We solve closed string theory in all regular homogeneous plane-wave backgrounds with homogeneous NS three-form field strength and a dilaton. The parameters of the model are constant symmetric and anti-symmetric matrices $k_{ij}$ and $f_{ij}$ associated with the metric, and a constant anti-symmetric matrix $h_{ij}$ associated with the NS field strength. In the light-cone gauge the rotation parameters $f_{ij}$ have a natural interpretation as a constant magnetic field. This is a generalisation of the standard Landau problem with oscillator energies now being non-trivial functions of the parameters $f_{ij}$ and $k_{ij}$. We develop a general procedure for solving linear but non-diagonal equations for string coordinates, and determine the corresponding oscillator frequencies, the light-cone Hamiltonian and level matching condition. We investigate the resulting string spectrum in detail in the four-dimensional case and compare the results with previously studied examples. Throughout we will find that the presence of the rotation parameter $f_{ij}$ can lead to certain unusual and unexpected features of the string spectrum like new massless states at non-zero string levels, stabilisation of otherwise unstable (tachyonic) modes, and discrete but not positive definite string oscillator spectra.
B. The commutation relations of the $\xi$'s

B.1 Imposing the canonical commutation relations

B.2 Determining the $C_J$ using the Vandermonde and Lagrange polynomials

C. Geometric Aspects of the Spectrum

C.1 The spectral hypersurface

C.2 Berry’s Connection

References
1 Introduction

A renewal of interest in plane wave backgrounds in string theory [1, 2, 3, 4] prompts one to study systematically various examples of solvable models of strings moving in such “null” spaces, extending early work on this subject [5, 6, 7]. Among the motivations (see [8] for a detailed discussion) is a desire to learn more about patterns of string spectra in non-trivial curved backgrounds.

Here we will investigate a seemingly simple but non-trivial class of plane wave models that escaped detailed attention until recently [9]. It is based on the plane wave metric

\[ ds^2 = 2dudv + k_{ij}x^i x^j du^2 + 2f_{ij}x^i dx^j du + dx^i dx^i , \]

(1.1)
supported by a null 2-form potential, \( B_{ui} = h_{ij}x_j \), and dilaton.

We shall mostly consider the model for which the matrices \( k_{ij} \) and \( f_{ij} \) are constant. In that case the metric (1.1) is regular and homogeneous [9]. If these matrices do not commute, \( f_{ij} \) cannot be eliminated by a coordinate transformation while keeping the \( du^2 \)-coefficient \( u \)-independent. The constant matrix \( f_{ij} \) can be interpreted as a “rotation” [9] or “magnetic field” [10] parameter. Indeed, the quantum mechanics of a relativistic particle propagating in the four-dimensional plane wave with \( k_{ij} = k_i \delta_{ij} \) and \( f_{ij} = f \epsilon_{ij} \) in the light-cone gauge is described by the Schrödinger equation for a two-dimensional non-relativistic oscillator coupled to a constant magnetic field of strength \( f \) with frequencies proportional to \( k_i \). This is a generalisation of the standard Landau problem (for which \( k_i = 0 \)). The solution of the more general problem has been given in [11] and references therein. For positive \( k_i \) the light-cone energy is unbounded from below, reflecting the possibility for a particle to escape to infinity in the corresponding direction. Switching on a non-zero \( f_{ij} \) tends to stabilise the motion, trapping the trajectories near the center of \( x \)-space.

We find the classical solutions of a closed string propagating in (1.1) in the presence of constant null NS 3-form field strength. For this we shall develop a formalism, based on what we will call the frequency base ansatz, which allows for the diagonalisation of the quantum mechanical problem. Then we quantise the theory in the light-cone gauge, compute the canonical commutation relations and determine the light-cone string spectrum. We shall show that the frequencies depend on the parameters \( k_{ij}, f_{ij} \) and \( h_{ij} \) in a more intricate way than in some previously discussed examples [2, 3, 4]. In particular, for certain values of the parameters there is a possibility that some states at non-zero string levels become massless. We shall compute the string Hamiltonian and investigate the level-matching condition.

Throughout we will find that the presence of the rotation parameter \( f_{ij} \) leads to certain...
unusual and unexpected features of the string spectrum. For instance, it may stabilise otherwise unstable (tachyonic) modes, the string oscillator spectrum may be discrete but not positive definite, and the level-matching condition for low-lying string modes can differ from the usual form of the level-matching condition by signs.

In some special cases, there is an analogy with string oscillator spectra found in the case of the “standard” \((f_{ij} = 0)\) plane waves supported by a non-constant null \(B_{MN}\) background in \([12]\). It is possible that the metric \([1,1]\) supplemented by appropriate null background fluxes is the Penrose limit of some complicated ten-dimensional geometries which are “deformations” of the \(AdS_5 \times S^5\) background. In that case the string spectrum we shall find below may be relevant for comparison with its gauge-theory counterpart along the lines of \([3]\) (cf. \([14]\)).

The paper is organised as follows. In section two, we review some aspects of string backgrounds with pp-wave or plane wave metrics. Then we turn to the special case of string backgrounds with homogeneous plane wave metrics supported by other homogeneous fluxes like for example a homogeneous three-form NS-NS field strength. We discuss the solution of the Klein-Gordon equation for the corresponding four-dimensional non-singular homogeneous plane wave metric, explaining its equivalence (in the light-cone frame) to the “2-dimensional oscillator in magnetic field” problem. We investigate the properties of the spectrum of the corresponding Hamiltonian as a function of the parameters of the plane wave metric.

In section three, we give the most general classical solution of a closed string in a non-singular homogeneous plane wave background using the so called frequency base ansatz. We illustrate the procedure in the case of four-dimensional plane waves and then go on to determine the canonical commutation relations, the light-cone Hamiltonian and the closed string level matching conditions in general.

In section four, we apply the above results to study some aspects of string theory in the case of four-dimensional non-singular homogeneous plane wave with non-vanishing homogeneous NS-NS three-form field strength (and/or a dilaton). In particular, we find the string spectrum explicitly for various special cases, including the anti-Mach model \([9]\). We also discuss the explicit form of the level matching condition and compute the zero-point energy part of the string Hamiltonian.

In appendix A, we summarise some facts about matrices with null vectors and their minors which are relevant for solving string theory in a homogeneous plane wave. In appendix B, we determine the closed string canonical commutation relations. In appendix C, we present a geometric description of a degeneracy of the spectrum of some string models in terms singularities of a hypersurface and an associated fibration. We also compute the Berry connection of some quantum mechanical models that arise in
the context of strings in homogeneous plane waves and we find that it vanishes.

2 Homogeneous plane wave backgrounds as a subclass of pp-wave string-theory solutions

2.1 pp-wave and plane wave backgrounds

We shall begin with a review of a class of string backgrounds with metrics admitting a covariantly constant null Killing vector, and explain how the homogeneous plane waves discussed in \cite{8,9} appear as special cases. We shall also describe the models that we shall focus on later in this paper. Let us start with the most general null Brinkmann metric with flat transverse space (for a review see, e.g., \cite{15})\footnote{If the transverse metric is taken to be $g_{ij}(u,x)dx^i dx^j$, then $K$ and $A_i$ can be eliminated by transforming $x^i$ and $v$; this is no longer the case if we set $g_{ij} = \delta_{ij}$ (see, e.g., \cite{10}).}

$$ds^2 = 2dudv + K(u,x)du^2 + 2A_i(u,x)dx^i du + dx^i dx^j ,$$  \hspace{1cm} (2.1)

where $i = 1, \ldots, d$. This is the metric of a plane-fronted wave ($dx^i dx^j$) with parallel rays ($\partial_v$ is covariantly constant and null), or a pp-wave for short.

String backgrounds (conformal sigma-models) with this metric may be constructed by switching on a combination of “null” p-form field strengths and the dilaton. For example, one can consider string backgrounds with metric (2.1), and non-vanishing NS-NS 2-form gauge potential and dilaton

$$B_{iu} = B_i(u,x) , \hspace{1cm} \phi = \phi(u) ,$$  \hspace{1cm} (2.2)

respectively. The one-loop sigma model conformal invariance conditions or, equivalently, the common-sector supergravity field equations are

$$R_{MN} - \frac{1}{4} H^R_{MLH} H^L_{RN} + 2\nabla_M \partial_N \phi = 0 ,$$

$$\nabla_L (e^{-2\phi} H^L_{MN}) = 0 ,$$  \hspace{1cm} (2.3)

where $M, N, L = 0, \ldots, 9$ and $H$ is the NS-NS field strength, $H = dB$.

For the pp-wave ansatz above, we find

$$-\frac{1}{2} \partial^2_i K + \partial_u \partial^j A_i + \frac{1}{4} F_{ij} F^{ij} - \frac{1}{4} H_{ij} H^{ij} + 2 \partial^2_u \phi = 0 ,$$

$$\partial^j F_{ij} = 0 , \hspace{1cm} \partial^i H_{ij} = 0 ,$$  \hspace{1cm} (2.4)

where $F_{ij} = \partial_i A_j - \partial_j A_i$ and $H_{ij} = \partial_i B_j - \partial_j B_i$. The general solution of these equations does not define an exact conformal sigma model, i.e. there will be non-trivial
\( \alpha' \)-corrections to the background fields. However, all \( \alpha' \)-corrections may be absent in some special cases (like that of \( A_i = 0, B_i = 0 \) in [5, 6]).

One such special choice is
\[
A_i = \pm B_i .
\] (2.5)

It leads to a chiral null model [17] which is an exact string solution to all orders in \( \alpha' \) provided the above 1-loop conditions,
\[
- \frac{1}{2} \partial_i^2 K + \partial_u \partial^i A_i + 2 \partial_u^2 \phi = 0 , \quad \partial^i F_{ij} = 0 ,
\] (2.6)
are satisfied. Models with \( K = 0, A_i = -\frac{1}{2} F_{ij} x^j, F_{ij} = \text{const}, \phi = \text{const} \) can be interpreted as WZW models for non-semisimple groups [18].

All higher-loop corrections vanish also in another special (now “non-chiral”) case [19, 20, 16]
\[
A_i = -\frac{1}{2} F_{ij}(u) x^j , \quad B_i = -\frac{1}{2} H_{ij}(u) x^j ,
\] (2.7)
where the string sigma model conformal invariance conditions (2.3) reduce to
\[
- \frac{1}{2} \partial_i^2 K + \frac{1}{4} F_{ij} F_{ij} - \frac{1}{4} H_{ij} H_{ij} + 2 \partial_u^2 \phi = 0 .
\] (2.8)
A simple special solution of the latter is
\[
K = K_{ij}(u) x^i x^j , \quad K_{ii} = \frac{1}{4} F_{ij} F_{ij} - \frac{1}{4} H_{ij} H_{ij} + 2 \partial_u^2 \phi .
\] (2.9)
For this choice of \( K \), the Brinkmann metric (2.1) is a plane wave,
\[
\text{ds}^2 = 2dudv + K_{ij}(u) x^i x^j du^2 + F_{ij}(u) x^i dx^j du + dx^i dx^i .
\] (2.10)
This metric is parametrised by a symmetric matrix \( K_{ij}(u) \) and antisymmetric matrix \( F_{ij}(u) \). However, there is a freedom of coordinate transformations that in general reduces the amount of independent data to one symmetric matrix function of \( u \).

Indeed, we can make an orthogonal rotation
\[
x^i = M^i_j(u) y^j , \quad \delta_{ij} M^i_k(u) M^j_l(u) = \delta_{kl}
\] (2.11)
that “absorbs” \( F_{ij} \) into \( K_{ij} \), giving the plane wave metric in Brinkmann coordinates,
\[
\text{ds}^2 = 2dudv + K_{ij}(u) y^i y^j du^2 + dy^i dy^j ,
\] (2.12)
where (a prime denotes a \( u \)-derivative)
\[
M^i_j = \frac{1}{2} F_{ik} M^k_j , \quad \bar{K}_{ij} = M^i_j M^j_k (K_{kl} - \frac{1}{4} F_{kn} F_{ln}) .
\] (2.13)
Note that $\tilde{K}_{i\bar{i}} = K_{i\bar{i}} - \frac{1}{4} F_{ik} F_{i\bar{k}}$ which is in agreement with the expression for the Ricci tensor component $R_{uu}$ in (2.8), (2.9). If $K_{ij}$ were constant, $\tilde{K}_{ij}$ would be $u$-dependent even for a constant $F_{ij}$. The rotation (2.11) also affects the expression for $B_i$ in (2.7).

We can also transform the metric (2.10) or (2.12) to Rosen coordinates $(u, V, X^a)$

$$x^i = L_i^a(u)X^a, \quad v = V + \frac{1}{2}s_{ab}(u)X^aX^b, \quad g_{ab} = L_a^i(u)L_b^i(u).$$

(2.14)

Starting with (2.12), we find (2.15) provided

$$s'_{ab} + \tilde{K}_{ij}L_i^aL_j^b + L_i^aL_j^b = 0, \quad s_{ab} + L_i^aL_i^b = 0, \quad \text{i.e.} \quad L_i^a'' = \tilde{K}_{ij}L_j^i. \quad (2.16)$$

For other aspects of the relation between Brinkmann and Rosen coordinates for plane waves, see, e.g., the discussion in [9].

### 2.2 Homogeneous Plane Wave Backgrounds

Amongst the various conformal plane wave backgrounds (2.10) with $F_{ij} \neq 0$, there are two special classes associated with the homogeneous plane wave metrics of [9], namely

$$I: \quad K_{ij}(u) = k_{ij} = \text{const}, \quad F_{ij}(u) = 2f_{ij} = \text{const}, \quad H_{ij}(u) = 2h_{ij} = \text{const}, \quad (2.17)$$

$$II: \quad K_{ij}(u) = \frac{1}{u^2}k_{ij}, \quad F_{ij}(u) = \frac{2}{u}f_{ij}, \quad H_{ij}(u) = \frac{2}{u}h_{ij}. \quad (2.18)$$

The homogeneity of the plane wave metrics is not always respected by the full string background. The NS-NS fluxes are not homogeneous unless the matrices $f$ and $h$ commute, $[f, h] = 0$. The presence of a non-constant dilaton also breaks homogeneity. The metric of the class I background is smooth while that of the class II background is singular, i.e. these two classes can be distinguished as non-singular and singular homogeneous plane wave backgrounds, respectively.

According to (2.13), the matrix $f \neq 0$ can be associated with a coordinate rotation. For that reason, we shall sometimes call $f$ a “rotation” matrix; its other natural interpretation (in the light-cone gauge) is as an effective magnetic field parameter [10].

We can rewrite the metrics for the two backgrounds (2.17) and (2.18) in Brinkmann coordinates (2.12) as

$$ds^2 = 2dudv + (e^{-fu}\tilde{K}e^{fu})_{ij}y^i\bar{y}^jdu^2 + dy^i\bar{dy}^i, \quad (2.19)$$

$$ds^2 = 2dudv + u^{-2}(e^{-f\ln u}\tilde{K}e^{f\ln u})_{ij}y^i\bar{y}^jdu^2 + dy^i\bar{dy}^i, \quad (2.20)$$
where \( \tilde{k}_{ij} = k_{ij} - f_{ik} f_{jk} \), and \( y_i = (e^{-f u})_{ij} x_j \), \( y_i = (e^{-f \ln u})_{ij} x_j \), respectively. Among all the metrics of the form (2.12), these backgrounds are singled out by the existence of an isometry with a non-trivial component in the \( u \)-direction. In the “stationary” coordinates (2.10), the corresponding Killing vectors are simply \( \partial_u \) and \( u \partial_u - v \partial_v \) respectively.

It is clear that if \( f \) commutes with \( k \), \([f, k] = 0\), the \( u \)-dependent rotation factors cancel out, i.e. we can simply absorb the dependence on the matrix \( f \) into the shift \( k_{ij} \to \tilde{k}_{ij} \).

This transformation leaves the two-form NS gauge potential \( B \) invariant. The models of class I with \( f = 0 \) have been widely investigated in the literature (see [8] for references and [25] for a more recent investigation of some particular models with \( f = 0 \)).

For generic matrices \( k, f, h \), there is still some redundancy in the parametrisation of the backgrounds. In both cases I and II, it is always possible to diagonalise \( k \) with an \( O(d) \) orthogonal transformation of the coordinates \( x \). In addition, under the scaling transformation \( u \to \ell u \), \( v \to \ell^{-1} v \), the parameters of the models of class I transform as \( k \to \ell^2 k \), \( f \to \ell f \) and \( h \to \ell h \). Thus the eigenvalues of \( k \) should be specified up to a positive constant. The remaining \( Z_2 \) symmetry (\( \ell = \pm 1 \)) allows one to determine \( f, h \) up to a sign.

The scaling transformation \( u \to \ell u \), \( v \to \ell^{-1} v \) is a symmetry of the models of class II. In fact, the presence of this additional isometry is responsible for the homogeneity property of the corresponding metrics. These models are parametrised by the eigenvalues of \( k \) and the two skew-symmetric matrices \( f, h \). Models with \( f = 0 \) have been investigated in [8] and models with \( f \neq 0 \) have been considered in [9].

Our aim here is to investigate string theory on the backgrounds (2.17) and (2.18) with \( f_{ij} \neq 0 \) and \([k, f] \neq 0\). In both cases, the sigma model conformal invariance condition (2.8) can be solved in terms of the dilaton to give

\[
I : \quad \phi = \phi_0 + c u - \frac{1}{2} \mu u^2 \tag{2.21}
\]

\[
II : \quad \phi = \phi_0 + c u + \mu \ln u \tag{2.22}
\]

where

\[
\mu = -\frac{1}{2} (k^i_i - f_{ij} f^{ij} + h_{ij} h^{ij}) \tag{2.23}
\]

and \( \phi_0 \) and \( c \) are arbitrary integration constants.

An important condition on our models is the requirement that the string coupling \( e^\phi \) be small for all values of \( u \). In the case I, we find that this leads to the requirement

\[
\mu \geq 0 . \tag{2.24}
\]

For the models of class II with \( \mu \neq 0 \), small string coupling requires that \( c < 0 \) and \( \mu > 0 \) for \( u \geq 0 \). A more detailed discussion of this issue can be found in [8] for a similar model with \( f = 0 \).
The condition (2.24) is satisfied automatically if $f_{ij} \neq 0$ but $k_{ij} = 0$ and $h_{ij} = 0$: then $\mu = \frac{1}{2}f_{ij}f^{ij}$. Since in this case $[k, f] = 0$, we can perform a coordinate transformation to construct a model with $\tilde{k} = f^2$ and $f = 0$.\footnote{Note that only a subclass of models with $k \neq 0$ and $f = 0$ can be constructed in this way because not every symmetric matrix is the square of a skew-symmetric one. Conversely, if $k = f^2$ for some skew-symmetric matrix $f$ and $f = 0$, then after a coordinate transformation such a model is related to one with $k = 0$ and $f = f$.}

### 2.3 Four-dimensional homogeneous plane waves with rotation

The simplest examples of homogeneous plane wave backgrounds with rotation for which $k \neq 0$, $f \neq 0$ and $[k, f] \neq 0$ are four-dimensional ones, where $k$ and $f$ are $2 \times 2$ matrices. The most general 4-d model of that type (2.7), (2.10) with parameter functions in (2.17), (2.18) has

$$k_{ij} = \text{diag}(k_1, k_2), \quad f_{ij} = f\epsilon_{ij}, \quad h_{ij} = h\epsilon_{ij}. \quad (2.25)$$

If $k_1 = k_2$, then $[k, f] = 0$ and the rotation matrix $f$ can be set to zero by a coordinate transformation. For that reason we shall focus on the non-trivial case where

$$k_1 \neq k_2.$$ 

For models with constant dilaton, we have to restrict the parameters as

$$k_1 + k_2 = 2f^2 - 2h^2. \quad (2.26)$$

We may instead set the 2-form NS-NS gauge potential to zero ($h_{ij} = 0$) and introduce a dilaton, i.e. consider the following models

$$k_{ij} = \text{diag}(k_1, k_2), \quad f_{ij} = f\epsilon_{ij}, \quad h_{ij} = 0, \quad (2.27)$$

with the dilaton given by (2.21) or (2.22) with

$$\mu = -\frac{1}{2}(k_1 + k_2) + f^2. \quad (2.28)$$

### 2.4 Models with constant dilaton

A subclass of models for which the dilaton $\phi$ is constant is found for $\mu = 0$ and $c = 0$. For $h = 0$, the condition of $\mu = 0$ implies (in both cases I and II) that

$$k^i_i = f_{ij}f^{ij}. \quad (2.29)$$
A solution of (2.20) with constant parameters is

\[ k_{ij} = \text{diag}(2f^2, 0), \quad f_{ij} = f \epsilon_{ij}. \]  

(2.30)

The corresponding metric is the anti-Mach metric (see [9] and section 4.2 below). Here \( k = 2f^2 \) can be set equal to 1 by a rescaling of \( u \) and \( v \).

A ten-dimensional solution can be constructed by setting

\[ f = f(\epsilon \oplus \epsilon \oplus \epsilon \oplus \epsilon), \quad \tilde{k} = k + f^2 = 1 \oplus (-1). \]

This background is characterised by an additional \( U(2) \times U(2) \times \mathbb{Z}_2 \) symmetry acting on the coordinates \( y \) in (2.19). To see this we can write the transverse space as \( \mathbb{R}^8 = \mathbb{C}^2 \oplus \mathbb{C}^2 \) using the obvious complex structure defined by \( f \). Each \( U(2) \) acts as a holomorphic isometry on each \( \mathbb{C}^2 \). The non-trivial element of \( \mathbb{Z}_2 \) exchanges the two \( \mathbb{C}^2 \) subspaces.

Let us also mention that it is straightforward to construct similar plane wave models containing in addition null R-R background field strengths. The corresponding string models can be solved using the GS formalism in the light-cone gauge, as was done in [2] and [4]. The effect of the R-R parameters on the metric (for constant dilaton) can be taken into account by replacing \( H_{ij}^2 \) in (2.9) (and thus \( h_{ij}^2 \) in (2.26)) by the sum of the squares of all null field strengths.

2.5 **World-sheet fermionic couplings in the light-cone gauge**

To generalise the discussion to the superstring case, we note that the world-sheet (NSR) fermions decouple from the function \( K_{ij} \) of the plane-wave metric (2.10) in the light-cone gauge \( u = p_i \tau, \psi^u = 0 \) (2\( \alpha' = 1 \)). In the presence of the NS-NS 2-form field, the coupling to the functions \( F_{ij} \) and \( H_{ij} \) is (for details see, e.g., [4])

\[ L_F = \psi_L^j [\delta_{ij} \partial_- - \frac{1}{4} p_v (F_{ij} - H_{ij})] \psi_L^i + \psi_R^j [\delta_{ij} \partial_+ - \frac{1}{4} p_v (F_{ij} + H_{ij})] \psi_R^i. \]  

(2.31)

The dependence on \( F_{ij}(u) \) can be eliminated by a local Lorentz transformation (rotation of fermions) as in (2.13). Indeed, we can apply the coordinate transformation (2.11) to eliminate \( F_{ij} \) by transforming \( K_{ij} \) into \( \tilde{K}_{ij} \) (2.13) which will not couple to the fermions. Thus an equivalent fermionic Lagrangian is

\[ L_F = \psi_L^j (\delta_{ij} \partial_- + \frac{1}{4} p_v H_{ij}) \psi_L^i + \psi_R^j (\delta_{ij} \partial_+ - \frac{1}{4} p_v H_{ij}) \psi_R^i. \]  

(2.32)

To conclude, the fermions do not couple to the parameters of the metric (2.10) in the light-cone gauge. This is consistent with the fact that the fermions do not contribute to (1-loop) conformal invariance conditions in this case.
We can also apply another local Lorentz rotation to get another equivalent action where the left-moving fermions are free

\[ L_F = \psi^i_L \partial_- \psi^i_L + \psi^i_R (\delta_{ij} \partial_+ - \frac{1}{2} p_v H_{ij}) \psi^j_R . \]  

(2.33)

Similar light-cone gauge actions are found in the Green-Schwarz approach. For example, the analog of (2.32) is

\[ L_F = i S_L (\partial_- + \frac{1}{16} p_v H_{ij} \gamma^{ij}) S_L + i S_R (\partial_+ - \frac{1}{16} p_v H_{ij} \gamma^{ij}) S_R . \]  

(2.34)

### 2.6 Solution of the four-dimensional Klein-Gordon equation

In section 3 we will explain a general procedure for quantising particles and strings in regular homogeneous plane wave backgrounds. The case of the relativistic particle in four dimensions (two transverse dimensions) can also be solved directly, and for comparison purposes we present this solution here.

The quantisation of a relativistic particle propagating in the homogeneous non-singular plane wave background

\[ ds^2 = 2dudv + k_{ij} x^i x^j du^2 + 2 f_{ij} x^i dx^j du + dx^i dx^i , \]  

(2.35)

leads to the Klein-Gordon equation. In general, the massive Klein-Gordon equation \((\nabla^2 - M^2) \Psi = 0\) corresponding to the metric (2.1) has the form

\[ [2 \partial_\alpha \partial_\nu - K \partial_\nu^2 + (\partial_i - A_i \partial_\nu)^2 - M^2] \Psi(u,v,x) = 0 . \]  

(2.36)

When \( K \) and \( A_i \) do not depend on \( u \) (as is the case for the homogeneous plane wave background) it is natural to perform a Fourier transform in the \( u,v \) coordinates,

\[ \Psi(u,v,x) = \int dp_u dp_v e^{ip_u u + ip_v v} \psi(p_u, p_v; x) . \]

Then the equation (2.36) becomes the Schrödinger-type equation for \( \psi(p_u, p_v, x) \)

\[ H \psi = \mathcal{E} \psi , \quad \mathcal{E} \equiv -\frac{1}{2} M^2 - p_u p_v , \]  

(2.37)

where

\[ H = -\frac{1}{2} [(\partial_i - i p_v A_i)^2 + p_v^2 K(x)] \]  

(2.38)

may be interpreted as a non-relativistic Hamiltonian of a particle with charge \( p_v \) coupled to a magnetic field \( A_i \) and moving in a potential \( V = -p_v^2 K(x) \). The light-cone Hamiltonian is

\[ H^{(0)} = -p_u = \frac{M^2}{2 p_v} + p_v^{-1} H . \]  

(2.39)
In the case of the homogeneous plane wave in four dimensions (2.27), we have

\[ A_i = -f \delta_{ij} x^j, \quad K = k_1 x_1^2 + k_2 x_2^2. \]

Setting \( x = x^1 \) and \( y = x^2 \), we get the Hamiltonian operator

\[ H = -\frac{1}{2} [(\partial_x + ip_v f y)^2 + (\partial_y - ip_v f x)^2 + p_v^2 (k_1 x^2 + k_2 y^2)]. \quad (2.40) \]

This is recognised as a Hamiltonian describing the dynamics of a non-relativistic charged 2-dimensional oscillator (with masses \( m_i = 1 \), frequencies \( \omega_i^2 = -p_v^2 k_i \) and charge \( p_v \)) coupled to a constant magnetic field of strength \( f \). The solution of this problem is well known (see, e.g., [11] and references therein). Such a Hamiltonian can be related to a standard decoupled two-dimensional harmonic oscillator Hamiltonian by a unitary transformation,

\[ H = U(\theta_1, \theta_2)(H_1 + H_2)U^+(\theta_1, \theta_2), \quad (2.41) \]

where

\[ U(\theta_1, \theta_2) = e^{-i\theta_1 x y} e^{-i\theta_2 p_x p_y}, \quad p_x = -i\partial_x, \quad p_y = -i\partial_y, \quad (2.42) \]

and

\[ H_i = \frac{1}{2m_i} p_i^2 + \frac{m_i}{2} \tilde{\omega}_i^2 x_i^2. \quad (2.43) \]

The parameters \( \theta_1, \theta_2 \) are

\[ \theta_1 = -p_v \frac{(k_1 - k_2) + D}{4f}, \quad \theta_2 = -\frac{2f}{p_v D}, \quad (2.44) \]

where

\[ D^2 = (k_1 - k_2)^2 - 8f^2 (k_1 + k_2) + 16f^4 = (4f^2 - k_1 - k_2)^2 - 4k_1 k_2. \quad (2.45) \]

The frequencies and masses of the free harmonic oscillators are \( \tilde{\omega}_i = p_v \omega_i \), with

\[ \omega_{1,2}^2 = \frac{1}{2} (4f^2 - k_1 - k_2 \mp D), \quad (2.46) \]

and

\[ m_{1,2} = \frac{2D}{D + k_1 - k_2 \mp 4f^2}. \quad (2.47) \]

We are assuming that \( D \) is real and positive and that \( k_1 > k_2 \); otherwise the labels of the two frequencies should be interchanged (for \( f = 0 \) we have \( \omega_i^2 = -k_i \)). The above expressions apply when \( \theta_i \) and \( m_i \) are finite. When the \( m_i \) are finite, they can be eliminated by a canonical transformation generated by \( x_i \to x_i |m_i|^{1/2} \), so the spectrum only depends on the sign of \( m_i \). The situation when \( m_i = 0 \) or \( \infty \) requires special consideration.
We will rederive these frequencies in a different way in the next section. We will also see that in the frequency basis adopted there (see section 3.2) the particle or string Hamiltonian is automatically a sum of decoupled harmonic oscillators (section 3.6) without the need to perform explicitly the counterpart of the above unitary transformation. Various other features of this spectrum and its string mode counterpart will be discussed in section 4.

3 Strings in non-singular homogeneous plane-wave backgrounds

3.1 The classical string mode equations

Our aim is to solve the classical equations of string theory in non-singular (case I in \( (2.17) \)) \( 2 + d \) dimensional homogeneous plane-wave backgrounds with metric \( (2.35) \),

\[
\text{ds}^2 = 2\text{dudv} + k_{ij}x^i x^j \text{d}u^2 + 2f_{ij}x^i \text{d}x^j\text{d}u + \text{d}x^i \text{d}x^j , \tag{3.1}
\]

where \( k_{ij} \) and \( f_{ij} \) are constant, and \( B \)-field as in \( (2.2), (2.7) \) and \( (2.17) \), i.e.

\[
B_{iu} = -h_{ij}x^j . \tag{3.2}
\]

The dilaton background does not enter the classical string equations of motion but modifies the quantum stress tensor, so the discussion below will apply to cases of both constant and non-constant dilaton.

We shall denote the string embedding coordinates associated with the spacetime coordinates \((u, v, x^i)\) as \( X^M = (U, V, X^i) \). Choosing the orthogonal gauge for the world-sheet metric, the standard sigma model Lagrangian

\[
L = \frac{1}{4\pi \alpha'}(g + B)_{MN}\partial_+ X^M \partial_- X^N \tag{3.3}
\]

(we will set \( \alpha' = \frac{1}{2} \) in the following) leads to the following equations for \( U \) and \( X^i \):

\[
\partial_+ \partial_- U = 0 , \tag{3.4}
\]

\[- \partial_+ \partial_- X_i + (f + h)_{ij} \partial_- U \partial_+ X^j + (f - h)_{ij} \partial_+ U \partial_- X^j + k_{ij} X^j \partial_+ U \partial_- U = 0 , \tag{3.5}
\]

where \( \sigma^\pm = \tau \pm \sigma, \partial_\pm = \partial_\tau \pm \partial_\sigma \). The equation for \( U \) is solved by \( U = U(\sigma^+) + U(\sigma^-) \); after gauge fixing the world-sheet conformal transformations by the light-cone gauge this becomes

\[
U = p_+ \sigma^+ + p_- \sigma^- . \tag{3.6}
\]
In the case of non-compact $U$-direction (the case we shall consider below), the condition of periodicity of $U$ in $\sigma$ implies that $p_+ = p_-$, i.e. $U = 2p_+ \tau$. In most of what follows, we shall set
\[ p_+ = p_- = \frac{1}{2} p_v , \] (3.7)
except in some equations below which for generality we write down as if $p_+$ and $p_-$ were independent to allow for compact $U$ in which case $p_+$ and $p_-$ are quantised.

Substituting (3.6) into the equation (3.5) for $X^i$, we find
\[ - \partial_+ \partial_- X^i + 2p_- (f + h)_{ij} \partial_+ X^j + 2p_+ (f - h)_{ij} \partial_- X^j + 4p_+ p_- k_{ij} X^j = 0 \tag{3.8} \]
while $V$ can be expressed in terms of $X^i$ using the equations associated with the variation of the 2-d metric.

To solve (3.8), we set
\[ X^i = \sum_{n=-\infty}^{+\infty} X^i_n(\tau) e^{2i n \sigma} , \quad X^i_n = (X^i_{-n})^* , \quad (3.9) \]
where $0 < \sigma \leq \pi$ and $\alpha' = \frac{1}{2}$, which leads to
\[ -\ddot{X}^i_n + 2p_- (h_{ij} + f_{ij}) \dot{X}^j_n + 2p_+ (-h_{ij} + f_{ij}) \dot{X}^j_n + 4(p_+ p_- k_{ij} - n^2 \delta_{ij}) X^j_n \\
+ 4nip_- (h_{ij} + f_{ij}) X^j_n - 4nip_+ (-h_{ij} + f_{ij}) X^j_n = 0 . \tag{3.10} \]
Setting $p_+ = p_- = \frac{1}{2} p_v$, this equation becomes
\[ -\ddot{X}^i_n + 2p_v f_{ij} \dot{X}^j_n + (p_v^2 k_{ij} - 4n^2 \delta_{ij}) X^j_n + 4inp_v h_{ij} X^j_n = 0 . \tag{3.11} \]
For $n = 0$ this equation is that of a relativistic particle and, as expected, the dependence on $h_{ij}$ drops out.

For the metrics of class II in (2.18), the analogue of this equation is
\[ -\ddot{X}^i_n + \frac{2p_v}{\tau} f_{ij} \dot{X}^j_n + \left( \frac{p_v^2 k_{ij} + p_v f_{ij}}{\tau^2} - 4n^2 \delta_{ij} \right) X^j_n + \frac{4inp_v}{\tau} h_{ij} X^j_n = 0 . \tag{3.12} \]
For $n = 0$ and changing $\tau$ to $t = \log \tau$ this is identical to the equation of the relativistic particle in the metrics of class I. For general ($n \neq 0$) string modes, the class II metrics have two easily analysable limits:

- When $\tau \to 0$ for any $n$, the same change of variables to logarithmic time $t$ reduces these equations to the same form as the particle case of class I, containing, however, an interesting correction in the constant term proportional to $f_{ij}$. This case can be studied using the same methods that we develop below, and then this limit becomes the basis for an expansion of the solution around $\tau = 0$.  

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• When $\tau \to \infty$ all $\tau$ dependent terms go to zero and we find just a string in flat space.

In the remainder of this paper we will concentrate exclusively on the non-singular homogeneous plane wave solutions of class I.

As explained above, without loss of generality we can assume that $k_{ij}$ is diagonal, $k_{ij} = k_i \delta_{ij}$. Also, from now on, to simplify the notation, we will set

$$p_v = 1 .$$

The dependence of any equation on $p_v$ can be reinstated by scaling $k_{ij} \to p_v^2 k_{ij}, f_{ij} \to p_v f_{ij}, h_{ij} \to p_v h_{ij}$.

### 3.2 The Frequency Base ansatz

The general method to solve a system of $d$ coupled second order equations like (3.11) is to rewrite it as a set of $2d$ first-order equations. This first-order $(2d \times 2d)$ matrix differential equation with constant coefficients is then solved by exponentiating this matrix. In practice, however, this is rather unwieldy.

Fortunately, for generic values of the parameters appearing in (3.11) one can use a much simpler procedure. Namely, to solve these equations, one makes the ansatz

$$X^i_n = \sum_{J=1}^{2d} \xi^{(n)}_J a^{(n)}_{iJ} e^{i\omega^{(n)}_J \tau} ,$$

with the frequencies $\omega^{(n)}_J$ and their eigen-directions $a^{(n)}_{iJ}$ to be determined.\(^3\)

Plugging this into the differential equation (3.11), one obtains the matrix equation

$$M_{ik}(\omega^{(n)}_J, n) a^{(n)}_{kJ} = 0 ,$$

where

$$M_{ik}(\omega, n) = (\omega^2 + k_i - 4n^2)\delta_{ik} + 2i\omega f_{ik} + 4inh_{ik} .$$

A necessary condition for finding a solution to this matrix equation is that

$$\det M(\omega, n) = 0 .$$

\(^3\)At this stage the $\xi$’s are redundant as they could be absorbed into the $a_{iJ}$. They have been introduced here for later convenience, as they will be promoted to operators upon quantisation. Their commutation relations will be determined from the canonical commutation relations of the $X^i$ and their conjugate momenta.
This equation has $2d$ roots $\omega = \omega_j^{(n)}$, $J = 1, \ldots, 2d$, which are the frequencies entering the ansatz (3.13). If all the roots are found to be distinct, or if equal roots are associated with linearly independent null eigenvectors, then the frequency base ansatz is justified because the expansion (3.13) then involves all the $2d$ linearly independent solutions of the equation (3.11). The degenerate case when there are multiple roots associated with the same eigenvector requires separate considerations. Unless stated otherwise, in the following we will always assume that all the roots are distinct.

We will see below that in the generic case this frequency base ansatz is very efficient, in particular, because in this basis the canonical commutation relations and the string Hamiltonian will turn out to be automatically diagonal (a sum of decoupled harmonic oscillators).

We also want to mention that, in addition to the general $(2d \times 2d)$ matrix method and the frequency base ansatz, there is also a third, in some sense, intermediate, possibility: to make the ansatz

$$X_n(\tau) = e^{\tau \lambda_n} X_n(0)$$

(3.17)

with $\lambda_n$ being some $(d \times d)$ matrix. This leads to a quadratic matrix equation for $\lambda_n$, and works well in certain simple cases. However, this method is neither as general as the first one, nor as efficient as the second, and we will not make use of it in the following.

### 3.3 Warm-up: four-dimensional plane waves

Before explaining in general how to obtain the solutions for the string modes, we illustrate the procedure in the case of four-dimensional homogeneous plane waves. Various other properties of these four-dimensional plane waves will be discussed in detail in section 4.

For two transverse dimensions, the matrix $M$ takes the form

$$M = \begin{pmatrix} \omega^2 + k_1 - 4n^2 & 2ifw + 4inh \\ -2ifw - 4inh & \omega^2 + k_2 - 4n^2 \end{pmatrix}$$

(3.18)

where we set $f_{ij} = f \epsilon_{ij}$, $h_{ij} = h \epsilon_{ij}$ in (3.15). Thus the equation for the roots (frequencies) is

$$\omega^4 + (k_1 + k_2 - 4f^2 - 8n^2)\omega^2 - 16nhw + (k_1 - 4n^2)(k_2 - 4n^2) - 16n^2h^2 = 0 \ .$$

(3.19)

We see that this equation is invariant under $(\omega, n) \rightarrow (-\omega, -n)$. In particular, the roots $\omega_j^{(0)}$ for $n = 0$ come in pairs $\pm \omega_j$, and for each frequency $\omega_j^{(n)}$ of the $n$'th string mode there is a corresponding frequency $\omega_j^{(-n)} = -\omega_j^{(n)}$ of the $(-n)'$th mode.
For \( n = 0 \), this becomes a quadratic equation for \( \omega^2 \),

\[
\omega^4 + (k_1 + k_2 - 4f^2)\omega^2 + k_1k_2 = 0 ,
\]

(3.20)

with solutions\(^4\)

\[
\omega_{1,2}^2 = \frac{1}{2}(4f^2 - k_1 - k_2) \pm \frac{1}{2}\sqrt{(4f^2 - k_1 - k_2)^2 - 4k_1k_2} .
\]

(3.21)

Note that these are precisely the frequencies (2.46) obtained in section 2.6 from the explicit diagonalisation of the particle Hamiltonian. This will also be verified later when we compute the Hamiltonian operator using the classical solutions of the system.

We obtain four distinct frequencies unless either the discriminant \( D \) (given by the same expression as in (2.45))

\[
D^2 = (4f^2 - k_1 - k_2)^2 - 4k_1k_2 ,
\]

(3.22)

vanishes, or one has \( k_1k_2 = 0 \).

Similarly, for the string modes \((n \neq 0)\), and when either \( h = 0 \) or \( f = 0 \), one obtains a quadratic equation for \( \omega_J^2 \). In the former case the solutions can be obtained from the solutions for \( n = 0 \) by shifting \( k_i \to k_i - 4n^2 \), so that the frequencies are

\[
(\omega_J^{(n)})^2 = \frac{1}{2}(4f^2 - k_1 - k_2 + 8n^2) \mp \frac{1}{2}\sqrt{(4f^2 - k_1 - k_2 + 8n^2)^2 - 4(k_1 - 4n^2)(k_2 - 4n^2)} .
\]

(3.23)

In the latter case, we find that

\[
(\omega_J^{(n)})^2 = \frac{1}{2}(-k_1 - k_2 + 8n^2) \mp \frac{1}{2}\sqrt{(k_1 - k_2)^2 + 64n^2h^2} .
\]

(3.24)

The squares of the frequencies are real for physical values of the parameters.

In the general case with \( n \) \( h \) \( f \neq 0 \), the equation for the frequencies is a quartic (with vanishing cubic term), whose roots can of course be given in a closed (albeit complicated and unenlightning) form.

Having found the frequencies \( \omega_J \), we now need to determine the corresponding null eigenvectors \( \mathbf{a}_{i,j} \). These can readily be constructed explicitly on a case-by-case basis, but there is also a general method which we outline now (and which, suitably interpreted, immediately generalises to the higher-dimensional case). Namely, we observe that for any \((2 \times 2)\) matrix \( \mathbf{M} \) with vanishing determinant,

\[
M_{11}M_{22} - M_{21}M_{12} = 0 ,
\]

(3.25)

\(^4\)The frequencies have been labelled in such a way that for \( f = 0 \) one finds the standard result \( \omega_i^2 = -k_i \).
a possible choice for the corresponding null eigenvector $v_k$ with $M_{ik}v_k = 0$ is
\[
(v_1, v_2) = (-M_{22}, M_{21}).
\] (3.26)
Indeed, one then has
\[
M_{1k}v_k = -\det M = 0
\]
\[
M_{2k}v_k = -M_{21}M_{22} + M_{22}M_{21} = 0.
\] (3.27)
Therefore, we can choose the null eigenvectors $a_{k,J}$ to be, e.g.,
\[
(a_{1,J}, a_{2,J}) = (-M_{22}(\omega_J), M_{21}(\omega_J)).
\] (3.28)
This is an acceptable choice of null eigenvector unless $M_{21}(\omega_J) \equiv 0$, in which case one can use $M_{11}(\omega_J)$ instead. If $M(\omega_J)$ is identically zero, obviously the construction of null eigenvectors in terms of matrix elements of $M$ fails. But the construction of (two linearly independent) null eigendirections is trivial in this case.

We have now completely determined the frequency expansion (3.13) of the string modes, up to an overall scale encoded in the $\xi$'s parametrising the classical solutions.

### 3.4 The General Classical Solution

We now discuss the construction of the solutions (3.13) in the general case. First of all, an important property of $M$ in (3.15) is that
\[
M^T(\omega, n) = M(-\omega, -n).
\] (3.29)
Thus $M(\omega, n)$ and $M(-\omega, -n)$ have the same determinant and hence the same roots. This means that for $n = 0$ the frequencies come in $\pm$ pairs,
\[
n = 0: \quad \{\omega_J\} = \{\pm \omega_j, \quad j = 1, \ldots, d\}, \quad (3.30)
\]
as we had already seen in the four-dimensional case. It is then convenient to rewrite the expansion of the zero-mode as
\[
X_0^i = \sum_{j=1}^{d} \left[ e^{+i\omega_j \tau} + e^{-i\omega_j \tau} \right].
\] (3.31)
For $n \neq 0$, on the other hand, we have
\[
n \neq 0: \quad \{\omega_J^{(-n)}\} = \{-\omega_J^{(n)}\}, \quad (3.32)
\]
so that the ±n-modes are paired. We will thus label the frequencies \( \omega_{-n}^J \) in such a way that
\[
\omega_{-n}^J = -\omega_{n}^J , \quad J = 1, \ldots, 2d .
\]
Next, we observe that the expression for the null eigenvectors we have found for \( d = 2 \) can be written as
\[
v_k = (-1)^k m_{1k} ,
\]
where \( m_{ik} \) is the minor of \( M_{ik} \), i.e. the determinant of the matrix obtained from \( M \) by removing the \( i \)'th row and \( k \)'th column. In the \((2 \times 2)\) case, one obviously has \( m_{11} = M_{22} \) and \( m_{12} = M_{21} \).

As we show in Appendix A.1, this way of writing the null eigenvector generalises to arbitrary dimensions. Thus the eigen-directions \( a_{iJ}^{(n)} \) can for instance be chosen to be
\[
a_{iJ}^{(n)} = (-1)^i m_{1i}(\omega_{n}^J) ,
\]
where \( m_{ij}(\omega_{n}^J) \) means the minor \( m_{ij} \) of \( M(\omega,n) \) evaluated for \( \omega = \omega_{n}^J \). As in the case \( d = 2 \) discussed above, this is an acceptable choice of null eigenvector unless all the \( m_{1i} \) evaluated at \( \omega_{n}^J \) are zero. In that case, instead of using the \( m_{1i} \), one could of course also use the \( m_{ji} \) for any other value of \( j \). One can also use different \( j \) for different frequencies, and generically all these choices are equivalent.

Because the minor of a transpose is the transpose of the minor, for \( n = 0 \) one can choose
\[
a_{iJ}^{+} = (-1)^i m_{1i}(\omega_j) \\
\]
\[
a_{iJ}^{-} = (-1)^i m_{1i}(-\omega_j) = (-1)^i m_{i1}(\omega_j) .
\]
Similarly, for \( n \neq 0 \), one can choose the eigenvectors for \(-n\) to be given by the transposes of the minors for \(+n\).

Thus the solutions for the string modes are, explicitly,
\[
X_0^i = (-1)^i \sum_{J=1}^{d} \left[ \xi_{J}^{+} m_{1i}(\omega_j) e^{i\omega_j \tau} + \xi_{J}^{-} m_{i1}(\omega_j) e^{-i\omega_j \tau} \right] ,
\]
for \( n = 0 \), and
\[
X_n^i = (-1)^i \sum_{J=1}^{2d} \xi_{J}^{(n)} m_{1i}(\omega_{J}^{(n)}) e^{i\omega_{J}^{(n)} \tau}
\]
for \( n \neq 0 \).
3.5 The canonical commutation relations

We now promote the $\xi$’s to operators and impose the canonical commutation relations (CCRs)

$$\left[ X^i(\sigma, \tau), X^k(\sigma', \tau) \right] = \left[ \Pi^i(\sigma, \tau), \Pi^k(\sigma', \tau) \right] = 0$$

$$\left[ X^i(\sigma, \tau), \Pi^k(\sigma', \tau) \right] = i\delta^{ik}\delta(\sigma - \sigma') \quad , \quad (3.39)$$

where

$$\Pi_i = \frac{1}{\pi}(\dot{X}_i - f_{ij}X^j) \quad (3.40)$$

are the canonical momenta that follow from the string sigma model Lagrangian (3.3) for the metric (3.1). Note that the momenta $\Pi_i$ do not depend on the null NS three-form field strength (2.2) in the light-cone gauge. It is clear that in order to get time-independent CCRs, the only non-zero commutators can be between $\xi_j^+$ and $\xi_j^-$ (for $n = 0$) and $\xi_j^{(n)}$ and $\xi_j^{(-n)}$ for $n \neq 0$. Let us call these

$$C_j := \left[ \xi_j^-, \xi_j^+ \right]$$

$$C_j^{(n)} := \left[ \xi_j^{(-n)}, \xi_j^{(n)} \right] \quad . \quad (3.41)$$

Note that obviously $C_j^{(-n)} = -C_j^{(n)}$. We will show in Appendix B that imposing the canonical commutation relations determines the $C_j$ and $C_j^{(n)}$ uniquely to be

$$C_j = \frac{1}{2m_{11}(\omega_j) \omega_j \prod_{k \neq j}(\omega_j^2 - \omega_k^2)}$$

$$C_j^{(n)} = \frac{1}{m_{11}(\omega_j^{(n)}) \prod_{K \neq J}(\omega_J^{(n)} - \omega_K^{(n)})} \quad . \quad (3.42)$$

We now need to relate the oscillators $\xi_j^\pm$ with

$$\left[ \xi_j^-, \xi_k^+ \right] = C_j \delta_{jk} \quad (3.43)$$

to canonically normalised operators $a_j^\pm$ with

$$\left[ a_j^-, a_k^+ \right] = \delta_{jk} \quad . \quad (3.44)$$

Let us assume that the frequencies $\omega_j$ are real (and chosen to be positive). Then the $C_j$ are also real. Reality of the string modes requires that

$$(\xi_j^+)^\dagger = \xi_j^- \quad . \quad (3.45)$$

If $C_j$ is positive, then one can define

$$a_j^\pm = \xi_j^\pm / C_j^{1/2} \quad \text{for} \quad C_j > 0 \quad , \quad (3.46)$$
with \((a_j^\pm)^\dagger = a_j^-\). If \(C_j\) is negative, the situation is somewhat different. One could try to define \(a_j^\pm = \pm \xi_j^\pm/|C_j|^{1/2}\). Then (3.44) is satisfied, but the \(a_j^\pm\) are not the hermitian conjugates of each other. Rather, one should note that \(C_j < 0\) really means that the creation operator \(a_j^+\) is associated with the negative frequency \((-\omega_j)\) and vice versa. In other words, one should define hermitian conjugate operators by

\[
a_j^{\pm} = \xi_j^{\mp}/|C_j|^{1/2} \quad \text{for } C_j < 0 .
\]

With the choice (3.46, 3.47), the bilinear \(\xi_j^+\xi_j^- + \xi_j^-\xi_j^+\) is related to the number operator

\[
N_j = a_j^+ a_j^-
\]

(with spectrum \(n = 0, 1, 2, \ldots\) on the Fock space defined by the vacuum annihilated by \(a_j^-\)) by

\[
\frac{1}{2}(\xi_j^+\xi_j^- + \xi_j^-\xi_j^+) = |C_j|(N_j + \frac{1}{2}) .
\]

In the same way one can relate the \(\xi_j^{(n)}\) to canonically normalised oscillators \(a_j^{(n)}\) with, say,

\[
[a_j^{(n)}, a_K^{(m)}] = -\text{sign}(n)\delta_{n+m,0}\delta_{JK} ,
\]

so that one has

\[
\frac{1}{2}(\xi_j^{(n)}(\xi_j^{(n)})^\dagger + (\xi_j^{(n)})^\dagger\xi_j^{(n)}) = |C_j^{(n)}|(N_j^{(n)} + \frac{1}{2})
\]

with

\[
N_j^{(n)} = a_j^{(n)} a_j^{(-n)} .
\]

Hence, putting everything together one now has a completely explicit string mode expansion. All the data entering the mode expansion (3.13) in this frequency basis for the solutions are determined by the matrix \(M\) in (3.15):

- the frequencies are the roots of \(M\)
- the eigen-directions of the frequencies are constructed from the minors of \(M\)
- the commutators of the \(\xi\)'s are determined by the frequencies.

### 3.6 The String Hamiltonian and Its Oscillator Frequencies

The light-cone Hamiltonian is\(^5\)

\[
H = \frac{1}{2\pi} \int_0^\pi \! d\sigma \left[ \delta_{ij} (\dot{X}^i \dot{X}^j + X^i X^j - k_i X^i X^j) - 2h_{ij} X^i X^j \right] .
\]

\(^5\)Note that written in terms of the velocities \(\dot{X}^k\) rather than the momenta \(\Pi^k\), the Hamiltonian does not explicitly depend on \(f_{ij}\). Recall also that we have set \(\alpha' = \frac{1}{2}\), \(p_v = 1\), and \(H = -p_u\).
Here primes indicate \( \sigma \)-derivatives. Inserting the mode expansion \((3.37)-(3.35)\) for the \( X^i \), the Hamiltonian becomes a bilinear expression in the oscillators \( \xi^\pm_j \) and \( \xi^{(\pm \eta)}_j \). Since the light-cone Hamiltonian for a smooth homogeneous plane wave in the stationary coordinates \((2.35)\) is time-independent, it follows that the only non-zero terms are those proportional to the diagonal combinations \( \xi^+_j \xi^-_j \) or \( \xi^{(n)}_j \xi^{(-n)}_j \).

The vanishing of the other contributions can be expressed as certain identities about roots and minors. While these are guaranteed to hold, by virtue of the time independence of the light-cone Hamiltonian, in low dimensional cases one can also verify them explicitly.\(^6\)

Because of the pairing of the \( \pm n \) modes, this Hamiltonian can be written as a sum

\[
H = \sum_{n=0}^{\infty} H^{(n)} .
\]  

(3.54)

The zero-mode part or particle Hamiltonian is

\[
H^{(0)} = \frac{1}{2} \sum_{i=1}^{d} \left[ (\dot{X}_0^i)^2 - k_i (X_0^i)^2 \right] .
\]  

(3.55)

Inserting the mode expansion \((3.37)\) and retaining only the terms proportional to \( \xi^+_j \xi^-_j \), one finds

\[
H^{(0)} = \frac{1}{2} \sum_{j} \sum_{i} (\omega_j^2 - k_i) m_{i1}(\omega_j) m_{i1}(\omega_j) \left( \xi^+_j \xi^-_j + \xi^-_j \xi^+_j \right) .
\]  

(3.56)

As the oscillators corresponding to different frequencies \( \omega_j \) commute, \( H^{(0)} \) is a sum of operators without cross interactions. We see that in the frequency basis the Hamiltonian is automatically diagonal, and there is no need to perform explicitly the unitary transformation to diagonal form, as was done in section 2.6.

When the frequencies \( \omega_j \) are real, the \( \xi^\pm_j \) are related to standard oscillators in the manner described in the previous section and the above Hamiltonian is a sum of harmonic oscillators. On the other hand, when the frequencies are, say, imaginary, then the reality conditions require not \((3.45)\) but rather (assuming that the eigendirections are real) \( (\xi^+_j)^\dagger = \xi^+_j \). In this case the Hamiltonian \((3.56)\) also has the form of a harmonic oscillator one, but now with imaginary frequencies, as can be seen by identifying the \( \xi^\pm_j \) up to constants with hermitian operators \( p_j \pm q_j \). Similar considerations apply in the case of complex frequencies. From now on, unless stated otherwise, we will assume that all frequencies are real.

\(^6\)It might be desirable to have a general proof of these identities which does not invoke reference to time-independence of the string Hamiltonian. A proof of one of the required identities can be found in Appendix A.2.
Using (3.49) and the identity (A.5), one finds that the Hamiltonian takes the form

\[ H^{(0)} = \sum_{j=1}^{d} \text{sign}(C_j) \Omega_j \left( \mathcal{N}_j + \frac{1}{2} \right) , \]  

(3.57)

where

\[ \Omega_j = C_j m_{11}(\omega_j) \sum_i (\omega_j^2 - k_i) m_{ii}(\omega_j) . \]  

(3.58)

Here we have split \(|C_j| = \text{sign}(C_j) C_j\), to keep track of the sign of the spectrum – as we had seen, a negative \(C_j\) corresponds to an exchange of positive and negative frequencies.

Using the expression for \(C_j\) obtained in (B.19), one finds

\[ \Omega_j = \frac{\sum_i (\omega_j^2 - k_i) m_{ii}(\omega_j)}{2 \omega_j \prod_{k \neq j} (\omega_j^2 - \omega_k^2)} . \]  

(3.59)

We now claim that, despite appearance, the result for \(\Omega_j\) is actually very simple, namely that (up to \(\text{sign}(C_j)\)) the oscillator frequencies of the quantum string Hamiltonian are equal to the frequencies of the classical string modes,

\[ \Omega_j = \omega_j , \]  

(3.60)

This relation is easy to establish for four-dimensional plane waves, i.e. \(d = 2\). We thus have two frequencies \(\omega_1\) and \(\omega_2\), and e.g. \(\Omega_1\) is

\[ \Omega_1 = \frac{(\omega_1^2 - k_1)(\omega_1^2 + k_2) + (\omega_1^2 - k_2)(\omega_1^2 + k_1)}{2 \omega_1 (\omega_1^2 - \omega_2^2)} = \frac{(\omega_1^4 - k_1 k_2)}{\omega_1 (\omega_1^2 - \omega_2^2)} = \omega_1 , \]  

(3.61)

where

\[ k_1 k_2 = \omega_1^2 \omega_2^2 \]  

(3.62)

follows from the fact that \(k_1 k_2\) is the constant term in the quadratic equation (3.20) for \(\omega^2\). Combined with the explicit expression (3.21) for the classical frequencies, this reproduces the result of section 2.6 based on explicit diagonalisation of the Hamiltonian by a unitary transformation.

General validity of (3.60) is equivalent to the identity

\[ \sum_i (\omega_j^2 - k_i) m_{ii}(\omega_j) = 2 \omega_j^2 \prod_{k \neq j} (\omega_j^2 - \omega_k^2) . \]  

(3.63)

Using the same kind of manipulations as in the case \(d = 2\), it is straightforward to verify this identity directly for \(d = 3\), and we certainly expect it to be true in general.
If this is the case, then this ought to be obvious on a priori grounds.\footnote{This may be more apparent after doing this calculation in phase space variables.} Note, however, that neither from the present point of view nor from the analysis of section 2.6 this is completely manifest. This is a reflection of the non-diagonal nature of the original problem.

Likewise, for the string \((n \neq 0)\) modes, the Hamiltonian takes the form

\[ H^{(n)} = \sum_{J=1}^{2d} \text{sign}(C_{J}^{(n)}) \Omega_{J}^{(n)} (\mathcal{N}_{J}^{(n)} + \frac{1}{2}) , \]  

(3.64)

where \(\Omega_{J}^{(n)}\) is the sum of two terms – one coming from the metric, the other from the \(B\)-field,

\[ \Omega_{J}^{(n)} = C_{J}^{(n)} m_{11} (\omega_{J}^{(n)}) \sum_{i,j} [((\omega_{J}^{(n)})^2 - k_i + 4n^2) \delta_{ij} - 4in(-1)^{i+j} h_{ij}] m_{ij} (\omega_{J}^{(n)}) . \]  

(3.65)

The above expression for the frequencies can be simplified somewhat. Using (A.3), (3.65) can be rewritten as

\[ \Omega_{J}^{(n)} = 2\omega_{J}^{(n)} C_{J}^{(n)} m_{11} (\omega_{J}^{(n)}) \sum_{i,j} [\omega_{J}^{(n)} \delta_{ij} + i(-1)^{i+j} f_{ij}] m_{ij} (\omega_{J}^{(n)}) . \]  

(3.66)

As in the particle case (3.60), we expect the frequencies \(\Omega_{J}^{(n)}\) to be exactly equal to the string mode frequencies,

\[ \Omega_{J}^{(n)} = \omega_{J}^{(n)} . \]  

(3.67)

General validity of (3.67) is equivalent to the identity

\[ 2 \sum_{i,j} [\omega_{J}^{(n)} \delta_{ij} + i(-1)^{i+j} f_{ij}] m_{ij} (\omega_{J}^{(n)}) = \prod_{K \neq J} (\omega_{J}^{(n)} - \omega_{K}^{(n)}) \]  

(3.68)

which can be verified for \(d = 2\) after some computations which are explained in section 4.4.

### 3.7 The level matching condition

In general, the light-cone Hamiltonian should be supplemented by the condition of translational invariance along the closed string, i.e.

\[ \int_{0}^{\pi} d\sigma \, \Pi_{i} X^{\prime i} = 0 , \]  

(3.69)

which should be imposed on the string spectrum. Using the definition (3.40) of the momenta \(\Pi_{i}\) and the mode expansion (3.38), one finds that more explicitly this constraint takes the form

\[ -2i\pi \sum_{n>0} n (\Pi_{n}^i \dot{X}_{-n}^i - \Pi_{-n}^i \dot{X}_{n}^i : ) = 0 , \]  

(3.70)
where : : refers to the normal ordering prescription of the previous section and
\[
\Pi_n^i = \frac{(-1)^i}{\pi} \sum_{J=1}^{2d} \xi_J^{(n)} |i\omega_J^{(n)}\delta_{ik} - (-1)^{i+k} f_{jk}| m_{1k} (\omega_J^{(n)}) e^{i\omega_J^{(n)} \tau} .
\] (3.71)

The constraint is time-independent, as can be checked by using the equations of motion (3.11). Hence we can drop all the non-diagonal terms \(\xi_J^{(n)} \xi_K^{(-n)}\), \(J \neq K\), in the expansion of the constraint as these would enter with time-dependent phases.\(^8\) After some rearrangement, using the identity (A.5), we can write the level-matching constraint in terms of the number operators \(N_J^{(n)}\) as
\[
\sum_{n>0} 2n \sum_{J=1}^{2d} m_{11} (\omega_J^{(n)}) |C_J^{(n)}| \sum_{j,k=1}^{d} [\omega_J^{(n)} \delta_{jk} + i(-1)^{j+k} f_{jk}] m_{jk} (\omega_J^{(n)}) N_J^{(n)} = 0 .
\] (3.72)

Using the explicit expression (B.9) for the \(C_J^{(n)}\), we can rewrite this condition as
\[
\sum_{n>0} n \sum_{J=1}^{2d} S_J^{(n)} N_J^{(n)} = 0
\] (3.73)
where
\[
S_J^{(n)} = 2 \text{sign}(m_{11} (\omega_J^{(n)})) \frac{\sum_{j,k=1}^{d} [\omega_J^{(n)} \delta_{jk} + i(-1)^{j+k} f_{jk}] m_{jk} (\omega_J^{(n)})}{| \prod_{K \neq J} (\omega_J^{(n)} - \omega_K^{(n)}) |} .
\] (3.74)

Assuming the general validity of (3.68), we find that
\[
S_J^{(n)} = \text{sign}(m_{11} (\omega_J^{(n)})) \text{sign}(\prod_{K \neq J} (\omega_J^{(n)} - \omega_K^{(n)}))
\] (3.75)
and so
\[
S_J^{(n)} = \pm 1 .
\] (3.76)

Comparison of (3.75) and the expression for \(C_J^{(n)}\) (3.42) shows that
\[
S_J^{(n)} = \text{sign}(C_J^{(n)}) .
\] (3.77)

This means that the signs appearing in the string oscillator spectrum agree with the signs in the level-matching condition.

In the four-dimensional case we will show in section 4.4 that typically (but not necessarily), and in particular for large \(n\), we have
\[
S_J^{(n)} = \text{sign}(\omega_J^{(n)}) ,
\] (3.78)

\(^8\)Once again, vanishing of these contributions can be rephrased as certain identities about roots and minors.
as might have naively been expected for the level-matching condition. In particular, therefore, for large \( n \) the string Hamiltonian is a sum of harmonic oscillators with positive frequencies

\[
\text{sign}(C_j^{(n)})\omega_j^{(n)} = |\omega_j^{(n)}|.
\]  

(3.79)

However for small \( n \) (small compared to \( f_{ij} \), measured in units of \( p_v \)), there may be deviations from this and both the Hamiltonian and the level-matching conditions may involve some unusual signs.

4 Aspects of the string spectrum for four-dimensional plane waves

In the previous sections we have shown that, generically, the particle Hamiltonian in a homogeneous plane wave background takes the form of a set of decoupled harmonic oscillators, with real or possibly complex frequencies. In the following, we will analyse various aspects of the resulting particle and string spectra in the case of four-dimensional plane waves.

4.1 Analysis of the particle spectrum

In the particle case we found that the frequencies, masses and oscillator commutators (3.41) are (see (2.46), (3.21) for the frequencies)

\[
\omega_{1,2}^2 = \frac{1}{2}(4f^2 - k_1 - k_2 \mp D) , \quad D^2 = (4f^2 - k_1 - k_2)^2 - 4k_1k_2 ,
\]  

(4.1)

\[
m_{1,2} = \frac{2D}{D + k_1 - k_2 \mp 4f^2} , \quad C_{1,2} = \mp \frac{1}{2\omega_{1,2}(\omega_{1,2}^2 + k_2)D} .
\]  

(4.2)

We see that when \( D = 0 \), so that \( \omega_1^2 = \omega_2^2 \), the masses \( m_i \) vanish and correspondingly the \( C_i \) are infinite. This can happen only if \( k_1 \) and \( k_2 \) are both positive and \( \sqrt{k_1} \pm \sqrt{k_2} = 2f \). In that case the unitary transformation (2.42) which related the original system to the system of two decoupled harmonic oscillators breaks down because the phase \( \theta_2 \) becomes infinite. One of the \( C_i \) will also be infinite when one of the frequencies is zero. This implies that one of the \( k_i \) is zero and that one of the masses \( m_i \) is infinite.

When there is no rotation, \( f = 0 \), the frequencies are \( \omega_i^2 = -k_i \), with masses \( m_i = 1 \).

We see that in this case one of the \( C_i \) would blow up, but this is due to the fact that the minors \( m_{11} = m_{12} = 0 \) so that this is simply a reflection of a bad choice of a null eigendirection, which can be fixed by choosing different null vectors for different frequencies (see the discussion following (3.28)).

Assuming that both frequencies \( \omega_i \) are non-zero and real (we shall always assume that if the \( \omega_i \) are real they are chosen to be positive), the Hilbert space of the theory is that
of two harmonic oscillators with generically different frequencies. The spectrum of the Hamiltonian $H^{(0)}$ is given by the (almost) standard oscillator expression

$$E = \pm \omega_1 (n_1 + \frac{1}{2}) \pm \omega_2 (n_2 + \frac{1}{2}) ,$$

(4.3)

where $n_1, n_2 = 0, 1, 2, \ldots$. As explained in section 2.6 and section 3.6, the sign of the spectrum depends on the sign of the $m_i$ or $C_i$. We will come back to this issue below.

One finds that $\omega_i^2$ are positive if, e.g., $k_i < 0$ (then $0 < \omega_i^2 < \omega_2^2$). If $k_i > 0$, then the frequencies $\omega_i^2$ are positive provided $4f^2 > k_1 + k_2$. If the signs of $k_1$ and $k_2$ are different, then $\omega_1^2 < 0$, $\omega_2^2 > 0$ and thus the oscillator system is unstable in one direction.

Let us now take into account the conditions (see section 2.3) on the parameters $k_i$ and $f$ imposed by the conformal invariance conditions (i.e. by the Einstein equations). First, in the pure-metric case (constant dilaton and zero 3-form strength) with vanishing rotation parameter $f = 0$, the Ricci-flatness implies that

$$k_1 + k_2 = 0 , \quad \text{i.e.} \quad D = k_1 - k_2 , \quad \omega_1^2 = -k_1 , \quad \omega_2^2 = -k_2 , \quad m_{1,2} = 1 .$$

(4.4)

Thus one of the frequencies is necessarily imaginary, and we get an inverted oscillator problem in one of the two directions: the absence of a stable ground state just means that the particle “escapes” to infinity (the classical geodesics are also “pushed” to infinity).\(^9\)

For metrics with $f \neq 0$ the Ricci-flatness condition implies that (cf. (2.26))

$$k_1 + k_2 = 2f^2 , \quad \text{i.e.} \quad D = k_1 - k_2 , \quad \omega_1^2 = k_2 , \quad \omega_2^2 = k_1 , \quad m_1 = -\frac{k_1 - k_2}{2k_2} , \quad m_2 = \frac{k_1 - k_2}{2k_1} .$$

(4.5)

(4.6)

Compared to (4.4) here $\omega_1^2$ and $\omega_2^2$ are multiplied by -1 and their absolute values are interchanged. It appears as if the role of the rotation parameter $f$ is thus to reverse the signs of the two frequencies compared to the $f = 0$ case, i.e. one is tempted to conclude that (like in the case of non-Ricci-flat plane waves with extra fluxes) the Ricci-flat plane waves with non-zero rotation parameter $f$ may have localised point-like particle dynamics in transverse directions (both at the classical and quantum level): if $k_1$ and $k_2$ are positive, the particle is described by an effective Hamiltonian of two decoupled oscillators with real frequencies.

For generic non-vanishing $k_i > 0$ one of the two masses in (2.43) is negative. That means that one of the effective oscillator Hamiltonians $H_i$ enters their sum with the

\(^9\)As in the standard plane wave cases discussed in 22, the negative values of the light-cone gauge energies do not of course imply any real instability of the plane wave background but are rather related to the fact that some geodesics may escape to infinity.
negative sign, and thus one of the two oscillator energy spectrum terms in (4.3) should be taken with the negative sign. This is not surprising given that for positive \( k_1 \) and \( k_2 \) the original Hamiltonian (2.40) is not positive definite, and can also be understood from the fact that \( m_1 < 0 \) is possible iff \( C_1 < 0 \), so that the role of positive and negative frequencies is interchanged (see section 3.5). Thus we are in a novel situation where the light-cone energy spectrum is discrete but not positive,

\[
\mathcal{E} = -\omega_1(n_1 + \frac{1}{2}) + \omega_2(n_2 + \frac{1}{2}) .
\]  

(4.7)

As it is clear from (2.9), (2.26), the role of the 3-form or the \( h \)-parameter is to effectively reduce the value of \( f \): for non-zero \( h \) we have \( k_1 + k_2 = 2(f^2 - h^2) \) and thus

\[
\omega_{1,2}^2 = f^2 + h^2 \pm \frac{1}{2} D , \quad D^2 = (k_1 - k_2)^2 + 16f^2h^2 .
\]  

(4.8)

Depending on the value of \( k_1 - k_2 \) both frequencies may be real or one of them may be imaginary. In particular, in the chiral null model case with an additional \( SO(2) \) symmetry in \( x_i \)-directions, i.e. for \( f^2 = h^2 \) and \( k_1 = k_2 \), we get \( \omega_1^2 = 0 \), \( \omega_2^2 = 4f^2 \), \( m_1 = \infty \), \( m_2 = 1 \), i.e. we get again the Landau-type spectrum, in agreement with [10].

In the case of a non-trivial (\( u \)-dependent) dilatonic background, the dilaton can be eliminated from the Klein-Gordon equation by a redefinition \( \Psi \rightarrow e^{\phi(u)} \Psi \) of the field in (2.36) (see [8] for details). For a non-constant dilaton given by the case I of (2.21) the positivity (2.24) of the parameter \( \mu \) in (2.23) or (2.28) implies the restriction \( k_1 + k_2 < 2(f^2 - h^2) \). Interestingly, the latter condition ensures also the reality of \( D \) in (4.1).

4.2 **The anti-Mach metric**

A border-line case is when one of the parameters \( k_1 \) or \( k_2 \) vanishes. This corresponds to the anti-Mach metric discussed in [9]. If we choose, as in (2.30), \( k_1 = 2f^2, \ k_2 = 0 \), we get

\[
\omega_1^2 = 0 , \quad \omega_2^2 = 2f^2 , \quad m_1 = -\infty, \ m_2 = \frac{1}{2} .
\]  

(4.9)

In this case the phases (2.44) are \( \theta_1 = 0, \ \theta_2 = -f^{-1} \).

The vanishing of one of the frequencies suggests that the spectrum should contain a discrete oscillator contribution in one direction and a continuous free-particle contribution in another, in agreement with the previous analysis of [9]. However, since here one of the effective masses is infinite, this case needs special consideration.

In [9] this conclusion was reached by noting that the coordinate transformation

\[
v \rightarrow v + 2fxy
\]  

(4.10)
puts the anti-Mach metric (which corresponds to the choice of the parameters (2.30))

\[ ds^2 = 2dudv + 2f^2 x^2 du^2 + 2f(xdy - ydx)du + dx^2 + dy^2 \] (4.11)

into the form

\[ ds^2 = 2dudv + 2f^2 x^2 du^2 + 4f xdydu + dx^2 + dy^2 \] . (4.12)

In this coordinate system, there is clearly a translation invariance in the \( y \)-direction which gives rise to a continuous contribution to the spectrum.

This coordinate transformation can be equivalently regarded as a gauge transformation of the gauge field \( A_i \) in (2.1). Thus, to make the discussion of the \( k_2 = 0 \) case more transparent, it is useful to follow the analysis of the Landau problem in a different gauge for \( A_i \) than used in (2.7), (2.17), (2.25), namely,

\[ A_x = 0 \, , \quad A_y = 2fx \, . \] (4.13)

Then the Hamiltonian (2.38) takes the following form (cf. (2.40))

\[ H = -\frac{1}{2}\left[ \partial_x^2 + (\partial_y - 2ifx)^2 + k_1 x^2 \right] . \] (4.14)

The case \( k_1 = 0 \) corresponds to the standard Landau problem, and the case of \( k_1 = 2f^2 \) to the anti-Mach metric case. Since \( y \to y + c \) is a symmetry, we may use the Fourier transformation in \( y \), i.e. \( \partial_y \to ip_y \). Then

\[ H = -\frac{1}{2}\partial_x^2 + V \, , \quad V = \frac{1}{2}\left[ (2fx - py)^2 - k_1 x^2 \right] . \] (4.15)

The potential here is positive at large \( x \) both in the Landau model case \( k_1 = 0 \) and in the anti-Mach case \( k_1 = 2f^2 \). In the Landau model case we can “absorb” \( p_y \) into \( x \) by a constant shift and thus the resulting spectrum is the discrete spectrum of a one-dimensional harmonic oscillator with mass 1 and frequency \( \omega_1 = 2f \). In the anti-Mach case we get, diagonalising the quadratic form in the potential,

\[ V = \frac{1}{2}\left[ (2fx - py)^2 - k_1 x^2 \right] = (fx - py)^2 - \frac{1}{2}p_y^2 \, . \] (4.16)

The resulting analog of the spectrum (4.3) is indeed continuous but is not positive definite (cf. (4.3)):

\[ \mathcal{E} = \omega_2(n_2 + \frac{1}{2}) - \frac{1}{2}p_y^2 \, , \quad \omega_2 = \sqrt{2} f \, . \] (4.17)

This is, indeed, in agreement with the spectrum found in (9). As expected on the basis of one of the two effective masses going to minus infinity as the frequency goes to zero in (4.3), here the free-motion contribution enters with negative sign relative to
the mass term. This is the opposite to what happens in the flat space limit where (for \( p_v = 1 \)) one has \( H = -p_u = \frac{1}{2} p^2_x + \frac{1}{2} p^2_y \). This result is obviously a reflection of an inherent “unboundedness” of trajectories in this case – the light-cone energy can take any possible negative values.

It is clear now that similar features are shared by the general case of non-zero \( k_2 \) where again one of the masses is negative: there, as explained above (see (4.7)), the negative contribution to the light-cone spectrum is not continuous as in the anti-Mach case but discrete.

4.3 Analysis of the string spectrum

The string mode frequencies are the roots of the equation \( \omega^4 + (k_1 + k_2 - 4f^2 - 8n^2)\omega^2 - 16nf\omega + (k_1 - 4n^2)(k_2 - 4n^2) - 16n^2h^2 = 0 \) . (4.18)

In general the roots \( \omega^{(n)}_J, J = 1, 2, 3, 4 \), of this equation satisfy the identities

\[
\sum_{J=1}^{4} \omega^{(n)}_J = 0 \ , \quad (4.19)
\]

\[
\sum_{J<K} \omega^{(n)}_J \omega^{(n)}_K = k_1 + k_2 - 4f^2 - 8n^2 \ , \quad (4.20)
\]

\[
\sum_{J<K<L} \omega^{(n)}_J \omega^{(n)}_K \omega^{(n)}_L = 16nfh \ , \quad (4.21)
\]

\[
\omega^{(n)}_1 \omega^{(n)}_2 \omega^{(n)}_3 \omega^{(n)}_4 = (k_1 - 4n^2)(k_2 - 4n^2) - 16n^2h^2 \ . \quad (4.22)
\]

Note that for large \( n \), the product of the roots is positive. Since the sum of the roots is zero, this means that for sufficiently large \( n \) there will be two positive and two negative frequencies.

When \( h = 0 \), eq. (4.18) is a quadratic equation for \( \omega^2 \) and the frequencies \( \omega_{1,2}^{(n)} \) are

\[
\omega_{1,2}^{(n)^2} = \frac{1}{2} (4f^2 + 8n^2 - k_1 - k_2 \pm D_n) \ , \quad (4.23)
\]

where

\[
D^2_n = (4f^2 + 8n^2 - k_1 - k_2)^2 - 4(k_1 - 4n^2)(k_2 - 4n^2)
= (k_1 - k_2)^2 - 8(k_1 + k_2)f^2 + 16f^4 + 64f^2n^2 \ . \quad (4.24)
\]

We see that, regardless of the values of \( k_i \) and \( f \), the frequencies are real for large enough values of \( n \), i.e. both the “rotation” or “magnetic” parameter \( f \) and the string excitation level \( n \) work towards stabilising the string motion.
A simple example is $k_1 = k_2 = f^2$, which satisfies the Ricci flatness condition (2.26). In this case, $D_n^2 = 64f^2n^2$, and the four distinct real frequencies are

$$\omega_j^{(n)} = \pm f \pm 2n \ .$$

(4.25)

An interesting possibility in the case when the particle motion is unstable is to get new string states that have (nearly) zero frequencies, i.e. new light states. For example, in the case of the anti-Mach metric (4.12) with parameters (2.30), $k_1 = 2f^2, k_2 = 0$, we find

$$\omega_{1,2}^{(n)} = f^2 + 4n^2 \pm \sqrt{f^4 + 16f^2n^2} \ .$$

(4.26)

For the zero mode ($n = 0$) we reproduce the result [19] of the point-particle analysis that one of the two frequencies vanishes. Clearly, $\omega_1^{(n)}$ is always real. $\omega_2^{(n)}$, on the other hand, can be real, zero, or imaginary. In fact, for the special values $f = \sqrt{2}n$, the frequency vanishes, and one has a new massless state. For $f > \sqrt{2}n$ the frequency is imaginary, i.e. for fixed $f$ for sufficiently large $n$ all frequencies are real and the motion is oscillatory.

Note that compared to the homogeneous R-R plane wave model in [2] where the spectrum had the form $\omega^{(n)} = \sqrt{|k| + 4n^2}$ (with $k$ determined by the R-R flux) here we have more intricate (“double square root”) dependence on the oscillator level $n$. Similar string oscillator spectra were found previously in the case of “standard” (no rotation) plane waves supported by a null (and non-constant, $\sim e^{\mu u}$) $B_{MN}$ background in [12] (see also [13]). One may thus contemplate possible connection to deformations of conformal gauge theories via Penrose limits of the corresponding geometries (cf. [14]).

When $f = 0$, the parameter $h$ appears only in the constant term and one has

$$\omega_j^{(n)} = \frac{1}{2}(8n^2 - k_1 - k_2) \pm \frac{1}{2}\sqrt{(k_1 - k_2)^2 + 64n^2h^2} \ .$$

(4.27)

In particular, when $k_1 = k_2 \equiv k$ and $k = -h^2$ (so that (2.20) is satisfied for constant dilaton), one can take the square root and the frequencies are (cf. (4.25))

$$\omega_j^{(n)} = \pm (2n \pm h) \ .$$

(4.28)

Here for special values of $h$ one finds additional states with zero light-cone energy. This case (related by a coordinate transformation to a WZW [18] or chiral null model case) was considered previously in [10], [3], [4].

In general, when $f \neq 0$ and $h \neq 0$, the frequencies are complicated functions of the parameters. We can write the defining equation (4.18) as

$$(\omega^2 - 4n^2 + k_1)(\omega^2 - 4n^2 + k_2) = 4(f\omega + 2nh)^2 \ .$$

(4.29)
Eq. (4.29) simplifies in the case $k_1 = k_2 = k$, when the four roots are the solutions of the two quadratic equations

$$\omega^2 - 4n^2 + k = \pm (2f\omega + 4nh),$$

(namely

$$\omega_{1,2} = f \pm \sqrt{f^2 + 4n^2 - k + 4hn},$$

$$\omega_{3,4} = -f \pm \sqrt{f^2 + 4n^2 - k - 4hn}$$

Let us now consider the chiral null model case $f = h$ (cf. [10, 4]). With $f = h$ and $k_1 = k_2$, the conformal invariance condition (2.26) $k_1 + k_2 = 2f^2 - 2h^2$ implies that $k_1 = k_2 = 0$, so that the four frequencies are thus simply (cf. (4.28))

$$\omega_{1,4}^{(n)} = 2(n \pm f), \quad \omega_2^{(n)} = \omega_3^{(n)} = -2n.$$  

Note that the matrix $M(\omega, n)$ in (3.18) evaluated at $\omega_2^{(n)} = \omega_3^{(n)}$ is identically zero. Therefore, it has two linearly independent null eigen-directions, and so the frequency base ansatz (3.13) provides us with the full set of four linearly independent solutions, even though two of the frequencies are equal (and the construction of the null eigendirections in terms of the minors of $M$ is not applicable).

### 4.4 The Level Matching Condition

In section 3.7 we had derived the general level matching condition (3.73),

$$\sum_{n > 0} \sum_{J = 1}^{2d} S_J^{(n)} N_J^{(n)} = 0$$

where

$$S_J^{(n)} = 2 \text{sign}(m_{11}(\omega_J^{(n)})) \frac{\sum_{j,k=1}^{d} [\omega_J^{(n)} \delta_{jk} + i(-1)^{j+k} f_{jk}] m_{jk}(\omega_J^{(n)})}{| \prod_{K \neq J}(\omega_J^{(n)} - \omega_K^{(n)}) |}.$$  

To evaluate this in the present case of four-dimensional plane waves, it is cumbersome (and unnecessary) to use the explicit expressions for the roots. Rather, we will use the general identities (4.19-4.22) satisfied by the roots of any quartic equation (with zero cubic term).

It follows from the form of the matrix $M$ (3.18) that

$$\sum_{j,k=1}^{d} [\omega_J^{(n)} \delta_{jk} + i(-1)^{j+k} f_{jk}] m_{jk}(\omega_J^{(n)}) = \omega_J^{(n)} (2\omega_J^{(n)} 2 - 8n^2 + k_1 + k_2 - 4f^2) - 8nhf.$$  

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Using the quadratic and cubic identities (4.20, 4.21), we can write this as
\[ (4.35) = \omega(n)J(2\omega(n)^2 + \sum_{K<L} \omega_K \omega_L(n) - \frac{1}{2} \sum_{K<L<M} \omega_K \omega_L \omega_M). \] (4.36)

Using the linear identity (4.19), we can write
\[ 2\omega(n)^2 = \frac{1}{2} \omega(n)^2 - \frac{3}{2} \omega(n) \sum_{K \neq J} \omega(n)_K. \] (4.37)

Then one finds that the r.h.s. of (4.35) becomes
\[ \frac{1}{2} \prod_{K \neq J} (\omega(n)_K - \omega(n)_J). \] (4.38)

This has two immediate consequences. First of all, it establishes the identity (3.68) ensuring that, in the four-dimensional case, we indeed have equality of the string mode and oscillator frequencies, \( \omega(n)_J = \Omega(n)_J \). Furthermore, the same identity now implies (as anticipated in section 3.7) that
\[ S_J^{(n)} = \text{sign}(m_{11}(\omega_J^{(n)})) \text{ sign}(\prod_{K \neq J} (\omega_J^{(n)} - \omega_K^{(n)})) = \pm 1. \] (4.39)

For example, in the case \( f = 0 \), it is now easy to see that
\[ S_J^{(n)} = \text{sign}(\omega_J^{(n)}) \] (4.40)
so that (4.33) becomes the same as the standard flat space level-matching condition.

To show that this is not necessarily the case when \( f \neq 0 \), it suffices to consider the example \( k_1 = k_2 = f^2, h = 0 \), with spectrum \( \omega(n)_J = \pm f \pm 2n \) (4.25). In this case one finds
\[ S_J^{(n)} = \text{sign}(\pm 2n). \] (4.41)

Evidently this agrees with \( \text{sign}(\omega_J^{(n)}) \) for sufficiently large values of \( n \), \( 2n > f > 0 \), but deviates from this for \( 2n < f \) (recall that we have set \( p_v = 1 \) – in general the condition is \( 2n > p_vf \), which for any \( n \) can be violated for sufficiently large \( p_v \)).

To show this more generally consider the case where two of the frequencies are positive \( \omega_{1,2}^{(n)} \) and the other two \( \omega_{3,4}^{(n)} \) are negative. We also assume that \( -k_2 + 4n^2 > 0 \) (this reduces to the inequality \( 2n > f \) in the above example). We order the frequencies as \( \omega_1^{(n)} > \omega_2^{(n)} > \omega_3^{(n)} > \omega_4^{(n)} \). The polynomial \( P(\omega) \) which has the frequencies as roots (4.18) is positive for \( |\omega| \rightarrow \infty \). In addition as it can be seen from (4.29), it takes negative values at \( \omega_2^{(n)} = -k_2 + 4n^2 \) and \( \omega_2^{(n)} = -k_1 + 4n^2 \). Therefore there is at least one root between \(+\infty\) and \( \omega = \sqrt{-k_2 + 4n^2} \). Since \( \omega_1^{(n)} \) is the largest root,
\[ \prod_{J \neq 1} (\omega_1^{(n)} - \omega_J^{(n)}) > 0 \] (4.42)
and

\[ \omega_1^{(n)} > \sqrt{-k_2 + 4n^2}, \quad (4.43) \]

which imply that \( S_1^{(n)} = \text{sign}(\omega_1^{(n)}) = 1. \) Similarly, it can be shown that \( S_4^{(n)} = \text{sign}(\omega_4^{(n)}) = -1 \) by observing that \( \omega_4^{(n)} < -\sqrt{-k_2 + 4n^2} \) and so \( (\omega_4^{(n)})^2 > -k_2 + 4n^2. \) It remains to show the statement for the other two frequencies \( \omega_2^{(n)} \) and \( \omega_3^{(n)}. \) There are two possibilities, either \( \omega_2^{(n)} > \sqrt{-k_2 + 4n^2} \) or \( \omega_2^{(n)} < \sqrt{-k_2 + 4n^2}. \) We exclude the former case. If \( \omega_2^{(n)} > \sqrt{-k_2 + 4n^2}, \) then \( P(\omega) \) has to have another positive root which is larger than \( \sqrt{-k_2 + 4n^2}. \) If it did not, the value of \( P(\omega) \) at \( \sqrt{-k_2 + 4n^2} \) would have been positive. However this is not the case as we have shown. But we have two positive and two negative frequencies, so we have to take \( \omega_2^{(n)} < \sqrt{-k_2 + 4n^2}. \) In this case, it is easy to see that \( S_2^{(n)} = \text{sign}(\omega_2^{(n)}) = 1. \) Similarly, we can show that \( S_3^{(n)} = \text{sign}(\omega_3^{(n)}) = -1. \)

When there are three positive frequencies and one negative, the formula \( S_J^{(n)} = \text{sign}(\omega_J^{(n)}) \) is not valid as can be seen by an argument similar to the above. However, since we have seen that for sufficiently large values of \( n \) there are precisely two positive and two negative frequencies, this establishes that typically, and for large \( n, \) we indeed have \( S_J^{(n)} = \text{sign}(\omega_J^{(n)}) \).

### 4.5 The zero-point energy

It follows from (3.57, 3.64) that the zero-point energy or normal ordering constant \( \epsilon_0 \) has the form

\[ \epsilon_0 = \frac{1}{2} \sum_{j=1}^{d} \text{sign}(C_j)\omega_j + \sum_{n=1}^{\infty} s_n, \quad (4.44) \]

where

\[ s_n = \frac{1}{2} \sum_{J=1}^{2d} \text{sign}(C_J^{(n)})\omega_J^{(n)} - 2dn. \quad (4.45) \]

In \( s_n \) we have included the contribution \((-2dn)\) of the superstring fermions which are decoupled from the background in the light-cone gauge (see section 2.5).

We had already seen that, for large \( n, \) \( \text{sign}(C_J^{(n)})\omega_J^{(n)} = |\omega_J^{(n)}|, \) so that for sufficiently large oscillator number the contribution to the zero-point energy is given by the sum of the absolute values of the frequencies,

\[ s_n = \frac{1}{2} \sum_{J=1}^{2d} |\omega_J^{(n)}| - 2dn. \quad (4.46) \]

In particular, for \( d = 2 \) and \( f = 0 \) or \( h = 0 \) the four frequencies \( \omega_J^{(n)} \) come in two pairs \( \pm \omega_{1,2}^{(n)}, \) so that for sufficiently large \( n \) we have

\[ s_n = \omega_1^{(n)} + \omega_2^{(n)} - 4n. \quad (4.47) \]
To estimate $s_n$, one can expand $\omega_1^{(n)} + \omega_2^{(n)}$ at large $n$. For example, for $h = 0$ one can use

\[
(\omega_1^{(n)} + \omega_2^{(n)})^2 = \omega_1^{(n)} + \omega_2^{(n)} + 2(\omega_1^{(n)} + \omega_2^{(n)})^{1/2} = 4f^2 + 8n^2 - k_1 - k_2 + ((k_1 - 4n^2)(k_2 - 4n^2))^{1/2} .
\] (4.48)

Then one sees that the leading order $O(n)$ contribution cancels against that of the fermions, and that the lower order terms are

\[
s_n = -\frac{1}{4n}(k_1 + k_2 - 2f^2) - \frac{1}{64n^3}[k_1^2 + k_2^2 - 2f^2(k_1 + k_2) + 2f^4] + O\left(\frac{1}{n^5}\right) .
\] (4.49)

The first term here leads to a logarithmic divergence after summing over $n$: $\sum_{n=1}^{\infty} \frac{1}{n} e^{-\epsilon n} = -\ln \epsilon + O(\epsilon)$. Its coefficient $(2f^2 - k_1 - k_2)$ vanishes automatically for the Ricci-flat spaces (cf. (2.28), (2.28)) or otherwise is cancelled by the dilaton contribution (2.21), (2.28) as in the similar dilatonic background case considered in [8]. The resulting expression for $\epsilon_0$ is thus finite, in agreement with conformal invariance of the theory before light-cone gauge fixing. This provides a useful check of the above expressions for the frequencies.

For example, in the anti-Mach case (4.20) we find:

\[
s_n = -\frac{f^4}{32n^3} - \frac{f^6}{128n^5} - \frac{f^8}{8192n^7} + O\left(\frac{1}{n^9}\right) ,
\] (4.50)

and $\epsilon_0$ is automatically finite (and negative). The negative vacuum energy is analogous to the case of [8] or [21]; the sum can be evaluated using Epstein function as in [21].

Likewise, when $h \neq 0$ but $f = 0$, one finds

\[
s_n = -\frac{1}{4n}(k_1 + k_2 + 2h^2) + O\left(\frac{1}{n^3}\right) ,
\] (4.51)

so that the logarithmic divergence in the sum (4.44) is again absent when the conformal invariance condition (2.28) is satisfied.

This fact, and a somewhat surprising stronger statement, are true for general values of $f$, $h$ and $k_i$. Even though the closed form solution of [4.29] is unenlightening, we can find the explicit solution of (4.29) in a series expansion in $1/n$. It is clear that the leading behaviour of $\omega$ at large $n$ is its flat space value $\omega^{(n)} = \pm 2n + O(1)$. Then, expanding

\[
\omega^{(n)} = 2n + a + \frac{b}{n} + \frac{c}{n^2} + O\left(\frac{1}{n^3}\right) ,
\] (4.52)

one finds

\[
a = \pm (f + h) ,
\]

\[
b = \frac{1}{8}(2f^2 - 2h^2 - k_1 - k_2) ,
\]

\[
c = \pm \frac{1}{16} h(2f^2 - 2h^2 - k_1 - k_2) \pm \frac{(k_1 - k_2)^2}{128(f + h)} .
\] (4.53)
If we now impose the conformal invariance condition (2.26), i.e. \( 2f^2 - 2h^2 = k_1 + k_2 \), we obtain the simple expression

\[
\omega^{(n)} = 2n \pm (f + h) \pm \frac{(k_1 - k_2)^2}{128n^2(f + h)} + \mathcal{O}\left(\frac{1}{n^3}\right).
\] (4.54)

Likewise, noting that (4.29) is invariant under \( n \to -n \) and \( h \to -h \), the other two frequencies are

\[
\omega^{(n)} = -2n \pm (f - h) \pm \frac{(k_1 - k_2)^2}{128n^2(f - h)} + \mathcal{O}\left(\frac{1}{n^3}\right).
\] (4.55)

We conclude that the logarithmic divergence in the sum over \( n \) is absent for each series of the frequencies \( \omega_J^{(n)} \) independently of \( J \).

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A  

\section*{Some facts about minors}

\subsection*{A.1 Null vectors and minors}

Let $M$ be a $(d \times d)$ matrix and let $m_{ik}$ denote the minor of $M$, i.e. the determinant of the matrix one obtains from $M$ by removing the $i$'th row and $k$'th column. These minors satisfy the equations

$$\sum_k M_{ik} m_{ik} (-1)^{i+k} = \det M \quad \text{(A.1)}$$

for any $i$. Moreover, for any $M$ one has the identities

$$\sum_k M_{lk} m_{ik} (-1)^k = 0 \quad \text{for } l \neq i . \quad \text{(A.2)}$$

This can be shown by noting that, whatever $M$ may be, the left-hand side also calculates the determinant of a matrix with two equal rows, namely the determinant of the matrix one obtains from $M$ by replacing $M_{ik}$ by $M_{lk}$. The determinant of this new matrix is clearly zero and the identity follows.

In particular, therefore, when $\det M = 0$, one can choose the null eigenvector $v_k$, $M_{ik} v_k = 0$, to be

$$v_k^{(i)} = (-1)^k m_{ik} \quad \text{(A.3)}$$

for any $i$. If there are no degeneracies, there should be a unique null eigen-direction. Hence the vectors $v_k^{(i)}$ for different choices of $i$ should be proportional to each other, i.e.

$$\frac{v_k^{(i)}}{v_l^{(j)}} = \frac{v_l^{(i)}}{v_l^{(j)}} . \quad \text{(A.4)}$$

This can be rephrased as the useful identity

$$m_{ik} m_{jl} = m_{il} m_{jk} , \quad \text{(A.5)}$$

which is obviously true for $i = j$ or $k = l$ and is indeed a property of matrices with $\det M = 0$ for $i \neq j, k \neq l$ because of the general identity

$$m_{ik} m_{jl} - m_{il} m_{jk} = \pm m_{ij,kl} \det M , \quad \text{(A.6)}$$

where $m_{ij,kl}$ is the secondary minor obtained by removing the two rows and columns indicated.

Another useful identity can be obtained by using the explicit form of the matrix $M$. Let $\omega_j^{(n)}$ be one of the roots of $\det M = 0$, where $M$ is given in (3.15). Evaluating (A.1) at $\omega = \omega_j^{(n)}$ and summing over $i$, we get

$$\sum_{i,k} M_{ik} m_{ik} (-1)^{k+i} = 0 . \quad \text{(A.7)}$$
Substituting the expression for $M(\omega_j^{(n)})$, we find
\[ \sum_i ((\omega_j^{(n)})^2 + k_i - 4n^2) m_{ii}(\omega_j^{(n)}) + 2i\omega_j^{(n)} \sum_{i,k} (-1)^{i+k} f_{ik} m_{ik}(\omega_j^{(n)}) + 4i n \sum_{i,k} (-1)^{i+k} h_{ik} m_{ik}(\omega_j^{(n)}) = 0. \] (A.8)

A.2 Another identity for minors

The general structure of the particle light-cone Hamiltonian (before normal ordering) is
\[ H(0) = \sum_{j,k=1}^d \sum_{\alpha,\beta=\pm} \xi_j^\alpha \xi_k^\beta H^{\alpha\beta}(\omega_j,\omega_k) e^{i(\alpha\omega_j + \beta\omega_k)\tau} \] (A.9)

where
\[ H^{\alpha\beta}(\omega_j,\omega_k) = -\frac{1}{2} \sum_i (k_i + \alpha\beta\omega_j\omega_k)m_{11}(\alpha\omega_j)m_{11}(\beta\omega_k). \] (A.10)

Since in stationary coordinates the Hamiltonian is time-independent, it must be true for instance that
\[ H^{++}(\omega_j,\omega_j) \sim \sum_i (\omega_j^2 + k_i)m_{11}(\omega_j)m_{11}(\omega_j) = 0 \quad \forall \ j. \] (A.11)

Noting that $(\omega_j^2 + k_i)$ is just the $(ii)$-component of $M(\omega_j)$, we can also write this as
\[ \sum_{i=1}^d M_{ii}(\omega_j)m_{11}(\omega_j)m_{11}(\omega_j) = 0 \quad \forall \ j. \] (A.12)

Here is a general proof of (A.12) which only makes use of the two elementary identities (A.1), (A.2), which we will use in the form
\[ M_{11}m_{11} = \sum_{i=2}^d (-1)^i M_{1i}m_{1i} \] (A.13)

and
\[ \sum_{k=1}^d (-1)^k M_{ik}m_{1k} = 0 \quad i \neq 1 \]
\[ \Rightarrow (-1)^i M_{i1}m_{11} + (-1)^i M_{ii}m_{1i} = - \sum_{k \neq i, k \neq 1} (-1)^k M_{ik}m_{1k}. \] (A.14)

Then, using the antisymmetry of $M_{ik}$ for $i \neq k$, we can write
\[ \sum_{i=1}^d M_{ii}m_{1i}m_{1i} = m_{11} \sum_{i=2}^d (-1)^i M_{1i}m_{1i} + \sum_{i=2}^d M_{ii}m_{1i}m_{1i}. \]
\[
\begin{align*}
&= \sum_{i=2}^{d} m_{1i}(M_{ii}m_{1i} - (-1)^i M_{i1}m_{11}) \\
&= \sum_{i=2}^{d} m_{1i}((-1)^i M_{ii}m_{1i} - M_{i1}m_{11}) \\
&= -\sum_{i=2}^{d} \sum_{k \neq i, k=2}^{m} (-1)^{i+k} M_{ik}m_{1i}m_{1k} \\
&= 0 . \tag{A.15}
\end{align*}
\]

B The commutation relations of the $\xi$’s

B.1 Imposing the canonical commutation relations

Beginning with the string mode expansion (3.37), (3.38), we now promote the $\xi$’s to operators and impose the CCRs

\[
\begin{align*}
[X^i(\sigma, \tau), X^k(\sigma', \tau)] &= [\Pi^i(\sigma, \tau), \Pi^k(\sigma', \tau)] = 0 \\
[X^i(\sigma, \tau), \Pi^k(\sigma', \tau)] &= i\delta^{ik}\delta(\sigma - \sigma') , \tag{B.1}
\end{align*}
\]

where

\[
\delta(\sigma - \sigma') = \frac{1}{\pi} \sum_{n=-\infty}^{+\infty} e^{2in(\sigma - \sigma')} . \tag{B.2}
\]

It will actually be sufficient to just impose the conditions

\[
\begin{align*}
[X^i(\sigma, \tau), X^k(\sigma', \tau)] &= 0 \\
[X^i(\sigma, \tau), \Pi^k(\sigma', \tau)] &= -i\pi\delta^{ik}\delta(\sigma - \sigma') , \tag{B.3}
\end{align*}
\]

the rest following from the definition of the $\Pi^k$ in (3.40) and the equations of motion.

In terms of the diagonal commutators $C^i_j$ and $C^{(n)}_j$ defined in (3.41), the CCRs become, for the zero mode,

\[
\begin{align*}
\sum_{j=1}^{d} C^i_j(m_{1i}(\omega_j)m_{k1}(\omega_j) - m_{1k}(\omega_j)m_{i1}(\omega_j)) &= 0 \\
\sum_{j=1}^{d} C^j_\omega_j(m_{1i}(\omega_j)m_{k1}(\omega_j) + m_{1k}(\omega_j)m_{i1}(\omega_j)) &= \delta_{ik} . \tag{B.4}
\end{align*}
\]

Now we can make use of the identity (A.5) to rewrite

\[
m_{1i}m_{k1} \pm m_{1k}m_{i1} = m_{11}(m_{ki} \pm m_{ik}) . \tag{B.5}
\]
Thus the conditions become

\[ \sum_{j=1}^{d} C_j m_{11}(\omega_j)(m_{ki}(\omega_j) - m_{ik}(\omega_j)) = 0 \]
\[ \sum_{j=1}^{d} C_j \omega_j m_{11}(\omega_j)(m_{ki}(\omega_j) + m_{ik}(\omega_j)) = \delta_{ik} \quad . \]  

(B.6)

Similarly, for the string modes one obtains the conditions

\[ \sum_{J=1}^{2d} C_J^{(n)} m_{11}(\omega_J^{(n)})m_{ik}(\omega_J^{(n)}) = 0 \]
\[ \sum_{J=1}^{2d} C_J^{(n)} \omega_J^{(n)} m_{11}(\omega_J^{(n)})m_{ik}(\omega_J^{(n)}) = \delta_{ik} \quad . \]  

(B.7)

B.2 Determining the $C_J$ using the Vandermonde and Lagrange polynomials

We now explain how to determine the solutions to the equations (B.6), (B.7) for the coefficients $C_j$ and $C_J^{(n)}$. At first sight, there appear to be too many equations for the $d$ ($2d$) coefficients $C_j$ ($C_J$). However, a closer inspection of the minors reveals that all of these are satisfied provided that these coefficients satisfy $d$ ($2d$) conditions - thus the $C_j$ and $C_J$ are uniquely determined by these equations. The structure of the equations is slightly different for the zero-modes and the string modes, so we will discuss them separately, starting with the latter.

The key observation is that the highest power of $\omega$ that can appear in any minor of a matrix $M$ of the form (3.15) is $\omega^{2d-2}$, arising from the product of $d-1$ diagonal entries. Such a term can only arise for a diagonal minor $m_{kk}$. Furthermore, the coefficient of $\omega^{2d-2}$ in such a minor is $1$ for any one of the $m_{kk}$. Moreover, $m_{kk}$ only arises in the commutator $[\dot{X}^k, X^k]$. All other commutators will involve off-diagonal minors and hence lower powers of $\omega$, $\omega^i$ with $i = 0, 1, \ldots, 2d - 3$, and all these powers will appear. Hence all the $d^2$ equations (B.7) are satisfied provided that the $2d$ coefficients $C_J$ satisfy the $2d$ equations

\[ \sum_{J=1}^{2d} C_J m_{11}(\omega_J)\omega_J^i = 0 \quad \text{for} \quad i = 0, \ldots, 2d - 2 \]
\[ \sum_{J=1}^{2d} C_J m_{11}(\omega_J)\omega_J^{2d-1} = 1 \quad . \]  

(B.8)

the extra power of $\omega_J$ in the last equation arising from the additional $\omega_J$ in the second
equation of (B.7). These equations are solved by

$$C_J = \frac{1}{m_{11}(\omega_J) \prod_{K \neq J}(\omega_J - \omega_K)} . \quad (B.9)$$

One way to see that this is the solution is to use the Vandermonde matrix and its inverse, expressed in terms of Lagrange polynomials.

Let $x_i$, $i = 0, \ldots, p-1$ be $p$ distinct real or complex numbers. The Lagrange polynomials $P_i(x)$ are defined by

$$P_i(x) = \prod_{k \neq i} \frac{(x - x_k)}{(x_i - x_k)} . \quad (B.10)$$

They clearly satisfy

$$P_i(x_k) = \delta_{ik} . \quad (B.11)$$

Expanding the degree $(p-1)$ Lagrange polynomials as

$$P_i(x) = \sum_{l=0}^{p-1} P_{il} x^l , \quad (B.12)$$

the above identity can be written as

$$P_{il} x_k^l = \delta_{ik} . \quad (B.13)$$

In other words, $P_{il}$ are the components of the matrix inverse to the matrix $V$ with components

$$V_{lk} = x_k^l . \quad (B.14)$$

This is just the Vandermonde matrix, and hence the coefficients of the Lagrange polynomials give the inverse of the Vandermonde matrix. In particular,

$$P_{i,p-1} = \frac{1}{\prod_{k \neq i}(x_i - x_k)} . \quad (B.15)$$

Thus, going back to our problem of determining the $C_J$, the defining equations can be written as

$$V_{L,J} C_J m_{11}(\omega_J) = \delta_{L,2d-1} , \quad (B.16)$$

and therefore the solution is

$$C_J m_{11}(\omega_J) = P_{J,2d-1} = \frac{1}{\prod_{K \neq J}(\omega_J - \omega_K)} , \quad (B.17)$$

as claimed.

For the zero-modes, the situation is slightly different. First of all, it follows from the symmetry properties of the matrix $M$ that in this case $m_{ik} + m_{ki}$ has to be an even
function of $\omega$, and $m_{ik} - m_{ki}$ an odd function of $\omega$. Hence the equations (B.6) only involve odd powers of $\omega$. Once again, the highest power that can appear is $\omega^{2d-1}$ and it is these terms that should give the non-vanishing term $\delta_{ik}$ for $i = k$, whereas all other odd powers should sum to zero to satisfy the remaining equations. Thus the $d$ conditions on $d$ coefficients $C_j$ are

$$\sum_{j=1}^{d} C_j m_{11}(\omega_j) \omega_j^{2i-1} = 0 \quad \text{for } i = 1, \ldots, d - 1$$

$$\sum_{j=1}^{d} C_j m_{11}(\omega_j) \omega_j^{2d-1} = \frac{1}{2}.$$  \hfill (B.18)

These equations are solved, in the same way as above, by

$$C_j = \frac{1}{2m_{11}(\omega_j)\omega_j \prod_{k \neq j}(\omega_j^2 - \omega_k^2)}.$$  \hfill (B.19)

### C Geometric Aspects of the Spectrum

#### C.1 The spectral hypersurface

As we have seen, the frequencies of the string are given by the vanishing condition (3.16) of the determinant of the matrix $M$. This condition can be viewed as defining a hypersurface $S$ in the space of parameters and frequencies. For a generic choice of parameters, the hypersurface $S$ is smooth. Here we shall determine the values of the parameters (and thus the corresponding plane-wave models) which correspond to the singularities of this hypersurface. The singularities of the hypersurface are determined from the condition that the first differential of eq. (3.16) with respect to the frequency $\omega$ and the parameters $k, f$ and $h$ vanishes on $S$.

We shall consider string theory on the four-dimensional plane wave for which $M$ in (3.16) is given in (3.18). In this case $S$ is a hypersurface in $\mathbb{R}^5$; we shall take the frequencies to be real but the analysis can be easily extended to complex frequencies. For $n = 0$, we set $x = \omega^2$ and find that the first differential of $\det M$ is

$$(2x + k_1 + k_2 - 4f^2)dx + (x + k_2)dk_1 + (x + k_1)dk_2 - 8x df .$$ \hfill (C.1)

The vanishing condition for this differential gives

$$k_1 = k_2 , \quad f = 0 , \quad \omega^2 = -k_1 .$$ \hfill (C.2)

Generically, (B.18) has four distinct solutions for the frequency $\omega$. The model at the singularity has only two distinct frequencies (for $k_1 \neq 0$). Thus the singularity occurs
at a model with degenerate frequencies. However, the singularity does not describe all models with degenerate frequencies. Next, let us take $n \neq 0$; in this case the hypersurface $S$ is singular at

$$\begin{align*}
\omega^3 + \frac{1}{2}(k_1 + k_2 - 4f^2 - 8n^2)\omega - 4nfh &= 0 \\
f\omega + 2nh &= 0, \\
k_1 &= k_2, \\
\omega^2 &= 4n^2 - k_1.
\end{align*}$$

(C.3)

There is another equation which is implied by the above. There are two cases to consider. If $f = 0$, then $h = 0$ and the frequencies are given by $\omega^2 = 4n^2 - k_1$ ($k_1 = k_2$). If $f \neq 0$, then

$$\omega = -2nfh^{-1}, \\
4n^2h^2 + (k_1 - 4n^2)f^2 = 0.$$  

(C.4)

Since for $n \neq 0$ we expect generically four distinct frequencies, the singularity occurs at a model with degenerate frequencies.

To describe all the models with degenerate frequencies by singularities, we should consider $S$ as a fibration over the space of parameters. For strings in four-dimensional plane waves, $S$ should be thought of as a fibration over $\mathbb{R}^4$ spanned by the parameters $(k_1, k_2, f, h)$. The projection is the standard projection of $\mathbb{R}^5$ onto $\mathbb{R}^4$ restricted on $S$. The generic fibre of such a fibration has four points, the roots of the polynomial (3.18). It is clear now that $S$ as a fibration degenerates precisely when two or more roots become equal. Thus at the singularities of this fibration there are models with two or more degenerate frequencies.

There is a similar description for the hypersurface $S$ for strings on all smooth homogeneous plane-waves. In this case $S$ is a fibration over the space of parameters with fibre which has generically $2(d - 2)$ points. The fibration degenerates precisely when two or more frequencies become the same.

C.2 Berry’s Connection

It is well known that during an adiabatic evolution along a closed path in the space of parameters of a quantum mechanical system, the wave functions develop a phase which is the holonomy of the Berry connection [23, 24]. The quantum mechanical systems that we are investigating in connection to string theory in smooth homogeneous waves have such parameters. Adiabatic evolution in the present context means that we allow the parameters, like the rotation $f$ and the matrix $k$, to depend adiabatically on the light-cone time $u$. The resulting quantum mechanical set-up becomes precisely that which appears in the context of Berry’s connection. We shall not investigate here the general case. Instead, we shall focus on the quantum mechanical models that arise for strings in four-dimensional plane-waves, like those described in section 2.6. In addition,
we shall assume the generic case where there is no degeneracy of states. We shall find
that the curvature of the Berry connection vanishes.

Using the unitary transformation (2.42), we can construct a basis in the Hilbert space
of the quantum mechanical system described in section 2.6 as follows:

\[ |\psi_{n_1,n_2} > = U(\theta_1, \theta_2) |n_1,n_2 > , \]  

(C.5)

where

\[ |n_1,n_2 > = \frac{1}{\sqrt{n_1!n_2!}} (a_1^+)^{n_1} (a_2^+)^{n_2} |0 > \]  

(C.6)

is the standard basis in the Hilbert space of a two-dimensional Harmonic oscillator with
frequencies \(\omega_1, \omega_2\), masses \(m_1, m_2\) and creation (annihilation) operators \(a_1^+, a_2^+\) \((a_1, a_2)\). We follow
the notation of section 2.6. Under the assumptions we have made, these states are not
degenerate and depend on the parameters \(k_1, k_2\) and \(f\). The Berry connection is

\[ A_{n_1,n_2} = <\psi_{n_1,n_2}|d|\psi_{n_1,n_2}> , \]  

(C.7)

where the exterior derivative \(d\) is with respect to the parameters of the system. Using
(C.5), we can write

\[ A_{n_1,n_2} = <n_1,n_2|U^+(\theta_1, \theta_2)dU(\theta_1, \theta_2)|n_1,n_2> + <n_1,n_2|d|n_1,n_2> . \]  

(C.8)

We shall show that the first term in the right-hand-side of (C.8) vanishes. Indeed,

\[ <n_1,n_2|U^+(\theta_1, \theta_2)dU(\theta_1, \theta_2)|n_1,n_2> = -id\theta_1 <n_1,n_2|U^+(\theta_1, \theta_2)xyU(\theta_1, \theta_2)|n_1,n_2> - id\theta_2 <n_1,n_2|p_x p_y|n_1,n_2> \]

\[ = -id\theta_1 <n_1,n_2|U^+(\theta_1, \theta_2)xyU(\theta_1, \theta_2)|n_1,n_2> . \]  

(C.9)

However, as it was shown in [11], after adjusting for notation, we have

\[ U^+(\theta_1, \theta_2) x U(\theta_1, \theta_2) = x + \theta_2 p_y \]

\[ U^+(\theta_1, \theta_2) y U(\theta_1, \theta_2) = y + \theta_2 p_x . \]  

(C.10)

Thus we find

\[ -i\theta_2 d\theta_1 <n_1,n_2|(xp_x + p_y y)|n_1,n_2> \]

\[ = \frac{1}{2} \theta_2 d\theta_1 <n_1,n_2|[a_1^+, a_2^+] - [a_2, a_1^+]]|n_1,n_2> = 0 . \]  

(C.11)

It remains to commute the second term in the right-hand-side of (C.8). For this, we
first compute \(<0|d|0>\) using

\[ <x, y|0> = Ne^{-\frac{1}{2}(m_1 \omega_1 x^2 + m_2 \omega_2 y^2)} \]
and

\[ N = \left( \frac{m_1 m_2 \omega_1 \omega_2}{\pi^2} \right)^{\frac{1}{4}} \]

to find

\[ < 0 | d | 0 > = N^{-1} dN - \frac{1}{4} (m_1 \omega_1)^{-1} d(m_1 \omega_1) - \frac{1}{4} (m_2 \omega_2)^{-1} d(m_2 \omega_2) = 0. \quad (C.12) \]

Next, observe that

\[ da_1^+ = \frac{1}{2} (m_1 \omega_1)^{-1} d(m_1 \omega_1)a_1, \quad da_2^+ = \frac{1}{2} (m_2 \omega_2)^{-1} d(m_2 \omega_2)a_2. \]

Using these relations and also that \(|n_1, n_2 > = (n_1! n_2!)^{-\frac{1}{2}} (a_1^+)^{n_1} (a_2^+)^{n_2} |0 >,\) we finally find \(< n_1, n_2 | d | n_1, n_2 > = 0.\) Thus we conclude that \(A^{n_1, n_2} = 0\) and so the Berry connection vanishes. This may be due to the fact that the light cone Hamiltonian is associated with a Klein-Gordon equation which is real. The same applies also to the quantum mechanical system associated with the string at level \(n\) (for \(h = 0\)).

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