Probing analytical and numerical integrability: The curious case of $(AdS_5 \times S^5)_\eta$

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ABSTRACT: Motivated by recent studies related to integrability of string motion in various backgrounds via analytical and numerical procedures, we discuss these procedures for a well known integrable string background $(AdS_5 \times S^5)_\eta$. We start by revisiting conclusions from earlier studies on string motion in $(\mathbb{R} \times S^3)_\eta$ and $(AdS_3)_\eta$ and then move on to more complex problems of $(\mathbb{R} \times S^5)_\eta$ and $(AdS_5)_\eta$. Discussing both analytically and numerically, we deduce that while $(AdS_5)_\eta$ strings do not encounter any irregular trajectories, string motion in the deformed five-sphere can indeed, quite surprisingly, run into chaotic trajectories. We discuss the implications of these results both on the procedures used and the background itself.

KEYWORDS: Bosonic strings, Gauge-gravity correspondence, Integrable Field Theories.
1. Introduction

String motion in curved spaces, described by two-dimensional non-linear sigma models, have been studied extensively from the birth of the subject. This is extremely interesting due to the complicated non-linear equations of motion associated with the worldsheet fields. It comes as no surprise that these equations of motion are only ‘integrable’ for a select subclass of target space backgrounds, and hence this notion of integrability helps one to pick out the cases where a complete quantitative analysis of classical (and perhaps quantum) string motion can be performed and compared to the flat space case. One of the most widely known cases is of course that of type IIB strings in the $\text{AdS}_5 \times S^5$ space-time [1], which is dual to operators in maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory (sYM) via $\text{AdS}/\text{CFT}$ correspondence [2]. The integrability of these strings moving in the bulk $\text{AdS}_5 \times S^5$, in conjunction with the integrability of the dual sYM theory, makes an exceptional example to study the $\text{AdS}/\text{CFT}$ correspondence from the point of view of integrable systems [3]. Moreover, the finding that in the semiclassical limit, the dynamics of this correspondence indeed becomes tractable [4], has regenerated interest in the classical string solutions in $\text{AdS}$ and related geometries. Indeed, a lot of literature has been devoted to the subject of integrability in $\text{AdS}/\text{CFT}$ in last two decades\(^1\).

With this advent of integrability studies in the context of $\text{AdS}/\text{CFT}$, there has been much celebrated quests of deforming the symmetries on both sides of the correspondence

\(^1\)For a recent introduction to this subject, the reader is directed to [5] and references therein.
while keeping the integrable structure intact. Most of these relied on the use of target space duality symmetries to generate new integrable backgrounds [6, 7, 8, 9]. Very recently, based on Klimcik's pioneering work on novel integrable deformations of $\sigma$-models [10, 11, 12] have paved the way for their application to string $\sigma$-models and finding probable deformed versions of $AdS/CFT$ correspondence. Since then a larger family of integrable deformations of $AdS \times S$ geometries have been explored, where the deformation is given by a classical $r$-matrix solution to the (modified) Classical Yang-Baxter Equation (CYBE). The explicit geometry and NS-NS forms for such a ‘Yang-Baxter’ deformation of $AdS_5 \times S^5$ first appeared in [13, 14], was analysed in detailed in [15] and various consistent truncations have been discussed in [16]. In the Yang-Baxter case, the deformation works by deforming the supercoset associated to $AdS_5 \times S^5$ itself by a continuous parameter, which is often referred to as a $q$-deformation, or a quantum group deformation [17]. This replaces the lie algebra of the classical charges by its $q$-deformed version, which is then incorporated into the superstring action for $AdS_5 \times S^5$ having a real deformation parameter $\eta \in [0, 1)$ or equivalently another parameter called $\kappa$ with $\kappa \in [0, \infty)$ 2. For various avenues of exploratory works on Yang-Baxter deformations, one should have a look at [18]-[72].

As integrable string backgrounds by construction, these $(AdS_5 \times S^5)_\kappa$ strings in general satisfy the Lax equations. But in the case of a random string sigma model, where the existence of Lax pair is not particularly known, proving (non)-integrability is a rather complicated task. To this note, there have been a number of works to consistently truncate the two dimensional string equations of motion of particular circular strings into one dimensional mechanical systems and analyzing the (non)-integrability properties thereof. It has been argued that it is sufficient to show that there exists at least one truncated dynamical system of differential equations, where the corresponding string motion turns chaotic [73], i.e. small variations around the equations grow non-deterministically in time. Useful tools in these studies have mainly been the variational non-integrability techniques of Hamiltonian systems and numerical experiments in the associated phase space in general. This approach is often hailed as the equivalent of the algebraic approach of finding Lax pairs for the system and a large number of works have appeared along these lines, see for example [74]-[89].

In the following note, we seek to understand this equivalence by studying string motion in the extremely complicated but integrable background of $(AdS_5 \times S^5)_\kappa$. One should bear in mind, the process of Yang-Baxter deformation breaks the supersymmetries associated to $(AdS_5 \times S^5)$ and even at the bosonic level, the isometry group of $SO(2, 4) \times SO(6)$ breaks down to $U(1)^3 \times U(1)^3$ in this case, but still the $\kappa$-deformed background inherits the parent integrability. We must mention here [61], in which, using these hamiltonian analytical methods, it was claimed that the associated phase space encounters chaos as the differential equations of motion are not ‘integrable’. This certainly creates a tension between the different methods of studying (non-)integrability of string motion in curved backgrounds. Spearheaded by this, we revisit these claims of non-integrability of string motion in $(R \times S^3)_\kappa$ and $(AdS_3)_\kappa$ and then attack the larger and more complicated problem

2Notice that here the parameter $\kappa$ is related to the original deformation parameter $\eta$ as [14], $\kappa = \frac{2\eta}{1-\eta}$. In what follows, we would denote $\kappa$ as being the deformation parameter in our analysis.
of strings in $(\mathbb{R} \times S^5)_\kappa$ and $(AdS_5)_\kappa$ with antisymmetric $B$ fields included. We explicitly show that string motion in former cases do not have any non-integrable traits. However, to our surprise, we find that the phase space of the deformed five-sphere indeed contains chaotic string motion, as the equations describing the motion are non-integrable in nature. We emphasize that this phenomenon happens only for the dynamical phase space associated with the full five sphere, and the sub-sectors can be presumed integrable in this sense. For the case of $(AdS_5)_\kappa$ also, we find string trajectories remain regular throughout the motion.

The paper is organized in the following way, in the section 2, we give a review of the background and fluxes associated to the $(AdS_5 \times S^5)_\kappa$ (or, as we will actually use, $(AdS_5 \times S^5)_\kappa$) string background. In section 3, after revisiting the case of $(\mathbb{R} \times S^3)_\kappa$, we will have a detailed discussion of string motion in $(\mathbb{R} \times S^5)_\kappa$. By the use of Normal Variational Equations (NVE’s) for fluctuations around equations of motion for consistent string solutions, we would arrive at the fact that strings in $(\mathbb{R} \times S^3)_\kappa$ do not run into any chaotic trajectories. In the case of deformed five-sphere, we will, however, find chaotic trajectories as soon as we turn on a non-zero deformation parameter, a result that will be corroborated by both using NVE and studying its Poincare sections by numerical trajectories method. In section 4, we will essentially repeat the same exercise of the earlier chapter, instead in the case of deformed $AdS$ backgrounds. As in the earlier case, we show that there are no chaotic trajectories in $(AdS_3)_\kappa$. And following this, no chaotic motion is found in the case of $(AdS_5)_\kappa$ as well, which we confirm via both analytical and numerical calculations. We discuss the ramifications of our results and conclude this work in section 5.

2. Setup

Let us start by introducing the geometry and the general setup required for our study. We first write down the full deformed metric for the $\kappa$ deformed $AdS_5 \times S^5$ \cite{14},

$$
\begin{align*}
\text{ds}_{(AdS)_\kappa}^2 &= -\frac{1 + \rho^2}{1 - \kappa^2 \rho^2} dt^2 + \frac{d\rho^2}{(1 + \rho^2) (1 - \kappa^2 \rho^2)} \\
&\quad + \frac{\rho^2}{1 + \kappa^2 \rho^4 \sin^2 \zeta} \left( d\zeta^2 + \cos^2 \zeta \, d\psi_1^2 \right) + \rho^2 \sin^2 \zeta \, d\psi_2^2, \\
\text{ds}_{(S^5)_\kappa}^2 &= \frac{1 - r^2}{1 + \kappa^2 r^2} d\phi^2 + \frac{dr^2}{(1 - r^2) (1 + \kappa^2 r^2)} \\
&\quad + \frac{r^2}{1 + \kappa^2 r^4 \sin^2 \xi} \left( d\xi^2 + \cos^2 \xi \, d\phi_1^2 \right) + r^2 \sin^2 \xi \, d\phi_2^2.
\end{align*}
$$

(2.1)

Also we have the $B$-fields $B = \frac{1}{2} B_{MN} \, dX^M \wedge dX^N$ \cite{14} associated to the solution,

$$
\begin{align*}
\tilde{B}_{(AdS)_\kappa} &= + \kappa \left( \frac{\rho^4 \sin (2\zeta)}{1 + \kappa^2 \rho^4 \sin^2 \zeta} \, d\psi_1 \wedge d\zeta + \frac{2\rho}{1 - \kappa^2 \rho^2} dt \wedge d\rho \right), \\
\tilde{B}_{(S^5)_\kappa} &= - \kappa \left( \frac{r^4 \sin (2\xi)}{1 + \kappa^2 r^4 \sin^2 \xi} \, d\phi_1 \wedge d\xi + \frac{2r}{1 + \kappa^2 r^2} d\phi \wedge dr \right).
\end{align*}
$$

(2.2)
It is easy to see that the contributions of the components $B_{\mu r}$ and $B_{\phi r}$ to the Lagrangian turn out to be total derivatives, and hence can be ignored without loss of generality. We can put in $\rho = 0$ and $r = \cos \theta$ and perform the redefinition of the coordinates $\phi \rightarrow \phi_3$ and $\xi \rightarrow \psi$ to write the metric of $(\mathbb{R} \times S^5)_\kappa$ in the following form,

$$
\begin{align*}
\text{ds}^2_{(\mathbb{R} \times S^5)_\kappa} &= -dt^2 + \frac{\sin^2 \theta}{1 + \kappa^2 \cos^2 \theta} d\phi_3^2 + \frac{d\theta^2}{1 + \kappa^2 \cos^2 \theta} \\
&\quad + \frac{\cos^2 \theta}{1 + \kappa^2 \cos^4 \theta \sin^2 \psi} \left( d\psi^2 + \cos^2 \psi d\phi_1^2 \right) + \cos^2 \theta \sin^2 \psi d\phi_2^2.
\end{align*}
$$

(2.3)

And in this case, the single surviving component of NS-NS flux takes the form as,

$$
\tilde{B}_{(\mathbb{R} \times S^5)_\kappa} = -\frac{\kappa}{2} \left( \frac{\cos^4 \theta \sin(2\psi)}{1 + \kappa^2 \cos^4 \theta \sin^2 \psi} \right).
$$

(2.4)

It is worthwhile to note that the $(AdS)_\eta$ contains a singularity, but we won’t be bothered with that part in the present analysis. We write the deformed $(AdS_5)_\kappa$ part of the metric and $B$ field again with the redefinition $\rho \rightarrow \sinh \rho$,

$$
\begin{align*}
\text{ds}^2_{(AdS_5)_\kappa} &= \frac{1}{1 - \kappa^2 \sinh^2 \rho} \left[ -\cosh^2 \rho dt^2 + d\rho^2 \right] \\
&\quad + \frac{\sinh^2 \rho}{1 + \kappa^2 \sinh^4 \rho \sin^2 \zeta} \left( d\zeta^2 + \cos^2 \zeta d\psi_1^2 \right) + \sinh^2 \rho \sin^2 \zeta d\psi_2^2 \\
B &= \frac{\kappa}{2} \left( \frac{\sinh^4 \rho \sin(2\zeta)}{1 + \kappa^2 \sinh^4 \rho \sin^2 \zeta} \right).
\end{align*}
$$

(2.5)

The singularity surface in this coordinate system is located at a critical value of the radial coordinate

$$
\rho = \rho_s = \sinh^{-1} \frac{1}{\kappa},
$$

(2.6)

So that $\kappa \rightarrow 0$ signals the usual $AdS$ boundary at conformal infinity. One must emphasize, that this is a general singularity in the spacetime which can’t be dealt with by simple change of coordinates alone.

In this background, to study string solutions, we use the Polyakov action coupled to an antisymmetric B-field,

$$
S = \int d\sigma d\tau \left( \mathcal{L}_G + \mathcal{L}_B \right)
$$

(2.7)

$$
= -\frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau [\sqrt{-\gamma} \gamma^{\alpha\beta} g_{MN} \partial_\alpha X^M \partial_\beta X^N - \epsilon^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N B_{MN}] ,
$$

where $\sqrt{\lambda}$ is the modified ‘t Hooft coupling for this case, given by $\lambda = \lambda(1 + \kappa^2)^{1/2}$, $\gamma^{\alpha\beta}$ is the worldsheet metric and $\epsilon^{\alpha\beta}$ is the antisymmetric tensor defined as $\epsilon^{\tau\sigma} = -\epsilon^{\sigma\tau} = 1$.

Variation of the action with respect to $X^M$ gives us the following equations of motion

$$
\begin{align*}
2\partial_\alpha (\eta^{\alpha\beta} \partial_\beta X^N g_{KN}) - \eta^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N \partial_K g_{MN} - 2\partial_\alpha (\epsilon^{\alpha\beta} \partial_\beta X^N b_{KN}) \\
+ \epsilon^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N \partial_K b_{MN} &= 0 ,
\end{align*}
$$

(2.8)
and variation with respect to the metric gives the two Virasoro constraints,

\[ g_{MN}(\partial_\tau X^M \partial_\sigma X^N + \partial_\sigma X^M \partial_\tau X^N) = 0 \] (2.9)

\[ g_{MN}(\partial_\tau X^M \partial_\tau X^N) = 0. \] (2.10)

We use the conformal gauge (i.e. \( \sqrt{-\gamma} \alpha^\beta = \eta^{\alpha\beta} \)) with \( \eta_{\tau\tau} = -1, \eta_{\sigma\sigma} = 1 \) and \( \eta_{\tau\sigma} = \eta_{\sigma\tau} = 0 \) to solve these equations of motion.

3. Strings in deformed sphere

3.1 Revisiting a warm up example: The case of \((\mathbb{R} \times S^3)_\kappa\)

Although the simplest case of an extended string in \((\mathbb{R} \times S^3)_\kappa\) has been addressed already in [61], we would first start with taking another look at the findings. The metric for \((\mathbb{R} \times S^3)_\kappa\) is obtained by putting \( \psi = \frac{\pi}{2} \) and \( \phi_3 \rightarrow \varphi \), \( \phi_2 \rightarrow \phi \) in (2.3),

\[ ds^2_{(\mathbb{R} \times S^3)_\kappa} = -dt^2 + \frac{1}{1 + \kappa^2 \cos^2 \theta} (d\theta^2 + \sin^2 \theta d\varphi^2) + \cos^2 \theta d\phi^2. \] (3.1)

The NS-NS flux vanishes in this case. We now have to choose a consistent worldsheet embedding for the worldsheet coordinates. We can’t help but note here that in [61] the following string embedding was chosen,

\[ t = t(\tau), \quad \theta = \theta(\tau), \quad \varphi(\tau, \sigma) = \alpha_1 \sigma + q(\tau), \quad \phi = \alpha_2 \sigma. \] (3.2)

We must stress here that this is not a consistent string embedding, since for this choice, the second Virasoro constraint \( (T_{\tau\sigma} = 0) \) gives rise to the following condition,

\[ \frac{\sin^2 \theta}{1 + \kappa^2 \cos^2 \theta} \alpha_1 \dot{q} = 0, \] (3.3)

Which in turn can only be satisfied consistently if the winding number \( \alpha_1 = 0 \) or \( \dot{q} = 0 \). For our case, we choose the former and propose a refined ansatz for a circular string with additional angular momentum in the aforementioned geometry,

\[ t = t(\tau), \quad \theta = \theta(\tau), \quad \varphi = \varphi(\tau), \quad \phi = m\sigma. \] (3.4)

This is a completely consistent embedding and makes the second Virasoro constraint zero naturally. We can now write the effective Lagrangian of this theory as,

\[ L_S = -l^2 + \frac{\dot{\theta}^2}{1 + \kappa^2 \cos^2 \theta} + \frac{\sin^2 \theta \varphi^2}{1 + \kappa^2 \cos^2 \theta} - m^2 \cos^2 \theta. \] (3.5)

From the equations of motion, it can be seen that the \( t \) equation is easily satisfied by,

\[ t = E\tau, \] (3.6)

where \( E \) is a constant, and the \( \phi \) equation is trivially satisfied. The other two equations for \( \theta \) and \( \varphi \) then read as following,

\[ -\sin \theta \cos \theta \left[ m^2 (1 + \kappa^2 \cos^2 \theta) - \kappa^2 \dot{\theta}^2 + (1 + \kappa^2) \varphi^2 \right] + \ddot{\theta} (1 + \kappa^2 \cos^2 \theta) = 0 \]

\[ 2\dot{\varphi}\sin \theta \cos \theta (1 + \kappa^2) + \ddot{\varphi} \sin^2 \theta (1 + \kappa^2 \cos^2 \theta) = 0. \]
These equations are supplemented by the other Virasoro constraint implying the vanishing of the 2d Hamiltonian,
\[ E^2 = \frac{\dot{\theta}^2}{1 + \kappa^2 \cos^2 \theta} + \frac{\sin^2 \theta \; \dot{\varphi}^2}{1 + \kappa^2 \cos^2 \theta} + m^2 \cos^2 \theta. \]  
(3.7)

This is exactly equivalent to the time integrated version of the \( \theta \) equation of motion. From the above, we can see that \( \theta \to 0, \dot{\theta} \to 0 \) is a solution to the both equations of motion, i.e. is an invariant plane of the system. We can demand that the Hamiltonian constraint is satisfied on the invariant plane, just with the identification of the constants as \( E^2 = m^2 \).

Now we can consider small fluctuations around this invariant plane, with the form,
\[ \theta(\tau) = 0 + \epsilon(\tau), \quad |\epsilon| << 1. \]  
(3.8)

Expanding the \( \theta \) equation up to first order in \( \epsilon \), we can get the Normal Variational Equation (NVE) for \( \theta \),
\[ \ddot{\epsilon} - (m^2 (1 + \kappa^2) + \dot{\varphi}^2) \epsilon = 0. \]  
(3.9)

We now have to replace \( \dot{\varphi} \) to get a differential equation for \( \epsilon(\tau) \). We then note that from the equation for \( \varphi \) we can easily write,
\[ \partial_\tau \left[ \frac{\sin^2 \theta \; \dot{\varphi}}{1 + \kappa^2 \cos^2 \theta} \right] = 0 \Rightarrow \dot{\varphi} = J \left[ \frac{1 + \kappa^2 \cos^2 \theta}{\sin^2 \theta} \right] \]  
(3.10)

where \( J \) is a constant of motion, i.e. evolves independently of time. So, near the point \( \theta \to 0 \), we can write\(^3\),
\[ \dot{\varphi} \sim \frac{J(1 + \kappa^2)}{\epsilon^2}. \]  
(3.11)

With this replacement, we can now analyze the NVE to find that it offers well defined rational solutions of the form,
\[ \epsilon(\tau) = \pm \sqrt{\frac{4m^2 (1 + \kappa^2)^3 J^2 + e^{-4m\sqrt{1+\kappa^2} \tau}}{2m \sqrt{1 + \kappa^2} e^{-2m\sqrt{1+\kappa^2} \tau}}}. \]  
(3.12)

The above solution is completely well defined in the parameter space and we can conclude that the string motion in this case does not run into chaos anywhere. Note that if we had chosen \( \theta = \pi/2 \) as an invariant plane of the system, we would have simply gotten \( \dot{\varphi} \sim J \), which would also be sufficient to satisfy the equations of motion and the NVE would just take the form of a simple Harmonic Oscillator equation. The Hamiltonian constraint in that case would simply become \( E^2 = J^2 \), i.e that of a BPS point-like string.

For the sake of completeness let us also discuss the case of choosing the angular momentum along the other direction from the above one, i.e considering the changed ansatz,
\[ t = t(\tau), \quad \theta = \theta(\tau), \quad \varphi = m \sigma, \quad \phi = \varphi(\tau), \]  
(3.13)

\(^3\)We must point out here that solving the Hamiltonian constraint near the limit for \( \varphi \) also gives a leading term in \( O(\frac{1}{\tau}) \).

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\[ \textbf{- 6 -} \]
since the two-spheres inside the deformed three-sphere are not equivalent to each other
(as is the case for undeformed spheres) this case has to be addressed separately. In this
case, the equations of motion can be seen to be trivially satisfied by \( \theta \to \frac{\pi}{2}, \dot{\theta} \to 0 \), which
define the other invariant plane. Doing the above analysis again for this case (expanding
as \( \theta(\tau) = \frac{\pi}{2} + \tilde{\epsilon}(\tau) \)) yields the same form of NVE as in (3.9). The only difference comes
from the definition of the angular momenta \( \tilde{J} \), which near \( \theta \to \frac{\pi}{2} \) gives,
\[
\dot{\phi} \sim \frac{\tilde{J}}{\tilde{\epsilon}}. \tag{3.14}
\]
So we can safely say here that the expansion near invariant planes is not sensitive to the
choice of the angular momentum direction, and in both the cases integrability properties
of the equations of motion stay unchanged.

3.2 Strings in the five-sphere: Analytical

In this section, we would try to repeat the exercise done in the last section for a deformed
\( S^5 \). Due to the complexity of the equations of motion in this case, more emphasis will be
given to the numerical analysis here. Let us take a general spinning string ansatz in the
deformed five-sphere of the following form,
\[
t = t(\tau), \quad \theta = \theta(\tau), \quad \psi = \psi(\tau), \quad \phi_3 = \phi_2 = 0, \quad \phi_1 = m\sigma. \tag{3.15}
\]
The effective lagrangian of the theory is given by,
\[
L_S = -\dot{t}^2 + \frac{\dot{\theta}^2}{1 + \kappa^2 \cos^2 \theta} + \frac{\cos^2 \theta (\dot{\psi}^2 - \cos^2 \psi \; m^2)}{1 + \kappa^2 \cos^4 \theta \sin^2 \psi} + \frac{\kappa \cos^4 \theta \sin(2\psi) \; m \dot{\psi}}{1 + \kappa^2 \cos^4 \theta \sin^2 \psi}. \tag{3.16}
\]
The equation of motion for \( t \) is satisfied trivially by,
\[
t = E\tau, \tag{3.17}
\]
where \( E \) is a constant as shown in the last section. The equation of motion for \( \theta \) on the
other hand reads,
\[
\sin \theta \cos \theta \left( \dot{\psi}^2 - m^2 \cos^2 \psi \right) + \frac{2 \kappa^2 \sin \theta \cos^5 \theta \sin^2 \psi \left( m^2 \cos^2 \psi - \dot{\psi}^2 \right)}{\left( 1 + \kappa^2 \cos^4 \theta \sin^2 \psi \right)^2} + \frac{2 \kappa m \sin \theta \cos^3 \theta \; \dot{\psi} \sin(2\psi)}{1 + \kappa^2 \cos^4 \theta \sin^2 \psi} - \frac{4 \kappa^3 m \sin \theta \cos^7 \theta \; \psi \sin^3 \psi \cos \psi}{\left( 1 + \kappa^2 \cos^4 \theta \sin^2 \psi \right)^2} + \frac{\dot{\theta}}{1 + \kappa^2 \cos^2 \theta} + \frac{\kappa^2 \dot{\psi}^2 \sin \theta \cos \theta}{\left( 1 + \kappa^2 \cos^2 \theta \right)^2} = 0. \tag{3.18}
\]

Similarly, we can easily write the equation of motion for \( \psi \) as,
\[
\cos^2 \theta \left[ \sin(2\psi) \left( -m^2 \left( 3\kappa^2 + \kappa^2(4 \cos(2\theta) + \cos(4\theta)) + 8 \right) - 8 \kappa^2 \cos^4 \theta \dot{\psi}^2 \right) + \frac{16}{(1 + \kappa^2 \cos^4 \theta \sin^2 \psi)} \right] = 0. \tag{3.19}
\]
In these equations, we can see that $\theta = 0$ and $\psi = \frac{\pi}{2}$ are trivial solutions of the $\theta$ and $\psi$ equations respectively. The non-zero Virasoro constraint equation has a form

$$E^2 = \frac{\dot{\theta}^2}{1 + \kappa^2 \cos^2 \theta} + \frac{\cos^2 \theta (\dot{\psi}^2 + \cos^2 \psi m^2)}{1 + \kappa^2 \cos^4 \theta \sin^2 \psi}.$$  \hspace{1cm} (3.20)

This is equivalent to the Hamiltonian constraint for the 2d theory explicitly.

**Normal variational equations**

Consider the equation of motion for $\psi$. This is satisfied by the trivial solution as we showed earlier. If we now look carefully at $\psi = \frac{\pi}{2}$ solution, the Virasoro constraint on the invariant dynamical plane yields,

$$\dot{\theta}^2 = (1 + \kappa^2 \cos^2 \theta) E^2,$$  \hspace{1cm} (3.21)

thereby effectively eliminating one variable from the equation. For us, $\psi = \frac{\pi}{2}$ will be the aforementioned invariant plane, making the dynamics effectively only along the $\theta$ direction.

This special solution to the equations of motion is given by (3.21), which can be solved with appropriate initial conditions to get,

$$\theta(\tau) = \bar{\theta} = \text{sn} \left[ \sqrt{1 + \kappa^2 E \tau} \left| \frac{\kappa^2}{1 + \kappa^2} \right. \right].$$  \hspace{1cm} (3.22)

Now we have to study small fluctuations near this special solution in order to comment on the integrability of the system. To write the normal variation equation, we then start by expanding the equation of motion for $\psi$ using

$$\psi(\tau) = \frac{\pi}{2} + \eta(\tau), \hspace{0.5cm} |\eta| << 1.$$  \hspace{1cm} (3.23)

Upto first order in $\eta$ the expansion then reads,

$$\ddot{\eta}(1 + \kappa^2 \cos^4 \theta) + 2(\kappa^2 \cos^3 \theta \sin \theta - \tan \theta) \dot{\theta} \dot{\eta}$$

$$- \left[ m^2 (1 + \kappa^2 \cos^4 \theta) + 2 m \kappa \sin(2\theta) \dot{\theta} + 8 \kappa^2 \cos^4 \theta \eta^2 \right] \eta = 0.$$  \hspace{1cm} (3.24)

One should note since there is a division by $\cos^2 \theta$ involved, this expansion does not include the case of $\theta = \frac{\pi}{2}$. Also since we are working in strictly first order of $\eta$, we can discard the $\dot{\eta}^2 \eta$ term. Now, on the invariant plane we can simply use (3.21) to replace terms having derivatives of $\theta$ \(^4\). Then this can be put into an algebraic form using a simple replacement

$$\tau \rightarrow z = \tan \theta(\tau).$$  \hspace{1cm} (3.25)

Instead of working with the total complicated differential equation, we can concentrate on the $\kappa \rightarrow 0$ limit, i.e. that of small deformation. We would see that this would be enough for our case. We can write this differential equation upto the leading order of $\kappa$ in the following form,

$$\eta'' + \frac{m^2 \eta}{E^2 (1 + z^2)^2} + \frac{4 m \kappa z \eta}{E (1 + z^2)^3} + O(\kappa^2) = 0.$$  \hspace{1cm} (3.26)

\(^4\text{Solving for } \dot{\theta} \text{ from hamiltonian constraint near the limit gives another } O(\kappa^2) \text{ term in leading order, which can be discarded.}
This equation is already in the so-called “Normal” or Schrödinger form. It is easy to see that the $\kappa = 0$ solution, i.e. the solution for the undeformed sphere is quite simple and in a rational form

$$
\eta^{(0)} = \frac{1}{2} \sqrt{1 + z^2} e^{-i\sqrt{\frac{m^2}{E^2} + 1} \tan^{-1}(z)} \left( 2c_1 e^{2i\sqrt{\frac{m^2}{E^2} + 1} \tan^{-1}(z)} + \frac{ic_2}{\sqrt{\frac{m^2}{E^2} + 1}} \right)
$$

(3.27)

Where $c_i$ are constants. But the total solution $\eta(\kappa)$ can’t be written in terms of rational functions. Specifically, the solutions can be found in terms of Doubly Confluent Heun function, thereby making it clear that even in the small deformation limit, there is no Liouvillian solution for this dynamics, making it effectively non-integrable in this sense. This claim will be more elucidated in the next section when we talk about numerical simulation of these string trajectories.

Of course the above discussion does not encompass the whole story here. In general the integrability properties of classical hamiltonian systems are associated with behaviour of variations for the phase space curves. The usefulness of NVE’s come in handy when we systematically want to analyse existence of functionally independent integrals of motion. The symmetries leading to existence of such integrals of motion are usually given by transformations between space of solutions of the variational differential equations. These are often described in mathematical literature via Picard-Vessiot theory or differential Galois group techniques [90](Also see [78]). Since determining the Galois group for a general case is hard, a different route is provided via the Koavacic algorithm [91] to find existence of Liouvillian solutions. We will describe more about this in the appendix, and explicitly calculate the case of $\theta$ NVE in $(\mathbb{R} \times S^5)_{\kappa}$ for any finite value of $\kappa$ via this algorithm, which will undoubtedly point out that inclusion of non-zero $\kappa$ leads to solutions becoming non-Liouvillian in this case. For the time being, we will accept the above discussion and focus on the numerical analysis.

### 3.3 The hamiltonian and numerical trajectories

Here we supplement our previous analysis by probing more into the chaotic behaviour. We will find the Hamiltonian equation of motion and plot the constant energy surfaces. Then, by observing the behaviour of those trajectories we can get some insight into this chaotic behaviour by invoking the Kolomogorov-Arnold-Moser (KAM) theorem. For integrable systems essentially the number of conserved charge is equal to the number of degrees of freedom present in the system. The systems that we will consider are basically coupled harmonic oscillators with non-trivial potentials. They are characterized by a set of co-ordinates $q_i$ and their conjugate momenta $p_i$. Together they $\{q_i, p_i\}$ give the phase space $(i = 1, \cdots N)$. Now if the system is integrable then there will be exactly $N$ number of conserved charges. Then we can plot this $N$ dimensional surfaces and typically for the integrable system the shape of these surfaces are of like torus, which are known as KAM tori. In other words for each values of these conserved charges (one of them will be the energy which we will mainly consider in our subsequent analysis), the points of the phase space will lie on this KAM tori.
Now when one adds non-integrable terms to the integrable Hamiltonian these KAM tori get perturbed. According to the KAM theorem most of these tori will be deformed but if the strength of the non-integrable deformation terms is small then the trajectories will still be ordered and fall on an the surface of this deformed tori (only the resonant tori i.e those corresponding to the frequencies $\omega_i$ such that $\alpha^i \omega_i = 0$, where $\alpha^i \in Q$ will be completely destroyed). But if the strength of the non-integrable deformations is large, all these tori will be completely destroyed and the trajectories can probe the entire accessible phase space (determined by the total energy) in a completely arbitrary way and thus we will observe chaotic behaviour.

We will adopt the following strategy for our case. We first consider the string motion on $(\mathbb{R} \times S^5)_\kappa$ case as discussed in section (3.2) and use the profile mentioned in (3.15). We write down the Hamiltonian starting from the Lagrangian mentioned in (3.16) below.

$$H_{S^5_\kappa} = \frac{1}{4} \left[ p_\theta^2 \left( 1 + \kappa^2 \cos^2 \theta \right) + \frac{\kappa^2 \psi}{\cos^2 \theta} E^2 + \cos^2 \theta \left( \kappa p_\psi \sin \psi - 2 m \cos^2 \psi \right) \right], \quad (3.28)$$

where, the two conjugate momenta are defined as

$$p_\theta = \frac{2 \dot{\theta}}{1 + \kappa^2 \cos^2 \theta}, \quad p_\psi = \frac{\kappa m \cos^4 \theta \sin(2\psi) + 2 \cos^2 \theta \dot{\psi}}{1 + \kappa^2 \cos^4 \theta \sin^2 \psi}, \quad (3.29)$$

and we have identified the energy with, $E = -p_t = 2 \ell$. We next find the Hamiltonian equation of motion using the ansatz mentioned in (3.15). The phase space is defined by the four coordinates: $\{\theta, p_\theta, \psi, p_\psi\}$. The constant energy surfaces ($E$) are defined by the equation (3.20). Keeping this in mind we solve the Hamiltonian equation of motion together with the constraint (3.15) for different values of $E$ and plot the phase space trajectories for both the canonical pairs $\{\theta, p_\theta\}$ and $\{\psi, p_\psi\}$. Surprisingly, we observe that for generic initial conditions even in the presence of small $\kappa$ as we increase the energy the trajectories becomes chaotic. Initially we identify that there are some kind of deformed tori in the phase space when energy is small but as we increase the energy these tori are completely destroyed and the trajectories moves freely in the phase space, the motion is only bounded by the total energy. We give the representative plots showing this behaviour below. This further supports our claim stemming from the NVE analysis that even for small $\kappa$ the system shows some signature of chaos for $(\mathbb{R} \times S^5)_\kappa$.

First for consistency check we set the initial condition for $\{\psi, p_\psi\}$ as $\{\psi(0) = 0, p_\psi(0) = 0\}$. So there will be no non trivial phase space trajectories in the $\{\psi, p_\psi\}$ plane. Only we will have non trivial trajectories in $\{\theta, p_\theta\}$ plane. In this case effectively what we are left with a harmonic oscillator type system characterized by $\{\theta, p_\theta\}$ and we should not observe any chaotic behaviour for any values of $\kappa$ and $E$ (energy) (this is exactly what happens for $(\mathbb{R} \times S^3)_\kappa$ where we get exactly one harmonic oscillator with a $\kappa$ dependent mass and hence we observe no chaos whatever be the values of $\kappa$ and energy.)

Next we choose more general boundary conditions where both the canonical pairs evolve. We plot the trajectories for both $\{\theta, p_\theta\}$ and $\{\psi, p_\psi\}$ below for different values of $E$ and $\kappa$. In all cases, we have set the winding number $m = 2$ for simplicity.
Figure 1: We have shown the trajectories in \( \{\theta, p_\theta\} \) plane and we have chosen the boundary condition such that the other canonical pair \( \{\psi, p_\psi\} \) remain at zero throughout the evolution. We set \( m = 2 \) and \( E = 5 \) in these plots and different colours corresponds to different values of \( \kappa \). As we can see even when \( \kappa \) is \( \mathcal{O}(1) \) trajectories are ordered and the phase space is foliated by constant energy surfaces which are the various ellipses as shown in the figure.

\[ \kappa = 0 \]

\[ \kappa = 0.1 \]

All the figures in the left show the phase space plot for \( \{\theta, p_\theta\} \) and all the figures in right show the phase space plot for \( \{\psi, p_\psi\} \). As it is evident from these plots that \( \kappa = 0 \) trajectories are ordered as expected, but even for small \( \kappa \), for example if we consider \( \kappa = 0.1 \) the trajectories become chaotic for all the values of the energy. The trajectories move freely in the phase space but within each of the energy envelopes (upto some numerical errors).
Similarly we plot these phase space trajectories for higher values of $\kappa$ in subsequent plots. We see the chaos persists and for high values of $\kappa$, for example if we look at the $\kappa = 100$ all the points in the phase space seem to concentrate near the edges of each of the energy contour. It is expected because the oscillators become very massive for higher values of $\kappa$ and the points in the phase space do not move much. This is also in agreement with our physical intuition.

4. Strings in the deformed $AdS$

4.1 Revisiting the case of $(AdS_3)_\kappa$

We again revisit the case of strings in $(AdS_3)_\kappa$ as already discussed in [61], and start by putting $\zeta = \frac{\pi}{2}$ in the metric, which makes the NS-NS two-form zero. Moreover, identifying
\( \psi_2 = \psi \), we write the relevant metric in this case,

\[
\begin{align*}
\text{ds}_{(AdS_3)_\kappa}^2 &= \frac{1}{1 - \kappa^2 \sinh^2 \rho} \left[ -\cosh^2 \rho \ dt^2 + d\rho^2 \right] + \sinh^2 \rho \ d\psi^2. \\
\end{align*}
\]

(4.1)

Starting with a simple circular string ansatz as the following form,

\[
\begin{align*}
t &= t(\tau), \quad \rho &= \rho(\tau), \quad \psi = m\sigma,
\end{align*}
\]

(4.2)

we can write down the equations of motion in the following form,

\[
\begin{align*}
&\cosh \rho \left[ 2 \cosh \rho \ i \left( \kappa^2 - \kappa^2 \cosh(2\rho) + 2 \right) + 8 (1 + \kappa^2) \sinh \rho \ i \dot{\rho} \right] = 0, \\
&\sinh(2\rho) \left[ m^2 \left( 1 - \kappa^2 \sinh^2 \rho \right)^2 + \kappa^2 \dot{\rho}^2 + (1 + \kappa^2) \ i^2 \right] + 2 \dot{\rho} \left( 1 - \kappa^2 \sinh^2 \rho \right) = 0.
\end{align*}
\]

These equations are trivially satisfied by \( \rho = 0 \) and \( \dot{\rho} = 0 \), which give us an invariant plane to work with, provided we have the solution,

\[
t(\tau) = \alpha\tau.
\]

(4.3)

Also there is the Hamiltonian constraint to be satisfied,

\[
\frac{1}{1 - \kappa^2 \sinh^2 \rho} \left[ -\cosh^2 \rho \ i^2 + \dot{\rho}^2 \right] + m^2 \sinh^2 \rho = 0.
\]

(4.4)

This in turn means that at \( \rho = 0 \), we should have \( \dot{i} = 0 \). Using the expansion

\[
\rho(\tau) = 0 + r(\tau), \quad |r| << 1,
\]

(4.5)

the desired NVE simply has the following form

\[
\ddot{r} + \left[ m^2 + (1 + \kappa^2)\ i^2 \right] r = 0.
\]

(4.6)

As we have discussed above, this is simply a Harmonic Oscillator equation of motion and hence is completely solvable.

**A concrete example: Extended ‘Spiky’ strings**

For the sake of completeness, we here mention the case of ‘spiky’ strings [92] in \((AdS_3)_\kappa\) which, unlike circular strings, are extended object and has been well studied in the literature [38]. To discuss these strings in the Polyakov framework, the worldsheet embedding is quite general, and has been discussed in [93]. We start here with that particular ansatz\(^5\) of the form,

\[
\begin{align*}
t &= \tau + f(\sigma), \quad \rho = \rho(\sigma), \quad \psi = \omega \tau + g(\sigma).
\end{align*}
\]

(4.9)

\(^5\)Here we again note that the ansatz for such strings discussed in [61] i.e.

\[
\begin{align*}
t &= \tau, \quad \rho = \rho(\tau), \quad \psi(\sigma, \tau) = m\sigma + \psi(\tau),
\end{align*}
\]

(4.7)

is completely incompatible with the Virasoro constraints. Notably, the constraint \( T_{\tau \sigma} = 0 \) leads to the condition

\[
\sinh^2 \rho \ m\dot{\psi} = 0,
\]

(4.8)

which forces either \( m \) or \( \dot{\psi} \) to be zero, rendering the ansatz inconsistent.
Note here the AdS radial direction is not dependent on the worldsheet time coordinate. The equations of motion for \( t \) and \( \psi \) here gives rise to

\[
f'(\sigma) = \frac{C_1(1 - \kappa^2 \sinh^2 \rho)}{\cosh^2 \rho}, \quad g'(\sigma) = \frac{C_2}{\sinh^2 \rho}.
\] (4.10)

Where \( C_{1,2} \) are just constants. Also the \( \rho \) equation is given by,

\[
\sinh(2\rho) \left( \left( 1 + \kappa^2 \right) (f'^2 - 1) + \kappa^2 \rho^2 \right) + \sinh(2\rho) (\omega^2 - g'^2) + \frac{\rho''(\sigma)}{1 - \kappa^2 \sinh^2 \rho} = 0.
\]

Note that \( \rho = 0 \) and \( \rho' = 0 \) is a trivial solution for the above equation of motion, although \( g' \) seems to be diverging at this limit. This is, as usual, supplemented by the Virasoro constraints. Here the constraint \( T_{\tau\tau} + T_{\sigma\sigma} = 0 \) gives simply the Hamiltonian, which in turn is consistent with the \( \rho \) equation. The other constraint \( T_{\tau\sigma} = 0 \) gives a nice relation between the constants,

\[-C_1 + \omega C_2 = 0.\] (4.11)

So we can easily choose \( C_2 = \frac{C_1}{\omega} \) without loss of generality. With these inputs and explicit expressions of \( f' \) and \( g' \) at hand, we can now expand the \( \rho \) equation as \( \rho = 0 + R(\sigma) \), with \( R << 1 \), to get the following NVE up to first order in \( R \),

\[
R'' - \left( (1 - C_1^2)(1 + \kappa^2) - \frac{C_1^2 \omega^2}{15 \omega^2} + \omega^2 \right) R - \frac{C_1^2}{\omega^2 R^3} = 0.
\] (4.12)

This is certainly reminiscent of the NVE’s we have dealt with in earlier sections, and a representative rational solution can be given by,

\[
R(\sigma) = \frac{\sqrt{2KC_1^2 \omega^2 + e^{-2\sqrt{2\sqrt{K} \omega}}} \sqrt{2\sqrt{K} \omega}}{\sqrt{2\sqrt{K} \omega} \sqrt{e^{-2\sqrt{2\sqrt{K} \omega}}}},
\] (4.13)

where we have defined,

\[
K = \left( (1 - C_1^2)(1 + \kappa^2) - \frac{C_1^2 \omega^2}{15 \omega^2} + \omega^2 \right),
\] (4.14)

which determines the trajectory of the string along the \( \sigma \) direction. We note here that if we want to discuss dynamics of such extended string where the radial direction is time dependent, we can perform a \( \sigma \leftrightarrow \tau \) exchange in (4.9) to transform it to the ‘Dual Spike’ solution [94], without any change in the analysis.

### 4.2 Analytical strings on \((AdS_5)_\kappa\)

The most important exercise would be to study string motion in the full \((AdS_5)_\kappa\) space-time for reaching a concrete conclusion in our case. To study circular strings in this background, we choose a particular simple ansatz as follows,

\[
t = t(\tau), \quad \rho = \rho(\tau), \quad \zeta = \zeta(\tau), \quad \psi_1 = \psi_2 = m\sigma.
\] (4.15)
Note here we have put both winding numbers to be same for simplicity. With this choice, the Lagrangian for the system of strings take the expression,

\[ L_{AdS} = \frac{1}{1 - \kappa^2 \sinh^2 \rho} \left[ - \cosh^2 \rho \dot{t}^2 + \dot{\rho}^2 \right] + \frac{\sinh^2 \rho}{1 + \kappa^2 \sinh^2 \rho} \left( \dot{\zeta}^2 - m^2 \cos^2 \zeta \right) - m^2 \sinh^2 \rho \sin^2 \zeta \]

\[ - \frac{\kappa \sinh^4 \rho \sin(2\zeta)}{1 + \kappa^2 \sinh^2 \rho} \frac{m \dot{\zeta}}{\sin^2 \zeta}. \]  

(4.16)

We will now write the explicit equations of motion for \( \zeta \),

\[ \sinh \rho \left[ \kappa^2 m^2 \sinh^5 \rho \left( 4\kappa^2 \sin^5 \zeta \cos \zeta \sinh^4 \rho - \sin(4\zeta) \right) \right. \]

\[ - 8\dot{\rho} \cosh \rho \left( \kappa m \sin(2\zeta) \sinh^2 \rho + \dot{\zeta} \left( \kappa^2 \sin^2 \zeta \sinh^4 \rho - 1 \right) \right) \]

\[ + 4\ddot{\zeta} \sinh \rho \left( \kappa^2 \sin^2 \zeta \sinh^4 \rho + 1 \right) - 2\kappa^2 \dot{\zeta}^2 \sin(2\zeta) \sinh^5 \rho \right] = 0. \]  

(4.17)

Also the equation for \( \rho \) can be written as,

\[ \frac{m^2 \cot^2 \zeta \sinh(2\rho)}{\csc^2 \zeta + \kappa^2 \sinh^4 \rho} - \frac{\kappa^2 m^2 \sin^2(2\zeta) \sinh^5(\rho) \cosh \rho}{(1 + \kappa^2 \sinh^2 \rho)^2} + M^2 \sin^2 \zeta \sinh(2\rho) + \]

\[ \frac{4\kappa m \dot{\zeta} \sin(2\zeta) \sinh^4 \rho \cosh \rho}{(1 + \kappa^2 \sinh^2 \rho)^2} + \frac{\dot{\zeta}^2 \sinh(2\rho)}{(1 + \kappa^2 \sinh^2 \rho)^2} - \frac{2\dot{\rho}}{1 - \kappa^2 \sinh^2 \rho} + \]

\[ \frac{\kappa^2 \rho^2 \sinh(2\rho)}{(1 - \kappa^2 \sinh^2 \rho)^2} + \frac{(1 + \kappa^2) \sinh(2\rho) \dot{t}^2}{(1 - \kappa^2 \sinh^2 \rho)^2} = 0. \]  

(4.18)

Similarly, the \( t \) equation has a form,

\[ \cosh \rho \left( 2 \cosh \rho \dot{t} \left( \kappa^2 + \kappa^2 (- \cosh(2\rho)) + 2 \right) + 8 \left( 1 + \kappa^2 \right) \rho \sinh \rho \dot{t} \right) = 0. \]  

(4.19)

These equations are not at all illuminating as one can see. Instead we can notice that the energy of this circular string is given by,

\[ E = \frac{\cosh^2 \rho \dot{t}}{1 - \kappa^2 \sinh^2 \rho}. \]

(4.20)

We can show that the equations of motion in this case also vanish as in the previous one for \( \rho = 0 \) and \( \dot{\rho} = 0 \), giving us an invariant plane to work with. Using this, we expand the \( \rho \) equation of motion with,

\[ \rho(\tau) = 0 + \mathcal{R}(\tau), \quad |\mathcal{R}| \ll 1. \]  

(4.21)

Then, the \( \rho \) NVE can be written in the following simple form up to the first order in \( \mathcal{R} \),

\[ \ddot{\mathcal{R}} + \mathcal{R} \left( E^2 (\kappa^2 + 1) + m^2 - \dot{\zeta}^2 \right) = 0. \]  

(4.22)
This is a remarkably simple equation for such a complex string background. Note that the effect of the singularity surface apparently vanishes here, since we are considerably closer to centre of the AdS space. To find $\dot{\zeta}$ we notice that the conserved angular momentum associated to $\zeta$ can be written as

$$J_\zeta = \frac{\sinh^2 \rho}{1 + \kappa^2 \sinh^2 \rho \sin^2 \zeta} \dot{\zeta} - \frac{\kappa \sinh^4 \rho \sin(2\zeta)}{2(1 + \kappa^2 \sinh^4 \rho \sin^2 \zeta)}.$$  \hspace{1cm} (4.23)$$

Near the invariant plane $\rho = 0$, this can be shown to lead us to the expansion,

$$\dot{\zeta} = \frac{2J_\zeta}{R^2} + O(R^2).$$ \hspace{1cm} (4.24)$$

As we have done before, we use the above in conjunction with (4.22) to write down the full form of the NVE. This equation, as evident, is completely solvable with rational functions. A representative solution can be written as,

$$R(\tau) = \pm \sqrt{\frac{4J_\zeta^2 (E^2 (1 + \kappa^2) + m^2) - e^{-4i\tau} \sqrt{E^2 (1 + \kappa^2) + m^2}}{2\sqrt{E^2 (1 + \kappa^2) + m^2} \sqrt{e^{-2i\tau} \sqrt{E^2 (1 + \kappa^2) + m^2}}}}.$$ \hspace{1cm} (4.25)$$

This analysis is strongly justified by the numerical calculations in the next section, where we show that no non trivial dynamics appear for these AdS strings in the phase space. Note that, throughout this section we have seen that the NVE for string motion in deformed AdS spaces also become weakly deformed Harmonic Oscillator problems, indicating the inherent simplicity of the motion itself.

### 4.3 Explicit numerical hamiltonian analysis

We repeat the same analysis for this case as we have done in section (3.3) for the case of sphere. We will consider only the circular string profile as mentioned in (4.15). We can leave the analysis for the extended spiky string for future investigation and instead focus only on these simple strings for our numerical experiment. The total Hamiltonian in this case is, as expected, very complicated. And the two conjugate momenta are defined below,

$$p_\rho = \frac{2\dot{\rho}}{1 - \kappa^2 \sinh^2 \rho}, \quad p_\zeta = \frac{2\dot{\zeta} \sinh^2 \rho - \kappa m \sin(2\zeta) \sinh^4 \rho}{1 + \kappa^2 \sinh^2 \zeta \sinh^4 \rho}.$$ \hspace{1cm} (4.26)$$

and again the energy is identified as

$$E = p_t = \frac{2 \cosh^2 \rho \dot{t}}{1 - \kappa^2 \sinh^2 \rho}.$$ \hspace{1cm} (4.27)$$

As before we solve the Hamiltonian equation of motions and we show the plots for the phase space trajectories for various values of $\kappa$ and energy $E$. Also from the metric (4.1) we note that there is a singularity surface at $\rho = \sinh^{-1} \left( \frac{1}{\kappa} \right)$. So the larger the value of $\kappa$ lesser becomes the range of the $\rho$, and the phase space plot of the evolution of $\rho$ will be restricted for each of the values of $E$. Keeping this in mind we show the phase-space
trajectories for both \( \{\rho, p_\rho\} \) and \( \{\zeta, p_\zeta\} \) plane. Also we set the winding number \( m = 2 \) throughout.

At \( \kappa = 0 \) there is no chaos as expected and this provides a benchmark for our numerics. Then we proceed to analyze other cases with non vanishing of \( \kappa \).

From these plots we can easily infer that for AdS within the range of the values of \( \kappa \) and \( E \) that we have considered the trajectories remains always ordered (upto some numerical errors) and hence it is in good agreement with our conclusion from NVE analysis that unlike the \( S^5_\kappa \) case, there is no chaotic behaviour for the case of \( AdS_\kappa \).

5. Summary and Conclusion

Let us first summarize the paper briefly. In this note, we set out on a humble quest, to settle the issue of analytic/numeric integrability techniques clashing with well-known algebraic formulation for Yang-Baxter deformed \( AdS_5 \times S^5 \) case. Although the methods of studying classical integrability of an exact string background have been believed to be equivalent to each other, we find evidences suggesting the contrary. Starting from revisiting the calculations provided in [61], we conclude there is no such analytical/numerical evidence of chaotic motion appearing in the string phase space for \((AdS_5)_\kappa\) and \((\mathbb{R} \times S^5)_\kappa\) case. However, we find that for a rigidly spinning circular string moving in \((\mathbb{R} \times S^5)_\kappa\), the motion surprisingly runs into chaos when we turn on a non-zero value of \( \kappa \). We show both analytically via perturbations around the classical trajectories and via numerical experiments that irregular evolution of trajectories indeed occur in this case. Surprisingly, the case of \((AdS_5)_\kappa\), which has a prominent space-time singularity, doesn’t show any evidence of chaotic string motion.

This rather shocking revelation puts us in crossroads about how we view the notion of classical integrability in this case from different vantage points. If there exists one such dynamical model truncation for the string system, where the differential equations are not integrable, the phase space definitely has problems. Non-integrability often does not explicitly lead to chaos, but for our case, it is evident in the Poincare sections. There could easily be some added subtlety to the case of \((\mathbb{R} \times S^5)_\kappa\), which is not captured by our analysis here, and which could stabilize the solution against irregular perturbations. However, the idea of what that could be eludes us as of now. Another viable point that can be considered, comes from the discussion presented in [65]. There, it was explicitly showed that at the fast spinning string limit the equations of motion for strings in \((\mathbb{R} \times S^5)_\kappa\) maps to that of a complex \( \beta \)-deformed sphere. This background has been shown to be classically non-integrable via analytical/numerical techniques [81]. We speculate that this might have deeper implications that we have had thought earlier, although in this paper we are not exactly taking the fast spinning limit anywhere.

The general question about different methods to check classical (non)-integrability, however, persists strongly. This work has been a standalone example towards scratching the surface of this mystery. One might try to find an answer to this via exploring other well-known but non-trivial classical string backgrounds. A very useful exercise perhaps would be to study the BTZ black hole background. BTZ has been known to be classically
integrable [95] for few years now. But since this background contains event horizons, one
would easily guess that string motion becomes irregular near these horizons. One could
then investigate string motion in BTZ background using the procedures used in this paper.
This might give more insight into this apparent disparity of discussing string integrability in
this context. However, all of this still remains speculations, and certainly require rigorous
understanding. We plan to come back to these concerns in the near future.

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Appendix: Effect of $\kappa \neq 0$ in NVE via Kovacic algorithm

We here study the NVE for $\theta(\tau)$ for the case of $(R \times S^5)_\kappa$ analytically. We will apply
Kovacic’s algorithm to study in detail whether it will have a Liouvillian solution or not.

We start with the $\theta$ equation of motion as written in (3.18). As we have seen earlier,
$\theta = 0, \dot{\theta} = 0$ is a trivial solution for this equation of motion. So, to find the NVE we would
consider fluctuation around this solution. At this point, the Virasoro constraint gives us
the equation of $\psi$ on this invariant plane,

$$\dot{\psi}^2 = E^2(1 + \kappa^2 \sin^2 \psi) - m^2 \cos^2 \psi. \quad (5.1)$$

The solution for $\psi(\tau)$ on this invariant plane can then be written as a jacobi function,

$$\cos \psi(\tau) = \text{sn} \left[ \sqrt{1 + \kappa^2 E\tau} \left| \frac{E^2 \kappa^2 + m^2}{E^2(1 + \kappa^2)} \right. \right]. \quad (5.2)$$

Then expanding the equation of motion up to linear order in the fluctuation $\theta(\tau) = 0 + y(\tau)$,
we obtain the required NVE,

$$\frac{2\ddot{y}(\tau)}{1 + \kappa^2} + y(\tau) \left[ \frac{m^2 (\kappa^2 \sin^2(2\psi) - 4 \cos^2 \psi) + 8km\dot{\psi} \sin(2\psi) + 4\dot{\psi}^2 (1 - \kappa^2 \sin^2 \psi)}{2 (1 + \kappa^2 \sin^2 \psi)^2} \right] = 0. \quad (5.3)$$

Now we perform the change of variable as $\tau \rightarrow z = \cos \psi(\tau)$. After some involved
algebra, the above equation takes the following form.

$$y''(z) + \mathcal{B}(z)y'(z) + \mathcal{A}(z)y(z) = 0, \quad (5.4)$$

\[
\]
where the primes denote derivatives w.r.t $z$ and the coefficient functions are,

$$
B = z \left[ \frac{E^2 \kappa^2 + m^2}{z^2 (E^2 \kappa^2 + m^2) - E^2 (1 + \kappa^2)} + \frac{1}{z^2 - 1} \right], \quad A = \frac{\mathcal{C}}{\mathcal{D}};
$$

$$
C = (1 + \kappa^2) \left[ - 16 z \kappa \left( 1 - z^2 \right) \left( E^2 (1 + \kappa^2) - z^2 \left( E^2 \kappa^2 + m^2 \right) \right) + (1 - z^2) \left( 4 \kappa^2 \left( z^2 - 1 \right) + 4 \right) \left( E^2 (1 + \kappa^2) - z^2 \left( E^2 \kappa^2 + m^2 \right) \right)^2 
+ 4m^2 z^2 \left( \kappa^2 - \kappa^2 z^2 - 1 \right) \right],
$$

$$
\mathcal{D} = 4 \left( z^2 - 1 \right) \left( \kappa^2 (z^2 - 1) - 1 \right)^2 \left[ (E^2 \kappa^2 + m^2)z^2 - E^2(1 + \kappa^2) \right].
$$

Now we can see that $\mathcal{A}$ and $\mathcal{B}$ are rational polynomial of $z$. Then we can bring the equation in the following normal or Schrodinger form

$$
w''(z) + V(z)w(z) = 0, \quad (5.6)
$$

via the transformation of variables as,

$$
y(z) \rightarrow w(z) = y(z) \exp \left[ - \frac{1}{2} \int B(x) \, dx \right]. \quad (5.7)
$$

The potential for this equation can be given by,

$$
V(z) = - \frac{1}{4} \left( 2B' + B^2 - 4A \right). \quad (5.8)
$$

Explicitly,

$$
V(z) = - E^2 \kappa^2 + \frac{3z^2 \left( E^2 - m^2 \right)^2}{4 \left( z^2 - 1 \right)^2 \left( E^2 ( \kappa^2(z^2 - 1) - 1) + m^2 z^2 \right)^2} - \frac{2z^2 \left( mz \left( z^2 - 1 \right) (2\kappa + mz) + 1 \right)}{\left( \kappa^2 (z^2 - 1)^2 - z^2 + 1 \right)^2}
- \frac{\left( E^2 - m^2 \right) \left( 4E^2 - m^2 \left( 3z^2 + 1 \right) \right)}{2m^2 \left( z^2 - 1 \right)^2 \left( E^2 ( \kappa^2(z^2 - 1) - 1) + m^2 z^2 \right)} + \frac{z^4 \left( - (E^2 + m^2) \right) + E^2 + m^2 z^2 + 1}{\left( z^2 - 1 \right)^2}
+ \frac{E^2 \left( 2 - 2m^2 z^2 \left( z^2 - 1 \right) \right) + mz^3 \left( - (z - 1) \right)z(z + 1) \left( 4\kappa + mz \left( z^2 + 2 \right) \right) - m^2 \left( z^2 + 2 \right)}{m^2 \left( z^2 - 1 \right)^2 \left( \kappa^2 (z^2 - 1) - 1 \right)} \quad (5.9)
$$

Now (5.6) is a linear second order differential equation and also in the correct form for applying Kovacic algorithm [91] to test whether it admits Liouvillian solutions. Now according to Kovacic algorithm, the potential should at least satisfy one of the following three necessary (but not sufficient) criteria so that the differential equation (5.6) and hence (5.4) will admit Liouvillian solution.

- I: All the poles of $V(z)$ will be either of order 1 or even order and the order of $V(z)$ at infinity has to be either even or greater than 2. The order of $V(z)$ at infinity can computed by the subtraction of the highest power of $z$ in numerator from the highest power of $z$ in denominator.
• II: All the poles of $V(z)$ will be of odd order greater than 2 or it will posses just one pole of order 2. 6

• III: Order of all the poles of $V(z)$ are less than or equal to 2 and order of $V(z)$ at infinity has to be at least order 2.

These conditions can be proven to be equivalent to the differential-Galois group treatment for differential equations. Now we can check that the potential $V(z)$ mentioned in (5.6) violates all of these three conditions and hence our normal variation equation for the $\theta$ does not admit Liouvillian solutions. This is consistent with our numerical results presented in the section (3.2). Putting $\kappa = 0$ one can easily check that this process succeeds as one would expect for the undeformed five sphere.

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