Policy Choice in Time Series by Empirical Welfare Maximization

Toru Kitagawa†  Weining Wang‡  Mengshan Xu§

Abstract

This paper develops a novel method for policy choice in a dynamic setting where the available data is a multi-variate time series. Building on the statistical treatment choice framework, we propose Time-series Empirical Welfare Maximization (T-EWM) methods to estimate an optimal policy rule by maximizing an empirical welfare criterion constructed using nonparametric potential outcome time series. We characterize conditions under which T-EWM consistently learns a policy choice that is optimal in terms of conditional welfare given the time-series history. We derive a nonasymptotic upper bound for conditional welfare regret. To illustrate the implementation and uses of T-EWM, we perform simulation studies and apply the method to estimate optimal restriction rules against Covid-19.

Keywords: Causal inference, potential outcome time series, treatment choice, regret bounds, concentration inequalities.

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†Department of Economics, Brown University and Department of Economics, University College London. Email: toru.kitagawa@brown.edu

‡Department of Economics and Related Studies, University of York. Email: weining.wang@york.ac.uk

§Department of Economics, University of Mannheim. Email: mengshan.xu@uni-mannheim.de
1 Introduction

A central topic in economics is the nature of the causal relationships between economic outcomes and government policies, both within and across time periods. To investigate this, empirical research makes use of time-series data, with the aim of finding desirable policy rules. For instance, a government official may wish to use the past and current pandemic data to learn a policy of public restriction that is optimal in terms of a social welfare criterion. Building on the recent development of potential outcome time series (White and Lu (2010), Angrist et al. (2018), Bojinov and Shephard (2019), and Rambachan and Shepherd (2021)), this paper proposes a novel method to inform policy choice when the available data is a multi-variate time series.

In contrast to the structural and semi-structural approaches that are common in policy analysis in economics, such as structural vector autoregressions (SVAR), we set up the policy choice problem from the perspective of the statistical treatment choice proposed by Manski (2004). The existing statistical treatment choice literature typically focuses on microeconomic applications in a static setting, and the applicability of these methods to a time-series setting has yet to be explored. In this paper, we propose a novel statistical treatment choice framework for time-series data and study how to learn an optimal policy rule. Specifically, we consider extending the conditional empirical success (CES) rule of Manski (2004) and the empirical welfare maximization rule of Kitagawa and Tetenov (2018) to time-series policy choice, and characterize the conditions under which these approaches can inform welfare optimal policy. These conditions do not require functional form specifications for structural equations or the exact temporal dependence of the time-series observations, but can be connected to the structural approach under certain conditions.

In the standard microeconometric setting considered in the treatment choice literature, the planner has access to a random sample of cross-sectional units, and it is often assumed that the populations from which the sample was drawn and to which the policy will be applied are the same. These assumptions are not feasible or credible in the time-series context, which leads to several non-trivial challenges. First, the economic environment and the economy’s causal response to it may be time-varying. Assumptions are required to make it possible to learn an optimal policy rule for future periods based on available past data. In addition, the outcomes and policies observed in the available data can be statistically and causally dependent in a complex manner, and accordingly, the identifiability of social welfare under counterfactual policies becomes non-trivial and requires some conditions on how past policies were assigned over time. Second, to define an optimal policy in the time-series setting, it is reasonable to consider social welfare conditional on the history of observables at the time the policy decision is made. This conditional welfare contrasts with unconditional welfare, which averages conditional welfare with respect to hypothetical realizations of the history.
Third, when past data is used to inform policy, we have only a single realization of a time series in which the observations are dependent across the periods and possibly nonstationary. Such statistical dependence complicates the characterization of the statistical convergence of the welfare performance of an estimated policy. Fourth, if the planner wishes to learn a dynamic assignment policy, which prescribes a policy in each period over multiple periods on the basis of observable information available at the beginning of every period, the policy learning problem becomes substantially more involved. This is because a policy choice in the current period may affect subsequent policy choices through the current policy assignment and a realized outcome under the assigned treatment.

Taking into account these challenges, we propose time-series empirical welfare maximization (T-EWM) methods that construct an empirical welfare criterion based on a historical average of the outcomes and obtain a policy rule by maximizing the empirical welfare criterion over a class of policy rules. We then clarify the conditions on the causal structure and data-generating process under which T-EWM methods consistently estimate a policy rule that is optimal in terms of conditional welfare. Extending the regret bound analysis of Manski (2004) and Kitagawa and Tetenov (2018) to time-series dependent observations, we obtain a finite-sample uniform bound for welfare regret. We then characterize the convergence of welfare regret and establish the minimax rate optimality of the T-EWM rule.

Our development of T-EWM builds on the recent potential outcome time-series literature including White and Lu (2010), Angrist et al. (2018), Bojinov and Shephard (2019), and Rambachan and Shepherd (2021). In particular, to identify the counterfactual welfare criterion, we employ the sequential exogeneity restriction considered in Bojinov and Shephard (2019). These studies focus on retrospective evaluation of the causal impact of policies observed in historical data, and do not analyze how to perform future policy choices based on the historical evidence.

Since the seminal work of Manski (2004), statistical treatment choice and empirical welfare maximization have been active topics of research, e.g., Dehejia (2005), Stoye (2009, 2012), Qian and Murphy (2011), Tetenov (2012), Bhattacharya and Dupas (2012), Zhao et al. (2012), Kitagawa and Tetenov (2018, 2021), Kallus (2021), Athey and Wager (2021), Mbakop and Tabord-Meehan (2021), Kitagawa et al. (2021), among others. These works focus on a setting where the available data is a cross-sectional random sample obtained from an experimental or observational study with randomized treatment, possibly conditional on observable characteristics. Viviano (2021) and Ananth (2020) consider EWM approaches for treatment allocations where the training data features cross-sectional dependence due to network spillovers, while to our knowledge, this paper is the first to consider policy choice with time-series data. As a related but distinct problem, there is a large literature on the estimation of dynamic treatment regimes, Murphy (2003), Zhao et al. (2015), Han (2021), and Sakaguchi (2021). The problem of dynamic treatment regimes assumes that training data
is a short panel in which treatments have been randomized both among cross-sectional units and across time periods. Recently, Adusumilli et al. (2022) considers an optimal policy in a dynamic treatment assignment problem with a budget constraint where the planner allocates treatments to subjects arriving sequentially. The T-EWM framework, in contrast, assumes observations are drawn from a single time series as is common in empirical macroeconomics and empirical finance.

A large literature on multi-arm bandit algorithms analyzes learning and dynamic allocations of treatments when there is a trade-off between exploration and exploitation. See Lattimore and Szepesvári (2020) and references therein, Kock et al. (2020), Kasy and Sautmann (2019), and Adusumilli (2021) for recent works in econometrics. The setting in this paper differs from the standard multi-arm bandit setting in the following three respects. First, our framework treats the available past data as a training sample and focuses on optimizing short-run welfare. We are hence concerned with the performance of the method in terms of short-term regret rather than cumulative regret over a long horizon. Second, in the standard multi-arm bandit problem, subjects to be treated are assumed to differ across rounds, which implies that the outcome generating process is independent over time. This is not the case in our setting, and we include the realization of outcomes and policies in the past periods as contextual information for the current decision. Third, suppose that bandit algorithms can be adjusted to take into account the dependence of observations, our method is then analogous to the “pure exploration” class, involving a long exploration phase followed by a one-period exploitation at the very end. However, a major difference is that the bandit algorithm concerns data in a random experiment while our method is aimed at data in quasi-random experiments.

The analysis of welfare regret bounds is similar to the derivation of risk bounds in empirical risk minimization, as reviewed by Vapnik (1998) and Lugosi (2002). Risk bounds studied in the empirical risk minimization literature typically assume independent and identically distributed (i.i.d.) training data. A few exceptions, Jiang and Tanner (2010), Brownlees and Gudmundsson (2021), and Brownlees and Llorens-Terrazas (2021) obtain risk bounds for empirical risk minimizing predictions with time-series data, but they do not consider welfare regret bounds for causal policy learning.

The rest of the paper is organized as follows. Section 2 describes the setting using a simple illustrative model with a single discrete covariate. Section 3 discusses the general model with continuous covariates and presents the main theorems. In Section 4 we discuss extensions to our proposed framework, including a case of multi-period welfare functions, T-EWM’s links with structural VARs, Markov Decision Processes, and reinforcement learning. In Sections 5 and 6 we present simulation studies and an empirical application. Technical proofs and other details are presented in Appendices.
2 Model and illustrative example

In this section, we introduce the basic setting, notation, the conditional welfare criterion we aim to maximize, and conditions on the data-generating process that are important for the learnability of an optimal policy. Then, we illustrate the main analytical tools used to bound welfare regret through a heuristic model with a simple dynamic structure.

2.1 Notation, timing, and welfare

We suppose that the social planner (SP) is at the beginning of time \( T \). Let \( W_t \in \{0, 1\} \) denote a treatment or policy (e.g., nominal interest rate) implemented at time \( t = 0, 1, 2, \ldots \). To simplify the analysis, we assume that \( W_t \) is binary (e.g., a high or low interest rate regime). The planner sets \( W_T \in \{0, 1\}, T \geq 1 \), making use of the history of observable information up to period \( T \) to inform her decision. This observable information consists of an economic outcome (e.g., GDP, unemployment rate, etc.), \( Y_{0:T-1} = (Y_0, Y_1, Y_2, \ldots, Y_{T-1}) \), the history of implemented policies, \( W_{0:T-1} = (W_0, W_1, W_2, \ldots, W_{T-1}) \), and covariates other than the policies and the outcome (e.g., inflation), \( Z_{0:T-1} = (Z_0, Z_1, Z_2, \ldots, Z_{T-1}) \). \( Z_t \) can be a multidimensional vector, but both \( Y_t \) and \( W_t \) are assumed to be univariate.

Following Bojinov and Shephard (2019), we refer to a sequence of policies \( w_{0:t} = (w_0, w_1, \ldots, w_t) \in \{0, 1\}^{t+1}, t \geq 0 \), as a treatment path. A realized treatment path observed in the data \( 0 \leq t \leq T - 1 \) is a stochastic process \( W_{0:T-1} = (W_0, W_1, \ldots, W_{T-1}) \) drawn from the data generating process. Without loss of generality, we assume that \( Z_t \) is generated after the outcome \( Y_t \) is observed. The timing of realizations is therefore

\[
W_{t-1} \rightarrow Y_{t-1} \rightarrow Z_{t-1} \rightarrow W_t \rightarrow Y_t \rightarrow Z_t
\]

i.e., the transition between periods happens after \( Z_{t-1} \) is realised but before \( W_t \) is realised.\(^1\)

Let

\[
X_t = \{W_t, Y_t, Z_t'\}
\]

collect the observable variables for period \( t \), and let \( X_t \) be drawn from \( \mathcal{X} \) for \( t = 0, 1, 2, \ldots \). We define the filtration

\[
F_{t-1} = \sigma(X_{0:t-1}),
\]

where \( \sigma(\cdot) \) denotes the Borel \( \sigma \)-algebra generated by the variables specified in the argument. The filtration \( F_{T-1} \) corresponds to the planner’s information set at the time of making her decision.

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\(^1\)We do not allow \( Z_t \) to be realised between \( Y_t \) and \( W_t \). If there exists some \( Z \) that is realised after \( W_t \), before \( Y_t \), and is not causally effected by \( W_t \), the causal link is unaffected by placing \( Z \) before \( W_t \) and labeling it \( Z_{t-1} \); if this \( Z \) is realised after \( W_t \), before \( Y_t \), and is causally effected by \( W_t \), then \( Z \) is a bad control and should not be included in the model.
Following the framework of Bojinov and Shephard (2019), we introduce potential outcome time series. At each \( t = 0, 1, 2, \ldots \), and for every treatment path \( w_{0:t} \in \{0, 1\}^{t+1} \), let \( Y_t(w_{0:t}) \in \mathbb{R} \) be the realized period \( t \) outcome if the treatment path from 0 to period \( t \) were \( w_{0:t} \). Hence, we have a collection of potential outcome paths indexed by treatment path, \( \{Y_t(w_{0:t}) : w_{0:t} \in \{0, 1\}^{t+1}, t = 0, 1, 2, \ldots \} \), which defines \( 2^{t+1} \) potential outcomes in each period \( t \). This is an extension of the Neyman-Rubin causal model originally developed for cross-sectional causal inference. As maintained in Bojinov and Shephard (2019), the potential outcomes for each \( t \) are indexed by the current and past treatments \( w_{0:t} \) only. This imposes the restriction that any future treatment \( w_{t+p}, p \geq 1 \), does not causally affect the current outcome, i.e., an exclusion restriction for future treatments.

For a realized treatment path \( W_{0:t} \), the observed outcome \( Y_t \) and the potential outcomes satisfy
\[
Y_t = \sum_{w_{0:t} \in \{0, 1\}^t} 1\{W_{0:t} = w_{0:t}\}Y_t(w_{0:t})
\]
for all \( t \geq 0 \).

The baseline setting of the current paper considers the choice of policy \( W_T \) for a single period \( T \). We denote the policy choice based on observations up to period \( T - 1 \) by
\[
g : \mathcal{X}^T \to \{0, 1\}. \tag{2}
\]
The period-\( T \) treatment is \( W_T = g(X_{0:T-1}) \), and we refer to \( g(\cdot) \) as a decision rule. We also define the region in the space of the covariate vector for which the decision rule chooses \( W_T = 1 \) to be
\[
G = \{X_{0:T-1} : g(X_{0:T-1}) = 1\} \subset \mathcal{X}^T. \tag{3}
\]
We refer to \( G \) as a decision set.

We assume that the planner’s preferences for policies in period-\( T \) are embodied in a social welfare criterion. In particular, we define one-period welfare conditional on \( \mathcal{F}_{T-1} \) (conditional welfare, for short) to be\[3]
\[
W_T(g|\mathcal{F}_{T-1}) := \mathbb{E} [Y_T(W_{0:T-1}, 1)g(X_{0:T-1}) + Y_T(W_{0:T-1}, 0)(1 - g(X_{0:T-1}))|\mathcal{F}_{T-1}].
\]

\[2\] Section 4.1 discusses how to extend the single-period policy choice problem to multi-period settings.

\[3\] Throughout the paper, we acknowledge that the expectation, \( \mathbb{E} \), and the probability, \( \Pr \), are corresponding to the outer measure whenever a measurability issue is encountered.
With some abuse of notation, conditional welfare can be expressed with the decision set $G$ as its argument:

$$W_T(G|\mathcal{F}_{T-1}) := \mathbb{E}[Y_T(W_{0:T-1}, 1) 1\{X_{0:T-1} \in G\} + Y_T(W_{0:T-1}, 0) 1\{X_{0:T-1} \notin G\}|\mathcal{F}_{T-1}] . \quad (4)$$

This welfare criterion is conditional on the planner’s information set. This contrasts with the unconditional welfare criterion common in the cross-sectional treatment choice setting, where any conditioning variables (observable characteristics of a unit) are averaged out. In the time-series setting, it is natural for the planner’s preferences to be conditional on the realized history, rather than averaging over realized and unrealized histories, as would be the case if the unconditional criterion were used.

As we clarify in Appendix A.7, regret for conditional welfare and regret for unconditional welfare require different conditions for convergence, and their rates of convergence may differ. Hence, the existing results for regret convergence for unconditional welfare shown in Kitagawa and Tetenov (2018) do not immediately carry over to the time-series setting.

The planner’s optimal policy $g^*$ maximizes her one-period welfare,

$$g^* \in \operatorname{arg}
\max_g W_T(g|\mathcal{F}_{T-1}).$$

The planner does not know $g^*$, so she instead seeks a statistical treatment choice rule (Manski, 2004) $\hat{g}$, which is a decision rule selected on the basis of the available data $X_{0:T-1}$.

Our goal is to develop a way of obtaining $\hat{g}$ that performs well in terms of the conditional welfare criterion (4). Specifically, we assess the statistical performance of an estimated policy rule $\hat{g}$ in terms of the convergence of conditional welfare regret,

$$W_T(g^*|X_{0:T-1} = x_{0:T-1}) - W_T(\hat{g}|X_{0:T-1} = x_{0:T-1}), \quad (5)$$

and its convergence rate with respect to the sample size $T$. When evaluating realised regret, $X_{0:T-1}$ is set to its realized value in the data. On the other hand, when examining convergence, we accommodate statistical uncertainty over $\hat{g}$ by focusing on convergence with probability approaching one uniformly over a class of sampling distributions for $X_{0:T-1}$. A more precise characterization of the regret convergence results will be given below and in Section 3.

Manski (2004) also considers a conditional welfare criterion in the cross-sectional setting.
2.2 An illustrative model with a discrete covariate

We begin our analysis with a simple illustrative model, which provides a heuristic exposition of the main idea of T-EWM and its statistical properties. We cover more general settings and extensions in Sections 3 and 4.

Suppose that the data consists of a bivariate time series $X_{0:T-1} = ((Y_t, W_t) \in \mathbb{R} \times \{0, 1\} : t = 0, 1, \ldots, T-1)$ with no other covariates. To simplify exposition for the illustrative model, we impose the following restrictions on the dynamic causal structure and dependence of the observations.

**Assumption 2.1.** [Markov properties] The time series of potential outcomes and observable variables satisfy the following conditions:

(i) **Markovian exclusion:** for $t = 2, \ldots, T$ and for arbitrary treatment paths $(w_{0:t-2}, w_{t-1}, w_t)$ and $(w'_0: t-2, w_{t-1}, w_t)$, where $w_{0:t-2} \neq w'_0: t-2$,

$$Y_t(w_{0:t-2}, w_{t-1}, w_t) = Y_t(w'_0: t-2, w_{t-1}, w_t) := Y_t(w_{t-1}, w_t)$$

holds with probability one.

(ii) **Markovian exogeneity:** for $t = 1, \ldots, T$ and any treatment path $w_0:t$,

$$Y_t(w_0:t) \perp X_0:t-1|W_{t-1},$$

and for $t = 1, \ldots, T-1$,

$$W_t \perp X_0:t-1|W_{t-1}.$$  \hspace{1cm} (8)

These assumptions significantly simplify the dynamic structure of the problem. Markovian exclusion, Assumption 2.1 (i), says that only the current treatment $W_t$ and treatment in the previous period $W_{t-1}$ can have a causal impact on the current outcome. This allows the indices of the potential outcomes to be compressed to the latest two treatments $(w_{t-1}, w_t)$, as in (6). Markovian exogeneity, Assumption 2.1 (ii), states that once you condition on the policy implemented in the previous period $W_{t-1}$, the potential outcomes and treatment for the current period are statistically independent of any other past variables.

It is important to note that these assumptions do not impose stationarity: we allow the distribution of potential outcomes to vary across time periods. In addition, under Assumption 2.1, we can reduce the class of policy rules to those that map from $W_{T-1} \in \{0, 1\}$ to $W_T \in \{0, 1\}$, and

$$g : \{0, 1\} \rightarrow \{0, 1\}.$$
With no loss of conditional welfare, i.e., welfare conditional on $F_{T-1}$ can be simplified to

$$W_T(g|W_{T-1}) = E\{Y_T(W_{T-1}, 1)g(W_{T-1}) + Y_T(W_{T-1}, 0)(1 - g(W_{T-1}))|W_{T-1}\}. \quad (9)$$

To make sense of Assumption 2.1 and illustrate the relationship between the potential outcome time series and the standard structural equation modeling, we provide a toy example.

**Example 1.** Suppose the SP (monetary policy authority) is interested in setting a low or high interest rate at period $T$. Let $W_t$ denote the indicator for whether the interest rate in period $t$ is high ($W_t = 1$) or low ($W_t = 0$). $Y_t$ denotes a measure of social welfare, which can be a function of aggregate output, inflation, and other macroeconomic variables. Let $\varepsilon_t$ be i.i.d. shock that is statistically independent of $X_{0:t-1}$, and we assume the following structural equation for the causal relationship of $Y_t$ on $W_t$ (and its lag) and the regression dependence of $W_t$ on its lag

$$Y_t = \beta_0 + \beta_1 W_t + \beta_2 W_{t-1} + \varepsilon_t, \quad (10)$$

$$W_t = (1 - q) + \lambda W_{t-1} + V_t, \quad (11)$$

$$\lambda = p + q - 1,$$

$$\varepsilon_t \perp (W_t, X_{0:t-1}) \quad \forall 1 \leq t \leq T - 1 \quad \text{and} \quad \varepsilon_T \perp X_{0:T-1}. \quad (12)$$

If $W_{t-1} = 1$,

$$\begin{cases} V_t = 1 - p & \text{with probability } p \\ V_t = -p & \text{with probability } 1 - p, \end{cases} \quad (13)$$

if $W_{t-1} = 0$,

$$\begin{cases} V_t = -(1 - q) & \text{with probability } q \\ V_t = q & \text{with probability } 1 - q. \end{cases} \quad (14)$$

Compatibility with Assumption 2.1 can be seen as follows. Assumption 2.1(i) is implied by (10), where the structural equation of $Y_t$ involves only $(W_t, W_{t-1})$ as the factors of direct cause. Assumption 2.1(ii) is implied by (10), (12), and the fact that the distribution of $V_t$ depends solely on $W_{t-1}$, i.e., under (11), (13), and (14), we have $Pr(W_t|F_{t-1}) = Pr(W_t|W_{t-1})$.

To examine the learnability of the optimal policy rule, we further restrict the data generating process. First, we impose a strict overlap condition on the propensity score.

**Assumption 2.2.** [Strict overlap] Let $e_t(w) := Pr(W_t = 1|W_{t-1} = w)$ be the period-$t$ propensity score. There exists a constant $\kappa \in (0, 1/2)$, such that for any $t = 1, 2, \ldots, T - 1$

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5 The distribution of $W_t$ follows Hamilton (1989). However, the Markov switching model of Hamilton (1989) has unobserved $W_t$, which differs from this example.
and \( w \in \{0, 1\} \),
\[
\kappa \leq e_t(w) \leq 1 - \kappa.
\]

The next assumption imposes an unconfoundedness condition on observed policy assignment.

**Assumption 2.3.** [Unconfoundedness] For any \( t = 1, 2, \ldots, T - 1 \) and \( w \in \{0, 1\} \),
\[
Y_t(W_{t-1}, w) \perp W_t|X_{0:t-1}.
\]

This assumption states that the treatments observed in the data are sequentially randomized conditional on lagged observable variables. This is a key assumption to make unbiased estimation for the welfare feasible at each period in the sample, as employed in Bojinov and Shephard (2019) and others. It is worth noting that the above assumption together with Assumption 2.1(ii) implies \( Y_t(W_{t-1}, w) \perp W_t|W_{t-1} \).

Combining Assumption 2.1 with unconfoundedness (Assumption 2.3), we have that, for any measurable function \( f \) of the potential outcome \( Y_t(W_0:t-1) \) and treatment \( W_t \), it holds
\[
E(f(Y_t(W_0:t), W_t)|F_{t-1}) = E(f(Y_t(W_{t-1}, W_t), W_t)|W_{t-1}) = E(f(Y_t, W_t)|W_{t-1}). \tag{15}
\]

**Example 1 continued.** Assumption 2.2 is satisfied if \( 0 < p < 1 \) and \( 0 < q < 1 \); Assumption 2.3 is implied by (10) and (12).

Imposing Assumption 2.2 and 2.3 and assuming propensity scores are known, we consider constructing a sample analogue of (9) conditional on \( W_{T-1} = w \) based on the historical average of the inverse propensity score weighted outcomes,
\[
\hat{W}(g|W_{T-1} = w) = \frac{1}{T(w)} \sum_{1 \leq t \leq T-1: W_{t-1} = w} \left[ \frac{Y_t W_t g(W_{t-1})}{e_t(W_{t-1})} + \frac{Y_t (1 - W_t) \{1 - g(W_{t-1})\}}{1 - e_t(W_{t-1})} \right], \tag{16}
\]
where \( T(w) = \#\{1 \leq t \leq T - 1 : W_{t-1} = w\} \) is the number of observations where the policy in the previous period took value \( w \), i.e. the subsample corresponding to \( W_{t-1} = w \). Unlike the microeconometric setting considered in, e.g., Kitagawa and Tetenov (2018), we do not necessarily have \( \hat{W}(g|W_{T-1} = w) \) as a direct sample analogue for the SP’s social welfare objective, since we allow a non-stationary environment in which the historical average of the conditional welfare criterion can diverge from the conditional welfare in the current period. Nevertheless, we refer to \( \hat{W}(g|W_{T-1} = w) \) as the empirical welfare of the policy rule \( g \).

Denoting \( (\cdot|W_{T-1} = w) \) by \( (\cdot|w) \), we define the true optimal policy and its empirical
analogue to be,

\[ g^*(w) \in \arg\max_{g:\{w\} \rightarrow \{0,1\}} \mathcal{W}_T(g|w), \quad (17) \]

\[ \hat{g}(w) \in \arg\max_{g:\{w\} \rightarrow \{0,1\}} \hat{\mathcal{W}}(g|w), \quad (18) \]

where \( \hat{g} \) is constructed by maximizing empirical welfare over a class of policy rules (four policy rules in total). We call a policy rule constructed in this way the *Time-series Empirical Welfare Maximization* (T-EWM) rule. The construction of the T-EWM rule \( \hat{g} \) is analogous to the conditional empirical success rule with known propensity scores considered by Manski (2004) in the i.i.d. cross-sectional setting. In the time-series setting, however, the assumptions imposed so far do not guarantee that \( \hat{\mathcal{W}}(g|w) \) is an unbiased estimator of the true conditional welfare \( \mathcal{W}_T(g|w) \).

### 2.3 Bounding the conditional welfare regret of the T-EWM rule

A major contribution of this paper is characterizing conditions that justify the T-EWM rule \( \hat{g} \) in terms of the convergence of conditional welfare. This section clarifies these points in the context of our illustrative example.

To bound conditional welfare regret, our strategy is to decompose empirical welfare \( \hat{\mathcal{W}}(g|w) \) into a conditional mean component and a deviation from it. The deviation is the sum of a martingale difference sequence (MDS), and this allows us to apply concentration inequalities for the sum of MDS. Define an intermediate welfare function,

\[ \bar{\mathcal{W}}(g|w) = T(w)^{-1} \sum_{1 \leq t \leq T-1 : W_{t-1} = w} \mathbb{E} \{ Y_t(W_{t-1}, 1) g(W_{t-1}) + Y_t(W_{t-1}, 0) [1 - g(W_{t-1})] | W_{t-1} \}. \quad (19) \]

Under the strict overlap and unconfoundedness assumptions (i.e. Assumptions 2.2 and 2.3), the difference between empirical welfare and \( \bar{\mathcal{W}}(g|w) \) is a sum of MDS. Furthermore, we impose the assumption:

**Assumption 2.4.** [Invariance of the welfare ordering]

Given \( w \in \{0, 1\} \), let \( g^* = g^*(w) \) defined in (17). There exist a positive constant \( c > 0 \), such that for any \( g \in \{0, 1\} \),

\[ \mathcal{W}_T(g^*|w) - \mathcal{W}_T(g|w) \leq c[ \bar{\mathcal{W}}(g^*|w) - \hat{\mathcal{W}}(g|w)], \quad (20) \]

with probability one, i.e., \( P_T \) (inequality (20) holds) = 1, where \( P_T \) is the probability distribution for \( X_{0:T-1} \).

Noting that the left-hand side of (20) is nonnegative for any \( g \) by construction and \( c > 0 \),
this assumption implies that $\bar{W}(g^*|w) - \bar{W}(g|w) > 0$ must hold whenever $\mathcal{W}_T(g^*|w) - \mathcal{W}_T(g|w) > 0$. That is, the optimality of $g^*$ in terms of the conditional value of welfare at $T$ is maintained in the historical average of the conditional values of welfare. Under this assumption, having an estimated policy $\hat{g}$ that attains a convergence of $\bar{W}(g^*|w) - \bar{W}(\hat{g}|w)$ to zero guarantees that $\hat{g}$ is also consistent for the optimal policy $g^*$ in terms of the conditional welfare at $T$.

**Remark 1.** Assumption 2.4 can be restrictive in a situation where the dynamic causal structure of the current period is believed to be different from the past, but is weaker than stationarity. In the current example, if we were willing to assume, A2.4' The stochastic process

$$S_t(w) \equiv Y_T(W_{T-1}, 1)g(W_{T-1}) + Y_T(W_{T-1}, 0)[1 - g(W_{T-1})]|_{W_{T-1} = w}$$

is weakly stationary.

Under A2.4', $E\{Y_t(W_{t-1}, 1)g(W_{t-1}) + Y_t(W_{t-1}, 0) [1 - g(W_{t-1})]|_{W_{t-1} = w}\}$ is invariant for $2 \leq t \leq T$. Then, Assumption 2.4 will hold naturally.

Furthermore, Assumption 2.4 can be satisfied by many classic non-stationary processes in linear time-series models, including series with deterministic or stochastic time trends.

**Example 1 continued.**

(i) By Remark 1, Assumption 2.4 holds for Example 1 since

$$S_t(w) = \beta_0 + \beta_1 \cdot g + \beta_2 \cdot w + \varepsilon_t$$

is weakly stationary.

$\varepsilon_t$ remains an i.i.d. noise in the following settings.

(ii) If we replace (10) by

$$Y_t = \delta_t + \beta_1 W_t + \beta_2 W_{t-1} + \varepsilon_t,$$

where $\delta_t$ is an arbitrary deterministic time trend. The process $Y_t$ is trend stationary (non-stationary), but Assumption 2.4 still holds with $c = 1$ since those deterministic trends are canceled out by differences, i.e.,

$$\mathcal{W}_T(g^*|w) - \mathcal{W}_T(g|w) = \beta_1 (g^* - g) = \bar{W}(g^*|w) - \bar{W}(g|w).$$

(iii) If we replace (10) by

$$Y_t = \beta_0 + \beta_1 W_t + \beta_2 W_{t-1} + \sum_{i=0}^{t} \varepsilon_i,$$
the process $Y_t$ is non-stationary with stochastic trends, but Assumption 2.4 still holds with $c = 1$ since the stochastic trends are canceled out by differences. (iv) If we replace (10) by

$$Y_t = \delta_t + \beta_{1,t} W_t + \beta_{2,t} W_{t-1} + \varepsilon_t$$

to allow heterogeneous treatment effect. Then

$$W_T(g^*|w) - W_T(g|w) = \beta_{1,T} (g^* - g)$$

$$\bar{W}(g^*|w) - \bar{W}(g|w) = \bar{\beta}_w (g^* - g),$$

where $\bar{\beta}_w := \frac{1}{T(w)} \sum_{1 \leq t \leq T-1: W_{t-1} = w} \beta_{1,t}$. Since $W_T(g^*|w) - W_T(g|w)$ is non-negative by the definition of $g^*$ in (17), Assumption 2.4 holds if

$$\beta_{1,T} \text{ and } \bar{\beta}_w \text{ have the same sign and } c \geq \frac{|\beta_{1,T}|}{|\bar{\beta}_w|}. $$

Without loss of generality, we can assume that both $\beta_{1,T}$ and $\bar{\beta}_w$ are positive, then a sufficient condition for Assumption 2.4 is: There are positive numbers $l$ and $u$, such that $0 < l \leq \beta_{1,t} \leq u$ holds for all $t$. In this case, $c = \frac{u}{l}$.

Assumption 2.4 implies that $g^*$ also maximizes $\bar{W}$. Hence, if empirical welfare $\hat{W}(\cdot|w)$ can approximate $\bar{W}(\cdot|w)$ well, intuitively the T-EWM rule $\hat{g}$ should converge to $g^*$. The motivation for Assumption 2.4 is to create a bridge between $W_T(g^*|w) - W_T(\hat{g}|w)$, the population regret, and $\hat{W}(\cdot|w) - \bar{W}(\cdot|w)$, which is a sum of MDS with respect to filtration $\{\mathcal{F}_{t-1} : t = 1, 2, \ldots, T\}$. Specifically,

$$W_T(g^*|w) - W_T(\hat{g}|w)$$

$$\leq c \left[ \hat{W}(g^*|w) - \hat{W}(\hat{g}|w) \right]$$

$$= c \left[ \hat{W}(g^*|w) - \hat{W}(\hat{g}|w) + \hat{W}(\hat{g}|w) - \bar{W}(\hat{g}|w) \right]$$

$$\leq c \left[ \hat{W}(g^*|w) - \bar{W}(g^*|w) + \bar{W}(\hat{g}|w) - \bar{W}(\hat{g}|w) \right]$$

$$\leq 2c \sup_{g: \{w\} \to \{0,1\}} |\bar{W}(g|w) - \hat{W}(g|w)|,$$  \hspace{1cm} (21)

where the first inequality follows by Assumption 2.4. The second inequality follows from the definition of T-EWM rule $\hat{g}$ in (18).

To bound the right-hand side of (21), define

$$\hat{W}_t(g|w) = 1(W_{t-1} = w) \left[ \frac{Y_t W_t g(W_{t-1})}{e_t(W_{t-1})} + \frac{Y_t(1 - W_t)(1 - g(W_{t-1}))}{1 - e_t(W_{t-1})} \right],$$

\hspace{1cm} (Note that by (15), $E(\cdot|\mathcal{F}_{t-1}) = E(\cdot|W_{t-1})$.)

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and

$$\tilde{W}_t(g|w) = 1(W_{t-1} = w) E\{Y_t(W_{t-1}, 1)g(W_{t-1}) + Y_t(W_{t-1}, 0)(1 - g(W_{t-1}))|W_{t-1} = w\}.$$  

Then, we can express (16) and (19) as

$$c_W(g|w) = T - 1 \sum_{t=1}^{T-1} W_t(g|w),$$
$$\bar{W}_t(g|w) = T - 1 \sum_{t=1}^{T-1} \bar{W}_t(g|w),$$

and

$$\tilde{W}(g|w) - \bar{W}(g|w) = T - 1 \sum_{t=1}^{T-1} \left[ W_t(g|w) - \bar{W}_t(g|w) \right]$$

follows. By Assumptions 2.1 and 2.3, $$\tilde{W}_t(g|w) - \bar{W}_t(g|w)$$ is an MDS, so we can apply a concentration inequality for sums of MDS to obtain a high-probability bound for $$\tilde{W}(g|w) - \bar{W}(g|w)$$ that is uniform in $$g$$. Next, we impose

**Assumption 2.5.** There exists $$M < \infty$$ such that the support of outcome variable $$Y_t$$ is contained in $$[-M/2, M/2]$$.

This implies that the welfare functions are also bounded.

In Appendix A.1 we show that under Assumptions 2.1 to 2.5

$$\sup_{g: \{0,1\} \to \{0,1\}} E|\tilde{W}(g|w) - \bar{W}(g|w)| \leq \frac{C}{\sqrt{T-1}}, \quad (22)$$

where $$C$$ is a constant defined in Appendix A.1. Combining (21) and (22), we can conclude that the convergence rate of expected regret $$E|W_T(g^*|w) - W_T(\hat{g}|w)|$$ is upper-bounded by $$\frac{C}{\sqrt{T-1}}$$, and is uniform in the conditioning value of $$w$$.

## 3 Continuous covariates

This section extends the illustrative example of Section 2 by allowing $$X_t$$ to contain continuous variables. For simplicity of exposition, we maintain the first-order Markovian structure similarly to the illustrative example, but it is straightforward to incorporate a higher-order Markovian structure. In this section, for ease of exposition with continuous covariates, we switch our notation from a policy rule $$g$$ to its decision set $$G$$. The relationship between $$g$$ and $$G$$ is shown in (3).
3.1 Setting

In addition to \((Y_t, W_t)\), we incorporate general covariates \(Z_t\) into \(X_t \in \mathcal{X}\), which can be continuous. Now, \(X_t = (Y_t, W_t, Z_t)\). We maintain the Markovian dynamics, while modifying Assumptions 2.1, 2.2, and 2.3 as follows.

**Assumption 3.1.** [Markov properties] The time series of potential outcomes and observable variables satisfy the following conditions:

(i) **Markovian exclusion:** the same as Assumption 2.1 (i).

(ii) **Markovian exogeneity:** for \(t = 1, \ldots, T\) and any treatment path \(w_{0:t}\),

\[
Y_t(w_{0:t}) \perp X_{0:t-1}|X_{t-1},
\]

and for \(t = 1, \ldots, T-1\),

\[
W_t \perp X_{0:t-1}|X_{t-1}.
\]

Similarly to (9) in the illustrative example, Assumption 3.1 implies that we can reduce the conditioning information of \(F_{t-1}\) to only \(X_{t-1}\) and reduce the policy to a binary map of \(X_{t-1}\) without any loss of conditional welfare, i.e., we can partition the space of \(X_{t-1}\) into \(G\) and its complement. Following these reductions and considering the planner’s focus on the policy choice at period \(T\), we can formulate the planner’s objective function as follows:

\[
W_T(G|X_{T-1}) = \mathbb{E}\{Y_T(W_{T-1}, 1)1(X_{T-1} \in G) + Y_t(W_{T-1}, 0)1(X_{T-1} \notin G)|X_{T-1}\}.
\]

We assume the strict overlap and unconfoundedness restrictions under the general covariates as follows.

**Assumption 3.2** (Strict overlap). Let \(e_t(x) = \Pr(W_t = 1|X_{t-1} = x)\) be the propensity score at time \(t\). There exists \(\kappa \in (0, 1/2)\), such that

\[
\kappa \leq e_t(x) \leq 1 - \kappa
\]

holds for every \(t = 1, \ldots, T-1\) and each \(x \in \mathcal{X}\).

---

7It is worth noting that a direct extension of (19) to the case with continuous conditioning variables \(X_{T-1}\) is to discretize the domain of \(X_{T-1}\), i.e., to condition on one of the events that constitutes a finite partition of \(\mathcal{X}\),

\[
W_T(G|E) = \mathbb{E}\{Y_T(W_{T-1}, 1)1(X_{T-1} \in G) + Y_t(W_{T-1}, 0)1(X_{T-1} \notin G)|X_{T-1} \in E\},
\]

where \(E\) is a pre-specified set of conditioning variables (for example, one of the age groups). Then, it is straightforward to extend the result of (22).
Assumption 3.3 (Unconfoundedness). for any \( t = 1, 2, \ldots, T - 1 \) and \( w \in \{0, 1\} \)

\[
Y_t(W_{t-1}, w) \perp W_t | X_{1:t-1}.
\]

Under Assumptions 3.1 and 3.3, we can generalize (15) by including the set of covariates in the conditioning variables: for any measurable function \( f \),

\[
E(f(Y_t(W_0:t), W_t)|\mathcal{F}_{t-1}) = E(f(Y_t(W_{t-1}, W_t), W_t)|X_{t-1}) = E(f(Y_t, W_t)|X_{t-1}).
\] (26)

For continuous conditioning covariates \( X_{T-1} \), a simple sample analogue of the objective function is not available due to the lack of multiple observations at any single conditioning value of \( X_{T-1} \). One approach is to use nonparametric smoothing to construct an estimate for conditional welfare. For instance, with a kernel function \( K(\cdot) \) and a bandwidth \( h \)

\[
\hat{W}(G|x) = \sum_{t=1}^{T-1} K\left(\frac{X_{t-1} - x}{h}\right) \left[ \frac{Y_t W_t}{\pi_t(X_{t-1})} 1(X_{t-1} \in G) + \frac{Y_t (1-W_t)}{1-\pi_t(X_{t-1})} 1(X_{t-1} \notin G) \right],
\] (27)

where \( (\cdot|x) \) denotes \( (\cdot|X_{T-1} = x) \). Theorem A.1 in Appendix A.7 provides a regret bound for (27). The kernel method is a direct way to estimate an optimal policy with the conditional welfare criterion. However, the localization by bandwidth slows down the speed of learning; the regret of conditional welfare can only achieve a \( \frac{1}{\sqrt{T-1}} \)-rate of convergence rather than a \( \frac{1}{\sqrt{T-1}} \)-rate. We defer discussion of the statistical properties of the regret bound of the kernel approach to Appendix A.7. In the current section, we instead pursue an alternative approach that estimates an optimal policy rule by maximizing an empirical analogue of unconditional welfare over a specified class of decision sets \( \mathcal{G} \).

3.2 Bounding the conditional regret: continuous covariate case

We first clarify how a maximizer of conditional welfare (25) can be linked to a maximizer of unconditional welfare. With this result in hand, we can focus on estimating unconditional welfare and choosing a policy by maximizing it. In our setup, the complexity of the functional class needs to be specified by the user.

We will show that this approach can attain a \( \frac{1}{\sqrt{T-1}} \)-rate of convergence. Faster convergence relative to the kernel approach comes at the cost of imposing an additional restriction on the data generating process, as we spell out in Assumption 3.4 below.

A sufficient condition for the equivalence of maximizing conditional and unconditional welfare is that the specified class of policy rules \( \mathcal{G} \) includes the first best policy for the
unconditional problem. Unconditional welfare under policy $G$ is defined as

$$W_T(G) = \mathbb{E}\{Y_T(W_{T-1}, 1)1(X_{T-1} \in G) + Y_T(W_{T-1}, 0)1(X_{T-1} \notin G)\},$$

and an optimal policy within the class of feasible unconditional decision sets, $\mathcal{G}$, is

$$G_* \in \arg\max_{G \in \mathcal{G}} W_T(G). \quad (28)$$

We define the first-best decision set by

$$G_{FB}^* := \{x \in \mathcal{X} : \mathbb{E}[Y_T(W_{T-1}, 1) - Y_T(W_{T-1}, 0)|X_{T-1} = x] \geq 0\}. \quad (29)$$

To ensure that maximizing unconditional welfare corresponds to maximizing conditional welfare over $\mathcal{G}$, we impose the following key assumption:

**Assumption 3.4.** [Correct specification]

Let $W_T(G|x)$ be the conditional welfare as defined in (25), we have, at every $x \in \mathcal{X}$,

$$\arg\sup_{G \in \mathcal{G}} W_T(G) \subset \arg\sup_{G \in \mathcal{G}} W_T(G|x).$$

With this assumption, we can shift the focus to maximizing unconditional welfare, even when the planner’s ultimate objective function is conditional welfare. A condition that implies Assumption 3.4 holds is

$$G_{FB}^* \in \mathcal{G}.$$

This sufficient condition states that the class of policy rules over which unconditional empirical welfare is maximized contains the set of points in $\mathcal{X}$ where the conditional average treatment effect $\mathbb{E}[Y_T(W_{T-1}, 1) - Y_T(W_{T-1}, 0)|X_{T-1} = x]$ is positive. This assumption thus restricts the distribution of potential outcomes at $T$ and its dependence on $X_{T-1}$. We refer to Assumption 3.4 as “correct specification”[8] If the specified class of policies $\mathcal{G}$ is sufficiently rich, the correct specification assumption is credible. If the restrictions placed on $\mathcal{G}$ are motivated by some constraints that the planner has to meet for policy practice, the correct specification assumption is restrictive. It is possible that under specific circumstances, the planner can be confident that Assumption 3.4 is met. E.g., $\mathbb{E}[Y_T(W_{T-1}, 1) - Y_T(W_{T-1}, 0)|X_{T-1} = x]$ is believed to be monotonic in $x$ (element-wise) and the class $\mathcal{G}$ consists of decision sets with monotonic boundaries [Mbakop and Tabord-Meccan, 2021; Kitagawa et al., 2021].

The following proposition directly results from Assumptions 3.1 (i), Assumption 3.4, and the definition of $G_{FB}^*$. Under these conditions, the value of the conditional welfare function

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[8]Kitagawa and Tetenov (2018), Kitagawa et al. (2021), and Sakaguchi (2021) consider correct specification assumptions exclusively for unconditional welfare criteria. These assumptions correspond to $G_{FB}^* \in \mathcal{G}$.
evaluated at $G_*$ defined in (28) attains the maximum.

**Proposition 3.1.** Under Assumptions 3.1 (i) and 3.4, an optimal policy rule $G_*$ in terms of the unconditional welfare maximizes the conditional welfare function,

$$G_* \in \arg \max_{G \in \mathcal{G}} W_T(G|X_{T-1}).$$

Furthermore, if the first best solution belongs to the class of feasible policies rules, $G_{FB}^* \in \mathcal{G}$, then we have

$$G_{FB}^* \in \arg \max_{G \in \mathcal{G}} W_T(G|X_{T-1}).$$

Having assumed the relationship of optimal policies between the two welfare criteria, we now show how the unconditional welfare function can bound the conditional function. For $G \in \mathcal{G}$ and $X_{T-1} = x$, define conditional regret as

$$R_T(G|x) = W_T(G_*|x) - W_T(G|x).$$

Note that the unconditional regret can be expressed as an integral of the conditional regret,

$$W_T(G) = \int W_T(G|x) dF_{X_{T-1}}(x),$$

$$R_T(G) = W_T(G_*) - W_T(G) = \int R_T(G|x) dF_{X_{T-1}}(x).$$

For $x' \in \mathcal{X}$, define

$$A(x', G) = \{ x \in \mathcal{X} : R_T(G|x) \geq R_T(G|x') \},$$

$$p_{T-1}(x', G) = \Pr(X_{T-1} \in A(x', G)) = \frac{dF_{X_{T-1}}(x)}{\int_{x \in A(x', G)} dF_{X_{T-1}}(x)},$$

and let $x^{obs}$ denote the observed value of $X_{T-1}$. We assume the following:

**Assumption 3.5.** For $x^{obs} \in \mathcal{X}$ and any $G \in \mathcal{G}$, there exists a positive constant $p$ such that

$$p_{T-1}(x^{obs}, G) \geq p > 0. \quad (30)$$

**Remark 2.** This assumption is satisfied if $X_{T-1}$ is a discrete random variable taking a finite number of different values. In this case, $p_{T-1}(x^{obs}, G) \geq \min_{x \in \mathcal{X}} \Pr(X_{T-1} = x) > 0$, so we can set $p = \min_{x \in \mathcal{X}} \Pr(X_{T-1} = x)$. If $X_t$ is continuous, then we need to exclude a set of points around the maximum of the function $R_T(G|x)$ for the assumption to hold. Namely, we can assume that we focus on $x$ belonging to a compact subset $\tilde{\mathcal{X}} \subset \mathcal{X}$ such that $\arg \max_{x \in \mathcal{X}} R_T(G|x) \notin \tilde{\mathcal{X}}$. If we would like to include the whole support of $X_t$, we can
modify the proof by imposing an additional uniform continuity condition on $R_T(G|\cdot)$.

The following lemma provides a bound for conditional regret $R_T(G|^{x_{obs}})$ using unconditional regret $R_T(G)$.

**Lemma 3.1.** Under Assumptions [3.5]

$$R_T(G|^{x_{obs}}) \leq \frac{1}{p} R_T(G).$$

(31)

Using Assumption [3.5] and Lemma [3.1] to bound conditional regret, we proceed to construct an empirical analogue of the welfare function and provide theoretical results for the regret bound. The sample analogue of $W_T(G)$ can be expressed as

$$\hat{W}(G) = \frac{1}{T-1} \sum_{t=1}^{T-1} \left[ \frac{Y_t W_t}{e_t(X_{t-1})} 1(X_{t-1} \in G) + \frac{Y_t(1 - W_t)}{1 - e_t(X_{t-1})} 1(X_{t-1} \notin G) \right],$$

(32)

and we define

$$\hat{G} \in \text{argmax}_{G \in \mathcal{G}} \hat{W}(G).$$

(33)

In addition, define two intermediate welfare functions,

$$\bar{W}(G) := \frac{1}{T-1} \sum_{t=1}^{T-1} \mathbb{E} \{ Y_T(W_{T-1}, 1) 1(X_{T-1} \in G) + Y_T(W_{T-1}, 0) 1(X_{T-1} \notin G) | \mathcal{F}_{t-1} \},$$

$$\bar{W}(G) := \frac{1}{T-1} \sum_{t=1}^{T-1} \mathbb{E} \{ Y_T(W_{T-1}, 1) 1(X_{T-1} \in G) + Y_T(W_{T-1}, 0) 1(X_{T-1} \notin G) \}. \quad (34)$$

To obtain a regret bound for unconditional welfare, Assumption [2.4] is modified to

**Assumption 3.6.** For any $G \in \mathcal{G}$, there exists some constant $c$

$$W_T(G_*) - W_T(G) \leq c[\bar{W}(G_*) - \bar{W}(G)],$$

(35)

with probability one, i.e., $P_T$ (inequality [35] holds) = 1, where $P_T$ is the probability distribution for $X_{0:T-1}$.

Below, we bound the regret for conditional welfare $W_T(G_*|^{x_{obs}}) - W_T(\hat{G}|^{x_{obs}})$ by regret for unconditional welfare $[W_T(G_*) - W_T(\hat{G})]$, and further by $[\bar{W}(G) - \bar{W}(G)]$ (up to constant
factors).
\[
\mathcal{W}_T(G_*|x^{obs}) - \mathcal{W}_T(\hat{G}|x^{obs}) \leq \frac{1}{p} \left[ \mathcal{W}_T(G_*) - \mathcal{W}_T(\hat{G}) \right],
\]
\[
\leq \frac{c}{p} \left[ \widehat{W}(G_*) - \widehat{W}(G) \right],
\]
\[
\leq \frac{2c}{p} \sup_{G \in \mathcal{G}} \left| \widehat{W}(G) - \widehat{W}(G) \right|. \quad (36)
\]

The first inequality follows from Lemma 3.1 and Assumption 3.5. The second inequality follows from (35). The last inequality follows from an argument similar to the one below (21).

Note that \( \widehat{W}(G) - \widehat{W}(G) \) is not a sum of MDS. Instead, it can be decomposed as
\[
\widehat{W}(G) - \widehat{W}(G) = \widehat{W}(G) - \widehat{W}(G) + (\widehat{W}(G) - \widehat{W}(G)),
\]
\[
= I + II,
\]
where \( I := \widehat{W}(G) - \widehat{W}(G) \) and \( II := \widehat{W}(G) - \widehat{W}(G) \). Subject to assumptions specified later, Theorem 3.1 below shows that \( II \), which is a sum of MDS, converges at \( \frac{1}{\sqrt{T}} \)-rate, and Theorem 3.2 below shows that \( I \) converges at the same rate.

(37) reveals that our proof strategy is considerably more complicated than the proof for the EWM model with i.i.d. observations of Kitagawa and Tetenov (2018), although the rates are similar. Specifically, we need to derive a bound for the tail probability of the sum of martingale difference sequences. In addition, we need to handle complex functional classes induced by non-stationary processes. For the EWM model, the main task is to show the convergence rate of a sample analogue of \( II \), which can be achieved with standard empirical process theory for i.i.d. samples. In comparison, we not only have to treat our \( II \) more carefully due to time-series dependence, but we also have to deal with \( I \).

To proceed, we restate Assumption 2.5 as

**Assumption 3.7.** There exists \( M < \infty \) such that the support of outcome variable \( Y_t \) is contained in \([-M/2, M/2]\).

The bound of the conditional regret will be further established in the following two subsections.
3.2.1 Bounding II

Define empirical welfare at time $t$ and its population conditional expectation as follows,

$$
\hat{W}_t(G) = \frac{Y_t W_t}{e_t(X_{t-1})} 1(X_{t-1} \in G) + \frac{Y_t (1 - W_t)}{1 - e_t(X_{t-1})} 1(X_{t-1} \notin G),
$$

$$
\bar{W}_t(G) = \mathbb{E} \{ Y_t(W_{t-1}, 1) 1(X_{t-1} \in G) + Y_t(W_{t-1}, 0) 1(X_{t-1} \notin G) | \mathcal{F}_{t-1} \}. 
$$

Thus, we examine two summations

$$
\hat{W}(G) = \frac{1}{T-1} \sum_{t=1}^{T-1} \hat{W}_t(G),
$$

$$
\bar{W}(G) = \frac{1}{T-1} \sum_{t=1}^{T-1} \bar{W}_t(G).
$$

For each $t = 1, \ldots, T - 1$, define a functional class indexed by $G \in \mathcal{G}$,

$$
\mathcal{H}_t = \{ h_t(\cdot; G) = \hat{W}_t(G) - \bar{W}_t(G) : G \in \mathcal{G} \}. \tag{38}
$$

The arguments of the function $h_t(\cdot; G)$ are $Y_t$, $W_t$, and $X_{t-1}$.

Given the class of functions $\mathcal{H}_t$, we consider a martingale difference array $\{ h_t(Y_t, W_t, X_{t-1}; G) \}_{t=1}^{n}$ and denote its average by

$$
\mathbb{E}_n h \overset{\text{def}}{=} \frac{1}{n} \sum_{t=1}^{n} h_t(Y_t, W_t, X_{t-1}, G),
$$

where $h \overset{\text{def}}{=} \{ h_1(\cdot; G), h_2(\cdot; G), \ldots, h_n(\cdot; G) \}$, and we suppress $n$ and $G$ if there is no confusion in the context.

Since we do not restrict $X_t$ to be stationary, we shall handle a vector of functional classes which possibly varies over $t$. To this end, we define the following set of notations. Let $H_t$ denote the envelope for the function class $\mathcal{H}_t$, and $\overline{H}_n = (H_1, H_2, \ldots, H_n)'$, and $H_n = \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_n$. For a function $f$ supported on $\mathcal{X}$, define $\| f \|_{Q,r} \overset{\text{def}}{=} (\int_{x \in \mathcal{X}} |f(x)|^{r} dQ(x))^{1/r}$, and for an $n$-dimensional vector $\mathbf{v} = \{ v_1, \ldots, v_n \}$, its $l_2$ norm is denoted by $|\mathbf{v}|_2 \overset{\text{def}}{=} (\sum_{i=1}^{n} v_i^2)^{1/2}$.

The covering number of a function class $\mathcal{H}$ w.r.t. a metric $\rho$ is denoted by $\mathcal{N}(\varepsilon, \mathcal{H}, \rho(\cdot))$. For two series of functions $f = \{ f_i \}_{i=1}^{n}$ and $g = \{ g_i \}_{i=1}^{n}$, define the metrics $\rho_{2,n}(f, g) = (n^{-1} \sum_{i} |f_i - g_i|^2)^{1/2}$ and $\sigma_n(f, g) = (n^{-1} \sum_{i} \mathbb{E}[|f_i - g_i|^2 | \mathcal{F}_{t-1}])^{1/2}$. Let $\alpha_n$ denote an $n$-dimensional vector in $\mathbb{R}^n$ and $\circ$ denote the element-wise product. In the next assumption,

---

9 We use $n$ to represent the number of summands since the endpoints of samples may vary across different settings in this section, subsequent sections, and appendices. For example, in the case of multi-period welfare functions, the endpoint of a sample is no longer fixed at $T - 1$.  

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we want to bound the covering number of $\mathcal{N}(\delta|H_n|_2, H_n, \rho_{2,n})$ by the covering number of all its one-dimensional projection.

**Assumption 3.8.** Let $n = T - 1$. For any discrete measures $Q$, any $\alpha_n \in \mathbb{R}_+^n$, and all $\delta > 0$, we have

$$
\mathcal{N}(\delta |\tilde{\alpha}_n \circ H_n|_2, \tilde{\alpha}_n \circ H_n, |\cdot|_2) \leq \max_t \mathcal{N}(\delta \|H_t\|_{Q,2}, H_n, \|\cdot\|_{Q,2}) \lesssim K (v + 1)(4e)^{v+1}(\frac{2}{\delta})^{rv},
$$

where $K$, $v$, $r$, and $e$ are positive constants; $r$ is an integer, and $\tilde{\alpha}_{n,t} = \sqrt{\alpha_{n,t} \sqrt{P_t \alpha_{n,t}}}$. The above assumption restricts the complexity of the functional class to be of polynomial discrimination, and the complexity index $v$ appears in the derived regret bounds. See Appendix A.4 for a justification of Assumption 3.8.

**Assumption 3.9.** There exists a constant $L > 0$ such that $\Pr(\sigma_n(f, g)/\rho_{2,n}(f, g) > L)$ $\to 0$ as $n \to \infty$. Also, $\Pr((n^{-1} \sum_t \mathbb{E}[(f_t - g_t)^2|\mathcal{F}_{t-1}])^{1/2}/\rho_{2,n}(f, g) > L)$ $\to 0$ as $n \to \infty$.

$\rho_{2,n}(f, g)^2$ is the quadratic variation difference and $\sigma_n(f, g)^2$ is its conditional equivalent. It is evident that $\rho_{2,n}(f, g)^2 - \sigma_n(f, g)^2$ involves martingale difference sequences. In the special case of i.i.d. observations, $\sigma_n(f, g)^2$ is equivalent to the sample average of unconditional expectations. Assumption 3.9 can thus be viewed as specifying that $\rho_{2,n}(f, g)^2$ and $\sigma_n(f, g)^2$ are asymptotically equivalent in a probability sense. A similar condition can be seen, for example, in Theorem 2.23 in Hall and Heyde (2014).

Let $A \lesssim_p B$ denote $A = O_p(B)$. Then, we have for $II$:

**THEOREM 3.1.** Under Assumptions 3.1 to 3.3 and 3.7 to 3.9

$$
\sup_{G \in \mathcal{G}} |\tilde{W}(G) - \bar{W}(G)| \lesssim_p C \sqrt{\frac{v}{T - 1}},
$$

where $C$ is a constant that depends only on $M$ and $\kappa$.

The proof of Theorem 3.1 is presented in Appendix A.3.

### 3.2.2 Bounding I

Here, we complete the process of bounding unconditional regret. Let

$$
S_t(G) \overset{\text{def}}{=} Y_t(W_{t-1}, 1)1(X_{t-1} \in G) + Y_t(W_{t-1}, 0)1(X_{t-1} \notin G),
$$

and

$$
\overline{S}_t(G) \overset{\text{def}}{=} \mathbb{E}(S_t(G)|\mathcal{F}_{t-1}) - \mathbb{E}(S_t(G)|\mathcal{F}_{t-2}),
$$

(40)
\[ \hat{S}_{\ell}(G) \overset{\text{def}}{=} E(S_{\ell}(G)|F_{\ell-2}) - E(S_{\ell}(G)). \] (41)

We can apply a similar technique in Theorem 3.1 (See Lemma A.3 in Appendix A.3) to bound the sum of \( E(S_{\ell}(G)|F_{\ell-1}) - E(S_{\ell}(G)|F_{\ell-2}) \). The summand \( E(S_{\ell}(G)|F_{\ell-2}) - E(S_{\ell}(G)) \) is handled below. Recalling (34) and (37), we have \( I = \frac{1}{T-1} \sum_{t=1}^{T-1} \hat{S}_{\ell}(G) + \frac{1}{T-1} \sum_{t=1}^{T-1} S_{\ell}(G) \).

Define the functional class,
\[ \mathcal{S}_{\ell} = \{ f_{\ell} : f_{\ell} = E(S_{\ell}(G)|F_{\ell-1}) - E(S_{\ell}(G)|F_{\ell-2}) : G \in \mathcal{G} \}, \]
\[ \hat{\mathcal{S}}_{\ell} = \{ f_{\ell} : f_{\ell} = E(S_{\ell}(G)|F_{\ell-2}) - E(S_{\ell}(G)) : G \in \mathcal{G} \}. \]

**Definition 3.1.** Let \( \{\varepsilon_t\}_{t=-\infty}^{\infty} \) be a sequence of i.i.d. random variables, and \( \{g_t\}_{t=-\infty}^{\infty} \) is a sequence of measurable functions of \( \varepsilon \)'s, which might vary with time \( t \). For a process \( \xi \overset{\text{def}}{=} \{\xi_t\}_{t=-\infty}^{\infty} \) with \( \xi_{t} \overset{\text{def}}{=} g_t(\varepsilon_t, \varepsilon_{t-1}, \cdots) \) and integers \( l, q \geq 0 \), we define the dependence adjusted norm for an arbitrary process \( X_t \) as
\[ \theta_{\xi,q} \overset{\text{def}}{=} \max_{t} \sum_{l=0}^{\infty} \|\xi_{t,l} - \xi_{t,l}'\|_q, \] (42)
where \( \|\cdot\|_q \) denotes \( (E|\cdot|^q)^{1/q} \), and \( \xi_{t,l}' = g_t(\varepsilon_t, \cdots, \varepsilon_{t-l, \cdots}) \) is the random variable \( \xi_t \) with its \( l \)-th lag replaced by \( \varepsilon_{t-l} \), an independent copy of \( \varepsilon_{t-l} \). The subexponential/Gaussian dependence adjusted norm is denoted as,
\[ \Phi_{\phi_{\xi}}(\xi) = \sup_{q \geq 2} (\theta_{\xi,q}/q^{\hat{v}}), \] (43)
where \( \hat{v} = 1/2 \) (resp. 1) corresponds to the case that the process \( \xi_t \) is sub-Gaussian (resp. sub-exponential). For \( \hat{v} = 1/2 \) or 1, we define \( \gamma = 1/(1 + 2\hat{v}) \), and impose the following assumptions.

**Assumption 3.10.** \( \hat{S}_{\ell}(G) = g_t(\varepsilon_t, \varepsilon_{t-1}, \cdots) \) and \( X_t = \tilde{g}_t(\varepsilon_t, \varepsilon_{t-1}, \cdots) \), where \( \varepsilon_t \) is a sequence of i.i.d. random variables, and \( g \) and \( \tilde{g} \) are continuously differentiable measurable functions of \( \varepsilon \)'s.

**Assumption 3.11.** (i) \( Z_t \perp X_{0:t-1}|X_{t-1} \); (ii) \( \partial E(\hat{S}_{\ell}(G)|F_{\ell-1})/\partial\varepsilon_{t-1} \) has envelope \( F_{t,l}(X_{t-2}, \varepsilon_{t-1}) \), i.e., \( \sup_{G} |\partial E(\hat{S}_{\ell}(G)|F_{\ell-1})/\partial\varepsilon_{t-1}| \leq F_{t,l}(X_{t-2}, \varepsilon_{t-1}) \) for every \( t, l \).

**Assumption 3.12.** For \( \hat{v} = 1/2 \) or 1, \( \Phi_{\phi_{\xi,F}} \overset{\text{def}}{=} \sup_{q \geq 2} (\sum_{l \geq 0} \max_{l} \|F_{t,l}(X_{t-2}, \varepsilon_{t-1})(\varepsilon_{t-1} - \varepsilon_{t-1}^*)\|_q)/q^{\hat{v}} < \infty \).

**Assumption 3.13.** Suppose that \( F_t \) (resp. \( \hat{F}_t \)) is the envelope of the functional class \( \mathcal{S}_t \) (resp. \( \hat{\mathcal{S}}_t \)). Define \( \overline{F}_n = (F_1, F_2, \cdots, F_n) \) (resp. \( \overline{F}_n = (\hat{F}_1, \hat{F}_2, \cdots, \hat{F}_n) \)), and \( \mathbf{F}_n = \{\mathcal{S}_1, \mathcal{S}_2, \cdots, \mathcal{S}_n\} \) (resp. \( \hat{\mathbf{F}}_n = \{\hat{\mathcal{S}}_1, \hat{\mathcal{S}}_2, \cdots, \hat{\mathcal{S}}_n\} \)). Let \( Q \) denote a discrete measure over a finite number of \( n \)
points, and $V$ be a positive integer. For $n = T - 1$, some positive $V$ and $v$, and all $\delta > 0$,

$$\mathcal{N}(\delta|\tilde{\alpha}_n \circ \tilde{F}_n|_{2}, \tilde{\alpha}_n \circ \bar{F}_n, |.|_{2}) \leq \max_t \mathcal{N}(\delta\|\tilde{F}_t\|_{Q,2}, \tilde{S}_t, \|.|_{2}) \lesssim (1/\delta)^V. \quad (44)$$

$$\mathcal{N}(\delta|\tilde{\alpha}_n \circ \tilde{F}_n|_{2}, \tilde{\alpha}_n \circ \bar{F}_n, |.|_{2}) \leq \max_t \mathcal{N}(\delta\|\tilde{F}_t\|_{Q,2}, \tilde{S}_t, \|.|_{2}) \lesssim (1/\delta)^v. \quad (45)$$

Assumption 3.10 imposes that the time series $\tilde{S}_t(G)$ and $X_t$ can be expressed as measurable functions of i.i.d. innovations $\varepsilon_t$. Assumption 3.11(i) (combined with Assumption 3.1 together) implies that $E(S_t(G)|F_{t-2}) = E(S_t(G)|X_{t-2})$. Assumption 3.11(ii) is a standard envelope assumption on the functional class, and it states that the partial derivative is enveloped by a function of $X_{t-2}$ and $\varepsilon_{t-1}$. This is partly due to Assumption 3.1. Assumption 3.12 implies that $\Phi_{\phi_{\tilde{v}}}(\tilde{S}_t(G)) < \infty$ for $\tilde{v} = 1/2$ or 1. Assumption 3.13 restricts the complexity of the functional classes. Then, we have the following tail probability bound.

**THEOREM 3.2.** Under Assumptions 3.1 to 3.3, 3.7, and 3.9 to 3.13,

$$\sup_{P_T \in \mathcal{P}_T(M, \kappa)} \mathbb{E}_{P_T} \left[ \mathcal{W}_T(G^*) - \mathcal{W}_T(\hat{G}) \right] \lesssim c_T \frac{2V(\log T)v \gamma^{1/\gamma} \Phi_{\phi_{\tilde{v}}}}{\sqrt{T - 1}} + C \sqrt{\frac{v}{T - 1}},$$

where $c_T$ is a large enough constant. $\tilde{v} = 1/2$ or 1, and $\gamma = 1/(1 + 2\tilde{v})$. $V$ and $v$ are the constants defined in Assumption 3.13 and $C$ is the similar constant in Theorem 3.1 which depends only on $M$ and $\kappa$.

A proof is presented in Appendix A.5. The bound depends on the complexity of the functional class, $V$ and $v$, and the time-series dependency, $\Phi_{\phi_{\tilde{v}}}$, $F$. As we consider the sample analogue of unconditional welfare, all the observations are utilized, and we have a $1/\sqrt{T - 1}$ rate of convergence.

### 3.2.3 The regret bound

Now, we can obtain the overall bound for unconditional welfare. Let $P_T$ be a joint probability distribution of a sample path of length $(T - 1)$, $\mathcal{P}_T(M, \kappa)$ be the class of $P_T$, which satisfies Assumptions 3.1 to 3.7 and $E_{P_T}$ be the expectation taken over different realizations of random samples.

**THEOREM 3.3.** Under Assumptions 3.1 to 3.13,

$$\sup_{P_T \in \mathcal{P}_T(M, \kappa)} \mathbb{E}_{P_T} \left[ \mathcal{W}_T(G^*) - \mathcal{W}_T(\hat{G}) \right] \lesssim C \sqrt{\frac{v}{T - 1}} + \frac{c_T \frac{2V(\log T)v \gamma^{1/\gamma} \Phi_{\phi_{\tilde{v}}}}{\sqrt{T - 1}}} {\sqrt{T - 1}}, \quad (46)$$

where $G^*$ is defined in (28).
This theorem follows from (36), Theorems 3.1 and 3.2 and a similar reasoning in Appendix A.1.

**Remark.** We have presented our main results of welfare regret upper bounds under the first-order Markovian structure of Assumption 3.1. This is for ease of exposition, and it is straightforward to relax Assumption 3.1 by introducing a higher-order Markovian structure. For example, current observations can depend causally or statistically on the realized treatment and covariates over the last \( q \) periods, \( q \geq 1 \). The following Assumption 3.1* corresponds to \( q > 1 \).

**Assumption 3.1* [q-th order Markov properties]** For an integer \( q > 1 \), the time series of potential outcomes and observable variables satisfy the following conditions:

(i) \( q \)-th order Markovian exclusion: for \( t = q + 1, q + 2, \ldots, T \) and for arbitrary treatment paths \((w_{0:t-q-1}, w_{t-q:t})\) and \((w'_{0:t-q-1}, w_{t-q:t})\), where \( w_{0:t-q-1} \neq w'_{0:t-q-1} \),

\[
Y_t(w_{0:t-q-1}, w_{t-q:t}) = Y_t(w'_{0:t-q-1}, w_{t-q:t}) := Y_t(w_{t-q:t})
\]
holds with probability one.

(ii) \( q \)-th order Markovian exogeneity: for \( t = q, q + 1, \ldots, T \) and any treatment path \( w_{0:t} \),

\[
Y_t(w_{0:t}) \perp X_{0:t-1} \mid X_{t-q:t-1},
\]
and for \( t = q, q + 1, \ldots, T - 1 \),

\[
W_t \perp X_{0:t-1} \mid X_{t-q:t-1}.
\]

When the Markov properties are of order \( q \), the propensity score can be defined as \( e_t(x) = \Pr(W_t = 1 \mid X_{t-q:t-1} = x) \), where the vector \( x \) is of the same dimension as the random vector \( X_{t-q:t-1} \). Assumption 3.3 can be modified to: for any \( t = 1, 2, \ldots, T - 1 \) and \( w \in \{0, 1\} \),

\[
Y_t(W_{t-q:t-1}, w) \perp W_t \mid X_{t-q:t-1}.
\]

In addition, a policy \( G \) becomes a region defined on the space of the random vector \( X_{T-q:T-1} \), and

\[
\mathcal{W}_T(G) = \mathbb{E} \{ Y_T(W_{T-q:T-1}, 1)1(X_{T-q:T-1} \in G) + Y_T(W_{T-q:T-1}, 0)1(X_{T-q:T-1} \notin G) \},
\]

\[
\hat{\mathcal{W}}(G) = \frac{1}{T - q} \sum_{t=q}^{T-1} \left[ \frac{Y_t W_t}{e_t(X_{t-q:t-1})} 1(X_{t-q:t-1} \in G) + \frac{Y_t (1 - W_t)}{1 - e_t(X_{t-q:t-1})} 1(X_{t-q:t-1} \notin G) \right].
\]

\( \mathcal{W}(G), \hat{\mathcal{W}}(G) \), and all associated function classes can be similarly redefined.
4 Extensions and Discussion

In this section, we discuss a few possible extensions. Section 4.1 introduces a multi-period policy making framework. Section 4.2 discusses a few connections of T-EWM to the literature on optimal policy choice. More extensions are discussed in Appendix B.

4.1 Multi-period welfare

So far we have considered only the case of a one-period welfare function. This subsection extends the setting to a multiple-period policy framework. In the interest of space, we focus on cases with discrete covariates and extend the simple model in Section 2.2 to a two-period welfare function. Extension to cases with more than two periods is straightforward. Recall Assumption 2.1, which imposed Markov properties on the data-generating processes,

\[ Y_t(w_{0:t}) = Y_t(w_{t-1}, w_t), \]
\[ Y_t(w_{0:t}) \perp X_{0:t-1}|W_{t-1}, \]
\[ W_t \perp X_{0:t-1}|W_{t-1}. \]

The SP chooses policy rules for two periods, \( g_1(\cdot) \) and \( g_2(\cdot) : \{0,1\} \to \{0,1\} \), to maximise aggregate welfare over periods \( T \) and \( T+1 \). The decision in the second period, \( g_2(\cdot) \), is not contingent on the functional form of \( g_1(\cdot) \). However, we assume that period \( T+1 \) welfare is conditional on the treatment choice in the period \( T \), i.e. \( W_T = g_1(W_{T-1}) \). This means there is an information update in period \( T \). The two-period welfare function is

\[
W_{T:T+1}(g_1(\cdot), g_2(\cdot)|\mathcal{F}_{T-1}) = W_{T:T+1}(g_1(\cdot), g_2(\cdot)|W_{T-1}) \\
= W_T(g_1(\cdot)|W_{T-1}) + W_{T+1}(g_2(\cdot)|W_T = g_1(W_{T-1})). \quad (47)
\]

To be more specific with the analytical format, we suppress the \( \cdot \) in \( g_i(\cdot) \), when the meaning is clear from the context.

\[
W_{T:T+1}(g_1, g_2|W_{T-1} = w) \\
= E \{ Y_T(W_{T-1}, 1)g_1(W_{T-1}) + Y_T(W_{T-1}, 0)(1 - g_1(W_{T-1}))|W_{T-1} = w \} \\
+ E \{ Y_{T+1}(W_T, 1)g_2(W_T) + Y_{T+1}(W_T, 0)(1 - g_2(W_T))|W_T = g_1(w) \}. \quad (48)
\]
The second term of (48) follows from
\[
\mathbb{E} \{ Y_{T+1}(g_1(W_{T-1}), 1)g_2(W_T) + Y_{T+1}(g_1(W_{T-1}), 0)(1 - g_2(W_T)) | W_{T-1} = w \}
\]
\[
= \mathbb{E} \{ Y_{T+1}(W_T, 1)g_2(W_T) + Y_{T+1}(W_T, 0)(1 - g_2(W_T)) | W_T = g_1(W_{T-1}), W_{T-1} = w \}
\]
\[
= \mathbb{E} \{ Y_{T+1}(W_T, 1)g_2(W_T) + Y_{T+1}(W_T, 0)(1 - g_2(W_T)) | W_T = g_1(w) \},
\]
where the last equality follows from Assumption 2.1. To estimate the above welfare function, we recall the definition of \(T(w) = \#\{1 \leq t \leq T - 1 : W_t = w\}\), and we define \(T(g_1(w))\) similarly. Then the empirical analogue of (47) can be written as,
\[
\hat{W}_{T:T+1}(g_1, g_2 | w) = \frac{1}{T(w)} \sum_{t:W_{t-1}=w} \left\{ \frac{Y_t W_t g_1(W_{t-1})}{e_t(W_{t-1})} + \frac{Y_t (1 - W_t) (1 - g_1(W_{t-1}))}{1 - e_t(W_{t-1})} \right\}
\]
\[
+ \frac{1}{T(g_1(w))} \sum_{t:W_{t-1}=g_1(w)} \left\{ \frac{Y_t W_t g_2(W_{t-1})}{e_t(W_{t-1})} + \frac{Y_t (1 - W_t) (1 - g_2(W_{t-1}))}{1 - e_t(W_{t-1})} \right\}.
\]

The maximizer of (49), \(\hat{g}_1, \hat{g}_2\), can be obtained by backward induction, a technique widely applied in the Markov decision process (MDP) literature. See Section 4.2.1 and Appendix B.1 for more discussion on the relationship between T-EWM and MDP. To derive the theoretical property of the estimator, we also define
\[
\mathbb{W}_{T:T+1}(g_1, g_2 | w) = \frac{1}{T(w)} \sum_{t:W_{t-1}=w} \mathbb{E} \{ Y_t(1)g_1(W_{t-1}) + Y_t(0) [1 - g_1(W_{t-1})] | W_{t-1} = w \}
\]
\[
+ \frac{1}{T(g_1(w))} \sum_{t:W_{t-1}=g_1(w)} \mathbb{E} \{ Y_t(1)g_2(W_{t-1}) + Y_t(0) [1 - g_2(W_{t-1})] | W_{t-1} = g_1(w) \}.\]

Similarly to the derivation in the previous sections, \(\hat{W}_{T:T+1}(g_1, g_2 | w) - \hat{W}_{T:T+1}(g_1, g_2 | w)\) is a (weighted) sum of MDS. Its upper bound can be shown by the method of Section 2.3. We show in Appendix B.3 the extension to multi-period welfare with continuous conditioning covariate.

4.2 Connections to other policy choice models in the literature

In this section, we discuss T-EWM’s relation to the literature on treatment and optimal policy analysis.
4.2.1 Connection to MDP and Reinforcement Learning

Markov Decision processes (MDP) are popular models used for decision making. See, e.g., Kallenberg (2016) for a comprehensive introduction. The current T-EWM model can be viewed as a special case of a Markov decision process (MDP) with a finite horizon. Briefly, the conditioning variables $X_{t-1}$ at each time $t$ correspond to the Markov state at time $t$, and the welfare outcome $Y_t$ corresponds to the reward. Before the SP intervenes (at time $T$), the transition probability of the Markov state is described by a rule $P_T$, and the transition of policies is governed by the propensity score. After the SP intervenes, the transition of policies is governed by a deterministic rule described by the (estimated) optimal policy function (2). In this MDP, the expected reward $Y_T$ under policy $W_T$ and state $X_{T-1}$ is unknown, and optimizing the conditional empirical welfare function (e.g., (16)) implicitly estimates the expected reward at state $X_{T-1}$ (in the language of T-EWM, the expected potential outcomes conditioning on $X_{T-1}$). If the Markov transition probability (the propensity score) before time $T-1$ is unknown, then it needs to be estimated by the methods described in Section 4.3.2. See Appendix B.1 for a detailed description of the link between T-EWM and MDP. Note that, for multi-period welfare functions, if the welfare function has a finite horizon (as in the case discussed here), the optimal policy is often nonstationary. See Chapter 2 of Kallenberg (2016) for more details about finite horizon MDPs.

The reinforcement learning (RL) literature provides a rich toolbox to solve MDPs, so it is naturally linked to the T-EWM framework. There are three main differences between T-EWM and RL. First, T-EWM is built on the causal framework of the potential outcome time series, while RL literature mainly focuses on maximizing value functions and rarely discusses causal inference. Second, T-EWM is solved by estimating the optimal treatment regime, which is a subset of the domains of historical conditioning variables, while RL methods usually use iteration-type methods to evaluate value functions. Third, T-EWM considers optimal policy choice based on available data that are exogenously given to the planner, while the RL literature studies the decision maker’s joint strategies of sampling data and learning policies. In the class of offline and off-policy RL problems, the T-EWM methods proposed in this paper can be regarded as an estimation method for optimal policies in a finite-horizon MDP.

4.2.2 Connection to IRF

Sims (1980) proposes analysing "causal effects" using a structural equation framework. In particular, the vector autoregressive model (VAR) circumvents the endogeneity issues of ordinary least squares regression. It is common to measure "causal effects" using the impulse response function (IRF) induced by the structural equations; see Ramey (2016), Plagborg-Moller (2016), Stock and Watson (2017), among others. Hinging causal effect on a specific
structural equation has an advantage in terms of interpretability. However, it depends crucially on the belief that the structure model is not misspecified. Our approach is to optimize treatment choices using a potential outcome framework, which is not linked to a specific structure model. Bojinov and Shephard (2019) show the connection of the defined treatment with the impulse response function (IRF) within a structural Vector autoregression (VAR) framework. In particular, the lag \( s \) IRF at time \( t + s \) with shocks \((w_{0:t+s})\) and observations \((Y_{t+s}(.))\) can be written as,

\[
IRF_{t,s} = E \{ Y_{t+s} (w_{0:t+s}) \mid W_{0:t+s} = Y_{0:t-1} \} - E \{ Y_{t+s} (w'_{0:t+s}) \mid W_{0:t+s} = w'_{0:t+s}, Y_{0:t-1} \},
\]

where \( w_{0:t-1} = w'_{0:t-1} = 0, w_{t+1:t+s} = w'_{t+1:t+s} = 0, w_t = 1, w'_t = 0, \) and the expectation is with respect to the treatment path. Therefore, evaluating (observable) treatment effects does not require specifying a structural equation, leading to a flexible optimal treatment analysis. In addition, it provides an alternative robustness check for structural models. Since assuming linear Vector Auto regressive structure might be too strong for decision making, people rarely make policy decisions based on IRF. Moreover, the usual IRF implied from a linear VAR would not allow us to model the heterogeneous causal effects of multiple shocks.

4.3 Estimation with unknown propensity score

To this point, we have treated the propensity score function as known, but this is infeasible in many applications. Here we consider the case where the propensity score at each time \( t, e_t(\cdot) \), is an estimated unknown. Estimation can be either parametric or non-parametric. Let \( \hat{e}_t(\cdot) \) denote the estimator of the propensity score function, and \( \hat{G}_c \) denote the optimal policy obtained using \( \hat{e}_t \).

4.3.1 The convergence rate with estimated propensity scores

In this subsection, we adapt Theorem 2.5 of Kitagawa and Tetenov (2018) to our setting and obtain a new regret bound with estimated propensity scores. We show that, with estimated propensity scores, the convergence rate is determined by the slower one of the rate of convergence of \( \hat{e}_t(\cdot) \) and the rate of convergence of the estimated welfare loss (given in Theorem 3.3).

**THEOREM 4.1.** Let \( \hat{e}_t(\cdot) \) be an estimated propensity score of \( e_t(\cdot) \), and \( \hat{\tau}_t = \frac{Y_t W_t}{\hat{e}_t(W_{t-1})} - \frac{Y_t (1 - W_t)}{1 - \hat{e}_t(W_{t-1})} \) be a feasible estimator for \( \tau_t = \frac{Y_t W_t}{e_t(W_{t-1})} - \frac{Y_t (1 - W_t)}{1 - e_t(W_{t-1})} \). Given a class of data-generating processes \( \mathcal{P}_T(M, \kappa) \) defined above equation (46), we assume that there exists a sequence
\( \phi_T \to \infty \) such that the series of estimators \( \hat{\tau}_t \) satisfy

\[
\lim \sup_{T \to \infty} \sup_{P_T \in P_T(M, \kappa)} \phi_T E_{P_T} \left[ (T - 1)^{-1} \sum_{t=1}^{T-1} |\hat{\tau}_t - \tau_t| \right] < \infty. \tag{50}
\]

Then under Assumptions 3.1 to 3.13 we have

\[
\sup_{P_T \in P_T(M, \kappa)} E_{P_T}[W(G_\ast) - W(\hat{G}_e)] \lesssim (\phi_T^{-1} \vee \frac{1}{\sqrt{T - 1}}).
\]

A proof is presented in Appendix A.8. This theorem shows that if the propensity score is estimated with sufficient accuracy (a rate of \( \phi_T^{-1} \lesssim \sqrt{T^{-1}} \)), we obtain a similar regret bound to the previous sections. It is not surprising to see that the rate is affected by the estimation accuracy of the propensity score, and it is the maximum of \( \frac{1}{\sqrt{T - 1}} \) and \( \phi_T^{-1} \).

**Remark 3.** In the cross-sectional setting, Athey and Wager (2021) show an improved rate of welfare convergence when propensity scores are unknown and estimated. It is possible to extend their analysis to our time-series setting and assess whether or not the rate shown in Theorem 4.1 can be improved. However, this is not a trivial extension, and we leave it for further research.

### 4.3.2 Estimation of propensity scores

In this subsection, we briefly review various methods of estimating propensity score functions. The propensity score function \( e_t(\cdot) \) can be estimated parametrically or nonparametrically. An example of a parametric estimator is the (ordered) probit model, which is employed in Hamilton and Jorda (2002), Scotti (2018), and Angrist et al. (2018). Under Assumption 3.1, \( e_t \) can be expressed as a function of \( X_{t-1} \), with the structure of the propensity score given by a probit model

\[
e_t(X_{t-1}) \equiv P(W_t = 1|X_{t-1}) = \Phi(\beta'X_{t-1}),
\]

\[
1 - e_t(X_{t-1}) \equiv P(W_t = 0|X_{t-1}) = 1 - \Phi(\beta'X_{t-1}).
\]

A more complicated structure, such as the dynamic probit model (Eichengreen et al. (1985); Davutyan and Parke (1995)) can also be employed.

We can also use a nonparametric estimator to estimate \( e_t(\cdot) \). For example, Frölich (2006) and Park et al. (2017) extend the local polynomial regression of Fan and Gijbels (1995) to a dynamic setting. Their methods can be employed here. For simplicity, we assume that the propensity score function is invariant across times, i.e., \( e_t(\cdot) = e(\cdot) \) for any \( t \), and \( e(\cdot) \) is continuous, and we set \( \mathcal{X} \subset \mathbb{R}^1 \). For a local polynomial of order \( p = 1 \) and any \( x \in \mathcal{X} \), a
local likelihood logit model can be specified as

\[
\log \left[ \frac{e(x)}{1 - e(x)} \right] = \alpha_x,
\]

for a local parameter \( \alpha_x \). By the continuity of the propensity score function \( e(\cdot) \), for some \( X_{t-1} \) close to \( x \), we can find a local parameter \( \beta_x \), such that \( \log \left[ \frac{e(X_{t-1})}{1 - e(X_{t-1})} \right] \approx \alpha_x + \beta_x(X_{t-1} - x) \).

The estimated propensity score evaluated at \( x \), \( \hat{e}(x) \), can be obtained by solving

\[
(\hat{\alpha}_x, \hat{\beta}_x) = \arg\max_{\alpha, \beta} \frac{1}{T-1} \sum_{i=1}^{T-1} \left\{ W_i \log \left( \frac{\exp(\alpha + \beta(X_{t_i-1} - x))}{1 + \exp(\alpha + \beta(X_{t_i-1} - x))} \right) \right. \\
+ \left. (1 - W_i) \log \left( \frac{1}{1 + \exp(\alpha + \beta(X_{t_i-1} - x))} \right) \right\} K \left( \frac{X_{t_i-1} - x}{h} \right).
\]

where \( K(\cdot) \) is a kernel function, and \( h \) is bandwidth. Then, we have \( \hat{e}(x) = \frac{\exp(\hat{\alpha}_x)}{1 + \exp(\hat{\alpha}_x)} \).

5 Simulation

In this subsection, we illustrate the accuracy of our method through a simple simulation exercise. We consider the following model

\[
Y_t = W_t \cdot \mu(Y_{t-1}, Z_{t-1}) + \phi Y_{t-1} + \epsilon_t, \\
\mu(Y_{t-1}, Z_{t-1}) = 1(Y_{t-1} < B_1) \cdot 1(Z_{t-1} < B_2) - 1(Y_{t-1} > B_1 \lor Z_{t-1} > B_2),
\]

where \( \mu \) is a function determining the direction of the treatment effect. The treatment effect at time \( t \) is positive if both \( Y_{t-1} < B_1 \) and \( Z_{t-1} < B_2 \) and is negative otherwise. The optimal treatment rule is therefore \( G_* = \{(Y_{t-1}, Z_{t-1}) : Y_{t-1} < B_1 \text{ and } Z_{t-1} < B_2\} \). We set \( \epsilon_t \iid N(0, 1), Z_{t-1} \iid N(0, 1), \phi = 0.5, B_1 = 2.5, \text{ and } B_2 = 0.52 \) (approximately the 70% quantile of the standard normal distribution). The propensity score \( e_t(Y_{t-1}, Z_{t-1}) \) is set to 0.5. Our goal is to estimate \( G_* \). We consider the quadrant treatment rules:

\[
G \equiv \left\{ ((y_{t-1}, z_{t-1}) : s_1(y_{t-1} - b_1) > 0 \land s_2(z_{t-1} - b_2) > 0), \\
s_1, s_2 \in \{-1, 1\}, b_1, b_2 \in \mathbb{R} \right\}.
\]

It is immediate that \( G_{FB}^* \in G \). Therefore, we can directly estimate the unconditional treatment rule as described in Section 3.2. Figure 1 illustrates the estimated bound and true bounds for sample sizes \( n = 100 \) and \( n = 1000 \). In each case, we draw 100 Monte Carlo samples.
Figure 1. The estimated bound for $n = 100$ and $n = 1000$.

The blue lines are estimated bounds, and the red lines are the true bounds. For $n=100$, the majority of the blue lines are close to the red line. As the sample size increases from 100 to 1000, the blue lines become tightly concentrated around the red line. The results in Table 1 confirm this. This table presents the Monte Carlo averages ($\hat{\mu}_{B_1}, \hat{\mu}_{B_2}$), variances ($\hat{\sigma}^2_{B_1}, \hat{\sigma}^2_{B_2}$), and MSE of estimated $B_1$ and $B_2$. We multiply the variances and MSEs by the sample size $n$. The sample sizes are $n = 100, 500, 1000$ and, 2000. The number of Monte Carlo replications is 500.

|   | $\hat{\mu}_{B_1}$ | $n \cdot \hat{\sigma}^2_{B_1}$ | $n \cdot \text{MSE}_{B_1}$ | $\hat{\mu}_{B_2}$ | $n \cdot \hat{\sigma}^2_{B_2}$ | $n \cdot \text{MSE}_{B_2}$ |
|---|-------------------|-------------------------------|-----------------|-------------------|-------------------------------|-----------------|
| 100 | 2.4688            | 14.4331                       | 14.5016         | 0.6589            | 18.9382                       | 20.7080         |
| 500 | 2.4919            | 2.3881                        | 2.4162          | 0.5433            | 3.3775                        | 3.5490          |
| 1000| 2.4924            | 1.4676                        | 1.5224          | 0.5310            | 1.2620                        | 1.3027          |
| 2000| 2.4958            | 0.5981                        | 0.6327          | 0.5267            | 0.7672                        | 0.7767          |

As the sample size increases, both the $\hat{\mu}_{B_1}$ and $\hat{\mu}_{B_2}$ converge to their true values, 2.5 and 0.52, respectively. The variances and MSEs shrink, even after multiplying by the sample size $n$, which suggests that the convergence rate in this case is faster than $\frac{1}{\sqrt{n}}$.

6 Application

Over the past two years, policy makers around the world have faced the problem of making effective policies to respond to the Covid-19 pandemic. In this section, we illustrate the usage of T-EWM with an application to choosing the stringency of restrictions imposed in the United States during the pandemic. Throughout this section, we consider the following
setup: The treatment $W_t$ is a binary indicator for whether the government relaxes restrictions at week $t$. $W_t = 1$ means the stringency of restrictions is maintained or increased from time $t-1$. $W_t = 0$ means restrictions are relaxed. The stringency of restrictions is measured by the Oxford Stringency Index, which is described in Section 6.1. We assume that a change in the stringency of restrictions at time $t$ will have a lagged effect on deaths, and set the outcome variable $Y_t$ to be $-1 \times \text{two-week ahead deaths}$. The time $t$ information set is,

$$X_t = (\text{cases}_t, \text{deaths}_t, \text{change in cases}_t, \text{change in deaths}_t, \text{restriction stringency}_t, \text{vaccine coverage}_t, \text{economic conditions}_t).$$

(53)

The variables in $X_t$ are chosen to include the most important factors considered by policy makers when deciding the stringency of restriction. The inclusion of the economic conditions reflects policy makers concerns over the economic effect of restrictions. Finally, the propensity score is estimated as a logit model with a linear index,

$$\log \left( \frac{\Pr(\text{keeping or increasing restrictions at week } t)}{\Pr(\text{decreasing restriction at week } t)} \right) = \alpha + \beta X_{t-1}. \quad (54)$$

6.1 Data

The dataset consists of weekly data for the United States. It runs from April 2020 to January 2022 and contains 92 observations. Data on cases, deaths, and vaccine coverage was downloaded from the website of the Centers for Disease Control and Prevention (CDC). These series are plotted in Figures 3 and 5 in Appendix A.6. We use the Oxford Stringency Index as a measure of the stringency of restrictions. This index is taken from the Oxford COVID-19 Government Response Tracker (OxCGRT), which tracks policy interventions and constructs a suite of composite indices that measure governments’ responses. Figure 4 in Appendix A.6 plots the time series of the stringency index.

The economic impact of restrictions is a crucial factor that every policy maker has to consider. To measure economic conditions, we use the Lewis-Mertens-Stock weekly economic index (WEI). The right panel of Figure 4 in Appendix A.6 plots the WEI. In general, none of these time series seems stationary over the sample period. They exhibit both seasonal fluctuations and changes with respect to different stages of the pandemic.

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10 The index is calculated based on a dataset that is updated by a professional team of over one hundred students, alumni, staff, and project partners. It is a composite index covering different types of restriction, such as school and workplace closures, restrictions on the size of gatherings, internal and international movements, facial covering policies etc. See Hale et al. (2020) for more details.

11 The WEI is an index of ten indicators of real economic activity. It represents the common component of series covering consumer behavior, the labor market, and production. See Lewis et al. (2021) for more details.
6.2 On T-EWM assumptions

Before we proceed with the analysis, we discuss the validity of some of the T-EWM assumptions in this context. The Markov properties (Assumption 3.1) require that (i) the potential outcome at time $t$ depends only on the time $t$ and time $t-1$ treatments; (ii) conditional on $X_{t-1}$, the potential outcomes and treatment at time $t$ are not affected by the path $X_{0:t-2}$.

(i) can be justified here as the treatment is the direction of change in the stringency of restrictions. While the level before $t-1$ may affect the outcome at $t$, its directional change may not have such a long-existing impact. (ii) requires that conditional on current cases and deaths, cases and deaths one week ago (as $X_t$ includes the first difference of cases and deaths, it includes the level of cases and deaths from one week ago), current economic conditions, the stringency of restrictions, vaccine coverage, and deaths in two weeks will be independent of lags of these variables if the lag is greater than one. For this assumption to hold, it must be that current infections are unrelated to deaths in more than three weeks. In general, it is unclear whether this is true, although there is some support in the literature. For example, based on data collected during the early stage of the pandemic in China, Verity et al. (2020) calculate that the posterior mean time from infection to death is 17.8 days, with a 95%-confidence interval of [16.9, 19.2].

Strict overlap (Assumption 3.2) requires that, for all $x \in X$, the propensity score is strictly larger than 0 and smaller than 1. Figure 6 in Appendix A.6 shows the histogram of estimated propensity scores. The strict overlap assumption can be verified by visual inspection: the smallest (estimated) propensity score is larger than 0.3, and the largest is smaller than 0.9.

Sequential unconfoundedness (Assumption 3.3) requires that, conditional on $X_{t-1}$, treatment assignment at time $t$ is quasi-random. This assumption cannot be tested in general. However, in the past two years, policy makers have had very limited knowledge of Covid-19 beyond the observable data. After controlling for the observables in (53), it seems reasonable that the remaining random factors in two-week ahead deaths and changes in the stringency of restrictions are independent.

6.3 Estimation results and policy recommendation

In this subsection, we summarize the estimation results and discuss the policy recommendations. We show that T-EWM leads to sensible and robust policy decisions. After observing $X_{T-1}$, we aim to maximize expected welfare, $E(-1 \cdot \text{deaths}_{T+1})$, over a set of quadrant policies (a two-variable quadrant policy is defined in (52)). Let $X_{T-1}^P$ denote a vector of variables for policy choice. This can be any subvector of $X_{T-1}$. For our first set of results, we use $X_{T-1}^P = (\text{change in deaths}_{T-1}, \text{restriction stringency}_{T-1})$. Figure 2 presents the estimated treatment region. The estimated optimal decision rule states that restrictions should not
be relaxed (i.e. $W_T = 1$), if the weekly fall in deaths is below 745, and the current level of restrictions is lower than 62.6.

Figure 2. Optimal policy based on $X_{T-1}^P = (\text{change in cases}_{T-1}, \text{restriction stringency}_{T-1})$.

The $x$-axis is the change in deaths at $T - 1$, and the $y$-axis is the stringency of restrictions at $T - 1$.

We can examine the robustness of this result by expanding the set of variables for policy choice. We first add vaccine coverage: $X_{T-1}^P = (\text{change in deaths}_{T-1}, \text{restriction stringency}_{T-1}, \text{vaccine coverage}_{T-1})$. The optimal estimated quadrant policy is then a 3d-quadrant. Figure 4 in Appendix A.6 shows the projection of this 3d-quadrant onto the 2d-planes of $(\text{change in deaths}_{T-1}, \text{restriction stringency}_{T-1})$ and $(\text{restriction stringency}_{T-1}, \text{vaccine coverage}_{T-1})$. We further expand the set of variables for policy choice by adding the change in cases, so that $X_{T-1}^P = (\text{change of deaths}_{T-1}, \text{restriction stringency}_{T-1}, \text{vaccine coverage}_{T-1}, \text{change in cases}_{T-1})$. The optimal estimated quadrant policy in this case is a 4d-quadrant. Figure 8 in Appendix A.6 shows projections of this 4d-quadrant onto the 2d-planes of $(\text{change in deaths}_{T-1}, \text{restriction stringency}_{T-1})$ and $(\text{vaccine coverage}_{T-1}, \text{change in cases}_{T-1})$.

Table 2 summarises the estimation results of Figures 2, 7, and 8. It shows that the T-EWM policy recommendation is consistent irrespective of which set of variables for policy choice we use. As the number of variables increases, the threshold of the change in weekly deaths decreases slightly from $-745$ to $-1007.5$, while the other thresholds remain stable. During the sample period, the mean of weekly cases is 594344.3. Therefore, the threshold in the last column and the last row corresponds to a 10% fall relative to the average number of cases during the sample period. In sum, T-EWM suggests that the policy maker should not relax restrictions ($W_T = 1$) if there are no significant drops in deaths and cases, current stringency is comparatively low, and vaccine coverage is comparatively low.
Table 2. Estimated optimal quadrant rules

| variables | change in deaths | restriction | vaccine (%) | change in cases |
|-----------|------------------|-------------|--------------|-----------------|
| Figure 2 2 | $> -745$ | $< 62.6$ | $-$ | $-$ |
| Figure 7 3 | $> -1007.5$ | $< 62.6$ | $< 115.2$ | $-$ |
| Figure 8 4 | $> -1007.5$ | $< 62.6$ | $< 115.2$ | $> -50842.5$ |

7 Conclusion

This article proposes T-EWM, a framework and method for choosing optimal policies based on time-series data. We characterise assumptions under which this method can learn an optimal policy. We evaluate its statistical properties by deriving non-asymptotic upper and lower bounds of the conditional welfare. We discuss its connections to the existing literature, including the Markov decision process and reinforcement learning. We present simulation results and empirical applications to illustrate the computational feasibility and applicability of T-EWM. As a benchmark formulation, this paper mainly focuses on a one-period social welfare function as a planner’s objective. Extensions of the analysis to policy choices for the middle-run and long-run cumulative social welfare are left for future research.
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A  Proofs of Lemmas and Theorems

The following two lemmas will be used in the proofs of the main results.

LEMMA A.1 (Freedman’s inequality). Let \( \xi_{a,i} \) be a martingale difference sequence indexed by \( a \in \mathcal{A} \), \( i = 1, \ldots, n \), \( \mathcal{F}_i \) be the filtration, \( V_a = \sum_{i=1}^{n} E(\xi_{a,i}^2 | \mathcal{F}_{i-1}) \), and \( M_a = \sum_{i=1}^{n} \xi_{a,i} \). For positive numbers \( A \) and \( B \), we have,

\[
\Pr(\max_{a \in \mathcal{A}} |M_a| \geq z) \leq \sum_{i=1}^{n} \Pr(\max_{a \in \mathcal{A}} \xi_{a,i} \geq A) + 2 \Pr(\max_{a \in \mathcal{A}} V_a \geq B) + 2 |\mathcal{A}| e^{-z^2/(2z^2+2B^2)}.
\]

(A.1)

The proof can be found in Freedman (1975). Next, let \( a \lesssim b \) indicate that there is a positive constant \( C \), such that \( a \leq C \cdot b \).

LEMMA A.2 (Maximal inequality based on Freedman’s inequality). Let \( \xi_{a,i} \) be a martingale difference sequence indexed by \( a \in \mathcal{A} \) and \( i = 1, \ldots, n \). If, for some positive constants \( A \) and \( B \), \( \max_{a \in \mathcal{A}} \xi_{a,i} \leq A \), \( V_a = \sum_{i=1}^{n} E(\xi_{a,i}^2 | \mathcal{F}_{i-1}) \leq B \), and \( M_a = \sum_{i=1}^{n} \xi_{a,i} \) we have,

\[
E(\max_{a \in \mathcal{A}} |M_a|) \lesssim A \log(1 + |\mathcal{A}|) + \sqrt{B} \sqrt{\log(1 + |\mathcal{A}|)}.
\]

(A.2)

Proof. This follows from Lemma 19.33 of Van der Vaart (2000) and Lemma A.1. From Freedman’s inequality we have

\[
\Pr(\max_{a} |M_a| \geq z) \leq 2 |\mathcal{A}| \exp(-z^2/4A) \quad \text{for} \quad z > B/A \quad \text{(A.3)}
\]
\[
\leq 2 |\mathcal{A}| \exp(-z^2/4B) \quad \text{Otherwise.} \quad \text{(A.4)}
\]

We truncate \( M_a \) into \( C_a = M_a 1\{M_a > B/A\} \) and \( D_a = M_a 1\{M_a \leq B/A\} \). Transforming \( C_a \) and \( D_a \) with \( \phi_p(x) = \exp(x^p) - 1 \) \((p = 1, 2)\) and applying Fubini’s Theorem, we have

\[
E(\exp |C_a/4A|) \leq 2, \quad \text{(A.5)}
\]
and

\[
E(\exp |D_a/\sqrt{4B}|^2) \leq 2. \quad \text{(A.6)}
\]

For completeness, we show the derivations of these inequalities.

\[
E(\exp |C_a/4A|) \leq \int_{0}^{\infty} P(|C_a/4A| \geq x) dx
\]
\[
\leq 2 \int_{0}^{\infty} \exp(-4Ax/4A) dx \leq 2,
\]

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the first equality follows from Fubini’s inequality and the second is due to Lemma A.1.

\[
\mathbb{E} \exp \left| \frac{D_a^2}{4B} \right| \leq \int_0^\infty P(\left| \frac{D_a^2}{4B} \right| \geq x) \, dx \\
\leq 2 \int_0^\infty \exp(-4Bx/4B) \, dx \leq 2.
\]

Again, the first inequality follows from Fubini’s inequality and the second is due to Lemma A.1. Now we show that due to Jensen’s inequality,

\[
\phi_1[\mathbb{E}(\max_a |C_a|/4A)] \leq \sum_a \mathbb{E}[\phi_1(|C_a|/4A)] \leq |A|.
\]

and similarly,

\[
\phi_2[\mathbb{E}(\max_a |D_a|/\sqrt{4B})] \leq \sum_a \mathbb{E}[\phi_2(|D_a|/\sqrt{4B})] \leq |A|.
\]

Then we have \( \mathbb{E}(\max_a |C_a|/4A) \leq \log(|A| + 1) \) and \( \mathbb{E}(\max_a |D_a|/\sqrt{4B}) \leq \sqrt{\log(|A| + 1)} \). The result thus follows.

A.1 Proof of (22) in the main paper

Proof. Define \( \tilde{p}_w \overset{\text{def}}{=} \frac{1}{T-1} \sum_{t=1}^{T-1} \Pr(W_{t-1} = w|\mathcal{F}_{t-2}) \). Note that \( \tilde{p}_w \) is only a device for the proof, so it might contain elements that are not observed in the dataset.

\[
\sup_{g:\{0,1\} \rightarrow \{0,1\}} \left[ \hat{W}(g|w) - \hat{\mathcal{W}}(g|w) \right] = \sup_{g:\{0,1\} \rightarrow \{0,1\}} \frac{1}{T-1} \frac{1}{T(w)} \sum_{t=1}^{T-1} \left[ \hat{W}_t(g|w) - \hat{\mathcal{W}}_t(g|w) \right] 
= \sup_{g:\{0,1\} \rightarrow \{0,1\}} \left( \frac{T(w)}{T-1} - \tilde{p}_w + \tilde{p}_w \right)^{-1} \left( T-1 \right)^{-1} \sum_{t=1}^{T-1} \left[ \hat{W}_t(g|w) - \hat{\mathcal{W}}_t(g|w) \right].
\]

And

\[
\frac{T(w)}{T-1} - \tilde{p}_w = \frac{1}{T-1} \sum_{t=1}^{T-1} \left[ \mathbb{1}(W_{t-1} = w) - \Pr(W_{t-1} = w|\mathcal{F}_{t-2}) \right] 
= \frac{1}{T-1} \sum_{t=1}^{T-1} \xi_t,
\]

where \( \xi_t \overset{\text{def}}{=} \sum_{t} \left[ \mathbb{1}(W_{t-1} = w) - \Pr(W_{t-1} = w|\mathcal{F}_{t-2}) \right] \). This is a sum of MDS.
Applying Freedman’s inequality with $|\mathcal{A}| = 1$ and

$$V_a = \sum_{t=1}^{T-1} \text{var} (\xi_t | \mathcal{F}_{t-2})$$

$$= \sum_{t=1}^{T-1} e(w | \mathcal{F}_{t-2}) [1 - e(w | \mathcal{F}_{t-2})] < T - 1 \equiv B,$$

where the last inequality is by Assumption 2.2. By the same assumption, we also have $\max_t \xi_t < 1$. Therefore, we can set $A = 1$, then the first term of (A.1) is reduced to

$$\Pr \left( \frac{1}{T - 1} \sum_{t=1}^{T-1} \xi_t \geq z \right) = \Pr \left( \sum_{t=1}^{T-1} \xi_t \geq (T - 1)z \right) \leq 2 \cdot \exp \left[ \frac{-z^2(T - 1)^2}{2A(T - 1)z + 2B} \right] = 2 \cdot \exp \left[ \frac{-z^2(T - 1)^2}{2(T - 1)z + 2(T - 1)} \right] = o(1).$$

(A.8)

hold for any $z > 0$. Therefore, for sufficient large $T$, we have $\left| \frac{T(w)}{T - 1} - \tilde{p}_w \right| \lesssim_p \kappa/2$, where $\kappa$ is the constant defined in Assumption 2.2. With probability approaching 1,

$$\left( \frac{T(w)}{T - 1} - \tilde{p}_w \right)^{-1} \leq \left( -\left| \frac{T(w)}{T - 1} - \tilde{p}_w \right| + \tilde{p}_w \right)^{-1} \leq (-\kappa/2 + \kappa)^{-1} = (\kappa/2)^{-1}. \quad (A.9)$$

Now we bound $\sum_{t=1}^{T-1} \left[ \hat{\mathcal{W}}_t(g|w) - \hat{\mathcal{W}}(g|w) \right]$, which is also a sum of MDS.

To apply Lemma A.2, we define $\mathcal{G} \equiv \{g : \{0, 1\} \rightarrow \{0, 1\}\}$ and $\xi_{g,t} \equiv \hat{\mathcal{W}}_t(g|w) - \hat{\mathcal{W}}(g|w)$. (With some abuse of notations, this $\xi_{g,t}$ is different from $\xi_t$ defined above). Note that $|\mathcal{G}|$ is finite, and we can find a constant $C_A$, such that for any $t$, $\sup_{g \in \mathcal{G}} \xi_{g,t} \leq (M + \frac{M}{\kappa}) < C_A$, where $\kappa$ and $M$ are constants defined in Assumption 2.2 and 2.5 respectively. By these two assumptions, we also can find a constant $C_B$ depending only on $M$ and $\kappa$, such that $V_g = \sum_{t=1}^{T-1} \mathbb{E} \left( \xi_{g,t}^2 | \mathcal{F}_{t-1} \right) < C_B(T - 1)$. Then by Lemma A.2

$$\mathbb{E} \left( \sup_{g \in \mathcal{G}} \left| \sum_{t=1}^{T-1} \xi_{g,t} \right| \right) \lesssim C_A \log (1 + |\mathcal{G}|) + \sqrt{B} \sqrt{\log (1 + |\mathcal{G}|)} \quad (A.10)$$

$$\lesssim C_A \log (1 + |\mathcal{G}|) + \sqrt{C_B(T - 1)} \log (1 + |\mathcal{G}|)$$

$$\lesssim \sqrt{(T - 1)} \sqrt{C_B \log (1 + |\mathcal{G}|)}.$$

By Jensen’s inequality $\sup_{g \in \mathcal{G}} \mathbb{E} \left| \sum_{t=1}^{T-1} \xi_{g,t} \right| \leq \mathbb{E} \left( \sup_{g \in \mathcal{G}} \left| \sum_{t=1}^{T-1} \xi_{g,t} \right| \right)$. Then, combining
\[ \sup_{g \in G} \mathbb{E} \left( \left| \hat{W}(g|w) - \hat{W}(g|w) \right| \right) = \sup_{g \in G} \mathbb{E} \left( \left| \hat{W}(g|w) - \hat{W}(g|w) \right| \right) \]
\[ \lesssim \sup_{g \in G} \mathbb{E} \left\{ \frac{1}{\kappa/2} \left( \sum_{t=1}^{T-1} \xi_{g,t} \right) \right\} \]
\[ \lesssim \frac{1}{\kappa/2} \sqrt{C_B \log (1 + |G|)} \sqrt{(T-1)} \]
\[ = \frac{C}{\sqrt{(T-1)}}, \]

where \( C \) depends only on \( \kappa, |G|, \) and \( M \). The first curved inequality follows from the exponential tail probability of (A.8).

\[ \square \]

### A.2 Proof of Lemma 3.1

**Proof.**

\[ R_T(G) = \int R_T(G|x) dF_{X_{T-1}}(x), \]
\[ = \int_{x \in A(x^{\text{obs}}, G)} R_T(G|x) dF_{X_{T-1}}(x) + \int_{x \notin A(x^{\text{obs}}, G)} R_T(G|x) dF_{X_{T-1}}(x), \]
\[ \geq R_T(G|x^{\text{obs}}) \cdot p_{T-1}(x^{\text{obs}}, G) + 0, \]
\[ = R_T(G|x^{\text{obs}}) \cdot p_{T-1}(x^{\text{obs}}, G). \]  

(A.11)

The first inequality follows from the definition of \( A(x', G) \) and \( R_T(G|x) \) being non-negative (since the first best policy is feasible). Then, Assumption 3.5 yields \( R_T(G|x^{\text{obs}}) \leq \frac{1}{p} R_T(G) \).

\[ \square \]

### A.3 Proof of Theorem 3.1

We first show the following lemma.

**Lemma A.3.** Under Assumptions 3.1 to 3.3, and 3.7 to 3.9 we have

\[ \mathbb{E}[\mathbb{E}_n h|H_n] \lesssim M \sqrt{v/n}. \]

It shall be noted that the result of the lemma above is of the maximal inequality type and has a standard \( \sqrt{n}^{-1} \) rate. The complexity of the function class \( v \) also plays a role. This is
in line with other results in the literature, such as Kitagawa and Tetenov (2018).

Proof. \( h_t \) denotes a function belonging to the functional class \( \mathcal{H}_t \), and \( h = \{ h_1, h_2, \ldots, h_n \} \).

\( J_k \) is a cover of the functional class \( H_n \) with radius \( 2^{-k}M \) with respect to the \( \rho_{2,n}(.) \) norm, and \( k = 1, \cdots , K \). We set \( 2^{-\frac{K}{2}} \simeq \sqrt{n}^{-1} \), then \( K \simeq \log(n) \). Recall that \( M \) is the constant defined in Assumption 3.7, which implies \( \max_t |h_t| \leq M \). We define \( h^* = \arg\max_{h \in H_n} E_n h \).

Let \( h^{(k)} = \min_{h \in J_k} \rho_{2,n}(h, h^*) \) and \( h^{(0)} = (0, \cdots , 0) \in \mathbb{R}^n \), then \( \rho_{2,n}(h^{(k)}, h^*) \leq 2^{-k}M \) holds by the definition of \( J_k \), and

\[
\rho_{2,n}(h^{(k-1)}, h^{(k)}) \leq \rho_{2,n}(h^{(k-1)}, h^*) + \rho_{2,n}(h^{(k)}, h^*) \leq 3 \cdot 2^{-k}M. \tag{A.12}
\]

By a standard chaining argument, we express any partial sum of \( h \in H_n \) as a telescoping sum,

\[
\sum_{t=1}^{n} h_t \leq |\sum_{t=1}^{n} h_t^{(0)}| + \sum_{k=1}^{K} \sum_{t=1}^{n} (h_t^{(k)} - h_t^{(k-1)})| + |\sum_{t=1}^{n} (h_t^{(K)} - h_t^{(1)})|. \tag{A.13}
\]

The inequality \( |\sum_{t} a_t| \leq \sum_{t} |a_t| \leq \sum_{t} a_t^2 \sqrt{n} \) can be applied to the third term. Notice that, by the definition of the \( h^{(K)} \)

\[
|\sum_{t=1}^{n} (h_t^{(K)} - h_t^{(1)})| \leq \left( \sum_{t=1}^{n} (h_t^{(K)} - h_t^{(1)})^2 \right)^{1/2} \sqrt{n} \leq n 2^{-K}M. \tag{A.14}
\]

Thus,

\[
E(\|E_n h|_{H_n}\|) \leq \sum_{k} E \max_{f \in J_k, \rho_{2,n}(f, g) \leq 3 \cdot 2^{-k}M} |E_n (f - g)| + 2^{-K}M. \tag{A.15}
\]

Apply Lemma A.2 and Assumption 3.9 to (A.15). The maximal inequality (A.2) of Lemma A.2 is reproduced here:

\[
E(\max_{a \in A} |M_a|) \lesssim A \log(1 + |A|) + \sqrt{B} \sqrt{\log(1 + |A|)},
\]

where for the first term of (A.15), we have \( A = \{ f - g : f \in J_k, g \in J_k-1, \rho_{2,n}(f, g) \leq 3 \cdot 2^{-k}M \} \), \( |A| = |J_k||J_{k-1}| \leq 2N^2(2^{-k}M, H_n, \rho_{2,n}(.)) \lesssim_p 2 \max_t \sup Q^2(2^{-k}M, \mathcal{H}_t, \|\|_Q) \), and \( A \leq 3M \). \( B \) in (A.2) is an upper bound of the sum of conditional variances of an MDS.

By Assumption 3.9 we have \( B = \sum_{t} E[(f_t - g_t)^2|F_{t-1}] \leq nL^2 \rho_{2,n}(f, g)^2 \leq nL^2(3 \cdot 2^{-k}M)^2 \) for any pair \( (f, g) \) satisfying \( f - g \in A \).
Therefore, by Lemma A.2

\[ n \mathbb{E}(|E_n h|_{H_n}) \lesssim \sum_{k=1}^{K} (L \ast 3 \ast 2^{-k} M \sqrt{n}) \sqrt{\log(1 + 2 \max_{t} \mathcal{N}^2(2^{-k} M, \mathcal{H}_t, \|\cdot\|_{Q,2}))} \]
\[ + 3 \ast M \sum_{k=1}^{K} \log(1 + 2 \ast \max_{t} \mathcal{N}^2(2^{-k} M, \mathcal{H}_t, \|\cdot\|_{Q,2})) + o_p(\sqrt{n}) \]
\[ \lesssim 6 \sqrt{n} \int \log(2^{1/2} \max_{t} \mathcal{N}(\varepsilon M, \mathcal{H}_t, \|\cdot\|_{Q,2})) d\varepsilon. \]

By Assumption 3.8, \[ \max_{t} \log \sup_{Q} \mathcal{N}(\varepsilon M, \mathcal{H}_t, \|\cdot\|_{Q,2}) \lesssim \log(K) + \log(v+1) + (v+1)(\log 4 + 1) + (rv) \log(\frac{2}{\varepsilon M}). \] Thus, the integral in the last row is finite by a standard argument for bracketing numbers (see, e.g., the comment following Theorem 19.4 in [Van der Vaart, 2000]).

Then, we have

\[ \mathbb{E}(|E_n h|_{H_n}) \lesssim M \sqrt{v/n}. \]

The next lemma concerns the tail probability bound. It states that, under certain regularity conditions, \[ |E_n h|_{H_n} \] is very close to \[ \mathbb{E}(|E_n h|_{H_n}). \]

**Lemma A.4.** Under Assumptions 3.1 to 3.3 and 3.7 to 3.9

\[ |E_n h|_{H_n} - \mathbb{E}(|E_n h|_{H_n}) \lesssim_p M c_n \sqrt{v/n}, \]

where \( c_n \) is an arbitrarily slowly growing sequence.
Proof. Similar to the above derivation, for a positive constant \( \eta_k \), with \( \sum_k \eta_k \leq 1 \),

\[
\Pr(n^{-1} \sum_{t=1}^{n} h_t \geq x) 
\leq \Pr(n^{-1} \sum_{k=1}^{K} \sum_{t=1}^{n} h_t^{(k)} - h_t^{(k-1)} \geq x - \sqrt{n^{-1}2^{-K} M}) 
\leq \sum_{k=1}^{K} \Pr(|n^{-1} \sum_{t=1}^{n} h_t^{(k)} - h_t^{(k-1)} \geq \eta_k(x - \sqrt{n^{-1}2^{-K} M})) 
\leq \sum_{k=1}^{K} \exp\{\log \max_{Q} \mathcal{N}^{2}(2^{-k} M, \mathcal{H}_t, \|\cdot\|_{Q,2}) - \eta_k^2(nx - \sqrt{n2^{-K} M})^2 / \left[2\{(nx - \sqrt{n2^{-K} M}) + 2((3 \cdot 2^{-k} \cdot M)^2n)\}\right]\}
\leq \sum_{k=1}^{K} \exp(\log(K) + \log(v + 1) + (v + 1)(\log 4 + 1) + (2v) \log(\frac{2}{2^{-k} M})) - \eta_k^2(nx - \sqrt{n2^{-K} M})^2 / \left(\{(nx - \sqrt{n2^{-K} M}) + 2((3 \cdot 2^{-k} \cdot M)^2n)\}\right),
\]

where the above derivation is due to the tail probability in Lemma A.1. We pick \( \eta_k \) and \( x \) to ensure the right-hand side converges to zero and \( \sum_k \eta_k \leq 1 \).

We take \( b_k = \log(K) + \log(v + 1) + (v + 1)(\log 4 + 1) + (2v) \log(\frac{2}{2^{-k} M}) \), \( a_k = 2^{-1}(nx - \sqrt{n2^{-K} M})^2 / \left(\{(nx - \sqrt{n2^{-K} M}) + 2((3 \cdot 2^{-k} \cdot M)^2n)\}\right) \). We pick \( \eta_k \geq \sqrt{a_k / b_k} \), so that \( b_k \leq \eta_k^2 a_k \). We also need to choose \( x \) to ensure that \( \sum_k \eta_k \leq 1 \) and \( \sum_k \exp(b_k - \eta_k^2 a_k) \to 0 \). We pick \( c_n \sqrt{v/n} \lesssim x \), and \( \eta_k = c'_n \sqrt{b_k / a_k} \), with two slowly growing functions \( c_n \) and \( c'_n \) such that \( c'_n \ll c_n \). We set \( x = \mathbb{E}(|E_n h|_{\mathcal{H}_n}) + c_n \sqrt{v/n} \). The result then follows.

Finally, Theorem 3.1 follows by combining Lemma A.3, Lemma A.4, and \( n = T - 1 \).

\[ \square \]

A.4 Justifying Assumption 3.8

Here we interpret the entropy condition in Assumption 3.8. We follow the argument of Chapter 11 in [Kosorok (2008)] for the functional class related to non-i.i.d. data.

First, we illustrate the case that the random stochastic process

\[
\{D_t\}_{t=-\infty}^{\infty} \overset{\text{def}}{=} \{Y_t, W_t, X_{t-1}\}_{t=-\infty}^{\infty}
\]
is assumed to be stationary. In this case, we have for all \(t \in \{1, 2, \ldots, n\}\),

\[
h_t(\cdot; G) = h(\cdot; G)
\]

and \(\mathcal{H} = \{h(\cdot; G) = \tilde{W}_t(G) - \tilde{W}_t(G) : G \in \mathcal{G}\}\).

For an \(n\)-dimensional non-negative vector \(\alpha_n = \{\alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,n}\}\), define \(Q_{\alpha_n}\) as a discrete measure with probability mass \(\sum_{t=1}^{n} \alpha_{n,t}\) on the value \(D_t\). Recall for a function \(f\), its \(L_r(Q)\)-norm is denoted by \(\|f\|_{Q,r} \overset{\text{def}}{=} \left[\int_\mathcal{H} |f(\nu)|^r dQ(\nu)\right]^{1/r}\). Thus, given a sample \(\{D_t\}_{t=1}^n\) and \(h(\cdot; G)\), we have \(\|h(\cdot; G)\|_{Q_{\alpha_n}, 2} = \left[\sum_{t=1}^{n} \alpha_{n,t} |h(D_t; G)|^2\right]^{1/2}\).

In the stationary case, we define the \textit{restricted} function class \(\mathcal{H}_n = \{h_{1:n} : h_{1:n} \in \mathcal{H} \times \mathcal{H} \times \cdots \times \mathcal{H}, h_1 = h_2 = \cdots = h_n\}\) and the envelope function \(\overline{\mathcal{H}}_n = (H_1, H_2, \ldots, H_n)'\). Furthermore, let \(\mathcal{Q}\) denote the class of all discrete probability measures on the domain of the random vector \(D_t = \{Y_t, W_t, X_{t-1}\}\). Recall for an \(n\)-dimensional vector \(v = \{v_1, \ldots, v_n\}\), its \(l_2\)-norm is denoted by \(\|v\|_2 = (\sum_{i=1}^{n} v_i^2)^{1/2}\). Then, for any \(\alpha_n \in \mathbb{R}_+^n\) and \(\tilde{\alpha}_{n,t} = \frac{\sqrt{\alpha_{n,t}}}{\sqrt{\sum_{i=1}^{n} \alpha_{n,i}}}\), we have

\[
N(\delta|\tilde{\alpha}_n \circ \overline{\mathcal{H}}_n|_2, \tilde{\alpha}_n \circ \mathcal{H}_n, \cdot|_2) = N(\delta||H||_{Q_{\alpha_n}, 2}, \mathcal{H}, ||\cdot||_{Q_{\alpha_n}, 2}) \leq \sup_{\mathcal{Q} \in \mathcal{Q}} N(\delta||H||_{Q, 2}, \mathcal{H}, ||\cdot||_{Q, 2}).
\]

In light of this relationship in the stationary case, we generalize the setup. Let \(\mathcal{K}\) be a subset of \(\{1, \ldots, n\}\), and its dimension is \(|\mathcal{K}| = |\mathcal{K}|\). Let \(\alpha_{n,K}\) denote the \(K\)-dimensional sub-vector of \(\alpha_n\) corresponding to the index set \(\mathcal{K}\). Recall that \(H_t\) denotes the envelope function of \(\mathcal{H}_t\), the functional class corresponding to \(\{h_t(\cdot, G), G \in \mathcal{G}\}\), and \(\mathcal{H}_n = \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_n\). Then, for the subset \(\mathcal{K}\), we similarly define \(\mathcal{H}_{n,K}\) as the corresponding functional class, and the vector of its envelope functions as \(\overline{\mathcal{H}}_{n,K}\).

Let us consider the following assumption: for any fixed \(K\), the covering number can effectively be reduced to \(K\) dimension, i.e.,

\[
N(\delta|\tilde{\alpha}_n \circ \overline{\mathcal{H}}_n|_2, \tilde{\alpha}_n \circ \mathcal{H}_n, \cdot|_2) \leq \max_{\mathcal{K} \subseteq \{1:n\}} N(\delta|\tilde{\alpha}_{n,K} \circ \overline{\mathcal{H}}_{n,K}|_2, \tilde{\alpha}_{n,K} \circ \mathcal{H}_{n,K}, \cdot|_2),
\]

\[
\leq \max_{\mathcal{K} \subseteq \{1:n\}} \prod_{t \in \mathcal{K}} N(\delta|\tilde{\alpha}_{n,t} \cdot H_t|_2, \tilde{\alpha}_{n,t} \cdot \mathcal{H}_t, \cdot|_2),
\]

\[
\leq \sup_{\mathcal{Q} \in \mathcal{Q}} \max_{\mathcal{K} \subseteq \{1:n\}} \max_{t \in \mathcal{K}} N(\delta||H_t||_{Q, 2}, \mathcal{H}_t, ||\cdot||_{Q, 2})^K.
\]

The first inequality of (39) (in the main paper) in Assumption 3.8 holds by setting \(K = |\mathcal{K}| = 1\). Under this assumption, it suffices to examine the one-dimensional covering number \(N(\delta||H||_{Q, 2}, \mathcal{H}_t, \cdot||_{Q, 2})\).

To provide an example for the second inequality of (39) in the main paper, we let \(1(X_{t-1} \in G) = 1(X_{t-1}^T \theta \leq 0)\) for some \(\theta \in \Theta\), where \(\Theta\) is a compact set in \(\mathbb{R}^d\). Without loss of
generality, we assume that $\Theta$ is the unit ball in $\mathbb{R}^d$, i.e., for all $\theta \in \Theta$: $|\theta|_2 \leq 1$. For $w = 0$ and $1$, define $S_{t,w} \overset{\text{def}}{=} Y_t(w)1(W_t = w)/\Pr(W_t = w|X_{t-1})$, where $Y_t(w)$ is the abbreviation of $Y_t(W_{t-1}, w)$. Under Assumption 3.7, $|Y_t(1)|, |Y_t(0)| \leq M/2$, so we have $\|H_t\|_{Q,r} \leq M + M/\kappa \overset{\text{def}}{=} M'$, and

$$h_t^\theta = E(Y_t(1)1(X_{t-1}^\top \theta \leq 0)|X_{t-1}) + E(Y_t(0)1(X_{t-1}^\top \theta > 0)|X_{t-1}) - S_{t,1}1(X_{t-1}^\top \theta \leq 0) - S_{t,0}1(X_{t-1}^\top \theta > 0).$$

The corresponding functional class can be written as

$$\mathcal{H}_t = \{h : (y_t, w_t, x_{t-1}) \rightarrow f_{1,1}^\theta + f_{1,0}^\theta + f_{0,1}^\theta + f_{0,0}^\theta, \theta \in \Theta\},$$

where $f_{1,1}^\theta$ (resp. $f_{1,0}^\theta$, $f_{0,1}^\theta$, and $f_{0,0}^\theta$) corresponds to $E(Y_t(1)1(X_{t-1}^\top \theta \leq 0)|X_{t-1})$ (resp. $E(Y_t(0)1(X_{t-1}^\top \theta > 0)|X_{t-1})$, $-S_{t,1}1(X_{t-1}^\top \theta \leq 0)$, and $-S_{t,0}1(X_{t-1}^\top \theta > 0)$). Let the corresponding functional class be denoted by $\mathcal{F}_{1,1}$ (resp. $\mathcal{F}_{1,0}, \mathcal{F}_{0,1}, \text{and} \mathcal{F}_{0,0}$). For all finitely discrete norms $Q$ and any positive $\varepsilon$, we know that

$$\sup_Q \mathcal{N}(\varepsilon, \mathcal{H}_t, \|\cdot\|_{Q,r}) \leq \sup_Q \mathcal{N}(\varepsilon/4, \mathcal{F}_{1,1}, \|\cdot\|_{Q,r}) \mathcal{N}(\varepsilon/4, \mathcal{F}_{1,0}, \|\cdot\|_{Q,r}) \mathcal{N}(\varepsilon/4, \mathcal{F}_{0,1}, \|\cdot\|_{Q,r}) \mathcal{N}(\varepsilon/4, \mathcal{F}_{0,0}, \|\cdot\|_{Q,r}).$$

We look at the covering number of the respective functional class. According to Lemma 9.8 of Kosorok (2008), the subgraph of the function $1(X_{t-1}^\top \theta \leq 0)$ is of VC dimension less than $d + 2$ since the class $\{x \in \mathbb{R}^d, x^\top \theta \leq 0, \theta \in \Theta\}$ is of VC dimension less than $d + 2$ (see the proof of Lemma 9.6 of Kosorok (2008)). Therefore, we have $\sup_Q \mathcal{N}(\varepsilon/4, \mathcal{F}_{0,0}, \|\cdot\|_{Q,r}) \lesssim (4/(\varepsilon M'))^{d+2}$.

Moreover, we impose the following Lipschitz condition on functions $f_{1,1}^\theta$ and $f_{1,0}^\theta$: For any distinct points $\theta, \theta' \in \Theta$ and a positive constant $M_d$, it holds that $|f_{1,1}^\theta - f_{1,1}^{\theta'}|_{Q,r} \leq M_d |\theta - \theta'|$, (a similar equality holds for $f_{1,0}^\theta$). Then, it falls within the type II class defined in Andrews (1994), so according to the derivation of the (A.2) Andrews (1994) we have

$$\sup_Q \mathcal{N}(\varepsilon M', \mathcal{F}_{1,1}, \|\cdot\|_{Q,r}) \leq \sup_Q \mathcal{N}(\varepsilon M'/M_d, \Theta, \|\cdot\|_{Q,r}),$$

where the latter is the covering number of a Euclidean ball under the norm $\|\cdot\|_{Q,r}$. Thus, according to Equation (5.9) in Wainwright (2019), $\sup_Q \mathcal{N}(\varepsilon M'/M_d, \Theta, \|\cdot\|_{Q,r}) \lesssim (1 + 2M_d/\varepsilon M')^d$.

Combining the above results, we have $\sup_Q \mathcal{N}(\varepsilon M', \mathcal{H}_t, \|\cdot\|_{Q,r}) \lesssim (4/(\varepsilon M'))^{2(d+2)}(1 + 2M_d/\varepsilon M')^{2d}$. Finally, with some rearrangement and redefinition of constant terms, we can obtain (39) in the main paper.
A.5 Proof of Theorem 3.2

In this proof, we show the concentration of \( I = \frac{1}{T} \sum_{t=1}^{T-1} \tilde{S}_t(G) + \frac{1}{T} \sum_{t=1}^{T-1} \overline{S}_t(G) \). Under Assumptions 3.8 and 3.13 and with similar arguments as those for Lemma A.4, we have
\[
\sup_{G \in \mathcal{G}} \frac{1}{T-1} \sum_{t=1}^{T-1} \tilde{S}_t(G) \preceq_p M \sqrt{v}/\sqrt{T-1}. \tag{A.18}
\]

For \( \frac{1}{T-1} \sum_{t=1}^{T-1} \tilde{S}_t(G) \), we first show

(i) For a finite function class \( \mathcal{G} \) with \( |\mathcal{G}| = \tilde{M} < \infty \) and under Assumptions 3.10-3.12
\[
\sup_{G \in \mathcal{G}} \frac{1}{T-1} \sum_{t=1}^{T-1} \tilde{S}_t(G) \preceq_p \frac{c_T [\log \tilde{M} 2e\gamma]^{1/2} \Phi_{\tilde{\phi},F}}{\sqrt{T-1}}, \tag{A.19}
\]
with probability \( 1 - \exp(-c_T^2) \), where \( c_T \) is a large enough constant, and \( \Phi_{\tilde{\phi},F} \) is defined in Assumption 3.12.

Then, we extend the above result to obtain Theorem 3.2:

(ii) Let \( \mathcal{G} \) be a function class with infinite elements, and its complexity is subject to Assumption 3.13. Under Assumptions 3.10-3.13, we have
\[
\sup_{G \in \mathcal{G}} \frac{1}{T-1} \sum_{t=1}^{T-1} \tilde{S}_t(G) \preceq_p \frac{c_T [V \log T 2e\gamma]^{1/2} \Phi_{\tilde{\phi},F}}{\sqrt{T-1}}.
\]

Proof. By Assumption 3.1 and 3.11(i), we have \( E(S_t(G)|\mathcal{F}_{t-2}) = E(S_t(G)|X_{t-2}) \). By Assumption 3.10, \( E(S_t(G)|X_{t-2}) \) is continuously differentiable with respect to the underlying i.i.d. innovations. Denote \( \tilde{\varepsilon}_{t-1-2} \) as a point between \( \varepsilon_{t-1-2} \) and \( \varepsilon^*_{t-1-2} \) such that the following equation holds.

\[
\tilde{S}_t(G) - \tilde{S}_{t,t}^*(G) = E(S_t(G)|X_{t-2}) - E(S_t(G)|X_{t-2,t}) = \partial E(S_t(G)|X_{t-2}) / \partial \varepsilon_{t-1-2} = \varepsilon_{t-1-2} - \varepsilon^*_{t-1-2},
\]
\[
\leq |F_{t,t}(X_{t-2}, \varepsilon_{t-1-1})(\varepsilon_{t-1-2} - \varepsilon^*_{t-1-2})|.
\]

Thus, the dependence adjusted norm satisfies \( \theta_{x,q} \leq \sum_{t \geq 0} \max_t \| F_{t,t}(X_{t-2}, \varepsilon_{t-1-1})(\varepsilon_{t-1-2} - \varepsilon^*_{t-1-2}) \|_q \), and
\[
\Phi_{\tilde{\phi}}(\tilde{S}_t(G)) \leq \sup_{q \geq 2} \left( \sum_{t \geq 0} \max_t \| F_{t,t}(X_{t-2}, \varepsilon_{t-1-1})(\varepsilon_{t-1-2} - \varepsilon^*_{t-1-2}) \|_q / q \right) = \Phi_{\tilde{\phi},F}, \tag{A.20}
\]
holds for any \( G \in \mathcal{G} \). Following Assumption 3.12, we have \( \Phi_{\tilde{\phi}}(\tilde{S}_t(G)) < \infty \). Recall \( \gamma = \ldots \)
and for a finite \( G \) with \( |G| = \tilde{M} \), the next exponential bound follows by Theorem 3 of Wu and Wu (2016).

\[
\Pr\left( \sup_{G \in \mathcal{G}} \sum_{t=1}^{T-1} \tilde{S}_t(G) \geq x \right) \leq \tilde{M} \exp\left[ - \left( \frac{x}{\sqrt{T-1} \Phi_{\tilde{v}}(\tilde{S}(.)(G))} \right)^\gamma \frac{1}{2e\gamma} \right].
\] (A.21)

Set \( x = c_T [\log \tilde{M} 2e\gamma]^{1/\gamma} \Phi_{\tilde{v}}(\tilde{S}(.)(G)) \sqrt{T-1} \), where \( c_T \) is a sufficiently large constant, then

\[
\sup_{G \in \mathcal{G}} \frac{1}{T-1} \sum_{t=1}^{T-1} \tilde{S}_t(G) \lesssim c_T [\log \tilde{M} 2e\gamma]^{1/\gamma} \Phi_{\tilde{v},F} / \sqrt{T-1},
\] (A.22)

with probability \( 1 - \exp(-c_T^\gamma) \).

We now verify (ii), where \( G \) is not finite. We define \( \mathcal{G}^{(1)\delta} \) to be a \( \delta \) max, sup\( Q \|\tilde{F}_i\|_{Q,2} \)-net of \( \mathcal{G} \). We denote \( \pi(G) \) as the closest component of \( G \) in the net \( \mathcal{G}^{(1)\delta} \). Then,

\[
\sup_G \frac{1}{T-1} \sum_i \tilde{S}_i(G) \leq \sup_{G \in \mathcal{G}} \frac{1}{T-1} \sum_{t=1}^T (\tilde{S}_t(G) - \tilde{S}_t(\pi(G))) + \sup_{G \in \mathcal{G}^{(1)\delta}} \left| \frac{1}{T-1} \sum_{t=1}^{T-1} (\tilde{S}_t(G)) \right|
\leq \delta \max \sup_Q \|\tilde{F}_i\|_{Q,2} + \sup_{G \in \mathcal{G}^{(1)\delta}} \left| \frac{1}{T-1} \sum_{t=1}^{T-1} (\tilde{S}_t(G)) \right|
\lesssim_p \delta \max \sup_Q \|\tilde{F}_i\|_{Q,2} + c_T [V \log(1/\delta) 4e\gamma]^{1/\gamma} \Phi_{\tilde{v},F} / \sqrt{T-1},
\]

where \( V \) and \( \delta \) following the first \( \lesssim_p \) are the constants in the first statement of Assumption 3.13. \( \Phi_{\tilde{v},F} \) is defined in Assumption 3.12. Finally, by setting \( \delta = \frac{1}{T} \),

\[
\sup_G \frac{1}{T-1} \sum_{t=1} \tilde{S}_t(G) \lesssim_p c_T [V \log T 2e\gamma]^{1/\gamma} \Phi_{\tilde{v},F} / \sqrt{T-1}.
\] (A.23)
A.6 Additional Figures for the empirical application

A.6.1 Time series plots of the raw data

Figure 3. Weekly cases and deaths from 4-2020 to 1-2022

Figure 4. Restriction level and economic condition from 4-2020 to 1-2022
Figure 5. Vaccine coverage from 4-2020 to 1-2022

The vaccine coverage is calculated by summing up the coverage rate of each dose.

A.6.2 Estimated propensity scores

Figure 6. Estimated propensity scores
A.6.3 Policy choices based on an increased number of variables

Figure 7. Optimal policies based on
\[ X_{T-1}^P = (\text{change in cases}_{T-1}, \text{restriction stringency}_{T-1}, \text{vaccine coverage}_{T-1}) \]

In the left panel, the \( x \)-axis is the change in deaths at week \( T - 1 \), and the \( y \)-axis is the stringency of restrictions at week \( T - 1 \); in the right panel, the \( x \)-axis is the stringency of restrictions at week \( T - 1 \), and the \( y \)-axis is the vaccine coverage at week \( T - 1 \).

Figure 8. Optimal policy based on
\[ X_{T-1}^P = (\text{change in deaths}_{T-1}, \text{restriction stringency}_{T-1}, \text{vaccine coverage}_{T-1}, \text{change in cases}_{T-1}) \]

In the left panel, the \( x \)-axis is the change in deaths at week \( T - 1 \), and the \( y \)-axis is the stringency of restrictions at week \( T - 1 \); in the right panel, the \( x \)-axis is the vaccine coverage at week \( T - 1 \), and the \( y \)-axis is the change in cases at week \( T - 1 \).
A.7 Nonparametric method to bound the conditional regret

In Section 3.2, we have shown that we can bound conditional regret by unconditional regret if the unconditional first best policy is feasible. In Appendix A.7.1 we present examples where this method is applicable, and examples where it is not. More examples can be found in Appendix B.2. When Assumption 3.4 does not hold, we proceed to the nonparametric method discussed in Appendix A.7.2.

A.7.1 Motivation

In this subsection, we discuss the relationship between the optimal policy solutions in terms of conditional and unconditional welfare. This relationship is not straightforward. In some cases, unconditional welfare does bound conditional welfare, and the methods and results in Section 3.2 directly apply. However, in other cases, when the first best policy is not feasible, we shall use the kernel estimator. This motivates us to present the kernel estimator as an important alternative to the method of Section 3.2.

Example A.1. Observing $X_{T-1} = (Y_{T-1}, W_{T-1}, Z_{T-1})' \in \mathbb{R} \times \{0, 1\} \times \mathbb{R}^2$ at time $T - 1$, the planner chooses $W_T$ based on the last two continuous variables. The feasible policy class is rectangles in $\mathbb{R}^2$:

$$G = \{ z \in [a_1, a_2] \times [b_1, b_2] : a_1, a_2, b_1, b_2 \in \mathbb{R} \}. $$

The corresponding unconditional problem is $\max_{G \in G} W_T(G)$. Suppose the planner is interested in maximizing the welfare conditional on $Z_{T-1} = z := (z^{(1)}, z^{(2)})$. The conditional problem is to find

$$\arg \max_{G \in G} W_T(G | Z_{T-1} = z). \quad (A.24)$$

We illustrate how the policy solutions can differ between conditional and unconditional welfare functions. In Figures 9 and 10, the shaded area represents the region where the conditional average treatment effect is positive. The first best unconditional policy assigns $W_T = 1$ to any value of $Z_{T-1}$ inside this region, and $W_T = 0$ to any point outside it. This policy is also the solution to (A.24). The red rectangle is the best feasible (i.e. rectangular) unconditional policy. This policy assigns $W_T = 1$ to any realization of $Z_{T-1}$ inside the rectangle. The conditional policy concerns what policy to assign only at a particular value of $Z_{T-1}$ corresponding to its realized value in the data (blue point in the right-hand side panel of Figure 9). If the best feasible unconditional policy (red rectangle) agrees with the first best unconditional policy (shared area), then the policy choice informed by the unconditional policy is guaranteed to be optimal in terms of the conditional policy at any conditioning value of $Z_{T-1}$.
Figure 9. $G_{FB}^* \in \mathcal{G}$

Unconditional policy

Conditional policy

Figure 10. $G_{FB}^* \notin \mathcal{G}$

Unconditional policy

Conditional policy

Figure 10 shows a case where the first best unconditional policy is not contained in the class of feasible policies: it is not possible to implement the policy choice that coincides with the shaded area. In this case, the policy chosen by the unconditionally optimal feasible policy (red rectangle in the left-hand side panel) does not coincide with the optimal policy choice of the conditional policy. The highlighted point $(z^{(1)}, z^{(2)})$ lies outside the red rectangle, so the best feasible unconditional policy would set $W_t = 0$. However, it lies within the shaded region, so the conditional average treatment effect given $Z_{T-1} = (z^{(1)}, z^{(2)})$ is positive, and the optimal policy conditional on $Z_{T-1} = (z^{(1)}, z^{(2)})$ is to set $W_t = 1$.

These two examples show the importance of the first best unconditional policy: when it is included in the set of feasible unconditional policies, the solution to the conditional problem corresponds to the solution to the unconditional problem. When it is not included, we do not have this correspondence. In Appendix B.2.1 we show that the feasibility of the first best solution to the unconditional problem is a sufficient condition. A sufficient and necessary condition is given by Assumption 3.4.
A.7.2 Nonparametric estimator of the optimal conditional policy

If the solution to the conditional problem cannot be obtained from the unconditional problem, the conditional problem must be solved directly. That is, the SP should estimate an optimal policy from the empirical analogue of the conditional welfare function. If the conditioning variables are continuous, some type of nonparametric smoothing is unavoidable. Here we use a kernel-based method to estimate the optimal conditional policy. Recall that welfare conditional on \( X_{T-1} = x \) can be written as

\[
W_T(G|x) = \mathbb{E}\{Y_T(W_{T-1}, 1)1(X_{T-1} \in G) + Y_t(W_{T-1}, 0)1(X_{T-1} \notin G)|X_{T-1} = x\}.
\]

For simplicity, we let \( X_{T-1} \in \mathbb{R} \). Then (27) can be rewritten as \(^{12}\)

\[
\hat{W}(G|x) = \frac{\sum_{t=1}^{T-1} K_h(X_{t-1}, x) \hat{W}_t(G)}{\sum_{t=1}^{T-1} K_h(X_{t-1}, x)},
\]

(A.25)

where \( \hat{W}_t(G) := \frac{Y_t W_t}{e_t(X_{t-1})}1(X_{t-1} \in G) + \frac{Y_t(1-W_t)}{1-e_t(X_{t-1})}1(X_{t-1} \notin G) \), and \( K_h(a, b) := \frac{1}{h}K\left(\frac{a-b}{h}\right) \) with \( K(\cdot) \) being assumed to be a bounded kernel function with a bounded support.

Recall that \( G \) is the set of feasible policies conditional on \( X_{T-1} = x \). We define

\[
G^*_x \in \arg\max_{G \in G} W(G|x),
\]

\[
\hat{G}_x \in \arg\max_{G \in G} \hat{W}(G|x),
\]

to be the maximizers of \( W(G|x) \) and \( \hat{W}(G|x) \), and

\[
\hat{W}_h(G|x) = \frac{\sum_{t=1}^{T-1} K_h(X_{t-1}, x) W_t(G|x)}{\sum_{t=1}^{T-1} K_h(X_{t-1}, x)},
\]

(A.26)

where the second equality follows from Assumption 3.3.

The invariance of welfare ordering assumption is modified to:

**Assumption A.1** (Invariance of welfare ordering). For any \( G \in G \) and \( x \in X \), there exists some constant \( c \)

\[
W_T(G^*_x|x) - W_T(G|x) \leq c[\hat{W}_h(G^*_x|x) - \hat{W}_h(G|x)].
\]

(A.27)

Similar to Assumption 2.4, (A.27) holds if the stochastic process \( S_t(x) := Y_T(W_{T-1}, 1)1(X_{T-1} \in G) + Y_t(W_{T-1}, 0)1(X_{T-1} \notin G)|X_{T-1}=x \) is weakly stationary.

The following theorem shows an upper bound for conditional regret in the one-dimensional

\(^{12}\)If the set of conditioning variables \( X_{T-1} \) contains both continuous and discrete components, we can adapt a hyper method to construct a valid sample analogue combing kernel-smoothing (for continuous variables) and subsamples (for discrete variables). In this section, we focus on the case where the target welfare function is conditional on a univariate continuous variable.
covariate case (i.e., $X_t \in \mathbb{R}$). This result can be readily extended to the multiple-covariate case.

**THEOREM A.1.** Under Assumptions A.1 and A.2 specified in Appendix A.7.3, we will have

\[
\sup_{P_T \in P(T,M,\kappa)} \sup_{G \in \mathcal{G}} \sup_{x \in \mathcal{X}} \mathbb{E}_{P_T} \left[ W_T(G|x) - W_T(\hat{G}_x|x) \right] \leq c_1(\sqrt{(T-1)h^{-1}} + (T-1)^{-1} + h^2).
\]

Setting $h = O(T^{-1/5})$, the right-hand side bound is $O(T^{-2/5})$.

A proof is presented in Appendix A.7.3.

### A.7.3 Proof of Theorem A.1

For simplicity, we maintain Assumption 3.1, and one of its implications is equation (26).

In addition to (A.25) and (A.26), we define

\[
\bar{W}_h(G, x) = \frac{\sum_{t=1}^{T-1} K_h(X_{t-1}, x) \mathbb{E} \left[ \frac{Y_t W_t}{\alpha_t(X_{t-1})} 1(X_{t-1} \in G) + \frac{Y_t(1-W_t)}{1-\alpha_t(X_{t-1})} 1(X_{t-1} \notin G) | \mathcal{F}_{t-1} \right]}{\sum_{t=1}^{T-1} K_h(X_{t-1}, x)},
\]

where $K(\cdot)$ is a bounded kernel with a bounded support, $K_h(a, b) := \frac{1}{h} K(\frac{a-b}{h})$. The second equality follows from Assumption 3.3.

Our strategy is to show for any $x \in \mathcal{X}$ and any $G \in \mathcal{G}$, such that

\[
\begin{align*}
W_T(G|x) - W_T(\hat{G}|x) &\leq c [\bar{W}_h(G|x) - \bar{W}_h(\hat{G}|x)] \\
&\leq c [\bar{W}_h(G|x) - \bar{W}_h(\hat{G}, x)] + O_p(h^2) + O_p(c_w^{-1}(\sqrt{(T-1)h})^{-1}) \\
&\leq \sup_{\mathcal{G} \in \mathcal{G}} 2c |\bar{W}_h(G, x) - \bar{W}G(x)| + O_p(h^2) + O_p(c_w^{-1}(\sqrt{(T-1)h})^{-1}) \\
&= O_p(h^2) + O_p(c_w^{-1}(\sqrt{(T-1)h})^{-1}),
\end{align*}
\]

where the first inequality follows from Assumption A.1. The second inequality follows from Lemma A.5 below. The third inequality follows from similar arguments to (21). The last equality follows from Lemma A.6 stated below.

We present these two lemmas and their proofs. First, let us impose

**Assumption A.2.** $X_t$ is a one-dimensional covariate. Let $c_m$, $c_k$, and $c_w$ be positive constants. (i) The kernel function $K(x)$ is bounded and has bounded support: $K(x) = 0$ if
LEMMA A.5. Under Assumption A.2, for any $G \in \mathcal{G}$ and $x \in \mathcal{X}$,

$$\tilde{W}_h(G, x) - \hat{W}_h(G|x) = O_p(h^2) + O_p(c_w^{-1}(\sqrt{(T-1)h})^{-1}).$$

**Proof.** Under Assumption A.2,

$$\left[\tilde{W}_h(G, x) - \hat{W}_h(G|x)\right] \left[\frac{1}{T-1} \sum_{t=1}^{T-1} K_h(X_{t-1}, x)\right]$$

$$= \frac{1}{T-1} \sum_{t=1}^{T-1} K_h(X_{t-1}, x)(W_t(G|X_{t-1}) - W_t(G|x))$$

$$= \frac{1}{T-1} \sum_{t=1}^{T-1} \left\{ K_h(X_{t-1}, x)(W_t(G|X_{t-1}) - W_t(G|x)) \right\}$$

$$- \mathbb{E}[K_h(X_{t-1}, x)(W_t(G|X_{t-1}) - W_t(G|x)) | \mathcal{F}_{t-2}]$$

$$+ \frac{1}{T-1} \sum_{t=1}^{T-1} \left\{ \mathbb{E}[K_h(X_{t-1}, x)(W_t(G|X_{t-1}) - W_t(G|x)) | \mathcal{F}_{t-2}] \right\}.$$

Rearranging the equation we have

$$\tilde{W}_h(G, x) - \hat{W}_h(G|x) = O_p(c_w^{-1}(\sqrt{(T-1)h})^{-1}) + O_p(h^2),$$

where the first term on the right-hand side follows from Theorem 3.1 and the second term follows from the standard result concerning the bias of the kernel estimator.

LEMMA A.6. Under Assumption A.2,

$$\sup_{G \in \mathcal{G}} \sup_{x \in \mathcal{X}} |\tilde{W}(G|x) - \hat{W}_h(G, x)| \lesssim_p c_w^{-1}(\sqrt{(T-1)h})^{-1}. \tag{A.28}$$

**Proof.** Note

$$\sup_{G \in \mathcal{G}} \sup_{x \in \mathcal{X}} |\tilde{W}(G|x) - \hat{W}_h(G, x)|$$

$$\leq \sup_{G \in \mathcal{G}} \sup_{x \in \mathcal{X}} \sum_{t=1}^{T-1} K_h(X_{t-1}, x)(\tilde{W}_t(G) - W_t(G|\mathcal{F}_{t-1}))/\sum_{t=1}^{T-1} K_h(X_{t-1}, x).$$

We first look at the numerator, $\sup_{G \in \mathcal{G}} \sup_{x \in \mathcal{X}} \sum_{t=1}^{T-1} K_h(X_{t-1}, x)(\tilde{W}_t(G) - W_t(G|\mathcal{F}_{t-1})).$

Suppose the order statistics of $\{X_t\}_{t=1}^{T-1}$ is $X_{(1)}, \ldots, X_{(T-1)}$, and $B_{x,h} = \{t : \|x - x_t\|/h \leq 1\}$.
Because of summation by part, the numerator is bounded by
\[
\sup_{G \in \mathcal{G}} \sup_{x \in \mathcal{X}} \left| \sum_{t=1}^{T-1} K_h(X(t), x) - K_h(X(t-1), x) \right|
\]
\[
\max_{1 \leq t \leq T-1, t \in B_{x,h}} \left| \sum_{t=1}^{T-1} (\hat{W}_t(G) - W_t(G|\mathcal{F}_{t-1})) \right|
\]
\[
+ \sup_{G \in \mathcal{G}} \sup_{x \in \mathcal{X}} \left| \sum_{t \in B_{x,h}} (\hat{W}_t(G) - W_t(G|\mathcal{F}_{t-1})) \right|
\]
\[
\lesssim_p h^{-1} \sup_{t \in B_{x,h}} \left| \sum_{t \in B_{x,h}} (\hat{W}_t(G) - W_t(G|\mathcal{F}_{t-1})) \right|,
\]
where \( \left| \sum_{t=1}^{T-1} [K_h(X(t), x) - K_h(X(t-1), x)] \right| \lesssim_p h^{-1} \) if the total variation of the function \( hK_h(\cdot, x) \) is bounded. We also have \( \sup_{G \in \mathcal{G}} (T-1)^{-1} \left| \sum_{t \in B_{x,h}} (\hat{W}_t(G) - W_t(G|\mathcal{F}_{t-1})) \right| \lesssim_p \sqrt{h/(\sqrt{T} - 1)} \), where \( \lesssim_p \) follows from \( |\hat{W}_t(G)| \leq M \).

For the denominator, \((T-1)^{-1} \sum_{t=1}^{T-1} \{K_h(X_{t-1}, x) - E(K_h(X_{t-1}, x)|\mathcal{F}_{t-2})\} \lesssim_p 1/\sqrt{h(T-1)} + (T-1)^{-1}, \)
where \( \lesssim_p \) follows from \( c_m, \int K(u)^2 du \) and the bound on the kernel function as assumed in Assumption A.2. Due to the boundedness of \( E(K_h(X_{t-1}, x)|\mathcal{F}_{t-2}) \) from the Assumption A.2 by following similar steps to the proof of (22) in the main paper, we have
\[
\sup_{G \in \mathcal{G}} \sup_{x \in \mathcal{X}} |\hat{W}(G|X_{T-1} = x) - \hat{W}(G|X_{T-1} = x)| \lesssim_p c_w^{-1}(\sqrt{(T-1)h^{-1}}).
\]
(A.29)

\section{Proof of Theorem 4.1}

\textbf{Proof.} Under policy \( G \), we use the following notation: \( \hat{W}(G) \) is defined in (34) in the main paper; \( \hat{W}(G) \) represents the estimated welfare defined in (32) in the main paper; \( \hat{W}^\hat{e}(G) \) represents the estimated welfare, with the estimated propensity score \( \hat{e}(\cdot) \); and
\[
\hat{W}^\hat{e}(G) = \frac{1}{T-1} \sum_{t=1}^{T-1} \left[ \frac{Y_t W_t}{\hat{e}_t(X_{t-1})} 1(X_{t-1} \in G) + \frac{Y_t (1 - W_t)}{1 - \hat{e}_t(X_{t-1})} 1(X_{t-1} \notin G) \right].
\]
Recall that \( G^* \) is the optimal policy defined in (28). Let \( \hat{G}^e \) be the optimal policy estimated using the estimated propensity score \( \hat{e}(\cdot) \),
\[
\hat{G}^e \in \text{argmax}_{G \in \mathcal{G}} \hat{W}^\hat{e}(G).
\]
(A.30)
Recall $\tau_t = \frac{Y_t W_t}{\epsilon_t(W_{t-1})} - \frac{Y_t (1 - W_t)}{1 - \epsilon_t(W_{t-1})}$ and $\hat{\tau}_t = \frac{Y_t W_t}{\hat{\epsilon}_t(W_{t-1})} - \frac{Y_t (1 - W_t)}{1 - \hat{\epsilon}_t(W_{t-1})}$. Similar to (A.29) in the supplementary material for Kitagawa and Tetenov (2018), we have

$$\hat{\mathcal{W}}(G_s) - \hat{\mathcal{W}}(\hat{G}^c_s),$$

$$= \hat{\mathcal{W}}(G_s) - \hat{\mathcal{W}}(\hat{G}^c_s) + \left[ \hat{\mathcal{W}}^c(G_s) - \hat{\mathcal{W}}^c(\hat{G}^c_s) \right] + \left[ \hat{\mathcal{W}}^c(\hat{G}^c_s) - \hat{\mathcal{W}}^c(G_s) \right] + \left[ \hat{\mathcal{W}}(G_s) - \hat{\mathcal{W}}(G_s) \right]$$

$$\leq \hat{\mathcal{W}}(G_s) - \hat{\mathcal{W}}(\hat{G}^c_s) + \left[ \hat{\mathcal{W}}^c(\hat{G}^c_s) - \hat{\mathcal{W}}^c(G_s) \right] + \left[ \hat{\mathcal{W}}(G_s) - \hat{\mathcal{W}}(G_s) \right]$$

$$= \left[ \hat{\mathcal{W}}(G_s) - \hat{\mathcal{W}}^c(G_s) - \hat{\mathcal{W}}(\hat{G}^c_s) + \hat{\mathcal{W}}^c(\hat{G}^c_s) \right] + \left[ \hat{\mathcal{W}}(G_s) - \hat{\mathcal{W}}(\hat{G}^c_s) - \hat{\mathcal{W}}(G_s) + \hat{\mathcal{W}}(\hat{G}^c_s) \right]$$

$$= I^\hat{c} + II^\hat{c}, \quad (A.31)$$

where the first inequality comes from $\hat{\mathcal{W}}^c(\hat{G}^c_s) \geq \hat{\mathcal{W}}^c(G_s)$, which is implied by the definition \[\text{(A.30)\].

For $II^\hat{c}$, we know $II^\hat{c} \leq 2 \sup_{G \in \mathcal{G}} |\mathcal{W}(G) - \hat{\mathcal{W}}(G)|$. Similar arguments to Section 3.2 can then be used to bound it.

For $I^\hat{c}$, Note that for any $G \in \mathcal{G}$

$$\hat{\mathcal{W}}(G) = \frac{1}{T-1} \sum_{t=1}^{T-1} \left[ \frac{Y_t W_t}{\epsilon_t(X_{t-1})} 1(X_{t-1} \in G) + \frac{Y_t (1 - W_t)}{1 - \epsilon_t(X_{t-1})} 1(X_{t-1} \notin G) \right]$$

$$= \frac{1}{T-1} \sum_{t=1}^{T-1} \left[ \tau_t 1(X_{t-1} \in G) + \frac{Y_t (1 - W_t)}{1 - \epsilon_t(X_{t-1})} \right]. \quad (A.32)$$

Similarly,

$$\hat{\mathcal{W}}^c(G) = \frac{1}{T-1} \sum_{t=1}^{T-1} \left[ \hat{\tau}_t 1(X_{t-1} \in G) + \frac{Y_t (1 - W_t)}{1 - \hat{\epsilon}_t(X_{t-1})} \right]. \quad (A.33)$$

Combining (A.32) and (A.33) with $I^\hat{c}$,

$$\hat{\mathcal{W}}(G_s) - \hat{\mathcal{W}}(\hat{G}^c_s) = \frac{1}{T-1} \sum_{t=1}^{T-1} \left[ \tau_t 1(X_{t-1} \in G_s) - 1(X_{t-1} \in \hat{G}^c_s) \right]$$

$$\hat{\mathcal{W}}^c(G_s) - \hat{\mathcal{W}}^c(\hat{G}^c_s) = \frac{1}{T-1} \sum_{t=1}^{T-1} \left[ \hat{\tau}_t 1(X_{t-1} \in G_s) - 1(X_{t-1} \in \hat{G}^c_s) \right].$$
Then,

\[ I^\delta = \hat{W}(G_s) - \hat{W}(\hat{G}^\delta) - \hat{W}^\delta(G_s) - \hat{W}^\delta(\hat{G}^\delta), \]

\[ = \frac{1}{T-1} \sum_{t=1}^{T-1} \left[ (\tau_t - \hat{\tau}_t) \cdot 1\{X_i \in G_s\} - (\tau_t - \hat{\tau}_t) \cdot 1\{X_i \in \hat{G}^\delta\} \right], \]

\[ = \frac{1}{T-1} \sum_{t=1}^{T-1} \left[ (\tau_t - \hat{\tau}_t) \left( 1\{X_i \in G_s\} - 1\{X_i \in \hat{G}^\delta\} \right) \right], \]

\[ \leq \frac{1}{T-1} \sum_{t=1}^{T-1} |\tau_t - \hat{\tau}_t|. \]

Finally, we have that the rate of convergence is bounded by the accuracy of propensity score estimation and the bound with known propensity scores,

\[ \mathbb{E}_{p_T} \left| W_T(G_s) - W_T(\hat{G}^\delta) \right| \leq \mathbb{E}_{p_T} \left[ \frac{1}{T-1} \sum_{t=1}^{T-1} |\tau_t - \hat{\tau}_t| \right] \]

\[ + 2 \mathbb{E}_{p_T} \left[ \sup_{G \in \mathcal{G}} |\hat{W}(G) - \hat{W}(\hat{G})| \right]. \]

The statement of Theorem 4.1 follows by (50) and Theorem 3.3.
B Other Results and Extensions

B.1 Link to Markov decision problems

In this section, we show the connection between our T-EWM setup and models of the Markov Decision Process (MDP). For MDP, we adapt the notation of Kallenberg (2016) (LK hereafter), an online lecture note by Lodewijk Kallenberg. As described in LK, the MDP is the set of models for making decisions for dependent data. An MDP typically has components \{p_{ij}(a)\}_{i,j}, r_t^i(a), W_{t-1}\}. In period t, \(W_{t-1}\) is the state and \(a\) is the action. The agent chooses their decision according to a policy (a map from the state \(W_{t-1}\) to an action \(a\)). They then receive a reward \(r_t^i(a)\). The reward function depends on the transition probabilities of a Markov process, which are determined by the action \(a\). Thus their action affects the reward via its effect on the transition probability matrix. The optimal policy is estimated by optimizing an aggregated reward function. In this subsection, we show a formal link between our T-EWM framework and an MDP. In particular, we show that the MDP’s reward function corresponds to our welfare function, and the optimal mapping between states and actions corresponds to the EWM policy in our framework.

In the following equations, the left-hand sides are the notations for the MDP in LK, and the right-hand sides are notations for T-EWM from this paper. We consider the model of Section 2.2. First, we link the transition probability with the propensity score: For \(i, j, a \in \{0, 1\}\) at time \(t\),

\[ p_t^{ij}(a) = \Pr(W_t = j|W_{t-1} = i; \text{choosing } W_t = a), \]

The left-hand side is the Markov transition probability between states \(i\) and \(j\) under policy \(a\). The right-hand side is a propensity score under policy \(a\): the probability \(W_t = j\), conditional on \(W_{t-1} = i\), given \(W_t = a\). Note that, in this simple model, the state at time \(t\) is the previous treatment \(W_{t-1}\), and the current policy and the next-period state are both \(W_t\). In our setting, after time \(T - 1\), the SP implements a deterministic policy, so the probability only takes values in \(\{0, 1\}\), i.e.,

\[ p_t^{ij}(a) = 1 \text{ if } j = a. \]
\[ p_t^{ij}(a) = 0 \text{ if } j \neq a. \] (B.1)

Secondly, we connect the reward function with the expected conditional counterfactual outcome. We denote the reward associated with action \(a\) for state \(i\) at time \(t\) as

\[ r_t^i(a) = \mathbb{E}[Y_t(a)|W_{t-1} = i], \]

(B.2)
The left-hand side is the reward in state $i$ under action $a$. The right-hand side is the conditional expected counterfactual outcome of $Y_t(a)$, (recall $a \in \{0,1\}$) conditional on $W_{t-1} = i$.

Thirdly, we link the expected reward function and the expected unconditional counterfactual outcome.

$$\sum_i \beta_i r_i^1(a) = r^1(a) = \mathbb{E}[Y_t(a)],$$

The left-hand side is the expected reward for action $a$, with $\beta_i$ as the initial probability of state $i$. The right-hand side is the unconditional expected counterfactual outcome.

Finally, we show the link between the total expected reward over a finite horizon (of length 2) and the finite-period welfare function

$$v_i^{T:T+1}(R) = \mathbb{E}_{i,R} \left[ \sum_{k=T}^{T+1} r_t^k(W_k) \right]$$

$$= \mathbb{E} \left\{ Y_T(1)p^T_{g1}(g_1(W_{T-1})) + Y_T(0)p^T_{g0}(g_1(W_{T-1})) + Y_{T+1}(1)p^{T+1}_{g1}(g_2(W_T)) + Y_{T+1}(0)p^{T+1}_{g0}(g_2(W_T)) \mid W_{T-1} = i \right\}$$

$$= \mathbb{E} \left\{ Y_T(1)g_1(W_{T-1}) + Y_T(0)[1 - g_1(W_{T-1})] \mid W_{T-1} = i \right\}$$

$$+ \mathbb{E} \left\{ Y_{T+1}(1)g_2(W_T) + Y_{T+1}(0)[1 - g_2(W_T)] \mid W_T = g_2(W_{T-1}) \right\}, \quad (B.3)$$

The left-hand side is the total expected reward over the planning horizon from $T$ to $T+1$ under the policy $R = (g_1, g_2)$, with the initial state $i$. The last equality follows from (B.1) and the exclusion condition.

With these connections established, we can regard EMW as an MDP with a finite period reward and a non-stationary solution. According to LK, in this case, the policy $R$ is usually obtained by using a backward induction algorithm.

Note that, for multi-period welfare functions, if the welfare function has a finite horizon (as in the case discussed here), the optimal policy is often nonstationary. See Chapter 2 of LK for more details about finite horizon MDPs.

### B.2 The relationship between the conditional and unconditional cases

In this section, we present further examples to illustrate that $G^*_FB \in \mathcal{G}$ is a sufficient but not necessary condition for equivalence between the unconditional and conditional problems.
Finally, we extend Example A.1 to show how Assumption 3.4 ensures this equivalence.

B.2.1 \( G^*_\text{FB} \in \mathcal{G} \) is sufficient: a discrete and a continuous case

Let \( X_{t-1} = W_{t-1} \in \{0, 1\} \), and \( \mathcal{G} \) be a subclass of the power set of \( \{0, 1\} \), \( \mathcal{P} = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\} \).

For compactness, suppress \( W_{T-1} \) in \( Y_T(W_{T-1}, 1) \). Unconditional welfare can be written as:

\[
W_T(G) = \mathbb{E} \{ Y_T(1)1(W_{T-1} \in G) + Y_T(0)1(W_{T-1} \notin G) \} \\
= \mathbb{E} \{ [Y_T(1) - Y_T(0)]1(W_{T-1} \in G) + Y_T(0) \} \\
= \mathbb{E} [\tau(W_{T-1})1(W_{T-1} \in G)] + \mathbb{E} [Y_T(0)], \tag{B.4}
\]

where \( \tau(w_{T-1}) = \mathbb{E} [Y_T(1) - Y_T(0)|W_{T-1} = w_{T-1}] \), and the last equality follows from the law of iterated expectations.

The first best unconditional policy is

\[
G^*_\text{FB} \equiv \{ w_{T-1} \in \{0, 1\} : \tau(w_{T-1}) \geq 0 \}. \tag{B.5}
\]

By the assumption that \( G^*_\text{FB} \in \mathcal{G} \)

\[
G^*_\text{FB} = \text{argmax}_{G \in \mathcal{G}} W_T(G). \tag{B.6}
\]

The SP’s conditional objective function can be written as

\[
W_T(G|W_{T-1}) = \mathbb{E} \{ Y_T(1)1(W_{T-1} \in G) + Y_T(0)1(W_{T-1} \notin G)|W_{T-1} \} \\
= \mathbb{E} \{ [Y_T(1) - Y_T(0)]1(W_{T-1} \in G) + Y_T(0)|W_{T-1} \} \\
= \mathbb{E} \{ [Y_T(1) - Y_T(0)]1(W_{T-1} \in G)|W_{T-1} \} + \mathbb{E} [Y_T(0)|W_{T-1}] \\
= \mathbb{E} [\tau(W_{T-1})1(W_{T-1} \in G)|W_{T-1}] + \mathbb{E} [Y_T(0)|W_{T-1}]. \tag{B.7}
\]

To check whether \( G^*_\text{FB} \) is optimal in the conditional problem, we need to study this problem w.r.t. \( \mathcal{P} \)

\[
\max_{G \in \mathcal{P}} W_T(G|W_{T-1} = w_{T-1}) = \max_{G \in \mathcal{P}} \mathbb{E} [\tau(W_{T-1})1(W_{T-1} \in G)|W_{T-1} = w_{T-1}] \\
= \max_{G \in \mathcal{P}} \tau(w_{T-1})1(w_{T-1} \in G) \\
= \tau(w_{T-1})1(w_{T-1} \in G^*_\text{FB}), \tag{B.8}
\]

where the first equality follows from (B.7), and the last one follows from the definition of
$G^*_{FB}$ in (B.5). The equivalence follows by combining (B.6) and (B.8).

We now turn to the continuous conditioning variable case.

The SP’s unconditional welfare function can be rewritten as (suppressing $W_{T-1}$ in $Y_T(W_{T-1}, 1)$.)

\[
W_T(G) = \mathbb{E} \{ Y_T(1) \mathbf{1}(X_{T-1} \in G) + Y_T(0) \mathbf{1}(X_{T-1} \not\in G) \} \\
= \mathbb{E} \{ [Y_T(1) - Y_T(0)] \mathbf{1}(X_{T-1} \in G) + Y_T(0) \} \\
= \mathbb{E} \{ \tau(X_{T-1}) \mathbf{1}(X_{T-1} \in G) + \mathbb{E} \{ Y_T(0) \} ,
\]

where $\tau(x_{T-1}) = \mathbb{E} \{ Y_T(1) - Y_T(0)|X_{T-1} = x_{T-1} \}$, and the last equality follows from the law of iterated expectations.

The first best policy is

\[
G^*_{FB} \equiv \{ x_{T-1} \in \mathbb{R}^2 : \tau(x_{T-1}) \geq 0 \}. \tag{B.9}
\]

(For simplicity, we assume that $G^*_{FB}$ is measurable, i.e., $G^*_{FB} \in \mathcal{B}(\mathbb{R}^2) \subset \mathcal{P}(\mathbb{R}^2)$. This means that we don’t have to deal with the outer probability and expectation.) By assumption $G^*_{FB} \in \mathcal{G}$,

\[
G^*_{FB} \in \operatorname{argmax}_{G \in \mathcal{G}} W_T(G). \tag{B.10}
\]

We now introduce some notations. Let $X_{t-1} = [X_{t-1}^{(1)}, X_{t-1}^{(2)}]' \in \mathbb{R}^2$. $X_{t-1}^{(1)}$ and $X_{t-1}^{(2)}$ can be continuous or discrete, e.g., one of them can be the treatment $W_{T-1}$. Let $\mathcal{G}$ be a class of subsets of $\mathbb{R}^2$, and $\mathcal{G}^{(1)}$ a class of subsets of $\mathbb{R}$. The SP’s conditional objective function can be written as:

\[
W_T(G^{(1)}|X_{T-1}^{(2)}) = \mathbb{E} \left\{ Y_T(1) \mathbf{1}(X_{T-1}^{(1)} \in G^{(1)}) + Y_T(0) \mathbf{1}(X_{T-1}^{(1)} \not\in G^{(1)}|X_{T-1}^{(2)}) \right\} \\
= \mathbb{E} \left\{ [Y_T(1) - Y_T(0)] \mathbf{1}(X_{T-1}^{(1)} \in G^{(1)}) + Y_T(0)|X_{T-1}^{(2)} \right\} \\
= \mathbb{E} \left\{ \tau(X_{T-1}) \mathbf{1}(X_{T-1}^{(1)} \in G^{(1)})|X_{T-1}^{(2)} \right\} + \mathbb{E} \left\{ Y_T(0)|X_{T-1}^{(2)} \right\}, \tag{B.11}
\]

where the last equality follows from the law of iterated expectations.

We also assume that the conditional first-best policy $G^*_{CFB}$ is measurable, i.e., $G^*_{CFB}(X_{T-1}^{(2)} = x_{T-1}^{(2)}) \in \mathcal{B}(\mathbb{R})$, for every $x_{T-1}^{(2)} \in \mathbb{R}$. Note that $\tau(x_{t-1}) = \tau(x_{t-1}^{(1)}, x_{t-1}^{(2)})$. Following the last row
of (B.11), the optimal conditional policy is defined to be
\[
\arg\max_{G(1) \in B(\mathbb{R})} \mathcal{W}_T(G(1)|X_{T-1}^{(2)} = x_{T-1}^{(2)})
= \arg\max_{G(1) \in \mathbb{R}} \mathbb{E} \left\{ \mathbb{E} \left[ \tau(X_{T-1}) 1(X_{T-1}^{(1)} \in G(1)) | X_{T-1} \right] | X_{T-1}^{(2)} = x_{T-1}^{(2)} \right\}
= \arg\max_{G(1) \in \mathbb{R}} \mathbb{E} \left\{ \mathbb{E} \left[ \tau(X_{T-1}^{(1)}, x_{T-1}^{(2)}) 1(X_{T-1}^{(1)} \in G(1)) | X_{T-1}^{(1)} \right] \right\}
= \arg\max_{G(1) \in \mathbb{R}} \mathbb{E} \left[ \tau(X_{T-1}^{(1)}, x_{T-1}^{(2)}) 1(X_{T-1}^{(1)} \in G(1)) \right].
\]

The optimal policy conditional on \(X_{T-1}^{(2)} = x_{T-1}^{(2)}\) is
\[
G_{*_{CFB}}^{*}(X_{T-1}^{(2)} = x_{T-1}^{(2)}) = \{x_{T-1}^{(1)} \in \mathbb{R} : \tau(x_{T-1}^{(1)}, x_{T-1}^{(2)}) \geq 0\}. \tag{B.12}
\]

Comparing (B.9) and (B.12), we see \(G_{*_{CFB}}^{*}(X_{T-1}^{(2)} = x_{T-1}^{(2)})\) is given by the intersection of \(G_{*_{FB}}^{*}\) with the line \(X_{T-1}^{(2)} = x_{T-1}^{(2)}\).

**B.2.2 \(G_{*_{FB}}^{*} \in \mathcal{G}\) is not a necessary condition**

Here we show that \(G_{*_{FB}}^{*} \in \mathcal{G}\) is sufficient but not necessary for correspondence between the optimal conditional policy and the first-best unconditional policy.

**Example B.1. Univariate covariates**

We set \(X_{t-1} = W_{t-1} \in \{0, 1\} \) and \(\mathcal{G} = \{\emptyset, \{1\}\} \subset \mathcal{P} = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}\). Suppose
\[
\tau(1) = \tau(0) = 1 > 0.
\]

For the unconditional problem (B.4), the first best policy is then
\[
G_{*_{FB}}^{*} = \{0, 1\}.
\]

Note that \(G_{*_{FB}}^{*} \notin \mathcal{G}\), so the solution to the unconditional problem is
\[
G_* = \arg\max_{G \in \mathcal{G}} \mathcal{W}_T(G) = \{1\}.
\]
Consider the conditional problem for \( W_{T-1} = 1 \).

\[
\begin{align*}
\max_{G \in \mathcal{P}} W_T(G | W_{T-1} = 1) \\
= & \max_{G \in \mathcal{P}} \mathbb{E}[\tau(W_{T-1}) 1(W_{T-1} \in G) | W_{T-1} = 1] \\
= & \max_{G \in \mathcal{P}} \tau(1) 1(1 \in G) \\
= & \tau(1) 1(1 \in G^*). \quad \text{(B.13)}
\end{align*}
\]

Therefore, the solution to the unconditional problem is also the solution to the conditional problem. (This is only the case for \( W_{T-1} = 1 \).)

**Example B.2. Two-dimensional discrete covariates**

Set \( X_{t-1} = (W_{t-1}, Z_{t-1})' \in \{0, 1\} \times \{i\}_{i=0}^{10}. \) Suppose

\[
G_{FB}^* = \{0, 1\} \times \{1, 3, 5, 7, 9\}.
\]

and

\[
\mathcal{G} = \left\{ \left\{ (w, z) : w \in \{0, 1\}, z \in \{i\}_{i=0}^{10}, \text{ and } z \in [0, a]\right\} \right\},
\]

\( a \in \mathbb{R}^+ \).

For example, if \( G \in \mathcal{G} \) and \( a = 4.5 \), \((0, 4) \in G\), so \((0, 0), (0,1), (0,2), \) and \((0,3)\) are also in \( G \).

Note that \( G_{FB}^* \notin \mathcal{G} \). Suppose the best feasible unconditional policy is

\[
G_* \equiv \arg\max_{G \in \mathcal{G}} W_T(G) = \{0, 1\} \times \{0, 1, 2, 3, 4, 5\}, \quad \text{(B.14)}
\]

We can always construct a data-generating process with a certain type of conditional treatment effect \( \tau \), such that \textbf{(B.14)} is the best feasible unconditional policy. For example, let \( \tau(x_{t-1}) = \tau(w_{t-1}, z_{t-1}) \), set \( \Pr(Z_{t-1} = i) = 1/10, \ Z_{t-1} \perp W_{t-1} \), and for any \( w \in \{0, 1\} \) assume

\[
\tau(w, z) = \begin{cases} 
2 & z \in \{1, 3, 5\} \\
0.1 & z \in \{7, 9\} \\
-1.5 & z \text{ is even.}
\end{cases}
\]

For example, a policy \( G_a \) with \( a = 5.5 \) includes \( \{w, 2\} \) and \( \{w, 4\} \), which has a welfare cost of \( 2 \times -1.5 \).
The conditional problem is
\[
\max_{G^* \in \mathcal{G}^*} \mathcal{W}_T(G^*|W_{T-1} = w)
\]
where \(\mathcal{G}^* = \{z : z \in \{i\}_{i=1}^{10}, \text{ and } z \in [0, a), a \in \mathbb{R}^+\}\). Here
\[
\arg\max_{G^* \in \mathcal{G}^*} \mathcal{W}_T(G^*|W_{T-1} = w) = \{1, 2, 3, 4, 5\},
\]
which is the intersection of \(G_*\) with \(\text{Supp}(Z_{T-1})\). We have the conclusion.

\subsection*{B.2.3 An illustration of Assumption 3.4}
Recall Assumption 3.4
\[
\arg\sup_{G \in \mathcal{G}} \mathcal{W}_T(G) \subset \arg\sup_{G \in \mathcal{G}} \mathcal{W}_T(G|x).
\]
We extend Example A.1 to show why Assumption 3.4 ensures equivalence between the unconditional and conditional problems.

\textbf{Example A.1} continued.

Figure 11. \(G_{FB}^* \notin \mathcal{G}_0\) but Assumption 3.4 is satisfied

Figure 11 shows a case where the solutions coincide even though the first best unconditional policy is not available. The red square differs from the shaded area (the first best), but the red point is inside the red square, and the blue point is inside the shaded area. The social planner will set \(W_T = 1\) in both cases. Hence, the feasibility of the first best solution is sufficient but not necessary for conditional and unconditional welfare to coincide.

Thus, there exist situations where the first best solution is not feasible, but we can still achieve correspondence. This example confirms the validity of Assumption 3.4.
B.3 Improved rates with margin assumption

In this section, we show special cases where we can improve the rate by imposing additional margin assumptions. Here we maintain the setting and method of Section 3.2. In particular, we assume that the first best policy is feasible, and we bound conditional regret with unconditional regret.

For simplicity, we assume that $X_t = \{Y_t, W_t\}$ are strictly stationary. We also use the compact notation $Y_t(W_{1:t-1}, 1) = Y_t(1)$ and $Y_t(W_{1:t-1}, 0) = Y_t(0)$. Define

$$\tau(W_{t-1}) = E((Y_t(1) - Y_t(0)|W_{t-1}).$$

If we are willing to make stronger assumptions, such that observations have very small probabilities to fall into a small region near the margin of the above optimal classification region, we can potentially achieve a better rate. For a detailed insight in the i.i.d. case, see, for example, Boucheron et al. (2005). Let

$$\bar{W}(g, W_{t-2}) = (T - 1)^{-1} \sum_{1 \leq t \leq T-1} E \{Y_t(W_{t-1}, 1)g(W_{t-1}) + Y_t(W_{t-1}, 0) [1 - g(W_{t-1})]|W_{t-2}\}.$$  

$$\bar{W}(g) = (T - 1)^{-1} \sum_{1 \leq t \leq T-1} E \{Y_t(W_{t-1}, 1)g(W_{t-1}) + Y_t(W_{t-1}, 0) [1 - g(W_{t-1})]\}.$$  

$$\widehat{W}(g) = (T - 1)^{-1} \sum_{1 \leq t \leq T-1} \{Y_t(W_{t-1}, 1)g(W_{t-1}) + Y_t(W_{t-1}, 0) [1 - g(W_{t-1})]\}.$$  

We shall assume that

**Assumption B.1.** The first best treatment rule $g^*_{FB}$ belongs to the class of candidate treatment rules $g$, i.e., $g^*_{FB} = g^*$.

**Assumption B.2.** The class of treatment rules is finite and countable with $|G| = 2 \lesssim n^V$, where $V$ is a fixed integer.

**Assumption B.3.** $Y_t(0), Y_t(1)$ belong to $[-C, C]$, where $C$ is a constant.

**Assumption B.4.** The following margin condition is assumed. Let $Pr_{t-1}$ be $Pr(\cdot|F_{t-2})$. There exists a constant $0 \leq \eta \leq C$, $0 < \alpha < \infty$ and $0 \leq u \leq \eta$, such that

$$\max_i Pr_{t-1}(|\tau(W_{t-1})| \leq u) \leq (u/\eta)^\alpha, \quad \forall 0 \leq u \leq \eta. \quad (B.15)$$

The above assumption can be implied in the discrete case by the condition that there exists a positive constant $c$ such that $\tau(0), \tau(1) > c$. Pick $\eta = c$. Then, we have $\alpha = \infty$, $Pr_{t-1}(|\tau(W_{t-1})| \leq u) = 0$, $\forall 0 \leq u \leq \eta$, and $Pr_{t-1}(\cdot) = Pr(\cdot|F_{t-2})$. Recall that there exists a positive constant $\overline{P}$ such that $\frac{1}{T-1} \sum_t Pr_{t-1}(g^*(W_{t-1}) \neq g(W_{t-1})) \geq \overline{P}$.
**Assumption B.5.** Pr(\(\cdot|W_{t-2}\)) and Pr(\(\cdot|W_{t-1}\)) are strictly positive.

**THEOREM B.1.** Under Assumption B.1-B.5 and Assumption 3.3, for a small enough positive \(\delta > 0\), we have the following bound with probability \(1 - \delta\),

\[
\bar{W}(g^*) - \bar{W}(\hat{g}) \lesssim C_{\alpha,\eta,V} \left( \frac{\sqrt{\log(2T/\delta)}}{p^2} \sqrt{T^{-1}} \right)^{2(\alpha+1)/(\alpha+2)},
\]

where \(C_{\alpha,\eta,V}\) is a constant depending only on \(\alpha\), \(\eta\), and \(V\).

We can see that when \(\alpha = 0\), the rate becomes \(\frac{1}{\sqrt{T^{-1}}p\log 2T/\delta}\), which means that it has almost no constraint. When \(\alpha \to \infty\), it is approaching \(\frac{1}{T^{-1}}p\log 2T/\delta\) (the best rate).

Then, with the arguments in Section 3.2 we can bound the conditional regret. A proof is in Appendix B.3.1.

**B.3.1 Proof of Theorem B.1**

*Proof.* Recall the definitions,

\[
\bar{W}(g|w) = T(w)^{-1} \sum_{1 \leq t \leq T-1; W_{t-1} = w} \mathbb{E} \{ Y_t(W_{t-1}, 1)g(W_{t-1}) + Y_t(W_{t-1}, 0) [1 - g(W_{t-1})] | W_{t-1} \}.
\]

and

\[
g^*(w) = \arg \max_{g(w) \to 0,1} \bar{W}(g|w).
\]

It is simple to show that,

\[
g^*(.) = \arg \max_{g(w) \to 0,1} \bar{W}(g, W_{t-2}).
\]

By a similar derivation to (36) and assumption B.5, there exists positive constants \(c_w, c'\) such that

\[
\bar{W}(g^*|w) - \bar{W}(g|w) \leq c_w(\bar{W}(g^*, W_{t-2}) - \bar{W}(g, W_{t-2})),
\]

\[
(\bar{W}(g^*, W_{t-2}) - \bar{W}(g, W_{t-2})) \leq c'(\bar{W}(g^*) - \bar{W}(g)).
\]

Therefore, it suffices to look at \(\hat{g} = \arg \max_{g(w) \to 0,1} \bar{W}(g)\). Throughout this section we take \(E_{t-1}(\cdot) = \mathbb{E}(\cdot|\mathcal{F}_{t-2})\), and \(P_{t-1}(\cdot) = \Pr(\cdot|\mathcal{F}_{t-2})\). \(\hat{g}\) is defined as in Section 2.2.

The first best rule is \(\{x: \tau(x) \geq 0\}\). Recall that there exists a positive constant \(\bar{P}\) such that \(\frac{1}{T-1} \sum_t P_{t-1}(g^*(W_{t-1}) \neq g(W_{t-1})) \geq \bar{P}\) happens with probability 1, where \(P_{t-1}\) is the
probability conditional on \( F_{t-2} \). We define the set of events \( \{ A : \min_t |\tau(W_{t-1})| > u \} \). Then,

\[
\hat{W}(g^*, W_{t-2}) - \hat{W}(g, W_{t-2}) = \sum_{1 \leq t \leq T-1} \frac{1}{T-1} E_{t-1}\{ |\tau(W_{t-1})|I(g^*(W_{t-1}) \neq g(W_{t-1})) \} \\
\geq u \frac{1}{T-1} \sum_{1 \leq t \leq T-1} P_{t-1}(g^*(W_{t-1}) \neq g(W_{t-1}) \cap A) \\
\geq u \sum_{1 \leq t \leq T-1} \frac{1}{T-1} [P_{t-1}(g^*(W_{t-1}) \neq g(W_{t-1})) - (u/\eta)^\alpha],
\]

where the last inequality is due to Assumption B.4.

Set \( u = \eta(1 + \alpha)^{-1/\alpha}P^{1/\alpha} \leq \eta \), then we have

\[
\hat{W}(g^*, W_{t-2}) - \hat{W}(g, W_{t-2}) \geq P^{(1+\alpha)/\alpha}(1 + \alpha)^{-(1+\alpha)/\alpha} \alpha \eta. \tag{B.17}
\]

Let \( \text{Var}_{t-1}(.) = E_{t-1}(.) - (E_{t-1}(.))^2 \). From this definition,

\[
\frac{1}{T-1} \sum_{t} \text{Var}_{t-1}\{ (Y_t(1) - Y_t(0))I(g^*(W_{t-1}) \neq g(W_{t-1})) \}, \\
\leq \frac{1}{T-1} \sum_{t} 2 \ E_{t-1}\{ (Y_t^2(1) + Y_t^2(0))I(g^*(W_{t-1}) \neq g(W_{t-1})) \} \\
- \frac{1}{T-1} \sum_{t} [E_{t-1}\{ (Y_t(1) - Y_t(0))I(g^*(W_{t-1}) \neq g(W_{t-1})) \}]^2 \\
\leq \frac{1}{T-1} \sum_{t} 2 \ E_{t-1}\{ (Y_t^2(1) + Y_t^2(0))I(g^*(W_{t-1}) \neq g(W_{t-1})) \} \\
\leq (4C^2)(P) \\
\leq (4C^2)(1/\eta\alpha)^{\alpha/(1+\alpha)}(1 + \alpha)(\hat{W}(g^*, W_{t-2}) - \hat{W}(g, W_{t-2}))^{\alpha/(1+\alpha)}, \tag{B.18}
\]

where the second to last line follows from the bounds imposed on \( Y_t(0), Y_t(1) \) in Assumption B.3 and Assumption B.5. The last line is due to Assumption B.4.

Next from the Freedman inequality we have, with probability \( 1 - \delta \), for a positive constant \( C > 2 \),

\[
\hat{W}(g^*, W_{t-2}) - \hat{W}(g, W_{t-2}) \\
\leq (\hat{W}(g^*) - \hat{W}(g)) + \max_g |\hat{W}(g^*, W_{t-2}) - \hat{W}(g, W_{t-2}) - (\hat{W}(g^*) - \hat{W}(g))| \\
\leq 2 \max_g \sqrt{\frac{1}{T-1} \sum_{t} C \text{Var}_{t-1}\{ (Y_t(1) - Y_t(0))I(g^*(W_{t-1}) \neq g(W_{t-1})) \}} \sqrt{\log(2M/\delta)} \\\n+ \log(2M/\delta)2C/(T-1), \\
\leq 2 \sqrt{(4C^2)(1/\eta\alpha)^{\alpha/(1+\alpha)}(1 + \alpha)} \max_g |\hat{W}(g^*, W_{t-2}) - \hat{W}(g, W_{t-2})|^{\alpha/(1+\alpha)} \sqrt{\log(2M/\delta)/\sqrt{T-1}} \\\n+ \log(2M/\delta)4C/(T-1).
\]
The second inequality is due to $\hat{\mathcal{W}}(g^*) - \hat{\mathcal{W}}(\hat{g}) \leq 0$, which follows from the definition of $\hat{g}$. The third line is based on (B.18). Solving the inequality on both sides with respect to $\hat{\mathcal{W}}(g^*, W_{t-2}) - \hat{\mathcal{W}}(\hat{g}, W_{t-2})$, we have that, with probability $1 - \delta$, there exists a positive constant $c_w$, such that

$$\hat{\mathcal{W}}(g^*) - \hat{\mathcal{W}}(\hat{g}) \leq c_w(\hat{\mathcal{W}}(g^*, W_{t-2}) - \hat{\mathcal{W}}(\hat{g}, W_{t-2})) \leq_p C_{\alpha, \eta, V}(\sqrt{\log(2/\delta)})^{2(\alpha + 1)/2(\alpha + 2)}.$$

Under the stationarity of $X_t = \{Y_t, W_t\}$, $C_{\alpha, \eta, V}$ is a constant depending only on $\alpha, \eta, C$, and $V$.

\section*{B.4 Multi-period welfare with continuous covariates}

In this section, we extend the results in the previous sections to a multi-period setup. We focus on a simple offline decision problem with deterministic treatment rules. Namely, we do not update policy after time $T$, but we allow the welfare function to include the updated realized observations. A social planner is faced with a finite multi-period welfare target and a set of continuous policy variables is available. Without loss of generality, we focus on the case of a two-period policy assignment.

The SP’s decision making procedure can be described as follows. Based on a sample collected from time 0 to $T - 1$, the SP chooses (or estimates) a decision rule that will be implemented on $T$ and $T + 1$. The rule is characterized by two sets defined on $\mathcal{X}$, and we write them as $G_{1:2} = \{G_1, G_2\} \in \mathcal{X}^2$. At the beginning of time $T$, the SP makes the decision by $W_T = 1(X_{T-1} \in G_1)$, then she/he will observe the outcome $Y_T$ as well as $X_T = (Y_T, W_T, Z_T)$ at the end of time $T$. At the beginning of time $T + 1$, the SP will make the decision by $W_{T+1} = 1(X_T \in G_2)$. Since we focus on an offline problem, $G_2$ is chosen (estimated) at the end of $T - 1$ and implemented at $T + 1$ after $X_T$ is revealed. Also, the class of subsets of $\mathcal{X}^2$ where $G_{1:2}$ is chosen from, i.e. $\mathcal{G}_{1:2}$, is assumed to be of polynomial classification and with finite VC dimension.

Recall the notations:

$$S_t(G) = Y_t(W_{t-1}, 1)1(X_{t-1} \in G) + Y_t(W_{t-1}, 0)1(X_{t-1} \notin G),$$

$$\mathcal{W}_t(G|X_{t-1}) = \mathbb{E}[S_t(G)|X_{t-1}],$$

$$\hat{\mathcal{W}}_t(G) = \frac{Y_t W_t}{e_t(X_{t-1})}1(X_{t-1} \in G) + \frac{Y_t(1 - W_t)}{1 - e_t(X_{t-1})}1(X_{t-1} \notin G).$$

Let us define

$$\hat{\mathcal{W}}_t(G) = \mathbb{E}[S_t(G)|\mathcal{F}_{t-1}].$$
We maintain Assumption 3.1, the Markovian condition of order 1. Then, the conditional two-period welfare function is defined as

\[
W_{T:T+1}(G_{1:2}|\mathcal{F}_{T-1}) = W_T(G_1|X_{T-1}) + \mathbb{E}[S_{T+1}(G_2)|\mathcal{F}_{T-1}]
= W_T(G_1|X_{T-1}) + \mathbb{E}(\mathbb{E}[S_{T+1}(G_2)|\mathcal{F}_T]|\mathcal{F}_{T-1})
= W_T(G_1|X_{T-1}) + \mathbb{E}(\mathbb{E}[S_{T+1}(G_2)|X_T]|X_{T-1})
= W_T(G_1|X_{T-1}) + \mathbb{E}(\mathbb{E}[S_{T+1}(G_2)|Y_T(W_{T-1}, 1(X_{T-1} \in G_1)), W_T = 1(X_{T-1} \in G_1)]|X_{T-1}),
\]

where the third equality follows from Assumption 3.1. It shall be noted that in this case, the welfare function is conditional on the information available up to time \(T - 1\), and we fixed the conditioning value of \(X_{T-1}\) at the observed one.

**B.4.1 Direct estimation of the conditional welfare function.**

With the defined two-period welfare function on hand, we can discuss how to estimate the welfare function. Similar to that discussed in Section 3.2 for the single-period welfare target, we can either directly estimate the conditional welfare functions or the regret bounds by the unconditional welfare function. In this subsection, we directly estimate the two-period conditional welfare function proposed. Without loss of generality, we start with the case \(X_t = (Y_t, W_t) \in \mathbb{R} \times \{0, 1\}\), i.e, \(Y_t\) is the only continuous variable in \(X_t\). It can be easily extended to the case with other continuous variables \(X_t = (Y_t, W_t, Z_t) \in \mathbb{R} \times \{0, 1\} \times \mathbb{R}^k\).

Again, we apply the abbreviation: \((\cdot|x) = (\cdot|X_{T-1} = x)\), where \(x = (y, w)\). Now, we have

\[
W_{T:T+1}(G_{1:2}|x) = W_T(G_1|x) + \mathbb{E}(W_{T+1}(G_2|X_T, W_T = 1(X_{T-1} \in G_1))|X_{T-1} = x).
\]

Let \(\mathcal{G}_{1:2}\) be the class of feasible policies, which is a sub-class of the class of all the measurable functions defined on \(\mathcal{X} \times \mathcal{X} \to \{0, 1\}^2\). Conditional on \(X_{T-1} = x\), we define

\[
G_{1:2}^* \in \arg\max_{G_{1:2} \in \mathcal{G}_{1:2}} W_{T:T+1}(G_{1:2}|x).
\]

Note \(G_{1:2}^*\) can depend on \(x\), but we suppress this dependence in the notation.
Similarly to (27), for any policy $G_1$ and $G_2$, define

$$
\hat{W}(G_{1:2}|x) = \frac{\sum_{t=1}^{T-1} \mathbf{1}(W_{t-1} = w) \ K_h(Y_{t-1}, y) \hat{W}_t(G_1)}{\sum_{t=1}^{T-1} \mathbf{1}(W_{t-1} = w) \ K_h(Y_{t-1}, y)} \\
+ \frac{\sum_{t=1}^{T-2} \ K_h(Y_{t-1}, y) \mathbf{1}(W_{t-1} = w) \mathbf{1}(W_t = 1(X_{t-1} \in G_1)) \hat{W}_{t+1}(G_2)}{\sum_{t=1}^{T-2} \ K_h(Y_{t-1}, y) \mathbf{1}(W_{t-1} = w) \mathbf{1}(W_t = 1(X_{t-1} \in G_1))},
$$

where $K_h(a, b) = \frac{1}{h} K(\frac{a-b}{h})$ with $K(\cdot)$ assumed to be a bounded kernel function with a bounded support. Since we have $\mathcal{F}_{t-1} \subset \mathcal{F}_t$, we have the $\mathbf{1}(W_t = 1(X_{t-1} \in G_1))$ can be removed from the above sum since $W_t$ is determined by $X_{t-1}$.

Let $\mathcal{G}^2$ denote the class of feasible unconditional decision sets, which is a class of subsets of $\mathcal{X} \times \mathcal{X}$, and

$$
\hat{G}_{1:2} \in \arg\max_{G_{1:2} \in \mathcal{G}^2} \hat{W}(G_{1:2}|x).
$$

Define $\mathcal{E}W_{t+1}(G_{1:2}|x) = E \{E [S_{T+1}(G_2)|Y_T(W_{t-1}, 1(X_{T-1} \in G_1)), W_T = 1(X_{T-1} \in G_1)] | X_{T-1} = x \}$ and $\mathcal{E}W_{t+1}(G_{1:2}|X_{T-1}) = E \{E [S_{T+1}(G_2)|Y_T(W_{t-1}, 1(X_{T-1} \in G_1)), W_T = 1(X_{T-1} \in G_1)] | X_{T-1} \}$. First of all, $x$ is a vector of values consisting of $w$ and $y$. To construct an MDS, we shall define an intermediate counterpart,

$$
\hat{W}_h(G_{1:2}|x) = \frac{\sum_{t=1}^{T-1} \mathbf{1}(W_{t-1} = w) \ K_h(Y_{t-1}, y)W_t(G_1|x)}{\sum_{t=1}^{T-1} \mathbf{1}(W_{t-1} = w) \ K_h(Y_{t-1}, y)} \\
+ \frac{\sum_{t=1}^{T-2} \ K_h(Y_{t-1}, y) \mathbf{1}(W_{t-1} = w) \mathbf{1}(W_t = 1(X_{t-1} \in G_1)) \mathcal{E}W_{t+1}(G_{1:2}|x)}{\sum_{t=1}^{T-2} \ K_h(Y_{t-1}, y) \mathbf{1}(W_{t-1} = w) \mathbf{1}(W_t = 1(X_{t-1} \in G_1))}.
$$

Note that

$$
E(K_h(Y_{t-1}, y) \mathbf{1}(W_{t-1} = w) \mathbf{1}(W_t = 1(X_{t-1} \in G_1)) \hat{W}_{t+1}(G_2)|\mathcal{F}_{t-1}) \\
= K_h(Y_{t-1}, y) \mathbf{1}(W_{t-1} = w) \mathbf{1}(W_t = 1(X_{t-1} \in G_1)) E(\hat{W}_{t+1}(G_2)|X_{t-1}),
$$

so $K_h(Y_{t-1}, y) \mathbf{1}(W_{t-1} = w) \mathbf{1}(W_t = 1(X_{t-1} \in G_1)) (\hat{W}_{t+1}(G_2) - \mathcal{E}W_{t+1}(G_2|X_{T-1}))$ is an MDS with respect to $\mathcal{F}_{t-1}$. Then with similar steps to those in the proof of Theorem [A.1] we can achieve the closeness between $K_h(Y_{t-1}, y) \mathbf{1}(W_{t-1} = w) \mathbf{1}(W_t = 1(X_{t-1} \in G_1)) \mathcal{E}W_{t+1}(G_2|X_{T-1})$ and $K_h(Y_{t-1}, y) \mathbf{1}(W_{t-1} = w) \mathbf{1}(W_t = 1(X_{t-1} \in G_1)) \mathcal{E}W_{t+1}(G_2|X_{T-1} = x)$ by a bias term of order $O_p(h^2)$. Thus, with similar assumptions assumed in Theorem [A.1] we can achieve the regret bound of the rate $\sqrt{(T-1)h^{-1} + (T-1)^{-1} + h^2}$.

**B.4.2 Bounding the conditional regret by the unconditional one**

Similar to Section [3.2] under the correct specification assumption, we can bound the conditional regret by unconditional regret. For the multivariate case, the unconditional welfare is
defined as

\[ W_{T:T+1}(G_{1:2}) = W_T(G_1) + \mathbb{E}\{E[S_{T+1}(G_2)|W_T = 1(X_{T-1} \in G_1)]\}. \]  
(B.20)

Note that we have

\[ \mathbb{E}\{E[S_{T+1}(G_2)|W_T = 1(X_{T-1} \in G_1)]\} = \mathbb{E}[S_{T+1}(G_2)1(W_T = 1(X_{T-1} \in G_1))]/\Pr(W_T = 1(X_{T-1} \in G_1)). \]

Note that slightly different from the one-period welfare function, in this unconditional welfare, the second part is still conditioning on \( W_T = 1(X_{T-1} \in G_1) \) since the treatment \( W_T \) is determined by the SP’s policy \( G_1 \).

The optimal unconditional policy within the class \( G_{1:2} \) is defined as

\[ G_{1:2}^* \in \argmax_{G_{1:2} \in G_{1:2}} W_{T:T+1}(G_{1:2}). \]  
(B.21)

To bound the conditional regret with the unconditional one we also need to impose the following assumption.

**Assumption B.6.** Let \( W_{T:T+1}(G_{1:2}|x) \) be the conditional welfare defined in (B.19),

\[ \arg\sup_{G_{1:2} \in G_{1:2}} W_{T:T+1}(G_{1:2}) \subset \arg\sup_{G_{1:2} \in G_{1:2}} W_{T:T+1}(G_{1:2}|x). \]

**Proposition B.1.** Under Assumption 3.1 and (B.6),

\[ G_{1:2}^* \in \arg\sup_{G_{1:2} \in G_{1:2}} W_{T:T+1}(G_{1:2}|X_{T-1}). \]

And the conditional regret is

\[ R_{T:T+1}(G_{1:2}|x) := W_{T:T+1}(G_{1:2}^*|x) - W_{T:T+1}(G_{1:2}|x). \]

Similarly, unconditional regret can be expressed as an integral of conditional regret. Thus, the unconditional welfare and regret are

\[ W_{T:T+1}(G_{1:2}) = \int W_{T:T+1}(G_{1:2}|x)dF_{X_{T-1}}(x), \]

\[ R_{T:T+1}(G_{1:2}) = W_{T:T+1}(G_{1:2}^*|x) - W_{T:T+1}(G_{1:2}) = \int R_{T:T+1}(G_{1:2}|x)dF_{X_{T-1}}(x). \]
For \( x' \in \mathcal{X} \)

\[
A(x', G_{1:2}) = \{ x : x \in \mathcal{X} \text{ and } R_{T:T+1}(G_{1:2}|x) \geq R_{T:T+1}(x', G_{1:2}) \},
\]

\[
p_{T-1}(x', G_{1:2}) = \Pr(X_{T-1} \in A(x', G_{1:2})) = \int_{x \in A(x', G_{1:2})} dF_{X_{T-1}}(x).
\]

Now, we impose the following assumption.

**Assumption B.7.** For \( x^{\text{obs}} \in \mathcal{X} \) and any \( G_{1:2} \in \mathcal{G}_{1:2} \)

\[
p_{T-1}(x^{\text{obs}}, G_{1:2}) \geq p > 0.
\]

For some positive constant \( p \).

**LEMMA B.1.** Under Assumption B.7

\[
R_{T:T+1}(G_{1:2}|x^{\text{obs}}) \leq \frac{1}{p} R_{T:T+1}(G_{1:2}).
\]

Now under Assumptions B.7 and Lemma B.1, we can specify the sample analogue

\[
\hat{W}(G_{1:2}) = \frac{1}{T} \sum_{t=1}^{T-1} \hat{W}_t(G_1) + \frac{1}{T(G_1)} \sum_{t=1}^{T-2} 1(W_t = 1(X_{t-1} \in G_1)) S_{t+1}(G_2),
\]

where \( T(G_1) \) is a random number defined as \( T(G_1) = \#\{1 \leq t \leq T - 1 : W_t = 1(X_{t-1} \in G_1)\} \). We define the estimated welfare policy,

\[
\hat{G}_{1:2} \in \arg\max_{G_{1:2} \in \mathcal{G}_{1:2}} \hat{W}(G_{1:2}).
\]

To prove the bound we can proceed with similar steps as in the proof of Theorem 3.1. Namely, we define a conditional welfare function to form an MDS as follows,

\[
\hat{W}(G_{1:2}) = \frac{1}{T} \sum_{t=1}^{T-1} W_t(G_1|\mathcal{F}_{t-2})
\]

\[
+ \frac{1}{\mathbb{E}(T(G_1))} \sum_{t=1}^{T-1} \mathbb{E}\{[1(W_t = 1(X_{t-1} \in G_1)) S_{t+1}(G_2)|W_t = 1(X_{t-1} \in G_1)] |\mathcal{F}_{t-2}\}.
\]
Then the unconditional population counterpart is as follows,

\[
\widetilde{W}(G_{1:2}) = \frac{1}{T} \sum_{t=1}^{T-1} W_t(G_1) \\
+ \frac{1}{E(T(G_1))} \sum_{t=1}^{T-1} E(1 [W_t = 1(X_{t-1} \in G_1)S_{t+1}(G_2)]) .
\] 

(B.23)

Now, similar to Assumption 3.6, we impose Assumption 3.6'. For \( G^*_{1:2} \) defined in (B.21) and any \( G_{1:2} \in G_{1:2} \), there exists some constant \( c \), such that

\[
W_{T:T+1}(G^*_1) - W_{T:T+1}(G_{1:2}) \leq c \left( \widetilde{W}(G^*_1) - \widetilde{W}(G_{1:2}) \right) .
\]

Then, the conditional regret can be bounded by

\[
W_{T:T+1}(G^*_1|x^{obs}) - W_{T:T+1}(G_{1:2}|x^{obs}) \leq \frac{1}{p} \left[ W_{T:T+1}(G^*_1) - W_{T:T+1}(G_{1:2}) \right] \\
\leq \frac{c}{p} \left[ \widetilde{W}(G^*_1) - \widetilde{W}(G_{1:2}) \right] \\
\leq \frac{c}{p} \sup_{G_{1:2} \in G_{1:2}} \left[ \widetilde{W}(G_{1:2}) - \widetilde{W}(G_{1:2}) \right] .
\]

And

\[
\widetilde{W}(G_{1:2}) - \widetilde{W}(G_{1:2}) = \left[ \widetilde{W}(G_{1:2}) - \widetilde{W}(G_{1:2}) \right] + \left[ \widetilde{W}(G_{1:2}) - \widetilde{W}(G_{1:2}) \right] \\
= I + II
\]

Similar to Section 3 we can drive upper bounds for \( I \) and \( II \).

### B.5 Accounting for Lucas critique

An interesting question is the extent to which T-EWM can deal with the Lucas critique. In this section, we attempt to answer the question by showing the link between T-EWM and the conventional SVAR approach. In VAR analysis, economists address the Lucas critique by explicitly modelling the conditional expectations of economic variables. In this way, the optimal policy implied by a structural VAR takes into account changes in the data-generating process in response to a policy decision. In most cases, the policy function and dynamics of the state variables share some common deep parameter(s). In comparison, our approach has less structure. Nevertheless, we discuss the possibility of linking our framework with the structural approach. The key to extending our method is to make the agent’s response function dependent on the policy parameter(s).
We start with a three-equation new Keynesian model. (See, e.g., Chapter 8 of Walsh (2010).) At time \( t \), let \( \pi_t \) denote inflation, \( x_t \) the output gap, and \( i_t \) the interest rate.

Phillips curve: \( \pi_t = \beta E_t \pi_{t+1} + \kappa x_t + \varepsilon_t \),
IS curve: \( x_t = E_t x_{t+1} - \sigma^{-1} (i_t - E_t \pi_{t+1}) \),
Taylor rule: \( i_t = \delta \pi_t + v_t \), \( B.24 \)

Here \( v_t \) is the treatment variable. It represents the baseline target rate and is assumed to follow an AR(1) process \( v_t = \rho v_{t-1} + \epsilon_t \). We also assume that \( \epsilon_t = \gamma \varepsilon_{t-1} + \delta_t \). Define \( d_t = \begin{pmatrix} v_t \\ \varepsilon_t \end{pmatrix} \), \( F = \begin{pmatrix} \rho & 0 \\ 0 & \gamma \end{pmatrix} \), and a vector of noises \( \eta_t = \begin{pmatrix} \epsilon_t \\ \delta_t \end{pmatrix} \). Then the process for \( \begin{pmatrix} v_t \\ \varepsilon_t \end{pmatrix} \) can be written as \( d_t = Fd_{t-1} + \eta_t \).

Define the outcome variables of the system \( B.24 \) to be \( \tilde{Y}_t = \begin{pmatrix} x_t \\ \pi_t \end{pmatrix} \). At the end of time \( T - 1 \), the goal of the SP is minimizing (or maximizing) the expectation of some function of \( \tilde{Y}_T \). For example, an objective function that balances the time \( T \) output gap and inflation, \( Y_T = |x_T|^2 + |\pi_T - \pi_0|^2 \), where the \( \pi_0 \) is the inflation target.

Appendix \( B.5.1 \) shows that the VAR-reduced form of the system \( B.24 \) can be solved to obtain

\[ \tilde{Y}_t = AC(I - F)^{-1}d_t, \quad B.25 \]

where \( A \) and \( C \) are non-random matrices defined in Appendix \( B.5.1 \). If the model \( B.24 \) is correctly specified, the solution to \( B.25 \) takes the Lucas critique into account since it solves for a deep parameter \( \rho \), which reflects both the direct effect of the treatment (\( \rho \) in \( d_t \)) and private agent’s anticipation of the policy change (\( \rho \) in \( AC(I - F)^{-1} \)).

Now we show how this is related to the T-EWM framework. The treatment \( v_t \) in \( B.24 \) corresponds to \( W_t \) in the previous sections. Since \( F \) contains \( \rho \), we can write \( AC(I - F)^{-1} = M(\rho) = \begin{pmatrix} m_{11}(\rho) & m_{12}(\rho) \\ m_{21}(\rho) & m_{22}(\rho) \end{pmatrix} \). When \( v_t \) is a binary variable (e.g., high target rate and low target rate), we have the potential outcomes: \( \tilde{Y}_t(1) = \begin{pmatrix} |m_{11}(\rho)| + |m_{12}(\rho)\varepsilon_t| \\ |m_{21}(\rho)| + |m_{22}(\rho)\varepsilon_t| \end{pmatrix} \) and \( \tilde{Y}_t(0) = \begin{pmatrix} |m_{12}(\rho)\varepsilon_t| \\ |m_{22}(\rho)\varepsilon_t| \end{pmatrix} \). Both \( \tilde{Y}_t(1) \) and \( \tilde{Y}_t(0) \) depend on \( \rho \), so we write them as \( \tilde{Y}_t(1; \rho) \) and \( \tilde{Y}_t(0; \rho) \). The dependence of \( \tilde{Y}_t(1) \) and \( \tilde{Y}_t(0) \) on \( \rho \) remains when we move back to the continuous \( v_t \) case, as \( Y_t \) is a transformation of \( \tilde{Y}_t \), the potential outcomes with respect to \( Y_t \) should also depend on \( \rho \). Denote \( f_{\sigma_T|F_{T-1}}(v; \rho) \) (resp. \( f_{W_T|F_{T-1}}(w) \)) as the conditional density of \( v_t \) (resp. \( W_T \)) on the filtration \( F_{T-1} \) evaluated at \( v, \rho \) (resp. \( w \)). Therefore, the
SP’s problem can is to choose \( \rho \) to minimize the following welfare function,

\[
E \left[ Y_T(v_T; \rho) \big| \mathcal{F}_{T-1} \right] = \int E \left( Y_T(v; \rho) \big| \mathcal{F}_{T-1} \right) f_{v_T|\mathcal{F}_{T-1}}(v; \rho) dv. \tag{B.26}
\]

This formula is similar to (9). To make the link clear, recall the welfare function (9):

\[
W_T(g|\mathcal{F}_{T-1}) = E \left[ Y_T(W_T) \big| \mathcal{F}_{T-1} \right] = E \left[ Y_T(1)g(W_{T-1}) + Y_T(0)(1 - g(W_{T-1})) \big| \mathcal{F}_{T-1} \right],
\]

where \( E(\cdot|\mathcal{F}_{T-1}) = E(\cdot|W_{T-1}) \) by Assumption 2.1. Since \( g(\cdot) \) is a deterministic policy function, by definition \( g(W_{T-1}) = \Pr(W_T = 1|W_{T-1}) = \Pr(W_T = 1|\mathcal{F}_{T-1}) \in \{0, 1\} \). (9) can then be written as

\[
E \left[ Y_T(W_T) \big| \mathcal{F}_{T-1} \right] = \sum_{w \in \{0, 1\}} E \left[ Y_T(w) \big| \mathcal{F}_{T-1} \right] \Pr(W_T = w|\mathcal{F}_{T-1}),
\]

\[
= \int_{w \in \{0, 1\}} E \left( Y_T(w) \big| \mathcal{F}_{T-1} \right) f_{W_T|\mathcal{F}_{T-1}}(w) dw. \tag{B.27}
\]

The treatment variable is \( w \) in (B.27), and \( v \) in (B.26). There are two differences between the two formulas. First, in (B.27) we have a deterministic treatment rule, while in (B.26), the SP chooses \( \rho \) to change the conditional probability density function of the treatment. They are essentially similar since a deterministic treatment rule can be regarded as a degenerate probability distribution. Second, we see that in (B.27), the policy only affects the outcome \( Y_T \) through the value of the treatment value \( w \), while in (B.26), the policy affects the outcome through both the treatment value \( v \) and the policy parameter \( \rho \). Note that in (B.26), the same parameter \( \rho \) appears in both \( Y_T(\cdot; \rho) \) and \( f_{v_T|\mathcal{F}_{T-1}}(v; \rho) \), which corresponds to the case of the SVAR model. Thus, the deep policy parameter \( \rho \) must be solved for, taking into account the change of data generating process \( Y_T(\cdot; \rho) \) in response to a change in policy.

The link between (9) and (B.26) reveals how the Lucas critique can be accounted for within the T-EWM framework. We extend the analytical form of the welfare function to allow potential outcomes to depend on policy through channels other than the implemented treatment. In (B.26), this dependence is represented by the policy parameter \( \rho \), which appears not only in the conditional policy distribution function \( f_{v_T|\mathcal{F}_{T-1}}(\cdot; \rho) \), but also in the potential outcomes \( Y_T(\cdot; \rho) \). In the same spirit, (9) can be modified to

\[
W_T(g, \rho|\mathcal{F}_{T-1}) = E[Y_T(1; \rho)g(v_{T-1}; \rho) + Y_T(0; \rho)(1 - g(v_{T-1}; \rho))|\mathcal{F}_{T-1}]. \tag{B.28}
\]

If we have a valid sample counterpart to (B.28), T-EWM allows us to obtain an optimal policy from it. Let \( \rho_t \) be a random process over time. Assume \( \rho_t \) is either observable or estimable. For simplicity, we assume that \( \rho_t \) is a discrete random process taking finitely many values. Recall from (9) that we have already \( E(\cdot|\mathcal{F}_{T-1}) = E(\cdot|W_{T-1}) \). Conditional on
\( W_{T-1} = w \), the sample counterpart to (B.28) is:

\[
\hat{W}(g(\cdot; \rho)|w) = T(w, \rho)^{-1} \sum_{t: W_{t-1} = w \text{ and } \rho_t = \rho} \left[ Y_t(v_{t-1}; 1; \rho_t) v_t e_t(W_{t-1}) g(v_{t-1}; \rho_t) + \frac{Y_t(v_{t-1}; 1; \rho_t) (1 - v_t)}{1 - e_t(W_{t-1})} (1 - g(v_{t-1}; \rho_t)) \right],
\]

where \( T(w, \rho) := \# \{ W_{t-1} = w \text{ and } \rho_t = \rho \} \). In addition, we note the following,

(i) As long as \( \rho_t \) is an observable or estimable random process, the functional forms of \( Y_t(v_{t-1}; 1; \rho_t) \) and \( g(\cdot; \rho_t) \) w.r.t. \( \rho_t \) can remain to be unspecified. The framework remains mostly model-free.

(ii) \( \rho_t \) can be estimated by the method proposed in Schorfheide (2005), assuming that monetary policy follows a nominal interest rate rule that is subject to regime shifts.

(iii) Under the assumption that private agents have knowledge of \( \rho_t \) only, other components of the functional form \( g(\cdot; \cdot) \) are not known by general society. This may be because private agents are myopic. In this framework, we can estimate \( g(\cdot; \cdot) \) using the usual EWM framework.

**Remark 4 (Continuous treatment).** Equation (B.26) is a welfare function that combines a continuous treatment dose \( \rho \) with randomized treatment outcomes. In this paper, we do not focus on continuous treatment choice, but there is a sizable literature concerning this topic, see, e.g., Hirano and Imbens (2004), Kennedy et al. (2017), Kallus and Zhou (2018), and Colangelo and Lee (2021) for discussion of the i.i.d. case. We briefly comment on how T-EWM can be applied to this problem.

Let \( W_t \) be a continuous treatment variable at time \( t \) taking values in \([0, 1]\), and \( g : \mathcal{X} \to [0, 1] \) be a continuous policy function. For simplicity, we assume \( g(\cdot) \) is a deterministic policy (treatment dose) function. For simplicity, let \( X_t \) be a univariate discrete random variable taking finitely many values. Under Assumption 2.1, the SP’s problem is to maximize

\[
\mathbb{E} \left[ Y_T(g(X_{T-1})) | X_{T-1} \right],
\]

where \( Y_t(w) \) is the potential outcome under the treatment dose \( w \in [0, 1] \). Based on the set up for the i.i.d. case in Kallus and Zhou (2018), a sample analogue of \( \mathbb{E} \left[ Y_T(g(X_{T-1})) | X_{T-1} = x \right] \) is

\[
\frac{1}{T(x)} \sum_{t: X_{t-1} = x} \frac{Y_t}{f_{W|X_t}(g(X_t))} K_h(g(X_{t-1}), W_t), \tag{B.29}
\]

where \( K_h(a, b) = \frac{1}{h} K(\frac{a-b}{h}) \) is a kernel function. The conditional density \( f_{W|X_t}(g(X_t)) \) also needs to be estimated. See Colangelo and Lee (2021) for different estimation methods. The best policy function can be obtained by maximizing (B.29) at each point of \( \mathcal{X} \). We expect
that the rate of convergence for regret will be determined by the rate of convergence of the 
nonparametric estimator.

B.5.1 On the reduced form \([B.25]\)

Here we solve the VAR reduced form of the three-equation New Keynesian model discussed in Section \([B.5]\).

We seek the solution of

\[
\begin{pmatrix}
E_t x_{t+1} \\
E_t \pi_{t+1}
\end{pmatrix} = \begin{pmatrix}
k/\beta + 1 & \delta/\sigma - 1/(\sigma \beta) \\
-\kappa/\beta & 1/\beta
\end{pmatrix} \begin{pmatrix}
x_t \\
\pi_t
\end{pmatrix} + \begin{pmatrix}
v_t/\sigma \\
\varepsilon_t/\beta
\end{pmatrix}.
\]

Denote

\[
N = \begin{pmatrix}
k/\beta + 1 & \delta/\sigma - 1/(\sigma \beta) \\
-\kappa/\beta & 1/\beta
\end{pmatrix},
\]

\[
\tilde{Y}_t = \begin{pmatrix}
x_t \\
\pi_t
\end{pmatrix}.
\]

Assuming \(N\) is invertible, we set

\[
A = N^{-1}.
\]

Let \(\Gamma_t = N^{-1} \begin{pmatrix}
v_t/\sigma \\
\varepsilon_t/\beta
\end{pmatrix} = ACd_t, \text{ with } C = \text{diag}[\sigma^{-1}, \beta^{-1}]\). In addition, define

\[
d_t = \begin{pmatrix}
v_t \\
\varepsilon_t
\end{pmatrix},
\]

and

\[
d_t = Fd_{t-1} + \eta_t, \quad \text{(B.30)}
\]

where \(F = \begin{pmatrix}
\rho & 0 \\
0 & \gamma
\end{pmatrix}\) for \((\rho, \gamma \neq 0)\). Then,

\[
\tilde{Y}_t = A E_t \tilde{Y}_{t+1} + \Gamma_t.
\]

Solving forward, we obtain

\[
\tilde{Y}_t = \lim_{L \to \infty} (A^L E_t (\tilde{Y}_{t+L}) + \sum_{l \geq 0} A^l E_t \Gamma_{t+l}).
\]
Let $\rho(A) < 1$, $\rho(F) < 1$, then

$$\lim_{L \to \infty} A^L E_t(\tilde{Y}_{t+L}) \to_{a.s.} 0,$$

and

$$E_t \Gamma_{t+1} = AF^d d_t.$$

Thus,

$$\tilde{Y}_t = AC(I - F)^{-1} d_t. \quad \text{(B.31)}$$

Combining (B.30) and (B.31), we can solve the VAR reduced form

$$\tilde{Y}_{t+1} = AC(I - F)^{-1} F(AC(I - F)^{-1})^{-1} \tilde{Y}_t + AC(I - F)^{-1} \eta_{t+1}.$$