Supporting Information for:

Cortical composition hierarchy driven by spine proportion economical maximization or wire volume minimization

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Supporting Figure

Figure A. (see below)
Euclidean distance ED for heavy-tailed distributions of spine sizes as a function of exponent $\gamma_2$ for spine economical maximization. (A) ED for Log-logistic distribution with different values of $\beta$. (B) ED for Log-normal distribution with different values of $\sigma$. In both panels $\theta = 0.321 \, \mu m^3$. 
THEORETICAL MODELS

Below, the details of calculations are provided for the three principles considered in the main text.

1 Wire minimization principle.

1.1 The system of basic equations for optimal solution.

Explicit form of the fitness function.

The explicit dependence of the fitness function $F_w$ on the three parameters $x, y, \overline{\pi}$ is given as

$$F_w = \frac{rx + y}{\overline{\pi}^\gamma} + \lambda_1 \left( x + y + Pxy + \frac{a(Pxy)^{2/3}}{\overline{\pi}^{2/3}} + \frac{a(Pxy)^{5/3}}{\overline{\pi}^{2/3}} - 1 \right),$$

where the parameter $a = (3\pi^2/256)^{1/3}d_{\alpha S}^2 = 0.352 \, \mu m^2$.

The basic optimal equations.

The optimal values of fractional volumes of axons and dendrites $x, y$, average spine volume $\overline{\pi}$, and the Lagrange multiplier $\lambda_1$ are found by differentiating the benefit-cost function $F_w$ with respect to $x, y, \overline{\pi}$, and $\lambda_1$, and requiring that

$$\partial F_w/\partial x = \partial F_w/\partial y = \partial F_w/\partial \overline{\pi} = \partial F_w/\partial \lambda_1 = 0,$$

which corresponds to a critical point of $F_w$. Consequently, we obtain the following set of four nonlinear equations:
\[ r + \lambda_1 \pi^{\gamma_1} \left( 1 + P y + \frac{a}{3} \left( \frac{P^2 y^2}{x a^2} \right)^{1/3} [2 + 5Pxy] \right) = 0, \]  

(2)

\[ 1 + \lambda_1 \pi^{\gamma_1} \left( 1 + P x + \frac{a}{3} \left( \frac{P^2 x^2}{y a^2} \right)^{1/3} [2 + 5Pxy] \right) = 0, \]  

(3)

\[ \gamma_1 (r x + y) = \lambda_1 \pi^{\gamma_1 + 1} \left[ \frac{\partial P}{\partial \pi} \left( xy + \frac{a(xy)^{2/3} (2 + 5Pxy)}{3P^{1/3} \pi^{2/3}} - \frac{2a(Pxy)^{2/3}}{3\pi^{5/3}} (1 + Pxy) \right) \right] \]  

(4)

and

\[ x + y + Pxy + \frac{a(Pxy)^{2/3}}{\pi^{2/3}} + \frac{a(Pxy)^{5/3}}{\pi^{2/3}} = 1. \]  

(5)

Reduction of dimensionality in the system of basic equations.

Next, we show that we can decrease the number of basic equations from 4 to 3. First, we can get rid of \( \lambda_1 \), since it always appears in the first power. The parameter \( \lambda_1 \) can be determined from Eq. (3) and it reads:
\[ \lambda_1 = -\frac{1}{\pi^1 \left(1 + P x + \frac{a}{3} \left(\frac{p^2 x^2}{\pi y}\right)^{1/3} \left[2 + 5Pxy\right]\right)}. \] (6)

Next, we can insert \( \lambda_1 \) into Eqs. (2) and (4). As a result we obtain Eqs. (29) and (30) in the main text.

1.2 Proof of the local minimum for optimal solution related to wire minimization.

Let us introduce the following notation: \( x_1 \equiv x, x_2 \equiv y, x_3 \equiv \pi \). Then the function \( F_w \) (Eq. 1) can be rewritten as:

\[ F_w = \frac{r x_1 + x_2}{x_3} + \lambda_1 g(x_1, x_2, x_3), \] (7)

where \( g(x_1, x_2, x_3) \) denotes the constraint term present in Eq. (1). Let us define partial derivatives: \( F_{ij} = \frac{\partial^2 F_w}{\partial x_i \partial x_j} \) and \( g_i = \frac{\partial g}{\partial x_i} \), which are determined at the critical point represented by optimal values of \( x_1, x_2, x_3 \). Using these definitions we can construct a matrix called bordered Hessian for our constraint optimization problem as [2]:

\[ \begin{bmatrix}
  0 & g_1 & g_2 & g_3 \\
  g_1 & F_{11} & F_{12} & F_{13} \\
  g_2 & F_{21} & F_{22} & F_{23} \\
  g_3 & F_{31} & F_{32} & F_{33}
\end{bmatrix} \]

This is a symmetric matrix, i.e. \( F_{ij} = F_{ji} \).
A sufficient condition for $F_w$ to have a local minimum at the critical point represented by the optimal values $x_1, x_2, x_3$ is that two principal minors, i.e. determinants of the upper-left sub-matrices 3x3 (called $D_1$) and 4x4 (determinant of the entire bordered Hessian called $D_2$), have negative signs [2]. The explicit forms of these determinants are as follows:

$$D_1 = -g_1^2 F_{22} - g_2^2 F_{11} + g_1 g_2 (F_{12} + F_{21})$$  \hspace{1cm} (8)$$

and

$$D_2 = g_1^2 (F_{23}^2 - F_{22} F_{33}) + g_2^2 (F_{13}^2 - F_{11} F_{33}) + g_3^2 (F_{12} - F_{11} F_{22}) + 2 g_1 g_2 (F_{12} F_{33} - F_{13} F_{23}) + 2 g_1 g_3 (F_{13} F_{22} - F_{12} F_{23}) + 2 g_2 g_3 (F_{11} F_{23} - F_{12} F_{13})$$  \hspace{1cm} (9)$$

Exact numerical values of the minors $D_1$ and $D_2$ are presented in Table A (below) together with the values of $F_{ij}$. These results indicate that indeed we have local minima at the critical points.
Table A: “Wire minimization” approach. The numerical values of the elements of the bordered Hessian and principal minors for each type of the spine volume distribution.

| Spine size distribution | θ     | $F_{11}$ | $F_{12}$ | $F_{22}$ | $F_{13}$ | $F_{23}$ | $F_{33}$ | $D_1$  | $D_2$  |
|-------------------------|-------|----------|----------|----------|----------|----------|----------|--------|--------|
| Exponential             | 0.100 (ED) | 0.117    | -0.905   | 0.162    | -0.196   | -0.230   | 2.476    | -5.529 | -13.69 |
|                         | 0.100 (MD) | 0.107    | -0.890   | 0.174    | -0.185   | -0.236   | 2.433    | -5.449 | -13.25 |
|                         | 0.321 (ED) | 0.031    | -0.748   | 0.041    | -0.020   | -0.023   | 0.026    | -3.365 | -0.087 |
|                         | 0.321 (MD) | 0.036    | -0.764   | 0.036    | -0.022   | -0.022   | 0.026    | -3.433 | -0.090 |
| Gamma (n=1)             | 0.100 (ED) | 0.110    | -1.045   | 0.156    | -0.191   | -0.227   | 3.205    | -7.020 | -22.49 |
|                         | 0.100 (MD) | 0.074    | -0.967   | 0.216    | -0.146   | -0.249   | 2.919    | -6.579 | -19.19 |
|                         | 0.321 (ED) | 0.033    | -0.843   | 0.045    | -0.026   | -0.031   | 0.082    | -4.148 | -0.341 |
|                         | 0.321 (MD) | 0.037    | -0.856   | 0.041    | -0.028   | -0.030   | 0.084    | -4.211 | -0.352 |
| Gamma (n=2)             | 0.100 (ED) | 0.112    | -1.118   | 0.156    | -0.201   | -0.238   | 4.440    | -7.925 | -35.19 |
|                         | 0.100 (MD) | 0.063    | -0.993   | 0.254    | -0.134   | -0.269   | 3.813    | -7.196 | -27.42 |
|                         | 0.321 (ED) | 0.036    | -0.891   | 0.047    | -0.031   | -0.036   | 0.155    | -4.585 | -0.712 |
|                         | 0.321 (MD) | 0.038    | -0.901   | 0.045    | -0.033   | -0.035   | 0.157    | -4.632 | -0.727 |
| Rayleigh                | 0.100 (ED) | 0.117    | -1.121   | 0.171    | -0.219   | -0.265   | 5.876    | -8.101 | -47.60 |
|                         | 0.100 (MD) | 0.063    | -0.987   | 0.288    | -0.141   | -0.301   | 4.979    | -7.327 | -36.45 |
|                         | 0.321 (ED) | 0.040    | -0.890   | 0.052    | -0.036   | -0.042   | 0.199    | -4.608 | -0.918 |
|                         | 0.321 (MD) | 0.042    | -0.899   | 0.050    | -0.038   | -0.041   | 0.201    | -4.655 | -0.938 |
| Log-logistic            | 0.100 (ED) | 0.060    | -0.856   | 0.080    | -0.065   | -0.075   | 0.243    | -4.482 | -1.088 |
|                         | 0.100 (MD) | 0.062    | -0.860   | 0.078    | -0.067   | -0.075   | 0.244    | -4.505 | -1.099 |
|                         | 0.321 (ED) | 0.037    | -0.936   | 0.048    | -0.035   | -0.040   | 0.310    | -5.012 | -1.555 |
|                         | 0.321 (MD) | 0.039    | -0.967   | 0.057    | -0.042   | -0.050   | 0.594    | -5.394 | -3.205 |
| Log-normal              | 0.100 (ED) | 0.073    | -0.696   | 0.095    | -0.079   | -0.090   | 1.323    | -3.290 | -4.354 |
|                         | 0.100 (MD) | 0.089    | -0.791   | 0.113    | -0.114   | -0.129   | 3.283    | -4.140 | -13.59 |
|                         | 0.321 (ED) | 0.045    | -0.838   | 0.061    | -0.042   | -0.049   | 0.985    | -4.206 | -4.141 |
|                         | 0.321 (MD) | 0.050    | -0.925   | 0.069    | -0.055   | -0.064   | 2.458    | -5.061 | -12.44 |

For each value of $\theta$ there are two values of Hessian matrix and Minors corresponding to minimal Euclidean (ED) and Mahalanobis (MD) distances.
2 Spine economical maximization principle.

2.1 The system of basic equations for optimal solution.

Explicit form of the fitness function.

The explicit dependence of the benefit-cost function $F_s$ on the three parameters $x, y, \pi$ is given as

$$F_s = \frac{Pxy}{u^2} + \lambda_2 \left( x + y + Pxy + \frac{a(\frac{Pxy}{u^2})^{2/3}}{3} + \frac{a(\frac{Pxy}{u^2})^{5/3}}{3} - 1 \right). \quad (10)$$

The basic optimal equations.

The optimal values of $x, y, \pi, \lambda_2$ are found by differentiating the benefit-cost function $F_s$ (Eq. 10) with respect to $x, y, \pi$, and $\lambda_2$, and requiring that appropriate derivatives are zero. As a result, we obtain the following set of four nonlinear equations:

$$P_y + \lambda_2 u^{-2} \left( 1 + P_y + \frac{a}{3} \left( \frac{P^2 y^2}{u^2} \right)^{1/3} [2 + 5Pxy] \right) = 0, \quad (11)$$

$$P_x + \lambda_2 \pi^{-2} \left( 1 + P_x + \frac{a}{3} \left( \frac{P^2 x^2}{\pi^2} \right)^{1/3} [2 + 5Pxy] \right) = 0, \quad (12)$$
\[
\left[ \frac{1}{u^2} + \lambda_2(xy)^{1/3} + \frac{a\lambda_2(2+5Pxy)}{3(Pu^2)^{1/3}} \right] \frac{\partial P}{\partial u} = \left[ \frac{\gamma_2(xy)^{1/3}}{\pi^{\gamma_2+1}} + \frac{2a\lambda_2(1+Pxy)}{3(Pu^2)^{1/3}} \right] P, \tag{13}
\]

and

\[
x + y + Pxy + \frac{a(Pxy)^{2/3}}{u^{2/3}} + \frac{a(Pxy)^{5/3}}{u^{2/3}} = 1. \tag{14}
\]

Note that from Eqs. (11) and (12) it follows that \(\lambda_2\) must be negative, since all other terms on the left hand side are positive. This observation is used below for determination of the type of extremum.

**Proof that** \(x = y\).

First, we show that for optimal \(x\) and \(y\) we have \(x = y\). To do this, we subtract Eqs. (11) and (12). As a result we get:

\[
(y - x) \left[ P(1 + \lambda_2 u^{\gamma_2}) + \lambda_2 u^{\gamma_2} \frac{a}{3} \left( \frac{P^2}{xyu^2} \right)^{1/3} (2 + 5Pxy) \right] = 0, \tag{15}
\]

where the expression in the [...] bracket is equal either to \(-\lambda_2 u^{\gamma_2}/y\) (from Eq. 11) or to \(-\lambda_2 u^{\gamma_2}/x\) (from Eq. 12). Thus, Eq. (15) is equivalent to the following equation:
\[(y - x) \frac{\lambda_2 \pi x^2}{y} = 0, \quad (16)\]

which implies that for nonzero \(\lambda_2\) and \(\pi\) we must have \(x = y\). (The benefit-cost function \(F_s\) is defined only for \(\pi > 0\), see Eq. 10). If however, \(\lambda_2 = 0\), then from Eqs. (11) and (12) we get that \(P_x = P_y = 0\). The case \(P = 0\) implies \(\pi = 0\) (see eqs relating \(P\) and \(\pi\) in the Methods), which however is forbidden. Thus \(P \neq 0\), and in this case we must have \(x = y = 0\), i.e. \(x\) and \(y\) are still equal to each other.

**Reduction of dimensionality in the system of basic equations.**

Next, we show that we can decrease the number of basic equations. Because \(x = y\), we can reduce the system of 4 equations to the system of 3 equations with unknowns \(x, \pi, \lambda_2\) (Eqs. 11 and 12 are in fact the same equation). Moreover, we can get rid of \(\lambda_2\), since it always appears in the first power, which additionally allows us to reduce the system dimensionality to 2. The parameter \(\lambda_2\) can be determined from Eq. (11) (with the substitution \(y = x\)) and it reads:

\[
\lambda_2 = -\frac{P_x}{\pi x^2 \left(1 + P_x + \frac{a}{3} \left(\frac{\lambda_2}{\pi} \right)^{1/3} [2 + 5P_x^2]\right)}. \quad (17)
\]

Next, we can insert \(\lambda_2\) into Eq. (13). After this procedure Eq.(13) becomes

\[
\pi^{2/3} \frac{\partial P}{\partial \pi} = \frac{P}{\pi} \left(\gamma_2 \pi^{2/3} (1 + P_x) + \frac{a}{3} P^{2/3} x^{1/3} [2(\gamma_2 - 1) + (5\gamma_2 - 2) P x^2]\right) \quad (18)
\]
and Eq. (14) after the substitution $y = x$ becomes

$$2x + P x^2 + \frac{a P^{2/3} x^{4/3}}{\pi^{2/3}} + \frac{a P^{5/3} x^{10/3}}{\pi^{2/3}} = 1. \quad (19)$$

The derivatives of $P$ with respect to $\pi$ have different forms depending on the type of density probability of spine volumes $H(u)$ (see the main text).

Eqs. (18) and (19) constitute the reduced system of basic equations, which is used for computations of two independent variables $x$ and $\pi$. This two-dimensional system can be solved by a handful of numerical techniques (e.g. [1]).

### 2.2 Proof of the local maximum for optimal solution related to spine economy.

As before, let us introduce the following notation: $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv \pi$. Then the fitness function $F_s$ (Eq. 10) can be rewritten as:

$$F_s = \frac{P x_1 x_2}{x_3^2} + \lambda_2 g(x_1, x_2, x_3), \quad (20)$$

where the probability $P$ is a function of $x_3$, and $g(x_1, x_2, x_3)$ denotes the constraint term present in Eq. (10). Let us define partial derivatives: $F_{ij} = \partial^2 F_s / \partial x_i \partial x_j$ and $g_i = \partial g / \partial x_i$. 

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which are determined at the critical point represented by optimal values of $x_1, x_2, x_3$. Using these definitions we can construct a matrix called bordered Hessian for our constraint optimization problem as [2]:

$$
\begin{bmatrix}
0 & g_1 & g_2 & g_3 \\
g_1 & F_{11} & F_{12} & F_{13} \\
g_2 & F_{21} & F_{22} & F_{23} \\
g_3 & F_{31} & F_{32} & F_{33}
\end{bmatrix}
$$

This particular matrix has a high degree of symmetry, since: $g_1 = g_2$, $F_{ij} = F_{ji}$, and additionally $F_{11} = F_{22}, F_{13} = F_{23}$.

A sufficient condition for $F_*$ to have a local maximum at the critical point represented by the optimal values $x_1, x_2, x_3$ is that two principal minors, i.e. determinants of the upper-left sub-matrices 3x3 (called $D_1$) and 4x4 (determinant of the entire bordered Hessian called $D_2$), alternate in sign. Specifically, the principal minors must have respectively positive ($D_1$) and negative ($D_2$) signs [2]. Using the high symmetry in the Hessian matrix, the explicit forms of these determinants are as follows:

$$
D_1 = 2g_1^2(F_{12} - F_{11})
$$

and

$$
D_2 = g_1^2(F_{12} - F_{11}) \left[ 2F_{33} - 4\epsilon F_{13} + \epsilon^2(F_{12} + F_{11}) \right],
$$

where $\epsilon \equiv g_3/g_1$. It is relatively easy to show that $g_1 \geq 1$, and the expression for $\epsilon$ reads
\[ \epsilon = \frac{x}{P} \left( \frac{\partial P}{\partial u} - \gamma_2 P \right). \]  

(23)

In general for all considered distributions of spine volume, the numerical value of \( \epsilon \) is very small at the critical point, i.e. \( |\epsilon| \ll 1 \). Typical values of \( F_{ij} \) are in the range \((-1.7, 1)\). Thus, approximately the sign of \( D_2 \) is determined by the sign of the product \( F_{33}(F_{12} - F_{11}) \), since other terms in Eq. (22) are much smaller and thus can be neglected. Of these two factors, \( F_{33} \) is always negative (which comes from a numerical calculation) and \( (F_{12} - F_{11}) = -\lambda_2/x \) is always positive (\( \lambda_2 < 0 \)). This implies that \( D_2 \) is negative, and \( D_1 \) is positive, which is sufficient for the benefit-cost function \( F_s \) (Eq. 10) to have maximum. Exact numerical values of the rescaled minors \( D_1/g_1^2 \) and \( D_2/g_1^2 \) are presented in Table B (below) together with the values of \( \epsilon \) and \( F_{ij} \).
Table B: “Spine economical maximization”. The numerical values of the elements of the bordered Hessian and principal minors for each type of the spine volume distribution.

| Spine size distribution | $\theta$ | $\epsilon$ | $F_{11}$ | $F_{12}$ | $F_{13}$ | $F_{33}$ | $D_1/g_1^2$ | $D_2/g_1^2$ |
|-------------------------|----------|------------|----------|----------|----------|----------|-------------|-------------|
| Exponential             | 0.100 (ED) | -0.053 | 0.017 | 0.621 | -0.018 | -0.059 | 1.207 | -0.072 |
|                        | 0.100 (MD) | -0.053 | 0.017 | 0.621 | -0.018 | -0.059 | 1.207 | -0.072 |
|                        | 0.321 (ED) | 0.024 | 0.016 | 0.540 | -0.010 | -0.145 | 1.048 | -0.151 |
|                        | 0.321 (MD) | 0.014 | 0.014 | 0.526 | -0.009 | -0.098 | 1.023 | -0.100 |
| Gamma (n=1)            | 0.100 (ED) | -0.072 | 0.017 | 0.657 | -0.020 | -0.055 | 1.279 | -0.072 |
|                        | 0.100 (MD) | -0.072 | 0.017 | 0.657 | -0.020 | -0.055 | 1.279 | -0.072 |
|                        | 0.321 (ED) | 0.061 | 0.022 | 0.664 | -0.014 | -0.397 | 1.284 | -0.506 |
|                        | 0.321 (MD) | 0.017 | 0.016 | 0.606 | -0.012 | -0.165 | 1.180 | -0.194 |
| Gamma (n=2)            | 0.100 (ED) | -0.062 | 0.013 | 0.645 | -0.014 | -0.024 | 1.263 | -0.031 |
|                        | 0.100 (MD) | -0.062 | 0.013 | 0.645 | -0.014 | -0.024 | 1.263 | -0.031 |
|                        | 0.321 (ED) | 0.085 | 0.025 | 0.739 | -0.016 | -0.653 | 1.427 | -0.924 |
|                        | 0.321 (MD) | 0.033 | 0.020 | 0.669 | -0.015 | -0.315 | 1.298 | -0.408 |
| Rayleigh               | 0.100 (ED) | -0.098 | 0.020 | 0.677 | -0.027 | -0.072 | 1.314 | -0.097 |
|                        | 0.100 (MD) | -0.058 | 0.013 | 0.641 | -0.013 | -0.021 | 1.256 | -0.028 |
|                        | 0.321 (ED) | 0.067 | 0.025 | 0.734 | -0.018 | -0.654 | 1.418 | -0.921 |
|                        | 0.321 (MD) | 0.015 | 0.019 | 0.662 | -0.016 | -0.283 | 1.285 | -0.362 |
| Log-logistic           | 0.100 (ED) | -0.016 | 0.022 | 0.654 | -0.022 | -0.189 | 1.264 | -0.240 |
|                        | 0.100 (MD) | -0.016 | 0.022 | 0.654 | -0.022 | -0.189 | 1.264 | -0.240 |
|                        | 0.321 (ED) | 0.124 | 0.026 | 0.778 | -0.014 | -0.895 | 1.505 | -1.332 |
|                        | 0.321 (MD) | 0.075 | 0.025 | 0.784 | -0.019 | -0.949 | 1.518 | -1.432 |
| Log-normal             | 0.100 (ED) | -0.074 | 0.021 | 0.582 | -0.025 | -0.247 | 1.121 | -0.279 |
|                        | 0.100 (MD) | -0.074 | 0.017 | 0.616 | -0.019 | -0.076 | 1.199 | -0.092 |
|                        | 0.321 (ED) | 0.044 | 0.023 | 0.683 | -0.018 | -1.737 | 1.321 | -2.292 |
|                        | 0.321 (MD) | -0.030 | 0.021 | 0.666 | -0.022 | -0.797 | 1.290 | -1.029 |

For each value of $\theta$ there are two values of Hessian matrix and Minors corresponding to minimal Euclidean (ED) and Mahalanobis (MD) distances.
3 Combined “wire minimization” and “spine economy maximization” principle.

3.1 The system of basic equations for optimal solution.

The optimal values of \( x, y, \overline{\pi}, \) and \( \lambda \) are found by differentiating the meta fitness function \( F \) (Eq. 1 in the main text) with respect to \( x, y, \overline{\pi}, \) and \( \lambda \), and requiring that appropriate derivatives are zero. As a result, we obtain the following set of four nonlinear equations:

\[
\frac{f_{rx}}{\overline{\pi}^{\gamma_1}} - \frac{(1-f)s}{\overline{\pi}^{\gamma_2}} + \lambda \left( x + s + \frac{2}{3}g + \frac{5}{3}c \right) = 0, (24)
\]

\[
\frac{f_{yy}}{\overline{\pi}^{\gamma_1}} - \frac{(1-f)s}{\overline{\pi}^{\gamma_2}} + \lambda \left( y + s + \frac{2}{3}g + \frac{5}{3}c \right) = 0, (25)
\]

\[
\left[ \lambda - \frac{(1-f)}{\overline{\pi}^{\gamma_2}} \right] s + \frac{\lambda g}{3} \left( 2 + 5s \right) \frac{\overline{\pi}}{P} \frac{\partial P}{\partial \overline{\pi}} - \frac{2}{3} \lambda g(1+s) = \frac{\gamma_1 f}{\overline{\pi}^{\gamma_1}} \left( r x + y \right) - \frac{\gamma_2 s}{\overline{\pi}^{\gamma_2}} (1-f) \quad (26)
\]

and
\[ x + y + s + g + c = 1. \quad (27) \]

**Reduction of dimensionality in the system of basic equations.**

As before we can reduce the number of equations from 4 to 3. For this purpose, we determine \( \lambda \) from Eq. (25):

\[
\lambda = \frac{(1 - f) \frac{s}{g} - f \frac{y}{c}}{y + s + \frac{2}{3} g + \frac{5}{3} c} \quad (28)
\]

and next, we insert this equation into Eqs. (24) and (26). As a result, we get Eqs. (26-27) in the main text.

**3.2 Proof of the local maximum for optimal solution related to spine economy.**

The bordered Hessian matrix can be determined similarly as in the previous two cases. A sufficient condition for the meta fitness function \( F \) (Eq. 1 in the main text) to have a local minimum is that two principal minors are negative. Exact numerical values of the minors and elements of the bordered Hessian are displayed in Tables C-E, for different mixing ratio \( f \) in \( F \). These tables correspond to Tables 4-6 in the main text.
Table C: Combined “Wire min + spine max” approach for $f = 0.1$.

| Principle type/ spine distr. | Bordered Hessian | Minors |
|------------------------------|------------------|--------|
|                              | $F_{11}$  | $F_{12}$ | $F_{22}$ | $F_{13}$ | $F_{23}$ | $F_{33}$ | $D_1$ | $D_2$ |
| wire length min + spine max  | Exponential     | -0.010 | -0.694  | -0.010 | 0.141 | 0.141 | 0.163 | -2.870 | -0.455 |
|                              | Gamma (n=1)     | -0.013 | -0.809  | -0.014 | 0.154 | 0.153 | 0.324 | -3.664 | -1.154 |
|                              | Gamma (n=2)     | -0.015 | -0.895  | -0.017 | 0.172 | 0.168 | 0.540 | -4.228 | -2.219 |
|                              | Rayleigh        | -0.015 | -0.898  | -0.018 | 0.177 | 0.171 | 0.535 | -4.289 | -2.252 |
|                              | Log-logistic    | -0.017 | -0.963  | -0.022 | 0.173 | 0.156 | 1.056 | -4.867 | -5.045 |
|                              | Log-normal      | -0.019 | -1.000  | -0.022 | 0.228 | 0.222 | 2.370 | -5.103 | -12.13 |
| wire surface min + spine max | Exponential     | -0.008 | -0.598  | -0.009 | 0.058 | 0.058 | 0.104 | -2.488 | -0.254 |
|                              | Gamma (n=1)     | -0.011 | -0.707  | -0.012 | 0.068 | 0.067 | 0.230 | -3.211 | -0.726 |
|                              | Gamma (n=2)     | -0.013 | -0.764  | -0.015 | 0.073 | 0.070 | 0.368 | -3.626 | -1.316 |
|                              | Rayleigh        | -0.013 | -0.760  | -0.016 | 0.074 | 0.071 | 0.349 | -3.645 | -1.262 |
|                              | Log-logistic    | -0.015 | -0.843  | -0.020 | 0.078 | 0.071 | 0.822 | -4.264 | -3.472 |
|                              | Log-normal      | -0.023 | -0.988  | -0.030 | 0.112 | 0.100 | 6.943 | -4.898 | -33.89 |
| wire volume min + spine max  | Exponential     | -0.008 | -0.545  | -0.008 | 0.005 | 0.005 | 0.090 | -2.267 | -0.204 |
|                              | Gamma (n=1)     | -0.011 | -0.637  | -0.012 | 0.008 | 0.008 | 0.209 | -2.886 | -0.601 |
|                              | Gamma (n=2)     | -0.012 | -0.681  | -0.014 | 0.010 | 0.010 | 0.296 | -3.239 | -0.955 |
|                              | Rayleigh        | -0.012 | -0.674  | -0.014 | 0.010 | 0.011 | 0.271 | -3.240 | -0.878 |
|                              | Log-logistic    | -0.014 | -0.759  | -0.019 | 0.013 | 0.015 | 0.734 | -3.833 | -2.804 |
|                              | Log-normal      | -0.015 | -0.726  | -0.021 | 0.016 | 0.019 | 1.938 | -3.632 | -7.038 |
| delays min + spine max       | Exponential     | -0.011 | -0.751  | -0.011 | 0.195 | 0.195 | 0.202 | -3.102 | -0.603 |
|                              | Gamma (n=1)     | -0.014 | -0.888  | -0.015 | 0.220 | 0.219 | 0.411 | -4.012 | -1.589 |
|                              | Gamma (n=2)     | -0.015 | -0.941  | -0.018 | 0.224 | 0.219 | 0.562 | -4.461 | -2.433 |
|                              | Rayleigh        | -0.018 | -1.004  | -0.020 | 0.258 | 0.253 | 0.712 | -4.778 | -3.313 |
|                              | Log-logistic    | -0.019 | -1.076  | -0.025 | 0.250 | 0.231 | 1.389 | -5.407 | -7.316 |
|                              | Log-normal      | -0.022 | -1.127  | -0.026 | 0.326 | 0.316 | 3.596 | -5.570 | -19.96 |
Table D: Combined “Wire min + spine max” approach for \( f = 0.5 \).

| Principle type/ spine distr. | \( F_{11} \) | \( F_{12} \) | \( F_{22} \) | \( F_{13} \) | \( F_{23} \) | \( F_{33} \) | \( D_1 \) | \( D_2 \) |
|-----------------------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| **wire length min + spine max** |                |                |                |                |                |                |                |                |
| Exponential                 | -0.432         | -16.79         | -0.432         | 9.602          | 9.604          | 858.9          | -43.24         | -35899.2       |
| Gamma (n=1)                 | -0.132         | -6.323         | -0.126         | 4.242          | 4.230          | 85.09          | -19.83         | -1542.5        |
| Gamma (n=2)                 | -0.055         | -4.055         | -0.051         | 2.668          | 2.612          | 24.29          | -15.13         | -320.69        |
| Rayleigh                    | -0.054         | -4.001         | -0.050         | 2.546          | 2.491          | 27.10          | -15.66         | -378.75        |
| Log-logistic                | -0.017         | -2.778         | -0.021         | 1.360          | 1.459          | 11.86          | -13.58         | -146.30        |
| Log-normal                  | 0.008          | -1.679         | 0.010          | 0.871          | 1.000          | 1.299          | -8.369         | -10.948        |
| **wire surface min + spine max** |                |                |                |                |                |                |                |                |
| Exponential                 | -0.017         | -2.345         | -0.017         | 1.280          | 1.280          | 12.93          | -7.794         | -91.96         |
| Gamma (n=1)                 | 0.008          | -1.536         | 0.008          | 0.583          | 0.583          | 1.576          | -6.470         | -8.808         |
| Gamma (n=2)                 | 0.012          | -1.206         | 0.012          | 0.353          | 0.353          | 0.231          | -5.844         | -1.155         |
| Rayleigh                    | 0.010          | -1.320         | 0.011          | 0.402          | 0.433          | 0.645          | -6.353         | -3.725         |
| Log-logistic                | 0.005          | -1.377         | 0.008          | 0.344          | 0.424          | 1.764          | -7.203         | -12.18         |
| Log-normal                  | 0.010          | -1.317         | 0.013          | 0.384          | 0.448          | 3.027          | -6.831         | -20.67         |
| **wire volume min + spine max** |                |                |                |                |                |                |                |                |
| Exponential                 | 0.017          | -0.639         | 0.017          | -0.011         | -0.011         | 0.098          | -2.754         | -0.272         |
| Gamma (n=1)                 | 0.014          | -0.733         | 0.016          | -0.010         | -0.011         | 0.208          | -3.451         | -0.720         |
| Gamma (n=2)                 | 0.013          | -0.786         | 0.016          | -0.011         | -0.012         | 0.308          | -3.877         | -1.196         |
| Rayleigh                    | 0.014          | -0.788         | 0.016          | -0.012         | -0.013         | 0.336          | -3.921         | -1.320         |
| Log-logistic                | 0.011          | -0.872         | 0.014          | -0.009         | -0.011         | 0.837          | -4.561         | -3.817         |
| Log-normal                  | 0.014          | -0.810         | 0.019          | -0.016         | -0.018         | 1.980          | -4.282         | -8.477         |
| **delays min + spine max**  |                |                |                |                |                |                |                |                |
| Exponential                 | -1.173         | -47.71         | -1.174         | 27.45          | 27.48          | 4239.5         | -115.2         | -4.7·10^5      |
| Gamma (n=1)                 | -0.290         | -11.87         | -0.290         | 8.223          | 8.223          | 280.65         | -34.14         | -8.9·10^3      |
| Gamma (n=2)                 | -0.142         | -7.007         | -0.142         | 5.057          | 5.057          | 82.31          | -23.48         | -1.7·10^3      |
| Rayleigh                    | -0.133         | -6.587         | -0.133         | 4.521          | 4.521          | 83.58          | -23.60         | -1.8·10^3      |
| Log-logistic                | -0.017         | -3.069         | -0.022         | 1.785          | 1.994          | 12.03          | -15.63         | -170.03        |
| Log-normal                  | -0.022         | -3.329         | -0.031         | 2.025          | 2.332          | 70.05          | -16.83         | -1156.2        |
Table E: Combined “Wire min + spine max” approach for \( f = 0.9 \).

| Principle type/ spine distr. | \( F_{11} \) | \( F_{12} \) | \( F_{22} \) | \( F_{13} \) | \( F_{23} \) | \( F_{33} \) | \( D_1 \) | \( D_2 \) |
|-----------------------------|------------|------------|------------|------------|------------|------------|-------|-------|
| **wire length min + spine max** |            |            |            |            |            |            |       |       |
| Exponential                 | -2*10^-7  | -0.027     | -2*10^-7  | 3*10^-8    | 3*10^-8    | 7*10^-13   | -0.107| -8*10^-14 |
| Gamma (n=1)                 | -28.19     | -3134.4    | -28.19     | -854.0     | -854.0     | 8*10^5     | -6590.6| -5*10^9  |
| Gamma (n=2)                 | -8.533     | -666.9     | -8.533     | -169.3     | -169.3     | 10^5       | -1440.5| -10^8   |
| Rayleigh                    | -6.136     | -283.8     | -6.136     | -87.28     | -87.28     | 4*10^4     | -660.35| -2*10^7  |
| Log-logistic                | -2*10^-7   | -0.027     | -2*10^-7   | 3*10^-8    | 3*10^-8    | 7*10^-13   | -0.107| -8*10^-14 |
| Log-normal                  | -0.677     | -20.66     | -0.944     | -1.887     | -4.444     | 17389.8    | -100.47| -2*10^6  |
| **wire surface min + spine max** |            |            |            |            |            |            |       |       |
| Exponential                 | -11.59     | -1045.9    | -11.59     | -397.0     | -397.0     | 4*10^5     | -2222.0| -8*10^8  |
| Gamma (n=1)                 | -1.608     | -94.90     | -1.608     | -23.57     | -23.57     | 12275.2    | -211.3 | -2*10^6  |
| Gamma (n=2)                 | -0.606     | -29.01     | -0.606     | -3.631     | -3.631     | 2166.6     | -68.4  | -10^5    |
| Rayleigh                    | -0.528     | -20.09     | -0.528     | -2.217     | -2.217     | 1437.8     | -51.8  | -7*10^4  |
| Log-logistic                | -0.084     | -5.240     | -0.118     | 0.747      | 0.745      | 329.89     | -26.1  | -8*10^3  |
| Log-normal                  | -0.013     | -3.494     | -0.017     | 1.029      | 1.174      | 477.03     | -17.12 | -8*10^3  |
| **wire volume min + spine max** |            |            |            |            |            |            |       |       |
| Exponential                 | 0.043      | -0.725     | 0.043      | -0.028     | -0.028     | 0.079      | -3.220 | -0.257   |
| Gamma (n=1)                 | 0.042      | -0.840     | 0.042      | -0.031     | -0.031     | 0.177      | -4.074 | -0.724   |
| Gamma (n=2)                 | 0.036      | -0.878     | 0.047      | -0.030     | -0.034     | 0.284      | -4.456 | -1.273   |
| Rayleigh                    | 0.038      | -0.879     | 0.049      | -0.034     | -0.038     | 0.325      | -4.501 | -1.470   |
| Log-logistic                | 0.038      | -0.986     | 0.049      | -0.032     | -0.036     | 0.948      | -5.277 | -5.015   |
| Log-normal                  | 0.044      | -0.913     | 0.057      | -0.046     | -0.052     | 2.645      | -4.907 | -12.99   |
| **delays min + spine max**  |            |            |            |            |            |            |       |       |
| Exponential                 | -2*10^-6   | -0.034     | -2*10^-6   | 5*10^-7    | 3*10^-7    | 10^-10     | -0.135 | -2*10^-11|
| Gamma (n=1)                 | -2*10^-6   | -0.034     | -2*10^-6   | 5*10^-7    | 3*10^-7    | 10^-10     | -0.135 | -2*10^-11|
| Gamma (n=2)                 | -20.49     | -1961.9    | -20.49     | -488.7     | -488.7     | 3*10^5     | -4169.7| -2*10^9  |
| Rayleigh                    | -16.33     | -865.41    | -16.33     | -287.2     | -287.2     | 10^5       | -1961.9| -3*10^8  |
| Log-logistic                | -2*10^-6   | -0.034     | -2*10^-6   | 5*10^-7    | 3*10^-7    | 10^-10     | -0.135 | -2*10^-11|
| Log-normal                  | -0.746     | -23.22     | -1.035     | -0.794     | -3.932     | 19583.9    | -112.99| -2*10^6  |
References

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[2] Neudecker H, Magnus JR (1988) Matrix Differential Calculus with Applications in Statistics and Econometrics. New York: Wiley.