High-Temperature Dynamics of Spin Glasses

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We develop a systematic expansion method of physical quantities for the SK model and the finite-dimensional ±J model of spin glasses in non-equilibrium states. The dynamical probability distribution function is derived from the master equation using a high temperature expansion. We calculate the expectation values of physical quantities from the dynamical probability distribution function. The theoretical curves show satisfactory agreement with Monte Carlo simulation results in the appropriate temperature and time regions. A comparison is made with the results of a dynamics theory by Coolen, Laughton and Sherrington.

KEYWORDS: dynamics, SK model, ±J model, dynamical probability distribution function, high-temperature expansion, CLS theory

§1. Introduction

Dynamics plays essential roles in understanding experimental observations on spin glasses. It is very difficult, however, to develop systematic theoretical methods to investigate dynamical behavior of spin glass models, and numerical approaches have been the major source of information on dynamical properties of finite-ranged spin glasses until recently. Recent activities in analytical studies on the dynamics of the Sherrington-Kirkpatrick (SK) model of spin glasses include closed-form solutions for dynamical correlation functions and construction of evolution equations for single-time macroscopic quantities.

In the latter approach, Coolen, Laughton and Sherrington (CLS) derived a closed-form evolution equation for the averaged single-site spin-field distribution function under a few assumptions on microscopic properties of the system. The averaged spin-field distribution function thus obtained provides sufficient information to determine the time evolution of single-time macroscopic quantities. It gives quite good agreement with the evolution of macroscopic quantities obtained by computer simulation of the microdynamics, although a high-temperature expansion of the dynamical microscopic probability distribution involved quantities not expressible solely in terms of the...
spin-field distribution of CLS.

The purpose of the present paper is to further develop the high-temperature expansion technique\cite{5} and derive explicit expressions of physical quantities to the third order in the inverse temperature. The obtained series results are still too short to discuss critical properties around the spin glass transition temperature. However, our method has an advantage that it works not only for the infinite-range SK model but also for finite-dimensional systems, in particular the nearest neighbor $\pm J$ model. Thus the present approach is a first step toward a systematic investigation of dynamics of finite-dimensional spin glasses.

In the next section the problem and the method are formulated. The explicit expression of the dynamical probability distribution function is derived there to the third order of the inverse temperature. A method to evaluate expectation values of physical quantities using the dynamical probability distribution function is formulated in §3. This formulation is used in §4 to determine the coefficients of series expansions of several physical quantities. In §5 the averaged spin-field distribution function is evaluated by the expansion method. The results in these two sections are used in §6 to discuss the limit of applicability of the CLS theory. The final section is devoted to general discussions.

§2. Dynamical probability distribution function

The essence of our theory is a high-temperature expansion of the microscopic probability distribution function. The basic formulation of this expansion was developed in Ref.\cite{5} for the SK model of spin glasses. We describe here briefly the method generalized to include finite-dimensional models.

The model we consider is an Ising spin glass with Hamiltonian

$$H(\sigma) = - \sum_{\langle ij \rangle} J_{ij} \sigma_i \sigma_j,$$

where $\sigma$ denotes the spin configuration and the $\{J_{ij}\}$ are quenched randomly distributed; $\langle ij \rangle$ denotes a pair of sites. The probability distribution function $p_t(\sigma)$ obeys the master equation:

$$\frac{1}{p_t(\sigma)} \frac{d}{dt} p_t(\sigma) = \frac{1}{p_t(\sigma)} \sum_k p_t(F_k \sigma) w_k(F_k \sigma) - \sum_k w_k(\sigma),$$

where $F_k$ is a single spin flip operator, $F_k \Phi(\sigma) \equiv \Phi(\sigma_1, \cdots, -\sigma_k, \cdots, \sigma_N)$, $N$ being the system size, and we use a transition rate of the heat bath method

$$w_k(\sigma) = \frac{1}{2} \{1 - \sigma_k \tanh \beta h_k(\sigma)\}.$$  \hspace{1cm} (2.3)

Here $\beta$ is the inverse temperature and $h_k(\sigma)$ denotes the local field on site $k$.

Let us solve the master equation (2.2) by a high-temperature expansion in the form

$$p_t(\sigma) = \exp \left\{ \beta f_t(\sigma) + \beta^2 g_t(\sigma) + \beta^3 u_t(\sigma) + \cdots \right\}.$$  \hspace{1cm} (2.4)
Inserting eq. (2.4) into the master equation (2.2) and expanding the result in powers of $\beta$, we obtain differential equations for the functions appearing in the exponent of eq. (2.4):

\[
\frac{df_t}{dt} = \frac{1}{2} \sum_k (\Delta_k f_t + 2\sigma_k h_k) \tag{2.5}
\]

\[
\frac{dg_t}{dt} = \frac{1}{2} \sum_k \left( \frac{1}{2}(\Delta_k f_t)^2 + \Delta_k g_t + \sigma_k h_k \Delta_k f_t \right) \tag{2.6}
\]

\[
\frac{du_t}{dt} = \frac{1}{2} \sum_k \left\{ \frac{1}{6}(\Delta_k f_t)^3 + \Delta_k f_t \Delta_k g_t + \Delta_k u_t + \sigma_k h_k \left( \frac{1}{2}(\Delta_k f_t)^2 + \Delta_k g_t \right) - \frac{2}{3}\sigma_k h_k^3 \right\}, \tag{2.7}
\]

where $\Delta_k f_t = f_t(F_k\sigma) - f_t(\sigma)$.

We now restrict ourselves to the nearest neighbor $\pm J$ model on a finite-dimensional lattice. The SK model is recovered by taking an appropriate limit of infinite coordination number. The system is assumed to be initially in equilibrium at the inverse temperature $\beta_0$:

\[
p_{t=0}(\sigma) = \exp\left[ -\beta_0 H(\sigma) \right]. \tag{2.8}
\]

Equation (2.5) is easily solved to yield

\[
f_t(\sigma) = a(t)H(\sigma), \tag{2.9}
\]

where

\[
a(t) = \alpha e^{-2t} - 1. \tag{2.10}
\]

The parameter $\alpha$ denotes $1 - \beta_0/\beta$. It is necessary that $\beta_0$ and $\beta$ are smaller than $\beta_c$, the inverse of the critical temperature, because the high-temperature expansion is valid only when the system stays in the paramagnetic phase.

Using eq. (2.9), the next order differential equation (2.6) is written explicitly as

\[
\frac{dg_t}{dt} = \frac{1}{2} \sum_k \Delta_k g_t + (\alpha^2 e^{-4t} - \alpha e^{-2t}) \sum_k h_k^2, \tag{2.11}
\]

This equation has the following solution

\[
g_t(\sigma) = b_1(t) \sum_i h_i^2 + b_2(t), \tag{2.12}
\]

where

\[
b_1(t) = -\frac{\alpha^2}{2} e^{-4t} - \left( \alpha t - \frac{\alpha^2}{2} \right) e^{-2t} \tag{2.13}
\]

and

\[
b_2(t) = NzJ^2 \left\{ \frac{\alpha^2}{4} e^{-4t} + \left( \alpha t - \frac{\alpha^2}{2} + \frac{\alpha}{2} \right) e^{-2t} + \frac{\alpha^2}{4} - \frac{\alpha}{2} \right\} \tag{2.14}
\]
with $z$ being the coordination number. Since eq. (2.14) is independent on the spin configuration, this term does not affect the following argument and shall be ignored hereafter.

Similarly, using eqs. (2.9), (2.10), (2.12) and (2.13) for $f_t$ and $g_t$, eq. (2.7) is rewritten as

$$\frac{du_t}{dt} = \frac{1}{2} \sum_k \Delta_k u_t - 4z J^2 (2\alpha e^{-2t} - 1) b_1(t) H(\sigma)$$
$$\quad + \left( \frac{2}{3} \alpha^3 e^{-6t} - \alpha^2 e^{-4t} \right) \sum_k \sigma_k h_k^3$$
$$\quad - 2(2\alpha e^{-2t} - 1) b_1(t) \sum_{k,l} J_{kl} h_k h_l. \quad (2.15)$$

The solution of this equation is

$$u_t(\sigma) = c_1(t) H(\sigma) + c_2(t) \sum_i \sigma_i h_i^3 + c_3(t) \sum_{i,j} J_{ij} h_i h_j, \quad (2.16)$$

where the coefficients satisfy

$$\dot{c}_1(t) = -2c_1(t) - (12z - 8) J^2 c_2(t)$$
$$\quad - 4z J^2 b_1(t) (2\alpha e^{-2t} - 1)$$
$$\dot{c}_2(t) = -4c_2(t) + \frac{2}{3} \alpha^3 e^{-6t} - \alpha^2 e^{-4t} \quad (2.17)$$
$$\dot{c}_3(t) = -2c_3(t) - 2b_1(t) (2\alpha e^{-2t} - 1).$$

In deriving these equations we have assumed that there are no triangular loops on the lattice in the sense

$$J_{kl} J_{lm} J_{mk} = 0,$$

where indices denote neighboring sites. This assumption excludes several types of lattices from our considerations such as the triangular lattice. Although it is possible to remove this simplifying assumption, resulting complicated formulas do not lead to new physics.

Equations (2.17) are solved under the initial condition $c_1 = c_2 = c_3 = 0$ as

$$c_1(t) = J^2 \left[ - \left( \frac{2}{3} \alpha^3 e^{-6t} + \left( \frac{4}{3} \alpha^3 \right) \right) \right.$$
$$\left. - (12z - 2) \alpha^2 - (10z - 4) \alpha^2 t \right] e^{-4t} + \left( \frac{2}{3} \alpha^2 t \right)$$
$$\left. - 2z \alpha^2 t - \left( \frac{2}{3} \alpha^3 + (4z - 2) \alpha^2 \right) \right] e^{-2t} \quad (2.18)$$
$$c_2(t) = \frac{1}{3} \alpha^3 (e^{-6t} - e^{-4t}) - \alpha^2 t e^{-4t} \quad (2.19)$$
$$c_3(t) = \frac{1}{2} \alpha^3 e^{-6t} + \left( \alpha^3 - \frac{1}{2} \alpha^2 - 2 \alpha^2 t \right) e^{-4t}$$
$$\quad + \left( \alpha^2 t - \alpha t^2 - \frac{1}{2} \alpha^3 + \frac{1}{2} \alpha^2 \right) e^{-2t}. \quad (2.20)$$
Since eqs. (2.5), (2.6) and (2.7) are first-order differential equations, the above expressions (2.9), (2.12) and (2.16) with coefficients obtained above represent the unique solution. We have therefore obtained the dynamical probability distribution function to the third-order in $\beta$ for the $\pm J$ model as

$$p_t(\sigma) = \exp \left[ \beta a(t) H(\sigma) + \beta^2 b_1(t) \sum_i h_i^2 + \beta^3 \left( c_1(t) H(\sigma) + c_2(t) \sum_i \sigma_i h_i^3 + c_3(t) \sum_{ij} J_{ij} h_i h_j \right) + \cdots \right] ,$$  

(2.21)

where $a(t), b_1(t), c_1(t), c_2(t)$ and $c_3(t)$ are given in eqs. (2.10), (2.13) and (2.18) - (2.20).

We have considered the $\pm J$ model on a finite-dimensional lattice with $z$ nearest neighbors. The results derived previously for the SK model are recovered by taking the limit $N \to \infty$ with $z = N - 1$ and $J = \tilde{J}/\sqrt{N}$ ($\tilde{J} \sim O(1)$).

The above solution (2.21) is invariant under the gauge transformation $\sigma_i \to \sigma_i \tau_i$, $J_{ij} \to J_{ij} \tau_i \tau_j$ ($\tau_i = \pm 1$). This invariance is naturally expected from gauge invariance of the master equation (2.2). In general, any term in the expansion can be expressed in terms of $I_{ij} \equiv J_{ij} \sigma_i \sigma_j$ which is the basic building block of all gauge invariant quantities. More explicitly, terms to the third order are written as

$$H = - \sum_{(ij)} I_{ij} ,$$  

(2.22)

$$\sum_i h_i^2 = \sum_{ilm} I_{il} I_{im} ,$$  

(2.23)

$$\sum_i \sigma_i h_i^3 = \sum_{ilmn} I_{il} I_{im} I_{in} ,$$  

(2.24)

$$\sum_{ij} J_{ij} h_i h_j = \sum_{ijlm} I_{ij} I_{il} I_{jm} ,$$  

(2.25)

where all the indices of the $I$'s stand for nearest neighbor sites. Higher-order terms should be able to be written similarly.

The series-expansion solution obtained in the present section forms the basis of estimation of physical quantities in the following sections.

§3. High-temperature expansion of physical quantities

We now apply the series expansion solution for the probability distribution function to the expectation value of a physical quantity $O$:

$$\langle O(\sigma, \{J_{ij}\}) \rangle_t = \sum_{\sigma} O(\sigma, \{J_{ij}\}) p_t(\sigma) ,$$  

(3.1)
The first-order solution (2.9) with eq. (2.10) gives

\[ p_t(\sigma) = \exp\left[-\beta(1 - \alpha e^{-2t})H(\sigma)\right] \equiv \exp[-\beta_{\text{eff}} H(\sigma)]. \tag{3.2} \]

This equation shows that the dynamical expectation value (3.1) can be calculated using a time-dependent effective inverse temperature \( \beta_{\text{eff}} \). Therefore we may regard the system as being in equilibrium at temperature \( T/(1 - \alpha e^{-2t}) \) at any given time \( t \). The expectation value with respect to this effective Boltzmann factor (3.2) will be denoted by \( \langle \cdots \rangle_1 \) in the following.

The higher-order terms in the series expansion are taken into account by the expansion

\[
\exp\left\{\beta a(t)H + \beta^2 b_1(t) \sum_i h_i^2 + \beta^3 (c_1(t)H + \cdots)\right\} \\
\approx e^{-\beta a_H} \left\{1 + \beta^2 b_1(t) \sum_i h_i^2 + \beta^3 (c_1(t)H + \cdots)\right\}. \tag{3.3}
\]

The thermodynamic average of a physical quantity \( O(\sigma, \{J_{ij}\}) \) then becomes

\[
\langle O(\sigma, \{J_{ij}\}) \rangle_t \simeq \frac{\langle O + \beta^2 b_1(t)O \sum_i h_i^2 + \beta^3 (c_1(t)OH(\sigma) + \cdots) \rangle_1}{\langle 1 + \beta^2 b_1(t) \sum_i h_i^2 + \beta^3 (c_1(t)H(\sigma) + \cdots) \rangle_1} \\
\simeq \langle O \rangle_1 + \beta^2 b_1(t) \left(\langle O \sum_i h_i^2 \rangle_1 - \langle O \rangle_1 \langle \sum_i h_i^2 \rangle_1 \right) + \beta^3 \left\{ c_1(t) \left(\langle OH(\sigma) \rangle_1 - \langle O \rangle_1 \langle H(\sigma) \rangle_1 \right) \right. \\
+ c_2(t) \left(\langle O \sum_i \sigma_i h_i^3 \rangle_1 - \langle O \rangle_1 \langle \sum_i \sigma_i h_i^3 \rangle_1 \right) + c_3(t) \left(\langle O \sum_{i,j} J_{ij} h_i h_j \rangle_1 - \langle O \rangle_1 \langle \sum_{i,j} J_{ij} h_i h_j \rangle_1 \right) \right. \\
+ \cdots. \tag{3.4}
\]

The configurational average of this equation is expressed as

\[
\left[\langle O \rangle_3\right] = \left[\langle O \rangle_1\right] + \beta^2 b_1(t) \text{Cov}(O \sum_i h_i^2)_1 \\
+ \beta^3 \left\{ c_1(t) \text{Cov}(OH(\sigma))_1 + c_2(t) \text{Cov}(O \sum_i \sigma_i h_i^3)_1 \\
+ c_3(t) \text{Cov}(O \sum_{i,j} J_{ij} h_i h_j)_1 \right\}. \tag{3.5}
\]

Here \( \langle O \rangle_n \) stands for the thermodynamic average obtained by truncating the exponent of eq. (2.4) at the \( n \)th order in \( \beta \), and \( \text{Cov}(AB)_1 = \left[\langle AB \rangle_1\right] - \left[\langle A \rangle_1 \langle B \rangle_1\right] \) represents the covariance of \( A \) and \( B \) at the effective inverse temperature \( \beta_{\text{eff}} \).

Generally it is quite difficult to evaluate the expectation value \( \left[\langle O \rangle_1\right] \) and covariances on the right-hand side of eq. (3.5) for the finite-dimensional \( \pm J \) model. An exception is the first-order term \( \left[\langle O \rangle_1\right] \) evaluated for a gauge-invariant quantity \( O \) on the Nishimori line defined by \( \tanh \beta J = 2p - 1 \), where \( p \) is the probability for \( J_{ij} \) to be positive. In the case of the SK model, we can derive explicit expressions for all terms in eq. (3.5) by the replica method.
Therefore we have to introduce an approximation to proceed further for the $\pm J$ model. For this purpose, we first focus our attention on the equilibrium expectation values of terms appearing in the expansion (3.5) evaluated on the Nishimori line:

\[
\langle H(\sigma) \rangle_e = -\frac{1}{2} NJ z \tanh \beta J
\] (3.6)

\[
\langle \sum_i h_i^2 \rangle_e = NJ^2 \left\{ z + z(z - 1)(\tanh \beta J)^2 \right\}
\] (3.7)

\[
\langle \sum_i \sigma_i h_i^3 \rangle_e = NJ^3 \left\{ z(3z - 2) \tanh \beta J + z(z - 1)(2z - 1) \tanh \beta J \right\}
\] (3.8)

\[
\langle \sum_{ij} J_{ij} h_i h_j \rangle_e = NJ^3 \left\{ z(2z - 1) \tanh \beta J + z(z - 1)^2 \tanh \beta J \right\}
\] (3.9)

where $\langle \cdots \rangle_e$ denotes the equilibrium expectation value. The first of these equations, eq. (3.6), indicates that the factor $\tanh \beta J$ can be replaced by the expectation value of the Hamiltonian. For instance,

\[
\left[ \langle \sum_i h_i^2 \rangle_e \right] = NJ^2 z - 2J(z - 1) \tanh \beta J \left[ \langle H(\sigma) \rangle \right].
\] (3.10)

In equilibrium, both sides of eq. (3.10) are of the order of $N$, and fluctuations around the expectation value are of order $\sqrt{N}$. Thus, given a typical equilibrium spin configuration, the value of the quantity

\[
\sum_i h_i^2
\]

is almost certainly equal to that of

\[
NJ^2 z - 2J(z - 1)(\tanh \beta J)H(\sigma).
\]

This argument does not apply to temperatures near the critical point where fluctuations are not negligible.

The situation should not change drastically if the system is close to equilibrium if not in equilibrium precisely. Thus the following replacement may be a reasonable approximation if the system is not far from equilibrium,

\[
\sum_i h_i^2 \rightarrow NJ^2 z - 2J(z - 1)(\tanh \beta J)H(\sigma).
\]

Further \textit{a posteriori} justification of this approximate replacement comes from agreement of the resulting values with simulations as shown in the following.
We therefore apply similar replacements to all terms in the exponent of eq. (2.21) using the relations (3.7) to (3.9). The result is

\[ p_t(\sigma) \simeq \exp \left( \beta a(t) H(\sigma) + \beta^2 b_1(t) \left\{ NJ^2 z - 2J(z-1)(\tanh \beta J)H(\sigma) \right\} 
+ \beta^3 \left\{ c_1(t) - 2c_2(t)J^2(3z-2) - 2c_3(t)J^2(2z-1) \right\} \right) \]

This approximation is valid for a system near equilibrium on the Nishimori line.

This result (3.11) indicates that the dynamical probability distribution is approximated by the equilibrium Boltzmann factor with effective inverse temperature

\[ \beta_{\text{eff}}^{(3)} = -\beta a(t) + 2\beta^2 b_1(t)J(z-1) \tanh \beta J - \beta^3 \times \left\{ c_1(t) - 2c_2(t)J^2(3z-2) - 2c_3(t)J^2(2z-1) \right\} \]

In the case of the SK model, the thermal expectation values on the left-hand side of eqs. (3.6) - (3.9) can be evaluated using the replica method. The corresponding \( \beta_{\text{eff}}^{(3)} \) for the SK model is then obtained as

\[ \beta_{\text{eff}}^{(3)}_{\text{SK}} = -\beta a(t) + \beta^3 \left( 2J^2 b_1(t) - c_1(t) \right) + 6J^2 c_2(t)J + 4J^2 c_3(t)J \]

where \( \tilde{J}^2/N \) represents the variance of distribution of \( J_{ij} \). The above formula for the SK model can also be derived from eq. (3.12), which is valid on the Nishimori line, by taking the limit \( N \rightarrow \infty \), \( z = N - 1 \) and \( J = \tilde{J}/\sqrt{N} \) and applying the approximation \( \tanh \beta J \simeq \beta J \) in the \( \pm J \) model. Note that equilibrium properties of the SK model in the paramagnetic phase are independent of \( J_0 \), the center of distribution of \( J_{ij} \). Therefore the results valid on the Nishimori line, eqs. (3.6) - (3.9), which passes through the paramagnetic phase toward the multicritical point, remain true in the whole region of the paramagnetic phase of the SK model. This is the reason why eq. (3.13) is valid in the whole paramagnetic phase of the SK model whereas eq. (3.12) for the finite-dimensional nearest-neighbor \( \pm J \) Ising model can be used only on the Nishimori line.

In the form exhibited in eq. (3.5) the direct expansion is useful in practice only for the infinite-ranged SK model or to the level of the first-order term. Quantitative estimation of validity of these methods will be studied in the next section.

§4. Explicit evaluation of expansion terms

We now apply the formulas derived in the previous section to the following physical quantities, \( H, \sum_i h_i^2, \sum_i \sigma_i h_i^3 \) and \( \sum_{ij} J_{ij} h_i h_j \). These four quantities constitute the expansion of the dynamical
probability distribution function to the third order in $\beta$ as in eq. (2.21). One of the purposes to choose these quantities for a test ground of the method developed in the previous section is to estimate the order of magnitude of various terms in eq. (2.21) for consistency check of the theory of Coolen, Laughton and Sherrington\(^3\),\(^4\) as discussed in §6.

The first-order approximation for the $\pm J$ model is obtained by replacing $\beta$ by $\beta_{\text{eff}}$ in eqs. (3.6), (3.7), (3.8) and (3.9). The corresponding expressions for the SK model are found from those of the $\pm J$ model by taking the limit of infinite dimensionality, or alternatively, by applying the replica method as described in Appendix A. The results are

\[
\left\langle \sigma \right\rangle_1 = \frac{1}{2} N \bar{J}^2 \beta_{\text{eff}} \tag{4.1}
\]

\[
\left\langle \sum_i h_i^2 \right\rangle_1 = N(\bar{J}^2 + \bar{J}^4 \beta_{\text{eff}}^2) \tag{4.2}
\]

\[
\left\langle \sum_i \sigma_i h_i^3 \right\rangle_1 = N(3\bar{J} \beta_{\text{eff}} + \bar{J}^6 \beta_{\text{eff}}^3) \tag{4.3}
\]

\[
\left\langle \sum_{ij} J_{ij} h_i h_j \right\rangle_1 = N(2\bar{J}^4 \beta_{\text{eff}} + \bar{J}^6 \beta_{\text{eff}}^3) \tag{4.4}
\]

The third-order formula (3.5) can be explicitly evaluated only for the SK model as mentioned before. The necessary covariances are calculated in Appendix A using the replica method:

\[
\text{Cov}(H^2) = \frac{1}{2} N \bar{J}^2
\]

\[
\text{Cov}\left( \sum_i h_i^2 \right)^2 = N \bar{J}^4(2 + 8 \bar{J}^2 \beta^2)
\]

\[
\text{Cov}\left( \sum_i \sigma_i h_i^3 \right)^2 = N \bar{J}^6(24 + 54 \bar{J}^2 \beta^2 + 18 \bar{J}^4 \beta^4)
\]

\[
\text{Cov}\left( \sum_{ij} J_{ij} h_i h_j \right)^2 = N \bar{J}^6(10 + 24 \bar{J}^2 \beta^2 + 26 \bar{J}^4 \beta^4)
\]

\[
\text{Cov}(H \sum_i h_i^2) = -2 N \bar{J}^4 \beta
\]

\[
\text{Cov}(H \sum_i \sigma_i h_i^3) = -N \bar{J}^4(3 + 3 \bar{J}^2 \beta^2)
\]

\[
\text{Cov}(H \sum_{ij} J_{ij} h_i h_j) = -N \bar{J}^4(2 + 3 \bar{J}^2 \beta^2)
\]

\[
\text{Cov}\left( \sum_i h_i^2 \right)\left( \sum_i \sigma_i h_i^3 \right) = N \bar{J}^6(18 + 12 \bar{J}^2 \beta^2)
\]

\[
\text{Cov}\left( \sum_i h_i^2 \right)\left( \sum_{ij} J_{ij} h_i h_j \right) = N \bar{J}^6(12 + 12 \bar{J}^2 \beta^2)
\]
\[
\text{Cov}\left(\sum_i \sigma_i h_i^3 \left(\sum_{ij} J_{ij} h_i h_j\right)\right) = N \tilde{J}^6 (12 + 42 \tilde{J}^2 \beta^2 + 18 \tilde{J}^4 \beta^4).
\]

Inserting these results into eq. (3.5), we obtain
\[
\langle H(\sigma) \rangle_3 = \left[ \langle H(\sigma) \rangle_1 \right] - 2N \tilde{J}^4 b_1(t) \beta \beta^2 + N \left( \frac{1}{2} \tilde{J}^2 c_1(t) - 3 \tilde{J}^4 c_2(t) - 2 \tilde{J}^4 c_3(t) \right) \beta^3
\]
\[
\left[ \sum_i h_i^2 \right]_3 = \left[ \sum_i h_i^2 \right]_1 + 2N \tilde{J}^4 b_1(t) \beta^2 + N \left( -3 \tilde{J}^4 c_1(t) + 24 \tilde{J}^6 c_2(t) + 12 \tilde{J}^6 c_3(t) \right) \beta^3
\]
\[
\left[ \sum_i \sigma_i h_i^3 \right]_3 = \left[ \sum_i \sigma_i h_i^3 \right]_1 + 18N \tilde{J}^6 b_1(t) \beta \beta^2 + N \left( -3 \tilde{J}^4 c_1(t) + 24 \tilde{J}^6 c_2(t) + 12 \tilde{J}^6 c_3(t) \right) \beta^3
\]
\[
\times \beta \beta^2 + N \left( -2 \tilde{J}^4 c_1(t) + 12 \tilde{J}^6 c_2(t) + 10 \tilde{J}^6 c_3(t) \right) \beta^3.
\]

The approximation of a Boltzmann form with the inverse temperature, eqs. (3.12) and (3.13), is also used in the following analysis.

We have compared the theoretical curves obtained by the approximations \(\langle \cdots \rangle_1, \langle \cdots \rangle_3\) and \(\beta^{(3)}\) with each other and with Monte Carlo simulations for the SK model. Figures 1 to 4 show the results for \(H, \sum_i h_i^2, \sum_i \sigma_i h_i^3\) and \(\sum_{ij} J_{ij} h_i h_j\), respectively. The initial temperature was \(T_0 = \infty\) and the final temperatures were set to \(T = 2\) and 5 in each figure. The system size of simulations was \(N = 1000\) with 50-sample averages and standard deviations indicated. The center of distribution is \(J_0 = 0\), and the temperature is expressed in units of \(J/k_B\) or \(\tilde{J}/k_B\).

For all quantities the theoretical curves follow simulations faithfully at relatively high temperature, \(T = 5\). When the temperature is lower, \(T = 2\), the first-order approximation deviates in the intermediate time region, \(t = 1 \sim 2\), from simulations as well as from other theoretical curves.

We next investigate the nearest neighbor \(\pm J\) model. The straightforward expansion (3.5) cannot be used in this case because the covariances are difficult to evaluate explicitly. We therefore compare the first-order term and the \(\beta^{(3)}\)-approximation with simulations.

The results for four quantities of the \(\pm J\) model on the square lattice are shown in Figs. 5 to 8. We have set \(T_0 = \infty\) and the final temperatures were \(T = 2\) and 5. Monte Carlo simulations were carried out for the system size \(N = 50^2\) on the Nishimori line. The plots show results averaged over 50 samples with standard deviations indicated.

The critical temperature is \(T_c \simeq 0.96\) for the two-dimensional \(\pm J\) model on the Nishimori line, while the SK model has \(T_c = 1\). In addition, the two-dimensional model has the problem of
Griffiths singularity at $T_G = 2.27\frac{\hbar}{k_B}$. Therefore it is not surprising that the $T = 2$ results do not reproduce simulation data very well while the $T = 5$ case does. The slightly better situation at $T = 2$ for the SK model than in the $\pm J$ model may be related to the absence of Griffiths singularities in the SK model.

§5. Spin-field distribution function

We now calculate the dynamical single-site spin-field distribution function by high-temperature series expansion. This distribution function has more information than individual physical quantities in the sense that the physical quantities such as $\langle H(\sigma) \rangle$, $\langle \sum_i h_i^2 \rangle$ and $\langle \sum_i \sigma_i h_i^3 \rangle$ can be derived from the distribution function. It should be noted that $\langle \sum_{ij} J_{ij} h_i h_j \rangle$ cannot be calculated from the present distribution function, because this distribution function does not carry information on correlations between fields at different sites.

The averaged single-site spin-field distribution function is defined by

$$ P(h, \sigma) = \left[ \frac{1}{N} \left\langle \sum_i \delta_{\sigma \sigma_i} \delta(h - h_i) \right\rangle_t \right], \quad (5.1) $$

where $\sigma = \pm 1$ should not be confused with the microscopic spin configuration of the whole system $\sigma$. The equilibrium spin-field distribution has been calculated by Laughton et al. as

$$ P(h, \sigma) = \frac{1}{2\sqrt{2\pi J}} \exp \left( -\frac{(h - \sigma J^2 \beta^2)}{2J^2} \right). \quad (5.2) $$

Now we consider the dynamical single-site spin-field distribution function. This function should reduce to eq. (5.2) in the equilibrium limit. It is convenient to introduce the characteristic function (Fourier transform):

$$ P(h, \sigma) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} G(k, \sigma) \exp(-ikh), \quad (5.3) $$

or

$$ G(k, \sigma) = \left[ \frac{1}{N} \left\langle \sum_i \frac{1 + \sigma \sigma_i}{2} \exp(ik \sum_j J_{ij} \sigma_j) \right\rangle_t \right]. \quad (5.4) $$

The method of high-temperature series expansion developed in the previous sections is applicable to the evaluation of eq. (5.4). Detailed calculations are described in Appendix B. The result is

$$ G(k, \sigma) = \frac{1}{2} \exp \left( -\frac{k^2 J^2}{2} \right) \left\{ \cos(kJ^2 \beta_{\text{eff}}) + i\sigma \sin(kJ^2 \beta_{\text{eff}}) \right\} $$

$$ + \beta^2 \left\{ -4k \tilde{J}^4 \beta_{\text{eff}} \sin(kJ^2 \beta_{\text{eff}}) - k^2 \tilde{J}^4 \cos(kJ^2 \beta_{\text{eff}}) + i\sigma \left( 4k \tilde{J}^4 \beta_{\text{eff}} \cos(kJ^2 \beta_{\text{eff}}) - k^2 \tilde{J}^4 \sin(kJ^2 \beta_{\text{eff}}) \right) \right\} $$

$$ + \beta^3 \left\{ c_1 \left( kJ^2 \sin(kJ^2 \beta_{\text{eff}}) + i\sigma \left( -kJ^2 \cos(kJ^2 \beta_{\text{eff}}) \right) \right) + c_2 \left( -3k^2 \tilde{J}^6 \beta_{\text{eff}} \cos(kJ^2 \beta_{\text{eff}}) ight) 

- (\tilde{J}^6(6k \beta_{\text{eff}} - k^3) + 6k \tilde{J}^4) \sin(kJ^2 \beta_{\text{eff}}) 

+ i\sigma \left( -3k^2 \tilde{J}^6 \beta_{\text{eff}} \sin(kJ^2 \beta_{\text{eff}}) + (\tilde{J}^6(6k \beta_{\text{eff}} - k^3) + 6k \tilde{J}^4) \cos(kJ^2 \beta_{\text{eff}}) \right) \right\} + c_3 \left\{ -2k^2 \tilde{J}^6 \beta_{\text{eff}} \cos(kJ^2 \beta_{\text{eff}}) \right\}. $$
\[-(6k_0 J^2_{\text{eff}} + 4k_0 J^4) \sin(k_0 J^2 \beta_{\text{eff}})\]
\[+ i\sigma \left(-2k^2 J_0 J^2_{\text{eff}} \sin(k_0 J^2 \beta_{\text{eff}}) + (6k_0 J^2_{\text{eff}} + 4k_0 J^4) \cos(k_0 J^2 \beta_{\text{eff}})\right)\}\].

(5.5)

Substituting eq. (5.5) into eq. (5.3), we arrive at the expression of the spin-field distribution:

\[
P(h, \sigma) = \frac{1}{2\sqrt{2\pi} J} \left[1 + \beta^2 b_1 \left\{4\sigma J^2_{\text{eff}} (h - \sigma J^2 \beta_{\text{eff}}) - (J^2 - (h - \sigma J^2 \beta_{\text{eff}})^2)\right\}\right.
\[+ \beta^3 \left\{c_1 \left(-\sigma (h - \sigma J^2 \beta_{\text{eff}})\right)\right.
\[+ c_2 \left(-3\sigma J^2_{\text{eff}} (J^2 - (h - \sigma J^2 \beta_{\text{eff}})^2 - 3\sigma J^4 (h - \sigma J^2 \beta_{\text{eff}}) + \sigma (h - \sigma J^2 \beta_{\text{eff}})^3\right.
\[+ 6\sigma J^2 (1 + J^2 \beta_{\text{eff}}) (h - \sigma J^2 \beta_{\text{eff}})\right\}\right.
\[+ c_3 \left(-2\sigma J^2_{\text{eff}} (J^2 - (h - \sigma J^2 \beta_{\text{eff}})^2 + 6\sigma J^4 (2 + 3\sigma J^2 \beta_{\text{eff}}) (h - \sigma J^2 \beta_{\text{eff}})\right)\right\}\left]\times \exp\left(-\frac{(h - \sigma J^2 \beta_{\text{eff}})^2}{2J^2}\right).\]

(5.6)

It is straightforward to check that the expansion results of \[\langle H(\sigma)\rangle, \langle \sum_i h_i^2 \rangle\] and \[\langle \sum_i \sigma_i h_i^3 \rangle\] given in §4 are recovered from eq. (5.6) by appropriate integrations.

In equilibrium the distribution (5.6) becomes Gaussian. However it is not Gaussian in general even in the paramagnetic phase. As the temperature is lowered, the distribution deviates from Gaussian more significantly.

The results of the analytical calculation and computer simulations are compared in Fig. 9. Simulations were performed for \(N = 5000\) with 100 samples at \(T = 2\). We may conclude that the third-order result (5.6) agrees with simulations in a rather satisfactory manner at this temperature.

§6. Discussions on the CLS theory

Coolen, Laughton and Sherrington (CLS)\(^3\),\(^4\) developed a theory of dynamics of the SK model. They derived the closed-form evolution equation of a physical quantity which determines macroscopic behavior of the system. An important assumption in their derivation was that the average of a macroscopic physical quantity can be calculated under the ansatz of equipartitioning of the dynamical probability distribution function. Here equipartitioning means that the probability distribution function is constant once the value of the single-site spin-field distribution function \(P(h, \sigma)\) is given.

The existence of the correlation term of fields at different sites, \(\sum_{ij} J_{ij} h_i h_j\), in the probability
distribution function (2.21) shows that the equipartitioning ansatz of CLS is not generally true and suggests that the CLS theory is an approximate one.\footnote{We investigate in the present section the effects of the field-correlation term on physical quantities to determine the degree of applicability of the CLS theory.}

Figure 10 shows the third-order series-expansion curves and simulation results, superimposed for comparison, of various physical quantities. This figure indicates that the field-correlation term is not necessarily small compared with other terms appearing in the series expansion. The time evolution of the correlation coefficients of $\sum_{ij} J_{ij} h_i h_j$ and the other quantities is shown in Fig. 11 obtained by the $\beta^{(3)}_{\text{eff}}$-approximation. The correlation coefficient between $A(\sigma; \{J_{ij}\})$ and $\sum_{ij} J_{ij} h_i h_j$ is defined as

$$\text{Corr} \left( A(\sigma; \{J_{ij}\}) \left( \sum_{ij} J_{ij} h_i h_j \right) \right) = \frac{\text{Cov} \left( A \left( \sum_{ij} J_{ij} h_i h_j \right) \right)}{\sqrt{\text{Cov} \left( A^2 \right)} \sqrt{\text{Cov} \left( \left( \sum_{ij} J_{ij} h_i h_j \right)^2 \right)}}. \quad (6.1)$$

Figure 11 shows that $\sum_{ij} J_{ij} h_i h_j$ is closely correlated with $H$ and $\sum_i \sigma_i h_i^3$ but is not with $\sum_i h_i^2$ in the initial time region. Since the absolute value of correlation coefficients with $H$ and $\sum_i \sigma_i h_i^3$ is close to unity, one may suppose that the field-correlation term $\sum_{ij} J_{ij} h_i h_j$ is approximately replaced by $H$ or $\sum_i \sigma_i h_i^3$, which may be taken as a support for the CLS theory as an approximation. We can also obtain results similar to Fig. 10 for the nearest neighbor $\pm J$ model.

It may be useful to see the behavior of various terms in the series expansion with the time-dependent coefficients $a(t), b_1(t), \ldots, c_3(t)$ taken into account as appearing in eq. (2.21). Third-order series estimates are given for the SK model in Figs. 12 and 13 and in Fig. 15 for the two-dimensional $\pm J$ model in the high temperature region. Simulation results run almost along the same curves and are omitted in these figures. Only simulation results are shown at lower temperatures, $T = 1$ for the SK model and $T = 2$ for the $\pm J$ model, because the expansion breaks down at these temperatures; see Figs. 14 and 16.

Our observation is that the field-correlation term multiplied by the coefficient, $\beta^3 c_3(t) \sum_{ij} J_{ij} h_i h_j$, is very small compared to first and second order terms in the high-temperature region, $T = 5$, both for the SK and $\pm J$ models. However, this is not necessarily the case in intermediate time regions at lower temperatures; see Figs. 14 and 16 which show effects around $(t \sim 2)$. Naively one might therefore expect to need to choose appropriate parameter regions when applying the CLS theory. However, trends of strong correlations for $\sum_{ij} J_{ij} h_i h_j$ with $H$ and $\sum_i \sigma_i h_i^3$ as seen in Fig. 11 allow $\sum_{ij} J_{ij} h_i h_j$ to be effectively approximated by $H$ or $\sum_i \sigma_i h_i^3$, giving a support for the CLS theory as an approximation irrespective of the size of the term $\sum_{ij} J_{ij} h_i h_j$. 
It is useful to compare the macroscopic spin-field distribution \( P(h, \sigma) \) obtained by the CLS theory and that by the high-temperature expansion as shown in Figs. 17 and 18. Figure 17 shows the first-order cumulant, or the average, of \( P(h, 1), P(h, -1) \) and \( P(h, 1) + P(h, -1) \) at \( T = 2 \). The simulation data are also displayed which do not seem to give a clear advantage either to the CLS theory or to the high-temperature expansion. The second-order cumulants are shown in Fig. 18. The upper set of curves are for \( P(h, 1) + P(h, -1) \) and the lower curves correspond to \( P(h, 1) \) and \( P(h, -1) \). This figure indicates that the simulation results follow the CLS theory more closely than the high-temperature expansion in this temperature region. Therefore the CLS theory serves as quite a good approximation for the macroscopic spin-field distribution function.

§7. Discussions

We have derived explicitly the time dependence of macroscopic physical quantities for the SK model and the nearest neighbor \( \pm J \) model on the Nishimori line. Our formulation gives the microscopic dynamical probability distribution function in the form of a high-temperature series expansion. Evaluation of physical quantities has then been reduced to calculations of various expectation values in equilibrium. Such expectation values can be calculated to any order in the inverse temperature \( \beta \) in the case of the SK model. For the \( \pm J \) model on a finite-dimensional lattice, the situation is more complicated and explicit evaluation is possible only to the first order in \( \beta \) on the Nishimori line. We therefore introduced an approximation in which the dynamical probability distribution has the same form as the equilibrium Boltzmann factor with a time-dependent effective temperature.

The resulting expressions show excellent agreement with numerical simulations at high temperatures. Deviations are observed at lower temperatures in the intermediate time region. We anyway think it a significant progress that a systematic method to evaluate explicitly the time dependence of physical quantities has been formulated.

We have also analyzed the averaged spin-field distribution functions obtained by the high-temperature expansion and the CLS theory for the SK model. Cumulants of the distribution functions show that the CLS theory gives excellent agreement with simulations at a relative low temperature where the high-temperature expansion is not very useful. Therefore the CLS theory, if not exact, serves as a reliable tool to analyze dynamics of the SK model.

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Appendix A: Covariances

We show in this Appendix the procedure to calculate the covariance between two physical quantities $A(\sigma, \{J_{ij}\})$ and $B(\sigma, \{J_{ij}\})$ for the SK model defined as

$$\text{Cov}(AB) \equiv \langle AB \rangle - \langle A \rangle \langle B \rangle.$$ 

Here $\langle \rangle$ is the thermal average and $[\ ]$ represents the configurational average. The exchange interaction obeys the Gaussian distribution with vanishing mean and variance $\tilde{J}^2/N$,

$$P(J_{ij}) = \sqrt{\frac{N}{2\pi \tilde{J}^2}} \exp\left(-\frac{N}{2\tilde{J}^2} J_{ij}^2\right).$$

We use the replica method to carry out the sample average

$$\langle AB \rangle = \lim_{n \to 0} \left[ \sum_{\sigma} A(\sigma, \{J_{ij}\}) B(\sigma, \{J_{ij}\}) \times \exp\left(\beta \sum_{i<j} J_{ij} \sum_{\alpha} \sigma_{i}^{\alpha} \sigma_{j}^{\alpha}\right) \right],$$

$$\langle A \rangle \langle B \rangle = \lim_{n \to 0} \left[ \sum_{\sigma} A(\sigma, \{J_{ij}\}) B(\sigma, \{J_{ij}\}) \times \exp\left(\beta \sum_{i<j} J_{ij} \sum_{\alpha} \sigma_{i}^{\alpha} \sigma_{j}^{\alpha}\right) \right],$$

where replica indices ($\alpha = 1, 2, \cdots, n$) are introduced.

It is necessary to calculate the average of various powers of exchange interactions to derive the expressions of covariances in §4. It is straightforward to show by integration that

$$\left[ J_{ab} \exp\left(\beta J_{ab} \sum_{\alpha} \sigma_{a}^{\alpha} \sigma_{b}^{\alpha}\right) \right] = \frac{\tilde{J}^2}{N} \beta \sum_{\alpha} \sigma_{a}^{\alpha} \sigma_{b}^{\alpha} \exp\left\{ \frac{\tilde{J}^2 \beta^2}{2N} \left( \sum_{\alpha} \sigma_{a}^{\alpha} \sigma_{b}^{\alpha} \right)^2 \right\} \quad (A.3)$$

and

$$\left[ J_{ab}^k \exp\left(\beta J_{ab} \sum_{\alpha} \sigma_{a}^{\alpha} \sigma_{b}^{\alpha}\right) \right] = \frac{\tilde{J}^2}{N} \exp\left\{ \frac{\tilde{J}^2 \beta^2}{2N} \left( \sum_{\alpha} \sigma_{a}^{\alpha} \sigma_{b}^{\alpha} \right)^2 \right\}, \quad (A.4)$$

where we have omitted terms which vanish in the $n \to 0$ limit. Higher powers of the interaction lead to contributions with higher-power dependence on $1/N$. That is, for $k \geq 3$,

$$\left[ J_{ab}^k \exp\left(\beta J_{ab} \sum_{\alpha} \sigma_{a}^{\alpha} \sigma_{b}^{\alpha}\right) \right]$$
\[ \leq \frac{3\tilde{J}^4\beta}{N^2} \left| \sum_{\alpha} \sigma_{a}^{\alpha} \sigma_{b}^{\alpha} \exp \left\{ \frac{j^2}{2N} \left( \sum_{\alpha} \sigma_{a}^{\alpha} \sigma_{b}^{\alpha} \right)^2 \right\} \right|. \]  

(A.5)

Therefore we can neglect such higher-order terms in the limit \( N \to \infty \).

Let us now work on \( \text{Cov}(H \sum_i h_i^2) \) as an example. Explicitly written,

\[
\text{Cov} \left( H \sum_i h_i^2 \right) = \lim_{n \to 0} \left( \left[ \sum_{\sigma} \left( -\frac{1}{2} \sum_{ab} J_{ab} \sigma_{a}^{1} \sigma_{b}^{1} \right) \right] \times \left( \sum_{cde} J_{cd} J_{de} \sigma_{c}^{1} \sigma_{e}^{1} \right) \exp \left( \beta \sum_{i<j} J_{ij} \sum_{\alpha} \sigma_{i}^{\alpha} \sigma_{j}^{\alpha} \right) \right) \times \left( \sum_{\sigma} \left( -\frac{1}{2} \sum_{ab} J_{ab} \sigma_{a}^{1} \sigma_{b}^{1} \right) \left( \sum_{cde} J_{cd} J_{de} \sigma_{c}^{2} \sigma_{e}^{2} \right) \right)
\]

It is straightforward to check that we need only terms in which there are overlaps between two interactions. The two quantities \( H \) and \( \sum_i h_i^2 \) have four such cases, i.e., \( ab = cd, \ ab = dc, \ ab = de \) and \( ab = ed \). From the symmetry under the exchange of site indices, these four cases give identical contributions

\[
\text{Cov} \left( H \sum_i h_i^2 \right) = \lim_{n \to 0} \left( \left[ \sum_{\sigma} \left( -\frac{1}{2} \sum_{ab} J_{ab} \sigma_{a}^{1} \sigma_{b}^{1} \right) \right] \times \left( \sum_{cde} J_{cd} J_{de} \sigma_{c}^{1} \sigma_{e}^{1} \right) \exp \left( \beta \sum_{i<j} J_{ij} \sum_{\alpha} \sigma_{i}^{\alpha} \sigma_{j}^{\alpha} \right) \right) \times \left( \sum_{\sigma} \left( -\frac{1}{2} \sum_{ab} J_{ab} \sigma_{a}^{1} \sigma_{b}^{1} \right) \left( \sum_{cde} J_{cd} J_{de} \sigma_{c}^{2} \sigma_{e}^{2} \right) \right)
\]

The averages over \( J_{ab}^2 \) and \( J_{be} \) are carried out independently. Using eqs. (A.3) and (A.4), the above equation is found to be

\[
\lim_{n \to 0} \left( \left[ \sum_{\sigma} \left( -\frac{1}{2} \sum_{ab} J_{ab} \sigma_{a}^{1} \sigma_{b}^{1} \right) \right] \times \left( \sum_{cde} J_{cd} J_{de} \sigma_{c}^{1} \sigma_{e}^{1} \right) \exp \left( \beta \sum_{i<j} J_{ij} \sum_{\alpha} \sigma_{i}^{\alpha} \sigma_{j}^{\alpha} \right) \right) \times \left( \sum_{\sigma} \left( -\frac{1}{2} \sum_{ab} J_{ab} \sigma_{a}^{1} \sigma_{b}^{1} \right) \left( \sum_{cde} J_{cd} J_{de} \sigma_{c}^{2} \sigma_{e}^{2} \right) \right)
\]

The expression in the exponent has been rearranged so that the problem reduces to that of a single-site system after the Hubbard-Stratonovitch transformation. The result of spin trace is, in
the paramagnetic phase,

$$\text{Cov}\left(H \sum_i h_i^2\right) = -2N\bar{J}^4 \beta \left( \sum_\alpha \delta_{\alpha 1} - \sum_\alpha \delta_{\alpha 1} \delta_{\alpha 2} \delta_{12} \right) = -2N\bar{J}^4 \beta.$$  \hspace{1cm} (A.7)

The other covariances can be calculated in the same way.

**Appendix B: Spin-field distribution function**

The dynamical spin-field distribution function is expressed by the characteristic function defined in eq. (5.4). In this Appendix, we give detailed calculations of this function.

The method of high-temperature series expansion developed in §2 is applicable to the evaluation of the characteristic function

$$G(k, \sigma) = \left[ \frac{1}{N} \left\langle \sum_i \frac{1 + \sigma \sigma_i}{2} \exp\left( ik \sum_j J_{ij} \sigma_j \right) \right\rangle \right].$$  \hspace{1cm} (B.1)

The first-order approximation in $\beta$ is obtained in the ordinary way used in equilibrium calculations with $\beta$ replaced by $\beta_{\text{eff}}$. This procedure is almost the same as the study by Thomsen et al. except for the factor $(1 + \sigma \sigma_i)/2$

$$G^{(1)}(k, \sigma) = \left[ \frac{1}{N} \left\langle \sum_i \frac{1 + \sigma \sigma_i}{2} \exp\left( ik \sum_j J_{ij} \sigma_j \right) \right\rangle \right].$$  \hspace{1cm} (B.2)

We use the replica method and carry out the average with respect to the distribution of $\{J_{ij}\}$, so that the function is written as

$$G^{(1)}(k, \sigma) = \lim_{n \to 0} \left\{ \frac{1}{N} \sum_{\sigma} \sum_i \frac{1 + \sigma \sigma_i}{2} \exp\left( ik \sum_j J_{ij} \sigma_j \right) \right\} \exp\left( \beta_{\text{eff}} \sum_{(jl)} J_{jl} u_{jl} \right)$$

$$= \lim_{n \to 0} \left\{ \frac{1}{N} \sum_{\sigma} \sum_i \frac{1 + \sigma \sigma_i}{2} \exp\left( \frac{j^2 \beta_{\text{eff}}^2}{2N} \sum_{(jl)} u_{jl}^2 - \frac{ik \sum_j J_{ij} \sigma_j}{N} \sum_{l} u_{il} \sigma_l^1 - \frac{k^2 j^2}{2} \right) \right\}$$

$$= \lim_{n \to 0} \left\{ \frac{1}{N} \sum_{\sigma} \sum_i \frac{1 + \sigma \sigma_i}{2} \exp\left( \frac{j^2 n N \beta_{\text{eff}}^2}{4} - \frac{k^2 j^2}{2} + \frac{ik \sum_j J_{ij} \sigma_j}{N} \sum_{l} u_{il} \sigma_l^1 - \frac{k^2 j^2}{2} \right) \right\}$$

$$+ \frac{j^2 \beta_{\text{eff}}^2}{2N} \sum_{(a1)} \sum_l \sigma_{l}^a \sigma_l^1 + \frac{ik \sigma_{l}^a}{\beta_{\text{eff}}} \frac{k^2 j^2}{N} \right\},$$  \hspace{1cm} (B.3)

where $u_{ab} = \sum_{\alpha=1}^n \sigma_{a}^\alpha \sigma_{b}^\alpha$. Application of the Hubbard-Stratonovitch transformation leads to

$$G^{(1)}(k, \sigma) = \lim_{n \to 0} \left\{ \exp\left( \frac{j^2 n N \beta_{\text{eff}}^2}{4} - \frac{k^2 j^2}{2} \right) \right\} \left\{ \prod_{(a1)} \left\langle \sqrt{\frac{N}{2\pi}} J_{\beta_{\text{eff}}^2} \right\rangle d\sigma_{a1} \right\} \exp\left( -\frac{N^2 j^2 \beta_{\text{eff}}^2}{2} \sum_{(a1)} \sigma_{a1}^2 \right).$$
\[
\times \sum_{\sigma} \frac{1}{N} \sum_{i} \frac{1 + \sigma \sigma_{i}}{2} \exp \left( \tilde{J}^{2} \beta_{\text{eff}}^{2} \sum_{(\alpha \gamma) \neq (\alpha 1)} \sigma_{\alpha} \sigma_{\gamma} \right) \\
+ \tilde{J}^{2} \beta_{\text{eff}}^{2} \sum_{(\alpha 1)} q_{\alpha} \left( \sum_{l} \sigma_{l}^{\alpha} \sigma_{l}^{\gamma} + \frac{i k \sigma_{l}^{\alpha}}{\beta_{\text{eff}}} \right) \\
\left( \sum_{l} \sigma_{l}^{\alpha} \sigma_{l}^{\gamma} + \frac{i k \sigma_{l}^{\alpha}}{\beta_{\text{eff}}} \right) \right). \tag{B.4}
\]

We evaluate the integrals at the saddle point of \( q_{\alpha \gamma} \):

\[
G^{(1)}(k, \sigma) = \lim_{n \to 0} \left\{ \exp \left( \frac{\tilde{J}^{2} n \beta_{\text{eff}}^{2}}{4} - \frac{k^{2} J^{2}}{2} \left( 1 - \frac{2}{N} \right) \right) \right\}^{n(n-1)/2} \exp \left( - \frac{N^{2} \tilde{J}^{2} \beta_{\text{eff}}^{2}}{2} \sum_{(\alpha \gamma)} q_{\alpha}^{2} \right) \\
\times \sum_{\sigma} \frac{1}{N} \sum_{i} \frac{1 + \sigma \sigma_{i}}{2} \exp \left( \frac{\tilde{J}^{2} \beta_{\text{eff}}^{2}}{2} \sum_{(\alpha \gamma) \neq (\alpha 1)} q_{\alpha} \sum_{l} \sigma_{l}^{\alpha} \sigma_{l}^{\gamma} \right) \\
+ \tilde{J}^{2} \beta_{\text{eff}}^{2} \sum_{(\alpha 1)} q_{\alpha} \left( \sum_{l} \sigma_{l}^{\alpha} \sigma_{l}^{\gamma} + \frac{i k \sigma_{l}^{\alpha}}{\beta_{\text{eff}}} \right) \right). \tag{B.5}
\]

Furthermore, we assume the system in the paramagnetic phase, that is \( q_{\alpha \gamma} = \delta_{\alpha \gamma} \). Finally, we obtain the first-order characteristic function as

\[
G^{(1)}(k, \sigma) = \lim_{n \to 0} \left\{ \exp \left( - \frac{k^{2} J^{2}}{2} \right) \right\}^{n(n-1)/2} \exp \left( - \frac{N^{2} \tilde{J}^{2} \beta_{\text{eff}}^{2}}{2} \sum_{(\alpha \gamma)} q_{\alpha}^{2} \right) \\
\times \sum_{\sigma} \frac{1}{N} \sum_{i} \frac{1 + \sigma \sigma_{i}}{2} \exp \left( \frac{i k \tilde{J}^{2} \beta_{\text{eff}}^{2}}{2} \right) \left( \cos \left( k \tilde{J}^{2} \beta_{\text{eff}}^{2} \right) \right. \\
+ i \sigma \sin \left( k \tilde{J}^{2} \beta_{\text{eff}}^{2} \right) \right). \tag{B.6}
\]

The higher order approximation needs covariances between \( \left( \frac{1}{N} \right) \sum_{i} (1 + \sigma \sigma_{i}) / 2 \exp(i k \sum_{j} J_{ij} \sigma_{j}) \) and \( A(\sigma, \{ J_{ij} \}) \), where \( A \) stands for the quantities appearing in the expansion of the dynamical probability distribution function. The difference from the calculations of other quantities is the existence of \( (i k \sum_{j} J_{ij} \sigma_{j}) \) in the exponent. For this reason, the following term is included in equations:

\[
\left[ J_{ab}^{l} \exp \left( \beta_{\text{eff}} J_{ab} \sum_{(ij)} u_{ab} + i k \sum_{c} J_{ac} \sigma_{c}^{2} \right) \right] \\
= \left[ J_{ab}^{l} \exp \left( \beta_{\text{eff}} J_{ab} u_{ab} + i k J_{ab} \sigma_{b}^{2} \right) \right]
\]
\[
\times \left[ \exp \left( \beta_{\text{eff}} J_{ab} \sum_{(ij) \neq (ab)} u_{ab} + ik \sum_{c \neq b} J_{ac} \sigma_c^2 \right) \right], \tag{B.7}
\]

where the coefficient \( J'_{ab} \) comes from the quantity \( A(\sigma, \{ J_{ij} \}) \). The index \( l \) is smaller than or equal to three in this expansion, because the quantity \( A \) contains \( J^3_{ab} \) at most.

The following relations are useful for sample averages:

\[
\left[ \exp \left( \beta_{\text{eff}} J_{ab} u_{ab} + ik J_{ab} \sigma_b^2 \right) \right] = \exp \left( \frac{\tilde{J}_2 ^2}{2N} (\beta_{\text{eff}} u_{ab}^2 + 2ik \beta_{\text{eff}} \sigma_b^1 u_{ab} - k^2) \right) \tag{B.8}
\]

\[
\left[ J_{ab} \exp \left( \beta_{\text{eff}} J_{ab} u_{ab} + ik J_{ab} \sigma_b^2 \right) \right] = \frac{\tilde{J}_2 ^2}{N} \left( \beta_{\text{eff}} u_{ab} + ik \sigma_b^1 \right)
\times \exp \left( \frac{\tilde{J}_2 ^2}{2N} (\beta_{\text{eff}} u_{ab}^2 + 2ik \beta_{\text{eff}} \sigma_b^1 u_{ab} - k^2) \right) \tag{B.9}
\]

\[
\left[ \tilde{J}_{ab}^2 \exp \left( \beta_{\text{eff}} J_{ab} u_{ab} + ik J_{ab} \sigma_b^2 \right) \right] = \frac{\tilde{J}_2 ^2}{N} + \frac{\tilde{J}_4 ^4}{N} \left( \beta_{\text{eff}} u_{ab} + ik \sigma_b^1 \right)^2
\times \exp \left( \frac{\tilde{J}_2 ^2}{2N} (\beta_{\text{eff}} u_{ab}^2 + 2ik \beta_{\text{eff}} \sigma_b^1 u_{ab} - k^2) \right). \tag{B.10}
\]

In the case of \( l = 3 \), the leading term is of order \( 1/N^2 \). Therefore there is no contribution in the thermodynamic limit and we pay attention only to the case of \( l \leq 2 \).

We explain a part of the calculation of the second-order approximation. The second-order term is expressed as

\[
G^{(2)}(k, \sigma) = \beta^2 b_1 \text{Cov} \left( \sum_i h_i^2, \sum_i \frac{1 + \sigma \sigma_i}{2} \right)
\times \exp \left( ik \sum_j J_{ij} \sigma_j \right)
= \beta^2 b_1 \lim_{n \to \infty} \left\{ \frac{1}{N} \sum_{\sigma} \sum_{abcd} J_{ab} J_{ac} \sigma_b^1 \sigma_c^1 \right. \right.
\times \exp \left( ik \sum_e J_{de} \sigma_e^1 + \beta_{\text{eff}} \sum_{(ij)} J_{ij} u_{ij} \right) \]
\[
- \left[ \frac{1}{N} \sum_{\sigma} \sum_{abcd} J_{ab} J_{ac}\sigma_1^b\sigma_1^c \right. \\
\times \exp \left( ik \sum_e J_{ae} \sigma_e^2 + \beta_{\text{eff}} \sum_{(ij)} J_{ij} u_{ij} \right) \right]. 
\]

(B.11)

The overlap between the interactions makes the covariance non-vanishing, but this does not mean that all overlaps give finite contributions. In the present case, three configurations \(d = a, b\) and \(c\) have non-vanishing covariances and others are zero.

Let us consider the configuration \(d = a\) as an example. The difference between the first and the second terms is only in the replica index of \(\sigma_e\) in the exponent. Thus we explain only the procedure to treat the second term. The procedure is similar to the equilibrium calculation\(^9\) and the second term is found to be expressed as

\[
\beta^2 b_1 \lim_{n \to 0} \left\{ \left[ \frac{1}{N} \sum_{\sigma} \sum_{abc} J_{ab} J_{ac}\sigma_1^b\sigma_1^c \exp \left( ik \sum_e J_{ae} \sigma_e^2 + \beta_{\text{eff}} \sum_{(ij)} J_{ij} u_{ij} \right) \right] \\
\times \left[ \exp \left( ik \sum_{e \neq b,c} J_{ab} \sigma_e^2 + \beta_{\text{eff}} \sum_{(ij) \neq (ab),(ac)} J_{ij} u_{ij} \right) \right] \right\} 
\]

\[
= \beta^2 b_1 \lim_{n \to 0} \left\{ \frac{\tilde{J}_4}{N^2} \sum_{\sigma} \sum_{abc} \sigma_1^b\sigma_1^c \left( \beta_{\text{eff}} u_{ab} + i k \sigma_e^2 \right) \left( \beta_{\text{eff}} u_{ac} + i k \sigma_e^2 \right) \right. \\
\times \exp \left( \frac{\tilde{J}_2^2}{2N} \sum_{(ij)} \beta_{\text{eff}}^2 u_{ij}^2 + \sum_e 2ik \beta_{\text{eff}} \sigma_e^2 u_{ae} \right) - \frac{k^2 \tilde{J}_2^2}{2} \right\}. 
\]

(B.12)

Applying the Hubbard-Stratonovitch transformation and the saddle-point method, we obtain in the paramagnetic phase

\[
\beta^2 b_1 \frac{\tilde{J}_4}{N^2} \sum_{\sigma} \sum_{abc} \left\{ \beta_{\text{eff}}^2 \sum_{\gamma\delta} \sigma_\gamma^a \sigma_\delta^a \sigma_\gamma^b \sigma_\delta^b \sigma_\gamma^c \sigma_\delta^c \\
+ ik \beta_{\text{eff}} \left( \sum_{\gamma} \sigma_\gamma^a \sigma_\gamma^b \sigma_\gamma^c \sigma_\gamma^2 + \sum_{\delta} \sigma_\delta^a \sigma_\delta^b \sigma_\delta^c \sigma_\delta^2 \right) \right. \\
\left. - k^2 \sigma_1^a \sigma_1^b \sigma_1^c \sigma_1^2 \right\} \exp \left( - \frac{k^2 \tilde{J}_2^2}{2} + ik \tilde{J}_2 \beta_{\text{eff}} \sigma_1^2 \right). 
\]

(B.13)

By tracing out the spin configurations, we find the only non-vanishing contribution in the first term
for the case $\gamma = \delta = 1$. In this way the second term of eq. (B.11) is written as
\[
\beta^2 b_1 \left\{ \tilde{J}^4 \beta_{\text{eff}}^2 \exp \left( -\frac{k^2 \tilde{J}^2}{2} \cos(k \tilde{J}^2 \beta_{\text{eff}}) \right) \right\}, \tag{B.15}
\]
in the limit of $N \to \infty$.

The first term is obtained by the exchange $\sigma^2 \to \sigma^1$ as
\[
\beta^2 b_1 \frac{\tilde{J}^4}{N^3} \sum_{\sigma} \sum_{abc} \left\{ \beta_{\text{eff}}^2 \sum_{\gamma \delta} \sigma_a^2 \gamma_a^\delta \sigma_b^\gamma \sigma_c^\delta \right.
\]
\[
+ ik \beta_{\text{eff}} \left( \sum_{\gamma} \sigma_a^\gamma \sigma_b^\gamma \sigma_c^\gamma \sigma_1^\gamma + i k \beta_{\text{eff}} \sum_{\delta} \sigma_1^\delta \sigma_b^\delta \sigma_c^\delta \right)
\]
\[
- k^2 \sigma_b^1 \sigma_c^1 \sigma_1^1 \exp \left( -\frac{k^2 \tilde{J}^2}{2} + ik \tilde{J}^2 \beta_{\text{eff}} \sigma_a^1 \right). \tag{B.16}
\]

Non-vanishing contributions come from terms satisfying $\gamma = \delta = 1$:
\[
\beta^2 b_1 \left\{ \tilde{J}^4 \beta_{\text{eff}}^2 \exp \left( -\frac{k^2 \tilde{J}^2}{2} \cos(k \tilde{J}^2 \beta_{\text{eff}}) \right) \right\}
\]
\[
- 2k \tilde{J} \beta_{\text{eff}} \exp \left( -\frac{k^2 \tilde{J}^2}{2} \sin(k \tilde{J}^2 \beta_{\text{eff}}) \right)
\]
\[
- k^2 \tilde{J} \exp \left( -\frac{k^2 \tilde{J}^2}{2} \sin(k \tilde{J}^2 \beta_{\text{eff}}) \right) \right\}. \tag{B.17}
\]

Subtraction of eq. (B.15) from eq. (B.17) gives the covariance of the configuration $d = a$ in eq. (B.12) as
\[
\beta^2 b_1 \left\{ -2k \tilde{J} \beta_{\text{eff}} \exp \left( -\frac{k^2 \tilde{J}^2}{2} \sin(k \tilde{J}^2 \beta_{\text{eff}}) \right) \right.
\]
\[
- k^2 \tilde{J} \exp \left( -\frac{k^2 \tilde{J}^2}{2} \sin(k \tilde{J}^2 \beta_{\text{eff}}) \right) \right\}. \tag{B.18}
\]

For the configurations of $d = b$ and $c$, the covariances are identical and the result is
\[
\beta^2 b_1 \left\{ -k \tilde{J} \beta_{\text{eff}} \exp \left( -\frac{k^2 \tilde{J}^2}{2} \sin(k \tilde{J}^2 \beta_{\text{eff}}) \right) \right\}. \tag{B.19}
\]

Contributions of all configurations are summed up to yield
\[
G^{(2)}(k, \sigma) = \beta^2 b_1 \left\{ -4k \tilde{J} \beta_{\text{eff}} \exp \left( -\frac{k^2 \tilde{J}^2}{2} \sin(k \tilde{J}^2 \beta_{\text{eff}}) \right) \right\}.
\]
\[-k^2 \bar{J}^4 \exp \left( -\frac{k^2 \bar{J}^2}{2} \right) \sin(k \bar{J}^2 \beta_{\text{eff}}) \\right\]. \hspace{1cm} (B.20)

The third-order term can be calculated similarly.

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Figure Captions

Fig. 1. The relaxation of \([\langle H \rangle]/N\) of the SK model by Monte Carlo simulation, high-temperature expansions, and the approximation of a Boltzmann form with the inverse temperature \(\beta_{\text{eff}}^{(3)}\). Average and standard deviation are shown for 50 samples simulated. The system size is \(N = 1000\) in simulations.

Fig. 2. The relaxation of the expectation value of \(\sum h_i^2\) per spin for the SK model.

Fig. 3. The relaxation of the expectation value of \(\sum \sigma_i h_i^3\) per spin for the SK model.

Fig. 4. The relaxation of the expectation value of \(\sum J_{ij} h_i h_j\) per spin for the SK model.

Fig. 5. The relaxation of \([\langle H \rangle]/N\) of the two-dimensional \(\pm J\) model by Monte Carlo simulation, the first order approximation and the \(\beta_{\text{eff}}^{(3)}\)-approximation. Average and standard deviation are shown for 50 samples simulated. The system size is \(N = 50^2\) in simulations.

Fig. 6. The relaxation of the expectation value of \(\sum h_i^2\) per spin for the two-dimensional \(\pm J\) model.

Fig. 7. The relaxation of the expectation value of \(\sum \sigma_i h_i^3\) per spin for the two-dimensional \(\pm J\) model.

Fig. 8. The relaxation of the expectation value of \(\sum J_{ij} h_i h_j\) per spin for the two-dimensional \(\pm J\) model.

Fig. 9. Time evolution of the spin-field distribution functions \(P(h, \pm 1)\) at \(T = 2\). Circles and squares are results of simulations. Solid lines denote the theoretical prediction of the third-order approximation.

Fig. 10. Time evolution of physical quantities by the third-order series expansions (full curves) and simulations (dotted curves, standard deviations omitted, \(N = 1000\) and 50 samples) when the SK model is quenched from \(T_0 = \infty\) to \(T = 2\).

Fig. 11. Time evolution of correlation coefficients with \(\sum J_{ij} h_i h_j\) when the system is quenched from \(T_0 = \infty\) to \(T = 2\) obtained by using \(\beta_{\text{eff}}^{(3)}\).

Fig. 12. Time evolution of the terms in the dynamical probability distribution function evaluated by the third-order series expansions for the SK model quenched from \(T_0 = \infty\) to \(T = 5\).

Fig. 13. Time evolution of the terms in the dynamical probability distribution function evaluated by the third-order series-expansions for the SK model quenched from \(T_0 = \infty\) to \(T = 2\).

Fig. 14. Time evolution of the terms in the dynamical probability distribution function for the SK model quenched from \(T_0 = \infty\) to \(T = 1 = T_c\) by simulations. The system size is \(N = 1000\).
The results are 50-sample averaged.

Fig. 15. Time evolution of the terms in the dynamical probability distribution function for the two-dimensional $±\, J$ model quenched from $T_0 = \infty$ to $T = 5$ by the $\beta_{\text{eff}}^{(3)}$-approximations.

Fig. 16. Time evolution of the terms in the dynamical probability distribution function for the two-dimensional $±\, J$ model quenched from $T_0 = \infty$ to $T = 2$ by simulations. The system size is $N = 50^2$ and the results are averaged over 50 samples.

Fig. 17. First-order cumulants of the spin-field distribution functions $P(h, \pm 1)$ at $T = 2$. The dotted line is by the high-temperature expansion and the full line is for the CLS theory. Simulation results ($N = 5000$, 100 samples) are also shown for comparison.

Fig. 18. Second-order cumulants of the spin-field distribution functions $P(h, \pm 1)$ at $T = 2$. Symbols are the same as in Fig. 17.