A Novel Way of Detecting and Correcting Transmitted Errors using Fuzzy Logic

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Abstract. Transmission and receival of messages always play an important role on various channels. Besides the transmission it is also necessary to be sure that only the exactly transmitted is transferred and the original message is not trampled due to the presence of some disturbance sources in the channel. Inspite of precautions if there is an error in transmission it is mandatory to recover the original message with the aid of several decoding measures. One among the decoding measures is the method of implementing the notion of fuzzy logic, which has its wide applications in several fields. Thus in this paper with the help of the concept of relative weights, the codes are studied in the sense of fuzzy logic to decode the transmitted message, detect errors in transmission and to correct them. Hence the technique of detecting and correcting errors presented in this paper provides a better outcome in comparison with several existing such methods. The prime aim of studying these concepts in fuzzy logic, is it's precision and accuracy in terms of membership values which avoids the disparities that arises in other methods.

1. Introduction
Message disruption has always been a cause of concern on various transmission channels. Disruption results in errors which makes the receiver to obtain a totally different message sent by the sender. The study of such disruptions and their corrections in transmission is the sole purpose of coding theory. The notion of coding theory was first initially proposed by [2], in which he proposed the transmission of additional bits in a much smaller way. This was further developed by [6] who developed the ingenious method of error detecting and error correcting codes [5, 7-10]. The notion of fuzzy sets was initially proposed in [11]. A fuzzy set can be defined mathematically by assigning to each possible individual in the universe of discourse a value representing its grade of membership in the fuzzy set. These membership grades are represented by real number values ranging in the closed interval between 0 and 1 [3] . A new perspective of coding theory based on fuzzy sets was studied in [1]. In this paper, the notions of Fuzzy Hamming distance of Fuzzy Codewords, Fuzzy Hamming Weight of Fuzzy Codewords, Linear Fuzzy codewords and Equivalent Fuzzy codewords are studied along with some of their properties.

2. Preliminaries
The preliminaries required for the study of this paper are as follows:

Definition 2.1: [4] The Exclusive Or is a basic computer operation denoted by XOR or ⊕, which takes two individual bits $\beta \in \{0,1\}$ and $\beta' \in \{0,1\}$ and yields
\[ \beta \oplus \beta' = \begin{cases} 0 & \text{if } \beta \text{ and } \beta' \text{ are same} \\ 1 & \text{if } \beta \text{ and } \beta' \text{ are different} \end{cases} \]

**Definition 2.2:** [1] A q-ary code is a set of sequences of symbols where each symbol is chosen from a set \( F_q = \{A_1, A_2, \ldots, A_q\} \) of q distinct elements.

**Definition 2.3:** [1] A Binary Code is a sequences of 0 and 1 which are called codewords.

**Definition 2.4:** [1] Let \( F_q^n \) denote the set of all ordered n-tuples \( a = a_1, a_2, \ldots, a_n \). The elements of \( F_q^n \) are called vectors or words.

**Definition 2.5:** [1] The weight \( w(x) \) of a vector \( x \) in \( F_q^n \) is defined to be the number of non-zero entries of \( x \).

**Definition 2.6:** [1] Suppose \( x \) is a codeword of \( C \). If \( w_1, w_2, \ldots, w_k \) are defined to be the positions of 1 in \( x \), then \( w_1 + w_2 + \ldots + w_k \) are called relative weight of codeword \( x \). Since \( 11\ldots1 \) is a codeword of \( C \), then its relative weight is \( 1 + 2 + \ldots + n = \frac{n(n+1)}{2} \). Thus this weight is called a Maximum Relative Weight of codeword \( x \) of \( C \).

**Definition 2.7:** [1] The Relative Weight of a codeword \( x \) in \( (F_2)^n \) is denoted by \( J(x) \) and it is defined by

\[ J(x) = \frac{w(x)}{\text{Maximum Relative Weight}} \]

**Definition 2.8:** [1] Let \( C \) be a Code. The function \( J : C \rightarrow [0,1] \) is said to be a Fuzzy Code if it satisfies the following conditions:

1. \( J(x + y) \geq \min \{J(x), J(y)\} \)
2. \( J(-x) = J(x) \)
3. \( J(xy) \leq \max \{J(x), J(y)\} \), for all \( x, y \in C \)

Here, the members of the fuzzy code \( J(C) \) are called as fuzzy codewords.

### 3. Fuzzy Linear Codes

In this section, the concepts of Fuzzy Hamming Weight, Fuzzy Hamming Distance for fuzzy codewords, Lower Bound and Linearity of a Fuzzy code and the decoding and error correction of fuzzy codewords are studied with suitable examples along with some of their properties.

Throughout this section, \( C \) is a code, \( \{C_1, C_2, \ldots, C_n\} \) are the collection of codewords in \( C \) and \( J(C) = \{J(C_1), J(C_2), \ldots, J(C_n)\} \) is the respective collection of fuzzy codewords associated with \( C \).

**Definition 3.1** The Fuzzy Hamming Weight of any codeword \( C_i \), \( 1 \leq i \leq n \) in \( C \) (denoted by \( FHW(C_i) \)) is a function from \( \{C_1, C_2, \ldots, C_n\} \rightarrow [0,1] \) and it is defined as the relative weight of the number of non-zero entries in the codeword \( C_i \) of \( C \).

**Example 3.2** Let \( C = \{11001, 101100, 11010, 10110\} \). The Maximum Relative Weight of the codewords in \( C \) is 15. For the codeword 11001, the number of non-zero entries is 3 and hence the Fuzzy Hamming Weight for 11001 is 3/15. Similarly the Fuzzy Hamming Weight for 10100, 11010, 10110 are 2/15, 3/15, 3/15 respectively.

**Definition 3.3** The Fuzzy Hamming Distance between two codewords \( C_i \) and \( C_j \), \( i \neq j, 1 \leq i, j \leq n \) in \( C \) (denoted by \( FH(D(C_i, C_j)) \)) is a function from \( \{C_1, C_2, \ldots, C_n\} \rightarrow [0,1] \) and it is defined as the relative weight of the number of places of the vectors by which the two codewords differ.

Suppose for any two codewords \( C_i \) and \( C_k \), \( i \neq k, 1 \leq i, k \leq n \) in \( C \), the number of vectors is 1, then in this case the Fuzzy Hamming Distance is considered for the vectors which is denoted by \( FH(D(x_i, x_k)) \), where \( x_i, x_k \) are single vectors of \( C_i \) and \( C_k \) respectively and it is defined as
\[
FHD(x_i, x_j) = \begin{cases} 
0 & \text{if } x_i = x_j \\
1 & \text{if } x_i \neq x_j 
\end{cases}
\]

**Example 3.4** Let \( C = \{10001, 10100\} \). Now, the number of places on which the vectors differ are 2, and the maximum relative weight of codewords in \( C \) is 15. Hence, the Fuzzy Hamming Distance is 2/15.

**Remark 3.5** For any three codewords \( C_i, C_j, C_k \) \((i \neq j \neq k, i, j, k = 1,2,\ldots,n)\) in \( C \), the fuzzy hamming distance between them satisfy the triangle inequality (i.e.,) the fuzzy codewords \( J(C_i), J(C_j) \) and \( J(C_k) \) satisfy the inequality

\[
FHD(C_i, C_k) \leq FHD(C_i, C_j) + FHD(C_j, C_k). 
\]

**Proof.** By Definition 3.3,

\[
FHD(C_i, C_j) = \sum_{i,j=1}^{n} FHD(x_i, x_j)
\]

where \( x_i, x_j \) \((1 \leq i, j \leq n)\) are the vectors in \( C_i \) and \( C_j \) respectively, and

\[
FHD(x_i, x_j) = \begin{cases} 
0 & \text{if } x_i = x_j \\
1 & \text{if } x_i \neq x_j 
\end{cases}
\]

For \( n = 1 \), if \( C_i = C_k \), then \( FHD(C_i, C_k) = 0 \) and hence \( FHD(C_i, C_i) \leq FHD(C_i, C_j) + FHD(C_j, C_k) \).

For \( n = 1 \), if \( C_i \neq C_k \), then \( FHD(C_i, C_k) = 1 \) and we have either \( C_i \neq C_j \) or \( C_j \neq C_k \), which implies \( FHD(C_i, C_j) + FHD(C_j, C_k) \) is atleast 1. Hence, \( FHD(C_i, C_k) \leq FHD(C_i, C_j) + FHD(C_j, C_k) \).

We now prove for \( n > 1 \), by the previous case, for each vector in the codeword we have,

\[
FHD(x_i, x_k) \leq FHD(x_i, x_j) + FHD(x_j, x_k),
\]

where \( x_i, x_j, x_k \) \((i,j,k = 1,2,\ldots,n)\) are the vectors of the codewords \( C_i, C_j \) and \( C_k \) \((i,j,k = 1,2,\ldots,n)\) of \( C \) respectively. Thus,

\[
FHD(C_i, C_k) = \sum_{i,k=1}^{n} FHD(x_i, x_k) \leq \sum_{i,j,k=1}^{n} [FHD(x_i, x_j) + FHD(x_j, x_k)] 
\]

\[
= \sum_{i,j=1}^{n} FHD(x_i, x_j) + \sum_{j,k=1}^{n} FHD(x_j, x_k) 
\]

\[
= FHD(C_i, C_j) + FHD(C_j, C_k)
\]

which is the triangle inequality.

**Remark 3.6** For any two codewords \( C_i \& C_j \), (\( 1 \leq i, j \leq n \)) in \( C \), \( FHD(C_i, C_j) = FHW(C_i \oplus C_j) \).

**Proof.** For any two codewords \( C_i \) and \( C_j \), \((i \neq j, i, j = 1,2,\ldots,n)\) in \( C \), let ‘\( x \)’ denote the number of places of the vectors by which the codewords \( C_i \) and \( C_j \) differ and let the maximum relative weight of the codewords in \( C \) be \( d \). Hence, by Definition 3.3, the Fuzzy Hamming Distance between \( C_i \) and \( C_j \) is \( x/d \).

Now, by Definition 2, \( C_i \oplus C_j \) gives ‘\( 1 \)’ only at places where the vectors of the codewords differ. Hence, for the codewords in \( C \) with maximum relative weight \( d \), the Fuzzy Hamming Weight of \( C_i \oplus C_j \) is \( x/d \), where \( x \) is the number of ‘\( 1 \)’ in \( C_i \oplus C_j \).

**Example 3.7** Let \( C = \{0010, 0001\} \). By Definition 3.3, \( FHD(0010, 0001) = 2/10 \). Also, \( 0010 \oplus 0001 = 0011 \). Thus, \( FHW(0010 \oplus 0001) = FHW(0011) = 2/10 \).

**Definition 3.8** The Lower Bound of a fuzzy code \( J(C) \) (denoted by \( LB(J(C)) \)) is the function (or operator) \( FHD: \{C_1, C_2,\ldots,C_n\} \rightarrow I = [0,1] \) and it is defined as the least Fuzzy Hamming Distance between any two fuzzy codewords in \( J(C) \).
Example 3.9 Let $\mathcal{C} = \{0100, 0101, 1100, 1110\}$. By Definition 3.3, $FH\mathcal{D}(0100, 0101) = 1/10$, $FH\mathcal{D}(0100, 1100) = 1/10$, $FH\mathcal{D}(0101, 1100) = 2/10$, $FH\mathcal{D}(0101, 1110) = 2/10$, $FH\mathcal{D}(0101, 1110) = 3/10$, $FH\mathcal{D}(1100, 1110) = 2/10$. Thus the Fuzzy Hamming Distance between the collection of codewords in $\mathcal{C}$ are $\{1/10, 2/10, 3/10\}$. Hence, the Lower Bound of the fuzzy code $J(\mathcal{C})$ is $1/10$.

3.1 Maximum Likelihood Decoding of Fuzzy Codewords

It is not always possible for the receiver to exactly receive the original message. When a message is sent, some disturbances can alter the original message to some other message. Whenever the original message is not received by the receiver, we say that there is an Error in the message. Due to the presence of errors, the original codeword, that is sent gets changed and a new codeword reaches the receiver. This new codeword is called the Transmitted Codeword. This transmitted codeword will not give the original message that was intended to be sent, which results in miscommunication thus leading to adverse effects. Hence, it is necessary to identify the error in transmission and rectify the mistake. The principle of Maximum Likelihood Decoding in Fuzzy Codes compares the Fuzzy Hamming Distance of the transmitted fuzzy codeword with the original fuzzy codewords. Among this, the original fuzzy codeword that gives the Lower Bound of the fuzzy code with the transmitted fuzzy codeword is the error rectified fuzzy codeword. Thus, we have the following Definition 3.10.

Definition 3.10 Let the a codeword $C_j$ be the transmitted codeword, such that $C_j \notin \mathcal{C}$. Then the Maximum Likelihood Decoder between every pair $(C_i, C_j)$ $(1 \leq i \leq n)$ of fuzzy codewords denoted by $(J(\mathcal{D}_{MLD}))C_i, C_j$ is a function from $\{C_1, C_2, ..., C_n\} \rightarrow [0, 1]$, and it is defined as

$$J(\mathcal{D}_{MLD})) = \underset{i=1}{\overset{n}{\Lambda}} [FH\mathcal{D}(C_i, C_j)]$$

for all $C_i \in \mathcal{C}$ $(i = 1, 2, ..., n)$ and $C_j \notin \mathcal{C}$, such that $C_i \neq C_j$.

Example 3.11 Let $\mathcal{C} = \{00100, 01110, 10001, 11000\}$ and let the received codeword be 11111. Since the received does not belong to the collection $\mathcal{C}$, the Fuzzy Hamming Distance between the transmitted codeword and the other codewords in the collection are $FH\mathcal{D}[00100, 11111] = 4/15$. This Fuzzy Hamming Distance indicates that the original codeword varies from the transmitted codeword by four bits. Similarly $FH\mathcal{D}[01110, 11111] = 2/15$, $FH\mathcal{D}[10001, 11111] = 3/15$ and $FH\mathcal{D}[11000, 11111] = 3/15$. The respective Fuzzy Hamming Distance between the transmitted codeword and the original codewords are $2/15, 3/15, 4/15$. The Minimum Fuzzy Hamming Distance is 2/15. Hence, the transmitted codeword must be 01110.

3.2 Fuzzy Erasures

Another important parameter, which is taken into account while decoding the original transmitted fuzzy codewords is the fuzzy erasure. Presence of a fuzzy erasure signifies the loss of relative weight of a particular bit or several bits. In other words, the relative weights are not altered due to the change in codewords, rather it goes missing.

Definition 3.12 Let the fuzzy codewords be transmitted to the receiver and during the transmission owing to the disturbances some relative weights of the fuzzy codewords are erased and a blank space is received by the receiver instead of the relative weights. These blank spaces are called the Fuzzy Erasures.

Example 3.13 Let $\mathcal{C} = \{1100, 0101, 1110\}$ and hence $J(\mathcal{C}) = \left\{ \frac{11+2, 2+4, 4+2+3}{10, 10, 10}, \frac{1+4+2+3}{10}, \frac{1+3+4+2+3}{10}, \frac{1+4+2+3}{10} \right\}$. Let the transmitted codewords to the receiver be $\{1, -0, -0, 1, -1\}$, where ’-’ denotes the missing vectors. Now $J(\mathcal{C}) = \left\{ \frac{1+4+2+3}{10}, \frac{1+4+2+3}{10}, \frac{1+4+2+3}{10} \right\}$, where ’-’ denotes the relative weight of the missing vectors in the codewords. Thus it is not possible to determine the relative weight without knowing the values of the
missing spaces. These missing spaces are the Fuzzy Erasures. Based on the Maximum Likelihood Decoding of Fuzzy Codewords and Fuzzy Erasures, we now propose the following equivalent conditions:

**Proposition 3.14** Let \( \mathcal{C} \) be a code, \( \{C_1, C_2, \ldots, C_n\} \) are the collection of codewords in \( \mathcal{C} \) and \( J(\mathcal{C}) = \{J(C_1), J(C_2), \ldots, J(C_n)\} \) is the respective collection of fuzzy codewords associated with \( \mathcal{C} \). Then the following statements are equivalent:

1. For the fuzzy code \( J(\mathcal{C}) \), \( LB(J(\mathcal{C})) \geq 2/n \), where \( 'n' \) is the maximum relative weight of the collection of fuzzy codewords in \( J(\mathcal{C}) \).
2. If \( LB(J(\mathcal{C})) = \frac{x}{n} \), such that \( 'x' \) is odd, then \( J(\mathcal{C}) \) can correct \( \frac{x-1}{2n} \) errors.
3. If \( 'e' \) denotes the errors in transmission, then the fuzzy code \( J(\mathcal{C}) \) can detect \( \frac{e}{n} \) errors only if \( \frac{x}{n} \geq \frac{e+1}{n} \), where \( \frac{x}{n} \) is the Lower Bound of \( J(\mathcal{C}) \).
4. The fuzzy code \( J(\mathcal{C}) \) can correct \( \frac{x-1}{n} \) fuzzy erasures, where \( \frac{x}{n} \) is the Lower Bound of \( J(\mathcal{C}) \).

**Proof.** (1) \( \Rightarrow \) (2). Suppose that \( LB(J(\mathcal{C})) = \frac{2e+1}{n} \). We now show that it is always possible to decode the output of a transmitted codeword with the aid of Maximum Likelihood Decoding of Fuzzy Codewords, if the error in the transmitted codeword is atmost \( \frac{e}{n} \). Let us assume that this is not true. Let \( U \) be the transmitted codeword with relative weight \( J(U) \) and let \( V \) be the received codeword with relative weight \( J(V) \). then,

\[
FHD(U, V) \leq \frac{e}{n}
\]

Since the Maximum Likelihood Decoder does not retrieve the originally transmitted codeword by our assumption, it decodes another codeword \( W \) with relative weight \( J(W) \), such that, \( J(D_{MLD})(U, V) = J(W) \), and \( J(W) \neq J(U) \). Hence by Definition 3.10,

\[
FHD(V, W) \leq FHD(V, U)
\]

By Remark 3.5, the Fuzzy Hamming Distance between the codewords satisfy the triangle inequality and hence we have

\[
FHD(U, W) \leq FHD(U, V) + FHD(V, W)
\]

\[
\leq FHD(V, U) + FHD(V, W)
\]

\[
\leq 2FHD(V, U)
\]

\[
\leq 2\frac{2e}{n}
\]

\[
= LB(J(\mathcal{C})) - \frac{1}{n}
\]

Thus, the Fuzzy Hamming Distance of two codewords is atmost \( LB(J(\mathcal{C})) - \frac{1}{n} \), which is not possible.

(1) \( \Rightarrow \) (3) Suppose that \( LB(J(\mathcal{C})) \geq \frac{e}{n} + \frac{1}{n} \). If suppose there are only \( \frac{e}{n} \) or a fewer errors in a transmitted codeword \( C_i \) \((i = 1, 2, \ldots, n) \) in \( \mathcal{C} \), then this transmitted codeword differ from the other codewords by atleast \( \frac{e}{n} + \frac{1}{n} \) bits, which makes it possible for the errors to be detected.

(1) \( \Rightarrow \) (4) Let the transmitted codeword be \( V \), which has some vectors missing on it. Then the relative weight of this transmitted codeword is, \( J(V) = x \), where \( 'x' \) denotes the relative weight of the missing bits.

Let \( w_i, v_i, u_i \) \((1 \leq i, k, l \leq n) \) denote the vectors in the codewords \( W, V \) and \( U \) respectively. We claim that there exists a unique codeword \( W \) such that for the vectors \( J(w_i) = J(v_k) \) \((i, k = 1, 2, \ldots, n) \), at every place except at the erasure part. Here \( J(w_i) \) and \( J(v_k) \) denote the relative weight of the vectors on the codeword \( W \) and \( V \) respectively. Assume the contrary that it is not unique. Then there exists another distinct codeword \( U \) such that \( J(u_j) = J(v_k) \), at every places except at the erasure part. This implies that \( J(w_i) = J(u_j) \), at every place except at the erasure part. Let \( N(x) \) denote the number of fuzzy codewords in the fuzzy erasure part. Then

\[
FHD(W, U) \leq N(x) \leq LB(J(\mathcal{C})) - \frac{1}{n}
\]
which is not possible as $LB(J(\mathcal{C}))$ is the Lower Bound of $J(\mathcal{C})$.

**Definition 3.15** Any fuzzy code $J(\mathcal{C})$ is said to be a linear fuzzy code, if it has the following properties:

(i) $J(0_c) = 0$ is always a member of $J(\mathcal{C})$.

(ii) The Lower Bound of a Fuzzy Code $J(\mathcal{C})$ is equal to the Minimum Fuzzy Hamming Weight of any non-zero codeword in the collection $J(\mathcal{C})$.

**Example 3.16** Let $\mathcal{C} = \{0000, 1000, 0001, 1001\}$. Now, the corresponding collection of fuzzy codewords associated with $\mathcal{C}$ is $J(\mathcal{C}) = \{0, 1/10, 4/10, 5/10\}$. From Definition 3.1, $FHW(1000) = 1/10$, $FHW(0001) = 1/10$, $FHW(1001) = 2/10$. The Minimum Fuzzy Hamming Weight is $1/10$. Now, from Definition 3.3, $FHD(0000, 1000) = 1/10$, $FHD(0000, 0001) = 1/10$, $FHD(0000, 1001) = 2/10$, $FHD(1000, 0001) = 1/10$, $FHD(1000, 1001) = 1/10$. The Fuzzy Hamming Distance between the codewords are $\{1/10, 2/10\}$. The Lower Bound of the Fuzzy Code is $1/10$.

Also $J(0_c) = 0 \in J(\mathcal{C})$. Hence $J(\mathcal{C})$ is a linear fuzzy code.

The following property given in Proposition 3.17, holds for a linear fuzzy code.

**Proposition 3.17** Suppose that $J(\mathcal{C})$ is a linear fuzzy code. Then, the lower bound to the minimum Fuzzy Hamming weight of any non-zero codeword in the collection $J(\mathcal{C})$, (i.e.,) $LB(J(\mathcal{C})) = \min_{c \in \mathcal{C}}\{FHW(C_i)\}$ for any two distinct codewords $C_i, C_j \in \mathcal{C}$. Thus, for any two distinct codewords $C_i, C_j \in \mathcal{C}$, we have $FHD(C_i, C_j) = FHW(C_i \oplus C_j)$. Hence we have $FHD(C_i, C_j) = FHW(C_k) = LB(J(\mathcal{C}))$ and thus the minimum Fuzzy Hamming weight of two nonzero codewords equals $LB(J(\mathcal{C}))$.

### 3.3 Construction of Extended Fuzzy Code for length 3

Suppose that $J(\mathcal{C})$ is a linear fuzzy code. Let a codeword $C_i$ in $\mathcal{C}$ (i = 1, 2, ..., n) be indexed with vectors as $c_1c_2c_3$. Now a vector is added to $C_i$ such that,

$$C_i = \begin{cases} c_1c_2c_30 & \text{if } FHW(C_i) \text{ is an even number} \\ c_1c_2c_31 & \text{if } FHW(C_i) \text{ is an odd number} \end{cases}$$

This new code which is of length 4 is the extended code and the relative weight of the extended fuzzy code is denoted by $J_E(\mathcal{C})$.

**Proposition 3.18** Suppose that $J(\mathcal{C})$ is a linear fuzzy code. Let $LB(J(\mathcal{C})) = \frac{c}{6}$ where 'c' is any positive integer. If $LB(J(\mathcal{C}))$ is odd, then the Least Bound of the Extended Fuzzy Code $LB(J_E(\mathcal{C}))$ is $\frac{c+1}{10}$ and if $LB(J(\mathcal{C}))$ is even, then the Least Bound of the Extended Fuzzy Code $LB(J_E(\mathcal{C}))$ is $\frac{c}{10}$.

**Proof.** Let $LB(J_E(\mathcal{C}))$ be the relative weight of the extended linear fuzzy codeword of length 4. Let the least Fuzzy Hamming Weight of $J(\mathcal{C})$, denoted by $LFHW(J(\mathcal{C}))$ be $c/6$. Then the least Fuzzy Hamming Weight of the extended fuzzy code denoted by $LFHW(J_E(\mathcal{C}))$ has the value

$$LFHW(J_E(\mathcal{C})) = \begin{cases} \frac{c+1}{10} & \text{if } w(c)/6 \text{ is odd} \\ \frac{c}{10} & \text{if } w(c)/6 \text{ is even} \end{cases}$$

where $w(c)$ is the number of non-zero entries of each codeword. Now by Proposition 3.17, since $J(\mathcal{C})$ is a linear fuzzy code, the minimum fuzzy hamming weight of $LB(J_E(\mathcal{C}))$ will be equal to $LB(J_E(\mathcal{C}))$ and hence we have
\[
LB(J_E(\mathcal{C})) = \begin{cases} 
\frac{c+1}{10} & \text{if } LB(J(\mathcal{C})) \text{ is odd}
\frac{c}{10} & \text{if } LB(J(\mathcal{C})) \text{ is even}
\end{cases}
\]

4. Permutation and Equivalence of Fuzzy Codewords

In this section, the notion of permutation and equivalence of fuzzy codewords are studied with suitable examples along with an interesting property of them.

**Definition 3.19** A permutation on a code \(\mathcal{C}\) is a one-to-one mapping from the collection of fuzzy codes \(J(\mathcal{C})\) onto itself. (i.e.,) if for a function, both the domain and codomain are same then the function is called a permutation of \(D\).

**Example 3.20** Let \(\mathcal{C} = \{10, 01, 010, 101\}\). Now \(J(\mathcal{C}) = \{\frac{1}{3}, \frac{2}{3}, \frac{2}{6}, \frac{4}{6}\}\). Thus if we arrange the codewords of the same length along one row, and the other codewords of the same length in the next row, we obtain \((\frac{2}{6} \ 4/6 \ 1/3 \ 2/3)\) in the sense \(J(\frac{2}{6}) = \frac{1}{3}\) and \(J(\frac{4}{6}) = \frac{2}{3}\) \(\in J(\mathcal{C})\).

**Definition 3.21** The Minimum Relative Weight of a fuzzy code \(J(\mathcal{C})\) (denoted by \(MRW(J(\mathcal{C}))\)) is the smallest relative weight of any non-zero codeword in the collection.

**Example 3.22** Let \(\mathcal{C} = \{0100, 0101, 1100, 1110\}\). Now \(J(\mathcal{C}) = \{\frac{2}{10}, \frac{6}{10}, \frac{3}{10}, \frac{6}{10}\}\). Hence \(MRW(J(\mathcal{C})) = \frac{2}{10}\) which corresponds to the codeword 0100.

**Definition 3.23** Two fuzzy codes are said to be equivalent if one can be obtained from the other by permutation. (i.e.,) fixing a condition that permutes or changes the vectors of the codewords. The Equivalent fuzzy code is denoted by \(J(\mathcal{P}(x_i))\), where \(x_i (i = 1, 2, \ldots, n)\) are the vectors on each codewords on which the permutation is applied.

**Example 3.24** Let \(\mathcal{C} = \{1000, 0101, 1110\}\). Fix a parameter \(p\) which denotes the Minimum Relative Weight of the collection \(J(\mathcal{C})\). Here \(p = \frac{2}{10}\). Let us now define the permutation for each vector on the codewords as,

\[
\mathcal{P}(x_i) = \begin{cases} 
0 & \text{if } J(x_i) \leq p \\
1 & \text{otherwise}
\end{cases}
\]

on the codewords of \(\mathcal{C}\). We now apply this permutation on each vector of the codeword. Thus in the codeword 0100, \(J(0) = 0/10 < 1/10; J(1) = 2/10 = 2/10; J(0) = 0/10 < 2/10; J(0) = 0/10 < 2/10\). Hence \(\mathcal{P}(0100) = \{0001\}\). Similarly, we get, \(\mathcal{P}(0101) = \{0001\}; \mathcal{P}(1110) = \{0010\}\). Hence \(\mathcal{P}(\mathcal{C}) = \{0000, 0001, 0010\}\) and \(J(\mathcal{P}(x_i)) = \{0, 4, 10, 3, 10\}\) and \(J(\mathcal{P}(x_i))\) is the equivalent fuzzy code of \(J(\mathcal{C})\).

**Notation 3.1** A code \(\mathcal{C} = \{C_1, C_2, \ldots, C_N\}\) of length 'n' and having 'N' number of codewords with minimum Fuzzy Hamming Distance \(d(C_i)\) between any two codewords \((C_i, C_k)\) is represented by \((n, N, d(C_i))\) fuzzy code.

**Proposition 3.25** Let \(\mathcal{C}\) be a code, \(\{C_1, C_2, \ldots, C_n\}\) are the collection of codewords in \(\mathcal{C}\) having same length \(n\). Let \(N\) denote the number of codewords in \(\mathcal{C}\) and \(d(C_i)\) \((1 \leq i, k \leq n)\) denote the minimum Fuzzy Hamming Distance between them. Any \((n, N, d(C_i))\) fuzzy code is equivalent to another \((n, N, d(C_i))\) fuzzy code which contains the fuzzy codeword having the relative weight zero.

**Proof.** Let \(x_{11} x_{12} \ldots x_{1n}\) be the vectors or letters in \(C_1\), \(x_{21} x_{22} \ldots x_{2n}\) be the vectors or letters in \(C_2\) and so on. For each non-zero vector \(x_{ij} (i = 1, 2, \ldots, n, j = 1, 2, \ldots, n)\) in the codewords \(C_k\) \((k = 1, 2, \ldots, n)\),
we now perform the permutation by fixing the parameter 'p', that is applied to all vectors of each codewords of ℶ thus resulting in an equivalent fuzzy code.

Thus, any vector having relative weight lesser then the parameter will become zero and the other vectors remain intact. Performing the mapping, we have

\[
\begin{align*}
    x_{ij} &\rightarrow 0 \quad \text{whenever } |x_i| \leq p \\
    x_{ij} &\rightarrow x_{ij} \quad \text{otherwise}
\end{align*}
\]

Eventually, the codeword with the minimum relative weight will be permuted to the codeword 0000 whose relative weight is 0. Thus, the original codeword is equivalent to a codeword whose relative weight is zero.

### 4 Conclusion

The main aim of this paper is the application of relative weight of codewords on various aspects to develop and study the concepts of error correction and error detection of codewords using relative weight of the codewords through the notion of fuzzy logic as it helps in obtaining more accuracy. Future works include the implementation of the notions of Maximum Likelihood Decoding of Fuzzy Codewords in a decision making problems which aids in the identification of erroneous message.

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