HOMOCLINIC POINTS FOR AREA-PRESERVING SURFACE DIFFEOMORPHISMS

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Abstract. We show a $C^r$ connecting lemma for area-preserving surface diffeomorphisms and for periodic Hamiltonian on surfaces. We prove that for a generic $C^r$, $r = 1, 2, \ldots, \infty$, area-preserving diffeomorphism on a compact orientable surface, homotopic to identity, every hyperbolic periodic point has a transversal homoclinic point. We also show that for a $C^r$, $r = 1, 2, \ldots, \infty$ generic time periodic Hamiltonian vector field in a compact orientable surface, every hyperbolic periodic trajectory has a transversal homoclinic point. The proof explores the special properties of diffeomorphisms that are generated by Hamiltonian flows.

1. Introduction and statement of main results

Let $M^{2n}$ be a compact $2n$ dimensional symplectic manifold with a symplectic form $\omega$ and let $\mathbf{Diff}_c^r(M)$ be the set of all $C^r$, $r = 1, 2, \ldots, \infty$, diffeomorphisms that preserves the symplectic form $\omega$. For $f \in \mathbf{Diff}_c^r(M)$, a point $p \in M$ is said to be a periodic point of $f$ with period $k$ if $f^k(p) = p$. A periodic point $p$ is said to be hyperbolic if all the eigenvalues of $df^k(p)$ are away from the unit circle. For any hyperbolic periodic point, there stable manifold, where points approaches $p$ under forward iterations of $f^k$, and an unstable manifold, where points approaches $p$ under backward iterations of $f^k$. For symplectic diffeomorphisms, both the stable and unstable manifolds are $n$ dimensional. We write the stable and unstable manifold of $p$ as, respectively, $W^s_f(p)$ and $W^u_f(p)$.

A point $q \in M$ is said to be a homoclinic point to a hyperbolic periodic point $p$ if $q \in (W^s_f(p) \cap W^u_f(p)) \setminus \{p\}$, i.e., homoclinic points are nontrivial intersections of stable and unstable manifolds. A homoclinic point $q$ is said to be transversal if $W^s_f(p)$ intersects $W^u_f(p)$ transversally at $q$.

Transversal homoclinic points are responsible for very complicated and chaotic dynamics. Poincaré discovered the homoclinic phenomenon and its associated chaotic dynamics in his study of the restricted three body problem in celestial mechanics [13]. Poincaré conjectured that transversal homoclinic points occurs generically in Hamiltonian systems. Proving the generic existence of homoclinic points to a hyperbolic periodic trajectory is a classical problem in Hamiltonian dynamics. Our main result in this paper positively answers Poincaré’s question for lower dimensional Hamiltonian systems.

Our first result is on area-preserving diffeomorphisms on compact surfaces that are homotopic to identity. Recall that a subset is said to be residual if it contains a countable intersection of open and dense subset. A property is said to be generic if it holds on a residual set.

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Theorem 1.1. Let $M$ be a compact orientable surface and let $\text{Diff}^{r}_{\omega,0}(M)$ be the set of all $C^r$ diffeomorphisms on $M$ preserving an area form $\omega$ on $M$ and isotropic to identity. Then for any $r = 1, 2, \ldots, \infty$, there is a residual subset $R_1 \subset \text{Diff}^{r}_{\omega,0}(M)$ such that if $f \in R_1$ and $p$ is a hyperbolic periodic point, then

$$W^s(p) \cap W^u(p) \backslash \{p\} \neq \emptyset$$

Our result naturally applies to Hamiltonian systems. Again let $M^{2n}$ be a symplectic manifold with the symplectic form $\omega$. The nondegenerate closed two-form $\omega$ on $M^{2n}$ defines an isomorphism between the tangent bundle and the cotangent bundle of $M$ with $J : T^*M \to TM$, where the isomorphism $J$ is uniquely determined by the following equality: $\omega(\ast, J\alpha) = \alpha(\ast)$ for any one-form $\alpha$.

Let $H : M^{2n} \to \mathbb{R}$ be a $C^{r+1}$ real valued function. $JdH$ defines a $C^r$ vector field on $M$. This vector field is called a Hamiltonian vector field and the function $H$ is called a Hamiltonian function, or Hamiltonian. The Hamiltonian is a constant of motion under the Hamiltonian flow. A periodic trajectory, also called closed trajectory is said to be hyperbolic if all its characteristic multipliers are away from the unit circle, except two. The characteristic multipliers are one in the flow direction and in the normal direction of the Hamiltonian function. The stable manifold and unstable manifold of a hyperbolic closed trajectory both have dimension $n$.

We will consider time dependent Hamiltonian systems, where the Hamiltonian function can be written as $H = H(x, t)$, with $x \in M$ and $t \in \mathbb{R}$. The resulting Hamiltonian vector field is no longer autonomous. We are particularly interested in time periodic Hamiltonian systems where there is a positive real number $T$ such that $H(x, t + T) \equiv H(x, t)$ for all $x \in M$ and $t \in \mathbb{R}$. Time periodic Hamiltonians define Hamiltonian flow on $M \times S^1$. Time periodic Hamiltonian can be reduced to a symplectic diffeomorphisms on $M$ by considering its Poincaré map. Hyperbolic periodic points and their stable and unstable manifold can be defined in the same way as those of symplectic diffeomorphisms.

We can now state our theorem for the Hamiltonian case.

Theorem 1.2. Let $M$ be a compact orientable surface with an area form $\omega$. For any $r = 1, 2, \ldots, \infty$, there is a residual subset $R_2 \subset C^{r+1}(M \times S^1)$ such that if $H \in R_2$ and $\gamma$ is a hyperbolic periodic orbit for the time periodic Hamiltonian vector field $JdH(x, t)$, then

$$W^s(\gamma) \cap W^u(\gamma) \backslash \{\gamma\} \neq \emptyset.$$
Xia [21] provided some further evidence supporting $C^r$ closing lemma for area-preserving surface diffeomorphisms.

Proving the existence of homoclinic points, or connecting stable and unstable manifolds, is also known as a connecting lemma. Takes [17] proved that $C^1$ generically every hyperbolic periodic points has a transversal homoclinic point for symplectic and volume-preserving diffeomorphisms. Takes results were extended by Xia [20], using closing lemma types of techniques of Hayashi. Hayashi established the first connecting lemma for hyperbolic invariant set of general diffeomorphisms (Hayashi [5]). A general version of $C^1$ connecting lemma was obtained by Wen and Xia [18] [19]. We remark that his method for all these $C^1$ results are local perturbation methods and they are definitely restricted to $C^1$ topology (cf. Gutierrez [4]).

The $C^r$ connecting lemma for $r > 1$ is a much more difficult problem. Using the idea of “closing gates”, Robinson [16], following an idea of Newhouse, showed that one can connect the stable and the unstable manifolds of a hyperbolic fixed point on two sphere by a $C^r$ small perturbation, if the stable manifold accumulates on the unstable manifold. The accumulation condition is a generic condition for area-preserving diffeomorphisms. Pixton [12] extended Robinson’s result to hyperbolic periodic points on two sphere. Using a more topological approach, Oliveira [9] showed the $C^r$ generic existence of homoclinic points for area-preserving diffeomorphisms on two torus $T^2$. More recently, Oliveira [10] showed the generic existence of homoclinic points on surfaces of higher genus for certain homotopic classes of area-preserving diffeomorphisms, where the actions on first homology, if complicated enough, simply forces intersections of stable and unstable manifolds. His result does not apply to the important cases, including Hamiltonians, where the action on homology is trivial.

To obtain our results, we explore the special properties of the Hamiltonian flow and Hamiltonian diffeomorphisms. Our result uses the concept of flux, a dual concept to the mean rotation vectors, for area preserving diffeomorphisms that are homotopic to identity. We show that the maps with rational flux has certain special properties. These special properties enable us to show the existence of homoclinic points.

We are motivated by the Arnold conjecture for Hamiltonian diffeomorphisms on compact surfaces. Hamiltonian diffeomorphisms are the ones generated by time periodic Hamiltonian flow. The number of fixed points for these diffeomorphisms are larger than what is predicted by the Lefschitz fixed point theorem, as Arnold conjectured and later proved in this case by Floer [2]. Likewise, we show that typically the stable and unstable manifolds for hyperbolic fixed point do intersect. We remark that there are easy examples of non-Hamiltonian diffeomorphisms with stable manifold and unstable manifold accumulating on each other, but never intersecting.

Combining with generic existence of hyperbolic periodic points for area-preserving surface diffeomorphisms (cf. Xia [21]), we have the following corollary.

**Corollary 1.3.** For $r = 1, 2, \ldots, \infty$, an open and dense set of $C^{r+1}$ time periodic Hamiltonian systems on compact surfaces have positive topological entropy.

Similar statement is true for area-preserving diffeomorphisms in $\text{Diff}_{r,0}^\omega(M)$.
2. Flux and mean rotation numbers

Let $M_g$, $g \geq 1$, be the orientable compact surface of genus $g$ and let $(a_i, b_i)$, $i = 1, 2, \ldots, k$, be the canonical generators of its first homology $H_1(M_g, \mathbb{R})$ and let $(\alpha_i, \beta_j)$, $i = 1, 2, \ldots, k$, be the dual basis for the first De Rham-cohomology $H^1(M_g, \mathbb{R})$. By normalize the area form $\omega$ on $M_g$, we may assume that

$$\int_M \omega = 1.$$ 

We consider the cases where $f$ is homotopic to the identity map. Let $l$ be an oriented closed curve in $M = M_g$, since $f$ is isotopic to identity, there is a oriented disk $D \subset M$ such that the boundary $\partial D = f(l) - l$. We define the flux of $f$ across $l$ to be

$$F_f(l) = \int_D d\mu \mod 1,$$

Where $\mu$ is the area element given by the two form $\omega$. The above quantity is independent of the choice of $D$. If $D'$ is another disk such that $\partial D' = f(l) - l$, then $\partial(D - D') = \partial D - \partial D' = 0$. Since $M$ is two dimensional, the boundaryless disk $D - D'$ is either the whole manifold or the empty set. Therefore $\int_D d\mu = 0 \mod 1$. We remark that one can always choose $D' = -(M \setminus D)$, therefore the flux can only be defined up to mod 1.

In fact, the flux $F_f(l)$ depends only on the homology class of $l$. i.e., if $l'$ is homologous to $l$, then $F_f(l') = F_f(l)$. This is because that there will be is a disk $A \subset M$ such that $\partial A = l - l'$. Let $D$ and $D'$ be the disks such that $\partial D = f(l) - l$ and $\partial D' = f(l') - l'$, then $\partial f(A) = f(\partial A) = f(l) - f(l')$ and $\partial D - \partial D' = \partial f(A) - \partial A$, therefore, $D - D' = f(A) - A$. Consequently, $\int_D d\mu = \int_{D'} d\mu = \int_{f(A)} d\mu - \int_A d\mu = 0$.

The flux $F_f$ defined above can be extended to a linear map on the first homology

$$F_f : H_1(M) \to \mathbb{R} \mod 1,$$

and thus it can be represented by a cohomology vector.

The flux has a nice additive property under the iterations of the map $f$. One easily verifies that $F_{f^k}(l) = kF_f(l)$, mod 1, for all integers $k$ and for all closed curves $l$.

The flux is closely related to the mean rotation vector of $f$. Let $f \in \text{Diff}_{c,0}^\omega(M)$ be a area-preserving map homotopic to identity and let $F_s : M \to M$, $s \in [0, 1]$ be the homotopy: $F_0 = \text{Id}_M$, the identity map on $M$, and $F_1 = f$. Let $F_s$ be a lift of $F_s$ in the universal covering space $\tilde{M}$ of $M$. Let $\alpha$ be a differential one-form on $M$. To simplify the notation, we also use $\alpha$ to denote its pull-back on $\tilde{M}$. Let $\tilde{M}_0$ be a fundamental domain in $\tilde{M}$ and let

$$R(\alpha) = \int_{\tilde{M}_0} \left( \int_{F_{s}(x); s \in [0, 1]} \alpha \right) d\mu.$$ 

The function $R(\alpha)$ is linear on $\alpha$. One can easily show that, when $\alpha$ is exact, $R(\alpha)$ is zero. Therefore $R(\alpha)$ induces a linear map on the first cohomology of $M$, $H^1(M, \mathbb{R})$. This gives a homology vector in $H_1(M, \mathbb{R})$. This homology vector is called the mean rotation vector. A different lift of $F_s$ will give a different mean rotation vector $R(\alpha)$, but the difference is an integer.

We remark that the flux is more intuitive than the mean rotation vector and it can be easily extended to the some cases where the map is not isotopic to identity.
For example, if a closed curve is homologous to its image, then their flux across this closed curve can be defined in the same way.

Let \( v_f = (F_f(a_1), F_f(b_1), F_f(a_2), F_f(b_2), \ldots, F_f(a_g), F_f(b_g)) \) be the vector in \( \mathbb{T}^g \). We call \( v_f \) the flux vector.

Let \( f \in \text{Diff}_w(M) \) be a map with a rational flux vector, where all its components are rational numbers. Then there exists an integer \( k \) such that \( f^k \) has zero flux as its flux vector. We will show that any map can be approximated by a map with rational flux vector. Later in the paper, we will discuss special properties of maps with zero flux vector.

It is easy exercise to show that if a map is isotopic to identity and has flux zero, then its mean rotation vector is zero, modulo \( \mathbb{Z}^n \).

As we will see later, all maps defined by Hamiltonian flows have zero flux.

### 3. Basic Perturbations and Some Generic Properties

In this section, we will do a sequence of initial perturbations so that the diffeomorphisms we consider satisfy certain properties. These properties will later guarantee the existence of homoclinic points for every hyperbolic periodic point.

Our first perturbation is to change the flux vector of the maps that are homotopic to identity. We will show that the maps with rational flux vector form a dense subset in \( \text{Diff}_w(M) \).

**Lemma 3.1.** Let \( f \in \text{Diff}_w^r(M) \) be a area preserving diffeomorphism, homotopic to identity. Then for any neighborhood \( V \) of \( f \) in \( \text{Diff}_w(M) \), there is a map \( g \in V \) such that the flux vector \( v_g \) is rational.

**Proof.** Let \((a_i, b_i)\) be the canonical generators of the first homology \( H_1(M, \mathbb{Z}) \). For any \( i = 1, 2, \ldots, g \), we may assume that \( a_i \) and \( b_i \) are simple closed curves such that these curves don’t intersect each other except \( a_i \) and \( b_i \); \( a_i \) and \( b_i \) intersect at only one point and the intersection is transversal.

Fix \( i \), let \( \delta_{h_i} \) be a small tubular neighborhood of \( b_i \). We can parametrize this tubular neighborhood \( \delta_{h_i} \) by \( \delta_{h_i} : S^1 \times [-\delta, \delta] \to M \) for some small \( \delta > 0 \). In fact, for convenience, we can even assume, without loss of generality, that the parametrization \( \delta_{h_i} \) is area-preserving.

Let \( \beta : [-\delta, \delta] \to \mathbb{R} \) be a \( C^\infty \) function such that \( \beta(t) > 0 \) for all \( -\delta < t < \delta \) and \( \beta(-\delta) = \beta(\delta) = 0 \) and \( \beta \) is \( C^\infty \) flat at \( \pm \delta \). i.e., all the derivatives of \( \beta(t) \) at \( \pm \delta \) are zero.

Let \( h_z : M \to AM \) be a \( C^\infty \) diffeomorphism such that if \( z \notin \delta_{h_i}, h_z(z) = z \) and if \( z \in \delta_{h_i}, h_z = \delta_{a_i} \circ T_z \circ (\delta_{a_i})^{-1} \) where \( T_z(\theta, t) = (\theta + \epsilon \beta(t), t) \) for all \( \theta \in S^1 \) and \( t \in [-\delta, \delta] \). We remark that \( h_z \to \text{Id}_M \) in \( C^\infty \) topology as \( \epsilon \to 0 \).

It is easy to see that the map \( h_z \) with \( a_j \) and \( b_j \) are all zero for \( j \neq i \) and the flux across \( b_i \) is also zero. For \( \epsilon > 0 \) small enough, we have \( F_{h_{\epsilon}}(a_i) \neq 0 \).

Since \( F_{f_{\epsilon h}(l)}(l) = F_f(l) + F_{h_{\epsilon}}(l) \) for any closed curve \( l \), we have that for sufficiently small \( \epsilon > 0 \), \( F_{f_{\epsilon h}}(a_i) = F_f(a_i) \neq 0 \). There are infinitely many choices of small \( \epsilon \) such that \( F_{f_{\epsilon h}}(a_i) \) is rational.

We can do similar perturbations to every closed curves \( a_i \) and \( b_i \) so that the flux across \( b_i \) and \( a_i \) are all rational.

This proves the lemma. \( \square \)

We now state a well-known local perturbation lemma (cf. Newhouse [8], Robinson [15], Takens [17], and Xia [20]).
Let $d$ be a metric on $M^n$ induced from some Riemannian structure and let $B_δ(x)$ denote the set of $y \in M^n$ with $d(x, y) < δ$. We also let $\overline{B}_δ(x)$ denote the closure of $B_δ(x)$.

**Lemma 3.2.** (perturbation lemma) Let $M^n$ be an $n$-dimensional compact manifold. Fix $φ \in \text{Diff}_r(M)$, $r \geq 1$. There exist constants $ε_0 > 0$ and $c > 0$, depending on $φ$, such that for any $x \in M^n$, and any $ψ \in \text{Diff}_r(M)$ such that $∥φ - ψ∥_{C^r} < ε_0$, and any positive numbers $0 < δ \leq ε_0$, $0 < ε \leq ε_0$, the following facts hold.

1. If $d(y, x) < cδ^rε$, then there is a $ψ_1 \in \text{Diff}_r(M)$, $∥ψ_1 - ψ∥_{C^r} < ε$ such that $ψ_1^{-1}(x) = y$, $ψ_1(z) = ψ(z)$ for all $z \notin ψ^{-1}(B_δ(x))$, and $ψ^{-1}(z) = ψ_1^{-1}(z)$ for all $z \notin B_δ(x)$.

The proof of this lemma uses the generating functions, which provide a convenient tool in the study of Hamiltonian systems and symplectic diffeomorphisms.

We remark that the local perturbation provided in the perturbation lemma does not change the flux vector or the mean rotation vector of the map.

Now we use the perturbation lemma to prove the following simple result:

**Lemma 3.3.** Let $φ \in \text{Diff}_r(M)$ and let $p \in M^n$ be a hyperbolic periodic points of period $k$, with respect to $φ$. For any $ε > 0$, any $q \in W^u_p(p)$ and any neighborhood $U$ of $q$, there exist a $φ' \in \text{Diff}_r(M)$, $∥φ - φ'∥_{C^r} < ε$ such that

1. $\text{Support}(φ' - φ') \subset U$ and hence, $p$ is a hyperbolic periodic point of period $k$ for $φ'$,
2. $q \in W^u_p(p)$,
3. $q$ is a recurrent point under $φ'$.

Recall that a point $q \in M^n$ is a recurrent point under $φ'$ if there exists a sequence of positive integers $\{n_i\}_{i=1}^\infty$, $n_i \to \infty$ as $i \to \infty$ such that $(φ')^{n_i}(q) \to q$ as $i \to \infty$.

**Proof.** Let $ε > 0$ be given. For any $q \in W^u_p(p)$ and $U \subset M^n$, a small neighborhood of $q$, choose $δ_1$ small so that $B_{δ_1}(q) \subset U$. Consider the small ball $B^1 = B_{δ_1}(q)$, where $c$ is given by the perturbation lemma and $0 < ε_1 < ε/2$. Since $φ$ preserves the volume and $M^n$ is compact, there exists a positive integer $j_1$ such that $φ^{j_1}(B^1) \cap B_{δ_1}(q) \neq \emptyset$ and $φ^i(B^1) \cap B_{δ_1}(q) = \emptyset$ for all $i = 1, 2, \ldots, j_1 - 1$. This implies that there exists a point $q_1 \in B_{δ_1}(q)$ such that $φ^{j_1}(q_1) \in B_{δ_1}(q)$ and $φ^i(q_1) \notin B_{δ_1}(q)$, for all $i = 1, 2, \ldots, j_1 - 1$. Now we apply the perturbation lemma to obtain $φ_1 = φ \in \text{Diff}_r(M)$ with $∥φ - φ_1∥_{C^r} < ε_1 < ε/2$ and $\text{Support}(φ - φ_1) \subset B_{δ_1}(q)$ such that $φ_1(q) = φ(q_1)$. Thus $(φ_1)^{j_1}(q) \in B_{δ_1}(q)$.

Now, since $(φ(q)_1)^{j_1}(q) \neq q$, we may choose $0 < δ_2 < δ_1/2$ so that $(φ(q)_1)^{j_1}(q) \notin B_{δ_2}(q)$. Let $B^2 = B_{δ_2}(q)$, where $0 < δ_2 < δ_1/2$. We choose $δ_2$ and $c_2$ so small such that for all $φ' \in \text{Diff}_r(M)$, $∥φ' - φ∥_{C^r} < ε_2$, and any point $x \in B^2$, we have $(φ')^{j_1}(x) \in B_{δ_1}(q) \setminus B_{δ_2}(q)$ and $(φ')^{j_1}(x) \notin B_{δ_2}(q)$ for all $i = 1, 2, \ldots, j_1 - 1$.

With the same argument, we see that there is a point $q_2 \in B^2$ and an integer $j_2 > j_1$ such that $(φ_1)^{j_2}(q_2) \in B_{δ_2}(q)$ and $(φ_1)^{j_2}(q_2) \notin B_{δ_2}(q)$, for all $i = 1, 2, \ldots, j_2 - 1$. Again we apply the perturbation lemma to obtain $φ_2 = φ \in \text{Diff}_r(M)$ with $∥φ - φ_2∥_{C^r} < ε_2 < ε/4$ and $\text{Support}(φ - φ_1) \subset B_{δ_2}(q)$ such that $φ_2(q) = φ_1(q)$. Observe that $(φ_2)^{j_1}(q) \in B_{δ_1}(q)$ and $(φ_2)^{j_1}(q) \notin B_{δ_2}(q)$.

Continue the above process, we obtain a sequence of real positive numbers $δ_1$, $δ_2$, $\ldots$, a sequence of integers $0 < j_1 < j_2 < \ldots$, and a sequence of functions $φ_1$, $φ_2$, $\ldots$, $∈ \text{Diff}_r(M)$ such that $(φ_i)^{j_k} \in B_{δ_k}(q)$ for all $k = 1, 2, \ldots, i$.,
Let $\phi' = \lim_{t \to \infty} \phi_t \in \text{Diff}^r_c(M)$, then $\|\phi' - \phi\|_{C^r} < \epsilon$ and $q$ is a recurrent point of $\phi'$.

This proves the lemma. \qed

The above lemma can also be applied to stable branches to obtain backward recurrent point. In fact, the proof of the above lemma yields a stronger result, which we will state next. For every hyperbolic periodic point on a surface, the stable manifold (and unstable manifold) is separated by the periodic point itself into two branches. We have the following lemma.

**Lemma 3.4.** There is a residual subset $R \subset \text{Diff}^r_c(M)$ such that if $f \in R$, and $p$ is a hyperbolic periodic point, then every branch of stable and unstable manifolds of $p$ has a forward or backward recurrent point.

As we will prove in the next sections, rational flux vector and the existence of recurrent points on each branch of stable and unstable manifolds imply the existence of homoclinic points.

Before we proceed, we state another lemma on accumulations of stable and unstable manifolds.

**Lemma 3.5.** Let $f \subset \text{Diff}^r_c(M)$ be an area preserving diffeomorphism and let $p$ be a hyperbolic periodic point. Let $B_1$ and $B_2$ be branches of stable or unstable manifolds of $p$. $B_1$ and $B_2$ may be the same branch. If $B_1 \cap L(B_2) \neq \emptyset$, then $B_1 \subset L(B_2)$. Where $L(B_2)$ is the $\alpha$-limit set of $B_2$ if $B_2$ is the stable branch and $L(B_2)$ is the $\omega$-limit set of $B_2$ if $B_2$ is the unstable branch.

This follows from a theorem of Mather [17]. See also Oliveira [10] for a proof.

Let $p$ be a hyperbolic periodic point of period $k$ for $f \in \text{Diff}^r_c(M)$. We may assume, by iterating $f^k$ twice, that each branch of the stable or unstable manifold is invariant under $f^k$. We take a linearization near $p$ such that $p$ is the origin and locally the $x$-axis is unstable manifold, $y$-axis is the stable manifold. We denote the stable and unstable branches by $B^+_u$ and $B^+_s$.

We first consider the branch of unstable manifold from the positive $x$-axis. Let $q \in B^+_u$ be a recurrent point $B^+_u$. The orbit of $q$ accumulates on $\{q\}$ itself. This implies that the orbit of $q$ accumulates at some point on the stable manifold of $p$. The orbit of $q$ can accumulate the $y$ axis in two ways: from the first quadrant or from the fourth quadrant, or possibly both. We may assume that it is the former case, the orbit of $q$ accumulates from the first quadrant. Let $B^+_y$ be the branch of the stable manifold from the positive $y$-axis. By Lemma 3.5 we have that $B^+_s \subset L(B^+_u)$ and $B^+_u \subset L(B^+_s)$.

By Lemma 3.5 there is a backward recurrent point on $B^+_u$. The orbit of this recurrent point approaches accumulate on itself either from the first quadrant or the third quadrant. If it is the former case, then we have an adjacent pair of stable and unstable branches $B^+_u$ and $B^-_u$, that accumulate on each other. See Figure 1.

We claim that there is always an adjacent pair of stable and unstable branches accumulate on each other. In the above case, if $B^+_s$ accumulates on itself through the third quadrant, then we can consider the unstable branch $B^-_u$ on the negative x-axis and the stable branch $B^-_s$ on negative y-axis. Again each branch has two ways of accumulating on itself. Either there are two adjacent branches accumulating on each other, or we have a cyclic accumulation: $B^+_u$, $B^+_s$, $B^-_u$, $B^-_s$. However, by Lemma 3.5 the above accumulations are transitive,
$B_u^+$ would have to accumulate on $B_s^-$ and hence $B_u^+$ and therefore, $B_u^+$ and $B_s^+$ are accumulating adjacent pair.

4. Maps with zero flux vector

In this section, we assume that $f \in \text{Diff}^r_{\omega, 0}(M)$ is homotopic to identity and $v_f = 0$.

Let $p$ be a hyperbolic periodic point of period $k$ and let $W^s(p)$ and $W^u(p)$ be, respectively, the stable manifold and the unstable manifold of $p$. For simplicity and without loss of generality, we choose a local coordinate around $p$,

$$D = \{(x, y) \in \mathbb{R}^2 \mid |x| \leq 2\lambda^k \eta, |y| \leq 2\lambda^k \eta\}$$

with some $\lambda > 0$ and some small $\eta > 0$ such that $p$ is the origin in $D$ and $f^k$ is linear $D$,

$$f^k(x, y) = (\lambda^k x, \lambda^{-k} y)$$

for some $\lambda > 1$. Let $B_u$ be the branch of the unstable manifold of $p$ containing the positive $x$-axis and Let $B_s$ be the branch of the stable manifold of $p$ containing the positive $y$-axis. We further assume that $B_s$ accumulates in $B_u$ from the first quadrant in $D$ and $B_u$ accumulates in $B_u$ from the first quadrant in $D$, as Figure 1 shows. The main result of this section is the following theorem.

![Figure 1. Accumulation of stable and unstable branch](image)

**Theorem 4.1.** Let $f \in \text{Diff}^r_{\omega, 0}(M)$ be an area preserving diffeomorphism. Assume that $f$ is isotopic to identity and $f$ has zero flux, $v_f = 0$. Let $p$ be a hyperbolic periodic points with $B_s$ and $B_u$ as described above. Then $B_u \cap B_s \{p\} \neq \emptyset$. i.e., there are homoclinic points for the hyperbolic periodic point $p$.

We remark that the assumption on the flux is crucial. For surface $M_g$ with genus $g \geq 2$, one can easily construct examples of area preserving flows with exactly $2g - 2$ hyperbolic fixed points and each branch of the stable and unstable manifolds is dense in the whole manifold and there is neither homoclinic point nor heteroclinic point. See Figure 2.
We will prove the theorem in a sequence of lemmas. For a positive integer \( m \), let \( \delta \) be a positive real number such that \( \lambda^{km}\delta = \eta \). Let \( S \subset D \) be a region defined by

\[
S_\delta = \{(x, y) \in D \mid xy \leq \delta \eta, \ 0 \leq x \leq \eta, \ 0 \leq y \leq \eta\}
\]

We use the natural order for the points on \( B_u \) and \( B_s \). For any \( z \in B_u \), we denote the segment of \( B_u \) from \( p \) to \( z \) by \( B_u[0, z] \). We also write \( B_u = B_u[0, \infty) \) and likewise, \( B_s = B_s(-\infty, 0] \). Let \( u_1 \) be the smallest point (in other words, the first point starting from \( p \)) on \( B_u \) that re-enters the region \( S_\delta \). The point \( u_1 \) is on the boundary of \( S_\delta \). Moreover, it enters either from the line \( \{0 \leq x \leq \delta, \ y = \eta\} \) or from the curve \( \{xy = \eta \delta, \ \delta \leq x \leq \lambda^k \delta\} \). For simplicity, we assume that \( u_1 \) is on the line \( \{y = \eta\} \), the other case can be dealt in a similar way.

For any two points \( z_1, z_2 \in D \), let \( l[z_1, z_2] \) be the oriented straight line segment from \( z_1 \) to \( z_2 \). Obviously \( B_u[0, u_1] * l[u_1, 0] \) is an oriented simple closed curve. Similarly, we let \( z_1 \) be the first point on the stable branch \( B_s \) that enters \( S_\delta \). Joining two line segment \( B_s[0, s_1] \) and \( l[s_1, 0] \) we also obtain an oriented simple closed curve. If our manifold \( M \) is \( S^2 \), we already see that \( B_s(0, s_1) \) and \( B_u(0, u_1) \) have to intersect, for otherwise we have two contractible simple closed curves \( B_u[0, u_1] * l[u_1, 0] \) and \( B_s[0, s_1] * l[s_1, 0] \) cross each other only once, at the origin. This is impossible.

For general surfaces, closed curves may not necessarily be contractible. The situation is much more complicated, this simple argument no longer work.

**Lemma 4.2.** Assume that \( f \in \text{Diff}_{\omega, 0}(M) \) is isotopic to identity and \( f \) has zero flux. Suppose that the hyperbolic periodic point has no homoclinic point. Then there is a sequence of points \( u_1 < u_2 < u_3 < \ldots \), on \( B_u \) such that

(a) \( \pi_y(u_i) = \eta \) and \( 0 < \ldots < \pi_x(u_3) < \pi_x(u_2) < \pi_x(u_1) \leq \delta \), where \( \pi_x \) and \( \pi_y \) are projections into respective coordinates;

(b) The length of the curve \( B_u[0, u_i], \ |B_u[0, u_i]| \to \infty \) as \( i \to \infty \); and

(c) The closed curves \( B_u[0, u_i] * l[u_i, 0], \ i \in \mathbb{N}, \) are all in the same homotopy class.
Proof. To simplify the notations, we assume that $p$ is a fixed point. i.e., $k = 1$. The general case with $k > 1$ is exactly the same.

Let $s = (0, \eta)$ be the point on the stable manifold $B_s$ and, as in the statement of the lemma, $u_1$ be the first intersection of $B_u$ in $S_\delta$. The curve

$$l_1 = B_u(0, u_1) \ast l[u_1, s] \ast l[s, 0]$$

form a simple closed curve. Let $s_1 = f(s) = (0, \lambda^{-1} \eta) \in B_s$ be the image of $s$. The image of $l_1$,

$$f(l_1) = B_u(0, f(u_1)) \ast l[f(u_1), s_1] \ast B_s[s_1, 0]$$

is a closed curve, homotopic to $l_1$. Moreover, since the flux of $f$ is zero, the signed area enclosed by $l_1$ and $f(l_1)$ is zero. This implies that the piece of the unstable manifold $B_u[u_1, f(u_1)]$ intersects the line $l[u_1, s_1]$ at least at two distinct points. There may be, and will be, many points of intersections $z$ such that $B[0, z] \ast l[z, 0]$ is not in the same homotopy class as that of $B[0, u_1] \ast l[u_1, 0]$. But at least two distinct points will have this property, since $l_1$ and $f(l_1)$ are homotopic. The first point $u_1$ certainly has this property. Let $u_2$ be the last point on $B[u_1, f(u_1)]$ such that $u_2$ is on the line $l(u_1, s)$ and $B[0, u_1] \ast l[u_1, 0]$ is homotopic to $B[0, u_1] \ast l[u_1, 0]$. Clearly, $u_2 \neq u_1$. A better way to understand the choice of $u_2$ is from the universal covering space of surface. The point $u_2$ is simply the last intersection point of the proper lifts of $l_1$ and $f(l_1)$. The lifts of $l_1$ are, of course, no longer necessarily closed. See Figure 3.

![Figure 3. The sequence $u_1, u_2, \cdots$](image-url)

We can now consider the closed curve $B_u(0, u_2) \ast l[u_2, s] \ast B_s[s, 0]$ and its image. In the same way, we obtain an intersection point $u_3$. Continue this process, we obtain a sequence, $u_1 < u_2 < u_3 < \ldots$, satisfying the properties of the lemma.

This proves the lemma. □

Let $\{u_i\}, i \in \mathbb{N}$ be the sequence from the above lemma. The sequence of real numbers $\{\pi_x(u_i)\}, i \in \mathbb{N}$ is monotonically decreasing and bounded below by 0, so it
converges. It is not obvious that the sequence \( \{u_i\}_{i \in \mathbb{N}} \) should converge to \( s \) in the stable manifold. However, we will show that it always does. We have the following lemma.

**Lemma 4.3.** Let \( u_1 < u_2 < u_3 < \ldots \), be the sequence of points on the unstable manifold \( B_u \) given by Lemma 4.2. Then \( \pi_x(u_i) \to 0 \) as \( i \to \infty \).

The proof of this lemma is quite technical, we postpone it to the next section. Here we explain some basic ideas involved in the proof. Suppose that there is curve \( l \) such that it enters the region \( S_\delta \) twice at different points, as shown in Figure 4. Further suppose that the curve is invariant in the sense that the segment \( (abcde) \) is mapped to \( (bcdef) \). We can join two points on the curve by a horizontal line to form a closed curve \( (abcdefa) \) as in the figure. Since the curve is invariant, this closed curve has nonzero flux, equal to the area of \( (abfea) \), which is impossible for our map. Suppose the sequence \( u_i, i \in \mathbb{N} \) does not converge to the stable manifold, then the limit is in an invariant set formed by taking the limit of pieces of unstable manifold. Since the unstable manifold accumulates on the stable manifold, this invariant set can get arbitrarily close to the stable manifold. If this invariant set is simply a curve, then above argument can derive a contradiction. However, the limit set of a sequence of curves can be very complicated, we can not directly apply the above argument. But nevertheless, we can only use curves to approximate the invariant set, as long as we can control the area lost by the approximation.

![Figure 4. Nonzero flux](image)

**Proof of Theorem 4.1.** We are ready to prove the main theorem of this section. By Lemma 4.3, there is a sequence of points \( u_i, i \in \mathbb{N} \) on the unstable manifold \( B_u \) such that \( u_i \to (0, \eta) \) and \( B_u[0, u_i] * l[u_i, 0] \) is in the same homotopy class for all \( i \in \mathbb{N} \). Likewise there is a sequence \( s_i, i \in \mathbb{N} \) in the stable manifold \( B_s \) such that \( s_i \to (\eta, 0) \) and \( B_s[s_i, 0] * l[0, s_i] \) are in the same homotopy class for all \( i \in \mathbb{N} \). For any \( \epsilon > 0 \), let

\[
S_\epsilon = \{(x, y) \in D \mid xy \leq \epsilon \eta, \ 0 \leq x \leq \eta, \ 0 \leq y \leq \eta\}
\]

We claim that there is point \( z \) in \( B_u \) such that \( z \in S_\epsilon \) and the closed curve \( B_u[0, z] * l[z, 0] \) is homotopic to the closed curve

\[
B_u[0, u_1] * l[u_1, 0] * B_s[0, s_1] * l[s_1, 0].
\]
This can be shown by the following: Let \( z' \) be a point in \( B_s \) which is close to some \( s_i \) and inside of \( S_c \). \( B_s[0, z'] \) is homotopic to \( B_s[0, s_i] \). There exists an integer \( j \) such that \( f^j(z') \) is between the points \( (0, \eta) \) and \( (0, \lambda^{-1}\eta) \) on the stable branch \( B_s \). By choosing a different \( \eta \), we may just assume that \( f^j(z') = (0, \eta) \). For large \( j \), the point \( u_i \) is close to \( (0, \eta) \), therefore we have that \( f^j(u_i) \in S_c \). Since \( B_s[0, u_i] \) is homotopic to \( B_s[0, s_i] \), so is \( f^{-j}(B_s[0, u_i]) \). This implies that \( B_s[0, f^{-j}(u_i)] \) is homotopic to \( B_s[0, s_i] \). We may choose \( z = f^{-j}(u_i) \). This proves the claim.

In fact, for later convenience, we will choose the point \( z \) differently. The trajectory of \( f^{-j}(u_i) \) enters the set \( S_c \) before \( f^{-j}(u_i) \). Let \( z \) be the first point on \( B_s \) such that \( z \) is in the set \( S_c \) and \( B_s[0, z] \) is homotopic to \( B_s[0, f^{-j}(u_i)] \). Obviously, \( z \) must be either on the line \( \{0 < x < \epsilon \}, y = \eta \) or on the curve \( \{xy = \epsilon \eta, \lambda^{-1}\eta \leq y \leq \eta \} \).

Similarly, there is a point \( w \) in \( B_s \) such that \( w \) is the first point in \( B_s \) to intersect \( S_c \) such that the closed curve \( B_s[0, w] \) is homotopic to the closed curve \( B_s[0, s_i] \). Similarly, \( w \) must be either on the line \( \{0 < y < \epsilon, x = \eta \} \) or on the curve \( \{xy = \epsilon \eta, \lambda^{-1}\eta \leq y \leq \eta \} \).

We have obtained two closed curves \( B_s[0, z] \) and \( B_s[0, w] \). They are not homotopic to each other, but they are homologous to each other, since \( B_s[0, u_i] \) is homotopic to \( B_s[0, s_i] \) and \( B_s[0, s_i] \) and \( B_s[0, u_i] \) are homologous. These two curves cross each other at the origin. This crossing at the origin has an intersection number \( \pm 1 \). The sign in \( \pm 1 \) depends on the orientation we pick for these two curves. There are possibly many other intersection between \( B_s[0, w] \) and \( B_s[0, z] \). However, we want to show that all other intersections contribute to a total of zero intersection number. Since any two homologous curves have a total of zero intersection number, we reach a contradiction and hence the theorem is proved.

Let \( z_1 = z \), apply Lemma 4.2 and Lemma 4.3 to the point \( z_1 \), we obtain a sequence of points \( z_i, i \in \mathbb{N} \) on the line \( y = \eta \). Such that the following is true:

1. the sequence \( z_i, i \in \mathbb{N} \) is on the unstable branch \( B_u \) and \( z_1 < z_2 < z_3 < \cdots \) with the order on the unstable branch.
2. \( \pi_x(z_1) > \pi_x(z_2) > \cdots \) and \( \lim_{x \to \infty} \pi_x(z_i) = 0 \).
3. For any \( i \in \mathbb{N} \), the closed curve \( B_u[0, z_i] \) is homotopic to \( B_s[0, z] \).

Similarly, we obtain a sequence of points \( \{w_i\}_{i \in \mathbb{N}} \) on the stable branch \( B_s \) with \( w_1 = w \).

For any \( i_0 \in \mathbb{N} \), the curve \( B_s[0, z_{i_0}] \) is finite and therefore does not intersect with \( l[0, w] \) for sufficiently large \( i \). For small \( i \), the curve \( B_u[0, z_{i_0}] \) may have intersection with \( l[0, w] \). However, the curve \( l[0, w] * B_s(w_i, w_j) * l[w, 0] \) is a simply closed curve, therefore the total intersection number between \( B_s[0, z_{i_0}] \) and \( l[w, 0] \) is zero for any \( i \). In other words, the total intersection number between \( B_s[0, z] \) and \( l[w, 0] \) is zero. Likewise the total intersection number between \( B_u[0, w] \) and \( l[z, 0] \). This shows that the total intersection number between \( B_s[0, z] \) and \( l[w, 0] \) is \( \pm 1 \), contradicting to the fact that these to curves are homologous. This contradiction show that the stable and unstable branches \( B_s \) and \( B_u \) have to intersect.

This proves the theorem.
5. Proof of Lemma 4.3

In this section, we give a proof of Lemma 4.3. We will prove by contradiction. Suppose on the contrary that \( \lim_{i \to \infty} \pi_x(u_i) = q_x > 0 \). We write \( q = (q_x, \eta) = \lim_{i \to \infty} u_i. \)

Let \( \epsilon \) be a positive real number such that \( \epsilon < q_x/2 \) and let

\[
S_\epsilon = \{(x, y) \in D \mid xy \leq \epsilon \eta, \ 0 \leq x \leq \eta, \ 0 \leq y \leq \eta\}
\]

Since the unstable branch \( B_u \) accumulates on the stable branch \( B_s \) from the first quadrant, there is a point \( v_1 \) on \( B_u \) such that \( v_1 \) is the first intersection of \( B[0, v_1] \) with \( S_\epsilon \). Again, we assume that \( v_1 \) is on the line \( \{0 \leq x \leq \delta, \ y = \eta\} \). The case where \( v_1 \) is on the curve \( \{xy = \eta \epsilon, \ \epsilon \leq x \leq \lambda \epsilon\} \) can be dealt in a similar way.

Similar to the choices of the sequence \( \{u_i\}_{i \in \mathbb{N}} \) in Lemma 4.2, we obtain a sequence of \( \{v_i\}_{i \in \mathbb{N}} \) having the same properties as in Lemma 4.2, except of course that \( B_u[0, v_i] \ast l[v_i, 0] \) is in a different homotopy class as that of \( B_u[0, v_i] \ast l[u_i, 0] \).

For each \( v_i, \ i \in \mathbb{N} \), there is a point \( u'_i \in B_u \) such that

1. \( \pi_y(u'_i) = \eta \) and \( 0 < \pi_x(u'_i) < \pi_x(u_i) \);
2. \( B_u[0, u'_i] \ast l[u'_i, 0] \) is in the same homotopy class as \( B_u[0, u_i] \ast l[u_i, 0] \);
3. the point \( u'_i \) is the last such point on \( B_u[0, v_i] \) to have the above properties.

Again, the choice of \( u'_i \) is easy if one looks from the universal covering space.

Now we have two sequences of points, \( \{u'_i\}_{i \in \mathbb{N}} \) and \( \{v'_i\}_{i \in \mathbb{N}} \). To simplify our notations, we will assume that our original choices of \( u_i \) and \( v_i \) already satisfy the properties for \( u'_i \) and \( v'_i \) we have stated.

We let \( q' = (q'_x, \eta) = \lim_{i \to \infty} v_i \). We have \( 0 \leq q'_x < \pi_x(v_i) \), for \( i \in \mathbb{N} \).

Now we have a sequence of lines \( B_u[u_i, v_i], \ i \in \mathbb{N} \). Every two consecutive lines \( B_u[u_i, v_i] \) and \( B_u[u_{i+1}, v_{i+1}] \), together with the line segments \( l[u_i, u_{i+1}] \) and \( l[v_i, v_{i+1}] \) bound an area. On the universal cover with a fixed lift, the area bounded

\[\text{Figure 5. Sequences } u'_i \text{ and } v'_i.\]
by these four curves do not have self intersections and it doesn’t intersect other such areas, except at the boundary. However, when these areas are projected down to the manifold, there might be multiple covers at some open sets. Since we will use the finiteness of the total area to show certain convergence properties of the sequence of the curves $B_\eta[u_i, v_i]$, We need to changes the curves $B_\eta[u_i, u_{i+1}]$ so that there will be no multiple covers. We will do it in the following way.

For any $i > 1$, let $A_i$ be the strip bounded by $B_\eta[u_i, v_i]$, $B_\eta[u_{i+1}, v_{i+1}]$, $\ell[u_i, u_{i+1}]$ and $\ell[v_i, v_{i+1}]$ and let $A = \cup_{i \in \mathbb{N}} A_i$. We will construct a sequence of new curves $d_i$, $i \in \mathbb{N}$, to replace $B_\eta[u_i, v_i]$. In the universal covering space, the strip $A_i$ and the set $A$ have infinitely many lifts. Fix one of the lift of $A_i$, say, $\tilde{A}_i$. The set $\tilde{A}_i$ may intersect other lifts of itself, or lifts of other strips, $A_j$, $j \in \mathbb{N}$ in the interior. If this does not happen, we will simply let $d_i = B_\eta[u_i, v_i]$ and $d_{i+1} = B_\eta[u_{i+1}, v_{i+1}]$. Otherwise, we will remove the interior of any part of $A_i$ that intersects with any other lift of $A_i$ and any lifts of $A_j$, $j \neq i$.

To understand the structure of the sets removed from $A_i$, the following fact is useful: if any curve $B_\eta[u_i, v_i]$ enters a strip $A_j$, $j \neq i$, it has to enter through the end of the strips, either through the line segment $[u_j, u_{j+1}]$ or the line segment $[v_j, v_{j+1}]$. Moreover, it has to eventually exit through the same segment. This is from our construction that $B_\eta[u_i, v_i]$ contains no proper segment crossing the strips.

On the manifold, the areas we removed therefore are disks with boundaries on the line $\{y = \eta, 0 < x < u_1\}$. We now let $d_i$ and $d_{i+1}$ be the new boundary pieces of $A_i$. The new boundaries $d_i$ and $d_{i+1}$ are homotopic to, respectively, $B_\eta[u_i, v_i]$ and $B_\eta[u_{i+1}, v_{i+1}]$, relative to their end points.

We now have a sequence of curves $d_i$, $i \in \mathbb{N}$ and a sequence of strips $B_i$ on the surface, bounded by $d_i$, $d_{i+1}$, $\ell[u_i, u_{i+1}]$ and $\ell[v_i, v_{i+1}]$. The sequence of strips $B_i$ does not intersect each other except at the boundaries. Since the total area of the surface is bounded, we conclude that the area of $B_i$ approaches zero, as $i \to \infty$. This will be an important fact.

We now consider the image of $d_i$ under the map $f$. Recall that by our assumption, $\lim_{i \to \infty} u_i = q = (q_x, \eta)$ and $\lim_{i \to \infty} v_i = q = (q_x, \eta)$. Reduce $\delta$ if necessary, we observe that the curves $d_i$ has to either intersect the line $y = \lambda \eta$ or the line $y = \lambda^{-1} \eta$, right after the point $u_i$. For definiteness, we may, by choosing a subsequence if necessary, assume that all $d_i$ intersect the line $y = \lambda^{-1} \eta$ right after the point $u_i$. With this choice, by the orientation preserving property, $d_i$ intersects the line $y = \lambda \eta$ right before the point $v_i$.

Let $u_i^1$ be the intersection point of $d_i$ with the line $y = \lambda^{-1} \eta$ right after $u_i$, i.e., $u_i^1$ is the first point on $B_\eta[u_i, v_i]$ that is on $y = \lambda^{-1} \eta$. Clearly, $f(u_i)$ is on the right hand side of $u_i^1$, but left hand side of $u_i^{1-1}$. This implies that

$$\lim_{i \to \infty} u_i^1 = \lim_{i \to \infty} f(u_i) = f(q).$$

This suggests certain invariance properties of the limit set of the curves $B_\eta[u_i, v_i]$ and $d_i$. Indeed, let $z$ be a limit point of the curves $B_\eta[u_i, v_i]$. i.e., there is a sequence of the points $z_i \in B_\eta[u_i, v_i]$ such that $\lim_{i \to \infty} z_i = z$, then $f(z_i)$ is between two curves $B_\eta[u_i, v_i]$ and $B_\eta[u_{i-1}, v_{i-1}]$, as long as $f(z_i)$ stays inside of the strip $A_i$ bounded by $B_\eta[u_i, v_i]$, $B_\eta[u_{i+1}, v_{i+1}]$, $\ell[u_i, u_{i+1}]$ and $\ell[v_i, v_{i+1}]$. Let $z'_i$ be a point on the intersection of the line $\ell[f(z), f(z_i)]$ and the curve $B_\eta[u_i, v_i]$, we have $f(z) = \lim_{i \to \infty} z'_i$, i.e., $f(z)$ is also a limit point of the curves $B_\eta[u_i, v_i]$. 

This shows that the limit set of the sequence of curves $B_u[u_i, v_i]$ is forward and backward invariant except near the ends of the curves. This limit set is connected and, in general, is expected to be extremely complicated. In certain sense, this limit set starts at $q$ and ends at $q'$, the limit of the sequence $v_i$. Near these two end points, the limit set is quite easy to describe. Between the two points $q$ and $f(q)$, this limit set is simply the limit of the curves $B_u[u_i, u_i^1]$. Similarly, between $f^{-1}(q')$ and $q'$, this limit set is the limit of the curves $B_u[v_i^1, v_i]$, where $v_i^1$ is defined in the similar way as $u_i^1$.

For any $\epsilon' > 0$, there is a integer $i_0 > 0$ such that, if $i \geq i_0$, the area of the strip $B_i$ bounded by $d_i, d_{i+1}, l[u_i, u_{i+1}]$ and $l[v_i, v_{i+1}]$ is smaller than $\epsilon'$. Let $E_i$ be the square bounded by $B_u[u_i, u_i^1], B_u[u_i+1, u_i^1+1], l[u_i, u_{i+1}]$ and $l[u_i^1, u_i^1+1]$ and let $E = \bigcup_{i=1}^{\infty} E_i$. Similarly, let $E_i'$ be the square bounded by $B_u[u_i^1, v_i], B_u[v_i^1+1, v_i+1], l[u_i, u_{i+1}]$ and $l[u_i^1, u_i^1+1]$ and let $E' = \bigcup_{i=1}^{\infty} E_i'$. By dropping first few terms of the sequence $u_i$ and $v_i$, we may assume that the area of $E$ and $E'$ are both smaller than $\epsilon'$.

Next, we consider the image of $B_u[u_{i+1}, v_{i+1}]$ under $f$ from the universal covering space. For most part, the image is in the strip $A_i$, except near the end point $v_{i+1}$, where it goes out of the strip $A_i$. The curve $d_{i+1}$ is the shortened version of $B_u[u_{i+1}, v_{i+1}]$, its image are mostly in the strip $B_i$, bounded by $d_{i+1}, d_{i+1}, l[u_i, u_{i+1}]$ and $l[v_i, v_{i+1}]$, except near the cut off points, where the image goes out of the strip $B_i$ near $v_j$, $j \in \mathbb{N}$.

Now, we consider the close curve $C_{i+1} = d_{i+1} * l[u_{i+1}, v_{i+1}]$. The image of the curve, $f(C_{i+1})$ is homologous to $C_{i+1}$, together they bound an oriented disk chain, say $D_{i+1}$. i.e., $\partial D_{i+1} = f(C_{i+1}) - C_{i+1}$. By the zero flux property, this disk chain has total area zero. We will show that this is impossible, hence a contradiction.

The disk chain $D_{i+1}$ can be divided into several parts. The first part is inside the strip $B_i$. Let $D^1_{i+1} = D_{i+1} \cap B_i$, this covers the part of $f(C_{i+1})$ inside the strip $B_i$. For choices of large $i$, we have that

$$|\int_{D^1_{i+1}} d\mu| < |\int_{B_i} d\mu| < \epsilon'. \quad (1)$$

The second part of $D_{i+1}$ covers the part of $f(C_{i+1})$ near the cut off points of $d_{i+1}$. Let $D^2_{i+1} = f(E') \cap D_{i+1}$, where $E'$ is the union of the squares near $v_i$. We also have that

$$|\int_{D^2_{i+1}} d\mu| < |\int_{E'} d\mu| < \epsilon'. \quad (2)$$

Let $D^3_{i+1}$ be the rest of $D_{i+1}$, i.e., $D^3_{i+1} = D_{i+1} \setminus (D^1_{i+1} \cup D^2_{i+1})$. The set $D^3_{i+1}$, plus or minus an area of size $\epsilon'$, contains the square bounded by $B_u[u_{i+1}, u_{i+1}^1], B_u[v_{i+1}, v_{i+1}^1], l[u_{i+1}, l[u_{i+1}^1, f(v_{i+1})]]$ and $l[u_i^1, f(v_{i+1})]$. Therefore, there is a constant $m > 0$, independent of $\epsilon'$ such that if $i$ is large enough,

$$\int_{D^3_{i+1}} d\mu > m > 0$$

This implies that the absolute value of signed area of $D_{i+1}$ is larger than $m - 2\epsilon' > 0$, if $\epsilon'$ is chosen sufficiently small. This contradicts to the fact that every closed curve has zero flux.

This contradiction proves the lemma. □
6. Hamiltonian flows and the proof of the main theorem

First, we give a proof of Theorem 1.1.

Let $\text{Diff}_{c,0}^r(M)$ be the set of area-preserving diffeomorphism on $M$ that are homotopic to identity and let $f \in \text{Diff}_{c,0}^r(M)$. Let $p$ be a hyperbolic periodic point. By Lemma 3.2 there is $f_1 \in \text{Diff}_{c,0}^r(M)$, $C^r$ close to $f$ such that $p_1$, slighted perturbed from $p$, is a periodic point for $f_1$ with the same period and the flux vector for $f_1$ is rational. By Lemma 3.3 there is map $f_3$, $C^r$ close to $f_2$, such that each branch of stable manifold and unstable manifold of $p_2$ has a forward or backward recurrent point. This implies that for any positive integer $k > 0$, the stable and unstable branches accumulate on stable and unstable branches under $f_3^k$, even though the original recurrent point may no longer be recurrent. Moreover, there is an adjacent pair of stable and unstable branches accumulating on each other. Let $k$ be the positive integer such that the flux for $f_3^k$ is zero. By Theorem 4.1, the stable and unstable manifolds of $p_2$ under $f_3^k$ intersects and this intersection can be made transversal by an arbitrary $C^r$ small perturbation to $f_3$. This implies that there is an open set of diffeomorphisms in $\text{Diff}_{c,0}^r(M)$, arbitrarily close to $f$, such that the perturbed periodic point from $p$ has a transversal homoclinic point. Since $C^r$ generically in $\text{Diff}_{c,0}^r(M)$ there are only countably many periodic point, this implies that there is a residual set $R_1 \subset \text{Diff}_{c,0}^r(M)$ such that for every $\phi \in R_1$ and every hyperbolic periodic point for $\phi$, there is a homoclinic point.

This proves Theorem 1.1. \qed

To prove Theorem 1.2 we need some preliminary results on Hamiltonian flow. Let $H : M \times S^1 \to \mathbb{R}$ be a time-periodic Hamiltonian function on $M$. Let $\phi : M \times \mathbb{R} \to M$, be the Hamiltonian flow on $M$, given by the $C^r$ Hamiltonian vector field $JdH$. We write $\phi_t(*) = \phi(*,t)$. For any $t_0 \in S^1$, let $f = \phi_t : M \times \{t_0\} \to M \times \{t_0\}$ be the Poincaré map of the Hamiltonian flow. We will identify $M \times \{t_0\}$ with $M$ in the obvious way and regard $f$ as a map on $M$. Such map $f$ is called Hamiltonian diffeomorphism. Obviously, Hamiltonian diffeomorphisms preserve symplectic forms and therefore, on surfaces, they are area-preserving. In fact we can say more: All Hamiltonian diffeomorphisms have zero flux.

**Lemma 6.1.** Let $f$ be a Hamiltonian diffeomorphism, i.e., $f$ is the time-one map of a 1-periodic Hamiltonian flow on compact surface $M$, then the flux vector for $f$ is zero.

**Proof.** Let $H$ be a 1-periodic Hamiltonian function on $M$ and let $\phi_t$ be its Hamiltonian flow such that $f = \phi_1$. Let $l$ be a simple closed curve on $M$. For any real number $s > 0$, let the set $D \subset M \times \mathbb{R}$ be a flow tube defined by

$$D = \{(x,t) \in M \times \mathbb{R} \mid x = \phi_t(x_0), 0 \leq t \leq s\}$$

By the definition of Hamiltonian flow, we have

$$\int_D \omega - dH \wedge dt = 0$$

Where $\omega$ is the symplectic form on $M$. Let $\pi_M(D)$ be the natural projection of $D$ into $M$. For $s$ small, this projection is in a small neighborhood of the curve $l$, there is, locally, a one-form $\alpha$ such that $d\alpha = \omega$. We remark that this one-form is only
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defined locally. Such one-form does not exist globally. By Stokes’ theorem, we have

\[ 0 = \int_D \omega - dH \wedge dt = \int_{\phi_s(t) \times \{s\} - t \times \{0\}} \alpha - H \wedge dt \]

\[ = \int_{\phi_s(t) \times \{s\} - t \times \{0\}} \alpha = \int_{\phi_s(t) - t} \alpha = \int_{\pi_M(D)} \omega \]

This is true for any simple closed curve \( l \). One can reach \( s = 1 \) in finitely many steps. This shows that the flux across \( l \) is zero.

This proves the lemma. □

The next question is whether we can perturb the Hamiltonians to create recurrent points on the stable and unstable manifolds. This is the same as asking whether the perturbation lemma (Lemma 3.2) is valid for Hamiltonian diffeomorphisms. This is an easy and standard exercise using an flow box along a segment in a giving trajectory. We omit the details.

The proof of Theorem 1.2 now follows from that of Theorem 1.1.

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