On $k$-tree Containment Graphs of Paths in a Tree

Liliana Alcón$^{1,2}$ · Noemí Gudiño$^{1,2}$ · Marisa Gutierrez$^{1,2}$

Received: 18 January 2019 / Accepted: 20 July 2020 / Published online: 22 August 2020
© Springer Nature B.V. 2020

Abstract

A $k$-tree is either a complete graph on $k$ vertices or a graph that contains a vertex whose neighborhood induces a complete graph on $k$ vertices and whose removal results in a $k$-tree. If the comparability graph of a poset $P$ is a $k$-tree, we say that $P$ is a $k$-tree poset. In the present work, we study and characterize by forbidden subposets the $k$-tree posets that admit a containment model mapping vertices into paths of a tree ($CPT$ $k$-tree posets). Furthermore, we characterize the dually-$CPT$ and strong-$CPT$ $k$-tree posets and their comparability graphs. The characterizations lead to efficient recognition algorithms for the respective classes.

Keywords

Containment models · Comparability graphs · $k$-trees · $CPT$ posets

1 Introduction and Definitions

Different classes of posets have been defined by imposing geometric restrictions to the sets used in their containment models [2, 4, 5, 8, 16]. Posets admitting a containment model using intervals of a line, which are called $CI$ posets and are known to be the posets with dimension at most 2, are exactly the posets whose comparability graphs belong to the well understood class of permutation graphs [3]. In [1, 2], we have initiated the study of those posets that admit a containment model mapping vertices into paths of a tree, which are called $CPT$ posets and clearly constitute a superclass of $CI$ posets. We have found remarkable differences between $CI$ and $CPT$ posets. First, the dimension of $CI$ posets is bounded above by 2, but the dimension of $CPT$ posets is unbounded, this means that for every positive integer $d$ there exists some $CPT$ poset with dimension greater than $d$. Second, any
CI poset admits a CI model using non trivial paths, however, there exist CPT posets that require the use of trivial paths in any CPT model. And third, the fact of being a CI poset is a comparability invariant, but the fact of being CPT is not. Figure 1 illustrates the previous observations, the poset N^d is CPT. In any CPT representation of N^d the vertex labelled b has to be represented by a trivial path. The dual poset N is non CPT [2]. Determining classes of posets in which being CPT is an invariant of comparability and understanding the structure behind it is a challenging problem. In Section 4, we solve this problem within a subclass of CPT posets.

As opposed to the previous differences, both classes are hereditary, meaning that any subposet of a CI (resp. CPT) poset is also CI (resp. CPT). Consequently, they admit a characterization by a family of minimal forbidden subposets. It is well known that the forbidden structures for being CI are the 3-irreducible posets, i.e. the minimal posets with dimension 3 [9, 14]. Do the 3-irreducible posets admit a CPT representation? In other words, can the obstacle that does not allow a 3-irreducible poset to have a containment model using paths of line be overcome by relaxing the host structure of the model? In Section 2, answering this question, we determine which of the 3-irreducible posets are CPT. In addition, using them, we describe an infinite family of posets which are minimal forbidden subposets for being CPT. The complete family of minimal forbidden subposets needed to characterize the class of CPT posets is unknown.

Finally, focusing in the algorithmic aspects of the recognition problems, CI poset can be recognized in linear time [15]. Determining the time complexity of recognizing CPT posets is an open problem.

Accordingly, it is natural to ask whether there are interesting classes of posets where these problems can be solved efficiently. In [2], we consider successfully two classes: the posets whose comparability graphs are split (the vertices can be partitioned into a clique and a stable set), and the posets whose comparability graphs are trees. Continuing with that work, in Section 3 of the present paper, we investigate (and solve) the characterization and the recognition problem in the class of posets whose comparability graphs are k-trees. In opposition to the small diameter, unbounded treewidth case of split graphs, we turn here to k-trees that are also chordal graphs, but with unbounded diameter and bounded treewidth [3]. In Section 5, we pose several open problems.

1.1 Definitions

A partially ordered set or poset is a pair P = (X, P) where X is a finite non-empty set and P is a reflexive, antisymmetric, and transitive binary relation on X. As usual, we write

Fig. 1  Poset N is non CPT. Its dual N^d is CPT. In any CPT representation of N^d, the vertex labelled b has to be represented by a trivial path. Thus, the poset M obtained from N^d by adding b' is non CPT.
x ≤ y in P for (x, y) ∈ P, and x < y in P when (x, y) ∈ P and x ≠ y. If x < y or y < x, we say that x and y are comparable in P and write x ⊥ y. An element x is covered by y in P, denoted by x < y in P, when x < y and there is no element z ∈ X for which x < z and z < y. The down-set {x ∈ X : x < z} and the up-set {x ∈ X : z < x} of an element z are denoted by D(z) and U(z), respectively. We let D[z] = D(z) ∪ {z} and U[z] = U(z) ∪ {z}. When D(z) = ∅, we say that z is a minimal element of P, and that z is maximal when U(z) = ∅.

A chain in P is a subposet whose vertices are pairwise comparable. The height of P is the number of vertices in its maximum chain. The restriction of the relation P to a subset Y of X is denoted by P(Y). We use P(Y) to refer to the subposet (Y, P(Y)) of P. The dual of a poset P = (X, P) is the poset Pd = (X, Pd) where x < y in Pd if and only if y < x in P.

A containment model M_P of a poset P = (X, P) maps each element x of X into a set M_x in such a way that x < y in P if and only if M_x is a proper subset of M_y. We identify the containment model M_P with the set family (M_x)_{x ∈ X}.

A poset P = (X, P) is a containment order of paths in a tree, or CPT poset for brevity, if it admits a containment model where every W_x is a path of a tree T, which is called the host tree of the model. When T is a path, P is said to be a containment order of intervals or CI poset for short.

The comparability graph G_P of a poset P = (X, P) is the simple graph with vertex set V(G_P) = X and edge set E(G_P) = {xy : x ⊥ y}. A graph G is a comparability graph if there exists some poset P such that G = G_P. An undirected graph G = (V, E) admits a transitive orientation if and only if G admits a transitive orientation E if x̅y ∈ E and x̅z ∈ E, then x̅z ∈ E. The graphs whose edges can be transitively oriented are exactly the comparability graphs [7].

A set M ⊆ V is a module of a graph G = (V, E), (homogeneous set [6]) if and only if N(x) − M = N(y) − M for all x, y ∈ M. The whole set V and the singleton sets {x}, for any x ∈ V, are modules of G. These modules are called trivial modules. A graph G is prime if all its modules are trivial. Otherwise G is decomposable or degenerate.

A set M ⊆ X is a module of a poset P = (X, P) if M is a module of G_P. So, for all x, y ∈ M and v ∈ X − M, it is true that x ⊥ v in P if and only if y ⊥ v in P. The trivial modules of G_P, X and {x} with x ∈ X, are also the trivial modules of P. If all modules of a poset are trivial, we say that it is a prime poset. Otherwise, we say that it is decomposable.

If two posets are isomorphic, then their comparability graphs are also isomorphic. In general, the converse does not hold. We say that two posets are associated if their comparability graphs are isomorphic. In this paper, we do not distinguish between isomorphic posets (or graphs).

**Theorem 1 ([6])** Let P and P’ be associated posets. Then the following statements hold.

1. If P is prime, then P’ = P or P’ = Pd.
2. If S is a subposet of P and P is prime, then S or Sd is a subposet of P’.
3. If S is a subposet of P and S is prime, then S or Sd is a subposet of P’.

**2 Forbidden Structures for CPT Posets**

The following necessary condition for being a CPT poset was stated in [2].

**Lemma 2** If z is a vertex of a CPT poset P, then the subposet P(D(z)) induced by the down-set of z is CI.
The posets in Figs. 2 and 3 and their dual posets are said to be 3-irreducible. They form the family of forbidden induced subposets for being CI. This family was independently determined by Kelly [9], and Trotter and Moore [14]. A consequence of the previous lemma is that every 3-irreducible poset plus a least upper bound is a non CPT poset. This motivates the following definition.

**Definition 3** Given a poset $P$, we let $\hat{P}$ be the poset obtained by adding a maximum element to $P$. We name $Top_{3-irred}$ the set \{ $\hat{P}$ | with $P$ any 3-irreducible poset \}.

Therefore, the $Top_{3-irred}$ posets are forbidden subposets for being CPT. A question that naturally arises is whether the $Top_{3-irred}$ posets are minimal forbidden subposets. Furthermore, are the 3-irreducible posets CPT? In a first inspection, we found without much difficulty that almost all 3-irreducible posets are CPT.

**Lemma 4** Every 3-irreducible poset except $I_n$ with $n \geq 0$ is CPT.

**Proof** Let $P = (X, P)$ be a 3-irreducible poset, see Figs. 2 and 3. First we consider the case when $P$ has a maximal element $x$ covering a unique element $x'$. Observe that in such case $P$ is one of the following posets: $B^d, C, C^d, CX_1, CX_1^d, CX_2, EX_1, EX_1^d, EX_2 = EX_2^d, FX_1^d, E_n, F_n^d, F_n = F_n^d$ or $H_n = H_n^d$, with $n \geq 0$.

Since $P$ is a minimal non CI poset, $P - x$ admits a CI model $M$ on a host path $T$. Let $q \in V(T)$ be an end vertex of $W_x$. Let $T'$ be the tree obtained from $T$ by adding a pendant vertex $q'$ adjacent to $q$. Also, let $W_x$ be the path of $T'$ obtained by adding the vertex $q'$ to the path $W_x$. Clearly, $M$ plus the path $W_x$ is a CPT model of $P$ on the host tree $T'$.

When $P$ is one of the posets $B$ or $CX_2^d$, a similar argument can be applied using a minimal vertex $x$ covered by a unique vertex $x'$. Otherwise, if $P$ is none of the posets cited above nor

![Fig. 2](image-url) These posets, the ones in Fig. 3, and their dual posets constitute the 3-irreducible family
These posets, the ones in Fig. 2, and their dual posets constitute the 3-irreducible family $I_n$ for $n \geq 0$, then $P$ is one of the following posets: $D, D^d, CX_3, CX_3^d, FX_1, FX_2 = FX_2^d, A_n = A_n^d, I_n^d, G_n = G_n^d$ or $J_n = J_n^d$, with $n \geq 0$. A $CPT$ model of each one of these posets is shown in Figs. 4 and 5.

![Fig. 3](image)

$CPT$ models of 3-irreducible posets $D, D^d, CX_3, CX_3^d, FX_1, FX_2 = FX_2^d$

![Fig. 4](image)
Remark 5 Let $P$ be a poset whose comparability graph is the chordless path $[v_1, v_2, \ldots, v_n]$, $n \geq 4$. Consider any CI model $(W_{vi})_{1 \leq i \leq n}$ of $P$, and let $W_{vi}$ be the interval $[l_i, r_i]$ for $1 \leq i \leq n$. Then

- $l_1 < l_3 < l_5 < \ldots$, $r_1 < r_3 < r_5 \ldots$, $l_2 < l_4 < l_6 < \ldots$, and $r_2 < r_4 < r_6 < \ldots$; or

Fig. 6 Construction used in Lemma 6 to prove that the posets $I_n$ are non-CPT
• \( l_1 > l_3 > l_5 \ldots, \ r_1 > r_3 > r_5 > \ldots, \ l_2 > l_4 > l_6 > \ldots, \) and \( r_2 > r_4 > r_6 > \ldots. \)

In addition, when \( n \) is odd and \( v_1 < v_2 \), in the first case (second case, resp.), if \( I \) is any interval contained in \([l_2, r_{n-1}]\) (\([I_{n-1}, r_2]\), resp.), then \( I \) properly contains or is properly contained in some of the intervals \( W_{v_i} \) of the model.

**Lemma 6** For every \( n \geq 0 \), the 3-irreducible poset \( I_n \) in Fig. 3 is not CPT.

**Proof** In order to derive a contradiction, suppose there exists a CPT model \( (W_v)_{v \in V(I_n)} \) of \( I_n \). Label the vertices of \( I_n \) as in Fig. 6, and let \( A \) be the vertex set \( D(d_1) \cap D(d_2) \). The subposet \( I_n(A) \) has a CI model contained in the path \( U = W_{d_1} \cap W_{d_2} \). Let \( v \) and \( l_v \) be the right and left extremes, respectively, of the path \( W_v \) with \( v \in A \). Without loss of generality, we assume that \( l_{t_2} \) is nearer to \( l_U \) than \( l_{n+2} \) (see Fig. 6).

By Remark 5 and the fact that none of the paths \( W_v \) with \( v \in A - \{c\} \) can either contain or be contained in \( W_c \), we can conclude that \( l_c \) is between \( l_U \) and \( l_{t_2} \), or \( r_c \) is between \( r_{n+2} \) and \( r_U \). By symmetry, we can conclude without loss of generality that \( l_c \) is between \( l_U \) and \( l_{t_2} \). This implies that \( r_c \) is between \( l_c \) and \( r_{s_1} \).

Moreover, the path \( W_{t_1} \) must contain \( W_{d_1} \), contain \( W_{s_1} \) and not be contained in \( U \), so \( l_U \) is an interior point of the path \( W_{t_1} \). Thus, \( W_c \) turns out to be contained in \( W_{t_1} \), contradicting the fact that \( c \) and \( t_1 \) are incomparable. \( \square \)

**Lemma 7** The Top\(_{3-irred} \) posets are forbidden subposets for being CPT. All of them except \( \hat{I}_n \) with \( n \geq 0 \) are minimal.

**Proof** The first statement is a direct consequence of Lemma 2. Let \( P = (X, P) \) be a 3-irreducible poset other than \( \hat{I}_n \) and let \( u \) be the maximum of \( \hat{P} \). By Lemma 4, \( \hat{P} - u = P \) is CPT. Now, let \( x \in X \). Since \( P \) is 3-irreducible, then \( P - x \) is CI. Thus, \( P - x \) plus any maximum element is CI, which implies that \( \hat{P} - x \) is CPT. \( \square \)

In addition to the examples in the previous lemma, we have proved that the posets \( I_n \) for \( n \geq 0 \) and the posets \( N^d \) and \( M \) in Fig. 1 are forbidden subposets for the class CPT. While describing the complete list of forbidden subposets remains as an open problem, the next theorem shows that the family Top\(_{3-irred} \) provides a complete characterization by forbidden subposets of CPT posets within any class in which the given necessary condition to be CPT is also sufficient. Later, we will use it in the particular case of k-tree posets.

**Theorem 8** Let \( \mathcal{A} \) be any class of posets in which the necessary condition to be CPT given by Lemma 2 is also sufficient. A poset \( P \in \mathcal{A} \) is CPT if and only if \( P \) is Top\(_{3-irred} \)-free.

**Proof** By Lemma 7, any CPT poset is Top\(_{3-irred} \)-free. Conversely, if \( P \) is Top\(_{3-irred} \)-free, then the down-set of each vertex is CI, i.e. \( P \) satisfies the necessary condition to be CPT. Since the necessary condition is also sufficient in \( \mathcal{A} \), we have that \( P \) is CPT. \( \square \)

Notice that if \( \mathcal{A} \) is a hereditary class, then in Theorem 8, Top\(_{3-irred} \) can be replaced by Top\(_{3-irred} \cap \mathcal{A} \).
2.1 CPT Representations with no Trivial Paths

As we noted in the introduction, there exist posets that do not admit a CPT model unless some vertex is represented by a trivial path. Actually, the only minimal example of such a poset that we have found is the poset $N$ in Fig. 1. Sometimes, in the process of modifying a CPT representation to justify the existence of some other particular representation, the possible existence of such vertices demands a special treatment. The following technical lemma will be used in the proof of Theorem 17.

Lemma 9 Let $P$ be a CPT poset. If $P$ admits a CPT model in which no path is reduced to a vertex, then $P$ admits a CPT model in which no path is reduced to a vertex and no two paths of the model have a common end vertex.

Proof Let $M = (W_x)_{x \in X}$ be a CPT model of $P$, and assume that each $W_x$ contains at least two vertices of the host tree $T$. We will proceed by induction on the number of coincident ends. If there are no common ends, there is nothing to be proved.

Let $W_v$ be a shortest path between those sharing at least one of its end vertices with other path of the model. Let $q \in V(T)$ be one shared end vertex of $W_v$. Let $q_1, q_2, \ldots, q_k$ denote the neighbors of $q$ in $T$. Since no path is reduced to a single vertex, we have that $W_v$ must contain some other vertex $q_i$, say $q_1$. To obtain a new CPT model of $P$, we will consider two cases.

First assume no other path of the model is equal to $W_x$. Then, we proceed by subdividing the edge $qq_1$ of $T$ with a vertex $q'_1$ and doing the same subdivision in all the paths of $M$ containing that edge. After that, replacing the path $W_v$ with the path $W'_v$ which is obtained by removing from $W_v$ the extreme vertex $q$ (in this case the new end of $W'_v$ turns out to be $q'_1$; $W'_v$ is not reduced into a vertex). The other paths of $M$ are not modified.

Since in the new model, the only path that contains the vertex $q'_1$ is $W'_v$, it is clear that the number of paths with coincident ends has decreased. We claim that the proper containment relation between the paths has not been modified. Indeed, the only changes that could occur are that either $W'_v$ is properly contained in a path in which $W_v$ was not, which clearly is not possible; or that $W'_v$ does not contain properly a path $W_x$ that $W_v$ did contain. This can not happen because in that case $q$ has to be end vertex of $W_x$ and $W_x$ has to be shorter than $W_v$, contradicting its choice.

Now suppose that some paths of the model are exactly like $W_v$. By identifying all them, we can assume that $W_u$ is the only such path. Let $p \in V(T)$ be the other end vertex of $W_v$. By subdividing edges if necessary we can assume that $W_v$ contains at least two edges $qq_1$ and $p_1p$. Then, we proceed by subdividing the edges $qq_1$ and $p_1p$ of $T$ with new vertex $q'_1$ and a new vertex $p'_1$, respectively, and doing the same subdivision in all the paths of $M$ containing that edges. Thereafter, replacing the path $W_v$ with the path $W'_v$ which is obtained by removing from $W_v$ the extreme vertex $q$ (in this case the new end of $W'_v$ turns out to be $q'_1$ and $W'_v$ is not reduced into a vertex), and also replacing the path $W_u$ with the path $W'_u$ which is obtained by removing from $W_u$ the extreme vertex $p$ (in this case the new end of $W'_u$ turns out to be $p'_1$ and $W'_u$ is not reduced into a vertex). The other paths of $M$ are not modified.

Clearly, the number of common end vertices between the paths of this new model is less than in the model $M$. Let us show that the proper containment relation did not change. This is clear between $W'_v$ and $W'_u$. Then, by symmetry and because the paths $W'_v$ and $W'_u$ are the only ones that were modified, it is enough to see what happen between $W'_v$ and $W'_x$ for $x \neq v, u$. Since $W_v$ and $W_u$ were shortened to $W'_v$ and $W'_u$ respectively, we have that any
path of the model that properly contains \( W_v \) (resp., \( W_u \)) also contains \( W'_v \) (resp., \( W'_u \)). On the other hand, if \( W_v \) properly contains a path \( W_x \), by election of \( v \), \( W_x \) and \( W_v \) do not share an end vertex, so \( W'_v \) contains \( W'_x \), and the proof is complete. \( \square \)

### 3 The Class of \( k \)-tree Posets

A graph \( G \) is a \( k \)-tree if it can be built recursively from a complete graph on \( k \) vertices by adding in each step a new vertex adjacent to exactly \( k \) neighbors which in turn induce a complete subgraph [12]. The 1-tree graphs are the trees, i.e. the connected graphs without cycles. The class of \( k \)-tree graphs has been widely studied [3]. In this paper, we introduce the naturally related notion of \( k \)-tree poset.

**Definition 10** A poset \( P \) is a \( k \)-tree if its comparability graph \( G_P \) is a \( k \)-tree graph.

Just like \( k \)-tree graphs, \( k \)-tree posets can be defined recursively.

**Construction process 11** Given a poset \( P' = (X', P') \), a new poset \( P \) is obtained by adding a vertex \( z \) to \( X' \) through one of the following procedures.

\( (u) \) Chose \( a \in X' \) such that \( D[a] \) induces a chain of size \( k \) and do \( a <_{z}: z \); or
\( (d) \) chose \( a \in X' \) such that \( U[a] \) induces a chain of size \( k \) and do \( z <_{a}: a \); or
\( (m) \) chose \( a, c \in X' \) such that \( D[c] \cup U[a] \) induces a chain of size \( k \) and do \( c <_{z}: z <_{a} \).

Clearly the given poset \( P' \) is a subposet of the new poset \( P \).

**Lemma 12** A poset \( P \) is a \( k \)-tree if and only if \( P \) can be obtained from a chain of \( k \) elements by repeatedly applying the construction process 11.

**Proof** Clearly any poset built from a \( k \)-chain using the described procedure is a \( k \)-tree. Thus, let \( P = (X, P) \) be any \( k \)-tree poset and let us show by induction on \( |X| \) that \( P \) can be obtained using the construction process 11.

If \( |X| = k \), then \( P \) itself is a chain on \( k \) vertices; and if \( |X| = k + 1 \), then \( P \) is obtain using procedure \( (i) \). So let \( |X| > k + 1 \).

Since \( G_P \) is a \( k \)-tree, there is a vertex \( z \in X \) such that \( G' = G_P - z \) is a \( k \)-tree, and \( N(z) = \{z_1, z_2, \ldots, z_k\} \) is a clique. Therefore the \( k \)-tree poset \( P' = (X - \{z\}, P(X - \{z\})) \), can be obtained using the construction process 11; in what follows we will show that \( P \) can be obtained from \( P' \) in one more step. Since every maximal clique of a \( k \)-tree on at least \( k + 1 \) vertices has size \( k + 1 \), then there exists \( u \in X - N[z] \) such that \( N(z) \cup \{u\} = \{z_1, z_2, \ldots, z_k, u\} \) induces a maximal chain in \( P' \). We will consider three cases.

Vertex \( u \) is maximal in \( P' \): in such a case we can assume \( z_1 < z_2 < \ldots < z_k < u \). Since \( z \parallel u \) and \( z_k \perp z \), we have \( z_k < z \); which implies that \( D[z_k] \) is exactly the \( k \)-chain \( z_1 < z_2 < \ldots < z_k \). We conclude that \( z \) can be added by procedure \( (u) \) doing \( z_k <_{z} \).

Vertex \( u \) is minimal in \( P' \): in such a case we can assume \( u < z_1 < z_2 < \ldots < z_k \). Since \( z \parallel u \) and \( z \perp z_1 \), we have that \( z < z_1 \); which implies that \( U[z_1] \) is exactly the \( k \)-chain \( z_1 < z_2 < \ldots < z_k \). Hence, \( z \) can be added by procedure \( (d) \) doing \( z <_{z} z_1 \).

Otherwise, we can assume \( z_1 < z_2 < \ldots < z_{\ell} < u < z_{\ell+1} < \ldots < z_k \) for some \( 1 \leq \ell \leq k \). Since \( z \parallel u \), \( z \perp z_\ell \) and \( z \perp z_{\ell+1} \), we have that \( z_\ell < z < z_{\ell+1} \), which implies that \( D[z_\ell] \cup U[z_{\ell+1}] \) is exactly the \( k \)-chain \( z_1 < z_2 < \ldots < z_\ell < z_{\ell+1} < \ldots < z_k \). Therefore, \( z \) can be added by procedure \( (m) \) doing \( z_{\ell} <_{z} z <_{z} z_{\ell+1} \). \( \square \)
Definition 13 The elements of a $k$-tree poset $P$ can be totally ordered as $z_1, z_2, \ldots, z_n$ in accordance with the order in which they appear in a given construction process. The first $k$ elements are the ones in the initial chain, so $z_1 < z_2 < \ldots < z_k$ in $P$. Such a total order of the vertices is called a construction sequence of $P$. The construction sequence is called good when $z_j < z_r$ in $P$ for all $j$ with $r < j \leq n$, where $z_r$ is the last element in the sequence that is a maximal element of $P$. 

The following remark is a direct consequence of the construction process itself and the fact that every maximal clique of a $k$-tree graph on at least $k + 1$ vertices has size $k + 1$.

Remark 14 Let $z_1, z_2, \ldots, z_k, \ldots, z_n$ be a construction sequence of $P$. Then,

(1) Since the first vertex $z_{k+1}$ can always be added using procedure $(u)$, we will assume $z_k < z_{k+1}$.

(2) An element $z_j$ is maximal in $P$ if and only if it was added by procedure $(u)$.

(3) An element $z_j \neq z_1$ is minimal in $P$ if and only if it was added by procedure $(d)$.

(4) If $z_j$ was added by procedure $(d)$, then there is a unique $i < j$ such that $z_i$ is maximal and $z_j < z_i$ in $P$.

(5) If $z_j$ was added by procedure $(m)$, then there is a unique $i < j$ such that $z_i$ is maximal and $z_j < z_i$ in $P$.

Next, we will show that every $k$-tree poset admits a good construction sequence. An example is shown in Fig. 7.

Lemma 15 Let $P$ be a $k$-tree poset. There exists a construction sequence $z_1, z_2, \ldots, z_n$ of $P$ such that if $r = \max \{i : z_i$ is maximal in $P\}$, then $z_i \in D(z_r)$ for all $i \geq r$.

Proof We will proceed by induction on $n$, the number of elements of $P$. Let $z_1, z_2, \ldots, z_n$ be any construction sequence of $P$. If $z_n$ is maximal, then the result is trivial. Hence, we can assume that $n \geq k + 3$ and that $z_n$ was added using procedure $(m)$ or $(d)$. In any case, $z_n$ is comparable with exactly $k$ elements which together with an element $u \parallel z_n$ induce a maximal $k+1$-chain $C$ of $P$. Let $w$ be the maximum of $C$, we have $u < w$ and $z_n < w$. Let $z_1', z_2', \ldots, z_{n-1}'$ be a construction sequence of $P' = P - \{z_n\}$ such that $z_i' \in D(z_i')$ for all $i > t$ where $z_t'$ is the last maximal element in this construction sequence. Keep in mind that the down and the up-set of every vertex is totally independent of the construction sequence. Clearly, if $z_n < z_t'$, then $z_n$ can be added at the end of the construction process of $P'$ and the proof follows.

Thus assume $z_n \notin D(z_i')$ and let $I' \subseteq \{1, 2, \ldots, n - 1\}$ such that the vertices $z_i'$ with $i \in I'$ are the ones in the maximal chain $C$. We claim that $i < t$ for every $i \in I'$, what allows to add in the process of $P'$ the vertex $z_n$ immediately before $z_i'$, finishing the proof. Indeed, let $s \in I'$ such that $w = z_s'$. Since $z_n < w$ and $z_n \notin z_i'$, then $w \neq z_i'$ and so $s < t$. In addition, any vertex $z_i'$ with $i \in I' - \{s\}$ was added using procedure $(m)$ or $(d)$. Thus, if $i > t$, then $s < t < i$ and the two maximal elements ($w = z_s'$ and $z_i'$) are greater than $z_i'$ in contradiction with either (4) or (5) of Remark 14. We concluded that $i < t$ and the proof follows.

The following lemma supplies the key to the proof of the main Theorem 17.

Lemma 16 Let $z_r$ be the last maximal element in a good construction sequence $z_1, z_2, \ldots, z_n$ of $P$; $n \geq k + 1$. Then,
Clearly if one of the vertices belong to poset $P$, then $z_s < z_j$ and $z_s \in C$.

Proof Since $z_r$ is maximal, then it has label $(u)$. Therefore, (i) and (ii) for $j = r$ hold trivially.

To prove (ii) for $r < j$, first recall that in a good order, $r < j$ implies $z_j < z_r$. Let $t \leq s$ be such that $z_t$ is maximal and $z_s \leq z_t$. If $z_j < z_s$, then $z_j < z_t$, which implies that $z_t$ and $z_r$ are maximal elements greater than $z_j$ in contradiction with either (4) or (5) of Remark 14. Thus $z_s < z_j$, and so $z_s \in D(z_r)$. By item (i), $z_s \in C$.

**Theorem 17** A $k$-tree poset $P$ is CPT if and only if $P(D[z])$ is CI for each maximal element $z$ of $P$. Moreover, in such a case, $P$ admits a CPT model with no path reduced to a single vertex and no two paths with a common end vertex.

Proof The necessary condition follows from Lemma 2. So let $P = (X, P)$ be a $k$-tree such that $D[z]$ induces a CI poset for any maximal element $z$. We will prove by induction on $|X|$ that $P$ admits a CPT model such that no path is reduced to a vertex and no two paths have a common end vertex.

If $|X| = k$ or $k + 1$, then $P$ is a chain and the proof is trivial. Accordingly, we assume $|X| \geq k + 2$.

By Lemma 15, there is a good construction sequence $z_1, z_2, \ldots, z_n$ of $P$. Let $z_r$ be the last maximal element in this order and let $X'$ be the vertex set $\{z_1, z_2, \ldots, z_{r-1}\}$. The $k$-tree poset $P' = P(X')$ admits a CPT model $M' = (W'_x)_{x \in X'}$ such that no path is reduced to a single vertex and no two paths have a common end vertex. On the other hand, by hypothesis, $P'' = P(D[z_r])$ is C1. Thus $P''$ admits a C1 model $M'' = (W''_x)_{x \in D[z_r]}$ such that no path is reduced to a vertex and no two paths have coincident ends [4].

Next, we will obtain a CPT model of $P$ using $M'$ and $M''$. By Lemma 16 (i), $X' \cap D[z_r]$ is a chain $C$ of size $k$. Therefore, the vertices of $C$ are represented in $M'$ and in $M''$. Notice that in both models the paths do not have coincident ends and no path is reduced to a vertex. Consequently, by subdividing edges where necessary, we can make sure that for each vertex $x$ of the chain $C$, the paths $W'_x$ and $W''_x$ in $M'$ and $M''$, respectively, are equal.

Let $T'$ be the host tree of the model $M'$. Let $T$ denote the tree obtained from $T'$ by adding two new branches, one in each end vertex of $W''_w$, where $w$ is the top element of the chain $C$. Let $M$ be the model on the host tree $T$ obtained from $M'$ by adding (along the path defined by the two new branches) the model $M''$ in such a way that, for each vertex $x$ in the chain $C$, the paths $W'_x$ and $W''_x$ are coincident and become a single path which will be denoted by $W_x$. See an example in Fig. 7.

Thus we have the path family $M = (W_x)_{x \in X}$ on the host tree $T$ with

$$W_x = \begin{cases} W'_x, & \text{if } x \in X' - C; \\ W''_x, & \text{if } x \in D[z_r] - C; \\ W'_x = W''_x, & \text{if } x \in C. \end{cases}$$

Now we will prove that $M$ is a CPT model of $P$. Let $x$ and $y$ be any two vertices of $P$. Clearly if one of the vertices belong to $C$, or both vertices belong to $X' - C$, or both vertices belong to $D[z_r] - C$, then $x < y$ in $P$ if and only if $W_x \subseteq W_y$. So we just need to see what happens when one vertex is in $X' - C$ and the other is in $D[z_r] - C$, say $x = X' - C$ and $y \in D[z_r] - C$. Therefore, $x = z_s$ for some $s < r$ and $y = z_t$ for some $t \geq r$. By Lemma 16, $z_s \parallel z_r$ in $P$.

Suppose $W_{z_s} \subseteq W_{z_t}$, then $W'_{z_s} \subseteq W''_{z_t}$, which implies $W_{z_s} \subseteq W''_w$ (see Fig. 7); and so $s < w$ in $P$, in contradiction with the fact that $z_s \notin C$. 

© Springer
Now we will show that $W_{zt} \subset W_{zs}$. If $t = r$, then the proof is clear since $W_{zr}$ is contained in no path of the model $M$.

If $t > r$ and $W_{zt} \subset W_{zs}$, then $W''_{zt} \subset W''_{zs}$, which implies $W''_{zt} \subset W''_{w}$ and so $z_t < w$ in $P$ in contradiction with Lemma 16.

Clearly, no path of the model $M$ is reduced to a single vertex. Therefore by the Lemma 9, $P$ admits a CPT model such that no path is reduced to a vertex and no two paths have a common end vertex; which concludes the inductive proof. 

The following corollary, stated for emphasis, follows immediately from Theorems 8 and 17.

**Corollary 18** A $k$-tree poset is CPT if and only if it is Top$_3$-irred-free.

Another significant consequence of the previous theorem is the existence of an efficient algorithm for recognizing CPT $k$-tree posets.

**Corollary 19** Determining whether a given poset is CPT and $k$-tree can be done in polynomial time.
Lemma 21  Let it is not known whether the two classes are distinct [2].

CPT respectively. It is clear that strong-class is contained in dually-
graphs CPT

P

iv)

Theorem 20  Let B, C, E\textsubscript{n} and F\textsubscript{n} be the posets in Figs. 2 and 3.

i) A 2-tree poset P is CPT if and only if P does not have \( \widehat{B} \) or \( \widehat{B}^d \) as subposet.

ii) A 3-tree poset P is CPT if and only if P does not have \( \widehat{B}, \widehat{B}^d, \widehat{C}, \widehat{C}^d, E\textsubscript{n}, \) or \( \widehat{E}\textsubscript{n} \) with \( n \geq 0 \) as subposet.

iii) A \( k \)-tree poset P with \( k \geq 4 \) is CPT if and only if P does not have \( \widehat{B}, \widehat{B}^d, \widehat{C}, \widehat{C}^d, E\textsubscript{n}, \) \( \widehat{E}\textsubscript{n} \) with \( n \geq 0 \) as subposet.

Proof  Let \( \widehat{H} \) be a \( Top_{3-irred} \) subposet of a \( k \)-tree poset P. Since \( k \)-tree graphs are chordal, we have that \( k \)-tree posets are acyclic. Therefore, \( H \) is none of the following posets nor their respective dual posets: \( D, CX_1, CX_2, CX_3, EX_1, EX_2, FX_1, FX_2, A_n, I_n, G_n, J_n, H_n \), with \( n \geq 0 \).

If P is 2-tree, then \( h(P) = 3 \), thus \( h(H) < 3 \). It implies that H is either B or \( B^d \).

Analogously, if P is 3-tree, then \( h(H) < 4 \) and the proof follows. \( \square \)

4  Dually and Strong-CPT Subclasses

A poset P is dually-CPT if it is CPT and \( P^d \) is also CPT. If P is CPT and every poset associated with P is also CPT, then P is strong-CPT. The dually-CPT graphs and strong-CPT graphs are the comparability graphs of dually-CPT posets and strong-CPT posets, respectively. It is clear that strong-CPT class is contained in dually-CPT class. However, it is not known whether the two classes are distinct [2].

Lemma 21  Let P and \( P' \) be associated posets. If H is a prime poset and \( \widehat{H} \) is subposet of P, then \( \widehat{H}, \widehat{H}^d, \widehat{H}^d \) or \( \widehat{H}^{d-d} \) is subposet of \( P' \).

Proof  By item 3 of Theorem 1, either H or \( H^d \) is subposet of \( P' \). In the former case, since \( \widehat{H} \) is subposet of P, we have that in \( P' \) there is a vertex comparable with every vertex of H, say u. Since H is prime, then either u is greater than every vertex of H or u is smaller that every vertex of H, which means that \( \widehat{H} \) or \( \widehat{H}^{d-d} \) is subposet of \( P' \). In the latter case, in an analogous way, we have that \( \widehat{H}^d \) or \( \widehat{H}^{d-d} \) is subposet of \( P' \). \( \square \)

Theorem 22  Let \( P = (X, P) \) be a \( k \)-tree poset with \( k \geq 2 \). The following statements are equivalent.

i) P is strong-CPT.

ii) P is dually-CPT.

iii) P does not contain \( \widehat{B}, \widehat{B}^d, \widehat{C}, \widehat{C}^d, E\textsubscript{n}, \widehat{E}\textsubscript{n}, \widehat{F}\textsubscript{n} \) with \( n \geq 0 \), or their respective dual posets as subposet.
For \( k = 2, 3 \) the given family of forbidden subposets is not minimal. For \( k = 2 \), the list of posets in iii) can be reduced to only \( \widehat{B} \) and \( \widehat{B}^d \). For \( k = 3 \), \( \widehat{F}_n \), with \( n \geq 0 \), can be removed from the list.

**Proof** Clearly, any strong-\( CPT \) poset is dually-\( CPT \). If \( P \) is dually-\( CPT \), then, by definition of dually-\( CPT \) and Theorem 20, none of the posets \( \widehat{B}, \widehat{B}^d, \widehat{C}, \widehat{C}^d, \widehat{E}_n, \widehat{E}_n^d, \widehat{F}_n \), with \( n \geq 0 \), nor their respective dual posets can be subposets of \( P \). So we have proved iii) implies iii).

Let \( P \) be a \( k \)-tree poset satisfying iii). By Theorem 20, \( P \) is \( CPT \). Assume, to derive a contradiction, that \( P \) is non strong-\( CPT \), thus there exist a poset \( P' \) associated with \( P \) which is non \( CPT \). Notice that \( P' \) is a \( k \)-tree, then, by Theorem 20, \( P' \) contains \( \widehat{B}, \widehat{B}^d, \widehat{C}, \widehat{C}^d, \widehat{E}_n, \widehat{E}_n^d, \widehat{F}_n \), with \( n \geq 0 \), as subposet. Since every 3-irreducible poset is prime [9], the proof follows by Lemma 21.

**Theorem 23** Let \( G \) be a comparability \( k \)-tree graph. The following conditions are equivalents.

i) \( G \) is strong-\( CPT \).

ii) \( G \) is dually-\( CPT \).

iii) \( G \) does not contain neither \( G_{\widehat{B}}, G_{\widehat{C}}, G_{\widehat{E}_n} \) nor \( G_{\widehat{F}_n} \) with \( n \geq 0 \) as induced subgraph.

For \( k = 2, 3 \) the given family of forbidden subgraphs is not minimal. For \( k = 2 \), the list of graphs in iii) can be reduced to \( G_{\widehat{B}} \) and \( G_{\widehat{B}^d} \). For \( k = 3 \), \( G_{\widehat{F}_n} \), with \( n \geq 0 \), can be removed from the list.

**Proof** By Theorem 22, i) and ii) are equivalent, and iii) implies i).

Let \( G = G_P \) with \( P \) strong-\( CPT \). Assume, to derive a contradiction, that \( G_{\widehat{H}} \) is an induced subgraph of \( G \), where \( H \) is any one of the posets \( B, C, E_n \) or \( F_n \). Then \( P \) contain a subposet associated with \( \widehat{H} \). Since \( H \) is prime the only posets associated with \( \widehat{H} \) are \( \widehat{H}, \widehat{H}^d, \widehat{H}^d \) and \( \widehat{H}^{dd} \). In any case, since \( P \) is strong-\( CPT \), we have a contradiction with Theorem 22. We have proved that i) implies iii).

**5 Open Problems**

Several problems related to \( CPT \) representations of posets remain open. The time complexity of the recognition problem of \( CPT \) posets is unknown.

Although a few minimal non \( CPT \) posets have been depicted, the whole family that would allow a total characterization of \( CPT \) posets by forbidden subposets has not been fully described. We have solved these issues restricted to the class of split posets and to the class of \( k \)-tree posets. However, we have not obtained a characterization by forbidden subgraphs of \( CPT \) split graphs or \( CPT \) \( k \)-tree graphs. An extra complexity in characterizing \( CPT \) graphs by forbidden structures is the fact that being \( CPT \) is not a comparability invariant. This means that a same graph may be the comparability graph of a poset that is \( CPT \) and of another poset which is non \( CPT \).

In the process of reviewing the present paper, one of the anonymous referees provided a simple proof of the following lemma.
Lemma Let $x$ and $y$ be distinct maximal elements in a poset $P$ with $D_P(x) \subseteq D_P(y)$. Suppose $D_P(x) \not\subseteq D_P(z)$ for every element $z \neq x, y$. Let $P'$ be the poset obtained from $P$ by removing $y$ and adding a new maximal element $y'$ so that $D_{P'}(y') = \{x\} \cup D_P(y)$. If $P$ is a CPT poset, then $P'$ is also a CPT poset.

Let $P$ be the poset obtained from the 3-irreducible poset $A_n$ by adding a new maximal element $y$ whose down-set contains all the minimal elements of $A_n$ and any arbitrary subset of the maximal elements of $A_n$. By applying the previous lemma as many times as necessary, the non CPT poset $A_n$ is obtained. Therefore, $P$ is non CPT. Even more, $P$ is minimal non CPT. Notice that this implies that there are exponentially many different minimal forbidden subposets for the class CPT that have the same structure as the poset $N$.

In addition, we have observed that new forbidden subposets can be obtained by applying the previous reasoning to some other 3-irreducible posets. However, for instance, the minimal non CPT poset $M$ in Fig. 1 cannot be obtained from a 3-irreducible poset by applying the previous lemma. We conjecture there are other forbidden subposet that are obtained by adding an element below some other element which need to be represented by a trivial path. For short, call singular the CPT posets that require the use of some trivial path in any of its CPT representation. A challenging open problem is determining the minimal singular CPT posets.

It is not known whether the classes of strong-CPT posets and dually-CPT posets are distinct. We have shown they are equal when intersected with the split class or with the $k$-tree class. Also, we have been able to demonstrate that a non singular CPT poset is strong if and only if it is dually (manuscript in preparation).

In [11] it is proved that any poset whose comparability graph is chordal has dimension at most 4. The fact that the inequality is tight was proved by Kierstead and Trotter in [10]. It is not known whether there are 4-dimensional $k$-tree posets.

Finally, in [2], we proved that for every positive integer $d$, there exists a CPT poset with dimension $d$. As opposed, we also showed that the dimension of a CPT poset is bounded above by the number of leaves of the host tree used in any of its CPT representations. It is not known whether dimension is bounded for either the class of dually-CPT posets or the class of strong-CPT posets.

5.1 Acknowledgement

Acknowledgement We want to express our deep gratitude to the anonymous reviewer who carefully read our work. Their comments and suggestions have been very useful to enrich the present manuscript.

References

1. Alcón, L., Gudiño, N., Gutierrez, M.: On containment graphs of paths in a tree. Electron. Notes Discret. Math. 50, 175–180 (2015)
2. Alcón, L., Gudiño, N., Gutierrez, M.: Recent results on containment graphs of paths in a tree. Discret. Appl. Math. 245, 139–147 (2018)
3. Brandstädt, A., Le, V., Spinrad, J.: Graph Classes: a Survey. Society for Industrial and Applied Mathematics, Philadelphia (1999)
4. Dushnik, B., Miller, E.: Partially ordered sets. Am. J. Math. 63, 600–610 (1941)
5. Fishburn, P.C., Trotter, W.T.: Geometric containment orders: a survey. Order 15, 167–182 (1999)
6. Gallai, T.: Transitiv orientierbare graphen. Acta Mathematica Hungarica 18, 25–66 (1967)
7. Ghouila-Houri, A.: Caractérisation des graphes non orientés dont on peut orienter les arêtes de manière à obtenir le graphe d’une relation d’ordre. C.R. Acad. Sci. Paris 254, 1370–1371 (1962)
8. Golumbic, M.C., Scheinerman, E., Graphs, Containment: Posets, and related classes of graphs. Ann. N. Y. Acad. Sci 555, 192–204 (1989)
9. Kelly, D.: The 3-irreducible partially ordered sets. Can. J. Math. 29, 367–383 (1977)
10. Kierstead, H.A., Trotter, W.T.: The dimension of cycle-free orders. Order 9, 103–110 (1992)
11. Ma, T., Spinrad, J.P.: Cycle-free partial orders and chordal comparability graphs. Order 8, 49–61 (1991)
12. Rose, D.J.: On simple characterizations of k-trees, Discret. Math. 7, 317–322 (1974)
13. Rose, D.J., Tarjan, R.E., Lueker, G.: Algorithmic aspects of vertex elimination on graphs. SIAM J. Comput. 5, 266–283 (1976)
14. Trotter, W.T., Moore, J.I.: Characterization problems for graphs, partially ordered sets, lattices and families of sets. Discret. Math. 16, 361–381 (1976)
15. Spinrad, J.P.: On comparability and permutation graphs. SIAM J. Comput. 14, 658–670 (1985)
16. Urrutia, J.: Partial orders and euclidean geometry. In: Rival, I. (ed.) Algorithms and Order, pp. 327–346. Kluwer Academic, Dordrecht (1989)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.