phi$. Then $G_\Phi$ acts naturally on $B^\Phi$, and hence on the singular homology $H_*(B^\Phi, \mathbb{C})$. Then $\rho$ is an irreducible representation of $G_\Phi$ which appears in the action of $G_\Phi$ on $H_*(B^\Phi, \mathbb{C})$.

A Reeder parameter $(\Phi, \rho)$ determines a Kazhdan-Lusztig parameter $(\sigma, u, \rho)$ in the following way. Let

$$u_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T_x = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$$

and set

$$u = \Phi(1, u_0), \quad \sigma = \Phi(\text{Frob}, T_{\sqrt{q}})$$

where $q$ is the cardinality of the residue field $k_F$. Then the triple $(\sigma, u, \rho)$ is a Kazhdan-Lusztig parameter. Since $\Phi$ is a homomorphism and

$$T_{\sqrt{q}} u_0 T_{\sqrt{q}}^{-1} = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} = u_0^q$$

it follows that

$$\sigma u \sigma^{-1} = u^q.$$

It is worth noting that the set $\mathfrak{P}(G)$ is $q$-independent.

We now move on to the rest of the principal series. We recall that $G$ denotes a reductive split $p$-adic group with connected centre, maximal split torus $T$, and $G, T$ denote the Langlands dual of $G, T$. We assume in addition that the residual characteristic of $F$ is not a torsion prime for $G$.

Let $\mathfrak{Q}(G)$ denote the set of conjugacy classes in $G$ of pairs $(\Phi, \rho)$ such that $\Phi$ is a continuous morphism

$$\Phi: W_F \times \text{SL}(2, \mathbb{C}) \rightarrow G$$

which is rational on $\text{SL}(2, \mathbb{C})$ and such that $\Phi(W_F)$ consists of semisimple element in $G$, and $\rho$ is defined in the following way.

Choose a Borel subgroup $B_2$ in $\text{SL}(2, \mathbb{C})$ and let $S_\Phi = \Phi(W_F \times B_2)$. Let $B^\Phi$ denote the variety of Borel subgroups of $G$ containing $S_\Phi$. The variety $B^\Phi$ is non-empty if and only if $\Phi$ factors through the topological abelianization
$\mathcal{W}^{\text{ab}} := \mathcal{W}_F / \left[ \mathcal{W}_F, \mathcal{W}_F \right]$ of $\mathcal{W}_F$ (see [16, § 4.2]). We will assume that $B^\Phi$ is non-empty, and we will still denote by $\Phi$ the homomorphism

$$\Phi : \mathcal{W}^{\text{ab}}_F \times \text{SL}(2, \mathbb{C}) \to G.$$  

Let $I^{\text{ab}}_F$ denote the image of $I_F$ in $\mathcal{W}^{\text{ab}}_F$. The choice of Frobenius $\text{Frob}$ determines a splitting

$$\mathcal{W}^{\text{ab}}_F = I^{\text{ab}}_F \times \langle \text{Frob} \rangle.$$  

(2)

Let $G_\Phi$ be the centralizer in $G$ of the image of $\Phi$. Then $G_\Phi$ acts naturally on $B^\Phi$, and hence on the singular homology of $H_\ast(B^\Phi, \mathbb{C})$. Then $\rho$ is an irreducible representation of $G_\Phi$ which appears in the action of $G_\Phi$ on $H_\ast(B^\Phi, \mathbb{C})$.

Let $\chi$ be a smooth quasicharacter of $T$ and let $s = [T, \chi]_G$ be the point in the Bernstein spectrum $\mathfrak{B}(G)$ determined by $\chi$. Let

$$W^s = \{ w \in W : w \cdot s = s \}.$$  

(3)

Let $X$ denote the rational co-character group of $T$, identified with the rational character group of $T$. Let $T_0$ be the maximal compact subgroup of $T$. By choosing a uniformizer in $F$, we obtain a splitting

$$T = T_0 \times X,$$

according to which

$$\chi = \lambda \otimes t,$$

where $\lambda$ is a character of $T_0$, and $t \in T$. Let $r_F : \mathcal{W}^{\text{ab}}_F \to F^\times$ denote the reciprocity isomorphism of abelian class field theory, and let

$$\hat{\lambda} : I^{\text{ab}}_F \to T$$  

(4)

be the unique homomorphism satisfying

$$\eta \circ \hat{\lambda} = \lambda \circ \eta \circ r_F, \quad \text{for all } \eta \in X,$$

(5)

where $\eta$ is viewed as a character of $T$ on the left side and as a co-character of $T$ on the right side of (5).

Let $H$ denote the centralizer in $G$ of the image of $\hat{\lambda}$:

$$H = G_{\hat{\lambda}}.$$  

(6)

The assumption that $G$ has simply-connected derived group implies that the group $H$ is connected (see [17, p. 396]). Note that $H$ itself does not have
simply-connected derived group in general (for instance, if $G$ is the exceptional group of type $G_2$, and $\sigma$ is the tensor square of a ramified quadratic character of $F^\times$ then $H = \text{SO}(4, \mathbb{C})$).

Let $\mathfrak{Q}(G)_\lambda$ be the subset of $\mathfrak{Q}(G)$ consisting of the $G$-conjugacy classes of all the pairs $(\Phi, \rho)$ such that $\Phi$ factors through $W_{F}^{\mathfrak{ab}}$ and

$$\Phi|_{I_F^{\mathfrak{ab}}} = \hat{\lambda}.$$ 

The group $W^s$ defined in (3) is a Weyl group: it is the Weyl group of $H$ (indeed, in the decomposition of [17, Lemma 8.1 (i)] the group $C_\chi$ is trivial as proven on [17, p. 396]):

$$W^s = W_H.$$

4 Main result

**Theorem 4.1.** There is a canonical bijection of the extended quotient of the second kind $(T//W^s)_2$ onto the set $\mathfrak{Q}(G)_\lambda$ of conjugacy classes of Reeder parameters attached to the point $s$ in the Bernstein spectrum of $G$. It follows that there is a bijection

$$\mu^s : T//W^s \simeq \mathfrak{Q}(G)_{\lambda}$$

so that the extended quotient $T//W^s$ is a model for the Reeder parameters attached to the point $s$.

The proof of this theorem requires a series of Lemmas. We recall that

$$W^s = W_H.$$ 

The plan of our proof is to begin with an element in the extended quotient of the second kind $(T//W_H)_2$. Lemmas 4.2 and 4.3 allow us to infer that $W_H(t)$ is a semidirect product $W_{\Phi(t)} \rtimes A_H(t)$. We now combine the Springer correspondence for $W_{\Phi(t)}$ with Clifford theory for semidirect products (Clifford theory is a noncommutative version of the Mackey machine). This creates 4 parameters $(t, x, \varrho, \psi)$. With this data, and the character $\lambda$ determined by the point $s$, we construct a Reeder parameter $(\Phi, \rho)$ such that $\Phi(\text{Frob}, 1) = t$, $\Phi(1, u_0) = \exp x$ and the restriction of $\rho$ contains $\varrho$.

**Lemma 4.2.** Let $M$ be a reductive algebraic group. Let $M^0$ denote the connected component of the identity in $M$. Let $T$ be a maximal torus of $M^0$ and let $B$ be a Borel subgroup of $M^0$ containing $T$. Let

$$W_{M^0}(T) := N_{M^0}(T)/T$$
denote the Weyl group of $M^0$ with respect to $T$. We set
\[ W_M(T) := N_M(T)/T. \]

(1) The group $W_M(T)$ has the semidirect product decomposition:
\[ W_M(T) = W_{M^0}(T) \rtimes (N_M(T, B)/T), \]
where $N_M(T, B)$ denotes the normalizer in $M$ of the pair $(T, B)$.

(2) We have
\[ N_M(T, B)/T \simeq M/M^0 = \pi_0(M). \]

Proof. The group $W_{M^0}(T)$ is a normal subgroup of $W_M(T)$. Indeed, let $n \in N_{M^0}(T)$ and let $n' \in N_M(T)$, then $n'n^{-1}$ belongs to $M^0$ (since the latter is normal in $M$) and normalizes $T$, that is, $n'n^{-1} \in N_{M^0}(T)$. On the other hand, $n'(nT)n'^{-1} = n'n^{-1}(n'Tn'^{-1}) = n'n^{-1}T$.

Let $w \in W_M(T)$. Then $wBw^{-1}$ is a Borel subgroup of $M^0$ (since, by definition, the Borel subgroups of an algebraic group are the maximal closed connected solvable subgroups). Moreover, $wBw^{-1}$ contains $T$. In a connected reductive algebraic group, the intersection of two Borel subgroups always contains a maximal torus and the two Borel subgroups are conjugate by a element of the normalizer of that torus. Hence $B$ and $wBw^{-1}$ are conjugate by an element $w_1$ of $W_{M^0}(T)$. It follows that $w_1^{-1}w$ normalises $B$. Hence
\[ w_1^{-1}w \in W_M(T) \cap N_M(B) = N_M(T, B)/T, \]
that is,
\[ W_M(T) = W_{M^0}(T) \cdot (N_M(T, B)/T). \]

Finally, we have
\[ W_{M^0}(T) \cap (N_M(T, B)/T) = N_{M^0}(T, B)/T = \{1\}, \]

since $N_{M^0}(B) = B$ and $B \cap N_{M^0}(T) = T$. This proves (1).

We will now prove (2). We consider the following map:
\[ N_M(T, B)/T \to M/M^0 \quad mT \mapsto mM^0. \]

It is injective. Indeed, let $m, m' \in N_M(T, B)$ such that $mM^0 = m'M^0$. Then $m^{-1}m' \in M^0 \cap N_M(T, B) = N_{M^0}(T, B) = T$ (as we have seen above). Hence $mT = m'T$.

On the other hand, let $m$ be an element in $M$. Then $m^{-1}Bm$ is a Borel subgroup of $M^0$, hence there exists $m_1 \in M^0$ such that $m^{-1}Bm = m_1^{-1}Bm_1$. 

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It follows that \(m_1m^{-1} \in N_M(B)\). Also \(m_1m^{-1}Tmm_1^{-1}\) is a torus of \(M^0\) which is contained in \(m_1m^{-1}Bmm_1^{-1} = B\). Hence \(T\) and \(m_1m^{-1}Tmm_1^{-1}\) are conjugate in \(B\): there is \(b \in B\) such that \(m_1m^{-1}Tmm_1^{-1} = b^{-1}Tb\). Then \(n := bm_1m^{-1} \in N_M(T, B)\). It gives \(m = n^{-1}bm_1\). Since \(bm_1 \in M^0\), we obtain \(mM^0 = n^{-1}M^0\). Hence the map (*) is surjective.

In order to approach the notation in [4, p.471], we let \(\mathfrak{g}(t)\) denote the identity component of the centralizer \(C_H(t)\):

\[
\mathfrak{g}(t) := C^0_h(t).
\]

Let \(W_{\mathfrak{g}(t)}\) denote the Weyl group of \(\mathfrak{g}(t)\).

**Lemma 4.3.** Let \(t \in T\). The isotropy subgroup \(W_H(t)\) is the group of \(N_{C_H(t)}(T)/T\), and we have

\[
W_H(t) = W_{\mathfrak{g}(t)} \rtimes A_H(t) \quad \text{with} \quad A_H(t) := \pi_0(C_H(t)).
\]

In the case when \(H\) has simply-connected derived group, the group \(C_H(t)\) is connected and \(W_H(t)\) is then the Weyl group of \(C_H(t) = \mathfrak{g}(t)\).

**Proof.** Let \(t \in T\). Note that

\[
W_H(t) = \{w \in W_H : w \cdot t = t\} \\
= \{w \in W_H : wtw^{-1} = t\} \\
= \{w \in W_H : wt = tw\} \\
= W \cap C_H(t).
\]

Note that \(H\) and \(C_H(t)\) have a common maximal torus \(T\). Now

\[
W_H \cap C_H(t) = N_H(T)/T \cap C_H(t) \\
= N_{C_H(t)}(T)/T \\
= W_{C_H(t)}(T).
\]

The result follows by applying Lemma [4,2] with \(M = C_H(t)\).

If \(H\) has simply-connected derived group, then the centralizer \(C_H(t)\) is connected by Steinberg’s theorem [4] §8.8.7.

Let \(\tau\) be an irreducible representation of \(W_{\mathfrak{g}(t)}\). Now we apply the Springer correspondence to \(\tau\). Note: the Springer correspondence that we are considering here coincides with that constructed by Springer for a reductive group over a field of positive characteristic and is obtained from the
correspondence constructed by Lusztig by tensoring the latter by the sign representation of $W_{\mathfrak{g}(t)}$ (see [6]).

Let $\mathfrak{c}(t)$ denote the Lie algebra of $\mathfrak{g}(t)$, for $x \in \mathfrak{c}(t)$, let $Z_{\mathfrak{g}(t)}(x)$ denote the centralizer of $x$ in $\mathfrak{g}(t)$, via the adjoint representation of $\mathfrak{g}(t)$ on $\mathfrak{c}(t)$, and let

$$A_x = \pi_0(Z_{\mathfrak{g}(t)}(x)) \quad (7)$$

Let $\mathbf{B}_x$ denote the variety of Borel subalgebras of $\mathfrak{c}(t)$ that contain $x$.

All the irreducible components of $\mathbf{B}_x$ have the same dimension $d(x)$ over $\mathbb{R}$, see [4, Corollary 3.3.24]. The finite group $A_x$ acts on the set of irreducible components of $\mathbf{B}_x$ [4, p. 161].

**Definition 4.4.** If a group $A$ acts on the variety $\mathbf{X}$, let $\mathcal{R}(A, \mathbf{X})$ denote the set of irreducible representations of $A$ appearing in the homology $H_*(\mathbf{X})$, as in [18a, p.118]. Let $\mathcal{R}_{\text{top}}(A, \mathbf{X})$ denote the set of irreducible representations of $A$ appearing in the top homology of $\mathbf{X}$.

The Springer correspondence yields a one-to-one correspondence

$$(x, \varrho) \mapsto \tau(x, \varrho) \quad (8)$$

between the set of $\mathfrak{g}(t)$-conjugacy classes of pairs $(x, \varrho)$ formed by a nilpotent element $x \in \mathfrak{c}(t)$ and an irreducible representation $\varrho$ of $A = A_x$ which occurs in $H_{d(x)}(\mathbf{B}_x, \mathbb{C})$ (that is, $\varrho \in \mathcal{R}_{\text{top}}(A_x, \mathbf{B}_x)$) and the set of isomorphism classes of irreducible representations of the Weyl group $W_{\mathfrak{g}(t)}$.

We now work with the Jacobson-Morozov theorem [4, p. 183]. Let $e_0$ be the standard nilpotent matrix in $\mathfrak{sl}(2, \mathbb{C})$:

$$e_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

There exists a rational homomorphism $\gamma : \text{SL}(2, \mathbb{C}) \to \mathfrak{g}(t)$ such that its differential $\mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{c}(t)$ sends $e_0$ to $x$, see [4, §3.7.4].

Define

$$\Phi : \mathcal{W}_F^{ab} \times \text{SL}(2, \mathbb{C}) \to G, \quad (w, \text{Frob}, Y) \mapsto \hat{\lambda}(w) \cdot t \cdot \gamma(Y) \quad (9)$$

$$\Upsilon : \mathcal{W}_F^{ab} \times \text{SL}(2, \mathbb{C}) \to H, \quad (w, \text{Frob}, Y) \mapsto \hat{\lambda}(w) \cdot t \cdot \gamma(Y) \quad (10)$$

$$\Psi : \mathcal{W}_F^{ab} \times \text{SL}(2, \mathbb{C}) \to \mathfrak{g}(t), \quad (w, \text{Frob}, Y) \mapsto \hat{\lambda}(w) \cdot t \cdot \gamma(Y) \quad (11)$$
\( \Xi : \mathcal{W}_{F}^{ab} \times \text{SL}(2, \mathbb{C}) \to \mathcal{G}(t), \quad (w, \text{Frob}, Y) \mapsto \hat{\lambda}(w) \cdot \gamma(Y). \) \hspace{1cm} (12)

where \( w \) is any element in \( I_{F}^{ab} \).

Note that \( \text{im} \Phi \subset H \) (see [16, § 4.2]) and that \( C(\text{im} \Psi) = C(\text{im} \Upsilon) \), for any element in \( C(\text{im} \Upsilon) \) must commute with \( \Upsilon(\text{Frob}) = t \). We also have \( C(\text{im} \Xi) = C(\text{im} \Psi) \subset \mathcal{C}(t) \). Let

\[
A_{\Psi} = \pi_{0}(C(\text{im} \Psi)), \quad A_{\Xi} = \pi_{0}(C(\text{im} \Xi)).
\]

**Lemma 4.5.** We have

\[ A_{x} = A_{\Xi} = A_{\Psi}. \]

**Proof.** According to [11, §3.7.23], we have

\[ Z_{\Phi(t)}(x) = C(\text{im} \Xi) \cdot U \]

with \( U \) the unipotent radical of \( Z_{\Phi(t)}(x) \). Now \( U \) is contractible via the map

\[ [0, 1] \times U \to U, \quad (\lambda, \exp Y) \mapsto \exp(\lambda Y) \]

for all \( Y \in n \) with \( \exp n = U \). \( \square \)

Lemma 4.5 allows us to define

\[ A := A_{x} = A_{\Psi} = A_{\Xi}. \]

Let \( \mathcal{C}(t) \) denote a predual of \( \mathfrak{G}(t) \), i.e., \( \mathfrak{G}(t) \) is the Langlands dual of \( \mathcal{C}(t) \). Let \( \mathcal{B}_{\Psi} \) (resp. \( \mathcal{B}_{\Xi} \)) denote the variety of the Borel subgroups of \( \mathfrak{G}(t) \) which contain \( S_{\Psi} := \Psi(\mathcal{W}_{F} \times B_{2}) \) (resp. \( S_{\Xi} := \Xi(\mathcal{W}_{F} \times B_{2}) = \gamma(B_{2}) \)).

**Lemma 4.6.** We have

\[ \mathcal{R}_{\top}(A, \mathcal{B}_{x}) = \mathcal{R}(A, \mathcal{B}_{\Xi}). \]

**Proof.** Let, as before, \( \tau \) be an irreducible representation of \( \mathcal{W}_{\Phi(t)} \). Let \( (x, \varrho) \) be the Springer parameter attached to \( \tau \) by the inverse bijection of (8). Define \( \Xi \) as in Eqn.12. Note that \( \Xi \) depends on the morphism \( \gamma \), which in turn depends on the nilpotent element \( x \in c(t) \).

Then \( \Xi \) is a real tempered \( L \)-parameter for the \( p \)-adic group \( \mathcal{C}(t) \), see [2, 3.18]. According to several sources, see [11, §10.13], [2], there is a bijection between Springer parameters and Reeder parameters:

\[ (d\gamma(e_{0}), \varrho) \mapsto (\Xi, \varrho). \] \hspace{1cm} (13)

Now \( \varrho \) is an irreducible representation of \( A \) which appears simultaneously in \( H_{d(x)}(\mathcal{B}_{x}, \mathbb{C}) \) and \( H_{*}(\mathcal{B}_{\Xi}, \mathbb{C}) \). \( \square \)
We will recall below a result of Ram and Ramagge, which is based on Clifford theoretic results developed by MacDonald and Green.

Let \( H \) be a finite dimensional \( \mathbb{C} \)-algebra and let \( A \) be a finite group acting by automorphisms on \( H \). If \( V \) is a finite dimensional module for \( H \) and \( a \in A \), let \( {}^aV \) denote the \( H \)-module with the action \( f \cdot v := a^{-1}(f)v, \ f \in H \) and \( v \in V \). Then \( V \) is simple if and only if \( {}^aV \) is. Let \( V \) be a simple \( H \)-module. Define the inertia subgroup of \( V \) to be

\[
A_V := \{ a \in A : V \simeq {}^aV \}.
\]

Let \( a \in A_V \). Since both \( V \) and \( {}^aV \) are simple, Schur’s lemma implies that the isomorphism \( V \to {}^aV \) is unique up to a scalar multiple. For each \( a \in A_V \) we fix an isomorphism

\[
\phi_a : V \to {}^aV.
\]

Then, as operators on \( V \),

\[
\phi_a v = a(r)\phi_a, \quad \text{and} \quad \phi_a \phi_{a'} = \eta_V(a,a')^{-1}\phi_{aa'},
\]

where \( \eta_V(a,a') \in \mathbb{C}^\times \). The resulting function

\[
\eta_V : A_V \times A_V \to \mathbb{C}^\times,
\]

is a cocycle. The isomorphism class of \( \eta_V \) is independent of the choice of the isomorphism \( \phi_a \).

Let \( \mathbb{C}[A_V]_{\eta_V} \) be the algebra with basis \( \{ c_a : a \in A_V \} \) and multiplication given by

\[
c_a \cdot c_{a'} = \eta_V(a,a')c_{aa'}, \quad \text{for} \ a, a' \in A_V.
\]

Let \( \psi \) be a simple \( \mathbb{C}[A_V]_{\eta_V} \)-module. Then putting

\[
(fa) : (v \otimes z) = f \phi_a v \otimes c_a z, \quad \text{for} \ f \in H, a \in A_V, \ v \in V, \ z \in \psi,
\]

defines an action of \( H \times A_V \) on \( V \otimes \psi \). Define the induced module

\[
V \rtimes \psi := \text{Ind}_{H \rtimes A_V}^{H \rtimes A}(V \otimes \psi).
\]

**Theorem 4.7.** (Ram-Ramagge, [14, Theorem A.6], Reeder, [16, (1.5.1)])

The induced module \( V \rtimes \psi \) is a simple \( H \rtimes A \)-module, every simple \( H \rtimes A \)-module occurs in this way, and if \( V \rtimes \psi \simeq V' \rtimes \psi' \), then \( V, V' \) are \( A \)-conjugate, and \( \psi \simeq \psi' \) as \( \mathbb{C}[A_V]_{\eta_V} \)-modules.

One the other hand, it follows from Lemma 4.3 that the isotropy group of \( t \) in \( W_H \) admits the following semidirect product decomposition:

\[
W_H(t) = W_{\phi(t)} \rtimes A_H(t) \quad \text{with} \ A_H(t) := \pi_0(C_H(t)).
\]
Hence the group algebra \( \mathbb{C}[W_H(t)] \) is a crossed-product algebra
\[
\mathbb{C}[W_H(t)] = \mathbb{C}[W_{\phi(t)}] \rtimes A_H(t).
\]

By applying Theorem 4.7 with \( H = \mathbb{C}[W_{\phi(t)}] \) and \( A = A_H(t) \), we see that the irreducible representations of \( W_H(t) \) are the
\[
\tau(x, \varrho) \rtimes \psi,
\]
with \( \psi \) any simple \( \mathbb{C}[A_{\tau}]_{\eta_{\tau}} \)-module and \( \tau = \tau(x, \varrho) \).

Let \( I \) be a standard Iwahori subgroup in \( \mathbb{C}(t) \), and let \( H(\mathbb{C}(t), I) \) denote the corresponding Iwahori-Hecke algebra. Recall that \( x = d_\gamma(e_0) \). We will denote by \( V = V(x, \varrho) \) the real tempered simple module of \( H(\mathbb{C}(t), I) \) which corresponds to \( (x, \varrho) \). Here “real” means that the central character of \( V \) is real.

By applying Theorem 4.7 with \( H = H(\mathbb{C}(t), I) \) and \( A = A_H(t) \), we obtain the following subset of simple modules for \( H(\mathbb{C}(t), I) \rtimes A_H(t) \):
\[
V(x, \varrho) \rtimes \psi,
\]
with \( \psi \) any simple \( \mathbb{C}[A_V]_{\eta_{V}} \)-module and \( V = V(x, \varrho) \).

**Lemma 4.8.** We have
\[
A_{\tau(x, \varrho)} = A_{V(x, \varrho)}.
\]
Moreover, the cocycles \( \eta_{\tau(x, \varrho)} \) and \( \eta_{V(x, \varrho)} \) can be chosen to be equal.

**Proof.** Recall that the *closure order on nilpotent adjoint orbits* is defined as follows
\[
\mathcal{O}_1 \leq \mathcal{O}_2 \quad \text{when} \quad \mathcal{O}_1 \subset \overline{\mathcal{O}_2},
\]
\[
\mathcal{O}_1 \leq \mathcal{O}_2 \quad \text{when} \quad \mathcal{O}_1 \subset \overline{\mathcal{O}_2}.
\]

For \( x \) a nilpotent element of \( \mathfrak{c}(t) \), we will denote by \( \mathcal{O}_x \) the nilpotent adjoint orbit which contains \( x \). Then as in [2 (6.5)], we define a *partial order on the representations of \( W_{\phi(t)} \) by*
\[
\tau(x_1, \varrho_1) \leq \tau(x_2, \varrho_2) \quad \text{when} \quad \mathcal{O}_{x_1} \leq \mathcal{O}_{x_2}.
\]

In this partial order, the trivial representation of \( W(t) \) is a minimal element and the sign representation of \( W(t) \) is a maximal element.

The \( W_{\phi(t)} \)-structure of \( V(x, \varrho) \) is
\[
V(x, \varrho)|_{W_{\phi(t)}} = \tau(x, \varrho) \oplus \bigoplus_{\tau(x, \varrho) < \tau(x_1, \varrho_1)} m_{(x_1, \varrho_1)} \tau(x_1, \varrho_1),
\]

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where the $m(x_1, q_1)$ are non-negative integers. (In case $C(t)$ has connected centre, \cite{15} is implied by \cite{2} Theorem 6.3 (1), the proof in the general case follows the same lines.) In particular, it follows from \cite{15} that

\[
\dim \mathbb{C} \text{Hom}_{W_{\Phi(t)}}(\tau(x, q), V(x, q)) = 1. \tag{16}
\]

Let $a \in A_H(t)$. Since the action of $A_H(t)$ on $W_{\Phi(t)}$ comes from its action on the root datum, we have (see \cite{16} 2.6.1, 2.7.3):

\[
a \tau(x, q) = \tau(a \cdot x, a q_1).
\]

Then

\[
a V(x, q)|_{W_{\Phi(t)}} = \tau(a \cdot x, a q_1) \oplus \bigoplus_{(x_1, q_1) \in \tau(x, q)} m(x_1, q_1) \tau(a \cdot x, a q_1).
\]

Since $\tau(x, q) \leq \tau(x_1, q_1)$ if and only if $\chi(a \cdot x, a q_1) \leq \tau(a \cdot x_1, a q_1)$, it follows that $a V(x, q)$ corresponds to the $\Phi(t)$-conjugacy class of $(a \cdot x, a q_1)$ via the bijection induced by \cite{13}.

Hence

\[
a V(x, q) \simeq V(x, q) \quad \text{if and only if} \quad a \tau(x, q) \simeq \tau(x, q).
\]

The equality of the inertia subgroups

\[
A_H(t)_{V(x, q)} = A_H(t)_{\tau(x, q)} =: A_H(t)_{x, q}
\]

follows.

Let $\{ \phi^V_a : a \in A_H(t)_{x, q} \}$ (resp. $\{ \phi^\tau_a : a \in A_H(t)_{x, q} \}$) a family of isomorphisms for $V = V(x, q)$ (resp. $\tau = \tau(x, q)$) which determines the cocycle $\eta_V$ (resp. $\eta_\tau$). We have

\[
\text{Hom}_{W_{\Phi(t)}}(\tau, V) \xrightarrow{\phi^V_a} \text{Hom}_{W_{\Phi(t)}}(\tau, a^{-1}V) \xrightarrow{\phi^\tau_a} \text{Hom}_{W_{\Phi(t)}}(a^{-1}\tau, a^{-1}V).
\]

The composed map is given by a scalar, since by Eqn. \cite{16} these spaces are one-dimensional. We normalize $\phi^V_a$ so that this scalar equals to one. This forces $\eta_V$ and $\eta_\tau$ to be equal.

\begin{lemma}

There is a bijection between Springer parameters and Reeder parameters for the group $C_H(t)$:

\[
(x, q, \psi) \mapsto (\Xi, q, \psi).
\]

\end{lemma}
Proof. Lemma 4.8 allows us to extend the bijection (13) from $G(t)$ to $C_H(t)$. \hfill \Box

Lemma 4.10. We have

$$B^\Psi = B^\Xi.$$ 

Proof. We note that

$$S^\Psi = < t > \gamma(B_2), \quad S^\Xi = \gamma(B_2)$$

Let $b$ denote a Borel subgroup of the reductive group $C_H(t)$. Since $b$ is maximal among the connected solvable subgroups of $C_H(t)$, we have $b \subset \mathfrak{g}(t)$. Then we have $b = T_b U_b$ with $T_b$ a maximal torus in $\mathfrak{g}(t)$, and $U_b$ the unipotent radical of $b$. Note that $T_b \subset \mathfrak{g}(t)$. Therefore $yt = ty$ for all $y \in T_b$. This means that $t$ centralizes $T_b$, i.e. $t \in Z(T_b)$. In a connected Lie group such as $\mathfrak{g}(t)$, we have

$$Z(T_b) = T_b$$

so that $t \in T_b$. Since $T_b$ is a group, it follows that $< t > \subset T_b$.

As a consequence, we have

$$b \supset < t > \gamma(B_2) \iff b \supset \gamma(B_2).$$

\hfill \Box

Let $S_Y = \Upsilon(W_F \times B_2)$, a solvable subgroup of $H$. Let $B^\Upsilon$ denote the variety of Borel subgroups of $H$ containing $S_Y$.

Lemma 4.11. We have

$$\mathcal{R}(A, B^\Upsilon) = \mathcal{R}(A, B^\Psi)$$

Proof. We denote the Lie algebra of $\mathfrak{g}(t)$ by $\mathfrak{g}(t)$, and the Lie algebra of $C_H(t)$ by $\mathfrak{c}_H(t)$ so that

$$\mathfrak{g}(t) = \mathfrak{c}_H(t).$$

We note that the codomain of $\Psi$ is $\mathfrak{g}(t)$.

Let $B^t$ denote the variety of all Borel subgroups of $G$ which contain $t$. Let $B \in B^t$. Then $B \cap \mathfrak{g}(t)$ is a Borel subgroup of $\mathfrak{g}(t)$.

The proof in [11, p.471] depends on the fact that $\mathfrak{g}(t)$ is connected, and also on a triangular decomposition of $\text{Lie}(\mathfrak{g}(t))$:

$$\text{Lie} \mathfrak{g}(t) = \mathfrak{n}^t \oplus t \oplus \mathfrak{n}^t$$

from which it follows that $\text{Lie} B \cap \text{Lie} \mathfrak{g}(t) = \mathfrak{n}^t \oplus t$ is a Borel subalgebra in $\text{Lie} \mathfrak{g}(t)$. The superscript “$t$” stands for the centralizer of $t$. 

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There is a canonical map

$$B^t \to \text{Flag} \mathsf{G}(t), \quad B \mapsto \mathsf{B} \cap \mathsf{G}(t) \quad (17)$$

Now $\mathsf{G}(t)$ acts by conjugation on $B^t$. We have

$$B^t = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_m \quad (18)$$

a disjoint union of $\mathsf{G}(t)$-orbits, see [4, Prop. 8.8.7]. These orbits are the connected components of $B^t$, and the irreducible components of the projective variety $B^t$. The above map (17), restricted to any one of these orbits, is a bijection from the $\mathsf{G}(t)$-orbit onto $\text{Flag} \mathsf{G}(t)$ and is $\mathsf{G}(t)$-equivariant. It is then clear that

$$\mathsf{B}^t_j \simeq \text{Flag} \mathsf{G}(t)^\Psi$$

for each $1 \leq j \leq m$. We also have $t \in S_{\mathsf{B}^t} = S_{\Psi}$. Now

$$\mathsf{B}^t = (\mathsf{B}^t)^\Psi = (\mathsf{B}^t)^\Psi$$

and then

$$H_*(\mathsf{B}^t, \mathbb{C}) = H_*(\mathsf{B}^t_1, \mathbb{C}) \oplus \cdots \oplus H_*(\mathsf{B}^t_m, \mathbb{C})$$

a direct sum of equivalent $A$-modules. Hence $\varrho$ occurs in $H_*(\mathsf{B}^t, \mathbb{C})$ if and only if it occurs $H_*(\mathsf{B}^\Psi, \mathbb{C})$. \hfill $\square$

Recall that $x$ is a nilpotent element in $\mathfrak{c}(t)$ (the Lie algebra of $\mathsf{G}(t)$). Define

$$A^+ := \pi_0(Z_{C_H(t)}(x)).$$

**Lemma 4.12.** We have

$$\mathcal{R}(A, \mathsf{B}^t) = \mathcal{R}(A^+, \mathsf{B}^t).$$

**Proof.** Choose an isogeny $\iota: \widetilde{H} \to H$ with $\widetilde{H}_{\text{der}}$ simply connected (as in [16, Theorem 3.5.4]) such that $H = \widetilde{H}/Z$ where $Z$ is a finite subgroup of the centre of $\widetilde{H}$ (see [16, § 3]). Let $\tilde{t}$ be a lift of $t$ in $\widetilde{H}$, that is, $\iota(\tilde{t}) = t$. Then we have (see [16, § 3.1]):

$$\iota(C_{\tilde{H}}(\tilde{t})) = C_{\tilde{H}}^{\text{der}}(t) = \mathsf{G}(t). \quad (19)$$

Let $u := \exp(x)$, a unipotent element in $\mathsf{G}(t)$. It follows from Eqn. (19) that there exists $\tilde{u} \in C_{\tilde{H}}(\tilde{t})$ such that $u = \iota(\tilde{u})$. Recall that $A = \pi_0(Z_{\mathsf{G}(t)}(x))$. Then

$$A \simeq \pi_0(Z_{\mathsf{G}(t)}(u)) = \pi_0(Z_{\iota(C_{\tilde{H}}(\tilde{t}))}(\iota(\tilde{u}))) \simeq \pi_0(Z_{C_{\tilde{H}}}(\tilde{t}, \tilde{u})), \quad (19)$$

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and $A$ is a subgroup of $\pi_0(Z_{\mathcal{H}}(u)) \simeq A^+$ (see [16 § 3.2–3.3]).

Recall from [16 Lemma 3.5.3] that

$$(\tilde{t}, \tilde{u}, g, \psi) \mapsto (t, u, \rho)$$

induces a bijection between $G$-conjugacy classes of quadruples $(\tilde{t}, \tilde{u}, g, \psi)$ and $G$-conjugacy classes of triples $(t, u, \rho)$, where $\rho \in \mathcal{R}(A^+, B^\Psi)$ is such that the restriction of $\rho$ to $A$ contains $g$.

Lemma 4.13. We have

$$\mathcal{R}(A^+, B^\Psi) = \mathcal{R}(A^+, B^\Phi).$$

Proof. It follows from [16 Lemma 4.4.1].

The proof can be reversed. Here is the reason for this claim: Lemmas 4.5, 4.6, 4.8 4.10 – 4.13 are all equalities, and Lemma 4.9 is a bijection.

This creates a canonical bijection between the extended quotient of the second kind $(T//W^g)_2$ and $\Omega(G)_{\hat{\lambda}}$:

$$\mu: (T//W^g)_2 \longrightarrow \Omega(G)_{\hat{\lambda}}, \quad (t, x, g, \psi) \mapsto (\Phi, \rho). \quad (20)$$

This in turn creates a bijection

$$T//W^g \longrightarrow \Omega(G)_{\hat{\lambda}}. \quad (21)$$

This bijection is not canonical in general, depending as it does on a choice of bijection between the set of conjugacy classes in $W_H(t)$ and the set of irreducible characters of $W_H(t)$. When $G = \text{GL}(n)$, the finite group $W_H(t)$ is a product of symmetric groups: in this case there is a canonical bijection between the set of conjugacy classes in $W_H(t)$ and the set of irreducible characters of $W_H(t)$, by the classical theory of Young tableaux.

To close this section, we will consider the case of $\text{GL}(n, F)$, and the Iwahori point $i$ in the Bernstein spectrum of $\text{GL}(n, F)$. The Langlands dual of $\text{GL}(n, F)$ is $\text{GL}(n, C)$, and we will take $T$ to be the standard maximal torus in $\text{GL}(n, C)$. The Weyl group is the symmetric group $S_n$. We will denote our bijection, in this case canonical, as follows:

$$\mu^i_F : T//W \longrightarrow \mathfrak{P}(\text{GL}(n, F))$$

Let $E/F$ be a finite Galois extension of the local field $F$. According to [12 Theorem 4.3], we have a commutative diagram

$$\begin{array}{ccc}
T//W \xrightarrow{\mu^i_F} \mathfrak{P}(\text{GL}(n, F)) & \xrightarrow{\text{BC}_{E/F}} & \mathfrak{P}(\text{GL}(n, E)) \\
\downarrow & & \downarrow \\
T//W & \xrightarrow{\mu^i_E} & \mathfrak{P}(\text{GL}(n, E))
\end{array}$$

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In this diagram, the right vertical map $BC_{E/F}$ is the standard base change map sending one Reeder parameter to another as follows:

$$(\Phi, 1) \mapsto (\Phi|_{W_E}, 1).$$

Let

$$f = f(E, F)$$

denote the residue degree of the extension $E/F$. We proceed to describe the left vertical map. We note that the action of $W$ on $T$ is as automorphisms of the algebraic group $T$. Since $T$ is a group, the map

$$T \to T, \ t \mapsto t^f$$

is well-defined for any positive integer $f$. The map

$$\tilde{T} \to \tilde{T}, \ (t, w) \mapsto (t^f, w)$$

is also well-defined, since

$$w \cdot t^f = wt^fw^{-1} = wtwd\cdots wtdw^{-1} = t^f.$$ 

Since

$$\alpha \cdot (t^f) = (\alpha \cdot t)^f$$

for all $\alpha \in W$, this induces a map

$$T//W \to T//W$$

which is an endomorphism (as algebraic variety) of the extended quotient $T//W$. We shall refer to this endomorphism as the base change endomorphism of degree $f$. The left vertical map is the base change endomorphism of degree $f$, according to [12, Theorem 4.3]. That is, our bijection $\mu^i$ is compatible with base change for $GL(n)$.

When we restrict our base change endomorphism from the extended quotient $T//W$ to the ordinary quotient $T/W$, we see that the commutative diagram containing $BC_{E/F}$ is consistent with [5, Lemma 4.2.1].

## 5 Interpolation

We will now provide details for the interpolation procedure described in §1. We will focus on the Iwahori point $i \in \mathcal{P}(G)$, i.e., on the smooth irreducible representations of $G$ which admit nonzero Iwahori fixed vectors. To simplify notation, we will write $\mu = \mu^i$. Let $\mathcal{P}(G)$ denote the set of conjugacy classes

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in $G$ of Kazhdan-Lusztig parameters. For each $s \in \mathbb{C}^*$, we construct a commutative diagram:

$$
\begin{array}{ccc}
T//W & \xrightarrow{\mu} & \mathfrak{P}(G) \\
\pi_s & & \downarrow i_s \\
T/W & \xrightarrow{i_s} & T/W
\end{array}
$$

in which the map $\mu$ is bijective. In the top row of this diagram, the set $T//W$, the set $\mathfrak{P}(G)$, and the map $\mu$ are independent of the parameter $s$.

We start by defining the vertical maps $i_s$, $\pi_s$ in the diagram. Let $s \in \mathbb{C}^*$. We will define

$$i_s : \mathfrak{P}(G) \to T/W, \quad (\Phi, \rho) \mapsto \Phi(\text{Frob}, T_s) \tag{22}$$

$$\pi_s : T//W \to T/W, \quad (t, w) \mapsto t \cdot \gamma(T_s) \tag{23}$$

where $(\Phi, \rho)$ is a Reeder parameter, and $(t, w) \in T//W$. We note that

$$\Phi(\text{Frob}, T_s) = t \cdot \gamma(T_s)$$

so that the diagram is commutative.

- Let $s = 1$, and assume, for the moment, that $C_H(t)$ is connected. The map $\mu$ in Theorem 4.1 sends $(t, \tau)$ to $(\Phi, \rho)$. We note that

$$t = \Phi(\text{Frob}, T_1) = \Phi(\text{Frob}, 1).$$

The map $\mu$ determines the map

$$(t, \tau) \mapsto (t, \Phi(1, u_0), \rho)$$

which, in turn, determines the map

$$\tau \mapsto (\exp(x), \rho)$$

which is the Springer correspondence for the Weyl group $W_H(t)$.

- Now let $s = \sqrt{q}$ where $q$ is the cardinality of the residue field $k_F$ of $F$. We now link our result to the representation theory of the $p$-adic group $G$ as follows. As in §3, let

$$\sigma := \Phi(\text{Frob}, T_{\sqrt{q}}), \quad u := \Phi(1, u_0).$$

Then we have

$$\sigma u \sigma^{-1} = u^q$$

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and the triple \((\sigma, u, \rho)\) is a Kazhdan-Lusztig triple.

The correspondence \(\sigma \mapsto \chi_\sigma\) between points in \(T\) and unramified quasi-characters of \(T\) can be fixed by the relation

\[\chi_\sigma(\lambda(\varpi_F)) = \lambda(\sigma)\]

where \(\varpi_F\) is a uniformizer in \(F\), and \(\lambda \in X_*(T) = X^*(T)\). The Kazhdan-Lusztig triples \((\sigma, u, \rho)\) parametrize the irreducible constituents of the (unitarily) induced representation

\[\text{Ind}_B^G(\chi_\sigma \otimes 1)\].

Note that

\[i_{\sqrt{q}} : (\Phi, \rho) \mapsto \sigma\]

so that \(i_{\sqrt{q}}\) is the infinitesimal character. The infinitesimal character is denoted \(\text{Sc}\) in [15, VI.7.1.1] (\(\text{Sc}\) for support cuspidal)

Since \(\mu\) is bijective and the diagram is commutative, the number of points in the fibre of the \(q\)-projection \(\pi_{\sqrt{q}}\) equals the number of inequivalent irreducible constituents of \(\text{Ind}_B^G(\chi_\sigma \otimes 1)\):

\[|\pi_{\sqrt{q}}^{-1}(\sigma)| = |\text{Ind}_B^G(\chi_\sigma \otimes 1)|\]  

(24)

The \(q\)-projection \(\pi_{\sqrt{q}}\) is a model of the infinitesimal character \(\text{Sc}\).

Our formulation leads to Eqn.(24), which appears to have some predictive power. Note that the definition of the \(q\)-projection \(\pi_{\sqrt{q}}\) depends only on the \(L\)-parameter \(\Phi\). An \(L\)-parameter determines an \(L\)-packet, and does not determine the number of irreducible constituents of the \(L\)-packet.

6 Examples

**Example 1.** Realization of the ordinary quotient \(T/W\). Consider an \(L\)-parameter \(\Phi\) for which \(\Phi|_{\text{SL}(2,\mathbb{C})} = 1\). Let \(t = \Phi(\text{Frob})\). Then

\[G_\Phi := C(\text{im} \Phi) = C(t)\]

so that \(G_\Phi\) is connected and acts trivially in homology. Therefore \(\rho\) is the unit representation 1.

Now \(t\) is a semisimple element in \(G\), and all such semisimple elements arise. Modulo conjugacy in \(G\), the set of such \(L\)-parameters \(\Phi\) is parametrized by the quotient \(T/W\). Explicitly, let

\[\mathcal{P}_1(G) := \{ \Phi \in \mathcal{P}(G) : \Phi|_{\text{SL}(2,\mathbb{C})} = 1 \}.$$
Then we have a canonical bijection
\[ \mathfrak{P}_1(G) \rightarrow T/W, \quad (\Phi, 1) \mapsto \Phi(\text{Frob}, 1) \]
which fits into the commutative diagram
\[
\begin{array}{ccc}
\mathfrak{P}_1(G) & \longrightarrow & T/W \\
\downarrow & & \downarrow \\
\mathfrak{P}(G) & \longrightarrow & T//W
\end{array}
\]
where the vertical maps are inclusions.

**Example 2.** *The general linear group.* Let \( G = \text{GL}(n), G = \text{GL}(n, \mathbb{C}). \) Let
\[ \Phi = \chi \otimes \tau(n) \]
where \( \chi \) is an unramified quasicharacter of \( \mathcal{W}_F \) and \( \tau(n) \) is the irreducible \( n \)-dimensional representation of \( \text{SL}(2, \mathbb{C}). \) By local classfield theory, the quasicharacter \( \chi \) factors through \( F^\times. \) In the local Langlands correspondence for \( \text{GL}(n), \) the image of \( \Phi \) is the unramified twist \( \chi \circ \det \) of the Steinberg representation \( \text{St}(n). \)

The sign representation \( sgn \) of the Weyl group \( W \) has Springer parameters \( (\mathcal{O}_{\text{prin}}, 1) \), where \( \mathcal{O}_{\text{prin}} \) is the principal orbit in \( \mathfrak{gl}(n, \mathbb{C}). \) In the canonical correspondence between irreducible representations of \( S_n \) and conjugacy classes in \( S_n, \) the trivial representation of \( W \) corresponds to the conjugacy class containing the \( n \)-cycle \( w_0 = (123\cdots n) \).

Now \( G_{\Phi} = C(\text{im} \Phi) \) is connected [4, §3.6.3], and so acts trivially in homology. Therefore \( \rho \) is the unit representation 1. The image \( \Phi(1, w_0) \) is a regular nilpotent, i.e. a nilpotent with one Jordan block (given by the partition of \( n \) with one part). The corresponding conjugacy class in \( W \) is \( \{w_0\}. \) The corresponding irreducible component of the extended quotient is
\[ T^{w_0}/Z(w_0) = \{(z, z, \ldots, z) : z \in \mathbb{C}^\times \} \simeq \mathbb{C}^\times. \]

This is our model, in the extended quotient picture, of the complex 1-torus of all unramified twists of the Steinberg representation \( \text{St}(n). \) The map from \( L \)-parameters to pairs \((w, t) \in T//W \) is given by
\[ \chi \otimes \tau(n) \mapsto (w_0, \chi(\text{Frob}), \ldots, \chi(\text{Frob})). \]

Among these representations, there is one real tempered representation, namely \( \text{St}(n), \) with \( L \)-parameter \( 1 \otimes \tau(n), \) attached to the principal orbit \( \mathcal{O}_{\text{prin}} \subset G. \)
More generally, let
\[ \Phi = \chi_1 \otimes \tau(n_1) \oplus \cdots \oplus \chi_k \otimes \tau(n_k) \]
where \( n_1 + \cdots + n_k = n \) is a partition of \( n \). This determines the unipotent orbit \( O(n_1, \ldots, n_k) \subset G \). There is a conjugacy class in \( W \) attached canonically to this orbit: it contains the product of disjoint cycles of lengths \( n_1, \ldots, n_k \). The fixed set is a complex torus, and the component in \( T//W \) is a product of symmetric products of complex 1-tori.

**Example 3.** The exceptional group of type \( G_2 \). This example contains a Reeder parameter \((\Phi, \rho)\) with \( \rho \neq 1 \). The torus \( T \) is identified with \( F^\times \times F^\times \). We take \( \lambda = \chi \otimes \chi \) where \( \chi \) is a nontrivial quadratic character of \( o^\times \times F \).

We have \( H = SO(4, \mathbb{C}) \cong \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C})/\{\pm I\} \). This complex reductive Lie group is neither simply-connected nor of adjoint type. We have \( W^s = W_H = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). We will write
\[
\text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \longrightarrow H^s, \quad (x, y) \mapsto [x, y],
\]
\[
T_{s,s'} = [T_s, T_{s'}], \quad s, s' \in \mathbb{C}^\times.
\]

We have
\[
\Omega(G)_\lambda \rightarrow T//W_H \cong A^1 \sqcup A^1 \sqcup pt_1 \sqcup pt_2 \sqcup pt_s \sqcup T/W_H,
\]
where
- one \( A^1 \) corresponds to \((\Phi, 1)\) with \( \Phi(\text{Frob}, 1) = [I, T_s] \) and \( \Phi(1, u_0) = [u_0, I] \),
- the other \( A^1 \) corresponds to \((\Phi, 1)\) with \( \Phi(\text{Frob}, 1) = [T_s, I] \) and \( \Phi(1, u_0) = [I, u_0] \),
- \( pt_1 \) corresponds to \((\Phi, 1)\) with \( \Phi(\text{Frob}, 1) = T_{1,1} \) and \( \Phi(1, u_0) = [u_0, u_0] \),
- \( pt_2 \) corresponds to \((\Phi, 1)\) with \( \Phi(\text{Frob}, 1) = T_{1,-1} \) and \( \Phi(1, u_0) = [u_0, u_0] \),
- \( T/W_H \) corresponds to \((\Phi, 1)\) with \( \Phi(\text{Frob}, 1) = T_{s,s'} \) \( s, s' \in \mathbb{C}^\times \), and \( \Phi(1, u_0) = [I, I] \),
- \( pt_s \) corresponds to \((\Phi, \text{sgn})\) with \( \Phi(\text{Frob}, 1) = T_{s,s'} \) \( s, s' \in \mathbb{C}^\times \), and \( \Phi(1, u_0) = [I, I] \).

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