On the Necessity and Sufficiency of the Zames-Falb Multipliers for Bounded Operators*

Sei Zhen Khong\textsuperscript{a}, Lanlan Su\textsuperscript{b}

\textsuperscript{a}Independent researcher
\textsuperscript{b}School of Engineering, University of Leicester, Leicester, LE1 7RH, UK

Abstract

This paper analyzes the robust feedback stability of a single-input-single-output stable linear time-invariant (LTI) system against four different classes of nonlinear systems using the Zames-Falb multipliers. The contribution is fourfold. Firstly, we present a generalised S-procedure lossless theorem that involves a countably infinite number of quadratic forms. Secondly, we identify a class of uncertain systems over which the robust feedback stability implies the existence of an appropriate Zames-Falb multiplier based on the generalised S-procedure lossless theorem. Meanwhile, we show that the existence of such a Zames-Falb multiplier is sufficient for the robust feedback stability over a smaller class of uncertain systems. Thirdly, when restricted to be static (a.k.a. memoryless), the second class of systems coincides with the class of sloped-restricted monotone nonlinearities, and the classical result of using the Zames-Falb multipliers to ensure feedback stability is recovered. Lastly, when restricted to be LTI, the second class is demonstrated to be a subset of the third, and the existence of a Zames-Falb multiplier is shown to be sufficient but not necessary for the robust feedback stability.

Key words: Zames-Falb multipliers, robust stability, integral quadratic constraints, nonlinear systems, uncertainty

1 Introduction

The robust stability analysis of the feedback interconnection of a single-input-single-output stable linear time-invariant (LTI) system $G$ and a nonlinear system $\Delta$ belonging to a specified uncertainty class, as depicted in Figure 1, is a fundamental object of study in the field of control theory. It is often termed absolute stability analysis in the nonlinear systems literature [21,27]. When the class of $\Delta$'s considered is static (i.e. memoryless) and sector bounded, a variety of multiplier-based methodologies including the renowned circle criterion and Popov criterion [11] have been proposed to establish the input-output feedback stability. For static, time-invariant, and monotone $\Delta$'s, the Zames-Falb multipliers are currently the most general class of time-invariant multipliers known for the study of these feedback systems [2].

The Zames-Falb multipliers were first introduced by O'Shea in [19,20] (see [4] for a survey), and formalized later by Zames and Falb in [28]. Unfortunately, its applicability was limited due to computational constraints. Motivated by the rapid development of computational capability in the 80s, the multiplier-based theory has regained the interests of researchers. Importantly, the seminal work [17] demonstrates that the Zames-Falb multipliers fit nicely to the framework of integral quadratic constrains (IQCs); see also [12]. A recent research line on numerically searching the Zames-Falb multipliers satisfying a mixture of time and frequency domain conditions can be located in ([3,24,25]). The phase limitation of the Zames-Falb multipliers has been studied in ([16,10,26]), thereby providing a interesting...
perspective from which to understand the Zames-Falb multipliers.

It is well known [28] that the robust feedback stability of an LTI system $G$ to static monotone nonlinearities $\Delta$'s with a slope restriction $b$ can be ensured by the existence of an appropriate Zames-Falb multiplier $M(j\omega)$ satisfying

$$\text{Re} \left\{ M(j\omega)(b^{-1} - G(j\omega)) \right\} > 0, \forall \omega \in \mathbb{R} \cup \infty. \quad (1)$$

In [4,26], it is conjectured by Carrasco that if there is no appropriate Zames-Falb multiplier, then the feedback system is not robustly stable. The Carrasco’s conjecture remains unsolved, i.e. it is unclear whether condition (1) is necessary for the robust feedback stability over the class of static monotone $\Delta$’s. Particular discrete-time results on the conjecture can be found in [22,29].

The purpose of this work is to investigate both the necessity of various robust stability conditions over the class of static monotone $\Delta$’s. Full discrete-time results on the conjecture can be found in [22,29]. The purpose of this work is to investigate both the necessity and sufficiency of the Zames-Falb multipliers in the continuous time over different uncertainty classes of $\Delta$’s, with the goal of enhancing our understanding of the conservatism of the Zames-Falb multipliers in the robust stability analysis of the feedback system in Figure 1. The necessity of various robust stability conditions has been studied in control over the years, since the pioneering work on that of the small-gain theorem for LTI systems [6,30]. Related converse results for IQCs can be found in [14,13].

This paper is concerned with the robust stability analysis of the feedback interconnection shown in Figure 1 over four uncertainty classes of $\Delta$’s using the Zames-Falb multipliers. While one might be inclined to yearn for a condition that guarantees the robust feedback stability over as large a class of $\Delta$’s as possible, this is unwise from the perspective of establishing the necessity of the condition. In fact, there is a subtle trade-off between the size of the class of $\Delta$’s and the strictness of the condition on $G$ when it comes to robust stability, the latter of which obviously affects its necessity. Understandably, a larger class of $\Delta$’s leads to a stricter condition on $G$ in order to ensure robust stability. This is well illustrated by our results involving classes of $\Delta$’s of different sizes. Firstly, we identify a set of nonlinear dynamic $\Delta$’s and show that the existence of a Zames-Falb multiplier satisfying condition (1) is implied by the robust stability against this uncertainty set. This is established using the generalised S-procedure lossless theorem that is presented in Section 2. Moreover, we show that the existence of such a Zames-Falb multiplier is sufficient for the robust feedback stability over a smaller class of uncertain systems. Secondly, when this set is restricted to be static, it coincides with the class of static monotone nonlinearities, and the existence of a Zames-Falb multiplier is sufficient for the robust stability whereas its necessity remains unknown. Thirdly, when the same set is restricted to consist of only LTI dynamics, it is shown to be a subset of the class of static monotone nonlinearities, and the existence of a Zames-Falb multiplier is sufficient but not necessary to establish the uniform stability. For the ease of exposition, the main results centered around monotone nonlinearities $\Delta$’s is presented in Section 3, and Section 4 is dedicated to its extension to the general two-sided slope restrictions. Specifically, in Section 4, the sufficiency direction considered in [28] is importantly generalised in order to make the condition consistent with the necessity direction. A crucial part of our results shows that the ‘monotonicity’ in the set of monotone static time-invariant nonlinearities is completely characterizable via the Zames-Falb IQCs. Therefore, while there exists a larger class of IQCs that such nonlinearities satisfy [15], using them does not deliver additional benefits as far as establishing robust feedback stability is concerned.

The remainder of this section sets up the notation and mathematical preliminaries to the rest of the paper. Section 2 generalises the S-procedure lossless theorem from [18] to involve a countably infinite number of quadratic forms. This generalised S-procedure lossless theorem is of independent interest. Some final remarks are described in Section 5.

**Notation and Preliminaries**

Let $\mathbb{R}$ denote the set of real numbers. For a vector $v$, its Euclidean norm is denoted by $|v|$. Given a matrix $M$, the transpose and conjugate transpose are denoted respectively as $M^T$ and $M^*$. We use $\text{Re}\{\lambda\}$ to denote the real part of a complex number $\lambda$.

Define $\mathcal{L}_1(-\infty, \infty) := \{ z : \mathbb{R} \to \mathbb{R} | \|z\|_1 := \int_{-\infty}^{\infty} |z(t)|dt < \infty \}$. We use $\mathcal{L}_1^+(\infty, \infty)$ to denote the set of all $z(t) \in \mathcal{L}_1(-\infty, \infty)$ satisfying $z(t) \geq 0, \forall t$. Given a signal $z(t) \in \mathcal{L}_1(-\infty, \infty)$, denote by $Z(j\omega)$ its Fourier transform and $Z$ the convolution operator whose kernel is $z(t)$. Let $\mathcal{L}_2(-\infty, \infty) := \{ x : \mathbb{R} \to \mathbb{R}^n | \|x\|^2 := \int_{-\infty}^{\infty} |x(t)|^2dt < \infty \}$ and $\mathcal{L}_2[0, \infty) := \{ x \in \mathcal{L}_2(-\infty, \infty) : x(t) = 0 \text{ for } t < 0 \}$. Given $x(t), y(t) \in \mathcal{L}_2(-\infty, \infty)$, their inner product is given by $\langle x, y \rangle := \int_{-\infty}^{\infty} x(t)y(t)dt$. A bounded linear operator $\Pi$ mapping $\mathcal{L}_2(-\infty, \infty)$ into $\mathcal{L}_2(-\infty, \infty)$ is said to be self-adjoint if $\langle x, \Pi y \rangle = \langle y, \Pi x \rangle$ for all $x, y \in \mathcal{L}_2(-\infty, \infty)$. A self-adjoint $\Pi$ is said to be positive if $\langle x, \Pi x \rangle \geq 0$ for all $x \in \mathcal{L}_2(-\infty, \infty)$. The Fourier transform of $x(t) \in \mathcal{L}_2(-\infty, \infty)$ is denoted as $\hat{x}(j\omega)$. For any $x : \mathbb{R} \to \mathbb{R}^n$, define the truncation operator $(P_T x)(t) := x(t)$ for $t \in (-\infty, T]$ and $(P_T x)(t) := 0$ for $t > T$, and the extended $\mathcal{L}_2$-space $\mathcal{L}_2[0, \infty) := \{ x : \mathbb{R} \to \mathbb{R}^n | P_T x \in \mathcal{L}_2[0, \infty) \forall T \in [0, \infty) \}$.

An operator $H : \mathcal{L}_2[0, \infty) \to \mathcal{L}_2[0, \infty)$ is said to be causal if $P_T H P_T = P_T H$ for all $T > 0$ and anti-causal if $(I - P_T)H(I - P_T) = (I - P_T)H$ for all $T > 0$. 


It is said to be static (a.k.a. memoryless) if it is simultaneously causal and anticausal. Let the shift operator \( S_\tau : L_2(0, \infty) \to L_2(0, \infty) \) be defined by \((S_\tau f)(t) = f(t-\tau)\) for \( \tau \in \mathbb{R} \). An operator \( H \) is said to be time-invariant if \( H S_\tau = S_\tau H \) for all \( \tau \in \mathbb{R} \). A causal operator \( H : L_2(0, \infty) \to L_2(0, \infty) \) is said to be bounded if
\[
\|H\| := \sup_{T>0: \|Pr_u\| \neq 0} \frac{\|Pr_Hu\|}{\|Pr_u\|} = \sup_{0 \neq u \in L_2} \frac{\|Hu\|}{\|u\|} < \infty.
\]

For a single-input single-output system \( H \), note that \( H \) is static and time-invariant if there exists \( N : \mathbb{R} \to \mathbb{R} \) such that \((Hv)(t) = N(v(t))\) for all \( t \in [0, \infty) \). In addition, \( H \) is said to be monotone if \( N \) is monotone, i.e. \( x_1 \geq x_2 \) implies \( N(x_1) \geq N(x_2) \). It is odd if \( N(-x) = -N(x) \) for all \( x \in \mathbb{R} \). For notational convenience, we do not distinguish between \( H \) and \( N \) when \( H \) is static and time-invariant.

Denote by \( L_\infty \) the set of transfer functions that are essentially bounded on the imaginary axis. Every element in \( L_\infty \) may be associated with a bounded LTI operator \( H \) mapping from \( L_2(-\infty, \infty) \) into \( L_2(-\infty, \infty) \), and its induced norm is denoted by
\[
\|H\| := \sup_{u \in L_2(-\infty, \infty)} \frac{\|Hu\|}{\|u\|}.
\]

Denote \( \mathcal{RH}_\infty \) as the space of proper real-rational transfer functions with no poles in the closed right half plane. Every element \( G \in \mathcal{RH}_\infty \) is associated with a causal bounded LTI operator \( G : L_2[0, \infty) \to L_2[0, \infty) \), which we do not differentiate for notational convenience [7].

The main object of study in this work is the feedback interconnection of a \( G \in \mathcal{RH}_\infty \) and a causal bounded \( \Delta : L_{2e} \to L_{2e} \), as illustrated in Figure 1. Denote the feedback system as \( [G, \Delta] \).

**Definition 1** \([G, \Delta]\) is said to be well-posed if the map
\[
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} \mapsto \begin{bmatrix} d_1 \\
d_2
\end{bmatrix}
\]
in Figure 1 has a causal inverse on \( L_{2e}[0, \infty) \). It is said to be stable if it is well-posed and the inverse is bounded, in which case \([G, \Delta]\) is also used to denote the map
\[
\begin{bmatrix} d_1 \\
d_2
\end{bmatrix} \mapsto \begin{bmatrix} u_1 \\
u_2
\end{bmatrix}.
\]

For the feedback system \([G, \Delta]\) as shown in Figure 1, its norm is defined as
\[
\| [G, \Delta] \| := \sup_{T>0: \|Pr_d\| \neq 0} \frac{\|Pr_u\|}{\|Pr_d\|},
\]
where \( d := \begin{bmatrix} d_1 \\
d_2
\end{bmatrix} \) and \( u := \begin{bmatrix} u_1 \\
u_2
\end{bmatrix} \).

We define uniform feedback stability as follows.

**Definition 2** The feedback system \([G, \Delta]\) is said to be uniformly stable over \( \Delta \) if \([G, \Delta]\) is stable for all \( \Delta \in \Delta \), and there exists \( \gamma > 0 \) such that
\[
\| [G, \Delta] \| < \gamma.
\]

This work is concerned with the uniform stability analysis of the interconnection system shown in Figure 1 over different classes of \( \Delta \)'s using the Zames-Falb multipliers.

**Definition 3** A pair \( v, w \in L_2[0, \infty) \) is said to satisfy the IQC defined by \( \Pi \in L_\infty \) satisfying \( \Pi(j\omega)^* = \Pi(j\omega) \) for all \( \omega \in \mathbb{R} \) if
\[
\sigma_\Pi(v, w) = \int_{-\infty}^{\infty} \left[ \dot{\hat{v}}(j\omega) \right]^* \Pi(j\omega) \left[ \dot{\hat{w}}(j\omega) \right] d\omega \geq 0.
\]

A bounded causal system \( \Delta \) is said to satisfy the IQC defined by \( \Pi \in L_\infty \), denoted by \( \Delta \in IQC(\Pi) \), if (2) holds for all \( v \in L_2(0, \infty) \) and \( w = \Delta v \).

### 2 S-procedure lossless theorem

In this section, the S-procedure lossless theorem in [18, Theorem 3.1] (see also [9, Theorem 7]) is first generalised to involve a countably infinite number of quadratic forms. This is then applied to generalising [9, Proposition 6] under a countably infinite number of IQCs.

Denote by \( Z_0^+ \) and \( Z^+ \) the sets of nonnegative integers and positive integers, respectively. Define
\[
\ell_\infty := \left\{ v : Z_0^+ \to \mathbb{R} : \|v\|_\infty := \sup_{i \in Z_0^+} |v_i| < \infty \right\}.
\]

Let \((X, \Sigma, \mu)\) be a measure space and \( L_\infty(X, \Sigma, \mu) \) denote the Banach space of essentially bounded measurable functions equipped with the norm
\[
\|u\|_\infty = \inf \{ c > 0 : |u(x)| \leq c \mu\text{-almost everywhere} \}.
\]

Note that one may identify \( \ell_\infty \) with \( L_\infty(Z_0^+, P(Z_0^+), \lambda) \), where \( P(S) \) denotes the power set of \( S \) and \( \lambda \) is the counting measure satisfying \( \lambda(S) = |S| \) is the cardinality of \( S \) if \( S \) is finite and \( \lambda(S) = \infty \) otherwise. Obviously, \( \lambda(S) = 0 \) if and only if \( S = \emptyset \). For notational convenience, we write \( \ell_\infty = L_\infty(Z_0^+, P(Z_0^+), \lambda) \).

Note that one may identify \( \ell_\infty \) with \( L_\infty(Z_0^+, P(Z_0^+), \lambda) \), i.e. we do not differentiate between sequences in \( \ell_\infty \) and functions in \( L_\infty(Z_0^+, P(Z_0^+), \lambda) \).
A bounded, finitely additive signed measure on $P(\mathbb{Z}_0^+)$ is a signed measure $\nu : P(\mathbb{Z}_0^+) \to \mathbb{R}$ satisfying $\nu(\emptyset) = 0$, sup \( \|\nu(A)\| < \infty \), and

$$\nu(A \cup B) = \nu(A) + \nu(B)$$

for all $A, B \in P(\mathbb{Z}_0^+)$ such that $A \cap B = \emptyset$. Denote by $\text{ba}(P(\mathbb{Z}_0^+))$ the set of bounded, finitely additive signed measures on $P(\mathbb{Z}_0^+)$. The dual space of $\ell_\infty = L_\infty(\mathbb{Z}_0^+, P(\mathbb{Z}_0^+), \lambda)$ is $\text{ba}(P(\mathbb{Z}_0^+))$ [23, Theorem 3.1], i.e. for every bounded linear functional $g$ on $\ell_\infty$ there exists $\nu \in \text{ba}(P(\mathbb{Z}_0^+))$ such that

$$g(u) = \int_{\mathbb{Z}_0^+} u \, d\nu \quad \forall u \in \ell_\infty.$$

Define the quadratic forms $\sigma_k : \mathcal{L}_2(0, \infty) \to \mathbb{R}$ as

$$\sigma_k(f) = \langle f, \Pi_k f \rangle, \quad k = 0, 1, \ldots,$$

where $\Pi_k : \mathcal{L}_2(-\infty, \infty) \to \mathcal{L}_2(-\infty, \infty)$. Let $\mathcal{H} \subset \mathcal{L}_2([0, \infty))$ be such that $S_{\tau} \mathcal{H} \subset \mathcal{H}$ for all $\tau > 0$.

**Assumption 4** Assume that

1. $\Pi_k : \mathcal{L}_2(-\infty, \infty) \to \mathcal{L}_2(-\infty, \infty)$ is bounded LTI self-adjoint for all $k \in \mathbb{Z}_0^+$;
2. $\sup_k \|\Pi_k\| < \infty$;
3. for any $f_1, f_2 \in \mathcal{H}$, $\sup_k \|\Pi_k f_1, S_{\tau} f_2\| \to 0$ as $\tau \to \infty$;
4. there exists $f^* \in \mathcal{H}$ and $\epsilon > 0$ such that $\sigma_k(f^*) \geq \epsilon$ for $k = 1, 2, \ldots$.

**Theorem 5** Suppose that Assumption 4 holds. Then the following are equivalent:

1. $\sigma_0(f) \leq 0$ for all $f \in \mathcal{H}$ such that $\sigma_k(f) \geq 0$ for all $k = 1, 2, \ldots$;
2. There exists $\nu \in \text{ba}(P(\mathbb{Z}_0^+))$ such that $\nu(S) \geq 0$ for all $S \in P(\mathbb{Z}_0^+)$ and

$$\sigma_0(f) + \int_{\mathbb{Z}_0^+} \sigma(f) \, d\nu \leq 0$$

for all $f \in \mathcal{H}$, where $\sigma(f) := (\sigma_1(f), \ldots, \sigma_k(f), \ldots) \in \ell_\infty$.

**Proof.** That (ii) implies (i) is obvious. To see that (i) implies (ii), first note that since $\sup_k \|\Pi_k\| < \infty$ by Assumption 4, $(\sigma_0(f), \sigma_1(f), \ldots, \sigma_k(f), \ldots) \in \ell_\infty$ for all $f \in \mathcal{L}_2(0, \infty)$. Define

$$\mathcal{K} = \{ (\sigma_0(f), \sigma_1(f), \ldots, \sigma_k(f), \ldots) \in \ell_\infty : f \in \mathcal{H} \}$$

and

$$\mathcal{N} = \{ (n_0, n_1, \ldots) \in \ell_\infty : \exists \epsilon > 0 \text{ s.t. } n_k > \epsilon \forall k \in \mathbb{Z}_0^+ \}.$$

Note that since every $\Pi_k$ is bounded LTI self-adjoint,

$$\sigma_0(S_{\tau} f_1, f_2), \sigma_1(S_{\tau} f_2, f_2), \ldots, \sigma_k(S_{\tau} f_2, f_2), \ldots) = k_2$$

for all $\tau > 0$. Furthermore, for all $\lambda \in [0, 1]$,

$$\sigma_k(\sqrt{\lambda} f_1 + \sqrt{1 - \lambda} S_{\tau} f_2) = \lambda \sigma_k(f_1) + (1 - \lambda) \sigma_k(f_2) + 2 \sqrt{\lambda(1 - \lambda)}(\langle \Pi_k f_1, S_{\tau} f_2 \rangle).$$

Since $\sup_{k \in \mathbb{Z}_0^+} \|\Pi_k f_1, S_{\tau} f_2\| \to 0$ as $\tau \to \infty$ by Assumption 4, it follows that for all $\epsilon > 0$ there exists sufficiently large $\alpha \in \mathbb{Z}^+$ such that

$$\sup_{k \in \mathbb{Z}_0^+} \|\Pi_k f_1, S_{\tau} f_2\| \leq \epsilon, \forall \tau \geq \alpha.$$

In other words, for all $\epsilon > 0$, there exists $\tau \in \mathbb{Z}_0^+$ such that

$$\left\| (\sigma_0(\sqrt{\lambda} f_1 + \sqrt{1 - \lambda} S_{\tau} f_2), \sigma_1(\sqrt{\lambda} f_1 + \sqrt{1 - \lambda} S_{\tau} f_2), \ldots, \sigma_k(\sqrt{\lambda} f_1 + \sqrt{1 - \lambda} S_{\tau} f_2), \ldots) - (\lambda k_1 + (1 - \lambda) k_2) \right\|_\infty < \epsilon$$

whereby $\lambda k_1 + (1 - \lambda) k_2 \in \mathcal{K}$. That is, $\mathcal{K}$ is convex.

Since $\mathcal{N}$ is open and (i) implies that $\mathcal{K} \cap \mathcal{N} = \emptyset$, it follows from the Hahn-Banach theorem [1, Theorem 1.6] that there exists a closed hyperplane that separates $\mathcal{K}$ and $\mathcal{N}$. That is, there exists $\nu_0 \in \text{ba}(P(\mathbb{Z}_0^+))$ such that

$$\int_{\mathbb{Z}_0^+} n \, d\nu_0 > 0 \quad \forall n \in \mathcal{N} \quad (3)$$

and

$$\int_{\mathbb{Z}_0^+} \kappa \, d\nu_0 \leq 0 \quad \forall \kappa \in \mathcal{K}. \quad (4)$$

Note that (3) being true for all $n \in \mathcal{N}$ implies that $\nu_0$ is a nonnegative measure, i.e., $\nu_0(S) \geq 0$ for all $S \in P(\mathbb{Z}_0^+)$. To see this, suppose for some $S \in P(\mathbb{Z}_0^+)$, $\nu_0(S) < 0$. Let $n \in \mathcal{N}$ be $d > 0$ on $S$ and $1/d$ on $\mathbb{Z}_0^+ \setminus S$. Then since $\nu_0$ is bounded,

$$\int_{\mathbb{Z}_0^+} n \, d\nu_0 = d\nu_0(S) + \frac{1}{d} \nu_0(\mathbb{Z}_0^+ \setminus S),$$

which is negative for sufficiently large $d$. 
Finally, let $\kappa_0 = \sigma_0(f^*)$ and $\kappa_k = \sigma_k(f^*)$ for $k = 0, 1, \ldots$. Note that $\kappa_k \geq \epsilon > 0$ for $k = 1, 2, \ldots$ by Assumption 4. It follows from (4) that

$$\int_{z^+} \kappa \, d\nu_0 = \nu_0(\{0\})\kappa_0 + \int_{z^+} (\kappa_1, \kappa_2, \ldots) \, d\nu_0 \leq 0.$$  

Suppose to the contrapositive that $\nu_0(\{0\}) = 0$, then $\int_{z^+} (\kappa_1, \kappa_2, \ldots) \, d\nu_0 = 0$. Letting $n := (1, \kappa_1, \kappa_2, \ldots)$ then yields $\int_{z^+} n \, d\nu_0 = 0$, which violates (3). Hence, $\nu_0(\{0\}) > 0$. Define $\nu \in \text{ba}(P(Z^+))$ as

$$\nu(S) := \frac{\nu_0(S)}{\nu_0(\{0\})}$$

for all $S \in P(Z^+)$. Dividing (4) by $\nu_0(\{0\})$ and substituting $\nu$ into it then yields (ii), as required. \qed

The necessity condition in Theorem 5 may be simplified when the multipliers $\Pi_k$, $k \in \mathbb{Z}^+$, belong to a certain convex cone.

**Assumption 6** Let $\mathcal{C} \subset \mathcal{L}_\infty$ have the following properties:

- $\mathcal{C}$ takes the form

$$\mathcal{C} = \left\{ \begin{bmatrix} 0 & M \\ M^* & 0 \end{bmatrix} : M \in \mathcal{M} \right\};$$

- there exists $S \subset \mathcal{L}_2(-\infty, \infty)$ such that $M \in \mathcal{M}$ if and only if $\langle f_1, Mf_2 \rangle \geq 0$ for all $(f_1, f_2) \in S$.

**Lemma 7** Suppose that Assumption 4 holds and let $\mathcal{C} \subset \mathcal{L}_\infty$ satisfy Assumption 6. If $\Pi_k \notin \mathcal{C}$ for all $k \in \mathbb{Z}^+$, then $\sigma_0(f) \leq 0$ for all $f \in \mathcal{H}$ such that $\sigma_k(f) \geq 0$ for all $k = 1, 2, \ldots$ implies there exists $\Pi \in \mathcal{C}$ such that

$$\sigma_0(f) + \langle f, \Pi f \rangle \leq 0$$

for all $f \in \mathcal{H}$.

**Proof.** First, by applying Theorem 5, it holds that there exists $\nu \in \text{ba}(P(Z^+))$ such that $\nu(S) \geq 0$ for all $S \in P(Z^+)$ and

$$\sigma_0(f) + \int_{z^+} \sigma(f) \, d\nu \leq 0$$  \hspace{1cm} (5)

for all $f \in \mathcal{H}$, where $\sigma(f) := (\sigma_1(f), \ldots, \sigma_k(f), \ldots) \in \ell_\infty$.

Since $\Pi_k \notin \mathcal{C}$ for all $k \in \mathbb{Z}^+$, it follows from Assumption 6 that there exists $M_k \in \mathcal{M}$ such that

$$\Pi_k = \begin{bmatrix} 0 & M_k \\ M_k^* & 0 \end{bmatrix}.$$  

Now let $F : \mathbb{Z}^+ \rightarrow \mathcal{L}_\infty$ satisfy $F(k) = M_k$ for $k \in \mathbb{Z}^+$. Given $f_2 \in \mathcal{L}_2(-\infty, \infty)$, set

$$H(f_1) := \int_{z^+} \langle f_1, F(\cdot)f_2 \rangle \, d\nu = \int_{z^+} s(f_1, f_2) \, d\nu,$$

where

$$s(f_1, f_2) := (\langle f_1, M_1f_2 \rangle, \ldots, \langle f_1, M_kf_2 \rangle, \ldots) \in \ell_\infty.$$

Let $\alpha := \sup_{k} \|M_k\|$ and since $\nu$ defines a bounded linear functional on $\ell_\infty$ with bound $\gamma > 0$, we have

$$|H(f_1)| \leq \int_{z^+} |\langle f_1, F(\cdot)f_2 \rangle| \, d\nu \leq \alpha \gamma \|f_2\| \|f_1\|.$$

That is, $H$ is bounded. Applying the Riesz representation theorem [5, Theorem A.3.52] yields the existence of a unique $f_F(f_2) \in \mathcal{L}_2(-\infty, \infty)$ such that

$$\langle f_1, f_F(f_2) \rangle = H(f_1) = \int_{z^+} \langle f_1, F(\cdot)f_2 \rangle \, d\nu.$$  

By repeating the arguments above, we also have that there exists a unique $f_F^*(f_1) \in \mathcal{L}_2(-\infty, \infty)$ such that

$$\langle f_2, f_F^*(f_1) \rangle = \int_{z^+} \langle f_2, F(\cdot)^*f_1 \rangle \, d\nu.$$  

Consider now the mapping $f_2 \mapsto f_F(f_2)$. By mimicking the arguments in [5, Lemma A.5.9], we next show that $f_2 \mapsto f_F(f_2)$ is linear and bounded. First note that $f_F(f_2)$ is linear in $f_2$ from the uniqueness of $f_F(f_2)$. We next establish using the closed graph theorem that $f_2 \mapsto f_F(z_2)$ is bounded. If $f_2^n \rightarrow f_2$ and $f_F(f_2^n) \rightarrow h$ (as $n \rightarrow \infty$), then for all $f_1 \in \mathcal{L}_2(-\infty, \infty)$

$$\langle f_1, f_F(f_2^n) \rangle \rightarrow \langle f_1, h \rangle$$  

and

$$\langle f_1, f_F(f_2^n) \rangle = \int_{z^+} \langle f_1, F(\cdot)f_2^n \rangle \, d\nu$$  

$$= \int_{z^+} \langle f_2^n, F(\cdot)^*f_1 \rangle \, d\nu$$  

$$= \langle f_2^n, f_F^*(f_1) \rangle \rightarrow \langle f_2, f_F^*(f_1) \rangle$$  

$$= \int_{z^+} \langle f_2, F(\cdot)^*f_1 \rangle \, d\nu$$  

$$= \langle f_1, f_F(f_2) \rangle.$$  


Thus, $h = f_P(f_2)$ and $f_2 \rightarrow f_P(z_2)$ has a closed graph. By the closed graph theorem [5, Theorem A.3.49], $f_2 \rightarrow f_P(z_2)$ is bounded.

Let $M := f_2 \rightarrow f_P(z_2)$, we then have $M \in \mathcal{L}_\infty$ and

$$\langle f_1, M f_2 \rangle = \int_{\mathbb{Z}^+} s(f_1, f_2) d\nu$$

(6)

for all $f_1, f_2 \in \mathcal{L}_2(-\infty, \infty)$. Suppose to the contrapositive that $M \notin \mathcal{M}$, then there exists, by Assumption 6, $(f_1, f_2) \in \mathcal{S}$ such that $\langle f_1, M f_2 \rangle < 0$ However, this violates (6) because $\langle f_1, M f_2 \rangle \geq 0$ for all $k \in \mathbb{Z}^+$ and $\nu$ is a nonnegative measure. The claim of the theorem then follows by noting that

$$\Pi := \left[ \begin{array}{cc} 0 & M \\ M^* & 0 \end{array} \right] \in \mathcal{C}$$

and substituting (6) into (5).

By exploiting Lemma 7 and mimicking the proof for [9, Proposition 6], one may obtain the following result.

**Theorem 8** Let $\mathcal{C} \subset \mathcal{L}_\infty$ satisfy Assumption 6 and

$$\mathcal{H} := \{ \Pi \in \mathcal{C} : \Pi_k \in \mathcal{C} \ \text{for} \ k = 1, 2, \ldots, \ \text{let} \ \Pi_k \ \text{such that} \ \sup_k \|\Pi_k\| < \infty, \ \sup_k \|\Pi_k f_1, S \mathcal{P}_f_2\| \rightarrow 0 \ \text{as} \ \tau \rightarrow \infty \ \text{for all} \ f_1, f_2 \in \mathcal{H}, \ \text{and there exist} \ (v', w') \in \mathcal{H}, \ \epsilon > 0 \ \text{such that} \ \sigma_{P_k}(v', w') \geq \epsilon \ \text{for all} \ k = 1, 2, \ldots, \ \text{Suppose} \ G, \Delta \ \text{is well-posed for all} \ \Delta \in \mathcal{S}, \ \text{then} \ G, \Delta \ \text{is uniformly stable over} \ \mathcal{S} \ \text{only if there exists} \ \Pi \in \mathcal{C} \ \text{such that} \ \text{for all} \ \omega \in \mathbb{R},$$

$$\left[ \begin{array}{c} G(j\omega) \\ 1 \end{array} \right]^* \Pi(j\omega) \left[ \begin{array}{c} G(j\omega) \\ 1 \end{array} \right] \leq -1.$$  

Proof. $G, \Delta$ in Figure 1 is uniformly stable over $\mathcal{S}$ implies that when $d_1 = 0$ (i.e., $u_1 = y_2$), there exists $\gamma > 0$ such that

$$\sup_{T > 0; \|F_T y_2\| \neq 0} \|F_T y_2\| < \gamma.$$ 

Let

$$\sigma_0(u_2, y_2) := ||y_2||^2 + ||Gy_2||^2 - \gamma ||u_2 - Gy_2||^2;$$

$$\sigma_k(u_2, y_2) := \sigma_{P_k}(u_2, y_2).$$

The uniform stability of $[G, \Delta]$ over $\mathcal{S}$ means that $\sigma_0(u_2, y_2) \leq 0$ for all $(u_2, y_2) \in \mathcal{H}$ such that $\sigma_k(u_2, y_2) \geq 0, k = 1, 2, \ldots$. By Lemma 7, this implies that there exists $\Pi \in \mathcal{C}$ such that

$$\sigma_0(u_2, y_2) + \left( \begin{array}{c} u_2 \\ y_2 \end{array} \right) \Pi \left( \begin{array}{c} u_2 \\ y_2 \end{array} \right) \leq 0$$

for all $(u_2, y_2) \in \mathcal{H}$. Setting $u_2 = Gy_2$ and applying [9, Proposition 4] then yields (7). In particular, note that since $G \in \mathcal{R}_\mathcal{H}_\infty$,

$$\left[ \begin{array}{c} Gy_2 \\ y_2 \end{array} \right] \in \mathcal{H},$$

as there always exist bounded causal (nonlinear) $(\Delta, \epsilon)$ for which $\|\Delta, Gy_2 - y\| \rightarrow 0$ given any $y \in \mathcal{L}_2(0, \infty)$. To see this, it suffices to consider $G(j\omega) \neq 0$ and let $\Delta(Gy_2) := y_2$. Furthermore, for all $u_2 \in \mathcal{L}_2(0, \infty)$ such that $u_2(t) = (Gy_2)(t)$ on some interval $[0, T)$ and $u(t) \neq (Gy_2)(t)$ on $[T, T + \epsilon)$ for some $\epsilon > 0$, let $(\Delta u_2)(t) := y_2(t)$ on $[0, T)$ and $(\Delta u)(t) := 0$ otherwise. Repeating the above procedure and setting to 0 the output of $\Delta$ to all input $u(t) = Gy_2(t)$ on any interval $[0, T)$, it may be verified that such a $\Delta$ is bounded and causal.

3 Robust stability against monotone nonlinearity

Let $\Delta_0$ consist of all causal bounded system $\Delta$ that maps $0$ into $0$ and satisfies $x(t), (\Delta x)(t) \geq 0, \forall x \in \mathcal{L}_2(0, \infty)$. For $\beta \in (1, \infty)$, consider the inequality

$$\int_0^\infty x(t + \tau)(\Delta x)(t) dt \leq \int_0^\infty x(t)(\Delta x)(t) dt,$$

$$\forall \tau \in (-\beta, -1/\beta) \cup (1/\beta, \beta), x \in \mathcal{L}_2(0, \infty)$$

and define

$$\Delta_\beta := \{ \Delta \in \Delta_0 \ | \ \Delta \text{ satisfies (8)} \}.  \tag{9}$$

Specifically, $\Delta_\infty$ consists of $\Delta \in \Delta_0$ that satisfies (8) for all $\tau \in \mathbb{R}$.

3.1 Nonlinear dynamic uncertainty

Denote by $\mathcal{Z}$ the convolution operator whose kernel is $z \in \mathcal{L}_1(-\infty, \infty)$. Let $\mathcal{L}_1(\beta)$ (respectively, $\mathcal{L}_1^+(-\infty, \infty)$) be the subspace of $\mathcal{L}_1(-\infty, \infty)$ (respectively, $\mathcal{L}_1^+(-\infty, \infty)$) which consists of all $z(t) \in \mathcal{L}_1(-\infty, \infty)$ (respectively, $z(t) \in \mathcal{L}_1^+(-\infty, \infty)$).
The following lemma characterizes the input-output relations of all elements in the set $\Delta_3$ by a class of IQCs defined by the Zames-Falb multipliers.

**Lemma 9** Given a $\Delta \in \Delta_0$, $\Delta$ satisfies (8) if and only if
\[
\langle x(\cdot), (1 - Z)(\Delta x)(\cdot) \rangle \geq 0, \forall x \in \mathcal{L}_2[0, \infty)
\] (10)
for all $z \in \mathcal{L}_1^+(\beta)$ such that $\|z\|_1 \leq 1$.

**PROOF.** "$\Rightarrow$" First note that $\int_0^\infty x(t)(\Delta x)(t)dt \geq 0$ for all $x \in \mathcal{L}_2[0, \infty)$ as $\Delta \in \Delta_0$. Since $x(\cdot)$ is in $\mathcal{L}_2[0, \infty)$, we have
\[
\langle x(\cdot), Z(\Delta x)(\cdot) \rangle = \int_{-\infty}^\infty z(\tau) \int_0^\infty x(t)(\Delta x)(t - \tau)dt d\tau.
\]
It follows from (8), $z(t) \geq 0$, and $z(t) = 0$ for all $t \notin \{(-\beta, -1/\beta) \cup (1/\beta, \beta)\}$ that
\[
\int_{-\infty}^\infty z(\tau) \int_0^\infty x(t)(\Delta x)(t - \tau)dt d\tau \leq \int_{-\infty}^\infty z(\tau) \int_0^\infty x(t)(\Delta x)(t)dt d\tau.
\]
Since $\|z\|_1 \leq 1$ and $\int_0^\infty x(t)(\Delta x)(t)dt \geq 0$, it follows that
\[
\int_{-\infty}^\infty z(\tau) \int_0^\infty x(t)(\Delta x)(t)dt d\tau \leq \int_0^\infty x(t)(\Delta x)(t)dt.
\]
Combining the two inequalities above yields
\[
\langle x(\cdot), Z(\Delta x)(\cdot) \rangle \leq \langle x(\cdot), (\Delta x)(\cdot) \rangle.
\]
Hence, (10) is shown.

"$\Leftarrow$" Given $\epsilon > 0$ and $\bar{\tau} \in \{(-\beta, -1/\beta) \cup (1/\beta, \beta)\}$, let $z(t) := 1/\epsilon$ if $t \in [\bar{\tau} - \frac{\epsilon}{2}, \bar{\tau} + \frac{\epsilon}{2}]$ and $z(t) := 0$ otherwise, whereby $\|z\|_1 = 1$. Then inequality (10) gives
\[
\int_{-\infty}^\infty z(\tau) \int_0^\infty x(t)(\Delta x)(t - \tau)dt d\tau \leq \int_0^\infty x(t)(\Delta x)(t)dt.
\]
Since $\int_0^\infty x(t)(\Delta x)(t - \tau)dt = \lim_{\tau \to 0} \int_{-\infty}^\infty z(\tau) \int_0^\infty x(t)(\Delta x)(t - \tau)dt d\tau$ and $\bar{\tau}$ can be arbitrary number in $\{(-\beta, -1/\beta) \cup (1/\beta, \beta)\}$, this implies that $\Delta$ satisfies (8).

Next, consider the inequality
\[
\int_0^\infty x(t + \tau)(\Delta x)(t)dt \leq \int_0^\infty x(t)(\Delta x)(t)dt, \forall \tau \in (-\beta, -1/\beta) \cup (1/\beta, \beta), \forall x \in \mathcal{L}_2[0, \infty)
\] (11)
and define
\[
\bar{\Delta}_\beta := \{\Delta \in \Delta_0 \mid \Delta \text{satisfies (11)}\}.
\] (12)
Specifically, $\bar{\Delta}_\beta$ consists of $\Delta \in \Delta_0$ that satisfies (11) for all $\tau \in \mathbb{R}$. Evidently $\bar{\Delta}_\beta$ is a subset of $\Delta_\beta$. The next lemma shows that as far as $\bar{\Delta}_\beta$ is concerned, a more restrictive necessary and sufficient condition than the one presented in Lemma 9 can be provided.

**Lemma 10** Given a $\Delta \in \Delta_0$, $\Delta$ satisfies (11) if and only if (10) holds for all $z \in \mathcal{L}_1(\beta)$ such that $\|z\|_1 \leq 1$.

**PROOF.** Sufficiency can be proved by following the same argument of the proof for Lemma 9 with $\pm z(t)$ constructed therein.

To prove necessity, observe that for any $z \in \mathcal{L}_1(\beta)$,
\[
\langle x(\cdot), Z(\Delta x)(\cdot) \rangle \leq \int_{-\infty}^\infty |z(\tau)| \int_0^\infty x(t)(\Delta x)(t - \tau)dt d\tau.
\]
It then follows from (11) that
\[
\langle x(\cdot), Z(\Delta x)(\cdot) \rangle \leq \int_{-\infty}^\infty |z(\tau)| \int_0^\infty x(t)(\Delta x)(t)dt d\tau.
\]
Since $\|z\|_1 \leq 1$ and $\int_0^\infty x(t)(\Delta x)(t)dt \geq 0$, it follows that
\[
\langle x(\cdot), Z(\Delta x)(\cdot) \rangle \leq \langle x(\cdot), (\Delta x)(\cdot) \rangle.
\]
Hence, (10) is shown.

The main result showing the necessary and sufficient condition for the existence of an appropriate Zames-Falb multiplier is presented below.

**Theorem 11** Consider Figure 1 with $G \in \mathcal{RH}_\infty$. Suppose $[G, \Delta]$ is well-posed for all $\Delta \in \Delta_\infty$, then $[G, \Delta]$ is uniformly stable over $\Delta_\infty$ if
\[
\exists z \in \mathcal{L}_1^+(-\infty, \infty), \epsilon > 0 \text{ s.t. } \|z\|_1 \leq 1 \text{ and } \Re \{(1 - Z(j\omega))(-G(j\omega))\} \geq \epsilon, \forall \omega \in \mathbb{R}.
\] (13)
Moreover, $[G, \Delta]$ is uniformly stable over $\Delta_\beta$ for any $\beta \in (1, \infty)$ only if (13) holds.
\textbf{PROOF.} \(\Leftarrow\) Letting \(\beta = \infty\), it is shown by the necessity direction of Lemma 9 that for all \(\Delta \in \Delta_\infty\),
\[
\langle u_2(\cdot), (1 - Z)\Delta u_2(\cdot) \rangle \geq 0, \forall u_2 \in \mathcal{L}_2[0, \infty)
\] (14)
for all \(z \in \mathcal{L}^+(-\infty, \infty)\) such that \(\|z\|_1 \leq 1\). Denote \(\Delta u_2 = y_2\), and note that condition (14) is equivalent to \((u_2, y_2)\) satisfying the IQC
\[
\int_{-\infty}^{\infty} \begin{bmatrix} \hat{y}_2(j\omega) \\ \hat{y}_2(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{y}_2(j\omega) \\ \hat{y}_2(j\omega) \end{bmatrix} d\omega \geq 0, \forall u_2 \in \mathcal{L}_2[0, \infty)
\] (15)
where the Zames-Falb multiplier \(\Pi\) is given by
\[
\Pi(j\omega) = \begin{bmatrix} 0 & 1 - Z(j\omega)^* \\ 1 - Z(j\omega) & 0 \end{bmatrix}.
\] (16)
Observe that (13) can be rewritten as
\[
\begin{bmatrix} G(j\omega)^* \\ 1 \end{bmatrix} \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ 1 \end{bmatrix} \leq -2\epsilon, \forall \omega \in \mathbb{R}.
\] (17)
Noting that \([G, \Delta]\) is well-posed for all \(\Delta \in \Delta_\infty\) and \(\Pi\) is anti-diagonal, the robust stability of \([G, \Delta]\) over \(\Delta \in \Delta_\infty\) follows from [17, Theorem 1].

To show the uniform stability, define the graphs of \(G\) and \(\Delta\) as, respectively,
\[
\mathcal{G}(G) := \left\{ \begin{bmatrix} y_1 & u_1 \end{bmatrix}^T \mid y_1 = Gu_1, \forall u_1 \in \mathcal{L}_2[0, \infty) \right\}
\]
and
\[
\mathcal{G}(\Delta) := \left\{ \begin{bmatrix} u_2 & y_2 \end{bmatrix}^T \mid y_2 = \Delta u_2, \forall u_2 \in \mathcal{L}_2[0, \infty) \right\}.
\]
Since \(G \in \mathcal{RH}_\infty\), it follows that \(\langle \nu_2, \Pi \nu_2 \rangle \geq 0, \forall \nu_2 \in \mathcal{G}(\Delta)\) and \(\langle \nu_1, \Pi \nu_1 \rangle \leq -\epsilon\|\nu_1\|^2, \forall \nu_1 \in \mathcal{G}(G)\) for some \(\epsilon > 0\). Summing gives
\[
\epsilon\|\nu_1\|^2 \leq \langle \nu_2, \Pi \nu_2 \rangle - \langle \nu_1, \Pi \nu_1 \rangle
= \langle \nu_2 - \nu_1, \Pi \nu_2 \rangle + \langle \nu_1, \Pi \nu_2 - \Pi \nu_1 \rangle + \langle \nu_2 - \nu_1, \Pi \nu_1 \rangle
\leq \|\Pi\| \cdot \|\nu_2 - \nu_1\|^2 + 2\|\Pi\| \cdot \|\nu_2 - \nu_1\| \cdot \|\nu_1\|
\leq \|\Pi\| \cdot \|\nu_2 - \nu_1\|^2 + \frac{2\|\Pi\|^2 \cdot \|\nu_2 - \nu_1\|^2}{\epsilon} + \frac{\epsilon}{2} \|\nu_1\|^2.
\]
It follows that
\[
\frac{\epsilon}{2} \|\nu_1\|^2 \leq \left( \|\Pi\| + \frac{2\|\Pi\|^2}{\epsilon} \right) \|\nu_2 - \nu_1\|^2.
\]
Moreover, \(\|\nu_2\|^2 = \|\nu_2 - \nu_1 + \nu_1\|^2 \leq (\|\nu_2 - \nu_1\| + \|\nu_1\|)^2 \leq 2\|\nu_2 - \nu_1\|^2 + \|\nu_1\|^2\). Combining the above inequalities yields the existence of \(\gamma > 0\) such that
\[
\gamma \left( \|\nu_1\|^2 + \|\nu_2\|^2 \right) \leq \|\nu_2 - \nu_1\|^2.
\]
Since \(\nu_1 \in \mathcal{G}(G), \nu_2 \in \mathcal{G}(\Delta),\) and \([G, \Delta]\) is robustly stable over \(\Delta \in \Delta_\infty\), the above inequality leads to
\[
\gamma \left( \|y_1\|^2 + \|u_1\|^2 + \|u_2\|^2 + \|y_2\|^2 \right) \leq \|d_1\|^2 + \|d_2\|^2
\]
for all \(d_1, d_2 \in \mathcal{L}_2[0, \infty), \forall \Delta \in \Delta_\infty\).

Therefore, \(\mathcal{G}(G), \mathcal{G}(\Delta), \mathcal{G}(\Delta)\) is uniformly stable over \(\Delta_\infty\).

\(\Rightarrow\) By defining \(v(t) = w(t) = 1\) for \(t \in [0, 1]\) and \(v(t) = w(t) = 0\) otherwise, we first show that there exists \(\epsilon > 0\) such that \(\langle v(\cdot), (1 - Z)w(\cdot) \rangle > \epsilon \) for all \(z \in \mathcal{L}_1(\beta)\) satisfying \(\|z\|_1 \leq 1\). Observe that \(\langle v(\cdot), w(\cdot) \rangle = \int_{-\infty}^{\infty} v(t)w(t - \tau)d\tau = \int_{-\infty}^{\infty} z(\tau)(1 - |\tau|)d\tau\). Since \(z(t) = 0\) for all \(t \notin \{(-1, 1) \cup (1, 1)\}\), it follows that \(\int_{-\infty}^{\infty} z(\tau)(1 - |\tau|)d\tau = \int_{-1}^{1} z(\tau)((1 - |\tau|)d\tau + \int_{1}^{\|z\|_1} z(\tau)(1 - |\tau|)d\tau \leq (1 - 1) \int_{-1}^{1} z(\tau)d\tau\). Together with \(\|z\|_1 \leq 1\), we have \(\int_{-1}^{1} z(\tau)(1 - |\tau|)d\tau \leq 1 - \|z\|_1\). Therefore, we obtain that \(\langle v(\cdot), (1 - Z)w(\cdot) \rangle > \beta > 0\).

Let \(z = 0\) and \(\{z_k\}_{k=1}^{\infty}\) be a normalized Schauder basis for \(\mathcal{L}_1(\beta)\) consisting of non-negative functions \(\{\bar{z}_k\}_{k=1}^{\infty}\). That is, \(\|z_k\|_1 = 1, k = 2, 3, \ldots\) and for all \(z \in \mathcal{L}_1^+(\beta)\), there exist \(\tau_k \geq 0, k = 2, 3, \ldots\) for which \(z = \sum_{k=1}^{\infty} \tau_k z_k\). Define
\[
\Pi_k(j\omega) := \begin{bmatrix} 0 & 1 - Z_k(j\omega)^* \\ 1 - Z_k(j\omega) & 0 \end{bmatrix}, k \in \mathbb{Z}_+.
\]
Note that \(\Pi_k, k \in \mathbb{Z}_+\) is bounded LTI and self-adjoint. It is clear that \(\sup_{k \in \mathbb{Z}_+} \|\Pi_k\| < \infty\). Moreover, for any \(f_1, f_2 \in \mathcal{L}_2[0, \infty), \sup_{k \in \mathbb{Z}_+} |\langle \Pi_k f_1, S f_2 \rangle| \to 0\) as \(\tau \to \infty\) since \(\beta < \infty\). Hence, Assumption 4 holds for \(\Pi_k, k \in \mathbb{Z}_+\).

By Lemma 9, every element in \(\Delta_\beta\) satisfies the IQCs defined by \(\Pi_k\)’s for all \(k \in \mathbb{Z}_+\). Let \(\mathcal{C} \subset \mathcal{L}_\infty\) be of the form:
\[
\mathcal{C} := \left\{ \begin{bmatrix} 0 & 1 - Z(j\omega)^* \\ 1 - Z(j\omega) & 0 \end{bmatrix} : \alpha \geq 0, \right. \left. z \in \mathcal{L}_1^+(-\infty, \infty), \|z\|_1 \leq 1 \right\}.
\]
Evidently, \(\Pi_k \in \mathcal{C}, \forall k \in \mathbb{Z}_+\). By [15, Theorem 1], \(\langle v, Mw \rangle \geq 0\) for all \((v, w) \in \{\langle v, w \rangle \in \mathcal{L}_2(-\infty, \infty) : \|

Given a static and time-invariant $\Delta \in \Delta_0$ satisfying the IQCs defined by $\Pi_k$ for $k = 1, 2, \ldots$ implies by Theorem 8 that there exists $\Pi \in \mathcal{C}$ such that for all $\omega \in \mathbb{R}$,

$$
\begin{bmatrix}
G(j\omega) \\
1
\end{bmatrix}^* \Pi(j\omega) \begin{bmatrix}
G(j\omega) \\
1
\end{bmatrix} \leq -1.
$$

This is equivalent to (13).

**Theorem 12** Consider Figure 1 with $G \in \mathcal{RH}_\infty$. Suppose $[G, \Delta]$ is well-posed for all $\Delta \in \Delta_\infty$, then $[G, \Delta]$ is uniformly stable over $\Delta_\infty$ if

$$\exists z \in \mathcal{L}_1(-\infty, \infty), \epsilon > 0 \text{ s.t. } \|z\|_1 \leq 1 \text{ and } \Re \{(1 - Z(j\omega))(-G(j\omega))\} \geq \epsilon, \forall \omega \in \mathbb{R}. \tag{18}
$$

Moreover, $[G, \Delta]$ is uniformly stable over $\Delta_\beta$ for any $\beta \in (1, \infty)$ only if (18) holds.

**PROOF.** “$\Leftarrow$” Sufficiency follows by similar arguments in the sufficiency proof of Theorem 11.

“$\Rightarrow$” Let $z_1 = 0$ and $\{z_i\}_{i=2}^\infty$ be such that $\|z_i\|_1 = 1$ and for all $z \in \mathcal{L}_1(\beta)$, there exist $\tau_i \geq 0$ for which $z = \sum_{i=1}^\infty \tau_i z_i$ [8]. This can be obtained by taking a union of a normalized Schauder basis for $\mathcal{L}_1(\beta)$ consisting of non-negative functions and the negatives of its elements, with the functions with flipped signs ordered next to each other. One can then readily establish necessity by applying Lemma 10 and the same arguments as those in the necessity proof of Theorem 11.

### 3.2 Static uncertainty

Next, we consider the class of static (i.e., memoryless) $\Delta$’s in the set $\Delta_\infty$. The following lemma explains the links of the set $\Delta_\infty$ to the type of static nonlinearity that is frequently encountered in the control literature.

**Lemma 13** Given a static and time-invariant $\Delta \in \Delta_0$, $\Delta \in \Delta_\infty$ if and only if $\Delta$ is monotone nondecreasing.

**PROOF.** “$\Leftarrow$” Since $\Delta$ is assumed to be static time-invariant and bounded, there exists a constant $C > 0$ such that for all $x \in \mathbb{R}$, $|\Delta(x)| \leq Cx$. Consequently, sufficiency follows [28, Lemma 8].

“$\Rightarrow$” We prove necessity by contraposition. Given a static and time-invariant $\Delta \in \Delta_0$ that satisfies (8), suppose to the contrapositive that there exist $x_1, x_2 \in \mathbb{R}$ such that $x_1 < x_2$ and $\Delta(x_1) > \Delta(x_2)$, i.e., $\Delta$ is not monotone nondecreasing. It can be easily verified that

$$x_2 \Delta(x_1) + x_1 \Delta(x_2) > x_1 \Delta(x_1) + x_2 \Delta(x_2).$$

Define $x(t) \in \mathcal{L}_2[0, \infty)$ as

$$x(t) \begin{cases}
x_1 & \text{if } t \in [0, 1) \cup \cdots \cup [2L, 2L + 1) \\
x_2 & \text{if } t \in [1, 2) \cup \cdots \cup [2L - 1, 2L) \\
-2x_2 \Delta(x_2) & \text{if } t \in [2L + 1, 2L + 2) \\
0 & \text{otherwise}
\end{cases} \tag{19}
$$

for some $L \in \mathbb{Z}^+$. Then it follows that

$$x(t)\Delta(x(t + 1)) - x(t)\Delta(x(t)) = x_1 \Delta(x_2) - x_1 \Delta(x_1) \text{ if } t \in [0, 1) \cup \cdots \cup [2L, 2L + 1)
$$

$$= x_2 \Delta(x_1) - x_2 \Delta(x_2) \text{ if } t \in [1, 2) \cup \cdots \cup [2L - 1, 2L)
$$

$$-2x_2 \Delta(x_2) \text{ if } t \in [2L + 1, 2L + 2)
$$

When $L$ is chosen to be sufficiently large, we have that

$$\int_0^\infty x(t)\Delta(x(t + 1))dt > \int_0^\infty x(t)\Delta(x(t))dt.
$$

Hence, from (9) we have $\Delta \notin \Delta_\infty$, which completes the proof.

Lemmas 9 and 13 demonstrate that the Zames-Falb IQCs used in this paper completely characterize the ‘monotonicity’ property in the set of monotonic static time-invariant nonlinearities. Whereas there exist richer classes of IQCs, defined by multipliers consisting of convolutional impulses as in [28] or linear time-varying multipliers [15], that such nonlinearities satisfy, these IQCs do not provide additional benefits from a theoretical perspective in that they do not capture any extra structures in the uncertainty. We caution that for computational or mathematical convenience, such IQCs may still be used in practice.

**Lemma 14** Given a static and time-invariant $\Delta \in \Delta_0$, $\Delta \in \Delta_\infty$ if and only if $\Delta$ is odd almost everywhere (a.e.) and monotone nondecreasing.

**PROOF.** “$\Leftarrow$” The sufficiency can be proved by a similar argument in [28, Lemma 8].

“$\Rightarrow$” Given a static and time-invariant $\Delta \in \Delta_0$ that belongs to $\Delta_\infty$, obviously $\Delta$ belongs also to $\Delta_\infty$. According to Lemma 13, one has that $\Delta$ is monotone nondecreasing. Moreover, since a monotone function can only
have countable jump discontinuities, \( \Delta \) is continuous a.e.. What remains to be shown is that when \( \Delta \) is static, \( \Delta \in \Delta_\infty \) only if it is odd a.e.. To this end, let \( x_1, x_2 \) be any scalars in \( \mathbb{R} \) such that \( \Delta \) is continuous at \( x_1 \) and \( x_2 > x_1 > 0 \). Since \( \Delta(0) = 0 \), it follows from Lemma 13 that \( \Delta(x_2) \geq \Delta(x_1) \geq 0 \) and \( \Delta(-x_2) \leq \Delta(-x_1) \leq 0 \).

Now construct a signal \( x(t) \in L_2[0, \infty) \) as

\[
x(t) = \begin{cases} 
    x_1 & \text{if } t \in [0, 1) \cup \cdots \cup [2L, 2L+1) \\
    -x_2 & \text{if } t \in [1, 2) \cup \cdots \cup [2L+1, 2L+2) \\
    0 & \text{otherwise} 
\end{cases}
\]

for some \( L \in \mathbb{Z}^+ \). Then it follows that

\[
x(t)\Delta(x(t+1)) + x(t)\Delta(x(t)) = \begin{cases} 
    x_1\Delta(-x_2) + x_1\Delta(x_1) & \text{if } t \in [0, 1) \cup \cdots \cup [2L, 2L+1) \\
    -x_2\Delta(x_1) - x_2\Delta(-x_2) & \text{if } t \in [1, 2) \cup \cdots \cup [2L-1, 2L) \\
    -x_2\Delta(-x_2) & \text{if } t \in [2L+1, 2L+2) \\
    0 & \text{otherwise} 
\end{cases}
\]

Since \( x(t)\Delta(x(t+1)) \leq 0 \) for all \( t \geq 0 \), it can be inferred from \( \Delta \in \Delta_\infty \) that

\[
\int_0^\infty x(t)\Delta(x(t+1))dt \geq -\int_0^\infty x(t)\Delta(x(t))dt,
\]

which by (21) with sufficiently large \( L \) implies that

\[
x_1\Delta(-x_2) + x_1\Delta(x_1) - x_2\Delta(x_1) - x_2\Delta(-x_2)
= (x_2-x_1)(-\Delta(x_1) - \Delta(-x_2)) \geq 0.
\]

Hence, one has \( \Delta(-x_2) \leq -\Delta(x_1) \).

If we construct the signal \( x(t) \) differently as

\[
x(t) = \begin{cases} 
    x_2 & \text{if } t \in [0, 1) \cup \cdots \cup [2L, 2L+1) \\
    -x_1 & \text{if } t \in [1, 2) \cup \cdots \cup [2L+1, 2L+2) \\
    0 & \text{otherwise} 
\end{cases}
\]

then by a similar deduction, it can be verified that \( \Delta(x_2) \geq -\Delta(-x_1) \). Since \( \Delta \) is continuous at \( x_1 \), taking the limit \( x_2 \to x_1^+ \) yields \( \Delta(-x_2) \to \Delta(-x_1) \) and \( \Delta(x_2) \to \Delta(x_1) \). Together with the two inequalities obtained, it can be concluded that \( \Delta(x_1) = -\Delta(-x_1) \), as required. \( \square \)

Now let us define

\[
\Delta_{\text{static}} := \{ \Delta \in \Delta_0 \mid \Delta \text{ is a monotone nondecreasing function} \}.
\]

It follows from Lemma 13 that \( \Delta_{\text{static}} = \{ \Delta \in \Delta_\infty \mid \Delta \text{ is static and time-invariant} \} \). Therefore, it is clear that \( \Delta_{\text{static}} \subset \Delta_\infty \), and the following result follows directly from Theorem 11.

**Corollary 15** Consider Figure 1 where \( G \in RH_\infty \) and \( \Delta \in \Delta_{\text{static}} \). Then, \( [G, \Delta] \) is uniformly stable over \( \Delta_{\text{static}} \) if (13) holds.

Analogously, define

\[
\tilde{\Delta}_{\text{static}} := \{ \Delta \in \Delta_{\text{static}} \mid \Delta \text{ is odd a.e.} \}.
\]

By Lemma 14,

\[
\tilde{\Delta}_{\text{static}} = \{ \Delta \in \tilde{\Delta}_\infty \mid \Delta \text{ is static and time-invariant} \}.
\]

Hence, the following result follows from Theorem 12 as \( \Delta_{\text{static}} \subset \tilde{\Delta}_{\text{static}} \subset \Delta_\infty \).

**Corollary 16** Consider Figure 1 where \( G \in RH_\infty \) and \( \Delta \in \Delta_{\text{static}} \). Then, \( [G, \Delta] \) is uniformly stable over \( \Delta_{\text{static}} \) if (18) holds.

**Remark 17** The preceding corollaries derive from Theorems 11 and 12 the classical results of applying the Zames-Falb multipliers to establishing stability of the feedback interconnection depicted in Figure 1 with \( G \in RH_\infty \) and \( \Delta \) being a static time-invariant monotone nonlinearity. As \( \Delta_{\text{static}} \) is a subset of \( \Delta_\beta \) (respectively, \( \tilde{\Delta}_{\text{static}} \subset \tilde{\Delta}_\beta \)), it is unknown if condition (13) is necessary for uniform stability when the class of \( \Delta \)’s is restricted to be memoryless.

In next subsection, we will show that when the set of \( \Delta \)’s is further confined, condition (13) is sufficient but not necessary.

### 3.3 LTI uncertainty

We consider here the case where \( \Delta \)’s are LTI systems.

**Lemma 18** Given an LTI \( \Delta \in \Delta_0 \), \( \Delta \in \Delta_\infty \) if and only if \( \Delta \) is a nonnegative constant.

**Proof.** “\( \Leftarrow \)” When \( \Delta \) is a nonnegative constant, it is clear that \( \Delta \) is also monotone nondecreasing. Then, it follows from Lemma 13 that \( \Delta \in \Delta_\infty \).

“\( \Rightarrow \)” Since \( \Delta \) is a bounded causal LTI system on \( L_2[0, \infty) \), it admits a transfer function representation \( \hat{\Delta}(-) \). By the Plancherel theorem, the inequality in (8)
can be expressed in the frequency domain as
\[
\text{Re}\left\{ \int_{-\infty}^{\infty} \Delta(j\omega) \Delta(j\omega) d\omega \right\}
\leq \text{Re}\left\{ \int_{-\infty}^{\infty} \hat{\Delta}(j\omega) \hat{\Delta}(j\omega) d\omega \right\}
\forall \tau \in \mathbb{R}, \forall x \in L_2[0, \infty),
\]
which implies for all \( \tau \in \mathbb{R}, \omega \in \mathbb{R} \)
\[
\text{Re}\left\{ \hat{\Delta}(j\omega)(1 + e^{j\omega\tau}) \right\} \geq 0. \quad (23)
\]
As the phase of \( (1 + e^{j\omega\tau}) \) can take any value in \(-\frac{\pi}{2}, \frac{\pi}{2}\) at \( \omega \neq 0 \), \( \hat{\Delta}(j\omega) \) would need to be zero or have zero phase in order to preserve the nonnegativity in (23). In other words, \( \Delta \) has to be a nonnegative constant. \( \Box \)

Define \( \Delta_{\text{LTI}} := \{ \Delta \in \Delta_{\infty} \mid \Delta \text{ is LTI} \} \).
By Lemma 18, \( \Delta_{\text{LTI}} = \{ \Delta \in \mathbb{R} \mid \Delta \geq 0 \} \).

**Theorem 19** Consider Figure 1 with \( G \in \mathcal{RH}_{\infty} \) and \( \Delta \in \Delta_{\text{LTI}}, \) and suppose \( [G, \Delta] \) is well-posed for all \( \Delta \in \Delta_{\text{LTI}}. \) Then, \([G, \Delta] \) is uniformly stable over \( \Delta_{\text{LTI}} \) if (18) holds. Moreover, the converse is not true.

**Proof.** The first part follows from Corollary 16 since \( \Delta_{\text{LTI}} \subset \Delta_{\text{static}} \) by Lemma 18. To prove that the converse is not true, consider the following O'Shea counterexample adapted from [26, (42)]. Let \( H(s) := -\frac{s^2}{(s^2 + 2s + 1)^2} \) and \( G := H - \varepsilon \in \mathcal{RH}_{\infty}, \) where \( \xi \in (0, 0.25) \) and \( \varepsilon > 0 \) is sufficiently small. Since the Nyquist plot of \( G \) does not intersect with the nonnegative real line, it follows from the Nyquist stability theorem that \([G, \Delta] \) is uniformly stable over all nonnegative constant \( \Delta. \) By applying the results about the phase limitation on the Zames-Falb multipliers in [26, Section III-C], it follows that there exists no \( z \in \mathcal{L}_1 \) such that \( \|z\|_1 \leq 1 \) and (13) holds true. \( \Box \)

The results presented in this section are concluded in Figure 2. As shown in the diagram, there is a subtle trade-off between the size of the class of \( \Delta \)'s and the strictness of the condition on \( G \) under which the robust feedback stability is guaranteed.

4 Extensions to Slope-Restricted Uncertainty

In this section, we continue investigating the uniform stability of the feedback system in Figure 1 with a smaller set of \( \Delta \)'s by taking into account an additional ‘slope’ restriction. The restriction is parameterized by the pair \((a, b)\) with \(0 \leq a < b \leq \infty.\)

4.1 Nonlinear dynamic uncertainty

Let \( y(t) = \langle \Delta x \rangle(t) \) and consider the following inequality on the operator \( \Delta: \)
\[
\int_0^\infty [x - b^{-1}y(t + \tau)] [-ax + y(t)] dt 
\leq \int_0^\infty [x - b^{-1}y(t)] [-ax + y(t)] dt,
\forall \tau \in (-\beta, -1/\beta) \cup (1/\beta, \beta), \forall x \in L_2[0, \infty)
\]
for some constants \( b > a \geq 0 \) and \( \beta \in (1, \infty). \) Also let
\[
\Delta^{[a,b]} := \{ \Delta \in \Delta_0 \mid \Delta \text{ satisfies (24)} \}.
\]

Before presenting the main theorem, we establish some supporting lemmas.

**Lemma 20** Given a \( \Delta \in \Delta_0, \Delta \text{ satisfies (24) if and only if} \)
\[
\langle x - b^{-1}y, (1 - \mathcal{Z})(-ax + y) \rangle \geq 0, \forall x \in L_2[0, \infty)
\]
with \( y = \Delta x \) for all \( z \in \mathcal{L}_1^+(\beta) \) such that \( \|z\|_1 = 1. \)

**Proof.** This can be proved by rewriting (26) as \( \langle x - b^{-1}y, \mathcal{Z}(-ax + y) \rangle \leq \langle x - b^{-1}y, (1 - \mathcal{Z})(-ax + y) \rangle \) and by using the same arguments in the proof of Lemma 9. \( \Box \)

Let \( M(j\omega) \) denote \(-1 - Z(j\omega). \) Lemma 20 shows that given a \( \Delta \in \Delta_0, \) a necessary and sufficient condition for \( \Delta \in \Delta^{[a,b]} \) is that \( \Delta \) satisfies the class of IQCs defined by
\[
\Pi^{[a,b]}(j\omega) = \begin{bmatrix}
-a(M(j\omega) + M(j\omega)^*) & ab^{-1}M(j\omega) + M(j\omega)^*
ab^{-1}M(j\omega)^* + M(j\omega) - b^{-1}(M(j\omega) + M(j\omega)^*)
\end{bmatrix}
\]
for all \( z \in \mathcal{L}_1^+(\beta) \) such that \( \|z\|_1 \leq 1. \)

**Lemma 21** If \( \Delta \in \Delta^{[a,b]} \), then for all \( \theta \in [0, 1], \theta \Delta + (1 - \theta)a \in \Delta^{[a,b]}. \)

**Proof.** Given any \( \Delta \in \Delta^{[a,b]}, \) (24) holds by definition, and note that \( \theta \Delta + (1 - \theta)a \in \Delta_0 \) for all \( \theta \in [0, 1]. \) It remains to verify that \( \theta \Delta + (1 - \theta)a \) also satisfies (24) for any \( \theta \in [0, 1]. \) Thus, it suffices to show that (24) with
Next, consider the inequality

$$
\left| \int_{0}^{\infty} [x - b^{-1}y] (t + \tau) [-ax + y] (t) dt \right| 
\leq \int_{0}^{\infty} [x - b^{-1}y] (t) [-ax + y] (t) dt,
$$

$$
\forall \tau \in (-\beta, -1/\beta) \cup (1/\beta, \beta), \forall x \in L_{2}[0, \infty).
$$

and define

$$
\Delta^{[a,b]} := \{ \Delta \in \Delta_{0} | \Delta \text{ satisfies (30)} \}.
$$

Lemma 22 Given a $\Delta \in \Delta_{0}$, $\Delta$ satisfies (30) if and only if (26) holds with $y = \Delta x$ for all $z \in L_{1}(\beta)$ such that $\|z\|_{1} \leq 1$. If $\Delta \in \Delta_{[a,b]}$, then for all $\theta \in [0, 1]$, $\theta \Delta + (1 - \theta)a \in \Delta^{[a,b]}$.

The proof to the lemma above is omitted since it follows the same lines of reasoning in Lemma 10 and Lemma 21.

A variant of the IQC stability theorem tailored to our purpose is stated next.

Proposition 23 Given $\Pi \in L_{\infty}, G \in RH_{\infty}$ and bounded causal $\Delta$, assume that

(i) for $\theta \in [0, 1]$ the interconnection $[G, \theta \Delta + (1 - \theta)a]$ is well-posed,
(ii) the interconnection \([G, a]\) is stable,
(iii) for \(\theta \in [0, 1]\), \(\theta \Delta + (1 - \theta) a \in IQC(\Pi),\)
(iv) there exists \(\epsilon > 0\) such that
\[
\begin{bmatrix}
G(j\omega)
\end{bmatrix}^* 
\Pi(j\omega) \begin{bmatrix}
G(j\omega)
\end{bmatrix} 
\leq -\epsilon, \quad \forall \omega \in \mathbb{R}.
\]

Then the system \([G, \Delta]\) is stable.

**PROOF.** The result can be established using the same line of reasoning of [17, Theorem 1], \(\square\)

**Theorem 24** Consider Figure 1 with \(G \in \mathcal{R}\mathcal{H}_\infty\). Suppose that \([G, a]\) is stable and \([G, \Delta]\) is well-posed for all \(\Delta \in \Delta_{\infty}^{[a, b]}\) (respectively, \(\Delta \in \widetilde{\Delta}_{\infty}^{[a, b]}\)). Then, \([G, \Delta]\) is uniformly stable over \(\Delta_{\infty}^{[a, b]}\) (respectively, \(\widetilde{\Delta}_{\infty}^{[a, b]}\)) if

\[
\exists \epsilon > 0 \quad \text{s.t.} \quad \|z\|_1 \leq 1 \quad \text{and} \quad \forall \omega \in \mathbb{R},
\]

\[
\text{Re} \{ (1 - Z(j\omega))(G(j\omega) - b^{-1})(aG(j\omega)^* - 1) \} \geq \epsilon.
\]

Moreover, \([G, \Delta]\) is uniformly stable over \(\Delta_{\beta}^{[a, b]}\) (respectively, \(\widetilde{\Delta}_{\beta}^{[a, b]}\)) for any \(\beta \in (1, \infty)\) only if (32) holds.

**PROOF.** First consider the case where \(\Delta \in \Delta_{\beta}^{[a, b]}\).

“\(\Rightarrow\)” By letting \(\beta = \infty\), we have from Lemma 20 and Lemma 21 that for each \(\Delta \in \Delta_{\infty}^{[a, b]}\), \(\theta \Delta + (1 - \theta) a \in \Delta_{\infty}^{[a, b]}\) for \(\theta \in [0, 1]\), which implies (iii) in Proposition 23 with \(\Pi\) defined by (27) with all \(z \in L_1^\infty(-\infty, \infty)\) such that \(\|z\|_1 \leq 1\) holds for all \(\Delta \in \Delta_{\infty}^{[a, b]}\). Moreover, it also implies that (i) in Proposition 23 holds for all \(\Delta \in \Delta_{\infty}^{[a, b]}\).

Now observe that condition (32) can be equivalently transformed into

\[
\begin{bmatrix}
G(j\omega)
\end{bmatrix}^* 
\Pi^{[a, b]}(j\omega) \begin{bmatrix}
G(j\omega)
\end{bmatrix} 
\leq -2\epsilon, \forall \omega \in \mathbb{R}.
\]

By exploiting Proposition 23, it can be concluded that \([G, \Delta]\) with \(\Delta \in \Delta_{\infty}^{[a, b]}\) is robustly stable. The uniform stability of \([G, \Delta]\) can be further established using the same arguments in the proof of Theorem 11.

“\(\Leftarrow\)” The necessity can be proved following the same line of the necessity proof in Theorem 11.

The case of \(\Delta \in \widetilde{\Delta}_{\beta}^{[a, b]}\) can be proved analogously based on Lemma 22. \(\square\)

**Remark 25** It can be easily verified that when \(a = 0\) and \(b \to \infty\), Theorem 24 specializes to Theorems 11 and 12. We note that it is challenging to show the sufficiency of Theorem 24 by applying loop-transformation techniques to the results in Section 3. The difficulties lie in the unboundedness of \(\bar{\Delta} := (\Delta - a)(b - \Delta)^{-1}\) or showing \(\Delta_{\beta}^{[a, b]} \subset \Delta_{\beta}^{[a, b + \delta]}\) where \(\delta\) is some arbitrarily small positive number.

4.2 Static uncertainty

Next, we consider the class of static (i.e. memoryless) \(\Delta\)'s in the set \(\Delta_{\infty}^{[a, b]}\) (respectively, \(\widetilde{\Delta}_{\infty}^{[a, b]}\)). The following lemma explains the links between these sets with certain types of slope-restricted static nonlinearity that are widely considered. Note that the necessity directions in the next lemma cannot be obtained by applying loop transformations [28] to the \(\Delta\) in Lemmas 13 and 14, since doing so would result in unbounded operators. As such, a different construction to those in Lemmas 13 and 14 is needed.

**Lemma 26** Given a static and time-invariant \(\Delta \in \Delta_0\), then

(i) \(\Delta \in \Delta_{\infty}^{[a, b]}\) if and only if

\[
a \leq \frac{\Delta(x_1) - \Delta(x_2)}{x_1 - x_2} \leq b, \forall x_1, x_2 \in \mathbb{R}, x_1 \neq x_2.
\]

(ii) \(\Delta \in \widetilde{\Delta}_{\infty}^{[a, b]}\) if and only if \(\Delta\) is odd and satisfies (33).

**PROOF.** Throughout the proof, let \(\bar{x} := x - b^{-1} \Delta(x)\) and \(\bar{y} := -ax + \Delta(x)\).

(i) Note that \(\Delta \in \Delta_{\infty}^{[a, b]}\) can be rewritten as

\[
\int_0^\infty \bar{x}(t + \tau)\bar{y}(t)d\tau \leq \int_0^\infty \bar{x}(t)\bar{y}(t)d\tau
\]

\[
\forall \tau \in \mathbb{R}, x \in L_2[0, \infty).
\]

“\(\Rightarrow\)” Given any \(x_1, x_2 \in \mathbb{R}\) such that \(x_1 < x_2\). If \(\bar{x} = \bar{x}_2\), it follows that \(\Delta(x_1) - \Delta(x_2) = b(x_1 - x_2)\), which satisfies (33). If \(\bar{x} \neq \bar{x}_2\), define \(\chi(t) \in L_2[0, \infty)\) as in (19) for some \(L \in \mathbb{Z}^+\). Then it follows that

\[
\bar{x}(t + 1)\bar{y}(t) - \bar{x}(t)\bar{y}(t) = \begin{cases} \bar{x}_2\bar{y}_1 - \bar{x}_1\bar{y}_1 & \text{if } t \in [0, 1) \cup \ldots \cup [2L, 2L + 1) \\ \bar{x}_1\bar{y}_2 - \bar{x}_2\bar{y}_2 & \text{if } t \in [1, 2) \cup \ldots \cup [2L - 1, 2L) \\ -\bar{x}_2\bar{y}_2 & \text{if } t \in [2L + 1, 2L + 2) \\ 0 & \text{otherwise}. \end{cases}
\]
To satisfy (34) when $\tau = 1$ and $L$ is sufficiently large, it is thus required that
\[
\bar{x}_1 \bar{y}_1 - \bar{x}_1 \bar{y}_2 + \bar{x}_2 \bar{y}_2 \leq 0. \tag{35}
\]
If $\bar{x}_1 > \bar{x}_2$, then \(\Delta(x_1) - \Delta(x_2) > 0\) at \(x_1 - x_2\), and (35) implies that \(\bar{y}_1 \geq \bar{y}_2\), which gives \(\Delta(x_1) - \Delta(x_2) \leq \alpha\). This gives rise to a contradiction since $b > a$. Hence, it must hold that $\bar{x}_1 < \bar{x}_2$. In this case, it follows from (35) that $\bar{y}_1 \leq \bar{y}_2$. Together, they imply $a \leq \Delta(x_1) - \Delta(x_2) < b$. Consequently, (33) follows.

“\(\Rightarrow\)” Suppose $a \leq \Delta(x_1) - \Delta(x_2) < b$. Then it is proved in [28, Section 7] that the function mapping $\bar{x}$ to $\bar{y}$ is static, bounded and monotone nondecreasing. Therefore it follows from Lemma 13 that (34) holds true. Note that the set of $\Delta$ satisfying (33) is the closure of the set of $\Delta$'s satisfying $a \leq \Delta(x_1) - \Delta(x_2) < b$. It is then straightforward to see that (24) holds for all $\Delta$ satisfying (33) with $\beta = \infty$ via a limiting argument.

(ii) “\(\Rightarrow\)” Given a static $\Delta \in \Delta[a,b]$, since $\Delta[a,b] \subseteq \Delta[a,b]_{\infty}$, it follows from (i) that $\Delta$ satisfies (33), wherein $\Delta$ is continuous. It remains to show that $\Delta$ is odd. First, note from (33) that both $x - b^{-1} \Delta(x)$ and $-ax + \Delta(x)$ are monotone nondecreasing.

If there exist $x_1, x_2$ such that $0 < \bar{x}_1 < \bar{x}_2$, then one has $0 < \bar{x}_1 < \bar{x}_2$ and $0 < \bar{y}_1 \leq \bar{y}_2$. In this case, let $\bar{x}_1 = -x_1 - b^{-1}\Delta(-x_1), \bar{x}_2 = -x_2 - b^{-1}\Delta(-x_2)$ and $\bar{y}_1 = ax_1 + \Delta(-x_1), \bar{y}_2 = ax_2 + \Delta(-x_2)$. By defining $x(t)$ as in (20) and (22) respectively for a sufficiently large $L \in \mathbb{Z}^+$ and investigating the term $\Delta(x_1) - \Delta(x_2) = -\Delta(x_1)$, it can be shown using the same argument of the necessity proof of Lemma 14 that $y(t) = -\Delta(t)$. As a result, $-ax_1 + \Delta(x_1) = -ax_1 - \Delta(x_1)$, which gives $\Delta(x_1) = -\Delta(x_1)$. In other words, $\Delta$ is odd.

If there exist no $x_1, x_2$ such that $0 < \bar{x}_1 < \bar{x}_2$, then it can be implied that $\Delta(x) = bx$, and hence $\Delta$ is odd.

“\(\Leftarrow\)” This can be shown based on the sufficiency proof for (i) and Lemma 14.

Define
\[
\Delta[a,b]_{\text{static}} := \\{ \Delta \in \Delta_0 \mid \Delta \text{ is static and satisfies (33)} \} \tag{36}
\]
and
\[
\bar{\Delta}[a,b]_{\text{static}} := \\{ \Delta \in \Delta[a,b]_{\text{static}} \mid \Delta \text{ is odd} \} \tag{37}
\]
By Lemma 26,
\[
\Delta[a,b]_{\text{static}} = \{ \Delta \in \Delta[a,b] \mid \Delta \text{ is static and time-invariant} \}
\]
and
\[
\Delta[a,b]_{\text{static}} = \{ \Delta \in \Delta[a,b] \mid \Delta \text{ is static and time-invariant} \}.
\]
Hence, $\Delta[a,b]_{\text{static}} \subset \Delta[a,b]_{\infty}$ and $\bar{\Delta}[a,b]_{\text{static}} \subset \bar{\Delta}[a,b]_{\infty}$. This together with Theorem 24 enables us to provide the following result.

**Corollary 27** Consider Figure 1 with $G \in \mathcal{R}_\infty$ and $\Delta \in \Delta[a,b]_{\text{static}}$ (respectively, $\bar{\Delta}[a,b]_{\text{static}}$). Suppose that $[G, a]$ is stable and $[G, \Delta]$ is well-posed for all $\Delta \in \Delta[a,b]_{\text{static}}$ (respectively, $\Delta \in \bar{\Delta}[a,b]_{\text{static}}$). Then, $[G, \Delta]$ is uniformly stable over $\Delta[a,b]_{\text{static}}$ (respectively, $\bar{\Delta}[a,b]_{\text{static}}$) if (32) holds.

In next subsection, we will show that when the set of $\Delta$'s is further confined, condition (32) is sufficient but not necessary.

**4.3 LTI uncertainty**

Lastly we consider the case where $\Delta$'s are LTI systems.

**Lemma 28** Given an LTI $\Delta \in \Delta_0, \Delta \in \Delta[a,b]_\infty$ if and only if $\Delta$ is a constant in $[a,b]$.

**PROOF.** The sufficiency follows from 26 and the fact that a constant $\Delta$ in $[a,b]$ satisfies (33). The necessity can be shown following the same line of argument in the proof of Lemma 18 with $\Delta(j\omega)$ in (23) replaced by $(1 - b^{-1} \Delta(j\omega)\omega^2)(-a + \Delta(j\omega)^2)$. In particular, it can be verified that $(1 - b^{-1} \Delta(j\omega)\omega^2)(-a + \Delta(j\omega)^2)$ is nonnegative real-valued, from which it follows that $\Delta(j\omega)$ is $[a,b]$-valued, which in turn implies that $\Delta$ is a constant in $[a,b]$.

Define
\[
\Delta[a,b]_{\text{LTI}} := \{ \Delta \in \Delta_\infty \mid \Delta \text{ is LTI} \}.
\]
By Lemma 28, $\Delta[a,b]_{\text{LTI}} = \{ \Delta \in \mathbb{R} \mid a \leq \Delta \leq b \}$. 

**Corollary 29** Consider Figure 1 with $G \in \mathcal{R}_\infty$ and $\Delta \in \Delta[a,b]_{\text{LTI}}$. Suppose that $[G, a]$ is stable and $[G, \Delta]$ is well-posed for all $\Delta \in \Delta[a,b]_{\text{LTI}}$. Then, $[G, \Delta]$ is uniformly stable over $\Delta[a,b]_{\text{LTI}}$ if (32) holds. Moreover, the converse is not true.

**PROOF.** The sufficiency can be proved in a straightforward manner with Corollary 27 since $\Delta[a,b]_{\text{T}}$
To prove the converse is not true, first observe that condition (32) can be rewritten as

\[ \operatorname{Re} \left\{ (1 - Z(j\omega))(G(j\omega) - b^{-1}) \right\} \geq \hat{\epsilon} \quad \forall \omega \in \mathbb{R} \tag{38} \]

for some \( \hat{\epsilon} > 0 \), in which the stability of \( [G,a] \) ensures that \( (aG - 1)^{-1} \in \mathcal{RH}_{\infty} \).

If \( a = 0 \), let \(-G = \frac{\xi^2 + \varepsilon - b^{-1}}{s + 2\xi - 1} + \varepsilon - b^{-1} \), where \( \xi \in (0,0.25) \) and \( \varepsilon > 0 \) is sufficiently small. Then we have \( G \in \mathcal{RH}_{\infty} \) and \( (G(s) - b^{-1})(aG(s) - 1)^{-1} = \frac{s^2 + 2\xi + 1}{s^2 + 2\xi + 1} + \varepsilon \). Since the Nyquist plot of \( \frac{s^2 + 2\xi + 1}{s^2 + 2\xi + 1} + \varepsilon \) does not intersect with nonpositive real line, it can be implied from the above equation that the Nyquist plot of \( G \) does not intersect the interval \( [b^{-1}, a^{-1}] \). This, together with the fact that \( [G,a] \) is stable, indicates that \( [G,\Delta] \) is uniformly stable over \( \Delta_{\text{LTI}}^{[a,b]} \). Following the same line of reasoning in the proof of Theorem 19, it can be shown that there is no \( z \in \mathcal{L}_1 \) such that \( \|z\|_1 \leq 1 \) and (38) holds true.

If \( a > 0 \), let \( G(s) = \frac{(b^{-1} - a)(s^2 + 2\xi + 1)^2 - a^{-1} \xi^2 s^2}{(1 - a\varepsilon)(s^2 + 2\xi + 1)^2 - \xi^2 s^2} \), where \( \xi \in (0,0.25) \) and \( \varepsilon > 0 \) is sufficiently small. One can verify that \( G \in \mathcal{RH}_{\infty} \) by, say, the Routh-Hurwitz stability criterion and that \( (G(s) - b^{-1})(aG(s) - 1)^{-1} = \frac{a^{-1} \xi^2 s^2}{s^2 + 2\xi + 1} + \varepsilon \). Similarly, since the Nyquist plot of \( \frac{a^{-1} \xi^2 s^2}{s^2 + 2\xi + 1} + \varepsilon \) does not intersect with nonpositive real line, it can be implied from the above equation that the Nyquist plot of \( G \) does not intersect the interval \( [b^{-1}, a^{-1}] \). This and the fact that \( [G,a] \) is stable imply that \( [G,\Delta] \) is uniformly stable over \( \Delta_{\text{LTI}}^{[a,b]} \). We note that \( a^{-1} \xi^2 s^2 \) differs from \( [26, (42)] \) only by a positive factor and hence we can conclude that there is no \( z \in \mathcal{L}_1 \) such that \( \|z\|_1 \leq 1 \) and (38) holds true.

\section{5 Conclusion}

In the systems and control literature, the Zames-Falb multipliers are widely used as a classical tool to establish the input-output stability of a feedback interconnection of an LTI system and a static monotone nonlinearity. Not much attention has been paid to investigating the conservatism of using the Zames-Falb multipliers. This paper identifies a class of uncertain systems over which the robust feedback stability implies the existence of an appropriate Zames-Falb multiplier based on the generalised S-procedure lossless theorem. Meanwhile, it is shown that the existence of such a Zames-Falb multiplier is sufficient for the robust feedback stability over a smaller class of uncertain systems. When restricted to be static (a.k.a. memoryless), this class of systems coincides with the class of monotone nonlinearities with possible slope-restrictions considered in the classical paper [28], and the classical result of using the Zames-Falb multipliers to ensure feedback stability is recovered. When the same class of systems is restricted to be LTI, the existence of a Zames-Falb multiplier is shown to be sufficient but not necessary for the robust feedback stability. Nevertheless it remains unknown whether the existence of a Zames-Falb multiplier is necessary for the uniform feedback stability over these classes. This gives rise to an interesting future research direction.

\section{Acknowledgements}

The authors gratefully acknowledge Peter Seiler and Andrey Kharitonko for many useful discussions.

\section{References}

[1] H. Brezis. \textit{Functional Analysis, Sobolev Spaces, and Partial Differential Equations}. Springer, 2011.

[2] J. Carrasco, W. P. Heath, and A. Lanzon. Equivalence between classes of multipliers for slope-restricted nonlinearities. \textit{Automatica}, 49(6):1732–1740, 2013.

[3] J. Carrasco, M. Maya-Gonzalez, A. Lanzon, and W. P. Heath. LMI searches for anticausal and noncausal rational Zames-Falb multipliers. \textit{Systems and Control Letters}, 70:17–22, 2014.

[4] J. Carrasco, M. C. Turner, and W. P. Heath. Zames-Falb multipliers for absolute stability: From O'Shea’s contribution to convex searches. \textit{European Journal of Control}, 28:1–19, 2016.

[5] R. F. Curtain and H. Zwart. \textit{An introduction to infinite-dimensional linear systems theory}, volume 21. Springer Science & Business Media, 1995.

[6] J. C. Doyle. \textit{Lecture Notes in Advances in Multivariable Control}. ONR/Honeywell Workshop, Minneapolis, 1984.

[7] G. E. Dullerud and F. Paganini. \textit{A course in robust control theory: a convex approach}, volume 36. Springer Science & Business Media, 2013.

[8] W. B. Johnson and G. Schechtman. A Schauder basis theory: a convex approach. \textit{Illinois Journal of Mathematics}, 59(2):337–344, 2015.

[9] U. Jönsson. Lecture notes on integral quadratic constraints. Department of Mathematics, Royal Institute of Technology (KTH), Stockholm, Sweden, 2001.

[10] U. Jönsson and M. C. Laiou. Stability analysis of systems with nonlinearities. In \textit{Proc. 35th IEEE Conf. Decision Control}, volume 2, pages 2145–2150, 1996.

[11] H. K. Khalil. \textit{Nonlinear Systems}. Prentice Hall, 3rd edition, 2002.

[12] S. Z. Khong. On integral quadratic constraints. \textit{IEEE Trans. Autom. Contr.}, 2022. In press.

[13] S. Z. Khong and C.-Y. Kao. Converse theorems for integral quadratic constraints. \textit{IEEE Trans. Autom. Contr.}, 2021. In press.
