Minimal graphs in the hyperbolic space with singular asymptotic boundaries

Qing Han · Weiming Shen · Yue Wang

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Abstract

We study asymptotic behaviors of solutions $f$ to the Dirichlet problem for minimal graphs in the hyperbolic space with singular asymptotic boundaries under the assumption that the boundaries are piecewise regular with positive curvatures, a case which also arises in the study of a Chaplygin gas. We derive an estimate of such solutions by the corresponding solutions in the intersections of interior tangent balls. The positivity of curvatures plays an important role.

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1 Introduction

Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain. Lin [5] studied the Dirichlet problem of the form

$$\Delta f - \frac{f_i f_j}{1 + |\nabla f|^2} f_{ij} + \frac{n}{f} = 0 \quad \text{in } \Omega,$$

$$f > 0 \quad \text{in } \Omega, \quad f = 0 \quad \text{on } \partial \Omega. \quad (1.1)$$

Geometrically, the graph of $f$ is a minimal surface in $\mathbb{H}^{n+1}$ with its asymptotic boundary at infinity given by $\partial \Omega$. The equation in (1.1) is a quasilinear non-uniformly elliptic equation.
and becomes singular on $\partial \Omega$ since $f = 0$ there. Lin [5] proved that (1.1) admits a unique solution $f \in C(\Omega) \cap C^\infty(\Omega)$ if $\Omega \subset \mathbb{R}^n$ is a $C^2$-domain with a nonnegative boundary mean curvature $H_{\partial \Omega} \geq 0$ with respect to the inward normal direction of $\partial \Omega$. Concerning the higher global regularity, Lin proved $f \in C^{1/2}(\Omega)$ if $H_{\partial \Omega} > 0$.

Two-dimensional version of (1.1) also appears in the study of a Chaplygin gas. Serre [6] analyzed the multidimensional shocks in such a fluid, which is sonic. For some two-dimensional Riemann problems, the subsonic domain is convex and bounded by four arcs of circles. Serre proved that (1.1) admits a solution $f \in C(\Omega) \cap C^\infty(\Omega)$ if $\Omega \subset \mathbb{R}^2$ is a piecewise $C^2$ convex domains, with positive boundary curvature. (Theorem 6.1 [6].)

In [4], for a bounded $C^2$-domain $\Omega \subset \mathbb{R}^n$ with $H_{\partial \Omega} \geq 0$, we proved that the solution $f$ of (1.1) satisfies $f \in C^{1/\pi^\tau}(\Omega)$ and

$$|f|_{C^{1/\pi^\tau}(\Omega)} \leq [(n + 1) \text{diam}(\Omega)^n]^{1/\pi^\tau}. \tag{1.2}$$

This estimate does not depend on the regularity of the domain, which permits us to discuss (1.1) in domains with singularity. In [4], we also proved that (1.1) admits a unique solution $f \in C^{1/2}(\Omega) \cap C^\infty(\Omega)$ if $\Omega$ is a bounded domain which is the intersection of finitely many bounded convex $C^2$-domains $\Omega_i$ with $H_{\partial \Omega_i} > 0$. The global Hölder regularity with an index $1/2$ is optimal in general.

Concerning asymptotic behaviors of solutions $f$ of (1.1), we have the following result. Let $\Omega$ be a bounded $C^{2,\alpha}$-domain with $H_{\partial \Omega} > 0$, for some $\alpha \in (0, 1)$. Then,

$$\left| \left( \frac{H_{\partial \Omega}}{2d} \right)^{1/2} f - 1 \right| \leq C d^{\frac{\alpha}{2}}, \tag{1.3}$$

where $d$ is the distance function to $\partial \Omega$. We point out that the estimate (1.3) is sharp under the present regularity assumption. For details, refer to [1].

A natural question arises, also motivated by the study in [6]. What is the asymptotic behavior of solutions of (1.1) near singular boundary points?

There have been only a few results concerning boundary behaviors of solutions of geometric PDEs in singular domains. This is partly due to the diversity of singularity and complexity of the relevant geometric problems. The first two authors studied the asymptotic behaviors of solutions of the Liouville equation in [2] and solutions of the Loewner–Nirenberg problem in [3] in singular domains and proved that the solutions are well approximated by the corresponding solutions in tangent cones at singular points on the boundary.

Asymptotic behaviors of solutions of (1.1) are more complicated than those of solutions of the Liouville equation and solutions of the Loewner–Nirenberg problem. As the estimate (1.3) illustrates, the positivity of the boundary mean curvature plays an important role in the estimates of solutions near $C^{2,\alpha}$-boundary. When we attempt to generalize (1.3) to domains with singularity, we cannot compare solutions $f$ of (1.1) with the corresponding solutions in tangent cones if the tangent cones have zero mean curvature wherever they are smooth. This is the case if the tangent cones are bounded by finitely many hyperplanes. We need a “model” domain to preserve the positivity of the boundary mean curvature.

We prove the following result for $n = 2$.

**Theorem 1.1** Let $\Omega$ be a bounded convex domain in $\mathbb{R}^2$ and, for some $x_0 \in \partial \Omega$ and $R > 0$, let $\partial \Omega \cap B_R(x_0)$ consist of two $C^{2,\alpha}$-curves $\sigma_1$ and $\sigma_2$ intersecting at $x_0$ with an angle $\mu \pi$, for some constants $\alpha$, $\mu \in (0, 1)$, such that $\sigma_i$ has a positive curvature $k_i$ at $x_0$, for $i = 1, 2$. Suppose $f \in C(\Omega) \cap C^\infty(\Omega)$ is the solution of (1.1) and $f_{\mu, R_1, R_2}$ is the corresponding
solution in
\[ \Omega_{\mu, R_1, R_2} = B_{R_1}(x_0 + R_1 v_1) \cap B_{R_2}(x_0 + R_2 v_2), \]  
(1.4)
where \( R_1 = 1/\kappa_1 \) and \( v_1 \) is the unit inner normal vector of \( \sigma_1 \) at \( x_0 \). Then, there exist a constant \( r \) and a \( C^{2,\alpha} \)-diffeomorphism \( \mathcal{T} : B_r(x_0) \to \mathcal{T}(B_r(x_0)) \subseteq \mathbb{R}^2 \), with \( \mathcal{T}(\Omega \cap B_r(x_0)) = \Omega_{\mu, R_1, R_2} \cap \mathcal{T}(B_r(x_0)) \) and \( \mathcal{T}(\partial \Omega \cap B_r(x_0)) = \partial \Omega_{\mu, R_1, R_2} \cap \mathcal{T}(B_r(x_0)) \), such that, for any \( x \in \Omega \cap B_r(x_0) \),
\[
\left| \frac{f(x)}{f_{\mu, R_1, R_2}(\mathcal{T}x)} - 1 \right| \leq C|x - x_0|^\beta, \tag{1.5}
\]
where \( \beta \) is a constant in \((0, \alpha/2)\) and \( C \) is a positive constant depending only on \( R, \alpha, \mu \), and the \( C^{2,\alpha} \)-norms of \( \sigma_1 \) and \( \sigma_2 \) in \( B_R \).

The “model” domain for approximations is the intersection of two tangent balls at the singular point, the shaded region in Fig. 1 as defined by (1.4). If \( \alpha \) is sufficiently small, we can take \( \beta \) in (1.5) to be \( \alpha/2 \), which is optimal.

The estimate (1.5) generalizes (1.3) to singular boundaries. To demonstrate this, consider a bounded \( C^{2,\alpha} \)-domain \( \Omega \) such that \( \partial \Omega \) has a positive curvature \( \kappa \) near \( x_0 \in \partial \Omega \). Set \( R = 1/\kappa(x_0) \) and \( x_* = x_0 + R\nu(x_0) \), where \( \nu(x_0) \) is the unit inner normal of \( \partial \Omega \) at \( x_0 \). Then, \( B_R(x_*) \) is a ball tangent to \( \partial \Omega \) at \( x_0 \). The unique solution on \( B_R(x_*) \) is given by
\[
f_{B_R(x_*)}(x) = \left[ R^2 - |x - x_*|^2 \right]^{1/2} = \left[ d_B(2R - d_B) \right]^{1/2} = \left[ d_B(2\kappa^{-1}(x_0) - d_B) \right]^{1/2},
\]
where \( d_B(x) \) is the distance from \( x \) to \( \partial B_R(x_*) \). We note that \( f_{B_R(x_*)} \) depends only on \( d_B \).

For any \( x \), denote by \( d(x) \) the distance from \( x \) to \( \partial \Omega \). For \( x \) close to \( \partial \Omega \), take \( y(x) \in \partial \Omega \) to be the unique point such that \( |xy(x)| = d(x) \), and then take \( p(x) \in \partial B_R(x_*) \) to be the unique point such that the oriented curve length from \( x_0 \) to \( y(x) \) along \( \partial \Omega \) is the same as the oriented curve length from \( x_0 \) to \( p(x) \) along \( \partial B_R(x_*) \). Let \( v_B \) be the unit inner normal of \( \partial B_R(x_*) \). Set
\[
\mathcal{T}x = p(x) + d(x) v_B(p(x)).
\]

Then, \( d(x) = d_B(\mathcal{T}x) \). By (1.3), we have, for \( x \in \Omega \) sufficiently close to \( \partial \Omega \) near \( x_0 \),
\[
\left| \frac{f(x)}{f_{B_R(x_*)}(\mathcal{T}x)} - 1 \right| \leq C|x - x_0|^{\alpha},
\]
where we used the estimate \( |\kappa(y) - \kappa(x_0)| \leq C|y - x_0|^{\alpha} \), for any \( y \in \partial \Omega \).

The proof of Theorem 1.1 is based on a combination of isometric transforms and the maximum principle. Usually, when we discuss asymptotic behaviors of solutions \( f \) in the domain \( \Omega \) with a singularity at \( x_0 \), we compare such solutions with the corresponding solutions in tangent cones at \( x_0 \). However, the positivity of curvatures is not preserved for tangent cones.
bounded by rays. Instead, we use the solution \( f_{\mu, R_1, R_2} \) in \( \Omega_{\mu, R_1, R_2} \) defined as intersections of tangent balls as in (1.4). Our goal is to compare the solution \( f \) in \( \Omega \) near \( x_0 \) with the solution \( f_{\mu, R_1, R_2} \) in \( \Omega_{\mu, R_1, R_2} \). We note that a given point \( x \in \Omega \) may not necessarily be a point in \( \Omega_{\mu, R_1, R_2} \). So as a part of the comparison of \( f \) with \( f_{\mu, R_1, R_2} \), we need to construct a map \( T \), which maps \( \Omega \) near \( x_0 \) onto \( \Omega_{\mu, R_1, R_2} \) near \( x_0 \), and to compare \( f(x) \) with \( f_{\mu, R_1, R_2}(Tx) \). The proof of Theorem 1.1 consists of two steps.

In the first step, we construct two sets \( \hat{B} \) and \( \tilde{B} \) with the property \( \hat{B} \subseteq \Omega \subseteq \tilde{B} \) near \( x \). To this end, we first place two balls tangent to \( \sigma_i \) at \( p_i \), the closest point to \( x \) on \( \sigma_i \), one inside \( \Omega \) and the other containing \( \Omega \), for each \( i = 1, 2 \). We can form \( \hat{B} \) and \( \tilde{B} \) as intersections of smaller balls and the larger balls, respectively. In the second step, we compare the solution \( f \) in \( \Omega \) near \( x_0 \) with the solution \( f_{\mu, R_1, R_2} \) in \( \Omega_{\mu, R_1, R_2} \). To this end, we first compare \( f \) with the solutions \( \hat{f} \) and \( \tilde{f} \) in \( \hat{B} \) and \( \tilde{B} \), respectively, and then compare \( \hat{f} \) and \( \tilde{f} \) with \( f_{\mu, R_1, R_2} \). We note that the sets \( \hat{B}, \tilde{B}, \Omega_{\mu, R_1, R_2} \) have the same structure; namely, they are the intersections of two balls. Comparisons of solutions in these sets are aided by a special class of isometric transforms in the hyperbolic space.

These isometric transforms on \( H^3 \) induce conformal transforms restricted to the asymptotic infinity, and play a fundamental role. Due to these transforms, we can transform the optimal estimates for (1.1) in infinite cones back to the intersections of balls.

The paper is organized as follows. In Sect. 2, we prove the existence of solutions of (1.1) in infinite cones and prove some basic estimates for these solutions. In Sect. 3, we prove that asymptotic expansions near singular boundary points up to certain orders are local properties. In Sect. 4, we study the asymptotic expansions near singular points with positive curvatures and prove Theorem 1.1.

## 2 Solutions in cones

In this section, we discuss (1.1) in infinite cones and prove the existence and uniqueness of its solutions. We also derive some basic estimates. In the following, we assume \( n = 2 \).

For some constant \( \mu \in (0, 1) \), define

\[
V_{\mu} = \{(r, \theta) \mid r \in (0, \infty), \theta \in (0, \mu \pi)\}.
\]  
(2.1)

This is an infinite convex cone in \( \mathbb{R}^2 \), expressed in polar coordinates. Our goal is to find a solution \( f \) of (1.1) in the form

\[
f = rh(\theta) \quad \text{in } V_{\mu}.
\]  
(2.2)

Here, \( h \) is a function on \( (0, \mu \pi) \). We point out that the form (2.2) has a geometric meaning. In fact, the minimal graph \((x, y, f(x, y))\), with \( f \) given by (2.2) and with \( V_{\mu} \) as the asymptotic boundary, is invariant under the isometry of \( H^3 \): \((x, y, z) \mapsto (\lambda x, \lambda y, \lambda z)\), for any positive \( \lambda \). By a straightforward calculation, (1.1) reduces to

\[
\begin{align*}
h'' + h + \frac{2h^2}{h(1 + h^2)} + \frac{2}{h} & = 0 \quad \text{on } (0, \mu \pi), \\
h(0) & = h(\mu \pi) = 0.
\end{align*}
\]  
(2.3)

In view of the equation in (2.3), we set

\[
\mathcal{F}(h) = h(1 + h^2)(h'' + h) + 2(1 + h^2 + h^2).
\]  
(2.4)

First, we construct supersolutions of \( \mathcal{F} \).
Lemma 2.1 For some constant $\mu \in (0, 1)$, there exists a function $\overline{h} \in C([0, \mu \pi]) \cap C^2((0, \mu \pi))$ such that $\mathcal{F}(\overline{h}) \leq 0$ and, for any $\gamma \in (0, 1/3)$,

$$0 \leq \overline{h} \leq C \left( \sin \frac{\theta}{\mu} \right)^\gamma \text{ on } (0, \mu \pi),$$

where $C$ is a positive constant depending only $\mu$ and $\gamma$.

**Proof** For some $\alpha \geq 2$, set

$$\varphi(\theta) = \left( \sin \frac{\theta}{\mu} \right)^{\frac{1}{1+\alpha}}.$$

A straightforward computation yields, for any positive constant $A$,

$$\mathcal{F}(A\varphi) = 2 + A^2 \left[ 3 \left( 1 - \frac{1}{\mu^2(1+\alpha)^2} \right) \varphi^2 - \frac{\alpha - 2}{\mu^2(1+\alpha)^2} \varphi^{-2\alpha} \right]$$

$$+ A^4 \left[ \left( 1 - \frac{1}{\mu^2(1+\alpha)^2} \right) \varphi^4 - \frac{\alpha}{\mu^2(1+\alpha)^2} \varphi^{2-2\alpha} \right].$$

With $\alpha = 2$, we have

$$\mathcal{F}(A\varphi) = 2 + 3A^2 \left( 1 - \frac{1}{9\mu^2} \right) \varphi^2 + A^4 \left[ \left( 1 - \frac{1}{9\mu^2} \right) \varphi^4 - \frac{2}{9\mu^2} \varphi^{-2} \right].$$

If $\mu \leq 1/3$, then

$$\mathcal{F}(\sqrt{3}\mu\varphi) \leq 2 - 2\varphi^{-2} \leq 0.$$

If $1/3 < \mu < 1/\sqrt{3}$, we have, by choosing $A$ large,

$$\mathcal{F}(A\varphi) \leq 2 + 3A^2 \left[ 1 - \frac{1}{9\mu^2} \right] + A^4 \left[ 1 - \frac{1}{3\mu^2} \right] \leq 0.$$

Next, we consider the case $\mu \geq 1/\sqrt{3}$. Set, for some positive $\beta \in (0, 1)$,

$$\psi(\theta) = \varphi(\theta)^{\frac{1}{1+\beta}} = \left( \sin \frac{\theta}{\mu} \right)^{\frac{1}{1+\beta}}.$$

Then, $\psi \leq \varphi$. For some positive constants $A$ and $B$, a straightforward computation yields

$$\mathcal{F}(A\varphi + B\psi) = I + II,$$

where

$$I = 1 + A^2 \varphi^2 + B^2 \psi^2 + 2AB \varphi \psi \right] [A\varphi + B\psi]$$

$$\cdot \left\{ A \left[ \left( 1 - \frac{1}{\mu^2(1+\alpha)^2} \right) \varphi - \frac{\alpha}{\mu^2(1+\alpha)^2} \varphi^{-1-2\alpha} \right]$$

$$+ B \left[ \left( 1 - \frac{1}{\mu^2(1+\beta)^2} \right) \psi - \frac{\beta}{\mu^2(1+\beta)^2} \psi^{-1-2\beta} \right] \right\},$$

and

$$II = 2 \left[ 1 + A^2 \varphi^2 + B^2 \psi^2 + 2AB \varphi \psi \right]$$

$$+ A^2 \frac{\varphi^{-2\alpha}}{\mu^2(1+\alpha)^2} \left( 1 - \varphi^{2+2\alpha} \right) + B^2 \frac{\psi^{-2\beta}}{\mu^2(1+\beta)^2} \left( 1 - \psi^{2+2\beta} \right).$$
\[ + 2AB \frac{\varphi^{-\alpha}}{\mu(1+\alpha)} \frac{\psi^{-\beta}}{\mu(1+\beta)} \sqrt{(1-\varphi^{2+2\alpha})(1-\psi^{2+2\beta})}. \]

Fix an \( \alpha > 2 \) and take
\[ \beta = \min \left\{ \frac{1}{2} \left( \frac{1}{\mu} - 1 \right), \frac{1}{100} \right\}. \]

With \( \psi \leq 1 \), it is easy to check
\[ \left( 1 - \frac{1}{\mu^2(1+\beta)^2} \right) \psi - \frac{\beta}{\mu^2(1+\beta)^2} \psi^{-1-2\beta} < 0. \] (2.8)

In fact, we only need to require \( \beta < 1/\mu - 1 \). Next, if \( \sin \frac{\vartheta}{\mu} \leq \frac{1}{1+\alpha} \), we have \( \varphi \leq \left( \frac{1}{1+\alpha} \right)^{1+\alpha} \), and hence
\[ \left( 1 - \frac{1}{\mu^2(1+\alpha)^2} \right) \varphi - \frac{\alpha}{\mu^2(1+\alpha)^2} \varphi^{-1-2\alpha} < 0. \] (2.9)

If \( \sin \frac{\vartheta}{\mu} \in [\frac{1}{1+\alpha}, 1] \), we have \( \psi \geq \left( \frac{1}{1+\alpha} \right)^{1+\beta} \) and hence
\[
\begin{align*}
&\left( 1 - \frac{1}{\mu^2(1+\alpha)^2} \right) \varphi - \frac{\alpha}{\mu^2(1+\alpha)^2} \varphi^{-1-2\alpha} \\
&\quad + C \left[ \left( 1 - \frac{1}{\mu^2(1+\beta)^2} \right) \psi - \frac{\beta}{\mu^2(1+\beta)^2} \psi^{-1-2\beta} \right] \\
&\leq 1 - C \left( \frac{1}{\mu^2(1+\beta)^2} - 1 \right) \psi \leq -1,
\end{align*}
\]
by choosing \( C > 0 \) large. By combining with (2.8) and (2.9), we have, on \([0, \mu \pi]\),
\[
\begin{align*}
&\left( 1 - \frac{1}{\mu^2(1+\alpha)^2} \right) \varphi - \frac{\alpha}{\mu^2(1+\alpha)^2} \varphi^{-1-2\alpha} \\
&\quad + C \left[ \left( 1 - \frac{1}{\mu^2(1+\beta)^2} \right) \psi - \frac{\beta}{\mu^2(1+\beta)^2} \psi^{-1-2\beta} \right] \leq -\eta,
\end{align*}
\] (2.10)
for some positive constant \( \eta \). With \( A \) to be determined, we set
\[ B = CA. \]

We first consider \( \sin \frac{\vartheta}{\mu} < \frac{1}{1+\alpha} \). By (2.8) and (2.9), we have
\[
I \leq 1 \cdot (A\varphi) \cdot A \left[ \left( 1 - \frac{1}{\mu^2(1+\alpha)^2} \right) \varphi - \frac{\alpha}{\mu^2(1+\alpha)^2} \varphi^{-1-2\alpha} \right] \\
\leq -\frac{A^2 \alpha}{\mu^2(1+\alpha)^2} \varphi^{-2\alpha} + C_1 A^2 \varphi^{-2\alpha+\tau},
\]
for some positive constant \( \tau \). We note that the omitted terms in \( I \) are all nonpositive. Similarly, we have
\[
II \leq \frac{2A^2}{\mu^2(1+\alpha)^2} \varphi^{-2\alpha} + 2 + C_2 A^2 \varphi^{-2\alpha+\tau}.
\]
Hence, by (2.7),
\[
\mathcal{F}(A\varphi + B\psi) \leq -\frac{A^2 (\alpha - 2)}{\mu^2(1+\alpha)^2} \varphi^{-2\alpha} + 2 + C_0 A^2 \varphi^{-2\alpha+\tau}.
\]
Then, there exists a small $\delta$, independent of $A \geq 1$, such that
\[
F(A\varphi + B\psi) < 0 \quad \text{if} \quad \frac{\theta}{\mu} \leq \delta.
\]
For $\sin \frac{\theta}{\mu} > \delta$, we have, by (2.10),
\[
I \leq -(A^2\varphi^2) \cdot (A\varphi) \cdot A\eta = -A^4\varphi^3\eta \leq -A^4\eta^3 \frac{\gamma}{\mu}.
\]
On the other hand,
\[
II \leq C_2 A^2.
\]
Hence, by choosing $A$ sufficiently large, depending on $\delta$, we have
\[
F(A\varphi + B\psi) = I + II < 0 \quad \text{if} \quad \sin \frac{\theta}{\mu} \geq \delta.
\]
We have the desired result.  \(\square\)

According to the proof, we can actually take $\gamma = 1/3$ if $\mu < 1/\sqrt{3}$. We point out that $\gamma = 1/3$ yields the same Hölder exponent as in (1.2) for $n = 2$. The present form of Lemma 2.1 is sufficient for the existence of solutions in cones. Later on, we will prove such solutions indeed admit an upper bound given by the function in (2.5) with $\gamma = 1/3$, for any $\mu \in (0, 1)$.

Next, we introduce an important transform. For any $L > 0$, we define the operator $T_L$ by
\[
T_L(x_1, x_2, x_3) = \frac{L}{(x_1 - L)^2 + x_2^2 + x_3^2} (L^2 - x_1^2 - x_2^2 - x_3^2, 2Lx_2, 2Lx_3). \tag{2.11}
\]
Then, $T_L$ is an isometric automorphism on $\mathbb{H}^3$ and, restricted to $\mathbb{R}^2 \times \{x_3 = 0\}$, $T_L$ is a conformal transform, which maps the point $(L, 0, 0)$ to infinity. In fact, $T_L(x_1, x_2, x_3)$ is a composition of the following transformations. First, consider
\[
G_1 : (x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 + L).
\]
Then, $G_1$ maps $\{(x_1, x_2, x_3) | x_3 > 0\}$ to $\{(x_1, x_2, x_3) | x_3 > L\}$. Second, consider
\[
G_2 : (x_1, x_2, x_3) \mapsto \frac{2L^2}{(x_1^2 + x_2^2 + x_3^2)}(x_1, x_2, x_3).
\]
Then, $G_2$ maps $\{(x_1, x_2, x_3) | x_3 > L\}$ to $\{(x_1, x_2, x_3) | x_1^2 + x_2^2 + (x_3 - L)^2 < L^2\}$ and maps $\{(x_1, x_2, x_3) | x_1^2 + x_2^2 + (x_3 - L)^2 < L^2\}$ to $\{(x_1, x_2, x_3) | x_3 > L\}$. Next, consider
\[
G_3 : (x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 - L).
\]
Then, $G_3$ maps $\{(x_1, x_2, x_3) | x_3 > L\}$ to $\{(x_1, x_2, x_3) | x_3 > 0\}$. Last, consider
\[
G_4 : (x_1, x_2, x_3) \mapsto (x_3, x_2, -x_1).
\]
Then, $G_4$ is an orthogonal transform which rotates the $x_1x_3$-plane by $\pi/2$ clockwise. Then,
\[
T_L = G_3G_2G_1G_4G_3G_2G_1.
\]
It is easy to see
\[
\frac{2L^2x_3}{(x_1 - L)^2 + x_2^2 + x_3^2} \to 0 \quad \text{as} \quad x_1^2 + x_2^2 + x_3^2 \to \infty.
\]
The transform $T_L$ introduced in (2.11) serves three purposes. First, $T_L$ is an isometric automorphism on $H^3$. Second, $T_L$ restricted to $\mathbb{R}^2 \times \{x_3 = 0\}$ is conformal. In other words, it maps domains in $\mathbb{R}^2 \times \{x_3 = 0\}$ conformally to domains in $\mathbb{R}^2 \times \{x_3 = 0\}$. Third, $T_L$ transforms (minimal) graphs over special domains $\Omega \subset \mathbb{R}^2 \times \{x_3 = 0\}$ to (minimal) graphs over $T_L |_{\{x_3=0\}}(\Omega) \subset \mathbb{R}^2 \times \{x_3 = 0\}$. In general, after applying an isometry to minimal graphs, we may get minimal surfaces which are not necessarily graphs. The preservation of graphs is important for us, and allows us to use PDE techniques for new graphs after the transform.

Now, we are ready prove the existence and uniqueness of solutions of (1.1) in cones.

**Theorem 2.2** Let $V_\mu$ be the cone as in (2.1), for some $\mu \in (0, 1)$. Then, there exists a unique solution $f$ of (1.1) in $V_\mu$, and $f = rh(\theta)$, with $h$ satisfying (2.3).

**Proof** We first prove the existence. For any $R > 0$, set

$$V_{\mu,R} = \{(r, \theta) | r \in (0, R), \ \theta \in (0, \mu \pi)\}.$$

By Theorem 3.1 [4], there exists a unique solution $f_{\mu,R}$ of (1.1) for $V_{\mu,R}$ and, by the maximum principle and Lemma 2.1,

$$f_{\mu,R} \leq C_{\alpha, \mu} r \left( \sin \frac{\theta}{\mu} \right)^{\gamma} \text{ in } V_{\mu,R}. \quad (2.12)$$

By the maximum principle again, we have, for any positive $R_1$ and $R_2$ with $R_1 < R_2$,

$$f_{\mu,R_1} \leq f_{\mu,R_2} \text{ in } V_{\mu,R_1}. \quad (2.13)$$

Next, the uniqueness and scaling invariance imply

$$f_{\mu,k}(x) = kf_{\mu,1} \left( \frac{x}{k} \right).$$

For any positive $\delta$ sufficiently small, set

$$W_{\mu,\delta} = \{(r, \theta) \in V_\mu | r \in (0, \infty), \ \theta \in (\delta \pi, (\mu - \delta) \pi)\}.$$

Then for any $x \in W_{\mu,\delta}$, we have

$$d \geq (\sin \delta \pi) r.$$

By employing the method in the proof of Theorem 3.2 in [4], we can prove, for any $x \in W_{\mu,\delta}$ and any $k \geq 2(|x| + 1),

$$|\nabla f_{\mu,k}(x)| \leq C(\mu, \delta, x). \quad (2.14)$$

In fact, for any $k \geq 2(|x| + 1)$, $B_{d_{k}/4}(x) \subseteq V_{\mu,k}$. By Step 1 in the proof of Theorem 3.2 in [4], we have

$$\|\nabla f_{\mu,k}\|_{L^p(B_{d_{k}/4}(x))} \leq C(p, \mu, \delta) d_{k}^\frac{n}{2}. \quad (2.15)$$

Note that $f_{\mu,k} \geq f_{\mu,2(|x|+1)}$ for any $k \geq 2(|x| + 1)$, by (2.13). By combining with (2.15) and proceeding as in Step 2 in the proof of Theorem 3.2 in [4], we can get (2.14) by applying the $W^{2,p}$-estimate. Now for any $x \in V_\mu$, there exists a positive small $\delta$ such that $x \in W_{\mu,\delta}$. Then for any $y \in B_{d_{k}/4}(x)$ and large positive $k$, we have

$$|f_{\mu,k}(x) - f_{\mu,k}(y)| \leq C(\mu, \delta, x)|x - y|.$$
Therefore, by (2.13) and the interior estimate, for any \( x \in V_\mu \), we have that \( f_{\mu,R}(x) \) converges to some \( f_\mu(x) \) as \( R \to \infty \) and \( f_\mu \in C^\infty (V_\mu) \) is a solution of

\[
\Delta f - \frac{f_i f_j}{1 + |\nabla f|^2} f_{ij} + \frac{2}{f} = 0 \quad \text{in } V_\mu.
\]

By (2.12), \( f_\mu \) is continuous up to the boundary of \( V_\mu \) and \( f_\mu = 0 \) on \( \partial V_\mu \). In summary, \( f_\mu \) is a solution of (1.1) for \( V_\mu \). Moreover, for any positive integer \( k \), \( f_{\mu,k,R}(x) \) converges to \( f_\mu(x) \) as \( R \to \infty \). By the property

\[
f_{\mu,k,R}(x) = k f_{\mu,R}(\frac{x}{k}),
\]

we obtain

\[
f_\mu(x) = k f_{\mu}(\frac{x}{k}).
\]

Therefore, we can write \( f_\mu = rh(\theta) \) for some function \( h \) on \((0, \mu \pi)\).

We now prove the uniqueness. For convenience, we rotate \( \mathbb{R}^2 \) and assume

\[
V_\mu = \left\{ (r, \theta) | r \in (0, \infty), \theta \in \left( -\frac{\mu \pi}{2}, \frac{\mu \pi}{2} \right) \right\}.
\]

Let \( f_1 \) and \( f_2 \) be two solutions of (1.1) for \( V_\mu \). By Remark 2.3 [5], \( (x, f_1(x)) \) and \( (x, f_2(x)) \) are two absolutely area-minimizing hypersurfaces with the same asymptotic boundary \( \partial V_\mu \). Let \( T_1 \) be the map defined in (2.11) with \( L = 1 \). Then, \( T_1^{-1}|_{\{x_3 = 0\}} \) maps \( V_\mu \) conformally to

\[
\tilde{\Omega} = B_{\sin \frac{\mu \pi}{2}} \left( (0, -\cot \frac{\mu \pi}{2}) \right) \cap B_{\sin \frac{\mu \pi}{2}} \left( (0, \cot \frac{\mu \pi}{2}) \right),
\]

and maps the absolutely area-minimizing hypersurface \( \{(x, f_1(x))\} \) with the asymptotic boundary \( \partial V_\mu \) to the absolutely area-minimizing hypersurface \( \{(y, \tilde{f}_1(y))\} \) with the asymptotic boundary \( \partial \tilde{\Omega}, i = 1, 2 \). By Corollary 2.4 [5], \( f_1 = f_2 \). Hence, \( f_1 = f_2 \). \( \square \)

By Theorem 2.2, the unique solution of (1.1) in \( V_\mu \) is given by \( f = rh(\theta) \). Here, \( h \) is the solution of (2.3). In the rest of this section, we discuss properties of \( h \).

**Lemma 2.3** Let \( h \) be the solution of (2.3), for some \( \mu \in (0, 1) \). Then,

\[
\lim_{\theta \to 0} \theta^{-1/3} h(\theta) > 0, \quad \lim_{\theta \to 0} \theta^{2/3} h'(\theta) > 0.
\]

A similar result holds for \( \theta = \mu \pi \).

**Proof** We first recall a general procedure in [5]. Let \( f \) be a solution of (1.1) in \( \Omega \). The graph of \( f \) can be represented by a function over its vertical tangent plane locally near a \( C^1 \) portion of the boundary. Specifically, we fix a boundary point of \( \Omega \), say the origin, and assume that the vector \( e_n = (0, \ldots, 0, 1) \) is the interior normal vector to \( \partial \Omega \) at the origin. Then, with \( x = (x', x_n) \), the \( x' \)-hyperplane is the tangent plane of \( \partial \Omega \) at the origin and the boundary \( \partial \Omega \) can be expressed in a neighborhood of the origin as a graph of a function, as regular as \( \partial \Omega \), over \( \mathbb{R}^{n-1} \times \{0\} \), say

\[
x_n = \varphi(x').
\]

We denote points in \( \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \) by \((x', x_n, t)\). The vertical hyperplane given by \( x_n = 0 \) is the tangent plane to the graph of \( f \) at the origin in \( \mathbb{R}^{n+1} \). We can represent the graph of \( f \)
as a graph of a new function $u$ defined in terms of $(x', 0, t)$ for small $x'$ and $t$, with $t > 0$. In other words, we treat $\mathbb{R}^n = \mathbb{R}^{n-1} \times \{0\} \times \mathbb{R}$ as our new base space and write $u = u(x', t)$. Then, for some $R > 0$, $u$ satisfies

$$
\Delta u - \frac{u_j u_j}{1 + |\nabla u|^2} u_j - \frac{n u_t}{t} = 0 \quad \text{in } B_R^+,\n$$

$$
u(\cdot, 0) = \varphi \quad \text{on } B_R'.
$$

(2.18)

We note that $u$ and $f$ are related by

$$
x_n = u(x', t), \quad t = f(x', x_n).
$$

(2.19)

The solution $u$ admits a formal expansion in the form

$$
u(\cdot, t) = \varphi + c_2 t^2 + \cdots + c_n t^n + c_{n+1,1} t^{n+1} \log t + c_{n+1} t^{n+1} + \cdots,
$$

(2.20)

where $c_2, \ldots, c_n, c_{n+1,1}, c_{n+1}$ are functions on $B_R'$. In fact, $c_2, \ldots, c_n$ and $c_{n+1,1}$ are coefficients for local terms and have explicit expressions in terms of $\varphi$, and $c_{n+1}$ is the coefficient of the first nonlocal term.

In our case, $n = 2$ and we consider (2.20) at $(r, \theta) = (1, 0) \in \partial V_\mu$. Then, $\varphi = 0$, and hence $c_2 = c_{3,1} = 0$. By applying Theorem 1.1 [1] with $\ell = k = 3$, we obtain, by renaming the coefficient for $t^3$,

$$
|u - a_\mu t^3| \leq C t^4, \quad |u_t - 3 a_\mu t^2| \leq C t^3.
$$

(2.21)

We point out that $a_\mu$ is a constant and, along the inner normal direction of $\partial V_\mu$ at $(r, \theta) = (1, 0)$,

$$
u = \tan \theta, \quad t = f = \frac{1}{\cos \theta} h(\theta).
$$

(2.22)

We write the coefficient of $t^3$ as $a_\mu$ to emphasize its dependence on $\mu$.

Fix an $\alpha \in (2, 3)$ and write $\gamma = 1/(1 + \alpha)$. By Lemma 2.1, we have

$$
f \leq C_{\alpha, \mu} r \left( \sin \frac{\theta}{\mu} \right)^{\frac{1}{1+\alpha}} \text{ in } V_\mu.
$$

Therefore, if $t > 0$ is small, we have

$$
v \geq \tilde{C}_\alpha t^{1+\alpha},
$$

for some positive constant $\tilde{C}_\alpha$. With (2.21), this implies $a_\mu > 0$ and $c_1 \leq u/t^3 \leq c_2$, for $t$ small. Therefore, for $\theta > 0$ small, we have

$$
f \leq Cr \left( \sin \frac{\theta}{\mu} \right)^{\frac{1}{3}},
$$

and

$$
\left| h(\theta) - \frac{1}{\sqrt[3]{a_\mu}} \theta^{\frac{1}{3}} \right| \leq C \theta^{\frac{2}{3}}.
$$

(2.23)

Next, we note

$$
u_t = 3 a_\mu t^2 + O(t^3).
$$

By (2.22) and $u_t = u_t t \theta$, we have

$$
\frac{1}{\cos^2 \theta} = u_t \left( \frac{\sin \theta}{\cos^2 \theta} h(\theta) + \frac{1}{\cos \theta} h'(\theta) \right).
$$
With (2.23), we have
\[ \left| h'(\theta) - \frac{1}{3\sqrt{\alpha_\mu}} \theta^{-\frac{3}{2}} \right| \leq C \theta^{-\frac{1}{2}}. \] (2.24)

We have the desired results. \( \square \)

In the following, we denote by \( h_\mu \) the solution of (2.3) on \((0, \mu \pi)\) to emphasize the dependence on \( \mu \). We now compare \( h_\mu \) for different \( \mu \).

**Lemma 2.4** Let \( \mu_1 \) and \( \mu_2 \) be two distinct constants in \((0, 1)\) and \( h_\mu \) be the solution of (2.3) on \((0, \mu \pi)\), for \( i = 1, 2 \). Then, for \( \mu_1 < \mu_2 < \mu_1 + \delta(\mu_1) \),
\[ h_{\mu_1}(\mu_1 \theta) \leq h_{\mu_2}(\mu_2 \theta) \leq C_{\mu_1, \mu_2} h_{\mu_1}(\mu_1 \theta) \] for any \( \theta \in (0, \pi) \), (2.25)
where \( \delta(\mu_1) \) and \( C_{\mu_1, \mu_2} \) are positive constants given by
\[ \delta(\mu_1) = \left( \frac{1}{8b_{\mu_1}} + 1 \right)^{\frac{1}{2}} - 1 \mu_1, \]
and
\[ C_{\mu_1, \mu_2} = \left( 1 + \frac{b_{\mu_1}}{\mu_1^2} (\mu_2^2 - \mu_1^2) \right)^{\frac{1}{2}}, \] (2.26)
with
\[ b_{\mu_1} = \max \left\{ \frac{81}{128} \sup_{\theta \in (0, \mu_1 \pi)} h_{\mu_1}^4, \sup_{\theta \in (0, \mu_1 \pi)} \left[ \frac{3h_{\mu_1}^2 + 2}{-(h''_{\mu_1} h_{\mu_1}^3 + h_{\mu_1}^4)} \right] \right\}. \] (2.27)

**Proof** By (2.3), we have
\[ h''_{\mu_i} h_{\mu_i} + h''_{\mu_i} h_{\mu_i}^3 + 3h_{\mu_i}^2 + h_{\mu_i}^4 + 2 + 2h_{\mu_i}^2 = 0 \] on \((0, \mu_1 \pi)\). (2.28)
For convenience, we set
\[ \tilde{h}_{\mu_i}(\theta) = h_{\mu_i}(\mu_i \theta) \] for \( \theta \in (0, \pi) \).

Note
\[ \tilde{h}'_{\mu_1}(\theta) = \mu_i h'_{\mu_1}(\mu_i \theta), \quad \tilde{h}''_{\mu_1}(\theta) = \mu_i^2 h''_{\mu_1}(\mu_i \theta) \] for \( \theta \in (0, \pi) \).

Then, (2.28) implies
\[ \frac{1}{\mu_1^2} \tilde{h}''_{\mu_1} \tilde{h}_{\mu_1} + \frac{1}{\mu_1} \tilde{h}'_{\mu_1} \tilde{h}_{\mu_1}^3 + 3\tilde{h}_{\mu_1}^2 + \tilde{h}_{\mu_1}^4 + 2 + 2\tilde{h}_{\mu_1}^2 = 0 \] on \((0, \pi)\).

In view of this equation, we set
\[ \mathcal{F}_{\mu_2}(h) = \frac{1}{\mu_2^2} h'' h + \frac{1}{\mu_2} h'' h^3 + 3h^2 + h^4 + 2 + \frac{2}{\mu_2^2} h^2. \]

We now prove the second inequality in (2.25). We claim, for the positive constant \( C = C_{\mu_1, \mu_2} \) as in (2.26),
\[ \mathcal{F}_{\mu_2}(\tilde{h}_{\mu_1}) \leq 0 \] on \((0, \pi)\). (2.29)
Assuming (2.29), we proceed as follows. Set \( Q \) by
\[
Q(h) = \Delta h - \frac{h_i h_j h_{ij}}{1 + |\nabla h|^2} + \frac{2}{h}.
\]
Comparing \( Q \) and \( \mathcal{F}_{\mu_2} \), we note that (2.29) implies
\[
Q \left( C r h_{\mu_1} \left( \frac{\mu_1}{\mu_2} \theta \right) \right) \leq 0 \quad \text{in} \quad V_{\mu_2}.
\]
As shown in the proof of Theorem 2.2, we can take a sequence \( f_{\mu_2,k} \) such that
\[
Q(f_{\mu_2,k}) = 0 \quad \text{in} \quad V_{\mu_2,k},
\]
and
\[
f_{\mu_2,k} = 0 \quad \text{on} \quad \partial V_{\mu_2,k}.
\]
Then, \( f_{\mu_2,k} \to f_{\mu_2} = r h_{\mu_2}(\theta) \) as \( k \to \infty \) and, by the maximum principle,
\[
f_{\mu_2,k} \leq C r h_{\mu_1} \left( \frac{\mu_1}{\mu_2} \theta \right) \quad \text{in} \quad V_{\mu_2,k}.
\]
Letting \( k \to \infty \), we obtain
\[
r h_{\mu_2}(\theta) = f_{\mu_2} \leq C r h_{\mu_1} \left( \frac{\mu_1}{\mu_2} \theta \right) \quad \text{in} \quad V_{\mu_2}.
\]
This is the desired conclusion.

Now, we proceed to prove (2.29). Note
\[
\mathcal{F}_{\mu_2}(\tilde{C} \tilde{h}_{\mu_1}(\mu_1 \theta)) = \left[ C^2 \left( \left( h_{\mu_1}'' + h_{\mu_1} h_{\mu_1}'' \right) h_{\mu_1} + C^4 \frac{\mu_1}{\mu_2} h_{\mu_1}'' h_{\mu_1}^3 \right) + 3C^2 h_{\mu_1}^2 + C^4 h_{\mu_1}^4 + 2 + 2C^2 \frac{\mu_1}{\mu_2} \right] (\mu_1 \theta).
\]
Set \( \tau = \mu_2/\mu_1 \). By (2.28) with \( i = 1 \), we have
\[
\mathcal{F}_{\mu_2}(\tilde{C} \tilde{h}_{\mu_1}(\mu_1 \theta)) = \frac{C^2}{\tau^2} \left( -(h_{\mu_1}'' h_{\mu_1} + h_{\mu_1}''') \right) \cdot \left\{ -(C^2 - 1) + \left[ 3(\tau^2 - 1) h_{\mu_1}^2 + C^2(\tau^2 - 1) h_{\mu_1}^4 + \frac{2}{C^2} \left( \tau^2 - 1 \right) + \frac{2}{C^2} \right] \right\}.
\]
By (2.3), we have
\[
h_{\mu_1}'' + h_{\mu_1} < 0.
\]
To prove (2.29), it is equivalent to verify
\[
C^2 - 1 \geq \left[ 3(\tau^2 - 1) h_{\mu_1}^2 + C^2(\tau^2 - 1) h_{\mu_1}^4 + \frac{2}{C^2} \left( \tau^2 - 1 \right) + \frac{2}{C^2} \right] \cdot \frac{1}{-(h_{\mu_1}'' h_{\mu_1} + h_{\mu_1}''')}. \tag{2.30}
\]
First, (2.26) implies
\[
C^2 = 1 + b_{\mu_1} (\tau^2 - 1).
\]
By $\mu_2 \in (\mu_1, \mu_1 + \delta(\mu_1))$ and the definition of $\delta(\mu_1)$, we have

$$C^4 \leq \left(\frac{9}{8}\right)^2.$$ 

Then, using the definition of $b_{\mu_1}$ in (2.27), we get

$$C^2(\tau^2 - 1)h_{\mu_1}^4 \leq C^2(\tau^2 - 1) \max(h_{\mu_1}^4) \leq \frac{2b_{\mu_1}(\tau^2 - 1)}{C^2} = -\left(\frac{2}{C^2} - 2\right).$$

Hence,

$$C^2(\tau^2 - 1)h_{\mu_1}^4 + \frac{2}{C^2} - 2 \leq 0.$$ 

Now we verify

$$C^2 - 1 \geq \left(3h_{\mu_1}^2 + \frac{2}{C^2}\right)(\tau^2 - 1) - \frac{1}{-(h''_{\mu_1} h_{\mu_1}^3 + h_{\mu_1}^4)}.$$ 

Note that $2/C^2 \leq 2$ and $-(h''_{\mu_1} h_{\mu_1}^3 + h_{\mu_1}^4)^{-1}$ is bounded. By the definition of $b_{\mu_1}$, we have

$$C^2 - 1 = b_{\mu_1}(\tau^2 - 1) \geq \left(3h_{\mu_1}^2 + \frac{2}{C^2}\right)(\tau^2 - 1) - \frac{1}{-(h''_{\mu_1} h_{\mu_1}^3 + h_{\mu_1}^4)}.$$ 

This ends the proof of (2.29).

Next, we prove the first inequality in (2.25). We aim to verify $F_{\mu_2}(\tilde{h}_{\mu_1}) \geq 0$ and then proceed similarly as in the first part of the present proof. By earlier calculations and (2.28), we have

$$F_{\mu_2}(\tilde{h}_{\mu_1})(\theta) = \left[\frac{\mu_2}{\mu_1} h''_{\mu_1} h_{\mu_1} + \frac{\mu_1^2}{\mu_2^2} h'' h_{\mu_1}^3 + 3h_{\mu_1}^2 + h_{\mu_1}^4 + 2 + \frac{2\mu_1}{\mu_2} h_{\mu_1}^2\right](\mu_1 \theta)$$

$$= \left[\left(1 - \frac{\mu_1^2}{\mu_2^2}\right)(3h_{\mu_1}^2 + h_{\mu_1}^4 + 2)\right](\mu_1 \theta) \geq 0,$$

where we used $\mu_1 < \mu_2$ in the last inequality.

Recall the constant $a_\mu$ in (2.21). By (2.23), we can compute it alternatively by

$$\lim_{\theta \to 0} \frac{h_{\mu}(\theta)}{\sqrt[3]{\theta}} = \frac{1}{\sqrt[3]{\alpha_\mu}}.$$ 

We now compare $a_\mu$ for different $\mu$.

**Lemma 2.5** Let $\mu_1$ and $\mu_2$ be two distinct constants in $(0, 1)$ and $a_{\mu_i}$ be defined as in (2.21) for $\mu = \mu_i$, $i = 1, 2$. Then, for $\mu_1 < \mu_2 < \mu_1 + \delta(\mu_1)$,

$$\frac{\mu_2}{\mu_1} C_{\mu_1,\mu_2} a_{\mu_1} \leq a_{\mu_2} \leq \frac{\mu_2}{\mu_1} a_{\mu_1},$$

where $\delta(\mu_1)$ and $C_{\mu_1,\mu_2}$ are given as in Lemma 2.4.

**Proof** By (2.21), we have

$$\left|\tan \theta - a_{\mu_1} \left(\frac{h_{\mu_1}(\theta)}{\cos \theta}\right)^3\right| \leq C \left(\frac{h_{\mu_1}(\theta)}{\cos \theta}\right)^4.$$
\[
|\tan \theta' - a_{\mu_2} \left( \frac{h_{\mu_2}(\theta')}{\cos \theta'} \right)^3| \leq C \left( \frac{h_{\mu_2}(\theta')}{\cos \theta'} \right)^4.
\]

Take \(\theta' = \frac{\mu_2}{\mu_1} \theta\). Then,

\[
\lim_{\theta \to 0} \frac{a_{\mu_1} h_{\mu_1}^3(\theta)}{a_{\mu_2} h_{\mu_2}^3\left(\frac{\mu_2}{\mu_1} \theta\right)} = \frac{\mu_1}{\mu_2}.
\]

By Lemma 2.4, for any \(\mu_2 \in (\mu_1, \mu_1 + \delta(\mu_1))\), we have

\[
h_{\mu_1}(\theta) \leq h_{\mu_2}\left(\frac{\mu_2}{\mu_1} \theta\right) \leq C_{\mu_1, \mu_2} h_{\mu_1}(\theta).
\]

This implies the desired result by letting \(\theta \to 0\). □

### 3 Local asymptotic expansions

In this section, we prove that asymptotic expansions near singular boundary points up to certain orders are local properties.

**Lemma 3.1** Let \(\Omega\) and \(\Omega_*\) be two convex domains in \(\mathbb{R}^2\) such that, for some \(x_0 \in \partial \Omega\) and \(R_0 > 0\),

\[
\Omega \cap B_{R_0}(x_0) = \Omega_* \cap B_{R_0}(x_0),
\]

and that \(\partial \Omega \cap B_{R_0}(x_0)\) consists of two \(C^{1,1}\)-curves \(\sigma_1, \sigma_2\) intersecting at \(x_0\) with the angle between the tangent lines of \(\sigma_1\) and \(\sigma_2\) given by \(\mu \pi\), for some \(\mu \in (0, 1)\). Suppose that \(f\) and \(f_*\) are solutions of (1.1) for \(\Omega\) and \(\Omega_*\), respectively. Then, for some \(\tau \in (0, 1)\),

\[
|f(x) - f_*(x)| \leq C f(x) \left( \frac{|x - x_0|}{r_0} \right) \tau \quad \text{for any } x \in \Omega \cap B_{r_0}(x_0), \tag{3.1}
\]

where \(r_0\) and \(C\) are positive constants depending only on \(R_0, \mu\) and the \(C^{1,1}\)-norms of \(\sigma_1\) and \(\sigma_2\) in \(B_{R_0}(x_0)\).

**Proof** Set \(v_i\) to be the unit inner normal vector to \(\sigma_i\) at \(x_0\), for \(i = 1, 2\). Note \(\Omega \subseteq V_{x_0}\), where \(V_{x_0}\) is the tangent cone of \(\Omega\) at \(x_0\), since \(\Omega\) is convex. By the maximum principle, we have, for any \(x \in \Omega\),

\[
f(x) \leq f_{\mu}(x) = |x - x_0| h_{\mu}(\theta), \tag{3.2}
\]

where \(f_{\mu}\) is the solution of (1.1) for \(\Omega = V_{x_0}\). For any fixed \(x \in \Omega\) sufficiently close to \(x_0\), we denote by \(d\) the distance from \(x\) to \(\partial \Omega\). We consider two cases.

**Case 1** We first prove (3.1) in the region \(\{d \geq |x - x_0|^{\frac{3}{2}}\}\), with \(\tau = 1/2\).

Since both \(\sigma_1\) and \(\sigma_2\) are \(C^{1,1}\), there exists a positive constant \(R\), depending only on \(R_0, \mu\), and the \(C^{1,1}\)-norms of \(\sigma_1\) and \(\sigma_2\) in \(B_{R_0}(x_0)\), such that

\[
\tilde{\Omega} \equiv B_R(x_0 + R v_1) \cap B_R(x_0 + R v_2) \subseteq \Omega.
\]

Let \(\tilde{f}\) be the solution of (1.1) for \(\tilde{\Omega}\). The maximum principle implies

\[
f \geq \tilde{f} \quad \text{in } \tilde{\Omega}. \tag{3.3}
\]
We note that the tangent cone of $\Omega$ at $x_0$ is also the tangent cone of $\tilde{\Omega}$ at $x_0$. It is easy to see that $\partial B_R(x_0 + Rv_1)$ and $\partial B_R(x_0 + Rv_2)$ intersect at two points, one of which is $x_0$ and the other denoted by $q$. A simple calculation yields

$$|x_0q| = 2R \sin \frac{\mu\pi}{2}.$$ 

Set $L = R \sin \frac{\mu\pi}{2}$. For convenience, we assume

$$x_0 = (-L, 0), \quad q = (L, 0).$$

We consider the map $T_L$ introduced in (2.11). Then, $T_L$ maps the minimal surface $\{(x, \tilde{f}(x))\}$ in $H^3$ to the minimal surface $\{y, \tilde{f}_\mu(y)\}$ in $H^3$ and maps conformally $\Omega$ to an infinite cone $\tilde{V}$, which conjugates $V_{x_0}$. Note

$$\tilde{V} = V_{x_0} + \frac{1}{2}x_0q.$$ 

By (2.11) and (3.2), we have

$$JT_L|_{(x_0, 0)} = \frac{1}{2} I_{3 \times 3},$$

and, for $|x - x_0|$ small,

$$\left| y_1 - \frac{1}{2}(x_1 + L) \right| \leq C|x - x_0|^2, \quad \left| y_2 - \frac{1}{2}x_2 \right| \leq C_2|x - x_0|,$$

and

$$\left| \tilde{f}_\mu(y) - \frac{1}{2}\tilde{f}(x) \right| \leq C\tilde{f}(x)|x - x_0|.$$ 

Lemma 2.3 implies

$$|h'_\mu(\theta)| \leq C_* \delta^{-\frac{3}{2}} \text{ for any } \theta \in \left[-\frac{1}{2}\mu\pi + \frac{\delta\pi}{2}, \frac{1}{2}\mu\pi - \frac{\delta\pi}{2}\right],$$

where $C_*$ is some positive constant depending only on $\mu$. With $\tilde{f}_\mu(y) = |y|h_\mu(\theta)$, we obtain

$$|\nabla \tilde{f}_\mu(\theta)| \leq C\delta^{-\frac{3}{2}} \text{ for any } \theta \in \left[-\frac{1}{2}\mu\pi + \frac{\delta\pi}{2}, \frac{1}{2}\mu\pi - \frac{\delta\pi}{2}\right].$$

If $|x - x_0|$ is small and $d \geq |x - x_0|^{\frac{3}{2}}$, then the angle $\delta$ between $x_0 \overline{x_0}$ and $l_i$ is greater than $|x - x_0|^{\frac{1}{2}}/2$, for $i = 1, 2$, where $l_i$ is the tangent line of $\sigma_i$ at $x_0$. By (2.23) and (2.24), we have

$$\tilde{f}_\mu(y) \geq \tilde{f}_\mu\left(\frac{1}{2}(x_1 + L), \frac{1}{2}x_2\right) - C\delta^{-\frac{3}{2}}|x - x_0|^2$$

$$\geq \frac{1}{2}\tilde{f}_\mu(x_1 + L, x_2) - C|x - x_0|^\frac{5}{2}$$

$$\geq \frac{1}{2}|x - x_0|h_\mu(\theta)(1 - C|x - x_0|^{\frac{1}{2}}).$$

Therefore, in the region $\{d \geq |x - x_0|^{\frac{3}{2}}\}$, we have

$$\tilde{f}(x) \geq |x - x_0|h_\mu(\theta)(1 - C|x - x_0|^{\frac{1}{2}}).$$
By combining with (3.3), we get, for any \( x \) with small \( |x - x_0| \) and \( d \geq |x - x_0|^2 \),

\[
f(x) \geq |x - x_0|h_\mu(\theta)(1 - C|x - x_0|^2).
\]

(3.4)

By (3.2) and (3.4), we obtain, for such \( x \),

\[
|x - x_0|h_\mu(\theta)(1 - C|x - x_0|^2) \leq f(x) \leq |x - x_0|h_\mu(\theta).
\]

(3.5)

Similar estimates also hold for \( f_* \). Hence, for such \( x \),

\[
|f(x) - f_*(x)| \leq C_0 f(x)|x - x_0|^2.
\]

(3.6)

**Case 2** Next, we prove (3.1) in the region \( \{d \leq |x - x_0|^2\} \). Assume \( x_0 = 0 \). Set \( \Omega_0 = \{x \in \Omega \cap B_{r_0}; d(x) \leq |x|^3, \ 1 + f(x)^\alpha - C|x|^\alpha \geq 0\} \),

(3.7)

where we take, for the constant \( C_0 \) as in (3.6),

\[
C = \frac{1}{r_0^\alpha} + 1, \quad \alpha = \frac{1}{100}, \quad \frac{1}{r_0^\alpha} \geq 2C_0.
\]

(3.8)

Define

\[
\tilde{Q}(w) = \frac{w(1 + |\nabla w|^2)}{\Delta w - \frac{w_i w_j w_{ij}}{1 + |\nabla w|^2} + \frac{n}{w}}
\]

\[
= w\Delta w + w(|\nabla w|^2\Delta w - w_i w_j w_{ij}) + n + n|\nabla w|^2.
\]

We claim

\[
\tilde{Q}(1 + f^\alpha - C|x|^\alpha) \geq 0 = \tilde{Q}(f_*) \text{ in } \Omega_0,
\]

(3.9)

and

\[
f_* \geq f(1 + f^\alpha - C|x|^\alpha) \text{ on } \partial\Omega_0.
\]

(3.10)

Then, the maximum principle implies

\[
f_* \geq f(1 + f^\alpha - C|x|^\alpha) \text{ in } \Omega_0.
\]

(3.11)

Note that (3.11) holds automatically if \( 1 + f^\alpha - C|x|^\alpha < 0 \). Hence, for any \( x \in \Omega \cap B_{r_0} \) with \( d \leq |x|^2 \),

\[
f(x) - f_*(x) \leq C f(x)|x|^\alpha.
\]

A similar estimate holds if we exchange \( f(x) \) and \( f_*(x) \).

We proceed to prove (3.9) and (3.10). We first prove (3.10). Since \( \Omega \) is convex, \( \Omega \) is in the tangent cone of \( \Omega \) at 0. By (3.2), we have

\[
f \leq \frac{C_\mu|x|}{\sin \frac{\theta}{\mu}}\left(\frac{\theta}{\mu}\right)^{\frac{1}{3}}.
\]

A simple geometric argument yields

\[
f \leq \frac{C_\mu|x|}{\left(\frac{d + C|x|^2}{|x|}\right)^{\frac{1}{3}}},
\]

and hence,

\[
\frac{f}{|x|} \leq C_\mu|x|^\frac{1}{3} \text{ in } \Omega_0.
\]

(3.12)
Note that, for \( r_0 \) small,
\[
f^{\alpha} \leq \frac{|x|^\alpha}{100}. \tag{3.13}
\]
Therefore, by (3.6) and (3.13), we have (3.10). We note that we need to discuss \( |x| = r_0 \) and \( d = |x|^\frac{3}{2} \) separately.

Now we proceed to prove (3.9). We will do this for general \( n \) under the conditions (3.12) and (3.13). Set
\[
h_0 = 1 + f^{\alpha} - C|x|^{\alpha}, \quad h = 1 + (1 + \alpha)f^{\alpha} - C|x|^{\alpha},
\]
and
\[
g = fh_0.
\]
A lengthy but straightforward calculation yields
\[
\widetilde{Q}(g) = I + II + III,
\]
where
\[
I = f \Delta fh_0 + f(|\nabla f|^2\Delta f - f_i f_j f_{ij})h^3h_0 + n + n|\nabla f|^2h^2,
\]
\[
II = -2C\alpha|x|^{\alpha-2}f^2h^2h_0[(x \cdot \nabla f)\Delta f - x_i f_j f_{ij}]
+ C^2\alpha^2|x|^{\alpha-4}f^3h_0[|x|^2\Delta f - x_i x_j f_{ij}]
+ 2C^2\alpha^2|x|^{\alpha-4}f^2h_0[(x \cdot \nabla f)^2 - |\nabla f|^2]
- C\alpha(n-1)|x|^{\alpha-2}f^2h_0|\nabla f|^2
+ 2C^2\alpha^2(n-1)|x|^{\alpha-4}f^3h_0(x \cdot \nabla f)
- 2C\alpha|x|^{\alpha-2}h_0(x \cdot \nabla f) - 2nC\alpha|x|^{\alpha-2}fh(x \cdot \nabla f)
- C^3\alpha^3(n-1)|x|^{3\alpha-4}f^4h_0 - C\alpha(\alpha - 2 + n)|x|^{\alpha-2}f^2h_0,
\]
and
\[
III = C\alpha(2 - \alpha)|x|^{\alpha-4}f^2h^2h_0(|x|^2|\nabla f|^2 - (x \cdot \nabla f)^2)
+ C^2\alpha^2(1 + \alpha)|x|^{\alpha-4}f^{\alpha+2}h_0(|x|^2|\nabla f|^2 - (x \cdot \nabla f)^2)
+ \alpha(1 + \alpha)f^{\alpha}h_0|\nabla f|^2 + nC^2\alpha^2|x|^{2\alpha-2}h^2.
\]
First, we note that \( III \geq 0 \). Next by \( \widetilde{Q}(f) = 0 \), we have,
\[
f \Delta f + f(|\nabla f|^2\Delta f - f_i f_j f_{ij}) + n + n|\nabla f|^2 = 0. \tag{3.14}
\]
Then,
\[
I = f \Delta fh_0 + f(|\nabla f|^2\Delta f - f_i f_j f_{ij})h_3h_0 + n + n|\nabla f|^2h^2
= -(hh_0 - h^3h_0)f(|\nabla f|^2\Delta f - f_i f_j f_{ij}) + n(h^2 - hh_0)|\nabla f|^2 + n(1 - hh_0)
= -hh_0(1 - h^2)f(|\nabla f|^2\Delta f - f_i f_j f_{ij}) + n\alpha f^{\alpha}h|\nabla f|^2 + n(1 - hh_0).
\]
By (3.13), we have
\[
0 \leq h_0 < h \leq 1 \quad \text{in } \Omega_0.
\]
and
\[ 1 - h^2 \geq \frac{99}{100} C|x|^\alpha, \quad 1 - hh_0 \geq \frac{99}{100} C|x|^\alpha \quad \text{in } \Omega_0. \]

We note that \( \tilde{Q}(w) \) is invariant under orthogonal transforms. Fix a point \( p \in \Omega_0 \) and assume, by a rotation, that \( f_{ij}(p) = 0 \) for \( i \neq j \). In the following, we calculate \( Q(g) \) at \( p \). First,
\[
I = -hh_0(1 - h^2)f \sum_{i=1}^{n}(|\nabla f|^2 - f_i^2)f_{ii} + n\alpha f^\alpha h|\nabla f|^2 + n(1 - hh_0).
\]

Since \( f \) is concave by Theorem 3.1 \[4\], then
\[
I \geq \frac{99}{100} C|x|^\alpha hh_0f \sum_{i=1}^{n} \sum_{k \neq i} f_k^2|f_{ii}| + n\alpha f^\alpha h|\nabla f|^2 + \frac{99}{100} n C|x|^\alpha. \quad (3.15)
\]

We now consider terms in \( II \). For illustrations, we consider the following three terms:
\[
II_1 = -2C\alpha|x|^{\alpha-2}f^2h^2h_0[(x \cdot \nabla f) \Delta f - x_i f_j f_{ij}],
\]
\[
II_2 = -C\alpha(n - 1)|x|^{\alpha-2}f^2h^2h_0|\nabla f|^2,
\]
\[
II_3 = -C^3\alpha^3(n - 1)|x|^{3\alpha-4}f^4h_0.
\]

For \( II_1 \), we write
\[
II_1 = -2C\alpha \left( \frac{f}{|x|} \right)^{1-\alpha} f^{1+\alpha}h^2h_0 \sum_{i=1}^{n} \sum_{k \neq i} x_k \left| \frac{f_k}{|x|} \right| f_k f_{ii}.
\]

By (3.12), we have
\[
C \left( \frac{f}{|x|} \right)^{1-\alpha} \leq \left( \frac{1}{r_0^{\alpha}} + 1 \right) (C_{\mu}|x|^{\frac{1}{2}})^{1-\alpha} \leq \frac{1}{100}.
\]

Hence,
\[
|II_1| \leq \frac{\alpha}{100} f^{1+\alpha}h^2h_0 \sum_{i=1}^{n} \sum_{k \neq i} f_k^2|f_{ii}| + \frac{\alpha}{100} f^\alpha h^2h_0f \sum_{i=1}^{n} |f_{ii}| \leq \frac{\alpha}{100} f^\alpha h^2h_0f \sum_{i=1}^{n} \sum_{k \neq i} f_k^2|f_{ii}| + \frac{n\alpha}{100} f^\alpha h(1 + |\nabla f|^2),
\]

where we used (3.14) with \( f_{ii} \leq 0 \). For \( II_2 \), we write
\[
II_2 = -C\alpha(n - 1) \left( \frac{f}{|x|} \right)^{2-\alpha} f^\alpha h^2h_0|\nabla f|^2.
\]

Then,
\[
|II_2| \leq (n - 1)\alpha \left( \frac{1}{r_0^{\alpha}} + 1 \right) (C_{\mu}|x|^{\frac{1}{2}})^{2-\alpha} f^\alpha h^2h_0|\nabla f|^2 \leq \frac{\alpha}{100} f^\alpha h|\nabla f|^2.
\]

For \( II_3 \), we write
\[
II_3 = -C^3\alpha^3(n - 1) \left( \frac{f}{|x|} \right)^{4-3\alpha} f^{3\alpha}h_0.
\]
Then,
\[ |III| \leq \left( \frac{1}{r_0^2} + 1 \right)^3 \alpha^3 (n-1)(C_\mu|x|^{\frac{1}{\beta}})^{4-3\alpha} f^\alpha h_0 \leq \frac{\alpha}{100} f^\alpha h. \]

We can consider other terms in \( II \) similarly. Therefore, with (3.15), we obtain \( \tilde{Q}(g) \geq 0 \) at \( p \in \Omega_0. \) Since \( p \) is arbitrary, we have (3.9).

The proof of Lemma 3.1 demonstrates that it is much harder to discuss points close to boundary. The choice of the auxiliary functions \( h_0 \) and \( g \) in Case 2 is not trivial. We will encounter a similar situation in the proof of Theorem 1.1 in the next section.

### 4 Singular points with positive curvatures

In this section, we will prove Theorem 1.1. The transformations introduced in (2.11) play a fundamental role. These transformations are isometric on \( \mathbb{H}^3 \) and induce conformal transforms when restricted to the asymptotic infinity of \( \mathbb{H}^3. \) Due to these transformations, we can transform the optimal estimates for (1.1) in infinite cones, derived in Sect. 2, back to intersections of balls.

Let \( \Omega \) be a bounded convex domain in \( \mathbb{H}^2 \) and, for some \( x_0 \in \partial \Omega \) and \( R > 0, \) let \( \partial \Omega \cap B_R(x_0) \) consist of two \( C^{2,\alpha} \)-curves \( \sigma_1 \) and \( \sigma_2 \) intersecting at \( x_0 \) at an angle \( \mu \pi, \) for some constants \( \alpha, \mu \in (0,1). \) Assume the curvature \( \kappa_i \) of \( \sigma_i \) at \( x_0 \) is positive. Set \( R_i = 1/\kappa_i \) and define \( \Omega_{\mu,R_1,R_2} \) as in (1.4), i.e.,
\[ \Omega_{\mu,R_1,R_2} = B_{R_1}(x_0 + R_1v_1) \cap B_{R_2}(x_0 + R_2v_2), \]
where \( v_i \) is the unit inner normal vector of \( \sigma_i \) at \( x_0. \) Any \( x^* \in \Omega_{\mu,R_1,R_2} \) near \( x_0 \) is uniquely determined by a pair of numbers \( d_i > 0 \) with \( i = 1,2, \) the distance from \( x^* \) to \( \partial B_{R_i}(x_0 + R_i v_i). \)

With such a one-to-one correspondence between \( x^* \in \Omega_{\mu,R_1,R_2} \) near \( x_0 \) and \( (d_1,d_2) \) with \( d_1 > 0 \) and \( d_2 > 0 \) small, we rewrite the solution of (1.1) for \( \Omega_{\mu,R_1,R_2} \) as
\[ f_{\mu,R_1,R_2}(d_1,d_2). \tag{4.1} \]

We first prove the following result in this section.

**Theorem 4.1** Let \( \Omega \) be a bounded convex domain in \( \mathbb{H}^2 \) and, for some \( x_0 \in \partial \Omega \) and \( R > 0, \) let \( \partial \Omega \cap B_R(x_0) \) consist of two \( C^{2,\alpha} \)-curves \( \sigma_1 \) and \( \sigma_2 \) intersecting at \( x_0 \) at an angle \( \mu \pi, \) for some constants \( \alpha, \mu \in (0,1), \) such that the curvature \( \kappa_i \) of \( \sigma_i \) at \( x_0 \) is positive. Suppose \( f \in C(\tilde{\Omega}) \cap C^\infty(\Omega) \) is the solution of (1.1) in \( \Omega \) and \( f_{\mu,R_1,R_2} \) is the corresponding solution in \( \Omega_{\mu,R_1,R_2} \) in terms of \( d_1 \) and \( d_2 \) as in (4.1), with \( R_i = 1/\kappa_i. \) Then, for any \( x \in \Omega \) close to \( x_0, \)
\[ |f(x) - f_{\mu,R_1,R_2}(d_1(x),d_2(x))| \leq C f(x)|x - x_0|^{\beta}, \tag{4.2} \]
where \( d_1(x) \) is the distance from \( x \) to \( \sigma_1, \beta \) is a constant in \( (0,\alpha/2), \) and \( C \) is a positive constant depending only on \( R, \mu, \alpha, \) and the \( C^{2,\alpha} \)-norms of \( \sigma_1 \) and \( \sigma_2 \) in \( B_R. \)

**Proof** We assume \( x_0 = 0. \) Fix an \( x \in \Omega \) close to the origin, and write \( d_i \) instead of \( d_i(x) \) for brevity. Without loss of generality, we assume \( d_1 \leq d_2. \) Then,
\[ d_2 \leq |x| \leq C d_2. \]

We consider two cases.
Therefore, without loss of generality, we denote by $V$.

Note that by the mean value theorem and $\theta \geq |x|^{\frac{1}{2}}$, we have

$$|f(x) - f_{\mu}(x)| \leq C f_{\mu}(x) |x|^{\frac{1}{2}}, \quad (4.3)$$

where $f_{\mu}(x)$ is the corresponding solution of (1.1) in the tangent cone $V_{\mu}$ of $\Omega$ at 0. Let $x^*$ be the unique point in $\Omega_{\mu,R_1,R_2}$ determined by $d_1$, $d_2$; namely the distance from $x^*$ to $\partial B_{R_i} (R_i \nu_i)$ is given by $d_i$, $i = 1, 2$. Then,

$$|x^* - x| \leq C |x|^{2+\alpha}.$$

Note that $V_{\mu}$ is also the tangent cone of $\Omega_{\mu,R_1,R_2}$ at 0. Hence,

$$|f_{\mu,R_1,R_2}(d_1, d_2) - f_{\mu}(x^*)| \leq C f_{\mu}(x^*) |x^*|^{\frac{1}{2}}.$$

By the mean value theorem and $\theta \geq |x|^{\frac{1}{2}}$, we get

$$|f_{\mu}(x) - f_{\mu}(x^*)| \leq C |x|^{-\frac{1}{2}} |x|^{2+\alpha} \leq C f_{\mu}(x) |x|^{\frac{1}{2}+\alpha}.$$

Therefore,

$$|f(x) - f_{\mu,R_1,R_2}(d_1, d_2)| \leq C f(x) |x|^{\frac{1}{2}}, \quad (4.4)$$

**Case 1** We first consider the case $d_1 \geq |x|^{\frac{3}{2}}$. By Case 1 in the proof of Lemma 3.1, specifically (3.5), we have

$$|f(x) - f_{\mu}(x)| \leq C f_{\mu}(x) |x|^{\frac{1}{2}},$$

where $f_{\mu}(x)$ is the corresponding solution of (1.1) in the tangent cone $V_{\mu}$ of $\Omega$ at 0. Let $x^*$ be the unique point in $\Omega_{\mu,R_1,R_2}$ determined by $d_1$, $d_2$; namely the distance from $x^*$ to $\partial B_{R_i} (R_i \nu_i)$ is given by $d_i$, $i = 1, 2$. Then,

$$|x^* - x| \leq C |x|^{2+\alpha}.$$

Note that $V_{\mu}$ is also the tangent cone of $\Omega_{\mu,R_1,R_2}$ at 0. Hence,

$$|f_{\mu,R_1,R_2}(d_1, d_2) - f_{\mu}(x^*)| \leq C f_{\mu}(x^*) |x^*|^{\frac{1}{2}}.$$

By the mean value theorem and $\theta \geq |x|^{\frac{1}{2}}$, we get

$$|f_{\mu}(x) - f_{\mu}(x^*)| \leq C |x|^{-\frac{1}{2}} |x|^{2+\alpha} \leq C f_{\mu}(x) |x|^{\frac{1}{2}+\alpha}.$$

Therefore,

$$|f(x) - f_{\mu,R_1,R_2}(d_1, d_2)| \leq C f(x) |x|^{\frac{1}{2}}, \quad (4.4)$$

**Case 2** We consider $d_1 \leq |x|^{\frac{3}{2}}$. Denote by $p_i$ the point on $\sigma_i$ closest to $x$ and by $v_{p_i}$ the unit inner normal vector to $\sigma_i$ at $p_i$. Set, for $i = 1, 2$,

$$\tilde{R}_i = R_i + |x|^2 \nu, \quad \tilde{R}_i = R_i - |x|^2 \nu,$$

and

$$\tilde{\Omega} = \bigcap_{i=1}^{2} B_{\tilde{R}_i}(p_i + \tilde{R}_i v_{p_i}), \quad \tilde{\Omega} = \bigcap_{i=1}^{2} B_{\tilde{R}_i}(p_i + \tilde{R}_i v_{p_i}).$$

For each $i = 1, 2$, both balls $B_{\tilde{R}_i}(p_i + \tilde{R}_i v_{p_i})$ and $B_{\tilde{R}_i}(p_i + \tilde{R}_i v_{p_i})$ have centers on the ray starting from $p_i$ along the normal direction $v_{p_i}$ and are tangent to the curve $\sigma_i$ at $p_i$. Note that $B_{\tilde{R}_i}(p_i + \tilde{R}_i v_{p_i}) \subset B_{\tilde{R}_i}(p_i + \tilde{R}_i v_{p_i})$. The shaded region in Fig. 2 is $\tilde{\Omega}$.

For $|x|$ small, it is straightforward to verify

$$\Omega \cap B_{C_0|x|^\frac{1}{2}} \subset \tilde{\Omega}, \quad \tilde{\Omega} \cap B_{C_0|x|^\frac{1}{2}} \subset \Omega.$$

In fact, for $|x|$ small, $\partial B_{\tilde{R}_1}(p_1 + \tilde{R}_1 v_{p_1})$ and $\partial B_{\tilde{R}_2}(p_2 + \tilde{R}_2 v_{p_2})$ intersect at points $\tilde{p}, \tilde{q}$. Without loss of generality, we denote by $\tilde{p}$ the point closer to 0. Similarly, $\partial B_{\tilde{R}_1}(p_1 + \tilde{R}_1 v_{p_1})$
and \( \partial B_{R_2}(p_2 + \tilde{R}_1 v_{p_2}) \) intersect at points \( \tilde{p}, \tilde{q} \). We denote by \( \tilde{p} \) the point closer to 0. It is easy to verify

\[
|\tilde{p}| \leq C|x|^2, \quad |\tilde{p}| \leq C|x|^2.
\]

Let \( \tilde{f}, \tilde{f}, f', \tilde{f}' \) be the solutions of (1.1) for \( \tilde{\Omega}, \tilde{\Omega}, \Omega \cap B_{C_0|x|^2}, \tilde{\Omega} \cap B_{C_0|x|^2}(\tilde{p}) \), respectively. By Lemma 3.1, we have

\[
|f'(x) - f(x)| \leq Cf(x) \left( \frac{|x|}{C_0|x|^2} \right)^{\tau} \leq Cf(x)|x|^{\frac{\tau}{2}},
\]

and

\[
|\tilde{f}'(x) - \tilde{f}(x)| \leq C \tilde{f}(x) \left( \frac{|x - \tilde{p}|}{C_0|x|^2} \right)^{\tau} \leq C \tilde{f}(x)|x|^{\frac{\tau}{2}},
\]

where we took \( r_0 = C_0|x|^\frac{1}{2} \) in (3.1). By the maximum principle, we have

\[
f'(x) \leq \tilde{f}(x), \quad \tilde{f}'(x) \leq f(x).
\]

Hence,

\[
\tilde{f}(x)(1 - C|x|^{\frac{\tau}{2}}) \leq f(x), \quad f(x)(1 - C|x|^{\frac{\tau}{2}}) \leq \tilde{f}(x).
\]

Therefore,

\[
\tilde{f}(x)(1 - C|x|^{\frac{\tau}{2}}) \leq f(x) \leq \tilde{f}(x)(1 + C|x|^{\frac{\tau}{2}}). \tag{4.5}
\]

Let \( \tilde{\mu}_\tau \) be the opening angle of the tangent cone of \( \tilde{\Omega} \) at \( \tilde{p} \) and \( \tilde{\mu}_\tau \) be the opening angle of the tangent cone of \( \tilde{\Omega} \) at \( \tilde{p} \). It is easy to check that

\[
|\tilde{\mu} - \mu| \leq C|x|^{1+\frac{\tau}{2}}, \quad |\tilde{\mu} - \mu| \leq C|x|^{1+\frac{\tau}{2}}. \tag{4.6}
\]

We note that \( \partial B_{R_1}(R_1v_{p_1}) \) and \( \partial B_{R_2}(R_2v_{p_2}) \) intersect at two points, one of which is 0 and another denoted by \( q \). By a straightforward computation or Lemma 6.1 in [3], we have

\[
|q| = \frac{2R_1R_2 \sin(\pi - \mu_\tau)}{\sqrt{R_1^2 + R_2^2 - 2R_1R_2 \cos(\pi - \mu_\tau)}},
\]

and similar formulas for \( |\tilde{p} - \tilde{q}| \) and \( |\tilde{p} - q| \). Hence,

\[
|\tilde{p} - \tilde{q}| - |q| \leq C|x|^{\frac{\tau}{2}}, \quad |\tilde{p} - q| - |q| \leq C|x|^{\frac{\tau}{2}}.
\]

We also note that in \( \tilde{\Omega} \), the distance of \( x \) to \( \partial B_{\tilde{R}_i}(\tilde{R}_i v_{p_i}) \) is \( d_i \) for \( i = 1, 2 \) and that in \( \tilde{\Omega} \), the distance of \( x \) to \( \partial B_{\tilde{R}_i}(\tilde{R}_i v_{p_i}) \) is \( d_i \) for \( i = 1, 2 \). Hence,

\[
\tilde{f}(x) = f_{\tilde{\mu}, \tilde{R}_1, \tilde{R}_2}(d_1, d_2), \quad \tilde{f}(x) = f_{\tilde{\mu}, \tilde{R}_1, \tilde{R}_2}(d_1, d_2).
\]

Next, we will prove, for some constant \( \gamma \),

\[
|\tilde{f}(x) - f_{\tilde{\mu}, R_1, R_2}(d_1, d_2)| \leq Cf_{\tilde{\mu}, R_1, R_2}(d_1, d_2)|x|^{\gamma}, \tag{4.7}
\]

and

\[
|\tilde{f}(x) - f_{\tilde{\mu}, R_1, R_2}(d_1, d_2)| \leq Cf_{\tilde{\mu}, R_1, R_2}(d_1, d_2)|x|^{\gamma}. \tag{4.8}
\]
We have the desired result by combining (4.5), (4.7), and (4.8).

Set \( \tilde{L} = |\tilde{p} - \tilde{q}|/2 \). By a translation and a rotation, we assume

\[
\tilde{p} = (-\tilde{L}, 0), \quad \tilde{q} = (\tilde{L}, 0).
\]

Then, \( T_{\tilde{L}|_{x_3=0}} \) transforms \( \tilde{\Omega} \) conformally to an infinite cone \( V_{\tilde{L}} \) and \( T_{\tilde{L}} \) transforms the minimal graph \( \{ (\tilde{x}, \tilde{f}(\tilde{x})) \} \) with the asymptotic boundary \( \partial \tilde{\Omega} \) to the minimal graph \( \{ (\tilde{y}, f_{\tilde{L}}(\tilde{y})) \} \) with the asymptotic boundary \( \partial V_{\tilde{L}} \). With \( x = (x_1, x_2) \), set \( y = (y_1, y_2) \) such that \( (y, f_{\tilde{L}}(y)) = T_{\tilde{L}}(x, \tilde{f}(x)) \). For brevity, set

\[
T_{\tilde{L},0} = T_{\tilde{L}|_{x_3=0}}.
\]

We have

\[
\tilde{f}(x) \leq C|x| \left( \frac{|x|^2}{|x|} \right)^{\frac{1}{2}} \leq C|x|^\frac{3}{2}.
\]

Moreover,

\[
J T_{\tilde{L}}|_{(\tilde{p},0)} = \frac{1}{2} I_{3 \times 3},
\]

and

\[
\tilde{f}_\mu(y) = \frac{2\tilde{L}^2 \tilde{f}(x)}{(x_1 - \tilde{L})^2 + x_2^2 + \tilde{f}^2(x)},
\]

\[
(y_1, y_2) = \frac{(x_1 - \tilde{L})^2 + x_2^2}{(x_1 - \tilde{L})^2 + x_2^2 + \tilde{f}^2(x)} T_{\tilde{L},0}(x_1, x_2) - \frac{\tilde{L}(\tilde{f}^2(x), 0)}{(x_1 - \tilde{L})^2 + x_2^2 + \tilde{f}^2(x)}.
\]

Hence,

\[
\left| \tilde{f}_\mu(y) - \frac{1}{2} \tilde{f}(x) \right| \leq C \tilde{f}(x)|x|.
\]

Set

\[
\tilde{t}_1 = T_{\tilde{L},0}(\partial B_{\tilde{R}_1}(p_1 + \tilde{R}_1 v_{p_1}))
\]

and write, for a unique vector \( e_{\tilde{t}_1} \),

\[
\partial V_{\tilde{L}} \cap \tilde{t}_1 = \{ t e_{\tilde{t}_1} | t \geq 0 \}.
\]

By the definition of \( d_1 \), we have

\[
\text{dist}(x, \partial B_{\tilde{R}_1}(p_1 + \tilde{R}_1 v_{p_1})) = d_1.
\]

Hence,

\[
\text{dist}(T_{\tilde{L},0}(x), \tilde{t}_1) = \left( \frac{1}{2} + O(|x|) \right) d_1.
\]

Here and hereafter, we adopt the standard notation \( f = g + O(h) \) meaning \( |f - g| \leq Ch \).

A simple geometric argument yields

\[
\text{dist}(y, \tilde{t}_1) = (1 + O(|x|))\text{dist}(T_{\tilde{L},0}(x), \tilde{t}_1) - \frac{\tilde{L}\tilde{f}^2}{(x_1 - \tilde{L})^2 + x_2^2 + \tilde{f}^2} \sin \angle \left( \frac{\tilde{p}}{|\tilde{p}|}, e_{\tilde{t}_1} \right)
\]

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Hence,

\[ \gamma \text{ where } \langle \cdot, e_{\tilde{f}_1} \rangle = |y| \cos \theta. \]

By (2.21), we have

\[ \text{dist}(y, \tilde{l}_1) = a_{\tilde{\mu}} \left( \frac{f_{\tilde{\mu}}}{\langle y, e_{\tilde{f}_1} \rangle} \right)^3 + O \left( \left( \frac{f_{\tilde{\mu}}}{\langle y, e_{\tilde{f}_1} \rangle} \right)^4 \right). \]

We point out that the left-hand side is simply \( \tan \theta \). The presence of the factor \( \langle y, e_{\tilde{f}_1} \rangle \) in the right-hand side is due to a scaling since (2.21) is expanded at \((r, \theta) = (1, 0)\). Note that

\[ \langle y, e_{\tilde{f}_1} \rangle = |y| \cos \theta = (1 + O(|x|))|y| = \left( \frac{1}{2} + O(|x|) \right)|x|. \]

Hence,

\[ (1 + O(|x|))d_1 = \frac{1}{4R_1} (2 + O(|x|)) \tilde{f}^2 + [a_{\tilde{\mu}} + O(|x|^{1/3})] \tilde{f}^3 \frac{|x|^2}{|x|^2}, \]

where we substituted \( f_{\tilde{\mu}} \) by \( \tilde{f} \). By Lemma 2.5, we have

\[ |a_{\tilde{\mu}} - a_\mu| \leq C|x|^{1+\frac{\alpha}{2}}, \]

and hence

\[ d_1 = \frac{1}{4R_1} \left[ 2 + O(|x|^\frac{\alpha}{2}) \right] \tilde{f}^2(x) + [a_\mu + O(|x|^{1/3})] \tilde{f}^3(x) \frac{|x|^2}{|x|^2}, \]

where we used the relation between \( R_1 \) and \( \tilde{R}_1 \). A similar argument holds for \( f_{\mu,R_1,R_2} \) as defined in \( \Omega_{\mu,R_1,R_2} \). Then,

\[ d_1 = \frac{1}{4R_1} \left[ 2 + O(|x|^{\frac{\alpha}{2}}) \right] f_{\mu,R_1,R_2}(d_1, d_2)^2 + [a_\mu + O(|x|^{1/3})] \frac{f_{\mu,R_1,R_2}(d_1, d_2)^3}{|x|^2}. \]

Hence,

\[ \tilde{f}(x) = f_{\mu,R_1,R_2}(d_1, d_2)(1 + O(|x|^{\gamma})), \]

where \( \gamma = \min\{\frac{\alpha}{2}, \frac{1}{6}\} \). This implies (4.7). We can prove (4.8) similarly. \( \square \)

If \( \alpha \leq \min\{\tau, 1/3\} \) for \( \tau \) in Lemma 3.1, we can take \( \beta = \alpha/2 \) by (4.5), (4.7), and (4.8). Moreover, (4.4) demonstrates that we can take \( \beta = 1/2 \) when the point \( x \) is away from boundary. It is more difficult to prove the desired estimate when \( x \) is close to boundary. Compare with the remark at the end of Sect. 3. Refer also to an estimate for points away from boundary for general dimension in [7].

We are ready to prove Theorem 1.1.

**Proof of Theorem 1.1** We take \( x_0 = 0 \) and adopt the notations from Theorem 4.1 and its proof. For any \( x \) sufficiently small, we define

\[ \mathcal{F}_{[\sigma_1]} x = (d_1(x), d_2(x)), \]

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where \( d_i(x) \) is the signed distance from \( x \) to \( \sigma_i \) with respect to \( v_i \), \( i = 1, 2 \), positive if \( x \) is on the side of \( v_i \) containing \( \Omega \) and negative if on the other side. Refer to [3] for details. We emphasize that \( \mathcal{T}_{\{\sigma_i\}} \) is defined in a full neighborhood of the origin instead of only in \( \Omega \) and that the signed distance is used instead of its absolute value. Then, \( \mathcal{T}_{\{\sigma_i\}} \) is \( C^{2,\alpha} \) near the origin and its Jacobi matrix at the origin is nonsingular by the linear independence of \( v_1 \) and \( v_2 \). Therefore, \( \mathcal{T}_{\{\sigma_i\}} \) is a \( C^{2,\alpha} \)-diffeomorphism in a neighborhood of the origin. We have a similar result for \( \mathcal{T}_{\partial B_{R_i}(R_i v_i)} \), with \( \partial B_{R_i}(R_i v_i) \) replacing \( \sigma_i \), \( i = 1, 2 \). Then, the map \( \mathcal{T} = \mathcal{T}_{\partial B_{R_i}(R_i v_i)}^{-1} \circ \mathcal{T}_{\{\sigma_i\}} \) is a \( C^{2,\alpha} \)-diffeomorphism near the origin and has the property that the signed distance from \( x \) to \( \sigma_i \) is the same as that from \( \mathcal{T} x \) to \( \partial B_{R_i}(R_i v_i) \), for \( i = 1, 2 \). Therefore, Theorem 1.1 follows from Theorem 4.1.

\[ \square \]

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