Bäcklund transformations for the second Painlevé hierarchy: a modified truncation approach

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Abstract.

The second Painlevé hierarchy is defined as the hierarchy of ordinary differential equations obtained by similarity reduction from the modified Korteweg-de Vries hierarchy. Its first member is the well-known second Painlevé equation, $P_{II}$.

In this paper we use this hierarchy in order to illustrate our application of the truncation procedure in Painlevé analysis to ordinary differential equations. We extend these techniques in order to derive auto-Bäcklund transformations for the second Painlevé hierarchy. We also derive a number of other Bäcklund transformations, including a Bäcklund transformation onto a hierarchy of $P_{34}$ equations, and a little known Bäcklund transformation for $P_{II}$ itself.

We then use our results on Bäcklund transformations to obtain, for each member of the $P_{II}$ hierarchy, a sequence of special integrals.

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1. Introduction

At the turn of the century there was an interest in finding new functions defined by differential equations; this led to the discovery of the six Painlevé equations \([1, 2, 3]\), \(P_1 - P_{VI}\). These six ordinary differential equations (ODEs) were discovered through a classification of second order ODEs of the form

\[ V''(X) = F(V'(X), V(X), X), \quad ' \equiv d/dX \]

where \(F\) is rational in \(V'(X)\), algebraic in \(V(X)\) and analytic in \(X\), whose solutions have no movable branch points. Here a singularity is said to be movable if its location in the complex plane depends on constants of integration. The requirement made was thus one of single-valuedness of solutions except at fixed singularities; this property is today referred to as the Painlevé property.

In addition to defining new transcendental functions, the six Painlevé equations have a number of other remarkable properties. For example, they can each be written as the compatibility condition of a corresponding linear system, which then allows them to be solved using the Inverse Monodromy Transform Method \([4]\). Also, at least for \(P_1 - P_{VI}\), they have a variety of Bäcklund transformations and special integrals (see, for example, \([5, 6, 7, 8, 9, 10, 11, 12]\) and references therein). It is the derivation of these last two properties, but for a hierarchy of higher order ODEs, that we are interested in here.

As an example, consider the second Painlevé equation, \(P_2\),

\[ P_2[V, \alpha] \equiv V'' - 2V^3 - VX - \alpha = 0. \tag{1} \]

This has the well-known Bäcklund transformation

\[ V = \tilde{V} + \frac{2\tilde{\alpha} - \varepsilon}{X + 2V^2 - 2\varepsilon V'}, \quad \alpha = \varepsilon - \tilde{\alpha} \tag{2} \]

where \(\varepsilon = \pm 1\), and where \(\tilde{V}\) satisfies \(P_2\) with parameter \(\tilde{\alpha}\), that is

\[ P_2[\tilde{V}, \tilde{\alpha}] \equiv \tilde{V}'' - 2\tilde{V}^3 - \tilde{V}X - \tilde{\alpha} = 0. \]

The second Painlevé equation \(P_2\) \([3]\) also has special integrals for certain choices of the parameter \(\alpha\). For example, with \(\alpha = \frac{1}{2}\), we have the special integral \([3]\)

\[ I_{1/2} \equiv V' - V^2 - \frac{1}{2}X = 0 \tag{3} \]

which, modulo \(P_2\) with \(\alpha = \frac{1}{2}\), satisfies the relation

\[ \left( \frac{d}{dX} + 2V \right) I_{1/2} = 0. \]

The Riccati equation \([3]\) can be linearised via \(V = -\psi'/\psi\) to yield

\[ \psi'' + \frac{1}{2}X\psi = 0, \tag{4} \]

which thus gives the well-known Airy function solutions of \(P_2\) \([3]\).

The Bäcklund transformation \([2]\), together with the discrete symmetry \((V, \alpha) \rightarrow (-V, -\alpha)\), then allows us to derive special integrals \(I_\alpha\) with \(\alpha = N + \frac{1}{2}\), for any integer \(N\). Similarly, rational solutions can be constructed for integer values of \(\alpha\), beginning with the trivial solution \(V = 0\) for \(\alpha = 0\). Later we will see a little known connection between the sequence of special integrals (Airy function solutions), and this sequence of rational solutions.

Modern-day interest in the Painlevé equations was to a large extent sparked by the observation of Ablowitz and Segur \([3]\) (see also \([4, 5, 6, 10]\)) that they arise as similarity reductions of many completely integrable partial differential equations (PDEs). The generalised symmetries of such PDEs then give immediately a means of writing down higher order ODEs that may define new transcendental functions. These form hierarchies of ODEs obtained as similarity reductions of hierarchies of completely integrable PDEs. However, although examples have been written down (see for example \([4, 5, 11]\)), it is only recently that much interest has been shown in such ODEs \([13]\). The question of whether some of the higher order ODEs obtained in this way define new functions remains to be answered. Answering this question is made more difficult by the fact that since the time of Painlevé only restricted progress has been made in the systematic (order by order) classification of ODEs having the Painlevé property.

We do not address this particular question in the present work. We are interested instead in the question of how to obtain Bäcklund transformations and special integrals for hierarchies of ODEs. Here we consider
one particular hierarchy, the $P_H$ hierarchy \cite{4,17}, which is obtained by similarity reduction from the modified Korteweg-de Vries (mKdV) hierarchy. We show that the so-called Painlevé truncation method, a technique commonly applied to PDEs \cite{18,19}, cannot be applied directly to ODEs. We show how these difficulties can be overcome, and accordingly modify the PDE approach in order to obtain systematically the auto-Bäcklund transformations presented (without derivation) in \cite{3}; these results provide a generalisation of (2) above. Using these results we are able to find sequences of special integrals for the $P_H$ hierarchy. Such special integrals have not been presented before for higher members of this hierarchy. We also give a Bäcklund transformation relating the $P_H$ hierarchy to another hierarchy of ODEs, namely a $P_{34}$ hierarchy (so called because its first member is equivalent to equation XXXIV of Chapter 14 in \cite{20}, which is commonly referred to as $P_{34}$). In addition we show how to obtain a little known Bäcklund transformation (due to Gambier \cite{3}) for $P_H$ itself. This last allows us to relate solutions of $P_H$ with parameter value $\alpha = \frac{1}{2}$ to solutions of $P_H$ with parameter value $\alpha = 0$.

2. The $P_H$ hierarchy

The Korteweg-de Vries (KdV) hierarchy can be written as
\begin{equation}
U_{t_{2n+1}} + \partial_x \mathcal{L}_{n+1}[U] = 0, \quad n = 0, 1, 2, \ldots ,
\end{equation}
where $\partial_x = \partial/\partial x$, and the sequence $\mathcal{L}_n$ satisfies the Lenard recursion relation \cite{21}
\begin{equation}
\partial_x \mathcal{L}_{n+1} = (\partial_x^2 + 4U \partial_x + 2U_x) \mathcal{L}_n.
\end{equation}
Beginning with $\mathcal{L}_0[U] = \frac{1}{2}$, this then gives
\begin{align*}
\mathcal{L}_1[U] &= U, \\
\mathcal{L}_2[U] &= U_{xx} + 3U^2, \\
\mathcal{L}_3[U] &= U_{xxxx} + 10UU_{xx} + 5U_x^2 + 10U^3,
\end{align*}
and so on. The mKdV hierarchy is obtained from the KdV hierarchy via the Miura map $U = W_x - W^2$, and can be written as
\begin{equation}
W_{t_{2n+1}} + \partial_x (\partial_x + 2W) \mathcal{L}_n[W_x - W^2] = 0, \quad n = 0, 1, 2, \ldots
\end{equation}
The $P_H$ hierarchy \cite{4,18,17} is obtained from this equation via the similarity reduction
\begin{align*}
W &= \frac{V(X)}{[(2n + 1)\ell_{2n+1}]^{1/2n+1}}, \\
X &= \frac{x}{[(2n + 1)\ell_{2n+1}]^{1/2n+1}},
\end{align*}
which gives
\begin{equation}
P_H^{(n)}[V, \alpha_n] \equiv \left( \frac{d}{dX} + 2V \right) \mathcal{L}_n[V' - V^2] - V X - \alpha_n = 0, \quad n = 1, 2, 3, \ldots
\end{equation}
where we have excluded the trivial case $n = 0$ from consideration. We note for future reference that the $P_H$ hierarchy has the discrete symmetry $(V, \alpha_n) \rightarrow (-V, -\alpha_n)$, inherited from the associated discrete symmetry $W \rightarrow -W$ of the mKdV hierarchy.

Since $\mathcal{L}_1[U] = U$ we see that the first member of this hierarchy is the second Painlevé equation \cite{1}. The next member of this hierarchy is the fourth order equation
\begin{equation}
P_H^{(2)}[V, \alpha_2] \equiv V''' - 10V^2V'' - 10V (V')^2 + 6V^3 - XV - \alpha_2 = 0.
\end{equation}
In what follows, our results on Bäcklund transformations and special integrals will all be presented in general, i.e. for any member of the $P_H$ hierarchy. However, in order to provide concrete examples, we will also give specific results in the special cases $n = 1$, i.e. $P_H^{(1)}$ \cite{3}, and $n = 2$, i.e. $P_H^{(2)}$ \cite{18}.

3. Bäcklund transformation for the $P_H$ hierarchy

3.1. Painlevé truncation for the $P_H$ hierarchy

For PDEs important information can be obtained using a so-called truncated Painlevé expansion \cite{18,19}. Here we consider the application of this technique to ODEs, and the modifications that must be made in order to extract information about Bäcklund transformations. We take as our example the $P_H$ hierarchy \cite{3}. We begin by making the change of variables
\begin{equation}
V = \frac{1}{2} \sigma''',
\end{equation}
which gives
\[
\left( \frac{d}{dX} + \frac{\sigma''}{\sigma'} \right) L_n \left[ \frac{1}{2} S(\sigma) \right] - \frac{1}{2} \frac{\sigma''}{\sigma'} X - \alpha_n = 0, \quad n = 1, 2, \ldots
\]  
(10)

where
\[
S(\sigma) = \frac{d}{dX} \left( \frac{\sigma''}{\sigma'} \right) - \frac{1}{2} \left( \frac{\sigma''}{\sigma'} \right)^2
\]
is the Schwarzian derivative of \( \sigma \). Since the Schwarzian derivative is invariant under the action of the Möbius group, setting \( \sigma = -1/\varphi \) gives
\[
V = -\frac{\varphi'}{\varphi} + \frac{1}{2} \frac{\varphi''}{\varphi'},
\]
which is a solution of the \( P_H \) hierarchy provided that
\[
\left( \frac{d}{dX} + \frac{\varphi''}{\varphi'} - \frac{2 \varphi'}{\varphi^2} \right) L_n \left[ \frac{1}{2} S(\varphi) \right] - \frac{1}{2} \left( \frac{\varphi''}{\varphi'} - \frac{2 \varphi'}{\varphi^2} \right) X - \alpha_n = 0, \quad n = 1, 2, \ldots
\]  
(12)

It then follows that \( V \) given by
\[
V = -\frac{\varphi'}{\varphi} + \tilde{V},
\]
where
\[
\tilde{V} = \frac{1}{2} \frac{\varphi''}{\varphi'},
\]
is a solution of the \( P_H \) hierarchy provided that
\[
\left( \frac{d}{dX} + 2 \tilde{V} - \frac{2 \varphi'}{\varphi} \right) L_n [\tilde{V}' - \tilde{V}^2] - \left( \tilde{V} - \frac{\varphi'}{\varphi} \right) X - \alpha_n = 0, \quad n = 1, 2, \ldots
\]  
(15)

holds.

Equation (15), or equivalently equation (13), is the result of substituting a truncated Painlevé expansion into the \( P_H \) hierarchy. The Painlevé expansion for the principal family \( V = -\varphi'/\varphi + \cdots \) has resonances at \(-1\), at \( 2, 3, 4, \ldots, 2n - 2, 2n - 1 \), and at \( 2n + 2 \) (these are the same as those for the corresponding family of the mKdV hierarchy [23], with the omission of that at \( 2n + 1 \), which arises from the extra differentiation of the dominant terms in [1]). Substitution of the truncated expansion (13) into the \( P_H \) hierarchy therefore leads at \( \varphi^{-2n} \) to the determination of \( \tilde{V} \) as given by (14), and then at \( \varphi^{-2n+1}, \ldots, \varphi^{-2} \) we find that all coefficients vanish since these correspond to the resonances \( 2, 3, 4, \ldots, 2n - 2, 2n - 1 \). We are then left with the terms at \( \varphi^{-1} \) and \( \varphi^{0} \), and it is these terms that are given by (15).

Setting coefficients of different powers of \( \varphi \) to zero independently then gives
\[
L_n [\tilde{V}' - \tilde{V}^2] - \frac{1}{2} X = 0
\]  
(16)

and
\[
\left( \frac{d}{dX} + 2 \tilde{V} \right) L_n [\tilde{V}' - \tilde{V}^2] - \tilde{V} X - \alpha_n = 0.
\]  
(17)

Together these two equations imply that we must have \( \alpha_n = \frac{1}{2} \). The result of using a truncated Painlevé expansion is therefore that if \( \tilde{V} \) satisfies (16), then \( V \) given by (13) and (14) is a solution of (15) for \( \alpha_n = \frac{1}{2} \). That is, we obtain a map from (16) to (15), though not an auto-Bäcklund transformation for the \( P_H \) hierarchy.

We do, however, obtain information about special integrals of the \( P_H \) hierarchy. Since \( \tilde{V}' - \tilde{V}^2 = V' - V^2 \), our result is that any solution of
\[
L_n [V' - V^2] - \frac{1}{2} X = 0
\]
provides a solution of (15) for \( \alpha_n = \frac{1}{2} \). Therefore this defines the special integral \( I_{1/2}^{(n)} \)
\[
I_{1/2}^{(n)} = L_n [V' - V^2] - \frac{1}{2} X = 0.
\]  
(18)
This can also be seen from the fact that the $P_I$ hierarchy \( \Phi \) can be written
\[
\left( \frac{d}{dX} + 2V \right) (L_n[V' - V^2] - \frac{1}{2}X) + (\frac{1}{2} - \alpha_n) = 0.
\]

From the point of view of the truncated Painlevé expansion, since
\[
\bar{V}' - \bar{V}^2 = \frac{1}{2}S(\varphi) = \frac{1}{2}S(\sigma),
\]
our result is that if $\varphi$ satisfies
\[
L_n \left[ \frac{1}{2}S(\varphi) \right] - \frac{1}{2}X = 0,
\]
or equivalently $\sigma$ satisfies
\[
L_n \left[ \frac{1}{2}S(\sigma) \right] - \frac{1}{2}X = 0,
\]
then
\[
V = -\frac{d}{dX} \left( \log \left[ \varphi(\varphi')^{-1/2} \right] \right) = -\frac{d}{dX} (\log \sigma')
\]
is a solution of the $P_I$ hierarchy for $\alpha_n = \frac{1}{2}$. The change of variables
\[
\sigma' = \frac{\varphi'}{\varphi^2} = \frac{1}{\psi^2}
\]
then gives that
\[
V = -\frac{\psi'}{\psi}
\]
is a solution of (18) for any (nonzero) $\psi$ a solution of
\[
L_n [-\psi''/\psi] - \frac{1}{2}X = 0.
\]
In the special case $n = 1$, (18) becomes (8), and (20) becomes (19). For $n = 2$ we obtain the special integral
\[
I_{1/2}^{(2)} \equiv V''' - 2VV'' + (V')^2 - 6V^2V' + 3V^4 - \frac{1}{2}X
\]
which provides solutions of (8) when $\alpha_2 = \frac{1}{2}$.

We note that in the case $n = 1$, this result that the truncated Painlevé expansion leads only to special integrals of $P_I$ appears in [23]. Weiss, again for $n = 1$, attempted to overcome this problem [24] by considering the symmetries of a particular integral (the integration constant is set equal to zero) of equation (11). In this way he was able to obtain, albeit rather implicitly, the auto-Bäcklund transformation for $P_I$. In what follows we give an alternative and altogether much more explicit derivation of the auto-Bäcklund transformations of the entire $P_I$ hierarchy.

### 3.2. Our approach: derivation of the Bäcklund transformation

Thus far we have obtained the special integrals (18) of the $P_I$ hierarchy. Special integrals in the case $\alpha_n = \frac{1}{2}$ are obtained using the discrete symmetry $(V, \alpha_n) \rightarrow (V, -\alpha_n)$. However our interest is in finding auto-Bäcklund transformations for the $P_I$ hierarchy. It is clear that in order to do so we need to modify the above truncation approach.

The first point to notice is that equations (8) and (17) are copies of the $P_I$ hierarchy in $V$ and $\bar{V}$, but with the same value of the parameter $\alpha_n$. Yet it is known that auto-Bäcklund transformations for ODEs, in particular the Painlevé equations, may involve changes in the value of any parameters in the ODE. Our first step therefore is to assume that $\bar{V}$ is a solution of the $P_I$ hierarchy for a different choice of parameter, that is
\[
P_I^{(n)}[\bar{V}, \bar{\alpha}_n] \equiv \left( \frac{d}{dX} + 2\bar{V} \right) L_n[\bar{V}' - \bar{V}^2] - \bar{V}X - \bar{\alpha}_n = 0, \quad n = 1, 2, \ldots.
\]

In the special case $n = 1$, (22) becomes (8), and (21) becomes (20). For $n = 2$ we obtain the special integral
\[
l^{(2)}_{1/2} \equiv V''' - 2VV'' + (V')^2 - 6V^2V' + 3V^4 - \frac{1}{2}X
\]
which provides solutions of (8) when $\alpha_2 = \frac{1}{2}$.
We now return to equation (13). Our added assumption that \( \tilde{V} \) satisfies (22) means that we now have a Bäcklund transformation (13) from the \( P_I \) hierarchy in \((\tilde{V}, \tilde{\alpha}_n)\), (22), to the \( P_I \) hierarchy in \((V, \alpha_n)\), (7), provided that (14) and

\[
\left(2\mathcal{L}_n[\tilde{V}' - \tilde{V}^2] - X\right)\frac{\varphi'}{\varphi} + (\alpha_n - \tilde{\alpha}_n) = 0, \quad n = 1, 2, \ldots
\]  

(23)

hold. We note now that we cannot set all coefficients of different powers of \( \varphi \) equal to zero in (23), since this would force \( \alpha_n = \tilde{\alpha}_n \) and we would be back where we were before, i.e. with the standard Painlevé truncation. What we do therefore is to use (23) to eliminate \( \varphi \) from our Bäcklund transformation. This is the most important difference between our approach and that usually used for PDEs. Our truncated Painlevé expansion can then be rewritten in terms of \( \tilde{V} \), a second solution of our ODE, only. For PDEs emphasis is usually placed on the singular manifold equation (14), (22), but we do not actually solve for \( \varphi \) as we do here.

The compatibility of (22) with (14) requires

\[
\left(\frac{d}{dX} + 2\tilde{V}\right)\mathcal{L}_n[\tilde{V}' - \tilde{V}^2] - \tilde{V}X + \frac{1}{2}(\alpha_n - \tilde{\alpha}_n - 1) = 0, \quad n = 1, 2, \ldots,
\]

and this last is consistent with (22) provided that

\( \alpha_n = 1 - \tilde{\alpha}_n \).

Using (23), we can now rewrite our truncated Painlevé expansion (13) as

\[ V = \tilde{V} + \frac{2\tilde{\alpha}_n - 1}{X - 2\mathcal{L}_n[\tilde{V}' - \tilde{V}^2]} \]

It is this last pair of equations that form the Bäcklund transformation for the \( P_I \) hierarchy, mapping the pair \((\tilde{V}, \tilde{\alpha}_n)\) to the pair \((V, \alpha_n)\). Taking into account also the discrete symmetry of the \( P_I \) hierarchy, \((V, \alpha_n) \rightarrow (-V, -\alpha_n)\), we can write our Bäcklund transformation as

\[ V = \tilde{V} + \frac{2\alpha_n - \varepsilon}{X - 2\mathcal{L}_n[\varepsilon V' - V^2]}, \quad \alpha_n = \varepsilon - \tilde{\alpha}_n, \]  

(24)

where \( \varepsilon = \pm 1 \). Using this discrete symmetry is equivalent to using the truncation \( V = (\varphi' / \varphi) + \tilde{V} \), that is, the truncation for the second principal family of the \( P_I \) hierarchy. We note that this Bäcklund transformation is an involution, if we iterate keeping the same value of \( \varepsilon \). We also note that it preserves the quantity \( \varepsilon V' - V^2 \); for \( \tilde{V} \) a solution of (22), we have \( \varepsilon V' - V^2 = \varepsilon \tilde{V}' - \tilde{V}^2 \).

Thus we have obtained, using a modification of the truncated Painlevé expansion technique, the auto-Bäcklund transformations of Airault [6], one for each member of the \( P_I \) hierarchy. These could now be used to generate sequences of rational solutions for members of the \( P_I \) hierarchy or, as we shall see later, sequences of special integrals. We note that an alternative approach to the construction of rational solutions for members of the \( P_I \) hierarchy can be found in [27].

For \( n = 1 \), equations (24) give the Bäcklund transformation (3) for \( P_{I34} \). If \( n = 2 \) then it gives

\[ V = \tilde{V} + \frac{2\alpha_2 - \varepsilon}{X - 2[\varepsilon \tilde{V}''' - 2\tilde{V}V'' + (\tilde{V}')^2 - 6\varepsilon \tilde{V}^2 \tilde{V}' + 3\tilde{V}^4]}, \quad \alpha_2 = \varepsilon - \tilde{\alpha}_2 \]

(25)

with \( \varepsilon = \pm 1 \). It is easy to verify that this is indeed a Bäcklund transformation for equation (8). The Bäcklund transformation in the special case \( n = 2 \) can also be found in (14), (28).

4. Bäcklund transformation to a \( P_{34} \) hierarchy

4.1. \( P_{34} \) hierarchy

Consider the \( P_I \) hierarchy (7), and we write

\[ Y = V' - V^2 \]

(26)

to obtain

\[
\left(\frac{d}{dX} + 2V\right)\mathcal{L}_n[Y] - VX - \alpha_n = 0.
\]

(27)
Equations (26) and (27) then provide a Bäcklund transformation. The result of eliminating $Y$ between these two equations is that $V$ satisfies (31). However since $\mathcal{L}_n[Y]$ contains $d^{2n-2}Y/dX^{2n-2}$, we see that the elimination of $V$, by solving (27) for $V$ and substituting into (26), also yields an ODE of order $2n$. That is, assuming that $\mathcal{L}_n[Y] - \frac{1}{2}X \neq 0$, we have an invertible transformation

$$Y = V' - V^2,$$

$$V = -\frac{1}{2\mathcal{L}_n[Y] - X} \left[ \frac{d}{dX} (\mathcal{L}_n[Y]) - \alpha_n \right]$$

between (7) and (28). For this case we obtain the Bäcklund transformation

$$(2\mathcal{L}_n[Y] - X) \frac{d^2}{dX^2} (\mathcal{L}_n[Y]) - \left( \frac{d}{dX} (\mathcal{L}_n[Y]) \right)^2 + \frac{d}{dX} (\mathcal{L}_n[Y])$$

$$+ (2\mathcal{L}_n[Y] - X)^2 Y - \alpha_n (1 - \alpha_n) = 0.$$  

(30)

This sequence of ODEs (30) is a hierarchy of higher order $P_{34}$ equations, and shall henceforth be referred to as the $P_{34}$ hierarchy (see [28] for a different derivation). Differentiating (28) gives

$$\left( \frac{d^3}{dX^3} + 4Y \frac{d}{dX} + 2Y' \right) \mathcal{L}_n[Y] - XY' - 2Y = 0,$$

that is,

$$\frac{d}{dX} (\mathcal{L}_{n+1}[Y]) - XY' - 2Y = 0.$$  

(31)

This last is just the similarity reduction of the KdV hierarchy (3) obtained via

$$U = \frac{Y(X)}{[(2n+1)t_{2n+1}]^{2/(2n+1)}}, \quad X = \frac{x}{[(2n+1)t_{2n+1}]^{1/(2n+1)}},$$

which is as should be expected for a $P_{34}$ hierarchy.

The Bäcklund transformation (28) appears in [3], but the $P_{34}$ hierarchy (30) does not. Instead the result of eliminating $V$ was found to be (31). In [17], the result that for any solution $V$ of (7), $Y$ defined by (28) gives a solution of (31) was also obtained, but in the special case $\alpha_n = \frac{1}{2}$.

For $n = 1$ we obtain the Bäcklund transformation

$$Y = V' - V^2, \quad V = -\frac{Y' - \alpha_1}{2Y - X}$$

between $P_I$, i.e. equation (7) with parameter $\alpha = \alpha_1$, and

$$(2Y - X)Y'' - (Y')^2 + Y' + (2Y - X)^2 Y - \alpha_1 (1 - \alpha_1) = 0.$$  

(33)

This equation can be mapped to $P_{34}$ of [3]. The Bäcklund transformation (28) for this case $n = 1$, can be found in [3] (with $(V, \alpha_1) \to (\alpha_1, -V_1)$).

For $n = 2$ we obtain the invertible transformation

$$Y = V' - V^2, \quad V = -\frac{Y''' + 6Y'Y'' - \alpha_2}{2(Y'' + 3Y^2) - X}$$

between equation (3) and

$$\left[ 2(Y'' + 3Y^2) - X \right] \frac{d^2}{dX^2} (Y'' + 3Y^2) - \left( \frac{d}{dX} (Y'' + 3Y^2) \right)^2 + \frac{d}{dX} (Y'' + 3Y^2)$$

$$+ \left[ 2(Y'' + 3Y^2) - X \right] Y - \alpha_2 (1 - \alpha_2) = 0.$$  

This is the fourth order equation in the $P_{34}$ hierarchy, which we give here for the first time.

Note that by considering what happens when the above Bäcklund transformation breaks down, we find that:

(i) if $Y$ is a solution of $2\mathcal{L}_n[Y] - X = 0$ then it is also a solution of (31);

(ii) from such a solution of $2\mathcal{L}_n[Y] - X = 0$ a solution $V$ of (7) in the special case $\alpha_n = \frac{1}{2}$ can be obtained from (28).

These two results can in fact be found in [17]; the first appears also in [3] for the special cases $n = 1$ and $n = 2$. Note that the hierarchy $\mathcal{L}_n[Y] - (X/2) = 0$, which gives rise to special integrals of the $P_I$ hierarchy, is just the $P_I$ hierarchy.
4.2. An alternative formulation

Here we give an alternative description of the $P_{34}$ hierarchy $[P]$. This we do by writing down a second hierarchy of ODEs which has $[33]$ as its first member, and then giving an invertible transformation between this second hierarchy and the hierarchy $[30]$. Let us consider once again the hierarchy of ODEs which has (33) as its first member, and then giving an invertible transformation between (28) we write

$$Z = L_n[V' - V^2]$$

(34)

to obtain

$$Z' + 2VZ - VX - \alpha_n = 0.$$  

(35)

Equations (34) and (35) again provide a Bäcklund transformation such that the equation satisfied by $V$ is $[7]$. Since $L_n[V' - V^2]$ contains $d^{2n-1}Z/dX^{2n-1}$, we see that the elimination of $V$ yields an ODE in $Z$ of order $2n$. That is, we have the invertible transformation

$$Z = L_n[V' - V^2],\quad V = -\frac{Z' - \alpha_n}{2Z - X}$$

between $[3]$ and

$$L_n\left[\frac{(Z')^2 - Z' - Z''(2Z - X) + \alpha_n(1 - \alpha_n)}{(2Z - X)^2}\right] - Z = 0.$$  

(36)

It is easy to see that for $n = 1$ equation (36) reduces to (33), and that for $n = 2$ it gives

$$\frac{d^2}{dX^2}\left[\frac{(Z')^2 - Z' - Z''(2Z - X) + \alpha_n(1 - \alpha_n)}{(2Z - X)^2}\right] + 3\left[\frac{(Z')^2 - Z' - (2Z - X)Z'' + \alpha_n(1 - \alpha_n)}{(2Z - X)^2}\right]^2 - Z = 0.$$

The hierarchy (36) is in fact an alternative formulation of the $P_{34}$ hierarchy (30). This is easily seen by considering the Bäcklund transformation

$$Z = L_n[Y],\quad Y = \frac{(Z')^2 - Z' - (2Z - X)Z'' + \alpha_n(1 - \alpha_n)}{2Z - X},$$

which provides an invertible transformation between (30) and (33).

5. Bäcklund transformations for the case $\alpha_n = \frac{1}{2}$

We now consider once again the Bäcklund transformation $[28][29]$, in the special case $\alpha_n = \frac{1}{2}$. Just because we take $\alpha_n = \frac{1}{2}$ does not mean that $[28][29]$ breaks down. Writing $L_n[Y] - \frac{1}{2}X = \delta \psi^2$, where $\delta$ is a constant, then gives a Bäcklund transformation

$$L_n[V' - V^2] - \frac{1}{2}X - \delta \psi^2 = 0,$$

$$\psi' + V\psi = 0.$$  

(37)

(38)

Elimination of $\psi$ shows that $V$ satisfies

$$\left(\frac{d}{dX} + 2V\right)L_n[V' - V^2] - VX - \frac{1}{2} = 0,\quad n = 1, 2, 3, \ldots,$$

(39)

as should be expected. Conversely, elimination of $V$ shows that $\psi$ satisfies

$$L_n[-\psi''/\psi] - \frac{1}{2}X - \delta \psi^2 = 0,\quad n = 1, 2, 3, \ldots.$$  

(40)

For $\delta = 0$ we obtain from (37) the special integrals $[18]$, with equation (10) becoming (27). However, for $\delta \neq 0$, what we have is a Bäcklund transformation between the two equations (28) and (40), each of which is of order $2n$. We note that the Bäcklund transformation $[28][29]$ can also be obtained by considering the general integral of (10) in the special case $\alpha_n = \frac{1}{2}$ (this is the case where the constant of integration cannot be removed by a simple linear transformation $\sigma \rightarrow \sigma + c$, for some constant $c$).
Equations (41) are of course related to those of the \( P_{44} \) hierarchy (30) with \( \alpha_n = \frac{1}{2} \) through the Bäcklund transformation
\[
\mathcal{L}_n[Y] - \frac{1}{2}X - \delta \psi^2 = 0, \tag{41}
\]
\[
\psi'' + Y \psi = 0, \tag{42}
\]
as is easily shown by the elimination of \( Y \) and \( \psi \) respectively from (41,42). We note that when \( \delta = 0 \) the system (41,42) provides an alternative formulation of the special integrals (18). It is straightforward, using the symmetry \((V, \alpha_n) \to (-V, -\alpha_n)\), to give Bäcklund transformations corresponding to (37,38) and (41,42) in the case \( \alpha_n = -\frac{1}{2} \).

The reason for our interest in this special case \( \alpha_n = \frac{1}{2} \), and the hierarchy (40), will now become clear.

5.1. A special auto-Bäcklund transformation for \( P_H \)

For \( n = 1 \) we obtain from the above the Bäcklund transformation
\[
V' - V^2 - \frac{1}{2}X - \delta \psi^2 = 0, \tag{43}
\]
\[
\psi' + V \psi = 0, \tag{44}
\]
between
\[
V'' - 2V^3 - VX - \frac{1}{2} = 0
\]
and
\[
\psi'' + \delta \psi^3 + \frac{X}{2} \psi = 0. \tag{45}
\]

For \( \delta = 0 \), equation (43) becomes (3), the first integral \( I^{(1)}_{1/2} \), and equation (45) becomes (4), giving rise to the Airy function solutions of \( P_H \).

However, for \( \delta \neq 0 \), then (43) is equivalent to \( P_H \) with parameter \( \alpha_1 = 0 \). Thus (43) and (44) provide a special auto-Bäcklund transformation for \( P_H \), mapping between solutions of \( P_H \) for \( \alpha_1 = 0 \) and solutions for \( \alpha_1 = \frac{1}{2} \). This Bäcklund transformation is not well known though it was first written down by Gambier[8]. It is easy to see from (43) and (44) that this Bäcklund transformation provides a connection between the simplest Airy function solution of \( P_H \) (for \( \alpha_1 = \frac{1}{2} \)) and the zero solution of \( P_H \) (for \( \alpha_1 = 0 \)), and thus between the Airy function solution hierarchy of \( P_H \) (for \( \alpha_1 = n + 1/2 \), with \( n \) an integer) and the rational solution hierarchy of \( P_H \) (for \( \alpha_1 \) an integer).

5.2. A Bäcklund transformation for \( \alpha_2 = \frac{1}{2} \)

For \( n = 2 \) we obtain from (37,38) the Bäcklund transformation
\[
V''' - 2VV'' + (V')^2 - 6V^2V' + 3V^4 - \frac{1}{2}X - \delta \psi^2 = 0, \tag{46}
\]
\[
\psi' + V \psi = 0, \tag{47}
\]
between
\[
V'''' - 10V^2V''' - 10V (V')^2 + 6V^5 - VX - \frac{1}{2} = 0 \tag{48}
\]
and
\[
\psi^2 \psi''' - 2\psi \psi' \psi'' - 4\psi(\psi'')^2 + 2(\psi'')^2 \psi'' + \frac{1}{2}X \psi^3 + \delta \psi^5 = 0.
\]

For \( \delta = 0 \), (46) gives the special integral \( I^{(2)}_{1/2} \) (21) of (48). This special integral is in fact equivalent to a special case of Chazy Class XI \((k = 3)\) (29), the additional non-dominant terms being such that it has, according to Chazy, the Painlevé property. In the variables used by Chazy \((y = -\frac{1}{2}V, z = -\frac{1}{2}V, a = 0, b = -\frac{1}{2}X)\), this special integral can be written (compare with (11,42) for \( n = 2 \) and \( \delta = 0 \)),
\[
z'' = 6z^2 - \frac{1}{2}X, \tag{49}
\]
\[
\psi'' = 2x \psi \tag{50}
\]
the second of which is a generalised Lamé equation, the usual elliptic function being replaced by a solution of the first Painlevé equation \( P_1 \) (a suitable rescaling of \( z \) and \( X \) brings (12,42) to standard form). The special integrals (18) then give a whole hierarchy of such equations; the system (37,38), or equivalently the system (41,42), then represents a further generalisation. In each case \( V \) satisfies a member of the \( P_H \) hierarchy.

† We are grateful to Chris Cosgrove for this information.
6. Special integrals of the $P_H$ hierarchy

In this section we consider the derivation of special integrals for the $P_H$ hierarchy. We do this by choosing $\varepsilon = 1$ and then using the mapping $(V, \alpha_n) \to (-V, -\alpha_n)$ so that our Bäcklund transformation (24) becomes

$$V = -\tilde{V} - \frac{2\tilde{\alpha}_n - 1}{X - 2\tilde{\mathcal{L}}_n[V' - V^2]}, \quad \alpha_n = -1 + \tilde{\alpha}_n,$$

(51)

Beginning with the special integrals (18),

$$I^{(n)}_{1/2} = \mathcal{L}_n[V' - V^2] - \frac{1}{4} X = 0,$$

(52)

which define solutions of (4) for $\alpha_n = \frac{1}{2}$, the Bäcklund transformation (51) then allows us to express these special integrals in terms of solutions of (4) but now for $\alpha_n = \frac{3}{2}, \frac{5}{2}, \ldots$ (so for the first step we take $\alpha_n = \frac{1}{2}$ and $\tilde{\alpha}_n = \frac{1}{2}$ and express (52) in terms of $\tilde{V}$). Special integrals for $\alpha_n = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$ are obtained using the discrete symmetry $(V, \alpha_n) \to (-V, -\alpha_n)$. In this way we are able to give for each member of the $P_H$ hierarchy a sequence of special integrals, one for each half odd integer value of $\alpha_n$.

For $n = 1$, the Bäcklund transformation (51), is

$$V = -\tilde{V} - \frac{2\tilde{\alpha}_n - 1}{X - 2(V' - V^2)}, \quad \alpha_n = -1 + \tilde{\alpha}_n.$$

Beginning with the special integral $I^{(1)}_{1/2}$ (3), that is,

$$I^{(1)}_{1/2} = V' - V^2 - \frac{1}{2} X = 0,$$

we obtain immediately the corresponding special integral for $\alpha_1 = -\frac{1}{2}$,

$$I^{(1)}_{-1/2} = V' + V^2 + \frac{1}{2} X = 0,$$

and then using the above procedure we obtain

$$I^{(1)}_{\pm 3/2} = (V')^3 \equiv (V^2 + \frac{1}{2} X) (V')^2 - (V^4 + XV^2 \pm 4V + \frac{1}{2} X^2) V'
\pm V^6 \pm \frac{1}{4} X V^4 + 4V^3 \pm \frac{1}{2} X^2 V^2 + 2X V \pm \frac{1}{8} X^3 \pm 2 = 0,$$

and

$$I^{(1)}_{\pm 5/2} = (V')^5 \equiv \left(V^2 + \frac{1}{2} X\right) (V')^4 - \left(2V^4 + 2XV^2 \pm 12V + \frac{1}{2} X^2\right) (V')^3
\pm 2V^6 \pm 3X V^4 + 12V^3 \pm \frac{3}{2} X^2 V^2 + 6X V \pm \frac{1}{4} X^3 \pm 6 (V')^2
\pm \left(V^8 + 2X V^6 \pm 12V^5 \pm \frac{3}{2} X^2 V^4 \pm 12X V^3 + \frac{1}{2} X^3 V^2 \pm 36V^2
\pm 3X^2 V + \frac{1}{16} X^4 \pm 2X\right) V' \equiv V^{10} \pm \frac{5}{2} X V^8 - 12V^7 \pm \frac{5}{2} X^2 V^6
\pm 18X V^5 \pm \frac{5}{4} X^3 V^4 \pm 24V^4 - 9X^2 V^3 \pm \frac{5}{16} X^4 V^2 \pm 26X V^2
\pm \frac{3}{2} X^3 V \pm 32V \equiv \frac{1}{32} X^5 \pm \frac{5}{2} X^2.$$

It was shown in [3] that in this case of $P_H$ itself, these special integrals $I^{(1)}_{\pm \alpha_1}$, for $\alpha_1 = n + 1/2$ with $n$ an integer, are the only possible special integrals of polynomial type, and that they satisfy the equation

$$\left(\frac{d}{dX} \pm 2V\right) I^{(1)}_{\pm \alpha_1} = 0 \quad \text{(modulo $P_H^{(1)}[V, \pm \alpha_1] = 0$)},$$

as can be easily checked for those examples given above.

For $n = 2$ we take our Bäcklund transformation in the form

$$V = -\tilde{V} - \frac{2\tilde{\alpha}_2 - 1}{X - 2(\tilde{V}'' - 2\tilde{V}'V' + (V')^2 - 6\tilde{V}^2 V' + 3V^4)}, \quad \alpha_2 = -1 + \tilde{\alpha}_2,$$
and, beginning with $I_{1/2}^{(2)}$ (21),
\[ I_{1/2}^{(2)} \equiv V''' - 2VV'' + (V')^2 - 6V^2V' + 3V^4 - \frac{1}{2}X = 0, \]
we obtain immediately
\[ I_{-1/2}^{(2)} \equiv V''' + 2VV'' - (V')^2 - 6V^2V' - 3V^4 + \frac{1}{2}X = 0. \]
Then, by following the above procedure, we obtain a sequence of special integrals, one for each half odd integer value of $\alpha_2$. For $\alpha_2 = \pm \frac{3}{2}$ we get
\[ I_{\pm 3/2}^{(2)} = (V''' - 2VV'' + (V')^2 + 18V^2V' + 3V^4 + \frac{1}{2}X)(V'')^2 - \left[ 4V^2(V'')^2 - 4V(V')^2V'' + 24V^3V'V'' - 12V^5V'' + 2XVV'' + 4V''' + (V')^4 + 12V^2(V')^3 - 102V^4(V')^2 - X(V')^2 + 36V^6V' + 6X^2V' + 9V^8 + 3XV^4 + 8V^3 + \frac{1}{4}X^2 \right] V''' + 8V^3(V'')^3 - \left[ 12V^2(V'')^2 - 24V^4V' + 36V^6 + 6X^2V^2 - 8V \right] (V'')^2 + \left[ 6V(V')^4 - 24V^3(V')^3 + 36V^5(V')^2 + 6XV(V')^2 - 4(V')^2 - 72V^7V' + 12XV^3V' + 24V^2V' + 54V^9 + 18XV^5 - 28V^4 + \frac{3}{2}X^2V + 2X \right] V'' + (V')^6 + 6V^2(V')^5 + 27V^4(V')^4 + \frac{3}{2}X(V')^4 - 180V^6(V')^3 - 6X^2V^2(V')^3 + 81V^8(V')^2 + 9XV^4(V')^2 + 8V^3(V')^2 + \frac{3}{4}X^2(V')^2 + 54V^{10}V' - 18XV^6V' + 48V^8V' + \frac{3}{2}X^2V^2V' + 4V' + 27V^{12} + 27XV^8 + 24V^7 + \frac{9}{4}X^2V^4 - 4X^3V^3 + 4V^2 + \frac{1}{8}X^3 = 0. \]

Unfortunately the length of successive special integrals prevents us from giving them explicitly here. For example, $I_{\pm 5/2}^{(2)}$ contains 318 terms and the first new special integral obtained at $n = 3$, $I_{\pm 3/2}^{(2)}$, contains 281 terms. We have checked that
\[ \left( \frac{d}{dX} \pm 2V \right) I_{\pm \alpha_2}^{(2)} = 0 \quad (\text{modulo } P_{H}^{(2)}|V, \pm \alpha_2] = 0) \]
for $\alpha_2 = \frac{1}{2}$, $\alpha_2 = \frac{3}{2}$ and $\alpha_2 = \frac{5}{2}$, and also that
\[ \left( \frac{d}{dX} \pm 2V \right) I_{\pm \alpha_3}^{(3)} = 0 \quad (\text{modulo } P_{H}^{(3)}|V, \pm \alpha_3] = 0) \]
for $\alpha_3 = \frac{1}{2}$ and $\alpha_3 = \frac{3}{2}$. In general, the special integral $I_{\pm \alpha_n}^{(n)}$, $\alpha_n = \frac{1}{2}, \frac{3}{2}, \ldots$, is a polynomial of degree $2\alpha_n$ in $d^{n-1}V/dX^{2n-1}$ and satisfies
\[ \left( \frac{d}{dX} \pm 2V \right) I_{\pm \alpha_n}^{(n)} = 0 \quad (\text{modulo } P_{H}^{(n)}|V, \pm \alpha_n] = 0). \]

We note that although we have given a method of constructing special integrals for higher members of the $P_H$ hierarchy for $\alpha_n$ any half odd integer, the question of completeness, as studied in (23) for $P_H$, remains open.

7. Conclusions

We have shown how to modify the truncation procedure in Painlevé analysis so that it may be used to find auto-Bäcklund transformations of ODEs. Our main observations are that we have to take into account the possibility of changes in the values of parameters in the ODE, and that this then implies that we cannot
set all coefficients of $\varphi$ equal to zero independently. The main difference between our approach and that for PDEs is that we eliminate the singular manifold from the truncated expansion, which we then rewrite in terms of a second solution of the ODE. This approach is further extended in [30].

We have also given a variety of other Bäcklund transformations, including one between the $P_1$ hierarchy and a $P_{2k}$ hierarchy, and Gambier’s special auto-Bäcklund transformation for $P_1$ itself, this last relating $P_1$ with $\alpha_1 = 0$ to $P_1$ with $\alpha_1 = \frac{1}{2}$. In addition we have also considered the construction of special integrals for the $P_1$ hierarchy, and have shown how to construct such integrals for $\alpha_1$ any half odd integer. Special integrals have not previously been given for the higher members of this hierarchy.

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