A parabolic transform and averaging methods for integro-partial differential equations

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Abstract

Averaging methods of the integro-partial differential equation is studied, without any restrictions on the characteristic form of the partial differential operators. By using the parabolic transform and the averaging methods, the integro-partial differential equation can be solved.

Keywords: Averaging method, integro-partial differential equation, parabolic transform, existence and uniqueness of solutions.

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1. Introduction

Consider the following integro-partial differential equation

\[
\frac{\partial u(x,t)}{\partial t} = \psi(x,t) + \varepsilon \int_0^t L(x,t,\theta,D)u(x,\theta)d\theta,
\]

(1.1)

\[
u(x,0) = \varphi(x),
\]

(1.2)

where

\[
L(x,t,\theta,D) = \sum_{|q| \leq m} a_q(x,t,\theta)D^q,
\]

\[
\varepsilon > 0, \; q = (q_1, \cdots, q_n) \text{ is an n-dimensional multi index, } |q| = q_1 + \cdots + q_n, \; D^q = D_1^{q_1} \cdots D_n^{q_n}, \; D_j = \frac{\partial}{\partial x_j},
\]

\[
j = 1, \cdots, n, \; x = (x_1, \cdots, x_n) \in \mathbb{R}^n, \; \mathbb{R}^n \text{ is the n-dimensional Euclidean space, } 0 \leq \theta \leq t \leq T.
\]

Let

\[
S = \{(x,t,\theta) : x \in \mathbb{R}^n, 0 \leq \theta \leq t \leq T\},
\]

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C_{b}(S) is the set of all bounded continuous functions on S and the coefficients \(a_q \in C_{b}(S)\) for all \(q, |q| \leq m\).

Consider the following Cauchy problem [8]
\[
\frac{\partial u(x,t)}{\partial t} = (D_1^{2} + \cdots + D_n^{2})^{2M+1}u(x,t),
\]
\[
u(x,0) = \varphi(x) \in C_{b}(\mathbb{R}^n),
\]
where \(M\) is a sufficiently large positive integer.

Define the operator \(P(t)\) by
\[
P(t)\varphi = \int_{\mathbb{R}^n} G(x-y,t)\varphi(y)dy.
\]

The solution of the Cauchy problem (1.3), (1.4) is given by
\[
u(x,t) = G(t)\varphi = \int_{\mathbb{R}^n} G(x-y,t)\varphi(y)dy,
\]
where the function \(G\) is the fundamental solution of the Cauchy problem (1.3), (1.4) and \(dy = dy_1 \cdots dy_n\).

For sufficiently large \(M\), we find \(\gamma \in (0,1)\) and a constant \(N > 0\) such that
\[
\max_{x} |D^q\nu(x,t)| \leq \frac{N}{t^\gamma} \max_{x} |\varphi(x)|,
\]
for all \(|q| \leq m, m < M, t > 0\).

A parabolic transform of a function \(Q\) is a function \(\tilde{Q}\) defined by [8]
\[
\tilde{Q}(x,t_1,\cdots,t_r,c_1 t + c_2) = \int_{\mathbb{R}^n} G(x-y,t_1 + c_2)Q(y,t_1,\cdots,t_r)dy,
\]
where \(c_1 \geq 0, c_2 \geq 0, t_j, t \in [0,T], j = 1,\cdots,r\) and \(Q(y,t_1,\cdots,t_r) \in C_{b}(\mathbb{R}^n \times [0,T]^r)\).

From (1.1), we have
\[
u(x,t) = \varphi(x) + \int_{0}^{t} \psi(x,t,\theta)\theta d\theta + \epsilon \int_{0}^{t} \sum_{|q| \leq m} b_q(x,t,\theta)D^q\nu(x,\theta)d\theta,
\]
where
\[
b_q(x,t,\theta) = \int_{0}^{t} a_q(x,s,\theta)ds,
\]
let
\[
L_1(x,t,\theta, D) = \sum_{|q| \leq m} b_q(x,t,\theta)D^q.
\]

In Section 2, we study the averaging of the linear operator of the integro-partial differential equation (1.1), (1.2) by using the parabolic transform where we generalize some known results due to Krol [14]. Compare also [1–13, 16].

2. The averaging of the linear operator

Consider the following equation [8]
\[
w(x,t) = \tilde{\varphi}(x,c_1 t) + \int_{0}^{t} \tilde{\psi}(x,\theta,c_1 t)\theta d\theta + \epsilon \int_{0}^{t} \sum_{|q| \leq m} \tilde{b}_q(x,t,\theta,c_1 t)D^q\tilde{w}(x,\theta,c_2 - c_1 \theta)d\theta,
\]
where \(c_1, c_2\) are positive constants and \(c_2 \geq c_1 T\).
If \( \psi, b \in C_b(\mathbb{R}^n \times [0, T]) \), then (2.1) can be solved [8]. Suppose that \( w(x, t, \frac{1}{nT}, \frac{1}{nT}) \) is the solution of (2.1) with \( c_1 = \frac{1}{nT} \) and \( c_2 = \frac{1}{nT} \). Consider the sequence

\[
u_n(x, t) = \int_{\mathbb{R}^n} G(x - y, \frac{1}{nT} - \frac{t}{nT}) w(y, t, \frac{1}{nT}, \frac{1}{nT}) dy.
\] (2.2)

The sequence \( \{\nu_n(x, t)\} \) satisfies the equation

\[
u_n(x, t) = G(\frac{1}{n})\psi + G(\frac{1}{n})\psi + G(\frac{1}{n}) \int_0^t \sum_{|q| \leq m} b_q(x, t, \theta) D^q \nu_n(x, \theta) d\theta.
\]

Let

\[
\bar{L}(x, t, \theta, c_1 t, D) = \sum_{|q| \leq m} a_q(x, t, \theta, c_1 t) D^q,
\]

\[
\bar{L}_1(x, t, \theta, c_1 t, D) = \sum_{|q| \leq m} b_q(x, t, \theta, c_1 t) D^q,
\]

\[
F(x, t) = \int_0^t \bar{\psi}(x, \theta, c_1 t) d\theta,
\]

\[
W(x, t) = \epsilon \int_0^t \sum_{|q| \leq m} b_q(x, t, \theta, c_1 t) D^q \bar{w}(x, \theta, c_2 - c_1 \theta) d\theta.
\]

We have

\[
w(x, t) = \bar{\phi}(x, c_1 t) + F(x, t) + W(x, t),
\]

\[
w(x, 0) = \phi(x, 0).
\]

By averaging the coefficients \( b_q(x, t, \theta) \) and \( \bar{b}_q(x, t, \theta, c_1 t) \) over \( t \), we can average the operators \( L_1(x, t, \theta, D) \) and \( \bar{L}_1(x, t, \theta, c_1 t, D) \),

\[
b_q(x, \theta) = \frac{1}{T} \int_0^T b_q(x, t, \theta) dt,
\]

\[
\bar{b}_q(x, \theta) = \frac{1}{T} \int_0^T \bar{b}_q(x, t, \theta, c_1 t) dt,
\]

for all \( (x, t, \theta), x \in \mathbb{R}^n \) the averaged operators \( L_1(x, \theta, D), \bar{L}_1(x, \theta, D) \) can be produced. As an approximating problem for (1.1), (1.2), we consider the following equation

\[
\frac{\partial u^*(x, t)}{\partial t} = \bar{\psi}(x) + \epsilon \int_0^t \bar{L}(x, \theta, D) u^*(x, \theta) d\theta.
\] (2.3)

With the initial condition

\[
u^*(x, 0) = \varphi(x),
\] (2.4)

we get,

\[
u^*(x, t) = \varphi(x) + t\bar{\psi}(x) + \epsilon \int_0^t \bar{L}_1(x, \theta, D) u^*(x, \theta) d\theta.
\]

As an approximating problem for (2.1), we consider also the following equation

\[
w^*(x, t) = \bar{\phi}(x) + \int_0^t \bar{\psi}(x, \theta) d\theta + \epsilon \int_0^t \bar{L}_1(x, \theta, D) w^*(x, \theta, c_2 - c_1 \theta) d\theta,
\] (2.5)

where

\[
\bar{\phi}(x) = \frac{1}{T} \int_0^T \bar{\phi}(x, c_1 t) dt,
\]
\[
\bar{\psi}(x, \theta) = \frac{1}{T} \int_0^T \tilde{\psi}(x, \theta, c_1 t) \, dt.
\]

If \( \bar{\psi}, \tilde{\psi}_q \in C_b(\mathbb{R}^n) \), then (2.5) can be solved and it is clear that all the derivatives \( D^q w^* \in C_b(\mathbb{R}^n \times [0, T]) \), for all \( |q| \leq m \) [8].

Suppose that \( w^*(x, t, 1/nT, 1/n) \) is the solution of (2.5) with \( c_1 = 1/nT \) and \( c_2 = 1/n \). Let the sequence

\[
u_n^*(x, t) = \int_{\mathbb{R}^n} G(x - y, 1/n - t/nT) w^*(y, t, 1/nT, 1/n) \, dy.
\]

The sequence \( \{u_n^*(x, t)\} \) satisfies the equation

\[
u_n^*(x, t) = G(1/n)\bar{\psi} + G(1/n)\tilde{\psi} + G(1/n) \int_0^t \sum_{|q| \leq m} b_q(x, \theta) D^q u_n^*(x, \theta) \, d\theta.
\]

Let

\[
F_1(x, t) = \int_0^t \bar{\psi}(x, \theta) \, d\theta,
\]

\[
W^*(x, t) = \epsilon \int_0^t L_1(x, \theta, D) \tilde{\psi}(x, \theta, c_2 - c_1 \theta) \, d\theta.
\]

We have

\[
w^*(x, t) = \tilde{\psi}(x, c_1 t) + F_1(x, t) + W^*(x, t),
\]

\[
w^*(x, 0) = \bar{\psi}(x),
\]

another straightforward analysis displays the existence and uniqueness of the solutions of the problems (1.1), (1.2), (2.1), (2.3), (2.4) and (2.5) on the time-scale \( 1/\epsilon \).

We consider the domain \( B = \mathbb{R}^n \times [0, T] \). The norm \( ||u(x, t)||_\infty \) is defined by the supremum norm on \( B \) and denoted by \( ||u(x, t)||_\infty = \sup_B |u(x, t)| \).

**Theorem 2.1.** There exist two sequences \( \{u_n(x, t)\} \) and \( \{u_n^*(x, t)\} \) with the initial conditions \( u_n(x, 0) = \varphi_n(x) \), \( u_n^*(x, 0) = \varphi_n(x) \). If the sequence \( \{\varphi_n(x)\} \) converges to \( \varphi(x) \), then we have the estimate

\[
||u_n(x, t) - u_n^*(x, t)||_\infty = O(\epsilon),
\]

on the time-scale \( 1/\epsilon \).

**Proof.** We consider the following near-identity transformation

\[
\hat{w}(x, t) = w^*(x, t) + \epsilon \int_0^t (L_1(x, t, \theta, c_1 t, D) - \tilde{L}_1(x, \theta, D)) \, d\theta w^*(x, t).
\]

We get

\[
||\hat{w}(x, t) - w^*(x, t)||_\infty = \epsilon \int_0^t (L_1(x, t, \theta, c_1 t, D) - \tilde{L}_1(x, \theta, D)) \, d\theta w^*(x, t) \, ||_\infty = O(\epsilon), \quad \text{on the time-scale} \quad 1/\epsilon.
\]

By differentiating of the near-identity transformation (2.8) and using (2.7), (2.8), we obtain

\[
\frac{\partial \hat{w}(x, t)}{\partial t} = \frac{\partial w^*(x, t)}{\partial t} + \epsilon \int_0^t (L_1(x, t, \theta, c_1 t, D) - \tilde{L}_1(x, \theta, D)) \, d\theta \frac{\partial w^*(x, t)}{\partial t}
\]
where

$$\text{We have}$$

$$\text{We use the barrier functions see [15]. Let the barrier function}$$

$$\left( M_{x}(B_{w}) \right) = \left( w_{x} x_{x} x_{x} \epsilon \epsilon_{t}, 0 \right)$$

$$\text{with initial value } \hat{w}(x, 0) = \tilde{\phi}(x). \text{ Let}$$

$$\frac{\partial}{\partial t} - \epsilon \int_{0}^{t} \frac{\partial}{\partial t} \hat{L}_{1}(x, t, \theta, c_{1} t, D) d\theta = \mathcal{L}.$$ 

$$\text{We have}$$

$$\mathcal{L}(\hat{w} - w^{*}) = O(\epsilon) \text{ on the time-scale } \frac{1}{\epsilon}.$$ 

$$\text{Moreover } \hat{w}(x, 0) - w^{*}(x, 0) = 0.$$ 

$$\text{We use the barrier functions see [15]. Let the barrier function}$$

$$B(x, t) = \epsilon \| M(x, t) \|_{\infty} t + \| J(x, t) \|_{\infty} t$$

$$+ \| \frac{\partial F_{1}(x, t)}{\partial t} - \frac{\partial F(x, t)}{\partial t} \|_{\infty} t^{2}$$

$$+ \frac{1}{2} \epsilon \| \left( t \frac{\partial}{\partial t} \hat{L}_{1}(x, t, \theta, c_{1} t, D) \right) d\theta \left( \frac{\partial F_{1}(x, t)}{\partial t} - \frac{\partial F(x, t)}{\partial t} \right) \|_{\infty} t^{2}$$

$$+ \frac{1}{6} \epsilon^{2} \left( \left( t \frac{\partial}{\partial t} \hat{L}_{1}(x, t, \theta, c_{1} t, D) \right) d\theta \right)^{2} \left( \frac{\partial F_{1}(x, t)}{\partial t} - \frac{\partial F(x, t)}{\partial t} \right) \|_{\infty} t^{3},$$

where

$$M(x, t) = \int_{0}^{t} (\hat{L}_{1}(x, t, \theta, c_{1} t, D) - \tilde{L}_{1}(x, \theta, D)) d\theta \frac{\partial W^{*}(x, t)}{\partial t}$$

$$- \epsilon \int_{0}^{t} \frac{\partial}{\partial t} \hat{L}_{1}(x, t, \theta, c_{1} t, D) d\theta \int_{0}^{t} (\hat{L}_{1}(x, t, \theta, c_{1} t, D) - \tilde{L}_{1}(x, \theta, D)) d\theta w^{*}(x, t),$$

and

$$J(x, t) = \frac{\partial W^{*}(x, t)}{\partial t} - \frac{\partial W(x, t)}{\partial t} + \epsilon \int_{0}^{t} \frac{\partial}{\partial t} \hat{L}_{1}(x, t, \theta, c_{1} t, D) d\theta w$$

$$+ \epsilon \int_{0}^{t} (\hat{L}_{1}(x, t, \theta, c_{1} t, D) - \tilde{L}_{1}(x, \theta, D)) d\theta \frac{\partial F_{1}(x, t)}{\partial t},$$

and the functions (we omit the arguments)

$$Q_{1}(x, t) = \hat{w}(x, t) - w(x, t) - B(x, t), \quad Q_{2}(x, t) = \hat{w}(x, t) - w(x, t) + B(x, t).$$
We get
\[
\mathcal{L} Q_1(x,t) = (\frac{\partial}{\partial t} - \varepsilon \frac{\partial}{\partial t} \mathcal{L}_1(x,t,\theta,c_1,D) \, d\theta) [\hat{w}(x,t) - w(x,t) - B(x,t)]
\]
\[
= J(x,t) - \| J(x,t) \|_\infty + \varepsilon M(x,t) - \varepsilon \| M(x,t) \|_\infty
\]
\[
+ \frac{\partial F_1(x,t)}{\partial t} - \frac{\partial F(x,t)}{\partial t} - \| \frac{\partial F_1(x,t)}{\partial t} - \frac{\partial F(x,t)}{\partial t} \|_\infty
\]
\[
+ \varepsilon \left( \int_0^t \frac{\partial}{\partial t} \mathcal{L}_1(x,t,\theta,c_1,D) \, d\theta \| \frac{\partial F_1(x,t)}{\partial t} - \frac{\partial F(x,t)}{\partial t} \|_\infty \right) t
\]
\[
- \varepsilon \| \int_0^t \frac{\partial}{\partial t} \mathcal{L}_1(x,t,\theta,c_1,D) \, d\theta \| J(x,t) \|_\infty t
\]
\[
+ \frac{1}{2} \varepsilon^2 \left( \int_0^t \frac{\partial}{\partial t} \mathcal{L}_1(x,t,\theta,c_1,D) \, d\theta \| \frac{\partial F_1(x,t)}{\partial t} - \frac{\partial F(x,t)}{\partial t} \|_\infty \right) t^2
\]
\[
- \frac{1}{2} \varepsilon^2 \left( \left( \int_0^t \frac{\partial}{\partial t} \mathcal{L}_1(x,t,\theta,c_1,D) \, d\theta \right)^2 \| \frac{\partial F_1(x,t)}{\partial t} - \frac{\partial F(x,t)}{\partial t} \|_\infty \right) t^2
\]
\[
+ \varepsilon \| \int_0^t \frac{\partial}{\partial t} \mathcal{L}_1(x,t,\theta,c_1,D) \, d\theta \| M(x,t) \|_\infty t
\]
\[
+ \frac{1}{6} \varepsilon^3 \left( \int_0^t \frac{\partial}{\partial t} \mathcal{L}_1(x,t,\theta,c_1,D) \, d\theta \right) \| \int_0^t \frac{\partial}{\partial t} \mathcal{L}_1(x,t,\theta,c_1,D) \, d\theta \| t^3
\]
\[
\times \left( \frac{\partial F_1(x,t)}{\partial t} - \frac{\partial F(x,t)}{\partial t} \right) \| J(x,t) \|_\infty t^2
\]
\[
\leq 0,
\]

Q_1(x,0) = \tilde{\phi}(x) - \tilde{\phi}(x,0),\, \text{similarly},\, \mathcal{L} Q_2(x,t) \geq 0, \, Q_2(x,0) = \tilde{\phi}(x) - \tilde{\phi}(x,0). \, Q_1(x,t) \text{ and } Q_2(x,t) \text{ are bounded, resulting in } Q_1(x,t) \leq 0 \text{ and } Q_2(x,t) \geq 0, \text{ we have}

- B(x,t) \leq \hat{w}(x,t) - w(x,t) \leq B(x,t),

so we can estimate

\[
\| \hat{w}(x,t) - w(x,t) \|_\infty \leq \| B(x,t) \|_\infty = O(\varepsilon),
\]
on the time-scale \( \frac{1}{\varepsilon} \). We apply the triangle inequality to have

\[
\| w(x,t) - w^*(x,t) \|_\infty \leq \| \hat{w}(x,t) - w^*(x,t) \|_\infty + \| \hat{w}(x,t) - w(x,t) \|_\infty
\]
\[
= O(\varepsilon), \, \text{on the time-scale} \frac{1}{\varepsilon}.
\] (2.9)

From (2.2), (2.6) and (2.9), we have

\[
\| u_n(x,t) - u_n^*(x,t) \|_\infty \leq \int_{\mathbb{R}^n} |G(x - y, \frac{1}{n} - \frac{t}{nT})| \| w(y,t, \frac{1}{n}, \frac{1}{n})\|_\infty.
\]
\[
-w^*(y, t, \frac{1}{nT}, \frac{1}{n}) \|_\infty \, dy = O(\varepsilon), \text{ on the time-scale } \frac{1}{\varepsilon}.
\]

3. Conclusion

The integro-partial differential equation can be solved without any restrictions on the characteristic form by using the parabolic transform and the averaging methods.

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