Exact Calculation of Ellipse Perimeter by Analytical Method

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Abstract

As you know, we do not have any exact equation for calculating the perimeter of an ellipse. In this article, we obtain this equation analytically.

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Introduction

Consider the following equation:

\[ r = \frac{ed}{1 - e\cos \theta} \] (1)

This equation is the equation of a conic section in the polar coordinate system, which is obtained using the directrix-focus property of conic sections based on Figure 1 [1],[2].

Fig. 1. This figure shows an ellipse as a conic section. Based on the directrix-focus property, we have: \(|OQ|/|OP| = e\). In an ellipse because of \(0 < e < 1\) we have: \(re_0 \cos \theta < r \cos \theta\), which you can observe it in the figure.

In equation 1, \(d\) is the distance between a conic section directrix line and the focal point close to it [1],[2]. For an ellipse with Semi major axis \(a = a_0\) and \(e = e_0\) we have: \(d = \frac{a_0}{e_0} - e_0 a_0\) [1]. Substituting \(d\), equation 1 for this ellipse is as follows [1]:

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\[ r = \frac{a_0(1 - e_0^2)}{1 - e_0 \cos \theta} \quad (2) \]

Now we want to obtain the ellipse perimeter using equation 2. Using the formula

\[ L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \quad (3) \]

Which is the equation of calculation of the length of a polar curve [1],[3] and according to equation 2, we have:

\[ L_{\text{Ellipse}} = a_0(1 - e_0^2) \int_{0}^{2\pi} \sqrt{\frac{(1 - e_0 \cos \theta)^2 + e_0^2 \sin^2 \theta}{(1 - e_0 \cos \theta)^4}} \, d\theta \quad (4) \]

As you can see, the above integral is clearly difficult to solve (probably the above integral is a kind of elliptic integral). So we need to find the perimeter of the ellipse in another way. First we write equation 2 as follows:

\[ r - re_0 \cos \theta = a_0(1 - e_0^2) \quad (5) \]

If we consider the left side of equation 5 equal to \( \eta \), then we have

\[ \eta = a_0(1 - e_0^2) \quad (6) \]

It is clear, in the above equation, \( \eta \) is a function of \( \theta \): \( \eta = \eta(\theta) \) like \( r = r(\theta) \). Inasmuch as \( r^2(Sin^2 \theta + Cos^2 \theta) = a^2 \) or \( r = a \) is the equation of a circle, we can conclude that equation 6 is the equation of a circle with radius \( a_0(1 - e_0^2) \). This means that equation 2 is both the equation of an ellipse and the equation of a circle. So to calculate the perimeter of an ellipse, we compute the perimeter of its equivalent circle namely equation 6, instead of calculating integral 4. Using integral 3 and equation 6, we have:

\[ L_{\text{Ellipse}} = L_{\text{Equivalent Circle}} = \int_{0}^{2\pi} \eta d\theta = 2\pi a_0(1 - e_0^2) = 2\pi a_0 - 2\pi a_0 e_0^2 \quad (7) \]

The above equation is our ellipse perimeter equation. If you look at equation 7, according to figure 2, you will see that the first sentence of the right side (namely \( 2\pi a_0 \)), is the perimeter of a circle with radius \( a_0 \) \( (L_{c_2}) \). This circle surrounds the ellipse as shown in figure 2, and its center coincides with the center of the ellipse. Equation 7 shows that the perimeter of the ellipse is less than the perimeter of the circle \( C_2 \), as expected.
We can write equation 7 in another way

\[ b_0 = a_0 \sqrt{1 - e_0^2} \Rightarrow L_{Ellipse} = 2\pi \frac{b_0^2}{a_0} \] (8)

Where \( b_0 \) is the Semi minor axis of the ellipse.

Where is the location of \( C_{equivalent} \) in Fig. 2? In an ellipse we have: \( 0 < e_0 < 1 \). Therefore:

\[ a_0 \sqrt{1 - e_0^2} < a_0 (1 - e_0^2) < a_0 \]

\[ \Rightarrow 2\pi a_0 \sqrt{1 - e_0^2} < 2\pi a_0 (1 - e_0^2) < 2\pi a_0 \Rightarrow L_{c_1} < (L_{c_{equivalent}} = L_{Ellipse}) < L_{c_2} \] (9)

As shown in Figure 2, four tangent lines to the ellipse can be drawn from points A and B. These four lines are also tangent to the red circle. On the other hand, there is a green circle that is tangent to the orange square faces. As shown in Figure 2, the perimeters of both green and red circles are greater than the perimeter of circle \( C_1 \) and less than the perimeter of circle \( C_2 \). So the inequality 9 is true about them, and therefore, probably, one of the red or green circles is the circle \( C_{equivalent} \).
Of course, **maybe none** of them. Many other circles can be drawn with center of \( O \) to satisfy inequality 9. We only guessed here that maybe one of the two green and red circles is our circle \( C_{equivalent} \).

Finally, I need to point out that the area of the circle \( C_{equivalent} \) and its corresponding ellipse are not equal:

\[
A_E = \pi a_0 b_0 = \pi a_0^2 \sqrt{1 - e_0^2} \quad \text{and} \quad A_{C_{equivalent}} = \pi \eta^2 = \pi a_0^2 (1 - e_0^2)^2 \quad \Rightarrow \quad A_{C_{equivalent}} \neq A_E
\]

**Conclusion**

It seems that after more than 300 years, we have been able to obtain the exact equation of the perimeter of an ellipse. I think using the method of this article can also lead us to the exact equation of the area of an ellipsoid.

**References:**

[1]. Silverman, R. *Calculus with Analytic Geometry* (Prentice-Hall, Inc. New Jersey. 1985), pp. 628-641
[2]. Johnson, R. *Calculus with Analytic Geometry* (Allyn and Bacon. Inc. Boston, ed. 4), pp. 536-539
[3]. Sherman, S. *Calculus with Analytic Geometry* (McGraw-Hill, Inc., ed. 3, 1982), pp. 548-554