Deformations of Calogero-Moser Systems

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Abstract. Recent results are surveyed pertaining to the complete integrability of some novel \(n\)-particle models in dimension one. These models generalize the Calogero-Moser systems related to classical root systems.

1. Introduction

The Hamiltonian of the celebrated Calogero-Moser (CM) system \([1]\) is given by

\[
H_{cm} = \frac{1}{2} \sum_{1 \leq j \leq n} \theta_j^2 + g^2 \sum_{1 \leq j < k \leq n} \wp(x_j - x_k),
\]

where \(\wp(z)\) denotes the Weierstraß \(\wp\)-function \([2]\) or a degeneration thereof \((1/z^2, 1/sh^2(z)\) or \(1/\sin^2(z)\)). The integrability of \(H_{cm}\) was proved with the aid of a Lax matrix \([1]\).

Some years ago a relativistic generalization of \(H_{cm}\) was introduced \([3, 4, 5]\). The Hamiltonian of the relativistic system \((RCM)\) reads

\[
H_{rcm} = \sum_{1 \leq j \leq n} \cosh(\beta \theta_j) \prod_{k \neq j} \left[ 1 + \beta^2 g^2 \wp(x_j - x_k) \right]^{1/2}.
\]

One can look upon the RCM system as a one-parameter deformation of the CM model, with \(\beta \sim 1/c\) (the inverse of the speed of light) acting as deformation parameter. For \(\beta \to 0\), which corresponds to the nonrelativistic limit, \(\beta^{-2}(H_{rcm}(\beta) - n)\) converges to \(H_{cm}\). The relativistic system is also integrable; explicit formulas have been found for a complete set of integrals in involution:

\[
H_{l,rcm} = \sum_{J \subset \{1, \ldots, n\}} e^{-\beta \sum_{j \in J} \theta_j} \prod_{j \in J} \prod_{k \in J^C} \left[ 1 + \beta^2 g^2 \wp(x_j - x_k) \right]^{1/2}, \quad l = 1, \ldots, n.
\]

From a Lie-theoretic perspective the above \(n\)-particle models are connected with the root system \(A_{n-1}\). Here, we will take a look at similar deformations of the CM systems related to classical root systems other than \(A_{n-1}\) (i.e. \(B_n, C_n, D_n\) and \(BC_n\)). A more detailed discussion of the material covered below (including proofs) can be found in the papers \([6, 7]\).

2. Trigonometric Potentials

In the case of trigonometric potentials our system is characterized by the Hamiltonian

\[
H = \sum_{1 \leq j \leq n} \left( \cosh(\beta \theta_j) V_j^{1/2} V_{-j}^{1/2} - (V_j + V_{-j})/2 \right)
\]

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with
\[
V_{\varepsilon j} = w(\varepsilon x_j) \prod_{k \neq j} v(\varepsilon x_j + x_k) v(\varepsilon x_j - x_k), \quad \varepsilon = \pm 1, \quad (5)
\]
\[
v(z) = \frac{\sin \alpha(\mu + z)}{\sin(\alpha z)}, \quad (6)
\]
\[
w(z) = \frac{\sin \alpha(\mu_0 + z) \cos \alpha(\mu_1 + z) \sin \alpha(\mu_0' + z) \cos \alpha(\mu_1' + z)}{\cos(\alpha z)} \cdot (7)
\]
One can look upon the functions \(v\) and \(w\) as potentials: \(v\) governs the interaction between the particles and \(w\) models an external field. The parameters \(\mu, \mu'_r\) and \(\mu'_r\) \((r = 0, 1)\) act as coupling constants; after setting them equal to zero the particles become free \((v, w = 1)\).

Just as for the RCM system, explicit formulas have been found that constitute a complete set of integrals in involution for the Hamiltonian \(H\) \((4)-(7)\)
\[
H_l = \sum_{j \subset \{1, \ldots, n\}, |J| \leq l, \varepsilon_j = \pm 1, j \in J} \text{ch}(\beta \theta_{\varepsilon J}) V_{\varepsilon J, J}^{1/2} V_{-\varepsilon J, J}^{1/2} U_{J, l - |J|}, \quad l = 1, \ldots, n, \quad (8)
\]
with
\[
\theta_{\varepsilon J} = \sum_{j \in J} \varepsilon_j \theta_j, \quad (9)
\]
\[
V_{\varepsilon J, K} = \prod_{j \in J} w(\varepsilon_j x_j) \prod_{j, j' \in J, j < j'} v^2(\varepsilon_j x_j + \varepsilon_j' x_j') \prod_{j' \in J, k \in K} v(\varepsilon_j x_j + x_k) v(\varepsilon_j x_j - x_k), \quad (10)
\]
\[
U_{l, p} = \sum_{\varepsilon_i = \pm 1, i \in I} (-1)^q \sum_{0 \leq I_1 \subset \cdots \subset I_q \subset I} \prod_{1 \leq q' \leq q} V_{\varepsilon(I_{q'} \setminus I_{q'-1}) : I_q \setminus I_{q'}} \quad (11)
\]
\((U_{1,0} = 1, I_0 = \emptyset)\). Notice that \(H_1\) coincides with the Hamiltonian \(H\) \((4)\) up to a factor two.

**Theorem 1 (Liouville integrability):** The functions \(H_1, \ldots, H_n\) are in involution (with respect to the standard Poisson bracket induced by the symplectic form \(\omega = \sum_j dx_j \wedge d\theta_j\)).

After reparametrization according to
\[
\mu = i \beta g, \quad \mu_r = i \beta g_r, \quad \mu'_r = i \beta g'_r, \quad (12)
\]
the asymptotics for \(\beta \to 0\) leads to the CM system associated with the root system \(BC_n\):
\[
H_1(\beta) = H_{1,0} \beta^2 + o(\beta^2), \quad (13)
\]
with
\[
H_{1,0} = \sum_{1 \leq j \leq n} \theta_j^2 + \beta^2 \sum_{1 \leq j \neq k \leq n} \left(\frac{1}{\sin^2 \alpha(x_j + x_k)} + \frac{1}{\sin^2 \alpha(x_j - x_k)}\right) + \alpha^2 \sum_{1 \leq j \leq n} \left(\frac{(g_0 + g'_0)^2}{\sin^2(\alpha x_j)} + \frac{(g_1 + g'_1)^2}{\cos^2(\alpha x_j)}\right) + \text{const.} \quad (14)
\]
More generally, one has
\[
H_l(\beta) = H_{1,0} \beta^{2l} + o(\beta^{2l}), \quad (15)
\]
with
\[ H_{l,0} = \sum_{J \subset \{1, \ldots, n\}, |J|=l} \prod_{j \in J} \theta_j^2 + \text{l.o.} \]  \hspace{1cm} (16)
(l.o. stands for terms of lower order in the momenta \( \theta_j \)). An immediate consequence of Theorem 1 and the above expansion formulas is the integrability of the CM Hamiltonian \( H_{1,0} \) \cite{14}.

**Theorem 2** (transition to the BC\(_n\)-type CM system): The limits
\[ H_{l,0} = \lim_{\beta \to 0} \beta^{-2l} H_l(\beta), \quad l = 1, \ldots, n, \]  \hspace{1cm} (17)
exist and the resulting functions \( H_{1,0}, \ldots, H_{n,0} \) are in involution.

3. Elliptic Potentials
As elliptic counterpart of \( H(4)-(7) \), I propose the following Hamiltonian:
\[ H = \sum_{1 \leq j \leq n} \text{ch}(\beta \theta_j) V_{1/2}^j V_{-1/2}^j + U \]  \hspace{1cm} (18)
where \( V_{\epsilon_j} \) is again of the form (4), but now with \( v \) and \( w \) given by
\[ v(z) = \frac{\sigma(\mu + z)}{\sigma(z)}, \quad w(z) = \prod_{0 \leq r \leq 3} \frac{\sigma_r(\mu_r + z) \sigma_r(\mu'_r + z)}{\sigma_r(z) \sigma_r(z)}. \]  \hspace{1cm} (19)

The function \( U \) is defined by
\[ U = \sum_{0 \leq r \leq 3} c_r \prod_{1 \leq j \leq n} v(\omega_r + x_j) v(-\omega_r - x_j), \]  \hspace{1cm} (20)
\[ c_r = \sigma(\mu)^{-2} \prod_{0 \leq s \leq 3} \sigma_s(\mu_{\pi_r(s)}) \sigma_s(\mu'_{\pi_r(s)}), \]  \hspace{1cm} (21)
where the following permutations have been introduced: \( \pi_0 = id, \pi_1 = (01)(23), \pi_2 = (02)(13) \) and \( \pi_3 = (03)(12) \). In the above expressions \( \sigma(z) \) denotes the Weierstraß \( \sigma \)-function with quasi-periods \( 2\omega_r, r = 1, 2, 3 \), and the \( \sigma_r \) are the associated functions \cite{4}
\[ \sigma_r(z) = \exp(-\eta_r z) \sigma(\omega_r + z)/\sigma(\omega_r), \quad r = 1, 2, 3. \]  \hspace{1cm} (22)

(By convention \( \omega_0 \equiv 0 \) and \( \sigma_0(z) \equiv \sigma(z) \)).

Although the integrability of \( H (18)-(21) \) has not been demonstrated yet, some partial results have been obtained \cite{4}: \( i. \) I found an additional Hamiltonian, which commutes with \( H (18)-(21) \) if the coupling constants of the external field satisfy the condition \( \sum_{0 \leq r \leq 3} (\mu_r + \mu'_r) = 0; \ ii. \) for special values of the coupling constants \( H (18)-(21) \) can be seen as a reduction of the RCM Hamiltonian \cite{3}; the integrability of the model then follows from \cite{4}; \( iii. \) if \( \mu \) equals a half-period \( \omega_r \), then straightforward generalization of the expressions in Section 2 results in an ansatz for the higher integrals; their commutativity has been verified for \( n \leq 4 \).

After setting parameters as in Eq. (12), one has
\[ H(\beta) = \text{const} + H_0 \beta^2/2 + o(\beta^2) \]  \hspace{1cm} (23)
with

$$H_0 = \sum_{1 \leq j \leq n} \theta_j^2 + g^2 \sum_{1 \leq j \neq k \leq n} (\wp(x_j + x_k) + \wp(x_j - x_k)) + \sum_{1 \leq j \leq n} (g_r + g'_r)^2 \wp(\omega_r + x_j). \quad (24)$$

A Lax pair for the flow generated by $H_0$ (24) has been presented by Inozemtsev [8].

Remarks: i. For special values of the parameters $g_r, g'_r$, the Hamiltonians $H_{1,0}$ (14) and $H_0$ (24) reduce to CM Hamiltonians that are associated with the root systems $B_n, C_n$ and $D_n$.

ii. Quantization of the Hamiltonians for $\beta \neq 0$ gives rise to difference operators instead of the usual partial differential operators. The reason is the occurrence of exponentials of the form $\exp(\pm \beta i \partial_{x_j})$ in the quantized Hamiltonian. In the case of trigonometric potentials the eigenfunctions of the quantum system turn out to be the product of a factorized ground-state wave function and recently discovered multivariable generalizations of the Askey-Wilson polynomials [6, 7].

iii. Further limits of the Hamiltonian with elliptic potentials in Eqs. (18)-(21) lead to novel $n$-particle models with nearest neighbor interaction [9]. These models generalize the nonperiodic relativistic Toda chain [10], and form a deformation of known Toda chains with very general boundary conditions [11].

References

[1] M. A. Olshanetsky, A. M. Perelomov, Classical integrable finite-dimensional systems related to Lie algebras, Phys. Reps. 71, 313 (1981).
[2] E. T. Whittaker, G. N. Watson, A course of modern analysis (Cambridge U. P., Cambridge, 1986).
[3] S. N. M. Ruijsenaars, H. Schneider, A new class of integrable systems and its relation to solitons, Ann. Phys. (N.Y.) 170, 370 (1986).
[4] S. N. M. Ruijsenaars, Complete integrability of relativistic Calogero-Moser systems and elliptic function identities, Commun. Math. Phys. 110, 191 (1987).
[5] S. N. M. Ruijsenaars, Finite-dimensional soliton systems, in Integrable and superintegrable systems, ed. B. Kupershmidt (World Scientific, Singapore, 1990), p. 165.
[6] J. F. van Diejen, Commuting difference operators with polynomial eigenfunctions, math. prepr. University of Amsterdam nr. 93-10, funct-an/9306002 (1993).
[7] J. F. van Diejen, Integrability of difference Calogero-Moser systems, math. prepr. University of Amsterdam nr. 93-19 (1993).
[8] V. I. Inozemtsev, Lax representation with spectral parameter on a torus for integrable particle systems, Lett. Math. Phys. 17, 11 (1989).
[9] J. F. van Diejen, Difference Calogero-Moser systems and finite Toda chains, in preparation.
[10] S. N. M. Ruijsenaars, Relativistic Toda systems, Commun. Math. Phys. 133, 217 (1990).
[11] V. I. Inozemtsev, The Finite Toda Lattices, Commun. Math. Phys. 121, 629 (1989).