1. Outline.

For a finitely generated Coxeter group $\Gamma$, its virtual cohomological dimension over a (non-zero, associative) ring $R$, denoted $vcd_R \Gamma$, is finite and has been described [8,1,11,13]. In [8], M. Davis introduced a contractible $\Gamma$-simplicial complex with finite stabilisers. The dimension of such a complex gives an upper bound for $vcd_R \Gamma$. In [1], M. Bestvina gave an algorithm for constructing an $R$-acyclic $\Gamma$-simplicial complex with finite stabilisers of dimension exactly $vcd_R \Gamma$, for $R$ the integers or a prime field; he used this to exhibit a group whose cohomological dimension over the integers is finite but strictly greater than its cohomological dimension over the rationals. For the same rings, and for right-angled Coxeter groups, J. Harlander and H. Meinert [13] have shown that $vcd_R \Gamma$ is determined by the local structure of Davis’ complex and that Davis’ construction can be generalised to graph products of finite groups.

Our contribution splits into three parts. Firstly, Davis’ complex may be defined for infinitely generated Coxeter groups (and infinite graph products of finite groups). We determine which such groups $\Gamma$ have finite virtual cohomological dimension over the integers, and give partial information concerning $vcd_{\mathbb{Z}} \Gamma$. We discuss a form of Poincaré duality for simplicial complexes that are like manifolds from the point of view of $R$-homology, and give conditions for a (finite-index subgroup of a) Coxeter group to be a Poincaré duality group over $R$. We give three classes of examples: we recover Bestvina’s examples (and give more information about their cohomology); we exhibit a group whose virtual cohomological dimension over the integers is finite but strictly greater than its virtual cohomological dimension over any field; we exhibit a torsion-free rational Poincaré duality group which is not an integral Poincaré duality group.

Secondly, we discuss presentations for torsion-free subgroups of low index in right-angled Coxeter groups. In some cases (depending on the local structure of Davis’ complex) we determine the minimum number of generators for any torsion-free normal subgroup of minimal index. Using the computer package GAP [17] we find good presentations for one of Bestvina’s examples, where ‘good’ means having as few generators and relations as possible.

Finally, we give an 8-generator 12-relator presentation of a group $\Delta$ and a construction of $\Delta$ as a tower of amalgamated free products, which allows us to describe a good CW-structure for an Eilenberg-Mac Lane space $K(\Delta,1)$ and explicitly show that $\Delta$ has cohomological dimension three over the integers and cohomological dimension two over the
rational. (In fact $\Delta$ is isomorphic to a finite-index subgroup of a Coxeter group, but our proofs do not rely on this.) The starting point of the work contained in this paper was the desire to see an explicit Eilenberg-Mac Lane space for an example like $\Delta$.

2. Introduction.

A Coxeter system $(\Gamma, V)$ is a group $\Gamma$ and a set of generators $V$ for $\Gamma$ such that $\Gamma$ has a presentation of the form

$$\Gamma = \langle V \mid (vw)^{m(v,w)} = 1 \ (v, w \in V) \rangle,$$

where $m(v, v) = 1$, and if $v \neq w$ then $m(v, w) = m(w, v)$ is either an integer greater than or equal to 2, or is infinity (in which case this relation has no significance and may be omitted). Note that we do not require that $V$ should be finite. The group $\Gamma$ is called a Coxeter group, and in the special case when each $m(v, w)$ is either 1, 2 or $\infty$, $\Gamma$ is called a right-angled Coxeter group.

Remark. Let $(\Gamma, V)$ be a Coxeter system, and let $m : V \times V \to \mathbb{N} \cup \{\infty\}$ be the function occurring in the Coxeter presentation for $\Gamma$. If $W$ is any subset of $V$ and $\Delta$ the subgroup of $\Gamma$ generated by $W$, then it may be shown that $(\Delta, W)$ is a Coxeter system, with $m_W$ being the restriction of $m_V$ to $W \times W$ [5]. The function $m$ is determined by $(\Gamma, V)$ because $m(v, w)$ is the order of $vw$ (which is half the order of the subgroup of $\Gamma$ generated by $v$ and $w$).

Definition. A graph is a 1-dimensional simplicial complex (i.e. our graphs contain no loops or multiple edges). A labelled graph is a graph with a function from its edge set to a set of ‘labels’. A morphism of graphs is a simplicial map which does not collapse any edges. A morphism of labelled graphs is a graph morphism such that the image of each edge is an edge having the same label. A colouring of a graph $X$ is a function from its vertex set to a set of ‘colours’ such that the two ends of any edge have different images. Colourings of a graph $X$ with colour set $C$ are in 1-1-correspondence with graph morphisms from $X$ to the complete graph with vertex set $C$.

Definition. For a Coxeter system $(\Gamma, V)$, the simplicial complex $K(\Gamma, V)$ is defined to have as $n$-simplices the $(n+1)$-element subsets of $V$ that generate finite subgroups of $\Gamma$. Note that our $K(\Gamma, V)$ is Davis’ $K_0(\Gamma, V)$ in [8]. The graph $K^1(\Gamma, V)$ is by definition the 1-skeleton of this complex. The graph $K^1(\Gamma, V)$ has a labelling with labels the integers greater than or equal to 2, which takes the edge $\{v, w\}$ to $m(v, w)$. This labelled graph is different from, but carries the same information as, the Coxeter diagram. The labelled graph $K^1(\Gamma, V)$ determines the Coxeter system $(\Gamma, V)$ up to isomorphism, and any graph labelled by the integers greater than or equal to 2 may arise in this way. A morphism of labelled graphs from $K^1(\Gamma, V)$ to $K^1(\Delta, W)$ gives rise to a group homomorphism from $\Gamma$ to $\Delta$.

Call a subgroup of $\Gamma$ special if it is generated by a (possibly empty) subset of $V$. Thus the simplices of $K(\Gamma, V)$ are in bijective correspondence with the non-trivial finite special subgroups of $\Gamma$. Let $D(\Gamma, V)$ be the simplicial complex associated to the poset of (left) cosets of finite special subgroups of $\Gamma$. By construction $\Gamma$ acts on $D$, and the stabiliser
of each simplex is conjugate to a finite special subgroup of $\Gamma$. In [8], Davis showed that $D(\Gamma, V)$ is contractible if $V$ is finite, and the general case follows easily (for example because any cycle (resp. based loop) in $D(\Gamma, V)$ is contained in a subcomplex isomorphic to $D(\langle V' \rangle, V')$ for some finite subset $V'$ of $V$, so a fortiori bounds (resp. bounds a disc) in $D(\Gamma, V)$). Note that $K(\Gamma, V)$ is finite-dimensional if and only if $D(\Gamma, V)$ is, and in this case the dimension of $D(\Gamma, V)$ is one more than the dimension of $K(\Gamma, V)$.

A graph product $\Gamma$ of finite groups in the sense of E. R. Green [12] is the quotient of the free product of a family $\{G_v \mid v \in V\}$ of finite groups by the normal subgroup generated by the sets $\{[g, h] \mid g \in G_v, h \in G_w\}$ for some pairs $v \neq w$ of elements of $V$. A graph product of groups of order two is a right-angled Coxeter group. If a special subgroup of a graph product is defined to be a subgroup generated by some subset of the given family of finite groups, then the above definitions of $K(\Gamma, V)$ and $D(\Gamma, V)$ go through unchanged. In [13] it is proved that for a graph product of finite groups, $D(\Gamma, V)$ is contractible. (As in [8] only the case when $V$ is finite is considered, but the general case follows easily.) The group algebra for a graph product $\mathbb{Z} \Gamma$ is isomorphic to the quotient of the free coproduct of the $\mathbb{Z} G_v$ by relations that ensure that the pairs $\mathbb{Z} G_v$ and $\mathbb{Z} G_w$ generate their tensor product whenever $G_v$ and $G_w$ commute. Theorem 4.1 of [11] is a result for algebras formed in this way which in the case of the group algebra of a graph product is equivalent to the acyclicity of $D(\Gamma, V)$.

3. Virtual cohomology of Coxeter groups.

Henceforth we shall make use of the abbreviations $\text{vcd}$ and $\text{cd}$ to denote the phrases ‘virtual cohomological dimension’ and ‘cohomological dimension’, respectively, and when no ring is specified, these dimensions are understood to be over the ring of integers.

**Theorem 1.** The Coxeter group $\Gamma$ has finite $\text{vcd}$ if and only if there is a labelled graph morphism from $K^1(\Gamma, V)$ to some finite labelled graph.

**Proof.** The complex $K = K(\Gamma, V)$ has simplices of arbitrarily large dimension if and only if $V$ contains arbitrarily large finite subsets generating finite subgroups of $\Gamma$. In this case $\Gamma$ cannot have a torsion-free subgroup of finite index, and there can be no graph morphism from $K^1$ to any finite graph. Thus we may assume that $K$ and hence also $D$ are finite-dimensional. Any torsion-free subgroup of $\Gamma$ acts freely on $D$, and so it remains to show that if $D$ is finite-dimensional then there is a labelled graph morphism from $K^1$ to a finite graph if and only if $\Gamma$ has a finite-index torsion-free subgroup.

As remarked above, a morphism from $K^1(\Gamma, V)$ to $K^1(\Delta, W)$ gives rise to a group homomorphism from $\Gamma$ to $\Delta$ in an obvious way. Moreover, if $v, v'$ have product of order $m(v, v')$, then so do their images in $W$, because the edge $\{v, v'\}$ and its image in $K^1(\Delta, W)$ are both labelled by $m(v, v')$. Now if $V'$ is a finite subset of $V$ generating a finite subgroup of $\Gamma$, and $W'$ is its image in $W$, then it follows that $\langle V' \rangle$ and $\langle W' \rangle$ have identical Coxeter presentations, so are isomorphic. Thus a morphism from $K^1(\Gamma, V)$ to $K^1(\Delta, W)$ gives rise to a homomorphism from $\Gamma$ to $\Delta$ which is injective on every finite special subgroup of $\Gamma$. Now suppose that there is a morphism from $K^1(\Gamma, V)$ to $K^1(\Delta, W)$ for some finite $W$. The finitely generated Coxeter group $\Delta$ has a finite-index torsion-free subgroup $\Delta_1$, so let $\Gamma_1$ be the inverse image of this subgroup in $\Gamma$. Since $\Gamma_1$ intersects any conjugate of any finite special subgroup trivially, it follows that $\Gamma_1$ acts freely on $D(\Gamma, V)$ and is torsion-free.

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Conversely, if \( \Gamma \) has a finite-index torsion-free subgroup \( \Gamma_1 \), which we may assume to be normal, let \( Q \) be the quotient \( \Gamma/\Gamma_1 \), and build a labelled graph \( X \) with vertices the elements of \( Q \) of order two and all possible edges between them. Label the edge \( \{q, q'\} \) by the order of \( qq' \). Now the homomorphism from \( \Gamma \) onto \( Q \) induces a simplicial map from \( K^1(\Gamma, V) \) to \( X \) which is a labelled graph morphism because if \( vv' \) has finite order then its image in \( Q \) has the same order.

**Remark.** 1) If we are interested only in right-angled Coxeter groups then all the edges of \( K^1 \) have the same label, 2, and we may replace the condition that there is a morphism from \( K^1 \) to a finite labelled graph by the equivalent condition that \( K^1 \) admits a finite colouring. The above proof can be simplified slightly in this case, because the right-angled Coxeter group corresponding to a finite complete graph is a finite direct product of cyclic groups of order two.

2) An easy modification of the proof of Theorem 1 shows that a graph product \( \Gamma \) of finite groups has finite vcd if and only if there are only finitely many isomorphism types among the vertex groups \( G_v \), and the graph \( K^1(\Gamma, V) \) admits a finite colouring.

3) Let \( (\Gamma, V) \) be the Coxeter system corresponding to the complete graph on an infinite set, where each edge is labelled \( n \) for some fixed \( n \geq 3 \). Then \( K(\Gamma, V) \) is one-dimensional (since the Coxeter group on three generators such that the product of any two has order \( n \) is infinite), \( \Gamma \) has an action on a 2-dimensional contractible complex with stabilisers of orders 1, 2, and \( 2n \), but by the above theorem \( \Gamma \) does not have finite vcd. Similarly, if we take a triangle-free graph which cannot be finitely coloured, then the corresponding right-angled Coxeter group acts on a contractible 2-dimensional complex with stabilisers of orders 1, 2, and 4, but does not have finite vcd. In contrast, any group acting on a tree with finite stabilisers of bounded order has finite vcd; see for example [10], Theorem I.7.4.

If a Coxeter group \( \Gamma \) has finite vcd then \( D(\Gamma, V) \) is finite-dimensional and the dimension of \( D \) gives an upper bound for \( \text{vcd}\Gamma \). Parts a) and c) of the following theorem determine when this upper bound is attained. The information concerning the right \( \Gamma \)-module structure on various cohomology groups will be used only during the construction (in example 3 of the next section) of a torsion-free rational Poincaré duality group that is not a Poincaré duality group over the integers. To avoid cluttering the statement unnecessarily we first give some definitions that are used in it.

**Definition.** For a Coxeter system \( (\Gamma, V) \) and an abelian group \( A \), let \( A^\circ \) denote the \( \Gamma \)-bimodule with underlying additive group \( A \) and \( \Gamma \)-actions given by \( \nu a = av = -a \) for all \( v \in V \). This does define compatible actions of \( \Gamma \) because each of the relators in the Coxeter presentation for \( \Gamma \) has even length as a word in \( V \). For a \( \Gamma \)-module \( M \), let \( M_a \) denote the underlying abelian group. For a simplicial complex \( D \), let \( C_*(D) \) denote the simplicial chain complex of \( D \), let \( C_+^*(D) \) denote the augmented simplicial chain complex (having a \( -1 \)-simplex equal to the boundary of every 0-simplex) and let \( H^*(D; A) \) denote the reduced cohomology of \( D \) with coefficients in \( A \), i.e. the homology of the cochain complex \( \text{Hom}(C_+^*, D; A) \). All our \( \Gamma \)-modules (in particular, all our chain complexes of \( \Gamma \)-modules) are left modules unless otherwise stated.

**Theorem 2.** Let \( (\Gamma, V) \) be a Coxeter system such that \( \Gamma \) has finite vcd, let \( K = K(\Gamma, V) \) have dimension \( n \) (which implies that \( \text{vcd}\Gamma \leq n + 1 \)), let \( D = D(\Gamma, V) \), let \( \Gamma_1 \) be a
finite-index torsion-free subgroup of \( \Gamma \), and let \( A \) be an abelian group containing no elements of order two. Then

a) For any \( \Gamma_1 \)-module \( M \), \( H^{n+1}(\Gamma_1; M) \) is a quotient of a finite direct sum of copies of \( \tilde{H}^n(K; M_a) \).

b) For each \( j \), there is an isomorphism of right \( \Gamma \)-modules as follows.

\[
H^{j+1} \text{Hom}_{\Gamma_1}(C_*(D), A^\circ) \cong \tilde{H}^j(K; A)^\circ
\]

c) The right \( \Gamma \)-module \( H^{n+1}(\Gamma; A\Gamma) \) (which is isomorphic to \( H^{n+1}(\Gamma_1; A\Gamma_1) \) as a right \( \Gamma_1 \)-module) admits a surjective homomorphism onto \( \tilde{H}^n(K; A)^\circ \).

d) If multiplication by the order of each finite special subgroup of \( \Gamma \) induces an isomorphism of \( A \), then for each \( j \) the right \( \Gamma \)-modules \( H^{j+1}(\Gamma; A^\circ) \) and \( \tilde{H}^j(K; A)^\circ \) are isomorphic.

Proof. Let \( K' \) be the simplicial complex associated to the poset of non-trivial finite special subgroups of \( \Gamma \), so that \( K' \) is the barycentric subdivision of \( K \). Let \( D' \) be the complex associated to the poset of cosets of non-trivial finite special subgroups of \( \Gamma \). Then \( D' \) is a subcomplex of \( D \), and consists of all the simplices of \( D \) whose stabiliser is non-trivial. We obtain a short exact sequence of chain complexes of \( \mathbb{Z} \Gamma \)-modules

\[
0 \to C_*(D') \to C_*(D) \to C_*(D, D') \to 0,
\]

such that for each \( n \) the corresponding short exact sequence of \( \mathbb{Z} \Gamma \)-modules is split.

There is a chain complex isomorphism as shown below.

\[
C_*(D, D') \cong \mathbb{Z} \Gamma \otimes_{\mathbb{Z}} C^+_{*-1}(K')
\]

Topologically this is because the quotient semi-simplicial complex \( D/D' \) is isomorphic to a wedge of copies of the suspension of \( K' \), with \( \Gamma \) acting by permuting the copies freely and transitively. More explicitly, one may identify \( m \)-simplices of \( D \) with equivalence classes of \( (m+2) \)-tuples \( (\gamma, V_0, \ldots, V_m) \), where \( V_0 \subseteq \cdots \subseteq V_m \) are subsets of \( V \) generating finite subgroups of \( \Gamma \), \( \gamma \) is an element of \( \Gamma \), and two such expressions \( (\gamma, V_0, \ldots, V_m) \) and \( (\gamma', V'_0, \ldots, V'_m) \) are equivalent if \( V_i = V'_i \) for all \( i \) and the cosets \( \gamma \langle V_0 \rangle \) and \( \gamma' \langle V_0 \rangle \) are equal. A map from \( C_*(D) \) to \( \mathbb{Z} \Gamma \otimes_{\mathbb{Z}} C^+_{*-1}(K') \) may be defined by

\[
(\gamma, V_0, \ldots, V_m) \mapsto \begin{cases} 
0 & \text{if } V_0 \neq \emptyset, \\
\gamma \otimes (V_1, \ldots, V_m) & \text{if } V_0 = \emptyset,
\end{cases}
\]

and it may be checked that this is a surjective chain map with kernel \( C_*(D') \).

The claim of part a) now follows easily. Applying \( \text{Hom}_{\Gamma_1}(\cdot, M) \) to the sequence (*) and taking the cohomology long exact sequence for this short exact sequence of cochain complexes, one obtains the following sequence.

\[
H^{n+1} \text{Hom}_{\Gamma_1}(C_*(D, D'), M) \to H^{n+1} \text{Hom}_{\Gamma_1}(C_*(D), M) \to 0
\]
Now $H^{n+1}\text{Hom}_{\Gamma_1}(C_*(D), M) = H^{n+1}(\Gamma_1; M)$, and there is a chain of isomorphisms as below.

\[
H^{n+1}\text{Hom}_{\Gamma_1}(C_*(D, D'), M) \cong H^n\text{Hom}_{\Gamma_1}(Z\Gamma \otimes C_+(K'), M) \\
\cong \bigoplus_{\Gamma/\Gamma_1} H^n\text{Hom}(C_+(K'), M_{a}) \\
\cong \bigoplus_{\Gamma/\Gamma_1} \bar{H}^n(K; M_{a})
\]

To prove b), note that since $A$ has no elements of order two, there are no non-trivial $\Gamma$-module homomorphisms from the permutation module $Z\Gamma/\langle V' \rangle$ to $A^\circ$ for any non-empty subset $V'$ of $V$. Hence applying $\text{Hom}_{\Gamma}(\cdot, A^\circ)$ to the sequence ($*$) one obtains an isomorphism of cochain complexes of right $\Gamma$-modules

\[
\text{Hom}_{\Gamma}(C_*(D), A^\circ) \cong \text{Hom}_{\Gamma}(C_*(D, D'), A^\circ).
\]

Taking homology gives the following chain of isomorphisms.

\[
H^{j+1}\text{Hom}_{Z\Gamma}(C_*(D), A^\circ) \cong H^{j+1}\text{Hom}_{Z\Gamma}(C_*(D, D'), A^\circ) \\
\cong H^j\text{Hom}_{Z\Gamma}(Z\Gamma \otimes C_+(K'), A^\circ) \\
\cong H^j\text{Hom}_{Z}(C_+(K'), A^\circ) \\
\cong \bar{H}^j(K; A)^\circ
\]

Now d) follows easily. Let $R$ be the subring of $\mathbb{Q}$ generated by the inverses of the orders of the finite special subgroups of $\Gamma$. Now $\text{Hom}_{RG}(R \otimes C_*(D), A^\circ)$ is isomorphic to $\text{Hom}_{Z\Gamma}(C_*(D), A^\circ)$, and $R \otimes C_*(D)$ is a projective resolution for $R$ over $RG$, so d) follows from b).

For c), note that there is an equivalence of functors (defined on $\Gamma$-modules) between $\text{Hom}_{\Gamma}(\cdot, A\Gamma)$ and $\text{Hom}_{\Gamma_1}(\cdot, A\Gamma_1)$. In particular, $H^*(\Gamma_1; A\Gamma_1)$ and $H^*(\Gamma; A\Gamma)$ are both isomorphic to the homology of the cochain complex $\text{Hom}_{\Gamma}(C_*(D), A\Gamma)$.

There is a $\Gamma$-bimodule map $\phi$ from $A\Gamma$ to $A^\circ$ sending $a.w$ to $(-1)^l a$, where $w$ is any element of $\Gamma$ representable by a word of length $l$ in the elements of $V$. Consider the following commutative diagram of cochain complexes, where the vertical maps are induced by $\phi$:

\[
\begin{array}{ccc}
\text{Hom}_{\Gamma}(C_*(D), A\Gamma) & \rightarrow & \text{Hom}_{\Gamma}(C_*(D, D'), A\Gamma) \\
\downarrow & & \downarrow \\
\text{Hom}_{\Gamma}(C_*(D), A^\circ) & \rightarrow & \text{Hom}_{\Gamma}(C_*(D, D'), A^\circ).
\end{array}
\]

The horizontal maps are surjective because $C_i(D, D')$ is a direct summand of $C_i(D)$ for each $i$, and the lower horizontal map is an isomorphism as in the proof of b). The right-hand vertical map is surjective because $C_*(D, D')$ is $Z\Gamma$-free, and hence the left-hand vertical map is surjective.

Since each of the cochain complexes is trivial in degrees greater than $n+1$, one obtains a surjection

\[
H^{n+1}\text{Hom}_{\Gamma}(C_*(D), A\Gamma) \rightarrow H^{n+1}\text{Hom}_{\Gamma}(C_*(D), A^\circ),
\]

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and hence by b) a surjection of right $\Gamma$-modules from $H^{n+1}(\Gamma_1; A\Gamma_1)$ to $\tilde{H}^n(K; A)$.

\textbf{Remark.} Parts b) and d) of Theorem 2 do not generalise easily to graph products of finite groups having finite virtual cohomological dimension, and we have no application for these statements except in the Coxeter group case. We outline the generalisation of a) and a weaker version of c) below.

The statement and proof of part a) carry over verbatim, and there is a generalisation of part c). If $\Gamma$ is a graph product of finite groups with $l$ distinct isomorphism types of vertex group such that the graph $K^l(\Gamma, V)$ can be $m$-coloured, then the graph product version of Theorem 1 implies that $\Gamma$ admits a finite quotient $G = G_1 \times \cdots \times G_k$ for some $k \leq lm$, where each $G_i$ is isomorphic to a vertex group of $\Gamma$ and each finite special subgroup of $\Gamma$ is mapped injectively to $G$ with image of the form $G_{i(1)} \times \cdots \times G_{i(j)}$ for some subset $\{i(1), \ldots, i(j)\}$ of $\{1, \ldots, k\}$. Now for $1 \leq i \leq k$, let $x_i \in ZG$ be the sum of all the elements of $G_i$, and let $Z$ be the $\mathbb{Z}\Gamma$-module defined as the quotient of $ZG$ by the ideal generated by the $x_i$. This $Z$ is the appropriate generalisation of $\mathbb{Z}^\alpha$ to the case of a graph product, because it is a quotient of $\mathbb{Z}\Gamma$ of finite $\mathbb{Z}$-rank and contains no non-zero element fixed by any vertex group. To see this, note that

$$Z \cong \mathbb{Z}G_1/(x_1) \otimes \cdots \otimes \mathbb{Z}G_k/(x_k),$$

where $\Gamma$ acts on the $i$th factor via its quotient $G_i$. Each factor is $\mathbb{Z}$-free, and the action of $G_i$ on $\mathbb{Z}G_i/(x_i)$ has no fixed points, because for example $\mathbb{C} \otimes (\mathbb{Z}G_i/(x_i))$ does not contain the trivial $\mathbb{C}G_i$-module.

The arguments used in the proof of Theorem 2 may be adapted to prove a statement like that of part b) for the module $Z$, namely that for any $j$,

$$H^{j+1}\text{Hom}_\Gamma(C_*(D), Z) \cong \tilde{H}^j(K; Z_a).$$

From this it may be deduced that if $\Gamma_1$ is a torsion-free finite-index subgroup of $\Gamma$, then $H^{n+1}(\Gamma_1; \mathbb{Z}\Gamma_1)$ admits $\tilde{H}^n(K; Z_a)$ as a quotient. A similar result could then be deduced for any torsion-free abelian group $A$. A similar result could also be proved for $A$ an $\mathbb{F}_p$-vector space, for $p$ a prime not dividing the order of any of the vertex groups, by using the fact that $\mathbb{F}_pG$ is semisimple to deduce that $\mathbb{F}_p \otimes Z$ has no fixed points for the action of any $G_i$.

\textbf{Corollary 3.} If $(\Gamma, V)$ is a finite Coxeter system such that the topological realisation $|K|$ of $K = K(\Gamma, V)$ is the closure of a subspace which is a connected $n$-manifold, then for any finite-index torsion-free subgroup $\Gamma_1$ of $\Gamma$,

$$H^{n+1}(\Gamma_1; \mathbb{Z}\Gamma_1) \cong \tilde{H}^n(K; \mathbb{Z}).$$

\textbf{Proof.} We shall apply the condition on $|K|$ in the following equivalent form: Every simplex of the barycentric subdivision $K'$ of $K$ is contained in an $n$-simplex, and any two $n$-simplices of $K'$ may be joined by a path consisting of alternate $n$-simplices and $(n-1)$-simplices, each $(n-1)$-simplex being a face of its two neighbours in the path and of no other $n$-simplex. It suffices to show that under this hypothesis, $H^{n+1}(\Gamma_1; \mathbb{Z}\Gamma_1)$ is a cyclic group, because by Theorem 2 it admits $\tilde{H}^n(K; \mathbb{Z})$ as a quotient and has the same exponent as $\tilde{H}^n(K; \mathbb{Z})$. 

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Recall the description of the $m$-simplices of $D = D(\Gamma, V)$ as $(m + 2)$-tuples as in the proof of Theorem 2. The boundary of the simplex $\sigma = (\gamma, V_0, \ldots, V_m)$ is given by

$$d(\sigma) = \sum_{i=0}^{m} (-1)^i (\gamma, V_0, \ldots, V_{i-1}, V_{i+1}, \ldots, V_m),$$

and the action of $\Gamma$ by

$$\gamma' \sigma = (\gamma' \gamma, V_0, \ldots, V_m).$$

The stabiliser of $\sigma$ is $\langle V_0 \rangle \gamma^{-1}$. In the case when $m = n + 1$, $V_i$ must be a subset of $V$ of cardinality $i$, and $\sigma$ is therefore in a free $\Gamma$-orbit. For $\sigma$ an $(n + 1)$-simplex, define $f_\sigma \in \text{Hom}_\Gamma(C_{n+1}(D), \mathbb{Z}\Gamma)$ by the equations

$$f_\sigma(\sigma') = \begin{cases} \gamma' & \text{if } \sigma' = \gamma' \sigma \text{ for some } \gamma' \in \mathbb{Z}\Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

The $f_\sigma$ form a $\mathbb{Z}$-basis for $\text{Hom}_\Gamma(C_{n+1}(D), \mathbb{Z}\Gamma)$, so it suffices to show that for each $\sigma$ and $\sigma'$, $f_\sigma \pm f_{\sigma'}$ is a coboundary.

From now on we shall fix $\sigma = (\gamma, V_0, \ldots, V_{n+1})$, and show that $f_\sigma \pm f_{\sigma'}$ is a coboundary for various choices of $\sigma'$. If $\sigma' = (\gamma, V_0, \ldots, V_{i-1}, V'_i, V_{i+1}, \ldots, V_{n+1})$ for some $i > 0$, let $\tau$ be the $n$-simplex $(\gamma, V_0, \ldots, V_{i-1}, V'_i, V_{i+1}, \ldots, V_{n+1})$. There are exactly two $i$-element subsets of $V_{i+1}$ containing $V_{i-1}$, so $\sigma$ and $\sigma'$ are the only $(n+1)$-simplices of $D$ having $\tau$ as a face. Defining $f_\tau$ in the same way as $f_\sigma$ and $f_{\sigma'}$ (which we can do because $\tau$ is in a free $\Gamma$-orbit), we see that the coboundary of $f_\tau$ is $(-1)^i (f_\sigma + f_{\sigma'})$.

If $\sigma' = (\gamma, W_0, \ldots, W_{n+1})$, take a path in $K'$ between the simplices $(V_1, \ldots, V_{n+1})$ and $(W_1, \ldots, W_{n+1})$ of the form guaranteed by the hypothesis, and use this to make a similar path of $(n+1)$- and $n$-simplices between $\sigma$ and $\sigma'$, and use induction on the length of this path to reduce to the case considered above.

It will suffice now to consider the case when $\sigma' = (\gamma', V_0, \ldots, V_{n+1})$. By induction on the length of $\gamma'$ as a word in $V$, it suffices to consider the case when $\gamma' = v$. Using the cases done above and the fact that $\{v\}$ is a vertex of some $n$-simplex of $K'$, we may assume that $V_1 = \{v\}$. Now the $n$-simplex $\tau = (\gamma, V_1, \ldots, V_{n+1})$ of $D$ is a face of only $\sigma$ and $\sigma'$. The simplex $\tau$ has stabiliser in $\Gamma$ the subgroup $\gamma \langle v \rangle \gamma^{-1}$, so we may define

$$g_{\tau}(\tau') = \begin{cases} \gamma' \gamma (1 + v) \gamma^{-1} & \text{if } \tau' = \gamma' \tau, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that $d(g_{\tau}) = f_\sigma + f_{\sigma'}$, using the fact that $f_{\sigma'}(\gamma' \sigma) = \gamma' \gamma v \gamma^{-1}$. \hfill $\Box$

In the same vein we have the following.

**Proposition 4.** If $(\Gamma, V)$ is a finite Coxeter system such that the topological realisation $|K|$ of $K = K(\Gamma, V)$ is the closure of a subspace which is a connected $n$-manifold, and $\Gamma_1$ is a finite-index torsion-free subgroup of $\Gamma$, then the topological space $|D(\Gamma, V)|/\Gamma_1$ (which is an Eilenberg-Mac Lane space for $\Gamma_1$) is homeomorphic to a CW-complex with exactly one $(n+1)$-cell.

**Proof.** We shall give only a sketch. The complex $D(\Gamma, V)/\Gamma_1$ consists of copies of the cone on $K(\Gamma, V)'$ indexed by the cosets of $\Gamma_1$ in $\Gamma$, where each $n$-simplex not containing a cone
point is a face of exactly two \((n+1)\)-simplices. (The simplex \((\Gamma_1 \gamma, V_1, \ldots, V_{n+1})\), where \(V_1 = \{v\}\), is a face of \((\Gamma_1 \gamma, \emptyset, V_1, \ldots, V_{n+1})\) and \((\Gamma_1 \gamma v, \emptyset, V_1, \ldots, V_{n+1})\).) By hypothesis and this observation there exists a tree whose vertices consist of all the \((n+1)\)-simplices of \(D(\Gamma, V)/\Gamma_1\) and whose edges are \(n\)-simplices of \(D(\Gamma, V)/\Gamma_1\) which are faces of exactly two \((n+1)\)-simplices. The ends of an edge of the tree are of course the two \((n+1)\)-simplices containing it. The union of the (topological realisations of the) open simplices of such a tree is homeomorphic to an open \((n+1)\)-cell. The required CW-complex has \(n\)-skeleton the simplices of \(D(\Gamma, V)/\Gamma_1\) not in the tree, with a single \((n+1)\)-cell whose interior consists of the union of the open simplices of the tree. 

\[\Box\]

**Remark.** The condition on \(K(\Gamma, V)\) occurring in the statements of Corollary 3 and Proposition 4 is equivalent to \(\hat{K}(\Gamma, V)\) is a pseudo-manifold’ in the sense of [15]. Neither Corollary 3 nor Proposition 4 has a good analogue for graph products, because both rely on the fact that the \(n\)-simplices of \(D(\Gamma, V)\) in non-free \(\Gamma\)-orbits are faces of exactly two \((n+1)\)-simplices.

In Theorem 5 we summarize a version of Poincaré duality for simplicial complexes that look like manifolds from the point of view of \(R\)-homology for a commutative ring \(R\). Our treatment is an extension of that of J. R. Munkres book [15], which covers the case when \(R = \mathbb{Z}\). We also generalise the account in [15] by allowing a group to act on our ‘manifolds’. The proofs are very similar to those in [15] however, so we shall only sketch them here.

**Definition.** Let \(R\) be a commutative ring. An \(R\)-homology \(n\)-manifold is a locally finite simplicial complex \(L\) such that the link of every \(i\)-simplex of \(L\) has the same \(R\)-homology as an \((n-i-1)\)-sphere, where a sphere of negative dimension is empty. From this definition it follows that \(L\) is an \(n\)-dimensional complex, and that every \((n-1)\)-simplex of \(L\) is a face of exactly two \(n\)-simplices. Thus (the topological realisation of) every open \((n-1)\)-simplex of \(L\) has an open neighbourhood in \(|L|\) homeomorphic to a ball in \(\mathbb{R}^n\). Say that \(L\) is orientable if the \(n\)-simplices of \(L\) may be oriented consistently across every \((n-1)\)-simplex. Call such a choice of orientations for the \(n\)-simplices an orientation for \(L\). If \(L\) is connected and orientable then a choice of orientation for one of the \(n\)-simplices of \(L\), together with the consistency condition, determines a unique orientation for \(L\). In particular, a connected \(L\) has either two or zero orientations, and a simply connected \(L\) has two.

For any locally finite simplicial complex \(L\), the cohomology with compact supports of \(L\) with coefficients in \(R\), written \(H^*_c(L; R)\), is the cohomology of the subcomplex of the \(R\)-valued simplicial cochains on \(L\) consisting of the functions which vanish on all but finitely many simplices of \(L\). (This graded \(R\)-submodule may be defined for any \(L\), but is a subcomplex only when \(L\) is locally finite.)

**Theorem 5.** Fix a commutative ring \(R\), and let \(L\) be a connected \(R\)-homology \(n\)-manifold. Let \(\Gamma\) be a group acting freely and simplicially on \(L\). If \(L\) is orientable, let \(R^\circ\) stand for the right \(R\Gamma\)-module upon which an element \(\gamma\) of \(\Gamma\) acts as multiplication by \(-1\) if it exchanges the two orientations of \(L\) and as the identity if it preserves the orientations of \(L\). If \(R\) has characteristic two, let \(R^\circ\) be \(R\) with the trivial right \(\Gamma\)-action. Then if either \(L\) is orientable or \(R\) has characteristic two, there is for each \(i\) an isomorphism of right \(R\Gamma\)-modules

\[H^i_c(L; R) \otimes R^\circ \cong H_{n-i}(L; R).\]
Proof. The statements and proofs contained in sections 63–65 of [15] hold for $R$-homology manifolds provided that all (co)chain complexes and (co)homology are taken with coefficients in $R$. For each simplex $\sigma$ of $L$, one defines the dual block $D(\sigma)$ and its boundary exactly as in section 64 of [15]. From the point of view of $R$-homology, the dual block to an $i$-simplex of $L$ looks like an $(n - i)$-cell, and its boundary looks like the boundary of an $(n - i)$-cell. Thus as in Theorem 64.1 of [15], the homology of the dual block complex $D_*(L; R)$ is isomorphic to the $R$-homology of $L$. There is a natural bijection between the dual blocks of $L$ and the simplices of $L$. This is clearly preserved by the action of $\Gamma$. Each choice of orientation on $L$ gives rise to homomorphisms

$$
\psi : D_{n-i}(L; R) \otimes_R C_i(L; R) \rightarrow R
$$

which behave well with respect to the boundary maps, and have the property that for any simplex $\sigma$ with dual block $D(\sigma)$, and simplex $\sigma'$, $\psi(D(\sigma) \otimes \sigma') = \pm 1$ if $\sigma = \sigma'$, and 0 otherwise. This allows one to identify $D_{n-s}(L; R)$ with $C^*_c(L; R)$. With the diagonal action of $\Gamma$ on $D_{n-i}(L; R) \otimes C_i(L; R)$, $\psi$ is not $\Gamma$-equivariant, but gives rise to a $\Gamma$-equivariant map

$$
\psi' : D_{n-i}(L; R) \otimes_R C_i(L; R) \rightarrow R^\circ,
$$

and hence an $R\Gamma$-isomorphism between $D_{n-s}(L; R)$ and $C^*_c(L; R) \otimes R^\circ$. 

Remark. The referee pointed out that the sheaf-theoretic proof of Poincaré duality in G. E. Bredon’s book ([3], 207–211) also affords a proof of Theorem 5.

Corollary 6. Let $R$ be a commutative ring and let $L$ be a contractible $R$-homology $n$-manifold. Let $\Gamma$ be a group and assume that $\Gamma$ admits a free simplicial action on $L$ with finitely many orbits of simplices. Then $\Gamma$ is a Poincaré duality group of dimension $n$ over $R$, with orientation module the module $R^\circ$ defined in the statement of Theorem 5. The same result holds if $L$ is assumed only to be orientable and $R$-acyclic rather than contractible.

Proof. The simplicial $R$-chain complex for $L$ is a finite free $R\Gamma$-resolution for $R$, and hence $\Gamma$ is FP over $R$. Since $L$ has only finitely many $\Gamma$-orbits of simplices, the cochain complexes (of right $R\Gamma$-modules)

$$
\text{Hom}_{R\Gamma}(C_*(L), R\Gamma) \quad \text{and} \quad C^*_c(L; R)
$$

are isomorphic. Hence by Theorem 5, the graded right $R\Gamma$-module $H^*(\Gamma; R\Gamma)$ is isomorphic to $R^\circ$ concentrated in degree $n$. Thus $\Gamma$ satisfies condition d) of Definition V.3.3 of [10] and is a Poincaré duality group as claimed.

Corollary 7. Let $(\Gamma, V)$ be a Coxeter system, and let $R$ be a commutative ring. If $K(\Gamma, V)$ is an $R$-homology $n$-sphere (i.e. $K(\Gamma, V)$ is an $R$-homology $n$-manifold and $H_*(K(\Gamma, V); R)$ is isomorphic to the $R$-homology of an $n$-sphere), then any finite-index torsion-free subgroup of $\Gamma$ is a Poincaré duality group over $R$, of dimension $n + 1$.

Proof. It suffices to show that whenever $K = K(\Gamma, V)$ is an $R$-homology $n$-sphere, $D = D(\Gamma, V)$ is an $R$-homology $(n + 1)$-manifold, because then Corollary 6 may be applied
to the free action of the finite-index torsion-free subgroup of \( \Gamma \) on \( D \). We shall show that
the link of any simplex in \( D \) is isomorphic to either \( K' \) or a suspension (of the correct
dimension) of the link of some simplex in \( K' \). This implies that under the hypothesis
on \( K \), the link of every simplex of \( D \) is an \( \mathbb{R} \)-homology \( n \)-sphere.

Let \( \sigma = (\gamma, V_0, \ldots, V_m) \) be an \( m \)-simplex of \( D \), and without loss of generality we
assume that \( \gamma = 1 \). Then the link of \( \sigma \) is the collection of simplices \( \sigma' \) of \( D \) having no
vertex in common with \( \sigma \) but such that the union of the vertex sets of \( \sigma \) and \( \sigma' \) is the
vertex set of some simplex of \( D \). Thus the link of the simplex \( \sigma \) as above consists of those
simplices \( (\gamma', U_0, \ldots, U_l) \) of \( D \) such that the finite subsets \( U_0, \ldots, U_l, V_0, \ldots, V_m \) of \( V \) are
all distinct, generate finite subgroups of \( \Gamma \), and are linearly ordered by inclusion, where \( \gamma' \)
is an element of the subgroup \( \Gamma_0 \) of \( \Gamma \) generated by \( V_0 \). The link of \( \sigma \) decomposes as a
join of pieces corresponding to posets of the three types listed below, where we adopt the
convention that the join of a complex \( X \) with an empty complex is isomorphic to \( X \), and
spheres of dimension \(-1\) are empty.

1) The poset of all subsets \( U \) of \( V \) such that \( \langle U \rangle \) is finite and \( U \) properly contains \( V_m \). This is isomorphic to the poset of faces of the link in \( K' \) of any simplex of \( K' \) of dimension \( |V_m| - 1 \) of the form \( (V'_1, \ldots, V'_i, \ldots, V'_{m'}) \), where \( m' = |V_m| \) and \( V'_{m'} = V_m \). By
the hypothesis on \( K \), this is an \( \mathbb{R} \)-homology sphere of dimension \( n - |V_m| \).

2) For each \( i \) such that \( 0 \leq i < m \), the poset of all subsets of \( V \) properly containing \( V_i \)
and properly contained in \( V_{i+1} \). This is isomorphic to the poset of faces of the boundary
of a simplex with vertex set \( V_{i+1} - V_i \), so is a triangulation of a sphere of dimension
\( |V_{i+1}| - |V_i| - 2 \).

3) The poset of all cosets in \( \Gamma_0 \) of proper special subgroups of \( (\Gamma_0, V_0) \). (Recall that
we defined \( \Gamma_0 = \langle V_0 \rangle \).) This is a triangulation of a sphere of dimension \( |V_0| - 1 \) on which
the group \( \Gamma_0 \) acts with each \( v \in V_0 \) acting as a reflection in a hyperplane (see [5], I.5,
especially I.5H).

The link of the \( m \)-simplex \( \sigma \) consists of a join of one piece of type 1), \( m \) pieces of type
2), and one piece of type 3). All of these are spheres except that the piece of type 1) is
only an \( \mathbb{R} \)-homology sphere. It follows that the link of \( \sigma \) is an \( \mathbb{R} \)-homology sphere, whose
dimension is equal to the sum

\[
(n - |V_m|) + 1 + (|V_m| - |V_{m-1}| - 2) + 1 + \cdots \\
+ 1 + (|V_2| - |V_1| - 2) + 1 + (|V_1| - |V_0| - 2) + 1 + (|V_0| - 1) = n - m.
\]

This sum is obtained from the fact that the dimension of the join of two simplicial com-
plexes is equal to the sum of their dimensions plus one, which is correct in all cases,
provided that the empty complex is deemed to have dimension equal to \(-1\).

Remark. 1) Theorem 10 may also be used to prove Corollary 7.

2) A \( \mathbb{Z} \)-homology sphere is a \textit{generalised homology sphere} in the sense of [8]. Davis
shows that if \( (\Gamma, V) \) is a Coxeter group such that \( K(\Gamma, V) \) is a manifold and a \( \mathbb{Z} \)-homology
sphere, then \( \Gamma \) acts on an acyclic manifold with finite stabilisers (see Sections 12 and 17
of [8]).
4. Unusual cohomological behaviour.

Not every simplicial complex may be \( K(\Gamma, V) \) for some Coxeter system \((\Gamma, V)\); for example the 2-skeleton of a 6-simplex cannot occur. To see this, note that any labelling of the edges of a 6-simplex by the labelling set \{red, blue\} that contains no red triangle must contain a vertex incident with at least three blue edges. Now recall that the Coxeter group corresponding to a labelled triangle can be finite only if one of the edges has label two. Writing the integer two in blue, and other integers in red, one sees that any 7-generator Coxeter group with all 3-generator special subgroups finite has a 3-generator special subgroup which commutes with a fourth member of the generating set.

A condition equivalent to a complex \( K \) being equal to \( K(\Gamma, V) \) for some right-angled Coxeter system \((\Gamma, V)\) is that whenever \( K \) contains all possible edges between a finite set of vertices, this set should be the vertex set of some simplex of \( K \). Complexes satisfying this condition are called ‘full simplicial complexes’ or ‘flag complexes’ \cite{1}, \cite{5}. The barycentric subdivision of any complex satisfies this condition. The barycentric subdivision of an \( n \)-dimensional complex admits a colouring with \( n + 1 \) colours, where the barycentre \( \hat{\sigma} \) of an \( i \)-simplex \( \sigma \) is given the colour \( i \in \{0, \ldots, n\} \). This proves the following (see 11.3 of \cite{8}).

**Proposition 8.** The barycentric subdivision of any \( n \)-dimensional simplicial complex is isomorphic to \( K(\Gamma, V) \) for some right-angled Coxeter system \((\Gamma, V)\) such that \( \text{vcd}_\Gamma \) is finite.

We refer the reader to \cite{14} for a statement of the universal coefficient theorem and a calculation of the Ext-groups arising in the following examples.

**Example 1 (Bestvina).** (A group of finite cohomological dimension over the integers whose rational cohomological dimension is strictly less than its integral cohomological dimension.) Let \( X \) be the space obtained by attaching a disc to a circle by wrapping its edge around the circle \( n \) times, so that \( H_1(X) \cong \mathbb{Z}/(n) \) and \( H_2(X) = 0 \). Now let \((\Gamma, V)\) be any Coxeter system such that \( K(\Gamma, V) \) is a triangulation of \( X \). The generating set \( V \) will be finite since \( X \) is compact, so any such \( \Gamma \) will have finite vcd. Now if \( \Gamma_1 \) is a finite-index torsion-free subgroup of \( \Gamma \), \( \text{cd}_\Gamma \) is at most 3, and for any \( \mathbb{Z}\Gamma_1 \)-module \( M \), \( H^3(\Gamma_1; M) \) is a quotient of a finite sum of copies of \( H^2(X; M_o) \), which is in turn isomorphic to \( \text{Ext}(\mathbb{Z}/(n), M_o) \) by the universal coefficient theorem. In particular, \( nH^3(\Gamma_1; M) = 0 \) for any \( M \), and \( H^3(\Gamma_1; \mathbb{Z}\Gamma_1) \cong \text{Ext}(\mathbb{Z}/(n), \mathbb{Z}) \cong \mathbb{Z}/(n) \) by Corollary 3. Note that the methods used by Bestvina \cite{1} and by Harlander and Meinert \cite{13} seem to show only that \( H^3(\Gamma_1; \mathbb{Z}\Gamma_1) \) contains elements of order \( p \) for each prime \( p \) dividing \( n \), whereas our argument gives elements of order exactly \( n \).

**Example 2.** (A group whose cohomological dimension over the integers is finite but strictly greater than its cohomological dimension over any field.) Let \( X \) be a 2-dimensional CW-complex with \( H_1(X) \cong \mathbb{Q} \) and \( H_2(X) = 0 \), for example \( X \) could be an Eilenberg-MacLane space \( K(\mathbb{Q}, 1) \) built from a sequence \( C_1, C_2, \ldots \) of cylinders, where the end of the \( i \)th cylinder is attached to the start of the \((i + 1)\)st cylinder by a map of degree \( i \). Now let \((\Gamma, V)\) be a Coxeter system such that \( K(\Gamma, V) \) is a 2-dimensional simplicial complex homotopy equivalent to \( X \), and \( \Gamma \) has finite vcd. We shall show that \( \text{vcd}_\Gamma = 3 \), and that for any field \( \mathbb{F} \), \( \text{vcd}_\mathbb{F} \Gamma = 2 \). Let \( \Gamma_1 \) be a finite-index torsion-free subgroup of \( \Gamma \). Then
cd\Gamma_1 is at most 3, and for any \( M \), \( H^3(\Gamma_1; M) \) is a quotient of a finite sum of copies of \( H^2(X; M_0) \cong \text{Ext}(\mathbb{Q}, M_0) \). If \( M \) is an \( \mathbb{F}\Gamma_1 \)-module for a field of non-zero characteristic \( p \), then \( \text{Ext}(\mathbb{Q}, M_0) \) is an abelian group which is both divisible and annihilated by \( p \), so is trivial. If \( M \) is an \( \mathbb{F}\Gamma_1 \)-module for a field of characteristic zero, then \( M_0 \) is a divisible abelian group, so is \( \mathbb{Z} \)-injective, and so once again \( \text{Ext}(\mathbb{Q}, M_0) = 0 \). On the other hand, \( H^3(\Gamma_1; \mathbb{Z}\Gamma_1) \) is non-zero, because it admits \( \text{Ext}(\mathbb{Q}, \mathbb{Z}) \) as a quotient, and \( \text{Ext}(\mathbb{Q}, \mathbb{Z}) \) is a \( \mathbb{Q} \)-vector space of uncountable dimension.

The group \( \Gamma_1 \) requires infinitely many generators, but a 2-generator example may be constructed from \( \Gamma_1 \) using an embedding theorem of Higman Neumann and Neumann ([16], Theorem 6.4.7). They show that any countable group \( G \) may be embedded in a 2-generator group \( \hat{G} \) constructed as an HNN-extension with base group the free product of \( G \) and a free group of rank two, and associated subgroups free of infinite rank. An easy Mayer-Vietoris argument shows that for any ring \( R \),

\[
cd_R G \leq \cd_R \hat{G} \leq \max\{2, \cd_R G\}.
\]

Thus \( \hat{\Gamma}_1 \) is a 2-generator group with \( \cd \hat{\Gamma}_1 = 3 \) but \( \cd_\mathbb{F} \hat{\Gamma}_1 = 2 \) for any field \( \mathbb{F} \). We do not know whether there is a finitely presented group with this property, but the referee showed us the following Proposition (see also [2], 9.12).

**Proposition 9.** Let \( G \) be a group of type \( FP \). Then there is a prime field \( \mathbb{F} \) such that \( \cd_\mathbb{F} G = \cd G \).

**Proof.** Recall that if \( G \) is of type \( FP \), then for any ring \( R \), \( \cd_R G \) is equal to the maximum \( n \) such that \( H^n(G; RG) \) is non-zero, and that if \( \cd G = n \), then for any \( R \), \( H^n(G; RG) \) is isomorphic to \( H^n(G; \mathbb{Z}G) \otimes R \) ([4], p.199–203). Let \( M \) stand for \( H^n(G; \mathbb{Z}G) \), where \( n = \cd G \). Since \( \text{Hom}_G(P, \mathbb{Z}G) \) is a finitely generated right \( \mathbb{Z}G \)-module for any finitely generated projective \( P \), it follows that \( M \) is a finitely generated right \( \mathbb{Z}G \)-module.

If \( M \otimes \mathbb{F}_p \) is non-zero for some prime \( p \) we may take \( \mathbb{F} = \mathbb{F}_p \). If not, then \( M \) is divisible and hence, as an abelian group, \( M \) is a direct sum of a number of copies of \( \mathbb{Q} \) and a divisible torsion group ([16], Theorem 4.1.5). If \( M \otimes \mathbb{Q} \) is non-zero then we may take \( \mathbb{F} = \mathbb{Q} \). It remains to show that \( M \) cannot be a divisible torsion abelian group. Suppose that this is the case, and let \( m_1, \ldots, m_r \) be a generating set for \( M \) as a right \( \mathbb{Z}G \)-module. If \( N \) is the least common multiple of the additive orders of the elements \( m_1, \ldots, m_r \), multiplication by \( N \) annihilates \( M \), contradicting the divisibility of \( M \).

**Example 3.** (A torsion-free rational Poincaré duality group of dimension four which is not an integral Poincaré duality group.) Fix an odd prime \( q \), and let \( X \) be a lens space with fundamental group of order \( q \), i.e. \( X \) is a quotient of the 3-sphere by a free linear action of the cyclic group of order \( q \). It is easy to see that \( X \) is triangulable. The homology groups of \( X \) are (in ascending order) \( \mathbb{Z}, \mathbb{Z}/(q), \{0\} \) and \( \mathbb{Z} \). From the universal coefficient theorem it is easy to see that \( X \) has the same \( R \)-homology as the 3-sphere for any commutative ring \( R \) in which \( q \) is a unit. Now let \((\Gamma, V)\) be a right-angled Coxeter system such that \( K(\Gamma, V) \) is a triangulation of \( X \). By Corollary 7, any finite-index torsion-free subgroup \( \Gamma_1 \) of \( \Gamma \) is a Poincaré duality group of dimension four over any \( R \) in which \( q \) is a unit.
We claim however, that $\Gamma_1$ is not a Poincaré duality group (or PD-group for short) over the field $\mathbb{F}_q$, which implies that it cannot be a PD-group over the integers. Since all finite subgroups of $\Gamma$ have order a power of two and $q$ is an odd prime, it follows from Theorem V.5.5 of [10] that $\Gamma_1$ is a PD-group over $\mathbb{F}_q$ if and only if $\Gamma$ is. We shall assume that $\Gamma$ is a PD-group over $\mathbb{F}_q$ and obtain a contradiction.

Firstly, note that the $\mathbb{F}_q$-cohomology groups $H^0, \ldots, H^3$ of $X$ are all isomorphic to $\mathbb{F}_q$. Now it follows from Theorem 2 part c) that the right $\Gamma$-module $H^4(\Gamma; \mathbb{F}_q \Gamma)$ admits $\mathbb{F}_q$ (as defined just above the statement of Theorem 2) as a quotient. Thus $\Gamma$ has cohomological dimension four over $\mathbb{F}_q$, and if $\Gamma$ is a PD-group over $\mathbb{F}_q$, its orientation module must be $\mathbb{F}_q$. In particular, for any $\mathbb{F}_q$-$\Gamma$-module $M$, there should be an isomorphism for each $i$

$$H^i(\Gamma; M) = \text{Ext}^i_{\mathbb{F}_q \Gamma}(\mathbb{F}_q, M) \cong \text{Tor}^\mathbb{F}_q \Gamma_i(\mathbb{F}_q, M).$$

Now consider the case when $M$ is the $\Gamma$-bimodule $\mathbb{F}_q^\circ$, viewed as a left $\mathbb{F}_q \Gamma$-module. There is an $\mathbb{F}_q$-algebra automorphism $\phi$ of $\mathbb{F}_q \Gamma$ defined by $\phi(v) = -v$ for each $v \in V$, because the relators in $\Gamma$ have even length as words in $V$. The $\Gamma$-bimodule obtained from $\mathbb{F}_q^\circ$ by letting $\mathbb{F}_q \Gamma$ act via $\phi$ is the trivial bimodule $\mathbb{F}_q$. Thus for each $i$, $\phi$ induces an isomorphism

$$\text{Tor}^\mathbb{F}_q \Gamma_i(\mathbb{F}_q, \mathbb{F}_q^\circ) \cong \text{Tor}^\mathbb{F}_q \Gamma_i(\mathbb{F}_q, \mathbb{F}_q).$$

(Here we are viewing the bimodules as left modules when they appear as the right-hand argument in Tor, and as right modules when they appear as the left-hand argument.) Putting this together with the isomorphism obtained earlier, it follows that if $\Gamma$ is a PD-group over $\mathbb{F}_q$, then for each $i$, $\Gamma_1$ is not a PD-group over $\mathbb{F}_q^\circ$, and we have

$$H^i(\Gamma; \mathbb{F}_q^\circ) = \text{Ext}^i_{\mathbb{F}_q \Gamma}(\mathbb{F}_q, \mathbb{F}_q^\circ) \cong \text{Tor}^\mathbb{F}_q \Gamma_i(\mathbb{F}_q, \mathbb{F}_q) = H_{4-i}(\Gamma; \mathbb{F}_q). \quad (*)$$

The cohomology groups $H^0, \ldots, H^4$ of $\Gamma$ with coefficients in $\mathbb{F}_q^\circ$ are calculated in Theorem 2 part d); as vector spaces over $\mathbb{F}_q$ they have dimensions 0, 0, 1, 1, and 1 respectively. We claim now that any finitely generated right-angled Coxeter group is $\mathbb{F}_q$-acyclic, i.e. its homology with coefficients in the trivial module $\mathbb{F}_q$ is 1-dimensional and concentrated in degree zero. This leads to a contradiction because given the claim, the isomorphism (*) for $i = 2$ or 3 is between a 1-dimensional vector space and a 0-dimensional vector space. The claim follows from Theorem 4.11 of [9], which is proved using an elegant spectral sequence argument. It is also possible to provide a direct proof by induction on the number of generators using the fact (see [6] or [12]) that a finitely generated right-angled Coxeter group which is not a finite 2-group is a free product with amalgamation of two of its proper special subgroups, and applying the Mayer-Vietoris sequence.

**Remark.** 1) Note that the only properties of the space $X$ used in Example 3 are that $X$ be a compact manifold which is triangulable (in the weak sense that it is homeomorphic to the realisation of some simplicial complex), and that for some rings $R$, $X$ be an $R$-homology sphere, but that there be a prime field $\mathbb{F}_q$ for $q \neq 2$ such that $X$ is not an $\mathbb{F}_q$-homology sphere. These examples show that being a GD-group over $R$ (in the sense of [10], V.3.8) is not equivalent to being a PD-group over $R$. 

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2) If $\Gamma$ is a Coxeter group such that $K(\Gamma, V)$ is a triangulation of 3-dimensional real projective space, then Corollary 7 shows that any torsion-free finite-index subgroup of $\Gamma$ is a PD-group over any $R$ in which 2 is a unit. The methods used above do not show that such a group is not a PD-group over $R$ however. The results of the next section show that this is indeed the case.

5. Cohomology of Coxeter groups with free coefficients.

The results of this section were shown to us by the referee, although we are responsible for the proofs given here. The main result is Theorem 10, which computes $H^* (\Gamma, R\Gamma)$ for any Coxeter group $\Gamma$ such that $K(\Gamma, V)$ is a (triangulation of a) closed compact oriented manifold. This should be contrasted with Theorem 2, which applies to any Coxeter group, but gives only partial information concerning cohomology with free coefficients. Theorem 10 may be used to prove Corollary 7 and to give an alternative proof of the existence of Example 3 of the previous section.

Let $K$ be a locally finite simplicial complex, so that the cohomology of $K$ with compact supports, $H^c_*(K)$ is defined, and suppose that we are given a sequence

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$$

of subcomplexes such that each $K_i$ is cofinite (i.e., each $K_i$ contains all but finitely many simplices of $K$), and the intersection of the $K_i$’s is trivial. For $j > i$, the inclusion of the pair $(K, K_j)$ in $(K, K_i)$ gives a map from $H^* (K, K_i)$ to $H^* (K, K_j)$, and the direct limit is isomorphic to $H^c_*(K)$:

$$H^c_*(K) \cong \lim_{\to} (H^* (K, K_1) \to H^* (K, K_2) \to \cdots).$$

(To see this, it suffices to check that a similar isomorphism holds at the cochain level.)

**Theorem 10.** Let $R$ be a ring, let $(\Gamma, V)$ be a Coxeter system such that $K(\Gamma, V)$ is a triangulation of a closed compact connected $n$-manifold $X$, and suppose that either $X$ is orientable or $2R = 0$. Then as right $R\Gamma$-module, the cohomology of $\Gamma$ with free coefficients is:

$$H^i (\Gamma, R\Gamma) = \begin{cases} 0 & \text{for } i = 0 \text{ or } 1, \\ H^{i-1} (X; R) \otimes_R R\Gamma & \text{for } 2 \leq i \leq n, \\ R^\circ & \text{for } i = n + 1. \end{cases}$$

(Here $R^\circ$ is the $\Gamma$-module defined above Theorem 2.)

**Proof.** By hypothesis $D = D(\Gamma, V)$ is a locally finite complex, and so $H^* (\Gamma, R\Gamma)$ is isomorphic to $H^c_*(D; R)$. Recall that the complex $D$ may be built up from a union of a collection of cones on the barycentric subdivision $K'$ of $K(\Gamma, V)$ indexed by the elements of $\Gamma$. For $\gamma \in \Gamma$, let $C(\gamma)$ be the cone with apex the coset $\gamma \{1\}$. In terms of the description of $D$ in the proof of Theorem 2, $C(\gamma)$ consists of all simplices of $D$ which may be represented in the form $(\gamma, V_0, \ldots, V_m)$ for some $m$ and subsets $V_0, \ldots, V_m$ of $V$. For any enumeration $1 = \gamma_1, \gamma_2, \gamma_3, \ldots$ of the elements of $\Gamma$, define subcomplexes $E_i$ and $D_i$ of $D$ by

$$E_i = \bigcup_{j \leq i} C(\gamma_j), \quad D_i = \bigcup_{j > i} C(\gamma_j).$$
Dicks’ original proof that $D$ is contractible [8] uses the following argument. For $W$ a subset of $V$, define $K_{\sigma}(W)$ to be the subcomplex of $K' = K(\Gamma, V)'$ consisting of the simplices having a face in common with $K((W), W)' \subseteq K(\Gamma, V)$ and all of their faces. The simplicial interior, $\text{int} K_{\sigma}(W)$, of $K_{\sigma}(W)$ (i.e., the union of the topological realisations of the open simplices of $K_{\sigma}(W)$ which are not faces of any simplex of $K' - K_{\sigma}(W)$) deformation retracts onto $\tilde{K}((W), W)$ by a linear homotopy. In particular, this interior is contractible if $(W)$ is finite, and it may also be shown that in this case $K_{\sigma}(W)$ is itself contractible. In [8] an enumeration of the elements of $\Gamma$ is given such that for each $i$ there exists $W \subseteq V$ with $(W)$ a finite subgroup of $\Gamma$, and $E_{i} \cap C(\gamma_{i+1})$ is isomorphic to $K_{\sigma}(W) \subseteq K'$ by the restriction of the natural isomorphism $C(\gamma_{i+1}) \cong CK'$. By induction it follows that each $E_{i}$ is contractible, and hence that $D$ is.

Throughout the remainder of the proof fix an enumeration of $\Gamma$ as in the previous paragraph. From (***) it follows that

$$H^*(\Gamma; R\Gamma) \cong H^*_c(D; R) \cong \lim_{\rightarrow} (H^*(D, D_{i}; R)).$$

Let $F_{i} = D_{i} \cap E_{i}$, which could be thought of as the boundary of $E_{i}$. By excision, $H^*(D, D_{i}; R)$ is isomorphic to $H^*(E_{i}, F_{i}; R)$, and since $E_{i}$ is contractible, this is in turn isomorphic to the reduced cohomology group $\tilde{H}^{*-1}(F_{i}; R)$. So far we have used none of the conditions on $K(\Gamma, V)$ except that $V$ be finite.

The hypothesis that $K(\Gamma, V)$ be a closed $R$-oriented $n$-manifold is used to compute the limit of the groups $\tilde{H}^{*-1}(F_{i}; R)$. Roughly speaking, $F_{i}$ is a connected sum of $i$ copies of $K'$. More precisely, if $W_{i} \subseteq V$ is such that $E_{i} \cap C(\gamma_{i+1}) \cong K_{\sigma}(W_{i})$, then $F_{i+1}$ is obtained from the complexes $F_{i} - \text{int} K_{\sigma}(W_{i})$ and $K' - \text{int} K_{\sigma}(W_{i})$ by identifying the two copies of $K_{\sigma}(W_{i}) - \text{int} K_{\sigma}(W_{i})$. (We defined the simplicial interior $\text{int} L$ of a subcomplex $L$ of $K'$ to be a topological space, but it is easy to see how to define a subcomplex $M - \text{int} L$ of $M$ for any $L \subseteq M \subseteq K'$.) Given our hypotheses on $K(\Gamma, V)$, there are equalities

$$H^j(K' - \text{int} K_{\sigma}(W_{i}); R) = \begin{cases} H^j(K'; R) & \text{for } i \neq n, \\ 0 & \text{for } i = n, \end{cases}$$

while $K_{\sigma}(W_{i}) - \text{int} K_{\sigma}(W_{i})$ is a homology $(n - 1)$-sphere by Poincaré-Lefschetz duality for $K_{\sigma}(W_{i})$. From the usual argument used to compute the cohomology of a connected sum and induction it follows that

$$H^j(F_{i}; R) \cong \bigoplus_{k=1}^{i} H^j(K'; R) \text{ for } 0 < j < n, \quad \text{for } j = n.$$

Moreover, the map from $\tilde{H}^j(F_{i}; R)$ to $\tilde{H}^j(F_{i+1}; R)$ given by

$$\tilde{H}^j(F_{i}; R) \cong H^{j+1}(D, D_{i}; R) \to H^{j+1}(D, D_{i+1}; R) \cong \tilde{H}^j(F_{i+1}; R)$$

is the inclusion of the first $i$ direct summands for $j < n$ and the identity for $j = n$.

As a right $\Gamma$-module, $H^{n+1}_c(D; \mathbb{Z})$ is isomorphic to $\mathbb{Z}^\sigma$ by Theorem 2c), and the claim for general $R$ follows by the universal coefficient theorem. To verify the claimed right
vertices have colour \( w' \), \( w,w \) a colouring of \( K \).

**Proposition 11.** Let

i) If \( c \) is such that any two colours \( w, w' \) are adjacent in \( K^1(\Gamma, V) \), then \( S \) generates \( \Gamma_1 \) as a normal subgroup of \( \Gamma \).

ii) If \( c \) is such that for each \( w, w' \in W \) the graph \( K^1(\Gamma, V)(w, w') \) is connected, then \( S \) generates \( \Gamma_1 \) as a group.

**Proof.** Let \( Q \) be the quotient of \( \Gamma \) by the normal subgroup generated by \( S \). Then \( Q \) has \( C_2^W \) as a quotient, and the images of \( v \) and \( v' \) in \( Q \) are equal if \( c(v) = c(v') \), so \( Q \) is generated by a set of elements of order two bijective with \( W \). If the colours \( w, w' \) are adjacent in \( K^1(\Gamma, V) \), there exist \( v, v' \) with \( c(v) = w, c(v') = w' \) which commute in \( \Gamma \). Thus under the hypothesis in i), the relations of \( Q \) include relations saying that all pairs of generators commute, and so \( Q \) is isomorphic to \( C_2^W \). The hypothesis in ii) implies that in i), so it remains to prove that when each \( K^1(\Gamma, V)(w, w') \) is connected, the subgroup generated by \( S \) is normal. For this it suffices to show that if \( u, v, v' \) are elements of \( V \) with \( c(v) = c(v') \), then \( uvwv'u \) is in the subgroup generated by \( S \). Let

\[
v = v_1, u_1, v_2, u_2, \ldots, v_{n-1}, u, v_n = v'
\]
be the sequence of vertices on a path in the graph $K^1(\Gamma, V)(c(v), c(u))$ between $v$ and $v'$. Thus $c(v_i) = c(v)$, $c(u_i) = c(u)$, and for all $1 \leq i \leq n - 1$, $u_i$ commutes with $v_i$ and $v_{i+1}$. These commutation relations imply that $uvv'u$ is expressible as the following word in elements of $S$.

$$(uu_1)(v_1v_2)(u_1u_2)(v_2v_3)\cdots(v_{n-2}v_{n-1})(u_{n-2}u_{n-1})(v_{n-1}v_n)(u_{n-1}u) = uvv'u \quad \square$$

**Remark.** The hypothesis in i) is not very strong. Given a colouring of a graph in which there exist colours $w$ and $w'$ which are not adjacent, it is possible to identify the colours $w$ and $w'$ to produce a new colouring of the same graph using fewer colours.

**Corollary 12.** Let $L$ be an $n$-dimensional simplicial complex having $N$ simplices in total, such that every simplex of $L$ is a face of an $n$-simplex and $|L| - |L^{n-2}|$ is connected. If $(\Gamma, V)$ is such that $K(\Gamma, V)$ is the barycentric subdivision of $L$, then $\Gamma$ has a torsion-free normal subgroup $\Gamma_1$ of index $2^{n+1}$, which may be generated by $N - n - 1$ elements. $\Gamma$ has no torsion-free subgroups of lower index, and any normal subgroup of this index requires at least this number of generators.

**Proof.** Vertices of $K(\Gamma, V)$ correspond bijectively with simplices of $L$, and we may colour $K^1(\Gamma, V)$ with the set $\{0, \ldots, n\}$ by sending a vertex to the dimension of the corresponding simplex of $L$. Let $\Gamma_1$ be the kernel of the induced homomorphism onto $C_2^{n+1}$. Even without the extra conditions on $L$, $\Gamma_1$ is a torsion-free subgroup of $\Gamma$ of index $2^{n+1}$. Since the vertices of an $n$-simplex of $K(\Gamma, V)$ generate a subgroup of $\Gamma$ isomorphic to $C_2^{n+1}$, $\Gamma$ cannot have a torsion-free subgroup of lower index. The abelianisation of $\Gamma$ is isomorphic to $C_2^{N}$, and so $\Gamma$ cannot be generated by fewer than $N$ elements. If $\Gamma_2$ is any normal subgroup of $\Gamma$ of index $2^{n+1}$, then $\Gamma/\Gamma_2$ can be generated by $n+1$ elements, so $\Gamma_2$ cannot be generated by fewer than $N - n - 1$ elements.

Now we claim that the extra conditions on $L$ are equivalent to the condition that for any $i, j \in \{0, \ldots, n\}$, the graph $K^1(\Gamma, V)(i, j)$ (as defined in the statement of Proposition 11) is connected. Firstly, $|L| - |L^{n-2}|$ is connected if and only if $K^1(\Gamma, V)(n, n - 1)$ is connected. Now if $K^1(\Gamma, V)(n, i)$ is connected, every $i$-simplex of $L$ is a face of some $n$-simplex. For the converse, note first that in the special case when $L$ is a single $n$-simplex, $K^1(\Gamma, V)(i, j)$ is connected for each $i < j \leq n$. In the case when $j = n$ this is trivial, and for general $n$ follows by an easy induction. For the case of arbitrary $L$, to find a path between any two vertices $v, v'$ of $K^1(\Gamma, V)(i, j)$, first pick $n$-simplices $u, u'$ of $L$ such that $v$ is a face of $u$ and $v'$ is a face of $u'$. Then pick a path in $K^1(\Gamma, V)(n, n - 1)$ between $u$ and $u'$. For each $(n - 1)$-simplex occurring on this path, pick one of its $i$-simplices. This gives a sequence $w_1, \ldots, w_m$ of $i$-simplices such that $w_l$ and $w_{l+1}$ are faces of the same $n$-simplex for each $l$, and similarly for the pairs $v, w_1$ and $v', w_m$. By the special case already proved, there are paths in $K^1(\Gamma, V)(i, j)$ between each of these pairs, which concatenate to give a path from $v$ to $v'$.

Hence by Proposition 11, $\Gamma_1$ can be generated by the set of pairs $vv'$, where $v$ and $v'$ correspond to simplices of $L$ of the same dimension. If we fix for each dimension $i$ one generator $v_i$, then $vv' = (v_i^{-1}v_iv'_i)$, so we really need only the pairs $v_iv$ to generate $\Gamma_1$, and there are exactly $N - n - 1$ of these. \quad \square
Remark. 1) The conditions imposed on the simplicial complex $L$ are satisfied if $L$ is a triangulation of a connected $n$-manifold, or more generally if $|L|$ is the closure of a subset which is a connected $n$-manifold. These conditions cannot be omitted. Let $L$ be the simplicial bow tie, consisting of two triangles joined at a point, and let $\Gamma, \Gamma_1$ be the Coxeter group and subgroup of index 8 constructed from $L$ as in Corollary 12. Note that $|L| - |L^0|$ is not connected. Using GAP [17] it may be shown that the abelianisation of $\Gamma_1$ is free of rank 11, and so $\Gamma_1$ requires at least 11 generators, rather than the 10 which would suffice if Corollary 12 applied.

2) When (as in Corollary 12) the Coxeter group $\Gamma$ has a torsion-free normal subgroup $\Gamma_1$ of index equal to the order of the largest finite subgroup of $\Gamma$, this torsion-free subgroup will not usually be unique.

There are other ways to construct a torsion-free group having similar homological properties to a given right-angled Coxeter group. One generalisation of the construction given above is as follows. Given a right-angled Coxeter group $\Gamma$, and a homomorphism $\psi$ from $\Gamma$ onto a finite group $Q$ with torsion-free kernel $\Gamma_1$, let $\Delta$ be any torsion-free group and $\phi$ any homomorphism from $\Delta$ to $Q$. The group $P$ defined as the pullback of the following diagram is torsion-free and has finite index in $\Gamma \times \Delta$.

$$
\begin{array}{ccc}
P & \longrightarrow & \Delta \\
\downarrow & & \downarrow \phi \\
\Gamma & \rightarrow & Q
\end{array}
$$

The group $\Gamma_1$ occurs as such a pullback in the case when $\Delta$ is the trivial group. The point about taking different choices of $\Delta$ is that the resulting group may have a simpler presentation than $\Gamma_1$. One such result is the following.

Proposition 13. Let $(\Gamma, V)$ be a right-angled Coxeter system, and fix an $(n+1)$-colouring $c$ of $K^1(\Gamma, V)$. Let $K(\Gamma, V)$ have $M$ edges, and $N$ vertices. Suppose that the colouring of $K^1(\Gamma, V)$ has the property that for every $v \in V$, the star of $v$ contains vertices of all colours. Then there is a torsion-free group, $P$, of finite index in $\Gamma \times \mathbb{Z}^{n+1}$ having a presentation with $N$ generators and $M + N - n - 1$ relators, all of length four. Identifying the generators of $P$ with the set $V$, the relators are the following words:

i) For every edge with ends $v, v'$ in $K(\Gamma, V)$, the commutator $[v, v']$.

ii) For each colour $w \in W$, for some fixed $v_w$ with $c(v_w) = w$, and for each $v \neq v_w$ such that $c(v) = w$, the word $v^2v_w^{-2}$.

Proof. Let $P_1$ be the group presented by the above generators and relations. It suffices to show that $P_1$ is isomorphic to the pullback $P$ in the diagram below, where $W$ is the $(n + 1)$-element set of colours, $\phi$ is the natural projection and $\psi$ is the homomorphism induced by the colouring $c$.

$$
\begin{array}{ccc}
P & \longrightarrow & \mathbb{Z}^W \\
\downarrow & & \downarrow \phi \\
\Gamma & \rightarrow & C_2^W
\end{array}
$$

Identify the standard basis for $\mathbb{Z}^W$ with the set $W$. The elements $(v, c(v))$ of $\Gamma \times \mathbb{Z}^W$ are naturally bijective with $V$, and satisfy the relations given in the statement. Thus there
is a homomorphism from $P_1$ to $P \leq \Gamma \times \mathbb{Z}^W$ which sends $v$ to $(v, c(v))$. It remains to show that this homomorphism is injective and is onto $P$. The relations given between the elements of $V$ suffice to show that each $v^2$ is central in $P_1$, because given $v, v'$, there exists $v''$ such that $c(v) = c(v'')$ and there is an edge in $K(\Gamma, V)$ between $v'$ and $v''$. By applying the relations given, one obtains $[v^2, v'] = [(v'')^2, v] = 1$. Let $P_2$ be the subgroup of $P_1$ generated by the elements $v^2$. It is now easy to see that $P_2$ is central in $P_1$, and is free abelian of rank $n + 1$. Under the map from $P_1$ to $P$, $P_2$ is mapped isomorphically to the kernel of the map from $P$ to $\Gamma$. The quotient $P_1/P_2$ has the same presentation as $\Gamma$. It follows that $P_1$ is mapped isomorphically to $P$.

**Remark.** The condition on the colouring in the statement of Proposition 13 is weaker than the condition of part ii) of Proposition 11. It could be restated as saying that ‘every component of each $K(\Gamma, V)(w, w')$ contains vertices of both colours $w$ and $w'$.’ In the case when $K(\Gamma, V)$ is the barycentric subdivision of an $n$-dimensional simplicial complex $L$ and the colouring taken is the usual ‘colouring by dimension’, the condition is equivalent to the statement that every simplex of $L$ should be contained in an $n$-simplex.

7. Presentations of some of Bestvina’s examples.

In this section and the next we shall give some explicit presentations of groups whose cohomological dimensions differ over $\mathbb{Z}$ and $\mathbb{Q}$. To simplify notation slightly we shall adopt the convention that if $x$ is a generator in a group presentation, $\bar{x}$ denotes $x^{-1}$. Much of the section will be based around the eleven vertex full triangulation of the projective plane given in figure 1, where of course vertices and edges around the boundary are to be identified in pairs. Proposition 14 shows that this triangulation is minimal in some sense.

![Fig. 1](image_url)

**Proposition 14.** *There is no full triangulation of the projective plane having fewer than 11 vertices.*

**Proof.** The following statements are either trivial or followed by their proof. A triangulation of the projective plane with $N$ vertices has $3(N - 1)$ edges and $2(N - 1)$ faces. A full triangulation of a closed 2-manifold can have no vertex of valency 3 or less. The only 2-manifold having a full $N$-vertex triangulation with a vertex of valency $N - 1$ is the disc.
The only closed 2-manifold having a full triangulation with a vertex of valency \( N - 2 \) is the 2-sphere. There is no triangulation of the projective plane having 7, 8, or 9 vertices, each of which has valency 4 or 5. (Write an equation for the numbers of vertices of each valency and obtain a negative number of vertices of valency 4.) There are triangulations of the projective plane having 6 vertices, all of valency 5, but they are not full.

There is no 9 vertex full triangulation of the projective plane: Assume that there is such a triangulation. Then by the above we know that it has a vertex of valency 6. This vertex and its neighbours form a hexagon containing twelve edges. There are no further edges between the vertices of this hexagon, so all the remaining twelve edges contain at least one of the remaining two vertices. Hence at least one of these two vertices is joined to each of the boundary vertices of the hexagon, giving an eight vertex triangulation of the 2-sphere before adding the final vertex.

Any 10 vertex full triangulation of the projective plane has no vertex of valency seven: Assume that there is such a vertex. Then it and its neighbours form a heptagon containing 14 edges. There are 13 other edges, each of which must contain at least one of the other two vertices. Thus either one of these vertices is joined to all of the boundary vertices of the heptagon and this gives a 9 vertex triangulation of the 2-sphere before adding the final vertex, or the final two vertices are joined to each other and to six each of the boundary vertices of the heptagon, in which case the complex contains a tetrahedron (consisting of the two final vertices and any two of the boundary vertices of the heptagon adjacent to both of the final vertices).

Any 10-vertex full triangulation of the projective plane has at least four vertices of valency six (there are no vertices of valency higher than 6 by the above, and the sum of the 10 valencies is 54). If no two of these are adjacent, then they all have the same set of neighbours, but now all the other vertices have valency 4 and the total is wrong. Hence we may assume that a pattern of edges as in figure 2a occurs in the triangulation.

All of the vertices are already in the picture, and so the vertices marked \( V \) and \( W \) can have no other neighbours. Hence the triangulation contains the pattern of edges shown in figure 2b. With two vertices of valency 4, there must be at least six vertices of valency 6. Hence by symmetry it may be assumed that the vertex \( X \) is one such. The only possible
new neighbours for \(X\) are the vertices \(Y\) and \(Z\). Adding edges \(XY\) and \(XZ\), together with the faces implied by fullness, a triangulated disc whose boundary consists of four edges is obtained. There is only one way to close up this surface without adding new vertices or violating fullness, and this gives a triangulation of the 2-sphere (as may be seen either by calculating its Euler characteristic, or simply from the fact that after removing one face it may be embedded in the plane).

\[\text{Remark.}\] The smallest triangulation of the projective plane which is a barycentric subdivision has 31 vertices.

It is easy to see that the triangulation of figure 1 cannot be 3-coloured, and also that it has no 4-colourings in which every vertex has a neighbour of each of the three other colours (see Proposition 13). It does have a 4-colouring in which all but one of the vertices has neighbours of three colours, and even in which all but one of the colour-pair subgraphs (as defined in the statement of Proposition 11) are connected. The colouring with vertex classes

\[
\{a,c,e\}, \{b,d,f\}, \{g,h,j\}, \{i,k\}
\]

is one such. Let \((\Gamma,V)\) be the right-angled Coxeter system with \(K(\Gamma,V)\) as in figure 1. The above 4-colouring gives rise to an index 16 torsion-free subgroup of \(\Gamma\). It is easy to see that as a normal subgroup this group is generated by the eight elements

\[
ac, ae, bd, bf, gh, gj, ik, gigi = [g,i].
\]

Moreover, the techniques of the proof of Proposition 11 can be used to show that the subgroup generated by these elements is already normal, and hence that these eight elements generate an index 16 subgroup \(\Gamma_2\) of \(\Gamma\).

It is still possible to improve upon \(\Gamma_2\). Let elements \(l, m,\) and \(n\) generate a product of three cyclic groups of order two, and define a homomorphism \(\psi\) from \(\Gamma\) to this group by the following equations.

\[
\psi(a) = \psi(c) = \psi(e) = l \quad \psi(b) = \psi(d) = \psi(f) = m \\
\psi(g) = \psi(h) = \psi(j) = n \quad \psi(i) = \psi(k) = lmn
\]

The homomorphism \(\psi\) maps each of the maximal special subgroups of \(\Gamma\) isomorphically to the group generated by \(l, m\) and \(n\), and hence its kernel \(\Gamma_1\) is a torsion-free normal subgroup of \(\Gamma\) of index eight, which contains the index sixteen subgroup \(\Gamma_2\) of the previous paragraph. The element \(bcsi\) is in \(\Gamma_1\), but is not in \(\Gamma_2\). However, since both \(b\) and \(c\) commute with \(g\) and \(i\), its square is \((bcsi)^2 = gigi\). It follows that the eight elements

\[
ac, ae, bd, bf, gh, gj, ik, bcsi
\]

generate the group \(\Gamma_1\). Any normal subgroup of \(\Gamma\) of index eight requires at least eight generators (see Proposition 11), so there is a sense in which \(\Gamma_1\) is best possible.

The Euler characteristic \(\chi(\Gamma)\) may be calculated using I. M. Chiswell’s formula [7], and then \(\chi(\Gamma_1)\) is equal to \(|\Gamma : \Gamma_1|\chi(\Gamma)\). In fact, for a group having a finite Eilenberg-Mac Lane
space (such as $\Gamma_1$), this Euler characteristic is just the usual (topological) Euler characteristic of the Eilenberg-Mac Lane space. Indeed, Chiswell’s formula can be obtained by using the Davis complex to make a finite Eilenberg-Mac Lane space for a torsion-free subgroup of finite index in a Coxeter group, and then dividing by the index. Since the complex $K(\Gamma, V)$ has 11 vertices, 30 edges, and 20 2-simplices, while $\Gamma_1$ has index eight, the formula gives
\[
\chi(\Gamma_1) = 8(1 - \frac{11}{2} + \frac{30}{4} - \frac{20}{8}) = 4.
\]

Using the computer algebra package GAP [17], together with some adjustments suggested by V. Felsch, we were able to find the presentation for $\Gamma_1$ given below, which has 8 generators and 12 relations of total length 70 as words in the generators. (Recall our convention that $\bar{x}$ stands for $x^{-1}$.)

\[
\Gamma_1 = \langle s, t, u, v, w, x, y, z \mid \bar{y}s\bar{y}s, v\bar{x}\bar{x}, \bar{z}^2w\bar{x}^2w, \\
x^2wuw\bar{u}, \bar{y}\bar{u}\bar{z}\bar{y}z, \bar{w}\bar{u}\bar{w}z\bar{w}, \bar{z}\bar{y}\bar{z}\bar{y}z, uzsu \\
\bar{t}z\bar{w}\bar{t}u, v\bar{w}\bar{y}\bar{v}\bar{w}, \bar{w}\bar{z}\bar{w}z\bar{w}, v\bar{z}\bar{v}\bar{w}\bar{t} \rangle
\]

As words in the eleven generators of the Coxeter group $\Gamma$ the above generators are:

\[
s = ca, t = db, u = fb, v = ki, w = eibg, x = hbie, y = jg, z = cigb.
\]

We know that $\Gamma_1$ needs eight generators, and also that an Eilenberg-Mac Lane space $K(\Gamma_1, 1)$ must have at least one 3-cell (because $H^3(\Gamma_1; \mathbb{Z}_{\Gamma_1})$ is non-zero by Theorem 2 or Corollary 3). We also know that the Euler characteristic of $\Gamma_1$ is 4, and it follows that the above presentation has the minimum possible numbers of generators and relations. Proposition 4 implies that there is a $K(\Gamma_1, 1)$ of dimension three having exactly one 3-cell, but it does not follow that one may make a $K(\Gamma_1, 1)$ by attaching one 3-cell to the 2-complex for a presentation with 8 generators and 12 relations.

In the next section we shall give another presentation for $\Gamma_1$ (although we shall not prove this), also having the minimum numbers of generators and relations, but with the total length of the relations slightly longer than here. The advantage of the other presentation is that it shows how the group presented (which is in fact $\Gamma_1$) can be built up using free products with amalgamation from surface groups, and gives an independent proof that the cohomological dimension of $\Gamma_1$ over a ring $R$ depends on whether 2 is a unit in $R$. We also describe an attaching map for a 3-cell to make an Eilenberg-Mac Lane space $K(\Gamma_1, 1)$ from the 2-complex for the new presentation.

It is worth noting that the technique used in Proposition 13 may be applied to the group $\Gamma$ to give an 11 generator group of cohomological dimension five over any ring $R$ such that $2R = R$, and six over other rings, with thirty-seven relations of length four, and one relator of length eight. The relators are the thirty commutators corresponding to the edges in figure 1, together with the following words.

\[
a^2c^2, a^2e^2, b^2d^2, b^2f^2, g^2h^2, g^2j^2, i^2k^2, b^2c^2g^2i^2
\]

As in Proposition 13, one verifies that the subgroup generated by the squares of the eleven generators is central and free abelian of rank three, and that the quotient group is isomorphic to $\Gamma$. 

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8. A handmade Eilenberg-Mac Lane space.

**Theorem 15.** The group \( \Delta \) given by the presentation below has cohomological dimension three over rings \( R \) such that \( 2R \neq R \), and cohomological dimension two over rings \( R \) such that \( R = 2R \).

\[
\Delta = \langle s, t, u, v, w, x, y, z \mid \bar{s}v\bar{t}u\bar{w}, \bar{x}^2\bar{w}^2, \bar{x}\bar{w}\bar{s}\bar{w}, w^2v^2, \bar{w}\bar{x}\bar{w}\bar{x}\bar{w}, \bar{w}\bar{t}\bar{w}\bar{y}\bar{w}\bar{t}, \bar{y}\bar{w}\bar{w}\bar{t}\bar{u}, y^2\bar{s}\bar{v}, \bar{z}\bar{u}\bar{s}\bar{w}\bar{v}, \bar{z}\bar{w}\bar{v}\bar{x}\bar{w}\bar{x}\bar{w}\bar{x} \rangle
\]

There is an Eilenberg-Mac Lane space \( K(\Delta, 1) \), having only one 3-cell and whose 2-skeleton is given by the above presentation. The 2-sphere forming the boundary of the 3-cell is formed from the hemispheres depicted in figures 3a and 3b.

**Remark.** In fact this group is isomorphic to the index eight subgroup \( \Gamma_1 \) of the 11 generator Coxeter group \( \Gamma \) of the previous section. The following function \( \phi \) from the generating set for \( \Delta \) given above to \( \Gamma \) extends to a homomorphism from \( \Delta \) to \( \Gamma \), because the image of each relator is the identity element. Moreover, it is easy to see that the image of \( \phi \) is equal to \( \Gamma_1 \). We shall not prove that the kernel of \( \phi \) is trivial.

\[
\phi(s) = ca \quad \phi(t) = hbia \quad \phi(u) = akia \quad \phi(v) = bgia \\
\phi(w) = gi fa \quad \phi(x) = fija \quad \phi(y) = fgie \quad \phi(z) = dgkc
\]

**Proof of Theorem 15.** We shall build the group \( \Delta \) in stages via a tower of free products with amalgamation, obtained by applying an algorithm due to Chiswell [6]. We shall use the same argument at each stage to justify this process, but shall give less detail as the steps become more complicated. Let \( \Delta_0 \) be the group given by the following presentation.

\[
\Delta_0 = \langle s, t, u, v \mid \bar{s}v\bar{t}u\bar{w} \rangle
\]

\( \Delta_0 \) is the fundamental group of a closed non-orientable surface of Euler characteristic \(-2\), and the 2-complex corresponding to the above presentation is a CW-complex homeomorphic to this surface. In particular \( \Delta_0 \) is torsion-free, and the 2-complex corresponding to the given presentation is an Eilenberg-Mac Lane space \( K(\Delta_0, 1) \).

Now let \( F_0 \) be the free group on generators \( w \) and \( w' \). Define automorphisms \( a \) and \( g \) of \( F_0 \) by the equations

\[
w^a = w\bar{w}' \quad w'^a = \bar{w}' \quad w^g = w'\bar{w} \quad w'^g = w\bar{w}'\bar{w}
\]

It is easy to check that \( a \) and \( g \) have order two and commute with each other, so that they generate a subgroup of \( \text{Aut}(F_0) \) isomorphic to the direct product of two cyclic groups of order two. Let \( G_0 \) be the subgroup of \( \text{Aut}(F_0) \) generated by \( F_0, a \) and \( g \), which is isomorphic to the split extension with kernel \( F_0 \) and quotient of order four generated by \( a \) and \( g \). Now define a homomorphism \( \psi_0 \) from \( \Delta_0 \) to \( G_0 \) by

\[
\psi_0(s) = a, \quad \psi_0(t) = gw, \quad \psi_0(u) = agw, \quad \psi_0(v) = w.
\]
This does define a homomorphism from $\Delta_0$ to $G_0$, because the image of the relator is the identity element, as shown below.

$$
\psi_0(\bar{svstuv}) = awa(gw)(agw)(\bar{wg})(agw)\bar{w} \\
= awagwg \\
= w\bar{w}'w'\bar{w} \\
= 1
$$

Now the images of $s\bar{s}v$, $v^2$ and $\bar{v}s\bar{s}u$ under $\psi_0$ are $w'$, $w^2$ and $\bar{w}w'\bar{w}$ respectively. These elements generate a normal subgroup of $F_0$ of index two, which is therefore free of rank three. (Subgroups of index two are always normal, but what one does to check that the subgroup is normal, and then show that it has index two by calculating the order of the quotient.) It follows that the elements $s\bar{s}v$, $v^2$ and $\bar{v}s\bar{s}u$ freely generate a free subgroup of $\Delta_0$, and that this subgroup is mapped isomorphically by $\psi_0$ to the subgroup of $F_0$ generated by $w'$, $w^2$ and $\bar{w}w'\bar{w}$. Hence a free product with amalgamation may be made from $\Delta_0$ and $F_0$ by taking the free product and adding the relations $w' = s\bar{s}v$, $w^2 = v^2$, and $\bar{w}w'\bar{w} = \bar{v}s\bar{s}u$. This gives a group $\Delta_1$. Using the first of the three new relations to eliminate the generator $w'$, it follows that $\Delta_1$ has a presentation as below.

$$
\Delta_1 = \langle s, t, u, v, w \mid \bar{svstuv}, w^2v^2, s\bar{s}v\bar{s}\bar{u}\bar{s}v\bar{u}\rangle
$$

Moreover, the 2-complex corresponding to this presentation is a $K(\Delta_1, 1)$.

Now take a free group $F_1$ of rank three generated by elements $x$, $x'$ and $x''$. Define an automorphism $f$ of $F_1$ by

$$
x^f = x, \quad x'^f = xx'x, \quad x''^f = xx''x.
$$

It may be seen that $f$ has order two. Let $G_1$ be the split extension with kernel $F_1$ and quotient the group of order two generated by $f$, or equivalently the subgroup of $\text{Aut}(F_1)$ generated by $F_1$ and $f$. Define a homomorphism $\psi_1$ from $\Delta_1$ to $G_1$ as below.

$$
\psi_1(s) = x', \quad \psi_1(t) = xf, \quad \psi_1(u) = x'', \quad \psi_1(v) = xf, \quad \psi_1(w) = x
$$

As before, to check that this does define a homomorphism it suffices to verify that $\psi_1$ sends each relator to the identity in $G_1$. The images of $s$, $u$, $s\bar{s}v$, $\bar{v}s\bar{s}u\bar{v}$, and $\bar{v}wv\bar{w}$ under $\psi_1$ are $x'$, $x''$, $x^2$, $xx'x$, and $xx''x$ respectively. These five elements generate a normal subgroup of $F_1$ of index two, which is therefore a free group on five generators. It follows that the subgroup of $\Delta_1$ generated by $s$, $u$, $s\bar{s}v$, $\bar{v}s\bar{s}u\bar{v}$, and $\bar{v}wv\bar{w}$ is freely generated by these elements and is mapped isomorphically to a free subgroup of $F_1$ by $\psi_1$. Hence we may form an amalgamated free product of $\Delta_1$ and $F_1$, identifying the two five generator free subgroups via $\psi_1$. Call the resulting group $\Delta_2$. After eliminating the generators $x'$ and $x''$ and the relations $x' = s$, $x'' = u$, the group $\Delta_2$ has the presentation given below. Once again, because the amalgamating subgroup is free, the 2-complex for this presentation is a $K(\Delta_2, 1)$.

$$
\Delta_2 = \langle s, t, u, v, w, x \mid \bar{svstuv}, w^2v^2, \\
\bar{sv}\bar{s}v\bar{s}u\bar{v}, x^2\bar{s}v, \\
xw\bar{w}s\bar{w}x, uv\bar{w}ux\bar{w}\rangle
$$

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Regions are to be read counter-clockwise.

Edge labels are on the left of the edge, and on the right of the inverse edge.

Pins indicate base points for regions.

Fig. 3a
Fig. 3b
Take a free group $F_2$ of rank three with generators $y, y', y''$, and define automorphisms $f$ and $i$ of $F_2$ by the following equations.

$$
y^f = y \quad y'^f = \bar{y}y'y \quad y''^f = \bar{y}y'y'
y^i = \bar{y} \quad y'^i = y'y^2 \quad y''^i = \bar{y}'
$$

It may be checked that $i$ and $f$ have order two and commute, so they generate a subgroup of $\text{Aut}(F_2)$ isomorphic to a direct product of two cyclic groups of order two. Let $G_2$ be the subgroup of $\text{Aut}(F_2)$ generated by $F_2$, $f$ and $i$. Define a homomorphism $\psi_2$ from $\Delta_2$ to $G_2$ by checking that the function defined as follows on the generators sends each relator to the identity in $G_2$.

$$\psi_2(s) = 1 \quad \psi_2(t) = y'yf \quad \psi_2(u) = y''$$

$$\psi_2(v) = yf \quad \psi_2(w) = y \quad \psi_2(x) = fi$$

Now $\psi_2$ sends the elements $t\bar{v}, \bar{t}u\bar{f}, v\bar{s}v\bar{s}, w\bar{v}\bar{w}$, and $wuw$ to $y'$, $y''$, $y^2$, $yy'y$ and $yy''\bar{y}$ respectively, and these five elements generate a normal subgroup of $F_2$ of index two, which they therefore freely generate. Hence we may make an amalgamated free product $\Delta_3$ from the free product of $\Delta_2$ and $F_2$ by adding the five relations $t\bar{v} = y', \ldots, wuw = yy''\bar{y}$. After eliminating the generators $y'$ and $y''$, we obtain the following presentation for $\Delta_3$, such that the corresponding 2-complex is a $K(\Delta_3, 1)$.

$$\Delta_3 = \langle s, t, u, v, w, x, y, z \mid s\bar{v}tu\bar{f}u, w^2\bar{v}^2, s\bar{v}v\bar{s}s\bar{v}s\bar{v},
\bar{x}^2\bar{v}^2s\bar{v}s, x\bar{w}w\bar{x}\bar{w}s, w\bar{v}w\bar{x}\bar{w}v, \bar{w}t\bar{v}w\bar{t},
\bar{v}wu\bar{w}tu\bar{f}, y^2s\bar{v}s \rangle$$

Now take a 1-relator group $F_3$, with presentation

$$F_3 = \langle z, z', z'' \mid z'z\bar{z}'z''z\bar{z}' \rangle.$$  

$F_3$ is the fundamental group of the closed nonorientable surface of Euler characteristic $-1$. The 2-complex corresponding to this presentation is a cellular decomposition of this surface, so in particular is a $K(F_3, 1)$. Define automorphisms $c$ and $g$ of $F_3$ by the following equations.

$$z^c = \bar{z}'z\bar{z}' \quad z'^c = \bar{z}' \quad z''^c = \bar{z}''$$

$$z^g = \bar{z}'z\bar{z}' \quad z'^g = \bar{z}' \quad z''^g = \bar{z}''$$

To check that $c$ and $g$ as defined above do extend to homomorphisms from $F_3$ to itself, note that

$$(z'z\bar{z}'z''z) \bar{c} = z'z\bar{z}'z'z''z\bar{z}'z''z = z'^c \bar{z}'z\bar{z}'z''z$$

is equal (in the free group) to a conjugate of the relator, and similarly $(z'z\bar{z}'z''z) \bar{g} = z'^g$ is equal to a conjugate of the inverse of the relator. It is easy to check that $c$ and $g$ define commuting involutions in $\text{Aut}(F_3)$ (and even on the free group with the same generating set). Now let $G_3$ be the split extension with kernel $F_3$ and quotient the subgroup of
Let $H$ be the subgroup of $\Delta_3$ generated by $c$ and $g$. (Since $F_3$ has trivial centre, $G_3$ is isomorphic to a subgroup of $\text{Aut}(F_3)$.)

Define a homomorphism $\psi_3$ from $\Delta_3$ to $G_3$ by taking the following specification on the generators, and checking that the image of each relator is equal to the identity in $G_3$.

$$\psi_3(s) = c \quad \psi_3(t) = 1 \quad \psi_3(u) = gzc \quad \psi_3(v) = g$$

$$\psi_3(w) = g \quad \psi_3(x) = z''g \quad \psi_3(y) = g'c$$

Let $H$ be the subgroup of $\Delta_3$ generated by $\bar{yw}s$, $x\bar{w}$, $\bar{v}vsv\bar{s}$ and $w\bar{x}vu\bar{v}v\bar{v}$. The images of these elements under $\psi_3$ are $z'$, $z''$, $z^2$ and $zz''\bar{z}$ respectively. These four elements of $F_3$ generate a normal subgroup $\psi_3(H)$ of index two, which turns out to be an orientable surface group (necessarily of Euler characteristic $-2$). If we write $\alpha = z^2$ and $\beta = zz''\bar{z}$, then $\psi_3(H)$ may be presented as follows.

$$\psi_3(H) = \langle z', z'', \alpha, \beta \mid \bar{\alpha}\beta\alpha z''z'\bar{\beta}z''z' \rangle$$

We claim that $\psi_3$ restricted to $H$ is injective. For this it suffices to show that the word in the four generators for $H$ mapping to the relator in $\psi_3(H)$ is equal to the identity in $\Delta_3$. Expressed in terms of the generators for $\Delta_3$, the word is

$$su\bar{v}sv\bar{x}ux\bar{w}v\bar{v}v\bar{v}ux\bar{w}x\bar{v}w\bar{v}s,$$

so after cyclically reducing this word it suffices to show that in $\Delta_3$,

$$u\bar{v}sv\bar{w}v\bar{x}ux\bar{w}v\bar{v}v\bar{v}uyw = 1.$$
for any \(\Delta\)-module \(M\), \(H^3(\Delta; M)\) is isomorphic to \(M/I'M\), where \(I\) is the right ideal of \(\mathbb{Z}\Delta\) generated by the twelve elements \(1 \pm w_1, \ldots, 1 \pm w_{12}\), and the sign in \(1 \pm w_i\) is positive if the \(i\)th relator appears in figure 3 with the same orientation each time, and is negative otherwise. In figure 3a, the two copies of the third relator meet at their base points and have the same orientation. It follows that \(1 + w_3 = 2\), and hence that \(H^3(\Delta; M)\) is a quotient of \(M/2M\). This completes the proof of the statement. With a little more work it may be shown that \(I\) is equal to the ideal of \(\mathbb{Z}\Delta\) generated by 2 and the augmentation ideal, which implies that for any \(M\),

\[
H^3(\Delta; M) \cong M_{2\Delta}/2M_{\Delta}.
\]

We leave this as an exercise.

9. Further questions.

1) We also used GAP [17] to try to find good presentations of various other finite-index torsion-free subgroups of Coxeter groups. The examples that we tried include:

a) The index sixteen subgroup \(\Gamma_2\) of the right-angled Coxeter group \(\Gamma\) described in section 7. Comparison of the Euler characteristic (which is twice that of \(\Gamma_1\), or eight) with the known minimum number of generators, together with the fact that any \(K(\Gamma_2, 1)\) must have at least one 3-cell indicate that the minimum number of relators in a presentation of \(\Gamma_2\) must be at least sixteen. Using GAP we were able to get down only to an 8-generator 17-relator presentation, but by hand (and then checking the result using GAP) we were able to eliminate one of the relators. The sum of the lengths of the relators in our presentation is 152. We found a CW-complex \(K(\Gamma_2, 1)\) having eight 1-cells, sixteen 2-cells and one 3-cell.

b) Other index eight normal subgroups of the Coxeter group \(\Gamma\) of Section 7. If \(\psi'\) is any homomorphism from \(\Gamma\) onto a product of three cyclic groups of order two which restricts to an isomorphism on each maximal special subgroup of \(\Gamma\), then the kernel of \(\psi'\) is a torsion-free index eight normal subgroup of \(\Gamma\). One way to create such a \(\psi'\) is to take \(\psi\) (the homomorphism given earlier, with kernel \(\Gamma_1\)) and modify it slightly. We did not find a \(\psi'\) whose kernel had a smaller presentation than the one given for \(\Gamma_1\).

c) Take figure 1, remove the vertex \(h\) and all edges leaving it, and add a new edge between vertices \(i\) and \(k\). Label the three boundary edges with the label three, and give all other edges the label two. This gives a presentation of a ten-generator Coxeter system \((\Delta, V)\) such that \(K(\Delta, V)\) is a triangulation of the projective plane. \(\Delta\) has torsion-free normal subgroups of index 24, and clearly has no torsion-free subgroup of lower index, since it contains elements of order three and subgroups isomorphic to \(C_3^2\). The subgroup we looked at required nine generators.

d) In [1] Bestvina pointed out that a finite-index torsion-free subgroup \(\Gamma_1\) of a Coxeter group \(\Gamma\) such that \(K(\Gamma, V)\) is an acyclic 2-complex would have cohomological dimension two over any ring, but might not have a 2-dimensional Eilenberg-Mac Lane space. (A famous conjecture of Eilenberg and Ganea asserts that any group of cohomological dimension two has a 2-dimensional Eilenberg-Mac Lane space [4].) Let \(K\) be the barycentric subdivision of the acyclic 2-complex having five vertices, ten edges corresponding to the ten pairs of vertices, and six pentagonal faces corresponding to a conjugacy class in \(A_5\) of elements of
order five. The simplicial complex $K$ is full. If $(\Gamma, V)$ is the corresponding right-angled Coxeter system, then the easy extension of Corollary 12 to polyhedral complexes shows that colouring $K(\Gamma, V)$ by dimension gives rise to a torsion-free index eight normal subgroup $\Gamma_1$ of $\Gamma$, requiring exactly eighteen generators. The complex $K(\Gamma_1, V)$ has 21 vertices, 80 edges and 60 2-simplices, so by Chiswell's formula [7], the Euler characteristic of $\Gamma_1$ is

$$\chi(\Gamma_1) = 8(1 - \frac{21}{2} + \frac{80}{4} - \frac{60}{8}) = 24.$$ 

Hence if a presentation for $\Gamma_1$ could be found having twenty-three more relators than generators, the corresponding 2-complex would be a $K(\Gamma_1, 1)$. An argument similar to that sketched in the proof of Proposition 4 shows that there is a 3-dimensional $K(\Gamma_1, 1)$ having exactly six 3-cells. Presentations of $\Gamma_1$ arising from this complex will have 29 more relators than generators. Using GAP we found an 18-generator 52-relator presentation for $\Gamma_1$, but were unable to reduce the number of relators any further. The problem of whether there exists a $K(\Gamma_1, 1)$ with less than six 3-cells remains open.

2) The groups exhibited in Section 4 Example 2 (whose cohomological dimension over the integers is three and whose cohomological dimension over any field is two) are finitely generated, but cannot be $FP$ by Proposition 9. The question of whether there can be similar examples which are $FP(2)$ or even finitely presented remains open.

3) The result proved in Proposition 14 does not really prove that the examples of Sections 7 and 8 are minimal, even in the sense of being finite-index subgroups of Coxeter groups with the least possible number of generators. It may be true that there can be no full simplicial complex having ten vertices or fewer whose highest non-zero homology group is non-free, which would suggest that no right-angled Coxeter group on less than eleven generators can have different virtual cohomological dimensions over $\mathbb{Z}$ and $\mathbb{Q}$.

4) Is there a simpler example of a group whose cohomological dimension over $\mathbb{Z}$ is finite and strictly greater than its cohomological dimension over $\mathbb{Q}$ than the group $\Gamma_1 \cong \Delta$ given in Sections 7 and 8? Applying the Higman-Neumann-Neumann embedding theorem to $\Delta$ we were able to construct a 2-generator 12-relator presentation of a group whose cohomological dimension over any ring is equal to that of $\Delta$. An Euler characteristic argument shows that this group requires at least 12 relators. The total length of the 12 relators we found was 1,130, so this group can hardly be said to be simpler than $\Delta$.

The two distinct 8-generator 12-relator presentations for $\Delta$ given in Sections 7 and 8 have short relators (i.e. simple attaching maps for the 2-cells) and a simple attaching map for the 3-cell respectively. Is it possible to make a presentation for $\Delta$ in which each 2-cell occurs only twice in the boundary of the 3-cell and such that the sum of the lengths of the relators is smaller than 96 (the sum of the lengths of the relators in the presentation given in Theorem 15)?

5) (P. H. Kropholler) Can there be a group $\Gamma$ with $\text{cd}_\mathbb{Z} \Gamma = 4$ and $\text{cd}_\mathbb{Q} \Gamma = 2$? Notice that taking direct products of copies of Bestvina’s examples gives groups with arbitrary finite differences between their cohomological dimensions over $\mathbb{Z}$ and over $\mathbb{Q}$.

6) What we call an $R$-homology manifold is really a simplicial $R$-homology manifold. One could give a similar definition and an analogue of Theorem 5 and Corollary 6 for general
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(locally compact Hausdorff) topological spaces. It may be the case that any torsion-free Poincaré duality group over \( R \) acts freely cocompactly and properly discontinuously on an orientable \( R \)-acyclic \( R \)-homology manifold.

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