Modified Friedmann Equations from Tsallis Entropy

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It was shown by Tsallis and Cirto that thermodynamical entropy of a gravitational system such as black hole must be generalized to the non-additive entropy, which is given by $S_h = \gamma A^3$, where $A$ is the horizon area and $\beta$ is the nonextensive parameter [1]. In this paper, by taking the entropy associated with the apparent horizon of the Friedmann-Robertson-Walker (FRW) Universe in the form of Tsallis entropy, and assuming the first law of thermodynamics, $dE = T_h dS_h + W dV$, holds on the apparent horizon, we are able to derive the corresponding Friedmann equations describing the dynamics of the universe with any spatial curvature. We also examine the time evolution of the total entropy and show that the generalized second law of thermodynamics is fulfilled in a region enclosed by the apparent horizon. Then, modifying the emergence proposal of gravity proposed by Padmanabhan and calculating the difference between the surface degrees of freedom and the bulk degrees of freedom in a region of space, we again arrive at the modified Friedmann equation of the FRW Universe with any spatial curvature which is the same as one obtained from the first law of thermodynamics. We also study the cosmological consequences of Tsallis cosmology. Interestingly enough, we find that this model can explain simultaneously the late time acceleration in the universe filled with pressureless matter without invoking dark energy, as well as the early deceleration. Besides, the age problem can be circumvented automatically for an accelerated universe and is estimated larger than $3/2$ age of the universe in standard cosmology. Taking $\beta = 2/5$, we find the age of the universe ranges as $13.12 \text{ Gyr} < t_0 < 16.32 \text{ Gyr}$, which is consistent with recent observations. Finally, using the Jeans’s analysis, we comment, in brief, on the density perturbation in the context of Tsallis cosmology and found that the growth of energy differs compared to the standard cosmology.

I. INTRODUCTION

Although gravity is the most universal forces of nature, understanding its origin has been a mystery for a long time. Einstein believed that gravity is just the spacetime curvature and regarded it as an emergent phenomenon which describes the dynamics of spacetime. In the past decades a lot of attempts have been done to disclose the nature of gravity. A great step in this direction put forwarded by Jacobson [2] who studied thermodynamics of spacetime and showed explicitly that Einstein’s equation of general relativity is just an equation of state for the spacetime. Combining the Clausius relation $\delta Q = TS$, together with the entropy expression, he derived Einstein field equations. This derivation is of great importance because it confirms that the Einstein field equations is nothing but the first law of thermodynamics for the spacetime. Following Jacobson, a lot of studies have been carried out to disclose the deep connection between gravity and thermodynamics [3,4]. The studies were also generalized to the cosmological setups [5,11], where it has been shown that the Friedmann equation of Friedmann-Robertson-Walker (FRW) universe can be written in the form of the first law of thermodynamics on the apparent horizon. Although Jacobson’s derivation is logically clear and theoretically sound, the statistical mechanical origin of the thermodynamic nature of general relativity remains obscure.

In 2010 Verlinde [15] put forwarded the next great step toward understanding the nature of gravity who claimed that gravity is not a fundamental force and can be interpreted as an entropic force caused by changes of entropy associated with the information on the holographic screen. Verlinde’s proposal is based on two principles, namely the holographic principle and the equipartition law of energy. Using these principles he derived the Newton’s law of gravitation, the Poisson equation and in the relativistic regime the Einstein field equations [15] (see also [16]). The investigation on the entropic origin of gravity have been extended in different setups [17–24] and references therein). Although Verlinde’s proposal has changed our understanding on the origin and nature of gravity, but it considers the gravitational field equations as the equations of emergent phenomenon and leave the spacetime as a pre-existed background geometric manifold. A new perspective towards emergence of spacetime dynamics was suggested in 2012 by Padmanabhan [25]. He argued that the spatial expansion of our Universe can be regarded as the consequence of emergence of space and the cosmic space is emergent as the cosmic time progresses. By calculating the difference between the number of degrees of freedom in the bulk and on the boundary, Padmanabhan [25] derived the Friedmann equation of the flat FRW Universe. This proposal was also applied for deriving the Friedmann equations of a higher dimensional FRW Universe in Einstein–Gauss–Bonnet and more general Lovelock cosmology [26,27]. By modification the Padmanabhan’s proposal, one may extract the Friedmann equation of FRW Universe with

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any spatial curvature not only in Einstein gravity, but also in Gauss-Bonnet and more general Lovelock gravity \[28\]. The novel idea of Padmanabhan has got a lot of attentions in the literatures \[29\]-\[35\].

It is important to note that in order to rewrite the Friedmann equations, in any gravity theory, in the from of the first law of thermodynamics, \(dE = T_h dS_h + W dV\), on the apparent horizon and vice versa, one should consider the entropy expression of the black hole in each gravity theory. The only change one should done is replacing the black hole horizon radius \(r_+\) by the apparent horizon radius \(r_A\). However, the entropy expression associated with the black hole horizon get modified from the inclusion of quantum effects. Several type of quantum corrections to the area law have been introduced in the literatures, among them are logarithmic and power-law corrections. Logarithmic corrections, arises from the loop quantum gravity due to thermal equilibrium fluctuations and quantum fluctuations \[36\]-\[38\]. The logarithmic term also appears in a model of entropic cosmology which unifies the inflation and late time acceleration \[39\]. Another form of correction to area law, namely the power-law correction, appears in dealing with the entanglement of quantum fields inside and outside the horizon \[40\]-\[43\]. Another modification for the area law of entropy comes from the Gibbs arguments who pointed out that in systems with divergency in the partition function, like gravitational system, the Boltzmann-Gibbs (BG) theory cannot be applied. As a result thermodynamical entropy of such nonstandard systems is not described by an additive entropy but must be generalized to the non-additive entropy \[44\]. Based on this, and using the statistical arguments, Tsallis and Cirto argued that the microscopic mathematical expression of the thermodynamical entropy of a black hole does not obey the area law and can be modified as \[45\],

\[
S_h = \gamma A^\beta,
\]

where \(A\) is the black hole horizon area, \(\gamma\) is an unknown constant and \(\beta\) known as Tsallis parameter or nonextensive parameter, which is a real parameter which quantifies the degree of nonextensivity \[41\]. It is obvious that the area law of entropy is restored for \(\beta = 1\) and \(\gamma = 1/(4E_p^2)\). Through this paper we set \(k_B = 1 = c = \hbar\) for simplicity. In fact, at this limit, the power-law distribution of probability becomes useless, and the system is describable by the ordinary distribution of probability \[1\]-\[45\].

It is worth mentioning that in deriving Friedmann equations from the first law of thermodynamics, the entropy expression associated with the horizon plays a crucial role \[12\]. Thus, it is interesting to see how the Friedmann equation get modified if the entropy-area relation gets corrections by some reasons. Starting from the first law of thermodynamics at apparent horizon of a FRW universe, and assuming that the associated entropy with apparent horizon has a logarithmic quantum corrected relation, the modified Friedmann equations were derived in \[46\]. Also, taking the associated entropy with apparent horizon as the power-law-corrected relation, one is able to obtain the corrected Friedmann equation by using the first law of thermodynamics at the apparent horizon \[17\]. Besides, if thermodynamical interpretation of gravity near apparent horizon is generic feature, one should also not only check the first law of thermodynamics but also the generalized second law of thermodynamics. The latter is a universal principle governing the evolution of the total entropy of the Universe. In the context of the accelerating Universe, the generalized second law of thermodynamics has been explored in \[48\]-\[51\]. It should be noted that dark energy and a modified Friedmann equations in the context of Tsalllis entropy and from different perspective, were first investigated in \[49\],\[52\]. It was argued that Tsalllis entropy parameter change the strength of the gravitational constant and consequently the energy density of the dark components of the universe, requiring more (less) dark energy to provide the observed late time universe acceleration \[52\]. They also explored some phenomenological aspects as well as some observational constraints from a modified Friedmann equations induced by Tsalllis entropy \[43\],\[52\]. In the context of the nonextensive Kaniadakis statistics \[53\], the Jeans length was investigated and the results were compared with the Jeans length obtained in the non-extensive Tsalllis statistics \[54\]. Recently, modified cosmology through nonextensive Tsalllis entropy have been investigated in \[55\]. It was shown that the universe exhibits the usual thermal history, with the sequence of matter and dark energy eras, and depending on the value of nonextensive parameter \(\beta\) the equation of state of dark energy can even cross the phantom-line \[55\]. In this work, we shall derive the modified Friedmann equation in a universe which its entropy is given by the nonextensive Tsalllis entropy. Then, we investigate the cosmological implications of the obtained modified Friedmann equations in the matter and radiation dominated era. In our study, we do not need to invoke the dark energy component and the early deceleration as well as the late time acceleration of the universe expansion can be achieved in the presence of radiation and pressureless matter. Besides, we shall show that our model can solve the age problem of the universe. Our approach and the obtained Friedmann equations, from the first law of thermodynamics completely differ from those investigated in \[49\],\[52\],\[55\].

This paper is organized as follows. In the next section, we derive the modified Friedmann equations by applying the first law of thermodynamics, \(dE = T_h dS_h + W dV\), at apparent horizon of a FRW universe and assuming the entropy associated with apparent horizon is in the form of Tsalllis entropy \[1\]. In section \[III\] we examine the generalized second law of thermodynamics for the total entropy including the corrected entropy-area relation together with the matter field entropy inside the apparent horizon. In section \[IV\] we adopt the modified version of Padmanabhan's proposal \[22\] for deriving the Friedmann equation corresponding to Tsalllis entropy \[1\]. By assuming the difference between the number of degrees of
freedom in the bulk and on the boundary is proportional to the volume change of the spacetime, we find the modified Friedmann equation which coincides with the one obtained from the first law of thermodynamics. In section we investigate the cosmological consequences of the modified Friedmann equations derived from the non-extensive Tsallis entropy. The last section is devoted to conclusion and discussion.

II. MODIFIED FRIEDMAN EQUATION FROM THE FIRST LAW OF THERMODYNAMICS

We assume the background spacetime is spatially homogeneous and isotropic which is described by the line element

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu + \tilde{r}^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]

where \( \tilde{r} = a(t)r \), \( x^0 = t, x^1 = r \), and \( g_{\mu\nu} = \text{diag} (-1, a^2/(1 - k r^2)) \) represents the two dimensional metric. The open, flat, and closed universes corresponds to \( k = 0, 1, -1 \), respectively. We also assume the physical boundary of the Universe, which is consistent with laws of thermodynamics, is the apparent horizon with radius

\[ \tilde{r}_A = \frac{1}{\sqrt{H^2 + k/a^2}}. \]

The associated temperature with the apparent horizon can be defined as

\[ T_h = \frac{\kappa}{2\pi} = -\frac{1}{2\pi \tilde{r}_A} \left( 1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A} \right), \]

where \( \kappa \) is the surface gravity. For \( \tilde{r}_A \ll 2H\tilde{r}_A \), the temperature becomes \( T \ll 0 \). To avoid the negative temperature one may define \( T = |\kappa|/2\pi \). Also, within an infinitesimal internal of time \( dt \) one may assume \( \dot{\tilde{r}}_A \ll 2H\tilde{r}_A \), which physically means that the apparent horizon radius is kept fixed. Thus there is no volume change in it and one may define \( T = 1/(2\pi\tilde{r}_A) \) \[8\]. The profound connection between temperature on the apparent horizon and the Hawking radiation has been considered in \[56\], which further confirms the existence of the temperature associated with the apparent horizon.

We assume the matter and energy content of the Universe is in the form of perfect fluid with stress-energy tensor

\[ T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \]

where \( \rho \) and \( p \) are the energy density and pressure, respectively. The conservation of the energy-stress tensor in the FRW background, \( \nabla_\mu T^{\mu\nu} = 0 \), leads to the continuity equation as

\[ \dot{\rho} + 3H(\rho + p) = 0, \]

where \( H = \dot{a}/a \) is the Hubble parameter. Following \[57\], we define the work density as

\[ W = \frac{1}{2} T^{\mu\nu} h_{\mu\nu}. \]

A simple calculation gives

\[ W = \frac{1}{2} (\rho - p). \]

The work density term is indeed the work done by the volume change of the Universe, which is due to the change in the apparent horizon radius. We assume the first law of thermodynamics on the apparent horizon is satisfied and has the form

\[ dE = T_h dS_h + W dV. \]

It is clear that this equation is similar to the standard first law of thermodynamics, unless the work term \(-pdV\) is replaced by \( W dV\). For a pure de Sitter space where \( \rho = -p \), the work term reduces to the standard \(-pdV\) and one arrives at the standard first law of thermodynamics.

We suppose the total energy content of the universe inside a 3-sphere of radius \( \tilde{r}_A \), is \( E = \rho V \) where \( V = 4\pi\tilde{r}_A^3 \) is the volume enveloped by 3-dimensional sphere with the area of apparent horizon \( A = 4\pi\tilde{r}_A^2 \). Taking differential form of the total matter and energy inside the apparent horizon, we find

\[ dE = 4\pi\tilde{r}_A^2 \rho d\tilde{r}_A + \frac{4\pi}{3} \tilde{r}_A^3 \rho dt. \]

Using the continuity equation \[6\], we obtain

\[ dE = 4\pi\tilde{r}_A^2 \rho d\tilde{r}_A - 4\pi H\tilde{r}_A^3 (\rho + p) dt. \]

We further assume the entropy associated with the apparent horizon is in the form of Tsallis entropy \[1\], where now \( A \) is the apparent horizon area of the Universe. Differentiating the modified entropy-area relation \[1\], we get

\[ dS_h = \gamma A^{\beta - 1} dA = 8\pi\gamma\beta (4\pi\tilde{r}_A^2)^{\beta - 1} \tilde{r}_A d\tilde{r}_A. \]

Substituting Eqs. \[5\], \[1\] and \[12\] in the first law \[9\] and using relation \[5\] we get the differential form of the Friedmann equation as

\[ \frac{\gamma \beta}{4\pi\tilde{r}_A^2} (4\pi\tilde{r}_A^2)^{\beta - 1} d\tilde{r}_A = H(\rho + p) dt. \]

Using the continuity equation \[8\], we can rewrite it as

\[ -\frac{2}{\tilde{r}_A^3} (4\pi\tilde{r}_A^2)^{\beta - 1} d\tilde{r}_A = \frac{2\pi}{3\gamma\beta} dp. \]

Next, we integrate Eq. \[14\],

\[ -2 (4\pi)^{\beta - 1} \int \tilde{r}_A^{2\beta - 5} d\tilde{r}_A = \frac{2\pi}{3\gamma\beta} p. \]
which can be written as
\[
\frac{1}{\tilde{r}_A^{2-2\beta}} = \frac{2\pi(2-\beta)}{3\gamma\beta} (4\pi)^{1-\beta} \rho,
\] (16)
where we have set the integration constant equal to zero. Substituting \(\dot{r}_A\) from Eq. (13) we obtain
\[
\left(\frac{1}{k} \right)^{2-\beta} = \frac{8\pi L_p^2}{3} \rho,
\] (17)
provided we define
\[
\gamma \equiv \frac{2-\beta}{4\beta L_p^2} (4\pi)^{1-\beta}.
\] (18)
Equation (17) is nothing but the modified Friedmann equation obtained through the non-additive Tsallis entropy. In this way we have not only derived the modified Friedmann equation by starting from the first law of thermodynamics at apparent horizon of a FRW universe, and assuming that the associated entropy with apparent horizon has a corrected relation (1), but also find a general definition for the unknown constant \(\gamma\) in the entropy expression (1). In the limiting case where \(\beta = 1\), we recover the standard Friedmann equation in Einstein gravity. Besides, from Eq. (15), we have always \(\beta < 2\) which puts an upper bound on the non-additive parameter of the Tsallis entropy.

III. GENERALIZED SECOND LAW OF THERMODYNAMICS

Now we want to examine the validity of the generalized second law of thermodynamics in a region enclosed by the apparent horizon. Combining Eq. (14) with Eq. (6) and using relation (11), we arrive at
\[
\frac{2}{r_A^2}(2-\beta)\dot{r}_A \tilde{r}_A^{2-2\beta} = 8\pi L_p^2 H (\rho + p).
\] (19)
Solving for \(\dot{r}_A\) we find
\[
\dot{r}_A = \frac{4\pi L_p^2 H}{2-\beta} \tilde{r}_A^{-5-2\beta} (\rho + p).
\] (20)
Since \(\beta < 2\), the sign of \(\dot{r}_A\) depends on the sign of \(\rho + p\). For \(\rho + p > 0\), which physically means the dominant energy condition holds, we have \(\dot{r}_A > 0\). Let us now turn to find out \(T_h \dot{S}_h\):
\[
T_h \dot{S}_h = \frac{1}{2\pi \tilde{r}_A} \left( 1 - \frac{\dot{r}_A}{2H \tilde{r}_A} \right) \frac{d}{dt} \left[ \gamma (4\pi \tilde{r}_A^2)^\beta \right].
\] (21)
After some simplifications and using Eq. (20) we obtain
\[
T_h \dot{S}_h = 4\pi H \tilde{r}_A^3 (\rho + p) \left( 1 - \frac{\dot{r}_A}{2H \tilde{r}_A} \right).
\] (22)
At present, our Universe is undergoing an acceleration phase and thus its equation of state parameter can cross the phantom line \((w_D = p/\rho < -1)\). This implies that in an accelerating universe the dominant energy condition may violate, \(\rho + p < 0\), implying that the second law of thermodynamics, \(\dot{S}_h \geq 0\), does not hold. Therefore, we should examine the validity of the generalized second law of thermodynamics, namely \(\dot{S}_h + \dot{S}_m \geq 0\).

From the Gibbs equation we have [55]
\[
T_m dS_m = d(\rho V) + p dV = V d\rho + (\rho + p) dV,
\] (23)
where \(T_m\) and \(S_m\) are, respectively, the temperature and the entropy of the matter fields inside the apparent horizon. We assume the local equilibrium hypothesis holds. Therefore, the thermal system bounded by the apparent horizon remains in equilibrium so that the temperature of the system must be uniform and the same as the temperature of its boundary, \(T_m = T_h\) [55]. Note that if the temperature of the fluid differs much from that of the horizon, there will be spontaneous heat flow between the horizon and the bulk fluid and the local equilibrium hypothesis will no longer hold. Thus, from the Gibbs equation (23) we can obtain
\[
T_h (\dot{S}_h + \dot{S}_m) = 4\pi \tilde{r}_A^2 \dot{r}_A (\rho + p) - 4\pi \tilde{r}_A^3 H (\rho + p).
\] (24)
In order to examine the generalized second law of thermodynamics, we should study the evolution of the total entropy \(\dot{S}_h + \dot{S}_m\). Adding equations (22) and (24), we get
\[
T_h (\dot{S}_h + \dot{S}_m) = 2\pi \tilde{r}_A^2 (\rho + p) \dot{r}_A = \frac{A}{2} (\rho + p) \dot{r}_A.
\] (25)
where \(A\) is the apparent horizon area. Inserting \(\dot{r}_A\) from Eq. (20) into (25) we find
\[
T_h (\dot{S}_h + \dot{S}_m) = \frac{8\pi^2}{2-\beta} L_p^2 H \tilde{r}_A^{\gamma-2\beta} (\rho + p)^2.
\] (26)
Since \(\beta < 2\), the right hand side of the above equation is always a non-negative function during the universe history, which means that \(\dot{S}_h + \dot{S}_m \geq 0\). This implies that for a universe with Tsallis entropy the total entropy namely the entropy of the boundary together with the matter entropy inside the bulk is a non decreasing function of time and hence the generalized second law of thermodynamics is fulfilled.

IV. MODIFIED FRIEDMANN EQUATION FROM EMERGENCE OF COSMIC SPACE

According to the Padmanabhan’s proposal [25], the basic equation governing the evolution of the Universe can be derived by relating the emergence of space to the difference between the number of degrees of freedom in the holographic surface and the one in the emerged bulk. In this viewpoint, the spatial expansion of our Universe can
be regarded as the consequence of emergence of space and the cosmic space is emergent, following the progressing in the cosmic time. Thus, he argued that in an infinitesimal interval \( dt \) of cosmic time, the increase \( dV \) of the cosmic volume, is given by 26

\[
d\bar{V} = L_p^2 \left( N_{\text{sur}} - N_{\text{bulk}} \right),
\]

(27)

where \( N_{\text{sur}} \) is the number of degrees of freedom on the boundary and \( N_{\text{bulk}} \) is the number of degrees of freedom in the bulk. Using this new idea, Padmanabhan derived the Friedmann equation of a flat FRW Universe 25. It was argued that this proposal failed to derive the Friedmann equation of a nonflat FRW universe in other gravity theories such as Gauss-Bonnet and Lovelock gravity 28. The modification of Padmanabhan’s proposal was done by the present author 28, who argued that in a nonflat Universe the proposal should be generalized as

\[
d\bar{V} = L_p^2 \frac{\bar{r}_A}{H} \left( N_{\text{sur}} - N_{\text{bulk}} \right).
\]

(28)

This implies that the volume increase, in a nonflat universe, is still proportional to the difference between the number of degrees of freedom on the apparent horizon and in the bulk, but the function of proportionality is not just a constant, and is equal to the ratio of the apparent horizon and Hubble radius. Clearly, for spatially flat universe, \( \bar{r}_A = H^{-1} \), and one recovers the proposal 27.

Now, we would like to see whether one can derive the modified Friedmann equation from the relation 28, when the entropy associated with the apparent horizon get modified to Tsallis entropy. First of all, we define the effective area of the holographic surface corresponding to the entropy 1 as

\[
\bar{A} = A^\beta = \left( 4\pi \bar{r}_A^2 \right)^\beta.
\]

(29)

Next, we calculate the increasing in the effective volume as

\[
\frac{d\bar{V}}{dt} = \frac{\bar{r}_A}{2} \frac{d\bar{A}}{dt} = \beta (4\pi \bar{r}_A^2)^\beta \frac{d\bar{r}_A}{dt} = \beta (4\pi)^\beta \frac{\bar{r}_A^3}{(2\beta - 4)} \frac{d}{dt} \left( 2\beta - 4 \right).
\]

(30)

Inspired by 30, we propose that the number of degrees of freedom on the apparent horizon with Tsallis entropy, is given by

\[
N_{\text{sur}} = \frac{4\gamma \beta}{2 - \beta} (4\pi \bar{r}_A^2)^\beta.
\]

(31)

We also assume the temperature associated with the apparent horizon is the Hawking temperature, which is given by 8

\[
T = \frac{1}{2\pi \bar{r}_A},
\]

(32)

and the energy contained inside the sphere with volume \( V = 4\pi \bar{r}_A^3/3 \) is the Komar energy

\[
E_{\text{Komar}} = \left( \rho + 3p \right) V.
\]

(33)

According to the equipartition law of energy, the bulk degrees of freedom obey

\[
N_{\text{bulk}} = \frac{2|E_{\text{Komar}}|}{T}.
\]

(34)

In order to have \( N_{\text{bulk}} > 0 \), we take \( \rho + 3p < 0 \) 28. Thus the number of degrees of freedom in the bulk is given by

\[
N_{\text{bulk}} = -\frac{16\pi^2}{3} \gamma^4 (\rho + 3p).
\]

(35)

We also replace \( L_p^2 \) with \( 1/(4\gamma) \) and \( V \) with \( \bar{V} \) in (28) and write it down as

\[
A = \frac{4\gamma \beta}{2 - \beta} (4\pi \bar{r}_A^2)^\beta, \quad \frac{d\bar{V}}{dt} = \frac{\bar{r}_A}{H\gamma} (N_{\text{sur}} - N_{\text{bulk}}).
\]

(36)

Substituting relations 30, 31 and 35 in Eq. 36, after simplifying, we arrive at

\[
(2 - \beta) \bar{r}_A^2 \gamma^5 \frac{\bar{r}_A}{H} - \bar{r}_A^{2\beta - 4} = \frac{4\pi L_p^2}{3} (\rho + 3p),
\]

(37)

where we have also used definition 32. Multiplying the both hand side of Eq. 37 by factor 2\( a a \), after using the continuity equation 6, we reach

\[
\frac{d}{dt} \left( a^2 \bar{r}_A^{2\beta - 4} \right) = \frac{8\pi L_p^2}{3} \frac{d}{dt} (\rho a^2).
\]

(38)

Integrating, yields

\[
\left( H^2 + \frac{k}{a^2} \right)^{2-\beta} = \frac{8\pi L_p^2}{3} \rho,
\]

(39)

where we have set the integration constant equal to zero. This is nothing but the modified Friedmann equation motivated from Tsallis entropy-area relation 1. We see that our result from the emergence approach coincides with the modified Friedmann equation derived from the first law of thermodynamics in section 11. Our study indicates that the approach presented here is enough powerful and further supports the viability of the Padmanabhan’s perspective of emergence gravity and its modification given by Eq. 36.

V. TSALLIS COSMOLOGY

In this section, we would like to investigate cosmological consequences of the modified Friedmann equation derived in Eqs. (17) and (32). Let us begin by deriving the second modified Friedmann equation in Tsallis cosmology. Taking the time derivative of the first Friedmann equation (39), we get

\[
2H(2 - \beta) \left( \dot{H} - \frac{k}{a^2} \right) \left( H^2 + \frac{k}{a^2} \right)^{1-\beta} = \frac{8\pi L_p^2}{3} \bar{V}.
\]

(40)
Using the continuity equation (49), we arrive at
\[
(2 - \beta) \left( \dot{H} - \frac{k}{a^2} \right) \left( H^2 + \frac{k}{a^2} \right)^{1-\beta} = -4\pi L_p^2 (\rho + p)
\]
\[
-\frac{3}{2} \left( H^2 + \frac{k}{a^2} \right)^{2-\beta} - 4\pi L_p^2 \rho,
\]
where we have also used Eq. (39) in the last step. Now using the fact that \(\dot{H} = \dot{a}/a - H^2\), after some calculations, we can rewrite the above equation as
\[
(4 - 2\beta) \frac{\ddot{a}}{a} + (2\beta - 1) \left( H^2 + \frac{k}{a^2} \right)^{2-\beta} = -4\pi L_p^2 \rho.
\]
This is indeed the second modified Friedmann equation governing the evolution of the Universe in Tsallis cosmology which is based on the nonextensive Tsallis entropy. When \(\beta = 1\), the above equation reduces to the second Friedmann equation in standard cosmology,
\[
2 \frac{\ddot{a}}{a} + \left( H^2 + \frac{k}{a^2} \right) = -8\pi L_p^2 \rho.
\]
We can also obtain the equation for the second time derivative of the scale factor. To this aim, we combine the first and second modified Friedmann equations (39) and (42). We find
\[
\frac{\ddot{a}}{a} = -\frac{4\pi L_p^2 \rho}{3(2 - \beta)} [(2\beta - 1)\rho + 3p].
\]
Since our Universe is currently undergoing an acceleration phase (\(\ddot{a} > 0\)), thus from the above equation we should have
\[
(2\beta - 1)\rho + 3p < 0 \quad \rightarrow \quad \omega < \frac{1 - 2\beta}{3},
\]
where \(\omega = p/\rho\) is the equation of state parameter. Again, for \(\beta = 1\) the above condition for the accelerated universe leads to the well-known inequality \(\omega < -1/3\) for the equation of state in standard cosmology. A close look on the inequality (11) show that for \(\beta \geq 1/2\), we should always have \(\omega < 0\). However, for \(\beta < 1/2\), we can have \(\omega > 0\) in an accelerated universe. For example, for \(\beta = 1/3\) the above inequality implies \(\omega < 1/9\). This is an interesting result, which indicates that in Tsallis cosmology, the late time accelerated universe can be achieved even in the presence of the ordinary matter with positive equation of state parameter. Thus, if we assume our Universe is now dominated with pressureless matter with (\(\omega = 0\)), then it can undergo an accelerated expansion provided we choose the nonextensive parameter \(\beta\) less than 1/2, without needing to an additional component of energy such as dark energy.

Next, we are going to find the scale factor as a function of time in Tsallis cosmology. For simplicity we only consider the flat universe with \(k = 0\), although the calculations can be easily extended to the nonflat universe (\(k \neq 0\)). We shall consider the radiation and matter dominated era, separately.

### A. Matter-dominated era

It is well-known that the pressure caused by random motions of galaxies and clusters of galaxies is negligible. This is due to the fact that the random velocities of the galaxies are of the order of \(10^8\) cm/s or less [59]. Besides, the average energy density of the universe is of the order of \(10^{-30}\) g cm\(^{-3}\). Thus, we can write
\[
p \sim \rho < v^2 \sim 10^{-5} \rho c^2 \ll \rho c^2.
\]
This implies that the pressure is negligible compared with \(\rho c^2\). Thus if the energy density of the universe is dominated by the nonrelativistic matter with negligible pressure, we can write \(p = 0\). In this case, the continuity equation (50) can be written \(\dot{\rho}(t) + 3H\rho(t) = 0\), which can be easily integrated to yield
\[
\frac{\rho(t)}{\rho(t_0)} = \left( \frac{a(t)}{a(t_0)} \right)^{-3},
\]
where \(t_0\) is the present time of the universe. Therefore, \(\rho a^3 = \text{constant}\), which is exactly the same as in standard cosmology in the matter dominated era. Substituting \(\rho\) from (47) in the first Friedmann equation (39), we arrive at
\[
\frac{\dot{a}}{a} = \frac{8\pi L_p^2 \rho_0 a_0^3}{3a^3},
\]
which can be rewritten as
\[
\left( \frac{da}{dt} \right)^{4 - 2\beta} = C_1 a^{1 - 2\beta},
\]
where we have defined
\[
C_1 = \frac{8\pi L_p^2 \rho_0 a_0^3}{3}.
\]
Eq. (49) has the following solution,
\[
a(t) = (C_2 t)^{(4 - 2\beta)/3},
\]
where
\[
C_2 = \frac{3 C_1^{1/(4 - 2\beta)}}{2 (2 - \beta)} > 0.
\]
In the limit of standard cosmology where \(\beta = 1\), the above solution restores
\[
a(t) = \left[ \frac{3}{2} \sqrt{C_1} \right]^{2/3} t^{2/3},
\]
Let us now calculate the second time derivative of the scale factor. We find
\[
\ddot{a}(t) = \frac{C_2^{(4 - 2\beta)/3}}{9} (4 - 2\beta)(1 - 2\beta) t^{-(2 + 2\beta)/3},
\]
Again, we see that in the matter dominated universe, the accelerated expansion \((\ddot{a}(t) > 0)\) can be achieved provided we take \(\beta < 1/2\), which is consistent with our previous discussion. Thus, in the framework of Tsallis cosmology, without invoking any kind of dark energy and only with pressureless matter, the late time acceleration of the universe expansion can be understood.

We can also calculate the evolution of the energy density, the Hubble and the deceleration parameters as

\[
\rho(t) \propto \frac{1}{\dot{a}^{4-2\beta}}, \quad (55)
\]
\[
H(t) = \frac{\dot{a}}{a} = \frac{4-2\beta}{3t}, \quad (56)
\]
\[
q(t) = \frac{a\ddot{a}}{\dot{a}^2} = \frac{2\beta - 1}{4-2\beta}, \quad (57)
\]

Again, all above parameters are reduced to the one in standard cosmology for \(\beta = 1\). From the deceleration parameter, we see that for \(\beta < 1/2\), we have \(q < 0\) and \(\ddot{a} > 0\), a result which confirm the accelerated universe.

Now, we look at the Hubble parameter which can be employed for estimating the age of the universe at the present time \((t = t_0)\). We have

\[
t_0 = \frac{4-2\beta}{3H_0}, \quad (58)
\]

where \(H_0 = H(t_0)\) is the Hubble constant. In an accelerated universe we have \(\beta < 1/2\), which implies \(4-2\beta > 3\), and thus

\[
t_0 > \frac{1}{H_0} = \frac{3}{2} \left(\frac{2}{3H_0}\right), \quad (59)
\]

Note that \(2/(3H_0)\) is the age of the universe in standard cosmology. Let us see how the above relation can alleviate the problem of age in standard cosmology. The Hubble constant is usually written as

\[
H_0 = 100h \text{ km sec}^{-1}\text{Mpc} = 2.1332 \times 10^{-42} \text{GeV}. \quad (60)
\]

where \(h\) describes the uncertainty on the value \(H_0\). The observations of the Hubble Key Project constrain this value to be \(h = 0.72 \pm 0.08\) \cite{60}. Thus the Hubble time is \(t_H = 1/H_0 = 9.78 \times 10^9\text{h}^{-1}\) years. Using this value for \(h\), the age of the Universe in standard cosmology is in the range 8.2 Gyr < \(t_0\) < 10.2 Gyr \cite{61}. Carretta et al. \cite{62} estimated the age of globular clusters in the Milky Way to be 12.9 ± 2.9 Gyr, whereas Jimenez et al. \cite{63} obtained the value 13.5 ± 2 Gyr. Hansen et al. \cite{64} constrained the age of the globular cluster M4 to be 12.7 ± 0.7 Gyr by using the method of the white dwarf cooling sequence. In most cases the ages of globular clusters are larger than 11 Gyr which indicates that the cosmic age estimated in standard cosmology is inconsistent with the ages of the oldest globular clusters. It was argued that, in standard model of cosmology, this problem cannot be circumvented unless the cosmological constant (dark energy) is taken into account \cite{61}. However, in the framework of Tsallis cosmology, the problem of age can be circumvented automatically for an accelerated universe. More precisely, from relation (59), the age of the universe in Tsallis cosmology is larger than 3/2 age of the universe in standard cosmology, namely

\[
t_0|_T > \frac{3}{2}t_0|_S, \quad (61)
\]

provided we choose \(\beta < 1/2\). Here subscript “T” and “S” stand for Tsallis and Standard cosmology, respectively. As an example, taking \(\beta = 2/5\), from relation (58) we find \(t_0|_T = 1.6 t_0|_S\). Consequently, the age of the accelerated universe in Tsallis cosmology is in the range 13.12 Gyr < \(t_0|_T\) < 16.32 Gyr, which is larger than the age of the oldest globular clusters. Therefore, the cosmic age problem can be properly alleviated in an accelerated universe in the framework of Tsallis cosmology without invoking additional component of energy.

\section*{B. Radiation-dominated era}

It is well-known that, in an expanding universe, the proper momenta of freely moving particles decreases in term of scale factor as \(1/a(t)\). Thus, small random velocities of particles seen today should have been large in the past when the scale factor was much smaller than its present value \cite{59}. As a result, the pressureless approximation is break down in the early universe. In this subsection, we are going to find the solution for the modified Friedmann equation in the framework of Tsallis cosmology by assuming that the universe is filled with a highly relativistic gas (radiation) with equation of state \(p = \rho/3\). In this case from the continuity equation, \(\dot{\rho}(t) + 4H\rho(t) = 0\), we can get

\[
\frac{\rho(t)}{\rho(t_1)} = \left(\frac{a(t)}{a(t_1)}\right)^{-4}, \quad (62)
\]

where \(t_1\) is an arbitrary reference time in the radiation dominated era. Thus, in this case \(\rho a^4 = \text{constant.}\), which is similar to the result obtained in the radiation dominated era in standard cosmology. Substituting \(\rho\) from (62) in the first Friedmann equation (39), we get

\[
\left(\frac{\dot{a}}{a}\right)^{4-2\beta} = \frac{8\pi L_p^2 \rho_1 a_1^4}{3a^4}, \quad (63)
\]

where \(\rho_1 = \rho(t_1), a_1 = a(t_1)\) and we have again considered the flat universe \((k = 0)\), although the \(k = \pm 1\) cases can also be easily solved. The above equation can be rewritten as

\[
\left(\frac{da}{dt}\right)^{4-2\beta} = B_1 a^{-2\beta}, \quad (64)
\]

where

\[
B_1 \equiv \frac{8\pi L_p^2 \rho_1 a_1^4}{3}, \quad (65)
\]
Eq. (64) admits a solution of the form,
\[ a(t) = B_1^{1/4} \left( \frac{2t}{2 - \beta} \right)^{1 - \beta/2}. \] (66)

When \( \beta = 1 \), we arrive at
\[ a(t) = \sqrt{2} B_1^{1/4} t^{1/2}, \] (67)
which is the corresponding solution in standard cosmology. The second time derivative of the scale factor is obtained as
\[ \ddot{a}(t) = -\beta B_1^{1/4} \left( \frac{2}{2 - \beta} \right)^{-\beta/2} t^{-(1+\beta/2)} < 0. \] (68)

This shows that in the radiation dominated era, the universe has been in an decelerated phase (\( \ddot{a}(t) < 0 \)). Now, we calculate the energy density, the Hubble and the deceleration parameters in the radiation dominated era. We find
\[
\rho(t) \propto \frac{1}{t^{4 - 2\beta}}, \quad H(t) = \frac{\dot{a}}{a} = \frac{2 - \beta}{2t}, \quad q(t) = -\frac{\ddot{a}}{a^2} = \frac{2}{2 - \beta}. \] (69-71)

Therefore, for \( \beta < 2 \), we have always \( q > 0 \) (\( \ddot{a}(t) < 0 \)), which confirms that we have a decelerated universe in the radiation dominated era. This implies that in the early stage of the universe where the relativistic particles have been dominated, our Universe has been in a decelerated phase and at the late time it undergoes an accelerated expansion. In summary, Tsallis cosmology can explain the evolution of the universe from the early deceleration to the late time acceleration, without invoking dark energy, which is consistent with recent observations.

### C. Density perturbation

In order to study the density perturbation, we follow the method which was first introduced by James Jeans [50]. Assuming a flat FRW universe in the matter-dominated era, the fundamental differential equation governing the fractional density perturbation \( \delta = \rho_1/\rho \) in an expanding universe becomes [50]

\[ \ddot{\delta} + \frac{2\dot{a}}{a} \dot{\delta} + \left( \frac{v_s^2 \kappa^2}{a^2} - 4\pi G \rho \right) \delta = 0, \] (72)

where \( v_s^2 = p_1/\rho_1 \) is the speed of sound, \( \rho_1 \) and \( p_1 \) stand for the perturbed part of the energy density and pressure, respectively, and \( \kappa \) is the comoving wavevector. The solution of the above equation depends on the sign of the function
\[ f(t) = \frac{v_s^2 \kappa^2}{a^2(t)} - 4\pi G \rho(t). \] (73)

For \( f(t) > 0 \), the solution for \( \delta(t) \) is periodic and the perturbation propagate in the medium as sound waves. For \( f(t) < 0 \), however, lead to imaginary frequency, indicating a growth of \( \delta(t) \) with time. In what follows we are interested in the second case where \( f(t) < 0 \) and further assume \( v_s^2 \kappa^2 \ll 4\pi G \rho \). In this case Eq. (72) can be written
\[ \ddot{\delta} + \frac{2\dot{a}}{a} \dot{\delta} - 4\pi G \rho \delta = 0. \] (74)

In the framework of standard model of cosmology where \( a(t) \propto t^{2/3} \), Eq. (74) can be further rewritten as
\[ \ddot{\delta} + \frac{4}{3t} \dot{\delta} - \frac{2}{3t^2} \delta = 0. \] (75)

which has a solution in the form \( \delta(t) = \delta_0 t^{2/3} \), and thus the perturbation grows with time. On the other hand, when the Friedmann equations are governed by the Tsallis cosmology, the scale factor in the matter dominated era is, \( a(t) \propto t^{(4-2\beta)/3} \). On substituting in Eq. (74), we arrive at
\[ \ddot{\delta} + \frac{4}{3t} (2 - \beta) \dot{\delta} - \frac{3}{2} \left( \frac{4 - 2\beta}{3t} \right)^{4-2\beta} \delta = 0. \] (76)

The solution of the above equation is given by
\[ \delta(t) = \delta_0 \times \]
\[ \text{Bessel}J \left( \frac{5 - 4\beta}{6 \beta - 6}, \frac{2(\beta - 2)^2}{9 \beta - 9} \sqrt{-6 \left( \frac{4 - 2\beta}{3} \right)^{4-2\beta} t^{\beta-1}} \right), \] (77)

To have an insight on the form of \( \delta(t) \), let us consider a special case where \( \beta = 5/4 \) at the early stage of the universe. In this case the expansion of solution (77) is given by
\[ \delta(t) = \delta_0 \left[ 1 - \alpha_1 t^{1/2} + \alpha_2 t + ... \right] \] (78)

where \( \alpha_i \) are constants. Thus, the growth of energy differs in Tsallis cosmology compared to the standard cosmology. It is important to note that this study is very brief and we leave the details of investigations on the density perturbation in Tsallis cosmology for future studies.

### VI. CONCLUSION AND DISCUSSION

It was already pointed out by Gibbs in 1902 that in systems whose partition function diverges, like gravitational system, the Boltzmann-Gibbs (BG) theory cannot be applied. As a result, the thermodynamical entropy of such nonstandard systems is not described by an additive entropy but with appropriately generalized nonadditive entropies. Based on this, and using the statistical arguments, Tsallis and Cirto [1] argued that the microscopic mathematical expression of the thermodynamical
entropy of a black hole does not obey the area law and can be modified as in Eq. (1). Besides, it is well-known that the entropy of the whole universe, considered as a system with apparent horizon radius has a similar expression to the entropy associated with black hole horizon. The only change is needed replacing the black hole horizon radius $r_+$ by the apparent horizon radius $\tilde{r}_A$ of the Universe.

In this paper, we have assumed the entropy associated with the apparent horizon of FRW universe is in the form of the Tsallis entropy and examined its consistency with the laws of thermodynamics. For this purpose, we first assumed the the first law of thermodynamics, $dE = T_h dS_h + W dV$, holds on the apparent horizon and the Tsallis entropy has the form (1). Then, we showed that the first law of thermodynamics on the apparent horizon can be rewritten in the form of the modified Friedmann equations of a FRW universe with any spatial curvature. We have also examined the generalized second law of thermodynamics, by studying the time evolution of the total entropy including the entropy of the matter and energy inside the Universe together with the Tsallis entropy associated with the apparent horizon. Assuming the local equilibrium hypothesis, we confirmed that the generalized second law of thermodynamics is fulfilled in a region enclosed by the apparent horizon.

We have also considered the idea of emergence of cosmic space proposed by Padmanabhan [25] and its modification given by Eq. (66) [28]. Assuming the entropy associated with the apparent horizon in the form of Tsallis entropy, we determine the number of degrees of freedom on the boundary. Then, by calculating the difference between the degrees of freedom on the boundary, $N_{\text{sur}}$, and the degrees of freedom in the bulk, $N_{\text{bulk}}$, we showed that one can extract the Friedmann equations corresponding to the Tsallis entropy-area relation (1) which coincides with the one obtained from the first law of thermodynamics. This study strongly supports the viability of the novel idea proposed in [25] [28]. Note that, we have only modified the number of degrees of freedom on the boundary due to the change in the entropy expression, while we assumed the bulk degrees of freedom, $N_{\text{bulk}}$, has the same expression as in standard cosmology. The reason comes from the fact that we have assumed $N_{\text{bulk}}$ depends only on the matter degrees of freedom, while $N_{\text{sur}}$ crucially depends on the entropy expression or/and underlying theory of gravity.

We have also investigated some cosmological consequences of the modified Friedmann equations. For this purpose, we first derived the second modified Friedmann equation as well as the equation for the second time derivative of the scale factor. Then, we considered the matter-dominated era and the radiation-dominated era, separately. We found that, in the matter dominated era which is filled with pressureless matter, this model admits an accelerated universe ($\ddot{a}(t) > 0$) provided $\beta < 1/2$. Besides, the problem of age of universe can be circumvented automatically for an accelerated universe. By calculating the Hubble parameter in the matter dominated era, we estimated the age of the universe larger than 3/2 age of the universe in standard cosmology. For example, taking $\beta = 2/5$, we estimated the age of the accelerated universe in Tsallis cosmology in the range $13.12 \text{ Gyr} < t_0 < 16.32 \text{ Gyr}$, which is larger than the age of the oldest globular clusters. Therefore, the cosmic age problem can be alleviated in an accelerated universe in the framework of Tsallis cosmology without invoking additional component of energy. In the radiation dominated era we have always $q > 0$ ($\ddot{a}(t) < 0$), which confirms the deceleration phase for the early stage of the universe. Therefore, Tsallis cosmology can explain the evolution of the universe from the early deceleration to the late time acceleration which is consistent with recent observations. Finally, employing the Jeans’s analysis, we explored in brief, density perturbation in the matter dominated era in the context of Tsallis cosmology. We showed that due to the modification of the Friedmann equations and hence the scale factor, the growth of energy is changed compared to the standard cosmology.

It is important to note that many issues remain to be studied. First, the details of the effects of the modified Friedmann equations on the growth of energy density in the early stage of the universe as well as its influences on the structure formation deserve further investigation. It is also interesting to consider other forms of entropies e.g., the generalized Kaniadakis entropy [53] to investigate the modified Friedmann equations as well as its cosmological consequences. These issues are under investigations and the results will be appeared in the future works.

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