Bounded composition operator on Lorentz spaces

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Abstract. We study a composition operator on Lorentz spaces. In particular we provide necessary and sufficient conditions under which a measurable mapping induces a bounded composition operator.

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1. Introduction

Lorentz spaces $L_{p,q}$ are a generalization of ordinary Lebesgue spaces $L_p$, and they coincide with $L_p$ when $q = p$. Some references as to basics on Lorentz spaces may be found in [1, 2, 3].

A composition operator induced by map $\varphi$ on some function space is quite a natural object which is defined as $C_{\varphi}f = f \circ \varphi$. Depending on the structure of particular function space various properties of a composition operator are under interest e.g. boundedness, compactness, inevitability and so on. The study of composition operators may be divided into three directions.

The first one could be referred to as classical and goes back to Littlewood’s Subordination Principle (1925). This principle states that a holomorphic self-mapping of the unit disk $D \subset \mathbb{C}$ preserving 0 induces a contractive composition operator on Hardy space $H^p(D)$, as well on Bergman and Dirichlet spaces. However, it is believed that the systematic study of composition operators induced by holomorphic maps started with the paper [4] by E. A. Nordgren in the mid 1960’s. Afterwards the study of composition operator developed at the juncture of analytic function theory and operator theory. We refer the reader to book [5] by J. Shapiro.
The second direction has a more operator flavor. Researchers raised all the questions about composition operators which could be posed regarding operators on normed spaces. One may find an exhaustive survey on the topic in the book [6] by R. K. Singh, J. S. Manhas and also in the proceedings [7]. The survey on composition operators on Sobolev spaces was motivated by the question, what change of variables does preserve a Sobolev class? Therefore the research was primarily focused on analytic and geometric properties of mappings, whereas operator theory was involved to a lesser extent. The first results in this area are due to S. L. Sobolev (1941), V. G. Maz’ya (1961), F. W. Gehring (1971). Subsequently S. K. Vodopjanov and V. M. Goldštein (1975-76) studied a lattice isomorphism on Sobolev spaces. Later on many more mathematicians contributed to this research, see details in [8, 9] and recent results on the subject in [10, 11]. We also mention here recent paper [12] on composition operator on Sobolev-Lorentz space.

Our work belongs to the second of the described directions. As of right now composition operators on $L_p$ have been investigated thoroughly enough (see [13, 6, 14]). In the case of Lorentz spaces most of the research has been concerned with composition operators from $L_{p,q}$ to $L_{p,q}$, domain and image spaces having the same parameters (see [15, 16, 17]). Here we initiate the study of a composition operator from $L_{r,s}$ to $L_{p,q}$, where the parameters may differ. The principal result of the paper is as follows.

**Theorem 1.1.** A measurable mapping $\varphi : X \to Y$ satisfying $\mathcal{N}^{-1}$-property induces a bounded composition operator

$$C_\varphi : L_{r,s}(Y) \to L_{p,q}(X), \quad s \leq q$$

if and only if

$$\int_B J_{\varphi^{-1}}(y) \, d\nu(y) \leq K^p(\nu(B))^\frac{1}{p}$$

for some constant $K < \infty$ and any measurable set $B$.

We prove the theorem above in section 3 while the range of composition operator and the case when composition operator is an isomorphism are studied in sections 4 and 5.

2. Lorentz spaces

Let $(X, \mathcal{A}, \mu)$ be a $\sigma$-finite measurable space. The Lorentz space $L_{p,q}(X, \mathcal{A}, \mu)$ is the set of all measurable functions $f : X \to \mathbb{C}$ for which

$$\|f\|_{p,q} = \left( \frac{q}{p} \int_0^{\infty} \left( t^{\frac{1}{q}} f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty, \quad \text{if} \ 1 < p < \infty, \ 1 \leq q < \infty,$$

or

$$\|f\|_{p,\infty} = \sup_{t>0} t^{\frac{1}{p}} f^*(t) < \infty, \quad \text{if} \ 1 < p < \infty, \ q = \infty.$$
The non-increasing rearrangement $f^*(t)$ of a function $f(x)$ is defined as

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\},$$

where

$$\mu_f(\lambda) = \mu\{x \in X : |f(x)| > \lambda\}$$

is the distribution function of $f(x)$.

Note that $\| \cdot \|_{p,q}$ is a norm if $1 < q \leq p$ and a quasi-norm if $p < q$. We will refer to $\| \cdot \|_{p,q}$ as the Lorentz norm. For brevity we will use $L_{p,q}(X)$ instead of $L_{p,q}(X, A, \mu)$.

In what follows we will need the next properties of Lorentz spaces.

**Lemma 2.1** ([18, Proposition 2.1.]). The Lorentz norm can be computed via distribution:

$$\|f\|_{p,q} = \left(\frac{q}{p} \int_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} = \left(\frac{q}{p} \int_0^\infty \left(\lambda \mu_f(\lambda)^{\frac{1}{p}}\right)^q \frac{d\lambda}{\lambda}\right)^{\frac{1}{q}} \quad (2.1)$$

and

$$\|f\|_{p,\infty} = \sup_{\lambda > 0} \lambda^{\frac{1}{p}} (\mu_f(\lambda))^{\frac{1}{p}}. \quad (2.2)$$

**Lemma 2.2** ([1, equation (2.10)]). Let $E \subset X$ be a measurable set. The Lorentz norm of its indicator is

$$\|\chi_E\|_{p,q} = (\mu(E))^{\frac{1}{p}}. \quad (2.3)$$

**Proof.** Observe that $\mu_{\chi_E}(\lambda) = \mu(E) \cdot \chi_{(0,1)}(\lambda)$. If $q < \infty$ we apply formula (2.1):

$$\|\chi_E\|_{p,q} = \left(q \int_0^1 \left(\lambda \mu(E)^{\frac{1}{p}}\right)^q \frac{d\lambda}{\lambda}\right)^{\frac{1}{q}} = (\mu(E))^{\frac{1}{p}}.$$

If $q = \infty$, we infer from (2.2) that

$$\|\chi_E\|_{p,\infty} = \sup_{0 < t < 1} t \cdot (\mu(E))^{\frac{1}{p}} = (\mu(E))^{\frac{1}{p}}.$$

\[\square\]

**Theorem 2.3** ([1, Theorem 3.11]). Suppose that $f \in L_{p,q_1}$ and $q_1 \leq q_2$, then

$$\|f\|_{p,q_2} \leq \|f\|_{p,q_1}.$$

### 3. Composition operator

Let $(X, A, \mu)$ and $(Y, B, \nu)$ be $\sigma$-finite measurable spaces and $\varphi : X \to Y$ be a measurable mapping.

**Lemma 3.1.** Let $s \leq q$ and $f \circ \varphi \in L_{p,q}(X)$ for all $f \in L_{r,s}(Y)$, then the following two statements are equivalent

1. $\|f \circ \varphi\|_{p,q} \leq K \|f\|_{r,s}$ for any $f \in L_{r,s}(Y)$;
2. $(\mu(\varphi^{-1}(B)))^{\frac{1}{r}} \leq K(\nu(B))^{\frac{1}{r}}$ for any set $B \in B$. 

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Proof. Let $B \in \mathcal{B}$ and $\nu(B) < \infty$. Plugging the indicator function $\chi_B(y)$ into statement 1 and using property (2.3), we obtain 2. If $\nu(B) = \infty$ the claim is trivial.

Suppose now that statement 2 holds. Let $f \in L_{r,s}(Y)$. First we find the expression for the distribution of the composition $f \circ \varphi$:

$$\mu_{f \circ \varphi}(\lambda) = \mu(\{x \in X : |f(\varphi(x))| > \lambda\}) = \mu(\varphi^{-1}(\{y \in Y : |f(y)| > \lambda\})).$$

Denote $E_\lambda = \{y \in Y : |f(y)| > \lambda\}$, then $\nu_f(\lambda) = \nu(E_\lambda)$ and $\mu_{f \circ \varphi}(\lambda) = \mu(\varphi^{-1}(E_\lambda)).$ From the inequality of statement 2 deduce

$$\left(\mu(\varphi^{-1}(E_\lambda))\right)^{\frac{1}{p}} \leq K \left(\nu(E_\lambda)\right)^{\frac{1}{r}}$$

and thus

$$\left(\mu_{f \circ \varphi}(\lambda)\right)^{\frac{1}{p}} \leq K \left(\nu_f(\lambda)\right)^{\frac{1}{r}}.$$

Consequently,

$$\|f \circ \varphi\|_{p,q} = \left(q \int_0^\infty \left(\frac{\lambda(\mu_{f \circ \varphi}(\lambda))^{\frac{1}{p}}}{\lambda}\right)^q \frac{d\lambda}{\lambda}\right)^{\frac{1}{q}}$$

$$\leq \left(q \int_0^\infty \left(\lambda K \left(\nu_f(\lambda)\right)^{\frac{1}{r}}\right)^q \frac{d\lambda}{\lambda}\right)^{\frac{1}{q}} = K\|f\|_{r,q} \leq K\|f\|_{r,s}$$

if $s < \infty$, and

$$\|f \circ \varphi\|_{p,\infty} = \sup_{\lambda>0} \lambda(\mu_{f \circ \varphi}(\lambda))^{\frac{1}{p}} \leq K \sup_{\lambda>0} \lambda \left(\nu_f(\lambda)\right)^{\frac{1}{r}} = K\|f\|_{r,\infty}$$

as desired. $\square$

Assuming $p = r, q = s$ obtain [15, Theorem 1] and [16, Theorem 2.1] as consequences of lemma 3.1.

**Definition 3.2.** A mapping $\varphi$ induces a composition operator on Lorentz spaces

$$C_\varphi : L_{r,s}(Y) \to L_{p,q}(X) \quad \text{by the rule} \quad C_\varphi f = f \circ \varphi$$

(3.1)

whenever $f \circ \varphi \in L_{p,q}(X)$.

Clearly that $C_\varphi$ is a linear operator between two vector spaces.

A composition operator $C_\varphi$ is bounded if

$$\|C_\varphi f\|_{p,q} \leq K\|f\|_{r,s}$$

(3.2)

for every function $f \in L_{r,s}(Y)$, the constant $K$ being independent of the choice of $f$.

Similarly, $C_\varphi$ is bounded below if

$$\|C_\varphi f\|_{p,q} \geq k\|f\|_{r,s}.$$ 

(3.3)

**Corollary 3.3.** If a measurable mapping $\varphi$ induces a bounded composition operator, then $\varphi$ enjoys Luzin $N^{-1}$-property (which means that $\mu(\varphi^{-1}(S)) = 0$ whenever $\nu(S) = 0$).
In particular, corollary 3.3 guarantees that if functions $f_1, f_2$ coincide a.e. on $Y$ then the images $C_\varphi f_1(x), C_\varphi f_2(x)$ coincide a.e. on $X$. On the other hand the a priori assumption of $\mathcal{N}^{-1}$-property enables us to consider (3.1) as an operator on equivalence classes.

Suppose we are given a measurable mapping $\varphi : X \to Y$ satisfying Luzin $\mathcal{N}^{-1}$-property. Then the measure $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to $\nu$. Thus the Radon–Nikodym theorem guarantees the existence of a measurable function $J_{\varphi^{-1}}(y)$ (the Radon–Nikodym derivative) such that

$$
\mu(\varphi^{-1}(E)) = \int_E J_{\varphi^{-1}}(y) \, d\nu(y).
$$

(3.4)

On account of (3.4), theorem 1.1 follows immediately from lemma 3.1.

**Example.** Let $X$ and $Y$ be subsets of $\mathbb{R}^n$ with Lebesgue measure $|\cdot|$. Consider a mapping $\varphi : X \to Y$ such that the Jacobian is bounded $J(x, \varphi) < M < \infty$ and the Banach indicatrix $^4$ is bounded $N(y, \varphi, X) < N$ as well. Therefore

$$
\frac{N}{M} < J_{\varphi^{-1}}(y).
$$

Suppose that $\varphi$ induces a bounded operator from $L_{r,s}(Y)$ to $L_{p,q}(X)$ then by theorem 1.1 and by the inequality above we obtain

$$
0 < \frac{N}{M} |B| < \int_B J_{\varphi^{-1}}(y) \, dy \leq K^p |B|^p
$$

and

$$
\frac{N}{MK^p} < |B|^p.
$$

If we take a sequence of sets $B_k$ such that $|B_k| \to 0$ we will derive the necessary condition $p \leq r$, which is usually taken for granted.

**Example.** Now let $X, Y \subset \mathbb{R}^2$. Examine a mapping $\varphi : X \to Y$ such that $\varphi(x_1, x_2) = (\frac{n}{2}, \frac{m}{2})$, where $n - 1 < x_1 < n$, $m - 1 < x_2 < m$, $n, m \in \mathbb{Z}$. Let $\mu$ be the Lebesgue measure on $\mathbb{R}^2$ while $\nu$ be a discrete measure with atoms in $((\frac{n}{2}, \frac{m}{2}))$, $n, m \in \mathbb{Z}$ and for the sake of simplicity we set $\nu((\frac{n}{2}, \frac{m}{2})) = 1$. Then $J_{\varphi^{-1}}(y) = 1$. In this case the mapping $\varphi$ could induce a bounded composition operator from $L_{r,s}(Y, B, \nu)$ to $L_{p,q}(X, A, \mu)$, even if $r < p$.

### 4. Properties of the image

In this section we exploit ideas from [16] to investigate the range of a composition operator. First we show that $J_{\varphi^{-1}}(y)$ may be assumed to be positive a.e. on $Y$. Let

$$
Z = \{y \in Y : J_{\varphi^{-1}}(y) = 0\},
$$

$^4 N(y, f, X) = \#\{x \in X \mid f(x) = y\}$ is the number of elements of $f^{-1}(y)$ in $X$. 
then
\[ \mu(\varphi^{-1}(Z)) = \int_Z J_{\varphi^{-1}}(y) \, d\nu(y) = 0. \]

Thus, after redefining the map \( \varphi \) on the set \( \mu(\varphi^{-1}(Z)) \) of measure zero we obtain the property \( J_{\varphi^{-1}}(y) > 0 \) a.e on \( Y \).

**Theorem 4.1.** A measurable mapping \( \varphi \) satisfying \( N^{-1} \)-property induces a bounded below composition operator
\[ C_{\varphi} : L_{r,s}(Y) \rightarrow L_{p,q}(X), \quad s \geq q \]
if and only if
\[ \int_B J_{\varphi^{-1}}(y) \, d\nu(y) \geq k^p(\nu(B))^\frac{1}{p} \] (4.1)
for any \( B \in \mathcal{B} \).

**Proof.** Applying (3.3) to the indicator function \( \chi_B(y) \) and using (2.3), (3.4) we derive
\[ \left( \int_B J_{\varphi^{-1}}(y) \, d\nu(y) \right)^{\frac{1}{p}} = k(\nu(B))^\frac{1}{p}. \]

Suppose now (4.1) holds. Then in view of (3.4)
\[ \mu_{f \circ \varphi}(\lambda) = \int_Y \chi_{E_{\lambda}}(y) J_{\varphi^{-1}}(y) \, d\nu(y) \geq k^p(\nu(E_{\lambda}))^\frac{1}{p} = k^p(\nu_f(\lambda))^\frac{1}{p}. \]
Thus \( (\mu_{f \circ \varphi}(\lambda))^\frac{1}{p} \geq k(\nu_f(\lambda))^\frac{1}{p} \) and
\[ \|C_{\varphi} f\|_{p,q} \geq k\|f\|_{r,q} \geq k\|f\|_{r,s}. \]

Let \( s = q \). Making use of the well known fact from functional analysis, which says that a linear bounded operator between Banach spaces is bounded below if and only if it is one-to-one and has closed range, we arrive to the following assertion.

**Theorem 4.2.** A bounded composition operator \( C_{\varphi} : L_{r,s}(Y) \rightarrow L_{p,s}(X) \) is injective and has the closed image if and only if there is a constant \( k > 0 \) such that
\[ \int_B J_{\varphi^{-1}}(y) \, d\nu(y) \geq k^p(\nu(B))^\frac{1}{p} \]
for any \( B \in \mathcal{B} \).

Next we discuss where a bounded composition operator has dense image.

**Theorem 4.3.** The image of a bounded composition operator \( C_{\varphi} : L_{r,s}(Y, B, \nu) \rightarrow L_{p,q}(X, A, \mu) \) is dense in \( L_{p,q}(X, \varphi^{-1}(B), \mu) \).
Proof. Let \( \chi_A \in L_{p,q}(X, \varphi^{-1}(B), \mu) \) be the indicator function of a set \( A = \varphi^{-1}(B), B \in \mathcal{B} \). It is easy to see that \( \chi_A(x) = \chi_B(\varphi(x)) \), though we cannot ensure \( \chi_B(y) \in L_{r,s}(Y, \mathcal{B}, \nu) \).

Let \( B = \bigcup B_k \), where \( \{B_k\} \) is an increasing sequence of sets of finite measure. Then \( \chi_{B_k}(y) \in L_{r,s}(Y, \mathcal{B}, \nu) \). Denote \( f_k = C_{\varphi} \chi_{B_k} \). Obviously \( f_k(x) \leq \chi_A(x) \) and \( f_k(x) \to \chi_A(x) \) as \( k \to \infty \) a.e. on \( X \). The similar inequality and convergence take place for distributions \((\mu_{f_k} \text{ and } \mu_{\chi_A})\), therefore from the Lebesgue theorem \( f_k(x) \to \chi_A(x) \) in \( L_{p,q}(X) \). The same arguments work for simple functions.

It follows that every simple function from \( L_{p,q}(X, \varphi^{-1}(B), \mu) \) is the limit of images. Since the set of simple functions is dense in \( L_{p,q}(X) \) we conclude that the image \( C_{\varphi}(L_{r,s}(Y, \mathcal{B}, \nu)) \) is dense in \( L_{p,q}(X, \varphi^{-1}(B), \mu) \).

\[ (5.2) \text{ is a straightforward consequence of (5.1).} \]

5. Isomorphism

We will say that a mapping \( \varphi : X \to Y \) induces an isomorphism of Lorentz spaces \( L_{p,q}(Y, \mathcal{B}, \nu), L_{p,q}(X, \mathcal{A}, \mu) \) whenever the composition operator \( C_{\varphi} \) is bijective and the inequalities

\[ \|f\|_{p,q} \leq \|C_{\varphi}f\|_{p,q} \leq K\|f\|_{p,q} \quad (5.1) \]

hold for every function \( f \in L_{p,q}(Y, \mathcal{B}, \nu) \) and for some constants \( 0 < k \leq K < \infty \) independent of the choice of \( f \).

**Theorem 5.1.** A measurable mapping satisfying \( N^{-1} \)-property induces an isomorphism of Lorentz spaces

\[ C_{\varphi} : L_{p,q}(Y, \mathcal{B}, \nu) \to L_{p,q}(X, \mathcal{A}, \mu) \]

if and only if

\[ k^p \leq J_{\varphi^{-1}}(y) \leq K^p \quad a.e. \ y \in Y \quad (5.2) \]

and \( \varphi^{-1}(\mathcal{B}) = \mathcal{A} \).

**Proof.** Let \( \varphi \) induce an isomorphism. Thanks to theorems 1.1, 4.1, inequalities (5.2) are a straightforward consequence of (5.1).

Show that \( \varphi^{-1}(\mathcal{B}) = \mathcal{A} \). Let \( A \in \mathcal{A} \) and \( \mu(A) < \infty \), then the indicator function \( \chi_A \in L_{p,q}(X, \mathcal{A}, \mu) \). Because of the surjectivity there is a function \( f \in L_{p,q}(Y, \mathcal{B}, \nu) \) such that \( \chi_A = C_{\varphi}f \). Observe that the set \( B = \{y \in Y : f(y) = 1\} \) is an element of \( \mathcal{B} \) and \( f = \chi_B \). This yields \( \chi_A = C_{\varphi} \chi_B = \chi_{\varphi^{-1}(B)} \) and hence \( A = \varphi^{-1}(B) \). Thus \( \mathcal{A} = \varphi^{-1}(\mathcal{B}) \).

Now assume that \( \mathcal{A} = \varphi^{-1}(\mathcal{B}) \) and (5.2) holds. Again (5.1) is equivalent to (5.2) owing to theorems 1.1, 4.1. From theorem 4.2 we infer that the operator \( C_{\varphi} \) is one-to-one and the image \( C_{\varphi}(L_{p,q}(Y, \mathcal{B}, \nu)) \) is closed, whereas theorem 4.3 implies the density of the image in \( L_{p,q}(X, \mathcal{A}, \mu) \). Consequently \( C_{\varphi}(L_{p,q}(Y, \mathcal{B}, \nu)) = L_{p,q}(X, \mathcal{A}, \mu) \). \( \square \)
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