Optimal Real-time Bidding Policies for Contract Fulfillment in Second Price Auctions

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Abstract

We study a real-time bidding problem resulting from a set of contractual obligations stipulating that a firm win a specified number of heterogeneous impressions or ad placements over a defined duration in a real-time auction. The contracts specify item targeting criteria (which may be overlapping), and a supply requirement. Using the Pontryagin maximum principle, we show that the resulting continuous time and time inhomogenous planning problem can be reduced into a finite dimensional convex optimization problem and solved to optimality. In addition, we provide algorithms to update the bidding plan over time via a receding horizon. Finally, we provide numerical results based on real data and show a connection to production-transportation problems.

keywords Computational Advertising; Realtime Bidding; Optimal Control; Auction Theory; Second Price Auction; Production Transportation Problem

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*Relevant code to be made available at github.com/RJTK
1 Introduction

Online advertising constitutes a significant part of today’s advertising landscape. The total amount of money spent directly on internet advertising (the largest advertising segment, far surpassing competitors like TV and print) in 2018, according to The Interactive Advertising Bureau [14] exceeded $100b, and display advertising alone (as opposed to sponsored search) accounted for roughly 40% of this total. Moreover, year on year growth rates remain extremely high: revenue in 2018 grew by 21.8% over 2017, and compounded annual growth rate between 2012-2018 exceeds 45%.

This advertising market consists primarily of two groups of players: publishers (e.g., website operators, apps) and advertisers. The goal of the advertisers in this setting is to get their messaging in front of the visitors to publisher websites or app users, generally for the purposes of generating either brand awareness in the long-term or immediate purchasing decisions in the short-term [7]. Matching ads to users is facilitated by ad exchanges like Google AdX [20], which allow publishers seeking to sell space on their website to solicit requests from advertisers. The exchanges implement a clearing mechanism in the form of an auction.

This paper focuses on a class of on-line advertising problems known as real-time bidding (RTB) auctions [7]. RTB constitutes about 35% of the entire display advertising market, with the remainder of the market consisting of traditional fixed advertising contracts which require a publisher to display an advertiser’s content for a pre-negotiated period of time [6]. RTB is characterized by auctions which occur every time a user visits a web page or opens an app, the “item” up for sale in the auction is an ad space that the winner of the auction fills with their advertising content. These items are also referred to as “impressions”. Bidders that lose the auction for a particular item need to wait for the next opportunity. The entire process beginning with the arrival of a user, to the decision about the winner of the auction and the display of the winner’s content, takes place in around 100ms.

Every auction taking place in RTB is sealed bid, and each bidder submits only a single bid. Moreover, bidding data is censored: bidders are not informed about the bid that won the auction, unless they are themselves the winner. For the winning bidder (which is always the highest bidder) their actual payment depends on the type of auction the exchange is running. The two most prevalent basic auction types are first price auctions wherein the winner pays what they bid, and second price, also known as Vickrey auctions [25], wherein the winner pays the second highest bid. The true mechanism in practice often incorporates both types with a first price auction below a (“soft”) floor [34] and a second price auction otherwise.

In practice, advertisers use the services of aggregators called Demand Side Platforms (DSPs) (see e.g. [27] for additional information) which participate in RTB on the advertiser’s behalf. It is common practice for a DSP to enter a contract with an advertiser which stipulates an up-front fee be paid for the guarantee that a minimum number of ads be displayed to targeted segments of the population (age, sex, location, other preferences, etc.). The segments are referred to as the targeting criteria and each such contract is referred to as a campaign. From the perspective of the DSP, the optimal bids should minimize the total cost to obtain the required number of impressions.

The problem that we address in this paper are optimal (i.e., cost minimizing) bidding strategies for DSPs to fulfill their contracts. Typically, DSPs handle hundreds of campaigns simultaneously and the targeting criteria of the campaigns may overlap. This induces a problem where on every bid request the DSP receives from the ad exchange, they must decide, based on the characteristics of the item, what price to bid and which contract the impression, if it is won, it should be allocated towards fulfilling. An important characteristic of our perspective is that it does not involve item valuations: the DSPs we consider seeks to fulfill acquisition contracts, not to maximize their valuation of items.
won. Once a contract has been agreed to, the DSP must fulfill its obligations.

1.1 Literature Review

Problems of optimal bidding have been addressed at different levels of generality and from various perspectives. Early works addressed the problem in the context of a single campaign and budget constraints \[13\] on an infinite horizon. They assumed that the prices of impressions arrive as an i.i.d. process from an unknown distribution and the goal was to maximize the utility (or valuation) subject to constraints on the budget for the ergodic and discounted cost criteria. The optimal strategy (which is stationary in this case) is to bid according to a shaded (i.e. reduced) item valuation where the shading factor arises from a Lagrange multiplier associated with the budget constraint and depends on the unknown distribution of the prices. In \[15\], they address the problem of determining the optimal shading factor by using a stochastic approximation algorithm in an i.i.d. price setting. The case when there are a large number of bidders was studied in \[2\] where a mean-field approach based on independence of the bidders was assumed. The optimal structure of the bids is similar to \[13\].

In \[36\] they consider a problem with many campaigns but with identical targeting criteria with the goal being to maximize the number of impressions subject to budget and risk constraints where risk is taken as a variance constraint on the total number of items. In \[38\] they consider the problem with multiple campaigns with non-overlapping targeting criteria in the distinct but related context of sponsored search.

Many other algorithms have been deployed on the problem of optimal bidding, including classical feedback control in the work of \[35, 16\] where they seek to track certain keep performance indicators and \[4, 12, 31\] which utilize the Markov Decision framework.

In the recent work of \[23\], the optimal contract management problem with multiple campaigns and overlapping targeting criteria was studied for the static case of optimizing over one duration. This is equivalent to apportioning equal impression requirements to each duration in the term of a contract. One of the key insights was the need for a supply curve for each targeting criterion (see also \[17\]). The supply curve is simply a right continuous increasing function indicating the average number of impressions (or estimate thereof) that will be won given a particular bid. We discuss this issue later.

An important attendant problem is that of estimating and forecasting market prices, referred to as “bid landscape forecasting” \[9, 32, 39, 28, 11\] which ultimately falls into the domain of statistics and machine learning applications. This important problem is not the focus of this paper.

1.2 Contributions

We formulate the contract fulfillment problem faced by a DSP as a continuous time optimal control problem and provide algorithms specifying how to bid on any given bid request. We show that the most general case can be reduced to solving the time homogeneous problem and that this solution can be computed through the solution of a convex optimization problem. This is a direct generalization of \[23\] to account for differing contract deadlines and time-inhomogenous supply curves. Moreover, we also show that the problem of \[23\] can be addressed via convex optimization.

We show that the time dependent problem addressed in this paper can be solved via the application of optimal control theory, which allows for bids at earlier periods to be appropriately readjusted to account for future anticipated changes in supply or price. In order to account for the
moment-to-moment adaptation, our solution may serve as a set point for the classical regulators studied for example by [16, 35]. Moreover, while our basic formulation doesn’t endogenously account for stochasticity in the environment, this is accounted for via a receding horizon [5].

Our formulation can also be seen as analogous to a continuous time Production-Transportation problem [18, 22] with transportation costs taking values of either 0 or \(\infty\).

1.3 Outline

We begin by discussing a simple market model and how the idea of a supply curve (Section 2) naturally arises and serves essentially as an information state for the problem. In Section 2.4 we introduce the relevant cost functions and auction mechanism. We focus is on second price auctions though many of our results can apply to some more general auctions that will be explored elsewhere.

Section 3 formally introduces our problem and Section 3.1 discusses segmenting the market according to the needs of the contract management problem.

Section 4 examines a simple special case and illustrates the receding horizon method through an analytically tractable example. Section 5 provides a concrete formulation as a continuous time optimal control problem, and rigorously establishes the existence and optimality of solutions to our problem along with the basic properties which enable the transformation into a finite convex optimization problem.

Our final result in Theorem 6.1 is that the entire continuous time portfolio management problem can be solved to optimality via a finite convex optimization problem.

In Section 7 we illustrate our methods through application on publicly available IPinYou dataset [19, 37], and compare against a solution obtained without direct consideration of time inhomogeneity. The cost savings by considering the dynamics is about 10%.

2 Market Model and Supply Curves

In this section we outline a simple market model that will lead to an understanding of the properties of what we will call the supply curve \(W_j(x,t)\), indicating the average instantaneous (at time \(t\)) rate of items accumulated by bidding \(x\) on every item of type \(j\). This function will describe what is analogous to an information state for a bidder participating in the market, i.e., \(W_j(x,t)\) encodes all of the information necessary for them to make informed decisions about bidding. The function \(W_j(x,t)\) naturally arises from the actions taken by bidders who are present in the market at time \(t\).

Since empirical data [33, 37] demonstrates clear cyclic and time varying behaviour in sale prices and item volumes, we explicitly consider time dependence in \(W_j(x,t)\) to capture these dynamics and allow bidders to plan for the future states of the market.

2.1 Market Model

Suppose that we have a generic real-time auction exchange dealing in heterogeneous items of type \(j \in [M]\). It may be that separate type \(j\) items are still distinguishable by participants in reality, but we do not model any intra-type distinctions. At any given time, there is a large group of bidders (or “participants”) who participate in the auction exchange, this group of bidders is subject to change.

\(^1\)Throughout, we use the notation \([N] = \{1, \ldots, N\}\)
over time. The items arrive (one by one) over time to the exchange, with each arrival triggering a bid to be submit by some (but not necessarily all) of the currently present participants.

Suppose that at fixed time $t$ there are $N(t)$ bidders ($i = 1, 2, \ldots, N(t)$) present at the auction exchange, and that for any bidder $i$, their behaviour is described by $M$ (bid, rate) tuples: $\{(b_{ij}, r_{ij})\}_{j=1}^{M}$. The number of participants $N$ in the model outline above, this function is given explicitly by $\phi(r_{ij})$. The quantity $r_{ij} \in (0, 1)$ indicates that if an item of type $j$ arrives, bidder $i$ will bid on it with probability $r_{ij}$, independently of all other bidders; possible interpretations being that items of type $j$ match $i$’s interests with probability $r_{ij}$, or that they only bid on a fraction of each item type in order to spread out their budget over time. The bid placed by $i$ for items of type $j$ is given by $b_{ij} \in \mathbb{R}_+$. The entire state of the market at time $t$ including $N(t)$ and $b, r$ will be denoted $\mathbf{M}_t$.

Since the winner of the auction is the individual submitting the largest bid, we denote by $\mathbf{W}_j(t)$ the price process which at time $t$ takes the value of the largest bid that would be submitted by the participants present in the auction exchange if at time $t$ an item arrived to be bid upon. The probability that an exogenous bidder would win an item of type $j$ arriving at the instant $t$ if they placed the bid $x$ is therefore a cumulative distribution function $\Pr\{\mathbf{W}_j(t) \leq x\}$. We can determine the exact form of this c.d.f. when the market state is fixed as follows:

**Proposition 2.1 (Win Probability Properties)** If at the fixed time $t$ the market $\mathbf{M}_t$ consists of $N$ participants, namely $\{(r_i, b_i)\}_{i=1}^{N}$, and suppose each $b_i$ is distinct, then the probability of an exogenous participant (denoted by $0$) winning an item with the bid $x$ and where ties are always settled in $0$’s favour, is a cumulative distribution function denoted by $\mathbf{W}_j^M(x, t) \overset{\Delta}{=} \Pr\{\mathbf{W}_j(t) \leq x \mid \mathbf{M}_t\}$. In the model outline above, this function is given explicitly by

$$W_j^M(x, t) = \exp\left(-\sum_{i=1}^{N} \phi(r_{ij})1_{(x, \infty)}(b_{ij})\right)1_{\mathbb{R}_+}(x), \quad (2.1)$$

where $\phi(r) \overset{\Delta}{=} -\ln(1 - r)$.

The proof can be found in the Appendix, see [A]

### 2.2 Demand Side

The number of participants $N(t)$ at any time is determined from the stochastic dynamics of how participants arrive and the time spent in the bidding process. Suppose participants arrive as a Poisson process of rate $\rho$ and let us assume that the amount of time in the bidding process is of unit duration (100ms in reality). Then, in equilibrium, the number of participants is given by the distribution of an $M/G/\infty$ model that is a Poisson distribution with parameter $\rho$ denoted by $\Pr(\rho)$, see for example [1].

Therefore, suppose that at an item arrival instant $t$, we have $N(t) \sim \Pr(\rho)$ bidders, where the parameter $\rho > 0$. Moreover, suppose that the (bid, rate) parameters of participants are drawn independently from distributions $F_{B_j}$ and $F_{R_j}$ respectively, that is, at any $t$, the parameters of the $N(t)$ participants are specified by $b_{ij} \overset{i.i.d.}{\sim} F_{B_j}, r_{ij} \overset{i.i.d.}{\sim} F_{R_j}$. This is sufficient to derive and motivate a supply curve $W_j(x, t)$.

We consider the average win probability $W_j^M(x) \overset{\Delta}{=} \mathbb{E}[W_j^M(x, t)]$ that can be characterized as follows where we suppress the index $j$. 
Proposition 2.2 (Expected Win Rate) If the bid distribution $F_B$ admits a probability density, and $N(t) \sim \text{Po}(\rho)$, then we have

$$W^{ss}(x) \triangleq \mathbb{E}[W^M(x, t)] = e^{-\rho_B(x)\mathbb{E}[r]}$$

(2.2)

where $\rho_B(x) \triangleq \rho(1 - F_B(x))$ and $\mathbb{E}[r] = \int_0^\infty (1 - F_R(x))dx$. Hence, $W^{ss}(x)$ is a cumulative distribution function.

Proof: See Appendix, see A

2.3 Supply Side

So far, we have focused on the demand side (i.e., the bidders) of the auction, leading to the win probability (for type $j$) function $W^M_j(x, t)$ and its average $W^{ss}_j(x)$ interpreted as the mean in steady state.

Turning attention now to the supply side (i.e. arrival of items), let us consider an arrival point process $A_j(t)$ (independent of $M_t$) with a time dependent intensity $\lambda_j(t)$. In the case of internet advertising, where item arrivals correspond to users visiting a web page, it is natural for the arrival rate to be time dependent, and can naturally be expected to exhibit daily and weekly cycles.

In this model then, if we are given a deterministic function $x : [0, T] \rightarrow \mathbb{R}_+$, the expected number of wins for an agent bidding according to $x$ can be calculated simply via Campbell’s formula (see [21] for example)

$$\mathbb{E} \int_0^T 1_{\{W_j(t) \leq x(t)\}} dA_j(t) = \int_0^T \lambda_j(t)W^{ss}_j(x(t))dt.$$  

(2.3)

It is in this sense that $W_j(x, t) = \lambda_j(t)W^{ss}_j(x)$ is the average number of items won instantaneously at time $t$ given a bid of $x(t)$. A time varying average win probability function $W_j(x, t)$ thus arises naturally in RTB, and integrating this function results naturally in the average number of items won with the fixed bid path $x(t)$. We will summarize these ideas later in Definition 2.1.

2.4 Cost Functions and Auctions

Before formulating our main problem (Section 5) we need to define a cost function. These functions arise most naturally from the rules of the auction, which will always be sealed-bid second price auctions wherein the item is sold immediately after a single round of bidding. Extensions to more general cost functions is possible but subtle and will be revisited in future work.

We will denote by $f(x, t)$ the (estimate of the) expected cost of bidding $x$ on an item arriving at time $t$. To explain the second price auction mechanism suppose that the bids among $N$ participants are denoted $b_1, ..., b_N$. Bidder $i$ will win the auction with bid $b_i$ if $b_i \geq 0$ and $b_i > \max_{j \neq i} b_j$, where we can break ties randomly. If $i$ is the winner, they pay $\max_{j \neq i} b_j$, which is in general less than their own bid.

From here we can see that if for a particular bidder, the maximum of competing bids is given by the random variable $Y \sim F_Y$, and they bid the value $x \geq 0$, their expected payment is

$$\mathbb{E}[Y 1_{0 < Y < x}] = \int_0^x u dF_Y(u).$$
In our context, the distribution of “Y” at time t for items of type j is given by \( W_j(x, t) \), modulo the supply rate normalization in \( W_j \). Therefore, the expected cost of bidding \( x \) on items of type \( j \) is, instantaneously at time \( t \),

\[
f_j(x, t) = 1_{\mathbb{R}_+}(x) \int_0^x u dW_j(u, t). \tag{2.4}
\]

We include \( 1_{\mathbb{R}_+}(x) \) since we will allow the domain of \( x \) to be all of \( \mathbb{R} \).

### 2.5 Randomized Bidding

Since there is a finite number of bidders participating, the nature of the auction mechanism makes it very natural for \( W_j(x, t) \) to exhibit discontinuous jumps (w.r.t. \( x \)). Such discontinuities are observed in real data, see e.g., [12, 17, 23]. Discontinuities may arise even in estimated supply curves e.g. if it is desirable to estimate directly the location of jumps in market prices, or if the estimates of \( W_j \) are carried out via an histogram, which is naturally discontinuous. However, for the purposes of deriving bidding strategies, it is desirable to work with continuous supply curves. To this end we will establish a means of implementing smooth approximations to discontinuous supply curves via randomized bidding. See also [12, 17] for earlier applications of this idea. An alternative approach is given in [23] wherein the authors work more directly with the discontinuous supply curves and establish a different type of randomization scheme which does not attempt to smooth out the entire curve.

In practice, randomization has the additional benefit of “hedging” against incorrectly estimating the locations of important jump discontinuities, as well as providing a parameter (the amount of bid noise) to probe the exploration-exploitation frontier if supply curve estimation is to take place simultaneously with bidding.

By choosing a parameter \( \sigma^2 > 0 \), define the function (suppressing the subscript \( j \)) \( \overline{W}_\sigma(x, t) = \mathbb{E}W(x + \sigma \mathcal{X}, t) \), where \( \mathcal{X} \sim \mathcal{N}(0, 1) \). A DSP can implement the function \( \overline{W}_\sigma \), which is a \( C_\infty \) function w.r.t. \( x \) (this follows directly from Leibniz’s integral formula), by using randomized bids: instead of placing the nominal bid \( x \), sample a \( \mathcal{N}(0, 1) \) variable \( \mathcal{X} \) and then place the bid \( x + \sigma \mathcal{X} \). This approximation has the secondary benefit of ensuring that \( \overline{W}_\sigma \) is strictly monotone increasing (hence invertible). We point out that we will generally have \( \overline{W}_\sigma(x, t) \neq W(x, t) \), but that as long as \( \sigma \) is small, the difference is slight. We formalize these notions in the following proposition, a complete proof is relegated to the Appendix.

**Proposition 2.3 (Smooth and Monotone Supply Curve)** Let \( W(x, t), t \geq 0 \) be \( L \)-Lipschitz in \( x \) at all but at most \( n \) points (\( n \) that does not depend on \( t \)), and that \( \sup W(x, t) = B(t) < \infty \).

Then, for any \( \epsilon > 0 \) and any compact set \( I \subset \mathbb{R} \), there exists \( \sigma^2 > 0 \) such that \( \int_I |\overline{W}_\sigma(x, t) - W(x, t)| dx < \epsilon \) and if \( W \) does not contain any jumps, such that \( ||\overline{W}_\sigma(\cdot, t) - W(\cdot, t)||_{\infty} < \epsilon \).

Moreover, \( \overline{W}_\sigma \) is a \( C_\infty \), strictly monotone increasing function and \( x \overline{W}_\sigma(x) \to 0 \) as \( x \to -\infty \). In particular, \( \text{cl range} \overline{W}_\sigma(\cdot, t) = [0, B(t)] \).

**Proof:** See the Appendix, see [A].

The effect of randomization should also be accounted for in the cost function. Formally, the true average cost is given by \( \overline{f}_\sigma(x) \equiv \mathbb{E}f(x + \sigma \mathcal{X}) \). However, for small \( \sigma \), we can make analogous statements as in Proposition 2.3, i.e., that \( \overline{f}_\sigma(x) \approx 1_{\mathbb{R}_+}(x) \int_0^x u \overline{W}_\sigma'(u) du \), which justifies the use of \( \overline{W}_\sigma \) as if it were the true supply curve for a second price auction.
Henceforth, we will posit existence of estimated supply curves and assume that they are smooth and strictly monotone, keeping in mind that these properties can be obtained from much less well behaved curves through randomized bidding. We can summarize the previous notions as follows:

**Definition 2.1 (Bid Path, Supply Curve, Cost Curve)** Let $x(t)$ denote the bid at time $t$. We refer to the sample-path $\{x(t)\}_{t \in [0,T]}$ as the bid path where $[0,T]$ denotes the duration of the contract. The bid path $x(.)$ thus represents a bidding policy.

For a particular item of type $j$, the supply curve $W_j(x,t)$ is the function such that for a fixed bidding path $x(t)$, the expected number of items won over the period $[0,T]$ is

$$\int_0^T W_j(x(t),t) dt.$$ 

We assume that the range of $W_j$ satisfies cl range $W_j(\cdot,t) = [0,B_j(t)]$ for some $B_j(t) < \infty$, and for every $t \geq 0$, $W(x,t)$ is strictly monotone increasing and twice differentiable in $x$ (recall Proposition 2.3). The derivative of the function $x \mapsto W_j(x,t)$ will be denoted $W_j'(x,t)$, as we have no need to refer to derivatives w.r.t. $t$. Finally, $W_j(x,t) \geq 0$ and $xW_j(x,t) \to 0$ as $x \to -\infty$.

The function $f(x,t)$ is the average cost of bidding $x$ at time $t$ and satisfies $f(x,t) = 0$ for any $x \leq 0$. It is continuously differentiable and strictly monotone for $x \geq 0$.

### 2.6 The Cost of Acquisition

Having defined a cost function $f_j(x,t)$ and a supply curve $W_j(x,t)$, it is natural to ask: What is the cost of acquiring a given supply $s_j$ of type $j$? Using the monotonicity of $W_j$, the lowest bid necessary to obtain $s_j$ supply is $x_j = \min\{x \in \mathbb{R} \mid W_j(x) \geq s_j\}$. After applying randomized bidding, since $W_j$ is then strictly monotone, this is simply $x_j = W_j^{-1}(s_j,t)$, where the inverse is w.r.t. $x$. The cost of acquiring $s_j$ units of type $j$ instantaneously at time $t$, which we will denote by $\Lambda_j(s_j,t)$, is therefore

$$\Lambda_j(s_j,t) \triangleq f_j(W_j^{-1}(s_j),t). \quad (2.5)$$

For second price auctions, this function turns out to be convex. We suppress the $t$ argument and the index $j$ in the following.

**Proposition 2.4 (Convex Acquisition Costs)** In a second price auction, the acquisition function $\Lambda(s) = f \circ W^{-1}(s)$ is convex. Moreover, $\Lambda(0) = 0$, and $\Lambda'(s) = 1[s \geq W(0)]W^{-1}(s)$.

**Proof:**

Consider the integral representation of the cost function in Equation (2.4) and make the substitution $y = W(u) \implies dy = W'(u)du$ to obtain

$$\Lambda(x) = 1[W^{-1}(x) \geq 0] \int_0^{W^{-1}(x)} uW'(u)du = 1[x \geq W(0)] \int_{W(0)}^x W^{-1}(y)dy.$$ 

It is seen here $\Lambda(x) = 0$ for $x \leq W(0)$, in particular, $\Lambda(0) = 0$. Moreover, this function is differentiable on $x > W(0)$ and $\Lambda'(x) = W^{-1}(x)$. On $x < W(0)$, we have $\Lambda'(x) = 0$. Since
$W^{-1}(W(0)) = 0$, $\Lambda$ is continuously differentiable on $\mathbb{R}$. Since $W$ is monotone, so is $W^{-1}$, and it is well known that functions with monotone derivatives are convex.

An analogous proof can be given for non-smooth (even discontinuous) supply curves to show that $\Lambda_j$ is convex and differentiable by using the generalized inverse

$$W^{-1}(x) = \min\{y \in \mathbb{R} \mid W(y) \geq x\},$$

and the substitution rule for the Lebesgue-Stieltjes integral (see e.g. [10]).

**Remark 2.1 (General Supply Curves)** The work of [23] studies the second price auction in the case where $W$ may be any right-continuous and non-decreasing supply curve. Proposition [2.4] does not rely on the smoothness of $W$ and in fact applies to this more general case.

### 3 Time Constrained Impression Contracts

We consider a DSP tasked with managing contracts (or “campaigns”) of the form $C = (T, C, S)$, where $T \in \mathbb{R}^+$ is a time deadline, $C \in \mathbb{N}$ is the number of items that must be won in auction by the deadline, and $S$ is a set of targeting criteria specifying the characteristics of impressions that can be used to satisfy the terms of the contract.

Suppose that we have a finite set $\Omega$ of possible impression characteristics (i.e. sex, age, location, publisher, etc.), where we note that our DSP is essentially free to construct this set. For instance, we may have $\Omega = \{0, 1\}^L$ where each dimension indicates the presence or absence of a particular characteristic. We will allow for any set $S \subseteq \Omega$ which satisfies some natural consistency rules (e.g., $S = \{\text{sex} = \text{female}, \text{sex} \neq \text{female}\}$ is inconsistent) to be associated with a contract. Then, any impression with characteristics $I \in \Omega$ won in an RTB auction is allowed to count towards satisfying the contract if $I \in S$. That is, if $I$ matches the type specification given by $S$.

#### 3.1 Target Criteria Decomposition

In this section, we discuss a target set partitioning important for the formulation of our main problem as in [23].

Suppose that we have $N$ contracts $\{C_i\}_{i=1}^N$, where the targeting sets $S \triangleq \{S_i\}_{i=1}^N$ may be overlapping.

It is clear that there exists some minimal $M$ and disjoint sets $R_j$ such that:

$$\bigcup_{j=1}^M R_j = \bigcup_{i=1}^N S_i. \tag{3.6}$$

Moreover for each $i \in [N]$ there exists a unique set $A_i \subseteq [M]$ such that:

$$\bigcup_{j \in A_i} R_j = S_i. \tag{3.7}$$

This in turn also induces a set $B_j \subseteq [N]$ such that

$$i \in B_j \iff R_j \subseteq S_i.$$
And, moreover,

\[ j \in A_i \iff i \in B_j. \]

The interpretation is that \( R_j \) represents a targeting criterion while \( B_j \) is the set of campaigns that require impressions satisfying criteria \( R_j \). With this decomposition the supply curve \( W_j(t, x) \) will denote the supply curve for the impressions that match \( R_j \). An example of such a partition is provided by Figure 1.

Figure 1: Set Partitioning Example

An example of set partitioning, best viewed in colour. In this case, \( S = \{S_1, S_2, S_3\} \), \( M = 6 \) and \( R = \{R_m\}_{m=1}^M \) contains subsets such that \( R \) is a partition of \( \bigcup S \). Moreover, for any \( S_i \in S \) we have some \( A_i \subseteq [M] \) such that \( \bigcup_{j \in A_i} R_j = S_i \). For example, \( S_2 = R_3 \cup R_5 \cup R_6 \). That is, \( A_2 = \{3, 5, 6\} \). Likewise, we have sets \( B_j \) such that \( j \in A_i \iff i \in B_j \). For example, \( B_1 = \{1\} \) and \( B_6 = \{1, 2, 3\} \).

### 4 Optimal Management of Impression Contracts: Preliminaries

In this section we will formulate optimal control problems for fulfilling impression contracts. We begin with the simplest case where there is a single type of item (we don’t distinguish between bid requests, or, \( M = 1 \)) and a single contract (i.e., \( N = 1 \)) stipulating that we must obtain \( C \) impressions by time \( T \). We will start with a time-homogeneous problem wherein \( W(x,t) = W(x) \) for every \( t \), and similarly for the cost. The simplicity ensures that this problem has what is essentially a closed form solution. We use this example to illustrate methods for revising the bid over time as more information becomes available via a receding horizon.

**Remark 4.1 (Notation)** For the optimization problems presented in this paper, we follow a convention for constraints where indices (e.g., \( i,j,t \)) that do not appear explicitly in summation or integration indicate that there is one constraint for every combination of valid indices. For example,

\[ \gamma_{ij}(t) \geq 0; \forall i \in [N], \forall j \in A_i, \forall t \in [0, T], \]

will be written simply as \( \gamma_{ij}(t) \geq 0 \). Some attention must also be given to the combinations of indices which are valid, e.g., we do not refer to any \( \gamma_{ij} \) for which \( j \notin A_i \).
4.1 A Single Item Type \((M = 1)\)

We begin with the case where we are obliged simply to fulfill a single contract \((T, C, S)\). Firstly, suppose that the structure of \(S\) is simple enough that we are satisfied with the estimate of a single supply curve \(W(x, t)\), i.e., all items satisfying \(S\) are estimated as having the same \(W\). Furthermore, suppose for now that the supply curve does not depend on \(t\), i.e., \(W(x, t) = \lambda W(x)\). We will see later that this assumption is not restrictive. The function here \(W(x)\) is now a bona fide cumulative distribution function, and \(\lambda > 0\) is the average rate of supply. Our problem is then

\[
\begin{align*}
\text{minimize } & 
\int_0^T f(x(t))dt \\
\text{subject to } & 
\lambda \int_0^T W(x(t))dt \geq C.
\end{align*}
\]

(4.8)

Making the substitution \(s(t) = W(x(t))\) we can rewrite this as a convex problem:

\[
\begin{align*}
\text{minimize } & 
\int_0^T \Lambda(s(t))dt \\
\text{subject to } & 
\lambda \int_0^T s(t)dt \geq C,
\end{align*}
\]

(4.9)

where we recall that \(\Lambda = f \circ W^{-1}\). This is a classical calculus of variations problem with integral constraints (see [8, Theorem 14.12]). We have the Lagrangian with \(\mu \geq 0\):

\[
\mathcal{L}(s, \mu) = \Lambda(s) - \mu [\lambda s - C].
\]

Any \(C_1\) solution \(s(t)\) necessarily satisfies the Euler-Lagrange equation

\[
\Lambda'(s(t)) = \mu \lambda \forall t \in [0, T].
\]

Since \(\Lambda' = W^{-1}\), it is necessary that \(x(t) = \mu \lambda\). That is, \(x(t)\) must be a constant.

Substituting this into the cost and impression constraints given by (4.8), we see that since \(f(x)\) and \(W(x)\) are monotone increasing functions, the optimal \(x\) is the smallest feasible bid:

\[
x^*(C, T) = \begin{cases} W^{-1}\left(\frac{C}{\lambda T}\right) & \frac{C}{\lambda T} < 1 \\ \lim_{x \to 1} W^{-1}(x) & \text{otherwise} \end{cases}.
\]

(4.10)

We define \(x^*\) through a “best effort” limit if the problem is not feasible in order to define a complete bidding strategy for the DSP. Note also that the inverse of \(W\) is guaranteed to exist by the strict monotonicity of \(W\), see Definition 2.1.

4.1.1 Receding Horizon Control.

The bid path (4.10) does not take into account any of the information gained during the course of bidding. In this case, it is natural to convert our solution into a receding horizon (RH) (see [5]) algorithm where if after time \(t\) has elapsed, we have accumulated \(c(t)\) supply, we can modify the
The intuition that good acquisition paths are simply straight lines when prices are time-independent is reinforced by examining the curves and their relative costs in Figure 2.

For the receding horizon case, the supply actually attained can be described by the differential equation

\[
x_{rh}(t) = \begin{cases} 
- \frac{1}{\gamma} \ln \left[ 1 - \frac{1}{\lambda_0} \frac{C-c(t)}{T-t} \right] & \frac{1}{\lambda_0} \frac{C-c(t)}{T-t} < 1 \\
\infty & \text{otherwise}
\end{cases}.
\] (4.12)

Example 4.1 We consider an illustrative example where the DSP forecasts supply with the parametric form \( W(x) = 1 - e^{-\gamma x} \), and constant supply \( \lambda_0 \). Clearly, \( W^{-1}(s) = -\frac{1}{\gamma} \ln(1 - s) \) for \( s \in [0, 1] \), from which the optimal bids, including receding horizon are immediately derived

\[
x_{rh}(t) = \begin{cases} 
- \frac{1}{\gamma} \ln \left[ 1 - \frac{1}{\lambda_0} \frac{C-c(t)}{T-t} \right] & \frac{1}{\lambda_0} \frac{C-c(t)}{T-t} < 1 \\
\infty & \text{otherwise}
\end{cases}.
\] (4.12)

Suppose now that the realized supply over the period \([0, T]\) obeyed the law \( \lambda(t)W(x) \), i.e., the DSP’s estimate of supply is in error by \( \lambda(t) - \lambda_0 \). Figure 2 illustrates the behaviour of the static and receding horizon algorithms for the case of undersupply: \( \int_0^T \lambda(t)dt < \lambda_0 \), and oversupply \( \frac{1}{T} \int_0^T \lambda(t)dt > \lambda_0 \).

For the receding horizon case, the supply actually attained can be described by the differential equation

\[
\dot{c}_{rh}(t) = \lambda(t)W(x^*(C - c_{rh}(t), T - t)); \quad c_{rh}(0) = 0,
\] (4.13)

and the analogous equations for the static case \( c(t) \). Since the optimal bid \( x^* \) involves the inverse of the win probability \( W^{-1} \), substituting it into Equation (4.13) results in a separable ordinary differential equation

\[
\dot{c}_{rh}(t) = \frac{\lambda(t)C - c_{rh}(t)}{\lambda_0(T-t)}; \quad c_{rh}(0) = 0
\]

\[\implies c_{rh}(t) = C \left[ 1 - \exp \left( -\frac{1}{\lambda_0} \int_0^t \frac{\lambda(s)ds}{T-s} \right) \right],\]

which reduces simply to the straight line \( c_{rh}(t) = \frac{Ct}{T} \) if the estimate is accurate and \( \lambda(t) = \lambda_0 \).

The intuition that good acquisition paths are simply straight lines when prices are time-independent is reinforced by examining the curves and their relative costs in Figure 2.

This solution corresponds to estimates of an average behaviour; further simulation results including discrete event simulations with real market data are developed in section 7.

5 Optimal Management of Impression Contracts

Following the decomposition of Section 3.1, we have a collection of \( N \) contracts indexed by \( i \) with differing deadlines \( 0 < T_1 \leq \cdots \leq T_N = T \) and another set of \( M \) item types indexed by \( j \). This induces a problem where we need to calculate an array of bids \( x(t) \in \mathbb{R}^{N \times M}_+ \), as well as an array of allocations \( \gamma(t) \in [0, 1]^{N \times M} \). The interpretation is that if an item of type \( j \) arrives at time \( t \) the
Simulated acquisition paths $c(t)$ for the case $M = N = 1$ comparing the behaviour of different algorithms in the presence of supply shortages or surpluses in comparison to expectation $\lambda_0$. Best viewed in colour. Qualitatively, when there is oversupply, the receding horizon smooths the acquisition rate to reduce costs, and when there is undersupply, it increases the bid in reaction to the shortage.

quantity $\sum_{i \in B_j} \gamma_{ij}(t)$ indicates the probability of bidding on the item, and $\gamma_{ij}(t)$ is the probability of allocating that item (if won) to fulfill contract $i$. The bid which is submit is given by $x_{ij}(t)$.

We naturally have the constraints $\gamma_{ij}(t) = 0$ if $i \notin B_j$, or equivalently $j \notin A_i$. Indeed, we may think of $\gamma$ as weights on the edges of a bipartite graph with nodes $[N] \times [M]$ and an edge $i, j$ if $i \in B_j$.

The contract deadlines are an important detail of the problem, and induce a set of times $T^1, \ldots, T^M$ for the item types where $T^j = \max_{i \in B_j} T_i$ is the last instant that an item of type $j$ is useful. Moreover, we will see that the sets $T^N_i = \{ i \in [N] \mid t < T_i \}$ of contracts active up to (but not including) time $t$ and the set $T^M_j = \{ j \in [M] \mid t < T^j \}$ of items useful up to (but not including) time $t$ will arise naturally.

Finally, since $\gamma_{ij}(t)$ is the allocation proportion from $j$ to $i$, we must have that $\gamma_{ij}(t) = 0$ for any $t \geq T_i$ or $t \geq T^j$ and that $\sum_{i \in B_j} \gamma_{ij}(t) \leq 1$. In fact, we can see that, necessarily, $\sum_{i \in B_j \cap T^N} \gamma_{ij}(t) \leq 1[t < T^j]$.

We formulate the joint problem for fulfilling the contracts as an optimal control problem as follows:

$$\min_{x, \gamma} \sum_{i=1}^N \int_0^{T_i} \left[ \sum_{j \in A_i \cap T^M_t} \gamma_{ij}(t)f_j(x_{ij}(t), t) \right] dt$$

subject to  
$$\dot{c}_i(t) = 1[t < T_i] \sum_{j \in A_i \cap T^M_j} \gamma_{ij}(t)W_j(x_{ij}(t), t)$$

$$\sum_{i \in B_j \cap T^N} \gamma_{ij}(t) \leq 1[t < T^j]$$

$$c_i(0) = 0, c_i(T) \geq C_i, \gamma_{ij}(t) \geq 0,$$

where the state $c_i(t)$ indicates the expected supply obtained by time $t$ for contract $i$. 

Figure 2: Receding Horizon Acquisition Paths
This is a direct generalization of [23] to the case where supply curves are time-dependent, and crucially, where there may be differing contract deadlines.

**Remark 5.1 (Single Contract, Multiple Item Types (N = 1, M > 1))** The special case of \([P]\) when \(N = 1\) may be of interest since the \(N > 1\) case could be approached by solving \(N\) instances of this special case. However, this would have the obvious drawback of putting the DSP in competition with itself.

### 5.1 The Convex Reformulation

Since the cost of acquisition function \(\Lambda_j\) is convex, it suggests that Problem \([P]\) can be reformulated into a convex problem. In order to carry out this transformation, we first show that the bids \(x_{ij}(t)\) can be chosen independently of \(i\), that is, \(x_{ij}(t) = x_j(t)\).

**Proposition 5.1 (Uniform Bid Principle (UBP))** Any solution \((x, \gamma)\) of Problem \([P]\) can be transformed into another solution \((\tilde{x}, \tilde{\gamma})\) such that \(x_{uj}(t) = \tilde{x}_{uj}(t)\) for every \(u, v \in [N]\), and moreover, such that \(\forall t < T^j \sum_{i \in B_j \cap T^n} \tilde{\gamma}_{ij}(t) \in \{0, 1\}\).

**Proof:**

Suppose \((x, \gamma)\) is a solution of Problem \([P]\), and with total cost \(J\) and where

\[
\tilde{x}_j(t) = W_j^{-1} \left( \sum_{i \in B_j \cap T^n} \gamma_{ij}(t) x_{ij}(t), t \right),
\]

\[
\tilde{\gamma}_{ij}(t) = \frac{\gamma_{ij}(t) W_j(x_{ij}(t), t)}{\sum_{u \in B_j \cap T^n} \gamma_{uj}(t) W_j(x_{uj}(t), t)},
\]

where \(0/0 \equiv 0\) in the definition of \(\tilde{\gamma}\).

It is clear that \(\tilde{\gamma}_{ij}(t)\) is feasible since \(\tilde{\gamma}_{ij}(t) \geq 0\) and \(\sum_{i \in B_j \cap T^n} \tilde{\gamma}_{ij}(t) \leq 1[t < T^j]\) by definition. Indeed, \(\forall t < T^j \sum_{i \in B_j \cap T^n} \tilde{\gamma}_{ij}(t) \in \{0, 1\}\).

The cost of \((\tilde{x}, \tilde{\gamma})\), instantaneously at time \(t\), then satisfies \(\tilde{J} = J\) since \(J\) is the minimal cost and

\[
\tilde{J} = \sum_{i \in T^n \setminus j \in A_i} \tilde{\gamma}_{ij}(t) f_j(\tilde{x}_j)
\]

\[
= (a) \sum_{i \in T^n \setminus j \in A_i} \tilde{\gamma}_{ij}(t) \Lambda_j \left( \sum_{u \in B_j \cap T^n} \gamma_{uj}(t) W_j(x_{uj}(t), t), t \right)
\]

\[
\leq (b) \sum_{i \in T^n \setminus j \in A_i} \tilde{\gamma}_{ij}(t) \sum_{u \in B_j \cap T^n} \gamma_{uj}(t) \Lambda_j(W_j(x_{uj}(t), t), t)
\]

\[
= (c) \sum_{j \in T^M} \sum_{u \in B_j \cap T^n} \gamma_{uj}(t) f_j(x_{uj}(t), t) \sum_{i \in B_j \cap T^n} \tilde{\gamma}_{ij}(t) = J
\]

where \((a)\) is just the definition of \(\Lambda_j\) (c.f. Proposition 2.4), \((b)\) follows by the convexity of \(\Lambda_j\) and that \(\Lambda_j(0) = 0\) (since \(\tilde{\gamma}_{ij}\) need not necessarily sum to 1), and \((c)\) since \(\Lambda_j = f_j \circ W_j^{-1}\) and then by swapping the order of summation using \(i \in B_j \iff j \in A_i\).
With this proposition in hand, there is no reason to consider solutions where the bids depend on \(i\). This fact enables us to make significant simplifications to Problem \([P]\). Rather than optimizing over the bid and allocation pair \((x, \gamma)\), we can instead optimize over a supply and unnormalized allocation \((s, r)\) where \(s_j(t) = W_j(x_j(t), t)\) and \(r_{ij}(t) = \gamma_{ij}(t)s_j(t)\). We summarize this idea in the following proposition, with a detailed description of the transformation provided in the Appendix.

**Proposition 5.2 (Convex Formulation)** Problem \([P]\) can be equivalently reformulated as the following convex optimization problem

\[
\begin{align*}
\text{minimize} & \quad \sum_{j=1}^{M} \int_{0}^{T_j} \Lambda_j(s_j(t), t) dt \\
\text{subject to} & \quad \dot{c}_i(t) = 1[t < T_i] \sum_{j \in A_i} r_{ij}(t) \\
& \quad \sum_{i \in B_j \cap \tau_i} r_{ij}(t) = s_j(t) 1[t < T_j] \\
& \quad c_i(0) = 0, c_i(T) \geq C_i, r_{ij}(t) \geq 0.
\end{align*}
\]

\((P_{cvx})\)

A solution to the original problem is obtained via \(x_j(t) = W_j^{-1}(s_j(t), t)\) and \(\gamma_{ij}(t) = r_{ij}(t)/s_j(t)\).

**Proof:**

Recall the original problem \([P]\), and apply Proposition 5.1 to eliminate the dependence of the bid on \(i\):

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{N} \int_{0}^{T_i} \left[ \sum_{j \in A_i \cap \tau_i} \gamma_{ij}(t)f_j(x_j(t), t) \right] dt \\
\text{subject to} & \quad \dot{c}_i(t) = 1[t < T_i] \sum_{j \in A_i \cap \tau_i} \gamma_{ij}(t)W_j(x_j(t), t) \\
& \quad \sum_{i \in B_j \cap \tau_i} \gamma_{ij}(t) \leq 1[t < T_j] \\
& \quad c_i(0) = 0, c_i(T) \geq C_i, \gamma_{ij}(t) \geq 0.
\end{align*}
\]

\((P)\)

Due to the bid’s independence of \(i\), we can rearrange the objective by swapping the order of summation:

\[
\sum_{i=1}^{N} \int_{0}^{T_i} \left[ \sum_{j \in A_i \cap \tau_i} \gamma_{ij}(t)f_j(x_j(t), t) \right] dt = \sum_{j=1}^{M} \int_{0}^{T_j} \left[ f_j(x_j(t), t) \sum_{i \in B_j \cap \tau_i} \gamma_{ij}(t) \right] dt, \quad (5.14)
\]
which, after making the substitution \( s_j(t) = W_j(x_j(t), t) \), results in

\[
\begin{align*}
\text{minimize} & \quad \sum_{j=1}^{M} \int_{0}^{T_j} \left[ \Lambda_j(s_j(t), t) \sum_{i \in B_j \cap T^N_i} \gamma_{ij}(t) \right] dt \\
\text{subject to} & \quad \dot{c}_i(t) = 1[t < T_i] \sum_{j \in A_i \cap T^M_i} r_{ij}(t) \\
& \quad \sum_{i \in B_j \cap T^N_i} r_{ij}(t) \leq 1[t < T_j] \\
& \quad \sum_{i \in B_j \cap T^N_i} r_{ij}(t) = s_j(t)1[t < T_j] \\
& \quad c_i(0) = 0, c_i(T) \geq C_i, r_{ij}(t) \geq 0.
\end{align*}
\] (5.15)

Now, make the substitution \( r_{ij}(t) \overset{\Delta}{=} \gamma_{ij}(t)s_j(t) \). Notice that Proposition 5.1 also ensures that we have a solution where \( \sum_{i \in B_j \cap T^N_i} \gamma_{ij}(t) \in \{0, 1\} \), and if this summation is 0, then necessarily \( s_j(t) = 0 \) which in turn implies that \( \Lambda_j(s_j(t), t) = 0 \). Therefore we can write

\[
\begin{align*}
\text{minimize} & \quad \sum_{j=1}^{M} \int_{0}^{T_j} \Lambda_j(s_j(t), t) dt \\
\text{subject to} & \quad \dot{c}_i(t) = 1[t < T_i] \sum_{j \in A_i \cap T^M_i} r_{ij}(t) \\
& \quad \sum_{i \in B_j \cap T^N_i} r_{ij}(t) = s_j(t)1[t < T_j] \\
& \quad c_i(0) = 0, c_i(T) \geq C_i, r_{ij}(t) \geq 0.
\end{align*}
\] (5.16) (P_{cvx})

Problem \((P_{cvx})\) has linear constraints and a convex objective, and is therefore itself a convex optimization problem.

**Remark 5.2** In this formulation, we see a close connection to the Production-Transportation problem [18, 22], where \( \Lambda_j \) are the production costs, and the transportation costs belong to the set \( \{0, \infty\} \) encoding the set \( A_i, B_j \). The proof of Proposition 2.4 is essentially establishing that the marginal production costs, \( \Lambda_j' \), are monotone: a key aspect of the analysis of [18, 22].

### 5.2 Necessary Conditions

We characterize the necessary properties of \( (s(t), r(t)) \) (and \( (x(t), \gamma(t)) \) by extension) via the Pontryagin Maximum Principle (8). Define the Hamiltonian

\[
H(t, p, c, s, r) = \sum_{i \in T^N_i} p_i \sum_{j \in A_i} r_{ij} - \sum_{j \in T^M_i} \Lambda_j(s_j, t).
\] (5.16)

The question of constraint qualifications and the existence of solutions is of technical importance, but we will defer these issues to Section 5.3, assuming for now that a solution sufficiently regular to allow the application of the maximum principle does, in fact, exist.
The maximum principle ensures that there exists some absolutely continuous function \( p : [0, T] \to \mathbb{R}^N \) that satisfies the adjoint equation
\[
\dot{p}(t)^T = -D_c \mathcal{H}(t, p, c, s, r)
\]  
(5.17)

But, since \( \mathcal{H} \) does not depend on \( c \) explicitly, \( \dot{p}(t) = 0 \), and the adjoint is a constant \( p \in \mathbb{R}^N \).

Denote \( E_i = [C_i, \infty) \) and

\[
N_{E_i}(x) = \begin{cases} \mathbb{R}_- & x = C_i \\ \{0\} & x > C_i \\ \end{cases}
\]  
(5.18)

the normal cone. The maximum principle requires \(-p_i(T) \in N_{E_i}(c_i(T))\) and therefore that for any optimal state \( c_i(t) \), we must have \( p_i = 0 \) for \( c_i(T) > C_i \) (i.e., in the case of over fulfillment) and \( p_i \geq 0 \) for \( c_i(T) = C_i \). Ultimately, this implies that \( p_i \geq 0 \).

**Remark 5.3 (Pathological Cases)** The unusual case of an optimal solution satisfying \( c_i(T) > C_i \) is in fact possible. This may arise from the randomized bidding and that we may have \( W_j(0) > 0 \) even though \( f_j(0) = 0 \). If the supply requirements \( C_i \) are extremely small, then our model allows the attainment of this supply at 0 cost. This is an artifact of the technical assumptions necessary to rigorously establish our results, but is not of practical relevance: for bids well within the interior of \( \mathbb{R}_+ \), the approximation error in the cost function is negligible.

Finally, a solution must satisfy the maximum condition
\[
\mathcal{H}(t, p, c(t), s(t), r(t)) = \sup_{(s,r) \in U(t)} \mathcal{H}(t, p, c(t), s, r),
\]  
(5.19)

where \( U(t) \) encodes the constraints. Since we are already asserting the attainment of the above suprema, we can formulate the problem of extremizing the Hamiltonian at time \( t \):

\[
\begin{align*}
\text{maximize} & \quad \sum_{s,r} \sum_{i \in \mathcal{T}^N} p_i r_{ij} - \sum_{j \in \mathcal{T}_M^M} \Lambda_j(s_j, t) \\
\text{subject to} & \quad \sum_{i \in \mathcal{B}_j \cap \mathcal{T}^N_t} r_{ij} = s_j(t), r_{ij} \geq 0.
\end{align*}
\]  
(5.20)

We will see that there is a tight relationship between the bids \( x_j(t) \) and the adjoint vector \( p \), such that the entire continuous time path \( x(t) \) will be fully determined by the finite vector \( p \) – for this reason, we refer to \( p \) as the vector of pseudo-bids. Moreover, this pseudo-bid vector determines some key aspects of the support (i.e., indices of non-zero entries) of \( r_{ij}(t) \). Introducing notation for the maximum pseudo-bid over the set \( \mathcal{B}_j \cap \mathcal{T}^N_t \)

\[
p_j^*(t) = \max_{i \in \mathcal{B}_j \cap \mathcal{T}^N_t} p_i,
\]  
(5.21)

we have the following proposition:

**Proposition 5.3 (Optimal Allocation)** Any (regular) solution \((r, s)\) of \([P]_{\text{cvx}}\) and the corresponding acquisition path \( c_i(t) \) and vector of pseudo-bids \( p \) must satisfy \( c_i(T) \geq C_i \) and \( p_i \geq 0 \) for every \( i \). Moreover, \((s, r)\) maximizes the Hamiltonian at time \( t \) if and only if

\[
\begin{align*}
i \in \mathcal{B}_j \cap \mathcal{T}^N_t, p_i < p_j^*(t) & \implies r_{ij}(t) = 0. & (5.22a) \\
 s_j(t) = W_j(p_j^*(t), t), & (5.22b)
\end{align*}
\]

For solutions \((x, \gamma)\) of \([P]\), this implies that \( x_j(t) = p_j^*(t) \) and \( p_i < p_j^*(t) \implies \gamma_{ij}(t) = 0.\)
The proof can be found in the Appendix, see [A].

5.3 Existence and Optimality

In this section we address two important technical questions: whether a solution to our problem actually does exist, and whether the necessary conditions studied in Section 5.2 are sufficient. Both questions are answered in the affirmative, and concrete methods for calculating such an optimal solution are provided in Section 6.

5.3.1 Existence.

That there exists solutions to the problem \((P)\) intuitively rests on the assumption that there is a sufficient amount of supply available to fulfill the contracts. In the context of our main application, this is often easily taken for granted due to the ubiquity of the internet and internet advertising resulting in large volumes of available impressions. However, in order to provide an explicit and interpretable condition, we consider the following assumption (a version of which also appears in [18]).

Assumption 5.1 (Adequate Supply) We say that an adequate supply condition holds if for every \(j \in [M]\) we have

\[
\int_0^{\tau_j} B_j(t) dt > \sum_{i \in B_j} C_i,
\]

where \(\tau_j = \min\{T_i \mid i \in B_j\}\) and \(B_j(t) = \max_{x \geq 0} W(x, t) \leq B_W < \infty\).

The above assumption implies that every item type individually has enough supply to fulfill each of the contracts to which it’s items may be assigned.

With this assumption in hand, we are able to address some important technical aspects concerning the existence of regular solutions, as well as the smoothness of such solutions. Recall that for a solution to be regular means, essentially, that the constraints are not so stringent as to completely determine the solution. Our application in Section 5.2 of the maximum principle requires the a-priori knowledge that a regular solution does in fact exist. Consult [8] for further detail.

Proposition 5.4 (Existence) If there exists a feasible point for Problem \((P_{cvx})\), then it admits a solution \((s(t), r(t))\). Moreover, if Assumption 5.1 holds, then there exists a feasible point and any solution is regular.

Proof: See Appendix, see [A].

Remark 5.4 A solution \((x, \gamma)\) to \((P)\) can be obtained from \((s(t), r(t))\) via

\[
\forall i \in [N] \quad x_{ij}(t) = W_{j}^{-1}(s_{j}(t), t),
\]

\[
\gamma_{ij}(t) = r_{ij}(t)/s_{j}(t) \quad \text{where } 0/0 = 0 \text{ by convention. This is clear from transformations applied to obtain } (P_{cvx}) \text{ from } (P). \text{ This solution inherits the regularity and normality of } (s(t), r(t)).
5.3.2 Optimality.

The maximum principle we applied in Section 5.2 is in essence a manifestation of Fermat’s rule: if \(x^\star\) minimizes the smooth function \(f\) we must necessarily have \(\nabla f(x^\star) = 0\). If it is known that the function \(f\) is convex, then this condition is also sufficient, and any stationary point is a global minimum. The following proposition (a corollary of \([8]\) Theorem 24.1) asserts the analogous result for our problem.

**Proposition 5.5 (Global Optimality)** If a regular solution exists (a sufficient condition being Assumption 5.1), then any pair \((s,r)\) satisfying the necessary conditions of Proposition 5.3 is globally optimal for Problem \((P_{cvx})\). By extension, the pair \((x,\gamma)\) derived from \((s,r)\) is globally optimal for \((P)\).

**Proof:** This is a corollary of \([8]\) Theorem 24.1 and 5.4 since the Hamiltonian does not depend on the state and since the objective function and constraint region is convex.

6 Solution Methods

The Problem \((P)\) can be reformulated as a convex problem, but with an uncountable infinite number of variables. In this section, we establish the fact that a piecewise constant solution exists, and therefore that the entire problem can be reduced into a finite dimensional optimization problem, and again formulated as a finite convex problem and solved by well known methods. We focus back on Problem \((P)\) because the upcoming Proposition 6.1 is easier to state and to understand than the equivalent statement for \((P_{cvx})\).

Combining the results of Propositions 5.1 and 5.3, as well as \(f_j(0, t) = 0\), we can narrow down the properties of the optimal solution, and reformulate Problem \((P)\) as

\[
\begin{align*}
\text{minimize} & \quad \sum_{j=1}^{M} \int_{0}^{T_j} f_j(q_j(t), t)dt \\
\text{subject to} & \quad \int_{0}^{T_i} \left[ \sum_{j \in A_i} \gamma_{ij}(t)W_j(q_j(t), t) \right] dt \geq C_i \\
& \quad \sum_{i \in B_j \cap T_i} \gamma_{ij}(t) = 1 \left[ t < T_j \right] \\
& \quad i \in B_j, p_i < q_j(t) \implies \gamma_{ij}(t) = 0, \\
& \quad q_j(t) = \max_{i \in B_j \cap T_i} p_i, p_i \geq 0, \gamma_{ij}(t) \geq 0.
\end{align*}
\]

(6.23)

It is convenient introduce the discrete analog of the set \(T_i^N\), namely,

\[
\mathcal{T}_k^N \equiv \{ i \mid T_i \leq T_k \}.
\]

We are now able to establish the existence of a piecewise constant solution.

**Proposition 6.1 (Piecewise Constant Allocation)** There exists piecewise constant functions \(q(t), \gamma(t)\) taking values \(q_j[k], \gamma_{ij}[k]\) for times \(t \in [T_{k-1}, T_k)\) which are optimal for Problem \((6.23)\).
Proof:

Let \((p, q, \gamma)\) be a solution to \((6.23)\).

First, any \(q\) solving \((6.23)\) is already piecewise constant by the definition of \(T^N_t\), and the constraint \(q_j(t) = \max_{i \in B_j \cap T^N_t} p_i\).

Since the objective does not depend on \(\gamma(t)\), we only need to find a feasible piecewise constant \(\gamma(t)\). To do so, define

\[
H_{ij}(k) \triangleq \int_{T_{k-1}}^{T_k} \gamma_{ij}(t) W_j(q_j(t), t) dt.
\]

Then, since \(\gamma(t)\) forms part of a solution, for each \(i \in [N]\) we have

\[
\sum_{j \in A_i} \sum_{k : T_k \leq T_i} H_{ij}(k) \geq C_i.
\]

Now, for \(k : T_k \leq T^j\) let

\[
H_j(k) \triangleq \sum_{i \in B_j} H_{ij}(k) \overset{(a)}{=} \int_{T_{k-1}}^{T_k} W_j(q_j(t), t) dt,
\]

where the latter equality follows since \(\sum_{i \in B_j \cap T^N_t} \gamma_{ij}(t) = 1; \forall t : B_j \cap T^N_t \neq \emptyset\).

Define now the piecewise constant allocation

\[
t \in [T_{k-1}, T_k), i \in T^N_t \implies \tilde{\gamma}_{ij}(t) = \gamma_{ij}[k] = \frac{H_{ij}(k)}{H_j(k)} 1[T_k \leq T^j].
\]

We see that this function is feasible firstly since \(\tilde{\gamma}_{ij}(t) \geq 0\) and \(\sum_{i \in B_j \cap T^N_t} \gamma_{ij}[k] = 1[T_k \leq T^j]\) by construction. Moreover, \(i \in B_j, p_i < q_j(t) \implies \tilde{\gamma}_{ij}(t) = 0\) is a property inherited from \(\gamma(t)\) by definition of \(H_j, H_{ij}\). Finally

\[
\sum_{j \in A_i} \sum_{k : T_k \leq T_i} \gamma_{ij}[k] \int_{T_{k-1}}^{T_k} W_j(q_j(t), t) dt = \sum_{j \in A_i} \sum_{k : T_k \leq T_i} H_{ij}(k) \geq C_i.
\]

by (a) above.

By incorporating the results of Proposition \(6.1\), Problem \(6.23\) can be written as a finite optimization problem. However, the final two constraints are, as written, still problematic. Our main theorem shows that these constraints can simply be dropped, and serves to summarize our developments by establishing that solutions of the resulting problem can be converted into globally optimal solutions of the original optimal control problem \((P)\). It will be seen that the general problem considered in this paper, can in fact be reduced exactly to an instance of the seemingly less general static problem considered in \([23]\).
Theorem 6.1 (Optimal Solution) Let \( \tilde{f}_{jk}(x) = \int_{T_{k-1}}^{T_k} f_j(x, t) dt \), \( \tilde{W}_{jk}(x) = \int_{T_{k-1}}^{T_k} W_j(x, t) dt \), and \( \Lambda_{jk} = \tilde{f}_{jk} \circ \tilde{W}_{jk}^{-1} \). Consider the following optimization problem

\[
\begin{align*}
\text{minimize} & \quad \sum_{j=1}^{M} \sum_{k:T_k \leq T_j} \Lambda_{jk}(s_j[k]) \\
\text{subject to} & \quad \sum_{j \in A_i, k:T_k \leq T_i} r_{ij}[k] \geq C_i \quad (P^\star) \\
& \quad \sum_{i \in B_j \cap \Gamma_k^N} r_{ij}[k] = s_j[k] \\
& \quad r_{ij}[k] \geq 0.
\end{align*}
\]

This problem is convex, and if its solutions \( s_j[k], r_{ij}[k] \) are transformed into functions \( s_j(t), r_{ij}(t) \) of \( t \in [0, T] \) according to \( s_j(t) = s_j[k] \) and \( r_{ij}(t) = r_{ij}[k] \) if \( t \in [T_{k-1}, T_k) \), results in a globally optimal solution to Problem \( P_{\text{conv}} \).

Similarly, by transforming \( \gamma_{ij}[k] \triangleq r_{ij}[k]/s_j[k] \) and \( x_j[k] \triangleq \tilde{W}_{jk}^{-1}(s_j[k]) \) into continuous functions results in a globally optimal solution to Problem \( P \).

Proof: The details can be found in the Appendix, see \( \Lambda \).

Remark 6.1 (Implementation) The Problem \( P^\star \) is equivalent to the time-homogeneous version of the problem studied by [23] with compound item types \((j, k)\) and supply curves \( \tilde{W}_{jk} \). The valid (compound) types for contract \( i \): \( \mathcal{A}_i \triangleq A_i \times \{ k \mid T_k \leq T_i \} \) and set of valid contracts for type \((j, k)\): \( \mathcal{B}_{jk} \triangleq B_j \cap \Gamma_k^N \). Hence one can use the algorithm in [23] that does not involve derivatives of \( W_j \). Problem \( P^\star \) can also be solved by standard convex optimization software. Indeed, we used CVXOPT [24] for the numerical results in Section 7.

In practice, if the problem \( P^\star \) is infeasible (i.e., the supply is not adequate c.f. Assumption 5.1), a “best-effort” set of bids can be computed by instead using a penalty formulation, e.g. with cost function

\[
\sum_{j=1}^{M} \sum_{k:T_k \leq T_j} \Lambda_{jk}(s_j[k]) + \rho \left[ \sum_{i=1}^{N} C_i - \sum_{j \in A_i, k:T_k \leq T_i} r_{ij}[k] \right],
\]

and a very large value of \( \rho \).

7 Simulation

To evaluate and illustrate the performance of algorithms derived in this paper we carried out a set of numerical simulations on data derived from the iPinYou dataset [19, 37]. All of the computations have been carried out with Python’s scientific computing ecosystem [26].

Using the top 5 highest rate item types from the iPinYou dataset (i.e., \( j \in [5] \)), we constructed 6 contracts with time deadlines, supply requirements, and targeting sets according to Table 1. Without loss of generality we assumed that all contracts began at \( t = 0 \).

A sliding window was used in the simulations for the one week of data available: each simulation period spanned a 72 hour long window beginning every 12 hours. Thus, a simulation was run on
Table 1: Simulation Contract Specifications

|   |   |   |   |
|---|---|---|---|
| i |  $T_i$ (hours) | $C_i$ | $A_i$ |
| 1 | 28 | 4500 | {0, 2} |
| 2 | 31 | 3240 | {0, 4} |
| 3 | 43 | 6300 | {1, 2, 4} |
| 4 | 56 | 3600 | {0, 3} |
| 5 | 63 | 1800 | {2} |
| 6 | 71 | 3600 | {2, 4} |

hours 0 through 72, 12 through 84, 24 through 96, etc. The purpose of the sliding window is to capture the variance in bid paths resulting from day to day changes in market conditions. We additionally repeated each simulation 4 times in order to capture the variance arising from the randomness inherent in the bidding strategy. There are a total of 9 unique periods and therefore 36 simulations per algorithm in total.

To facilitate interpretation and plotting, we re-normalize the simulation results by scaling time and supply requirements. If $c_i(t)$ is the total supply attained for contract $i$ by time $t$, we define $\tilde{c}_i(t) \triangleq c_i(tT_i)/C_i$ so that $\tilde{c}_i(1) \geq 1$ indicates that contract $i$ has been fulfilled by its deadline. Finally, we can average each of these curves into a single function $\tilde{c}(t) \triangleq \frac{1}{N} \sum_{i=1}^{N} \tilde{c}_i(t)$ so that $\tilde{c}(1) \geq 1$ indicates that every contract has been fulfilled by their deadlines. Our figures depict this averaged and re-normalized curve.

We compared the results using 6.1 referred to as the dynamic solution with a static approach based on averaging supply and impression constraints over the contract duration. The supply curve for the averaged system is taken to be $W_j(x) = \frac{1}{T} \int_{0}^{T} W_j(x,t)dt$. Such a time homogenous solution is a natural heuristic and provides a baseline.

The whole dataset was used to estimate the supply curves as 24hr-periodic functions by taking the product of a Gaussian kernel density estimate of the price (for each hour) and the average arrival rate by hour. This was extended via periodicity to the entire week-long period. In practice, simple average prices and arrival rates could be estimated from historical data available to any DSP. The estimated supply curves were then used for bidding with the impression arrivals sampled from the dataset. Details are provided in Appendix B.1.

Figure 3 provides a comparison between these two approaches. For both algorithms, we recompute a new bid every 1 hour of simulated time. In the case of the the dynamic solution, updating the bids at time $\tau \in [0, T)$ requires re-aggregating active (i.e. $\tau \in [T_{k-1}, T_k)$) supply curves via $W_{jk}(x) = \int_{\tau}^{T_k} W_j(x,t)dt$ before recomputing new bids, c.f. Theorem 6.1. Figure 3b depicts the average bid across campaigns $\frac{1}{N} \sum_{i=1}^{N} p_i(t)$ with the time axis being real time and where the hourly bid updates are clearly discernible.

The results demonstrate the benefits conferred by accounting for time dynamics and different durations: the simulation results for the dynamic and static algorithms are $3.58 \times 10^5$ and $3.93 \times 10^5$ respectively, an average improvement for the dynamic policy of about 10% for the iPinYou dataset.
Figure 3: Contract Management Discrete Event Simulation

(a) Normalized Acquisition Paths $\tilde{c}(t)$

(b) Bid Paths $\rho(t)$

Discrete event simulations with IPinYou data. Thin and lightly shaded lines depict a single simulation with the thick dark line being the mean. (a) Plots of the averaged and re-normalized acquisition paths $\tilde{c}(t)$ for the contracts described in Table 1. Bids for the Dynamic solution (blue) are calculated according to Problem $P^*$ and the Static solution (red) is calculated similarly, except averages are taken over the entire $[0,T]$ period. In both cases, a receding horizon of one hour is employed to update bids over time. The average cost across simulations is denoted $J_{avg}$ (in the legend) and is in abstract currency units. (b) The average (across contracts) bid path $\rho(t)$ corresponding to the simulations of (a). Large discontinuities correspond to contract fulfillment times, and small adjustments to the hourly receding horizon.

8 Conclusion

This paper has studied a control problem faced by a DSP obligated to acquire, on the RTB market, a certain number of items by a given time deadline. We have shown that the notion of a supply curve emerges naturally from the market dynamics and that the optimal contract management problem can be formulated in terms of these supply curves, which act essentially as information states for the DSP.

Using the Pontryagin maximum principle as our primary tool, we have analyzed how the structure of the optimal bids depends upon a targeting set decomposition, the set of campaigns, and the time deadlines. We used these results to derive a globally optimal bidding algorithm from the solution of a convex optimization problem. The resulting optimization problem turns out to be a generalization of the simpler 1-period problem of [23] and the Transportation-Production problems of [18, 22].

Our algorithm has been illustrated through simulation with real auction data, demonstrating the potential for improvement over and above a strictly average case time-homogeneous method.
References

[1] I. Adan and J. Resing. *Queueing Systems*. 2002. URL: https://www.win.tue.nl/~iadan/queueing.pdf

[2] S. R. Balseiro, O. Besbes, and G. Y. Weintraub. “Repeated auctions with budgets in ad exchanges: Approximations and design”. In: *Management Science* 61.4 (2015), pp. 864–884.

[3] D. P. Bertsekas. “Nonlinear programming”. In: *Journal of the Operational Research Society* 48.3 (1997), pp. 334–334.

[4] H. Cai, K. Ren, W. Zhang, K. Malialis, J. Wang, Y. Yu, and D. Guo. “Real-time bidding by reinforcement learning in display advertising”. In: *Proceedings of the Tenth ACM International Conference on Web Search and Data Mining*. 2017, pp. 661–670.

[5] M. Canon and B. Kouvaritakis. *Model Predictive Control—Classical, Robust and Stochastic*. New York, NY, USA: Springer, 2016.

[6] B. Chen, S. Yuan, and J. Wang. “A dynamic pricing model for unifying programmatic guarantee and real-time bidding in display advertising”. In: *Proceedings of the Eighth International Workshop on Data Mining for Online Advertising*. 2014, pp. 1–9.

[7] H. Choi, C. F. Mela, S. R. Balseiro, and A. Leary. “Online display advertising markets: A literature review and future directions”. In: *Information Systems Research* (2020).

[8] F. Clarke. *Functional analysis, calculus of variations and optimal control*. Vol. 264. Springer Science & Business Media, 2013.

[9] Y. Cui, R. Zhang, W. Li, and J. Mao. “Bid landscape forecasting in online ad exchange marketplace”. In: *Proceedings of the 17th ACM SIGKDD international conference on Knowledge discovery and data mining*. 2011, pp. 265–273.

[10] N. Falkner and G. Teschl. “On the substitution rule for Lebesgue–Stieltjes integrals”. In: *Expositiones Mathematicae* 30.4 (2012), pp. 412–418.

[11] A. Ghosh, S. Mitra, S. Sarkhel, J. Xie, G. Wu, and V. Swaminathan. “Scalable Bid Landscape Forecasting in Real-time Bidding”. In: *arXiv preprint arXiv:2001.06587* (2020).

[12] N. Grislain, N. Perrin, and A. Thabault. “Recurrent Neural Networks for Stochastic Control in Real-Time Bidding”. In: *Proceedings of the 25th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining*. 2019, pp. 2801–2809.

[13] R. Gummadi, P. Key, and A. Proutiere. “Optimal bidding strategies and equilibria in dynamic auctions with budget constraints”. In: *Available at SSRN 2066175* (2013).

[14] IAB. *IAB internet advertising revenue report conducted by PricewaterhouseCoopers (PWC)*. 2018. URL: https://www.iab.com/insights

[15] C. Jiang, C. L. Beck, and R. Srikant. “Bidding with limited statistical knowledge in online auctions”. In: *ACM SIGMETRICS Performance Evaluation Review* 41.4 (2014), pp. 38–41.

[16] N. Karlsson. “Adaptive control using Heisenberg bidding”. In: *2014 American Control Conference*. IEEE, 2014, pp. 1304–1309.

[17] N. Karlsson. “Control problems in online advertising and benefits of randomized bidding strategies”. In: *European Journal of Control* 30 (2016), pp. 31–49.
[18] L. J. Leblanc and L. Cooper. “The transportation-production problem”. In: Transportation Science 8.4 (1974), pp. 344–354.

[19] H. Liao, L. Peng, Z. Liu, and X. Shen. “iPinYou global rtb bidding algorithm competition dataset”. In: Proceedings of the Eighth International Workshop on Data Mining for Online Advertising. 2014, pp. 1–6.

[20] Y. Mansour, S. Muthukrishnan, and N. Nisan. “Doubleclick ad exchange auction”. In: arXiv preprint arXiv:1204.0535 (2012).

[21] R. R. Mazumdar. “Performance modeling, loss networks, and statistical multiplexing”. In: Synthesis Lectures on Communication Networks 2.1 (2009), pp. 1–151.

[22] J. F. Sharp, J. C. Snyder, and J. H. Greene. “A decomposition algorithm for solving the multifacility production-transportation problem with nonlinear production costs”. In: Econometrica: Journal of the Econometric Society (1970), pp. 490–506.

[23] E. Tillberg, P. Marbach, and R. Mazumdar. “An Optimal Bidding Algorithm for Online Ad Auctions with Overlapping Targeting Criteria”. In: Proc. ACM Meas. Anal. Comput. Syst. 4.2 (June 2020). DOI: 10.1145/3366707.

[24] L. Vandenberghe. “The CVXOPT linear and quadratic cone program solvers”. In: Online: http://cvxopt.org/documentation/coneprog.pdf (2010).

[25] W. Vickrey. “Counterspeculation, auctions, and competitive sealed tenders”. In: The Journal of finance 16.1 (1961), pp. 8–37.

[26] P. Virtanen, R. Gommers, T. E. Oliphant, M. Haberland, T. Reddy, D. Cournapeau, E. Burovski, P. Peterson, W. Weckesser, J. Bright, S. J. van der Walt, M. Brett, J. Wilson, K. Jarrod Millman, N. Mayorov, A. R. J. Nelson, E. Jones, R. Kern, E. Larson, C. Carey, İ. Polat, Y. Feng, E. W. Moore, J. Vand erPlas, D. Laxalde, J. Perktold, R. Cimrman, I. Henriksen, E. A. Quintero, C. R. Harris, A. M. Archibald, A. H. Ribeiro, F. Pedregosa, P. van Mulbregt, and S. 1. 0. Contributors. “SciPy 1.0: Fundamental Algorithms for Scientific Computing in Python”. In: Nature Methods 17 (2020), pp. 261–272.

[27] J. Wang, W. Zhang, and S. Yuan. “Display advertising with real-time bidding (RTB) and behavioural targeting”. In: Foundations and Trends® in Information Retrieval 11.4-5 (2017), pp. 297–435.

[28] Y. Wang, K. Ren, W. Zhang, J. Wang, and Y. Yu. “Functional bid landscape forecasting for display advertising”. In: Joint European Conference on Machine Learning and Knowledge Discovery in Databases. Springer. 2016, pp. 115–131.

[29] L. Wasserman. All of nonparametric statistics. Springer Science & Business Media, 2006.

[30] A. Winkelbauer. “Moments and absolute moments of the normal distribution”. In: arXiv preprint arXiv:1209.4340 (2012).

[31] D. Wu, X. Chen, X. Yang, H. Wang, Q. Tan, X. Zhang, J. Xu, and K. Gai. “Budget constrained bidding by model-free reinforcement learning in display advertising”. In: Proceedings of the 27th ACM International Conference on Information and Knowledge Management. 2018, pp. 1443–1451.

[32] W. C.-H. Wu, M.-Y. Yeh, and M.-S. Chen. “Predicting winning price in real time bidding with censored data”. In: Proceedings of the 21th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining. 2015, pp. 1305–1314.
[33] S. Yuan, J. Wang, and X. Zhao. “Real-time bidding for online advertising: measurement and analysis”. In: *Proceedings of the Seventh International Workshop on Data Mining for Online Advertising*. 2013, pp. 1–8.

[34] R. Zeithammer. “Soft floors in auctions”. In: *Management Science* 65.9 (2019), pp. 4204–4221.

[35] W. Zhang, Y. Rong, J. Wang, T. Zhu, and X. Wang. “Feedback control of real-time display advertising”. In: *Proceedings of the Ninth ACM International Conference on Web Search and Data Mining*. 2016, pp. 407–416.

[36] W. Zhang, S. Yuan, and J. Wang. “Optimal real-time bidding for display advertising”. In: *Proceedings of the 20th ACM SIGKDD international conference on Knowledge discovery and data mining*. 2014, pp. 1077–1086.

[37] W. Zhang, S. Yuan, J. Wang, and X. Shen. “Real-time bidding benchmarking with ipinyou dataset”. In: *arXiv preprint arXiv:1407.7073* (2014).

[38] W. Zhang, Y. Zhang, B. Gao, Y. Yu, X. Yuan, and T.-Y. Liu. “Joint optimization of bid and budget allocation in sponsored search”. In: *Proceedings of the 18th ACM SIGKDD international conference on Knowledge discovery and data mining*. 2012, pp. 1177–1185.

[39] W. Zhang, T. Zhou, J. Wang, and J. Xu. “Bid-aware gradient descent for unbiased learning with censored data in display advertising”. In: *Proceedings of the 22nd ACM SIGKDD international conference on Knowledge discovery and data mining*. 2016, pp. 665–674.
A Proofs

In this section we present the proofs of technical results in the paper.

Proof of Proposition 2.1

That the formula for $W_j^M$ is a CDF is clear since it is monotone non-decreasing, right-continuous, and $W_j(x, t) \to 1$ as $x \to \infty$.

Now when an item arrives, since each of the participants choose independently with probabilities $r_{ij}$ whether or not to bid on the item, the probability that bidder 0 wins the item with bid $x \geq 0$ is given by the probability that no one with a bid greater than $x$ decides to bid (we use $\prod_{k \in S} \Delta = 1$):

$$W_j^M(x, t) = \prod_{k : b_{kj} > x} (1 - r_{kj}).$$

The result then follows by simple algebra:

$$\prod_{k : b_{kj} > x} (1 - r_{kj}) = \prod_{k : b_{kj} > x} \exp(\ln(1 - r_{kj}))$$

$$= \exp(\sum_{k : b_{kj} > x} \phi(r_{kj}))$$

$$= \exp(-\sum_{i=1}^N \phi(r_{ij}) \mathbf{1}_{(x, \infty)}(b_{ij})),$$

where we define $W_j^M(x) = 0$ for any $x < 0$.

Proof of Proposition 2.2

Consider the summation

$$\sum_{i=1}^{N(t)} \phi(r_i) \mathbf{1}_{(x, \infty)}(b_i).$$

Firstly, since $\mathbf{1}_{(x, \infty)}(b_i) \sim \text{Ber}(1 - F_B(x))$ the number of non-zero elements in the summation is $\text{Bin}(N(t), 1 - F_B(x))$ distributed. Then, the Poisson thinning property implies that the number of terms in the summation remains Poisson with parameter $\rho_B(x)$.

Now, following from Proposition 2.1 using the independence of $N(t), r_i, b_i$, and conditioning on the number of summation terms, we obtain:
\[ \mathbb{E}[W_{\mathcal{M}}(x,t)] = \mathbb{E}\exp\left( - \sum_{i=1}^{N_B} \Phi(r_i) 1_{(x,\infty)}(b_i) \right) \]

\[ = \sum_{n=0}^{\infty} \frac{e^{-\rho_B(x)} \rho_B(x)^n}{n!} \mathbb{E}\exp\left( - \sum_{i=1}^{n} \Phi(r_i) \right). \]

\[ = \sum_{n=0}^{\infty} \frac{e^{-\rho_B(x)} \rho_B(x)^n}{n!} \prod_{i=1}^{n} \mathbb{E}[e^{-\Phi(r_i)}] \]

\[ = \sum_{n=0}^{\infty} \frac{e^{-\rho_B(x)} \rho_B(x)^n}{n!} \prod_{i=1}^{n} (1 - \mathbb{E}[r_i]) \]

\[ = e^{-\rho_B(x) \mathbb{E}[r]} \sum_{n=0}^{\infty} \frac{e^{-\rho_B(x)(1-\mathbb{E}[r])} (\rho_B(x)(1 - \mathbb{E}[r]))^n}{n!} \]

Noting that \( \rho_B(x) = \rho(1 - F_B(x)) \) we see that \( W(x) \) is a right-continuous increasing function and \( W(x) \to 1 \) as \( x \to \infty \), hence it is a distribution.

**Proof of Proposition 2.3**

Fix some \( \epsilon > 0 \), and compact set \( I \subset \mathbb{R} \), as well as a parameter \( \delta > 0 \). Then, denoting \( \phi_\sigma(x) \overset{\Delta}{=} \frac{\mathbf{e}^{-x^2/2\sigma^2}}{\sqrt{2\pi}\sigma^2} \) the standard Gaussian density and \( \mu_\alpha \overset{\Delta}{=} \mathbb{E}[|X - \mathbb{E}[X]|^\alpha] \) the centered \( \alpha \)-moment of a standard Gaussian\(^2\). We then have,

\[ \int_I |W_\sigma(x) - W(x)| \, dx = \int_I \left| \int_\mathbb{R} W(t) \phi_\sigma(x-t) \, dt - W(x) \right| \, dx \]

\[ \overset{(a)}{=} \int_I \left| \int_\mathbb{R} (W(t) - W(x)) \phi_\sigma(x-t) \, dt \right| \, dx \]

\[ \overset{(b)}{=} \sum_{k=1}^{n} \int_{I \cap B(x_k; \delta)} \left| \int_\mathbb{R} (W(t) - W(x)) \phi_\sigma(x-t) \, dt \right| \, dx, \]

\[ + \int_{I \setminus \bigcup_{k=1}^{n} B(x_k; \delta)} \left| \int_\mathbb{R} (W(t) - W(x)) \phi_\sigma(x-t) \, dt \right| \, dx, \]

where \((a)\) follows since \( \int_\mathbb{R} \phi_\sigma(x) \, dx = 1 \), and \((b)\) by breaking up the first integral with balls

\(^2\)precisely, \( \mu_\alpha = 2^{\alpha/2} \Gamma(\frac{\alpha+1}{2}) / \sqrt{\pi} \) according to [30].
\[ B(x_k, \delta) \] at the jump points \( x_1, \ldots, x_n \). Then,

\[
\int_I |\overline{W}_\sigma(x) - W(x)| \, dx \leq \sum_{k=1}^n \int_{I \cap B(x_k; \delta)} \int_{\mathbb{R}} |W(t) - W(x)| \phi_\sigma(x - t) \, dt \, dx \\
+ \int_{I \setminus \bigcup_{k=1}^n B(x_k; \delta)} \int_{\mathbb{R}} |W(t) - W(x)| \phi_\sigma(x - t) \, dt \, dx \\
\overset{(a)}{\leq} \sum_{k=1}^n \int_{I \cap B(x_k; \delta)} \int_{\mathbb{R}} (L|t - x|^\alpha + B) \phi_\sigma(x - t) \, dt \, dx \\
+ \int_I \int_{\mathbb{R}} L|t - x|^\alpha \phi_\sigma(x - t) \, dt \, dx,
\]

where \( (a) \) follows via the definition of \((L, \alpha)\)-Hölder continuity. We then evaluate the centered moments to obtain:

\[
\int_I |\overline{W}_\sigma(x) - W(x)| \, dx \leq \sum_{k=1}^n \int_{I \cap B(x_k; \delta)} (L\mu_\alpha \sigma^\alpha + B) \, dx + \int_I L\mu_\alpha \sigma^\alpha \, dx \\
\leq nB\delta + (n\delta + |I|)L\mu_\alpha \sigma^\alpha.
\]

Finally, let \( \sigma = \delta^{1/2} \) and take the limit:

\[
nB\delta + (n\delta + |I|)L\mu_\alpha \sigma^\alpha = nB\delta + nL\mu_\alpha \delta^{\alpha/2} + |I|L\mu_\alpha \delta^{1/2} \rightarrow 0 \quad \text{as} \quad \delta \rightarrow 0.
\]

When there are no jumps, (or at regions outside the jumps) we similarly have a uniform approximation simply by applying Hölder continuity and the triangle inequality.

That \( \overline{W}_\sigma \) is \( C_\infty \) simply follows by Leibniz’s integral rule, the smoothness of \( e^{-x^2} \), and boundedness of \( W \):

\[
\frac{d\overline{W}_\sigma(x)}{dx} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^\infty W(t) \frac{d}{dx} \left[ \exp\left( -\frac{(t - x)^2}{2\sigma^2} \right) \right] \, dt \\
= \frac{1}{\sigma^2 \sqrt{2\pi}} \int_{-\infty}^\infty W(t)(t - x) \exp\left( -\frac{(t - x)^2}{2\sigma^2} \right) \, dt.
\]
Finally, we see from here that \( \overline{W}'(x) > 0 \) since for any \( x \) s.t. \( W(x) > 0 \) we have

\[
\overline{W}'(x) = \frac{1}{\sigma^2 \sqrt{2\pi}} \int_0^\infty W(t)(t-x)\phi_\sigma(t-x)dt = \frac{1}{\sigma^2 \sqrt{2\pi}} \int_x^\infty W(t)(t-x)\phi_\sigma(t-x)dt + \frac{1}{\sigma^2 \sqrt{2\pi}} \int_0^x W(t)(t-x)\phi_\sigma(t-x)dt
\]

\[
\geq \frac{1}{\sigma^2 \sqrt{2\pi}} \int_0^x W(t)(t-x)\phi_\sigma(t-x)dt + \frac{1}{\sigma^2 \sqrt{2\pi}} \int_0^\infty W(t)(t-x)\phi_\sigma(t-x)dt = W(x) \frac{1}{\sigma^2 \sqrt{2\pi}} \int_0^\infty (t-x)\phi_\sigma(t-x)dt > 0,
\]

where \((a)\) follows from the (weak-) monotonicity of \( W \) and noticing that \( t-x < 0 \) in the first integral and \( t-x > 0 \) in the second. If \( W(x) = 0 \) we can replace the inequality in \((a)\) with

\[
\geq \frac{1}{\sigma^2 \sqrt{2\pi}} \int_0^\infty W(t)(t-x)\phi_\sigma(t-x)dt > 0,
\]

and therefore \( \overline{W}_\sigma \) is strictly monotone increasing.

The final statement that \( x\overline{W}_\sigma(x) \to 0 \) as \( x \to -\infty \) follows since for \( x < 0 \) we have \( W(x) = 0 \) and the Mills’ ratio (ratio of the complementary c.d.f. to the p.d.f.) of \( \mathcal{N}(0,1) \) is asymptotically \( 1/x \).

**Proof of Proposition 5.3**

The statement concerning \( c_i(T_i) \geq C_i \) and \( p_i \geq 0 \) has already been established in the main text.

Now, look at a fixed time \( t \), let \( s \) form part of an optimal solution to Problem \((5.20)\). We consider optimization over \( r \) alone as

\[
\text{maximize} \quad \sum_{i\in \mathcal{T}_i^N} \sum_{j\in \mathcal{A}_i} p_i r_{ij} - \sum_{j\in \mathcal{T}_i^M} \Lambda_j(s_j, t) \\
\text{subject to} \quad \sum_{i\in \mathcal{B}_j \cap \mathcal{T}_i^N} r_{ij} = s_j, r_{ij} \geq 0.
\]

(A.24)

We first swap the order of summation in the objective:

\[
\sum_{i\in \mathcal{T}_i^N} \sum_{j\in \mathcal{A}_i} p_i r_{ij} = \sum_{j\in \mathcal{T}_i^M} \sum_{i\in \mathcal{B}_j \cap \mathcal{T}_i^N} p_i r_{ij},
\]

which is valid since \((i,j) \in \mathcal{T}_i^N \times \mathcal{A}_i \iff (j,i) \in \mathcal{T}_i^M \times (\mathcal{B}_j \cap \mathcal{T}_i^N)\). To see this, note that \( j \in \mathcal{A}_i \iff i \in \mathcal{B}_j \). and if \( i \in \mathcal{T}_i^N \), then for any \( j \in \mathcal{A}_i \) we necessarily have \( t < T^j \) and therefore \( j \in \mathcal{T}_i^M \).

Applying Hölder’s inequality to the second summation and using the problem constraints we have the inequality

\[
\sum_{j\in \mathcal{T}_i^M} \sum_{i\in \mathcal{B}_j \cap \mathcal{T}_i^N} p_i r_{ij} \leq \sum_{j\in \mathcal{T}_i^N} p^*_j(t) s_j(t).
\]

If Condition \((5.22a)\) is satisfied, then this value is achieved.
Now, suppose that \( r_{ij} > 0 \), but \( p_i < p^*_j(t) \). Then, the objective value can be increased by reassigning \( r_{ij} \leftarrow 0 \) and \( r_{ij} \leftarrow r_{ij} + r_{ij} \), where \( r^* \) is such that \( p_i = p^*_j(t) \). If \( p^*_j(t) = 0 \), the statement is vacuous. Therefore, any \( r \) which fails to satisfy (5.22a) cannot be a solution.

We turn to the Lagrangian necessary conditions (see e.g. [8, Chap. 9] or [3, Chap. 3]) for Problem (5.20). We consider the Lagrangian of the equivalent minimization problem (swapping the sign of 2.4) which is consistent. Equation (5.22b) follows, and the final statement is immediate by the proof of Theorem 6.1:

\[ \mu, \theta \]

must have least one subject to the constraint \( \sum_{i=1}^{N} \mu_i \sum_{j \in A_i} r_{ij} = s_j \) (i.e., the solution is independent of the cost function), subject to the constraint \( \sum_{i \in B_j} r_{ij} = s_j \), which is independent of the cost. Since there must be at least one \( p_i > 0 \) (otherwise we would violate the non-triviality condition [8, Theorem 22.26]) we must have \( s_j(t) = B_j(t) \) for at least one \( j \in A_i \). However, by the assumption, this must oversupply contract \( i \) (and others), and therefore by the requirement that \( -p_i(T) \in N_{E_1}(c_i(T)) \) (see Equation (5.18)) we must have \( p_i = 0 \), which is a contradiction. Therefore, the solution must be regular.

**Proof of Theorem 6.1**

If we augment Problem \( P_{cvx} \) with the constraint \( s_j(t) \leq B_j(t) \), then the existence of a solution follows from [8, Theorem 23.11] using convexity and the boundedness of the control set, as well as the (assumed) existence of a feasible point.

Supposing henceforth that Assumption 5.1 holds, in this case the existence of a feasible point is evident. Suppose by way of contradiction that a solution \( (s(t), r(t)) \) is non-regular solution with corresponding adjoint \( p \in \mathbb{R}_+^N \) (see Section 5.2). Then, at a particular point in time \( \tau \), where \( s, r = s(\tau), r(\tau) \) maximizes \( \sum_{i=1}^{N} p_i \sum_{j \in A_i} r_{ij} \) (i.e., the solution is independent of the cost function), subject to the constraint \( \sum_{i \in B_j} r_{ij} = s_j \), which is independent of the cost. Since there must be at least one \( p_i > 0 \) (otherwise we would violate the non-triviality condition [8, Theorem 22.26]) we must have \( s_j(t) = B_j(t) \) for at least one \( j \in A_i \). However, by the assumption, this must oversupply contract \( i \) (and others), and therefore by the requirement that \( -p_i(T) \in N_{E_1}(c_i(T)) \) (see Equation (5.18)) we must have \( p_i = 0 \), which is a contradiction. Therefore, the solution must be regular.
We first establish convexity, and then use Proposition 6.1 to show that solutions of \((P^\star)\) can be converted into solutions of \((6.23)\) and therefore to solutions of \((P_{\text{cvx}})\) and \((P)\).

Calculate, for \(x \geq 0\):

\[
\mathcal{F}_{jk}(x) = \int_{T_{k-1}}^{T_k} f_j(x, t)\,dt \\
= \int_{T_{k-1}}^{T_k} \int_0^x uW_j'(u, t)\,du\,dt \\
= \int_0^x u \int_{T_{k-1}}^{T_k} W_j'(u, t)\,dt\,du \\
= \int_0^x u \bar{W}_{jk}(u)\,du,
\]

so \(\mathcal{F}_{jk}\) is just the cost function for a second price auction with strictly monotone supply curve \(W_{jk}\). Therefore, the results of Proposition 2.4 hold for \(\Lambda_{jk}\).

Now, using Proposition 6.1, convert Problem \((6.23)\) into the finite optimization problem

\[
\begin{array}{l}
\text{minimize} \quad \frac{1}{M} \sum_{j=1}^{M} \sum_{k: T_k \leq T^j} \mathcal{F}_{jk}(q_j[k]) \\
\text{subject to} \quad \sum_{j \in A_i} \sum_{k: T_k \leq T_i} \gamma_{ij}[k]W_{jk}(q_j[k]) \geq C_i \\
\sum_{i \in B_j \cap T^j_k} \gamma_{ij}[k] \leq 1 [T_k \leq T^j] \\
i \in B_j, p_i < q_j[k] \implies \gamma_{ij}[k] = 0, \\
q_j[k] = \max_{i \in B_j \cap T^j_k} p_i, p_i \geq 0, \gamma_{ij}[k] \geq 0.
\end{array}
\]

Applying similar transformations as in Section 5.1, this is equivalent to

\[
\begin{array}{l}
\text{minimize} \quad \frac{1}{M} \sum_{j=1}^{M} \sum_{k: T_k \leq T^j} \bar{\Lambda}_{jk}(s_j[k]) \\
\text{subject to} \quad \sum_{j \in A_i} \sum_{k: T_k \leq T_i} r_{ij}[k] \geq C_i \\
\sum_{i \in B_j \cap T^j_k} r_{ij}[k] = s_j[k], r_{ij}[k] \geq 0 \\
i \in B_j, p_i < q_j[k] \implies r_{ij}[k] = 0, \\
q_j[k] = \max_{i \in B_j \cap T^j_k} p_i, s_j[k] = W_{jk}(q_j[k]), q_j[k] \geq 0, p_i \geq 0.
\end{array}
\]

As written, the final two lines of constraints are intractable. However, the cost function is independent of these constraints, and omitting them completely results in Problem \((P^\star)\). We show that for solutions of Problem \((P^\star)\), there necessarily exists variables \(p, q\) satisfying these additional constraints, and therefore that they can be omitted without affecting optimality.
Since \( P^* \) is convex with linear constraints, the first order Lagrangian conditions are necessary and sufficient. We consider multipliers \( \rho_i, \mu_{jk}, \theta_{ijk} \) and Lagrangian

\[
\mathcal{L}(s, r, \rho, \mu, \theta) = \sum_{j=1}^{M} \sum_{k:T_k \leq T_j} X_{jk}(s_j[k]) + \sum_{i=1}^{N} \rho_i \left( C_i - \sum_{j \in A_i} \sum_{k:T_k \leq T_i} r_{ij}[k] \right) \\
+ \sum_{j=1}^{M} \sum_{k:T_k \leq T_j} \mu_{jk} \left( \sum_{i \in B_j \cap T_k^N} r_{ij}[k] - s_j[k] \right) - \sum_{i=1}^{N} \sum_{j \in A_i} \sum_{k:T_k \leq T_i} \theta_{ijk} r_{ij}[k].
\]

Suppose that \( s, r \) are optimal primal solutions – we require multipliers satisfying \( \rho_i \geq 0, \theta_{ijk} \geq 0 \), complementary slackness, and the stationarity conditions \( \frac{\partial \mathcal{L}}{\partial r_{ij}[k]} = \frac{\partial \mathcal{L}}{\partial s_j[k]} = 0 \):

\[
\mu_{jk} = \theta_{ijk} + \rho_i \\
\bar{X}'_{jk}(s_j[k]) = \mu_{jk},
\]

where it is implicit that \((i, j, k)\) must satisfy \( j \in A_i, T_k \leq T_i \). From the first equation, it must be that \( \mu_{jk} \geq 0 \) and therefore we can solve the second equation to obtain \( s_j[k] = \bar{W}_{jk}(\mu_{jk}) \). We will write the dual problem and deduce that \( \mu_{jk} = \max_{i \in B_j \cap T_k^N} \rho_i \Delta = \rho^*_j[k] \). Substituting the above stationarity conditions into the Lagrangian we have the dual:

\[
\begin{align*}
\text{maximize} & \quad \sum_{j=1}^{M} \sum_{k:T_k \leq T_j} \left( f_{jk}(\mu_{jk}) - \mu_{jk} \bar{W}_{jk}(\mu_{jk}) \right) + \sum_{i=1}^{N} \rho_i C_i \\
\text{subject to} & \quad \mu_{jk} = \theta_{ijk} + \rho_i \\
& \quad \theta_{ijk} \geq 0, \rho_i \geq 0.
\end{align*}
\]

The objective function has derivative w.r.t. \( \mu_{jk} \) of simply \( -\bar{W}_{jk}(\mu_{jk}) < 0 \) and is therefore monotone decreasing. Assuming \( \rho_i \) is optimal for \( D^* \), the variable \( \theta_{ijk} \) is simply a slack variable which requires \( \mu_{jk} \geq \rho_i \) for \( i, j, k \) satisfying \( j \in A_i, T_k \leq T_i \). Therefore, by the monotonicity of the objective, the optimal \( \mu_{jk} \) must be the smallest feasible, which is \( \mu_{jk} = \max_{i \in B_j \cap T_k^N} \rho_i \). It follows that \( \theta_{ijk} = \rho^*_j[k] - \rho_i \).

These dual variables necessarily satisfy the constraints of Problem (A.29) with \( p_i = \rho_i \geq 0 \), \( q_j[k] = \rho^*_j[k] \) since by complementary slackness and the form of \( \theta_{ijk} \) above, we have \( p_i < \rho^*_j[k] \implies r_{ij}[k] = 0 \).

That the solution is globally optimal follows from the fact that solutions of \( P^* \), transformed to continuous solutions as described, satisfy the conditions of Proposition 5.3 and are therefore globally optimal for \( P_{cvx} \) by Proposition 5.5. These solutions can be transformed into solutions of the equivalent problem \( P \) as seen in Section 5.1.

\[\text{B \ Simulation (Additional Details)}\]

In this section we provide additional details on the methods used to produce the results of Section 7.
Cost and supply curves estimated from iPinYou data from the first 3 days of season 2. (a) Item arrival rates and the corresponding forecasts. The hatched region indicates an in-sample period with the remainder being out-of-sample. Our simulations run on similar 3 day periods with a 12 hour sliding window for a total of 9 periods of 72 hours each. (b) Estimates of the win probability function $\tilde{W}(x, t)$ for $t = 00:00:00$ (blue) and $t = 09:00:00$ (red). We compare Gaussian KDE (solid line) with a parametric Exponential CDF (dashed line).

B.1 Estimating Supply Curves

The iPinYou dataset consists of impression data derived from a real DSP and includes information about bidding prices, market prices, and user characteristics. We focus on the season two data (a week long period 2013-06-06 to 2013-06-12). In all cases, our supply curve estimates are 24h-periodic in time and therefore account for the natural daily (but not weekly) cycles in prices and arrival rates. Since this paper does not focus on the estimation of supply curves, we apply a simple estimation procedure using the entire dataset as input. Though this has the effect of leaking some information from the future, the estimation of supply curves is not subject to optimization, limiting the impact of this leakage. Moreover, the dataset is averaged into a single 24h periodic function and extended through periodicity. It is reasonable to believe that the previous week’s (out of sample) data would provide similar results. Figure 4 provides an illustration of estimated supply curves where only 72h of data is used to forecast the remaining 96h for purposes of illustration.

Provided with the iPinYou dataset is a user_tag which, according to [37] is “[a segment] in iPinYou’s proprietary audience database”. We therefore use the user_tag property as the “item types”, focusing on the five most common tags: 10063, 10006, 13866, 10024, and 10083.

B.2 Estimating the Supply Rate

In order to estimate the supply rate $\lambda_j(t)$ for each user tag, we have taken the inverse of the average of the time differences between arrival instants in each hour of the day, after removing outliers. Calculating the number of arrivals over an hour long period is not adequate as there appear to be large consistent gaps in arrival times: we suspect that the dataset was subsampled prior to being released.

This calculation results in estimates $\lambda_j[0], \ldots, \lambda_j[23]$ with time denoted in hours. The continuous
estimate was subsequently formed by smoothly interpolating between these points with a 24–periodic boundary, resulting in a function \( \tilde{\lambda}_j(t) \) defined on \([0, 24]\). A forecast for the average supply rate at time \( t \) is obtained via \( \lambda_j(t) = \tilde{\lambda}(t \mod 24) \).

An illustrative example for tags 10063, 10006, and 13866 is provided in Figure 4a.

### B.3 Estimating Win Probabilities

Similarly to the supply rate estimates, we estimate an average win probability function for each \( t \in [0, \ldots, 23] \) and then smoothly interpolate along \( t \) to estimate a 24–periodic function \( \tilde{W}_j(x,t) \) indicating the probability of winning an impression of type (user tag) \( j \) arriving at time \( t \) given a bid \( x \).

The estimate of \( x \mapsto \tilde{W}_j(x,t) \) is obtained by smoothing the histogram with a Gaussian kernel (bandwidth chosen simply by the Normal Reference Rule [29, Chap. 6.3]) for each \texttt{market_price} data point falling into the hour long window. The results of this procedure, as well as a comparison to a parametric estimate with an Exponential density are given in Figure 4b.

The \texttt{market_price} attribute in the dataset corresponds to the price actually paid in the second price auction. We have not accounted for the affects of censoring – since the DSP collected the dataset with large bids intended to win most impressions that were bid on, this isn’t a significant factor.

### B.4 Cost and Supply Curves

The supply curve \( W_j(x,t) \) is simply the product of the supply rate \( \lambda_j(t) \) and the win probability \( \tilde{W}_j(x,t) \). The cost curve \( f(x,t) \triangleq \int_0^x aW'(u,t) du \) is derived from the supply function where we have used numerical integration and differentiation to estimate \( f \) on a grid and subsequently extended \( f \) to the entire surface via interpolation.

### B.5 Simulating the Bidding Process

The simulations of Section 7 are obtained by storing the hour-by-hour inter-arrival and price data for each item type \( j \in [M] \) and sampling uniformly from these datasets. At simulation time \( t \in \mathbb{R}_+ \) we sample an inter-arrival time \( \Delta t \) and price \( P \) from the data for hour \( \lfloor t \rfloor + 1 \) with probability \( t - \lfloor t \rfloor \) and otherwise from the data for hour \( \lfloor t \rfloor \). A bid is solicited from a bidder (an implementation of \( \bar{P} \) or the algorithm of [23]) and if the bid exceeds \( P \) the bidder allocates that item to the fulfillment of a contract. The simulation time is then updated to \( t + \Delta t \) and the process continues.
**input**: A Bidder derived from Section 3 and solution to Problem $(P^*)$

**output**: Recording of Bidder’s item allocations to process into normalized acquisition curves $\tilde{c}(t)$

1 // Initialize:
2 $Q \leftarrow \text{Priority-Queue}([\])$ // Sort by time
3 $t \leftarrow 0$ // The “current” time
4 for $j \in [M]$ do
5     // Sample an interarrival time and a price
6     $(\Delta t, P) \leftarrow \text{Sample-Dataset}(t, j)$
7     $Q.\text{push}((t + \Delta t, P, j))$

8 // Simulate bidding process:
9 while $t < T_{\text{end}}$ do
10    $t, P, j \leftarrow Q.\text{pop}()$
11    $b \leftarrow \text{Bidder.solicit_bid}(t, j)$ // Ask for a bid on type $j$ at time $t$
12    if $b \geq P$ then
13        $\text{Bidder.award_item}(t, j)$ // Allocate items for winning bids
14    $(\Delta t, P) \leftarrow \text{Sample-Dataset}(t, j)$ // Append next $(t, P)$ pair for $j$ to $Q$
15    $Q.\text{push}((t + \Delta t, P, j))$

16 Function Sample-Dataset$(t, j)$:
17    $p \leftarrow t - \lfloor t \rfloor$
18    $U \sim U(0, 1)$ // Interpolate between hours
19    if $p \leq U$ then
20        $h \leftarrow \lfloor t \rfloor$
21    else
22        $h \leftarrow \lfloor t \rfloor + 1$
23    $\Delta t \leftarrow \text{Sample-Interarrivals}(\text{hour}=h, \text{type}=j)$
24    $P \leftarrow \text{Sample-Prices}(\text{hour}=h, \text{type}=j)$
25    return $(\Delta t, P)$

Algorithm 1: Bidding Simulation