A toy model for “elementariness”

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Abstract

Motivated by recent efforts to analyze corrections to Weinberg’s relations for the scattering length and effective range in the presence of a near-threshold bound state, we play around with an instructive toy model for non-relativistic scattering in a central potential. The model allows to interpolate between bound-state configurations of high “compositeness”, where the wave function is spread over a wide region beyond the range of the interaction, and compact configurations of high “elementariness”, where the wave function is confined to a small region around the center of the potential.
I. INTRODUCTION

There has been some renewed interest recently [1–4] in refinements of the concepts of “compositeness” and “elementariness” of bound states and resonances [5–20], which can be extracted from scattering data, e.g. through the Weinberg relations [21] for the scattering length and effective range in the case of near-threshold bound states. In particular, the impact of a non-zero range of the interaction has been under discussion. Having this in mind, it might be amusing, and possibly also instructive, to study these problems in a simple framework familiar from a first course in quantum-mechanical scattering theory, namely, the scattering of non-relativistic particles in a fixed potential \( V(r) \) of a finite range (see also [15] for previous work in this direction).

Before we specify a model potential, let us shortly recapitulate the formalism, in particular the extraction of “compositeness” from partial-wave scattering amplitudes and the Weinberg relations, adapted to our intended model setting.

We consider elastic s-wave scattering of spinless point particles of mass \( \mu \) and energy \( E \) in a local, spherically symmetric, energy-independent potential \( V(\vec{x}) = V(r) \) of finite range. In this case, we can write the s-wave scattering amplitude in the form [22]

\[
    f_0(E) = \left[ (K_0(E))^{-1} - i k(E) \right]^{-1} = \left[ (K_0(E))^{-1} + \kappa(E) \right]^{-1}.
\]

Here we employ the notation \( k(E) = +\sqrt{2\mu E} =: k, \kappa(E) = -ik(E) =: \kappa \). For a bound state \( B \) at \( E = E_B < 0 \), \( k_B = i\kappa_B = i\sqrt{-2\mu E_B} \). \( K_0(E) \) is a meromorphic function of the energy, and real for real \( E \). It is well-known from the beginning of S-matrix theory that the residues of scattering amplitudes at the poles pertaining to bound states are related to normalization factors of the bound-state wave functions [23]. In our notation,

\[
    f_0(E) \rightarrow \text{Res}_B f_0 \frac{E - E_B}{E - E_B} = -\frac{N^2_B}{2\mu(E - E_B)} \quad \text{for} \quad E \rightarrow E_B,
\]

when the wave function outside of the range of the potential assumes its asymptotic form \( N_B e^{-\kappa_B r} Y_{00}(\theta_x, \varphi_x)/r \) (of course, \( Y_{00}(\theta_x, \varphi_x) = 1/\sqrt{4\pi} \), and we use units where \( \hbar = c = 1 \)).

See e.g. [15] for a demonstration of this relation.

If the bound state is located very close to the threshold \( E = 0 \), and \( K_0^{-1} \) has no poles in the region of such small energies \( E \sim E_B \), we can match two linear approximations of \( (K_0(E))^{-1} \),

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namely a truncated Taylor expansion around $E = E_B$ and the first two terms in the effective range expansion,

\[
(K_0(E_B))^{-1} + (E - E_B) \frac{dK_0^{-1}}{dE} \bigg|_{E_B} \approx -\frac{1}{a_0} + r_0 \mu E,
\]

which (using $(K_0(E_B))^{-1} + \kappa_B = 0$ and Eq. (2)) leads to

\[
\kappa_B a_0 \approx \frac{2C^0_B}{1 + C^0_B}, \quad \kappa_B r_0 \approx \frac{C^0_B - 1}{C^0_B},
\]

where the “compositeness” $C^0_B$ is given by

\[
C^0_B := \frac{-\mu}{\kappa_B} \mathrm{Res}_{E_B} f_0 \frac{dk}{dE} \bigg|_{E_B} + \frac{ds}{dE} \bigg|_{E_B},
\]

or $C^0_B = N^2_B/(2\kappa_B)$, while the “elementariness” $E^0_B$ is

\[
E^0_B := 1 - C^0_B = \frac{-\mu}{\kappa_B} \mathrm{Res}_{E_B} f_0 \frac{dk}{dE} \bigg|_{E_B} + \frac{ds}{dE} \bigg|_{E_B}.
\]

Generalizations of Eqs. (5) and (6) are extensively discussed in the literature [1–14], but for the simple setting studied here, the above relations are adequate. To assess the validity of the linear approximations in Eq. (3), we also introduce

\[
X_a := \frac{\kappa_B a_0}{2 - \kappa_B a_0}, \quad X_r := \frac{1}{1 - \kappa_B r_0}.
\]

For a sufficiently small $E_B$, we should expect $X_a \approx X_r \approx C^0_B$. Of course, it will depend on the details of the potential what “sufficiently small” exactly means here.

II. TOY MODEL POTENTIAL

The toy model we shall examine consists of a “spherical well” of radius $d$, surrounded by a “spherical wall” of thickness $\delta$, i.e.

\[
V(r) = V_0 \theta(d-r) + W_0 \theta(r-d) \theta(d+\delta-r), \quad V_0 < 0, \quad W_0 \geq 0, \quad d > 0, \quad \delta > 0,
\]

where $\theta(\cdot)$ are Heaviside step functions, so the potential vanishes for $r > d + \delta$. It is straightforward to find s-wave bound-state solutions $\psi_{B00}(\vec{x}) = \frac{u_{B00}(r)}{r} \mathcal{Y}_{00}(\theta_x, \varphi_x)$ for the Schrödinger equation with this potential:

\[
\begin{align*}
&u_{B00}(r \leq d) = N_B e^{-\kappa_B(d+\delta)} \frac{\sin(\xi_B r)}{\sin(\xi_B d)} \left( \left( \frac{\omega + \kappa_B}{2\omega} \right) e^{\omega \delta} + \left( \frac{\omega - \kappa_B}{2\omega} \right) e^{-\omega \delta} \right), \\
&u_{B00}(d \leq r \leq d + \delta) = N_B \left( \left( \frac{\omega + \kappa_B}{2\omega} \right) e^{(\omega - \kappa_B)(d+\delta)} e^{-\omega r} + \left( \frac{\omega - \kappa_B}{2\omega} \right) e^{-(\omega + \kappa_B)(d+\delta)} e^{\omega r} \right), \\
&u_{B00}(r \geq d + \delta) = N_B e^{-\kappa_B r}, \quad \omega := \sqrt{2\mu W_0 + \kappa_B^2}, \quad \xi_B := \sqrt{-2\mu V_0 - \kappa_B^2}.
\end{align*}
\]
The normalization factor \( N_B \) is determined by \( \int d^3 x |\psi_{B00}(\vec{x})|^2 = 1 \) (as usual, the phase is chosen so that it is real). The bound-state energies must obey the condition

\[
\frac{\omega}{\xi_B} \tan(\xi_B d) + \frac{(\omega + \kappa_B)e^{i\omega\delta} + (\omega - \kappa_B)e^{-i\omega\delta}}{(\omega + \kappa_B)e^{i\omega\delta} - (\omega - \kappa_B)e^{-i\omega\delta}} = 0,
\]

and can be found numerically. For the s-wave amplitude \( f_0 = [K_0^{-1} - ik]^{-1} \), one obtains

\[
K_0 = \frac{1}{k} \frac{g_1 (\xi_V + \xi_W \tan(\xi_V d) \tan(\xi_W d)) + g_2 (\xi_W \tan(\xi_V d) - \xi_V \tan(\xi_W d))}{h_1 (\xi_V + \xi_W \tan(\xi_V d) \tan(\xi_W d)) + h_2 (\xi_W \tan(\xi_V d) - \xi_V \tan(\xi_W d))},
\]

where \( \xi_V := \sqrt{k^2 - 2\mu V_0} \), \( \xi_W := \sqrt{k^2 - 2\mu W_0} \) (\( \xi_V \to \xi_B \), \( \xi_W \to i\omega \) for a bound state). The following table should give an impression of the properties of near-threshold bound states for a chosen parameter set \((V_0, W_0, \mu, d, \delta)\).

| \( W_0 \) | \( V_0 \) | \( \kappa_B \) | \( C_B^0 \) | \( X_a \) | \( X_r \) | \( 2\kappa_B \langle r \rangle_B \) | \( P(r > d + \delta) \) |
|---|---|---|---|---|---|---|---|
| 0 | -1.284 | 0.050 | 1.051 | 1.051 | 1.051 | 1.050 | 0.933 |
| 3 | -2.159 | 0.051 | 1.004 | 1.004 | 1.004 | 1.006 | 0.889 |
| 15 | -3.397 | 0.047 | 0.692 | 0.695 | 0.698 | 0.710 | 0.619 |
| 50 | -4.074 | 0.057 | 0.094 | 0.121 | 0.154 | 0.151 | 0.082 |

The examples in the table are for \((\mu, d, \delta) = (1, 1, 0.2)\). For a chosen \( W_0 \), we have adjusted \( V_0 \) so that a bound state with \( \kappa_B \approx 0.05\mu \) emerges. \( P(r > d + \delta) \) is the probability to find the particle outside of the range of the potential, and \( \langle r \rangle_B := \int_0^\infty dr \ r |u_{B00}(r)|^2 \). The “elementariness” \( 1 - C_B^0 \) will approach one when the “wall height” \( W_0 \to +\infty \). Then,

\[
u_{B00}(r) \to \sqrt{\frac{2}{d}} \theta(d - r) \sin \left( \frac{\pi r}{d} \right), \quad \langle r \rangle_B \to \frac{d}{2},
\]

the bound particle becomes confined to the region \( r < d \) and is shielded from the outside world, while the s-wave amplitude approaches that for scattering on a hard sphere. This appearance of a small impermeable shell is the closest we can get to “pure elementariness” in a potential model.
FIG. 1: The wave functions $\psi_{B00}$ pertaining to the states from the previous table, versus $|\vec{x}| = r$. Gray: $W_0 = 0$, green: $W_0 = 3$, blue: $W_0 = 15$, red: $W_0 = 50$.

To make sure that these observations do not depend crucially on peculiar features of the potential like e.g. the discontinuities at $r = d$ and $r = d + \delta$, we consider in App. A a continuous potential of a similar “well+wall” structure, assembled from harmonic-oscillator potentials. It is seen that the properties of the bound states qualitatively agree with those of the model studied above.

III. CONCLUSIONS

While it might at first seem counterintuitive that one can have a noticeable “elementariness” in a potential-scattering model, it is easily explained. Due to the relation \[ C_0^B = e^{2\kappa_B R} P(r > R) \]
for the “compositeness” (as defined in Eq. (5)), where $R$ is the finite range of the potential (in our case, $R = d + \delta$), it is clear that one can find states of arbitrarily small $C_0^B$ in a potential that prevents the wave function of the low-energy bound state from “leaking out” to the outside region where $V(r) = 0$. The toy model presented in this contribution is a simple example where this can be accomplished. Put differently, the model potential simulates the presence of a real elementary state. The challenge would be to distinguish one possibility from the other for a given set of scattering data. It is conceivable that the model discussed here could serve as a good testing ground for the extraction and interpretation of “elementariness” from scattering amplitudes, before one goes on to deal with more realistic, and much more difficult, problems involving relativistic dynamics, energy-dependent potentials, inelasticity, resonances and so on. For these more subtle questions we refer to the cited literature.
Appendix A: Harmonic well with a harmonic wall

Potential:  
\[ V(r) = \theta(d-r)V_0 \left( 1 - \frac{r^2}{d^2} \right) + \theta(r-d)\theta(d+\delta-r)W_0 \left( 1 - \frac{4}{\delta^2}(r-(d+(\delta/2))^2) \right), \]

with \( V_0 < 0 \), \( W_0 \geq 0 \). We employ the abbreviations \( \kappa_B = \sqrt{-2\mu E_B} \) and
\[
\zeta_V = \sqrt{-2\mu V_0 - \kappa_B^2}, \quad \zeta_W = \sqrt{-2\mu W_0 - \kappa_B^2}, \quad \lambda_V = \frac{\sqrt{-2\mu V_0}}{d}, \quad \lambda_W = \frac{2\sqrt{-2\mu W_0}}{\delta}.
\]

The s-wave solutions can be found as
\[
\psi_{B00}(\vec{x}) = \frac{u_{B00}(r)}{r} Y_{00}(\theta_x, \varphi_x), \quad \text{with}
\begin{align*}
\psi_{B00}(r \leq d) &= C_B re^{-\frac{1}{2} \lambda_V r^2} \text{F}_1 \left( \frac{1}{4} \left( 3 - \frac{\zeta_V^2}{\lambda_V} \right), \frac{3}{2}, \lambda_V r^2 \right), \\
\psi_{B00}(d \leq r \leq d + \delta) &= C_1 \tilde{r} e^{-\frac{1}{2} \lambda_W \tilde{r}^2} \text{F}_1 \left( \frac{1}{4} \left( 3 - \frac{\zeta_W^2}{\lambda_W} \right), \frac{3}{2}, \lambda_W \tilde{r}^2 \right) \\
&\quad + C_2 e^{-\frac{1}{2} \lambda_W \tilde{r}^2} \text{F}_1 \left( \frac{1}{4} \left( 1 - \frac{\zeta_W^2}{\lambda_W} \right), \frac{1}{2}, \lambda_W \tilde{r}^2 \right),
\end{align*}
\]
\[
\psi_{B00}(r \geq d + \delta) = \mathcal{N}_B e^{-\kappa_B r}.
\]

As usual, the constants \( C_{1,2} \) and \( C_B \) as well as possible values of \( \kappa_B \) are found via the constraints of continuity and differentiability at \( r = d \) and \( r = d + \delta \); \( \mathcal{N}_B \) is subsequently fixed by the normalization condition. The following table displays some properties of the least-bound state, for different “wall heights” \( W_0 = 0, 3, 15 \) and \( 50 \), and \( d = 1, \delta = 0.2 \).

| \( W_0 \) | \( V_0 \) | \( \kappa_B \) | \( C^0_B \) | \( X_\alpha \) | \( X_\tau \) | \( 2\kappa_B \langle r \rangle_B \) | \( P(r > d + \delta) \) |
|---|---|---|---|---|---|---|---|
| 0 | -2.647 | 0.050 | 1.042 | 1.042 | 1.042 | 1.041 | 0.924 |
| 3 | -3.613 | 0.050 | 1.000 | 1.000 | 1.000 | 1.002 | 0.887 |
| 15 | -4.984 | 0.049 | 0.752 | 0.754 | 0.756 | 0.767 | 0.668 |
| 50 | -5.740 | 0.048 | 0.206 | 0.223 | 0.242 | 0.247 | 0.184 |
Appendix B: The case of the deuteron

Just for the time being, we shall assume that the deuteron can be described with a finite-range potential of the kind considered in this work. For the deuteron, we have

$$\mu = \frac{m_n m_p}{m_n + m_p} \approx 0.47 \text{ GeV}, \quad \kappa_B \approx \frac{\mu}{10.272}, \quad X_n \approx 1.68, \quad X_r \approx 1.69. \quad (B.1)$$

We can already say that our treatment will at least be more realistic than with a zero-range interaction, because from our formula for $C_B^0$ (see Sec. [III]) we can derive the following estimate for the lower bound of the range,

$$R \gtrapprox \frac{\ln (1.68)}{2 \kappa_B} \approx 2.66 \mu^{-1} \approx 0.79 M^{-1}_\pi, \quad (B.2)$$

since $P(r > R) \leq 1$. Compare also [3] for a very recent study underlining the importance of the interaction range for the deuteron case. More concretely, let us adopt the harmonic potential of App. [A] with $W_0 = 0$, so that effectively $R = d$. One then finds that one can reproduce the data of Eq. (B.1) with

$$V_0 = -0.098 \mu, \quad d = 6.168 \mu^{-1} \quad \Rightarrow \quad C_B^0 = 1.666 \quad \Rightarrow \quad P(r > d) = 0.502. \quad (B.3)$$

This would really be “evidence that the deuteron is not an elementary particle” [21].

Why could one say this? Here is an (admittedly crude) analogy: Why am I confident that a couple in my neighborhood, named Smetana, is composed of two individuals labeled as “Mr. Smetana” and “Mrs. Smetana”, and is not just a single organism one could call e.g. a “Smetanovi”? Well, in the majority of times I see them, I perceive two objects with the typical characteristics of a human being, and I estimate their distance to be more than two arm lengths,

$$P(r > 2\ell(\text{arm})) > 0.5.$$ 

In such an instance, one can in principle examine one of the components individually (put it in a magnetic field, weigh it, turn it upside down, bombard it with antineutrinos, etc.). Sometimes, however, quite rarely, I see them from a distance being so close together that I could suspect that they have merged into one two-headed creature with four legs...

But most likely I just need a better telescope.
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