ARE BISEPARABLE EXTENSIONS FROBENIUS?

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Abstract. In Section 1 we describe what is known of the extent to which a separable extension of unital associative rings is a Frobenius extension. A problem of this kind is suggested by asking if three algebraic axioms for finite Jones index subfactors are dependent. In Section 2 the problem in the title is formulated in terms of separable bimodules. In Section 3 we specialize the problem to ring extensions, noting that a biseparable extension is a two-sided finitely generated projective, split, separable extension. Some reductions of the problem are discussed and solutions in special cases are provided. In Section 4 various examples are provided of projective separable extensions that are neither finitely generated nor Frobenius and which give obstructions to weakening the hypotheses of the question in the title. We show in Section 5 that characterizations of the separable extensions among the Frobenius extensions in \cite{HS, K, K99} are special cases of a result for adjoint functors.

1. Introduction

An old problem is the extent to which separable algebras are Frobenius algebras. By a Frobenius algebra we mean a finite dimensional algebra $A$ with a non-degenerate linear functional, which induces an $A$-module isomorphism $A \cong A^*$; symmetric algebra if this isomorphism is an $A$-bimodule map. Eilenberg and Nakayama observed in \cite{EN} that the (reduced) trace of a central simple algebra over a field is non-degenerate, which implies that a finite dimensional semisimple algebra is symmetric. Passing to a commutative ground ring $k$, Hattori \cite{H} and DeMeyer \cite{D} showed that a $k$-projective separable $k$-algebra $A$ is symmetric as well if the Hattori-Stallings rank of $A$ over its center $C$ is an invertible element in $C$. Endo and Watanabe extended this result to $k$-projective separable faithful $k$-algebras essentially by using the Auslander-Goldman Galois theory for commutative rings to define a more general notion of reduced trace \cite{EW}.

The main theorem in \cite{EW} led to several general results by Sugano \cite{S70} for when separable extensions \cite{HS} are Frobenius extensions \cite{K60}. These are noncommutative ring extensions and are natural objects for study from the point of view of induced representations \cite{Hoch}. Sugano shows that a centrally projective separable extension $R/S$ is Frobenius since it satisfies $R \cong S \otimes_{Z(S)} C_R(S)$ where the centralizer $C_R(S)$ is faithfully projective and separable over the center $Z(S)$, whence Frobenius. Somewhat similarly, it is shown that a split one-sided finite projective $H$-separable extension $R/S$ is Frobenius, since in this case the endomorphism ring $\text{End}(R_S) \cong R \otimes_{Z(R)} C_R(S)$ with $C_R(S)$ again separable, the result following

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from the endomorphism ring theorem as developed in [K60, Mu64, M67]: if \( RS \) is generator module, \( R/S \) is Frobenius if \( \text{End}(R_S)/R \) is Frobenius. However, it is implicit in the literature that there are several cautionary examples showing separable extensions are not always Frobenius extensions in the ordinary untwisted sense [K91]: in Section 4 we show that a non-finite ring leads to an example of split separable, two-sided projective extension which is not finitely generated, whence not Frobenius.

As an independent line of inquiry, algebraic axioms for finite Jones index subfactors have been investigated in [K95, K96, NK]. If we simplify the discussion somewhat, we may start with an irreducible subfactor \( M/N \) of finite index: from the Pimsner-Popa orthonormal base, the natural modules \( M_N \) and \( N \) are finite projective [JS], and the algebra extension \( M/N \) is (1) split, (2) separable, and (3) Frobenius. A ring extension \( R/S \) is said to be split if there is a bimodule projection \( E : R \to S \). At the same time, Axiom (2) yields a Casimir element \( e = \sum_i x_i \otimes y_i \) such that \( \sum_i x_i y_i = 1 \). A problem in the independence of the axioms above becomes whether Axioms (1) and (2) imply Axiom (3) in the presence of the assumption of two-sided finite projectivity of the ring extension: i.e., whether a bimodule map \( E : A \to S \) and Casimir element \( e \in A \otimes S A \) may be chosen such that \( \sum_i E(x_i)y_i = 1 = \sum_i x_i E(y_i) \). Equivalently, can a bimodule map \( E \) be found such that the \( E \)-multiplication on \( R \otimes S R \) is unital? That Axiom (3) in combination with (1) or (2) does not imply the other are easy examples disposed of in Section 3. Many other algebraic examples of split separable Frobenius can be found [K95] but none thus far that are finite projective split separable and not Frobenius.

In this paper, we will formulate the problem of independence of axioms for subfactors in several algebraic ways. In Section 2 we first formulate the problem using separable bimodules [S71], a theory in which separable extension and split extension become dual notions [K96, K99]. In analogy with “bialgebra,” we will baptise finite projective split separable extensions as biseparable extensions. Posed in the negative, our question then becomes if biseparable extensions are Frobenius. This question will be formulated in several other ways in Section 3, with one special case being answered in the affirmative. We point out here that the problem has many interesting sub-problems if restrictions are placed on the rings (e.g., “finite dimensional algebras,” “Hopf algebras,” etc.).

In Section 5 we discuss a type of converse to the considerations above. We find a common feature of the theorems in [HS, K, K99] on when a Frobenius extension or bimodule is separable: in each case, it is a specific example of a known theorem on adjoint functors, which we expose in this last section.

**Preliminaries.** A ring \( R \) will mean a unital associative ring. A ring homomorphism sends 1 into 1. A right module \( M_R \) or left module \( M_L \) is always unitary. Bimodules are associative with respect to the left and right actions.

An \( R-S \)-bimodule \( M \) is denoted by \( _RM_S \). Its right dual is defined by \( M^* := \text{Hom}(M_S, S_S) \), an \( S-R \)-bimodule where \( sf_r(m) := sf(rm) \). The left dual of \( M \) is \( ^*M := \text{Hom}(R_M, R_R) \) is also \( S-R \)-bimodule where \( (m)(sf_r) := [(ms)|f]r \). Both \( M \mapsto M^* \) and \( M \mapsto ^*M \) are contravariant functors of bimodule categories, sending \( _RM_S \to _SM_R \).

If \( R = S \) in the last paragraph, denote \( \hat{M} := \text{Hom}_{S-S}(M, S) \). Define the group of \( S \)-central or Casimir elements by \( M^S := \{ m \in M \mid ms = sm, \forall s \in S \} \). Note that \( (M^*)^S = (^*M)^S = \hat{M} \).
If $rM_S$, $rN_T$, $rQ_S$ and $sP_T$ are bimodules, then $M \otimes S P$ receives the natural $R$-$T$-bimodule structure indicated by $r(m \otimes n)t := rm \otimes nt$, and the group of right module homomorphisms $\text{Hom}(M_S, Q_S)$ receives the natural $T$-$R$-bimodule structure indicated by $(tfr)(m) := t(f rm))$. The group of left module homomorphisms $\text{Hom}(pM, pN)$ receives the natural $S$-$T$-bimodule structure indicated by $(m)(sft) = ((ms)f)t$. All bimodules arising from Hom and tensor in this paper are the natural ones unless otherwise indicated.

A ring extension $R/S$ is a ring homomorphism $S \rightarrowtail R$. A ring extension is an algebra if $S$ is commutative and $\iota$ factors into $S \rightarrow Z(R) \hookrightarrow R$ where $Z(R) := R^R$ is the center of $R$. A ring extension is proper if $\iota$ is 1-1, in which case identification is made.

The natural bimodule $sRs$ is given by $s \cdot a \cdot s' := \iota(s)a(s')$. In particular, we consider the natural modules $Rs$ and $Sr$. An adjective, such as right projective or projective, for the ring extension $S$ refers to the same adjective for one or both of these natural modules. The structure map $\iota$ is usually suppressed.

Separable extensions are studied in [HS, K97, K, RAFAEL, S70] among others. A ring extension $R/S$ is separable if the natural (multiplication) map $R \otimes_S R \rightarrow R$ is a split epimorphism of $R$-bimodules. Examples are abundant among finite dimensional algebras since a separable algebra is a separable extension of any of its subalgebras. The next proposition, whose proof follows Sugano [S82, Prop. 1], is important to keep in mind when finding examples of separable extensions from the class of finite dimensional algebras.

**Proposition 1.1.** If $0 \rightarrow J \rightarrow A \rightarrow[\pi] S \rightarrow 0$ is a split exact sequence of algebras and $A/S$ is a separable extension, then $J$ is an idempotent ideal (i.e., $J^2 = J$).

**Proof.** We assume with no loss of generality that $S \subseteq A$ and $\pi|_S = \text{id}_S$. Let $\sum_i x_i \otimes y_i$ be a separability element in $A \otimes_S A$. Let $e = \sum_i \pi(x_i)y_i$. Then $e$ satisfies $xe = \pi(x)e$ for all $x \in A$. Since $\pi(e) = 1$, it follows that $e$ is idempotent. Similarly, $f = \sum_i x_i\pi(y_i)$ satisfies $xf = f\pi(x), \pi(f) = 1$ and is idempotent. Then $e = \pi(f)e = ef = f$.

Then $xe = e\pi(x)e = ex$ and $e$ is central. Then $J = (1 - e)A$, whence $J$ is idempotent. \hfill $\square$

An example of a ring epi splitting in the next corollary would be the one implicit in the Wedderburn Principal Theorem for finite dimensional algebras.

**Corollary 1.2.** If $A/S$ is a split, separable extension with splitting map $\pi : A \rightarrow S$ a ring epimorphism with nilpotent kernel $J$, then $J = 0$ and $A = S$.

Indeed a separable finitely generated (f.g.) extension of a separable algebra is itself separable [HS]. The next proposition builds new separable extensions from old using multiplicative bimodules [K].

**Proposition 1.3.** Suppose $R/S$ is a ring extension and $I$ is a multiplicative $R$-bimodule. Then $R/I$ is a separable extension if and only if $A = R \otimes I$ is a separable extension of $T = S \otimes I$.

**Proof.** ($\Rightarrow$) Let $f = \sum_i x_i \otimes y_i \in R \otimes_S R$ be a separability element for $R/S$. Let $e$ be its image in $A \otimes_T A$ induced by $R \hookrightarrow A$. If $x \in I$, then:

$$xe = \sum_i 1 \otimes xx_i y_i = \sum_i x_i y_i x \otimes 1 = cx.$$
We easily conclude that $e$ is a separability element for $A/T$. ($\Leftarrow$) Since $I$ is an ideal in both $A$ and $T$, the canonical epimorphism $\pi : A \to R$ sends the separable extension $A/T$ onto a separable extension $R/S$.

As our final preliminary topic we recall Frobenius and QF extensions. A ring extension $R/S$ is Frobenius if $R_S$ is f.g. projective and $R \cong R^*$ as $S$-$R$-bimodules: note that this extends the notion of Frobenius algebra. We recall also the Morita characterization of Frobenius extensions [M65]: an extension $R/S$ is Frobenius iff induction and co-induction (of $S$-modules to $R$-modules) are naturally isomorphic (cf. [K]).

A ring extension $R/S$ is a left Quasi-Frobenius (QF) extension if $S_R$ is finitely generated projective and $R_S$ is isomorphic to a direct summand of a finite direct product of $S_R^*$ with itself. Equivalently, $R_S$ and $S_R$ are finitely generated projectives and $S_R^*$ is a direct summand of a finite direct sum of copies of $S_R$. We similarly define right QF extensions [Mu64, Mu65]. There is no published example of a right QF extension that is not left QF.

2. Biseparable Bimodules

In this section we pose our question in the more general terms of bimodules rather than ring extensions. There are two reasons for this. First, the problem has a more attractive symmetrical formulation in terms of bimodules. Second, Morita has shown in [M67] how to generate interesting examples of ring extensions from bimodules via the endomorphism ring.

Let $R$ and $T$ be rings. Given a bimodule $T*R$, there is a natural $T$-bimodule homomorphism,

$$\mu_M : M \otimes_R^*M \to T, \ m \otimes f \mapsto (m)f.$$  

We next recall the definition of a separable bimodule [S71].

**Definition 2.1.** $M$ is separable, or $T$ is $M$-separable over $R$, if $\mu_M$ is a split $T$-epimorphism.

It follows trivially that $M_R$ is a generator module [AF]. By applying a splitting map to $1_T$, we note that $M$ is separable iff there is an element

$$e = \sum_i m_i \otimes f_i \in M \otimes_R^*M,$$

called an $M$-separability element, which satisfies $\mu_M(e) = 1_T$ and $te = et$ for all $t \in T$. As is the case with separability elements and idempotents [D, DMI], $M$-separability elements are usually not unique.

Retaining this notation, we recall a useful proposition and its proof [S71, Proposition 3]. But first a lemma which does not require $M$ to be separable.

**Lemma 2.2.** If $M_R$ is finitely generated projective, then $\alpha_M : M \otimes_R^*M \to \text{Hom}(R^*M, R^*M)$ given by

$$m \otimes f \mapsto (g \mapsto g(m)f)$$  

is a $T$-bimodule isomorphism. Similarly, $TM$ f.g. projective implies that

$$\eta_M : M^* \otimes_T M \to \text{Hom}(M^*T, M^*_T), \ f \otimes m \mapsto (h \mapsto f[(m)h])$$

is an $R$-bimodule isomorphism.
The proof of this and the next two propositions are left to the reader. Now define \( \beta_M : \text{Hom}(R^*, R^*) \to T \) by
\[
\beta_M(G) = \sum_i (x_i)(f_i)G.
\]

**Proposition 2.3.** We have \( \mu_M = \beta_M \circ \alpha_M \); i.e., the diagram below is commutative. Whence \( M \) is separable iff there is an \( R \)-\( T \)-bimodule homomorphism \( \gamma_M : M^* \to *M \) such that \( (x_i)(f_i)\gamma_M = 1T \).

\[
\begin{array}{ccc}
M \otimes_R *M & \xrightarrow{\mu_M} & \text{Hom}(R^*, R^*) \\
& \downarrow \alpha_M \downarrow & \\
& T & \\
& \uparrow \beta_M \uparrow & \\
& R & 
\end{array}
\]

\( \gamma_M \) is an \( M \)-separability element.

**Definition 2.4.** A bimodule \( \tau_M \) is said to be biseparable if \( M \) and \( M^* \) are separable and \( \tau_M, \tau_M^* \) are finite projective modules.

We derive some consequences of assuming \( \tau_M \) f.g. projective and \( M^* \) separable. First, since \( M^* \) is reflexive, it follows that \( *M \cong M \) via the “evaluation mapping” from \( M \) to \( \text{Hom}(M^*, R) \). It follows that \( \mu_M : M^* \otimes T \to R \) under this identification is the evaluation map given by \( f \otimes m \mapsto f(m) \). Let \( \{g_i\} \subset *M, \{y_j\} \subset M \) be finite dual bases for the f.g. projective module \( \tau_M \). We have the following easy analog of Proposition 2.3.

**Proposition 2.5.** The triangle below is commutative. Whence if \( M^* \) is separable there is a \( R-T \)-bimodule homomorphism \( \rho_M : *M \to M^* \) such that \( \sum_j \rho_M(g_i)(y_j) = 1R \).

\[
\begin{array}{ccc}
M^* \otimes_T M & \xrightarrow{\mu_M} & \text{Hom}(M^*, M^*_T) \\
& \downarrow \eta_M \downarrow & \\
& R & \\
& \uparrow \mu_M \uparrow & \\
& \tau_M & 
\end{array}
\]

The downward map to the right is given by \( G \mapsto \sum_j G(g_i)(y_j) \).

For example, a bimodule \( \tau_M \) yielding a Morita equivalence of \( T \) and \( R \) is biseparable, since \( *M \cong M^* \) as \( R-T \)-bimodules \( [M67] \) while \( \mu_M \) and \( \mu_M^* \) are isomorphisms. There have been various studies of properties shared by rings \( R \) and \( T \) related by a bimodule \( \tau_M \) in a Morita context and generalizations of this \( [AR, C, M, M65] \). A precursor of these studies is the theorem of D.G. Higman \( [H] \) that a finite group has finite representation type (f.r.t.) in characteristic \( p \) iff its Sylow \( p \)-subgroup is cyclic, which later became a corollary of the theorem of J.P. Jans \( [J] \) that for Artinian algebras \( R \subseteq T \) in a split separable extension, \( R \) has f.r.t. iff \( T \) has f.r.t. (cf. \( [P] \)). It is in this spirit that the next theorem offers a sample of shared properties of \( R \) and \( T \) linked by a biseparable bimodule \( \tau_M \).
Theorem 2.6. If $T M_R$ is biseparable, then
1. (Cf. [ST1], Prop. 5, Theorem 2) $T$ is a QF ring if and only if $R$ is a QF ring;
2. (Cf. Ibid.) $T$ is semisimple if and only if $R$ is semisimple;
3. weak global dimension $D(R) = D(T)$.

Proof. (QF). Assume $T$ is QF and $P_R$ is injective. By the Faith-Walker theorem, it will suffice to show that $P_R$ is projective. Since $T M$ is projective, then flat, we note that $H_T := \text{Hom}_R(M, P)$ is injective, then projective. Since $M_R$ is projective, we note that $H \otimes T M_R$ is projective. But the evaluation mapping

$$ev : H \otimes T M_R \to P_R, \ f \otimes m \mapsto f(m)$$

is a split epi, for if $\sum_j f_j \otimes m_j$ is an $M^*$-separability element, then $p \mapsto pf_j \otimes m_j$ defines a splitting $R$-monic, where $pf_j : m \mapsto pf_j(m)$. Hence $P_R$ is isomorphic to a direct summand in $H \otimes T M_R$ and is projective.

Assuming that $R$ is QF, and $T Q$ is injective, we argue similarly that $R H' := \text{Hom}_T(M, Q)$ is injective-projective and that $Q$ is isomorphic to direct summand in the projective $T$-module $M \otimes_R H'$.

(SEMISIMPLICITY). Suppose $T$ is semisimple and $P_R$ is a module. It suffices to note that $P_R$ is projective. Since $H_T := \text{Hom}_R(M, P)$ is projective and the map $ev$ defined as above is a split $R$-epimorphism, it follows that $P_R$ is isomorphic to a direct summand of the projective module $H \otimes T M_R$. Similarly, we argue that given $R$ semisimple and module $T Q$, $T Q$ is projective.

(WEAK GLOBAL DIMENSION.) If $X_1 \to N_T$ is a projective resolution of $N_T$, then $X_1 \otimes T M_R \to N \otimes T M_R$ is a projective resolution as well since $T M$ is flat and $M_R$ is projective. Recall that $\text{Tor}^R_n(N, Q)$ is the $n$'th homology group of the chain complex $X_1 \otimes T Q$ for each non-negative integer $n$, so $\text{Tor}^R_n(N \otimes T M, * M \otimes T Q)$ is the $n$'th homology group of $X_1 \otimes T M \otimes_R * M \otimes T Q$. At the level of chain complex, there is a split epi

$$X_n \otimes T M \otimes_R * M \otimes T Q \twoheadrightarrow X_n \otimes T Q, \ x \otimes m \otimes g \otimes q \mapsto x(mg) \otimes q$$

which implies that $\text{Tor}^R_n(N, Q)$ is isomorphic to a direct summand in $\text{Tor}^R_n(N \otimes T M, * M \otimes T Q)$ for each $n$. This shows that $D(R) \geq D(T)$. A similar argument with left modules shows that $D(T) \geq D(R)$.

We remark that in trying to prove other shared homological properties of biseparable $T M_R$, particularly one-sided notions, one may run into the following complications: although the modules $*M_T$ and $R M^*$ are (quite easily seen to be) f.g. projective, one should avoid assuming the same of $\gamma_M M$ and $M_T^*$.

We next recall the definition of Frobenius bimodule $\text{AF}$, $\text{K}$.

Definition 2.7. A bimodule $T M_R$ is Frobenius if $M_R$, $T M$ are f.g. projective and $* M \cong M^*$ as $R-T$-bimodules.

Based on the many examples in [K93] and elsewhere, we propose the following problem, which turns out to be almost equivalent to the ring extension formulation in the title:

Problem 2.8. Is a biseparable bimodule $T M_R$ a Frobenius bimodule?

For example, can we choose $\gamma_M$ and $\rho_M$ such that they are inverses to one another? The problem above subsumes many interesting questions in various restricted cases. For example, what can be said for the problem above if $T$ and $R$
are finite dimensional algebras? There is an affirmative answer in the next section if one algebra is separable.

A generalization of Frobenius bimodule is a twisted Frobenius bimodule \(\alpha M \beta\) where \(\alpha : T \to T\) and \(\beta : R \to R\) are ring automorphisms, and the bimodule structure is now given by \(t \cdot m \cdot r := \alpha(t)m\beta(r)\) (for the definition see [K99]). We might ask more widely

**Problem 2.9.** Is a biseparable bimodule a twisted Frobenius bimodule?

However, this problem is the same as the previous one if the following question has an affirmative answer:

**Problem 2.10.** If a twisted Frobenius bimodule \(\alpha M \beta\) is biseparable, does this imply that \(\alpha M \beta \cong TM\) ?

We say that a twisted Frobenius bimodule is nontrivial if it not isomorphic to an untwisted Frobenius bimodule; for a \(\beta\)-Frobenius extension \(R/S\) non triviality means that \(\beta : S \to S\) is not an extended inner automorphism in the sense that there is a unit \(u \in R\) such that \(\beta\) is conjugation by \(u\) [NT]. We pose the last question since we have never observed a nontrivial \(\beta\)-Frobenius extension (e.g. in [FMS, NT]) which was simultaneously split and separable (cf. next section). In this more limited setting, which covers Hopf subalgebras of finite dimensional Hopf algebras, the question becomes:

**Problem 2.11.** If a \(\beta\)-Frobenius extension is split and separable, is \(\beta\) an extended inner automorphism?

We will return to a discussion of this problem in the next section.

3. Biseparable Extensions

Suppose \(R/S\) is a ring extension. Letting \(M = RS\) in the definition of separable bimodule, we observe the following lemma [S71].

**Lemma 3.1.** \(R/S\) is a separable extension iff \(RS\) is a separable bimodule.

Dually, we let \(M = ^*(RS) \cong SR\) and observe the following lemma [K96].

**Lemma 3.2.** \(R/S\) is a split extension iff \(SR\) is a separable bimodule iff \(^*(RS)^*\) is a separable bimodule.

The \(M\)-separability element in this case is a bimodule projection \(E : SR \to SS\), which implies \(R/S\) is a proper extension. \(E\) is also called a conditional expectation if it satisfies additional properties in subfactor theory. From the last two lemmas, it follows that:

**Lemma 3.3.** \(R/S\) is a split, separable, two-sided finite projective extension iff \(RS\) and \(SR\) are biseparable bimodules iff \(RS\) is biseparable and \(SR\) is f.g. projective iff \(SR\) is biseparable and \(RS\) is f.g. projective.

We call \(R/S\) a biseparable extension if any of the equivalent conditions in Lemma 3.3 are satisfied. Additionally, we have the following lemma [K].

**Lemma 3.4.** \(R/S\) is a Frobenius extension iff \(RS\) or \(SR\) is a Frobenius bimodule.

The last two lemmas lead to the seemingly restricted formulation of Problem 2.8 also the title of this article.
Problem 3.5. Are biseparable extensions Frobenius?

Surprisingly, this problem is almost equivalent to Problem 2.8 because of the endomorphism ring theorems for Frobenius bimodules [K, Chapter 2]. Suppose we knew an affirmative answer to the somewhat weaker problem where b iseparable extension includes left f.g. projective, split separable extensions. Given a biseparable bimodule $T_{MR}$, we know from Sugano that $E = \text{End}(M_R)$ is a left f.g. projective, split, separable extension of $T$ (whose elements are identified in $E$ with left multiplication operators) [S71, Theorem 1, Prop. 2] (cf. [K99, Theorem 3.1]). Then $E/T$ is a Frobenius extension by our affirmative answer to the weak Problem 3.5. Since $E/M_R$ is faithfully balanced by Morita’s Lemma, it follows from [M67, Theorem 1.1] that $E/M_R \cong M^*$ and then from the endomorphism ring theorem-converse [K, Theorem 2.8] that $T_{MR}$ is a Frobenius bimodule.

What evidence do we have then for proposing Problem 3.5? First, if $R/S$ is an $S$-algebra, we are in the situation of a faithfully projective separable algebra, which is Frobenius by the Endo-Watanabe Theorem [EW] discussed in the introduction. This implies by elementary considerations that $k$-algebra extensions of the form $R \otimes_k A$ over $R$ are Frobenius if $A$ is faithfully projective $k$-separable.

Second, there are the many examples of split, separable, Frobenius extensions [K95, K96] and apparently none that contradict in the literature for noncommutative rings and ring extensions [JS, L, M67, NT]. We recall from [K95, K96] that some of the examples of $R/S$ split, separable Frobenius are the following:

1. Let $R = E$ a field of characteristic $p$ (zero or prime) and $S = F$ a subfield such that $E/F$ is a finite separable extension where $p$ does not divide $[E : F]$. The classical trace map from $E$ into $F$ is a Frobenius homomorphism in this example.

2. Let $R$ be the group algebra $k[G]$ for a discrete group $G$, $k$ a field, and $S = k[H]$, where $H$ is a subgroup of $G$ with finite index not divisible by the characteristic of $k$ (e.g., char $k = p$ and $H$ a Sylow $p$-subgroup). Note that if the characteristic of $k$ divides $[G : H]$ we have an example of a split Frobenius extension which is not separable. By [S71] the endomorphism ring extension $\text{End}(R_S)/R$ is a separable Frobenius extension which is not split.

3. Let $R$ and $S$ be algebras over a commutative ring $k$ such that $R/S$ is an Hopf-Galois extension over the f.g. projective Hopf $k$-algebra $H$ which is separable and coseparable over $k$.

4. Let $R$ be a type $II_1$ factor, $S$ a subfactor of $R$ of finite Jones index, as discussed in the introduction.

Third, Sugano’s result [S71] for when H-separable extensions are Frobenius is evidence for biseparable implies Frobenius. This is because an H-separable extension is a strong type of separable extension [I]: see Section 4 for a separable extension which is not H-separable. Thus the result that a (one-sided) f.g. projective split H-separable extension is a (symmetric) Frobenius extension is a particular case of a biseparable extension which is Frobenius (cf. [K, Section 2.6] [I]).

The next proposition shows that a biseparable extension is almost a two-sided QF extension in a certain sense. If a module $M_R$ is isomorphic to a direct summand in another module $P_R$, we denote this by $M_R \leq_P P_R$.

\[2\text{[K, Theorem 2.25]}\] should read “Suppose $A/S$ is a right progenerator split H-separable...” as the proof clearly shows.
Proposition 3.6. Suppose \( R/S \) is a biseparable extension. Then all \( R \)-modules are \( S \)-relative injective and \( S \)-relative projective; moreover, \( R^*_R \) and \( ^*R_R \) are generator modules.

Proof. The first statement follows from the fact that a separable extension is both right and left semisimple extension and properties of these [HS].

For the second statement, we first establish an interesting isomorphism below involving \( R \) and its dual \( R^* \). On the one hand, since \( R/S \) is a separable extension, \( \mathcal{E} = \text{Hom}(R_S, R_S) \) is a split extension of \( R \), for if \( \sum_i x_i \otimes y_i \) is a separability element we define a bimodule projection by \( \mathcal{E} \to R \) by \( f \mapsto \sum_i f(x_i)y_i \) (cf. [Mu65]). Then as \( R \)-bimodules, \( \mathcal{E} \cong R \oplus M \) for some \( M \): moreover, by restriction this is true as \( S-R \)-bimodules. On the other hand, since \( R/S \) is split, it follows that for some \( S \)-bimodule \( N \), which is left and right projective \( S \)-module, \( R \cong S \oplus N \) as \( S \)-bimodules; whence

\[
\mathcal{E} \cong R^* \oplus \text{Hom}(R_S, N_S)
\]
as \( S-R \)-bimodules. Putting together the two isomorphisms for \( \mathcal{E} \), we obtain

\[
(1) \quad R \oplus M \cong \text{Hom}(R_S, S_S) \oplus \text{Hom}(R_S, N_S).
\]

Since \( N_S \) is f.g. projective, there is a module \( P_S \) such that \( N_S \oplus P_S \cong S_S^* \). Then applying \( \text{Hom}(S_R, -) \) to this:

\[
\text{Hom}(R_S, N_S) \oplus \text{Hom}(R_S, P_S) \cong \text{Hom}(R_S, S_S)^*.
\]

Combining this with Eq. (1), we obtain

\[
(2) \quad R_R \oplus M_R \oplus \text{Hom}(R_S, P_S) \cong R_r^{*r+1}.
\]

This establishes that \( R_R \cong R_r^{*r+1} \).

We similarly conclude \( rR \cong R_r^{*r+1} \) by combining the split extension \( \mathcal{E}' := \text{Hom}(S, S) \otimes R \) with the \( S \)-bimodule isomorphism \( R \cong S \oplus N \) and the existence of \( sQ \) such that \( sQ \otimes S N \cong sS^t \).

¿From the proof just completed, we obtain a corollary worth noting for its relatively easy proof. \( A_M A \) is said to be centrally projective if \( A_M A \cong A_A^n \) for some positive integer \( n \).

Corollary 3.7. (Cf. [S70], Theorem 2) If \( R/S \) is centrally projective biseparable extension, then it is a QF extension.

Proof. Since there is a bimodule \( sP_S' \) such that \( sR_S \oplus sP_S' \cong sS_S^* \), we combine this with \( sR_S \cong SS_S \oplus S N_S \) to see that Eq. (1) is an \( S-R \)-isomorphism, whence \( R/S \) is a right QF extension [Mu64]. Similarly, we show \( R/S \) to be a left QF extension. 

The proposition and corollary above lead naturally to the problem below, a weakening of Problem 3.3.

Problem 3.8. Are biseparable extensions QF?

The next theorem provides a solution of Problem 3.3 in case \( A \) or \( S \) is a separable algebra. We assume our algebras to be faithful.

\[\text{In terms of finite depth for subfactors [PS] and ring extensions [NK, SzK], the corollary states that a depth one biseparable extension is QF. A depth two biseparable extension is QF as well [SzK], where depth two is the condition that } R \otimes_S R \oplus * \cong \oplus^n R \text{ as } S-R \text{ and } R-S \text{-bimodules.}\]
**Theorem 3.9.** Suppose $A/S$ is a biseparable extension of $k$-algebras with $k$ a commutative ring. If either $S$ is a $k$-projective separable $k$-algebra, or $A$ is a separable $k$-algebra with $k$ a field, then $A/S$ is a Frobenius extension.

**Proof.** The proof does not make use of $S$ being f.g. projective. Suppose $k$ is a field and $A$ is $k$-separable. Let $D(R)$ denote the right global dimension of a ring $R$ and $d(M)$ denote the projective dimension of a module $M$. Then $D(A) = 0$ since $A$ is finite dimensional semisimple. By Cohen’s Theorem for split extensions [Kap],

$$D(S) \leq D(A) + d(A_S)$$

whence $D(S) = 0$ and $S$ is semisimple [Kap]. Then $A$ and $S$ are finite dimensional semisimple algebras. It follows from [EN] that $S$ and $A$ are symmetric algebras.

Similarly, we arrive at symmetric algebras $S$ and $A$ via [EW] under the assumption that $S$ is $k$-projectively $k$-separable with no restriction on $k$. For then $A$ is $k$-projective and $k$-separable by transitivity for projectivity and separability.

Now we compute using the bimodule isomorphisms $A \cong A^*$ and $S \cong S^*$ and the hom-tensor adjunction:

$$S_A A \cong S\text{Hom}(A, k)_A$$

$$\cong \text{Hom}_k(A \otimes S, k)_A$$

$$\cong S\text{Hom}_S(A, \text{Hom}_k(S, k))_A$$

$$\cong S\text{Hom}_S(A, S)_A.$$ 

Then, since $A_S$ is f.g. projective, $A/S$ is a Frobenius extension. $\square$

Part of the theorem is true without the hypothesis of biseparable extension for a finite-dimensional Hopf subalgebra pair $H \supseteq K$: if $H$ is semisimple, then $K$ is semisimple [M, 2.2.2], and $H/K$ is a Frobenius extension (cf. [FMS, 1.8]).

Finally, Problem (3.8) can be widened to twisted extensions, as Problem (2.8) was widened to twisted Frobenius bimodules in Problem (2.9).

**Problem 3.10.** Are biseparable extensions $\alpha$-$\beta$-Frobenius?

We refer the reader to [M65, K99] for the definition of these twisted extensions, which are more general than the usual $\beta$-Frobenius extensions. Also, Problem (3.8) has a twisted enlargement.

### 4. Examples and Counterexamples

In this section, we consider weakening the definition of biseparability in various ways, and find examples of non-Frobenius extension for each such case. We will see an example of non-finitely generated projective separable extension, which is an obstruction to extending Villamayor’s theorem [F, Prop. 10.3] and Tominaga’s theorem [T].

**Lemma 4.1.** Suppose $k$ is a commutative ring and $R$ is a $k$-algebra with $xy = 1$ but $yx \neq 1$. Then $R$ is a separable extension over $S = k1 + yRx$.

**Proof.** We note that $e = yx$ is a nontrivial idempotent in $S$. Consider $f = x \otimes y \in R \otimes_S R$. Of course, $\mu(f) = 1$. We compute with $r \in R$:

$$rx \otimes y = xyrx \otimes y = x \otimes yrxy = x \otimes yr.$$ $\square$
Now if $R$ is a finitely generated, projective $k$-algebra, it is well-known that $xy = 1$ implies $yx = 1$. So we let $V$ be a countably infinite rank free $k$-module and

$$R = \text{End}_k(V) = \{ \begin{pmatrix} a & b \\ c^t & D \end{pmatrix} | a \in k, D \in M_\infty(k), b \in M_1 \times \infty(k) \},$$

where $c^t$ and $D$ are column-finite and $X \mapsto X^t$ denotes transpose. $R$ is a ring satisfying the hypothesis in the lemma with elements $x, y \in R$ given in terms of the matrix units $e_{i,j}$ by

$$x = \sum_{n=1}^{\infty} e_{n,n+1}, \quad y = \sum_{n=1}^{\infty} e_{n+1,n}.$$ 

Clearly, $xy = 1$ but $e = yx = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$ and the $k$-subalgebra,

$$S = k1_R + yRx = \{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} | a \in k, D \in M_\infty(k) \}$$

**Proposition 4.2.** $R/S$ is a split, separable, projective and non-finitely generated extension.

**Proof.** We have seen in the lemma that $R/S$ is separable. It is split since we easily check that $E : R \to S$ below is a bimodule projection:

$$E \begin{pmatrix} a & b \\ c^t & D \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & D \end{pmatrix}$$

$R_S$ is countably generated projective, since $f_n : R_S \to S_S$ defined by

$$f_1 \begin{pmatrix} a & b \\ c^t & D \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & (b/0) \end{pmatrix} \quad f_2 \begin{pmatrix} a & b \\ c^t & D \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$$

and $f_{2+n} \begin{pmatrix} a & b \\ c^t & D \end{pmatrix} = \begin{pmatrix} c_n & 0 \\ 0 & 0 \end{pmatrix}$, where $c = (c_1, c_2, \ldots)$, which satisfies the dual bases equation,

$$A = \begin{pmatrix} a & b \\ c^t & D \end{pmatrix} = \begin{pmatrix} 1 & (0/0) \\ 0 & 0 \end{pmatrix} f_1(A) + \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} f_2(A) + \sum_{n=1}^{\infty} e_{n+1,1}f_{2+n}(A).$$

By tranposing the elements and maps above, we similarly find a countable projective base for $S_R$. The rest of the proof is now clear.

**Corollary 4.3.** $R/S$ is not Frobenius and not H-separable.

**Proof.** For $R/S$ to be a Frobenius extension, we must have $R_S$ finitely generated from the very start. A right projective H-separable extension is right f.g. by [1].

Finally, we consider various weakenings of Problem 3.5 and note that they all have known counterexamples. There are easy examples of separable extensions which are not Frobenius, such as the rationals $Q$ extending the integers $Z$. There are even f.g. free separable extensions that are not Frobenius in the ordinary sense, but are $\beta$-Frobenius [K99]. As for a f.g. split, separable extension that is not Frobenius, here is one that is not flat, whence not projective: consider $Z \oplus Z_2$ as a
$\mathcal{Z}$-algebra, which has projection onto its first factor as splitting bimodule projection, is a direct sum of separable algebras – whence separable – and its second factor is of course not flat over $\mathcal{Z}$.

The following example of a f.g. projective H-separable (therefore separable \cite{H1}) extension $A/S$ which is not even twisted QF is certainly worth a mention: let $A$ be the $n$ by $n$ matrices over a field with $S$ the upper triangular matrices (including diagonal matrices). Although this example is not Frobenius in any sense that we have mentioned in this paper (left as an exercise to the reader), we note that $\text{End}A_S \cong A \otimes S A(\cong A)$. However, there is no published example of a one-sided progenerator H-separable extension which is not Frobenius.

There are clearly many examples of split extensions that are not Frobenius, let alone f.g. Asking for a split, f.g. projective extension that fails to be Frobenius is not hard: for example, let $R$ be the upper triangular $n \times n$ matrix algebra with splitting $S \oplus I$ where $S$ is the subalgebra of diagonal matrices and $I^n = 0$. It is well-known that $R$ is not a QF-algebra, and certainly not Frobenius \cite{H1}, but $S$ is semisimple, so $R_S$ and $_S R$ are f.g. projective; moreover, $R/S$ cannot be Frobenius by the transitivity property of Frobenius extension (cf. \cite{Sk2}). This is an example too of $\text{Hom}(R_S, S_S)$ not being a projective right $R$-module.

As a last cautionary example, we consider $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $S = \mathbb{Z}_2$. It is easy to check that $R/S$ is split, separable and Frobenius, even f.g. free. But there are only two bimodule projections $E : R \rightarrow S$, neither of which is a Frobenius homomorphism, i.e. in possession of dual bases \cite{K}. The Frobenius homomorphism in this example is unique, since the group of units in $C_R(S)$ consists only of the identity \cite{K}.

5. Categorical interpretation

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a contravariant functor. $F$ induces a natural transformation $\mathcal{F} : \text{Hom}_C(\bullet, \bullet) \rightarrow \text{Hom}_D(F(\bullet), F(\bullet))$, $\mathcal{F}_{C,C'}(f) = F(f)$. $F$ is called a separable functor \cite{NVO} if $\mathcal{F}$ splits, i.e. there exists a natural transformation $\mathcal{P} : \text{Hom}_D(F(\bullet), F(\bullet)) \rightarrow \text{Hom}_C(\bullet, \bullet)$ such that $\mathcal{P} \circ \mathcal{F}$ is the identity natural transformation on $\text{Hom}_C(\bullet, \bullet)$.

**Proposition 5.1.** \cite{RAFAEL} Assume that $F$ has a right adjoint $G$, and let $\eta : 1_\mathcal{C} \rightarrow GF$ and $\varepsilon : FG \rightarrow 1_\mathcal{D}$ be the unit and counit of the adjunction.

$F$ is separable if and only if there exists a natural transformation $\nu : GF \rightarrow 1_\mathcal{C}$ such that $\nu \circ \eta$ is the identity natural transformation on $\mathcal{C}$.

$G$ is separable if and only if there exists a natural transformation $\zeta : 1_\mathcal{D} \rightarrow FG$ such that $\varepsilon \circ \zeta$ is the identity natural transformation on $\mathcal{D}$.

The terminology stems from the fact that, for a ring homomorphism $i : R \rightarrow S$, the restriction of scalars functor is separable if and only if $S/R$ is separable (see \cite{NVO}, \cite{RAFAEL}); Separable functors satisfy the following version of Maschke’s Theorem: if a morphism $f$ in $\mathcal{C}$ is such that $F(f)$ has a left or right inverse in $\mathcal{D}$, then $f$ has a left or right inverse in $\mathcal{C}$. Separable functors have been studied in several particular cases recently, see e.g. \cite{CGN97}, \cite{CGN98}, \cite{CIMZ98}.

The functor $F$ is called Frobenius if $F$ has a right adjoint $G$ that is at the same time a right adjoint. We will then say that $(F,G)$ is a Frobenius pair. Now the terminology is inspired by the property that a ring homomorphism $i : R \rightarrow S$ is Frobenius if and only if the restriction of scalars functor is Frobenius. Frobenius pairs were introduced in \cite{M65}, and studied more recently in \cite{CMZ97} and \cite{CGN99}.
For more details and examples of separable functors and Frobenius functors, we refer the reader to [CMZ].

Suppose we know that \((F, G)\) is a Frobenius pair. Then Rafael’s Theorem can be simplified: we can give an easier criterion for \(F\) or \(G\) to be separable. First we need a result on adjoint functors. Let \(\varepsilon\) and \(\eta\) be the counit and unit of an adjunction \((F, G)\), and recall that
\[
\varepsilon_{F(C)} \circ F(\eta_C) = I_{F(C)} \quad \text{and} \quad G(\varepsilon_D) \circ \eta_{G(D)} = I_{G(D)}
\]
for all \(C \in \mathcal{C}\) and \(D \in \mathcal{D}\).

**Lemma 5.2.** Let \((F, G)\) be an adjoint pair functors, then we have isomorphisms
\[
\text{Nat}(F, F) \cong \text{Nat}(G, G) \cong \text{Nat}(1_C, GF) \cong \text{Nat}(FG, 1_D)
\]

**Proof.** In fact this is an easy consequence of [Pa, Section 1.15, Lemma 3 and Section 2.1, Corollary 1 and Lemma]. We restrict to giving a description of the isomorphism \(\text{Nat}(G, G) \cong \text{Nat}(1_C, GF)\), the other isomorphisms are similar. For a natural transformation \(\theta : 1_C \rightarrow GF\), we define \(\alpha = X(\theta) : G \rightarrow G\) by
\[
\alpha_D = G(\varepsilon_D) \circ \theta_{G(D)}
\]
Conversely, for \(\alpha : G \rightarrow G\), \(\theta = X^{-1}(\alpha) : 1_C \rightarrow GF\) is defined by
\[
\theta_C = \alpha_{F(C)} \circ \eta_C
\]
The fact that \(X\) and \(X^{-1}\) are each others inverses is proved using (3). \(\square\)

**Corollary 5.3.** Let \((F, G)\) be a Frobenius pair of functors, then we have isomorphisms
\[
\text{Nat}(F, F) \cong \text{Nat}(G, G) \cong \text{Nat}(1_C, GF) \cong \text{Nat}(FG, 1_D) \\
\cong \text{Nat}(GF, 1_C) \cong \text{Nat}(1_D, FG)
\]

For a Frobenius pair \((F, G)\), we will write \(\nu : GF \rightarrow 1_C\) and \(\zeta : 1_D \rightarrow FG\) for the counit and unit of the adjunction \((G, F)\). For all \(C \in \mathcal{C}\) and \(D \in \mathcal{D}\), we then have
\[
F(\nu_C) \circ \zeta_{F(C)} = I_{F(C)} \quad \text{and} \quad \nu_{G(D)} \circ G(\zeta_D) = I_{G(D)}
\]

**Proposition 5.4.** Let \((F, G)\) be a Frobenius pair, and let \(\eta, \varepsilon, \nu\) and \(\zeta\) be as above. The following statements are equivalent:

- \(F\) is separable;
- \(\exists \alpha \in \text{Nat}(F, F) : \nu_C \circ G(\alpha_C) \circ \eta_C = I_C\) for all \(C \in \mathcal{C}\);
- \(\exists \beta \in \text{Nat}(G, G) : \nu_C \circ \beta_{F(C)} \circ \eta_C = I_C\) for all \(C \in \mathcal{C}\).

We have a similar characterization for the separability of \(G\): the statements

- \(G\) is separable;
- \(\exists \alpha \in \text{Nat}(F, F) : \varepsilon_D \circ \alpha_{G(D)} \circ \zeta_D = I_D\) for all \(D \in \mathcal{D}\);
- \(\exists \beta \in \text{Nat}(G, G) : \varepsilon_D \circ F(\beta_D) \circ \zeta_D = I_D\) for all \(D \in \mathcal{D}\).

are equivalent.

**Proof.** Assume that \(F\) is separable. By Rafael’s Theorem, there exists \(\tilde{\nu} \in \text{Nat}(GF, 1_C)\) such that \(\tilde{\nu}_C \circ \eta_C = I_C\) for all \(C \in \mathcal{C}\). Let \(\alpha : F \rightarrow F\) be the corresponding natural transformation of Corollary 5.3, i.e. \(\alpha_C = F(\tilde{\nu}_C) \circ \zeta_{F(C)}\), and \(\tilde{\nu}_C = \nu_C \circ G(\alpha_C)\), and the first implication of the Proposition follows. The converse follows trivially from Rafael’s Theorem. All the other equivalences can be proved in a similar way. \(\square\)
Application to bimodules. We use the notation of Section 2: let $R$ and $T$ be rings, and $M$ a $(T,R)$-bimodule. We have already seen that $M^* = \text{Hom}(M_R,R_R)$ and $^*M = \text{Hom}(T M, T)$ are $(R,T)$-bimodules. As given in the preliminaries, $\text{Hom}(T M, T M)$ and $\text{Hom}(M_R, M_R)$ are respectively the natural $(R,R)$-bimodule and $(T,T)$-bimodule. Furthermore, the induction functor

$$F = M \otimes_R \cdot : R M \to T M$$

has a right adjoint

$$G = \text{Hom}(T M, T \cdot) : T M \to R M$$

For $Q \in T M$, $\text{Hom}(T M, T Q) \in R M$ via $(m)(rf) = (mr)(f)$. For $\kappa : Q \to Q'$ in $T M$, we put $G(\kappa)(f) = \kappa \circ f$. We also have a functor $G' = ^* M \otimes_T \cdot : T M \to R M$

and a natural transformation $\gamma : G' \to G$ given by

$$\gamma_Q : \cdot G \cdot T Q \to \text{Hom}(T M, T Q), (m)(\gamma_Q(f \otimes_T q)) = (m)fq$$

We will also write $\gamma_Q(f \otimes_T q) = f \cdot q$. $\gamma_Q$ is well-defined on the tensor product, and left $R$-linear, so

$$(rt) \cdot q = r(f \cdot tq)$$

for all $r \in R$ and $t \in T$. If $Q$ is a $(T,R)$-bimodule (for example, $T = M$), then $\gamma_Q$ is also right $R$-linear, and we have

$$f \cdot (qr) = (f \cdot q)r$$

Now assume that $T M$ is finitely generated and projective, and consider a dual basis $\{ n_j \in M, g_j \in ^* M \}$, i.e.

$$m = \sum_j (m)g_j n_j$$

for all $m \in M$, or $I_M = \sum_j \gamma_M(g_j \otimes_T n_j)$. Then $\gamma : G \to G'$ is a natural isomorphism, and for all left $T$-linear $f : M \to Q$, we have

$$\gamma_Q^{-1}(f) = \sum_j g_j \otimes_T f(n_j)$$

In order to decide whether $F$ or $G$ is separable, or whether $(F,G)$ is a Frobenius pair, we have to investigate natural transformations $GF \to 1_{rM}$ and $1_{T M} \to FG$.

This is done in the next two Propositions. We thank the referee for pointing out to us that Proposition 5.5 and 5.6 can be derived from [Po, Theorem 3.5.6].

**Proposition 5.5.** Let $F$ and $G$ be as above, and consider

$$V = \text{Nat}(GF, 1_{rM}), \quad V_1 = \text{Hom}(r \text{Hom}(T M, T M)_R, R R_R)$$

$$V_2 = \text{Hom}(^* M, ^* M)_T$$

Then we have maps

$$V \xrightarrow{\alpha} V_1 \xrightarrow{\alpha_1} V_2$$

which are isomorphisms if $M$ is finitely generated and projective as a left $T$-module.
Proof. For $\nu \in V$, we put $\alpha(\nu) = \nu_R$. By definition, $\nu_R$ is left $R$-linear. For any $s \in R$, we consider the left $R$-linear map $m_s : R \to R$, $m_s(r) = rs$. The naturality of $\nu$ implies that $\nu_R(fs) = \nu_R(GF(m_s)(f)) = \nu_R(f s)$, so $\nu_R$ is also right $R$-linear. 

$\alpha_1 : V_1 \to V_2$ is given by $\alpha_1(\overline{\nu}) = \phi$, with 

$$\overline{\phi}(f)(m) = \overline{\nu}(\gamma_M(f \otimes m)) = \overline{\nu}(f \cdot m)$$

Using (4) and (5), we easily deduce that $\overline{\phi}$ is right $R$-linear, and that $\overline{\phi}$ is left $R$-linear and right $T$-linear.

Assume that $T_M$ is finitely generated projective, and, as above, assume that $\{n_j \in M, g_j \in *M\}$ is a dual basis. We can then define the inverse $\alpha^{-1}$ of $\alpha$ as follows. We view $\overline{\nu} \in V_1$ as a map $\overline{\nu} : *M \otimes_T M \to R$, and we identify $G$ and $G'$. We then define $\nu \in V$ by 

$$\nu_p = \overline{\nu} \otimes R P : GF(P) \cong *M \otimes_T M \otimes_R P \to R \otimes_R P \cong P$$

It is clear that $\nu$ is natural and that $\alpha$ and $\alpha^{-1}$ are each others inverses. For $\overline{\phi} \in V_2$, we define $\alpha_1^{-1}((\overline{\phi}) = \Phi$ by 

$$\Phi(\varphi) = \sum f \in \overline{\phi}(g_j)((n_j)\varphi)$$

for all $\varphi : M \to M$ in $T_M$. Straightforward computations yield that $\alpha_1^{-1}$ is well-defined, and is indeed an inverse of $\alpha_1$. \qed

In a similar fashion, we have:

**Proposition 5.6.** Let $F$ and $G$ be as above, and consider

$$W = \text{Nat}(1_{\mathcal{M}}, FG), \ W_1 = \{e \in M \otimes_R *M \mid te = ct, \ \text{for all} \ t \in T\}$$

$$W_2 = \text{Hom}(R(\mathcal{M}^*)_T, (\mathcal{M})_T)$$

We have maps

$$W \xrightarrow{\beta} W_1 \xrightarrow{\beta_1} W_2$$

$\beta$ is an isomorphism, and $\beta_1$ is an isomorphism if $M$ is finitely generated as a right $R$-module.

Proof. The proof is similar to the previous one, so we restrict to giving the connecting maps. For a natural transformation $\zeta : 1_{\mathcal{M}} \to FG$, we define $e = \beta(\zeta) = \zeta_T(1) \in FG(T) = M \otimes_R *M$. Conversely, take $e = \sum_i m_i \otimes_R f_i \in W_1$. $e \in W$ is defined as follows: for all $Q \in \mathcal{M}$, we put 

$$\zeta_Q : Q \to M \otimes_R \text{Hom}(T_M, T, Q), \ \zeta_Q(q) = \sum m_i \otimes_R f_i \cdot q$$

Now we define $\beta_1 : W_1 \to W_2$. For $e = \sum_i m_i \otimes_R f_i \in W_1$, we let

$$\beta_1(e) = \phi : M^* \to *M$$

be given by

$$\phi(h) = \sum h(m_i) f_i \text{ or } (m)\phi(h) = \sum_i (mh(m_i)) f_i$$

Straightforward computations show that $\phi(h) \in *M$, and that $\phi$ is $(R, T)$-bilinear. \qed

The two previous results can be used to decide when the induction functor $F$ and its adjoint $G$ are separable or Frobenius. Let us first look at separability.
Corollary 5.7. Let $M$ be a $(T, R)$-bimodule, and assume that $M$ is finitely generated projective as a left $T$-module, with finite dual basis $\{n_j, g_j\}$. Then the following assertions are equivalent:

- $F = M \otimes_R \bullet$ is a separable functor;
- there exists $\varpi \in V_1$ such that $\varpi(I_M) = 1_R$;
- there exists $\varphi \in V_2$ such that $\sum_j \varphi(g_j)(n_j) = 1_R$.

Corollary 5.8. Let $M$ be a $(T, R)$-bimodule. The functor $G = \text{Hom}(\tau M, \tau \bullet)$ is separable if and only if $T$ is $M$-separable over $R$, in the sense of Definition 2.4. If $M$ is finitely generated and projective as a left $R$-module, with finite dual basis $\{h_k, p_k\}$, then this is also equivalent to the existence of $\phi : M^* \to {}^*M$ in $W_2$ such that

$$\sum_k (p_k)\phi(h_k) = 1$$

The Frobenius analog of Corollary 5.8 is that $\tau M_R$ is a Frobenius bimodule if and only if $M \otimes_R \bullet$ is a Frobenius functor. This was stated explicitely in [CGN99], where it is also proved that any additive Frobenius functor between module categories is of this type. Let us show how this result can be deduced easily from Propositions 5.5 and Proposition 5.6.

Proposition 5.9. Let $M$ be a $(T, R)$-bimodule, and consider the functors $F = M \otimes_R \bullet$ and $G = \text{Hom}(\tau M, \tau \bullet)$. The following assertions are equivalent.

1. $(F, G)$ is a Frobenius pair;
2. $\tau M$ is finitely generated and projective, and there exist $e = \sum_i m_i \otimes_R f_i \in W_1$ and $\varpi \in V_1$ such that

$$m = \sum_i m_i \varpi(f_i \cdot m)$$

$$f = \sum_i \varpi(f \cdot m_i) f_i$$

for all $m \in M$ and $f \in {}^*M$;
3. $\tau M_R$ is Frobenius in the sense of Definition 2.7.

Proof. 1) $\Rightarrow$ 2). If $(F, G)$ is Frobenius, then $G = \text{Hom}(\tau M, \bullet)$ is a left adjoint, and therefore right exact and preserving direct limits, so $\tau M$ is necessarily finitely generated and projective. Let $\zeta$ and $\nu$ be the counit and unit of the adjunction $(G, F)$, i.e.

$$F(\nu_P) \circ \zeta_{F(P)} = I_{F(P)}$$

and $\nu_{G(Q)} \circ G(\zeta_Q) = I_{G(Q)}$

for all $P \in R \mathcal{M}$ and $Q \in \tau \mathcal{M}$. Let $\varpi = \nu_R \in V_1$ and $e = \sum_i m_i \otimes_R f_i = \zeta_T(1) \in W_1$. Putting $P = R$ in (11), we find (12). Then take $Q = T$ in (11). Making the identification $G' = G(\tau M$ is finitely generated projective), we find

$$G(\zeta_T) : {}^*M \to {}^*M \otimes_R \tau M \otimes_R {}^*M$$

and $\nu_{G(T)}(G(\zeta_T)(f)) = \sum_i \varpi(f \cdot m_i) f_i$, so (10) follows.

2) $\Rightarrow$ 1). Let $\nu = \alpha^{-1}(\varpi)$ and $\zeta = \beta^{-1}(e)$. $\nu$ and $\zeta$ satisfy (11), so $(F, G)$ is Frobenius.

2) $\Rightarrow$ 3). (12) implies that $M_R$ is finitely generated projective. Let $\varphi = \alpha_1(\varpi)$ and $\phi = \beta_1(e)$. We easily compute that $\varphi$ and $\phi$ are each others inverses.
If $^\ast M$ and $M^\ast$ are isomorphic as $(R,T)$-bimodules, then there exist $\phi \in V_2$ and $\phi \in W_2$ that are each others inverses. Put $\varphi = \alpha_1^{-1}(\phi)$ and $e = \beta_1^{-1}(\phi)$. Straightforward computations show that $\nu$ and $e$ satisfy (9-10).

Obviously our results also hold for functors between categories of right modules. As before, let $M$ be a $(T,R)$-bimodule, and consider the functors

$$\tilde{\mathcal{F}} = \bullet \otimes_T M : \mathcal{M}_T \rightarrow \mathcal{M}_R$$

and

$$\tilde{\mathcal{G}} = \text{Hom}(M_R, \bullet) : \mathcal{M}_R \rightarrow \mathcal{M}_T$$

Then $(\tilde{\mathcal{F}}, \tilde{\mathcal{G}})$ is an adjoint pair. We have a natural transformation

$$\tilde{\alpha} = \text{Nat}(\tilde{\mathcal{G}}, \tilde{\mathcal{F}})$$

and

$$\tilde{\beta} = \text{Nat}(1_{\mathcal{M}_R}, \tilde{\mathcal{F}} \tilde{\mathcal{G}})$$

where $\tilde{\alpha}_1 = \{ e \in M^\ast \otimes_T M \mid re = er \text{ for all } r \in R \}$. $\tilde{\beta}$ is always an isomorphism, $\tilde{\alpha}$ and $\tilde{\alpha}_1$ are isomorphisms if $M_R$ is finitely generated projective, and $\tilde{\beta}_1$ is an isomorphism if $T^\ast M$ is finitely generated projective.

**Proposition 5.10.** With notation as above, we have maps

$$\tilde{V} = \text{Nat}(\tilde{\mathcal{G}} \tilde{\mathcal{F}}, 1_{\mathcal{M}_T}) \xrightarrow{\tilde{\alpha}} \tilde{V}_1 = \text{Hom}(\text{Hom}(M_R, M_R) \otimes_T T, T) \xrightarrow{\tilde{\alpha}_1} W_2$$

and

$$\tilde{W} = \text{Nat}(1_{\mathcal{M}_R}, \tilde{\mathcal{F}} \tilde{\mathcal{G}}) \xrightarrow{\tilde{\beta}} \tilde{W}_1 \xrightarrow{\tilde{\beta}_1} V_2$$

where $\tilde{W}_1 = \{ e \in M^\ast \otimes_T M \mid re = er \text{ for all } r \in R \}$. $\tilde{\beta}$ is always an isomorphism, $\tilde{\alpha}$ and $\tilde{\alpha}_1$ are isomorphisms if $M_R$ is finitely generated projective, and $\tilde{\beta}_1$ is an isomorphism if $T^\ast M$ is finitely generated projective.

**Proof.** Completely similar to the proof of Propositions 5.3 and 5.4. Let us mention that

$$\alpha_1(\varphi) = \phi \text{ with } (m)\phi(h) = \varphi(m \otimes h))$$

$$\beta_1(\sum_i k_i \otimes m_i) = \varphi \text{ with } \varphi(f) = \sum_i k_i ((m_i) f)$$

As a consequence, we obtain relations between the separability and Frobenius properties of $F$, $G$, $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$.

**Corollary 5.11.** Let $M$ be a $(T,R)$-bimodule, and assume that $M_R$ is finitely generated projective. Then the following assertions are equivalent:

- $\tilde{\mathcal{F}}$ is separable;
- there exists $\varphi \in \tilde{V}_1$: $\varphi(I_M) = 1_T$;
- there exists $\phi \in W_2$: $\sum_k (p_k) \phi(h_k) = 1$;
- $G$ is separable.

Here $\{ p_k, h_k \}$ is a finite dual basis of $M$ as a right $R$-module.

**Proof.** The equivalence of the first three statements is obtained in exactly the same way as Corollary 5.7. The equivalence of the third and the fourth statement is one of the equivalences in Corollary 5.8.
Corollary 5.12. Let $M$ be a $(T, R)$-bimodule. The following statements are equivalent:

- $\tilde{G}$ is separable;
- there exists $e = \sum_i k_i \otimes_T m_i \in \tilde{W}_1$ such that $\sum_i k_i(m_i) = 1_R$.

If $T M$ is finitely generated projective, then they are also equivalent to

- There exists $\bar{\phi} \in V_2$ such that $\sum_j \bar{\phi}(g_j)n_j = 1_R$;
- $F$ is separable.

Here $\{n_j, g_j\}$ is a finite dual basis of $M$ as a left $T$-module.

Corollary 5.13. Let $M$ be a $(T, R)$-bimodule. The following statements are equivalent:

- $(\tilde{F}, \tilde{G})$ is a Frobenius pair and $M_R$ is finitely generated projective;
- $M_R$ is finitely generated projective and there exist $e = \sum_i k_i \otimes_T m_i \in \tilde{W}_1$ and $\bar{\pi} \in \tilde{V}_1$ such that
  \begin{align}
  m &= \sum_i \bar{\pi}(g \cdot k_i)m_i \quad \text{and} \quad h = \sum_i k_i\bar{\pi}(m_i \cdot f) \\
  \end{align}
  for all $m \in M$ and $f \in M^*$;
- $\tau M_R$ is a Frobenius bimodule.

We now address the following problem: assume that we know that $T M_R$ is a Frobenius bimodule. When is $T M$-separable over $R$? In view of the above considerations, it suffices to apply Proposition 5.4. We first need a Lemma.

Lemma 5.14. Let $M$ be a $(T, R)$-bimodule, and $F = M \otimes_R \cdot$. Then

$$\text{Nat}(F, F) \cong \text{Hom}(T M_R, T M_R)$$

Proof. Let $\alpha : F \to F$ be a natural transformation. Then $\alpha_R : M \to M$ is left $T$-linear, and right $R$-linear since $\alpha$ is natural. Given a $(T, R)$-linear map $\alpha_R : M \to M$, we define a natural transformation $\alpha : F \to F$ as follows: $\alpha_F = \alpha_R \otimes I_P$. \[\square\]

Proposition 5.15. Assume that $M$ is a Frobenius $(T, R)$-bimodule, and let $e = \sum_i m_i \otimes_R f_i \in W_1$ and $\bar{\pi} \in V_1$ be as in the second statement of Proposition 5.4. Then $T$ is $M$-separable over $R$ (i.e. $G$ is separable) if and only if there exists a $(T, R)$-linear map $\bar{\pi} : M \to M$ such that

$$\sum_i (\bar{\pi}(m_i))f_i = 1_T$$

This proposition recovers [K, Theorem 2.15]. One can similarly recover [K99, Theorem 4.1].

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