Ground-state of fractional and integral quantum Hall systems at $\nu \leq 1$ and it excitations

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Many-body variational ground-state wave function of two-dimensional electron system (2DES), localized in the main strip (MS) $L_x^\square \times L_y^\square$ of the finite width $L_x^\square = \sqrt{2\pi m\ell_0}$ (and the periodic boundary condition (PBC) imposed along $x$-direction), is presented at the fractional and the integral filling factors $\nu = 1/m$ for two different ion backgrounds, giving homogeneous ion density: microscopical uniform ion background (UIB) and classical ion jellium background (IJB); $\ell_0$ is the magnetic length, $m = 2\ell + 1$ and $\ell = 0, 1, 2, \ldots$ It is shown that the ground-state and the lowest excited-state can correspond to partial crystal-like correlation order among $N$ electrons of the main region (MR) $L_x \times L_y$; then the study of 2DES of $N$ electrons within MR is exactly reduced to the treatment of 2DES of $\tilde{N} = NL_x^\square/L_x$ electrons localized within MS, with PBC along $x$. Both for UIB and IJB, the ground-state wave function of the present symmetry liquid-crystal state with 2DES density that is periodic along the $y-$direction, with the period $L_y^\square/m$, and independent of $x$. The difference between the ground-state energy per electron, for these two backgrounds is only due to the difference between the energies of UIB-IUB and IJB-IJB interactions. For IJB, at $m = 3, 5$, the ground-state has essentially lower energy per electron than the Laughlin, uniform liquid, ground-state (the Laughlin model uses IJB); the same holds at $m = 1$. At $m \geq 3$, the compound form of the many-body ground-state wave function leads to the compound structure of each electron already within MS (due to important similarities between the present $m = 1$ and $m \geq 3$ states, and to simplify notations, the term “compound” is often used as well at $m = 1$); these compound electrons play important role in the properties of the many-body excited-states. Obtained compound exciton (without the change of spin of the excited compound electron) and compound spin-exciton (with the change of spin of the excited compound electron) states show finite excitation gaps, for $m = 1, 3, 5, 7$. The excited compound electron (hole) is composed, within MS, from $m$ strongly correlated quasiholes (quasiholes of the total charge $e/m (-e/m)$ each; this charge is fractionally quantized at $m \geq 3$. The activation gap, experimentally observable from the activation behavior of the direct current magnetotransport coefficients, is obtained: it is given by the excitation gap of relevant compound exciton, at $m \geq 3$, and by the gap of pertinent compound spin-exciton, at $m = 1$. Quantized Hall conductance $\sigma_H = e^2/(2m\pi \hbar)$ is obtained; it is fractional at $m \geq 3$. The theory is in good agreement with experiments.

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I. INTRODUCTION

The discoveries of the integer\textsuperscript{1} and the fractional\textsuperscript{2} quantum Hall effects in two-dimensional electron systems (2DES) of a semiconductor based samples had born strong ongoing interest to the ground-state of 2DES, and it elementary excitations, in a quantum Hall regime at a filling factor $\nu \leq 1/3, 1/5$ fractional quantum Hall effect (FQHE) states.\textsuperscript{3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24} Since seminal work of Laughlin,\textsuperscript{2} a particular attention in past years is given to properties of the $\nu = 1/3$ and $1/5$ fractional quantum Hall effect (FQHE) states.\textsuperscript{4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26} At $\nu = 1$, corresponding to fully occupied the lowest Landau level (LLL), the Laughlin wave function gives the same total energy per electron as a Hartree-Fock wave function (if the latter is built from the symmetric gauge the LLL single-particle wave functions it coincides with the former one, see, e.g., Refs.\textsuperscript{22}–\textsuperscript{24}}

$$\epsilon_{HF} = -\frac{1}{2} \frac{\sqrt{\pi}}{2 \varepsilon \ell_0^2} \frac{e^2}{\hbar c}$$

where $\ell_0 = \sqrt{\hbar c/|e|B}$ is the magnetic length and $\varepsilon$ is the background dielectric constant. Point out, for $\nu = 1$ the Hartree-Fock approximation (HFA) study that uses the Landau gauge single-particle wave functions also gives $\epsilon_{HF}$, see, e.g., Refs.\textsuperscript{22,23} However, the HFA result, obtained for $0 < \nu \leq 1/5$,\textsuperscript{25}

$$\epsilon_{HF}(\nu) = -\frac{\nu}{2} \sqrt{\pi} \frac{e^2}{\varepsilon \ell_0^2}$$

gives for $\nu = 1/3$ and $1/5$ the total energy per electron substantially higher than that of Laughlin model\textsuperscript{2} and even the energy of corresponding charge-density wave or Wigner crystal states.\textsuperscript{25,26} To date understanding is that for $m = 3, 5$ the Laughlin wave function\textsuperscript{2} gives the best known analytical approximation of exact many-body ground-state wave
function \( f \). Point out, there were many attempts (some of them are quite recent) to find out a ground-state, at \( \nu = 1/3 \), with an energy lower than in the Laughlin theory \( \nu = 1/3 \), see, e.g., References 6,10,11, and references cited therein. In all pertinent previous theoretical works neutralizing ion background is treated as a “classical” uniform ion density; see, e.g., References 6,11,22, and 27,28, where discrete nature of ions and specific form of single-ion wave functions does not appear. In addition, in these theoretical works the 2DES is placed in the 2D-plane of the neutralizing ion background, e.g., see References 6,14,22,27,28, where we call this model of neutralizing ion background as classical ion jellium background (IJB). Besides the IJB model, I treat the other theoretical model of the ion background (also localized within the plane of 2DES) where I assume, in good agreement with typical experimental conditions, that electric charge of each ion is totally localized within the finite square unit cell, \( L_x^0 \times L_y^0 \). The latter model also gives exactly homogeneous ion density; we call this model of the ion background as microscopical uniform ion background (UIB). The latter model treats the neutralizing ion background in more correct manner (it excludes, in particular, interaction of an ion with itself). However, as IJB model typically is used,\( 3,14,22 \), to make comparison with previous studies we present the results for IJB as well.

At \( \nu = 1/m \), I present variational many-body wave functions of ground-states for UIB and IJB as \( \Psi_{N,N}^{(m),e} \), Sec. IV A, and \( \Psi_{N,\tilde{N}}^{(m),JB} \), Sec. IV B, respectively. In these ground-states the total lowering per electron due to many-body interactions is given (in units of \( e^2/\varepsilon \ell_0 \)) by (i) \( U^{(m)}_{UIB}(m) \), Eq. (15), and (ii) \( U^{(m)}_{IJB}(m) \), Eq. (12), respectively. In particular, \( U^{(m)}_{IJB}(m) \) presents, for \( m = 1, 3, 5 \), substantially stronger lowering than the lowest total lowering for the Laughlin variational wave function.\( 3 \) In addition, \( U^{(m)}_{UIB}(m) \) presents, for \( m = 1, 3, 5 \), much stronger lowering than the total lowering \( U^{(m)}_{IJB}(m) \). For IJB model I obtain that \( U^{(1)}_{IJB} \approx -0.66510 \), \( U^{(3)}_{IJB} \approx -0.42854 \), and \( U^{(5)}_{IJB} \approx -0.33885 \) are well below of pertinent total lowerings for the Laughlin variational wave function that are given as \(-\sqrt{\pi}/8 \approx -0.6267, -0.4156 \pm 0.0012 \) (notice, more accurate calculations for the Laughlin model show here the lowering \( -0.410 \pm 0.001 \), and \(-0.3340 \pm 0.0028 \), respectively.

For treated electron-ion systems many-body effects are essentially related with \( N \) electrons of 2DES localized within the main strip (MS) \( L_x^0 \times L_y \), to which periodic boundary condition (PBC) along \( x \)-direction is imposed. Of course, the “images” of MS (cf. with Refs. 8,10,11), periodically repeated with the period \( L_x^0 \) along \( x \)-direction of the main region (MR) \( L_x \times L_y \), are taken into account as well. We assume that within MR there are present \( N \) electrons and \( N \) ions such that \( N/N = L_x/L_x^0 \). It is important that more adequate, physically, sets of single-electron wave functions are used than previously. These wave functions are localized mainly within the square unit cell \( L_x^0 \times L_y^0 \), where \( L_x^0 = \sqrt{2\pi m\ell_0} \); e.g., see Eqs. (18) – (27). This choice helps to reflect the tendency (proven by present results) of each electron: a) to be present mainly within one such relatively “localized” (around its centre point) unit cell and, in addition, b) to occupy all these \( N \) unit cells of MS with equal probability. The latter is achieved by proper construction of many-body wave functions \( \Psi_{N,N}^{(m),e} = \Psi_{N,\tilde{N}}^{(m)} \), etc. Notice, in the present study it is assumed that \( N \rightarrow \infty \).

It is shown that the ground-state and the lowest excited-state can correspond to partial crystal-like correlation order, Eq. (19), among \( N \) electrons of MR. As a result the study of 2DES of \( N \) electrons within MR is exactly reduced to the treatment of 2DES of \( N \) electrons localized within MS of the finite width (along \( x \)) to which PBC is imposed along \( x \)-direction. I.e., present study shows that proper PBC can be totally relevant to symmetry, periodicity, correlations, etc. properties, e.g., of a sought ground-state. Then it will not lead to any over-simplification or nonphysical “boundary effects”. Present below study of ground-state (e.g., the trial wave function of ground-state with the energy, per electron, lower than pertinent energy of Ref. 3) confirms assumption Eq. (19) and relevant PBC.

Notice that for an infinite MR any specific orientation of \( x \)- (or \( y \)) axis, within 2D-plane, is not defined until the quantum phase transition to broken symmetry liquid-crystal state will take place. Point out, all main formulas of the work are obtained by exact analytical calculations. In this work I further develop main physical ideas outlined in Ref. 22. All present physical results for the electron-ion system with UIB coincide with those for the electron-ion system with IJB, except that for IJB the energies of the ground-state and the excited-states are shifted upwards on the same value, for given \( m \), with respect to the energies of relevant many-body states for UIB.

The paper is organized as follows. In Sec. II we present many-body Hamiltonian of the electron-ion system as for UIB, Sec. II A, so for IJB, Sec. II B. In addition, in Sec. II C we present exact obtaining of the model Hamiltonians of Secs. II A, II B from first principles and physical conditions involved. In Sec. III we introduce complete set of single-body wave functions. At \( \nu = 1/m \) and odd integer \( m \), in Sec. IV we present two variational ground-state wave functions: one for UIB, Sec. IV A, and other for IJB, Sec. IV B. In Sec. V A and Sec. V B we calculate the ground-state energy of electron-ion system for each of these two ground-state wave functions. In Sec. V C we give more details and remarks on partial crystal-like correlation order and energy of ground-state. In Sec. VI we study excited-states of the ground-states of Secs. IV A, IV B, and calculate their energies of excitation. In Secs. VI A we present (both for UIB and IJB) the many-body wave functions both for the compound exciton and the compound spin-exciton; in addition, we obtain com-
pound structure for the charge density of these excitons. In Sec. VI B, at \( m \geq 3 \), we obtain the energies of the compound excitons and treat them, both for UBI and LJB. In Sec. VI C, at \( m \geq 1 \), we obtain the energies of the compound spin-excitons (both for UBI and LJB) and treat them. In Sec. VII quantized Hall conductance is calculated for present two ground-states, at \( \nu = 1/m \). Finally, we make concluding remarks in Sec. VIII.

II. HAMILTONIAN OF THE ELECTRON-ION SYSTEM

A. For a microscopical model of ion background

First we consider the Hamiltonian of the electron-ion system that will allow to introduce correctly microscopical ion background, in particular, UIB.

We consider a zero-thickness 2DES localized within the main strip (MS) of the finite width \( L_x^\square = \sqrt{2\pi m t_0} \) \((L_x^{\square} n_{xx}^i x > L_x^{\square} (n_{xx}^i - 1))\); where \( n_{xx}^i \) is a finite integer) and of very large (in principle, infinite) length \( L_y \) \((L_y/2 > y > -L_y/2)\) in the presence of a strong perpendicular magnetic field \( \mathbf{B} = B\hat{z} \). The Landau gauge for the vector potential \( \mathbf{A}(\mathbf{r}) = (-By, 0, 0) \) is used. We assume that both \( \tilde{N} \) electrons of a 2DES and \( N \) ions (single-charged, with the charge \(|e|\)) of a neutralizing background are located, within MS, at the same \((z = 0)\)-plane (it is \((Z = 0)\)-plane, for ions); i.e., at the same 2D-plane. Further, we assume PBC only along \( x \)-direction (cf. with Refs.\(^{10,11}\), for review see Ref.\(^4\)). Notice, already there is essential difference from the approach of Refs.\(^{10,11}\), where a rectangular cell is considered and in both directions PBCs are imposed; however, still some important analogies between, e.g., the forms of the Coulomb interaction, properties of matrix elements will be seen. We assume that the ions are very heavy such that their kinetic energy can be neglected; similar approximation is widely expected, physical results will not depend on \( N_C \) \(\to\) \(\infty\). In Eq. \((1)\) the electron-ion potential

\[
V_{ei} = -\sum_{i=1}^{\tilde{N}} \sum_{j=1, j\neq i}^{N} e^2 |\mathbf{r}_i - \mathbf{r}_j - kL_x^\square \mathbf{\hat{x}}|,
\]

where \( \tilde{N} \) is a very large natural number; as it is expected, physical results will not depend on \( N_C \to \infty \). In Eq. \((1)\) the electron-ion potential

\[
V_{ei} = -\sum_{i=1}^{\tilde{N}} \sum_{j=1, j\neq i}^{N} \frac{e^2}{|\mathbf{r}_i - \mathbf{r}_j - kL_x^\square \mathbf{\hat{x}}|},
\]

and the ion-ion potential

\[
V_{ii} = \frac{1}{2} \sum_{i=1}^{\tilde{N}} \sum_{j=1, j\neq i}^{N} \sum_{k=-N_C}^{N_C} \frac{e^2}{|\mathbf{R}_i - \mathbf{R}_j - kL_x^\square \mathbf{\hat{x}}|},
\]

Notice, the modified form of the Coulomb interaction, due to PBC, that appears in Eqs. \((3)\) and \((4)\) is quite similar with the one given in Refs.\(^{10,11}\), see also Ref.\(^4\). Further, it is seen that the second (constant) term in Eq. \((3)\) gives the total contribution due to the interaction of each electron, within MS, with its images (that appear in other strips due to PBC); here the final sum over all \( \tilde{N} \) electrons of MS leads to the factor \( \tilde{N} \). Point out, that the second term in Eq. \((3)\) it follows from the first term in Eq. \((3)\) if formally to assume that \( j = i \) (i.e., formally neglecting the important condition \( j \neq i \)) and excluding \( k = 0 \) term from the sum over \( k \); it clearly should be absent as now it gives self-interaction of an electron with itself, not with its image. In addition, if formally to discard PBC then only electrons within MS are present and respectively in Eq. \((3)\); the second term should be dropped and in the first term the sum over \( k \) must be reduced to only one term, \( k = 0 \). I.e., in this limiting case Eq. \((3)\) reduces to the correct form of the electron-electron potential.

Point out, the form of the electron-electron interaction Eq. \((3)\), modified due to PBC, can be proven by a detailed consideration. The latter is mainly omitted as it final result Eq. \((3)\) is rather natural, as we have shown above. Notice, such detailed consideration should exclude double counting of the interactions between an \( i^- \) electron, of MS, with all other \( j \neq i \) electrons of the main strip and their images; the same is valid for the interactions of this \( i^- \) electron with its images. In particular, it
should be kept in mind that \( \tilde{N} \) electrons of MS represent only very small fraction of the \( N \) electrons of the main region and for any other strip \( n_{z_s}^0 \neq n_{z_s}^0 \), the electrons of such \( \beta \)–strip will interact with electrons of the \( \alpha \)–strip (i.e., our MS) as with their images in the \( \alpha \)–strip.

As it should be, the form of the ion-ion potential Eq. (5) is totally analogous to the one of the electron-electron potential Eq. (6). The form of the electron-ion interaction Eq. (4), modified due to PBC, can be proven by a detailed consideration; the latter is omitted as it final result Eq. (1) is quite natural. Notice, this consideration takes into account, without double counting, as interactions of any electron of MS with all ions and their images so interactions of any ion of MS with all electrons and their images. Point out, if formally to discard PBC then only electrons and ions within MS are present and in Eq. (4) in the sum over \( k \) should be left only one term, \( k = 0 \); i.e., Eq. (4) in this limiting case reduces to the correct form of the electron-electron potential.

Further, for implicit limit \( N_C \rightarrow \infty \) (assuming that MS is repeated \( N/\tilde{N} = L_x/L_x^0 \rightarrow \infty \) times within MR \( L_x \times L_y \)), it is easy to see that the Hamiltonian \( \hat{H}_{\tilde{N},\tilde{N}} \), defined by Eqs. (1)-5, is periodic with the period \( L_x^0 \) along any of it \( 2\tilde{N} \) variables \( x_i \) and \( x_j \), where \( i, j = 1, \cdots, \tilde{N} \). Then a many-body wave function that describes a state pertinent to the Hamiltonian \( \hat{H}_{\tilde{N},\tilde{N}} \), Eq. (1), should satisfy the same property, i.e., to be periodic with the period \( L_x^0 \) along any of \( 2\tilde{N} \) variables \( x_i \) and \( x_j \). Point out, present problem has the translational symmetry along the \( x \) axis very similar with pertinent translational symmetry of Refs. 10,11 (i.e., along the \( y \) axis in Refs. 10,11), due to another form of the Landau gauge used by 10,11.

As a good approximation of typical experimental conditions, in present study we will assume for UIB model that each ion is located totally (as, e.g., in Sec. IV A) in a square unit cell \( L_x^0 \times L_y^0 \), such that \( L_x^0 L_y^0/(L_x^0)^2 = \tilde{N} \). Further, for a ground-state we will assume that any particular electron tends to be with equal probability in all these \( \tilde{N} \) unit cells. This in turn leads to some important conditions for an optimal set of single-electron wave functions.

\section*{B. For classical jellium model of ion background}

For widely used\textsuperscript{1,14,22} model of the classical jellium background (IJB), we need to modify the Hamiltonian Eq. (1). Then the relevant many-electron Hamiltonian of electron-ion system, \( \hat{H}_{\tilde{N}}^{IB} \), is given as

\begin{equation}
\hat{H}_{\tilde{N}}^{IB} = \hat{H}_0 + V_{ee} + V_{cb} + V_{bb},
\end{equation}

where \( \hat{H}_0 \) and \( V_{ee} \) are defined by Eqs. 2 and 3, respectively. Here the electron-ion system, localized within MS \( L_x^0 L_y^0 > x > L_x^0 (n_{z_s}^0 - 1) \), consists: i) from \( N \) electrons, of MS, interacting with IJB of MR and ii) from the uniform positive charge density of the ion background, \( |e| n_0 \), localized within MS that interacts with IJB of the main region. Respectively, in Eq. (10) we have, cf. with Refs. 14,22, that

\begin{equation}
V_{cb} = -\sum_{i=1}^{\tilde{N}} \int_{MR} dR \frac{e^2 n_0(R)}{|r_i - R|},
\end{equation}

where the subscript “MR” shows that integration is carried out over MR, \( L_x \times L_y \), and

\begin{equation}
V_{bb} = \frac{1}{2} \int_{MS} dR \int_{MR} dR' \frac{e^2 n_0(R) n_0(R')}{|R - R'|},
\end{equation}

where the subscript “MS” shows that integration is carried out over MS. Notice, it is assumed that within the main region \( n_0(R) = \text{const}(R) = n_0 \) and \( \int_{MS} dR n_0 = \tilde{N} \).

Assuming that MS is repeated \( N/\tilde{N} = L_x/L_x^0 \rightarrow \infty \) times within MR, it is easy to see that the Hamiltonian \( \hat{H}_{\tilde{N}}^{IB} \), defined by Eqs. (10), (8), (10), is periodic with the period \( L_x^0 \) along any of it variable \( x_i, i = 1, \cdots, \tilde{N} \).

Then a many-body wave function that describes a state pertinent to the Hamiltonian \( \hat{H}_{\tilde{N}}^{IB} \), Eq. (10), should satisfy the same property, i.e., to be periodic with the period \( L_x^0 \) along any of \( \tilde{N} \) variables \( x_i \).

\section*{C. Hamiltonian \( \hat{H}_{\tilde{N}}^{IB} \): straightforward obtaining from first principles}

For definiteness, here we will present straightforward justification, starting from first principles, of the model Hamiltonian \( \hat{H}_{\tilde{N}}^{IB} \). Point out, very similar treatment will justify \( \hat{H}_{\tilde{N}}^{IB} \) model Hamiltonian.

Here we will start with the same many-electron Hamiltonian, \( \hat{H}(r_1, \ldots, r_N) \), for 2DES of \( N \) electrons as in 10,14, only for the Landau vector potential gauge \( \text{A} = -B \mathbf{k} \). We assume that \( N \) electrons are localized in MR; \( N/L_x L_y = \nu/2\pi \ell^2 \). As in 10,14, the IJB-model of neutralizing ion background is assumed. Eigenstates \( \Psi(r_1, \ldots, r_N) \) of the Hamiltonian \( \hat{H}(r_1, \ldots, r_N) \equiv \hat{H}_{MR} \) and their energies are determined by

\begin{equation}
\hat{H}_{MR} \Psi(r_1, \ldots, r_N) = E_N \Psi(r_1, \ldots, r_N),
\end{equation}

here

\begin{equation}
\hat{H}_{MR} = \hat{H}_0^{MR} + V_{ee}^{MR} + V_{cb}^{MR} + V_{bb}^{MR},
\end{equation}

where

\begin{equation}
\hat{H}_0^{MR} = \frac{1}{2m^*} \sum_{i=1}^{N} \left[ p_i - \frac{e}{c} \text{A}(r_i) \right]^2,
\end{equation}

\begin{equation}
V_{ee}^{MR} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{e^2}{|r_i - r_j|},
\end{equation}

\begin{equation}
V_{cb}^{MR} = \sum_{i=1}^{N} \int_{MR} dR \frac{e^2 n_0(R)}{|r_i - R|},
\end{equation}

\begin{equation}
V_{bb}^{MR} = \frac{1}{2} \int_{MS} dR \int_{MR} dR' \frac{e^2 n_0(R) n_0(R')}{|R - R'|},
\end{equation}

\begin{equation}
V_{ee}^{MR} = \frac{1}{2} \int_{MS} dR \int_{MR} dR' \frac{e^2 n_0(R) n_0(R')}{|R - R'|},
\end{equation}

\begin{equation}
V_{cb}^{MR} = -\sum_{i=1}^{N} \int_{MR} dR \frac{e^2 n_0(R)}{|r_i - R|},
\end{equation}

\begin{equation}
V_{bb}^{MR} = \frac{1}{2} \int_{MS} dR \int_{MR} dR' \frac{e^2 n_0(R) n_0(R')}{|R - R'|},
\end{equation}

\begin{equation}
V_{ee}^{MR} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1,j\neq i}^{N} \frac{e^2}{|r_i - r_j|},
\end{equation}

\begin{equation}
V_{cb}^{MR} = -\sum_{i=1}^{N} \int_{MR} dR \frac{e^2 n_0(R)}{|r_i - R|},
\end{equation}

\begin{equation}
V_{bb}^{MR} = \frac{1}{2} \int_{MS} dR \int_{MR} dR' \frac{e^2 n_0(R) n_0(R')}{|R - R'|},
\end{equation}

\begin{equation}
V_{ee}^{MR} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{e^2}{|r_i - r_j|},
\end{equation}

\begin{equation}
V_{cb}^{MR} = -\sum_{i=1}^{N} \int_{MR} dR \frac{e^2 n_0(R)}{|r_i - R|},
\end{equation}

\begin{equation}
V_{bb}^{MR} = \frac{1}{2} \int_{MS} dR \int_{MR} dR' \frac{e^2 n_0(R) n_0(R')}{|R - R'|}.
\end{equation}
further,
\[ V_{eb}^{MR} = -\sum_{i=1}^{N} \int_{MR} dR \frac{\varepsilon^2 \mu_b(R)}{\varepsilon [r_i - R]}, \tag{13} \]
and
\[ V_{bb}^{MR} = \frac{1}{2} \int_{MR} dR \int_{MR} dR' \frac{\varepsilon^2 n_b(R)n_b(R')}{\varepsilon [r_i - R - R']}. \tag{14} \]

We can assume that 2DES, with IJB, is located within the ribbon of width \( L_y \) bent into loop of radius \( L_x/2\pi \). Then Born-Carman PBCs \( r_i \pm L_x \hat{x} = r_i \) are holded, where \( i = 1, \ldots, N \). It is seen that the area of MR per electron, \( L_x L_y/N = (L_x^0)^2 \), where \( L_x^0 = \sqrt{2m\pi^2} \). Then the strip of the width \( L_x^0 \), along \( x \)-direction, and of the length \( L_y \) contains \( N = L_y/L_x^0 \) of the (square) unit cells, \( L_x^0 \times L_x^0 \). The integer number of such strips within MR is given as \( n_{x,\text{max}} = L_x/L_x^0 = N/N \); for definiteness, odd (as \( N \)). Further, we assume that for Eq. (9) the ground-state and, at the least, the lowest excited-state correspond to partial crystal-like correlation order among \( N \) electrons of MR as
\[ r_{1+k_1N} = r_1 + k_1 L_x \hat{x}, \quad r_{2+k_2N} = r_2 + k_2 L_x \hat{x}, \nonumber \]
\[ \cdots, \quad r_{N+k_NN} = r_N + k_N L_x \hat{x}, \tag{15} \]
where \( k_i = 1, 2, \ldots, n_{x,\text{max}} \); \( n_{x,\text{max}} = L_x/L_x^0 \). Then using Eq. (15) in Eqs. (10)-(13) it is easy to see that exact many-body Hamiltonian \( H^{MR} \) becomes: i) dependent only on \( N \) \( r_i \) \((i, y_i)\) and ii) periodic, with period \( L_x^0 \), on any \( x_i; \ i = 1, \ldots, N \). Hence, for a wave function in Eq. (9) the properties (i) and (ii) also will be valid. Then the study (e.g., calculation of the total energy per electron, etc.) within MR of Eq. (9) for 2DES of \( N \) electrons with many-electron wave functions orthonormal within MR, can be exactly reduced to the treatment of the Schrödinger equation for 2DES of \( N \) “compound” electrons within MS as
\[ H^{1B}_{N}\Psi_N = E_{N}^{1B}\Psi_N(r_1, \ldots, r_N), \tag{16} \]
with many-electron wave functions orthonormal within MS; also for pertinent discussion see Sec. V.C. Notice, here we formally obtain that \( N_C = (n_{x,\text{max}}^2 - 1)/2 \) in Eq. (9). However, as physical results will be practically independent of \( N_C \) for \( N_C \gg 1 \), we can assume \( N_C \) in Eq. (9) very large, however, such that \( N_C \ll (n_{x,\text{max}}^2 - 1)/2 \rightarrow \infty \); notice, the same conditions on \( N_C \) will be obtained for \( H^{MR}_{N} \) model Hamiltonian. Conditions Eq. (15) can be thought as assumed physical constraints that are justified only if they will lead to a ground-state with the energy, per electron, lower than obtained in Ref. 32.

I.e., we assume that there are some low energy eigenstates for which holds partial crystal-like correlation order, Eq. (15); due to the latter some many-body correlations are implicitly included in the Hamiltonian Eq. (9). Present below study of the low energy eigenstates (in particular, the trial wave function of ground-state with the energy lower than the energy for the Laughlin’s trial wave function of ground-state) confirms this physical assumption, Eq. (15).

### III. SINGLE-BODY WAVE FUNCTIONS

Let us for MS (\( L_x^0 n_{y,\text{gs}}^a > x > L_x^0 (n_{y,\text{gs}}^a - 1) \); \( L_y/2 > y > -L_y/2 \)) introduce normalized solutions of the single-electron Schrödinger (\( \omega_c = e|B|/m^*c \)) equation
\[ \hat{h}_0 \psi_{\nu; n, k, \alpha}^{L_x^0}(r) = \hbar \omega_c (\nu + 1/2) \psi_{\nu; n, k, \alpha}^{L_x^0}(r), \tag{17} \]
that satisfy PBC \((y_0(k_x, \alpha) = L_y^0 k_x, k_x = (2\pi/L_x^0)n_{y,\text{gs}}^a)\), of the form
\[ \psi_{\nu; n, k, \alpha}^{L_x^0}(r) = \psi_{\nu; n, k, \alpha}^{L_x^0}(y - y_0(k_x, \alpha))/\sqrt{L_x^0}, \tag{18} \]
where \( \Psi_n(y) \) is the harmonic oscillator function, \( n_{y,\text{gs}}^a = 0, \pm 1, \ldots, \pm (N_L - 1)/2; N_L \) is the odd integer such that \((2\pi/L_x^0)N_L \ell_0 = L_y^0 \). As we see \( n_{y,\text{gs}}^a \) gives the number to a “bare” cell of the width \( \Delta y_0 = 2\pi \ell_0/L_x^0 \), the quantum of \( y_0(k_x, \alpha) \), and of the length \( L_x^0 \), along \( x \). So within each “bare” cell \( L_x^0 \times \Delta y_0 \) there is only one quantized \( y_0(k_x, \alpha) \) (or \( k_x, \alpha) \) at the \( n_a \)-th Landau level; i.e., there is only one state Eq. (15) per “bare” cell at the latter level. Respectively, the total number of the bare cells, per Landau level, within MS is equal to \( N_L \).

It can be shown that, within MS, the set of single-body wave functions Eq. (15) is complete. In addition, they are orthonormal within the strip as
\[ \int_{L_x^0 (n_{y,\text{gs}}^a - 1)}^{L_x^0 n_{y,\text{gs}}^a} dx \int_{-\infty}^{\infty} dy \psi_{\nu; n, k, \alpha}^{L_x^0}(r) \psi_{\nu; n, k, \alpha}^{L_x^0}(r) \equiv \langle \psi_{\nu; n, k, \alpha}^{L_x^0}| \psi_{\nu; n, k, \alpha}^{L_x^0} \rangle = \delta_{n, n_a} \delta_{k, k_a}, \tag{19} \]
which can be also rewritten in the equivalent form by formally changing in Eq. (15) of \( k_x \rightarrow n_{y,\text{gs}}^a \) and \( k_x \rightarrow n_{y,\text{gs}}^b \).

If formally to assume \( L_x^0 = L_x, \) Eq. (19) reduces to well known result\(^{32}\) for “usual” wave functions, \( \psi_{\nu; n, k, \alpha}^{L_x}(r) \).

Now looking for an optimal set of single-electron wave functions Eq. (15) at \( \nu \leq 1 \) and taking into account that the total ion charge within the unit cell \( L_x^0 \times L_x^0 \) is equal to \( |e| \), we assume that the total electron charge within the unit cell must be exactly equal to \( e(0) \). In some way this square unit cell is “dressed” by one electron charge \( e \). We call it the unit cell: these cells cannot be confused with the “bare” cells. It follows that
\[ L_x^0 = \frac{L_y}{N} = \frac{L_y}{\nu N_L}, \tag{20} \]
It is natural to assume that $\tilde{N}$ is fixed for a given sample while the filling factor $\nu$ can obtain different values. Then $L_x^{\square}$ is independent of $\nu$ (or $B$) for the given system.

Using Eqs. (20), we obtain

$$\frac{L_x^{\square}}{\Delta y_0} = \frac{1}{\nu}, \tag{21}$$

so the quantum $\Delta y_0 \neq L_x^{\square}$ for $\nu < 1$. More importantly, with the help of Eq. (21) it is seen that within each unit cell can appear only an odd integer number, $m = 1, 3, 5, \ldots$, of the quantized oscillator centres $y_0(k_{x\alpha})$ of the states Eq. (13) at the given $n_{\alpha}$-th Landau level. Indeed, for an even integer number $m = m_0$ of the states per unit cell only $m_0 - 1$ states from them have $y_0(k_{x\alpha}) = \ell \frac{\pi}{L} k_{x\alpha}$ inside the unit cell. As exactly at the boundaries of unit cells, separating them at $y = \pm L_x^{\square}/2, \pm(1 + 1/2) L_x^{\square}/2, \pm(2 + 1/2) L_x^{\square}/2, \ldots$, there is one state (i.e., $y_0(k_{x\alpha})$) per such boundary. So the case of even $m$ is a special one and it is not treated in this work. Then the left hand side (LHS) of Eq. (21) must be an odd integer, $m = 2\ell + 1$, and it follows that

$$\frac{1}{\nu} = m, \tag{22}$$

where $\ell = 0, 1, 2, \ldots$.

According to PBC, the wave functions Eq. (18) are periodic along $x$ with period $L_x^{\square}$, cf. with Refs. 10, 11. So if multiply Eq. (17), from the left, by $\psi_{n_2,k_{x\beta}}^{L_x^{\square}}(r)$ and then integrate over $r$ (as $\int_{L_x^{\square}}^{n_{x_2}^0} d\xi \int_{-\infty}^{\infty} dy \ldots$) within MS, it follows that

$$\langle \psi_{n_2,k_{x\beta}}^{L_x^{\square}} | \hat{h}_0 | \psi_{n_1,k_{x\alpha}}^{L_x^{\square}} \rangle = \hbar \omega_c (n_\alpha + \ell/2) \delta_{n_\alpha n_\beta} \delta_{k_{x\beta},k_{x\alpha}}, \tag{23}$$

where the right-hand side (RHS) have been obtained by using Eq. (19).

Notice, from Eqs. (21), (22) it follows that

$$L_x^{\square} = \sqrt{2 \pi m} \ell_0, \tag{24}$$

this form is useful for present study. Notice, as $L_x^{\square}$ is actually independent of $m$, a superscript ($m$) in $L_x^{\square}$ is not used.

Further, as for $\nu = 1/m$ there are $m = 2\ell + 1$ quantized values of $y_0(k_{x\alpha})$ (or $k_{x\alpha}$) within an $i$-th unit cell and each of them has a particular spatially within the unit cell, we separate all $\tilde{N}_L$ states Eq. (13), of the $n_{\alpha}$-th Landau level, into the $m$ sets of wave functions. Within any such $n$-th set of states (here $n$ can obtain $m$ different values) the difference $|y_0(k_{x\alpha}^{(n)}) - y_0(k_{x\alpha}^{(n)})| = kL_x^{\square}$, where $k$ is an integer. Here $j(i)$ is the number of a unit cell; it can be any integer from 1 to $\tilde{N}$. Point out, this $i$–number unambiguously defines the $i$–th unit cell, among all $\tilde{N}$ unit cells of MS. The superscript in $k_{x\alpha}^{(n)}$ is given to distinguish the $k_{x\alpha}$ pertinent to the $n$–th set of states; the subscript (superscript) “i” in $k_{x\alpha}^{(n)}$, $n_{ys}^{(i)}$, etc. indicates belonging to the $i$–th unit cell. For definiteness, we will choose the values of $n$ as follows: $n = 0, \ldots, \pm \ell$. In particular, for $m = 1$ it follows $\ell = 0$ and $n = 0$, for $m = 3$ it follows $\ell = 1$ and $n = 0, \pm 1$. We define $k_{x\alpha}^{(n)}$, for $m = 2\ell + 1$, as follows

$$k_{x\alpha}^{(0)} = (2\pi m/L_x^{\square}) n_{ys}^{(i)}, \ldots, k_{x\alpha}^{(\pm \ell)} = k_{x\alpha}^{(0)} \pm 2\pi \ell/L_x^{\square}, \tag{25}$$

where $n_{ys}^{(i)} = 0, \pm 1, \ldots, (\tilde{N} - 1)/2$, and $\tilde{N} = \tilde{N}_L/m$. It is seen that for the given $n$–th set the total number of different $k_{x\alpha}^{(n)}$ within MS is equal to $\tilde{N}$, as it should be. So all $m$ sets of $k_{x\alpha}^{(n)}$ give altogether $\tilde{N}_L = m \times \tilde{N}$ different values, the same as for the $k_{x\alpha}$ in Eq. (13).

Point out, that the choice of $k_{x\alpha}^{(n)}$ in the form Eq. (25) is quite natural as here: i) all $k_{x\alpha}^{(n)}$ are symmetrical with respect of the $y$-centre of MS $y = 0$; ii) the smallest $|k_{x\alpha}^{(n)}|$ is given by $k_{x\alpha}^{(0)} = 0$, for $n_{ys}^{(i)} = 0$; iii) within an $i$–th unit cell all $m$ states have $k_{x\alpha}^{(n)}$ symmetric with the respect of $k_{x\alpha}^{(0)}$, the centre of this cell; iv) this choice leads to symmetric and more homogeneous electron charge density within a unit cell, along $y$–direction.

To simplify writing, we will use notation $\psi_{n_{ys};k_{x\alpha}}(r) \equiv \psi_{n_{ys};k_{x\alpha}}(r)$. Widely used below wave functions Eq. (18) of the $n_{\alpha} = 0$ Landau level $\psi_{0;k_{x\alpha}}(r) \equiv \psi_{0,n_{ys}}(r)$ we denote, at $\nu = 1/m$, as well as

$$\varphi_{(m)}^{(i)}(r) \equiv \psi_{0;k_{x\alpha}^{(n)}}(r), \tag{26}$$

where $i = 1, 2, \ldots, \tilde{N}$ is the number of a unit cell and $n = 0, \pm 1, \ldots, \pm \ell$ the "set" number; they unambiguously define $k_{x\alpha}^{(n)}$. For these wave functions Eq. (19) reduces to

$$\int_{L_x^{\square}}^{n_{x_2}^0} dx \int_{-\infty}^{\infty} dy \varphi_{(m)}^{(i)}(r) \varphi_{(m)}^{(n)}(r) \equiv \langle \varphi_{(m)}^{(i)} | \varphi_{(m)}^{(n)} \rangle = \delta_{k,n} \delta_{k_{x\beta},k_{x\alpha}}^{(n)} \tag{27}$$

i.e., the single-body wave functions of the same $n$–th set are orthonormal and they are orthogonal to any wave function from another set $k \neq n$. From Eq. (23) we have

$$\langle \varphi_{(m)}^{(i)} | \hat{h}_0 | \varphi_{(m)}^{(n)} \rangle = \frac{\hbar \omega_c}{2} \delta_{k,n} \delta_{k_{x\beta},k_{x\alpha}}^{(n)}, \tag{28}$$

Point out, Eq. (28) is very similar to pertinent result of Refs. 10, 11, for their finite rectangular main cell.

**IV. GROUND-STATE WAVE FUNCTION OF ELECTRON-ION SYSTEM AT $\nu = 1/m$**

At $\nu = 1/m$ ($m = 2\ell + 1; \ell = 0, 1, \ldots$), we look for the ground-state many-body wave function of the electron-ion system Eq. (1), Sec. IV A, and Eq. (9), Sec. IV B.
Sec. IV A we consider electron-ion system for UIB with the total wave function \( \Psi_{N,N}^{(m),eh} (r_1, \ldots, r_N; R_1, \ldots, R_N) \), that corresponds to totally homogeneous ion density \( n_{io,eh} \), Eq. (43). It is important to point out that for UIB model (in difference of typically used IJBJ model) it is absent, e.g., self-interaction of an ion with itself. In Sec. IV B we treat the ground-state, \( \Psi_{N,N}^{(m),JB} (r_1, \ldots, r_N) \), of electron-ion system for IJB model. To simplify notations, arguments \( r_i, R_i \) in the wave functions \( \Psi_{N,N}^{(m)}, \Psi_{N,N}^{(m),eh} \), etc. are often suppressed.

A. Ground-state, \( \Psi_{N,N}^{(m),eh} \), for UIB model. Compound electrons.

Now we will consider a ground-state, \( \Psi_{N,N}^{(m),eh} \), of electron-ion system which gives exactly homogeneous ion density in MS. Notice, due to PBC here the ion density is homogeneous for the whole MR. We assume that \( \Psi_{N,N}^{(m),eh} \) has the “compound” form

\[
\Psi_{N,N}^{(m),eh} (r_1, \ldots, r_N; R_1, \ldots, R_N) = \prod_{i=1}^{\tilde{N}} \phi_{n_{io,eh}}^{(i)} (R_i)
\]

\[
\times \sum_{n=-\ell}^{\ell} C_n(m) \psi_{N}^{n,(m)} (r_1, \ldots, r_N), \tag{29}
\]

where

\[
|C_n(m)|^2 = 1/m, \tag{30}
\]

the “partial” many-electron wave function \( \psi_{N}^{n,(m)} (r_1, r_2, \ldots, r_N) \) (or the \( n \)-th set many-electron wave function) is an \( \tilde{N} \)-dimensional Slater determinant of the wave functions Eq. (28) given as

\[
\psi_{N}^{n,(m)} = \frac{1}{\sqrt{N!}} \begin{vmatrix}
\varphi_{x_1}^{k,(m)} (r_1) & \cdots & \varphi_{x_1}^{k,(m)} (r_N) \\
\varphi_{x_2}^{k,(m)} (r_1) & \cdots & \varphi_{x_2}^{k,(m)} (r_N) \\
\vdots & \ddots & \vdots \\
\varphi_{x_{\tilde{N}}}^{k,(m)} (r_1) & \cdots & \varphi_{x_{\tilde{N}}}^{k,(m)} (r_N)
\end{vmatrix}, \tag{31}
\]

i.e., in the “Hartree-Fock”-alike form. Point out that all \( m \)-many-electron wave functions Eq. (31) form the orthonormal assembly as

\[
(\psi_{N}^{k,(m)} (r_1, \ldots, r_N) | \psi_{N}^{n,(m)} (r_1, \ldots, r_N) ) = \delta_{k,n}. \tag{32}
\]

It is readily seen that due to PBC satisfied by the single-electron wave functions, Eqs. (18), (28), the many-body wave function, Eq. (29), and the “partial” many-electron wave functions, Eq. (31), are periodic with period \( L_x^{\square} \) with respect to any \( x_i; i = 1, \ldots, \tilde{N} \).

Further, in Eq. (29) the “partial” many-ion wave function

\[
\prod_{i=1}^{\tilde{N}} \phi_{n_{io,eh}}^{(i)} (R_i)
\]

is given in the “Hartree”-alike form, where a “single-ion” wave function \( \phi_{n_{io,eh}}^{(i)} (R) \), localized in the \( i \)-th unit cell of the main strip, is introduced as follows. For both \( X \in (L_x^{\square}(N_x^2 - 1), L_x^{\square}N_x^2) \) and \( Y \in (L_x^{\square}(n_{ys}^2 - 1/2), L_x^{\square}(n_{ys}^2 + 1/2)) \), we have

\[
|\phi_{n_{io,eh}}^{(i)} (R)|^2 = 1/(L_x^{\square})^2, \tag{33}
\]

if \( Y \) is outside of the \( i \)-th unit cell then \( \phi_{n_{io,eh}}^{(i)} (R) = 0 \).

The set of these single-body wave functions is orthonormal, within MS, as we have

\[
\int_{L_x^{\square}(N_x^2 - 1)}^{L_x^{\square}N_x^2} dx \int_{-\infty}^{\infty} dy \phi_{n_{io,eh}}^{(i)} (R) \phi_{n_{io,eh}}^{(j)} (R) = \delta_{i,j}. \tag{34}
\]

Point out, PBC is also applied to the single-ion wave functions \( \phi_{n_{io,eh}}^{(i)} (R) \); i.e., the many-body wave function Eq. (29) is periodic with period \( L_x^{\square} \) with respect to any \( X_i \) as well. It is seen that wave function Eq. (29) is normalized, \( (\psi_{N,N}^{(m),eh} | \psi_{N,N}^{(m),eh} ) = 1 \). Also we will need to use a shorter notation for the integral over MS (given, e.g., in Eqs. (27), (31)) as \( \int dR \ldots = \int_{L_x^{\square}(N_x^2 - 1)}^{L_x^{\square}N_x^2} dX \int_{-\infty}^{\infty} dY \ldots \).

Point out that, due to the quantized according to Eq. (30) contributions from the partial many-electron wave functions Eq. (31), for \( m \geq 3 \) the compound form of the ground-state wave function Eq. (29) leads to the compound structure of each electron within MS. In particular, this compound structure of the electrons plays important role in the treatment of excited-states of the present ground-state, as it is shown in Sec. VI.

Now we consider the electron charge density \( \rho_{eh}^{(r)}(r) = en_{eh}(r) \) in the state Eq. (29), in MS. We have

\[
n_{eh}^{(r)} = (\psi_{N,N}^{(m),eh} | \sum_{j=1}^{\tilde{N}} \delta (r - r_j) | \psi_{N,N}^{(m),eh} ), \tag{35}
\]

where in the RHS integration is taken over all \( r_i \) and \( R_i \). Notice, the matrix element of Eq. (35) cannot be mixed with the matrix elements of Eqs. (27), (28). Using Eqs. (27), (29) + (31), from Eq. (35) it follows

\[
n_{eh}^{(r)} = \frac{1}{m} \sum_{n=-\ell}^{\ell} (\psi_{N}^{n,(m)} | \sum_{j=1}^{\tilde{N}} \delta (r - r_j) | \psi_{N}^{n,(m)} ), \tag{36}
\]

where arguments \( r_i \) in the \( n \)-th set many-electron wave function \( \psi_{N}^{n,(m)} \), over which the integration holds in the RHS of Eq. (36), are suppressed. Eq. (36), after using
Eq. (31), gives
\[ n_{ch}(r) = \frac{1}{m} \sum_{n=-\ell}^{\ell} \sum_{k_{xi}^{(n)}} |(\chi_{k_{xi}^{(n)}}|)^2 = \frac{1}{\sqrt{2\pi}m^{3/2}\ell_0} \times \sum_{n=-\ell}^{\ell} \sum_{k_{xi}^{(n)}} \Psi_0^2(y - y_0(k_{xi}^{(n)})), \]

where the \( k_{xi}^{(n)} \) are given by Eq. (24). Due to PBC, using Eq. (37) we conclude that both in MS and MR \( n_{ch}(r) \equiv n_{ch}(y) \), i.e., it is independent of \( x \).

To study \( n_{ch}(y) \), we apply to Eq. (37) the Fourier transformation over \( y \), \( n(q_y) = \int_{-\infty}^{\infty} dy \, n(y) \exp(-iq_y y) \); we obtain
\[ n_{ch}(q_y) = \frac{e^{-q_y^2\ell_0^2/4}}{\sqrt{2\pi}m^{3/2}\ell_0} \sum_{n=-\ell}^{\ell} e^{-i\sqrt{2\pi}m/\ell_0} \sum_{n=-\ell}^{\ell} e^{-i\sqrt{2\pi}m/\ell_0} \sum_{n=-\ell}^{\ell} e^{-i\sqrt{2\pi}m/\ell_0} \sum_{n=-\ell}^{\ell} \psi_0^2(y - y_0(k_{xi}^{(n)})), \]

where in the RHS, using \( \tilde{N} \to \infty \) assumed in the present study, the last sum is given by \( \sum_{k=-\infty}^{\infty} \exp(-i\sqrt{2\pi}m|q_y|\ell_0) = \sum_{k=-\infty}^{\infty} \exp(-ikq_y L_x) \). Using the latter Poisson’s summation formula, we obtain (it is well known result) that
\[ \sum_{k=-\infty}^{\infty} \exp[-ikq_y L_x] = \frac{2\pi}{L_x} \sum_{M=-\infty}^{\infty} \delta(q_y + M \frac{2\pi}{L_x}). \]

By making use of Eq. (39) in Eq. (38), we calculate
\[ n_{ch}(q_y) = \frac{1}{m^2 L_x^2} \sum_{M=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} e^{-q_y^2\ell_0^2/4} \delta(q_y + M \frac{2\pi}{m L_x}) \times \sum_{n=-\ell}^{\ell} \exp(i2\pi Mn/m) \]
\[ = \frac{1}{m^2 L_x^2} \sum_{k=-\infty}^{\infty} e^{-q_y^2\ell_0^2/4} \delta(q_y + M \frac{2\pi}{m L_x}), \]

where it is used (remind, \( m = 2\ell + 1 \)) that i) \( \sum_{n=-\ell}^{\ell} \exp(i2\pi Mn/m) = M = 0, \pm 2m, \pm 2m, \ldots \) and ii) this sum is equal to zero for any other \( M \) (by definition, \( M \) is an integer); then after using the notation \( M = m \times k \), where \( k = 0, \pm 1, \pm 2, \ldots \), we readily arrive to the final form in Eq. (40). Applying inverse Fourier transformation to Eq. (40), \( n(y) = (1/2\pi) \int_{-\infty}^{\infty} dq_y n(q_y) \exp(iq_y y) \), we obtain
\[ n_{ch}(y) = \frac{1}{2\pi m L_x} [1 + 2 \sum_{k=1}^{\infty} e^{-\pi mk^2/2} \cos(\sqrt{2\pi}m\ell_0 ky)]. \]

Now we consider the ion charge density \( \rho_{ch}^i(r) = -e\rho_{ch}^i(r) \) in the state Eq. (29). The ion density, within MS, is given as
\[ n_{io}^i(r) = \langle \Psi^{(m),ch}_{N,\tilde{N}} | \sum_{j=1}^{\tilde{N}} \delta(r - R_j) | \Psi^{(m),ch}_{N,\tilde{N}} \rangle \]
\[ = \sum_{n_{io}^{(i)}} |\phi_{n_{io}^{(i)}}(r)|^2, \]

where, using Eq. (33), it follows that \( n_{io}^i(r) = const(x, y) = n_{io}^i \), has the form
\[ n_{io}^i = \frac{1}{(L_x)^2} = \frac{1}{2\pi m L_x^2}. \]

I.e., in MS (and, due to PBC, in the main region \( L_x \times L_y \)) the ion density Eq. (43) is spatially homogeneous and independent of \( m \). Point out that \( \sum_{n_{io}^{(i)}=-(\tilde{N}-1)/2}^{(\tilde{N}-1)/2} \approx \sum_{n_{io}^{(i)}=1}^{\tilde{N}} \delta_{n_{io}^{(i)}} \); equivalently ”\( \ell \)” can be understood as \( i = \{k^{(i)} \} \).

Relative inhomogeneity \( \delta n_{ch}^i \) of \( n_{ch}(y) \), Eq. (11), is very well approximated by the amplitude of \( k = 1 \) oscillating term, i.e., \( \delta n_{ch}^i \approx 2 \times \exp(-\pi m/2) \). We have that for \( m = 1, 3, \) and \( 5 \) \( \delta n_{ch}^i \approx 0.416, 1.8 \times 10^{-2}, \) and \( 7.8 \times 10^{-3} \). So the relative inhomogeneity of the electron density \( n_{ch}(y) \), Eq. (41), pertinent to the homogeneous ion background Eq. (43), is not very small only for \( m = 1 \), while for \( m \geq 3 \) the inhomogeneity is very small.

Point out that for the electron-ion system described by the wave function Eq. (29), the electron charge density does not exactly cancels the ion charge density, \( \rho_{ch}^e(r) + \rho_{ch}^i(r) \neq 0 \), i.e., the system is not exactly electrically neutral, even though within each unit cell the total electron charge, \( e \), exactly cancels the total ion charge, \(-e\).

B. Ground-state, \( \Psi^{(m),JB} \) for IJB model.

Compounded electrons.

Now we consider a ground-state, \( \Psi^{(m),JB} \), of electron-ion system for typically used form of the ion background, i.e., the continuous homogeneous one that we call as IJB. It is exactly homogeneous as well within MS. From the Hamiltonian Eq. (6) it is clear that here PBC is applied only to the electrons coordinates and their functions. Based on Eq. (29), it is natural to assume that \( \Psi_{\tilde{N}}^{(m),JB} \) has the “compound” form as follows
\[ \Psi_{\tilde{N}}^{(m),JB}(r_1, \ldots, r_{\tilde{N}}) = \sum_{n=-\ell}^{\ell} C_n(m) \Psi^{(m,n)}_{\tilde{N}}(r_1, \ldots, r_{\tilde{N}}). \]
It is readily seen that due to PBC satisfied by the single-electron wave functions, Eqs. (18), (29), the many-body wave function, Eq. (44), is periodic with period \( \hat{L}_x^2 \) with respect to any \( x_i; \ i = 1, \ldots, \tilde{N} \). Respectively, the electron density in the state Eq. (44), within the main strip,

\[
n_{JB}^e(r) = \langle \Psi_{\tilde{N}}^{(m),JB} | \sum_{j=1}^{\tilde{N}} \delta(r - r_j) | \Psi_{\tilde{N}}^{(m),JB} \rangle,
\]

(45)

coinsides with \( n_{eh}^e(r) \), see Eqs. (30)-(41); i.e., \( n_{JB}^e(r) \equiv n_{JB}^e(y) \) both in MS and MR.

For the ion density, \( n_b \), it follows that \( n_b = n_{io}^{eh} \).

V. GROUND-STATE ENERGY OF ELECTRON-ION SYSTEM AT \( \nu = 1/m \)

In Sec. IV A, for UIB model Eqs. (1)–(5), we study the energy of the ground-state Eq. (23). In Sec. IV B, for IJB model Eqs. (9)–(31), (23), (26), we treat the energy of the ground-state Eq. (44). As only for the latter model of the ion background we can directly compare the total lowering per electron due to many-body interactions (it includes any electron-electron, electron-ion and ion-ion contributions) with pertinent total lowering for the Laughlin variational wave function. As we will show, the difference between the ground-state energy per electron \( U_{JB}(m) \), for UIB, and \( U_{JB}(m) \), for IJB, is related only with the difference between the contributions from the ion-ion interaction \( v_{ii} \), Eq. (5), and \( v_{ab} \), Eq. (8), respectively. In Sec. V C we present additional remarks on partial crystal-like correlation and energy of ground-state.

Point out, all analytical results obtained in Sec. V are exact.

A. Energy of ground-state \( \Psi_{\tilde{N},\tilde{N}}^{(m),eh} \) for UIB

Using the Hamiltonian Eq. (1), we calculate the total energy of electron-ion system in the state Eq. (29) as

\[
E_{kin}^m = \frac{1}{m\tilde{N}} \sum_{n=-\ell}^{\ell} \langle \Psi_{\tilde{N}}^{n,(m)} | \hat{H}_0 | \Psi_{\tilde{N}}^{n,(m)} \rangle
\]

(46)

where in the RHS for the kinetic energy term

\[
\langle \Psi_{\tilde{N},\tilde{N}}^{(m),eh} | \hat{H}_0 | \Psi_{\tilde{N},\tilde{N}}^{(m),eh} \rangle = \tilde{N} E_{kin}
\]

(47)

we obtain (remind, all matrix elements should be calculated within MS) the kinetic energy per electron (or per electron-ion pair) as

\[
E_{kin} = \frac{1}{m\tilde{N}} \sum_{n=-\ell}^{\ell} \langle \Psi_{\tilde{N}}^{n,(m)} | \hat{H}_0 | \Psi_{\tilde{N}}^{n,(m)} \rangle
\]

(48)

In Eq. (48) the RHS of the first line contains usual in the Hartree-Fock theory\(^{30,31}\) matrix elements, on the “Hartree-Fock”-alike many-electron wave functions Eq. (41); this RHS it follows straightforwardly from the LHS, quite similar with transition from Eq. (35) to Eq. (30). Then transition to the second line of Eq. (48) is analogous to well known one in the Hartree-Fock theory\(^{30,31}\). Notice, in Eq. (48) index “j” distinguishes electrons. Further, the second line in Eq. (48) is simplified by using Eq. (28). Point out, from Sec. IV as well as Eq. (48) it is seen that within the subspace of the \( n \)-th set of single-electron states (from which the \( n \)-th set many-electron wave function Eq. (31) is constructed) an \( j \)-th electron is equally distributed (present) over (in) all these \( \tilde{N} \) states. Point out, the kinetic energy \( E_{kin}^m = \hbar \omega_c / 2 \) coincides with the pertinent result of Refs.\(^ {30,31} \) for their finite rectangular main cell.

In the RHS of Eq. (48) the term related with ion-ion interaction, we call it also UIB-UIB interaction, is given as

\[
\langle \Psi_{\tilde{N},\tilde{N}}^{(m),eh} | v_{ii} | \Psi_{\tilde{N},\tilde{N}}^{(m),eh} \rangle = \tilde{N}(E_{ii}^m + E_{ii}^b),
\]

(49)

where, due to the second term in the RHS of Eq. (50),

\[
E_{ii}^b = \sum_{k=1}^{N_C} \frac{e^2}{\varepsilon L_x^2 k}
\]

(50)

and, due to the first term in the RHS of Eq. (50),

\[
E_{ii}^b = \frac{1}{2\tilde{N}} \sum_{i=1}^{\tilde{N}} \sum_{j=1, j \neq i}^{\tilde{N}} \sum_{k=-N_C}^{N_C} \left[ \langle \Psi_{\tilde{N},\tilde{N}}^{(m),eh} | v_{kk} \hat{R}_{ij} - k \hat{L}_x^2 \hat{x} | \Psi_{\tilde{N},\tilde{N}}^{(m),eh} \rangle \right]
\]

(51)

In Eq. (51) we have used Eqs. (29)–(32). Finally, by
standard transformations, Eq. (51) is rewritten as
\[
E_{\ell i}^b = \frac{1}{2N} \sum_{k=-N_C}^{N_C} \sum_{j=1, j \neq \ell} \sum_{j=1, j \neq \ell} \int \int \frac{e^2 dR dR'}{\varepsilon |R - R' - kL_x^{|}}
\times |\phi_{n_y}(R)|^2 |\phi_{n_y}(R')|^2.
\]

(52)

In the RHS of Eq. (46) the term related with electron-ion interaction, we call it also electron-UB interaction, is given as
\[
\langle \Psi_{N,N}^{(m),eh} | V_{ei} | \Psi_{N,N}^{(m),eh} \rangle = \tilde{N} E_{ei},
\]

(53)

where
\[
E_{ei} = -\frac{1}{N} \sum_{i=1}^{\tilde{N}} \sum_{j=1}^{\tilde{N}} \sum_{j=1}^{N} \sum_{j=1}^{N} \langle \Psi_{N,N}^{(m),eh} | \Psi_{N,N}^{(m),eh} \rangle
\times |\phi_{n_y}(R)|^2 |\phi_{n_y}(R')|^2
\times |\phi_{n_y}(R)|^2 |\phi_{n_y}(R')|^2.
\]

(54)

In Eq. (54) we have used Eqs. (29)-(31) and (cf. with Eq. (30), (31)) the property
\[
\langle \Psi_{N,N}^{(m),eh} | V_{ei} | \Psi_{N,N}^{(m),eh} \rangle = \delta_{N,N} \langle \Psi_{N,N}^{(m),eh} | V_{ei} | \Psi_{N,N}^{(m),eh} \rangle
\times |\phi_{n_y}(R)|^2 |\phi_{n_y}(R')|^2.
\]

(55)

which is valid for \( \tilde{N} \geq 2 \). By making calculations analogous to ones of the HFA and the Hartree approximation, from Eq. (54) we obtain
\[
E_{ei} = -\frac{1}{mN} \sum_{n=-\ell}^{\ell} \sum_{n=-\ell}^{\ell} \sum_{n=-\ell}^{\ell} \sum_{n=-\ell}^{\ell} \int d\mathbf{r} \int d\mathbf{r}'
\times |\phi_{n_y}(R)|^2 |\phi_{n_y}(R')|^2.
\]

(56)

In the RHS of Eq. (36) the term related with electron-electron interaction
\[
\langle \Psi_{N,N}^{(m),eh} | V_{ee} | \Psi_{N,N}^{(m),eh} \rangle = \tilde{N} (E_{ee} + E_{ee}^b),
\]

(57)

where, due to the second term in the RHS of Eq. (38),
\[
E_{ee} = \sum_{k=1}^{N_C} \sum_{k=1}^{N_C} \frac{e^2}{\varepsilon |R - R' - kL_x^{|}}
\times |\phi_{n_y}(R)|^2 |\phi_{n_y}(R')|^2.
\]

(58)

and, due to the first term in the RHS of Eq. (3),
\[
E_{ee}^b = \frac{1}{2N} \sum_{i=1}^{\tilde{N}} \sum_{j=1}^{\tilde{N}} \sum_{j=1}^{N} \sum_{j=1}^{N} \langle \Psi_{N,N}^{(m),eh} | V_{ee} | \Psi_{N,N}^{(m),eh} \rangle
\times |\phi_{n_y}(R)|^2 |\phi_{n_y}(R')|^2.
\]

(59)

In Eq. (59) we have used Eqs. (29), (30) and (cf. with Eq. (54)) the property
\[
\langle \Psi_{N,N}^{(m),eh} | V_{ei} | \Psi_{N,N}^{(m),eh} \rangle = \delta_{k,n} \langle \Psi_{N,N}^{(m),eh} | V_{ee} | \Psi_{N,N}^{(m),eh} \rangle
\times |\phi_{n_y}(R)|^2 |\phi_{n_y}(R')|^2.
\]

(60)

which is valid for \( \tilde{N} \geq 3 \) (remind, in the present study it is assumed that \( \tilde{N} \to \infty \)). As in the RHS of Eq. (59) the matrix elements are calculated on many-electron wave functions Eq. (31) of “Hartree-Fock”-alike form, by making typical HFA calculations, of the matrix elements, we rewrite Eq. (59) as
\[
E_{ee}^b = E_{ee} + E_{ee}^c.
\]

(61)

where the direct-like (or the Hartree-like) contribution
\[
E_{ee}^d = \frac{1}{2mN} \sum_{n=-\ell}^{\ell} \sum_{n=-\ell}^{\ell} \sum_{n=-\ell}^{\ell} \sum_{n=-\ell}^{\ell} \int d\mathbf{r} \int d\mathbf{r}'
\times |\phi_{n_y}(R)|^2 |\phi_{n_y}(R')|^2.
\]

(62)

and the exchange-like (or the Fock-like) contribution
\[
E_{ee}^e = \frac{1}{2mN} \sum_{n=-\ell}^{\ell} \sum_{n=-\ell}^{\ell} \sum_{n=-\ell}^{\ell} \sum_{n=-\ell}^{\ell} \int d\mathbf{r} \int d\mathbf{r}'
\times |\phi_{n_y}(R)|^2 |\phi_{n_y}(R')|^2.
\]

(63)

Notice, the direct-like term, Eq. (62), is always positive and the exchange-like term, Eq. (63), is always negative. Point out that strictly speaking these two terms cannot be called as the direct and the exchange ones because some correlations are already taken into account in the form of many-body electron-ion wave function \( \Psi_{N,N}^{(m),eh} \).

Point out, in final expressions of Eqs. (52), (62), (63) the sums over \( i \) and \( j \) have the additional condition, \( j \neq i; \).
however, it is important that this condition in Eq. (56) is absent. Remind that $\int dr \ldots = \int L_{x}^{2} n_{x}^{2} dx \int_{-\infty}^{\infty} dy \ldots$

The sum of direct and direct-alike terms

$$E^{da} = E_{ce} + E_{ti} + P_{ii} + E_{eci} + E_{ei},$$

we rewrite (by, quite obvious, exact transformations) as follows

$$E^{da} = E_{1}^{(m)} + E_{2}^{(m)} + E_{3}^{(m)},$$

where $E_{ce}, E_{ti}$, and the "diagonal" terms, $j = i$, from $E_{eci}, Eq. (56)$, for $k \neq 0$, have contributed to

$$E_{1}^{(m)} = \frac{2e^{2}}{\epsilon L_{x}^{2}} \sum_{k=1}^{N_{c}} \sum_{\ell=0}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{k=-N_{c}}^{N_{c}} \int dr \int dR$$

$$\times \frac{e^{2}}{\epsilon |r - R - kL_{x}^{2x}|} \left| \phi_{k}^{(m)}(r) \right|^{2} \left| \phi_{n}^{(i)}(R) \right|^{2},$$

further, the only left, however, very important "diagonal" term, $j = i$, from $E_{eci}, Eq. (56)$, for $k = 0$, have contributed to

$$E_{2}^{(m)} = \frac{1}{m \epsilon N} \sum_{n=-\infty}^{\infty} \sum_{\ell=1}^{\infty} \int dr \int dR$$

$$\times \frac{e^{2}}{\epsilon |r - R - kL_{x}^{2x}|} \left| \phi_{k}^{(m)}(r) \right|^{2} \left| \phi_{n}^{(i)}(R) \right|^{2},$$

and $E_{ti}, E_{eci}$ and all the rest "nondiagonal" terms, $j \neq i$, from $E_{eci}, Eq. (56)$, for any $k$ from $-N_{c}$ to $N_{c}$, have given

$$E_{3}^{(m)} = \frac{1}{2m \epsilon} \sum_{n=-\infty}^{\infty} \sum_{\ell=1}^{\infty} \sum_{k=-N_{c}}^{N_{c}} \int dr \int dR$$

$$\times \frac{e^{2}}{\epsilon |r - R - kL_{x}^{2x}|} \left| \phi_{k}^{(m)}(r) \right|^{2} \left| \phi_{n}^{(i)}(R) \right|^{2} \left| \phi_{n}^{(n)}(R) \right|^{2}.$$

As an $i$--th term in the sums over $i$ in Eqs. (66)-(68) is independent of $i = n_{(j)s}$, these equations can be rewritten as

$$E_{1}^{(m)} = \frac{2e^{2}}{\epsilon L_{x}^{2}} \sum_{k=1}^{N_{c}} \sum_{\ell=0}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{k=-N_{c}}^{N_{c}} \int dr \int dR$$

$$\times \frac{e^{2}}{\epsilon |r - R - kL_{x}^{2x}|} \left| \phi_{k}^{(m)}(r) \right|^{2} \left| \phi_{n}^{(i)}(R) \right|^{2},$$

$$E_{2}^{(m)} = \frac{1}{m \epsilon} \sum_{n=-\infty}^{\infty} \sum_{\ell=1}^{\infty} \int dr \int dR$$

$$\times \frac{e^{2}}{\epsilon |r - R - kL_{x}^{2x}|} \left| \phi_{k}^{(m)}(r) \right|^{2} \left| \phi_{n}^{(i)}(R) \right|^{2},$$

and

$$E_{3}^{(m)} = \frac{1}{2m} \sum_{n=-\infty}^{\infty} \sum_{\ell=1}^{\infty} \sum_{k=-N_{c}}^{N_{c}} \int dr \int dR$$

$$\times \frac{e^{2}}{\epsilon |r - R - kL_{x}^{2x}|} \left| \phi_{k}^{(m)}(r) \right|^{2} \left| \phi_{n}^{(i)}(R) \right|^{2} \left| \phi_{n}^{(n)}(R) \right|^{2},$$

As it can be expected, in particular, we will see that in Eqs. (66)-(71) their RHS are actually independent of $n_{x}^{2}$. Point out, it easy to see that a weak logarithmic divergence of the first sum over $k$ in the RHS of Eq. (59), that appears due to $k \gg 1$, is exactly canceled by the main contributions to the second sum over $k$ in the RHS Eq. (69), at $|k| \gg 1$. Such that the total sum over $k$ in the RHS of Eq. (69) will quickly converge with the increase of $|k|$. Indeed, for $|k| \rightarrow \infty$ in the second term of the RHS of Eq. (69) as $|r - R| \sim L_{x}^{2}$, due to properties of the factors given by the squares of the single-electron and the single-ion wave functions involved, we have in a very good approximation $e^{2} |(r - R - kL_{x}^{2x}| \approx e^{2} |(r - kL_{x}^{2x})|$. Using the latter constant expression in the second term of Eq. (69) leads to the mutual canceling with the relevant $k$--term of the first contribution to the RHS of Eq. (69). By similar considerations, it is easy to see that the sum over $k$ in the RHS of Eq. (71) is quickly convergent. As a result, we can in Eqs. (69), (71) and, respectively, (66), (68) to assume that $N_{c} \rightarrow \infty$; it is in agreement with pertinent discussion in Sec. II.

To calculate the RHS of Eqs. (69), (70) we will use that

$$\frac{e^{2}}{\epsilon r - R} = \frac{e^{2}}{2\pi \epsilon} \int_{-\infty}^{\infty} dq_{x} \int_{-\infty}^{\infty} dq_{y} \frac{e^{iq(r-R)}}{\sqrt{q_{x}^{2} + q_{y}^{2} + \delta^{2}/\epsilon_{0}^{2}}},$$

where, e.g., cf. with $\delta^{2}$ a dimensionless $\delta \rightarrow 0$ and it is implicit that $L_{\mu} \rightarrow \infty$ such that $0_{f} / (\delta \times L_{\mu}) \rightarrow 0$, $0 \leq x \leq y$. Respectively, we have that $L_{x}^{2} / (\delta \times L_{x}) \rightarrow 0$ as only finite $m$ are treated. In addition, we will use the matrix elements

$$\int dr \ e^{iqr} \left| \psi_{k}^{(m)}(r) \right|^{2} = \exp(iq_{k}k_{x}^{(m)} \ell_{0}^{2} - q_{y}^{2} \ell_{0}^{2}/4),$$

$$\exp(iq_{x}L_{x}^{(n_{x})}) \left[ 1 - \exp(-i\ell_{x}^{2}L^{\Delta}_{x}) \right],$$

and

$$\int dR \ e^{-iqR} \left| \phi_{n}^{(i)}(R) \right|^{2} = \exp(-iq_{y}L_{x}^{(n_{x})} \left| S_{m}(q_{y}L_{x}^{\Delta}) \right| \left[ 1 - \exp(iq_{x}L_{x}^{\Delta}) \right],$$

where

$S_{m}(q_{x}L_{x}^{\Delta}) = \sin(q_{x}L_{x}^{\Delta}/2) / (q_{x}L_{x}^{\Delta}/2) = \sin(\sqrt{\pi m/2} q_{x}L_{x}^{\Delta}/2) / (\sqrt{\pi m/2} q_{x}L_{x}^{\Delta}/2) = \exp(\sqrt{\pi m/2} q_{x}L_{x}^{\Delta}/2) / (\sqrt{\pi m/2} q_{x}L_{x}^{\Delta}/2)$; it is even
function. In particular, in the LHS of Eq. (73) i) the integral over \(x\) is calculated as
\[
\int_{L_x^0(m_x^0)}^\infty e^{in_x} \, dx/L_x = e^{i\eta_x} L_x^0 \int_{-\infty}^{\infty} \frac{[1 - e^{-\imath \eta_x L_x^0}]}{(i\eta_x L_x^0)}
\]
and ii) the integral over \(y\) is calculated as
\[
\int_{-\infty}^{\infty} dy \, e^{i\eta_y} \Psi_0^2(y - y_0(k_{x1}^{(n)}) = \exp(-\frac{1}{4} \eta_y^2 + i\eta_y (k_{x1}^{(n)})^2).
\]
After using of Eqs. (72)-(74), we rewrite Eq. (70) as
\[
E_2^{(m)} = \frac{e^2}{\varepsilon_0} F_1^C(m),
\]
where (this, always negative, function is introduced in Ref. 29), for \(\xi = q_x \varepsilon_0\) and \(\eta = q_y \varepsilon_0\), we have that
\[
F_1^C(m) = -\frac{2}{\pi} \int_0^\infty \, d\xi \int_0^\infty \, d\eta \, \frac{e^{-\eta^2/4}}{\eta^2 + \xi^2} f_m(\eta)
\times S_m(\xi) S_m^0(\xi),
\]
where, as there is no any finite contribution (or divergence) in the RHS for \(\delta \to 0\), we have neglected by \(\delta^2\) in the factor \((\xi^2 + \eta^2 + \delta^2)^{-1/2}\). Here (cf. with Ref. 29) \(f_1(\eta) = 1\) and, for \(m \geq 3\), the even function
\[
f_m(\eta) = \frac{1}{2} \left[ 1 + 2 \sum_{n=1}^{\ell} \cos(\sqrt{2 \pi / m} \eta) \right].
\]
From Eq. (70) we calculate: \(F_1^C(1) \approx -1.184878\), \(F_1^C(3) \approx -0.665565\), \(F_1^C(5) \approx -0.518796\), and \(F_1^C(7) \approx -0.440366\); notice, these values for \(F_1^C(m)\) were previously obtained in Ref. 29.

Using Eqs. (72)-(74), we rewrite Eq. (68) as
\[
E_1^{(m)} = \frac{e^2}{\varepsilon_0} D(m),
\]
where
\[
D(m) = \sum_{k=1}^{\infty} D_1^{(m)}(k),
\]
and
\[
D_1^{(m)}(k) = \frac{2}{\pi m} \int_0^\infty \, d\xi \int_0^\infty \, d\eta \, \frac{f_m(\eta)}{\sqrt{\xi^2 + \eta^2}}
\times e^{-\eta^2/4} \frac{S_m(\eta)}{S_m^0(\eta)} \cos(k \sqrt{2 \pi / m} \xi).
\]
Point out that the sum in the RHS of Eq. (72) is rapidly convergent as, e.g., \(D_1^{(m)}(2)/D_1^{(m)}(1) < 0.1\) and \(D_1^{(m)}(5)/D_1^{(m)}(2) < 0.1\), for \(m = 1, 3, 5, 7\); see also the paragraph above Eq. (72). From Eqs. (79), (80) we calculate: \(D(1) \approx -0.10661\) (for this precision it is enough to include the first seventeen terms in the sum of Eq. (79)), \(D(3) \approx -0.05386\) (for this precision it is enough to include the first fourteen terms of the sum), \(D(5) \approx -0.04282\) (for this precision it is enough to include the first fourteen terms of the sum), and \(D(7) \approx -0.03678\) (for this precision it is enough to include the first twelve terms of the sum).

Using Eqs. (72)-(74), we rewrite Eq. (71) as
\[
E_3^{(m)} = \frac{e^2}{4 \pi \varepsilon_0} \int_{-\infty}^{\infty} dq_x \int_{-\infty}^{\infty} dq_y \frac{g_m(q_x \varepsilon_0)}{\sqrt{q_x^2 + q_y^2 + \delta^2 / \ell_0^2}}
\times S_0(q_x \ell_0) \sum_{k=-\infty}^{\infty} e^{-ikq_x L_x^0}
\times \left\{ \sum_{m_y=-\infty, m_y \neq 0} \sum_{m_z=-\infty}^{\infty} e^{-im_y \eta} S_m(\xi) \right\},
\]
where (this even function is introduced in Ref. 29)
\[
g_m(\eta) = S_m^2(\eta) + e^{-\eta^2/2} - 2e^{-\eta^2/4} f_m(\eta) S_m(\eta).
\]
Notice, \(g_1(\eta) = [\exp(-\eta^2/4) - S_1(\eta)]^2\). Further, using Eq. (83) we express in the RHS of Eq. (81) the product of the sums in the square and the curly brackets as
\[
\frac{2\pi}{L_x^0} \sum_{M_x=-\infty}^{\infty} \delta(q_x + M_x) \frac{2\pi}{L_x^0}
\times \left\{ \left[ \frac{2\pi}{L_x} \sum_{M_y=-\infty}^{\infty} \delta(q_y + M_y \frac{2\pi}{L_x}) \right] - 1 \right\}.
\]
Using Eq. (83) in Eq. (82) and calculating the integrals with the help of delta-functions we obtain
\[
E_3^{(m)} = E_3^{(m),a} + E_3^{(m),b},
\]
where
\[
E_3^{(m),a} = \frac{e^2}{2\varepsilon_0 L_x^0} \sum_{M_x=-\infty}^{\infty} \sum_{M_y=-\infty}^{\infty} S_m^2(\sqrt{2\pi/m} M_x)
\times \frac{g_m(\sqrt{2\pi/m} M_y)}{\sqrt{M_x^2 + M_y^2 + (m/2\pi)\delta^2}},
\]
and
\[
E_3^{(m),b} = -\frac{e^2}{2\varepsilon_0 L_x^0} \sum_{M_x=-\infty}^{\infty} \sum_{M_y=-\infty}^{\infty} S_m^2(\sqrt{2\pi/m} M_x) \int_{-\infty}^{\infty} dq_y
\times \frac{g_m(q_y \ell_0)}{\sqrt{(2\pi/L_x^0)^2 M_x^2 + q_y^2 + \delta^2 / \ell_0^2}}.
\]
Taking into account that \(S_m(\sqrt{2\pi/m} M) = \sin(\pi M)/(\pi M) = 0\) for \(M \neq 0\) and, in addition, that \(S_m(0) = 1\) for \(M = 0\), we obtain that in the RHS of Eqs. (83) only the \(M_y = 0\) term, of the sums over \(M_x\), is contributed. Then Eq. (86) obtains the form
\[
E_3^{(m),b} = -\frac{e^2}{2\pi \varepsilon_0 \ell_0} \int_0^{\infty} dq_y \frac{g_m(q_y \ell_0)}{q_y},
\]
where it is used that under the integral in $g_m(\eta)/\sqrt{\eta^2 + \delta^2}$ it is safe to neglect by $\delta^2 \to 0$ as, for $\eta \ll 1$, $g_m(\eta)/\eta \propto m = 3, 5, \ldots$ and $g_m(\eta)/\eta \propto m = 1$. Further, in the sum over $M_y$ in the RHS of Eq. (85) it is easy to see that $M_y = 0$ term vanish as it is $\propto g_m(0)/\sqrt{m/2\pi \delta} = 0$, where $g_m(0) = 0$ and $\delta \neq 0$. Then finally we rewrite Eq. (85) as

$$E_3^{(m),a} = \frac{e^2}{2\sqrt{2\pi m}} \sum_{k=1}^{\infty} \frac{1}{k} e^{-\pi k^2/m}, \quad (88)$$

where it is used that $g_m(\sqrt{2\pi/m} k) = \exp(-\pi k^2/m)$, for $k \neq 0$. Then using Eqs. (87), (88) we rewrite Eq. (85) in the form

$$E_3^{(m)} = \frac{e^2}{\varepsilon \varepsilon_0} \Delta F_C^{(m)}, \quad (89)$$

where

$$\Delta F_C^{(m)} = \frac{1}{\sqrt{2\pi m}} \left( \sum_{k=1}^{\infty} \frac{1}{k} e^{-\pi k^2/m} - \int_0^{\infty} dq_m g_m(q_m) \right). \quad (90)$$

From Eqs. (91) simple numerical calculations give that: $\Delta F_C^{(1)} \approx -0.0021047$, $\Delta F_C^{(3)} \approx -0.0812376$, $\Delta F_C^{(5)} \approx -0.0654775$, and $\Delta F_C^{(7)} \approx -0.0552258$. Now, Eqs. (85), (77), (80), (89) give for $E^{(a)}(m) = (e^2/\varepsilon \varepsilon_0)U_C^{(m)}(m)$ the analytical expression, $E^{(a)}(m) \equiv E^{(a)}$. Then above given numerical results shows that $U^{(a)}_C(1) \approx -1.29350, U^{(a)}_C(3) \approx -0.80066, U^{(a)}_C(5) \approx -0.62709, and U^{(a)}_C(7) \approx 0.53237$.

To calculate $E^{(e)}_{ee}$, in the RHS of Eq. (69) we use that an $i$-th term in the sum over $i$ is independent of $i = n_y(i)$. I.e., in the RHS of Eq. (63) $N^{-1} \sum_{i=1}^{N} A_i \to A_i$. Then Eq. (69) can be rewritten by using Eq. (72) as

$$E^{(e)}_{ee} = -\frac{e^2}{4\pi \varepsilon_0} \sum_{n=-\ell}^{\ell} \sum_{k=1}^{\infty} \int_0^{\infty} dq_x \int_0^{\infty} dq_y \times \frac{e^{-iq_2L_x^2}}{\sqrt{q_x^2 + q_y^2 + \delta^2 / \ell_0^2}} |M(\mathbf{q}; k_x^{(n)}, k_x^{(n)})|^2, \quad (91)$$

where the matrix element

$$M(\mathbf{q}; k_x^{(n)}, k_x^{(n)}) = \int d\mathbf{r} \exp[i\mathbf{q}\cdot\mathbf{r}] \phi^{(m)}_{k_x^{(n)}}(\mathbf{r}) \phi^{(m)}_{k_x^{(n)}}(\mathbf{r}) = \exp[2iq_y(k_x^{(n)} + k_x^{(n)}) - (k_x^{(n)} - k_x^{(n)})^2 - q_y^2/\ell_0^2/4] \times \exp[i(q_x + k_x^{(n)} - k_x^{(n)})L_x^2/n_x^2] \times [1 - \exp(-i(q_x + k_x^{(n)} - k_x^{(n)})L_x^2)]. \quad (92)$$

After using Eq. (92) in Eq. (91), it follows that the factor $|M(\mathbf{q}; k_x^{(n)}, k_x^{(n)})|^2$ in Eq. (91) is independent of $n_x^2$ and $k_x^{(n)}$. However, it is dependent on $\mathbf{q}$ and $\Delta k = k_x^{(n)} - k_x^{(n)} = (\sqrt{2m/\ell_0})(n_y^{(n)} - n_y^{(n)})$; the latter is independent of $n$. Introducing $m_y = n_y^{(n)} - n_y^{(n)}$, we have that $\Delta k = (\sqrt{2m/\ell_0})m_y$, where $m_y \neq 0$. It is important to point out that from above it follows that in the RHS of Eq. (91) the $n$-th term of the sum over $n$ is independent of $n$. Further, using Eq. (90) we readily rewrite Eq. (91) as follows

$$E^{(e)}_{ee} = -\frac{e^2}{\varepsilon \varepsilon_0} \sum_{n=-\ell}^{\ell} \sum_{k=1}^{\infty} e^{-\pi m m_y^2} \times \frac{\sin^2[\pi(M_x - m m_y)]}{[\pi(M_x - m m_y)]^2} \int_0^{\infty} dq_y \times e^{-q_y^2/\ell_0^2} \times \int_0^{\infty} dq_x \times \sqrt{q_x^2 + q_y^2 + \delta^2 / \ell_0^2}, \quad (93)$$

where the integral over $q_x$ is already carried out, using obtained delta-functions. It is readily seen that, due to $m_y \neq 0$, the term $M_y = 0$ does not contribute to the RHS of Eq. (93). I.e., the sum over all $M_x$ can be substituted only by the sum over $M_x \neq 0$, i.e., by $M_x = \pm 1, \pm 2, \ldots$. Further, in the RHS of Eq. (87) we obtain that

$$\frac{\sin^2[\pi(M_x - m m_y)]}{[\pi(M_x - m m_y)]^2} = \delta_{M_x, m m_y}, \quad (94)$$

i.e., reduces to the Kronecker delta symbol. Then using Eq. (94) in Eq. (93) we readily obtain

$$E^{(e)}_{ee} = -\frac{e^2}{\varepsilon \varepsilon_0} F_2(m), \quad (95)$$

where

$$F_2(m) = \frac{2}{m \pi} \sum_{k=1}^{\infty} e^{-\pi m k^2} \int_0^{\infty} dq_y e^{-q_y^2/\ell_0^2}. \quad (96)$$

Notice, the sum over $k$ in the RHS of Eq. (90) is very rapidly convergent already for $m = 1$: in the $k = 1$ term $\approx 0.01618$ the $k = 2$ term $\approx 6.85 \times 10^{-7}$. Numerical calculations (obviously, quite simple) give that $F_2(1) \approx 0.016183$, $F_2(3) \approx 1.0475 \times 10^{-5}$, $F_2(5) \approx 1.18 \times 10^{-8}$, and $F_2(7) \approx 1.6 \times 10^{-11}$. Then using Eqs. (46), (50), (53), (57), (58), (61), (64), (63), (75), (80), (89), (90), (95), (96) we can rewrite Eq. (46) as

$$E^{(m),ch}_N = \frac{\hbar \omega_c \hat{N}}{2} + \frac{e^2 \hat{N}}{\varepsilon \varepsilon_0} U^{UB}(m), \quad (97)$$

where $U^{UB}(m) = D(m) + F_C^{(m)}(m) - \Delta F^{(C)}(m) - F_2(m)$. Notice that $U^{UB}(m)$ is negative, due to many-body interactions, and gives lowering of the total energy per electron in the units of $e^2/\varepsilon \varepsilon_0$, i.e., $U^{UB}(m) = [E^{(m),ch}_N - \hbar \omega_c]/2(e^2/\varepsilon \varepsilon_0)$. Finally, we calculate

$$U^{UB}(1) \approx -1.30968, \quad U^{UB}(3) \approx -0.80067, \quad U^{UB}(5) \approx -0.62709, \quad U^{UB}(7) \approx -0.53237. \quad (98)$$
where after the decimal point only first five digits are kept. Even though the values of $U^{UB}(1)$, $U^{UB}(3)$ and $U^{UB}(5)$ given by Eq. (12) are much lower than pertinent total lowering at $\nu = 1$, 1/3, and 1/5 for the Laughlin variational wave function $\Psi^\nu(r)$ (i.e., $-\sqrt{\frac{\pi}{8}} \approx -0.6267$, $-0.4156 \pm 0.0012$, and $-0.3340 \pm 0.0028$, respectively), the comparison of these results by using their face value is not too useful. In particular, due to the absence in $U^{UB}_e(m)$ of any contribution from the self-interaction of an ion with itself while for IJB model used in Ref. [2] (and many others studies) pertinent contribution is present.

It is interesting that the results Eq. (12) are quite close to pertinent numerical results obtained, for $U^C(m)$, in Ref. [2] within framework that, however, contains some oversimplifications the influence of which it is difficult to estimate beforehand.

### B. Energy of ground-state $\Psi^{(m),IJB}_N$, for IJB

Using the Hamiltonian Eq. (6), we calculate the total energy of electron-ion system (matrix elements should be calculated within the main strip) in the state Eq. (44) as

$$E^{(m),IJB}_N = \langle \Psi^{(m),IJB}_N | \hat{H}^{IJB}_N | \Psi^{(m),IJB}_N \rangle,$$  \hspace{1cm} (99)

where in the RHS for the kinetic energy term, cf. with Eqs. (14) - (18), it follows

$$\langle \Psi^{(m),IJB}_N | \hat{\dot{H}}_0 | \Psi^{(m),IJB}_N \rangle = N \frac{\hbar \omega_c}{2}. \hspace{1cm} (100)$$

In the RHS of Eq. (99) the term related with the $V_{bb}$, Eq. (5), interaction (we call it also IJB-IJB interaction) is given as

$$\langle \Psi^{(m),IJB}_N | V_{bb} | \Psi^{(m),IJB}_N \rangle \equiv V_{bb} = \tilde{N} (E^I_{bb} + E^{II}_{bb}), \hspace{1cm} (101)$$

where it follows (now in the RHS of Eq. 101) all integrations we reduce to MS that

$$E^I_{bb} = \frac{1}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} \int \int \frac{e^2 dR dR'}{\varepsilon |R - R'|} \times |\phi_{n_{gs}}^{(i)}(R)|^2 |\phi_{n_{gs}}^{(j)}(R')|^2, \hspace{1cm} (102)$$

and

$$E^{II}_{bb} = \frac{1}{2N} \sum_{k=-N_C}^{N_C} \sum_{i=1}^{N} \sum_{j=1}^{N} \int \int \frac{e^2 dR dR'}{\varepsilon |R - R' - kL_x|^2} \times |\phi_{n_{gs}}^{(i)}(R)|^2 |\phi_{n_{gs}}^{(j)}(R')|^2, \hspace{1cm} (103)$$

where we have used that, e.g., within MS given by Eq. (42) $n^{eh}_{ia}(R) = n_{ia}(R)$, according to Sec. IV B. Point out, in Eqs. (102), (103) the integrals are carried out within MS as, e.g., instead of $\int_{MS} dR$ we write $\int dR$.

In the RHS of Eq. (99) the term related with electron-ion interaction, we call it also electron-IJB interaction, is given as

$$\langle \Psi^{(m),IJB}_N | V_{ee} | \Psi^{(m),IJB}_N \rangle = \tilde{N} (E^e_{ee} + E^{II}_{ee}), \hspace{1cm} (104)$$

where using Eqs. (7), (42) and $n_{ia}(R) = n_{ia}^{eh}(R)$ we calculate that

$$E_{ee} = -\frac{1}{mN} \sum_{\ell=0}^{N} \sum_{j=1}^{N} \sum_{k=-N_C}^{N_C} \int d\mathbf{r} \int d\mathbf{R} \times \frac{e^2 \varepsilon}{|\mathbf{r} - \mathbf{R} - kL_x|^2} |\phi_{\ell_{gs}}^{(m)}(\mathbf{r})|^2 |\phi_{n_{gs}}^{(j)}(\mathbf{R})|^2, \hspace{1cm} (105)$$

and comparing with Eq. (54), it follows that

$$E_{ee} = E_{ee}. \hspace{1cm} (106)$$

In the RHS of Eq. (99) the term related with electron-electron interaction

$$\langle \Psi^{(m),IJB}_N | V_{ee} | \Psi^{(m),IJB}_N \rangle \equiv \langle \Psi^{(m),eh} | V_{ee} | \Psi^{(m),eh} \rangle = \tilde{N} (E^e_{ee} + E^{II}_{ee}), \hspace{1cm} (106)$$

where $E^e_{ee}$ and $E^{II}_{ee}$ are calculated in Sec. V A.

As in the RHS of Eqs. (102), (103), and (52) the $i$-th term is independent of $i = n_{gs}^{(i)}$, it follows that

$$E^I_{bb} = \frac{1}{2} \sum_{k=-N_C}^{N_C} \sum_{i=1}^{N} \int \int \frac{e^2 dR dR'}{\varepsilon |R - R' - kL_x|^2} \times |\phi_{n_{gs}}^{(i)}(R)|^2 |\phi_{n_{gs}}^{(j)}(R')|^2, \hspace{1cm} (107)$$

and

$$E^{II}_{bb} = \frac{1}{2} \sum_{k=-N_C}^{N_C} \sum_{i=1}^{N} \int \int \frac{e^2 dR dR'}{\varepsilon |R - R' - kL_x|^2} \times |\phi_{n_{gs}}^{(i)}(R)|^2 |\phi_{n_{gs}}^{(j)}(R')|^2, \hspace{1cm} (108)$$

Further, using Eqs. (46), (99) and above results, we have that

$$\langle \Psi^{(m),IJB}_N | E^{(m),IJB}_N | \Psi^{(m),IJB}_N \rangle = E^I_{bb} + E^{II}_{bb} - E^I_{ii} - E^{II}_{ii}, \hspace{1cm} (110)$$

First, in the RHS of Eq. (110) we rewrite $E^I_{bb} - E^I_{ii}$, by separating in $E^I_{ii}$ the $k = 0$ term and mutually canceling it with all $j \neq i$ terms in $E^I_{bb}$, as follows

$$E^I_{bb} - E^I_{ii} = \frac{1}{2} \int \int \frac{e^2 dR dR'}{\varepsilon |R - R'|} |\phi_{n_{gs}}^{(i)}(R)|^2 |\phi_{n_{gs}}^{(i)}(R')|^2$$

$$\hspace{3cm} - \frac{1}{2} \sum_{k=-N_C}^{N_C} \sum_{i=1}^{N} \int \int \frac{e^2 dR dR'}{\varepsilon |R - R' - kL_x|^2} \times |\phi_{n_{gs}}^{(i)}(R)|^2 |\phi_{n_{gs}}^{(j)}(R')|^2. \hspace{1cm} (111)$$
Now it is easy to see from Eqs. (111) and (108) that

\[ E_{bb}^I - E_{ii}^b + E_{bb}^{II} = \frac{1}{2} \int \int \frac{e^2 dR dR'}{|R - R'|} |\phi_{n_y}(R')|^2 \]

\[ \times |\phi_{n_y}(R)|^2 |\phi_{n_y}(R')|^2 \]

\[ \times |\phi_{n_y}(R)|^2 |\phi_{n_y}(R')|^2 \],

(112)

as the second term in the RHS of Eq. (111) mutually cancels all terms in the RHS of Eq. (108) in the sum over \( j \) except one, \( j = i \). Then using Eq. (112) and Eq. (50) in Eq. (113), we obtain that

\[ (E^{(m)J}_{N} - E^{(m)k}_{N})/\tilde{N} = \Delta E_{J,B,U,B}^{I} + \Delta E_{J,B,U,B}^{II} \]

where

\[ \Delta E_{J,B,U,B}^{I} = \frac{1}{2} \int \int \frac{e^2 dR dR'}{|R - R'|} \]

\[ \times |\phi_{n_y}(R)|^2 |\phi_{n_y}(R')|^2 \],

(114)

and

\[ \Delta E_{J,B,U,B}^{II} = \frac{1}{2} \sum_{k=-N_C k \neq 0}^{-N_C} \left( \int \int \frac{e^2 dR dR'}{|R - R' - kL_s^x x|} \right) \]

\[ \times |\phi_{n_y}(R)|^2 |\phi_{n_y}(R')|^2 - \frac{e^2}{\epsilon L_s^x k} \].

(115)

Using Eqs. (122), (124) in Eqs. (114) and (115), we calculate that

\[ \Delta E_{J,B,U,B} = \frac{e^2}{\epsilon l_0} F_{J,B,U,B}^{I}(m) \]

(116)

and

\[ \Delta E_{J,B,U,B}^{II} = \frac{e^2}{\epsilon l_0} F_{J,B,U,B}^{II}(m) \]

(118)

We obtain that

\[ F_{J,B,U,B}^{I}(m) = \sum_{k=1}^{\infty} D_{II}^{(m)}(k) \]

(119)

where

\[ D_{II}^{(m)}(k) = \frac{2}{\pi} \int_0^{\infty} d\xi \int_0^{\infty} d\eta \frac{1}{\sqrt{\xi^2 + \eta^2}} \cos(k \sqrt{2m \pi} \xi) \]

\[ \times S_2^2(\eta) S_2^2(\xi) - \frac{1}{\sqrt{2m \pi} k} \]

(120)

Point out that the sum in the RHS of Eq. (119) is rapidly convergent as, e.g., \( D_{II}^{(m)}(2)/D_{II}^{(1)}(1) < 0.1 \) and \( D_{II}^{(m)}(5)/D_{II}^{(m)}(2) < 0.1 \), for \( m = 1, 3, 5, 7 \).

From Eqs. (119), (120) we calculate: \( F_{J,B,U,B}^{I}(1) \approx 0.05151 \) (for this precision it is enough to include the first eighteen terms in the sum of Eq. (129)), \( F_{J,B,U,B}^{I}(3) \approx 0.02972 \) (for this precision it is enough to include the first fourteen terms of the sum), and \( F_{J,B,U,B}^{I}(7) \approx 0.01944 \) (for this precision it is enough to include the first thirteen terms of the sum). Further, from Eq. (117) we calculate: \( F_{J,B,U,B}^{I}(1) \approx 0.593068 \), \( F_{J,B,U,B}^{I}(3) \approx 0.342408 \), \( F_{J,B,U,B}^{I}(5) \approx 0.265228 \), and \( F_{J,B,U,B}^{I}(7) \approx 0.224518 \).

As the result we have that \( F_{J,B,U,B}^{I}(1) + F_{J,B,U,B}^{I}(3) \approx 0.64458 \), \( F_{J,B,U,B}^{I}(3) + F_{J,B,U,B}^{I}(5) \approx 0.37213 \), \( F_{J,B,U,B}^{I}(5) + F_{J,B,U,B}^{I}(7) \approx 0.28824 \), and \( F_{J,B,U,B}^{I}(7) \approx 0.24360 \). Now using Eqs. (177-183) and Eqs. (113-120), we obtain the lowering of the total energy per electron (in the units of \( e^2/\epsilon l_0 \)) for JIB model, \( U_{J,B}^{I}(m) \), as

\[ U_{J,B}^{I}(m) = U_{J,B}^{II}(m) + F_{J,B,U,B}^{I}(m) + F_{J,B,U,B}^{II}(m) \]

(121)

Finally, we calculate

\[ U_{J,B}^{I}(1) \approx -0.66510, \quad U_{J,B}^{I}(3) \approx -0.42854 \]

\[ U_{J,B}^{I}(5) \approx -0.33885, \quad U_{J,B}^{I}(7) \approx -0.28877 \].

(122)

Given by Eq. (122), values of \( U_{J,B}^{I}(1), U_{J,B}^{I}(3) \) and \( U_{J,B}^{I}(5) \) are substantially lower than pertinent total lowering at \( \nu = 1, 1/3, \) and \( 1/5 \) for the Laughlin variational wave function, i.e., \( \Delta \sqrt{\pi/8} \approx -0.2667, -0.4156 \pm 0.0012, \) and \(-0.3340 \pm 0.0028 \), respectively. Point out that comparison of \( U_{J,B}^{I}(m) \) with the results of Ref.2 is totally justified. As for the JIB model the ion background is totally equivalent to the model of ion background used in Ref.2. Notice, more accurate numerical calculations for the Laughlin trial function, e.g., at \( \nu = 1/3 \) show the lowering \( \Delta \approx -0.410 \pm 0.001 \).

C. Remarks on partial crystal-like correlation order and energy of ground-state

Here, for JIB, we will make additional remarks on partial crystal-like correlation order Eq. (15) and some effects of it modification on the energy of ground-state.

First, it is easy to see that using Eq. (15) in Eq. (10) we obtain that, due assumed partial crystal-like correlation order among \( N \) electrons of MR, the Hamiltonian \( H_{MR}^{II} \), Eq. (10), obtains the form \( n_{xz}^{max} \times H_{J,B}^{II} \). The latter is i) dependent only on \( \tilde{N} \) “compound” electrons radius vectors and ii) periodic, with the period \( L_s^x \) on any \( x_i, i = 1, \ldots, \tilde{N} \); so far these radius vectors are defined within whole MR. Then we can rewrite Eq. (9) as follows

\[ n_{xz}^{max} \tilde{H}_{J,B}^{II}(r_{1,\ldots,\tilde{N}}) = E_{N} \tilde{\Psi}(r_{1,\ldots,\tilde{N}}) \]

(123)
where $\Psi(r_1, \ldots, r_N)$, after using Eq. (15), is transformed to $\tilde{\Psi}(r_1, \ldots, r_N)$. Now from Eq. (123) it is clear that its wave functions $\tilde{\Psi}(r_1, \ldots, r_N)$ have the same properties, (i) and (ii), as the Hamiltonian $\tilde{H}_{N}^{IB}(r_1, \ldots, r_N)$. So we can assume the ground-state trial wave function $\tilde{\Psi}(r_1, \ldots, r_N)$, normalized within MR, as

$$\tilde{\Psi}(r_1, \ldots, r_N) = \frac{1}{(n_{xs}^{\max})^{N/2}} \tilde{\Psi}_{N}^{(m),IB}(r_1, \ldots, r_N), \quad (124)$$

where $\tilde{\Psi}_{N}^{(m),IB}(r_1, \ldots, r_N)$ is given by Eq. (41) and $r_i$ are defined within whole MR. It is clear that $\tilde{\Psi}(r_1, \ldots, r_N)$ is periodic in MR, with period $L_{x}^N$, over any its $x_i$. It is easy to see that

$$\int_{MR} dr_1 \ldots \int_{MR} dr_N |\tilde{\Psi}(r_1, \ldots, r_N)|^2 = 1, \quad (125)$$

indeed, due to its periodicity we have that

$$\int_{MR} dr_i |\tilde{\Psi}(r_1, \ldots, r_N)|^2 = n_{xs}^{\max} \int_{MS} dr_i |\tilde{\Psi}(r_1, \ldots, r_N)|^2, \quad (126)$$

Then using Eq. (126) in Eq. (125), its LHS we rewrite as

$$\left( n_{xs}^{\max} \right)^N \int_{MS} dr_1 \ldots \int_{MS} dr_N |\tilde{\Psi}(r_1, \ldots, r_N)|^2 = \int_{MS} dr_1 \ldots \int_{MS} dr_N |\tilde{\Psi}_{N}^{(m),IB}|^2 = 1, \quad (127)$$

i.e., the normalization of the wave function $\tilde{\Psi}(r_1, \ldots, r_N)$. Eq. (123), within MR is reduced to the normalization of the wave function $\tilde{\Psi}_{N}^{(m),IB}$ within MS.

Further, from Eq. (123) the ground-state energy of $N$ electrons within MR, for IJB, is given as

$$E_N = n_{xs}^{\max} \int_{MR} dr_1 \ldots \int_{MR} dr_N \tilde{\Psi}^* \tilde{H}_{N}^{IB} \tilde{\Psi}, \quad (128)$$

where, using the periodicity of the Hamiltonian $\tilde{H}_{N}^{IB}(r_1, \ldots, r_N)$ and property similar to Eq. (126), we calculate

$$\frac{E_N}{n_{xs}^{\max}} = \left( n_{xs}^{\max} \right)^N \int_{MS} dr_1 \ldots \int_{MS} dr_N \tilde{\Psi}^* \tilde{H}_{N}^{IB} \tilde{\Psi}, \quad (129)$$

or

$$\frac{E_N}{n_{xs}^{\max}} = E_{N}^{(m),IB}, \quad (130)$$

where, in agreement with with Eq. (129), we have

$$E_{N}^{(m),IB} = \int_{MS} dr_1 \ldots \int_{MS} dr_N \tilde{\Psi}_{N}^{(m),IB} \tilde{H}_{N}^{IB} \tilde{\Psi}_{N}^{(m),IB}. \quad (131)$$

So it is shown that the ground-state can correspond to partial crystal-like correlation order, Eq. (15), among $N$ electrons of MR. Then the study of 2DES of $N$ electrons within MR is exactly reduced to the treatment of 2DES of $\tilde{N}$ electrons localized within MS to which PBC is imposed along $x$-direction. In addition, similar justification can be applied for the lowest excited-state; i.e., the excited-states that we study in Sec. VI.

In Sec. VIII, Concluding Remarks, we will discuss effect on the energy of ground-state, treated in Sec. V, of the change of the period $L_{x}^N$ in Eqs. (15)-(16) by period $L_{x}^N = n_{a} \times L_{x}^M$ of arbitrary value. The area of unit cell is fixed as it is equal to the area of MR per electron. It is seen that $n_{a} \ll 1$ correspond to very short period of PBC (and much stronger crystal-like correlation order than for $n_{a} = 1$) while $n_{a} \rightarrow \infty$ correspond to (practical) absence of both the PBC and the crystal-like correlation order.

VI. EXCITED-STATES OF THE GROUND-STATE $\Psi_{N,NS}^{(m),eh}$ AND OF THE GROUND-STATE $\Psi_{N,NS}^{(m),IB}$

In this section we are looking for an excited-state of the lowest energy, with respect to the ground-state $\Psi_{N,NS}^{(m),eh}$, Eq. (29): i.e., for the quantum Hall system with UIB at $\nu = 1/m$ (remind that $m = 2\ell + 1, \ell = 0, 1, 2, \ldots$). The treatment should include excitations that occur: i) as without the change of spin of the excited compound electron (the compound exciton), ii) so with the change of spin of the excited compound electron (the compound spin-exciton). The treatment of the compound spin-exciton, at $m = 1, 3, 5, \ldots$, should be partly different from the treatment of the compound exciton, at $m = 3, 5, \ldots$. Indeed, in $\Psi_{N,NS}^{(m),r_1,\ldots,r_N}$, Eq. (23), it is implicit that single-electron wave function $\tilde{\psi}_{k_{r_1}}^{(m)}(r_1)$ is multiplied by the spin wave function $|\sigma > = \tilde{\psi}_{\sigma}(\sigma_j) = \delta_{\sigma,\sigma_j}$, where spin eigenvalue $\sigma = \pm 1$ is pertinent to spin up LLL. Hence, for the treatment of the compound spin-exciton (in particular, the excited-state at $\nu = 1$) we will need to take into account as well, at the least partly, the subspace of the lowest (empty) spin down ($\sigma = -1$) Landau level: with the same spatial single-electron wave functions and the spin wave function $| -1 > = \tilde{\psi}_{\sigma = -1}(\sigma_j)$.

It follows that all results obtained for the excited-states of the ground-state $\Psi_{N,NS}^{(m),eh}$, for UIB, are very simply extended on the excited-states of the ground-state $\Psi_{N,NS}^{(m),IB}$, for IJB. In particular, for IJB the energy of any excited-state, counted from the energy of the ground-state, $\Psi_{N,NS}^{(m),IB}$, coincides with the energy of relevant excited-state for UIB, counted from the energy of its ground-state, $\Psi_{N,NS}^{(m),eh}$. I.e., here the only difference is that for IJB model the energies of the ground-state and of its excited-states are shifted upwards on the same value (for given $m$) with respect to the relevant energies
for UIB model.

A. Compound exciton and compound spin-exciton. Compound electrons and compound hole

We assume, for \( m \geq 3 \), the compound exciton wave function \( \Psi_{N,N}(m) \) it follows from the “ground state” partial many-electron wave function \( \Psi_{N,N}(m) \) that the following (cf. with Eq. (19) of Ref. 22) form

\[
\Psi_{N,N}(m) = \prod_{i=1}^{N} \phi_{n_{gs}}(\mathbf{r}_i) \sum_{n=-\ell}^{\ell} C_n(m) \times \Phi_{N,N}(m) \langle \mathbf{r}_1, \ldots, \mathbf{r}_N \rangle,
\]

where an excited “partial” many-electron wave function \( \Phi_{N,N}(m) \) is also valid; in particular, for \( m = 1 \), we have that the second term in its square brackets, \( \langle n + \hat{n} \rangle \), is zero. For definiteness, we call the spin-exciton Eq. (136) as compound one as well for \( m = 1 \), even though here there is no any compound structure of electrons or the hole resembling pertinent structure for \( m = 3, 5, \ldots \).

Point out, in Eqs. (132)-(133) we have that \( n_{gs} = n_{gs}^0 \) obtain any finite integer value (i.e., \( n_{gs}^0 = 0, \pm 1, \pm 2, \ldots \) as for \( m = 3 \) so for \( m = 1 \).

Point out, the excited-state wave functions \( \Psi_{N,N}(m) \) are periodic with the period \( L_{x,j} \) along any of \( 2N \) variables \( x_i \) and \( x_j \) remind the same properties have the ground-state wave function \( \Psi_{N,N}(m) \), Eq. (29). Point out that

\[
\langle \Psi_{N,N}(m) | \Psi_{N,N}(m) \rangle = \langle \Psi_{N,N}(m) | \Psi_{N,N}(m) \rangle = 0,
\]

i.e., as required, the excited-state wave functions \( \Psi_{N,N}(m) \) are orthogonal to the ground-state wave function \( \Psi_{N,N}(m) \). These excited-state wave functions are orthonormalized as it is seen, e.g., that

\[
\langle \Psi_{N,N}(m) | \Psi_{N,N}(m) \rangle = \delta_{m,m'} \delta_{N,N'}.
\]

\[
\text{Eq. (137) is also valid if in its LHS to substitute} \Psi_{N,N}(m) \text{ by } \Psi_{N,N}(m), \text{ i.e., } i = 1, 2.
\]

It is seen that the compound exciton wave function \( \Psi_{N,N}(m) \) describes an exciton like excitation of the compound (composite) structure, cf. with Refs. 14, 15, 37, 38, 39.

In particular, the form of the exciton charge (cf. with Eqs. 29, 33, 37, 132) density, \( \delta_{m,m'}(\mathbf{r}) \), given as

\[
\delta_{m,m'}(\mathbf{r}) = e \times \delta_{m,m'}(\mathbf{r})
\]

\[
\langle \Psi_{N,N}(m) | \Psi_{N,N}(m) \rangle = \delta_{m,m'} \delta_{N,N'}.
\]

\[
\text{Eq. (139) is also valid if in its LHS to substitute} \Psi_{N,N}(m) \text{ by } \Psi_{N,N}(m), \text{ i.e., } i = 1, 2.
\]

It is seen that the compound exciton wave function \( \Psi_{N,N}(m) \) describes an exciton like excitation of the compound (composite) structure, cf. with Refs. 14, 15, 37, 38, 39.

In particular, the form of the exciton charge (cf. with Eqs. 29, 33, 37, 132) density, \( \delta_{m,m'}(\mathbf{r}) \), given as

\[
\langle \Psi_{N,N}(m) | \Psi_{N,N}(m) \rangle = \delta_{m,m'} \delta_{N,N'}.
\]
is readily reduced to the form
\[
\delta \rho_{(m),n}^{(m)}(\mathbf{r}) = \sum_{n=-\ell}^{\ell} \delta \rho_{(m),n}^{(m),n}(\mathbf{r}),
\]
where the charge density of the \(n\)-th quasiexciton
\[
\delta \rho_{(m),n}^{(m),n}(\mathbf{r}) = \frac{e}{m} \left( \tilde{\varphi}_{x_0}^{(m)}(\mathbf{r}) \right)^2 - \frac{e}{m} \left( \varphi_{x_0}^{(m)}(\mathbf{r}) \right)^2.
\]
I.e., for \(m \geq 3\), the compound exciton charge density is the superposition of the charge densities of \(m\) quasiexcitons, counted by the subscript \(n\). The first and the second terms in the RHS of Eq. (140) present the \((n\)-th\) quasielectron and the \((n\)-th\) quasihole charge densities, respectively, of the \((m\)-th\) quasiexciton. The former is mainly localized, within MS, at \(y \approx \ell_0^2 k_{x_0}^{(n+\hat{n})}\) (i.e., almost within the \(j_0\)-th unit cell) and the latter is mainly localized at \(y \approx k_{x_0}^{(n)} \ell_0^2\) (i.e., almost within the \(i_0\)-th unit cell). Point out, the RHS of Eq. (139) is calculated from the RHS of Eq. (138) without any approximations. Integrating the quasielectron or the quasihole charge densities, from Eq. (140), over \(\mathbf{r}\) within the total area of MS we readily obtain that the total quasiexciton or quasihole charge, within MS, is given as \(e/m\) or \(-e/m\), respectively; for given \(n\)-th quasiexciton. I.e., in the fractional quantum Hall regime, at \(\nu = 1/m\), these charges are fractional and have the same values as quasielectron and quasihole fractionally charged excitations within the Laughlin model.\(^1\) Point out that in our model the total charge of the quasiexciton (the quasiholes) is independent from it "specific" intra-unit-cell quantum numbers \(n, j_0, \hat{n}\), or \(n, k_{x_0}^{(n+\hat{n})}, i_0, j_0, i_0, j_0, k_{x_0}^{(n)}\).

Due to PBC (in particular, the periodicity of single-electron wave functions) it is clear that the charge density of the \(n\)-th quasiexciton, Eq. (130), has its images that are periodic with the period \(L_x^\prime = \sqrt{2m\pi\ell_0}\) along the \(x\)-direction, for \(x\) outside MS; i.e., for \(x > L_x^\prime\) or \(x \leq L_x^\prime\) \((\ell_0^2 - 1)\). The same property of the periodicity holds for the charge density of the \(n\)-th quasiexciton and of the \(n\)-th quasiholes. Notice, these properties of the quasielectron and the quasiholes periodicity does not have a counterpart among the properties of the quasiparticles, of fractionally charged excitations, in the Laughlin model\(^2\), see also, e.g., (133). In addition, in the present model all \(m\) different quasiholes (quasiexcitons) and quasiexcitons are strongly correlated among themselves. Such that \(m\) different quasiexcitons (however, strongly correlated) compose the compound exciton excited-state Eq. (132), according to Eqs. (138)- (140).

Point out that for the compound spin-exciton wave function \(\Psi_{N,N_1,N}(m)\), the form of the spin-exciton charge density, \(\delta \rho_{(m),s}^{(m)}(\mathbf{r})\), coincides with that given by the RHS of Eqs. (138), (139) and Eq. (140). With the only difference that here are allowed all possible \(m\) and \(\hat{n}\): \(m = 1, 3, 5, \ldots\) and \(\hat{n} = 0, \pm 1, \ldots, \pm \ell\). I.e., the compound spin-exciton will have the charge density \(\delta \rho_{(m),s}^{(m)}(\mathbf{r}) = 0\), if \(\hat{n} = 0\) and \(n_{y_0}^{(m)} = 0\). Notice, for the compound exciton we have that \(\delta \rho_{(m),s}^{(m)}(\mathbf{r}) \neq 0\), at any allowed \(n, n_{y_0}^{(m)}\). Hence, \(m\) strongly correlated spin-quasieexcitons, for \(m \geq 3\), compose the compound spin-exciton Eq. (135).

We are interested in calculation of the energy gaps \(\Delta E_{(m),s}^{(m)}\) and \(\Delta E_{(m),s}^{(m)}\), for the creation of the compound exciton and the compound spin-exciton, respectively, within MS, cf. with Refs.\(^3,14,15,37,38,39\). We also call \(\Delta E_{(m),s}^{(m)}\) and \(\Delta E_{(m),s}^{(m)}\) as the energy of the compound exciton (or the energy of the compound exciton excitation) and the energy of the compound spin-exciton (or the energy of the compound spin-exciton excitation), respectively. Moreover, we are mainly interested in the calculation of the minimal value of \(\Delta E_{(m),s}^{(m)}\), at \(m \geq 3\), and \(\Delta E_{(m),s}^{(m)}\), at \(m = 1\), that defines the activation gap at \(m \geq 3\) and \(m = 1\), respectively. The activation gap is experimentally observable from the activation behavior of the direct current magnetotransport coefficients related with dissipation, i.e., typically, the diagonal resistance or resistivity \(\rho_{yy}, \rho_{xx}\) and the diagonal conductance \(\sigma_{yy}\). As we will show, for \(m = 1\) defined in such manner the activation gap (given by one of the smallest values of \(\Delta E_{(1),s}^{(1)},\) however, does not by the smallest one) is much larger than simply the minimal value of \(\Delta E_{(1),s}^{(1)},\) because for the latter gap pertinent transitions does not contribute to any relevant dissipative kinetic coefficient. Respectively, this gap should be here discarded.

So far we have assumed UIB. Now, for IJB, instead of the wave function Eq. (132), we obtain the pertinent compound exciton wave function, \(\Psi_{N,N_1,m}^{(m)}\), of the ground-state Eq. (141) as
\[
\Psi_{N,N_1,m}^{(m)} = \sum_{n=-\ell}^{\ell} C_n(m) \Phi_{N_1,m;}^{(m)}(r_1, \ldots, r_N),
\]
and instead of the wave function Eq. (139), we obtain the pertinent compound spin-exciton wave function, \(\Psi_{N,N_1,m}^{(m),s}\), of the ground-state Eq. (141), in the following form
\[
\Psi_{N,N_1,m}^{(m),s} = \sum_{n=-\ell}^{\ell} C_n(m) \Phi_{N_1,m;}^{(m),s}(r_1, \ldots, r_N).
\]
It is easy to see that all above results, e.g., Eqs. (136)- (140), are valid for the excited-states Eqs. (141), (142) (and their ground-state).

### B. Energy of the compound exciton

For UIB, by using the Hamiltonian Eq. (11), we calculate the total energy of the electron-ion system in the compound exciton state Eq. (132) as
\[
E_{N,N_1,m}^{(m)} = \langle \Psi_{N,N_1,m}^{(m)} | \hat{H}_{N,N_1} | \Psi_{N,N_1,m}^{(m)} \rangle,
\]
where \(\hat{H}_{N,N_1} \) is the Hamiltonian of the exciton including the interactions of the exciton with the ion and the electron-ion system.
where in the RHS for the kinetic energy term we, similar to Eq. (148), obtain
\[ (\psi_{i_0,j_0;\hat{n}}^{\text{h},\text{g},i_0,j_0;\hat{n}} | \hat{H}_0 | \psi_{i_0,j_0;\hat{n}}^{\text{h},\text{g},i_0,j_0;\hat{n}}) = (\hbar \omega_c - |g_0| \mu_B B \hat{N} / 2, \] (144)
where \( \mu_B \) is the Bohr magneton; here the Zeeman energy is included in \( \hat{h}_0 \) explicitly (then the RHS of Eq. (47) should be changed by the RHS of Eq. (144)).

Point out, for IJB in the RHS of Eq. (143) \( \psi_{i_0,j_0;\hat{n}}^{\text{h},\text{g},i_0,j_0;\hat{n}} \) is changed on \( \psi_{i_0,j_0;\hat{n}}^{\text{h},\text{g},i_0,j_0;\hat{n}} \) and \( \hat{H}_0 \) on \( \hat{H}_N^{\text{IB}} \). Then Eq. (144) is correct after making the former change in its LHS.

Using Eqs. (143, 46), we obtain that the energy of the excited-state Eq. (143) with respect to the energy of the ground-state Eq. (46), \( \Delta E^{(m)}_{\text{exc}}(m) \), or the energy of the compound exciton, is given as
\[ \Delta E^{(m)}_{\text{exc}}(m) = E^{(m),\text{ch}}_{\text{exc}} - E^{(m)}_{\hat{N}}. \] (145)

Eq. (145), after using Eqs. (47, 144), obtains the form
\[ \Delta E^{(m)}_{\text{exc}}(m) = \Delta E^{(m),\text{IB}}_{\text{exc}}(m) + \Delta E^{(m)}_{\text{exc}}(m), \] (146)
where in the RHS the term related with the contributions from Eqs. (17, 144) is equal to zero. In addition, it is taken into account that in the RHS of Eq. (146) the term related with the ion-ion (or UIB-UIB interaction) potential \( V_{ii} \), \( \Delta E^{(m),\text{IB}}_{\text{exc}}(m) = 0 \); indeed, in the RHS of Eq. (143) the term \( \langle \psi_{i_0,j_0;\hat{n}}^{\text{h},\text{g},i_0,j_0;\hat{n}} | V_{ii} | \psi_{i_0,j_0;\hat{n}}^{\text{h},\text{g},i_0,j_0;\hat{n}} \rangle \) coincides with the RHS of Eq. (49).

Point out, for IJB in the RHS of Eq. (145) \( E^{(m),\text{ch}}_{\hat{N}} \) is changed on \( E^{(m)}_{\text{exc}} \) and in the RHS of Eq. (146) \( \Delta E^{(m),\text{IB}}_{\text{exc}}(m) \) is changed on \( \Delta E^{(m)}_{\text{exc}}(m) \). It is taken into account that now in the RHS of Eq. (146) the term related with IJB-IJB interaction potential \( V_{bb} \), \( \Delta E^{(m)}_{\text{exc}}(m) = 0 \).

In the RHS of Eq. (146) the term related with electron-ion potential,
\[ \Delta E^{(m)}_{\text{exc}}(m) = \langle \psi_{i_0,j_0;\hat{n}}^{\text{h},\text{g},i_0,j_0;\hat{n}} | V_{i\ell} | \psi_{i_0,j_0;\hat{n}}^{\text{h},\text{g},i_0,j_0;\hat{n}} \rangle \] (147)
i.e., electron-UIB interaction, obtains (cf. with Eq. (50)) the form
\[ \Delta E^{(m),\text{IB}}_{\text{exc}}(m) = -\frac{1}{m} \sum_{n=-\ell}^{\ell} \sum_{k=-\infty}^{\infty} \int \frac{e^2 \dd r}{|r - R - kL_{\hat{X}}|^2} \times |\psi^{(m)}_{k,j_0}(r)|^2 |\phi^{(i)}_{n,j_0}(R)|^2, \] (148)
where, due to the fast convergence of the sum over \( k \), in the limits of the sum \( N_C \) is substituted by \( \infty \).

Point out, for IJB in Eq. (147) \( \Delta E^{(m),\text{IB}}_{\text{exc}}(m) \) is changed on \( \Delta E^{(m)}_{\text{exc}}(m) \) \( V_{i\ell} \) on \( \psi_{i_0,j_0;\hat{n}}^{\text{h},\text{g},i_0,j_0;\hat{n}} \) is changed on \( \psi_{i_0,j_0;\hat{n}}^{\text{h},\text{g},i_0,j_0;\hat{n}} \), and \( \psi^{(m),\text{ch}}_{\hat{N},\hat{N}} \) on \( \psi^{(m),\text{ch}}_{\hat{N},\hat{N}} \). It is easy to see that
\[ \Delta E^{(m)}_{\text{exc}}(m) = \Delta E^{(m),\text{IB}}_{\text{exc}}(m). \] (149)

Further, in the RHS of Eq. (146) the term related with electron-electron potential,
\[ \Delta E^{(m),\text{IB}}_{\text{exc}}(m) = \langle \psi_{i_0,j_0;\hat{n}}^{\text{h},\text{g},i_0,j_0;\hat{n}} | V_{ee} | \psi_{i_0,j_0;\hat{n}}^{\text{h},\text{g},i_0,j_0;\hat{n}} \rangle \] (150)

obtains (cf. with Eqs. (57)–(63)) the form
\[ \Delta E^{(m),\text{IB}}_{\text{exc}}(m) = \Delta E^{(m),\text{IB}}_{\text{exc}}(m) (i_0,j_0; \hat{n}) + \Delta E^{(m),\text{IB}}_{\text{exc}}(m) (i_0,j_0; \hat{n}), \] (151)
where the direct-like (or the Hartree-like) contribution, cf. with Eq. (62), is given as
\[ \Delta E^{(m),\text{IB}}_{\text{exc}}(m) (i_0,j_0; \hat{n}) = \frac{1}{m} \sum_{n=-\ell}^{\ell} \sum_{k=-\infty}^{\infty} \int \frac{e^2 \dd r}{|r - R - kL_{\hat{X}}|^2} \times |\phi^{(m)}_{k,j_0}(r)|^2 |\phi^{(m),\text{ch}}_{n,j_0}(r')|^2, \] (152)
and the exchange-like (or the Fock-like) contribution, cf. with Eq. (63), is given (\( m \geq 3 \)) as
\[ \Delta E^{(m),\text{IB}}_{\text{exc}}(m) (i_0,j_0; \hat{n}) = \frac{1}{m} \sum_{n=-\ell}^{\ell} \sum_{k=-\infty}^{\infty} \int \frac{e^2 \dd r}{|r - R - kL_{\hat{X}}|^2} \times |\phi^{(m),\text{ch}}_{k,j_0}(r)|^2 |\phi^{(m),\text{ch}}_{k,j_0}(r')|^2. \] (153)
Notice, due to the fast convergence of the sums over \( k \) in the RHS of Eqs. (152)–(153), in the limits of these sums \( N_C \) is substituted by \( \infty \).

Point out, for IJB in Eq. (150) \( \psi^{(m),\text{IB}}_{\hat{N},\hat{N}} \) is changed on \( \psi^{(m),\text{IB}}_{\hat{N},\hat{N}} \) and \( \psi^{(m),\text{IB}}_{\hat{N},\hat{N}} \) on \( \psi^{(m),\text{IB}}_{\hat{N},\hat{N}} \). Then it is easy to see that Eqs. (151)–(153) are obtained again.

Changing in the RHS of Eq. (148) \( \mathbf{r} \rightarrow \mathbf{r}' \) and in the RHS of Eq. (152) \( \mathbf{r} \rightarrow -\mathbf{r}' \) and \( k \rightarrow -k \), we obtain
\[ \Delta E^{(m),\text{IB}}_{\text{exc}}(m) + \Delta E^{(m),\text{IB}}_{\text{exc}}(m) (i_0,j_0; \hat{n}) = \sum_{i=1}^{2} E^{(m)}_{i} (i_0,j_0; \hat{n}), \] (154)
where
\[ E^{(m)}_{i} (i_0,j_0; \hat{n}) = \frac{1}{m} \sum_{n=-\ell}^{\ell} \sum_{k=-\infty}^{\infty} \int \frac{e^2 \dd r}{|r - R - kL_{\hat{X}}|^2} \times |\phi^{(m)}_{k,j_0}(r)|^2 |\phi^{(m)}_{k,j_0}(r')|^2, \] (155)
and

$$E_2^{(m)}(i_0, j_0; \tilde{n}) = -\frac{1}{m} \sum_{n=1}^{\ell} \sum_{k=0}^{\tilde{N}} \sum_{k=-\infty}^{\infty} \int dr \int dr'$$

$$\times \frac{e^2}{\varepsilon_0 |r - r' - kL_{x}^0|} [\|\varphi_{k_z \pm i_0}^{(m)}(r')\|^2 - |\varphi_{k_{z_i} i_0}^{(m)}(r')|^2]$$

$$\times [\|\phi_{n_{y_0}}^{(m)}(r')\|^2 - |\phi_{n_{y_0}}^{(m)}(r')|^2]. \quad (156)$$

It is shown in Appendix A that Eq. (156) is rewritten as

$$E_1^{(m)}(i_0, j_0; \tilde{n}) = \frac{4\varepsilon_0^2}{\varepsilon L_x^0} \int_0^\infty \frac{d\eta}{\eta} e^{-\eta^2/4} f_m(\eta) S_m(\eta) \times \sin^2 \left(\sqrt{\frac{m\pi}{2}} (n_{y_0} + \frac{\tilde{n}}{m}) \eta\right). \quad (157)$$

Remind that $n_{y_0} = n_{y_0}^{(i_0)} - n_{y_0}^{(j_0)}$. Further, in Appendix A it is shown that Eq. (156) obtains the form

$$E_2^{(m)}(i_0, j_0; \tilde{n}) = \frac{4\varepsilon_0^2}{\varepsilon L_x^0} \int_0^\infty \frac{d\eta}{\eta} G_m(\eta) \times \sin^2 \left(\sqrt{\frac{m\pi}{2}} (n_{y_0} + \frac{\tilde{n}}{m}) \eta\right)$$

$$- \sum_{k=1}^\infty \frac{1}{k} e^{-\pi k^2/m} \sin^2 \left(\frac{\pi k \tilde{n}}{m}\right). \quad (158)$$

where it is taken into account that

$$\sin^2 \left(\pi k (n_{y_0} + \frac{\tilde{n}}{m})\right) = \sin^2 \left(\pi k \tilde{n}/m\right); \quad G_m(\eta)$$

is defined in Appendix A. After mutual cancelling of $E_1^{(m)}(i_0, j_0; \tilde{n})$ with the pertinent part of $E_2^{(m)}(i_0, j_0; \tilde{n})$, we obtain from Eqs. (157), (158) that

$$\sum_{i=1}^{2} E_i^{(m)}(i_0, j_0; \tilde{n}) = \frac{e^2}{\varepsilon_0} F_{di}^{(m)}(n_{y_0}^{(i_0)}; \tilde{n}), \quad (159)$$

where

$$F_{di}^{(m)}(n_{y_0}^{(i_0)}; \tilde{n}) = 2 \sqrt{\frac{2}{m\pi}} \left\{ \int_0^\infty \frac{d\eta}{\eta} e^{-\eta^2/2} \times \sin^2 \left(\sqrt{\frac{m\pi}{2}} (n_{y_0} + \frac{\tilde{n}}{m}) \eta\right) - \sum_{k=1}^\infty \frac{1}{k} \right\}$$

$$\times e^{-\pi k^2/m} \sin^2 \left(\frac{\pi k \tilde{n}}{m}\right). \quad (160)$$

In addition, in Appendix A it is shown that from Eq. (158) it follows that

$$\Delta E_{exc, (m)}(i_0, j_0; \tilde{n}) = \frac{e^2}{\varepsilon_0} \left( F_{\chi x}^{(m)}(n_{y_0}; \tilde{n}) + 2F_2(m) \right), \quad (161)$$

where we have

$$F_{\chi x}^{(m)}(n_{y_0}; \tilde{n}) = -\sqrt{\frac{2}{m\pi}} \left\{ \sum_{k=-\infty}^{\infty} e^{-\pi m(k + \tilde{n})/m^2} \times \int_0^\infty \frac{d\eta}{\sqrt{\eta^2 + 2m(k + \tilde{n})/m^2}} - e^{-\pi m(n_0 + \tilde{n})/m^2} \right\} \times \int_0^\infty \frac{d\eta}{\sqrt{\eta^2 + 2m(n_{y_0} + \tilde{n})/m^2}}. \quad (162)$$

Using Eqs. (147), (162), we rewrite the energy of the compound exciton Eq. (146) as follows

$$\Delta E_{io, j_0; \tilde{n}}^{(m)} = \frac{e^2}{\varepsilon_0} \left( F_{di}^{(m)}(n_{y_0}^{(i_0)}; \tilde{n}) + F_{\chi x}^{(m)}(n_{y_0}; \tilde{n}) + 2F_2(m) \right), \quad (163)$$

where $\tilde{n} = \pm 1, \ldots, \pm \ell$, and, remind, $m \geq 3$.

Now we will study contributions to the RHS of Eq. (163). Analytical and numerical treatment shows that the main contribution to the RHS of Eq. (163) is given by $F_{di}^{(m)}(n_{y_0}^{(i_0)}; \tilde{n}) > 0$, where in turn the main contribution (obviously, positive) is related with the first term in the curly brackets of the RHS of Eq. (163). It is seen that this integral contribution grows monotonically with the increase of $|n_{y_0}^{(i_0)} + \tilde{n}/m|$. In particular, for $|n_{y_0}^{(i_0)}| \gg 1$ it is given as $\approx \sqrt{2/(m\pi)} \ln \left[ \sqrt{2m\pi |n_{y_0}^{(i_0)}|} - (\ln(2) - \gamma)/2 \right]$, where $\gamma$ is the Euler constant. Moreover, numerical study shows that both $F_{di}^{(m)}(n_{y_0}^{(i_0)}; \tilde{n})$ and the total value, always positive, of the RHS of Eq. (163) grow monotonically with the increase of $|n_{y_0}^{(i_0)} + \tilde{n}/m|$. Point out, these monotonic increases, of $F_{di}^{(m)}(n_{y_0}^{(i_0)}; \tilde{n})$ and the RHS of Eq. (163), take place starting from the min{|$n_{y_0}^{(i_0)} + \tilde{n}/m|} = 1/m; remind, for $m \geq 3$, |$n_{y_0}^{(i_0)} + \tilde{n}/m| \geq 1/m$. I.e., for $m \geq 3$ the minimal value of $\Delta E_{exc, (m)}^{(i_0, j_0; \tilde{n})}$ (here it defines the activation gap of the compound exciton, $E_{exc}^{(m)}$) is given by the RHS of Eq. (163), for $n_{y_0}^{(i_0)} = 0$ and $\tilde{n} = 1$ (notice, $\tilde{n} = -1$ shows the same result), as follows

$$E_{ac}^{(m)} = \frac{e^2}{\varepsilon_0} \left( F_{di}^{(m)}(0; 1) + F_{\chi x}^{(m)}(0; 1) + 2F_2(m) \right). \quad (164)$$

Using Eq. (160) and Eq. (162), we calculate numerically that $F_{di}^{(3)}(0; 1) \approx 0.10453$, $F_{di}^{(5)}(0; 1) \approx 0.02372$, $F_{\chi x}^{(7)}(0; 1) \approx 0.010725$ and $F_{\chi x}^{(5)}(0; 1) \approx -0.002878$, $F_{\chi x}^{(5)}(0; 1) \approx -0.42 \times 10^{-5}$, $F_{\chi x}^{(7)}(0; 1) \approx -0.63 \times 10^{-8}$. Then substituting these numerical results, along with given above numerical values of $F_2(3)$, $F_2(5)$, $F_2(7)$, in Eq. (164) we obtain that dimensionless activation gap of the compound exciton $\Delta_{ac}^{(m)} = E_{ac}^{(m)}/(e^2/\varepsilon_0)$ is given, for $m = 3, 5, 7$, as

$$\Delta_{ac}^{(3)} \approx 0.101596, \quad \Delta_{ac}^{(5)} \approx 0.025719, \quad \Delta_{ac}^{(7)} \approx 0.010725. \quad (165)$$
It is seen that the main contribution to these $\Delta_{(m)}$ comes from the direct-alike term $F_{(m)}^{(0)}(0;1)$. Point out, this direct-alike contribution should not be understood as strictly the direct (the Hartree term) because some important correlations are already taken into account in the form of many-body electron-ion wave functions, $\Psi_{N,N,m}$ and $\Psi_{e^{i},N,N(m)}^{eh}$, cf. with the discussion below Eq. (63). Notice, from Eqs. (160), (163) for the compound exciton energy larger than $\Delta_{(m)}^{(3)}$, however, that is still the smallest one if not count $\Delta_{(m)}^{(2)}$, we obtain that: i) for $m = 3$ it is 3.72 times larger than $\Delta_{(m)}^{(2)}$ (it corresponds, e.g., to $n = 1$ and $n_{ys}^{(i)} = -1$), ii) for $m = 5$, it is 4.26 times larger than $\Delta_{(m)}^{(2)}$ (it corresponds, e.g., to $n = 2$ and $n_{ys}^{(i)} = 0$), and iii) for $m = 7$, it is 4.11 times larger than $\Delta_{(m)}^{(2)}$ (it corresponds, e.g., to $n = 2$ and $n_{ys}^{(i)} = 0$).

Point out, it is shown above that the excitation energy of the compound exciton for IJB coincides with the excitation energy of the relevant compound exciton for UIB. In particular, for IJB the results Eqs. (155)–(165) are also valid.

C. Energy of the compound spin-exciton

For UIB, using the Hamiltonian Eq. (1), we calculate the total energy of the electron-ion system in the compound spin-exciton (remind, here $m \geq 1$) state Eq. (132) as

$$E_{N,N(m)}^{(i)} = \langle \Psi_{N,N(m)}^{(i)} | \hat{H}_{\mu} | \Psi_{N,N(m)}^{(i)} \rangle,$$

where in the RHS the kinetic energy term we, similar to Eq. (141), obtain

$$\langle \Psi_{N,N(m)}^{(i)} | \hat{H}_{\mu} | \Psi_{N,N(m)}^{(i)} \rangle = \hbar \omega_{c} - |g_{0}| \mu_{B} B \hat{N} / 2 + |g_{0}| \mu_{B} B,$$

here the Zeeman energy is included in $\hat{h}_{0}$ explicitly (then the RHS of Eq. (147) should be changed by the RHS of Eq. (144)).

Point out, for IJB in the RHS of Eq. (166) $\Psi_{N,N(m)}^{(i)}$ is changed on $\hat{N}_{\mu}^{(i)}$, $\hat{N}_{\mu}^{(i)}$, and $\hat{H}_{\mu}^{(i)}$ on $\hat{H}_{\mu}^{(i)}$. Then Eq. (167) is correct after making the former change in its LHS.

Using Eqs. (160), (166), we obtain that the energy of the excited-state Eq. (166) with respect to the energy of the ground-state Eq. (160), i.e., the energy of the compound spin-exciton $\Delta E_{(m)}^{(i)}$ is given as

$$\Delta E_{(m)}^{(i)} = E_{N,N(m)}^{(i)} - E_{N,N(m)}.$$

The energy of the compound spin-exciton, Eq. (168), after using Eqs. (173), (174), obtains the form

$$\Delta E_{(m)}^{(i)} = |g_{0}| \mu_{B} B + \Delta E_{ei,m}^{N,N},$$

where in the RHS the first term is related with contributions from Eqs. (47), (167). In addition, it is taken into account that in the RHS of Eq. (169) the term, related with the ion-ion potential $V_{ii}$, $\Delta E_{ei,m}^{N,N}, = 0$. Indeed, in the RHS of Eq. (166) the term $\langle \Psi_{N,N(m)}^{(i)} | V_{ii} | \Psi_{N,N(m)}^{(i)} \rangle$ coincides with the RHS of Eq. (49).

Point out, for IJB in the RHS of Eq. (168) $E_{N,N(m)}^{(m),eh}$ is changed on $E_{N,N(m)}^{(m),IB}$ and in the RHS of Eq. (169) $\Delta E_{ei,m}^{N,N}$ is changed on $\Delta E_{ei,m}^{N,N}$. It is taken into account that now in the RHS of Eq. (169) the term related with IJB-IJB interaction potential $V_{ee}$, $\Delta E_{ei,m}^{N,N} = 0$.

The treatment shows that in the RHS of Eq. (169) the term related with electron-electron potential,

$$\Delta E_{ei,m}^{N,N} = \Delta E_{ei,m}^{N,N},$$

is equal to the RHS of Eq. (169), i.e.

$$\Delta E_{ei,m}^{N,N} = \Delta E_{ei,m}^{N,N},.$$

Point out, for IJB in Eq. (170) $\Delta E_{ei,m}^{N,N}$ is changed on $\Delta E_{ei,m}^{N,N},$, $V_{ee}$ on $V_{ee}$, $\Psi_{N,N(m)}^{(i),N,N(m)}$ is changed on $\Psi_{N,N(m)}^{(i),N,N(m)}$, and $\Psi_{N,N(m)}^{(m),eh}$ on $\Psi_{N,N(m)}^{(m),IB}$. It is easy to see that

$$\Delta E_{ei,m}^{N,N} = \Delta E_{ei,m}^{N,N}.$$

Further, in the RHS of Eq. (169) the term related with electron-electron potential,

$$\Delta E_{ee,m}^{(i)} = \langle \Psi_{N,N(m)}^{(i)} | V_{ee} | \Psi_{N,N(m)}^{(i)} \rangle - \langle \Psi_{N,N(m)}^{(i)} | V_{ee} | \Psi_{N,N(m)}^{(i)} \rangle,$$

obtains (cf. with Eqs. (150)–(152)) the form

$$\Delta E_{ee,m}^{(i)} = \Delta E_{ee,m}^{(i)} + \Delta E_{ee,m}^{(i)}(i_{0}, j_{0}; \hat{n}),$$

where the direct-alike (or the Hartree-alike) contribution

$$\Delta E_{ee,m}^{(i)}(i_{0}, j_{0}; \hat{n}) = \Delta E_{ee,m}^{(i)}(i_{0}, j_{0}; \hat{n}),$$

i.e., $\Delta E_{ee,m}^{(i)}(i_{0}, j_{0}; \hat{n})$ is given by the RHS of Eq. (152). Now the exchange-alike (or the Fock-alike) contribution, cf. with Eq. (153), is given as

$$\Delta E_{ee,m}^{(i)}(i_{0}, j_{0}; \hat{n}) = \frac{1}{m} \sum_{n = -\ell}^{\ell} \sum_{\ell} \sum_{k = -\ell}^{\ell} \sum_{\ell} \sum_{k = -\ell}^{\ell} \int dx \int dr \int dr' \frac{r^{2}}{\varepsilon |x - r - kLx|} \varphi_{s_{xi}}^{(m)}(r) \varphi_{k_{xi}}^{(m)}(r') \times \varphi_{s_{xi}}^{(m)}(r) \varphi_{k_{xi}}^{(m)}(r'),$$

(176)
Notice, Eq. (176) it follows from the RHS of Eq. (153) after formally omitting the first term in the square brackets. Indeed, for the compound spin-exciton this term vanishes, as it includes both the spin up, $|1\rangle$, and the spin down, $|−1\rangle$, spin wave functions.

Point out, for JLB in Eq. (173) $\Psi_{i0,j0;\hat{n}}^{\text{fs}}$ on $N;N(m)$ is changed on $\Psi_{iN,jN;\hat{n}}^{\text{fs}}$ and $\Psi^{\text{m,fs}}$ on $\Psi^{(m);JN}_{N;\hat{n}}$. Then it is easy to see that Eqs. (174)-(176) are obtained again.

Comparing Eqs. (177)-(179) with Eqs. (148)-(160), we readily obtain that

$$\Delta E^{(m),s}_{i0,j0;\hat{n}} = |g_0|\mu_B B + \frac{\epsilon^2}{\varepsilon\ell_0} F^{(m)}_{\text{xc}}(n_{ys};\hat{n}), \quad (177)$$

In addition, in Appendix A it is shown that from Eqs. (176) it follows (cf. with Eq. (161)) that

$$\Delta E^{(s),m}_{i0,j0;\hat{n}} = \frac{2\epsilon^2}{\varepsilon\ell_0} F_2(m), \quad (178)$$

where $F_2(m)$ is given by Eq. (96).

Using Eqs. (172)-(178), we rewrite the energy of the compound spin-exciton Eq. (159), for $m \geq 1$, as follows

$$\Delta E^{(m),s}_{i0,j0;\hat{n}} = |g_0|\mu_B B + \frac{\epsilon^2}{\varepsilon\ell_0} \left( F^{(m)}_{\text{di}}(n_{ys};\hat{n}) + 2F_2(m) \right), \quad (179)$$

where $\hat{n} = 0, \pm 1, \ldots, \pm \ell$; $F^{(m)}_{\text{di}}(n_{ys};\hat{n})$ is given by Eq. (160) both for $m = 1$ and $m \geq 3$.

First, we obtain the difference between the energy of the compound spin-exciton, Eq. (179), and the energy of the compound exciton, Eq. (163), for the same $n_{ys}$, $p_{ys}$, and $\hat{n}$ (i.e., here $m \geq 3, \hat{n} = \pm 1, \ldots, \pm \ell$), as follows

$$\Delta E^{(m),s}_{i0,j0;\hat{n}} - \Delta E^{(m),s}_{i0,j0;\hat{n}} = \frac{\epsilon^2}{\varepsilon\ell_0} F^{(m)}_{\text{xc}}(n_{ys};\hat{n}), \quad (180)$$

where the RHS is always positive as $|g_0|\mu_B B > 0$ and $F^{(m)}_{\text{xc}}(n_{ys};\hat{n}) < 0$. Moreover, it is seen (for given $m \geq 3, \hat{n} = \pm 1, \ldots, \pm \ell$) that $F^{(m)}_{\text{xc}}(n_{ys};\hat{n})$ is minimal for $\hat{n} = 0$ and $\hat{n} = 1$ (for $\hat{n} = -1$, its value is the same). I.e., from Eqs. (163), (164), (180) we obtain that

$$\min\{\Delta E^{(m),s}_{i0,j0;\hat{n}} - \Delta E^{(m),s}_{i0,j0;\hat{n}}\} = \min\{\Delta E^{(m),s}_{i0,j0;\hat{n}} - F^{(m)}_{\text{xc}}(0;1)\}, \quad (181)$$

where the numerical values of $F^{(m)}_{\text{xc}}(0;1)$, for $m = 3, 5, 7$, are given above. Then it is seen that for $m \geq 5$ in the RHS of Eq. (181) typically (estimations are made for conditions relevant to GaAs-based samples, in particular: $g_0 = -0.44, \varepsilon = 12.5, m^*/m_0 = 0.067$) obtain that only the first term, due to the “bare” Zeeman spin splitting energy, is essential. In addition, even though for $m = 3$ the relative role of the bare Zeeman spin splitting energy in the RHS of Eq. (181) is much smaller than for $m \geq 5$, however, the Zeeman term is still dominant for $m = 3$ as well. Indeed, for $n_{ch} \approx 1.26 \times 10^{11} \text{cm}^{-2}$, $B \approx 15.6T$ we have (here $\epsilon^2/\varepsilon\ell_0 \approx 200K, h\omega_c \approx 300K$) that $|g_0|\mu_B B \approx h\omega_c/68 \approx 4.4K$ is more than seven times greater than $(\epsilon^2/\varepsilon\ell_0)|F^{(3)}_{\text{xc}}(0;1)| \approx 0.6K$.

Now we will study, both at $m \geq 3$ and $m = 1$, the energy of the compound spin-exciton for $\hat{n} = 0$ and $n_{ys}^{\text{bo}} = 0$. Then from Eq. (160) it follows that $F^{(m)}_{\text{di}}(0;0) = 0$, i.e., the result that on physical grounds is well understood, so we obtain from Eq. (179) that

$$\Delta E^{(m),s}_{i0,j0;\hat{n}} = |g_0|\mu_B B + \frac{\epsilon^2}{\varepsilon\ell_0} F_2(m), \quad (182)$$

where the numerical values of $F_2(m)$, for $m = 1, 3, 5, 7$, are given below Eq. (96). In particular, for GaAs-based sample at $m = 1$ regime we can make typical estimations in the RHS of Eq. (182) as: $|g_0|\mu_B B \approx 0.015h\omega_c$ and $2F_2(1)(\epsilon^2/\varepsilon\ell_0) \approx 2F_2(1)h\omega_c \approx 0.032h\omega_c$. I.e., the RHS of Eq. (182) is roughly equal to $3|g_0|\mu_B B$: so the exchange-alike contribution in Eq. (182) strongly enhances, at $m = 1$ (and $n_{ys}^{\text{bo}} = 0; \hat{n} = 0$), the energy of the compound spin-exciton in comparison to the bare Zeeman spin splitting. However, for $m \geq 3$ the exchange-alike term in the RHS of Eq. (182) can be already neglected as it is much smaller than $|g_0|\mu_B B$, for a typical GaAs-based sample: e.g., at $m = 3, \epsilon^2/\varepsilon\ell_0 \approx 200K, h\omega_c \approx 300K$ we have in the RHS of Eq. (182) that $2F_2(3)(\epsilon^2/\varepsilon\ell_0)/(|g_0|\mu_B B) \approx 10^{-3}$.

In addition, it is important to treat the compound spin-exciton energy Eq. (179) for $m = 1$ (here $\hat{n} = 0$) as well for $n_{ys}^{\text{bo}} \neq 0$, we have here that

$$\Delta E^{(1),s}_{i0,j0;\hat{n}} = |g_0|\mu_B B + \frac{\epsilon^2}{\varepsilon\ell_0} \left( F^{(1)}_{\text{di}}(n_{ys};0) + 2F_2(1) \right), \quad (183)$$

where $F^{(1)}_{\text{di}}(n_{ys};0)$ grows monotonically with the increase of $|n_{ys}^{\text{bo}}|$. In particular, for $|n_{ys}^{\text{bo}}| \gg 1$ we have that $F^{(1)}_{\text{di}}(n_{ys};0) \approx \sqrt{2/\pi}\ln(\sqrt{2\pi|n_{ys}^{\text{bo}}|}) - (\ln(2) - \gamma)/2$, as the second term in the RHS of Eq. (160) is equal to zero. Notice, $F_{\text{di}}^{(1)}(n_{ys}^{\text{bo}};0) = F_{\text{di}}^{(1)}(-n_{ys}^{\text{bo}};0)$, i.e., $F_{\text{di}}^{(1)}(n_{ys}^{\text{bo}};0)$ is even function over its argument $n_{ys}^{\text{bo}}$. In addition, notice that Eq. (183) is valid also for $n_{ys}^{\text{bo}} = 0$, where $F_{\text{di}}^{(1)}(0;0) = 0$ and, respectively, Eq. (182) reduces to the form given by Eq. (183), at $m = 1$. Point out, the monotonic increase of $F_{\text{di}}^{(1)}(n_{ys}^{\text{bo}};0)$, and the RHS of Eq. (183), take place starting from $n_{ys}^{\text{bo}} = 0$. As we pointed out in the end of Sec. VI.A, for more details see below discussions, the minimal value of $\Delta E^{(1),s}_{i0,j0;\hat{n}}$ given by Eq. (182) (or Eq. (183), for $n_{ys}^{\text{bo}} = 0$) do not correspond to any relevant kinetic coefficient (e.g., $\sigma_{yy}$) of the steady state (or direct current) magnetotransport. So we need to calculate $\Delta E^{(1),s}_{i0,j0;\hat{n}}$, given by Eq. (183), for $n_{ys}^{\text{bo}} = 1$ (for $n_{ys}^{\text{bo}} = -1$ its value is the same), as

$$E^{(1),s}_{\text{ac}} = |g_0|\mu_B B + \frac{\epsilon^2}{\varepsilon\ell_0} \left( F^{(1)}_{\text{di}}(1;0) + 2F_2(1) \right), \quad (184)$$
where, using Eq. (160), we calculate numerically that $E_{ac}^{(1)}(1; 0) \approx 1.15194$ and, finally, obtain that

$$E_{ac}^{(1), s} = |y_0| \mu_B B + 1.18431 \frac{e^2}{\varepsilon \ell_0}. \quad (185)$$

Notice that many-body contribution in the RHS of Eq. (185), $\approx 1.18431 e^2/\varepsilon \ell_0$ is a bit smaller than relevant result of HFA ($\sqrt{2/\pi} (e^2/\varepsilon \ell_0)$; also cf. with pertinent result of Refs. 157, 158, 160).

Point out that even the energy of the compound spin-exciton Eq. (182), at $m \geq 1$ (and $\hat{n} = 0$, $n_{ys}^{(1)} = 0$), is very small in respect with the relevant activation gaps $E_{ac}^{(3)}$, $E_{ac}^{(5)}$, $E_{ac}^{(7)}$, at $m = 3$, 5, 7 (given by Eqs. (164)-(165)), and $E_{ac}^{(1), s}$, at $m = 1$ (given by Eqs. (184)-(185)), it can be seen that such compound spin-excitons, with $\hat{n} = 0$ and $n_{ys}^{(1)} = 0$, will not contribute (to my best knowledge; pertinent complete treatment is beyond the scope of the present study) to any pertinent magneto-transport coefficient (e.g., the diagonal electrical conductance $\sigma_{yy}$ and, respectively, to the pertinent activation gap. Hence, in the limit of small but nonzero impurity concentrations, I expect that $\sigma_{yy}$ in the Hall bar sample can be related with the electrical resistivity $\rho_{xx}$, Ref. 122, will be thermally activated with the activation energy $E_{ac}^{(m)}/2$, at $m \geq 3$, and $E_{ac}^{(1), s}/2$, at $m = 1$.

Indeed, typically it is assumed (cf. with Ref. 124) that, to obtain the activation gap, the observation of the dissipative conductance $\sigma_{yy}$ ($\rho_{yx}$ or $\rho_{xx}$) should be made in the limit of small but nonzero impurity concentrations. I.e., it is assumed (or implicitly) that mainly elastic scattering of electrons by impurities in the “bulk” of the channel (e.g., the Hall bar channel) contributes to $\sigma_{yy}$; i.e., the contributions related with scattering by edge states are assumed negligible, cf. with Ref. 124 and references cited therein. Then, e.g., we can speculate (by taking into account only elastic scattering between the compound exciton potential, etc.) that $\sigma_{yy}$, calculated in the linear response approximation within MS, should be given (cf., e.g., with Ref. 124), as follows

$$\sigma_{yy} \propto \frac{1}{m} \frac{\ell}{\ell_{\text{H}} \ell_{\text{I}}} \sum_{\hat{n}} \sum_{n_{ys}^{(1)} \geq 1} \int_{-\infty}^{\infty} \frac{\partial f}{\partial E} dE_{\epsilon_i}^{(m)\hat{n}} \left| y_0 (k_{xi}^{(n+\hat{n})}) - y_0 (k_{xi}^{(n-\hat{n})}) \right|^2 < U^2 > q \times |M(q; e_{\xi, \eta}^{(n+\hat{n})}, e_{\eta, \xi}^{(n-\hat{n})})|^2 G_{\xi, \eta, n, n}(q), \quad (186)$$

where $\epsilon_i^{(m)\hat{n}} = \Delta E_{\epsilon_i}^{(m)\hat{n}, \xi, \eta}$, $< U^2 > q$ is the impurity potential (or only its short-range part) correlation function, $U(r)$ is the (single-particle) impurity fluctuation potential within MS (or/and due to a weak randomness of actual ion distribution), $f(E)$ is the Fermi-Dirac function, $G_{\xi, \eta, n, n}(q)$ includes the energy $\delta$-function that expresses the conservation of energy in the process of scattering between two compound exciton states $\hat{n} = \pm \hat{n}_0$ (here it is essential only $\hat{n}_0 = 1$ and some effects of impurity potential (or only its long-range, in comparison with $\ell_0$, part). It is crucial the presence in the RHS of Eq. (186) of the factor $|y_0 (k_{xi}^{(n+\hat{n})}) - y_0 (k_{xi}^{(n-\hat{n})})|^2$, which shows that a finite contribution to $\sigma_{yy}$ can be given only by processes of scattering between such two states of the same energy that involve the real space transition (that leads to usual diffusion alike form for $\sigma_{yy}$) of an electron charge between the quantized positions of $y_0 (k_{x0})$ within MS. In addition, it is important that in Eq. (186) the value of the characteristic matrix element of the transition, $|M(q; k_{xi}^{(n+1)}, k_{xi}^{(n-1)})|^2 \approx \exp(-4\pi/m)$, is not too small. However, the processes of elastic scattering, at $m \geq 1$, that involves the spacial transition (quantum diffusion) only of the electron spin from one spin-exciton state with $\hat{n} = 0$ and $n_{ys}^{(1)} = n_{ys}^{(1)}$ (i.e., $n_{ys}^{(1)} = 0$) to any another spin-exciton state with $\hat{n} = 0$ and $n_{ys}^{(1)} = n_{ys}^{(1)} \neq n_{ys}^{(1)}$ does not involve any real space transfer (or quantum diffusion) of the electron charge and, respectively, does not contribute to $\sigma_{yy}$. More general study, also justifies the above claim: that at $m \geq 3$ the activation gap is given by the compound exciton activation gap $E_{ac}^{(m)}$, Eq. (164) (at $m = 3$, 5, 7, the dimensionless activation gap is given by Eq. (165), and at $m = 1$ the activation gap is given by the compound spin-exciton activation gap $E_{ac}^{(1), s}$, Eqs. (184)-(185).

Point out, it is shown above that the excitation energy of the compound spin-exciton for IJB coincides with the excitation energy of the relevant compound spin-exciton for UIB. In particular, for IJB the results Eqs. (177)-(185) are also valid; for IJB Eq. (180) is also correct. Notice, except Eq. (180), all expressions of the present study are given for the zero temperature, $T = 0$.

VII. QUANTIZED HALL CONDUCTANCE

For UIB, assuming that the Fermi level is located within the finite energy gap between the ground-state and excited-states, the Hall conductance, $\sigma_{xy} = -\sigma_{xy}$, can be calculated, within MS, from the Kubo formula (notice, it readily gives, cf. Ref. 12, that $\sigma_{xy} = 0$) as

$$\sigma_{xy} = \frac{ie^2 \hbar}{L_x L_y} \sum_{k>0} \langle 0 | \hat{v}_x | k \rangle \langle k | \hat{v}_y | 0 \rangle - \langle 0 | \hat{v}_y | k \rangle \langle k | \hat{v}_x | 0 \rangle (E_k - E_0)^2, \quad (187)$$

where $k = 0$ and $k = 1, 2, ...$ correspond to the ground-state and excited states of the Hamiltonian $\hat{H} = \hat{H}_{\hat{N}, \hat{N}} + \sum_{i=1}^{\hat{N}} \hat{H}(r_i); \hat{H}(k) = E_k |k\rangle$. Here the "ideal" many-body Hamiltonian $\hat{H}_{\hat{N}, \hat{N}}$ is given by Eq. 11 and $U(r)$ is a static fluctuation potential (e.g., due to a weak randomness of actual ion distribution). The velocity operators $\hat{v}_\mu$, $\mu = x, y$; hence $v_{ix} = (-i\hbar/m^*) \partial/\partial x_i - \omega_i y_i$, $v_{iy} = (-i\hbar/m^*) \partial/\partial y_i$. To calculate the RHS of Eq. 167, we will use the many-body operator identities, that generalize the single-electron identities (8a), (8b)
then \[ \tilde{v}_x = \frac{\ell_0^2}{\hbar} \sum_{i=1}^{N} \frac{\partial}{\partial y_i} V_{ef}(r_i) - \frac{i}{\hbar \omega_c} [\tilde{v}_y, \hat{H}] \]

where \( V_{ef}(r) = U(r) - \sum_{j=1}^{N} \sum_{k_1=-\infty}^{\infty} e^2/(\varepsilon |r - R_j - k_1 L_2^x x|) \). Notice that the last term in the RHS of \( V_{ef}(r) \) will give the interaction of an electron at \( r \) with the homogeneous ion background. Then from Eq. (188) to assume that \( V_{ef}(r) \equiv const(r) \), i.e., independent of \( r \), than Eq. (188) readily follows from Eq. (5) of Ref. 45.

Further, we will neglect by the fluctuation potential \( U(r) \), until it is not stated otherwise. Then taking the ground-state wave function as given by \( \Psi_{N,S}^{(m),eh} \), Eq. (20), it is natural to assume that for all many-body wave functions \( |k\rangle \) the part related with ions has the same form,

\[ \tilde{v}_x = \left( \begin{array}{c} 
\frac{\ell_0^2}{\hbar} \sum_{i=1}^{N} \frac{\partial}{\partial y_i} V_{ef}(r_i) \\frac{i}{\hbar \omega_c} [\tilde{v}_y, \hat{H}] \end{array} \right) \]

where, due to the properties (they follow from as from Eq. (189) so from direct calculation of these matrix elements) \( (0)\tilde{v}_y|0\rangle = 0, \mu = x, y \), the value of the RHS will not be changed by adding \( k = 0 \) term to the sum. I.e., in the RHS of Eq. (192) we can change \( \sum_{k(>0)} \) to \( \sum_k \). Then Eq. (192) gives, cf. with Ref. 44, that

\[ \sigma_H = \frac{i e^2}{\hbar \omega_c L_2^x L_y} \langle 0 | [\tilde{v}_x, \tilde{v}_y] | 0 \rangle, \]

where the many-body commutator can be further simplified as \( \langle \tilde{v}_x, \tilde{v}_y \rangle = \sum_i \phi_{y_i}^{(m),eh} (r_i) \). Further, applying the single-electron commutator \( [\tilde{v}_x, \tilde{v}_y] = -i\hbar \omega_c/m^* \), we obtain that \( \langle \tilde{v}_x, \tilde{v}_y \rangle = -i\hbar \omega_c N/m^* \). Using the latter exact result in Eq. (193) we obtain that

\[ \sigma_H = \frac{e^2 N}{\omega_c m^* L_2^x L_y} = \frac{e^2}{\omega_c m^*(L_2^x L_y)}, \]

where the last form is obtained by using Eq. (20). Finally, using Eq. (21) in Eq. (194) we have

\[ \sigma_H = \frac{e^2}{\omega_c m^* 2\pi m^2 e_0^2} = \frac{e^2}{2 m \pi \hbar}, \]

i.e., for \( m = 3, 5, 7, \ldots \) the ground-state \( \Psi_{N,S}^{(m),eh} \) corresponds to the fractional quantum Hall effect, \( \nu = 1/m \).

Now, again neglecting by a weak random potential \( U(r) \), we will calculate the Hall conductance of the quantum Hall system pertinent to the ground-state \( \Psi_{N,S}^{(m),eh} \) in a different manner. Here, similar to Refs. 46,47, we assume that a static electric field \( E = E_H \hat{y} \) is applied; it is implicit that an adiabatic process of turning on of this electric field is already over. Then the total many-body Hamiltonian \( \hat{H}_{N,S}^{E} \), it follows from \( \hat{H}_{N,S} \), Eq. (1), after changing of \( \hat{H}_0 \) on \( \hat{H}_0^{E} \). Where in the latter \( \hat{H}_0 \) is changed on \( \hat{H}_0 - eE_Hy \). Then in Eqs. (17), (18): \( \hbar \omega_c(n_\alpha + 1/2) \) is changed on \( \hbar \omega_c(n_\alpha + 1/2) - (eE_H/m^* \omega_c)[\hat{k}_{\alpha x} + eE_H/2\omega_c], y_0(k_{\alpha x}) = \ell_0^2 k_{\alpha x} \) is changed on \( y_0^{E_H}(k_{\alpha x}) = \ell_0^2 k_{\alpha x} + E_H/m^* \omega_c \) and, respectively, \( \psi_{n_\alpha,k_{\alpha x}}^{E_H}(r) \) on \( \psi_{n_\alpha,k_{\alpha x}}^{E_H}(r) \). Then \( \varphi_{k_{\alpha x}}^{(m)}(r) \), Eq. (20), is changed on \( \varphi_{k_{\alpha x}}^{(m);E_H}(r) \) and pertinent change should be done in \( \Psi_{N,S}^{(m),eh} \), Eq. (29), transforming it to nonequilibrium many-body wave function \( \Psi_{N,S}^{(m),eh;E_H} \). Further, the net current, \( I_x \), in the latter state is obtained as

\[ I_x = \frac{e}{L_x^0 \Psi_{N,S}^{(m),eh;E_H}} \sum_{j=1}^{N} \hat{v}_x(j) \Psi_{N,S}^{(m),eh;E_H} \]

\[ \times \varphi_{k_{\alpha x}}^{(m);E_H}(r) \hat{v}_x \varphi_{k_{\alpha x}}^{(m);E_H}(r), \]

where, due to the properties (they follow from as from Eq. (189) so from direct calculation of these matrix elements) \( (0)\tilde{v}_y|0\rangle = 0, \mu = x, y \), the value of the RHS will not be changed by adding \( k = 0 \) term to the sum. I.e., in the RHS of Eq. (192) we can change \( \sum_{k(>0)} \) to \( \sum_k \). Then Eq. (192) gives, cf. with Ref. 44, that

\[ \sigma_H = \frac{i e^2}{\hbar \omega_c L_2^x L_y} \langle 0 | [\tilde{v}_x, \tilde{v}_y] | 0 \rangle, \]

where the many-body commutator can be further simplified as \( \langle \tilde{v}_x, \tilde{v}_y \rangle = \sum_i \phi_{y_i}^{(m),eh} (r_i) \). Further, applying the single-electron commutator \( [\tilde{v}_x, \tilde{v}_y] = -i\hbar \omega_c/m^* \), we obtain that \( \langle \tilde{v}_x, \tilde{v}_y \rangle = -i\hbar \omega_c N/m^* \). Using the latter exact result in Eq. (193) we obtain that

\[ \sigma_H = \frac{e^2 N}{\omega_c m^* L_2^x L_y} = \frac{e^2}{\omega_c m^*(L_2^x L_y)}, \]

where the last form is obtained by using Eq. (20). Finally, using Eq. (21) in Eq. (194) we have

\[ \sigma_H = \frac{e^2}{\omega_c m^* 2\pi m^2 e_0^2} = \frac{e^2}{2 m \pi \hbar}, \]
where the matrix element
\[ \langle \phi_{\mathbf{k}}^{(m)}:EH | \hat{\mathbf{v}}_{x} | \phi_{\mathbf{k}}^{(m)}:EH \rangle = -eE_{H}/m^{*}\omega_{c}. \]
Using the latter, Eq. (190) gives
\[ \frac{I_{x}}{V_{H}} = -\frac{e^{2}N}{L_{x}^{2}L_{y}m^{*}\omega_{c}}, \]
where \[ V_{H} = E_{H} \times L_{y} \] is the Hall voltage. As \[ I_{x}/V_{H} = \sigma_{xy} = -\sigma_{H}, \] from Eq. (197) it follow Eqs. (194)-(195).

Point out, the above treatment of Sec. VII can be readily extended as well on IJB, i.e., the quantum Hall system at ground-state \[ \Psi_{(m),JB} \].

Rather similar to Refs. 343, we can speculate that for a weak disorder if the Fermi level still lies in a gap or mobility gap the Hall conductance should be quantized in agreement with Eq. (195) as for IJB so for UIB.

VIII. CONCLUDING REMARKS

Present study shows (see, in particular, Secs. II C, V C) that the ground-state and the lowest excited-state can correspond to partial crystal-like correlation order, Eq. (15), among \( N \) electrons of MR. As a result, the treatment of 2DES of \( N \) electrons within MR is reduced straightforwardly to the study of 2DES of \( \tilde{N} \) electrons (\( \tilde{N} \to \infty \) and \( \tilde{N}/N \to 0 \)) localized within MS; with the PBC, of the period \( L_{x}^{\Box} \), imposed along \( x \).

In particular, present study shows that proper PBC can be totally relevant to symmetry, periodicity, correlations, and etc. properties of a sought state. So it will not lead to any oversimplifications or artificial “boundary effects”.

In this work, I have presented many-body ground-state wave functions Eq. (20) and Eq. (11) of the quantum Hall systems (at \( \nu = 1/m \), \( m = 3, 5, \ldots \) and \( m = 1 \)) for UIB model and IJB model, respectively. For both these models the charge density of ion background is exactly homogeneous. However, only for IJB model the ion background is totally equivalent to the model of ion background used in Ref. 3. So only the results obtained for IJB can be compared with the results of Laughlin3 (or studies based on model of Ref. 3) directly. For UIB the ground-state energy is much lower than for IJB due to the difference between the terms related with ion-ion interaction in these two models. The “electron” part in the energy (that includes contributions from electron-electron and electron-ion interactions) is the same for UIB and IJB. Above it is shown that, per electron, for IJB the ground-state energy, at \( \nu = 1/3 \), \( 1/5 \) and \( 1 \), is substantially lower than obtained in well known studies. It is important that in the present study (both for IJB and UIB), similar with the model of Ref. 3, the ions are localized within the same 2D-plane as 2DES; see also, e.g., Refs. 14-27,28.

Point out that in UIB model I assume ideally periodic distribution of ions (it follows from the form of the many-body wave functions, e.g., Eqs. (29), (132), and the Hamiltonian Eq. (11)) with homogeneous ion density, within 2D-plane. Notice that partly similar model of impurities that form a regular crystallic lattice (so-called, impurity sublattice) having a much larger period than the host lattice is widely used to study the impurity band, etc. of doped semiconductors, see, e.g., Ref.49. Notice, definition of very similar model (with UIB model) of ion background is given by Mahan50 as “One can think of taking the positive charge of the ions and spreading it uniformly about unit cell of the crystal”. Indeed, if we have one ion of the charge \( |e| \) within unit cell and will spread it uniformly about our square unit cell and then take into account that direct interaction of any ion with itself must be excluded (independently if it “shape”) we arrive to UIB model. On the other hand, if we assume that the ion is spread within rectangular unit cell \( L_{x}^{ap} \times L_{y}^{ap} \) of the same area (\( L_{x}^{\Box} \)^2 but with, e.g., \( L_{x}^{ap} / L_{y}^{ap} \to \infty \) it is easy to see that we arrive to IJB model. Notice, direct interaction of ion with itself tends to zero for spreading within above rectangular unit cell although it is finite for spreading within our square unit cell.

Point out that, due to the quantized according to Eq. (30) contributions from the partial many-electron wave functions Eq. (31), for \( m \geq 3 \) the compound form of the ground-state wave function Eq. (29) leads to the compound structure of each electron already within MS. In particular, this compound structure of the electrons plays important role in the present treatment of the excited-states. Due to PBC, the charge density of the excited compound electron and hole (given within MS as the superposition of \( m \), strongly correlated, quasielectrons and quasiholes, e.g., cf. Eqs. (113)-(140) have periodical images for \( x \) outside MS, with the period \( L_{x}^{\Box} \). Respectively, the same property of the periodicity holds for the charge density of the compound exciton and spin-exciton or the charge density of the \( n \)-th quasiexciton, Eq. (140). Point out, these properties of the quasielectron and the quasihole periodicity are different from the properties of the fractionally charged elementary excitations of the Laughlin models. Notice, that in the present model, at \( m \geq 3 \), it is impossible to create only one quasielectron (quasi-hole) without simultaneous excitation another \( m - 1 \) strongly correlated quasielectrons (quasiholes). Notice, above it is shown that the same properties of periodicity, compound structure of electrons, holes, compound excitons and spin-excitations, etc. also are valid for the model with IJB.

Further, from the exact analytical result, Eq. (140), for the activation gap, at \( m \geq 3 \), pertinent to the compound exciton, I have obtained numerically that the activation gap at \( m = 3, 5, 7 \) is given (in units of \( e^{2}/\varepsilon_{0}a_{0} \), according to Eq. (105), by (i) \( \Delta_{ac}^{(3)} \approx 0.1016 \), (ii) \( \Delta_{ac}^{(5)} \approx 0.0257 \), and (iii) \( \Delta_{ac}^{(7)} \approx 0.0107 \), respectively. Notice, \( \Delta_{ac}^{(5)} \) is very close with typically calculated for the Laughlin liquid pertinent excitation gap13,14,15 \( \Delta_{ac,1/3} \approx 0.10 \div 0.11 \). In addition, obtained here \( \Delta_{ac}^{(5)} \) is also rather close to the
calculated for the Laughlin liquid pertinent activation gap, \( \Delta_{ac,1/5} \approx 0.031 \). At \( m = 1 \), from the exact analytical result, Eq. (153), the activation gap, pertinent to the compound spin-exciton, I have obtained it value, Eq. (155), as \( E_{ac}^{(1),s} \approx |g_0| \mu_B B + 1.1843 e^2/\epsilon \ell_0 \), where the many-body contribution is a bit smaller than the relevant result of HFA, \( \sqrt{\pi}/2 \) \( (e^2/\epsilon \ell_0) \), see also Refs. 57–38,39,40. Notice, for LJB above it is shown that any activation gap coincides with pertinent gap for UIB. Moreover, for given \( m \) all calculated energies for LJB (of the ground-state and of the excited-states) can be obtained by upward shift, on the same value, of the pertinent energies for UIB.

For a detailed comparison of the activation gaps with experiments it is known that a finite thickness of 2DES should be taken into account as well as effects of disorder, see, e.g., 19,48 and references cited therein. In addition, I speculate that many-body effects similar to those studied in 58 (though for more “traditional” \( \nu = 1 \) state than the one presented above) and related with edge states possibly here also can lead to highly asymmetric pinning of the Fermi level within the energy gap, as well as fractional \( \nu \); at least in high quality samples with weak disorder and for very low temperatures, typical for experiments on the fractional quantum Hall effect. Then, similar with 51 actual activation energy can be substantially smaller than, e.g., \( E_{ac}^{(m)/2} \). However, pertinent calculations should involve effects due to channel edges and there are beyond the scope of present study. Other effect that will lead to a decrease of excitation gaps can be related with a finite separation, along \( z \)-direction, due to finite spacer layer of the neutralizing ion background plane from the 2D-plane of 2DES; we, analogous to Ref. 52, did not study this effect here.

Now we will outline effects of the changing in Eqs. (134)–(136) of the period \( L_x \) on the period \( L_x^{ap} \) of arbitrary value. Point out, in Eq. (15) we actually should assume arbitrary period \( L_x^{ap} = \eta_x \times L_x^{ap} \) instead of the period \( L_x^{ap} \); now a unit cell becomes rectangular \( L_x^{ap} \times L_y^{ap} \), where \( L^{ap} = L_x^{ap}/\eta_x \), etc. Then we need to find an optimal value of \( \eta_x \) that we notate as \( \eta_{x}^{min}(m) \), for which the trial wave function of a ground-state at \( \nu = 1/m \) will give the lowest energy. Study shows 36 that \( \eta_{x}^{min}(m) \) are very close to 1, at the least, for \( m = 1, 3, 5, 7 \). The energy very slowly (parabolically) is dependent on actual small deviations, \( |\eta_x - \eta_{x}^{min}(m)| \ll 1 \). Point out, at \( m = 3, 5 \) for \( \eta_x \gg 1 \) and \( \eta_x \ll 1 \) this form of crystal-like correlation order shows that the lowest energy per electron is substantially higher than relevant ground-state energies of Ref. 52. In particular, for \( \eta_x \to \infty \) instead of \( U^{JB}(m; \eta_x = 1) \), given for \( m = 1, 3, 5, 7 \) by Eq. (122), we obtain \( U^{JB}(m; \eta_x) \to \infty = -m^{-1} \sqrt{\pi}/8 \); the latter coincides with the HFA result \( \epsilon_{HF}(\nu = 1/m)/(e^2/\epsilon \ell_0) \). If \( \eta_x \to 0 \) then it is seen that \( U^{JB}(m; \eta_x) \) becomes positive and divergent (faster than \( \eta_x^{-1} \)). This result is due to localization of electron charge along, e.g., few “lines” of length \( L_x \to \infty \), of a typical width (along \( y \) \( \ell_0 \), infinitely separated from each other along \( y \), within LJB. In particular, \( U^{JB}(m; \eta_x \ll 1) \) gives much higher energy for \( m = 1, 3, 5 \) than obtained in Ref. 52. It is seen that \( \eta_x \ll 1 \) correspond to very short period of PBC (and much stronger crystal-like correlation order than for \( \eta_x = 1 \)) while \( \eta_x \to \infty \) correspond to the (practical) absence of both the PBC and the crystal-like correlation order. Notice, we can speculate that, e.g., for Hall bar sample effect of lateral confinement potential can define optimal value of \( \eta_x \) slightly different from \( \eta_{x}^{min}(m) \); however, anyway it will be very close to 1, if not equal. Point out, that some conditions used in well known studies Refs. 10,11 do not allow realization of the ground-state, at \( \nu = 1/3 \), of present type 36, with partial crystal-like correlation order. In particular, in the case assumed as rather favorable in Refs. 10,11 when the number of electrons within the rectangular unit cell, \( n \), of \( \nu = 1/11 \) tends to infinity (then \( n \) will correspond to our \( \tilde{N} \); notice, \( \nu \) and \( \nu = 1/11 \) will correspond to \( L_x^{ap} \) and \( L_y \)), while keeping \( a/b = n/4 \). Indeed, then for the trial wave function with most relevant to Refs. 10,11 partial crystal-like correlation order the energy is higher 36 than even for the charge-density wave or Wigner crystal states 6,8,9,24,26, at \( \nu = 1/3 \). So it is not too big surprise that numerical study of Refs. 12,14 (see also Ref. 52) strongly indicates that for \( n \to \infty \) in their model the ground-state energy tends to the result \( \approx -0.410 \) pertinent to the Laughlin’s trial wave function 12,14; also it manifests a liquid-like ground-state similar to the one of Ref. 52.

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APPENDIX A: TRANSFORMATION OF THE COMPOUND EXCITON ENERGY CONTRIBUTIONS

Here we will transform the RHS of Eqs. (153)–(155), i.e., the contributions to the compound exciton energy, to rather simple analytical forms by carrying out explicitly exact analytical calculations. In particular, obtained analytical expressions are very suitable even for quite simple numerical treatment. First, using Eqs. (22), (24), we rewrite Eq. (155) as

\[
E_{L_x^{ap}}^{(m)}(i_0, j_0; \bar{n}) = \frac{e^2}{2\pi \varepsilon} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-q^2 \ell^2/4} dq_x dq_y \left[ \frac{q^2}{q_x^2 + q_y^2 + \delta^2/\ell_0^2} \right. \\
\times \left. S_m(q_y \ell_0) S^2_m(q_y \ell_0) \left( \sum_{k=-\infty}^{\infty} e^{-i k q_y L_x^q} \right) \right] \\
\times f_m(q_y \ell_0) \left( 1 - e^{i \eta_x L_x^q (n y_0 + \hat{\phi})} \right), \quad (A1)
\]
where the sum over \( n \) is carried out. Further, using Eq. (A1), the sum over \( k \) in the square brackets obtains the form given by the first line of Eq. (33). Then the integral over \( q_y \) is calculated with the help of delta-functions and as a result the factor \( S_m^2(q_x \ell_0) \to S_m^2(\sqrt{2\pi M_x}) \). The latter there is equal to 1 for \( M_x = 0 \) while it is equal to zero for any \( M_x \neq 0 \); see also the paragraph below Eq. (36). Therefore we readily arrive from Eq. (A1) to the form given by Eq. (157).

Further, using Eqs. (72)–(74), we rewrite Eq. (156) as follows

\[
E_{2}^{(m)}(i_0, j_0 ; \bar{n}) = -\frac{e^2}{2\pi \varepsilon} \int_{-\infty}^{\infty} \frac{d\eta}{\eta} G_m^x(\eta) [1 - \cos \left( q_y L_x^2 \eta_j y_{j0} + \bar{n} \right) \right] - \frac{2\pi e^2}{\varepsilon (L_x^2)^2} \times \sum_{n_y} G_m^x(\sqrt{2\pi M_y}) \frac{\xi_i |0\rangle}{(2\pi/L_x^2)^2 M_x^2 + \delta^2/\ell_0^2} \times \left( 1 - e^{-i\pi/2 M_y (n_y^0 y_{j0} + \bar{n})} \right),
\]

where it is taken into account that the factor \( S_m^2(q_x \ell_0) \to S_m^2(\sqrt{2\pi M_x}) \) is equal to 1 for \( M_x = 0 \) while it is zero for any \( M_x \neq 0 \). Now, using that the contribution from the \( M_y = 0 \) term of the sum over \( M_y \) in the RHS of Eq. (A2) is exactly equal to zero (it is easy to see as, for \( M_y = 0 \), both \( G_m^x(0) \) is zero and the last factor, within the round brackets, are exactly equal to zero and the denominator, \( \delta/\ell_0 \), does not equal to zero even though it is very small), we readily obtain Eq. (A3) in the form given by Eq. (158).

For a transformation of the exchange-alike contribution Eq. (159), first we use its RHS Eqs. (72), (74). Then, for \( m \geq 3 \), we obtain that

\[
\Delta E_{cc}^{ex}(m ; i_0, j_0) = -\frac{e^2}{2\pi \varepsilon} \sum_{i = 1}^{N} \int_{-\infty}^{\infty} dq_x \int_{-\infty}^{\infty} dq_y \frac{e^{-\frac{q_x^2 \ell_0^2}{2}}}{\sqrt{q_x^2 + q_y^2 + \delta^2/\ell_0^2} \left( q_x + k_{x0} + k_{xj0} \right) \ell_0} - \frac{e^{-\frac{q_x^2 \ell_0^2}{2}}}{\sqrt{q_x^2 + q_y^2 + \delta^2/\ell_0^2} \left( q_x + k_{x0} - k_{xj0} \right) \ell_0} \times \sum_{k_{x0}} e^{-ik_{x0} L_x^2} \left( k_{xj0} \right) \ell_0 \right) \}
\]

where it is used that \((k_{xj}^{(n)} - k_{xj0}^{(n+\bar{n})}) = (k_{xj0}^{(0)} - k_{xj0}^{(\bar{n})})\) and \((k_{xj}^{(n)} - k_{xj0}^{(n-\bar{n})}) = (k_{xj0}^{(0)} - k_{xj0}^{(\bar{n})}),\) to carry out the sum over \( n \). Further, the sum over \( k \) in Eq. (A4) we will rewrite, cf. with Eq. (39), in the form given by the first line of Eq. (33). Then by using the delta-functions we calculate the integral over \( q_x \) and obtain Eq. (A4) in the following form

\[
\Delta E_{cc}^{ex}(m ; i_0, j_0) = -\frac{e^2}{2\pi \varepsilon} \sum_{i = 1}^{N} \int_{-\infty}^{\infty} dq_x \int_{-\infty}^{\infty} dq_y \frac{e^{-\frac{q_x^2 \ell_0^2}{2}}}{\sqrt{q_x^2 + q_y^2 + \delta^2/\ell_0^2} \left( q_x + k_{xj0}^{(0)} \right) \ell_0} - \frac{e^{-\frac{q_x^2 \ell_0^2}{2}}}{\sqrt{q_x^2 + q_y^2 + \delta^2/\ell_0^2} \left( q_x + k_{xj0}^{(0)} \right) \ell_0} \times \sum_{k_{x0}} e^{-ik_{x0} L_x^2} \left( k_{xj0}^{(0)} - k_{xj0}^{(\bar{n})} \right) \ell_0 \right) \}
\]

Point out that Eqs. (A4), (A5) will give the results obtained from Eq. (176), i.e., for the compound spin-exciton, if formally neglect by the first term in the curly brackets. Now if take into account that only for \( M_x = m(n_{j0}^{(0)} - n_{j0}^{(0)}) + \bar{n} \) the factor

\[
S_m \left( \left( k_{xj0}^{(0)} - k_{xj0}^{(0)} \right) \ell_0 \right) = \sin \left( \pi \left( m(n_{j0}^{(0)} - n_{j0}^{(0)}) + \bar{n} - M_x \right) \right)
\]

in the first term of the curly brackets in the RHS of Eq. (A5), is equal to 1 and for any other \( M_x \) its value is zero, in addition, treating in a similar manner the second term of these curly brackets we readily rewrite Eq. (A5) in the form given by Eqs. (101), (102) for \( m \geq 3 \). Respectively, for the compound spin-exciton we obtain from Eq. (176), cf. with Eq. (101).
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