HOLOMORPHIC ANOMALY EQUATIONS FOR
THE FORMAL QUINTIC

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Abstract. We define a formal Gromov-Witten theory of the quintic 3-fold via localization on \( \mathbb{P}^4 \). Our main result is a direct geometric proof of holomorphic anomaly equations for the formal quintic in precisely the same form as predicted by B-model physics for the true Gromov-Witten theory of the quintic 3-fold. The results suggest that the formal quintic and the true quintic theories should be related by transformations which respect the holomorphic anomaly equations. Such a relationship has been recently found by Q. Chen, S. Guo, F. Janda, and Y. Ruan via the geometry of new moduli spaces.

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0. Introduction

0.1. GW/SQ. Let \( X_5 \subset \mathbb{P}^4 \) be a nonsingular quintic Calabi-Yau 3-fold. The moduli space of stable maps to the quintic of genus \( g \) and degree \( d \),

\[
\overline{M}_g(X_5, d) \subset \overline{M}_g(\mathbb{P}^4, d),
\]

has virtual dimension 0. The Gromov-Witten invariants,

\[
N_{g,d}^{GW} = \langle 1 \rangle_{g,d}^{GW} = \int_{[\overline{M}_g(X_5, d)]^{vir}} 1,
\]

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have been studied for more than 20 years, see [12, 13, 19] for an introduction to the subject.

The theory of stable quotients developed in [22] was partially inspired by the question of finding a geometric approach to a higher genus linear sigma model. The moduli space of stable quotients for the quintic,

$$
\overline{Q}_g(X_5, d) \subset \overline{Q}_g(\mathbb{P}^4, d),
$$

was defined in [22, Section 9]. The existence of a natural obstruction theory on $$\overline{Q}_g(X_5, d)$$ and a virtual fundamental class $$[\overline{Q}_g(X_5, d)]_{vir}$$ is easily seen in genus 0 and 1. A proposal in higher genus for the obstruction theory and virtual class was made in [22] and was carried out in significantly greater generality in the setting of quasimaps in [9]. The associated integral theory is defined by

(2) $$N_{g,d}^{SQ} = \langle 1 \rangle_{g,d}^{SQ} = \int_{[\overline{Q}_g(X_5,d)]_{vir}} 1.$$ 

In genus 0 and 1, the invariants (2) were calculated in [11] and [18] respectively. The answers on the stable quotient side exactly match the string theoretic B-model for the quintic in genus 0 and 1.

A relationship in every genus between the Gromov-Witten and stable quotient invariants of the quintic has been proven by Ciocan-Fontanine and Kim [7].

Let $$H \in H^2(X_5, \mathbb{Z})$$ be the hyperplane class of the quintic, and let

$$F_{g,n}^{GW}(Q) = \langle H, \ldots, H \rangle_{g,n}^{GW} = \sum_{d=0}^{\infty} Q^d \int_{[\overline{M}_{g,n}(X_5,d)]_{vir}} \prod_{i=1}^{n} \text{ev}_i^*(H),$$

$$F_{g,n}^{SQ}(q) = \langle H, \ldots, H \rangle_{g,n}^{SQ} = \sum_{d=0}^{\infty} q^d \int_{[\overline{Q}_g(X_5,d)]_{vir}} \prod_{i=1}^{n} \text{ev}_i^*(H)$$

be the Gromov-Witten and stable quotient series respectively (involving the pointed moduli spaces and the evaluation morphisms at the markings). Let

$$I_0(q) = \sum_{d=0}^{\infty} q^d \frac{(5d)!}{(d!)^5}, \quad I_1(q) = \log(q) I_0(q) + 5 \sum_{d=1}^{\infty} q^d \frac{(5d)!}{(d!)^5} \left( \sum_{r=d+1}^{5d} \frac{1}{r} \right).$$

\(^1\)For stability, marked points are required in genus 0 and positive degree is required in genus 1.

\(^2\)A second proof (in most cases) can be found in [10].
The mirror map is defined by
\[ Q(q) = \exp \left( \frac{I_1(q)}{I_0(q)} \right) = q \cdot \exp \left( \frac{5 \sum_{d=1}^{\infty} q^d (5d)! (\sum_{r=d+1}^{\infty} \frac{1}{r})}{\sum_{d=0}^{\infty} q^d (5d)!} \right). \]

The relationship between the Gromov-Witten and stable quotient invariants of the quintic in case
\[ 2g - 2 + n > 0 \]
is given by the following result [7]:
\[ \mathcal{F}^{GW}_{g,n}(Q(q)) = I_0(q)^{2g-2+n} \cdot \mathcal{F}^{SQ}_{g,n}(q). \]

The transformation (3) shows the stable quotient theory matches the string theoretic B-model series for the quintic \( X_5 \).

0.2. **Formal quintic invariants.** The (conjectural) holomorphic anomaly equation is a beautiful property of the string theoretic B-model series which has been used effectively since [4]. Since the stable quotients invariants provide a geometric proposal for the B-model series, we should look for the geometry of the holomorphic anomaly equation in the moduli space of stable quotients.

A particular twisted theory on \( \mathbb{P}^4 \) is related to the quintic 3-fold. Let the algebraic torus
\[ T = (\mathbb{C}^*)^5 \]
act with the standard linearization on \( \mathbb{P}^4 \) with weights \( \lambda_0, \ldots, \lambda_4 \) on the vector space \( H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)) \). Let
\[ C \rightarrow \overline{M}_g(\mathbb{P}^4, d), \quad f : C \rightarrow \mathbb{P}^4, \quad S = f^*\mathcal{O}_{\mathbb{P}^4}(-1) \rightarrow C \]
be the universal curve, the universal map, and the universal bundle over the moduli space of stable maps — all equipped with canonical \( T \)-actions. We define the formal quintic invariants following [21] by
\[ \mathcal{N}_{g,d}^{GW} = \int_{[\overline{M}_g(\mathbb{P}^4, d)]^{vir}} e(R\pi_*(S^{-5}))^{vir}, \]
where \( e(R\pi_*(S^{-5})) \) is the equivariant Euler class defined after localization. More precisely, on each \( T \)-fixed locus of \( \overline{M}_g(\mathbb{P}^4, d) \), both
\[ R^0\pi_*(S^{-5}) \quad \text{and} \quad R^1\pi_*(S^{-5}) \]
are vector bundles with moving weights, so
\[ e(R\pi_*(S^{-5})) = \frac{c_{\text{top}}(R^0\pi_*(S^{-5}))}{c_{\text{top}}(R^1\pi_*(S^{-5}))} \]

\(^3\)The negative exponent denotes the dual: \( S \) is a line bundle and \( S^{-5} = (S^*)^{\otimes 5} \).
is well-defined. The integral (5) is homogeneous of degree 0 in localized equivariant cohomology,
\[
\int_{[\overline{M}_g(P^4, d)]^{vir}} e(R\pi_*(S^{-5})) \in \mathbb{Q}(\lambda_0, \ldots, \lambda_4),
\]
and defines a rational number \( \tilde{N}_{g,d}^{GW} \in \mathbb{Q} \) after the specialization
\[
\lambda_i = \zeta^i
\]
for a primitive fifth root of unity \( \zeta^5 = 1 \).

Our main result here is that the holomorphic anomaly equations conjectured for the true quintic theory (1) are satisfied by the formal quintic theory (5). In particular, the formal quintic theory and the true quintic theory should be related by transformations which respect the holomorphic anomaly equations. In a recent breakthrough by Q. Chen, S. Guo, F. Janda, and Y. Ruan, precisely such a transformation was found via the virtual geometry of new moduli spaces intertwining the formal and true theories of the quintic.

0.3. Holomorphic anomaly for the formal quintic. We state here the precise form of the holomorphic anomaly equations for the formal quintic.

Let \( H \in H^2(\mathbb{P}^4, \mathbb{Z}) \) be the hyperplane class on \( \mathbb{P}^4 \), and let
\[
\tilde{F}_g^{GW}(Q) = \sum_{d=0}^{\infty} Q^d \int_{[\overline{M}_g(P^4, d)]^{vir}} e(R\pi_*(S^{-5})) ,
\]
\[
\tilde{F}_g^{SQ}(Q) = \sum_{d=0}^{\infty} Q^d \int_{[\overline{M}_g(P^4, d)]^{vir}} e(R\pi_*(S^{-5}))
\]
be the formal Gromov-Witten and formal stable quotient series respectively (involving the evaluation morphisms at the markings). The relationship between the formal Gromov-Witten and formal stable quotient invariants of quintic in case of \( 2g - 2 + n > 0 \) follows from (8):
\[
\tilde{F}_g^{GW}(Q(q)) = I_0(q)^{2g-2} \cdot \tilde{F}_g^{SQ}(q)
\]
with respect to the true quintic mirror map
\[
Q(q) = \exp \left( \frac{I_1(q)}{I_0(q)} \right) = q \cdot \exp \left( 5 \sum_{d=1}^{\infty} \frac{q^d (5d)!}{(5d)^d} \frac{1}{r^{d+1}} \right).
\]

In order to state the holomorphic anomaly equations, we require several series in \( q \). First, let
\[
L(q) = (1 - 5^5 q)^{-\frac{1}{5}} = 1 + 625q + 117185q^2 + \ldots.
\]
Let $D = q \frac{d}{dq}$, and let

$$C_0(q) = I_0, \quad C_1(q) = D \left( \frac{I_1}{I_0} \right),$$

where $I_0$ and $I_1$ are the hypergeometric series appearing in the mirror map for the true quintic theory. We define

$$K_2(q) = -\frac{1}{L^5} \frac{DC_0}{C_0},$$

$$A_2(q) = \frac{1}{L^5} \left( -\frac{1}{5} \frac{DC_1}{C_1} - \frac{2}{5} \frac{DC_0}{C_0} - \frac{3}{25} \right),$$

$$A_4(q) = \frac{1}{L^{10}} \left( -\frac{1}{25} \left( \frac{DC_0}{C_0} \right)^2 - \frac{1}{25} \left( \frac{DC_0}{C_0} \right) \left( \frac{DC_1}{C_1} \right) + \frac{1}{25} D \left( \frac{DC_0}{C_0} \right) + \frac{2}{25^2} \right),$$

$$A_6(q) = \frac{1}{31250 L^{15}} \left( 4 + 125 D \left( \frac{DC_0}{C_0} \right) + 50 \left( \frac{DC_0}{C_0} \right) \left( 1 + 10 D \left( \frac{DC_0}{C_0} \right) \right) - 5L^5 \left( 1 + 10 \left( \frac{DC_0}{C_0} \right) + 25 \left( \frac{DC_0}{C_0} \right)^2 + 25D \left( \frac{q dq}{C_0} \right) \right) + 125D^2 \left( \frac{DC_0}{C_0} \right) - 125 \left( \frac{DC_0}{C_0} \right)^2 \left( \left( \frac{DC_1}{C_1} \right) - 1 \right) \right).$$

Let $T$ be the standard coordinate mirror to $t = \log(q)$,

$$T = \frac{I_1(q)}{I_0(q)}.$$

Then $Q(q) = \exp(T)$ is the mirror map.

Define a new series

$$\tilde{F}_g^B = i_0^{2g-2} \cdot \tilde{F}_g^{SQ}$$

motivated by (6). The superscript $B$ here is for the $B$-model. Let

$$\mathbb{C}[L^\pm][A_2, A_4, A_6, C_0^\pm, C_1^\pm, K_2]$$

be the free polynomial ring over $\mathbb{C}[L^\pm]$.

**Theorem 1.** For the series $\tilde{F}_g^B$ associated to the formal quintic,

(i) $\tilde{F}_g^B(q) \in \mathbb{C}[L^\pm][A_2, A_4, A_6, C_0^\pm, C_1^\pm, K_2]$ for $g \geq 2$,

(ii) $\frac{\partial^k \tilde{F}_g^B(q)}{\partial T^k} \in \mathbb{C}[L^\pm][A_2, A_4, A_6, C_0^\pm, C_1^\pm, K_2]$ for $g \geq 1, k \geq 1$. 

(iii) \( \frac{\partial^k \tilde{F}_B}{\partial T^k} \) is homogeneous with respect to \( C_1^{-1} \) of degree \( k \).  

We follow here the canonical lift convention of [21, Section 0.4]. When we write \( \tilde{F}_B(q) \in \mathbb{C}[L^\pm][A_2, A_4, A_6, C_0^\pm, C_1^{-1}, K_2] \), we mean that the series \( \tilde{F}_B(q) \) has a canonical lift to the free algebra. The question of uniqueness of the lift has to do with the algebraic independence of the series

\[ A_2(q), A_4(q), A_6(q), C_0^+(q), C_1^{-1}(q), K_2(q) \]

which we do not address nor require.

**Theorem 2.** The holomorphic anomaly equations for the series \( \tilde{F}_B \) associated to the formal quintic hold for \( g \geq 2 \):

\[
\frac{1}{C_0^2 C_1^2} \frac{\partial \tilde{F}_B}{\partial A_2} - \frac{1}{5C_0^2 C_1^2} \frac{\partial \tilde{F}_B}{\partial A_4} K_2^2 + \frac{1}{50C_0^2 C_1^2} \frac{\partial \tilde{F}_B}{\partial A_6} K_2^2 = \frac{1}{2} \sum_{i=1}^{g-1} \frac{\partial \tilde{F}_B}{\partial T} \frac{\partial \tilde{F}_B}{\partial T} + \frac{1}{2} \frac{\partial^2 \tilde{F}_B}{\partial T^2},
\]

\[
\frac{\partial \tilde{F}_B}{\partial K_2} = 0.
\]

The equality of Theorem 2 holds in the ring

\[
\mathbb{C}[L^\pm][A_2, A_4, A_6, C_0^\pm, C_1^{-1}, K_2].
\]

Theorem 2 exactly matches[4] the conjectural holomorphic anomaly equations [11 (2.52)] for the true quintic theory \( I_0^{2g-2} \cdot F_{SQ}^5 \).

The first holomorphic anomaly equation of Theorem 2 was announced in our paper [21] in February 2017 where a parallel study of the toric Calabi-Yau \( K\mathbb{P}^2 \) was developed. In January 2018 at the Workshop on higher genus at ETH Zürich, Shuai Guo of Peking University informed us that our same argument also yields the second holomorphic anomaly equation

\[
\frac{\partial \tilde{F}_B}{\partial K_2} = 0.
\]

In fact, we had incorrectly thought (7) would fail in the formal theory and would not have included (7) without communication with Guo, so Guo should be credited with the first proof of (7).

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[4]Our functions \( K_2 \) and \( A_{2k} \) are normalized differently with respect to \( C_0 \) and \( C_1 \). The dictionary to exactly match the notation of [11 (2.52)] is to multiply our \( K_2 \) by \((C_0 C_1)^2\) and our \( A_{2k} \) by \((C_0 C_1)^{2k}\).
0.4. **Constants.** Theorem 2 and a few observations determine \( \tilde{F}_g^B \) from the lower genus data 
\[
\left\{ \begin{array}{c} h < g \\
\tilde{F}_h^B 
\end{array} \right\}
\]
and finitely many constants of integration. The additional observations required are:

(i) The proof of part (i) of Theorem 1 shows that \( \tilde{F}_g^B \in \mathbb{C}[L^{\pm 1}][A_2, A_4, A_6, C_0^{\pm 1}, C_1^{-1}, K_2] \) does not depend on \( C_1^{-1} \). Hence, every term (both on the left and right) in the first holomorphic anomaly equation of Theorem 2 is of degree 2 in \( C_1^{-1} \). After multiplying by \( C_1^2 \), no \( C_1 \) dependence remains.

(ii) The proof of Theorem 1 shows that all terms in the first equation are homogeneous of degree \( 2g - 4 \) with respect to \( C_0 \). After dividing by \( C_0^{2g-4} \), no \( C_0 \) dependence remains.

Therefore, the first holomorphic anomaly equation of Theorem 2 may be viewed as holding in \( \mathbb{C}[L^{\pm 1}][A_2, A_4, A_6, K_2] \).

Since the second holomorphic anomaly equation (7) implies \( \tilde{F}_g^B \) has no \( K_2 \) dependence, the first holomorphic anomaly equation determines the each of the three derivatives
\[
\frac{\partial \tilde{F}_g^B}{\partial A_2}, \quad \frac{\partial \tilde{F}_g^B}{\partial A_4}, \quad \frac{\partial \tilde{F}_g^B}{\partial A_6}.
\]
Hence, Theorem 2 determines
\[
C_0^{2-2g} \cdot \tilde{F}_g^B
\]
uniquely as a polynomial in \( A_2, A_4, A_6 \) up to a constant term in \( \mathbb{C}[L^{\pm 1}] \).

In fact, the degree of the constant term can be bounded (as will be seen in Section 6.5). So Theorem 2 determines \( \tilde{F}_g^B \) from lower genus data together with finitely many constants of integration.

The constants of integration for the formal quintic can be effectively computed via the localization formula, but whether there exists a closed formula determining the constants is an interesting open question.

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\[^5\text{See Remark 27.}\]
Janda, S. Guo, and Y. Ruan as presented at the Workshop on higher genus is crucial for the wider study of the formal (and true) quintic. We are very grateful to them for sharing their ideas with us.

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1. Localization graphs

1.1. Torus action. Let $T = (\mathbb{C}^*)^{m+1}$ act diagonally on the vector space $\mathbb{C}^{m+1}$ with weights $-\lambda_0, \ldots, -\lambda_m$.

Denote the $T$-fixed points of the induced $T$-action on $\mathbb{P}^m$ by $p_0, \ldots, p_m$.

The weights of $T$ on the tangent space $T_{p_j}(\mathbb{P}^m)$ are $\lambda_j - \lambda_0, \ldots, \lambda_j - \lambda_j, \ldots, \lambda_j - \lambda_m$.

There is an induced $T$-action on the moduli space $\overline{Q}_{g,n}(\mathbb{P}^m, d)$. The localization formula of [17] applied to the virtual fundamental class $[\overline{Q}_{g,n}(\mathbb{P}^m, d)]^{vir}$ will play a fundamental role in our paper. The $T$-fixed loci are represented in terms of dual graphs, and the contributions of the $T$-fixed loci are given by tautological classes. The formulas here are standard. We precisely follow the notation of [21, Section 2].

1.2. Graphs. Let the genus $g$ and the number of markings $n$ for the moduli space be in the stable range

$2g - 2 + n > 0$.

We can organize the $T$-fixed loci of $\overline{Q}_{g,n}(\mathbb{P}^m, d)$ according to decorated graphs. A decorated graph $\Gamma \in \mathcal{G}_{g,n}(\mathbb{P}^m)$ consists of the data $(V, E, N, g, p)$ where

(i) $V$ is the vertex set,

(ii) $E$ is the edge set (including possible self-edges),

(iii) $N : \{1, 2, \ldots, n\} \rightarrow V$ is the marking assignment,

(iv) $g : V \rightarrow \mathbb{Z}_{\geq 0}$ is a genus assignment satisfying $g = \sum_{v \in V} g(v) + h^1(\Gamma)$

and for which $(V, E, N, g)$ is stable graph\(^6\).

\(^6\)Corresponding to a stratum of the moduli space of stable curves $\overline{M}_{g,n}$. 
(v) \( \mathfrak{p} : \mathcal{V} \to (\mathbb{P}^m)^T \) is an assignment of a \( \mathfrak{T} \)-fixed point \( \mathfrak{p}(v) \) to each vertex \( v \in \mathcal{V} \).

The markings \( \mathfrak{L} = \{1, \ldots, n\} \) are often called legs.

To each decorated graph \( \Gamma \in \mathcal{G}_{g,n}(\mathbb{P}^m) \), we associate the set of fixed loci of

\[
\sum_{d \geq 0} \left[ \overline{Q}_{g,n}(\mathbb{P}^m, d) \right]^{\text{vir}} q^d
\]

with elements described as follows:

(a) If \( \{v_{i_1}, \ldots, v_{i_k}\} = \{v \mid \mathfrak{p}(v) = p_i\} \), then \( f^{-1}(p_i) \) is a disjoint union of connected stable curves of genera \( \mathfrak{g}(v_{i_1}), \ldots, \mathfrak{g}(v_{i_k}) \) and finitely many points.

(b) There is a bijective correspondence between the connected components of \( \mathcal{C} \setminus \mathcal{D} \) and the set of edges and legs of \( \Gamma \) respecting vertex incidence where \( \mathcal{C} \) is domain curve and \( \mathcal{D} \) is union of all subcurves of \( \mathcal{C} \) which appear in (a).

We write the localization formula as

\[
\sum_{d \geq 1} \left[ \overline{Q}_{g,n}(\mathbb{P}^m, d) \right]^{\text{vir}} q^d = \sum_{\Gamma \in \mathcal{G}_{g,n}(\mathbb{P}^m)} \text{Cont}_\Gamma .
\]

While \( \mathcal{G}_{g,n}(\mathbb{P}^m) \) is a finite set, each contribution \( \text{Cont}_\Gamma \) is a series in \( q \) obtained from an infinite sum over all edge possibilities (b).

1.3. Unstable graphs. The moduli spaces of stable quotients

\[
\overline{Q}_{0,2}(\mathbb{P}^m, d) \quad \text{and} \quad \overline{Q}_{1,0}(\mathbb{P}^m, d)
\]

for \( d > 0 \) are the only cases where the pair \( (g, n) \) does not satisfy the Deligne-Mumford stability condition (8).

An appropriate set of decorated graphs \( \mathcal{G}_{0,2}(\mathbb{P}^m) \) is easily defined: The graphs \( \Gamma \in \mathcal{G}_{0,2}(\mathbb{P}^m) \) all have 2 vertices connected by a single edge. Each vertex carries a marking. All of the conditions (i)-(v) of Section 1.2 are satisfied except for the stability of \( (\mathcal{V}, \mathcal{E}, \mathcal{N}, \gamma) \). The localization formula holds,

\[
\sum_{d \geq 1} \left[ \overline{Q}_{0,2}(\mathbb{P}^m, d) \right]^{\text{vir}} q^d = \sum_{\Gamma \in \mathcal{G}_{0,2}(\mathbb{P}^m)} \text{Cont}_\Gamma ,
\]

For \( \overline{Q}_{1,0}(\mathbb{P}^m, d) \), the matter is more problematic — usually a marking is introduced to break the symmetry.

\[\text{The moduli spaces } \overline{Q}_{0,0}(\mathbb{P}^m, d) \text{ and } \overline{Q}_{0,1}(\mathbb{P}^m, d) \text{ are empty by the definition of a stable quotient.}\]
2. Basic correlators

2.1. Overview. We review here basic generating series in $q$ which arise in the genus 0 theory of quasimap invariants. The series will play a fundamental role in the calculations of Sections 3 - 6 related to the holomorphic anomaly equation for formal quintic invariants.

We fix a torus action $T = (\mathbb{C}^*)^5$ on $\mathbb{P}^4$ with weights 
\[-\lambda_0, -\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4\]
on the vector space $\mathbb{C}^5$. The $T$-weight on the fiber over $p_i$ of the canonical bundle
\[(10)\quad \mathcal{O}_{\mathbb{P}^4}(5) \to \mathbb{P}^4\]
is $5\lambda_i$.

For our formal quintic theory, we will use the specialization
\[(11)\quad \lambda_i = \zeta^i\]
where $\zeta$ is the primitive fifth root of unity. Of course, we then have
\[
\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0, \\
\sum_{i \neq j} \lambda_i \lambda_j = 0, \\
\sum_{i \neq j \neq k} \lambda_i \lambda_j \lambda_k = 0, \\
\sum_{i \neq j \neq k \neq l} \lambda_i \lambda_j \lambda_k \lambda_l = 0.
\]

2.2. First correlators. We will require several correlators defined via the Euler class\[9\]
\[(12)\quad e(\text{Obs}) = e(R\pi_*(-S^5)),\]
associated to the formal quintic geometry on the moduli space $\overline{Q}_{g,n}(\mathbb{P}^4, d)$. The first two are obtained from standard stable quotient invariants. For $\gamma_i \in H^*_T(\mathbb{P}^4)$, let
\[
\langle \gamma_1 \psi^{a_1}, ..., \gamma_n \psi^{a_n} \rangle_{g,n,d}^{\text{SQ}} = \int_{[\overline{Q}_{g,n}(\mathbb{P}^4,d)]_{\text{vir}}} e(\text{Obs}) \cdot \prod_{i=1}^{n} \text{ev}_i^*(\gamma_i) \psi_i^{a_i},
\]
\[
\langle \langle \gamma_1 \psi^{a_1}, ..., \gamma_n \psi^{a_n} \rangle \rangle_{0,n}^{\text{SQ}} = \sum_{d \geq 0} \sum_{k \geq 0} \frac{q^d}{k!} \langle \gamma_1 \psi^{a_1}, ..., \gamma_n \psi^{a_n}, t, ..., t \rangle_{0,n+k,d}^{\text{SQ}}.
\]

\[8\]The associated weights on $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ are $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ and so match the conventions of Section 0.2.

\[9\]Equation (12) is the definition of $e(\text{Obs})$. The right side of (12) is defined after localization as explained in Section 0.2.
where, in the second series, \( t \in H^*_T(\mathbb{P}^4) \). We will systematically use the quasimap notation 0+ for stable quotients,

\[
\left\langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n} \right\rangle_{g,n,d}^{0+} = \left\langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n} \right\rangle_{g,n,d}^{SQ}
\]

\[
\left\langle \left\langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n} \right\rangle \right\rangle_{0,n}^{0+} = \left\langle \left\langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n} \right\rangle \right\rangle_{0,n}^{SQ}.
\]

2.3. Light markings. Moduli of quasimaps can be considered with \( n \) ordinary (weight 1) markings and \( k \) light (weight \( \epsilon \)) markings \[^{10}\] 

\[
Q_{g,n,k}^{0+,0+} (\mathbb{P}^4, d).
\]

Let \( \gamma_i \in H^*_T(\mathbb{P}^4) \) be equivariant cohomology classes, and let \( \delta_j \in H^*_T([\mathbb{C}^5/\mathbb{C}^*]) \) be classes on the stack quotient. Following the notation of \[^{10}\], we define series for the formal quintic geometry,

\[
\left\langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n}; \delta_1, \ldots, \delta_k \right\rangle_{g,n,k,d}^{0+,0+} = \int_{Q_{g,n,k}^{0+,0+} (\mathbb{P}^4, d)\text{vir}} e(Obs) \cdot \prod_{i=1}^n ev^*_i(\gamma_i) \psi^{a_i} \cdot \prod_{j=1}^k \hat{ev}^*_j(\delta_j),
\]

\[
\left\langle \left\langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n} \right\rangle \right\rangle_{0,n}^{0+,0+} = \sum_{d \geq 0} \sum_{k \geq 0} \frac{q^d}{k!} \left\langle \left\langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n}; t, \ldots, t \right\rangle \right\rangle_{0,n|k,d}^{0+,0+},
\]

where, in the second series, \( t \in H^*_T([\mathbb{C}^5/\mathbb{C}^*]) \).

For each \( T \)-fixed point \( p_i \in \mathbb{P}^4 \), let

\[
e_i = \frac{e(T_{p_i}(\mathbb{P}^4))}{5\lambda_i}
\]

be the equivariant Euler class of the tangent space of \( \mathbb{P}^4 \) at \( p_i \) with twist by \( O_{\mathbb{P}^4}(5) \). Let

\[
\phi_i = \frac{\prod_{j \neq i} (H - \lambda_j)}{5\lambda_i e_i}, \quad \phi^i = e_i \phi_i \in H^*_T(\mathbb{P}^4)
\]

\[^{10}\]See Sections 2 and 5 of \[^{6}\].
be cycle classes. Crucial for us are the series
\[ S_i(\gamma) = e_i \left\langle \left\langle \frac{\phi_i}{z - \psi}, \gamma \right\rangle \right\rangle_{0,2}^{0+,0+}, \]
\[ V_{ij} = \left\langle \left\langle \frac{\phi_i}{x - \psi} - \frac{\phi_j}{y - \psi} \right\rangle \right\rangle_{0,2}^{0+,0+}. \]

Unstable degree 0 terms are included by hand in the above formulas. For \( S_i(\gamma) \), the unstable degree 0 term is \( \gamma |_{p_i} \). For \( V_{ij} \), the unstable degree 0 term is \( \delta_{ij} e_i(x+y) \).

We also write
\[ S(\gamma) = \sum_{i=0}^{2} \phi_i S_i(\gamma). \]

The series \( S_i \) and \( V_{ij} \) satisfy the basic relation
\[ e_i V_{ij}(x,y)e_j = \sum_{k=0}^{2} S_i(\phi_k)|_{z=x} S_j(\phi^k)|_{z=y} \]
\[ \frac{1}{x+y} \]
proven\(^{11}\) in [8].

Associated to each \( T \)-fixed point \( p_i \in \mathbb{P}^4 \), there is a special \( T \)-fixed point locus,
\[ Q_{0,0}^{0+,0+}(\mathbb{P}^4,d)^{T,p_i} \subset Q_{0,k}^{0+,0+}(\mathbb{P}^4,d), \]
where all markings lie on a single connected genus 0 domain component contracted to \( p_i \). Let \( \text{Nor} \) denote the equivariant normal bundle of \( Q_{0,n|k}^{0+,0+}(\mathbb{P}^4,d)^{T,p_i} \) with respect to the embedding (14). Define
\[ \left\langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n}; \delta_1, \ldots, \delta_k \right\rangle_{0,n|k,d}^{0+,0+p_i} = \]
\[ \int_{Q_{0,n|k}^{0+,0+}(\mathbb{P}^4,d)^{T,p_i}} e(\text{Obs}) \cdot e(\text{Nor}) \cdot \prod_{i=1}^{n} \text{ev}^*\left(\gamma_i\right) \psi^{a_i} \cdot \prod_{j=1}^{k} \hat{\text{ev}}^*\left(\delta_j\right), \]
\[ \left\langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n} \right\rangle_{0,n}^{0+,0+p_i} = \]
\[ \sum_{d \geq 0} \sum_{k \geq 0} \frac{q^d}{k!} \left\langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n}; t, \ldots, t \right\rangle_{0,n|k,\beta}^{0+,0+p_i}. \]

2.4. Graph spaces and I-functions.

\(^{11}\)In Gromov-Witten theory, a parallel relation is obtained immediately from the WDDV equation and the string equation. Since the map forgetting a point is not always well-defined for quasimaps, a different argument is needed here [8].
2.4.1. \textit{Graph spaces}. The big I-function is defined in \cite{6} via the geometry of weighted quasimap graph spaces. We briefly summarize the constructions of \cite{6} in the special case of \((0+,0+)-\)stability. The more general weightings discussed in \cite{6} will not be needed here.

As in Section 2.3, we consider the quotient

\[ C^5 / \mathbb{C}^* \]

associated to \( \mathbb{P}^4 \). Following \cite{6}, there is a \((0+,0+)-\)stable quasimap graph space

\[ QG_{g,n|k,d}^{0+,0+}([C^5 / \mathbb{C}^*]) . \]

A \( \mathbb{C} \)-point of the graph space is described by data

\[ ((C, x, y), (f, \varphi) : C \longrightarrow [C^5 / \mathbb{C}^*] \times [C^2 / \mathbb{C}^*]). \]

By the definition of stability, \( \varphi \) is a regular map to \( \mathbb{P}^1 = \mathbb{C}^2 / \mathbb{C}^* \) of class 1. Hence, the domain curve \( C \) has a distinguished irreducible component \( C_0 \) canonically isomorphic to \( \mathbb{P}^1 \) via \( \varphi \). The \textit{standard} \( \mathbb{C}^* \)-action,

\[ t \cdot [\xi_0, \xi_1] = [t \xi_0, \xi_1], \text{ for } t \in \mathbb{C}^*, [\xi_0, \xi_1] \in \mathbb{P}^1, \]

induces a \( \mathbb{C}^* \)-action on the graph space.

The \( \mathbb{C}^* \)-equivariant cohomology of a point is a free algebra with generator \( z \),

\[ H^*_{\mathbb{C}^*} (\text{Spec}(\mathbb{C})) = \mathbb{Q}[z]. \]

Our convention is to define \( z \) as the \( \mathbb{C}^* \)-equivariant first Chern class of the tangent line \( T_0 \mathbb{P}^1 \) at \( 0 \in \mathbb{P}^1 \) with respect to the action \( \text{(16)} \),

\[ z = c_1(T_0 \mathbb{P}^1). \]

The \( T \)-action on \( C^5 \) lifts to a \( T \)-action on the graph space \( \text{(15)} \) which commutes with the \( \mathbb{C}^* \)-action obtained from the distinguished domain component. As a result, we have a \( T \times \mathbb{C}^* \)-action on the graph space and \( T \times \mathbb{C}^* \)-equivariant evaluation morphisms

\[ \text{ev}_i : QG_{g,n|k,d}^{0+,0+}([C^5 / \mathbb{C}^*]) \rightarrow \mathbb{P}^4, \quad i = 1, \ldots, n, \]

\[ \tilde{\text{ev}}_j : QG_{g,n|k,d}^{0+,0+}([C^5 / \mathbb{C}^*]) \rightarrow [C^5 / \mathbb{C}^*], \quad j = 1, \ldots, k. \]

Since a morphism

\[ f : C \rightarrow [C^5 / \mathbb{C}^*] \]

is equivalent to the data of a principal \( G \)-bundle \( P \) on \( C \) and a section \( u \) of \( P \times_{\mathbb{C}^*} C^5 \), there is a natural morphism

\[ C \rightarrow E \mathbb{C}^* \times_{\mathbb{C}^*} C^5. \]
and hence a pull-back map
\[ f^*: H^*_C([C^5/C^*]) \to H^*(C). \]
The above construction applied to the universal curve over the moduli space and the universal morphism to \([C^5/C^*]\) is \(T\)-equivariant. Hence, we obtain a pull-back map
\[ \hat{\nu}_j^*: H^*_T(C^5, Q) \otimes Q[z] \to H^*_T \times_{C^*} (QG_{g,n,k,\beta}^0([C^5/C^*]), Q) \]
associated to the evaluation map \(\hat{\nu}_j\).

### 2.4.2. \(I\)-functions.

The description of the fixed loci for the \(C^*\)-action on
\[ \big(QG_{g,0,0,d}^0([C^5/C^*]) \big) \]
is parallel to the description in \([5, \S 4.1]\) for the unweighted case. In particular, there is a distinguished subset \(M_{k,d}\) of the \(C^*\)-fixed locus for which all the markings and the entire curve class \(d\) lie over \(0 \in \mathbb{P}^1\). The locus \(M_{k,d}\) comes with a natural proper evaluation map \(ev_\bullet\) obtained from the generic point of \(\mathbb{P}^1\):
\[ ev_\bullet: M_{k,d} \to C^5/C^* = \mathbb{P}^4. \]

We can explicitly write
\[ M_{k,d} \cong M_d \times 0^k \subset M_d \times (\mathbb{P}^1)^k, \]
where \(M_d\) is the \(C^*\)-fixed locus in \(QG_{0,0,d}^0([C^5/C^*])\) for which the class \(d\) is concentrated over \(0 \in \mathbb{P}^1\). The locus \(M_{k,d}\) parameterizes quasimaps of class \(d\),
\[ f: \mathbb{P}^1 \longrightarrow [C^5/C^*], \]
with a base-point of length \(d\) at \(0 \in \mathbb{P}^1\). The restriction of \(f\) to \(\mathbb{P}^1 \setminus \{0\}\) is a constant map to \(\mathbb{P}^4\) defining the evaluation map \(ev_\bullet\).

As in \([4, 5, 9]\), we define the big \(I\)-function as the generating function for the push-forward via \(ev_\bullet\) of localization residue contributions of \(M_{k,d}\). For \(t \in H^*_T([C^5/C^*], Q) \otimes Q[z]\), let
\[
\text{Res}_{M_{k,d}}(t^k) = \prod_{j=1}^k \hat{\nu}_j^*(t) \cap \text{Res}_{M_{k,d}}(QG_{g,0,k,d}^0([C^5/C^*]))^{\text{vir}}
= \prod_{j=1}^k \hat{\nu}_j^*(t) \cap [M_{k,d}]^{\text{vir}}_{\text{vir}}
eq, 
\]
where \(\text{Nor}_{M_{k,d}}^{\text{vir}}\) is the virtual normal bundle.
Definition 3. The big I-function for the \((0^+,0^+)-\)stability condition, as a formal function in \(t\), is

\[
\mathbb{I}(q,t,z) = \sum_{d\geq 0} \sum_{k\geq 0} \frac{q^d}{k!} e^{t d (H + dz)/z} \left( \operatorname{Res}_{M_{h,d}}(t^k) \right).
\]

2.4.3. Evaluations. Let \(\tilde{H} \in H^*_T(\mathbb{C}^5/\mathbb{C}^\times)\) and \(H \in H^*_T(\mathbb{P}^4)\) denote the respective hyperplane classes. The I-function of Definition 3 is evaluated in \([6]\).

Proposition 4. For \(t = \tilde{t}H \in H^*_T(\mathbb{C}^5/\mathbb{C}^\times, \mathbb{Q})\),

\[
\mathbb{I}(t) = \sum_{d=0}^{\infty} q^d e^{t (H + dz)/z} \frac{\prod_{k=0}^{5d} (5H + kz)}{\prod_{i=0}^{4} \prod_{k=1}^{d} (H - \lambda_i + kz)}.
\]

We return now to the functions \(S_i(\gamma)\) defined in Section 2.3. Using Birkhoff factorization, an evaluation of the series \(S(H^3)\) can be obtained from the I-function, see \([13]\):

\[
\begin{align*}
S(1) &= \frac{\mathbb{I}}{\mathbb{I}|_{t=0,H=1,z=\infty}}, \\
S(H) &= \frac{z \frac{d}{dt} S(1)}{z \frac{d}{dt} S(1)|_{t=0,H=1,z=\infty}}, \\
S(H^2) &= \frac{z \frac{d}{dt} S(H)}{z \frac{d}{dt} S(H)|_{t=0,H=1,z=\infty}}, \\
S(H^3) &= \frac{z \frac{d}{dt} S(H^2)}{z \frac{d}{dt} S(H^2)|_{t=0,H=1,z=\infty}}, \\
S(H^4) &= \frac{z \frac{d}{dt} S(H^3)}{z \frac{d}{dt} S(H^3)|_{t=0,H=1,z=\infty}}, \\
S(1) &= \frac{z \frac{d}{dt} S(H^4)}{z \frac{d}{dt} S(H^4)|_{t=0,H=1,z=\infty}}.
\end{align*}
\]

For a series \(F \in \mathbb{C}[[\frac{1}{z}]]\), the specialization \(F|_{z=\infty}\) denotes constant term of \(F\) with respect to \(\frac{1}{z}\).

2.4.4. Further calculations. Define small I-function

\[
\mathbb{I}(q) \in H^*_T(\mathbb{P}^4, \mathbb{Q})[[q]]
\]

by the restriction

\[
\mathbb{I}(q) = \mathbb{I}(q,t)|_{t=0}.
\]
Define differential operators

\[ D = q \frac{d}{dq}, \quad M = H + z \frac{d}{dq}. \]

Applying \( z \frac{d}{dq} \) to \( I \) and then restricting to \( t = 0 \) has the same effect as applying \( M \) to \( I \)

\[
\left[ \left( z \frac{d}{dq} \right)^k I \right]_{t=0} = M^k I.
\]

The function \( I \) satisfies the following Picard-Fuchs equation

\[
(M^5 - 1 - q(5M + z)(5M + 2z)(5M + 3z)(5M + 4z)(5M + 5z)) I = 0
\]

implied by the Picard-Fuchs equation for \( I \),

\[
\left( \left( z \frac{d}{dt} \right)^5 - 1 - q \prod_{k=1}^{5} \left( \left( z \frac{d}{dt} \right) + kz \right) \right) I = 0.
\]

The restriction \( I|_{H=\lambda_i} \) admits the following asymptotic form

\[
(18) \quad I|_{H=\lambda_i} = e^\mu \lambda_i/z \left( R_0 + R_1 \left( \frac{z}{\lambda_i} \right) + R_2 \left( \frac{z}{\lambda_i} \right)^2 + \ldots \right)
\]

with series \( \mu, R_k \in \mathbb{C}[[q]] \).

A derivation of (18) is obtained in [25] via the Picard-Fuchs equation for \( I|_{H=\lambda_i} \). The series \( \mu \) and \( R_k \) are found by solving differential equations obtained from the coefficient of \( z^k \). For example,

\[
1 + D \mu = L, \\
R_0 = L, \\
R_1 = \frac{3}{20} (L - L^5), \\
R_2 = \frac{9L}{800} (1 - L^4)^2,
\]

where \( L(q) = (1 - 5^5 q)^{-1/5} \). The specialization (11) is used for these results.

Define the series \( C_i \) by the equations

\[
(19) \quad C_0 = I|_{z=\infty, t=0, H=1},
\]

\[
(20) \quad C_i = z \frac{d}{dt} \left( S(H^{i-1}) \right)|_{z=\infty, t=0, H=1}, \quad \text{for } i = 1, 2, 3, 4.
\]
The following relations were proven in [25],

\[ C_0 C_1 C_2 C_3 C_4 = L^5, \]
\[ C_i = C_{4-i} \text{ for } i = 0, 1, 2, 3, 4. \]

From the equations (17) and (18), we can show the series

\[ S_i(H^k) = S_i(H^k)|_{H=\lambda_i} \]

have the following asymptotic expansion:

\[
S_i(1) = e^{\frac{\mu_i}{2}} \frac{1}{C_0} \left( R_{00} + R_{01} \left( \frac{z}{\lambda_i} \right) + R_{02} \left( \frac{z}{\lambda_i} \right)^2 + \ldots \right),
\]
\[
S_i(H) = e^{\frac{\mu_i}{2}} \frac{L \lambda_i}{C_0 C_1} \left( R_{10} + R_{11} \left( \frac{z}{\lambda_i} \right) + R_{12} \left( \frac{z}{\lambda_i} \right)^2 + \ldots \right),
\]
\[
S_i(H^2) = e^{\frac{\mu_i}{2}} \frac{L^2 \lambda_i^2}{C_0 C_1 C_2} \left( R_{20} + R_{21} \left( \frac{z}{\lambda_i} \right) + R_{22} \left( \frac{z}{\lambda_i} \right)^2 + \ldots \right),
\]
\[
S_i(H^3) = e^{\frac{\mu_i}{2}} \frac{L^3 \lambda_i^3}{C_0 C_1 C_2 C_3} \left( R_{30} + R_{31} \left( \frac{z}{\lambda_i} \right) + R_{32} \left( \frac{z}{\lambda_i} \right)^2 + \ldots \right),
\]
\[
S_i(H^4) = e^{\frac{\mu_i}{2}} \frac{L^4 \lambda_i^4}{C_0 C_1 C_2 C_3 C_4} \left( R_{40} + R_{41} \left( \frac{z}{\lambda_i} \right) + R_{42} \left( \frac{z}{\lambda_i} \right)^2 + \ldots \right).
\]

We follow here the normalization of [25]. Note 

\[ R_{0k} = R_k. \]

As in [25] Theorem 4, we obtain the following constraints.

**Proposition 5.** (Zagier-Zinger [25]) For all \( k \geq 0 \), we have

\[ R_k \in \mathbb{C}[L^{\pm 1}]. \]

Define generators

\[ \mathcal{X} = \frac{DC_0}{C_0}, \quad \mathcal{X}_1 = D \mathcal{X}, \quad \mathcal{X}_2 = D \mathcal{X}_1, \quad \mathcal{Y} = \frac{DC_1}{C_1}. \]

From (17), we obtain the following result.
Lemma 6. For \( k \geq 0 \) we have
\[
R_{1k+1} = R_{0k+1} + \frac{DR_{0k}}{L} - \frac{\chi}{L} R_{0k},
\]
\[
R_{2k+1} = R_{1k+1} + \frac{DR_{1k}}{L} - \frac{\chi}{L} R_{1k} - \frac{\gamma}{L} R_{1k} + \frac{DL}{L^2} R_{1k},
\]
\[
R_{3k+1} = R_{2k+1} + \frac{DR_{2k}}{L} + \frac{\chi}{L} R_{2k} + \frac{\gamma}{L} R_{2k} - \frac{3DL}{L^2} R_{2k},
\]
\[
R_{4k+1} = R_{3k+1} + \frac{DR_{3k}}{L} + \frac{\chi}{L} R_{3k} - \frac{2DL}{L^2} R_{3k},
\]
\[
R_{0k+1} = R_{4k+1} + \frac{DL}{L^2} R_{4k}.
\]

Applying Lemma 6 for \( k = 0, 1 \), we obtain the following two equations among above generators which were also proven in \([24, \text{Section 3.1}]\). First,
\[
D\gamma = \frac{2}{5}(L^5 - 1) + 2(L^5 - 1)\chi - 2\chi^2 - 4\chi_1
\]
\[
+ (L^5 - 1)\gamma - \gamma^2 - 2\chi\gamma.
\]

For the second equation, define\(^{12}\)
\[
B_1 = -5\chi,
\]
\[
B_2 = 5^2(\chi_1 + \chi^2),
\]
\[
B_3 = -5^3(\chi_2 + 3\chi\chi_1 + \chi^3),
\]
\[
B_4 = 5^4(D\chi_2 + 4\chi\chi_2 + 3\chi_1^2 + 6\chi^2\chi_1 + \chi^4).
\]

Then, we have
\[
B_4 = -(L^5 - 1)(10B_3 - 35B_2 + 50B_1 - 24).
\]

For the proof of first holomorphic anomaly equation, we will require the following generalization of Proposition 5.

Proposition 7. For all \( k \geq 0 \), we have
\[
\begin{align*}
\text{(i)} & \quad R_{1k} \in \mathbb{C}[[L^{\pm 1}]][\chi], \\
\text{(ii)} & \quad R_{2k} = Q_{2k} - \frac{R_{1k+1}}{L} \gamma, \quad \text{with} \quad Q_{2k} \in \mathbb{C}[[L^{\pm 1}]][\chi, \chi_1], \\
\text{(iii)} & \quad R_{3k}, R_{4k} \in \mathbb{C}[[L^{\pm 1}]][\chi, \chi_1, \chi_2].
\end{align*}
\]

Proof. (i) Using Lemma 6 we can calculate
\[
R_{1k+1} = \frac{DR_{0k}}{L} + R_{0k+1} - \frac{R_{0k}}{L} \chi.
\]

\(^{12}\) We follow here the notation of \([24]\) for \( B_k \).
(ii) Using Lemma 6 and relations (23), we can calculate

\[ R_{2k+2} = \frac{D^2 R_{0k}}{L^2} - \frac{R_{0k+1}^2}{5L} + \frac{L^4 R_{0k+1}}{5} + \frac{2D R_{0k+1}}{L} + R_{0k+2} \]
\[ - \frac{2D R_{0k} X}{L^2} - \frac{2R_{0k+1} X}{L} + \frac{R_{0k} X^2}{L^2} - \frac{R_{0k} X_1}{L^2} \]
\[ + \frac{-D R_{0k} - LR_{0k+1} + R_{0k} X Y}{L^2} . \]

(iii) We can also explicitly calculate \( R_{3k} \) and \( R_{4k} \) in terms of \( R_{0k}, R_{0k-1}, R_{0k-2}, X, X_1, X_2, Y \)
using Lemma 6 and relations (23) and (24). We can check (iii) using these explicit calculations and Proposition 7. We leave the details to the reader.

\[ \square \]

For the proof of second holomorphic anomaly equation, we will require the following result.

**Proposition 8.** For all \( k \geq 0 \), we have

(i) \( R_{1k} = P_{0k} - \frac{R_{0k-1}}{L} X \), with \( P_{0k} \in \mathbb{C}[L^\pm 1] \),

(ii) \( R_{2k} \in \mathbb{C}[L^\pm 1][A_2, A_4] \),

(iii) \( R_{3k} = P_{3k} - \frac{R_{2k-1}}{L} X \) with \( P_{3k} \in \mathbb{C}[L^\pm 1][A_2, A_4, A_6] \),

(iv) \( R_{4k} \in \mathbb{C}[L^\pm 1][A_2, A_4, A_6] \).

**Proof.** The proof follows from the explicit calculations in the proof of Proposition 7 and the definition of \( A_2, A_4, A_6 \). \[ \square \]

3. Higher genus series on \( \overline{M}_{g,n} \)

3.1. Intersection theory on \( \overline{M}_{g,n} \). We review here the now standard method used by Givental [15, 16, 20] to express genus \( g \) descendent correlators in terms of genus 0 data. We refer the reader to [21, Section 4.1] for a more leisurely treatment.

Let \( t_0, t_1, t_2, \ldots \) be formal variables. The series

\[ T(c) = t_0 + t_1 c + t_2 c^2 + \ldots \]

in the additional variable \( c \) plays a basic role. The variable \( c \) will later be replaced by the first Chern class \( \psi_i \) of a cotangent line over \( \overline{M}_{g,n} \),

\[ T(\psi_i) = t_0 + t_1 \psi_i + t_2 \psi_i^2 + \ldots , \]

with the index \( i \) depending on the position of the series \( T \) in the correlator.
Let \(2g - 2 + n > 0\). For \(a_i \in \mathbb{Z}_{\geq 0}\) and \(\gamma \in H^*(\overline{M}_{g,n})\), define the correlator
\[
\langle \langle \psi^{a_1}, \ldots, \psi^{a_n} | \gamma \rangle \rangle_{g,n} = \sum_{k \geq 0} \frac{1}{k!} \int_{\overline{M}_{g,n+k}} \gamma \psi^{a_1} \cdots \psi^{a_n} \prod_{i=1}^{k} T(\psi_{n+i}) .
\]
In the above summation, the \(k = 0\) term is
\[
\int_{\overline{M}_{g,n}} \gamma \psi^{a_1} \cdots \psi^{a_n} .
\]
We also need the following correlator defined for the unstable case,
\[
\langle \langle 1, 1 \rangle \rangle_{0,2} = \sum_{k > 0} \frac{1}{k!} \int_{\overline{M}_{0,2+k}} \prod_{i=1}^{k} T(\psi_{2+i}) .
\]
For formal variables \(x_1, \ldots, x_n\), we also define the correlator
\[
\langle \langle \frac{1}{x_1 - \psi}, \ldots, \frac{1}{x_n - \psi} | \gamma \rangle \rangle_{g,n}
\]
in the standard way by expanding \(\frac{1}{x_i - \psi}\) as a geometric series.

Denote by \(L\) the differential operator
\[
L = \frac{\partial}{\partial t_0} - \sum_{i=1}^{\infty} t_i \frac{\partial}{\partial t_{i-1}} = \frac{\partial}{\partial t_0} - t_1 \frac{\partial}{\partial t_0} - t_2 \frac{\partial}{\partial t_1} - \ldots .
\]

The string equation yields the following result.

**Lemma 9.** For \(2g - 2 + n > 0\), we have \(L \langle \langle 1, \ldots, 1 | \gamma \rangle \rangle_{g,n} = 0\) and
\[
L \langle \langle \frac{1}{x_1 - \psi}, \ldots, \frac{1}{x_n - \psi} | \gamma \rangle \rangle_{g,n} = \left( \frac{1}{x_1} + \ldots + \frac{1}{x_n} \right) \langle \langle \frac{1}{x_1 - \psi}, \ldots, \frac{1}{x_n - \psi} | \gamma \rangle \rangle_{g,n} .
\]

We consider \(\mathbb{C}(t_1)[t_2, t_3, \ldots]\) as \(\mathbb{Z}\)-graded ring over \(\mathbb{C}(t_1)\) with \(\deg(t_i) = i - 1\) for \(i \geq 2\).

Define a subspace of homogeneous elements by
\[
\mathbb{C} \left[ \frac{1}{1 - t_1} \right] [t_2, t_3, \ldots]_{\text{Hom}} \subset \mathbb{C}(t_1)[t_2, t_3, \ldots].
\]
We easily see
\[
\langle \langle \psi^{a_1}, \ldots, \psi^{a_n} | \gamma \rangle \rangle_{g,n} \big|_{t_0 = 0} \in \mathbb{C} \left[ \frac{1}{1 - t_1} \right] [t_2, t_3, \ldots]_{\text{Hom}} .
\]
Using the leading terms (of lowest degree in \( \frac{1}{1-t_1^i} \)), we obtain the following result.

**Lemma 10.** The set of genus 0 correlators

\[
\left\{ \langle \langle 1, \ldots, 1 \rangle \rangle_{0,n} \mid t_0 = 0 \right\}_{n \geq 4}
\]

freely generate the ring \( \mathbb{C}(t_1)[t_2, t_3, \ldots] \) over \( \mathbb{C}(t_1) \).

**Definition 11.** For \( \gamma \in H^*(\overline{M}_{g,k}) \), let

\[
P_{g,n}^{a_1, \ldots, a_n, \gamma}(s_0, s_1, s_2, \ldots) \in \mathbb{Q}(s_0, s_1, \ldots)
\]

be the unique rational function satisfying the condition

\[
\langle \langle \psi^{a_1}, \ldots, \psi^{a_n} \mid \gamma \rangle \rangle_{g,n} |_{t_0 = 0} = P_{g,n}^{a_1, a_2, \ldots, a_n, \gamma} |_{s_i = \langle \langle 1, \ldots, 1 \rangle \rangle_{0,i+3} | t_0 = 0}.
\]

By applying Lemma 10, we obtain the two following results, see [21, Section 4.1].

**Proposition 12.** For \( 2g - 2 + n > 0 \), we have

\[
\langle \langle 1, \ldots, 1 \mid \gamma \rangle \rangle_{g,n} = P_{g,n}^{0, \ldots, 0, \gamma} |_{s_i = \langle \langle 1, \ldots, 1 \rangle \rangle_{0,i+3}}.
\]

**Proposition 13.** For \( 2g - 2 + n > 0 \),

\[
\left\langle \left\langle \frac{1}{x_1 - \psi_1}, \ldots, \frac{1}{x_n - \psi_n} \mid \gamma \right\rangle \right\rangle_{g,n} =
\]

\[
e^{\langle \langle 1, 1 \rangle \rangle_{0,2}} \sum_{a_1, \ldots, a_n} \frac{P_{g,n}^{a_1, \ldots, a_n, \gamma} |_{s_i = \langle \langle 1, \ldots, 1 \rangle \rangle_{0,i+3}}}{x_1^{a_1+1} \cdots x_n^{a_n+1}}.
\]

\( \mathbb{L} \langle \langle 1, 1 \rangle \rangle_{0,2} = 1 \), \( \langle \langle 1, 1 \rangle \rangle_{0,2} |_{t_0 = 0} = 0 \).

The definition given in (25) of the correlator is valid in the stable range

\( 2g - 2 + n > 0 \).

The unstable case \((g, n) = (0, 2)\) plays a special role. We define

\[
\left\langle \left\langle \frac{1}{x_1 - \psi_1}, \frac{1}{x_2 - \psi_2} \right\rangle \right\rangle_{0,2}
\]

by adding the degenerate term

\[
\frac{1}{x_1 + x_2}
\]

to the terms obtained by the expansion of \( \frac{1}{x_i - \psi_i} \) as a geometric series.

The degenerate term is associated to the (unstable) moduli space of genus 0 with 2 markings. By [21, Section 4.2], we have.
Proposition 14. We have
\[
\left< \frac{1}{x_1 - \psi_1}, \frac{1}{x_2 - \psi_2} \right>_{0,2} = e^{\langle (1,1) \rangle_{0,2} \left( \frac{1}{x_1 + 1} + \frac{1}{x_2} \right)} \left( \frac{1}{x_1 + x_2} \right).
\]

3.2. Local invariants and wall crossing. The torus $T$ acts on the moduli spaces $\overline{M}_{g,n}(\mathbb{P}^4, d)$ and $\overline{Q}_{g,n}(\mathbb{P}^4, d)$. We consider here special localization contributions associated to the fixed points $p_i \in \mathbb{P}^4$.

Consider first the moduli of stable maps. Let $\overline{M}_{g,n}(\mathbb{P}^4, d)^{T,p_i} \subset \overline{M}_{g,n}(\mathbb{P}^4, d)$ be the union of $T$-fixed loci which parameterize stable maps obtained by attaching $T$-fixed rational tails to a genus $g$, $n$-pointed Deligne-Mumford stable curve contracted to the point $p_i \in \mathbb{P}^4$. Similarly, let $\overline{Q}_{g,n}(\mathbb{P}^4, d)^{T,p_i} \subset \overline{Q}_{g,n}(\mathbb{P}^4, d)$ be the parallel $T$-fixed locus parameterizing stable quotients obtained by attaching base points to a genus $g$, $n$-pointed Deligne-Mumford stable curve contracted to the point $p_i \in \mathbb{P}^4$.

Let $\Lambda_i$ denote the localization of the ring
\[
\mathbb{C}[\lambda_0^\pm 1, \ldots, \lambda_4^\pm 1]
\]
at the five tangent weights at $p_i \in \mathbb{P}^4$. Using the virtual localization formula [17], there exist unique series
\[
S_{p_i} \in \Lambda_i[\psi][[Q]]
\]
for which the localization contribution of the $T$-fixed locus $\overline{M}_{g,n}(\mathbb{P}^4, d)^{T,p_i}$ to the equivariant Gromov-Witten invariants of formal quintic can be written as
\[
\sum_{d=0}^{\infty} Q^d \int_{[\overline{M}_{g,n}(\mathbb{P}^4, d)^{T,p_i}]_{vir}} \frac{e(\text{Obs})}{e(\text{Nor})} \psi_1^{a_1} \cdots \psi_n^{a_n} =
\sum_{k=0}^{\infty} \frac{1}{k!} \int_{\overline{M}_{g,n+k}} H^p_{g} \psi_1^{a_1} \cdots \psi_n^{a_n} \prod_{j=1}^{k} S_{p_i}(\psi_{n+j}).
\]
Here, $H^p_{g}$ is the standard vertex class,
\[
e(\mathbb{E}_g^* \otimes T_{p_i}(\mathbb{P}^4))) = e(T_{p_i}(\mathbb{P}^4)} \otimes (5 \lambda_i)) \cdot \frac{e(\mathbb{E}_g^* \otimes (5 \lambda_i))}{e(\mathbb{E}_g^* \otimes T_{p_i}(\mathbb{P}^4))},
\]
obtained the Hodge bundle $\mathbb{E}_g \to \overline{M}_{g,n+k}$.

Similarly, the application of the virtual localization formula to the moduli of stable quotients yields classes
\[
F_{p_i,k} \in H^*(\overline{M}_{g,n+k}) \otimes \mathbb{C} \Lambda_i
\]
for which the contribution of $\mathcal{Q}_{g,n}(\mathbb{P}^4, d)^{T,p_i}$ is given by
\[
\sum_{d=0}^{\infty} q^d \int_{[\mathcal{Q}_{g,n}(\mathbb{P}^4, d)^{T,p_i}]^{\text{vir}}} \frac{e(\text{Obs})}{e(\text{Nor})} \psi_1^{a_1} \cdots \psi_n^{a_n} = 
\sum_{k=0}^{\infty} \frac{q^k}{k!} \int_{\overline{M}_{g,n}|k}} \mathcal{H}_{g}^{p_i} \psi_1^{a_1} \cdots \psi_n^{a_n} F_{p_i,k}.
\]

Here $\overline{M}_{g,n}|k$ is the moduli space of genus $g$ curves with markings 
\[\{p_1, \ldots, p_n\} \cup \{\hat{p}_1 \cdots \hat{p}_k\} \in C_{\text{ns}} \subset C\]
satisfying the conditions

(i) the points $p_i$ are distinct,

(ii) the points $\hat{p}_j$ are distinct from the points $p_i$,

with stability given by the ampleness of
\[\omega_C(\sum_{i=1}^{m} p_i + \epsilon \sum_{j=1}^{k} \hat{p}_j)\]
for every strictly positive $\epsilon \in \mathbb{Q}$.

The Hodge class $\mathcal{H}_{g}^{p_i}$ is given again by formula (26) using the Hodge bundle 
\[\mathcal{E}_g \to \overline{M}_{g,n}|k\).

**Definition 15.** For $\gamma \in H^*(\overline{M}_{g,n})$, let
\[
\langle \langle \psi_1^{a_1}, \ldots, \psi_n^{a_n} \mid \gamma \rangle \rangle_{g,n}^{p_i,\infty} = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\overline{M}_{g,n+k}} \gamma_1^{a_1} \cdots \gamma_n^{a_n} \prod_{j=1}^{k} S_{p_i}(\psi_n^+),
\]
\[
\langle \langle \psi_1^{a_1}, \ldots, \psi_n^{a_n} \mid \gamma \rangle \rangle_{g,n}^{p_i,0+} = \sum_{k=0}^{\infty} \frac{q^k}{k!} \int_{\overline{M}_{g,n+k}} \gamma_1^{a_1} \cdots \gamma_n^{a_n} F_{p_i,k}.
\]

**Proposition 16** (Ciocan-Fontanine, Kim [8]). For $2g - 2 + n > 0$, we have the wall crossing relation
\[
\langle \langle \psi_1^{a_1}, \ldots, \psi_n^{a_n} \mid \gamma \rangle \rangle_{g,n}^{p_i,\infty}(Q(q)) = (I_0^Q)^{2g-2+n}\langle \langle \psi_1^{a_1}, \ldots, \psi_n^{a_n} \mid \gamma \rangle \rangle_{g,n}^{p_i,0+}(q)
\]
where $Q(q)$ is the mirror map
\[Q(q) = \exp \left( \frac{I_1^Q(q)}{I_0^Q(q)} \right).\]
Proposition 16 is a consequence of [8, Lemma 5.5.1]. The mirror map here is the mirror map for quintic discussed in Section 0.1. Propositions 12 and 16 together yield

\[
\langle\langle 1, \ldots, 1 | \gamma \rangle \rangle_{g,n} = \sum_{a_1, \ldots, a_n} \text{Vol}_{a_1, \ldots, a_n, \gamma} \left( \langle\langle 1, 1, 1 \rangle \rangle_{0,3}^{p_i,\infty}, \langle\langle 1, 1, 1 \rangle \rangle_{0,4}^{p_i,\infty} \right)
\]

Similarly, using Propositions 13 and 16, we obtain

\[
\langle\langle 1, \ldots, 1 | \gamma \rangle \rangle_{g,n} = \sum_{a_1, \ldots, a_n} \text{Vol}_{a_1, \ldots, a_n, \gamma} \left( \langle\langle 1, 1, 1 \rangle \rangle_{0,3}^{p_i,0+}, \langle\langle 1, 1, 1 \rangle \rangle_{0,4}^{p_i,0+} \right)
\]

4. Higher genus series on the formal quintic

4.1. Overview. We apply Givental’s localization strategy [15, 16, 20] for Gromov-Witten theory to the stable quotient invariants of formal quintic. The contribution \( C_{\Gamma}(q) \) discussed in Section 1 of a graph \( \Gamma \in G_g(\mathbb{P}^4) \) can be separated into vertex and edge contributions. We express the vertex and edge contributions in terms of the series \( S_i \) and \( V_{ij} \) of Section 2.3. Our treatment here follows our study of \( K \mathbb{P}^2 \) in [21, Section 5]

4.2. Edge terms. Recall the definition\(^{13}\) of \( V_{ij} \) given in Section 2.3

\[
V_{ij} = \langle\langle \phi_i \phi_j \rangle \rangle_{0,2}^{0+,0+}
\]

Let \( V_{ij} \) denote the restriction of \( V_{ij} \) to \( t = 0 \). Via formula (9), \( V_{ij} \) is a summation of contributions of fixed loci indexed by a graph \( \Gamma \) consisting of two vertices connected by a unique edge. Let \( w_1 \) and \( w_2 \) be T-weights. Denote by

\[ V_{ij}^{w_1,w_2} \]
the summation of contributions of $T$-fixed loci with tangent weights precisely $w_1$ and $w_2$ on the first rational components which exit the vertex components over $p_i$ and $p_j$.

The series $\nabla_{ij}^{w_1, w_2}$ includes both vertex and edge contributions. By definition (28) and the virtual localization formula, we find the following relationship between $\nabla_{ij}^{w_1, w_2}$ and the corresponding pure edge contribution $E_{ij}^{w_1, w_2}$,

$$e_i \nabla_{ij}^{w_1, w_2} e_j = \left\langle \left\langle \frac{1}{w_1 - \psi}, \frac{1}{x_1 - \psi} \right\rangle \right\rangle_{p_i,0}^{p_i,0+} E_{ij}^{w_1, w_2} \frac{\left\langle \left\langle \frac{1}{w_2 - \psi}, \frac{1}{x_2 - \psi} \right\rangle \right\rangle_{p_j,0}^{p_j,0+}}{w_1 + x_1} \frac{\left\langle \left\langle (1,1)_{0,2}^{p_i,0+}, (1,1)_{0,2}^{p_j,0+} \right\rangle \right\rangle_{w_1}^{w_2}}{w_1 + x_2} \psi \left(1+\frac{1}{w_1} \right) \left(1+\frac{1}{w_2} \right) \left(1+\frac{1}{x_1} \right) \left(1+\frac{1}{x_2} \right).$$

After summing over all possible weights, we obtain

$$e_i \left( \nabla_{ij} - \frac{\delta_{ij}}{e_i(x+y)} \right) e_j = \sum_{w_1, w_2} e_i \nabla_{ij}^{w_1, w_2} e_j.$$

The above calculations immediately yield the following result.

**Lemma 17.** We have

$$e_i \left( \nabla_{ij} - \frac{\delta_{ij}}{e_i(x+y)} \right) e_j = \sum_{w_1, w_2} e_i \nabla_{ij}^{w_1, w_2} e_j \frac{\left\langle \left\langle (1,1)_{0,2}^{p_i,0+}, (1,1)_{0,2}^{p_j,0+} \right\rangle \right\rangle_{w_1}^{w_2}}{w_1 + x_1} \left(1+\frac{1}{w_1} \right) \left(1+\frac{1}{w_2} \right) \left(1+\frac{1}{x_1} \right) \left(1+\frac{1}{x_2} \right).$$

The notation $[\ldots]_{x_1^{-1} x_2^{-1}}$ in Lemma 17 denotes the coefficient of $x_1^{-1} x_2^{-1}$ in the series expansion of the argument.

4.3. **A simple graph.** Before treating the general case, we present the localization formula for a simple graph$^{14}$ Let $\Gamma \in \mathcal{G}_g(\mathbb{P}^1)$ consist of two vertices and one edge,

$$v_1, v_2 \in \Gamma(V), \quad e \in \Gamma(E)$$

$^{14}$We follow here the notation of Section 1.
with genus and $T$-fixed point assignments

$$g(v_i) = g_i, \quad p(v_i) = p_i.$$  

Let $w_1$ and $w_2$ be tangent weights at the vertices $p_1$ and $p_2$ respectively. Denote by $\text{Cont}_{\Gamma, w_1, w_2}$ the summation of contributions to

(29) $$\sum_{d>0} q^d (e(\text{Obs}) \cap [Q_g(\mathbb{P}^4, d)]^\text{vir})$$

of $T$-fixed loci with tangent weights precisely $w_1$ and $w_2$ on the first rational components which exit the vertex components over $p_1$ and $p_2$. We can express the localization formula for (29) as

$$\left\langle \left\langle \frac{1}{w_1 - \psi} \right| H^p_{g_1} \right\rangle_{w_1, w_2} E^{w_1, w_2}_{12} \left\langle \left\langle \frac{1}{w_2 - \psi} \right| H^p_{g_2} \right\rangle_{w_2}$$

which equals

$$\sum_{a_1, a_2} e^{\langle(1,1)\rangle_{0,2}^{p_1,0+} w_1} P_{w_1}^{a_1, a_2} e^{\langle(1,1)\rangle_{0,2}^{p_2,0+} w_2} E^{a_1, a_2}_{12} P_{w_2}^{a_1, a_2}$$

where $H^p_{g_i}$ is the Hodge class (26). We have used here the notation

$$P_{h, n}^{k_1, \ldots, k_n, H^p} = P_{h, n}^{k_1, \ldots, k_n, H^p} \left( \langle(1,1,1)\rangle_{0,3}^{p_1,0+}, \langle(1,1,1)\rangle_{0,4}^{p_1,0+}, \ldots \right)$$

and applied (27).

After summing over all possible weights $w_1, w_2$ and applying Lemma 17, we obtain the following result for the full contribution

$$\text{Cont}_\Gamma = \sum_{w_1, w_2} \text{Cont}_{\Gamma, w_1, w_2}$$

of $\Gamma$ to $\sum_{d>0} q^d (e(\text{Obs}) \cap [Q_g(\mathbb{P}^4, d)]^\text{vir})$.

**Proposition 18.** We have

$$\text{Cont}_\Gamma = \sum_{a_1, a_2} P_{a_1, a_2}^{p_1,0+} P_{a_1, a_2}^{p_2,0+}$$

\begin{align*}
\cdot (-1)^{a_1 + a_2} e^{-\frac{\langle(1,1)\rangle_{0,2}^{p_1,0+}}{s_1}} e^{-\frac{\langle(1,1)\rangle_{0,2}^{p_2,0+}}{s_2}} e_i \left( \nabla_{ij} - \frac{\delta_{ij}}{e_i(x_1 + x_2)} \right) e_j \cdot x_1^{a_1-1} x_2^{a_2-1}.
\end{align*}
4.4. A general graph. We apply the argument of Section 4.3 to obtain a contribution formula for a general graph $\Gamma$.

Let $\Gamma \in G_{g,0}(\mathbb{P}^4)$ be a decorated graph as defined in Section 1. The flags of $\Gamma$ are the half-edges$^{15}$ Let $F$ be the set of flags. Let $w : F \to \text{Hom}(T, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q}$ be a fixed assignment of $T$-weights to each flag.

We first consider the contribution $\text{Cont}_{\Gamma,w}$ to

$$
\sum_{d \geq 0} q^d (e(\text{Obs}) \cap [Q_g(\mathbb{P}^4, d)]_{\text{vir}})
$$

of the $T$-fixed loci associated $\Gamma$ satisfying the following property: the tangent weight on the first rational component corresponding to each $f \in F$ is exactly given by $w(f)$. We have

$$
(30) \quad \text{Cont}_{\Gamma,w} = \frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}^F > 0} \prod_{v \in V} \text{Cont}^A_{G,w}(v) \prod_{e \in E} \text{Cont}_{\Gamma,w}(e).
$$

The terms on the right side of (30) require definition:

- The sum on the right is over the set $\mathbb{Z}^F > 0$ of all maps $A : F \to \mathbb{Z}_{>0}$ corresponding to the sum over $a_1, a_2$ in Proposition 18.
- For $v \in V$ with $n$ incident flags with $w$-values $(w_1, \ldots, w_n)$ and $A$-values $(a_1, a_2, \ldots, a_n)$,

$$
\text{Cont}^A_{\Gamma,w}(v) = \frac{P \left[ \psi_1^{a_1-1}, \ldots, \psi_n^{a_n-1} \mid H_{g(v)}^{p(v),0+} \right]_{g(v),n}}{w_1^{a_1} \cdots w_n^{a_n}}.
$$

- For $e \in E$ with assignments $(p(v_1), p(v_2))$ for the two associated vertices$^{16}$ and $w$-values $(w_1, w_2)$ for the two associated flags,

$$
\text{Cont}_{\Gamma,w}(e) = e^{\langle (1,1) \rangle^{p(v_1),0+}_{0,2}} e^{\langle (1,1) \rangle^{p(v_2),0+}_{0,2}} E^{w_1, w_2}_{p(v_1), p(v_2)}.
$$

The localization formula then yields (30) just as in the simple case of Section 4.3.

By summing the contribution (30) of $\Gamma$ over all the weight functions $w$ and applying Lemma 17, we obtain the following result which generalizes Proposition 18.

---

$^{15}$Flags are either half-edges or markings.

$^{16}$In case $e$ is self-edge, $v_1 = v_2$. 
Proposition 19. We have

\[ \text{Cont}_\Gamma = \frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}_{>0}} \prod_{v \in V} \text{Cont}^A(v) \prod_{e \in E} \text{Cont}^A(e), \]

where the vertex and edge contributions with incident flag A-values 
\((a_1, \ldots, a_n)\) and \((b_1, b_2)\) respectively are

\[
\begin{align*}
\text{Cont}^A(v) &= p \left[ \psi_1^{a_1-1}, \ldots, \psi_n^{a_n-1} \right] H^{p(v)} \left[ g(v), n \right], \\
\text{Cont}^A(e) &= (-1)^{b_1+b_2} \left[ e^{-\frac{\langle(1,1)\rangle_{0,2}^{p(v_1),0+}}{x_1} - \frac{\langle(1,1)\rangle_{0,2}^{p(v_2),0+}}{x_2}} e_i \left( \nabla_{ij} - \frac{1}{e_i(x+y)} \right) e_j \right] x_1^{b_1-1} x_2^{b_2-1},
\end{align*}
\]

where \(p(v_1) = p_i\) and \(p(v_2) = p_j\) in the second equation.

4.5. Legs. Let \(\Gamma \in G_{g,n}(\mathbb{P}^4)\) be a decorated graph with markings. While no markings are needed to define the stable quotient invariants of formal quintic, the contributions of decorated graphs with markings will appear in the proof of the holomorphic anomaly equation. The formula for the contribution \(\text{Cont}_\Gamma(H^{k_1}, \ldots, H^{k_n})\) of \(\Gamma\) to

\[
\sum_{d \geq 0} q^d \prod_{j=0}^n e v^*(H^{k_j}) \cdot e(\text{Obs}) \cap [\mathcal{Q}_g(\mathbb{P}^4, d)]^{\text{vir}}
\]

is given by the following result.

Proposition 20. We have

\[ \text{Cont}_\Gamma(H^{k_1}, \ldots, H^{k_n}) = \frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}_{>0}} \prod_{v \in V} \text{Cont}^A(v) \prod_{e \in E} \text{Cont}^A(e) \prod_{l \in L} \text{Cont}^A(l), \]

where the leg contribution is

\[
\text{Cont}^A(l) = (-1)^{A(l)-1} \left[ e^{-\frac{\langle(1,1)\rangle_{0,2}^{p(l),0+}}{z} S_p(l)(H^{k_l})} \right] z^{A(l)-1}.
\]

The vertex and edge contributions are same as before.

The proof of Proposition 20 follows the vertex and edge analysis. We leave the details as an exercise for the reader. The parallel statement for Gromov-Witten theory can be found in [15, 16, 20].
5. Vertices, edges, and legs

5.1. Overview. Following the analysis of $K\mathbb{P}^2$ in [21, Section 6] which uses results of Givental [15, 16, 20] and the wall-crossing of [8], we calculate here the vertex and edge contributions in terms of the function $R_k$ of Section 2.4.4.

5.2. Calculations in genus 0. We follow the notation introduced in Section 3.1. Recall the series

$$T(c) = t_0 + t_1 c + t_2 c^2 + \ldots.$$  

Proposition 21. (Givental [15, 16, 20]) For $n \geq 3$, we have

$$\langle\langle 1, \ldots, 1 \rangle\rangle_{0,n}^{p_i,0} = \left(\sqrt{\Delta_i}\right)^{2g-2+n} \left(\sum_{k \geq 0} \frac{1}{k!} \int_{M_{0,n+k}} T(\psi_{n+1}) \cdots T(\psi_{n+k}) \right) \bigg|_{t_0=0, t_1=0, t_j \geq 2 = (-1)^{j} Q_j \lambda_j^{-1}}$$

where the functions $\sqrt{\Delta_i}, Q_l$ are defined by

$$S_i^{\infty}(1) = e^{\langle \phi_i, \frac{1}{z-\psi}, 1 \rangle}_{p_i,0}^{\infty} = e^{\langle (1,1), 0, 2 \rangle_{0,2}^{p_i,0,\infty}} \left(1 + \sum_{l=1}^{\infty} Q_l \left(\frac{z}{\lambda_i}\right)^l \right).$$

From [21] and Proposition 16 we have

$$\sqrt{\Delta_i} = \frac{C_0}{R_0},$$

$$Q_k = \frac{R_k}{R_0}.$$

Using Proposition 16 again, we have proven the following result.

Proposition 22. For $n \geq 3$, we have

$$\langle\langle 1, \ldots, 1 \rangle\rangle_{0,n}^{p_i,0} = R_0^{2g-2-n} \left(\sum_{k \geq 0} \frac{1}{k!} \int_{M_{0,n+k}} T(\psi_{n+1}) \cdots T(\psi_{n+k}) \right) \bigg|_{t_0=0, t_1=0, t_j \geq 2 = (-1)^{j} \frac{R_j \lambda_j^{-1}}{\lambda_j}}.$$

Proposition 22 immediately implies the evaluation

$$\langle\langle 1, 1, 1 \rangle\rangle_{0,3}^{p_i,0} = \frac{1}{R_0}.$$  

Another simple consequence of Proposition 22 is the following basic property.

Corollary 23. For $n \geq 3$, we have $\langle\langle 1, \ldots, 1 \rangle\rangle_{0,n}^{p_i,0} \in \mathbb{C}[R_0^{\pm 1}, R_1, R_2, \ldots][\lambda_i^{-1}].$
5.3. **Vertex and edge analysis.** By Proposition 19, we have decomposition of the contribution to $\Gamma \in G_g(\mathbb{P}^4)$ to the stable quotient theory of formal quintic into vertex terms and edge terms

$$\text{Cont}_\Gamma = \frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}_{>0}} \prod_{v \in V} \text{Cont}_\Gamma^{A}(v) \prod_{e \in E} \text{Cont}_\Gamma^{A}(e).$$

**Lemma 24.** We have $\text{Cont}_\Gamma^{A}(v) \in \mathbb{C}(\lambda_0, \ldots, \lambda_4)[L^{\pm 1}]$.

**Proof.** By Proposition 19,

$$\text{Cont}_\Gamma^{A}(v) = P \left[ \psi^{a_1-1}, \ldots, \psi^{a_n-1} \left| \frac{H_{g(v)}}{g(v)} \right. \right].$$

The right side of the above formula is a polynomial in the variables $1\langle 1, 1, 1 \rangle$, $0^+, 0^-, 0^+$ with coefficients in $\mathbb{C}(\lambda_0, \ldots, \lambda_4)$. The Lemma then follows from the evaluation (31), Corollary 23, and Proposition 5. □

Let $e \in E$ be an edge connecting the $T$-fixed points $p_i, p_j \in \mathbb{P}^4$. Let the $A$-values of the respective half-edges be $(k, l)$.

**Lemma 25.** We have $\text{Cont}_\Gamma^{A}(e) \in \mathbb{C}(\lambda_0, \ldots, \lambda_4)[L^{\pm 1}, X, X_1, X_2, Y]$ and

- the degree of $\text{Cont}_\Gamma^{A}(e)$ with respect to $Y$ is 1,
- the coefficient of $Y$ in $\text{Cont}_\Gamma^{A}(e)$ is

$$(-1)^{k+l+1} \frac{R_1 R_{l-1}}{5 L^3 \lambda_i^{k-2} \lambda_j^{l-2}}.$$

**Proof.** By Proposition 19

$$\text{Cont}_\Gamma^{A}(e) = (-1)^{k+l} e^{\frac{-\mu \lambda_i}{x} - \frac{\mu \lambda_j}{y}} e_i \left( \nabla_{ij} - \frac{\delta_{ij}}{e_i(x+y)} \right) e_j \right]_{x^{k-1} y^{l-1}}.$$

Using also the equation

$$e_i \nabla_{ij}(x, y)e_j = \sum_{r=0}^{2} \overline{S}_i(\phi_r)|_{z=x} \overline{S}_j(\phi_r)|_{z=y},$$

we write $\text{Cont}_\Gamma^{A}(e)$ as

$$\left[ (-1)^{k+l} e^{\frac{-\mu \lambda_i}{x} - \frac{\mu \lambda_j}{y}} \sum_{r=0}^{2} \overline{S}_i(\phi_r)|_{z=x} \overline{S}_j(\phi_r)|_{z=y} \right]_{x^{k-1} y^{l-1}}.$$

where the subscript signifies a (signed) sum of the respective coefficients. If we substitute the asymptotic expansions (21) for

$\overline{S}_i(1), \overline{S}_i(H), \overline{S}_i(H^2), \overline{S}_i(H^3), \overline{S}_i(H^4)$
in the above expression, the Lemma follows from Proposition 7.

Similarly, we obtain the following result using Proposition 8.

**Lemma 26.** We have

\[ \text{Cont}_A^A(e) \in \mathbb{C}([\lambda_0, \ldots, \lambda_4]; L^{\pm 1}, \mathcal{X}, A_2, A_4, A_6) \]

- the degree of \( \text{Cont}_A^A \) with respect to \( \mathcal{X} \) is 1,
- the coefficient of \( \mathcal{X} \) in \( \text{Cont}_A^A \) is

\[
(-1)^{k+l+1} \left( \frac{R_{0k-1}R_{2l-1}}{5L^3\lambda_i^{k-1}\lambda_j^{l-3}} + \frac{R_{2k-1}R_{0l-1}}{5L^3\lambda_i^{k-3}\lambda_j^{l-1}} \right)
\]

5.4. **Legs.** Using the contribution formula of Proposition 20,

\[ \text{Cont}_A^A(l) = (-1)^{A(l)-1} \left[ e^{-\frac{(l(1,1))^{p(l),0+}}{2} S_p(l)(H^{k_l})} \right] z^{A(l)-1}, \]

we easily conclude
- when the insertion at the marking \( l \) is \( H^0 \),
  \[ C_0 \cdot \text{Cont}_A^A(l) \in \mathbb{C}([\lambda_0, \ldots, \lambda_4]; L^{\pm 1}, \mathcal{X}) \],
- when the insertion at the marking \( l \) is \( H^1 \),
  \[ C_0C_1 \cdot \text{Cont}_A^A(l) \in \mathbb{C}([\lambda_0, \ldots, \lambda_4]; L^{\pm 1}, \mathcal{X}, \mathcal{X}_1, \mathcal{X}) \],
- when the insertion at the marking \( l \) is \( H^2 \),
  \[ C_0C_1C_2 \cdot \text{Cont}_A^A(l) \in \mathbb{C}([\lambda_0, \ldots, \lambda_4]; L^{\pm 1}, \mathcal{X}, \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3) \],
- when the insertion at the marking \( l \) is \( H^3 \),
  \[ C_0C_1C_2C_3 \cdot \text{Cont}_A^A(l) \in \mathbb{C}([\lambda_0, \ldots, \lambda_4]; L^{\pm 1}, \mathcal{X}, \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3) \],
- when the insertion at the marking \( l \) is \( H^4 \),
  \[ C_0C_1C_2C_3C_4 \cdot \text{Cont}_A^A(l) \in \mathbb{C}([\lambda_0, \ldots, \lambda_4]; L^{\pm 1}, \mathcal{X}, \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4) \].

6. **holomorphic anomaly for the formal quintic**

6.1. **Proof of Theorem 1.** By definition, we have

\[ K_2(q) = -\frac{1}{L^5} \mathcal{X}, \]
\[ A_2(q) = \frac{1}{L^5} \left( -\frac{1}{5} \mathcal{Y} - \frac{2}{5} \mathcal{X} - \frac{3}{25} \right), \]
\[ A_4(q) = \frac{1}{L^{10}} \left( -\frac{1}{25} \mathcal{X}^2 - \frac{1}{25} \mathcal{Y} \mathcal{X} + \frac{1}{25} \mathcal{X}_1 + \frac{2}{25^2} \right), \]
\[ A_6(q) = \frac{1}{10 \cdot 5^5 L^{15}} \left( 4 + 125 \mathcal{X}_1 + 50 \mathcal{X}(1 + 10 \mathcal{X}_1) - 5L^5(1 + 10 \mathcal{X} + 25 \mathcal{X}^2 + 25 \mathcal{X}_1) + 125 \mathcal{X}_2 - 125 \mathcal{X}^2(\mathcal{Y} - 1) \right). \]
Hence, statement (i),

$$\tilde{\mathcal{F}}_g^B(q) \in \mathbb{C}[L^{\pm 1}][C_0^{\pm 1}, K_2, A_2, A_4, A_6],$$

follows from Proposition 19 and Lemmas 24 - 25.

Since

$$\frac{\partial}{\partial T} = q \frac{\partial}{\partial q},$$

statement (ii),

$$(32) \quad \frac{\partial^k \tilde{\mathcal{F}}_g^B}{\partial T^k}(q) \in \mathbb{C}[L^{\pm 1}][C_0^{\pm 1}, C_1^{-1}, K_2, A_2, A_4, A_6],$$

follows since the ring

$$\mathbb{C}[L^{\pm 1}][C_0^{\pm 1}, C_1^{-1}, K_2, A_2, A_4, A_6] = \mathbb{C}[L^{\pm 1}][C_0^{\pm 1}, C_1^{-1}, \mathcal{X}, \mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}]$$

is closed under the action of the differential operator

$$D = q \frac{\partial}{\partial q}$$

by (23). The degree of $C_1^{-1}$ in (32) is 1 which yields statement (iii). □

**Remark 27.** The proof of Theorem 1 actually yields:

$$C_0^{2-2g} \cdot \tilde{\mathcal{F}}_g^B, \quad C_0^{2-2g} C_1^k \cdot \frac{\partial^k \tilde{\mathcal{F}}_g^B}{\partial T^k} \in \mathbb{C}[L^{\pm 1}][A_2, A_4, A_6, K_2].$$

### 6.2. Proof of Theorem 2: first equation.

Let $\Gamma \in \mathcal{G}_g(\mathbb{P}^4)$ be a decorated graph. Let us fix an edge $f \in E(\Gamma)$:

- if $\Gamma$ is connected after deleting $f$, denote the resulting graph by $\Gamma^0_f \in \mathcal{G}_{g-1,2}(\mathbb{P}^4)$,

- if $\Gamma$ is disconnected after deleting $f$, denote the resulting two graphs by $\Gamma^1_f \in \mathcal{G}_{g_1,1}(\mathbb{P}^4)$ and $\Gamma^2_f \in \mathcal{G}_{g_2,1}(\mathbb{P}^4)$

where $g = g_1 + g_2$.

There is no canonical order for the 2 new markings. We will always sum over the 2 labellings. So more precisely, the graph $\Gamma^0_f$ in case • should be viewed as sum of 2 graphs

$$\Gamma^0_{f,(1,2)} + \Gamma^0_{f,(2,1)}.$$ 

Similarly, in case ••, we will sum over the ordering of $g_1$ and $g_2$. As usually, the summation will be later compensated by a factor of $\frac{1}{2}$ in the formulas.
By Proposition 19, we have the following formula for the contribution of the graph $\Gamma$ to the stable quotient theory of $K\mathbb{P}^2$,

$$\text{Cont}_\Gamma = \frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}_{\geq 0}} \prod_{v \in V} \text{Cont}_\Gamma^A(v) \prod_{e \in E} \text{Cont}_\Gamma^A(e).$$

Let $f$ connect the $T$-fixed points $p_i, p_j \in \mathbb{P}^4$. Let the $A$-values of the respective half-edges be $(k, l)$. By Lemma 25, we have

$$\frac{\partial \text{Cont}_\Gamma^A(f)}{\partial Y} = (-1)^{k+l+1} \frac{R_{1-k-1} R_{1-l-1}}{5L^3 \lambda_i^{k-2} \lambda_j^{l-2}}.$$  

If $\Gamma$ is connected after deleting $f$, we have

$$\frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}_{\geq 0}} \left( -\frac{5L^5}{C_0^2 C_1^2} \right) \frac{\partial \text{Cont}_\Gamma^A(f)}{\partial Y} \prod_{v \in V} \text{Cont}_\Gamma^A(v) \prod_{e \in E, e \neq f} \text{Cont}_\Gamma^A(e) = \text{Cont}_\Gamma^A(H, H).$$

The derivation is simply by using (33) on the left and Proposition 20 on the right.

If $\Gamma$ is disconnected after deleting $f$, we obtain

$$\frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}_{\geq 0}} \left( -\frac{5L^5}{C_0^2 C_1^2} \right) \frac{\partial \text{Cont}_\Gamma^A(f)}{\partial Y} \prod_{v \in V} \text{Cont}_\Gamma^A(v) \prod_{e \in E, e \neq f} \text{Cont}_\Gamma^A(e) = \text{Cont}_\Gamma^A(H) \text{Cont}_\Gamma^A(H)$$

by the same method.

By combining the above two equations for all the edges of all the graphs $\Gamma \in G_g(\mathbb{P}^4, d)$ and using the vanishing

$$\frac{\partial \text{Cont}_\Gamma^A(v)}{\partial Y} = 0$$

of Lemma 24, we obtain

$$\left( -\frac{L^5}{5C_0^2 C_1^2} \right) \frac{\partial \langle \rangle_{g, 0}^{\text{SQ}}}{\partial Y} = \frac{1}{2} \sum_{i=1}^{g-1} \langle H \rangle^\text{SQ}_{g-i, i} \langle H \rangle^\text{SQ}_{i, i} + \frac{1}{2} \langle H, H \rangle^\text{SQ}_{g-1, 2}.$$  

We have followed here the notation of Section 0.1.
By definition of $A_2, A_1, A_6$, we have following equations.

\[
\frac{1}{C_0^2 C_1^2} \frac{\partial}{\partial A_2} - \frac{K_2}{5C_1^2} \frac{\partial}{\partial A_4} + \frac{K_2^2}{50C_1^2} \frac{\partial}{\partial A_6} \mathcal{Y} = -5L^5,
\]

\[
\frac{1}{C_0^2 C_1^2} \frac{\partial}{\partial A_2} - \frac{K_2}{5C_1^2} \frac{\partial}{\partial A_4} + \frac{K_2^2}{50C_1^2} \frac{\partial}{\partial A_6} \mathcal{X}_1 = 0,
\]

\[
\frac{1}{C_0^2 C_1^2} \frac{\partial}{\partial A_2} - \frac{K_2}{5C_1^2} \frac{\partial}{\partial A_4} + \frac{K_2^2}{50C_1^2} \frac{\partial}{\partial A_6} \mathcal{X}_2 = 0.
\]

Since $I_0^{2g-2} \langle q \rangle_{g-i}^{\text{SQ}} = \tilde{F}_g^B$, the left side of (34) after multiplication by $I_0^{2g-2}$ is, by the chain rule,

\[
\frac{1}{C_0^2 C_1^2} \frac{\partial}{\partial A_2} - \frac{K_2}{5C_1^2} \frac{\partial}{\partial A_4} + \frac{K_2^2}{50C_1^2} \frac{\partial}{\partial A_6} \in \mathbb{C}[L^\pm][C_0^\pm, C_1^{-1}, K_2, A_2, A_4, A_6].
\]

On the right side of (34), we have

\[
I_0^{2(g-i)-2+1} \langle H \rangle_{g-i,1}^{\text{SQ}} = \tilde{F}_g^B_{g-i,1}(q) = \tilde{F}_{g-i,1}^\text{GW}(Q(q)),
\]

where the first equality is by definition and the second is by wall-crossing (6). Then,

\[
\tilde{F}_{g-i,1}^\text{GW}(Q(q)) = \frac{\partial \tilde{F}_{g-i}^\text{GW}}{\partial T}(Q(q)) = \frac{\partial \tilde{F}_g^B}{\partial T}(q)
\]

where the first equality is by the divisor equation in Gromov-Witten theory and the second is again by wall-crossing (6). So we conclude

(35) \[ I_0^{2(g-i)-2+1} \langle H \rangle_{g-i,1}^{\text{SQ}} = \frac{\partial \tilde{F}_g^B}{\partial T}(q) \in \mathbb{C}[[q]]. \]

Similarly, we obtain

(36) \[ I_0^{2(g-i)-2+1} \langle H \rangle_{i,1}^{\text{SQ}} = \frac{\partial \tilde{F}_g^B}{\partial T}(q) \in \mathbb{C}[[q]], \]

(37) \[ I_0^{2(g-1)-2+2} \langle H, H \rangle_{g-1,2}^{\text{SQ}} = \frac{\partial^2 \tilde{F}_g^B}{\partial T^2}(q) \in \mathbb{C}[[q]]. \]

The above equations transform (34), after multiplication by $I_0^{2g-2}$, to exactly the first holomorphic anomaly equation of Theorem 2

\[
\frac{1}{C_0^2 C_1^2} \frac{\partial}{\partial A_2} - \frac{1}{5C_0^2 C_1^2} \frac{\partial}{\partial A_4} K_2 + \frac{1}{50C_0^2 C_1^2} \frac{\partial}{\partial A_6} K_2^2 = \frac{1}{2} \sum_{i=1}^{g-1} \frac{\partial \tilde{F}_g^B}{\partial T} \frac{\partial \tilde{F}_g^B}{\partial T} + \frac{1}{2} \frac{\partial^2 \tilde{F}_g^B}{\partial T^2}
\]

as an equality in $\mathbb{C}[[q]]$.

In order to lift the first holomorphic anomaly equation to the ring

$\mathbb{C}[L^\pm][A_2, A_4, A_6, C_0^\pm, C_1^{-1}, K_2]$, 

we must lift the equalities (35)-(37). The proof is identical to the parallel lifting for \(K\mathbb{P}^2\) given in [21, Section 7.3].

6.3. **Proof of Theorem 2: second equation.** By Proposition 19, we have the following formula for the contribution of the graph \(\Gamma\) to the stable quotient theory of formal quintic,

\[
\text{Cont}_\Gamma = \frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}_{\geq 0}} \prod_{v \in V} \text{Cont}^A_\Gamma(v) \prod_{e \in E} \text{Cont}^A_\Gamma(e).
\]

Let \(f\) connect the \(T\)-fixed points \(p_i, p_j \in \mathbb{P}^4\). Let the \(A\)-values of the respective half-edges be \((k, l)\). By Lemma 26, we have

\[
(38) \quad \frac{\partial \text{Cont}^A_\Gamma(f)}{\partial X} = (-1)^{k+l+1} \left( \frac{R_0 R_{k-1} R_{2l-1}}{5 L^3 \lambda_{k-1} \lambda_{l-3}} + \frac{R_{2k-1} R_{0l-1}}{5 L^3 \lambda_{k-3} \lambda_{l-1}} \right).
\]

- If \(\Gamma\) is connected after deleting \(f\), we have

\[
\frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}_{\geq 0}} \left( -\frac{5 L^5}{C^2_1} \right) \frac{\partial \text{Cont}^A_\Gamma(f)}{\partial X} \prod_{v \in V} \text{Cont}^A_\Gamma(v) \prod_{e \in E, e \neq f} \text{Cont}^A_\Gamma(e)
\]

\[= \text{Cont}_{\Gamma^f}(1, H^2).\]

The derivation is simply by using (38) on the left and Proposition 20 on the right.

- If \(\Gamma\) is disconnected after deleting \(f\), we obtain

\[
\frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}_{\geq 0}} \left( -\frac{5 L^5}{C^2_1} \right) \frac{\partial \text{Cont}^A_\Gamma(f)}{\partial X} \prod_{v \in V} \text{Cont}^A_\Gamma(v) \prod_{e \in E, e \neq f} \text{Cont}^A_\Gamma(e)
\]

\[= \text{Cont}_{\Gamma^f}(1) \text{Cont}_{\Gamma^f}(H^2)\]

by the same method.

By combining the above two equations for all the edges of all the graphs \(\Gamma \in \mathcal{G}_g(\mathbb{P}^4, d)\) and using the vanishing

\[
\frac{\partial \text{Cont}^A_\Gamma(v)}{\partial X} = 0
\]

of Lemma 24 we obtain

\[
(39) \quad \left( -\frac{5 L^5}{C^2_1} \right) \frac{\partial}{\partial X} \langle \mathbb{S}^Q_{g,0} \rangle = \sum_{i=1}^{g-1} \langle 1 \rangle_{g-i,1} \langle H^2 \rangle_{i,1}^{\mathbb{S}Q} + \langle 1, H^2 \rangle_{g-1,2}^{\mathbb{S}Q} = 0.
\]
The second equality in the above equations follow from the string equation for formal stable quotient invariants. Since 

\[ K_2 = -\frac{1}{L^5} \mathcal{X}, \]

the equation (39) is equivalent to the second holomorphic anomaly equation of Theorem 2 as an equality in \( C[[q]] \).

In order to lift the second holomorphic anomaly equation to the ring 

\[ \mathbb{C}[L^{\pm 1}][A_2, A_4, A_6, C_0^{\pm 1}, C_1^{-1}, K_2], \]

we must lift the equalities

\[ \langle 1 \rangle_{g-1,1}^{\text{SQ}} = 0, \]
\[ \langle 1, H^2 \rangle_{g-1,2}^{\text{SQ}} = 0. \]

The proof follows from the properties of the unit 1 in a CohFT. Specifically, the method of the proof of [23, Proposition 2.12] is used. We leave the details to the reader.

6.4. Genus one invariants. We do not study the genus 1 unpointed series \( \tilde{F}_B^g(q) \) in the paper, so we take

\[ \begin{align*}
I_0 \langle H \rangle_{1,1}^{\text{SQ}} &= \frac{\partial \tilde{F}_B^1}{\partial T}, \\
I_0^2 \cdot \langle H, H \rangle_{1,2}^{\text{SQ}} &= \frac{\partial^2 \tilde{F}_B^1}{\partial T^2}.
\end{align*} \]

as definitions of the right side in the genus 1 case. There is no difficulty in calculating these series explicitly using Proposition 20

\[ \begin{align*}
\frac{\partial \tilde{F}_B^1}{\partial T} &= \frac{L^5}{C_1} \left( \frac{1}{2} A_2 + \frac{19}{24} K_2 + \frac{1}{12} - \frac{19}{120} \frac{1}{L^5} \right), \\
\frac{\partial^2 \tilde{F}_B^1}{\partial T^2} &= \frac{1}{C_1} D \left( \frac{L^5}{C_1} \left( \frac{1}{2} A_2 + \frac{19}{24} K_2 + \frac{1}{12} - \frac{19}{120} \frac{1}{L^5} \right) \right).
\end{align*} \]

6.5. Bounding the degree. For the holomorphic anomaly equation for \( K\mathbb{P}^2 \), the integration constants can be bounded [21, Section 7.5]. A parallel result hold for the formal quintic.

The degrees in \( L \) of the term of 

\[ \tilde{F}_g^{\text{SQ}} \in \mathbb{C}[L^{\pm 1}][A_2, A_4, A_6, K_2] \]

for formal quintic always fall in the range

\[ \begin{align*}
15 - 15g, 10g - 10.
\end{align*} \]
In particular, the constant (in $A_2, A_4, A_6, K_2$) term of $\tilde{F}^{SQ}_{g,missed by the holomorphic anomaly equation for formal quintic is a Laurent polynomial in $L$ with degree in the range (40). The bound (40) is a consequence of Proposition 19, the vertex and edge analysis of Section 5, and the following result.

**Lemma 28.** The degrees in $L$ of $R_{ip}$ fall in the range 

$$[-i, 4p + 1].$$

**Proof.** The proof for the functions $R_{0p}$ follows from the arguments of [25]. The proof for the other $R_{ip}$ follows from Lemma 17. □

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