Exact Analysis of Non-Linear Electro-Osmotic Flow of Generalized Maxwell Nanofluid: Applications in Concrete Based Nano-Materials

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Abstract

To find the solutions to the equations containing non-linear terms is a very tough task for the researchers. Even it becomes more challenging when someone wants to find the exact solutions. The exact solutions play a vital role which can be used as a benchmark for numerical and empirical solutions. Therefore, the present article aims to investigate the amalgamated effect of viscous dissipation and joule heating on the electro-osmotic flow of generalized Maxwell nanofluid along with heat transfer in a channel. Nanofluid is formed by the uniform dispersion of ultra-fine nano-sized solid particles of clay in concrete which is considered as a base fluid. The classical model of the Maxwell nanofluid is generalized by using the new definition of the Atangana-Baleanu time-fractional derivative. Fourier sine transform and the Laplace transform techniques are used to evaluate the exact expressions for the velocity and temperature distributions.

The impact of various embedded parameters such as fractional parameter, Maxwell fluid parameter, volume fraction parameter, zeta potential parameters, electro-kinetic parameter, Brinkman number, joule heating parameter and Prandtl number on velocity and temperature profiles are drawn and illustrated graphically.

It is interesting to see that by using nano-clay in concrete the rate of heat transfer increases with 25.5%.

Index Terms

Maxwell nanofluid, electro-osmosis, viscous dissipation, Atangana-Baleanu fractional derivative, Fourier sine transform, laplace transform.

I. INTRODUCTION

Cement is one of the most widely used materials in the construction industry. It is used broadly in roads, buildings, dams, and bridges. Its uses extend from the basic applications to pipes and drains. Despite being broadly used, materials of cement-based, have poor mechanical properties and are profoundly penetrable to water and other fierce chemicals, which diminishes their durability. Additionally, the concrete industry is one of the noteworthy wellsprings of $\text{CO}_2$ discharges, which records for 6% of worldwide man-made $\text{CO}_2$ emission yearly. However, the increasing demand for high performance of the structural material and components has prompted the rapid advancement of the new classes of materials. Nowadays, it becomes a challengeable issue for the researchers on how to augment these mechanical properties up to the mark.

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to increase the tensile strength, reduced oxygen permeability and enhanced heat resistance. Clay nanoparticles show an improvement in the mechanical properties of nano-enabled cement. Some interesting properties regarding nano-clay can be seen in [2]. Morsy et al. [3] studied the experimentally hybrid effect of CNTs and nano clay on the physic mechanical properties of cement mortar and also show the compressive strength, phase composition, and microstructure of blended cement. Khan et al. [4], conducted a theoretical attempt on MHD entropy optimized viscous hybrid nanomaterial flow between two parallel plates. Keeping the applications of hybrid nanofluid, Afridi et al. [5], studied heat transfer analysis of Blasius and Sakiadis flows by using two types of hybrid nanofluids namely, Cu–Al2O3–water and Cu–Al2O3–kerosene oil.

Fractional calculus becomes a central focus of researchers from the last 30 years. Fractional calculus is attracting the attention of the researcher’s day by day all over the world [6]. At the start, many fractional models have been presented by researchers and mathematicians like Riemann-Liouville, Caputo, and Caputo-Fabrizio fractional model. But all these fractional models have some limitations and due to this, these fractional models were not able to solve the entire real-world problems. The ideas of the Power-law model for the justification of fractional derivative in solving everyday problems fail when the exponential decay law is observed in a real-world problem. To overcome all these difficulties and also keeping in mind the concept of exponential law, a new concept regarding time-fractional derivatives were introduced by Atangana and Baleanu [7]. The models with fractional-order derivatives are very handy and more generalized as compared to classical models. Fractional derivatives are used in different fields of science including biology, physics, fractals, engineering, and applied mathematics. The concept of AB time-fractional derivative was first used by Sheikh et al. [8] and acquired the solutions for the Casson fluid flow. By making use of the Atangana-Baleanu approach, Koca and Atangana [9] found the solution of the Cattanuto-Hristov model. By making it simpler some recent and important work done by using Atangana-Baleanu and Caputo-Fabrizio time-fractional models has been done by Hristov [10] and Khan et al. [11]. Ali et al. [12] developed the mathematical model for Jeffrey nanofluid flow in a rotating frame by utilizing the approach of the Atangana-Baleanu fractional derivative. They model the governing equations via Tiwari and Das nanofluid model and Boussinesq’s approximation. In their study, they found that the fractional-order parameter α boosts up the fluid motion. A highly complicated class of non-Newtonian fluid which is known as Fractionalized Burger’s fluid is investigated by Alqahtani and Khan [13]. Darcy’s law along with the constitutive equations has been used to develop the system of PDEs that describe the fluid motion on the plate. The solution of the governing equations has been acquired via the Laplace Transform technique (LTT). For the validity of the work, they reduced their solution to the well-known published work. The implementation of fractional derivative in DC motor is dealt Abro et al. [14]. The authors carried out the comparative examination of two different fractional schemes that is Atangana-Baleanu and Caputo-Fabrizio. Furthermore, they solved the system of governing PDEs by using the LTT. They concluded that the outcomes propose that the speed of turn depends on the voltage which definitely controls the precise situation of servomechanism through the two sorts of partial separation. In continuation of the application of fractional derivative Abro et al. [15] discussed the fractionalized Burger fluid on a permeable plate. They developed the governing equations using the corresponding constitutive equation and Darcy’s law. They had shown the acquired outcomes in terms of Gamma and Fox-H functions. The results of this article are very fruitful and meaningful for future research work in fractional calculus. The unsteady flow of an incompressible Maxwell fluid with fractional derivative induced by a sudden moved plate has been studied by Sadiq et al. [16]. Some interesting research work regarding fractional Maxwell fluid can be found in [17], [18].

From the last few years, electro-osmotic flow (EoF) in channels gained much attraction of scientists and researchers because of its practical applications in biomedical lab-on-a-chip apparatus, for example, separations of species, analysis of DNA and involvement of different biochemical reactions. Menni et al. [19], studied numerically the dynamic and thermal behavior of fluid with a constant property and flowing turbulently through a two-dimensional horizontal rectangular channel. Shashikumar et al. [20], investigated the effects of different alloy nanoparticles on the microchannel flow of magneto-nano liquids with partial slip and convective boundary conditions. Some more interesting work on channel flow done by Chamkha [21] & Chamkha et al. [22]. Basically, electro-osmotic flow is a mechanism of transport. EoF is the motion of fluid generated when an external electric field is applied. For the first time, Reuss [23] showed experimentally the EoF by making use of the porous clay. The authors showed that water can be forced to pass through clay by applying an electric voltage. EoF is frequently used nowadays because of its significance that it does require any moving part to control the flow behavior in microchannel without making any noise. Many scientists and researchers did some important and meaningful studies regarding EoF. The theoretical study of electro-osmotic second-grade fluid in a thin channel is done by Misra and Chandra [24]. The study of the EoF of a viscoelastic fluid with the effect of Zeta potential is carried out by Afonso et al. [25]. By using the slit microchannel of power-law fluids the analysis of electro-osmosis is done by Zhao et al. [26]. The MHD flow of Casson fluid in a microchannel along with the effect of electro-osmosis is studied by Reddy et al. [27]. In their study, they also assumed the viscous dissipation, wall slip, and Joule heating effects. Escandón et al. [28] examined the transient EoF of the Maxwell fluid in a slit channel with zeta potential. In this article, they considered only the momentum equation for their study. In the continuation of electro-osmosis applications, Ali et al. [29] developed a non-integer order model
for the analysis of Walter’s-B liquid. In their study, they used relative and corresponding constitutive equations along with Darcy’s law to make the system of governing PDEs. To gain the solution of governing equations fractional approach of AB along with LTT has been implemented. In their study, they showed that fluid motion retards in the response of larger values of the electro-osmosis parameter. Furthermore, Kim et al. [30] presented worldwide and point-wise examinations of exploratory and numerical simulations consequences of electro-osmotically determined flow in a microchannel. Transparent channels have been made by using poly-dimethyl siloxane, which was permeable for various types of fluids electro-osmotically. It can be examined that the Correlations of exploratory and numerical outcomes show great understandings, approving both numerical and test approaches.

A literature survey reveals that no endeavor has ever been made to investigate the exact solutions for the electro-osmotic flow of fractionalized Maxwell nanofluid with the combined effect of viscous dissipation and joule heating using the concept of the modern definition of the fractional derivative with the non-local and non-singular kernel. The derived model is generalized using the concept of the Atangana-Baleanu fractional derivative. The obtained fractional model is solved by the joint applications of the mathematical tools that are Fourier sine transform and the Laplace Transform technique. More precisely, the present work intends to highlight the applications of clay nanoparticles in cement concrete to augment the mechanical properties of concrete like poor cracking resistance, long restoring time, low rigidity, high water retention, low pliability, and numerous other mechanical exhibitions.

II. MATHEMATICAL MODELING

This section concerns the formulation of Cauchy’s momentum equation for the fractional Maxwell model. Assume the EoF of viscoelastic Maxwell nanofluid in the presence of an electric field $E_0$ flowing in a channel between two parallel plates. Nanofluid is formed by spreading clay nanoparticles in concrete, which is assumed as base fluid. Initially, both the plates and fluid are at rest. The effect of viscous dissipation is also assumed in the energy equation. The geometrical description of the flow is shown in figure 1.

According to the above consideration velocity field will be as;

$$\vec{v} = \{u(y, t), 0, 0\}. \quad (1)$$

According to the above velocity field and assumptions the Cauchy’s momentum equation for the electro-osmotic flow of Maxwell nanofluid is given as[28];

$$\rho_{nf} \left[ \frac{\partial u(y, t)}{\partial t} \right] = - \frac{\partial \tau_{xy}}{\partial y} + P_e E_x, \quad (2)$$

where $\rho_{nf}$ shows the density of nanofluid, $\tau_{xy}$ represents shear stress, $P_e$ denotes the net charge density of the electrolyte solution.

Physical initial and boundary conditions are;

$$u(y, 0) = 0, \quad 0 \leq y \leq 2H, \quad \frac{\partial u(y, 0)}{\partial t} = 0, \quad 0 \leq y \leq 2H, \quad u(0, t) = 0, \quad t \geq 0, \quad u(\infty, t) = 0, \quad t \geq 0. \quad (3)$$

The nanofluid expression for $\rho_{nf}$ is [28];

$$\rho_{nf} = (1 - \varphi) \rho_f + \varphi \rho_s. \quad (4)$$

The constitutive equation for the fractional Maxwell model is given by [28];

$$\tau_{xy} = -\xi_0 \frac{\partial u(y, t)}{\partial y} \left( \frac{1}{1 + \lambda_1 D_t^\alpha} \right), \quad (5)$$

here $\lambda_1$ is the material parameter of the Maxwell fluid and the term $P_e$ is defined as [28];

$$P_e = \varepsilon k^2 \xi_1 \left( A e^{\lambda y} + B e^{-\lambda y} \right), \quad (6)$$

where

$$A = \left[ \frac{R_\zeta - e^{-2\bar{k}}}{2 \sinh(2\bar{k})} \right], \quad B = (1 - A), \quad R_\zeta = \frac{\xi_2}{\xi_1}, \quad \bar{k} = kH.$$  

Here, $k^2$ is the Debye-Hückel parameter, $\varepsilon$ denotes dielectric permittivity of the solvent, $R_\zeta$ shows the ratio of zeta potentials and $\bar{k}$ represents the electro-kinetic parameter.

Incorporate equation (5) and (6) in equation (2), we get;

$$\rho_{nf} \frac{\partial u(y, t)}{\partial t} = \xi_0 \left( \frac{1}{1 + \lambda_1 D_t^\alpha} \right) \frac{\partial^2 u(y, t)}{\partial y^2} - E_x \varepsilon k^2 \xi_1 \left( A e^{\lambda y} + B e^{-\lambda y} \right) \quad (7)$$

multiplying both sides by $(1 + \lambda_1 D_t^\alpha)$ we have;

$$\rho_{nf} \left( 1 + \lambda_1 D_t^\alpha \right) \frac{\partial u(y, t)}{\partial t} = \xi_0 \left( \frac{\partial^2 u(y, t)}{\partial y^2} \right) - \left( 1 + \lambda_1 D_t^\alpha \right) \times \left[ E_x \varepsilon k^2 \xi_1 \left( A e^{\lambda y} + B e^{-\lambda y} \right) \right]. \quad (8)$$

### Table 1. The thermal transport properties of the nano-clay and concrete [29].

| Properties | $\rho$ (kg/m$^3$) | $C_p$ (J/kg K) | $K$ (W/m K) | $\beta \times 10^3$ (K$^{-1}$) |
|------------|-------------------|----------------|-------------|------------------|
| Concrete   | 5.61              | 1.160          | 41.086      | 1.57             |
| Clay       | 6320              | 531.8          | 76.5        | 1.80             |
| Nanoparticle |                 |                |             |                  |
after simplification, Cauchy’s momentum equation for Maxwell nanofluid will take the form;

\[
\rho_{nf} \left(1 + \lambda_1 D_y^2 \right) \frac{\partial u (y, t)}{\partial t} = \xi_0 \frac{\partial^2 u (y, t)}{\partial y^2} - E_{x, \xi} k^2 \xi_1 \left(A e^{\xi y} + B e^{-\xi y} \right) \tag{9}
\]

To make the above system dimensionless, the following dimensionless variables are used;

\[
\begin{align*}
    u^* &= \frac{u}{U_{Hs}}, \ y^* &= \frac{y}{H}, \ t^* &= \xi_0 t, \\
    k^* &= kH, \ U_{Hs} &= \frac{-E_{x,\xi} \xi_1}{\xi_0}. \tag{10}
\end{align*}
\]

Here \(U_{Hs} = \frac{-E_{x,\xi} \xi_1}{\xi_0} \) is the Helmholtz-Smoluchowski velocity \([30]\). Using the above dimensionless variables mentioned in Eq (10) and dropping \((*)\) notation, the governing equation defined in eq (9) takes the form;

\[
\left(1 + \lambda_1 D_y^2 \right) \frac{\partial u (y, t)}{\partial t} = a_1 \frac{\partial^2 u (y, t)}{\partial y^2} + g (y), \tag{11}
\]

where the constants produced in the depersonalization process are;

\[
\begin{align*}
    a_0 &= (1 - \varphi) + \varphi \frac{\rho_g}{\rho}, \ a_1 &= a_0^{-1} \quad \text{and} \\
    g (y) &= a_1 k^2 \left(A e^{\xi y} + B e^{-\xi y} \right).
\end{align*}
\]

The dimensionless initial and boundary conditions are;

\[
\begin{align*}
    u (0, t) = 0, \ u (0, t) = 0, \ u (2, t) = 0. \tag{12}
\end{align*}
\]

To acquire the exact solution of Cauchy’s momentum equation, taking Fourier sine transform on equation (11),

\[
\left(1 + \lambda_1 D_y^2 \right) F_s \left[ \frac{\partial u (y, t)}{\partial t} \right] = a_1 F_s \left[ \frac{\partial^2 u (y, t)}{\partial y^2} \right] + F_s [g (y)], \tag{13}
\]

by applying FST \([34]\) along with boundary condition on equation (13), we have;

\[
\left(1 + \lambda_1 D_y^2 \right) \frac{\partial u_s (w, t)}{\partial t} = -w^2 u_s (w, t) + g_s (w). \tag{14}
\]

Now utilizing the Laplace transform \([34]\) of equation (14), we have;

\[
p \bar{u}_s (w, p) + \lambda_1 \left[ p \cdot \frac{N (\alpha)}{1 - \alpha} + \frac{p^\alpha}{\alpha} \bar{u}_s (w, p) \right] = a_1 w^2 \bar{u}_s (w, p) + \frac{g_s (w)}{p}, \tag{15}
\]

here \(N (\alpha)\) represents normalization function such that \(N (0) = N (1) = 1\) and \(P\) is the Laplace transform variable.

Re-arrange equation (15), we get;

\[
\bar{u}_s (w, p) \left[ p^{\alpha+1} \frac{(1-\alpha) + \lambda_1 N (\alpha)}{\alpha} + p + a_1 w^2 + a_1 w^2 p^\alpha \frac{\alpha-\alpha}{\alpha} \right] = \frac{g_s (w)}{p}. \tag{16}
\]

Assume that \(\gamma = \frac{1-\alpha}{\alpha}\) and \(\gamma_r = \frac{1-\alpha + \lambda_1 N (\alpha)}{\alpha}\), equation (16) takes the form,

\[
\bar{u}_s (w, p) \left[ p^{\alpha+1} \gamma_r + p + a_1 w^2 + a_1 w^2 p^\alpha \right] \frac{\gamma_r}{\alpha} + 1 = \frac{\gamma_r}{\alpha} \frac{g_s (w)}{p}. \tag{17}
\]

For determining inverse Laplace transform, we introduce series in equation (18), we have;

\[
\bar{u}_s (w, p) = g_s (w) \sum_{k=0}^{\infty} \left[ \frac{\gamma_r p + a_1 \gamma w^2}{\alpha} \right]^k \frac{p^{-k+1}}{k+1} \times \left( \frac{\gamma p^{\alpha-1} + p^{-1}}{\alpha} \right). \tag{19}
\]

By further simplification of the numerator, Eq (19) takes the form;

\[
\bar{u}_s (w, p) = g_s (w) \sum_{k=0}^{\infty} \sum_{m,l \geq 0} \left( -1 \right)^k \frac{\gamma_r m \gamma w^2 a_1^2 \gamma^l k!}{m! l!} \times \left( \frac{\gamma p^{\alpha-1} + p^{-1}}{\alpha} \right). \tag{20}
\]

In simplified form equation (20) can be written as;

\[
\bar{u}_s (w, p) = g_s (w) \sum_{k=0}^{\infty} \sum_{m,l \geq 0} \left( -1 \right)^k \frac{\gamma_r m \gamma w^2 a_1^2 \gamma^l k!}{m! l!} \times \left[ \bar{u}_1 (w, p) + \bar{u}_2 (w, p) \right], \tag{21}
\]

here \(\bar{u}_1 (w, p) = \frac{\gamma p^{\alpha-1} + p^{-1}}{\alpha} \left( \frac{p^m a_1^2}{(p + a_1 w^2)^{m+1}} \right)\) and \(\bar{u}_2 (w, p) = \frac{p^{\alpha-1} + p^{-1}}{\alpha} \left( \frac{p^m a_1^2}{(p + a_1 w^2)^{m+1}} \right)\).

Applying the inverse operator of the Laplace transform technique of equation (21), we arrived at;

\[
\bar{u}_s (w, t) = g_s (w) \sum_{k=0}^{\infty} \sum_{m,l \geq 0} \left( -1 \right)^k \frac{\gamma_r m \gamma w^2 a_1^2 \gamma^l k!}{m! l!} \times \left[ \bar{u}_1 (w, t) + \bar{u}_2 (w, t) \right]. \tag{22}
\]

where \(\bar{u}_1 (w, t) = \gamma t^{k+1 - \alpha - 1} r a_1^2 E_1 (k \alpha - m + 1) (w^2 a_1 t)\) and \(\bar{u}_2 (w, t) = \gamma t^{k+1 - \alpha - 1} r a_1^2 E_1 (k \alpha - m + 1) (w^2 a_1 t)\).

Now taking inverse Fourier sine transform, we get the closed-form solution;

\[
u (y, t) = \frac{2}{\pi} \int_0^\infty \sin (wy) h (w, t) dw. \tag{23}
\]

where

\[
\begin{align*}
    h (w, t) &= g_s (w) \sum_{k=0}^{\infty} \sum_{m,l \geq 0} \left( -1 \right)^k \frac{\gamma_r m \gamma w^2 a_1^2 \gamma^l k!}{m! l!} \times \left[ \bar{u}_1 (w, t) + \bar{u}_2 (w, t) \right].
\end{align*}
\]
III. ENERGY EQUATION

Closed-form solution of the fractional thermal equation with the amalgamated influence of electro-osmosis and viscous dissipation is acquired in this portion of the article. For temperature distribution the energy equation is defined as;

\[
\frac{\partial T}{\partial t} (y, t) = K_{nf} \frac{\partial^2 T}{\partial y^2} + \sigma_{nf} \left( E_y \right)^2 + \xi_0 \left( \frac{\partial u}{\partial y} \right)^2 \left( \frac{1}{1 + \lambda_1 D_t^\alpha} \right). \tag{24}
\]

where \((\rho c_p)_{nf}\) and \(K_{nf}\) shows the specific heat capacitance and thermal conductivity of nanofluid respectively, while \(\sigma_{nf}\) and \(E_y\) represents the electric conductivity of nanofluid and electric field respectively.

Multiplying both sides by \((1 + \lambda_1 D_t^\alpha)\), equation (24) becomes;

\[
(\rho c_p)_{nf} (1 + \lambda_1 D_t^\alpha) \frac{\partial T}{\partial t} (y, t) = K_{nf} (1 + \lambda_1 D_t^\alpha) \frac{\partial^2 T}{\partial y^2} + \sigma_{nf} \left( E_y \right)^2 + \xi_0 \left( \frac{\partial u}{\partial y} \right)^2. \tag{25}
\]

Physical initial and boundary conditions for temperature profile are given as;

\[
T(y, 0) = T_\infty; \quad y > 0, \quad T(0, t) = T_w, \quad t > 0. \tag{26}
\]

Nanofluid expressions involved in the energy equation for temperature distribution is given as [40].

\[
\begin{align*}
(\rho c_p)_{nf} &= (1 - \varphi) (\rho c_p)_{f} + \varphi (\rho c_p)_{p}, \\
\sigma_{nf} &= \sigma_f \left( 1 + \frac{3 (\sigma - 1) \varphi}{\sigma + 2 - (\sigma - 1) \varphi} \right), \\
k_{nf} &= k_f \left( \frac{k_s + 2k_f - 2\varphi (k_f - k_s)}{k_s + 2k_f + \varphi (k_f - k_s)} \right), \quad \sigma = \frac{\sigma_f}{\sigma_s}.
\end{align*} \tag{27}
\]

The non-dimensional variables are;

\[
u^* = \frac{u^*}{U_{hs}}, \quad y^* = \frac{y}{H}, \quad t^* = \frac{\xi_0}{\rho H^2}, \quad T^* = \frac{T - T_\infty}{T_w - T_\infty}. \tag{28}
\]

Incorporating expressions defined in equation (27) and non-dimensional variables given in equation (28), the energy equation defined in (25) takes the form;

\[
\text{Pr} \left(1 + \lambda_1 D_t^\alpha\right) \frac{\partial T}{\partial t} (y, t) = \alpha_S \left(1 + \lambda_1 D_t^\alpha\right) \frac{\partial^2 T}{\partial y^2} + \sigma_\gamma + j(y, t), \tag{29}
\]

with non-dimensional conditions are;

\[
T(y, 0) = 0; \quad y > 0, \quad T(0, t) = 1; \quad t > 0. \tag{30}
\]

where

\[
\begin{align*}
a_2 &= (1 - \varphi) + \varphi (\rho c_p)_s, \\
a_3 &= \frac{K_s + 2K_f - 2\varphi (K_f - K_s)}{K_s + 2K_f + 2\varphi (K_f - K_s)}, \\
a_4 &= 1 + \frac{3 (\sigma - 1) \varphi}{(\sigma + 2 - (\sigma - 1) \varphi)} \varphi a_3 a_2, \\
a_6 &= \frac{a_2}{a_2}, \quad \sigma = a_2^{-1}, \\
Pr &= \frac{\xi_0 c_p}{K_f}, \quad Br = \frac{a_1 \xi_0 (U_{hs})^2}{(T_w - T_\infty) K_f}, \quad \delta_\gamma = \frac{\sigma_f H^2 a_6 (E_y)^2}{(T_w - T_\infty) K_f}.
\end{align*}
\]

Here \(Br\) shows Brinkman’s number and \(\delta_\gamma\) represent the Joule heating parameter.

For exact expression of the energy equation, first, we apply FST to equation (29) along with boundary conditions, we get;

\[
\text{Pr} \left(1 + \lambda_1 D_t^\alpha\right) \frac{\partial T_s}{\partial t} (w, t) = \alpha_S \left(1 + \lambda_1 D_t^\alpha\right) \left[ -w^2 T_s (w, t) \right] \delta_\gamma + j_s (w, t). \tag{31}
\]

Now apply the Laplace transform to equation (31), we obtain;

\[
\text{Pr} \left(\frac{p T_s}{w, p} + \lambda_1 N (\alpha) p^{\alpha+1} T_s (w, p) \right) = \alpha_S \left(1 + \lambda_1 D_t^\alpha\right) \left[ -w^2 T_s (w, t) \right] \delta_\gamma + j_s (w, p). \tag{32}
\]

in simplified form equation (32) can be expressed as (33) and (34), as shown at the bottom of this page. By taking the inverse Laplace transform, we introduce series in equation (34);

\[
\bar{T}_s (w, p) = \frac{\delta_\gamma}{\text{Pr}} \sum_{k=0}^{\infty} \sum_{m,l \geq 0} (-1)^k \left( \frac{\gamma r^m}{\text{Pr}} + \frac{\alpha \omega^2 r^m}{\text{Pr}} \right) \frac{p^{-k\alpha}}{\left( p + \frac{\alpha \omega^2}{\text{Pr}} \right)^{k+1} m!} \times \left( p^{\alpha+1} \gamma + p^{-1} \right) \tag{35}
\]

Re-arranging equation (35), we get;

\[
\bar{T}_s (w, p) = \frac{\delta_\gamma}{\text{Pr}} \sum_{k=0}^{\infty} \sum_{m,l \geq 0} (-1)^k \left( \frac{\gamma r^m}{\text{Pr}} \right)^m \frac{\left( \frac{\alpha \omega^2 r^m}{\text{Pr}} \right)^{k\alpha} \gamma}{\left( p + \frac{\alpha \omega^2}{\text{Pr}} \right)^{k+1} m!} \times \left( p^{\alpha+1} \gamma + p^{-1} \right) + \frac{j_s (w, p)}{\text{Pr}} \sum_{k=0}^{\infty} \frac{\delta_\gamma}{\text{Pr}} \tag{33}
\]

\[
\bar{T}_s (w, p) = \left[ \frac{p^{\alpha+1} [ (1 - \alpha) + \lambda_1 N (\alpha) ] + \frac{w^2 \gamma r^m}{\text{Pr}} (p^{\alpha+1} (1 - \alpha)) + \frac{\alpha \omega^2 r^m}{\text{Pr}} + \lambda_1 \omega^2 N (\alpha) p^{\alpha+1} }{p^{\alpha+1} (1 - \alpha)} \right] \tag{34}
\]
Utilizing the inverse operator of the Laplace transform to equation (36) can be written as:

\[
\mathcal{T} \left( w, p \right) = \frac{\delta_w}{Pr} \sum_{k=0}^{\infty} \sum_{m,l \geq 0} \frac{(-1)^k \gamma_r^{m+l} \left( \frac{\alpha s w^2 \gamma_r}{Pr} \right)^l}{m!!} \frac{k! \gamma p^{m-ka+a}}{\left( p + \frac{\alpha s w^2}{Pr} \right)^{k+1}} m!!
\]

In simplified form equation (36) can be written as:

\[
\mathcal{T}_{i} \left( w, p \right) = \frac{\delta_w}{Pr} \sum_{k=0}^{\infty} \sum_{m,l \geq 0} \frac{(-1)^k \gamma_r^{m+l} \left( \frac{\alpha s w^2}{Pr} \right)^l}{m!!} \frac{k! \gamma p^{m-ka+a}}{\left( p + \frac{\alpha s w^2}{Pr} \right)^{k+1}}
\]

Utilizing the inverse operator of the Laplace transform to equation (37), we get:

\[
\mathcal{T} \left( w, t \right) = \frac{\delta_w}{Pr} \sum_{k=0}^{\infty} \sum_{m,l \geq 0} \frac{(-1)^k \gamma_r^{m+l} \left( \frac{\alpha s w^2}{Pr} \right)^l}{m!!} \gamma t^{k(a+1)-m+a+1}
\]

Now taking the inverse operator FST to equation (38) we get;

\[
T \left( y, t \right) = T_1 \left( y, t \right) + T_2 \left( y, t \right) + T_3 \left( y, t \right) + T_4 \left( y, t \right), \tag{39}
\]

where

\[
T_1 \left( y, t \right) = \frac{2}{\alpha} \int_0^\infty \frac{\delta_w}{Pr} \sum_{k=0}^{\infty} \sum_{m,l \geq 0} \frac{(-1)^k \gamma_r^{m+l} \left( \frac{\alpha s w^2}{Pr} \right)^l}{m!!} \gamma t^{k(a+1)-m+a+1}
\]

\[
T_2 \left( y, t \right) = \frac{2}{\alpha} \int_0^\infty \frac{\delta_w}{Pr} \sum_{k=0}^{\infty} \sum_{m,l \geq 0} \frac{(-1)^k \gamma_r^{m+l} \left( \frac{\alpha s w^2}{Pr} \right)^l}{m!!} \gamma t^{k(a+1)-m+a+1}
\]

\[
T_3 \left( y, t \right) = \frac{2}{\alpha} \int_0^\infty \frac{\delta_w}{Pr} \sum_{k=0}^{\infty} \sum_{m,l \geq 0} \frac{(-1)^k \gamma_r^{m+l} \left( \frac{\alpha s w^2}{Pr} \right)^l}{m!!} \gamma t^{k(a+1)-m+a+1}
\]

\[
T_4 \left( y, t \right) = \frac{2}{\alpha} \int_0^\infty \frac{\delta_w}{Pr} \sum_{k=0}^{\infty} \sum_{m,l \geq 0} \frac{(-1)^k \gamma_r^{m+l} \left( \frac{\alpha s w^2}{Pr} \right)^l}{m!!} \gamma t^{k(a+1)-m+a+1}
\]

IV. NUSSELT NUMBER

A dimensionless number that shows the ratio between convective and conductive heat transfer, which was mathematically developed by Wilhelm Nusselt who was a German engineer. In practical life, Nusselt number is actually the heat transfer rate found at the surface of the moving fluid and in the non-dimensional form it is addressed as:

\[
Nu = -k_{nf} \frac{\partial \theta}{\partial y} \bigg|_{y=0}. \tag{40}
\]

V. SKIN FRICTION

Skin Friction is a kind of a drag force that is provided by the plate to the fluid moving on it. The skin friction arises due to the viscous forces found in fluids and it varies from laminar to turbulent flow concerned to the motion of the fluid.

Skin friction in non-dimensional form is given as:

\[
S_f = \left( 1 + \lambda_1 \frac{\partial u}{\partial t} \right) \frac{\partial u}{\partial y} \bigg|_{y=0}. \tag{41}
\]
VI. RESULT AND DISCUSSION

This section concerns the physical interpretations of the considered flow model. The classical model of fractionalized Maxwell nanofluid in a channel is transformed into the AB time-fractional model. Appropriate non-dimensional variables have been introduced to make the system of PDEs dimensionless. Nanofluid has been shaped by uniform scattering of clay nanoparticles in the cementitious material. Exact solutions have been analyzed for the velocity and temperature profiles by the joint application of the FST and the LTT. The numerical values of thermo-physical properties of clay nano-particles and concrete which is chosen as base fluid are mentioned in Table 1. In order to discuss the effect of the relative constructive parameters such as fractional parameter $\alpha$, Maxwell fluid parameter $\lambda_1$, zeta potential parameter $R_\xi$, electro-kinetic parameter $k$, Brinkman number $\varepsilon$, Joule heating parameter $\delta_{\gamma}$ on velocity and temperature profile, various graphs are sketched.

The influence of the volume fraction $\varphi$ of clay nanoparticle on the velocity profile can be checked out in figure 2. It is noted from the sketch that nanoparticles decrease the velocity of the fluid because nanoparticles enhance the resistive forces between the liquid layers and accordingly the fluid flow impedes. This result is of vital interest. From this result, it is noticed that suspending clay-nanoparticles in the concrete enhances the cohesive forces which result in the binding strength of concrete.

Fig.3 and Fig.4 show enlargement in velocity profile for the effects of zeta potential ratio $R_\xi$ and electro-kinetic parameter $k$ respectively. Both the parameters are related to the electric double layer (EDL) occurring in the fluid. As we increase the values of $R_\xi$ and $k$ the EDL becomes thin due to which the drag forces decrease and consequently the flow of the concrete accelerates. In the present study, we are discussing cementitious materials, in which lower values of $R_\xi$ and $k$ will

| TABLE 2. Variation in Nusselt number. |  |
|---|---|---|---|---|---|
| $\varphi$ | $\alpha$ | $T$ | $\text{Nu}$ | $\%$ |
| 0.00 | 0.5 | 1 | 0.235 | — |
| 0.02 | 0.5 | 1 | 0.301 | 6.6 |
| 0.03 | 0.5 | 1 | 0.368 | 13.3 |
| 0.04 | 0.5 | 1 | 0.421 | 18.6 |

| TABLE 3. Variation in skin friction. |  |
|---|---|---|---|---|
| $\alpha$ | $\lambda_1$ | $k$ | $R_{\xi}$ | $\varphi$ | $S_I$ |
| 0.2 | 1 | 1 | 0.02 | 0.49 |
| 0.2 | 1 | 1 | 0.02 | 0.68 |
| 0.2 | 1 | 1.5 | 0.02 | 0.33 |
| 0.2 | 1 | 1 | 0.02 | 0.30 |
| 0.2 | 1 | 1 | 0.04 | 0.72 |

FIGURE 2. Effect of Volume Fraction Parameter $\varphi$ on the Velocity Profile.

FIGURE 3. Effect of Zeta Potential Parameter $R_\xi$ on the Velocity Profile.

FIGURE 4. Effect of the Electro-Kinetic Parameter $k$ on the Velocity Profile.
be helpful to keep the viscosity of the fluid high. Higher the viscous forces will lead to higher binding strength.

Fig. 5 shows the influence of the material parameter $\lambda_1$ of Maxwell fluid on the velocity profile. A fall in the velocity profile is observed in the response of higher values $\lambda_1$. Physically, it is genuine in light of the fact that the material parameter of the Maxwell fluid is related to the viscosity of the fluid. As the magnitude of $\lambda_1$ increase the cohesive forces in the fluid will also increase, which will expand the life expectancy of nano-bond.

Fig. 6 is plotted to show variation in the velocity profile of Maxwell nanofluid in the response of fractional parameter $\alpha$. It is interesting to see that the velocity profile decreases with an increase in the value of the fractional parameter.

Figures 7 and 8 are drawn to highlight the behavior of temperature profile under the different dimensionless parameters. The impact of $Br$ the temperature profile is shown in figure 7. Augmentation in the temperature profile of Maxwell fluid can be observed for higher values of $Br$. As Brinkman number has an inverse relation to the thermal conduction of the fluid, so larger the value of $Br$, slower the conduction of heat generated by viscous dissipation and overall increase is observed in the profile of the temperature.

Fig 8 shows the impact of the joule heating parameter $\delta_y$ on the temperature profile. From figure 7, it can be noticed that the temperature profile enhances with an increase in the value of $\delta_y$. This is due to the fact of additional heat energy generated in the fluid by the Lorentz forces caused by the electric field.

Variation in the rate of heat transfer due to clay nanoparticles is shown in table 2. 25.5% enhancement in the rate of heat transfer of cementitious material is noted in the response of clay nanoparticles. Table 3 shows the involvement of different relative parameters in skin friction of the fluid on the plate. From table 3 it is worth noting that the effect of volume fraction parameter which is quite in the agreement of Fig. 2. By adding the nano clay particles, the skin friction increases due to the increase in cohesive forces, which strengthens the binding forces of cement materials and the same behavior are also reported for higher values $\lambda_1$. Fall in the skin friction of the fluid is noticed for the higher value of $k$ and $R_e$. It is happened due to decreases in the thickness of the EDL to which consequently decreases the cohesive forces between the fluid particles, as a result, skin friction decrease.
Furthermore, the present problem can be extended in the future by suspending different nanoparticles in base fluids, different geometry can be used i.e. cylindrical coordinate or infinite plate, taking slip conditions, ramped wall temperature, Newtonian heating, and thermo-diffusion.

VII. CONCLUDING REMARKS

Time fractional analysis has been discussed for fractionalized Maxwell nanofluid with clay-nanoparticles in a channel with the combined effect of viscous dissipation and joule heating. The analysis is carried out to study the applications of clay nanoparticles in cement concrete. The closed-form solutions have been gotten via the Fourier sine transform and the Laplace transform technique. The key results of the present work have been abridged as pursues:

- The cohesive and binding forces of the fluid are increase due to the suspension of clay nanoparticles.
- The profile of the velocity of the cement-based concrete enhanced for larger values of $R_{f}$ and $k$, while retardation is noticed for the values of $\sigma$ and $\lambda_{1}$.
- The temperature profile of the cementitious material is upgraded for a larger magnitude of the joule heating parameter $\delta_{t}$ and Brinkman number $Br$.
- The heat transfer rate augmented by 25.5% as $\varphi$ is increasing from 0 to 0.04.
- Skin friction for the larger values of volume fraction $\varphi$ and $\lambda_{1}$, while opposite trends are noticed for the higher values of the ratio of zeta potential $R_{f}$ and electrokinetic parameter $k$.

APPENDIX

1) $£^{-1} \left[ \mathcal{H}_{1}(w, \rho) \right] = £^{-1} \left[ \frac{\gamma \rho^{-m-k\alpha+1}}{(p + a_{1}w^{12})^{1+k+1}} \right]
\mathcal{H}_{1}(w, t) = \gamma t^{k(\alpha+1)-k-\alpha+1}
\times E(1, k_{a}-m+2) \left(-w^{2} a_{1} t\right)$

2) $£^{-1} \left[ \mathcal{H}_{2}(w, \rho) \right] = £^{-1} \left[ \frac{\rho^{-m-k\alpha-1}}{(p + w^{2} a_{1})^{1+k+1}} \right]
\mathcal{H}_{2}(w, t) = \gamma t^{k(\alpha+1)-m+1}
\times E(1, k_{a}-m+2) \left(-w^{2} a_{1} t\right)$

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S. Murtaza et al.: Exact Analysis of Non-Linear EoF of Generalized Maxwell Nanofluid

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