A table of short-period Tausworthe generators for Markov chain quasi-Monte Carlo

Shin Harase\textsuperscript{a,}\textsuperscript{*}

\textsuperscript{a}College of Science and Engineering, Ritsumeikan University, 1-1-1 Nojihigashi, Kusatsu, Shiga, 525-8577, Japan.

Abstract

We consider the problem of estimating expectations by using Markov chain Monte Carlo methods and improving the accuracy by replacing IID uniform random points with quasi-Monte Carlo (QMC) points. Recently, it has been shown that Markov chain QMC remains consistent when the driving sequences are completely uniformly distributed (CUD). However, the definition of CUD sequences is not constructive, so an implementation method using short-period Tausworthe generators (i.e., linear feedback shift register generators over the two-element field) that approximate CUD sequences has been proposed. In this paper, we conduct an exhaustive search of short-period Tausworthe generators for Markov chain QMC in terms of the $t$-value, which is a criterion of uniformity widely used in the study of QMC methods. We provide a parameter table of Tausworthe generators and show the effectiveness in a numerical example using Gibbs sampling.

Keywords: Pseudorandom number generation, Quasi-Monte Carlo, Markov chain Monte Carlo, Polynomial lattice point set, Continued fraction expansion

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1. Introduction

We consider the problem of estimating the expectation $E_{\pi}[f(X)]$ by using Markov chain Monte Carlo (MCMC) methods for a target distribution $\pi$
and some function $f$. For this problem, we want to improve the accuracy by replacing independent and identically distributed (IID) uniform random points with quasi-Monte Carlo (QMC) points. However, typical QMC points (e.g., Sobol', Faure, and Niederreiter–Xing) are not applicable in general. Motivated by a simulation study by Liao [1], Owen and Tribble [2] and Chen et al. [3] proved that Markov chain QMC remains consistent when the driving sequences are completely uniformly distributed (CUD). Here, a sequence $u_0, u_1, u_2, \ldots \in [0,1)$ is said to be CUD if overlapping $s$-blocks $(u_i, u_{i+1}, \ldots, u_{i+s-1})$, $i = 0, 1, 2, \ldots$, are uniformly distributed for every dimension $s \geq 1$.

Levin [4] provided several constructions for CUD sequences, but they are not convenient to implement. Instead, to construct CUD sequences approximately, Tribble and Owen [5] and Tribble [6] proposed an implementation method using short-period linear congruential and Tausworthe generators (i.e., linear feedback shift register generators over the two-element field $\mathbb{F}_2 := \{0,1\}$) that run for the entire period. Chen et al. [7] implemented short-period Tausworthe generators optimized in terms of the equidistribution property, which is a coarse criterion used in the area of pseudorandom number generation (see [5, §8.1] for the complete parameter table). In QMC theory, the $t$-value is a central criterion of uniformity. In fact, typical QMC points (e.g., Sobol', Faure, and Niederreiter–Xing) are optimized in terms of the $t$-value (see [9, 10]).

The aim of this paper is to conduct an exhaustive search of short-period Tausworthe generators for Markov chain QMC in terms of the $t$-value and to provide a parameter table of Tausworthe generators. It is known that Tausworthe generators can be viewed as polynomial Korobov lattice point sets with a denominator polynomial $p(x)$ and a numerator polynomial $q(x)$ over $\mathbb{F}_2$ (e.g., see [11, 12]). For dimension $s = 2$, there is a connection between the $t$-value and continued fraction expansions, that is, the $t$-value is optimal (i.e., the $t$-value is zero) if and only if the partial quotients in the continued fraction of $q(x)/p(x)$ are all of degree one. To satisfy the definition of CUD sequences approximately, we want to search parameters $(p(x), q(x))$ whose $t$-values are optimal for $s = 2$ and as small as possible for $s \geq 3$. As a previous study, in 1993, Tezuka and Fushimi [13] proposed an algorithm for searching such parameters using a polynomial analogue of Fibonacci numbers from the viewpoint of continued fraction expansions. Thus, we refine their algorithm on modern computers, and conduct an exhaustive search again. In addition, we report a numerical example using Gibbs sampling in which the resulting
QMC point sets perform better than the existing point sets developed by Chen et al. [7].

One might consider searching parameters \((p(x), q(x))\) with \(t\)-value zero for \(s = 3\). Kajiura et al. [14] proved that there exists no maximal-period Tausworthe generator with this property.

The remainder of this paper is organized as follows: In Section 2 we briefly recall the definition of CUD sequences, Tausworthe generators, and the \(t\)-value and equidistribution property. Section 3 is devoted to our main results: we describe an exhaustive search algorithm and provide a table of short-period Tausworthe generators for Markov chain QMC. We also compare our new generators with existing generators developed by Chen et al. [7] in terms of the \(t\)-value and equidistribution property. In Section 4 we present a numerical example using two-dimensional Gaussian Gibbs sampling. In Section 5 we conclude this paper.

2. Preliminaries

We refer the reader to [9, 10, 11, 15] for general information.

2.1. Discrepancy and completely uniformly distributed sequences

Let \(P_s = \{u_0, u_1, \ldots, u_{N-1}\} \subset [0, 1)^s\) be an \(s\)-dimensional point set of \(N\) elements in the sense of a “multiset.” We recall the definition of the discrepancy as a criterion of uniformity of \(P_s\).

**Definition 1 (Discrepancy).** For a point set \(P_s = \{u_0, u_1, \ldots, u_{N-1}\} \subset [0, 1)^s\), the (star) discrepancy is defined as

\[
D^{*s}_N(P_s) := \sup_J \left| \frac{\nu(J; P_s)}{N} - \text{vol}(J) \right|
\]

where the supremum is taken over every sub–interval \(J = [0, t_1) \times \cdots \times [0, t_s) \subset [0, 1)^s\), \(\nu(J; P_s)\) is the number of points that belong to \(J\), and \(\text{vol}(J) := t_1 \cdots t_s\) is the volume of \(J\).

If \(D^{*s}_N(P_s)\) is close to zero, we regard \(P_s\) as highly uniformly distributed.

Next, we define the CUD property for a one-dimensional infinite sequence \(\{u_i\}_{i=0}^\infty \subset [0, 1)\).
Definition 2 (CUD sequences). A one-dimensional infinite sequence $u_0, u_1, u_2, \ldots \in [0, 1)$ is said to be completely uniformly distributed (CUD) if overlapping $s$-blocks satisfy
\[
\lim_{N \to \infty} D^s_N \left((u_0, \ldots, u_{s-1}), (u_1, \ldots, u_s), \ldots, (u_{N-1}, \ldots, u_{N+s-2})\right) = 0
\]
for every dimension $s \geq 1$, that is, $s$-blocks $(u_i, \ldots, u_{i+s-1})$, $i = 0, 1, \ldots$, are uniformly distributed in $[0, 1)^s$ for all $s \geq 1$.

This is one of the definitions of a random sequence from Knuth [16]. From the viewpoint of QMC, it is desirable that $D^s_N$ converges to zero fast if $N \to \infty$; see [17, 18] for details. As a necessary and sufficient condition of Definition 2, Chentsov [19] showed that non-overlapping blocks satisfy
\[
\lim_{N \to \infty} D^s_N \left((u_0, \ldots, u_{s-1}), (u_s, \ldots, u_{2s-1}), \ldots, (u_{s(N-1)}, \ldots, u_{sN-1})\right) = 0
\]
for every dimension $s \geq 1$. Thus, we use a sequence $\{u_i\}_{i=0}^\infty \subset [0, 1)$ for Markov chain QMC in this order.

2.2. Tausworthe generators

We recall some results of Tausworthe generators. Let $\mathbb{F}_2 := \{0, 1\}$ be the two-element field, and perform addition and multiplication over $\mathbb{F}_2$ (or modulo 2).

Definition 3 (Tausworthe generators [20, 21, 22]). Let $p(x) := x^m - c_1x^{m-1} - \cdots - c_{m-1}x - c_m \in \mathbb{F}_2[x]$. Consider the linear recurrence
\[
a_i := c_1a_{i-1} + \cdots + c_ma_{i-m} \in \mathbb{F}_2, \quad (1)
\]
whose characteristic polynomial is $p(x)$. Let $\sigma$ be a step size and
\[
u_{j} := \sum_{j=0}^{w-1} a_{i\sigma+j}2^{-j-1} \in [0, 1) \quad (2)
\]
be the output at step $i$, where $w$ is the word size of the intended machine. If $p(x)$ is primitive, $\{a_0, \ldots, a_{m-1}\} \neq \{0, \ldots, 0\}$, and $\gcd(\sigma, 2^m-1) = 1$, then the sequences (1) and (2) are both purely periodic with maximal period $2^m - 1$. Assume the maximal periodicity and $\sigma \geq w$. A generator in such a class is called a Tausworthe generator (or a linear feedback shift register generator).
Let $N = 2^m$ and consider a sequence
\begin{equation}
    u_0, u_1, \ldots, u_{N-2}, u_{N-1} = u_0, \ldots \in [0, 1)
\end{equation}
generated from a Tausworthe generator with the period length $N - 1$. We consider $s$-dimensional overlapping points $u_i = (u_{i}, \ldots, u_{i+s-1})$ for $i = 0, 1, \ldots, N-2$, that is, $u_0 = (u_0, \ldots, u_{s-1}), u_1 = (u_1, \ldots, u_{s}), \ldots, u_{N-2} = (u_{N-2}, u_0, \ldots, u_{s-2})$. Adding the origin $\{0\}$, we regard a point set
\begin{equation}
    P_s = \{0\} \cup \{u_i\}_{i=0}^{N-2} \subset [0, 1)^s
\end{equation}
as a QMC point set. Note that the cardinality is $|P_s| = 2^m$.

Here, Tausworthe generators can be represented as a polynomial analogue of linear congruential generators:
\begin{align}
    q(x) &:= x^\sigma \mod p(x) \quad (5) \\
    X_i(x) &:= q(x)X_{i-1}(x) \mod p(x) \quad (6) \\
    X_i(x)/p(x) &= a_{i\sigma}x^{-1} + a_{i\sigma+1}x^{-2} + a_{i\sigma+2}x^{-3} + \ldots \in F_2((x^{-1})). \quad (7)
\end{align}

Then, the sequence (2) is expressed as $u_i = \nu_w(X_i(x)/p(x))$, where a map $\nu_w : F_2((x^{-1})) \to [0, 1)$ is given by $\sum_{j=j_0}^{\infty} k_j x^{-j-1} \mapsto \sum_{j=\max\{0, j_0\}}^{w-1} k_j 2^{-j-1}$, which is obtained by substituting $x = 2$ into (7) and truncating the value with the word size $w$. Furthermore, a point set $P_s$ in (4) can also be represented as polynomial Korobov lattice point sets:
\begin{equation}
    P_s = \left\{ \nu_w \left( \frac{h(x)}{p(x)}(1, q(x), q(x)^2, \ldots, q(x)^{s-1}) \right) \bigg| \deg(h(x)) < m \right\}, \quad (8)
\end{equation}

where $m = \deg(p(x))$ and the map $\nu_w$ is obtained for each component. A pair of polynomials $(p(x), q(x))$ is a parameter set of $P_s$. Thus, to satisfy Definition [2] approximately, we want to find a pair $(p(x), q(x))$ with small discrepancies $D_{N_s}^s(P_s)$ for each $s \geq 1$.

### 2.3. Criteria of uniformity

Generally, calculating $D_{N_s}^s(P_s)$ is NP-hard [23]. A point set $P_s$ in (4) generated from a Tausworthe generator is a digital net, so we can compute the $t$-value closely related to $D_{N_s}^s(P_s)$ for $N = 2^m$. 
Definition 4 \((t, m, s)\)-nets). Let \(s \geq 1\) and \(0 \leq t \leq m\) be integers. Then, a point set \(P_s\) consisting of \(2^m\) points in \([0, 1)^s\) is called a \((t, m, s)\)-net (in base 2) if every subinterval \(E = \prod_{j=1}^{s} [r_j/2^{d_j}, (r_j + 1)/2^{d_j})\) in \([0, 1)^s\) with integers \(d_j \geq 0\) and \(0 \leq r_j < 2^{d_j}\) for \(1 \leq j \leq s\) and of volume \(2^{t-m}\) contains exactly \(2^t\) points of \(P_s\).

For dimension \(s\), the smallest value \(t\) for which \(P_s\) is a \((t, m, s)\)-net is called the \(t\)-value. \(D^*_N(P_s) = O(2^t(\log N)^{s-1}/N)\) holds, so a small \(t\)-value is desirable. Thus, we want to find Tausworthe generators with pairs of polynomials \((p(x), q(x))\) whose \(t\)-values are optimal (i.e., \(t = 0\)) for \(s = 2\) and as small as possible for \(s \geq 3\). Note that all Tausworthe generators have the \(t\)-value zero for \(s = 1\).

Conversely, Chen et al. \cite{7} used the following equidistribution property as a criterion of uniformity:

Definition 5 \((s\)-dimensional equidistribution with \(l\)-bit accuracy). For \(1 \leq s \leq m\) and \(1 \leq l \leq m\), a point set \(P_s\) consisting of \(2^m\) points in \([0, 1)^s\) is said to be \(s\)-dimensionally equidistributed with \(l\)-bit accuracy if we can partition the \(s\)-dimensional unit cube \([0, 1)^s\) into congruent cubic boxes of volume \(2^{-sl}\) by dividing each axis \([0, 1)\) into \(2^l\) intervals, and can obtain an equal number of points from \(P_s\) in each box.

For dimension \(s\), the largest value of \(l\) for which this definition holds is called the resolution of \(P_s\) and denoted by \(l_s\). We have a trivial upper bound \(l_s \leq \lfloor m/s \rfloor\). As a criterion of uniformity, the high resolution \(l_s\) is desirable. Thus, we define the resolution gap \(d_s = \lfloor m/s \rfloor - l_s\) and the sum of resolution gaps \(\Delta = \sum_{s=1}^{m} d_s\). If \(\Delta = 0\), the generator is said to be fully equidistributed (FE). Note that \(P_s\) consists of the origin \(\{0\}\) and the output values of a Tausworthe generator for the entire period of \(2^m - 1\). Chen et al. \cite{7} searched FE Tausworthe generators for Markov chain QMC.

3. Main result

3.1. An exhaustive search algorithm using Fibonacci polynomials

To satisfy Definition \cite{2} approximately, we search pairs of polynomials \((p(x), q(x))\) whose \(t\)-values are optimal for \(s = 2\) and as small as possible for \(s \geq 3\). Thus, we refine the algorithm of Tezuka and Fushimi \cite{13}.
For dimension \( s = 2 \), there is a connection between the \( t \)-value of polynomial Korobov lattice point sets \(^8\) and continued fraction expansion of \( q(x)/p(x) \). Let

\[
\frac{q(x)}{p(x)} = A_0(x) + \frac{1}{A_1(x) + \frac{1}{A_2(x) + \frac{1}{\ddots + \frac{1}{A_v(x)}}}} =: [A_0(x); A_1(x), A_2(x), \ldots, A_v(x)]
\]

be the continued fraction expansion of the rational function \( q(x)/p(x) \) with a polynomial part \( A_0(x) \in \mathbb{F}_2[x] \) and partial quotients \( A_k(x) \in \mathbb{F}_2[x] \) satisfying \( \deg(A_k(x)) \geq 1 \) for \( 1 \leq k \leq v \).

**Theorem 1** \(^9, 13\). Let \( p(x) \in \mathbb{F}_2[x] \) with \( m = \deg(p(x)) \) and \( q(x) \in \mathbb{F}_2[x] \) with \( \deg(q(x)) < m \). Assume \( \gcd(p(x), q(x)) = 1 \). Then, the two-dimensional point set

\[
P_2 = \left\{ \nu_w \left( \frac{h(x)}{p(x)}(1, q(x)) \right) \mid \deg(h(x)) < m \right\}
\]

is a \((0, m, 2)\)-net (i.e., the \( t \)-value is zero) if and only if the partial quotients in the continued fraction expansion \([0; A_1(x), A_2(x), \ldots, A_v(x)]\) of \( q(x)/p(x) \) all have degree one, so \( v = m \).

The next theorem asserts the existence of \( q(x) \) with the above property for every irreducible polynomial \( p(x) \).

**Theorem 2** \(^{24}\). Let \( p(x) \) be an irreducible polynomial with \( m = \deg(p(x)) \) and \( q(x) \in \mathbb{F}_2[x] \) with \( \deg(q(x)) < m \). For each \( p(x) \), there are exactly two polynomials \( q(x) \) for which the partial quotients of the continued fraction expansion of \( q(x)/p(x) \) all have degree one.

In fact, the two polynomials are \( q(x) \) and \( q^{-1}(x) \mod p(x) \), which means that we generate Tausworthe generators in normal order and reverse order, respectively. Hence, they yield essentially the same lattice point set \( P_s \).

To obtain \((p(x), q(x))\) satisfying the above theorems, Tezuka and Fushimi \(^{13}\) defined a polynomial analogue of Fibonacci numbers as follows:

\[
F_k(x) = A_k(x)F_{k-1}(x) + F_{k-2}(x) \quad (k \geq 2), \\
F_0(x) = 1, \quad F_1(x) = A_1(x), \\
A_k(x) = x \text{ or } x + 1 \quad (k \geq 1).
\]
They called a pair of polynomials \((F_k(x), F_{k-1}(x))\) a pair of “Fibonacci polynomials” because the partial quotients in the continued fraction of \(F_{k-1}(x)/F_k(x)\) are all of degree one. Figure 1 shows the initial part of a tree of Fibonacci polynomials (see also [25, Figure 4.5]). Note that there are \(2^m\) different pairs \((F_m(x), F_{m-1}(x))\) for Fibonacci polynomials with degree \(m\). From them, we choose suitable pairs \((p(x), q(x))\) satisfying Definition 2 approximately.

![A tree of Fibonacci polynomials.](image)

Here, we refine the algorithm of Tezuka and Fushimi [13]. Our exhaustive search algorithm proceeds as follows:

**Algorithm 1** An exhaustive search algorithm

1. Generate all the pairs \((F_m(x), F_{m-1}(x))\) using the recurrence relation of Fibonacci polynomials [9–11].
2. Check the primitivity of \(F_m(x)\).
3. Find \(\sigma\) such that \(x^\sigma \equiv F_{m-1}(x) \mod F_m(x)\). Check \(\gcd(\sigma, 2^m - 1) = 1\) and \(\sigma \geq w\).
4. Choose pairs \((F_m(x), F_{m-1}(x))\) whose \(t\)-value is 2 or 3 for \(s = 3\).
5. Record good pairs \((F_m(x), F_{m-1}(x))\) in lexicographic order of \(t\)-values for \(s = 4, 5, 6, \ldots, m\).
6. Choose the best pairs \((F_m(x), F_{m-1}(x))\).
7. Set \((p(x), q(x)) \leftarrow (F_m(x), F_{m-1}(x))\).

In Step 4, this criterion means that the \(t\)-value is sufficiently small for \(s = 3\) by rule of thumb. In Steps 4 and 5, we calculate the \(t\)-values by using
Gaussian elimination [26] instead of solving Diophantine equations in [13, Theorem 1].

**Remark 1.** In the original paper [13], before Step 2, Tezuka and Fushimi checked the condition

\[ F_{m-1}(x)^m + F_{m-1}(x)^n + 1 \equiv 0 \mod F_m(x), \]

where \(0 < n < m\), to obtain fast Tausworthe generators using trinomial generalized feedback shift register generators. They also restricted the calculation of the \(t\)-values to only \(3 \leq s \leq 5\). A reason for these conditions might be the difficulty of checking from Steps 2–5 on computers around 1990. As a result, in the range \(3 \leq m \leq 32\), there exist pairs \((F_m(x), F_{m-1}(x))\) only for \(m = 3, 5, 7, 15, 17, 18, 20, 22, 23, 25, 28,\) and \(31\); otherwise, there exists no pair. In the related paper [27], the authors found pairs \((F_m(x), F_{m-1}(x))\) for all \(5 \leq m \leq 22\) under a pentanomial condition. Currently, it is not difficult to remove these conditions when we conduct an exhaustive search on modern computers. In Remark 2, we note a reasonably fast generation method instead of the direct use of Definition 3.

### 3.2. Specific parameters

Table 1 lists specific parameters for \(w = 32, 64\) and \(10 \leq m \leq 32\). In Table 1, each first and second row shows the coefficients of \(p(x)\) and \(q(x)\) respectively; for example, \(1 1 0 1\) means \(1 + x + x^3\). We also note the step size \(\sigma\) corresponding to \(q(x)\). For \(m = 21\) and \(28\), we obtained the pairs of polynomials \((p(x), q(x))\) with somewhat large defects \(\Delta = 6\) and 4, respectively, so we replaced them by the second-best pairs. Table 2 summarizes the \(t\)-values and sum of resolution gaps \(\Delta\) for our new Tausworthe generators (labeled “New”) and the existing Tausworthe generators developed by Chen et al. [7] (labeled “Chen”) in the range of \(2 \leq s \leq 20\). For \(2 \leq s \leq 5\), our new generators have the \(t\)-values equal to or smaller than the existing generators (except for \(m = 32\)). It is known that QMC are successful in high-dimensional problems, particularly in the case in which problems are dominated by the first few variables, so we focus on the optimization of leading dimensions. Conversely, from the viewpoint of the FE property, our generators are not FE. We can also optimize both the \(t\)-values and FE property, but the \(t\)-values slightly increase. Thus, we prioritized the \(t\)-values over the FE property for simplicity. The code in C is available at [https://github.com/sharase/cud](https://github.com/sharase/cud).
| $m$ | $p(x)$ | $q(x)$ | $\sigma$ |
|-----|--------|--------|-------|
| 10  | 100000011011 | 01011110101 | 70    |
| 11  | 11001001111111110111 | 010000111101 | 179   |
| 12  | 1111101001001111 | 0010011110111 | 146   |
| 13  | 11101000101111110 | 1010111111001 | 139   |
| 14  | 1010110111110111110111 | 1011110100101111 | 20984 |
| 15  | 1101100111101011111 | 0011011110000111 | 12749 |
| 16  | 11010111110010010111 | 10011110100110111 | 1028  |
| 17  | 101110000101110001 | 111101011110111011 | 226826 |
| 18  | 1101011011100111001 | 010001111001100101 | 179263 |
| 19  | 10110111110001100101 | 0000111110001110101 | 2947446 |
| 20  | 11010101011100101101 | 010001111001100101 | 1127911 |
| 21  | 1111110111001010111011 | 010111100110100101 | 226826 |
| 22  | 1100100011001000110111 | 0011010000100110111 | 629680 |
| 23  | 1110011001011001100110 | 10100100110011110001 | 1796311 |
| 24  | 111100011101011000010111101 | 110000111111110101010111 | 7017398 |
| 25  | 1110101100110011100110001110111 | 010100010001100111001110 | 2947446 |
| 26  | 111010110110111011100000111111 | 11011101100011011101100001 | 19101221 |
| 27  | 11000100010001010001110110101 | 010001000111110010100101111 | 4397933 |
| 28  | 1000101100011101010100100101111 | 00011010011000111101100011001 | 16771336 |
| 29  | 1010000001101010011011001101 | 111110100110001011110101101 | 8318917 |
| 30  | 100001011001001100111100000100101 | 010111110100000011100011101110 | 315800840 |
| 31  | 10111011100010000111110110110101 | 000011110010011101111011001101 | 36109125 |
| 32  | 1000101010111111111110000010010101 | 0100001111011011110101111111111 (σ = 686019401) |
Table 2: Comparison of the $t$-values and $\Delta$ for our new Tausworthe generators and the existing Tausworthe generators developed by Chen et al.\(^\text{[7]}\).

| $m$ | dim | New $t$-values | Chen $t$-values |
|-----|-----|-----------------|-----------------|
| 10  |     | 0 3 4 5 6 6 6 6 6 6 6 6 6 6 6 6 6 6 6 7 | 2 5 5 5 6 6 6 7 7 7 7 7 7 7 7 7 7 7 7 0 |
| 11  |     | 0 3 3 5 6 6 6 6 7 7 7 7 7 7 7 7 7 7 7 1 | 2 5 5 6 6 6 7 7 7 7 7 7 8 8 8 8 8 8 0 |
| 12  |     | 0 3 4 5 6 6 6 6 6 8 8 8 8 8 8 8 8 8 8 2 | 2 3 5 7 7 7 7 7 7 8 8 8 8 8 8 8 8 8 8 0 |
| 13  |     | 0 2 3 5 6 6 7 7 7 8 8 8 8 9 9 9 9 9 9 0 | 1 5 5 5 6 8 9 9 9 9 9 9 9 9 9 9 9 9 0 |
| 14  |     | 0 3 4 5 7 7 7 7 8 9 9 9 9 9 9 9 9 9 9 9 1 | 1 6 7 7 7 2 8 9 9 9 9 9 9 9 9 10 10 10 0 |
| 15  |     | 0 3 4 6 7 8 8 9 9 9 9 9 9 9 9 9 9 9 9 0 1 | 2 4 5 7 7 8 9 9 9 9 9 9 9 9 9 0 |
| 16  |     | 0 3 4 7 7 8 10 10 10 11 11 11 11 11 11 11 11 11 11 1 | 3 4 5 7 8 8 8 8 10 10 10 10 10 10 10 10 10 0 |
| 17  |     | 0 3 4 7 7 8 10 10 10 10 11 11 11 11 11 11 12 12 12 1 | 2 5 6 10 10 10 10 10 10 10 10 11 11 11 11 11 11 11 0 |
| 18  |     | 0 3 5 6 7 9 9 9 10 10 10 10 11 11 12 12 13 13 13 2 | 3 4 5 7 8 9 12 12 12 12 12 12 12 12 12 12 12 12 0 |
| 19  |     | 0 3 5 6 7 12 12 12 12 12 12 13 13 13 13 13 13 13 13 1 | 2 4 5 7 7 2 8 9 9 9 9 9 9 9 9 10 10 10 0 |
| 20  |     | 0 3 5 7 7 10 10 11 11 12 12 13 13 13 13 13 13 13 13 2 | 3 4 5 8 8 13 13 13 13 13 14 14 14 14 14 14 0 |
| 21  |     | 0 3 5 8 8 9 10 10 10 13 13 13 13 13 13 13 13 14 14 1 | 3 6 8 8 8 11 11 11 12 12 12 12 12 12 13 13 15 0 |
| 22  |     | 0 3 5 7 10 10 12 12 12 12 13 13 13 13 13 15 15 15 15 1 | 7 7 7 8 8 14 14 14 14 14 14 14 14 14 14 14 15 0 |
| 23  |     | 0 3 5 9 9 11 12 13 13 13 13 13 13 13 15 15 15 15 15 1 | 5 5 9 9 9 11 15 15 15 15 15 15 15 15 15 15 15 0 |
| 24  |     | 0 3 6 8 10 11 12 13 14 14 14 14 14 15 15 17 17 17 17 17 3 | 5 5 8 8 11 11 11 12 12 14 14 14 14 14 14 14 15 15 16 16 0 |
| 25  |     | 0 3 6 8 12 12 12 13 13 13 13 13 13 16 16 16 18 18 18 3 | 4 6 8 9 9 10 11 12 12 12 14 16 16 16 16 16 16 16 0 |
| 26  |     | 0 3 6 8 12 12 12 13 13 13 13 13 14 15 15 16 16 16 16 18 2 | 6 7 7 9 11 11 12 12 13 13 14 15 15 16 16 16 16 17 17 0 |
| 27  |     | 0 3 7 7 7 11 12 13 13 13 13 14 14 14 16 16 16 16 16 16 3 | 3 6 8 11 12 12 14 14 14 15 15 15 15 16 16 16 16 17 17 0 |
| 28  |     | 0 3 7 9 9 13 13 13 13 13 14 15 15 17 17 17 17 17 17 17 17 2 | 4 5 13 13 13 13 13 14 15 15 15 15 16 16 17 17 18 18 0 |
| 29  |     | 0 3 6 9 11 13 14 14 14 20 20 20 20 20 20 20 20 20 20 20 1 | 5 5 12 12 12 12 14 14 15 17 17 17 17 17 17 17 17 17 17 0 |
| 30  |     | 0 3 7 9 12 13 14 14 16 16 16 16 17 17 17 17 17 17 18 18 18 3 | 2 7 7 10 13 13 13 14 17 17 17 17 17 17 18 18 18 18 18 0 |
| 31  |     | 0 3 7 9 12 15 15 16 18 19 19 19 19 19 19 19 19 19 20 1 | 2 5 9 10 13 13 13 15 15 15 15 15 15 15 15 15 15 15 15 0 |
| 32  |     | 0 3 7 10 13 14 14 15 15 15 17 17 17 18 18 20 20 20 20 20 4 | 5 5 9 9 13 13 15 15 15 16 16 17 18 18 18 19 19 20 20 0 |
Remark 2. We note a reasonably fast generation method for Tausworthe generators. Let \( \mathbf{x}_i = (a_{i\sigma}, a_{i\sigma+1}, \ldots, a_{i\sigma+m-1}, a_{i\sigma+m}, \ldots, a_{i\sigma+w-1})^T \) be a \( w \)-bit state vector at step \( i \) for \( m \leq w \). We can define a state transition \( \mathbf{x}_{i+1} = B\mathbf{x}_i \), where \( B := (b_0 \ldots b_{m-1} 0 \ldots 0) \) is a \( w \times w \) state transition matrix consisting of \( w \)-bit column vectors \( b_0, \ldots, b_{m-1} \) and \( w-m \)-bit zero column vectors \( 0 \). Then, we have the recurrence relation \( \mathbf{x}_{i+1} = a_{i\sigma}b_0 \oplus a_{i\sigma+1}b_1 \oplus \cdots \oplus a_{i\sigma+m-1}b_{m-1} \), which can be calculated by adding column vectors \( b_j \) if \( a_{i\sigma+j} = 1 \) holds for \( j = 0, \ldots, m-1 \), where the symbol \( \oplus \) denotes the bitwise exclusive-or operation. Using this method, we can generate \( \{u_i\}_{i=0}^{\infty} \) in (2) with reasonable speed. See [15, §3 and 5.1] for the construction of \( B \).

4. Numerical example

In this section, we provide a numerical example to confirm the performance of Markov chain QMC. We use a systematic Gibbs sampler to generate the two-dimensional Gaussian distribution

\[
\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathcal{N}\left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right)
\]

for correlation \( \rho \in (-1, 1) \). This can be implemented as

\[
X_{i,1} \leftarrow \rho X_{i-1,2} + \sqrt{1-\rho^2}\Phi^{-1}(u_{2i-2}),
\]
\[
X_{i,2} \leftarrow \rho X_{i,1} + \sqrt{1-\rho^2}\Phi^{-1}(u_{2i-1}),
\]

where \( \Phi \) is the cumulative distribution function for the standard normal distribution. For the output values (3) generated from Tausworthe generators, we define two-dimensional non-overlapping points starting from the origin:

\[
(0, 0), (u_0, u_1), (u_2, u_3), \ldots, (u_{N-2}, u_0), (u_1, u_2), \ldots, (u_{N-3}, u_{N-2}),
\]

where \( N = 2^m \). We apply digital shifts, that is, we add \((z_1, z_2)\) to each point in (12) using bitwise exclusive-or \( \oplus \), where \( z_1 \) and \( z_2 \) are IID samples from \( U(0, 1) \).

We estimate \( E(X_1) \) and \( E(X_2) \) by taking a sample average. Hence, the true values are zero. We compare the following driving sequences:

1. New: our new Tausworthe generators;
2. Chen et al. (2012): Tausworthe generators developed by Chen et al. [7];

and
3. IID: Mersenne Twister \cite{28}.

Figure 4 shows a summary of standard deviations (in log$_2$ scale) for $\rho = 0, 0.3$ and 0.9 and $12 \leq m \leq 25$ using 100 digital shifts. Our new generators outperformed Chen’s generators for no correlation $\rho = 0$ and weak correlation $\rho = 0.3$. Even for strong correlation $\rho = 0.9$, our new generators were still better than Chen’s generators. In Figure 4, we generated scatter plots of sampling $(X_1, X_2)$ from our new and Chen’s Tausworthe generators for $\rho = 0$ and $m = 12$. In the scatter plots, Chen’s generator has a pattern of wiggly strips of points, which is optimized in terms of $64 \times 64$ grids for $s = 2$, but our generator seems to be highly balanced both for $X_1$ and $X_2$. Therefore, it can be expected that our new generators have better marginal distributions than the existing generators.

**Remark 3.** In our experiments, we set $w = 32$. In fact, Chen et al. \cite{7} originally defined Tausworthe generators in (2) with $m$-bit precision, that is, $u_i = \sum_{j=0}^{m-1} a_{i+j} 2^{-j-1} \in [0, 1)$. In this definition, we could not observe clear differences between our new generators and Chen’s generators. However, we increased the precision of points and redefined Tausworthe generators with $w$ bits as in (2), and then the differences became clear.

5. Conclusion

We conducted an exhaustive search of short-period Tausworthe generators for Markov chain QMC in terms of the $t$-value. Our key technique was to use the continued fraction expansion of $q(x)/p(x)$ by refining the algorithm of Tezuka and Fushimi \cite{13} on modern computers. As a result, we obtained the point sets with $t$-values optimal for $s = 2$ and small for $s \geq 3$. We also reported a numerical example using Gibbs sampling in which our new generators performed better than the existing generators of Chen et al. \cite{7}. The code in C is available at [https://github.com/sharase/cud](https://github.com/sharase/cud).

As a future work, we will attempt more realistic numerical examples, such as Bayesian inference for hierarchical Poisson–Gamma models \cite{29} and probit models \cite{30} as in \cite{2, 5, 6}. For this purpose, we believe that the next task is to embed our new and existing generators into several programming languages for statistical computing; for example, R, Stan, and Python. Thus, we are now planning a software implementation of Markov chain QMC.
Figure 2: Estimation of $E(X_1)$ and $E(X_2)$ for $\rho = 0, 0.3$ and 0.9.
Figure 3: Scatter plots of sampling \((X_1, X_2)\) for \(\rho = 0\) and \(m = 12\).
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