The Next-to-Simplest Quantum Field Theories

Suvrat Raju

Harish-Chandra Institute

TIFR Theory Colloquium
6 November 2009

based on arXiv:0910.0930 – Shailesh Lal, S.R.
In the past few years, there has been a revival of interest in S-matrix techniques.

The idea is that if we look at scattering amplitudes in gauge theories or gravity, they show a lot of interesting structure.

They have properties that are not at all manifest from the Lagrangian but appear when you look at on-shell amplitudes (Parke, Taylor 1986, Bern et al. 1990s, Witten 2003, Britto, Cachazo et al. 2005, Arkani-Hamed et al. 2008)

This talk is about these properties and the techniques that have been developed around them.
All this is work in progress but over the past few years, the community has succeeded in putting together a coherent set of S-matrix techniques.

Not only are these techniques teaching us new things about perturbative quantum field theory, they are also useful for calculating amplitudes relevant at the LHC.

These properties are most useful for gravity, very useful for Yang-Mills and not so useful for the scalar $\phi^4$ theory.

**Usefulness**: Gravity > Yang – Mills > Scalars
Outline

PROGRESS IN SCATTERING AMPLITUDES
   Tree Amplitudes
   Loop Amplitudes

SIMPLE QUANTUM FIELD THEORIES

MOTIVATION

PURE $\mathcal{N} = 1, 2$ THEORIES
   New Recursion Relations
   Structure

THEORIES WITH MATTER
   The Next-to-Simplest Quantum Field Theories

SUMMARY
We look at scattering amplitudes of the elementary particles of the theory – gluons for gauge theories, gravitons for gravity etc.

S-matrix elements are distinct from correlation functions. We get them by putting external legs on-shell.

Scattering amplitudes, but not correlation functions of gravitons and gluons have nice properties.
Consider a tree-level gluon scattering amplitude where 2 gluons have negative helicity have positive helicity?

One might suspect that this amplitude is a mess. If we have 100,000 gluons, then this amplitude is related to the 100,000 pt correlation function in YM theory. This is ugly even at tree-level!

Answer is very beautiful and very simple. Called The Parke-Taylor Formula:

\[ |M^{--++\ldots}|^2 = \frac{(p_1 \cdot p_2)^4}{(p_1 \cdot p_2)(p_2 \cdot p_3)\ldots(p_n \cdot p_1)} \]
BCFW RELATIONS

Consider a n-point gluon amplitude.

**Figure: BCFW EXTENSION**

- Extend *any* two momenta on shell

\[ p_4 \rightarrow p_4 + qz; \quad p_n \rightarrow p_n - qz \]

\[ q^2 = q \cdot p_4 = q \cdot p_n = 0 \]

- For each \( p \), one of two gauge boson polarization vectors also grows as \( O(z) \).
LARGE Z BEHAVIOUR

- How do these amplitudes behave at large z?
- Naive guess:
  - Independent of z for scalars
  - grow fast for gauge theories \( O(z^3) \).
  - grow even faster for gravity \( O(z^6) \)
- Correct Answer: For 3 out of 4 possible polarizations:
  - \( M \sim O(1/z) \) for gauge theories
  - \( M \sim O(1/z^2) \) for gravity
This property is more than nice. It is very useful.

The scattering amplitude is a holomorphic function of $z$. If a holomorphic function dies off at infty, we can reconstruct it from its poles.

Poles in the amplitude occur when an internal line goes on shell. Residues are lower pt on-shell amplitudes.

So,

$$M(z) \sim \sum_{\text{partitions}} M_{\text{left}} \frac{1}{P_L(z)} M_{\text{right}}$$

These are the BCFW recursion relations. (Britto et al. ’05) They allow us to reconstruct all tree amplitudes from a knowledge of the 3 pt amplitude!
SCHEMATIC BCFW

Figure: Recursion Relations

\[ p_n - 1 \]
\[ p_n \]
\[ p_2 \]
\[ p_3 \]
\[ p_1 \]
\[ p_1 \]
\[ p_l \]
\[ \frac{1}{p_l^2} \times \]
\[ + \ldots \]
WHAT USE IS THIS?

- These recursion relations are more than a curiosity.
- They are actually used for calculating scattering amplitudes at the LHC. Far superior to standard Feynman diagram techniques. Large Industry around this (Berger, Bern, Dixon, Forde ...).
- For gravity, these recursion relations are even more useful. Perturbative gravity is a mess!. It has an infinite set of interaction vertices and already there are more than a 1000 terms in the 4-pt interaction.
- Here, everything comes from a 3-pt on-shell function that is determined by Lorentz invariance.
WHY DOES THIS WORK?

▶ Why do these techniques work and moreover why are they particularly useful for gauge theories and gravity?

▶ Why are gauge theories and gravity so complicated?

▶ One reason is that to write a manifestly local description of the theory, we need to introduce redundant degrees of freedom.

▶ A gluon/graviton has only 2-physical degrees of freedom. To keep locality manifest, we introduce additional degrees of freedom and then try and project them out.
STAY ON-SHELL

- Perhaps, if we can work directly with the physical degrees of freedom, life might be simpler.
- The BCFW recursion relations do this for the classical theory.
- However, they are not manifestly local
- Can these techniques teach you to move away from locality?
Natural question (precedes grand dreams!): can we generalize on-shell techniques to loop-amplitudes.

- At one-loop, this has been completely worked out.
- At higher loops, we have made some progress but we don’t have a complete answer yet.
ANY one loop amplitude in any quantum field theory can be written as a sum of scalar boxes triangles and bubbles with rational coefficients and a possible rational remainder.

Figure: ONE LOOP DECOMPOSITION

\[
\begin{align*}
\text{Diagram} &= \text{Box} + \text{Triangle} + \text{Bubble} + R_n
\end{align*}
\]

This is surprising. A 1-loop diagram might have 1000 propagators in the denominator. How can we reduce it to something that has at most 4 factors in the denominator.
ANALYTIC PROPERTIES OF ONE LOOP AMPLITUDES

- The reason this works is that the analytic structure of the one loop amplitude is tightly constrained.
  - could have branch cut discontinuities: the 2-cut gives us the discontinuity across the branch cut
  - This discontinuity may itself have a discontinuity: cutting three lines gives us the discontinuity of the discontinuity.
  - In 4 dimensions, not more than 4 lines can go on shell.
- So a box plus triangle plus bubble can reproduce the most general branch cut singularities that can appear at one loop.
- A possible rational remainder is accounted for explicitly
To find the S-matrix at one-loop, we need to find the box, triangle and bubble coefficients.

These coefficients can be found in terms of products of BCFW extended tree amplitudes.

So, the structure at one-loop is intimately related to the structure of tree-amplitudes under BCFW deformations.
ON-SHELL SUPERSYMMETRY

\[\mathcal{N} = 1\] Multiplet

\[\begin{array}{c}
1 \\
\frac{1}{2} \\
-\frac{1}{2} \\
-1
\end{array}\]

\[\mathcal{N} = 2\] Multiplet

\[\begin{array}{cc}
1 \\
\frac{1}{2} & \frac{1}{2} \\
0 & 0 \\
-\frac{1}{2} & -\frac{1}{2} \\
-1
\end{array}\]

\[\mathcal{N} = 4\] Multiplet

\[\begin{array}{c}
1 \\
4 (\frac{1}{2}) \\
6 (0) \\
4 (\frac{-1}{2}) \\
-1
\end{array}\]

Figure: Representations of SUSY algebra
HOW DOES SUSY HELP?

- In the study of scattering amplitudes, supersymmetry helps in an unusual way.
- Maximal susy implies that every scattering amplitude can be related to a scattering amplitude involving at least two gluons!
- In fact, for maximal susy, we can take both these gluons to be negative helicity gluons.
- So, tree amplitudes for $\mathcal{N} = 4$ Super-Yang-Mills and $\mathcal{N} = 8$ Supergravity are the nicest of all! These theories are the Simplest Quantum Field Theories despite having very complicated Lagrangians.
WHAT DOES SIMPLICITY MEAN

Figure: TREE LEVEL

Figure: ONE LOOP
At tree-level, in the $\mathcal{N} = 4$ theory, all amplitudes die off at large $z$ die off under an appropriate BCFW extension.

The $\mathcal{N} = 4$ theory and $\mathcal{N} = 8$ theory have only boxes at one-loop. Called the no-triangle property.

This one-loop property is directly linked to the nice tree-level properties of the theory.

So, the scattering amplitudes of these theories have the simplest analytic structure.
THE POINT OF SIMPLICITY

► The fact that $\mathcal{N} = 4$ and $\mathcal{N} = 8$ have such complicated Lagrangians and such simple scattering amplitudes tells us to look for an alternate formulation of these theories.

► Are there any symmetries that can guide us?

► The $\mathcal{N} = 4$ SYM S-matrix has a remarkable symmetry called dual-superconformal invariance which doesn’t come from an invariance of the Lagrangian at all!

► Is there any structure at higher loops?

► Many concrete calculations have been done for $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA. This leads to the Leading Singularity Conjecture (Arkani-Hamed et. al ’08, ’09) for the all-loop S-matrix.
AN EXAMPLE SHOULD BE GENERALIZABLE

▷ Evidently, $\mathcal{N} = 4$ SYM is the best test-bed to study these S-matrix techniques further. This S-matrix has many wonderful properties, especially when the gauge group becomes large.

▷ However, it would be disappointing if all this program ended up doing is determining planar gluon amplitudes in maximally supersymmetric SYM.

▷ We would like to gain some perspective on perturbative quantum field theories. What we learn from these simple theories should be applicable elsewhere.

▷ This is not a futile hope! The tree-level and one-loop techniques I’ve described above work for non-supersymmetric gauge theories as well. They even work for noncommutative theories (S.R 2009)
So, it makes sense to look for the Next-to-Simplest Quantum Field Theories.

i.e. theories that are in-between the $\mathcal{N} = 4$ theory and $\mathcal{N} = 0$ theory in terms of complexity.

It is natural to look at theories with $\mathcal{N} = 1, 2$ supersymmetry.

It turns out that we can find some next-to-simplest quantum field theories!

These theories share many of the nice properties of the $\mathcal{N} = 4$ theory. So, they open up new vistas in our study of amplitudes.
ON-SHELL SUSY

- Look again at the structure of these multiplets.

\[ \mathcal{N} = 1 \text{ Multiplet} \]

\[ \begin{pmatrix} 1 \\ \frac{1}{2} \\ -\frac{1}{2} \\ -1 \end{pmatrix} \]

\[ \mathcal{N} = 2 \text{ Multiplet} \]

\[ \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \]

\[ \mathcal{N} = 4 \text{ Multiplet} \]

\[ \begin{pmatrix} 1 \\ 4 \left(\frac{1}{2}\right) \\ 6 \left(0\right) \\ 4 \left(-\frac{1}{2}\right) \\ -1 \end{pmatrix} \]

Figure: Representations of SUSY algebra
In $\mathcal{N} = 1, 2$ theories, every scattering amplitude can be related either to one where there are two positive helicity gluons or to one with two negative helicity gluons.

So, tree-level BCFW relations generalize to $\mathcal{N} = 1, 2$ theories.

Structurally, these recursion relations are similar to non-supersymmetric YM.

The two separate multiplets are like the two gluons (of positive and negative helicity) in pure YM.

Just like pure YM (but unlike $\mathcal{N} = 4$ SYM), not all BCFW extensions lead to good behavior.
ONE LOOP: STRUCTURAL SIMILARITY TO PURE YM

- What happens at one-loop? Do pure $\mathcal{N} = 1, 2$ theories see any simplifications?
- Unfortunately, both triangles and bubbles occur in the one-loop S-matrix of $\mathcal{N} = 1, 2$ theories.
- We should have expected this. The presence of bubbles relates to UV-divergences in the theory.
- We know pure $\mathcal{N} = 1, 2$ theories have UV-divergences, so we should expect bubbles.
This leads to an interesting possibility. What if we look at theories that have a vanishing $\beta$ function! We need to add matter for this. Do commonly studied superconformal theories like the Seiberg-Witten theory ($\mathcal{N} = 2$ SU(N) theory with $2N$ hypermultiplets) have simple S-matrices? These theories do see some simplifications, but not as simple as the $\mathcal{N} = 4$ theory. But, by adding different kinds of matter, we can find theories that are even better. Gluon scattering at one-loop is as good as in the $\mathcal{N} = 4$ theory.
Setting

- Consider gauge theories coupled to matter. Let's look at both supersymmetric and non-supersymmetric theories.
- For simplicity, we will focus on gluon amplitudes.
- At tree-level, these are the same in pure YM, YM with matter or $\mathcal{N} = 4$ SYM. So, the question is whether we see structural simplicity at one-loop.
Figure: Are Matter Contributions Very Complicated?
One Loop Possibilities

- At first sight, the contribution of matter seems very complicated. A 100-gluon amplitude can get a contribution from 100 generators.

- On the other hand, the contribution of matter to the $\beta$ function is very simple. Matter gives a universal contribution that is proportional to the quadratic index of the matter-representation.

- Turns out that the truth about matter contributions at one-loop is neither very complicated nor very simple. In between the two in a beautiful way!
INDICES: A DETOUR INTO GROUP THEORY

- Recall, that for any representation

\[ \text{Tr}_R(T^a T^b) = l_2(R) \kappa^{ab} \]

- Similarly, for \( SU(N) \), we can define

\[ \frac{1}{2} \text{Tr}_R \left( T^a \{ T^b, T^c \} \right) = l_3(R) d^{abc} \]

\( l_3 \) is called the anomaly.

- At higher orders also,

\[ \text{Tr}_R \left( T^{(a T^b T^c T^d)} \right) = l_4(R) d^{abcd} + l_{2,2}(R) \kappa^{(ab} \kappa^{cd)} \]

- There are as many independent indices as the rank of the algebra.
MATTER CONTRIBUTIONS AT ONE-LOOP: NON-SUPERSYMMETRIC MATTER

For non-supersymmetric theories

- **Triangle** coefficients depend on the higher order indices up to the **sixth** order index.
- **Bubble** coefficients depend on the higher order indices up to the **fourth** order index.
- The Box-coefficient is sensitive to the entire character.
MATTER CONTRIBUTIONS AT ONE-LOOP: SUPERSYMMETRIC MATTER

For supersymmetric theories

- **Triangle** coefficients depend on the higher order indices up to the fifth order index.
- **Bubble** coefficients depend only on the quadratic index.
- The Box-coefficient is sensitive to the entire character.
MIMICKING THE ADJOINT

- The $\mathcal{N} = 4$ theory can also be thought of as a gauge theory with matter in the adjoint representation.
- What if we find a representation whose indices mimic the first few indices of the adjoint?
- Since triangle and bubble coefficients are sensitive only to these indices, such a theory would have a simple S-matrix as well.
CONDITIONS FOR THE S-MATRIX TO SIMPLIFY

- **Condition (C):** \( \text{Tr}_R(\prod_{i=1}^n T^{a_i}) = m \text{Tr}_{\text{adj}}(\prod_{i=1}^n T^{a_i}), \ n \leq p \)

|                | Non-susy theories have | Only boxes | no bubbles |
|----------------|------------------------|------------|------------|
| if \( R_f \) satisfies C with | p=6, m=4             | p=4, m=4   |            |
| and \( R_s \) satisfies C with | p=6, m=6             | p=4, m=6   |            |
| Susy theories have | only boxes             | no bubbles |            |
| if \( R_\chi \) satisfies C with | p=5, m=3             | p=2, m=3   |            |

**Table:** Conditions for the S-matrix to simplify

- Since, any representation can be reduced into irreducible representations,

\[
R = \bigoplus n_i R_i,
\]

this leads to **Linear Diophantine Equations** in the \( n_i \).
THEORIES WITH ONLY BOXES

1. \( \mathcal{N} = 2, SU(N) \) (for \( N \geq 3 \)) theory with a symmetric tensor hypermultiplet and an antisymmetric tensor hypermultiplet.

2. More exotic example! Theory based on the gauge-group \( G_2 \). Adjoint has dimension 14. The \( \mathcal{N} = 1 \) theory, with a chiral multiplet in the representation

\[
R_\chi = 3 \cdot [7] \oplus [27],
\]

3. Commonly studied superconformal theories like the \( \mathcal{N} = 2, SU(N) \) theory with \( 2N \) hypermultiplets are not simple in this sense.
THEORIES WITHOUT BUBBLES

- Much easier to find theories without bubbles.
- Any supersymmetric theory with vanishing one-loop $\beta$ function is free of bubbles at one-loop.
- Can also find non-supersymmetric examples of theories that do not have bubbles.
- Example: Consider $SU(2)$ theory with 7 complex scalar doublets, a pseudo-real scalar in the representation $4$ and $4$ adjoint fermions.
- Another Example: $SU(N)$ theory with scalar content of a symmetric and anti-symmetric hypermultiplet but $4$ adjoint fermions.
OUR SPECIAL $\mathcal{N} = 2$ THEORY

- The $\mathcal{N} = 4$ theory has many nice properties at large $N$. It has a gravity dual, it shows dual superconformal invariance etc. We hoped that this theory would have these properties too.

- It does! However, this is somewhat trivial. This theory is an orientifold of $\text{AdS}_5 \times S^5$. (Park, Uranga ’98, Ennes et al. 2000) Put another way, at large $N$, it can be obtained from a truncation of the $\mathcal{N} = 4$, $SU(2N)$ theory.

- Such a theory is called a daughter of $\mathcal{N} = 4$ and inherits the large $N$ properties of $\mathcal{N} = 4$.

- I emphasize that our results go beyond this parent-daughter relationship, since they do not rely on large $N$. 

SUMMARY I

- On-shell techniques hold out the promise of a new formulation of perturbation theory in QFTs.
- We have learned many new and surprising things about scattering amplitudes.
- These beautiful structures are most pronounced in the $\mathcal{N} = 4$ SYM theory and $\mathcal{N} = 8$ supergravity. These theories have the simplest scattering amplitudes despite having complicated Lagrangians.
SUMMARY II

► However, these are not exclusive properties of some very special theories.

► Described the generalization of these techniques to theories with less supersymmetry and theories with matter.

► This led us to next-to-simplest Quantum Field Theories. Offer us new test-beds for furthering our study of scattering amplitudes.

► We would like to use all this to understand the reasons for this structure. Holds tremendous promise – sheds new light on QFT, might help us understand gravity and might allow us to move away from locality!