Projective completions of affine varieties via degree-like functions
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Abstract
We study projective completions of affine algebraic varieties induced by filtrations on their coordinate rings. In particular, we study the effect of the ‘multiplicative’ property of filtrations on the corresponding completions and introduce a class of projective completions (of arbitrary affine varieties) which generalizes the construction of toric varieties from convex rational polytopes. As an application we recover (and generalize to varieties over algebraically closed fields of arbitrary characteristics) a ‘finiteness’ property of divisorial valuations over complex affine varieties proved in [dFEI08]. We also find a formula for the pull-back of the ‘divisor at infinity’ and apply it to compute the matrix of intersection numbers of the curves at infinity on a class of compactifications of certain affine surfaces.

1 Introduction
1.1 Summary
It is a standard fact in algebraic geometry that filtrations (satisfying some natural conditions) on the ring of regular functions of affine varieties correspond to their projective compactifications. In this article we study the connection between properties of filtrations and the corresponding completions. We primarily focus on (completions corresponding to) filtrations induced by subdegrees - a notion motivated by the theory of toric varieties (in particular completions of the $n$-torus $(\mathbb{C}^*)^n$ induced by convex rational polytopes). In short the main results are as follows:

1. We develop a structure theorem for completions determined by subdegrees and show that in a (precise) sense the correspondence between completions determined by general filtrations and completions determined by subdegrees is analogous to the correspondence between varieties and their normalizations.

2. As an application we show that every divisorial valuations over an affine variety is ‘determined’ by its values on a finite number of functions. This result was proved for varieties over $\mathbb{C}$ in [dFEI08] using desingularization and geometry of arc spaces. Our proof uses essentially elementary (but perhaps somewhat involved) commutative algebra and algebraic geometry.

3. We prove a formula for the pull-back (under a dominating morphism) of the ‘divisor at infinity’ on completions determined by subdegrees in terms of the linking numbers of valuations centered at infinity. As applications we compute the matrix of intersection numbers of curves at infinity on a class of completions of certain affine surfaces and find a necessary
condition for the finite generation of a subdegree on the coordinate ring of these surfaces. The former computation motivates a somewhat amusing conjecture on the invertibility of the matrix of the maximum ratios of a finite collection of $n$-tuples of positive real numbers.

1.2 More Elaborate Discussion of Main Results

Notation 1.1. Throughout this article $X$ denotes an affine variety over an algebraically closed field $k$. The ring of regular functions (resp. the field of rational functions) on $X$ will be denoted by $k[X]$ (resp. $k(X)$). Given an ideal $I$ of a ring $A$ (resp. a graded ring $B$), the zero set of $I$ in Spec $A$ (resp. $Proj B$) will be denoted by $V(I)$.

Basic Construction and Guiding Examples: We consider the following ‘rough’ correspondence (here we avoid some technical assumptions - see Proposition 2.5 and Remark-Definition 2.6 for the rigorous version)

\[
\begin{align*}
\text{Filtrations on } k[X] & \quad \longleftrightarrow \quad \text{Closed immersions of } X \text{ into weighted projective spaces.}
\end{align*}
\]

In our case a filtration $\mathcal{F}$ on the ring $k[X]$ of regular functions on $X$ is a family $\{F_k : k \in \mathbb{Z}\}$ of vector subspaces of the ring $k[X]$ of regular functions on $X$ such that $1 \in F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq \bigcup_{d\geq 0} F_d = k[X]$, and $F_d F_e \subseteq F_{d+e}$ for all $d, e \in \mathbb{Z}$. The above correspondence is induced via the natural open immersion $X \hookrightarrow \bar{X}^\mathcal{F} := \text{Proj} \left( \sum_{d\geq 0} F_d t^d \right)$, where $t$ is an indeterminate over $k[X]$ (Proposition 2.5). The complement of $X$ in $\bar{X}^\mathcal{F}$ is the zero set of the ideal of $t$.

In most classical settings filtrations appear in the guise of degree-like functions, i.e. functions $\delta : k[X] \rightarrow \mathbb{Z}$ which satisfy:

1. $\delta(f+g) \leq \max\{\delta(f), \delta(g)\}$ for all $f, g \in k[X]$, with $<$ in the preceding equation implying $\delta(f) = \delta(g)$.

2. $\delta(fg) \leq \delta(f) + \delta(g)$ for all $f, g \in k[X]$.

The filtration $\mathcal{F}^\delta := \{F^\delta_d\}_{d \in \mathbb{Z}}$ induced by $\delta$ is given by $F^\delta_d := \{f \in k[X] : \delta(f) \leq d\}$ and the completion of $X$ corresponding to $\delta$ is $\bar{X}^\delta := \text{Proj} k[X]^\delta$, where $k[X]^\delta := \sum_{d\geq 0} F^\delta_d t^d$ is the graded ring associated to the filtration constructed as in the preceding paragraph. For example, if $X := \mathbb{P}^n$ with coordinates $(x_1, \ldots, x_n)$ and $\delta$ is the degree of polynomials in $k[x_1, \ldots, x_n]$, then $\bar{X}^\delta \cong \mathbb{P}^n(k)$. Similarly, if $\delta$ is the weighted degree on $k[x_1, \ldots, x_n]$ which assigns a positive integral weight $d_j$ to $x_j$, $1 \leq j \leq n$, then $\bar{X}^\delta$ is the weighted projective space $\mathbb{P}^n(k; 1, d_1, \ldots, d_n)$.

A bigger class of completions induced by degree-like functions are projective toric varieties. Indeed, let $\mathcal{P}$ be an $n$-dimensional convex polytope in $\mathbb{R}^n$ which contains the origin in the interior and whose vertices have rational coordinates. Then $\mathcal{P}$ corresponds to an $n$-dimensional projective normal toric variety $X_{\mathcal{P}}$. We claim that $X_{\mathcal{P}}$ is the completion of the $n$-torus $X := (\mathbb{K}^*)^n$ induced by a degree-like function. Indeed, define $\delta_{\mathcal{P}} : k[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}] \setminus \{0\} \rightarrow \mathbb{Q}$ such that

$$
\delta_{\mathcal{P}}(\sum a_\alpha x^\alpha) := \inf\{r \in \mathbb{R} : \alpha \in r\mathcal{P} \text{ for all } \alpha \in \mathbb{Z}^n \text{ such that } a_\alpha \neq 0\}.
$$

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(see Figure 1 for an example). Pick any positive integer $e$ such that $e\delta$ is integer-valued. Then $e\delta$ is a degree-like function on $K[X]$ and it is not hard to see that $X_P \cong X^{e\delta}$.

Note that our first two examples of degree-like functions, namely degree and weighted degrees on polynomial rings, satisfy the multiplicative property (i.e. property 2) of degree-like functions with exact equality. On the other hand, it is straightforward to see that even though the degree-like functions $\delta$ of the preceding paragraph in general do not satisfy property 2, they are the maxima of finitely many degree-like functions which do satisfy it, namely the weighted degrees determined by the facets (i.e. $(n-1)$-dimensional faces) of $P$ (see Figure 1 and Example 3.6). Motivated by these examples we define two special classes of degree-like functions: semidegrees, which satisfy property 2 of degree-like functions with exact equality, and subdegrees, which are the maxima of finitely many semidegrees. The primary motive of this article is to (start to) explore to what extent properties of toric varieties remain true for completions of affine varieties determined by general subdegrees.

**Structure Theorem for Subdegrees and its Corollaries:** Let $\delta$ be a subdegree on $K[X]$ with $\delta = \max\{\delta_1, \ldots, \delta_k\}$. Getting rid of the ‘extra’ $\delta_j$’s if necessary, we may assume the presentation is minimal, i.e. for every $j$, $1 \leq j \leq k$, there exists $f \in K[X]$ such that $\delta_j(f) > \delta_i(f)$ for all $i \neq j$, $1 \leq i \leq k$. Our first main result is the ‘structure theorem’ for subdegrees:

**Theorem 1.2** (Informal version of Theorem 4.1 and Proposition 5.2). Every subdegree $\delta$ has a unique minimal presentation. A degree-like function $\delta$ is a subdegree iff the ideal $I$ of $K[X]$ generated by $t$ is a decomposable radical ideal. If $\delta$ is a subdegree then the non-zero semidegrees in the minimal presentation of $\delta$ are multiples of the orders of pole (of rational functions on $X$) along the irreducible components of $V(I) = X^\delta \setminus X$.

An immediate corollary of Theorem 1.2 gives a characterization of subdegrees in the case that $K[X]^\delta$ is Noetherian, namely: $\delta$ is a subdegree iff $\delta(f^k) = k\delta(f)$ for all $f \in K[X]$ and $k \geq 0$. Another corollary of the arguments in the proof of Theorem 1.2 is a ‘finiteness property’ of divisorial valuations. Recall that divisorial valuations on $K(X)$ are valuations of the form $\nu := q \circ \text{ord}_E : K(X)^* \to \mathbb{Z}$, where $E$ is a prime divisor on a normal variety $Y$ equipped with a proper birational morphism $f : Y \to X$, $q = q(v)$ is a positive integer number called the multiplicity of $\nu$, and for every $h \in K(X)^*$ which is regular at the generic point of $f(E)$, $\text{ord}_E(h)$ is the order of vanishing of $h \circ f$ at the generic point of $E$. Given a divisorial valuation
\(\nu\) over \(X\), we may consider that \(\nu\) is centered at infinity with respect to some affine variety \(Z\) which is birational to \(X\) and therefore we may assume by means of Theorem 1.2 that \(-\nu\) is a semidegree associated to some subdegree on \(K[Z]\). Applying arguments of the proof of Theorem 1.2 to this context yields the following result which was proved in [dFE10b, Theorem 0.2] for \(K = \mathbb{C}\).

**Theorem** (Theorem 5.14). Let \(X\) be an irreducible affine variety and \(\nu\) be a divisorial valuation over \(X\). Then there exist elements \(f_1, \ldots, f_r \in K[X] \setminus \{0\}\) such that for every \(f \in K[X] \setminus \{0\}\),

\[
\nu(f) = \min \{\nu'(f) : \nu' \text{ is a divisorial valuation over } X, \nu'(f_i) = \nu(f_i) \text{ for } 1 \leq i \leq r\} \tag{*}
\]

\[
= \min \{\nu'(f) : \nu' \text{ is a valuation on } K(X) \text{ such that the value group of } \nu \text{ contains the integers and } \nu'(f_i) = \nu(f_i) \text{ for } 1 \leq i \leq r\} \tag{*'}
\]

**Normalization at infinity of completions:** In order to study a given completion \(\bar{X}\) of an affine variety \(X\), it is natural to try to find a ‘simpler’ model, i.e. a ‘simpler’ completion \(X’\) of \(X\) which dominates \(X\), i.e. there is a morphism \(\pi : X' \to X\) which restricts to the identity on \(X\). If \(X\) is normal or non-singular, a natural choice for the ‘simpler’ model is the normalization or a desingularization of \(X\) (provided the latter exists of course). For a general \(X\), we introduce the normalization at infinity of \(X\) with respect to \(X\), which is the unique minimal completion \(X'\) dominating \(X\) such that \(X'\) is normal at infinity with respect to \(X\), i.e. for all open subset \(U\) of \(X'\), the ring of regular functions on \(U\) is integrally closed in the ring of regular functions on \(U \cap X\). In particular, if \(X\) is normal, then \(X' \cong X\). Using the theory of Rees valuations ([Ree56a, Ree56b]) we show that

**Theorem** (A reformulation of Theorem 5.10). Let \(\bar{X}\) be a projective completion of \(X\) such that \(\bar{X} \setminus X\) is the support of an effective ample divisor. Then there is a subdegree \(\delta'\) on \(K[X]\) such that \(\bar{X}^{\delta'}\) is the normalization at infinity of \(\bar{X}\) with respect to \(X\). If \(\delta\) is any degree-like function such that \(\bar{X}^{\delta} \cong \bar{X}\), then one such \(\delta'\) is given by:

\[
\delta'(f) := e \lim_{n \to \infty} \frac{\delta(f^n)}{n} \text{ for all } f \in K[X],
\]

where \(e\) is a sufficiently divisible integer.

**Note 1.3.** A completion \(\bar{X}\) of \(X\) is induced by a degree-like function iff \(\bar{X} \setminus X\) is the support of an effective ample divisor (Proposition 2.5).

**Divisor at Infinity and its Multiplicities:** Let \(\delta\) be a subdegree and \(\delta = \max\{\delta_1, \ldots, \delta_k\}\) be the minimal presentation of \(\delta\) as the maximum of semidegrees. It follows from Theorem 1.2 that \(\delta_j = -d_j \operatorname{ord}_{E_j}\), where \(E_1, \ldots, E_k\) are irreducible components of \(X^{\delta} \setminus X\), and for each \(j\), \(\operatorname{ord}_{E_j}\) is the order of vanishing along \(E_j\) and \(d_j\) is some positive integer. This motivates a natural question about the roles of the numbers \(d_j\). More precisely, it is natural to ask

**Question 1.4.** If \(\bar{X}_1\) and \(\bar{X}_2\) are two completions of \(X\) such that the set of divisorial valuations associated to the irreducible complements of \(\bar{X}_j \setminus X\) are identical, then is it true that \(\bar{X}_1 \cong \bar{X}_2?\)

If \(\dim X \leq 2\), then it is easy to see that the answer to Question 1.4 is positive. However, straightforward examples from toric geometry shows that it is not true in higher dimensions.
Indeed, for \( n \geq 3 \), it is easy to find two convex rational polytopes (‘rational’ means that the vertices have rational coordinates) \( P_1, P_2 \) in \( \mathbb{R}^n \) such that the following properties are satisfied (see Figure 2):

1. there is a one-to-one correspondence between the facets of \( P_i \)’s such that for each facet \( Q_1 \) of \( P_1 \), the corresponding facet \( Q_2 \) of \( P_2 \) has the same outer normal (i.e. \( Q_1 \) and \( Q_2 \) are parallel and on the ‘same side’ of the origin), but
2. \( P_1 \) and \( P_2 \) are not combinatorially isomorphic (i.e. the face lattice of \( P_i \)’s are not isomorphic).

Figure 2: Polytopes corresponding to toric varieties giving negative answer to Question 1.4

Then it follows from basic theory of toric varieties (e.g. as in [Ful93]) that the toric varieties \( X_{P_1} \) and \( X_{P_2} \) are not isomorphic, even though the irreducible components of the complement of the \( n \)-torus in \( X_{P_j} \)’s determine the same set of divisorial valuations.

The preceding example shows that \( q_j \)’s play a non-trivial role in the structure of \( \bar{X}_{\delta} \). One of the places they manifest is in the divisor on \( \bar{X}_{\delta} \) determined by \( t \in \mathbb{K}[X]=\sum_{d \geq 0} F_{\delta}^d t^d \).

Indeed, the \( Q \)-Cartier divisor of \( t \) on \( \bar{X}_{\delta} \) is precisely \( \sum_{k=1}^j 1_{\delta} E_j \) (Lemma 6.2). We call this divisor the divisor at infinity and denote it by \( D_{\delta}^{\infty} \). The toric analogue for the divisor at infinity is the divisor of a polytope ([CLS11, Section 7.1]). We prove a formula for its pull-back under a dominant morphism:

**Proposition** (Proposition 6.4). Let \( \phi : Z \to \bar{X}_{\delta} \) be a dominant morphism. Then

\[
\phi^*(D_{\delta}^{\infty}) = \sum_{W} l^0_{\infty}(\delta, \text{pole}_W)[W],
\]

where the sum is over codimension one irreducible subvarieties \( W \) of \( Z \), and for each such \( W \), the function \( \text{pole}_W \) is the negative of the order \( \text{ord}_W \) of vanishing along \( W \) and

\[
l^0_{\infty}(\delta, \text{pole}_W) := \max \left\{ \frac{\text{pole}_W(\phi^*(f))}{\delta(f)} : f \in \mathbb{K}[X], \delta(f) > 0 \right\}.
\]

**Remark 1.5.** Identity \( \square \) in particular implies that \( l^0_{\infty}(\delta, \text{pole}_W) \) exists. Note also that the sum in identity \( \square \) is finite, since \( l^0_{\infty}(\delta, \text{pole}_W) = 0 \) for all codimension one irreducible subvariety \( W \) of \( Z \) such that \( W \cap \phi^{-1}(X) \neq \emptyset \).

**Remark-Definition 1.6.** Given a Krull local ring \( o \), a valuation \( \omega \) on \( o \) and a height one prime ideal \( p \) of \( o \), the linking number [Huc70] of \( \omega \) and the valuation \( \nu_p \) on \( o \) corresponding to \( p \) is

\[
l(\omega, \nu_p) := \inf_{f \in p, f \neq 0} \omega(f) / \nu_p(f).
\]
The linking number was introduced in [Sam59] in connection with defining the pull-back of Weil divisors under a birational regular mapping. More precisely, if \( \phi : Z \to Y \) is a birational map and \( \omega \) (resp. \( \nu_p \)) is the order of vanishing along a codimension one subvariety \( W \) of \( Z \) (resp. \( V \) of \( Y \)), then \( l(\omega, \nu_p) \) is a ‘candidate’ for the coefficient of \([W]\) in \( \phi^*([V]) \). Now assume \( Y := \bar{X}_\delta \) for a semidegree \( \delta \) (i.e. \( Y \backslash X \) is irreducible) and \( V := Y \backslash X \). Then \( \delta = -\nu_p \) and \( \text{pole}_W = -\omega \) and identity (1) implies that

\[
l^\infty(\delta, \text{pole}_W) = l(\omega, \nu_p).
\]

It is perhaps instructive that \( l(\omega, \nu_p) \) and \( l^\infty(\delta, \text{pole}_W) \) measure the same quantity, but the inf of the local computation is replaced by a sup in the global computation. We call \( l^\infty(\delta, \text{pole}_W) \) the linking number at infinity (relative to \( \phi \)) of \( \delta \) and \( \text{pole}_W \). In the case that \( \phi \) is birational (so that \( \phi^* \) is identity on the function field), we simply write \( l^\infty(\delta, \text{pole}_W) \) for \( l^\infty(\delta, \text{pole}_W) \).

**Intersection numbers of curves at infinity:** Let \( X \) be a normal affine surface with no non-constant invertible regular functions (e.g. \( K^2 \)) and \( \bar{X}_1, \ldots, \bar{X}_k \) be normal algebraic completions of \( X \) such that for each \( j \), the complement \( C_j \) of \( X \) in \( \bar{X}_j \) is an irreducible curve. Let \( \bar{X} \) be the normalization of the diagonal embedding of \( X \) into \( \bar{X}_1 \times \cdots \times \bar{X}_k \). We apply Proposition [6.4] to compute the matrix of intersection numbers of the curves at infinity on \( \bar{X} \) in terms of the linking numbers at infinity of associated semidegrees (Lemma [6.6]) - in particular, it turns out that in this case the matrix of linking numbers at infinity of the semidegrees is invertible. Taking the semidegrees to be weighted degrees yields the following statement (which is not hard to verify directly, see e.g. [mat]):

**Corollary 1.7** (cf. Corollary [6.7]). Let \( n = 2 \). Let \( v_1, \ldots, v_k \) be pairwise non-proportional elements in \( \mathbb{Q}^n \) with positive coordinates and \( L \) be the \( k \times k \) matrix with entries

\[
ll_{ij} := \max\{ v_m \, v_n : 1 \leq m \leq n \}.
\]

Then \( L \) is invertible.

Motivated by Corollary [1.7] and a lot of computation (done mostly for \( n = 3 \)), we propose the following

**Conjecture 1.8.** The conclusion of Corollary [1.7] remains true for all \( n \geq 2 \).

Since \( l_{ij} \)'s are precisely the linking numbers at infinity of weighted degrees determined by \( v_i \) and \( v_j \), it seems plausible that Conjecture [1.8] can be proved/disproved using the geometry of toric varieties associated to polytopes with facets determined by \( v_j \)'s.

**Finite generation of subdegrees:** For a subdegree \( \delta \) on \( K[X] \) to correspond to a completion of \( X \), it is necessary that \( \delta \) is finitely generated, i.e. the graded ring \( K[X]^\delta \) is finitely generated as an algebra over \( K \). It is therefore natural to try to find conditions which guarantee that \( \delta \) is finitely generated. Note that if \( \delta_1, \ldots, \delta_k \) are semidegrees such that \( \delta = \max\{ \delta_1, \ldots, \delta_k \} \), then \( K[X]^\delta = \bigcap_{j=1}^k K[X]^\delta_j \). It is therefore natural to ask:

**Question 1.9.** If each \( \delta_j \) is finitely generated, then is it true that \( \delta \) is also finitely generated?
If \( X = (\mathbb{K}^*)^n \) and each \( \delta_j \) is a weighted degree in a fixed set of coordinates on \( X \), then the answer to Question 1.9 is positive and is an implication of Gordan’s Lemma (which states that the integral points in a closed rational cone of \( \mathbb{R}^n \) form a finitely generated semigroup). But as the following example of [Mon11] shows, the answer is in general false, even for \( n = 2 \).

**Example 1.10.** Consider elements \( y_1 := y - x^5 - x^{-2} \) and \( y_2 := y + x^5 - x^{-2} \) of \( \mathbb{C}(x, y) \).

Let \( \delta_1 \) (resp. \( \delta_2 \)) be the weighted degree on \( \mathbb{C}(x, y) \) with respect to coordinates \( (x, y_1) \) (resp. \( (x, y_2) \)) corresponding to weights 1 for \( x \) and \(-3\) for \( y_1 \) (resp. \( y_2 \)). Then it can be shown that \( \mathbb{C}[x, y]^{\delta_1} \) is finitely generated for each \( j \) and \( \delta := \max\{\delta_1, \delta_j\} \) is strictly positive on non-constant elements of \( \mathbb{C}[x, y] \), but \( \delta \) is not finitely generated. In fact, as it is shown in [Mon11], \( \delta \) corresponds to a normal analytic completion of \( \mathbb{C}^2 \) which is not algebraic.

For \( n = 2 \), we give a positive answer to Question 1.9 under the additional hypothesis that each \( \delta_j \) is strictly positive on every non-constant regular function on \( X \) (Corollary 6.9). Indeed, this additional hypothesis is equivalent to saying that each \( \dot{X}^{\delta_j} \) is a projective completion of \( X \). Let \( \dot{X} \) be the normalization of the diagonal embedding of \( X \) into \( \dot{X}^{\delta_1} \times \cdots \times \dot{X}^{\delta_k} \). Then Question 1.9 reduces to a question about ampleness of some divisors on \( \dot{X} \), which we answer using Proposition 6.4.

**Remark-Question 1.11.** Counter-examples to Question 1.9 that we know of (e.g. Example 1.10) require the characteristic of the field to be zero. Therefore we ask: is the answer to Question 1.9 positive if \( \mathbb{K} \) has positive characteristic?

### 1.3 Organization

In the next section we introduce and give some examples of the correspondence between filtrations on the coordinate ring of an affine variety \( X \) and projective completions of \( X \). In Section 3 we reformulate the notion of filtrations in terms of degree-like functions and introduce two special classes of degree-like functions which are protagonists of this article, namely semidegree and subdegree.

The remaining sections contain our main results. In Section 4 we establish the structure theorem of subdegrees (Theorem 4.1) and some of its immediate corollaries. Section 5 is devoted to the study of ‘normality’ properties of completions determined by subdegrees. In this section we establish the relation between semidegrees in the minimal presentation of a subdegree and orders of pole along components of the hypersurface at infinity on the completion induced by the subdegree (Proposition 5.2). We also introduce the notion of relative normality at infinity, and show that the compactification \( \dot{X}^{\delta} \) of an affine variety \( X \) determined by a subdegree \( \delta \) on \( \mathbb{K}[X] \) is relatively normal at infinity with respect to \( X \) (Proposition 5.6). We then establish the existence of a subdegree \( \delta \) which normalizes a given degree-like function \( \delta \) in the sense that \( \dot{X}^{\delta} \) is the normalization at infinity (with respect to \( X \)) of \( \dot{X}^{\delta} \) (Theorem 5.10). We end Section 5 with a proof of the ‘finiteness property’ of divisorial valuations discovered (in the case of complex varieties) in [dFEI08] (Theorem 5.14). In Section 6 we prove the ‘pull-back formula’ of the divisor at infinity (Proposition 6.4) and apply it to give a geometric proof of Corollary 1.7 and find a sufficient condition for the answer to Question 1.9 to be positive.

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2 Compactifications from Filtrations

In this section we set up the correspondence between filtrations and completions. The material is basic and well-known (except perhaps for assertion 2 of Proposition 2.5).

Definition 2.1. A filtration $\mathcal{F}$ on an $\mathbb{K}$-algebra $A$ is a family $\{F_i : i \in \mathbb{Z}\}$ of $\mathbb{K}$-vector subspaces of $A$ such that

1. $F_i \subseteq F_{i+1}$ for all $i \in \mathbb{Z}$
2. $1 \in F_0 \setminus F_{-1}$
3. $A = \bigcup_{i \in \mathbb{Z}} F_i$ and
4. $F_iF_j \subseteq F_{i+j}$ for all $i, j \in \mathbb{Z}$.

Remark 2.2. We introduce the condition $1 \notin F_{-1}$ in order to exclude the trivial filtration, i.e. the filtration $\{F_i := A\}_{i \in \mathbb{Z}}$.

To each filtration $\mathcal{F}$ on a $\mathbb{K}$-algebra $A$ we associate two graded rings:

$$A^F := \bigoplus_{d \geq 0} F_d, \text{ and } \text{gr} A^F := \bigoplus_{d \geq 0} (F_d/F_{d-1}).$$

For each $f \in F_d$, we denote the ‘copy’ of $f$ in the $d$-th graded component of $A^F$ by $(f)_d$. The ring $A^F$ has a natural graded $\mathbb{K}$-algebra structure with the multiplication defined by:

$$\left( \sum_d (f_d)_d \right) \left( \sum_e (g_e)_e \right) := \sum_k \sum_{d+e=k} (f dg_e)_k.$$

We say that $\mathcal{F}$ is a finitely generated filtration if $A^F$ is a finitely generated $\mathbb{K}$-algebra.

Remark 2.3. Note that even though it is possible for a filtration to have non-trivial components with negative degree, when forming the graded rings we take the direct sum only over components of non-negative degree. The reason is geometric: we are interested in the Proj of these graded rings (see Proposition 2.5 below) and all the standard texts we are aware of defines Proj only for rings graded over non-negative integers. However, we note that there exist at least two ‘reasonable’ generalizations of Proj to rings graded over $\mathbb{Z}$, and that given a ring $S = \bigoplus_{k \in \mathbb{Z}} S_k$ graded over $\mathbb{Z}$ such that $S_+ := \bigoplus_{k > 0} S_k$ is non-zero, both of these definitions imply $\text{Proj} S \cong \text{Proj} S_\geq 0$, where $S_\geq 0 := \bigoplus_{k \geq 0} S_k$.

Remark 2.4. Let $t$ be an indeterminate over $A$. Then there is an isomorphism

$$A^F \cong \sum_{d \geq 0} F_d t^d \subseteq A[t, t^{-1}]$$

which maps $(f)_d \mapsto ft^d$ for all $f \in A$ (in particular, $(1)_1 \mapsto t$). It follows immediately from this isomorphism that $A$ is a domain iff $A^F$ is also a domain.
The following proposition establishes the basic dictionary between completions of an affine variety and filtrations on its coordinate ring.

**Proposition 2.5.** Let $X$ be a (possibly reducible) affine variety.

1. If $\mathcal{F} := \{ F_d : d \in \mathbb{Z} \}$ is a finitely generated filtration on $\mathbb{K}[X]$ such that $F_0 = \mathbb{K}$, then then $\bar{X}^\mathcal{F} := \text{Proj} \mathbb{K}[X]^\mathcal{F}$ is a projective variety containing $X$ as a (Zariski) open dense subset. The embedding $X \hookrightarrow \bar{X}^\mathcal{F}$ is induced via the map:

$$X = \text{Spec} \mathbb{K}[X] \ni p \mapsto \bigoplus_{d \geq 0} (p \cap F_d) \in \text{Proj} \mathbb{K}[X]^\mathcal{F} = \bar{X}^\mathcal{F}.$$  

The complement of $X$ in $\bar{X}^\mathcal{F}$ is the support of the ample $\mathbb{Q}$-Cartier divisor determined by (1) and is isomorphic (as a variety) to $\text{Proj} \text{gr} \mathcal{F}^\mathbb{Z}$.

2. If $Y$ is a projective variety containing $X$ such that $Y \setminus X$ is the support of an ample divisor, then there is a filtration $\mathcal{F}$ on $\mathbb{K}[X]$ satisfying the hypothesis of assertion 1 which $\bar{X}^\mathcal{F} \cong Y$ as algebraic varieties.

**Proof.** We omit the proof of assertion 1 since it follows in a straightforward manner from the definitions. For assertion 2 w.l.o.g. we may assume that $Y = \text{Proj} S$ for some graded $\mathbb{K}$-algebra $S$ and $Y \setminus X = V(h)$ for some homogeneous $h \in S$ of degree $d > 0$. Recall that $Y \setminus V(h) \cong \text{Spec} S(h)$, where $S(h)$ is the subring of the localization $S_h$ of $S$ consisting of homogeneous elements of degree zero. By assumption there is an isomorphism $\phi : S_h \cong \mathbb{K}[X]$. For each $k \geq 0$, define

$$F_k := \phi(S_{kd}/h^{kd}) = \{ g \in \mathbb{K}[X] : \phi^{-1}(g) \in S_{kd}/h^{kd} \}.$$ 

Then it is straightforward to check that the $\mathcal{F} := \{ F_i \}_{i \geq 0}$ is a filtration on $\mathbb{K}[X]$ which satisfies the claim of assertion 2.

**Remark-Definition 2.6.** We say that a filtration $\mathcal{F}$ on the coordinate ring of an affine variety $X$ is **projective** if it satisfies the hypothesis of assertion 1 of Proposition 2.5. Note that most of the conclusions of Proposition 2.5 remains true (with the same proof!) even if $\mathcal{F}$ is not projective. More precisely, if $\mathcal{F}$ is an arbitrary filtration, assertion 1 remains true with the only modification that $\bar{X}^\mathcal{F}$ may not be a projective variety (but the $\mathbb{Q}$-Cartier divisor determined by (1) remains ample!). Similarly, if $S$ is an arbitrary graded $\mathbb{K}$-algebra such that $\text{Proj} S$ contains $X$ as a dense open subset and $\text{Proj} S \setminus X = V(h)$ for some homogeneous $h \in X$ of degree $d > 0$, then there is a filtration $\mathcal{F}$ on $\mathbb{K}[X]$ such that $\bar{X}^\mathcal{F} \cong \text{Proj} S$.

**Example 2.7.** Let $X := \mathbb{K}^n$ with coordinate ring $A := \mathbb{K}[x_1, \ldots, x_n]$ and $\omega$ be the weighted degree on $A$ which assigns a positive integer $d_j$ to $x_j$, $1 \leq j \leq n$. Then $\{ f \in A : \omega(f) \leq d \}_{d \geq 0}$ defines a filtration $\mathcal{F}_\omega$ on $A$, and, as noted in the beginning of Section 1.2, $\bar{X}^\mathcal{F}_\omega$ is the weighted projective space $\mathbb{P}^n(\mathbb{K}, 1, d_1, \ldots, d_n)$.

**Example 2.8.** Let $X$ and $A$ be as in 2.7. Assume $n \geq 2$. Pick $k \in \mathbb{Z}$, $1 \leq k < n$, and define a filtration $\mathcal{F} := \{ F_d \}_{d \geq 0}$ on $A$ as follows: take $F_1$ to be the $\mathbb{K}$-linear span of all monomials of degree less than or equal to two excluding those of the form $x_ix_j$ with $i \geq j > k$. Let $F_d = (F_1)^d$ for $d \geq 1$. Then $\bar{X}^\mathcal{F}$ is isomorphic to the blow-up of $\mathbb{P}^n(\mathbb{K})$ along the subspace $V := V(x_0, \ldots, x_k)$. 

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Example 2.9. If \( X \) is a normal affine variety with trivial Picard group and \( \bar{X} \subseteq \mathbb{P}^N(\mathbb{K}) \) be a normal projective completion of \( X \) such that \( X_{\infty} := \bar{X} \setminus X \) is irreducible, then \( X_{\infty} \) is the support of an effective ample divisor and therefore \( \bar{X} \) is isomorphic to \( \bar{X}^F \) for some filtration \( F \) on \( \mathbb{K}[X] \) (Proposition 2.5).

Example 2.10. If \( \dim X = 1 \), then every projective completion \( Y \) of \( X \) is isomorphic to \( \bar{X}^F \) for some filtration \( F \) on \( \mathbb{K}[X] \). Indeed, it follows from the Nakai-Moishezon criterion of ampleness (or the Riemann-Roch theorem) that \( Y \setminus X \) is the support of an effective ample Cartier divisor \([\text{Gro69}, \text{Proposition 5}]\), so that the claim follows from Proposition 2.5.

Example 2.11. Every non-singular projective completion of affine surfaces comes from a filtration. More generally, it follows from \([\text{Gro69}, \text{Theorem 1}]\) and the Nakai-Moishezon criterion that if \( X \) is an arbitrary affine surface and \( \bar{X} \) is a complete surface containing \( X \) such that \( \bar{X} \setminus X \) is the support of an effective Cartier divisor, then in fact \( \bar{X} \setminus X \) is the support of an effective ample divisor, so that \( \bar{X} \) is determined by a filtration on \( \mathbb{K}[X] \). Note that there exist complete normal surfaces \( \bar{X} \) and open affine subsets \( X \) of \( \bar{X} \) such that \( \bar{X} \setminus X \) is not the support of any Cartier divisor (see e.g. \([\text{art}]\)) - so that \( \bar{X} \) is not determined by any filtration on \( \mathbb{K}[X] \).

Example 2.12. There are non-singular projective completions of (non-singular) affine varieties of dimension \( \geq 3 \) which are not determined by a filtration, as the following example of Hironaka shows. Let \( \phi : V' \to V \) be a morphism of 3-dimensional projective varieties, with \( V' \) non-singular and \( V, V' \) non-singular except at a point \( P \), such that \( \phi^{-1}(P) \) is a curve and \( \phi \) is an isomorphism on the rest of \( V' \). Let \( W \) be a hyperplane section of \( V \) through \( P \). Then \( X := V' \setminus \phi^{-1}(W) \cong V \setminus W \) is affine, but \( \phi^{-1}(W) \) is not the support of any effective ample divisor on \( V \) (see \([\text{Gro69}, \text{Section 1}]\)). Therefore Proposition 2.5 implies that there is no filtration \( F \) on \( A \) such that \( \bar{X}^F \cong V' \).

3 Semidegree and Subdegree

Definition 3.1. A degree-like function on \( A \) is a map \( \delta : A \setminus \{0\} \to \mathbb{Z} \cup \{-\infty\} \) such that:

1. \( \delta(\mathbb{K}) = 0 \).
2. \( \delta(f + g) \leq \max\{\delta(f), \delta(g)\} \) for all \( f, g \in A \), with \( < \) in the preceding equation implying \( \delta(f) = \delta(g) \).
3. \( \delta(fg) \leq \delta(f) + \delta(g) \) for all \( f, g \in A \).

Remark 3.2. We allow degree-like functions to have \( -\infty \) as a value, since they do correspond to some filtrations (via the correspondence defined below). On the other hand, we will see (from Proposition 5.1 and Theorem 5.10 below) that the degree-like functions associated to completions of affine varieties are always integer-valued.

There is a one-to-one correspondence between degree-like functions and filtrations:

\[
\begin{align*}
\text{Filtrations} & \quad \longleftrightarrow \quad \text{degree-like functions} \\
F = \{F_d\}_{d \in \mathbb{Z}} & \quad \mapsto \quad \delta_F : f \in A \mapsto \inf\{d : f \in F_d\} \\
F_\delta := \{F_d := \{f \in A : \delta(f) \leq d\}\}_{d \in \mathbb{Z}} & \quad \longleftrightarrow \quad \delta
\end{align*}
\]

In the remainder of this article we identify degree-like functions with the corresponding filtrations. In particular, we refer to a degree-like function \( \delta \) as projective (resp. finitely generated) iff the corresponding filtration \( F_\delta \) is projective (resp. finitely generated). Similarly,
\[ K[X]^\delta, \text{ gr } K[X]^\delta \text{ and } X^\delta \text{ will be shorthand notations respectively for } K[X]^F, \text{ gr } K[X]^F \text{ and } X^F. \]

The protagonists of this article are two classes of degree-like functions which satisfy stronger versions of the multiplicative property (i.e. property \([\text{above}]\) above).

**Definition 3.3.**
- A degree-like function \( \delta \) on \( A \) is a *semidegree* iff \( \delta(fg) = \delta(f) + \delta(g) \) for all \( f, g \in A \setminus \{0\} \).
- We say that \( \delta \) is a *subdegree* if there are subdegrees \( \delta_1, \ldots, \delta_N \) such that
  \[ \delta(f) = \max_{1 \leq i \leq N} \delta_i(f) \text{ for all } f \in A \setminus \{0\}. \]

Given a subdegree \( \delta \) as in \([\text{2}]\), we may assume by getting rid of some \( \delta_i \)’s if necessary that every \( \delta_i \) that appears in \([\text{2}]\) is not redundant, i.e. for every \( i \), there is an \( f \in A \) such that \( \delta_i(f) > \delta_j(f) \) for all \( j \neq i \). In this case we will say that \([\text{2}]\) is a minimal presentation of \( \delta \).

**Remark 3.4.** Note that the negative of a semidegree is a *discrete valuation*. The negative of a degree-like function is usually called a (discrete) pseudo-valuation and the negative of a subdegree is a (discrete) subvaluation (see e.g. \([\text{Huc70}]\)).

**Example 3.5** (Weighted Degree). As noted in the introduction, every weighted degree \( \delta \) on the polynomial ring \( K[x_1, \ldots, x_n] \) is a semidegree.

**Example 3.6** (Subdegrees determined by rational polytopes). Let \( X \) be the \( n \)-torus \((\mathbb{K}^*)^n\) and \( A := K[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}] \) be its coordinate ring. Let \( \mathcal{P} \) be a convex rational polytope (i.e. a convex polytope in \( \mathbb{R}^n \) with vertices in \( \mathbb{Q}^n \)) of dimension \( n \) containing origin in its interior. Recall (from introduction) that \( \mathcal{P} \) determines a degree-like function on \( A \). More precisely, recall that if \( \delta_{\mathcal{P}} : A \setminus \{0\} \to \mathbb{Q} \) is defined as

\[
\delta_{\mathcal{P}}(\sum a_\alpha x^\alpha) := \inf \{ r \in \mathbb{R} : \alpha \in r\mathcal{P} \text{ for all } \alpha \in \mathbb{Z}^n \text{ such that } a_\alpha \neq 0 \},
\]

and \( e \) is any positive integer such that \( e\delta_{\mathcal{P}} \) is integer-valued, then \( e\delta_{\mathcal{P}} \) is a degree-like function on \( A \). We claim that \( e\delta_{\mathcal{P}} \) is in fact a subdegree on \( A \). Indeed, for each facet \( \mathcal{Q} \) of \( \mathcal{P} \), let \( \omega_{\mathcal{Q}} \) be the smallest ‘outward pointing’ integral vector normal to \( \mathcal{Q} \) and let \( c_{\mathcal{Q}} := \langle \omega_{\mathcal{Q}}, \alpha_{\mathcal{Q}} \rangle \), where \( \alpha_{\mathcal{Q}} \) is any element of the hyperplane that contains \( \mathcal{Q} \). Define

\[
\delta_{\mathcal{Q}}(\sum a_\alpha x^\alpha) := \max_{a_\alpha \neq 0} \frac{\langle \omega_{\mathcal{Q}}, \alpha \rangle}{c_{\mathcal{Q}}}.
\]

Then it is straightforward to see that \( e\delta_{\mathcal{Q}} \) is a weighted degree on \( A \) for each facet \( \mathcal{Q} \) of \( \mathcal{P} \), and \( e\delta_{\mathcal{P}} = \max\{e\delta_{\mathcal{Q}} : \mathcal{Q} \text{ is a facet of } \mathcal{P} \} \), so that \( e\delta_{\mathcal{P}} \) is a subdegree, as claimed. It is not hard to see that \( X^{\delta_{\mathcal{P}}} \) is isomorphic to the toric variety \( X_{\mathcal{P}} \) determined by \( \mathcal{P} \).

**Example 3.7** (An iterated semidegree). Let \( X := \mathbb{K}^2 \) and \( A := K[x_1, x_2] \). Define a filtration \( \mathcal{F} := \{ F_d : d \geq 0 \} \) on \( A \) by setting \( F_0 := K \), \( F_1 := K \langle 1, x_1^2 - x_2^2 \rangle \), \( F_2 := (F_1)^2 + K(x_2) \), \( F_3 := F_1 F_2 + K(x_1) \) and \( F_d := \sum_{j=1}^{d-1} F_d F_{d-j} \) for \( d \geq 4 \). Let \( \delta := \delta_{\mathcal{F}} \). Note that \( \delta(x_1) = 3 \) and \( \delta(x_2) = 2 \), but \( \delta(x_1^2 - x_2^2) = 1 \). We claim that \( \delta \) is a semidegree. Indeed,

\[
A^\delta = K[(1), (x_1)_3, (x_2)_2, (x_1^2 - x_2^2)_1]\cong K[X_1, X_2, Y, Z]/(Y Z^3 - X_1^2 + X_2^3),
\]

(3)
where the last isomorphism is induced by a \(\mathbb{K}\)-algebra homomorphism which sends \(X_1 \mapsto (x_1)_3\), \(X_2 \mapsto (x_2)_2\), \(Y \mapsto (x_2^2 - x_3)_1\) and \(Z \mapsto (1)_1\). The inverse image of the ideal \(I := \langle (1)_1 \rangle\) of \(A^3\) under this isomorphism coincides with ideal \(\langle Z, YZ^5 - X_1^2 + X_2^3 \rangle = \langle Z, X_1^2 - X_2^3 \rangle\). Since the latter is a prime ideal of \(\mathbb{K}[X_1, X_2, Y, Z]\), it follows that \(I\) is a prime ideal of \(A^3\) as well. Theorem 4.1 then implies that \(\delta\) is a semidegree. Identity (3) implies that \(X_\delta\) is isomorphic to the hypersurface \(V(YZ^5 - X_1^2 + X_2^3)\) of the weighted projective space \(\mathbb{P}^3(\mathbb{K}; 1, 3, 2, 1)\) with (weighted homogeneous) coordinates \([Z : X_1 : X_2 : Y]\). The semidegree \(\delta\) is an example of an iterated semidegree (to be introduced in the forthcoming continuation of this article).

4 Basic Structure of Subdegrees

The goal of this section is to prove the following theorem and its corollaries. Recall that an ideal \(I\) of a ring \(R\) is decomposable if it is the intersection of finitely many primary ideals of \(R\).

**Theorem 4.1** (Structure theorem for subdegrees). Let \(\delta\) be a degree-like function on an integral domain \(A\) and \(I\) be the ideal of \(A^\delta\) generated by \((1)_1\) (recall the notation from Section 2) that each for \(f \in A\) and each \(d \geq \delta(f)\), we denote the ‘copy’ of \(f\) in the \(d\)-th graded component of \(A^\delta\) by \((f)_d\).

1. \(\delta\) is a semidegree if and only if \(I\) is a prime ideal.
2. \(\delta\) is a subdegree if and only if \(I\) is a decomposable radical ideal.
3. If \(\delta\) is a subdegree, then semidegrees \(\delta_1, \ldots, \delta_N\) of a minimal presentation \(\delta = \max_{1 \leq i \leq N} \delta_i\) of \(\delta\) are unique.
4. If \(A^\delta\) is Noetherian, then \(\delta\) is a subdegree if and only if \(I\) is a radical ideal.

Theorem 4.1 was proven in my Ph.D. thesis [Mon10, Section 2.2]. Later I came to know of the article [Szp66] (I heartily thank Professor Bernard Teissier for bringing it to my attention) - from which all the assertions of Theorem 4.1 except for the uniqueness of the minimal presentation (i.e. assertion 3) follow (see Remark 4.3 below). I present here a full proof of the result, since proving only assertion 3 takes almost the same amount of work. The essential ingredients are Lemmas 4.1.1 and 4.1.2.

**Corollary 4.2.** Assume \(A^\delta\) is Noetherian. Then \(\delta\) is a subdegree on \(A\) iff \(\delta(f^k) = k\delta(f)\) for all \(f \in A\) and \(k \geq 0\).

**Proof.** Let \(I\) be the ideal of \(A^\delta\) from the statement of Theorem 4.1. Note that for every \(f \in A\) and \(d \in \mathbb{Z}\), \((f)_d \in I\) iff \(\delta(f) < d\). It follows that \(I\) is radical iff \(\delta(f^k) = k\delta(f)\) for all \(f \in A\) and \(k \geq 0\). Now the corollary follows from assertion 4 of Theorem 4.1. \(\square\)

**Remark 4.3.** [Szp66] considers order functions, which are negative of our degree-like functions with the exception that the values are allowed to be in \(\mathbb{R} \cup \infty\) (instead of \(\mathbb{Z} \cup \infty\)). An order function \(\nu\) on a ring \(A\) is homogeneous if \(\nu(f^k) = k\delta(f)\) for all \(k \geq 0\) and \(f \in A\). [Szp66] Theorem 1] states that \(\nu\) is homogeneous iff \(\nu\) is the maximum of a (possibly infinite) collection of valuations. Assertions 1, 2 and 4 of Theorem 4.1 and Corollary 4.2 follow from this result. We however did not manage to prove assertion 3 of Theorem 4.1 from statements or proofs of the results of [Szp66].

**Proof of Theorem 4.1.** Assertion 1 follows from the definition of semidegrees and the observation that for all \(f \in A\) and \(d \in \mathbb{Z}\), \(\delta(f) = d\) iff \((f)_d \in A^\delta \setminus I\). Assertion 4 is a consequence
Assume $\delta$ is a subdegree. Let $\delta = \max_{1 \leq i \leq N} \delta_i$ be a minimal presentation of $\delta$. For each $i$ with $0 \leq i \leq N$, fix an $f_i \in A$ such that
\[ d_i := \delta_i(f_i) > \delta_j(f_i) \text{ for } 1 \leq j \neq i \leq N. \quad (4) \]
Then, in particular, $\delta(f_i) = d_i \in \mathbb{Z}$. Assertion 2 of the following lemma implies the ‘only if’ direction of assertion 2 of the theorem.

**Lemma 4.1.1.** For each $i$, $1 \leq i \leq N$, let $p_i := (I : (f_i)_{d_i}) = \{ F \in A^\delta : F(f_i)_{d_i} \in I \}$. 
(1) The collection of homogeneous elements of $p_i$ is $L_i := \{ (f)_{d} : f \in A, \delta_i(f) < \delta(f) \leq d \}$. 
(2) For each $i$, $p_i$ is a distinct minimal prime ideal of $A^\delta$ containing $I$. In particular, $(f_i)_{d_i} \in \left( \bigcap_{j \neq i} p_j \right) \setminus p_i$ for each $i$. Moreover, $I = \bigcap_{i=1}^{N} p_i$.

**Proof.** We first prove assertion 1. Fix an $i$, $1 \leq i \leq N$. Let $(f)_{d} \in I$ be an arbitrary homogeneous element of $p_i$. We will show that $(f)_{d} \in L_i$. If $\delta(f) < d$, then $\delta_i(f) \leq \delta(f) < d$ and $(f)_{d} \in L_i$. So assume $\delta(f) = d$. Since $(f)_{d}(f_i)_{d_i} = (f_i)_{d+d_i} \in I$, it follows that $\delta((f_i)_{d_i}) < d + d_i$. But then $\delta_i(f_i) = \delta_i(f) + \delta_i(f_i) < d + d_i$, so that $\delta_i(f) < d = \delta(f)$, and thus $(f)_{d} \in L_i$. To summarize, all homogeneous elements of $p_i$ belong to $L_i$.

Now we show that $L_i \subseteq p_i$. Pick $(f)_{d} \in L_i$. If $d > \delta(f)$, then $(f)_{d} \in I \subseteq p_i$. So assume $d = \delta(f)$. Then $\delta_i(f) < d = \delta(f)$, and thus $\delta_i(f_i) = \delta_i(f) + \delta_i(f_i) < d + d_i$. Also, for each $j \neq i$, $\delta_j(f_i) < d_i$, so that $\delta_j((f_i)_{d_i}) = \delta_j(f) + \delta_j((f_i)_{d_i}) < d + d_i$. It follows that $\delta((f_i)_{d_i}) < d + d_i$, so that $(f)_{d}(f_i)_{d_i} = (f_i)_{d+d_i} \in I$. Therefore $(f)_{d} \in p_i$, which proves that $L_i \subseteq p_i$ and completes the proof of assertion 1 of the lemma.

To see that $p_i$ is prime, let $(g_1)_{e_1},(g_2)_{e_2}$ be homogeneous elements of $A^\delta$ such that $(g_1)_{e_1}(g_2)_{e_2} \in p_i$. Since $p_i$ is a homogeneous ideal, it suffices to show that one of the $(g_j)_{e_j}$’s belong to $p_i$. Due to assertion 1 of the lemma, $(g_1)_{e_1}(g_2)_{e_2} = (g_1)_{e_1}(g_2)_{e_2} \in L_i$, which means that $\delta_i(g_1) = \delta_i(g_1) + \delta_i(g_2) < e_1 + e_2$. Since $e_j \geq \delta(g_j) \geq \delta_i(g_1)$ for each $j$, it follows that there is at least one $j$ such that $\delta_i(g_j) < e_j$. Then assertion 1 of the lemma implies that $(g_j)_{e_j} \in p_i$, as required.

Next we show that $p_i$’s are distinct minimal prime ideals of $A^\delta$ containing $I$. Indeed, that $p_i$’s are distinct follows from assertion 1 and the observation that $(f_i)_{d_i} \in \left( \bigcap_{j \neq i} L_j \right) \setminus L_i$ for $1 \leq i \leq N$. Now fix $i$, $1 \leq i \leq N$, and a prime ideal $p$ of $A^\delta$ such that $I \subseteq p \subseteq p_i$. Then $(f_i)_{d_i} \notin p$ (since assertion 1 implies $(f_i)_{d_i} \notin p_i$). But if $(g)_{e} \in p_i$, then $(g)_{e} \in p$ and it follows that $(g)_{e} \in p$. Hence $p_i \subseteq p$, and therefore $p_i = p$. Consequently, $p_i$ is a minimal prime ideal of $A^\delta$ containing $I$, as required.

Finally, pick any homogeneous $(g)_{e} \in \bigcap_{i=1}^{N} p_i$. Then by assertion 1 for each $i$, $\delta_i(g) < e$. Therefore $\delta(g) = \max_{1 \leq i \leq N} \delta_i(g) < e$, and hence $(g)_{e} \in I$. It follows that $\bigcap_{i=1}^{N} p_i = I$, which concludes the proof of the lemma.

Now we prove assertion 2 of the theorem (the ‘if’ direction of assertion 2 will be proven at the end). Let $\delta = \max_{1 \leq i \leq N} \delta_i$ be another minimal presentation of $\delta$. Then there exist
Fix an $i$, $1 \leq i \leq N$. Lemma 4.1 implies that

$$(f'_i)_{d'_i} \in \bigcap_{j \neq i} L'_j \setminus L'_i = \bigcap_{j \neq i} L_j \setminus L_i,$$

where $L_j$'s (resp. $L'_j$'s) are the set of homogeneous elements of $p_j$'s (resp. $p'_j$'s). It follows from the description of $L_j$'s and $L'_j$'s from assertion 1 of Lemma 4.1 that for each $i$, $1 \leq i \leq N$,

$$d'_i = \delta_i(f'_i) = \delta(f'_i) = \delta_i(f'_i).$$

(5) \hspace{1cm} \delta_j(f'_i) < \delta_i(f'_i) \text{ for all } j \neq i. \hspace{1cm} (6)

Identity (5) and inequality (6) imply that property (4) remains true even after replacing $f_i$'s and $d_i$'s by $f'_i$'s and $d'_i$'s. Since $f_i$'s were assumed to be arbitrary elements in $A$ such that (4) is true, we may assume without loss of generality that $f_i = f'_i$ for each $i$. But then the following lemma implies that $\delta_i = \delta'_i$ for each $i$, as required.

**Lemma 4.1.2.** Suppose $\delta = \max_{1 \leq i \leq N} \delta_i$ is a minimal presentation for $\delta$ and $f_1, \ldots, f_N \in A$ satisfy (4), then

$$\delta_i(f) = \lim_{k \to \infty} \delta((f_i)^k f) - \delta((f_i)^k)$$

(7) for all $f \in A$ and all $i$, $1 \leq i \leq N$.

**Proof.** Fix an $i$, $1 \leq i \leq N$, and define $\tilde{\delta}_i(f) := \lim_{k \to \infty} \delta((f_i)^k f) - \delta((f_i)^k)$ for all $f \in A$. We first show that $\tilde{\delta}_i$ is well defined. Indeed, let $f \in A$ and $k \geq 1$. Since $(f_i)_d_i \notin I$ and $I$ is radical (assertion 2 of Lemma 4.1), it follows that $((f_i)_d_i)^k = ((f_i)^k)_{kd_i} \notin I$, so that $\delta((f_i)^k) = kd_i$. Therefore,

$$\delta((f_i)^{k+1}) - \delta((f_i)^k) \leq \delta((f_i)^k f) + \delta(f_i) - \delta((f_i)^k + 1)$$

$$= \delta((f_i)^k f) + d_i - (k + 1)d_i$$

$$= \delta((f_i)^k f) - kd_i$$

$$= \delta((f_i)^k f) - \delta((f_i)^k).$$

(8)

It follows that $\tilde{\delta}_i(f)$ is a well defined element in $\mathbb{Z} \cup \{-\infty\}$.

We now show that $\tilde{\delta}_i = \delta_i$. Note that for all $k \geq 0$ and all $f \in A$, $\delta((f_i)^k f) - \delta((f_i)^k) \geq \delta_i((f_i)^k f) - \delta((f_i)^k) = \delta_i(f)$ (the last equality uses that $\delta(f_i) = \delta_i(f_i)$). So, that $\delta_i \geq \delta_i$. To see the opposite inequality, let $f \in A$ and $d \in \mathbb{Z}$ be such that $d \geq \delta_i(f)$. Then $kd_i + d \geq \delta_i((f_i)^k f)$ for all $k$. Moreover, (4) implies that for sufficiently large $k$, $kd_i + d > kd_i(f_i) + \delta_j(f) = \delta_j(f)$.
\[ \delta_i((f_i)^k f) \] for all \( j \neq i \). It follows that for sufficiently large \( k \), \( \delta((f_i)^k f) \leq kd_i + d \), and hence
\[ \delta((f_i)^k f) - \delta((f_i)^k) \leq d. \] Since \( \{ \delta((f_i)^k f) - \delta((f_i)^k) \} \) is a decreasing sequence due to (5), it follows that \( \delta_i(f) \leq d \). Summarizing, we have proved that \( \delta_i(f) \leq d \) whenever \( \delta_i(f) \leq d \).

Applying the preceding statement to \( d = \delta_i(f) \) when \( \delta_i(f) \in \mathbb{Z} \), and otherwise letting \( d \) converge to \( \delta_i(f) = -\infty \) from above, we see that \( \delta_i(f) \leq \delta_i(f) \). It follows that \( \delta_i(f) = \tilde{\delta}_i(f) \), which completes the proof of the lemma and therefore, assertion 3 of Theorem 4.1.

It remains to prove the ‘if’ direction of assertion 2 of the theorem. So assume \( I \) is a decomposable radical ideal of \( A^\delta \). We have to show that \( \delta \) is a subdegree. It follows from the general theory of primary decomposition (e.g. as in [AM69] Chapter 4) that there exist \( (f_i)_{d_1}, \ldots, (f_N)_{d_N} \in A^\delta \setminus I \) such that \( p_i := (I : (f_i)_{d_i}) \) is a prime ideal containing \( I \) for each \( i \) and \( I = \bigcap_{i=1}^N p_i \). Since \( (f_i)_{d_i} \notin I \), it follows that \( \delta(f_i) = d_i \in \mathbb{Z} \) for each \( i \). For each \( i = 1, \ldots, N \), define \( \delta_i : A \to \mathbb{Z} \cup \{-\infty\} \) as in (7), i.e.
\[ \delta_i(f) := \lim_{k \to \infty} \delta((f_i)^k f) - \delta((f_i)^k) \]
for all \( f \in A \). The same arguments as in the first part of the proof of Lemma 4.1.2 imply that the relations of (5) remain true and consequently \( \delta_i \) is well defined for all \( i \), \( 1 \leq i \leq N \). It follows from elementary but somewhat long computations that \( \delta_i \)'s satisfy the properties of a subdegree - we refer to the proof of [Mon10] Theorem 2.2.1 for details. We now show that \( \delta = \max_{i=1}^N \delta_i \). Indeed, fix \( f \in A \) and \( 1 \leq i \leq N \). Then \( \delta(f) \geq \delta(f_i f) - \delta(f_i) \geq \delta(f) \) (the first inequality follows from the multiplicative property of degree-like functions and the second one is a consequence of (5)). It follows that \( \delta \geq \max_{i=1}^N \delta_i \).

For the opposite inequality, pick any \( f \in A \). If \( \delta(f) = -\infty \), then for all \( i \) and for all \( k \geq 0 \), \( \delta((f_i)^k f) = -\infty \), and therefore \( \delta_i(f) = -\infty = \delta(f) \). So assume \( \delta(f) \in \mathbb{Z} \). Since \( (f_i)_{\delta(f)} \notin I \), there is an \( i \) such that \( (f_i)_{\delta(f)} \notin p_i \). Note that \( (f_i)_{d_i} \) also does not lie in \( p_i \), for otherwise \( ((f_i)_{d_i})^2 \) would be an element of \( I \) and this would imply that \( (f_i)_{d_i} \in I \) (since \( I \) is a radical ideal), contradicting our choice of \( (f_i)_{d_i} \). Thus neither of \( (f)_{\delta(f)} \) and \( (f_i)_{d_i} \) is an element of \( p_i \). Since \( p_i \) is prime, it follows that for all \( k \geq 0 \), \( (f_i)_{d_i}((f_i)^k f)_{\delta(f)} = ((f_i)^k f)_{kd_i+\delta(f)} \notin p_i \). Consequently \( \delta((f_i)^k f) = \delta(f) + kd_i = \delta(f) + \delta((f_i)^k) \) for all \( k \geq 0 \). It then follows that \( \delta_i(f) = \delta(f) \). Combining with the inequality proved in the preceding paragraph, this shows that \( \delta = \max_{i=1}^N \delta_i \) and therefore \( \delta \) is indeed a subdegree. This completes the proof of the theorem.

**Corollary 4.4.** Let \( \delta \) be a non-negative subdegree on \( A \) and \( \delta = \max_{i=1}^N \delta_i \) be its minimal presentation. Let \( X := \text{Spec} A \). Then the number of the irreducible components of \( X_\infty := X^\delta \setminus X \) is \( N \) if none of the \( \delta_i \)'s is the zero degree-like function and \( N - 1 \) otherwise.

**Proof.** Proposition 2.20 and Remark-Definition 2.6 imply that \( X_\infty = V(I) \subseteq \text{Proj} A^\delta \), where \( I \) is the ideal in \( A^\delta \) generated by (1). According to Lemma 4.1.1 \( I \) has a minimal prime decomposition of the form \( I = \bigcap_{i=1}^N p_i \), with each \( p_i \) being a prime ideal corresponding to \( \delta_i \), \( 1 \leq i \leq N \). Consequently \( X_\infty = V(I) = \bigcup_{i=1}^N V(p_i) \). The corollary now follows from the following

**Claim.** For each \( i \), \( 1 \leq i \leq N \), the semidegree \( \delta_i \) is identically zero on \( A \) iff \( V(p_i) = \emptyset \).

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Proof. Let $A^\delta_+ = \bigoplus_{d \geq 1} \{(f)_d : f \in A\}$ be the irrelevant ideal of $A^\delta$. Fix $i$, $1 \leq i \leq N$. Then

\begin{align*}
V(p_i) = \emptyset & \iff p_i \supseteq A^\delta_+ \\
& \iff \{(f)_d : f \in A, \; d > \delta_i(f)\} \supseteq \{(f)_d : f \in A, \; d \geq 1\} \text{ (according to Lemma 4.1.1)} \\
& \iff \delta_i(f) \leq 0 \text{ for all } f \in A \\
& \iff \delta = \max \{\delta_j : j \neq i, \; 1 \leq j \leq N\} \cup \{\delta_0\} \text{ (since } \delta \text{ is non-negative)},
\end{align*}

where $\delta_0$ is the zero degree-like function on $A$, i.e. $\delta_0(f) := 0$ for all $f \in A$. Since $\delta_0$ is a semidegree and the minimal decomposition of $\delta$ is unique, it follows that $V(p_i) = \emptyset \iff \delta_i = \delta_0$, which completes the proof of the claim.

Example 4.5. Applied to the subdegree of Example 3.6 determined by a convex rational polytope $\mathcal{P}$, Corollary 4.4 implies the (standard) fact that the irreducible components of the complement of the torus in the toric variety $X_{\mathcal{P}}$ are in a one-to-one correspondence with the facets of $\mathcal{P}$.

5 Normality at infinity

In this section using the theory of Rees’ valuations we work out the precise relation between the semidegrees in the minimal presentation of a subdegree and the orders of vanishing along the components of the hypersurface at infinity on the corresponding completion. We also establish our ‘Main Existence Theorem’ which states that given a completion $\bar{X}$ of an affine variety $X$ determined by an arbitrary degree-like function, there is a subdegree on $\mathbb{K}[X]$ which determines the normalization at infinity of $\bar{X}$ with respect to $X$. As a consequence we prove a ‘finiteness’ property of divisorial valuations. We start with a presentation (following [McA83, chapter XI]) of the relevant results of Rees (starting with a reminder of the notion of a Krull domain).

Definition. A domain $B$ is a Krull domain iff

1. $B_p$ is a discrete valuation ring for all height one prime ideals $p$ of $B$, and
2. every non-zero principal ideal of $B$ is the intersection of a finite number of primary ideals of height one.

Every normal Noetherian domain is a Krull domain [Mat80, Section 41]. In particular, the integral closure of $A^\delta$ is a Krull domain provided that $A^\delta$ is finitely generated.

For an ideal $I$ of a ring $R$ define $\nu_I : R \to \mathbb{N} \cup \{\infty\}$ and $\bar{\nu}_I : R \to \mathbb{Q}_+ \cup \{\infty\}$ by:

\[
\nu_I(x) := \sup \{m : x \in I^m\}, \quad \text{and} \quad \bar{\nu}_I(x) := \lim_{m \to \infty} \frac{\nu_I(x^m)}{m},
\]

for all $x \in R$. Recall that the integral closure $\bar{J}$ of an ideal $J$ of $R$ is the ideal defined by: $\bar{J} := \{x \in R : x \text{ satisfies an equation of the form: } x^s + j_1 x^{s-1} + \cdots + j_s = 0 \text{ with } j_k \in J^k \text{ for all } k = 1, \ldots, s\}$.  

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Theorem 5.0 (Rees’ Theorem - see [McA83 Propositions 11.1 – 11.6]). For any ring $R$ and any ideal $I$ of $R$, $\hat{\nu}_I$ is well defined. Assume $R$ is a Noetherian domain. Then

(1) there is a positive integer $e$ such that for all $x \in R$, $\hat{\nu}_I(x) \in \frac{1}{e}\mathbb{N}$, and

(2) if $k \geq 0$ is an integer then $\hat{\nu}_I(x) \geq k$ if and only if $x \in \bar{I}^k$, where $\bar{I}^k$ is the integral closure of $I^k$ in $R$.

(3) Assume in addition that $I$ is a principal ideal generated by $u$ and $\bar{R}$ is an integral extension of $R$ which is a Krull domain. Let $p_1, \ldots, p_r$ be the height 1 prime ideals of $\bar{R}$ containing $u$. Then for all $x \in R$, $\hat{\nu}_I(x) = \min\{(\nu_i(x) / e_i) : i = 1, \ldots, r\}$, where for each $i = 1, \ldots, r$, $\nu_i$ is the valuation associated with the discrete valuation ring $\bar{R}_{p_i}$, and $e_i := \nu_i(u)$.

Let $\delta$ be a finitely generated subdegree on a $K$-algebra $A$ with a minimal presentation $\delta = \max_{1 \leq i \leq N} \delta_i$. It is not hard to see that the Noetherian-ness of $A^\delta$ implies that $\delta$ is integer-valued, i.e., there is no non-zero $f \in A$ such that $\delta(f) = -\infty$. On the other hand, it is not obvious (e.g. from identity (7)) whether $\delta_i$’s can take $-\infty$ as a value or not. But the following corollary states that in fact they can not.

Proposition 5.1. Let $A$ and $\delta$ be as above. Let $B$ be any Krull domain which is also an integral extension of $A^\delta$ and $p_1, \ldots, p_r$ be the height one primes of $B$ containing $(1)_1$. For each $j$, $1 \leq j \leq r$, define a function $\delta_j$ on $A \setminus \{0\}$ by

$$\delta_j(f) = \delta(f) - \frac{\nu_j((f)\delta(f))}{e_j}$$

where $\nu_j$ is the discrete valuation of the discrete valuation ring $B_{p_j}$ and $e_j := \nu_j((1)_1)$. Then for each $i$, $1 \leq i \leq N$, $\delta_i \equiv \delta_j$ for some $j$, $1 \leq j \leq r$. In particular, for every $i$, the semidegree $\delta_i$ is integer valued, and consequently $-\delta_i$ is a discrete valuation.

Sketch of a proof. Let $I$ be the ideal generated by $(1)_1$ in $A^\delta$ and $\hat{\nu}_I$ be the Rees’ valuation corresponding to $I$. Let $f \in A$. Since $\delta(f) \in \mathbb{Z}$ (from the remark following Theorem 5.0) and $\delta$ is a subdegree, it follows that $\hat{\nu}_I((f)\delta(f)) = 0$. Assertion 3 of Rees’ theorem then implies that $\min_{i=1}^r \frac{\nu_i((f)\delta_i)}{e_i} = 0$, where $e_i := \nu_i((1)_1)$ for every $j$. Therefore $\delta(f) = \delta(f) - \min_{i=1}^r \frac{\nu_i((f)\delta_i)}{e_i} = \max_{j=1}^r \delta_j(f)$ for all $f \in A$. The assertions of Proposition 5.1 then follow from the uniqueness of the minimal presentation of subdegrees (assertion 3 of Theorem 4.1) provided we can show that each $\delta_j$ is a subdegree. The latter assertion follows from case by case computations - we refer the reader to [Mon10 Theorem 2.21] for details.$\square$

Let $X$ be an affine algebraic variety and $\delta$ be a finitely generated subdegree on $A := K[X]$ with the minimal presentation $\delta = \max_{1 \leq i \leq N} \delta_i$. Pick $i$ such that $\delta_i \neq 0$. It follows from (the proof of) Corollary 5.1 that $\delta_i$ corresponds to a component $V_i$ of the hypersurface at infinity on $X^\delta$. Using Proposition 5.1, the following proposition establishes the relation between $\delta_i$ and the orders of vanishing along $V_i$.

Proposition 5.2. Let $X$, $A$, $\delta$, $\delta_i$ and $V_i$ be as above. For every $f \in K(X) \setminus \{0\}$, the order of vanishing of $f$ along $V_i$ is $-\delta_i(f)/d_i$, where $d_i := \gcd\{\delta_i(g) : g \in K(X), \delta_i(g) > 0\}$. In particular, the local ring $\mathcal{O}_{V_i,X^\delta}$ of $X^\delta$ at $V_i$ is regular.
Proof. Let $p_i$ be the prime ideal of $A^6$ corresponding to $\delta_i$. Then $O_{V_i, X^6}$ is the degree zero part of the local ring $A^6_{p_i}$. As in Remark 5.2, let us identify $A^6$ with $\sum F_d t^d$. Then
\[
O_{V_i, X^6} = \left\{ \frac{f^{kd}}{(gt)^k} : \text{gt} \notin p_i, \ d \geq \delta(g), \ k \geq 0 \right\}
\]
\[= \left\{ \frac{f}{g^k} : (g)\delta(g) \notin p_i, \ \delta(f) \leq k\delta(g), \ k \geq 0 \right\}. \tag{10}
\]

Recall from Proposition 5.1 that $\nu_i := -\delta_i/d_i$ is a discrete valuation on $K(X)$. Let $R_i \subseteq K(X)$ be the valuation ring of $\nu_i$. It suffices to show that $O_{V_i, X^6} = R_i$. Recall that $R_i := \{g_1/g_2 : g_1, g_2 \in A, \ g_2 \neq 0, \ \nu_i(g_1/g_2) \geq 0\}$. Pick $g_1, g_2 \in A$ such that $g_1/g_2 \in R_i$. Then $\nu_i(g_1/g_2) = \nu_i(g_1) - \nu_i(g_2) \geq 0$ and hence $\delta_i(g_1) \leq \delta_i(g_2)$. Pick $f_i \in A$ such that $\delta_i(f_i) > \delta_i(f_i)$ for all $j \neq i$. It follows due to (8) that there is $k \geq 1$ such that $\delta(g_i f_i^k) = \delta_i(g_2 f_i^k)$ for $l = 1, 2$. Then $\delta(g_i f_i^k) = \delta_i(g_1 f_i^k) = \delta_i(g_1) + \delta_i(f_i) \leq \delta_i(g_2) + \delta_i(f_i) = \delta_i(g_2 f_i^k) = \delta(g_2 f_i^k)$. Moreover, Lemma 4.1.1 implies that $(g_i f_i^k)\delta_i(g_2 f_i^k) \notin p_i$ for $l = 1, 2$. Then, according to (10), $g_i f_i^k / (g_2 f_i^k) = g_1/g_2 \in O_{V_i, X^6}$. Consequently $R_i \subseteq O_{V_i, X^6}$. Therefore $O_{V_i, X^6} = R_i$ due to

**Lemma 5.2.1.** Let $R$ be a discrete valuation ring and $K$ be the quotient field of $R$. If $S$ is a proper subring of $K$ such that $R \subseteq S$, then $R = S$.

Proof. Let $\nu$ be the discrete valuation associated to $R$ and $h \in R$ be a parameter for $\nu$, in particular $\nu(h) = 1$. Assume contrary to the claim that $R \neq S$. Let $f \in S \setminus R$. Then $f = u/h^k$ for some unit $u$ of $R$ and $k > 0$. It follows that $h^{-1} = u^{-1}f h^{k-1} \in S$. Let $g \in K \setminus \{0\}$. Then $\nu(gh^{-\nu(g)}) = 0$ and therefore $gh^{-\nu(g)} \in R \subseteq S$ and also $g = gh^{-\nu(g)} \cdot h^{\nu(g)} \in S$. Therefore $S = K$, contrary to the assumptions, which completes the proof. \hfill $\square$

To summarize, we have proved that $O_{V_i, X^6}$ is precisely the valuation ring of $\nu_i$ (and therefore regular). Since valuations are completely determined by their valuation rings [ZS73, Section VI.8], it follows that $\nu_i$ is the valuation corresponding to $O_{V_i, X^6}$, which completes the proof of the proposition. \hfill $\square$

**Remark 5.3.** If $V$ is a codimension one irreducible subvariety a variety $Y$, then $O_{V, Y}$ is a regular local ring iff $V \subseteq \text{Sing} Y$ iff codimension of $\text{Sing} Y \cap V$ in $Y$ is at least 2 (where $\text{Sing} Y$ is the set of singular points of $Y$). Therefore, an equivalent formulation of Proposition 5.2 is to say that in a completion $\bar{X}^\delta$ of an affine variety $X$ determined by a subdegree $\delta$, the codimension of the ‘singular points at infinity’ is at least two, or in other words, $\bar{X}^\delta$ is non-singular in codimension one at infinity.

**Example 5.4.** Let $P$ be a convex polytope of dimension $n$ containing the origin in its interior and $Q$ be a facet of $P$. Recall from Example 3.8 that the semidegree corresponding to $Q$ associated with the subdegree $c\delta_P$ (where $c$ is an appropriate integer to ensure $c\delta_P$ is integer valued) is $e\delta_Q$, where $\delta_Q(x^\alpha) := \langle \omega_Q, \alpha \rangle / c\delta_Q$. Since the greatest common divisor of the coordinates of $\omega_Q$ is 1, Proposition 5.2 implies the familiar fact that the order of zero of $x^\alpha$ along the component of the hypersurface at infinity corresponding to $Q$ is $-\langle \omega_Q, \alpha \rangle$ [Ful93, Section 3.3].

Let $X$ and $\delta$ be as in proposition 5.2. If $X$ is normal, then Proposition 5.2 implies that $\bar{X}^\delta$ is non-singular in codimension one, i.e. $\bar{X}^\delta$ satisfies one of the two criteria of Serre for normality (see, e.g. [Mat80, Theorem 39]). It is not hard to see that if $X$ is normal, then $\bar{X}^\delta$ is in fact normal (it follows from the fact that in this case $K[X]^\delta$ is integrally closed - see
\textbf{Definition 5.5.} Let $Y$ be an algebraic variety containing $X$ as a (Zariski) dense open subset and $\psi : Z \to Y$ be a morphism of algebraic varieties. We say that $\psi$ is \textit{relatively normal at infinity with respect to $X$}.

If on the other hand $X$ is not normal, then clearly $\tilde{X}^\delta$ is not normal. But we will see below that all is not lost: $\tilde{X}^\delta$ is \textit{relatively normal at infinity with respect to $X$}.

\textbf{Proposition 5.6.} Let $\delta$ be a finitely generated subdegree on the ring of regular functions on an affine variety $X$. Then $\tilde{X}^\delta$ is relatively normal at infinity with respect to $X$.

\textit{Proof.} Let $A := \mathbb{K}[X]$ and $D(G) := \{Q \in \tilde{X}^\delta | G \not\in Q\} = \text{Spec}(A^\delta)_{(G)}$ be a basic affine open subset of $\tilde{X}^\delta$, where $G$ is a homogeneous element in $A^d$ of \textit{positive} degree $d$ and $(A^\delta)_{(G)}$ is the subring of $(A^\delta)_G := A^\delta[\frac{1}{d}]$ consisting of degree zero homogeneous elements. Then $G = (g)_d$ for some $g \in A$ with $d \geq \delta(g)$ and therefore $(A^\delta)_{(G)} = \{\frac{f}{g^k} : f \in A, \delta(f) \leq kd\}$. In particular, $\frac{1}{g} \in (A^\delta)_{(G)}$. Consequently, the regular functions on $D(G) \cap X$ are generated as a $\mathbb{K}$-algebra by $A$ and $\frac{1}{g}$, i.e. $D(G) \cap X = \text{Spec}A_g$. We now show that $(A^\delta)_{(G)}$ is integrally closed relative to $A_g$.

If $d > \delta(g)$, then $G = (g)_d((1)_1)^{d-\delta(g)}$, so that $\text{Spec}(A^\delta)_{(G)} \subseteq \text{Spec}(A^\delta)_{((1)_1)} = \text{Spec}A = X$ (Proposition 2.5). It follows that $(A^\delta)_{(G)} = A_g$, so that $(A^\delta)_{(G)}$ is obviously integrally closed relative to $A_g$. So assume $d = \delta(g)$. Let $\delta_1, \ldots, \delta_N$ be the semidegrees associated to $\delta$. W.l.o.g. we may assume that there exists $m, 1 \leq m \leq N$, such that $d = \delta_1(g) = \cdots = \delta_m(g) > \delta_i(g)$ for all $i, m < i \leq N$.

\textbf{Claim 5.6.1.} $(A^\delta)_{(G)} = A_g \cap \bigcap_{i=1}^m \mathcal{O}_{V_i, \tilde{X}^\delta}$, where for every $i, 1 \leq i \leq m, V_i$ is the component of the hypersurface at infinity of $\tilde{X}^\delta$ corresponding to $\delta_i$.

\textit{Proof.} At first note that

\[(A^\delta)_{(G)} = \left\{ \frac{f}{(g)_d} : \delta(f) \leq kd, \ k \geq 0 \right\}
= \left\{ \frac{f}{g^k} : f \in A_g, \delta(f) \leq kd, \ k \geq 0 \right\}
= A_g \cap \bigcap_{i=1}^m \left\{ \frac{f}{g^k} : f \in A, \delta_i(f) \leq kd, \ k \geq 0 \right\}
= A_g \cap \bigcap_{i=1}^m \mathcal{O}_{V_i, \tilde{X}^\delta} \cap \bigcap_{i=m+1}^N \left\{ \frac{f}{g^k} : f \in A, \delta_i(f) \leq kd, \ k \geq 0 \right\},\]
where the last equality is a consequence of Proposition 5.2. It follows that \((A^\delta)_{(G)} \subseteq A_g \cap \bigcap_{i=1}^m O_{V_i, \tilde{X}^s_i}\). It remains to show the opposite inclusion. Let \(f \in A\) and \(\frac{l}{g^x} \in \bigcap_{i=1}^m O_{V_i, \tilde{X}^s_i}\). Then \(\delta_i(f) \leq k\delta_i(g) = kd\) for all \(i, 1 \leq i \leq m\). Recall that \(\delta_j(g) < d\) for all \(j\) with \(m+1 \leq j \leq N\). It follows that if \(l\) is a sufficiently large integer, then for all \(j, m+1 \leq j \leq N\), \(\delta_j(fg^l) = \delta_j(f) + l\delta_j(g) \leq (l + k)d\). Pick an integer \(l\) as in the preceding sentence. Then \(\delta(fg^l) = \max_{j=1}^N \delta_j(fg^l) \leq (l + k)d\), and therefore \(\frac{l}{g^x} = \frac{fg^l}{g^{l+1}} \in (A^\delta)_{(G)}\). This implies that \((A^\delta)_{(G)} \supseteq A_g \cap \bigcap_{i=1}^m O_{V_i, \tilde{X}^s_i}\) and completes the proof of the claim.

Recall (from Proposition 5.2) that each \(O_{V_i, X^s_i}\) is a discrete valuation ring, therefore in particular is integrally closed. It then follows that \((A^\delta)_{(G)}\) is integrally closed relative to \(A_g\), as required to prove the proposition.

**Remark 5.7.** In the same way that normality implies non-singularity in codimension one, it can be shown that for a completion \(\tilde{X}\) of \(X\), if \(\tilde{X}\) is normal at infinity with respect to \(X\), then \(\tilde{X}\) is also non-singular at infinity in codimension one (with respect to \(X\)). Therefore Proposition 5.9 strengthens Proposition 5.2.

In general, completions corresponding to degree-like functions may have arbitrarily bad singularities at infinity. Below we introduce the notion of normalization at infinity, which makes the hypersurface at infinity of a given completion a bit ‘less singular’, in the same sense that normalization of a singular variety is less singular than itself.

**Definition 5.8.** Let \(Y\) be a variety containing \(X\) as a dense open subset. The **normalization of \(Y\) at infinity with respect to \(X\)** is another variety \(\tilde{Y}\) containing \(X\) as a dense open subset and a finite morphism \(\phi : \tilde{Y} \to Y\) such that \(\tilde{Y}\) is normal at infinity with respect to \(X\) and \(\phi|_X\) is the identity map.

**Proposition 5.9.** Let \(X\) be an arbitrary irreducible variety and \(Y\) be a variety containing \(X\) as a Zariski dense open subset. Then the normalization \(\tilde{Y}\) of \(Y\) at infinity (with respect to \(X\)) exists and is unique. Moreover, \(\tilde{Y}\) has the following universal properties:

1. If \(\psi : Z \to Y\) is a dominant morphism of algebraic varieties such that \(Z\) is normal at infinity with respect to \(X\) via \(\psi\), then there exists a unique morphism \(\theta : Z \to \tilde{Y}\) such that the following diagram commutes.

   \[
   \begin{array}{ccc}
   Z & \xrightarrow{\psi} & Y \\
   \downarrow{\theta} & & \downarrow{\phi} \\
   \tilde{Y} & \xrightarrow{\phi} & Y
   \end{array}
   \]

2. If \(Z\) is another variety containing \(X\) as an open subset and \(\psi : Z \to Y\) is a finite morphism such that \(\psi|_X\) is the identity map, then there exists a unique morphism \(\theta : \tilde{Y} \to Z\) such that the following diagram commutes.

   \[
   \begin{array}{ccc}
   Z & \xrightarrow{\psi} & Y \\
   \downarrow{\theta} & & \downarrow{\phi} \\
   \tilde{Y} & \xrightarrow{\phi} & Y
   \end{array}
   \]

**Proof.** The universal properties of the normalization at infinity are completely analogous to those of normalization and the standard proofs of the latter (e.g. the proof in [Sha94]) apply almost word by word to prove the former. We therefore skip the details.\[\Box\]
Let \( X \) be a Zariski open subset of a variety \( Y \). If \( Y \) is affine (resp. projective), then it can be shown that the normalization \( \tilde{Y} \) of \( Y \) at infinity with respect to \( X \) is also affine (resp. projective). If \( X \) is affine and \( Y \) is a completion of \( X \) determined by a degree-like function \( \delta \) on \( \mathbb{K}[X] \), then similarly it turns out that \( \tilde{Y} \) is also the completion of \( X \) determined by a degree-like function \( \delta \). In fact, as the next theorem shows, \( \delta \) can be taken to be a subdegree.

**Theorem 5.10 (Main Existence Theorem).** Let \( X \) be an affine variety and \( \delta \) be a finitely generated degree-like function on \( A := \mathbb{K}[X] \). Then

1. There is a positive integer \( e \) and a subdegree \( \delta \) on \( A \) such that for all \( h \in A \),
   \[
   \tilde{\delta}(h) := e \lim_{m \to \infty} \frac{\delta(h^m)}{m},
   \]

2. There is a natural inclusion \( A^{e\delta} \subseteq A^{\delta} \) of graded rings such that \( A^{\delta} \) is integral over \( A^{e\delta} \).
   In particular, \( \delta \) is finitely generated. If \( \delta \) is non-negative (resp. projective), then \( \delta \) is also non-negative (resp. projective).

3. The variety \( \hat{X}^{\delta} \) is the normalization at infinity of \( \hat{X}^{\delta} \) with respect to \( X \).

**Proof.** Let \( I \) be the ideal generated by \((1)_1\) in \( A^{\delta} \). Fix \( h \in A \) and \( m \in \mathbb{N} \). Since \( \delta(h^m) \leq m\delta(h) \), it follows that \( k := m\delta(h) - \delta(h^m) \) is the largest integer such that \((h^m)_{\delta(h)} \subseteq I^k \), and consequently \( \nu_I(((h)_{\delta(h)})^m) = k = m\delta(h) - \delta(h^m) \) (where \( \nu_I \) is defined as in \( \Box \)). Therefore \( \delta(h^m)/m = \delta(h) - \nu_I(((h)_{\delta(h)})^m)/m \). Assertion 1 of Theorem 5.0 then implies that \( \delta(h) := \lim_{m \to \infty} \delta(h^m)/m = \delta(h) - \nu_I(((h)_{\delta(h)})^m)/m \) is well defined and there exists a positive integer \( e \) (independent of \( h \)) such that \( \delta(h) \in \frac{1}{e} \mathbb{Z} \). Taking \( \tilde{\delta} := e\delta \) proves the displayed formula of assertion 1. Moreover, note that \( \delta(h^m) = m\delta(h) \) for all \( h \) and \( m \). Therefore, if we show that \( A^{\tilde{\delta}} \) is finitely generated, then it follows via Corollary 4.2 that \( \delta \) is a subdegree. Hence it suffices to prove assertion 2 to complete the proof of assertion 1.

We now prove assertion 2. Let \( F := \{ F_d \}_{d \in \mathbb{N}} \) be the filtration on \( A \) corresponding to \( \delta \) and identify \( A^{\delta} \) with \( \bigoplus_{i \geq 0} F_it^i \). Let \( \tilde{F}_n := \{ h \in A : \delta(h) \leq \frac{m}{e} \} \) for all \( m \geq 0 \), and define \( A^{\tilde{\delta}} := \bigoplus_{m \geq 0} \tilde{F}_m t^m \). Since \( \delta \leq \delta \), it follows that \( F_k \subseteq \tilde{F}_k \) for each \( k \in \mathbb{Z} \). Therefore \( A^{\delta} \subseteq A^{\tilde{\delta}} \).

**Claim 5.10.1.** \( A^{\tilde{\delta}} \) is integral over \( A^{\delta} \).

**Proof.** It suffices to show that \( h^i\delta(h) \) is integral over \( A^{\delta} \) for each \( h \in A \) such that \( \delta(h) \geq 0 \). Let \( h \) be as in the preceding sentence. Then \( h^i\delta(h) \) is integral over \( A^{\delta} \) if and only if \( \hat{H} := (h^i\delta(h))^e \) is integral over \( A^{\delta} \). Note that \( e\delta(h) = \delta(h^e) \) by construction of \( \delta \), and therefore \( \hat{H} = h^e\delta(h^e) \). Let \( \hat{H} := h^e\delta(h^e) \subseteq A^{\delta} \) and \( k := \nu_I(H) \), where \( I \) is the ideal generated by \((1)_1\) in \( A^{\delta} \). Since \( \nu_I(H^m) = m\delta(H) - \delta(H^m) \) and \( \delta(H) = \delta(h^e) \), it follows that \( k = \delta(h^e) - \delta(h^e) = \delta(h^e) - e\delta(h) \). Then \( k \) is an integer. Hence according to assertion 2 of Theorem 5.0, \( H \) is in the integral closure of \( I^k \) in \( A^{\delta} \), i.e. \( H \) satisfies an equation of the form \( H^e + G_1H^{e-1} + \cdots + G_l = 0 \), where \( G_i \subseteq I^k \) for each \( i \). Comparing the coefficients at \( t^{e\delta(h^e)} \) in the above equation, we may assume w.l.o.g. that each \( G_i \) is of the form \( g_it^{i\delta(h^e)} \) for some \( g_i \in A \). Since \( G_i \subseteq I^k \), it follows that \( i\delta(h^e) \geq \delta(g_i) + ik \), implying that \( g_it^{i\delta(h^e) - k} \) is an element of \( A^{\delta} \). Moreover, regarding \( H \) as an element of the ring \( A^{\delta} \) (via the embedding \( A^{\delta} \hookrightarrow A^{\tilde{\delta}} \)) yields \( H = h^e\delta(h^e) = h^e\delta(h^e + k) = h^e\delta(h^e + k) t^k = t^k \hat{H} \). Substituting these values of
Let \( \tilde{F} := \{ \tilde{F}_d | d \geq 0 \} \) be the filtration corresponding to \( \tilde{\delta} = e\tilde{\delta} \). Observe that \( \tilde{F}_d = \{ f : e\tilde{\delta}(h) \leq d \} = \tilde{F}_d \), so that the homomorphism \( \chi : A^\delta := \bigoplus_{d \geq 0} \tilde{F}_d t^d \longrightarrow \bigoplus_{d \geq 0} \tilde{F}_d e^d \cong A^\tilde{\delta} \) sends \( t \mapsto s^e \) and is the identity map on the coefficients (i.e. on \( \tilde{F}_d \) for \( d \geq 0 \)) is in fact an isomorphism of \( \mathbb{K} \)-algebras. Therefore, it follows due to Claim 5.10.1 that \( A^\tilde{\delta} = \chi(A^\delta) \) is integral over \( \chi(A^\delta) \). On the other hand, since \( \tilde{\delta} \leq e\delta \), there is a natural inclusion \( A^{e\delta} \subseteq A^\tilde{\delta} \) of graded rings and \( \chi(A^\delta) \subseteq A^{e\delta} \). Therefore \( A^\tilde{\delta} \) is integral over \( A^{e\delta} \). Since \( A^\delta \) (and therefore also \( A^{e\delta} \)) is a finitely generated \( \mathbb{K} \)-algebra, it follows that \( A^\tilde{\delta} \) is a finitely generated \( \mathbb{K} \)-algebra.

If \( \delta \) is non-negative (resp. complete), then by construction \( \tilde{\delta} \) is also non-negative (resp. complete). This completes the proof of assertion 2 and therefore also assertion 1 of the theorem. Moreover, Proposition 5.6 implies that \( X^\delta \) is normal at infinity with respect to \( X \). Since the natural morphism \( X^\delta \rightarrow X^{\tilde{\delta}} \) induced by the integral inclusion \( A^{e\delta} \subseteq A^\tilde{\delta} \) is finite, it follows that \( X^{\tilde{\delta}} \) is the normalization of \( X^\delta \) at infinity with respect to \( X \). This completes the proof of the theorem.

Remark-Definition 5.11. Let \( X, A, \delta \) and \( \tilde{\delta} \) be as in Theorem 5.10. If in addition \( X \) is normal, then \( X^{\tilde{\delta}} \) is also normal (cf. the remark following Example 5.4). The universal property of the normalization then implies that \( X^{\tilde{\delta}} \) is in fact the normalization of \( X^\delta \). Motivated by this, we will refer to \( \tilde{\delta} \) as a normalization of \( \delta \). (Note that the normalization of degree-like functions is unique only up to a constant factor, depending on the choice of \( e \) of assertion 1 of Theorem 5.10).

Example 5.12. Let \( A \) be a finite subset of \( \mathbb{Z}^n \) such that \( \mathbb{Z}^n = \mathbb{Z}A \) and the convex hull \( P \) of \( A \) in \( \mathbb{R}^n \) contains the origin in its interior. Let \( X_A \) be the closure of the image of the map \( \phi_A : (\mathbb{K}^*)^n \rightarrow \mathbb{P}^{|A|-1}(\mathbb{K}) \) whose components are the monomials \( x^a \) with \( a \in A \). \( X_A \) is the (possibly non-normal) toric variety corresponding to \( A \) (see e.g. [GKZ94, Section 5.1]). Let \( X := (\mathbb{K}^*)^n \) and \( \eta \) be the degree-like function on \( \mathbb{K}[X] \) corresponding to the completion \( X \hookrightarrow X_A \) and \( e\eta_P \) be a subdegree associated to \( P \) as in Example 2.6. Then it is not hard to see that \( \delta_P \) is a normalization of \( \eta \). This implies (following Remark-Definition 5.11) that \( X_P \) is the normalization of \( X_A \) (cf. [GKZ94, Proposition 2.8(a)]).

Remark 5.13. Let \( \delta \) be a finitely generated subdegree on \( \mathbb{K}[X] \) and \( \tilde{\delta} \) be a normalization of \( \delta \). Theorem 5.10 states that applying \( \tilde{\delta} \) to \( \mathbb{K}[X] \) produces the normalization at infinity of \( X^{\tilde{\delta}} \) with respect to \( X \). In light of this statement, it is natural to ask if applying \( \tilde{\delta} \) to the coordinate ring of the normalization of \( X \) would produce the normalization of \( X^{\tilde{\delta}} \). This indeed turns out to be true, i.e. if we denote normalization by the symbol ‘tilde’, then we have:

\[ \tilde{X}^{\delta} \cong \tilde{X}^{\tilde{\delta}}. \]

We end this section with an application of our results. Namely, we prove the following ‘finiteness-property’ of divisorial valuations (defined in the paragraph following Theorem 1.2) proved in [dFEI08] for \( \mathbb{K} = \mathbb{C} \).
Theorem 5.14. Let \( X \) be an irreducible affine variety and \( \nu \) be a divisorial valuation over \( X \). Then there exist elements \( f_1, \ldots, f_r \in \mathbb{K}[X] \setminus \{0\} \) such that for every \( f \in \mathbb{K}[X] \setminus \{0\} \),

\[
\nu(f) = \min \{ \nu'(f) : \nu' \text{ is a divisorial valuation over } X, \nu'(f_i) = \nu(f_i) \text{ for } 1 \leq i \leq r \} \quad (*)
\]

\[
= \min \{ \nu'(f) : \nu' \text{ is a valuation on } \mathbb{K}(X) \text{ such that the value group of } \nu \text{ contains the integers and } \nu'(f_i) = \nu(f_i) \text{ for } 1 \leq i \leq r \} \quad (**')
\]

Proof. There exists a normal variety \( Y \) equipped with a birational morphism \( \pi : Y \to X \) and a prime divisor \( E \) on \( Y \) such that \( \nu = q \text{ord}_E \) for some positive integer \( q \). Let \( U \) be an open affine subset of \( Y \) such that \( E \cap U \neq \emptyset \) and \( \tilde{U} \) be an arbitrary projective completion of \( U \). Pick an effective ample divisor \( D \) on \( \tilde{U} \) such that \( E \cap \tilde{U} \subseteq \text{Supp}(D) \) and set \( Z := \tilde{U} \setminus \text{Supp}(D) \). Then \( \tilde{U} \cong \mathbb{P}^2 \) for some degree-like function \( \delta \) on \( \mathbb{K}[Z] \) (Proposition 5.2). Let \( \delta \) be a normalization of \( \delta \) with minimal presentation \( \tilde{\delta} = \max_{i=1}^{N} \delta_i \). Proposition 5.2 implies that there exists \( i \) such that \( \delta_i = -\frac{1}{q} \nu \) for some positive integer \( d \). W.l.o.g. assume \( i = 1 \). Pick \( g_1 \in \mathbb{K}[Z] \setminus \{0\} \) such that \( \delta(g_1) = \delta_1(g_1) > \delta_i(g_1) \) for all \( i > 1 \). Finally, pick \( g_2, \ldots, g_s \in \mathbb{K}[Z] \setminus \{0\} \) such that \( (g_1)_{\delta g_1}, \ldots, (g_s)_{\delta g_s} \) generate \( \mathbb{K}[Z]^\delta \) as a \( \mathbb{K} \)-algebra. We claim that for all \( g \in \mathbb{K}[Z] \setminus \{0\} \),

\[
\delta_1(g) = \max \{ \delta'(g) : \delta' \text{ is a semidegree on } \mathbb{K}[Z], \delta'(g_i) = \delta_i(g_i) \text{ for } 1 \leq i \leq s \} \quad (**)
\]

Indeed, pick a semidegree \( \delta' \) on \( \mathbb{K}[Z] \) such that \( \delta'(g_i) = \delta_i(g_i) \) for all \( i, 1 \leq i \leq s \). Since \( \delta_1 \leq \delta \), it follows from the choice of \( g_i \)’s that \( \delta' \leq \delta \). Now pick any \( g \in \mathbb{K}[Z] \setminus \{0\} \). Then for all sufficiently large \( k \), we have that \( \delta_1(gg_1^k) = \delta(gg_1^k) \geq \delta'(gg_1^k) \), and consequently, \( \delta_1(g) \geq \delta'(g) \). This proves identity (**).

Let \( g_0 \in \mathbb{K}[Z] \setminus \{0\} \) such that \( Z \setminus V(g_0) \subseteq X \) and let \( f_1, \ldots, f_r \in \mathbb{K}[X] \setminus \{0\} \) such that for each \( i, 0 \leq i \leq s \), \( g_i = f_j / f_k \) for some \( j, k \). We show that \( f_1, \ldots, f_r \) satisfies the claim of the theorem. Indeed, pick any \( \nu' \) as in the right hand side of the identity (**') and an arbitrary \( f \in \mathbb{K}[X] \setminus \{0\} \). Then \( f = g/g_0^k \) for some \( k \geq 0 \) and \( g \in \mathbb{K}[Z] \). Since \( \nu'(g_j) = \nu(g_j) = -\frac{1}{q} \delta_i(g_j) \) for all \( j, 0 \leq j \leq s \), applying identity (**') to \( \delta' := -\frac{1}{q} \nu' \) yields that \( \nu(f) \leq \nu'(f) \), as required. \( \square \)

6 Divisor at infinity

Associated to every degree-like function \( \delta \) there is a canonical ample \( \mathbb{Q} \)-Cartier divisor supported at infinity on the corresponding completion. In this section we study this divisor for the case that \( \delta \) is a subdegree and establish a formula for its pull-back under a dominant morphism. As an application we compute the matrix of intersection numbers of the curves at infinity on a class of completions of certain affine surfaces. We also compute the nef cone of the latter completions and as a consequence give a positive answer to Question 1.9 in the case of surfaces under the additional hypothesis that each \( \delta_j \) is strictly positive on every non-constant regular function on \( X \) (Corollary 6.9).

6.1 Pull-back formula

Remark-Definition 6.1. Let \( X \) be an affine variety and \( \delta \) be a degree-like function on \( \mathbb{K}[X] \). The divisor at infinity \( D^\delta_{\infty} \) on \( X^\delta \) is the \( \mathbb{Q} \)-Cartier divisor defined by (1). \( D^\delta_{\infty} \) is Cartier if there is a subset \( S \) of \( \mathbb{K}[X]^\delta \) consisting of homogeneous elements of degree one such that the
radical of the ideal generated by $S$ is the irrelevant ideal of $\mathbb{K}[X]$. It follows from Remark 2.6 that $D^\delta_{\infty}$ is ample and supported at infinity, i.e. $\text{Supp }D^\delta_{\infty} = \bar{X}^\delta \setminus X =: X^\delta_{\infty}$.

**Notation.** Let $Z$ be an algebraic variety. For a Cartier divisor $D$ on $Z$ (resp. an irreducible codimension one subvariety $V$ of $Z$), we denote the corresponding Weil divisor by $[D]$ (resp. $[V]$). Moreover, given a $\mathbb{Q}$-Weil or $\mathbb{Q}$-Cartier divisor $D$ on $Z$, we write $D \geq 0$ if $D'$ is effective. For convenience of the reader we recall the definition of ‘effectiveness’ of $\mathbb{Q}$-Weil and $\mathbb{Q}$-Cartier divisors: if $D$ is a $\mathbb{Q}$-Weil divisor, then $D$ is effective iff for each irreducible codimension one subvariety $V$ of $Z$, the coefficient of $[V]$ in $D$ is zero. On the other hand, if $D$ is Cartier, then $D$ is said to be effective iff all the local equations of $D$ are regular functions. Finally, a $\mathbb{Q}$-Cartier divisor is effective iff it has a multiple which is effective and Cartier.

**Lemma 6.2.** Let $X$ be an irreducible affine variety, $\delta$ be a finitely generated subdegree on $\mathbb{K}[X]$, and $\delta_1, \ldots, \delta_N$ be the non-zero semidegrees associated to $\delta$. Then the $\mathbb{Q}$-Weil divisor associated to $D^\delta_{\infty}$ is

$$[D^\delta_{\infty}] = \sum_{j=1}^N \frac{1}{d_j} [V_j]$$

where for each $j$, $1 \leq j \leq N$, $d_j := \gcd\{\delta_j(g) : g \in \mathbb{K}(X), \delta_j(g) > 0\}$ and $V_j$ is the irreducible component of $X^\delta_{\infty}$ corresponding to $\delta_j$.

**Proof.** Fix $j$, $1 \leq j \leq N$ and $g \in \mathbb{K}[X]$ such that $d := \delta(g) = \delta_j(g)$. Let $U := \bar{X}^\delta \setminus V((g)_d)$. Assertion [1] of Lemma 3.1 implies that $U \cap V_j \neq \emptyset$. Recall that for each $f \in \mathbb{K}(X) \setminus \{0\}$, the order of vanishing of $f$ at $V_j$ is precisely $-\delta_j(f)/d_j$ (Proposition 5.2). Since $1/g$ is a local equation of $dD^\delta_{\infty}$ on $U$, it follows that the coefficient of $[V_j]$ in the expression for $[dD^\delta_{\infty}]$ is $-\frac{1}{d_j} \delta_j(1/g) = d/d_j$, as required to prove the lemma.

**Example 6.3.** Let $P$ be an $n$-dimensional convex rational polytope as in Example 3.6 and $X_P$ be the corresponding toric variety. We use the notation of Example 3.6. With $X := (\mathbb{K}^*)^n$ and $\delta := c \delta_P$, it follows that $[D^\delta_{\infty}] = \sum_{Q} \frac{c}{2^h} [X_Q]$, where the sum is over all facets $Q$ of $P$ and $X_Q$ are the components of $X_P \setminus X$ corresponding to $\delta_Q$ (see Example 4.5). In particular, $D^\delta_{\infty} = \frac{c}{2} D_P$, where $D_P$ is the divisor on $X_P$ corresponding to $P$ [Ful93, Section 3.4].

Next we establish the pull-back formula for the divisor at infinity. To state it we use from [Ful98, Section 1.2] the notion of order of vanishing $\text{ord}_W(f)$ of a non-zero rational function $f$ on an arbitrary irreducible variety $Z$ along a codimension one irreducible subvariety $W$.

**Proposition 6.4.** Let $X$ be an irreducible affine variety, $\delta$ be a finitely generated non-negative (non-zero) subdegree on $\mathbb{K}[X]$ and $\phi : Z \to \bar{X}^\delta$ be a dominant morphism from an irreducible variety $Z$. Then

$$[\phi^*(D^\delta_{\infty})] = \sum_W l^\phi_{\infty}(\delta, \text{pole}_W)[W],$$

(11)

where the sum is over codimension one irreducible subvarieties $W$ of $Z$, and for each such $W$, the function $\text{pole}_W$ is the negative of $\text{ord}_W$, and

$$l^\phi_{\infty}(\delta, \text{pole}_W) := \max \left\{ \frac{\text{pole}_W(\phi^*(f))}{\delta(f)} : f \in \mathbb{K}[X], \delta(f) > 0 \right\}.$$
Remark 6.5. Recall (from Remark-Definition 1.6) that \( t^0_{\infty}(\delta, \text{pole}_W) \) is the linking number at infinity (relative to \( \phi \)) of \( \delta \) and \( \text{pole}_W \).

Proof. Let \( \delta_1, \ldots, \delta_N \) be the non-zero semidegrees associated to \( \delta \). For each \( j, 1 \leq j \leq N \), let \( V_j \) be the component of the hypersurface at infinity on \( X^\delta \) corresponding to \( \delta_j \) and \( d_j := \gcd\{\delta_j(g) : g \in \mathbb{K}[X], \delta_j(g) > 0\} \). Pick \( f \in \mathbb{K}[X] \) such that \( \delta(f) > 0 \) and let \( \text{div}_{X^\delta}(f) \) be the principal Cartier divisor of \( f \) on \( X^\delta \). Then Proposition 5.2 implies that

\[
[\text{div}_{X^\delta}(f)] + \sum_{j=1}^{N} \frac{\delta_j(f)}{d_j} [V_j] \geq 0
\]

\[
\Rightarrow [\text{div}_{X^\delta}(f)] + \delta(f) \sum_{j=1}^{N} \frac{1}{d_j} [V_j] \geq 0 \quad \text{(since \( \delta(f) \geq \delta_j(f) \) for all \( j, 1 \leq j \leq N \))}
\]

\[
\Rightarrow \text{div}_{X^\delta}(f) + \delta(f) D^\delta_{\infty} \geq 0 \quad \text{(due to Lemma 6.2 and Claim 5.6.1).} \quad (12)
\]

Let \( W \) be an irreducible codimension one subvariety of \( Z \) and \( c_W \) be the coefficient of \( [W] \) in \( [\phi^*(D^\delta_{\infty})] \). Since the pull-back (under a dominant map) of an effective \( \mathbb{Q} \)-Cartier divisor is also effective, inequality (12) implies that \( \text{ord}_W(\phi^*(f)) + \delta(f)c_W \geq 0 \), or in other words, \( c_W \geq \text{pole}_W(\phi^*(f))/\delta(f) \). It follows that \( c_W \geq t^0_{\infty}(\delta, \text{pole}_W) \).

To complete the proof of the proposition, it suffices to show that there exists \( f \in \mathbb{K}[X] \) such that \( \delta(f) > 0 \) and \( c_W = \text{pole}_W(\phi^*(f))/\delta(f) \). We divide the proof into two cases:

Case 1: \( \phi(W) \subseteq X \). In this case \( c_W = 0 \), since \( \phi(W) \cap \text{Supp} D^\delta_{\infty} = \emptyset \). On the other hand, for all \( g \in \mathbb{K}[X] \), \( \phi^*(g) \) is regular on a neighbourhood of \( W \), so that \( \text{pole}_W(\phi^*(g)) \leq 0 \). Pick any \( g \in \mathbb{K}[X] \) such that \( \delta(g) > 0 \). Then setting \( f := g - \alpha \) for generic \( \alpha \in \mathbb{K} \) yields that \( \text{pole}_W(\phi^*(f))/\delta(f) = 0 \), as required.

Case 2: \( \phi(W) \not\subseteq X \). In this case there exists \( w \in W \) and \( j, 1 \leq j \leq N \), such that \( \phi(w) \in V_j \). Since \( D^\delta_{\infty} \) is ample, there exists \( d > 0 \) such that \( \mathcal{O}_{X^\delta}(dD^\delta_{\infty}) \) is globally generated. Note that the global sections of \( \mathcal{O}_{X^\delta}(dD^\delta_{\infty}) \) are precisely \( \{f \in \mathbb{K}[X] : \delta(f) \leq d\} \). In particular, there exists \( f \in \mathbb{K}[X] \) such that \( \delta(f) \leq d \) and \( fh \) is invertible near \( \phi(w) \), where \( h \) is a local equation for \( D^\delta_{\infty} \) near \( \phi(w) \). Then \( \delta_j(f) = -d_j \text{ord}_{V_j}(f) = d_j \text{ord}_{V_j}(h) = d \). Since \( \delta(f) \geq \delta_j(f) \), it follows that \( \delta(f) = d \); in particular, \( \delta(f) > 0 \). Moreover, since \( \phi^*(fh) \) is invertible near \( w \), it follows that \( 0 = \text{ord}_W(\phi^*(fh)) = \text{ord}_W(\phi^*(f)) + \text{ord}_W(\phi^*(h)) = -\text{pole}_W(\phi^*(f)) + dc_W \). Taken together, the preceding two sentences imply that \( c_W = \text{pole}_W(\phi^*(f))/\delta(f) \), as required.

6.2 Applications of the pull-back formula

Throughout this subsection we set \( X \) to be an affine surface such that the only invertible regular functions on \( X \) are (non-zero) constants (the most typical example being \( X = \mathbb{K}^2 \)). Let \( X_1, \ldots, X_k \) be non-isomorphic normal projective completions of \( X \) such that the complement \( C_j \) of \( X \) in each \( X_j \) is an irreducible curve and let \( \tilde{X} \) be the normalization of the closure in \( X_1 \times \cdots \times X_k \) of the image of \( X \) under the diagonal mapping. As the first application of Proposition 6.4, we compute the matrix of intersection numbers of the curves at infinity on \( \tilde{X} \).
Lemma 6.6. \( LI \)

Proof. \( \) Then each \( \delta \) implies that \( C \) that \( (\) Since intersection numbers are preserved by pull-backs under birational morphisms, it follows that \( (\) \)

intersection numbers (\( \) completions of \( K \) k

consequently is simplicial of dimension \( \delta \) for normal complete surfaces [Mum61]). Let \( \) be the torus invariant curve associated to \( e \)

to a toric surface \( \bar{X} \)

axes and the other edge is parallel to \( \)

Corollary 6.7. \( \)

I := (j)

be the

k

be the torus invariant curve associated to \( e \)

to a toric surface \( \bar{X} \)

axes and the other edge is parallel to \( e \). Then an application of Lemma 6.6 to \( \bar{X} \)

list the non-axis edges of \( P \)

ut yields:

1. one vertex at the origin,
2. two edges along the axes, and
3. the other edges being line segments with negative rational slopes.

List the non-axis edges of \( P \) counter-clockwise by \( e_1, \ldots, e_k \) and for each \( j, 1 \leq j \leq k \), let \( v_j := (v_{j1}, v_{j2}) \) be the smallest integral vector on the outward pointing normal to \( e_j \) and \( \bar{C}_j \) be the torus invariant curve associated to \( e_j \) on the toric surface \( X_P \) corresponding to \( P \). Let \( I \) be the \( k \times k \) matrix of intersection numbers \( (\bar{C}_i, \bar{C}_j) \) for \( 1 \leq i, j \leq k \). Each \( v_j \) corresponds to a toric surface \( \bar{X}_j \) corresponding to a triangle \( P_j \) which has two edges along the positive axes and the other edge is parallel to \( e_j \). Then an application of Lemma 6.6 to \( \bar{X}_1, \ldots, \bar{X}_k \) and \( \bar{X} := X_P \) yields:

Corollary 6.7.

\[
I = \begin{pmatrix}
1 & v_{12} & v_{13} & \cdots & v_{1k} \\
v_{21} & 1 & v_{23} & \cdots & v_{2k} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
v_{k1} & v_{k2} & v_{k3} & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
\]

Let \( X \) and \( \bar{X} \) be as in the first paragraph of this subsection. Lemma 6.6 implies that the cone \( \text{NE}(\bar{X}) \) of curves on \( \bar{X} \) is generated by (the equivalence classes of) \( \bar{C}_1, \ldots, \bar{C}_k \) and consequently is simplicial of dimension \( k \). The nef cone \( \text{Nef}(\bar{X}) \) of \( \bar{X} \) is dual to \( \text{NE}(\bar{X}) \) under the intersection product. We next give another description of \( \text{Nef}(\bar{X}) \).
Lemma 6.8. \( \text{Nef}(\overline{X}) = \{0\} \cup \{ \sum_{i=1}^{k} m_i[\overline{C}_i] : m_i > 0 \} \) and there exist non-constant regular functions \( f_i \) on \( X \) such that \( \delta_i(f_i)/m_i \geq \delta_j(f_j)/m_j \) for all \( i, j, 1 \leq i, j \leq k \).

Proof. Let \( C \) be the set of divisors on the right hand side of the equality to be proved. Since \( \delta \)'s are semidegrees, it follows that \( C \) is indeed a cone. At first we show that \( C \subseteq \text{Nef}(\overline{X}) \). Pick positive integers \( m_1, \ldots, m_k \) such that \( D := \sum_{j=1}^{k} m_j[\overline{C}_j] \in C \). Fix \( i, 1 \leq i \leq k \), and \( f_i \in \mathbb{K}[X] \setminus \mathbb{K} \) such that \( \delta_i(f_i)/m_i \geq \delta_j(f_j)/m_j \) for all \( j, 1 \leq j \leq k \). The coefficient of \( [\overline{C}_j] \) in \( D_i := \delta_i(f_i)D + \text{div}_X(f_i^{m_i}) \) is zero for \( j = i \) and non-negative for \( j \neq i \), \( 1 \leq j \leq k \). It follows that \( (D, \overline{C}_i) = (D_i, \overline{C}_i)/\delta_i(f_i) \geq 0 \) and therefore \( D \) is nef, as required.

For the opposite inclusion, pick positive rational numbers \( m_1, \ldots, m_k \) such that \( D := \sum_{j=1}^{k} m_j[\overline{C}_j] \) is in the interior of \( \text{Nef}(\overline{X}) \). Then \( D \) is ample, and therefore, replacing \( D \) by some of its multiple if necessary we may assume that each \( m_i \) is an integer and \( D \) is very ample. Since \( O_{\overline{X}}(D) \) is globally generated, for each \( i, 1 \leq i \leq k \), there exists \( f_i \in \mathbb{K}[X] \) such that \( D_i := [\text{div}_X(f_i)] + D \geq 0 \) and the coefficient of \( [\overline{C}_j] \) in \( D_i \) is zero. Then \( \delta_i(f_i) = m_i \) and \( \delta_j(f_i) \leq m_j \) for all \( j \neq i \), so that the interior of \( \text{Nef}(\overline{X}) \) is contained in \( C \). Since \( C \) is closed, we see that \( \text{Nef}(\overline{X}) \subseteq C \).

Using Lemma 6.8 we give a positive answer to Question 1.9 in a special case. Recall that a degree-like function \( \delta \) on a \( \mathbb{K} \)-algebra is called projective if \( \delta \) is finitely generated and \( \delta(f) > 0 \) for all \( f \in A \setminus \mathbb{K} \).

Corollary 6.9. Let \( X \) be an affine surface and \( \tilde{\delta}_1, \ldots, \tilde{\delta}_k \) be projective semidegrees on \( \mathbb{K}[X] \). Then \( \delta := \max\{\tilde{\delta}_1, \ldots, \tilde{\delta}_k\} \) is finitely generated (and therefore also projective).

Remark 6.10. Not all affine varieties \( X \) admit projective semidegrees on \( \mathbb{K}[X] \); a necessary condition for this to hold is the non-existence of any invertible regular functions on \( X \) other than (non-zero) constants.

Proof. W.l.o.g. we may assume that \( \tilde{\delta} = \max\{\tilde{\delta}_1, \ldots, \tilde{\delta}_k\} \) is the minimal presentation of \( \tilde{\delta} \). At first assume that \( X \) is normal and let \( \overline{X} \) be the normalization of the closure in \( \overline{X}_1 \times \cdots \times \overline{X}_k \) of the image of \( X \) under the diagonal mapping. For each \( j, 1 \leq j \leq k \), define \( d_j := \gcd\{\tilde{\delta}(f) : f \in \mathbb{K}[X]\} \), \( \delta_j := \tilde{\delta}_j/d_j \) and \( m_j := 1/d_j \). Lemma 6.8 then implies that \( D := \sum_{j=1}^{k} m_j[\overline{C}_j] \) is in the interior of the nef cone of \( \overline{X} \), and therefore \( D \) is ample. Since \( D = D_{\infty}^\delta \), it follows that \( \delta \) is finitely generated, as required to prove the corollary in the case that \( X \) is normal. The general case follows (due to Remark 6.10) via applying the corollary to the normalization of \( X \).

References

[AM69] M. F. Atiyah and I. G. Macdonald. Introduction to commutative algebra. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. Toric varieties, volume 124 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2011.

[dFEI08] Tommaso de Fernex, Lawrence Ein, and Shihoko Ishii. Divisorial valuations via arcs. Publ. Res. Inst. Math. Sci., 44(2):425–448, 2008.
[Ful93] William Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

[Ful98] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[GKZ94] I. M. Gel’fand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants, and multidimensional determinants*. Mathematics: Theory & Applications. Birkhäuser Boston Inc., Boston, MA, 1994.

[Goo69] Jacob Eli Goodman. Affine open subsets of algebraic varieties and ample divisors. *Ann. of Math. (2)*, 89:160–183, 1969.

[Huc70] James A. Huckaba. Some results on pseudo valuations. *Duke Math. J.*, 37:1–9, 1970.

[Kho] A. G. Khovanskii. Private communications. April, 2009.

[mat] http://mathoverflow.net/questions/37082/appropriate-journal-to-publish-a-det}\textdollar$\text{erminantal-inequality}.

[Mat80] Hideyuki Matsumura. *Commutative algebra*, volume 56 of *Mathematics Lecture Note Series*. Benjamin/Cummings Publishing Co., Inc., Reading, Mass., second edition, 1980.

[McA83] Stephen McAdam. *Asymptotic prime divisors*, volume 1023 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.

[Mon10] Pinaki Mondal. *Towards a Bezout-type theory of affine varieties*. PhD thesis, University of Toronto, 2010.

[Mon11] Pinaki Mondal. Primitive normal compactifications of the affine plane II. arXiv: 1100.0014, 2011.

[Mum61] David Mumford. The topology of normal singularities of an algebraic surface and a criterion for simplicity. *Inst. Hautes Études Sci. Publ. Math.*, (9):5–22, 1961.

[ove] http://mathoverflow.net/questions/78305/ample-divisors-on-projective-surfaces (see Jason Starr’s comments on the question).

[Ree56a] D. Rees. Valuations associated with ideals. *Proc. London Math. Soc. (3)*, 6:161–174, 1956.

[Ree56b] D. Rees. Valuations associated with ideals. II. *J. London Math. Soc.*, 31:221–228, 1956.

[Sam59] Pierre Samuel. Multiplicités de certaines composantes singulières. *Illinois J. Math.*, 3:319–327, 1959.
[Sha94] Igor R. Shafarevich. *Basic algebraic geometry. 1*. Springer-Verlag, Berlin, second edition, 1994. Varieties in projective space, Translated from the 1988 Russian edition and with notes by Miles Reid.

[Szp66] Lucien Szpiro. Fonctions d’ordre et valuations. *Bull. Soc. Math. France*, 94:301–311, 1966.

[ZS75] Oscar Zariski and Pierre Samuel. *Commutative algebra. Vol. II*. Springer-Verlag, New York, 1975. Reprint of the 1960 edition, Graduate Texts in Mathematics, Vol. 29.