Nonlinear Optical Vector Amplitude Equations. Polarization and Vortex Solutions

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Abstract

We investigate two kind of polarization of localized optical waves in nonlinear Kerr type media, linear and combination of linear and circular. In the first case of linear polarized components we have obtained the vector version of 3D+1 Nonlinear Schrödinger Equation (VNSE). We show that these equations admit exact vortex solutions with spin $\ell = 1$. We have determined the dispersion region and medium parameters necessary for experimental observation of these vortices in the conclusion. In the second case we represent the electric and magnetic fields as a sum of circular and linear components. We suppose also that our nonlinear media in this case admit linear magnetic polarization. This allows us to reduce the Maxwell’s equations to a set of amplitude Nonlinear Dirac Equations (NDE). We find two representations on NDE- spherical and spinor. In the spherical representation we obtain optical vortices with spin $\ell = 1$ and in the spinor representation, vortices with spin $j = 1/2$.
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1 Introduction

The scalar theory of optical vortices was based on the well-known 2D+1 paraxial Nonlinear Schrödinger equation (NSE) [1, 2]. A generalization of the scalar paraxial theory of optical vortices based on the investigation of co-called spatio-temporal evolution equations [3, 4, 5] has also been performed. The polarization and the vector character of the electric field play an important role in dynamics and the stabilization of the vortices. In [6] this problem was discussed for the first time and the existence of polarized vortices was predicted. To investigate this phenomenon more completely, it is necessary to investigated the corresponding vector amplitude equations. In this paper, we investigated two type of vector amplitude equations, VNSE for three linear orthogonal polarized component, and a Dirac representation of Nonlinear Maxwell’s equations for a combination of linear and circular polarized components.

2 Vector NSE

The vector version of 3D+1 amplitude equation of electrical field, describing the propagation of light in dispersion media with Kerr nonlinearity in a dimensionless coordinate system, moving with group velocity may be written:

\[
2i\alpha \frac{\partial \vec{A}}{\partial t} + \Delta_{\perp} \vec{A} - \beta \frac{\partial^2 \vec{A}}{\partial z^2} + \gamma |\vec{A}|^2 \vec{A} = 0,
\]

(1)

where \( \vec{A} \) is the normed amplitude of the electrical field, \( \alpha = kr_0^2/t_0v; \beta = v_i^2k^n k; \gamma = k^2r_0^2n_2|A_0|^2; \) are corresponding normed constant, \( v \) is group velocity, \( k \) is carrying wave number, \( r_0 \) and \( t_0 \) are the spatial and time dimensions of amplitude field, \( k^n \) is the dispersion parameter, \( n_2 \) is the nonlinear refractive index. We investigate the case \( \beta = -1; \gamma = 1. \) The constant \( \alpha \) has a typical value of \( \alpha \approx 10^2 - 10^3 \) in the optical region \( (\alpha \approx r_0k). \)

To solve the vector equation (1), we use the method of separation of variables. The amplitudes of three linear polarized orthogonal vector fields \( \vec{A}(x, y, z, t) \) are represented as:

\[
\vec{A}(x, y, z, t) = \sum_{\vec{j}=\vec{x},\vec{y},\vec{z}} \vec{j}j A^j(x, y, z) \exp(-i\omega t).
\]

(2)
In a Cartesian coordinate system, for solutions of the form of (2), the vector equation (1) is reduced to a scalar system of three nonlinear wave equations:

$$\alpha^2 A_l + \Delta A_l + \sum_{j=x,y,z} \left| A_j \right|^2 A_l = 0; \quad l = x, y, z. \quad (3)$$

The system (3) can be written in spherical variables:

$$\alpha^2 A_l + \Delta_r A_l + \frac{1}{r^2} \Delta_{\theta,\phi} A_l + \sum_{j=x,y,z} \left| A_j \right|^2 A_l = 0, \quad (4)$$

where \( l = x, y, z \). Next we represent the components of the field as a product of a radial and an angular part:

$$A_i(r, \theta, \phi) = R(r) Y_i(\theta, \phi); \quad i = x, y, z, \quad (5)$$

with the additional constraint on the angular parts:

$$|Y_x(\theta, \phi)|^2 + |Y_y(\theta, \phi)|^2 + |Y_z(\theta, \phi)|^2 = \text{const.} \quad (6)$$

Multiplying each of the equations (4) by the \( \frac{r^2}{R Y_i} \), and bearing in mind the constraint expressed in (6), we obtain:

$$r^2 \Delta_r R + r^2 \left( \alpha^2 + \left| R \right|^2 \right) R = \frac{\Delta_{\theta,\phi} Y_j}{Y_j} = \ell(\ell + 1), \quad (7)$$

where \( \ell \) is one number. Thus the following equations for the radial and the angular parts of the wave functions are obtained:

$$\Delta_r R + \alpha^2 R + \left| R \right|^2 R - \frac{\ell(\ell + 1)}{r^2} = 0 \quad (8)$$

$$\Delta_{\theta,\phi} Y_j + \ell(\ell + 1) = 0; \quad j = x, y, z. \quad (9)$$

As equations (8),(9) shows, the nonlinear term occurs only in the radial components of the fields, while for the angular part we have the usual linear eigenvalue problem. Going back to set (4), we see that the separation of variables for the spherical functions, which satisfied condition (6), is possible only for \( \ell = 1 \).

$$Y_x = Y_1^{-1} = \sin \theta \cos \phi; \quad Y_y = Y_1^1 = \sin \theta \sin \phi; \quad Y_z = Y_1^0 = \cos \theta. \quad (10)$$
By choosing one of these angular components for each of the field components we see that the eigenfunctions (10) are solutions to the angular part of the set of equations (4). The nonlinear radial part of equations admit exact ”de Brogile soliton” solutions (10)

$$R = \frac{\sqrt{2} e^{i\alpha r}}{r}. \quad (11)$$

The real solutions of the vector amplitude equation (4) in a fixed basis is then:

$$A_x = \Re (R(r)Y_x) = \sqrt{2} \frac{\sin(\alpha r)}{r} \sin \theta \cos \varphi \cos(\alpha t)$$

$$A_y = \Re (R(r)Y_y) = \sqrt{2} \frac{\sin(\alpha r)}{r} \sin \theta \sin \varphi \cos(\alpha t) \quad (12)$$

$$A_z = \Re (R(r)Y_z) = \sqrt{2} \frac{\sin(\alpha r)}{r} \cos \theta \cos(\alpha t)$$

3 Nonlinear Maxwell’s Equations

We consider the Maxwell’s equations for the case a source-free nonlinear medium with linear and nonlinear electric polarization and linear magnetic polarization (magnetization) [7], [8]:

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}; \nabla \times \vec{B} = \frac{1}{c} \frac{\partial \vec{D}}{\partial t} \quad (13)$$

$$\nabla \cdot \vec{D} = 0; \nabla \cdot \vec{B} = \nabla \cdot \vec{H} = 0 \quad (14)$$

$$\vec{D} = \vec{P}_l + 4\pi \vec{P}_{nl} \quad (15)$$

$$\vec{B} = \vec{H} + 4\pi \vec{M}_l, \quad (16)$$

where $\vec{E}$ and $\vec{H}$ are the electric and magnetic intensity fields, $\vec{D}$ and $\vec{B}$ are the electric and magnetic induction fields, $\vec{P}_l$, $\vec{P}_{nl}$ are the linear and nonlinear polarization of the medium respectively and $\vec{M}_l$ is the linear magnetic
polarization. We apply the slowly varying amplitude approximation to the Maxwell equations (13)-(16):

\[ \vec{E}(x, y, z, t) = \vec{A}(x, y, z, t)e^{i\omega_0 t} \]  

(17)

\[ \vec{H}(x, y, z, t) = \vec{C}(x, y, z, t)e^{-i\omega_0 t}, \]  

(18)

where \( \vec{A}, \vec{C} \) and \( \omega_0 \) are the amplitudes of the electric and magnetic fields and the optical frequency respectively. After using the Fourier representation and condition of slowly-varying amplitudes we obtained the next system of Nonlinear Maxwell amplitude Equations (NME), written in rescaled variables:

\[ \nabla \times \vec{A} = i\alpha_{2}\vec{C} - \delta \frac{\partial \vec{C}}{\partial t} - \beta_{2} \frac{\partial^{2} \vec{C}}{\partial t^{2}} \]  

(19)

\[ \nabla \times \vec{C} = i\alpha_{1}\vec{A} + \frac{\partial \vec{A}}{\partial t} - i\beta_{1} \frac{\partial^{2} \vec{A}}{\partial t^{2}} + i\gamma |\vec{A}|^{2} \vec{A} \]  

(20)

\[ \nabla \cdot \vec{A} = 0; \nabla \cdot \vec{C} = 0, \]  

(21)

where the constants are \( k_{1} = \frac{\omega\varepsilon(\omega)}{c}; \) \( k_{2} = \frac{\omega\mu(\omega)}{c}; \) \( \frac{1}{v_{i}} = \frac{\partial k_{i}}{\varepsilon_{i}}; \) \( \alpha_{i} = k_{i}^{0} r_{0}; \) \( \beta_{i} = \frac{\gamma}{r_{0}} k_{i} n_{2} |A_{0}|^{2}; \) \( \delta = \frac{\gamma}{v_{2}}. \) While the nonlinear condition \( \gamma = 1 \) can be satisfied, the constants \( \alpha_{i}; i = 1, 2 \) have typical values \( \alpha_{i} \approx 10^{2}(\alpha_{i} \approx r_{0} k_{i}). \) The constants \( \beta_{i} \) have typical values \( \beta_{1} \approx 10^{-5} - 10^{-6} << 1; \beta_{2} < \beta_{1} \) and for picosecond and sub-picosecond pulses in the transparency region of nonlinear optical media they may be neglected.

## 4 Dirac representation of NME

We will neglecting the second derivative in time terms of the eqn.(19)-(21), as \( \beta_{i} << 1. \) We solve the NME (19)-(21), by the method of separation of variables. The slowly varying amplitude vectors of the electric field \( \vec{A} \) and the magnetic field \( \vec{C} \) are represented as:

\[ \vec{A}(x, y, z, t) = \vec{F}(x, y, z, t)e^{i\Delta at} \]  

(22)

\[ \vec{C}(x, y, z, t) = \vec{G}(x, y, z, t)e^{i\Delta at}. \]  

(23)
Substituting these forms into (19)-(21), we obtain the stationary version of the NME:

\[ \nabla \times \vec{F} = -i \vartheta_2 \vec{G}; \nabla \times \vec{G} = i \vartheta_1 \vec{F} + i \gamma |\vec{F}|^2 \vec{F} \] (24)

\[ \nabla \cdot \vec{F} = 0; \nabla \cdot \vec{G} = 0, \] (25)

where \( \vartheta_1 = \alpha_1 + \Delta \alpha; \vartheta_2 = \delta \Delta \alpha - \alpha_2 > 0 \). When the electric and magnetic fields are represented as a sum of a linear and circular polarized component it is possible to reduce eqns. (24) - (25) to a system of four nonlinear equations. Substituting \[19\]:

\[ \Psi_1 = i F_z; \Psi_2 = i F_x - F_y; \Psi_3 = -G_z; \Psi_4 = -G_x - i G_y \] (26)

into the nonlinear system (24)-(25) we obtain a stationary nonlinear Dirac system (NDE):

\[ \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \Psi_4 + \frac{\partial}{\partial z} \Psi_3 = -i(\vartheta_1 + \gamma \sum_{j=1}^{2} |\Psi_j|^2) \Psi_1 \] (27)

\[ \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \Psi_3 - \frac{\partial}{\partial z} \Psi_4 = -i(\vartheta_1 + \gamma \sum_{j=1}^{2} |\Psi_j|^2) \Psi_2 \] (28)

\[ \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \Psi_2 + \frac{\partial}{\partial z} \Psi_1 = -i \vartheta_2 \Psi_3 \] (29)

\[ \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \Psi_1 - \frac{\partial}{\partial z} \Psi_2 = -i \vartheta_2 \Psi_4. \] (30)

As may be noted, the optical NDE are significantly different from the NDE in field theory. The nonlinear part appears only in the first two coupled equations of the system.

5 Vortex solutions with spin \( l = 1 \)

The NDE (27)-(30) have both spherical and spinor symmetry only in the case where the nonlinear part does not manifest an angular dependence on the radial variable \( \sum_{j=1}^{2} |\Psi_j|^2 = F(r) \). This type of solution may be found
using with the following technique: using Pauli matrices, we write the NDE system (27)-(30) as:

\[
(\vec{\sigma} \cdot \vec{P}) \phi = -i(\vartheta_1 + \gamma \sum_{j=1}^{2} |\eta_j|^2) \eta
\]

(31)

\[
(\vec{\sigma} \cdot \vec{P}) \eta = -i\vartheta_2 \phi,
\]

(32)

where \(\vec{\sigma}\) are the Pauli matrices, \(\vec{P} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\) are the differential operator and \(\eta\) and \(\phi\) are the corresponding spinors. After substituting eqn. (32) into eqn. (31) we obtain:

\[
(\vec{\sigma} \cdot \vec{P})(\vec{\sigma} \cdot \vec{P}) \eta = -\vartheta_2(\vartheta_1 + \gamma \sum_{j=1}^{2} |\eta_j|^2) \eta
\]

(33)

When there is no external electric or magnetic field, the operator on the left-hand side of eqn. (33) is simply the Laplacian operator \(\triangle\). From (33) and we obtain:

\[
\vartheta_1 \vartheta_2 \tilde{\eta} + \vartheta_2 \gamma \sum_{j=1}^{2} |\tilde{\eta}_j|^2 \eta + \triangle \eta = 0
\]

(34)

The scalar variant of these equations has been investigated in many papers but exact localized solutions have not been found. In the case \(l = 1\) we look for spinors of the form:

\[
\eta_1 = \tilde{\eta}(r) \cos \theta; \eta_2 = \tilde{\eta}(r) \sin \theta \exp i\phi.
\]

(35)

After substituting eqn. (33) into equations (34) the following equation describing the radial dependence is obtained:

\[
\vartheta_1 \vartheta_2 \tilde{\eta} + \vartheta_2 \gamma \sum_{j=1}^{2} |\tilde{\eta}_j|^2 \eta + \frac{\partial^2 \tilde{\eta}}{\partial r^2} + \frac{2}{r} \frac{\partial \tilde{\eta}}{\partial r} - \frac{2}{r^2} \tilde{\eta} = 0.
\]

(36)

The angular parts are the standard spherical harmonics with \(l = 1\). This system has exact vortex de Broglie soliton solutions of the form:

\[
\tilde{\eta}(r) = \frac{\sqrt{2}}{i} \exp(i \sqrt{\vartheta_1 \vartheta_2} r)
\]

(37)

if \(\theta_2 \gamma = 1\).
6 Hamiltonian representation of the NDE: Orbital momentum, first integrals and vortex solutions with spin $j = 1/2$

It is not difficult to show that for the NDE system of eqns. (27)-(30) a Hamiltonian of the form:

$$H = (\vec{\sigma} \cdot \vec{P}) + \sum_{j=1}^{2} |\Psi_j|^2$$  \hspace{1cm} (38)

may be written. Using this, NDE may be rewritten in the form:

$$H\Psi = \varepsilon \Psi,$$  \hspace{1cm} (39)

where $\varepsilon = (-iv_1, -iv_1, -iv_2, -iv_2)$ is the energy operator. Here we investigate the case where the nonlinear part, as a sum of square module of spinors, depends only on the radial component $\sum_{j=1}^{2} |\Psi_j|^2 = F(r)$. We also introduce here the well known orbital momentum operator $\vec{L}$, spin momentum $\vec{S}$, and the full momentum $\vec{J}$, as well as:

$$\vec{L} = \vec{r} \times \vec{P}; \vec{J} = \vec{L} + \frac{1}{2} \vec{S}.$$  \hspace{1cm} (40)

It straightforward to show that the Hamiltonian of eqn. (38) commutes with the operators $\vec{J}^2$ and $J_z$ (the z- projections must be x or y). Using these symmetries and the condition that the nonlinearity be centrosymmetric, we can solve the NDE equations (39) by a separation of variables technique. To find vortex solution of the NDE (39) with spin $j = 1/2$ we use the spinor representation of these equations. We look for solutions of the form:

$$\Psi_1 = a(r)\Omega_{jlm}; \Psi_2 = a(r)\Omega_{jlm}$$  \hspace{1cm} (41)

$$\Psi_3 = ib(r)\Omega_{j'lm}; \Psi_4 = ib(r)\Omega_{j'lm}$$  \hspace{1cm} (42)

where $\Omega_{jlm}$ is the spherical spinor, $l + l' = 1$, and $a(r)$ and $b(r)$ are arbitrary radial functions. Using the symmetries of (39) and the fact, that the nonlinear parts depend only on $r$, we separate variables and obtain the following system of equations for the radial part:
\[
\frac{\partial a(r)}{\partial r} + \frac{1 + \sigma}{r} a(r) = -\vartheta_2 b(r) \tag{43}
\]

\[
\frac{\partial b(r)}{\partial r} + \frac{1 - \sigma}{r} b(r) = \vartheta_1 a(r) + \gamma |a(r)|^2 a(r), \tag{44}
\]

where \(\sigma = l(l + 1) - j(j + 1) - 1/4\). We find localized solutions of these equations only for the angular component corresponding to \(l = 1\) and \(j = 1/2\). In this case the system (43)-(44) becomes:

\[
\frac{\partial a(r)}{\partial r} + \frac{2}{r} a(r) = -\vartheta_2 b(r) \tag{45}
\]

\[
\frac{\partial b(r)}{\partial r} = \vartheta_1 a(r) + \gamma |a(r)|^2 a(r). \tag{46}
\]

As was shown above, this system has exact radial solutions of the form:

\[
a(r) = \frac{\sqrt{2}}{i} \exp \frac{i \sqrt{\vartheta_1 \vartheta_2} r}{r} \tag{47}
\]

\[
b(r) = -\frac{\sqrt{2}}{i \vartheta_2} \left( \frac{\exp i \sqrt{\vartheta_1 \vartheta_2} r}{r^2} + \frac{i \sqrt{\vartheta_1 \vartheta_2} \exp i \sqrt{\vartheta_1 \vartheta_2} r}{r} \right). \tag{48}
\]

The vortex solutions of the spinor representation have four components and this type of solutions is found only for the case where the spin \(j = 1/2\).

7 Conclusion

In this paper we investigated two cases of vector amplitude equations of electromagnetic field in nonlinear Kerr media. In the first case, for three orthogonal linear polarized component of electrical field we obtained exact vortex solutions of VNSE with spin \(l = 1\). The numerical analysis shows, that these solutions are stable at distances comparable to those where localized waves with the same amplitudes of the scalar 3D+1 NSE self-focussed rapidly. The conditions \(\beta = -1\) is satisfied only near to Langmuir frequency in plasma or near to some of the electron resonances. In the second case we have derived a set of Nonlinear Maxwell amplitude Equations (NME) for nonlinear optical media with dispersion of the magnetic susceptibility. We show that for the case of linear and circularly polarized components of the electric and magnetic
fields, the NME is reduced to the Nonlinear Dirac system of equations (NDE). Using the method of separation of variables, we have obtained exact vortex solutions for this case. We have investigated the NDE in two representations: spherical and spinor. In the spherical presentation we obtain optical vortices with spin $l = 1$ and in the spinor representation vortices with spin $j = 1/2$. These vortices are Lorenz invariant, and periodically pulsating with the group velocity of solutions of the NME.

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