Canonical form of the Evolution Operator of a
Time–Dependent Hamiltonian in the Three Level
System

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Abstract

In this paper we study the evolution operator of a time–dependent Hamiltonian in
the three level system. The evolution operator is based on $SU(3)$ and its dimension
is 8, so we obtain three complex Riccati differential equations interacting with one
another (which have been obtained by Fujii and Oike) and two real phase equations.
This is a canonical form of the evolution operator.

1 Introduction

In this paper we treat a finite dimensional quantum model and study the evolution operator
of a time–dependent Hamiltonian from a geometrical point of view. Then we must solve the

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time–dependent Schrödinger equation, which is a very hard task. Our method is based on one in [1], [2] and [3].

Let us start by setting the problem. By $H_0(n; C)$ we show the set of all $n \times n$–hermitian matrices with zero trace. The general form of $H$ is given by

$$H = H(t) = \begin{pmatrix} h_1(t) & \bar{v}_{21}(t) & \bar{v}_{31}(t) & \cdots & \bar{v}_{n-1,1}(t) & \bar{v}_{n1}(t) \\ v_{21}(t) & h_2(t) & \bar{v}_{32}(t) & \cdots & \bar{v}_{n-1,2}(t) & \bar{v}_{n2}(t) \\ v_{31}(t) & v_{32}(t) & h_3(t) & \cdots & \bar{v}_{n-1,3}(t) & \bar{v}_{n3}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{n-1,1}(t) & v_{n-1,2}(t) & v_{n-1,3}(t) & \cdots & h_{n-1}(t) & \bar{v}_{n,n-1}(t) \\ v_{n1}(t) & v_{n2}(t) & v_{n3}(t) & \cdots & v_{n,n-1}(t) & h_n(t) \end{pmatrix} \in H_0(n; C)$$

(1)

with $h_1(t) + h_2(t) + h_3(t) + \cdots + h_{n-1}(t) + h_n(t) = 0$.

Under this Hamiltonian we want to find the evolution operator $U = U(t)$ satisfying the Schrödinger equation

$$i \dot{U} \equiv i \frac{dU}{dt} = HU$$

(2)

where we have set $\hbar = 1$ for simplicity. The wave function $\Psi = \Psi(t)$ is of course related to the evolution operator like $\Psi(t) = U(t)\Psi(0)$, $U(0) = E$ (: identity). However, it is almost impossible to solve (2) exactly except for a few examples.

We decompose $U(t)$ into two unitary parts

$$U(t) = U_1(Z(t))U_2(t)$$

(3)

where $Z$ is a local coordinate of some symmetric space like Grassmann manifolds or (generalized) flag manifolds, see for example [4], [5], [6] and [7]. In the case of flag manifolds this decomposition is based on the approximation

$$SU(n) \approx \{SU(n)/U(1)^{n-1}\} \times U(1)^{n-1}.$$ 

(4)

In the following sections we treat the Bloch sphere $SU(2)/U(1) \cong \mathbb{C}P^1 \cong S^2$ and the flag manifold $SU(3)/U(1)^2$ as interesting examples.
$U_2(t)$ is a small unitary part and in the paper it is set as the phase part

$$U_2(t) = \begin{pmatrix} e^{i \phi_1(t)} \\ e^{i \phi_2(t)} \\ \vdots \\ e^{i \phi_n(t)} \end{pmatrix}; \quad \phi_1(t) + \phi_2(t) + \cdots + \phi_n(t) = 0. \quad (5)$$

Then the equation (2) is reduced to a complicated combination of equations on $Z(t) = (z_{ij}(t))$ and $\{\phi_k(t)\}$. We want to call it a canonical form in our sense. Unfortunately, it is very hard to write down the general case explicitly.

In this paper we transform the equation (2) into a canonical form in the case of $n = 2$ and $n = 3$. Namely, since $\text{dim}_R SU(2) = 3$ for $n = 2$ we obtain one complex Riccati differential equation and one real phase equation. See for example [1] and its references.

On the other hand, since $\text{dim}_R SU(3) = 8$ for $n = 3$ we obtain three complex Riccati differential equations interacting with one another (which have been obtained by Fujii and Oike [7]) and two real phase equations.

## 2 Two Level System

In this section we treat the two level system ($n = 2$) in detail. The Hamiltonian is

$$H = \begin{pmatrix} h(t) & \bar{v}(t) \\ v(t) & -h(t) \end{pmatrix} \quad (6)$$

from (1). From the fact

$$SU(2) \cong SU(2)/U(1) \times U(1) \cong S^2 \times U(1)$$

(see [7]), $U(t)$ can be parametrized as

$$U(t) = U_1(Z(t)) U_2(t) = \frac{1}{\sqrt{1 + |z(t)|^2}} \begin{pmatrix} 1 & -\bar{z}(t) \\ z(t) & 1 \end{pmatrix} \begin{pmatrix} e^{i \phi(t)} \\ e^{-i \phi(t)} \end{pmatrix}. \quad (7)$$

From the equation (2) we have only to calculate

$$H = i \dot{U} U^\dagger = i \dot{U} U^{-1}. \quad (8)$$
Some calculation gives

\[
\dot{U}U^\dagger = -\frac{\ddot{z}\bar{z} + \dot{z}\bar{z}}{2(1 + |z|^2)}1_2 + \frac{1}{1 + |z|^2} \left( \begin{array}{cc} 0 & -\dot{z} \\ \dot{z} & 0 \end{array} \right) \left( \begin{array}{cc} 1 & \bar{z} \\ -\bar{z} & 1 \end{array} \right) + i\dot{\phi} \left( \begin{array}{c} 1 \\ -1 \end{array} \right) U^\dagger
\]

\[
= \frac{1}{1 + |z|^2} \left( \begin{array}{cc} \frac{\ddot{z}\bar{z}}{2} - \frac{\dot{z}}{2} & -\bar{z} \\ \frac{\ddot{z}}{2} - \frac{\dot{z}\bar{z}}{2} & \ddot{z} + 2i\dot{\phi}\bar{z} \end{array} \right) + i\dot{\phi} \frac{1}{1 + |z|^2} \left( \begin{array}{cc} 1 - |z|^2 & 2\bar{z} \\ 2z & -1 + |z|^2 \end{array} \right)
\]

\[
= \frac{1}{1 + |z|^2} \left( \begin{array}{cc} \frac{\ddot{z}\bar{z}}{2} + i\dot{\phi}(1 - |z|^2) & -\bar{z} + 2i\dot{\phi}\bar{z} \\ \ddot{z} + 2i\dot{\phi}z & \frac{\ddot{z}\bar{z}}{2} - i\dot{\phi}(1 - |z|^2) \end{array} \right)
\]

and from (8) we have

\[
\begin{cases}
\frac{i}{1 + |z|^2} (\ddot{z} + 2i\dot{\phi}z) = v, \\
\frac{i}{1 + |z|^2} \left( \frac{\ddot{z}\bar{z}}{2} - i\dot{\phi}(1 - |z|^2) \right) = -h
\end{cases}
\]

or

\[
\begin{cases}
\ddot{z} + 2i\dot{\phi}z = -iv(1 + |z|^2), \\
\dot{\phi} - 2i\dot{\phi}\bar{z} = i\bar{v}(1 + |z|^2), \\
\frac{\ddot{z}\bar{z}}{2} - i\dot{\phi}(1 - |z|^2) = ih(1 + |z|^2).
\end{cases}
\]

From this it is not difficult to show

\[
\begin{cases}
\ddot{z} + 2i\dot{\phi}z = -iv(1 + |z|^2), \\
\dot{\phi} = -\frac{\bar{v}\bar{z} + \bar{v}z + 2h}{2}
\end{cases}
\]

and finally we obtain

\[
\begin{align*}
\ddot{z} - i(\bar{v}z^2 + 2hz - v) &= 0, \\
\dot{\phi} &= -\frac{v\bar{z} + \bar{v}z + 2h}{2}.
\end{align*}
\]

The equation (9) is a Riccati equation and (10) is a phase equation. This is our canonical form of (8). We believe that our derivation is smarter than that of [1].

A comment is in order. Unfortunately, it is impossible to solve (9) exactly, so we need some numerical computational method.

### 3 Three Level System

In this section we treat the three level system \((n = 3)\) in detail, which is the aim of the paper. However, the calculation is not easy compared to the two level one.
The Hamiltonian is
\[
H = \begin{pmatrix}
  h_1(t) & \bar{v}_1(t) & \bar{v}_2(t) \\
  v_1(t) & h_2(t) & \bar{v}_3(t) \\
  v_2(t) & v_3(t) & h_3(t)
\end{pmatrix}; \quad h_1(t) + h_2(t) + h_3(t) = 0
\] (11)
from (1). From the fact
\[SU(3) \approx SU(3) \times U(1)^2,\]
\[U(t)\] can be parametrized as
\[
U(t) = U_1(Z(t))U_2(t)
\] (12)
where
\[
U_1(Z(t)) = \begin{pmatrix}
  1 & \frac{z(t) + \bar{y}(t)z(t)}{\Delta_1} & \frac{\bar{x}(t)z(t) - y(t)}{\Delta_2} \\
  x(t) & 1 - \frac{x(t)(\bar{x}(t) + \bar{y}(t)z(t))}{\Delta_1} & \frac{-\bar{z}(t)}{\Delta_2} \\
  y(t) & z(t) - \frac{y(t)(\bar{x}(t) + \bar{y}(t)z(t))}{\Delta_1} & \frac{1}{\Delta_2}
\end{pmatrix}
\] (13)
and
\[
\Delta_1 = 1 + |x(t)|^2 + |y(t)|^2, \quad \Delta_2 = 1 + |z(t)|^2 + |x(t)z(t) - y(t)|^2
\] (14)
and
\[
U_2(t) = \begin{pmatrix}
  e^{i\phi_1(t)} \\
  e^{i\phi_2(t)} \\
  e^{-i(\phi_1(t) + \phi_2(t))}
\end{pmatrix}
\] (15)
This form is convenient in the latter calculation, though it is a variant of that of [7], [8], [9].
Moreover, for the latter convenience we set
\[
V(t) = \begin{pmatrix}
  1 & \frac{z(t) + \bar{y}(t)z(t)}{\Delta_1} & \frac{\bar{x}(t)z(t) - y(t)}{\Delta_2} \\
  x(t) & 1 - \frac{x(t)(\bar{x}(t) + \bar{y}(t)z(t))}{\Delta_1} & \frac{-\bar{z}(t)}{\Delta_2} \\
  y(t) & z(t) - \frac{y(t)(\bar{x}(t) + \bar{y}(t)z(t))}{\Delta_1} & \frac{1}{\Delta_2}
\end{pmatrix}
\] (16)
\[
D_\Delta(t) = \begin{pmatrix}
  \frac{1}{\sqrt{\Delta_1}} & 0 & 0 \\
  0 & \sqrt{\frac{\Delta_1}{\Delta_2}} & 0 \\
  0 & 0 & \sqrt{\frac{\Delta_1}{\Delta_2}}
\end{pmatrix}
\] (17)
Then $U(t)$ in (12) can be written as

$$U(t) = V(t)D_{\Delta}(t)U_2(t).$$

In the following we omit $t$ from all variables ($x(t) \rightarrow x$, etc) for simplicity. From the equation (2) we have

$$i\left(\dot{V}D_{\Delta}U_2 + V\dot{D}_{\Delta}U_2 + VD_{\Delta}\dot{U}_2\right) = HVD_{\Delta}U_2$$

and

$$i\left(V + V\dot{D}_{\Delta}D^{-1}_{\Delta} + V\dot{U}_2U_2^{-1}\right) = HV$$

because $D_{\Delta}$ and $U_2$ are diagonal. This form is better as shown in the following calculation.

A comment is in order. From (18) we have a clear form

$$H = i\left(\dot{V}V^{-1} + V\dot{D}_{\Delta}D^{-1}_{\Delta}V^{-1} + V\dot{U}_2U_2^{-1}V^{-1}\right).$$

However, we don’t recommend readers to calculate the right hand side because the calculation becomes very complicated.

The calculation is truly complicated. The (11), (21) and (31)–components of the matrix equation (18) read

$$i\dot{x} - i\frac{\dot{\Delta}_1}{2\Delta_1}x - \dot{\phi}_1x = v_1 + h_2x + \bar{v}_3y,$$

$$i\dot{y} - i\frac{\dot{\Delta}_1}{2\Delta_1}y - \dot{\phi}_1y = v_2 + v_3x + h_3y,$$

$$-i\frac{\dot{\Delta}_1}{2\Delta_1} - \dot{\phi}_1 = h_1 + \bar{v}_1x + \bar{v}_2y.$$

From these equations we have

$$i\dot{x} = (v_1 + h_2x + \bar{v}_3y) - (h_1 + \bar{v}_1x + \bar{v}_2y)x$$

$$= v_1 + (h_2 - h_1)x - \bar{v}_1x^2 + \bar{v}_3y - \bar{v}_2xy$$

(19)

and

$$i\dot{y} = (v_2 + v_3x + h_3y) - (h_1 + \bar{v}_1x + \bar{v}_2y)y$$

$$= v_2 + (h_3 - h_1)y - \bar{v}_2y^2 + v_3x - \bar{v}_1xy$$

(20)
and
\[ \dot{\phi}_1 = -\frac{1}{2} \frac{i \hat{\Delta}_1}{\Delta_1} - (h_1 + \bar{v}_1 x + \bar{v}_2 y). \] (21)

Next, we must calculate the term
\[ i \hat{\Delta}_1 = i \frac{d}{dt} (1 + |x|^2 + |y|^2) = i(\dot{x}\bar{x} + \dot{\bar{x}}\bar{x} + \dot{y}\bar{y} + \dot{\bar{y}}\bar{y}). \]

Then from (19) and (20) it is not difficult to show
\[ i \hat{\Delta}_1 = \{ (h_1 + v_1 \bar{x} + v_2 \bar{y}) - (h_1 + \bar{v}_1 x + \bar{v}_2 y) \} \Delta_1, \]
so that we finally obtain
\[ \dot{\phi}_1 = -\frac{(h_1 + \bar{v}_1 x + \bar{v}_2 y) + (h_1 + \bar{v}_1 x + \bar{v}_2 y)}{2} \] (22)
from (21).

The (12), (22) and (32)–components of the matrix equation (18) read
\[ -i \frac{d}{dt} \left( \begin{array}{c} \bar{x} + \bar{y}z \\ \Delta_1 \end{array} \right) - i \frac{\bar{x} + \bar{y}z}{\Delta_1} \times \frac{1}{2} \left( \frac{\Delta_1}{\Delta_1} - \frac{\Delta_2}{\Delta_2} \right) - \frac{\bar{x} + \bar{y}z}{\Delta_1} (-\dot{\phi}_2) \]
\[ = -h_1 \frac{\bar{x} + \bar{y}z}{\Delta_1} + \bar{v}_1 \left\{ 1 - \frac{x(\bar{x} + \bar{y}z)}{\Delta_1} \right\} + \bar{v}_2 \left\{ z - \frac{y(\bar{x} + \bar{y}z)}{\Delta_1} \right\}, \]
\[ -i \bar{z} \frac{\bar{x} + \bar{y}z}{\Delta_1} - i \bar{z} \frac{d}{dt} \left( \begin{array}{c} \bar{x} + \bar{y}z \\ \Delta_1 \end{array} \right) + i \left\{ 1 - \frac{x(\bar{x} + \bar{y}z)}{\Delta_1} \right\} \times \frac{1}{2} \left( \frac{\Delta_1}{\Delta_1} - \frac{\Delta_2}{\Delta_2} \right) + \left\{ 1 - \frac{x(\bar{x} + \bar{y}z)}{\Delta_1} \right\} (-\dot{\phi}_2) \]
\[ = -v_1 \frac{\bar{x} + \bar{y}z}{\Delta_1} + h_2 \left\{ 1 - \frac{x(\bar{x} + \bar{y}z)}{\Delta_1} \right\} + \bar{v}_3 \left\{ z - \frac{y(\bar{x} + \bar{y}z)}{\Delta_1} \right\}, \]
\[ i \bar{z} - i \bar{y} \frac{\bar{x} + \bar{y}z}{\Delta_1} - i \bar{y} \frac{d}{dt} \left( \begin{array}{c} \bar{x} + \bar{y}z \\ \Delta_1 \end{array} \right) + i \left\{ z - \frac{y(\bar{x} + \bar{y}z)}{\Delta_1} \right\} \times \frac{1}{2} \left( \frac{\Delta_1}{\Delta_1} - \frac{\Delta_2}{\Delta_2} \right) + \left\{ z - \frac{y(\bar{x} + \bar{y}z)}{\Delta_1} \right\} (-\dot{\phi}_2) \]
\[ = -v_2 \frac{\bar{x} + \bar{y}z}{\Delta_1} + v_3 \left\{ 1 - \frac{x(\bar{x} + \bar{y}z)}{\Delta_1} \right\} + h_3 \left\{ z - \frac{y(\bar{x} + \bar{y}z)}{\Delta_1} \right\}. \]

From these equations it is not difficult to show
\[ i \bar{z} = v_3 + h_3 \bar{z} - (h_2 + \bar{v}_3 z)z = (xz - y)(\bar{v}_1 + \bar{v}_2 z) \]
\[ = v_3 + (h_3 - h_2)z - \bar{v}_3 z^2 + (xz - y)(\bar{v}_1 + \bar{v}_2 z) \] (23)
and
\[ \frac{i}{2} \left( \frac{\dot{\Delta}_1}{\Delta_1} - \frac{\dot{\Delta}_2}{\Delta_2} \right) - \dot{\phi}_2 = h_2 + \bar{v}_3 z - x(\bar{v}_1 + \bar{v}_2) \]
or
\[ \dot{\phi}_2 = \frac{1}{2} \left( i \frac{\dot{\Delta}_1}{\Delta_1} - i \frac{\dot{\Delta}_2}{\Delta_2} \right) - (h_2 + \bar{v}_3 z) + x(\bar{v}_1 + \bar{v}_2). \] (24)
Since
\[ \frac{i \dot{\Delta}_1}{\Delta_1} = (h_1 + v_1 \bar{x} + v_2 \bar{y}) - (h_1 + \bar{v}_1 x + \bar{v}_2 y) \]
the remaining one is to calculate $\dot{\Delta}_2/\Delta_2$. However, it is very troublesome.

From
\[ \frac{i \dot{\Delta}_2}{\Delta_2} = \frac{d}{dt} \left( 1 + |z|^2 + |xz - y|^2 \right) \]
\[ = i \left\{ \ddot{z} \bar{z} + \dot{z} \right\} \]
long calculation gives
\[ \frac{i \dot{\Delta}_2}{\Delta_2} = -\bar{v}_3 z + v_3 \bar{z} + \bar{v}_2 (xz - y) - v_2 (\bar{x} \bar{z} - \bar{y}) \]
by use of (19), (20) and (23). Therefore we finally obtain
\[ \dot{\phi}_2 = \frac{(-h_2 - \bar{v}_3 z + v_3 \bar{z} + v_1 \bar{x} + v_2 \bar{x} \bar{z})}{2} \] (25)
from (24).

Let us summarize the result. Our canonical form is
\[ i \dot{x} = v_1 + (h_2 - h_1) x - \bar{v}_1 x^2 + \bar{v}_3 y - \bar{v}_2 xy, \]
\[ i \dot{y} = v_2 + (h_3 - h_1) y - \bar{v}_2 y^2 + v_3 x - \bar{v}_1 xy, \]
\[ i \dot{z} = v_3 + (h_3 - h_2) z - \bar{v}_3 z^2 + (xz - y)(\bar{v}_1 + \bar{v}_2) \]
and
\[ \dot{\phi}_1 = \frac{-(h_1 + \bar{v}_1 x + \bar{v}_2 y) + (h_1 + v_1 \bar{x} + v_2 \bar{y})}{2}, \]
\[ \dot{\phi}_2 = \frac{(-h_2 - \bar{v}_3 z + v_3 \bar{z} + v_1 \bar{x} + v_2 \bar{x} \bar{z})}{2} \]
where \( h_1 + h_2 + h_3 = 0 \).

A comment is in order. We have a set of complex Riccati differential equations interacting with one another (7) and real phase equations. Unfortunately, to solve them exactly is impossible, so we must use some numerical computational method.

4 Discussion

In this paper we obtained a canonical form of the evolution operator of a time-dependent Hamiltonian in the three level system \((n = 3)\), namely three complex Riccati differential equations interacting with one another and two real phase equations. For the case of \( n \geq 4 \) it is very hard to calculate, so we will leave calculation to (young) readers.

The result looks good and may be applied to a wide class of problems in Quantum Physics or Mathematical Physics. Further work will be needed.

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\(^1\)the author expects the result to solve important problems in Quantum Computation
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