ON A FORMULA FOR THE PI-EXponent OF LIE ALGEBRAS

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Abstract. We prove that one of the conditions in M. V. Zaicev’s formula for the PI-exponent and in its natural generalization for the Hopf PI-exponent, can be weakened. Using the modification of the formula, we prove that if a finite dimensional semisimple Lie algebra acts by derivations on a finite dimensional Lie algebra over a field of characteristic 0, then the differential PI-exponent coincides with the ordinary one. Analogously, the exponent of polynomial $G$-identities of a finite dimensional Lie algebra with a rational action of a connected reductive affine algebraic group $G$ by automorphisms, coincides with the ordinary PI-exponent. In addition, we provide a simple formula for the Hopf PI-exponent and prove the existence of the Hopf PI-exponent itself for $H$-module Lie algebras whose solvable radical is nilpotent, assuming only the $H$-invariance of the radical, i.e. under weaker assumptions on the $H$-action, than in the general case. As a consequence, we show that the analog of Amitsur’s conjecture holds for $G$-codimensions of all finite dimensional Lie $G$-algebras whose solvable radical is nilpotent, for an arbitrary group $G$.

1. Introduction

The intensive study of polynomial identities and their numeric invariants revealed the strong connection of the invariants with the structure of an algebra [2, 7, 8, 20]. If an algebra is endowed with a grading, an action of a Lie algebra by derivations, an action of a group by automorphisms and anti-automorphisms, or an action of a Hopf algebra, it is natural to consider graded, differential, $G$- or $H$-identities [3, 4, 5, 17].

In 2002, M. V. Zaicev [20] proved a formula for the PI-exponent of finite dimensional Lie algebras over an algebraically closed field of characteristic 0. It can be shown [9, 11, 12] that, under some assumptions, the natural generalization of the formula (see Subsection 3.2) holds for the exponent of graded, differential, $G$-, and $H$-identities too. In Subsection 3.3 we prove that one of the conditions can be weakened, which makes the formula easier to apply.

In [12], the authors showed that if a connected reductive affine algebraic group $G$ acts on a finite dimensional associative algebra $A$ rationally by automorphisms, then the exponent of $G$-identities coincides with the ordinary PI-exponent of $A$. Also, if a finite dimensional semisimple Lie algebra acts on a finite dimensional associative algebra by derivations, then the differential PI-exponent coincides with the ordinary one. Using the modification of M. V. Zaicev’s formula, we prove the analogous results for finite dimensional Lie algebras (Theorems 4 and 5 in Section 4).

In Section 5 we consider finite dimensional $H$-module Lie algebras $L$ such that the solvable radical of $L$ is nilpotent and $H$-invariant. We prove the analog of Amitsur’s conjecture for such algebras $L$ and provide a simple formula for the Hopf PI-exponent of $L$.

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2. Polynomial $H$-identities and their codimensions

Let $H$ be a Hopf algebra over a field $F$. An algebra $A$ over $F$ is an $H$-module algebra or an algebra with an $H$-action, if $A$ is endowed with a homomorphism $H \to \text{End}_F(A)$ such that $h(ab) = (h(1)a)(h(2)b)$ for all $h \in H$, $a, b \in A$. Here we use Sweedler’s notation $\Delta h = h(1) \otimes h(2)$ where $\Delta$ is the comultiplication in $H$.

**Example 1.** If $M$ is an $H$-module, then $\text{End}_F(M)$ is an associative $H$-module algebra where $(h \psi)(v) = h(1)\psi((Sh(2))v)$ for all $h \in H$, $\psi \in \text{End}_F(M)$, and $v \in M$. (Here $S$ is the antipode of $H$.)

We refer the reader to [11, 16, 18, 19] for an account of Hopf algebras and algebras with Hopf algebra actions.

Let $F\{X\}$ be the free nonassociative algebra over the set $X := \{x_1, x_2, x_3, \ldots\}$. Then $F\{X\} = \bigoplus_{n=1}^{\infty} F\{X\}(n)$ where $F\{X\}(n)$ is the linear span of all monomials of total degree $n$. Let $H$ be a Hopf algebra over a field $F$. Consider the algebra

$$F\{X|H\} := \bigoplus_{n=1}^{\infty} H^{\otimes n} \otimes F\{X\}(n)$$

with the multiplication $$(u_1 \otimes w_1)(u_2 \otimes w_2) := (u_1 \otimes u_2) \otimes w_1 w_2$$ for all $u_1 \in H^{\otimes j}$, $u_2 \in H^{\otimes k}$, $w_1 \in F\{X\}(j)$, $w_2 \in F\{X\}(k)$. We use the notation

$$x_{i_1}^{h_{i_1}}x_{i_2}^{h_{i_2}} \ldots x_{i_n}^{h_{i_n}} := (h_1 \otimes h_2 \otimes \ldots \otimes h_n) \otimes x_{i_1}x_{i_2} \ldots x_{i_n}$$

(the arrangements of brackets on $x_{i_j}$ and on $x_{i_j}^{h_{i_j}}$ are the same). Here $h_1 \otimes h_2 \otimes \ldots \otimes h_n \in H^{\otimes n}$, $x_{i_1}x_{i_2} \ldots x_{i_n} \in F\{X\}(n)$.

Note that if $(\gamma\beta)_{\beta \in \Lambda}$ is a basis in $H$, then $F\{X|H\}$ is isomorphic to the absolutely free nonassociative algebra over $F$ with free formal generators $x_i^{\gamma\beta}$, $\beta \in \Lambda$, $i \in \mathbb{N}$.

Define on $F\{X|H\}$ the structure of a left $H$-module by

$$h\left(x_{i_1}^{h_{i_1}}x_{i_2}^{h_{i_2}} \ldots x_{i_n}^{h_{i_n}}\right) = x_{i_1}^{h(1)h_{i_1}}x_{i_2}^{h(2)h_{i_2}} \ldots x_{i_n}^{h(n)h_{i_n}},$$

where $h(1) \otimes h(2) \otimes \ldots \otimes h(n)$ is the image of $h$ under the comultiplication $\Delta$ applied $(n - 1)$ times, $h \in H$. Then $F\{X|H\}$ is the absolutely free $H$-module nonassociative algebra on $X$, i.e. for each map $\psi: X \to A$ where $A$ is an $H$-module algebra, there exists a unique homomorphism $\tilde{\psi}: F\{X|H\} \to A$ of algebras and $H$-modules, such that $\tilde{\psi}|_X = \psi$. Here we identify $X$ with the set $\{x_i^j \mid j \in \mathbb{N}\} \subset F\{X|H\}$.

Consider the $H$-invariant ideal $I$ in $F\{X|H\}$ generated by the set

$$\{u(vw) + v(wu) + w(vu) \mid u, v, w \in F\{X|H\}\} \cup \{u^2 \mid u \in F\{X|H\}\}. \quad (1)$$

Then $L(X|H) := F\{X|H\}/I$ is the free $H$-module Lie algebra on $X$, i.e. for any $H$-module Lie algebra $L$ and a map $\psi: X \to L$, there exists a unique homomorphism $\tilde{\psi}: L(X|H) \to L$ of algebras and $H$-modules such that $\tilde{\psi}|_X = \psi$. We refer to the elements of $L(X|H)$ as $H$-polynomials and use the commutator notation for the multiplication.

**Remark.** If $H$ is cocommutative and $\text{char } F \neq 2$, then $L(X|H)$ is the ordinary free Lie algebra with free generators $x_i^{\beta}$, $\beta \in \Lambda$, $i \in \mathbb{N}$ where $(\gamma\beta)_{\beta \in \Lambda}$ is a basis in $H$, since the ordinary ideal of $F\{X|H\}$ generated by $\{1\}$ is already $H$-invariant. However, if $h(1) \otimes h(2) \neq h(2) \otimes h(1)$ for some $h \in H$, we still have

$$[x_i^{h(1)}, x_j^{h(2)}] = h[x_i, x_j] = -h[x_j, x_i] = -[x_j^{h(1)}, x_i^{h(2)}] = [x_j^{h(2)}, x_i^{h(1)}]$$

in $L(X|H)$ for all $i, j \in \mathbb{N}$, i.e. in the case $h(1) \otimes h(2) \neq h(2) \otimes h(1)$ the algebra $L(X|H)$ is not free as an ordinary Lie algebra.
Let $L$ be an $H$-module Lie algebra for some Hopf algebra $H$ over a field $F$. An $H$-polynomial $f \in L(X|H)$ is a $H$-identity of $L$ if $\psi(f) = 0$ for all homomorphisms $\psi : L(X|H) \to L$ of algebras and $H$-modules. In other words, $f(x_1, x_2, \ldots, x_n)$ is a polynomial $H$-identity of $L$ if and only if $f(a_1, a_2, \ldots, a_n) = 0$ for any $a_i \in L$. In this case we write $f \equiv 0$. The set $\text{Id}^H(L)$ of all polynomial $H$-identities of $L$ is an $H$-invariant ideal of $L(X|H)$.

Denote by $V^H_n$ the space of all multilinear $H$-polynomials in $x_1, \ldots, x_n$, $n \in \mathbb{N}$, i.e.

$$V^H_n = \langle [x^h_1, x^h_2], \ldots, [x^h_n] \rangle_{H} \quad h_i \in H, \sigma \in S_n \rangle_F \subset L(X|H).$$

(All long commutators in the article are left-normed, although this is not important in this particular case in virtue of the Jacobi identity.) The number $c^H_n(L) := \text{dim} \left( \frac{V^H_n}{\text{Id}^H_n(L)} \right)$ is called the $n$th codimension of polynomial $H$-identities or the $n$th $H$-codimension of $L$.

The analog of Amitsur’s conjecture for $H$-codimensions of $L$ can be formulated as follows.

**Conjecture.** There exists $\text{Plexp}^H(L) := \lim_{n \to \infty} \sqrt[n]{c^H_n(L)} \in \mathbb{Z}_+$.  

We call $\text{Plexp}^H(L)$ the Hopf PI-exponent of $L$.

Here we list three important particular cases:

**Example 2.** Every algebra $L$ is an $H$-module algebra for $H = F$. In this case the $H$-action is trivial and we get ordinary polynomial identities and their codimensions. (See the original definition e.g. in [2].) We write $c_n(L) := c^F_n(L)$, $\text{Id}(L) := \text{Id}^F(L)$, $V_n(L) := V^F_n(L)$, $\text{Plexp}(L) = \text{Plexp}^F(L)$.

**Example 3.** If $H = FG$ where $FG$ is the group algebra of a group $G$, then an $H$-module algebra $L$ is an algebra with a $G$-action by automorphisms. In this case we get polynomial $G$-identities and $G$-codimensions. We write $c^G_n(L) := c^{FG}_n(L)$, $\text{Id}^G(L) := \text{Id}^{FG}(L)$, $V^G_n(L) := V^{FG}_n(L)$, $\text{Plexp}^G(L) = \text{Plexp}^{FG}(L)$. Note that one can consider $G$-actions not only by automorphisms, but by anti-automorphisms too and define polynomial $G$-identities and $G$-codimensions in this case as well. (See e.g. [11] Section 1.2.)

**Example 4.** If $H = U(\mathfrak{g})$ where $U(\mathfrak{g})$ is the universal enveloping algebra of a Lie algebra $\mathfrak{g}$, then an $H$-module algebra is an algebra with a $\mathfrak{g}$-action by derivations. The corresponding $H$-identities are called differential identities or polynomial identities with derivations.

### 3. Two Formulas for the Hopf PI-Exponent

**3.1. $H$-nice Lie algebras.** The analog of Amitsur’s conjecture was proved [11] for a wide class of $H$-module Lie algebras that we call $H$-nice (see the definition below). The class of $H$-nice algebras includes finite dimensional semisimple $H$-module Lie algebras, finite dimensional $H$-module Lie algebras for finite dimensional semisimple Hopf algebras $H$, finite dimensional Lie algebras with a rational action of a reductive affine algebraic group by automorphisms, and finite dimensional Lie algebras graded by an Abelian group (see [11]).

Let $L$ be a finite dimensional $H$-module Lie algebra where $H$ is a Hopf algebra over an algebraically closed field $F$ of characteristic 0. We say that $L$ is $H$-nice if either $L$ is semisimple or the following conditions hold:

1. the nilpotent radical $N$ and the solvable radical $R$ of $L$ are $H$-invariant;
2. (Levi decomposition) there exists an $H$-invariant maximal semisimple subalgebra $B \subseteq L$ such that $L = B \oplus R$ (direct sum of $H$-modules);
3. (Wedderburn — Mal’cev decompositions) for any $H$-submodule $W \subseteq L$ and associative $H$-module subalgebra $A_1 \subseteq \text{End}_F(W)$, the Jacobson radical $J(A_1)$ is $H$-invariant and there exists an $H$-invariant maximal semisimple associative subalgebra $\bar{A}_1 \subseteq A_1$ such that $A_1 = \bar{A}_1 \oplus J(A_1)$ (direct sum of $H$-submodules);
(4) for any $H$-invariant Lie subalgebra $L_0 \subseteq \mathfrak{gl}(L)$ such that $L_0$ is an $H$-module algebra and $L$ is a completely reducible $L_0$-module disregarding $H$-action, $L$ is a completely reducible $(H, L_0)$-module.

3.2. Original formula. Let $L$ be an $H$-nice Lie algebra over an algebraically closed field $F$ of characteristic 0. Fix some Levi decomposition $L = B \oplus R$ (direct sum of $H$-submodules).

Consider $H$-invariant ideals $I_1, I_2, \ldots, I_r, J_1, J_2, \ldots, J_r, r \in \mathbb{Z}_+$, of the algebra $L$ such that $J_k \subseteq I_k$, satisfying the conditions

1. $I_k/J_k$ is an irreducible $(H, L)$-module;
2. for any $H$-invariant $B$-submodules $T_k$ such that $I_k = J_k \oplus T_k$, there exist numbers $q_i > 0$ such that

$$[[T_{1 \text{q}_1}, L, \ldots, L], [T_{2 \text{q}_2}, L, \ldots, L], \ldots, [T_{r \text{q}_r}, L, \ldots, L]] \neq 0.$$ 

Let $M$ be an $L$-module. Denote by $\text{Ann} M$ its annihilator in $L$. Let

$$d(L, H) := \max \left( \dim \frac{L}{\text{Ann}(I_1/J_1) \cap \cdots \cap \text{Ann}(I_r/J_r)} \right)$$

where the maximum is found among all $r \in \mathbb{Z}_+$ and all $I_1, \ldots, I_r, J_1, \ldots, J_r$ satisfying Conditions 1–2.

In [11] Theorem 9, see also Section 1.8] the following theorem is proved:

**Theorem 1.** Let $L$ be a non-nilpotent $H$-nice Lie algebra over an algebraically closed field $F$ of characteristic 0. Then there exist constants $C_1, C_2 > 0, r_1, r_2 \in \mathbb{R}$ such that

$$C_1 n^{r_1} d^n \leq c_n^H (L) \leq C_2 n^{r_2} d^n \text{ for all } n \in \mathbb{N}.$$ 

Here $d := d(L, H)$.

In particular, there exists $\text{PExp}^H(L) = d(L, H) \in \mathbb{Z}_+$.

3.3. Modification. Let $L$ be an $H$-nice Lie algebra. By [11] Lemma 10], $L = B \oplus S \oplus N$ for some $H$-submodule $S \subseteq R$ such that $[B, S] = 0$. Consider the associative subalgebra $A_0$ in $\text{End}_F(L)$ generated by $\text{ad} S$. Note that $A_0$ is an $H$-module algebra since $S$ is $H$-invariant. By Condition 3 of Subsection 3.1 $A_0 = \tilde{A}_0 \oplus J(A_0)$ (direct sum of $H$-submodules) where $\tilde{A}_0$ is a maximal semisimple subalgebra of $A_0$. (If $L$ is semisimple, $A_0 = \tilde{A}_0 = 0$.)

**Lemma 1.** $\tilde{A}_0 = Fe_1 \oplus \cdots \oplus Fe_q$ (direct sum of ideals) for some idempotents $e_i \in A_0$.

**Proof.** Since $R$ is solvable, by Lie’s theorem, there exists a basis of $L$ such that the matrices of all operators $\text{ad} a, a \in R$, are upper triangular. Denote the corresponding isomorphism $\text{End}_F(L) \to M_s(F)$ of algebras by $\psi$ where $s := \dim L$. Since $\psi(\text{ad} R) \subseteq UT_s(F)$, we have $\psi(A_0) \subseteq UT_s(F)$ where $UT_s(F)$ is the associative algebra of upper triangular $s \times s$ matrices. However,

$$UT_s(F) = Fe_{11} \oplus Fe_{22} \oplus \cdots \oplus Fe_{ss} \oplus \tilde{N}$$

where

$$\tilde{N} := \langle e_{ij} \mid 1 \leq i < j \leq s \rangle_F$$

is a nilpotent ideal. Since $\psi$ is an isomorphism, there is no subalgebras in $A_0$ isomorphic to $M_s^2(F)$, and $\tilde{A}_0 = Fe_1 \oplus \cdots \oplus Fe_q$ (direct sum of ideals) for some idempotents $e_i \in A_0$. 

Since $[B, S] = 0$ and $e_i$ are polynomials in $\text{ad} a, a \in S$, we have $[\text{ad} B, \tilde{A}_0] = 0$. The semisimplicity of $B$ implies $(\text{ad} B) \cap \tilde{A}_0 = \{0\}$. Now we treat $(\text{ad} B) \oplus \tilde{A}_0$ as an $H$-module Lie algebra.

**Lemma 2.** $L$ is a completely reducible $(\text{ad} B) \oplus \tilde{A}_0$- and $(H, (\text{ad} B) \oplus \tilde{A}_0)$-module.
Proof. If $L$ is semisimple, then $L = B_1 \oplus \ldots \oplus B_s$ (direct sum of $H$-invariant ideals) for some $H$-simple Lie algebras $B_i$ (see \cite[Theorem 6]{10}), and $L$ is a completely reducible $(H, (ad B) \oplus \tilde{A}_0)$-module.

Suppose now that $L$ satisfies Conditions 1-4 of Subsection 3.1. Note that $e_i$ are commuting diagonalizable operators on $L$. Hence they have a common basis of eigenvectors, and $L = \bigoplus_j W_j$ where $W_j$ are the intersections of eigenspaces of $e_i$. Each $e_i$ commutes with the operators from $ad B$. Thus $W_j$ are $(ad B)$-submodules. Recall that $B$ is semisimple. Therefore, $W_j$ is the direct sum of irreducible $(ad B)$-submodules. Since $e_i$ act on each $W_j$ as scalar operators, $L$ is the direct sum of irreducible $(ad B) \oplus \tilde{A}_0$-submodules. Now Condition 3 of Subsection 3.1 implies the lemma.

We replace Condition 2 of Subsection 3.2 with Condition 2' below:

(2') there exist $H$-invariant $(ad B) \oplus \tilde{A}_0$-submodules $T_k, I_k = J_k \oplus T_k$, and numbers $q_i \geq 0$ such that

\[ [T_1, L, \ldots, L], [T_2, L, \ldots, L], \ldots, [T_r, L, \ldots, L] \neq 0. \]

Define

\[ d'(L, H) := \max \left( \frac{\dim L}{\dim(\text{Ann}(I_1/J_1) \cap \cdots \cap \text{Ann}(I_r/J_r))} \right) \]

where the maximum is found among all $r \in \mathbb{Z}_+$ and all $I_1, \ldots, I_r, J_1, \ldots, J_r$ satisfying Conditions 1 and 2'.

**Theorem 2.** Let $L$ be an $H$-nice Lie algebra over an algebraically closed field $F$ of characteristic 0. Then $\text{PExp}^H(L) = d'(L, H)$.

**Proof.** Clearly, $d'(L, H) \geq d(L, H) = \text{PExp}^H(L)$ since, by Lemma 2, $L$ is a completely reducible $(H, (ad B) \oplus \tilde{A}_0)$-module and we can always choose $H$-invariant $(ad B) \oplus \tilde{A}_0$-submodules $T_k$ such that $I_k = J_k \oplus T_k$.

If $L$ is semisimple, then \cite[Example 7]{11} implies $d'(L, H) = d(L, H)$. Hence we may assume that $L$ satisfies Conditions 1-4 of Subsection 3.1.

We prove that there exist $r \in \mathbb{R}, C > 0$ such that $c_n^H(L) \geq C n^*(d'(L, H))^n$ for all $n \in \mathbb{N}$. We take $H$-invariant ideals $I_1, \ldots, I_r$ and $J_1, \ldots, J_r$ satisfying Conditions 1 and 2' such that

\[ \dim \frac{L}{\text{Ann}(I_1/J_1) \cap \cdots \cap \text{Ann}(I_r/J_r)} = d'(L, H). \]

Then we choose $H$-invariant $(ad B) \oplus \tilde{A}_0$-submodules $\tilde{T}_k, I_k = J_k \oplus \tilde{T}_k$, such that

\[ [\tilde{T}_1, L, \ldots, L], [\tilde{T}_2, L, \ldots, L], \ldots, [\tilde{T}_r, L, \ldots, L] \neq 0 \]

for some numbers $q_i \geq 0$. Now we repeat the arguments of \cite[Section 6]{11} with the following changes. (We use the notation from \cite[Section 6]{11}.) Instead of using Lemma 15, we choose $c_{ij} \in \tilde{A}_0$ and $d_{ij} \in J(A_0)$ such that each $ad a_{ij} = c_{ij} + d_{ij}$. Note that, by the second part of the proof of \cite[Lemma 5]{11} for $W = S$ and $M = L$, we have $J(A_0) \subseteq J(A)$ where $A$ is the associative subalgebra of $\text{End}_F(L)$ generated by the operators from $H$ and $ad L$. Hence $d_{ij} \in J(A)$. Moreover, $\tilde{T}_k$ that we have chosen by Condition 2', are $H$-invariant $B$-submodules, and we use them in \cite[Lemma 17]{11}. The rest of the proof is the same as in \cite[Section 6]{11}. Finally, we have $\text{PExp}^H(L) \geq d'(L, H)$, and the theorem is proved.

4. Lie $G$-algebras and Lie algebras with derivations

In \cite[Theorem 7]{12}, the authors proved the existence of the differential PI-exponent for finite dimensional Lie algebras with an action of a finite dimensional semisimple Lie algebra
by derivations. Here we prove that the differential PI-exponent coincides with the ordinary one.

**Theorem 3.** Let $L$ be a finite dimensional Lie algebra over an algebraically closed field $F$ of characteristic 0. Suppose a Lie algebra $\mathfrak{g}$ is acting on $L$ by derivations, and $L$ is an $U(\mathfrak{g})$-nice algebra. Then $\Pi_{\exp}(L) = \Pi_{\exp}U(\mathfrak{g})(L)$.

**Remark.** If a reductive affine algebraic group $G$ is rationally acting on $L$ by automorphisms, then $L$ is an $FG$-nice algebra [11 Example 6]. Hence if $G$ is connected and $\mathfrak{g}$ is the Lie algebra of $G$, then by [15 Theorems 13.1 and 13.2], $L$ is an $U(\mathfrak{g})$-nice algebra. In particular, a finite dimensional Lie algebra $L$ with an action of a finite dimensional semisimple Lie algebra $\mathfrak{g}$ by derivations is always an $U(\mathfrak{g})$-nice algebra, since there exists a simply connected semisimple affine algebraic group $G$ rationally acting on $L$ by automorphisms, such that $\mathfrak{g}$ is the Lie algebra of $G$ and the $\mathfrak{g}$-action is the differential of the $G$-action (see e.g. [13 Chapter XVIII, Theorem 5.1] and [12 Theorem 3]).

**Proof of Theorem 3** By Theorems 1 and 2, there exist $\Pi_{\exp}(L) = d'(L,F)$ and $\Pi_{\exp}U(\mathfrak{g})(L) = d'(L,U(\mathfrak{g}))$. If we treat differential and ordinary multilinear Lie polynomials as multilinear functions on $L$, we obtain $c_{n}(L) = c_{n}(U(\mathfrak{g})(L))$ for all $n \in \mathbb{N}$. Hence $\Pi_{\exp}(L) \leq \Pi_{\exp}U(\mathfrak{g})(L)$.

Suppose $\mathfrak{g}$-invariant ideals $I_1, I_2, \ldots, I_r, J_1, J_2, \ldots, J_r$, $r \in \mathbb{Z}_{+}$, of the algebra $L$ such that $J_k \subseteq I_k$, satisfy Conditions 1 and 2' for $H = U(\mathfrak{g})$. By Condition 2', there exist $\mathfrak{g}$-invariant (ad $B$) $\oplus \tilde{A}_0$-submodules $T_k$, $I_k = J_k \oplus T_k$, and numbers $q_i \geq 0$ such that

$$[[T_{1i_1}, L, \ldots, L], [T_{2j_2}, L, \ldots, L], \ldots, [T_{ri_r}, L, \ldots, L]] \neq 0.$$ 

By Lemma 2, $L$ is a completely reducible (ad $B$) $\oplus \tilde{A}_0$-module. Hence $T_k = T_{k_1} \oplus T_{k_2} \oplus \ldots \oplus T_{k_{n_k}}$ for some irreducible (ad $B$) $\oplus \tilde{A}_0$-submodules $T_{k_j}$. Therefore, we can choose $1 \leq j_k \leq n_k$ such that

$$[[T_{1j_1}, L, \ldots, L], [T_{2j_2}, L, \ldots, L], \ldots, [T_{rij_r}, L, \ldots, L]] \neq 0.$$ 

Let $\tilde{I}_k = T_{k_{j_k}} \oplus J_k$.

We claim that $\tilde{I}_k$ is an ideal in $L$ and Ann($\tilde{I}_k/J_k$) = Ann($I_k/J_k$) for all $1 \leq k \leq r$. Denote by $L_0$, $B_0$, $R_0$, $\mathfrak{g}_0$, respectively, the images of $L$, $B$, $R$, $\mathfrak{g}$ in $\mathfrak{gl}(I_k/J_k)$. Note that $B_0$ and $R_0$ are, respectively, semisimple and solvable. Hence $L_0 = B_0 \oplus R_0$ (direct sum of $\mathfrak{g}$-submodules) where $\mathfrak{g}$-action on $\mathfrak{gl}(I_k/J_k)$ is induced from the $\mathfrak{g}$-action on $I_k/J_k$ and corresponds to the adjoint action of $\mathfrak{g}_0$ on $\mathfrak{gl}(I_k/J_k)$. In particular, $R_0$ is a solvable ideal of $(L_0 + \mathfrak{g}_0)$ and $B_0$ is an ideal of $(B_0 + \mathfrak{g}_0)$. Note that $I_k/J_k$ is an irreducible $(L_0 + \mathfrak{g}_0)$-module. By E. Cartan’s theorem [13 Proposition 1.4.11], $L_0 + \mathfrak{g}_0 = B_1 \oplus R_1$ (direct sum of ideals) where $B_1$ is semisimple and $R_1$ is either zero or equal to the center $Z(\mathfrak{gl}(I_k/J_k))$ consisting of scalar operators. Considering the resulting projection $(L_0 + \mathfrak{g}_0) \rightarrow R_1$, we obtain $B_0 \subseteq B_1$. Since $R_0 \subseteq R_1$ consists of scalar operators, $B_0$ is an ideal of $(L_0 + \mathfrak{g}_0)$ and $B_1$.

Since $\tilde{I}_k/J_k$ is an irreducible (ad $B$) $\oplus \tilde{A}_0$-module and $\tilde{A}_0$ is acting on $I_k/J_k$ by scalar operators, $\tilde{I}_k/J_k$ is an irreducible $B_0$- and $L$-module. In particular, $\tilde{I}_k$ is an ideal.

If Ann($\tilde{I}_k/J_k$) $\neq$ Ann($I_k/J_k$), then $a \tilde{I}_k/J_k = 0$ for some $a \in L_0 \cong L$/Ann($I_k/J_k$), $a \neq 0$. Let $\varphi: L_0 \rightarrow \mathfrak{gl}(\tilde{I}_k/J_k)$ be the corresponding action and $a = b + c$ where $b \in B_0$, $c \in R_0$. Then $\varphi(b) = -\varphi(c)$ is a scalar operator on $I_k/J_k$. Hence $\varphi(b)$ belongs to the center of the semisimple algebra $\varphi(B_0)$. Thus $\varphi(b) = \varphi(c) = 0$, $b \neq 0$. Recall that $B_1$ is a semisimple algebra. Therefore $B_1 = B_0 \oplus B_2$ (direct sum of ideals) for some $B_2$. Since $R_1$ consists of
scalar operators, $I_k/J_k$ is an irreducible $B_1$-module and we have
\[ I_k/J_k = \sum_{\alpha \in B_2, \alpha \in \mathbb{Z}_+} a_1 \ldots a_\alpha \tilde{I}_k/J_k. \]

Now $[b, B_2] = 0$ and $bI_k/J_k = 0$ implies $bI_k/J_k = 0$ and $b = 0$. We get a contradiction. Hence
\[ \text{Ann}(\tilde{I}_k/J_k) = \text{Ann}(I_k/J_k). \]

Note that $\tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_r, J_1, J_2, \ldots, J_r$ satisfy Conditions 1 and 2 for $H = F$, i.e. for the case of ordinary polynomial identities. Moreover,
\[ \dim \frac{L}{\text{Ann}(I_1/J_1) \cap \cdots \cap \text{Ann}(I_r/J_r)} = \dim \frac{L}{\text{Ann}(\tilde{I}_1/\tilde{J}_1) \cap \cdots \cap \text{Ann}(\tilde{I}_r/\tilde{J}_r)}. \]

Hence $\text{PExp}^U(\theta)(L) = \text{PExp}(L)$. □

Analog for associative algebras of Theorems 4 and 5 below were proved in [12 Theorems 15 and 16].

**Theorem 4.** Let $L$ be a finite dimensional Lie algebra over a field $F$ of characteristic 0. Suppose a finite dimensional semisimple Lie algebra $g$ acts on $L$ by derivations. Then $\text{PExp}^U(\theta)(L) = \text{PExp}(L)$.

**Proof.** $H$-codimensions do not change upon an extension of the base field. The proof is analogous to the cases of ordinary codimensions of associative [3, Theorem 4.1.9] and Lie algebras [20 Section 2]. Thus without loss of generality we may assume $F$ to be algebraically closed. Now we use Theorem 3 and the remark after it. □

**Remark.** Theorem 4 implies similar asymptotic behavior of ordinary and differential codimensions, however the codimensions themselves may be different. Consider the adjoint action of $\mathfrak{sl}_2(F)$ on itself. Then $c_1(\mathfrak{sl}_2(F)) = 1 < c_1^U(\mathfrak{sl}_2(F))$ since $x_1^{e_{11}}x_2^{e_{22}}$ and $x_1^{e_{12}}$ are linearly independent modulo $\text{Id}_n^U(\mathfrak{sl}_2(F))$.

**Theorem 5.** Let $L$ be a finite dimensional Lie algebra over an algebraically closed field $F$ of characteristic 0. Suppose a connected reductive affine algebraic group $G$ is rationally acting on $L$ by automorphisms. Then $\text{PExp}^G(L) = \text{PExp}(L)$.

**Proof.** Note that the Lie algebra $g$ of the group $G$ is acting on $L$ by derivations. By [12 Lemma 5], $c_n^U(g)(L) = c_n^G(L)$ for all $n \in \mathbb{N}$. Hence Theorem 3 implies $\text{PExp}^G(L) = \text{PExp}(L)$. □

**Remark.** In Theorem 5 one could consider the case when $G$ is acting by anti-automorphisms too. However, in this case $G = G_0 \cup G_1$, $G_0 \cap G_1 = \emptyset$, where the elements of $G_0$ are acting on $L$ by automorphisms and the elements of $G_1$ are acting by anti-automorphisms. Since $G_0$ and $G_1$ are defined by polynomial equations, they are closed subsets in $G$. Recall that $G$ is connected. Therefore $G_1 = \emptyset$ and $G$ must act by automorphisms only.

5. Lie algebras with $R = N$

5.1. **Formulation of the theorem.** If the solvable radical of an $H$-module Lie algebra $L$ is nilpotent, we do not require from $L$ to satisfy Conditions 2–4 in the definition of an $H$-nice algebra (see Subsection 5.1). Moreover, the formula for the Hopf PI-exponent is simpler, than in the general case (Subsections 5.2 and 5.3).

**Theorem 6.** Let $L$ be a finite dimensional non-nilpotent $H$-module Lie algebra where $H$ is a Hopf algebra over a field $F$ of characteristic 0. Suppose that the solvable radical of $L$...
coincides with the nilpotent radical $N$ of $L$ and $N$ is an $H$-submodule. Then there exist constants $d \in \mathbb{N}$, $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$ such that

$$C_1 n^{r_1} d^n \leq c_n^H(L) \leq C_2 n^{r_2} d^n \text{ for all } n \in \mathbb{N}.$$  

Moreover, if $F$ is algebraically closed, the constant $d$ is defined as follows. Let

$$L/N = B_1 \oplus \ldots \oplus B_q \text{ (direct sum of } H\text{-invariant ideals})$$

where $B_i$ are $H$-simple Lie algebras and let $\pi : L/N \to L$ be any homomorphism of algebras (not necessarily $H$-linear) such that $\pi \varphi = \text{id}_{L/N}$ where $\pi : L \to L/N$ is the natural projection. Then

$$d = \max \left( B_{i_1} + B_{i_2} + \ldots + B_{i_r} \mid r \geq 1, \left[ \left( \prod_{q_1} H \varphi(B_{i_1}), \ldots, L \right), \ldots, \left[ \left( \prod_{q_r} H \varphi(B_{i_r}), \ldots, L \right) \right] \right] \neq 0 \text{ for some } q_i \geq 0 \right). \quad (2)$$

Remark. If $L$ is nilpotent, i.e. $[x_1, \ldots, x_p] \equiv 0$ for some $p \in \mathbb{N}$, then $V_n \subseteq \text{Id}^H(L)$ and $c_n^H(L) = 0$ for all $n \geq p$.

Theorem 6 will be proved at the end of Subsection 5.3.

**Corollary.** The analog of Amitsur’s conjecture holds for such codimensions.

Remark. The existence of a decomposition $L/N = B_1 \oplus \ldots \oplus B_q$ (direct sum of $H$-invariant ideals) where $B_i$ are $H$-simple Lie algebras, follows from [10, Theorem 6]. The existence of the map $\varphi$ follows from the ordinary Levi theorem.

Remark. Note that by [12, Lemma 9], every differential simple algebra is simple. By [12, Lemma 10], a $G$-simple algebra is simple for a rational action of a connected affine algebraic group $G$. Therefore, Theorem 6 yields another proof of Theorems 4 and 5 for the case $R = N$ since in the conditions of the latter theorems there exists an $H$-invariant Levi decomposition and we can choose $\varphi$ to be a homomorphism of $H$-modules.

**Corollary.** Let $L$ be a finite dimensional non-nilpotent Lie algebra over a field $F$ of characteristic 0 with an action of a group $G$ by automorphisms and anti-automorphisms. Suppose that the solvable radical of $L$ coincides with the nilpotent radical $N$ of $L$. Then there exist constants $d \in \mathbb{N}$, $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$ such that

$$C_1 n^{r_1} d^n \leq c_n^U(L) \leq C_2 n^{r_2} d^n \text{ for all } n \in \mathbb{N}.$$  

Proof. By [11, Lemma 28], we may assume that $G$ is acting by automorphisms only. Now we notice that radicals are invariant under all automorphisms. Hence we may apply Theorem 6.

**Corollary.** Let $L$ be a finite dimensional non-nilpotent Lie algebra over a field $F$ of characteristic 0 with an action of a Lie algebra $\mathfrak{g}$ by derivations. Suppose that the solvable radical of $L$ coincides with the nilpotent radical $N$ of $L$. Then there exist constants $d \in \mathbb{N}$, $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$ such that

$$C_1 n^{r_1} d^n \leq c_n^{U(\mathfrak{g})}(L) \leq C_2 n^{r_2} d^n \text{ for all } n \in \mathbb{N}.$$  

Proof. By [16, Chapter III, Section 6, Theorem 7], the radical is invariant under all derivations. Hence we may apply Theorem 6.

The algebra in the example below has no $G$-invariant Levi decomposition (see [10, Example 12]), however it satisfies the analog of Amitsur’s conjecture.
Example 5 (Yuri Bahturin). Let $F$ be a field of characteristic 0 and let

$$L = \left\{ \begin{pmatrix} C & D \\ 0 & 0 \end{pmatrix} \right| C \in \mathfrak{sl}_m(F), D \in M_m(F) \right\} \subseteq \mathfrak{sl}_2m(F), \quad m \geq 2.$$ 

Consider $\varphi \in \text{Aut}(L)$ where

$$\varphi \left( \begin{pmatrix} C & D \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} C & C + D \\ 0 & 0 \end{pmatrix}.$$ 

Then $L$ is a Lie algebra with an action of the group $G = \langle \varphi \rangle \cong \mathbb{Z}$ by automorphisms and there exist constants $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$ such that

$$C_1 n^{r_1} (m^2 - 1)^n \leq c_n^U(L) \leq C_2 n^{r_2} (m^2 - 1)^n \quad \text{for all } n \in \mathbb{N}.$$ 

Proof. $G$-codimensions do not change upon an extension of the base field. The proof is analogous to the cases of ordinary codimensions of associative [8, Theorem 4.1.9] and Lie algebras [20, Section 2]. Moreover, upon an extension of $F$, $L$ remains the algebra of the same type. Thus without loss of generality we may assume $F$ to be algebraically closed.

Note that

$$N = \left\{ \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \right| D \in M_m(F) \right\}$$

is the solvable (and nilpotent) radical of $L$ and $L/N \cong \mathfrak{sl}_m(F)$ is a simple Lie algebra. Hence $\text{Pexp}^U(L) = \dim \mathfrak{sl}_m(F) = m^2 - 1$ by Theorem 6.

The algebra in the example below has no $L$-invariant Levi decomposition (see [10, Example 13]), however it satisfies the analog of Amitsur’s conjecture.

Example 6. Let $L$ be the Lie algebra from Example 5. Consider the adjoint action of $L$ on itself by derivations. Then there exist constants $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$ such that

$$C_1 n^{r_1} (m^2 - 1)^n \leq c_n^U(L) \leq C_2 n^{r_2} (m^2 - 1)^n \quad \text{for all } n \in \mathbb{N}.$$ 

Proof. Again, without loss of generality we may assume $F$ to be algebraically closed. Since $L/N \cong \mathfrak{sl}_m(F)$ is a simple Lie algebra, $\text{Pexp}^U(L) = \dim \mathfrak{sl}_m(F) = m^2 - 1$ by Theorem 6.

5.2. $S_n$-cocharacters and upper bound. One of the main tools in the investigation of polynomial identities is provided by the representation theory of symmetric groups.

Let $L$ be an $H$-module Lie algebra over a field $F$ of characteristic 0. The symmetric group $S_n$ acts on the spaces $V_n^H$ by permuting the variables. Irreducible $FS_n$-modules are described by partitions $\lambda = (\lambda_1, \ldots, \lambda_s) \vdash n$ and their Young diagrams $D_\lambda$. The character $\chi_n^H(L)$ of the $FS_n$-module $V_n^H$ is called the $n$th cocharacter of polynomial $H$-identities of $L$. We can rewrite $\chi_n^H(L)$ as a sum

$$\chi_n^H(L) = \sum_{\lambda \vdash n} m(L, H, \lambda) \chi(\lambda)$$

of irreducible characters $\chi(\lambda)$. Let $e_{T\lambda} = a_{T\lambda} b_{T\lambda}$ and $e_{T\lambda}^* = b_{T\lambda} a_{T\lambda}$ where $a_{T\lambda} = \sum_{\pi \in R_{T\lambda}} \pi$ and $b_{T\lambda} = \sum_{\sigma \in C_{T\lambda}} \text{sign } \sigma \sigma$, be Young symmetrizers corresponding to a Young tableau $T\lambda$. Then $M(\lambda) = FSe_{T\lambda} \cong FSe_{T\lambda}^*$ is an irreducible $FS_n$-module corresponding to a partition $\lambda \vdash n$. We refer the reader to [2, 7, 8] for an account of $S_n$-representations and their applications to polynomial identities.

In the next two lemmas we consider a finite dimensional $H$-module Lie algebra $L$ with an $H$-invariant nilpotent ideal $N$ where $H$ is a Hopf algebra over a field $F$ of characteristic 0 and $N^p = 0$ for some $p \in \mathbb{N}$. Fix a decomposition $L/N = B_1 \oplus \ldots \oplus B_q$ where $B_i$ are some
subspaces. Let $\kappa: L/N \rightarrow L$ be an $F$-linear map such that $\pi \kappa = \text{id}_{L/N}$ where $\pi: L \rightarrow L/N$ is the natural projection. Define the number $d$ by (2).

**Lemma 3.** Let $n \in \mathbb{N}$ and $\lambda = (\lambda_1, \ldots, \lambda_s) \vdash n$. Then if $\sum_{k=d}^{s} \lambda_k \geq p$, we have $m(L, H, \lambda) = 0$.

**Proof.** It is sufficient to prove that $e_{T_{\lambda}}^* f \in \text{Id}^H(L)$ for all $f \in V_n$ and for all Young tableaux $T_{\lambda}$ corresponding to $\lambda$.

Fix a basis in $L$ that is a union of bases of $\kappa(B_1), \ldots, \kappa(B_q)$ and $N$. Since $e_{T_{\lambda}}^* f$ is multilinear, it is sufficient to prove that $e_{T_{\lambda}}^* f$ vanishes under all evaluations on basis elements. Fix some substitution of basis elements and choose 1 $\leq i_1, \ldots, i_r \leq q$ such that all the elements substituted belong to $\kappa(B_{i_1}) \oplus \cdots \oplus \kappa(B_{i_r}) \oplus N$, and for each $j$ we have an element being substituted from $\kappa(B_{i_j})$. Then we may assume that $\dim(B_{i_1} \oplus \cdots \oplus B_{i_r}) \leq d$, since otherwise $e_{T_{\lambda}}^* f$ is zero by the definition of $d$. Note that $e_{T_{\lambda}}^* f = b_{T_{\lambda}} a_{T_{\lambda}}$ and $b_{T_{\lambda}}$ alternates the variables of each column of $T_{\lambda}$. Hence if $e_{T_{\lambda}}^* f$ does not vanish, this implies that different basis elements are substituted for the variables of each column. Therefore, at least $\sum_{k=d+1}^{s} \lambda_k \geq p$ elements must be taken from $N$. Since $N^p = 0$, we have $e_{T_{\lambda}}^* f \in \text{Id}^H(L)$.

**Lemma 4.** If $d > 0$, then there exist constants $C_2 > 0$, $r_2 \in \mathbb{R}$ such that $c_n^H(L) \leq C_2 n^{r_2} d^n$ for all $n \in \mathbb{N}$. In the case $d = 0$, the algebra $L$ is nilpotent.

**Proof.** Lemma 3 and 8 Lemmas 6.2.4, 6.2.5] imply

$$\sum_{m(L, H, \lambda) \neq 0} \dim M(\lambda) \leq C_3 n^{r_3} d^n$$

for some constants $C_3, r_3 > 0$. Together with [11, Theorem 12] this inequality yields the upper bound. $\square$

5.3. Lower bound. Lemma 5 below is a version of [11, Lemma 20] adapted for our case.

**Lemma 5.** Suppose that $F$ is an algebraically closed field of characteristic 0 and let $L$, $N$, $\kappa$, $B_1$, and $d$ be the same as in Theorem 0. If $d > 0$, then there exists a number $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exist disjoint subsets $X_1, \ldots, X_{2k} \subseteq \{x_1, \ldots, x_n\}$, $k := [\frac{n-n_0}{2d}]$, $|X_1| = \ldots = |X_{2k}| = d$ and a polynomial $f \in V_n^H \setminus \text{Id}^H(L)$ alternating in the variables of each set $X_j$.

**Proof.** Without loss of generality, we may assume that $d = \dim(B_1 \oplus B_2 \oplus \cdots \oplus B_r)$ where $[[H \kappa(B_1), a_{i_1}, a_{i_2}], [H \kappa(B_2), a_{21}, a_{22}], \ldots, [H \kappa(B_r), a_{r1}, a_{r2}]] \neq 0$ for some $q_i \geq 0$ and $a_{kj} \in L$. Since $N$ is nilpotent, we can increase $q_i$ adding to $\{a_{ij}\}$ sufficiently many elements of $N$ such that

$$[[\gamma_1 \kappa(b_1), a_{i_1}, a_{i_2}], [\gamma_2 \kappa(b_2), a_{21}, a_{22}], \ldots, [\gamma_r \kappa(b_r), a_{r1}, a_{r2}]] \neq 0$$

for some $q_i \geq 0$, $b_i \in B_i$, $\gamma_i \in H$, however

$$[[\hat{b}_1, a_{11}, a_{12}], [\hat{b}_2, a_{21}, a_{22}], \ldots, [\hat{b}_r, a_{r1}, a_{r2}]] = 0$$

for all $t_i \geq 0$, $\hat{b}_i \in [H \kappa(B_i), L, \ldots, L]$ such that $b_j \in [H \kappa(B_j), L, \ldots, L]$ for at least one $j$.

Recall that $\kappa$ is a homomorphism of algebras. Moreover $\pi(h \kappa(a) - \kappa(ha)) = 0$ implies $h \kappa(a) - \kappa(ha) \in N$ for all $a \in L$ and $h \in H$. Hence, by (2), if we replace $\kappa(b_i)$ in

$$[[\gamma_1 \kappa(b_1), a_{i_1}, a_{i_2}], [\gamma_2 \kappa(b_2), a_{21}, a_{22}], \ldots, [\gamma_r \kappa(b_r), a_{r1}, a_{r2}]]$$

with the commutator of $\kappa(b_i)$ and an expression involving $\kappa$, the map $\kappa$ will behave like a homomorphism of $H$-modules. We will exploit this property further.
In virtue of [11] Theorem 11, there exist constants \( m_i \in \mathbb{Z}_+ \) such that for any \( k \) there exist multilinear associative \( H \)-polynomials \( f_i \), of degree \( (2kd_i + m_i) \), \( d_i := \dim B_i \), alternating in the variables from disjoint sets \( X_i^{(j)}, 1 \leq \ell \leq 2k \), \( |X_i^{(j)}| = d_i \), such that each \( f_i \) does not vanish under some evaluation in \((ad B_i)\).

Since \( B_i \) is an irreducible \((H, \text{ad } B_i)\)-module, by the Density Theorem, \( \text{End}_F(B_i) \) is generated by the operators from \( H \) and \( \text{ad } B_i \). Note that \( \text{End}_F(B_i) \cong M_{d_i}(F) \). Thus every matrix unit \( e^{(i)}_{j \ell} \in M_{d_i}(F) \) can be represented as a polynomial in operators from \( H \) and \( \text{ad } B_i \). Choose such polynomials for all \( i \) and all matrix units. Denote by \( m_0 \) the maximal degree of those polynomials.

Let \( n_0 := r(2m_0 + 1) + \sum_{i=1}^{r}(m_i + q_i) \). Now we choose \( f_i \) for \( k = \left\lfloor \frac{n - m_0}{2d} \right\rfloor \). In addition, we choose \( \tilde{f}_i \) for \( \tilde{k} = \left[ \frac{n - 2kd - m_1}{2d} \right] + 1 \) and \( B_i \) using [11] Theorem 11 once again. The polynomials \( f_i \) will deliver us the required alternations. However, the total degree of the product may be less than \( n \). We will use \( \tilde{f}_i \) to increase the number of variables and obtain a polynomial of degree \( n \).

By [11] Theorem 11, there exist \( \bar{x}_{11}, \ldots, \bar{x}_{i,2kd_i+m_i} \in B_i \) such that

\[
f_i(ad \bar{x}_{11}, \ldots, ad \bar{x}_{i,2kd_i+m_i}) \neq 0,
\]

and \( \bar{x}_1, \ldots, \bar{x}_{2kd_1+m_1} \in B_1 \) such that \( \tilde{f}_1(ad \bar{x}_1, \ldots, ad \bar{x}_{2kd_1+m_1}) \neq 0 \). Hence

\[
e_{\ell \ell, i}(ad \bar{x}_1, \ldots, ad \bar{x}_{i,2kd_i+m_i})e_{s,s_i}^{(i)} \neq 0
\]

and

\[
e_{\ell \ell}^{(i)} \tilde{f}_1(ad \bar{x}_1, \ldots, ad \bar{x}_{2kd_1+m_1})e_{s,s}^{(i)} \neq 0
\]

for some matrix units \( e_{\ell \ell, i}, e_{s,s_i}^{(i)} \in \text{End}_F(B_i), 1 \leq \ell, s_i \leq d_i, e_{\ell \ell}^{(i)}, e_{s,s}^{(i)} \in \text{End}_F(B_1), 1 \leq \ell, s \leq d_1 \). Thus

\[
\sum_{\ell=1}^{d_1} e_{\ell \ell}^{(i)} f_1(ad \bar{x}_1, \ldots, ad \bar{x}_{i,2kd_i+m_i}) e_{s,s}^{(i)}
\]

is a nonzero scalar operator in \( \text{End}_F(B_1) \).

Hence

\[
[\gamma_1 \times \left( \sum_{\ell=1}^{d_1} e_{\ell \ell}^{(i)} f_1(ad \bar{x}_1, \ldots, ad \bar{x}_{i,2kd_i+m_1}) e_{s,s}^{(i)} \tilde{f}_1(ad \bar{x}_1, \ldots, ad \bar{x}_{2kd_1+m_1}) e_{s,s}^{(i)} b_1 \right), a_{11}, \ldots, a_{1 q_1}],
\]

\[
[\gamma_2 \times \left( \sum_{\ell=1}^{d_2} e_{\ell \ell}^{(2)} f_2(ad \bar{x}_1, \ldots, ad \bar{x}_{i,2kd_2+m_2}) e_{s,s}^{(2)} b_2 \right), a_{21}, \ldots, a_{2 q_2}],
\]

\[
[\gamma_r \times \left( \sum_{\ell=1}^{d_r} e_{\ell \ell}^{(r)} f_r(ad \bar{x}_1, \ldots, ad \bar{x}_{i,2kd_r+m_r}) e_{s,s}^{(r)} b_r \right), a_{r1}, \ldots, a_{r q_r}],
\]

\( \neq 0 \).

Now we rewrite \( e_{ij}^{(i)} \) as polynomials in elements of \( \text{ad } B_i \) and \( H \). Using linearity of the expression in \( e_{ij}^{(i)} \), we can replace \( e_{ij}^{(i)} \) with the products of elements from \( \text{ad } B_i \) and \( H \), and the expression will not vanish for some choice of the products. By the definition of an \( H \)-module algebra, \( h(ad a)b = \text{ad}(h(a))\text{ad}(b) \) for all \( h \in H \) and \( a, b \in L \). Hence we can move all elements from \( H \) to the right. As we have mentioned, \( \times \) is a homomorphism of algebras and, by (3), behaves like a homomorphism of \( H \)-modules. Hence we get

\[
a_0 := \left[ \gamma_1 \left[ \bar{y}_{11}, [\bar{y}_{12}, \ldots [\bar{y}_{1 a_1},
\right.
\]

\[
(f_1(ad \times(\bar{x}_{11}), \ldots, ad \times(\bar{x}_{i,2kd_i+m_i}))) h_1(\bar{w}_{11}, [\bar{w}_{12}, \ldots, [\bar{w}_{r \theta_1},
\]
\[(f_1(\text{ad } \kappa(\bar{x}_1), \ldots, \text{ad } \kappa(\bar{x}_{2kd_1+m_1})))^h[w_1, [\bar{w}_2, \ldots, [\bar{w}_g, \kappa(h'_b)]], \ldots, a_{1n}, \ldots, a_{1q}],\]
\[
\begin{bmatrix}
\gamma_2 \tilde{y}_{21}, \tilde{y}_{22}, \ldots \tilde{y}_{2a_2},
\end{bmatrix}
\]
\[(f_2(\text{ad } \kappa(\bar{x}_{21}), \ldots, \text{ad } \kappa(\bar{x}_{2,2kd_2+m_2})))^{h_2}[\bar{w}_{21}, \bar{w}_{22}, \ldots, [\bar{w}_{2g_2}, \kappa(h'_b)], \ldots, a_{21}, \ldots, a_{2q}], \ldots,\]
\[
\begin{bmatrix}
\gamma_2 \bar{y}_{21}, \bar{y}_{22}, \ldots \bar{y}_{2a_2},
\end{bmatrix}
\]
\[(f_r(\text{ad } \kappa(\bar{x}_r), \ldots, \text{ad } \kappa(\bar{x}_{r,2kd_r+m_r})))^{h_r}[\bar{w}_{r1}, \bar{w}_{r2}, \ldots, [\bar{w}_{r\theta_r}, \kappa(h'_b)], \ldots, a_{r1}, \ldots, a_{rql}],\]
\[
\begin{bmatrix}
\gamma_r \bar{y}_{r1}, \bar{y}_{r2}, \ldots \bar{y}_{r\alpha_r},
\end{bmatrix}
\]
for some \(0 \leq \alpha_i, \theta_i, \tilde{\theta} \leq m_0, \ h_i, h'_i, \tilde{h} \in H, \ y_{ij}, \bar{y}_{ij} \in \kappa(B_i), \ w_j \in \kappa(B_1)\).

We assume that each \(f_i\) is a polynomial in \(x_{1i}, \ldots, x_{2kd_i+m_i}\) and \(\tilde{f}_i\) is a polynomial in \(x_{1i}, \ldots, x_{2kd_i+m_i}\). Denote \(X_\ell := \bigcup_{i=1}^r X^{(i)}_\ell\) where \(f_i\) is alternating in the variables of each \(X^{(i)}_\ell\). Let \(\text{Alt}_\ell\) be the operator of alternation in the variables from \(X_\ell\).

Consider
\[
\hat{f} := \text{Alt}_1 \text{Alt}_2 \ldots \text{Alt}_{2k} \begin{bmatrix}
\gamma_1[y_{11}, y_{12}, \ldots, y_{1a_1}],
\end{bmatrix}
\]
\[
(f_1(\text{ad } x_{11}, \ldots, \text{ad } x_{1,2kd_1+m_1}))^{h_1}[w_{11}, [w_{12}, \ldots, [w_{1\theta_1}, z_1]], \ldots, u_{11}, \ldots, u_{1q}],
\]
\[
\begin{bmatrix}
\gamma_2[y_{21}, y_{22}, \ldots, y_{2a_2}],
\end{bmatrix}
\]
\[(f_2(\text{ad } x_{21}, \ldots, \text{ad } x_{2,2kd_2+m_2}))^{h_2}[w_{21}, [w_{22}, \ldots, [w_{2g_2}, z_2]], \ldots, u_{21}, \ldots, u_{2q}], \ldots,\]
\[
\begin{bmatrix}
\gamma_2[y_{21}, y_{22}, \ldots, y_{2a_2}],
\end{bmatrix}
\]
\[(f_r(\text{ad } x_{r1}, \ldots, \text{ad } x_{r,2kd_r+m_r}))^{h_r}[w_{r1}, [w_{r2}, \ldots, [w_{r\theta_r}, z_r]], \ldots, u_{r1}, \ldots, u_{rql}],\]
\[
\begin{bmatrix}
\gamma_r[y_{r1}, y_{r2}, \ldots, y_{r\alpha_r}],
\end{bmatrix}
\]
Then the value of \(\hat{f}\) under the substitution \(z_i = \kappa(h'_b), u_{id} = a_{id}, x_{id} = \kappa(\bar{x}_{id}), x_i = \kappa(\bar{x}_i), y_{id} = \bar{y}_{id}, w_{id} = w_{id}, w_i = \bar{w}_i\) equals \((d_1)^{2k} \ldots (d_r)^{2k} a_0 \neq 0\) since \(f_i\) are alternating in the variables of each \(X^{(i)}_\ell, [B_i, B_\ell] = 0\) for \(i \neq \ell\), and \(\kappa\) is a homomorphism of algebras.

Hence
\[
f_0 := \text{Alt}_1 \text{Alt}_2 \ldots \text{Alt}_{2k} \begin{bmatrix}
\gamma_1[y_{11}, y_{12}, \ldots, y_{1a_1}],
\end{bmatrix}
\]
\[(f_1(\text{ad } x_{11}, \ldots, \text{ad } x_{1,2kd_1+m_1}))^{h_1}[w_{11}, [w_{12}, \ldots, [w_{1\theta_1}, z_1]], \ldots, u_{11}, \ldots, u_{1q}],\]
\[
\begin{bmatrix}
\gamma_2[y_{21}, y_{22}, \ldots, y_{2a_2}],
\end{bmatrix}
\]
\[(f_2(\text{ad } x_{21}, \ldots, \text{ad } x_{2,2kd_2+m_2}))^{h_2}[w_{21}, [w_{22}, \ldots, [w_{2g_2}, z_2]], \ldots, u_{21}, \ldots, u_{2q}], \ldots,\]
\[
\begin{bmatrix}
\gamma_2[y_{21}, y_{22}, \ldots, y_{2a_2}],
\end{bmatrix}
\]
\[(f_r(\text{ad } x_{r1}, \ldots, \text{ad } x_{r,2kd_r+m_r}))^{h_r}[w_{r1}, [w_{r2}, \ldots, [w_{r\theta_r}, z_r]], \ldots, u_{r1}, \ldots, u_{rql}],\]
\[
\begin{bmatrix}
\gamma_r[y_{r1}, y_{r2}, \ldots, y_{r\alpha_r}],
\end{bmatrix}
\]
does not vanish under the substitution
\[
z_i = \kappa(h'_b) \text{ for } 2 \leq i \leq r; \ u_{id} = a_{id}, x_{id} = \kappa(\bar{x}_{id}), \ y_{id} = \bar{y}_{id}, w_{id} = \bar{w}_{id}.
\]

Note that \(f_0 \in V_n^H, \ n := 2kd_r, \sum_{i=1}^r (m_i + q_i + \alpha_i + \theta_i) \leq n. \) If \(n = \bar{n}\), then we take \(f := f_0\).

Suppose \(n > \bar{n}\). Note that \((\tilde{f}_1(\text{ad } \kappa(\bar{x}_1), \ldots, \text{ad } \kappa(\bar{x}_{2kd_1+m_1}))^{h_1}[\bar{w}_1, [\bar{w}_2, \ldots, [\bar{w}_g, \kappa(h'_b)], \ldots,\]
\[
\begin{bmatrix}
\gamma_2 \bar{y}_{21}, \bar{y}_{22}, \ldots \bar{y}_{2a_2},
\end{bmatrix}
\]
is a linear combination of long commutators. Each of these commutators contains at least $2kd_i + m_1 + 1 > n - \tilde{n} + 1$ elements of $L$. Hence $f_0$ does not vanish under a substitution $z_i = [v_1, \ldots, v_{n-\tilde{n}}, \ldots]$ for some $0 \geq n - \tilde{n}$, $v_i \in L$; $z_i = \alpha(h'_ib)$ for $2 \leq i \leq r$; $u_{id} = a_{id}$, $x_{id} = \alpha(x_{id})$, $y_{id} = \tilde{y}_{id}$, $w_{id} = \tilde{w}_{id}$. Therefore,

$$f := \text{Alt}_1 \text{Alt}_2 \ldots \text{Alt}_k \left[ \gamma_1 [y_{11}, [y_{12}, \ldots, y_{1\alpha_1}],ight.$$\n
$$(f_1(ad x_{11}, \ldots, ad x_{12kd_1 + m_1}))^{k_1}[w_{11}, [w_{12}, \ldots, w_{1\theta_1}],

[[v_1, [v_2, \ldots, [v_{n-\tilde{n}}, z_1], \ldots]], u_{11}, \ldots, u_{1q_1}],

\gamma_2 [y_{21}, [y_{22}, \ldots, y_{2\alpha_2}],

(f_2(ad x_{21}, \ldots, ad x_{22kd_2 + m_2}))^{k_2}[w_{21}, [w_{22}, \ldots, w_{2\theta_2}, z_2], \ldots], u_{21}, \ldots, u_{2q_2}],

\gamma_r [y_{r1}, [y_{r2}, \ldots, y_{r\alpha_r}],

(f_r(ad x_{r1}, \ldots, ad x_{r2kd_r + m_r}))^{k_r}[w_{r1}, [w_{r2}, \ldots, w_{r\theta_r}, z_r], \ldots], u_{r1}, \ldots, u_{rq_r}]]$$

does not vanish under the substitution $v_i = \tilde{v}_i$, $1 \leq \ell \leq n - \tilde{n}$,

$$z_i = [\tilde{v}_{n-\tilde{n}+1}, [\tilde{v}_{n-\tilde{n}+2}, \ldots, [\tilde{v}_b, \alpha(h'_ib)]], \ldots];$$

$$z_{i} = \alpha(h'_ib)$$

for $2 \leq i \leq r$; $u_{id} = a_{id}$, $x_{id} = \alpha(x_{id})$, $y_{id} = \tilde{y}_{id}$, $w_{id} = \tilde{w}_{id}$. Note that $f \in V_n^H$ and satisfies all the conditions of the lemma.

Lemma 6 is an analog of Lemma 21.

**Lemma 6.** Let $k, n_0$ be the numbers from Lemma 5. Then for every $n \geq n_0$ there exists a partition $\lambda = (\lambda_1, \ldots, \lambda_d) \vdash n$, $\lambda_i \geq 2k - p$ for every $1 \leq i \leq d$, with $m(L, H, \lambda) \neq 0$. Here $p \in \mathbb{N}$ is such a number that $N^p = 0$.

**Proof.** Consider the polynomial $f$ from Lemma 5. It is sufficient to prove that $e_T^*f \notin \text{Id}^H(L)$ for some tableau $T_\lambda$ of the desired shape $\lambda$. It is known that $FS_n = \bigoplus_{\lambda, T_\lambda} FS_n e_{T_\lambda}$, where the summation runs over the set of all standard tableaux $T_\lambda$, $\lambda \vdash n$. Thus $FS_n f = \sum_{\lambda, T_\lambda} FS_n e_{T_\lambda} f \nsubseteq \text{Id}^H(L)$ and $e_{T_\lambda}^* f \notin \text{Id}^H(L)$ for some $\lambda \vdash n$. We claim that $\lambda$ is of the desired shape. It is sufficient to prove that $\lambda_d \geq 2k - p$, since $\lambda_i \geq \lambda_d$ for every $1 \leq i \leq d$. Each row of $T_\lambda$ includes numbers of no more than one variable from each $X_i$, since $e_{T_\lambda}^* = b_{T_\lambda} a_{T_\lambda}$ and $a_{T_\lambda}$ is symmetrizing the variables of each row. Thus $\sum_{i=1}^{d-1} \lambda_i \leq 2k(d - 1) + (n - 2kd) = n - 2k$. In virtue of Lemma 5, $\sum_{i=1}^{d} \lambda_i \geq n - p$. Therefore $\lambda_d \geq 2k - p$. \qed

**Proof of Theorem 6.** Let $K \supset F$ be an extension of the field $F$. Then

$$(L \otimes_F K) / (N \otimes_F K) \cong (L/N) \otimes_F K$$

is again a semisimple Lie algebra and $N \otimes_F K$ is still nilpotent. As we have already mentioned, $H$-codimensions do not change upon an extension of $F$. Hence we may assume $F$ to be algebraically closed.

The Young diagram $D_\lambda$ from Lemma 6 contains the rectangular subdiagram $D_\mu$, $\mu = (2k - p, \ldots, 2k - p)$. The branching rule for $S_n$ implies that if we consider the restriction of $S_n$-action on $M(\lambda)$ to $S_{n-1}$, then $M(\lambda)$ becomes the direct sum of all non-isomorphic $FS_{n-1}$-modules $M(\nu)$, $\nu \vdash (n - 1)$, where each $D_\nu$ is obtained from $D_\lambda$ by deleting one box.
In particular, \( \dim M(\nu) \leq \dim M(\lambda) \). Applying the rule \((n - d(2k - p))\) times, we obtain \( \dim M(\mu) \leq \dim M(\lambda) \). By the hook formula,

\[
\dim M(\mu) = \frac{(d(2k - p))!}{\prod_{i,j} h_{ij}}
\]

where \( h_{ij} \) is the length of the hook with edge in \((i, j)\). By Stirling formula,

\[
c_n^H(L) \geq \dim M(\lambda) \geq \dim M(\mu) \geq \frac{(d(2k - p))!}{((2k - p + d)!)^d} \sim \frac{\sqrt{2\pi d(2k - p)}}{(2k - p + d)^{2k - p + d}} \sim C_4 k^{r_4} d^{2kd}
\]

for some constants \( C_4 > 0, r_4 \in \mathbb{Q} \), as \( k \to \infty \). Since \( k = \left[ \frac{n - n_0}{2d} \right] \), this gives the lower bound. The upper bound has been proved in Lemma [H].

\[ \square \]

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