On regular hypersemigroups

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Abstract. It is shown that an hypersemigroup \((S, \circ)\) is regular if and only if the set of all quasi-ideals of \(S\) with the operation \(\ast\) is a von Neumann regular semigroup. It is both regular and intra-regular if and only if the set of all quasi-ideals of \(S\) with the operation \(\ast\) is a band.

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It has been shown in Semigroup Forum [2] that an le-semigroup \((S, \cdot, \leq)\) is regular if and only if the set \(Q\) of all quasi-ideal elements of \(S\) with the multiplication \(\cdot\) of \(S\) is a von Neumann regular semigroup. Moreover, it has been proved that if \(S\) is both regular and intra-regular, then \((Q, \cdot)\) is a band. “Conversely”, if the quasi-ideal elements of \(S\) are idempotent, then \(S\) is both regular and intra-regular. As a consequence, an le-semigroup \(S\) is both regular and intra-regular if and only if \((Q, \cdot)\) is a band.

As an example to the paper in Turkish J. Math. [7], we examine the above results on lattice ordered semigroups in case of an hypersemigroup. An hypersemigroup \((S, \circ)\) is called regular if for every \(a \in S\) there exists \(x \in S\) such that \(a \in (a \circ x) \ast \{a\}\); that is, for every \(a \in S\) there exists \(y \in a \circ x\) such that \(a \in y \circ a\). It is called intra-regular if for every \(a \in S\) there exist \(x, y \in S\) such that \(a \in (x \circ a) \ast (a \circ y)\); that is, for every \(a \in S\) there exist \(x, y \in S\), \(u \in x \circ a\) and \(v \in a \circ y\) such that \(a \in u \circ v\). A subset \(A\) of an hypersemigroup \((S, \circ)\) is called idempotent if \(A \ast A = A\). For notations and definitions not given in the present paper we refer to [7].

Lemma 1 [3] Let \((S, \circ)\) be an hypersemigroup. If \(S\) is regular, then the right ideals and the left ideals of \(S\) are idempotent and for every right ideal \(A\) and every left ideal \(B\) of \(S\), the product \(A \ast B\) is a quasi-ideal of \(S\).

Lemma 2 [4, 5] An hypersemigroup \((S, \circ)\) is regular if and only if, for any nonempty subset \(A\) of \(S\), we have \(A \subseteq A \ast S \ast A\).

Lemma 3 Let \((S, \circ)\) be an hypersemigroup, \(A\) a right ideal and \(B\) a left ideal of \(S\). Then the intersection \(A \cap B\) is a quasi-ideal of \(S\).
Lemma 1, they are idempotent and we have

\[ \text{In fact: Since } S \text{ and } A \subseteq S \]

**Proof** First of all, since \( A \) is a right ideal and \( B \) is a left ideal of \( S \), the intersection \( A \cap B \) is nonempty. Indeed: Take an element \( a \in A \) and an element \( b \in B \) \((A, B \neq \emptyset)\); then \( a \circ b \subseteq A \circ B \subseteq A \subseteq A \circ S \subseteq A \) and \( a \circ b \subseteq A \circ B \subseteq S \circ B \subseteq B \), so \( a \circ b \subseteq A \cap B \). Since \( a \circ b \) is a nonempty set, the set \( A \cap B \) is nonempty as well (see also \[5\]). We also have

\[ \left((A \cap B) \circ S\right) \cap \left(S \circ (A \cap B)\right) \subseteq (A \circ S) \cap (S \circ B) \subseteq A \cap B, \]

thus \( A \cap B \) is a quasi-ideal of \( S \).

Lemma 4 \([4, 5]\) An hypersemigroup \((S, \circ)\) is regular if and only if, for every right ideal \( A \) and every left ideal \( B \) of \( S \), we have \( A \cap B \subseteq A \circ B \) (equivalently, \( A \cap B = A \circ B \)).

**Lemma 5** If \((S, \circ)\) is a regular hypersemigroup, then \( S \circ S = S \).

**Proof** Since \( S \) is regular, for every nonempty subset \( A \) of \( S \), by Lemma 2, we have \( A \subseteq A \circ S \circ A \). Thus we have \( S \subseteq (S \circ S) \circ S \subseteq S \circ S \subseteq S \) and so \( S \circ S = S \).

A semigroup \((S, \cdot)\) is called von Neumann regular (or just regular) if for each \( a \in S \) there exists \( x \in S \) such that \( a = axa \) \([1, 8]\).

As always, \( \mathcal{P}^*(S) \) denotes the set of all nonempty subsets of \( S \).

**Theorem 6** An hypersemigroup \((S, \circ)\) is regular if and only if the set \( Q \) of all quasi-ideals of \( S \) with the multiplication “\( \ast \)” of \( \mathcal{P}^*(S) \) is a von Neumann regular semigroup.

**Proof** \( \Rightarrow \). First of all, for every quasi-ideal \( Q \) of \( S \), we have

\[ Q = (Q \ast S) \cap (S \ast Q) \quad (1) \]

In fact: Since \( S \) is regular, \( R(Q) \) is a right ideal and \( L(Q) \) is a left ideal of \( (S, \circ) \), by Lemma 1, they are idempotent and we have

\[ Q \subseteq Q \cup (Q \ast S) = R(Q) = R(Q) \ast R(Q) = \left(Q \cup (Q \ast S)\right) \ast \left(Q \cup (Q \ast S)\right) \]

\[ = Q \ast Q \cup Q \ast S \cup Q \ast Q \ast S \cup Q \ast S \cup S \ast Q \ast S \ast Q \ast S \subseteq Q \ast S \]

and

\[ Q \subseteq Q \cup (S \ast Q) = L(Q) = L(Q) \ast L(Q) = \left(Q \cup (S \ast Q)\right) \ast \left(Q \cup (S \ast Q)\right) \]

\[ = Q \ast Q \cup S \ast Q \cup Q \ast Q \cup S \ast Q \cup S \ast Q \ast S \ast Q \ast S \subseteq S \ast Q. \]

Thus we have \( Q \subseteq (Q \ast S) \cap (S \ast Q) \subseteq Q \), then \( Q = (Q \ast S) \cap (S \ast Q) \) and property \((1)\) is satisfied.

In addition, since \( S \) is regular, \( A \) is a right ideal and \( B \) is a left ideal of \( S \), by Lemmas 3 and 4, \( A \ast B \) is a quasi-ideal of \( S \). So, by \((1)\), we have

\[ A \ast B = (A \ast B \ast S) \cap (S \ast A \ast B) \quad (2) \]

We are ready now to prove that \((Q, \ast)\) is a von Neumann regular semigroup. In this respect, we prove the following:
\((Q, \ast)\) is semigroup. Indeed: First of all, in an hypersemigroup, the operation “\ast” is associative (see [5], also [6; p. 22]). Let now \(Q_1, Q_2\) be quasi-ideals of \(S\). Then \(Q_1 \ast Q_2\) is a quasi-ideal of \(S\). Indeed: Since \(S\) is regular, \(Q_1 \ast Q_2 \ast S\) is a right ideal and \(S \ast Q_1 \ast Q_2\) is a left ideal of \(S\), by Lemma 1, they are idempotent and we have

\[
(Q_1 \ast Q_2) \ast S \subseteq (S \ast (Q_1 \ast Q_2))
\]

Thus we have

\[
\begin{align*}
(Q_1 \ast Q_2 \ast S) & \ast (Q_1 \ast Q_2 \ast S) \ast (Q_1 \ast Q_2) \ast (S \ast Q_1 \ast Q_2) \\
& \ast (Q_1 \ast Q_2 \ast S) \ast (Q_1 \ast Q_2) \ast (S \ast Q_1 \ast Q_2) \\
& \ast (Q_1 \ast Q_2 \ast S) \ast (Q_1 \ast Q_2) \ast (S \ast Q_1 \ast Q_2) \\
& \ast (Q_1 \ast Q_2 \ast S) \ast (Q_1 \ast Q_2) \ast (S \ast Q_1 \ast Q_2) \\
& \subseteq (Q_1 \ast (Q_2 \ast S \ast Q_2)) \\
& \subseteq Q_1 \ast (Q_2 \ast S \ast Q_2) \\
& \subseteq Q_1 \ast Q_2 \ast S \ast Q_2 \\
& \subseteq Q_1 \ast Q_2 \ast Q_2 \ast S \\
& \subseteq Q_1 \ast Q_2 \ast Q_2 \ast S \\
& \subseteq Q_1 \ast Q_2 \ast Q_2 \ast S \\
& \subseteq Q_1 \ast Q_2 \ast Q_2 \ast S
\end{align*}
\]

Hence \(Q_1 \ast Q_2\) is a quasi-ideal of \(S\). Thus \((Q, \ast)\) is semigroup.

The semigroup \((Q, \ast)\) is a von Neumann regular semigroup. In fact: Let \(Q \subseteq Q \ast S \ast Q \subseteq (Q \ast S) \cap (S \ast Q) \subseteq Q\).

Then \(Q = Q \ast S \ast Q\), where \(S \subseteq Q\) and so \((Q, \ast)\) is a von Neumann regular semigroup.

\(\iff\). We remark first that for each quasi-ideal \(Q\) of \(S\), we have

\[
Q = Q \ast S \ast Q
\]

In fact: Let \(Q\) be a quasi-ideal of \(S\). Since \((Q, \ast)\) is von Neumann regular semigroup, there exists \(X \in Q\) such that \(Q = Q \ast X \ast Q\). Then

\[
Q = Q \ast X \ast Q \subseteq Q \ast S \ast Q \subseteq (Q \ast S) \cap (S \ast Q) \subseteq Q
\]

Thus we have \(Q = Q \ast S \ast Q\) and property (3) holds.

We are ready now to prove that \((S, \circ)\) is regular. For this, let \(A\) be a nonempty subset of \(S\). By Lemma 2, it is enough to prove that \(A \subseteq A \ast S \ast A\).

Since \(R(A)\) is a right ideal and \(L(A)\) is a left ideal of \(S\), by Lemma 3, \(R(A) \cap L(A)\) is a quasi-ideal of \(S\). Then, by (3), we have

\[
A \subseteq R(A) \cap L(A) = \left( R(A) \cap L(A) \right) \ast S \ast \left( R(A) \cap L(A) \right)
\]

\[
\subseteq \left( R(A) \ast S \right) \ast L(A) \subseteq R(A) \ast L(A)
\]

\[
= \left( A \cup (A \ast S) \right) \ast \left( A \cup (S \ast A) \right)
\]
then $A \ast A \subseteq A \ast A \cup A \ast S \ast A \ast S \ast A$, thus we obtain $A \subseteq A \ast S \ast A$ and so the hypersemigroup $(S, \circ)$ is regular. \hfill $\square$

**Lemma 7** \cite{4,5} An hypersemigroup $(S, \circ)$ is intra-regular if and only if, for every right ideal $A$ and every left ideal $B$ of $S$, we have $A \cap B \subseteq B \ast A$.

An element $a$ of a semigroup $S$ is called idempotent if $a^2 = a$. An idempotent semigroup or shorter a band is a semigroup in which all elements are idempotent.

**Theorem 8** Let $(S, \circ)$ be an hypersemigroup. If $(S, \circ)$ is both regular and intra-regular, then the set $Q$ of all quasi-ideals of $S$ with the operation “$\ast$” is a band. “Conversely”, if the quasi-ideals of $(S, \circ)$ are idempotent, then $S$ is both regular and intra-regular.

**Proof** $\implies$. Let $(S, \circ)$ be both regular and intra-regular. Since $(S, \circ)$ is regular, by Theorem 6, $(Q, \ast)$ is a semigroup. Moreover, the elements of the semigroup $Q$ are idempotent.

In fact: Let $Q$ be a quasi-ideal of $S$. Since $S$ is regular, we have $Q = Q \ast S \ast Q$ (cf. the proof of Theorem 6). Hence we have

$$Q = Q \ast S \ast Q = (Q \ast S \ast Q) \ast S \ast (Q \ast S \ast Q)$$

$$= (Q \ast S \ast Q) \ast S \ast (Q \ast S \ast Q) \quad (\text{by Lemma 5})$$

$$= (Q \ast S) \ast (Q \ast S) \ast (S \ast Q) \ast (S \ast Q).$$

Since $S$ is intra-regular and $Q \ast S$ is a right ideal and $S \ast Q$ is a left ideal of $S$, by Lemma 7, we have $(Q \ast S) \cap (S \ast Q) \subseteq (S \ast Q) \ast (Q \ast S)$. Thus we have

$$Q = (Q \ast S) \ast (Q \ast S) \ast (S \ast Q) \ast (S \ast Q)$$

$$\subseteq (Q \ast S) \ast (S \ast Q) \ast (Q \ast S) \ast (S \ast Q)$$

$$= (Q \ast S \ast S \ast Q) \ast (Q \ast S \ast S \ast Q)$$

$$= (Q \ast S \ast Q) \ast (Q \ast S \ast Q) \quad (\text{by Lemma 5})$$

$$= Q \ast Q \subseteq (Q \ast S) \cap (S \ast Q) \subseteq Q,$$

and $Q \ast Q = Q$. Hence $(Q, \ast)$ is an idempotent semigroup and so is a band.

$\impliedby$. Let $A$ be a right ideal and $B$ a left ideal of $S$. By Lemma 3, $A \cap B$ is a quasi-ideal of $S$. By hypothesis, we have $A \cap B = (A \cap B) \ast (A \cap B) \subseteq A \ast B$, $B \ast A$. Since $A \cap B \subseteq A \ast B$, by Lemma 4, $S$ is regular. Since $A \cap B \subseteq B \ast A$, by Lemma 7, $S$ is intra-regular. \hfill $\square$

**Corollary 9** An hypersemigroup $(S, \circ)$ is both regular and intra-regular if and only if the set $Q$ of all quasi-ideals of $S$ with the operation “$\ast$” is a band.

**Proof** If $(Q, \ast)$ is a band, that is an idempotent semigroup, then for every $Q \in Q$, we have $Q \ast Q = Q$, that means that the quasi-ideals of $(S, \circ)$ are idempotent so, by Theorem 8, $S$ is both regular and intra-regular. \hfill $\square$
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