PROPERTIES OF GENERALIZED FORCHHEIMER FLOWS IN POROUS MEDIA

LUAN T. HOANG†,*, THINH T. KIEU†, AND TUOC V. PHAN‡

ABSTRACT. The nonlinear Forchheimer equations are used to describe the dynamics of fluid flows in porous media when Darcy’s law is not applicable. In this article, we consider the generalized Forchheimer flows for slightly compressible fluids and study the initial boundary value problem for the resulting degenerate parabolic equation for pressure with the time-dependent flux boundary condition. We estimate \( L^\infty \)-norm for pressure and its time derivative, as well as other Lebesgue norms for its gradient and second spatial derivatives. The asymptotic estimates as time tends to infinity are emphasized. We then show that the solution (in interior \( L^\infty \)-norms) and its gradient (in interior \( L^{2-\delta} \)-norms) depend continuously on the initial and boundary data, and coefficients of the Forchheimer polynomials. These are proved for both finite time intervals and time infinity. The De Giorgi and Ladyzhenskaya-Uraltseva iteration techniques are combined with uniform Gronwall-type estimates, specific monotonicity properties, suitable parabolic Sobolev embeddings and a new fast geometric convergence result.

CONTENTS

1. Introduction 2
2. Preliminaries 3
3. Estimates of solutions 8
3.1. Estimates for pressure 9
3.2. Estimates for pressure’s gradient 17
3.3. Estimates for pressure’s time derivative 24
3.4. Estimates for pressure’s second derivatives 33
4. Dependence on initial and boundary data 36
4.1. Results for pressure 36
4.2. Results for pressure gradient 48
5. Dependence on the Forchheimer polynomial 51
5.1. Results for pressure 52
5.2. Results for pressure gradient 58
Appendix A. Auxiliaries 61
References 63

Date: May 28, 2014.
1. INTRODUCTION

In literature, Darcy’s equation is considered as law of hydrodynamics in porous media. This linear relation between the fluid velocity and pressure gradient was derived by Darcy in [7]. However, even in their early works, Darcy and Dupuit already observed deviations of fluid flows from this linear equation [7, 11]. In early 1900s, Forchheimer proposed three models for nonlinear flows, the so-called two-term, three-term and power laws (cf. [13, 14]). More experiments gave rise to more nonlinear models during 1940-60s (cf. [2, 22]). Despite that fact, most mathematical papers on fluids in porous media deal only with Darcy’s law starting from 1960s. The mathematical investigations of Forchheimer equations and related Brinkman-Forchheimer equation started much later in 1990s and have been growing ever since (cf. [4–6, 15, 16, 23–25], see also [29]). Even so, most of these papers consider incompressible fluids only. In our previous works [1, 17–19], we proposed and studied generalized Forchheimer equations for slightly compressible fluids in porous media. Such mathematical generalization is appropriate and useful. This is due to the nature of Forchheimer equations which are derived from experiments and have their physical parameters found by fitting real life data. From mathematical point of view, it introduces a new class of degenerate parabolic equations into studies of porous medium equations [31]. In our mentioned papers, we study the properties of pressure in space $L^\alpha$ ($1 \leq \alpha < \infty$), of pressure gradient in space $L^{2-a}$ and of pressure’s time derivative in $L^2$. Here $a$ is a number between 0 and 1 defined in terms the degree of the polynomial in the generalized Forchheimer equations (see (2.9)). In this paper, we focus on properties of fluid flows in higher regularity spaces. Specifically, we will study the pressure and its time derivative in space $L^\infty$, the pressure gradient in $L^s$ for any $1 \leq s < \infty$ and the pressure Hessian in $L^{2-\delta}$ for $\delta > 0$. Moreover, our high priority is the long-time dynamical properties, including uniform estimates in time, asymptotic bounds and asymptotic stability. Such topics of long-time dynamics of degenerate parabolic equation, particularly in $L^\infty$, is important, and the specific results are usually hard to obtain. (See, for e.g., chapters 18–20 of [31] for porous medium equations, [26, 27] for degenerate equations with Dirichlet boundary condition, [10] for systems.) In order to work in these much higher regularity spaces and obtain estimates for large time, more sophisticated techniques are called for. We combine iteration techniques by De Giorgi [8] and Ladyzhenskaya-Uraltseva [20], which were primarily used for studying local properties of solutions to elliptic and parabolic problems, with those from long-time dynamics studies for nonlinear partial differential equations such as Navier-Stokes equations [12]. For our degenerate equations, we also use and refine relevant techniques in DiBenedetto’s book [9]. Such a combination gives fruitful results on the estimates of solutions for large time as well as detailed continuous dependence of the solutions on time-dependent boundary data and coefficients of the Forchheimer polynomials. (The latter results are called structural stability.) We also emphasize that the mentioned general techniques from parabolic equations must be used in accordance with the structure of our equation, in this case, the important monotonicity and perturbed monotonicity in Lemma 2.2 below.

In the current work, we focus on the case of Degree Condition, see (DC) in the next section, for the following reasons. First, it already covers most commonly used Forchheimer equations in practice, namely, the two-term, three-term and power laws. Second, to take advantage of available estimates in our previous work [18]. Third, to make clear our ideas and techniques without involving much more complicated technical details in case that the Degree Condition is not met (see [19]); such case will be investigated in our future work.
The paper is organized as follows. In section 2, we present the formulation of generalized Forchheimer equations and consequently obtain a degenerate parabolic equation for pressure $p$. Basic properties of this equation are reviewed and suitable parabolic Sobolev embeddings are presented. In section 3, we study the initial boundary value problem (IBVP) for pressure with the time-dependent flux boundary data $\psi(x,t)$. We derive various estimates for the shifted solution $\bar{p}$ (see (3.6)). This section is divided into four subsections. Subsection 3.1 deals with interior $L^\infty$-estimates for $\bar{p}$, where De Giorgi's technique is applied to the time derivative with weighted parabolic Sobolev embedding. The main estimates are in Theorem 3.16. Subsection 3.2 deals with interior $L^s$-estimates for $\nabla p$ for any $s \geq 1$, see Theorems 3.10 and 3.12. The key Sobolev embedding is Lemma 3.7 with specific weight $K(|\nabla w|)$ related to our degenerate structure. The main iteration step is (3.46). Subsection 3.3 deals with interior $L^\infty$-estimates for $\bar{p}_t$, where De Giorgi's technique is applied to the time derivative with weighted parabolic Sobolev embedding. The main estimates are in Theorem 3.16. Other particular large time and asymptotic estimates are in Theorems 3.17 and 3.18. Subsection 3.4 deals with interior $L^{2^{-\delta}}$-estimates for $\bar{p}$, where De Giorgi's technique is applied to the time derivative with weighted parabolic Sobolev embedding. The main estimates are in Theorems 3.21 and 3.22. Subsection 3.5 deals with second derivatives were not considered in our previous works. Sections 4 and 5 are devoted to the structural stability issues. In section 4, we prove the continuous dependence of the solutions on the boundary data. Specifically, it is established for $\bar{p}$ in interior $L^\infty$-norms (see Theorem 4.4) and for $\nabla p$ in interior $L^{2^{-\delta}}$-norms (see Theorems 4.12 and 4.15 for finite time intervals, and Theorems 4.6 and 4.9 for $t \to \infty$). The results show that even when each individual flux $\psi_1, \psi_2$ grows unbounded as time $t \to \infty$, the difference between two corresponding solutions $\bar{p}_1, \bar{p}_2$ can be small provided the difference $\Psi = \psi_1 - \psi_2$ is small. In order to obtain this, De Giorgi's iteration is combined, in Proposition 4.1, with the monotonicity available for our degenerate equation. In section 5, we prove the continuous dependence of the solutions on the Forchheimer polynomials (see Theorems 5.3, 5.4, 5.7 and 5.9). Here, the perturbed monotonicity is combined with De Giorgi's technique as presented in Proposition 5.1. It is proved in Theorems 5.5 and 5.8 that the smallness of the difference $\bar{P}(x,t)$ between two solutions corresponding to two Forchheimer equations, when $t \to \infty$, can be controlled by the difference between coefficient vectors of the two Forchheimer polynomials.

2. Preliminaries

We consider a fluid in a porous medium having velocity $u(x,t) \in \mathbb{R}^n$, pressure $p(x,t) \in \mathbb{R}$ and density $\rho(x,t) \in \mathbb{R}^+ = [0, \infty)$, with the spatial variable $x \in \mathbb{R}^n$ and time variable $t \in \mathbb{R}$. The space dimension $n = 3$ in physics problems, but here we consider any $n \geq 2$. Generalized Forchheimer equations, studied in [1,17], are of the form:

$$g(|u|)u = -\nabla p, \quad (2.1)$$

where $g(s) \geq 0$ is a function defined on $[0, \infty)$. When

$$g(s) = \alpha, \alpha + \beta s, \alpha + \beta s + \gamma s^2, \alpha + \gamma_m s^{m-1},$$

where $\alpha, \beta, \gamma, m, \gamma_m$ are empirical constants, we have Darcy’s law, Forchheimer’s two-term, three-term and power laws, respectively. In this paper, the function $g$ in (2.1) is a generalized polynomial with
non-negative coefficients, that is,
\[ g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + \ldots + a_N s^{\alpha_N}, \ s \geq 0, \] (2.2)
where \( N \geq 1, \alpha_0 = 0 < \alpha_1 < \ldots < \alpha_N \) are real (not necessarily integral) numbers, the coefficients satisfy \( a_0, a_N > 0 \) and \( a_1, \ldots, a_{N-1} \geq 0 \). The number \( \alpha_N \) is the degree of \( g \) and is denoted by \( \deg(g) \). Denote the vectors of powers and coefficients by \( \vec{\alpha} = (\alpha_0, \ldots, \alpha_N) \) and \( \vec{a} = (a_0, \ldots, a_N) \). The class of functions \( g(s) \) as in (2.2) is denoted by \( \text{FP}(N, \vec{\alpha}) \), which is the abbreviation of “Forchheimer polynomials”. When the function \( g \) in (2.1) belongs to \( \text{FP}(N, \vec{\alpha}) \), it is referred to as the Forchheimer polynomial.

From relation (2.1) one can solve velocity \( u \) in terms of pressure gradient \( \nabla p \) and derives a nonlinear version of Darcy’s equation:
\[ u = -K(|\nabla p|) \nabla p. \] (2.3)
The function \( K : \mathbb{R}_+ \to \mathbb{R}_+ \) is defined by
\[ K(\xi) = \frac{1}{g(s(\xi))}, \] where \( s(\xi) \geq 0 \) satisfies \( sg(s) = \xi, \) for \( \xi \geq 0. \) (2.4)
We will use notation \( g(s, \vec{a}), K(\xi, \vec{a}), s(\xi, \vec{a}) \) to denote the corresponding functions in (2.2) and (2.4) when the dependence on \( \vec{a} \) needs be specified.

Equation (2.1) replaces the momentum equation in fluid mechanics. In addition to this, we have the equation of continuity
\[ \frac{\partial p}{\partial t} + \nabla \cdot (\rho u) = 0, \] (2.5)
and the equation of state which, for slightly compressible fluids, is
\[ \frac{d\rho}{dp} = \frac{\rho}{\kappa}, \ \kappa > 0. \] (2.6)
From equations (2.3), (2.5) and (2.6) one derives an equation for the pressure:
\[ \frac{\partial p}{\partial t} = \kappa \nabla \cdot (K(|\nabla p|) \nabla p) + K(|\nabla p|)|\nabla p|^2. \] (2.7)
Since the constant \( \kappa \) is very large for most slightly compressible fluids in porous media \([21]\), we neglect the second term on the right-hand side of (2.7). This results in the following reduced equation
\[ \frac{\partial p}{\partial t} = \kappa \nabla \cdot (K(|\nabla p|) \nabla p). \] (2.8)
Note that this reduction is commonly used in engineering. By rescaling the time variable, hereafter we assume that \( \kappa = 1. \)
Let \( g = g(s, \vec{a}) \) in \( \text{FP}(N, \vec{a}) \). The following numbers are in our calculations:
\[ a = \frac{\alpha_N}{1 + \alpha_N} \in (0, 1), \ b = \frac{a}{2 - a} = \frac{\alpha_N}{2 + \alpha_N} \in (0, 1), \] (2.9)
\[ \chi(\vec{a}) = \max \left\{ a_0, a_1, \ldots, a_N, \frac{1}{a_0}, \frac{1}{a_N} \right\} \in [1, \infty). \] (2.10)
The following properties for \( K(\xi, a) \) are proved Lemmas III.5 and III.9 of \([1]\). For \( \xi \geq 0, \) we have
\[ -a K(\xi, \vec{a}) \leq \xi \frac{\partial K(\xi, \vec{a})}{\partial \xi} \leq 0. \] (2.11)
Consequently, \( K(\xi, a) \) is decreasing in the variable \( \xi \) and hence
\[
K(\xi, a) \leq K(0, a) = a_0^{-1} \leq \chi(a),
\] (2.12)

Moreover, \( K(\xi, a)\xi^m \) is increasing in \( \xi \) for all \( m \geq 1 \).

For the type of degeneracy of \( K(\xi, a) \) in \( \xi \), we recall the following facts.

**Lemma 2.1** (cf. [17], Lemma 2.1). Let \( g(s, a) \) be in class \( FP(N, a) \). One has for any \( \xi \geq 0 \) that
\[
\frac{c_0^{-1} \chi(a)^{1-a}}{(1 + \xi)a} \leq K(\xi, a) \leq \frac{c_0 \chi(a)^{1+a}}{(1 + \xi)a},
\] (2.13)
and for any \( m \geq 1, \delta > 0 \) that
\[
c_0^{-1} \chi(a)^{1-a} \frac{\delta^a}{(1 + \delta)^a} (\xi^{m-a} - \delta^{m-a}) \leq K(\xi, a)\xi^m \leq c_0 \chi(a)^{1+a} \xi^{m-a},
\] (2.14)
where \( c_0 = c_0(N, \alpha_N) > 0 \) depends on \( N \) and \( \alpha_N \) only. In particular, when \( m = 2, \delta = 1, \) one has
\[
2^{-a}c_0^{-1} \chi(a)^{1-a}(\xi^2 - 1) \leq K(\xi, a)\xi^2 \leq c_0 \chi(a)^{1+a} \xi^{2-a}.
\] (2.15)

Same as in [17], we define
\[
H(\xi, a) = \int_0^{\xi^2} K(\sqrt{s}, a)ds \quad \text{for } \xi \geq 0.
\] (2.16)

When vector \( a \) is fixed, we denote \( K(\cdot, a) \) and \( H(\cdot, a) \) by \( K(\cdot) \) and \( H(\cdot) \), respectively. The function \( H(\xi) \) can be compared with \( \xi \) and \( K(\xi) \) by
\[
K(\xi)\xi^2 \leq H(\xi) \leq 2K(\xi)\xi^2, \quad c_1(\xi^{2-a} - 1) \leq H(\xi) \leq c_2\xi^{2-a},
\] (2.17)
where \( c_1, c_2 > 0 \) depend on \( \chi(a) \).

Next, we recall important monotonicity properties. For convenience, we use the following notation:
let \( \bar{x} = (x_1, x_2, \ldots) \) and \( \bar{x}' = (x_1', x_2', \ldots) \) be two arbitrary vectors of the same length, including possible length 1. We denote by \( \bar{x} \lor \bar{x}' \) and \( \bar{x} \land \bar{x}' \) their maximum and minimum vectors, respectively, with components \( (\bar{x} \lor \bar{x'})_j = \max\{x_j, x'_j\} \) and \( (\bar{x} \land \bar{x'})_j = \min\{x_j, x'_j\} \).

**Lemma 2.2** (cf. [17], Lemma 5.2). Let \( g(s, a) \) and \( g(s, a') \) belong to class \( FP(N, a) \). Then for any \( y, y' \) in \( \mathbb{R}^n \), one has
\[
(K(|y|, a)y - K(|y'|, a')y') \cdot (y - y') \geq (1 - a) \cdot K(|y| \lor |y'|, a \lor a') \cdot |y - y'|^2
- N \cdot \max\{\chi(a), \chi(a')\} \cdot |a - a'| \cdot K(||y| \lor |y'|, a \land a') \cdot (|y| \lor |y'|) \cdot |y - y'|,
\] (2.18)
where \( a \in (0, 1) \) is defined in (2.9). Particularly, when \( a = a' \), we have
\[
(K(|y|, a)y - K(|y'|, a)y') \cdot (y - y') \geq (1 - a) \cdot K(|y| \lor |y'|, a) \cdot |y - y'|^2.
\] (2.19)

When the fluid is confined in an open, bounded domain \( U \) of \( \mathbb{R}^n \), the Sobolev embeddings play an important role in our study. For \( 0 < r < n \), we denote by \( r^* \) the critical Sobolev exponent, i.e. \( r^* = \frac{nr}{n-r} \).

The following fact is used frequently
\[
r^* > 2 \iff \frac{nr}{n-r} > 2 \iff r > \frac{2n}{n+2} \iff (2-r)n < 2r.
\] (2.20)

With this notation, we define the **Degree Condition** as one of the following equivalent conditions:
\[
\deg(g) \leq \frac{4}{n-2}, \quad a \leq \frac{4}{n+2}, \quad 2 \leq (2-a)^*, \quad 2 - a \geq \frac{2n}{n+2}.
\] (DC)
Similarly, we define the **Strict Degree Condition** as one of the following equivalent conditions:

\[
\deg(g) < \frac{4}{n - 2}, \quad a < \frac{4}{n + 2}, \quad 2 < (2 - a)^*, \quad 2 - a > \frac{2n}{n + 2}.
\]  

(SDC)

We will assume the Strict Degree Condition very often in this paper, but not always. Whenever this condition is met, the Sobolev space \(W^{1,2-a}(U)\) is continuously embedded into \(L^2(U)\).

We now consider parabolic Sobolev embeddings. Let us denote throughout \(Q_T = U \times (0, T)\).

**Lemma 2.3.** If \(2n/(n + 2) \leq r \leq 2, \ r < n \text{ and } p = (n + 2)/n\) then

\[
\|u\|_{L^p(Q_T)} \leq C(1 + T^{1/p})[[u]]_{2,r,T},
\]  

where \(C = C(U, n, r) > 0\) is independent of \(T\) and

\[
[[u]]_{2,r,T} = \text{ess sup}_{t \in [0, T]} \|u(t)\|_{L^2(U)} + \|\nabla u\|_{L^r(Q_T)}.
\]

In (2.21) above, we can remove \(T^{1/p}\) whenever \(u\) vanishes on the boundary of \(U\).

**Proof.** The proof is standard, cf. [9, 20]. The short proof is presented here for the sake of completeness, and also serves the next lemma. For convenience we denote \([[:]] = [[:]]_{2,r,T}\. Note that \(2 \leq p \leq r^*\). We write

\[
\frac{1}{p} = \frac{\alpha}{2} + \frac{\beta}{r^*}, \quad \alpha, \beta \geq 0, \quad \alpha + \beta = 1.
\]  

(2.23)

By interpolation inequality and Sobolev embedding, we have

\[
\|u(t)\|_{L^p(U)} \leq \|u(t)\|^{\alpha}_{L^2(U)} \|u(t)\|^{\beta}_{L^r(U)} \leq C\|u(t)\|^{\alpha}_{L^2(U)} \delta \|u(t)\|^{\beta}_{L^r(U)} + \|\nabla u(t)\|^{\beta}_{L^r(U)},
\]

where \(\delta = 1\) in general, and \(\delta = 0\) in case \(u\) vanishes on the boundary \(\partial U\). Note that \(r \leq 2\) then

\[
\|u(t)\|_{L^r(U)} \leq C\|u(t)\|_{L^2(U)}.
\]

Thus,

\[
\|u(t)\|_{L^p(U)} \leq C\delta \|u(t)\|_{L^2(U)} + C\|u(t)\|^{\alpha}_{L^2(U)} \|\nabla u(t)\|^{\beta}_{L^r(U)}.
\]  

(2.24)

Raising (2.24) to power \(p\), integrating it in \(t\) from 0 to \(T\) and using the fact that \(\|u(t)\|_{L^2(U)} \leq [[u]]\) a.e. in \([0, T]\), we obtain

\[
\|u\|_{L^p(Q_T)}^p \leq C\delta T[[u]]^p + C[[u]]^{\alpha p}\int_0^T \|\nabla u(t)\|^{\beta p}_{L^r(U)} dt.
\]  

(2.25)

From (2.23), we find \(\beta = \frac{n}{n+2}\). Thus, \(\beta p = r\) and we have from (2.25) that

\[
\|u\|_{L^p(Q_T)}^p \leq C\delta T[[u]]^p + C[[u]]^{\alpha p} \left(\int_0^T \int_U |\nabla u(x,t)|^r dx dt\right)^{\frac{\beta p}{r}}
\]

\[
\leq C\delta T[[u]]^p + C[[u]]^{\alpha p} [[u]]^{\beta p} = C[[u]]^p (\delta T + 1).
\]  

(2.26)

Therefore, we obtain (2.21). \(\square\)

The following parabolic Sobolev embedding with spatial weights will be useful in this paper.

**Lemma 2.4.** Given \(W(x,t) > 0 \text{ on } Q_T\). Suppose two numbers \(m\) and \(r\) satisfy \(2n/(n + 2) \leq r \leq 2, \ r < n \text{ and } r < m < r^*\). Let

\[
p = 2 + m - \frac{2m}{r^*}.
\]  

(2.27)

Then

\[
\|u\|_{L^p(Q_T)} \leq C[[u]]_{2,m,W;T} \left\{T^{1/p} + \text{ess sup}_{t \in [0, T]} \left(\int_U W(x,t)^{-\frac{r}{m-r}} dx\right)^{\frac{m-r}{pr}}\right\},
\]  

(2.28)
Proof. Thus, we obtain (2.28). In particular, when

\begin{equation}
\|u\|_{L^p(Q_T)} \leq C[\|u\|_{2,2,W;T}]^{1/p} + \left( \int_0^T \int_U W(x,t) |\nabla u|^m \ dx \ dt \right)^{1/m}
\end{equation}

Consequently, under the Strict Degree Condition, when \(m = 2\) and \(r = 2 - a\) we have

\begin{equation}
\|u\|_{L^p(Q_T)} \leq C[\|u\|_{2,2,W;T}]^{1/p} + \left( \int_0^T \int_U W(x,t)^{-\frac{2-a}{a}} \ dx \right)^{\frac{2}{p(2-a)}}
\end{equation}

where \(2 < p < (2 - a)^*\) given explicitly by

\begin{equation}
p = 4 - \left( \frac{1}{2 - a} \right).
\end{equation}

In (2.28) and (2.30) above, we can remove \(T^{1/p}\) whenever \(u\) vanishes on the boundary of \(U\).

**Proof.** In this proof we denote \([\cdot] = [\cdot]_{2,2,W;T}\). By Hölder’s inequality,

\begin{equation}
\|\nabla u\|_{L^r(U)} = \left( \int_U W^\sigma |\nabla u|^r \cdot W^{-\frac{r}{m}} \ dx \right)^{1/2} \leq \left( \int_U W |\nabla u|^m \ dx \right)^{1/2} \left( \int_U W^{-\frac{m-r}{m}} \ dx \right)^{1/2 - \frac{1}{m}}.
\end{equation}

By (2.20) and the condition on \(r\) we have \(r^* \geq 2\). With exponent \(p\) in (2.27), we write

\begin{equation}
p - 2 = \frac{m(r - 2)}{r^*}, \quad p - r^* = \frac{(r - 2)(m - r^*)}{r^*}, \quad p - m = \frac{2(r^* - m)}{r^*},
\end{equation}

then we have \(2 \leq p \leq r^*\) and \(p > m\). Let \(\alpha\) and \(\beta\) be defined as in (2.23). Then applying (2.24) and (2.32) yields

\begin{equation}
\|u\|_{L^p(Q_T)}^p \leq C \delta T [\|u\|]^{p} + C[\|u\|]^{\alpha p} \left( \int_0^T \left( \int_U W |\nabla u|^m \ dx \right)^{\frac{m}{m-r}} \ dx \right)^{\frac{1}{1-\frac{m}{m-r}}} \cdot \text{ess sup}_{[0,T]} \left( \int_U W(x,t)^{-\frac{r}{m-r}} \ dx \right)^{1-\frac{1}{m}}.
\end{equation}

From (2.23) and (2.33), we have \(\beta = \frac{r^*(p-2)}{p(r^*-2)} = \frac{m}{p}\). Therefore, we rewrite (2.34) as

\begin{equation}
\|u\|_{L^p(Q_T)}^p \leq C \delta T [\|u\|]^{p} + C[\|u\|]^{\alpha p} \left( \int_0^T \int_U W |\nabla u|^m \ dx \right)^{\frac{m}{m-r}} \cdot \text{ess sup}_{[0,T]} \left( \int_U W(x,t)^{-\frac{r}{m-r}} \ dx \right)^{\frac{1}{1-\frac{m}{m-r}}}.
\end{equation}

Thus, we obtain (2.28). In particular, when \(m = 2\) and \(r = 2 - a\) then (2.27) becomes (2.31). Under the Strict Degree Condition, requirements on \(r\) and \(m\) are met. Then (2.30) follows (2.28). \(\square\)

Another type of embedding will be proved in Lemma 3.7.
3. ESTIMATES OF SOLUTIONS

Let $U$ be a bounded, open, connected subset of $\mathbb{R}^n$ with boundary $\Gamma$ of class $C^2$. We consider a fluid flow in $U$ that satisfies the generalized Forchheimer equation \( 2.1 \) with a fixed $g(s) = g(s, \vec{a}) \in FP(N, \vec{a})$. We study the resulting parabolic equation for the pressure $p = p(x, t)$:

$$\frac{\partial p}{\partial t} = \nabla \cdot (K(|\nabla p|)\nabla p), \quad x \in U, \quad t > 0. \tag{3.1}$$

In addition, we assume the flux condition on the boundary:

$$u \cdot \vec{v} = \psi(x, t), \quad x \in \Gamma, \quad t > 0,$$

where $\vec{v}$ is the outward normal vector on $\Gamma$ and the flux $\psi(x, t)$ is known. Hence, by \( 2.3 \) we have

$$-K(|\nabla p|)\nabla p \cdot \vec{v} = \psi \quad \text{on} \quad \Gamma \times (0, \infty). \tag{3.2}$$

The initial data

$$p(x, 0) = p_0(x) \text{ is given.} \tag{3.3}$$

We will focus on the IBVP \( 3.1, 3.2 \) and \( 3.3 \). By integrating \( 3.1 \) over $U$, we easily find

$$\frac{d}{dt} \int_U p(x, t)dx = \int_U K(|\nabla p|)\nabla p \cdot \vec{v} d\sigma = -\int_\Gamma \psi(x, t)d\sigma, \quad t > 0. \tag{3.4}$$

(Here $d\sigma$ is the surface area element.) By the continuity of $\int_U p(x, t)dx$ and $\int_\Gamma \psi(x, t)d\sigma$ on $[0, \infty)$, we assert

$$\int_U p(x, t)dx = \int_U p(x, 0)dx - \int_0^t \int_\Gamma \psi(x, \tau)d\sigma d\tau, \quad t \geq 0. \tag{3.5}$$

Let $\bar{p}(x, t) = p(x, t) - \frac{1}{|U|} \int_U p(x, t)dx$, for $x \in U$ and $t \geq 0$, where $|U|$ denotes the volume of $U$. Then $\bar{p}$ satisfies the zero average condition

$$\int_U \bar{p}(x, t)dx = 0 \quad \text{for all} \quad t \geq 0. \tag{3.6}$$

It follows from \( 3.4 \) that for $t \geq 0$,

$$\bar{p}(x, t) = p(x, t) - \frac{1}{|U|} \int_U p_0(x)dx + \frac{1}{|U|} \int_0^t \int_\Gamma \psi(x, \tau)d\sigma d\tau. \tag{3.7}$$

We call $\bar{p}(x, t)$ the shifted solution. Then $\bar{p}$ satisfies the following IBVP

$$\begin{align*}
\frac{\partial \bar{p}}{\partial t} &= \nabla \cdot (K(|\nabla \bar{p}|)\nabla \bar{p}) + \frac{1}{|U|} \int_\Gamma \psi(x, t)d\sigma, \quad x \in U, \quad t > 0, \\
-K(|\nabla \bar{p}|)\nabla \bar{p} \cdot \vec{v} &= \psi(x, t), \quad x \in \Gamma, \quad t > 0, \\
\bar{p}(x, 0) &= p_0(x) - \frac{1}{|U|} \int_U p_0(x)dx, \quad x \in U.
\end{align*} \tag{3.7}$$

Note that even in the linear case, i.e., $K(\xi) \equiv const.$ and $\psi(x, t)$ is uniformly bounded on $\Gamma \times [0, \infty)$, the solution $p(x, t)$ can still be unbounded as $t \to \infty$. Therefore instead of estimating $p(x, t)$ directly, we will estimate $\bar{p}(x, t)$. Thanks to the explicit relation \( 3.6 \) between $p$ and $\bar{p}$, the obtained results will have clear physical interpretations and the estimates for $p(x, t)$ can be easily retrieved from those for $\bar{p}(x, t)$.

The existence of weak solutions can be treated by the theory of nonlinear monotone operators (cf. \( 3, 28, 32 \), see also section 3 of \( 19 \) for our proof for the Dirichlet boundary condition). The regularity of weak solutions is treated in \( 19 \). For simplicity, we always assume that the initial data $p_0$ and boundary data $\psi$ are sufficiently smooth and the solution of \( 3.7 \) exists for all $t \geq 0$. Also, sufficient regularity of the solution is assumed. For example, when we estimate $L^\infty$-norm of $\bar{p}$ we require $\bar{p} \in C(\bar{U} \times [0, \infty))$;
when we estimate $L^\infty$-norm of $\bar{p}_t$ we require $\bar{p}_t \in C(U \times (0, \infty))$; when we estimate $L^s$-norm of $\nabla p$ we require $\nabla p \in C([0, \infty), L^s_{\text{loc}}(U))$.

In the following subsections, we derive various estimates for pressure and its derivatives. These estimates are important by themselves and for the next sections when we study the stability of the solutions.

**Notation.** In estimates below, constants $C$’s always depend on the dimension $n$, domain $U$ and the Forchheimer polynomial $g(s, \tilde{a})$. Additional dependence will be specified as needed.

We use short-hand notation $\|f\|_{L^2}, \|f\|_{L^\infty}$ for the norms $\|f\|_{L^2(U)}, \|f\|_{L^\infty(U)}$ whenever $f$ is defined on $U$. Similarly, if a function $\phi$ is defined on $\Gamma$, we use short-hand notation $\|\phi\|_{L^2}, \|\phi\|_{L^\infty}$ for the norms $\|\phi\|_{L^2(\Gamma)}, \|\phi\|_{L^\infty(\Gamma)}$.

If $f(x, t)$ is a function of two variables, we denote by $f(t)$ the function $t \to f(\cdot, t)$, therefore $\|f(t)\|_{L^2}$ means $\{\|f(\cdot, t)\|_{L^2} : t \in [0, T]\}$.

Throughout, we denote $Q_T = U \times (0, T)$. Also, we use the following notation for partial derivatives: $\partial p/\partial t = p_t$ and $\partial/\partial x_i = \partial_i$.

### 3.1. Estimates for pressure

We begin our subsection by introducing the following result on the $L^\infty$-estimates of for the $\bar{p}$ of the IBVP (3.7). Throughout the paper, we denote

$$\mu_1 = (2 - a)(n + 2)/n.$$  \hfill (3.8)

**Proposition 3.1.** Assume $(SDC)$. Then, there is a constant $C > 0$ such that any $T > 0$, the following inequality holds

$$\sup_{[0, T]} \|\bar{p}\|_{L^\infty} \leq C\left\{ \|\bar{p}_0\|_{L^\infty} + (1 + T)^{\frac{2}{n} - a} (\sup_{[0, T]} \|\psi\|_{L^\infty} + 1)^{\frac{1}{n}} \right\}.$$  \hfill (3.9)

**Proof.** We follow the celebrated De Giorgi’s technique. Although, the method is standard (see, for example, [20]), calculations are tedious and new. Thus, for completeness, we provide its details here.

For any $k \geq 0$, let us denote

$$\bar{p}^{(k)}(x, t) = \sup_{t \in [0, T]} \int_U |\bar{p}^{(k)}(x, t)|^2 \, dx + \int_0^T \int_U |\nabla \bar{p}^{(k)}(x, t)|^{2 - a} \, dx \, dt.$$  \hfill (3.10)

Assume $k \geq \|\bar{p}_0\|_{L^\infty}$, then $\bar{p}^{(k)}(x) = 0$ a.e.. Multiplying the first equation of (3.7) by $\bar{p}^{(k)}$ and integrating the resultant over the domain $U$, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_U |\bar{p}^{(k)}|^2 \, dx + \int_U K(|\nabla \bar{p}^{(k)}|)|\nabla \bar{p}^{(k)}|^2 \, dx = \int_\Gamma \psi \bar{p}^{(k)} \sigma + \frac{1}{|\Gamma|} \int_\Gamma \psi(x, t) \, d\sigma \int_U \bar{p}^{(k)} \, dx.$$  \hfill (3.11)

We bound $|\psi(x, t)|$ and $|\int_\Gamma \psi(x, t) \, d\sigma|$ by $C \|\psi(t)\|_{L^\infty}$, and apply the trace theorem to the boundary integral, and then Hölder’s inequality to have

$$\frac{1}{2} \frac{d}{dt} \int_U |\bar{p}^{(k)}|^2 \, dx + \int_U K(|\nabla \bar{p}^{(k)}|)|\nabla \bar{p}^{(k)}|^2 \, dx \leq C \|\psi(t)\|_{L^\infty} \int_U \left( |\bar{p}^{(k)}| + |\nabla \bar{p}^{(k)}| \right) \, dx$$

$$\leq C \|\psi(t)\|_{L^\infty} \left\{ \left( \int_U |\bar{p}^{(k)}|^2 \, dx \right)^{1/2} |S_k(t)|^{1/2} + \left( \int_{S_k(t)} |\nabla \bar{p}^{(k)}|^{2 - a} \, dx \right)^{\frac{1}{2 - a}} |S_k(t)|^{\frac{1}{2 - a}} \right\}.$$  \hfill (3.12)
Using (2.15) to compare $|\nabla p_k|^2$ with $K(|\nabla p_k|) |\nabla p_k|^2 + 1$, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_U |p_k|^2 dx + \int_U K(|\nabla p_k|) |\nabla p_k|^2 dx \leq C \|\psi(t)\|_{L^\infty} \left\{ \left( \int_U |p_k|^2 dx \right)^{1/2} |S_k(t)|^{1/2} \right. + \\
+ \left. \left( \int_{S_k(t)} K(|\nabla p_k|) |\nabla p_k|^2 dx \right)^{1/2} |S_k(t)|^{1-\alpha} + |S_k(t)| \right\}. $$

Let $\varepsilon > 0$. By Young’s inequality, we have

$$\frac{1}{2} \frac{d}{dt} \int_U |p_k|^2 dx + \frac{1}{2} \int_U K(|\nabla p_k|) |\nabla p_k|^2 dx \leq \varepsilon \int_U |p_k|^2 dx + C \left\{ \varepsilon^{-1} \|\psi(t)\|_{L^\infty}^2 + \|\psi(t)\|_{L^\infty}^{2-\alpha} + \|\psi(t)\|_{L^\infty} \right\}|S_k(t)|.$$

Note that $p_k(0) = 0$. For each $t \in [0, T)$, integrating the previous estimate on $(0, t)$, and taking the supremum in $t$ yield

$$\sup_{[0, T]} \int_U |p_k|^2 dx + \int_0^T \int_U K(|\nabla p_k|) |\nabla p_k|^2 dx dt \leq 4\varepsilon \int_0^T \int_U |p_k|^2 dx + C \int_0^T \left( \varepsilon^{-1} \|\psi(t)\|_{L^\infty}^2 + \|\psi(t)\|_{L^\infty}^{2-\alpha} + \|\psi(t)\|_{L^\infty} \right)|S_k(t)| dt \leq 4\varepsilon T \sup_{[0, T]} \int_U |p_k|^2 dx + C \int_0^T \left( \varepsilon^{-1} \|\psi(t)\|_{L^\infty}^2 + \|\psi(t)\|_{L^\infty}^{2-\alpha} + \|\psi(t)\|_{L^\infty} \right)|S_k(t)| dt.$$

By taking $\varepsilon = 1/(8T)$, we obtain

$$\sup_{[0, T]} \int_U |p_k|^2 dx + \int_0^T \int_U K(|\nabla p_k|) |\nabla p_k|^2 dx dt \leq C \left[ T \|\psi(t)\|_{L^\infty}^2 + \|\psi(t)\|_{L^\infty}^{2-\alpha} + 1 \right]|S_k(t)| dt \leq C \left[ T \|\psi(t)\|_{L^\infty}^2 + \|\psi(t)\|_{L^\infty}^{2-\alpha} + 1 \right]|S_k(t)| dt.$$

Again, using (2.15) to compare $K(|\nabla p_k|) |\nabla p_k|^2$ with $|\nabla p_k|^{2-\alpha}$ and using Young’s inequality, one gets

$$F_k \overset{\text{def}}{=} \sup_{[0, T]} \int_U |p_k|^2 dx + \int_0^T \int_U |\nabla p_k|^{2-\alpha} dx dt \leq C_0 \alpha T \sigma_k,$$

where $C_0$ is a constant depending on $a$, $n$ and $U$, and $\alpha T = T \|\psi(t)\|_{L^\infty}^2 + \|\psi(t)\|_{L^\infty}^{2-\alpha} + 1$.

We now iterate (3.10) to derive our desired estimate. Under the Strict Degree Condition (SDC), $\mu_1 > 2$, hence, the parabolic Sobolev embedding in Lemma 2.3 with $r = 2-\alpha$ implies

$$\|\tilde{p}(t)\|_{L^{\mu_1}(Q_T)} \leq C (1 + T)^{1/\mu_1} (F_k^{1/2} + F_k^{1/(2-\alpha)}).$$

Next, let $M_0 \geq \|\tilde{p}(\cdot, 0)\|_{L^\infty}$ be a fixed number which will be determined. For each $i \in \mathbb{N} \cup \{0\}$, let $k_i = M_0(2^{-i})$. Then $k_i$ is increasing in $i$, and therefore $S_{k_i}, \sigma_{k_i}$ are decreasing. Also, note that $k_i \geq M_0 \geq \|\tilde{p}\|_{L^\infty}$ for all $i \geq 0$. We now define $Q_k = \{(x, t) \in U \times (0, T) : \tilde{p}(x, t) > k\}$.

From (3.11) and the fact that

$$\|\tilde{p}(k_i)\|_{L^{\mu_1}(Q_T)} \geq \|\tilde{p}(k_i)\|_{L^{\mu_1}(Q_{k_{i+1}})} \geq \|k_{i+1} - k_i\|_{L^{\mu_1}(Q_{k_{i+1}})} \geq (k_{i+1} - k_i)\sigma_{k_{i+1}}^{1/\mu_1},$$

(3.12)
it follows
\[
\sigma^{1/\mu_1}_{k_i + 1} \leq C(1 + T)^{1/\mu_1} \frac{\rho_i}{M_0} \left[ F_{k_i}^{1/2} + F_{k_i}^{1/(2-a)} \right].
\]
This together with (3.10) yield
\[
\sigma^{1/\mu_1}_{k_i + 1} \leq C(1 + T)^{1/\mu_1} \frac{\rho_i}{M_0} \left[ (\alpha_T \sigma_{k_i})^{1/2} + (\alpha_T \sigma_{k_i})^{1/(2-a)} \right].
\]
Equivalently,
\[
\sigma_{k_i + 1} \leq C(1 + T)\alpha_T^{\mu_1/(2-a)} M_0^{\mu_1} \frac{\rho_i}{M_0} 2^{\mu_1} (\sigma_{k_i}^{1/2} + \sigma_{k_i}^{1/(2-a)}).
\]
Let us denote \( Y_i = \sigma_{k_i}, D = \frac{C(1+T)\alpha_T^{\mu_1/(2-a)} M_0^{\mu_1}}{\rho_i}, \theta_1 = \mu_1/(2-a) - 1 > 0 \) and \( \theta_2 = \mu_1/2 - 1 > 0 \). We now obtain
\[
Y_{i+1} \leq D 2^{\mu_1} [Y_i^{1+\theta_1} + Y_i^{1+\theta_2}].
\]
Note that \( k_0 = M_0 \geq \|\bar{p}_0\|_L^\infty \) and
\[
Y_0 = \sigma_{M_0} \leq |Q_T| = |U| T \leq |U|(T + 1).
\]
To apply (A.3) with \( m = 2 \) in Lemma A.2 we choose \( M_0 \) sufficiently large such that
\[
|U|(T + 1) \leq C \min \left\{ (1 + T)\alpha_T^{\mu_1/(2-a)} M_0^{-\mu_1} - 1/\theta_2, ((1 + T)\alpha_T^{\mu_1/(2-a)} M_0^{-\mu_1})^{-1/\theta_1} \right\}.
\]
It suffices to have
\[
M_0 \geq C(1 + T)\alpha_T^{\mu_1/(2-a)} = C(1 + T)^{1/(2-a)} \quad \text{and} \quad M_0 \geq C(1 + T)\alpha_T^{\mu_1/(2-a)} = C(1 + T)^{1/(2-a)} \alpha_T^{1/(2-a)}.
\]
These and the initial requirement \( M_0 \geq \|\bar{p}_0\|_L^\infty \) are satisfied if we choose
\[
M_0 \geq C \left\{ \|\bar{p}_0\|_L^\infty + (1 + T)\alpha_T^{1/(2-a)} \right\}.
\]
Note that \( (1 + T)\alpha_T \leq C(T + 1)^2 (\sup_{[0,T]} \|\psi\|_L^\infty + 1)^{2/(2-a)} \). Therefore we finally select
\[
M_0 = C_1 \left\{ \|\bar{p}_0\|_L^\infty + (1 + T)^{2/(2-a)} (\sup_{[0,T]} \|\psi\|_L^\infty + 1)^{1/(2-a)} \right\}.
\]
Now, applying (A.3) in Lemma A.2 we have \( \sigma_{2M_0} = \lim Y_i = 0 \). Thus, \( \bar{p}(x, t) \leq 2M_0 \) in \( Q_T \). Note that we used the continuity of \( \bar{p}(x, t) \) on \( Q_T \). By using the same argument with \( p \) replaced by \( -p \) and \( \psi \) replaced by \( -\psi \), one can show that \( -\bar{p} \leq 2M_0 \) in \( Q_T \). Therefore, (3.9) follows and the proof is complete.

We emphasize that the \( L^\infty \)-estimate of \( \bar{p} \) in Proposition 3.1 depends on \( L^\infty \)-norm of the initial data and on \( T \). In the next proposition, we give a \( L^\infty \)-estimate which removes the former dependence. The result will be used flexibly to derive the time-uniform \( L^\infty \)-estimate of \( \bar{p} \).
Proposition 3.2. Assume (SDE). Then, there is a constant $C > 0$ such that for any $T_0 \geq 0$, $T > 0$, $\delta \in (0, 1]$ and $\theta \in (0, 1)$, the following inequality holds

$$
\sup_{[T_0, T_0 + T]} \|\bar{p}\|_{L^\infty} \leq C \left\{ \sqrt{E} + (T + 1)^{\frac{4 - \alpha}{2(\alpha - 1)}} \left( 1 + \frac{1}{\delta^\theta \theta T} \right)^{\frac{4}{4 - (n + 2)\alpha}} \cdot \left( \|\bar{p}\|_{L^2(U \times (T_0, T_0 + T))} + \|\bar{p}\|_{L^2(U \times (T_0, T_0 + T))}^\frac{4}{4 - (n + 2)\alpha} \right) \right\},
$$

where

$$
E = E_{T_0, T} \overset{\text{def}}{=} \delta^{2 - \alpha} + T \delta^{\alpha - 1} \sup_{[T_0, T_0 + T]} \|\psi\|_{L^\infty}^{\frac{4}{4 - (n + 2)\alpha}} \sup_{[T_0, T_0 + T]} \|\psi\|_{L^\infty}^{\frac{2 - \alpha}{4 - (n + 2)\alpha}}.
$$

Proof. Without loss of generality, we assume $T_0 = 0$. Again, we use De Giorgi’s iteration technique. But, unlike in the previous one, we will iterate using the $L^2$-norm of $\bar{p}$. The proof is therefore significantly different from that of Proposition 3.1 and we give it here in full details.

For each $k \geq 0$, let $\bar{p}^{(k)}$ be as in the proof of the previous theorem. Also, let $\zeta = \zeta(t)$ be a non-negative cut-off function on $[0, T]$ with $\zeta(0) = 0$. Multiplying (3.7) by $\bar{p}^{(k)} \zeta$ and integrating over $U$, we have

$$
\frac{1}{2} \int_U \frac{d}{dt} |\bar{p}^{(k)}|^2 \zeta dx + \int_U K(|\nabla \bar{p}^{(k)}|) |\nabla \bar{p}^{(k)}|^2 \zeta dx = \int_U \psi \bar{p}^{(k)} \zeta d\bar{\sigma} + \frac{1}{|U|} \int_U \psi(x, t) d\bar{\sigma} \int_U \bar{p}^{(k)} \zeta dx.
$$

(3.14)

Denote by $\chi_k(x, t)$ the characteristic function of the set $\{(x, t) \in U \times (0, T) : \bar{p}^{(k)}(x, t) > 0\}$. Then, using (2.14) in Lemma 2.1 to estimate $K(|\nabla \bar{p}^{(k)}|) |\nabla \bar{p}^{(k)}|^2 \geq C \delta^\alpha |\nabla \bar{p}^{(k)}|^{2 - \alpha} - C \delta^2$, and estimating the right-hand side of (3.14) the same way as in the proof of Proposition 3.1, we have

$$
\frac{1}{2} \frac{d}{dt} \int_U |\bar{p}^{(k)}|^2 \zeta dx - \frac{1}{2} \int_U |\bar{p}^{(k)}|^2 \zeta dx + C_1 \delta^\alpha \int_U |\nabla \bar{p}^{(k)}|^{2 - \alpha} \zeta dx

\leq C \|\psi(t)\|_{L^\infty} \int_U (|\bar{p}^{(k)}| \zeta + |\nabla \bar{p}^{(k)}| \zeta) dx.
$$

Let $\varepsilon > 0$. Then it follows from Young’s inequality that

$$
\frac{1}{2} \frac{d}{dt} \int_U |\bar{p}^{(k)}|^2 \zeta dx + C_1 \delta^\alpha \int_U |\nabla \bar{p}^{(k)}|^{2 - \alpha} \zeta dx

\leq \frac{1}{2} \int_U |\bar{p}^{(k)}|^2 \zeta dx + C_2 \delta^2 \int_U \chi_k(t) \zeta dx + \varepsilon \int_U |\bar{p}^{(k)}|^2 \zeta dx + C_3 \varepsilon^{-1} \|\psi\|_{L^\infty} \int_U \chi_k \zeta dx

+ \frac{C_1 \delta^\alpha}{2} \int_U |\nabla \bar{p}^{(k)}|^{2 - \alpha} \zeta dx + C \delta^{\frac{\alpha}{1 - \alpha}} \|\psi(t)\|_{L^\infty}^{\frac{2 - \alpha}{4 - (n + 2)\alpha}} \int_U \chi_k \zeta dx.
$$

Now, integrating this inequality on $(0, t)$ and taking supremum of the resultant on $[0, T]$, we obtain

$$
\frac{1}{2} \sup_{[0, T]} \int_U |\bar{p}^{(k)}|^2 \zeta dx + C_1 \delta^\alpha \int_0^T \int_U |\nabla \bar{p}^{(k)}|^{2 - \alpha} \zeta dx dt

\leq \int_0^T \int_U |\bar{p}^{(k)}|^2 \zeta dx dt + 2 \varepsilon T \sup_{[0, T]} \int_U |\bar{p}^{(k)}|^2 \zeta dx

+ C \left[ \delta^2 + \varepsilon^{-1} \|\psi\|_{L^\infty(0, T; L^\infty)}^2 + \delta^{\frac{\alpha}{1 - \alpha}} \|\psi\|_{L^\infty(0, T; L^\infty)}^{\frac{2 - \alpha}{4 - (n + 2)\alpha}} \right] \int_0^T \int_U \chi_k \zeta dx dt.
$$
By choosing \( \varepsilon = 1/(8T) \), we obtain

\[
\sup_{[0,T]} \left\| \int_U |p^{(k)}|^{2\alpha} \zeta \, dx + \int_0^T \int_U |\nabla p^{(k)}|^{2-a} \zeta \, dx \right\| \leq C \delta^{-a} \sup_{[0,T]} \left\| \int_U |p^{(k)}|^{2\alpha} \zeta \, dx + \int_0^T \int_U |\nabla p^{(k)}|^{2-a} \zeta \, dx \right\|
\]

(3.15)

where \( \mathcal{E} = \delta^{2-a} + T \delta^{-a} \sup_{[0,T]} \| \phi \|^2_{L_\infty} + \delta \frac{a(2-a)}{1-a} \sup_{[0,T]} \| \psi \|^2_{L_\infty} \). We will iterate this relation with \( k = k_{i+1} \) and \( \zeta = \zeta_i \). The choice of \( k_i \) and \( \zeta_i(t) \) is as follows. Let \( t_i = \theta T (1 - 2^{-i}) \) for all \( i \geq 0 \), and let \( \zeta_i \) be a piecewise linear function with \( \zeta_i(t) = 0 \) for \( t \leq t_i \), \( \zeta_i(t) = 1 \) for \( t \geq t_{i+1} \) and

\[
|\zeta_i(t)| \leq \frac{1}{t_{i+1} - t_i} = \frac{2^{i+1}}{\theta T} \quad \forall \ t \in [0,T].
\]

Next, let \( M_0 \) be a fixed positive number which will be determined, and let \( k_i = M_0 (1 - 2^{-i}) \), for \( i \geq 0 \). Note that, because of our choices, \( k_0 = 0 \) and \( k_{i+1} - k_i = 2^{-i-1} M_0 \) and \( t_0 = 0 < t_1 < \ldots < \theta T \). We also denote \( A_{i,j} = \{(x,t) : u(x,t) > k_i, \ t \in (t_j, T)\} \) and \( A_i = A_{i,i} \). Then, by applying (3.15) with \( k = k_{i+1} \) and \( \zeta = \zeta_i \), we obtain

\[
\sup_{[0,T]} \left\| \int_U |p^{(k_{i+1})}|^{2\alpha} \zeta_i \, dx + \int_0^T \int_U |\nabla p^{(k_{i+1})}|^{2-a} \zeta_i \, dx \right\| \leq \delta^{-a} \sup_{[0,T]} \left\| \int_U |p^{(k_{i+1})}|^{2\alpha} \zeta_i \, dx + \int_0^T \int_U |\nabla p^{(k_{i+1})}|^{2-a} \zeta_i \, dx \right\|
\]

(3.16)

Now, we define \( F_i = \sup_{[t_i+1,T]} \left\| \int_U |p^{(k_{i+1})}|^{2\alpha} \zeta_i \, dx + \int_0^T \int_U |\nabla p^{(k_{i+1})}|^{2-a} \zeta_i \, dx \right\| \). Then it follows from (3.16) that

\[
F_i \leq \delta^{-a} \int_{t_i}^{t_{i+1}} \int_U |p^{(k_{i+1})}|^{2\alpha} (\zeta_i) \, dx + \mathcal{E} \left( \int_0^T \int_U |\nabla p^{(k_{i+1})}|^{2-a} \zeta_i \, dx \right).
\]

Therefore,

\[
F_i \leq C 2^i (\theta T)^{-1} \delta^{-a} \left\| p^{(k_{i+1})} \right\|_{L^2(A_{i+1,i})}^2 + C \mathcal{E} |A_{i+1,i}|.
\]

(3.17)

Since \( \left\| p^{(k_i)} \right\|_{L^2(A_i)} \geq \left\| p^{(k_{i+1})} \right\|_{L^2(A_{i+1,i})} \geq (k_{i+1} - k_i) |A_{i+1,i}|^{1/2} \), we see that

\[
|A_{i+1,i}| \leq (k_{i+1} - k_i)^{-2} \left\| p^{(k_i)} \right\|_{L^2(A_i)} = 4^i M_0^{-2} \left\| p^{(k_i)} \right\|_{L^2(A_i)}^2.
\]

(3.18)

From (3.17) and (3.18), we get

\[
F_i \leq C \delta^{-a} (\theta T)^{-1} 2^i \left\| p^{(k_{i+1})} \right\|_{L^2(A_{i+1,i})}^2 + C \mathcal{E} |A_{i+1,i}| \leq C C_\delta 4^i \left\| p^{(k_i)} \right\|_{L^2(A_i)}^2,
\]

(3.19)

where \( C_\delta = \delta^{-a} (\theta T)^{-1} + \mathcal{E} M_0^{-2} \). Under the Strict Degree Condition, \( 2 < \mu_1 < (2-a)^\alpha \), then applying Lemma 2.3 with \( r = 2-a \) yields

\[
\left\| p^{(k_{i+1})} \right\|_{L^\mu_1(A_{i+1,i+1})} = \left( \int_{t_{i+1}}^{T} \int_U |p^{(k_{i+1})}|^{\mu_1} \, dx \right)^{1/\mu_1} \leq C (1 + (T - t_{i+1})^{1/\mu_1} \left[ \sup_{[t_{i+1},T]} \left\| p^{(k_{i+1})} \right\|_{L^2(U)} + \left( \int_{t_{i+1}}^{T} \int_U |\nabla p^{(k_{i+1})}|^{2-a} \, dx \right)^{1/(2-a)} \right] \leq C (1 + T)^{1/\mu_1} \left[ \sup_{[t_{i+1},T]} \left\| p^{(k_{i+1})} \right\|_{L^2(U)} + \left( \int_{t_{i+1}}^{T} \int_U |\nabla p^{(k_{i+1})}|^{2-a} \zeta_i \, dx \right)^{1/(2-a)} \right].
\]
Since \( t_i \leq t_{i+1} \), it follows that

\[
\|\bar{p}^{(k_{i+1})}\|_{L^1(t_{i+1}, t_{i+1} + \delta)} \leq C(T + 1)^{1/\mu_1} (F_{i+1}^{1/2} + F_{i+1}^{1/(2-a)}) .
\] (3.20)

Then, it follows from Hölder’s inequality and (3.20) that

\[
\|\bar{p}^{(k_{i+1})}\|_{L^2(A_{i+1}, t_{i+1} + \delta)} \leq \|\bar{p}^{(k_{i+1})}\|_{L^1(A_{i+1}, t_{i+1} + \delta)} \|A_{i+1, i+1}\|^{1/2 - 1/\mu_1} \\
\leq \|\bar{p}^{(k_{i+1})}\|_{L^1(A_{i+1}, t_{i+1} + \delta)} \|A_{i+1, i+1}\|^{1/2 - 1/\mu_1} \leq C(T + 1)^{1/\mu_1} (F_{i+1}^{1/2} + F_{i+1}^{1/(2-a)}) \|A_{i+1, i+1}\|^{1/2 - 1/\mu_1} .
\]

This and (3.18), (3.19) give

\[
\|\bar{p}^{(k_{i+1})}\|_{L^2(A_{i+1}, t_{i+1} + \delta)} \leq 4C(T + 1)^{1/\mu_1} \left( C_{\delta}^{1/2} \|\bar{p}^{(k_i)}\|_{L^2(A_{i})} + C_{\delta}^{1/(2-a)} \|\bar{p}^{(k_i)}\|_{L^2(A_{i})}^{2/(2-a)} \right) M_0^{-2/\mu_1} \|\bar{p}^{(k_i)}\|_{L^2(A_{i})}^{1-2/\mu_1} \\
\leq 4C(T + 1)^{1/\mu_1} M_0^{-2/\mu_1} \left( C_{\delta}^{1/2} \|\bar{p}^{(k_i)}\|_{L^2(A_{i})}^{2-2/\mu_1} + C_{\delta}^{1/(2-a)} \|\bar{p}^{(k_i)}\|_{L^2(A_{i})}^{1-2/\mu_1 + 2/(2-a)} \right) .
\]

Now, let \( Y_i = \|\bar{p}^{(k_i)}\|_{L^2(A_{i})} \), we get

\[
Y_{i+1} \leq D 4^i \left( Y_{i-2/\mu_1} + Y_{i-2/\mu_1 + 2/(2-a)} \right) = D 4^i \left( Y_{i+\mu_2} + Y_{i+\mu_3} \right),
\] (3.21)

where \( D = C(T + 1)^{1/\mu_1} (C_{\delta}^{1/2} + C_{\delta}^{1/(2-a)}) M_0^{-\mu_2} \), and

\[
\mu_2 = 1 - \frac{2}{\mu_1} = \frac{4 - a(n + 2)}{(2 - a)(n + 2)}, \quad \mu_3 = 2 \left( \frac{1}{2 - a} - \frac{1}{\mu_1} \right) = \frac{4}{(2 - a)(n + 2)}. \] (3.22)

To be able to apply (A.5) with \( m = 2 \) in Lemma A.2, we will choose \( M_0 \) sufficiently large such that

\[
Y_0 \leq C \min\{D^{-1/\mu_2}, D^{-1/\mu_3}\} .
\] (3.23)

Since \( Y_0 \leq \|\bar{p}\|_{L^2(U \times (0,T))} \), it suffices to choose \( M_0 \) such that

\[
M_0 \geq C(T+1)^{1/\mu_2} (C_{\delta}^{1/2} + C_{\delta}^{1/(2-a)}) \|\bar{p}\|_{L^2(U \times (0,T))} , \quad M_0 \geq C(T+1)^{1/\mu_2} (C_{\delta}^{1/2} + C_{\delta}^{1/(2-a)}) \|\bar{p}\|_{L^2(U \times (0,T))} .
\]

Observe that, if \( M_0 \geq \sqrt{C} \) then \( C_{\delta} \leq 1 + \frac{1}{\delta^a \theta T} \in (1, \infty). \) Thus, (3.23) holds for

\[
M_0 = C \left\{ \sqrt{C} + (T + 1)^{1/\mu_2} (1 + \frac{1}{\delta^a \theta T}) \|\bar{p}\|_{L^2(U \times (0,T))} + \|\bar{p}\|_{L^2(U \times (0,T))} \right\} .
\]

With this choice of \( M_0 \), applying (A.5) with \( m = 2 \) in Lemma A.2 to (3.21), we obtain \( \lim_{i \to \infty} Y_i = 0 \).

Consequently, \( \int_{T}^{T} \int_{U} |\bar{p}^{(M_0)}|^2 \, dx \, dt = 0 \). This implies \( \bar{p}(x, t) \leq M_0 \) in \( U \times (\theta T, T) \). To prove the lower bound \( \bar{p} \geq -M_0 \), we replace \( p \) by \(-p\) and \( \psi \) by \(-\psi\) and use the same argument. We conclude

\[
|\bar{p}(x, t)| \leq M_0 \quad \text{in } U \times (\theta T, T).
\]

The proof of (3.13) is complete.

Our next goal is to combine Propositions 3.1 and 3.2 with \( L^2 \)-estimates of \( \bar{p} \) in [18] to derive the uniform (in time) \( L^\infty \)-estimate of \( \bar{p} \). For our purpose, we need to introduce some notation. Firstly, let us define two functions \( f(t) \) and \( \tilde{f}(t) \) for \( t \geq 0 \) by

\[
f(t) = \|\psi(t)\|_{L^\infty} + \|\psi(t)\|_{L^\infty}^{2-a} \quad \text{and} \quad \tilde{f}(t) = \|\psi(t)\|_{L^\infty}^2 + \|\psi(t)\|_{L^\infty}^{2-a} .
\] (3.24)
Assume throughout that \( \psi(\cdot, t) \) and \( \psi_t(\cdot, t) \) belong \( C([0, \infty), L^\infty(\Gamma)) \), hence \( f(t) \) and \( \tilde{f}(t) \) belong to \( C([0, \infty)) \). Whenever \( f'(t) \) is mentioned, we implicitly assume that \( f \in C^1([0, \infty)) \). Let \( M_f(t) \) be a continuous, increasing majorant of \( f(t) \) on \([0, \infty)\). Let

\[
A = \limsup_{t \to \infty} f(t) \quad \text{and} \quad \beta = \limsup_{t \to \infty} [f'(t)]^-.
\]

(3.25)

Again, whenever \( \beta \) is used in subsequently statements, it is understood that \( f(t) \in C^1([0, \infty)) \).

Also, for a function \( u(x, t) \) defined on \( U \times [0, \infty) \) we denote

\[
J_H[u](t) = \int_U H(|\nabla u(x, t)|)dx,
\]

where \( H \) is the function defined by (2.16). Note from (2.17) that

\[
c_1 \int_U |\nabla u(x, t)|^{2-a} dx - c_3 \leq J_H[u](t) \leq c_2 \int_U |\nabla u(x, t)|^{2-a} dx, \quad c_3 > 0.
\]

(3.27)

We now recall relevant estimates from [18].

**Theorem 3.3** (cf. [18], Theorem 4.3). The following estimates hold

(i) For \( t \geq 0 \),

\[
\|\tilde{p}(t)\|_{L^2}^2 + \int_0^t J_H[p](\tau)d\tau \leq \|\tilde{p}_0\|_{L^2}^2 + C \int_0^t f(\tau)d\tau.
\]

(3.28)

(ii) Assume that the Degree Condition holds, then

\[
\|\tilde{p}(t)\|_{L^2}^2 \leq \|\tilde{p}_0\|_{L^2}^2 + C(1 + M_f(t)\frac{2}{2-a}) \quad \text{for all } t \geq 0.
\]

Moreover, if \( A < \infty \) then

\[
\limsup_{t \to \infty} \|\tilde{p}(t)\|_{L^2}^2 \leq C(A + A^\frac{2}{2-a}),
\]

and if \( \beta < \infty \) then there is \( T > 0 \) such that

\[
\|\tilde{p}(t)\|_{L^2}^2 \leq C\left(1 + \beta^\frac{1}{2-a} + f(t)^\frac{2}{2-a}\right) \quad \text{for all } t > T.
\]

(3.31)

We finalize our main estimates for \( L^\infty \)-norm in the following theorem. The exponent \( \mu_2 \) appearing below is already defined in (3.22), and we also denote

\[
\mu_4 = \frac{4}{(2-a)(4-a(n+2))}.
\]

(3.32)

**Theorem 3.4.** Assume (SDC). Then there is a constant \( C > 0 \) such that the following statements hold true.

(i) For \( t > 0 \),

\[
\|\tilde{p}(t)\|_{L^\infty} \leq C\left(1 + t^{-\frac{1}{2(2-a)}}\right)\left\{1 + \|\tilde{p}_0\|_{L^2}^{\mu_4(2-a)} + M_f(t)^{\mu_4}\right\},
\]

(3.33)

(ii) If \( A < \infty \) then

\[
\limsup_{t \to \infty} \|\tilde{p}(t)\|_{L^\infty} \leq C(1 + A^{\mu_4}).
\]

(3.35)

(iii) If \( \beta < \infty \) then there is \( T > 0 \) such that

\[
\|\tilde{p}(t)\|_{L^\infty} \leq C\left\{1 + \beta^{\frac{\mu_4(2-a)}{4(2-a)}} + \sup_{[t-T, t]} \|\psi\|_{L^\infty}^{\frac{\mu_4(2-a)}{4(2-a)}}\right\} \quad \text{for all } t > T.
\]

(3.36)
Proof. (i) We prove (3.33) first. For \( t \in (0, 1) \), applying (3.13) to \( T_0 = 0 \), \( T = t \), \( \delta = 1 \) and \( \theta = 1/2 \) we obtain
\[
\| \bar{p}(t) \|_{L_\infty} \leq C \left\{ (t + 1)^{1/2} \left( 1 + \sup_{[0,t]} \| \psi \|_{L_\infty} \right)^{\frac{2-a}{2(1-a)}} \right\} + (t + 1)^{1/2} \left( 1 + 2t^{-1} \right)^{\frac{1}{2-a}} \left( \| \bar{p} \|_{L^2(U \times (0,t))} + \| \bar{p} \|_{L^2(U \times (0,t))}^4 \right) \}
\leq C \left\{ (1 + t^{-1})^{\frac{1}{2(2-a)}} \left( 1 + \sup_{[0,t]} \| \psi \|_{L^\infty}^{\frac{2-a}{2(1-a)}} + \sup_{[0,t]} \| \bar{p} \|_{L^2}^{4/(4-a(\alpha+2))} \right) \right\}.
\]
By estimate (3.29),
\[
\| \bar{p}(t) \|_{L_\infty} \leq C t^{\frac{1}{\mu_2(2-a)}} \left\{ 1 + \| M(t) \|_{L_\infty}^{\frac{1}{2}} + \left( 1 + \| \bar{p}_0 \|_{L_2} + M(t)^{\frac{1}{2}} \right)^{4/(4-a(\alpha+2))} \right\} \leq C t^{\frac{1}{\mu_2(2-a)}} \left\{ 1 + \| \bar{p}_0 \|_{L_2}^{4/(4-a(\alpha+2))} + M(t)^{4/(4-a(\alpha+2))} \right\}.
\]
Therefore, we obtain (3.33) for \( t \in (0, 1) \). For \( t \in [1, \infty) \), applying (3.13) to \( T_0 = t - 1 \), \( T = 1 \), \( \delta = 1 \), and \( \theta = 1/2 \), we have
\[
\| \bar{p}(t) \|_{L_\infty} \leq C \left\{ 1 + \sup_{[t-1,t]} \| \psi \|_{L^\infty}^{\frac{2-a}{2(1-a)}} + \sup_{[t-1,t]} \| \bar{p} \|_{L^2} + \sup_{[t-1,t]} (\bar{p}(t))^{\mu_4(2-a)} \right\}.
\]
Note that \( \mu_4(2-a) > 1 \), then Hölder’s inequality gives
\[
\| \bar{p}(t) \|_{L_\infty} \leq C \left\{ 1 + \sup_{[t-1,t]} \| \psi \|_{L^\infty}^{\frac{2-a}{2(1-a)}} + \sup_{[t-1,t]} (\bar{p}(t))^{\mu_4(2-a)} \right\}.
\]
(3.37)
Then, again, combining (3.37) with (3.29) and using Hölder’s inequality yield (3.33) for \( t \in [1, \infty) \).

To prove (3.34), we note that in the case \( t \geq 1 \), (3.33) implies (3.34). Now, for \( t \in (0, 1) \), we have from inequality (3.9) that
\[
\| \bar{p}(t) \|_{L_\infty} \leq C \left( 1 + \| \bar{p}_0 \|_{L_\infty} + \sup_{[0,t]} \| \psi \|_{L^\infty}^{\frac{1}{2(1-a)}} \right).
\]
(3.38)
Since \( \| \psi(t) \|_{L_\infty} \leq M_f(t)^{\frac{1}{2-a}} \leq M_f(t)^{\mu_4} \), then (3.34) follows (3.38).

(ii) Since \( A < \infty \), we see that
\[
\limsup_{t \to \infty} \sup_{[t-1,t]} (\| \psi \|_{L_\infty})^{\frac{2-a}{2(1-a)}} \leq \limsup_{t \to \infty} f(t)^{\frac{1}{2}} = A^{\frac{1}{2}},
\]
(3.39)
It follows (3.37), (3.40) and (3.39) that
\[
\limsup_{t \to \infty} \| \bar{p}(t) \|_{L_\infty} \leq C \left\{ 1 + A^{\frac{1}{2}} + (A^{\frac{1}{2}} + A^{\frac{1}{2(1-a)}})^{\mu_4(2-a)} \right\},
\]
which implies (3.35).

(iii) Because \( \beta < \infty \), using (3.31) in (3.37) there is \( T > 0 \) such that for all \( t > T \),
\[
\| \bar{p}(t) \|_{L_\infty} \leq C \left\{ 1 + \sup_{[t-1,t]} (\| \psi \|_{L^\infty})^{\frac{2-a}{2(1-a)}} + \left[ 1 + \beta^{\frac{1}{2(1-a)}} + \sup_{[t-1,t]} (\| \psi \|_{L^\infty})^{\frac{1}{2(1-a)}} \right]^{\mu_4(2-a)} \right\},
\]
and (3.36) follows. □

Concerning \( \bar{p}(t) \) being small, as \( t \to \infty \), rather than just being bounded, we have the following result.
Theorem 3.5. Assume (SDFC). For any \( \varepsilon > 0 \), there is \( \delta_0 > 0 \) such that

\[
\text{if } \lim_{t \to \infty} \| \psi(t) \|_{L^\infty} < \delta_0 \quad \text{then} \quad \lim_{t \to \infty} \| \bar{p}(t) \|_{L^\infty} < \varepsilon.
\]

Consequently,

\[
\text{if } \lim_{t \to \infty} \| \psi(t) \|_{L^\infty} = 0 \quad \text{then} \quad \lim_{t \to \infty} \| \bar{p}(t) \|_{L^\infty} = 0.
\]

Proof. Applying (3.13) to \( T_0 = t - 1, T = 1 \) and \( \theta = 1/2 \), we have

\[
\| \bar{p}(t) \|_{L^\infty} \leq C \left\{ \sqrt{E(t)} + (1 + \delta^{-a}) \frac{1}{(2-a)\mu_2} \left( \| \bar{p} \|_{L^2(U \times (t-1,t))} + \| \bar{p} \|_{L^2(U \times (t-1,t))} \right) \right\},
\]

where \( E(t) = \delta^{2-a} + \delta^{-a} \sup_{[t-1,t]} \| \psi \|_{L^\infty}^2 - \delta^{-\frac{a}{1-a}} \sup_{[t-1,t]} \| \psi \|_{L^\infty}^2 \). Then, it follows that

\[
\| \bar{p}(t) \|_{L^\infty} \leq C \sqrt{E(t)} + C(1 + \delta^{-a}) \frac{1}{(2-a)\mu_2} \left\{ \sup_{[t-1,t]} \| \bar{p} \|_{L^2} + \sup_{[t-1,t]} \| \bar{p} \|_{L^2} \right\}.
\]

Let \( \delta_0 \in (0,1) \), then \( A \leq \delta_0^{\frac{2-a}{2}} \frac{\delta_0^{2-a}}{\delta_0^{1-a}} \leq 2\delta_0^2 \), and therefore, by (3.30),

\[
\limsup_{t \to \infty} \left( \sup_{[t-1,t]} \| \bar{p} \|_{L^2} \right) \leq C(A^{\frac{1}{2}} + A^{1/2}) \leq C\delta_0.
\]

Hence, we obtain

\[
\limsup_{t \to \infty} (1 + \delta^{-a}) \frac{1}{(2-a)\mu_2} \left\{ \sup_{[t-1,t]} \| \bar{p} \|_{L^2} + \sup_{[t-1,t]} \| \bar{p} \|_{L^2} \right\} \leq C(1 + \delta^{-a}) \frac{1}{(2-a)\mu_2} \delta_0.
\]

Also,

\[
\lim_{t \to \infty} E(t) \leq \delta^{2-a} + \delta^{-a} \delta_0^2 + \delta^{-\frac{a}{1-a}} \delta_0^2 \leq \delta^{2-a} + 2\delta^{-\frac{a}{1-a}} \delta_0^2.
\]

Therefore, for any \( \delta > 0 \), we have from (3.42) that

\[
\lim_{t \to \infty} \| \bar{p}(t) \|_{L^\infty} \leq C \left( \delta^{1-a} \frac{1}{2} + \delta^{-\frac{a}{1-a}} \frac{1}{2-1} \delta_0 \right).
\]

Now, choose \( \delta \) sufficiently small such that \( 2C \delta^{1-a/2} < \varepsilon \). Then we can choose \( \delta_0 > 0 \) even sufficiently smaller so that \( \delta^{-\frac{a}{1-a}} \frac{1}{2-a} \delta_0 \leq \delta^{1-a/2} \). From this, the desired estimates (3.40) follows (3.43). The statement (3.41) obviously is a consequence of (3.40). The proof is complete.

\[
\text{□}
\]

3.2. Estimates for pressure’s gradient. In this subsection, we establish the interior \( L^s \)-estimate of \( \nabla p \), for all \( s > 0 \). We follow the approach in (20). First is a basic estimate for \( \nabla p \) which prepares for our iteration later.

Lemma 3.6. For each \( s \geq 0 \), there is a constant \( C > 0 \) depending on \( s \) such that for any \( T > 0 \) and smooth cut-off function \( \zeta(x) \in C_c^\infty(U) \), the following estimate holds

\[
\sup_{[0,T]} \int_U |\nabla p(x,t)|^{2s+2} \zeta^2 dx + C \int_0^T \int_U K(|\nabla p|)|\nabla^2 p|^2 |\nabla p|^{2s} \zeta^2 dx dt \leq \int_U |\nabla p_0(x)|^{2s+2} \zeta^2 dx + C \int_0^T \int_U K(|\nabla p|)|\nabla p|^{2s+2} |\nabla \zeta|^2 dx dt.
\]
Proof. In this proof, we use Einstein’s summation convention, that is, when an index variable appears twice in a single term it implies summation of that term over all the values of the index. Multiplying the equation (3.1) by $-\nabla \cdot (|\nabla p|^{2s} \nabla p \zeta^2)$ and integrating the resultant over $U$, we obtain

$$
\frac{1}{2s + 2} \frac{d}{dt} \int_U |\nabla p|^{2s+2} \zeta^2 dx = - \int_U \partial_i (K(|\nabla p|) \partial_i p) \partial_j (|\nabla p|^{2s} \partial_j p \zeta^2) dx.
$$

This equality and the integration by parts yield

$$
\frac{1}{2s + 2} \frac{d}{dt} \int_U |\nabla p|^{2s+2} \zeta^2 dx = - \int_U \partial_j [K(|\nabla p|) \partial_i p] \partial_i \partial_j p |\nabla p|^{2s} \zeta^2 dx - \int_U \partial_j [K(|\nabla p|) \partial_i p] \partial_j p |\nabla p|^{2s} \partial_i [\zeta^2] dx
$$
$$
- \int_U \partial_j [K(|\nabla p|) \partial_i p] \partial_j p [2s |\nabla p|^{2s-2} \partial_i \partial_i m p \partial_i m p] \zeta^2 dx.
$$

We rewrite it in the following form

$$
\frac{1}{2s + 2} \frac{d}{dt} \int_U |\nabla p|^{2s+2} \zeta^2 dx = - \int_U \left[ \partial_k (K(|y|) y_i) \bigg|_{y=\nabla p} \partial_j \partial_i p \right] \partial_j \partial_i p |\nabla p|^{2s} \zeta^2 dx
$$
$$
- 2 \int_U \left[ \partial_k (K(|y|) y_i) \bigg|_{y=\nabla p} \partial_j \partial_i p \right] \partial_j p |\nabla p|^{2s} \zeta^2 \partial_i \zeta dx
$$
$$
- 2s \int_U \left[ \partial_k (K(|y|) y_i) \bigg|_{y=\nabla p} \partial_j \partial_i p \right] \partial_j p (|\nabla p|^{2s-2} \partial_i \partial_i m p \partial_i m p) \zeta^2 dx.
$$

We denote the three terms on the right-hand side by $I_1$, $I_2$ and $I_3$, and estimate each of them. By (2.11), one can easily prove for any $y, z \in \mathbb{R}^n$ that

$$
z^T \nabla (K(|y|) y) z \geq (1 - a) K(|y|) |z|^2 \quad \text{and} \quad |\nabla (K(|y|) y)| \leq (1 + a) K(|y|).
$$

It follows that

$$
I_1 = - \int_U \partial_k [K(|y|) y_i] \bigg|_{y=\nabla p} \partial_i (\partial_j p) \partial_i (\partial_j p) |\nabla p|^{2s} \zeta^2 dx
$$

$$
\leq -(1 - a) \sum_j \int_U K(|\nabla p|) |\nabla (\partial_j p)|^2 |\nabla p|^{2s} \zeta^2 dx
$$

$$
= -(1 - a) \int_U K(|\nabla p|) |\nabla^2 p|^2 |\nabla p|^{2s} \zeta^2 dx.
$$

Moreover, we have

$$
|I_2| \leq 2(1 + a) \int_U K(|\nabla p|) |\nabla^2 p| |\nabla p|^{2s+1} \zeta |\nabla \zeta| dx,
$$

and

$$
I_3 = -2s \int_U \partial_k [K(|y|) y_i] \bigg|_{y=\nabla p} \partial_i (\partial_j p) \partial_i (\partial_j p) \partial_i (\partial_i m p \partial_i m p) |\nabla p|^{2s-2} \zeta^2 dx
$$

$$
= -2s \int_U \partial_k [K(|y|) y_i] \bigg|_{y=\nabla p} \left( \partial_i \frac{1}{2} |\nabla p|^2 \right) \left( \partial_i \frac{1}{2} |\nabla p|^2 \right) |\nabla p|^{2s-2} \zeta^2 dx
$$

$$
\leq -2(1 - a) s \int_U K(|\nabla p|) \left( \frac{1}{2} |\nabla p|^2 \right)^2 |\nabla p|^{2s-2} \zeta^2 dx \leq 0.
$$
Combining these estimates together with Young’s inequality, we see that
\[
\frac{1}{2s+2} \frac{d}{dt} \int_U |\nabla p|^{2s+2} \zeta^2 dx + (1-a) \int_U K(|\nabla p|)|\nabla^2 p|^2 |\nabla p|^{2s} \zeta^2 dx
\leq 2(1+a) \int_U K(|\nabla p|)|\nabla^2 p||\nabla p|^{2s+1} \zeta |\nabla \zeta| dx
\leq \frac{1-a}{2} \int_U K(|\nabla p|)|\nabla^2 p|^2 |\nabla p|^{2s} \zeta^2 dx + C \int_U K(|\nabla p|)|\nabla p|^{2s+2} |\nabla \zeta|^2 dx.
\]
Thus,
\[
\frac{1}{2s+2} \frac{d}{dt} \int_U |\nabla p|^{2s+2} \zeta^2 dx + \frac{1-a}{2} \int_U K(|\nabla p|)|\nabla^2 p|^2 |\nabla p|^{2s} \zeta^2 dx 
\leq C \int_U K(|\nabla p|)|\nabla p|^{2s+2} |\nabla \zeta|^2 dx. \tag{3.45}
\]
Integrating this inequality in time yields \((3.44)\).

In order to iterate \((3.44)\) in \(s\) we need an embedding similar to Lemma 5.4 on page 93 in \([20]\). For our degenerate equation, the following version has the key weight function \(K(|\nabla w|)\).

**Lemma 3.7.** For each \(s \geq 1\), there exists a constant \(C > 0\) depending on \(s\) such that for each smooth cut-off function \(\zeta(x) \in C_c^\infty(U)\), the following inequality holds
\[
\int_U K(|\nabla w|)|\nabla w|^{2s+2} \zeta^2 dx \leq C \sup_{\text{supp} \zeta} |w|^2 \left\{ \int_U K(|\nabla w|)|\nabla w|^{2s-2} |\nabla^2 w|^2 \zeta^2 dx 
+ \int_U K(|\nabla w|)|\nabla w|^2 |\nabla \zeta|^2 dx \right\},
\]
for every sufficiently regular function \(w(x)\) such that the right hand side is well-defined.

**Proof.** Again, the Einstein’s summation convention is used in this proof.

Let \(I = \int_U K(|\nabla w|)|\nabla w|^{2s+2} \zeta^2 dx\). From direct calculations, we see that
\[
I = \int_U K(|\nabla w|) |\nabla w|^{2s} \partial_i w \partial_j w \zeta^2 dx = -\int_U \partial_i (K(|\nabla w|)|\nabla w|^{2s} \partial_j w \zeta^2) \cdot w dx 
- \int_U \left( K'(|\nabla w|) \frac{\partial_i \partial_j w \partial_j w}{|\nabla w|^2} \right) |\nabla w|^{2s} \partial_i w \cdot \zeta^2 \cdot w dx
- \int_U \left( K(|\nabla w|) |\nabla w|^{2s} \partial_i \partial_j w \partial_j w \right) \cdot \partial_i w \cdot \zeta^2 \cdot w dx
- 2s \int_U K(|\nabla w|) \left( |\nabla w|^{2s-2} \partial_i \partial_j w \partial_j w \right) \cdot \partial_i \zeta \cdot \zeta^2 \cdot w dx
- 2 \int_U K(|\nabla w|) |\nabla w|^{2s} \zeta \partial_i w \partial_i \zeta \cdot w dx.
\]
From this and \((2.11)\), it follows that
\[
I \leq a \int_U K(|\nabla w|)|\nabla w|^{2s} |\nabla w|^2 |\Delta w|^2 |\zeta| |w| dx 
+ 2s \int_U K(|\nabla w|)|\nabla w|^{2s-2} |\nabla^2 w| |\nabla w|^2 |\zeta|^2 |w| dx
+ 2 \int_U K(|\nabla w|)|\nabla w|^{2s} |\nabla w| |\zeta|^2 |\zeta| |w| dx
\leq C \int_U K(|\nabla w|)|\nabla w|^{2s} |\nabla^2 w|^2 |\zeta|^2 |w| dx
+ C \int_U K(|\nabla w|)|\nabla w|^{2s+1} |\Delta w| |\zeta|^2 |w| dx.
\]
This last inequality and the Young’s inequality imply that
\[
I \leq \frac{1}{2} I + C \left\{ \int_U K(|\nabla w|)|\nabla w|^{2s-2} |\nabla^2 w|^2 |\zeta|^2 |w|^2 dx 
+ \int_U K(|\nabla w|)|\nabla w|^{2s} |\zeta|^2 |w|^2 dx \right\}. 
\]
Therefore, we obtain
\[ I \leq C \sup_{\text{supp } \zeta} |w|^2 \left\{ \int_U K(|\nabla w|)|\nabla w|^{2s-2}|\nabla^2 w|^2 \zeta^2 \, dx + \int_U K(|\nabla w|) |\nabla w|^{2s} |\nabla \zeta|^2 \, dx \right\}. \]
This completes the proof. \qed

We combine Lemmas 3.6 and 3.7 now. By applying Lemma 3.7 to \( \bar{p} \) with \( s + 1 \) in place of \( s \), we have
\[
\int_0^T \int_U K(|\nabla p|)|\nabla p|^{2s+4} \zeta^2 \, dx \, dt \leq C \sup_{\text{supp } \zeta} |\bar{p}|^2 \left\{ \int_0^T \int_U K(|\nabla p|)|\nabla p|^{2s} |\nabla^2 p|^2 \zeta^2 \, dx \, dt \right. \\
+ \left. \int_0^T \int_U K(|\nabla p|)|\nabla p|^{2s+2} |\nabla \zeta|^2 \, dx \, dt \right\}. \tag{3.46}
\]
This last inequality and (3.44) imply for \( s \geq 0 \) that
\[
\int_0^T \int_U K(|\nabla p|)|\nabla p|^{2s+4} \zeta^2 \, dx \, dt \leq C \sup_{\text{supp } \zeta} |\bar{p}|^2 \left\{ \int_U |\nabla p_0(x)|^{2s+2} \zeta^2 \, dx \\
+ \int_0^T \int_U K(|\nabla p|)|\nabla p|^{2s+2} |\nabla \zeta|^2 \, dx \, dt \right\}. \\
\text{Also, from (3.44), we have for } s \geq 2 \text{ that}
\sup_{[0,T]} |\nabla p(x,t)|^s \zeta^2 \, dx \leq \int_U |\nabla p_0(x)|^s \zeta^2 \, dx + C \int_0^T \int_U K(|\nabla p|)|\nabla p|^s |\nabla \zeta|^2 \, dx \, dt. \tag{3.47}
\]

We are ready to iterate the relation (3.46). Hereafterward, we consider \( U' \Subset U \), that is, \( U' \) is an open set compactly contained in \( U \).

**Proposition 3.8.** For \( U' \Subset V \Subset U \) and \( s \geq 2 \) there exists a constant \( C > 0 \) depending on \( U' \), \( V \) and \( s \) such that for any \( T > 0 \), we have
\[
\int_0^T \int_{U'} K(|\nabla p|)|\nabla p|^s \, dx \, dt \leq C \left( 1 + \sup_{[0,T]} \|\bar{p}\|_{L^\infty(V)} \right)^{s-2} \left\{ \int_U |\nabla p_0(x)|^2 \, dx + \int_U |\nabla p_0(x)|^{s-2} \, dx + \int_0^T \int_U K(|\nabla p|)|\nabla p|^2 \, dx \, dt \right\}. \tag{3.48}
\]

**Proof.** Let \( m \in \mathbb{N}, m \geq 2 \) be a fixed number. Then, let \( \{U_k\}_{k=0}^m \) be a family of open, smooth domains in \( U \) such that \( U' = U_m \Subset U_{m-1} \Subset U_{m-2} \Subset \ldots \Subset U_1 \Subset U_0 = V \Subset U \). For each \( k = 1, 2, \ldots, m \), let \( \zeta_k(x) \) be a smooth cut-off function which is equal to one on \( U_k \) and zero on \( U \setminus U_{k-1} \). There is a positive constant \( C > 0 \) depending on \( \zeta_k \), \( k = 1, 2, \ldots, m \), such that \( |\nabla \zeta_k| \leq C \), for all \( k = 1, 2, \ldots, m \). Also, for each integer \( k \geq 1 \) and \( s_0 \geq 0 \), we define
\[X_k = \int_0^T \int_{U_k} K(|\nabla p|)|\nabla p|^{2k+s_0} \, dx \, dt, \quad A_k = \int_{U_k} |\nabla p_0(x)|^{2k+s_0} \, dx. \]
Then, by applying (3.46) with \( \zeta = \zeta_k \) and \( 2s+2 = 2k+s_0 \), we see that
\[X_{k+1} \leq C_{k,m,s_0} N_0(A_k + X_k), \quad k = 1, 2, \ldots, m - 1, \tag{3.49}\]
where \( N_0 = \sup_{V \times [0,T]} |\bar{p}|^2 = \sup_{[0,T]} \|\bar{p}\|_{L^\infty(V)}^2 \). Letting \( C_m = \max\{C_{k,m,s_0} : k = 1, 2, \ldots, m - 1\} \), we have from (3.49) that
\[X_{k+1} \leq C_m N_0(A_k + X_k), \quad k = 1, 2, \ldots, m - 1, \]
This inequality particularly yields
\[ X_m \leq (C_mN_0)A_{m-1} + (C_mN_0)A_{m-2} + \cdots + (C_mN_0)^{m-1}A_1 + (C_mN_0)^{m-1}X_1 \]
\[ \leq C[(N_0A_{m-1} + N_0^2A_{m-2} + \cdots + N_0^{m-1}A_1) + N_0^{m-1}X_1], \]
with \( C = C(m) \) depending on all sets \( U_k \) and functions \( \zeta_k \), for \( k = 1, 2, \ldots, m - 1 \). In other words, we have proved that for each integer \( m \geq 2 \) and real number \( s \geq 0 \), there is \( C(m, s_0) \) such that
\[ \int_0^T \int_{U'} K(|\nabla p|)|\nabla p|^{2m+s_0} \, dx \, dt \leq C \sum_{i=1}^{m-1} N_0^i \int_{U'_{m-i}} |\nabla p_0(x)|^{2(m-i)+s_0} \, dx \]
\[ + C N_0^{m-1} \int_0^T \int_{U_1} K(|\nabla p|)|\nabla p|^{2+s_0} \, dx \, dt. \]

By using Young’s inequality, one can rewrite this inequality as
\[ \int_0^T \int_{U'} K(|\nabla p|)|\nabla p|^{2m+s_0} \, dx \, dt \leq C(N_0 + N_0^{m-1}) \]
\[ \cdot \left\{ \int_U |\nabla p_0(x)|^{2+s_0} \, dx + \int_U |\nabla p_0(x)|^{2m+s_0-2} \, dx + \int_0^T \int_{U_1} K(|\nabla p|)|\nabla p|^{2+s_0} \, dx \, dt \right\}. \quad (3.50) \]

In particular, with \( m = 2 \) and \( s_0 = 0 \), (3.50) becomes
\[ \int_0^T \int_{U'} K(|\nabla p|)|\nabla p|^4 \, dx \, dt \leq C N_0 \left\{ \int_U |\nabla p_0(x)|^2 \, dx + \int_0^T \int_{U_1} K(|\nabla p|)|\nabla p|^2 \, dx \, dt \right\}. \quad (3.51) \]

This implies (3.48) when \( s = 4 \). In case \( s \in (2, 4) \), let \( \alpha \) and \( \beta \) be two positive numbers such that
\[ \frac{1}{s} = \frac{\alpha}{2} + \frac{\beta}{4} \quad \text{and} \quad \alpha + \beta = 1. \quad (3.52) \]

Then, using interpolation inequality, we get
\[ \left( \int_0^T \int_{U'} K(|\nabla p|)|\nabla p|^s \, dx \, dt \right)^\frac{1}{s} \leq \left( \int_0^T \int_{U_1} K(|\nabla p|)|\nabla p|^2 \, dx \, dt \right)^\frac{\alpha}{\alpha + \beta} \left( \int_0^T \int_{U_1} K(|\nabla p|)|\nabla p|^4 \, dx \, dt \right)^\frac{\beta}{\alpha + \beta}. \]

From this and (3.51), it follows that
\[ \int_0^T \int_{U'} K(|\nabla p|)|\nabla p|^4 \, dx \, dt \leq C N_0^{\frac{\alpha}{\alpha + \beta}} \left\{ \int_U |\nabla p_0(x)|^2 \, dx + \int_0^T \int_{U_1} K(|\nabla p|)|\nabla p|^2 \, dx \, dt \right\}. \]

From (3.52) follows \( \beta s/4 = s/2 - 1 \). Thus, we have
\[ \int_0^T \int_{U'} K(|\nabla p|)|\nabla p|^s \, dx \, dt \leq C N_0^{\frac{\alpha}{\alpha + \beta} - 1} \left\{ \int_U |\nabla p_0(x)|^2 \, dx + \int_0^T \int_{U_1} K(|\nabla p|)|\nabla p|^2 \, dx \, dt \right\}. \]

This implies (3.48) for \( s \in (2, 4) \). Therefore, we have proved (3.48) with \( s \in (2, 4] \). Now, for \( s > 4 \), there is a number \( s_0 \in (0, 2] \) and an integer \( m \geq 2 \) such that \( s = s_0 + 2m \). From (3.50), we have
\[ \int_0^T \int_{U'} K(|\nabla p|)|\nabla p|^s \, dx \, dt \leq \left\{ \int_U |\nabla p_0(x)|^{2+s_0} \, dx + \int_U |\nabla p_0(x)|^{2m+s_0-2} \, dx \right\} \]
\[ + \int_0^T \int_{U_1} K(|\nabla p|)|\nabla p|^{2+s_0} \, dx \, dt \}
\[ \leq C(N_0 + N_0^{\frac{s-s_0-2}{2}}) \left\{ \int_U |\nabla p_0(x)|^{2+s_0} \, dx + \int_U |\nabla p_0(x)|^{s-2} \, dx + \int_0^T \int_{U_1} K(|\nabla p|)|\nabla p|^{2+s_0} \, dx \, dt \right\}. \]
Since \( s_0 + 2 \in (2, 4) \), it follows from the last inequality and estimate (3.48) already proved for the case \( s \in (2, 4) \) with \( U_1 \) in place of \( U' \) and \( s_0 + 2 \) in place of \( s \) that
\[
\int_0^T \int_U K(|\nabla p|)|\nabla p|^s dxdt \leq C(N_0 + N_0^{s_0-2}) \left\{ \int_U |\nabla p_0(x)|^{2+s_0} dx + \int_U |\nabla p_0(x)|^{s_0-2} dx \right. \nonumber
\]
\[
+ N_0^{\frac{2}{s_0}} \left( \int_U |\nabla p_0(x)|^2 dx + \int_0^T \int_U K(|\nabla p|)|\nabla p|^2 dxdt \right) \} \nonumber
\]
\[
\leq C(1 + N_0^{\frac{2}{s_0}}) \left\{ \int_U |\nabla p_0(x)|^2 dx + \int_U |\nabla p_0(x)|^{s_0-2} dx + \int_0^T \int_U K(|\nabla p|)|\nabla p|^2 dxdt \right\}. \nonumber
\]
From this, the desired estimate (3.48) follows and the proof is complete. \( \square \)

Now, we give specific estimates for the \( L^s \)-norm of \( \nabla p \) in terms of initial and boundary data. We denote
\[
\mathcal{K}_{1,T} = 1 + \sup_{[0,T]} \| \psi \|_{L^\infty}. \tag{3.53}
\]

**Theorem 3.9.** Assume (SDC). Then for \( s \geq 2 \), there is a positive constant \( C \) depending on \( U' \) and \( s \) such that for any \( T > 0 \)
\[
\int_0^T \int_U K(|\nabla p|)|\nabla p|^s dxdt \leq C L_1(s)(T + 1)^{\frac{2(s-2)}{2-a}} + \frac{s-a}{1-a} \mathcal{K}_{1,T}, \tag{3.54}
\]
\[
\sup_{[0,T]} \int_U \nabla p(x,t)|^s dx \leq C L_2(s)(T + 1)^{\frac{2(s-2)}{2-a}} + \frac{s-a}{1-a} \mathcal{K}_{1,T}, \tag{3.55}
\]
where
\[
L_1(s) = L_1(s; [p_0]) \overset{\text{def}}{=} (1 + \| p_0 \|_{L^\infty})^{s-2} \left( 1 + \int_U |\nabla p_0(x)|^{\max\{2,s-2\}} dx \right), \nonumber
\]
\[
L_2(s) = L_2(s; [p_0]) \overset{\text{def}}{=} (1 + \| p_0 \|_{L^\infty})^{s-2} \left( 1 + \int_U |\nabla p_0(x)|^s dx \right). \nonumber
\]

**Proof.** From the inequality (3.53) in Proposition 3.1, it follows that
\[
\sup_{[0,T]} \| p \|_{L^\infty} \leq C \left\{ \| p_0 \|_{L^\infty} + (T + 1)^{\frac{2}{2-a}} \mathcal{K}_{1,T} \right\}. \nonumber
\]
Thanks to (3.28) of Theorem 3.3 we have
\[
\int_0^T \int_U K(|\nabla p|)|\nabla p|^2 dxdt \leq \| \nabla p_0 \|^2_{L^2} + C \int_0^T f(\tau) d\tau. \tag{3.56}
\]
These inequalities together with (3.48) of Proposition 3.8 yield
\[
\int_0^T \int_U K(|\nabla p|)|\nabla p|^s dxdt \leq C \left( \| \nabla p_0 \|_{L^\infty} + (T + 1)^{\frac{2}{2-a}} \mathcal{K}_{1,T} \right)^{s-2} \nonumber
\]
\[
\cdot \left( \| \nabla p_0 \|^2_{L^2} + \int_U (|\nabla p_0(x)|^2 + |\nabla p_0(x)|^{s-2}) dx + \int_0^T f(\tau) d\tau \right). \nonumber
\]
Applying Poincaré’s inequality to \( \tilde{p}_0 \), and using Young’s inequality, we obtain
\[
\int_0^T \int_U K(|\nabla p|)|\nabla p|^s dxdt \leq C \left( \| \nabla p_0 \|_{L^\infty} + (T + 1)^{\frac{2}{2-a}} \mathcal{K}_{1,T} \right)^{s-2} \nonumber
\]
\[
\cdot \left( 1 + \int_U |\nabla p_0(x)|^{\max\{2,s-2\}} dx + \int_0^T f(\tau) d\tau \right), \tag{3.57}
\]
Then (3.54) follows. It follows from (3.47) and (3.54) that

$$\sup_{[0,T]} \int_{U'} |\nabla p(x,t)|^s \, dx \leq \int_{U'} |\nabla p_0(x)|^s \, dx + C \left( \|\bar{p}_0\|_{L^\infty} + \left( T + 1 \right)^{2/\sigma} K_{1,T}^{1/\sigma} \right)^{s-2} \left( 1 + \int_U |\nabla p_0(x)|^{\max\{2, s-2\}} \, dx + \int_0^T f(\tau) \, d\tau \right),$$

Then by Young’s inequality we obtain

$$\sup_{[0,T]} \int_{U'} |\nabla p(x,t)|^s \, dx \leq C \left( \|\bar{p}_0\|_{L^\infty} + \left( T + 1 \right)^{2/\sigma} K_{1,T}^{1/\sigma} \right)^{s-2} \left( 1 + \int_U |\nabla p_0(x)|^{\max\{2, s-2\}} \, dx + \int_0^T f(\tau) \, d\tau \right).$$

Then (3.55) follows. □

Recall that $f$ is defined in (3.24) and $M_f$ is a continuous increasing majorant of $f$. Our next result is similar to Theorem 3.9 but the estimates do not contain the power growth in $T$.

**Theorem 3.10.** Assume (SDC). For $s \geq 2$, there is a positive constant $C$ depending on $U'$ and $s$ such that for any $T > 1$ we have

$$\int_0^T \int_{U'} K(|\nabla p|)|\nabla p|^s \, dx \, dt \leq C L_3(s) \left( 1 + M_f(T) \right)^{\mu_4(s-2)} \left\{ 1 + \int_0^T f(t) \, dt \right\},$$

where

$$L_3(s) = L_3(s; [p_0]) \overset{\text{def}}{=} \left( 1 + \|\bar{p}_0\|_{L^\infty} + \|\bar{p}_0\|_{L^2}^{\mu_4(2-a)} \right)^{s-2} \left\{ 1 + \int_U |\nabla p_0(x)|^{\max\{2, s-2\}} \, dx \right\},$$

$$L_4(s) = L_4(s; [p_0]) \overset{\text{def}}{=} \left( 1 + \|\bar{p}_0\|_{L^\infty} + \|\bar{p}_0\|_{L^2}^{\mu_4(2-a)} \right)^{s-2} \left\{ 1 + \int_U |\nabla p_0(x)|^{s} \, dx \right\}.$$

**Proof:** The proof is the same as in Theorem 3.9 with the use of estimate (3.34) in place of (3.9). Instead of (3.54) and (3.55), we have, respectively,

$$\int_0^T \int_{U'} K(|\nabla p|)|\nabla p|^s \, dx \, dt \leq C \left( 1 + \|\bar{p}_0\|_{L^\infty} + \|\bar{p}_0\|_{L^2}^{\mu_4(2-a)} + M_f(T)^{\mu_4} \right)^{s-2} \left\{ 1 + \int_U |\nabla p_0(x)|^{\max\{2, s-2\}} \, dx + \int_0^T f(t) \, dt \right\},$$

$$\sup_{[0,T]} \int_{U'} |\nabla p(x,t)|^s \, dx \leq C \left( 1 + \|\bar{p}_0\|_{L^\infty} + \|\bar{p}_0\|_{L^2}^{\mu_4(2-a)} + M_f(T)^{\mu_4} \right)^{s-2} \left\{ 1 + \int_U |\nabla p_0(x)|^s \, dx + \int_0^T f(t) \, dt \right\}.$$

Then (3.59) and (3.60) follow, respectively. □

In (3.59) and (3.60), letting $T \to \infty$, we obtain the following.
Corollary 3.11. Under the Strict Degree Condition, if
\[ \Psi_1 \equiv 1 + \sup_{[0, \infty)} f < \infty \quad \text{and} \quad \Psi_2 \equiv 1 + \int_0^\infty f(t) \, dt < \infty, \] (3.63)
then for any \( s \geq 2 \), there is a constant \( C = C(U', s) > 0 \) such that
\[ \int_0^\infty \int_{U'} K(|\nabla p|) |\nabla p|^s \, dx \, dt \leq C \mathcal{L}_3(s) \Psi_1^{\mu_4(s-2)} \Psi_2, \] (3.64)
\[ \sup_{[0, \infty)} \int_{U'} |\nabla p(x, t)|^s \, dx \leq C \mathcal{L}_4(s) \Psi_1^{\mu_4(s-2)} \Psi_2. \] (3.65)

Using the preceding theorems and the properties of function \( K(\cdot) \), we can derive the following direct estimates for \( \int_0^T \int_{U'} |\nabla p|^s \, dx \, dt \).

Theorem 3.12. Assume (SDC). For \( s \geq 2 - a \), there is a constant \( C > 0 \) such that for any \( T > 0 \) we have
\[ \int_0^T \int_{U'} |\nabla p|^s \, dx \, dt \leq C \mathcal{L}_1(s + a)(T + 1)^{\frac{2s-a}{2}} K^\mu \frac{s}{1,T}, \] (3.66)
and, alternatively,
\[ \int_0^T \int_{U'} |\nabla p|^s \, dx \, dt \leq CT + C \mathcal{L}_3(s + a) \left(1 + M_f(T)\right)^{\mu_4(s+a-2)} \left\{1 + \int_0^T f(t) \, dt\right\}, \] (3.67)
where the positive numbers \( \mathcal{L}_1(\cdot) \) and \( \mathcal{L}_3(\cdot) \) are defined in Theorems 3.9 and 3.10.

Proof. By relation (2.14),
\[ \int_0^T \int_{U'} |\nabla p|^s \, dx \, dt \leq CT + \int_0^T \int_{U'} K(|\nabla p|) |\nabla p|^{s+a} \, dx \, dt. \] (3.68)
The last integral is estimated by applying (3.54) with \( s + a \) replacing \( s \). As a result, we obtain (3.66).

Now, using relation (3.68) and applying (3.59) for \( s + a \) in place of \( s \), we obtain (3.67). \( \square \)

3.3. Estimates for pressure’s time derivative. In this subsection, we derive the interior \( L^\infty \)-norm of \( \dot{p}_t \). Throughout this subsection \( U' \subset U \). Under the Strict Degree Condition, let
\[ \mu_5 = 4 \left(1 - \frac{1}{(2 - a)^s}\right), \quad \mu_6 = 1 - \frac{2}{\mu_5}, \quad \mu_7 = \frac{2}{\mu_5(2-a)}. \] (3.69)
Then \( \mu_5 \in (2, (2-a)^s) \) and \( \mu_6, \mu_7 \in (0, 1) \).

Proposition 3.13. Assume (SDC). There is a constant \( C = C(U') > 0 \) such that for any \( T_0 \geq 0, T > 0 \) and \( \theta \in (0, 1) \), we have
\[ \sup_{[T_0 + \theta T, T_0 + T]} \|\dot{p}_t\|_{L^\infty(U')} \leq C \left\{\lambda \left(1 + \theta T'^{-1/2}\right)^{-1/2} + \lambda T'^{1/2} \sup_{[T_0, T_0 + T]} \|\psi_t\|_{L^\infty(U')} \right\} \left(\|\dot{p}_t\|_{L^2(U \times (T_0, T_0 + T))} + \|\ddot{p}_t\|_{L^{2-a}(U)}^{\frac{2}{\mu_5+2}}\right), \] (3.70)
where \( \lambda = 1 + \sup_{[T_0, T_0 + T]} \left(\int_U |\nabla p(x, t)|^{2-a} \, dx\right)^{\mu_7}, \) and
\[ \sup_{[T_0 + \theta T, T_0 + T]} \|p_t\|_{L^\infty(U')} \leq C \left(1 + \frac{1}{\theta T}\right)^{\frac{1}{\mu_6}} \left(1 + \sup_{[T_0, T_0 + T]} \|\nabla p\|_{L^{2-a}(U)}^{2}\right)\|p_t\|_{L^2(U \times (T_0, T_0 + T))}. \] (3.71)
Proof. Without loss of generality, we assume $T_0 = 0$. We prove (3.10) first. Let $q = p_t$ and
\[
\bar{q} = q - \frac{1}{|U|} \int_U q dx = p_t - \frac{1}{|U|} \frac{d}{dt} \int_U p dx = p_t + \frac{1}{|U|} \int_G \psi(x, t) d\sigma = \frac{\partial}{\partial t} \bar{p} = \bar{p}_t.
\]
Then it follows from (3.11) that $\bar{q}$ solves
\[
\frac{\partial \bar{q}}{\partial t} = \nabla \cdot (K(|\nabla p|) \nabla p) + \frac{1}{|U|} \int_G \psi \, d\sigma. \tag{3.72}
\]

For $k \geq 0$, let $\bar{q}(k) = \max \{ \bar{q} - k, 0 \}$ and $\chi_k(x, t)$ be the characteristic function of set $\{(x, t) \in U \times (0, T) : \bar{q}(x, t) > k\}$. On $S_k(t)$, we have $(\nabla p)_t = (\nabla \bar{p})_t = \nabla \bar{q} = \nabla \bar{q}(k)$.

Let $\zeta = \zeta(x, t)$ be the cut-off function on $U \times [0, T]$ satisfying $\zeta(\cdot, 0) = 0$ and $\zeta(\cdot, t)$ having compact support in $U$. We will use test function $\bar{q}(k)\zeta^2$, noting that $\nabla (\bar{q}(k)\zeta^2) = \zeta [\nabla (\bar{q}(k)\zeta) + \bar{q}(k)\nabla \zeta]$.

Multiplying (3.72) by $\bar{q}(k)\zeta^2$ and integrating the resultant on $U$, we get
\[
\frac{1}{2} \frac{d}{dt} \int_U |\bar{q}(k)\zeta|^2 dx = \int_U |\bar{q}(k)|^2 \zeta \zeta_t dx - \int_U (K(|\nabla p|))_t \nabla p \cdot [\nabla (\bar{q}(k)\zeta) + \bar{q}(k)\nabla \zeta] \zeta dx - \int_U K(|\nabla p|)(\nabla p)_t \cdot [\nabla (\bar{q}(k)\zeta) + \bar{q}(k)\zeta \nabla] \zeta dx + \frac{1}{|U|} \int_G \psi \, d\sigma \int_U \bar{q}(k)\zeta^2 dx.
\]

Put $z = \zeta [\nabla (\bar{q}(k)\zeta) + \bar{q}(k)\nabla \zeta]$. We simplify the third term on the right-hand side of the last inequality as
\[
(p_t \cdot z = \zeta \nabla \bar{q}(k) \cdot [\nabla (\bar{q}(k)\zeta) + \bar{q}(k)\nabla \zeta] = [\nabla (\bar{q}(k)\zeta) - \bar{q}(k)\nabla \zeta] \cdot [\nabla (\bar{q}(k)\zeta) + \bar{q}(k)\nabla \zeta] = |\nabla (\bar{q}(k)\zeta)|^2 - |\bar{q}(k)\nabla \zeta|^2.
\]

For the second term of right-hand side, using (2.11) we have
\[
|(K(|\nabla p|))_t \nabla p \cdot z| = |K'(|\nabla p|)|\frac{|\nabla p \cdot p_t|}{|\nabla p|} |\nabla p \cdot z| \leq aK(|\nabla p|)|\nabla \bar{q}||z|.
\]

Moreover,
\[
|\nabla \bar{q}|z| = |\zeta \nabla \bar{q}(k)||\nabla (\bar{q}(k)\zeta) + \bar{q}(k)\nabla \zeta| \leq (|\nabla (\bar{q}(k)\zeta)| + |\bar{q}(k)||\nabla \zeta|)^2 = |\nabla (\bar{q}(k)\zeta)|^2 + 2|\bar{q}(k)||\nabla \zeta||\nabla (\bar{q}(k)\zeta)| + |\bar{q}(k)||\nabla \zeta|^2.
\]

It follows that
\[
\frac{1}{2} \frac{d}{dt} \int_U |\bar{q}(k)\zeta|^2 dx + (1 - a) \int_U K(|\nabla p|)|\nabla (\bar{q}(k)\zeta)|^2 dx \\
\leq \int_U |\bar{q}(k)|^2 \zeta_t dx + (1 + a) \int_U K(|\nabla p|)|\bar{q}(k)||\nabla \zeta|^2 dx + 2a \int U K(|\nabla p|)|\bar{q}(k)||\nabla \zeta||\nabla (\bar{q}(k)\zeta)| + C \|\psi_t(t)\|_{L^\infty} \int_U |\bar{q}(k)||\zeta^2 dx.
\]

Let $\varepsilon > 0$. By Cauchy’s inequality,
\[
2aK(|\nabla p|)|\bar{q}(k)||\nabla (\bar{q}(k)\zeta)| \leq \frac{1 - a}{2} K(|\nabla p|)|\nabla (\bar{q}(k)\zeta)|^2 + \frac{2a^2}{1 - a} K(|\nabla p|)|\bar{q}(k)||\nabla \zeta|^2,
\]
\[
C \|\psi_t(t)\|_{L^\infty} |\bar{q}(k)| \zeta^2 \leq \varepsilon |\bar{q}(k)| \zeta^2 + C \varepsilon^{-1} \|\psi_t(t)\|_{L^\infty}^2 \zeta^2.
\]
Applying inequality (2.30) in Lemma 2.4 to Lemma 2.5, we have

\[
\frac{1}{2} \frac{d}{dt} \int_U |\bar{q}^{(k)}_t \zeta|^2 dx + \frac{1-a}{2} \int_U K(|\nabla p|)|\nabla (\bar{q}^{(k)}_t \zeta)|^2 dx \
+ C \int_U K(|\nabla p|)|\bar{q}^{(k)}_t \nabla \zeta|^2 dx + \varepsilon \int_U |\bar{q}^{(k)}_t \zeta|^2 dx + C \varepsilon^{-1} \int_U \psi_t(t)^2 dx \
\leq \int_U |\bar{q}^{(k)}_t \zeta|^2 dx.
\]

Therefore, by (3.73)

\[
\sup_{[0,T]} \int_U |\bar{q}^{(k)}_t \zeta|^2 dx + \int_0^T \int_U K(|\nabla p|)|\nabla (\bar{q}^{(k)}_t \zeta)|^2 dx dt \
\leq C \left( \int_U |\bar{q}^{(k)}_t \zeta|^2 dx \right)^{1/2} + C T \sup_{[0,T]} \int_U |\nabla \zeta|^2 dx dt.
\]

Applying inequality (2.30) to Lemma 2.4 with the weight \( W(x,t) = K(|\nabla p(x,t)|) \), we have

\[
\|\bar{q}^{(k)}_t \zeta\|_{L^{p_5}(Q_T)} \leq C \left\{ \sup_{[0,T]} \left( \int_U K(|\nabla p|)^{-\frac{2a}{a-1}} dx \right)^{\frac{a-1}{2}} \right\} \sup_{[0,T]} \left( \int_U |\bar{q}^{(k)}_t \zeta|^2 dx \right)^{1/2} + C \sup_{[0,T]} \int_U |\nabla \zeta|^2 dx dt.
\]

Therefore, by (3.73)

\[
\|\bar{q}^{(k)}_t \zeta\|_{L^{p_5}(Q_T)} \leq C \lambda \left( \int_U |\bar{q}^{(k)}_t \zeta|^2 dx + T \sup_{[0,T]} \int_U |\nabla \zeta|^2 dx dt \right)^{1/2}.
\]

Let \( x_0 \) be any given point in \( U \). Denote \( \rho = \text{dist}(x_0, \partial U) > 0 \). Let \( M_0 > 0 \) be fixed which will be determined later. For \( i \geq 0 \), define

\[
k_i = M_0 (1 - 2^{-i}), \quad t_i = \theta T (1 - 2^{-i}), \quad \rho_i = \frac{1}{4} \rho (1 + 2^{-i}).
\]

Then \( t_0 = 0 < t_1 < \ldots < \theta T \) and \( \rho_0 = \rho/2 > \rho_1 > \ldots > \rho/4 > 0 \). Note that

\[
\lim_{i \to \infty} t_i = \theta T \quad \text{and} \quad \lim_{i \to \infty} \rho_i = \rho/4.
\]

Let \( U_i = \{ x : \| x - x_0 \| < \rho_i \} \) then \( U_{i+1} \subseteq U_i \) for \( i = 0, 1, 2, \ldots \). For \( i, j \geq 0 \), we denote

\[
Q_i = \{ (x,t) : x \in U_i, t \in (t_i, T) \}, \quad A_{i,j} = \{ (x,t) \in Q_i : \bar{q}(x,t) > k_i, t \in (t_j, T) \}.
\]
For each $Q_i$, we use a cut-off function $\zeta_i(x,t)$ which is piecewise linear in $t$ and satisfies $\zeta_i \equiv 1$ on $Q_{i+1}$ and $\zeta_i \equiv 0$ on $Q_T \setminus Q_i$. Then there is $C > 0$ such that
\[
\|\zeta_i\|_t \leq \frac{C}{t_{i+1} - t_i} = \frac{C^2i+1}{\theta T} \quad \text{and} \quad |\nabla \zeta_i| \leq \frac{C}{\rho_i - \rho_{i+1}} = \frac{C^2i+1}{4\rho} \quad \text{for all } i \geq 0.
\] (3.76)

Define $F_i = \|q^{(k+1)}(k)i\|_{L^{\mu_5}(A_{i+1},1)}$. Applying (3.74) with $k = k_{i+1}$ and $\zeta = \zeta_i$ gives
\[
F_i \leq C\lambda \left\{ \int_0^T \int_0^\infty \left( \zeta_i |\zeta_i| + |\nabla \zeta_i|^2 \right) dx dt + T \sup_{[0,T]} \|\psi_t(t)\|_\infty \right\} \frac{2i+1}{\mu_5} \leq C\lambda \left\{ \int_0^T \int_0^\infty \left( \zeta_i |\zeta_i| + |\nabla \zeta_i|^2 \right) dx dt + T \sup_{[0,T]} \|\psi_t(t)\|_\infty \right\} \frac{2i+1}{\mu_5}.
\] (3.77)

Using derivative estimates (3.76) for $\zeta_i$, we obtain
\[
F_i \leq C\lambda \left[ \left( \psi_t(t)^{-1/2} + 2 \right) \|q^{(k+1)}(k)i\|_{L^2(A_{i+1},1)} + T^{1/2} \sup_{[0,T]} \|\psi_t(t)\|_\infty \right] \frac{2i+1}{\mu_5} \leq C\lambda \left[ \left( \psi_t(t)^{-1/2} + 2 \right) \|q^{(k+1)}(k)i\|_{L^2(A_{i+1},1)} + T^{1/2} \sup_{[0,T]} \|\psi_t(t)\|_\infty \right] \frac{2i+1}{\mu_5}.
\] (3.78)

Then, it follows from Hölder’s inequality, (3.77) and (3.78) that
\[
\|q^{(k+1)}(k)i\|_{L^2(A_{i+1},1,1)} \leq \|q^{(k+1)}(k)i\|_{L^{\mu_5}(A_{i+1},1,1)} \|A_{i+1,1,1}\|_\infty \frac{2i+1}{\mu_5} \leq C\lambda \left[ \left( \psi_t(t)^{-1/2} + 2 \right) \|q^{(k+1)}(k)i\|_{L^2(A_{i+1},1)} + T^{1/2} \sup_{[0,T]} \|\psi_t(t)\|_\infty \right] \frac{2i+1}{\mu_5} \leq C\lambda \left[ \left( \psi_t(t)^{-1/2} + 2 \right) \|q^{(k+1)}(k)i\|_{L^2(A_{i+1},1)} + T^{1/2} \sup_{[0,T]} \|\psi_t(t)\|_\infty \right] \frac{2i+1}{\mu_5}.
\] (3.79)

Note that $\|q^{(k+1)}(k)i\|_{L^2(A_{i+1},1)} \geq \|q^{(k+1)}(k)i\|_{L^2(A_{i+1},1)} \geq (k_{i+1} - k_i)\|A_{i+1,1}\|_\infty \frac{2i+1}{\mu_5}$. Thus,
\[
\|A_{i+1,1}\|_\infty \leq \left( k_{i+1} - k_i \right) \|q^{(k+1)}(k)i\|_{L^2(A_{i+1},1)} \leq C4^\lambda M_0^{-2} \|q^{(k+1)}(k)i\|_{L^2(A_{i+1},1)} \leq C4^\lambda M_0^{-2} \|q^{(k+1)}(k)i\|_{L^2(A_{i+1},1)}.
\] (3.80)

Then it follows (3.79), (3.78) and (3.80) that
\[
\|q^{(k+1)}(k)i\|_{L^2(A_{i+1},1,1)} \leq \|q^{(k+1)}(k)i\|_{L^2(A_{i+1},1,1)} \|A_{i+1,1,1}\|_\infty \frac{2i+1}{\mu_5} \leq C\lambda \left[ \left( \psi_t(t)^{-1/2} + 2 \right) \|q^{(k+1)}(k)i\|_{L^2(A_{i+1},1)} + T^{1/2} \sup_{[0,T]} \|\psi_t(t)\|_\infty \right] \frac{2i+1}{\mu_5} \leq C\lambda \left[ \left( \psi_t(t)^{-1/2} + 2 \right) \|q^{(k+1)}(k)i\|_{L^2(A_{i+1},1)} + T^{1/2} \sup_{[0,T]} \|\psi_t(t)\|_\infty \right] \frac{2i+1}{\mu_5}.
\] (3.79)

Let $Y_i = \|q^{(k+1)}(k)i\|_{L^2(A_{i+1},1)}$, $B = 4$ and
\[
D_1 = C\lambda \left( \psi_t(t)^{-1/2} + 2 \right) \|q^{(k+1)}(k)i\|_{L^2(A_{i+1},1)} + T^{1/2} \sup_{[0,T]} \|\psi_t(t)\|_\infty \right\}_\infty \frac{2i+1}{\mu_5} \leq C4^\lambda \left[ \left( \psi_t(t)^{-1/2} + 2 \right) \|q^{(k+1)}(k)i\|_{L^2(A_{i+1},1)} + T^{1/2} \sup_{[0,T]} \|\psi_t(t)\|_\infty \right] \frac{2i+1}{\mu_5}.
\] (3.81)

We obtain $Y_{i+1} \leq B^3(D_1 Y_i^1 + D_2 Y_i^1)$ for all $i \geq 0$. We now determine $M_0$ so that
\[
Y_0 \leq (2D_1)^{-1/\mu_6} B^{-1/\mu_6}, \quad Y_0 \leq (2D_2)^{-1/\mu_6} B^{-1/\mu_6}.
\] (3.81)

This condition is met if
\[
M_0 \geq C \left[ \lambda \left( 1 + (\theta T)^{-1/2} \right) \right]^{1/\mu_6} Y_0, \quad M_0 \geq C \left[ \lambda T^{1/2} \sup_{[0,T]} \|\psi_t(t)\|_\infty \right] \frac{1}{\mu_6} Y_0^\mu_6^{1 + 1/\mu_6}.
\] (3.82)
Then by condition (3.81), applying (A.5) in Lemma A.2 for \( m = 2 \) gives \( \lim_{t \to \infty} Y_t = 0 \). (Alternatively, Lemma A.1 can be used in this case.) Hence,

\[
\int_{\theta T}^T \int_{B(x_0, \rho/4)} |\tilde{q}(M_0)|^2 dx dt = 0.
\] (3.83)

Since \( \tilde{q}(x, t) \in C(U \times (0, \infty)) \), it follows from (3.83) that \( \tilde{q}(x, t) \leq M_0 \) in \( B(x_0, \rho/4) \times (\theta T, T) \). Replace \( q \) by \(-q\) and \( \psi \) by \(-\psi\) and use the same argument we obtain \( |\tilde{q}(x, t)| \leq M_0 \) in \( B(x_0, \rho/4) \times (\theta T, T) \). Now by covering \( U' \) by finitely many such balls \( B(x_0, \rho/4) \), we come to conclusion

\[
|\tilde{q}(x, t)| \leq M_0 \quad \text{in } U' \times (\theta T, T).
\] (3.84)

By the choice of \( M_0 \), we obtain (3.70) from (3.84).

In the above proof of (3.70), we can work with \( \tilde{q} \) instead of \( \bar{q} \), with

\[
\frac{\partial \tilde{q}}{\partial t} = \nabla \cdot \left( K(\nabla \tilde{p}) \nabla \tilde{p} \right),
\]

instead of equation (3.72), then the term \( \| \psi_t \|_{L^\infty} \) can be removed and we obtain the desired estimate (3.71) for \( \bar{p}_t \).

**Remark 3.14.** The main difference between estimate of \( \bar{p}_t \) in (3.70) and the estimate of \( p_t \) in (3.71) is the involvement of \( \psi_t \). Though we cannot derive (3.71) from (3.70), in the following development we will focus on \( \bar{p}_t \) only.

The following estimates can be easily derived (by the mean of Young’s inequality) from corresponding ones in [18]. They will be used in finding \( L^\infty \)-estimates for \( \bar{p}_t \) in terms of initial data and boundary data. We use the following notation:

\[
m_1(t) = 1 + \| \tilde{p}_0 \|_{L^2}^2 + M_f(t) \frac{2}{2-a} + \int_{t-1}^t \tilde{f}(\tau) d\tau, \quad m_2(t) = 1 + A \frac{2-a}{2-a} + \int_{t-1}^t \tilde{f}(\tau) d\tau,
\]

\[
m_3(t) = 1 + \beta \frac{1}{2-a} + \sup_{t \in [t-1, t]} f \frac{2-a}{2-a} + \int_{t-1}^t \tilde{f}(\tau) d\tau, \quad A_1 = A + A \frac{2-a}{2-a} + \limsup_{t \to \infty} \int_{t-1}^t \tilde{f}(\tau) d\tau.
\]

**Theorem 3.15** (cf. [18] Theorems 4.4 and 4.5). (i) One has for all \( t \geq 0 \) that

\[
J_H[p](t) + \| \bar{p}_t \|^2_{L^2(U \times (0, t))} \leq C \left( \| \tilde{p}_0 \|^2_{L^2} + J_H[p](0) + (t+1) \sup_{[0, t]} f + \int_0^t \tilde{f}(\tau) d\tau \right),
\] (3.88)

and for all \( t \geq 1 \) that

\[
J_H[p](t) + \| \bar{p}_t \|^2_{L^2(U \times (t-1/2, t))} \leq C \left( \| \tilde{p}(t-1) \|^2_{L^2} + \sup_{[t-1, t]} f + \int_{t-1}^t \tilde{f}(\tau) d\tau \right).
\] (3.89)

Now, assume that the Degree Condition holds.

(ii) For \( 0 < t_0 < 1 \) and \( t \geq t_0 \), one has

\[
\| \bar{p}_t(t) \|^2_{L^2} \leq C t_0^{-1} L_5(t_0) + C \int_0^t f(\tau) + \tilde{f}(\tau) d\tau,
\] (3.90)

where \( L_5(t_0) = L_5(t_0; [p_0, \psi]) \overset{\text{def}}{=} 1 + \| \tilde{p}_0 \|^2_{L^2} + \| \nabla p_0 \|^2_{L^{2-a}} + M_f(t_0) \frac{2-a}{2-a} + \int_0^{t_0} \tilde{f}(\tau) d\tau.

For all \( t \geq 1 \), one has

\[
J_H[p](t), \| \bar{p}_t(t) \|^2_{L^2} \leq C m_1(t) \quad \text{for all } t \geq 1.
\] (3.91)
(iii) If \( A < \infty \) then there is \( T > 1 \) such that

\[
J_H[p](t), \; \|p(t)\|^2_{L^2} \leq C m_2(t) \quad \text{for all } t > T, 
\]

\[
\limsup_{t \to \infty} J_H[p](t), \; \limsup_{t \to \infty} \|p(t)\|^2_{L^2} \leq C A_1. 
\]

(iv) If \( \beta < \infty \) then there is \( T > 1 \) such that

\[
J_H[p](t), \; \|p(t)\|^2_{L^2} \leq C m_3(t) \quad \text{for all } t > T. 
\]

We now state our main estimates of \( \bar{p}_t \). In addition to \( \mathcal{K}_{1,t} \) defined by (3.53), we will also use

\[
\mathcal{K}_{2,t} = 1 + \int_0^t \bar{f}(\tau) d\tau. 
\]

**Theorem 3.16.** Assume (SDC). For \( t > 0 \), one has

\[
\|\tilde{p}_t(t)\|_{L^\infty(U')} \leq C \left\{ \|\tilde{p}_0\|^2_{L^2} + \|\nabla p_0\|_{L^{2-a}} + (1 + t) \mathcal{K}_{1,t}^{2-a} + \mathcal{K}_{2,t} \right\}^{\frac{\mu^2}{\nu_0}} + \frac{1}{2} \left\{ 1 + t^{-\frac{1}{2\nu_0}} + t^{1/(\nu_0 + 1)} \sup_{[0,t]} \|\psi_t\|_{L^\infty} \right\} \left( 1 + \sup_{[t-1/2,t]} \|\psi_t\|_{L^\infty} \right). 
\]

For \( t \geq 3/2 \), one has

\[
\|\tilde{p}_t(t)\|_{L^\infty(U')} \leq C \left\{ 1 + \|\tilde{p}_0\|^2_{L^2} + M_f(t)^{2-a} + \int_{t-3/2}^t \bar{f}(\tau) d\tau \right\}^{\frac{\mu^2}{\nu_0}} + \frac{1}{2} \left( 1 + \sup_{[t-1/2,t]} \|\psi_t\|_{L^\infty} \right). 
\]

Above, the positive constant \( C \) depends on \( U' \).

**Proof.** Applying (3.70) with \( T_0 = 0 \), \( T = t \) and \( \theta = 1/2 \), an then applying Young’s inequality to the term \( \|\tilde{p}_t\|_{L^2(U \times (0,t))} \), we have

\[
\|\tilde{p}_t(t)\|_{L^\infty(U')} \leq C \left( 1 + \sup_{[0,t]} \|\nabla p\|_{L^{2-a}} \right)^{\frac{\mu^2}{\nu_0}} \left\{ 1 + \frac{\sqrt{2t}}{\nu_0} + \left( t^{1/2} \sup_{[0,t]} \|\psi_t\|_{L^\infty} \right) \right\} \left( 1 + \|\tilde{p}_0\|^2_{L^2} + \|\nabla p_0\|_{L^{2-a}} + (1 + t) \mathcal{K}_{1,t}^{2-a} + \mathcal{K}_{2,t} \right)^{\frac{\mu^2}{\nu_0}} + \frac{1}{2} \left\{ 1 + t^{-\frac{1}{2\nu_0}} + t^{1/(\nu_0 + 1)} \sup_{[0,t]} \|\psi_t\|_{L^\infty} \right\} \left( 1 + \|\tilde{p}_0\|^2_{L^2} + \|\nabla p_0\|_{L^{2-a}} + (1 + t) \mathcal{K}_{1,t}^{2-a} + \mathcal{K}_{2,t} \right)^{\frac{\mu^2}{\nu_0}} + \frac{1}{2} \left\{ 1 + \|\tilde{p}_0\|^2_{L^2} + \|\nabla p_0\|_{L^{2-a}} + (1 + t) \mathcal{K}_{1,t}^{2-a} + \mathcal{K}_{2,t} \right\}^\frac{\mu^2}{\nu_0} + \frac{1}{2} \left\{ 1 + \|\tilde{p}_0\|^2_{L^2} + \|\nabla p_0\|_{L^{2-a}} + (1 + t) \mathcal{K}_{1,t}^{2-a} + \mathcal{K}_{2,t} \right\}. 
\]

Using (3.88) to estimate \( \sup_{[0,t]} \|\nabla p\|_{L^{2-a}} \) and \( \|\tilde{p}_t\|_{L^2(U \times (0,t))} \) in the previous inequality, we find that

\[
\|\tilde{p}_t(t)\|_{L^\infty(U')} \leq C \left\{ 1 + \|\tilde{p}_0\|^2_{L^2} + \|\nabla p_0\|_{L^{2-a}} + (1 + t) \mathcal{K}_{1,t}^{2-a} + \mathcal{K}_{2,t} \right\}^{\frac{\mu^2}{\nu_0}} + \frac{1}{2} \left\{ 1 + t^{-\frac{1}{2\nu_0}} + t^{1/(\nu_0 + 1)} \sup_{[0,t]} \|\psi_t\|_{L^\infty} \right\} \left( 1 + \|\tilde{p}_0\|^2_{L^2} + \|\nabla p_0\|_{L^{2-a}} + (1 + t) \mathcal{K}_{1,t}^{2-a} + \mathcal{K}_{2,t} \right)^{\frac{\mu^2}{\nu_0}} + \frac{1}{2} \left\{ 1 + \|\tilde{p}_0\|^2_{L^2} + \|\nabla p_0\|_{L^{2-a}} + (1 + t) \mathcal{K}_{1,t}^{2-a} + \mathcal{K}_{2,t} \right\}. 
\]

Then inequality (3.96) follows.

Now, consider \( t \geq 3/2 \). Applying estimate (3.70) to the interval \([t-1/2,t]\), i.e., \( T_0 = t - 1/2 \) and \( T = 1/2 \), with \( \theta = 1/2 \), we obtain

\[
\|\tilde{p}_t(t)\|_{L^\infty(U')} \leq C \left( 1 + \sup_{[t-1/2,t]} \|\nabla p\|_{L^{2-a}(U')} \right)^{\frac{\mu^2}{\nu_0}} \left( 1 + \sup_{[t-1/2,t]} \|\psi(t)\|_{L^\infty} \right) \left( 1 + \|\tilde{p}_0\|^2_{L^2(U \times (t-1/2,t))} + \|\tilde{p}_t\|_{L^2(U \times (t-1/2,t))} \right). 
\]
Let $m(t) = \sup_{[t-3/2,t]} f + \int_{t-3/2}^t \tilde{f}(\tau) d\tau$. Utilizing estimate (3.89), we have

$$\|\tilde{p}(t)\|_{L^\infty(U')} \leq C \left(1 + \sup_{t\in[t-3/2,t-1]} \|\tilde{p}(\tau)\|_{L^2} + m(t) \right)^{\frac{\mu_6}{\mu_6} + \frac{1}{2}} \left(1 + \sup_{t\in[t-1/2,t]} \|\psi_t(t)\|_{L^{\mu_6+1}}^{\frac{1}{\mu_6+1}} \right).$$

By Young's inequality,

$$\|\tilde{p}(x,t)\| \leq C \left(1 + \sup_{t\in[t-3/2,t-1]} \|\tilde{p}(\tau)\|_{L^2} + m(t) \right)^{\frac{\mu_6}{\mu_6} + \frac{1}{2}} \left(1 + \sup_{t\in[t-1/2,t]} \|\psi_t(t)\|_{L^{\mu_6+1}}^{\frac{1}{\mu_6+1}} \right).$$

Then inequality (3.97) is obtained by using (3.29) to estimate $\|\tilde{p}(\tau)\|_{L^2}$ for $\tau \leq t - 1$. \hfill \Box

Let

$$A_2 = \limsup_{t\to\infty} m_2(t) = 1 + A^{\frac{2}{2-a}} + \limsup_{t\to\infty} \int_{t-1}^t \tilde{f}(\tau) d\tau,$$

$$A_3 = 1 + A^{\frac{2}{2-a}} + \limsup_{t\to\infty} \|\psi_t(t)\|_{L^{\frac{2}{2-a}}}^{\frac{2-a}{1-a}}.$$  

We have from (3.87), (3.99) and (3.100) the relation:

$$A_1 \leq CA_2 \leq C'A_3.\tag{3.101}$$

**Theorem 3.17.** Assume (SDC).

(i) If $A < \infty$ then

$$\limsup_{t\to\infty} \|\tilde{p}(t)\|_{L^\infty(U')} \leq C \left(1 + \frac{1}{A_3^{\frac{2}{2-a}}} + \sup_{t\in[t-3/2,t-1]} \|\tilde{p}(\tau)\|_{L^2} + \int_{t-3/2}^t \tilde{f}(\tau) d\tau \right)^{\frac{\mu_6}{\mu_6} + \frac{1}{2}} \left(1 + \sup_{t\in[t-1/2,t]} \|\psi_t(t)\|_{L^{\mu_6+1}}^{\frac{1}{\mu_6+1}} \right).\tag{3.102}$$

Consequently, denoting $\mu_8 = \frac{\mu_6}{\mu_6} + \frac{1}{2} + \frac{1}{(2-a)(\mu_6+1)}$, one has

$$\limsup_{t\to\infty} \|\tilde{p}(t)\|_{L^\infty(U')} \leq CA_3^{\mu_8}.\tag{3.103}$$

(ii) If $\beta < \infty$, then there is $T > 0$ such that for all $t > T$,

$$\|\tilde{p}(t)\|_{L^\infty(U')} \leq C \left(1 + \beta^{\frac{1}{1-a}} + \sup_{t\in[t-3/2,t-1]} \|f(\tau)\|_{L^{\frac{2}{2-a}}} + \int_{t-3/2}^t \tilde{f}(\tau) d\tau \right)^{\frac{\mu_6}{\mu_6} + \frac{1}{2}} \left(1 + \sup_{t\in[t-1/2,t]} \|\psi_t(t)\|_{L^{\mu_6+1}}^{\frac{1}{\mu_6+1}} \right).\tag{3.104}$$

Above, the positive constant $C$ depends on $U'$.

**Proof.** (i) We follow the proof in Theorem 3.16. Observe that

$$\limsup_{t\to\infty} \int_{t-3/2}^t \tilde{f}(\tau) d\tau \leq \limsup_{t\to\infty} \int_{t-1}^t \tilde{f}(\tau) d\tau + \limsup_{t\to\infty} \int_{t-1}^t \tilde{f}(\tau) d\tau \leq 2 \limsup_{t\to\infty} \int_{t-1}^t \tilde{f}(\tau) d\tau.\tag{3.105}$$

The limit estimates (3.30) and (3.105) yield

$$\limsup_{t\to\infty} \left(\sup_{t\in[t-3/2,t-1]} \|\tilde{p}\|_{L^2} + m(t) \right) \leq CA_1.$$

Combining this with (3.98), we obtain

$$\limsup_{t\to\infty} \|\tilde{p}(t)\|_{L^\infty(U')} \leq C \left(1 + A_1 \right)^{\frac{\mu_6}{\mu_6}} \left(1 + \sup_{t\in[t-1/2,t]} \|\psi_t(t)\|_{L^{\mu_6+1}}^{\frac{1}{\mu_6+1}} \right) \left(A_1^{\frac{\mu_6}{\mu_6+1}} + A_1 \right)^{\frac{1}{2}}.\tag{3.106}$$

Then (3.102) follows this and relation (3.101). Elementary calculations will give (3.103) from (3.102).
(ii) We use (3.98) and Young’s inequality again, this time, with estimate (3.31); it results in (3.104).

While (3.102) gives an asymptotic estimate for \( \bar{p}_t \) in the case \( A < \infty \), the next result covers the case \( A = \infty \).

**Theorem 3.18.** Assume (SDC). Let \( L_6 = \| \bar{p}_0 \|^2_{L^2} \) in general case, and \( L_6 = \beta \frac{1}{1-a} \) in case \( \beta < \infty \), and define

\[
N(t) = 1 + L_6 + M_f(t) \frac{2}{3-a} + \int_{t-2}^t \tilde{f}(\tau) d\tau, \quad h(t) = N(t)\frac{\mu}{\nu_6} \left( 1 + \sup_{[-1,1]} \| \psi_t \|_{L^\infty}^{\frac{1}{\nu_6+1}} \right). \tag{3.106}
\]

If

\[
\lim_{t \to \infty} h(t) \left( \frac{2(\nu_6+1)}{\beta} e^{-d_1 \int_0^t N(\tau)^{-b} d\tau} \right) = 0 \quad \text{and} \quad \lim_{t \to \infty} h'(t) N^b(t) = 0, \tag{3.107}
\]

where \( b \) is defined in (2.9) and \( d_1 > 0 \) appears in (3.111) below, then

\[
\lim_{t \to \infty} \sup_{[t-1,1]} \| \bar{p}_t(t) \|_{L^\infty(U')} \leq C \lim_{t \to \infty} \left\{ h(t) \left( N(t)^b \| \psi_t(t) \|_{L^\infty} + \left[ N(t)^b \| \psi_t(t) \|_{L^\infty} \right] \frac{\mu}{\nu_6+1} \right) \right\}, \tag{3.108}
\]

where \( C > 0 \) depends on \( U' \).

**Proof.** Applying (3.70) to the interval \([t-1,1] \) with \( \theta = \frac{1}{2} \), we have

\[
\| \bar{p}_t(t) \|_{L^\infty(U')} \leq C \left[ \left( 1 + \sup_{[t-1,1]} J_H[p]{(\tau)} \right)^{\frac{\mu}{\nu_6}} \left( 1 + \sup_{[0,T]} \| \psi_t(\tau) \|_{L^\infty}^{\frac{1}{\nu_6+1}} \right) \right] \cdot \left( \sup_{[t-1,1]} \| \bar{p}_t \|_{L^2(U)} + \sup_{[t-1,1]} \| \bar{p}_t \|_{L^2(U)} \right). \tag{3.109}
\]

In the following \( T > 2 \) is sufficiently large. By (3.91) in the general case, and by (3.94) in case \( \beta < \infty \), we have

\[
J_H[p]\{\tau\} \leq C \left( 1 + L_6 + M_f(t) \frac{2}{3-a} + \int_{t-1}^t \tilde{f}(\tau) d\tau \right), \quad \tau > T.
\]

Therefore,

\[
\sup_{[t-1,1]} J_H[p]{(\tau)} \leq CN(t). \tag{3.110}
\]

Recall inequality (5.13) in [18]; there is \( d_1 > 0 \) such that for \( t > 0 \) we have

\[
\frac{d}{dt} \| \bar{p}_t(t) \|^2 \leq -d_1 N(t)^{-b} \| \bar{p}_t(t) \|^2 + C \| \psi_t(t) \|_{L^\infty}^2 \| N(t) \|^b. \tag{3.111}
\]

Therefore, for \( t' \in (T, \infty) \)

\[
\| \bar{p}_t(t') \|^2 \leq e^{-d_1 \int_T^{t'} N(\tau)^{-b} d\tau} \| \bar{p}_t(T) \|^2 + \int_T^{t'} e^{-d_1 \int_T^s N(\tau)^{-b} d\tau} \| \psi_t(\tau) \|_{L^\infty}^2 N(\tau)^b d\tau.
\]

Then taking supremum in \( t' \) over the interval \([t-1,1] \) yields

\[
\sup_{[t-1,1]} \| \bar{p}_t(t') \|^2 \leq e^{-d_1 \int_T^{t-1} N(\tau)^{-b} d\tau} \| \bar{p}_t(T) \|^2 + e^{-d_1 \int_T^{t-1} N(\tau)^{-b} d\tau} \int_T^{t} e^{d_1 \int_T^s N(\tau)^{-b} d\tau} \| \psi_t(\tau) \|_{L^\infty}^2 N(\tau)^b d\tau.
\]

Thanks to the fact \( N(t) \geq 1 \), we have

\[
e^{-d_1 \int_T^{t-1} N(\tau)^{-b} d\tau} e^{-d_1 \int_T^{t-1} N(\tau)^{-b} d\tau} e^{d_1 \int_T^{t-1} N(\tau)^{-b} d\tau} \leq e^{d_1 \int_T^{t-1} N(\tau)^{-b} d\tau}.
\]
Hence,

\[
\sup_{[t-1,t]} \|\bar{p}_t\|^2 \leq C \left( e^{-d_1 \int_T^t N^{-b}(\tau) d\tau} \|\bar{p}_t(T)\|^2 + e^{-d_1 \int_T^t N^{-b}(\tau) d\tau} \int_T^t e^{-d_1 \int_T^\tau N^{-b}(\theta) d\theta} \|\psi_t(\tau)\|_{L^\infty(\tau) b}^2 d\tau \right)
\]

\[
= C \left( e^{-d_1 \int_T^t N^{-b}(\tau) d\tau} \|\bar{p}_t(T)\|^2 + \int_T^t e^{-d_1 \int_T^\tau N^{-b}(\theta) d\theta} \|\psi_t(\tau)\|_{L^\infty(\tau) b}^2 d\tau \right). \tag{3.112}
\]

By (3.109), (3.110) and (3.112), we obtain

\[
\|\bar{p}_t(t)\|_{L^\infty(U')} \leq C \left( \left( e^{-d_1 \int_T^t N^{-b}(\tau) d\tau} \|\bar{p}_t(T)\|^2 + \int_T^t e^{-d_1 \int_T^\tau N^{-b}(\theta) d\theta} \|\psi_t(\tau)\|_{L^\infty(\tau) b}^2 d\tau \right) \right)^{\frac{\mu_6}{2(\mu_6 + 1)}}
\]

Thus,

\[
\|\bar{p}_t(t)\|_{L^\infty(U')} \leq C \left( \left( h(t) e^{-d_1 \int_T^t N^{-b}(\tau) d\tau} \|\bar{p}_t(T)\|^2 + \int_T^t e^{-d_1 \int_T^\tau N^{-b}(\theta) d\theta} \|\psi_t(\tau)\|_{L^\infty(\tau) b}^2 d\tau \right) \right)^{\frac{\mu_6}{2(\mu_6 + 1)}}
\]

Under condition (3.107), applying Lemma A.3, we obtain

\[
\lim_{t \to \infty} \sup_{t \to \infty} \|\bar{p}_t(t)\|_{L^\infty} \leq C \left( \lim_{t \to \infty} \sup_{t \to \infty} \left[ \|\psi_t(\tau)\|_{L^\infty(\tau) b}^2 \right] \right)^{\frac{\mu_6}{2(\mu_6 + 1)}}
\]

\[
+ C \left( \lim_{t \to \infty} \sup_{t \to \infty} \left[ \|\psi_t(\tau)\|_{L^\infty(\tau) b}^2 \right] \right)^{\frac{1}{2}}.
\]

Therefore, we obtain (3.108). □

Note from (3.102) that \( \lim_{t \to \infty} \|\bar{p}_t(t)\|_{L^\infty(U')} = 0 \) provided

\[
\lim_{t \to \infty} \|\psi(t)\|_{L^\infty} = 0 \quad \text{and} \quad \lim_{t \to \infty} \|\psi_t(t)\|_{L^\infty} = 0.
\]

In the following, we can drop the first limit condition.

**Corollary 3.19.** Under the Strict Degree Condition, if \( \|\psi(t)\|_{L^\infty} \) is uniformly bounded on \([0, \infty)\) and \( \|\psi_t(t)\|_{L^\infty} \to 0 \) as \( t \to \infty \), then

\[
\|\bar{p}_t(t)\|_{L^\infty(U')} \to 0 \quad \text{as} \quad t \to \infty. \tag{3.113}
\]

**Proof.** In this case, \( N(t) \) and \( h(t) \) are uniformly bounded on \([2, \infty)\) by a constant \( C_0 \). In the Theorem 3.18, we can replace \( N(t) \) and \( h(t) \) by this \( C_0 \). Then conditions in (3.107) are met and (3.113) follows (3.108). □
3.4. Estimates for pressure’s second derivatives. We estimate the Hessian $\nabla^2 p = \left(\frac{\partial^2 p}{\partial x_i \partial x_j}\right)_{i,j=1,2,\ldots,n}$.

Throughout this subsection $U' \Subset U$.

**Proposition 3.20.** Let $U' \Subset V \Subset U$ and $\delta \in (0, a]$. Then for $t > 0$,

$$
\int_{U'} |\nabla^2 p(x,t)|^{2-\delta} \, dx \leq C \left(1 + \int_{U} |\nabla p(x,t)|^{2-\delta} \, dx\right)^{1/2} \left(1 + \|\bar{p}_t(t)\|_{L^\infty(V)}^2\right),
$$

(3.114)

where the positive constant $C$ depends on $U'$, $V$ and $\delta$.

**Proof.** Young’s inequality gives

$$
\int_{U'} |\nabla^2 p|^2 \, dx \leq \int_{U'} |\nabla p|^2 \, dx + \int_{U'} K(|\nabla p|)|\nabla^2 p|^2 \, dx
\leq \int_{U'} |\nabla p|^2 \, dx + C \int_{U'} |\nabla p|^{2-\delta} \, dx.
$$

(3.115)

Note that for $s \geq 0$, we have

$$
\frac{1}{2s+2} \frac{d}{dt} \int U |\nabla \bar{p}|^{2s+2} \, dx = - \int U \bar{p}_t \nabla \cdot (|\nabla \bar{p}|^{2s} \nabla \bar{p} \zeta) \, dx.
$$

(3.116)

Since $\nabla \bar{p} = \nabla p$, replacing $p$ by $\bar{p}$ in (3.45) and using (3.116) with $s = 0$, we have for any $\zeta(x) \in C_c^\infty(U)$ that

$$
\int U K(|\nabla p|)|\nabla^2 p|^2 \zeta^2 \, dx \leq C \int U K(|\nabla p|)|\nabla p|^2 |\nabla \zeta|^2 \, dx + C \int U \bar{p}_t \nabla \cdot (\nabla p \zeta^2) \, dx.
$$

Let $\varepsilon > 0$. For the last integral, we have

$$
\left|\int U \bar{p}_t \nabla \cdot (\nabla p \zeta^2) \, dx\right| \leq C \int U |\bar{p}_t||\nabla^2 p||\zeta|^2 + |\bar{p}_t||\nabla p||\zeta||\nabla \zeta| \, dx
\leq \varepsilon \int U K(|\nabla p|)|\nabla^2 p|^2 \zeta^2 \, dx + C \varepsilon^{-1} \int U |\bar{p}_t|^2 K(|\nabla p|)^{-1} \zeta^2 \, dx + C \int U |\bar{p}_t||\nabla p||\zeta||\nabla \zeta| \, dx.
$$

Therefore,

$$
(1 - \varepsilon) \int U K(|\nabla p|)|\nabla^2 p|^2 \zeta^2 \, dx \leq C \int U K(|\nabla p|)|\nabla p|^2 |\nabla \zeta|^2 \, dx
$$

$$
+ C \varepsilon^{-1} \int U |\bar{p}_t|^2 (1 + |\nabla p|)^a \zeta^2 \, dx + C \int U |\bar{p}_t||\nabla p||\zeta||\nabla \zeta| \, dx.
$$

(3.117)

Constructing appropriate $\zeta$ with $\zeta \equiv 1$ on $U'$ and supp $\zeta \subset V$, we obtain from (3.117) that

$$
\int_{U'} K(|\nabla p|)|\nabla^2 p|^2 \, dx \leq C \int_V |\nabla p|^{2-a} \, dx + C\|\bar{p}_t(t)\|_{L^\infty(V)}^2 \int_V (1 + |\nabla p|)^a \, dx
$$

$$
+ C\|\bar{p}_t(t)\|_{L^\infty(V)} \int_V |\nabla p| \, dx \leq C \int_V |\nabla p|^{2-a} \, dx + C \left(1 + \int_V |\nabla p| \, dx\right) \left(1 + \|\bar{p}_t\|_{L^\infty(V)}^2\right).
$$

Thus, by Young’s inequality, we have

$$
\int_{U'} K(|\nabla p|)|\nabla^2 p|^2 \, dx \leq C \left(1 + \int_V |\nabla p|^{2-a} \, dx\right) \left(1 + \|\bar{p}_t\|_{L^\infty(V)}^2\right).
$$

(3.118)

Combining (3.115) with (3.118), we obtain

$$
\int_{U'} |\nabla^2 p(x,t)|^{2-\delta} \, dx \leq C \left(1 + \int_U |\nabla p(x,t)|^{2-a} \, dx\right)^{1/2} \left(1 + \|\bar{p}_t(t)\|_{L^\infty(V)}^2\right)
$$

$$
+ C \int_{U'} |\nabla p(x,t)|^{(2-\delta)a/\delta} \, dx.
$$
Then (3.114) follows.

We consider two cases $\delta = a$ and $\delta < a$ separately. We define two exponents
\[ \mu_9 = \frac{2\mu_7}{\mu_6} + 2 + \frac{1}{\mu_6 + 1} \quad \text{and} \quad \mu_{10} = \frac{2 - a}{1 - a} \left( \frac{2\mu_7}{\mu_6} + 2 \right) + \frac{2}{\mu_6 + 1}. \] (3.119)

**Theorem 3.21.** Assume (SDC).

(i) If $t > 0$ then
\[ \int_{U} |\nabla^2 p(x,t)|^{2-a} dx \leq C L_7 (1 + t^{-\frac{1}{\mu_6}})(t + 1)^{\mu_9} K_{1,t} \left( \frac{2\mu_7}{\mu_6} + 2 \right) \left\{ 1 + \sup_{[0,t]} \|\psi_t\|_{L^\infty} \right\}^\mu_{10}, \] (3.120)
where $L_7 = (1 + \|\tilde{p}_0\|_{L^2}^2 + \|\nabla p_0\|_{L^{2-a}}^2 )^\frac{2\mu_7}{\mu_6} + 2$ and $K_{1,t}$ is defined in (3.53).

If $t > 3/2$ then
\[ \int_{U} |\nabla^2 p(x,t)|^{2-a} dx \leq C L_8 \{ 1 + M_f(t) \left( \frac{2\mu_7}{\mu_6} + 2 \right) \} \left\{ 1 + \sup_{[t-3/2,t]} \|\psi_t\|_{L^\infty} \right\}^\mu_{10}, \] (3.121)
where $L_8 = (1 + \|\tilde{p}_0\|_{L^2}^2 )^\frac{2\mu_7}{\mu_6} + 2$.

(ii) If $A < \infty$ then
\[ \limsup_{t \to \infty} \int_{U} |\nabla^2 p(x,t)|^{2-a} dx \leq C A_3^{2\mu_8 + 1}, \] (3.122)
where $A_3$ is defined by (3.100).

(iii) If $\beta < \infty$, then there is $T > 0$ such that for all $t > T$,
\[ \int_{U} |\nabla^2 p(x,t)|^{2-a} dx \leq C \left( 1 + \beta^{\frac{1}{4}} t^{\frac{1}{2}} \sup_{[t-3/2,t]} f \right)^{\frac{2\mu_7}{\mu_6} + 2} \left( 1 + \sup_{[t-3/2,t]} \|\psi_t\|_{L^\infty} \right)^\mu_{10}. \] (3.123)

Above, the positive constant $C$ depends on $U'$.

**Proof.** (i) For $t > 0$, we obtain from (3.114) with $\delta = a$ that
\[ \int_{U} |\nabla^2 p(x,t)|^{2-a} dx \leq C (1 + \|\nabla p\|_{L^{2-a}}^2) (1 + \|\tilde{p}_t\|_{L^\infty(V)}^2). \] (3.124)

Then using estimates (3.88) and (3.96) in (3.124), we obtain
\[ \int_{U} |\nabla^2 p(x,t)|^{2-a} dx \leq C \left\{ 1 + \|\tilde{p}_0\|_{L^2}^2 + \|\nabla p_0\|_{L^{2-a}}^2 + (1 + t) \sup_{[0,t]} f + \int_0^t \tilde{f}(\tau) d\tau \right\}^{\frac{2\mu_7}{\mu_6} + 2}
\cdot \left\{ 1 + t^{-\frac{1}{4\mu_6}} + t^{\frac{1}{2(\mu_6 + 1)}} \sup_{[0,t]} \|\psi_t\|_{L^{\mu_8 + 1}} \right\}^\mu_{10}.
\]

Then (3.120) follows. If $t \geq 3/2$ then using (3.91) and (3.97) in (3.124) we obtain
\[ \int_{U} |\nabla^2 p(x,t)|^{2-a} dx \leq C \left\{ 1 + \|\tilde{p}_0\|_{L^2}^2 + M_f(t) \frac{2}{\mu_6} + \int_{t-3/2}^t \tilde{f}(\tau) d\tau \right\}^{\frac{2\mu_7}{\mu_6} + 2}
\cdot \left\{ 1 + \sup_{[t-1/2,t]} \|\psi_t\|_{L^\infty(t)} \right\}^2.
\]

Then (3.121) follows.

(ii) If $A < \infty$ then using (3.93) and (3.103) in (3.124) we obtain
\[ \limsup_{t \to \infty} \int_{U} |\nabla^2 p(x,t)|^{2-a} dx \leq C A_2 A_3^{2\mu_8}.
\]
This yields (3.122).

(iii) If $\beta < \infty$ then using (3.94) and (3.104) in (3.124) we obtain (3.126).

Next, we treat the case $\delta \in (0, a)$ for which we define the exponents

$$
\nu_1 = \max \{2, \frac{(2 - \delta)a}{\delta} \}, \quad \nu_2 = 2 + \frac{2\mu_7}{\mu_6} + \frac{1}{\mu_6 + 1} + \frac{2(\nu_1 - 2)}{2 - a},
$$

(3.125)

$$
\nu_3 = \frac{2 - a}{1 - a} \left( \frac{2 \mu_7}{\mu_6} + 1 \right) + \frac{\nu_1 - a}{1 - a}, \quad \nu_4 = \frac{2}{2 - a} \left( \frac{2 \mu_7}{\mu_6} + 1 \right) + \mu_4(\nu_1 - 2),
$$

(3.126)

$$
\mu_{11} = \frac{2 - a}{1 - a} \left( \frac{2 \mu_7}{\mu_6} + 1 \right) + \frac{2}{\mu_6 + 1}, \quad \mu_{12} = \frac{2 \mu_7}{\mu_6} + 1 + \frac{1 - a}{2 - a} \frac{2}{\mu_6 + 1}.
$$

(3.127)

Theorem 3.22. Assume (SDC). Let $\delta$ be any number in $(0, a)$.

For $t > 0$, we have

$$
\int_{U'} |\nabla^2 p(x, t)|^{2-\delta} dx \leq CL_0(1 + t^{-\frac{1}{\mu_6}})(t + 1)^\nu_2 K_1(t_2, \sup_{[0, t]} \|\psi_t\|_{L^\infty} + 1)^{\mu_{11}},
$$

(3.128)

where $L_0 = (1 + \|\tilde{p}_0\|^2_{L^2} + \|\nabla p_0\|^2_{L^2-a})L_2(\nu_1)$.

For $t \geq 3/2$, we have

$$
\int_{U'} |\nabla^2 p(x, t)|^{2-\delta} dx \leq CL_{10}(1 + M_f(t))^{\nu_1} \left( 1 + \sup_{[t-3/2, t]} \tilde{f} \right)^{\mu_{12}} \left\{ 1 + \int_{[0, t]} f(\tau) d\tau \right\},
$$

(3.129)

where $L_{10} = (1 + \|\tilde{p}_0\|^2_{L^2})L_4(\nu_1)$.

Above, the positive constant $C$ depends on $U'$ and $\delta$, and $L_2(\cdot)$, $L_4(\cdot)$ are defined in Theorems 3.9 and 3.10.

Proof. We use estimate (3.114) and utilizing (3.55) with $s = \nu_1$ and (3.96) to obtain

$$
\int_{U'} |\nabla^2 p(x, t)|^{2-\delta} dx \leq C \left\{ 1 + \|\tilde{p}_0\|^2_{L^2} + \|\nabla p_0\|^2_{L^2-a} + (1 + t)^{\frac{2 - a}{1 - a} K_1 t + K_2 t} \right\}^{\frac{2 \mu_7}{\mu_6} + 1}
$$

$$
\times \left\{ 1 + t^{-\frac{1}{\mu_6}} + t^{\frac{1}{\mu_6 - \nu_1 + 1}} \sup_{[0, t]} \|\psi_t\|_{L^\infty} \right\}^\nu_1 \left( 1 + t^{-\frac{1}{\mu_6}} + t^{\frac{1}{\mu_6 - \nu_1 + 1}} \sup_{[0, t]} \|\psi_t\|_{L^\infty} \right)^{\frac{2 - a}{1 - a}}
$$

$$
\cdot L_2(\nu_1)(t + 1)^{\frac{2(\nu_1 - 2)}{2 - a} + 1 + \frac{1}{\mu_6 + 1}}.
$$

Then (3.128) follows. For $t \geq 3/2$, using (3.60) for $s = \nu_1$ instead of (3.55) and using (3.97) instead of (3.96), we obtain

$$
\int_{U'} |\nabla^2 p(x, t)|^{2-\delta} dx \leq C \left\{ 1 + \|\tilde{p}_0\|^2_{L^2} + M_f(t)^{\frac{2 - a}{\mu_6}} + \int_{[t-3/2, t]} \tilde{f}(\tau) d\tau \right\}^{\frac{2 \mu_7}{\mu_6} + 1}
$$

$$
\times \left\{ 1 + \sup_{[t-1/2, t]} \|\psi_t\|_{L^\infty} \right\}^\nu_1 \left( 1 + \sup_{[t-1/2, t]} \|\psi_t\|_{L^\infty} \right)^{\frac{2 - a}{\mu_6 + 1}}
$$

$$
\cdot L_4(\nu_1)(1 + M_f(t))^{\mu_4(\nu_1 - 2)} \left\{ 1 + \int_{[0, t]} f(\tau) d\tau \right\}.
$$

Note that $\|\psi_t(\cdot)\|_{L^\infty} \leq C(1 + \tilde{f}(\cdot))^{\frac{1-a}{2-a}}$. Then we obtain (3.129). \qed

For asymptotic estimates, we have the following.

Theorem 3.23. Assume (SDC). Suppose $\Upsilon_1$ and $\Upsilon_2$ defined in Corollary 3.11 are finite numbers. Then for any $\delta \in (0, a)$, we have

$$
\limsup_{t \to \infty} \int_{U'} |\nabla^2 p(x, t)|^{2-\delta} dx \leq CL_4(\nu_1)^{\Upsilon_1(\nu_1 - 2)} \Upsilon_2 A_3^{2 \mu_8},
$$

(3.130)

where $C > 0$ depends on $U'$ and $\delta$. 

Proof. Taking limit superior of (3.114) as $t \to \infty$ and using (3.65) for $s = \nu_1$, and using (3.103), we obtain (3.130). \hfill \square

4. DEPENDENCE ON INITIAL AND BOUNDARY DATA

In this section, we prove the continuous dependence of solutions $p(x,t)$ of the IBVP (3.1), (3.2) and (3.4) with respect to the $L^\infty$-norm on the initial data and boundary data. The results are established for either finite time intervals or at time infinity.

Let $p_1(x,t)$ and $p_2(x,t)$ be two solutions of the IBVP (3.7) having fluxes $\psi_1$ and $\psi_2$, and initial data $p_1(x,0)$ and $p_2(x,0)$, respectively. Let $\Psi = \psi_1 - \psi_2$, $P = p_1 - p_2$, and $\bar{P} = P - |U|^{-1} \int_U Pdx$. Then by (3.6),

$$\bar{P}(x,t) = \bar{p}_1(x,t) - \bar{p}_2(x,t) = P(x,t) - \frac{1}{|U|} \int_U P(x,0)dx + \frac{1}{|U|} \int_0^t \int_\Gamma \Psi(x,\tau)d\sigma d\tau.$$

We will estimate $\|\bar{P}\|_{L^\infty(U')}$ where the subset $U'$ satisfies $U' \subseteq U$ throughout the section. From (3.7) follows

$$\frac{\partial \bar{P}}{\partial t} = \nabla \cdot (K(\nabla \bar{p}_1) \nabla \bar{p}_1) - \nabla \cdot (K(\nabla \bar{p}_2) \nabla \bar{p}_2) + \frac{1}{|U|} \int_\Gamma \Psi(x, t)d\sigma. \quad (4.1)$$

The following quantity will be used throughout this section:

$$\Lambda(t) = 1 + \int_U \left( |\nabla p_1(x,t)|^{2-\alpha} + |\nabla p_2(x,t)|^{2-\alpha} \right) dx,$$

4.1. Results for pressure. First we establish $L^\infty$-estimates for $\bar{P}$ in terms of its $L^2$- and $W^{1,\infty}$-norms. We recall that the exponents $\mu_5$, $\mu_6$, and $\mu_7$ are defined in (3.69).

**Proposition 4.1.** Assume [SDC]. Let $U' \subseteq U$. Let $\mu > \mu_6^{-1} = \frac{\mu_5}{\mu_5 - 2}$ and denote

$$\gamma_1 = \mu_6 - \frac{1}{\mu} = 1 - \frac{2}{\mu_5} - \frac{1}{\mu} \in (0,1). \quad (4.2)$$

Then we have for any $T_0 \geq 0$, $T > 0$ and $\theta \in (0,1)$ that

$$\sup_{[T_0 + \theta T, T_0 + T]} \|\bar{P}\|_{L^\infty(U')} \leq C_{T_0, T, \theta} \left( \left( \frac{2^{\gamma_1}}{\gamma_1} \right)^{\frac{1}{\gamma_1 + \tau}} + \left( \lambda \left[ 1 + \frac{1}{\theta T} \right]^{1/2} \right)^{\frac{1}{\mu_6}} + \left( \lambda T^{1/2} \sup_{[T_0, T_0 + T]} \|\Psi\| \right)^{\frac{1}{\mu_6 + \tau}} \right), \quad (4.3)$$

where the positive constant $C$ is independent of $T_0$, $T$ and $\theta$,

$$C_{T_0, T, \theta} = (\lambda T^{1/2} \theta)^{\frac{1}{\gamma_1 + \tau}} + \left( \lambda \left[ 1 + \frac{1}{\theta T} \right]^{1/2} \right)^{\frac{1}{\mu_6}} + \left( \lambda T^{1/2} \sup_{[T_0, T_0 + T]} \|\Psi\| \right)^{\frac{1}{\mu_6 + \tau}}, \quad (4.4)$$

with

$$\lambda = \lambda_{T_0, T} \overset{\text{def}}{=} \sup_{t \in [T_0, T_0 + T]} \Lambda(t)^{\mu_7}, \quad (4.5)$$

$$\theta = \theta_{T_0, T} \overset{\text{def}}{=} \left[ \int_{T_0}^{T_0 + T} \int_V \left( |\nabla \bar{p}_1|^{2\mu(1-\alpha)} + |\nabla \bar{p}_2|^{2\mu(1-\alpha)} \right) dx dt \right]^{\frac{1}{2\mu}}. \quad (4.6)$$

**Proof.** Without loss of generality, we assume $T_0 = 0$. Let $\zeta(x,t) = \phi(x)\varphi(t)$ be a cut-off function with $\varphi(0) = 0$ and $\text{supp } \phi \subseteq V$. Same as in Proposition 3.1 we define

$$\bar{P}^{(k)} = \max\{\bar{P} - k, 0\}, \quad S_k(t) = \{ x \in U : \bar{P}^{(k)}(x,t) \geq 0 \},$$

and denote by $\chi_k(x,t)$ the characteristic function of $S_k(t)$. \hfill \square
Multiplying equation \((4.1)\) by \(\bar{P}^{(k)}\zeta^2\), integrating it over \(U\) and using integration by parts, we have

\[
\frac{1}{2} \frac{d}{dt} \int_U |\bar{P}^{(k)}\zeta|^2 dx = \int_U |\bar{P}^{(k)}\zeta|^2 \zeta dx - \int_U \left( K(\nabla \bar{p}_1) \nabla \bar{p}_1 - K(|\nabla \bar{p}_2|) \nabla \bar{p}_2 \right) \cdot \nabla (\bar{P}^{(k)}\zeta^2) dx + \frac{1}{|U|} \int_G \Psi(x, t) d\sigma \int_U \bar{P}^{(k)}\zeta^2 dx.
\]

Elementary calculations yield

\[
\frac{1}{2} \frac{d}{dt} \int_U |\bar{P}^{(k)}\zeta|^2 dx \leq \int_U |\bar{P}^{(k)}\zeta| |\zeta_i| dx - \int_U \left( K(\nabla \bar{p}_1) \nabla \bar{p}_1 - K(|\nabla \bar{p}_2|) \nabla \bar{p}_2 \right) \cdot \nabla \bar{P}^{(k)}\zeta^2 dx - 2 \int_U \left( K(\nabla \bar{p}_1) \nabla \bar{p}_1 - K(|\nabla \bar{p}_2|) \nabla \bar{p}_2 \right) \cdot \bar{P}^{(k)}\zeta \nabla \zeta dx + \frac{1}{|U|} \int_G \Psi(x, t) d\sigma \int_U \bar{P}^{(k)}\zeta^2 dx.
\]

Denoting the last four summands by \(I_i, i = 1, 2, 3, 4\), respectively, we rewrite the above as

\[
\frac{1}{2} \frac{d}{dt} \int_U |\bar{P}^{(k)}\zeta|^2 dx \leq I_1 + I_2 + I_3 + I_4. \tag{4.7}
\]

Let \(\xi(x, t) = |\nabla \bar{p}_1| \vee |\nabla \bar{p}_2|\). We consider \(I_2\). Let \(J(x, t) = \left( K(|\nabla \bar{p}_1|) \nabla \bar{p}_1 - K(|\nabla \bar{p}_2|) \nabla \bar{p}_2 \right) \cdot \nabla \bar{P}^{(k)}\). On the set \(U \setminus S_k(t)\), since \(\nabla P^{(k)} = 0\) a.e., we have \(J(x, t) = 0\) a.e.. On the set \(S_k(t)\), we have \(\nabla P^{(k)} = \nabla \bar{P}\) a.e., and, by the monotonicity \((2.19)\), we have for almost all \(x \in S_k(t)\) that

\[
J(x, t) = \left( K(|\nabla \bar{p}_1|) \nabla \bar{p}_1 - K(|\nabla \bar{p}_2|) \nabla \bar{p}_2 \right) \cdot (\nabla \bar{p}_1 - \nabla \bar{p}_2) \geq (1 - a) K(|\nabla \bar{p}_1| \vee |\nabla \bar{p}_2|) |\nabla \bar{P}|^2 = (1 - a) K(\xi) |\nabla P^{(k)}|^2.
\]

Therefore,

\[
I_2 \leq -(1 - a) \int_U K(\xi) |\nabla \bar{P}^{(k)}|^2 \zeta^2 dx. \tag{4.8}
\]

Note that

\[
|\zeta \nabla \bar{P}^{(k)}|^2 = |\nabla (\bar{P}^{(k)}\zeta) - \bar{P}^{(k)} \nabla \zeta|^2 = |\nabla (\bar{P}^{(k)}\zeta)|^2 - 2 \nabla (\bar{P}^{(k)}\zeta) \cdot (\bar{P}^{(k)} \nabla \zeta) + |\bar{P}^{(k)} \nabla \zeta|^2.
\]

Thus, \((4.8)\) gives

\[
I_2 \leq -(1 - a) \int_U K(\xi) |\nabla (\bar{P}^{(k)}\zeta)|^2 dx + 2(1 - a) \int_U K(\xi) |\nabla (\bar{P}^{(k)}\zeta)||P^{(k)} \nabla \zeta| dx
\]

\[
- (1 - a) \int_U K(\xi) |P^{(k)} \nabla \zeta|^2 dx.
\]

Using Cauchy’s inequality for the second term on the right-hand side of the previous inequality gives

\[
I_2 \leq -\frac{1 - a}{2} \int_U K(\xi) |\nabla (\bar{P}^{(k)}\zeta)|^2 dx + (1 - a) \int_U K(\xi) |\bar{P}^{(k)} \nabla \zeta|^2 dx
\]

\[
\leq -\frac{1 - a}{2} \int_U K(\xi) |\nabla (\bar{P}^{(k)}\zeta)|^2 dx + C \int_U |\bar{P}^{(k)} \nabla \zeta|^2 dx. \tag{4.9}
\]
For the last inequality, we used the fact that \( K(\xi) \) is bounded above. For \( I_3 \) and \( I_4 \), we have for any \( \varepsilon > 0 \) that

\[
I_3 \leq C \int_U |\bar{P}^{(k)}|(|\nabla \bar{p}_1|^{1-a} + |\nabla \bar{p}_2|^{1-a})\xi|\nabla \xi| \, dx \\
\leq \varepsilon \int_U |\bar{P}^{(k)}| \xi^2 \, dx + C\varepsilon^{-1} \int_U (|\nabla \bar{p}_1|^{1-a} + |\nabla \bar{p}_2|^{1-a})^2 \chi_k|\nabla \xi|^2 \, dx, \tag{4.10}
\]

\[
I_4 \leq \varepsilon \int_U |\bar{P}^{(k)}| \xi^2 \, dx + C\varepsilon^{-1} \|\Psi(t)\|_{L^\infty}^2 \int_U \chi_k \xi^2 \, dx. \tag{4.11}
\]

Combining (4.7), (4.9), (4.10) and (4.11) yields

\[
\frac{1}{2} \frac{d}{dt} \int_U |\bar{P}^{(k)}| \xi^2 \, dx + \frac{1-a}{2} \int_U K(\xi)|\nabla \bar{P}^{(k)}| \xi^2 \, dx \leq C \int_U |\bar{P}^{(k)}|^2 (|\xi| \xi + |\nabla \xi|) \, dx \\
+ 2\varepsilon \int_U |\bar{P}^{(k)}| \xi^2 \, dx + C\varepsilon^{-1} \|\Psi(t)\|_{L^\infty}^2 \int_U \chi_k \xi^2 \, dx + C\varepsilon^{-1} \int_U (|\nabla \bar{p}_1|^{1-a} + |\nabla \bar{p}_2|^{1-a})^2 \chi_k|\nabla \xi|^2 \, dx.
\]

Integrating in time, taking the supremum in \( t \) over \([0, T]\), and selecting \( \varepsilon = 1/(16T) \) we find that

\[
\sup_{[0,T]} \|\bar{P}^{(k)}| \xi|^2 \|_{L^2(U)} + \int_0^T \int_U K(\xi)|\nabla \bar{P}^{(k)}| \xi| \, dx \, dt \leq C \int_0^T \int_U |\bar{P}^{(k)}|^2 (|\xi| \xi + |\nabla \xi|) \, dx \, dt \\
+ CT \sup_{[0,T]} \|\Psi\|_{L^\infty}^2 \int_0^T \int_U \chi_k \xi^2 \, dx \, dt + CT \int_0^T \int_U (|\nabla \bar{p}_1|^{1-a} + |\nabla \bar{p}_2|^{1-a})^2 \chi_k|\nabla \xi|^2 \, dx \, dt.
\]

Applying Hölder’s inequality to the last double integral yields

\[
\sup_{[0,T]} \|\bar{P}^{(k)}| \xi|^2 \|_{L^2(U)} + \int_0^T \int_U K(\xi)|\nabla \bar{P}^{(k)}| \xi|^2 \, dx \, dt \\
\leq C \int_0^T \int_U |\bar{P}^{(k)}|^2 (|\xi| \xi + |\nabla \xi|) \, dx \, dt + CT \sup_{[0,T]} \|\Psi\|_{L^\infty}^2 \int_0^T \int_U \chi_k \xi^2 \, dx \, dt \\
+ CT \left( \int_0^T \int_U (|\nabla \bar{p}_1|^{1-a} + |\nabla \bar{p}_2|^{1-a})^2 \mu \chi_2 \xi^2 \, dx \, dt \right)^{1/2} \left( \int_{Q_T \cap \text{supp} \xi} \chi_k \, dx \right)^{-1/2}.
\tag{4.12}
\]

Under the Strict Degree Condition, by applying Sobolev inequality \( 2, 30 \) with \( W(x, t) = K(\xi(x, t)) \), we have

\[
\|\bar{P}^{(k)}| \xi| \|_{L^\mu(Q_T)} \leq C\lambda \left( \sup_{[0,T]} \|\bar{P}^{(k)}| \xi|^2 \|_{L^2(U)} + \int_0^T \int_U K(\xi)|\nabla \bar{P}^{(k)}| \xi|^2 \, dx \, dt \right)^{1/2},
\]

where \( \lambda \) is defined by (4.5). Combining with (4.12), we have

\[
\|\bar{P}^{(k)}| \xi| \|_{L^\mu(Q_T)} \leq C\lambda \left( \left( \int_0^T \int_U |\bar{P}^{(k)}|^2 (|\xi| \xi + |\nabla \xi|)^2 \, dx \, dt \right)^{1/2} \\
+ T^{1/2} \sup_{[0,T]} \|\Psi\|_{L^\infty} \left( \int_{Q_T \cap \text{supp} \xi} \chi_k \, dx \right)^{1/2} + T^{1/2} \vartheta \left( \int_{Q_T \cap \text{supp} \xi} \chi_k \, dx \right)^{1/2} \right), \tag{4.13}
\]

where \( \vartheta \) is defined by (4.6).

Let \( M_0 > 0 \) be fixed which we will determine later. For integers \( i \geq 0 \), let \( k_i = M_0(1 - 2^{-i}) \) and where \( \xi_t \) be defined as in (3.7) and the sets \( Q_i, A_{i,j} \) be defined similar to (3.75) replacing \( p_t \) by \( P \).
Define $F_i = \| \bar{P}^{(k_{i+1})} \zeta_i \|_{L^{p_0}(A_{i+1}, \cdot)}$. Applying (4.13) with $k = k_{i+1}$ and $\zeta = \zeta_i$ we have

$$F_i \leq C\lambda \left[ \left( 2^{k_{i+1}}(\theta T)^{-1/2} + 2 \right) \left( \| \bar{P}^{(k_{i+1})} \|_{L^2(A_{i+1}, \cdot)} + 2T^{1/2}\| \partial_i \Psi \|_{L^\infty[A_{i+1}, \cdot]} \right)^{1/2} \right] + T^{1/2} \| A_{i+1,i} \|_{L^\infty[A_{i+1}, \cdot]}^{1/2}. \quad (4.14)$$

Estimating the same way as in (5.79), we have

$$\| \bar{P}^{(k_{i+1})} \zeta_i \|_{L^2(A_{i+1}, \cdot)} \leq CF_i \| A_{i+1,i} \|_{L^\infty[A_{i+1}, \cdot]}^{1/2 - 1/\mu_5}. \quad (4.15)$$

Also, similar to (3.80), we have

$$|A_{i+1,i}| \leq C4^{i}M_0^{-2}\| \bar{P}^{(k_i)} \|_{L^2(A_i)}^{2}. \quad (4.16)$$

From estimates (4.14), (4.15), and the boundedness of $\zeta_i$, we obtain

$$\| \bar{P}^{(k_{i+1})} \|_{L^2(A_{i+1}, \cdot)} = \| \bar{P}^{(k_{i+1})} \zeta_i \|_{L^2(A_{i+1}, \cdot)} \leq C\lambda \left[ \left( 2^{k_i}(\theta T)^{-1/2} + 1 \right) \| \bar{P}^{(k_{i+1})} \|_{L^2(A_{i+1}, \cdot)} + 4T^{1/2}\| \partial_i \Psi \|_{L^\infty[A_{i+1}, \cdot]} \right] + T^{1/2}2^{i}M_0^{-1} \sup_{[0,T]} \| \Psi \|_{L^\infty[A_{i+1}, \cdot]}\| \bar{P}^{(k_i)} \|_{L^2(A_i)}^{-1/\mu} \left( \| \bar{P}^{(k_i)} \|_{L^2(A_i)}^{1/\mu} \right) \left( \| \bar{P}^{(k_i)} \|_{L^2(A_i)}^{1/\mu} \right).$$

Using (4.16) gives

$$\| \bar{P}^{(k_{i+1})} \|_{L^2(A_{i+1}, \cdot)} \leq C\lambda \left[ \left( 2^{k_i}(\theta T)^{-1/2} + 1 \right) \| \bar{P}^{(k_{i+1})} \|_{L^2(A_{i+1}, \cdot)} + 4T^{1/2}\| \partial_i \Psi \|_{L^\infty[A_{i+1}, \cdot]} \right] + T^{1/2}2^{i}M_0^{-1} \sup_{[0,T]} \| \Psi \|_{L^\infty[A_{i+1}, \cdot]}\| \bar{P}^{(k_i)} \|_{L^2(A_i)}^{-1/\mu} \left( \| \bar{P}^{(k_i)} \|_{L^2(A_i)}^{1/\mu} \right) \left( \| \bar{P}^{(k_i)} \|_{L^2(A_i)}^{1/\mu} \right).$$

Put $Y_i = \| \bar{P}^{(k_{i+1})} \|_{L^2(A_i)}$, the previous inequality gives

$$Y_{i+1} \leq C\lambda M_0^{-1}Y_i^{-1/\mu} + \left( 1 + \frac{1}{\theta T} \right)Y_i^{-1/\mu} + \left( 1 + \frac{1}{\theta T} \right)Y_i^{-1/\mu} \sup_{[0,T]} \| \Psi \|_{L^\infty[A_{i+1}, \cdot]} \left( \| \bar{P}^{(k_i)} \|_{L^2(A_i)} \right).$$

Since $Y_0 \leq \| \bar{P} \|_{L^2(U \times (0,T))}$, aiming at applying Lemma A.2 with $m = 3$ we choose $M_0$ sufficiently large such that

$$\| \bar{P} \|_{L^2(U \times (0,T))} \leq C \min \left\{ \lambda T^{1/2}(\theta T)^{-1/\gamma_i}M_0^{1/\gamma_i}, \lambda \left( 1 + \frac{1}{\theta T} \right)^{1/\mu_6}M_0^{1/\mu_6} \right\}. \quad (4.16)$$

Thus, we require

$$M_0 \geq C(\lambda T^{1/2}(\theta T)^{-1/\gamma_i}M_0^{1/\gamma_i}) \quad \text{and} \quad M_0 \geq C(\lambda \left( 1 + \frac{1}{\theta T} \right)^{1/\mu_6}M_0^{1/\mu_6}).$$

We choose

$$M_0 \geq C \left( \lambda T^{1/2}(\theta T)^{-1/\gamma_i} \right) \left( \lambda \left( 1 + \frac{1}{\theta T} \right)^{1/\mu_6} \right) \left( \lambda T^{1/2} \sup_{[0,T]} \| \Psi \|_{L^\infty[A_{i+1}, \cdot]} \right).$$

(4.17)
Applying Lemma A.2, we have

\[
M_0 = C \left( (\lambda T^{1/2} \vartheta)^{\frac{1}{\gamma_1+1}} + (\lambda(1 + \frac{1}{\theta T})^{1/2})^{1/\mu_6} + (\lambda T^{1/2} \sup_{[0,T]} \|\Psi\|)^{\frac{1}{\mu_6+1}} \right) 
\cdot \left( \|\bar{P}\|_{L^2(U \times (0,T))} + \|\bar{F}\|_{L^2(U \times (0,T))} \right).
\]

(4.18)

Applying Lemma A.2 and using the same arguments as in Proposition 3.2, we obtain (4.3).

We now derive a variation of Proposition 4.1. Let \( T_0 = 0 \). In the proof of Proposition 4.1, let \( \zeta = \zeta(x) \) be the cut-off function such that \( \zeta \) vanishes in neighborhood of the boundary \( \Gamma \). If \( k \geq \|\bar{P}_0\|_{L^\infty} \), then \( \bar{P}^{(k)}(0) = 0 \) and, hence, (4.13) holds true with \( \zeta_t = 0 \). In the iteration process, we choose \( M_0 \geq \|\bar{P}(\cdot, 0)\|_{L^\infty} \) and \( k_i = M_0(2 - 2^{-i}) \) for \( i \geq 0 \). This way, we have \( k_i \geq M_0 \geq \|\bar{P}_0\|_{L^\infty} \), thus, \( \bar{P}^{(k_i)}(0) = 0 \). Therefore, we can replace \( \frac{1}{\theta T} \) by 0 in (4.17), which becomes

\[
M_0 \geq C \left( (\lambda T^{1/2} \vartheta)^{\frac{1}{\gamma_1+1}} + (\lambda T^{1/2} \sup_{[0,T]} \|\Psi\|)^{\frac{1}{\mu_6+1}} \right) 
\cdot \left( \|\bar{P}\|_{L^2(U \times (0,T))} + \|\bar{P}\|_{L^2(U \times (0,T))} \right).
\]

Therefore, instead of (4.18), we can select

\[
M_0 = \|\bar{P}(0)\|_{L^\infty} 
+ C \left( (\lambda T^{1/2} \vartheta)^{\frac{1}{\gamma_1+1}} + (\lambda T^{1/2} \sup_{[0,T]} \|\Psi\|)^{\frac{1}{\mu_6+1}} \right) \left( \|\bar{P}\|_{L^2(U \times (0,T))} + \|\bar{P}\|_{L^2(U \times (0,T))} \right).
\]

Applying Lemma A.2, we have \( \lim_{i \to \infty} Y_i = 0 \) and \( \bar{p}(x, t) \leq 2M_0 \) a.e. in \( B(x_0, \rho/4) \times (\theta T, T) \). Proceeding the proof as in Proposition 3.2, we obtain the following result.

**Proposition 4.2.** Assume the same as in Proposition 4.1. Then we have for any \( T > 0 \) that

\[
\sup_{[0,T]} \|\bar{P}\|_{L^\infty(U)} \leq 2\|\bar{P}(0)\|_{L^\infty} + C C_T \left( \|\bar{P}\|_{L^2(U \times (0,T))} + \|\bar{P}\|_{L^2(U \times (0,T))} \right),
\]

(4.19)

where

\[
C_T = (\lambda_T T^{1/2} \vartheta_T)^{\frac{1}{\gamma_1+1}} + \lambda_T^{\frac{1}{\mu_6}} + \left( \lambda_T T^{1/2} \sup_{[0,T]} \|\Psi\| \right)^{\frac{1}{\mu_6+1}},
\]

(4.20)

with \( \lambda_T = \lambda_{0,T} \) and \( \vartheta_T = \vartheta_{0,T} \) defined in Theorem 4.1 for \( T_0 = 0 \).

We will use Propositions 4.1 and 4.2 to obtain specific \( L^\infty \)-estimates of \( \bar{P} \) in terms of initial and boundary data. First, we introduce some quantities and parameters.

Same as (3.24), we define for \( i = 1, 2 \),

\[
f_i(t) = \|\psi_i(t)\|_{L^\infty} + \|\psi_i(t)\|_{L^\infty}^{\frac{2-a}{2-a}} \quad \text{and} \quad \tilde{f}_i(t) = \|\psi_{i t}(t)\|_{L^\infty} + \|\psi_{i t}(t)\|_{L^\infty}^{\frac{2-a}{2-a}}.
\]

For \( i = 1, 2 \), we assume \( f_i(t), \tilde{f}_i(t) \in C([0, \infty)) \) and when needed \( f_i(t) \in C^1((0, \infty)) \). Let

\[
A_i = \lim_{t \to \infty} \sup_{t \to \infty} f_i(t) \quad \text{and} \quad \beta_i = \lim_{t \to \infty} \sup_{t \to \infty} \left[ f'_i(t) \right],
\]

\[
\bar{A} = A_1 + A_2 \quad \text{and} \quad \bar{\beta} = \beta_1 + \beta_2.
\]

Let \( M_{f_i}(t), i = 1, 2 \), be a continuous increasing majorant of \( f_i(t) \) on \([0, \infty)\). Set

\[
F(t) = f_1(t) + f_2(t), \quad M_{F}(t) = M_{f_1}(t) + M_{f_2}(t) \quad \text{and} \quad \tilde{F}(t) = \tilde{f}_1(t) + \tilde{f}_2(t).
\]
For initial data, set
\[ A_0 = \|\tilde{p}_1(0)\|_{L^2}^2 + \|\tilde{p}_2(0)\|_{L^2}^2 \quad \text{and} \quad B_0 = J_H[p_1](0) + J_H[p_2](0). \] (4.21)

We recall results from [18].

**Theorem 4.3** (cf. [18], Lemma 5.5 and Theorem 5.6). (i) For \( t \geq 0 \),
\[ \|\tilde{P}(t)\|_{L^2}^2 \leq \|\tilde{P}(0)\|_{L^2}^2 + C \int_0^t \|\Psi(\tau)\|_{L^\infty}^2 \Lambda(\tau)^b d\tau, \] (4.22)
where \( b \) is defined in (2.9). Consequently, for any \( T > 0 \),
\[ \sup_{[0,T]} \|\tilde{P}\|_{L^2}^2 \leq \|\tilde{P}(0)\|_{L^2}^2 + C : M_{1,T} \sup_{[0,T]} \|\Psi(t)\|_{L^\infty}^2, \] (4.23)
where \( M_{1,T} = A_0 + T + \int_0^T \left[ f_1(\tau) + f_2(\tau) \right] d\tau. \)

(ii) Assume the Degree Condition. Then for \( t \geq 0 \)
\[ \|\tilde{P}(t)\|_{L^2}^2 \leq e^{-d_2 \int_0^t \Lambda_-(\tau)^b d\tau} \|\tilde{P}(0)\|_{L^2}^2 + \int_0^t e^{-d_2 \int_0^\tau \Lambda_-(\theta)^b d\theta} \|\Psi(\tau)\|_{L^\infty}^2 \Lambda(\tau)^b d\tau. \] (4.24)
Moreover, if \( \tilde{A} < \infty \) and \( \int_1^\infty (1 + \int_{\tau-1}^{\tau} \tilde{F}(s) ds)^{-b} d\tau = \infty \) then
\[ \limsup_{t \to \infty} \|\tilde{P}(t)\|_{L^2}^2 \leq C \limsup_{t \to \infty} \left\{ \|\Psi(t)\|_{L^\infty}^2 \left( 1 + \tilde{A} \frac{2}{\tau-a} + \int_{t-1}^{t} \tilde{F}(\tau) d\tau \right)^{2b} \right\}. \] (4.25)

We now state the continuous dependence in interior \( L^\infty \)-norm. We use similar quantities to \( K_{1,t} \) in (3.53) and \( K_{2,t} \) (3.55), namely,
\[ \tilde{K}_{1,T} = 1 + \sum_{i=1,2} \sup_{[0,T]} \|\psi_i\|_{L^\infty} \quad \text{and} \quad \tilde{K}_{2,T} = 1 + \int_0^T \tilde{F}(\tau) d\tau. \] (4.26)

**Theorem 4.4.** Assume (SDC). Let \( \mu \) be any number satisfying
\[ \mu > \frac{\mu_5}{\mu_5 - 2} \quad \text{and} \quad \mu \geq \frac{2 - a}{2(1 - a)}, \] (4.27)
For \( T > 0 \), we have
\[ \sup_{[0,T]} \|\tilde{P}\|_{L^\infty(U')} \leq 2\|\tilde{P}(0)\|_{L^\infty} + C L_{11} M_{2,T} \left( \|\tilde{P}(0)\|_{L^2} + \sup_{[0,T]} \|\Psi(t)\|_{L^\infty} + \|\tilde{P}(0)\|_{L^2} + \sup_{[0,T]} \|\Psi(t)\|_{L^\infty} \right)^{\frac{\gamma_1}{\gamma_1 + 1}}, \] (4.28)
where \( \gamma_1 \) is defined by (4.2), number \( L_{11} > 0 \) depends on the initial data and is defined by (4.44) below, number \( M_{2,T} > 0 \) depends on the boundary data and is defined by (4.32) below.

**Proof.** Many exponents will be needed in our proof and are defined here:
\[ \gamma_2 = 2(1 - a) \mu \geq 2 - a, \quad \gamma_3 = \frac{\gamma_2}{2\mu(1 - a)} = 1, \] (4.29)
\[ \gamma_4 = \frac{1}{2\mu} \left( \frac{2\gamma_2}{2 - a} - 1 \right) = \frac{2(1 - a)}{2 - a} - \frac{1}{2\mu}, \quad \gamma_5 = \frac{\mu_5}{\mu_6} + \frac{2\gamma_4 + 1}{2(\gamma_1 + 1)}, \] (4.30)
\[ \gamma_6 = \frac{\mu_7(2 - a)}{\mu_6(1 - a)} + \frac{\gamma_3}{\gamma_1 + 1}. \] (4.31)
We will use notation $\gamma_3$ in calculations below instead of its explicit value for the sake of generality which will be needed in section 5. We prove (4.28) with $M_{2,T}$ explicitly given by

$$M_{2,T} = T^{\mu_{20}}K_{1,T}^{\gamma_6 + \frac{2-a}{2(1-a)}}K_{2,T}^{\lambda_7},$$

where the exponent $\mu_{20}$ is $\frac{\gamma_7}{2(1-a)}$ in case $T \leq 1$ and is $\gamma_5 + 1$ in case $T > 1$.

We will apply Proposition 4.1 for $T_0 = 0$ and Proposition 4.2. Fix a subset $V$ of $U$ such that $U' \subseteq V \subseteq U$. First, for $M_{1,T}$ in (4.23), we note that

$$1 + M_{1,T} \leq C\left\{ 1 + \sum_{i=1}^{2} \|\bar{p}_i(0)\|_{L^2}^2 + (T + 1)^{\frac{\gamma_7}{2(1-a)}} \sum_{i=1}^{2} \sup_{[0,T]} \|\psi_i\|^{\frac{2-a}{2(1-a)}} + 1 \right\} \leq C\ell_0(T + 1)K_{1,T}^{\frac{2-a}{2(1-a)}},$$

(4.33)

where $\ell_0 = 1 + A_0$. By (4.23) and (4.33), we have

$$\|\bar{P}\|_{L^2(U \times (0,T))} + \|\bar{P}\|_{L^2(U \times (0,T))} \leq C\ell_0^{1/2}(T + 1)^{1/2}K_{1,T}^{\frac{2-a}{2(1-a)}} \cdot \left( (T^{1/2}(\|\bar{P}(0)\|_{L^2} + \sup_{[0,T]} \|\Psi(t)\|_{L^\infty})^\frac{\gamma_7}{2(1-a)} + T^{1/2}(\|\bar{P}(0)\|_{L^2} + \sup_{[0,T]} \|\Psi(t)\|_{L^\infty})^\frac{\gamma_7}{2(1-a)} \right)^{1/2}.$$

(4.34)

Second, we will estimate $C_{T_0,T,\theta}$ in (4.3) with $T_0 = 0$, and $C_T$ in (4.20). To simplify our calculations, we will replace $C_{T_0,T,\theta}$ in (4.4) by the following upper bound

$$C_{T,\theta} = \lambda_T \left\{ (T^{1/2}\vartheta_T)^{\frac{1}{2(1-a)}} + (1 + \frac{1}{(\theta T)^{1/2}})^{\frac{1}{\mu_6}} + (T^{1/2}\sup_{[0,T]} \|\Psi\|_{L^\infty})^{\frac{1}{\mu_6+1}} \right\},$$

(4.35)

and replace $C_T$ in (4.20) by

$$C_T = \lambda_T \left\{ 1 + (T^{1/2}\vartheta_T)^{\frac{1}{2(1-a)}} + (T^{1/2}\sup_{[0,T]} \|\Psi\|_{L^\infty})^{\frac{1}{\mu_6+1}} \right\}.$$

(4.36)

We now find bounds for involved quantities in (4.35) and (4.36). We have from estimate (3.88) that

$$\lambda_T = \sup_{t \in [0,T]} \Lambda(t)^{\mu_7} \leq C \sum_{i=1}^{2} \left\{ 1 + \|\bar{p}_i(0)\|_{L^2}^2 + J_H(\bar{p}_i(0)) + (T + 1)^{\sup_{[0,T]} \|\psi_i\|^{\frac{2-a}{2(1-a)}}} + \int_{0}^{T} f_1(t)\vartheta_T \right\}^{\mu_7}.$$

Therefore,

$$\lambda_T \leq C\ell_1(T + 1)^{\mu_7}K_{1,T}^{\frac{\gamma_7(2-a)}{2(1-a)}}K_{2,T}^{\lambda_7},$$

(4.37)

where $\ell_1 = (1 + A_0 + B_0)^{2-a}$. Applying (3.66) to $s = \gamma_2$ and $V$ replacing $U'$, we have

$$\vartheta_T = \left( \int_{0}^{T} \int_{V} \left[ 1 + |\nabla\bar{p}_1|^2 + |\nabla\bar{p}_2|^2 \right] dx dt \right)^{\frac{1}{\mu_7}} \leq C\ell_2(T + 1)^{\gamma_4}K_{1,T}^{\gamma_7},$$

(4.38)

where $\ell_2 = \left\{ \sum_{i=1,2} \gamma_1(\gamma_2 + a; [p_i(0)]) \right\}^{\frac{1}{\mu_7}}$. Also,

$$\sup_{[0,T]} \|\Psi\|_{L^\infty} \leq (\sum_{i=1}^{2} \sup_{[0,T]} \|\psi_i\|_{L^\infty}) \leq K_{1,T}.$$

(4.39)

We denote $D_T = \|\bar{P}(0)\|_{L^2(U \times (0,T))} + \|\bar{P}(0)\|_{L^2(U \times (0,T))} + \sup_{[0,T]} \|\Psi(t)\|_{L^\infty} + \sup_{[0,T]} \|\Psi(t)\|_{L^\infty}$.

We consider $0 < T \leq 1$ first. By (4.34) and (4.38), we respectively have

$$\|\bar{P}\|_{L^2(U \times (0,T))} \leq C\ell_0^{1/2}K_{1,T}^{\frac{2-a}{2(1-a)}} D_T,$$

$$\vartheta_T \leq C\ell_2K_{1,T}^{\gamma_7}.$$
By \((4.36), (4.37), (4.41)\) and \((4.39)\), we have

\[ C_T \leq C \ell_0^{1/6} (T + 1)^\frac{\mu_2}{\mu_6} K_{1,T}^{\frac{\mu_2(2-a)}{\mu_6}} K_{2,T}^{1/6} \left\{ 1 + C \ell_2^{1/6} \tilde{K}_{1,T}^{\frac{\gamma_3}{1}}, \ell_2^{1/6} \tilde{K}_{1,T}^{\frac{1}{1}} \right\} \leq C \ell_3 K_{1,T}^{\gamma_6} K_{2,T}^{\mu_2}, \]  

where \(\ell_3 = \ell_1^{\frac{\mu_2}{\mu_6}} \ell_2^{1-\frac{a}{1}}\). Note that we used the facts \(\gamma_3 \geq 1\) and \(\gamma_1 < \mu_6 < \gamma_1 + 1\). Applying \((4.19)\) with the use of \((4.40)\) and \((4.42)\), we have

\[
\sup_{[0,T]} \|\tilde{P}(t)\|_{L^\infty(U')} \leq 2\|\tilde{P}(0)\|_{L^\infty} + C \ell_0^{1/2} T^{\gamma_6} K_{1,T}^{\frac{2-a}{1}} D_T, 
\]

hence, obtaining \((4.28)\) for \(T \leq 1\) with

\[
L_{11} = \ell_0^2 \ell_3 = \ell_0^2 \ell_1^2 \ell_2^{1-\frac{a}{1}}.
\]

Consider \(T > 1\) now. By \((4.34)\),

\[
\|\tilde{P}\|_{L^1(0,T)} + \|\tilde{P}\|_{L^2(0,T)} \leq C \ell_1^{1/2} T^{\gamma_6} D_T, 
\]

Using \(C_{T,\theta}\) in \((4.35)\) with bounds \((4.37), (4.38)\) and \((4.39)\), we have

\[ C_{T,\theta} \leq C \ell_1^{1/2} (T + 1)^\frac{\mu_2}{\mu_6} K_{1,T}^{\frac{\mu_2(2-a)}{\mu_6}} K_{2,T}^{\mu_2} \left\{ 1 + T^{2(\gamma_1+1)} \ell_2^{1/6} (T + 1)^\frac{1}{1} \tilde{K}_{1,T}^{\frac{1}{1}} + T^{2(\gamma_6+1)} \tilde{K}_{1,T}^{\frac{1}{1}} \right\}. \]

Thus,

\[ C_{T,\theta} \leq C \ell_3 T^{\gamma_6} K_{1,T}^{\gamma_6} \tilde{K}_{2,T}^{\mu_2}. \]

Therefore, combining \((4.3), (4.45)\) and \((4.46)\) yields

\[
\sup_{[1,T]} \|\tilde{P}\|_{L^\infty(U')} \leq C L_{11} T^{\gamma_6+1} K_{1,T}^{\gamma_6} \tilde{K}_{2,T}^{\mu_2} D_T. 
\]

Then combining \((4.43)\) for \(T = 1\) with \((4.47)\), noting that \(\gamma_5 \geq \frac{\gamma_1}{2(\gamma_1 + 1)}\), we obtain \((4.28)\) for \(T > 1\). The proof is complete. \(\square\)

Now, we derive asymptotic estimates as \(t \to \infty\). Let \(t > 2\). Applying Proposition \(4.1\) to \(T_0 = t - 1, T = 1\) and \(\theta = 1/2\), we have

\[
\|\tilde{P}(t)\|_{L^\infty(U')} \leq C \tilde{C}(t) \left(\|\tilde{P}\|_{L^2(U \times (t-1,t))}^{\frac{1}{\gamma_1+1}} + \|\tilde{P}\|_{L^2(U \times (t-1,t))} \right), 
\]

where

\[
\tilde{C}(t) = [\tilde{\lambda}(t) \tilde{\gamma}(t)]^{\frac{1}{\gamma_1+1}} + \tilde{\lambda}(t) + \tilde{\gamma}(t) \sup_{[t-1,t]} \|\nu\|_{L^\infty} \right) \frac{1}{\gamma_6+1}, 
\]

with

\[
\tilde{\lambda}(t) = \sup_{\tau \in [t-1,t]} \Lambda(\tau)^{\mu_7}, 
\]

\[ \tilde{\gamma}(t) = \left[ \int_{t-1}^{t} \int_{V} \left( |\nabla p_1|^{2\mu(1-a)} + |\nabla p_2|^{2\mu(1-a)} \right) dx \right]^{\frac{1}{2\mu}}. 
\]

Before going into specific estimates, we state a general result on the limit superior of \(\|\tilde{P}(t)\|_{L^\infty(U')}\) as \(t \to \infty\). It is of the same spirit as of the \(L^2\)-result \((4.25)\).
Lemma 4.5. Assume (SDC). Suppose
\[
\sup_{\tau \in [t-1, t]} \Lambda(\tau) \leq \kappa_0 g(t), \quad \hat{C}(t) \leq CB(t)
\] (4.52)
for sufficiently large \( t \), with some functions \( g(t), B(t) \geq 1 \). Let \( \kappa_1 = d_2 \kappa_0^{-b} \), where \( d_2 \) is the positive constant in (4.24). If
\[
\lim_{t \to \infty} B(t) \left( \frac{2^{(\kappa_1 + 1)}}{\kappa_1} e^{-\kappa_1 f_2} g(\gamma)^{-b} \right) = 0 \quad \text{and} \quad \lim_{t \to \infty} g(t)^b \frac{B'(t)}{B(t)} = 0,
\] (4.53)
then
\[
\limsup_{t \to \infty} \| \tilde{P}(t) \|_{L^\infty(U')} \leq C \limsup_{t \to \infty} \left\{ B(t) \left( g(t)^{\frac{b}{2}} \| \Psi(t) \|_{L^\infty} + [g(t)^b \| \Psi(t) \|_{L^\infty}]^{\frac{1}{\kappa_1 + 1}} \right) \right\}.
\] (4.54)

Proof. Let \( T > 2 \) be sufficiently large such that (4.52) holds for all \( t > T \). By (4.24) and (4.52), we have for \( t' > T \) that
\[
\| \tilde{P}(t') \|_{L^2} \leq C \left( e^{-\kappa_1 f_2} g(\gamma)^{-b} \| \tilde{P}(T) \|_{L^2} + \int_T^{t'} e^{-\kappa_1 f_2} g(\gamma)^{-b} \| \tilde{P}(T) \|_{L^2} \right).
\] (4.55)
Combining (4.48) with (4.52) and estimate (4.55), we obtain
\[
\| \tilde{P}(t) \|_{L^\infty(U')} \leq CB(t) \left\{ e^{-\kappa_1 f_2} g(\gamma)^{-b} \| \tilde{P}(T) \|_{L^2} + \int_T^t e^{-\kappa_1 f_2} g(\gamma)^{-b} \| \tilde{P}(T) \|_{L^2} \right\}^{\frac{\kappa_1}{2}} + CB(t) \left\{ e^{-\kappa_1 f_2} g(\gamma)^{-b} \| \tilde{P}(T) \|_{L^2} + \int_T^t e^{-\kappa_1 f_2} g(\gamma)^{-b} \| \tilde{P}(T) \|_{L^2} \right\}^{\frac{1}{2}},
\]
where \( \varrho_1 = \frac{\gamma_1}{\kappa_1 + 1} \). Note that \( \varrho_1 < 1 \), then by condition (4.53), we have
\[
\limsup_{t \to \infty} \| \tilde{P}(t) \|_{L^\infty(U')} \leq C \limsup_{t \to \infty} \left\{ B(t)^{\frac{2}{\varrho_1}} \int_T^t e^{-\kappa_1 f_2} g(\gamma)^{-b} \| \tilde{P}(T) \|_{L^2} \right\}^{\frac{\varrho_1}{2}} + C \limsup_{t \to \infty} \left\{ B(t)^2 \int_T^t e^{-\kappa_1 f_2} g(\gamma)^{-b} \| \tilde{P}(T) \|_{L^2} \right\}^{1/2}.
\] (4.56)
The first condition in (4.53) and the fact that \( B(t) \geq 1 \) imply \( \int_T^t g(\gamma)^{-b} d\tau = \infty \). With this and the second condition in (4.53), we apply Lemma A.3 to each limit in (4.56) and obtain
\[
\limsup_{t \to \infty} \| \tilde{P}(t) \|_{L^\infty(U')} \leq C \limsup_{t \to \infty} \left( B(t)^{\frac{2}{\varrho_1}} \| \tilde{P}(T) \|_{L^\infty} + B(t)^2 \right) \| \tilde{P}(T) \|_{L^\infty} \right\}^{\frac{1}{2}},
\]
thus, (4.54) follows.
By (4.52), there is hence, for \( d \) where (4.58), we have (4.57) that
\[
\hat{\theta}(t) \leq C + C \left( \int_{t-1}^{t} \int_{V} K(|\nabla \bar{p}_1|)|\nabla \bar{p}_1|^\gamma_2 + a + K(|\nabla \bar{p}_2|)|\nabla \bar{p}_2|^\gamma_2 + a \, dx \, dt \right)^{\frac{1}{2p}},
\]
\[
\leq C + C \left( \int_{0}^{t} \int_{V} K(|\nabla \bar{p}_1|)|\nabla \bar{p}_1|^\gamma_2 + a + K(|\nabla \bar{p}_2|)|\nabla \bar{p}_2|^\gamma_2 + a \, dx \, dt \right)^{\frac{1}{2p}}. 
\]
(4.57)

By (4.57) and (3.59),
\[
\hat{\theta}(t) \leq C L_{12} B_1(t)^{\frac{\mu \nu (\gamma_2 + a - 2)}{2p}} B_2(t)^\frac{1}{2p}, 
\]
(4.58)
where \( L_{12} = \left\{ \sum_{i=1,2} L_3 (\gamma_2 + a; |p_i(0)|) \right\}^{\frac{1}{2p}}, \)
\[
B_1(t) = 1 + M_F(t) \quad \text{and} \quad B_2(t) = 1 + \int_{0}^{t} F(\tau) \, d\tau. 
\]

**Theorem 4.6.** Assume \([SDC]\). Suppose \( \bar{A} < \infty \). Define
\[
\bar{\Upsilon}_3 = 1 + \sup_{[0,\infty)} (\|\psi_1\|_{L^\infty} + \|\psi_2\|_{L^\infty}), \quad \omega(t) = 1 + \int_{t-2}^{t} \tilde{F}(\tau) \, d\tau, \quad B_3(t) = \omega(t)^{\frac{\mu \nu}{2p}} B_2(t)^{\frac{1}{2p(\gamma_1 + 1)}}. 
\]
Let \( d_4 = d_2 d_3^b \), where \( d_2 \) is in (4.42) and \( d_3 \) is in (4.61) below. If
\[
\lim_{t \to \infty} B_3(t) \left( \frac{2(\gamma_2 + 1)}{\gamma_1} \right) e^{-d_4 \bar{\Upsilon}_3^{\frac{2}{1-a}}} \int_{0}^{t} \omega(t)^{-b} \, d\tau = 0, \quad \lim_{t \to \infty} (\omega(t)^{b} F(t)) = 0, \quad (4.59)
\]
then
\[
\limsup_{t \to \infty} \|\bar{P}(t)\|_{L^\infty(\Omega)} \leq C \bar{\Upsilon}_4 \limsup_{t \to \infty} \left\{ B_3(t) \left( \omega(t)^{b}\|\Psi(t)\|_{L^\infty} + [\omega(t)^{b}\|\Psi(t)\|_{L^\infty}] \right) \right\}, 
\]
(4.60)
where \( \bar{\Upsilon}_4 \) is defined by (4.64) below.

**Proof.** Since \( \bar{A} < \infty \), we have \( \bar{\Upsilon}_3 < \infty \). Then
\[
\bar{A} \leq C (\bar{\Upsilon}_3^2 + \bar{\Upsilon}_3^{\frac{2-a}{1-a}}) \leq C \bar{\Upsilon}_3^{\frac{2-a}{1-a}} < \infty. 
\]
By (3.92), there is \( T_1 > 0 \) such that for \( \tau > T_1 \), we have
\[
\Lambda(\tau) \leq C \left( \bar{\Upsilon}_3^{\frac{2-a}{1-a}} + \int_{\tau-1}^{\tau} \tilde{F}(\tau) \, d\tau \right), 
\]
hence, for \( t > T_1 + 1, \)
\[
\sup_{\tau \in [t-1, t]} \Lambda(\tau) \leq C \left( \bar{\Upsilon}_3^{\frac{2-a}{1-a}} + \int_{t-2}^{t} \tilde{F}(\tau) \, d\tau \right) \leq d_3 \bar{\Upsilon}_3^{\frac{2-a}{1-a}} \omega(t), \quad \bar{A}(t) \leq C \bar{\Upsilon}_3^{\frac{2-a}{1-a}} \omega(t)^{\mu \nu}, \quad (4.61)
\]
where \( d_3 \) is a positive constant. Note also that \( \|\Psi(t)\|_{L^\infty} \leq \bar{\Upsilon}_3 \) for all \( t \geq 0 \). Then by (4.49), (4.61) and (4.58), we have
\[
\tilde{C}(t) \leq C \bar{\Upsilon}_3^{\frac{2-a}{1-a} \mu \nu} \omega(t)^{\frac{\mu \nu}{2p}} \left( 1 + L_{12}^{\frac{1}{7}} \bar{\Upsilon}_3^{\frac{\mu \nu (\gamma_2 + a - 2)}{2p(\gamma_1 + 1)}} B_2(t)^{\frac{1}{2p(\gamma_1 + 1)}} + \bar{\Upsilon}_3^{\frac{1}{2p(\gamma_1 + 1)}} \right) 
\]
(4.62)
Thus,
\[
\tilde{C}(t) \leq C \eta_1 \omega(t)^{\frac{\mu \nu}{2p}} B_2(t)^{\frac{1}{2p(\gamma_1 + 1)}} = C \eta_1 B_3(t), \quad (4.63)
\]
where $\eta_1 = L^{-1/2} \left( \frac{2^{\mu_\tau}}{\gamma_1 + 1} \right)^{\frac{1}{2}} + \left[ \frac{2^{\mu_\tau} + a}{\gamma_1 + 1} + \frac{1}{\gamma_1 + 1} \right]$. Using (4.61) and (4.63), we apply Lemma 4.5 with $g(t) = \frac{2^{\mu_\tau}}{\gamma_1 + 1} \omega(t)$ and $B(t) = \eta_1 B_3(t)$. Note that the last two limits in (4.59) imply the second limit in (4.53). As a result, we obtain from (4.54) that

$$\limsup_{t \to \infty} \|P(t; \cdot)\|_{L^\infty} \leq C \limsup_{t \to \infty} \left\{ \eta_1 B_3(t) \left[ \left( \frac{2^{\mu_\tau}}{\gamma_1 + 1} \omega(t) \right)^b \|\Psi(t)\|_{L^\infty} \right]^{p_1} + \left[ \frac{2^{\mu_\tau} + a}{\gamma_1 + 1} \omega(t) \right]^b \|\Psi(t)\|_{L^\infty} \right\}.$$  

Hence (4.60) follows with

$$\gamma_4 = \frac{2^{\mu_\tau}}{\gamma_1 + 1} \eta_1 = L_{1/2} \left( \frac{2^{\mu_\tau}}{\gamma_1 + 1} \right)^{\frac{1}{2}} + \left[ \frac{2^{\mu_\tau} + a}{\gamma_1 + 1} + \frac{1}{\gamma_1 + 1} \right].$$  

The proof is complete.

The next, we will treat the case $A = \infty$. For that we recall some estimates from [18].

**Lemma 4.7 (cf. [18], Lemma 5.8).** Assume the Degree Condition and $A = \infty$. Define

$$W_1(t) = 1 + M_F(t) \frac{2}{\gamma_1 + 1} + \int_{t-1}^t \tilde{F}(\tau) d\tau,$$

$$W_2(t) = 1 + \tilde{\beta} t^{\frac{1}{\gamma_1 - a}} + F(t - 1) t^{\frac{2}{\gamma_1 - a}} + F(t) + \int_{t-1}^t F(\tau) + \tilde{F}(\tau) d\tau. \tag{4.52}$$

(i) There is $T_1 > 0$ such that $\Lambda(t) \leq CW_1(t)$ for all $t > T_1$.

(ii) If $\tilde{\beta} < 0$ then there is $T_2 > 0$ such that $\Lambda(t) \leq W_2(t)$ for all $t > T_2$.

Let $W(t)$ be defined, in the general case, by

$$W(t) = 1 + M_F(t) \frac{2}{\gamma_1 + 1} + \int_{t-2}^t \tilde{F}(\tau) d\tau,$$

and, in case $\tilde{\beta} < 0$, by

$$W(t) = 1 + \tilde{\beta} t^{\frac{1}{\gamma_1 - a}} + \sup_{[t-2, t]} F(t) t^{\frac{2}{\gamma_1 - a}} + \int_{t-2}^t \tilde{F}(\tau) d\tau. \tag{4.53}$$

Then for large $t$, we have from Lemma 4.7 that

$$\sup_{\tau \in [t-1, t]} \Lambda(\tau) \leq d_5 W(t) \quad \text{and} \quad \tilde{\lambda}(t) \leq CW(t)^{\mu_\tau}, \tag{4.54}$$

where $d_5$ is a positive constant. With (4.55), we restate Lemma 4.5 as the following.

**Lemma 4.8.** Assume (SDC). Suppose $A = \infty$ and $\tilde{C}(t) \leq CB(t)$ for sufficient large $t$, with some function $B(t) \geq 1$. Let $d_6 = d_2 d_5^{-b}$, where $d_2$ is in (4.24) and $d_5$ is in (4.55). If

$$\lim_{t \to \infty} B(t)^{\frac{2(\gamma_2 + a - 1)}{\gamma_1 + 1}} e^{-d_6 \int_2^t W(\tau)^{-b} d\tau} = 0 \quad \text{and} \quad \lim_{t \to \infty} W(t)^b \frac{\tilde{B}(t)}{B(t)} = 0, \tag{4.56}$$

then

$$\limsup_{t \to \infty} \|P(t; \cdot)\|_{L^\infty(U)} \leq C \limsup_{t \to \infty} \left\{ \tilde{B}(t) \left( W(t)^b \|\Psi(t)\|_{L^\infty} + [W(t)^b \|\Psi(t)\|_{L^\infty}]^{\frac{\gamma_7}{\gamma_1 + 1}} \right) \right\}. \tag{4.57}$$

Denote

$$\mu_{13} = \max \left\{ \frac{\mu_7}{\mu_6}, \frac{\mu_7}{\mu_6 + 1} + \frac{1 - a}{2(\mu_6 + 1)} \right\} \quad \text{and} \quad \gamma_7 = \frac{\mu_4(\gamma_2 + a - 2)}{2\mu_7}. \tag{4.58}$$
Theorem 4.9. Assume \( SDC \). Suppose \( \bar{\theta} = \infty \) and \( \int_0^\infty F(\tau) d\tau = \infty \). Define
\[
B_4(t) = \int_0^t F(\tau) d\tau \quad \text{and} \quad B_5(t) = M_F(t)^{\gamma_7} B_4(t)^{\frac{1}{\gamma_7}}.
\]
Let \( d_0 \) be defined as in Lemma 4.8. If
\[
\lim_{t \to \infty} \frac{2\mu_{13}^{\gamma_1+1}}{\gamma_1} B_5(t) \frac{1}{\gamma_7} e^{-d_0 \int_0^t W^b(\tau) d\tau} = 0,
\]
then
\[
\lim_{t \to \infty} (W^b(t))' = 0, \quad \lim_{t \to \infty} \frac{M_F'(t)}{M_F(t)} = 0 \quad \text{and} \quad \lim_{t \to \infty} \frac{F(t)}{\int_0^t F(\tau) d\tau} = 0,
\]
and
\[
\limsup_{t \to \infty} \|\tilde{P}(t)\|_{L^\infty(U')} \leq C \sup_{t \to \infty} \left\{ W(t)^{\mu_{13}} B_5(t)^{\frac{1}{\gamma_7+1}} \left( W(t)^{\mu_7} \|\Psi(t)\|_{L^\infty} + [W(t)^{\mu_7} \|\Psi(t)\|_{L^\infty}]^\frac{1}{\gamma_7+1} \right) \right\}.
\]

Proof. Let \( B_6(t) = 1 + \sup_{0 \leq \tau \leq t} (\|\psi_1\|_{L^\infty} + \|\psi_2\|_{L^\infty}) \). Then \( \lim_{t \to \infty} B_6(t) = \infty \) and \( \lim_{t \to \infty} B_4(t) = \infty \).

We estimate \( \tilde{C}(t) \). By (4.57) and (3.61), we have for \( t \) sufficiently large that
\[
\tilde{\theta}(t) \leq C M_F(t)^{\frac{\mu_4 (\gamma_2 + a - 2) \theta_3 + \theta_1 + 1}{2\mu}} B_5(t)^{\frac{1}{\gamma_7}} = CB_5(t).
\]
By (4.49), (4.65), (4.73) and (4.72),
\[
\tilde{C}(t) \leq C \left\{ W(t)^{\mu_7} + B_5(t)^{\frac{1}{\gamma_7+1}} W(t)^{\mu_7} + W(t)^{\mu_7} W(t)^{\frac{1}{\gamma_7+1}} \right\} \leq CB_7(t),
\]
where \( B_7(t) = W(t)^{\mu_{13}} B_5(t)^{\frac{1}{\gamma_7+1}} \). We apply Lemma 4.8 with \( \tilde{B}(t) = B_7(t) \). The first condition in (4.66) is replaced by
\[
\lim_{t \to \infty} (W(t)^{\mu_{13}} B_5(t)^{\frac{1}{\gamma_7+1}}) \frac{2}{\gamma_7} e^{-d_0 \int_0^t W^b(\tau) d\tau} = 0,
\]
which is (4.69). The second condition in (4.66) is replaced by (4.70). Then we obtain (4.71) directly from (4.67).

Example 4.10. Suppose that for \( t \) sufficiently large, we have
\[
F(t) \leq M_F(t) \leq Ct^{\theta_1}, \quad W(t) \leq Ct^{\theta_2/b},
\]
for some \( \theta_1 > 0 \) and \( 0 < \theta_2 < 1 \). Following the proof of Theorem 4.9, we can see that the statements still hold true if the functions \( F(t), M_F(t), W(t) \) are replaced by their upper bounds \( C t^{\theta_1}, C t^{\theta_1}, C t^{\theta_2/b}, \) respectively. With such replacements, conditions (4.69) and (4.70) are met, and \( B_5(t) = C t^{\theta_3} \), where
\[
\theta_3 = \frac{\mu_4 (\gamma_2 + a - 2) \theta_1 + \theta_1 + 1}{2\mu}.
\]
Therefore, from (4.71), it follows
\[
\limsup_{t \to \infty} \|\tilde{P}(t)\|_{L^\infty(U')} \leq C \sup_{t \to \infty} \left\{ t^{\theta_4} \|\Psi(t)\|_{L^\infty} + [t^{\theta_4} \|\Psi(t)\|_{L^\infty}]^\frac{1}{\gamma_7+1} \right\},
\]
where
\[
\theta_4 = \frac{\mu_4 (\gamma_2 + a - 2) \theta_1 + \theta_1 + 1}{\gamma_7} + \frac{\theta_2 \mu_{13} (\gamma_1 + 1)}{\gamma_7} + \theta_2 \quad \text{and} \quad \theta_5 = \frac{\mu_4 (\gamma_2 + a - 2) \theta_1 + \theta_1 + 1}{2\mu (\gamma_7 + 1)} + \frac{\theta_2 \mu_{13}}{b} + \theta_2.
\]
4.2. Results for pressure gradient. In \([18]\), the norm \(\|\nabla P(t)\|_{L^2(U')}\) is estimated for all \(t > 0\). Now we estimate \(\|\nabla P(t)\|_{L^s(U')}\) and \(\|\nabla P\|_{L^s(U' \times (0,T))}\) for any \(s \in (2 - \alpha, 2)\).

Proposition 4.11. Let \(\delta \in (0, \alpha)\). Then for all \(t > 0\),

\[
\|\nabla P(t)\|_{L^{2-\delta}(U')} \leq C\|\tilde{P}(t)\|_{L^2}^\frac{\delta}{2} \left\{ 1 + \sum_{i=1,2} \left[ \|\tilde{p}_{it}(t)\|_{L^2}^\delta + \|\nabla p_i(t)\|_{L^{2-\alpha}}^\delta + \|\psi_i(t)\|_{L^\infty}^\delta \right] \right\}^\frac{\alpha}{\delta},
\]

and for any \(T > 0\),

\[
\|\nabla P\|_{L^{2-\delta}(U' \times (0,T))} \leq C\|\tilde{P}\|_{L^2(U' \times (0,T))}^{1/2} \left[ T + \sum_{i=1,2} \left( \|\tilde{p}_{it}\|_{L^2(U' \times (0,T))}^\delta + \int_0^T \int_U |\nabla p_i|^{2-\alpha} dx dt + \int_0^T \|\psi_i(t)\|_{L^\infty} dx dt \right) \right]^\frac{\delta}{2} \cdot \left[ T + \sum_{i=1,2} \int_0^T \int_U |\nabla p_i|^\frac{2(\delta-\alpha)}{\delta} dx dt \right]^\frac{\delta}{2(2-\alpha)}. \tag{4.75}
\]

Here, constant \(C > 0\) depends on \(U, U'\) and \(\delta\).

Proof. Note that \(\nabla \tilde{P} = \nabla P\). Let \(\zeta = \zeta(x)\) be a cut-off function such that \(\zeta\) vanishes in neighborhood of \(\Gamma\). Multiplying equation (4.1) by \(\tilde{P}\zeta^2\) and integrating over \(U\), using integration by parts, we have

\[
\int_U \tilde{P}_t \tilde{P} \zeta^2 dx = -\int_U \left( K(\nabla p_1) |\nabla p_1 - K(\nabla p_2)| \nabla p_2 \right) \cdot (\nabla P \zeta^2 + 2P \zeta \nabla \zeta) dx + \frac{1}{|U|} \int_{\Gamma} \Psi(x,t) d\sigma \int_U \tilde{P} \zeta^2 dx.
\]

Let \(\xi(x,t) = |\nabla p_1| \vee |\nabla p_2|\). By the monotonicity of \(\psi\) in Lemma 2.2, we obtain

\[
\int_U \tilde{P}_t \tilde{P} \zeta^2 dx \leq -(1 - \alpha) \int_U K(\xi) |\nabla P \zeta|^2 dx + 2 \int_U (|\nabla p_1| + |\nabla p_2|)^1 - \alpha |P| |\xi| |\nabla \xi| dx + \|\Psi(t)\|_{L^\infty} \int_U |\tilde{P}| \zeta^2 dx. \tag{4.76}
\]

Let \(V \subseteq U\) such that \(U' \subseteq V\). We select \(\zeta\) such that \(\zeta \equiv 1\) on \(U'\) and \(\text{supp } \zeta \subseteq V\). We obtain from (4.76) that

\[
(1 - \alpha) \int_U K(\xi) |\nabla P \zeta|^2 dx \leq \int_U |\tilde{P}_t| |\tilde{P}| dx + C \int_V (|\nabla p_1| + |\nabla p_2|)^{1-\alpha} |P| dx + C \|\Psi(t)\|_{L^\infty} \|\tilde{P}\|_{L^2} \leq C \left( \sum_{i=1,2} \|\tilde{p}_{it}\|_{L^2} \right) \|\tilde{P}\|_{L^2} + C \left( \sum_{i=1,2} \|\nabla p_i|^{2-\alpha} \right)^{1/2} \|\tilde{P}\|_{L^2} + C \sum_{i=1,2} \|\psi_i(t)\|_{L^\infty} \|\tilde{P}\|_{L^2}.
\]

Hence

\[
\int_{U'} K(\xi) |\nabla P|^2 dx \leq C \|\tilde{P}\|_{L^2} \left[ \sum_{i=1,2} \|\tilde{p}_{it}\|_{L^2}^2 + \left( \int_U (1 + \sum_{i=1,2} |\nabla p_i|^{2-\alpha}) dx \right)^{1/2} + \sum_{i=1,2} \|\psi_i(t)\|_{L^\infty} \right]. \tag{4.77}
\]

By H"older’s inequality and property (2.13), we have

\[
\int_{U'} |\nabla P|^{2-\delta} dx \leq C \left( \int_{U'} K(\xi) |\nabla P|^2 dx \right)^\frac{2-\delta}{2} \left( \int_U (1 + |\nabla p_1| + |\nabla p_2|)^{\frac{2(2-\delta)}{\delta}} dx \right)^\frac{\delta}{2}. \tag{4.78}
\]
Combining (4.78) with (4.77) yields
\[ \|\nabla P(t)\|_{L^{2-\delta}(U)} \leq C\|\bar{P}(t)\|_{L^2}^{1/2} \left\{ 1 + \sum_{i=1,2} \left[ ||\bar{\psi}_i(t)||_{L^2}^2 + \int_U |\nabla p_i(x,t)|^{2-a} \, dx + ||\bar{\psi}_i(t)||_{L^\infty}^2 \right] \right\}^{1/2} \]
\[ \cdot \left( \int_U (1 + |\nabla p_1(x,t)| + |\nabla p_2(x,t)|)^{a(2-\delta)/s} \, dx \right)^{\frac{s}{2(2-s)}}, \]
then we obtain (4.74). We now prove (4.75). Integrating (4.77) in \( t \) over \([0, T]\) and applying Hölder’s inequality yield
\[ \int_0^T \int_U K(\xi)\nabla P(t)^2 \, dx \leq C \left( \int_0^T \|\bar{P}(t)\|_{L^2}^2 \, dt \right)^{1/2} \cdot \left( \int_0^T \|\bar{\psi}_1(t)\|_{L^2}^2 + \|\bar{\psi}_2(t)\|_{L^2}^2 \right)^{1/2} dt 
\]
\[ + \int_0^T \int_U (|\nabla p_1| + |\nabla p_2|)^{2-2a} \, dx \, dt + \int_0^T \|\psi(t)\|_{L^\infty}^2 \, dt \right)^{1/2}. \]
Then
\[ \int_0^T \int_U K(\xi)\nabla P(t)^2 \, dx \leq C\|\bar{P}\|_{L^2(U \times (0,T))} \]
\[ \cdot \left[ \sum_{i=1,2} \left( ||\bar{\psi}_i||_{L^2(U \times (0,T))}^2 + \int_0^T \int_U (1 + |\nabla p_1|^{2-a}) \, dx \, dt + \int_0^T \|\psi(t)\|_{L^\infty}^2 \, dt \right) \right]^{1/2}. \]

Again, by Hölder’s inequality and property (2.13), we have
\[ \int_0^T \int_U |\nabla P|^{2-\delta} \, dx \, dt \leq C \left( \int_0^T \int_U K(\xi)|\nabla P|^2 \, dx \, dt \right)^{2/2-\delta} \cdot \left( \int_0^T \int_U (1 + |\nabla p_1| + |\nabla p_2|)^{a(2-\delta)/s} \, dx \, dt \right)^{\frac{s}{2}}. \]

Using (4.79) in (4.80) and taking the power \( 1/(2-\delta) \), we obtain (4.75). \( \square \)

We now have explicit estimates in terms of initial and boundary data.

**Theorem 4.12.** Assume (SDC). For \( \delta \in (0, a), 0 < t_0 < 1 \) and \( T > t_0 \) we have
\[ \sup_{[t_0, T]} \|\nabla P(t)\|_{L^{2-\delta}(U)} \leq CM_{3,t_0,T}(\|\bar{P}(0)\|_{L^2} + \sup_{[0, T]} \|\psi\|_{L^\infty})^{\frac{1}{2}}, \]
where \( M_{3,t_0,T} \) is defined in (4.85) below.

**Proof.** We use estimate (4.74). We bound \( \|\bar{P}(t)\|_{L^2} \) by (4.23):
\[ \sup_{[0, T]} \|\bar{P}(t)\|_{L^2} \leq C \cdot M_{1,T}(\|\bar{P}(0)\|_{L^2} + \sup_{[0, T]} \|\psi\|_{L^\infty}^2), \]
where \( M_{1,T} = A_0 + T + \int_0^T [f_1(\tau) + f_2(\tau)] \, d\tau \). We estimate time derivative by (3.90), estimate \( \int_U |\nabla p_i|^{2-a} \, dx \) by (3.88), then we have
\[ \sum_{i=1,2} \left[ ||\bar{\psi}_i(t)||_{L^2}^2 + \int_U |\nabla p_i(x,t)|^{2-a} \, dx + ||\psi_i(t)||_{L^\infty}^2 \right] \]
\[ \leq Ct_0 \left( \sum_{i=1,2} L_5(t_0; [p_i(0), \psi_i]) \right) + (T + 1)K_{1,T}^{2-a} + \bar{K}_{2,T}. \]
Recalling \( \nu_1 = \max\{2, \frac{a(2-\delta)}{\delta}\} \), we apply (3.55) with \( s = \nu_1 \) and obtain
\[
\sum_{i=1,2} \int_{U^i} |\nabla p_i(x,t)|^{a(2-\delta)} \frac{dx}{s} \leq \sum_{i=1,2} \int_{U^i} 1 + |\nabla p_i(x,t)|^{\nu_1} dx
\]
\[
\leq C \left[ \sum_{i=1,2} L_2(\nu_1; [p_i(0)]) \right] (T + 1) \frac{2(\nu_1-2)}{2-a} + 1 K_1^{\frac{\nu_1}{2-a}}. \tag{4.84}
\]
Combining (4.82), (4.83) and (4.84) with (4.74), we have (4.81) with
\[
M_{3,t_0,T} = (1 + t_0^{-\frac{1}{4}}) M_{1,T} \left[ \sum_{i=1,2} L_5(t_0; [p_i(0), \psi_i]) \right]^{\frac{1}{4}} \left[ \sum_{i=1,2} L_2(\nu_1; [p_i(0)]) \right]^{\frac{\delta}{2(2-\delta)}} (T + 1)^{\gamma_8} K_{1,T}^{4} K_{2,T}^{\frac{1}{2}},
\]
where \( \gamma_8 = \frac{1}{4} + \frac{\delta}{2(2-\delta)} (\frac{2(\nu_1-2)}{2-a} + 1) \) and \( \gamma_9 = \frac{2-a}{4(1-a)} + \frac{\delta}{2(2-\delta)} (\frac{\nu_1}{1-a}) \). The proof is complete. \( \square \)

As \( t \to \infty \), we have the following asymptotic estimate.

**Theorem 4.13.** Assume (SDC). Let
\[
\bar{\Upsilon}_1 = 1 + \sup_{[0, \infty)} F, \quad \bar{\Upsilon}_2 = 1 + \int_0^\infty F(t) dt \quad \text{and} \quad \bar{A}_2 = 1 + \bar{A}_2^{\frac{2}{a-2}} + \limsup_{t \to \infty} \int_{t-1}^{t} \bar{F}(\tau) d\tau.
\]
If \( \bar{\Upsilon}_1, \bar{\Upsilon}_2 \) and \( \bar{A}_2 \) are finite then
\[
\limsup_{t \to \infty} \|\nabla P(t)\|_{L^{2-\delta}(U^i)} \leq C \bar{\Upsilon}_5 \limsup_{t \to \infty} \|\Psi\|_{L^{\infty}}, \tag{4.86}
\]
where \( \bar{\Upsilon}_5 \) is defined by (4.87) below.

**Proof.** Taking limsup of (4.74) and making use the limits (4.25), (3.93) and estimate (3.65) give
\[
\limsup_{t \to \infty} \|\nabla P(t)\|_{L^{2-\delta}(U^i)} \leq C \limsup_{t \to \infty} \|\Psi\|_{L^{\infty}}^{\frac{1}{2}} \bar{A}_2^{\frac{1}{2}} \sum_{i=1,2} L_4(\nu_1; [p_i(0)]) \bar{\Upsilon}_1^{\frac{\nu_1}{2-a}} \bar{\Upsilon}_2^{\frac{\nu_1}{2-a}}.
\]
Therefore, we obtain (4.86) with
\[
\bar{\Upsilon}_5 = \bar{A}_2^{\frac{b}{2} + \frac{1}{4}} \left\{ \bar{\Upsilon}_1^{\frac{\nu_1}{2-a}} \bar{\Upsilon}_2 \sum_{i=1,2} L_4(\nu_1; [p_i(0)]) \right\}^{\frac{\delta}{2(2-\delta)}}. \tag{4.87}
\]
\( \square \)

**Remark 4.14.** Even though the limit estimate in (4.86) still depends on the initial data presented in \( \bar{\Upsilon}_5 \), the smallness of the estimate can be controlled by the difference \( \Psi(t) \) for large \( t \).

The estimate in Theorem 4.12 blows up when \( t_0 \to 0 \). To overcome this, we consider the Lebesgue norm in both \( x \) and \( t \).

**Theorem 4.15.** Assume (SDC). Let \( \delta \in (0, a) \). For any \( T > 0 \), we have
\[
\|\nabla P\|_{L^{2-\delta}(U^i \times (0, T))} \leq C M_{4,T} \left( \|P(0)\|_{L^2}^2 + \int_0^T \|\Psi(t)\|_{L^{\infty}}^2 dt \right)^{\frac{1}{2}}, \tag{4.88}
\]
where \( M_{4,T} \) is defined in (4.92) below.
Proof. Applying (3.66) to \( s = \nu_1 \geq 2 \) we have
\[
T + \int_0^T \int_{U'} (1 + |\nabla p_1| + |\nabla p_2|)^{\frac{a(2-\delta)}{2}} dx dt \leq T + \int_0^T \int_{U'} (1 + |\nabla p_1| + |\nabla p_2|)^{\nu_1} dx dt
\]
\[
\leq C \left[ \sum_{i=1,2} L_1(\nu_1 + a, [p_i(0)]) \right] (T + 1)^{\frac{b+1}{b-\delta} - 1} \mathcal{K}_{1,T}^{\frac{b+1}{b-\delta}}. \tag{4.89}
\]

Inequality (3.88) provides
\[
\sum_{i=1}^2 \|\bar{P}_i\|^2_{L^2(U \times (0,T))} \leq C(1 + A_0 + B_0)(T + 1)\mathcal{K}_{1,T}^{\frac{2-a}{2-a}} \mathcal{K}_{2,T}.
\]
Using (3.28) we see that
\[
T + \sum_{i=1}^2 \int_0^T \int_U |\nabla p_i|^2 dx dt \leq C(1 + A_0)(T + 1)\mathcal{K}_{1,T}^{\frac{2-a}{2-a}}.
\]

Hence,
\[
T + \sum_{i=1}^2 \left\{ \left\| \bar{P}_i \right\|_{L^2(U \times (0,T))}^2 + \int_0^T \int_U |\nabla p_i(x, t)|^{2-2a} dx dt + \int_0^T \|\psi_i(t)\|_{L^\infty}^2 dt \right\} \leq C N_{1,T}, \tag{4.90}
\]
where \( N_{1,T} = (1 + A_0 + B_0)(T + 1)\mathcal{K}_{1,T}^{\frac{2-a}{2-a}} \mathcal{K}_{2,T} \). According to (4.22) and (3.88),
\[
\int_0^T \|\bar{P}(t)\|_{L^2}^2 dt \leq CT \sup_{t \in [0, T]} A(t)^b \left( \|\bar{P}(0)\|_{L^2}^2 + \int_0^T \|\Psi(t)\|_{L^\infty}^2 dt \right)
\]
\[
\leq CT N_{1,T}^b \left( \|\bar{P}(0)\|_{L^2}^2 + \int_0^T \|\Psi(t)\|_{L^\infty}^2 dt \right). \tag{4.91}
\]

Combining (4.75), (4.89), (4.90) and (4.91) we obtain
\[
\|\nabla P\|_{L^{2-\delta}(U \times (0,T))} \leq CT^{\frac{1}{4}} N_{1,T}^{\frac{b}{4}} \left( \|\bar{P}(0)\|_{L^2}^2 + \int_0^T \|\Psi(t)\|_{L^\infty}^2 dt \right)^{\frac{\delta}{4}} \cdot N_{1,T}^{\frac{b}{4}} \cdot \left[ \sum_{i=1,2} L_1(\nu_1 + a, [p_i(0)]) \right] (T + 1)^{\frac{b+1}{b-\delta} - 1} \mathcal{K}_{1,T}^{\frac{b+1}{b-\delta}}.
\]

Therefore, we obtain (4.88) with
\[
M_{4,T} = (1 + A_0 + B_0)^{\frac{b+1}{b-\delta}} \left[ \sum_{i=1,2} L_1(\nu_1 + a, [p_i(0)]) \right]^{\frac{\delta}{2(2-\delta)}} T^{\frac{\delta}{4}} (T + 1)^{\gamma_{10} \mathcal{K}_{1,T}^{\gamma_{11} \mathcal{K}_{2,T}^{\frac{b+1}{b-\delta}}}}, \tag{4.92}
\]
where \( \gamma_{10} = \frac{\delta}{2(2-\delta)} \left( \frac{2\nu_1}{2-a} - 1 \right) + \frac{b+1}{4} \) and \( \gamma_{11} = \frac{\nu_1}{1-a} \frac{\delta}{2(2-\delta)} + \frac{(b+1)(2-a)}{4(1-a)}. \)

5. Dependence on the Forchheimer Polynomial

In this section we study the dependence of solutions of IBVP (3.7) on the coefficients of the Forchheimer polynomial \( g(s) \) in (2.2). Let \( N \geq 1 \), the exponent vector \( \bar{\alpha} = (0, \alpha_1, \ldots, \alpha_N) \) and the boundary data \( \psi(x, t) \) be fixed.

Let \( \mathcal{D} \) be a compact subset of \( \{\bar{\alpha} = (a_0, a_1, \ldots, a_N) : a_0, a_N > 0, a_1, \ldots, a_{N-1} \geq 0\} \). Set \( \hat{\chi}(\mathcal{D}) = \max\{\chi(\bar{\alpha}) : \bar{\alpha} \in \mathcal{D}\} \). Then \( \hat{\chi}(\mathcal{D}) \) is a number in \([1, \infty)\).
Let \( g_1(s) = g(s, \tilde{a}^{(1)}) \) and \( g_2(s) = g(s, \tilde{a}^{(2)}) \) be two functions of class \( FP(N, \tilde{a}) \), where \( \tilde{a}^{(1)} \) and \( \tilde{a}^{(2)} \) belong to \( D \). For \( k = 1, 2 \), let \( p_k = p_k(x, t; \tilde{a}^{(k)}) \) be the solution of (3.1) and (3.2) with \( K = K(\xi, \tilde{a}) \) and the same boundary flux \( \psi \).

Let \( P = p_1 - p_2 \) and \( \bar{P} = P - |U|^{-1} \int_U P \, dx \).

\[
\bar{P}_t = \nabla \cdot (K(|\nabla \bar{p}_1|, \tilde{a}^{(1)}) \nabla \bar{p}_1) - \nabla \cdot (K(|\nabla \bar{p}_2|, \tilde{a}^{(2)}) \nabla \bar{p}_2) \quad \text{in} \ U \times (0, \infty) \tag{5.1}
\]

As shown in \([17]\), all constants \( d_j, c_j, C_j \) and \( C \) appearing in estimates in the previous sections when \( \tilde{a} \) varies among the vectors \( \tilde{a}^{(1)}, \tilde{a}^{(2)}, \tilde{a}^{(1)} \vee \tilde{a}^{(2)} \) and \( \tilde{a}^{(1)} \wedge \tilde{a}^{(2)} \), can be made dependent of \( \chi(D) \), but independent of \( \tilde{a} \). We still denote them by \( d_j, c_j, C_j \) and \( C \), respectively, in this section.

Throughout this section, we continue to have \( U' \subset U \). The flux-related quantities \( f(t), \tilde{f}(t), M_f(t) \) \( A \) and \( \beta \) are defined in section \([3]\) from (3.24) to (3.25). Also, \( K_{1,t} \) and \( K_{2,t} \) are defined in (3.53) and (3.95), respectively. Regarding initial data, \( A_0 \) is defined in (4.21).

### 5.1 Results for pressure

We will estimate the interior \( L^\infty \)-norm for \( \bar{P}(x, t) \) in terms of \( \bar{P}(x, 0) \) and \( |\tilde{a}^{(1)} - \tilde{a}^{(2)}| \).

We start with the following general estimates which are counter parts of Propositions 4.1 and 4.2.

**Proposition 5.1.** We use the same assumptions and notation as in Proposition 4.1.

(i) There exists a positive constant \( C = C(U, V, U') \) such that

\[
\sup_{[T_0 + \theta T; T_0 + T]} \| \bar{P} \|_{L^\infty(U')} \leq C \tilde{C}_{T_0, T, \theta} \left( \| \bar{P} \|_{L^2(U \times (T_0, T_0 + T))} + \| \bar{P} \|_{L^\infty(U \times (T_0, T_0 + T))} \right), \tag{5.2}
\]

where

\[
\tilde{C}_{T_0, T, \theta} = \left[ \lambda \left( 1 + (\theta T)^{-\frac{1}{2}} \right) \right]^{\frac{1}{\nu_6}} + \left[ \lambda \left( T^{1/2} \vartheta_1 + \vartheta_2 \right) \right]^{\frac{1}{\gamma_1 + 1}}, \tag{5.3}
\]

with \( \lambda = \lambda_{T_0, T} \) defined in (4.4), and

\[
\vartheta_1 = \vartheta_1, T_0, T \overset{\text{def}}{=} \left( \int_{T_0}^{T_0 + T} \int_V |\nabla \bar{P}_1|^2 (1 - a)^{\mu} + |\nabla \bar{P}_2|^2 (1 - a)^{\mu} \, dx \, dt \right)^{\frac{1}{2\mu}}, \tag{5.4}
\]

\[
\vartheta_2 = \vartheta_2, T_0, T \overset{\text{def}}{=} \left( \int_{T_0}^{T_0 + T} \int_V |\nabla \bar{P}_1|^2 (2 - a)^{\mu} + |\nabla \bar{P}_2|^2 (2 - a)^{\mu} \, dx \, dt \right)^{\frac{1}{2\mu}}. \tag{5.5}
\]

(ii) There exists a positive constant \( C = C(U, V, U') \) such that

\[
\sup_{[0, T]} \| \bar{P} \|_{L^\infty(U')} \leq 2 \| \bar{P}(0) \|_{L^\infty} + C \tilde{C}_T \left( \| \bar{P} \|_{L^2(U \times (0, T))} + \| \bar{P} \|_{L^\infty(U \times (0, T))} \right), \tag{5.6}
\]

where

\[
\tilde{C}_T = \lambda^{\frac{1}{\nu_6}}_T + \left[ \lambda_T \left( T^{1/2} \vartheta_1, T_0 + T + \vartheta_2, T \right) \right]^{\frac{1}{\gamma_1 + 1}}, \tag{5.7}
\]

with \( \lambda_T = \lambda_{0, T}, \vartheta_1, T_0 + T = \vartheta_1, 0, T \) and \( \vartheta_2, T = \vartheta_2, 0, T \).

**Proof.** (i) Let \( \zeta(x, t) \) be a cut-off function such that it is piecewise linear continuous in \( t \), \( \zeta(\cdot, 0) = 0 \) and for each \( t \), supp \( \zeta(\cdot, t) \subset V \). Let \( \bar{P}^{(k)} = \max\{ \bar{P} - k, 0 \} \) and \( \chi_k \) be characteristic function of the set \( \{(x, t) : \bar{P}^{(k)} > 0\} \).
Let $\xi = |\nabla \tilde{p}_1| \vee |\nabla \tilde{p}_2|$. Multiplying equation (2.11) by $\tilde{P}^{(k)} \zeta^2$ and integrating over $U$, we have

$$\frac{1}{2} \frac{d}{dt} \int_U |\tilde{P}^{(k)} \zeta|^2 \, dx + \int_U |\tilde{P}^{(k)}|^2 \zeta_t \, dx = \int_U (K(|\nabla \tilde{p}_1|, \tilde{a}^{(1)}) \nabla \tilde{p}_1 - K(|\nabla \tilde{p}_2|, \tilde{a}^{(2)}) \nabla \tilde{p}_2) \cdot \nabla \tilde{P}^{(k)} \zeta^2 \, dx$$

$$+ \int_U (K(|\nabla \tilde{p}_1|, \tilde{a}^{(1)}) \nabla \tilde{p}_1 - K(|\nabla \tilde{p}_2|, \tilde{a}^{(2)}) \nabla \tilde{p}_2) \cdot [2P^{(k)} \zeta \nabla \zeta] \, dx = I_1 + I_2.$$

Let $\epsilon > 0$. For $I_2$, we use \ref{2.13} to estimate

$$I_2 \leq C \int_U (|\nabla \tilde{p}_1|^{1-a} + |\nabla \tilde{p}_2|^{1-a})|\tilde{P}^{(k)}| \zeta |\nabla \zeta| \, dx$$

$$\leq \epsilon \int_{S_k} |\tilde{P}^{(k)}|^2 \zeta^2 \, dx + C\epsilon^{-1} \int_U \left(|\nabla \tilde{p}_1| + |\nabla \tilde{p}_2|\right)^{2(1-a)} \chi_k |\nabla \zeta|^2 \, dx.$$

For $I_1$, similar to estimate (4.8) of term $I_2$ in Proposition 4.1 but using monotonicity \ref{2.17} in place of \ref{2.19}, we obtain

$$I_1 \leq -(1-a) \int_U K(\xi) |\nabla \tilde{P}^{(k)} \zeta|^2 \, dx + C|\tilde{a}^{(1)} - \tilde{a}^{(2)}| \int_U (|\nabla \tilde{p}_1|^{1-a} + |\nabla \tilde{p}_2|^{1-a}) |\nabla \tilde{P}^{(k)}| \zeta^2 \, dx.$$

Similar to the way (4.9) was derived from (4.8), one obtains for $\epsilon > 0$ that

$$I_1 \leq -\frac{1-a}{2} \int_U K(\xi) |\nabla (\tilde{P}^{(k)} \zeta)|^2 \, dx + C \int_U |\tilde{P}^{(k)} \nabla \zeta|^2 \, dx$$

$$+ C|\tilde{a}^{(1)} - \tilde{a}^{(2)}| \int_U (|\nabla \tilde{p}_1|^{2-a} + |\nabla \tilde{p}_2|^{2-a}) \chi_k \zeta^2 \, dx.$$

Therefore,

$$\frac{1}{2} \frac{d}{dt} \int_U |\tilde{P}^{(k)} \zeta|^2 \, dx + \frac{1-a}{2} \int_U K(\xi) |\nabla (\tilde{P}^{(k)} \zeta)|^2 \, dx$$

$$\leq \epsilon \int_{S_k} |\tilde{P}^{(k)}|^2 \zeta_t \, dx + C \int_U K(\xi) |\tilde{P}^{(k)} \nabla \zeta|^2 \, dx + \epsilon \int_U |\tilde{P}^{(k)} \zeta|^2 \, dx$$

$$+ C\epsilon^{-1} \left(|\nabla \tilde{p}_1| + |\nabla \tilde{p}_2|\right)^{2(1-a)} \chi_k |\nabla \zeta|^2 \, dx + C|\tilde{a}^{(1)} - \tilde{a}^{(2)}| \int_U (|\nabla \tilde{p}_1|^{2-a} + |\nabla \tilde{p}_2|^{2-a}) \chi_k \zeta^2 \, dx.$$

Let $J = \sup_{[0,T]} \int_U |\tilde{P}^{(k)} \zeta|^2 \, dx + \int_0^T \int_U K(\xi) |\nabla (\tilde{P}^{(k)} \zeta)|^2 \, dx \, dt$. Integrating the preceding inequality in time, taking supremum in $t$ over $[0, T]$, selecting $\epsilon = 1/(8T)$ and applying Hölder’s inequality yield

$$J \leq C \int_0^T \int_U |\tilde{P}^{(k)} \zeta|^2 \zeta |\zeta_t| \, dx \, dt + C \int_0^T \int_U K(\xi) |\tilde{P}^{(k)} \nabla \zeta|^2 \, dx \, dt$$

$$+ CT \left\{ \int_0^T \int_U \left(|\nabla \tilde{p}_1| + |\nabla \tilde{p}_2|\right)^{2(1-a)} \chi_k \zeta^2 \, dx \, dt \right\}^{\frac{1}{2}} \left\{ \int_{Q_T \cap \text{supp} \zeta} \chi_k \zeta \, dx \, dt \right\}^{\frac{1}{2}}$$

$$+ C|\tilde{a}^{(1)} - \tilde{a}^{(2)}| \left\{ \int_0^T \int_{S_k} \left(|\nabla \tilde{p}_1| + |\nabla \tilde{p}_2|\right)^{(2-a)} \chi_k \zeta^2 \, dx \, dt \right\}^{\frac{1}{2}} \left\{ \int_0^T \int_U \chi_k \zeta^2 \, dx \, dt \right\}^{\frac{1}{2}}.$$
Note that $\bar{a}^{(i)}$ and $\bar{a}^{(i)}$ belong to the compact set $\mathbb{D}$, hence $|\bar{a}^{(1)} - \bar{a}^{(2)}| \leq C$. Then
\[
J \leq C \int_0^T \int_U |\bar{P}(k)|^2 (|\zeta_i| + |\nabla \zeta|) \, dx \, dt \\
+ CT \left\{ \int_0^T \int_U \left( |\nabla \bar{p}_1| + |\nabla \bar{p}_2| \right)^{2(1-a)\mu} |\nabla \zeta|^{2\mu} \, dx \, dt \right\}^{1/\mu} \left\{ \int_{Q_T \cap \text{supp} \zeta} \lambda dx \, dt \right\}^{1-1/\mu} \\
+ C \left\{ \int_0^T \int_{S_N} \left( |\nabla \bar{p}_1| + |\nabla \bar{p}_2| \right)^{(2-a)\mu} \, dx \, dt \right\}^{1/\mu} \left\{ \int_{Q_T \cap \text{supp} \zeta} \lambda dx \, dt \right\}^{1-1/\mu}.
\]
Using inequality (2.30) in Lemma 2.4 with $W(x, t) = K(\xi(x, t))$, we have
\[
\|P(k)\|_{L^\infty(Q_T)} \leq C \lambda J \leq C \lambda \left\{ \int_0^T \int_U |\bar{P}(k)|^2 (|\zeta_i| + |\nabla \zeta|) \, dx \, dt \right\}^{1/2} \\
+ T^{1/2} \left\{ \int_0^T \int_U \left( |\nabla \bar{p}_1| + |\nabla \bar{p}_2| \right)^{2(1-a)\mu} |\nabla \zeta|^{2\mu} \, dx \, dt \right\}^{1/2} \left\{ \int_{Q_T \cap \text{supp} \zeta} \lambda dx \, dt \right\}^{1/2-1/\mu} \\
+ \left( \int_0^T \int_U \left( |\nabla \bar{p}_1| + |\nabla \bar{p}_2| \right)^{(2-a)\mu} \, dx \, dt \right)^{1/2} \left\{ \int_{Q_T \cap \text{supp} \zeta} \lambda dx \, dt \right\}^{1/2-1/\mu}.
\]
Let $Y_i = \|\bar{P}(k)\|_{L^2(A_i)}$. In the same way as in proof of Proposition 4.1 we find that
\[
Y_{i+1} \leq C \lambda M_0 \int_0^T \int_U \left( |\nabla \bar{p}_1| + |\nabla \bar{p}_2| \right)^{(2-a)\mu} \, dx \, dt \left\{ \int_{Q_T \cap \text{supp} \zeta} \lambda dx \, dt \right\}^{1-1/\mu} \\
\leq C \lambda 2^{i} \left\{ \int_0^T \int_U \left( |\nabla \bar{p}_1| + |\nabla \bar{p}_2| \right)^{(2-a)\mu} \, dx \, dt \right\}^{1-1/\mu}.
\]
where the exponents $\gamma_1, \mu_6$ are defined in Theorem 4.1. Since $Y_0 \leq \|\bar{P}\|_{L^2(U \times (0, T))}$ we choose $M_0$ sufficiently large such that
\[
\|\bar{P}\|_{L^2(U \times (0, T))} \leq C \min \left\{ \left( \left( \frac{1}{\lambda (1 + (\theta T)^{-1})} \right)^{-1/\mu} M_0, (\lambda \theta T^{1/2})^{-1/\mu} M_0^{-1/\mu} \right), \left( \left( \frac{1}{\lambda (1 + (\theta T)^{-1})} \right)^{-1/\mu} M_0^{-1/\mu}, \left( \frac{1}{\lambda (1 + (\theta T)^{-1})} \right)^{-1/\mu} M_0^{-1/\mu} \right) \right\},
\]
thus
\[
M_0 \geq C \max \left\{ \left( \lambda (1 + (\theta T)^{-1}) \right)^{-1/\mu} M_0^{1/\mu}, \left( \lambda \theta T^{1/2} \right)^{-1/\mu} M_0^{1/\mu}, \left( \lambda \theta T^{1/2} \right)^{-1/\mu} M_0^{1/\mu} \right\},
\]
Using the fact that $\left( \lambda (1 + (\theta T)^{-1}) \right)^{-1/\mu}, (\lambda \theta T^{1/2})^{-1/\mu}, (\lambda \theta T^{1/2})^{-1/\mu}$ are less than or equal to $C \bar{C}_{T_0, T, \theta}$, we then choose
\[
M_0 = C \bar{C}_{T_0, T, \theta} \left( \|\bar{P}\|_{L^2(U \times (0, T))} + \|\bar{P}\|_{L^2(U \times (0, T))} \right).
\]
Then applying Lemma A.2 for $m = 3$ to sequence $\{Y_i\}$, and using the same argument as in Theorem 4.1 we obtain $|\bar{p}(x, t)| \leq M_0$ in $U' \times (\theta T, T)$.

(ii) Using same arguments as for deriving Proposition 4.2 from the proof of Proposition 4.1 we can modify the proof in part (i) to obtain (5.6).

We recall some results from [18] which will be needed in subsequent developments.

**Theorem 5.2** (cf. [18], Theorems 5.16 and 5.17). (i) For $0 < T < \infty$, we have
\[
\sup_{[0, T]} \|\bar{P}(t)\|_{L^2}^2 \leq \|\bar{P}(0)\|_{L^2}^2 + C M_5(\bar{a}^{(1)} - \bar{a}^{(2)}),
\]
where $M_5(T) = A_0 + T + \int_0^T f(\tau) \, d\tau$. 

54 Luan T. Hoang, Thinh T. Kieu and Tuoc V. Phan
Theorem 5.3. Assume the Degree Condition. Suppose $\sup_{[0,\infty)} f(t) < \infty$ and $\sup_{[1,\infty)} \int_{t-1}^{t} \tilde{f}(\tau) d\tau < \infty$. Then

\[
\sup_{[1,\infty)} \| \tilde{P}(t) \|_{L^2}^2 \leq \| \tilde{P}(0) \|_{L^2}^2 + C T \gamma_{12} \| \hat{\varphi}_1 \|_{H^1} + C T \gamma_{12} \| \hat{\varphi}_2 \|_{H^1},
\]

where $T = 1 + A_0 + \sup_{[0,\infty)} f^{2-a}(t) + \sup_{[1,\infty)} \int_{t-1}^{t} \tilde{f}(\tau) d\tau$, and $A_2$ is defined by (3.9).

We will take advantage of calculations in Theorem 4.4 However the exponents will be changed. The new counterparts of exponents $\gamma_2, \gamma_3, \ldots, \gamma_6$ in (4.29)–(4.31) are

\[
\begin{align*}
\gamma_2 &= (2 - a)\mu, \\
\gamma_3 &= \frac{2 - a}{2(1 - a)}, \\
\gamma_4 &= \frac{1}{2(1 - a)} - \frac{1}{2}, \\
\gamma_5 &= \frac{\mu_7}{\mu_6} + \frac{1}{2(1 - a)}, \\
\gamma_6 &= \frac{\mu_7}{\mu_6} + \frac{1}{1 + \gamma_1}.
\end{align*}
\]

Firstly, we replace $\gamma_2$ by $\gamma_2'$ in estimate (4.38) to obtain

\[
\begin{align*}
\varphi_{2,T} &\leq C \varphi_{2,T} T + \gamma_{12} K_{1,T}\varphi_{12,T}, \\
\varphi_{1,T} &\leq C \varphi_{1,T} T + \varphi_{2,T} T + \gamma_{12} K_{1,T}\varphi_{12,T}.
\end{align*}
\]

where $\gamma_2 = \gamma_{12} = \frac{\gamma_2}{\varphi_{12,T}}$. Secondly, since $\gamma_4 \geq 1/(2\mu)$, it follows

\[
\begin{align*}
\lambda_{T} &\leq C \ell_1 \varphi_{12,T} + \lambda_{T} \leq C \ell_1 \varphi_{12,T} + \gamma_{12} K_{1,T}\varphi_{12,T},
\end{align*}
\]

where $\lambda_{T} = 1 + A_0 + \| \nabla p_1(0) \|_{L^2-a} + \| \nabla p_2(0) \|_{L^2-a}$.

For finite time intervals, we have the following estimates.

Theorem 5.3. Assume SDC.

(i) For $T \in (0, 1]$, we have

\[
\sup_{[0,T]} \| \tilde{P} \|_{L^\infty(U')} \leq 2 \| \tilde{P}(0) \|_{L^\infty} + C \bar{M}_{6,T}(A + A^{\gamma_1}),
\]

where $M_{6,T}$ is defined in (5.17) below and $A = \| \tilde{P}(0) \|_{L^2} + \| \hat{\varphi}_1 \|_{H^1} + \| \hat{\varphi}_2 \|_{H^1}$.

(ii) For $T > 1$, we have

\[
\sup_{[1,T]} \| \tilde{P} \|_{L^\infty(U')} \leq C \bar{M}_{7,T}(A + A^{\gamma_1}),
\]

where $M_{7,T}$ is defined in (5.18) below.

Proof. (i) Let $T \in (0, 1]$. Applying (5.6) of Proposition 5.1 gives

\[
\| \tilde{P}(t) \|_{L^\infty(U')} \leq 2 \| \tilde{P}(0) \|_{L^\infty} + C \bar{C}_T \left( T^{-1/2} \sup_{[0,t]} \| \tilde{P} \|_{L^2} + (T^{1/2} \sup_{[0,t]} \| \tilde{P} \|_{L^2})^{1/2} \right),
\]

where $T = 1 + A_0 + \sup_{[0,\infty)} f^{2-a}(t) + \sup_{[1,\infty)} \int_{t-1}^{t} \tilde{f}(\tau) d\tau$, and $A_2$ is defined by (3.9).
Using (5.11), (5.12) and (5.13) to estimate corresponding terms in (5.7), we have
\[
\tilde{C}_T \leq C \Lambda_T^{\mu_0} \left( \varphi_{1,T} + \varphi_{2,T} \right)^{\frac{1}{\gamma_1+1}} = C \ell_3^{\gamma_0} \gamma_0^{\mu_0} \kappa_{1,T}^{\mu_0} \kappa_{2,T}^{\mu_0},
\]
where \( \ell_3 = \left( \ell_1^{\frac{1}{\mu_0}} \ell_2^{\frac{1}{\gamma_1+1}} \right)^{-1} \). Moreover, by (5.8)
\[
\sup_{[0,T]} \| \tilde{P} \|_{L^2} \leq C \left( \| \tilde{P}(0) \|_{L^2} + |\tilde{a}^{(1)} - \tilde{a}^{(2)}| \right) \left[ 1 + A_0 + \int_0^T f(\tau) d\tau \right]^{\frac{1}{2}} \leq C A (1 + A_0) \frac{1}{2} K_{1,T}^{\frac{2-a}{1-a}}. \tag{5.16}
\]
Combining the above, we obtain (5.14) with
\[
M_{6,T} = \ell_3^{\frac{1}{\mu_0}} (1 + A_0)^{1/2} T^{\frac{\gamma_0}{(1+a) \mu_0}} K_{1,T}^{\frac{\mu_0}{\gamma_0}} K_{2,T}^{\frac{\mu_0}{\gamma_0}}. \tag{5.17}
\]
(ii) Let \( T > 1 \). We apply (5.2) with \( T_0 = 0 \) and \( \theta T = 1 \). Note that
\[
\tilde{C}_{0,T,0} \leq C \Lambda_T^{\mu_0} \left( 1 + [T^{1/2}(\varphi_{1,T} + \varphi_{2,T})]^{-\frac{1}{\gamma_1+1}} \right) \leq C \ell_3^{\mu_0} T^{\gamma_0} K_{1,T}^{\mu_0} K_{2,T}^{\mu_0}.
\]
Then similar calculations to those for proving (4.47) in Theorem 4.4 lead to (5.15) with
\[
M_{7,T} = (1 + A_0)^{1/2} \ell_3 T^{\gamma_0 + \frac{1}{\gamma_1+1}} K_{1,T}^{\mu_0} K_{2,T}^{\mu_0}. \tag{5.18}
\]
The proof is complete.

We now consider estimates when \( t \to \infty \). We use the same notation as in subsection 4.1. For \( t \geq 2 \), we apply Proposition 5.1(i) to the interval \([t - 1, t]\), that is \( T_0 = t - 1 \) and \( T = 1 \), and use \( \theta = 1/2 \). Then we have
\[
\| \tilde{P}(t) \|_{L^\infty(U')} \leq C \tilde{C}(t) \left\{ \sup_{[t-1,t]} \| \tilde{P} \|_{L^2} + \sup_{[t-1,t]} \| \tilde{P} \|_{L^2}^{\frac{1}{\gamma_1+1}} \right\}, \tag{5.19}
\]
where
\[
\tilde{C}(t) = \tilde{\lambda}(t)^{\frac{1}{\mu_0}} (1 + \tilde{\vartheta}(t))^{\frac{1}{\gamma_1+1}}, \tag{5.20}
\]
with \( \tilde{\lambda}(t) \) defined by (4.50), and
\[
\tilde{\vartheta}(t) = 1 + \left( \int_{t-1}^t \int_U \left| \nabla \tilde{p}_1 \right|^{(2-a)\mu} + \left| \nabla \tilde{p}_2 \right|^{(2-a)\mu} dx dt \right)^{\frac{1}{2\mu}}. \tag{5.21}
\]
Note that \( \tilde{\vartheta}(t) \) is the same as \( \tilde{\vartheta}(t) \) in (4.51) with exponent \( \gamma_2' = (2-a)\mu \) replacing \( \gamma_2 = 2(1-a)\mu \).

By (5.53) in [13], there is \( d_T > 0 \) such that
\[
\frac{1}{2} \frac{d}{dt} \int_U \tilde{P}^2 dx \leq -d_T \Lambda^{-b}(t) \int_U \tilde{P}^2 dx + C |\tilde{a}^{(1)} - \tilde{a}^{(2)}| \Lambda(t).
\]
Hence, for \( t' \geq 1 \)
\[
\| \tilde{P}(t') \|_{L^2}^2 \leq e^{-d_T \int_1^{t'} \Lambda(\tau)^{-b} d\tau} \| \tilde{P}(1) \|_{L^2}^2 + C |\tilde{a}^{(1)} - \tilde{a}^{(2)}| \int_1^{t'} e^{-d_T \int_1^{\tau} \Lambda(\theta)^{-b} d\theta} \Lambda(t) d\tau. \tag{5.23}
\]
Similar to (4.55), we obtain
\[
\sup_{[t-1,t]} \| \tilde{P}(t') \|_{L^2}^2 \leq C e^{-d_T \int_0^{t_2} \Lambda(\tau)^{-b} d\tau} \| \tilde{P}(1) \|_{L^2}^2 + C |\tilde{a}^{(1)} - \tilde{a}^{(2)}| \int_1^{t_2} e^{-d_T \int_1^{\tau} \Lambda(\theta)^{-b} d\theta} \Lambda(t) d\tau. \tag{5.24}
\]
Let \( \Upsilon_2 \) be defined as in Corollary 3.11 and
\[
\Upsilon_3 = 1 + \sup_{[0,\infty)} \| \psi \|_{L^\infty}, \quad \Upsilon_7 = 1 + \sup_{[2,\infty)} \int_{t-2}^t \tilde{f}(\tau) d\tau.
\]
We have the following result for unbounded time intervals.
Theorem 5.4. Assume (SDE). Suppose \( \Upsilon_2, \Upsilon_3 \) and \( \Upsilon_7 \) are finite numbers. Then we have
\[
\sup_{[2, \infty)} \| \tilde{P} \|_{L^\infty(U')} \leq C \Upsilon_8 (A + A^{\frac{1}{\alpha + 1}}),
\]
(5.25)
where \( \Upsilon_8 \) is defined by (5.30) below, and \( A \) is defined in Theorem 5.3.

Proof. Use (4.61) to bound \( \Lambda(t) \) and \( \tilde{\lambda}(t) \), we have for all \( t \geq 1 \) that
\[
\sup_{\tau \in [t^{-1}, t]} \Lambda(\tau) \leq C \Upsilon_7 \Upsilon_3^{\frac{2}{1+\alpha}}, \quad \tilde{\lambda}(t) \leq C \Upsilon_7^{\frac{2\alpha}{1+\alpha}} \Upsilon_3^{\frac{1}{1+\alpha}}.
\]
(5.26)
Moreover, using (4.58) to bound \( \tilde{\vartheta}(t) \) with \( \gamma' \) replacing \( \gamma_2 \), we obtain
\[
\tilde{\vartheta}(t) \leq CL'_{12} B_1(t) \frac{\mu_4(\gamma' + a - 2)}{2\alpha} B_2(t) \frac{1}{2\alpha} \leq CL'_{12} \Upsilon_3^{\frac{1}{2\alpha}} \Upsilon_2^{\frac{1}{2\alpha}},
\]
(5.27)
where \( L'_{12} = \{ \sum_{i=1}^3 \Lambda(\gamma' + a, [p_i(0)]) \}^{\frac{1}{2\alpha}} \). It follows from (5.20), (5.26) and (5.27) that
\[
\tilde{\vartheta}(t) \leq C \eta_2, \quad \text{where } \eta_2 = \Upsilon_7^{\frac{\mu_4}{2\alpha}} \Upsilon_2^{\frac{1}{2\alpha}} \Upsilon_3^{\frac{1}{2\alpha}} \Upsilon_4^{\frac{1}{2\alpha}} \Upsilon_5^{\frac{1}{2\alpha}} \Upsilon_6^{\frac{1}{2\alpha}} \Upsilon_7^{\frac{2}{1+\alpha}} \Upsilon_3^{\frac{1}{1+\alpha}}.
\]
(5.28)
Then we have from (5.19) and (5.28) that
\[
\| \tilde{P}(t) \|_{L^\infty(U')} \leq C \eta_2 \left\{ \sup_{[t^{-1}, t]} \| \tilde{P} \|_{L^2} + \sup_{[t^{-1}, t]} \| \tilde{P} \|_{L^\infty(U')} \right\}.
\]
(5.29)
By (5.24) and (5.26), we have
\[
\sup_{[t^{-1}, t]} \| \tilde{P} \|_{L^2} \leq C \| \tilde{P}(1) \|_{L^2}^2 + C |\tilde{\vartheta}(1) - \tilde{\vartheta}(2)| (\Upsilon_7 \Upsilon_3^{\frac{2}{1+\alpha}})^{b+1}.
\]
We estimate \( \| \tilde{P}(1) \|_{L^2} \) by (5.16) with \( T = 1 \) and obtain
\[
\sup_{[t^{-1}, t]} \| \tilde{P} \|_{L^2} \leq C \eta_3 A, \quad \text{where } \eta_3 = (1 + A_0)^{\frac{2-a}{2}},
\]
with \( K_{1,1} \) defined by (5.53). Combining this with (5.29), we obtain (5.25) with
\[
\Upsilon_8 = \eta_2 \eta_3.
\]
(5.30)
The proof is complete. □

For results as \( t \to \infty \), we have:

Theorem 5.5. Assume the same as in Theorem 5.4. Then
\[
\limsup_{t \to \infty} \| \tilde{P}(t) \|_{L^\infty(U')} \leq C \Upsilon_9 \left( |\tilde{\vartheta}(1) - \tilde{\vartheta}(2)| + |\tilde{\vartheta}(1) - \tilde{\vartheta}(2)|^{\frac{1}{\gamma_1 + 1}} \right)^{1/2},
\]
(5.31)
where \( \Upsilon_9 \) is defined by (5.33) below.

Proof. Define \( B_7(t) = 1 + A^{\frac{2}{2\alpha}} + \int_{t^{-1}}^t \tilde{\vartheta}(\tau) d\tau \). Note that \( \limsup_{t \to \infty} B_7(t) \leq 2A_2 \), where \( A_2 \) is defined by (3.99). By (3.92), for \( t \) sufficiently large we have
\[
\sup_{\tau \in [t^{-1}, t]} \Lambda(\tau) \leq C B_7(t) \quad \text{and} \quad \tilde{\lambda}(t) \leq C B_7(t)^{\mu_7}.
\]
(5.32)
Using (5.32) and (5.27) to estimate \( \tilde{\vartheta}(t) \) in (5.20), we derive from (5.19) for large \( t \) that
\[
\| \tilde{P}(t) \|_{L^\infty(U')} \leq C \eta_4 B_7(t)^{\mu_4} \left\{ \sup_{[t^{-1}, t]} \| \tilde{P} \|_{L^2} + \sup_{[t^{-1}, t]} \| \tilde{P} \|_{L^\infty(U')} \right\},
\]
(5.33)
where \( \eta_4 = \left[ L^2 \gamma_7 \frac{1}{2} \gamma_3 \frac{\mu_4 (\gamma_2 + a - 2)}{2 \mu} \right]^{1/\gamma}. \) Similar to (4.55) and (5.24), for large \( T \) and \( t > T \), we have

\[
\sup_{[t-1,t]} \| \bar{P}(t) \|_{L^2(U')}^2 \leq C e^{-C \int_T^t B_T(\tau)^{-b} d\tau} \| \bar{P}(T) \|_{L^2(U')}^2 + C |\bar{a}(1) - \bar{a}(2)| \int_T^t e^{-C \int_T^\tau B_T(\theta)^{-b} d\theta} B_T(\tau) d\tau. \tag{5.34}
\]

Since \( B_T(t) \) is bounded, one can easily see from (5.33) and (5.34) that

\[
\limsup_{t \to \infty} \| \bar{P}(t) \|_{L^\infty(U')} \leq C \eta_4 \left\{ \limsup_{t \to \infty} B_T(t) \left( \frac{\gamma}{\gamma - \delta} \right)^{\beta} \left[ |\bar{a}(1) - \bar{a}(2)| \frac{1}{2} \gamma A_2^{-\delta} + \gamma A_2^{-(\gamma - \delta)} \right] \right\}. \tag{5.35}
\]

Applying Lemma \ref{lemma} with \( \eta = 1 \) to the last two limits, we obtain

\[
\limsup_{t \to \infty} \| \bar{P}(t) \|_{L^\infty(U')} \leq C \eta_4 \left\{ |\bar{a}(1) - \bar{a}(2)| \frac{1}{2} \gamma A_2^{-\delta} + |\bar{a}(1) - \bar{a}(2)| \gamma A_2^{-(\gamma - \delta)} \right\}. \tag{5.36}
\]

Then (5.31) follows with
\[
\gamma_9 = \eta_4 A_2^{-\alpha_0 + \frac{\alpha(2)}{\gamma}}. \tag{5.37}
\]

The proof is complete. \( \Box \)

Similar to Remark 4.14, estimate (5.31) shows that the smallness of \( \| \bar{P}(t) \|_{L^\infty(U')} \) as \( t \to \infty \) can be controlled by \( |\bar{a}(1) - \bar{a}(2)| \).

5.2. Results for pressure gradient. We now study the dependence for pressure gradient. The results are parallel to those in subsection 4.2. For \( i = 1, 2 \), we denote \( H_i(\xi) = H(\xi, a^{(i)}) \) defined in (2.16), and recall that the functional \( J_{H[i]} \) is defined by (3.26).

**Proposition 5.6.** Let \( \delta \in (0, a) \) and \( U' \subseteq V \subseteq U \). There exists a constant \( C > 0 \) depending on \( U, U', V \) and \( \delta \) such that for any \( t > 0 \), we have

\[
\| \nabla P(t) \|_{L^{2-\delta}(U')} \leq C \frac{1}{\gamma} \left[ 1 + \sum_{i=1,2} \| \bar{P}_i(t) \|_{L^2}^2 + \sum_{i=1,2} J_{H_i}[p_i](t) \right]^{\frac{1}{2}} \left[ 1 + \sum_{i=1,2} \int_V |\nabla p_i(x,t)| \frac{a(2-\delta)}{\gamma} dx \right]^{\frac{\delta}{2(2-\delta)}}, \tag{5.38}
\]

and for any \( T > 0 \), we have

\[
\| \nabla P \|_{L^{2-\delta}(U' \times (0,T))} \leq C \left\{ \| \bar{P} \|_{L^2(U' \times (0,T))}^2 \left( T + \sum_{i=1,2} \| \bar{P}_i(t) \|_{L^2(U' \times (0,T))}^2 + \sum_{i=1,2} \int_0^T J_{H_i}[p_i](t) dt \right) \right\}^{\frac{1}{2}} + \left\{ \sum_{i=1,2} \int_0^T J_{H_i}[p_i](t) dt \right\}^{\frac{1}{2}} \left( T + \sum_{i=1,2} \int_0^T \int_V |\nabla p_i(x,t)| \frac{a(2-\delta)}{\gamma} dx dt \right)^{\frac{\delta}{2(2-\delta)}}. \tag{5.39}
\]

**Proof.** Let \( \zeta(x, t) \) be the same cut-off function as in Proposition 4.11. Multiplying equation (5.1) by \( \bar{P} \zeta^2 \), integrating over \( U \) and using integration by parts, we obtain

\[
\int_U \bar{P}_1 \bar{P} \zeta^2 dx = - \int_U \left( K(\nabla p_1, \bar{a}(1)) \nabla p_1 - K(|\nabla p_1|, \bar{a}(2)) \nabla p_2 \right) \cdot (\nabla P \zeta^2 + 2 \bar{P} \zeta \nabla \zeta) dx.
\]
Again, the fact $\nabla \tilde{P} = \nabla P$ was used in the above. Let $\xi(x) = |\nabla p_1| \lor |\nabla p_2|$. By the monotonicity (2.18),

$$
\int_U \tilde{P}_t \tilde{P} \zeta^2 dx \leq -(1-a) \int_U K(\xi, \tilde{a}^{(1)} \lor a^{(2)}) |\nabla P \zeta|^2 dx
$$

$$
\quad + C|\tilde{a}^{(1)} - \tilde{a}^{(2)}| \int_U (|\nabla p_1|^{-a} + |\nabla p_2|^{-a}) |\nabla P \zeta|^2 dx + C \int_U (|\nabla p_1| + |\nabla p_2|)^{-a} |\tilde{P}| \zeta dx.
$$

Hence,

$$
(1-a) \int_U K(\xi, \tilde{a}^{(1)} \lor a^{(2)}) |\nabla P \zeta|^2 dx \leq C \|\tilde{P}\|_{L^2} \|\tilde{P}\|_{L^2}
$$

$$
\quad + C \left( \int_U (|\nabla p_1| + |\nabla p_2|)^{2-2a} dx \right)^{1/2} \|\tilde{P}\|_{L^2} + C|\tilde{a}^{(1)} - \tilde{a}^{(2)}| \int_U (|\nabla p_1|^{2-a} + |\nabla p_2|^{2-a}) dx.
$$

By constructing appropriate function $\zeta(x) \in [0, 1]$ with $\zeta \equiv 1$ on $V$, we have

$$
\int_V K(\xi, \tilde{a}^{(1)} \lor a^{(2)}) |\nabla P|^2 dx \leq \int_U K(\xi, \tilde{a}^{(1)} \lor a^{(2)}) |\nabla P|^2 dx
$$

$$
\quad \leq C \|\tilde{P}\|_{L^2} \left\{ \|\tilde{p}_1\|_{L^2} + \|\tilde{p}_2\|_{L^2} + \left( \int_U (|\nabla p_1| + |\nabla p_2|)^{2-2a} dx \right)^{1/2} \right\}
$$

$$
\quad + C|\tilde{a}^{(1)} - \tilde{a}^{(2)}| \int_U (|\nabla p_1|^{2-a} + |\nabla p_2|^{2-a}) dx. \quad (5.38)
$$

Using (4.78) with $K(\xi) = K(\xi, \tilde{a}^{(1)} \lor a^{(2)})$ and (5.38) we obtain

$$
|\nabla P(t)|_{L^{2-\delta}(U')} \leq C \|\tilde{P}(t)\|_{L^2} \left\{ \left( 1 + \|\tilde{p}_1(t)\|_{L^2}^2 + \|\tilde{p}_2(t)\|_{L^2}^2 + J_{H_1}[p_1] + J_{H_2}[p_2] \right)^{1/4}
$$

$$
\quad + |\tilde{a}^{(1)} - \tilde{a}^{(2)}|^{1/2} \left( 1 + J_{H_1}[p_1] + J_{H_2}[p_2] \right)^{1/2} \left( \int_V (1 + |\nabla p_1| + |\nabla p_2|) \frac{a(2-\delta)}{\delta} dx \right)^{\frac{\delta}{2(2-\delta)}}.
$$

Then simply due to $|\tilde{a}^{(1)} - \tilde{a}^{(2)}| \leq C$, inequality (5.36) follows.

To prove (5.37), we have from (5.38) that

$$
\int_0^T \int_V K(\xi) |\nabla P|^2 dx dt
$$

$$
\leq C \left( \int_0^T \|\tilde{P}\|_{L^2}^2 dt \right)^{1/2} \left( \int_0^T \|\tilde{p}_1(t)\|_{L^2}^2 + \|\tilde{p}_2(t)\|_{L^2}^2 dt + \int_0^T \int_U (|\nabla p_1| + |\nabla p_2|)^{2-2a} dx dt \right)^{1/2}
$$

$$
+ C|\tilde{a}^{(1)} - \tilde{a}^{(2)}| \int_0^T \int_U (|\nabla p_1|^{2-a} + |\nabla p_2|^{2-a}) dx dt. \quad (5.39)
$$

According to (4.80),

$$
\int_0^T \int_U |\nabla P|^{2-\delta} dx dt \leq C \left( \int_0^T \int_V K(\xi) |\nabla P \zeta|^2 dx dt \right)^{\frac{2-\delta}{2}} \left( \int_0^T \int_V (1 + |\nabla p_1| + |\nabla p_2|) \frac{a(2-\delta)}{\delta} dx dt \right)^{\frac{\delta}{2}}.
$$

(5.40)
Using (5.38) in (5.40), we obtain
\[
\int_0^T \int_U |\nabla P|^{2-\delta} \, dx \, dt \\
\leq C \left\{ \left( \int_0^T \| \bar{P} \|_{L^2}^2 \, dt \right)^{1/2} \left( \int_0^T \| \bar{p}_1 \|_{L^2}^2 + \| \bar{p}_2 \|_{L^2}^2 \, dt + \int_0^T \int U (|\nabla p_1| + |\nabla p_2|)^{2-2\alpha} \, dx \, dt \right)^{1/2} + |\bar{a}^{(1)} - \bar{a}^{(2)}| \int_0^T \int_U (|\nabla p_1|^{2-\alpha} + |\nabla p_2|^{2-\alpha}) \, dx \, dt \right\}^{2\alpha} \left( \int_0^T \int_V (1 + |\nabla p_1| + |\nabla p_2|)^{\alpha(2-\delta)} \, dx \, dt \right)^{\frac{\delta}{2}}.
\]

Thus, we have (5.37).

**Theorem 5.7.** Assume (SDC). For \( \delta \in (0, a) \), \( 0 < t_0 < 1 \), and \( T > t_0 \) we have
\[
\sup_{[t_0, T]} \| \nabla P(t) \|_{L^{2-\delta}(U')} \leq CM_{8, t_0, T}(\| \bar{P}(0) \|_{L^2} + |\bar{a}^{(1)} - \bar{a}^{(2)}|^{1/2})^{\frac{\delta}{2}},
\]
where \( M_{8, t_0, T} \) is defined in (5.42) below.

**Proof.** Using estimates (5.8), (3.90), (3.88) and (4.84) in (5.36), we obtain (5.41) with
\[
M_{8, t_0, T} = M_{5, T}^{1/4} \left( t_0^{-1} \sum_{i=1,2} L_5(t_0; [p_i(0)]) + (T + 1)K_{1, T}^{\frac{2-a}{1-a}} + K_{2, T} \right)^{1/2} \cdot \left\{ \sum_{i=1,2} L_2(\nu_1; [p_i(0)]) (T + 1) \frac{2(\nu_1 - 2)}{2-\alpha} + 1 \frac{\nu_1 - a}{1-a} \right\}^{\frac{\delta}{2(2-\delta)}}.
\]

**Theorem 5.8.** Assume (SDC). Suppose \( \Upsilon_1, \Upsilon_2 \) (defined in (3.63)) and \( A_2 \) (defined by (3.99)) are finite. Then for any \( \delta \in (0, a) \), we have
\[
\limsup_{t \to \infty} \| \nabla P(t) \|_{L^{2-\delta}(U')} \leq C\Upsilon_{10}|\bar{a}^{(1)} - \bar{a}^{(2)}|^{1/4},
\]
where \( \Upsilon_{10} \) is defined by (5.47) below.

**Proof.** Taking limsup of (5.36) and making use the limits (5.10), (3.93) and estimate (3.65) with \( s = \nu_1 \) yield
\[
\limsup_{t \to \infty} \| \nabla P(t) \|_{L^{2-\delta}} \leq C \left( A_2^{\frac{b+1}{4}} |\bar{a}^{(1)} - \bar{a}^{(2)}| \right)^{\frac{1}{4}} A_2^{1/2} \left\{ \sum_{i=1,2} L_4(\nu_1; [p_i(0)]) \Upsilon_1^{\nu_1 - 2} \Upsilon_2 \right\}^{\frac{\delta}{2(2-\delta)}}.
\]
Therefore, we obtain (5.43) with
\[
\Upsilon_{10} = A_2^{\frac{b+3}{4}} \left\{ \Upsilon_1^{\nu_1 - 2} \Upsilon_2 \sum_{i=1,2} L_4(\nu_1; [p_i(0)]) \right\}^{\frac{\delta}{2(2-\delta)}}.
\]

**Theorem 5.9.** Assume (SDC). Let \( \delta \in (0, a) \) and \( T > 0 \). Then there exists a constant \( C > 0 \) depending on \( U, U', \delta \) such that
\[
\| \nabla P \|_{L^{2-\delta}(U' \times (0, T))} \leq CM_{9, T} \left( \| \bar{P}(0) \|_{L^2}^{1/2} + |\bar{a}^{(1)} - \bar{a}^{(2)}|^{1/2} + |\bar{a}^{(1)} - \bar{a}^{(2)}|^{1/4} \right),
\]
where \( M_{9, T} \) is defined by (5.48) below.
Lemma A.2. Define $N_{2,T} = \sum_{i=1}^{2} \|\bar{\mu}_i(0)\|_{L^2}^2 + \sum_{i=1}^{2} \|\nabla p_i(0)\|_{L^{2-a}}^2 - (T+1)K_1^{2-a} + K_{2,T}$. From (3.28) and (3.88) we have
\[
\sum_{i=1}^{2} \|\bar{\mu}_i\|_{L^2(U \times (0,T))}^2 + \sum_{i=1}^{2} \int_0^T \int_U |\nabla p_i(x,t)|^{2-a} \, dx \, dt \leq CN_{2,T}. \tag{5.46}
\]
Neglecting the negative term on the right-hand side of (5.22) and integrating in $t$ twice yield
\[
\|\bar{P}\|_{L^2(U \times (0,T))} \leq CT\left(\|\bar{P}(0)\|_{L^2}^2 + |\bar{a}^{(1)}(t) - \bar{a}^{(2)}(t)|^2 \int_0^T \Lambda(t) \, dt \right) \leq CN_{2,T} \left(\|\bar{P}(0)\|_{L^2}^2 + |\bar{a}^{(1)} - \bar{a}^{(2)}| \right). \tag{5.47}
\]
Combining (5.46), (5.47), (4.89) and (5.37) we get
\[
\|\nabla P\|_{L^{2-a}(U \times (0,T))} \leq C \left\{ T^{1/4} N_{2,T}^{1/4} \left(\|\bar{P}(0)\|_{L^2}^2 + |\bar{a}^{(1)}(t) - \bar{a}^{(2)}(t)|\right)^{1/4} \cdot N_{2,T}^{1/4} + |\bar{a}^{(1)}(t) - \bar{a}^{(2)}(t)|^{1/2} N_{2,T}^{1/2} \right\} \cdot \left\{ \sum_{i=1,2} L_1(\nu_1 + a; [p_i(0)]) \right\} (T + 1)^{2/\nu_1 - 1} k_1^{\nu_1} \right\} \cdot \frac{\delta}{2(2-\delta)}. \tag{5.48}
\]
Therefore, we obtain (5.45) with
\[
M_{0,T} = (T+1)^{\frac{\delta}{2}} N_{2,T}^{\frac{1}{2}} \left\{ \sum_{i=1,2} L_1(\nu_1 + a; [p_i(0)]) \right\} (T + 1)^{2/\nu_1 - 1} k_1^{\nu_1} \right\} \cdot \frac{\delta}{2(2-\delta)}. \tag{5.49}
\]
The proof is complete. \hfill \qed

**APPENDIX A. AUXILIARIES**

We recall a classical result on fast decaying geometry sequences.

**Lemma A.1** (cf. [20], Chapter II, Lemma 5.6). Let $\{Y_i\}_{i=0}^\infty$ be a sequence of non-negative numbers satisfying
\[
Y_{i+1} \leq AB^i Y_i^{1+\mu}, \quad i = 0, 1, 2, \ldots, \tag{A.1}
\]
where $A > 0$, $B > 1$ and $\mu > 0$. Then
\[
Y_i \leq A \left(\frac{1+\mu}{\nu} \right)^i B \left(\frac{1+\mu}{\nu} \right)^i \frac{1}{\nu} Y_0^{(1+\mu)i}, \quad i = 0, 1, 2, \ldots \tag{A.2}
\]
Consequently, if $Y_0 \leq A^{-1/\mu} B^{-1/\nu}$ then $\lim_{i \to \infty} Y_i = 0$.

The following generalization is an important ingredient for our iteration technique used in this paper.

**Lemma A.2.** Let $\{Y_i\}_{i=0}^\infty$ be a sequence of non-negative numbers satisfying
\[
Y_{i+1} \leq \sum_{k=1}^{m} A_k B_k Y_i^{1+\mu_k}, \quad i = 0, 1, 2, \ldots, \tag{A.3}
\]
where $A_k > 0$, $B_k > 1$ and $\mu_k > 0$ for $k = 1, 2, \ldots, m$. Let $B = \max\{B_k : 1 \leq k \leq m\}$ and $\mu = \min\{\mu_k : 1 \leq k \leq m\}$. Then the following statement holds true.
\[
\text{If } \sum_{k=1}^{m} A_k Y_0^{\mu_k} \leq B^{-1/\mu} \text{ then } \lim_{i \to \infty} Y_i = 0. \tag{A.4}
\]
In particular,
\[
\text{if } Y_0 \leq \min\{(m^{-1} A^{-1/\mu} B^{-1/\nu})^{1/\mu_k} : 1 \leq k \leq m\} \text{ then } \lim_{i \to \infty} Y_i = 0. \tag{A.5}
\]
Proof. It is obvious that (A.5) is a consequence of (A.4) when we require \( A_k Y_0^{\mu_k} \leq m^{-1} B^{-1/\mu} \) for each \( k = 1, 2, \ldots, m \). We now prove (A.4). To make a clear presentation, we consider \( m = 2 \). The proof can be robustly modified to cover general \( m \). Therefore, we consider a sequence \( \{Y_i\}_{i=0}^\infty \) of non-negative numbers that satisfies
\[
Y_{i+1} \leq B^i (A_1 Y_i^{1+\mu_1} + A_2 Y_i^{1+\mu_2}) \quad \text{for all } i \geq 0,
\]
where \( A_1, A_2 > 0, B > 1 \) and \( 0 < \mu_1 \leq \mu_2 \), and \( Y_0 \) satisfies
\[
A_1 Y_0^{\mu_1} + A_2 Y_0^{\mu_2} \leq B^{-1/\mu_1}.
\] (A.6)
Note, in this case, that \( \mu = \mu_1 \).

Claim. If there is \( D > 0 \) such that
\[
Y_0 \leq D (A_1 D^{\mu_1} + A_2 D^{\mu_2})^{-1/\mu_1} B^{-1/\mu_1} \leq D,
\] (A.7)
then
\[
Y_i \leq D \quad \text{for all } i \geq 0,
\] (A.8)
and
\[
\lim_{i \to \infty} Y_i = 0.
\] (A.9)

Proof of the Claim. First, we prove (A.8) by induction. By condition (A.7), the inequality (A.8) holds for \( i = 0 \). Let \( Y_j \leq D \) for all \( 0 \leq j \leq i \). We then have for \( 0 \leq j \leq i \) that
\[
Y_{j+1} \leq B^j (A_1 Y_j^{1+\mu_1} + A_2 Y_j^{1+\mu_2}) = B^j \left[ A_1 D^{1+\mu_1} \left( \frac{Y_j}{D} \right)^{1+\mu_1} + A_2 D^{1+\mu_2} \left( \frac{Y_j}{D} \right)^{1+\mu_2} \right] \leq B^j \left[ A_1 D^{1+\mu_1} + A_2 D^{1+\mu_2} \right] \left( \frac{Y_j}{D} \right)^{1+\mu_1} \leq B^j \left[ A_1 D^{1+\mu_1} + A_2 D^{1+\mu_2} \right] Y_j^{1+\mu_1}.
\]
Then \( Y_{j+1} \leq \tilde{A} B^j Y_j^{1+\mu_1} \) for all \( 0 \leq j \leq i \), where \( \tilde{A} = (A_1 D^{\mu_1} + A_2 D^{\mu_2})/D^{\mu_1} \). Note from (A.7) that
\[
Y_0 \leq \tilde{A}^{1-1/\mu_1} B^{-1/\mu_1} \leq D.
\] (A.10)
Applying Lemma (A.1) to sequence \( \{Y_0, \ldots, Y_{i+1}, 0, 0, \ldots\} \), we have
\[
Y_{i+1} \leq \tilde{A}^{\frac{(1+\mu_1)+(1+\mu_2)}{\mu_1}} B^{\frac{(1+\mu_1)+(1+\mu_2)}{\mu_1}} \frac{Y_0^{(1+\mu_1)+(1+\mu_2)}}{D^{\mu_1}}.
\]
This and the first inequality in (A.10) give
\[
Y_{i+1} \leq \tilde{A}^{-1/\mu_1} B^{-1/\mu_1} B^{1-1/\mu_1} = D.
\] (A.11)
It follows from this and the second inequality in (A.10) that \( Y_{i+1} \leq \tilde{A}^{-1/\mu_1} B^{-1/\mu_1} \leq D \). Thus we obtain (A.8) for \( i + 1 \), and by the induction principle, statement (A.8) must hold true for all \( i \geq 0 \). Since (A.8) now holds, the estimate (A.11) also holds for all \( i \), thus the limit (A.9) follows. The proof of Claim is complete.

By the virtue of the Claim, it suffices to find \( D > 0 \) that satisfies two inequalities in (A.7). Let \( D > 0 \) be the unique positive number such that \( A_1 D^{\mu_1} + A_2 D^{\mu_2} = B^{-1/\mu_1} \). Obviously, \( D(A_1 D^{\mu_1} + A_2 D^{\mu_2})^{-\frac{1}{\mu_1}} B^{-\frac{1}{\mu_1}} = D \). Then the second inequality in (A.7) is satisfied. Since (A.6) implies \( D \geq Y_0 \), the first inequality in (A.7) is also satisfied. Our proof is complete. \( \square \)

The following is a useful inequality in estimating limits superior of solutions.
Lemma A.3. For \( f(t) \geq 0, g(t), h(t) > 0, y(t) \geq 0, t > T, \) let
\[
y(t) = h(t) \int_{T}^{t} e^{-\int_{T}^{\tau} y(\theta) d\theta} f(\tau) d\tau.
\]
Assume
\[
\int_{T}^{\infty} g(t) dt = \infty \quad \text{and} \quad \lim_{t \to \infty} \frac{h'(t)}{h(t) g(t)} = 0. \tag{A.12}
\]
Then
\[
\limsup_{t \to \infty} y(t) \leq \limsup_{t \to \infty} \frac{h(t) f(t)}{g(t)}. \tag{A.13}
\]
Proof. Let \( \delta > 0 \). We have for \( t > T \) that
\[
y'(t) = \left( -g + \frac{h'}{h} \right) y + hf = -\left( 1 - \frac{h'}{hg} \right) gy + hf.
\]
By the second condition in (A.12), for sufficiently large \( t \), we have
\[
y'(t) \leq -(1 - \delta)gy + hf.
\]
With the first condition in (A.12), we apply Lemma A.1(ii) in [19] and obtain
\[
\limsup_{t \to \infty} y(t) \leq \frac{1}{1 - \delta} \limsup_{t \to \infty} \frac{h(t) f(t)}{g(t)}.
\]
Letting \( \delta \to 0 \), we obtain (A.13). \( \square \)

References

[1] Aurila, E., Bloshanskaya, L., Hoang, L., and Ibragimov, A. Analysis of generalized Forchheimer flows of compressible fluids in porous media. J. Math. Phys. 50, 10 (2009), 103102, 44.
[2] Bear, J. Dynamics of Fluids in Porous Media. Dover, New York, 1972.
[3] Brézis, H. Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert. North-Holland Publishing Co., Amsterdam, 1973. North-Holland Mathematics Studies, No. 5. Notas de Matemática (50).
[4] Çelebi, A. O., Kalantarov, V. K., and Üğurlu, D. Continuous dependence for the convective Brinkman-Forchheimer equations. Appl. Anal. 84, 9 (2005), 877–888.
[5] Çelebi, A. O., Kalantarov, V. K., and Üğurlu, D. On continuous dependence on coefficients of the Brinkman-Forchheimer equations. Appl. Math. Lett. 19, 8 (2006), 801–807.
[6] Chadam, J., and Qin, Y. Spatial decay estimates for flow in a porous medium. SIAM J. Math. Anal. 28, 4 (1997), 808–830.
[7] Darcy, H. Les Fontaines Publiques de la Ville de Dijon. Dalmont, Paris, 1856.
[8] De Giorgi, E. Sulla differenziabilità e l’analiticità delle estremali degli integrali multipli regolari. Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) 3 (1957), 25–43.
[9] DiBenedetto, E. Degenerate parabolic equations. Universitext. Springer-Verlag, New York, 1993.
[10] Dung, L. Ultimately uniform boundedness of solutions and gradients for degenerate parabolic systems. Nonlinear Anal. 39, 2, Ser. A: Theory Methods (2000), 157–171.
[11] Dupuit, J. Mouvement de l’eau a travers les terrains permeables. C. R. Hebd. Seances Acad. Sci. 45 (1857), 92–96.
[12] Foias, C., Manley, O., Rosa, R., and Temam, R. Navier-Stokes equations and turbulence, vol. 83 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2001.
[13] Forchheimer, P. Wasserbewegung durch Boden Zeit, vol. 45. Ver. Deut. Ing., 1901.
[14] Forchheimer, P. Hydraulik. No. Leipzig, Berlin, B. G. Teubner. 1930. 3rd edition.
[15] Franchi, F., and Straughan, B. Continuous dependence and decay for the Forchheimer equations. R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 459, 2040 (2003), 3195–3202.
[16] Gentile, M., and Straughan, B. Structural stability in resonant penetrative convection in a Forchheimer porous material. Nonlinear Anal. Real World Appl. 14, 1 (2013), 397–401.
[17] Hoang, L., and Ibragimov, A. Structural stability of generalized Forchheimer equations for compressible fluids in porous media. *Nonlinearity* 24, 1 (2011), 1–41.
[18] Hoang, L., and Ibragimov, A. Qualitative study of generalized forchheimer flows with the flux boundary condition. *Adv. Diff. Eq.* 17, 5–6 (2012), 511–556.
[19] Hoang, L., Ibragimov, A., Kieu, T., and Sobol, Z. Stability of solutions to generalized forchheimer equations of any degree. submitted.
[20] Ladyženskaja, O. A., Solonnikov, V. A., and Ural’ceva, N. N. *Linear and quasilinear equations of parabolic type*. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.
[21] Muskat, M. *The flow of homogeneous fluids through porous media*. McGraw-Hill Book Company, inc., 1937.
[22] Nield, D. A., and Bejan, A. *Convection in porous media*, second ed. Springer-Verlag, New York, 1999.
[23] Payne, L. E., Song, J. C., and Straughan, B. Continuous dependence and convergence results for Brinkman and Forchheimer models with variable viscosity. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* 455, 1986 (1999), 2173–2190.
[24] Payne, L. E., and Straughan, B. Convergence and continuous dependence for the Brinkman-Forchheimer equations. *Stud. Appl. Math.* 102, 4 (1999), 419–439.
[25] Qin, Y., and Kaloni, P. N. Spatial decay estimates for plane flow in Brinkman-Forchheimer model. *Quart. Appl. Math.* 56, 1 (1998), 71–87.
[26] Ragnedda, F., Vernier Piro, S., and Vespri, V. Asymptotic time behaviour for non-autonomous degenerate parabolic problems with forcing term. *Nonlinear Anal.* 71, 12 (2009), e2316–e2321.
[27] Ragnedda, F., Vernier Piro, S., and Vespri, V. Large time behaviour of solutions to a class of non-autonomous, degenerate parabolic equations. *Math. Ann.* 348, 4 (2010), 779–795.
[28] Showalter, R. E. *Monotone operators in Banach space and nonlinear partial differential equations*, vol. 49 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.
[29] Straughan, B. *Stability and wave motion in porous media*, vol. 165 of *Applied Mathematical Sciences*. Springer, New York, 2008.
[30] Vázquez, J. L. *Smoothing and decay estimates for nonlinear diffusion equations*, vol. 33 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2006. Equations of porous medium type.
[31] Vázquez, J. L. *The porous medium equation*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, Oxford, 2007. Mathematical theory.
[32] Zeidler, E. *Nonlinear functional analysis and its applications. II/B*. Springer-Verlag, New York, 1990. Nonlinear monotone operators, Translated from the German by the author and Leo F. Boron.

† Department of Mathematics and Statistics, Texas Tech University, Box 41042, Lubbock, TX 79409–1042, U. S. A.  
*E-mail address: luan.hoang@ttu.edu  
E-mail address: thinh.kieu@ttu.edu  
‡ Department of Mathematics, University of Tennessee, Knoxville, 227 Ayress Hall, 1403 Circle Drive, Knoxville, TN 37996  
E-mail address: phan@math.utk.edu  
∗Corresponding Author