The Absolute Definition of the Phase-Shift in Potential Scattering

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Dedicated to Professor Shinsho Oryu for his 60th anniversary

Abstract

The variable phase approach to potential scattering with regular spherically symmetric potentials satisfying (1), and studied by Calogero in his book\textsuperscript{5}, is revisited, and we show directly that it gives the absolute definition of the phase-shifts, i.e. the one which defines $\delta_\ell(k)$ as a continuous function of $k$ for all $k \geq 0$, up to infinity, where $\delta_\ell(\infty) = 0$ is automatically satisfied. This removes the usual ambiguity $\pm n\pi$, $n$ integer, attached to the definition of the phase-shifts through the partial wave scattering amplitudes obtained from the Lippmann-Schwinger integral equation, or via the phase of the Jost functions. It is then shown rigorously, and also on several examples, that this definition of the phase-shifts is very general, and applies as well to all potentials which have a strong repulsive singularity at the origin, for instance those which behave like $gr^{-m}$, $g > 0$, $m \geq 2$, etc. We also give an example of application to the low-energy behaviour of the $S$-wave scattering amplitude in two dimensions, which leads to an interesting result.
1 Introduction

In quantum scattering theory with a spherically symmetric potential \( V(r) \), the phase-shift, defined for each partial-wave by the asymptotic behaviour of the radial wave function at large distances, has the usual ambiguity of \( \pm \pi n \), \( n \) integer. When the potential is regular, i.e. is \( L^1(a, \infty) \), and satisfies the Bargmann-Jost-Kohn condition\(^1\)\(^2\)\(^3\)

\[
\int_0^\infty r|V(r)|dr < \infty, \tag{1}
\]

one can show that the phase-shift \( \delta_\ell(k) \) at infinite energy satisfies the condition

\[
tg \delta_\ell(\infty) = 0, \quad \text{or} \quad \sin \delta_\ell(\infty) = 0, \tag{2}
\]

so that one can make the “canonical” choice

\[
\delta_\ell(\infty) = 0, \tag{3}
\]

and then proceed downwards by continuity for finite values of \( k \).

The same ambiguity exists, of course, when the phase-shift is defined through the phase of the Jost function\(^1\)\(^2\)\(^3\). The problem becomes even more serious when the potential is singular and repulsive near \( r = 0 \), and short-range otherwise:

\[
V(r) \begin{cases} \approx r^{-m}, & g > 0, \ m > 2; \\ \approx \exp(g/r), & g > 0, \ \alpha > 0; \end{cases} \tag{4}
\]

In these cases, one can still define the \( S \)-matrix, \( S_\ell = \exp(2i\delta_\ell(k)) \), with a continuous phase-shift \( \delta_\ell(k) \), which is again given by the asymptotic behaviour of the wave function for \( r \to \infty \). However, now, one has\(^7\)\(^8\)\(^9\)

\[
\delta_\ell(\infty) = -\infty, \tag{5}
\]

and this adds to the difficulty of finding a unique phase-shift.

From the above remarks, it would seem therefore satisfactory to find a unique definition of the phase-shift itself by some formula containing \( V(r) \) and the wave function, and such that (3) is automatically satisfied when the potential satisfies (1). And then, to see whether such a definition can be extended, eventually with some modification, to potentials which are singular at \( r = 0 \) or at \( r = \infty \) (long range).
The answer to the above quest has already been found, although not written down explicitly in the form (20) we give below. It is given by the variable phase method\textsuperscript{5}. To simplify the algebra, and see what is essential, let us take the case of the $S$-wave, $\ell = 0$. We have the radial Schrödinger equation

\[
\begin{cases}
\varphi''(k, r) + k^2 \varphi(k, r) = V(r) \varphi(k, r) \\
r \in [0, \infty) , \quad \varphi(0) = 0 , \quad \varphi'(0) = 1
\end{cases}
\]  

If we write now the wave function as

\[
\varphi = A(k, r) \sin[kr + \delta(k, r)] ,
\]

\[
A(k, r) \geq 0 , \quad A(k, 0) \neq 0 , \quad \delta(k, 0) = 0
\]

it can be shown that one has formally the differential equation\textsuperscript{5}

\[
\begin{cases}
\frac{d}{dr} \delta(k, r) = -\frac{1}{k} V(r) \sin^2[kr + \delta(k, r)] \\
\delta(k, 0) = 0
\end{cases}
\]

and then

\[
A(k, r) = \frac{1}{k} e^{\int_0^r V(t) \sin^2[kt + \delta(k, t)] dt} ,
\]

provided (8) has a solution, and the integral in (9) is finite. We shall see this in a moment. In essence, $A$ is the amplitude of the wave function $\varphi$, whose oscillations are given by $\sin[kr + \delta(k, r)]$. The function $\delta(k, r)$ can be interpreted as the local phase-shift since $\delta(k, R)$ is the phase-shift due to the cut potential $V(r)\theta(R - r)$. The total phase-shift is defined as

\[
\delta(k) = \lim_{r \to \infty} \delta(k, r) .
\]

In order to see whether (8) has a unique solution, we can write it as

\[
\delta(k, r) = -\frac{1}{k} \int_0^r V(t) \sin^2[kt + \delta(k, t)] dt .
\]

One can then try to solve this nonlinear integral equation by iteration, starting from the zeroth order approximation
\[ \delta^{(0)}(k, r) = 0 \]  \hspace{1cm} (12)

Numerical calculations show that this process is fast converging for usual regular potentials used in Nuclear Physics, and indeed has been used in practice.

From (11), it is obvious that the phase-shift is given by

\[ \delta(k) = -\frac{1}{k} \int_0^\infty V(r) \sin^2 [kr + \delta(k, r)] dr \]  \hspace{1cm} (13)

This formula shows that, for short-range potentials, the tail of the integral:

\[ \left| \frac{-1}{k} \int_R^\infty V(r) \sin^2 [kr + \delta(k, r)] dr \right| \leq \frac{1}{k} \int_R^\infty |V(r)| dr \]  \hspace{1cm} (14)

can be made very small if we choose \( R \) larger than the “range” of the potential.

In order to study the nonlinear integral equation (11) in a rigorous way, we can use the inequality\(^1,2\)

\[ |\sin x| \leq C \frac{|x|}{1 + |x|}, \quad x \text{ real}, \]  \hspace{1cm} (15)

where \( C \) is an appropriate constant. Noting that \( x/(1 + x) \) is an increasing function of \( x \) for \( x \) positive, we have

\[ |\sin(x + y)| \leq C \frac{|x + y|}{1 + |x + y|} \leq C \frac{|x| + |y|}{1 + |x| + |y|}. \]  \hspace{1cm} (16)

Using this in the integral equation (11), we find

\[ |\delta(k, r)| \leq \frac{C^2}{k} \int_0^r |V(t)| \left[ \frac{kt + |\delta(k, t)|}{1 + kt + |\delta(k, t)|} \right]^2 dt, \quad k > 0. \]  \hspace{1cm} (17)

It is then obvious that an upper bound \( \Delta(k, r) \) for \( |\delta(k, r)| \) is obtained from the solution of the integral equation

\[ \Delta(k, r) = \frac{C^2}{k} \int_0^r |V(t)| \left[ \frac{kt + \Delta(k, t)}{1 + kt + \Delta(k, t)} \right]^2 dt. \]  \hspace{1cm} (18)

We can again solve this integral equation by iteration, starting from \( \Delta^{(0)} = 0 \). It is obvious here that the solution cannot become infinite at any point on \([0, \infty)\).

Indeed, if \( \Delta(k, r) \) becomes infinite at \( r = r_1 \), then, as \( r \uparrow r_1 \), the fraction in (18) becomes one, and since \( V \) is supposed to be \( L^1 \), we get a contradiction.
We show in Appendix A that the equation (18) has always a unique solution for all values of $k > 0$, provided $V$ satisfies (1), and therefore that (11) also has a unique solution. We show also that $\lim_{k \to \infty} \Delta(k, \infty) = 0$, from which we conclude then directly that we have (3). However, it is not really necessary to use the integral equation. Indeed, as we shall see in the next section, we can express the right-hand side of (11) in terms of the regular solution $\varphi$ given by (3), and its derivative $\varphi'$, to obtain the new formulae

$$\delta(k, r) = -k \int_0^r V(t) \frac{\varphi^2(k, t)}{\varphi'^2(k, t) + k^2 \varphi^2(k, t)} \, dt$$

and

$$\delta(k) = -k \int_0^\infty V(r) \frac{\varphi^2}{\varphi'^2 + k^2 \varphi^2} \, dr$$

We shall see in the next section how to generalize these equations to higher $\ell$. Likewise, the amplitude $A(k, r)$ can also be written as

$$A(k, r) = \frac{1}{k} \sqrt{\varphi'^2 + k^2 \varphi^2}$$

Now, we know, quite generally, that for potentials satisfying (1), the wave function $\varphi(k, r)$ exists for all values of $k$, real or complex, and all values of $r \geq 0$. Likewise for $\varphi'(k, r)$, and we have, of course

$$\varphi'(k, 0) = 1 \quad \text{and} \quad \varphi(k, r) \underset{r \to 0}{=} r + r \, o(1).$$

Moreover, we have also that $\varphi$ and $\varphi'$, for any fixed real $k > 0$, are real and continuous function of $r$, and bounded as $r \to \infty$. Also, from the general theory of differential equations, we know that, for any $k$, $\varphi$ and $\varphi'$ cannot vanish simultaneously at any point $r = r_0 \geq 0$, since this would entail that $\varphi \equiv 0$, a contradiction with $\varphi'(k, 0) = 1$. It follows that, because of (1), the integrals in (19) and (20) are absolutely convergent for all real values of $k > 0$, and define continuous functions of $k$. Therefore, (19) and (20) define, in a very nice and simple way the “local” and the total phase-shift, respectively. We shall come in the third section to the case $k = 0$.

Several remarks are now in order. First of all, (20) shows that, for a potential of a given sign, the phase-shift has the opposite sign, something well-known, and also
obvious on (11) and (13). Secondly, it is obvious on (19) and (20) that the phase-shift is independent of the amplitude of $\varphi$. Multiplying $\varphi$ by a constant factor $\lambda(k)$ independent of $r$ leaves (19) and (20) invariant. This is as expected, of course. Finally, from (21) we find

$$A(k, r) > 0 \ , \ \ A(k, 0) = \frac{1}{k} \neq 0$$

which was assumed in (7). Now, for $r \to \infty$, the asymptotic behaviour of $\varphi$ and $\varphi'$ are given by\(^1\)\(^-\)\(^4\)

$$\varphi(k, r) \underset{r \to \infty}{\simeq} \frac{|F(k)|}{k} \sin(kr + \delta) + o(1) \ ,$$

and

$$\varphi'(k, r) \underset{r \to \infty}{\simeq} |F(k)| \cos(kr + \delta) + o(1) \ ,$$

where $F(k)$ is the Jost function, which never vanishes for $k > 0\(^1\)\(^-\)\(^4\)$. Using now (24) and (25) in (21), we find

$$A(k, \infty) = \frac{|F(k)|}{k}$$

and this, used in (7), leads again to (24), a consistency check.

We have now to see whether, for regular potentials satisfying (1), the phase-shift defined by (20) satisfies (3):\(\delta(\infty) = 0\). This is very easy to check. Indeed, for $k$ real and going to infinity, we have uniformly in $r\(^1\)\(^-\)\(^3\)\(^-\)\(^4\)$, for any $r \in [0, R], \ R < \infty$,

$$\varphi(k, r) \underset{k \to \infty}{=} \frac{\sin kr}{k} + \frac{1}{k} o(1) \ ,$$

and

$$\varphi'(k, r) \underset{k \to \infty}{=} \cos kr + o(1) \ .$$

Using these in (20), together with (14), leads to

$$\delta(k) \underset{k \to \infty}{\simeq} -k \int_0^R \frac{V(r) \sin^2 kr}{k^2} \, dr = -\int_0^R V(r) \frac{1 - \cos 2kr}{2k} \, dr$$

$$= -\int_0^R V(r) \, dr \int_0^r \sin 2kt \, dt = -\int_0^R \sin 2kt \, dt \left( \int_0^R V(r) \, dr \right) \ .$$

(29)
Now, in general, \( W(t) = \int_t^\infty V(r) \, dr \) is \( L^1(0, \infty) \). Indeed

\[
\int_0^\infty |W(t)| \, dt = \int_0^\infty dt \, \left| \int_t^\infty V(r) \, dr \right| \leq \int_0^\infty dt \, \int_t^\infty |V(r)| \, dr
\]

\[
= \int_0^\infty |V(r)| \, dr \int_0^r dt = \int_0^\infty r |V(r)| \, dr < \infty
\]

by virtue of (4). Therefore, in the last integral in (29) we have a Fourier sine transform of an \( L^1 \) function. From the Riemann-Lebesgue lemma, it follows that, as \( k \to \infty \), it vanishes. Therefore, \( \delta(k \to \infty) = 0 \). This argument, which is quite general, can be made more precise (see Appendix B). It follows that the definition of the phase-shift given by (20) in terms of the well-defined regular solution \( \varphi \), (3), is indeed an absolute definition of \( \delta(k) \) for all \( k > 0 \) satisfying automatically (3).

We shall see later that (20) can be extended to potentials which are outside the Bargmann-Jost-Kohn class, especially to those potentials which are strongly repulsive at \( r = 0 \). We are going now to give first the derivation of (8), (9) and (21). These are found essentially in 5, except for the integral forms (19) and (20), which are the basic equations of our paper. We reproduce the proofs for the convenience of the reader.

2 Derivation of (19) and (20)

We follow the usual method given in Calogero 5 for deriving the differential equation (8). Differentiating (7), we find

\[
\varphi' = A' \sin(kr + \delta) + A(k + \delta') \cos(kr + \delta) \quad .
\]

(31)

We have here two unknown functions \( A \) and \( \delta \), and only one equation, the Schrödinger equation (3), at our disposal. We can therefore impose a relation between \( A \) and \( \delta \). We impose

\[
A' \sin(kr + \delta) + A\delta' \cos(kr + \delta) = 0 \quad .
\]

(32)

It follows that (31) becomes simply

\[
\varphi' = Ak \cos(kr + \delta) \quad .
\]

(33)
Differentiating now this, and using the Schrödinger equation (6), we find the new equation

\[ A'k \cos(kr + \delta) - Ak'\sin(kr + \delta) = V A \sin(kr + \delta) \quad . \]  \hspace{1cm} (34)

Combining this with (32), we find the differential phase equation (8) and the amplitude equation (9), a complete set equivalent to the Schrödinger equation (6).

However, we can go back again to \( \varphi \) and \( \varphi' \), using (7) and (33). Adding their squares, we find

\[ A^2(k, r) = \frac{1}{k^2} \left( \varphi'^2 + k^2 \varphi^2 \right) \quad . \]  \hspace{1cm} (35)

It follows that (8) can be written

\[ \delta' = - \frac{1}{k} V \sin^2(kr + \delta) = - \frac{1}{k} V \frac{\varphi^2}{A^2} \]
\[ = -k V \frac{\varphi^2}{\varphi'^2 + k^2 \varphi^2} \quad . \]  \hspace{1cm} (36)

This is the basic equation, giving the "local" phase-shift \( \delta(k, r) \) in terms of \( \varphi \) and \( \varphi' \). Integrating it, we find

\[ \delta(k, r) = -k \int_0^r V(t) \frac{\varphi^2(k, t)}{\varphi'^2(k, t) + k^2 \varphi^2(k, t)} \, dt \quad , \]  \hspace{1cm} (37)

where the initial condition \( \delta(k, 0) = 0 \) has been used. Making now \( r \to \infty \), we get the total phase-shift, \( (20) \). We have already checked that in \( (19) \) and \( (20) \) the integrals are absolutely convergent, and define continuous functions of \( k \) for all \( k > 0 \).

**Higher waves.**

The method for \( \ell \neq 0 \) is quite similar\(^5\). We have to deal now with the radial Schrödinger equation

\[ \varphi''(k, r) + k^2 \varphi(k, r) = \left[ \frac{\ell(\ell + 1)}{r^2} + V(r) \right] \varphi(k, r) \quad , \]  \hspace{1cm} (38. a)

and its free counterpart when \( V = 0 \):
\[
\frac{d^2}{dr^2}(u_\ell) + k^2 \left( \frac{u_\ell}{v_\ell} \right) = \frac{\ell(\ell + 1)}{r^2} \left( \frac{u_\ell}{v_\ell} \right) .
\] (38.b)

These free solutions are given by\(^{1-6}\)

\[
\begin{align*}
&u_\ell(kr) = \sqrt{\frac{\pi kr}{2}} J_{\ell + \frac{1}{2}}(kr) \equiv \frac{(kr)^{\ell + 1}}{(2\ell + 1)!!} + \cdots , \\
v_\ell(kr) = \sqrt{\frac{\pi kr}{2}} N_{\ell + \frac{1}{2}}(kr) \rightarrow 0 \frac{(2\ell - 1)!!}{(kr)^{\ell}} + \cdots , \\
(2\ell + 1)!! &= \frac{\Gamma(2\ell + 2)}{2^{\ell} \Gamma(\ell + 1)} , \quad \ell > -\frac{1}{2} .
\end{align*}
\] (39.a)

We have also

\[
\begin{align*}
&u_\ell(kr) \rightarrow_{r \rightarrow \infty} \sin(kr - \frac{1}{2} \ell \pi) + \cdots , \\
v_\ell(kr) \rightarrow_{r \rightarrow \infty} \cos(kr - \frac{1}{2} \ell \pi) + \cdots .
\end{align*}
\] (39.b)

Their Wronkian is given by

\[
W(u_\ell, v_\ell) \equiv u_\ell' v_\ell - u_\ell v_\ell' = k .
\] (40)

As for the regular solution \(\varphi_\ell(k, r)\), it is customary to normalize it such that\(^{1-4}\)

\[
\varphi_\ell(k, r) \rightarrow_{r \rightarrow 0} \frac{r^{\ell + 1}}{(2\ell + 1)!!} + \cdots .
\] (41)

Now, again, under the condition \([\text{I}]\) on the potential, one can show that both \(\varphi\) and \(\varphi'\) exist for all real or complex values of \(k\), all \(r \geq 0\), and are continuous in both variables\(^{1-4}\). Moreover, one has the asymptotic behaviours

\[
\begin{align*}
&\varphi_\ell(k, r) \rightarrow_{r \rightarrow \infty} \frac{|F_{\ell}(k)|}{k^{\ell + 1}} \sin \left( kr - \frac{1}{2} \ell \pi + \delta_\ell(k) \right) + o(1) , \\
&\varphi'_\ell(k, r) \rightarrow_{r \rightarrow \infty} \frac{|F_{\ell}(k)|}{k^{\ell}} \cos \left( kr - \frac{1}{2} \ell \pi + \delta_\ell(k) \right) + o(1) ,
\end{align*}
\] (42)

where \(F_{\ell}(k)\) is the Jost function, well-defined and continuous for all \(k \geq 0\). For \(k\) real, one has also, uniformly in \(r\) for any \(r \in [0, R], R < \infty\),
\[
\begin{align*}
\varphi_\ell(k, r) & = \frac{\sin kr}{k^{\ell+1}} + o(1) \frac{1}{k^{\ell+1}}, \\
\varphi'_\ell(k, r) & = \frac{\cos kr}{k^{\ell}} + o(1) \frac{1}{k^{\ell}}.
\end{align*}
\] (43)

We write now
\[
\varphi_\ell(k, r) = A_\ell(k, r)[u_\ell(kr) \cos \delta_\ell(k, r) + v_\ell(kr) \sin \delta_\ell(k, r)],
\] (44)

\(A_\ell(k, r) \neq 0\) for all \(r \geq 0\), \(\delta_\ell(k, 0) = 0\). We have now again two unknown functions \(A_\ell\) and \(\delta_\ell\), and only one (differential) equation to determine them. We can therefore impose a relation between \(A_\ell\) and \(\delta_\ell\). Simplifying the writing, we impose
\[
A'[u \cos \delta + v \sin \delta] + A[-u \sin \delta + v \cos \delta]\delta' = 0
\] (45)

Differentiating now (44) and taking into account (45), we find
\[
\varphi' = A[u' \cos \delta + v' \sin \delta] .
\] (46)

One more differentiation gives us now
\[
\varphi'' = A'[u' \cos \delta + v' \sin \delta] + A[u'' \cos \delta + v'' \sin \delta] + A[-u' \sin \delta + v' \cos \delta]\delta' .
\] (47)

Using now (38.a) for \(\varphi''\), and (38.b) for \(u''\) and \(v''\), we find from (47)
\[
A'[u' \cos \delta + v' \sin \delta] + A[-u' \sin \delta + v' \cos \delta]\delta' = VA[u \cos \delta + v \sin \delta] .
\] (48)

We have now two equations, namely (45) and (48) to determine \(A\) and \(\delta\). We can write them, symbolically, as
\[
aA' + bA\delta' = 0
\] (49)

\[
cA' + dA\delta' = aVA .
\] (50)

Eliminating \(A'\), and using \(A \neq 0\) and (40), we find
\[ ad - bc = -(u'v - uv') = -k \quad , \]

and

\[ \delta_\ell' = -\frac{1}{k} V(r) [u_\ell \cos \delta_\ell + v_\ell \sin \delta_\ell]^2 \quad , \]

to which we have to add \( \delta_\ell(k, 0) = 0 \). Likewise, eliminating \( \delta' \), we find

\[ kA'_\ell = V(r) A_\ell [u_\ell \cos \delta_\ell + v_\ell \sin \delta_\ell][-u_\ell \sin \delta_\ell + v_\ell \cos \delta_\ell] \quad . \]

As it is easily seen, we have again equations very similar to the equations for \( \ell = 0 \). Once (52) is solved, with the boundary condition \( \delta_\ell(k, 0) = 0 \), we can replace its solution \( \delta_\ell \) in (53), and integrate it to get \( A_\ell \) :

\[ A_\ell(k, r) = A_\ell(k, 0) \exp \left[ \frac{1}{k} \int_0^r \cdots \right] \quad . \]

However, as we saw previously, we can write both \( \delta_\ell \) and \( A_\ell \) in terms of \( \varphi_\ell \). From (44) and (46), we can calculate \( \sin \delta_\ell \) and \( \cos \delta_\ell \):

\[ \sin \delta = \frac{u'\varphi - u\varphi'}{kA} \quad (55) \]

\[ \cos \delta = \frac{v\varphi' - v'\varphi}{kA} \quad . \]

Using now \( \sin^2 \delta + \cos^2 \delta = 1 \), we get

\[ A^2_\ell = \frac{1}{k^2} \left[ (u_\ell \varphi'_\ell - u'\varphi_\ell)^2 + (v_\ell \varphi'_\ell - v'\varphi_\ell)^2 \right] \quad . \]

Using also (55) and (56) in (52) and remembering (40), we get

\[ \delta_\ell' = -\frac{1}{k} V(r) \frac{\varphi^2_\ell(k, r)}{A^2_\ell(k, r)} \quad . \]

Integrating this now, we have

\[ \delta_\ell(k, r) = -k \int_0^r V(t) \frac{\varphi^2_\ell(k, t)}{[(u_\ell \varphi'_\ell - u'\varphi_\ell)^2 + (v_\ell \varphi'_\ell - v'\varphi_\ell)^2]} \quad dt \quad . \]

All these formulae, so far purely formal, are very similar to those for \( \ell = 0 \). However, using now the behaviours of \( \varphi, \varphi', u, u', v, v' \) for \( r \to 0 \), we can check that all our
above formulae are meaningful, i.e. the integrals are convergent at \( r = 0 \). Again, it is clear from (57) that \( A_\ell(k, r) \neq 0 \) for all \( r \geq 0 \). Indeed, if \( A_\ell \) is zero at \( r = r_0 \), we must have, according to (57), \( u\varphi' - u'\varphi = 0 \) and \( v\varphi' - v'\varphi = 0 \) at this point. Calculating again \( \varphi \) and \( \varphi' \) from these two equations, we find that, if \( k \neq 0 \), \( \varphi = \varphi' = 0 \) at \( r = r_0 \). And this entails that \( \varphi \equiv 0 \) everywhere, a contradiction with (41). Therefore, our assumption on \( A_\ell \) is satisfied: \( A_\ell(k, r) \), for \( k > 0 \), never vanishes for \( r \in [0, \infty) \).

Let us now look at the behaviour of \( \delta_\ell(k, r) \), (59), \( k > 0 \) for \( r \to 0 \), \( \ell > -1/2 \). From (39.a) and (41), it is easily seen that the numerator behaves like \( r^{2\ell+2} \) whereas the denominator becomes a positive constant. Therefore, we have, for \( k > 0 \),

\[
\delta_\ell(k, r) \underset{r \to 0}{\sim} \text{constant} \int_0^r t^{2\ell+2} V(t) dt = r^{2\ell+1} + o(1) \quad .
\]  

(60)

This shows that our assumption on \( \delta_\ell(k, r) \) is also satisfied: \( \delta_\ell(k, 0) = 0 \). Using (60) in (53), we find also that we have a convergent integral in (54). This completes the validity of the method and its consistency.

We have now to look at (57) and (59) for \( r \to \infty \). From (39.b) and (42), we find easily, for all \( k > 0 \), that we have

\[
A^2_\ell(k, \infty) = \frac{|F_\ell(k)|^2}{k^{2\ell+2}} \Rightarrow A_\ell(k, \infty) = \frac{|F_\ell(k)|}{k^{\ell+1}} \quad ,
\]  

(61)

where \( F_\ell(k) \) is the Jost function, finite and continuous for all \( k \geq 0 \). From (59) for \( r \to \infty \), we find

\[
\delta_\ell(k) = -k \int_0^\infty V(r) \frac{\varphi^2_\ell(k, r)}{[(u_\ell' \varphi_\ell - u_\ell \varphi'_\ell)^2 + (v_\ell' \varphi_\ell - v_\ell \varphi'_\ell)^2]} dr \quad .
\]  

(62)

and it is easily checked from (39.b) and (42) that the integral here is well-defined and absolutely convergent, and since \( \varphi, \varphi', u, u', v \) and \( v' \) are all continuous functions of \( k \) for all \( k > 0 \), the same is true for \( \delta_\ell(k) \) : the phase-shift is a continuous function of \( k \) for all \( k > 0 \). To check (3), we can use (43) in (62), and we find, as for the case \( \ell = 0 \), that \( \delta_\ell(\infty) = 0 \). Moreover, \( k^{\ell+1}A_\ell \) and \( \delta_\ell \) are also continuous functions of \( k \) for all \( k > 0 \). They are given, respectively, by (57) and (59). The physical phase-shift is given by (62), an absolutely convergent integral for all \( k > 0 \) under the assumption (1), and one has also (3). Therefore, (62) gives an absolute definition of the phase-shift \( \delta_\ell(k) \) in general for all \( k \), and for potentials satisfying (1). We shall see in the next section that (20) and (62) are valid also for singular
repulsive potentials as well. The case $k = 0$ will be considered at the end of the next section.

Remark. Making $\ell = 0$, one finds, as expected, all the formulae we found previously for the $S$-wave. Before ending this section, let us mention that by combining (38.a) and (38.b), and integrating from 0 to $r$, using the appropriate boundary conditions at $r = 0$, namely (39.a), (40), and (41), to evaluate the integrated terms, one can calculate $u' \varphi - u \varphi'$ and $v' \varphi - v \varphi'$, so that (62) can be written, for $\ell > -\frac{1}{2}$, also as

$$\delta_\ell(k) = -k \int_0^\infty V(r) \frac{\varphi^2_\ell(k, r)}{(\int_0^r u_\ell \varphi_k V dt)^2 + (k^{-\ell} + \int_0^r v_\ell \varphi_k V dt)^2} dr . \quad (63)$$

Note here that we have now well precised boundary conditions at $r = 0$ for all the functions entering in (63), so that care must be taken in using it, whereas in (20) and (62), the normalization of $\varphi$ at $r = 0$ does not matter. For $-\frac{1}{2} \leq \ell < 0$, as we shall see in the next section, (62) and (63) are valid provided we use there the “distinguished” pure Bessel solution for $\varphi$, given by the integral equation (68). Indeed, now, both free solutions $u_\ell = \sqrt{r} J_{\ell+\frac{1}{2}}(kr)$ and $v_\ell = \sqrt{r} N_{\ell+\frac{1}{2}}(kr)$ vanish at $r = 0$, and so, as free solution, we can start from any combination $au_\ell + bv_\ell$, and use it as the inhomogeneous term in (68). We get then always a solution $\varphi_\ell$ with $\varphi(k, 0) = 0$. The “distinguished” solution is the one with $b = 0$.

3 Domain of validity of (20) and (62)

As we have seen, formulae (20) for the $S$-wave, or its generalization (62) for higher waves, are valid for all $k > 0$ and all $\ell \geq 0$, provided the potential satisfies the integrability condition (1): $rV(r) \in L^1(0, \infty)$. Roughly speaking, this means that $V$ is less singular than $r^{-2}$ at the origin. We may now ask whether they are also valid for potentials having stronger singularity there, for instance $V(r) \sim gr^{-m}$, $m \geq 2$, $g > 0$, or $g \exp(\alpha r^n)$, $g > 0$, $\alpha > 0$, $n > 0$, etc., as $r \to 0$. Rather than developing the general formalism for such general singular potentials (singular and repulsive at the origin), we shall consider explicit examples, and leave the full theory for a forthcoming paper.
i) We consider the formula (20) for the S-wave, and take boldly the potential to be the centrifugal barrier

\[ V(r) = \frac{\ell (\ell + 1)}{r^2}, \quad \ell > 0, \]

in the Schrödinger equation (6). The wave function is now just, up to an unimportant constant multiplicative factor, \( \varphi = \sqrt{kr} J_{\ell+1/2}(kr) \) which is of the form \( \phi(kr) \). It is easily seen that (20) is well-defined because the integral is convergent (absolutely) both at \( r = 0 \) and \( r = \infty \). If we make the change of variable \( z = kr \) in it, we find, by writing \( \varphi' = k \frac{dz}{dr} \phi = k \dot{\phi}(z), \dot{\cdot} = d/dz, \)

\[ \delta = -\ell (\ell + 1) \int_0^\infty \frac{\phi'^2(z)}{[\phi^2(z) + \phi'^2(z)]} \, dz, \]

where \( \phi = \sqrt{z} J_{\ell+1/2}(z) \). This last formula is now independent of \( k \). The same is therefore true for the original formula (20) with our \( \varphi(k,r) \). We can therefore calculate it at any value of \( k \), for instance for \( k = 0 \). Using (39.a), we find

\[ \delta = -\ell (\ell + 1) \int_0^\infty \frac{dz}{[(\ell + 1)^2 + z^2]} = -\ell \int_0^\infty \frac{dt}{1 + \ell t^2} = -\ell \frac{\pi}{2}. \]

This is exactly the phase-shift of the centrifugal barrier \( \ell (\ell + 1)/r^2 \) since, without this potential, the wave-function is \( \sin kr \), and with the potential, \( \varphi \approx \text{constant} \times \sin(kr - \frac{1}{2} \ell \pi) \), as \( r \to \infty \), according to (42).

In conclusion, our formula (20) is valid for repulsive singular potentials \( g/r^2, \ g > 0 \), which violate (1) both at \( r = 0 \) and \( r = \infty \).

ii) We consider now the previous example, but with \( -1/2 \leq \ell < 0 \). Here, proceeding as before, we find, as expected, again (66). Note that \( \delta \) is now positive because the potential \( \ell (\ell + 1)/r^2 \) is negative. For \( \ell = -\frac{1}{2} \), the full solution is \( \varphi = \sqrt{kr} J_0(kr) \), so that \( \varphi \sim \sqrt{r} \) and \( \varphi' \sim 1/(2\sqrt{r}) \) as \( r \to 0 \).

**Remark.** Formula (20) was proved for regular potentials. In all rigor, in order to apply it to the centrifugal barrier potential \( \ell (\ell + 1)/r^2 \), we must first regularize this at the origin, for instance by cutting it by \( \theta(r - \varepsilon) \), and then make \( \varepsilon \downarrow 0 \), or by replacing \( r \) in the denominator by \( (r + \varepsilon) \), and again take the limit \( \varepsilon \downarrow 0 \). However, as we saw, at the limit, we have already an absolutely convergent integral for all
\( \ell \geq -\frac{1}{2} \). Also, the derivation of (20) was based on the assumption \( \varphi'(0) = 1 \), i.e. \( \varphi(r) \simeq r + \cdots \) for \( r \to 0 \). However, for \( \ell \neq 0 \), we have rather \( \varphi_\ell \simeq r^{\ell+1} + \cdots \). The extra factor \( r^\ell \) comes from the regularized formula for the amplitude \( A \) in the limit \( \varepsilon \downarrow 0 \), as can easily be seen\(^5\).

iii) We consider now (13) with

\[
V(r) = \frac{\ell(\ell + 1)}{r^2} + V_1(r) , \quad \ell \geq -\frac{1}{2} ,
\]

assuming that \( rV_1(r) \in L^1(0, \infty) \). Consider first \( \ell > -\frac{1}{2} \). As we saw, the presence of this “weak perturbation”, as compared to \( \ell(\ell + 1)/r^2 \), does not modify the behaviour of the regular solution \( \varphi_\ell \) at \( r = 0 \), given by (41), to be compared with (39.a)\(^1-4\). And since (20), as we just saw, works with \( \ell(\ell + 1)/r^2 , \ell \geq -1/2 \), applied to the regular solution \( \varphi_\ell \), it should work also when we add \( V_1 \), provided we use always pure Bessel functions as free solutions. This means that the Volterra integral equation which combines the Schrödinger equation and the boundary condition at \( r = 0 \) is

\[
\left\{
\begin{array}{l}
\varphi_\ell(k, r) = \frac{1}{kr} u_\ell(kr) + \int_0^r G_\ell(k; r, r') V(r') \varphi_\ell(k, r') \, dr' , \\
G_\ell(k; r, r') = \frac{1}{k} [u_\ell(kr)v_\ell(kr') - u_\ell(kr')v_\ell(kr)] .
\end{array}
\right.
\]

We consider now \( \ell = -\frac{1}{2} \). As has been shown in reference\(^10\), in order to formulate a decent scattering theory leading to the asymptotic form (42) with a well-defined Jost function \( F_\ell(k) \) and a well-defined phase-shift \( \delta_\ell \), one has to make stronger assumptions on \( V \) than (1), namely

\[
\left\{
\begin{array}{l}
\int_0^\infty r|V(r)|(1 + |\log r|)dr < \infty , \\
\int_a^\infty r|V(r)|(\log r)^2dr < \infty , \quad a > 0 .
\end{array}
\right.
\]

This will be used later for \( \ell = -1/2 \), and \( k \to 0 \), where we give more details.

In conclusion, formula (20) is valid for (67) and the regular solution \( \varphi_\ell \). Therefore, we have now two methods to deal with (67). The first one is to apply (62) to \( V_1 \), and the second one to apply (20) to the full potential (67). In any case, we get, of course, the full phase-shift
\[ \delta^\text{total}_\ell = \delta_\ell - \frac{1}{2} \ell \pi \quad , \]  

\( \delta_\ell \) being the physical phase-shift due to \( V_1 \), which is what interests us in scattering theory. Remember that, in (20) or (62), \( \varphi \) is always the full solution: solution of (38.a) with \( V = V_1 \), or (6) with (67), always together with (41).

**iv)** We consider now more singular potentials, namely \( V(r) = g/r^m, \ m > 2, \ g > 0 \). Here, it is known that the solution of the Schrödinger equation (1) which vanishes at \( r = 0 \) behaves there as\(^7-9\)

\[
\varphi_\ell(k, r) \underset{r \to 0}{\simeq} [V(r)]^{-1/4} \exp \left( - \int_r^{\infty} [V(t)]^{1/2} dt \right) = \phi_0(r) = g^{-1/4} r^{m/4} \exp \left[ -\sqrt{g} \frac{2}{m-2} r^{-\frac{m-2}{2}} \right] ,
\]

(71)

independent of \( k \) and \( \ell \). The wave function \( \varphi \) and all of its derivatives \( \varphi^{(n)} \) vanish exponentially at \( r = 0 \). Notice that we can omit the factor \( g^{-1/4} \) in front of the last expression since our formulae for the phase-shifts are homogeneous in \( \varphi \). In fact, at \( k = 0 \), the Schrödinger equation is soluble exactly, and its solution is\(^7\), up to an unimportant constant multiplicative factor,

\[
\varphi_\ell(k = 0, r) = \sqrt{r} K_{\frac{m+1}{2}} \left( \frac{2\sqrt{g}}{m-2} r^{-\frac{m-2}{2}} \right) ,
\]

(72)

where \( K_{\nu} \) is the modified Hankel function\(^6\). Using the asymptotic behaviour of \( K_{\nu}(x) \) for \( x \to \infty \), we find indeed, up to constant multiplicative factors, the behaviour shown in (71). On the other hand, we know that now, because of the strong singularity of \( V \) at \( r = 0 \), the phase-shifts does not go to zero as \( k \to \infty \), contrary to the case of regular potentials satisfying (1). Rather, one has the high energy behaviour\(^8,9\)

\[
\delta_\ell(k) \underset{k \to \infty}{=} -A \ g^\frac{1}{m} \ k^{\frac{m-2}{m}} + \cdots
\]

(73)

where

\[
A = \frac{\sqrt{\pi}}{2} \ \frac{\Gamma(1-1/m)}{\Gamma(3/2-1/m)} .
\]

(74)
Since the main term in (73) is independent of $\ell$, we shall consider the case $\ell = 0$ to simplify the algebra, and therefore use (20). In this formula, the integral can be split into $\int_0^R + \int_R^\infty$. For the second integral, we have

$$\left| k \int_R^{\infty} \cdots \right| < \frac{g}{k} \int_R^{\infty} \frac{dr}{r^m} = O \left( k^{\frac{m-2}{m}} + \varepsilon \right),$$

(75)

$\varepsilon$ very small ($> 0$), as $k \to \infty$, provided $R = O(k^{-\frac{2}{m} + \varepsilon})$. The contribution of (75) can therefore be neglected if we compare with (73). Note that $R \to 0$ as $k \to \infty$.

In the first integral, we can therefore replace, in first approximation, $\phi$ by $\phi_0$ given in (71), and independent of $k$. We find then

$$-gk \int_0^R \frac{1}{r^m} \frac{\phi_0^2}{\phi_0^2 + k^2 \phi_0^2} \, dr = -gk \int_0^R \frac{1}{\left( \frac{m}{2} r^{\frac{m}{2} - 1} + \sqrt{g} \right)^2 + k^2 r^m} \, dr.$$  

(76)

Making now the change of variable $x = k^{2/m} r$, letting $k \to \infty$, and noting that $k^{2/m} R \to \infty$, we find

$$-gk \int_0^{\infty} \cdots \approx -g k^{-\frac{m-2}{m}} \int_0^{\infty} \frac{dx}{g + x^m} = -g^{1/m} k^{-\frac{m-2}{m}} \int_0^{\infty} \frac{dt}{1 + t^m}.$$  

(77)

It follows that the approximate value of $\delta(k)$ behaves asymptotically as

$$\delta_{app}(k) \approx -B \frac{g^{\frac{1}{m}} k^{-\frac{m-2}{m}}}{1 + t^m}, \quad B = \int_0^{\infty} \frac{dt}{1 + t^m},$$  

(78)

This coincides with (73) up to a numerical factor, and is obtained without much effort, as we see.

The above argument to obtain (78) is, of course, heuristic because (71) is uniform in $k$ only for $k \in [0, K]$, $K < \infty$. But it can be made more precise and quite rigorous. In a forthcoming paper, we shall develop a technique to deal with all these problems in a unified way.

**Remark.** The remark at the end of ii) applies here too. We must regularize first the potential, and then let $\varepsilon \downarrow 0$. The exponential decrease of $\varphi$ as $r \to 0$ comes then from the amplitude $A$ in the limit $\varepsilon \downarrow 0$. 

17
A low-energy example.

Our last example is the low energy behaviour of the phase-shift for $\ell = -1/2$, where the potential is assumed to be repulsive ($V \geq 0$), and to satisfy the integrability conditions

$$\int_0^\infty r|V(r)|(1 + |\log r|)dr < \infty , \quad (79)$$

and

$$\int_a^\infty r|V(r)|(\log r)^2 dr < \infty , \quad a > 0 . \quad (80)$$

This corresponds to the $S$-wave Schrödinger equation in two space dimensions, and is interesting to study$^{10}$. As we saw in $\text{iii)$, we must use here only the pure “Bessel” solutions. This means that the solution $\varphi$ is the solution of the integral equation (68) for $\ell = -1/2$:

$$\varphi(k, r) = \sqrt{kr} J_0(kr) + \int_0^r G(k, r, r') V(r') \varphi(k, r') \, dr' , \quad (81)$$

where the Green’s function is given by

$$G = \frac{\pi}{2} \sqrt{rr'} \left[ J_0(kr) N_0(kr') - J_0(kr') N_0(kr) \right] . \quad (82)$$

The occurrence of $\log r$ in (79) is due to the presence of $\log(kr)$ in $N_0$. Here, $\varphi$ is normalized somewhat differently, but we know that the normalization of $\varphi$ does not matter in (20), (62). In (63), $\varphi$ is normalized now as to $\varphi \simeq \sqrt{r}$ for $r \to 0$.

It is then shown in the above reference that the phase-shift, i.e. the phase-shift $\delta_0(k)$ of the $S$-wave in the two dimensional space problem, has the universal behaviour

$$\delta_0(k) \underset{k \to 0}{\sim} \frac{-\pi}{2|\log k|} + \cdots , \quad (83)$$

i.e. the main term is independent of the potential.

We are going to find (83) by using the formula (63) for $\ell = -1/2$, in which $\varphi_\ell$ is given as above, and, for $z \to 0$, $^6$
\[
\begin{align*}
J_0(z) &= 1 - \frac{z^2}{4} + \cdots \\
N_0(z) &= 2 \pi \left[ \log \frac{\sqrt{z}}{2} + \gamma \right] J_0(z) + O(z^2) .
\end{align*}
\] (84)

From the integral equation for \( \varphi \), it can be easily shown that, under the assumptions (79) and (80), the low-energy behaviour of \( \varphi \) is given by
\[
\varphi \approx \sqrt{kr} J_0(kr) \approx \sqrt{kr} + \cdots ,
\]
which we have to normalize to \( \varphi \approx \sqrt{r} \), and, for \( u \) and \( v \), in order to comply with (40), we must take
\[
\begin{align*}
u &\approx -\sqrt{kr} \log kr + \cdots ,
\end{align*}
\] (85)

Using now the above low-energy behaviours in (63), we find
\[
\delta(k) \approx k \to 0 - \int_0^\infty \frac{rV(r)}{(\int_0^r tV(t)dt)^2 + (1 - \int_0^r tV(t) \log kt dt)^2} dr .
\] (86)

Since \( V \) was assumed to be positive, we can introduce the new variable \( X = X(r) \) by
\[
\begin{align*}
X(r) &= |\log k| \int_0^r tV(t)dt , & dX &= |\log k|rV dr , \quad \text{(87)}
\end{align*}
\]
which is a one-to-one mapping from \( r \in [0, \infty) \) to \( X \in [0, |\log k| A] \), where \( A = \int_0^\infty r|V(r)|dr \). Letting now \( k \to 0 \), we find easily from (86)
\[
\delta(k) \approx k \to 0 - \frac{1}{|\log k|} + \cdots .
\] (88)

Here again, as for the case of singular potentials, we do not get the exact constant \( \pi/2 \) because we use \( \varphi \approx \sqrt{r} \), which is not uniform on the entire \( r \)-axis. We shall come back to this in more detail in the forthcoming paper.

**The case \( k = 0 \).**

Let us consider now \( k = 0 \) in (20). It is known that, under (1), one has\(^{1-4}\)
\[
\varphi(0, r) \to \infty \varphi'(0, \infty) r + D + o(1) ,
\] (89)
where $\varphi'(0, \infty)$ is finite or zero, and $D$ is finite also. Making now $k = 0$ in (20), we find, for the $S$-wave scattering length

$$a_0 = \lim_{k \to 0} \frac{-\delta_0(k)}{k} = \int_0^\infty \frac{\varphi^2(0, r)}{\varphi''(0, r)} V(r) dr . \quad (90)$$

At the origin $r = 0$, there is no convergence problem since $\varphi'(k, 0) = 1$ for all $k$. However, at $r = \infty$, because of (89), we must assume

$$\int_R^\infty r^2 |V(r)| dr < \infty \quad (91)$$

in order to secure proper convergence, which is also well-known\textsuperscript{1,2}. But this is not yet the end. We must also be sure that $\varphi'(0, r)$ does not vanish for $r > 0$. This is surely the case if $V$ is positive\textsuperscript{1,2}, but cannot be guaranteed otherwise. In conclusion, (91) is valid only when $\varphi'(0, r) \neq 0$ for all $r > 0$. For higher waves, in order to have proper convergence at $r = \infty$, one needs\textsuperscript{1,2}

$$\int_0^\infty r^{2\ell+2} |V(r)| dr < \infty \quad (92)$$

and again the non-vanishing of the denominators in (20) or (62). We shall see in a forthcoming paper, how to modify these formulae in the presence of bound states. However, we know that in all cases, $\delta_\ell(k)$ is continuous down to $k + 0$, and one has the Levinson theorem $\delta(+0) = n\pi$, where $n$ is the number of bound states\textsuperscript{1−5}.

**Two-potential case.**

Formulae (62) and (63), as it is obvious, can of course be applied in the case where we have two potentials:

$$V = V_1 + V_2 \quad , \quad (93)$$

both satisfying (1). Here, we can apply either (20) to $V$, or (62) and (63) to $V_2$, where $u_\ell$ and $v_\ell$ are now replaced by two appropriate independent solutions of the Schrödinger equation $\varphi_1$ and $\psi_1$, normalized according to (40):

$$W[\varphi_1, \psi_1] = \varphi'_1 \psi_1 - \varphi_1 \psi'_1 = k . \quad (94)$$
With (20), we would get the full phase-shift, and with (62) or (63), the phase-shift \( \delta_2 \) due to \( V_2 \).
Appendix A

We have to study here the integral equation (18):

\[ \Delta(k, r) = \frac{D}{k} \int_0^r |V(t)| \left[ \frac{kt + \Delta(k, t)}{1 + kt + \Delta(k, t)} \right]^2 dt . \]  

(A.1)

We solve it by iteration, starting from

\[ \Delta^{(0)}(k, r) = 0 . \]  

(A.2)

Since \( x/(1 + x) \) is an increasing function of \( x \) for \( x \geq 0 \), we get the increasing sequence of iterations

\[ \Delta^{(1)}(k, r) < \Delta^{(2)}(k, r) < \cdots , \]  

(A.3)

where

\[ \begin{cases} 
\Delta^{(1)}(k, r) = \frac{D}{k} \int_0^r |V(t)| \left( \frac{kt}{1 + kt} \right)^2 dt , \\
\Delta^{(n)}(k, r) = \frac{D}{k} \int_0^r |V(t)| \left( \frac{kt + \Delta^{(n-1)}(k, t)}{1 + kt + \Delta^{(n-1)}(k, t)} \right)^2 dt .
\end{cases} \]  

(A.4)

i) Assume now first that \( V \) is integrable at \( r = 0 \) and therefore is \( L^1(0, \infty) \). It is then obvious that the increasing sequence of iterations is bounded by

\[ \overline{\Delta}(k, r) = \frac{D}{k} \int_0^r |V(t)| dt , \quad r \geq 0 , \quad k > 0 . \]  

(A.5)

It has therefore a limit, and one has the solution

\[ \Delta(k, r) = \lim_{n \to \infty} \Delta^{(n)}(k, r) \leq \overline{\Delta}(k, r) \]  

(A.6a)

for all \( r \geq 0 \), and all \( k > 0 \). For \( r \to \infty \), the same statement is valid. Indeed, if we note that both \( \Delta(k, r) \) and \( \overline{\Delta}(k, r) \) are increasing functions of \( r \), it follows that we have, when \( r \to \infty \),

\[ \Delta(k) = \lim_{r \to \infty} \Delta(k, r) \leq \overline{\Delta}(k) = \lim_{r \to \infty} \overline{\Delta}(k, r) = \frac{D}{k} \int_0^\infty |V(t)| dt . \]  

(A.6b)
It is then obvious from (A.5) that we have $\Delta(k \to \infty) = 0$, which gives in turn (3), as expected.

**ii)** If we have only (1) : $rV(r) \in L^1$, we must refine slightly our argument. Since our problem is now the convergence of the integral in $\Delta^{(n)}(k, r)$ at $r = 0$, we consider $r$ very small. Now, as it is obvious, another sequence of upper bounds for $\Delta^{(n)}(k, r)$ is obtained by using the drastic inequality $x/(1 + x) < x$ in (A.1). We obtain in this way a sequence of upper bounds for $\Delta^{(n)}$ from the sequence of iterations of

$$
\overline{\Delta}(k, r) = \frac{D}{k} \int_0^r |V(t)| \left[ kt + \overline{\Delta}(k, t) \right]^2 dt .
$$

(A.7)

Putting $\overline{\Delta} = k\omega$, we get

$$
\omega(k, r) = D \int_0^r |V(t)|(t + \omega)^2 dt ,
$$

(A.8)

which is in fact independent of $k$, and must be iterated now. The above equation is nothing else than the integral equation for minus the local scattering length

$$
a(r) = \lim_{k \to 0} \frac{-\delta(k, r)}{k}
$$

(A.9)

for the potential $-D|V(r)|$, and has been studied in the book of Calogero\textsuperscript{5}, chapters 11 and 12, where it is shown that the iteration of (A.8) leads to an absolutely convergent series expansion for the solution, provided $r$ is small enough and $rV(r) \in L^1$. Alternatively, (A.8) is nothing else than the Riccati equation

$$
\omega'(r) = D|V(r)|(r + \omega(r))^2
$$

(A.10)

with $\omega(0) = 0$, which has been also thoroughly studied in the books of Hille\textsuperscript{12} and Coddington and Levinson\textsuperscript{13}, to which we refer the reader, with the same conclusion.

Once we secure the solution of (A.7) in a small interval $[0, r_0]$, with $\Delta(k, 0) = 0$, we can then start at $r = r_0$, consider instead of (A.1) the integral equation

$$
\Delta(k, r) = \Delta(k, r_0) + \frac{D}{k} \int_{r_0}^r |V(t)| \left[ \frac{kt + \Delta(k, t)}{1 + kt + \Delta(k, t)} \right]^2 dt ,
$$

(A.11)

and proceed as before, by iteration. Here, we need only $V \in L^1(r_0, \infty)$. We get then again an increasing sequence bounded by

$$
\Delta(k, r_0) + \frac{D}{k} \int_{r_0}^\infty |V(t)| dt ,
$$

(A.12)
with the same conclusions as before for the existence of the limit of the iterations, ... etc. However, the high energy behaviour of \( \delta(k, r) \) cannot be obtained from the above analysis.

A different method, which superseeds the above considerations, and provides at the same time the high energy limit of \( \delta(k, r) \) and of \( \delta(k) = \delta(k, \infty) \) is as follows. It consists in neglecting \( \Delta \) in the denominator of the right-hand side of (A.1):

\[
\Delta(k, r) \leq \frac{D}{k} \int_0^r |V(t)| \left[ \frac{kt + \Delta(k, t)}{1 + kt} \right]^2 dt \leq \frac{D}{k} \int_0^r \frac{r}{t} |V(t)| \left[ \frac{kt + \Delta}{1 + kt} \right]^2 dt . \tag{A.13}
\]

Writing now \( \Delta = r\omega \), the two ends of (A.13) lead us to the integral equation

\[
\omega = \frac{D}{k} \int_0^r t |V(t)| \frac{(k + \omega)^2}{(1 + kt)^2} dt \tag{A.14}
\]

whose solution provides still a stronger upper bound for \( \Delta(k, r) \). Now, it is trivial to solve (A.14). We just differentiate it, and integrate the differential equation, taking into account \( \omega(k, 0) = 0 \). This solution is just:

\[
\begin{aligned}
\omega(k, r) &= \frac{k}{1 - I(k, r)} I(k, r) , \\
I(k, r) &= D \int_0^r \frac{t |V(t)|}{(1 + kt)^2} dt .
\end{aligned} \tag{A.15}
\]

The solutions exists as long as \( I(k, r) < 1 \), that is, in the interval \([0, r_0)\), where \( r_0(k) \) is given by

\[
D \int_0^{r_0} \frac{r |V(r)|}{(1 + kr)^2} dr = 1 . \tag{A.16}
\]

Note that, in any fixed interval \([0, R]\), \( R \leq \infty \), we have

\[
\lim_{k \to \infty} I(k, r) = 0 . \tag{A.17}
\]

Indeed, from (A.15), and using \( 1 + kt \geq 1 \) and \( 1 + kt > kt \), we have

\[
\begin{aligned}
I(k, r) &\leq I(k, \infty) \leq D \int_0^\infty \frac{t |V(t)|}{1 + kt} dt = J(k, \infty) \leq \int_0^\varepsilon t |V(t)| dt + \frac{1}{k} D \int_{\varepsilon}^{\infty} |V(t)| dt ,
\end{aligned} \tag{A.18}
\]
where, in obvious notations, $J(k, r)$ is defined by $D \int_0^r [t|V(t)|/(1 + kt)]dt$. Now, we can make first $\varepsilon$ small enough, independent of $k$, in order to make the first integral in the right-hand side as small as we wish. Once $\varepsilon$ is fixed, we can then make $k$ large enough in order to make also the second integral as small as we wish. This proves (A.17).

**Remark.** It is obvious that (A.17) is uniform in any finite interval $0 \leq r \leq R$.

As a consequence of (A.17), $r_0(k)$ defined by (A.16) satisfies

$$
\lim_{k \to \infty} r_0(k) = \infty , \quad (A.19)
$$

so that, the larger $k$ is, the larger is the domain of validity of (A.15). In any case, by making $k$ large enough, and combining (A.15) and (A.17), we can have, uniformly in $r$ in $[0, R]$,

$$
\omega(k, r) \leq 2k I(k, r) \quad , \quad k > K \quad , \quad (A.20)
$$

and we know that $r\omega(k, r)$ is an upper bound for $\Delta(k, r)$. We can therefore, for $k$ large enough, use (A.20) in (A.1):

$$
\Delta(k, r) = \frac{D}{k} \int_0^r |V(t)||\cdots|^2 dt \leq \frac{D}{k} \int_0^r |V(t)||\cdots|^1 dt \leq \frac{D}{k} \int_0^r |V(t)| \\
\frac{kt + 2kt I(k, t)}{1 + kt} dt = J(k, r) + 2D \int_0^r \frac{t|V(t)|}{1 + kt} I(k, t) dt \leq J(k, r) + 2I(k, r) J(k, r) \quad . \quad (A.21)
$$

Now, as we saw for (A.17), both $I$ and $J$ go to zero as $k \to \infty$, uniformly in $r$ for $r$ in any finite interval $[0, R]$. Therefore

$$
\lim_{k \to \infty} \Delta(k, r) = 0 \quad , \quad 0 \leq r \leq R \quad . \quad (A.22)
$$

Once we have shown the existence of the solution $\Delta(k, r)$ in $[0, R]$, we can proceed as for (A.11), write, for $r > R$,

$$
\Delta(k, r) = \Delta(k, R) + \frac{D}{k} \int_R^r |V(t)| \left[ \frac{kt + \Delta(k, t)}{1 + kt + \Delta(k, t)} \right]^2 dt \quad , \quad (A.23)
$$
and proceed again by iteration. This way of proceeding is legitimate since (A.1), or (18), was obtained from the differential equation (8) and the bound (16). We can therefore start at any point \( r = R \), and integrate after, provided we know \( \delta(k, R) \). And we obtain a bigger upper bound if we replace \(|\delta(k, R)|\) by \( \Delta(k, R) \). The process is now very similar to what we had in i). We get an increasing sequence of iterations \( \Delta^{(n)}(k, r) \), with \( \Delta^{(0)}(k, r) = \Delta(k, R) \), and a global upper bound for all \( \Delta^{(n)}(k, r) \) :

\[
\Delta^{(n)}(k, r) \leq \Delta_0(k) = \Delta(k, R) + \frac{D}{k} \int_R^\infty |V(t)| dt , \tag{A.24}
\]

valid for all \( r > R \), including \( r = \infty \), and all \( n \). The sequence has therefore a limit, and this limit provides the solution \( \Delta(k, r) \) of (A.1) for all \( r \geq R \), including \( r = \infty \), with \( R < r_0(k) \), and \( r_0(k) \) given by (A.16). Since we had also proved the existence of the solution in \([0, r_0] \), we have therefore proved the existence of the solution of (A.1) for all \( r \geq 0 \). Obviously, we have

\[
\Delta(k, 0) = 0 . \tag{A.25}
\]

Therefore, we have, for \( r > R \),

\[
\Delta(k, r) \leq \Delta(k, R) + \frac{D}{k} \int_R^\infty |V(t)| dt . \tag{A.25}
\]

If we note also that \( \Delta(k, r) \) is an increasing function of \( r \), we secure the existence of \( \Delta(k) = \Delta(k, \infty) \), and from (A.22) and (A.24), we obviously have, for the phase-shift

\[
\lim_{k \to \infty} \delta(k) \leq \lim_{k \to \infty} \Delta(k) = \lim_{k \to \infty} \Delta(k, \infty) = 0 . \tag{A.26}
\]

This completes our proof for the general case where \( rV(r) \in L^1(0, \infty) \).
Appendix B

Here, we shall show that, for regular potentials satisfying (1), one can make (3) more precise if one knows the detail behaviour of the potential when \( r \to 0 \). One has, indeed, the following:

**Theorem 1.** Let us assume that \( V \) is continuous and bounded, away from the origin, in \([0, R], \ R < \infty\), and is such that

\[
\lim_{r \to 0} r^{1+\alpha} V(r) = V_0, \quad 0 < \alpha < 1. \quad \text{(B.1)}
\]

We have then, as \( k \to \infty \),

\[
\delta(k) = -\frac{V_0}{\alpha} \cos \left(\frac{\pi \alpha}{2}\right) \Gamma(1-\alpha) \frac{1}{(2k)^{1-\alpha}} + \cdots, \quad \text{(B.2)}
\]

and conversely: (B.2) entails (B.1).

We shall give the proof for the \( S \)-wave, \( \ell = 0 \). The proof for higher waves is quite similar. Because of (14), we can just limit ourselves to (29):

\[
\delta(k) \sim -\int_0^R dt \sin 2kt \left( \int_t^R V(r) \, dr \right). \quad \text{(B.3)}
\]

For the asymptotic behaviour of this integral (as \( k \to \infty \)), we can now use the following theorem of Titchmarsh:

**Theorem 2.** Let \( f(x) \) and \( f'(x) \) be integrable over any finite interval not ending at \( x = 0 \); let \( x^{1+\alpha} f'(x) \) be bounded for all \( x \), and let \( f(x) \sim x^{-\alpha} \) as \( x \to 0 \). Then, denoting by \( F_0 \) the limit of \( x^{1+\alpha} f'(x) \) as \( x \to 0 \), we have

\[
\int_0^\infty f(x) \sin kx \, dx = F_0 \, \Gamma(1-\alpha) \cos \frac{\pi \alpha}{2} \frac{1}{k^{1-\alpha}} (1 + o(1)), \quad \text{(B.4)}
\]

as \( k \to \infty \). The converse theorem is also true if we deal with finite intervals in (B.4) since we can define the Fourier inverse transforms in a straightforward manner. This is indeed the case in (B.3).

Our theorem 1 follows now immediately from the theorem of Titchmarsh applied to (B.3). Moreover, it is a Tauberian kind theorem, i.e. its converse is also true if we remember that the Fourier transform of an \( L^1 \) function \( f(x) \) is a continuous
function $F(k)$ of $k$. More refined theorems containing logarithmic terms can also be proved. Examples are treated in Calogero’s book\textsuperscript{5}. For higher waves, the proof is similar by using (39.b) and (43) in (62).
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